Abstract. Using Dunkl theory, we introduce into consideration some weighted $L^p$-spaces on $[-1,1]$ and on the unit Euclidean sphere $S^{d-1}$, $d \geq 2$. Then we define a family of linear bounded operators $\{V^\kappa_p(x): x \in S^{d-1}\}$ acting from the $L^p$-space on $[-1,1]$ to the $L^p$-space on $S^{d-1}$, $1 \leq p < \infty$. We establish a necessary and sufficient condition for a function $g$ belonging to the $L^p$-space on $[-1,1]$ such that the family of functions $\{V^\kappa_p(x,g): x \in S^{d-1}\}$ is fundamental in the $L^p$-space on $S^{d-1}$.

Key words and phrases: fundamental set, unit sphere, Dunkl theory, Dunkl intertwining operator, Funk–Hecke formula for $\kappa$-spherical harmonics

MSC 2010: 42B35, 42C05, 42C10

1. Introduction and preliminaries

In this section we introduce some basic definitions and notions of general Dunkl theory (see, e.g., [2, 6]); for a background on reflection groups and root systems the reader is referred to [6, 8].

Let \( \mathbb{N}_0 \) be the set of nonnegative integers, let \( \mathbb{R}^d \) be the \( d \)-dimensional real Euclidean space of all \( d \)-tuples of real numbers. For \( x \in \mathbb{R}^d \), we write \( x = (x_1, \ldots, x_d) \). The inner product of \( x, y \in \mathbb{R}^d \) is denoted by \( \langle x, y \rangle = \sum_{i=1}^d x_i y_i \), and the norm of \( x \) is denoted by \( \|x\| = \sqrt{\langle x, x \rangle} \).

Let \( S^{d-1} = \{x: \|x\| = 1\} \) be the unit sphere in \( \mathbb{R}^d \), \( d \geq 2 \). Denote by \( d\omega \) the usual Lebesgue measure on \( S^{d-1} \).

For a nonzero vector \( v \in \mathbb{R}^d \), define the reflection \( s_v \) by

\[
s_v(x) = x - 2 \frac{\langle x, v \rangle}{\|v\|^2} v, \quad x \in \mathbb{R}^d.
\]

Each reflection \( s_v \) is contained in the orthogonal group \( O(\mathbb{R}^d) \).

Recall that a finite set \( R \subset \mathbb{R}^d \setminus \{0\} \) is called a root system if the following conditions are satisfied:

1. \( R \cap \mathbb{R}v = \{ \pm v \} \) for all \( v \in R \);  
2. \( s_v(R) = R \) for all \( v \in R \).

The subgroup \( G = G(R) \subset O(\mathbb{R}^d) \) which is generated by the reflections \( \{s_v: v \in R\} \) is called the reflection group associated with \( R \). It is known that the reflection group \( G \) is finite and the set of reflections contained in \( G \) is exactly \( \{s_v: v \in R\} \).

Each root system \( R \) can be written as a disjoint union \( R = R_+ \cup (-R_+) \), where \( R_+ \) and \( -R_+ \) are separated by a hyperplane through the origin. Such a set \( R_+ \) is called a positive subsystem. Its choice is not unique.

A nonnegative function \( \kappa \) on a root system \( R \) is called a multiplicity function if it is \( G \)-invariant, i.e. \( \kappa(v) = \kappa(g(v)) \) for all \( v \in R, g \in G \).

Note that definitions given below do not depend on the special choice of \( R_+ \), thanks to the \( G \)-invariance of \( \kappa \).
The Dunkl operators are defined by

\[ D_i f(x) = \frac{\partial f(x)}{\partial x_i} + \sum_{v \in R_+} \kappa(v) \frac{f(x) - f(s_v(x))}{\langle v, x \rangle} v_i, \quad 1 \leq i \leq d. \]

In case \( \kappa = 0 \), the Dunkl operators reduce to the corresponding partial derivatives. These operators were introduced and first studied by C. F. Dunkl.

Let

\[ \lambda = \gamma + \frac{d - 2}{2}, \quad \gamma = \sum_{v \in R_+} \kappa(v), \]

let \( w_\kappa \) denote the weight function on \( S^{d-1} \) defined by

\[ w_\kappa(x) = \prod_{v \in R_+} |\langle v, x \rangle|^{2\kappa(v)}, \quad x \in S^{d-1}. \]

The weight function \( w_\kappa \) is a positively homogeneous \( G \)-invariant function of degree \( 2\gamma \). In case \( \kappa = 0 \), \( w_\kappa \) is identically equal to 1.

Suppose \( \Pi^d \) is the space of all polynomials in \( d \) variables with complex coefficients, \( P_n^d \) is the subspace of homogeneous polynomials of degree \( n \in \mathbb{N}_0 \) in \( d \) variables.

C. F. Dunkl has proved in [5] that there exists a unique linear isomorphism \( V_\kappa \) of \( \Pi^d \) such that

\[ V_\kappa(P_n^d) = P_n^d, \quad n \in \mathbb{N}_0, \quad V_\kappa 1 = 1, \quad D_i V_\kappa = V_\kappa \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq d. \]

This operator is called the Dunkl intertwining operator. The operator \( V_\kappa \) was studied by many mathematicians (for example, C. F. Dunkl, M. Rösler, K. Trimèche, Y. Xu). If \( \kappa = 0 \), then \( V_\kappa \) is the identity operator.

Throughout this paper, we assume that \( p \in [1, \infty) \) and \( \lambda \) is strictly positive. In particular, it follows that \( \gamma > 0 \) if \( d = 2 \).

To explain our main result of the present paper, we need to introduce some weighted \( L_p \)-spaces and one family of linear operators.

Denote by \( L_{\kappa, p}(S^{d-1}) \) the space of complex-valued Lebesgue measurable functions \( f \) on \( S^{d-1} \) with finite norm

\[ \| f \|_{\kappa, p, S^{d-1}} = \left( \int_{S^{d-1}} |f(x)|^p \, d\sigma_\kappa(x) \right)^{1/p}, \quad d\sigma_\kappa(x) = a_\kappa w_\kappa(x) \, d\omega(x), \]

where the normalizing constant \( a_\kappa \) satisfies \( a_\kappa \int_{S^{d-1}} w_\kappa \, d\omega = 1 \). The space \( L_{\kappa, 2}(S^{d-1}) \) is a complex Hilbert space with the inner product

\[ \langle f, h \rangle_{\kappa, S^{d-1}} = \int_{S^{d-1}} f(x) \overline{h(x)} \, d\sigma_\kappa(x). \]

We also introduce the space \( L_{\kappa, \infty}(S^{d-1}) \) composed of all complex-valued Lebesgue measurable functions defined on \( S^{d-1} \) which are \( \sigma_\kappa \)-measurable and \( \sigma_\kappa \)-essentially bounded. Because \( w_\kappa \) is \( \omega \)-a.e. nonzero on \( S^{d-1} \), the above notions coincides with the one of \( w_\kappa \)-measurable and \( w_\kappa \)-essentially bounded function, respectively.
Let $\lambda > 0$. Suppose $L_{p,\lambda}[-1,1]$ is the space of complex-valued Lebesgue measurable functions $g$ on the segment $[-1,1]$ with finite norm

$$
\|g\|_{p,\lambda,[-1,1]} = \left( c_\lambda \int_{-1}^{1} |g(t)|^p (1-t^2)^{\lambda-1/2} dt \right)^{1/p}, \quad c_\lambda = \left( \int_{-1}^{1} (1-t^2)^{\lambda-1/2} dt \right)^{-1}.
$$

The Gegenbauer polynomials $C_n^\lambda$ (see, e.g., [1, p. 302]) are orthogonal with respect to the weight function $(1-t^2)^{\lambda-1/2}$. For a function $g \in L_{p,\lambda}[-1,1]$, its Gegenbauer expansion takes the form

$$
g(t) \sim \sum_{n=0}^{\infty} A_{n,\lambda}(g) \frac{n+\lambda}{\lambda} C_n^\lambda(t) \quad \text{with} \quad A_{n,\lambda}(g) = \frac{c_\lambda}{C_n^\lambda(1)} \int_{-1}^{1} g(t) C_n^\lambda(t)(1-t^2)^{\lambda-1/2} dt, \quad (1)
$$

since $\|C_n^\lambda\|_{2,[-1,1]}^2 = C_n^\lambda(1)\lambda/(n+\lambda)$.

Theorem 13.17 in [7] says that $L_{p,\lambda}[-1,1] \subset L_{p,\lambda}[-1,1]$. This fact is formulated and proved in section 3.

Recall that a set $\mathcal{F}$ in a Banach space $E$ is said to be fundamental if the linear span of $\mathcal{F}$ is dense in $E$. To prove the main result, we use a consequence of the Hahn–Banach theorem related to fundamentality of sets in normed linear spaces. We include it as a separate lemma for convenience.

Lemma 1. Let $\mathcal{F}$ be a subset of a Banach space $E$. In order that $\mathcal{F}$ be fundamental in $E$, it is necessary and sufficient that $\mathcal{F}$ not be annihilated by a nonzero bounded linear functional on $E$.

2. Some facts of Dunkl harmonic analysis on the unit sphere

The Dunkl Laplacian $\Delta_\kappa$ is defined by

$$
\Delta_\kappa = \sum_{i=1}^{d} D_i^2
$$
and it plays the role similar to that of the ordinary Laplacian. It reduces to the ordinary Laplacian provided that $\kappa = 0$.

A $\kappa$-harmonic polynomial $P$ of degree $n \in \mathbb{N}_0$ in $d$ variables is a homogeneous polynomial $P \in \mathcal{P}_n^d$ such that $\Delta_\kappa P = 0$. Its restriction to the unit sphere is called the $\kappa$-spherical harmonic of degree $n$ in $d$ variables. Denote by $\mathcal{A}_n^d(\kappa)$ the space of $\kappa$-spherical harmonics of degree $n$ in $d$ variables. The $\kappa$-spherical harmonics of different degrees turn out to be orthogonal with respect to the weighted inner product $\langle \cdot, \cdot \rangle_{\kappa,S^{d-1}}$ [3 Theorem 1.6].

Let $C(S^{d-1})$ be the space of complex-valued continuous functions on $S^{d-1}$.

**Lemma 2.** The set $\bigcup_{n=0}^{\infty} \mathcal{A}_n^d(\kappa)$ is fundamental in $C(S^{d-1})$ and in $L_{\kappa,p}(S^{d-1})$, $1 \leq p < \infty$.

**Proof.** Theorem 3.14 in [11] states that the space $C(S^{d-1})$ is dense in $L_{\kappa,p}(S^{d-1})$ for $1 \leq p < \infty$. So it is sufficient to show that $\bigcup_{n=0}^{\infty} \mathcal{A}_n^d(\kappa)$ is fundamental in $C(S^{d-1})$.

By the Weierstrass approximation theorem, if $f$ is continuous on $S^{d-1}$, then it can be uniformly approximated by polynomials restricted to $S^{d-1}$. According to [3 Theorem 1.7], these restrictions belong to the linear span of $\bigcup_{n=0}^{\infty} \mathcal{A}_n^d(\kappa)$. Thus, $\bigcup_{n=0}^{\infty} \mathcal{A}_n^d(\kappa)$ is fundamental in $C(S^{d-1})$.

The above proof is analogous to that of Corollary 2.3 in [12].

**Lemma 3.** Let $g \in L_{p,\lambda_n}[-1,1]$, $1 \leq p < \infty$. Then for every $Y_n^\kappa \in \mathcal{A}_n^d(\kappa)$,

$$
\int_{S^{d-1}} V_\kappa^p(x; g, y) Y_n^\kappa(y) \, d\sigma_\kappa(y) = \Lambda_{n,\lambda_n}(g) Y_n^\kappa(x), \quad x \in S^{d-1},
$$

(3)

where the constant $\Lambda_{n,\lambda_n}(g)$ is defined from (1).

Equality (3) is the Funk–Hecke formula for $\kappa$-spherical harmonics written in our setting and designations (cf. [2 Theorem 7.2.7], [13 Theorem 2.1]).

3. Main result: proof and its consequence

We can now state and prove the main theorem.

**Theorem 1.** Let $d \geq 2$, $1 \leq p < \infty$. Fix a root system $R$ in $\mathbb{R}^d$ and a multiplicity function $\kappa$ on $R$. Let $g \in L_{p,\lambda_n}[-1,1]$. In order that the set $\mathcal{M}_n^p(g)$ (2) be fundamental in $L_{\kappa,p}(S^{d-1})$, it is necessary and sufficient that $\Lambda_{n,\lambda_n}(g) \neq 0$ (1) for every $n \in \mathbb{N}_0$.

**Proof.** We first prove that the condition is sufficient. Let $\Phi$ be a bounded linear functional on $L_{\kappa,p}(S^{d-1})$ which annihilates $\mathcal{M}_n^p(g)$. According to the Riesz representation theorem [11 Theorem 6.16], $\Phi$ can be written as follows: $\Phi(\cdot) = \langle \cdot, h \rangle_{\kappa,S^{d-1}}$, where $h \in L_{\kappa,q}(S^{d-1})$ and $q$ is the exponent conjugate to $p$ ($p^{-1} + q^{-1} = 1$; $q = \infty$ whenever $p = 1$). Then the annihilating property of $\Phi$ reduces to

$$
\int_{S^{d-1}} V_\kappa^p(x; g, y) \overline{h(y)} \, d\sigma_\kappa(y) = 0, \quad x \in S^{d-1}.
$$
Next, we multiply both sides of the previous equality by $Y_n^\kappa(x) \in \mathcal{A}_n^d(\kappa)$, $n \in \mathbb{N}_0$, and integrate the resulting expression with respect to the measure $d\sigma\kappa$. Hölder’s inequality implies that $V_\kappa^p(x; g, y) \overline{h}(y) Y_n^\kappa(x)$ is $\sigma_\kappa \times \sigma_\kappa$-integrable over $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$, and hence, using the Fubini theorem to interchange the order of integration, we get

$$
\int_{\mathbb{S}^{d-1}} \overline{h}(y) \left( \int_{\mathbb{S}^{d-1}} V_\kappa^p(x; g, y) Y_n^\kappa(x) \, d\sigma_\kappa(x) \right) \, d\sigma_\kappa(y) = 0.
$$

Using the symmetric relation \[9, \text{formula (7)}\]

$$
V_\kappa [g(\langle x, \cdot \rangle)](y) = V_\kappa [g(\langle y, \cdot \rangle)](x) \quad \sigma_\kappa \times \sigma_\kappa\text{-a.e. on } \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}
$$

and the Funk–Hecke formula \[3\], we obtain

$$
\Lambda_{n,\lambda}(g) \langle Y_n^\kappa, h \rangle_{\kappa,\mathbb{S}^{d-1}} = 0, \quad Y_n^\kappa \in \mathcal{A}_n^d(\kappa), \quad n \in \mathbb{N}_0.
$$

It follows from the condition that

$$
\langle Y_n^\kappa, h \rangle_{\kappa,\mathbb{S}^{d-1}} = 0, \quad Y_n^\kappa \in \mathcal{A}_n^d(\kappa), \quad n \in \mathbb{N}_0.
$$

Thus, $\Phi$ annihilates $\bigcup_{n=0}^{\infty} \mathcal{A}_n^d(\kappa)$. By continuity of $\Phi$ and Lemma \[2\], $\Phi = 0$ on $L_{\kappa, p}(\mathbb{S}^{d-1})$.

Therefore, the set $\mathcal{M}_p^\kappa(g)$ is fundamental in $L_{\kappa, p}(\mathbb{S}^{d-1})$ by Lemma \[1\].

Let us now prove that the condition described in the theorem is necessary. Assume, to reach a contradiction, that there exists an index $m \in \mathbb{N}_0$ such that $\Lambda_{m,\lambda}(g) = 0$. Select any nontrivial $\kappa$-spherical harmonic $Y_m^\kappa \in \mathcal{A}_m^d(\kappa)$ and consider a measure $\mu$ defined on the Lebesgue subsets $\mathcal{L}$ of $\mathbb{S}^{d-1}$ by the rule

$$
\mu(B) = \int_B Y_m^\kappa(x) \, \sigma_\kappa(x), \quad B \in \mathcal{L}.
$$

This measure is nontrivial by its definition.

Using the Funk–Hecke formula \[3\], we obtain

$$
\int_{\mathbb{S}^{d-1}} V_\kappa^p(x; g, y) \, d\mu(y) = \int_{\mathbb{S}^{d-1}} V_\kappa^p(x; g, y) Y_m^\kappa(y) \, d\sigma_\kappa(y) = \Lambda_{m,\lambda}(g) Y_m^\kappa(x) = 0, \quad x \in \mathbb{S}^{d-1}.
$$

Thus, the nontrivial bounded linear functional $\Phi_1$ on $L_{\kappa, p}(\mathbb{S}^{d-1})$ given by $\Phi_1(f) = \int_{\mathbb{S}^{d-1}} f \, d\mu$ annihilates $\mathcal{M}_p^\kappa(g)$. By Lemma \[1\], $\mathcal{M}_p^\kappa(g)$ is not fundamental in $L_{\kappa, p}(\mathbb{S}^{d-1})$. This contradicts our assumption.

The above proof is exactly like that of Theorem 2.4 in \[10\]. Using the scheme of the proof of the theorem, one can prove the following result.

**Corollary 1.** Let $d \geq 2$, $s \geq 1$, $1 \leq p < \infty$. Fix a root system $R$ in $\mathbb{R}^d$ and a multiplicity function $\kappa$ on $R$. Let $g_1, \ldots, g_s \in L_{p, \lambda}[\mathbb{S}^d]$. In order that the set $\bigcup_{i=1}^{s} \mathcal{M}_p^\kappa(g_i)$ be fundamental in $L_{\kappa, p}(\mathbb{S}^{d-1})$, it is necessary and sufficient that $\sum_{i=1}^{s} |\Lambda_{n,\lambda}(g_i)| \neq 0$ for every $n \in \mathbb{N}_0$. 
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