A Conjecture on the Amount of Non-Locality

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Abstract

Imagine a world in which there exist physical resources for non-local correlations whose CHSH value lies between 2 and $X \leq 4$. Assume that such resources can be mixed in some sense. Using Connes’s result on the extension of characteristic 1 semi-rings, we conjecture a possible motivation for quantum mechanical resources obeying the Tsirelson bound $X = 2\sqrt{2}$.

1 The CHSH inequality and the Tsirelson bound

In order to define the amount of correlations made available by some resource, assume that Alice and Bob share a box that takes binary inputs $x, y \in \{0, 1\}$ and produces binary outputs $a, b \in \{0, 1\}$. The workings of the box are fully described by a set of joint probabilities $P(ab|xy)$. The correlators $E_{xy} = P(a = b|xy) - P(a \neq b|xy)$ appear in the Clauser-Horne-Shimony-Holt inequality for classical, i.e. local, resources [5]:

$$\text{CHSH} = |E_{00} + E_{01} + E_{10} - E_{11}| \leq 2.$$  \hspace{1cm} (1)

Mathematics suggests that the CHSH value can be as high as 4, therefore realizing full non-locality as described by Popescu and Rorlich [14]. However, quantum mechanical resources only provide a maximum value $2\sqrt{2}$, called the Tsirelson bound [3]. One is tempted to find a compelling reason for this value among various physical or information-theoretic principles [13, 15, 2, 1, 12].

We assume that available resources can be mixed in some sense (Section 2), so that if Alice and Bob can reach a given CHSH value $X \in (2, 4]$, they can combine these non-local resources with local ones and obtain any CHSH value between 2 and $X$. Such resources can then be used simultaneously. No explicit model or expression is given of the physical resources that would lead to an arbitrary CHSH limit in the interval $(2, 4]$. We conjecture that a low-level description of the physics of such resources is irrelevant for the analysis of correlations that Alice and Bob can build by using them. The combination rule alone is sufficient in order to obtain the Tsirelson bound (Section 3).
Combining non-local resources

If available resources permit $\text{CHSH} \leq X$, then any combination of these or less non-local resources must at most permit the same CHSH value. Let us write $R_X$ for a model possessing such resources and use ‘+’ notation to denote a combination of models, i.e., $R_X + R_Y = R_Z$ is a new model whose physical content may be quite different from that of $R_X$ and $R_Y$. The assumption that non-locality does not increase when we add resources more local than a given threshold means that the two-element set $\mathcal{R} = \{R_2, R_X\}$ can be supplied with idempotent addition:

\[ R_2 + R_2 = R_2, \quad R_2 + R_X = R_X = R_2 + R_X = R_2 = R_X. \]  

(2)

We identify this set with a $\mathbb{B}$-module, i.e., a commutative monoid with an external multiplication by the elements of a finite semi-field $\mathbb{B} = \{0, 1\}$. This produces a map $(\lambda, x) \to \lambda x$, where $\lambda \in \mathbb{B}$ and $x \in \mathcal{R}$, with the following properties:

\[ (\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x, \quad \lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2, \quad 1x = x, \quad 0x = 0. \]  

(3)

A few surprises can be expected from this notation in the case $\lambda \in \mathbb{B}$. Our goal is to ‘lift’ this trivial structure on $\mathcal{R}$ to a new structure that would allow to consider simultaneously the combinations of models with different CHSH limits or, in other words, that would extend the notion of sum to all $\lambda \in [0, 1]$. This extension relies on the interpretation of $\mathcal{R}$ as a semi-ring of characteristic 1 [6]. We then build a required extension through another extension known in the literature:

\[ \mathbb{B} \subset \mathbb{R}^{\text{max}}, \]  

(4)

of $\mathbb{B}$ by the semi-field $\mathbb{R}^{\text{max}}$, which plays a central role in idempotent analysis [9] and tropical geometry [11, 16].

If $\mathcal{R}$ is a semi-ring, then we must explain the meaning of multiplication $R_X \cdot R_Y$. We identify $R_X \cdot R_Y$ with a box shared between Alice and Bob that is a sequential wiring of two boxes corresponding to $R_X$ and $R_Y$. Only resources of $R_X$ are available in the first part, which receives the initial input $(x, y)$. Only the output of the first part then serves as input of the second part, and the only available resources are from $R_Y$. Final output $(a, b)$ is the output of the second stage only. The resulting correlations arise from this ‘chain’ of boxes, which is different from the wirings with a memory of the initial input that can distill entanglement and raise the CHSH value [8, 3, 1]. The multiplicative structure on $\mathcal{R}$ is a monoid and distributivity law holds. One can see that

\[ R_2 \cdot R_X = R_X \cdot R_2 = R_2 \quad \forall X, \]  

(5)

because inserting a local stage into any chain of wired boxes reduces final correlations to purely local. This semi-ring is multiplicatively cancellative:

\[ X \neq 2, \quad R_X \cdot R_Y = R_X \cdot R_Z \quad \Rightarrow \quad R_Y = R_Z. \]  

(6)
Further assumptions are needed in order to apply to $R$ Connes’s result \[7\]. We formally assume that the semi-ring is perfect, i.e., the map $R_X \to R_X^n$ is a bijection for any $n \in \mathbb{N}$, and the multiplicative group is uniquely divisible, so that expressions $\theta_\alpha(R_X) = R_X^\alpha$ make sense for all rational $\alpha$. In this case we obtain a partially ordered vector space over $\mathbb{Q}$ and, following Connes \[6, Section 5.2\], we assume that it is a partially ordered space over $\mathbb{R}$. Then there exists a deformation of $R$ into a semi-ring of characteristic zero that allows us to express the sum of multiplicative ‘lifts’ of $R_X$ and $R_Y$ as:

$$R_X + \omega R_Y = \sum_{\alpha \in I} \omega(\alpha, T) R_X^\alpha R_Y^{1-\alpha},$$  \hspace{1cm} (7)

where $I = \mathbb{Q} \cap (0, 1)$, $T > 0$ and the coefficients

$$\omega(\alpha, T) = e^{-T \alpha} \alpha^{1-\alpha} e^{T(1-\alpha)} = e^{TS(\alpha)},$$ \hspace{1cm} (8)

where $S(\alpha)$ is the entropy of a simple binary mixing:

$$S(\alpha) = -\alpha \log \alpha - (1-\alpha) \log(1-\alpha).$$ \hspace{1cm} (9)

Formula (7) resembles the extension of $\mathbb{B}$ to the tropical field $\mathbb{R}_{+}^{\max}$ and it helps in our case to study models $R_Y$, $2 \leq Y \leq X$, whose maximum CHSH values lie between two given extremes. Most importantly, although we may know the physical content of $R_2$ and $R_X$, nothing can be said about the physical content of $R_Y$: these mixed models are studied purely formally and only with respect to their CHSH limit.

3 Main conjecture

We have conjectured in (7) how to build models whose resources reach intermediate CHSH limit values. This construction depends on the choice of parameter $T$, for which the value $T \sim h$ was suggested \[9\] providing an analogy with the generating functional in quantum field theory \[6, Section 7.7\]. This makes sense if we study the field of real numbers but is misleading in the case of semi-field $R$ composed of models. Another analogy might be developed with free energy in thermodynamics, suggesting that $T$ should be linked with the Boltzmann constant. However, we focus here on a simple case described qualitatively in (9). If only binary mixings of $R_X$ and $R_Y$ with different ‘weights’ are taken into account, it is reasonable to see an analogy between the exponent in (8) and the partition function of a binary distribution, therefore taking $T = 1$. As a result, the model combining $R_2$ and $R_X$ is written as:

$$R_2 + \omega R_X = \sum_{\alpha \in I} \omega(\alpha, 1) R_2^\alpha R_X^{1-\alpha} = \int_0^1 \omega(\alpha, 1) R_2^\alpha R_X^{1-\alpha} d\alpha,$$ \hspace{1cm} (10)

where we have formally replaced the sum by the idempotent integral over $[0, 1]$ and extended $\omega(0, T) = \omega(1, T) = 0.$
Our final conjecture is a bold one. Once the combination of $R_2$ and $R_X$ has been defined, we assume that formula \[ (11) \] becomes a formula for the CHSH limit value of this combination as a function of the CHSH limit values of the summands, equal to 2 and $X$ respectively, when the idempotent integral gets replaced by an ordinary one. If we write $R_Z = R_2 + \omega R_X$ with idempotent addition, then

\[
Z = \int_0^1 \omega(\alpha, 1)2^\alpha X^{1-\alpha}d\alpha = \int_0^1 e^{-\alpha \log \alpha - (1-\alpha) \log(1-\alpha)} 2^\alpha X^{1-\alpha}d\alpha. \tag{11}
\]

But $Z$ is a sum of four correlators; hence it cannot exceed 4. This gives an upper bound on $X$:

\[
\int_0^1 \omega(\alpha, 1)2^\alpha X_{\text{max}}^{1-\alpha} = \int_0^1 e^{-\alpha \log \alpha - (1-\alpha) \log(1-\alpha)} 2^\alpha X_{\text{max}}^{1-\alpha}d\alpha = 4. \tag{12}
\]

If we now solve this equation numerically, we obtain

\[
X_{\text{max}} = 2.82355 \ldots, \tag{13}
\]

which is very close to $2\sqrt{2} = 2.82843 \ldots$. This numeric coincidence may serve as a foundation for the following conjecture:

**Conjecture 3.1.** Let the physical world provide local as well as non-local resources that can be combined and their different combinations used simultaneously. Then the maximum allowed CHSH limit is slightly less than the theoretical Tsirelson bound $2\sqrt{2}$ with an error of 0.17%.

### 4 Conclusion

Many highly speculative, conjectural steps were involved in getting to (3.1). However, a few meaningful assumptions may merit special attention:

1. Models with different CHSH limit values form a $\mathbb{B}$-module with idempotent addition and therefore possess a partial order, which is somewhat analogous to Birkhoff’s results in lattice theory \[ [11] \]. However, it is not clear whether there exists a further analogy between quantum logic and the category of $\mathbb{B}$-modules.

2. In order to define a combination of models with different CHSH limit values, it is not mandatory to give their low-level physical description. Global properties of such models, e.g., the amount of correlations that one can obtain by using them, can be studied independently.

3. The multiplicative structure of models, i.e., their wirings in chains of boxes, is a powerful tool for approaching the problem of model combination. These chains alone suffice in order to give an extension of the original finite semi-ring into a rich structure of models, whose physical meaning remains unknown in spite of the possibility to study the amount of correlations that they provide.
Finally, our conjecture suggests a small but finite difference between the maximum CHSH value in the physical world and the theoretical Tsirelson bound. Although it is beyond reach of today’s technology, this difference may eventually become observable. This will either disprove or confirm our conjecture.

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References

[1] J. Allcock, N. Brunner, N. Linden, S. Popescu, P. Skrzypczyk, and T. Vértesi. Closed sets of nonlocal correlations. Phys. Rev. A, 80(6):062107, 2009.
[2] G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger. Limit on nonlocality in any world in which communication complexity is not trivial. Phys. Rev. Lett., 96(25):250401, 2006.
[3] N. Brunner and P. Skrzypczyk. Nonlocality distillation and postquantum theories with trivial communication complexity. Phys. Rev. Lett., 102:160403, 2009.
[4] B. S. Cirel’son. Quantum generalizations of Bell’s inequality. Lett. Math. Phys., 4(2):93–100, 1980.
[5] J.F. Clauser, R.A. Holt, M.A. Horne, and A. Shimony. Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett., 23:880–884, 1969.
[6] A. Connes. The Witt construction in characteristic one and quantization. arXiv:1009.1769.
[7] A. Connes and C. Consani. Characteristic one, entropy and the absolute point. In Proceedings of the JAMI Conference 2009. Johns Hopkins University Press, 2011. in press.
[8] M. Forster, S. Winkler, and S. Wolf. Distilling nonlocality. Phys. Rev. Lett., 102:120401, 2009.
[9] V.N. Kolokol’tsov and V. P. Maslov. Idempotent analysis as a tool of control theory and optimal synthesis. 2. Functional Analysis and Its Applications, 23:300–307, 1989.
[10] P. Lescot. Algèbre absolue. Ann. Sci. Math. Québec, 33(1):63–82, 2009.
[11] G. L. Litvinov. Maslov dequantization, idempotent and tropical mathematics: A brief introduction. Journal of Mathematical Sciences, 140(3):426–444, 2007.
[12] M. Navascués and H. Wunderlich. A glance beyond the quantum model. Proceedings of the Royal Society A, 466:881–890, 2010.
[13] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Żukowski. Information causality as a physical principle. Nature, 461:1101–1104, 2009.
[14] S. Popescu and D. Rohrlich. Nonlocality as an axiom for quantum theory. Foundations of Physics, 24:379, 1994. quant-ph/9508009
[15] W. van Dam. Implausible consequences of superstrong nonlocality. arXiv: quant-ph/0501159
[16] O. Viro. Basic notions of tropical geometry. In Russian, http://www.pdmi.ras.ru/~olegviro/tc-rus.pdf.