Probing strong gravitational lensing by Bardeen black holes in 4D Einstein–Gauss–Bonnet gravity

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In the strong deflection limits, we investigate gravitational lensing by the Bardeen black holes in four-dimensional Einstein-Gauss-Bonnet (4D EGB) gravity and study the effect of coupling constant $\alpha$ and charge $q$. Interestingly, the spherical photon orbit radius $r_{\text{m}}$, the critical impact parameter $u_{\text{m}}$, the lensing coefficient $b$ are decreasing functions of $q$ and $\alpha$ whereas $\bar{a}$ shows opposite behaviour with these parameters. The deflection angle $\alpha_D(\theta)$, angular position of innermost image $\theta_{\alpha}$ decrease, but angular separation $s$ increases with $\alpha$ and $q$. Investigating the astrophysical consequences of gravitational lensing for supermassive black holes implies that 4D EGB Bardeen black holes can be distinguished from the Bardeen and Schwarzschild black hole. The deviations of the angular positions from the Schwarzschild black hole and Bardeen black holes becomes significant for large $\alpha$ and $q$ and are not more than $2.45016 \, \mu s$ for Sgr A$^*$ and $1.84084 \, \mu s$ for M87$^*$. Hence, Bardeen black holes in 4D EGB gravity is potentially different from the Schwarzschild black hole or Bardeen black holes. For Sgr A$^*$ and M87$^*$, the time delay can reach $\sim 10.6004 \, \text{min}$ and $\sim 15161.11 \, \text{min}$ and deviate from Bardeen black holes by $\sim 0.1076 \, \text{min}$ and $\sim 153.93 \, \text{min}$, respectively. These deviations are inconsequential for Sgr A$^*$ being too small, but for M87$^*$ and some other black holes, they are enough for an astronomical observation to test the 4D EGB gravity.

I. INTRODUCTION

The singularities in Einstein’s theory of general relativity (GR) appears to be a property inherent in most physically relevant exact solutions [1]. Penrose [2], assuming the weak energy condition and global hyperbolicity, proposed that formation of singularities in spacetime is inevitable. According to the Penrose cosmic censorship conjecture [3], these singularities must be surrounded by event horizons, with no causal connection between the interior of a black hole and the exterior fields such that the physics outside would be well behaved. Although it is believed that the singularities in a black hole can be removed in a quantum theory of gravity [4–6] and alternative gravities [7], some regular models have also been proposed [8, 9] to avoid this problem. Bardeen [8], was the first to introduce the regular black hole model, which has an event horizon with no singularities rather has a de-Sitter core.

The Bardeen’s regular black holes is compared in various applications which include thermonucler properties [10], geodesics equations [11], quasinormal modes [12], Hawking evaporation [13] and black hole’s remnant [13]. Lately, Bardeen’s solution has been extended to higher-dimensional spacetime [14], and to rotating counterpart [15].

Einstein-Gauss-Bonnet (EGB) theory, is one of the simplest natural extension of GR to higher dimensions $D \geq 5$. The GB correction to the Einstein Hilbert action in $D = 4$ is a total derivative and gives a non trivial contribution to the gravitational dynamics. The 4D EGB gravity in [16] is on equal footing with the GR and has received an enthralling attention which include a Vaidya-like radiating black holes [17], charged spherically symmetric black holes [18], rotating black holes [19, 20], gravitational lensing [21–23], thermodynamical properties of anti-de Sitter black holes [24] and other contributions [25–27].

Gravitational lensing, as an important application of GR, is a general term used to account for all the effects induced by the gravitational field on the propagation of electromagnetic radiation. Light propagates along a straight path unless intervened by an object with gravitational field. The effects induced by the gravitational field as observed in several scenarios ranging from our solar system to massive cluster of galaxies include change of apparent position of source, magnification, distortion, time delays, and multiple imaging. Gravitational lensing proved to be an important astrophysical tool to extract information about the distant stars, highly red shifted galaxies, quasars, supermassive black holes, exoplanets which were otherwise too dim to be observed. It has been used to to determine the Hubble constant [28], probe the structure of galaxies [29], dark matter and dark energy in Galactic halo [30], measure the density of cosmic string[31].

The weak-field limit, based on the assumption that the deflection angle is small, has been extensively studied [32–34] and has successfully explained the experimental tests done on GR. However, in the vicinity of compact objects like a black hole where the lensing will have a rich structure, GR is yet to be tested. Darwin [35] was the first to notice that light rays would make one or more loops around the black hole resulting in an infinite sequence of exotic images. Some years later, in what could be considered as the beginning of
black hole imaging, Walsh, Carswell and Weymann [36] discovered the first example of gravitational lensing in which they reported multiple images of a binary Quasar. Following the remarkable discovery of Quasars, gravitational lensing in strong deflection limit (SDL) was extensively studied [37]. The gravitational lensing in SDL was though resurrected by Virbhadra and Ellis [38], who found the exact lens equation with a large deflection angle for galactic supermassive black hole in an asymptotically flat background. Frittelli, Kling and Newman [39] used an approach to construct the lens equation and time of arrival without reference to a background metric for a Schwarzschild black hole. Bozza et al. [40], developed an analytical method to investigate Schwarzschild black hole lensing in SDL and found a logarithmic divergence of the deflection angle. This technique has been extended to Reissner black holes [41], and more so to an arbitrary static spherically symmetric metric [42]. The studies in gravitational lensing in SDL achieved a real boost when the first image of black hole M87* [43–45] was captured by Event Horizon Telescope. It also opens the gateway to investigate the region near the black hole and test the alternate theories of gravity. This has motivated us to investigate the gravitational lensing by Bardeen black holes in 4D EGB gravity ∼ 4D EGB Bardeen black holes. We calculate the lensing observables in terms of lensing coefficients and determine the effect of charge and coupling constant on the observables.

The rest of the paper is organized as follows: in Sec. II, we introduce the 4D EGB Bardeen black holes necessary to efficiently calculate properties of photon in SDL, including a discussion on parameter space for black holes and horizon structures. A discussion on the gravitational lensing of light in SDL for 4D EGB Bardeen black holes is the subject in Sec. III. The Sec. IV is devoted to the evaluation of lensing observables by 4D EGB Bardeen black holes, including the image positions, separation and magnifications by supermassive black holes Sgr A* and M87*. By taking the supermassive black holes as the lens, we numerically estimate time delays of the images in Sec. V. In Sec. VI we estimate and compare the lensing observables with the GR counterparts. Finally, we summarize our main findings in Sec. VII.

Throughout this paper, unless otherwise stated, we adopt natural units (8πG = c = 1)

II. 4D EGB BARDEEN BLACK HOLES

Lovelock [46] demonstrated the Einstein gravity could be extended by series of higher order curvature terms such that the equation of motion still remains second order. The simplest such extension is the Einstein Gauss Bonnet gravity when coupled with the non linear electrodynamics, the action reads [16, 47, 48]

\[ \mathcal{I}_{G} = \frac{1}{2} \int_{\mathcal{M}} d^{D}x \sqrt{-g} [R + \alpha \mathcal{L}_{GB} + \mathcal{L}(\mathcal{F})], \]

where \( \alpha \) is the coupling constant, \( g \) is the determinant of metric tensor \( g_{\mu\nu} \) and \( R \) is the Ricci Scalar. The Gauss-Bonnet Lagrangian is a combination of Ricci tensor \( R_{\mu\nu} \) and Riemann tensor \( R_{\mu\nu\rho\sigma} \)

\[ \mathcal{L}_{GB} = R^{2} - 4R\mathcal{R}_{\mu\nu} - R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \]

(2)

and \( \mathcal{L}(\mathcal{F}) \) is an arbitrary function of invariant \( \mathcal{F} = F_{\mu\nu}F_{\mu\nu}, \) where \( F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \) is the electromagnetic field tensor for gauge potential \( A_{\mu} \). When \( D = 4 \), the \( \mathcal{L}_{GB} \) becomes topological invariant and hence does not contribute to the dynamics [16]. The variation of action (1) yields field equations as

\[ G_{\mu\nu} + \alpha H_{\mu\nu} = T_{\mu\nu} \equiv 2\left[ \mathcal{L}_{F} F_{\mu\sigma} F_{\nu} - \frac{1}{4} g_{\mu\nu} \mathcal{L}(\mathcal{F}) \right], \]

(3)

where the Einstein tensor \( G_{\mu\nu} \) and Lanczos tensors \( H_{\mu\nu} \) respectively, are given by

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \]

(4)

\[ H_{\mu\nu} = 2(R R_{\mu\nu} - 2R_{\nu\mu} R^{\nu} - 2R_{\mu\sigma\rho\delta}R^{\sigma\rho\delta} - R_{\mu\sigma\rho\delta}R^{\sigma\rho\delta}) - \frac{1}{2} g_{\mu\nu} \mathcal{L}_{GB}. \]

(5)

We wish to obtain the Bardeen-like black holes to Eq. (3) for which the Lagrangian density reads [14, 50]

\[ \mathcal{L}(\mathcal{F}) = \frac{(D - 1)(D - 2)}{4g^{D-1}} \left( \sqrt{2g^{2}\mathcal{F}} \right)^{\frac{2-D}{D-3}}, \]

(6)

with \( \mathcal{F} = g^{2(D-3)/2} \). Here we choose general static, spherically symmetric metric anstaz in arbitrary dimensions [47, 50] as

\[ ds^{2} = -A(x)dt^{2} + \frac{dx^{2}}{A(x)} + x^{2}d\Omega_{D-2}^{2}, \]

(7)

where \( d\Omega_{D-2}^{2} \) is the line element of a \( (D-2) \)-dimensional constant curvature space [51]. Using field Eqs. (3) for

FIG. 1. The parameter space for 4D EGB Bardeen black holes. The dark-blue line correspond to values of parameters for which we have extremal 4D EGB Bardeen black holes.
and \( u > u \) corresponds to the value of impact parameter \( u < u \), by appropriately relating the constant of integration with \( u \rightarrow 0 \) encompasses those of Bardeen black holes, those of Schwarzschild black hole when \( \tilde{\alpha} \rightarrow 0 \), \( q = 0 \) and if only \( q = 0 \) we obtain the 4D EGB black holes.

\[ A_-(x) = 1 - \frac{x^2}{(x^2 + q^2)^{3/2}} + O \left( \frac{1}{x^4} \right), \]
\[ A_+(x) = 1 + \frac{x^2}{(x^2 + q^2)^{3/2}} + \frac{4x^2}{\tilde{\alpha}} + O \left( \frac{1}{x^4} \right). \]

The \(-\)ve branch corresponds to Bardeen black holes \([8]\), whereas \(+\)ve branch lead to an unphysical solution and hence, we shall restrict ourselves to \(-\)ve branch. Further, in the limit \( x \rightarrow \infty \), the \(-\)ve branch is asymptotically flat. Interestingly, the gravity with quantum corrections \([6]\) and the semi-classical Einstein’s equations with conformal anomaly \([52]\) also admit the solution of the form Eq. (8). The 4D EGB Bardeen black holes (8) encompasses the Bardeen black holes when \( \tilde{\alpha} \rightarrow 0 \) and 4D EGB black holes \([16]\) when \( q = 0 \). When \( \tilde{\alpha} \rightarrow 0 \) and \( q = 0 \), it resembles the Schwarzschild black hole.

The 4D EGB Bardeen black holes (8) is characterised by the parameters \((\tilde{\alpha}, q)\) and its horizon are the zeroes of \( g^{xx} = A(x) = 0 \), which for \( q = 0 \) admits

\[ x_{\pm} = \frac{1}{2} \left( 1 \pm \sqrt{1 - \frac{1}{\tilde{\alpha}}} \right). \]

In general, \( A(x) = 0 \) admits two roots \( x_{\pm} \) when parameters \((\tilde{\alpha}, q)\) are in the blue region of parameter space (cf. Fig. 1) which correspond to the inner Cauchy horizon \((x-)\) and outer event horizon \((x+ > x-)\) (cf. Fig. 2). When the parameters \((\tilde{\alpha}, q)\) are on the dark-blue line in the Fig. 1, we have extremal 4D EGB Bardeen black holes with \( x_e = x_+ = x_- \). In Fig. 2, we have shown the dependence of event horizon and Cauchy horizon on \( \tilde{\alpha} \) and \( q \).

It is evident from Fig. 2 that there are extremal values of \( q_c \) \((\tilde{\alpha}_c)\) represented by bullet points on the \( x\)-axis for given values of \( \tilde{\alpha} (q) \) such that for \( q < q_c \) \((\tilde{\alpha} < \tilde{\alpha}_c)\) there exists two horizons \((x_{\pm})\).

Before we starting our discussion on gravitational lensing in SDL, we would like to clarify that the regularisation proposed in \([6, 16]\), is subject to dispute and many
The authors raised questions [53–58]. Many alternative regularisation have also been suggested [54, 59–61]. However, the spherically symmetric 4D black hole solution obtained in [6, 16, 25] still remains valid in these regularised theories [54, 59, 60]. Hence these regularisation procedure lead to exactly the same black hole solutions [6, 16] at least for the case of 4D spherically symmetric spacetimes. We can confirm that our solution (8) can be obtained by the regularisation proposed in Ref. [54].

III. GRAVITATIONAL LENSING IN SDL

After a discussion on the spherically symmetric 4D EGB Bardeen black holes, we now discuss the gravitational lensing in SDL. The black hole significantly influences the motion of photon in its neighbourhood (cf. Fig. 4). The photon from a source approaches the black hole to a minimum distance $x_0$ and is deflected by its gravitational field to be received by the observer at infinity (cf. Fig. 5). The light ray trajectory is described by $k^\mu k_\mu = 0$, where $k^\mu = \dot{x}^\mu$ is a wave number of light and the overdot denotes the differentiation with respect to the affine parameter along the trajectory. The trajectories of the light rays are depicted in Fig. 4. The energy $\mathcal{E} \equiv -g_{\mu\nu}k^\mu k^\nu = A(x)\dot{t}$ and angular momentum $\mathcal{L} \equiv g_{\mu\nu}\phi^\mu k^\nu = x^2\dot{\phi}$ are constant along the light trajectory. From $k^\mu k_\mu = 0$, we obtain the trajectories of the light as

$$-A(x)\dot{t}^2 + \frac{\dot{x}^2}{A(x)} + C(x)\dot{\phi}^2 = 0. \quad (11)$$

where $C(x) = x^2$. Using the definition of energy $\mathcal{E}$, angular momentum $\mathcal{L}$ and introducing the impact parameter

FIG. 4. The trajectory of the light ray for (a) $\tilde{\alpha} = 0.4$, $q = 0.15$ (left) (b) $\tilde{\alpha} = 0.5$, $q = 0.2$ (right), in the polar coordinates ($r, \phi$). The red line corresponds to value of $u$ close to $u_m$. The black hole is shown as a solid disk and the photon sphere as a dashed black circle.

FIG. 5. Formation of primary images and relativistic images of source(S) in case of gravitational lensing. Light rays are deviated by black hole to be observed at an angular position $\theta$ by the observer(O).
\[ u = \mathcal{L}/\mathcal{E}, \text{ Eq. (11) can be rewritten as} \]
\[ \dot{x}^2 + V(x) = 1, \quad V(x) = u^2 \frac{A(x)}{C(x)}, \quad (12) \]
where \( V(x) \) is its effective potential (cf. Fig. 3 for its behaviour). The unstable spherical photon orbits radius which correspond to the distance where deflection angle diverges, are timelike hypersurfaces obtained by simultaneously solving \( \dot{x} = \dot{z} = 0 \) for \( V'(x) < 0 \) giving \( C'(x)/C(x) = A'(x)/A(x) \), whose radius is given by the largest positive root of
\[ 3x^4 - 2(q^2 + x^2)^{7/4} \sqrt{\alpha} + (q^2 + x^2)^{3/2} = 0. \quad (13) \]
We define the unstable photon sphere radius by \( x_m \) and depict it for 4D EGB Bardeen black holes with varying \( \alpha \) (q) in Fig. 6 right (left). The photon orbit radius is a monotonically decreasing function of \( \alpha \) and \( q \), particularly, \( x_m \) of Bardeen black holes as well as 4D EGB black holes is always larger than the 4D EGB Bardeen black holes. Moreover, the radius \( x_m \) peaks to a value of 1.5 when \( \alpha \to 0 \) and \( q = 0 \). Without loss of generality, we set \( \mathcal{E} = 1 \) such that at the distance of minimum approach \( x_0 \), which is by definition the turning point of the photon trajectory i.e., \( \dot{x} = 0 \), the impact parameter reads
\[ u^2 = \frac{x_0^2}{2x_0^2(1 - \sqrt{1 + \alpha/(q^2 + x_0^2)}^{3/2}) + \alpha}. \quad (14) \]
The above equation in the limit \( \alpha \to 0 \) gives the impact parameter of Bardeen black holes as
\[ u^2 = \frac{x_0^2}{1 - x_0^2/(q^2 + x_0^2)^{3/2}}. \quad (15) \]
\( u_m = u(x_m) \) is the critical impact parameter whose behaviour is shown in Fig. 7. The dependence of \( u_m \) on \( \alpha \) and \( q \) are qualitatively same as that of \( x_m \). The light rays with impact parameter \( u < u_m \) fall into the black hole whereas get scattered by black hole when \( u > u_m \) (cf. Fig. 3). The deflection caused by the black hole increases as impact parameter decreases (cf. Fig. 4). The deflection angle for null geodesic can be found to be [42, 62]
\[ \alpha_D(x_0) = -\pi + 2 \int_{x_0}^{\infty} \frac{1}{\sqrt{A(x)C(x)}} \left( \frac{C(x)A(x_0)}{C(x_0)A(x)} - 1 \right) dx. \quad (16) \]
The deflection angle increases as \( x_0 \to x_m \) and diverges at \( x_0 = x_m \). We expand the integral near the photon sphere [38, 42, 63] by defining a new variable \( z = 1 - x_0/x \) in SDL. This technique not only shows the behaviour of photons near the photon sphere but also provides an analytical representation of deflection angle, which for 4D EGB Bardeen black holes reads [42]
\[ \alpha_D(u) = \bar{a} \log \left( \frac{u}{u_m} - 1 \right) + \bar{b} + \mathcal{O}(u - u_m), \quad (17) \]
where
\[ \bar{a} = \frac{1}{1 - \frac{x_0}{\beta_m}}, \quad \bar{b} = -\pi + J_m + \bar{a} \log(x_m^2), \quad (18) \]
\[ \beta_m = \frac{x_m^5 \left( 60Q^4 + 9\alpha \sqrt{q^2 + x_m^2} \right) + 3Qq^2x_m^3 + 30Qq^2x_m^3}{4Q^{3/2} \left( q^2 + x_m^2 \right)^{11/2}}, \]
\[ I_m = \int_0^1 [R(z, x_m)f(z, x_m) - R(0, x_m)f_0(z, x_m)] dz, \quad (19) \]
\[ R(z, x_0) = \frac{2x_0^2 \sqrt{C(x_0)}}{x_0 \sqrt{C^2(x)}} \frac{C(x_0)}{C^2(x)} \frac{1}{\sqrt{A(x_0) - A(x) \frac{C(x_0)}{C(x)}}}, \quad \gamma = \frac{2}{x_m^2} \left( \frac{4 - 4\sqrt{\Omega}}{\alpha} + \frac{9\alpha x_m^4}{21Q^2 \left( q^2 + x_m^2 \right)^3} + \frac{15\alpha x_m^2}{\sqrt{(q^2 + x_m^2)^{5/2}}}, \right), \quad \Omega = \sqrt{1 + \frac{\alpha}{(q^2 + x_m^2)^{3/2}}}. \quad (20) \]
FIG. 7. The behavior of critical impact parameter $u_m$ (upper panel), lensing coefficients $\bar{a}$ (middle panel) and $\bar{b}$ (lower panel) as a function of $q$ for different values of $\tilde{\alpha}$ (left) and as a function of $\tilde{\alpha}$ for different values of $q$ (right). Our results in limits $\tilde{\alpha} \to 0$ encompasses those of Bardeen black holes, those of Schwarzschild black hole when $\tilde{\alpha} \to 0$, $q = 0$ and if only $q = 0$ we obtain the 4D EGB black holes.

and $u \approx \theta D_{OL}$. Here, the function $f_0(z, x)$ is obtained by series expansion of $f(z, x)$ up to second order. $\bar{a}$ and $\bar{b}$ are the lensing coefficients for 4D EGB Bardeen black holes which are depicted in Fig. 7 and also tabulated in Table 1. Our results coincide with Schwarzschild black hole of $\bar{a} = 1$ and $\bar{b} = -0.4004$ at $\tilde{\alpha} \to 0$ and $q = 0$. For other values of $\tilde{\alpha}$ and $q$, $\bar{a}$ is always larger whereas $\bar{b}$ is smaller than the corresponding values of Bardeen as well as 4D EGB black holes (cf. Table 1).

The deflection angle is a monotonically decreasing function of $\tilde{\alpha}$ and $q$ (cf. Fig. 8) and is sensitive to impact parameter as $u \to u_m$, for instance, at $\tilde{\alpha} = 0.5$ and $q = 0.2$, $\alpha_D(\theta) = 2\pi$ (first loop) for $u = 2.40566$, which deviates from $u_m$ by 0.42%. Thus, depending on the impact parameter, a light ray can make one, two or several loops around the black hole before reaching the observer resulting in addition to the primary and secondary images ($|\alpha_D(\theta)| < 2\pi$), two infinite sequences of relativistic images (cf. Fig. 5), one produced by clockwise winding of photon and other by counterclockwise winding of photon around the black hole. These images, respectively, are located on the same and opposite side of source.

IV. IMAGE POSITION, ANGULAR SEPARATION AND MAGNIFICATION

Assuming the observer and source are almost aligned along the optical axis and are placed in flat spacetime while the curvature affects the deflection angle near the
FIG. 8. The deflection angle is plotted against the impact parameter for a given value of \( q \) and different values of \( \tilde{\alpha} \) (left) and given value of \( \tilde{\alpha} \) for different values of \( q \) (right). The coloured bullet points on the \( x \)-axis correspond to the impact parameter at which deflection angle diverges.

FIG. 9. Formation of outer most relativistic Einstein ring for a given value of \( q \) and different values of \( \tilde{\alpha} \) (left) and given value of \( \tilde{\alpha} \) for different values of \( q \) (right). The outer dotted and dashed rings, respectively, correspond to the case when Sgr A* and M87* are considered as Schwarzschild black holes \( (\tilde{\alpha} \rightarrow 0 \text{ and } q = 0) \). In the limits \( \tilde{\alpha} \rightarrow 0 \text{ and } q = 0 \), our results, respectively, encompasses Bardeen black holes and 4D EGB black holes.

lens only. Further, we consider the source is located behind the lens and the lens equation reads [64]

\[
\beta = \theta - \frac{D_{LS}}{D_{OS}} \Delta \alpha_n, \tag{21}
\]

where \( \Delta \alpha_n = \alpha_D - 2n\pi \), is the extra deflection angle with \( 0 < \Delta \alpha_n < 1 \) and \( n \in \mathbb{N} \). Also, \( \theta \) and \( \beta \) are the angular separation of image and source from the optic axis whereas \( D_{LS} \) and \( D_{OS} \), respectively, are the distances of lens and observer from the source (cf. Fig. 5). Using the relation \( u = \frac{D_{OL}}{D_{OS}} \tan(\theta) \approx \theta D_{OL} \), Eq. (17) together with Eq. (21) yields the angular position of \( n \)-th relativistic image \( \theta_n \) in terms of lensing coefficients as [42]

\[
\theta_n = \theta_0^n + \frac{(D_{OL} + D_{LS})}{D_{LS}} \frac{u_m e_n}{D_{OL} \tilde{\alpha}} (\beta - \theta_0^n), \tag{22}
\]

where

\[
\theta_0^n = \frac{u_m}{D_{OL}} (1 + e_n), \quad e_n = \exp \left( \frac{-b}{2n\pi \tilde{\alpha}} \right). \tag{23}
\]

The quantity \( \theta_0^n \) corresponds to value of \( \theta \) when photon travels \( 2n\pi \) around the black hole and second term are the corrections to \( \theta_0^n \). In the limit \( n \to \infty \), we find that \( e_n \to 0 \) such that \( u_m = \theta_\infty D_{OL} \). Moreover, the case \( \beta = 0 \) corresponds to the perfect alignment and taking \( D_{OS} = 2D_{OL} \) with \( D_{OL} \gg u_m \), Eq. (22) reduces to

\[
\theta_n^E = \frac{u_m}{D_{OL}} (1 + e_n), \tag{24}
\]

which is the angular radius of \( n \)-th relativistic Einstein ring. Note that \( n = 1 \) corresponds to the outermost ring.
The size of the ring decreases with $\tilde{\alpha}$ and $q$ (cf. Fig. 9) and is smaller than the Bardeen as well as 4D EGB black holes. $\theta_1^B$ takes maximum value in the limit $\tilde{\alpha} \to 0$ and $q = 0$.

Another important observable is the magnification of the image which is ratio of the solid angle onto the observer subtended by the image to solid angle by the source i.e., $\mu = \sin \theta d\theta / \sin \beta d\beta$. Using Eq. (22), we deduce the magnification of $n$-loop images as \[\mu_n = \frac{1}{\beta} \left[ \frac{u_m}{D_{OL}} (1 + e_n) \left( \frac{D_{OS}}{D_{LS}} \frac{u_m e_n}{D_{OL} a} \right) \right]. \tag{25}\]

Clearly, in the limit $\beta \to 0$, $\mu_n \to \infty$ implying that the magnification of the images is maximum in case of the perfect alignment. The Eq. (25) relates the magnification and angular position of the source to the lensing coefficients. As can be seen the magnification decreases with $n$ resulting in higher order images to become less visible.

Finally, in order to obtain the lensing coefficients, we consider a case where $\theta_1$ can be separated as a single image and remaining images are packed together at $\theta_\infty$. Then, we can define three observable characteristic as \[\theta_\infty = \frac{u_m}{D_{OL}}, \tag{26}\]
\[s = \theta_1 - \theta_\infty \approx \theta_\infty \exp \left( \frac{\bar{b}}{\bar{a}} - \frac{2\pi}{\bar{a}} \right), \tag{27}\]
\[r_{\text{mag}} = \frac{\mu_1}{\sum_{n=2}^{\infty} \mu_n} = \exp \left( \frac{2\pi}{\bar{a}} \right). \tag{28}\]

Here $\theta_\infty$ is the angular position acquired by the set of images in the limit $n \to \infty$ or angular radius of photon sphere, $\theta_1$ is the angular position of the outermost image, $s$ is the angular separation between outermost image ($n = 1$) and innermost image ($n = \infty$) and $r_{\text{mag}}$ is the magnitude of ratio of flux between the outermost image and remaining images. Measuring the lensing observables $\theta_\infty$, $s$ and $r_{\text{mag}}$ from the observation, one can find the coefficients $\bar{a}$ and $\bar{b}$ in SDL [42]. Comparing the calculated values with those predicted by the theoretical models, we can get the information about the parameters of the lens (black hole).

\section{V. TIME DELAY IN SDL}

Further, we derive the time delay between different relativistic images following the method developed by Bozza and Manchini [65]. Time difference is caused by the fact...
that photon takes different paths while winding the black hole so there is a time delay between different images which generally depends upon which side of the lens are the images formed.

The time taken for a photon to travel from source to an observer at infinity is given by \[ \tilde{T}(u) = \tilde{\alpha} \log \left( \frac{u}{u_m} - 1 \right) + \tilde{b} + \mathcal{O}(u - u_m), \quad (29) \]

which can be used to calculate the time difference between two relativistic images. The images are highly demagnified and the separation between the images is of the order of \( \mu \text{as} \), so we must at least distinguish the outermost relativistic image from the rest. We assume the source to be variable, which generally are abundant in all galaxies, otherwise there is no time delay to measure, then for spherically symmetric black holes the time delay between the first and second relativistic image when the two images are on the same side of the source is given by

FIG. 11. The behavior of lensing observables \( \theta_{\infty} \) (upper panel) and \( s \) (lower panel) with \( q \) for different values of \( \tilde{\alpha} \) (left) and with \( \tilde{\alpha} \) for different values of \( q \) (right) for M87*. Our results in limits \( \tilde{\alpha} \to 0 \) encompasses those of Bardeen black holes, those of Schwarzschild black hole when \( \tilde{\alpha} \to 0 \), \( q = 0 \) and if only \( q = 0 \) we obtain the 4D EGB black holes.

FIG. 12. The behavior of lensing observable \( r_{\text{mag}} \) as function of \( q \) for different values of \( \tilde{\alpha} \) (left) and as a function of \( \tilde{\alpha} \) for different values of \( q \) (right). Our results in limits \( \tilde{\alpha} \to 0 \) encompasses those of Bardeen black holes, those of Schwarzschild black hole when \( \tilde{\alpha} \to 0 \), \( q = 0 \) and if only \( q = 0 \) we obtain the 4D EGB black holes.
and the same set of observables when the lens is

\[ \Delta T^2_{1,1} = 2\pi u_m = 2\pi D_{OL}\theta_\infty. \]  

(30)

We calculated the time delay for Sgr A*, M87*, and 19 other supermassive black holes. Our aim is to compare the time delays between the first and second relativistic image by 4D EGB Bardeen black holes with those of GR. Using the metric of 4D EGB and GR, we have tabulated the numerical results in Table II. For Sgr A* and M87*, the time delay can reach ~ 10.6004 min and ~ 1516.11 min at \( \bar{\alpha} = 0.5 \) and \( q = 0.2 \) and hence deviate from the Bardeen black holes (\( \bar{\alpha} \rightarrow 0, q = 0.3 \)) by ~ 0.1076 min and ~ 153.93 min, respectively. If the black is considered a 4D EGB black hole with \( \bar{\alpha} = 0.95 \), the deviation from 4D EGB Bardeen black hole (\( \bar{\alpha} = 0.5, q = 0.2 \)) can reach up to ~ 0.1328 min and ~ 197.724 min. These deviations are insignificant for Sgr A* but for M87* and some other black holes are sufficient values to test the 4D EGB gravity.

**TABLE I.** Estimates for lensing observables and lensing coefficients for the black hole Sgr A* and M87* for different values of \( \bar{\alpha} \) and \( q \). \( R_s = 2GM/c^2 \) is the Schwarzschild radius. Our results in limits \( \bar{\alpha} \rightarrow 0 \) encompasses those of Bardeen black holes, those of Schwarzschild black hole when \( \bar{\alpha} \rightarrow 0, \ q = 0 \) and if only \( q = 0 \) we obtain the 4D EGB black holes.

| \( \bar{\alpha} \) | \( q \) | \( \theta_\infty \) (\( \mu \)as) | Sgr A* | \( \theta_\infty \) (\( \mu \)as) | M87* | \( r_{mag} \) | \( \bar{\alpha} \rightarrow 0 \) | Lensing Coefficients |
|-----------------|-----------------|-----------------|-------|-----------------|-------|---------------|-----------------|-------------------|
| 0.00            | 0.00            | 26.3297         | 0.0329556 | 19.7819         | 0.0247601 | 6.82173       | 1.00002         | -0.400252         | 2.59806          |
| 0.00            | 0.10            | 26.1519         | 0.0351171 | 19.6483         | 0.026384  | 6.74409       | 1.01153         | -0.40608          | 2.58052          |
| 0.00            | 0.20            | 25.5894         | 0.0432767 | 19.2257         | 0.0325144 | 6.48576       | 1.05182         | -0.429893         | 2.52502          |
| 0.00            | 0.36            | 23.4991         | 0.107608  | 17.6553         | 0.0808477 | 5.27011       | 1.29445         | -0.688999         | 2.31876          |
| 0.00            | 0.001           | 26.1311         | 0.0371955 | 19.6327         | 0.0279455 | 6.66617       | 1.02336         | -0.424612         | 2.57846          |
| 0.15            | 0.15            | 25.7076         | 0.0436746 | 19.3145         | 0.0328134 | 6.46588       | 1.05506         | -0.445738         | 2.53668          |
| 0.20            | 0.25            | 24.8764         | 0.0612974 | 18.869          | 0.0460536 | 6.0308        | 1.13117         | -0.510569         | 2.45466          |
| 0.35            | 0.35            | 23.3186         | 0.12908   | 17.5196         | 0.0969797 | 4.96981       | 1.37266         | -0.84997          | 2.30094          |
| 0.00            | 0.00            | 25.2446         | 0.0645993 | 18.9667         | 0.0485345 | 5.9365        | 1.14914         | -0.575084         | 2.49099          |
| 0.10            | 0.10            | 25.0197         | 0.0822556 | 18.5756         | 0.0617999 | 5.60461       | 1.21719         | -0.661753         | 2.43964          |
| 0.20            | 0.20            | 24.2771         | 0.102748  | 18.2398         | 0.0771963 | 5.28221       | 1.29148         | -0.774778         | 2.39552          |
| 0.23            | 0.23            | 23.9154         | 0.124375  | 17.968          | 0.0934446 | 4.98769       | 1.36799         | -0.911054         | 2.35983          |
| 0.00            | 0.00            | 24.1275         | 0.133564  | 18.1273         | 0.100349  | 4.88771       | 1.39572         | -0.969719         | 2.38076          |
| 0.04            | 0.04            | 24.0798         | 0.137123  | 18.0915         | 0.103022  | 4.84519       | 1.40797         | -0.993561         | 2.37606          |
| 0.06            | 0.06            | 24.0194         | 0.141807  | 18.0461         | 0.106542  | 4.79013       | 1.42415         | -1.02578          | 2.37009          |
| 0.09            | 0.09            | 23.8795         | 0.153419  | 17.941          | 0.115266  | 4.65751       | 1.46471         | -1.11007          | 2.35629          |

[65]

\[ \begin{align*}
\Delta T^2_{1,1} &= 2\pi u_m = 2\pi D_{OL}\theta_\infty. 
\end{align*} \]

VI. LENSING BY SUPERMASSIVE BLACK HOLES

We here model the supermassive black holes Sgr A* in our galactic center, M87* in Messier 87 galaxy and several other supermassive black holes as the 4D EGB Bardeen black holes for numerical estimation of lensing observables. With the source distance \( D_{OS} = 2D_{OL} \), we estimate the angular position of innermost image \( \theta_\infty \), the angular separation of outermost image with the remaining bunch of relativistic images \( s \) and the relative magnification \( r_{mag} \) in order to get some information about the charge \( q \) and coupling constant \( \bar{\alpha} \) for 4D EGB Bardeen black holes. The behaviour of \( \theta_\infty \) and \( s \) with the parameters \( \bar{\alpha} \) and \( q \) for Sgr A* and M87* has been depicted in Fig. 10 and Fig. 11, respectively. These observables for Sgr A* and M87* are tabulated in Table I and the same set of observables when the lens is a Schwarzschild black hole, Bardeen black holes or 4D EGB black holes are also enlisted for comparison. Our results in limits \( \bar{\alpha} \rightarrow 0 \) encompasses those of Bardeen black holes [66] and of Schwarzschild black hole [38] when \( \bar{\alpha} \rightarrow 0, \ q = 0 \) but if only \( q = 0 \) we obtain the results of 4D EGB black holes [21]. It is evident from Table I that for small values of \( \bar{\alpha} \) and \( q \), the observational predictions of 4D EGB Bardeen black holes are indistinguishable from the Schwarzschild black hole, Bardeen black holes or 4D EGB black holes. However, the deviation becomes significant for large \( \bar{\alpha} \) and \( q \) as Table I indicates that \( \theta_\infty \) for 4D EGB Bardeen black holes is always smaller and hence can be potentially differentiated from
TABLE II. Estimation of Time Delay for SMBHs: Estimation of time delay for Supermassive black holes at the center of nearby galaxies considering them Schwarzschild, Bardeen black hole ($q = 0.3$) and 4D EGB Bardeen black hole with $\alpha = 0.5$ and $q = 0.2$. Mass ($M$) and distance ($D_{OL}$) are given in the units of solar mass and Mpc, respectively. Time Delays are expressed in minutes.

| Galaxy    | $M(M_\odot)$ | $D_{OL}$ (Mpc) | $M/D_{OL}$ | $\Delta T_{12}^{\alpha}$ Schwarzschild | $\Delta T_{12}^{\alpha}$ Bardeen | $\Delta T_{12}^{\alpha}$ 4D EGB Bardeen |
|-----------|---------------|----------------|------------|----------------------------------------|----------------------------------|--------------------------------------|
| Milky Way | $4.3 \times 10^6$ | 0.0083         | $2.471 \times 10^{-11}$ | 11.4967                                | 10.7081                          | 10.6004                              |
| M87       | $6.15 \times 10^9$ | 16.68          | $1.758 \times 10^{-11}$ | 16442.9                                | 15315.0                          | 15161.1                              |
| NGC 4472  | $2.54 \times 10^9$ | 16.72          | $7.246 \times 10^{-12}$ | 6791.06                                | 6325.24                          | 6261.66                              |
| NGC 1332  | $1.47 \times 10^9$ | 22.66          | $3.094 \times 10^{-12}$ | 3930.26                                | 3660.67                          | 3623.87                              |
| NGC 4374  | $9.25 \times 10^8$ | 18.31         | $2.383 \times 10^{-12}$ | 2473.12                                | 2303.48                          | 2280.33                              |
| NGC 1399  | $8.81 \times 10^8$ | 20.85          | $2.015 \times 10^{-12}$ | 2355.48                                | 2193.91                          | 2171.86                              |
| NGC 3779  | $4.16 \times 10^8$ | 10.70          | $1.854 \times 10^{-12}$ | 1112.24                                | 1035.94                          | 1025.53                              |
| NGC 4846B | $6 \times 10^8$   | 16.26          | $1.760 \times 10^{-12}$ | 1604.91                                | 1494.15                          | 1479.13                              |
| NGC 1374  | $5.90 \times 10^8$ | 19.57          | $1.438 \times 10^{-12}$ | 1577.45                                | 1469.25                          | 1454.48                              |
| NGC 4649  | $4.72 \times 10^8$ | 16.46          | $1.367 \times 10^{-12}$ | 12619.0                                | 1175.4                           | 1163.58                              |
| NGC 3608  | $4.65 \times 10^8$ | 22.75          | $9.750 \times 10^{-13}$ | 1243.25                                | 1157.97                          | 1146.33                              |
| NGC 3377  | $1.78 \times 10^8$ | 10.99          | $7.726 \times 10^{-13}$ | 475.909                                | 443.265                          | 438.809                               |
| NGC 4697  | $2.02 \times 10^8$ | 12.54          | $7.684 \times 10^{-13}$ | 540.077                                | 503.031                          | 497.975                               |
| NGC 5128  | $5.69 \times 10^7$ | 3.62           | $7.498 \times 10^{-13}$ | 152.131                                | 141.695                          | 140.271                               |
| NGC 1316  | $1.69 \times 10^8$ | 20.95          | $3.848 \times 10^{-13}$ | 451.816                                | 420.852                          | 416.622                               |
| NGC 3607  | $1.37 \times 10^8$ | 22.65          | $2.885 \times 10^{-13}$ | 366.265                                | 341.164                          | 337.735                               |
| NGC 4473  | $0.90 \times 10^8$ | 15.25          | $2.815 \times 10^{-13}$ | 240.628                                | 224.123                          | 221.87                                |
| NGC 4559  | $6.96 \times 10^7$ | 16.01          | $2.073 \times 10^{-13}$ | 186.086                                | 173.321                          | 171.579                               |
| M32       | $2.45 \times 10^6$ | 0.8057         | $1.450 \times 10^{-13}$ | 6.5504                                 | 6.10112                          | 6.03979                               |
| NGC 4846A | $1.44 \times 10^7$ | 18.36          | $3.741 \times 10^{-14}$ | 38.5005                                | 35.8596                          | 35.4992                               |
| NGC 4382  | $1.30 \times 10^7$ | 17.88          | $3.468 \times 10^{-14}$ | 34.7574                                | 32.3733                          | 32.0479                               |

FIG. 13. Left: The behavior of position of first primary $\theta_{1p}$ (solid black line) and secondary image $\theta_{1s}$ (dashed red line) with position of the source ($\beta$) for M87*. Right: The absolute magnification of first primary image is plotted against source position for $D_{OS} = 2D_{OL}$.

the GR counterparts provided we have a telescope with very strong resolving power. As an example, the deviation of $\theta_{1s}$ for Sgr A* if considered a 4D EGB Bardeen black hole ($\alpha = 0.5$, $q = 0.2$) from Bardeen black hole ($q = 0.3$) is $0.2464 \mu\text{as}$ whereas for M87*, the deviation is $0.18519 \mu\text{as}$. Similarly, the divation from 4D EGB black hole ($\alpha = 0.95$) can reach upto $0.316610 \mu\text{as}$ for Sgr A* and $0.237874 \mu\text{as}$ for M87*. In Table III, we give an estimation of $\theta_{1p}$ and $\theta_{2p}$, which are the angular positions of first and second order primary images for black holes in several nearby galaxies which are arranged according to the decreasing ratio of $M/D_{OL}$, by considering these black holes as Schwarzschild black hole, Bardeen black hole ($q = 0.3$) and 4D EGB Bardeen black hole with $\alpha = 0.5$ and $q = 0.2$. The deviation from GR are of the $O(\mu\text{as})$ and such a deviation in a realistic astrophysical environment is certainly not feasible in the near future, although if the images could be resolved, it would provide an excellent test of gravity in SDL. Also, as suggested from Fig. 13, $\theta_{1p} > |\theta_{1s}|$ for higher values of $\beta$. Further, the separation $s$ due to 4D EGB Bardeen black holes for Sgr A* and M87* range between 0.0329-0.153419 $\mu\text{as}$ and 0.0247-0.11526 $\mu\text{as}$, respectively. If these images could be resolved, it is possible
to calculate their magnification. The relative magnification of the first and second order images are estimated in Table IV for black holes in nearby galaxies by considering them 4D EGB Bardeen black hole with $\tilde{\alpha} = 0.5$ and $q = 0.2$ and compared with the Schwarzschild black hole and Bardeen black hole ($q = 0.3$). The first order images of 4D EGB Bardeen black holes are highly magnified than the second order images as well the corresponding images in GR. The ratio of the flux from the first image to all other images, however, decreases with $\tilde{\alpha}$ and $q$ (cf. Fig. 12).

| Galaxy          | Schwarzschild Black hole $\theta_{1p}$ | Schwarzschild Black hole $\theta_{2p}$ | Bardeen Black hole $\theta_{1p}$ | Bardeen Black hole $\theta_{2p}$ | 4D EGB Bardeen Black hole $\theta_{1p}$ | 4D EGB Bardeen Black hole $\theta_{2p}$ |
|-----------------|---------------------------------------|---------------------------------------|----------------------------------|----------------------------------|----------------------------------------|----------------------------------------|
| Milky Way       | 26.5214                               | 26.4883                               | 24.7385                          | 24.6716                          | 24.5207                                | 24.4242                                |
| M87             | 18.8749                               | 18.8514                               | 17.6061                          | 17.5585                          | 17.4553                                | 17.3823                                |
| NGC 4472        | 7.77686                               | 7.76715                               | 7.25406                          | 7.23444                          | 7.19195                                | 7.16187                                |
| NGC 1332        | 3.32906                               | 3.31682                               | 3.09771                          | 3.08933                          | 3.07119                                | 3.05835                                |
| NGC 4374        | 2.55824                               | 2.55505                               | 2.38627                          | 2.37981                          | 2.36584                                | 2.35594                                |
| NGC 1399        | 2.1631                                | 2.1604                                | 2.01769                          | 2.01223                          | 2.00041                                | 1.99904                                |
| NGC 3379        | 1.99029                               | 1.98781                               | 1.85649                          | 1.85147                          | 1.8406                                 | 1.8329                                 |
| NGC 4486B       | 1.88902                               | 1.88667                               | 1.76203                          | 1.75727                          | 1.74695                                | 1.73964                                |
| NGC 1374        | 1.54336                               | 1.54144                               | 1.43961                          | 1.43572                          | 1.42728                                | 1.42132                                |
| NGC 4649        | 14.6798                               | 14.6614                               | 13.6929                          | 13.6559                          | 13.5757                                | 13.5189                                |
| NGC 3608        | 1.04635                               | 1.04505                               | 0.97602                          | 0.973372                         | 0.967562                               | 0.963609                                |
| NGC 3377        | 0.829142                              | 0.828108                              | 0.773403                         | 0.771311                         | 0.766781                               | 0.763575                                |
| NGC 4697        | 0.824633                              | 0.823604                              | 0.769197                         | 0.767166                         | 0.762611                               | 0.759422                                |
| NGC 5128        | 0.804656                              | 0.803652                              | 0.750564                         | 0.748533                         | 0.744137                               | 0.741025                                |
| NGC 1316        | 0.412961                              | 0.412446                              | 0.3852                           | 0.384158                         | 0.381902                               | 0.380305                                |
| NGC 3607        | 0.309641                              | 0.309255                              | 0.288826                         | 0.288045                         | 0.286353                               | 0.285155                                |
| NGC 4473        | 0.30212                               | 0.301743                              | 0.28181                         | 0.281048                         | 0.279397                               | 0.278229                                |
| NGC 4459        | 0.222548                              | 0.222271                              | 0.207588                         | 0.207026                         | 0.20581                                | 0.20495                                 |
| M32             | 0.155668                              | 0.155474                              | 0.145203                         | 0.144811                         | 0.14396                                | 0.143358                                |
| NGC 4486A       | 0.040151                              | 0.0401009                             | 0.0374519                        | 0.0373506                        | 0.0371312                               | 0.0369759                                |
| NGC 4382        | 0.0372205                              | 0.0371741                             | 0.0347184                        | 0.0346245                        | 0.0344211                               | 0.0342772                                |

TABLE III. Image positions of SMBHs: Image positions of first and second relativistic image on the same side of the source for Supermassive black holes at the center of nearby galaxies considering them Schwarzschild black hole, Bardeen black hole ($q = 0.3$) and 4D EGB Bardeen black hole with $\tilde{\alpha} = 0.5$ and $q = 0.2$ at $\beta = 1$ arcsec.

VII. CONCLUSION

Lately, the researchers has devoted significant attention to several regularisations of EGB gravity to 4D. It turns out that the procedure of spherically symmetric 4D black hole solution [16] remains valid for these regularised theories [54, 59, 60]. It is a general impression that singularities do not exist in nature and are the artefact of general relativity. To resolve the issue, Bardeen [8] proposed one of the first regular black holes having horizons with no singularity. Recently, the 4D Bardeen-like black holes obtained as an exact solution of EGB with additional parameter $q$ because of minimal coupling with NED and the famous Galavau-Lin black holes contained as a particular case in the absence of NED ($q = 0$). Therefore, it is illuminating to probe the gravitational lensing to assess the reliance of observables, deflection angle and time delay on the parameter $q$ and compare the results with those for the GR counterparts Bardeen and Schwarzschild black holes. In this paper we studied the gravitational lensing in SDL by the 4D EGB Bardeen black holes and its dependence on Gauss-Bonnet coupling constant $\tilde{\alpha}$ and charge $q$. Beginning with the Euler-Lagrangian formalism, we show that the trajectory of the particle is indeed influenced by the coupling constant $\tilde{\alpha}$ and NED charge $q$. Interestingly, the 4D EGB Bardeen black holes make smaller photon spheres compared to the Bardeen black holes or 4D EGB black holes, and the unstable photon orbit radius is a decreasing function of $\tilde{\alpha}$ and $q$. Moreover, the dependence of critical impact parameters $u_m$ on $\tilde{\alpha}$ and $q$ has decreasing behaviours which are qualitatively the same as that of unstable photon orbit radius $r_m$ and its value is always smaller than the GR counterparts. The deflection angle $\alpha_\varphi(\theta)$, for fixed impact parameter $u$, is always higher for the Schwarzschild black hole. Further, the photon makes its first loop around the 4D EGB Bardeen black hole with $\tilde{\alpha} = 0.5$ and $q = 0.2$ at $u = 2.40566$ whereas the corresponding value for Schwarzschild black hole and Bardeen black hole ($q = 0.3$) is $u = 2.601309$ and 2.426437, respectively. The lensing coefficients $\tilde{a}$ increases, while as $b$ decreases with $\tilde{\alpha}$ and $q$.

The lensing observables i.e., angular position of innermost images $\theta_{1\infty}$, angular separation $s$ and $r_{mag}$ of the relativistic images for supermassive black holes, namely, Sgr A* and M87* are calculated by modelling them with the 4D EGB Bardeen black holes. The angular position $\theta_{1\infty}$ of the images in 4D EGB Bardeen black holes is more diminutive than Schwarzschild and Bardeen black holes.
The $\theta_\infty$ rapidly decreases, but the angular separation between the first and innermost image $s$ is higher when compared to the Schwarzschild black hole or Bardeen black holes. The $\theta_\infty$ ranges between 23.8795-26.3297 $\mu$as for Sgr A* and its maximum deviation from GR counterparts can reach up to 2.45016 $\mu$as. For M87*, it ranges between 17.941 - 19.7819 $\mu$as and deviation is as much as 1.84084$\mu$as. The separation $s$, an increasing function of $\alpha$ and $q$, due to 4D EGB Bardeen black holes for Sgr A* and M87* range between 0.0329-0.15341 $\mu$as and 0.0247-0.1152 $\mu$as, respectively. Although, the ratio of the flux from the first image to all other images, decreases with $\alpha$ and $q$ but the first order images of 4D EGB Bardeen black holes are highly magnified than the second order images as well the corresponding images in GR. Moreover, the time delay between the first and second-order images for black holes in 4D EGB Bardeen black holes, except Sgr A*, is significant enough for astronomical measurements, provided we have enough angular resolution separating two relativistic images.

The results presented here generalise previous discussions on the black holes lensing in GR viz. Schwarzschild, Bardeen and 4D EGB gravity black holes contained, respectively, in the limits, $\alpha, q \rightarrow 0$, $\alpha \rightarrow 0$, and $q \rightarrow 0$. Although it is tough to resolve the order estimated in SDL, the outlook of future observations looks bright. The Event Horizon Telescope observation of M87* has achieved angular resolution of 20 $\mu$as. Thus, it is essential to use GR and other alternate theories of gravity to give a realistic view of the observed images.

Finally, due to the complicated higher-order curvatures metric (8), in the present analysis, we have restricted to the spherically symmetric case, i.e., overlooked the spins. However, we can reasonably expect our results on lensing by supermassive black holes Sgr A* and M87* are valid, and the EHT observation can also be adopted to test these spherical black holes suitably. Meanwhile, some detailed investigation for the rotating counterpart will be a promising avenue for the future.

| Galaxy     | Schwarzschild Black hole | Bardeen Black hole | 4D EGB Bardeen Black hole |
|------------|--------------------------|--------------------|---------------------------|
| Milky Way  | $1.5936 \times 10^{-21}$  | $5.935189 \times 10^{-20}$ | $1.46162 \times 10^{-19}$ |
| M87        | $2.009908 \times 10^{-20}$ | $2.45016 \mu$as     | $2.275083 \times 10^{-22}$ |
| NGC 4472   | $1.818335 \times 10^{-20}$ | $1.84084 \mu$as     | $1.428561 \times 10^{-22}$ |
| NGC 3379   | $2.759071 \times 10^{-23}$ | $1.84084 \mu$as     | $1.428561 \times 10^{-22}$ |
| NGC 4486B  | $3.744962 \times 10^{-20}$ | $1.84084 \mu$as     | $1.428561 \times 10^{-22}$ |
| NGC 4382   | $3.744962 \times 10^{-20}$ | $1.84084 \mu$as     | $1.428561 \times 10^{-22}$ |

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