Abstract—By leveraging experience from previous tasks, meta-learning algorithms can achieve effective fast adaptation ability when encountering new tasks. However, it is unclear how the generalization property applies to new tasks. Therefore, a theoretically correct (PAC) Bayes bound theory provides a theoretical framework to analyze the generalization performance for meta-learning. We derive three novel generalisation error bounds for meta-learning based on PAC-Bayes relative entropy bound. Furthermore, using the empirical risk minimization (ERM) method, a PAC-Bayes bound for meta-learning with data-dependent prior is developed. Experiments illustrate that the proposed three PAC-Bayes bounds for meta-learning guarantee a competitive generalization performance guarantee, and the extended PAC-Bayes bound with data-dependent prior can achieve rapid convergence ability.

Index Terms—Meta-learning, statistical learning, generalization, PAC-Bayes bound, data dependent prior

I. INTRODUCTION

Machine learning models often require training with a large number of samples, for example, image classification issue [1]–[4]. Besides, traditional machine learning algorithms mainly focus on a single task. But it is generally difficult to collected so much labelled data. So how to train a model when only a small amount of data is available? More to the point, since humans can learn new skills much faster and more effectively, how can we build such a model, which can reflect aspects of human learning? That is what meta-learning sets out. Meta-learning — or “learning to learn” — is capable of learning very complex uncertainty structure. The core idea of black-box adaptation meta-learning is to train a neural network to represent a meta-learner. With the aim of achieving a fast adaptation ability, [7] a meta-learning algorithm is presented with memory-augmented neural networks, which can summarize and storage important knowledge. When facing new learning tasks, the memory-based method can extract certain skills it has experienced to assist in the current process. In order to adapt to access past experiences, a simple neural attentive-learner (SNAIL) is proposed by [8].

Non-parametric methods try to utilize a non-parametric learner as meta-learner instead of parametric models. Non-parametric methods are simple and perform well in few-shot learning. [9] proposes a siamese neural network, which contains two sub-networks with same weights. During training phase, the two sub-networks can extract features from two different input vectors, and then compute the distance between the two feature vectors. Matching networks are another non-parametric method, which is presented by [10]. In order to learn from a few examples, matching network framework learns a net structure that maps few labelled training datasets and an unlabelled instance to its label. Combined with recent advances in attention and memory, the matching networks enable rapid learning. Besides, [11] proposes a prototypical network, where classification problem is regarded as finding the prototype center of each category in the semantic space and then predict the category of the new sample by the nearest neighbor classifier. This method mainly combines the prototype network with clustering algorithm.

Different from two aforementioned algorithms, optimization-based meta-learning algorithms learn to train the parameter vector to represent the meta-learner through optimization. In the traditional gradient-descent approach, optimization updates rules, for example the learning step, are still hard to design. [12] considers this issue as a learning problem, allowing the optimization algorithms learn to exploit update rules structure in an automatic way. Furthermore, [13] propose another LSTM-based meta-learner model by combining gradient descent and LSTM algorithm, which is applied to train neural network. In order to extract common knowledge from previous task so as to achieve fast adaptation ability, [14] proposes a model agnostic meta-learning (MAML) algorithm. The key idea of MAML is to learn a set of initialization parameter that allows efficient learning of new tasks. However, MAML requires the computation of second-order derivative which may exhibit instabilities. Therefore [15] present a scalable meta-learning algorithm, called Reptile, which does not calculate any second derivatives. Besides, [16] addresses the training of MAML and propose several tricks to improve the stability of MAML.

One of a majority challenges in few-shot learning is task ambiguity. [17] proposes a probabilistic MAML, which tries to incorporate a parameter distribution with neural network that is trained via a variational lower bound. In order to improve the robustness of MAML, [18] propose a Bayesian MAML algorithm. Compared with a point estimate or a simple Gaussian approximation in fast adaptation phase, this algorithm is capable of learning very complex uncertainty structure. The
Bayesian MAML outperforms vanilla MAML in terms of accuracy and robustness. Furthermore, based on Bayesian inference framework and variational inference, [19] propose a new Bayesian task-adaptive meta-learning (Bayesian TAML) algorithm for imbalanced and out-of-distribution tasks. In addition, several improved MAML are also introduced, such as Alpha MAML [20], meta-learning with latent embedding optimization [21] and Bayesian hierarchical modeling based MAML [22].

Although meta-learning algorithms provide a powerful inductive biases based on various tasks, even with those which comprise only limited data, its generalization performance is poorly understood. PAC-Bayes theory, known as generalization error bounds theory, provides a theoretical framework for estimating the generalization performance of the machine learning model.

The first PAC-Bayes theory was established by McAllester [23], which provides generalization error upper bounds for the performance of randomized learning algorithms. Then this method was subsequently used to analyze the generalization-error bound of the stochastic neural network [24]. PAC-Bayes bound theories were meant for a wide range of approximate Bayesian GP classification issues [25], [26]. One source [27] tries to explain the generalization in neural network from the view of norm-based control, sharpness and robustness, and attempts to build a connection between sharpness and PAC-Bayes theory. The systemically undertaken study is addressed with a view to training stochastic neural networks based on the PAC-Bayes bounds in [28].

That PAC-Bayes theory is only suitable for bounded loss function and i.i.d data. PAC-Bayesian bounds tailored for the sub-Gaussian or sub-Gamma loss family, such as negative log-likelihood function, is also developed by [29] and [30]. However, those algorithms require a distribution parameter, such as a variance factor and a scale parameter. Therefore, [31] proposes an exponential bound under the assumption that the first three moments of the loss distribution are bounded. By introducing the special boundedness condition, [32] expands the PAC-Bayesian theory to learning problems with unbounded loss functions.

Recently, there has been a gradually increasing interest in research on overparameterized deep neural networks and SGD. [33] study the generalization of randomized learning algorithms trained with SGD. Inspired by [24], [34] obtains nonvacuous generalization numerical bounds for deep stochastic neural network classifiers with many more parameters than are present in the training data. The first non-vacuous generalization bound for compressed networks applied to the ImageNet classification problem is provide in [35]. Moreover, [36] further investigates the relationship between generalization performance and SGD.

As mentioned above, PAC-Bayesian bound is only valid for stochastic classifiers, although a growing body of literature illustrates efforts to construct PAC-Bayes bounds on deterministic classifiers. To fill this gap, [37] develops a PAC-Bayesian transportation bound, by unifying the PAC-Bayesian analysis and the chaining method. This generalization error bound relates the distance between any two predictors, both for stochastic classifiers and deterministic classifiers. A new perturbation bounds for feedforward neural networks is derived based on the sharpness of a model class by [38]. In addition, [39] presents a general PAC-Bayesian framework for the deterministic and uncompressed neural network by leveraging the noise-resilience of deep neural networks on training data.

In order to achieve tighter generalization error bounds, [40] proposes two alternative prior distributions: one is to learn a prior distribution from a separate training data set which is not used in computing the bound, and another is to consider an expectation prior. [41] further investigates that PAC-Bayes bound with localized prior distribution defined in terms of the data generating distribution. Under the stability of the hypothesis, a Gaussian prior distribution, informed by the data-generating distribution and centered at the expected output, is proposed for the SVM classifier [42]. More discussion can be seen in [43] and [44]. Furthermore, because data distribution is usually unknown, [45] develops a PAC-Bayes bound via ε-differentially private data-dependent prior.

PAC-Bayes theory provides a theoretical framework for the generalization performance analysis of meta-learning. This theory can be considered as a generalized framework which is more resistant to over-fitting and that yields a generalization error upper bound that holds with an arbitrarily high probability. For meta learning, [46] provides a generalization error bound within the PAC-Bayes framework for lifelong learning. Furthermore, two principled algorithms are implemented, including parameter and representation transfers. More recently, [47] develops a theoretical framework for meta-learning, allowing extended various PAC-Bayes bounds to meta-learning. To add to this, [48] considers the scenario in which a common model set is used for model averaging via a model selection procedure that accounts for the model’s uncertainty. Two data-based algorithms are proposed to obtain ideal priors for model averaging.

Specifically, a gradient-based algorithm which minimizes an objective function derived from PAC-Bayes bounds is also applied to training deep neural networks. The tighter bounds might achieve a better generalization performance. Besides, in PAC-Bayes theory, prior distribution is selected randomly before learning. Generally, with PAC-Bayes, the generalization error upper bound is primarily determined by the distance between prior and posterior distributions. Obviously the choice of prior distribution affects the performance of the PAC-Bayes bound significantly.

Motivated by the previous discussions, three novel generalization error bounds for meta-learning are presented. Furthermore, a data-based approach for adjusting prior distribution is developed, and the specific implementations of those algorithms are given. The main contributions are concluded as follows:

- In order to improve generalization performance, based on the PAC-Bayes relative entropy theory, meta-learning PAC-Bayes λ bound and meta-learning PAC-Bayes quadratic bound are proposed;
- Using the variational Kullback-Leibler (KL) bound, meta-learning PAC-Bayes variational bound is investigated, which can achieve tighter generalisation error bound by taking the piecewise combination of the two above-mentioned meta-learning bounds;
- Based on ERM method, a PAC-Bayes bound for meta-
learning with data-dependent prior is developed by adjusting the priors to attain fast convergence ability.

- Empirical demonstration illustrates that the proposed algorithms achieve competitive generalization guarantee and better convergence performance.

The rest of this paper is organized as follows. The classical PAC-Bayes bounds for both single task and meta-learning is introduced in Section II. Section III investigates three novel PAC-Bayes bounds for meta-learning, based on the PAC-Bayes relative entropy bound theory. A PAC-Bayes bound for meta-learning with data-dependent prior is developed in Section IV. The implementation details are described in Section V. Section VII draws some conclusions. Finally, Section VI provides numerical examples to verify the proposed algorithms.

II. PRELIMINARIES: PAC-BAYES THEOREM

In the classical statistical learning model setting, a set of dependent samples $S = \{z_i\}_{i=1}^m$ is randomly drawn from the unknown data distribution $\mathcal{D}$ over space $\mathcal{Z}$. In the supervised learning, $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X} \subset \mathbb{R}^d$ and $\mathcal{Y} \subset \mathbb{R}$, each sample $Z_i = (X_i, Y_i)$ consists of an input $X_i$ and its corresponding label $Y_i$. The learning objective is to find a classifier $h \in \mathcal{H}$ that predicts the label and minimizes the expected loss $\mathbb{E}\{\ell(h, z)\}$, where $\mathcal{H}$ is considered as the hypothesis space and $\ell(h, z) : \mathcal{H} \times \mathcal{Z} \rightarrow \mathbb{R}$ is the loss function which are used to measure the performance of prediction. For the classification problems, the loss function is always bounded in $[0, 1]$. In the statistical inference stage, the core idea of machine learning is to minimize the expected error $er(h, S)$ under the data distribution $\mathcal{D}$

$$er(h, \mathcal{D}) = \mathbb{E}_{z \sim \mathcal{D}} \ell(h, z).$$

Since the data distribution $\mathcal{D}$ is unknown, generalization error $er(h, \mathcal{D})$ cannot be calculated. Therefore, the empirical error $\hat{er}(h, S)$ gives an observable estimation

$$\hat{er}(h, S) = \frac{1}{n} \sum_{i=1}^n \ell(h, z_i).$$

Under certain neural network conditions, in order to minimize the empirical risk, a single classifier $h_w \in \mathcal{H}$ is selected. However, this may cause that the learned classifier $\hat{h}_w$ to form too close a fit to a limited set of data points $S$ — creating a case of over-fitting, which can be measured by $er(h, \mathcal{D}) - \hat{er}(h, S)$. Various methods are used to avoid over-fitting, various methods are used, including complexity regularization.

A. PAC-Bayes bounds for single task

PAC-Bayes theory, known as generalization error bound theory, is a framework for theoretical generalization performance analysis of a machine-learning model.

In the PAC-Bayes theory, in contrast with the classical neural network which aims to learn data-dependent parameter weights, the probability neural network is employed to learn a data-dependent distribution over weights. Specifically, based on the “prior” distribution $P \in \mathcal{M}$, PAC-Bayes bound tries to learn a posterior distribution $Q(S, P) \in \mathcal{M}$ from training data $S$, where $\mathcal{M}$ denotes the set of distributions over hypothesis space $\mathcal{H}$. Then generalization error $er(Q, S)$ and empirical error $\hat{er}(Q, S)$ are defined as the expectation over posterior distribution $Q$, such as $er(Q, \mathcal{D}) \triangleq \mathbb{E}_{h \sim Q} er(h, S)$ and $\hat{er}(Q, \mathcal{D}) \triangleq \mathbb{E}_{h \sim Q} er(h, S)$. The first PAC-Bayes generalization theory for single task learning issue has been proposed by [23].

Lemma 1 (McAllester’s single-task bound [23]). Let $P \in \mathcal{M}$ be some prior distribution over $\mathcal{H}$, then for any $\delta \in (0, 1]$, the following inequality holds uniformly for all posteriors distributions $Q \in \mathcal{M}$ with probability at least $1 - \delta$

$$er(Q, \mathcal{D}) \leq \hat{er}(Q, S) + \sqrt{\frac{D(Q||P) + \log \frac{1}{\delta}}{2(m-1)}}. \tag{3}$$

Here $D (\rho || \rho_0)$ is the KL divergence, which measures the difference between two distributions:

$$KL (\rho || \rho_0) \overset{\text{def}}{=} \mathbb{E}_{c \sim \rho} \left[ \ln \frac{d\rho(c)}{d\rho_0(c)} \right], \tag{4}$$

where $d\rho(c)$ is the Radon-Nikodym derivative of $\rho$ with respect to $\rho_0$. The Radon-Nikodym derivative can be substituted with the ratio of Probability Density Functions (PDF) if they exist.

Generally, a PAC-Bayes generalization theory attempts to balance the discrepancy between a prior distribution $P$ with posterior distribution $Q$, and empirical risk $\hat{er}(Q, S)$. Here, we should emphasize that prior distribution $P$ is selected randomly before learning, which must not be dependent on training data. Besides, posterior distribution $Q(S, P)$ does not necessarily have to be the traditional Bayesian posterior distribution. The prior distribution $P$ is used chiefly to measure the distance of hypothesis space $\mathcal{H}$. Obviously, the choice of prior distribution $P$ significantly affects the performance of PAC-Bayes bound significantly.

B. PAC-Bayes bounds for meta-learning

In this subsection, the PAC-Bayes bound for meta-learning is introduced. meta-learning comprises two parts: meta-learner extracts common knowledge (prior knowledge) from different observed tasks, and base learner aims to adapt new tasks. In the PAC-Bayes meta-learning framework, we assume that all tasks belong to the same distribution $\mathcal{T}$. Different tasks share the same sample space $\mathcal{Z}$ and loss function $\ell(h, z) : \mathcal{H} \times \mathcal{Z} \rightarrow \mathbb{R}$. For each observed task $\tau_i$, the corresponding $S_i$ are generated from the unknown distribution $S_i \sim D^{m_i}_\tau$, where $m_i$ is the number of training samples for task $i$. As mentioned before, the meta-learner tries to extract common knowledge $P \sim \mathcal{M}(\mathcal{H})$ from tasks $\tau$ before, based on the prior information $P$ and new task’s data $S$, base learner learns the posterior information $Q(S, P) : \mathcal{Z}^{m_i} \times \mathcal{M}(\mathcal{H}) \rightarrow \mathcal{M}(\mathcal{H})$ to infer the process applied when faced with new tasks. Here prior information $P$ and posterior information $Q(S, P)$ are represented as distribution of neural network weights $w$, characterized as the mean and co-variance.

For the meta-learning generalization error bound theory, based on the hyper prior $P$, which is a distribution over prior distribution $P$, meta-learner aims to learn a hyper posterior $Q(P)$ by utilizing the observed tasks. When encountering new tasks, base learner samples a prior distribution $P$ from the hyper-posterior $Q(P)$. Capitalizing on observed samples of the
new task, base learner infers a posterior distribution \( Q(S, P) \). The performance of hyper-posterior \( Q \) can be measured by the expectation loss of prior \( P \) when learning new tasks, the so-called generalization error

\[
er(Q, \tau) \triangleq \mathbb{E}_{P \sim Q} \left( \mathbb{E}_{h \sim Q(S, P)} \ell(h, z) \right). \tag{5}\]

While \( er(Q, \tau) \) is not commutable in practice, nevertheless, we can estimate this generalization error by the 

\[
\hat{er}(Q, S) \triangleq \mathbb{E}_{P \sim Q} \frac{1}{n} \sum_{i=1}^{n} \hat{er}(Q(S_i, P), S_i). \tag{6}\]

Different to the single-task PAC-Bayes bound, in meta-learning, meta-learner chooses a hyper-prior distribution \( \mathcal{P} \) over prior distribution \( P \), following observed samples for all training tasks, and updates it to hyper-posterior distribution \( Q \). Base learner selects a prior distribution \( P \sim \mathcal{P} \) and updates it to posterior distribution \( Q(S, P) \) when learning new tasks.

Firstly, we introduce the classical extended PAC-Bayes bound proposed by \cite{47} as follows

**Lemma 2** (Classical meta-learning PAC-Bayes bound \cite{47}). Let \( \mathcal{P} \in \mathcal{M} \) be some hyper-prior distribution over \( \mathcal{H} \), and \( Q \) be a base learner. Then for any \( \delta \in [0, 1] \), the following inequality holds uniformly for all hyper-posteriors distributions \( Q \in \mathcal{M} \) with probability at least \( 1 - \delta \)

\[
er(Q, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{P \sim \mathcal{P}} \hat{er}(Q(S_i, P), S_i) + \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{\left(D(Q||P) + \mathbb{E}_{P \sim \mathcal{P}} D(Q||P) + \log \frac{2n\lambda}{m(n-1)}\right)}{2\delta}} + \sqrt{\frac{1}{2(n-1)} D(Q||P) + \log \frac{2\delta}{\lambda}}. \tag{7}\]

It is obviously this PAC-Bayes meta-learning bound consists of empirical multi-task error plus two regularization terms. The first task-complexity term is the average of task-complexity terms of observed tasks, created by the finite number of samples in each observed tasks. This term converges to zero in the face of a large number of samples in each task. The second is an environment-complexity term, which is caused by a finite number of observed training tasks. Obviously, this term converges to zero if an infinite number of tasks is observed from the task environment.

### III. PAC-Bayes meta-learning bounds

In this section, based on the PAC-Bayes relative entropy theory, we propose three novel PAC-Bayes bounds for meta-learning, including meta-learning PAC-Bayes \( \lambda \) bound, meta-learning PAC-Bayes quadratic bound, and meta-learning PAC-Bayes variational bound. We begin by investigating the PAC-Bayes relative entropy bound and then extend those bounds to meta-learning.

**A. PAC-Bayes relative entropy bound**

With high probability \( 1 - \delta \), PAC-Bayes bound theory provides a generalization performance guarantee for the learned model. The generalization error upper bound depends on empirical loss and a regularization item involves the distance between prior distribution and posterior distribution.

First of all, the PAC-Bayes relative entropy bound and its corresponding variants based on different inequalities are introduced. Similarly, in Theorem \[4\] with high probability \( 1 - \delta \), the PAC-Bayes relative entropy bound holds that

\[
\text{kl}(er(Q, D)||\hat{er}(Q, S)) \leq \mathcal{D}(Q||P) + \log \frac{2\delta}{\lambda}. \tag{8}\]

Here, as shown in \[9\], kl is known as the binary KL divergence, which is the divergence of two Bernoulli distributions with parameters \( q, q' \in [0, 1] \)

\[
\text{kl}(q||q') = q \log \left( \frac{q}{q'} \right) + (1 - q) \log \left( \frac{1-q}{1-q'} \right). \tag{9}\]

Obviously, with an arbitrarily high probability, the generalization error of the learned model is bounded by the summation of empirical loss, and a regularization element involves the distance between prior distribution and posterior distributions.

Applying the refined Pinsker inequality \( \text{kl}(\hat{p}||p) \geq (p - \hat{p})^2 \), \( \hat{p}, p \in (0, 1) \), \( \hat{p} < p \) yields

\[
er(Q, D) - \hat{er}(Q, S) \leq \sqrt{2er(Q, D) \mathcal{D}(Q||P) + \log \frac{2\delta}{\lambda}}. \tag{10}\]

The PAC-Bayes relative entropy bound cannot be selected directly as the training objective function directly, because generalization error \( er(Q, D) \) appears in the right side of the bound which cannot be used as an optimization objective. Therefore, the following two PAC-Bayes bounds are proposed.

First, combining (10) with the inequality \( \sqrt{ab} \leq \frac{a}{2} + \frac{b}{2} \), \( a, b > 0 \) yields the PAC-Bayes \( \lambda \) bound.

**Lemma 3** (PAC-Bayes \( \lambda \) bound \cite{49}). Let \( P \in \mathcal{M} \) be some prior distribution over \( \mathcal{H} \). Then for any \( \delta \in [0, 1] \) and \( \lambda \in (0, 2) \), the following inequality holds uniformly for all posteriors distributions \( Q \in \mathcal{M} \) with probability at least \( 1 - \delta \)

\[
er(Q, \tau) \leq \frac{\hat{er}(Q, S)}{1 - \lambda/2} + \frac{\text{KL}(Q||P) + \log(2\sqrt{\lambda/\delta})}{n\lambda(1-\lambda/2)}. \tag{11}\]

Compared with classic generalization error upper bound (see Theorem \[4\]), with a reasonable selection of parameters \( \lambda \), we may obtain a smaller upper bound.

Alternatively, one may view inequality (10) as a quadratic inequality on \( \sqrt{er(Q, D)} \). Solving this inequality yields the following PAC-Bayes quadratic bound.

**Lemma 4** (PAC-Bayes quadratic bound \cite{28}). Let \( P \in \mathcal{M} \) be some prior distribution over \( \mathcal{H} \). Then for any \( \delta \in [0, 1] \), the following inequality holds uniformly for all posteriors distributions \( Q \in \mathcal{M} \) with probability at least \( 1 - \delta \)

\[
er(Q, \tau) \leq \left( \sqrt{\hat{er}(Q, S)} + \varepsilon + \sqrt{\varepsilon} \right)^2, \tag{12}\]

where \( \varepsilon = \frac{1}{2\delta} \left( D(Q||P) + \log \frac{2\sqrt{\delta}}{\delta} \right) \).

It can be seen that when the generalization error is smaller (especially \( er(Q, D) < 1/4 \)), this bound is tighter (see \[28\]) than the classical PAC-Bayes bound. When generalization error \( er(Q, D) \) varies, different bounds can enact alternative different generalization performances. Motivated by this, \[50\] proposes a variational KL bound.

**Lemma 5** (Variational PAC-Bayes bound \cite{50}). Let \( P \in \mathcal{M} \) be some prior distribution over \( \mathcal{H} \). Then for any \( \delta \in [0, 1] \),
the following inequality holds uniformly for all posteriors distributions \( Q \in \mathcal{M} \) with probability at least \( 1 - \delta \)

\[
er(h, D) \leq \min \left\{ \hat{\epsilon}r(Q, S) + \epsilon + \sqrt{\epsilon(\epsilon + 2\hat{\epsilon}r(Q, S))}, \right. \\
\left. \hat{\epsilon}r(Q, S) + \sqrt{\epsilon/2}, \right. \\
\text{where } \epsilon = \frac{1}{n}(D(Q||P) + \log \frac{2n}{\delta} \sqrt{\delta}).
\]

(13)

Contrasting with the two previously described PAC-Bayes bounds, PAC-Bayes variational bound can take a minimum of two bounds, which might achieve a tighter generalization error for upper bound.

B. Extended PAC-Bayes \( kl \) bounds for meta-learning

For meta-learning, careful design of the training tasks is required to prevent a subtle form of task overfitting, which means a learned meta-learner generalizes on training tasks but fails to adapt to new ones. This form of overfitting is known as the memorization problem in meta-learning [31]. PAC-Bayes theory provides a theoretical framework for the analysis of the generalization performance in meta-learning. By selecting the PAC-Bayes bound as the training objective, one not only can reduce overfitting but also develop a deep neural network with a guaranteed generalization performance. Here, the tighter bounds can achieve enhanced results.

Motivated by this, and based on the PAC-Bayes relative entropy theory, in this section we propose three novel PAC-Bayes bounds for meta-learning. We begin with examining meta-learning PAC-Bayes \( \lambda \) bound. First of all, based on Lemma 3 and Lemma 4 followed by investigations of meta-learning PAC-Bayes quadratic bound and meta-learning PAC-Bayes variational bound.

**Theorem 1** (meta-learning PAC-Bayes \( \lambda \) bound). Let \( \mathcal{P} \) be some hyper-prior distribution, and \( Q \) be the posterior distribution, which is also called as the base learner. Then for any \( \delta \in (0,1] \) and \( \lambda \in (0,2] \), the following inequality holds uniformly for all hyper-posteriors distributions \( Q \in \mathcal{M} \) with probability at least \( 1 - \delta \)

\[
er(Q, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} D(Q||P) + \frac{\mathbb{E}_{p \sim \mathcal{Q}} \hat{\epsilon}r(Q, S_i)}{m_i} + \frac{1}{2(n-1)} D(Q||P) + \log \frac{2n}{\delta}.
\]

(14)

With the reasonable selection of \( \lambda \), this meta-learning PAC-Bayes bound can attain a tighter generalization error bound. The proof of this meta-learning bound is shown as follows.

**Proof:** In this section, a briefly proof of the extended PAC-Bayes \( \lambda \) bound is introduced.

Let \( n \) be the number of training tasks. The samples of task \( i \) are \( z_{i,j}, j = 1, \ldots, K, K \triangleq m_i \), over the data distribution \( D_i \). The bounded loss function is defined as \( \ell(h, z) \). We define the prior distribution \( P \), which is sampled from hyper-prior distribution \( \mathcal{P} \). The posterior distribution is defined as \( Q = Q(S_i, P) \), which is sampled from hyper-posterior distribution \( \mathcal{Q} \). Here as exemplified in [47], ‘tuple hypothesis’ is defined as \( f = (P, h) \) where \( P \in \mathcal{M} \) and \( h \in \mathcal{H} \); ‘prior over hypothesis’ is defined as \( \pi \triangleq (P, P) \), where \( h \) is sampled from \( P \). We note that the ‘posterior over hypothesis’ can be any distribution (even sample dependent). In particular, the PAC-Bayes bound will hold for the following family of distributions over \( \mathcal{M} \times \mathcal{H} \), \( \rho \triangleq (Q(S_i, P), P) \), where \( P \) is sampled from \( \mathcal{Q} \) and \( h \) is sampled from \( Q = Q(S_i, P) \) respectively.

The KL-divergence term is

\[
D(\rho||\pi) = \mathbb{E}_{f \sim \rho} \log \frac{\rho(f)}{\pi(f)} = \mathbb{E}_{p \sim Q_h \sim Q(S_i, P)} \log \frac{Q(S_i, P)(h)}{P(P)} + \mathbb{E}_{Q} \log \frac{Q(S_i, P)(h)}{P(h)}
\]

(15)

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\]

(15)

Just as with classical extended PAC-Bayes theory, our proof also involves two steps:

**Step 1:** For the task \( i \), we use PAC-Bayes relative entropy bound to evaluate the generalization error for each observed task \( i \)

\[
er(Q, D_i) \leq \frac{1}{n} \sum_{i=1}^{n} D(Q||P) + \frac{\mathbb{E}_{p \sim \mathcal{Q}} \hat{\epsilon}r(Q, S_i)}{m_i} + \frac{1}{2(n-1)} D(Q||P) + \log \frac{2n}{\delta}.
\]

(16)

**Step 2:** We try to bound the environment-level generalization. Due to observing only a finite number of tasks from the environment, re-using the classical PAC-Bayes bound yields

\[
er(Q, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} D(Q||P) + \frac{\mathbb{E}_{p \sim \mathcal{Q}} \hat{\epsilon}r(Q, S_i)}{m_i} + \frac{1}{2(n-1)} D(Q||P) + \log \frac{2n}{\delta}.
\]

(17)

Combining Eq. (16) and Eq. (17) by union bound yields

\[
er(Q, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} D(Q||P) + \frac{\mathbb{E}_{p \sim \mathcal{Q}} \hat{\epsilon}r(Q, S_i)}{m_i} + \frac{1}{2(n-1)} D(Q||P) + \log \frac{2n}{\delta}.
\]

(19)

Assuming that \( \delta_0 = \frac{\delta}{2n} \), \( \delta_i = \frac{\delta}{2n} \), and \( \lambda_0 = \lambda_i = \lambda \), this then yields

\[
er(Q, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} D(Q||P) + \frac{\mathbb{E}_{p \sim \mathcal{Q}} \hat{\epsilon}r(Q, S_i)}{m_i} + \frac{1}{2(n-1)} D(Q||P) + \log \frac{2n}{\delta}.
\]

(20)
Furthermore, based on the Lemma 2 and Lemma 3, another PAC-Bayes meta-learning quadratic bound is proposed.

**Theorem 2 (meta-learning PAC-Bayes quadratic bound).** Let \( \mathcal{P} \) be some hyper-prior distribution, and \( Q \) be the posterior distribution, which is also called as the base learner. Then for any \( \delta \in (0, 1] \), the following inequality holds uniformly for all hyper-posteriors distributions \( Q \in \mathcal{M} \) with probability at least \( 1 - \delta \)

\[
er(Q, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} \left( \sqrt{\frac{1}{n} \left( \sum_{p \sim Q} \mathbb{E} \hat{e}_p(Q, S_i) + \epsilon_i + \sqrt{\epsilon_i} \right)^2} + \sqrt{\frac{1}{2(n-1)} (D(Q||P) + \log \frac{2\sqrt{n} \epsilon}{\delta})} \right).
\]

Here the \( \epsilon_i = \frac{1}{2m_i} \left( D(Q||P) + \mathbb{E}_{p \sim Q} D(Q||P) + \log \frac{4n \sqrt{m_i}}{\delta} \right) \).

**Proof:** For the task \( i \), applying PAC-Bayes relative entropy bound to evaluate the generalization error in each of the observed tasks \( i \) yields

\[
\mathbb{E}_{p \sim Q} er(Q, D_i) \leq \left( \sqrt{\frac{1}{n} \left( \sum_{p \sim Q} \mathbb{E} \hat{e}_p(Q, S_i) + \epsilon_i + \sqrt{\epsilon_i} \right)^2} + \sqrt{\frac{1}{2(n-1)} (D(Q||P) + \log \frac{2\sqrt{n} \epsilon}{\delta})} \right).
\]

By utilizing the first step of the proof in Theorem 1 and assuming that \( \delta_0 = \frac{\delta}{2} \), \( \delta_i = \frac{\delta}{2n} \), we can get the following meta-learning PAC-Bayes bound

\[
er(Q, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} \left( \sqrt{\frac{1}{n} \left( \sum_{p \sim Q} \mathbb{E} \hat{e}_p(Q, S_i) + \epsilon_i + \sqrt{\epsilon_i} \right)^2} + \sqrt{\frac{1}{2(n-1)} (D(Q||P) + \log \frac{2n \epsilon}{\delta})} \right).
\]

Here \( \epsilon_i = \frac{1}{m_i} \left( D(Q||P) + \mathbb{E}_{p \sim Q} D(Q||P) + \log \frac{4n \sqrt{m_i}}{\delta} \right) \).

With different situations, different bounds may lead to different generalization performances. One can combine the two above-mentioned meta-learning bounds by a function which is defined piecewise to improve performance. The variational KL divergence can take the minimum value of Theorem 1 and Theorem 2 ensuring it is tight in both regimes. Prompted by PAC-Bayes variational bound, meta-learning PAC-Bayes variational bound is derived as follows:

**Theorem 3 (meta-learning PAC-Bayes variational bound).** Let \( P \) be some hyper-prior distribution, and \( Q \) be the posterior distribution, which is also known as base learner. Then for any \( \delta \in (0, 1] \), the following inequality holds uniformly for all hyper-posteriors distributions \( Q \in \mathcal{M} \) with probability at least \( 1 - \delta \)

\[
er(Q, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} \left( \mathbb{E}_{p \sim Q} \hat{e}_p(Q, S_i) + \epsilon_i + \sqrt{\epsilon_i} \right)^2 + \sqrt{\frac{1}{2(n-1)} (D(Q||P) + \log \frac{2n \epsilon}{\delta})}.
\]

Here \( \epsilon_i = \frac{1}{m_i} \left( D(Q||P) + \mathbb{E}_{p \sim Q} D(Q||P) + \log \frac{4n \sqrt{m_i}}{\delta} \right) \).

**IV. PAC-BAYES BOUNDS WITH DATA-DEPENDENT PRIOR**

For the PAC-Bayes bound theory, the generalization error upper bound mainly depends mainly on the regularization item involving the distance between prior distribution \( P \) and posterior distribution \( Q \). However, the prior distribution is chosen randomly, with a view to measuring the parameter space. Especially in meta-learning, the generalization error bound involves both hyper-prior and hyper-posterior distributions, which are hard to converge. Seeking to solve this issue, in this section we aim to learn a localized prior distribution through the ERM approach on a part of the training samples. Then the remaining data will be used to optimize generalization error bound.

Akin to the classical PAC-Bayes bound with data-dependent prior, for the meta-learning, we try to propose a novel extended PAC-Bayes bound with data-dependent prior. Specifically, during the training phase of meta-learning, the corresponding dataset of training task \( i \) is also divided into two separate datasets. Based on the ERM approach, one can learn a data-dependent prior distribution over a section of the training samples

\[
P_0 = \arg \min_{h \in \mathcal{H}_0} \mathbb{E}_{h \in \mathcal{H}_0} er_{\text{emp}}(h, S),
\]

where \( er_{\text{emp}}(h) = \frac{1}{n} \sum_{i=1}^{n} \ell(h, R_i) \), \( n \) is the number of all training tasks, \( R_i \) is the sample subset of task \( i \) selected from the whole dataset \( S_i \), providing information which can be used to calculate data-dependent prior. The remaining data \( S_i \setminus R_i \) of task \( i \) is applied to evaluate the generalization error bound for meta-learning. In practice, expectations over distribution \( P \) are difficult to calculate. Therefore, the Monte Carlo method deemed most effective in obtaining the numerical results of [27]. Furthermore, prior distribution \( P \) is selected directly from hyper posterior distribution \( Q_0 \). So the learned parameters \( P_0 \) is designated as the initial mean parameter of \( Q_0 \). The extended PAC-Bayes bound with data-dependent prior can then be shown as follows:

**Theorem 4 (Meta-learning PAC-Bayes bound with data-dependent prior).** Let \( Q : \mathcal{Z}^m \times \mathcal{M} \to \mathcal{M} \) be a base learner, and let \( P \) be some predefined hyper-prior distribution. Then for any \( \delta \in (0, 1] \) the following inequality holds uniformly for
all hyper-posterior distributions $Q$ with probability at least $1 - \delta$, 
\[
  \epsilon_r(Q, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{D(Q)} \epsilon_r(Q, S_i) + \frac{1}{n} \sum_{i=1}^{n} \sqrt{D(Q_i||P) + \mathbb{E}_{D(Q)} D(Q_i||P) + \log \frac{4n}{\delta}} \frac{2m_i(n-1)}{n-1} 
\]
(28)

Similarly, in the testing phase, when encountering new tasks, one can also learn a data-dependent prior $P$ through the ERM approach.

V. Practical meta-learning PAC-Bayes methods

meta-learning PAC-Bayes methods tries to provide a generalization performance guarantee for the learned model with an arbitrarily high probability. In practice, one can train a probability neural network by minimizing the generalization error upper bound.

A. Training objectives

Based on the proposed three meta-learning PAC-Bayes bounds, the corresponding three training objectives are developed as:

Theorem. 1 and Theorem. 2 lead to the meta-learning PAC-Bayes $\lambda$ objective
\[
f_{\lambda}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{D(Q)} \epsilon_r(Q, S_i) + \frac{1}{n} \sum_{i=1}^{n} \sqrt{D(Q_i||P) + \mathbb{E}_{D(Q)} D(Q_i||P) + \log \frac{4n}{\delta}} \frac{2m_i(n-1)}{n-1} 
\]
(29)

and meta-learning quadratic PAC-Bayes objective
\[
f_{\text{quad}}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \left( \mathbb{E}_{D(Q)} \epsilon_r(Q, S_i) + \epsilon_i + \sqrt{\frac{\delta}{2}} \right)^2 + \sqrt{\frac{1}{2(n-1)}} \left( D(Q||P) + \log \frac{2n}{\delta} \right) 
\]
(30)

By comparison, the training objective from Theorem. 3 takes the following form
\[
f_{\text{varia}}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{D(Q)} \epsilon_r(Q, S_i) + \frac{1}{n} \sum_{i=1}^{n} \min \left( \epsilon_i + \mathbb{E}_{D(Q)} \epsilon_r(Q, S_i), \sqrt{\frac{\delta}{2}} \right) + \sqrt{\frac{1}{2(n-1)}} \left( D(Q||P) + \log \frac{2n}{\delta} \right) 
\]
(31)

where $\epsilon_i = \frac{1}{m_i} \left( D(Q||P) + \mathbb{E}_{D(Q)} D(Q||P) + \log \frac{4n}{\delta} \right)$. 

B. Loss function

As a rule, the standard loss function used on multi-class classification problems is the cross-entropy loss function $\ell : \mathbb{R}^k \times [k] \rightarrow \mathbb{R}$ defined by
\[
\ell_{CE} = - \sum_{c=1}^{k} y_{o,c} \log(p_{o,c}), 
\]
(32)

where $y_{o,c}$ is the binary indicator (0 or 1) if class label $c$ is the correct classification for observation $o$, $k$ is the number

Algorithm 1 Meta training phase, without data-dependent prior
Input: Datasets of $n$ training tasks: $S_1, ..., S_n$. 
Output: Learned meta-learner with parameter $\theta$.
1: Initializing hyper-prior $P$, hyper-posterior $Q$, prior model $\theta$, posterior model $\phi_i, i = 1, ..., n$;
2: while not done do
3: for task $i, i = 1, ..., n$ do
4: Sample mini-batch from datasets $S_i, i = 1, ..., n$
5: Calculate $D(Q_{\phi_i}||P_0)$
6: Calculate $\mathbb{E}_{P \sim Q} \mathbb{E}_{\epsilon_r(Q, S_i)}$ by Monte-Carlo method
7: end for
8: Compute the training objective $f$ (see [29] [30] or [31])
9: Gradient step using $\nabla_{\theta} f$
10: end while
11: return $\theta$.

of class and $p_{o,c}$ is the predicted probability observation $o$ of class $c$. It is obviously that this loss function is unbounded loss function. However, the proposed meta-learning PAC-Bayes bound is only available for bounded loss function. Here, a “bounded cross-entropy” loss function is applied (See [28]) as the surrogate loss for training in all experiments with $f_{\lambda}$, $f_{\text{quad}}$ and $f_{\text{varia}}$. Specifically, the loss function is clipped to $[0, \log(\frac{1}{p_{\min}})]$, where $p_{\min}$ is the lower bound of the network probabilities.

C. Gaussian weight distributions

In this section, the specific forms of hyper-prior distribution $P$, hyper-posterior distribution $Q$ and weights distribution of the stochastic neural network are selected.

For the meta-learning PAC-Bayes bound, the hyper-prior distribution $P$ is set as a zero-mean Gaussian distribution
\[
P \triangleq \mathcal{N}(0, \kappa_P^2 I_{N_P \times N_P}), 
\]
(33)

where $\kappa_P > 0$ is constant and $N_P$ is the number of neural network parameters $w$.

Correspondingly, the hyper-posterior distribution $Q$, which consists of all distributions over $\mathbb{R}^{N_P}$, is defined as a family of isotropic Gaussian distributions as follows
\[
Q_{\theta} \triangleq \mathcal{N}(\theta, \kappa_Q^2 I_{N_P \times N_P}), 
\]
(34)

where $\kappa_Q > 0$ is also a predefined constant. Therefore the KL divergence between the hyper-prior distribution $P$ and hyper-posterior distribution $Q$ equals
\[
D(Q_{\theta}||P) = \frac{1}{2} \left( \frac{\theta^2}{\kappa_P^2} + \frac{1}{2} \log 2 \pi e \kappa_Q^2 + 1 \right) 
\]
(35)

In the PAC-Bound theory, a probability neural network is applied, which means all weights $w$ are stochastic variables drawn from prior or posterior distribution. In this paper, we define that each weight $w_i$ in the neural network as it obeys Gaussian distribution. The prior $P_0$ and the posteriors $Q_{\phi_i}, i = 1, \ldots, n$, are defined as factorized Gaussian distributions
\[
P_0(w) = \prod_{k=1}^{d} \mathcal{N}(w_k; \mu_{P,k}, \sigma_{P,k}^2), 
\]
(36)
Algorithm 2 Meta training phase, with data-dependent prior

Input: Datasets of n training tasks: $S_1, \ldots, S_n$.
Output: Learned meta-learner with parameter $\theta$.
1: Initializing prior model $\theta$;
2: Separate training datasets $S_i$ into two parts $S_i/R_i$, $R_i, i = 1, \ldots, n$;
3: while not done do
4: Sample mini-batch from datasets $R_i$, $i = 1, \ldots, n$
5: Calculate $\mathbb{E}_{h \in P_r} E_{\text{emp}}(h, S)$ (27)
6: Gradient step using $\nabla_{\theta} f$
7: end while
8: Initializing hyper-posterior $Q$ with learned parameter $\theta$,
   prior model $P$, posterior model $Q$, $i = 1, \ldots, n$;
9: while not done do
10: for task $i, i = 1, \ldots, n$ do
11: Sample mini-batch from $S_i/R_i$, $i = 1, \ldots, n$
12: Calculate $D(Q_{\phi} \| P_r)$ (38)
13: Calculate $\mathbb{E}_{P \sim Q} \epsilon_k (Q_i, S_i)$ by Monte-Carlo method
14: end for
15: Compute the training objective $f$ (see 29, 30 or 31)
16: Gradient step using $\nabla_{\theta} f$
17: end while
18: return $\theta$;

$Q_{\phi_i}(w) = \prod_{k=1}^{d} \mathcal{N} \left( w_k; \mu_{i,k}, \sigma_{i,k}^2 \right)$, (37)
where $d$ is the number of neural network parameters and $n$ is the number of tasks.
The corresponding KL divergence between prior $P_0$ and the posteriors $Q_{\phi_i}, i = 1, \ldots, n$, is

$$D(Q_{\phi} \| P_0) = \frac{1}{2} \sum_{k=1}^{d} \left( \log \frac{\sigma_{i,k}^2}{\sigma_{i,k}^2} + \frac{\sigma_{i,k}^2 + (\mu_{i,k} - \mu_{0,k})^2}{\sigma_{i,k}^2} - 1 \right).$$ (38)

As we started earlier, the prior distribution $P_0$ is sampled from hyper-posterior distribution $Q_0$. Practically, it follows that the prior distribution parameters $\theta = \theta + \varepsilon_p, \varepsilon_p \sim \mathcal{N}(0, \kappa P, N_1, N_2)$. In other words, prior distribution $P_0$ sampling from hyper-posterior distribution $Q_0$ means adding Gaussian noise $\varepsilon_p$ to the parameters $\theta$ during training. The specific pseudo code is shown in Algorithm 1 and Algorithm 2 for both random prior and data-dependent prior respectively.

VI. EXPERIMENTS

In this section, the performance of our proposed meta-learning PAC-Bayes bounds is illustrated with image classification tasks solved by stochastic neural networks. Specifically, we conduct our procedure within two different environments based on the MNIST dataset, those being permuted pixels and permuted labels. For the permuted pixels environment, each task is constructed by a shuffle of image pixels with 60000 training samples and 10000 testing samples. For the permuted labels environment, each task is generated by a permutation of image labels with the same number of training and testing samples as produced in the permuted pixels environment.

For the shuffled pixels experiment, the neural network structure selected is a full connected neural network (FCN) with 4 layers (3 hidden layers and a linear output layer) and 400 units per layer. For the permuted labels experiment, the neural network structure is designated as a 4-layers convolutional neural network (CNN), comprising 2 convolution layers each with $5 \times 5$ kernels, and 2 full connected layers. For all experiments, ReLU activations are used. The optimizer is selected as Adam, with a learning rate of $10^{-3}$.

For both of two experiments, each initialized log-var $\log \sigma_p^2$ of weights is drawn from $\mathcal{N}(-10, 0.01)$. The hyper-prior and hyper-posterior parameters are $\kappa = 2000$ and $\kappa_0 = 0.001$ respectively. In the meta-learning PAC-Bayes bound, the confidence parameter chosen is $\delta = 0.1$. Source code is available at GitHub[1].

A. Comparison of various PAC-Bayes bounds

In this section, we focus on the performance of three proposed meta-learning PAC-Bayes bounds. For the meta-training phase, we run the total training of 50 epochs, maximal number of tasks in each meta-batch being 16, while 10 tasks are used for the meta-learner to learn. For the testing phase, we run a total testing of 20 epochs, using 20 tasks to confirm the meta-learner performance. We select 128 as the data batch size for training and testing.

First, we investigate the weights of the stochastic neural network. As shown in Figure 1(a), the average log-variance parameter of each layer’s weights is analyzed. The higher the average $\log(\sigma^2)$ is, the more flexible are the weights are. In the shuffled pixel experiment, the lower layers perform with high variance which can extract the feature of shuffled-pixels image robustly, while the higher layers perform with a low variance which corresponds with fixed labels. Contrastingly, in the permuted label experiments, as shown in Figure 1(b), the higher layers perform with high variance which can adapt robustly to the permutation of image label, and the lower layers’ low variance performance corresponds with fixed-image samples.

Next, the influence of different numbers of training tasks on performance is analyzed, in relation to generalization error bound, empirical loss and empirical error. In Figure 2, it is clear that, as the number of training tasks increases, the learned model achieves improved generalization performance and accuracy.

We also compare five meta-learning PAC-Bayes bounds on two different MNIST environments. Those consist of meta-learning McAllester PAC-Bayes bound ($f_{\text{classical}}$), meta-learning Seeger PAC-Bayes bound ($f_{\text{seeger}}$), meta-learning PAC-Bayes $\lambda$ bound ($f_{\lambda}$), meta-learning PAC-Bayes quadratic bound ($f_{\text{quad}}$) and meta-learning PAC-Bayes variational bound ($f_{\text{varia}}$). As shown in Table I, the performance of various training objectives in the training phase with a random prior model is analyzed, in terms of bound, task complexity, meta complexity, empirical loss and estimated error. Figure 3(a) demonstrates that the

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[1] Codes are available on https://github.com/tyliu22/Meta-learning-PAC-Bayes-bound-with-data-dependent-prior.git
The performance in the training phase of five meta-learning training objectives on the shuffled pixels and permuted labels MNIST environments with different prior models is analyzed in the above Table, in terms of PAC Bayes bound, task complexity, meta complexity, empirical loss and estimated error.

The performance in the testing phase of five meta-learning training objectives on the shuffled pixels and permuted labels MNIST environments with different prior models is analyzed in the above Table, in terms of PAC Bayes bound, task complexity, meta complexity, empirical loss and estimated error.

![Graphs](image-url)  
**Fig. 1.** Model parameter analysis by layers, $\log(\sigma^2)$ represents the weight uncertainty of each layer. (a) Prior model parameter analysis for shuffled pixels environment; (b) Prior model parameter analysis for permuted labels environment.

The performance in the testing phase of five meta-learning training objectives on the shuffled pixels and permuted labels MNIST environments with different prior models is analyzed in the above Table, in terms of test bound, test loss and test error.

Furthermore, the performance of a learned meta-learner on new tasks with five training objectives is also established. Table [II] shows the specific result of generalization performance and accuracy in the testing phase. As indicated in Figure [3(b)] the proposed meta-learning PAC-Bayes $\lambda$ bound and meta-learning PAC-Bayes variational bound perform tighter generalization error bound. In addition, these two training objectives lead to improved accuracy.
B. PAC-Bayes bounds with data-dependent prior

In this section, meta-learning PAC-Bayes bounds with data-dependent prior algorithms are verified on the permuted MNIST dataset. We experiment both with priors centered at randomly set weights and priors learnt by ERM on a part of dataset. Specifically, training data is randomly divided into two separate datasets: 30% is used to learn a prior model by ERM approach and the remaining data is applied to train the meta-learner. We run about 10 epochs in the training phase and 30 epochs in the testing phase to build the prior.

First, the discrepancy between the prior model with randomly initialized weights and the learned data-dependent prior model is compared in Figures 4(a) and Figure 4(b). It is obviously that, compared with the random prior model, the nature of the parameter learned in the data-dependent prior model is closer to the finally posterior model, which means it can achieve enhanced convergence performance. Besides, as shown in Figure 5, where the convergence performance between a random prior model and a data-dependent prior model during the training phase is analyzed. Obviously, the meta-learning PAC-Bayes bound with data-dependent prior demonstrates a faster convergence ability with a series of epochs, in terms of generalization error bound, accuracy, empiric loss, task complexity and meta-complexity.
Logarithmic scale for the KL divergence is used to balance between model accuracy and the KL term. Faster convergence ability and greater accuracy are achieved when optimizing this term. Bayes theory, and will explore efficient ways to optimize this term. The KL term dominates the generalization upper bound of PAC-Bayes. Meta-learning PAC-Bayes variational bound can achieve competitive performance guarantee and reduced overfitting. Next, in order to improve the convergence ability, combining the ERM approach, meta-learning PAC-Bayes bounds with data-dependent prior algorithms are also proposed. The results of our experiments on two different MNIST environments, including shuffled pixels and permuted labels, demonstrate that meta-learning PAC-Bayes bounds with data-dependent prior algorithms can achieve competitive performances in terms of generalization error upper bound and estimation accuracy in both training and testing phases. Moreover, meta-learning PAC-Bayes bounds with data-dependent prior can attain rapid convergence ability.

In future work, one could further investigate different prior distribution, such as distribution-dependent prior, to achieve faster convergence ability and greater accuracy. We note that the KL term dominates the generalization upper bound of PAC-Bayes theory, and will explore efficient ways to optimize this term.

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The results of our experiments on two different MNIST environments, including shuffled pixels and permuted labels, demonstrate that meta-learning PAC-Bayes bounds with data-dependent prior can achieve competitive performances in terms of generalization error upper bound and estimation accuracy in both training and testing phases. Moreover, meta-learning PAC-Bayes bounds with data-dependent prior can attain rapid convergence ability.

In future work, one could further investigate different prior distribution, such as distribution-dependent prior, to achieve faster convergence ability and greater accuracy. We note that the KL term dominates the generalization upper bound of PAC-Bayes theory, and will explore efficient ways to optimize this term.
Fig. 5. Performance analysis of meta-learning with data-dependent prior (training objective is PAC-Bayes quadratic bound).

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