The Kramers problem for SDEs driven by small, accelerated Lévy noise with exponentially light jumps

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Abstract

We establish Freidlin-Wentzell results for a nonlinear ordinary differential equation starting close to a stable state subject to a perturbation by a stochastic integral driven by an $\varepsilon$-small and $(1/\varepsilon)$-accelerated Lévy process with exponentially light jumps. For this purpose we first generalize the large deviations results by Budhiraja, Dupuis and collaborators (2011, 2013) to the case of multiplicative coefficients, unbounded jump size and infinite jump intensity. In the sequel we solve the Kramers problem, that is, the associated asymptotic first escape problem from the bounded neighborhood of a deterministic exponentially stable state in the limit for small $\varepsilon$. The result covers also linear systems driven by Lévy processes with strongly tempered jump measures, such as the multiplicative Gamma process.

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1 Introduction and main results

In this article we solve the Kramers problem for strong solutions $(X_\varepsilon)_{\varepsilon \in (0,1]}$ of the stochastic differential equation

\[ X_t^{\varepsilon,x} = x + \int_0^t b(X_s^{\varepsilon,x}) \, ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} G(X_s^{\varepsilon,x}, z) \varepsilon \tilde{N}^\varepsilon(ds, dz), \quad t \geq 0 \]  

from a domain $D$ of a stable state of the strictly monotone vector field $b$ where $\tilde{N}^\varepsilon$ is a compensated Poisson random measure with intensity $\frac{1}{\varepsilon} ds \nu(dz)$, for a Lévy measure $\nu$ of infinite intensity and with some exponential moments. The heart of our analysis is the establishment of large deviations results based on the recent weak convergence approach results by Maroulas, Budhiraja, Dupuis and collaborators [13, 14].

Historically, the Kramers problem, that is, the escape time and location of a randomly excited deterministic dynamical system from close to a stable state at small intensity arose in the context of chemical reaction kinetics [3], [24] and [40]. Nowadays this classical problem is virtually ubiquitous and has given since crucial insight in many diverse areas ranging from statistical mechanics, statistics, insurance mathematics, population dynamics, fluid dynamics to neurology. The mathematical theory of large deviations goes back to the seminal work by Crâmer [18] before taking off in the seventies with the fundamental works by [26, 27, 54, 56]. One main focus was the first exit problem for ordinary, delay and partial differential equations with small Gaussian noise in different settings and derived effects from it such as metastability and stochastic resonance. Classical texts with splendid expositions of the history of large deviations theory include [4, 5, 6, 9, 10, 11, 16, 20, 21, 22, 28, 53] among others.

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as well as the references therein. Furthermore there is a lot of active research in the field, see for instance [17, 29, 43, 44, 55]. The major part of the literature body studying large deviations and the Kramers law for stochastic differential equations with small noise are centered in the study of Gaussian dynamics.

For the dynamics of Markovian systems with jumps the literature is noticeably more fragmented, scattered and recent. It is due to the great variety of Lévy processes, including processes with heavy tails, and the resulting lack of moments, that there is no general large deviations theory for Lévy processes and diffusions with jumps. Large deviations results for certain classes of Lévy noise and Poisson random measures are given in [7, 12, 25, 30, 42, 45, 48, and 57]. The first exit problem for small jump Lévy processes starts with the seminal paper by [31] for α-stable processes and for more generally heavy-tailed processes by [19, 32, 34, 35, 47] without a large deviations principle though, but with the help of a parameter dependent flow decomposition. The first exit times in this setting changes from growing exponentially in the noise intensity to a polynomial growth, with the exponent representing the polynomial tail decay. In [39, 37] the authors show in one dimension a complete scale of asymptotic exit times as a function of $1/\varepsilon$ from polynomial via subexponential to exponential.

This article follows a rather new strain of research including not only an $\varepsilon$-dependent amplitude but also an $\varepsilon$-dependent acceleration of the jump intensity of the noise as can be seen by the term $\varepsilon\tilde{N}_1^\varepsilon$ in (1). Large deviations results for this type of noise go back to the fundamental papers on the weak convergence approach for such processes with the help of variational representation formulas for functionals of continuous time processes by Maroulas, Budhiraja, Dupuis and collaborators [13, 14] with a lot of ongoing research [15, 58, 59]. In [13] the reader finds an extensive and up-to-date introduction to this subject such that we will refrain from this at this point.

In this work we extend the large deviations and first exit results results in Section 4.1, p. 736, of [13], from uniformly bounded jumps to the analogue of the classical Freidlin-Wentzell case with unbounded jumps with exponential moments, infinite intensity and multiplicative diffusion coefficients. Instead of imposing the abstract sufficient Conditions 3.1. of [14], we state our results in terms of the coefficients of the SDE for a generic class of infinite intensity Lévy measures. For this purpose we verify Condition 4.1 in [13] to derive a LDP in the Skorokhod space. Our integrability conditions seem to be near optimal, since for subexponential tail integrability of the Lévy measure, that is of order $e|z|^{\alpha}$, $\alpha \in (0,1)$, the first author obtained a moderate deviations principle in [46]. The LDP in [13] is given as an optimization procedure of continuous controlled paths in a pseudo-potential framework analogously to the Freidlin-Wentzell theory, however for controls given in terms of a temporal and spatially local intensity density of the Poisson random measure. In other words, on an abstract level the physical intuition remains intact, however, since the control is given as a density w.r.t. the Lévy measure in the case of infinite intensity it is almost impossible to calculate the energy minimizing paths. We verify Condition 4.1 of [14] by several technical lemmas with the help of a particular Bernstein type inequality given in [23]. From this we infer Freidlin-Wentzell type results, in particular, we solve the associated first exit problem which to our knowledge is missing in the literature to date. For this purpose and due to the lack of continuity we construct for the lower bound of the first exit times a modified Markov chain approximation as given for instance in [21, 27] taking into account the topological particularities of the Skorokhod space.

The article is organized as follows. We start with the generic setting, we explain the specific hypothesis for the large deviations principle and the Kramers problem followed by an illustration of different classes of Lévy measures, which fall in our setting. Section 2 is dedicated to the establishment of the LDP on a finite time interval with several technical lemmas evacuated to the appendix. Section 3 deals with the upper and the respective lower bound of the LDP.

### 1.1 Object of study

The **deterministic dynamics:** Consider the following $C^1$ vector field $b : \mathbb{R}^d \to \mathbb{R}^d$, $x \in \mathbb{R}^d$ and the deterministic dynamical system given as the solution flow $t \mapsto X_t^{0,x}$ of the ordinary differential equation

$$\frac{dX_t^{0,x}}{dt} = b(X_t^{0,x}), \quad t \geq 0, \quad \text{and} \quad X_0^{0,x} = x, \quad (2)$$
subject to the following assumptions.

**Hypothesis A.** The vector field $b$ satisfies the following.

A.1: There is a constant $c_1 > 0$ such that
\[
(b(y_1) - b(y_2), y_1 - y_2) \leq -c_1|y_1 - y_2|^2, \quad \text{for all } y_1, y_2 \in \mathbb{R}^d.
\]  
(3)

A.2: The point $0 \in \mathbb{R}^d$ is critical in that $b(0) = 0$.

**Remark 1.**
1. It is well-known that under Hypothesis A every initial point $x \in \mathbb{R}^d$ there is a unique solution $t \mapsto X_{t}^{0,x}$ of (2) for all $t \geq 0$.

2. Hypothesis (A.1) implies that $Db(x)$ is strictly negative definite for all $x \in D$, that reads in the case of a gradient system $b = -\nabla U$ for some potential $U : \mathbb{R}^d \to [0, \infty]$ the implication of uniform convexity. As a consequence, $0$ is a hyperbolic stable fixed point of the dynamical system (2).

In the sequel we define formally the stochastic perturbation $\varepsilon \dot{L}^\varepsilon$ of (2), where we follow by and large the notation developed in [13] and [14].

**The noise perturbation $\varepsilon \dot{L}^\varepsilon$:**
Let $\mathcal{M}$ be the space of the locally finite measures defined on the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R}^d\setminus\{0\})$.

We fix a non-atomic measure $\nu \in \mathcal{M}$, that is, $\nu(\{z\}) = 0$ for all $z \in \mathbb{R}^d$ and $\nu(K) < \infty$ for every compact set $K \subset \mathbb{R}^d$ with $0 \notin K$. Theorem I.9.1 in [33] then shows that the measurable space $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ can be equipped with a unique non-atomic probability measure $\tilde{\mathbb{P}}$ such that the canonical map $N : \mathcal{M} \to \mathcal{M}$, $N(m) := m$ defines a Poisson random measure with intensity measure $ds \otimes \nu$ on $[0, \infty) \times \mathbb{R}^d \setminus \{0\}$, where $ds$ denotes the Lebesgue measure on the interval $[0, \infty)$. We also refer the reader to Proposition 19.4 in [50]. The compensated Poisson random measure of $N$ is defined by $\tilde{N}([0, s] \times A) := N([0, s] \times A) - s\nu(A)$, for all $s \geq 0$ and $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ such that $\nu(A) < \infty$. The expectation under $\tilde{\mathbb{P}}$ is denoted by $\tilde{E}$. For all $\varepsilon \in (0, 1]$ we denote by $N^\varepsilon$ the Poisson random measure defined on the probability space $(\mathcal{M}, \mathcal{B}(\mathcal{M}), \tilde{\mathbb{P}})$ with intensity measure $ds \otimes \frac{1}{\varepsilon} \nu$ and its compensated counterpart $\tilde{N}^\varepsilon$. In particular, we have $N = N^1$ and $\tilde{N} = \tilde{N}^1$.

Consider the space $[0, \infty) \times \mathbb{R}^d \setminus \{0\} \times [0, \infty)$ and denote by $\mathcal{M}$ the space of the locally finite measures defined on the Borel $\sigma$-algebra $\mathcal{B}([0, \infty) \times \mathbb{R}^d \setminus \{0\} \times [0, \infty))$. Analogously there is a unique probability measure $\bar{\mathbb{P}}$ defined on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ such that the canonical map $\bar{N} : \mathcal{M} \to \mathcal{M}$, $\bar{N}(\bar{m}) := \bar{m}$, is a Poisson random measure on the probability space $(\mathcal{M}, \mathcal{B}(\mathcal{M}), \bar{\mathbb{P}})$ with intensity measure $ds \otimes \nu \otimes dr$, where $dr$ denotes the Lebesgue measure on the interval $[0, \infty)$. We write $\bar{E}$ for the $\bar{\mathbb{P}}$ expectation.

**Remark 2.** The first component takes into account the time variable $t$, the second one is in the space of the jump increments $z$ of the underlying Lévy process associated to the Poisson random measure and the third component registers the frequency $r$ of the jump $z$.

For any $\varepsilon \in (0, 1]$ the Poisson random measure $N^\varepsilon$ has the following representation as a controlled random measure with respect to $\tilde{N}$ under $\tilde{\mathbb{P}}$. We have $\tilde{\mathbb{P}}$-almost surely for every $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ the identity
\[
N^\varepsilon([0, t] \times A) = \int_{0}^{t} \int_{A} \int_{0}^{\infty} 1_{[0, \frac{1}{\varepsilon}]}(r) \tilde{N}(ds, dz, dr).
\]  
(4)

For details we refer the reader to [13].

**Hypothesis B.** The measure $\nu \in \mathcal{M}$ is non-atomic and satisfies the following conditions.

B.1: $\nu \in \mathcal{M}$ is a Lévy measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ in that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \nu(dz) < \infty$.

B.2: There exists $\alpha > 0$ such that
\[
\int_{|z| > 1} e^{\alpha |z|} \nu(dz) < \infty.
\]  
(5)
Remark 3. Equation (6) determines the tail behavior of $\nu$
\[
\int_{\|z\|>1} e^{\alpha \|z\|} \nu(dz) = \int_{\|z\|>1} \int_0^\infty 1\{s \leq \|z\|\} \alpha e^{\alpha s} ds \nu(dz) = \int_0^\infty \nu((1 \land s)B_1^\alpha(0)) \alpha e^{\alpha s} ds = C_1 + \int_1^\infty \nu(sB_1^\alpha(0)) \alpha e^{\alpha s} ds,
\]
for $C_1 = \nu(B_1^\alpha(0))(e^\alpha - 1) < \infty$. Hence (4) is equivalent to the integral on the right-hand side of (6) and for any continuous function $f : [0, \infty) \to [0, \infty)$ such that $\lim_{t \to \infty} f(t)/e^{\alpha t} = 0$ we have
\[
\int_{\|z\|>1} f(\|z\|) \nu(dz) < \infty.
\]
In particular, it follows
\[
\int_{\mathbb{R}^d} |z|^r \nu(dz) < \infty \quad \text{for all } r \geq 2,
\]
given as condition (3.2) in Theorem 3.1 of [11].

The multiplicative noise coefficient $G$: The multiplicative noise coefficient $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following.

Hypothesis C. Let $r \geq 2$. There exists $L > 0$ such that for all $x, y, z \in \mathbb{R}^d$ we have
\[
|G(x, z) - G(y, z)| \leq L \|z\| |x - y| \quad \text{and} \quad |G(x, z)| \leq L \|z\| (1 + |x|).
\]

The stochastic differential equation: The main object of study is the effect of a stochastic perturbation $G(X, \varepsilon L^\varepsilon)$ on (2) under Hypotheses A - C in the following sense. For every $\varepsilon \in (0, 1]$ and $x \in \mathbb{R}^d$ we consider the following stochastic differential equation
\[
X^{\varepsilon, x}_t = x + \int_0^t b(X^{\varepsilon, x}_{s-}) ds + \int_0^t \int_{\mathbb{R}^d} G(X^{\varepsilon, x}_{s-}, z) \varepsilon \tilde{N}^\varepsilon(ds, dz), \quad t \geq 0.
\]
(7)

We denote the completed, natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of $\tilde{N}$ on $\tilde{\mathcal{M}}$ given by
\[
\mathcal{F}_t = \sigma\big(\{N^1([0, s] \times A \times C) \mid 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), C \in \mathcal{B}(\mathbb{R}, \mathbb{R})\}\big) \vee \mathcal{N}, \quad t \geq 0,
\]
where $\mathcal{N}$ is the collection of the $\mathbb{P}$-null sets in $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \times (0, \infty))$.

Let $\mathbb{D}([0, T], \mathbb{R}^d)$ be the vector space of càdlàg functions over the interval $[0, T]$, $T > 0$, with values in $\mathbb{R}^d$. It is well-known in the literature that the space $\mathbb{D}([0, T], \mathbb{R}^d)$ equipped with the topology generated by the $J_1$-metric $d_{J_1}$, known as the Skorokhod space is a Polish space (see for instance Theorem 12.1 and Theorem 12.2 in [8]). For the following theorem we cite Theorem III.2.3.2 of [8].

Theorem 1. Given $\varepsilon, T > 0$, $x \in \mathbb{R}^d$ and $\nu \in \mathcal{M}$ let Hypotheses A - C be satisfied. Then there is a unique strong solution $(X^{\varepsilon, x}_t)_{t \in [0, T]}$ of (7), that is a $(\mathcal{F}_t)_{t \in [0, T]}$-adapted stochastic process satisfying $\mathbb{P}$-a.s. (7) which takes $\mathbb{P}$-almost surely values in $\mathbb{D}([0, T], \mathbb{R}^d)$. In addition, the process $X^{\varepsilon, x} = (X^{\varepsilon, x}_t)_{t \in [0, T]}$ is a strong Markov process with respect to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. In particular, there is a measurable map $\mathcal{G}^{\varepsilon, x} : \mathcal{M} \to \mathbb{D}([0, T], \mathbb{R}^d)$ such that $X^{\varepsilon, x} := \mathcal{G}^{\varepsilon, x}(\varepsilon \tilde{N}^\varepsilon)$.

1.2 Specific hypotheses and statement of the main results

Let the standing assumptions of Subsection 1.1 in particular Hypotheses A - C be satisfied.

Hypothesis D. We consider a bounded domain $D \subset \mathbb{R}^d$ with $0 \in D$, $\partial D \in C^1$, and $b$ inward pointing on $\partial D$, that is
\[
(b(z), n(z)) < 0, \quad \text{for all } z \in \partial D,
\]
where the vector field $\partial D \ni z \mapsto n(z) \in \mathbb{R}^d$ denotes the outer normal on $\partial D$. 
Remark 4. Hypothesis[1] implies that the solution of (2) is positive invariant on $\bar{D}$, that is, for all $x \in D$ we have $X^0_{t,x} \in D$ for all $t \geq 0$ and $X^0_{t,x} \to 0$ as $t \to \infty$.

The main results of this article concern the asymptotics of the first exit time $\sigma^\varepsilon(x)$ of the solution $X^\varepsilon_{t,x}$ of (10) from $D$ and its first exit location $X^\varepsilon_{\sigma^\varepsilon(x),x}$ as $\varepsilon \to 0$. Given $\varepsilon \in (0,1]$, $x \in D$ and $\nu \in \mathcal{M}$ satisfying Hypotheses A - D we define the first exit time of the solution $X^\varepsilon_{t,x}$ of (9) from $D$ starting in some $x \in D$

$$\sigma^\varepsilon(x) := \inf\{ t \geq 0 \mid X^\varepsilon_{t,x} \notin D \}$$

and the first exit location $X^\varepsilon_{\sigma^\varepsilon(x)}$.

1.2.1 A large deviations principle for $(X^\varepsilon_{t,x})_{\varepsilon \in (0,1]}$.

We fix some necessary notation. Whenever possible without confusion we shall drop the index $\varepsilon$ or the initial condition $x$. Furthermore, we define the measure $\nu_1$ on $\mathcal{B}(\mathbb{R}^d)$ by $\nu_1(A) := \nu(A \cap B_1(0))$.

In the sequel we introduce the objects for the statement of the large deviations principle of $(X^\varepsilon_{t,x})_{\varepsilon \in (0,1]}$ following [13] and [14]. For a $T > 0$ fixed we define the entropy functional for any measurable function $g : [0,T] \times \mathbb{R}^d \to [0,\infty)$ by

$$\mathcal{E}_T(g) := \int_0^T \int_{\mathbb{R}^d} (g(s,z) \ln g(s,z) - g(s,z) + 1)\nu(dz)ds.$$  \hspace{1cm} (8)

For every $M \geq 0$ we define the sublevel sets of the functional $\mathcal{E}_T$ by

$$\mathcal{S}^M := \left\{ g : [0,T] \times \mathbb{R}^d \to [0,\infty) \mid \mathcal{E}_T(g) \leq M \right\}$$

and set $\mathcal{S} := \bigcup_{M \geq 0} \mathcal{S}^M$.

Given $T > 0$, $x \in \mathbb{R}^d$ and $g \in \mathcal{S}$ we consider the controlled integral equation

$$U^g(t; x) = x + \int_0^t b(U^g(s;x))ds + \int_0^t \int_{\mathbb{R}^d} G(U^g(s;x),z)(g(s,z) - 1)\nu(dz)ds, \hspace{0.5cm} t \in [0,T].$$  \hspace{1cm} (9)

It is standard in the literature (see Proposition A.2.1 in [10] and Theorem 3.7 in [11]) that the equation (9) has a unique solution $U^g \in C([0,T],\mathbb{R}^d)$ and it satisfies the uniform bound

$$\sup_{t \in [0,T]} \sup_{g \in \mathcal{S}^M} |U^g(t; x)| < \infty \hspace{1cm} \text{for all} \hspace{0.5cm} M > 0.$$  \hspace{1cm} (10)

In particular, for any fixed $x \in \mathbb{R}^d$ the map $G^{0,x} : \mathcal{S} \to \mathcal{D}([0,T],\mathbb{R}^d)$, $g \mapsto G^{0,x}(g) := U^g(\cdot; x)$ is well-defined. Under these assumptions we define for any $\varphi \in \mathcal{D}([0,T],\mathbb{R}^d)$ the pre-image set $\mathcal{S}_{\varphi,x} := \{ g \in \mathcal{S} \mid \varphi = G^{0,x}(g) \}$ under $G^0$. Let $\mathcal{J}_{x,T} : \mathcal{D}([0,T],\mathbb{R}^d) \to [0,\infty]$ be defined by

$$\mathcal{J}_{x,T}(\varphi) := \inf_{g \in \mathcal{S}_{\varphi,x}} \mathcal{E}_T(g)$$  \hspace{1cm} (11)

with the convention that $\inf \emptyset = \infty$.

**Theorem 2.** Let Hypotheses A - D be satisfied for some $\nu \in \mathcal{M}$, $T > 0$ and $x \in D$ fixed and let $X^\varepsilon = (X^\varepsilon_{t,x})_{t \in [0,T]}$ be the family obtained, for every $\varepsilon > 0$, by the strong solution of (9) given in Theorem 1 Then the family $(X^\varepsilon_{t,x})_{\varepsilon \in (0,1]}$ satisfies a large deviations principle with the good rate function $\mathcal{J}_{x,T}$ given by (11) in the Skorokhod space $\mathcal{D}([0,T],\mathbb{R}^d)$.
1.2.2 The asymptotic first exit problem of $X^\varepsilon$ from $D$ as $\varepsilon \to 0$.

The cost function $V$ quantifying the "cost" of shifting the intensity jump measure by a scalar control $g$ and steering $U^g(t; x)$ from its initial position $x$ to some $z \in \mathbb{R}^d$ in cheapest time is defined as

$$V(x, z) := \inf_{T > 0} \left\{ \int_{x, T} \varphi_{\xi} \mid \varphi \in D([0, T], \mathbb{R}^d) : \varphi(T) = z \right\} \quad \text{for } x, z \in \mathbb{R}^d. \tag{12}$$

The function $V(0, z)$ is called the quasi-potential of the stable state 0 with potential height $\bar{V} := \inf_{z \in \partial D} V(0, z). \tag{13}$

**Hypothesis E.** Let Hypotheses $[A] - [D]$ be satisfied. For every $\rho_0 > 0$ there exist a constant $M > 0$ and a non-decreasing function $\xi : [0, \rho_0] \to \mathbb{R}^+$ with $\lim_{\rho \to 0} \xi(\rho) = 0$ satisfying the following. For all $x_0, y_0 \in \mathbb{R}^d$ such that $|x_0 - y_0| \leq \rho_0$ there exist $\Phi \in C([0, \xi(\rho_0)], \mathbb{R}^d)$ and $g \in \mathcal{S}^M$ such that $\Phi(\xi(\rho_0)) = y_0$ and solving

$$\Phi(t) = x_0 + \int_0^t b(\Phi(s))ds + \int_0^t \int_{\mathbb{R}^d} G(\Phi(s), z)(g(s, z) - 1)\nu(dz)ds, \quad t \in [0, \xi(\rho_0)]. \tag{14}$$

**Remark 5.**

i) Hypothesis $[E]$ guarantees the (local) controllability of the dynamical system in small balls around the initial position.

ii) Hypothesis $[E]$ ensures that $\bar{V} < \infty$ excluding trivial cases. Due to Hypothesis $[E]$ we fix $z \in D^c$ and take $\rho_0 = |z|$. Consider $M < \infty$, $\xi : [0, \rho_0] \to \mathbb{R}^+$ and $g \in \mathcal{S}^M$ such that $[E]$ holds for some $\Phi \in C([0, \xi(\rho_0)], \mathbb{R}^d)$ and with $\Phi(\xi(\rho_0)) = z$. Therefore we have

$$\bar{V} := \inf_{z \in D^c} V(0, z) \leq \int_0^{\xi(\rho_0)} \int_{\mathbb{R}^d} \ell(g(s, z))\nu(dz)\leq M < \infty.$$

The second main result of this work has two parts and determines the asymptotics $\varepsilon \to 0$ of the first exit time $\sigma^\varepsilon(x)$ and the first exit location $X^\varepsilon_{\sigma^\varepsilon(x)}$.

**Theorem 3.** Let Hypotheses $[A] - [E]$ be satisfied.

1. Then for any $x \in D$ and $\delta > 0$ we have

$$\lim_{\varepsilon \to 0} \mathbb{P}\left( e^{\frac{\delta}{\varepsilon}} < \sigma^\varepsilon(x) < e^{\frac{\delta + \delta}{\varepsilon}} \right) = 1.$$

Furthermore for all $x \in D$ we have $\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{E}[\sigma^\varepsilon(x)] = \bar{V}$.

2. Then for any closed set $F \subset D^c$ satisfying $\inf_{z \in F} V(0, z) > \bar{V}$ and any $x \in D$ we have

$$\lim_{\varepsilon \to 0} \mathbb{P}\left( X^\varepsilon_{\sigma^\varepsilon(x)} \in F \right) = 0.$$

In particular, if $\bar{V}$ is taken by a unique point $z^* \in D^c$, it follows for any $x \in D$ and $\delta > 0$ that $\lim_{\varepsilon \to 0} \mathbb{P}(|X^\varepsilon_{\sigma^\varepsilon(x)} - z^*| < \delta) = 1$.

1.3 Examples

**Strongly tempered Lévy measures** Hypothesis $[E]$ covers a wide class of Lévy measures and we point out the following special benchmark cases.

1. Our setting covers the simplest case of finite intensity super-exponentially light jump measures, given by $\nu(dz) = e^{-|z|^\alpha}dz$, $\alpha \geq 1$. The study of the first exit problem for this explicit jump measure and the corresponding additive stochastic perturbation $(L^T_t)_{t \geq 0}$ that turns to be a compensated compound Poisson process can be found in the thesis $[46]$. 

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2. The first paradigmatic infinite intensity example is the compensated one-dimensional Gamma process with Lévy measure
\[ \nu_{a,b}(A) = b e^{-az} dz, \quad z > 0, \]
for some \( a, b > 0 \). For all \( \varepsilon > 0 \), the stochastic perturbation \((L^\varepsilon_t)_{t \geq 0}\) is the compensated version of a one dimensional Gamma process, given by
\[ \mathbb{E}[e^{iuL^\varepsilon_t}] := \exp \left( \frac{t}{\varepsilon} \int_0^\infty \frac{(e^{iu\varepsilon z} - 1) e^{-z b \varepsilon}}{z} \right), \quad t \geq 0. \]
We refer the reader to [2]-Example 1.3.22 and to [50]-Example 2.8.10. The Gamma process differs qualitatively from the compound Poisson process not only from the fact that the corresponding jump measure has infinite total mass but also from the fact that although a compound Poisson process with positive jumps has almost surely nondecreasing paths, it does not have paths that are almost surely strictly increasing.

3. More generally, Hypothesis \( B \) covers strongly tempered exponentially light measures introduced by Rosiński [49], which are given in polar coordinates \( r = |z| \) as
\[ \nu(A) = \int_{\mathbb{R}} \int_0^\infty 1_A(raz) e^{-r \frac{e^{r \alpha'}}{\alpha' + 1}} dr R(dz), \quad \alpha' \in (0, 2), \]
for a measure \( R \in \mathcal{M} \) satisfying for some \( \alpha' \in (0, 2) \) and \( \int_{\mathbb{R}} |z|^{\alpha'} R(dz) < \infty \) and \( R(B_{\theta^{-1}}(0)) = 0 \). In [49], Proposition 2.3 (iv) shows that
\[ \int_{|z| > 1} e^{\theta |z|} \nu(dz) < \infty \quad \iff \quad \text{supp} (\nu) \subset B_{\theta^{-1}}(0). \]

2 The large deviations principle

2.1 Verifying the sufficient criteria in [13]

Let Hypotheses \( A - C \) be satisfied for some \( \nu \in \mathcal{M} \). For every \( \varepsilon > 0 \), \( T > 0 \) and \( x \in \mathbb{R}^d \) we consider the strong solution \((X^{\varepsilon,x})_{t \in [0,T]}\) of the stochastic differential equation (7).

The proof of Theorem 2 uses the following crucial lemma. Although the statements are fundamental to derive the large deviations principle for the family \((X^{\varepsilon,x})_{\varepsilon > 0}\), the proof can be skipped in a first reading. For this reason the result is shown in the appendix, Subsection 4.1.

Lemma 6. Let \( \nu \in \mathcal{M} \) satisfy Hypothesis \( B \). For any \( M, T > 0 \), \( u \geq 2 \) and \( x \in \mathbb{R}^d \) we have the following statements
\[ \sup_{g \in \mathcal{G}^M} \int_{(0,T] \times \mathbb{R}^d} |z|^u g(s,z) \nu(dz) ds < \infty, \quad (15) \]
and
\[ \lim_{\delta \to 0} \sup_{g \in \mathcal{G}^M} \int_{(t,T] \times \mathbb{R}^d} |z|^u g(s,z) ds = 0. \quad (16) \]

Proof of Theorem 2. The proof consists of verifying two conditions given as Condition 2.2 in [14] which jointly imply the large deviations principle (Theorem 2.3 in [14]). It is organized in five consecutive steps. The first condition is the identification of a weak convergence limit point and the second one is a continuity statement. They are verified in Step 4 and Step 3, respectively.
Step 1: The setup. Fix $T > 0$, $\varepsilon \in (0, 1]$ and $x \in \mathbb{R}^d$. By Theorem 1.1 (w.r.t. $\nu_1$) the map
\[
G^{\varepsilon,x} : \mathcal{M} \to \mathbb{D}([0, T], \mathbb{R}^d), \quad G^{\varepsilon,x}(\varepsilon N) := X^{\varepsilon,x},
\]
is measurable with respect to the Borel sigma algebras associated to the topology given in $\mathcal{M}$ by the vague convergence and to the $J_1$ topology in $\mathbb{D}([0, T], \mathbb{R}^d)$. On the other hand, given $g \in \mathcal{S}$, the wellposedness of the integral equation (9) yields the existence of a measurable map
\[
G^{0,x} : \mathcal{S} \to C([0, T], \mathbb{R}^d), \quad G^{0,x}(\nu_\mathcal{P}) := \hat{U}^g(\cdot, x).
\]
For the sake of readability we essentially follow the notation introduced in [13] and in [14]. Denote by $\mathcal{P}$ the predictable $\sigma$-field on $[0, T] \times \mathcal{M}$ with respect to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. We define the space of positive (random) controls in $\mathcal{M}$
\[
\mathcal{A}^+ := \left\{ \varphi : [0, T] \times \mathbb{R}^d \times \mathcal{M} \to [0, \infty) \mid \text{\varphi is \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}([0, \infty)) measurable} \right\}.
\]
Given a covering $(K_n)_{n \in \mathbb{N}}, \mathbb{R}^d \subset \bigcup_{n \in \mathbb{N}} K_n$, of compact sets $K_n \subset \mathbb{R}^d$, we define the set of the $n$-cutoff positive (random) controls
\[
\mathcal{A}_{b,n} := \left\{ \varphi \in \mathcal{A}^+ \mid \varphi(t, x, \bar{\mu}) \in [\frac{1}{n}, n], \quad x \in K_n, \quad \bar{\mu} \in \mathcal{M}, \quad \text{for all } (t, \bar{\mu}) \in [0, T] \times \mathcal{M} \right\}.
\]
The set of positive bounded controls is then given by
\[
\mathcal{A}_{b} := \bigcup_{n \in \mathbb{N}} \mathcal{A}_{b,n} \quad \text{and} \quad \mathcal{M}_+ := \left\{ \varphi \in \mathcal{A}^+_b : \varphi(\cdot, \cdot, \bar{\mu}) \in \mathcal{S}_+ \quad \text{P-a.s} \right\}, \quad M > 0,
\]
is the set of positive bounded random controls whose entropy functional is $\mathcal{P}$-a.s. bounded by $M$. We associate to every $g \in \mathcal{S}_+$ the measure
\[
B([0, T] \times \mathbb{R}^d) \ni A \mapsto \nu_\mathcal{P}^+(A) := \int_A g(s, z)\nu_1(dz)ds
\]
and identify $\mathcal{S}_+$ with the space of associated measures $\{\nu_\mathcal{P}^+ \mid g \in \mathcal{S}_+\} \subset \mathcal{M}$ equipped with the topology induced by the vague convergence on $\mathcal{M}$. We refer the reader to the Lemma 5.1 in [14] which ensure that this identity produces a topology in $\mathcal{S}_+$ under which $\mathcal{S}_+$ turns out to be compact.

For any fixed $M > 0$ and a family $(\varphi_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)} \subset \mathcal{M}_+$ we set $\psi_\varepsilon := \frac{1}{\bar{\nu}_\varepsilon}$. The random measure $N^{\bar{\nu}_\varepsilon}$ is a controlled random measure given by
\[
N^{\bar{\nu}_\varepsilon}([0, t] \times A) := \int_0^t \int_U \int_0^\infty 1_{[0, \bar{\nu}_\varepsilon]}(N(ds, dx, dr) \quad \text{for all } t \in [0, T], A \in B(\mathbb{R}^d).
\]
Note that the canonical map $\bar{N} : 2\mathcal{M} \to \mathcal{M}, \bar{N}(\bar{\mu}) := \bar{\mu}$ is the Poisson random measure defined on $(2\mathcal{M}, B(2\mathcal{M}), \mathcal{P})$ with intensity measure $ds \otimes \nu \otimes dr$.

Since $\varphi_\varepsilon \in \mathcal{M}_+$ yields that $\varphi_\varepsilon$ is bounded from below and above on a compact set in $[0, T] \times \mathbb{R}^d$ and $\varphi_\varepsilon = 1$ outside of that compact, we can use the version of Girsanov’s theorem given [13] - Lemma 2.3. Therefore the Doleans-Dade exponential of $\psi_\varepsilon$ with respect to $N^{\bar{\nu}_\varepsilon}$ under $\mathcal{P}$ defined, for any $t \in [0, T]$, by
\[
\mathcal{E}(\psi_\varepsilon)(t) := \exp \left( \int_0^t \int_{\mathbb{R}^d} \int_0^{\frac{1}{\bar{\nu}_\varepsilon}} \ln \psi_\varepsilon(s, z)\bar{N}(ds \, dz \, dr) + \int_0^t \int_{\mathbb{R}^d} \int_0^{\frac{1}{\bar{\nu}_\varepsilon}} (-\psi_\varepsilon(s, z) + 1)\nu(dz)drds \right)
\]
is an $(\mathcal{F}_t)_{t \in [0, T]}$ - martingale under $\mathcal{P}$. In addition, the measure
\[
\mathcal{Q}_T^\varepsilon(G) := \int_G \mathcal{E}(\psi_\varepsilon)d\mathcal{P}(\bar{\mu}), \quad \text{for all } G \in B(\mathcal{M}),
\]
is a probability measure on \((\mathcal{M}, \mathcal{B}(\mathcal{M}))\). Furthermore, the measures \(\tilde{P}\) and \(Q_2^\varepsilon\) are mutually absolutely continuous and the controlled random measure \(\varepsilon N^{\varepsilon^+}\) under \(Q_2^\varepsilon\) has the same law as \(\varepsilon N^{\varepsilon^-}\) under \(\tilde{P}\) on \((\mathcal{M}, \mathcal{B}(\mathcal{M}))\). For more details we refer the reader to Lemma 2.3 in [13] and further references given there.

Denote by \(\tilde{X}_{\varepsilon,x} := G^{\varepsilon,x}(\varepsilon N^{\varepsilon}t)\) the unique strong solution of the following controlled SDE, for every \(t \in [0,T]\),

\[
\tilde{X}_{\varepsilon,x}^t = x + \int_0^t b(\tilde{X}_{\varepsilon,x}^s) ds + \int_0^t \int_{\mathbb{R}^d} G(\tilde{X}_{\varepsilon,x}^s, z)(\varepsilon N^{\varepsilon+}(ds, dz) - \nu(dz)ds) .
\]  

(17)
The process \(\tilde{X}_{\varepsilon,x}^t\) has the following localization property, which we use in Step 3.

**Step 2: A priori estimate:** The first step consists in proving the following.

**Proposition 7.** Let the hypotheses of Theorem [2] be satisfied. Then for any \(M > 0\), any family \((\varphi_\varepsilon)_{\varepsilon \in (0,1]}\), \(\varphi_\varepsilon \in \Phi_+^M\), any function \(R : (0,1] \to (0,\infty)\) satisfying the limits \(\lim_{\varepsilon \to 0^+} R(\varepsilon) = \infty\) and \(\lim_{\varepsilon \to 0^+} \varepsilon R^2(\varepsilon) = 0\), \(x \in \mathbb{R}^d\) and \(T > 0\) we have the following. There exists \(\varepsilon_0 \in (0,1]\) and a constant \(C > 0\) such that for all \(\varepsilon \in (0, \varepsilon_0]\) we have

\[
\tilde{P}\left( \sup_{s \in [0,T]} |\tilde{X}_{\varepsilon,x}^s| > R(\varepsilon) \right) \leq 2e^{-\frac{1}{2}R(\varepsilon)} + C\varepsilon R(\varepsilon).
\]  

(18)
The proof uses a Bernstein-type inequality from [29] but rather standard in nature and found in Subsection 4.2 of the appedix.

**Step 3: The LDP limit condition (Condition 2.2(b) in [14])**

**Proposition 8.** Given \(M > 0\) let \(\varphi \in \Phi_+^M\) and \((\varphi_\varepsilon)_{\varepsilon \in (0,1]}\) be a family of \(\varphi_\varepsilon \in \Phi_+^M\) such that \(\varphi_\varepsilon \overset{d}{\to} \varphi\) as \(\varepsilon \to 0^+\). Then for all \(x \in \mathbb{R}^d\) the law of \(G^{\varepsilon^+}(\nu_2^\varepsilon)\) is a limit point in law of \(G^{\varepsilon,x}(\varepsilon N^{\varepsilon\varphi_\varepsilon})\) in \(\mathcal{D}([0,T], \mathbb{R}^d)\).

**Proof.** Once again we drop the dependence on \(x\) of \(\tilde{X}_{\varepsilon,x}^t\). For every \(\varepsilon \in (0,1]\) and \(t \in [0,T]\) we define

\[
J_\varepsilon^t := \int_0^t b(\tilde{X}^t_\varepsilon) ds + \int_0^t \int_{\mathbb{R}^d} G(\tilde{X}^t_\varepsilon, z)(\varphi_\varepsilon(s, z) - 1)\nu(dz)ds,
\]

\[
\tilde{M}_\varepsilon^t := \int_0^t \int_{\mathbb{R}^d} G(\tilde{X}^t_\varepsilon, z)\varepsilon N^{\varepsilon\varphi_\varepsilon}(ds, dz).
\]  

(19)
We start by showing that the family of processes \((J_\varepsilon^t)_{\varepsilon \in (0,1]}\) is C-tight. By [19] and Proposition 4 there exists \(\varepsilon_1 \in (0,1]\) such that for every \(\rho > 0\) we have \(\delta = \delta_\rho > 0\), we have

\[
\sup_{\varepsilon \in (0,\varepsilon_1]} \tilde{P}\left( \sup_{s,t \in [0,T]} |J_\varepsilon^t - J_\varepsilon^s| > \rho \right) < \rho.
\]
Fix \(\rho > 0\) and the corresponding constant \(\varepsilon_1 \in (0,1]\). The set

\[
K_\rho := \bigcap_{m \in \mathbb{N}} \left\{ f \in C([0,T], \mathbb{R}^d) \mid f(0) = 0 \text{ and for all } s, t \in [0,T] \right\}
\]

\[
|t-s| \leq \delta_{2^{-m}} \text{ implies } |f(t) - f(s)| < \rho 2^{-m}
\]  

(20)
is the countable intersection of equicontinuous and pointwise bounded sets in \(C([0,T], \mathbb{R}^d)\) each of which is relatively compact in \(C([0,T], \mathbb{R}^d)\) by the Arzela-Ascoli theorem. Furthermore it is non-empty, relatively compact in \(C([0,T], \mathbb{R}^d)\) and satisfies

\[
\tilde{P}(J_\varepsilon \notin K_\rho) \leq \rho \sum_{m=1}^{\infty} 2^{-m} = \rho,
\]
which implies that \((J^r)_{c \in (0,1]}\) is C-tight. For the definition we refer to [38], Definition VI.3.25. We now show that the family of processes \((M^r)_{c \in (0,1]}\) is C-tight. As a first step we first show that the family of quadratic variations \((\langle M^r \rangle)_{c \in (0,1]}\) is C-tight. For the scale \(R(\varepsilon)\) and the constants \(x_0, C > 0\) given in Proposition [7] and the constants \(L > 0\) and \(r \geq 2\) given in Hypothesis [C] we have for any \(\kappa > 0\) and for some \(x_0 \in (0,1]\) sufficiently small such that \(\varepsilon \in (0,x_0],\)

\[
\bar{P}\left(\langle M^r \rangle_T > \kappa\right) \leq \bar{P}\left(\langle M^r \rangle_T > \kappa, \sup_{s \in [0,T]} |X^s_T| \leq R(\varepsilon)\right) + \bar{P}\left(\sup_{s \in [0,T]} |\tilde{X}^s_T| > R(\varepsilon)\right)
\]

\[
= \bar{P}\left(\varepsilon^2 \int_0^T \int_{\mathbb{R}^d} |G(\tilde{X}^s_T, z)|^2 N_{2r}^\frac{1}{2}(ds, dz) > \kappa, \sup_{s \in [0,T]} |\tilde{X}^s_T| \leq R(\varepsilon)\right) + 2e^{-R(\varepsilon)} + C\varepsilon R(\varepsilon)
\]

\[
\leq \bar{P}\left(2\varepsilon^2 L(1 + R^2(\varepsilon)) \int_0^T \int_{\mathbb{R}^d} \varepsilon^{2r} N_{2r}^\frac{1}{2}(ds, dz) > \kappa\right) + 2e^{-\frac{1}{2}R(\varepsilon)} + C\varepsilon R(\varepsilon)
\]

\[
\leq \frac{2\varepsilon(1 + R^2(\varepsilon))}{\kappa} \int_0^T \int_{\mathbb{R}^d} \varepsilon^{2r} |\varepsilon^r g(s, z)\nu_1(ds) + 2e^{-\frac{1}{2}R(\varepsilon)} + C\varepsilon R(\varepsilon)
\]

\[
= \frac{2\varepsilon R^2(\varepsilon)}{\kappa} \sup_{s \in [0,T]} \int \int_{\mathbb{R}^d} \varepsilon^{2r} |\varepsilon^r g(s, z)\nu_1(ds) + 2e^{-\frac{1}{2}R(\varepsilon)} + C\varepsilon R(\varepsilon)
\]

for a constant \(C_1 = \frac{2C_{M,2r,T}}{\kappa}\) with a constant \(C_{M,2r,T} > 0\) depending on \(M, r, T > 0\). In other words, \([M^r]_T \to 0\) as \(\varepsilon \to 0\) in probability and therefore in law, which implies that also \((\langle M^r \rangle)_{c \in (0,1]}\) is C-tight. Due to [39], Theorem 6.1.1, the laws of the family \(\tilde{Z}^r_t = x + J^r_t + M^r_t\) are tight in \(\mathcal{D}([0,T], \mathbb{R}^d)\). By Prokhorov’s Theorem there exists the weak limit of \((X^{r_n}, J^{r_n}, M^{r_n})\) for some subsequence \(r_n \to 0\). Skorokhod’s representation’s theorem implies that there exists a triplet of random variables \((X, \tilde{\varphi}, 0)\) defined on \((\mathcal{M}, \mathcal{B}(\mathcal{M}), \mathbb{P})\) such that \((X^{r_n}, J^{r_n}, M^{r_n})\) given by [17] and [10] converges to \((X, \tilde{\varphi}, 0)\) \(\mathbb{P}\)-a.s. as \(n \to \infty\). Due to [15] and the continuity of the functions \(b\) and \(G\) we can pass to the limit \(\tilde{X}^r_t \to \tilde{Z}^r\). Hence, we conclude that \((\tilde{X}_s)_{s \in [0,T]}\) satisfies \(\mathbb{P}\)-a.s.

\[
\tilde{X}_t = x + \int_0^t b(\tilde{X}_s)ds + \int_0^t \int_{\mathbb{R}^d} G(\tilde{X}_s, z)(\tilde{\varphi}(s, z) - 1)\nu(ds)ds, \quad t \in [0,T].
\]

Therefore we conclude that \(\tilde{Z} = \mathcal{G}^0(\nu^\varepsilon_T)\). Since \(\varphi\) and \(\tilde{\varphi}\) are indistinguishable in law and since \(\mathbb{P}\)-almost sure convergence implies convergence in law, we infer

\[
\mathcal{G}^0(\nu^\varepsilon_T) \quad \text{is a weak limit point of} \quad \mathcal{G}^\varepsilon N^{\frac{1}{2}\nu^\varepsilon}.
\]

\[
\square
\]

Step 4: The LDP continuity condition (Condition 2.2(a) in [14])

**Proposition 9.** For every \(M \geq 0\) and for every \(n \in \mathbb{N}\) let \(g_n, g \in \mathcal{G}^M\) such that \(\nu^\varepsilon_T \to \nu^\varepsilon_T\) in the vague topology of \(\mathcal{G}\) as \(n \to \infty\). Then there exists a subsequence \((g_{n_k})_{k \in \mathbb{N}}\subset (g_n)_{n \in \mathbb{N}}\) such that

\[
\mathcal{G}^0(\nu^\varepsilon_T^{g_{n_k}}) \to \mathcal{G}^0(\nu^\varepsilon_T),
\]

in the uniform topology on \(C([0,T], \mathbb{R}^d)\).

**Proof.** We set \(U^n := U^{g_n} = \mathcal{G}^0(\nu^\varepsilon_T^{g_n})\). Estimate [10] yields the existence of a constant \(K \in (0,\infty)\) such that \(\mathbb{P}\)-a.s.

\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} |U^n_t| \leq K.
\]

Due to [10] from Lemma [6] we conclude

\[
\lim_{\delta \to 0} \sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} |U^n_t - U^n_t| = 0.
\]

10
This implies that \((U^n)_{n \in \mathbb{N}}\) is a family of equicontinuous uniformly bounded functions in \(C([0, T], \mathbb{R}^d)\). Due to the Arzelà-Ascoli theorem there exists a limit point in the uniform topology \(U \in C([0, T], \mathbb{R}^d)\) for some subsequence. Since we have the uniform estimate \((21)\), due to the continuity of the functions \(b\) and \(G\) and \((15)\), dominated convergence yields
\[
U_t = x + \int_0^t b(U_s)ds + \int_0^t \int_{\mathbb{R}^d} G(U_s, z)(g(s, z) - 1)\nu_1(dz)ds \quad \text{for all } t \in [0, T].
\] (22)

The uniqueness of solution of \((9)\) implies that \(U = U^g = G^0(\nu_T^d)\).

**Step 5: Conclusion:** Proposition \([5]\) implies for \((X^{\varepsilon, x})_{\varepsilon \in (0, 1]}\) Condition 2.2(a) and Proposition \([9]\) yields Condition 2.2(b) in \([14]\). Hence Theorem 2.3 of \([14]\) implies the desired large deviations principle and we conclude the proof of Theorem 2.

\[\square\]

### 2.2 Some useful consequences

In the sequel it follows that the large deviations principle for \((X^{\varepsilon, x})_{\varepsilon \in (0, 1]}\) is continuous with respect to the initial condition \(x \in D\).

**Proposition 10.** Given \(T > 0\) and \(x \in D\), let \(F \subset \mathcal{D}([0, T], \mathbb{R}^d)\) be closed and \(G \subset \mathcal{D}([0, T], \mathbb{R}^d)\) open with respect to the Skorokhod topology. Then we have
\[
\begin{align*}
\limsup_{\varepsilon \to 0} \sup_{y \to x} \varepsilon \ln \bar{P}(X^{\varepsilon, y} \in F) & \leq -\inf_{f \in F} \mathcal{J}_{x,T}(f), \\
\liminf_{\varepsilon \to 0} \inf_{y \to x} \varepsilon \ln \bar{P}(X^{\varepsilon, y} \in G) & \geq -\inf_{g \in G} \mathcal{J}_{y,T}(g).
\end{align*}
\] (23) (24)

**Proof.** The strategy of the proof follows closely the arguments given in the proof of Theorem 2 and we omit its details. \[\square\]

As a consequence of Proposition \([10]\) follows the next result which is a uniform large deviations principle for \((X^{\varepsilon, x})_{\varepsilon \in (0, 1]}\) when the initial state \(x \in K\) for \(K \subset D\) a closed (and bounded) set. The proof is virtually the same as the one given in the Brownian case and we omit it. We refer the reader to Corollary 5.6.15 in \([21]\).

**Corollary 11.** Let \(T > 0\), \(K \subset D\) be closed, \(F \subset \mathcal{D}([0, T], \mathbb{R}^d)\) closed, \(G \subset \mathcal{D}([0, T], \mathbb{R}^d)\) open with respect to the \(J_1\) topology and \(x \in \mathbb{R}^d\). Then it follows
\[
\begin{align*}
\limsup_{\varepsilon \to 0} \sup_{y \in K} \varepsilon \ln \bar{P}(X^{\varepsilon, y} \in F) & \leq -\inf_{y \in K, f \in F} \mathcal{J}_{y,T}(f), \\
\liminf_{\varepsilon \to 0} \inf_{y \in K} \varepsilon \ln \bar{P}(X^{\varepsilon, y} \in G) & \geq -\inf_{y \in K, g \in G} \mathcal{J}_{y,T}(g).
\end{align*}
\]

In the sequel this result is applied to the first exit time problem of \(X^{\varepsilon, x}\) from \(D\).

### 3 The first exit problem in the small noise limit

In this section we fix the standing assumptions of the Hypotheses \([A] - [E]\) for some bounded domain \(D \subset \mathbb{R}^d\), \(x \in D\) and \(\nu \in \mathfrak{M}\). This section is the study of the exit time problem associated to \(\sigma^\varepsilon(x) = \inf\{t \geq 0 \mid X^{\varepsilon, x}_t \notin D\}\), where \(X^{\varepsilon, x}\) is the solution of \((7)\), in the limit of \(\varepsilon \to 0\).
We define the following cost function associated to the system (7), which measures the cost of steering $U^g$ given in (9) from its initial position $x \in D$ to some point $y \in \mathbb{R}^d$ in exactly time $t > 0$

$$V(x, y, t) := \inf \{ \xi(g) \mid g \in \mathcal{G}_\phi \quad \phi(s) = U^g(s, x), \quad s \in [0, t], \quad \phi(t) = y \}. $$

We start with elementary continuity properties shown in the Appendix, Subsection 1.3 which are essentially an easy consequence of Hypotheses D and 1.5

**Lemma 12.** Under Hypotheses D and 1.5 for any $\delta > 0$, there exists $\rho > 0$ such that

$$\sup_{x, y \in B_\rho(0)} \inf_{t \in [0, 1]} V(x, y, t) < \delta,$$

$$\sup_{x, y \in D} \inf_{t \in [0, 1]} V(x, y, t) < \delta.$$ 

**3.1 Upper bound in Theorem 3**

The proof of the upper bound in Theorem 3 shows a large deviations estimate for small initial values.

**Lemma 13.** Let $c > 0$. There exists a constants $\rho_0 > 0$ and $s_0$ such that for any $\rho \in (0, \rho_0]$ we have

$$\liminf \varepsilon \ln \inf_{x \in B_\rho(0)} \mathcal{P}(\sigma^\varepsilon(x) \leq s_0) \geq -(\bar{V} + c),$$

where the potential height $\bar{V}$ is given in equation (13).

**Proof.** Fix $\rho_0 > 0$ be small enough such that the inequalities of Lemma 12 are satisfied for $\delta = \frac{c}{2}$. Hence we may choose $x \in B_\rho(0)$ and a path $\phi_1^\varepsilon \in C([0, s_x], \mathbb{R}^d)$ satisfying $\phi_1^\varepsilon(0) = x$, $\phi_1^\varepsilon(s_x) = 0$ such that

$$J_{x, s_x}(\phi_1^\varepsilon) \leq \frac{c}{2}.$$ 

With the help of Lemma 12 and Hypothesis 1.5 we may choose $z \in D^c$, $s_z > 0$, $\phi_2^\varepsilon \in C([0, s_z], \mathbb{R}^d)$ such that $\phi_2^\varepsilon(0) = 0$, $\phi_2^\varepsilon(s_z) = z$ and

$$J_{0, s_z}(\phi_2^\varepsilon) \leq \bar{V} + \frac{c}{2}.$$ 

Let $\phi_3$ be the solution of the differential equation $\dot{\phi}_3 = b(\phi_3)$ with $\phi_3(0) = z$. We set $s_0 = s_x + s_z + \delta'$ with $\delta' > 0$ such that $\phi_3([0, \delta']) \subset D^c$ and define

$$\Phi^\varepsilon(t) = \begin{cases} 
\phi_1^\varepsilon(t) & \text{if } t \in [0, s_x], \\
\phi_2^\varepsilon(t - s_x) & \text{if } t \in (s_x, s_x + s_z], \\
\phi_3(t - s_x - s_z) & \text{if } t \in (s_x + s_z, s_0].
\end{cases}$$

Then the concatenation of the paths yields

$$J_{x, s_0}(\Phi^\varepsilon(t)) \leq J_{x, s_x}(\phi_1^\varepsilon) + J_{0, s_z}(\phi_2^\varepsilon) \leq \bar{V} + c.$$ 

Let $\Delta = d(z, D)$ and consider the open set

$$\mathcal{O} = \bigcup_{x \in B_{\rho_0}(0)} \{ \psi \in D([0, s_0], \mathbb{R}^d) : d_{L_1}(\psi, \Phi^\varepsilon) < \frac{\Delta}{2} \}.$$ 

Our constructed path $\Phi^\varepsilon$ visits $z$ by definition and stays outside of $D$ in the time interval $[s_x + s_z, s_0]$, due to the choice of $z \in D^c$ and the continuity of $\phi_3$. By definition of $\mathcal{O}$, every path $\psi \in \mathcal{O}$ exits
Lemma 14. We have $d(z, \psi([0, s_0])) > \Delta$. (25)

Since $\psi \in \mathcal{O}$ we have $d_{\mathcal{H}}(\psi, \Phi^x) < \frac{\Delta}{2}$, that is, there is an increasing homeomorphism $\lambda : [0, s_0] \to [0, s_0]$ such that

$$\sup_{t \in [0, s_0]} |\psi(\lambda(t)) - \Phi^x(t)| < \frac{\Delta}{2}. $$

In particular

$$|\psi(\lambda(s_z + s_x)) - \Phi^x(s_z + s_x)| = |\psi(\lambda(s_z + s_x)) - z| < \frac{\Delta}{2},$$

which contradicts (25). Due to Corollary 11 we have

$$\liminf_{\varepsilon \to 0} \inf_{x \in B_{\rho_0}(0)} \mathbb{P}(\sigma^\varepsilon(x) \leq s_0) \geq \liminf_{\varepsilon \to 0} \inf_{x \in B_{\rho_0}(0)} \mathbb{P}(X_{t, x}^\varepsilon \in \mathcal{O}) \geq -\sup_{x \in B_{\rho_0}(0)} \inf_{\psi \in \mathcal{O}} \mathbb{J}_{x,s_0}(\psi) = -(V + c),$$

which finishes the proof.

For fixed $x \in D$, we show next that the probability $X_{t, x}^\varepsilon$ staying inside $D$, but without hitting a small neighborhood of 0 is exponentially small. For given $\rho > 0$ such that $B_\rho(0) \subset D$, we define

$$\vartheta_\rho^x(x) := \inf\{t \geq 0 \mid |X_{t, x}^\varepsilon| \leq \rho \text{ or } X_{t, x}^\varepsilon \notin D^c\}. $$

Lemma 14. We have

$$\lim_{t \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{x \in D} \mathbb{P}(\vartheta_\rho^x(x) > t) = -\infty. $$

Proof. Let us fix $\rho > 0$. For $t \geq 0$, we define the subset of $\mathbb{D}([0, t], \mathbb{R}^d)$

$$\mathcal{G}_t := \left\{ \Phi \in \mathbb{D}([0, t], \mathbb{R}^d) : \Phi(s) \in \overline{D \setminus B_\rho(0)} \text{ for all } s \in [0, t] \right\}$$

and

$$\tilde{\mathcal{G}}_t := \left\{ \Phi \in \mathbb{D}([0, t], \mathbb{R}^d) : \Phi(s) \in \overline{D \setminus B_\rho(0)} \text{ for all } s \in [0, t] \right\}$$

except in a countable number of points).

It is a fact that $\tilde{\mathcal{G}}_t = \mathcal{G}_t$ and $\tilde{\mathcal{G}}_t$ is a closed set in $\mathbb{D}([0, t], \mathbb{R}^d)$ with respect to the Skorokhod topology. We refer the reader to Lemma 13 in the appendix, Subsection 4.4. By the definition of $\mathcal{G}_t$ and Corollary 11 we have

$$\sup_{x \in D} \sup_{t \geq 0} \mathbb{P}(\vartheta_\rho^x(x) > t) \leq \limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{x \in D \setminus B_\rho(0)} \mathbb{P}(\vartheta_\rho^x(x) > t) \leq -\inf_{x \in D \setminus B_\rho(0)} \inf_{\psi \in \tilde{\mathcal{G}}_t} \mathbb{J}_{x,t}(\psi) $$

$$= -\inf_{x \in D \setminus B_\rho(0)} \inf_{\psi \in \tilde{\mathcal{G}}_t} \mathbb{J}_{x,t}(\psi) = -\inf_{\psi \in \tilde{\mathcal{G}}_t} \mathbb{J}_{\psi(0),t}(\psi). $$

Next we show that

$$\lim_{t \to \infty} \inf_{\psi \in \tilde{\mathcal{G}}_t} \mathbb{J}_{\psi(0),t}(\psi) = \infty. $$

Let $(\varphi_\varepsilon)_{\varepsilon \geq 0}$ be the dynamical system associated to $\varphi_\varepsilon = b(\varphi_\varepsilon)$. Due to Hypothesis A, given $x \in D \setminus B_\rho(0)$, there exists $t_x \geq 0$ such that $\varphi(t_x) \in B_\rho(0)$. Define $O_x := \varphi^{-1}(B_\rho(0))$. $O_x$ is an open
neighborhood of $x$ in the usual topology of $\mathbb{R}^d$. Choose $x_1, \ldots, x_k \in D \setminus B_\rho(0)$ such that $\bigcup_{i=1}^k O_{x_i} \supset (D \setminus B_\rho(0))$ and define $s = t_{x_1} \vee \cdots \vee t_{x_k}$. Before time $s$, any path that solves $\dot{\varphi}_t = b(\varphi_t)$, with initial condition in $D \setminus B_\rho(0)$, hits $B_{\rho}$. We argue by contradiction. Assume that

$$\lim_{t \to +\infty} \inf_{\psi \in \mathcal{G}_t} I_{\varphi(0),s}(\psi) < \infty.$$  \hspace{1cm} (30)

Let us fix $M > 0$ such that, for any $n \in \mathbb{N}$, there exists $\varphi^n \in \mathcal{G}_{n,s}$ such that $I_{\varphi^n(0),ns}(\varphi^n) \leq M$. For $k = 0, \ldots, n-1$ let

$$\varphi^{n,k}(t) := \varphi^n(k(s - t)), t \in [0, s].$$

Hence $\varphi_{n,k} \in \mathcal{G}_s$ and

$$M \geq I_{\varphi^n(0),ns}(\varphi^n) = \sum_{i=0}^{n-1} I_{\varphi^n(ks),s}(\varphi^{n,k}) \geq n \min_{0 \leq k \leq n-1} I_{\varphi^n,ks}(\varphi^{n,k}).$$

We finally show the existence of a sequence $(\varphi^n)_{n \in \mathbb{N}}$ in $\mathcal{G}_t$ such that

$$\lim_{n \to \infty} I_{\varphi^n(0),s}(\varphi^n) = 0.$$

First we see that the set

$$\{ \varphi \in D([0, s], \mathbb{R}^d) \mid \varphi(0) \in D \setminus B_\rho(0), I_{\varphi(0),s}(\varphi) \leq 1 \}$$

is a closed subset of the compact set $\{ \varphi \in D([0, s], \mathbb{R}^d) \mid I_{\varphi(0),s}(\varphi) \leq 1 \}$. The compactness comes from the fact that since $I_{\varphi(0),s}$ is a good rate function with respect to the Skorokhod topology. Hence the sequence $(\varphi^n)_{n \in \mathbb{N}}$ has a limit point in $\mathcal{G}_s$ which we call $\bar{\varphi}$. Since $I_{\varphi(0),s} = \inf_{x \in \mathcal{R}^d} I_{\varphi,s,x}$ is lower semicontinuous, it follows that $I_{\varphi(0),s}(\bar{\varphi}) = 0$, which means that $\bar{\varphi}$ solves $\dot{\varphi}_t = b(\varphi_t)$ with $\bar{\varphi}(0) \in D \setminus B_\rho(0)$. Therefore $\bar{\varphi}$ reaches $B_\rho(0)$ before time $s$, which contradicts $\bar{\varphi} \in \mathcal{G}_s$ and thus the assumption \cite{[30]}. Combining inequality \cite{[28]} and \cite{[24]} yields the desired result \cite{[27]}.

**Theorem 4.** For $x \in D$ and $\delta > 0$ we have

$$\liminf_{\epsilon \to 0} \epsilon \ln \bar{\mathbb{P}}(\sigma^\epsilon(x) < \epsilon \frac{V + \delta}{4}) = 1.$$  \hspace{1cm} (31)

**Proof.** The proof consists of two steps.

**Claim 1.** For any $\delta > 0$ there are $T > 0$ and $\epsilon_0 \in (0, 1]$ such that $\epsilon \in (0, \epsilon_0]$ implies

$$\inf_{x \in D} \bar{\mathbb{P}}(\sigma^\epsilon(x) \leq T) \geq \epsilon^{-\frac{V + \delta}{4}}.$$  \hspace{1cm} (32)

We first observe that by Lemma \cite{[33]} for every $\delta > 0$ there are $\epsilon_0 > 0$ and $\rho > 0$ such that

$$\liminf_{\epsilon \to 0} \epsilon \ln \inf_{x \in B_\rho(0)} \bar{\mathbb{P}}(\sigma^\epsilon(x) \leq \epsilon_0) > -(V + \frac{\delta}{4}).$$

Lemma \cite{[14]} applied for the fixed value $\rho$ yields a time $t_1 > 0$ such that

$$\limsup_{\epsilon \to 0^+} \epsilon \ln \sup_{x \in D} \bar{\mathbb{P}}(\sigma^\epsilon(x) > t_1) < 0.$$  \hspace{1cm} (33)

This implies for any $r > 0$ the existence of $\epsilon_0 \in (0, 1]$ such that $\epsilon \in (0, \epsilon_0]$ gives

$$\epsilon \ln \sup_{x \in D} \bar{\mathbb{P}}(\sigma^\epsilon(x) > t_1) < -r.$$  \hspace{1cm} (34)

In addition, if we choose $\epsilon_0 \in (0, 1]$ small enough we have for $\epsilon \in (0, \epsilon_0]$ the inequality $1 - e^{-r} > e^{-\frac{r}{4}}$. Since $\{\sigma^\epsilon(x) < \sigma^\epsilon(x)\} = \{X_{\sigma^\epsilon(x)} \in B_\rho(0)\}$ it follows on this event

$$\sigma^\epsilon(x) = \sigma^\epsilon(x) + \sigma^\epsilon(X_{\sigma^\epsilon(x)}) \circ \Theta_{\sigma^\epsilon(x)},$$

where $\Theta_s$ is the characteristic function of the set $\{\sigma^\epsilon(x) < \sigma^\epsilon(x)\}$.
where \( \Theta_s \) is the shift by time \( s \) on the path space \( \mathbb{D}(\mathbb{R}^d) \). Using the homogeneous strong Markov property of \( X^{\varepsilon,x} \) we obtain for any fixed \( \varepsilon \in (0, \varepsilon_0) \) and \( x \in D \)

\[
\bar{\mathbb{P}}(\sigma^\varepsilon(x) \leq t_0 + t_1) \geq \bar{\mathbb{P}}(\sigma^\varepsilon(x) \leq t_1) \text{ and } \sigma^\varepsilon(X^{\varepsilon,x}_{\bar{\sigma}^\varepsilon(x)}) \leq t_0
\]

\[
= \bar{\mathbb{P}}(\sigma^\varepsilon(x) \leq t_1) \bar{\mathbb{P}}(\sigma^\varepsilon(X^{\varepsilon,x}_{\bar{\sigma}^\varepsilon(x)}) \leq t_0 | \sigma^\varepsilon(x) \leq t_1)
\]

\[
\geq \inf_{y \in D} \bar{\mathbb{P}}(\sigma^\varepsilon(y) \leq t_1) \inf_{x \in B(y,0)} \bar{\mathbb{P}}(\sigma^\varepsilon(x) \leq t_0)
\]

\[
\geq e^{-\frac{\varphi + \frac{4}{e}}{\varepsilon}} e^{-\frac{\varphi}{\varepsilon}} \geq e^{-\frac{\varphi + \frac{4}{e}}{\varepsilon}} (1 - e^{-\frac{\varphi}{\varepsilon}}) = e^{-\frac{\varphi + \frac{4}{e}}{\varepsilon}}.
\]

Setting \( T = t_0 + t_1 \) we finish the proof of Claim \( \mathbb{I} \).

**Step 2:** We follow with the proof of the limit \( \mathbb{I} \). We set \( q^\varepsilon := \inf_{x \in D} \bar{\mathbb{P}}(\sigma^\varepsilon(x) \leq T) \) for the time \( T > 0 \) given in Claim \( \mathbb{I} \). Claim \( \mathbb{I} \) yields \( q^\varepsilon > 0 \) for all \( \varepsilon \in (0, \varepsilon_0) \). For any \( k \in \mathbb{N} \) and \( x \in D \) we consider the family of events \( \{ \sigma^\varepsilon(x) > kT \} \) for which we derive the following recursion

\[
\bar{\mathbb{P}}(\sigma^\varepsilon(x) > (k+1)T) = \left( 1 - \bar{\mathbb{P}}(\sigma^\varepsilon(x) \leq (k+1)T | \sigma^\varepsilon(x) > kT) \right) \bar{\mathbb{P}}(\sigma^\varepsilon(x) > kT)
\]

\[
\leq \left( 1 - q^\varepsilon \right) \bar{\mathbb{P}}(\sigma^\varepsilon(x) > kT), \quad k \in \mathbb{N}.
\]

Solving the recursion above in \( k \in \mathbb{N} \) we obtain for any \( \varepsilon \in (0, \varepsilon_0) \)

\[
\sup_{x \in D} \bar{\mathbb{P}}(\sigma^\varepsilon(x) > kT) \leq (1 - q^\varepsilon)^k, \quad k \in \mathbb{N}.
\]

This implies the following bound

\[
\sup_{x \in D} \mathbb{E} [\sigma^\varepsilon(x)] = \sup_{x \in D} T \int_0^\infty \bar{\mathbb{P}}(\sigma^\varepsilon(x) > Ts) ds \leq T \sum_{k=0}^\infty \mathbb{P}(\sigma^\varepsilon(x) > kT) \leq T \sum_{k=0}^\infty (1 - q^\varepsilon)^k = T \frac{q^\varepsilon}{1 - q^\varepsilon}.
\]

Since we have \( q^\varepsilon \geq e^{-\frac{\varphi + \frac{4}{e}}{\varepsilon}} \) for \( \varepsilon \in (0, \varepsilon_0) \) we obtain

\[
\sup_{x \in D} \mathbb{E} [\sigma^\varepsilon(x)] \leq T e^{-\frac{\varphi + \frac{4}{e}}{\varepsilon}}.
\]

Chebyshev’s inequality implies, for all \( x \in D \) and \( \varepsilon \in (0, \varepsilon_0) \),

\[
\bar{\mathbb{P}}(\sigma^\varepsilon(x) \geq e^{-\frac{\varphi + \frac{4}{e}}{\varepsilon}}) \leq e^{-\frac{\varphi + \frac{4}{e}}{\varepsilon}} \mathbb{E} [\sigma^\varepsilon(x)] \leq e^{-\frac{\varphi}{\varepsilon}}.
\]

(32)

Sending \( \varepsilon \to 0 \) we conclude the proof of the lower bound. \( \square \)

### 3.2 The lower bound in Theorem 3

Let \( x \in D \) and \( \rho > 0 \) such that \( \bar{B}_\rho(0) \subset D \). We keep the notation of the last subsection.

**Lemma 15.** For any \( x \in D \) and \( \rho > 0 \) such that \( \bar{B}_\rho(0) \subset D \) we have

\[
\lim_{\varepsilon \to 0} \bar{\mathbb{P}}(X^{\varepsilon,x}_{\bar{\sigma}^\varepsilon(x)} \in \bar{B}_\rho(0)) = 1.
\]

**Proof.** We fix \( \rho > 0 \) and \( x \in D \setminus \bar{B}_\rho(0) \). Otherwise the result is trivial. We note that there exists a certain \( T < \infty \) such that \( X^0_t \in \bar{B}_\rho(0) \) for all \( t \geq T \) due to Hypothesis \( \Lambda \). Thanks to Hypothesis \( \mathbb{I} \)

\[
\Delta := \rho \wedge \text{dist}\left(\{X^0_t \mid t \in [0,T]\}, D^c\right) > 0
\]

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and it follows that
\[ \left\{ X^\varepsilon,x_{\theta^e}(x) \in D^c \right\} \subset \left\{ \sup_{t \in [0,T \land \theta^e_p(x)]} |X^\varepsilon,x_t - X^0,x_t| > \frac{\Delta}{2} \right\}. \]

Hence, for some \( \lambda = \lambda_\varepsilon > 0, \varepsilon > 0, \) fixed below, it follows that
\[
\bar{P}\left( X^\varepsilon,x_{\theta^e}(x) \in D^c \right) \leq \bar{P}\left( \sup_{t \in [0,T \land \theta^e_p(x)]} |X^\varepsilon,x_t - X^0,x_t| > \frac{\Delta}{2} \right)
\leq \bar{P}\left( \sup_{t \in [0,T \land \theta^e_p(x)]} |X^\varepsilon,x_t - X^0,x_t| > \frac{\Delta}{2}, [X^\varepsilon,x_t - X^0,x_t]_{T \land \theta^e_p(x)} \leq \lambda \right) + \bar{P}\left( [X^\varepsilon,x_t - X^0,x_t]_{T \land \theta^e_p(x)} > \lambda \right).
\]

In this case the Bernstein-type inequality given by Theorem 3.3. of [23] reads as
\[
\bar{P}\left( \sup_{t \in [0,T \land \theta^e_p(x)]} |X^\varepsilon,x_t - X^0,x_t| > \frac{\Delta}{2}, [X^\varepsilon,x_t - X^0,x_t]_{T \land \theta^e_p(x)} \leq \lambda \right) \leq 2 \exp\left( - \frac{\Delta^2}{2 \lambda} \right). \tag{34}
\]

Hypotheses [C] - [D] yield some constant \( C = C(D, L) > 0 \) such that for \( \varepsilon > 0 \) small enough
\[
[X^\varepsilon,x_t - X^0,x_t]_{T \land \theta^e_p(x)} \leq C\varepsilon^2 \int_{0}^{T} |z|^{2p} N^\frac{p}{2}(ds, dz).
\]

Therefore, we obtain for \( \varepsilon > 0 \) small enough
\[
\bar{P}\left( [X^\varepsilon,x_t - X^0,x_t]_{T \land \theta^e_p(x)} > \lambda \right) \leq \frac{\varepsilon^2 C}{\lambda} \left[ \int_{0}^{T} \int_{\mathbb{R}^d} |z|^{2p} N^\frac{p}{2}(ds, dz) \right] \leq C\varepsilon^2 T \frac{\varepsilon}{\lambda}. \tag{35}
\]

where \( c_{\nu}^{2r} := \int_{\mathbb{R}^d} |z|^{2r} \nu(dz) < \infty \) due to the fact that \( \nu \) is a Lévy measure respecting the integrability condition [M]. Hence, choosing \( \lambda_\varepsilon = \varepsilon^\frac{1}{2} \), the inequalities (33) and (35) imply with (33) that
\[
\bar{P}\left( X^\varepsilon,x_{\theta^e_p(x)} \notin D \right) \leq 2 \exp\left( - \frac{\Delta^2}{2 \lambda} \right) + CT c_{\nu}^{2r} \varepsilon^\frac{1}{2}.
\]

Sending \( \varepsilon \to 0 \) we infer the desired result. \( \square \)

The proof of the following lemma repeats similar arguments used above and omit it.

**Lemma 16.** For any \( \rho > 0 \) and \( c > 0 \), there exists \( \xi(\rho) > 0 \) such that \( t \in [0, \xi(\rho)] \) implies
\[
\limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{x \in D} \bar{P}(\sup_{t \in [0, \xi(\rho)]} |X^\varepsilon,x_t - x| \geq \rho) < -c.
\]

**Lemma 17.** Let \( F \subset D^c \) closed. Then
\[
\limsup_{\rho \to 0} \varepsilon \ln \sup_{x \in B_{2\rho}(0)} \bar{P}(X^\varepsilon,x_{\theta^e_p} \in F) \leq - \inf_{z \in F} V(0, z).
\]

**Proof.** Fix \( \delta > 0 \) and \( V_F(\delta) := \min\{((\inf_{z \in F} V(0, z) - \delta), \frac{1}{\delta}) \}. \) By definition of \( V \), we conclude
\[
V(x, z) \leq V(x, y) + V(y, z) \text{ for all } x, y, z \in \mathbb{R}^d.
\]

By Lemma 12 there is \( \rho_0 > 0 \) such that for \( \rho \in (0, \rho_0) \) we have
\[
\inf_{z \in F, y \in B_{2\rho}(0)} V(y, z) \geq \inf_{z \in F} V(0, z) - \sup_{y \in B_{2\rho}(0)} V(0, y) \geq V_F(\delta).
\]
Lemma \[\text{14}\] provides a constant \(T > 0\) such that, for any \(\rho \in (0, \rho_0]\), one has
\[
\limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{y \in B_{2\rho}(0)} \hat{P}(\vartheta^\varepsilon_\rho(y) > T) \leq \limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{y \in B_{2\rho}(0) \setminus B_{2\rho}(0)} \hat{P}(\vartheta^\varepsilon_\rho(y) > T) < -V_F(\delta).
\]
We consider the following subset of \(\mathbb{D}([0, T], \mathbb{R}^d)\),
\[
\mathcal{A} := \{ \varphi \in \mathbb{D}([0, T], \mathbb{R}^d) \mid \varphi(s) \in F \text{ for some } s \in [0, T]\}.
\]
We have that \(\mathcal{A}\) is a closed set of \(\mathbb{D}([0, T], \mathbb{R}^d)\) for the Skorokhod topology. For a proof of this simple fact we refer to Lemma \[\text{19}\] in the Appendix. Proposition \[\text{10}\] implies that there exists \(\rho_0 > 0\) such that, for \(0 < \rho < \rho_0\),
\[
\limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{y \in B_{2\rho}(0)} \hat{P}(X_{\vartheta^\varepsilon_\rho} \in \mathcal{A}) \leq - \inf_{y \in B_{2\rho}(0)} \inf_{\varphi \in \mathcal{A}} \sum_{y, t} \varphi(t) \leq - \inf_{y \in B_{2\rho}(0), z \in F} V(y, z) \leq -V_F(\delta).
\]
In conclusion,
\[
\limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{x \in B_{2\rho}(0)} \hat{P}(X_{\vartheta^\varepsilon_\rho} \in F) \leq \limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{x \in B_{2\rho}(0)} \hat{P}(\vartheta^\varepsilon_\rho(x) < \infty) \leq \limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{x \in B_{2\rho}(0)} \hat{P}(\{\vartheta^\varepsilon_\rho(x) > \bar{T}\} \cup \{\vartheta^\varepsilon_\rho(x) \leq \bar{T}\}) \leq \limsup_{\varepsilon \to 0} \varepsilon \ln \left( \sup_{y \in B_{2\rho}(0)} \hat{P}(\vartheta^\varepsilon_\rho(y) > T) + \sup_{y \in B_{2\rho}(0)} \hat{P}(X_{\vartheta^\varepsilon_\rho} \in \mathcal{A}) \right) \leq -V_F(\delta).
\]
The result follows sending \(\delta \to 0\).
\[\square\]
\textbf{Theorem 5.} Let \(\delta > 0, x \in D\). Then we have
\[
\lim_{\varepsilon \to 0^+} \hat{P}(\sigma^\varepsilon(x) \leq e^{\frac{V_{-1}}{\varepsilon}}) = 0.
\]
\textit{Proof.} The proof is organized in four consecutive steps.

\textbf{Step 1.} We assume \(\bar{V} > 0\). Choose \(\rho > 0\) such that \(B_{3\rho}(0) \subset D\). Define recursively, for \(x \in D\) and \(k \in \mathbb{N}\),
\[
\zeta_k^x := 0,
\vartheta^x_{k,\rho} := \inf\{t \geq \zeta_k^x \mid X_t^{x,\varepsilon} \in B_{2\rho}(0) \cup D^c\},
\zeta_{k+1}^x := \begin{cases}
\infty & \text{if } X_t^{x,\varepsilon} \in D^c \\
\inf\{t \geq \vartheta^x_{k,\rho} \mid X_t^{x,\varepsilon} \in B_{3\rho}(0) \setminus \bar{B}_{2\rho}(0)\} & \text{if } X_t^{x,\varepsilon} \in \bar{B}_{\rho}(0)
\end{cases}
\]
(36)
Due to the way \((\zeta_k^x)_{k \in \mathbb{N}}\) and \((\vartheta^x_{k,\rho})_{k \in \mathbb{N}}\) were defined we have, for all \(k \in \mathbb{N}\),
\[
\vartheta^x_{k,\rho} \leq \zeta_k^x \leq \vartheta^x_{k+1,\rho}.
\]
The facts that \((\vartheta^x_{k,\rho})_{k \in \mathbb{N}}\) is a sequence of stopping times and, for every \(\varepsilon \in (0, 1]\), \((X_t^{x,\varepsilon})_{t \geq 0}\) is a strong Markov process imply that \((X_t^{x,\varepsilon})_{t \geq 0}\) is a Markov chain, with the convention \(X_t^{x,\varepsilon} := X_{\sigma^\varepsilon(x)}^x\) if \(\vartheta^x_{k,\rho} = \infty\). We observe that, for every \(\varepsilon \in (0, 1]\) and \(x \in D\), \(\sigma^\varepsilon(x) = \vartheta^x_{k,\rho}\) for some \(k \in \mathbb{N}\).

\textbf{Claim 2.} For any \(x \in D, \varepsilon \in (0, 1], T > 0\) and \(k \in \mathbb{N}\) arbitrary it follows
\[
\{\sigma^\varepsilon(x) \geq \vartheta^x_{k,\rho}\} \cap \bigcap_{m=1}^k \{\zeta_m^x - \vartheta_{m-1,\rho}^x \geq T\} \subset \{\sigma^\varepsilon(x) \geq kT\}
\]
(37)
Proof. The inclusion of events \( [37] \) follows from noting that \( \sigma^x(x) \geq \vartheta_{k,\rho}^x \) and \( \xi_m^x - \vartheta_{m-1,\rho}^x \geq T \), for all \( m = 1, \ldots, k \), imply

\[
\vartheta_{k,\rho}^x = \sum_{m=1}^{k} (\vartheta_{m,\rho}^x - \vartheta_{m-1,\rho}^x) + \vartheta_{0,\rho}^x \geq \sum_{m=1}^{k} (\xi_m^x - \vartheta_{m,\rho}^x) \geq kT.
\]

\( \square \)

Fix \( \delta > 0 \). We set \( k(\varepsilon) = \left\lfloor \frac{4}{\varepsilon} \right\rfloor + 1 \) for some \( T > 0 \) which we determine below. For any \( x \in D \) this yields

\[
P(\sigma^x(x) \leq e^{-\frac{\varepsilon}{2}}) \leq P(\sigma^x(x) \leq kT).
\]

The inclusion of events \( [37] \) implies that

\[
\{\sigma^x(x) \leq k(\varepsilon)T\} \subset \{\sigma^x(x) \leq \vartheta_{0,\rho}^x\} \cup \bigcup_{m=1}^{k(\varepsilon)} \{\sigma^x(x) \leq \vartheta_{m,\rho}^x\} \cup \{\xi_m^x - \vartheta_{m-1,\rho}^x \leq T\}. \tag{38}
\]

Step 2. Using Lemma \([17] \) there exists \( \varepsilon_0 > 0 \) such that for \( \varepsilon \in (0, \varepsilon_0] \) we have

\[
\limsup_{x \to 0} \sup_{x \in B_{3\varepsilon}(0)} P(X_{\vartheta_{\rho,\rho}^x}^x \in D^c) \leq -\bar{V} + \frac{\delta}{2}.
\]

Let \( 0 < \rho < \rho_0 \) arbitrary for some \( \rho_0 > 0 \) fixed below. For any \( y \in D \), the estimate above yields, due to the strong Markov property, the following

\[
\sup_{x \in D} P(\sigma^x(x) \leq \vartheta_{m,\rho}^x) \leq \sup_{x \in D} P(X_{\vartheta_{\rho,\rho}^x}^x \in D^c) \leq \sup_{y \in B_{\rho}(0)} P(X_{\vartheta_{\rho,\rho}^x}^x \in D^c) \leq e^{-\frac{\varepsilon_0}{\rho}}. \tag{39}
\]

We fix the time \( T = \xi(\rho) > 0 \) accordingly to Lemma \([18] \). Then, there exists \( \rho_0 > 0 \) such that, for \( \rho \leq \rho_0 \), one has

\[
\sup_{x \in D} P(\xi^x_k - \vartheta_{k-1,\rho}^x \leq T) \leq \sup_{x \in B_{\rho}(0)} P(\sup_{t \in [0,T]} |X_{\vartheta_{\rho,\rho}^x}^x - x| \geq \rho) \\
\leq \sup_{x \in D} P(\sup_{t \in [0,T]} |X_{\vartheta_{\rho,\rho}^x}^x - x| \geq \rho) \leq e^{-\frac{\varepsilon_0}{\rho}}. \tag{40}
\]

Hence, for any \( k \in \mathbb{N} \) and \( x \in D \), \([38], [39] \) and \([40] \) yield

\[
P(\sigma^x(x) \leq kT) \leq P(\sigma^x(x) \leq \vartheta_{0,\rho}^x) + \sum_{m=1}^{k} \left( P(\sigma^x(x) \leq \vartheta_{m,\rho}^x) + P(\xi_m^x - \vartheta_{m-1,\rho}^x \leq T) \right) \\
\leq P(\sigma^x(x) \leq \vartheta_{0,\rho}^x) + 2ke^{-\frac{\varepsilon_0}{\rho}}.
\]

Step 3. Due to Lemma \([15] \) we have, for all \( x \in D \),

\[
P(\sigma^x(x) \leq e^{-\frac{\varepsilon_0}{\rho}}) \leq P(\sigma^x(x) \leq kT) \leq P(X_{\vartheta_{0,\rho}^x}^x \notin B_{\rho}(0)) + \frac{4}{T} e^{-\frac{T}{\varepsilon}} \to 0 \quad \text{as } \varepsilon \to 0. \tag{41}
\]

Chebyshev’s inequality implies that, for some \( c(T) > 0 \), we have

\[
\mathbb{E}[\sigma^x(x)] \geq e^{-\frac{\varepsilon_0}{\rho}} P(\sigma^x(x) \geq e^{-\frac{\varepsilon_0}{\rho}}) \geq c(T) e^{-\frac{T}{\varepsilon}}.
\]
Step 4. We treat the case $\bar{V} = 0$ in what follows. Let $\delta > 0$ and $x \in D$. Choose $\rho > 0$ such that $B_{\rho}(0) \subset D$. Assume $c > 0$ and by Lemma 15 and Lemma 17 combined with the strong Markov property, we can choose $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$

$$
\mathbb{P}(\sigma_{\varepsilon}(x) > e^{-\frac{1}{\varepsilon}}) \geq \mathbb{P}(X^x_{\sigma_{\varepsilon}^x} \in B_{\rho}(0)) \inf_{y \in B_{\rho}(0)} \mathbb{P}(\sup_{t \in [0, \xi(c, \rho)]} |X^y - y| \leq \rho) \to 1,
$$

as $\varepsilon \to 0$, which concludes the proof. \hfill \square

3.3 The exit location in Theorem 8

The proof of the statement 2. of Theorem 8 goes along the same line of reasoning that was done in the Brownian case and that is extensively documented, under different hypotheses, in the literature. We refer the reader for example to Theorem 4.2.4 in [27] in a more general setting for the deterministic dynamical system [22] but with an additive Brownian perturbation and to Theorem 5.7.11 in [21] for a multiplicative Brownian perturbation of (2) under hypotheses for the deterministic dynamical system (2) but with an additive Brownian perturbation and to Theorem 3.3 of [29] under hypotheses for the deterministic system that are closer to the ones we assume in this work. Our result is derived with analogous arguments used to prove the second statement of Theorem 2.4.6 in [52] (pp. 88-90). For this reason we omit the proof and refer the reader to [52].

4 Appendix

4.1 Proof of Lemma 6

Let $\nu \in \mathfrak{M}$ satisfying Hypothesis 6. We recall that $\nu_1$ is the restriction of the measure $\nu$ to $\mathcal{B}(B^c_1(0))$.

1. We start with the proof of (15). Let $g \in \mathcal{S}^M$. We have the immediate decomposition

$$
\int_0^T \int_{\mathbb{R}^d} |z|^nu(s, z)\nu(dz)ds \leq \int_0^T \int_{|z|\leq 1} |z|^nu(s, z)\nu(dz)ds + \int_0^T \int_{|z|> 1} |z|^nu(s, z)\nu(dz)ds. \tag{42}
$$

We estimate the first integral in the right hand side of (42) as follows. Young’s inequality reads for any $a, b \geq 0$ that $ab \leq e^a + b \ln b - b$. This implies immediately that

$$
\int_0^T \int_{|z|\leq 1} |z|^nu(s, z)\nu(dz)ds \leq \int_0^T \int_{|z|\leq 1} |z|^n(e + \ell(g(s, z))\nu(dz)ds \\
\leq eTc_\nu^a + \int_0^T \int_{|z|\leq 1} |z|^n\ell(g(s, z))\nu(dz)ds \leq eTc_\nu^a + M < \infty, \tag{43}
$$

since $\nu$ is a Lévy measure ($c_\nu^a := \int_{|z|\leq 1} |z|^n < \infty$). We proceed with the estimate of the second integral in the right hand side of (42). Young’s inequality for the entropy function $\ell(b) = b \ln b - b + 1, b \geq 0$ implies $ab \leq e^{a\sigma} + \frac{1}{\sigma^2}\ell(b)$, for all $a, b \geq 0$ and $\sigma \geq 1$. This together with $\nu_1(\mathbb{R}^d) < \infty$ yields

$$
\int_0^T \int_{\mathbb{R}^d} |z|^ng(s, z)\nu_1(dz)ds \leq e^{a\sigma}\nu_1(\mathbb{R}^d)T + \int_0^T \int_{\mathbb{R}^d} \ell(|z|^ng(s, z))\nu_1(dz)ds. \tag{44}
$$

We define the measurable set $E := \{(s, z) \in [0, T] \times \mathbb{R}^d \mid |z|^ng(s, z) \geq 1\}$ and split the remaining term

$$
\int_{[0, T] \times \mathbb{R}^d} \ell(|z|^ng(s, z))\nu_1(dz)ds \\
= \int_{([0, T] \times \mathbb{R}^d) \cap E} \ell(|z|^ng(s, z))\nu_1(dz)ds + \int_{([0, T] \times \mathbb{R}^d) \cap E^c} \ell(|z|^ng(s, z))\nu_1(dz)ds. \tag{45}
$$
On $E^c$ we have $|z|^u g(s, z) < 1$ which implies $\ell(|z|^u g(s, s)) \leq 1$. Therefore,

$$
\int_{(0, T] \times \mathbb{R}^d \cap E^c} \ell(|z|^u g(s, z)) \nu_1(dz) ds \leq \nu_1(\mathbb{R}^d) T < \infty. \quad (46)
$$

On $E$ we have $|z|^r g(s, z) \geq 1$. Young’s $L^p$ inequality, the monotonicity and the convexity of $\ell$ in $[1, +\infty)$ yield, for any conjugate exponents $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$
\ell(|z|^u g(s, z)) \leq \ell\left(\frac{1}{p} |z|^pu + \frac{1}{q} (g(s, z))^q\right) \leq \frac{1}{p} \ell(|z|^pu) + \frac{1}{q} \ell((g(s, z))^q).
$$

Consequently,

$$
\int_{(0, T] \times \mathbb{R}^d \cap E} \ell(|z|^u g(s, z)) \nu_1(dz) ds \\
\leq \frac{1}{p} \int_{(0, T] \times \mathbb{R}^d \cap E} \ell(|z|^pu) \nu_1(dz) ds + \frac{1}{q} \int_{(0, T] \times \mathbb{R}^d \cap E} \ell((g(s, z))^q) \nu_1(dz) ds. \quad (47)
$$

Due to Fatou’s lemma we have

$$
\limsup_{q \to 1^+} \frac{1}{q} \ell(g(s, z))^q \nu_1(dz) ds \leq \int_{(0, T] \times \mathbb{R}^d \cap E} \ell(g(s, z))^q \nu_1(dz) ds \leq M.
$$

This implies the existence of $q_0 > 1$ such that

$$
\frac{1}{q_0} \int_{(0, T] \times \mathbb{R}^d \cap E} \ell(g(s, z)) (q_0) \nu_1(dz) ds \leq M.
$$

Hence the corresponding convex conjugate is $p_0 := \frac{q_0}{q_0 - 1}$ and

$$
\int_{(0, T] \times \mathbb{R}^d \cap E} \ell(|z|^u g(s, z)) \nu(dz) ds \\
\leq \frac{1}{p_0} \int_{(0, T] \times \mathbb{R}^d \cap E} \ell(|z|^{p_0 u}) \nu_1(dz) ds + \frac{1}{q_0} \int_{(0, T] \times \mathbb{R}^d \cap E} \ell((g(s, z))^{q_0}) \nu_1(dz) ds \\
\leq \int_{(0, T] \times \mathbb{R}^d \cap E} \ell(|z|^{p_0 u}) \nu_1(dz) ds + M. \quad (48)
$$

Using the asymptotic properties of $\ell$ we have that there exists $\mathcal{R} > 0$ such that $\ell(|z|^{p_0 u}) \leq |z|^{p_0 u + 1}$ on $\{|z| > \mathcal{R}\}$. Therefore

$$
\int_{[0, T] \times \mathbb{R}^d} \ell(|z|^{p_0 u}) \nu_1(dz) ds \\
\leq \int_{[0, T] \times \{|z| \geq \mathcal{R}\}} |z|^{p_0 u + 1} \nu_1(dz) ds + \int_{[0, T] \times \{|z| < \mathcal{R}\}} \ell(|z|^{p_0 u}) \nu_1(dz) ds. \quad (49)
$$

Since $\ell$ is bounded on $\{|z| \leq \mathcal{R}\}$ and $\nu_1(\mathbb{R}^d) < \infty$ the second integral is finite. Using first the Cauchy-Schwarz inequality, secondly the generalized spherical change of coordinates in $\mathbb{R}^d$ and lastly the change of variables $t = |z|^u$ we obtain due to Remark 33 the estimate

$$
\int_{[0, T] \times \mathbb{R}^d} |z|^{p_0 u + 1} \nu_1(dz) ds < \infty. \quad (50)
$$

Collecting (43) - (49) and (50) and the result (15) follows.
2. We fix $M > 0$, $g \in \mathcal{S}^M$ and $\delta' > 0$. Let $t \leq t' \leq T$. Let $\delta > 0$ such that $|t - t'| \leq \delta$. For any $\lambda \geq 1$ it holds
\[
\int_t^{t'} \int_{|z| \leq 1} |z|^u g(s, z) - 1|\nu(dz) ds \leq \int_t^{t'} \int_{|z| \leq 1} |z|^u g(s, z)\nu(dz) ds + |t' - t|c_\nu^u
\]
\[
\leq e^{\lambda|t - t'|} c_\nu^u + \frac{1}{\lambda} \int_t^{t'} \int_{|z| \leq 1} \ell(g(s, z))\nu(dz) ds + |t' - t|c_\nu^u
\]
\[
\leq |t - t'|((c_\nu^u(e^\lambda + 1)) + \frac{\lambda}{M}).
\]

Let $\lambda \geq 1$ such that $\frac{1}{M} < \frac{\delta'}{6}$. Now let $\delta > 0$ sufficiently small such that $\delta < \frac{\delta'}{6(c_\nu^u(e^\lambda + 1))}$. Therefore the choices of $\lambda$ and $\delta$ that were fixed above and (51) yield
\[
\int_t^{t'} \int_{|z| \leq 1} |z|^u g(s, z) - 1|\nu(dz) ds \leq \frac{\delta'}{3}.
\]

Due to Young's inequality, for any $\sigma > 1$ it holds, for all $0 \leq t \leq t' \leq T$,
\[
\int_t^{t'} \int_{\mathbb{R}^d} |z|^u g(s, z) - 1|\nu_1(dz) ds
\]
\[
\leq \int_t^{t'} \int_{\mathbb{R}^d} |z|^u g(s, z)\nu_1(dz) ds + \int_t^{t'} \int_{\mathbb{R}^d} |z|^u \nu_1(dz) ds
\]
\[
\leq c_{\nu_1}^u|t' - t| + e^\sigma \nu_1(\mathbb{R}^d)|t' - t| + \frac{1}{\sigma} \int_t^{t'} \int_{\mathbb{R}^d} \ell(|z|^u g(s, z))\nu_1(dz) ds,
\]
where $c_{\nu_1}^u := \int_{\mathbb{R}^d} |z|^u \nu_1(dz) < \infty$, due to the exponential integrability $\|B\|$ of $\nu_1$. Combining the statements (45), (46), (48), (49) and (50) we have
\[
\int_0^T \int_{\mathbb{R}^d} \ell(|z|^u g(s, z))\nu_1(dz) ds < \infty.
\]

We choose $\sigma > 0$ such that
\[
\frac{1}{\sigma} \int_{[0, T] \times \mathbb{R}^d} \ell(|z|^u g(s, z))\nu_1(dz) ds < \frac{\delta'}{3}.
\]

If $\delta > 0$ satisfies additionally that
\[
\delta < \frac{\delta'}{3(c_{\nu_1}^u + e^\sigma \nu_1(\mathbb{R}^d))},
\]
we obtain that the estimates (53), (54) and (55) imply, for any $0 \leq t \leq t' \leq T$ such that $|t - t'| < \delta$,
\[
\int_t^{t'} \int_{\mathbb{R}^d} |z|^u g(s, z) - 1|\nu(dz) ds < \delta'.
\]

This concludes the proof of (10).

4.2 Proof of Proposition $7$

For convenience of notation we drop the dependence of $\hat{X}_{t,x}^\varepsilon$ on $x$. For every $\varepsilon \in (0, 1]$ let $R(\varepsilon) > 0$ such that $\mathcal{R}(\varepsilon) \to \infty$ and $\varepsilon\mathcal{R}^2(\varepsilon) \to 0$ as $\varepsilon \to 0$. For example $\mathcal{R}(\varepsilon) := |\ln \varepsilon|$ for every $\varepsilon \in (0, 1]$ will do the job. Let us define the $(\mathcal{F}_t)_{t \in [0, T]}$-stopping time
\[
\hat{\tau}_{\mathcal{R}(\varepsilon)} := \inf\{t \geq 0 \mid \hat{X}_{t,x}^\varepsilon \notin B_{\mathcal{R}(\varepsilon)}(0)\}.
\]
Due to the definition of \( \tilde{\mathcal{R}}_{\varepsilon} \) it is immediate that
\[
\mathbb{P}\left( \sup_{t \in [0,T]} |\tilde{X}^\varepsilon_t| > \mathcal{R}(\varepsilon) \right) \leq \mathbb{P}\left( \sup_{t \in [0,\tilde{\mathcal{R}}_{\varepsilon}]} |\tilde{X}^\varepsilon_t| > \mathcal{R}(\varepsilon) \right).
\]
We observe that for every \( \varepsilon > 0 \) the process \((\tilde{X}^\varepsilon_t)_{t \in [0,T]}\) is a locally square integrable martingale. Therefore we use the Bernstein-type inequality given by Theorem 3.3 of [23] and infer for some parameter \( \lambda = \lambda_\varepsilon > 0 \) that is fixed below,
\[
\mathbb{P}\left( \sup_{t \in [0,\tilde{\mathcal{R}}_{\varepsilon}]} |\tilde{X}^\varepsilon_t| > \mathcal{R}(\varepsilon) \right) \leq \mathbb{P}\left( \sup_{t \in [0,\tilde{\mathcal{R}}_{\varepsilon}]} |\tilde{X}^\varepsilon_t| > \mathcal{R}(\varepsilon), |\tilde{X}^\varepsilon_t|_{\tilde{\mathcal{R}}_{\varepsilon}} < \lambda \right)
+ \mathbb{P}\left( |\tilde{X}^\varepsilon|_{\tilde{\mathcal{R}}_{\varepsilon}} < \lambda \right)
\leq 2 \exp\left( - \frac{\mathcal{R}^2(\varepsilon)}{2 \lambda_\varepsilon} \right) + \mathbb{P}\left( |\tilde{X}^\varepsilon|_{\tilde{\mathcal{R}}_{\varepsilon}} > \lambda \right).
\]
(56)
For every \( \varepsilon > 0 \) the quadratic variation of the process \((\tilde{X}^\varepsilon_t)_{t \in [0,T]}\) is given, for every \( t \in [0,T] \), by
\[
|\tilde{X}^\varepsilon_t| = \varepsilon^2 \int_0^t \int_{\mathbb{R}^d} |G(\tilde{X}^\varepsilon_{\tau'},z)|^2 N_{\tilde{g}}(ds,dz).
\]
Due to Hypothesis \([\mathbb{C}]\) and Chebyshev’s inequality, the second probability of the last estimate of (56) is estimated as follows for \( \varepsilon \in (0,\varepsilon_0) \) for some \( \varepsilon_0 > 0 \) small enough,
\[
\mathbb{P}\left( |\tilde{X}^\varepsilon|_{\tilde{\mathcal{R}}_{\varepsilon}} > \lambda \right) \leq \mathbb{P}\left( \varepsilon^2 \int_0^{\tilde{\mathcal{R}}_{\varepsilon}} \int_{\mathbb{R}^d} |G(\tilde{X}^\varepsilon_{\tau'},z)|^2 N_{\tilde{g}} > \lambda \right)
\leq \mathbb{P}\left( L^2 \varepsilon(1 + \mathcal{R}(\varepsilon))^2 \int_0^T \int_{\mathbb{R}^d} |z|^2 \phi_\varepsilon(s,z) \nu(ds)ds > \lambda_\varepsilon \right)
\leq \frac{CL\varepsilon(1 + \mathcal{R}(\varepsilon))^2}{\lambda_\varepsilon},
\]
(57)
where \( \bar{C} := \sup_{g \in \mathcal{G}} \int_0^T \int_{\mathbb{R}^d} |z|^2 r g(s,z) \nu(ds)ds < \infty \) by \([14]\) of Proposition \([\mathbb{I}]\). Hence choosing \( \lambda = \lambda_\varepsilon = \mathcal{R}(\varepsilon) \) for every \( \varepsilon > 0 \) and combining (56) and (57) yields, for some \( C > 0 \),
\[
\mathbb{P}\left( \sup_{t \in [0,T]} |\tilde{X}^\varepsilon_t| > \mathcal{R}(\varepsilon) \right) \leq 2e^{-\frac{1}{2} \mathcal{R}(\varepsilon)} + C \varepsilon \mathcal{R}(\varepsilon) \to 0 \quad \text{as} \ \varepsilon \to 0.
\]
4.3 Proof of Lemma \([\mathbb{L}]\)
The two statements follow from a simple observation of the following fact. For any fixed \( M > 0 \), using Hypothesis \([\mathbb{E}]\) let \( g \in \mathcal{G}^M \) be a control function, \( \xi(\rho) \to 0 \) as \( \rho \to 0 \) a small time horizon and \( \Phi \in C([0,T],\mathbb{R}^d) \) the solution of the associated controlled ODE \([\mathbb{I}]\). Since for the function \( \ell(b) = b \ln b - b + 1, b \geq 0 \) we have
\[
\int_0^T \int_{\mathbb{R}^d} \ell(g(s,z))\nu(ds)ds \leq M.
\]
The monotone convergence theorem implies
\[
\lim_{\rho \to 0} \int_0^{\xi(\rho)} \int_{\mathbb{R}^d} \ell(g(s,z))\nu_\rho(ds)ds = \int_0^T \int_{\mathbb{R}^d} \lim_{\rho \to 0} 1_{[0,\xi(\rho)]}(s)\ell(g(s,z))\nu(ds)ds = 0.
\]
Therefore, given \( \delta > 0 \) there exists \( \rho_0 > 0 \) small enough such that \( \rho \in (0,\rho_0) \) implies
\[
V(x,y,t) \leq \xi(\rho)(g) \leq \delta.
\]
4.4 Topological properties of Skorokhod space used in Section 3

**Lemma 18.** Given $t > 0$, $D \subset \mathbb{R}^d$ a bounded domain, $\rho > 0$ and the sets

$$\mathcal{G}_t := \left\{ \Phi \in \mathbb{D}([0, t], \mathbb{R}^d) : \Phi(s) \in \overline{D \setminus B_{\rho}(0)} \text{ for all } s \in [0, t] \right\},$$

$$\tilde{\mathcal{G}}_t := \left\{ \Phi \in \mathbb{D}([0, t], \mathbb{R}^d) : \Phi(s) \in \overline{D - B_{\rho}(0)} \text{ for all } s \in [0, t],\right. $n$

$$\left. \text{except in a countable number of points} \right\},$$

we have that $\tilde{\mathcal{G}}_t = \mathcal{G}_t$ and $\tilde{\mathcal{G}}_t$ is a closed set in $\mathbb{D}([0, t], \mathbb{R}^d)$ with respect to the Skorokhod topology.

**Proof.**

**Step 1:** We prove that $\tilde{\mathcal{G}}_t$ is closed in $\mathbb{D}([0, t], \mathbb{R}^d)$ with respect to the Skorokhod topology. Let $(\Phi_n)_{n \in \mathbb{N}} \subset \tilde{\mathcal{G}}_t$ such that $d_{\mathcal{I}}(\Phi_n, \Phi) \to 0$ as $n \to \infty$, for some $\Phi \in \mathbb{D}([0, t], \mathbb{R}^d)$. We denote $(s_k)_{k \in \mathbb{N}}$ the countable set of discontinuity points of $\Phi$. For each $n \in \mathbb{N}$ we denote $(t^n_k)_{k \in \mathbb{N}}$ the countable set such that

$$\Phi_n(s) \in \overline{D \setminus B_{\rho}(0)} \quad \text{for all } s \in [0, t] - (t^n_k)_{k \in \mathbb{N}}.$$

For all $s \in [0, t] \setminus \left( \bigcup_{n=1}^{\infty} (t^n_k)_{k \in \mathbb{N}} \cup (s_k)_{k \in \mathbb{N}} \right)$ it is a standard property of càdlàg functions that

$$\Phi_n(s) \to \Phi(s) \quad \text{as } n \to \infty.$$

Since $\overline{D \setminus B_{\rho}(0)}$ is a compact set of $\mathbb{R}^d$, $\Phi(s) \in \overline{D \setminus B_{\rho}(0)}$, which concludes the proof that $\tilde{\mathcal{G}}_t$ is closed in $\mathbb{D}([0, T], \mathbb{R}^d)$.

**Step 2:** We prove next that $\tilde{\mathcal{G}}_t = \mathcal{G}_t$. The inclusion $\tilde{\mathcal{G}}_t \supset \mathcal{G}_t$ is obvious. Let $\Phi \in \tilde{\mathcal{G}}_t$. If there exists $s \in [0, t]$ such that $\Phi(s) \notin \text{cl} (\overline{D \setminus B_{\rho}(0)})$, since $(\overline{D})^c$ and $B_{\rho}(0)$ are open sets of $\mathbb{R}^d$, by right-continuity of $\Phi$, there exists $\delta > 0$ such that

$$\Phi[s, s + \delta) \subset (\overline{D})^c \cup B_{\rho}(0),$$

which violates $\Phi \in \tilde{\mathcal{G}}_t$.

**Lemma 19.** For any closed set $F \subset \mathcal{B}(\mathbb{R}^d)$ and $t > 0$ we consider the following subset of $\mathbb{D}([0, t], \mathbb{R}^d)$,

$$\mathcal{A} := \{ \varphi \in \mathbb{D}([0, t], \mathbb{R}^d) \mid \varphi(s) \in F \text{ for some } s \in [0, t] \}. $$

Then the set $\mathcal{A}$ is closed in $\mathbb{D}([0, t], \mathbb{R}^d)$ with respect to the Skorokhod topology.

**Proof.** Let $(\varphi_n)_{n \in \mathbb{N}}$ a sequence of elements of $\mathcal{A}$ and $\varphi \in \mathbb{D}([0, t], \mathbb{R}^d)$ such that $d_{\mathcal{I}}(\varphi_n, \varphi) \to 0$ as $n \to \infty$. For every $n \in \mathbb{N}$, let $s_n \in [0, t]$ such that $\varphi_n(s_n) \in F$. By right continuity of $\varphi_n$, there exists $\delta_n > 0$ such that $\varphi_n([s_n, s_n + \delta_n)) \subset F$. For every $n \in \mathbb{N}$, we denote $I_n := [s_n, s_n + \delta_n)$. For every $n \in \mathbb{N}$ let $(t^n_k)_{k \in \mathbb{N}}$ be the set of discontinuities of $\varphi$ in $I_n$. Therefore the fact that, for every $n \in \mathbb{N}$, $\varepsilon_n$ and $\varphi$ are càdlàg implies

$$\varphi_n(r) \to \varphi(r), \quad \text{for all } r \in \bigcup_{n \in \mathbb{N}} (I_n - (t^n_k)_{k \in \mathbb{N}}).$$

Since $F$ is a closed subset of $\mathbb{R}^d$, $\varphi(r) \in F$, for all $r \in \bigcup_{n \in \mathbb{N}} (I_n - (t^n_k)_{k \in \mathbb{N}})$. This proves that $\varphi \in \mathcal{A}$ and that $\mathcal{A}$ is closed in $\mathbb{D}([0, T], \mathbb{R}^d)$ with respect to the Skorokhod topology.

\[\square\]
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