PERSISTENCE AND NIP IN THE CHARACTERISTIC SEQUENCE

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Abstract. For a first-order formula \( \varphi(x; y) \) we introduce and study the characteristic sequence \( \langle P_n : n < \omega \rangle \) of hypergraphs defined by \( P_n(y_1, \ldots, y_n) := (\exists x) \bigwedge_{i \leq n} \varphi(x; y_i) \). We show that combinatorial and classification theoretic properties of the characteristic sequence reflect classification theoretic properties of \( \varphi \) and vice versa. Specifically, we show that some tree properties are detected by the presence of certain combinatorial configurations in the characteristic sequence while other properties such as instability and the independence property manifest themselves in the persistence of complicated configurations under localization.

1. Introduction

This article defines and develops the theory of characteristic sequences. The characteristic sequence \( \langle P_n : n < \omega \rangle \) associated to a first-order formula \( \varphi \) is a countable sequence of hypergraphs defined on the parameter space of \( \varphi \); this association allows for a new description of the combinatorial complexity of \( \varphi \)-types in terms of graph-theoretic complexity of the hypergraphs \( P_n \). The construction arose from work of the author on saturation of ultrapowers, as described briefly below, but is of independent interest. There is a model-theoretic sensibility throughout, but many of the arguments are combinatorial. In fact, if the reader is familiar with basic model theory and is willing to take on faith the interest of certain classification-theoretic dividing lines, the article is largely self-contained.

At first glance, the characteristic sequence gives a transparent language for the kinds of arguments which occur in many contexts where the fine structure of dividing or thorn-dividing is being analyzed. However, there is a power in our general framework which accrues from the fact that the formulas \( P_n \) are simultaneously:

1. graphs, so we can ask about their complexity in the sense of graph theory;
2. formulas definable in the background theory \( T \), so we can ask about their complexity in the sense of classification theory;
3. descriptors of the parameter space of the given formula \( \varphi \).

Leveraging these three contexts against each other puts strong restrictions on the behavior of the \( P_n \). We obtain, for instance, a description of NIP theories as theories in which any initial segment of the characteristic sequence is, after localization, essentially trivial: see §6 below. In some sense, then, the characteristic sequence is a tool for analyzing the fine structure of the independence property and gives a natural description of its complexity.

By way of describing the kinds of complexity we consider, let us briefly mention two background motivations for this work. The first is a deep question of Keisler about the structure of a preorder on countable theories which compares the difficulty of producing saturated regular ultrapowers [4]. Shelah described the structure of this so-called Keisler order on NIP theories in a series of surprising results, collected in [8] chapter VI, but its structure on theories with the independence property remains open. In [6], [7] we showed
that the structure of Keisler’s order on unstable theories (thus, on theories with the independence property) depends on a classification of $\varphi$-types, and specifically on an analysis of characteristic sequences.

The second motivation is the development of a new language for interactions between model theory and graph theory. An interest in the complexity of $\varphi$-types asks how the many finite fragments of configurations cluster in the characteristic sequence and how uniformly or regularly they are distributed. These are issues which graph theory is particularly articulate at describing. Insofar as properties like edge density and edge distribution and structural properties of the hypergraphs $P_n$ can be shown to have model-theoretic content, this opens up the possibility of using a deep collection of structure theorems for graphs to give model-theoretic information [7].

The organization of the article is as follows. Section 2 contains definitions and basic properties. Section 3 gives several motivating examples. Section 4 gives a series of “static” arguments relating configurations in the characteristic sequence to classification-theoretic dividing lines. Section 4 begins the work of separating out inessential complexity from the base set of a type under analysis via localization; persistence and its associated “dynamic” arguments are motivated and defined. Section 6 describes NIP, simplicity and stability in terms of persistence.

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2. The characteristic sequence

Definition 2.1. (Notation and conventions, I)

1. Throughout this article, if a variable or a tuple is written $x$ or a rather than $\overline{x}, \overline{a}$, this does not necessarily imply that $\ell(x), \ell(a) = 1$.

2. Unless otherwise stated, $T$ is a complete theory in the language $\mathcal{L}$.

3. A graph in which no two elements are connected is called an empty graph. A pair of elements which are not connected is an empty pair. When $R$ is an $n$-ary edge relation, to say that some $X$ is an $R$-empty graph means that $R$ does not hold on any $n$-tuple of elements of $X$. $X$ is an $R$-complete graph if $R$ holds on every $n$-tuple from $X$.

4. Important: $\varphi_n(x; y_1, \ldots , y_n)$ denotes the formula $\bigwedge_{i \leq n} \varphi(x; y_i)$.

5. In discussing graphs we will typically write concatenation for union, i.e. $A \cup \{c\}$.

6. A formula $\psi(x; y)$ of $\mathcal{L}$ will be called dividable if there exists an infinite set $C \subset P_1$ and $k < \omega$ such that $\{\psi(x; c) : c \in C\}$ is 1-consistent but $k$-inconsistent. (Thus, by compactness, some instance of $\psi$ divides.) When it is important to specify the arity $k$, write $k$-dividable.

7. A set is $k$-consistent if every $k$-element subset is consistent, and it is $k$-inconsistent if every $k$-element subset is inconsistent.

To each formula $\varphi$ we associate a countable sequence of hypergraphs, the characteristic sequence, which describe incidence relations on the parameter space of $\varphi$. The idea is to give
an analysis of \( \varphi \)-types by describing the way that certain distinguished sets \( A \) (the complete \( P_\infty \)-graphs, avatars of consistent partial \( \varphi \)-types) sit inside the ambient hypergraphs \( P_n \).

**Definition 2.2.** (Characteristic sequences) Let \( T \) be a first-order theory and \( \varphi \) a formula of the language of \( T \).

- For \( n < \omega \), \( P_n(z_1, \ldots z_n) := \exists x \wedge_{i \leq n} \varphi(x; z_i) \).
- The characteristic sequence of \( \varphi \) in \( T \) is \( \langle P_n : n < \omega \rangle \).
- Write \( (T, \varphi) \mapsto \langle P_n \rangle \) for this association.
- Convention: we assume that \( \exists \exists y \forall x (\varphi(x; z) \iff \neg \varphi(x; y)) \). If this does not already hold for some given \( \varphi \), replace \( \varphi \) with \( \theta(x; y, z) = \varphi(x; y) \land \neg \varphi(x; z) \).

**Convention 2.3.** Below, we will ask a series of questions about whether certain, possibly infinite, configurations appear as subgraphs of the \( P_n \), or of the \( P_n' \) in some finite localization, Definition 5.1 For our purposes, the existence of these configurations is a property of \( T \). That is, we may, as a way of speaking, ask if some configuration \( X \) appears, or is persistent, inside of some \( P_n \); however, we will always mean whether or not it is consistent with \( T \) that there are witnesses to \( X \) inside of \( P_n \) interpreted in some sufficiently saturated model. Certainly, one could ask the question of whether some given model of \( T \), expanded to model of the \( P_n \), must include witnesses to \( X \); we will not do so here. Thus, the formulas \( P_n \) will often w.l.o.g. be identified with their interpretations in some monster model.

**Definition 2.4.** (Notation and conventions, II)

1. \( P_\infty \) will be shorthand for the collection of predicates \( P_n \) when the context (of a given condition, not necessarily definable, which holds of \( P_n \) for all \( n \)) is clear, e.g. \( A \) is a \( P_\infty \)-complete graph meaning \( A \) is a \( P_n \)-complete graph for all \( n \).
2. The complete \( P_\infty \)-graph \( A \) will be called a positive base set when the emphasis is on its identification with some consistent partial \( \varphi \)-type under analysis, as described in Observation 2.5(5).
3. The sequence \( \langle P_n \rangle \) has support \( k \) if: \( P_n(y_1, \ldots y_n) \) iff \( P_k \) holds on every \( k \)-element subset of \( \{y_1, \ldots y_n\} \). See Remark 2.6.
4. The element \( a \in P_1 \) is a one-point extension of the \( P_n \)-complete graph \( A \) just in case \( Aa \) is also a \( P_n \)-complete graph. In most cases, \( n \) will be \( \infty \).

**Observation 2.5.** (Basic properties) Let \( \langle P_n : n < \omega \rangle \) be the characteristic sequence of \( (T, \varphi) \). Then, regardless of the choice of \( T \) and \( \varphi \), we will have:

1. (Reflexivity) \( \forall x (P_1(x) \rightarrow P_n(x, \ldots x)) \). In general, for each \( \ell \leq m < \omega \),

\[
\forall z_1, \ldots, z_\ell, y_1, \ldots y_m \left( \{\{z_1, \ldots z_\ell\} = \{y_1, \ldots y_m\} \implies (P_\ell(z_1, \ldots z_\ell) \iff P_m(y_1, \ldots y_m)) \right)
\]

2. (Symmetry) For any \( n < \omega \) and any bijection \( g : n \rightarrow n \),

\[
\forall y_1, \ldots y_n \left( P_n(y_1, \ldots y_n) \iff P_n(y_{g(1)}, \ldots y_{g(n)}) \right)
\]
(3) (Monotonicity) For each $\ell \leq m < \omega$,
$$\forall z_1, \ldots, z_{\ell}, y_1, \ldots, y_m \left( \{z_1, \ldots, z_{\ell}\} \subseteq \{y_1, \ldots, y_m\} \right) \Rightarrow \left( P_m(y_1, \ldots, y_m) \Rightarrow P_\ell(z_1, \ldots, z_{\ell}) \right)$$

So in particular, if $\models P_m(y_1, \ldots, y_m)$ and $\ell < m$ then $P_\ell$ holds on all $\ell$-element subsets of $\{y_1, \ldots, y_m\}$. The converse is usually not true; see Remark 2.2.

(4) (Dividing) Suppose that for some $n < \omega$, it is consistent with $T$ that there exists an infinite subset $Y \subseteq P_n$ such that $Y^k \cap P_{nk} = \emptyset$. Then in any sufficiently saturated model of $T$, some instance of the formula $\varphi_n(x; y_1, \ldots, y_n) = \bigwedge_{i < n} \varphi(x; y_i)$ $k$-divides.

(5) (Consistent types) Let $A \subset P_1$ be a set of parameters in some $M \models T$. Then $\{\varphi(x; a) : a \in A\}$ is a consistent partial $\varphi$-type iff $A^n \subset P_n$ for all $n < \omega$.

Proof. (4) By compactness, there exists an infinite indiscernible sequence of $n$-tuples $C = \langle c_1^n, \ldots, c^n_1 : i < \omega \rangle$ such that $C^k \cap P_{nk} = \emptyset$. The set $\{\varphi_n(x; c_1^i, \ldots, c^n_1) : i < \omega \}$ is therefore $k$-inconsistent. However, it is 1-consistent: for each $c_1^i, \ldots, c_1^n \in C$, $M \models P_n(c_1^i, \ldots, c_1^n)$, so $M \models \exists x \varphi_n(x; c_1^i, \ldots, c_1^n)$.

Convention 2.6. ($T_0$-configurations) Throughout this article, let $T_0$ denote the incomplete theory in the language $L_0 := \{P_n : n < \omega\} \cup \{=\}$ which describes (1)-(3) of Observation 2.1. Blueprints for hypergraphs in the language $L_0$ which are consistent with $T_0$ will be called $T_0$-configurations. That is: a finite $T_0$-configuration is a pair $X = (V_X, E_X)$ where $V_X = n < \omega$, $E_X \subseteq P(n)$ and the following is consistent with $T_0$:

\begin{itemize}
  \item (1) $\exists x_1, \ldots, x_n \left( \forall \sigma \subseteq n, |\sigma| = i, \sigma = \{\ell_1, \ldots, \ell_i\} \right) \left( P_i(x_{\ell_1}, \ldots, x_{\ell_i}) \iff \sigma \in E_X \right)$
\end{itemize}

In general, the domain of a $T_0$-configuration may be infinite; we simply require that its restriction to every finite subdomain satisfy (1). These are the graphs which can consistently occur as finite subgraphs of some characteristic sequence. That every such graph appears in some sequence follows from Example 3.3 below.

Convention 2.7. ($T_1$-configurations) Fix $T, \varphi$, and the associated sequence $\langle P_n : n < \omega \rangle$. Let $M \models T$; there is a unique expansion of $M$ to $L_0 = \{P_n : n < \omega\} \cup \{=\}$. Throughout this article, whenever $T, \varphi, \langle P_n \rangle$ are thus fixed, let $T_1$ denote the complete theory of $M$ in the language $L_0$. As the characteristic sequence is definable in $T$, when $T$ is complete this will not depend on the model chosen.

Hypergraphs in the language $L_0$ which are consistent with $T_1$ will be called $T_1$-configurations.

Recall that a formula $\varphi(x; y)$ has the finite cover property if for arbitrarily large $n < \omega$ there exist $a_0, \ldots, a_n$ such that $\{\varphi(x; a_0), \ldots, \varphi(x; a_n)\}$ is $n$-consistent but $(n+1)$-inconsistent.

Remark 2.8. The following are equivalent, for $(T, \varphi) \mapsto \langle P_n \rangle$:

1. There is $k < \omega$ such that the sequence $\langle P_n \rangle$ has support $k$.
2. $\varphi$ does not have the finite cover property.

In practice, when analyzing saturation of $\varphi$-types the finite cover property can often, but not always, be avoided by a judicious choice of formula. For instance, if $\varphi$ is unstable, some fixed finite conjunction $\theta$ of instances of $\varphi$ has the finite cover property (8.II.4); if
we choose to present \( \varphi \)-types as \( \theta \)-types the characteristic sequence would not have finite support. Nonetheless, it may happen even in unstable theories that there is a set \( \Sigma \subseteq \mathcal{L} \) of formulas without the fcp such that \( M \models T \) is \( \lambda^+ \)-saturated iff \( M \) realizes all \( \varphi_0 \)-types over sets of size \( \lambda \) for all \( \varphi_0 \in \Sigma \). This is true, for instance, of \( \Sigma = \{ \psi(x; y, z) := xRy \land \neg xRz \} \) in the random graph, and of \( \Sigma = \{ \psi(x; y, z) := y < x < z \} \) in \((\mathbb{Q},<)\).

3. Some examples

This section works out several motivating examples. We refer informally to localization and persistence, which will be defined in Definitions 12 and 15 below; the general definitions of \((\eta,\nu)\)-arrays and trees will be given in Definition 4.2.

Example 3.1. (The random graph)

\( T \) is the theory of the random graph, and \( R \) its binary edge relation. Let \( \varphi(x; y, z) = xRy \land \neg xRz \), with \((T,\varphi) \mapsto (P_n)\). Then:

- \( P_1((y, z)) \iff y \neq z \).
- \( P_n((y_1, z_1), \ldots, (y_n, z_n)) \iff \{y_1, \ldots, y_n\} \cap \{z_1, \ldots, z_n\} = \emptyset \).

Notice:

1. The sequence has support 2.
2. There is a uniform finite bound on the size of an empty graph \( C \subseteq P_1, C^2 \cap P_2 = \emptyset \): an analysis of the theory shows that \( \varphi \) is not dividable, and inspection reveals this bound to be 3.
3. \( P_n \) does not have the order property for any \( n \) and any partition of the \( y_1, \ldots, y_n \) into object and parameter variables. (Proof: The order property in \( P_n \) implies dividability of \( \varphi_{2n} \) by Observation 5.9. But none of the \( \varphi_x \) are dividable, as inconsistency only comes from equality.)
4. Of course, the formula \( \varphi \) has the independence property in \( T \). We can indeed find a configuration in \( P_2 \) which witnesses this: any \( C \) which models the \( T_0 \)-configuration having \( V_C = \omega \) and \( \{i, j\} \notin E_C \iff \exists n(i = 2n \land j = 2n + 1) \). Note that \( \varphi \) will have the independence property on any infinite \( P_2 \)-complete subgraph of the so-called \((\omega,2)\)-array \( C \) (see Observation 6.11 below).
5. As \( \varphi \) is unstable, \( \varphi \)-types are not necessarily definable in the sense of stability theory. However, we can obtain a kind of definability “modulo” the independence property, or more precisely, definability over the name for a maximal consistent subset of an \((\omega,2)\)-array as follows:

Definable types modulo independence. Let \( p \in S(M) \) be a consistent partial \( \varphi \)-type presented as a positive base set \( A \subseteq P_1 \). Let us suppose \( p \vdash \{xRc : c \in C\} \cup \{-xRd : d \in D\} \vdash p \), so that \( A \subseteq M^2 \) is a collection of pairs of the form \((c, d)\) which generate the type.

There is no definable (in \( T \) with or without parameters, so in particular not from \( P_2 \)) extension of the type \( A \), so we cannot expect to find a localization of \( P_1 \) around \( A \) which is a \( P_2 \)-complete graph. However:

Claim 3.2. In the theory of the random graph, with \( \varphi(x; y, z) = xRy \land \neg xRz \) as above, for any positive base set \( A \subseteq P_1 \) there exist a definable \((\omega,2)\)-array \( W \subseteq P_1 \), a solution \( S \) of \( W \) and an \( S \)-definable \( P_\infty \)-graph containing \( A \).
Proof. Work in $P_1$. Fix any element $(a, b)$ with $a, b \notin C, D$ and set $W_0 := \{(y, z) \in P_1 : \neg P_2(y, z), (a, b)\}$. Thus $W_0 = \{(b, z) : z \neq b\} \cup \{(y, a) : y \neq a\}$. So the only $P_2$-inconsistency among elements of $W_0$ comes from pairs of the form $(b, c), (c, a)$; thus, writing Greek letters for the elements of $P_1$,
\[(\forall \eta \in W_0)(\exists \nu \in W_0)(\forall \zeta \in W_0) (\neg P_2(\eta, \zeta) \rightarrow \zeta = \nu)\]

In other words, $W := W_0 \setminus \{(b, a)\}$ is an $(\omega, 2)$-array (Definition 4.2). Moreover:

1. $(y, z), (w, v) \in W$ and $\neg P_2((y, z), (w, v))$ implies $y = v$ or $z = w$, and
2. for any $c \neq a, b$, there are $d, e \in M$ such that $(d, c), (c, e) \in W$. Thus:
3. we may choose a maximal complete $P_2$-subgraph $C$ of $W$ such that $CA$ is a complete $P_{\omega}$-graph. For instance, let $C$ be any maximal consistent extension of $\{(b, d) : d \in D\} \cup \{(c, a) : c \in C\}$. Call any such $C$ a solution of the array $W$.

Let $S$ be a new predicate which names this solution $C$ of $W$. Then $\{y \in P_1 : z \in S \rightarrow P_2(y, z)\} \supset A$ is a $P_2$-complete graph, definable in $L \cup \{S\}$. Support 2 implies that it is a $P_{\omega}$-graph. Notice that by (2), we have in fact chosen a maximal consistent extension of $A$ (i.e. a complete global type).

Remark 3.3. The idiosyncracies of this proof, e.g. the choice of a definable $(\omega, 2)$-array, reflect an interest in structure which will be preserved in ultrapowers.

Example 3.4. (Coding complexity into the sequence)

It is often possible to choose a formula $\varphi$ so that some particular configuration appears in its characteristic sequence. For instance, by applying the template below when $\varphi$ has the independence property, we may choose a simple unstable $\theta$ whose $P_2$ is universal for finite bipartite graphs $(X, Y)$, provided we do not specify whether or not edges hold between $x, x' \in X$ or between $y, y' \in Y$. Nonetheless, Conclusion 5.10 below will show this is “inessential” structure in the case of simple theories: whatever complexity was added through coding can be removed through localization.

The construction. Fix a formula $\varphi$ of $T$. Let $\theta(x; y, z, w) := (z = w \land x = y) \lor (z \neq w \land \varphi(x; y))$. Write $(y, *)$ for $(y, z, w)$ when $z = w$, and $(y, -)$ for $(y, z, w)$ when $z \neq w$. Let $\langle P_n \rangle$ be the characteristic sequence of $\theta$, $\langle P_n^\varphi \rangle$ be the characteristic sequence of $\varphi$, and $\langle P_n^w \rangle$ be the characteristic sequence of $x = y$. Then $P_n$ can be described as follows:

- $P_n((y_1, -), \ldots, (y_n, -)) \leftrightarrow P_n^\varphi(y_1, \ldots, y_n)$.
- $P_n((y_1, *), \ldots, (y_n, *)) \leftrightarrow P_n^w(y_1, \ldots, y_n)$.
- Otherwise, the $n$-tuple $y := ((y_1, z_1), \ldots, (y_n, z_n))$ can contain (up to repetition) at most one $*$-pair, so $z_i = z_j = * \rightarrow y_i = y_j$. In this case the unique $y_*$ in the $*$-pair is the realization of some $\varphi$-type in the original model $M$ of $T$, and $P_{n+1}((y_*, *), (y_1, -), \ldots, (y_n, -))$ holds iff $M \models \wedge_{j \leq n} \varphi(y^*; y_j)$.

Remark 3.5. This highlights an important distinction: the fact that a characteristic sequence may contain a bipartite graph is not anywhere near as powerful as the fact of containing a random graph, see Example 3.8 below. In the coding just given we could not choose how elements within each side of the graph interrelated. This is quite restrictive, and eludes our coding for deep reasons: for instance, applying a consistency result of Shelah on the Keisler
order one can show that the order property in the characteristic sequence cannot imply the compatible order property in the characteristic sequence, Definition 4.8 below [7].

Example 3.6. (A theory with \( TP_2 \))

\( TP_2 \) is Shelah’s tree property of the second kind, to be defined and discussed in detail in Definition 4.11. Let \( T \) be the model completion of the following theory [12]. There are two infinite sorts \( X, Y \) and a single parametrized equivalence relation \( E_x(y, z) \), where \( x \in X \), and \( y, z \in Y \). Let \( \varphi_{eq} := \varphi(y; xzw) = E_x(y, z) \land \neg E_x(z, w) \). Then:

- \( P_1((xzw)) \iff z \neq w \).
- \( P_2((x_1z_1w_1), (x_2z_2w_2)) \iff \) each triple is in \( P_1 \) and furthermore:
  \[
  (x_1 = x_2) \to (E_x(z_1, z_2) \land \bigwedge_{i \neq j \leq 2} \neg E_x(w_i, z_j))
  \]

The sequence has support 2. There are many empty graphs; these persist under localization (Theorem 6.24). One way to see the trace of \( TP_2 \) is as follows. Fixing \( \alpha \), choose \( a_i \) (\( i < \omega \)) to be a set of representatives of equivalence classes in \( E_{\alpha_i} \), and choose \( b \) such that \( \neg E_{\alpha}(a_i, b) \) (\( i < \omega \)). Then \( \{ (\alpha, a_i, b) : i < \omega \} \subset P_1 \) is a \( P_2 \)-empty graph. We in fact have arrays \( \{ (\alpha^t, a_i^t, b_i^t) : i < \omega, t < \omega \} \) whose “columns” (fixing \( t \)) are \( P_2 \)-empty graphs and where every path which chooses exactly one element from each column is a \( P_2 \)-complete graph, thus a \( P_\infty \)-complete graph. The parameters in this so-called \( (\omega, \omega) \)-array describe \( TP_2 \) for \( \varphi_{eq} \) (Claim 4.6).

Note that a gap has appeared between the classification-theoretic complexity of \( \varphi \), which is not simple, and that of the formula \( P_2 \):

Claim 3.7. \( P_2 \) does not have the order property.

Proof. This is essentially because inconsistency requires the parameters \( x \) to coincide. Suppose that \( (a_i, b_i : i < \omega) \) were a witness to the order property for \( P_2 \). Fix any \( a_i = (\alpha_s, a_s, d_s) \). Now \( \neg P_2(b_j, a_i) \) for \( j < i \), where \( b_j = (\beta_i, b_i, c_i) \). \( P_2 \)-inconsistency requires \( \alpha_s = \beta_t \). As this is uniformly true, \( \alpha_t = \alpha_t = \beta_t = \beta_t \) for all \( s, t < \omega \) in the sequence. But now that we are in a single equivalence relation \( E_{\alpha} \), transitivity effectively blocks order: \( \neg P_2(b_j, a_i) \iff \neg E_{\alpha}(a_s, b_t) \).

Depending on whether at least one of the \( a \)- or \( b \)-sequences is an empty graph, we can find a contradiction to the order property with either three or four elements.

Example 3.8. (A maximally complicated theory)

In this example the sequence is universal for finite \( T_0 \)-configurations (Convention 2.6), a natural sufficient condition for “maximal complexity.”

Let the elements of \( M \) be all finite subsets of \( \omega \); the language has two binary relations, \( \subseteq \) and \( = \), with the natural interpretation. Set \( T = Th(M) \).

Choose \( \varphi_{\subseteq} := \varphi(x; y, z) = x \subseteq y \land x \not\subseteq z \). Then:

- \( P_1((y, z)) \iff \emptyset \subseteq y \not\subseteq z \).
- \( P_n((y_1, z_1), \ldots, (y_n, z_n)) \iff \emptyset \subseteq \bigcap_{i \leq n} y_i \not\subseteq \bigcup_{i \leq n} z_i \).

The sequence does not have finite support. Moreover:

Claim 3.9. Let \( \langle P_n \rangle \) be the characteristic sequence of \( \varphi_{\subseteq} \), \( k < \omega \), and let \( X \) be a finite \( T_0 \)-configuration. Then there exists a finite \( A \subseteq P_1 \) witnessing \( X \).
Proof. Write the elements of \( P_1 \) as \( w_i = (y_i, z_i) \); it suffices to choose the positive pieces \( y_i \) first, and afterwards take the \( z_i \) to be completely disjoint. More precisely, suppose \( X \) is given by \( V_X = m \) and \( E_X \subset \mathcal{P}(m) \). We need simply to choose \( y_1, \ldots, y_m \) such that for all \( \sigma \subseteq m \),

\[
\left( \bigcap_{j \in \sigma} y_j \neq \emptyset \right) \iff \sigma \in E_X
\]

which again, is possible by the downward closure of \( E_X \). \( \square \)

Corollary 3.10. This characteristic sequence is universal for finite \( T_0 \)-configurations.

Remark 3.11. That the sequence is universal for finite \( T_0 \)-configurations is sufficient, though not necessary, for maximal complexity in the Keisler order. By \([8]\.VI.3, \varphi(x; y, z) = y < x < z \text{ in } Th(\mathbb{Q}, <) \) is maximal. Its characteristic sequence has support 2, but its \( P_2 \) is clearly not universal.

4. Static configurations

This section establishes a series of correspondences between \( T_1 \)-configurations found in the characteristic sequence of \( \varphi \) and the classification-theoretic complexity of \( \varphi \) itself. It lays the groundwork for the next section, which will build “dynamic” arguments out of these “static” ones by asking what happens when certain configurations persist under all reasonable restrictions of the set \( P_1 \).

Here we describe configurations which signal that \( \varphi \) has the order property, the independence property, the tree property and SOP\(_2\). Recall that:

**Definition 4.1.** (Tree properties) Let \( \subseteq \) indicate initial segment. To simplify notation, say that the nodes \( \rho_1, \rho_2 \in \omega^{<\omega} \) are *incomparable* if

\[
\neg(\rho_1 \subseteq \rho_2) \land \neg(\rho_2 \subseteq \rho_1) \land \neg(\exists \nu \in \omega^{<\omega}, i, j \in \omega)(\rho_1 = \nu \land i, \rho_2 = \nu \land j)
\]

i.e., if they do not lie along the same branch and are not immediate successors of the same node.

Then the formula \( \varphi \) has:

- the \( k \)-tree property, where \( k < \omega \), if there is an \( \omega^{<\omega} \)-tree of instances of \( \varphi \) where paths are consistent and the immediate successors of any given node are \( k \)-inconsistent, i.e.
  \[ X = \{ \varphi(x; a_\eta) : \eta \in \omega^{<\omega} \}, \]
  and:
  1. for all \( \nu \in \omega^{\omega}, \{ \varphi(x; a_\eta) : \eta \subseteq \nu \} \) is a consistent partial type;
  2. for all \( \rho \in \omega^{<\omega}, \{ \varphi(x; a_{\rho^{-}i}) : i < \omega \} \) is \( k \)-inconsistent.

- the tree property if it has the \( k \)-tree property for some \( 2 \leq k < \omega \).
- the non-strict tree property \( TP_2 \) if there exists a \( \varphi \)-tree with \( k = 2 \) and for which, moreover:
  3. for any two *incomparable* \( \rho_1, \rho_2 \in \omega^{<\omega}, \exists x(\varphi(x; a_{\rho_1}) \land \varphi(x; a_{\rho_2})) \).
- the strict tree property, also known as \( TP_1 \) or SOP\(_2\), if there exists a \( \varphi \)-tree with \( k = 2 \) and for which, moreover:
  3. for any two *incomparable* \( \rho_1, \rho_2 \in \omega^{<\omega}, \neg \exists x(\varphi(x; a_{\rho_1}) \land \varphi(x; a_{\rho_2})) \).

**Theorem A.** (Shelah; see \([8]\.III.7\))

- \( T \) is simple iff no formula \( \varphi \) of \( T \) has the tree property, iff no \( \varphi \) has the 2-tree property.
• If \( \varphi \) has the 2-tree property then either \( \varphi \) has \( TP_1 \) or \( \varphi \) has \( TP_2 \).

We fix a monster model \( M \) from which the parameters are drawn; see Convention 2.3.

**Definition 4.2.** (Diagrams, arrays, trees) Let \( \lambda \geq \mu \) be finite cardinals or \( \omega \). Write \( \subseteq \) to indicate initial segment. The sequence \( \langle P_n \rangle \) has:

1. an \((\omega, 2)\)-diagram if there exist elements \( \{ a_\eta : \eta \in 2^{<\omega} \} \subseteq P_1 \) such that
   - for all \( \eta \in 2^{<\omega} \), \( \neg P_2(a_{\eta^0}, a_{\eta^-1}) \), and
   - for all \( n < \omega \) and \( \eta_1, \ldots, \eta_n \in 2^{<\omega} \), we have that \( \eta_1 \subseteq \cdots \subseteq \eta_n \implies P_n(a_{\eta_1}, \ldots, a_{\eta_n}) \)
   That is, sets of pairwise comparable elements are \( P_\infty \)-consistent, while immediate successors of the same node are \( P_2 \)-inconsistent.
2. a \((\lambda, \mu, 1)\)-array if there exists \( X = \{ a_\eta^m : l < \lambda, m < \mu \} \subseteq P_1 \) such that:
   - \( P_2(a_{l_1}^{m_1}, a_{l_2}^{m_2}) \iff (l_1 = l_2 \rightarrow m_1 = m_2) \)
   - For all \( i < \omega \),
     \[
     P_n(a_{l_1}^{m_1}, \ldots, a_{l_n}^{m_n}) \iff \bigwedge_{1 \leq i, j \leq n} P_2(a_{l_i}^{m_i}, a_{l_j}^{m_j})
     \]
   That is, any \( C \subseteq X \), possibly infinite, is a \( P_\infty \)-graph iff it contains no more than one element from each column. (We will relax this last condition in the more general Definition 6.9 below.)
3. a \((\lambda, \mu)\)-tree if there exist elements \( \{ a_\eta : \eta \in \mu^{<\lambda} \} \subseteq P_1 \) such that
   - for all \( \eta_2, \eta_3 \in \mu^{<\lambda} \),
     \[
     P_2(a_\eta, a_\nu) \iff (\eta_1 \subseteq \eta_2 \lor \eta_2 \subseteq \eta_1)
     \]
   i.e. only if the nodes are comparable; and
   - for all \( n < \omega \), \( \eta_1, \ldots, \eta_n \in \mu^{<\lambda} \),
     \[
     \eta_1 \subseteq \cdots \subseteq \eta_n \implies P_n(a_{\eta_1}, \ldots, a_{\eta_n})
     \]

**Remark 4.3.** Diagrams are prototypes which can give rise to either arrays or trees, in the case where the unstable formula \( \varphi \) has the independence property or \( SOP_2 \), respectively.

The arrays will be revisited in Definitions 6.1 and 6.9.

**Claim 4.4.** Let \( \varphi \) be a formula of \( T \) and set \( \theta(x; y, z) = \varphi(x; y) \land \neg \varphi(x; z) \). Let \( \langle P_n \rangle \) be the characteristic sequence of \((T, \theta)\). The following are equivalent:

1. \( \langle P_n \rangle \) has an \((\omega, 2)\)-diagram.
2. \( R(x = x, \varphi(x; y), 2) \geq \omega \), i.e. \( \varphi \) is unstable.
3. \( R(x = x, \theta(x; y, z), 2) \geq \omega \), i.e. \( \theta \) is unstable.

**Proof.** (2) \( \implies \) (1): We have in hand a tree of partial \( \varphi \)-types \( R = \{ p_\nu : \nu \in 2^{<\omega} \} \), partially ordered by inclusion, witnessing that \( R(x = x, \varphi, 2) \geq \omega \). Let us show that we can build an \((\omega, 2)\)-diagram. That is, we shall choose parameters \( \{ a_\eta : \eta \in 2^{<\omega} \} \subseteq P_1 \) satisfying Definition 4.2(1).

First, by the definition of the rank \( R \), which requires the partial types to be explicitly contradictory, we can associate to each \( \nu \) an element \( c_\nu \in M \), \( \ell(c_\nu) = \ell(y) \) such that:
- \( \varphi(x; c_\nu) \in p_{\nu^1} \setminus p_\nu \), and
- \( \neg \varphi(x; c_\nu) \in p_{\nu^0} \setminus p_\eta \).
i.e., the split after index $\nu$ is explained by $\varphi(x; c_\nu)$.

Second, choose a set of indices $S \subseteq 2^{<\omega}$ such that:

- $(\forall \eta \in 2^{<\omega}) \ (\exists s \in S)(\eta \subseteq s)$
- $(\forall s_1 \subseteq s_2 \in S) \ (\exists \eta \notin S) \ (s_1 \subseteq \eta \subseteq s_2)$

It will suffice to define $a_{s^{-}}$ for $s \in S$, $i \in \{0, 1\}$. (The sparseness of $S$ ensures the chosen parameters for $\varphi$ won’t overlap, which will make renumbering straightforward.) Recall that the $a_\eta$ will be parameters for $\theta(x; y, z) = \varphi(x; y) \land \lnot(x; z)$. So we define:

- $a_{s^{-}0} = (c_{s^{-}0}, c_s)$;
- $a_{s^{-}1} = (c_s, c_{s^{-}1})$.

The consistency of the paths through our $(\omega, 2)$-diagram is inherited from the tree $R$ of consistent partial types. However, $\lnot P_2(a_{s^{-}0}, a_{s^{-}1})$ because these contain an explicit contradiction:

$$\lnot \exists x \left( (\varphi(x; c_{s^{-}0}) \land \lnot \varphi(x; c_s)) \land (\varphi(x; c_s) \land \lnot \varphi(x; c_{s^{-}1})) \right)$$

$(1) \rightarrow (3)$: Reading off the parameters from the diagram we obtain a tree of consistent partial $\theta$-types $\{p_\eta : \eta \in 2^{<\omega}\}$, partially ordered by inclusion. For any $\eta \in 2^{<\omega}$, $\lnot P_2(a_{\eta^{-}0}, a_{\eta^{-}1})$, i.e., $\lnot \exists x (\theta(x; a_{\eta^{-}0}) \land \theta(x; a_{\eta^{-}1}))$. Furthermore, $\theta(x; a_{\eta^{-}0}) \in p_{\eta^{-}0} \setminus p_\eta$, while $\theta(x; a_{\eta^{-}1}) \in p_{\eta^{-}1} \setminus p_\eta$. So there is no harm in making the types explicitly inconsistent, as the rank $R$ requires, by adding $\lnot \theta(x; a_{\eta^{-}i})$ to $p_{\eta^{-}i}$ for $i \neq j < 2$.

$(2) \leftrightarrow (3)$: for all $A$, $|A| \geq 2$, $|S_\varphi(A)| = |S_\theta(A)|$. \hfill $\Box$

**Claim 4.5.** Let $\varphi$ be a formula of $T$ and set $\theta(x; y, z) = \varphi(x; y) \land \lnot \varphi(x; z)$. Let $\langle P_n \rangle$ be the characteristic sequence of $(T, \theta)$. The following are equivalent:

1. $\langle P_n \rangle$ has an $(\omega, 2, 1)$-array.
2. $\varphi$ has the independence property.
3. $\theta$ has the independence property.

**Proof.** $(1) \rightarrow (3)$: This is Observation 6.11. (Essentially, let $A_0$ be the top row of the array $A$, and $\sigma, \tau \subset A$ finite disjoint; let $B \subset A$ be a maximal positive base set, i.e. a maximal $P_\infty$-complete graph, in $A$ containing $\sigma$ and avoiding $\tau$. Then any realization of the type corresponding to $B$ is a witness to this instance of independence.)

$(2) \rightarrow (1)$: Let $\langle i_\ell : \ell < \omega \rangle$ be a sequence over which $\varphi$ has the independence property. For $t < 2$, $j < \omega$ set $a_j^t = (i_\ell, i_{\ell+1})$, $a_j^t = (i_{\ell+1}, i_\ell)$. Then $\{a_j^t : t < 2, j < \omega\}$ is an $(\omega, 2, 1)$-array for $P_\infty$.

$(3) \rightarrow (2)$: For any infinite $A$, $|S_\varphi(A)| = |S_\theta(A)|$, as any type on one side can be presented as a type on the other. The independence property can be characterized in terms of the cardinality of the space of types over finite sets (8 Theorem II.4.11). \hfill $\Box$

**Claim 4.6.** Let $\varphi$ be a formula of $T$ and set $\theta(x; y, z) = \varphi(x; y) \land \lnot \varphi(x; z)$. Let $\langle P_n \rangle$ be the characteristic sequence of $(T, \theta)$. Suppose that $T$ does not have SOP$_2$. Then the following are equivalent:

1. $\langle P_n \rangle$ has an $(\omega, \omega, 1)$-array.
2. $\varphi$ has the 2-tree property.

**Proof.** $(1) \rightarrow (2)$ Each column (=empty graph) of the array witnesses that $\varphi$ is 2-dividable, and the condition that any subset of the array containing no more than one element from each column is a $P_\infty$-complete graph ensures that the dividing can happen sequentially.
(2) → (1) By Theorem A above, NSOP₂ implies $\varphi$ has $TP_2$. That is, there is a tree of instances $\{ \varphi(x; a_\eta) : \eta \in \omega^{<\omega} \}$ such that first, for any finite $n$, $\eta_1 \subseteq \cdots \subseteq \eta_n$ implies that the partial type $\{ \varphi(x; a_{\eta_1}), \ldots, \varphi(x; a_{\eta_n}) \}$ is consistent; and second,

$$
\neg \exists x \left( \varphi(x; a_\eta) \land \varphi(x; a_\nu) \right) \iff \left( \exists \rho \in \omega^{<\omega} \right) \left( \exists i \neq j \in \omega \right) \left( \eta = \rho \cap i \land \nu = \rho \cap j \right)
$$

Thus the parameters $\{ a_\eta : \eta \in \omega^{<\omega} \} \subseteq P_1$ form an $(\omega, \omega, 1)$-array for $P_\infty$. \hfill \Box

It is straightforward to characterize the analogous $k$-tree properties in terms of arrays whose columns are $k$-consistent but $(k+1)$-inconsistent.

**Claim 4.7.** The following are equivalent:

1. $\langle P_n \rangle$ has an $(\omega, 2)$-tree.
2. $\varphi$ has $SOP_2$.

**Proof.** (2) → (1) This is a direct translation of Definition 4.1.

(1) → (2) It suffices to show that $\langle P_n \rangle$ has an $(\omega, \omega)$-tree, which is true by compactness, using the strictness of the tree. \hfill \Box

In the next definition, the power of the classification-theoretic order property on the $P_n$ is magnified when it can be taken to describe the interaction between complete graphs, i.e. base sets for partial $\varphi$-types. Compare Remark 3.3.

**Definition 4.8.** $P_\infty$ has the compatible order property if there exists a sequence $C = \{ a_i, b_i : i < \omega \} \subseteq P_1$ such that for any $n < \omega$ and any $a_1, b_1, \ldots, a_n, b_n \subseteq C$,

$$P_n((a_1, b_1), \ldots, (a_n, b_n)) \iff \left( \max\{a_1, \ldots, a_n\} < \min\{b_1, \ldots, b_n\} \right)$$

Say that $P_m$ has the compatible order property to indicate that this holds for all $P_n$, $n \leq m$.

**Observation 4.9.** Suppose $(T, \varphi) \hookrightarrow \langle P_n \rangle$, and that $\langle P_n \rangle$ has the compatible order property. Then $\varphi_2$ has the tree property, and in particular, $SOP_2$.

**Proof.** Let us build an $SOP_2$-tree $\{ \varphi_2(x; a_\eta, b_\eta) : \eta \in \omega^{<\omega} \}$ following Definition 4.1 above by specifying the corresponding tree of parameters $\{ c_\eta : \eta \in \omega^{<\omega} \} \subseteq P_1$, where each $c_\eta$ is a pair $(a_\eta, b_\eta)$. Let $S = \{ a_i b_i : i < Q \}$ be an indiscernible sequence witnessing the compatible order property. We will use two facts in our construction:

1. Let $\langle a_i b_i : \ell < \omega \rangle$ be any subsequence of $S$ such that $\ell < k \Rightarrow a_i < b_j < a_k < b_j$. Then $\{ \varphi_2(x; a_i, b_i) : j < \omega \}$ 2-divides by Observation 5.7.
2. Let $a_{i_1}, b_{j_1}, \ldots, a_{i_n}, b_{j_n} \subseteq S$. Then

$$P_n((a_{i_1}, b_{j_1}), \ldots, (a_{i_n}, b_{j_n})) \iff \max\{i_1, \ldots, i_n\} < \min\{j_1, \ldots, j_n\}$$

so in particular

$$P_2((a_{i_1}, b_{j_1}), (a_{i_2}, b_{j_2})) \iff \max\{i_1, i_2\} < \min\{j_1, j_2\}$$

Let $\eta \in \omega^{<\omega}$ be given and suppose that either $c_\eta$ has been defined or $\eta = \emptyset$. If $c_\eta$ has been defined, it will be $(a_i, b_j)$ for some $i < j \in Q$. Let $\langle k_\ell : \ell < \omega \rangle$ be any $\omega$-indexed subset of $(i, j) \cap Q$, or of $Q$ if $\eta = \emptyset$. Define $c_{\eta \cap \ell} = (a_{k_\ell}, b_{k_\ell+1})$. Now suppose we have defined the full tree of parameters $c_\eta$ in this way. By fact (1) we see that immediate successors of the same node are $P_2$-inconsistent. By fact (2), paths are consistent, while by fact (2), any two *incomparable* (Definition 4.1) elements $c_\varphi, c_\eta$ are $P_2$-inconsistent. \hfill \Box
Remark 4.10. The compatible order property in $P_\infty$ is in fact enough to imply maximality in the Keisler order [7].

5. Localization and persistence

The goal of these methods is to analyze $\varphi$-types, and thus to concentrate on the combinatorial structure which is “close to” or “inseparable from” the complete graph $A$ representing a consistent partial $\varphi$-type under analysis. Localization and persistence, defined in this section, are tools for honing in on this essential structure.

$P_n$ asks about incidence relations on a set of parameters; it will be useful to definably restrict the witness and parameter sets. For instance:

- we may ask that the witnesses lie inside certain instances of $\varphi$, e.g. by setting $P'_1(y) = \exists x (\varphi(x;y) \land \varphi(x;a))$, i.e. $P'_1 = P_2(y,a)$.
- we may ask that the parameters be consistent 1-point extensions (in the sense of some $P_n$) of certain finite graphs $C$. For instance, we might define $P''_1(y) = P_1(y) \land P_2(y,c_1) \land P_3(y,c_2,c_2)$.

The next definition gives the general form.

Definition 5.1. (Localization) Fix a characteristic sequence $(T,\varphi) \rightarrow (P_n)$, and choose $B,A \subset M \models T$ with $A$ a positive base set and $A = \emptyset$ possible.

1. (the localized predicate $P_n^f$) A localization $P_n^f$ of the predicate $P_n(y_1,\ldots,y_n)$ around the positive base set $A$ with parameters from $B$ is given by a finite sequence of triples $f : m \rightarrow \omega \times P_{\aleph_0}(y_1,\ldots,y_n) \times P_{\aleph_0}(B)$ where $m < \omega$ and:
   - writing $f(i) = (r_i,\sigma_i,\beta_i)$ and $s$ for the elements of the set $s$, we have:
     $$P_n^f(y_1,\ldots,y_n) := \bigwedge_{i \leq m} P_i(r_i,\sigma_i,\beta_i)$$
   - for each $\ell < \omega$, $T_1$ implies that there exists a $P_\ell$-complete graph $C_\ell$ such that $P_n^f$ holds on all $n$-tuples from $C_\ell$. If this last condition does not hold, $P_n^f$ is a trivial localization. By localization we will always mean nontrivial localization.
   - In any model of $T_1$ containing $A$ and $B$, $P_n^f$ holds on all $n$-tuples from $A$.

Write $\text{Loc}^B_n(A)$ for the set of localizations of $P_n$ around $A$ with parameters from $B$ (i.e. nontrivial localizations, even when $A = \emptyset$).

2. (the localized formula $\varphi^f$) For each localization $P_n^f$ of some predicate $P_n$ in the characteristic sequence of $\varphi$, define the corresponding formula

   $$\varphi^f_n(x;y_1,\ldots,y_n) := \varphi_n(x;y_1,\ldots,y_n) \land P_n^f(y_1,\ldots,y_n)$$

When $n = 1$, write $\varphi^f = \varphi^f_1$. Let $S_\varphi^f(N)$ denote the set of types $p \in S_\varphi(N)$ such that for all $\{\varphi_{i_1}(x;c_{i_1}),\ldots,\varphi_{i_n}(x;c_{i_n})\} \subset p$, $P_n^f(c_{i_1},\ldots,c_{i_n})$. Then there is a natural correspondence between the sets of types

   $$S_\varphi^f(N) \leftrightarrow S_\varphi^f(N)$$
We have thus far described localizations of the parameters of $\varphi$. We will also want to consider restrictions of the possible witnesses to $\varphi$ by adjoining instances of $\varphi_k$. That is, set

$$\varphi^{f+\overline{a}}(x; y) = \varphi^{f+a_1\ldots a_k}(x; y) := \varphi(x; y) \land P^f_1(y) \land \varphi_k(x; a_1, \ldots, a_k)$$

where, as indicated, $k = \ell(\overline{a})$. The * is to emphasize that this is really the construction from $\varphi$ of a new, though related, formula, which will have its own characteristic sequence, given by:

(4) (the *localized characteristic sequence $\langle P^f_n : n < \omega \rangle$) The sequence $\langle P^f_n : n < \omega \rangle$ associated to the formula $\varphi^{f+\overline{a}}$ is given by, for each $n < \omega$,

$$P^f_n(y_1, \ldots, y_n) = \bigwedge_{i \leq n} P^f_1(y_i) \land P^f_{n+k}(y_1, \ldots, y_n, a_1, \ldots a_k)$$

When $f$ or $\overline{a}$ are empty, we will omit them.

Remark 5.2. Convention 2.3 applies: that is, localization is not essentially dependent on the choice of model $M$. See Definition 5.14 (Persistence) and the observation following.

As a first example of the utility of localization, notice that when $\varphi$ is simple we can localize to avoid infinite empty graphs.

Observation 5.3. Fix a positive base set $A$ for the formula $\psi$, possibly empty. When $\psi$ does not have the tree property, then for every $k < \omega$ there is a finite set $C$ over which $\psi$ is not $k$-divisible. As a consequence, if $\psi$ does not have the tree property, then for each predicate $P_n$ there is a localization around $A$ on which there is a uniform finite bound on the size of a $P_n$-empty graph. We can clearly also choose the localizations so that none of $\psi_1, \ldots, \psi_{\ell}$ are $k$-divisible for any finite $k, \ell$ fixed in advance.

Proof. This is the proof that $D(x = x, \psi, k) < \omega$ for any simple formula $\psi$; see for instance [14]. (For a more direct argument, see the proof of Conclusion 5.10 below.)

The following important property of formulas was isolated by Buechler [1].

Definition 5.4. The formula $\varphi$ is low if there exists $k < \omega$ such that for every instance $\varphi(x; a)$ of $\varphi$, $\varphi(x; a)$ divides iff it $\leq k$-divides.

Observation 5.5. If $\varphi$ does not have the independence property then $\varphi$ is low.

Proof. To show that any non-low formula $\varphi$ has the independence property, it suffices to establish the consistency of the following schema. For $k < \omega$, $\Psi_k$ says that there there exist $y_1, \ldots, y_{2k}$ such that for every $\sigma \subset 2k$, $|\sigma| = k$,

$$\exists x \left( \varphi(x; y_i) \iff i \in \sigma \right)$$

But $\Psi_k$ will be true on any subset of size $2k$ of an indiscernible sequence on which $\varphi$ is $k$-consistent but $(k+1)$-inconsistent, and such sequences exist for arbitrarily large $k$ by hypothesis of non-lowness.

Corollary 5.6. When the formula $\psi$ of Observation 5.3 is simple and low, we can find a localization in which $\psi$ is not $k$-dividable, for any $k$. 13
5.1. Stability in the parameter space. The classification-theoretic complexity of the formulas $P_n$ is often strictly less than that of the original theory $T$. Note that the results here refer to the formulas $P_n(y_1, \ldots, y_n)$, not necessarily to their full theory $T_1$.

Observation 5.7. Suppose $(T, \varphi) \rightarrow \langle P_n \rangle$. If $P_2(x; y)$ has the order property then $\varphi(x; y) \land \varphi(x; z)$ is 2-divisible.

Proof. Let $(a_i, b_i : i < \omega)$ be a sequence witnessing the order property for $P_2$, so $P_2(a_i, b_i)$ iff $i < j$. This means that $\exists x(\varphi(x; a_i) \land \varphi(x; b_j))$ iff $i < j$. So $\varphi(x; a_i) \land \varphi(x; b_{i+1})$ are consistent for each $i$, but the set $\{ \varphi(x; a_i) \land \varphi(x; b_{i+1}) : i < \omega \}$ is 2-inconsistent. \hfill \Box

Remark 5.8. By compactness, without loss of generality the sequence of Observation 5.7 can be chosen to be $(T-)indiscernible, and so actually witnesses the dividing of some instance of $\varphi_2$.

Note that the converse of Observation 5.7 fails: for $\varphi(x; y) \land \varphi(x; z)$ to divide it is sufficient to have a disjoint sequence of “matchsticks” in $P_2$ (i.e. $(a_i, b_i) : i < \omega$ such that $P_2(a_i, b_j)$ iff $i = j$), without the additional consistency which the order property provides.

Nonetheless, work relating the characteristic sequence to Szemerédi regularity illuminates the role of the order \cite{7}.

Observation 5.9. Suppose that $(T, \varphi) \rightarrow \langle P_n \rangle$, and for some $n, k$ and some partition of $y_1, \ldots, y_n$ into $k$ object and $(n-k)$ parameter variables, $P_n(y_1, \ldots, y_k; y_{k+1}, \ldots, y_n)$ has the order property. Then $\varphi_n(x; y_1, \ldots, y_n)$ is 2-divisible.

Proof. The proof is analogous to that of Observation 5.7 replacing the $a_i$ by $k$-tuples and the $b_j$ by $(n-k)$-tuples. \hfill \Box

Thus in cases where we can locate to avoid dividing of $\varphi$, we can assume any initial segment of the associated predicates $P_n$ are stable:

Conclusion 5.10. For each formula $\varphi$ and for all $m < \omega$, if $\varphi_{2n}$ does not have the tree property, then for each positive base set $A$ there are a finite $B$ and $P_1^f \in \text{Loc}^B(A)$ over which $P_2, \ldots, P_n$ do not have the order property. In particular, this holds if $T$ is simple.

Proof. We proceed by asking: do there exist elements $\langle y_i z_i : i < \omega \rangle$ such that (1) each $y_i z_i$ is a 2-point extension of $A$ and (2) $\langle y_i z_i : i < \omega \rangle$ witnesses the order property for $P_2$? If not, localize using the finite set of conditions in (1) which prevent (2). Otherwise, let $a_1, b_1$ be the first pair in any such sequence, set $A_1 := A \cup \{ a_1, b_1 \}$ and repeat the argument using $A_1$ in place of $A$. By simplicity, there is a uniform finite bound on the number of times $\varphi_n$ (see Observation 5.9) can sequentially divide. Condition (1) ensures that the dividing is sequential, corresponding to choosing progressive forking extensions of the partial type corresponding to $A$. At some finite stage $t$ this will stop, meaning that (1) and (2) fail with $A_t$ in place of $A$; the finite fragment of (1) which prevents (2) gives the desired localization. \hfill \Box

By way of motivating the next subsection, let us prove the contrapositive: If the order property in $P_2$ persists under repeated localization, then $\varphi$ has the tree property. Compare the proof of Observation 4.9 above. Without the compatible order property, we cannot ensure the tree is strict. While that argument built a tree out of a set of parameters which were given all at once (a so-called “static” argument), the following “dynamic” argument must constantly localize to find subsequent parameters, so cannot ensure that elements in different localizations are inconsistent.
Lemma 5.11. Suppose that in every localization of \( P_1 \) (around \( A = \emptyset \)), \( P_2 \) has the order property. Then \( \varphi_2 \) has the tree property.

Proof. Let us describe a tree with nodes \((c_\eta, d_\eta)\), \((\eta \in \omega^\omega)\), such that:

1. for each \( \rho \in \omega^\omega \), \( \{c_\eta, d_\eta : \eta \subseteq \rho\} \) is a complete \( P_\infty \)-graph, where \( \subseteq \) means initial segment.
2. for any \( \nu \in \omega^\omega \), \( P_2(c_{\eta^{-1}}, d_{\eta^{-1}}) \iff i \leq j \).

For the base case \((\eta \in \omega^1)\), let \( \langle c_i, d_i : i \in \omega \rangle \) be an indiscernible sequence witnessing the order property (so \( P_2(c_i, d_i) \iff i \leq j \)) and assign the pair \((c_i, d_i)\) to node \( i \).

For the inductive step, suppose we have defined \((c_\eta, d_\eta)\) for \( \eta \in \omega^n \). Write \( E_\eta = \{(c_\nu, d_\nu) : \nu \leq \eta\} \) for the parameters used along the branch to \((c_\eta, d_\eta)\). Using \( \bar{x} \) to mean the elements of the set \( x \), let \( P_1^{f_n} \) be given by \( P_{n+1}((y, z), E_\eta) \). Let \( \langle a_j, b_j : j \in \omega \rangle \) be an indiscernible sequence witnessing the order property inside this localization, and define \((c_{\eta^{-1}}, d_{\eta^{-1}}) := (a_j, b_j)\).

Finally, let us check that this tree of parameters witnesses the tree property for \( \varphi_2 \). On one hand, the order property in \( P_2 \) ensures that for each \( n \in \omega^\omega \), the set

\[ \{\varphi_2(x; c_{\eta^{-1}}, d_{\eta^{-1}}) : i \in \omega\} \]

is \( 1 \)-consistent but \( 2 \)-inconsistent. On the other hand, the way we constructed each localization \( P_1^{f_n} \) ensured that each path was a complete \( P_\infty \)-graph, thus naturally a complete \( P_\infty' \)-graph, where \( (P_n') \) is the characteristic sequence of the conjunction \( \varphi_2 \).

Remark 5.12. Example 3.6 shows that the condition that \( \varphi \) has the tree property is necessary, but not sufficient, for the order property in \( P_2 \) to be persistent, Definition 5.14 below.

Question 5.13. Is \( SOP_2 \) sufficient?

Compare the issue of whether \( SOP_2 \Rightarrow SOP_3 \): see [2], [12].

5.2. Persistence. Localization, Definition 5.1 above, gives rise to a natural limit question: what happens when certain \( T_0 \)-configurations persist under all finite localizations?

Definition 5.14. (Persistence) Fix \((T, \varphi) \iff \langle P_n \rangle\), \( M \models T \) sufficiently saturated, and a positive base set \( A \), possibly \( \emptyset \). Let \( X \) be a \( T_0 \)-configuration, possibly infinite. Then \( X \) is persistent around the positive base set \( A \) if for all finite \( B \subset M \) and for all \( P_1^f \in \text{Loc}^B_1(A) \), \( P_1^B \) contains witnesses for \( X \).

We will write \( X \) is \( A \)-persistent to indicate that \( X \) is persistent around \( A \).

Note 5.15. Persistence asks whether all finite localizations around \( A \) contain witnesses for some \( T_0 \)-configuration \( X \). The predicates \( P_n \) mentioned in \( X \) are, however, not the localized versions. We have simply restricted the set from which witnesses can be drawn. This is an obvious but important point: for instance, in the proof of Lemma 5.6 below it is important that the sequence of \( P_2 \)-inconsistent pairs found inside of successive localizations \( P_1^{f_n} \) are \( P_2 \)-inconsistent in the sense of \( T_1 \).

Observation 5.16. (Persistence is a property of the theory \( T \)) The following are equivalent, fixing \( T, \varphi, \langle P_n \rangle \), \( A \subset M \) a small positive base set in the monster model, and a \( T_0 \)-configuration \( X \). Write \( P_1^f(M) \) for the set which \( P_1^f \) defines in the model \( M \).

1. In some sufficiently saturated model \( M \models T_1 \) which contains \( A \), \( X \) is persistent around \( A \) in \( M \). That is, for every finite \( B \subset M \) and every localization \( P_1^f \in \text{Loc}^B_1(A) \), there exist witnesses to \( X \) in \( P_1^f(M) \).
Corollary 5.17. Persistence around the positive base set $A$ remains a property of $T$ in the language with constants for $A$.

Finally, let us check the (easy) fact that persistence of some $T_0$-configuration around $\emptyset$ in some given sequence $\langle P_n \rangle$ implies its persistence around any positive base set $A$ for that sequence. Recall that all localizations are, by definition, non-trivial.

Fact 5.18. Suppose that $X$ is an $\emptyset$-persistent $T_0$-configuration in the characteristic sequence $\langle P_n \rangle$ and $A$ is a positive base set for $\langle P_n \rangle$. Then $X$ remains persistent around $A$.

Proof. Let $p(x_0, \ldots)$ in the language $\mathcal{L}(=, P_1, P_2, \ldots)$ describe the type, in $V_X$-many variables, of the configuration $X = (V_X, E_X)$. Let $q(y) \in S(A)$ be the type of a 1-point $P_\infty$-extension of $A$ in the language $\mathcal{L}_0 = \{P_n : n < \omega\} \cup \{=\}$. We would like to know that $q(x_0), q(x_1), \ldots, p(x_0, \ldots)$ is consistent, i.e., that we can find, in some given localization, witnesses for $X$ from among the elements which consistently extend $A$. If not, for some finite subset $A' \subset A$, some $n < \omega$, and some finite fragments $q'$ of $q|_{A'}$ and $p'$ of $p$,

$$q'(x_0) \cup \cdots \cup q'(x_n) \vdash \neg p'(x_0, \ldots x_n)$$

But now localizing $P_1$ according to the conditions on the lefthand side (which are all positive conditions involving the $P_n$ and finitely many parameters $A'$) shows that $X$ is not persistent, contradiction.

6. Dividing lines: Stability, Simplicity, NIP

The first natural question for persistence is: given $n$, when isn’t it possible to localize $P_1$ so that $P_n$ is a complete graph? The answer, surprisingly, is: in the presence of the independence property. This section gives the argument, using the language of persistence to give a new description of NIP and of simplicity, Theorem 6.19 and Theorem 6.24 below. Recall that a theory $T$ is NIP [10] if no formula of $T$ has the independence property; for more on this hypothesis, see [13].

6.1. NIP: the case of $P_2$. We will see that if $\varphi$ is NIP then we can localize around any fixed positive base set so that $P_2$ is a complete graph.

The argument in this technically simpler case will generalize without too much difficulty. We first revisit an avatar of the independence property.

Definition 6.1. $((\omega, 2)$-arrays revisited)
Remark 6.2. If $P_\infty$ is $(\omega, 2)$, then $\varphi$ has the independence property.

Proof. Let $X$ be a maximal path through the $(\omega, 2)$-array $A$. Choose any $\sigma, \tau \subset X$ finite and disjoint. Let $Y_{\sigma, \tau} \subset A$ be a maximal path such that $\sigma \subset Y_{\sigma, \tau}$ and $Y_{\sigma, \tau} \cap \tau = \emptyset$. $Y_{\sigma, \tau}$ is a positive base set, so any element $c$ realizing the corresponding $\varphi$-type will satisfy $a \in \sigma \rightarrow \varphi(c; a)$ and $b \in \tau \rightarrow \neg \varphi(c; b)$. Thus $\varphi$ has the independence property on $X$. \hfill $\Box$

Lemma 6.3. (Springboard lemma for 2) If $\varphi$ is stable then there is a finite localization $P^f_1$ for which TFAE:

1. There exists $X \subset P^f_1$, $X$ an $(\omega, 2)$-array wrt $P_2$
2. There exists $Y \subset P^f_1$, $Y$ an $(\omega, 2)$-array wrt $P_\infty$

Proof. Choose the localization $P^f_1$ according to Observation 5.3 so that neither $\varphi$ nor $\varphi_2$ are dividable using parameters from $P^f_1$. This is possible because stable formulas are simple and low, and $\varphi$ stable implies $\varphi_2$ stable. Let $Z = \langle c^i_t : t < 2, i < \omega \rangle \subset P^f_1$ be an indiscernible sequence of pairs which is an $(\omega, 2)$-array for $P_2$. Each of the sub-sequences $\langle c^i_0 : i < \omega \rangle$, $\langle c^i_1 : i < \omega \rangle$ is indiscernible, so will be either $P_2$-complete or $P_2$-empty; by choice of $P^f_1$, they cannot be empty.

It remains to show that any path $X \subset Z$ is a $P_\infty$-complete graph. Suppose not, and let $n$ be minimal so that the $n$-type of some increasing sequence of elements $z^1_1, \ldots, z^n_n$ implies $\neg \exists x (\bigwedge_{i<n} \forall \varphi(x; z^i_t))$. Choose an infinite indiscernible subsequence of pairs $Z' \subset Z^2$ of the form $\langle c^i_0, c^i_{i+1} : i \in W \subset \omega \rangle$. Then the set $\{ \varphi(x; c^i_0) \land \varphi(x; c^i_{i+1}) : i \in W \}$ will be $1$-consistent by definition but $n$-inconsistent by assumption (though not necessarily sharply $n$-inconsistent). This contradicts the assumption that $\varphi_2$ is not dividable in $P^f_1$. \hfill $\Box$

When the formula is low but not necessarily simple, bootstrapping up to $P_\infty$ is still possible but requires a stronger initial assumption on the array.

Corollary 6.4. Suppose the formulas $\varphi$ and $\varphi_2$ are low. Then there exists $k < \omega$ such that, in any localization $P^f_1$, TFAE:

1. There exists $X \subset P^f_1$, $X$ an $(\omega, 2)$-array wrt $P_k$
2. There exists $Y \subset P^f_1$, $Y$ an $(\omega, 2)$-array wrt $P_\infty$

Proof. Let $k_0$ be a uniform finite bound on the arity of dividing of instances of $\varphi$ and $\varphi_2$, using lowness; by the proof of the previous Lemma, any $k > 2k_0$ will do. \hfill $\Box$

Recall from Definition 2.1 that an “empty pair” is the $T_0$-configuration given by $V_x = 2, E_x = \{\{1\}, \{2\}\}$, i.e., a pair $y, z$ such that $P_1(y), P_1(z)$ but $\neg P_2(y, z)$.

Lemma 6.5. Suppose $\varphi$ is stable, and that every localization $P^f_1$ around some fixed positive base set $A$ contains an empty pair. Then $P_\infty$ is $(\omega, 2)$.
Proof. Choose $P_1^{f_n}$ to be a localization given by Lemma 6.3. We construct an $(\omega, 2)$-array as follows.

At stage 0, let $c_0^n, c_0$ be any pair of $P_2$-incompatible elements each of which is a consistent 1-point extension of $A$ in $P_1^{f_n}$. At stage $n + 1$, write $C_{n}$ for $\{c_i^t: t < 2, i \leq n\}$ and suppose we have defined $P_{1}^{f_n} \in \text{Loc}_{1}^{C_n}(A)$. By hypothesis, there are $c_{n+1}^0, c_{n+1}^1 \in P_{1}^{f_n}$ such that $\neg P_2(c_{n+1}^0, c_{n+1}^1)$ and such that each $c_{n+1}$ is a consistent 1-point extension of $A$ (Fact 5.18). Let $C_{n+1} = C_{n} \cup \{c_{n+1}^0, c_{n+1}^1\}$ and define $P_{1}^{f_{n+1}} \in \text{Loc}_{1}^{C_{n+1}}(A)$ by

$$P_{1}^{f_{n+1}}(y) = P_{1}^{f}(y) \land P_{2}(y; c_{n+1}^0) \land P_{2}(y; c_{n+1}^1)$$

Thus we construct an $(\omega, 2)$-array for $P_2$, as desired. Applying Lemma 6.3 we obtain an $(\omega, 2)$-array for $P_{\infty}$. \hfill $\square$

**Conclusion 6.6.** Suppose that $\varphi$ is stable, $(T, \varphi) \mapsto \langle P_n \rangle$ and $A$ is a positive base set. Then empty pairs are not persistent around $A$.

**Proof.** By stability, we may work inside the localization given by Lemma 6.3. Suppose empty pairs were persistent around $A$. By Lemma 6.5 $P_{\infty}$ is $(\omega, 2)$, which by Remark 6.2 implies that $\varphi$ has the independence property: contradiction. \hfill $\square$

In order to replace the hypothesis of stable with low, we will need to replace $P_2$-consistency in the proof of Lemma 6.5 with $P_{\infty}$-consistency. This argument is given in full generality in Lemma 6.16 but here we state the result:

**Corollary 6.7.** (to Corollary 6.4) Suppose $\varphi$ and $\varphi_2$ are both low. Suppose every localization $P_1^{f}$ around some fixed positive base set $A$ contains an empty pair. Then $P_{\infty}$ is $(\omega, 2)$.

In fact, modulo the proof of Lemma 6.16 we have shown:

**Conclusion 6.8.** Suppose that $\varphi$ is NIP, $(T, \varphi) \mapsto \langle P_n \rangle$ and $A$ is a positive base set. Then empty pairs are not persistent around $A$.

**Proof.** By Observation 6.5 all NIP formulas are low. By Fact 6.15 $\varphi$ NIP implies $\varphi_2$ is NIP and therefore low. By Corollary 6.7 and Remark 6.2 the persistence of empty pairs would imply $\varphi$ has the independence property, contradiction. \hfill $\square$

6.2. NIP: the case of $n$. We now build a more general framework, working towards Theorem 6.17 which generalizes Conclusions 6.6.6.8 to the case of arbitrary $n < \omega$: if $T$ is NIP then no $P_n$-empty tuple can be persistent. The basic strategy is as follows. If a $P_n$-empty tuple is persistent, Lemma 6.16 produces an $(\omega, n)$-array. In this higher-dimensional case, in order to extract the independence property from an $(\omega, n)$-array via Observation 6.11 we need the array to have an additional property called sharpness. The “sharpness lemma,” Lemma 6.14 returns an array of the correct form at the cost of possibly adding finitely many parameters. Fact 6.15 then pulls this down to the independence property for $\varphi$.

With some care, we are able to get quite strong control on the kind of localization used. When $T$ is stable in addition to NIP, the argument can be done with a uniform finite bound (as a function of $n$) on the arity of the predicates $P_m$ used in localization.

**Definition 6.9.** ($(\omega, n)$-arrays revisited) Assume $n \leq r < \omega$. Compare Definition 4.2: here, the possible ambiguity of the amount of consistency will be important.
Observation 6.11.

(1) The predicate $P_r$ is $(\omega, n)$ if there is $C = \{c^t_i : t < n, i < \omega\} \subseteq P_1$ such that, for all $c^t_{i_1}, \ldots, c^t_{i_r} \in C$,

- $r$-tuples from $r$ distinct columns are consistent, i.e.

\[
\bigwedge_{j,k \leq r} i_j \neq i_k \implies P_r(c^t_{i_1}, \ldots, c^t_{i_r})
\]

- and no column is entirely consistent, i.e. for all $\sigma \subseteq r$, $|\sigma| = n$,

\[
\bigwedge_{j,k \in \sigma} i_j = i_k \implies \neg P_r(c^t_{i_1}, \ldots, c^t_{i_r})
\]

Any such $C$ is an $(\omega, n)$-array. The precise arity of consistency is not specified, see condition (4).

(2) If for all $n \leq r < \omega$, $P_r$ is $(\omega, n)$, say that $P_\infty$ is $(\omega, n)$.

(3) A path through the $(\omega, n)$ array $C$ is a set $X \subseteq C$ which contains no more than $n$-1 elements from each column.

(4) $P_r$ is sharply $(\omega, n)$ if it contains an $(\omega, n)$-array $C$ on which, moreover, for all $\{c^t_{i_1}, \ldots, c^t_{i_r}\} \subseteq C$

\[
P_r(c^t_{i_1}, \ldots, c^t_{i_r}) \iff \bigwedge_{\sigma \subseteq r, |\sigma| = n} \left( \bigwedge_{j,k \in \sigma} i_j = i_k \implies \bigvee_{j \neq k \in \sigma} t_j = t_k \right)
\]

i.e., if every path is a $P_r$-complete graph.

(5) $P_\infty$ is sharply $(\omega, n)$ if $P_r$ is sharply $(\omega, n)$ for all $n \leq r < \omega$.

Remark 6.10.  

(1) Every $(\omega, 2)$-array is automatically sharp.

(2) Suppose $P_\infty$ has an $(\omega, n)$-array; this does not necessarily imply that $P_\infty$ has an $(\omega, m)$-array for $m < n$, because $m$ elements from a single column need not be inconsistent, e.g. if the $(\omega, n)$-array is sharp.

Observation 6.11. If $P_\infty$ is sharply $(\omega, k)$ then $\varphi_{k-1}$ has the independence property.

Proof. Let $X = \langle a^1_i, \ldots, a^k_i : i < \omega \rangle$ be the array in question; then $\varphi_{k-1}$ has the independence property on any maximal path, e.g. $B := \langle a^1_i, \ldots, a^{k-1}_i : i < \omega \rangle$. To see this, fix any $\sigma, \tau \subseteq \omega$ finite disjoint; then by the sharpness hypothesis $\{a^1_i, \ldots, a^{k-1}_i : i \in \sigma\} \cup \{a^j_\tau : j \in \tau\}$ is a $P_\infty$-complete graph and thus corresponds to a consistent partial $\varphi$-type $q$. But any realization $\alpha$ of $q$ cannot satisfy $\varphi(x; a^1_j)$ for any $j \in \tau$, because $P_k$ does not hold on the columns. A fortiori $\neg \varphi_k(\alpha; a^1_j, \ldots, a^{k-1}_j)$.

Let us write down some conventions for describing types in an array.

Definition 6.12. Let $x^t_{i_1}, x^t_{i_2}$ be elements of some $(\omega, n)$-array $X$.

(1) Let $[x] = \{x^t_j \in X : j = i\}$, i.e. the elements in the same column as $x^t_i$.

(2) Let $X_0 = \{x^t_{i_1}, \ldots, x^t_{i_r}\} \subseteq X$ be a finite subset. The column count of $\{x^t_{i_1}, \ldots, x^t_{i_r}\}$ is the unique tuple $(m_1, \ldots, m_\ell) \in \omega^\ell$ such that:

- $m_i \geq m_{i+1}$ for each $i \leq \ell$
- $\Sigma_i m_i = \ell$
• if \( Y_0 = \{ y_1, \ldots, y_r \} \) is a maximal subset of \( X_0 \) such that \( y, z \in Y_0, y \neq z \rightarrow y \notin [z] \), then some permutation of

\[
\left( \left| [y_1] \cap X_0 \right|, \ldots, \left| [y_r] \cap X_0 \right| \right)
\]

is equal to \((m_1, \ldots, m_r)\).

In other words, we count how many elements have been assigned to each column, and put these counts in descending order of size. Write \( \text{col-ct}(\mathfrak{p}) \) for this tuple.

(3) Let \( \leq \) be the lexicographic order on column counts, i.e. \((1, 1, \ldots) < (2, 1, \ldots)\). This is a discrete linear order, so we can define \((m_1, \ldots, m_r)^+ \) to be the immediate successor of \((m_1, \ldots, m_r)\) in this order. Define \( \text{gap}((m_1, \ldots, m_r)) = m_i \) where \(((n_1, \ldots, n_\ell)^+ = (m_1, \ldots, m_\ell) \) and \( \forall j \neq i \ m_j = n_j \), i.e. the value which has just incremented.

By analogy to Lemma 6.3 and its corollary,

Lemma 6.13. (Springboard lemma) Fix \( 2 \leq n < \omega \), and let \( \langle P_n \rangle \) be the characteristic sequence of \((T, \varphi)\). Suppose that the formulas \( \varphi, \varphi_2, \ldots, \varphi_{2^{n-2}} \) are low. Then there exist \( 1 \leq k_0 < \omega \) and a localization \( P^f_1 \) of \( P_1 \) in which the following are equivalent:

1. \( P^f_1 \) contains a sharp \((\omega, n)\)-array for \( P_\mu \), where \( \mu = (2n-2)k_0 \).
2. \( P^f_1 \) contains a sharp \((\omega, n)\)-array for \( P_\infty \).

Proof. Assume (1), so let \( C = \{ c^i_t : t < n, i < \omega \} \subset P^f_1 \) be sharply \((\omega, n)\) for \( P_\mu \), chosen without loss of generality to be an indiscernible sequence of \( n \)-tuples. Fix a path \( Y = y_1, \ldots, y_m \) of minimal size \( m > n \) such that \( \neg P_m(y_1, \ldots, y_m) \). Let \( S := \{ c^0_i, \ldots, c^i_1, \ldots, c^i_n : i < \omega \} \subset C^{2^{n-2}} \) be a sequence of pairs of offset \((n - 1)\)-tuples.

Note that \( S \) is 1-consistent as we assumed (1).

On the other hand, \( C \) is indiscernible, so any increasing sequence of \( m \) elements from \( S \) will cover all the possible \( m \)-types from \( C \). Since \( Y \) is inconsistent, this implies that \( S \) is \( m \)-inconsistent. These \( m \) elements will be distributed over at least \( \frac{m}{2^{n-2}} \) instances of \( \varphi_{2^{n-2}} \); by inductive hypothesis, one fewer element, thus one fewer instance, would be consistent. Thus \( \varphi_{2^{n-2}} \) is sharply \( m' \)-divisible for some \( m' \geq k_0 \).

The appropriate \( k_0 \) is thus a strict upper bound on the possible arity of dividing of each of the formulas \( \{ \varphi_{2^{n-2}} : 1 \leq \ell < k_0 \} \), which exists by lowness. When \( T \) is low but possibly unstable, determining \( k_0 \) is the important step; no localization is then necessary. When \( T \) is stable, however, w.l.o.g. \( k_0 = 2n - 2 \) as by Corollary 5.6 we can simply choose a localization in which the \( 2n - 2 \) formulas are not \( k \)-divisible for any \( k \). \( \square \)

We next give a lemma which will extract a sharp array from an array. Recall that \( P^\infty_\infty \) is the *localized sequence* from Definition 5.4 i.e. the characteristic sequence of the formula \( \varphi(x; y) \land \bigwedge_{a \in \mathfrak{p}} \varphi(x; a) \).

Lemma 6.14. (Sharpness lemma) Let \( \mathfrak{p} \subset P_1 \) be finite, \( n < \omega \). Suppose that \( P^\infty_\infty \) contains an \((\omega, n)\)-array. Then there exist \( \pi', \ell \) with \( \pi \subset \pi' \subset P_1 \) and \( 2 \leq \ell \leq n \) such that \( P^\infty_\infty \) contains a sharp \((\omega, \ell)\)-array.

Proof. Let us show that, given an \((\omega, n)\)-array for \( P^\infty_\infty \), either

• there is a sharp \((\omega, n)\) array for \( P^\infty_\infty \), or else
• by adding no more than finitely many parameters we can construct an \((\omega, \ell)\)-array for \( P^\infty_\infty \) and some \( \ell < n \).
Note that the second is nontrivial by Remark 6.10. As an \((\omega,2)\)-array is automatically sharp, we can then iterate the argument to obtain the lemma.

We have, then, some \((\omega,n)\)-array \(C\) in hand. Without loss of generality \(C\) is an indiscernible sequence of \(n\)-tuples. If every path through \(C\) is a \(P_\infty\)-complete graph then \(C\) is a sharp \((\omega,n)\)-array and we are done. Otherwise, choose some finite \(Z \subset C\) whose column count is as small as possible subject to the conditions:

1. \(Z\) is a path
2. There exists some \(Y \subset C\) such that
   (i) \(Y \cap [Z] = \emptyset\)
   (ii) \(y_1, y_2 \in Y \implies [y_1] \cap [y_2] = \emptyset\)
   but \(Z \cup Y\) is not a \(P_\infty\)-complete graph.

In other words, \(Z\) is a possible new parameter set which is just slightly too large: the subset of \(C\) which is consistent with \(Z\) fails to be an \((\omega,n)\)-array because some set of elements from distinct columns is not consistent relative to \(Z\). Set \(X := Z \cup Y\), where \(Y\) is the finite sequence from (2).

The assumption that \(C\) is not sharp gives an unspecified finite bound on \(|Z|\); in fact the springboard lemma gives a more informative bound \(k \geq 2n - 2\). On the other hand, by definition of \((\omega,n)\)-array, any such \(Z\) must contain at least two elements from the same column, so \(|Z| > 1\) and we can find our witness working upwards on column count. Because \(C\) is an indiscernible sequence of \(n\)-tuples, we may assume that the elements of \(Z\) are in columns which are infinitely far apart. Finally, if \(Z_0 \subsetneq Z\), then col-ct \((Z_0) <\) col-ct \((Z)\). So for any \(W \subset C\) satisfying conditions (2).(i)-(ii) just given, \(Z_0 \cup W\) is a \(P_\infty\)-complete graph.

In particular, we can choose a partition \(X = X_0 \cup X_1\) where

(I) \(X_0 \cap X_1 = \emptyset\), \(\emptyset \subsetneq X_1 \subset X\)
(II) \(x, x' \in X_1 \implies [x] = [x']\)
(III) \(n > \ell := |X_1| = \text{gap(col-ct}(Z)) > 1\)
(IV) For any \(W \subset C\) satisfying conditions (2).(i)-(ii), \(X_0 \cup W\) is a \(P_\infty\)-complete graph.

To finish, let \(a' = a \cup X_0\) and let \(C' \subset C\) be an infinite sequence of \(\ell\)-tuples which realize the same type as \(X_1\) over \(a \cup X_0\). (For instance, restrict \(C'\) to the rows containing elements of \(X_1\) and to infinitely many columns which do not contain elements of \(X_0\).) Since \(Z\) was chosen to be a path, \(\ell < n\) (condition (III)) and \(|a'| < |X| < \omega\). By condition (2), \(\neg P_\ell(\tau)\) for any column \(\tau\) of \(C'\). On the other hand, by condition (IV) any subset of \(C'\) containing no more than one element from each column is a \(P_\infty\)-complete graph. Thus \(C'\) is an \((\omega, \ell)\)-array for \(P_\infty\), as desired. If it is not sharp, repeat the argument. \(\square\)

**Fact 6.15.** The following are equivalent for a formula \(\varphi(x; y)\).

1. \(\varphi\) has the independence property.
2. For some \(n < \omega\), \(\varphi_n\) has the independence property.
3. For every \(n < \omega\), \(\varphi_n\) has the independence property.
4. Some *localization \(\varphi^\tau\) has the independence property.

**Proof.** (1) \(\rightarrow\) (3) \(\rightarrow\) (2) \(\rightarrow\) (1) \(\rightarrow\) (4) are straightforward: use the facts that the formulas \(\varphi_i\)

\(\varphi_j\) generate the same space of types, and that the independence property can be characterized in terms of counting types over finite sets (III.4II.4). Finally, (4) \(\rightarrow\) (2) as we have simply specified some of the parameters. \(\square\)
Lemma 6.16. Suppose that for some \( n < \omega \), every localization of \( P_1 \) around some fixed positive base set \( A \) contains an \( n \)-tuple on which \( P_n \) does not hold. Then \( P_\infty \) is \((\omega, n)\), though not necessarily sharply \((\omega, n)\).

Proof. Let us show that \( P_k \) is \((\omega, n)\) for any \( k \geq n \). This suffices as, by Convention 2.3, we may apply compactness.

Fix \( k \geq n \) and let \( P_1^f \) be any localization, for instance that of Lemma 6.13.

At stage 0, let \( c_0^n \subset P_1^{f_0} := P_1^f \) be an \( n \)-tuple of elements on which \( P_n \) does not hold, chosen by Fact 5.18 so that each \( c_0^n \) is a consistent 1-point extension [in the sense of \( P_n \)] of \( A \). Let \( X_0 = \{ c_0^n \} : i = n, \} \) be the set of these singletons. Write \( \bar{x} \) to denote the elements of \( x \).

Define
\[
P_1^{f_0}(y) = P_1^{f_0}(y) \land \bigwedge_{x \in X_0} P_2(y; \bar{x})
\]
which includes \( A \) by construction.

At stage \( m + 1 \), write \( C_m \) for \( \{ c_i^n : t < n, i \leq m \} \) and consider the localized set of elements \( P_1^{f_m} \in \text{Loc}_{C_m}(A) \). Let
\[
X_m := \{ x \subset C_m : |x| = k - 1 \text{ for all } i < m, |x \cap (C_{i+1} \setminus C_i)| \leq 1 \}
\]
i.e. sets which choose no more than one element from each stage in the construction.

By hypothesis, there are \( c_m^{n+1} \in P_1^{f_m} \) such that \( \lnot P_n(c_m^{n+1}) \) and such that for all \( x \in X_m \), each \( c_m^{n+1} \) is a consistent 1-point extension of \( A \cup x \), in the sense of \( P_n \). Let \( C_{m+1} = C_m \cup \{ c_m^{n+1} \} \), and let \( X_{m+1} \) be the sets from \( C_{m+1} \) which choose no more than one element from each stage in the construction. We now define \( P_1^{f_{m+1}} \in \text{Loc}_{C_{m+1}}(A) \) by
\[
P_1^{f_{m+1}}(y) = P_1^{f_m}(y) \land \bigwedge_{x \in X_{m+1}} P_k(y; \bar{x})
\]
(If \( m < k \), the parameters from \( \bar{x} \) need not necessarily be distinct.) Again, this localization contains \( A \) by construction. Thus we construct an \((\omega, n)\)-array for \( P_k \), as desired. As \( k \) was arbitrary, we finish. \( \square \)

Recall that a \( P_n \)-empty tuple is any \( T_0 \)-configuration for which \( X = n \), \( \{ 1, \ldots, n \} \notin E_x \), i.e. \( y_1, \ldots, y_n \in P_1 \) such that \( \lnot P_n(y_1, \ldots, y_n) \). We are now in a position to prove:

Theorem 6.17. Suppose that \( \varphi \) is NIP, \((T, \varphi) \mapsto \langle P_n \rangle\) and \( A \) is a positive base set for \( \varphi \). Then for each \( n < \omega \), \( P_n \)-empty tuples are not persistent around \( A \).

Proof. By lowness, we work inside the localization \( P_1^f \supset A \) given by the Springboard Lemma 6.13. Suppose that for some \( n < \omega \), \( P_n \)-empty tuples are persistent. Apply Lemma 6.16 to obtain an \((\omega, n)\)-array for \( P_\infty \), which is not necessarily sharp. The Sharpness Lemma 6.14 then gives a sharp \((\omega, \ell)\)-array for \( P_\infty^{\pi} \), where \( \pi \subset P_1 \) is a finite set of parameters. The sequence \( \langle P_\infty^{\pi} \rangle \) is just the characteristic sequence of the “localized formula \( \varphi_\pi \), that is, \( \varphi(x; y) \land \varphi_m(x; \bar{\pi}) \), where \( m = |\pi| \). By Observation 6.11 this means \( \varphi_{\pi}^m \) has the independence property. Now by (2) \( \rightarrow \) (1) of Fact 6.15 applied to \( \varphi_\pi \), \( \varphi_\pi \) has the independence property. By (4) \( \rightarrow \) (1) of the same fact, \( \varphi \) must also have the independence property, contradiction. \( \square \)

Corollary 6.18. Suppose that \( \varphi \) is NIP, \((T, \varphi) \mapsto \langle P_n \rangle\) and \( A \) is a positive base set for \( \varphi \). Then for each \( n < \omega \), there is a localization \( P_1^f \) such that:
Observation 6.22. Suppose that $\varphi$ has the independence property then $P^\varphi$ restriction of $P_1$ containing $A$ on which $P_n$ is a complete graph. In the other direction, if $\varphi$ has the independence property then $P_\infty$ is $(\omega,2)$ by Claim 4.5 so in particular $P_1$ is not a complete graph. In fact:

**Theorem 6.19.** Let $\varphi$ be a formula of $T$ and $\langle P_n : n < \omega \rangle$ its characteristic sequence. Then the following are equivalent for any positive base set $A$:

1. There exists a localization $\varphi^f$ of $\varphi$ such that $\varphi^f$ is NIP and $P_1^f \supset A$.
2. There exists a localization $\varphi^g$ of $\varphi$ such that $\varphi^g$ is stable and $P_1^g \supset A$.
3. For every $n < \omega$, there exists a localization $P_n^* \supset A$ which is a $P_n$-complete graph.

**Proof.** Note that $\varphi^f = \varphi$ and $\varphi^g = \varphi$ are possible.

(2) $\rightarrow$ (1) because stable implies NIP.

(1) $\rightarrow$ (3) is just Theorem 6.17 no $P_n$-empty tuple is persistent, so eventually one obtains a localization which is a complete graph.

(3) $\rightarrow$ (2) By Claim 4.4 if $\varphi^g$ has the order property its associated $P_1^g$ contains a diagram in the sense of Definition 4.2. Thus it contains an empty pair, and so a fortiori a $P_n$-empty tuple, for each $n$. \hfill $\square$

**Example 6.20.** Consider $(\mathbb{Q},<)$, let $\varphi(x;y,z) = y > x > z$ and let the positive base set $A$ be given by concentric intervals $\{(a_i,b_i) : i < \kappa\} \subset P_1$. Then there is indeed a $P_2$-empty pair $(c_1,c_2),(d_1,d_2)$ which are each consistent 1-point extensions of $A$ – namely, any pair of disjoint intervals lying in the cut described by the type corresponding to $A$. Localizing to require consistency with any such pair amounts to giving a definable complete graph containing $A$, i.e. realizing the type.

6.3. Simplicity. We have seen that the natural first question for persistence, whether there exist persistent empty tuples, characterizes stability: Theorem 6.19. Here we will show that a natural next question, whether there exist persistent infinite empty graphs, characterizes simplicity. Recall that a formula is simple if it does not have the tree property; see [5], [14].

Notice that we have an immediate proof of this fact by Observation 6.3 which appealed to finite $D(\varphi,k)$-rank for simple formulas to conclude that infinite empty graphs are not persistent. Let us sketch the framework for a different proof by analogy with the previous section. This amounts to deriving Observation 6.3 directly in the characteristic sequence.

**Remark 6.21.** In the case of stability, much of the work came in establishing sharpness of the $(\omega,\ell)$-array. Here, since the persistent configuration is infinite, we have compactness on our side; we may in fact always choose the persistent empty graphs to be indiscernible and uniformly $k$-consistent but $(k + 1)$-inconsistent, for some given $k < \omega$.

**Observation 6.22.** Suppose that $(T,\varphi) \mapsto \langle P_n \rangle$. Then the following are equivalent:

1. there is a set $T = \{a_\eta : \eta \in 2^{<\omega}\} \subset P_1$ such that, writing $\subseteq$ for initial segment:
   a. For each $\nu \in 2^\omega$, $\{a_\eta : \eta \subseteq \nu\}$ is a complete $P_\infty$-graph.
   b. For some $k < \omega$, and for all $p \in \omega^{<\omega}$, the set $\{a_{p \cdot i} : i < \omega\} \subset P_1$ is a $P_k$-empty graph.
2. $\varphi$ has the $k$-tree property.
Proof. This is a direct translation of Definition 4.1. □

Lemma 6.23. Let $X_k$ be the $T_0$-configuration describing a strict $(k+1)$-inconsistent sequence, i.e. $V_{X_k} = \omega$ and $E_{X_k} = \{\sigma : \sigma \subseteq \omega, |\sigma| \leq k\}$. Suppose that for some fixed $k < \omega$ and some formula $\varphi$, $X_k$ is persistent in the characteristic sequence $\langle P_n \rangle$ of $\varphi$. Then $\varphi$ is not simple.

Proof. Let us show that $\varphi$ has the tree property, around some positive base set $A$ if one is specified. At stage 0, by hypothesis there exists an infinite indiscernible sharply $(k+1)$-inconsistent sequence $Y_0 \subset P_1$, each of whose elements can be chosen to be a consistent 1-point extension of $A$ in the sense of $P_\infty$ by Fact 5.18. Set $a_i$ to be the $i$th element of this sequence, for $i < \omega$.

At stage $t+1$, suppose we have constructed a tree of height $n$, $T_n = \{a_\eta : \eta \in \omega^n\}$ such that, writing $\subseteq$ for initial segment:

- every path is a consistent $n$-point extension of $A$, i.e. $A \cup \{a_\eta : \eta \subseteq \nu\}$ is a complete $P_\infty$-graph, for each $\nu \in \omega^n$;
- for all $0 \leq k < n$ and all $\eta \in \omega^k$, $\{a_{\eta-i} : i < \omega\}$ is $P_k$-complete but $P_{k+1}$-empty.

We would like to extend the tree to level $n+1$, and it suffices to show that the extension of any given node $a_\nu$ (for $\nu \in \omega^n$) can be accomplished. But this amounts to repeating the argument for stage 0 in the case where $A = A \cup \{a_\eta : \eta \subseteq \nu\}$. By assumption and Fact 5.18 this remains possible, so we continue.

Notice that the threat of all possible localizations is what makes continuation possible. That is, the schema which says that “$x$ is a 1-point extension of $A$” simply says that $x$ remains (along with witnesses for $X_k$) in each of an infinite set of localizations of $P_1$ with parameters from $A$. If this schema is inconsistent, there will be a localization contradicting the hypothesis. □

We can now characterize simplicity in terms of persistence:

Theorem 6.24. Let $\varphi$ be a formula of $T$ and $\langle P_n \rangle$ its characteristic sequence.

(1) If the localization $\varphi^f$ of $\varphi$ is simple, then for each $P_\infty$-graph $A \subset P_1^f$ and for each $n < \omega$, there exists a localization $P_1^n \supset A$ of $P_1^f$ in which there is a uniform finite bound on the size of a $P_n$-empty graph, i.e. there exists $m_n$ such that $X \subset P_1^f$ and $X^n \cap P_n = \emptyset$ implies $|X| \leq m_n$.

(2) If localization $\varphi^g$ of $\varphi$ is not simple, then for all but finitely many $r < \omega$, $P_r^g$ contains an infinite $(r + 1)$-empty graph.

In other words, the following are equivalent for any positive base set $A$:

(i) There exists a localization $\varphi^f$ of $\varphi$ (with $\varphi^f = \varphi$ possible) such that $\varphi^f$ is simple and $P_1^f \supset A$.

(ii) For each $n < \omega$, there exists a localization $P_1^n \supset A$ in which there is a uniform finite bound on the size of a $P_n$-empty graph.

Proof. It suffices to show the first two statements. (1) is Lemma 6.23 applied to the formula $\varphi^f$. (2) is the second clause of Observation 6.22 where “almost all” means for $r$ above $k$, the arity of dividing. □
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