COMBINATORIAL SUMS AND IDENTITIES INVOLVING GENERALIZED DIVISOR FUNCTIONS WITH BOUNDED DIVISORS

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Abstract. We prove several new forms of the expansions of the higher-order derivatives of the Lambert series generating functions, \( F_\alpha(q) \), which enumerate the generalized sums of divisors functions, \( \sigma_\alpha(n) = \sum_{d \mid n} d^\alpha \), for all integers \( n \geq 1 \) and real-valued \( \alpha \geq 0 \). These results are combined to formulate new identities expanding the sums of divisors functions, including the notable classically defined special cases of \( d(n) \equiv \sigma_0(n) \) and \( \sigma(n) \equiv \sigma_1(n) \), given in terms of analogous divisor sums with bounded divisors. Equivalently, we expand these special number theoretic functions by the coefficients of corresponding transformed Lambert series expansions of the form \( \sum_{i \geq 1} i^\alpha q^m / (1 - q^i)^k \) for positive integers \( k, m \geq 1 \). Our results include new relations between finite sums involving the divisor functions and exact formulas for these divisor functions and their corresponding closely-related bounded-divisor variants over multiple values of \( m \) and the function parameter \( \alpha \).

1. INTRODUCTION

1.1. Generating the sums of divisors functions.

Generalized divisor functions. We are interested in obtaining new properties of the sums of divisors functions, \( \sigma_\alpha(n) = \sum_{d \mid n} d^\alpha \) for \( \alpha \geq 0 \), through the Lambert series generating functions whose expansions enumerate these special arithmetic functions for all \( n \geq 1 \). In particular, for real-valued \( \alpha \geq 0 \), these functions are generated by [1, §27.7]

\[
F_\alpha(q) := \sum_{i \geq 1} i^\alpha q^i / (1 - q^i) = \sum_{m \geq 1} \sigma_\alpha(m) q^m, \quad |q| < 1,
\]

where we have a symmetric identity for the negative-order cases of these functions given by \( \sigma_{-\alpha}(n) = \sigma_\alpha(n) / n^\alpha \). The divisor function, \( d(n) \equiv \sigma_0(n) \), and the sum of divisors function, \( \sigma(n) \equiv \sigma_1(n) \), are of great importance and interest in number theory. These functions are central to a number of famous open problems such as determining the form of the perfect numbers \( n \) such that \( \sigma(n) = 2n \), which are then linked to the Mersenne primes and Mersenne numbers, \( M_n \). We will not delve too deeply into a discussion of the applications and theoretical underpinnings of these special arithmetic function sequences in this article.

Approach through bounded-divisor sum functions and their relations to Lambert series. Instead, we choose a more combinatorial route and treat these special sequences to be of interest in the article mainly for their beautiful expansions, their implied relation to the primes, and for their remarkable generating functions given in the form of (1) that also satisfy a number of special algebraic properties and expansions which we will exploit in order to prove our new results. In particular, we prove several new expansions, identities, and recurrence relations satisfied between these functions for arbitrary \( \alpha \geq 0 \) and the modified divisor sums and binomial
convolution divisor sums respectively defined for all positive integers \( n, k, m \geq 1 \) by

\[
\sigma_{\alpha,m}(n) := \sum_{d|n} d^\alpha \quad \text{and} \quad B_{k,m}(\alpha;n) := \sum_{d|n} \binom{n/d - m + k}{k} d^\alpha.
\]  

(2)

More to the point, we encounter these functions in “tinkering” with the arithmetic of the Lambert series in (1) and their higher-order \( s \)-th derivatives as the coefficients

\[
[q^n] \sum_{i \geq 1} \frac{i^\alpha q^{mi}}{(1 - q^i)^{k+1}} = \sum_{d|n} \binom{n/d - m + k}{k} d^\alpha, \quad \alpha \geq 0, m \in \mathbb{Z}^+, k \in \mathbb{N}.
\]  

(3)

We recount some of our motivating explorations with these Lambert series generating functions in the next subsection.

1.2. Motivation and methods of exploration. We begin by noticing from the Lambert series generating functions in (1) we can generate the sums of divisors functions by considering only the terms in the truncated series identity

\[
\sigma_\alpha(n) = [q^n] \sum_{i=1}^{n} \frac{i^\alpha q^i}{1 - q^i}
\]

\[
= [q^n] \left( \sum_{i=1}^{n} i^{\alpha-1} q^i + \sum_{j=0}^{i-2} \frac{i^{\alpha-1} q^{i+j}(i-1-j)(1-q^i) 1-q^i}{1-q^i} \right),
\]

(4)

which forms the basis for all of our new expansions proved in the next sections of the article. In this most straightforward example, we see that we may evaluate the inner sum

\[
\sum_{j=0}^{i-2} q^{i+j}(i-1-j)(1-q) = \frac{q^{2i}}{1-q} + \frac{1-iq^{i+1}}{1-q},
\]

which quickly leads to the identity involving the bounded divisor sum functions from (2) providing that

\[
\sigma_\alpha(n) = \sum_{k=1}^{n} [k^{\alpha-1} + \sigma_{\alpha-1,k}(k) + \sigma_\alpha(k) - \sigma_{\alpha-1,k}(k) - \sigma_\alpha(k-1)].
\]

Unfortunately, we see that \( \sigma_{\alpha,2}(n) = \sigma_\alpha(n) - n^\alpha \) for all \( n \geq 1 \), so we are stuck with a result that falls out to identity in this initial trial case. However, we continue by expanding these partial sums using the next higher-order derivative identity involving the Stirling number triangles for any fixed integers \( s \geq 1 \):

\[
q^s D^{(s)} \left[ \frac{q^i}{1-q^i} \right] = \sum_{s=0}^{m} \sum_{k=0}^{m} \binom{m}{k} \left( -1 \right)^{s-k} k! \cdot i^m (1-q^i)^{k+1}
\]

\[
= \sum_{r=0}^{s} \sum_{s=0}^{m} \binom{m}{k} \left( -1 \right)^{s-k} r! \cdot i^m (1-q^i)^{k+1} \right) q^{(r+1)i}.
\]

(5)

Since we have the generating function coefficient identity for the Lambert series in (3), our intuition suggests that summing over the previous expansions of (5) in (4) with respect to \( j \) will lead to meaningful, non-trivial identities relating the ordinary sums of divisors functions, \( \sigma_\alpha(n) \), to the divisor sum functions with bounded divisors defined by (2). Our intuition is in fact correct here, and moreover, we are able to perform derivative-based and finite-sum-based arithmetic operations with the Lambert series expansions in (4) to prove new finite sum identities and recurrence relations between these special arithmetic function variants. We shall
be quick in introducing several multiple sum definitions and the primary statements of our new results in the next section.

2. Definitions and statements of main results

2.1. Definitions.

Definition 2.1 (Multiple Coefficient Sums). For integers \( s \geq 1 \) and \( r, p, m, u, n, w \geq 0 \), we define the following single and multiple coefficient sums expanded by

\[
C_{4s}^{(\alpha)}(r, p, m) := \sum_{k=0}^{m} \left( \begin{array}{c} s-r \\hline r \end{array} \right) \left( \begin{array}{c} m \\hline k \end{array} \right) \left( \begin{array}{c} s-r-k \\hline p \end{array} \right) (-1)^{s-r-k+p} k! \cdot r!
\]

\[
C_{8s}^{(\alpha)}(r, p, n, u) := \sum_{m=0}^{s} \sum_{k=0}^{m} \sum_{m_1=0}^{m} \sum_{m_2=0}^{m} \left( \begin{array}{c} s-r \\hline m \end{array} \right) \left( \begin{array}{c} m \\hline k \end{array} \right) \left( \begin{array}{c} s-r-k \\hline p \end{array} \right) \left( \begin{array}{c} r+1-n \\hline m_1 \end{array} \right) \left( \begin{array}{c} n \\hline m_2 \end{array} \right) \times
\]

\[
\left( \begin{array}{c} w+2-m \\hline m_1 \end{array} \right) (-1)^{s-r+k+p+n+m_1+m_2} k! \cdot (n-2-r)^{m+m_1+m_2-w}
\]

\[
C_{42s}^{(\alpha)}(x, k; p, m, u) := \sum_{r=0}^{s-r} \left( \begin{array}{c} x-k \\hline r \end{array} \right) \left( \begin{array}{c} s-r \\hline j \end{array} \right) \left( \begin{array}{c} j \\hline u \end{array} \right) (-1)^{s-r-u}(p+1-s+r)^{j-u} k^u \frac{C_{4s}^{(\alpha)}(r, p, m)}{(s-r)!}
\]

\[
C_{43s}^{(\alpha)}(x, k; p, m, u) := \sum_{r=0}^{s-r} \left( \begin{array}{c} x+1-k \\hline r+1 \end{array} \right) \left( \begin{array}{c} s-r \\hline j \end{array} \right) \left( \begin{array}{c} j \\hline u \end{array} \right) (-1)^{s-r-u}(r+1)(p+1-s+r)^{j-u} k^u \frac{C_{4s}^{(\alpha)}(r, p, m)}{(s-r)!}
\]

\[
C_{82s}^{(\alpha)}(x, k; w, p, u) := \sum_{r=0}^{s-r} \sum_{n=0}^{r} \sum_{j=0}^{s-r} \left( \begin{array}{c} x+1-n-k+r \\hline r \end{array} \right) \left( \begin{array}{c} r \\hline j \end{array} \right) \left( \begin{array}{c} j \\hline u \end{array} \right) \times
\]

\[
(-1)^{s-r-u}(p+2-s+r)^{j-u} k^u \frac{C_{8s}^{(\alpha)}(r, p, n, w)}{(s-r)!}
\]

\[
C_{83s}^{(\alpha)}(x, k; w, p, u) := \sum_{r=0}^{s-r} \sum_{n=0}^{r+1} \sum_{j=0}^{s-r} \left( \begin{array}{c} x+2-n-k+r \\hline r+1 \end{array} \right) \left( \begin{array}{c} r+1 \\hline n \end{array} \right) \left( \begin{array}{c} s-r \\hline j \end{array} \right) \left( \begin{array}{c} j \\hline u \end{array} \right) \times
\]

\[
(-1)^{s-r-u}(p+2-s+r)^{j-u} k^u \frac{C_{8s}^{(\alpha)}(r, p, n, w)}{(s-r)!}
\]

The fact that we can effectively abstract these complicated terms as the coefficients of other sums in our more important results suggests an approach to simplifications of these results by summing these multiple sum terms in closed-form, or at least partially closed-form. We do not consider the task of possible simplifications of these obscure coefficient terms in the article, though the interested reader is certainly encouraged to do so. We next move along to defining the multiple sum expansions of a few component series which characterize the Lambert series expansions in (4).

Definition 2.2 (Multiple Sum Expansions of Key Component Sums). For fixed integers \( s, i \geq 1 \) and an indeterminate series parameter \(|q| < 1\), we define the next two primary component sums used to state the results below:

\[
\text{Sum}_{4s}^{(\alpha)}(q, i) := \sum_{0 \leq r, p, m \leq s} q^{(p+1)i+r} \left( \frac{i^{m+1}}{(1-q)^{s+r+1}} - \frac{(r+1)i^m}{(1-q)^{r+2}} \right) C_{4s}^{(\alpha)}(r, p, m)
\]

\[
\text{Sum}_{8s}^{(\alpha)}(q, i) := \sum_{0 \leq r, p \leq s} \sum_{n=0}^{s} \sum_{w=0}^{r+1} q^{(p+1)i+n-1} \left( \frac{1}{(1-q)^{s+r+1}} - \frac{r+1}{n} \right) \frac{1}{(1-q)^{r+2}} \times
\]

\[
\left( \frac{1}{1-q} \right)
\]

\[
\left( \frac{1}{1-q} \right)
\]
Finally, we define shorthand notation for the coefficients of common sums and expansions that we will be working with to state our new results in the next subsection of the article.

**Definition 2.3** (Series Coefficients). For fixed real $\alpha \geq 0$ and integers $s, x \geq 1$ we define the next coefficients of several intermediate power series expansions of the Lambert series expansions from the introduction. We prove that these formulas are in fact the power series coefficients of these few key series as lemmas in the next section.

\[
L_{s,x}^{(\alpha)} := [q^x]q^s D^{(s)} \left[ \frac{F_\alpha(q)}{1-q} \right]
\]

\[
= \sum_{r=0}^{s} \sum_{k=1}^{x} \binom{s}{r} (x-k) (s-r)! k! \sigma_\alpha(k)
\]

\[
S_{00,s,x}^{(\alpha)} := [q^x]q^s D^{(s)} \left[ \sum_{i \geq 1} \frac{i^{\alpha-1} q^i}{(1-q)^2} \right]
\]

\[
= \sum_{r=0}^{s} \sum_{k=0}^{x} \binom{s}{r} (x-k+1) (s-r+1)! k^{\alpha-1} \cdot k! (k-r)!
\]

\[
S_{01,s,x}^{(\alpha)} := [q^x]q^s D^{(s)} \left[ \sum_{i \geq 1} \frac{i^{\alpha-1} q^i}{1-q} \right]
\]

\[
= \sum_{r=0}^{s} \sum_{k=0}^{x} \binom{s}{r} (x-k) (s-r)! k^{\alpha-1} \cdot k! (k-r)!
\]

\[
S_{4,s,x}^{(\alpha)} := [q^x] \sum_{i \geq 1} \text{Sum}_{4,s}(q,i) i^{\alpha-1}
\]

\[
S_{8,s,x}^{(\alpha)} := [q^x] \sum_{i \geq 1} \text{Sum}_{8,s}(q,i) i^{\alpha-1}
\]

In the corollaries of the new results given in later subsections of the article, we may also employ the shorthand notation of $S_{48,s,x}^{(\alpha)} := S_{4,s,x}^{(\alpha)} + S_{8,s,x}^{(\alpha)}$.

**2.2. Statements of main results.**

**Proposition 2.4** (Higher-Order Derivatives of Lambert Series Expansions). For a fixed indeterminate series parameter satisfying $|q| < 1$, a real-valued $\alpha \geq 0$, and any integers $s \geq 1$, we have the following identity for the expansions of the higher-order $s^{th}$ derivatives of the Lambert series generating functions in (1):

\[
q^s D^{(s)} \left[ \frac{1}{1-q} \times \sum_{i \geq 1} \frac{i^{\alpha} q^i}{1-q^i} \right] = q^s D^{(s)} \left[ \sum_{i \geq 1} \frac{i^{\alpha} q^i}{(1-q)^2} \right] + \sum_{i \geq 1} (\text{Sum}_{4,s}(q,i) + \text{Sum}_{8,s}(q,i)) i^{\alpha-1}.
\]

In the notation of Definition 2.3 above, this statement is equivalent to saying that for all integers $s, x \geq 1$, we have that

\[
L_{s,x}^{(\alpha)} = S_{00,s,x}^{(\alpha)} + S_{4,s,x}^{(\alpha)} + S_{8,s,x}^{(\alpha)};
\]

since the left-hand-side of the previous series equation is equal to $q^s$ times the $s^{th}$ derivative of the generating function, $F_\alpha(q)/(1-q)$.

The proof of Proposition 2.4 is somewhat involved and requires the machinery of several lemmas we will prove in the next section. For this reason we delay the proof of this key result until Section 3.2. With this result at our disposal, we are able to prove a few other series
coefficient expansions key to our goal of relating the ordinary sums of divisors functions to the modified divisor sums with bounded divisors defined in (2) of the introduction. In particular, the next corollary provides direct expansions of the coefficients of the series from the previous proposition in terms of sums over the convolved divisor sum functions, \( B_{k,m}(\alpha; n) \). Then by expanding the inner sum terms of the binomial coefficients by the Stirling numbers of the first kind and the binomial theorem, we are able to state and finally prove the second result of the theorem stated below which provides expansions of the series coefficients implicit to the proposition only in terms of the modified divisor sum functions. The significance of proving this result should be underscored in that it provides non-trivial, deeper expansions of the ordinary sums of divisors functions in terms of the same functions with upper divisor terms removed from these special sums over variable inputs of \( \alpha \).

**Corollary 2.5** (Relations to the Binomial Divisor Sums). For any integer \( x \geq 1 \), we have the next expansions of the series coefficients implicit to the identity from Proposition 2.4.

\[
S_{4,s,x}^{(\alpha)} = \sum_{0 \leq r,p,m \leq s} \sum_{k=1}^{x-r} \left( \binom{x-k}{r} B_{s-r,p+1}(m+\alpha;k) - \binom{x-1}{r+1} (r+1) B_{s-r,p+1}(m+\alpha-1;k) \right) C_{4,s}^{(\alpha)}(r,p,m) \times C_{8,s}(r,p,n,w) B_{s-r,p+2}(w+\alpha-1;k)
\]

The proofs of the two results are similar. We prove the second result and leave the details of the first proof as an exercise. Since we employ the result in (3) to justify these expansions in our argument, we first sketch a proof of this expansion.

**Proof of (3).** For fixed integers \( k \geq 0 \) and \( m \geq 1 \), consider the expansion of the next terms by the geometric series where \( |q|, |q^i| < 1 \) by assumption:

\[
L_{i,k,m}(q) := \frac{f(i)q^{mi}}{(1-q^i)^{k+1}} = f(i) \times \sum_{j=0}^{\infty} \binom{j+k}{k} q^{(m+j)i}.
\]

Since we sum over all \( i \geq 1 \), the coefficients of \( q^n \) in the full series expansion must satisfy the integer relation that \( (m+j)i = n \), which leads to a sum over the divisors \( i \) of \( n \). We must then 1) solve for the input \( j = \frac{n}{i} - m \) depending on the other fixed integer parameters to rewrite binomial coefficient terms in the resulting divisor sums and 2) notice that the divisors \( i \) are positive integers bounded by \( 0 < i = \frac{n}{m+j} \leq \frac{n}{m} \) given by

\[
[q^n] \sum_{d \geq 1} L_{d,k,m}(q) = \sum_{d \mid n} \left( \binom{\frac{n}{d} - m + k}{k} f(d) \right).
\]

**Proof of (ii) in the Corollary.** Thus we may apply (3) to our function, \( \text{Sum}_{8,s}(q,i) \), defined in the last subsection along with the known Cauchy product formula for the convolution of two generating functions, or in our case functions of \( q \), to obtain our result. More precisely, we see that for \( r_0 := 0,1 \) we have a difference of coefficients of the form

\[
\sum_{r,p,n,\omega \geq 0} \sum_{i \geq 1} \left( \binom{r + r_0}{n} \right) \frac{i^w + \alpha - 1 q^{(p+1)i+n-1}}{(1-q^i)^{s-r+p+1} \times (1-q)^{r+1+r_0}}
\]
from which we shift the indices of the coefficients of \(q^x\) by \(n-1\) to obtain the complete proof of our second expansion.

**Theorem 2.6** (Expansions by the Bounded Divisor Sum Functions). For all integers \(x \geq 1\), we have the following variants of the preceding corollary expanded only terms of the modified divisor sum functions, \(\sigma_{\alpha,m}(n)\):

\[
S_{4,s,x}^{(\alpha)} = \sum_{0 \leq p, m, u \leq s} \sum_{k=1}^{x} \left( C_{42,s}^{(\alpha)}(x, k; p, m, u) - C_{43,s}^{(\alpha)}(x, k; p, m + 1, u) \right) \sigma_{m+\alpha-u,p+1}(k) \\
- \sum_{0 \leq p, u \leq s} \sum_{k=1}^{x} C_{43,s}^{(\alpha)}(x, k; p, 0, u) \sigma_{\alpha-1-u,p+1}(k) \\
S_{8,s,x}^{(\alpha)} = \sum_{0 \leq p, u, w \leq s} \sum_{k=1}^{x+1} \left( C_{82,s}^{(\alpha)}(x, k; w, p, u) - C_{83,s}^{(\alpha)}(x, k; w, p, u) \right) \sigma_{w+\alpha-1-u,p+2}(k).
\]

**Proof.** We can expand the forms of the binomial coefficients implicit to the divisor terms for the functions, \(B_{k,m}(\alpha; n)\), in the previous corollary by the Stirling numbers of the first kind, which arise in the polynomial expansions of the single factorial function, and the binomial theorem to write the inner sum terms defining these functions in (2) as polynomials in \(n/d\). Suppose that \(t\) is an indeterminate and that the non-negative integers \(m, k\) are fixed. In particular, we see that we have the following identity:

\[
\binom{t-m+k}{k} = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{(t-m+k)^j}{j!}
\]

\[
= \sum_{r=0}^{k} \left( \sum_{j=0}^{k} \binom{k}{j} (j-r) \frac{(-1)^{k-r} (m-k)^{j-r}}{k!} \right) t^r.
\]

The expansions of the binomial divisor functions in the corollary according to the previous formula motivate the definitions of the second set of (last four) coefficient sums given in the previous subsection. The rest of the proof is rearranging terms and shifting indices in the first result so that \(m \mapsto m + 1\). \(\square\)

### 2.3. A few notable corollaries, new results, and applications.

**Corollary 2.7** (Exact Formulas for the Generalized Sums of Divisors Functions). For real-valued \(\alpha \geq 0\), any integer \(s \geq 2\), and all integers \(x \geq 1\), we have the following exact finite sum identity of the generalized sums of divisors functions:

\[
\binom{x}{s}^{(\alpha)} = \frac{1}{s!} \left( S_{01,s,x}^{(\alpha)} + S_{48,s,x}^{(\alpha)} - S_{48,s,x-1}^{(\alpha)} + s \cdot S_{48,s-1,x-1}^{(\alpha)} \right).
\]

**Proof (Sketch).** We first notice that given any function, \(f(q)\), that is \(k\)-order differentiable for all \(1 \leq k \leq j\) for some natural number \(j \geq 1\), we have the following derivative identity which is obtained easily by induction on \(j\):

\[
D^{(j)} [(1 - q)f(q)] = (1 - q)f^{(j)}(q) - jf^{(j-1)}(q), \quad j \geq 1.
\]

The formulas obtained in Theorem 2.6 proved starting from the results in Proposition 2.4 in the previous subsection, were initially grounded on the argument that dividing the Lambert series expanded as in (4) through by a factor of \((1 - q)\) before differentiating would lead to less complicated \(s^{th}\) derivative formulas. While this is true in some respects, we are only able to generate left-hand-side sums in the theorem involving \(\text{sums over}\) the generalized sums of
The motivation for defining the component functions of \( k \) derivatives of \((\text{fact, produce tangible formulas involving the bounded-divisor divisor sum functions from...})\) \(\sigma\) (given by the previous sections to exact, closed-form expressions involving \(\sigma\)) for simplifying the complicated nested sums defining these functions in their immediate forms we provide several examples for small cases of \(s\) not obtain any additional information from these Lambert series expansions when \(s \geq 2\). As explained in the introduction, we do not obtain any additional information from these Lambert series expansions when \(s := 0, 1\). When \(s \geq 2\), we begin to pick up divisor sum terms suggesting more interesting properties of the left-hand-side relations. The next few examples expanded below show the character of these relations more precisely. These examples tend to illustrate the complexity of the relations between the generalized sums of divisors functions at differing orders of \(\alpha\).

| \(n\) | \(m\) | 1   | 2   | 3   | 4   |
|------|------|-----|-----|-----|-----|
| 1    | 1    | 1   | 0   | 0   | 0   |
| 2    | \(2^\alpha + 1\) | 1   | 0   | 0   | 0   |
| 3    | \(3^\alpha + 1\) | 1   | 1   | 0   | 0   |
| 4    | \(2^\alpha + 4^\alpha + 1\) | \(2^\alpha + 1\) | 1   | 1   | 0   |
| 5    | \(5^\alpha + 1\) | 1   | 1   | 1   | 0   |
| 6    | \(2^\alpha + 3^\alpha + 6^\alpha + 1\) | \(2^\alpha + 3^\alpha + 1\) | \(2^\alpha + 1\) | 1   | 0   |
| 7    | \(7^\alpha + 1\) | 1   | 1   | 1   | 0   |
| 8    | \(2^\alpha + 4^\alpha + 8^\alpha + 1\) | \(2^\alpha + 4^\alpha + 1\) | \(2^\alpha + 1\) | \(2^\alpha + 1\) | 0   |
| 9    | \(3^\alpha + 9^\alpha + 1\) | \(3^\alpha + 1\) | \(3^\alpha + 1\) | 1   | 0   |
| 10   | \(2^\alpha + 5^\alpha + 10^\alpha + 1\) | \(2^\alpha + 5^\alpha + 1\) | \(2^\alpha + 1\) | \(2^\alpha + 1\) | 0   |
| 11   | \(11^\alpha + 1\) | 1   | 1   | 1   | 0   |
| 12   | \(2^\alpha + 3^\alpha + 4^\alpha + 6^\alpha + 12^\alpha + 1\) | \(2^\alpha + 3^\alpha + 4^\alpha + 6^\alpha + 1\) | \(2^\alpha + 3^\alpha + 4^\alpha + 1\) | \(2^\alpha + 3^\alpha + 1\) | 0   |
| 13   | \(13^\alpha + 1\) | 1   | 1   | 1   | 0   |
| 14   | \(2^\alpha + 7^\alpha + 14^\alpha + 1\) | \(2^\alpha + 7^\alpha + 1\) | \(2^\alpha + 1\) | \(2^\alpha + 1\) | 0   |
| 15   | \(3^\alpha + 5^\alpha + 15^\alpha + 1\) | \(3^\alpha + 5^\alpha + 1\) | \(3^\alpha + 5^\alpha + 1\) | \(3^\alpha + 1\) | 0   |
| 16   | \(2^\alpha + 4^\alpha + 8^\alpha + 16^\alpha + 1\) | \(2^\alpha + 4^\alpha + 8^\alpha + 1\) | \(2^\alpha + 4^\alpha + 1\) | \(2^\alpha + 4^\alpha + 1\) | 0   |
| 17   | \(17^\alpha + 1\) | 1   | 1   | 1   | 0   |
| 18   | \(2^\alpha + 3^\alpha + 6^\alpha + 9^\alpha + 18^\alpha + 1\) | \(2^\alpha + 3^\alpha + 6^\alpha + 9^\alpha + 1\) | \(2^\alpha + 3^\alpha + 6^\alpha + 1\) | \(2^\alpha + 3^\alpha + 1\) | 0   |
| 19   | \(19^\alpha + 1\) | 1   | 1   | 1   | 0   |
| 20   | \(2^\alpha + 4^\alpha + 5^\alpha + 10^\alpha + 20^\alpha + 1\) | \(2^\alpha + 4^\alpha + 5^\alpha + 10^\alpha + 1\) | \(2^\alpha + 4^\alpha + 5^\alpha + 1\) | \(2^\alpha + 4^\alpha + 5^\alpha + 1\) | 0   |
| 21   | \(3^\alpha + 7^\alpha + 21^\alpha + 1\) | \(3^\alpha + 7^\alpha + 1\) | \(3^\alpha + 7^\alpha + 1\) | \(3^\alpha + 1\) | 0   |

Table 1. The Bounded-Divisor Divisor Sum Functions, \(\sigma_{\alpha,m}(n)\)

divisors functions in place of an exact formula for these special functions. If we revisit the \(s^\text{th}\) derivatives of (4) and subsequently re-interpret our results using the formula in (6), we arrive at the stated exact formulas for \(\sigma\) when \(j = s\).

Expansions of the finite sum identities for small \(s\). To illustrate that our methods of expanding the higher-order derivatives of the Lambert series generating functions in (1), are “tractable”, that is to say that, our definitions of the expansions of these series defined by abstracting the two layers of constants starting with the somewhat complicated forms in Definition 2.1, do, in fact, produce tangible formulas involving the bounded-divisor divisor sum functions from (2), we provide several examples for small cases of \(s \geq 2\). As explained in the introduction, we do not obtain any additional information from these Lambert series expansions when \(s := 0, 1\). When \(s \geq 2\), we begin to pick up divisor sum terms suggesting more interesting properties of the left-hand-side relations. The next few examples expanded below show the character of these relations more precisely. These examples tend to illustrate the complexity of the relations between the generalized sums of divisors functions at differing orders of \(\alpha\).

For fixed \(x, k \in \mathbb{Z}\), we can give components of our results by defining

\[
\begin{align*}
S_{01,s,x,k}^{(\alpha)} &\mapsto S_{01,s,x}^{(\alpha)} \quad \text{omitting the sum over } k \\
S_{4,s,x,k}^{(\alpha)} &\mapsto S_{4,s,x}^{(\alpha)} \quad \text{omitting the sum over } k \\
S_{8,s,x,k}^{(\alpha)} &\mapsto S_{8,s,x}^{(\alpha)} \quad \text{omitting the sum over } k \\
S_{16,s,x,k}^{(\alpha)} := S_{16,s,x,k}^{(\alpha)} &= S_{4,s,x,k}^{(\alpha)} + S_{8,s,x,k}^{(\alpha)}.
\end{align*}
\]

The motivation for defining the component functions of \(k\) and \(x\) in (7) is to provide a mechanism for simplifying the complicated nested sums defining these functions in their immediate forms given by the previous sections to exact, closed-form expressions involving \(\sigma_{\alpha-j,m}(k)\) for integers...


\begin{align*}
\left( \frac{x}{s} \right) \sigma_\alpha(x) &= \sum_{k=1}^{s-1} \frac{1}{s!} \left( S_{1,s,k}^{(\alpha)} + S_{4,s,k}^{(\alpha)} - S_{1,s-1,k}^{(\alpha)} + S_{4,s-1,k}^{(\alpha)} + S_{4,s,k}^{(\alpha)} \right) \\
&= \sum_{k=1}^{s-1} \tau_{s,x}^{(\alpha)}(k) + \rho_{s,x}^{(\alpha)}(x).
\end{align*}

In general, we can prove inductively that for polynomials, \( p_{i,s,\beta}(k, x) \) and \( q_{i,s,\beta}(x) \) in their respective inputs, we have expansions of the component functions comprising the formulas in (8) of the form

\begin{align*}
\tau_{s,x}^{(\alpha)}(k) &= \sum_{\beta=-s}^{s+1} \sum_{m=1}^{s+1} p_{1,s,\beta}(k, x) \cdot \sigma_{\alpha+\beta}(k) + p_{2,s,\beta}(k, x) \cdot \sigma_{\alpha+\beta,m}(k) \\
\tau_{s,x}^{(\alpha)}(k) &= \sum_{\beta=-s}^{s+1} \sum_{m=1}^{s+1} q_{1,s,\beta}(k, x) \cdot \sigma_{\alpha+\beta}(k) + q_{2,s,\beta}(k) \cdot \sigma_{\alpha+\beta,m}(k),
\end{align*}

where for positive real \( \beta > 0 \) we have a negative-order identity for the generalized sum of divisors functions providing that \( \sigma_{-\beta}(n) = n^{-\beta} \cdot \sigma_\beta(n) \).

**Example 2.8** (Special Cases for Classical Divisor Functions). For the small special cases of \( \alpha := 0, 1 \) corresponding to the cases of the divisor function, \( d(n) \), and the ordinary sum of divisors functions, \( \sigma(n) \), respectively, on the left-hand-side of (8), we can simplify our resulting finite sum expansions using the identities that (see Table 1 on page 7)

\[ \sigma_{\alpha,1}(n) = \sigma_\alpha(n) \quad \text{and} \quad \sigma_{\alpha,2}(n) = \sigma_\alpha(n) - n^\alpha, \quad \text{for all } n \geq 1, \]

to find expansions of the following exact forms when \( s := 2 \):

\begin{align*}
\tau_{2,x}^{(0)}(k) &= \frac{1}{4} \left( (3k - 2)\sigma_0(k) - (k - 1)\sigma_1(k) - \sigma_2(k) \right) \\
&= \frac{1}{4} \left( k^2\sigma_{-2,3}(k) + k(k - 3)\sigma_{-1,3}(k) - (3k - 2)\sigma_{0,3}(k) + 2\sigma_{1,3}(k) \right) \\
\rho_{2}^{(0)}(x) &= \frac{1}{4} \left( (-2 + x + 2x^2) \sigma_0(x) - (x - 1)\sigma_1(x) - \sigma_2(x) \right) \\
&= \frac{1}{4} \left( x^2\sigma_{-2,3}(x) + x(x - 3)\sigma_{-1,3}(x) - (3x - 2)\sigma_{0,3}(x) + 2\sigma_{1,3}(x) \right) \\
\tau_{2,x}^{(1)}(k) &= \frac{1}{4} \left( (3 - k)k\sigma_0(k) + 2(k - 1)\sigma_1(k) - 2\sigma_2(k) \right) \\
&= \frac{1}{4} \left( k^2\sigma_{-1,3}(k) + (k - 3)k\sigma_{0,3}(k) - (3k - 2)\sigma_{1,3}(k) + 2\sigma_{2,3}(k) \right) \\
\rho_{2}^{(1)}(x) &= \frac{1}{4} \left( (x^2 - 1) \sigma_1(x) - x(x - 3)\sigma_0(x) - 2\sigma_2(x) \right)
\end{align*}

1 For positive integers \( \beta \geq 0 \), we have formulas for the polynomial power sum functions expanded by the Bernoulli numbers and Bernoulli polynomials given by [1, §24.4(iii)]

\[ \sum_{k=0}^{n} k^\beta = \frac{B_{\beta+1}(n+1) - B_{\beta+1}}{\beta + 1}, \]

which then provides alternate expressions for the leading terms in the next formulas stated in (8) below.

2 In general, however, we cannot further simplify the bounded-divisor divisor sum functions since (cf. Table 1 on page 7)

\[ \sigma_{-\beta,m}(n) \neq \frac{1}{n^\beta} \sigma_{\beta,m}(n), \ m \geq 2, \beta > 0, n \geq 1. \]
That is to say, the component functions defined in (10) immediately above define the classical cases of our multiplicative, number theoretic functions of interest according to the sums

\[
\left( \frac{x}{2} \right) d(x) = \sum_{k=1}^{x-1} \tau_{2,x}^{(0)}(k) + p_{2}^{(0)}(x)
\]

\[
\left( \frac{x}{2} \right) \sigma(x) = \sum_{k=1}^{x-1} \tau_{2,x}^{(1)}(k) + p_{2}^{(1)}(x).
\]

For the higher-order cases of \( s, \alpha \geq 2 \) these component functions have much wilder and more complicated expansions. The polynomial coefficient sum expansions for \( \tau_{s,x}^{(\alpha)}(k) \) and \( \rho_{s}^{(\alpha)}(x) \) satisfy the general form noted in (9) for subsequent cases of the \( s \geq 2 \) when some real \( \alpha \geq 0 \) is fixed. Due to the page length of the formulas necessary to provide the next few special cases of these expansions when \( s \geq 2 \), we provide only the explicit finite sum forms of the case where \( s := 2, 3 \) in the next equations, and leave the verification of the growing complexity of these finite sum expressions when \( s \geq 4 \) as an enlightening computational exercise.

\[
\tau_{2,x}^{(\alpha)}(k) = \frac{1}{4} \left( k^2 \sigma_{2-3}(k) + k(k-3)\sigma_{2-1,3}(k) - (3k-2)\sigma_{2,3}(k) + 2\sigma_{2,1,3}(k) \right)
\]

\[
\rho_{2}^{(\alpha)}(x) = \frac{1}{4} \left( x^2 \sigma_{2-3}(x) + x(x-3)\sigma_{2-1,3}(x) - (3x-2)\sigma_{2,3}(x) + 2\sigma_{2,1,3}(x) \right)
\]

\[
\tau_{3,x}^{(\alpha)}(k) = -\frac{1}{6} k^3 \sigma_{3-3,3}(k) + \frac{1}{18} k^3 \sigma_{3-3,4}(k) + \left( -\frac{k^2}{6} + \frac{11k}{12} - \frac{1}{3} \right) \sigma_{3,4}(k)
\]

\[
\rho_{3}^{(\alpha)}(x) = -\frac{1}{6} x^3 \sigma_{3-3,3}(x) + \frac{1}{18} x^3 \sigma_{3-3,4}(x) - \frac{1}{36} (x-18)x^2 \sigma_{3-2,3}(x) + \frac{1}{12} (x-4)x^2 \sigma_{3-2,4}(x)
\]
\[-\frac{1}{6}\sigma_{a+2,4}(x) + \frac{1}{9}x^3\sigma_{a-3}(x) - \frac{1}{18}(x + 3)x^2\sigma_{a-2}(x) - \frac{1}{36}(7x^2 - 6x + 10)x\sigma_{a-1}(x)
+ \frac{1}{36}(5x^3 - 3x^2 + 8x + 12)\sigma_a(x) + \frac{1}{36}(3x + 4)x\sigma_{a+1}(x) + \frac{1}{18}(-x - 6)\sigma_{a+2}(x)\].

3. Complete proofs of the new results

3.1. Statement and proofs of key lemmas. The statement of the first key lemma we prove below allows us to expand the Lambert series generating functions in (1) in the form of (4) as outlined in the introduction. As we will see when we prove Proposition 2.4 in the next subsection, this particular expansion of the Lambert series generating functions whose power series expansions in \(q\) enumerate the generalized sums of divisors functions is key to establishing the new results cited in the previous sections.

**Lemma 3.1.** For any formal indeterminate and integers \(i \geq 2\), we have the following expansion:

\[
\frac{1}{1 - q^i} = \frac{1}{i} \left( \frac{1}{1 - q} + \sum_{j=0}^{i-2} \frac{(i - 1 - j)q^j(1 - q)}{1 - q^i} \right)
\]

**Proof.** We start by noticing that \((1 - q^i)/(1 - q) = 1 + q + \cdots + q^{i-1}\) by a finite geometric-series-like expansion. If we combine denominators of the two fractions we obtain that

\[
1 + q + \cdots + q^{i-1} + (1 - q)\sum_{j=0}^{i-2}(i - 1 - j)q^j
\]

Then by simplifying the rightmost sum terms in the numerator of the above expansion we see that

\[
(1 - q)\sum_{j=0}^{i-2}(i - 1 - j)q^j = i - 1 - q^{-1} + \sum_{j=0}^{i-3}((i - 2 - j) - (i - 1 - j))q^{j+1} = i - \sum_{j=0}^{i-1}q^j,
\]

and so the result is proved. \(\square\)

Lemma 3.2 stated immediately below provides another Lambert series transformation identity which we will need to complete the proof of the proposition in the next subsection.

**Lemma 3.2** (Higher-Order Derivatives of Lambert Series). For any fixed non-zero \(q \in \mathbb{C}\), \(i \in \mathbb{Z}^+\), and prescribed integer \(s \geq 0\), we have the following two results stated in (5) of the introduction:

\[
q^sD^{(s)}\left[\frac{q^i}{1 - q^i}\right] = \sum_{m=0}^{s} \sum_{k=0}^{m}\left[\begin{array}{c}s \\ m \end{array}\right] \left[\begin{array}{c}m \\ k \end{array}\right] \frac{(-1)^{s-k}k! \cdot i^m}{(1 - q^i)^{k+1}} \]  

(i)

\[
q^sD^{(s)}\left[\frac{q^i}{1 - q^i}\right] = \sum_{r=0}^{s} \sum_{m=0}^{s} \sum_{k=0}^{m}\left[\begin{array}{c}s \\ m \end{array}\right] \left[\begin{array}{c}m \\ k \end{array}\right] \frac{(-1)^{s-k}r! \cdot i^m}{(1 - q^i)^{k+1}} q^{(r+1)i} \]  

(ii)

**Proof.** The expansion of the second identity in (ii) follows by applying the binomial theorem to the first identity and changing the index of summation. We seek to prove the first identity in (i) by induction on \(s\). When \(s := 0\), the Stirling number terms are identically equal to one, and so we obtain \(q^i/(1 - q^i)\) on both sides of the equation. Next, we suppose that (i) is correct for all \(s < t\) for some \(t \geq 1\). We then use our assumption to prove that (i) is correct when \(s = t\). We notice that

\[
q^{s+1} \cdot D\left[\frac{1}{q^i \cdot (1 - q^i)^{k+1}}\right] = \frac{i(k + 1)q^i}{(1 - q^i)^{k+2}} - \frac{s}{(1 - q^i)^{k+1}}.
\]

\[
q^{s+1} \cdot D\left[\frac{1}{q^i \cdot (1 - q^i)^{k+1}}\right] = \frac{i(k + 1)q^i}{(1 - q^i)^{k+2}} - \frac{s}{(1 - q^i)^{k+1}}.
\]
which by our inductive hypothesis implies that we have the following simplifications resulting from applying the triangular recurrence relations for the Stirling numbers to the inner sum terms, shifting summation indices, and since $\left[\frac{s+1}{2}\right] \equiv 0$ for all $s \geq 0$ [cf. §26.8]:

$$q^{s+1}D^{(s+1)} \left[ \frac{q^i}{1-q^i} \right] = \sum_{m=0}^{s} \sum_{k=0}^{m} \binom{s}{m} \binom{m}{k} \left( -\frac{i(k+1)q^i}{(1-q^i)^{k+2}} + \frac{s}{(1-q^i)^{k+1}} \right) (-1)^{s+1-k}k!m^n$$

$$= \sum_{m=0}^{s} \sum_{k=0}^{m} \binom{s}{m} \binom{m}{k} \left( \frac{(k+1)i^{m+1}}{(1-q^i)^{k+2}} + \frac{(k+1)i^{m+2}}{(1-q^i)^{k+1}} + \frac{s \cdot i^m}{(1-q^i)^{k+1}} \right) \times (-1)^{s+1-k}k!$$

$$= \sum_{m=0}^{s} \sum_{k=0}^{m+1} \binom{s+m+1}{m+1} \left( \frac{(-1)^{s+1-k}k!i^{m+1}}{(1-q^i)^{k+1}} \right)
+ \sum_{m=0}^{s+1} \sum_{k=0}^{m} \binom{s}{m} \binom{m}{k} \frac{s \cdot (-1)^{s+1-k}k!i^m}{(1-q^i)^{k+1}}$$

$$= \sum_{m=0}^{s+1} \sum_{k=0}^{m} \binom{s+1}{m} \binom{m}{k} \left( \frac{(-1)^{s+1-k}k!i^m}{(1-q^i)^{k+1}} \right).$$

The next lemma is used as another transformation of the higher-order derivatives of the Lambert series identity in (4). Despite being straightforward to state, the two right-hand-side formulas which we make particular use of in the proof in the next subsection are somewhat non-obvious to prove directly. We prove (i) in the lemma in complete detail and then leave the similar proof of (ii), which we note follows easily from the first, as an exercise.

**Lemma 3.3 (A Pair of Interesting Utility Sums).** For any fixed non-zero $q$ and integers $p \geq 1$ and $n \geq 0$, we have the next two identities.

$$\sum_{j=0}^{n} \frac{j^i}{(j-p)!} q^j = \frac{1}{(1-q)^{p+1}} \left( p! \cdot q^p + \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^{k+1}(n+1)!q^{n+k+1}}{(n-p)!(n+1-p+k)} \right)$$  \hspace{1cm} (i)

$$\sum_{j=0}^{n} \frac{(j+1)!}{(j+1-p)!} q^j = \frac{1}{(1-q)^{p+1}} \left( p! \cdot q^{p-1} + \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^{k+1}(n+2)!q^{n+k+1}}{(n+1-p)!(n+2-p+k)} \right)$$  \hspace{1cm} (ii)

**Proof of (i).** We first define the two sums, $S_{i,p,n}$, for $i = 1, 2$ and positive integers $p, n \geq 1$ by

$$S_{1,p,n} := \sum_{k=0}^{p} \frac{(-1)^{p-k+1}q^{n+1+p-k}(n+1)!}{(n-p)!(n+1-k)(1-q)^{p+1}}$$

$$S_{2,p,n} := \sum_{j=0}^{n} \frac{j!}{(j-p)!} q^j - \frac{p!q^p}{(1-q)^{p+1}},$$

where the generating function for the second sum follows by differentiation of the geometric series function in the form of

$$S_p(z) := \sum_{n \geq 0} S_{2,p,n} z^n = \frac{p!q^p}{(1-z)(1-qz)^{p+1}} - \frac{p!q^p}{(1-q)^{p+1}(1-z)}. \hspace{1cm} (iii)$$

Next, we can compute using Mathematica’s *Sigma* package that the first sums defined above satisfy the following homogeneous recurrence relation:

$$(n(p+1)q - p(p+1)q) S_{1,p,n} + (n(q-1) + p - 2q - 2pq) S_{1,p+1,n} + (1-q)S_{1,p+2,n} = 0. \hspace{1cm} (iv)$$
We complete the proof by showing that the sums, $S_{2,p,n}$ also satisfy the recurrence relation in (iv) for all integers $p \geq 1$, and that each of these sums produce the same formulas in $q$ and $n$ for the first few cases of $p \geq 1$. To show that the second sums satisfy the recurrence in (iv), we perform the following equivalent computations using the generating functions for the sequence in (iii) with Mathematica:

$$(p + 1)qzD - p(p + 1)q [S_p(z)] + ((q - 1)zD + p - 2q - 2pq) [S_{p+1}(z)] + (1 - q) S_p(z) = 0.$$ 

Finally, we compute the first few special cases of these sums in the form of

$$S_{12,1,n} = \frac{q^{n+1}}{(1 - q)^2} (qn - (n + 1))$$

$$S_{12,2,n} = \frac{q^{n+1}}{(1 - q)^3} (-n(n - 1)q^2 + 2(n + 1)(n - 1)q - (n + 1)n)$$

$$S_{12,3,n} = \frac{q^{n+1}}{(1 - q)^4} (n(n - 1)(n - 2)q^3 - 3(n + 1)(n - 1)(n - 2)q^2 + 3(n + 1)n(n - 2)q - (n + 1)n(n - 1)).$$

Thus we have shown that the two sums satisfy the same homogeneous recurrence relations and initial conditions, and so must be equal for all $p, n \geq 1$ giving the proof of our result. □

Finally, the last lemma we prove makes the justification of the series coefficient formulas we claimed in Definition 2.3 rigorous. This lemma is included for completeness even though the results are not difficult to obtain in each of the three cases.

**Lemma 3.4 (Formulas for Special Series Coefficients).** For any fixed $\alpha \geq 0$ and integers $s \geq 0$, we have the following series coefficient formulas cited in the statement of Definition 2.3 from Section 2.1:

$$[q^x]q^s D(s) \left[ \frac{F_\alpha(q)}{1 - q} \right] = \sum_{r=0}^{s} \sum_{k=1}^{\alpha} \binom{s}{r} \frac{(x - k)(s - r)k!}{(k - r)!} \sigma_\alpha(k)$$  \hspace{1cm} (i)

$$[q^x]q^s D(s) \left[ \sum_{i \geq 1} \frac{i\alpha - q^i}{1 - q^2} \right] = \sum_{r=0}^{s} \sum_{k=0}^{\alpha - 1} \binom{s}{r} \frac{(x - k + 1)(s - r + 1)k^{\alpha - 1} \cdot k!}{(k - r)!}$$  \hspace{1cm} (ii)

$$[q^x]q^s D(s) \left[ \sum_{i \geq 1} \frac{i\alpha - q^i}{1 - q} \right] = \sum_{r=0}^{s} \sum_{k=0}^{\alpha - 1} \binom{s}{r} \frac{(x - k)(s - r)k^{\alpha - 1} \cdot k!}{(k - r)!}.$$  \hspace{1cm} (iii)

**Proof of a More General Result.** The proofs of all three results are almost identical. We prove a more general result, of which the three identities are special cases. Namely, if we let $G(q) := \sum_n g_n q^n$ denote the ordinary generating function of an arbitrary sequence, and suppose that $m \geq 0$, then we have that

$$[q^x]q^s D(s) \left[ \frac{G(q)}{(1 - q)^{m+1}} \right] = \sum_{r=0}^{s} \sum_{k=0}^{m} \binom{s}{r} \frac{(x - k + m)(s - r - m)k!}{m! \cdot (k - r)!} g_k.$$  \hspace{1cm} (iv)

We need a few results which we will give in a laundry listing format below where $f(q)$ and $g(q)$ denote functions which are each differentiable up to order $s$ for some integers $s, m \geq 0$.

$$q^s \cdot D^{(s)} [f(q)g(q)] = \sum_{r=0}^{s} \binom{s}{r} q^r f^{(r)}(q) \times q^{s-r}g^{(s-r)}(q)$$ \hspace{1cm} (12)

$$q^s D^{(s)} \left[ \frac{1}{(1 - q)^{m+1}} \right] = \frac{q^s \cdot (m + 1 + s)!}{m! \cdot (1 - q)^{m+s+1}}$$
\[ [q^r] \frac{1}{(1 - q)^{k+1}} = \binom{x+k}{k} \]
\[ [q^r] q^s \cdot G^{(s)}(q) = \frac{x!}{(x-s)!} \cdot g_x \]

Once we have the results stated above, our generalized result in (iv) is only a matter of assembly of the double sums, and so we are done. These results are either well-known coefficients of power series (as in the last two cases), are established formulas, or are trivial to prove by induction (as in the second identity case). The identity in (i) corresponds to the special case where \((g_n, m) := (\sigma_\alpha(n), 0)\), (ii) to the special case where \((g_n, m) := (n^{\alpha-1}, 1)\), and (iii) to the case of (iv) where \((g_n, m) := (n^{\alpha-1}, 0)\).

3.2. Proof of the key proposition expanding Lambert series generating functions.

With the key lemmas from the previous subsection at our disposal, we are finally able to complete the more involved, and quite mechanical, proof of Proposition 2.4.

**Proof of Proposition 2.4.** We begin by defining the following shorthand for the higher-order \(s^{th}\) derivatives of inner summation terms in the expansion of our key Lambert series identity in (4), which we recall that we proved to be correct in Lemma 3.1 above:

\[
\text{Sum}_{s,i}(q) := \sum_{j=0}^{i-2} q^s D^{(s)} \left[ \frac{(i-j-1)q^{i+j}}{(1-q^i)} \right].
\]

If we then apply Lemma 3.2 in combination with the first generalized product rule identity stated in (12), we see that

\[
\text{Sum}_{s,i}(q) = \sum_{j=0}^{i-2} \sum_{r=0}^{s} \sum_{s-r}^{s-r} \sum_{m=0}^{s-r} \sum_{k=0}^{m} (i-j+1) \binom{s-r}{r} \binom{s-r}{m} \binom{s-r-k}{p} \frac{j!}{(j-r)!} \times
\]
\[
\times (-1)^{s-r-k-p} k! \cdot i^m q^{(p+1)i+j} \left(1 - q^i\right)^{k+1}.
\]

To simplify notation for the multiple sum formulas we obtain in the next few formulas, we define the following sums where we denote \(\Sigma_k\) to be the multiple sum defined by \(\Sigma_k\) over the function, \(f\), whose inputs are implicit to the indices of summation in \(\Sigma_k\):

\[
\Sigma_4 := \sum_{r=0}^{s} \sum_{s-r}^{s-r} \sum_{m=0}^{s-r} \sum_{k=0}^{m} \sum_{p=0}^{s-r} \sum_{m=0}^{s-r} \sum_{k=0}^{m} (i-j+1) \binom{s-r}{r} \binom{s-r}{m} \binom{s-r-k}{p} \frac{j!}{(j-r)!} \times
\]
\[
\times (-1)^{s-r-k-p} k! \cdot i^m q^{(p+1)i+j} \left(1 - q^i\right)^{k+1}.
\]

Then when we employ the notation defined in the previous equations to the last expansion of \(\text{Sum}_{s,i}(q)\), we can perform the two sums over \(j\) using the pair of results in Lemma 3.3 to obtain that

\[
\text{Sum}_{s,i}(q) = \Sigma_4 \binom{s}{r} \binom{s-r}{m} \binom{m}{k} \binom{s-r-k}{p} \frac{(-1)^{s-r-k-p} k! r! q^{(p+1)i+r}}{(1-q^i)^{k+1}} \times
\]
\[
\times \left( \frac{i^{m+1}}{(1-q)^{r+1}} - \frac{(r+1) i^m}{(1-q)^{r+2}} \right)
\]
\[
+ \Sigma_5 \binom{s}{r} \binom{s-r}{m} \binom{m}{k} \binom{s-r-k}{p} \frac{(-1)^{s-r-k-p+n+1} k! q^{(p+1)i+n-1}}{(1-q^i)^{s-r+1}} \times
\]
\[ \frac{i!}{(i - 1 - r)!(i - r + n)} = \prod_{j=0}^{r-n-1} (i - j) \times \prod_{j=r-n+1}^{r} (i - j) = \left( \sum_{m=0}^{r-n} \binom{r-n}{m} (-1)^{r-n-m} \right) \times \left( \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} (i + n - 1 - r)^m \right) = \sum_{v=0}^{r-n} \sum_{m_1=0}^{r-n} \sum_{m_2=0}^{n} \left[ r - n \right] \left[ \begin{array}{c} m \\ m_1 \end{array} \right] \left[ \begin{array}{c} n \\ m_2 \end{array} \right] (v - m_1) (-1)^{r+m_1+m_2} \times \times (n - 1 - r)^{m_1+m_2-v} \cdot i^v \]

Finally, we are now in a position to define equivalent forms of the summation functions depending on \( q \) and \( i \) in Definition 2.2 as

\[
\text{Sum}_{s}(q, i) = \text{Sum}_{s}(q, i) = \sum_{r} \binom{s-r}{m} \binom{m}{k} \binom{s-r-k}{p} (-1)^{s-r-k-p} k! r! q^{(p+1)i+r} \frac{1}{(1-q)^{s-r+1}} \times \left( \frac{i^{m+1}}{(1-q)^{r+1}} - \frac{(r+1)i^m}{(1-q)^{r+2}} \right) \]

and

\[
\text{Sum}_{s}(q, i) = \sum_{r} \binom{s-r}{m} \binom{m}{k} \binom{s-r-k}{p} (-1)^{s-r-k-p+n+m_1+m_2} k! r! q^{(p+2)i+n-1} (n - 2 - r)^{m_1+m_2-w} \times \frac{1}{(1-q)^{s-r+1}} \times \left( \frac{r}{n} \frac{1}{(1-q)^{r+1}} - \frac{r+1}{n} \frac{1}{(1-q)^{r+2}} \right) .
\]

Thus by defining the component sum coefficients as in Definition 2.1 and re-writing the above two sums, we can apply Lemma 3.4 to the remaining term in the sum from (4) arrive at the conclusion of the proposition, and so we are done. \( \square \)

4. Conclusions

4.1. Summary. The identities we have derived in the article are new and are derived from elementary series transformations, operations on Lambert series, and new expansions of the higher-order \( s^{th} \) derivatives of the Lambert series generating functions in (1) and in (4). These identities provide exact formulas for the generalized sum of divisors functions, \( \sigma(n) \), defined through polynomial sums over other sums of divisor functions and the bounded-divisor sum functions, \( \sigma_{\alpha,m}(n) \), that we defined as the series coefficients of generalized Lambert series expansions in (2) of the introduction. We remark that the requirements of the bounded divisors welcoming \textit{convolute} (rather than \textit{convolve}) our results for the ordinary divisor sum cases in the sense that these identities are complicated, or at least usefully obfuscated in their immediate forms, by the terms involving \( \sigma_{\alpha,m}(n) \) which do not in general have well-specified closed-form expression in terms of the \( \sigma_{\alpha}(n) \) for \( m \geq 3 \). There do not appear to be correspondingly elegant identities that can be derived from operations in more general Lambert series expansions over other multiplicative functions, \( f(i) \), when \( f(i) \neq i^\alpha \) for some \( \alpha \geq 0 \).
4.2. Comparison to existing results. There are several known forms of divisor sum function convolution identities of the form

$$\sigma_\alpha(n) = c_1 \left( (a_1 n + a_2)\sigma_\beta(n) + a_3 \sigma_\gamma(n) + a_4 \times \sum_{k=1}^{n-1} \sigma_\gamma(k)\sigma_\beta(n-k) \right),$$

found in the references for the triples \((\alpha, \beta, \gamma) := (3, 1, 1), (5, 3, 1), (7, 5, 1), (9, 7, 1), (9, 5, 3)\) which result from identities and functional equations satisfied by Eisenstein series [2, 3]. For example, the following two convolution identities are known relating the generalized sums of divisors functions over differing inputs to the \(\alpha\) parameter input to these functions:

\[
\begin{align*}
\sigma_3(n) &= \frac{1}{5} \left( 6n \cdot \sigma_1(n) - \sigma_1(n) + 12 \times \sum_{k=1}^{n-1} \sigma_1(k)\sigma_1(n-k) \right) \\
\sigma_5(n) &= \frac{1}{21} \left( 10(3n - 1) \cdot \sigma_3(n) + \sigma_1(n) + 240 \times \sum_{k=1}^{n-1} \sigma_1(k)\sigma_3(n-k) \right).
\end{align*}
\]

In contrast with these results, we have sums of polynomial multiples of generalized divisor sums and bounded-divisor divisor functions in place of the discrete convolutions of these functions in the pair of identities cited above. Our results, like the classes of divisor convolution sum identities expanded in the previous equations, do also imply exact generating function representations for the generalized divisor functions, \(\sigma_\alpha(n)\) and \(\sigma_{\alpha,m}(n)\), which are new and consequently may imply additional results in future work based on these expansions.

4.3. Other applications and future research topics. The subject of finding new interpretations of our results proved in Theorem 2.6 and Corollary 2.7 is key to future research investigating the properties and consequences of the new results presented in this article. For example, since we know exact multiplicative formulas for these generalized functions at prime powers as \(\sigma_\alpha(p^k) = (p^{(k+1)\alpha} - 1)/(p^\alpha - 1)\), we may attempt to use the formulas in Corollary 2.7, which must hold at any \(x \geq 1\) to write

$$\sigma_\alpha(x) = \sum_{d \leq x} c_\alpha(d, x)d^\alpha$$

and then use the prime powers formula requirement to attempt to determine non-trivial bounds on the \(c_\alpha(d, x)\) terms. Another application that is suggested by the results in Theorem 2.6 and Corollary 2.7 is to formulate new congruences for the generalized divisor functions, Finally, we would be totally remiss not to mention a new application that we may consider is the new expansions of the ordinary sum of divisors function, \(\sigma(2n)\), from the special cases examples cited in Example 2.8 to formulate new conditions for the cases of \(n\) corresponding to the \textit{perfect numbers}, the \textit{deficient numbers}, and the \textit{abundant numbers}, respectively, in terms of our new finite sum formulas involving the bounded-divisor sum functions, \(\sigma_{\alpha,m}(n)\).

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