$w$-FP-PROJECTIVE MODULES AND DIMENSION

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Abstract. Let $R$ be a ring. An $R$-module $M$ is said to be an absolutely $w$-pure module if and only if $\text{Ext}^1_R(F, M)$ is a GV-torsion module for any finitely presented module $F$. In this paper, we introduce and study the concept of $w$-FP-projective module which is in some way a generalization of the notion of FP-projective module. An $R$-module $M$ is said to be $w$-FP-projective if $\text{Ext}^1_R(M, N) = 0$ for any absolutely $w$-pure module $N$. This new class of modules will be used to characterize (Noetherian) DW-rings. Hence, we introduce the $w$-FP-projective dimensions of modules and rings. The relations between the introduced dimensions and other (classical) homological dimensions are discussed. Illustrative examples are given.

1. Introduction

Throughout, all rings considered are commutative with unity and all modules are unital. Let $R$ be a ring and $M$ be an $R$-module. As usual, we use $\text{pd}_R(M)$, $\text{id}_R(M)$, and $\text{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of $M$, and $\text{wdim}(R)$ and $\text{gldim}(R)$ to denote, respectively, the weak and global homological dimensions of $R$.

Now, we review some definitions and notation. Let $J$ be an ideal of $R$. Following [8], $J$ is called a Glaz-Vasconcelos ideal (a GV-ideal for short) if $J$ is finitely generated and the natural homomorphism $\varphi: R \rightarrow J^* = \text{Hom}_R(J, R)$ is an isomorphism. Let $M$ be an $R$-module and define $\text{tor}_{GV}(M) = \{ x \in M \mid Jx = 0 \text{ for some } J \in GV(R) \}$, where $GV(R)$ is the set of GV-ideals of $R$. It is clear that $\text{tor}_{GV}(M)$ is a submodule of $M$. Now $M$ is said to be GV-torsion (resp., GV-torsion-free) if $\text{tor}_{GV}(M) = M$ (resp., $\text{tor}_{GV}(M) = 0$). A GV-torsion-free module $M$ is called a $w$-module if $\text{Ext}^1_R(R/J, M) = 0$ for any $J \in GV(R)$. Projective modules and reflexive modules are $w$-modules. In the recent paper [12], it was shown that flat modules are $w$-modules. The notion of $w$-modules was introduced firstly over a domain [15] in the study of Strong Mori domains and was extended to commutative rings with zero divisors in [8]. Let $w-\text{Max}(R)$ denote the set of maximal $w$-ideals of $R$, i.e., $w$-ideals of $R$ maximal among proper integral $w$-ideals of $R$. Following [8 Proposition 3.8], every maximal $w$-ideal is prime. For any GV-torsion-free module $M$,

$$M_w := \{ x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R) \}$$

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is a \(w\)-submodule of \(E(M)\) containing \(M\) and is called the \(w\)-envelope of \(M\), where \(E(M)\) denotes the injective hull of \(M\). It is clear that a \(GV\)-torsion-free module \(M\) is a \(w\)-module if and only if \(M_w = M\). Let \(M\) and \(N\) be \(R\)-modules and let \(f : M \to N\) be a homomorphism. Following [17], \(f\) is called a \(w\)-monomorphism (resp., \(w\)-epimorphism, \(w\)-isomorphism) if \(f_p : M \to N_p\) is a monomorphism (resp., an epimorphism, an isomorphism) for all \(p \in w\)-Max\((R)\). A sequence \(0 \to A \to B \to C \to 0\) of \(R\)-modules is said to be \(w\)-exact if \(0 \to A_p \to B_p \to C_p \to 0\) is exact for any \(p \in w\)-Max\((R)\). An \(R\)-module \(M\) is called a \(w\)-flat module if the induced map \(1 \otimes f : M \otimes A \to M \otimes B\) is a \(w\)-monomorphism for any \(w\)-monomorphism \(f : A \to B\). Certainly flat modules are \(w\)-flat, but the converse implication is not true in general.

Recall from [11] that an \(R\)-module \(A\) is called absolutely pure if \(A\) is a pure submodule in every \(R\)-module which contains \(A\) as a submodule. C. Megibben showed in [19], that an \(R\)-module \(A\) is absolutely pure if and only if \(\text{Ext}^1_R(N, A) = 0\) for every finitely presented \(R\)-module \(N\). Hence, an absolutely pure module is precisely an \(FP\)-injective module in [20]. For more details about absolutely pure (or \(FP\)-injective) modules, see [11, 19, 20, 18, 3]. In a very recent paper [4], the authors introduced the notion of absolutely \(w\)-pure modules as generalization of absolutely pure (\(FP\)-injective) modules in the sense of the \(w\)-operation theory. As in [3], a \(w\)-exact sequence of \(R\)-modules \(0 \to A \to B \to C \to 0\) is said to be \(w\)-pure exact if, for any \(R\)-module \(M\), the induced sequence \(0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0\) is \(w\)-exact. In particular, if \(A\) is a submodule of \(B\) and \(0 \to A \to B \to B/A \to 0\) is a \(w\)-pure exact sequence of \(R\)-modules, then \(A\) is said to be a \(w\)-pure submodule of \(B\). If \(A\) is a \(w\)-pure submodule in every \(R\)-module which contains \(A\) as a submodule, then \(A\) is said to be an absolutely \(w\)-pure module. Following [4, Theorem 2.6], an \(R\)-module \(A\) is absolutely \(w\)-pure if and only if \(\text{Ext}^1_R(N, A) = 0\) for every finitely presented \(R\)-module \(N\). In [11], Ding and Mao introduced and studied the notion of \(FP\)-projective dimension of modules and rings; the \(FP\)-projective dimension of an \(R\)-module \(M\), denoted by \(\text{fpd}_R(M)\), is the smallest positive integer \(n\) for which \(\text{Ext}^{n+1}_R(M, A) = 0\) for all absolutely pure (\(FP\)-injective) \(R\)-modules \(A\), and \(FP\)-projective dimension of \(R\), denoted by \(\text{fpD}(R)\), is defined as the supremum of the \(FP\)-projective dimensions of finitely generated \(R\)-modules. These dimensions measures how far away a finitely generated module is from being finitely presented, and how far away a ring is from being Noetherian.

In Section 2, we introduce the concept of \(w\)-\(FP\)-projective modules. Hence, we prove that a ring \(R\) is \(DW\) ([13]) if and only if every \(FP\)-projective \(R\)-module is \(w\)-\(FP\)-projective if and only if every finitely presented \(R\)-module is \(w\)-\(FP\)-projective, and \(R\) is a coherent \(DW\) ring if and only if every finitely generated ideal is \(w\)-\(FP\)-projective.

Section 3 deals with the \(w\)-\(FP\) projective dimension of modules and rings. After a routine study of these dimensions, we prove that \(R\) is a Noetherian \(DW\) ring if and only if every \(R\)-module is \(w\)-\(FP\)-projective and \(R\) is \(FP\)-hereditary \(DW\)-ring if and only if every submodule of projective \(R\)-module is \(w\)-\(FP\)-projective.

### 2. \(w\)-\(FP\)-projective modules

We start with the following definition.
Definition 2.1. An $R$-module $M$ is said to be $w$-FP-projective if \( \text{Ext}^1_R(M, A) = 0 \) for any absolutely $w$-pure $R$-module $A$.

Since every absolutely pure module is absolutely $w$-pure ([4, Corollary 2.7]), we have the following inclusions:

\{Projective modules\} $\subseteq$ \{w-FP-projective modules\} $\subseteq$ \{FP-projective modules\}

Recall that a ring $R$ is called a DW-ring if every ideal of $R$ is a $w$-ideal, or equivalently every maximal ideal of $R$ is $w$-ideal. Examples of DW-rings are Prüfer domains, domains with Krull dimension one, and rings with Krull dimension zero. Hence, it is clear that if $R$ is a DW-ring, then $w$-FP-projective $R$-modules are just the FP-projective $R$-modules. Moreover, using [4, Corollary 2.9], it is easy to see that over a von Neumann regular ring, the three classes of modules above coincide.

Remark 2.2. It is proved in [14] that a finitely generated $R$-module $M$ is finitely presented if and only if \( \text{Ext}^1_R(M, A) = 0 \) for any absolutely pure (FP-injective) $R$-module $A$. Thus, every finitely generated $w$-FP-projective $R$-module is finitely presented.

We need the following lemma.

Lemma 2.3. Every GV-torsion $R$-module is absolutely $w$-pure.

Proof. Let $A$ be an arbitrary $R$-module and $N$ be a finitely presented $R$-module. For any maximal $w$-ideal $p$ of $R$, the naturel homomorphism

\[ \theta : \text{Hom}_R(N, A)_p \to \text{Hom}_{R_p}(N_p, A_p) \]

induces a homomorphism

\[ \theta_1 : \text{Ext}^1_R(N, A)_p \to \text{Ext}^1_{R_p}(N_p, A_p) \]

Following [6, Proposition 1.10], $\theta_1$ is a monomorphism. Suppose that $A$ is a GV-torsion $R$-module. Then, we get $(\text{Ext}^1_R(N, A))_p = 0$ since $A_p = 0$ (by [6, Lemma 0.1]). Hence, $\text{Ext}^1_R(N, A)$ is GV-torsion (by [6, Lemma 0.1]). Consequently, $A$ is an absolutely $w$-pure $R$-module (by [4, Theorem 2.6])

The first main result of this paper characterizes DW-rings in terms of $w$-FP-projective $R$-modules.

Proposition 2.4. Let $R$ be a ring. Then the following conditions are equivalent:

1. Every finitely presented $R$-module is $w$-FP-projective.
2. Every FP-projective $R$-module is $w$-FP-projective.
3. $R$ is a DW-ring.

Proof. (3) $\Rightarrow$ (2) It is obvious and (2) $\Rightarrow$ (1) follows from the fact that finitely presented modules are always FP-projective. 

(1) $\Rightarrow$ (3) Since $R$ is not a DW-ring. Then, by [7, Theorem 6.3.12], there exist maximal ideal $m$ of $R$ which is not $w$-ideal, and so by [7, Theorem 6.2.9], $m_w = R$. Hence, by [7, Proposition 6.2.5], $R/m$ is a GV-torsion $R$-module (since $m$ is a GV-torsion-free $R$-module, and so $R/m$ is an absolutely $w$-pure $R$-module (by Lemma 2.3). Hence, by hypothesis, for any $I$ finitely generated ideal $I$ of $R$, we get $\text{Ext}^1_R(R/I, R/m) = 0$. Using [9, Lemma 3.1], we obtain that $\text{Tor}^1_R(R/I, R/m) = 0$, which means that $R/m$ is flat. Accordingly, $m$ is a $w$-ideal, and then $m_w = m$, a contradiction with $m_w = R$. Consequently, $R$ is a DW-ring.

□
Remark 2.5. Let \((R, m)\) be a regular local ring with \(\text{gldim}(R) = n\) \((n \geq 2)\). By \cite{[2]} Example 2.6, \(R\) is not DW ring. Hence, there exists an FP-projective \(R\)-module \(M\) which is not \(w\)-FP-projective.

Next, we give some characterizations of \(w\)-FP-projective modules.

**Proposition 2.6.** Let \(M\) be an \(R\)-module. Then the following are equivalent:

1. \(M\) is \(w\)-FP-projective.
2. \(M\) is projective with respect to every exact sequence \(0 \to A \to B \to C \to 0\), where \(A\) is absolutely \(w\)-pure.
3. \(P \otimes M\) is \(w\)-FP-projective for any projective \(R\)-module \(P\).
4. \(\text{Hom}(P, M)\) is \(w\)-FP-projective for any finitely generated projective \(R\)-module \(P\).

**Proof.** (1) \(\Leftrightarrow\) (2). It is straightforward.

(1) \(\Rightarrow\) (3). Let \(A\) be any absolutely \(w\)-pure \(R\)-module and \(P\) be a projective \(R\)-module. Following \cite{[7], Theorem 3.3.10}, we have the isomorphism:

\[
\text{Ext}^1_{R}(P \otimes M, A) \cong \text{Hom}(P, \text{Ext}^1_{R}(M, A)).
\]

Since \(M\) is \(w\)-FP-projective, we have \(\text{Ext}^1_{R}(M, A) = 0\). Thus, \(\text{Ext}^1_{R}(P \otimes M, A) = 0\), and so \(P \otimes M\) is \(w\)-FP-projective.

(1) \(\Rightarrow\) (4). Let \(A\) be any absolutely \(w\)-pure \(R\)-module and \(P\) be a finitely generated projective \(R\)-module. Using \cite{[7], Theorem 3.3.12], we have the isomorphism:

\[
\text{Ext}^1_{R}(\text{Hom}(P, M), A) \cong P \otimes \text{Ext}^1_{R}(M, A) = 0.
\]

Hence, \(\text{Hom}(P, M)\) is a \(w\)-FP-projective \(R\)-module.

(3) \(\Rightarrow\) (1) and (4) \(\Rightarrow\) (1). Follow by letting \(P = R\). \(\Box\)

Recall that a fractional ideal \(I\) of a domain \(R\) is said to be \(w\)-invertible if \((I^{-1})_w = R\). A domain \(R\) is said to be a Prüfer \(v\)-multiplication domain (\(PvMD\)) when any nonzero finitely generated ideal of \(R\) is \(w\)-invertible. Equivalently, \(R\) is a \(PvMD\) if and only if \(R_q\) is a valuation domain for any maximal \(w\)-ideal \(p\) of \(R\) (\cite{[24], Theorem 2.1}). The class of \(PvMDs\) strictly contains the classes of Prüfer domains, Krull domains, and integrally closed coherent domains.

**Proposition 2.7.** Let \(R\) be a \(PvMD\). Then \(pd_R(M) \leq 1\) for any \(w\)-FP-projective \(R\)-module \(M\).

**Proof.** Let \(M\) be a \(w\)-FP-projective \(R\)-module. Following \cite{[4], Theorem 2.10], every \(h\)-divisible \(R\)-module is absolutely \(w\)-pure. Hence, \(\text{Ext}^1_{R}(M, D) = 0\) for any \(h\)-divisible \(R\)-module \(D\). Hence, by \cite{[21], vii, Proposition 2.5], \(pd_R(M) \leq 1\), as desired. \(\Box\)

**Proposition 2.8.** If \(M\) is a \(w\)-FP-projective \(R\)-module and \(\text{Ext}^1_{R}(M, G) = 0\) for any \(GV\)-torsion-free \(R\)-module \(G\), then \(M\) is projective.

**Proof.** Let \(A\) be an arbitrary \(R\)-module. The exact sequence

\[
0 \to \text{tor}_{GV}(A) \to A \to A/\text{tor}_{GV}(A) \to 0
\]

gives rise to the exact sequence

\[
0 = \text{Ext}^1_{R}(M, \text{tor}_{GV}(A)) \to \text{Ext}^1_{R}(M, A) \to \text{Ext}^1_{R}(M, A/\text{tor}_{GV}(A)) = 0
\]

Thus \(\text{Ext}^1_{R}(M, A) = 0\), and so \(M\) is projective. \(\Box\)
Lemma 2.11. Let \((R, m)\) be a local ring which not DW-ring (for example, regular local rings \(R\) with \(\text{gldim}(R) = n\) \((n \geq 2)\)). Then every finitely generated \(w\)-FP-projective \(R\)-module \(M\) is free.

Proof. Let \(M\) be a finitely generated \(w\)-FP-projective \(R\)-module. As in the proof of Proposition 2.3, we obtain that \(\text{Tor}^1_R(M, R/m) = 0\). But \(M\) is finitely generated, and so finitely presented (by Remark 2.2). Hence, by [10, Lemma 2.5.8], \(M\) is projective. Consequently, \(M\) is free since \(R\) is local. \(\square\)

Proposition 2.10. The class of all \(w\)-FP-projective modules is closed under arbitrary direct sums and under direct summands.

Proof. Follows from [7, Theorem 3.3.9(2)]. \(\square\)

Recall that a ring \(R\) is called coherent if every finitely generated ideal of \(R\) is finitely presented.

Lemma 2.11. Let \(R\) be a coherent ring and \(A\) be an \(R\)-module. Then \(A\) is absolutely \(w\)-pure if and only if \(\text{Ext}^{n+1}_R(N, A)\) is a \(GV\)-torsion \(R\)-module for any finitely presented module \(N\) and any integer \(n \geq 0\).

Proof. \((\Rightarrow)\) Suppose that \(A\) is absolutely \(w\)-pure \(R\)-module and let \(N\) be a finitely presented \(R\)-module. The case \(n = 0\) is obvious. Hence, assume that \(n > 0\). Consider an exact sequence

\[
0 \to N' \to F_{n-1} \to \cdots \to F_0 \to N \to 0
\]

where \(F_0, \ldots, F_{n-1}\) are finitely generated free \(R\)-module and \(N'\) is finitely presented. Such sequence exists since \(R\) is coherent. Thus, \((\text{Ext}^{n+1}_R(N, A))_p \cong (\text{Ext}^{n}_R(N', A))_p = 0\) for any \(w\)-maximal ideal \(p\) of \(R\). So, \(\text{Ext}^{n+1}_R(N, A)\) is a \(GV\)-torsion \(R\)-module.

\((\Leftarrow)\) Clear. \(\square\)

Lemma 2.12. Let \(R\) be a coherent ring and \(0 \to A \to B \to C \to 0\) be an exact sequence of \(R\)-modules, where \(A\) is absolutely \(w\)-pure. Then, \(B\) is absolutely \(w\)-pure if and only if \(C\) is absolutely \(w\)-pure.

Proof. Let \(N\) be a finitely presented \(R\)-module. We have

\[
\text{Ext}^1_R(N, A) \to \text{Ext}^1_R(N, B) \to \text{Ext}^1_R(N, C) \to \text{Ext}^2_R(N, A).
\]

By Lemma 2.11 for any maximal \(w\)-ideal \(p\), we get

\[
0 = \text{Ext}^1_R(N, A)_p \to \text{Ext}^1_R(N, B)_p \to \text{Ext}^1_R(N, C)_p \to \text{Ext}^2_R(N, A)_p = 0.
\]

Thus, \(\text{Ext}^1_R(N, B)_p \cong \text{Ext}^1_R(N, C)_p\). So, \(\text{Ext}^1_R(N, B)\) is a \(GV\)-torsion \(R\)-module if and only if \(\text{Ext}^1_R(N, C)\) is a \(GV\)-torsion \(R\)-module. Thus, \(B\) is absolutely \(w\)-pure if and only if \(C\) is absolutely \(w\)-pure. \(\square\)

Proposition 2.13. Let \(R\) be a coherent ring and \(M\) be an \(R\)-module. Then the following are equivalent:

1. \(M\) is \(w\)-FP-projective.
2. \(\text{Ext}^{n+1}_R(M, A) = 0\) for any absolutely \(w\)-pure module \(A\) and any integer \(n \geq 0\).
Proof. (1) ⇒ (2). Let \( A \) be an absolutely \( w \)-pure \( R \)-module. The case \( n = 0 \) is obvious. So, we may assume \( n > 0 \). Consider an exact sequence

\[
0 \to A \to E^0 \to \cdots \to E^{n-1} \to A' \to 0
\]

where \( E^0, \ldots, E^{n-1} \) are injective \( R \)-modules. By Lemma 2.12, \( A' \) is absolutely \( w \)-pure. Hence, \( \text{Ext}^{n+1}_R(M, A) \cong \text{Ext}^1_R(M, A') = 0 \).

(2) ⇒ (1). Obvious. \( \square \)

**Proposition 2.14.** Let \( R \) be a coherent ring and \( 0 \to M'' \to M' \to M \to 0 \) be an exact sequence of \( R \)-modules, where \( M \) is \( w \)-FP-projective. Then, \( M' \) is \( w \)-FP-projective if and only if \( M'' \) is \( w \)-FP-projective.

Proof. Follows from Proposition 2.13. \( \square \)

We end this section with the following characterizations of a coherent \( DW \)-rings.

**Proposition 2.15.** Let \( R \) be a ring. Then the following are equivalent:

1. \( R \) is a coherent \( DW \)-ring.
2. Every finitely generated submodule of a projective \( R \)-module is \( w \)-FP-projective.
3. Every finitely generated ideal of \( R \) is \( w \)-FP-projective.

Proof. (1) ⇒ (2). Follows immediately from [12, Theorem 3.7] since, over a \( DW \)-ring, the classes of \( w \)-FP-projective modules and FP-projective modules coincide.

(2) ⇒ (3). Obvious.

(3) ⇒ (1). \( R \) is coherent by Remark 2.2. Assume that \( R \) is not a \( DW \)-ring. As in the proof of Proposition 2.4, there exist a maximal ideal \( \mathfrak{m} \) of \( R \) such that \( R/\mathfrak{m} \) is absolutely \( w \)-pure and \( \mathfrak{m}w = R \). So, for any finitely generated ideal \( I \) of \( R \), we have

\[
0 = \text{Ext}^1_R(I, R/\mathfrak{m}) \to \text{Ext}^2_R(R/I, R/\mathfrak{m}) \to \text{Ext}^2_R(R, R/\mathfrak{m}) = 0,
\]

and then \( \text{Ext}^2_R(R/I, R/\mathfrak{m}) = 0 \). By [9, Lemma 3.1], \( \text{Tor}^2_R(R/I, R/\mathfrak{m}) = 0 \), which means that \( \text{fd}_R(R/\mathfrak{m}) \leq 1 \). Then, \( \mathfrak{m} \) is flat, and so a \( w \)-ideal, a contradiction. \( \square \)

**Corollary 2.16.** Let \( R \) be a domain. Then \( R \) is a coherent \( DW \)-domain if and only if every finitely generated torsion-free \( R \)-module is \( w \)-FP-projective.

Proof. Following [7, Theorem 1.6.15], every finitely generated torsion-free \( R \)-module can be embedded in a finitely generated free module (since \( R \) is a domain). Hence, \( (\Rightarrow) \) follows immediately from Proposition 2.15. For \( (\Leftarrow) \), it suffices to see that since \( R \) is a domain, every ideal is torsion-free, and then use Proposition 2.15. \( \square \)

### 3. The \( w \)-FP-projective dimension of modules and rings

In this section, we introduce and investigate the \( w \)-FP-projective dimension of modules and rings.

**Definition 3.1.** Let \( R \) be a ring. For any \( R \)-module \( M \), the \( w \)-FP-projective dimension of \( M \), denoted by \( w \text{-fpd}_R(M) \), is the smallest integer \( n \geq 0 \) such that \( \text{Ext}^{n+1}_R(M, A) = 0 \) for any absolutely \( w \)-pure \( R \)-module \( A \). If no such integer exists, set \( w \text{-fpd}_R(M) = \infty \).

The \( w \)-FP-projective dimension of \( R \) is defined by:

\[
w \text{-fpD}(R) = \sup\{w \text{-fpd}_R(M) : M \text{ is finitely generated } R \text{-module}\}\]
Clearly, an $R$-module $M$ is $w$-FP-projective if and only if $w$-$\text{fpd}_R(M) = 0$, and $\text{fpd}_R(M) \leq w$-$\text{fpd}_R(M)$, with equality when $R$ is a DW-ring. However, this inequality may be strict (Remark 2.5). Also, $\text{fpD}(R) \leq w$-$\text{fpD}(R)$ with equality when $R$ is a DW-ring, and this inequality may be strict. To see that, consider a regular local ring $(R, m)$ with $\text{gldim}(R) = n$ $(n \geq 2)$. Since $R$ is Noetherian, we get $\text{fpD}(R) = 0$ (by [1, Proposition 2.6]). Moreover, by Remark 2.5 there exists an (FP-projective) $R$-module $M$ which is not $w$-FP-projective. Thus, $w$-$\text{fpD}(R) > 0$.

First, we give a description of the $w$-FP-Projective dimension of modules over coherent ring.

**Proposition 3.2.** Let $R$ be a coherent ring. The following statements are equivalent for an $R$-module $M$.

1. $w$-$\text{fpd}(M) \leq n$.
2. $\text{Ext}^{n+1}_R(M, A) = 0$ for any absolutely $w$-pure $R$-module $A$.
3. $\text{Ext}^{n+1}_R(M, A) = 0$ for any absolutely $w$-pure $R$-module $A$ and any $j \geq 1$.
4. If the sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is exact with $P_0, \cdots, P_{n-1}$ are $w$-FP-projective $R$-modules, then $P_n$ is $w$-FP-projective.
5. If the sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is exact with $P_0, \cdots, P_{n-1}$ are projective $R$-modules, then $P_n$ is $w$-FP-projective.
6. There exists an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ where each $P_i$ is $w$-FP-projective.

**Proof.** (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6). Trivial.

(1) $\Rightarrow$ (4). Let $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be an exact sequence of $R$-modules with $P_0, \cdots, P_{n-1}$ are $w$-FP-projective, and set $K_0 = \text{Ker}(P_0 \to M)$ and $K_i = \text{Ker}(P_i \to P_{i-1})$, where $i = 1, \cdots, n-1$. Using Proposition 2.13 we get

$$0 = \text{Ext}^{n+1}_R(M, A) \cong \text{Ext}^n_R(K_0, A) \cong \cdots \cong \text{Ext}^1_R(P_n, A)$$

for all absolutely $w$-pure $R$-module $A$. Thus, $P_n$ is $w$-FP-projective.

(6) $\Rightarrow$ (3). We proceed by induction on $n \geq 0$. For the $n = 0$, $M$ is $w$-FP-projective module and so (3) holds by proposition 2.13. If $n \geq 1$, then there is an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ where each $P_i$ is $w$-FP-projective. Set $K_0 = \text{Ker}(P_0 \to M)$. Then, we have the following exact sequences

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to K_0 \to 0$$

and

$$0 \to K_0 \to P_0 \to M \to 0$$

Hence, by induction $\text{Ext}^{n-1+j}_R(K_0, A) = 0$ for all absolutely $w$-pure $R$-module $A$ and all $j \geq 1$. Thus, $\text{Ext}^{n+1}_R(M, A) = 0$, and so we have the desired result. \qed

The proof of the next proposition is standard homological algebra. Thus we omit its proof.

**Proposition 3.3.** Let $R$ be a coherent ring and $0 \to M'' \to M' \to M \to 0$ be an exact sequence of $R$-modules. If two of $w$-$\text{fpd}_R(M'')$, $w$-$\text{fpd}_R(M')$ and $w$-$\text{fpd}_R(M)$ are finite, so is the third. Moreover

1. $w$-$\text{fpd}_R(M'') \leq \text{sup}\{w$-$\text{fpd}_R(M'), w$-$\text{fpd}_R(M) - 1\}$.
2. $w$-$\text{fpd}_R(M') \leq \text{sup}\{w$-$\text{fpd}_R(M''), w$-$\text{fpd}_R(M)\}$.
3. $w$-$\text{fpd}_R(M) \leq \text{sup}\{w$-$\text{fpd}_R(M''), w$-$\text{fpd}_R(M') + 1\}$.
Corollary 3.4. Let $R$ be a coherent ring and $0 \to M' \to M \to 0$ be an exact sequence of $R$-modules. If $M'$ is $w$-FP-projective and $w\text{-f}pd_R(M) > 0$, then $w\text{-f}pd_R(M) = w\text{-f}pd_R(M') + 1$.

Proposition 3.5. Let $R$ be a coherent ring and $\{M_i\}$ be a family of $R$-modules. Then $w\text{-f}pd_R(\bigoplus M_i) = \sup\{w\text{-f}pd_R(M_i)\}$.

Proof. The proof is straightforward. \qed

Proposition 3.6. Let $R$ be a ring and $n \geq 0$ be an integer. Then the following statements are equivalent:

1. $w\text{-fpD}(R) \leq n$.
2. $w\text{-fpd}(M) \leq n$ for all $R$-modules $M$.
3. $w\text{-fpd}(R/I) \leq n$ for all ideals $I$ of $R$.
4. $\id_R(A) \leq n$ for all absolutely $w$-pure $R$-modules $A$.

Consequently, we have

\[
\begin{align*}
w\text{-fpD}(R) &= \sup\{w\text{-fpd}_R(M) \mid M \text{ is an } R\text{-module}\} \\
&= \sup\{w\text{-fpd}_R(R/I) \mid I \text{ is an ideal of } R\} \\
&= \sup\{\id_R(A) \mid A \text{ is an absolutely } w\text{-pure } R\text{-module}\}
\end{align*}
\]

Proof. (2) $\Rightarrow$ (1) $\Rightarrow$ (3). Trivial.
(3) $\Rightarrow$ (4). Let $A$ be an absolutely $w$-pure $R$-module. For any ideal $I$ of $R$, we have $\Ext^{n+1}_R(R/I, A) = 0$. Thus, $\id_R(A) \leq n$.
(4) $\Rightarrow$ (2). Let $M$ be an $R$-module. For any absolutely $w$-pure $R$-module $A$, we have $\Ext^{n+1}_R(M, A) = 0$. Hence, $w\text{-fpd}(M) \leq n$. \qed

Note that Noetherian rings need not to be DW (for example, a regular ring with global dimension 2), and DW-rings need not to be Noetherian (for example, a non Noetherian von Neumann regular ring). Next, we show that rings $R$ with $w\text{-fpD}(R) = 0$ are exactly Noetherian DW-rings.

Proposition 3.7. Let $R$ be a ring. Then the following are equivalent:

1. $w\text{-fpD}(R) = 0$.
2. Every $R$-module is $w$-FP-projective.
3. $R/I$ is $w$-FP-projective for every ideal $I$ of $R$.
4. Every absolutely $w$-pure $R$-module is injective.
5. $R$ is Noetherian DW-ring.

Proof. The equivalence of (1), (2), (3), and (4) follows from Proposition 3.6.
(2) $\Leftrightarrow$ (5). Follows from Proposition 2.4 and [1] Proposition 2.6. \qed

Recall from [12], that a ring $R$ is said $FP$-hereditary if every ideal of $R$ is $FP$-projective. Note that $FP$-hereditary rings need not to be DW (for example, a non DW Noetherian ring), and DW-rings need not to be $FP$-hereditary (for example, a non coherent DW-ring). Next, we show that rings $R$ with $w\text{-fpD}(R) \leq 1$ are exactly $FP$-hereditary DW-rings.

Proposition 3.8. Let $R$ be a ring. Then the following are equivalent:

1. $w\text{-fpD}(R) \leq 1$.
2. Every submodule of $w$-FP-projective $R$-module is $w$-FP-projective.
3. Every submodule of projective $R$-module is $w$-FP-projective.
(4) \( I \) is \( w\)-FP-projective for every ideal \( I \) of \( R \).
(5) \( \text{id}_R(A) \leq 1 \) for all absolutely \( w \)-pure \( R \)-module \( A \).
(6) \( R \) is a (coherent) \( F\)P-hereditary \( DW \)-ring.

**Proof.** The implications (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) are obvious.
(1) \( \Leftrightarrow \) (5). By Proposition 3.6.
(4) \( \Rightarrow \) (5). Let \( A \) be an absolutely \( w \)-pure \( R \)-module and \( I \) be an ideal of \( R \). The exact sequence \( 0 \to I \to R \to R/I \to 0 \) gives rise to the exact sequence
\[
0 = \text{Ext}^1_R(I, A) \to \text{Ext}^2_R(R/I, A) \to \text{Ext}^2_R(R, A) = 0.
\]
Thus, \( \text{Ext}^2_R(R/I, A) = 0 \), and so \( \text{id}_R(A) \leq 1 \).
(5) \( \Rightarrow \) (4). Let \( I \) be an ideal of \( R \). For any absolutely \( w \)-pure \( R \)-module \( A \), we have
\[
0 = \text{Ext}^2_R(R/I, A) = \text{Ext}^1_R(I, A).
\]
Thus, \( I \) is \( w \)-FP-projective.
(4) \( \Rightarrow \) (6). By hypothesis, \( R \) is \( F\)P-hereditary. Now, by Proposition 2.15 \( \Rightarrow \) (6).
(6) \( \Rightarrow \) (2). By [12, Theorem 3.16], since the \( w\)-FP-projective \( R \)-modules are just the FP-projective \( R \)-modules over a \( DW \)-ring.

\[\blacksquare\]

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