Improved direct fitting algorithm for elliptic parameters

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Abstract. For the ellipse fitting problem often encounter in visual measurement, this paper proposes a method combining algebraic distance fitting method based on least squares and characteristic root method for ellipse fitting. Firstly, the improved hierarchical agglomerative clustering is used to denoise the data. Then the algebraic distance fitting method based on least squares is used to calculate the initial iteration value of the parameters in the characteristic root method. Finally, the Gauss-Newton iteration method is used to solve the elliptic parameters. This algorithm is used to detect the out-of-roundness of the optical fiber and compare the test results with the high-precision optical fiber tester FGM-502. The results prove the validity and accuracy of the ellipse fitting algorithm studied in this paper and show that the algorithm studied in this paper meets the requirements of actual ellipse fitting measurement.

1. Introduction

There are a large number of elliptical objects in real life. In many fields, machine vision is required to quickly identify elliptical contours in images and fit the identified contours [1-2]. The methods for obtaining elliptic parameters generally fall into two categories. That is, the ellipse parameters is obtained indirectly, and the ellipse parameters is directly obtained. The specific fitting algorithm has an algebraic distance fitting method based on least squares [3-4], geometric distance fitting method based on least squares [5], Hough transform method [6-7] and Kalman filter method [8]. Algebraic distance fitting based on least squares and Kalman filtering are susceptible to noise points and isolated points and can lead to inaccurate results. The geometric distance fitting method based on least squares is complicated in solving the process and is not easy to be used in actual ellipse fitting. Hough transform method increases the time and space complexity of the algorithm when solving more parameters. Considering the actual visual measurement problem, this paper uses the first definition of ellipse to improve the hierarchical agglomerative clustering to remove noise points and isolated points. The effective data points are then elliptically fitted using the least squares method and the characteristic root method.

2. Remove noise and outliers from the data

In order to improve the accuracy of ellipse fitting and reduce the complexity of ellipse fitting, a method of improved hierarchical agglomerative clustering is proposed to achieve data denoising.
2.1. Hierarchical agglomerative clustering
Hierarchical agglomerative clustering is a bottom-up clustering algorithm. The idea is to treat each sample as a cluster and merge the two adjacent clusters according to specific distance metrics until all clusters cannot be merged [9]. Choosing merge points according to inter-cluster distance is the key to hierarchical agglomerative clustering. There are generally four methods for inter-cluster distance metrics:

- Minimum distance method: the minimum distance between two clustered data points is used to represent the distance between two clusters. As in equation (1).
  \[
  D_{\text{min}}(C_i, C_j) = \min_{p \in C_i, p' \in C_j} |p - p'| \tag{1}
  \]

- Maximum distance method: the maximum value between two cluster data points indicates the distance between two clusters. As in equation (2).
  \[
  D_{\text{max}}(C_i, C_j) = \max_{p \in C_i, p' \in C_j} |p - p'| \tag{2}
  \]

- Mean distance method: the distance between two clusters is represented by the distance between the two cluster mean values. As in Equation (3)
  \[
  D_{\text{mean}}(C_i, C_j) = |M_i - M_j| \tag{3}
  \]

- Average distance method: the average of the distances between all data points in two clusters represents the distance between the two clusters. As in equation (4).
  \[
  D_{\text{avg}}(C_i, C_j) = \frac{1}{n_i n_j} \sum_{p \in C_i} \sum_{p' \in C_j} |p - p'| \tag{4}
  \]

In the above equation (1)-(4): \( |p - p'| \) is the distance between the data points \( p \) and \( p' \) from two different clusters. \( M_i \) is the average of the class \( C_i \) and \( M_j \) is the average of the class \( C_j \). \( n_i, n_j \) are the number of data points in the classes \( C_i \) and \( C_j \), respectively. Hierarchical agglomerative clustering does not need to set the initial cluster center and cluster number in advance, and the distance and rule definition are simple. However, the above distance method cannot meet various clustering requirements. The minimum distance method is easy to cause class dispersion. The maximum distance method and the mean distance method are easy to form a close class of the same species. The cumbersome operation of the average distance method may lead to an inverse increase in clustering. Therefore, choosing the appropriate clustering conditions according to the actual situation is the key to hierarchical agglomerative clustering.

2.2. Improved hierarchical agglomerative clustering
In order to achieve the purpose of removing noise and obtaining valid data points, it is necessary to improve the similarity measure of the hierarchical agglomerative clustering. Similarity is the criterion for data clustering. In order to reduce the complexity of ellipse fitting, the elliptic definition method is proposed as the similarity measure.

2.2.1. Similarity distance metric
The trajectory of the moving point whose sum of the distances between the two fixed points \( F_1 \) and \( F_2 \) in the plane is equal to the constant is called an ellipse. According to the definition of the ellipse, the similarity between the two data points \( p \) and \( p' \) is defined as in equation (5).
  \[
  d(p, p') = |((p - F_1)^2 + (p - F_1)^2)^{1/2} - ((p' - F_2)^2 + (p' - F_2)^2)^{1/2}| \tag{5}
  \]

2.2.2. Steps to use the improved hierarchical agglomerative clustering
In order to be able to filter out valid data points, the threshold \( \delta \) is set to inter-class fusion criterion. The specific implementation steps of the algorithm are as follows:

- Set fixed points \( F_1, F_2 \) and threshold \( \delta \).
- Arbitrarily select two object points \( p, p' \) in the data and calculate the similarity \( d(p, p') \) Determine whether the threshold condition is met: \( d(p, p') < \delta \). If the condition is true, merge the two objects into one cluster \( C \).
• Calculate the center point \( q \) of the cluster and reselect the object point \( q' \). Calculate the similarity \( d(p, p') \) between the center point and the object point.

• Determine whether the threshold condition is met: \( d(q, q') < \delta \). If the condition is true, merge the two objects into one cluster \( C \).

• Repeat steps until all data points have been traversed.

The improved hierarchical agglomerative clustering method is used to denoise the raw data shown in figure 1. The denoising result is shown in figure 2. It can be seen from figure 2 that the improved hierarchical agglomerative clustering removes the noise points and isolated points of the original data and reduces the complexity of the ellipse fitting.

3. Ellipse fitting

3.1. Least squares algebraic fitting

In a two-dimensional plane coordinate system, an ellipse can generally be represented by two forms. One of this form is expressed in algebraic form using a conic curve equation, as in equation (6).

\[
Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0
\]

Another more intuitive method of ellipse is represented by the geometric parameters of the planar coordinate system. The elliptical geometric parameters mainly have a central coordinate \( X_{o} = (x_{o}, y_{o}) \), a long axis \( a \), a short axis \( b \), and a long axis rotation angle \( \theta \). Two forms of parameters can be converted from each other using equations (7)-(11).

\[
x_{c} = \frac{BE-2CD}{4AC-B^2}
\]

\[
y_{c} = \frac{AC-2BE}{4AC-B^2}
\]

\[
a^2 = \frac{A+C-(B^2+4AC)^{1/2}}{-2F}
\]

\[
b^2 = \frac{A+C+(B^2+4AC)^{1/2}}{-2F}
\]

\[
\theta = \frac{1}{2} \tan^{-1} \left( \frac{B}{A-C} \right)
\]

\( X_{i} = (x_{i}, y_{i}) \) \( (i = 1, 2, \ldots, n) \) represents the coordinates of \( n \) data points. The quadratic curve equation can be expressed as in equation (12).

\[
F(D, a) = Da = ax^2 + bxy + cy^2 + dx + ey + f = 0
\]

\( D = (x^2, xy, y^2, x, y, 1) \) is a coefficient matrix. \( a^T = [a, b, c, d, e, f] \) is the parameter to be tested. Minimize the algebraic distance, we get equation (13). The minimum value of equation (13) is solved under the constraint \( b^2 - 4ac < 0 \). The constraint can be written as equation (14).

\[
Q(\delta) = \sum_{i=1}^{n} V_i^2 = \sum_{i=1}^{n} a^T D^T D a = a^T S
\]
\[\mathbf{a}^T \mathbf{B} \mathbf{a} = -1 \quad (14)\]

Where \( S = \sum_{i=1}^{n} D_i^T D_i \), \( \mathbf{B} \) is a \( 6 \times 6 \) matrix, \( B_{1,3} = B_{3,1} = -2, B_{2,2} = -2 \).

Use the Lagrange multiplier method and calculate the Lagrange equation as in equation (15) by the equation (13) and equation (14).

\[ L(a) = \mathbf{a}^T S \mathbf{a} - \lambda (\mathbf{a}^T \mathbf{B} \mathbf{a} + 1) \quad (15)\]

Finally, the ellipse parameters obtained according to the equation (7)-(11) are not accurate, and the ellipse parameters need to be directly solved. The characteristic root method can directly fit the ellipse parameters, but the output is affected by the initial iteration value. Therefore, the least squares algebraic fitting method is used to obtain the initial iteration value of the characteristic root method.

3.2. Characteristic root method

The ellipse can be expressed as equation (16) in the elliptical standard coordinate system.

\[ f((x, y), \beta) = X_s \Lambda^2 X_s^T = 1 \quad (16)\]

In Equation (16), the diagonal matrix \( \Lambda = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \), the ellipse coordinate \( X_s = (x_s, y_s) \). The measurement coordinate system is rotated and translated into an elliptical standard coordinate system. The relationship between them is as in equation (17).

\[ X_s = R(\theta)(X - X_c) \quad (17)\]

Rotation matrix: \( R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \). Equation (17) is substituted into equation (16) and the error equation is listed for each pair of measurement points. As in equation (18).

\[ V_i = X_{si} \Lambda^2 X_{si}^T - 1 \quad (18)\]

The error equation corresponding to equation (18) is nonlinear and cannot be solved by the traditional least square method. The Gauss-Newton iteration method [10] is used to solve the least squares nonlinear fitting problem. The basic idea of the Gauss-Newton iteration method is to replace the nonlinear regression model with Taylor series expansion. After repeated iterations, the regression coefficients are continuously corrected, so that the regression coefficient is close to the optimal regression coefficient of the nonlinear regression model. It is to minimize the sum of the squares of the residuals of the original model. The initial value of the Gauss-Newton iteration method is selected by the least squares algebraic fitting method to fit the parameters. The iteration equation is equation (19).

Where \( J_F \) is the Jacobian matrix of the function \( f((x, y), \beta) \) on \( \beta \). As in equation (20).

\[ \beta^{(k+1)} = \beta^k + (J_F^T J_F)^{-1} J_F^T V(\beta^k) \quad (19)\]

\[ J_F = \frac{\partial f_i((x_i, y_i), \beta^k)}{\partial \beta} = \begin{bmatrix} \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial y_0} & \frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial y_0} & \ldots & \frac{\partial f_n}{\partial x_0} & \frac{\partial f_n}{\partial y_0} \\ \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \ldots & \frac{\partial f_n}{\partial a} & \frac{\partial f_n}{\partial b} \\ \frac{\partial f_1}{\partial \theta} & \frac{\partial f_2}{\partial \theta} & \ldots & \frac{\partial f_n}{\partial \theta} \end{bmatrix} \quad (20)\]

Characteristic root method is used to perform ellipse fitting on the data points corresponding to figure 2. The fitting results are consistent with the data distribution law, as shown in figure 3.
4. Out-of-roundness detection of optical

In order to better study the effectiveness of the proposed algorithm, the algorithm is applied to the experiment of optical fiber out-of-roundness detection. The optical fiber generally consists of a core, a cladding, and a coating layer. The periphery of the core is a cladding that acts to confine the light wave signal within the core. The outer layer of the cladding is a coating layer that serves to protect the fiber. Figure 4 shows the fiber image captured by the camera. The out-of-roundness formula of the fiber is as in equation (21).

\[ \gamma = 2 \times \frac{a - b}{a + b} \times 100\% \]  

(21)

Three types of optical fibers are measured using the proposed algorithm. Each type of optical fiber contains 10 samples. The algorithm of this paper is used to calculate the average out-of-roundness \( \alpha \) of the cladding, the average out-of-roundness \( \beta \) of the primary coating layer and the average roundness \( \gamma \) of the secondary coating layer. The measurement results are compared with the results of the high-precision FGM-502 fiber tester. Table 1 and table 2 show the test results of the FGM-502 fiber tester and the test results of the algorithm.

| Fiber number | \( \alpha \)(%) | \( \beta \)(%) | \( \gamma \)(%) |
|--------------|----------------|----------------|----------------|
| 1            | 0.095          | 0.313          | 0.173          |
| 2            | 0.160          | 0.142          | 0.314          |
| 3            | 0.148          | 0.130          | 0.727          |

Table 1. FGM-502 fiber tester measurement results

| Fiber number | \( \alpha \)(%) | \( \beta \)(%) | \( \gamma \)(%) |
|--------------|----------------|----------------|----------------|
| 1            | 0.111          | 0.317          | 0.195          |
| 2            | 0.157          | 0.163          | 0.354          |
| 3            | 0.142          | 0.151          | 0.755          |

Table 2. detection results of the algorithm in this paper
According to the measurement data of table 1 and table 2, it can be seen that the detection of the out-of-roundness of the primary coating of the optical fiber, the out-of-roundness of the secondary coating layer and the out-of-roundness of the cladding are close to the test results of the tester. The calculated cladding detection accuracy is about 6.8%, the detection accuracy of the primary coating layer is about 9.3% and the detection accuracy of the secondary coating layer is 8.7%. The experimental results show the feasibility and accuracy of the proposed algorithm.

5. Conclusion
In this paper, the clustering algorithm and curve fitting algorithm are studied. The traditional hierarchical agglomerative clustering cannot achieve effective data screening. An improved hierarchical agglomerative clustering is proposed, and the elliptic parameters are solved by the ellipse fitting algorithm. The algorithm is applied to the detection of fiber out-of-roundness. The experimental results verify the effectiveness and accuracy of the proposed algorithm and achieve the actual ellipse fitting requirements.

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