BROWNSTEIN’S WHOLE-PARTIAL DERIVATIVES: THE CASE OF THE LORENTZ GAUGE

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Abstract

In this brief note we show that the usual Lorentz gauge is not satisfied by the Lienard-Wiechert potentials, then, using Brownstein’s concept of whole-partial derivatives we introduce the generalized expression for the Lorentz gauge showing that it is satisfied by the LW-potentials.

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I. INTRODUCTION

In the paper of 1997 [1] a way was proposed to generalize the Maxwell equations using *total (whole)* derivatives instead of partial ones. However, the concept of total derivative in some special cases was criticized and refined by K. R. Brownstein [2] which introduced and rationalized so-called “whole-partial” derivative [3]. Some related ideas were discussed by A. E. Chubykalo and R. A. Flores [4] trying to give a theoretical rationale for the correct use of total derivative concept in mathematical analysis.

In this paper we shall show that the Lorentz gauge as written in the form:

\[
\sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0 \quad \text{or} \quad \sum_{i=1}^{3} \varphi \frac{\partial \beta_i}{\partial x_i} + \sum_{i=1}^{3} \beta_i \frac{\partial \varphi}{\partial x_i} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0 \quad (1)
\]

is not satisfied when \( A_i \) and \( \varphi \) are given by the Liénard-Wiechert (LW) potentials:

\[
\varphi = \frac{q}{R - R\beta}, \quad A_i = \varphi \beta_i, \quad (2)
\]

where \( q \) means electric charge, \( R = \sqrt{\sum_{i=1}^{3}(x_i - x_{qi})^2} \), \( \beta = \frac{\mathbf{v}}{c} \), \( R = |R| = |r - r_q| \), \( x_i \) are coordinates of the observation point, and \( \mathbf{v} \) is the velocity of the particle with coordinates \( x_{qi} \) at the instant \( t' \) (the earlier time for which the time of propagation of the signal from the point \( r_q(t') \), where the charge \( q \) was located, to the observation point \( r \) just coincides with the difference \( t - t' \), \( t' \) is determined by the equation \( t' - t = R/c \)), and that the correct way to write down the Lorentz gauge is:

\[
\sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i} + \frac{1}{c} \frac{\partial \varphi}{\partial t} \hat{=} 0 \quad \text{or} \quad \sum_{i=1}^{3} \varphi \frac{\partial \beta_i}{\partial x_i} + \sum_{i=1}^{3} \beta_i \frac{\partial \varphi}{\partial x_i} + \frac{1}{c} \frac{\partial \varphi}{\partial t} \hat{=} 0 \quad (3)
\]

with Brownstein’s *whole-partial* derivatives that are noted by the symbol \( \hat{\partial} \).

The strategy that we shall use is to show, in section II, that a calculation
of the Lorentz gauge for the LW-potentials using the form (1) leads to an incorrect result. Then in section III we show that the calculation using the equations (3) gives the correct result. In section IV we comment on the possibility of saving the use of *usual* partial derivatives, and we enumerate the reasons to discard it. In section V we present the conclusions.

II. THE LW-POTENTIALS DO NOT SATISFY THE LORENTZ GAUGE.

It is standard knowledge that the Lorentz gauge is satisfied in general by the retarded potentials (solutions of unhomogeneous D’Alembert equations) as a consequence of charge conservation, hence, in particular for the LW-potentials this must be the case.

However, let us show that traditional wisdom is wrong. In order to do it we proceed to calculate partial derivatives of the LW-potentials:

\[ \frac{\partial \varphi}{\partial t} = -\frac{q}{(R - R\beta)^2} \left( \frac{\partial R}{\partial t} - \frac{\partial}{\partial t} (R\beta) \right), \quad (4) \]

obviously, \( R \) does not depend explicitly on \( t \) because by definition \( R = \sqrt{\sum(x_i - x_{qi})^2} \), so \( \frac{\partial R}{\partial t} = 0 \). However, the term \( \frac{\partial}{\partial t} (R\beta) \) also is zero because \( \beta \) does not depend on \( t \) explicitly: the point is that the variables like \( x_{qi}(t') \) and \( \beta_i[v_i(t')] \) depend on \( t, x_i \) implicitly by the function \( t'(x_i, t) \).

Then

\[ \frac{\partial \varphi}{\partial t} = 0. \quad (5) \]

In turn

\[ \frac{\partial \varphi}{\partial x_i} = -\frac{q}{(R - R\beta)^2} \left( \frac{\partial R}{\partial x_i} - \frac{\partial}{\partial x_i} (R\beta) \right) \quad (6) \]
\[ \frac{\partial R}{\partial x_i} = \frac{x_i - x_{qi}}{R} \quad \text{and} \quad \frac{\partial}{\partial x_i}(R\beta) = \beta_i. \quad (7) \]

Therefore as a final result we get:

\[ \frac{\partial \varphi}{\partial x_i} = -\frac{q}{(R - R\beta)^2} \left( \frac{x_i - x_{qi}}{R} - \beta_i \right). \quad (8) \]

With the derivatives (5) and (8) we can write for the Lorentz gauge given by Eq. (1) the following expression:

\[ -\frac{q}{(R - R\beta)^2} \left( \sum_{i=1}^{3} \left[ \frac{x_i - x_{qi}(t')}{} \right] - \beta_i \right) \quad (9) \]

which is not identically zero.

An objection can be raised against our procedure of calculation: the LW-potentials are retarded potentials, that is, functions evaluated at the retarded time \( t' \) which depends on variables \( x_1, x_2, x_3, t \) and which was obtained as an implicit function from the equation:

\[ \sum_{i=1}^{3} (x_i - x_{qi}(t'))^2 = c^2(t - t')^2, \quad (10) \]

therefore, the objection continues, there are missing terms and the calculation is wrong. However, we can offer the following answer, which picks up the crux of the problem: for the calculation of the usual partial derivatives we must first suppose that LW-potentials are not evaluated at the retarded time \( t' \), only after the calculation the retarded is introduced, in the following form:

\[ \left. \frac{\partial \varphi}{\partial x_i} \right|_{t'} \quad \text{and in this way for all partial derivatives.} \quad (11) \]

With the use of this rule, which is consistent with the generally accepted concept of partial derivatives, the LW-potentials do not satisfy the Lorentz
gauge, because as we can see from the expression (9), even if we could find the explicit form of $t'$, it could not change the form of (9) in such a way to get a zero identically. We must also note that it is not only necessary to know the form of $t'$, we need the explicit path of the particle too.

Of course, the idea to consider the potentials as functions not evaluated at the retarded time has to seem wrong. However, it is common knowledge the difficulty that the functions evaluated at the retarded time are not easy to partially differentiate. Just consider the following comments in order to sustain the assertion:

1. “...in differentiating them (the potentials) to obtain the fields it must be noted that derivatives with respect to the position of the field point must be taken at constant observation time, and derivatives with respect to the observation time at fixed field points. Since the retarded time appears explicitly in the potentials, care must be taken to obtain the correct derivatives” [5]

In the case of a charge in the uniform motion it is possible to eliminate the retarded time. However, this is not an option for the case of an non-uniformly moving point charge. So, as in the quoted cite we have to maintain constant the field point, while the observation time does not have to be constant.

Probably W. Panofsky and M. Phillips [6] give a better statement of the problem:

2. “Partial differentiation with respect to $x_\alpha$ compares the potentials at neighboring points at the same time, but these potential signals originated from the charge at different times.
Similarly, the partial derivative with respect to $t$ implies constant $x_\alpha$, and hence refers to the comparison of potentials at a given field point over an interval of time during which the coordinates of the source will have changed."

The solution which was used by W. Panofsky and M. Phillips was clear: a transformation of coordinates involving the transformation of the operator $\frac{\partial}{\partial t}$, the generator of the time translations. So, they changed the coordinates to a system where the concept of partial differentiation can be applied, however, the quoted cite shows that the real problem is the use of the concept of partial differentiation defined on a given coordinate cover of the underlying manifold. Why? Because as it has been stated, a function of the form $f\{x_i, x_q, [t'(x_i, t)]\}$ cannot be partially differentiated with respect to spatial coordinates. Indeed, how one can maintain constant the function $t'(x_i, t)$ while the $x_i$ are varying? The answer is clear: in the fixed coordinate cover this is not possible!

The potentials are retarded functions, so, it is not possible the use of a usual partial derivative in a direct manner as it is indicated in the usual statement of the Lorentz gauge (1) because a clear contradiction with the usual concept of the partial differentiation is involved. So a solution must be proposed, and this solution must take into account the fact that a real contradiction is involved with the concept of the partial differentiation.

To this point, we have showed that in the usual coordinate cover in which the potentials $A_i, \varphi$ are written the Lorentz gauge equation (1) is not satisfied, and the reason for this must be clear: a function of the form $f\{x_i, x_q, [t'(x_i, t)]\}$ has been subjected to a coordinate transformation involving the retarded time $t'$, and the partial derivative is not an invariant concept of the underlying manifold, so Maxwell’s equations are not invari-
ant in front of the general group of diffeomorphisms. Of course, the retarded time is an essential ingredient in the LW-potentials, so we must always take it into account but in the proper manner: recognizing the limitation in the use of partial derivatives and generalizing the expression (1).

Then we have come to these results:

A. The partial differentiation of functions like the retarded potentials is not possible, because the partial differentiation is not an invariant operator.

B. When it is possible it is generally wrong. One can consider that the use of just a $t$ and not a $t'$ in the functions $A_i, \varphi$ is an approximation: $t \approx t'$. The well-known dipole approximation \[7\] is $t' \approx t + \sqrt{\sum x_i^2(t)}$ and it is equally wrong.

As we shall show in the next section the adequate way to deal with the transformation of involved coordinates is the use of the “whole-partial derivative operator”. In this case a function like $f\{x_i, x_{qi}[t'(x_i, t)]\}$ has the following whole-partial derivatives:

\[
\begin{align*}
\hat{\partial}f/\hat{\partial}t &= \partial f/\partial t \bigg|_{t'} + \sum_i \partial f/\partial x_{qi} \bigg|_{t'} \frac{dx_{qi}}{dt'} \bigg|_{t'} \hat{\partial}t'/\hat{\partial}t' \bigg|_{t'}, \\
\hat{\partial}f/\hat{\partial}x_i &= \partial f/\partial x_i \bigg|_{t'} + \sum_i \partial f/\partial x_{qi} \bigg|_{t'} \frac{dx_{qi}}{dt'} \bigg|_{t'} \hat{\partial}t'/\hat{\partial}x_i \bigg|_{t'},
\end{align*}
\]

(12)

(13)

where the involved partial derivatives are common partial derivatives taken with the function $f$ independent of the retarded time $t'$ and, once taken the derivative, evaluated at the retarded time. Of course, we can see that if the retarded time $t'$ is not a function of the coordinates and time $t$ we get the results: \(\hat{\partial}f/\hat{\partial}t = \partial f/\partial t, \hat{\partial}f/\hat{\partial}x_i = \partial f/\partial x_i\), so we have a genuine generalization.
III. THE LW-POTENTIALS SATISFY A GENERALIZED Lorentz Gauge

Against the previous background we can try a different way to calculate the Lorentz gauge for the LW-potentials.

First of all we must consider them as retarded functions, that is functions evaluated at the retarded time $t'$ before the process of differentiation takes place. So, we use whole-partial derivatives to made the calculations. In this way we shall show that LW-potentials satisfy the Lorentz gauge as given by Eq. (3), as must be the case if the rule of generalization used with the Maxwell’s equations is right. Now, let’s start to calculate the expression (3) (the symbols $\hat{\partial}$ denote the Brownstein’s whole-partial derivative, the symbols $d$ denote the usual whole derivative with respect to a given variable, the symbols $\partial$ denote the usual partial derivative, $\nabla_q$ denotes the usual operator “nabla” with respect to $x_qi$, $\nabla$ denotes the usual operator “nabla” with respect to $x_i$ and $\nabla_v$ denotes the usual operator “nabla” with respect to $v_i$). Besides we indicate the kind of a functional dependence for each involved function:

$$R = R\{x_i, x_qi[t'(x_i, t)]\}; \quad \beta = \beta\{v[t'(x_i, t)]\}; \quad \eta = \frac{d\beta}{dt'}; \quad (14)$$

$$\frac{1}{c} \frac{\hat{\partial} \varphi}{\hat{\partial}t} = -\frac{q}{(R - R\beta)^2} \left( \frac{1}{c} \frac{\hat{\partial} R}{\hat{\partial}t} - \frac{1}{c} \frac{\hat{\partial}}{\hat{\partial}t} (R\beta) \right); \quad (15)$$

where

$$\frac{1}{c} \frac{\hat{\partial} R}{\hat{\partial}t} = \frac{\partial t'}{\partial t} (\nabla_q R \cdot \beta) = -\frac{\partial t'}{\partial t} \left( \frac{R\beta}{R} \right); \quad (16)$$

$$\frac{1}{c} \frac{\hat{\partial}}{\hat{\partial}t} (R\beta) = \frac{\partial t'}{\partial t} \left[ \nabla_q (R\beta)\beta + \nabla_v (R\beta)\eta \right]. \quad (17)$$
Now, using well-known vector operations we obtain:
\[
\nabla_q(R\beta) = R \times (\nabla_q \times \beta) + \beta \times (\nabla_q \times R) + (R \cdot \nabla_q)\beta + (\beta \cdot \nabla_q)R = -\beta,
\]
(18)

and
\[
\nabla_v(R\beta) = R \times (\nabla_v \times \beta) + \beta \times (\nabla_v \times R) + (R \cdot \nabla_v)\beta + (\beta \cdot \nabla_v)R = \frac{R}{c},
\]
(19)

so, with the help of Eqs. (16)-(19) and taking into account that (see [10])
\[
\frac{\partial t'}{\partial t} = \frac{R}{R - R\beta},
\]
we can write (15) as:
\[
\frac{1}{c} \frac{\hat{\partial} \varphi}{\partial t} = -\frac{qR}{(R - R\beta)^3} \left( \frac{R\beta}{R} - \beta^2 + \frac{R\eta}{c} \right).
\]
(20)

Now:
\[
\frac{\hat{\partial} \varphi}{\partial x_i} = -\frac{q}{(R - R\beta)^2} \left( \frac{\hat{\partial} R}{\partial x_i} - \frac{\hat{\partial}}{\partial x_i}(R\beta) \right),
\]
(21)

where
\[
\frac{\hat{\partial} R}{\partial x_i} = \frac{\partial R}{\partial x_i} + (\nabla_q R \cdot \mathbf{v}) \frac{\partial t'}{\partial x_i},
\]
(22)

and
\[
\frac{\hat{\partial}}{\partial x_i}(R\beta) = \frac{\partial}{\partial x_i}(R\beta) + \left[ \nabla_q(R\beta) \cdot \mathbf{v} + \nabla_v(R\beta) \cdot \frac{d\mathbf{v}}{dt'} \right] \frac{\partial t'}{\partial x_i}.
\]
(23)

This allows us, taking into account that
\[
\nabla t' = -\frac{R}{c(R - R\beta)} \quad \text{(see Ref. 10, Eq. 63.7)},
\]
to write down (after a straightforward calculation):

\[
3 \sum_{i=1}^{3} \beta_i \frac{\hat{\partial} \varphi}{\hat{\partial} x_i} = -\frac{q}{(R - R\beta)^3} \left[ \left(1 + \frac{R\eta}{c}\right) R\beta - \beta^2 R \right].
\]  

(24)

The last term to be calculated in Eq. (3) is

\[
3 \sum_{i=1}^{3} \frac{\hat{\partial} \beta_i}{\hat{\partial} x_i} = \nabla t' \cdot \eta,
\]  

(25)

so

\[
3 \sum_{i=1}^{3} \frac{\varphi}{\varphi} \frac{\hat{\partial} \beta_i}{\hat{\partial} x_i} = -\frac{q}{(R - R\beta)^2} \left(\frac{R\eta}{c}\right).
\]  

(26)

Now let us substitute the results (20), (24) and (26) into lhs of (3):

\[
3 \sum_{i=1}^{3} \frac{\hat{\partial} \beta_i}{\hat{\partial} x_i} + 3 \sum_{i=1}^{3} \beta_i \frac{\hat{\partial} \varphi}{\hat{\partial} x_i} + \frac{1}{c} \frac{\hat{\partial} \varphi}{\hat{\partial} t},
\]  

(27)

and after a straightforward calculation one can easily make sure that the expression (27) is identically zero.

It means that the Lorentz gauge for the LW-potentials is satisfied if we use Brownstein’s \textit{whole-partial} derivatives.

**IV. DIVERSE INTERPRETATIONS**

There is a way of saving the use of partial differentiation, but nevertheless the Lorentz gauge must change.

If we have the retarded potentials given as

\[
\varphi = \varphi(x_k, t, x_{q_k}, t', v_k), k = 1, 2, 3,
\]  

(28)

\[
A_i = A_i(x_k, t, x_{q_k}, t', v_k), i = 1, 2, 3,
\]  

(29)
then if we know the functional dependence of each argument we can use an adequate process of whole-partial differentiation, as it was done in section III, but if we know the function \( t' \) as an explicit function of coordinates and time \( t \), then we can write the following functions (after substitution of the explicit form of the function \( t' = f(x, y, z, t) \) in Eqs. (28), (29):

\[
\varphi = \psi(x, y, z, t),
\]

(30)

\[
A_i = a_i(x, y, z, t),
\]

(31)

where \( \psi \) and \( a_i \) are explicit function of the coordinates and time. Obviously, if we have the explicit functions \( \psi \) and \( a_i \), we must have the following equalities:

\[
\frac{\partial \psi}{\partial t} = \frac{\hat{\partial} \varphi}{\hat{\partial} t},
\]

(32)

\[
\frac{\partial a_i}{\partial x_i} = \frac{\hat{\partial} A_i}{\hat{\partial} x_i};
\]

(33)

but we cannot use, in an interpretation of this kind which tries to save the use of partial derivatives, the \textit{rhs} of Eqs. (32), (33), because then the interpretation would not be an independent interpretation. The point is that in this interpretation we must try to find the explicit form of \( t' \), and to give an explicit path. This is the only way in which we have really an independently defined interpretation which saves the use of partial derivatives.

Now, in order to get the explicit form of \( t' \) we have to integrate the following differential form:

\[
\frac{\partial t'}{\partial t} dt + \nabla t' \cdot dr
\]

(34)
which does not seem like an easy task, because in fact, the differential form (34) is not an integrable 1-form. We can check this assertion calculating the cross partial derivatives as follows:

\[
\frac{\partial}{\partial x_i} \left( \frac{R}{R - R\beta} \right) = \left( -\frac{x_i - x_{qi}}{R} \right) \left( \frac{1}{R - R\beta} \right) - \left( \frac{R}{(R - R\beta)^2} \right) \left( -\frac{x_i - x_{qi}}{R} - \beta_i \right),
\]

(35)

So one can see the non-integrability of (34). In this case the general integral of the equation cannot be expressed using some single function, instead, in general, we require several ones.

Another way to get \( t' \) relies on two points:

- we explicitly know the path of the particles;
- if the prior point is given, then we can write \( t' \) as \( t' = t - \frac{R(t')}{c} \) and we can use the well-known Lagrange development to get

\[
t' = t - \sum_{n=1}^{\infty} \frac{c^{-n}}{n!} \frac{d^{n-1}R(t)}{dt^{n-1}}
\]

(37)

and test its convergence behavior. Clearly \( c^{-n} \) is decreasing, but we need to know if the trajectory is bounded or not. Yet we can find another way to save the use of partial derivatives in the correspondent literature, but introducing

\[\text{Why we must use partial derivatives? If the coefficients involve the retarded time this cannot be the case. However, the coefficients of the differential 1-form (34) does not involve the evaluation at the retarded time \( t' \) because the functional dependence of the retarded time is given by } t' = f(x, y, z, t) \text{ a functional dependence not involving the evaluation of } f \text{ (a function obtained by the implicit function theorem applied to } t' = t - \frac{R(t')}{c} \text{) at any retarded time.}\]
a new set of concepts (see Ref [8] pp. 47 and 69), for example, Faraday’s law is correctly written with partial derivatives if we leave aside the vector character of the magnetic field $\mathbf{B}$ and introduce a two-form $B_{\lambda\mu} dx_\lambda \wedge dx_\mu$, and then we write Faraday’s law as (Post’s notation):

$$2\partial_\nu E_\lambda = -\frac{\partial}{\partial t} B_{\nu\lambda}. \quad (38)$$

As it was emphasized by Post, the transformation behavior of $E_\lambda$ and $B_{\nu\lambda}$ comes from the transformation properties of the tensor $F_{\mu\nu}$ (which is the usual electromagnetic tensor) under holonomic transformations (see Ref. [8] pp. 57-58). Clearly Eq. (38) is not the usual Faraday’s law, but it is a generalized form.

However, if we retain the vector character we need the use of *whole-partial* derivatives and we can guess why: by definition partial derivatives cannot take into account transformation properties, and when we evaluate functions at the retarded time $t'$ we are making a non-holonomic transformation. We can explain this fact in more detail:

Let us consider the operator

$$X = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} \quad (39)$$

under the holonomic transformation: $\vec{r}_i = \vec{r}_i(x_1, \ldots, x_n), i = 1, \ldots, n$. It is a standard knowledge that the transformation rule for (39) is given by

$$\sum_i \xi_i(x) \frac{\partial}{\partial x_i} = \sum_{i,k} \xi_i[x(\vec{x})] \frac{\partial \vec{r}_k}{\partial x_i} \frac{\partial}{\partial \vec{r}_k} = \overline{X}. \quad (40)$$

Non-holonomic because we cannot integrate the expression (34) to get an explicit function $t' = f(x, y, z, t)$. And in this case Post’s proof of the invariance of the Maxwell’s equations given by $\partial_\nu F_{\lambda\mu} = 0, \partial_\nu G^{\lambda\nu} = 4\pi e^\lambda$ fails because it is only valid for holonomic transformations (see Ref. [8] pp. 57-61).
Now it is clear that (39) is just a *whole-partial* derivative\(^3\) \(\frac{\partial}{\partial s}\) if the integral curves of the vector field (39) are given by \(\frac{\partial s}{\partial x} = \xi_i(x)\). Under the holonomic transformation used, which does not affect the arbitrary parameter \(s\), the integral curves of the transformed vector field are \(\frac{\partial x_i}{\partial s} = \sum_k \xi_i[x(\tau)] \frac{\partial x_k}{\partial s}\), but we still have just a *whole-partial* derivative \(\frac{\partial}{\partial s}\) which remains invariant under the transformation. This is the reason, the underlying mathematical reason of the generalization realized in Ref. [1]. Besides, it is clear that the use of the transformation rule (40) is not limited to holonomic transformation\(^4\), instead we can use transformations of the form \(\sum_k A_i^k \xi_i\) where \(A_i^k\) does not arise from a holonomic transformation but, however, are non-singular transformations. So, in this sense, when we change our coordinates to coordinates including the retarded time \(t'\) we are changing to what is known not only as a non-holonomic reference frame, but as a set of quasi-parameters of the kind often found in dynamics [11].

\(\text{A *whole-partial* derivative is just a case of a *total* derivative. Just consider the operator (39), if we choose the coefficients } \xi_i \text{ in an adequate manner we can get, without doubt, the expressions (12), (13). It is possible to choose the invariant parameter by just choosing a coefficient: for some } i \text{ we put } \xi_i = 1. \text{ It is clear that the concept of a *whole-partial derivative* has some ground because the functions involve several variables, but the basic concept is that of a “total derivative”, as two authors of the present work have tried to show in Ref. [4].}

\(\text{It is possible to say that the differential 1-form (34) is of necessity the form } dt'. \text{ However, the criterion for decision if (34) is an integrable 1-form or not is only given by cross differentiation.}\)
V. CONCLUSIONS

We have showed that the correct way to write down the Lorentz gauge is

\[
\sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0
\]  

(41)

for the case of the LW-potentials. Hence previous generalization (see Ref. [1]) of the Maxwell’s equations is really a generalized form and we must take Eq. (3) as the general way to write down the Lorentz gauge and, of course, all gauges.

The underlying reasoning is clear: if the form of writing (1) would be general, it should be valid for all the cases. However we have shown at least one exception, then Eq. (1) is not general. And as we have shown the Brownstein’s generalization (3) gives the correct results.

We have discussed the possibilities to continue using partial derivatives, we have deduced that for that interpretation to be useful one requires either: (i) an explicitly defined functional form for the retarded time \(t'\) or (ii) according to Post change the vector formalism for the tensor formalism (change \(B\) for \(B_{\lambda\nu}\)) which is another kind of generalization.

However we can use the Post’s propositions\(^5\) to reject the use of partial derivatives: partial derivatives are always well-defined in a given set of coordinates, however, when we use changes of coordinates we use implicitly the transformation rule (40), which is just of a whole-partial derivative.

\(^5\)Let’s reproduce them here: 1) “Physicists cannot think without a cartesian frame”, 2) “Physicists cannot think without explicit reference to an inertial frame” (see Ref. [9] p. 77).
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