ADDENDUM TO “MORSE THEORY OF CAUSAL GEODESICS IN A STATIONARY SPACETIME VIA MORSE THEORY OF GEODESICS OF A FINSLER METRIC”, ANN. INST. H. POINCARÉ ANAL. NON LINÉAIRE, 27 (2010) 857–876

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Abstract. We give the details of the proof of equality (29) in [3]

1. Introduction

In [3, Eq. (29)], we claim that the relative homology groups $H_{\ast}(\tilde{E}_{1_X}^c \cap \tilde{O}^\ast, \tilde{E}_{1_X}^c \cap \tilde{O}^\ast \setminus \{0\})$ and $H_{\ast}(\tilde{E}^c \cap \tilde{O}^\ast, \tilde{E}^c \cap \tilde{O}^\ast \setminus \{0\})$ are isomorphic, where, we recall, $X = C^1_0([0,1], U)$, $U$ is a neighbourhood of $0 \in \mathbb{R}^n$, $\tilde{E}: H^1_0([0,1], U) \to \mathbb{R}$, $\tilde{E}(x) = \int_0^1 \tilde{G}(s, x, \dot{x})ds$, $0 \in H^1_0([0,1], U)$ is a non-degenerate critical point of $\tilde{E}$, $c = \tilde{E}(0), \tilde{E}^c = \{x \in H^1_0([0,1], U) | \tilde{E}(x) \leq c\}$ and $\tilde{O}^\ast$ is a neighbourhood of $0$ in $H^1_0([0,1], U)$.

For this we refer to the following result by Palais [12, Theorem 16]:

**Theorem 1.1** (Palais, [12]). Let $V_1$ and $V_2$ be two locally convex topological vector spaces, $f$ be a continuous linear map from $V_1$ onto a dense linear subspace of $V_2$ and let $O$ be an open subset of $V_2$ and $\tilde{O} = f^{-1}(O)$. If $V_1$ and $V_2$ are metrizable then $\tilde{f} = f_{|\tilde{O}}: \tilde{O} \to O$ is a homotopy equivalence.

As a consequence, if $E$ is a Banach space which is dense and continuously immersed in a Hilbert space $H$ and $(A, B)$ is a pair of open subsets of $H$ with $B \subset A$, then the relative homology groups $H_{\ast}(A, B)$ and $H_{\ast}(\tilde{A}, \tilde{B})$, where $\tilde{A} = A \cap E$ and $\tilde{B} = B \cap E$, are isomorphic.

In this addendum we would like to make clear how the above result can be applied to get

$$H_{\ast}(\tilde{E}_{1_X}^c \cap \tilde{O}^\ast, \tilde{E}_{1_X}^c \cap \tilde{O}^\ast \setminus \{0\}) \equiv H_{\ast}(\tilde{E}^c \cap \tilde{O}^\ast, \tilde{E}^c \cap \tilde{O}^\ast \setminus \{0\}).$$

Although it is not difficult to find some open subsets which are homotopically equivalent, with respect to the $H^1$ topology, to the ones involved in the computations of the critical groups (cf., for example, [5, Ch. III, Corollary 1.2]), it is not trivial to ensure, after applying Palais’ result, that the intersections of these subsets with $X$ continue to be homotopically equivalent in the $C^1$ topology.

Actually, the equality between the critical groups of a Dirichlet functional with respect to the $H^1$ and $C^1$ topology is not a novelty (cf. [5, 6, 10]). Anyway, there are some issues for the functional $\tilde{E}$ that we would like to point out. First, $\tilde{E}$ is not $C^2$ with respect to the $H^1$ topology (this is a very general phenomenon for smooth, at most quadratic in the velocities Lagrangians cf. [1, Prop. 3.2]); moreover, as $\tilde{G}$ is not everywhere twice differentiable, $\tilde{E}$ is also not twice Gateaux differentiable at any non-$G$-regular curve (see Definition 2.1). Secondly, although its flow is well defined on $X$, the gradient of $\tilde{E}$ is not of the type identity plus a compact operator, thus we cannot immediately state that it possesses the retractible property in [4, §III], which ensures that the deformation retracts involved in the computation of the critical groups are also continuous in $X$, where the Palais-Smale condition does not hold. To overcome this problem, we extend a result in...
constructing a smooth vector field, which is a pseudo-gradient in \( \mathcal{V} \setminus \overline{B}(0, r) \), where \( \mathcal{V} \) is a neighbourhood of 0 in \( H^1_0([0, 1], U) \) and \( \overline{B}(0, r) \) is the closure of a ball, and whose flow satisfies the retractive property.

The proof we give in the next section (without Lemma 2.3, which becomes superfluous) also holds for any smooth Lagrangian on \([0, 1] \times TM\), where \( M \) is a finite dimensional manifold, which is fiberwise strongly convex and has at most quadratic growth in each fibre. We can also consider, with minor modifications, more general boundary conditions as the curves joining two given submanifolds in \( M \). The Lagrangian action functional will be then defined on the Hilbert manifold of the \( H^1 \) curves between the two submanifolds. As we have already mentioned above, such functional is in general not \( C^2 \). Assuming that at least one of the submanifolds is compact and that all the critical points are non-degenerate, we can obtain, as in [3, Theorem 9], the Morse relations for the solutions of the corresponding Lagrangian system. In this case, the number of the conjugate instants along a geodesic, counted with their multiplicity, is replaced by the number of the "focal instants" with respect to one of the two submanifold (counted with multiplicities) along a solution plus the index of a bilinear symmetric form related to the other submanifold, [7]. We recall that a Morse complex for the action functional of such kind of Lagrangian, whose homology is isomorphic to the singular homology of the path space between the two submanifolds, has been obtained in [1].

2. Proof of the isomorphism between the critical groups in \( H^1 \) and \( C^1 \)

We recall that the Lagrangian \( \tilde{G}: [0, 1] \times U \times \mathbb{R}^n \to [0, +\infty) \) is given by

\[
\tilde{G}(t, q, y) = F^2(\varphi(t, q), d\varphi(t, q)[(1, y)]),
\]

where \( F \) is a Finsler metric on the \( n \)-dimensional smooth manifold \( M \) and \( \varphi: [0, 1] \times U \to M \) is defined as \( \varphi(t, q) = \exp_{y_0(t)} P_t(q) \); here, \( \exp \) is the exponential map with respect to any auxiliary Riemannian metric \( h \) on \( M \), \( y_0 \) is the geodesic of \( (M, F) \) in which we want to compute the critical groups, \( P_t: U \to T_{y_0(t)}M \) is given by \( P_t(q_1, \ldots, q_m) = \sum_{i=1}^m q_i E_i(t) \), where \( \{E_i\}_{i \in \{1, \ldots, n\}} \) are \( n \)-orthonormal smooth vector fields along \( y_0 \) and \( U \) is the Euclidean ball of radius \( \rho/2 \), where \( \rho \) is the minimum of the injectivity radii (with respect to the metric \( h \)) at the points \( y(t), t \in [0, 1] \).

The set \( Z \) where \( \tilde{G} \) is not twice differentiable is defined by the equation \( d\varphi(t, q)[1, y] = 0 \) and then it corresponds to the subset of \([0, 1] \times U \times \mathbb{R}^n \) where the Lagrangian \( \tilde{G}(t, q, y) = F^2(\varphi(t, q), d\varphi(t, q)[(1, y)]) \) vanishes. We recall also that for each \((t, q) \in [0, 1] \times U \) there is only one \( y \in \mathbb{R}^n \) such that \( d\varphi(t, q)[(1, y)] = 0 \). Indeed, \( d\varphi(t, q)[(1, y)] = \partial_t \varphi(t, q) + \partial_q \varphi(t, q)[y] \) and, as \( \partial_q \varphi(t, q) \) is one-to-one, \( y \in \mathbb{R}^n \) is the only vector such that

\[
\partial_q \varphi(t, q)[y] = -\partial_t \varphi(t, q).
\]

We recall also that the map \( \varphi_*: H^1_0([0, 1], U) \to \Omega_{p_0, q_0}(M), \varphi_*(x)(t) = \varphi(t, x(t)) \), such that \( \tilde{E} = E \circ \varphi_* \), where \( E \) is the energy functional of \( F \), i.e. \( E(y) = \frac{1}{2} \int_0^1 F^2(y, \dot{y}) \, dt \) and \( \Omega_{p_0, q_0} \) is the Hilbert manifold of the \( H^1 \) curves on \( M \) between \( p_0 \) and \( q_0 \). Observe that the curve of constant value 0 is mapped by \( \varphi_* \) to the geodesic \( y_0 \) (hence 0 is a critical point of \( \tilde{E} \)).

From the fact that \( F^2 \) is fiberwise positively homogeneous of degree 2 and \( \varphi \) is a smooth map, it follows that there exists a constant \( c_1 \), depending only on \( U \), such that

\[
\|\tilde{G}_{qq}(s, q, y)\| \leq c_1 (1 + |y|^2), \quad \|\tilde{G}_{qy}(s, q, y)\| \leq c_1 (1 + |y|), \quad \|\tilde{G}_{yy}(s, q, y)\| \leq c_1,
\]

for every \((s, q, y) \in [0, 1] \times U \times \mathbb{R}^n \setminus Z \), where \(|\cdot|\) and \(\|\cdot\|\) are, respectively, the euclidean norm and the norm of bilinear forms on \( \mathbb{R}^n \).
Moreover, since $F^2$ is fiberwise strongly convex, there exists a positive constant $c_2$ such that
\[
G_{yy}(s, q, y)[w, w] \geq c_2 |w|^2,
\]
for each $(s, q, y) \in [0, 1] \times U \times \mathbb{R}^n \setminus Z$ and $w \in \mathbb{R}^n$.

**Definition 2.1.** A curve $x \in H^0_0([0, 1], U)$ is said $\tilde{G}$-regular if the set of points $t \in [0, 1]$ where $(t, x(t), \dot{x}(t)) \in Z$ is negligible.

Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\alpha_{|U'} = 1$, $\alpha_{|U''} = 0$, where $U'$ is an open subset of $\mathbb{R}^n$ such that $0 \in U'$ and $U' \subset U$. Consider the Lagrangian $\mathcal{L} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, where $\mathcal{L}(t, q, y) = \alpha(q)^2 + (1 - \alpha(q))|y|^2$. Clearly, by the definition of $\alpha$, $0$ is also a critical point of the action functional $\mathcal{A}_x(x) = \frac{1}{2} \int_0^1 \mathcal{L}(s, x, \dot{x})\,ds$. Notice also that, like $\tilde{E}$, $\mathcal{A}_x : H^1_0([0, 1], \mathbb{R}^n) \to \mathbb{R}$ is a $C^1$ functional with locally Lipschitz differential.

Let $\mathcal{B}$ be a closed ball in $H^1_0([0, 1], \mathbb{R}^n)$, centred in 0 and containing curves that have support in $U'$.

As $\mathcal{L} = \tilde{G}$ on $\mathbb{R} \times \mathbb{U}' \times \mathbb{R}^n$, we have that $\mathcal{A}_x|_{\mathcal{B}} = \tilde{E}|_{\mathcal{B}}$. Since $\tilde{E}$ satisfies the Palais-Smale condition (see [2]), we also have that $\mathcal{A}_x$ satisfies the Palais-Smale condition in $\mathcal{B}$.

Moreover, from (1) it follows that $\tilde{E}$ is twice Gateaux differentiable at any $\tilde{G}$-regular curve $x \in H^1_0([0, 1], U)$ and then the same property is satisfied by $\mathcal{A}_x$.

Observe that, as the endpoints of the geodesic $\gamma_0$ are not conjugate, then we can assume that $\mathcal{B}$ is an isolating neighbourhood of the critical point 0. Moreover, the non-conjugacy assumption implies also that 0 is a non-degenerate critical point of $\tilde{E}$, that is, the kernel of the operator $A_t$, which represents the second Gateaux differential at 0 of both $\tilde{E}$ and $\mathcal{A}_x$, with respect to the scalar product $(\cdot, \cdot)$, is empty.

The following proposition has been obtained in [1, Lemma 4.1 formula 4.8] for the action functional of a $C^2$, time-dependent, fiberwise strongly convex, at most quadratic in the velocities, Lagrangian on $TM$.

**Proposition 2.2.** There exist a neighbourhood $\mathcal{U}'$ of 0 in $H^1_0([0, 1], \mathbb{R}^n)$ (that we can assume is contained in $\mathcal{B}$) and a positive constant $\mu_0$, such that the linear vector field $x \in \mathcal{U}' \to Ax$, satisfies the inequality
\[
\text{d}\mathcal{A}_x(x)|Ax| \geq \mu_0 \|\nabla \mathcal{A}_x(x)\|_0^2,
\]
for each $x \in \mathcal{U}'$.

Here $\|\cdot\|_0$ is the $H^1_0$ norm. In our setting, the Lagrangian $\mathcal{L}$ is not twice differentiable on $Z \subset TM$ and this leads to some differences between the proof of [1, Lemma 4.1] and ours, which we outline in lemmata 2.3.2.6 and 2.7.

**Lemma 2.3.** Let $x$ be a smooth curve (non necessarily $\tilde{G}$-regular ) in $H^1_0([0, 1], U)$. Then the curves $t \in [0, 1] \to sx(t)$ can be non-$\tilde{G}$-regular for $s$ in a subset of $[0, 1]$ which is at most countable.

**Proof.** We recall that $w \in H^1_0([0, 1], U)$, $w = w(t)$, is not $\tilde{G}$-regular if $(t, w(t), \dot{w}(t)) \in Z$ for each $t$ in a subset of positive Lebesgue measure in $[0, 1]$. Now, for $x : [0, 1] \to U$, smooth and $x(0) = x(1) = 0$, we use the map $f : [0, 1] \times [0, 1] \to M$ defined as $f(s, t) = \varphi(t, sx(t))$. Observe that for each $\bar{t} \in [0, 1]$, $s \to f(s, \bar{t})$ is the affinely parametrized geodesic $\gamma_{\bar{t}}$ of the Riemannian metric $h$ defined by $\sigma_{\bar{t}}(s) = \varphi(t, sx(t)) = \exp_{y_{\bar{t}}}(sx(t))$ (for $\bar{t} = 0$ and $\bar{t} = 1$ the geodesics are constant) while, for each $s \in [0, 1]$, $t \to f(\bar{s}, t)$ is the curve $\gamma_{\bar{s}}$ corresponding to $\bar{s}x$ by the map $\varphi_{\bar{s}}$. (for $\bar{s} = 0$ and $\bar{s} = 1$, we get respectively $\gamma_0$, the geodesic of $(M, F)$, and the curve $\gamma_1 = \varphi_s(x)$). Thus $f = f(s, t)$ defines a geodesic congruence and, then, $s \to \bar{f}(s) = \bar{\gamma}_f(s, t) = \hat{\gamma}_f(t)$ defines a Jacobi field along $\sigma_{\bar{t}}$ for each...
t ∈ (0, 1) where x(t) ≠ 0. Observe that at the instants \( \bar{t} \) where \( x(\bar{t}) = 0 \) (if they exist), \( \sigma_j \) is constant and equal to \( y_0(\bar{t}) \). Since there is only one \( y \in \mathbb{R}^n \) such that \( (\bar{t}, 0, y) \in Z \) and such \( y \) cannot be equal to 0 (otherwise \( 0 = \partial y(\bar{t}, 0, y) = \partial y(\bar{t}, 0, 0) + \partial y(\bar{t}, 0, 0)|0 = \partial y(\bar{t}, 0, 0) = y_0(\bar{t}) \neq 0 \)), there can be at most one \( s \in [0, 1] \) such that \( (\bar{t}, sx(\bar{t}), s\dot{x}(\bar{t})) = (\bar{t}, 0, s\dot{x}(\bar{t})) \in Z \). Now let us assume that for \( s, s' \in (0, 1), s ≠ s' \), the curves \( sx \) and \( s'x \) are not \( \bar{G} \)-regular.

From what we have recalled above, this is equivalent to the fact that the curve \( \gamma \) and \( \gamma' \) have velocity vector fields vanishing on, respectively, \( Z_s \subset [0, 1] \) and \( Z'_{s'} \subset [0, 1] \) with \( |Z_s|, |Z'_{s'}| > 0 \). We claim that \( Z_s \cap Z'_{s'} = \emptyset \). Indeed, if there exists \( \bar{t} \in Z_s \cap Z'_{s'} \), then \( x(\bar{t}) \) must be different from 0 and this implies that the Jacobi field \( J_\bar{t} \) is well defined and equal to 0 at the instants \( s \) and \( s' \). Thus the points \( \sigma_j(s) \) and \( \sigma_j(s') \) are conjugate along \( \sigma_j \), but this is impossible (see, e.g., [8, Prop. 2.2, p. 267]) because such geodesic has length less than the injectivity radius at \( y_0(\bar{t}) \). Therefore the set \( \mathcal{J} \) of \( s \in [0, 1] \) such that \( |Z_s| > 0 \) is at most countable. Indeed, by contradiction, assume that \( \mathcal{J} \) is uncountable and consider the set \( A_h = \{ s \in [0, 1] : |Z_s| > \frac{1}{h} \} \). Since \( \cup_{h \in \mathbb{N}} A_h = \mathcal{J} \), there must exist at least one \( k \in \mathbb{N} \) such that \( A_k \) is uncountable. Thus, for infinitely many \( s \in [0, 1] \), we would have disjoint subsets \( Z_s \subset [0, 1] \) having measure greater than \( \frac{1}{h} \), which is impossible.

**Remark 2.4.** From Lemma 2.3 it also follows that any smooth non-\( \bar{G} \)-regular curve \( x \in H^1([0, 1], U) \) is the limit, in the \( H^1 \) topology, of some sequence \( (x_k) \subset H^1([0, 1], U) \) of smooth \( \bar{G} \)-regular curves. Indeed, it is enough to consider a sequence \( (s_n) \subset [0, 1] \) such that \( s_n \to 1 \) and \( s_nx \) is \( \bar{G} \)-regular.

**Remark 2.5.** From (2), the second Gateaux differential of \( \bar{E} \) at a \( \bar{G} \)-regular curve \( x \) is represented by a linear bounded self-adjoint operator on \( H^1([0, 1], \mathbb{R}^n) \) of the type \( A_x = B_x + K_x \) where \( B_x \) is a strictly positive definite operator and \( K_x \) is compact. Moreover from (1), if a sequence of \( \bar{G} \)-regular curves \( (x_n) \) converges to a \( \bar{G} \)-regular curve \( x \) in the \( H^1 \) topology then \( K_{x_n} \) converges to \( K_x \) in the norm topology of the bounded operators and \( B_{x_n} \) converges strongly to \( B_x \), i.e. \( B_{x_n}[\xi] \to B_x[\xi] \) for each \( \xi \in H^1([0, 1], \mathbb{R}^n) \) (cf. claim 1 and 2 of the proof of Lemma 4.1 in [1]). We recall that from [3, Lemma 2], \( A \equiv A_0 \) is given by \( I + K \) (that is, \( B_0 \) is the identity operator).

The following two results are the analogous of, respectively, Eq. (4.5) and Claim 3 in [1].

**Lemma 2.6.** Let \( (x_n) \subset H^1([0, 1], U) \) be a sequence of smooth \( \bar{G} \)-regular curves such that \( x_n \to 0 \) in the \( H^1 \) topology. Then

\[
\frac{d\bar{E}(x_n)}{dt}|_{x_n} = \frac{1}{n} \left( \langle B_{x_n}^{1/2} + K_{x_n} \rangle x_n, x_n \rangle \right) ds + o(\|x_n\|_0^n), \quad \text{as } n \to \infty.
\]

**Proof.** Eqs. (1)-(2) imply that \( \bar{G}(t, q, y) \) satisfies assumptions \((L1') \) and \((L2') \) at page 605 of [1], for each \( (t, q, y) \in [0, 1] \times U \times \mathbb{R}^n \setminus Z \). Hence the lemma follows arguing as in [1, Lemma 4.1], taking into account that

\[
\frac{d\bar{E}(x)}{dt}|_{x} = \langle \nabla \bar{E}(x), x + K(x) \rangle = \left( \int_0^1 \langle \nabla \bar{E}(sx), x + K(x) \rangle ds \right)
\]

\[
= \int_0^1 \langle (B_{sx} + K_{sx})x, x + K(x) \rangle ds.
\]

In fact, \( \frac{d}{dt}\nabla \bar{E}(sx) = (B_{sx} + K_{sx})[x] \) at the points \( s \) where the curve \( t \in [0, 1] \to sx(t) \) is \( \bar{G} \)-regular. From Lemma 2.3, the set of points \( s \in [0, 1] \) where \( sx \) is not \( \bar{G} \)-regular is at most countable. □
The next lemma follows as in Claim 3 of [1, Lemma 4.1], recalling Remark 2.5 and the fact that $0$ is a non-degenerate critical point of $\tilde{E}$.

**Lemma 2.7.** There exist a number $\mu > 0$ and a neighbourhood $\mathcal{U}''$ of $0$ in $H^1(0,1,\mathbb{R})$ such that, for each smooth and $\tilde{G}$-regular curve $x \in \mathcal{U}''$, the spectrum of the self-adjoint operator $B^{1/2} + K_x$ is disjoint from $[-\mu, \mu]$.

**Proof of Proposition 2.2.** Since $\mathcal{A}_H|_{\mathcal{A}} = \tilde{E}|_{\mathcal{A}}$, it is enough to prove the proposition for the functional $\tilde{E}$. From Lemmata 2.6 and 2.7, we get that there exists a positive constant $\mu_1$, such that
\[
d\tilde{E}(x)[Ax] \geq \mu_1 \|x\|^2_0,
\]
for each smooth $\tilde{G}$-regular curve $x \in \mathcal{U}''$. From Remark 2.4 and the continuity of $d\tilde{E}$ and $A$ with respect to the $H^1$ topology, inequality (5) can be extended to any smooth curve in $\mathcal{U}''$ and then, since smooth curves are dense in $H^1(0,1,\mathbb{R})$, to any $x \in \mathcal{U}'$. As $\nabla \tilde{E}$ is a locally Lipschitz field and $\nabla \tilde{E}(0) = 0$, we get
\[
d\tilde{E}(x)[Ax] \geq \mu_0 \|\nabla \tilde{E}(x)\|^2_0,
\]
for some positive constant $\mu_0$ and for all $x$ in some neighbourhood $\mathcal{U}'$ of $0$.

Now let $\eta_0 : H^1(0,1,\mathbb{R}) \to [0,1]$ be a smooth bump function such that $\text{supp} \eta_0 \subset \mathcal{U}'$ and $\eta_0(x) = 1$, for all $x \in \mathcal{U}$, where $\mathcal{U}$ is an open neighbourhood of $0$ in $H^1(0,1,\mathbb{R})$ with $\overline{\mathcal{U}} \subset \mathcal{U}'$. Let us consider the vector field on $H^1(0,1,\mathbb{R})$ defined as
\[
Y(x) = -\eta_0(x)Ax - (1 - \eta_0(x))\nabla \mathcal{A}_H(x).
\]
We point out that we cannot state that $Y$ is a pseudo-gradient vector field because we are not able to prove that
\[
\|Ax\|_0 \leq \mu_2 \|d\mathcal{A}_H(x)\|_0,
\]
for some constant $\mu_2 > \mu_0$ and all $x$ in some neighbourhood of $0$.

1 Actually using that $\mathcal{A}_H$ satisfies the Palais-Smale condition and $0$ is an isolated critical point of $\mathcal{A}_H$, we can prove that $Y$ satisfies (6) in any open subset $\mathcal{U} \setminus \overline{B(0,r)}$, where $\overline{B(0,r)}$ is an open ball strictly contained in $\mathcal{U}$, for a constant $\mu_2$ depending on $\mathcal{U} \setminus \overline{B(0,r)}$.
Lemma 2.8. Let $V$ be a closed neighbourhood of $0$ contained in $\mathcal{U}$. Then there exist $\varepsilon > 0$ and an open neighbourhood $O' \subset V$ of $0$ in $H_0^1([0,1],\mathbb{R}^n)$ such that if $x \in O'$, then the solution $\psi(x, \cdot)$ of (8) either stays in $V$ for all $t \in [0, +\infty)$ or it stays in $V$ at least until $\mathcal{A}_E(\psi(x, t))$ becomes less than $c - \varepsilon$, (where $c = \mathcal{A}_E(0) = \tilde{E}(0)$).

Proof. Observe that, since $Y_{|V} = -A, \psi(x, \cdot)$ is defined for all times until it lies in $V$. Let $B(0, \rho)$ be the ball of radius $\rho$ centred at $0$ such that $\tilde{B}(0, \rho) \subset V$ and let

$$\mathcal{C} = \{ x \in H_0^1([0,1],\mathbb{R}^n) : \| x \|_0 \leq \rho \}. $$

Since $\mathcal{C} \subset \mathcal{B}$, it is free of critical points and then

$$\delta = \inf_{x \in \mathcal{C}} \| \nabla \mathcal{A}_E(x) \|_0 > 0, \quad \text{(10)}$$

because $\mathcal{A}_E$ satisfies the Palais-Smale condition on $\mathcal{C}$. Moreover

$$\| Y(x) \|_0 = \| Ax \|_0 \leq \rho \| A \|_0 \leq \frac{\rho \| A \|_0}{\delta} \| \nabla \mathcal{A}_E(x) \|_0, \quad \text{(11)}$$

for each $x \in \mathcal{C}$. Let $v := \frac{\partial \| A \|_0}{\delta}$ and $O' = B(0, \rho/2) \cap \mathcal{A}_E^{-1}(v \rho)$. If $x \in O'$ is such that $\psi(x, \bar{t})$ does not belong to $V$ for some $t > 0$, then there exist $0 < t_1 < t_2 < \omega^+(x)$ such that $\psi(x, t) \in \mathcal{C}$, for all $t \in (t_1, t_2)$ and $\| \psi(x, t_1) \|_0 = \rho/2, \| \psi(x, t_2) \|_0 = \rho$. It follows that

$$\mathcal{A}_E(\psi(x, t_2)) = \mathcal{A}_E(\psi(x, t_1)) + \int_{t_1}^{t_2} d\mathcal{A}_E(\psi(x, t)) \| Y(\psi(x, t)) \|_0^2 dt \leq \mathcal{A}_E(x) - \mu \int_{t_1}^{t_2} \| \nabla \mathcal{A}_E(\psi(x, t)) \|_0^2 dt \leq c + \frac{\mu \delta \rho}{4v} - \mu \delta \int_{t_1}^{t_2} \| \nabla \mathcal{A}_E(\psi(x, t)) \|_0 dt \leq c + \frac{\mu \delta \rho}{4v} - \frac{\mu \delta}{v} \int_{t_1}^{t_2} \| Y(\psi(x, t)) \|_0^2 dt \leq c + \frac{\mu \delta \rho}{4v} - \frac{\mu \delta}{v} (\| \psi(x, t_2) \|_0 - \| \psi(x, t_1) \|_0) = c + \frac{\mu \delta \rho}{4v} - \frac{\mu \delta \rho}{2v} = c - \frac{\mu \delta \rho}{4v}. \quad \text{(12)}$$

In the first inequality above, we have used the fact that $\mathcal{A}_E$ is decreasing in the flow of (8) and inequality (7); in the second one, the fact that $x \in O' \subset \mathcal{A}_E^{-1}(\frac{\mu \delta \rho}{4v})$ and (10); in the third one, inequality (11); in the last one, the following chain of inequalities:

$$\int_{t_1}^{t_2} \| Y(\psi(x, t)) \|_0^2 dt = \int_{t_1}^{t_2} \| \psi(x, t) \|_0^2 dt \geq \int_{t_1}^{t_2} \psi(x, t) \| dt \geq \| \psi(x, t_2) \|_0 - \| \psi(x, t_1) \|_0. $$

Thus the conclusion follows with $\varepsilon = \frac{\mu \delta \rho}{4v}$. □

Let $V$ be the subset of $H_0^1([0,1],\mathbb{R}^n)$ given as $V = \bigcup_{x \in O'} \psi(x, [0, \omega^+(x)))$, where $O'$ is the neighbourhood of $0$ associated to $V$ by Lemma 2.8. Since $O'$ is open, from standard results in ODE theory (cf. for example [9, Corollary 4.2.10]), $V$ is also an open subset of $H_0^1([0,1],\mathbb{R}^n)$. From Lemma 2.8, $\mathcal{A}_E^{-1}(\frac{c - \varepsilon, c + \varepsilon}{2}) \cap V \setminus \{ 0 \}$ is contained in $V \subset \mathcal{U}$ and it is free of critical points.
Lemma 2.9. For every $x \in \mathcal{A}_x \backslash \{(c, c+\epsilon)\} \cap V$, either there exists a unique $T(x) \in [0, \omega^+(x))$ such that $\mathcal{A}_x(\psi(x, T(x))) = c$ or $\omega^+(x) = +\infty$ and $\psi(x, t) \to 0$, in $H^1_0([0,1], \mathbb{R}^n)$, as $t \to +\infty$.

Proof. If $\mathcal{A}_x(\psi(x, t)) > c$, for all $t \in [0, \omega^+(x))$, then from Lemma 2.8, $\omega^+(x) = +\infty$ and $\psi(x, t) \in \mathcal{V}$, for each $t \in [0, +\infty)$. From inequality (12),

$$\int_0^{+\infty} \|\nabla \mathcal{A}_x(\psi(x, t))\|^2 dt \leq \frac{1}{\mu}(\mathcal{A}_x(x) - c) < +\infty,$$

hence $\liminf_{t \to +\infty} \|\nabla \mathcal{A}_x(\psi(x, t))\|^2 = 0$ and the Palais-Smale condition implies the existence of a sequence $\{t_n\}$ converging to $+\infty$ such that $\psi(x, t_n) \to 0$. Hence the conclusion follows from Lemma 2.8. $\Box$

By Lemmata 2.8 and 2.9, as in [11, Lemma 8.3], we get that $\mathcal{A}_x \cap V$ is a strong deformation retract of $\mathcal{A}_{c+\epsilon/2} \cap V$. Analogously, $\mathcal{A}_{c-\epsilon} \cap V$ is a strong deformation retract of both $\mathcal{A}_x \cap V \backslash \{0\}$ and $\mathcal{A}_{c-\epsilon/2} \cap V$. Using that, for $A \subset B \subset C$, if $B$ is a strong deformation retract of $C$, then $H_*(B, A) \cong H_*(C, A)$ and if $A$ is a strong deformation retract of $B$, then $H_*(C, A) \cong H_*(B, C)$ (for the last property, see for example [13, Property H0–β]), we obtain

$$H_*(\mathcal{A}_x \cap V, \mathcal{A}_x \cap V \backslash \{0\}) \cong H_*(\mathcal{A}_{c+\epsilon/2} \cap V, \mathcal{A}_{c+\epsilon/2} \cap V). \quad (13)$$

Let $O = \varphi_x(O')$ and $y_0 = \varphi_x(0)$, then

$$C_*(E, y_0) = H_*(E^c \cap O, E^c \cap O \backslash \{y_0\}) \cong H_*(E \circ \varphi_x)^c \cap O', (E \circ \varphi_x)^c \cap O' \backslash \{0\}) = H_*(E^c \cap O', E^c \cap O' \backslash \{0\}) = H_*(\mathcal{A}_x \cap O', \mathcal{A}_x \cap O' \backslash \{0\}) \cong H_*(\mathcal{A}_x \cap V, \mathcal{A}_x \cap V \backslash \{0\}), \quad (14)$$

last equivalence, by the excision property of the singular relative homology groups. By Palais’ theorem above we get

$$H_*(\mathcal{A}_{c+\epsilon/2} \cap V, \mathcal{A}_{c-\epsilon/2} \cap V) \cong H_*(\mathcal{A}_{c+\epsilon/2} \cap [0,1], \mathbb{R}^n) \cap V, \mathcal{A}_{c-\epsilon/2} \cap [0,1], \mathbb{R}^n) \cap V).$$

The above equivalence, together with (13) and (14), implies that

$$C_*(E, y_0) \cong H_*(\mathcal{A}_{c+\epsilon/2} \cap [0,1], \mathbb{R}^n) \cap V, \mathcal{A}_{c-\epsilon/2} \cap [0,1], \mathbb{R}^n) \cap V).$$

It remains to prove that these last relative homology groups are isomorphic to the critical groups in $X = C^1_0([0,1], U)$. To this end, let us consider the Cauchy problem (8), with $x \in C^1([0,1], \mathbb{R}^n) \cap \mathcal{A}_x^{-1}((c-\epsilon/2, c+\epsilon/2)) \cap V$. Since $\mathcal{A}_x^{-1}((c-\epsilon/2, c+\epsilon/2)) \cap V \subset \mathcal{U}$, it holds (9) and the orbit $\psi(x, \cdot)$, defined by $x$, is also in $C^1_0([0,1], \mathbb{R}^n)$.

As a consequence, the strong deformation retracts that we have considered above are well defined in $C^1_0([0,1], \mathbb{R}^n) \times [0,1]$ and by the continuity of the flow (9) with respect to the $C^1$ topology, we immediately deduce that they are also continuous at each point different from $(0,1)$. Clearly, the continuity at the point $(0,1)$ with respect to the product topology of $C^1_0([0,1], \mathbb{R}^n)$, with the $C^1$ topology, and $\mathbb{R}$, with the standard one, comes into play only for the deformation map $\eta: \mathcal{A}_{c+\epsilon/2} \cap [0,1] \to \mathcal{A}_{c+\epsilon/2} \cap V$ of $\mathcal{A}_{c+\epsilon/2} \cap V$ in $\mathcal{A}_x \cap V$, which is given by

$$\eta(x, t) = \begin{cases} \rho(x, \frac{t}{\omega(x)}) & \text{if } t \in [0,1), \\ \lim_{s \to +\infty} \rho(x, s) & \text{if } t = 1, \end{cases}$$
where $\rho: \mathcal{A}^{c+1/2}_L \cap V \times [0, +\infty) \to \mathcal{A}^{c+1/2}_L \cap V$ is the map defined as follows: if $\mathcal{A}^{c}_L(x) > c$ and there exists $T(x) > 0$ such that $\mathcal{A}^{c}_L(\psi(x, T(x))) = c$, then

$$
\rho(x, t) = \begin{cases} 
\psi(x, t) & \text{if } t \in [0, T(x)], \\
\psi(x, T(x)) & \text{if } t \in (T(x), +\infty), 
\end{cases}
$$

if $\psi(x, t) \to c$ as $t \to +\infty$, then $\rho(x, t) = \psi(x, t)$ and if $\mathcal{A}^{c}_L(x) \leq c$, then $\rho(x, t) = x$, for all $t \in [0, +\infty)$. Since the flow $\psi_1$ of the linear vector field $x \mapsto -Ax - Kx$ is given by (9) and $K$ is bounded from $H^1_0((0, 1], \mathbb{R}^n)$ to $C^1_0([0, 1], \mathbb{R}^n)$, we have

$$
\left\| \int_0^t e^{-t+s}K(\psi_1(x, s))ds \right\|_{C^1} \leq e^{-t} \int_0^t e^s \|K(\psi_1(x, s))\|_{C^1} ds \leq C e^{-t} \int_0^t e^s \|\psi_1(x, s)\|_{C^1} ds.
$$

Thus, if $\psi(x, t) \to 0$ in $H^1$, as $t \to +\infty$, then, from Lemmata 2.8 and 2.9, $\psi(x, t) = \psi_1(x, t)$. Hence, for every $\varepsilon > 0$, there exists $\tilde{t} > 0$ such that for all $t > \tilde{t}$, $\|\psi(x, t)\|_{C^1} < \varepsilon$ and then the last function in the above inequalities can be estimated, for $t > \tilde{t}$, as

$$
e^{-t} \int_0^t e^s \|\psi(x, s)\|_{C^1} ds = e^{-t} \int_0^{\tilde{t}} e^s \|\psi(x, s)\|_{C^1} ds + e^{-t} \int_{\tilde{t}}^t e^s \|\psi(x, s)\|_{C^1} ds
$$

$$
\leq e^{-t} \int_0^{\tilde{t}} e^s \|\psi(x, s)\|_{C^1} ds + \varepsilon(1 - e^{-t} e^{\tilde{t}}).
$$

Thus $\psi(x, t) \to 0$ also with respect to the $C^1$ topology, giving the continuity of the map $\eta$ at the point $(0, 1)$ also with respect to the product of such a topology and the Euclidean one on the interval $[0, 1]$.

In conclusion we have that the following groups are isomorphic

$$
H_*(\mathcal{A}^{c+1/2}_L|_{C^1_0([0,1], \mathbb{R}^n)} \cap V, \mathcal{A}^{c-1/2}_L|_{C^1_0([0,1], \mathbb{R}^n)} \cap V) \cong \

H_*(\mathcal{A}^{c}_L|_{C^1_0([0,1], \mathbb{R}^n)} \cap V, \mathcal{A}^{c-1}_L|_{C^1_0([0,1], \mathbb{R}^n)} \cap V \setminus \{0\}).
$$

By excision, these last relative homology groups are isomorphic to $H_* (\mathcal{A}^{c}_L|_{C^1_0([0,1], \mathbb{R}^n)} \cap O', \mathcal{A}^{c}_L|_{C^1_0([0,1], \mathbb{R}^n)} \cap O' \setminus \{0\})$ and then, since the curves in $O'$ have their support in $U$, to $H_* \hat{E}^{c+1}_L|_{C^1_0([0,1], U') \cap V' \setminus \{0\}).$

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