ON THE INVARIANCE PRINCIPLE FOR A CHARACTERISTIC FUNCTION

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In respectful memory of Sergei Nikolaevich Naboko

Abstract. We extend the invariance principle for a characteristic function of a dissipative operator with respect to the group of affine transformations of the real axis preserving the orientation to the case of general \( SL_2(\mathbb{R}) \) transformations.

1. Introduction

In \cite{18} we introduced the concept of a characteristic function associated with a triple of operators consisting of (i) a densely defined symmetric operator with deficiency indices \((1, 1)\), (ii) its quasi-self-adjoint dissipative extension and (iii) a (reference) self-adjoint extension. This concept turns out to be convenient in two respects. On the one hand, the issue of choosing an undetermined constant phase factor in the definition of the characteristic function of an unbounded dissipative operator (a quasi-self-adjoint extension), which is due to M. S. Livšic \cite{14} (also see \cite{1}), is cleared. On the other hand, the characteristic function of a triple determines the whole triple up to mutual unitary equivalence, provided that the underlying symmetric operator is prime (see the corresponding uniqueness theorem in \cite{18}).

From technical point of view, the characteristic function of a triple has shown itself as an adequate tool for solving certain problems in operator theory. For instance, the solution of the Jørgensen-Muhly problem \cite{12} presented in \cite{19, 20} (in the particular case of the deficiency indices \((1, 1)\)) led to the complete classification of simplest solutions of the classical commutation relations in the form

\[
U_t \hat{A} U_t^* = \hat{A} + tI \quad \text{on Dom}(\hat{A}), \quad t \in \mathbb{R},
\]

where \(\hat{A}\) is a dissipative quasi-self-adjoint extension of a symmetric operator \(\hat{A}\) with deficiency indices \((1, 1)\) and \(U_t\) is a strongly continuous one-parameter group of unitary operators. Recall that the Jørgensen-Muhly problem is to provide an intrinsic characterization of symmetric operators \(\hat{A}\) satisfying the commutation relation (1.1).

In this context it is worth mentioning that the solution of the problem was based on the study of the transformation properties of the characteristic function of a triple of operators (generated by the symmetric and dissipative solutions of (1.1) and augmented by a reference self-adjoint extension of \(\hat{A}\)) with respect to affine transformations of the operators of the triple (18, 23).

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As opposed to the familiar transformation law for the characteristic function $S_{\hat{A}}(z)$ of a bounded dissipative operator $\hat{A}$ introduced by M. S. Livšic in [15] (also see [17]),

\[ S_{f(\hat{A})} \circ f = S_{\hat{A}} \]

valid with respect to affine transformations $f(z) = az + b$, $a > 0$, $b \in \mathbb{R}$, the extension of the law (1.2) to the case of triples of unbounded operators requires appropriate modifications related to alignment of relevant phase factors (see [13, 20], Theorem F.1, Appendix F).

The goal of this Note is to extend the transformation law to the case of general automorphisms $f \in Aut(C_+)$ of the upper half-plane $C_+$. The corresponding main result of this Note is as follows, see Theorem 4.2.

Given $f \in Aut(C_+)$ and a triple $A = (\hat{A}, \hat{A}, A)$, with $\hat{A}$ a prime symmetric operator, $\hat{A}$ a maximal dissipative extension of $\hat{A}$ and $A$ its (reference) self-adjoint extension, we show that if the preimage $\omega = f^{-1}(\infty)$ belongs to the spectrum of $\hat{A}$, then the triple

\[ f(A) = (f(\hat{A}), f(\hat{A}), f(A)) \]

is well defined. Moreover, for an appropriately normalized $\hat{S}_A(z)$ characteristic function of the triple $A$ the invariance relation

\[ \hat{S}_{f(A)} \circ f = \hat{S}_A \]

holds (see (1.4) for the definition of the normalized characteristic function $\hat{S}_A(z)$).

If, instead, $\omega = f^{-1}(\infty)$ is a regular point of the dissipative operator $\hat{A}$, we relate the characteristic function $S_{f(\hat{A})}(z)$ of the bounded dissipative operator $f(\hat{A})$ to the characteristic function $S_A(z)$ of the triple as

\[ S_{f(\hat{A})} \circ f = \Theta_f S_A(z), \]

with $\Theta_f = S_\hat{A}(\omega + i0)$ a unimodular constant factor.

As an application of the extended invariance principle we obtain identities relating the Friedrichs and Krein-von Neumann extensions of model homogeneous non-negative symmetric operators and their inverses, see Theorem 7.5.

2. Preliminaries and Basic Definitions

Let $\hat{A}$ be a densely defined symmetric operator with deficiency indices $(1, 1)$ and $A$ its self-adjoint (reference) extension.

Following [1, 11, 13, 14] recall the concept of the Weyl-Titchmarsh and Livšic functions associated with the pair $(\hat{A}, A)$.

Suppose that (normalized) deficiency elements $g_{\pm}$,

\[ g_{\pm} \in \text{Ker}(\hat{A}^* + iI), \quad \|g_{\pm}\| = 1, \]

are chosen in such a way that

\[ g_+ - g_- \in \text{Dom}(A). \]
Consider the Weyl-Titchmarsh function

\[ M(z) = ((Az + I)(A - zI)^{-1}g_+, g_+), \quad z \in \mathbb{C}_+, \]

associated with the pair \((\hat{A}, A)\) and also the Livšic function

\[ s(z) = \frac{z - i}{z + i} \cdot \frac{(g_+, g_-)}{(g_+, g_+)} , \quad z \in \mathbb{C}_+, \]

\[ 0 \neq g_z \in \operatorname{Ker}(\hat{A}^* - zI), \quad z \in \mathbb{C}_+. \]

Recall the important relationship that links the Weyl-Titchmarsh and Livšic functions

\[ s(z) = M(z) - iM(z) + i , \quad z \in \mathbb{C}_+. \]

If \(\hat{A} \neq (\hat{A})^*\) is a maximal dissipative extension of \(\hat{A}\),

\[ \text{Im}(\hat{A}f, f) \geq 0, \quad f \in \text{Dom}(\hat{A}), \]

then \(\hat{A}\) is automatically quasi-self-adjoint \([3, 8, 21, 24]\) and therefore

\[ g_+ - \kappa g_- \in \text{Dom}(\hat{A}) \quad \text{for some } |\kappa| < 1. \]

By definition, we call \(\kappa\) the von Neumann parameter of the triple \(\mathfrak{A} = (\hat{A}, \hat{A}, A)\).

Given \((2.2)\) and \((2.4)\), define the characteristic function \(S_{\mathfrak{A}}(z)\) associated with the triple \(\mathfrak{A} = (\hat{A}, \hat{A}, A)\) as follows (see \([18, \text{ cf. } 13]\))

\[ S_{\mathfrak{A}}(z) = \frac{s(z) - \kappa}{\kappa s(z) - 1}, \quad z \in \mathbb{C}_+, \]

where \(s(z) = s(\hat{A}, A)(z)\) is the Livšic function associated with the pair \((\hat{A}, A)\).

We remark that the von Neumann parameter \(\kappa\) can explicitly be evaluated in terms of the characteristic function of the triple \((\hat{A}, \hat{A}, A)\) as

\[ \kappa = S_{(\hat{A}, \hat{A}, A)}(i). \]

Moreover, as it follows from \((2.8)\), the Livšic function associated with the pair \((\hat{A}, A)\) admits the representation

\[ s(z) = s(\hat{A}, A)(z) = \frac{S_{\mathfrak{A}}(z) - \kappa}{\kappa S_{\mathfrak{A}}(z) - 1}, \quad z \in \mathbb{C}_+, \]

Summing up, the calculation of the main characteristics (unitary invariants) of the triple \(\mathfrak{A} = (\hat{A}, \hat{A}, A)\) can be performed in accordance with the following algorithm: at the first step, deficiency elements \(g_{\pm}\) that satisfy the relation \((2.1)\) are to be found, then one calculates the Weyl-Titchmarsh \((2.3)\) or/and Livšic \((2.4)\) functions depending of whether the resolvent of the reference self-adjoint operator \(A\) or the “deficiency” field \(\mathbb{C}_+ \ni z \mapsto g_z (2.3)\) is available. In the next step, the von Neumann parameter \(\kappa\) of the triple \((2.7)\) is to be determined and finally, one arrives at the characteristic function \(S_{\mathfrak{A}}(z)\) of the triple given by \((2.8)\).

\(^1\)Paying tribute to historical justice it is worth mentioning that the function \(M(z)\) has been introduced by Donoghue in \([8]\). However, as one can see from \([8, \text{ eq. (5.42)}]\), it is elementary to express \(M(z)\) in terms of the classical Weyl-Titchmarsh function which explains the terminology we use.
One can take a different point of view as presented in \[\text{[18]}\] and in this way we come to a functional model of a prime dissipative triple in which the characteristic functions is considered as a parameter of the model.

3. A FUNCTIONAL MODEL OF A TRIPLE

Given a contractive analytic map \( S \),

\[
S(z) = \frac{s(z) - \varkappa}{z s(z) - 1}, \quad z \in \mathbb{C}_+,
\]

where \(|\varkappa| < 1\) and \(s(z)\) is an analytic, contractive function in \(\mathbb{C}_+\) satisfying the Livšic criterion \[\text{[14]}\] (also see \[\text{[18], Theorem 1.2}\]), that is,

\[
s(i) = 0 \quad \text{and} \quad \lim_{z \to \infty} z (s(z) - e^{2i\alpha}) = \infty \quad \text{for all} \quad \alpha \in [0, \pi),
\]

\[
0 < \varepsilon \leq \arg(z) \leq \pi - \varepsilon,
\]

introduce the function

\[
M(z) = \frac{1}{i} \frac{s(z) + 1}{s(z) - 1}, \quad z \in \mathbb{C}_+.
\]

In this case, the function \(M(z)\) admits the representation,

\[
M(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) \, d\mu(\lambda), \quad z \in \mathbb{C}_+,
\]

for some infinite Borel measure \(\mu(d\lambda)\),

\[
\mu(\mathbb{R}) = \infty,
\]

such that

\[
\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1.
\]

In the Hilbert space \(L^2(\mathbb{R}; d\mu)\) introduce the (self-adjoint) operator \(B\) of multiplication by independent variable on

\[
\text{Dom}(B) = \left\{ f \in L^2(\mathbb{R}; d\mu) \left| \int_{\mathbb{R}} \lambda^2 |f(\lambda)|^2 d\mu(\lambda) < \infty \right. \right\}
\]

and denote by \(\hat{B}\) its symmetric restriction on

\[
\text{Dom}(\hat{B}) = \left\{ f \in \text{Dom}(B) \left| \int_{\mathbb{R}} f(\lambda) d\mu(\lambda) = 0 \right. \right\}.
\]

Next, introduce \(\hat{B}\) as the dissipative quasi-self-adjoint extension of the symmetric operator \(\hat{B}\) on

\[
\text{Dom}(\hat{B}) = \text{Dom}(\hat{B}) + \text{lin span} \left\{ \frac{1}{\lambda - i} - \frac{\varkappa}{\lambda + i} \right\},
\]

where the von Neumann parameter \(\varkappa\) of the triple \((\hat{B}, \hat{B}, B)\) is given by

\[
\varkappa = S(i).
\]

Notice that in this case,

\[
\text{Dom}(B) = \text{Dom}(\hat{B}) + \text{lin span} \left\{ \frac{1}{\lambda - i} - \frac{1}{\lambda + i} \right\}.
\]
We will refer to the triple \((\mathcal{B}, \hat{\mathcal{B}}, \mathcal{B})\) as the model triple in the Hilbert space \(L^2(\mathbb{R}; d\mu)\) parameterized by the characteristic functions \(S_{\mathcal{B}}(z) = S(z)\).

**Remark 3.1.** Notice that the core of the spectrum \(\hat{\sigma}(\mathcal{B})\) of the symmetric operator \(\mathcal{B}\) can be characterized as

\[
\hat{\sigma}(\mathcal{B}) = \left\{ s \in \mathbb{R} \mid \int_{\mathbb{R}} \frac{d\mu(\lambda)}{(\lambda - s)^2} = \infty \quad \text{and} \quad \mu(\{s\}) = 0 \right\}.
\]

Here \(\mu(d\lambda)\) is the measure \((3.4)\) associated with the pair \((\mathcal{B}, \mathcal{B})\) from the model representation of the triple \(\mathcal{B}\).

Recall that the core \(\hat{\sigma}(\mathcal{B})\) of the spectrum of a prime symmetric operator \(\mathcal{B}\) is just the complement of the set of its quasi-regular points,

\[
\hat{\sigma}(\mathcal{B}) = \mathbb{R} \setminus \hat{\rho}(\mathcal{B})
\]

For the convenience of the reader we provide a short proof of \((3.11)\).

**Proof.** As in [23], introduce the set

\[
\mathcal{P} = \left\{ s \in \mathbb{R} \mid \int_{\mathbb{R}} \frac{d\mu(\lambda)}{(\lambda - s)^2} < \infty \quad \text{or} \quad \mu(\{s\}) > 0 \right\},
\]

which apparently is the complement of the right hand side of \((3.11)\).

We claim that \(\mathcal{P}\) coincides with the set of quasi-regular points \(\hat{\rho}(\mathcal{B})\) of the symmetric (prime) operator \(\mathcal{B}\).

Indeed, it is well known (see, e.g., [10, 23]) that the set \(\mathcal{P}\) does not depend on the choice of the self-adjoint (reference) extension \(\mathcal{B}\) of the symmetric operator \(\mathcal{B}\). Therefore, given \(s \in \mathcal{P}\), without loss we may assume that \(\ker(\mathcal{B} - sI) = \{0\}\).

Hence, \(s \in \rho(\mathcal{B}) \subseteq \hat{\rho}(\mathcal{B})\), which proves the inclusion

\[
\mathcal{P} \subseteq \hat{\rho}(\mathcal{B})
\]

On the other hand, if \(t \in \hat{\rho}(\mathcal{B})\), one can always choose a self-adjoint extension \(\mathcal{B}'\) of \(\hat{\mathcal{B}}\) such that \(t\) is an eigenvalue of \(\mathcal{B}'\). If \(d\mu'(d\lambda)\) is the representing measure associated with the corresponding model representation for the pair \((\hat{\mathcal{B}}, \mathcal{B}')\) in the space \(L^2(\mathbb{R}, d\mu')\), then

\[
\mu'(\{t\}) > 0
\]

and therefore, \(t \in \mathcal{P}\) (here we have again used the independence of the set \(\mathcal{P}\) from the choice of the (reference) extension \(\mathcal{B}'\)). Thus, \(\hat{\rho}(\mathcal{B}) \subseteq \mathcal{P}\) and hence

\[
\mathcal{P} = \hat{\rho}(\mathcal{B}),
\]

as stated and \((3.11)\) follows. \(\square\)

Let \(\hat{A}\) be a densely defined symmetric operator with deficiency indices \((1, 1)\), \(\hat{A}\) a maximal non-selfadjoint dissipative extension of \(\hat{A}\) and \(A\) its self-adjoint (reference) extension. Given the triple \(\mathfrak{A} = (\hat{A}, A, \hat{A})\), recall that if the symmetric operator \(\hat{A}\) is prime, then both the Weyl-Titchmarsh function \(M(z)\) and the Livšic function \(s(z)\) are complete unitary invariants of the pair \((\hat{A}, A)\), while the characteristic function \(S_{\mathfrak{A}}(z)\) is a complete unitary invariant of the triple \(\mathfrak{A}\) (see [18]). In particular, the von Neumann parameter \(\kappa\) is a unitary invariant of the triple \((\hat{A}, \hat{A}, A)\), not a complete unitary invariant though. Also notice that if the symmetric operator \(\hat{A}\)

\[\text{A symmetric operator } \hat{A} \text{ is called a prime operator if there is no (non-trivial) subspace invariant under } \hat{A} \text{ such that the restriction of } \hat{A} \text{ to this subspace is self-adjoint.}\]
from a triple in the Hilbert space $\mathcal{H}$ is not prime and $\dot{A}'$ is the prime part of $\dot{A}$ in a reducing subspace $\mathcal{H}' \subset \mathcal{H}$, then the triples $(\dot{A}, \dot{A}, A)$ and $(\dot{A}|_{\mathcal{H}'}, \dot{A}|_{\mathcal{H}'}, A|_{\mathcal{H}'})$ have the same characteristic function, which in many cases allows one to focus on the case where $\dot{A}$ is a prime operator.

**Theorem 3.2** ([15], Theorems 1.4, 4.1). Suppose that $\dot{A}$ and $\dot{B}$ are prime, closed, densely defined symmetric operators with deficiency indices $(1,1)$. Assume, in addition, that $A$ and $B$ are some self-adjoint extensions of $\dot{A}$ and $\dot{B}$ and that $\dot{A}$ and $\dot{B}$ are maximal dissipative extensions of $\dot{A}$ and $\dot{B}$, respectively ($\dot{A} \neq (\dot{A})^*$ and $\dot{B} \neq (\dot{B})^*$).

Then,

(i) the triples $(\dot{A}, \dot{A}, A)$ and $(\dot{B}, \dot{B}, B)$ are mutually unitarily equivalent if and only if, the corresponding characteristic functions of the triples coincide;

(ii) the triple $(\dot{A}, \dot{A}, A)$ is mutually unitarily equivalent to the model triple $(\dot{B}, \dot{B}, B)$ in the Hilbert space $L^2(\mathbb{R}; d\mu)$, where $\mu(d\lambda)$ is the representing measure for the Weyl-Titchmarsh function $M(z) = M_{(\dot{A}, A)}(z)$ associated with the pair $(\dot{A}, A)$.

In particular,

(iii) the pairs $(\dot{A}, A)$ and $(\dot{B}, B)$ are mutually unitary equivalent if and only if $M_{(\dot{A}, A)}(z) = M_{(\dot{B}, B)}(z)$.

For the further reference recall the following resolvent formula [15].

**Theorem 3.3.** Suppose that $\mathcal{B} = (\dot{B}, \dot{B}, B)$ is the model triple in the Hilbert space $L^2(\mathbb{R}; d\mu)$ given by \([\mathbf{7}]-\mathbf{9}\).

Then the resolvent of the model dissipative operator $\dot{B}$ in $L^2(\mathbb{R}; d\mu)$ has the form

\[
(\dot{B} - zI)^{-1} = (B - zI)^{-1} - p(z)(\cdot, g_z)g_z,
\]

where

\[
p(z) = \left( M_{(\dot{B}, B)}(z) + i\frac{z + 1}{z - 1} \right)^{-1},
\]

\[z \in \rho(\dot{B}) \cap \rho(B).
\]

Here $M_{(\dot{B}, B)}(z)$ is the Weyl-Titchmarsh function associated with the pair $(\dot{B}, B)$ continued to the lower half-plane by the Schwarz reflection principle, $\varpi$ is the von Neumann parameter of the triple $\mathcal{B}$, and the deficiency elements $g_z$,

\[g_z \in \text{Ker}((\dot{B})^* - zI), \quad z \in \mathbb{C} \setminus \mathbb{R},\]

are given by

\[
g_z(\lambda) = \frac{1}{\lambda - z} \quad \text{for } \mu\text{-almost all } \lambda \in \mathbb{R}.
\]

**Remark 3.4.** Notice that if $z = 0$ is a quasi-regular point for the symmetric operator $\dot{B}$, $0 \in \rho(\dot{B})$, and therefore

\[0 \in \rho(B) \cap \rho(\dot{B}),
\]

We say that triples of operators $(\dot{A}, \dot{A}, A)$ and $(\dot{B}, \dot{B}, B)$ in Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are mutually unitarily equivalent if there is a unitary map $U$ from $\mathcal{H}_A$ onto $\mathcal{H}_B$ such that $\dot{B} = U\dot{A}U^{-1}$, $B = U\dot{A}U^{-1}$, and $B = U\dot{A}U^{-1}$.
then the inverse $\tilde{B}^{-1}$ is a rank-one perturbation of the bounded self-adjoint operator $B^{-1}$ and the following resolvent formula

$$\tilde{B}^{-1} = B^{-1} - pQ$$

holds. Here

$$(3.16)$$

$$p = \left( M(0) + i \frac{x + 1}{x - 1} \right)^{-1},$$

$Q$ is a rank-one self-adjoint operator given by

$$(Qf)(\lambda) = \frac{1}{\lambda} \int_{\mathbb{R}} \frac{f(s)}{s} \mu(ds), \quad \mu \text{-a.e. } \lambda \in \mathbb{R},$$

and $M(0) = M(0 + i0)$ is the boundary value of the Weyl-Titchmarsh function associated with the pair $(\tilde{B}, B)$ at the point zero.

4. The invariance principle

Let $\hat{A}$ be a densely defined symmetric operator with deficiency indices $(1, 1)$, $A$ its self-adjoint (reference) extension and $\tilde{A}$ a maximal non-self-adjoint dissipative extension of $\hat{A}$. In this case, the dissipative operator $\tilde{A}$ is automatically a quasi-self-adjoint dissipative extension of $\hat{A}$ (see, e.g., [18]),

$$\hat{A} \subset \tilde{A} \subset (\hat{A})^*$$

and

$$\hat{A} = \tilde{A}|_{\text{Dom}(\tilde{A}) \cap \text{Dom}(\hat{A}^*)}.$$

Throughout this Note such kind of triples $\mathfrak{A} = (\hat{A}, \tilde{A}, A)$ will be called regular.

First, recall the concept of an invariance principle for the characteristic function of a triple $\mathfrak{A} = (\hat{A}, \tilde{A}, A)$ with respect to affine transformations of the operators $\hat{A}$, $\tilde{A}$ and $A$ from $\mathfrak{A}$.

Given an affine transformation of the upper half-plane, $f(z) = az + b$, $a, b \in \mathbb{R}$, $a > 0$, introduce the triple

$$f(\mathfrak{A}) = (f(\hat{A}), f(\tilde{A}), f(A)),$$

where

$$f(X) = aX + bI \quad \text{on} \quad \text{Dom}(X), \quad X = \hat{A}, \tilde{A}, A.$$

Clearly, $f(\hat{A})$ is a symmetric operator with deficiency indices $(1, 1)$, $f(A)$ is its self-adjoint extension, and $f(\tilde{A})$ is a quasi-self-adjoint dissipative extension of $f(\hat{A})$.

Therefore, the triple $f(\tilde{A})$ is regular and hence the characteristic function $S_{\mathfrak{A}}(z)$ of the triple $f(\mathfrak{A})$ is well defined.

Along with the characteristic function $S_{\mathfrak{A}}(z)$ introduce the normalized characteristic function $\hat{S}_{\mathfrak{A}}(z)$ of the triple $\mathfrak{A}$ as

$$(4.1)$$

$$\hat{S}_{\mathfrak{A}}(z) = \frac{1 - S_{\mathfrak{A}}(i)}{1 - S_{\mathfrak{A}}(i)} \cdot S_{\mathfrak{A}}(z), \quad z \in \mathbb{C}_+.$$

Theorem 4.1 ([3, 8, Theorem F.1]). Let $\mathfrak{A} = (\hat{A}, \tilde{A}, A)$ be a regular triple. Suppose that $f(z) = az + b$ with $a, b \in \mathbb{R}$, $a > 0$, is an affine transformation.

Then for the normalized characteristic functions of the triples $\mathfrak{A}$ and $f(\mathfrak{A})$ the following invariance principle

$$(4.2)$$

$$\hat{S}_{f(\mathfrak{A})} \circ f = \hat{S}_{\mathfrak{A}}$$
holds.

If $\mathfrak{A}$ is a regular triple and $f$ is a linear-fractional automorphism of the upper half-plane $\mathbb{C}^+$, $f \in SL_2(\mathbb{R})$, one can also introduce the triple $f(\mathfrak{A}) = (f(\dot{A}), f(\hat{A}), f(A))$, where the function $f(X)$ of a linear operator $X$ is understood as

$$f(X) = (aX + cI)(cX + dI)^{-1}$$

on $\operatorname{Ran}(cX + dI)$, $X = \dot{A}, \hat{A}, A$, provided that $\ker(cX + dI) = \{0\}$. However, if the preimage $f^{-1}(\infty)$ is a quasi-regular point for the symmetric operator $A$, then the operator $f(A)$ is not densely defined and therefore the triple $f(\mathfrak{A})$ is not regular.

In fact, if $\dot{A}$ is a prime symmetric operator and $f \in SL_2(\mathbb{R})$ is a linear-fractional automorphism, then we have the following alternative: either the triple $f(\mathfrak{A}) = (f(\dot{A}), f(\hat{A}), f(A))$ is regular or $f(\hat{A})$ is a bounded dissipative operator.

The following main result of the Note takes care of the two possible outcomes in the alternative mentioned above.

**Theorem 4.2.** Suppose that $\mathfrak{A} = (\dot{A}, \hat{A}, A)$ is a regular triple. Assume, in addition, that $\dot{A}$ is a prime symmetric operator. Suppose that $f \in SL_2(\mathbb{R})$ is a linear-fractional automorphism of the upper half-plane.

Then,

(i) if $\omega = f^{-1}(\infty)$ belongs to the spectrum of the dissipative operator $\hat{A}$, equivalently, to the core of the spectrum of the symmetric operator $\dot{A}$, then the triple $f(\mathfrak{A}) = (f(\dot{A}), f(\hat{A}), f(A))$ is regular.

In this case, for the normalized characteristic functions $\tilde{S}_A(z)$ and $\tilde{S}_{f(\mathfrak{A})}(z)$ of the triples $\mathfrak{A}$ and $f(\mathfrak{A})$ the following invariance principle

$$\tilde{S}_{f(\mathfrak{A})} \circ f = \tilde{S}_A$$

holds;

(ii) if $\omega = f^{-1}(\infty)$ is a regular point of the dissipative operator $\hat{A}$, equivalently, a quasi-regular point of the symmetric operator $\dot{A}$, then the operator $f(\hat{A})$ is well defined as a bounded dissipative operator.

In this case, for the normalized characteristic function $\tilde{S}_A(z)$ of the triple $\mathfrak{A}$ and the characteristic function $S_{f(\hat{A})}(z)$ of the bounded dissipative operator $f(\hat{A})$ the following invariance principle

$$S_{f(\hat{A})} \circ f = \frac{1}{S_A(\omega + i0)} \cdot S_A$$

holds.

**Remark 4.3.** Notice that the technical requirement that $\dot{A}$ is a prime operator can easily be relaxed with obvious modifications in the formulation.

We defer the proof of Theorem 4.2 to Section 6.
5. INVARIANCE PRINCIPLE FOR MODEL TRIPLES

Taking into account that each automorphism of the upper half-plane is either a linear transformation of \( \mathbb{C}_+ \) or a composition of linear transformations of \( \mathbb{C}_+ \) and the automorphism

\[
(5.1) \quad f(z) = -\frac{1}{z}, \quad z \in \mathbb{C}_+,
\]

we will concentrate first on the case where the relevant mapping is given by \((5.1)\). Under this assumption we will establish the corresponding invariance principle for the model triple of operators \( \mathfrak{B} = (\mathcal{B}, \mathbf{\hat{B}}, \mathcal{B}) \) given by \((3.7)-(3.9)\).

Notice, that in the situation in question, the symmetric operator \( \mathcal{B} \) is automatically a prime operator and hence we have the following alternative: either the point \( z = 0 \) belongs to the core of the spectrum of \( \mathcal{B} \) or zero is a quasi-regular point for that operator (see \([1]\)).

We start with the case where the point \( z = 0 \) belongs to the core of the spectrum of the model symmetric operator \( \mathcal{B} \).

**Theorem 5.1.** Suppose that \( \mathfrak{B} = (\mathcal{B}, \mathbf{\hat{B}}, \mathcal{B}) \) is the model triple in \( L^2(\mathbb{R}; d\mu) \) given by \((3.7)-(3.9)\). Assume that \( 0 \) belongs to the core of the spectrum \( \hat{\sigma}(\mathcal{B}) \) of the symmetric operator \( \mathcal{B} \).

Then,

(i) \( (\mathbf{\hat{B}})^{-1} \) is a symmetric operator with deficiency indices \((1,1)\);
(ii) \( \mathcal{B}^{-1} \) is a self-adjoint extension of \( (\mathbf{\hat{B}})^{-1} \);
(iii) \( (\mathbf{\hat{B}})^{-1} \) is a quasi-self-adjoint extension of \( (\mathcal{B})^{-1} \);

In particular, the triple

\[
(5.2) \quad \mathfrak{C} = (-\mathbf{\hat{B}})^{-1}, -\mathbf{\hat{B}}^{-1}, -\mathcal{B}^{-1})
\]

is well defined and regular.

Moreover,

(iv) the von Neumann parameters of the triples \( \mathfrak{B} \) and \( \mathfrak{C} \) coincide;
(v) the Livšic functions \( s_{(-\mathbf{\hat{B}})^{-1}, -\mathcal{B}^{-1}}(z) \) and \( s_{(\mathcal{B},\mathbf{\hat{B}})}(z) \) associated with the pairs \( (-\mathbf{\hat{B}})^{-1}, -\mathcal{B}^{-1} \) and \( (\mathcal{B},\mathbf{\hat{B}}) \) are related as

\[
(5.3) \quad s_{(-\mathbf{\hat{B}})^{-1}, -\mathcal{B}^{-1}} \left( -\frac{1}{z} \right) = s_{(\mathcal{B},\mathbf{\hat{B}})}(z), \quad z \in \mathbb{C}_+;
\]

(vi) the Weyl-Titchmarsh functions \( M_{(-\mathbf{\hat{B}})^{-1}, -\mathcal{B}^{-1}}(z) \) and \( M_{(\mathbf{\hat{B}},\mathcal{B})}(z) \) associated with the pairs \( (-\mathbf{\hat{B}})^{-1}, -\mathcal{B}^{-1} \) and \( (\mathbf{\hat{B}},\mathcal{B}) \) are related as

\[
(5.4) \quad M_{(-\mathbf{\hat{B}})^{-1}, -\mathcal{B}^{-1}} \left( -\frac{1}{z} \right) = M_{(\mathbf{\hat{B}},\mathcal{B})}(z), \quad z \in \mathbb{C}_+;
\]

(vii) the characteristic functions \( S_{\mathfrak{C}}(z) \) and \( S_{\mathfrak{B}}(z) \) of the triples

\[ \mathfrak{C} = (-\mathbf{\hat{B}})^{-1}, -\mathbf{\hat{B}}^{-1}, -\mathcal{B}^{-1} \quad \text{and} \quad \mathfrak{B} = (\mathbf{\hat{B}}, \mathbf{\hat{B}}, \mathcal{B}) \]

satisfy the invariance identity

\[
(5.5) \quad S_{\mathfrak{C}} \left( -\frac{1}{z} \right) = S_{\mathfrak{B}}(z), \quad z \in \mathbb{C}_+.
\]
Proof. Since $\hat{B}$ is a prime symmetric operator and the point $z = 0$ is not a quasi-regular point of $\hat{B}$, the subspace $\text{Ker}((\hat{B})^*)$ is trivial. For the convenience of the reader, we present the corresponding argument.

Indeed, suppose on the contrary that $\text{Ker}((\hat{B})^*) \neq \{0\}$. Then the restriction $\mathcal{B}'$ of $(\hat{B})^*$ on

$$\text{Dom}(\mathcal{B}') = \text{Dom}(\hat{B}) \cap \text{Ker}((\hat{B})^*)$$

is a self-adjoint operator. Let $\mu'(d\lambda)$ be the measure from the representation for the Weyl-Titchmarsh function $M_{(\hat{B},\mathcal{B}')} (z)$ associated with the pair $(\hat{B}, \mathcal{B}')$,

$$M_{(\hat{B},\mathcal{B}')} (z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu' (\lambda), \quad z \in \mathbb{C}_+.$$

Taking into account that $\text{Ker}(\mathcal{B}') = \text{Ker}((\hat{B})^*) \neq \{0\}$, we see that zero is an eigenvalue of $\mathcal{B}'$ and hence

$$\mu'(\{0\}) \neq 0.$$

However, since the right hand side of (3.11) after replacing $\mu(d\lambda)$ with $\mu'(d\lambda)$ remains invariant (see, e.g., [10], cf. Remark 3.1), the membership $0 \in \hat{\sigma}(\hat{B})$ implies

$$\mu(\{0\}) = \mu'(\{0\}) = 0.$$

The obtained contradiction shows that

$$(5.6) \quad \text{Ker}((\hat{B})^*) = \{0\}.$$  

From (5.6) it follows that $\text{Ran}(\hat{B})$ is dense in $L^2(\mathbb{R}; d\mu)$. In particular, $\hat{B} = (\hat{B})^{-1}$ is well defined as a densely-defined unbounded symmetric operator. Also,

$$\text{Ker}(\mathcal{B}) \subseteq \text{Ker}((\hat{B})^*) = \{0\}$$

and therefore the inverse $B = \mathcal{B}^{-1}$ of the self-adjoint (multiplication) operator $\mathcal{B}$ is well defined as a self-adjoint operator. Clearly, $\hat{B}$ coincides with the restriction of the unbounded self-adjoint operator $B = \mathcal{B}^{-1}$ on

$$(5.7) \quad \text{Dom}(\hat{B}) = \left\{ f \in \text{Dom}(B) \mid \int_{\mathbb{R}} \frac{f(\lambda)}{\lambda} d\mu(\lambda) = 0 \right\}.$$

Using (5.7), we see that for any $f \in \text{Dom}(\hat{B}) = \text{Dom}((\hat{B})^{-1}) \subset \text{Dom}((B)^{-1})$ we have

$$(\hat{B} - \pi I) f, h_z) = ((B^{-1} - \pi I) f, h_z) = \int_{\mathbb{R}} \left( \frac{f(\lambda)}{\lambda - \pi} \right) \overline{\frac{1}{1 - \lambda^2}} d\mu(\lambda)$$

$$= \int_{\mathbb{R}} \frac{f(\lambda)}{\lambda} d\mu(\lambda) = 0.$$

Therefore, the functions

$$(5.8) \quad h_z(\lambda) = \frac{1}{1 - \lambda^2}, \quad \text{Im}(z) \neq 0,$$

are deficiency elements of $\hat{B}$. That is,

$$(5.9) \quad h_z \in \text{Ker}((\hat{B})^* - zI), \quad \text{Im}(z) \neq 0.$$

In particular, the symmetric operator $\hat{B}$ has deficiency indices $(m, n)$ with $m, n \geq 1$.

To show that $m = n = 1$, observe that for any $f \in \text{Dom}(\hat{B}) = \text{Dom}(B^{-1})$ the function

$$(5.10) \quad \tilde{f}(\lambda) = f(\lambda) - \int_{\mathbb{R}} \frac{f(s)}{s} d\mu(s) \cdot \frac{\lambda}{\lambda^2 + 1}$$
has the property that
\[ \frac{\hat{f}(\lambda)}{\lambda} \, d\mu(\lambda) = \frac{f(\lambda)}{\lambda} \, d\mu(\lambda) - \frac{f(s)}{s} \, d\mu(s) \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda^2 + 1} = 0, \]
so that
\[ \hat{f} \in \text{Dom}(\hat{B}). \]
Here we have used the normalization condition (3.6).
From (5.8) and (5.9) it follows that the function
\[ r(\lambda) = \frac{\lambda}{\lambda^2 + 1} \]
belongs to the subspace \( \text{Ker}((\hat{B})^* - iI) + \text{Ker}((\hat{B})^* + iI) \). In particular, (5.10) along with (5.11) show that the quotient space \( \text{Dom}(\hat{B})/\text{Dom}(\hat{B}) \) is one-dimensional and hence
\[ m = n = 1. \]
Thus, \( \hat{B} \) is a symmetric restriction with deficiency indices \((1, 1)\) of the self-adjoint operator \( B \) and the proof of (i) and (ii) is complete.

To check (iii), first observe that \( \hat{\hat{B}} = (\hat{B})^{-1} \) is well-defined, since
\[ \text{Ker}(\hat{\hat{B}}) \subseteq \text{Ker}((\hat{B})^* - iI) \]
belongs to the subspace \( \text{Ker}((\hat{B})^* - iI) + \text{Ker}((\hat{B})^* + iI) \). In particular, (5.10) along with (5.11) show that the quotient space \( \text{Dom}(\hat{B})/\text{Dom}(\hat{B}) \) is one-dimensional and hence
\[ m = n = 1. \]
Next we claim that the operator \( \hat{\hat{B}} = (\hat{B})^{-1} \) is the quasi-self-adjoint extension of \( \hat{B} \) on
\[ \text{Dom}(\hat{\hat{B}}) = \text{Dom}(\hat{\hat{B}}) + \text{span}\left\{ \frac{1}{\lambda - i} + \kappa \frac{1}{\lambda + i} \right\}, \]
where \( \kappa \) is the von Neumann parameter of the model triple \( \mathfrak{B} = (\hat{B}, \hat{\hat{B}}, B) \).

Indeed, since (see (5.13))
\[ \text{Dom}(\hat{\hat{B}}) = \text{Dom}(\hat{\hat{B}}) + \text{span}\left\{ \frac{1}{\lambda - i} - \kappa \frac{1}{\lambda + i} \right\} \]
and \( \hat{\hat{B}} = (\hat{B})^{-1} \), it suffices to check the following two (mapping) properties for the operators \( \hat{B} \) and \( \hat{\hat{B}} \),
\[ \hat{\hat{B}} \left( \frac{1}{\lambda - i} - \kappa \frac{1}{\lambda + i} \right) = (\hat{B})^* \left( \frac{1}{\lambda - i} - \kappa \frac{1}{\lambda + i} \right) = \frac{i}{\lambda - i} + \kappa \frac{i}{\lambda + i} \]
and
\[ \hat{\hat{B}} \left( \frac{i}{\lambda - i} + \kappa \frac{i}{\lambda + i} \right) = (\hat{B})^* \left( \frac{i}{\lambda - i} + \kappa \frac{i}{\lambda + i} \right) = i \left( \frac{-i}{\lambda - i} + \kappa \frac{i}{\lambda + i} \right) \]
\[ = \frac{1}{\lambda - i} - \kappa \frac{1}{\lambda + i}. \]
Here we used that by (3.17),
\[ \text{Ker}((\hat{B})^* \mp iI) = \text{span}\{h_\pm\}, \]
and also that (see (5.8) and (5.9))
\[ \text{Ker}((\hat{B})^* \mp iI) = \text{span}\{h_\mp\}, \]
with
\[ h_\pm(\lambda) = \frac{1}{\lambda \mp i}. \]
The proof of (iii) is complete.

Finally, to calculate the von Neumann parameter of the triple $C$ given by (5.2) we proceed as follows.

Observing that
$$h_+ + (\lambda) + h_- = 1 = \frac{\lambda - i}{\lambda^2 + 1},$$
we see that
$$(5.16) h_+ + h_- \in \text{Dom}(B) = \text{Dom}(-B).$$

Moreover, from (5.12) it follows that
$$(5.17) h_+ + \kappa h_- \in \text{Dom}(\hat{B}) = \text{Dom}(-\hat{B}).$$

Since
$$\text{Ker}(-\dot{B}^* \mp iI) = \text{Ker}((\dot{B})^* \mp iI) = \text{span}\{h_{\pm}\},$$
and by (5.13)
$$\text{Ker}((\dot{B})^* \mp iI) = \text{span}\{h_{\pm}\},$$
the memberships (5.16) and (5.17) ensure that $\kappa$ is the von Neumann parameter of the triple
$$C = (\dot{B}, \hat{B}, B) = (-\dot{B}^{-1}, -\hat{B}^{-1}, -B^{-1}),$$
which completes the proof of (iv).

Next, we will evaluate the Livšic functions associated with the pairs $(\dot{B}, B)$ and $(-\dot{B}, -B)$.

Recall that by (3.15),
$$\text{Ker}((\dot{B})^* - zI) = \text{span}\{g_z\}, \quad \text{Im}(z) \neq 0,$$
where
$$g_z(\lambda) = \frac{1}{\lambda - z}, \quad \text{Im}(z) \neq 0.$$

Since
$$h_+ - h_- \in \text{Dom}(B), \quad h_{\pm} \in \text{Ker}((\dot{B})^* \mp iI),$$
for the Livšic function $s_{(\dot{B}, B)}(z)$ associated with the pair $(\dot{B}, B)$ we obtain the representation
$$(5.18) s_{(\dot{B}, B)}(z) = \frac{z - i}{z + i} \cdot \frac{(g_z, h_+)}{(g_z, h_-)} = \frac{z - i}{z + i} \cdot \frac{\int_{\lambda - z} \frac{du(\lambda)}{(\lambda - z)(\lambda - i)}}{\int_{\lambda - z} \frac{du(\lambda)}{(\lambda - z)(\lambda + i)}}.$$

From (5.7) and (5.8) it follows that
$$\text{Ker}(-((\dot{B})^* - zI) = \text{span}\{\hat{h}_z\}, \quad \text{Im}(z) \neq 0,$$
where
$$\hat{h}_z(\lambda) = \frac{1}{1 + \lambda z}, \quad \text{Im}(z) \neq 0.$$

Observing that
$$h_+ + h_- = h_+ - (-1)h_- \in \text{Dom}(B), \quad h_{\pm} \in \text{Ker}(-(\dot{B})^* \mp iI),$$
in accordance with the definition of the Livšic function \( s_{(-B,-B)}(z) \) associated with the pair \((-\dot{B}, -B)\) we obtain

\[
(5.19) \quad s_{(-\dot{B},-B)}(z) = \frac{z - i}{z + i} \cdot \left( \frac{\dot{h}_z, h_+}{(h_+(-1)h_-)} \right) = -\frac{z - i}{z + i} \cdot \left( \frac{\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1+\lambda^2(\lambda-i)}}{\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1+\lambda^2(\lambda+i)}} \right), \quad z \in \mathbb{C}_+.
\]

A simple computation using (5.18) and (5.19) shows that

\[
s_{(-\dot{B},-B)}\left(-\frac{1}{z}\right) = s_{(\dot{B},B)}(z), \quad z \in \mathbb{C}_+,
\]

thus proving (v). The proof of (v) is complete.

The assertion (vi) is a direct consequence of (v) and the relationship (3.3) linking the Livsic and Weyl-Titchmarsh functions.

The last assertion (vii) is a consequence of (iv) and (v) and the definition of the characteristic function of a triple.

\[\square\]

Remark 5.2. Since the point \( z_0 = i \) is a fixed point of the automorphism \( f(z) = \frac{-1}{z}, \quad z \in \mathbb{C}_+ \),
using (5.7) we see that

\[
S_\varepsilon(i) = S_\varepsilon\left(-\frac{1}{i}\right) = S_B(i).
\]

Therefore, for the normalized characteristic functions

\[
\hat{S}_f(B)(z) = \hat{S}_\varepsilon(z) = \frac{1 - S_\varepsilon(i)}{1 - S_\varepsilon(i)} \cdot S_\varepsilon(z), \quad z \in \mathbb{C}_+,
\]

and

\[
\hat{S}_B(z) = \frac{1 - S_B(i)}{1 - S_B(i)} \cdot S_B(z), \quad z \in \mathbb{C}_+,
\]

associated with the triples

\[
f(B) = \varepsilon = \left(-\dot{B}\right)^{-1}, -\left(\dot{B}\right)^{-1}, -B^{-1}\right) \quad \text{and} \quad B = (\dot{B}, \hat{B}, B),
\]

respectively, we also have the invariance equality

\[
\hat{S}_{f(B)} \circ f = \hat{S}_B.
\]

Notice that if the point \( z = 0 \) is a quasi-regular point of the symmetric operator \( \dot{B} \), then the operator \(-\dot{B}^{-1}\) is well defined as a bounded dissipative operator, while \(\text{Ker}(\dot{B})^*\) is non-trivial and therefore \(\dot{B}^{-1}\) is not densely defined (although \((\dot{B})^{-1}\) is continuous on \(\text{Ran}(\dot{B})\)). Therefore, the triple

\[
f(B) = \left(-\dot{B}^{-1}, -\left(\dot{B}\right)^{-1}, -B^{-1}\right)
\]

is not regular in this case. However, we have the following result (cf. [22], Theorem 8.4.4, [25] in system theory that relates the transfer functions of \(L\)-systems under the transformation \( z \mapsto \frac{1}{z} \) of the spectral parameter).
**Lemma 5.3** (cf. [2] Theorem 8.4.4). Suppose that \( \mathfrak{B} = (\hat{B}, \hat{B}, B) \) is the model triple in \( L^2(\mathbb{R}; d\mu) \) given by \((12)-(13)\). Assume that the point \( z = 0 \) is a quasi-regular point of the symmetric operator \( \hat{B} \).

In this case, the operator \( \hat{B} \) has a bounded inverse and the characteristic function \( S_{-(\hat{B})^{-1}}(z) \) of the dissipative operator \( -(\hat{B}) \) and the characteristic function \( S_{\mathfrak{B}}(z) \) of the triple \( \mathfrak{B} \) are related as

\[
S_{-(\hat{B})^{-1}} \left( -\frac{1}{z} \right) = \frac{S_{\mathfrak{B}}(z)}{S_{\mathfrak{B}}(0+i0)}, \quad z \in \mathbb{C}_+.
\]

**Proof.** Assume temporarily that \( \text{Ker}(B) = \{0\} \). Then under the hypothesis that \( 0 \) is a quasi-regular point of the symmetric operator \( \hat{B} \), we have

\[
0 \in \rho(B) \cap \rho(\hat{B}).
\]

By Corollary \([3]\), the inverse \( \hat{B}^{-1} \) is a rank-one perturbation of the self-adjoint operator \( B^{-1} \),

\[
\hat{B}^{-1} = B^{-1} - pQ.
\]

Here

\[
p = \left( M(0) + i \frac{\kappa + 1}{\kappa - 1} \right)^{-1},
\]

\( M(0) = M(0 + i0) \) is the value of the Weyl-Titchmarsh function \( M(z) = M_{(B,B)}(z) \) associated with the pair \( (\hat{B}, B) \) at the point zero, \( \kappa \) is the von Neumann parameter of the triple \( \mathfrak{B} = (\hat{B}, \hat{B}, B) \), and \( Q \) is a rank-one self-adjoint operator given by

\[
(Qh)(\lambda) = \frac{1}{\lambda} \int_{\mathbb{R}} \frac{h(\lambda)}{\lambda} d\mu(\lambda), \quad \mu \text{-a.e. } \lambda.
\]

In accordance with the definition \([13]\) of the characteristic function \( S_{-(\hat{B})^{-1}}(z) \) of the bounded dissipative operator \( -\hat{B}^{-1} \) we have

\[
S_{-(\hat{B})^{-1}}(z) = 1 + 2i \text{Im}(p) \text{tr} \left[ (-\hat{B}^{-1})^* - zI \right]^{-1} Q,
\]

\[
= 1 + 2i \text{Im}(p) \text{tr} \left[ (-B^{-1} + \overline{p}Q - zI)^{-1} Q \right],
\]

where we have used \((5.21)\) in the last step. Since \( \overline{p}Q \) is a rank-one perturbation of the self-adjoint operator \( -B^{-1} \), from the first resolvent identity it follows that (see, e.g., \([22]\))

\[
\text{tr} \left[ (-B^{-1} + \overline{p}Q - zI)^{-1} Q \right] = \frac{\text{tr} \left[ (-B^{-1} - zI)^{-1} Q \right]}{1 + \overline{p} \text{tr} \left[ (-B^{-1} - zI)^{-1} Q \right]},
\]

and hence

\[
S_{-(\hat{B})^{-1}}(z) = \frac{1 + p \text{tr} \left[ (-B^{-1} - zI)^{-1} Q \right]}{1 + \overline{p} \text{tr} \left[ (-B^{-1} - zI)^{-1} Q \right]} = \frac{1 - p \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda(\lambda + \kappa z)}}{1 - \overline{p} \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda(\lambda + \kappa z)}}, \quad z \in \mathbb{C}_+.
\]
On the other hand, using (2.6) and (2.8), for the characteristic function \( S_B(z) \) of the triple \( \mathcal{B} \) one obtains
\[
S_B(z) = \frac{1 - \kappa}{1 - \kappa} - M(z) - i \frac{1 + \kappa}{1 - \kappa}, \quad z \in \mathbb{C}_+,
\]
and therefore
\[
S_B \left( \frac{1}{z} \right) = \frac{1 - \kappa}{1 - \kappa} - M \left( \frac{1}{z} \right) - i \frac{1 + \kappa}{1 - \kappa}, \quad z \in \mathbb{C}_+.
\]
From (5.22) it also follows that
\[
p^{-1} - M(0) = -i \frac{\kappa + 1}{1 - \kappa}
\]
and hence
\[
S_B \left( \frac{1}{z} \right) = \frac{1 - \kappa}{1 - \kappa} - M \left( \frac{1}{z} \right) - M(0) + p^{-1}, \quad z \in \mathbb{C}_+.
\]
Here we used that \( M(0) \) is real for \( 0 \in \rho(\mathcal{B}) \).

Since
\[
\frac{M \left( \frac{1}{z} \right) - M(0) + p^{-1}}{M \left( \frac{1}{z} \right) - M(0) + p^{-1}} = -\frac{\int_{\mathbb{R}} \frac{dp(\lambda)}{\lambda(1+\lambda z)}}{p} \left( 1 - \frac{1}{1 - \kappa} \right) = \frac{1 - \kappa}{1 - \kappa} - M(0) + \frac{p^{-1}}{1 - \kappa - \frac{dp(\lambda)}{\lambda(1+\lambda z)}},
\]
one concludes that
\[
S_B \left( \frac{1}{z} \right) = \frac{1 - \kappa}{1 - \kappa} - M \left( \frac{1}{z} \right) - p \int_{\mathbb{R}} \frac{dp(\lambda)}{\lambda(1+\lambda z)} + \frac{1}{1 - \kappa}, \quad z \in \mathbb{C}_+.
\]
Thus, taking into account (5.23), one obtains
\[
(5.24) \quad S_B \left( \frac{1}{z} \right) = -\frac{1 - \kappa}{1 - \kappa} - \frac{\mathcal{P}}{p} \cdot S_{\hat{\mathcal{B}}^{-1}}(z), \quad z \in \mathbb{C}_+.
\]
Using (5.22) we get
\[
(5.25) \quad -\frac{1 - \kappa}{1 - \kappa} \cdot \frac{\mathcal{P}}{p} = -\frac{1 - \kappa}{1 - \kappa} - M(0) + i \frac{\kappa + 1}{1 - \kappa} = \frac{M(0) - i - \kappa(M(0) + i)}{M(0) - i - \kappa(M(0) - i)} = \frac{s_{\hat{\mathcal{B}}^{-1}}(0)}{\mathcal{P} S_{\hat{\mathcal{B}}^{-1}}(0 + i) - 1} = S_B(0 + i0).
\]
Combining (5.24) and (5.25) shows that
\[
S_{\hat{\mathcal{B}}^{-1}} \left( \frac{1}{z} \right) = (S_B(0 + i0))^{-1} S_B(z), \quad z \in \mathbb{C}_+,
\]
which proves the claim provided that \( \operatorname{Ker}(\mathcal{B}) = \{ 0 \} \).

To relax the requirement that \( \operatorname{Ker}(\mathcal{B}) = \{ 0 \} \), suppose that \( \mathcal{B}' \) is a self-adjoint extension of \( \hat{\mathcal{B}} \) such that \( \operatorname{Ker}(\mathcal{B}') = \{ 0 \} \). Such an extension is always available since \( 0 \in \rho(\hat{\mathcal{B}}) \). However,
\[
\frac{S_B(z)}{S_B(0 + i0)} = \frac{S_B'(z)}{S_B'(0 + i0)}, \quad z \in \mathbb{C}_+,
\]
where \( \mathcal{B}' = (\hat{\mathcal{B}}, \hat{\mathcal{B}}, \mathcal{B}') \) and hence (5.20) holds regardless of whether \( \operatorname{Ker}(\mathcal{B}) = \{ 0 \} \) or not.
6. Proof of Theorem 4.2

Now we are ready to establish the invariance principle also for the characteristic function of a triple for general linear-fractional automorphisms of the upper half-plane.

Proof. Any linear-fractional automorphism
\[ C_+ \ni z \mapsto f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \]
\[ ad - bc > 0, \quad c > 0 \]
can be represented as the composition
\[ f = h \circ \iota \circ g, \]
where \( h \) is a linear automorphism of \( C_+ \),
\[ \iota(z) = -\frac{1}{z} \quad \text{and} \quad g(z) = z - f^{-1}(\infty), \quad z \in C_+. \]

If \( \omega = f^{-1}(\infty) \) belongs to the core of the spectrum of \( \hat{A} \), the point 0 belongs to the core of the spectrum of the symmetric operator \( g(\hat{A}) = \hat{A} - \omega I \). Therefore, in the model representation of the triple \( \iota \circ g(\hat{A}) \) the hypotheses of Theorem 5.1 are satisfied. In particular, the triple \( \iota \circ g(\hat{A}) \) is regular, so is \( f(\hat{A}) \), since \( f = h \circ \iota \circ g \) and \( h \) is a linear isomorphism. By Theorem 4.1,
\[ \hat{S}_{f(\hat{A})} \circ f = \hat{S}_{\iota \circ g(\hat{A})} \circ (\iota \circ g). \]
Applying Theorem 5.3, we get
\[ \hat{S}_{\iota \circ g(\hat{A})} \circ (\iota \circ g) = \hat{S}_{g(\hat{A})} \circ g = \hat{S}_{\hat{A}}, \]
where we have used Theorem 4.1 one more time in the last step, thus proving (4.3).

If \( \omega = f^{-1}(\infty) \) is a quasi-regular point of \( \hat{A} \), both \( f(\hat{A}) \) and \( (\iota \circ g)(\hat{A}) \) are bounded dissipative operators. By the invariance principle in the bounded case (see (1.2)), we have
\[ S_f(\hat{A}) \circ f = S_{\iota \circ g(\hat{A})} \circ (\iota \circ g). \]
Applying Lemma 5.3, we get
\[ S_{\iota \circ g(\hat{A})} \circ (\iota \circ g) = \frac{1}{\hat{S}_{g(\hat{A})}(0 + i0)} \cdot \hat{S}_{g(\hat{A})} \circ g. \]
By Theorem 4.1,
\[ \frac{1}{\hat{S}_{g(\hat{A})}(0 + i0)} \cdot \hat{S}_{g(\hat{A})} \circ g = \frac{1}{S_{\hat{A}}(\omega + i0)} \cdot \hat{S}_{\hat{A}} = \frac{1}{S_{\hat{A}}(\omega + i0)} \cdot S_{\hat{A}}. \]
Therefore,
\[ S_f(\hat{A}) \circ f = \frac{1}{S_{\hat{A}}(\omega + i0)} \cdot S_{\hat{A}}, \]
completing the proof (4.4). \( \square \)
7. Applications to the Krein-von Neumann extensions theory

Recall that if $\dot{A}$ is a densely defined (closed) nonnegative operator, then the set of all nonnegative self-adjoint extensions of $\dot{A}$ has the minimal element $[\dot{A}]_K$, the Krein-von Neumann extension (different authors refer to the minimal extension by using different names, see, e.g., $[3, 4, 5, 6]$, and the maximal one $[\dot{A}]_F$, the Friedrichs extension. This means, in particular, that for any nonnegative self-adjoint extension $A$ of $\dot{A}$ the following operator inequality holds:

$$([\dot{A}]_F + \lambda I)^{-1} \leq (A + \lambda I)^{-1} \leq ([\dot{A}]_K + \lambda I)^{-1}, \quad \text{for all } \lambda > 0.$$ 

**Hypothesis 7.1.** Suppose that $\nu \in (-1, 1)$. Assume that $A(\nu)$ is the self-adjoint multiplication operator by independent variable in the Hilbert space $\mathcal{H}_+ = L^2((0, \infty); \lambda \nu d\lambda)$ and $\dot{A}(\nu)$ is its restriction on

$$\text{Dom}(\dot{A}(\nu)) = \left\{ f \in \text{Dom}(A(\nu)) \mid \int_0^\infty f(\lambda)\lambda^\nu d\lambda = 0 \right\}.$$

Analogously, suppose that $B(\nu)$ is the self-adjoint multiplication operator by independent variable in the Hilbert space $\mathcal{H}_- = L^2((-\infty, 0); |\lambda|^\nu d\lambda)$ and $\dot{B}(\nu)$ is its restriction on

$$\text{Dom}(\dot{B}(\nu)) = \left\{ f \in \text{Dom}(B(\nu)) \mid \int_{-\infty}^0 f(\lambda)|\lambda|^\nu d\lambda = 0 \right\}.$$

**Lemma 7.2.** Assume Hypothesis 7.1. Then the Weyl-Titchmarsh function $M_\nu(z)$ associated with the pair $(\dot{A}(\nu), A(\nu))$ admits the representation

$$M_\nu(z) = \begin{cases} 
(i - \cot \frac{\pi}{2} \nu)(\frac{z}{\pi})^{-\nu} + \cot \frac{\pi}{2} \nu, & \nu \neq 0 \\
\frac{2}{\pi} \log (-\frac{1}{z}), & \nu = 0 
\end{cases}, \quad z \in \mathbb{C}_+.
$$

Analogously, the Weyl-Titchmarsh function $N_\nu(z)$ associated with the pair $(\dot{B}(\nu), B(\nu))$ admits the representation

$$N_\nu(z) = \begin{cases} 
(i + \cot \frac{\pi}{2} \nu)(\frac{z}{\pi})^{\nu} - \cot \frac{\pi}{2} \nu, & \nu \neq 0 \\
\frac{2}{\pi} \log z, & \nu = 0 
\end{cases}, \quad z \in \mathbb{C}_+.
$$

Here $\log z$ denotes the principal brach of the logarithmic function with the cut on the negative semi-axis and

$$\left(\frac{z}{\pi}\right)^\nu = \exp \left[ \nu \left( \log z - i \frac{\pi}{2} \right) \right], \quad z \in \mathbb{C}_+.$$

**Proof.** Suppose that $\nu \neq 0$. We have

$$M_\nu(z) = \frac{1}{\|g_+\|^2} \langle (zA(\nu) + I)(A(\nu) - zI)^{-1}g_+, g_+ \rangle, \quad z \in \mathbb{C}_+,$$

where $g_+$ is a deficiency element from $\text{Ker}((\dot{A}(\nu))^* - iI)$. One can choose (see, e.g., $[8]$)

$$g_+(z) = \frac{1}{\lambda - z}, \quad z \in \mathbb{C}_+, \quad \lambda \in (0, \infty).$$
We have
\[
((zA(\nu) + I)(A(\nu) - zI)^{-1}g_+, g_+) = \int_0^\infty \frac{(z\lambda + 1)\lambda^\nu}{(\lambda - z)(1 + \lambda^2)} d\lambda
\]
and
\[
\|g_+\|^2 = \int_0^\infty \frac{\lambda^\nu}{1 + \lambda^2} d\lambda.
\]

Let \(\Gamma_\varepsilon, \varepsilon > 0\), denote the anti-clockwise oriented contour in the complex plane
\[
\Gamma_\varepsilon = \{ z \in \mathbb{C} \mid \text{dist}(z, [0, \infty)), |z| \leq \varepsilon^{-1} \}
\]
\[
\cup \{ z \in \mathbb{C} \mid |z| = \varepsilon^{-1}, \text{arg}(z) \in [\arcsin(\varepsilon^2), 2\pi - \arcsin(\varepsilon^2)] \},
\]
which consists of “an infinitely distant circle and an indentation round the cut along the positive real axis” [13, §129, Fig. 46] (as \(\varepsilon \to 0\)).

Given \(z \in \mathbb{C}_+\) and \(\varepsilon < \min(|z|, 1)\), by the Residue theorem we get
\[
\oint_{\Gamma_\varepsilon} \frac{(z\lambda + 1)\lambda^\nu}{(\lambda - z)(1 + \lambda^2)} d\lambda = 2\pi i \sum_{\zeta \in \{z, i, -i\}} \text{Res}(F, \zeta),
\]
where
\[
F(\lambda) = \frac{(z\lambda + 1)\lambda^\nu}{(\lambda - z)(1 + \lambda^2)}.
\]

Going to the limit as \(\varepsilon \to 0\) and taking into account that
\[
(\lambda - i0)^\nu = e^{i2\pi\nu} \lambda^\nu, \quad \lambda > 0,
\]
we arrive at the representation
\[
(1 - e^{i2\pi\nu}) \int_0^\infty \frac{(z\lambda + 1)\lambda^\nu}{(\lambda - z)(1 + \lambda^2)} d\lambda = \lim_{\varepsilon \to 0} \oint_{\Gamma_\varepsilon} \frac{(z\lambda + 1)\lambda^\nu}{(\lambda - z)(1 + \lambda^2)} d\lambda
\]
\[
\quad \quad = 2\pi i \sum_{\zeta \in \{z, i, -i\}} \text{Res}(F, \zeta).
\]

Analogously,
\[
(1 - e^{i2\pi\nu}) \int_0^\infty \frac{\lambda^\nu}{1 + \lambda^2} d\lambda = 2\pi i \sum_{\zeta \in \{i, -i\}} \text{Res}(G, \zeta),
\]
where
\[
G(\lambda) = \frac{\lambda^\nu}{1 + \lambda^2}.
\]

A direct computation shows that
\[
M_\nu(z) = \frac{\sum_{\zeta \in \{z, i, -i\}} \text{Res}(F, \zeta)}{\sum_{\zeta \in \{i, -i\}} \text{Res}(G, \zeta)} = \frac{z^\nu - i^{\nu+(-i)^\nu}}{-\nu^{\nu-(-i)^\nu}}
\]
\[
= \frac{2i}{1 - e^{i\pi\nu}} \left( \frac{z}{i} \right)^\nu - i \frac{1 + e^{i\pi\nu}}{1 - e^{i\pi\nu}}
\]
\[
= \left( i - \cot \frac{\pi\nu}{2} \right) \left( \frac{z}{i} \right)^\nu + \cot \frac{\pi\nu}{2}, \quad z \in \mathbb{C}_+.
\]

The case of \(\nu = 0\) can be justified by taking the limit
\[
M_0(z) = \lim_{\nu \to 0} M_\nu^+(z) = \frac{2}{\pi} \log \left( \frac{1}{z} \right), \quad z \in \mathbb{C}_+,
\]
which completes the proof of (7.3).
The proof of (7.2) is analogous. □

**Lemma 7.3.** Assume Hypothesis [7.1]

Then, if \( \nu \in [0, 1) \), then \( A(\nu) \) is the Friedrichs extension of \( \hat{A}(\nu) \). If \( \nu \in (-1, 0] \), then \( A(\nu) \) is the Krein-von Neumann extension of \( \hat{A}(\nu) \).

In particular, the Friedrichs and Krein-von Neumann extensions of \( \hat{A}(0) \) coincide.

**Proof.** By Lemma 7.2, the Weyl-Titchmarsh function associated with the pair \((\hat{A}(\nu), A(\nu))\) can be evaluated as

\[
M_{(\hat{A}(\nu), A(\nu))}(z) = M_{\nu}(z), \quad z \in \mathbb{C}_+.
\]

To complete the proof it remains to observe that

\[
\lim_{\lambda \downarrow -\infty} M_{\nu}(\lambda) = -\infty, \quad 0 \leq \nu < 1,
\]

and

\[
\lim_{\lambda \uparrow 0} M_{\nu}(\lambda) = \infty, \quad -1 < \nu \leq 0,
\]

and then apply [1], Theorem 4.4, a result that characterizes the Weyl-Titchmarsh function threshold behavior for the Friedrichs and Krein-von Neumann extensions, respectively. □

**Remark 7.4.** Notice that the Cayley transforms of \( M_{\nu}(z) \) and \( M_{-\nu}(z) \) coincide up to a constant unimodular factor, that is,

\[
\frac{M_{\nu}(z) - i}{M_{\nu}(z) + i} = \frac{(i - \cot \frac{i}{2} \nu) (\frac{i}{2} \nu)^{\nu} + \cot \frac{i}{2} \nu - i}{(i - \cot \frac{i}{2} \nu) (\frac{i}{2} \nu)^{\nu} + \cot \frac{i}{2} \nu + i} = \frac{\left(\frac{i}{2} \nu\right)^{\nu} - 1}{\left(\frac{i}{2} \nu\right)^{\nu} - e^{i\pi \nu}}.
\]

\[
\frac{M_{-\nu}(z) - i}{M_{-\nu}(z) + i} = \frac{(i + \cot \frac{i}{2} \nu) (\frac{i}{2} \nu)^{\nu} - \cot \frac{i}{2} \nu - i}{(i + \cot \frac{i}{2} \nu) (\frac{i}{2} \nu)^{\nu} - \cot \frac{i}{2} \nu + i} = \frac{\left(\frac{i}{2} \nu\right)^{-\nu} - 1}{\left(\frac{i}{2} \nu\right)^{-\nu} - e^{-i\pi \nu}} = e^{i\pi \nu} \frac{\left(\frac{i}{2} \nu\right)^{\nu} - 1}{\left(\frac{i}{2} \nu\right)^{\nu} - e^{i\pi \nu}}, \quad z \in \mathbb{C}_+.
\]

Therefore, by (2.4), the Livšic functions \( s_{\nu}(z) \) and \( s_{-\nu}(z) \) associated with the pairs \((\hat{A}(\nu), A(\nu))\) and \((\hat{A}(-\nu), A(-\nu))\), respectively, are related as

\[
s_{\nu}(z) = e^{i\pi \nu} s_{-\nu}(z), \quad z \in \mathbb{C}_+.
\]

Since the knowledge (up to a unimodular factor) of the Livšic function \( s_{(\hat{A}, A)}(z) \) of a pair \((\hat{A}, A)\), where \( \hat{A} \) is a prime symmetric operator and \( A \) its self-adjoint extension, determines the symmetric operator \( \hat{A} \) up to a unitary equivalence (see [1, 4, 13, 20]), and, moreover, \( A(\nu) \) and \( A(-\nu) \) are prime symmetric operators, we conclude that \( \hat{A}(\nu) \) and \( \hat{A}(-\nu) \) are unitarily equivalent.

The following theorem addresses “intertwining” properties of the Friedrichs and Krein-von Neumann extensions of an operator with respect to the inverse operation.

**Theorem 7.5.** Assume Hypothesis [7.1] and set \( \hat{A} = \hat{A}(\nu) \) and \( A = A(\nu) \). Then

\[
(\lfloor \hat{A}\rfloor)^{-1} = \lfloor (A)^{-1}\rfloor_K.
\]
and
\[
(\hat{A} K)^{-1} = \left[ (\hat{A})^{-1} \right]_F.
\]

**Remark 7.6.** Notice that the operators \( \hat{A}(\nu), A(\nu), \) etc., referred to in Hypothesis 7.4 are essentially coincide with the model multiplication operators in the weighted Hilbert space \( L^2((0, \infty); d\mu) \) given by \( (3.7) - (3.9) \) (after an appropriate renormalization of the weight \( d\mu(\lambda) = \lambda^{\nu} d\lambda \)), so that Theorem 7.1 applies. For instance, the inverse of \( A = A(\nu) \) is well defined as a prime symmetric operator with deficiency indices \((1,1).\)

**Proof.** Comparing \((7.3)\) and \((7.2)\) one observes that for \( \nu \in (-1,1) \) we have
\[
M_\nu \left( \frac{1}{z} \right) = N_{-\nu}(z), \quad z \in \mathbb{C}_+.
\]
By Theorem 7.1 (vi), the left hand side of \((7.7)\) is the Weyl-Titchmarsh function associated with the pair \((-A(\nu))^{-1}, -(A(\nu))^{-1}\) and therefore
\[
(-A(\nu))^{-1}, -(A(\nu))^{-1} \cong ((\hat{B}(\nu))^{-1}, (B(\nu))^{-1}).
\]
Here the symbol \( \cong \) denotes the mutual unitary equivalence of the corresponding pairs. From the definition of the operators \( A(\nu) \) and \( B(\nu) \) it follows that
\[
((\hat{B}(\nu))^{-1}, (B(\nu))^{-1}) \cong (-\hat{A}(\nu), -A(\nu))
\]
and therefore
\[
((\hat{A}(\nu))^{-1}, (A(\nu))^{-1}) \cong (\hat{A}(\nu), A(\nu)).
\]
Suppose that \( \nu \geq 0. \) By Lemma 7.2,
\[
A(\nu) = [\hat{A}(\nu)]_F.
\]
So that
\[
((\hat{A}(\nu))^{-1}, (A(\nu))^{-1}) = ((\hat{A}(\nu))^{-1}, ([\hat{A}(\nu)]_F)^{-1})
\]
and hence
\[
(7.8) \quad ((\hat{A}(\nu))^{-1}, ([\hat{A}(\nu)]_F)^{-1}) \cong (\hat{A}(\nu), A(\nu)).
\]
Again, by Lemma 7.3, the operator \( A(-\nu) \) is the Krein-von Neumann extension of \( \hat{A}(-\nu). \) From the mutual unitary equivalence of the pairs \((\hat{A}(\nu))\) it follows that the second operator from the left pair is the Krein-von Neumann extension of the first one. That is,
\[
([\hat{A}(\nu)]_F)^{-1} = [([\hat{A}(\nu)]^{-1})_K
\]
or, equivalently,
\[
(\hat{A})^{-1} = [([\hat{A}]^{-1})_K,
\]
which proves \((7.9)\) (for \( \nu \in [0,1)\)).

One also has that
\[
((\hat{A}(-\nu))^{-1}, (A(-\nu))^{-1}) \cong (\hat{A}(\nu), A(\nu)).
\]
The same reasoning shows that
\[
([\hat{A}(-\nu)]_F)^{-1} = ([\hat{A}(-\nu)]_K)^{-1}.
\]
However, by Remark 7.4, the operators \( \hat{A}(-\nu) \) and \( \hat{A}(\nu) \) are unitarily equivalent which implies that
\[
([\hat{A}(\nu)]_F)^{-1} = ([\hat{A}(\nu)]_K)^{-1}.
or, equivalently,

\[((\hat{A})^{-1})_F = ([\hat{A}]_K)^{-1},\]

and (7.6) follows (for \(\nu \in (0, 1)\)).

The proof for \(\nu \in (-1, 0)\) is analogous.

\[\square\]

**Remark 7.7.** In view of Remark 7.4 from (7.4) it also follows that the symmetric operators \(\hat{A}\) and \((\hat{A})^{-1}\) referred to in Theorem 7.5 are unitarily equivalent.

**Remark 7.8.** In connection with (7.6) it is worth mentioning that in a more general context the (Krein-)von Neumann extension \(S_N\) of a positive subspace \(S\) has been defined in (6) as

\[S_N = (S^{-1}_F)^{-1},\]

where \([S^{-1}_F]\) is the Friedrichs extensions of the relation \(S^{-1}\) (see also [1] and [2]).

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