Fractal vortex structure in the lattice model of superconductors

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Abstract

The problem of the vortex structure in the lattice model of superconductors has been reduced to the nonlinear map problem characteristic for the fractal theory.
I. INTRODUCTION

The lattice magnetic translation symmetry of superconductors near the upper critical magnetic field was shown to be crucially important for the critical thermodynamic properties \([1], [2], [3]\). Attempts to construct the incommensurate vortex structure were undertaken in \([4]\). The small parameter \(\beta = \Phi / \Phi_0\) (where \(\Phi = Ha^2\) and \(\Phi_0 = \frac{\hbar c}{2e}\) are respectively the magnetic flux through the lattice plaquette \(a^2\) and the London quantum of magnetic flux) was essentially used there that predetermined very weak dependence of the sample free energy on the vortex configuration. Here I formulate this problem in a different way, so that smallness of \(\beta\) will be not assumed.

II. BASIC EQUATION

Confining myself here by the mean field approximation I consider the two-dimensional model. Apart from neglecting of fluctuations this means that an entanglement of vortices will not be accounted for in this paper. The free energy functional for the lattice model reads:

\[
F = \sum_{n,m} J_{nm,nm} \exp \left[ i \frac{2e}{\hbar c} \int d\mathbf{l} \cdot \mathbf{A} \right] \phi_{nm}^* \phi_{nm} + \sum_{nm} \left[ \tau \phi_{nm} \phi_{nm} + g |\phi_{nm}|^2 \phi_{nm} \right],
\]

where \(J\) is the tunneling integral, the lattice site coordinates can be written as \(x = ma, y = na, \phi_{nm}\) stands for the complex conjugated order parameter, \(\mathbf{A}\) is the vector potential, integration of it is carried out along the straight line connecting the sites \(nm\) and \(n'm'\), \(\tau = \alpha \frac{T - T_c}{T_c}\). Differentiating eq. \((1)\) with respect to \(\phi_{nm}\) we obtain the lattice Ginzburg-Landau (LGL) equation for a superconductor at a strong magnetic field:

\[
\sum_{n'm'} J_{nm,n'm'} \exp \left[ i \frac{2e}{\hbar c} \int d\mathbf{l} \cdot \mathbf{A} \right] \phi_{n'm'} + \tau \phi_{nm} + g |\phi_{nm}|^2 \phi_{nm} = 0
\]

Choosing the Landau gauge \(\mathbf{A} = e_y H x\) and considering the simple square lattice case within the tight-binding approximation (with \(J\) standing for the nearest-neighbor tunneling integral) we can write eq. \((1)\) in the form

\[
J \{ \phi_{m+1,n} + \phi_{m-1,n} + \phi_{m,n+1} \exp [-i2\pi \alpha m] + \phi_{m,n-1} \exp [i2\pi \alpha m] \} + \\
\tau \phi_{nm} + g |\phi_{nm}|^2 \phi_{nm} = 0.
\]
Eq. (3) can be transformed to the simpler form. Notice that eq. (3) is a nonlinear equation; therefore, the Fourier transforming is not particularly useful since it will convert this equation into an integral-difference one. However, the specific form of nonlinearity characteristic for the Ginzburg-Landau equation allows us to make a substitution that is usually done in the theory of the linear Harper equation [5]:

$$\phi_{mn} = u_m \exp (ikm).$$  \hspace{1cm} (4)

Then eq. (3) takes the form

$$u_{m+1} + u_{m-1} + 2 \cos (2\pi m \beta - \kappa) u_m + \tau u_m + g |u_m|^2 u_m = 0.$$ \hspace{1cm} (5)

Eq. (3) can be named the nonlinear Harper equation (NHE) or the Harper-Ginzburg-Landau (HGL) equation. The rest of the paper will be devoted to a discussion of possible consequences of this equation.

III. ANALYSIS OF THE NONLINEAR HARPER EQUATION

Eq. (3) is very peculiar one. It contains two mechanisms, either of which can lead to forming of irregular self-similar structures that have fractal properties. If we shall proceed in the spirit of the Abricosov vortex structure theory, we have firstly to linearize the equation.

$$u_{m+1} + u_{m-1} + 2 \cos (2\pi m \beta - \kappa) u_m + \tau u_m = 0,$$ \hspace{1cm} (6)

This equation differs from the linear Harper equation [4] in the same way as the Ginzburg-Landau one for a superconductor at the magnetic field differs from the Schroedinger equation for a charged particle at the magnetic field. Therefore, the line $H_{c2}$ is determined by the condition $\tau = \epsilon_l$, where $\epsilon_l$ is the Harper multiband spectrum lower edge. The dimensional crossover takes place near the lowest band bottom at $\beta$ rational [1]. The flux structure is completely pinned by the crystal in this case. A subsequent decrease of temperature or magnetic field, i.e. an increase of $|\tau|$, leads to appearance of fractal structures. The Harper operator spectrum is known to form a Cantor set at irrational $\beta$ and, therefore, the dimensional crossover is absent in this case. Scaling properties of this operator have been studied using the renormalization group technique [6, 7]. Exact solutions with a use of the Bethe Ansatz and quantum groups can be found in [8]. Multifractal properties of the
Harper operator eigenfunctions and eigenvalues are discussed basing on numeric calculations and on the Bethe Ansatz in the work by Wiegmann with co-authors [9]. Interesting results have been also obtained in [10]. Unfortunately, the Bethe Ansatz rigorous approach is not extended to the nonlinear case; it is a challenge now for the mathematical physics community. An approximate solution of the linearized Harper equation can be presented as a chaotic distribution of the Wannier-type functions, which can be considered as nucleation centres of the superconducting state. The order parameter amplitude can be found perturbatively in this case. On the other hand, Eq. (5) can be written in the form of the two-dimensional nonlinear map on the complex field:

\[
\begin{pmatrix}
u_{m+1} \\ u_m
\end{pmatrix} = \begin{pmatrix} -\tau - g |u_m|^2 - 2 \cos(2\pi m \beta - \kappa) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_m \\ u_{m-1} \end{pmatrix}.
\] (7)

IV. CONCLUSION

The main inference from the work is that the fractal flux structure is controlled by two-dimensional maps on the complex field enumerated by the whole real numbers. This means that the two-dimensional flux lattice in the crystal lattice can be completely specified establishing the nucleation centre distribution along one say real axis. In opposite to the continuum model case, this distribution is not periodic in general. The following distribution can be taken as a first approximation

\[
\phi(x, y) = C \exp \left( -\frac{x^2 + y^2}{2l_H^2} \right) \times 
\sum_{n=-\infty}^{\infty} \exp \left[ -\pi (n + u_n)^2 + \frac{\sqrt{2\pi i} (x + iy)}{l_H} (n + u_n) \right],
\] (8)

where \(u_n\) are determined by some proper discrete map taking fractal effects into account. Notice that this sum is reduced at the condition \(u_m = 0\) into the elliptic theta function so that this formula can be considered as its natural generalization on special number fields similarly to introduction of the basis functions, for instance. Along with the discussed above map, one can try the Chirikov standard map, which is equivalent to the Frenkel-Kontorova
model that is widely used for description of the incommensurability effects:

\[
\begin{align*}
I_{n+1} &= I_n - \frac{2\pi V_0}{\lambda b} \sin \left(2\pi y_n/b\right) \\
y_{n+1} &= y_n + a + I_{n+1}
\end{align*}
\] (9)

or, if the interaction between vortices must be taken into account, the generalized standard map

\[
\begin{align*}
I_{n+1} &= I_n - \frac{2\pi V_0}{\lambda b} \sin \left(2\pi y_n/b\right) \\
y_{n+1} &= y_n + a - \frac{1}{\beta} \ln\left(1 - \beta I_{n+1}\right)
\end{align*}
\] (10a)

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