Topological Pressure
for Locally Compact Metrizable Systems

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Abstract

It is widely known that when $X$ is compact Hausdorff, and when $T: X \to X$ and $f: X \to \mathbb{R}$ are continuous,

$$P(T, f) = \sup_{\mu: \text{Radon probability}} \left( h_{\mu}(T) + \int f \, d\mu \right),$$

where $P(T, f)$ is the topological pressure and $h_{\mu}(T)$ is the measure theoretic entropy of $T$ with respect to $\mu$. This result is known as variational principle.

We generalize the concept of topological pressure for the case where $X$ is a separable locally compact metric space. Our definitions are quite similar to those used in the compact case. Our main result is the validity of the variational principle (Theorem 3.1).

1. Introduction

Traditionally, as happens with topological entropy, topological pressure has been applied for dynamical systems defined over compact spaces. As a special case, for compact metric spaces. In this paper we generalize the concept of topological pressure for locally compact subsystems of compact metric ones (Definition 2.30). Another way to state this condition is to say that the system is defined over a locally compact separable metric space. Or yet, that the refered space possesses a one-point metrizable compactification. Our definition is quite similar to the compact case, and our main result is the validity of the variational principle (Theorem 3.1). A consequence of having a one-point metrizable compactification is the fact that every Borel measure is in fact a Radon measure (see Chapter II, Theorem 3.2 from [Par67]).

We try to follow the same approach used for the compact case, presented in [VO16]. When $X$ is a locally compact separable metrizable space, it possesses a one-point compactification $X^*$ with a metric $d$. It does not mean that a system $T: X \to X$ can be continuously extended to a system $T^*: X^* \to X^*$. However, it can be extended to $S: Z \to Z$, where $Z \supset X$ is compact with
a metric $r$, and $X$ has the topology induced from $Z$ (Lemma 2.3). In this context, we have:

1. The spaces $X$, $X^*$ and $Z$.
2. The dynamical systems $T$ and $S$.
3. The metrics $d$, $d$ restricted to $X$, $r$ and $r$ restricted to $X$.
4. A weak* sequentially compact space of Borel probabilities over $Z$, and its restriction to $X$, giving a set of measures $\mu$ such that $0 \leq \mu(X) \leq 1$.
5. The spaces of continuous functions over $X$, $X^*$ and $Z$.

Instead of studying directly the system $T$, we shall look at $S : Z \to Z$. However, since we do not want to capture any complexity for $S$ over $Z \setminus X$, we shall not make use of the metric $r$. When defining topological pressure, instead of $r$, we make use of $d$ restricted to $X$. Also, for the measure theoretic pressure, Lemma 2.8 allows us to make calculations avoiding the complexity of $S$ outside $X$.

2. Preliminaries

This section is devoted to recalling some elementary definitions related to different types of pressure, and to proving some fundamental facts which are used in the sequel. We also extend the concept of topological pressure, originally defined only for compact systems.

A topological dynamical system — or simply a dynamical system — $T : X \to X$ is a continuous map $T$ defined over a topological space $X$. A measurable dynamical system $T : X \to X$ is a measurable map $T$ defined over a measurable space $X$. If we embed $X$ with the Borel $\sigma$-algebra, a topological dynamical system becomes also a measurable dynamical system.

Recall that a family $\mathcal{A}$ of subsets of $X$ is a cover of $X$ when

$$X = \bigcup_{A \in \mathcal{A}} A.$$ 

If the sets in $\mathcal{A}$ are disjoint, then we say that $\mathcal{A}$ is a partition of $X$. A subcover of $\mathcal{A}$ is a family $\mathcal{B} \subset \mathcal{A}$ which is itself a cover of $X$. If $\mathcal{A}$ is a cover of $X$ and $Y \subset X$, then we denote by $Y \cap \mathcal{A}$ the cover of $Y$ given by

$$Y \cap \mathcal{A} = \{ A \cap Y \mid A \in \mathcal{A} \}.$$ 

Given two covers $\mathcal{A}$ and $\mathcal{B}$ of an arbitrary set $X$, we say that $\mathcal{A}$ is finer than $\mathcal{B}$ or that $\mathcal{A}$ refines $\mathcal{B}$ — and write $\mathcal{B} \prec \mathcal{A}$ — when every element of $\mathcal{A}$ is a subset of some element of $\mathcal{B}$. We also say that $\mathcal{B}$ is coarser than $\mathcal{A}$. The relation $\prec$ is a preorder, and if we identify the symmetric covers (i.e.: covers $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \prec \mathcal{A}$), we have a lattice. As usual, $\mathcal{A} \lor \mathcal{B}$ denotes the representative of the coarsest covers of $X$ that refines both $\mathcal{A}$ and...
Given a dynamical system \( T : X \to X \) and a cover \( \mathcal{A} \), for each \( n \in \mathbb{N} \) we define

\[
\mathcal{A}^n = \mathcal{A} \cup T^{-1}(\mathcal{A}) \cup \cdots \cup T^{-(n-1)}(\mathcal{A})
\]

If we want to emphasise the dynamical system \( T \), we write \( \mathcal{A}^n_T \) instead. And if \( f : X \to \mathbb{R} \) is any function, we shall denote by

\[
f_n = \sum_{j=0}^{n-1} f \circ T^j.
\]

And if we really need to emphasise \( T \), we write \( f_{T,n} \) instead.

For any pseudometric space \((X, d)\), given \( \varepsilon > 0 \) and \( x \in X \), we denote by

\[
B(\varepsilon; d)x = \{ y \in X \mid d(x, y) < \varepsilon \}
\]

the open ball of radius \( \varepsilon \), centered at \( x \). And

\[
\mathcal{B}_d(\varepsilon) = \{ B(\varepsilon; d)x \mid x \in X \}
\]

is the cover of \( X \) composed of all open balls with radius \( \varepsilon \).

2.1. Compactification. We are mainly interested in dynamical systems defined over a metrizable locally compact separable space \( X \). This is the same as requiring \( X \) to have a metrizable one-point compactification. And in general, this is not the same as requiring that the system \( T \) can itself be continuously extended to the one-point compactification of \( X \). The topology of \( X \) can be induced by different metrics. In special, it can be induced by a metric restricted from its one-point compactification.

To demonstrate our extended version of the variational principle for pressure, we shall regard the dynamical system \( T : X \to X \) as a subsystem of a compact metrizable one. Please, refer to [CP15] for a detailed treatment of the results stated in this subsection.

**Definition 2.1 (Subsystem).** We say that a dynamical system \( T : X \to X \) is a subsystem of \( S : Z \to Z \) when \( X \subset Z \) has the induced topology and \( T(x) = S(x) \) for every \( x \in X \). We also say that \( S \) extends \( T \) to \( Z \).

**Lemma 2.2.** Suppose \( T : X \to X \) is a subsystem of \( S : Z \to Z \). If \( \mathcal{Z} \) is a covering of \( Z \) and \( \mathcal{C} = X \cap \mathcal{Z} \), then

\[
\mathcal{C}^n_T = X \cap \mathcal{Z}^n_S.
\]

**Proof.** This is Lemma 2.2 from [CP15].
Under the conditions of local compacity, separability and metrizability, the dynamical system \( T : X \to X \) can always be extended to a metrizable compact system.

**Lemma 2.3.** Suppose that \( X \) is a topological space with metrizable one-point compactification \( X^* = X \cup \{ \infty \} \). Then, any topological dynamical system \( T : X \to X \) can be extended to a dynamical system \( S : Z \to Z \), with \( Z \) compact metrizable, and such that the natural projection

\[
\pi : \ Z \to X^* \\
x \mapsto \pi(x) = \begin{cases} x, & x \in X \\ \infty, & x \notin X \end{cases}
\]

is continuous.

*Proof.* This is Lemma 2.3 from [CP15]. \( \Box \)

This projection \( \pi \), on Lemma 2.3, induces the pseudometric

\[
\tilde{d}(x, y) = d(\pi(x), \pi(y))
\]

over \( Z \). We denote by the same letter \( d \) the metric over \( X^* \) and its restriction to \( X \). Since \( \pi \) is continuous, this pseudometric \( \tilde{d} \) is such that the “open balls” are in fact open in the topology of \( Z \), although in general, \( \tilde{d} \) does not generate the topology. Denote by

\[
X^c = Z \setminus X
\]

the complement of \( X \) in \( Z \). Since the set \( X^c \) has zero diameter with respect to \( \tilde{d} \), the balls \( B(\varepsilon; \tilde{d}) \) either contain \( X^c \), or have empty intersection with it.

**Definition 2.4 (One-Point Metric).** Whenever \( X \) has a metrizable one-point compactification \( X^* \), we shall call the restriction of a metric \( d \) over \( X^* \) to \( X \) a one-point metric.

In particular, if we say that \( X \) has a one-point metric \( d \), this implies that \( X \) has a one-point compactification.

Under the conditions of Lemma 2.3, \( X \) is an open subset of \( Z \). In fact, \( X^c = \pi^{-1}(\infty) \) is closed. In this case, the Borel sets of \( X \) are Borel sets of \( Z \), and we may restrict Borel measures over \( Z \) to the Borel sets of \( X \) and produce a Borel measure over \( X \). On the other hand, if \( \mu \) is a Borel measure over \( X \), we can extend it to \( Z \) by declaring \( \mu(X^c) = 0 \). We shall use the same letter \( \mu \) to denote a measure over \( Z \) as well as its restriction to \( X \). If we want to make the distinction clear, we may write \( \mu|_X \) instead.

From now on, unless explicit mention to the contrary, \( T : X \to X \) will be a topological dynamical system, where \( X \) admits a one-point metrizable compactification \( X^* = X \cup \{ \infty \} \). Also, \( S : Z \to Z \) will be a continuous
extension of $T$ and $\pi : Z \to X^*$ the natural projection whose properties and existence are assured by Lemma 2.3. Also, $d$ will be a one-point metric and $\tilde{d}(x, y) = d(\pi(x), \pi(y))$ will be the pseudometric induced in $Z$ by $d$.

**Definition 2.5** (One-Point Uniformly Continuous). If $f : X \to \mathbb{R}$ is uniformly continuous with respect to some one-point metric (and therefore, every one-point metric), we shall say that $f$ is one-point uniformly continuous.

A one-point uniformly continuous $f : X \to \mathbb{R}$ is nothing more then the restriction to $X$ of a continuous $f : X^* \to \mathbb{R}$, which we shall denote by the same letter $f$. We can always write a one-point uniformly continuous function as a sum $f + c$, where $f$ vanishes at infinity (i.e.: $f \in C_0(X)$) and $c \in \mathbb{R}$ is a constant.

The one-point uniformly continuous $f$ induces the continuous $g = f \circ \pi$, from $Z$ to $\mathbb{R}$. Notice that since we have the dynamical systems $T : X \to X$ and $S : Z \to Z$, and the functions $f_n : X \to \mathbb{R}$ and $g_n : Z \to \mathbb{R}$ are defined. That is,

$$f_n = f + f \circ T + \cdots + f \circ T^{n-1}$$

$$g_n = g + g \circ S + \cdots + f \circ S^{n-1}.$$  

However, $f_n : X^* \to \mathbb{R}$ is, in principle, not defined.

2.2. Pressure with a Measure. Now, we shall define the concept of pressure of a measurable dynamical system with respect to an invariant finite measure. Traditionally, pressure has been defined only for probability measures. However, as shown in [CP15], extending this to finite measures is straightforward, and can be quite useful in topological dynamical systems where $X$ is not compact. First, we recall some definitions. More details can be found in [CP15].

**Definition 2.6** (Kolmogorov-Sinai Entropy). Consider the finite measure space $(X, \mathcal{B}, \mu)$ and a finite measurable partition $\mathcal{C}$. The partition entropy of $\mathcal{C}$ is

$$H_\mu(\mathcal{C}) = \sum_{C \in \mathcal{C}} \mu(C) \log \frac{1}{\mu(C)}.$$  

For the measurable dynamical system $T : X \to X$, if $\mu$ is a $T$-invariant finite measure, the partition entropy of $T$ with respect to $\mathcal{C}$ is

$$h_\mu(T, \mathcal{C}) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{C}^n),$$  

and the Kolmogorov-Sinai entropy of $T$ is

$$h_\mu(T) = \sup_{\mathcal{C} : \text{finite measurable partition}} h_\mu(T, \mathcal{C}).$$
Definition 2.7 (Pressure). Let $T : X \to X$ be a measurable dynamical system with a finite $T$-invariant measure $\mu$. Let $f : X \to \mathbb{R}$ be an integrable function. The quantity

$$P_\mu(T, f) = h_\mu(T) + \int f \, d\mu$$

is the pressure of $T$ with respect to the measure $\mu$ and potential $f$. To make the notation cleaner, also define

$$P_\mu(T, f, \mathcal{E}) = h_\mu(T, \mathcal{E}) + \int f \, d\mu.$$

2.2.1. Properties. In our way into showing the variational principle, we shall make use of some techniques that are already quite standard, and some that are not. The following fact is not a standard result. It was presented in [CP15], and relates the entropy of a system and the entropy of its extension.

Lemma 2.8. Let $S : Z \to Z$ be a measurable dynamical system and $T : X \to X$ a subsystem with $X \subset Z$ measurable. If $\mu$ is an $S$-invariant measure, and if

$$\mathcal{Z} = \{Z_0, \ldots, Z_k\}$$

is a measurable partition of $Z$ such that $X^c \subset Z_0$, then $\mu$ is $T$-invariant and

$$h_\mu(S, \mathcal{Z}) \leq h_\mu(T).$$

If $f \in C_0(X)$, then

$$P_\mu(S, f \circ \pi, \mathcal{Z}) \leq P_\mu(T, f).$$

Proof. The first part is Lemma 2.10 from [CP15]. The second part follows from the fact that

$$\int f \circ \pi \, d\mu = \int f \, d\mu|_X + \mu(X^c) f(\infty) = \int f \, d\mu|_X.$$

Since we do not require the $T$-invariant measure to be a probability, the following lemma can be quite handy.

Lemma 2.9. Given a measurable dynamical system $T : X \to X$ and a finite $T$-invariant measure $\mu$, then, for $\alpha \geq 0$,

$$h_{\alpha \mu}(T) = \alpha h_\mu(T).$$

Proof. This is Lemma 2.9 from [CP15].

Now, we list some properties of the pressure, most of them are just a consequence of some corresponding property of the Kolmogorov-Sinai entropy.
Proposition 2.10. Let $T : X \to X$ be a measurable dynamical system, $\mu$ a $T$-invariant finite measure, and $f : X \to \mathbb{R}$ an integrable function. Then,
\[ P_\mu(T^k, f) = kP_\mu(T, f). \]

Proof. Notice that since $\mu$ is $T$-invariant,
\[ \int f_k \, d\mu = k \int f \, d\mu. \]

Also, the equality
\[ h_\mu(T^k) = kh_\mu(T) \]

is usually stated for the case where $\mu$ is a probability measure (for example, Proposition 9.1.14 from [VO16]). For the general case, where $\mu$ is a finite measure, combine this with Lemma 2.9. Of course, the original demonstration for probability measure works verbatim for the more general finite measure case (see Remark 2.20 in [CP15]).

When defining topological pressure we shall use the concept of admissible cover (Definition 2.21). A measure theoretic counterpart is the admissible partition.

Definition 2.11 (Admissible Partition). In a topological space $X$, a finite (measurable) partition is said to be admissible when every element but one is compact.

The pressure can be calculated using admissible partitions. This fact is usually not explicitly stated as we did in Proposition 2.12. But it is not new, as it is usually embedded in the demonstrations of the variational principle for the compact case (see, for example, [VO16, Wal00, CP15]).

Proposition 2.12. If $T : X \to X$ is a topological dynamical system, $\mu$ is a $T$-invariant Radon probability measure, and $f : X \to \mathbb{R}$ is an integrable function. Then,
\[ P_\mu(T, f) = \sup_{K \text{ admissible partition}} P_\mu(T, f, K). \]

For the proof of Proposition 2.12, we need the concept of conditional entropy. The proof will be presented after some preparation.

Definition 2.13 (Conditional Entropy). Given a probability measure $\mu$ and two finite measurable partitions $\mathcal{E}$ and $\mathcal{D}$, the conditional entropy is defined as the expected value
\[ H_\mu(\mathcal{E} \mid \mathcal{D}) = \sum_{D \in \mathcal{D}} \mu(D) H_\mu(\cdot \mid D)(\mathcal{E}). \]

Conditional entropy possesses the following properties.
Lemma 2.14. Let $T: X \to X$ be a measurable dynamical system with $T$-invariant probability measure $\mu$. If $\mathcal{C}$ and $\mathcal{D}$ are two measurable finite partitions, then

$$h_\mu (T, \mathcal{C}) \leq h_\mu (T, \mathcal{D}) + H_\mu (\mathcal{C} | \mathcal{D}).$$

Proof. This is item (iv) of Theorem 4.12 from [Wal00]. Or Lemma 9.1.11 from [VO16].

We shall need to calculate the conditional entropy only for the following case.

Lemma 2.15. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a measurable partition. If $\mathcal{K} = \{K_0, K_1, \ldots, K_n\}$ is such that $K_j \subset C_j$ for every $j = 1, \ldots, n$, then

$$H_\mu (\mathcal{C} | \mathcal{K}) = \mu(K_0) H_\mu (\cdot | K_0) (\mathcal{C}) \leq \mu(K_0) \log n.$$

Proof. One just has to notice that for every $C \in \mathcal{C}$ and $j \neq 0$, $\mu(C | K_j)$ is either 0 or 1. Therefore, for $j = 1, \ldots, n$,

$$H_\mu (\cdot | K_j) (\mathcal{C}) = 0.$$

It is a very well known fact that $H_\mu (\mathcal{C}) \leq \log n$ (see, for example, Lemma 9.1.3 from [VO16]).

We are now, ready to demonstrate Proposition 2.12.

Proof (Proposition 2.12). If $\mathcal{K}$ is an admissible partition, it is finite by definition, and measurable because compact sets are measurable. From the definition of $h_\mu (T)$, it is evident that

$$\sup_{\mathcal{K}: \text{admissible partition}} h_\mu (\mathcal{K}, T) \leq \sup_{\mathcal{C}: \text{finite measurable partition}} h_\mu (\mathcal{C}, T) = h_\mu (T).$$

To finish the demonstration, we just have to find for any $\varepsilon > 0$ and any measurable finite partition $\mathcal{C} = \{C_1, \ldots, C_n\}$, an admissible partition $\mathcal{K}$ such that

$$h_\mu (\mathcal{C}, T) \leq h_\mu (\mathcal{K}, T) + \varepsilon.$$

To that end, let’s choose the partition $\mathcal{K} = \{K_0, \ldots, K_n\}$, where $K_j \subset C_j$ for $j = 1, \ldots, n$, and $\mu(K_0) \leq \frac{\varepsilon}{n \log n}$. For example, since $\mu$ is Radon, just choose a compact $K_j \subset C_j$ for each $j = 1, \ldots, n$, such that

$$\mu(C_j \setminus K_j) \leq \frac{\varepsilon}{n \log n}.$$
Since $K_0 = (K_1 \cup \cdots \cup K_n)^c$,
\[
\mu(K_0) = \sum_{j=1}^n \mu(C_j \setminus K_j) \leq \frac{\varepsilon}{\log n}.
\]
Now, using Lemmas 2.14 and 2.15,
\[
h_\mu(\mathcal{C}, T) \leq h_\mu(\mathcal{K}, T) + \frac{\varepsilon}{\log n} \log n
= h_\mu(\mathcal{K}, T) + \varepsilon.
\]

Next, we present an upper bound for calculating the pressure that also motivates the definition of topological pressure. First, notice that
\[
P_\mu(T, f) = \sup_{\mathcal{C}} \lim_{n \to \infty} \left( \int f \, d\mu + \frac{1}{n} H_\mu(\mathcal{C}^n) \right),
\]
where the supremum is taken over every measurable finite partition $\mathcal{C}$.

**Lemma 2.16.** Let $T : X \to X$ be a measurable dynamical system, $\mu$ a $T$-invariant probability measure, and $f : X \to \mathbb{R}$ an integrable function. Then, for every finite measurable partition $\mathcal{C}$,
\[
\int f \, d\mu + \frac{1}{n} H_\mu(\mathcal{C}^n) \leq \frac{1}{n} \log \sum_{C \in \mathcal{C}^n} \mu(C) \log \frac{1}{\mu(C)}.
\]

**Proof.** Notice that, from the $T$-invariance of $\mu$,
\[
\int f \, d\mu = \frac{1}{n} \int f_n \, d\mu.
\]
Therefore,
\[
\int f \, d\mu + \frac{1}{n} H_\mu(\mathcal{C}^n) = \frac{1}{n} \left( \int f_n \, d\mu + \sum_{C \in \mathcal{C}^n} \mu(C) \log \frac{1}{\mu(C)} \right)
\leq \frac{1}{n} \sum_{C \in \mathcal{C}^n} \left( \mu(C) \sup f_n(C) + \mu(C) \log \frac{1}{\mu(C)} \right).
\]
Now, the result follows from Lemma 10.4.4 from [VO16].

**2.3. Topological Pressure.** As it happens with topological entropy in the compact case, there are different equivalent ways to define topological pressure. For non-compact systems, those different definitions might not be equivalent. In the same spirit of that from [CP15], we shall adapt some of those definitions so they work in the non-compact case as well. As for the notation, we try to follow as closely as possible that of [VO16]. As in [CP15], we use admissible covers and covers of balls in order to define the different concepts of topological pressure.
Definition 2.17. Given \( f : X \to \mathbb{R} \) and a cover \( \mathcal{A} \) of a set \( X \), define

\[
Q_n(T, f, \mathcal{A}) = \inf \left\{ \sum_{A \in \mathcal{A}'} \inf_{f(A)} \mid \mathcal{A}' \text{ is a subcover of } \mathcal{A}^n \right\}
\]

\[
P_n(T, f, \mathcal{A}) = \inf \left\{ \sum_{A \in \mathcal{A}'} \sup_{f(A)} \mid \mathcal{A}' \text{ is a subcover of } \mathcal{A}^n \right\}.
\]

The role played by \( Q_n(T, f, \mathcal{A}) \) and \( P_n(T, f, \mathcal{A}) \) in Definition 2.17 is analogous to that of \( N(\mathcal{A}^n) \) when we define topological entropy. In fact,

\[
Q_n(T, 0, \mathcal{A}) = P_n(T, 0, \mathcal{A}) = N(\mathcal{A}^n).
\]

The following Lemma shows that \( Q_n(T, f, \mathcal{A}) \) has a property very similar to that of \( N(\mathcal{A}^n) \).

Lemma 2.18. If \( \mathcal{A} \prec \mathcal{B} \), then, for any \( f : X \to \mathbb{R} \), and any \( n = 1, 2, \ldots \),

\[
Q_n(T, f, \mathcal{A}) \leq Q_n(T, f, \mathcal{B}).
\]

Proof. Notice that \( \mathcal{A} \prec \mathcal{B} \) implies \( \mathcal{A}^n \prec \mathcal{B}^n \).

For every \( B \in \mathcal{B}^n \), there is an \( A_B \in \mathcal{A}^n \) such that \( B \subset A_B \). In this case,

\[
\inf_{f(A_B)} \leq \inf_{f(B)}.
\]

Notice that for every subcover \( \mathcal{B}' \) of \( \mathcal{B}^n \),

\[
\mathcal{A}' = \left\{ A_B \in \mathcal{A}^n \mid B \in \mathcal{B}' \right\}
\]

is a subcover of \( \mathcal{A}^n \). Therefore,

\[
Q_n(T, f, \mathcal{A}) \leq \sum_{A \in \mathcal{A}'} \inf_{f(A)} \leq \sum_{B \in \mathcal{B}'} \inf_{f(A_B)} \leq \sum_{B \in \mathcal{B}'} \inf_{f(B)}.
\]

The result follows if we take the infimum over every subcover \( \mathcal{B}' \) of \( \mathcal{B}^n \). \( \Box \)

While \( Q_n(T, f, \mathcal{A}) \) has the property stated in Lemma 2.18, the sequence \( P_n(T, f, \mathcal{A}) \) shares a different property with \( N(\mathcal{A}^n) \): it is submultiplicative. That is,

\[
P_{m+n}(T, f, \mathcal{A}) \leq P_m(T, f, \mathcal{A}) P_n(T, f, \mathcal{A}).
\]

And therefore,

\[
\lim_{n \to \infty} \frac{1}{n} \log P_n(T, f, \mathcal{A})
\]

exists.
Lemma 2.19. For any cover $\mathcal{A}$ of a set $X$,
\[
\lim_{n \to \infty} \frac{1}{n} \log P_n(T, f, \mathcal{A})
\]
eexists.

Proof. This is Lemma 9.3 from [Wal00]. But it is important to notice that although Walters assumes $X$ to be compact, Lemma 9.3 does not depend on this hypothesis. It is also worth noticing that the demonstration also does not depend on the fact that $\mathcal{A}$ is an open cover, or that $f : X \to \mathbb{R}$ is continuous, and these hypothesis could be removed from the statement of Lemma 9.3. □

Definition 2.20. Given $f : X \to \mathbb{R}$ and a cover $\mathcal{A}$ of a set $X$, define
\[
Q^{-}(T, f, \mathcal{A}) = \liminf_{n \to \infty} \frac{1}{n} \log Q_n(T, f, \mathcal{A})
\]
\[
Q^{+}(T, f, \mathcal{A}) = \limsup_{n \to \infty} \frac{1}{n} \log Q_n(T, f, \mathcal{A})
\]
\[
P(T, f, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \log P_n(T, f, \mathcal{A}).
\]

As in [CP15], we shall restrict our attention to admissible covers.

Definition 2.21 (Admissible Cover). In a topological space $X$, an open cover $\mathcal{A}$ is said to be admissible when at least one of its elements has compact complement. If every set has compact complement, $\mathcal{A}$ is said to be strongly admissible, or s-admissible for short.

Lemma 2.22. In a topological space $X$, if
\[
\mathcal{H} = \{K_0, \ldots, K_n\}
\]
is an admissible partition where $K_1, \ldots, K_n$ are all compact, then
\[
\mathcal{A} = \{K_0 \cup K_1, K_0 \cup K_2, \ldots, K_0 \cup K_n\}
\]
is a strongly admissible cover.

Proof. One just has to notice that $\mathcal{A}$ does cover $X$. And also, that
\[
(K_0 \cup K_j)^c = \bigcup_{i \in \{1, \ldots, n\} \setminus \{j\}} K_i
\]
is compact for every $j = 1, \ldots, n$. □

The following Lemma is usually embedded in the demonstration of the variational principle. It is not new, except for the fact that it is usually applied without being formally stated.

Lemma 2.23. In a topological space $X$, let
\[
\mathcal{H} = \{K_0, \ldots, K_n\}
\]
be an admissible partition where \(K_1, \ldots, K_n\) are all compact. Let
\[
\mathcal{A} = \{K_0 \cup K_1, K_0 \cup K_2, \ldots, K_0 \cup K_n\}.
\]

If \(\mathcal{B}\) refines \(\mathcal{A}\), then, for each \(B \in \mathcal{B}^k\), the number of elements of \(\mathcal{X}^k\) that \(B\) intersects is at most \(2^k\).

**Proof.** Since \(\mathcal{A} \prec \mathcal{B}\), \(\mathcal{A}^k \prec \mathcal{B}^k\). Therefore, \(B\) is contained in some \(A \in \mathcal{A}^k\). Now, \(A\) is of the form
\[
(K_0 \cup K_{\lambda_1}) \cap T^{-1}(K_0 \cup K_{\lambda_2}) \cap \cdots \cap T^{-(n-1)}(K_0 \cup K_{\lambda_k}),
\]
for some \(\lambda \in \{1, \ldots, n\}\).

Therefore,
\[
B \subset \bigcup_{\gamma \in \{0,1\}^k} \left(K_{\gamma_1 \lambda_1} \cap T^{-1}K_{\gamma_2 \lambda_2} \cap \cdots \cap T^{-(n-1)}K_{\gamma_n \lambda_n}\right).
\]
Since \(\mathcal{X}^k\) partitions \(X\), \(B\) intersects only the non empty sets in this union. And since there is one for each \(\gamma \in \{0,1\}^k\), the claim follows. \(\square\)

An important feature of admissible covers is that there is a Lebesgue Number associated to them.

**Lemma 2.24 (Lebesgue Number).** Let \(d\) be the restriction to \(X\) of a metric in some compactification, and let \(\mathcal{A}\) be an admissible cover. Then, there exists \(\varepsilon > 0\) such that
\[
\mathcal{A} \prec \mathcal{B}_d(\varepsilon).
\]

**Proof.** Remark 2.15 and Lemma 2.27, both from [CP15], lead to the desired result. \(\square\)

**Definition 2.25 (Topological Pressures).** For a dynamical system \(T : X \to X\), and a function \(f : X \to \mathbb{R}\), define
\[
Q^- (T, f) = \sup_{\mathcal{A} : \text{admissible cover}} Q^- (T, f, \mathcal{A})
\]
\[
Q^+ (T, f) = \sup_{\mathcal{A} : \text{admissible cover}} Q^+ (T, f, \mathcal{A}).
\]
And if \(d\) is a metric over \(X\), define
\[
P_d (T, f) = \limsup_{\varepsilon \to 0} P (T, f, \mathcal{B}_d(\varepsilon)).
\]

**Lemma 2.26.** If \(d\) is a one-point metric for \(X\), then \(\mathcal{B}_d(\varepsilon)\) is admissible for any \(\varepsilon > 0\). Also, for any \(f : X \to \mathbb{R}\),
\[
Q^- (T, f) = \sup_{\varepsilon > 0} Q^- (T, f, \mathcal{B}_d(\varepsilon))
\]
\[
Q^+ (T, f) = \sup_{\varepsilon > 0} Q^+ (T, f, \mathcal{B}_d(\varepsilon)).
\]
And if $f$ is uniformly continuous with respect to $d$,

$$P_d(T, f) \leq Q^-(T, f).$$

Proof. Just take any $x \in X$ such that $d(x, \infty) < \varepsilon$. Then,

$$X \setminus B_d(\varepsilon; x) = X^* \setminus B_d(\varepsilon; x)$$

is closed in $X^*$, and therefore, compact. Therefore, $\mathcal{B}_d(\varepsilon)$ is admissible.

In particular, the definition of $Q^-(T, f)$ and $Q^+(T, f)$ implies that

$$\sup_{\varepsilon > 0} Q^-(T, f, \mathcal{B}_d(\varepsilon)) \leq Q^-(T, f)$$

$$\sup_{\varepsilon > 0} Q^+(T, f, \mathcal{B}_d(\varepsilon)) \leq Q^+(T, f).$$

On the other hand, if $\mathcal{A}$ is admissible, Lemma 2.24 gives $\varepsilon > 0$ such that $\mathcal{A} \prec \mathcal{B}_d(\varepsilon)$. Therefore, Lemma 2.18 implies that

$$Q^-(T, f, \mathcal{A}) \leq \sup_{\varepsilon > 0} Q^-(T, f, \mathcal{B}_d(\varepsilon))$$

$$Q^+(T, f, \mathcal{A}) \leq \sup_{\varepsilon > 0} Q^+(T, f, \mathcal{B}_d(\varepsilon)).$$

By taking the supremum over all admissible covers $\mathcal{A}$,

$$Q^-(T, f) \leq \sup_{\varepsilon > 0} Q^-(T, f, \mathcal{B}_d(\varepsilon))$$

$$Q^+(T, f) \leq \sup_{\varepsilon > 0} Q^+(T, f, \mathcal{B}_d(\varepsilon)).$$

Finally, if $f$ is uniformly continuous with respect to $d$, then, for each $\eta > 0$, there exists $\varepsilon_0 > 0$ such that for every $n = 1, 2, \ldots$, every non null $\varepsilon \leq \varepsilon_0$, and every $B \in \mathcal{B}_d(\varepsilon)^n$,

$$\sup_{B \in \mathcal{B}} f_n(B) \leq \inf_{B \in \mathcal{B}} f_n(B) + n\eta.$$

One just has to choose $\varepsilon_0 > 0$ such that

$$d(x, y) < 2\varepsilon_0 \Rightarrow |f(x) - f(y)| < \eta.$$

In this case, for any subcover $\mathcal{B} \subset \mathcal{B}_d(\varepsilon)^n$

$$\sum_{B \in \mathcal{B}} \sup_{B \in \mathcal{B}} e^f_n(B) \leq e^{n\eta} \sum_{B \in \mathcal{B}} \inf_{B \in \mathcal{B}} e^f_n(B).$$

Taking the infimum for every subcover $\mathcal{B}$, taking the logarithm, dividing by $n$ and taking the lim inf,

$$P_d(T, f) \leq Q^-(T, f) + \eta.$$

Since $\eta$ is arbitrary, the result follows. \qed
As in the case of topological entropy, we can define yet another concept of topological pressure using \((n, \varepsilon)\)-separated and \((n, \varepsilon)\)-generating sets. In the compact case, those concepts are all equivalent to the ones we have already defined. Given a metric \(d\) over \(X\) and \(\varepsilon > 0\), we say that a set \(E_n\) is \((n, \varepsilon)\)-separated if for any \(x, y \in E_n\),

\[
\forall j = 0, \ldots, n - 1, \ d(T^j x, T^j y) < \varepsilon \Rightarrow x = y.
\]

And we say that \(G_n\) is \((n, \varepsilon)\)-generating if given any \(x \in X\), there is \(y \in G_n\) such that for any \(j = 0, \ldots, n - 1, \ d(T^j x, T^j y) < \varepsilon\).

More information about the relation between \((n, \varepsilon)\)-separated sets, \((n, \varepsilon)\)-generating sets and \(N(\mathcal{B}_d(\varepsilon)^n)\) can be found in [CP15].

**Definition 2.27.** For a dynamical system \(T : X \to X\), a function \(f : X \to \mathbb{R}\), \(n = 1, 2, \ldots\) and \(\varepsilon > 0\), define

\[
G_d^n(T, f, \varepsilon) = \inf \left\{ \sum_{x \in E} e^{f_n(x)} \middle| E \text{ is } (n, \varepsilon)\text{-generating} \right\}
\]

\[
S_d^n(T, f, \varepsilon) = \sup \left\{ \sum_{x \in E} e^{f_n(x)} \middle| E \text{ is } (n, \varepsilon)\text{-separated} \right\}
\]

and

\[
G_d(T, f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log G_d^n(T, f, \varepsilon)
\]

\[
S_d(T, f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log S_d^n(T, f, \varepsilon).
\]

Since those last two are monotonic in \(\varepsilon\), define

\[
G_d(T, f) = \lim_{\varepsilon \to 0} \sup_{\varepsilon > 0} G_d(T, f, \varepsilon)
\]

\[
S_d(T, f) = \lim_{\varepsilon \to 0} \sup_{\varepsilon > 0} S_d(T, f, \varepsilon).
\]

We now state some very basic properties satisfied by the different kinds of topological pressure we have defined. First, let’s relate them all.

**Lemma 2.28.** For a dynamical system \(T : X \to X\), any function \(f : X \to \mathbb{R}\) and any metric \(d\) over \(X\),

\[
Q^- (T, f) \leq Q^+ (T, f) \leq G_d(T, f) \leq S_d(T, f) \leq P_d(T, f).
\]

**Proof.** It is quite evident that \(Q^- (T, f) \leq Q^+ (T, f)\). The fact that \(G_d(T, f) \leq S_d(T, f)\) is a consequence of the fact that any \((n, \varepsilon)\)-separated set is contained in a maximal one. And a maximal \((n, \varepsilon)\)-separated set is in fact an \((n, \varepsilon)\)-generating set.
\[ Q^+(T, f) \leq G_d(T, f) \]

Let \( \mathcal{A} \) be an admissible cover of \( X \). Then, Lemma 2.24 gives us \( \varepsilon_0 > 0 \) such that \( \mathcal{A} \prec \mathcal{B}_d(\varepsilon) \) for any \( \varepsilon \leq \varepsilon_0 \). Let \( E \) be an \((n, \varepsilon)\)-generating set. In this case,
\[
\mathcal{B} = \left\{ B_d(\varepsilon; x) \cap \cdots \cap T^{-(n-1)} B_d(\varepsilon; T^{n-1}x) \bigg| x \in E \right\} \{ B_{dn}(\varepsilon; x) \big| x \in E \}
\]
is a subcover of \( \mathcal{B}_d(\varepsilon)^n \). Therefore,
\[
Q_n(T, f, \mathcal{A}) \leq Q_n(T, f, \mathcal{B}_d(\varepsilon)) \leq \sum_{B \in \mathcal{B}} \inf_{x \in B} e_{f_n}(x).
\]
Taking the infimum for every \((n, \varepsilon)\)-generating \( E \),
\[
Q_n(T, f, \mathcal{A}) \leq G^n_d(T, f, \varepsilon).
\]
Taking the logarithm, dividing by \( n \), and taking the \( \limsup \) for \( n \to \infty \), we get that
\[
Q^+(T, f, \mathcal{A}) \leq G_d(T, f, \varepsilon),
\]
for every \( \varepsilon < \varepsilon_0 \). Therefore, by making \( \varepsilon \to 0 \),
\[
Q^+(T, f, \mathcal{A}) \leq G_d(T, f).
\]
Finally, since \( \mathcal{A} \) was an arbitrary admissible cover,
\[
Q^+(T, f) \leq G_d(T, f).
\]

Let us demonstrate the last inequality.

\[ S_d(T, f) \leq P_d(T, f) \]

Given \( \varepsilon > 0 \), let \( E \) be any \((n, \varepsilon)\)-separated set, and let \( \mathcal{B} \) be any subcover of \( \mathcal{B}_d(\frac{\varepsilon}{n})^n \). Then, for each \( x \in E \), pick a \( B_x \in \mathcal{B} \), such that \( x \in B_x \). Notice that, since \( E \) is \((n, \varepsilon)\)-separated, \( x \neq y \Rightarrow B_x \neq B_y \). Therefore,
\[
\sum_{x \in E} e_{f_n}(x) \leq \sum_{x \in E} \sup_{B \in \mathcal{B}} e_{f_n}(B_x) \leq \sum_{B \in \mathcal{B}} \sup_{B \in \mathcal{B}} e_{f_n}(B).
\]
Taking the infimum for \( \mathcal{B} \subseteq \mathcal{B}_d(\frac{\varepsilon}{n}) \), and then the supremum for \((n, \varepsilon)\)-separated \( E \) gives
\[
S^n_d(T, f, \varepsilon) \leq P(T, f, \mathcal{B}_d(\frac{\varepsilon}{n})).
\]
And the result follows by taking the logarithm, dividing by \( n \), making \( n \to \infty \), and then, taking the lim inf for \( \varepsilon \to 0 \).

**Proposition 2.29.** For a dynamical system \( T : X \to X \), if \( d \) is a one-point metric for \( X \) and \( f : X \to \mathbb{R} \) is one-point uniformly continuous, then
\[
Q^- (T, f) = Q^+ (T, f) = G_d (T, f) = S_d (T, f) = P_d (T, f).
\]

**Proof.** This is a direct consequence of Lemmas 2.26 and 2.28. \( \square \)

**Definition 2.30 (Topological Pressure).** Let \( T : X \to X \) be a dynamical system that admits a metrizable one-point compactification. Suppose that \( f : X \to \mathbb{R} \) is one-point uniformly continuous. Then, the topological pressure is the quantity in Proposition 2.29, and is denoted by \( P (T, f) \).

When the space is compact and \( f : X \to \mathbb{R} \) is continuous, it is a simple fact that
\[
P (T^k, f_k) = kP (T, f).
\]
For the non-compact case, only one inequality follows from a similar argument, and only for \( Q^+ (T, f) \) and \( Q^- (T, f) \).

**Proposition 2.31.** Consider the dynamical system \( T : X \to X \), and a function \( f : X \to \mathbb{R} \). Then, for any \( k = 1, 2, \ldots \),
\[
Q^- (T^k, f_k) \leq kQ^- (T, f)
\]
\[
Q^+ (T^k, f_k) \leq kQ^+ (T, f).
\]

**Proof.** Let \( \mathcal{A} \) be an admissible cover of \( X \). Notice that \((\mathcal{A}^k_T)^n_T = \mathcal{A}^{kn}_T \). And also, \((f_k)_{T^k,n} = f_{nk} \). So,
\[
\frac{1}{n} \log Q_n (T^k, f_k, \mathcal{A}^k) = \frac{1}{kn} \log Q_{kn} (T, f, \mathcal{A}).
\]
Taking the lim inf and lim sup for \( n \to \infty \),
\[
Q^- (T^k, f_k, \mathcal{A}^k) = kQ^- (T, f, \mathcal{A})
\]
\[
Q^+ (T^k, f_k, \mathcal{A}^k) = kQ^+ (T, f, \mathcal{A}).
\]
And since \( \mathcal{A} \preceq \mathcal{A}^k \), Lemma 2.18 implies that
\[
Q^- (T^k, f_k, \mathcal{A}) \leq Q^- (T^k, f_k, \mathcal{A}^k) = kQ^- (T, f, \mathcal{A})
\]
\[
Q^+ (T^k, f_k, \mathcal{A}) \leq Q^+ (T^k, f_k, \mathcal{A}^k) = kQ^+ (T, f, \mathcal{A}).
\]
Now, we just have to take the supremum for every admissible cover \( \mathcal{A} \) to reach the desired conclusion. \( \square \)

Notice that the power of Proposition 2.31 is quite limited even in the case where \( f : X \to \mathbb{R} \) is one-point uniformly continuous. In this case, eventhough
Let us finally mention a feature that is common to every concept of pressure we have defined so far.

**Lemma 2.32.** If \( d \) is any metric over \( X \), \( f : X \rightarrow \mathbb{R} \) is any function and \( c \in \mathbb{R} \). Then,
\[
Q^+(T, f + c) = Q^+(T, f) + c
\]
\[
Q^-(T, f + c) = Q^-(T, f) + c
\]
\[
P_d(T, f + c) = P_d(T, f) + c
\]
\[
G_d(T, f + c) = G_d(T, f) + c
\]
\[
S_d(T, f + c) = S_d(T, f) + c.
\]
And if \( \mu \) is a \( T \)-invariant probability measure and \( f \) has a well defined integral,
\[
P_\mu(T, f + c) = P_\mu(T, f) + c.
\]

**Proof.** For \( P_\mu(T, f + c) \), this is an obvious consequence of
\[
\int (f + c) \, d\mu = \int f \, d\mu + c.
\]
The other equalities are easy consequences of the exponential function properties. \( \square \)

3. **Variational Principle**

Inspired by what has been done for the compact case, we demonstrate a **variational principle** for the pressure of a topological system \( T : X \rightarrow X \), where \( X \) is not assumed to be compact but it is just assumed to have a one-point compactification \( X^* \). This does not imply that \( T \) can be itself extended to a topological dynamical system over \( X^* \).

We use the preparations made in Section 2 in order to adapt Misiurewicz’s demonstration of the variational principle. Misiurewicz’s original article is [Mis76]. We shall follow the more didactic presentation of the variational principle presented in [VO16], Section 10.3 and Section 10.4. A similar presentation can also be found in [Wal00], Chapter 9.

We are concerned about the supremum of \( P_\mu(T, f) \) over all \( T \)-invariant Radon probability measures for a given one-point uniformly continuous \( f : X \rightarrow \mathbb{R} \). However, there might happen that no such a probability measure exists. In this case, we agree that
\[
\sup_{\mu} P_\mu(T, f) = 0.
\]
According to Lemma 2.8, this is the same as taking the supremum over all \( T \)-invariant Radon measures \( \mu \) with \( 0 \leq \mu(X) \leq 1 \). In this case, there is always an invariant measure. Namely, \( \mu = 0 \).
Theorem 3.1. Let $T : X \to X$ be a metrizable locally compact separable dynamical system, and let $f : X \to \mathbb{R} \in C_0(X)$. Then,

$$\sup_{\mu} P_\mu(T, f) = P(T, f),$$

where the supremum is taken over all $T$-invariant Radon probability measures. If there is no $T$-invariant Radon probability measure,

$$P(T, f) = 0.$$ 

Before the proof, let’s extend Theorem 3.1 to one-point uniformly continuous functions.

Corollary 3.2. Let $T : X \to X$ be a metrizable locally compact separable dynamical system, and let $f : X \to \mathbb{R}$ be one-point uniformly continuous. Then,

$$\sup_{\mu} P_\mu(T, f) = P(T, f),$$

where the supremum is taken over all $T$-invariant Radon probability measures. If there is no $T$-invariant Radon probability measure,

$$P(T, f) = f(\infty).$$

Proof. Use the theorem with $f - f(\infty)$ in place of $f$. Then, use Lemma 2.32. □

The theorem will be demonstrated if we show that:

1. For any $T$-invariant Radon probability $\mu$,

$$P_\mu(T, f) \leq P(T, f).$$

2. If we fix a one-point metric $d$, then, for any $\varepsilon > 0$, there is a $T$-invariant Radon measure $\mu$, with $0 \leq \mu(X) \leq 1$, such that

$$S_d(T, f, \varepsilon) \leq P_\mu(T, f).$$

These claims are the contents of the following two subsections.

3.1. Topological Pressure is an Upper Bound. This subsection is devoted to the proof of the following proposition. The technique we present is a mix of what is done for Lemma 3.2 of [CP15] and what is done in Section 10.4.1 in [VO16].

Proposition 3.3. Let $T : X \to X$ be a dynamical system such that $X$ has a metrizable one-point compactification, let $f : X \to \mathbb{R}$ be one-point uniformly continuous, and $\mu$ a $T$-invariant Radon probability measure. Then,

$$P_\mu(T, f) \leq P(T, f).$$
Proof. Let $\mu$ be any $T$-invariant Radon probability measure. We shall show that for any $n = 1, 2, \ldots,$

\begin{equation}
P_\mu(T^n, f_n) \leq nP(T, f) + 2 + \log 2.
\end{equation}

And then, Proposition 2.10 implies that

\[
P_\mu(T, f) = \frac{1}{n}P_\mu(T^n, f_n) \\
\leq P(T, f) + \frac{2 + \log 2}{n} \to P(T, f).
\]

And this will finish the demonstration. Notice that $f_n$ might not be one-point uniformly continuous, and therefore, we do not talk about $P(T^n, f_n)$. From now on, we fix $n$ and attempt to show the validity of inequation (1).

According to Proposition 2.12, we have to show that given an admissible partition $\mathcal{K}$,

\[
P_\mu(T^n, f_n, \mathcal{K}) \leq nP(T, f) + 2 + \log 2.
\]

To that end, let $d$ be a one-point metric, it is enough if we prove that there is an $\varepsilon > 0$ such that

\begin{equation}
P_\mu(T^n, f_n, \mathcal{K}) \leq nQ^+(T, f, \mathcal{B}_d(\varepsilon)) + 2 + \log 2.
\end{equation}

Let $\mathcal{A}$ be the strongly admissible cover from Lemma 2.22. Using the Lebesgue Number of Lemma 2.24, fix $\varepsilon > 0$ such that

\[
\mathcal{A} \prec \mathcal{B}_d(\varepsilon).
\]

Also, choose $\varepsilon$ small enough such that

\[
d(x, y) < 2\varepsilon \Rightarrow |f(y) - f(x)| \leq \frac{1}{n}.
\]

With $\varepsilon > 0$ properly chosen, we attempt at demonstrating the validity of inequality (2). Since we are working with $T$ and $T^n$ at the same time, let’s agree that whenever the transformation is omitted, it is assumed to be $T$.

Claim. For any $m = 1, 2, \ldots,$

\[
\int f_n \, d\mu + \frac{1}{m}H_\mu(\mathcal{K}^m) \leq \\
\leq m + \frac{m^2 + 2 + n}{n} \log Q_{mn}(T, f, \mathcal{B}_d(\varepsilon)).
\]

Let $\mathcal{B} \subset \mathcal{B}_d(\varepsilon)^{mn}$ be any subcover. And notice that

\[
\mathcal{K}^m_T \prec \mathcal{K}^{mn}
\]

\[
(f_n)_{T^n, m} = f_{mn}.
\]
Given \( C \in \mathcal{K}^m_{T^n} \), let \( x_C \in C \) be such that
\[
\sup f_{mn}(C) \leq f_{mn}(x_C) + 1.
\]
Also, for each \( C \in \mathcal{K}^m_{T^n} \), choose \( B_C \in \mathcal{B} \) such that \( x_C \in B_C \).
Notice that for any \( x \in B_C \) and \( j = 0, \ldots, mn - 1 \),
\[
d(T^j x_C, T^j x) < 2 \varepsilon.
\]
Therefore, by the choice of \( \varepsilon \),
\[
\sup f_{mn}(C) \leq f_{mn}(x_C) + 1 \leq \inf f_{mn}(B_C) + \frac{mn}{n} + 1 = \inf f_{mn}(B_C) + m + 1.
\]
For each \( B \in \mathcal{B} \), let \( c_B \) be the cardinality of
\[
\{ C \in \mathcal{K}^m_{T^n} \mid B_C = B \}.
\]
Since \( \mathcal{A} \prec \mathcal{B}_d(\varepsilon) \prec \mathcal{B} \), Lemma 2.23 implies that
\[
c_B \leq 2^m.
\]
Now, Lemma 2.16 with \( T^n \) in place of \( T \) and \( f_n \) in place of \( f \) implies that
\[
\int f_n \, d\mu + \frac{1}{m} H_\mu(\mathcal{K}^m_{T^n}) \leq \frac{1}{m} \log \sum_{C \in \mathcal{K}^m_{T^n}} e^{\sup f_{mn}(C)}
\leq \frac{1}{m} \log \left( e^{m+1} \sum_{C \in \mathcal{K}^m_{T^n}} e^{\inf f_{mn}(B_C)} \right)
= \frac{m+1}{m} + \frac{1}{m} \log \sum_{C \in \mathcal{K}^m_{T^n}} e^{\inf f_{mn}(B_C)}
= \frac{m+1}{m} + \frac{1}{m} \log \sum_{B \in \mathcal{B}} c_B e^{\inf f_{mn}(B)}
\leq \frac{m+1}{m} + \frac{1}{m} \log \left( 2^m \sum_{B \in \mathcal{B}} e^{\inf f_{mn}(B)} \right)
= \frac{m+1}{m} + \log 2 + \frac{n}{mn} \log \sum_{B \in \mathcal{B}} e^{\inf f_{mn}(B)}.
\]
Taking the infimum for every subcover \( \mathcal{B} \subset \mathcal{B}_d(\varepsilon)^{mn} \), gives the Claim.
Now, use the Claim and take the lim sup for \( m \to \infty \)
\[
\int f_n \, d\mu + h_\mu \left(T^n, \mathcal{X}\right) \leq 1 + \log 2 + \limsup_{m \to \infty} \frac{n}{mn} \log Q_{mn} \left(T, f, \mathcal{B}_d (\varepsilon)\right)
\leq 1 + \log 2 + n \limsup_{k \to \infty} \frac{1}{k} \log Q_k (T, f, \mathcal{B}_d (\varepsilon))
= 1 + \log 2 + n \left(1 + \log 2 + n \limsup_{k \to \infty} \frac{1}{k} \log Q_k (T, f, \mathcal{B}_d (\varepsilon))\right)
\leq 1 + \log 2 + n Q^+ (T, f)
= 1 + \log 2 + n P (T, f),
\]
to get inequality (2) and conclude the proof. \(\square\)

### 3.2. Topological Pressure is a Lower Bound

This subsection is devoted to the proof of the following proposition, which is nothing more than a straightforward adaption of what is done in Subsection 10.4.2 of [VO16], using the same technique applied for Theorem 3.1 in [CP15].

**Proposition 3.4.** Let \( T : X \to X \) be a dynamical system such that \( X \) admits a one-point compactification. Suppose \( f \in C_0 (X) \). Then, for any \( \varepsilon > 0 \), there exists a \( T \)-invariant Radon measure \( \mu \), with \( 0 \leq \mu (X) \leq 1 \), such that

\[
S_d (T, f, \varepsilon) \leq P_\mu (T, f).
\]

**Proof.** Use Lemma 2.3 to get a compact metrizable extension \( S : Z \to Z \) for \( T \). According to Lemma 2.8, the demonstration will be complete if we find a probability measure \( \mu \) over \( Z \) which is \( S \)-invariant, and a partition \( \mathcal{C} \) having a \( C \in \mathcal{C} \) such that \( X^c \subset C \), and such that

\[
S_d (T, f, \varepsilon) \leq P_\mu (S, g, \mathcal{C}),
\]

where \( g = f \circ \pi \), and \( \pi : Z \to X^* \) is the projection from Lemma 2.3. Notice that

\[
\int g \, d\mu = \int f \, d\mu,
\]

because \( g|_{X^c} = f(\infty) = 0 \).

Let \( \tilde{d} \) be a one-point metric for \( X \), and \( \tilde{d} \) be the pseudometric over \( Z \) induced by it. That is, considering \( \tilde{d} \) as a metric over \( X^* \),

\[
\tilde{d}(x, y) = d(\pi(x), \pi(y)).
\]

For each \( n = 1, 2, \ldots \), let \( E_n \subset X \) be an \((n, \varepsilon)\)-separated set such that

\[
\frac{1}{2} S_n^\mu (T, f, \varepsilon) \leq \sum_{x \in E_n} e^{f_n(x)}.
\]

Call the rightside quantity \( A_n \). That is,

\[
\frac{1}{2} S_n^\mu (T, f, \varepsilon) \leq A_n.
\]
Then, define over $\mathbb{Z}$ the measure 
\[
\sigma_n = \frac{1}{A_n} \sum_{x \in E_n} e^{g_n(x)} \delta_x,
\]
where $\delta_x$ is the Dirac measure with support in $x$. And notice that $\sigma_n$ is a probability measure. Also define 
\[
\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \sigma_n \circ S^{-j}.
\]

**Claim.** There is a subsequence $n_k$ and a Radon probability measure $\mu$ such that $\mu_{n_k} \to \mu$, and such that 
\[
\lim_{k \to \infty} \frac{1}{n_k} \log A_{n_k} = \limsup_{n \to \infty} \frac{1}{n} \log A_n.
\]
Also, for any measurable $C \subset \mathbb{Z}$ with $\mu(\partial C) = 0$, 
\[
\lim \mu_{n_k}(C) = \mu(C).
\]

It is clear that there is a subsequence $n_k$ such that 
\[
\lim_{k \to \infty} \frac{1}{n_k} \log A_{n_k} = \limsup_{n \to \infty} \frac{1}{n} \log A_n.
\]
In the weak-$*$ topology, the set of Radon probability measures $\mu$ over $\mathbb{Z}$ is easily seen to be sequentially compact (Proposition 2.1.6 from [VO16]).

The sequential compactness means that we can assume that $n_k$ is such that $\mu_{n_k}$ converges to some Radon probability $\mu$. The last assertion in our claim is a consequence of the Portmanteau Theorem, and can be found in [Bil99], Theorem 2.1, item (v).

**Claim.** The measure $\mu$ is $S$-invariant.

It is clear that $\mu_{n_k} \circ S^{-1} \to \mu \circ S^{-1}$. In fact, for any continuous $\phi : \mathbb{Z} \to \mathbb{R}$, $\phi \circ S$ is also continuous. Therefore, 
\[
\int \phi \, d(\mu_{n_k} \circ S^{-1}) = \int \phi \circ S \, d\mu_{n_k} \\
\to \int \phi \circ S \, d\mu \\
= \int \phi \, d(\mu \circ S^{-1}).
\]
On the other hand,
\[
\left| \int \phi \, d(\mu_{n_k} - \mu_{n_k} \circ S^{-1}) \right| = \left| \frac{1}{n_k} \int \phi \, d(\sigma_{n_k} - \sigma_{n_k} \circ S^{-n_k}) \right|
\]
\[
= \frac{1}{n_k} \left| \int (\phi - \phi \circ S^{n_k}) \, d\sigma_{n_k} \right|
\]
\[
\leq \frac{1}{n_k} \int \| \phi - \phi \circ S^{n_k} \|_\infty \, d\sigma_{n_k}
\]
\[
\leq \frac{1}{n_k} \int 2 \| \phi \|_\infty \, d\sigma_{n_k}
\]
\[
= \frac{1}{n_k} 2 \| \phi \|_\infty \to 0.
\]

This implies that
\[
\mu = \lim \mu_{n_k} = \lim \mu_{n_k} \circ S^{-1} = \mu \circ S^{-1}.
\]

Now, we construct a suitable measurable partition \(\mathcal{Z}\), so that inequation (3) holds. To that end, we use the pseudometric \(\tilde{d}\). For each \(z \in Z\), there exists a non null \(\varepsilon_z < \frac{\delta}{2}\) such that the ball \(B_z = B(z; \varepsilon_z)\), centered at \(z\) with radius \(\varepsilon_z\), is such that \(\mu(\partial B_z) = 0\). Such an \(\varepsilon_z\) exists because since the border of the balls \(B(z; \delta)\) are all disjoint, there is at most a countable number of reals \(\delta < \frac{\delta}{2}\) such that \(B(z; \delta)\) has border with non null measure. Now, since \(Z\) is compact and the balls are open, there is a finite number of such balls, \(B_0, \ldots, B_n\) covering \(Z\). We can assume that \(\{B_0, \ldots, B_n\}\) has no proper subcover. Let
\[
Z_j = B_j \setminus (B_1 \cup \cdots \cup B_{j-1}).
\]

Then, \(\mathcal{Z} = \{Z_0, \ldots, Z_k\}\) is a measurable partition. We can also assume that \(X^c \subset B_0 = Z_0\), because in the pseudometric \(\tilde{d}\), \(X^c\) has diameter equals to 0. That is, \(\mathcal{Z}\) satisfies the conditions of Lemma 2.8.

Also, notice that each \(C \in \mathcal{Z}^n\) is such that for any \(x, y \in C\),
\[
\tilde{d}(S^j x, S^j y) < \varepsilon
\]
for all \(j = 0, 1, \ldots, n - 1\).

**Claim.** For each \(C \in \mathcal{Z}^n\), \(\mu(\partial C) = 0\).

Notice that, since \(S\) is continuous, the border operator \(\partial\) possesses the following properties.

1. \(\partial A = \partial A^c\).
2. \(\partial(A_1 \cap \cdots \cap A_k) \subset \partial A_1 \cup \cdots \cup \partial A_k\).
3. \(\partial S^{-1}(A) \subset S^{-1}(\partial A)\).
From items (1) and (2), each $Z_j = B_j \cap B_1^c \cap \cdots \cap B_{j-1}^c$ in $\mathcal{F}$ has border with null measure. And from items (2) and (3), the same is true for the sets in $\mathcal{F}^n$.

Having constructed $\mu$ and $\mathcal{C}$, it remains to show that inequation (3) holds.

Claim. $\int g_n \, d\sigma_n = n \int g \, d\mu_n$.

In fact,
\[
\int g \, d\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \int g \circ S^{-j} \, d\sigma_n
= \frac{1}{n} \sum_{j=0}^{n-1} \int g \circ S^j \, d\sigma_n
= \frac{1}{n} \int \sum_{j=0}^{n-1} g \circ S^j \, d\sigma_n
= \frac{1}{n} \int g_n \, d\sigma_n.
\]

Claim. $H_{\sigma_n}(\mathcal{F}^n) + n \int g \, d\mu_n = \log A_n$.

Let $C \in \mathcal{F}^n$. Since each element of $\mathcal{F}$ has diameter less than $\varepsilon$, we have that $C$ can contain at most one element $x \in E_n$. That is, $\sigma_n(C) = 0$ or $\sigma_n(C) = e^{g_n(x)} A_n$. Therefore,
\[
H_{\sigma_n}(\mathcal{F}^n) + n \int g \, d\mu_n = H_{\sigma_n}(\mathcal{F}^n) + \int g_n \, d\sigma_n
= \sum_{x \in E_n} \sigma_n(\{x\}) \left( g_n(x) + \log \frac{1}{\sigma_n(\{x\})} \right)
= \sum_{x \in E_n} \frac{e^{g_n(x)}}{A_n} \log \frac{e^{g_n(x)}}{A_n}
= \sum_{x \in E_n} \frac{e^{g_n(x)}}{A_n} \log A_n
= \log A_n.
\]

Passing from $\sigma_n$ to $\mu_n$ is the same procedure as in the compact case, as we shall detail right now. Notice that for any measurable finite partition $\mathcal{D}$,
Lemma 2.7 from [CP15] implies that

$$\sum_{j=0}^{n-1} \frac{1}{n} H_{\sigma \circ S^{-j}} (\mathcal{D}) \leq H_{\mu} (\mathcal{D}).$$

For $n, q \in \mathbb{N}$ with $1 < q < n$, take an integer $m$ such that $mq \geq n > m(q-1)$. Then, for every $j = 0, \ldots, q-1$,

$$\mathcal{D}^n \prec \mathcal{D}^j \vee S^{-j} (\mathcal{D}^{qm}) = \mathcal{D}^j \vee S^{-j} (\mathcal{D}^q) \vee S^{-(j+q)} (\mathcal{D}^q) \vee \ldots \vee S^{-(j+(m-1)q)} (\mathcal{D}^q).$$

Therefore, using Lemma 2.6 from [CP15],

$$H_{\sigma_n} (\mathcal{D}^n) \leq H_{\sigma_n} (\mathcal{D}^j) + H_{\sigma_n \circ S^{-(j+0q)} (\mathcal{D}^q)} + \ldots + H_{\sigma_n \circ S^{-(j+(m-1)q)} (\mathcal{D}^q)}$$

$$\leq H_{\sigma_n} (\mathcal{D}^q) + H_{\sigma_n \circ S^{-(j+0q)} (\mathcal{D}^q)} + \ldots + H_{\sigma_n \circ S^{-(j+(m-1)q)} (\mathcal{D}^q)}$$

$$\leq \log \# \mathcal{D}^q + H_{\sigma_n \circ S^{-(j+0q)} (\mathcal{D}^q)} + \ldots + H_{\sigma_n \circ S^{-(j+(m-1)q)} (\mathcal{D}^q)}.$$

Summing up in $j = 0, \ldots, q-1$,

$$qH_{\sigma_n} (\mathcal{D}^n) \leq q \log \# \mathcal{D}^q + \sum_{j=0}^{q-1} \sum_{a=0}^{m-1} H_{\sigma_n \circ S^{-(j+aq)} (\mathcal{D}^q)}$$

$$= q \log \# \mathcal{D}^q + \sum_{p=0}^{n-1} H_{\sigma_n \circ S^{-p} (\mathcal{D}^q)} + \sum_{p=n}^{mq-1} H_{\sigma_n \circ S^{-p} (\mathcal{D}^q)}$$

$$\leq 2q \log \# \mathcal{D}^q + n \sum_{p=0}^{n-1} H_{\sigma_n \circ S^{-p} (\mathcal{D}^q)}$$

$$\leq 2q \log \# \mathcal{D}^q + nH_{\mu_n} (\mathcal{D}^q).$$

Since each element $C \in \mathcal{D}^q$ has border with null measure,

$$\lim_{k \to \infty} \mu_{n_k} (C) = \mu (C).$$

An this implies that

$$H_{\mu_{n_k}} (\mathcal{D}^q) \to H_{\mu} (\mathcal{D}^q).$$
Therefore,
\[
\limsup_{n \to \infty} \frac{1}{n} \log S^n_d (T, f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{2} S^n_d (T, f, \varepsilon) \\
\leq \limsup_{n \to \infty} \frac{1}{n} \log A_n \\
= \lim_{k \to \infty} \frac{1}{n_k} \log A_{n_k} \\
= \lim_{k \to \infty} \frac{1}{n_k} \left( H_{\sigma_{n_k}} (\mathcal{Z}^{n_k}) + n_k \int g \, d\mu_{n_k} \right) \\
= \lim_{k \to \infty} \left( \frac{q H_{\sigma_{n_k}} (\mathcal{Z}^{n_k})}{q n_k} + \int g \, d\mu_{n_k} \right) \\
\leq \lim_{k \to \infty} \left( \frac{1}{n_k} 2 \log \# \mathcal{Z}^{q} + \frac{1}{q} H_{\mu_{n_k}} (\mathcal{Z}^{q}) + \int g \, d\mu_{n_k} \right) \\
= 0 + \frac{1}{q} \int H_{\mu} (\mathcal{Z}^{q}) + \int g \, d\mu \\
\xrightarrow{q \to \infty} h_{\mu} (S, \mathcal{Z}) + \int g \, d\mu \\
= P_{\mu} (S, g, \mathcal{Z}) \\
\leq P_{\mu} (T, f).
\]

Where the last inequality is from Lemma 2.8. \hfill \Box

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