Distance Matrix of Multi-block Graphs: Determinant and Inverse

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Abstract

A connected graph is called a multi-block graph if each of its blocks is a \( m \)-partite graph, whenever \( m \geq 2 \). Building on the work of \([2, 10]\), we compute the determinant and inverse of the distance matrix for a class of multi-block graphs.

Keywords: \( m \)-partite graphs, Laplacian like matrix, distance matrix, determinant, cofactor.

MSC: 05C12, 05C50

1 Introduction and Motivation

Let \( G = (V(G), E(G)) \) be a finite, simple, connected graph with \( V(G) \) as the set of vertices and \( E(G) \subset V(G) \times V(G) \) as the set of edges in \( G \), we simply write \( G = (V, E) \) if there is no scope of confusion. We write \( i \sim j \) to indicate that the vertices \( i, j \in V \) are adjacent in \( G \).

Before proceeding further, we first introduce a few notations which will be used time and again throughout this article. Let \( I_n, 1_n \) and \( e_i \) denote the identity matrix, the column vector of all ones and the column vector with 1 at the \( i^{th} \) entry respectively. Further, \( J_{m \times n} \) denotes the \( m \times n \) matrix of all ones and if \( m = n \), we use the notation \( J_m \). We write \( 0_{m \times n} \) to represent zero matrix of order \( m \times n \) and simply write \( 0 \) if there is no scope of confusion with the order of the matrix.

A connected graph \( G \) is a metric space with respect to the metric \( d \), where \( d(i, j) \) equals the length of the shortest path from \( i \) and \( j \). Before proceeding further, we recall the definitions of the distance matrix and the Laplacian matrix of a graph \( G \).

Let \( G \) be a graph with \( n \) vertices. The distance matrix of graph \( G \) is an \( n \times n \) matrix, denoted by \( D(G) = [d_{ij}] \), where

\[
d_{ij} = \begin{cases} 
  d(i, j) & \text{if } i \neq j, \ i, j \in V, \\
  0 & \text{if } i = j, \ i, j \in V.
\end{cases}
\]

and the Laplacian matrix of \( G \) is an \( n \times n \) matrix, denoted as \( L(G) = [l_{ij}] \), where

\[
l_{ij} = \begin{cases} 
  \delta_i & \text{if } i = j \\
  -1 & \text{if } i \neq j, i \sim j \\
  0 & \text{otherwise,
\end{cases}
\]

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where $\delta_i$ denotes the degree of the vertex $i$. It is well known that $L(G)$ is a symmetric, positive semi-definite matrix. The constant vector $1$ is the eigenvector of $L(G)$ corresponding to the smallest eigenvalue 0 and hence satisfies $L(G)1 = 0$ and $1^T L(G) = 0$ (for details see [1]).

Let $T$ be a tree with $n$ vertices. In [5], the authors proved that the determinant of the distance matrix $D(T)$ of $T$ is given by $\det D(T) = (-1)^{n-1}(n-1)2^{n-2}$. Note that, the determinant does not depend on the structure of the tree but the number of vertices. In [7], it was shown that the inverse of the distance matrix of a tree is given by $D(T)^{-1} = -\frac{1}{2}L(T) + \frac{1}{2(n-1)}\tau \mu^T$, where, $\tau = (2-\delta_1, 2-\delta_2, ..., 2-\delta_n)^T$. The above expression gives an formula for inverse of distance matrix of a tree in terms of the Laplacian matrix.

A vertex $v$ of a graph $G$ is a cut-vertex of $G$ if $G - v$ is disconnected. In [2], the authors compute the determinant and the inverse of the distance matrix of graphs whenever each of its blocks is a complete graph, called block graphs. Further, in [10] the determinant and the inverse of the distance matrix of graphs whenever each of its blocks is a complete bipartite graph, such graphs are called bi-block graph. To be specific, the authors define a matrix $L$ called Laplacian like matrix satisfying $L1 = 0$ and $1^T L = 0$. Further, it was shown that for a given graph $G$ of the above classes if the determinant of the distance matrix $D(G)$ is not zero, then the inverse of $D(G)$ is given by

$$D(G)^{-1} = -L + \frac{1}{\lambda_G} \mu \mu^T$$

where $\mu$ is a column vector and $\lambda_G$ a suitable constant (for details see [2, 10]). Similar results obtained for other class of graphs, namely catcoid digraphs, cycle-clique graphs, weighted cactoid digraph (for details see [8, 9, 11]).

In this manuscript, our first objective is to extends the results in [2, 10]. To be precise, we aim to compute the determinant and inverse of the distance matrix of graphs such that each of its blocks is a complete $m$-partite graph, whenever $m \geq 2$, we call such graphs as a multi-block graph. Now we will define a few notations and recall a few elementary results useful for subsequent sections.

Let $A$ be an $m \times n$ matrix. We use the notation $A(i \mid j)$ to denote the submatrix obtained by deleting the $i^{th}$ row and the $j^{th}$ column. Given a matrix $A$, we use $A^t$ to denote the transpose of the matrix. Let $A$ be an $n \times n$ matrix. For $1 \leq i, j \leq n$, the cofactor $c_{i,j}$ is defined as $(-1)^{i+j} \det A(i \mid j)$. We use the notation cof $A$ to denote the sum of all cofactors of $A$.

**Lemma 1.1.** [1] Let $A$ be an $n \times n$ matrix. Let $M$ be the matrix obtained from $A$ by subtracting the first row from all other rows and then subtracting the first column from all other columns. Then

$$\text{cof } A = \det M(1|1).$$

A graph $G$ is said to be $k$-connected if there does not exist a set of $k-1$ vertices whose removal disconnects the graph $G$. The authors in [6] established the following result for connected graphs with 2-connected blocks.

**Theorem 1.2.** [6] Let $G$ be a connected graph with 2-connected blocks $G_1, G_2, \cdots, G_b$. Then

$$\text{cof } D(G) = \prod_{i=1}^{b} \text{cof } D(G_i),$$

$$\det D(G) = \sum_{i=1}^{b} \det D(G_i) \prod_{j \neq i} \text{cof } D(G_i).$$
We conclude this section with a standard result on computing determinant of block matrices.

**Proposition 1.3.** [12] Let \( A_{11} \) and \( A_{22} \) be square matrices. Then
\[
\det \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} = \det A_{11} \det A_{22}.
\]

This article is organized as follows. In Section 2, we find the determinant and cofactor of the distance matrix for complete \( m \)-partite graph, for \( m \geq 3 \) and compute its inverse whenever it exists. In Section 3, we compute the inverse of multi-block graphs subject to the condition, cofactor is nonzero. Finally, Section 4 deals with a class of multi-block graphs, not covered in Section 3.

## 2 Determinant and Inverse of \( D(K_{n_1,n_2,\ldots,n_m}) \)

Let \( D(K_{n_1,n_2,\ldots,n_m}) \) be the distance matrix of complete \( m \)-partite graph \( K_{n_1,n_2,\ldots,n_m} \). Then \( D(K_{n_1,n_2,\ldots,n_m}) \) can be expressed in the following block form
\[
D(K_{n_1,n_2,\ldots,n_m}) = \begin{bmatrix}
2(J_{n_1} - I_{n_1}) & J_{n_1 \times n_2} & \cdots & J_{n_1 \times n_m} \\
J_{n_2 \times n_1} & 2(J_{n_2} - I_{n_2}) & \cdots & J_{n_2 \times n_m} \\
\vdots & \vdots & \ddots & \vdots \\
J_{n_m \times n_1} & J_{n_m \times n_2} & \cdots & 2(J_{n_m} - I_{n_m})
\end{bmatrix}.
\]

We begin with few lemmas which will be used to compute the determinant of \( D(K_{n_1,n_2,\ldots,n_m}) \).

**Lemma 2.1.** Let \( A_m \) be square matrix of order \( m \) and is of the following form
\[
A_m = \begin{bmatrix}
p_1 & p_1 & \cdots & p_1 \\
p_2 & 2(p_2 - 1) & \cdots & p_2 \\
\vdots & \vdots & \ddots & \vdots \\
p_m & p_m & \cdots & 2(p_m - 1)
\end{bmatrix}.
\]

Then, the determinant of \( A_m \) is given by
\[
\det A_m = p_1 \prod_{j \neq 1} (p_j - 2).
\]

**Proof.** By subtracting the first columns from all the other columns, the resulting matrix is of the following form:
\[
\begin{bmatrix}
p_1 & 0 & \cdots & 0 \\
p_2 & p_2 - 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
p_m & 0 & \cdots & p_m - 2
\end{bmatrix}
\]
and hence the result follows.

**Lemma 2.2.** Let \( B_m \) be a square matrix of order \( m \) and is of the following form
\[
B_m = \begin{bmatrix}
2(n_1 - 1) & n_1 & \cdots & n_1 \\
n_2 & 2(n_2 - 1) & \cdots & n_2 \\
\vdots & \vdots & \ddots & \vdots \\
n_m & n_m & \cdots & 2(n_m - 1)
\end{bmatrix}.
\]

The determinant of the above matrix is given by
\[
\det B_m = \sum_{i=1}^{m} \left( n_i \prod_{j \neq i} (n_j - 2) \right) + \prod_{i=1}^{m} (n_i - 2).
\]
Proof. We prove the lemma using induction on the order of the matrix. For \( m = 1 \), the result is true. Let us assume the result is true for matrix order \( m - 1 \) of similar form. Now expanding along the first row and using Lemma 2.1, we get

\[
\det B_m = 2(n_1 - 1) \det B_{m-1} - n_1 \sum_{i=2}^{m} n_i \prod_{j \neq i} (n_j - 2)
\]

and hence the result follows.

**Theorem 2.3.** Let \( D(K_{n_1,n_2,\ldots,n_m}) \) be the distance matrix of complete \( m \)-partite graph \( K_{n_1,n_2,\ldots,n_m} \) on \( |V| = \sum_{i=1}^{m} n_i \) vertices. Then the determinant of the distance matrix is given by

\[
\det D(K_{n_1,n_2,\ldots,n_m}) = (-2)^{|V|-m} \sum_{i=1}^{m} \left( n_i \prod_{j \neq i} (n_j - 2) + \prod_{i=1}^{m} (n_i - 2) \right).
\]

**Proof.** We begin with few elementary row and column operation on \( D(K_{n_1,n_2,\ldots,n_m}) \), the distance matrix with the block form in Eqn (2.1). First for each partition we subtract the first column from all other columns, then add all the rows to the first row. Further we shift the first column of all the \( m \)-partition to the first \( m \) columns and repeat the same operation for the rows. Then the resulting matrix is of the following block form:

\[
\begin{bmatrix}
B_m & 0 \\
* & -2^{|V|-m}
\end{bmatrix},
\]

where \( B_m \) is the matrix as defined in Eqn (2.2). Thus by Proposition 1.3 and Lemma 2.2 the result follows.

Next, we will prove a lemma which will help us to compute the cof \( D(K_{n_1,n_2,\ldots,n_m}) \).

**Lemma 2.4.** Let \( C_m \) be a square matrix of order \( m \) and is of the following form

\[
C_m = \begin{bmatrix}
n_1 & 2(n_1 - 1) & 2(n_1 - 1) & \cdots & 2(n_1 - 1) \\
n_2 & 2 & n_2 & \cdots & n_2 \\
n_3 & n_3 & 2 & \cdots & n_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n_m & n_m & n_m & \cdots & 2
\end{bmatrix}.
\]

The determinant of the above matrix is given by

\[
\det C_m = (-1)^{m-1} \sum_{i=1}^{m} \left( n_i \prod_{j \neq i} (n_j - 2) \right).
\]

**Proof.** Subtracting the first column form all the remaining columns of \( C_m \) yields the following matrix:

\[
\begin{bmatrix}
n_1 & n_1 - 2 & n_1 - 2 & \cdots & n_1 - 2 \\
n_2 & 2 - n_2 & 0 & \cdots & 0 \\
n_3 & 0 & 2 - n_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n_m & 0 & 0 & \cdots & 2 - n_m
\end{bmatrix}.
\]

Now expanding along the first row, we get

\[
\det C_m = n_1 \prod_{j=2}^{m} (2 - n_m) - (n_1 - 2) \sum_{i=2}^{m} n_i \prod_{j \neq i} (2 - n_j)
\]

and the desired result follows.
Theorem 2.5. Let \( D(K_{n_1,n_2,\ldots,n_m}) \) be the distance matrix of complete \( m \)-partite graph \( K_{n_1,n_2,\ldots,n_m} \) on \( |V| = \sum_{i=1}^{m} n_i \) vertices. Then the cofactor of the distance matrix is given by

\[
\text{cof } D(K_{n_1,n_2,\ldots,n_m}) = (-2)^{|V| - m} \left[ \sum_{i=1}^{m} \left( n_i \prod_{j \neq i} (n_j - 2) \right) \right].
\]

Proof. For complete \( m \)-partite graph \( K_{n_1,n_2,\ldots,n_m} \) with \( n_i = 1 \) for all \( 1 \leq i \leq m \), we have \( K_{n_1,n_2,\ldots,n_m} = K_m \). Then, the result is true as \( \text{cof } D(K_m) = (-1)^{m-1}m \). For other cases, without loss of generality, let \( n_1 > 1 \) and \( M \) be the matrix obtained from \( D(K_{n_1,n_2,\ldots,n_m}) \) by subtracting the first row from all other rows and then subtracting the first column from all other columns. Then the block form of the matrix \( M(1\mid 1) \) is given by

\[
\begin{bmatrix}
-2(J_{n_1-1} + I_{n_1-1}) & -2J_{(n_1-1)\times n_2} & \cdots & -2J_{(n_1-1)\times n_m} \\
-2J_{n_2\times (n_1-1)} & -2I_{n_2} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots \\
-2J_{n_m\times (n_1-1)} & -J_{n_m \times n_2} & \cdots & -2I_{n_m}
\end{bmatrix}
\]

First for each partition of \( M(1\mid 1) \) we subtract the first column from all other columns, then add all the rows to the first row. Further we shift the first column of all the \( m \)-partition to the first \( m \) columns and repeat the same operation for the rows. Then the resulting matrix is of the following block form:

\[
\begin{bmatrix}
-\tilde{C}_m & 0 \\
* & -2I_{|V|-(m+1)}
\end{bmatrix},
\]

where

\[
\tilde{C}_m =
\begin{bmatrix}
2n_1 & 2(n_1 - 1) & 2(n_1 - 1) & \cdots & 2(n_1 - 1) \\
2n_2 & 2 & n_2 & \cdots & n_2 \\
2n_3 & n_3 & 2 & \cdots & n_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
2n_m & n_m & n_m & \cdots & 2
\end{bmatrix}.
\]

Using the Proposition 1.3, Lemmas 1.1 and 2.2 the result follows. \( \square \)

Now we are interested for the cases in which the determinant and cofactor of the distance matrix of \( G = K_{n_1,n_2,\ldots,n_m} \) are zero. Unlike the case of complete bipartite graphs, for complete \( m \)-partite graphs (with \( m \geq 3 \)) the determinant and cofactor vanishes for infinitely many partitions. As an immediate consequence to Theorems 2.3 and 2.5, we first list out few observations below:

1. If \( n_i > 2 \), for all \( i = 1, 2, \ldots, m \), then both \( \det D(G) \) and \( \text{cof } D(G) \) are positive.
2. For \( 1 \leq i \leq m \), if at least two \( n_i \)’s are 2, then \( \det D(G) = \text{cof } D(G) = 0 \).
3. For \( 1 \leq i \leq m \), if exactly one \( n_i \) is 2, then \( \det D(G) = \text{cof } D(G) \neq 0 \).
4. If \( n_i = 1 \), for all \( i = 1, 2, \ldots, m \), then \( G = K_m \) and for \( m > 1 \), \( \det D(G), \text{cof } D(G) \neq 0 \).

Next, let \( G = K_{n_1,n_2,\ldots,n_m} \) with \( n_i \neq 2 \) for all \( 1 \leq i \leq m \). Then \( \det D(G) \) or \( \text{cof } D(G) \) is zero only if some \( n_i \)’s are 1 and we presented these cases as theorems. Before stating these theorems we first prove a simple lemma and introduce few notations useful for the subsequent results.

Lemma 2.6. For any \( p \in \mathbb{N} \), the sum

\[
\sum_{i=1}^{p} \frac{1}{q_i} = \frac{r}{2}, \quad \text{for all } 1 \leq r \leq 2p, \quad r \in \mathbb{N}.
\]

have non-negative integer solution.
Proof. It is easy to see non-negative integer solution to the above sum may not be unique. For the sake completeness we provide one such set of solution.

If $r$ is even, that is, $r \in \{2, 4, \cdots, 2p\}$ choose $q_i = 1, 1 \leq i \leq \frac{r}{2} - 1$ and $q_i = p - \frac{r}{2} + 1, \frac{r}{2} \leq i \leq p$. Next if $r$ is odd, if $r = 1$ choose $q_i = 2p, 1 \leq i \leq p$, otherwise if $r \in \{3, 5, \cdots, 2p-1\}$ choose $q_i = 1, 1 \leq i \leq \frac{r-1}{2} - 1$ and $q_i = p - \frac{r-1}{2}, \frac{r-1}{2} \leq i \leq p$. \hfill $\square$

Let $n_i \in \mathbb{N}, 1 \leq i \leq m$ and let us denote

\[
\begin{cases}
\beta_{n_1 n_2 \cdots n_m} = \sum_{i=1}^{m} n_i^{j} \prod_{j \neq i} (n_j - 2) + \prod_{i=1}^{m} (n_i - 2), \\
\beta_{n_i} = \beta_{n_1 n_2 \cdots n_i-1 n_i+1 \cdots n_m},
\end{cases}
\]

and

\[
\begin{cases}
\gamma_{n_1 n_2 \cdots n_m} = \sum_{i=1}^{m} n_i^{j} \prod_{j \neq i} (n_j - 2), \\
\gamma_{n_i} = \gamma_{n_1 n_2 \cdots n_i-1 n_i+1 \cdots n_m}.
\end{cases}
\]

**Theorem 2.7.** Let $m, l \in \mathbb{N}$ and $l < m$.

(a) Let $G = K_{n_1, n_2, \cdots, n_m}$ with $n_i = 1, 1 \leq i \leq l$ and $n_i > 2, l + 1 \leq i \leq m$. If $\det(D(G)) = 0$, then

\[
\frac{m+1}{2} < l \leq \frac{3m+1}{4}.
\]

(b) Let $\frac{m+1}{2} < l \leq \frac{3m+1}{4}$. There exists $n_i > 2, l + 1 \leq i \leq m$ such that

\[
\det(D(G)) = 0, \text{ where } G = K_{n_1, n_2, \cdots, n_m} \text{ with } n_i = 1, 1 \leq i \leq l.
\]

Moreover, $2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2} = 2l - (m + 1)$.

**Proof.** Let $n_i = 1, 1 \leq i \leq l$ and $n_i > 2, l + 1 \leq i \leq m$. Using Eqn. (2.3), we get

\[
\beta_{n_1 n_2 \cdots n_m} = (-1)^l \prod_{i=l+1}^{m} (n_i - 2) \left[ 2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2} + (m + 1 - 2l) \right].
\]

Thus, by Theorem 2.3 we have $\det(D(G)) = 0$ iff $2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2} = 2l - (m + 1)$. Using $n_i > 2$ for $l + 1 \leq i \leq m$, we get $\frac{m + 1}{2} < l \leq \frac{3m + 1}{4}$. This proves (a). In view of Eqn. (2.5), to prove (b) it is enough to solve

\[
2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2} = 2l - (m + 1), \text{ whenever } \frac{m + 1}{2} < l \leq \frac{3m + 1}{4}.
\]

Therefore, the result follows from Lemma 2.6. \hfill $\square$

**Theorem 2.8.** Let $m, l \in \mathbb{N}$ and $l < m$.

(a) Let $G = K_{n_1, n_2, \cdots, n_m}$ with $n_i = 1, 1 \leq i \leq l$ and $n_i > 2, l + 1 \leq i \leq m$. If $\text{cof } D(G) = 0$, then

\[
\frac{m}{2} < l \leq \frac{3m}{4}.
\]
(b) Let \( \frac{m}{2} < l \leq \frac{3m}{4} \). There exists \( n_i > 2, l + 1 \leq i \leq m \) such that
\[
\text{cof } D(G) = 0, \text{ where } G = K_{n_1,n_2,\ldots,n_m} \text{ with } n_i = 1, 1 \leq i \leq l.
\]
Moreover, \( 2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2} = 2l - m \).

Proof. Let \( n_i = 1, 1 \leq i \leq l \) and \( n_i > 2, l + 1 \leq i \leq m \). Using Eqn. (2.4), we get
\[
\gamma_{n_1n_2\ldots n_m} = (-1)^l \prod_{i=l+1}^{m} (n_i - 2) \left[ 2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2} + (m - 2l) \right].
\]
(2.6)
Thus, by Theorem 2.5 we have \( \text{cof } D(G) = 0 \) if \( 2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2} = 2l - m \). In view of Eqn. (2.6), proceeding similar to the proof of Theorem 2.7 yields the result.

The next theorem gives us the inverse of the distance matrix of complete \( m \)-partite graph \( K_{n_1,n_2,\ldots,n_m} \) whenever it exists. We obtained this result using induction on \( m \) and Schur complement technique. We omit the proof as it involves long calculations, but it can be verified that the matrix given below is indeed the inverse by multiplying with the distance matrix.

**Theorem 2.9.** Let \( D(K_{n_1,n_2,\ldots,n_m}) \) be the distance matrix of complete \( m \)-partite graph \( K_{n_1,n_2,\ldots,n_m} \). If \( \det D(K_{n_1,n_2,\ldots,n_m}) \neq 0 \), then the inverse in \( m \times m \) block form is given by \( D(K_{n_1,n_2,\ldots,n_m})^{-1} = [\tilde{D}_{ij}] \), where
\[
\tilde{D}_{ij} = \begin{cases} 
\frac{2\beta_{n_1\ldots n_m} - \gamma_{n_i}}{2\Lambda_{n_1\ldots n_m}} J_{n_1} - \frac{1}{2} I_{n_i} & \text{if } i = j; \\
\prod_{l \neq i,j} (n_l - 2) \beta_{n_1\ldots n_m} J_{n_i \times n_j} & \text{if } i \neq j.
\end{cases}
\]

3 Distance Matrix of Multi-block Graphs

In this section, we aim to compute the determinant and inverse for multi-block graphs subject to the condition that \( \text{cof } D(G) \neq 0 \). Recall that, a connected graph is said to be a multi-block graph if each of its blocks is a complete \( m \)-partite graph, whenever \( m \geq 2 \). We will use induction on the number of blocks to achieve our goal, so we first define a few notions on single blocks and then extends these notions to multi-block graphs to prove the requisite results.

If \( G \) is a complete \( m \)-partite graph \( K_{n_1,n_2,\ldots,n_m} \) with \( \text{cof } D(G) \neq 0 \), then we define
\[
\lambda_G = \frac{\det D(G)}{\text{cof } D(G)} = \frac{\beta_{n_1n_2\ldots n_m}}{\gamma_{n_1n_2\ldots n_m}}.
\]
For \( m = 2 \), if \( K_{n_1,n_2} \) for \( (n_1,n_2) \neq (2,2) \) then \( \lambda_G > 0 \). Next let \( m \geq 3 \) and \( G = K_{n_1,n_2,\ldots,n_m} \). Then it can be seen that \( \lambda_G < 0 \), only if some of the \( n_i = 1 \). If \( G = K_m \), then \( \lambda_G > 0 \). So let \( l < m \) and considering \( G = K_{n_1,n_2,\ldots,n_m} \) with \( n_i = 1, 1 \leq i \leq l \) and \( n_i > 2, l + 1 \leq i \leq m \), we have
\[
\lambda_G = 1 + \frac{1}{(m - 2l) + 2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2}}.
\]
Using Theorems 2.7 and 2.8, let \((p_{l+1},\ldots,p_m)\) and \((p'_{l+1},\ldots,p'_m)\) be a solution, whenever the sum \((m - 2l) + 2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2}\) takes the value 0 and 1 respectively. Then
1. If $1 \leq t \leq \frac{m}{2}$ or $\frac{3m+1}{4} < l \leq m$, then $\lambda_G > 0$.

2. For $\frac{m}{2} < l \leq \frac{m+1}{2}$, if $(n_{t+1}, \ldots, n_m) \neq (p_{t+1}, \ldots, p_m)$ with $p_i \leq n_i$, then $\lambda_G < 0$.

3. For $\frac{3m}{4} < l \leq \frac{3m+1}{4}$, if $(n_{t+1}, \ldots, n_m) \neq (p_{t+1}, \ldots, p_m)$ with $2 < n_i \leq p_i$, then $\lambda_G < 0$.

4. For $\frac{m+1}{2} < l \leq \frac{3m}{4}$, by choosing solutions as in the proof of Lemma 2.6, it is possible to find $(p_{t+1}, \ldots, p_m)$ and $(p_{t+1}', \ldots, p_m')$ such that $p_i \leq p_i'$, for all $l + 1 \leq i \leq m$. Let

$$R = \{(n_{t+1}, \ldots, n_m) \mid p_i \leq n_i \leq p_i' \text{ for } 1 + 1 \leq i \leq m\}.$$ 

If $(n_{t+1}, \ldots, n_m) \in R \setminus \{(p_{t+1}, \ldots, p_m), (p_{t+1}', \ldots, p_m')\}$, then $\lambda_G < 0$.

**Remark 3.1.** For $m \geq 3$, there are infinitely many complete $m$-partite graph $G$ such that $\lambda_G = 0$, $\lambda_G > 0$ and as well as $\lambda_G < 0$.

In particular for $m = 3$, $\det D(G) = 0$ iff $\lambda_G = 0$ iff $G \cong K_{2,2,n}$ where $n \in \mathbb{N}$. Using Theorem 2.8, we have $\det D(G) = 0$ iff $G \cong K_{1,1,4}$ or $K_{2,2,n}$ where $n \in \mathbb{N}$. The complete tripartite graph $K_{1,1,n-1}$ appeared in the study of completely positive graphs, denoted by $T_n$ (for details see [3]) and for notational convenience we are continuing with $T_n$ instead of $K_{1,1,n-1}$.

For $n \geq 3$, $T_n$ can seen as a graph consisting of $(n - 2)$ triangles with a common base (see Figure 1). Note that $\det D(T_6) = 0$ and $\lambda_{T_n} = -\frac{2}{n-6}$. This implies that $\lambda_{T_n} < 0$ iff $G = T_n, n \geq 7$.

![Figure 1: $T_4, T_5, T_6$](image)

Let $G = (V, E)$ be a $m$-block graph on $|V|$ vertices with $G_t, 1 \leq t \leq b$. If $\det D(G_t) \neq 0$ for all $1 \leq t \leq b$, then define

$$\lambda_G = \sum_{t=1}^{b} \lambda_{G_t}. \quad (3.1)$$

The following result gives us the determinant of the distance matrix of multi-block graphs. We omit the proof as it follows from Theorem 1.2.

**Theorem 3.2.** Let $G = (V, E)$ be a multi-block graph with $b$ blocks. If $G$ contains $b_m$ blocks of complete $m$-partite graph i.e. $K_{n_1^{(j)}, n_2^{(j)}, \ldots, n_m^{(j)}}$, for $1 \leq j \leq b_m$ and $b = \sum_{m \geq 2} b_m$, then

$$\det D(G) = \lambda_G \left[ (-2)^{|V|+(b-1)-\sum_{m \geq 2} m b_m} \prod_{m \geq 2} \prod_{j=1}^{b_m} \gamma_{n_1^{(j)}, n_2^{(j)}, \ldots, n_m^{(j)}} \right].$$

Note that, in the above theorem the sum and product are finite as $G$ contains only finitely many blocks. By Theorem 3.2, given a multi-block graph $\det D(G) = 0$ iff $\lambda_G = 0$. By Remark 3.1 and Eqn. (3.1), we can choose choose the blocks such that $\lambda_G = 0$. Two such possible rearrangement is given in the following examples.
Example 3.3. Let $G$ be a multi-block graph with blocks $G_i = T_{n_i}$, $(n_i \neq 6)$ for all $1 \leq i \leq b$. Suppose $G$ contain $b_1$ blocks of $T_3$, $b_2$ blocks of $T_4$ and $b_3$ blocks of $T_5$. Now for each $T_3$ we can associate $x$ blocks of $T_n$, where $n = 3x + 6$. Then

$$\lambda_{T_3} + x\lambda_{T_n} = \frac{2}{3} - \frac{2x}{3x + 6} - 6 = 0.$$ 

Similarly, for each $T_4$ we can associate $y$ blocks of $T_n$, where $n = 2y + 6$ and for each $T_5$ we can associate $z$ blocks of $T_n$, where $n = z + 6$. Then $\lambda_G = \sum_i \lambda_{G_i} = 0$ and $D(G)$ is not invertible. Consequently, for different choices $x, y$ and $z$, we can generate infinitely many multi-block graphs $G$ (where $\text{cof} D(G_i) \neq 0; \forall i$ ) with $\det D(G) = 0$.

Example 3.4. Let $G$ be a multi-block graph consisting of a $K_4$, $K_{m,m}$ ($m \neq 2$) and $b$ blocks of $T_n$, where $b = 9m - 4$, $n = 6 + 8m$. Then

$$\lambda_G = \lambda_{K_4} + \lambda_{K_{m,m}} + b\lambda_{T_n} = \frac{2}{3} + \frac{3m - 2}{2m} - \frac{2b}{n - 6} = \frac{9m - 4}{4m} - \frac{2(9m - 4)}{6 + 8m} - 6 = 0,$$

and $D(G)$ is not invertible. Consequently for different values of $m \in \mathbb{N} \setminus \{2\}$, it is possible to produce infinitely many multi-block graphs $G$ (where $\text{cof} D(G_i) \neq 0; \forall i$ ) with $\det D(G) = 0$.

Let $G = (V, E)$ is a complete $m$-partite graph $K_{n_1,n_2,\ldots,n_m}$ with $\text{cof} D(G) \neq 0$. If a vertex $v$ of $G$, then we define a $|V|$-dimensional column vector $\mu_G$ as follows:

$$\mu_G(v) = \frac{1}{\gamma_{n_1,n_2,\ldots,n_m}} \sum_{i=1}^{m} \sum_{v \in V, j \neq i} \prod (n_j - 2)$$

Remark 3.5. Notice that, If $G = K_{n_1,n_2,\ldots,n_m}$, then

$$\sum_{v \in V(G)} \mu_G(v) = \frac{1}{\gamma_{n_1,n_2,\ldots,n_m}} \sum_{i=1}^{m} n_i \prod (n_j - 2) = 1.$$

Next, we extend the definition of $\mu_G$ for multi-block graphs. Let $G = (V, E)$ be a multi-block graph on $|V|$ vertices with blocks $G_t$, $1 \leq t \leq b$ such that $\text{cof} D(G_t) \neq 0$ for $1 \leq t \leq b$. Let $\mu_G$ (we will use $\mu$ if there is no scope of confusion) be a $|V|$-dimensional column vector defined as follows. If a vertex $v$ belongs to $k$ many blocks of $G$, then

$$\mu_G(v) = \sum_{t=1}^{b} \mu_{G_t}(v) - (k - 1). \quad (3.2)$$

Before proceeding further, it is easy to observe that the definition of $\lambda_G$ and $\mu$ as defined in Eqns (3.1) and (3.2) agrees with the same, in [2, 10]. The result below gives a property of $\mu$ needed in our subsequent results.

Lemma 3.6. Let $G = (V, E)$ be a multi-block graph and $\mu$ be the column vector defined in Eqn (3.2). Then

$$\sum_{v \in V} \mu(v) = 1.$$
Further, suitably rearranging the vertex indexing, the distance matrix result is true for all multi-block graph with \( b \) blocks. Let \( m \) be the number of blocks to prove this result. It is easy to check the result is true whenever \( b = 1 \). Assume the result is true for all multi-block graph with \((b-1)\) blocks. Let \( H \) be any leaf block connected to the graph \( G \) at a cut vertex \( c \in V(G) \) and \( F = G - (H - \{c\}) \) be the graph obtained from \( G \) by removing \( H - \{c\} \). By induction hypothesis, the results holds good for the multi-block graph \( F \) with \((b-1)\) blocks, that is \( D(F) \mu = \lambda_F \mathbb{1} \). If \( H = K_{n_1,n_2} \), then the result follows from [10, Lemma 5]. Let \( m \geq 3 \) and \( H = K_{n_1,n_2,...,n_m} \) with the \( m \)-partition of the vertex set as \( V_{n_i} \), \( 1 \leq i \leq m \). If \( H = K_m \), then the result follows from [2, Lemma 1]. For other cases with \( m \geq 3 \), without loss of generality assume \( n_1 > 1 \) and \( c \in V_{n_1} \). Then

\[
\mu_G(c) = \mu_F(c) + \frac{\prod_{j \geq 2}(n_j - 2)}{\gamma_{n_1n_2...n_m}} - 1.
\]

Therefore,

\[
\sum_{v \in V} \mu(v) = \sum_{v \in V(F - \{c\})} \mu(v) + \sum_{v \in V(H - \{c\})} \mu(v) + \mu(c)
\]
\[
= \sum_{v \in V(F)} \mu_F(v) + \frac{\prod_{j \geq 2}(n_j - 2)}{\gamma_{n_1n_2...n_m}} - 1 + \sum_{v \in V(H - \{c\})} \mu(v)
\]
\[
= \sum_{v \in V(F)} \mu_F(v) + \sum_{i=1}^{m} \frac{n_i \prod_{j \neq i}(n_j - 2)}{\gamma_{n_1n_2...n_m}} - 1
\]
\[
= 1.
\]

\( \square \)

**Lemma 3.7.** Let \( D(G) \) be the distance matrix of multi-block graph \( G \). Then \( D(G) \mu = \lambda_G \mathbb{1} \), where \( \lambda_G \) and \( \mu \) as defined in Eqns (3.1) and (3.2) respectively.

**Proof.** Let \( G \) be a multi-block graph with \( b \) number of blocks. We will use induction on the number of blocks to prove this result. It is easy to check the result is true whenever \( b = 1 \). Assume the result is true for all multi-block graph with \((b-1)\) blocks. Let \( H \) be any leaf block connected to the graph \( G \) at a cut vertex \( c \in V(G) \) and \( F = G - (H - \{c\}) \) be the graph obtained from \( G \) by removing \( H - \{c\} \). By induction hypothesis, the results holds good for the multi-block graph \( F \) with \((b-1)\) blocks, that is \( D(F) \mu = \lambda_F \mathbb{1} \). If \( H = K_{n_1,n_2} \), then the result follows from [10, Lemma 6]. Let \( m \geq 3 \) and \( H = K_{n_1,n_2,...,n_m} \) with the \( m \)-partition of the vertex set as \( V_{n_i} \), \( 1 \leq i \leq m \). If \( H = K_m \), then the result follows from [2, Lemma 2]. For other cases with \( m \geq 3 \), without loss of generality assume \( n_1 > 1 \) and \( c \in V_{n_1} \).

Let \( \rho = D(F)e_c \), where \( e_c \) denote the column vector with 1 at the \( c \)th entry, then \( \rho(v) = d(v,c) \).

Further, suitably rearranging the vertex indexing, the distance matrix \( D(G) \) can written as the following block matrix form:

\[
D(G) = \begin{bmatrix}
D(F) & \frac{\rho + 2\mathbb{1}^t}{\gamma_{n_1n_2...n_m}} & \frac{(\rho + 1)\mathbb{1}^t}{\gamma_{n_1n_2...n_m}} & \cdots & \frac{(\rho + 1)\mathbb{1}^t}{\gamma_{n_1n_2...n_m}} \\
\frac{\rho^t + 2\mathbb{1}}{\gamma_{n_1n_2...n_m}} & 2(J_{n_1-1} - I_{n_1-1}) & J_{(n_1-1)\times n_2} & \cdots & J_{(n_1-1)\times n_m} \\
\frac{\rho^t + 1\mathbb{1}}{\gamma_{n_1n_2...n_m}} & J_{n_2\times(n_1-1)} & 2(J_{n_2} - I_{n_2}) & \cdots & J_{n_2\times n_m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\rho^t + 1\mathbb{1}}{\gamma_{n_1n_2...n_m}} & J_{n_m\times(n_1-1)} & J_{n_m\times n_2} & \cdots & 2(J_{n_m} - I_{n_m})
\end{bmatrix}
\]

(3.3)

Let \( D(G) \mu = \nu \) and \( R_v \) be the \( v \)th row of \( D(F) \). Note that \( R_v(c) = \rho(v) \). For \( v \in F \), we have

\[
\nu(v) = \langle R_v, \mu_F \rangle + \left( \frac{\prod_{j \geq 2}(n_j - 2)}{\gamma_{n_1n_2...n_m}} - 1 \right) R_v(c) + (n_1 - 1)(\rho(v) + 2) \frac{\prod_{j \geq 2}(n_j - 2)}{\gamma_{n_1n_2...n_m}}
\]
\[
+ (\rho(v) + 1)(n_1 - 2) \sum_{i \geq 2} \frac{n_i \prod_{j \neq i}(n_j - 2)}{\gamma_{n_1n_2...n_m}}.
\]
Simplifying we get \( \nu(v) = \langle R_v, \mu_F \rangle + \frac{\beta_{n_1n_2...n_m}}{\gamma_{n_1n_2...n_m}} = \lambda_G \), where \( \beta_{n_1n_2...n_m} \) and \( \gamma_{n_1n_2...n_m} \) are the constants defined in Eqns (2.3) and (2.4) respectively. Now for \( v \in H - \{c\} \) and \( v \in V_{n_i} \), we have

\[
\nu(v) = \langle \rho', 2 L, \mu_F \rangle + 2 \left( \prod_{j \geq 2} (n_j - 2) \frac{\gamma_{n_1n_2...n_m}}{\gamma_{n_1n_2...n_m}} \right) + 2(n_1 - 2) \prod_{j \geq 2} (n_j - 2) \frac{\gamma_{n_1n_2...n_m}}{\gamma_{n_1n_2...n_m}} \\
+ (n_1 - 2) \sum_{i \geq 2} n_i \frac{\prod_{j \neq i} (n_j - 2)}{\gamma_{n_1n_2...n_m}} 
\]

Similarly it follows for the cases when \( v \in H - \{c\} \) and \( v \in V_{n_i} \) for \( 2 \leq i \leq m \). Thus for all \( v \in V \) we have \( \nu(v) = \lambda_G \). Hence the result follows.

Next, we defined Laplacian-like matrix complete \( m \)-partite graphs. For \( m = 2 \), the Laplacian-like matrix \( L(K_{n_1,n_2}) = [L(K_{n_1,n_2})_{uv}] \) for \( K_{n_1,n_2} \), where \( (n_1, n_2) \neq (2, 2) \) as defined in [10, Eqn.(4)]. For \( m \geq 3 \), given a complete \( m \)-partite graph \( K_{n_1,n_2,...,n_m} \) with \( m \)-partition of the vertex set as \( V_{n_i} \) for \( 1 \leq i \leq m \) respectively, we define a matrix \( L = L(K_{n_1,n_2,...,n_m}) = [L_{uv}] \), called Laplacian-like matrix of \( K_{n_1,n_2,...,n_m} \), where

\[
L_{uv} = \begin{cases} 
\frac{(n_i - 1)\beta_{n_i}}{2\gamma_{n_1n_2...n_m}} - 2\gamma_{n_i} & \text{if } u = v, u \in V_{n_i}, \text{for } 1 \leq i \leq m; \\
\frac{-\beta_{n_i}}{2\gamma_{n_1n_2...n_m}} & \text{if } u \neq v, u, v \in V_{n_i}, \text{for } 1 \leq i \leq m; \\
\frac{\prod_{j \neq i} (n_j - 2)}{\gamma_{n_1n_2...n_m}} & \text{if } u \sim v, u \in V_{n_i}, v \in V_{n_j}, \text{for } 1 \leq i, j \leq m.
\end{cases}
\]

For the single blocks, it can be easily checked that all row sums and column sums of \( L \) are zero.

Let \( G = (V, E) \) be a multi-block graph on \( |V| \) vertices with blocks \( G_t = (V_t, E_t), 1 \leq t \leq b \). Each block \( G_t \) is also considered as a graph on vertex set \( V \) with perhaps isolated vertices, and let its edge set be \( E_t \) (i.e, \( G_t \) is a graph on \( |V_t| \) vertices, consider it as a graph on vertex set \( V \)). Let \( L \) be the \( |V| \times |V| \) matrix defined as above for the vertices of \( G_t \) and 0 for others. Define

\[
L = \sum_{t=1}^{b} L(G_t).
\]

It can be seen that \( L_{uv} = 0 \), if \( u \) and \( v \) are not in the same block and \( L1 = 1^t L = 0 \).

**Lemma 3.8.** Let \( L \) be the Laplacian-like matrix and \( D(G) \) be the distance matrix of a multi-block graph \( G \) then, \( LD(G) + I = \mu 1^t \), where \( \mu \) is as defined in Eqns (3.2).

**Proof.** We will use induction on \( b \) the number of blocks of the graph \( G \) to prove the lemma. For \( b = 1 \), it is easy to show that the base case \( LD(G) + I = \mu 1^t \) whenever \( G \) is either a complete graph \( K_n \) or a complete \( m \)-partite graph \( G = K_{n_1,n_2,...,n_m} \) with \( m \geq 2 \). Assume that the result is true for any multi-block graph with \( (b - 1) \) blocks. Let \( G \) be a multi-block graph with \( b \) blocks. Let \( H \) be any leaf block connected to the graph \( G \) at a cut vertex \( c \in V(G) \) and \( F = G - (H - \{c\}) \) be the graph obtained from \( G \) by removing \( H - \{c\} \). By induction hypothesis, the results holds good for the multi-block graph \( F \) with \( (b - 1) \) blocks, that is, \( L_F D(F) + I_F = \mu_F 1^t \), where the notations are clear from the context.
If \( H = K_{n_1,n_2} \), then the result follows from [10, Lemma 6]. Let \( m \geq 3 \) and \( H = K_{n_1,n_2,\ldots,n_m} \) with the \( m \)-partition of the vertex set as \( V_n \), \( 1 \leq i \leq m \). If \( H = K_m \), then the result follows from [2, Lemma 2]. For other cases with \( m \geq 3 \), without loss of generality assume \( n_1 > 1 \) and \( c \in V_n \). Labelling the vertices suitably we have the Laplacian-like matrix \( L \) of \( G \) in the following block form:

\[
L = \begin{bmatrix}
L_F + a_1 e_c e^t_c & b_1 e_c e^t & \cdots & b_{1m} e_c e^t \\
b_1 e^t_c & b_1 J_{n_1-1} + (a_1 - b_1) I_{n_1-1} & \cdots & c_{1m} J_{n_1-1} x n_m \\
c_{21} e^t_c & c_{21} J_{n_2} x (n_1 - 1) & b_2 J_{n_2} + (a_2 - b_2) I_{n_2} & \cdots & c_{2m} J_{n_2} x n_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{m1} e^t_c & c_{m1} J_{r x (p-1)} & c_{m2} J_{r x q} & \cdots & b_m J_m + (a_m - b_m) I_m 
\end{bmatrix}
\]

Let \( \rho \) be the column vector defined by \( \rho = D(F)e_c \), that is, \( \rho(v) = d(v,c) \). Considering the distance matrix \( D(G) \) as in Eqn. (3.3), we need the following cases to complete the proof.

**Case 1:** \( u \in F - \{c\} \):
If \( v \in F \), then by the induction hypothesis gives us

\[
(LD(G) + I)_{uv} = (L_F D(F) + I_F)_{uv} = \mu_F(u) = \mu(u).
\]

Let \( R_u \) denotes the \( u \)th row of \( L_F \). Then \( (R_u, 1) = 0 \) and the induction hypothesis yields:

If \( v \in H - \{c\} \) and \( u \in V_n \), then

\[
(LD(G) + I)_{uv} = LD(G)_{uv} = (R_u, \rho + 21) = \mu_F(u) = \mu(u).
\]

For \( 2 \leq i \leq m \), if \( v \in H - \{c\} \) and \( u \in V_n \), then

\[
(LD(G) + I)_{uv} = LD(G)_{uv} = (R_u, \rho + 1) = \mu_F(u) = \mu(u).
\]

**Case 2:** \( u \in H - \{c\} \):

**Subcase 2.1:** \( u \in V_n - \{c\} \),
If \( v = u \), then

\[
(LD(G) + I)_{uv} = 1 + LD(G)_{uu} = 1 + 2(n_1 - 1)b_1 + \sum_{j \geq 2} n_j c_{1j} = \mu(u).
\]

If \( v \neq u \) and \( v \in V_n \), then

\[
(LD(G) + I)_{uv} = LD(G)_{uv} = 2a_1 + 2(n_1 - 2)b_1 + \sum_{j \geq 2} n_j c_{1j} = \mu(u).
\]

If \( v \in V_n \), for \( 2 \leq i \leq m \), then

\[
(LD(G) + I)_{uv} = LD(G)_{uv} = a_1 + (n_1 - 1)b_1 + 2(n_i - 2)c_{1i} + \sum_{j \geq 2, j \neq i} n_j c_{1j} = \mu(u).
\]

If \( v \in F - \{c\} \), then

\[
(LD(G) + I)_{uv} = LD(G)_{uv} = a_1 (\rho(v) + 2) + (n_1 - 2)(\rho(v) + 2)b_1 + \rho(v)b_1 + (\rho(v) + 1) \sum_{j \geq 2} n_j c_{1j} = \mu(u).
\]

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Similar calculations yields the result for sub cases \( u \in V_{n_i} \), for \( 2 \leq i \leq m \).

**Case 3: \( u = c \):**

**Subcase 3.1: \( v = c \):**

\[(\mathcal{L}D(G) + I)_{cc} = 1 + \mathcal{L}D(G)_{cc} = 1 + \sum_{w \in F} \mathcal{L}_{cw} \rho(w) + 2b_1(n_1 - 1) + \sum_{j \geq 2} n_j c_{1j} \]

\[
= \sum_{w \in F} \mathcal{L}_{cw} \rho(w) + \prod_{j \geq 2} (n_j - 2) \\
= \mu_F(c) + \prod_{j \geq 2} (n_j - 2) - 1 = \mu(c). 
\]

**Subcase 3.2: \( v \in H - \{c\} \):**

If \( v \in V_{n_1} \), then

\[
(\mathcal{L}D(G) + I)_{cv} = \mathcal{L}D(G)_{cv} = \sum_{w \in F} \mathcal{L}_{cw} (\rho(w) + 2) + 2a_1 + 2b_1(n_1 - 2) + \sum_{j \geq 2} n_j c_{1j} \\
= \sum_{w \in F} \mathcal{L}_{cw} \rho(w) + \prod_{j \geq 2} (n_j - 2) \\
= \mu_F(c) + \prod_{j \geq 2} (n_j - 2) - 1 = \mu(c). 
\]

If \( v \in V_{n_2} \), then

\[
(\mathcal{L}D(G) + I)_{cv} = \mathcal{L}D(G)_{cv} = \sum_{w \in F} \mathcal{L}_{cw} (\rho(w) + 1) + a_1 + b_1(n_1 - 1) + 2(n_2 - 1)c_{12} + \sum_{j \geq 2} n_j c_{1j} \\
= \sum_{w \in F} \mathcal{L}_{cw} \rho(w) + \prod_{j \geq 2} (n_j - 2) \\
= \mu_F(c) + \prod_{j \geq 2} (n_j - 2) - 1 = \mu(c). 
\]

And similar calculations yields the result for the sub cases whenever \( v \in V_{n_i} \), for \( 3 \leq i \leq m \).

**Subcase 3.3: \( v \in F - \{c\} \):**

\[
(\mathcal{L}D(G) + I)_{cv} = \sum_{w \in F} \mathcal{L}_{cw} D(G)_{wv} + a_1 D(G)_{cv} + (\rho(v) + 2)b_1(n_1 - 1) + (\rho(v) + 1) \sum_{j \geq 2} n_j c_{1j} \\
= \sum_{w \in F} \mathcal{L}_{cw} D(G)_{wv} + \rho(v) \left[ a_1 + b_1(n_1 - 1) + \sum_{j \geq 2} n_j c_{1j} \right] + 2b_1(n_1 - 1) + \sum_{j \geq 2} n_j c_{1j} \\
= \sum_{w \in F} \mathcal{L}_{cw} D(G)_{wv} + 2b_1(n_1 - 1) + \sum_{j \geq 2} n_j c_{1j} \\
= \sum_{w \in F} \mathcal{L}_{cw} D(G)_{wv} + \prod_{j \geq 2} (n_j - 2) - 1 = \mu(c). 
\]

Hence combining all the cases, the result follows.

**Theorem 3.9.** Let \( G \) be multi-block graph with blocks \( G_t, 1 \leq t \leq b \). Let \( \mathcal{L} \) be the Laplacian-like matrix and \( D(G) \) be the distance matrix of \( G \). If \( \det D(G) \neq 0 \) and \( \cof D(G_i) \neq 0, 1 \leq i \leq b \), then

\[
D(G)^{-1} = -\mathcal{L} + \frac{1}{\lambda_G} \mu t, 
\]

where \( \lambda_G \) and \( \mu \) are as defined in Eqns (3.1) and (3.2) respectively.
Proof. By Lemma 3.7, we have \( \mu^t D(G) = \lambda_G \mathbf{I}^t \), so \( \mu \mu^t D(G) = \lambda_G \mu \mathbf{I}^t \). By Theorem 3.2, \( \det D(G) \neq 0 \) implies that \( \lambda_G \neq 0 \). Thus using Lemma 3.8, we have \( \mathbf{L} D(G) + \mu \mathbf{I}^t = \frac{1}{\lambda_G} \mu \mu^t D(G) \). Hence the result follows.

Notice that, the strategy adopted in this section is not applicable for the graphs with one of its blocks is of cofactor zero. The next section deals one such class of multi-block containing exactly one block as \( T_6 \).

4 Determinant and Inverse of \( D(T_6 \otimes T_n^{(b)}) \)

Let \( T_6 \otimes T_n^{(b)} \) denote the multi-block graph with exactly one block as \( T_6 \) and \( b \geq 1 \) blocks of \( T_n \), for a fixed \( n \), where \( n \neq 6 \) and with a central cut vertex, which is not a base vertex (Figures 2 and 3). Let \( D(T_6 \otimes T_n^{(b)}) \) be the distance matrix \( T_6 \otimes T_n^{(b)} \). Using Theorem 3.2, the \( \det D(T_6 \otimes T_n^{(b)}) \neq 0 \) and hence invertible. For sake of completeness we state the result for determinant without proof.

**Theorem 4.1.** Let \( D(T_6 \otimes T_n^{(b)}) \) be the distance matrix of the graph \( T_6 \otimes T_n^{(b)} \). Then, the determinant of \( D(T_6 \otimes T_n^{(b)}) \) is given by

\[
\det D(T_6 \otimes T_n^{(b)}) = (-1)^{nb+1} 2^{(n-3)b+4}(n-6)^b.
\]

![Figure 2: \( T_6 \otimes T_3^{(3)} \)](image)

![Figure 3: \( T_6 \otimes T_3^{(3)} \)](image)

We have found that the inverse of \( D(T_6 \otimes T_n^{(b)}) \) is of a similar form as in [4, Theorem 3.3]. To be specific, we found a matrix \( \mathcal{R} \) such that \( D(T_6 \otimes T_n^{(b)})^{-1} \) is a linear combination of Laplacian matrix \( \mathbf{L}(T_6 \otimes T_n^{(b)}) \), rank one matrix \( \mathbf{J} \) and the matrix \( \mathcal{R} \). For notational convenience, we write \( D \) and \( \mathbf{L} \) to denote \( D(T_6 \otimes T_n^{(b)}) \) and \( \mathbf{L}(T_6 \otimes T_n^{(b)}) \) respectively. With suitable choice of vertex indexing, the block form of \( D \) and \( \mathbf{L} \) can be written as:

\[
D = \begin{bmatrix}
D_1 & D_2 & D_2 & \cdots & D_2 & d_3 \\
D_2 & D_4 & D_5 & \cdots & D_5 & d_6 \\
D_2 & D_5 & D_4 & \cdots & D_5 & d_6 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
D_2 & D_5 & D_5 & \cdots & D_4 & d_6 \\
d_3 & d_6^t & d_6^t & \cdots & d_6^t & 0
\end{bmatrix}, \quad
\mathbf{L} = \begin{bmatrix}
L_1 & 0 & 0 & \cdots & 0 & l_1 \\
0 & L_2 & 0 & \cdots & 0 & l_2 \\
0 & 0 & L_2 & \cdots & 0 & l_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & L_2 & l_2 \\
l_1 & l_2 & l_2 & \cdots & l_2 & 2(b+1)
\end{bmatrix},
\]

(4.1)

where

\[
D_1 = \begin{bmatrix}
J_2 - I_2 & J_2 \times 3 \\
J_3 \times 2 & 2(J_3 - I_3)
\end{bmatrix}, \quad
D_2 = \begin{bmatrix}
2J_2 & 3J_2 \times (n-3) \\
3J_3 \times 2 & 4J_3 \times (n-3)
\end{bmatrix}, \quad
d_3 = \begin{bmatrix}
\mathbf{I}_2 \\
2\mathbf{I}_3
\end{bmatrix},
\]

\[
D_4 = \begin{bmatrix}
J_2 - I_2 & J_2 \times (n-3) \\
J_3 \times (n-3) & 2(J_{n-3} - I_{n-3})
\end{bmatrix}, \quad
D_5 = \begin{bmatrix}
2J_2 & 3J_2 \times (n-3) \\
3J_3 \times (n-3) & 4J_{n-3}
\end{bmatrix}
\]

and \( d_6 = \begin{bmatrix}
\mathbf{I}_2 \\
2\mathbf{I}_{n-3}
\end{bmatrix} \).
\[ L_1 = \begin{bmatrix} 6I_2 - J_2 & -J_{2 \times 3} \\ -J_{3 \times 2} & 2I_3 \end{bmatrix}, \quad L_2 = \begin{bmatrix} nI_2 - J_2 & -J_{2 \times (n-3)} \\ -J_{(n-3) \times 2} & 2I_{n-3} \end{bmatrix}, \quad l_1 = \begin{bmatrix} -1_2 \\ 0_{3 \times 1} \end{bmatrix} \quad \text{and} \quad l_2 = \begin{bmatrix} -1_2 \\ 0_{(n-3) \times 1} \end{bmatrix}. \]

**Remark 4.2.** Note that for \( n = 3 \), some of the above block matrices contain submatrices of the form \( A_{p \times 0} \) or \( A_{0 \times p} \) and in these cases we will consider such matrices do not exist. For example with \( n = 3 \), \( D_2 = \begin{bmatrix} 2J_2 \\ 3J_{3 \times 2} \end{bmatrix} \), \( D_4 = J_2 - I_2 \), \( D_5 = 2J_2 \), \( d_6 = 1_2 \) etc. and we follow similar convention for rest of the section.

Next, we define the matrix \( \mathcal{R} \), whose block matrix is given by

\[
\mathcal{R} = \begin{bmatrix}
R_1 & R_2 & R_3 & \cdots & R_2 & r_1 \\
R_2 & R_3 & R_4 & \cdots & R_4 & r_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
R_2 & R_4 & R_3 & \cdots & R_3 & r_2 \\
R_1 & r_2 & r_2 & \cdots & r_2 & r_3 \\
\end{bmatrix}, \quad \text{where} \quad (4.2)
\]

\[
R_1 = \begin{bmatrix}
(4b(b+1) - (3b+4)(n-6))J_2 + 4(n-6)(b+1)I_2 \\
-2b(b+1) + (n-6)J_{3 \times 2} \\
(n-2 - n-6)(b+1)J_2 + (n-2)(n-6)I_2 \\
-2b(b+1) + (n-6)J_{3 \times 2} \\
(n-2 - n-6)(b+1)J_{3 \times 2} + (n-6)I_{3 \times 3}
\end{bmatrix}
\]

\[
R_2 = \begin{bmatrix}
-n-4 + n-6 \\
-4 + n-6 \\
-n-4 + n-6 \\
-4 + n-6 \\
-n-4 + n-6
\end{bmatrix}
\]

\[
R_3 = (b+1)\begin{bmatrix}
J_{2 \times (n-3)} \\
J_{2 \times (n-3)} \\
J_{2 \times (n-3)} \\
J_{2 \times (n-3)} \\
J_{2 \times (n-3)}
\end{bmatrix}
\]

\[
r_1 = (n-6)\begin{bmatrix}
2b(b+1) - 1_2 \\
-(b+1) + 1_3
\end{bmatrix}
\quad \text{and} \quad r_2 = -(n-6)1_{(n-1)}.
\]

Now we state the result which gives the inverse of \( D \), whenever \( n \neq 6 \) and \( b \geq 1 \). The proof is mostly computational and hence omitted (for details of the proof see Appendix A).

**Theorem 4.3.** Let \( b \geq 1 \) and \( n \neq 6 \). Let \( D \) be the distance matrix of \( T_b \circ T_n^{(b)} \). Then the inverse of \( D \) is given by

\[
D^{-1} = -\frac{1}{2}L + \frac{1}{2(b+1)}J + \frac{1}{2(b+1)(n-6)}\mathcal{R},
\]

where \( J \) is the matrix of all ones of conformal order, \( L \) and \( \mathcal{R} \) are the matrices as defined in Eqn (4.1) and Eqn (4.2) respectively.

### 5 Conclusion

In this article, we first compute the determinant and cofactor of the distance matrix for complete \( m \)-partite graphs \( (m \geq 3) \) and find its inverse whenever it exists. Unlike the case of complete bipartite graphs, for \( m \geq 3 \), the determinant and cofactor can be zero for infinitely many complete \( m \)-partite graphs and provide an equivalent condition to determine the same. Next, we consider the distance matrix of multi-block graphs with non-zero cofactor. For this case, if inverse exists, we find the inverse as a rank one perturbation of a multiple of the Laplacian-like matrix similar to trees, block graphs and bi-block graphs. We also provide the inverse of the distance matrix for a class of multi-block graphs with cofactor zero. Consequently, as a special case to multi-block graphs, we compute the determinant and inverse of the distance matrix for a class completely positive graphs, which improves the class studied in [4].
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Appendices

A Proof of the Theorem 4.3

Let us denote

\[ C = \frac{1}{2}L(T_6 \otimes T_n^{(b)}) + \frac{1}{2(b+1)}J + \frac{1}{2(b+1)(n-6)}R(T_6 \otimes T_n^{(b)}), \]

whose block form is given by

\[
\begin{bmatrix}
C_1 & C_2 & C_2 & C_2 & \cdots & C_2 & c_3 \\
C_2^T & C_4 & 0 & 0 & \cdots & 0 & c_6 \\
C_2^T & 0 & C_4 & 0 & \cdots & 0 & c_6 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
C_2^T & 0 & 0 & 0 & \cdots & C_4 & c_6 \\
c_3^T & c_6^T & c_6^T & c_6^T & \cdots & c_6^T & c_7 \\
\end{bmatrix}
\]

where,

\[
C_1 = \begin{bmatrix}
\frac{2(2b-(n-6))J_2 - 2(n-6)J_2}{-(2b-(n-6))J_{3 \times 2}} & \frac{-(2b-(n-6))J_{2 \times 3}}{bJ_3 - (n-6)I_3} \\
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
-\frac{2(n-4)J_2}{(n-4)J_{3 \times 2}} & \frac{2J_{2 \times (n-3)}}{-J_{3 \times (n-3)}} \\
\end{bmatrix},
\]

\[
C_4 = \begin{bmatrix}
\frac{2(n-4)J_2 - 2(n-6)J_2}{-2J_{(n-3) \times 2}} & -\frac{2J_{2 \times (n-3)}}{J_{n-3} - (n-6)I_{n-3}} \\
\end{bmatrix}, \quad c_3 = (n-6) \begin{bmatrix}
\frac{(2b+1)I_2}{bI_3} \\
\end{bmatrix},
\]

\[
c_6 = \begin{bmatrix}
(n-6)I_2 \\
0_{(n-3) \times 1} \\
\end{bmatrix} \quad \text{and} \quad c_7 = -(n-6)(3b+1).
\]
Consider the block matrix $Y = DC = (Y_{ij})$ of dimension $b+2$, where $Y_{ij}$ are block matrices of conformal order given by

$$Y_{ij} = \begin{cases} 
D_1 C_1 + bD_2 C_1^t + d_3 c_3^t, & \text{if } i = j = 1, \\
D_1 C_2 + D_2 C_4 + d_3 c_6^t, & \text{if } i = 1, j = 2, 3, ..., b + 1, \\
D_1 c_3 + bD_2 c_3 + c_7 d_3, & \text{if } i = 1, j = b + 2, \\
D_2 C_1 + D_4 C_2 + (b-1)D_5 C_2^t + d_6 c_4^t, & \text{if } i = 2, 3, ..., b + 1, j = 1, \\
D_2 C_2 + D_4 C_4 + d_6 c_6^t, & \text{if } i = j, i, j = 2, 3, ..., b + 1, \\
D_2 c_3 + D_4 c_6 + (b-1)D_5 c_6 + c_7 d_3, & \text{if } i = 2, 3, ..., b + 1, j = b + 2, \\
d_3^t C_1 + b d_1^t C_2, & \text{if } i = b + 2, j = 1, \\
d_3^t C_2 + d_5^t C_1, & \text{if } i = b + 2, j = 2, 3, ..., b + 1, \\
d_3^t c_3 + b d_1^t c_6, & \text{if } i = j = b + 2.
\end{cases}$$  \tag{A.2}

We will show $Y = I$ to complete the proof. For the sake of simplicity, we compute $Y_{ij}$ in different steps.

**Step 1 :** $Y_{ij}$, for $i = j = 1$

Note that,

$$D_1 C_1 = \begin{pmatrix}
-(2b + (n - 6)) J_2 + 2(n - 6) J_2 \\
-2(n - 6) J_3 \\
\end{pmatrix}, \quad d_3 c_3^t = (n - 6) \begin{pmatrix}
(2b + 1) J_2 \\
-2(2b + 1) J_3 \\
\end{pmatrix},$$

and

$$D_2 C_2 = \begin{pmatrix}
(n - 4) J_2 \\
-2(n - 5) J_2 \\
\end{pmatrix}, \quad D_2 C_4 = \begin{pmatrix}
-2(n - 5) J_2 \\
-2(n - 6) J_3 \\
\end{pmatrix}, \quad c_7 d_3 = -(n - 6)(3b + 1) \begin{pmatrix}
I_2 \\
3I_3 \\
\end{pmatrix}.$$

Thus, for $i = j = 1$, we have

$$Y_{ij} = \frac{1}{2(n - 6)} [D_1 C_1 + bD_2 C_1^t + d_3 c_3^t] = I_5.$$

**Step 2 :** $Y_{ij}$, for $i = 1, j = 2, 3, ..., b + 1$

Now, $D_1 C_2 = \begin{pmatrix}
(n - 4) J_2 \\
-2(n - 6) J_3 \\
\end{pmatrix}$, $D_2 C_1 = \begin{pmatrix}
J_2 \\
2(n - 6) J_3 \\
\end{pmatrix}$, and $D_2 C_6 = \begin{pmatrix}
-2(n - 5) J_2 \\
-2(n - 6) J_3 \\
\end{pmatrix}$ and

$d_3 c_6^t = (n - 6) \begin{pmatrix}
J_2 \\
2(n - 6) J_3 \\
\end{pmatrix}$. Thus, for $i = 1, j = 2, 3, ..., b + 1$, we have

$$Y_{ij} = \frac{1}{2(n - 6)} [D_1 C_2 + D_2 C_4 + d_3 c_6^t] = 0_{5 \times (n-1)}.$$

**Step 3 :** $Y_{ij}$, for $i = 1, j = b + 2$

Now, $D_1 c_3 = (n - 6) \begin{pmatrix}
-(b - 1) I_2 \\
2I_3 \\
\end{pmatrix}$, $D_2 c_6 = 2(n - 6) \begin{pmatrix}
2I_2 \\
3I_3 \\
\end{pmatrix}$ and $c_7 d_3 = -(n - 6)(3b + 1) \begin{pmatrix}
I_2 \\
2I_3 \\
\end{pmatrix}$.

Thus, for $i = 1, j = b + 2$, we have

$$Y_{ij} = \frac{1}{2(n - 6)} [D_1 c_3 + bD_2 c_6 + c_7 d_3] = 0_{5 \times 1}.$$

**Step 4 :** $Y_{ij}$, for $i = 2, 3, ..., b + 1, j = 1$

Next, $D_2^t C_1 = \begin{pmatrix}
-3(n - 6) + 2b J_2 \\
-6(n - 6) J_{(n-3)\times 2} \\
\end{pmatrix}$, $D_4 C_2 = \begin{pmatrix}
2J_2 \\
0_{(n-3)\times 2} \\
\end{pmatrix}$, and

$$D_5 C_2 = \begin{pmatrix}
-2(n - 7) J_2 \\
-4(n - 6) J_{(n-3)\times 2} \\
\end{pmatrix}, \quad d_6 c_3^t = (n - 6) \begin{pmatrix}
2(2b + 1) J_2 \\
-2b J_{(n-3)\times 3} \\
\end{pmatrix}$. Thus, for $i = 2, 3, ..., b + 1, j = 1$, we have

$$Y_{ij} = \frac{1}{2(n - 6)} [D_5^t C_1 + D_4^t C_2^t + (b-1)D_5 C_2^t + d_6 c_3^t] = 0_{(n-1)\times 5}.$$
Step 5: \( Y_{ij}, \text{for } i = j, i, j = 2, 3, ..., b + 1 \)

Next, \( D_2 C_2 = \begin{pmatrix} (n - 4) J_2 & -J_{2 \times (n-3)} \\ 0_{(n-3) \times 2} & 0_{(n-3)} \end{pmatrix} \), \( d_6 c_6^i = (n - 6) \begin{pmatrix} J_2 \\ 2J_{(n-3) \times 2} \\ 0_{(n-3)} \end{pmatrix} \) and

\[
D_4 C_4 = \begin{pmatrix} 4 - 2(n - 3) J_2 + 2(n - 6) J_2 & J_{2 \times (n-3)} \\ -2(n - 6) J_{2 \times (n-3)} & 2(n - 6) J_{(n-3)} \end{pmatrix} .
\]

Thus, for \( i = j = 2, 3, ..., b + 1 \), we have

\[
Y_{ij} = \frac{1}{2(n-6)} [D_2 C_2 + D_4 C_4 + d_6 c_6^i] = I_{(n-1)}.
\]

Step 6: \( Y_{ij}, \text{for } i \neq j, i, j = 2, 3, ..., b + 1 \)

Now, \( D_5 C_4 = \begin{pmatrix} -2(n - 5) J_2 \\ -2(n - 6) J_{2 \times (n-3)} \end{pmatrix} \). Thus, using calculations from Step 5, we have

\[
Y_{ij} = \frac{1}{2(n-6)} [D_2 C_2 + D_5 C_4 + d_6 c_6^i] = 0_{(n-1)}.
\]

Step 7: \( Y_{ij}, \text{for } i = 2, 3, ..., b + 1, j = b + 2 \)

Next, \( D_2 c_3 = (n - 6) \begin{pmatrix} -(b - 4) I_2 \\ 6 I_{(n-3)} \end{pmatrix} \), \( D_4 c_6 = (n - 6) \begin{pmatrix} I_2 \\ 2 I_{(n-3)} \end{pmatrix} \) and

\[
D_5 c_6 = 2(n - 6) \begin{pmatrix} 2 I_2 \\ 3 I_{(n-3)} \end{pmatrix} .
\]

Thus, for \( i = 2, 3, ..., b + 1, j = b + 2 \), we have

\[
Y_{ij} = \frac{1}{2(n-6)} [D_2 c_3 + D_4 c_6 + (b - 1) D_5 c_6 + c_7 d_3] = 0_{(n-1) \times 1}.
\]

Step 8: \( Y_{ij}, \text{for } i = b + 2, j = 1 \)

Here, \( d_3 c_1 = \begin{pmatrix} -4b I_2 \\ 2b I_{(n-3)} \end{pmatrix} \) and \( d_6 c_6^2 = \begin{pmatrix} 4 I_2 \\ -2 I_{(n-3)} \end{pmatrix} \). Thus, for \( i = b + 2, j = 1 \), we have

\[
Y_{ij} = \frac{1}{2(n-6)} [d_3 c_1 + b d_6 c_6^2] = 0_{1 \times 5}.
\]

Step 9: \( Y_{ij}, \text{for } i = b + 2, j = 2, 3, ..., b + 1 \)

Next, \( d_3 c_2 = \begin{pmatrix} 2(n - 4) I_2 \\ -2 I_{(n-3)} \end{pmatrix} \) and \( d_6 c_6^1 = \begin{pmatrix} -2(n - 4) I_2 \\ 2 I_{(n-3)} \end{pmatrix} \). Thus, for \( i = b + 2, j = 2, 3, ..., b + 1 \), we have

\[
Y_{ij} = \frac{1}{2(n-6)} [d_3 c_2 + d_6 c_6^1] = 0_{1 \times (n-1)}.
\]

Step 10: \( Y_{ij}, \text{for } i = j = b + 2 \)

Now, \( d_3 c_3 = 2(n - 6)(2b + 1) - 6b(n - 6) \) and \( d_6 c_6 = 2(n - 6) \). Thus, for \( i = j = b + 2 \), we have

\[
Y_{ij} = \frac{1}{2(n-6)} [d_3 c_3 + b d_6 c_6] = 1.
\]

From the above calculations, it is clear that \( Y = I \) and hence, the desired result follows. \( \square \)