The Tangent Space at a Special Symplectic Instanton Bundle on $\mathbb{P}_{2n+1}$

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Introduction

Mathematical instanton bundles on $\mathbb{P}_3$ have their analogues in rank--2n instanton bundles on odd dimensional projective spaces $\mathbb{P}_{2n+1}$. The families of special instanton bundles on these spaces, which generalize the special 'tHooft bundles on $\mathbb{P}_3$, were constructed and described in [OS] and [ST]. More general instanton bundles have recently been constructed in [AO2]. Let $MI_{2n+1}(k)$ denote the moduli space of all instanton bundles on $\mathbb{P}_{2n+1}$ with second Chern class $c_2 = k$. In order to obtain a first impression of this space it is important to know its tangent dimension $h^1\text{End}(\mathcal{E})$ at a stable bundle $\mathcal{E}$ and the dimension $h^2\text{End}(\mathcal{E})$ of the space of obstructions to smoothness.

In this paper we prove that for a special symplectic bundle $\mathcal{E} \in MI_{2n+1}(k)$

$$h^2\text{End}(\mathcal{E}) = (k - 2)^2 \binom{2n - 1}{2}.$$  

Such bundles are stable by [AO1]. So for $n \geq 2$ the situation is quite different to that of $\mathbb{P}_3$, where this number becomes zero, which was shown in [HN]. Since $H^i\text{End}(\mathcal{E}) = 0$ for $i \geq 3$, our result and the Hirzebruch–Riemann–Roch formula, see Remark 2.4,

$$h^1\text{End}(\mathcal{E}) - h^2\text{End}(\mathcal{E}) = -k^2 \binom{2n - 1}{2} + k(8n^2) + 1 - 4n^2$$

give

$$h^1\text{End}(\mathcal{E}) = 4(3n - 1)k + (2n - 5)(2n - 1).$$

Therefore the dimension of $MI_{2n+1}(k)$ grows linearly in $k$, whereas the difference $h^1\text{End}(\mathcal{E}) - h^2\text{End}(\mathcal{E})$ becomes negative for $n \geq 2$ and grows quadratically in $k$. A more important consequence, however, is that in general $MI_{2n+1}(k)$ cannot be smooth at special symplectic bundles, see section 4 and [AO2].

In order to derive our result we fix a 2–dimensional vector space $U$ and consider the natural action of $SL(2)$ on $\mathbb{P}_{2n+1} = \mathbb{P}(U \otimes S^nU)$ as in [ST]. The special instanton bundles are related to the $SL(2)$–homomorphisms $\beta$, see [T4], and are $SL(2)$–invariant. We prove that there is an isomorphism of $SL(2)$–representations

$$H^2(\text{End} \mathcal{E}) \cong S^{k-3}(U) \otimes S^{k-3}(U) \otimes S^2(U \otimes S^{n-2}U).$$

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Notation

- Throughout the paper $K$ denotes an algebraically closed ground field of characteristic 0.
- $U$ denotes a $2$–dimensional $K$–vector space, $S_n = S^n U$ its $n$th symmetric power and $V_n = U \otimes S_n$.
- There is the natural exact sequence of $GL(U)$–equivariant maps for any $k, n \geq 1$
  \[ 0 \to \Lambda^2 U \otimes S_{k-1} \otimes S_{n-1} \xrightarrow{\beta} S_k \otimes S_n \xrightarrow{\mu} S_{k+n} \to 0 \]
  where $\mu$ is the multiplication map and $\beta$ is defined by $(s \wedge t) \otimes f \otimes g \mapsto sf \otimes tg - tf \otimes sg$. This sequence splits and leads to the Clebsch–Gordan decomposition of $S_k \otimes S_n$ by induction. When we tensorize the sequence with $U$ we obtain the exact sequence
  \[ 0 \to \Lambda^2 U \otimes S_{k-1} \otimes V_{n-1} \xrightarrow{\beta} S_k \otimes V_n \xrightarrow{\mu} V_{k+n} \to 0. \]
- $\mathbb{P} = \mathbb{P}_{2n+1} = \mathbb{P} V_n$ is the projective space of one dimensional subspaces of $V_n$.
- The terms vector bundle and locally free sheaf are used synonymously.
- $\mathcal{O}(d)$ denotes the invertible sheaf of degree $d$ on $\mathbb{P}$, $\Omega^p$ the locally free sheaf of differential $p$–forms on $\mathbb{P}$, such that $\Omega^p(p) = \Lambda^p Q^\vee$ where $Q = \mathcal{T}(-1)$ is the canonical quotient bundle on $\mathbb{P}$.
- We use the abbreviations $\mathcal{F}(d) = \mathcal{F} \otimes_\mathcal{O} \mathcal{O}(d)$ for any sheaf $\mathcal{F}$ of $\mathcal{O}$–modules on $\mathbb{P}$, $H^i \mathcal{F} = H^i(\mathcal{F}) = H^i(\mathbb{P}, \mathcal{F})$, $h^i \mathcal{F} = \dim H^i \mathcal{F}$. If $E$ is a finite dimensional $K$–vector space, $E \otimes \mathcal{O}$ denotes the sheaf of sections of the trivial bundle $\mathbb{P} \times E$, and $E \otimes \mathcal{F} = (E \otimes \mathcal{O}) \otimes_\mathcal{O} \mathcal{F}$. We also write $m \mathcal{F} = K^m \otimes \mathcal{F}$.
- We use the Euler sequence $0 \to \Omega^1(1) \to V^\vee \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$ and the derived sequences in its Koszul complex $0 \to \Omega^p(p) \to \Lambda^p V^\vee \otimes \mathcal{O} \to \Omega^{p-1}(p) \to 0$ without extra mentioning.
- $Ext^i(\mathcal{F}, \mathcal{G}) = Ext^i_\mathcal{O}(\mathbb{P}, \mathcal{F}, \mathcal{G})$ for any two $\mathcal{O}$–modules $\mathcal{F}$ and $\mathcal{G}$.
1 Instanton bundles

1.1 An instanton bundle on \( \mathbb{P} = \mathbb{P}_{2n+1} \) with instanton number \( k \) or a \( k \)–instanton is an algebraic vector bundle \( \mathcal{E} \) on \( \mathbb{P} \) satisfying:

(i) \( \mathcal{E} \) has rank \( 2n \) and Chern polynomial \( c(\mathcal{E}) = (1 - h^2)^{-k} = 1 + kh^2 + \ldots \).

(ii) \( \mathcal{E} \) has natural cohomology in the range \(-2n - 1 \leq d \leq 0\), that is for any \( d \) in that range \( h^i \mathcal{E}(d) \neq 0 \) for at most one \( i \).

A \( k \)–instanton bundle \( \mathcal{E} \) is called symplectic if there is an isomorphism \( \mathcal{E} \xrightarrow{\varphi} \mathcal{E}^\vee \) satisfying \( \varphi^\vee = -\varphi \). In this case the spaces \( A \) and \( B \) below are Serre–duals of each other, since \( H^{2n}(\mathcal{E}(-2n-1))^\vee \cong H^1\mathcal{E}^\vee(-1) \cong H^1\mathcal{E}(-1) \).

Remark: In the original definition in [OS] the additional conditions

(iii) \( \mathcal{E} \) is simple, that is \( Hom(\mathcal{E}, \mathcal{E}) = K \),

(iv) the restriction of \( \mathcal{E} \) to a general line is trivial

are imposed. It was shown in [AO1] that (iii) is already a consequence of (i) and (ii). Condition (iv) seems to be independent but we do not need it in this paper. By [ST] special instantons satisfy (iv).

1.2 Let now \( A, B, C \) be vector spaces of dimensions \( k, k, 2n(k-1) \) respectively. A pair of linear maps

\[
A \xrightarrow{a} B \otimes \Lambda^2 V_n^\vee, \quad B \otimes V_n^\vee \xrightarrow{b} C
\]
corresponds to a pair of sheaf homomorphisms

\[
A \otimes \mathcal{O}(-1) \xrightarrow{\tilde{a}} B \otimes \Omega^1(1), \quad B \otimes \Omega^1(1) \xrightarrow{\tilde{b}} C \otimes \mathcal{O}.
\]

Here \( \tilde{a} \) is the composition of the induced homomorphisms \( A \otimes \mathcal{O}(-1) \rightarrow B \otimes \Lambda^2 V_n^\vee \otimes \mathcal{O}(-1) \rightarrow B \otimes \Omega^1(1) \) and \( \tilde{b} \) is the composition of the induced homomorphismus \( B \otimes \Omega^1(1) \rightarrow B \otimes V_n^\vee \otimes \mathcal{O} \rightarrow C \otimes \mathcal{O} \). Conversely, \( a \) and \( b \) are determined by \( \tilde{a} \) and \( \tilde{b} \) respectively as \( H^0(\tilde{a}(1)) \) and \( H^0(\tilde{b}^\vee)^\vee \). Moreover, the sequence

\[
A \otimes \mathcal{O}(-1) \xrightarrow{\tilde{a}} B \otimes \Omega^1(1) \xrightarrow{\tilde{b}} C \otimes \mathcal{O}
\]
is a complex if and only if the induced sequence

\[
A \rightarrow B \otimes \Lambda^2 V_n^\vee \rightarrow C \otimes V_n^\vee
\]
is a complex. We say that (1) is a monad if it is a complex and if in addition \( \tilde{a} \) is a subbundle and \( \tilde{b} \) is surjective.
Proposition 1.3 The cohomology sheaf $E = \text{Ker} \tilde{b}/\text{Im} \tilde{a}$ of a monad (1) is a $k$–instanton and conversely any $k$–instanton is the cohomology of a monad (1). Moreover, the spaces $A, B, C$ of such a monad can be identified with $H^{2n}E(-2n-1), H^{1}E(-1), H^{1}E$ respectively.

Sketch of a proof: if a monad (1) is given it is easy to derive the properties of the definition. Conversely using Beilinson’s spectral sequence, Riemann–Roch and in particular (ii), one obtains a monad with the identification of the vector spaces as in [US]. The map $b$ is then nothing but the natural map $H^{1}E(-1) \otimes V^{\vee}_{n} \to H^{1}E$ and the map $a$ is given as the composition of the cup product

$$H^{2n}E(-2n-1) \otimes \Lambda^{2}V_{n} \to H^{2n}E \otimes \Omega^{2n-1}(-1)$$

and the natural isomorphisms

$$H^{2n}E \otimes \Omega^{2n-1}(-1) \cong H^{2n-1}E \otimes \Omega^{2n-2}(-1) \cong \ldots \cong H^{1}E(-1)$$

arising from the Koszul sequences and condition (ii), see [V] in case of $P_{3}$.

1.4 Existence and special instanton bundles: Using the special structure $V_{n} = U \otimes S^{n}U$ and the Clebsch–Gordan type exact sequence

$$0 \longrightarrow \Lambda^{2}U \otimes S_{k-2} \otimes V_{n-1} \overset{\beta}{\longrightarrow} S_{k-1} \otimes V_{n} \overset{\mu}{\longrightarrow} V_{k+n+1} \longrightarrow 0,$$

see notation, we define the special homomorphism

$$S_{k-1}^{\vee} \otimes \Omega^{1}(1) \overset{\tilde{b}}{\longrightarrow} \Lambda^{2}U^{\vee} \otimes S_{k-2}^{\vee} \otimes V_{n-1}^{\vee} \otimes \mathcal{O}$$

by $b = \beta^{\vee}$. We denote $\mathcal{K} = \text{Ker} (\tilde{b})$. It was shown in [ST] that $\tilde{b}$ is surjective and that $H^{0}\mathcal{K}(1) \subset S_{k-1}^{\vee} \otimes H^{0}\Omega^{1}(2)$ can be identified with a canonical injective $GL(U)$–homomorphism

$$S_{2n+k-1}^{\vee} \otimes \Lambda^{2}U^{\vee} \overset{\delta}{\rightarrow} S_{k-1}^{\vee} \otimes \Lambda^{2}V_{n}^{\vee},$$

dual to the map

$$S_{k-1} \otimes \Lambda^{2}V_{n} \to S_{2n+k-1} \otimes \Lambda^{2}U$$

which is defined by $f \otimes (s \otimes g) \wedge (t \otimes h) \mapsto (fgh) \otimes (s \wedge t)$.

In order to construct instanton bundles we have to find $k$–dimensional subspaces

$$A \subset S_{2n+k-1}^{\vee} \otimes \Lambda^{2}U^{\vee} \subset S_{k-1}^{\vee} \otimes \Lambda^{2}V_{n}^{\vee}$$

such that the induced homomorphism $\tilde{a}$ is a subbundle. By [ST], Lemma 3.7.1, this is the case exactly when $\mathbb{P}A \subset \mathbb{P}(S_{2n+k-1}^{\vee})$ does not meet the secant variety $\text{Sec}_{n}(C_{2n+k-1})$ of $(n-1)$–dimensional secant planes of the canonical
rational curve $C_{2n+k-1}$ of $\mathbb{P}S^\vee_{2n+k-1}$, given by $u \mapsto u^{2n+k-1}$. By dimension reasons such subspaces exist, [ST], 3.7, and hence instanton bundles exist.

A $k$–instanton bundle $\mathcal{E}$ is called **special** if the map $b$ of its monad is isomorphic to the $GL(U)$–homomorphism $\beta^\vee$, that is if there are isomorphisms $\varphi$ and $\psi$ and $g \in GL(V_n)$ with the commutative diagram

\[
\begin{array}{ccc}
H^1\mathcal{E}(-1) \otimes V_n^\vee & \xrightarrow{b} & H^1\mathcal{E} \\
\varphi \otimes g^\vee \downarrow \llcorner & & \psi \downarrow \llcorner \\
S^\vee_{k-1} \otimes V_n^\vee & \xrightarrow{\beta^\vee} & \Lambda^2 U^\vee \otimes S^\vee_{k-2} \otimes V_{n-1}^\vee.
\end{array}
\]

Whereas in [ST] the family of all special $k$–instanton bundles was described, examples of different types of general instanton bundles were found in [AO2].

**Remark 1.5** If $\mathcal{E}$ is special and symplectic then, in addition to the special $GL(U)$–homomorphism $b = \beta^\vee$ of its monad, the map $a$ is given by an element $\alpha \in S^\vee_{2n+2k-2}$ as $a = \kappa \circ \tilde{\alpha}$ where $S_{k-1} \xrightarrow{\tilde{\alpha}} S^\vee_{2n+k-1}$ is defined by $\tilde{\alpha}(f)(g) = \alpha(fg)$ and $S^\vee_{2n+k-1} \xrightarrow{\kappa} S^\vee_{k-1} \otimes \Lambda^2 V_n^\vee$ is as above, [ST], 4.3 and 5.8. In particular $a$ is a $GL(U)$–homomorphism, too, and can be represented by a persymmetric matrix.

**Remark 1.6** It is shown in [AO1] that special symplectic instanton bundles are stable in the sense of Mumford–Takemoto.
2 Representing $\text{Ext}^2(\mathcal{E}, \mathcal{E})$

**Proposition 2.1** Let $\mathcal{E}$ be a symplectic $k$–instanton and let $\mathcal{N}$ be the kernel of the monad (1). Then $\text{Ext}^2(\mathcal{E}, \mathcal{E}) \cong H^2(\mathcal{N} \otimes \mathcal{N})$.

**Proof:** The monad (1) gives rise to the exact sequences

$$0 \to \mathcal{N} \to B \otimes \Omega^1(1) \xrightarrow{\bar{b}} C \otimes \mathcal{O} \to 0$$

and

$$0 \to A \otimes \mathcal{O}(-1) \to \mathcal{N} \to \mathcal{E} \to 0.$$  \hspace{1cm} (2)

After tensoring we have the exact sequences

$$0 \to A \otimes \mathcal{N}(-1) \to \mathcal{N} \otimes \mathcal{N} \to \mathcal{E} \otimes \mathcal{N} \to 0$$

and

$$0 \to A \otimes \mathcal{E}(-1) \to \mathcal{N} \otimes \mathcal{E} \to \mathcal{E} \otimes \mathcal{E} \to 0.$$  \hspace{1cm} (3)

Since $\mathcal{E} \cong \mathcal{E}^\vee$ we obtain $\text{Ext}^2(\mathcal{E}, \mathcal{E}) \cong H^2(\mathcal{E} \otimes \mathcal{E})$. Sequence (2) implies $h^2\mathcal{N}(-1) = h^3\mathcal{N}(-1) = 0$ and from this and (3) also $h^2\mathcal{E}(-1) = h^3\mathcal{E}(-1) = 0$. Now sequences (4) and (5) yield isomorphisms $H^2(\mathcal{E} \otimes \mathcal{E}) \cong H^2(\mathcal{N} \otimes \mathcal{E}) \cong H^2(\mathcal{N} \otimes \mathcal{N})$. \hspace{1cm} \(\square\)

**2.2** In order to represent $H^2(\mathcal{N} \otimes \mathcal{N})$ we note that the sequence (2) is part of the exact diagram

$$
\begin{array}{cccccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{N} & \to & B \otimes \Omega^1(1) & \xrightarrow{\bar{b}} & C \otimes \mathcal{O} & \to & 0 \\
\downarrow & & \downarrow & & \parallel & & \parallel \\
0 & \to & H \otimes \mathcal{O} & \to & B \otimes V^\vee \otimes \mathcal{O} & \xrightarrow{b} & C \otimes \mathcal{O} & \to & 0 \\
\downarrow & & \downarrow & & \parallel & & \parallel \\
B \otimes \mathcal{O}(1) & = & B \otimes \mathcal{O}(1) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

where $H$ is the kernel of the operator $b$, which is surjective because $\bar{b}$ is surjective. The left–hand column of (6) gives us after tensoring by $\Omega^1(1)$

$$B \otimes H^0\Omega^1(2) \xrightarrow{\delta} H^1(\mathcal{N} \otimes \Omega^1(1)) \text{ and } H^2(\mathcal{N} \otimes \Omega^1(1)) = 0.$$  \hspace{1cm} (7)

Since $\bar{b}$ is the Beilinson representation of $\mathcal{N}$, we have the commutative diagram

$$
\begin{array}{ccc}
H^1\mathcal{N}(-1) \otimes H^0\mathcal{O}(1) & \xrightarrow{\text{cup}} & H^1\mathcal{N} \\
\parallel & & \parallel \\
B \otimes V^\vee & \xrightarrow{b} & C.
\end{array}
$$

6
Moreover, \( \delta \) in (7) coincides also with cup:

\[
\begin{align*}
B \otimes H^0 \Omega^1(2) & \xrightarrow{\delta} H^1(\mathcal{N} \otimes \Omega^1(1)) \\
\approx & \quad \cup \quad \\
H^1 \mathcal{N}(-1) \otimes H^0 \Omega^1(2)
\end{align*}
\]  

(9)

Tensoring the top row of (6) with \( \mathcal{N} \) and using (7) we obtain the following diagram with exact row:

\[
\begin{align*}
0 & \to H^1(\mathcal{N} \otimes \mathcal{N}) \\
& \to B \otimes H^1(\mathcal{N} \otimes \Omega^1(1)) \\
& \to C \otimes H^1(\mathcal{N}) \\
& \to H^2(\mathcal{N} \otimes \mathcal{N}) \\
& \to 0
\end{align*}
\]

(10)

It follows that

\[
H^2(\mathcal{N} \otimes \mathcal{N}) = \text{Coker}(\Phi) = \text{Ker}(\Phi^\vee).
\]  

(11)

**Lemma 2.3** The induced operator \( \Phi \) is the composition \( B \otimes B \otimes \Lambda^2 V_n^\vee \overset{id \otimes \sigma}{\longrightarrow} B \otimes B \otimes V_n^\vee \otimes V_n^\vee \overset{\Phi}{\longrightarrow} C \otimes C \), where \( \sigma \) denotes the canonical desymmetrization.

Proof: The computation of \( \Phi \) is achieved by the diagram

\[
\begin{array}{ccccccccc}
B \otimes B \otimes \Lambda^2 V_n^\vee & \xrightarrow{id \otimes \sigma} & B \otimes B \otimes V_n^\vee \otimes V_n^\vee \\
\approx & & \approx \\
B \otimes H^1 \mathcal{N}(-1) \otimes H^0 \Omega^1(2) & \xrightarrow{id \otimes H^0(\iota(1))} & B \otimes H^1 \mathcal{N}(-1) \otimes V_n^\vee \otimes H^0 \mathcal{O}(1) \\
\approx & & \approx \\
B \otimes H^1(\mathcal{N} \otimes \Omega^1(1)) & \xrightarrow{id \otimes H^1(\iota) \otimes \iota} & B \otimes V_n^\vee \otimes H^1 \mathcal{N} \\
\approx & & \approx \\
C \otimes H^1 \mathcal{N} & \xrightarrow{b \otimes \iota} & C \otimes C \\
\approx & & \approx \\
C \otimes C & & \\
\end{array}
\]

In this diagram \( \iota \) denotes the canonical inclusion \( \Omega^1(1) \hookrightarrow V_n^\vee \otimes \mathcal{O} \), and up to \( \Lambda^2 V_n^\vee \cong H^0 \Omega^1(2) \) and \( V_n^\vee \cong H^0 \mathcal{O}(1) \) the map \( \sigma \) can be identified with \( H^0(\iota(1)) \). Therefore, the square I is commutative. Square II is a canonically
induced diagram of cup–operations and commutative using $B \cong H^1\mathcal{N}(-1)$. The triangle III is induced by the commutative triangle

$$
\begin{array}{ccc}
B \otimes \mathcal{N} \otimes \Omega^1(1) & \xrightarrow{id \otimes \iota} & B \otimes V^\vee \otimes \mathcal{N} \\
\downarrow b \otimes id & & \swarrow \sigma_{\otimes id} \\
C \otimes \mathcal{N} & & 
\end{array}
$$

and hence commutative, and the commutativity of IV results just from the identification $H^1\mathcal{N} \cong C$. Now by definition the composition of the left–hand column is $\Phi$ and the composition of the right–hand column is $id_B \otimes id_V \otimes b$ since $b$ is defined by (8).

It follows that $\Phi = (b \otimes id_C) \circ (id_B \otimes id_V \otimes b) \circ (id_B \otimes \sigma) = (b \otimes b) \circ (id \otimes \sigma)$.

**Remark 2.4** If $\mathcal{E}$ is a $k$–instanton bundle it is easily checked that $h^i\mathcal{E}(d) = h^i\mathcal{E}^\vee(d) = 0$ for $i \geq 2$ and $d \geq -1$. Using $\mathcal{E}^\vee \otimes \mathcal{N}$ again it follows that $Ext^i(\mathcal{E}, \mathcal{E}) = H^i(\mathcal{E}^\vee \otimes \mathcal{E}) = H^i(\mathcal{E}^\vee \otimes \mathcal{N}) = 0$ for $i \geq 3$. This and the Riemann–Roch formula, which can also ad hoc be derived from the monad representation, give

$$h^1(\mathcal{E}^\vee \otimes \mathcal{E}) - h^2(\mathcal{E}^\vee \otimes \mathcal{E}) = -k^2\left(\frac{2n - 1}{2}\right) + 8kn^2 - 4n^2 + 1.$$
3 Determination of Ext²(\mathcal{E}, \mathcal{E})

We are now able to determine \( Ext^2(\mathcal{E}, \mathcal{E}) \) as a \( GL(2) \)–representation space in case of a special instanton bundle. In that case \( b \) is the dual of the operator \( \beta : \Lambda^2 U \otimes S_{k-2} \otimes V_{n-1} \rightarrow S_{k-1} \otimes V_n \), see notation or [1.4]. Then \( \Phi^\vee \) is the composition of \( \beta \otimes \beta \) and the multiplication map \( V_n \otimes V_n \rightarrow \Lambda^2 V_n \). In order to simplify we choose a fixed basis \( s,t \in U \) and the isomorphism \( \Lambda^2 U \cong k \) given by \( s \wedge t \). Then

\[
S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1} \xrightarrow{\Phi^\vee} S_{k-1} \otimes S_{k-1} \otimes \Lambda^2 V_n
\]

is explicitly given by

\[
\Phi^\vee(g \otimes g' \otimes v \otimes v') = sg \otimes sg' \otimes (tv \wedge tv') - sg \otimes tg' \otimes (tv \wedge sv') -
- tg \otimes sg' \otimes (sv \wedge tv') + tg \otimes tg' \otimes (sv \wedge sv').
\]

In order to determine the kernel of \( \Phi^\vee \) we consider the \( GL(U) \)–homomorphism

\[
S_{k-3} \otimes S_{k-3} \otimes V_{n-2} \otimes V_{n-2} \xrightarrow{\epsilon'} S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1}
\]

defined similarly by

\[
\epsilon'(f \otimes f' \otimes u \otimes u') = sf \otimes sf' \otimes tu \otimes tu' - sf \otimes tf' \otimes su \otimes tu' -
- tf \otimes sf' \otimes tu \otimes su' + tf \otimes tf' \otimes su \otimes su'.
\]

Up to the order of factors the map \( \epsilon' \) is the tensor product \( \beta' \otimes \beta' \) where \( \beta' : S_{k-3} \otimes V_{n-2} \rightarrow S_{k-2} \otimes V_{n-1} \) is defined as \( \beta \). Hence, \( \epsilon' \) is injective. Finally, we define \( \epsilon \) as the composition

\[
S_{k-3} \otimes S_{k-3} \otimes S^2 V_{n-2} \xrightarrow{id \otimes i} S_{k-3} \otimes S_{k-3} \otimes V_{n-2} \otimes V_{n-2} \xrightarrow{\epsilon'} S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1}
\]

where \( i \) is the canonical desymmetrization. Then also \( \epsilon \) is injective.

**Proposition 3.1** \((S_{k-3} \otimes S_{k-3} \otimes S^2 V_{n-2}, \epsilon)\) is the kernel of \( \Phi^\vee \).

**Proof:** A straightforward computation shows that \( \text{Im}(\epsilon) \subset \text{Ker}(\Phi^\vee) \). In order to show equality we reduce \( \text{Ker}(\Phi^\vee) \) modulo \( \text{Im}(\epsilon) \) using canonical bases of the vector spaces. A more elegant proof using Clebsch–Gordan decompositions seems much harder to achieve. Let us denote the bases as follows:
basis of $S_{k-3}$: \( e_\alpha = s^{k-3-\alpha}t^\alpha \) for \( 0 \leq \alpha \leq k-3 \)

basis of $S_{k-2}$: \( f_\alpha = s^{k-2-\alpha}t^\alpha \) for \( 0 \leq \alpha \leq k-2 \)

basis of $S_{k-1}$: \( g_\alpha = s^{k-1-\alpha}t^\alpha \) for \( 0 \leq \alpha \leq k-1 \)

basis of $V_{n-2}$: \( u_\mu = s \otimes s^{n-2-\mu}t^\mu \) for \( 0 \leq \mu \leq n-2 \)

basis of $V_{n-1}$: \( x_\mu = s \otimes s^{n-1-\mu}t^\mu \) for \( 0 \leq \mu \leq n-1 \)

basis of $V_n$: \( y_\mu = s \otimes s^{n-\mu}t^\mu \) for \( 0 \leq \mu \leq n \)

For the basis \( f_\alpha \otimes f_\beta \otimes x_\mu \otimes x_\nu, \ f_\alpha \otimes f_\beta \otimes x_\mu \otimes x_\nu, \ f_\alpha \otimes f_\beta \otimes x_\mu \otimes x_\nu, \ f_\alpha \otimes f_\beta \otimes x_\mu \otimes x_\nu \) we use the index tuplets \((\alpha, \beta, \mu, \nu), (\alpha, \beta, \mu, \bar{\nu}), (\alpha, \beta, \bar{\mu}, \nu), (\alpha, \beta, \bar{\mu}, \bar{\nu})\) respectively. The set of these indices will be ordered lexicographically with the additional assumption that always \( \mu < \nu \). Then, for example, \((\alpha, \beta, \mu, \bar{\nu}) < (\alpha, \beta, \bar{\lambda}, \delta)\).

Accordingly, the coefficients of an element \( \xi \in S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1} \) will be denoted by \( c(\alpha, \beta, \mu, \nu), \ c(\alpha, \beta, \mu, \bar{\nu}), \ c(\alpha, \beta, \bar{\mu}, \nu), \ c(\alpha, \beta, \bar{\mu}, \bar{\nu})\).

By the formula for \( \Phi^\vee \) we obtain the

**Lemma 3.2** Let \( \xi \in S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1} \).

(i) The coefficient of \( \Phi^\vee(\xi) \) at the basis element \( g_\alpha \otimes g_\beta \otimes y_\mu \wedge \bar{y}_\nu \) in \( S_{k-1} \otimes S_{k-1} \otimes \Lambda^2 V_n \) is:

\[
\begin{align*}
&c(\alpha, \beta, \mu - 1, \nu - 1) - c(\alpha, \beta, \nu - 1, \mu - 1) \\
&-c(\alpha - 1, \beta - 1, \mu - 1, \nu) + c(\alpha - 1, \beta - 1, \nu - 1, \mu) \\
&-c(\alpha - 1, \beta, \mu, \nu - 1) + c(\alpha - 1, \beta, \nu, \mu) \\
&+c(\alpha - 1, \beta - 1, \mu, \nu) - c(\alpha - 1, \beta - 1, \nu, \mu).
\end{align*}
\]

Here we agree that each of these coefficients is 0 if one of \( \alpha, \alpha - 1, \beta, \beta - 1 \notin [0, k-2] \) or if one of \( \mu, \mu - 1, \nu, \nu - 1 \notin [0, n-1] \).

(ii) Analogous statements hold for the coefficient of \( \Phi^\vee(\xi) \) at \( g_\alpha \otimes g_\beta \otimes y_\mu \wedge y_\nu \) for \( \mu < \nu \) (without bars) and at \( g_\alpha \otimes g_\beta \otimes \bar{y}_\mu \wedge \bar{y}_\nu \) for \( \mu < \nu \) (with two bars).

**Lemma 3.3** Let the notation be as above. If \( \Phi^\vee(\xi) = 0 \) then:

(i) If \( c(\alpha, \beta, \mu, \nu) \) is the first non–zero coefficient of \( \xi \) (in the lexicographical order), then \( 0 < \mu \leq \nu \).

(ii) If \( c(\alpha, \beta, \mu, \bar{\nu}) \) is the first non–zero coefficient of \( \xi \), then \( \mu \neq 0, \nu \neq 0 \).

(iii) \( c(\alpha, \beta, \bar{\mu}, \nu) \) is never a first non–zero coefficient of \( \xi \).
(iv) If \(c(\alpha, \beta, \mu, \nu)\) is the first non-zero coefficient of \(\xi\), then \(0 < \mu \leq \nu\).

Proof: (i) Let \(c(\alpha, \beta, \mu, \nu)\) be the first coefficient of \(\xi\). Then, by Lemma \ref{lem:3.2} the coefficient of \(0 = \Phi^\vee(\xi)\) at \(g_\alpha \otimes g_\beta \otimes y_{\mu+1} \land y_{\nu+1}\) is

\[
0 = c(\alpha, \beta, \mu, \nu) - c(\alpha, \beta, \mu, \nu) - c(\alpha, \beta - 1, \mu, \nu + 1) + c(\alpha, \beta - 1, \nu, \mu + 1)
- c(\alpha - 1, \beta, \mu + 1, \nu) + c(\alpha - 1, \beta, \nu + 1, \mu) - \ldots
\]

Since \(c(\alpha, \beta, \mu, \nu)\) is the first coefficient, only the first two in this formula could be non-zero because the others have smaller index in the lexicographical order. Hence

\[
c(\alpha, \beta, \mu, \nu) = c(\alpha, \beta, \nu, \mu).
\]

If \(\mu > \nu\) then \(c(\alpha, \beta, \nu, \mu)\) would be earlier and non-zero. Hence, \(\mu \leq \nu\).

Assume now that \(\mu = 0\). The coefficient of \(\phi^\vee(\xi)\) of \(g_\alpha \otimes g_{\beta+1} \otimes y_0 \land y_{\nu+1}\) is

\[
0 = c(\alpha, \beta + 1, -1, \nu) - c(\alpha, \beta + 1, \nu, -1)
- c(\alpha, \beta, -1, \nu + 1) + c(\alpha, \beta, \nu, 0) ± \ldots
\]

In this sum all but \(c(\alpha, \beta, \nu, 0)\) are automatically zero because \((\alpha - 1, \beta, \ldots) \leq (\alpha, \beta, 0, \nu)\) and \(-1\) occurs. Hence, \(c(\alpha, \beta, 0, \nu) = c(\alpha, \beta, 0, 0) = 0\), contradiction.

The statements (ii), (iii), (iv) are proved analogously. \(\square\)

Now we continue the proof of Proposition \ref{prop:3.1}. We reduce an element \(\xi \in \text{Ker}(\Phi^\vee)\) to \(0 \mod \text{Im}(\epsilon)\) using Lemma \ref{lem:3.3}.

a) Assume that the first non-zero coefficient of \(\xi\) is

\[
c(\alpha, \beta, \mu, \nu).
\]

Then by Lemma \ref{lem:3.3} \(0 < \mu \leq \nu\). Then the element

\[
\xi' = \xi - c(\alpha, \beta, \mu, \nu)\epsilon(e_\alpha \otimes e_\beta \otimes u_{\mu-1} \cdot u_{\nu-1})
\]

belongs to \(\text{Ker}(\Phi^\vee)\). We have

\[
\epsilon(e_\alpha \otimes e_\beta \otimes u_{\mu-1} \cdot u_{\nu-1})
= f_\alpha \otimes f_\beta \otimes (x_\mu \otimes x_\nu + x_\nu \otimes x_\mu)
- f_\alpha \otimes f_{\beta+1} \otimes (x_{\mu-1} \otimes x_\nu + x_{\nu-1} \otimes x_\mu)
- f_{\alpha+1} \otimes f_\beta \otimes (x_\mu \otimes x_{\nu-1} + x_\nu \otimes x_{\mu-1})
+ f_{\alpha+1} \otimes f_{\beta+1} \otimes (x_{\mu-1} \otimes x_{\nu-1} + x_{\nu-1} \otimes x_{\mu-1})
\]

and therefore \(\xi'\) is a sum of monomials of index \(> (\alpha, \beta, \mu, \nu)\). Hence, we can assume that \(\xi \mod \text{Im}(\epsilon)\) has no coefficient with index \((\alpha, \beta, \mu, \nu)\).
b) By Lemma 3.3 we can assume that the first non-zero coefficient of $\xi$ has index $(\alpha, \beta, \mu, \bar{\nu})$ or $(\alpha, \beta, \bar{\mu}, \bar{\nu})$. In the first case we know by Lemma 3.3 that $0 < \mu, \nu$. When we consider again

$$\xi' = \xi - c(\alpha, \beta, \mu, \bar{\nu})e(\epsilon_{\alpha} \otimes \epsilon_{\beta} \otimes u_{\mu-1} \cdot \bar{u}_{\nu-1})$$

we have $\phi'(\xi') = 0$ and $\xi'$ is a sum of monomials of index $> (\alpha, \beta, \mu, \bar{\nu})$. Hence, we may assume that $\xi \mod \text{Im}(\epsilon)$ has $c(\alpha, \beta, \bar{\mu}, \bar{\nu})$ as first non-zero coefficient. Again by Lemma 3.3 $0 < \mu, \nu$ and

$$\xi' = \xi - c(\alpha, \beta, \bar{\mu}, \bar{\nu})e(\epsilon_{\alpha} \otimes \epsilon_{\beta} \otimes \bar{u}_{\mu-1} \cdot \bar{u}_{\nu-1})$$

is a sum of monomials of index $> (\alpha, \beta, \bar{\mu}, \bar{\nu})$. This finally shows that $\xi = 0 \mod \text{Im}(\epsilon)$.

This completes the proof of Proposition 3.1.
4 Conclusions

By Proposition 2.1, Proposition 3.1, (11) and Lemma 2.3 we have determined the space $\text{Ext}^2(E, E)$. Together with Remark 2.4 we obtain

**Theorem 4.1** For any special symplectic $k$–instanton bundle $E$ on $\mathbb{P}_{2n+1}$

\begin{enumerate}
\item $\text{Ext}^2(E, E) \simeq S_{k-3}^\vee \otimes S_{k-3}^\vee \otimes S^2 V_{n-2}^\vee$
\item $\dim \text{Ext}^2(E, E) = (k-2)^2 \binom{2n-1}{2}$
\item $\dim \text{Ext}^1(E, E) = 4k(3n-1) + (2n-5)(2n-1)$.
\end{enumerate}

Let $MI_{2n+1}(k)$ denote the open part of the Maruyama scheme of semi-stable coherent sheaves on $\mathbb{P}_{2n+1}$ with Chern polynomial $(1 - h^2)^{-k}$ consisting of instanton bundles. By [AO1] any special symplectic instanton bundle $E$ is stable. Therefore, $\text{Ext}^1(E, E)$ can be identified with the tangent space of $MI_{2n+1}(k)$ at $E$. In [AO2] deformations $E'$ of special symplectic instanton bundles in $MI_{2n+1}(k)$ have been found for $n = 2$ and $k = 3, 4$ which satisfy $\text{Ext}^2(E', E') = 0$. This shows that in these cases there are components $MI'_{2n+1}(k)$ of $MI_{2n+1}(k)$ of the expected dimension $4(3n-1)k + (2n-5)(2n-1)$ containing the set of special instanton bundles. In particular, see [AO2]:

for $k = 3, 4$ the moduli space $MI_5(k)$ is singular at least in special symplectic bundles.

However, in case $2n+1 = 3$ we obtain the vanishing result of [HN]:

any special $k$–instanton bundle $E$ on $\mathbb{P}_3$ satisfies $\text{Ext}^2(E, E) = 0$ and is a smooth point of $MI_3(k)$,

since any rank–2 instanton bundle is symplectic.
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