Fano manifolds obtained by blowing up along curves with maximal Picard number

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April 15, 2009

Abstract

The Picard number of a Fano manifold $X$ obtained by blowing up a curve in a smooth projective variety is known to be at most 5, in any dimension greater than or equal to 4. We show that the Picard number attains to the maximal if and only if $X$ is the blow-up of the projective space whose center consists of two points, the strict transform of the line joining them and a linear space or a quadric of codimension 2. This result is obtained as a consequence of a classification of special types of Fano manifolds.

1 Introduction

Let $X$ be a Fano manifold obtained by blowing up along a curve, i.e. there exists a pair $(Y, C)$ of a smooth projective variety $Y$ and a smooth connected curve $C \subset Y$ such that the anticanonical divisor $-K_X$ is ample. Using a recent result on Minimal Model Program due to [2], C. Casagrande shows that such a Fano manifold has Picard number at most 5 (see [5] Theorem 4.2 for a more general statement, and see [7] for the toric case).

The purpose of the present paper is to classify the maximal case:

Theorem 1. Let $Y$ be a smooth projective variety of dimension $n \geq 4$ defined over the field of complex numbers, $C$ a smooth curve on $Y$, and $X$ the blow-up of $Y$ along $C$. Assume that $X$ is a Fano manifold and has Picard number 5. Then, the pair $(Y, C)$ is exactly one of the following:

1. $Y$ is the blow-up of $\mathbb{P}^n$ whose center is the union of two points $p, q$ and $\mathbb{P}^{n-2}$ disjoint from $\overline{pq}$, and $C$ is the strict transform of $\overline{pq}$.

2. $Y$ is the blow-up of $\mathbb{P}^n$ whose center is the union of two points $p, q$ and a smooth quadric $Q_{n-2}$ disjoint from $\overline{pq}$, and $C$ is the strict transform of $\overline{pq}$.

Remark. We denote by $\overline{pq}$ the line passing through $p$ and $q$ in $\mathbb{P}^n$.

According to Casagrande’s result (see [5] Theorem 4.2 (ii)), if the assumption of Theorem 1 is satisfied, then there exists another structure of blow-up $\varphi : X \to Z$ with the following properties:
• $Z$ is a smooth projective variety, and the center of the blow-up $\varphi$ is a smooth subvariety of codimension 2
• $E \cdot f > 0$, where $E$ is the exceptional divisor of the blow-up $\pi : X \to Y$ and $f$ is a non trivial fiber of $\varphi$
• $F \cdot e = 0$, where $F$ is the exceptional divisor of $\varphi$ and $e$ is a line in a fiber of the $\mathbb{P}^{n-2}$-bundle $\pi|_E : E \to C$

Hence, our Theorem is a consequence of the following classification result (in which only two examples (8) and (9) have Picard number 5):

**Theorem 2.** Let $Y$ be a complex manifold of dimension $n \geq 4$. Assume that there exists a smooth curve $C \subset Y$ such that the blow-up $X$ of $Y$ along $C$ is a Fano manifold. Assume moreover that there exists a smooth projective variety $Z$ and a smooth subvariety $W \subset Z$ of codimension 2 such that the blow-up of $Z$ along $W$ is isomorphic to $X$. Let $E$ (resp. $F$) be the exceptional divisor of the blow-up $\pi : X \to Y$ (resp. $\varphi : X \to Z$). Let $e$ (resp. $f$) be a line in a fiber of the $\mathbb{P}^{n-2}$-bundle $\pi|_E : E \to C$ (resp. a fiber of the $\mathbb{P}^1$-bundle $\varphi|_F : F \to W$). If $E \cdot f > 0$ and $F \cdot e = 0$, then we have exactly one of the following:

1. $Y$ is the blow-up of $\mathbb{P}^n$ at a point $p$ and $C$ is the strict transform of the line passing through $p$,
2. $Y$ is the blow-up of $Q_n$ at a point $p$ and $C$ is the strict transform of a line passing through $p$,
3. $Y$ is the blow-up of $Q_n$ at a point $p$ and $C$ is the strict transform of a conic passing through $p$,
4. $Y$ is the blow-up of $\mathbb{P}^n$ whose center is the union of a point $p$ and a linear subspace $P \cong \mathbb{P}^{n-2}$ not containing $p$, and $C$ is the strict transform of a line passing through $p$ and disjoint from $P$,
5. $Y$ is the blow-up of $\mathbb{P}^n$ whose center is the union of a smooth quadric $Q \cong Q_{n-2}$ and a point $p$ not on the hyperplane containing $Q$, and $C$ is the strict transform of a line passing through $p$ and disjoint from $Q$,
6. $Y$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ at a point $p$ and $C$ is the strict transform of the fiber of the projection $\mathbb{P}^1 \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ passing through $p$,
7. $Y$ is the blow-up of $\mathbb{P}^n$ whose center is two distinct points $p$ and $q$, and $C$ is the strict transform of the line $\overline{pq}$,
8. $Y$ is the blow-up of $\mathbb{P}^n$ whose center is the union of two points $p, q$ and $\mathbb{P}^{n-2}$ disjoint from $\overline{pq}$, and $C$ is the strict transform of $\overline{pq}$,
9. $Y$ is the blow-up of $\mathbb{P}^n$ whose center is the union of two points $p, q$ and a smooth quadric $Q_{n-2}$ disjoint from $\overline{pq}$, and $C$ is the strict transform of $\overline{pq}$.

**Remark.** We do not assume the projectivity of $Y$ because it follows from the assumption (see Lemma below).
2 Preliminaries

We prove lemmas which will be needed for the proof of Theorem 2.

Lemma 1. We have $E \cdot f = 1$.

Proof. Since $F \cdot e = 0$, (the reduced part of) the intersection $E \cap F$ is a union of fibers of $\pi|_E : E \to C$. Hence $E \cap F$ is the exceptional locus of $\varphi|_F : F \to \pi(F)$. Since $\varphi|_F : F \to W$ is a $\mathbb{P}^1$-bundle, we see that $E \cap F$ is a section of $\varphi|_F$. Hence we can write $E|_F = mE_c$ where $m$ is a natural number and $E_c := \pi^{-1}(c)$ with $c \in C$ is a fiber of $\pi|_E$. Let $e_c$ be a line in $E_c \cong \mathbb{P}^{n-2}$. We have $mE_c \cdot e_c = E|_F \cdot e_c = E \cdot e_c = -1$, where the first and second intersection numbers are taken in $F$ and the last one is in $X$. Note that $(E_c \cdot e_c)$ is an integer because $F$ is smooth. Thus we get $m = 1$. It follows that $E \cap F$ is a reduced section of $\varphi|_F : F \to W$. Therefore, we have $E \cdot f = 1$. \hfill $\square$

Now we consider $F_Y := \pi(F) \subset Y$. Note that we have $\pi^*F_Y = F$.

Lemma 2. We have $F_Y \cdot C = 1$.

Proof. Let $\tilde{C}$ be a section of $\pi|_E : E \to C$. By (the proof of) Lemma 1 $F|_E$ is a reduced fiber of $\pi|_E$. Thus we have $F_Y \cdot C = F_Y \cdot \pi_*\tilde{C} = F \cdot \tilde{C} = 1$. \hfill $\square$

By the proof of Lemma 1 we see that the intersection number $E_c \cdot e_c$ (taken in $F$) is equal to $-1$. It follows that $\pi|_F : F \to F_Y$ is the blow-up at the point $c$ whose exceptional divisor is $E_c$ and $F_Y$ is smooth. By the lemma 1 we see that $W$ is isomorphic to $E_c \cong \mathbb{P}^{n-2}$. We have the diagram:

$$
\begin{align*}
F & \xrightarrow{\varphi|_F} W \cong \mathbb{P}^{n-2} \\
\pi|_F & \downarrow \quad \quad \downarrow \\
F_Y & 
\end{align*}
$$

where $\varphi|_F$ is a $\mathbb{P}^1$-bundle. Note that $F$ is a Fano manifold. Indeed, we have $\rho(F) = 2$ and $F$ has two extremal contractions $\pi|_F$ and $\varphi|_F$. According to the classification result from [3], this implies that $F_Y$ is isomorphic to $\mathbb{P}^{n-1}$. Furthermore, $f_Y := \pi_*f$ is a line passing through the point $c = F_Y \cap C$. Note that $F_Y \cdot f_Y = F \cdot f = -1$. Hence there exists a blow-down $\varphi' : Y \to Y'$ contracting $F_Y \cong \mathbb{P}^{n-1}$ to a smooth point $p$, $Y'$ being (a priori) a complex manifold. Hence we have the commutative diagram:

$$
\begin{align*}
X & \xrightarrow{\varphi} Z \\
\pi & \downarrow \quad \quad \downarrow \pi' \\
Y & \xrightarrow{\varphi'} Y' 
\end{align*}
$$

where $\pi' : Z \to Y'$ is the blow-up along the curve $C' := \varphi'(C)$.

Lemma 3. $Y$ is projective (hence, so is $Y'$).
Proof. Assume to the contrary that $Y$ is not projective. Then, by [4] the normal bundle $N_{C/Y}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus (n-1)}$. Since $\varphi' : Y \to Y'$ is a blow-up, $Y'$ is not projective either. Note that $Z$ is projective by the assumption of Theorem 2. Hence $N_{C/Y'} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus (n-1)}$ by [4] again. On the other hand, we have $N_{C/Y} \neq N_{C/Y'}$ because $F_Y \cdot C > 0$. Hence we get a contradiction.

We recall here the classification result due to [3] which is indispensable to the proof of our Theorem 2. Let $V_d$ denote the blow-up of $\mathbb{P}^n$ along a smooth complete intersection $U_d := H \cap D$ where $H$ is a hyperplane and $D$ is a hypersurface of degree $d$.

**Theorem 3** (Bonavero, Campana and Wiśniewski [3] Theorem 1.1). Let $Y'$ be a complex manifold of dimension $n \geq 3$. Let $\varphi' : Y \to Y'$ be the blow-up at a point $p \in Y'$. Then $Y$ is a Fano manifold if and only if $Y'$ is isomorphic to either $\mathbb{P}^n$, $Q_n$, or $V_d$ with $1 \leq d \leq n$ and $p$ is not on the hyperplane $H$ containing the center $U_d$.

### 3 Proof of Theorem 2

The proof is divided into two parts:

(A) If $Y$ is a Fano manifold, then $(Y, C)$ is one of the examples from (1) to (5).

(B) If $Y$ is not a Fano manifold, then $(Y, C)$ is one of the examples from (6) to (9).

Throughout the section, we frequently use the following:

**Lemma 4** (cf.[4] Proposition 7). Let $Y$ be a smooth projective variety of dimension $n \geq 3$, $C \subset Y$ a smooth subvariety of codimension $k \geq 2$, and $X$ the blow-up of $Y$ along $C$. Assume that $X$ is a Fano manifold. If $\Gamma \subset Y$ is a curve not contained in $C$ and $\Gamma \cap C \neq \emptyset$, then we have $(-K_Y) \cdot \Gamma \geq k$.

**Proof.** Let $\widetilde{\Gamma}$ be the strict transform of $\Gamma$ by the blow-up. For the exceptional divisor $E$, we have $E \cdot \widetilde{\Gamma} \geq 1$. Hence we have

$$0 < -K_X \cdot \widetilde{\Gamma} = -K_Y \cdot \Gamma - (k - 1)E \cdot \widetilde{\Gamma} \leq -K_Y \cdot \Gamma - (k - 1),$$

which gives the statement. \qed

In what follows, we use the notation of the diagram (1) in the previous section.

#### 3.1 Proof of (A)

We assume that $Y$ is a Fano manifold. Since $\varphi' : Y \to Y'$ is a blow-up at a point, we are exactly in the situation of Theorem 3. Consider the extremal contraction $\gamma : Y \to Y''$ of ray $\mathbb{R}^+ [g]$ such that $F_Y \cdot g > 0$ (see [3] Lemme 2.1 for the existence of such a contraction). Then, by [3] Proposition 2.2, $\gamma$ is either:

(A1) a $\mathbb{P}^1$-bundle, or

(A2) a blow-up of a smooth projective variety along a smooth subvariety of codimension 2.
In the case (A1), $Y'$ is isomorphic to $\mathbb{P}^n$. We shall determine the position of $C$ in $Y$. If $C$ is not a fiber of $\gamma$, then there exists a fiber $\Gamma \simeq \mathbb{P}^1$ of $\gamma$ such that $\Gamma \cap C \neq \emptyset$. Note that $-K_Y \cdot \Gamma = 2$. Hence, by Lemma 4, this is a contradiction. It follows that $C$ is a fiber of $\gamma$, i.e. the strict transform of a line in $Y' \simeq \mathbb{P}^n$ passing through $p$, the center of the blow-up $\varphi'$. So, we get the example (1).

Now we treat the case (A2). Let $W_\gamma$ be the center of the blow-up $\gamma : Y \rightarrow Y''$ and $G$ the exceptional divisor. Note that $\gamma|_G : G \rightarrow W_\gamma$ is a $\mathbb{P}^1$-bundle. By $[3]$, there are two possibilities:

(A2–1) $Y''$ is isomorphic to $\mathbb{P}^n$ and $W_\gamma$ is isomorphic to $Q_{n-2}$, or

(A2–2) $Y''$ is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(d-1))$ and $W_\gamma$ is a hypersurface in the section $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}) \simeq \mathbb{P}^{n-1}$ whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(d-1)$.

In the case (A2–1), $Y'$ is isomorphic to $V_d$ with $1 \leq d \leq n$. Let $\beta : Y' \simeq V_d \rightarrow \mathbb{P}^n$ denote the blow-up along the smooth complete intersection $U_d = H \cap D$ with $H \in |\mathcal{O}_{\mathbb{P}^n}(1)|$ and $D \in |\mathcal{O}_{\mathbb{P}^n}(d)|$. Consider the composite of the two blow-ups $\varepsilon := \varphi' \circ \beta : Y \rightarrow \mathbb{P}^n$. Note that the exceptional divisor $G$ of $\gamma$ is the strict transform by $\varepsilon$ of the cone over $U_d$ with vertex $\beta(p)$ (recall that $p$ is the center of the blow-up $\varphi' : Y \rightarrow Y'$). Let $H_Y$ be the strict transform by $\varepsilon$ of the hyperplane $H$ containing $U_d$. We have $H_Y \cap F_Y = \emptyset$ because $\varepsilon(F_Y) \not\in H$ (see the statement of Theorem $[3]$).

**Claim 1.** We have $H_Y \cdot C = 1$.

**Proof.** Let $M$ be the exceptional divisor of the blow-up $\beta$ and $M_Y$ its strict transform by $\varphi'$. Since $F_Y \cdot C = 1$ and $M_Y \cap F_Y = \emptyset$, we see that $C \not\subset M_Y$. In particular we have $\varepsilon_*C \not\equiv 0$. If $C \cap M_Y \neq \emptyset$, then there exists a fiber $\Gamma$ of the $\mathbb{P}^1$-bundle $M_Y \rightarrow U_d$ meeting $C$. Note that $-K_Y \cdot \Gamma = 1$. By Lemma $[3]$ this is a contradiction. Hence $M_Y \cdot C = 0$. Note that $\varepsilon*H = H_Y + M_Y$. We have

$$H_Y \cdot C = (H_Y + M_Y) \cdot C = \varepsilon*H \cdot C = H \cdot \varepsilon_*C > 0.$$ 

If $H_Y \cdot C \geq 2$ then there exists a line $h \subset H_Y \simeq \mathbb{P}^{n-1}$ whose strict transform $\widetilde{h}$ by the blow-up $\pi : X \rightarrow Y$ satisfies $E \cdot \widetilde{h} \geq 2$. Then we have

$$K_X \cdot \widetilde{h} = K_Y \cdot h + (n-2)E \cdot \widetilde{h} \geq -n + d - 1 + 2(n-2) = n + d - 5 \geq 0,$$

which is a contradiction because $X$ is a Fano manifold. Hence we are done. \qed

**Claim 2.** We have $d = 1$ or $2$.

**Proof.** Let $h$ be a line in $H_Y \simeq \mathbb{P}^{n-1}$ such that $E \cdot \widetilde{h} = 1$. Then we have

$$K_X \cdot \widetilde{h} = K_Y \cdot h + (n-2)E \cdot \widetilde{h} = d - 3.$$ 

Since $K_X \cdot \widetilde{h} < 0$, we get $d = 1$ or $2$. \qed

If $d = 1$, we get the example (4) and if $d = 2$, the example (5). The curve $C$ is determined by the condition $H_Y \cdot C = 1$ and $F_Y \cdot C = 1$.  

5
3.2 Proof of (B)

Assume that $Y$ is not a Fano manifold. By [9] Proposition 3.5, $E$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-2}$ and $E \cdot l = -1$ where $l$ is a fiber of the projection $E \simeq \mathbb{P}^1 \times \mathbb{P}^{n-2} \to \mathbb{P}^{n-2}$. Put $E_Z := \varphi(E)$, $e_Z := \varphi_* e$ and $l_Z := \varphi_* l$. Since $E \cdot t = 1$, $\varphi|_E : E \to E_Z$ is an isomorphism. Since $\varphi^* E_Z = E + F$, we have $E_Z \cdot e_Z = -1$ and $E_Z \cdot l_Z = 0$. Recall that $Y'$ is projective by Lemma [3].

**Lemma 5.** The projective varieties $Y'$ and $Z$ are Fano manifolds.

**Proof.** Since $E_Z \cdot l_Z = 0$, we have $\mathcal{N}_{C'/Y'} \simeq \mathcal{O}_{\mathbb{P}^1}((-1)) \not\cong \mathcal{O}_{\mathbb{P}^1}((-1))$. Therefore, $Y'$ is a Fano manifold by [9] Proposition 3.5. Since $K_X = \varphi^* K_Z + F$, we get

$$(-K_Z) \cdot e_Z = -K_X \cdot e + F \cdot e = -K_X \cdot e > 0.$$  

Note that the center $W$ of the blow-up $\varphi : X \to Z$ is a fiber of the projection $E_Z \simeq \mathbb{P}^1 \times \mathbb{P}^{n-2} \to \mathbb{P}^1$. Hence, any curve contained in $W$ is numerically proportional to a positive multiple of the line $e_Z$. By [8] Proposition 1, we conclude that $Z$ is a Fano manifold. □

Since $Z$ is a Fano manifold, there exists an extremal ray $\mathbb{R}^+ [m] \subset \text{NE}(Z)$ such that $E_Z \cdot m > 0$ (see [3] Lemme 2.1). We investigate the associated extremal contraction $\mu : Z \to Z'$.

**Lemma 6.** We have $\mu_* B \not\equiv 0$ for any curve $B$ contained in $E_Z$.

**Proof.** Assume to the contrary that there exists a curve $B \subset E_Z$ such that $\mu_* B \equiv 0$. Then, there exists $a > 0$ such that $B \equiv a m$. On the other hand, we can write $B \equiv b l_Z + c e_Z$ with $b, c \geq 0$ because $B$ is contained in $E_Z \simeq \mathbb{P}^1 \times \mathbb{P}^{n-2}$. So, we have $E_Z \cdot B = E_Z \cdot (am) > 0$ and $E_Z \cdot B = E_Z \cdot (bl_Z + ce_Z) = -c \leq 0$, a contradiction. □

If there exists $z' \in Z'$ such that $\dim \mu^{-1}(z') \geq 2$, then there exists a curve $B$ contained in $E_Z \cap \mu^{-1}(z')$. Hence, any non-trivial fiber of $\mu$ has dimension at most 1. By [11] (see [9] Theorem 1.2), the extremal contraction $\mu$ is either:

- (B1) a conic bundle, or
- (B2) a blow-up of a smooth projective variety along a smooth subvariety of codimension 2.

First we treat the case (B1). We show that $\mu$ has no singular fiber, i.e. $\mu$ is a $\mathbb{P}^1$-bundle. Let $\Gamma$ be a fiber of $\mu$. Note that $\Gamma$ is isomorphic to $\mathbb{P}^1$ or $\Gamma \simeq \Gamma_1 \cup \Gamma_2$ with $\Gamma_i \simeq \mathbb{P}^1$ ($i = 1, 2$).

**Claim 3.** If $\Gamma$ meets $W$, then $\Gamma$ is a smooth fiber and the intersection $\Gamma \cap W$ is one point with multiplicity one.

**Proof.** Assume $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap W \neq \emptyset$. Note that $-K_Z \cdot \Gamma_1 = 1$. By Lemma [4], this is a contradiction. Hence $\Gamma$ is a smooth fiber. Let $\tilde{\Gamma}$ be the strict transform by $\varphi$. We have

$$0 < -K_X \cdot \tilde{\Gamma} = -K_Z \cdot \Gamma - F \cdot \tilde{\Gamma} = 2 - F \cdot \tilde{\Gamma},$$

which gives $F \cdot \tilde{\Gamma} = 1$ and completes the proof. □
We conclude that $\mu|_W : W \to \mu(W)$ is an isomorphism. In particular, $\mu(W) \simeq \mathbb{P}^{n-2}$. We put $M := \mu^{-1}(\mu(W))$. Remark that $\mu|_M : M \to \mu(W)$ is a $\mathbb{P}^1$-bundle and $W$ is a section.

**Claim 4.** We have $E_Z \cdot \Gamma = 1$.

**Proof.** By Claim 3, it is sufficient to prove $E_Z \cap M = W$. Let $\Gamma$ be any fiber of $\mu|_M : M \to \mu(W)$. We show that $E_Z \cap \Gamma \subset W$. Assume to the contrary that there exists a point $z \in E_Z \cap \Gamma$ such that $z \notin W$. Let $\Phi$ be the fiber of the $\mathbb{P}^{n-2}$-bundle $\pi|_{E_Z} : E_Z \to C'$ containing the point $z$. Since $\dim M \cap \Phi = n - 3 \geq 1$, there exists a curve $A \subset M \cap \Phi$. Consider the ruled surface $S := \mu^{-1}(\mu(A))$. By Lemma 6 above, $\Gamma \notin E_Z$, hence $\Gamma' \notin C'$ and we have $\dim \pi'(S) = 2$. Therefore, $W \cap S$ and $A$ are exceptional curves on $S$. Note that $A \neq W \cap S$ because $\Phi \cap W = \emptyset$. Thus, we have a contradiction because $S$ is a ruled surface.

Now, we see that $\mu : Z \to Z'$ is a $\mathbb{P}^1$-bundle and $\mu|_{E_Z} : E_Z \to Z'$ is an isomorphism. It follows that $Z'$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-2}$. Pushing down the exact sequence:

$$0 \to O_Z \to O_{E_Z}(E_Z) \to O_{E_Z}(E_Z) \to 0,$$

we get

$$0 \to \mu_*O_Z \to \mu_*O_{E_Z}(E_Z) \to \mu_*O_{E_Z}(E_Z) \to R^1\mu_*O_Z = 0.$$

Since $\mu$ is an extremal contraction, we have $\mu_*O_Z \simeq O_{\mathbb{P}^1 \times \mathbb{P}^{n-2}}$. Recall that $O_{E_Z}(E_Z) \simeq O_{\mathbb{P}^1 \times \mathbb{P}^{n-2}}(0, -1)$. Since $\mu|_{E_Z}$ is an isomorphism, we have $\mu_*O_{E_Z}(E_Z) \simeq O_{\mathbb{P}^1 \times \mathbb{P}^{n-2}}(0, -1)$. Thus we get the splitting sequence

$$0 \to O_{\mathbb{P}^1 \times \mathbb{P}^{n-2}} \to \mu_*O_{E_Z}(E_Z) \to O_{\mathbb{P}^1 \times \mathbb{P}^{n-2}}(0, -1) \to 0,$$

which gives $\mu_*O_{E_Z}(E_Z) \simeq O_{\mathbb{P}^1 \times \mathbb{P}^{n-2}} \oplus O_{\mathbb{P}^1 \times \mathbb{P}^{n-2}}(0, -1)$. Thus we have

$$Z \simeq \mathbb{P}(O_{\mathbb{P}^1 \times \mathbb{P}^{n-2}} \oplus O_{\mathbb{P}^1 \times \mathbb{P}^{n-2}}(0, -1)) \simeq \mathbb{P}^1 \times \text{Bl}_p(\mathbb{P}^{n-1}),$$

where $\text{Bl}_p(\mathbb{P}^{n-1})$ denotes the blow-up of $\mathbb{P}^{n-1}$ at the point $p$. We see that $Y' \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1}$ and $C'$ is a fiber of the projection $Y' \to \mathbb{P}^{n-1}$. We obtain the example (6).

Now we consider the case (B2). Let $F_Z$ be the exceptional divisor of the blow-up $\mu : Z \to Z'$. Since $E_Z$ is strictly positive on the extremal ray $\mathbb{R}^+ [m]$, we have $E_Z \neq F_Z$, in particular $F_Z \cdot e_Z \geq 0$. If $F_Z \cdot e_Z > 0$, there exists a fiber $m_0$ of the $\mathbb{P}^1$-bundle $F_Z \to \mu(F_Z)$ such that $m_0 \cap W \neq \emptyset$ (recall that $W$ denote the center of the blow-up $\varphi : X \to Z$). Since $-K_Z \cdot m_0 = 1$, we get a contradiction by Lemma 4. Hence we have $F_Z \cdot e_Z = 0$.

Recall that $\pi' : Z \to Y'$ is the blow-up along $C'$ and $\mu : Z \to Z'$ is a blow-up along a center of codimension 2 with $F_Z \cdot e_Z = 0$. Since $Y'$ and $Z$ are Fano manifolds, we can use the statement (A) (already proved in the previous subsection) to classify the pairs $(Y', C')$. Moreover, we have the condition on the normal bundle: $N_{C'/Z'} \simeq O_{\mathbb{P}^1}^{\oplus (n-1)}$, which is satisfied for the following cases:

- $Y'$ is the blow-up of $\mathbb{P}^n$ at a point $q$ and $C'$ is the strict transform of the line passing through $q$
• $Y'$ is the blow-up of $\mathbb{P}^n$ at a point $q$ and a linear subspace $P \simeq \mathbb{P}^{n-2}$ and $C'$ is the strict transform of a line passing through $q$.

• $Y'$ is the blow-up of $\mathbb{P}^n$ at a point $q$ and a quadric $Q \simeq Q_{n-2}$ and $C'$ is the strict transform of a line passing through $q$.

Recall that in each case, $Y$ is the blow-up of $Y'$ at the point $p \in C'$. So, we get the examples (7), (8), and (9). Hence, the proof of the statement (B) is completed.

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