Conservation of Supergravity Currents from Matrix Theory

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Abstract

In recent work by Kabat and Taylor, certain Matrix theory quantities have been identified with the spatial moments of the supergravity stress-energy tensor, membrane current, and fivebrane current. In this note, we determine the relations between these moments required by current conservation, and prove that these relations hold as exact Matrix Theory identities at finite N. This establishes conservation of the effective supergravity currents (averaged over the compact circle). In addition, the constraints of current conservation allow us to deduce Matrix theory quantities corresponding to moments of the spatial current of the longitudinal fivebrane charge, not previously identified.

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1 Introduction

Since the original BFSS conjecture [1], a remarkable correspondence between matrix quantum mechanics and eleven-dimensional supergravity has emerged. Matrix theory states corresponding to gravitons, membranes, and fivebranes have been identified, and many aspects of the interaction between these objects calculated from Matrix theory have been shown to agree with supergravity predictions ([3, 2, 1] and refs. therein). Certain apparent discrepancies have also been pointed out, however (see, for example [10, 11]).

Recently, more general results involving the interaction potentials of arbitrary separated objects have been proven [5, 6, 7]. In particular, Kabat and Taylor [7] have shown that in a Matrix theory background corresponding to two arbitrary well separated objects, a series of terms in the one-loop matrix theory potential exactly reproduces the linearized supergravity potential arising from the exchange of quanta with zero longitudinal momentum. Central to this analysis was a proposal for the identification of certain Matrix theory quantities with the spatial moments of the stress-energy tensor, membrane current, and fivebrane current of the related supergravity theory. These definitions give a precise correspondence between a given Matrix theory background and the distribution of matter and charges in the transverse spatial dimensions of the supergravity background. Recently, this correspondence has been used to show that the statistical calculation of black hole entropy in the Matrix description of M-theory reduces to the original Gibbons-Hawking calculation of thermodynamic entropy [13].

Crucial to the consistency of eleven-dimensional supergravity is the conservation of the stress-energy tensor, membrane current, and fivebrane current, since these are the currents which couple to the massless particles of the theory. If Matrix theory actually does contain a description of supergravity, we expect that this current conservation should be derivable directly from matrix theory. Thus, an obvious consistency check for the identifications in [7] is to determine whether or not the moments, as defined in Matrix theory, obey the relations imposed by conservation of the respective currents. This is the aim of the present paper.

In section 2, the relations between the moments of a spacetime current dictated by current conservation are derived. In section 3, we show that all such relations involving the moments identified in [7] from Matrix theory are satisfied as precise, finite-N Matrix theory identities. This demonstrates that the definitions given are consistent with supergravity current conservation. Alternatively, our results prove the conservation of ten-dimensional spacetime currents which we associate with the supergravity stress energy tensor and membrane current averaged over the compact circle. In the case of the fivebrane current, previous work suggested only a definition for the static longitudinal fivebrane charge, but the relations we derive give a natural Matrix theory definition for moments of the spatial current associated with this charge. Section 4 contains a discussion of our results.
2 Predictions of conservation equations

In this section, we wish to derive the relations among the moments of a given spacetime current imposed by current conservation.

We consider a current $C^\mu(x)$ with compact support in a (for now) uncompactified spacetime with spatial directions $i = 1, ..., d$. Define the spatial moments of this current as

$$C^{\mu(l_1, \ldots, l_n)} = \int d^d x [C^\mu(x)x^{l_1} \cdots x^{l_n}]$$

Now, assuming the current is conserved,

$$\partial_\mu C^\mu(x) = 0$$

we have

$$0 = \int d^d x [\partial_\mu C^\mu x^{l_1} \cdots x^{l_n}] = \int d^d x [(\partial_t C^t + \partial_i C^i)x^{l_1} \cdots x^{l_n}]$$

so, integrating by parts and using the definition above, we find

$$\partial_t C^{t(l_1 \cdots l_n)} = C^{l_1(l_2 \cdots l_n)} + \cdots + C^{l_n(l_1 \cdots l_{n-1})}$$

This is our desired relation. It expressed the time derivative of the moments of a conserved charge in terms of moments of the associated spatial current.

In the case of a theory with a compact circular direction $x^-$, relevant to finite N Matrix theory, $x^-$ is not continuous around the circle, so we should define moments with a periodic variable in the integral (e.g. Fourier modes around the circle). Thus, we take

$$C^{\mu(l_1 \cdots l_n)} = \int d x^- \int d^d x [C^\mu(x)x^{l_1} \cdots x^{l_n} e^{imx^-/R}]$$

In this case, the relevant relations become:

$$\partial_+ C^{+l(l_2 \cdots l_n)} = C^{l_1(l_2 \cdots l_n)} + \cdots + C^{l_n(l_1 \cdots l_{n-1})}$$

where we have used indices appropriate to a lightlike compactified theory. The relations for $m = 0$,

$$\partial_+ C^{0l(l_2 \cdots l_n)} = C^{l_1(l_2 \cdots l_n)} + \cdots + C^{l_n(l_1 \cdots l_{n-1})}$$

simply express the fact that the averaged current $\int d x^- C^\mu(x), \mu \in \{+, 1, \ldots, 9\}$ obeys the conservation equations for uncompactified ten-dimensional spacetime (compare (2)).

\footnote{In this work we will always be dealing with expressions appropriate to linearized gravity with a flat background, thus it is not necessary to use a covariant derivative.}
In fact, it is these \( m = 0 \) moments, of the stress-energy tensor, membrane current and fivebrane current, that appear in the supergravity potential between widely separated objects arising from the exchange of quanta with zero longitudinal momentum. Hence, it is these objects for which corresponding Matrix theory quantities were deduced in [7]. In the following sections, we recall these identifications and show that these do in fact satisfy all relations \((3)\) implied by current conservation.

### 3 Proof of conservation relations

In this section, we review the identifications made in [7] between Matrix theory quantities and moments of the supergravity stress-energy tensor, membrane current, and fivebrane current, and show that these objects satisfy all relations required by conservation of the respective supergravity currents.

#### 3.1 Stress-Energy Tensor

In [7], the following Matrix theory quantities were identified as the transverse spatial moments of the stress-energy tensor (integrated around the compact circle) for a given Matrix theory background \( X^i \) satisfying the equations of motion. Defining \( F_{ij} = \{X^i, X^j\} \), we have (suppressing the subscript 0 which denotes the zeroeth fourier mode)

\[
\begin{align*}
\tau^{--(l_1 \cdots l_n)} &= \frac{1}{R} \text{STr} \left( \frac{1}{4} \dot{X}^i \dot{X}^j \dot{X}^i \dot{X}^j + \frac{1}{4} \dot{X}^i \dot{X}^k F^{jk} + \dot{X}^i \dot{X}^j F^{ik} F^{kj} + \frac{1}{4} F_{ij} F^{kl} F^{li} - \frac{1}{16} F_{ij} F^{ij} F^{kl} F^{kl} \right) \dot{X}^l_1 \cdots \dot{X}^l_n \\
\tau^{-i(l_1 \cdots l_n)} &= \frac{1}{R} \text{STr} \left( \frac{1}{2} \dot{X}^i \dot{X}^i \dot{X}^j + \frac{1}{4} \dot{X}^i F^{jk} F^{ik} + F^{ij} F^{ik} \dot{X}^k \dot{X}^l_1 \cdots \dot{X}^l_n \right) \\
\tau^{+-l_1 \cdots l_n)} &= \frac{1}{R} \text{STr} \left( \frac{1}{2} \dot{X}^i \dot{X}^i + \frac{1}{4} F^{ij} F^{ij} \dot{X}^l_1 \cdots \dot{X}^l_n \right) \\
\tau^{ij(l_1 \cdots l_n)} &= \frac{1}{R} \text{STr} \left( \dot{X}^i \dot{X}^j F^{ik} F^{kj} \dot{X}^l_1 \cdots \dot{X}^l_n \right) \\
\tau^{+i(l_1 \cdots l_n)} &= \frac{1}{R} \text{STr} \left( \dot{X}^i \dot{X}^l_1 \cdots \dot{X}^l_n \right) \\
\tau^{++(l_1 \cdots l_n)} &= \frac{1}{R} \text{STr} \left( \dot{X}^l_1 \cdots \dot{X}^l_n \right)
\end{align*}
\]

Here, \( \text{STr} \) is a symmetrized trace, in which for each term we take an average of all possible orderings of the independently written factors (i.e. \( F^{ij} \) is to be treated as a unit). From equation \((3)\) we find that current conservation implies the following relations among these moments

\[
\partial_i \tau^{++(l_1 \cdots l_n)} = \tau^{ij(l_1 \cdots l_n)} + \cdots + \tau^{l_n(l_1 \cdots l_{n-1})}
\]

for arbitrary \( n \) and \( \mu \in \{+, -, i\} \). (Recall that \( x^+ \) is the Matrix theory time). We now prove these as exact Matrix theory identities. For \( \mu = + \), we have (using a bar to denote an
omitted index)
\[
\sum_{j=1}^{n} T^{+i j_{1} \cdots j_{l_{1}} \cdots l_{n}} = \frac{1}{R} \sum_{j=1}^{n} \text{STr} \left( X^{l_{1}} \cdots X^{l_{j-1}} \dot{X}^{l_{j}} X^{l_{j+1}} \cdots X^{l_{n}} \right)
\]
\[
= \frac{1}{R} \partial_{t} \text{STr} (X^{l_{1}} \cdots X^{l_{n}})
\]
\[
= \partial_{t} T^{++(l_{1} \cdots l_{n})}
\]
In the next case ($\mu = i$), using equation (8) from the Appendix, we have
\[
\sum_{j=1}^{n} T^{i j_{1} \cdots j_{l_{1}} \cdots l_{n}} = \frac{1}{R} \sum_{j=1}^{n} \text{STr} \left( \dot{X}^{i} X^{l_{1}} \cdots X^{l_{j-1}} \dot{X}^{l_{j}} X^{l_{j+1}} \cdots X^{l_{n}} \right)
\]
\[
= \frac{1}{R} \text{STr} (\dot{X}^{i} \partial_{t} (X^{l_{1}} \cdots X^{l_{n}}) - [[X^{i}, X^{k}], X^{l}] X^{l_{j-1}} X^{l_{j+1}} \cdots X^{l_{n}})
\]
\[
= \frac{1}{R} \text{STr} (\dot{X}^{i} \partial_{t} (X^{l_{1}} \cdots X^{l_{n}}) + \dot{X}^{i} X^{l_{1}} \cdots X^{l_{n}})
\]
\[
= \frac{1}{R} \partial_{t} \text{STr} (\dot{X}^{i} X^{l_{1}} \cdots X^{l_{n}})
\]
\[
= \partial_{t} T^{++(l_{1} \cdots l_{n})}
\]
where we have used the equations of motion $\ddot{X}^{i} = -[[X^{i}, X^{k}], X^{k}]$ in the third line. We emphasize that each commutator is to be treated as a unit in the symmetrization, throughout this work. For the last case ($\mu = -$), using equation (9) from the appendix, we have:
\[
\sum_{j=1}^{n} T^{-i j_{1} \cdots j_{l_{1}} \cdots l_{n}} = \frac{1}{R} \sum_{j=1}^{n} \text{STr} \left( \dot{X}^{i} X^{l_{1}} \cdots X^{l_{j-1}} \dot{X}^{l_{j}} X^{l_{j+1}} \cdots X^{l_{n}} \right)
\]
\[
= \frac{1}{R} \text{STr} \left( \dot{X}^{i} X^{l_{1}} \dot{X}^{l_{1}} + \frac{1}{4} F^{k l} F^{k l j_{1} \cdots j_{l_{1}} \cdots l_{n}} \partial_{t} (X^{l_{1}} \cdots X^{l_{n}})
\]
\[
= \frac{1}{R} \partial_{t} \text{STr} \left( \dot{X}^{i} X^{l_{1}} \dot{X}^{l_{1}} + \frac{1}{4} F^{k l} F^{k l} X^{l_{1}} \cdots X^{l_{n}} \dot{X}^{i} \dot{X}^{i} + \frac{1}{2} F^{k l} F^{k l} X^{l_{1}} \cdots X^{l_{n}} \right)
\]
\[
= \partial_{t} T^{-++(l_{1} \cdots l_{n})},
\]
as desired.

Therefore, the Matrix theory quantities $T^{-i j_{1} \cdots j_{l_{1}} \cdots l_{n}}$ satisfy all relations expected for moments of a conserved current. We may thus define a ten-dimensional spacetime tensor
\[
T^{\mu \nu}(x) = \int \frac{d^{9} k}{(2\pi)^{9}} \left[ e^{-i k_{i} x_{i}} \sum_{n} \sum_{l_{1} \cdots l_{n}} T^{\mu \nu(l_{1} \cdots l_{n})} \frac{(i)^{n}}{n!} k_{l_{1}} \cdots k_{l_{n}} \right]
\]
interpreted as the eleven-dimensional stress-energy tensor integrated over the compact circle. The relations we have proven show that these ten-dimensional currents are conserved, \( \partial_{\mu} J^{\mu
u}(x) = 0 \) where \( \mu \) runs over \{+, 1, 2, ..., 9\} and \( \nu = 1, 2, ..., 9, +, - \). Note that the moments we have been discussing contain no information about the distribution of matter in the compact direction. This is encoded in the \( m > 0 \) moments in equation (2), and these did not appear in the supergravity potential arising from the exchange of quanta with no longitudinal momentum.

### 3.2 Membrane current

The membrane current is a totally antisymmetric tensor whose spatial moments (averaged over the compact circle) for a given Matrix theory background were deduced to be

\[
\mathcal{J}^{ij(l_1\cdots l_n)} = \frac{1}{6R} \text{Str} \left( (\dot{X}^i \dot{X}^k F^{kj} - \dot{X}^j \dot{X}^k F^{ki} - \frac{1}{2} \dot{X}^k \dot{X}^k F^{ij} + \frac{1}{4} F^{ij} F^{kl} + F^{ik} F^{lj} ) X^{l_1} \cdots X^{l_n} \right)
\]

\[
\mathcal{J}^{+i(l_1\cdots l_n)} = \frac{1}{6R} \text{Str} \left( F^{ij} \dot{X}^j X^{l_1} \cdots X^{l_n} \right)
\]

\[
\mathcal{J}^{ijkl(l_1\cdots l_n)} = -\frac{1}{6R} \text{Str} \left( (\dot{X}^k F^{kj} + \dot{X}^j F^{ki} + \dot{X}^k F^{ij}) X^{l_1} \cdots X^{l_n} \right)
\]

\[
\mathcal{J}^{+ij(l_1\cdots l_n)} = -\frac{1}{6R} \text{Str} \left( F^{ij} X^{l_1} \cdots X^{l_n} \right)
\]

Conservation of the membrane current requires the following relations between the moments (3):

\[
\partial_{\mu} \mathcal{J}^{+\mu(l_1\cdots l_n)} = \mathcal{J}^{\mu l_1(l_2\cdots l_n)} + \ldots + \mathcal{J}^{\mu l_n(l_1\cdots l_{n-1})}
\]

Since \( \mathcal{J} \) is an antisymmetric tensor, it suffices to give a proof for \((\mu, \nu) = (i, j)\) and \((\mu, \nu) = (-, i)\).

In the first case, we have

\[
\sum_{k=1}^{n} \mathcal{J}^{ijkl(l_k\cdots l_n)} = -\frac{1}{6R} \sum_{k=1}^{n} \text{Str} \left( F^{ij} X^{l_1} \cdots X^{l_{k-1}} \dot{X}^{l_k} X^{l_{k+1}} \cdots X^{l_n} \right)
\]

\[
\quad + i \dot{X}^i X^{l_1} \cdots X^{l_{k-1}} [X^j, X^{l_k}] X^{l_{k+1}} \cdots X^{l_n}
\]

\[
\quad - i \dot{X}^j X^{l_1} \cdots X^{l_{k-1}} [X^i, X^{l_k}] X^{l_{k+1}} \cdots X^{l_n}
\]

\[
= -\frac{1}{6R} \text{Str} \left( F^{ij} \partial_{l} (X^{l_1} \cdots X^{l_n}) + (i[X^i, X^j] + i[X^i, \dot{X}^j]) X^{l_1} \cdots X^{l_n} \right)
\]

\[
= -\frac{1}{6R} \partial_{l} \text{Str} \left( F^{ij} X^{l_1} \cdots X^{l_n} \right)
\]

\[
= \partial_{l} \mathcal{J}^{+ij(l_1\cdots l_n)}
\]

where we have used equation (3) from the appendix to arrive at the second line.
For \((\mu, \nu) = (-, i)\), we have

\[
\sum_{j=1}^{n} \mathcal{J}^{-i_l j \ldots \bar{l}_n} = \frac{1}{6R} \sum_{j=1}^{n} \text{STr} \left( F^{ik} \dot{X}^k X^{l_i} \ldots X^{l_{j-1}} \dot{X}^j X^{l_{j+1}} \ldots X^{l_n} \right)
+ \frac{i}{2} \dot{X}^k X^k X^{l_i} \ldots X^{l_{j-1}} \dot{X}^j X^{l_{j+1}} \ldots X^{l_n} - \frac{i}{4} \dot{X}^k \dot{X}^k X^{l_i} \ldots X^{l_{j-1}} \dot{X}^j X^{l_{j+1}} \ldots X^{l_n}
+ i F^{ik} F^{kl} X^{l_i} \ldots X^{l_{j-1}} [X^i, X^j] X^{l_{j+1}} \ldots X^{l_n} \right)
= \frac{1}{6R} \text{STr} \left( F^{ik} \dot{X}^k \partial_l (X^{l_i} \ldots X^{l_n}) + \left( \frac{i}{2} [F^{ik}, X^l] X^{l_i} \ldots X^{l_n} \right)
+ \left( i[F^{ik}, X^l] F^{kl} + \frac{i}{2} [F^{ik}, X^l] F^{kl} \right) X^{l_i} \ldots X^{l_n} \right)
= - \text{STr} \left( \frac{i}{2} F^{ik} \dot{X}^k \partial_l (X^{l_i} \ldots X^{l_n}) + \left( \frac{i}{2} [F^{ik}, X^l] X^{l_i} \ldots X^{l_n} \right)
+ \left( i[F^{ik}, X^l] F^{kl} + \frac{i}{2} [F^{ik}, X^l] F^{kl} \right) X^{l_i} \ldots X^{l_n} \right)
= 0,
\]

since the inner expression in brackets, treated as a unit in the symmetrization, is symmetric in \(k\) and \(l\) while \(F^{kl}\) is antisymmetric. Hence,

\[
\sum_{j=1}^{n} \mathcal{J}^{-i_l j \ldots \bar{l}_n} = \frac{1}{6R} \text{STr} \left( F^{ik} \dot{X}^k \partial_l (X^{l_i} \ldots X^{l_n}) + F^{ik} \dot{X}^k X^{l_i} \ldots X^{l_n} + F^{ik} \dot{X}^k X^{l_i} \ldots X^{l_n} \right)
= \frac{1}{6R} \partial_l \text{STr} \left( F^{ik} \dot{X}^k X^{l_i} \ldots X^{l_n} \right)
= \partial_l \mathcal{J}^{+i_l \ldots \bar{l}_n},
\]
as desired. In \([9]\), it was noted that the terms

\[
\frac{1}{6R} \text{STr} \left( \frac{1}{2} F^{ij} F^{kl} \dot{F}^{ij} + F^{ik} F^{kl} F^{lj} \right).
\]
appeared in the matrix expression for \(\mathcal{J}^{-ij}\) but vanished for a classical membrane. Their physical interpretation was therefore unclear, however, we see here that they are necessary for current conservation at finite \(N\).

Thus, we have shown that the matrix theory quantities \(\mathcal{J}^{\mu \nu \lambda (l_1 \ldots l_n)}\) satisfy all relations expected for moments of a conserved membrane current. As with the stress energy tensor,
we may define a ten-dimensional tensor

\[ J^{\mu\nu\lambda}(x) = \int \frac{d^9 k}{(2\pi)^9} \left[ e^{-i k \cdot x_i} \sum_n \sum_{l_1..l_n} J^{\mu\nu\lambda(l_1..l_n)} (i)^n k_{l_1} \ldots k_{l_n} \right] \]

interpreted as the eleven-dimensional membrane current integrated over the compact circle. The matrix identities we have shown amount to a proof of the conservation of these ten-dimensional currents, \( \partial_\mu J^{\mu\nu\lambda}(x) = 0 \) where \( \mu \) runs over \( \{+, 1, 2, ..., 9\} \) and \( \nu, \lambda = 1, 2, ..., 9, +, - \). Again, these moments do not give any information about the charge distribution in the compact direction.

### 3.3 Fivebrane current

In the case of the fivebrane current, not all components had moments appearing in the potential considered in \([7]\), so the authors were only able to determine expressions for the moments of the static longitudinally wrapped fivebrane charge,

\[ \mathcal{M}^{+ijkl(l_1..l_n)} = \frac{1}{12R} \text{STr} \left( (F^{ij} F^{kl} + F^{ik} F^{lj} + F^{il} F^{jk}) X^{l_1} \ldots X^{l_n} \right) . \]

However, these moments appear together with moments of \( \mathcal{M}^{-ijklm} \) in the relations imposed by conservation of fivebrane current. Requiring these relations to be satisfied in Matrix theory we are led to define

\[ \mathcal{M}^{-ijklm(l_1..l_n)} = \frac{1}{12R} \text{STr} \left( \left( \hat{X}^i (F^{jk} F^{lm} + F^{jl} F^{mk} + F^{jm} F^{kl}) + \hat{X}^j (F^{kl} F^{mi} + F^{km} F^{il} + F^{ki} F^{lm}) + \hat{X}^k (F^{lm} F^{ij} + F^{lj} F^{im} + F^{il} F^{mj}) + \hat{X}^l (F^{mi} F^{jk} + F^{mj} F^{ki} + F^{mk} F^{ij}) + \hat{X}^m (F^{ij} F^{kl} + F^{ik} F^{lj} + F^{il} F^{jk}) \right) X^{l_1} \ldots X^{l_n} \right) \]

With this definition, it may be checked using equation (7) that the conservation relation

\[ \partial_t \mathcal{M}^{+ijkl(l_1..l_n)} = \mathcal{M}^{-ijklm(l_2..l_n)} + \ldots + \mathcal{M}^{-ijklm(l_1..l_{n-1})} \]

is satisfied. The remaining components of the totally antisymmetric membrane current, \( \mathcal{M}^{+ijklm} \) and \( \mathcal{M}^{ijklmn} \) are the charge and spatial current of the transverse fivebrane, poorly understood and perhaps vanishing in Matrix theory.

### 4 Discussion

The results described here provide further evidence for the connection between Matrix theory and eleven-dimensional supergravity. Starting with the Matrix theory quantities identified
in [7] as corresponding to moments of the stress-energy tensor and membrane current, we have demonstrated that all relations required by current conservation hold as exact matrix identities at finite N. For the longitudinal fivebrane current, only the static charge had been identified previously in [7]. By requiring consistency with the fivebrane current conservation equations, we have been led to a definition for moments of the spatial current associated with this charge (equation (4)).

Our results show that for a given Matrix theory background, we may associate corresponding ten-dimensional stress-energy, membrane, and longitudinal fivebrane currents (the eleven-dimensional currents averaged around the compact circle) which are conserved in the ten-dimensional sense. This current conservation, manifested in Matrix theory as the series of non-trivial identities we have proven, is essential to the validity of the correspondence with supergravity.

Though convincing evidence has been given that finite-N Matrix theory is a description of DLCQ M theory [8, 9], it is not clear that DLCQ M-theory has a low energy description as DLCQ supergravity (see however, [12]). Many properties of supergravity, including many aspects of the long range interactions between objects, seem to be reproduced correctly in finite N matrix theory, however others, such as the equivalence principle [7] seem only to be recovered in the large-N limit of matrix theory. Thus, it is particularly interesting that our results hold exactly at finite N.

It is important to note that the moments we have been dealing with contain no information about the distribution of matter or charge in the longitudinal direction. In supergravity, this information appears as non-zero fourier modes of the current around the compact circle ($m > 0$ in equation (2)). These moments do not contribute to the supergravity potential arising from the exchange of quanta with zero longitudinal momenta, so if corresponding Matrix theory quantities were to be deduced in an analogous way it would probably require a study of processes involving longitudinal momentum transfer. Actually, it is unclear to what extent a distribution of matter and charge in the compact direction is encoded in the finite-N Matrix theory variables. If DLCQ supergravity is not an appropriate description of DLCQ M-theory, then it may be incorrect to speak classically of a full eleven-dimensional current. In certain sectors of Matrix theory, for example matrices which describe membranes with no winding around the compact direction, a classical current distribution in the longitudinal direction (up to an undefined constant) certainly is encoded in Matrix theory, at least in the large N limit. However, there may be some ambiguity at finite N, in particular for states with winding around the compact circle.

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Appendix

In this appendix, we prove the following matrix identity, useful in demonstrating the Matrix theory relations implied by current conservation. Let $A_i$, $B_j$, and $C$ be $N \times N$ matrices with $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. Then

$$ \sum_{j=1}^{m} \text{Str} \left( A_1 \cdots A_n B_1 \cdots B_{j-1} [C, B_j] B_{j+1} \cdots B_m \right) = \sum_{i=1}^{n} \text{Str} \left( A_1 \cdots A_{i-1} [A_i, C] A_{i+1} \cdots A_n B_1 \cdots B_m \right) $$

(5)

where the commutator is to be treated as a unit in the symmetrization.

The proof is straightforward. Consider the set of terms on the left side of (5) with a particular ordering of the $A$’s and $B$’s. Noting the simple identity

$$ \sum_{i} B_{l_1} \cdots B_{l_{i-1}} [C, B_{l_i}] B_{l_{i+1}} \cdots B_{l_k} = CB_{l_1} \cdots B_{l_k} - B_{l_1} \cdots B_{l_k} C, $$

we see that upon expansion of the commutators, all terms with $C$ sandwiched between two $B$’s will cancel. The resulting expression is the sum of all possible insertions of $C$ between an $A$ and a $B$ in the sequence of $A$’s and $B$’s in the original order, where terms containing $A_i CB_j$ appear with a positive sign, and terms containing $B_j CA_i$ appear with a negative sign. Applying the same reasoning to the set of terms on the right side of (5) with this same ordering of $A$’s and $B$’s, we arrive at an identical expression. Since we have considered an arbitrary ordering of $A$’s and $B$’s, the proof of (5) is complete, as the symmetrized trace is just the average of all such orderings.

In deriving the Matrix theory current conservation relations, we will use the following special cases of this identity. First, taking $n=1$ above, we have

$$ \sum_{i=1}^{n} \text{Str} \left( AX^{l_1} \cdots X^{l_{i-1}} [C, X^{l_i}] X^{l_{i+1}} \cdots X^{l_n} \right) = \text{Str} \left( [A, C] X^{l_1} \cdots X^{l_n} \right) $$

(6)

Taking $n=2$ above, we have

$$ \sum_{i=1}^{n} \text{Str} \left( ABX^{l_1} \cdots X^{l_{i-1}} [C, X^{l_i}] X^{l_{i+1}} \cdots X^{l_n} \right) = \text{Str} \left( ([A, C] B + [B, C] A) X^{l_1} \cdots X^{l_n} \right) $$

(7)

As above, the commutator is to be treated as a unit in the symmetrization for these equations.

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