Strict Left (Right)-Conjunctive Left (Right) Semi-Uninorms and Implications Satisfying the Order Property

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Abstract: We firstly give out the formulas for calculating the upper and lower approximation strict left (right)-conjunctive left (right) semi-uninorms of a binary operation. Then, we lay bare the formulas for calculating the upper and lower approximation implications, which satisfy the order property, of a binary operation. Finally, we reveal the relationships between the upper approximation strict left (right)-conjunctive left (right) arbitrary ∨-distributive left (right) semi-uninorms and lower approximation right arbitrary ∧-distributive implications which satisfy the order property.

Keywords: Fuzzy Logic, Fuzzy Connective, Left (Right) Semi-Uninorm, Implication, Strict Left (Right)-Conjunctive

1. Introduction

In fuzzy logic systems (see [1-2]), connectives “and”, “or” and “not” are usually modeled by t-norms, t-conorms, and strong negations on [0, 1] (see [3]), respectively. Based on these logical operators on [0, 1], the three fundamental classes of fuzzy implications on [0, 1], i.e., R-, S-, and QL-implications on [0, 1], were defined and extensively studied. But, as was pointed out by Fodor and Keresztfalvi [4], sometimes there is no need of the commutativity or associativity for the connectives “and” and “or”. Thus, many authors investigated implications based on some other operators like weak t-norms [5], pseudo t-norms [6], pseudo-uninorms [7], left and right uninorms [8], semi-uninorms [9], aggregation operators [10] and so on.

Uninorms, introduced by Yager and Rybalov [11], and studied by Fodor et al. [12], are special aggregation operators that have proven useful in many fields like fuzzy logic, expert systems, neural networks, aggregation, and fuzzy system modeling. This kind of operation is an important generalization of both t-norms and t-conorms and a special combination of t-norms and t-conorms. But, there are real-life situations when truth functions cannot be associative or commutative. By throwing away the commutativity from the axioms of uninorms, Mas et al. introduced the concepts of left and right uninorms on [0, 1] in [13] and later in a finite chain in [14], and Wang and Fang [8, 15] studied the left and right uninorms on a complete lattice. By removing the associativity and commutativity from the axioms of uninorms, Liu [9] introduced the concept of semi-uninorms, and Su et al. [16] discussed the notions of left and right semi-uninorms, on a complete lattice. On the other hand, it is well known that a uninorm (semi-uninorm) U can be conjunctive or disjunctive whenever U(0, 1) = 0 or 1, respectively. This fact allows us to use uninorms in defining fuzzy implications.

Constructing fuzzy connectives is an interesting topic. Recently, Jenei and Montagna [17] introduced several new types of constructions of left-continuous t-norms, Wang [18] laid bare the formulas for calculating the smallest pseudo-t-norm that is stronger than a binary operation and the largest implication that is weaker than a binary operation, Su et al. [16] studied the constructions of left and right semi-uninorms on a complete lattice, Su and Wang [19] investigated the constructions of implications and coimplications on a complete lattice. and Wang et al. [20-22] studied the relations among implications, coimplications and left (right) semi-uninorms, on a complete lattice. Moreover, Wang et al. [23-24] investigated the constructions of conjunctive left (right) semi-uninorms, disjunctive left (right)
semi-uninorms, strict left (right)-disjunctive left (right) semi-uninorm, implications and complications satisfying the neutrality principle.

This paper is a continuation of [16, 19, 23-24]. Motivated by these works in [16, 19, 23-24], we will further focus on this issue and investigate constructions of the upper and lower approximation strict left (right)-conjunctive left (right) semi-uninorms and the upper and lower approximation implications which satisfy the order property.

This paper is organized as follows. In Section 2, we give out the formulas for calculating the upper and lower approximation strict left (right)-conjunctive left (right) semi-uninorms of a binary operation. In Section 3, we lay bare the formulas for calculating the upper and lower approximation implications, which satisfy the order property, of a binary operation. In Section 4, we reveal the relationships between the upper approximation strict left (right)-conjunctive left (right) arbitrary left semi-uninorms and lower approximation right arbitrary left distributive left distributive implications which satisfy the order property, and find out some conditions such that the lower approximation strict left (right)-conjunctive left (right) arbitrary left semi-uninorm of a binary operation and upper approximation implication, which satisfies the order property, of left (right) residuum of the binary operation satisfy the generalized modus ponens rule.

The knowledge about lattices required in this paper can be found in [25].

Throughout this paper, unless otherwise stated, L always represents any given complete lattice with maximal element 1 and minimal element 0; J stands for any index set.

2. Strict Conjunctive Left and Right Semi-Uninorms

In this section, we firstly recall some necessary concepts about the strict conjunctive left (right) semi-uninorms on a complete lattice.

Definition 2.1 (Su et al. [16]). A binary operation $U$ on $L$ is called left (right) distributive if both $U(0,0)=0$ and $U(1,0)=0$ since it satisfies the classical boundary conditions of AND.

$U$ is said to be strict left-conjunctive and strict right-conjunctive if $U$ is conjunctive and for any $x \in L$, $U(x,1)=0$ and $U(1,x)=0$, respectively.

Definition 2.2 (Wang and Fang [8]). A binary operation $U$ on $L$ is called left (right) arbitrary distributive if $U(\vee_{J} x, y)=\vee_{J} U(x, y)$ for all $x, y \in L$.

Example 2.1. Let $e_{L} \in L$.

$U^{se}_{cs}(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0, \\ y & \text{if } 0 < x \leq e_{L}, y \neq 0, \\ 1 & \text{otherwise}, \\ \land \{a \in L \mid a \neq 0\} & \text{if } 0 < x \not\geq e_{L}, y = 1, \\ 0 & \text{otherwise}. \end{cases}$

where $x$ and $y$ are elements of $L$. By Example 2 and Theorem 8 in [20], we see that $U^{se}_{cs}(L)$ and $U^{se}_{cs}(L)$ are two join-semilattices with the greatest element $U^{se}_{cs}$. When
Also \( e_L \neq 0 \) and \( \land \{a \in L \mid a \neq 0\} \neq 0 \), it is straightforward to verify that \( U_{\text{str}}^{e_L} \) is the smallest element of \( U_{\text{str}}^{e_L}(L) \).

Moreover, assume that \( \lor \{a \in L \mid a \not\geq e_L^{\prime}\} \not\geq e_L \). 

\( U_{\text{str}}^{e_L} \) is left arbitrary \( \lor \)-distributive and the smallest element of \( U_{\text{str}}^{e_L}(L) \).

**Example 2.2.** Let \( e_R \in L \),

\[
U_{\text{str}}^{e_R}(x, y) = \begin{cases} 
0 & \text{if } x = 0 \text{ or } y = 0, \\
1 & \text{otherwise}, 
\end{cases}
\]

where \( x \) and \( y \) are elements of \( L \). By Example 3 and Theorem 8 in [20], we see that \( U_{\text{str}}^{e_L}(L) \) and \( U_{\text{str}}^{e_R}(L) \) are two join-semilattices with the greatest element \( U_{\text{str}}^{e_L} \).

Similarly, When \( e_L \neq 0 \) and \( \land \{a \in L \mid a \neq 0\} \neq 0 \), \( U_{\text{str}}^{e_L} \) is the smallest element of \( U_{\text{str}}^{e_L}(L) \). Moreover, if \( \lor \{a \in L \mid a \not\geq e_R^{\prime}\} \not\geq e_R \), then \( U_{\text{str}}^{e_R} \) is the smallest element of \( U_{\text{str}}^{e_R}(L) \).

Constructing aggregation operators is an interesting work. Recently, Jenei and Montagna [17] introduced several new types of constructions of left-continuous \( t \)-norms, Su et al. [16] studied the constructions of left and right semi-\( \text{uni} \)norms on a complete lattice, and Wang et al. [23-24] investigated the constructions of conjunctive left (right) semi-\( \text{uni} \)norms and disjunctive left (right) semi-\( \text{uni} \)norms on a complete lattice. Now, we continue this work and give out the formulas for calculating the upper and lower approximation strict left (right)-conjunctive left (right) semi-\( \text{uni} \)norms of a binary operation.

It is easy to verify that \( \lor \{j \in J \mid U_j \in U_{\text{str}}^{e_L}(L)\} \) for any nonempty subset {\( U_j \mid j \in J\)} of \( U_{\text{str}}^{e_L}(L) \). If \( e_L \neq 0 \) and \( \land \{a \in L \mid a \neq 0\} \neq 0 \), then \( U_{\text{str}}^{e_L}(L) \) is a complete lattice with the smallest element \( U_{\text{str}}^{e_L} \) and greatest element \( U_{\text{str}}^{e_L} \) by Example 2.1. Thus, for a binary operation \( A \) on \( L \), if there exists \( U \in U_{\text{str}}^{e_L}(L) \) such that \( A \leq U \), then

\[
\land \{U \mid A \leq U, U \in U_{\text{str}}^{e_L}(L)\}
\]

is the smallest strict left-conjunctive left semi-\( \text{uni} \)norm that is stronger than \( A \) on \( L \), we call it the upper approximation strict left-conjunctive left semi-\( \text{uni} \)norm of \( A \) and write as \( [A]_{\text{str}}^{e_L} \); if there exists \( U \in U_{\text{str}}^{e_L}(L) \) such that \( U \leq A \), then

\[
\lor \{U \mid U \leq A, U \in U_{\text{str}}^{e_L}(L)\}
\]

is the largest strict left-conjunctive left semi-\( \text{uni} \)norm that is weaker than \( A \) on \( L \), we call it the lower approximation strict left-conjunctive left semi-\( \text{uni} \)norm of \( A \) and write as \( [A]_{\text{str}}^{e_L} \).

Similarly, we introduce the following symbols:

\( [A]_{\text{str}}^{e_L} \): the upper approximation strict right-conjunctive right semi-\( \text{uni} \)norm of \( A \);

\( [A]_{\text{str}}^{e_R} \): the lower approximation strict right-conjunctive right semi-\( \text{uni} \)norm of \( A \);

\( [A]_{\text{str}}^{e_R} \): the upper approximation strict left-conjunctive left arbitrary \( \lor \)-distributive left semi-\( \text{uni} \)norm of \( A \);

\( [A]_{\text{str}}^{e_L} \): the lower approximation strict left-conjunctive left arbitrary \( \lor \)-distributive left semi-\( \text{uni} \)norm of \( A \);

\( [A]_{\text{str}}^{e_R} \): the upper approximation strict right-conjunctive right arbitrary \( \lor \)-distributive right semi-\( \text{uni} \)norm of \( A \);

\( [A]_{\text{str}}^{e_L} \): the lower approximation strict right-conjunctive right arbitrary \( \lor \)-distributive right semi-\( \text{uni} \)norm of \( A \).

**Definition 2.3** (Su et al. [16]). Let \( A \) be a binary operation on \( L \). Define the upper approximation aggregator \( A_{ua} \) and the lower approximation aggregator \( A_{la} \) of \( A \) as follows:

\[
A_{ua}(x, y) = \lor\{A(u, v) \mid u \leq x, v \leq y\} \quad \forall x, y \in L,
\]

\[
A_{la}(x, y) = \land\{A(u, v) \mid u \leq x, v \leq y\} \quad \forall x, y \in L.
\]

**Theorem 2.1** (Su et al. [16]). Let \( A, B \in L_{\text{\huge L}} \). Then the following statements hold:

\[
A_{la} \leq A \leq A_{ua}.
\]

\[
(A \lor B)_{ua} = A_{ua} \lor B_{ua} \quad \text{and} \quad (A \land B)_{la} = A_{la} \land B_{la}.
\]

\( A_{ua} \) and \( A_{la} \) are non-decreasing in its each variable.

If \( A \) is non-decreasing in its each variable, then

\[
A_{ua} = A_{la} = A.
\]

**Theorem 2.2.** Let \( A \in L_{\text{\huge L}} \).

(1) If \( A \) is left (right) arbitrary \( \lor \)-distributive, then \( A_{ua} \) is left (right) arbitrary \( \lor \)-distributive.

(2) If \( A \) is left (right) arbitrary \( \land \)-distributive, then \( A_{la} \) is left (right) arbitrary \( \land \)-distributive.

Proof. We only prove that statement (1) holds. If \( A \) is left arbitrary \( \lor \)-distributive, then \( A \) is non-decreasing in its first variable,

\[
A_{ua}(x, y) = \lor\{A(u, v) \mid u \leq x, v \leq y\} = \lor\{A(x, v) \mid v \leq y\} \quad \forall x, y \in L,
\]
\[ A_{u}(\vee_{j \neq j} x_{j}, y) = \vee \{ A(\vee_{j \neq j} x_{j}, v) \mid v \leq y \} \]
\[ = \vee \{ \chi_{\vee_{j \neq j} A(x_{j}, v) \mid v \leq y} \} = \vee_{j \neq j} A_{u}(x_{j}, y) ; x_{j}, y \in L \ (j \in J), \]
i.e., \( A_{u} \) is left arbitrary \( \vee \)-distributive.

Similarly, we can show that \( A_{u} \) is right arbitrary \( \vee \)-distributive when \( A \) is right arbitrary \( \vee \)-distributive.

The theorem is proved.

Below, we give out the formulas for calculating the upper and lower approximation strict left (right)-conjunctive left (right) semi-uninorms of a binary operation.

**Theorem 2.3.** Suppose that \( A \in L^{\times L} \), \( e_{L} \neq 0 \) and \( \wedge \{ a \in L \mid a \neq 0 \} \neq 0 \).

1. If \( A \leq U_{\text{ext}} \), then \( [A]_{\text{ext}} = U_{\text{ext}} \vee A_{u} \); if \( U_{\text{ext}} \leq A \), then \( [A]_{\text{ext}} = U_{\text{ext}} \wedge A_{u} \).

2. If \( \forall a \in L \mid a \geq e_{L} \) and \( A \) is left arbitrary \( \vee \)-distributive, then

\[ [A]_{\text{ext}} = U_{\text{ext}} \vee A_{u} \]  \hspace{1cm} (12)

Moreover, if \( A \) is non-decreasing in its second variable, then \( [A]_{\text{ext}} = U_{\text{ext}} \vee A \).

Proof. Assume that \( e_{L} \neq 0 \) and \( \wedge \{ a \in L \mid a \neq 0 \} \neq 0 \). Then \( U_{\text{ext}} \) and \( U_{\text{ext}} \) are, respectively, the smallest and greatest elements of \( U_{\text{ext}} \) by Example 2.1.

1. Let \( U_{j} = U_{\text{ext}} \vee A_{u} \). If \( A \leq U_{j} \), then \( A_{u} \leq U_{\text{ext}} \).

\[ U_{j} \leq U_{\text{ext}} \]  \hspace{1cm} (13)

It implies that \( U_{j}(1, 0) = U_{j}(0, 1) = 0 \) and \( U_{j}(e_{L}, y) = y \) for any \( y \in L \). If \( U_{j}(x, 1) = 0 \), then \( U_{\text{ext}}(x, 1) = 0 \) and so \( x = 0 \), i.e., \( U_{j} \) is strict left-conjunctive. By Theorem 2.1(3) and the monotonicity of \( U_{\text{ext}} \), we can see that \( U_{j} \) is non-decreasing in its each variable. So, \( U_{j} \in U_{\text{ext}} \) if \( A \leq U \) and \( U_{j} \in U_{\text{ext}} \).

Therefore,

\[ [A]_{\text{ext}} = U_{\text{ext}} \vee A_{u} \]  \hspace{1cm} (14)

Let \( U_{2} = U_{\text{ext}} \wedge A_{u} \) if \( U_{\text{ext}} \leq A \), then

\[ U_{\text{ext}} \leq U_{\text{ext}} \wedge A_{u} \]  \hspace{1cm} (15)

Thus, \( U_{2}(1, 0) = U_{2}(0, 1) = 0 \) and \( U_{2}(e_{L}, y) = y \) for any \( y \in L \) and \( U_{2} \) is strict left-conjunctive. By Theorem 2.1(3) and the monotonicity of \( U_{\text{ext}} \), we know that \( U_{2} \) is non-decreasing in its each variable. So, \( U_{2} \in U_{\text{ext}} \).

**3. Implications Satisfying the Order Property**

Recently, Su and Wang [19] have studied the constructions of implications and coimplications and Wang et al. [23-24] further investigated the constructions of implications and coimplications satisfying the neutrality principle on a complete lattice. This section is a continuation of [19, 23-24]. We will study the constructions of the upper and lower approximation implications which satisfy the order property.

**Definition 3.1** (Baczynski and Jayaram [26], Bustince et al. [27], De Baets and Fodor [28], Fodor and Roubens [1]). An implication \( I \) on \( L \) is a hybrid monotonous (with decreasing first and increasing second partial mappings) binary operation that satisfies the corner conditions \( I(0, 0) = I(1, 1) = 1 \) and \( I(1, 0) = 0 \).

An implication \( I \) is said to satisfy the order property with respect to \( e \) (w.r.t. \( e \) for short) when \( x \leq y \) if and
only if $I(x, y) \geq e$ for any $x, y \in L$.

Note that for any implication $I$ on $L$, due to the monotonicity, the absorption principle holds, i.e.,

$I(0, x) = I(x, 0) = 1$ for any $x \in L$.

For the sake of convenience, we introduce the following symbols:

$I^{\text{up}}_L(L)$: the set of all implications which satisfy the order property w. r. t. $e$ on $L$;

$I^{\text{up}}_L(L)$: the set of all right arbitrary $\land$-distributive implications which satisfy the order property w. r. t. $e$ on $L$.

Clearly, $I^{\text{up}}_L(L)$ and $I^{\text{up}}_L(L)$ are all meet-semilattices.

**Definition 3.2.** Let $U$ be a binary operation on $L$. Define $I^L_U, I^R_U \in L^{\times L}$ as follows:

$$I^L_U(x, y) = \lor \{ z \in U \mid U(z, x) \leq y \} \quad \forall x, y \in L, \quad (19)$$

$$I^R_U(x, y) = \lor \{ z \in U \mid U(x, z) \leq y \} \quad \forall x, y \in L. \quad (20)$$

Here, $I^L_U$ and $I^R_U$ are, respectively, called the left and right residue of the binary operation $U$.

When $U$ is non-decreasing in each variable, it is easy to see that $I^L_U$ and $I^R_U$ are all decreasing in the first variable and increasing in the second one by Definition 3.2.

**Example 3.1.** For some left and right semi-uninorms in Examples 2.1-2.2, a simple computation shows that

$$I^L_{\land}(x, y) = \begin{cases} 0 & \text{if } x = 1 \text{ and } y = 0, \\ 1 & \text{if } x \leq y, \\ \lor \{ a \in L \mid a \not\geq e \} & \text{otherwise,} \end{cases}$$

$$I^R_{\land}(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ e & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$I^L_{\lor}(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ e & \text{if } 0 < x < y < 1, \\ \lor \{ a \in L \mid a \not\geq e \} & \text{otherwise,} \end{cases}$$

$$I^R_{\lor}(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ e & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $x$ and $y$ are elements of $L$. By the virtue of Theorem 8 in [20], we see that $I^L_{\land}$ is the smallest element of both $I^{\text{up}}_{\land}(L)$ and $I^{\text{up}}_{\land}(L)$.

When $e \neq 0$ and $\lor \{ a \in L \mid a \not\geq e \} \not\geq e$, it is easy to see that $I^L_{\land}$ is the greatest element of $I^{\text{up}}_{\land}(L)$.

Moreover, assume that $\land \{ a \in L \mid a \neq 0 \} \neq 0$. $I^L_{\land}$ is the greatest element of $I^{\text{up}}_{\land}(L)$.

Similar conclusions hold for $I^{\text{up}}_{\lor}(L)$ and $I^{\text{up}}_{\lor}(L)$.

It is easy to verify that if $f \neq \Phi$, then

$$I^L_{\lor} \in I^{\text{up}}_{\lor}(L) \quad \forall f \in J \quad \land_{\lor} I^L_{\lor} \in I^{\text{up}}_{\lor}(L). \quad (21)$$

When $e \neq 0$ and $\lor \{ a \in L \mid a \not\geq e \} \not\geq e$, we see that $I^{\text{up}}_{\lor}(L)$ is also a complete lattice with the smallest element $I^L_{\lor}$ and greatest element $I^R_{\lor}$ by Example 3.1.

Thus, for a binary operation $A$ on $L$, if there exists $I \in I^{\text{up}}_{\lor}(L)$ such that $A \leq I$, then

$$\land \{ I \mid A \leq I, I \in I^{\text{up}}_{\lor}(L) \} \quad (22)$$

is the largest implication that is stronger than $A$ and satisfies the order property w. r. t. $e$, on $L$. Here, we call it the upper approximation implication, which satisfies the order property w. r. t. $e$, of $A$ and write as $[A]^{\text{up}}$. Similarly, if there exists $I \in I^{\text{up}}_{\lor}(L)$ such that $I \leq A$, then

$$\lor \{ I \mid I \leq A, I \in I^{\text{up}}_{\lor}(L) \} \quad (23)$$

is the largest implication that is weaker than $A$ and satisfies the order property w. r. t. $e$, on $L$. Here, we call it the lower approximation implication, which satisfies the order property w. r. t. $e$, of $A$ and write as $[A]^{\text{down}}$.

Likewise, for a binary operation $A$ on $L$, we may introduce the following symbols:

$[A]^{\text{up}}$: the upper approximation implication, which satisfies the order property w. r. t. $e$, of $A$;

$[A]^{\text{down}}$: the lower approximation implication, which satisfies the order property w. r. t. $e$, of $A$;

$[A]^{\text{up}}$: the upper approximation right arbitrary $\land$-distributive implication, which satisfies the order property w. r. t. $e$, of $A$;

$[A]^{\text{down}}$: the lower approximation right arbitrary $\land$-distributive implication, which satisfies the order property w. r. t. $e$, of $A$.

**Definition 3.3** (see Su and Wang [19]). Let $A$ be a binary operation on $L$. Define the upper approximation implicator $A_u$ and the lower approximation implicator $A_l$ of $A$ as follows:

$$A_u(x, y) = \lor \{ A(u, v) \mid u \geq x, v \leq y \} \quad \forall x, y \in L, \quad (24)$$

$$A_l(x, y) = \land \{ A(u, v) \mid u \leq x, v \geq y \} \quad \forall x, y \in L. \quad (25)$$

**Theorem 3.1** (see Su and Wang [19]). Let $A, B \in L^{\times L}$. Then the following statements hold:

$$A_{li} \leq A \leq A_{ui}. \quad (26)$$

$$(A \lor B)_{ui} = A_{ui} \lor B_{ui} \quad \text{and}$$
(A \wedge B)_i = A_i \land B_i.

(A \land B)_i = A_i \land B_i.

A_{\text{ab}} and A_{\text{by}} are hybrid monotonous.

If A is hybrid monotonous, then A_{\text{ab}} = A_{\text{by}} = A.

Theorem 3.2. Let \( A \in L^{\text{LST}}. \)

(1) If A is right arbitrary \( \lor \)-distributive, then \( A_{\text{ab}} \) is also right arbitrary \( \lor \)-distributive,

\((I^R_\lambda)_i = I^R_{\lambda,ab}, (I^R_\lambda)_u = I^R_{\lambda,by}, \)

\(A_{\text{ab}}(x, (I^R_\lambda)_b(x, y)) \leq y \quad \forall x, y \in L. \) \hfill (28)

(2) If A is right arbitrary \( \land \)-distributive, then \( A_{\text{by}} \) is also right arbitrary \( \land \)-distributive.

(3) If A is left arbitrary \( \lor \)-distributive, then,

\((I^L_\lambda)_b = I^L_{\lambda,by}, (I^L_\lambda)_u \leq I^L_{\lambda,ab}, \)

\(A_{\text{by}}((I^L_\lambda)_b(x, y), x) \leq y \quad \forall x, y \in L. \)

Proof. We only prove that statement (1) holds.

Assume that A is a right arbitrary \( \lor \)-distributive binary operation on L. Clearly, \( A_{\text{ab}} \) is also right arbitrary \( \lor \)-distributive. By Definition 3.3, the monotonicity of A and \( I^R_\lambda \), and the right residual principle, we have that

\(I^R_\lambda(x, y) = \lor\{ z \in L \mid A_{\text{ab}}(x, z) \leq y \} \)

\(= \lor\{ z \in L \mid \lor\{ A(u, v) \mid u \leq x, v \leq z \} \leq y \} \)

\(= \lor\{ z \in L \mid \lor\{ A(u, z) \mid u \leq x \} \leq y \} \)

\(= \lor\{ z \in L \mid A(u, z) \leq y \quad \forall u \leq x \} \)

\(= \lor\{ z \in L \mid z \leq I^R_\lambda(u, y) \quad \forall u \leq x \} \)

\(= \lor\{ z \in L \mid z \leq \land_{\text{ab}} I^R_\lambda(u, y) \} \)

\(= \land_{\text{ab}} I^R_\lambda(u, y) \quad \forall x, y \in L. \)

Thus, \((I^R_\lambda)_b = I^R_{\lambda,by}. \) Similarly, we have that

\((I^L_\lambda)_b = I^L_{\lambda,by}, \)

\(A_{\text{by}}(x, z) = \land\{ A(u, v) \mid u \geq x, v \geq z \} \)

\(= \land\{ A(u, v) \mid u \geq x, v \geq z \} \quad \forall x, z \in L, \)

\((I^L_\lambda)_b(x, y) = \land\{ I^L_\lambda(u, y) \mid u \leq x \} \quad \forall x, y \in L. \)

If \( u \geq x, \) let \( z = I^R_\lambda(u, y), \) then

\[ A(u, z) = A(u, \lor\{ c \in L \mid A(u, c) \leq y \} \) \]

\[ = \lor\{ A(u, c) \mid A(u, c) \leq y \} \leq y, \]

\[ \land\{ A(u, z) \mid u_1 \geq x \} \leq A(u, z) \leq y. \]

So, \((I^R_\lambda)_b(x, y) \leq (I^R_\lambda)_b(x, y)\) for any \( x, y \in L, \) i.e.,

\((I^R_\lambda)_b \leq I^R_{\lambda,ab}. \) Moreover, we know that \( A_{\text{ab}} \) is right arbitrary \( \lor \)-distributive and hence

\[ A_{\text{ab}}(x, (I^R_\lambda)_b(x, y)) = A_{\text{ab}}(x, I^R_{\lambda,ab}(x, y)) \]

\[ = \land\{ A_{\text{ab}}(x, z) \mid A_{\text{ab}}(x, z) \leq y \} \leq y \quad \forall x, y \in L. \]

The theorem is proved.

Below, we give out the formulas for calculating the upper and lower approximation implications which satisfy the order property.

Theorem 3.3. Suppose that \( A \in L^{\text{LST}}, \) \( e_1 \neq 0 \) and \( \lor\{ a \in L \mid a \geq e_1 \} \neq \lor\{ a \in L \mid a \geq e_1 \}. \)

(1) If \( A \leq I^L_{\lambda,ab} \), then \( [A]^\text{ppw}_{\lambda} = I^L_{\lambda,ab} \lor A_i; \)

if \( A \geq I^L_{\lambda,ab} \), then \( [A]^\text{ppw}_{\lambda} = I^L_{\lambda,ab} \land A_i; \)

(2) If \( \land\{ a \in L \mid a \neq 0 \} \neq 0, \) \( A \geq I^L_{\lambda,by} \) and \( A \) is right arbitrary \( \land \)-distributive, then

\( [A]^\text{ppw}_{\lambda} = I^L_{\lambda,by} \lor A_i. \)

Moreover, if \( A \) is non-decreasing in its first variable, then \( [A]^\text{ppw}_{\lambda} = I^L_{\lambda,by} \land A_i. \)

Proof. Assume that \( \lor\{ a \in L \mid a \geq e_1 \} \neq \lor\{ a \in L \mid a \geq e_1 \} \).

(1) If \( A \leq I^L_{\lambda,ab} \), let \( I_i = I^L_{\lambda,ab} \lor A_i, \) then \( A \leq I_i \) and

\[ I^L_{\lambda,by} \leq I_i \leq I^L_{\lambda,ab}. \]

Thus, \( I_i(0, 0) = I_i(1, 1) = 1 \) and \( I_i(0, 1) = 0 \). If \( x \leq y \), then \( I_i(x, y) \geq I^L_{\lambda,ab}(x, y) \geq e_i; \) if \( I_i(x, y) \geq e_i \), then \( I^L_{\lambda,by}(x, y) \geq I_i(x, y) \geq e_i \) and so \( x \leq y \), i.e., \( I_i \) satisfies the order property w. r. t. \( e_i \). By Theorem 3.1(3) and the hybrid monotonicity of \( I^L_{\lambda,by} \), we know that \( I_i \) is hybrid monotonous. So, \( I_i \in I^\text{ppw}_{\lambda}(L). \) If \( A \leq I \) and \( A \in I^\text{ppw}_{\lambda}(L), \) then \( A_{\text{ab}} \leq I_{\text{ab}} = I \) and \( I_i = I^L_{\lambda,by} \lor A_i \leq I. \) Therefore,

\[ [A]^\text{ppw}_{\lambda} = I^L_{\lambda,by} \lor A_i. \]
If \( A \geq I_{\cup_{\text{csW}}} \), let \( I_2 = I_{\cap_{\text{csW}}} \wedge A_i \), then \( I_2 \leq A \),

\[
A_i \geq (I_{\cup_{\text{csW}}}^L)_L = I_{\cap_{\text{csW}}}^L I_{\cup_{\text{csW}}}^L \leq I_2 \leq I_{\cup_{\text{csW}}}^L .
\] (42)

Thus, we can prove in an analogous way that \( I_2 \in I_{\cup_{\text{csW}}}^R (L) \) and \( (A)^{\text{ppw}, \wedge}_{\cup_{\text{csW}}} = I_{\cap_{\text{csW}}}^L \wedge A_i \).

(2) When \( \wedge \{a \in L | a \neq 0 \} \neq 0 \), \( I^L_{\cup_{\text{csW}}} \) and \( I^L_{\cap_{\text{csW}}} \) are, respectively, the smallest and greatest elements of \( I_{\cup_{\text{csW}}}^R (L) \) by Example 3.1. Let \( I_2 = I_{\cap_{\text{csW}}}^L \wedge A_i \). If \( A \geq I_{\cup_{\text{csW}}}^L \), then \( I_2 \in I_{\cup_{\text{csW}}}^R (L) \) by statement (1). Noting that \( A_i \) is right arbitrary \( \wedge \)-distributive, we can see that \( A_i \) is also right arbitrary \( \wedge \)-distributive by Theorem 3.2(2). So \( I_2 \) is right arbitrary \( \wedge \)-distributive, i.e. \( I_2 \in I_{\cup_{\text{csW}}}^R (L) \). By the proof of statement (1), we know that \( (A)^{\text{ppw}, \wedge}_{\cup_{\text{csW}}} = I_{\cap_{\text{csW}}}^L \wedge A \).

Moreover, if \( A \) is non-decreasing in its first variable, then \( A_i = A \) by Theorem 3.1(4) and so

\[
(A)^{\text{ppw}, \wedge}_{\cup_{\text{csW}}} = I_{\cap_{\text{csW}}}^L \wedge A .
\] (43)

The theorem is proved.

Analogous to Theorem 3.3, we have the following theorem.

**Theorem 3.4.** Suppose that \( A \in L^{\text{L x L}} \), \( e_R \neq 0 \) and \( \lor \{a \in L | a \neq 0 \} \neq 0 \).

(1) If \( A \leq I_{\cup_{\text{csW}}}^L \), then \( (A)^{\text{ppw}, \lor}_{\cup_{\text{csW}}} = I_{\cup_{\text{csW}}}^L \lor A_i \);

(2) If \( A \geq I_{\cup_{\text{csW}}}^L \), then \( (A)^{\text{ppw}, \lor}_{\cup_{\text{csW}}} = I_{\cup_{\text{csW}}}^L \lor A_i \).

Moreover, if \( A \) is non-decreasing in its first variable, then

\[
(A)^{\text{ppw}, \lor}_{\cup_{\text{csW}}} = I_{\cup_{\text{csW}}}^L \lor A .
\] (44)

\[\begin{align*}
\text{Proof.} & \quad \text{We only prove the statement (1) holds.} \\
& \text{Assume that } A \leq U^R_{\text{csW}} \text{ and } A \text{ is left arbitrary } \lor \text{-distributive. Then it follows from Theorem 4.6 in [8] and} \\
& \text{Definition 3.2 that } I_{\cup_{\text{csW}}}^L \leq I \text{ and } I_{\downarrow}^L \text{ is right arbitrary } \wedge \text{-distributive. Thus, } (I_{\downarrow}^L)^{\text{ppw}, \wedge}_{\cup_{\text{csW}}} = I_{\cap_{\text{csW}}}^L \wedge (I_{\downarrow}^L)_R \text{ by Theorem 3.3(2). Moreover, it follows from Theorems 2.2(1) and 2.3(2)} \\
& \text{and the left residual principle that}
\end{align*}\]

\[
I_{\downarrow_{\text{csW}}}^L(x, y) = \lor \{z \in L | (A)^{\text{ppw}, \lor}_{\cup_{\text{csW}}} (z, x) \leq y \}
\]

\[
= \lor \{z \in L | U^R_{\text{csW}} \lor A (z, x) \leq y \}
\]

\[
= \lor \{z \in L | U^R_{\text{csW}} (z, x) \leq y, A (z, x) \leq y \}
\]

\[
= \lor \{z \in L | z \leq I_{\downarrow_{\text{csW}}}^L (x, y), z \leq I_{\downarrow_{\text{csW}}}^L (y, x) \}
\]

\[
= \lor \{z \in L | z \leq I^L_{\downarrow_{\text{csW}}} (x, y) \wedge I^L_{\downarrow_{\text{csW}}} (y, x) \}
\]

\[
= (I^L_{\downarrow_{\text{csW}}} \wedge I^L_{\downarrow_{\text{csW}}}) (x, y) \forall x, y \in L,
\]

i.e., \( I_{\downarrow_{\text{csW}}}^L = I_{\downarrow_{\text{csW}}}^L \wedge I^L_{\downarrow_{\text{csW}}} \). By Theorem 3.2(3), we know that \( (I^L_{\downarrow})_R = I^L_{\downarrow_{\text{csW}}} \). Therefore,

\[
(I^L_{\downarrow})^{\text{ppw}, \vee}_{\text{csW}} = I^L_{\downarrow_{\text{csW}}} \wedge (I^L_{\downarrow})_R = I^L_{\downarrow_{\text{csW}}} \wedge I^L_{\downarrow_{\text{csW}}} = I^L_{\downarrow_{\text{csW}}} .
\] (48)

The theorem is proved.

Finally, we give out some conditions such that the lower approximation strict left (right)-conjunctive left (right) semi-uninorm of a binary operation and upper approximation implication, which satisfies the order property, of left (right) residuum of the binary operation satisfy the GMP rule.

**Theorem 4.2.** Suppose that \( A \in L^{\text{L x L}} \), \( e_L, e_R \neq 0 \) and \( \lor \{a \in L | a \neq 0 \} \neq 0 \).

(1) If \( \lor \{a \in L | a \neq 0 \} \neq 0 \), \( U^R_{\text{csW}} \leq A \) and \( A \) is non-decreasing in its second variable and left arbitrary \( \lor \)-distributive and \( I_{\downarrow_{\text{csW}}}^L \) and \( e_R \) are comparable (see [25]) when \( 0 < x \leq y < 1 \), then \( (A)^{\text{ppw}, \lor}_{\text{csW}} \) and \( (I_{\downarrow_{\text{csW}}}^L)^{\text{ppw}, \lor}_{\text{csW}} \) satisfy the GMP rule in the form

\[
(A)^{\text{ppw}, \lor}_{\text{csW}} (I_{\downarrow_{\text{csW}}}^L \lor (x, y), x) \leq y \forall x, y \in L .
\] (49)

(2) If \( \lor \{a \in L | a \neq 0 \} \neq 0 \), \( U^R_{\text{csW}} \leq A \) and \( A \) is non-decreasing in its first variable and right arbitrary \( \lor \)-distributive and \( I_{\downarrow_{\text{csW}}}^L \) and \( e_L \) are comparable (see [25]) when \( 0 < x \leq y < 1 \), then \( (A)^{\text{ppw}, \lor}_{\text{csW}} \) and \( (I_{\downarrow_{\text{csW}}}^L)^{\text{ppw}, \lor}_{\text{csW}} \) satisfy the GMP rule in the form

\[
\begin{align*}
\text{4. The Relations Between Strict} \\
\text{(Right)-Conjunctive Left (Right) Semi-Uninorms and Implications}
\end{align*}\]
investigated the constructions of implications and coimplications on a complete lattice. In this paper, motivated by these works, we give out the formulas for calculating the upper and lower approximation strict left (right)-conjunctive left (right) semi-uninorms of a binary operation; lay bare the formulas for calculating the upper and lower approximation implications, which satisfy the order property, of a binary operation; reveal the relationships between the upper approximation strict left (right)-conjunctive left (right) arbitrary ∨ -distributive left (right) semi-uninorms and lower approximation right arbitrary ∧ -distributive implications which satisfy the order property.

In a forthcoming paper, we will further investigate the constructions of left (right) semi-uninorms and coimplications on a complete lattice.

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