ADMISSIBLE POISSON BIALGEBRAS

JINTING LIANG, JIEFENG LIU, AND CHENGMING BAI

Abstract. An admissible Poisson algebra (or briefly, an adm-Poisson algebra) gives an equivalent presentation with only one operation for a Poisson algebra. We establish a bialgebra theory for adm-Poisson algebras independently and systematically, including but beyond the corresponding results on Poisson bialgebras given in [27]. Explicitly, we introduce the notion of adm-Poisson bialgebras which are equivalent to Manin triples of adm-Poisson algebras as well as Poisson bialgebras. The direct correspondence between adm-Poisson bialgebras with one comultiplication and Poisson bialgebras with one cocommutative and one anti-cocommutative comultiplications generalizes and illustrates the polarization-depolarization process in the context of bialgebras. The study of a special class of adm-Poisson bialgebras which include the known coboundary Poisson bialgebras in [27] as a proper subclass in general, illustrating an advantage in terms of the presentation with one operation, leads to the introduction of adm-Poisson Yang-Baxter equation in an adm-Poisson algebra. It is an unexpected consequence that both the adm-Poisson Yang-Baxter equation and the associative Yang-Baxter equation have the same form and thus it motivates and simplifies the involved study from the study of the associative Yang-Baxter equation, which is another advantage in terms of the presentation with one operation. A skew-symmetric solution of adm-Poisson Yang-Baxter equation gives an adm-Poisson bialgebra. Finally the notions of an $O$-operator of an adm-Poisson algebra and a pre-adm-Poisson algebra are introduced to construct skew-symmetric solutions of adm-Poisson Yang-Baxter equation and hence adm-Poisson bialgebras. Note that a pre-adm-Poisson algebra gives an equivalent presentation for a pre-Poisson algebra introduced by Aguiar.

Contents

1. Introduction
2. Some facts on Poisson bialgebras and coboundary Poisson bialgebras
3. Representations and matched pairs of adm-Poisson algebras
4. Admissible Poisson bialgebras
5. A special class of adm-Poisson bialgebras
6. $O$-operators of adm-Poisson algebras and pre-adm-Poisson algebras

1. Introduction

A Poisson algebra whose name comes from the French mathematician Siméon Poisson, is an algebra with a Lie algebra structure and a commutative associative algebra structure which are entwined by the Leibniz rule. Poisson algebras appear in a lot of fields such as Poisson geometry [14, 27], classical and quantum mechanics [1, 10, 28], algebraic geometry [14, 27], quantization theory [11, 18] and quantum groups [1, 12].

2010 Mathematics Subject Classification. 16T10, 16T25, 17B63.
Key words and phrases. Poisson algebra, bialgebra, classical Yang-Baxter equation, $O$-operator.
Definition 1.1. (\cite{21,33}) Let $P$ be a vector space equipped with two bilinear operations $[\ , \], \circ : P \otimes P \to P$. ($P,[\ , \], \circ$) is called a Poisson algebra if $(P,[\ , \])$ is a Lie algebra, $(P,\circ)$ is a commutative associative algebra and

$$[x, y \circ z] = [x, y] \circ z + y \circ [x, z], \ \forall x, y, z \in P.$$  \hspace{1cm} (1.1)

It is remarkable that there is an equivalent presentation for a Poisson algebra $(P,[\ , \], \circ)$ with one operation as follows.

Definition 1.2. (\cite{21}) Let $P$ be a vector space equipped with one bilinear operation $\star : P \otimes P \to P$. We call $(P, \star)$ an admissible Poisson algebra if the following equation holds:

$$(x \star y) \star z = x \star (y \star z) - \frac{1}{3}(-x \star (z \star y) + z \star (x \star y) + y \star (x \star z) - y \star (z \star x)), \ \forall x, y, z \in P.$$ \hspace{1cm} (1.2)

Proposition 1.3. (\cite{21}) If $(P, [\ , \], \circ)$ is a Poisson algebra, then $(P, \star)$ is an admissible Poisson algebra, which is called the corresponding admissible Poisson algebra, where the multiplication $\star$ is defined by

$$x \star y = x \circ y + [x, y], \ \forall x, y \in P.$$ \hspace{1cm} (1.3)

Conversely, if $(P, \star)$ is an admissible Poisson algebra, then $(P, [\ , \], \circ)$ is a Poisson algebra, which is called the corresponding Poisson algebra, where the multiplication $\circ$ and the bracket operation $[\ , \]$ are respectively defined by

$$x \circ y = \frac{1}{2}(x \star y + y \star x), \ [x, y] = \frac{1}{2}(x \star y - y \star x), \ \forall x, y \in P.$$ \hspace{1cm} (1.4)

The structure of an admissible Poisson algebra was given explicitly in \cite{26} by a polarization-depolarization trick which was also independently employed in \cite{22}, that is, polarization interprets structures with one operation as structures with one commutative and anticommutative operations, whereas conversely depolarization interprets structures with one commutative and anticommutative operations as structures with one operation (\cite{22, 28}). There has not been a formal notion for such a structure until the present notion was given in \cite{10}. To avoid confusion, we call an admissible Poisson algebra briefly an adm-Poisson algebra, whereas an ordinary Poisson algebra is still called a Poisson algebra. Moreover, same presentations for Poisson superalgebras were given in \cite{31} and there is a work on nonassociative Poisson algebras by presenting an approach of the nonassociative admissible Poisson algebra which is called weakly associative algebra in \cite{32}.

Although the two presentations are equivalent, as pointed out in \cite{26}, “this change of perspective might sometimes lead to new insights and results”, and hence there might be respective advantages when we study some properties or apply them in other topics in terms of different presentations.

In this paper we establish a bialgebra theory for adm-Poisson algebras. We would like to point out that there is a bialgebra theory for Poisson algebras in terms of the usual presentation in \cite{27}. Note that there is also a bialgebra theory for noncommutative Poisson algebras (\cite{24}). The study of the bialgebra theory in terms of adm-Poisson algebras is completely independent and systematic, including but beyond the corresponding results given in \cite{27}.

Explicitly, we still take a similar approach as of the study on Lie bialgebras (\cite{3, 11}), that is, the compatibility condition is still decided by an analogue of a Manin triple of Lie algebras, which we call a Manin triple of adm-Poisson algebras. The notion of an adm-Poisson bialgebra is thus introduced as an equivalent structure of a Manin triple of adm-Poisson algebras, which is interpreted in terms of matched pairs of adm-Poisson algebras. These results are parallel to their counterparts in terms of the usual presentation of Poisson algebras and hence an adm-Poisson
bialgebra is exactly a Poisson bialgebra given in [27]. Furthermore, there is a direct correspondence between adm-Poisson bialgebras with one comultiplication and Poisson bialgebras with one cocommutative and one anti-cocommutative comultiplications, generalizing and illustrating the polarization-depolarization process in the context of bialgebras.

The results in terms of the presentation with one operation are “beyond” the usual presentation appear in the study of a special class of adm-Poisson bialgebras which is similar to the study of coboundary Lie bialgebras for Lie algebras ([4], [11]) or coboundary infinitesimal bialgebras for associative algebras ([4], [5]). The corresponding Poisson bialgebras of such adm-Poisson bialgebras include the coboundary Poisson bialgebras introduced in [27] as a proper subclass in general, that is, there might exist some new examples of non-coboundary Poisson bialgebras from such adm-Poisson bialgebras, although we have not obtained such an example explicitly. This feature illustrates an advantage in terms of the presentation with one operation, as well as might help demonstrate more implications in some applications.

The study of such adm-Poisson bialgebras also leads to the introduction of adm-Poisson Yang-Baxter equation in an adm-Poisson equation which is an analogue of the classical Yang-Baxter equation in a Lie algebra or the associative Yang-Baxter equation in an associative algebra. It is also interesting to see that the set of solutions of adm-Poisson Yang-Baxter equation in an adm-Poisson algebra includes the set of solutions of Poisson Yang-Baxter equation given in [27] in the corresponding Poisson algebra as a subset in general, whereas the two sets coincide under certain conditions including the skew-symmetric case. Moreover, a skew-symmetric solution of adm-Poisson Yang-Baxter equation gives an adm-Poisson bialgebra.

There is an unexpected consequence that both the adm-Poisson Yang-Baxter equation and the associative Yang-Baxter equation have the same form. It is partly due to the fact that both adm-Poisson and associative algebras have the same forms of dual representations. Therefore some properties of adm-Poisson Yang-Baxter equation can be obtained directly from the corresponding ones of associative Yang-Baxter equation. Consequently, the study on adm-Poisson algebras involving them would get more motivations or simplicities from the study of the associative Yang-Baxter equation, which is another advantage in terms of the presentation with one operation.

In particular, as for the study on the associative Yang-Baxter equation ([4], [5], [6]), in order to obtain skew-symmetric solutions of adm-Poisson Yang-Baxter equation, we introduce the notion of an O-operator of an adm-Poisson algebra which is an analogue of an O-operator of a Lie algebra introduced by Kupershmidt in [13] as a natural generalization of the classical Yang-Baxter equation in a Lie algebra (also see [4]), and a pre-adm-Poisson algebra as an analogue of a dendriform algebra introduced by Loday ([24]). The former gives a construction of skew-symmetric solutions of adm-Poisson Yang-Baxter equation in a semi-direct product adm-Poisson algebra, whereas the latter gives a representation of the sub-adjacent adm-Poisson algebra such that the identity is a natural O-operator associated to it. Therefore a construction of skew-symmetric solutions of adm-Poisson Yang-Baxter equation and hence adm-Poisson bialgebras from pre-adm-Poisson algebras is given. Note that a pre-adm-Poisson algebra gives an equivalent presentation for a pre-Poisson algebra introduced by Aguiar in [2].

This paper is organized as follows. In Section 2, we recall some facts on Poisson bialgebras and coboundary Poisson bialgebras given in [27]. In Section 3, we introduce the notions of representations and matched pairs of adm-Poisson algebras. The dual representation of a representation of an adm-Poisson algebra is also given. In Section 4, the notions of Manin triples of adm-Poisson algebras and adm-Poisson bialgebras are introduced. Their equivalence is interpreted in terms of matched pairs of adm-Poisson algebras. We also give a direct correspondence between adm-Poisson
bialgebras and Poisson bialgebras. In Section 5, we consider a special class of adm-Poisson bialgebras. They include coboundary Poisson bialgebras as a proper subclass. This study also leads to the introduction of adm-Poisson Yang-Baxter equation whose skew-symmetric solutions give adm-Poisson bialgebras. In Section 6, we introduce the notions of $\Theta$-operators of adm-Poisson algebras and pre-adm-Poisson algebras, and give constructions of skew-symmetric solutions of adm-Poisson Yang-Baxter equation from these structures.

We adopt the following conventions and notations.

1. Let $(A, \circ)$ be a vector space with a bilinear operation $\circ : A \otimes A \to A$. Let $r = \sum_i x_i \otimes y_i \in A \otimes A$. Set

$$r_{12} = \sum_i x_i \otimes y_i \otimes 1, \quad r_{13} = \sum_i x_i \otimes 1 \otimes y_i, \quad r_{23} = \sum_i 1 \otimes x_i \otimes y_i,$$

where $1$ is the unit if $(A, \circ)$ is unital or a symbol playing a similar role as the unit for the non-unital cases. The operation between two $r$s is given in an obvious way. For example,

$$r_{12} \circ r_{13} = \sum_{i,j} x_i \otimes x_j \otimes y_i \otimes y_j, \quad r_{13} \circ r_{23} = \sum_{i,j} x_i \otimes x_j \otimes y_i \otimes y_j,$$

$$r_{23} \circ r_{12} = \sum_{i,j} x_i \otimes x_j \otimes y_i \otimes y_j.$$ (1.6)

2. Let $V$ be a vector space. Let $\tau : V \otimes V \to V \otimes V$ be the twisting operator defined as

$$\tau(v \otimes u) = v \otimes u, \quad \forall v, u \in V.$$ (1.7)

3. Let $V$ be a vector space. For an $r \in V \otimes V$, the linear map $r^\sharp : V^* \to V$ is given by

$$\langle r^\sharp(u^*), v^* \rangle = \langle r, u^* \otimes v^* \rangle, \quad \forall u^*, v^* \in P^*.$$ (1.8)

We say that $r \in V \otimes V$ is nondegenerate, if the linear map $r^\sharp$ is an isomorphism.

4. Let $V_1, V_2$ be two vector spaces and $T : V_1 \to V_2$ be a linear map. Denote the dual map by $T^* : V_2^* \to V_1^*$, defined by

$$\langle v_1, T^*(v_2^*) \rangle = \langle T(v_1), v_2^* \rangle, \quad \forall v_1 \in V_1, v_2^* \in V_2^*.$$ (1.9)

5. Let $V$ be a vector space and $A$ be a vector space (usually with some bilinear operations). For a linear map $\rho : A \to \text{End}_F(V)$, define a linear map $\rho^* : A \to \text{End}_F(V^*)$ by

$$\langle \rho^*(x)v^*, u \rangle = -\langle v^*, \rho(x)u \rangle, \quad \forall x \in A, u \in V, v^* \in V^*.$$ (2.1)

Throughout this paper, all vector spaces are finite-dimensional over a base field $F$, although many results still hold in the infinite dimension.

2. Some facts on Poisson bialgebras and coboundary Poisson bialgebras

In this section, we recall some facts on Poisson bialgebras and coboundary Poisson bialgebras given in [7].

Let $(\mathfrak{g}, [\ , \ ])$ be a Lie algebra. Let $\text{ad}(x)$ denote the adjoint operator, that is, $\text{ad}(x)(y) = [x, y]$ for all $x, y \in \mathfrak{g}$. Let $\text{ad} : \mathfrak{g} \to \text{End}_F(\mathfrak{g})$ with $x \to \text{ad}(x)$ be the adjoint representation of $\mathfrak{g}$. A Lie bialgebra structure on $\mathfrak{g}$ is a linear map $\delta : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$ such that $\delta^* : \Lambda^2 \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie algebra structure on $\mathfrak{g}^*$ and $\delta$ is a 1-cocycle of $\mathfrak{g}$ with the coefficient in the representation $(\mathfrak{g} \otimes \mathfrak{g}; \text{ad} \otimes \text{id} + \text{id} \otimes \text{ad})$, i.e.

$$\delta([x, y]) = (\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))\delta(y) - (\text{ad}(y) \otimes \text{id} + \text{id} \otimes \text{ad}(y))\delta(x), \quad \forall x, y \in \mathfrak{g}.$$ (2.1)
Let \((A, \circ)\) be an associative algebra. Let \(L_o(x)\) and \(R_o(x)\) be the left and the right multiplication operators respectively, that is, \(L_o(x)(y) = R_o(y)(x) = x \circ y\) for all \(x, y \in A\). Let \(L_o, R_o : A \to \text{End}_F(A)\) be two linear maps with \(x \to L_o(x)\) and \(x \to R_o(x)\) respectively. An \textbf{infinitesimal bialgebra} structure on \(A\) is a linear map \(\Delta : A \to A \otimes A\) such that \(\Delta^* : A^* \otimes A^* \to A^*\) defines an associative algebra structure on \(A^*\) and \(\Delta\) satisfies
\[
\Delta(x \circ y) = (\text{id} \otimes L_o(x))\Delta(y) + (R_o(y) \otimes \text{id})\Delta(x), \quad \forall x, y \in A. \tag{2.2}
\]

\textbf{Definition 2.1.} Let \((P, [ , ], \circ)\) be a Poisson algebra. Let \(\delta : P \to \wedge^2 P\) and \(\Delta : P \to P \otimes P\) be two linear maps such that \(\delta^* : \wedge^2 P^* \to P^*\) defines a Lie algebra structure on \(P^*\), \(\Delta^* : P^* \otimes P^* \to P^*\) defines a commutative associative algebra structure on \(P^*\), and they satisfy the following compatible condition:
\[
(id \otimes \Delta)\delta(x) = (\delta \otimes \text{id})\Delta(x) + (\tau \otimes \text{id})(\text{id} \otimes \delta)\Delta(x), \quad \forall x \in P. \tag{2.3}
\]

If in addition, \((P, [ , ], \circ, \delta, \Delta)\) is a Lie bialgebra, \((P, \circ, \Delta)\) is an infinitesimal bialgebra, and \(\delta\) and \(\Delta\) are compatible in the following sense
\[
\delta(x \circ y) = (L_o(x) \otimes \text{id})\delta(y) + (L_o(y) \otimes \text{id})\delta(x) + (\text{id} \otimes \text{ad}(x))\Delta(y)
+ (\text{id} \otimes \text{ad}(y))\Delta(x), \tag{2.4}
\]
\[
\Delta([x, y]) = (\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))\Delta(y) + (L_o(y) \otimes \text{id} - \text{id} \otimes L_o(y))\delta(x) \tag{2.5}
\]
for all \(x, y \in P\), then \((P, [ , ], \circ, \delta, \Delta)\) is called a \textbf{Poisson bialgebra}.

A Lie bialgebra \((g, \delta)\) is called \textbf{coboundary} if there exists an \(r \in g \otimes g\) such that
\[
\delta(x) = (\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))r, \quad \forall x \in g. \tag{2.6}
\]
In this case, \(\delta\) automatically satisfies Eq. \((2.14)\).

Let \(g\) be a Lie algebra and \(r \in g \otimes g\). The linear map \(\delta\) defined by Eq. \((2.4)\) makes \((g, \delta)\) become a Lie bialgebra if and only if the following conditions are satisfied:
\[
(\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))(r + \tau(r)) = 0, \tag{2.7}
\]
\[
(\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{id} \otimes \text{ad}(x))C(r) = 0 \tag{2.8}
\]
for all \(x \in g\), where \(C(r) = [r_{23}, r_{12}] + [r_{23}, r_{13}] + [r_{13}, r_{12}]\).

In particular, the following equation:
\[
C(r) = [r_{23}, r_{12}] + [r_{23}, r_{13}] + [r_{13}, r_{12}] = 0 \tag{2.9}
\]
is called \textbf{classical Yang-Baxter equation (CYBE)}.

An infinitesimal bialgebra \((A, \Delta)\) is called \textbf{coboundary} if there exists an \(r \in A \otimes A\) such that
\[
\Delta(x) = (\text{id} \otimes L_o(x) - R_o(x) \otimes \text{id})r, \quad \forall x \in A. \tag{2.10}
\]
In this case, \(\Delta\) automatically satisfies Eq. \((2.2)\).

Let \((A, \circ)\) be a commutative associative algebra and \(r \in A \otimes A\). The linear map \(\Delta\) defined by Eq. \((2.11)\) makes \((A, \Delta)\) become an infinitesimal bialgebra such that \(\Delta^*\) defines a commutative associative algebra structure on \(A^*\) if and only if the following conditions are satisfied:
\[
(L_o(x) \otimes \text{id} - \text{id} \otimes L_o(x))(r + \tau(r)) = 0, \tag{2.11}
\]
\[
(L_o(x) \otimes \text{id} - \text{id} \otimes L_o(x))A(r) = 0 \tag{2.12}
\]
for all \(x \in A\), where \(A(r) = r_{23} \circ r_{12} - r_{13} \circ r_{23} - r_{12} \circ r_{13}\).

In particular, the following equation:
\[
A(r) = r_{23} \circ r_{12} - r_{13} \circ r_{23} - r_{12} \circ r_{13} = 0 \tag{2.13}
\]
is called **associative Yang-Baxter equation (AYBE)**.

**Definition 2.2.** A Poisson bialgebra \((P, [,], \circ, \delta, \Delta)\) is called **coboundary** if \(\delta\) and \(\Delta\) satisfy
\[
\delta(x) = (\text{ad}(x) \otimes \text{id}) + \text{id} \otimes \text{ad}(x))r, \\
\Delta(x) = (\text{id} \otimes L_\circ(x) - L_\circ(x) \otimes \text{id})r
\]
for all \(x \in P\) and some \(r \in P \otimes P\). We sometimes denote it by \((P, [,], \circ, r)\).

**Theorem 2.3.** Let \((P, [,], \circ)\) be a Poisson algebra and \(r \in P \otimes P\). Let \(\delta : P \to \wedge^2 P\) and \(\Delta : P \to P \otimes P\) be two linear maps defined by Eqs. (2.14) and (2.15) respectively. Then \((P, [,], \circ, \delta, \Delta)\) is a Poisson bialgebra if and only if for all \(x \in P\), the following conditions are satisfied:

1. \((\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))(r + \tau(r)) = 0;
2. \((L_\circ(x) \otimes \text{id} - \text{id} \otimes L_\circ(x))(r + \tau(r)) = 0;
3. \((\text{ad}(x) \otimes \text{id} \otimes \text{id} + \text{id} \otimes \text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{id} \otimes \text{ad}(x))C(r) = 0;
4. \((L_\circ(x) \otimes \text{id} \otimes \text{id} - \text{id} \otimes \text{id} \otimes L_\circ(x))A(r) = 0;
5. \((\text{ad}(x) \otimes \text{id} \otimes \text{id})A(r) - (\text{id} \otimes L_\circ(x) \otimes \text{id} - \text{id} \otimes \text{id} \otimes L_\circ(x))C(r) = 0.

**Definition 2.4.** Let \((P, [,], \circ)\) be a Poisson algebra and \(r \in P \otimes P\). \(r\) is a solution of **Poisson Yang-Baxter equation (PYBE)** in \((P, [,], \circ)\) if \(r\) is a solution of both CYBE and AYBE, that is, \(C(r) = A(r) = 0\).

**Corollary 2.5.** Let \((P, [,], \circ)\) be a Poisson algebra and \(r \in P \otimes P\). Let \(\delta : P \to \wedge^2 P\) and \(\Delta : P \to P \otimes P\) be two linear maps defined by Eqs. (2.14) and (2.15) respectively. If \(r\) is a skew-symmetric solution of PYBE in \((P, [,], \circ)\), then \((P, [,], \circ, \delta, \Delta)\) is a Poisson bialgebra.

### 3. Representations and matched pairs of adm-Poisson algebras

In this section, we introduce the notions of representations and matched pairs of adm-Poisson algebras and then give some properties.

**Definition 3.1.** Let \((P, \ast)\) be an adm-Poisson algebra and \(V\) be a vector space. Let \(l, r : P \to \text{End}_F(V)\) be two linear maps. The triple \((l, r, V)\) is called a **representation** of \((P, \ast)\) if
\[
l(x \ast y) = l(x)l(y) - \frac{1}{3}( - l(x)l(y) + r(x \ast y) + l(y)l(x) - l(y)l(x)), \\
r(y)l(x) = l(x)r(y) - \frac{1}{3}( - l(x)l(y) + l(y)l(x) + r(x \ast y) - r(y \ast x)), \\
l(y)r(x) = r(x \ast y) - \frac{1}{3}( - r(y \ast x) + l(y)l(x) + l(x)r(y) - l(x)l(y))
\]
for all \(x, y \in P\). Two representations \((l_1, r_1, V_1)\) and \((l_2, r_2, V_2)\) of an adm-Poisson algebra \(P\) are called **equivalent** if there exists an isomorphism \(\varphi : V_1 \to V_2\) satisfying
\[
\varphi l_1(x) = l_2(x)\varphi, \quad \varphi r_1(x) = r_2(x)\varphi, \quad \forall x \in P.
\]

**Lemma 3.2.** Let \((l, r, V)\) be a representation of an adm-Poisson algebra \((P, \ast)\). Then the following equation holds:
\[
l(x \ast y) + r(x)r(y) = l(x)l(y) + r(y \ast x), \quad \forall x, y \in P.
\]

**Proof.** On the one hand, by Eq. (3.1), we have
\[
\frac{1}{3}( - l(x)l(y) + r(x \ast y) + l(y)l(x) - l(y)r(x)) = l(x)l(y) - l(x \ast y), \quad \forall x, y \in P.
\]
On the other hand, by Eq. (3.3), we have
\[ \frac{1}{3}(-\ell(x)\tau(y) + \ell(x \star y) + \ell(y)\ell(x) - \ell(y)\tau(x)) = \tau(x)\tau(y) - \tau(x \star y), \quad \forall x, y \in P. \]
Hence Eq. (3.5) holds. \(\square\)

**Proposition 3.3.** Let \((P, \star)\) be an adm-Poisson algebra. Let \(V\) be a vector space and \(\ell, \tau : P \to \text{End}_F(V)\) be two linear maps. Define a bilinear operation \(\ast_{\ell, \tau} : (P \oplus V) \otimes (P \oplus V) \to P \oplus V\) on \(P \oplus V\) by
\[
(x + u) \ast_{\ell, \tau} (y + v) = x \star y + \ell(x)v + \tau(y)u, \quad \forall x, y \in P, u, v \in V.
\]
Then \((\ell, \tau, V)\) is a representation of \((P, \star)\) if and only if \((P \oplus V, \ast_{\ell, \tau})\) is an adm-Poisson algebra, which is called the semi-direct product of \(P\) by \(V\) and denoted by \(P \ltimes_{\ell, \tau} V\) or simply \(P \ltimes V\).

**Proof.** It is due to [33] with a straightforward proof. \(\square\)

**Remark 3.4.** Recall that a representation of a Poisson algebra \((P, [\ , ]\), \(\circ)\) is a triple \((S_{[\ , ]}, S_\circ, V)\) such that \((S_{[\ , ]}, V)\) is a representation of the Lie algebra \((P, [\ , ]))\) and \((S_\circ, V)\) is a representation of the commutative associative algebra \((P, \circ)\) satisfying some compatible conditions ([2]). Let \((\ell, \tau, V)\) be a representation of an adm-Poisson algebra \((P, \star_P)\). Then \(\left(\frac{1}{3}(1 - \ell), \frac{1}{3}I + \tau\right), V)\) is a representation of the corresponding Poisson algebra \((P, [\ , ]\), \(\circ)\). Conversely, if \((S_{[\ , ]}, S_\circ, V)\) is a representation of a Poisson algebra \((P, [\ , ]\), \(\circ)\), then \((S_{[\ , ]} + S_\circ, S_\circ - S_{[\ , ]}, V)\) is a representation of the corresponding adm-Poisson algebra \((P, \star_P)\). Thus the representations of an adm-Poisson algebra are in a one-to-one correspondence with the representations of the corresponding Poisson algebra.

**Proposition 3.5.** Let \((\ell, \tau, V)\) be a representation of an adm-Poisson algebra \((P, \star)\). Then \((-\tau^*, -\ell^*, V^*)\) is a representation of \((P, \star)\). We call it the dual representation of \((\ell, \tau, V)\).

**Proof.** We only show that Eq. (3.1) holds for \((-\tau^*, -\ell^*, V^*)\) as an example. The proofs for the holding of Eqs. (3.2) and (3.3) are similar. Since \((\ell, \tau, V)\) is a representation of the adm-Poisson algebra \((P, \star)\), by Lemma 3.3, for all \(x, y \in P, u \in V, v^* \in V^*,\) we have
\[
\langle (\tau^*(x \star y) + \tau^*(x)\tau^*(y) + \frac{1}{3}(\ell^*(x)\tau^*(y) + \ell^*(x \star y) - \tau^*(y)\tau^*(x) + \tau^*(y)\ell^*(x)))v^*, u \rangle
\]
\[
= \langle v^*, (\ell^*(x \star y) + \tau^*(y)\tau(x) + \frac{1}{3}(\ell^*(y)\tau(x) + \ell^*(x)\tau(y) - \ell^*(x \star y) - \tau^*(x)\ell^*(y)))u \rangle
\]
\[
= \langle v^*, (\ell^*(x \star y) + \tau^*(y)\tau(x) + \frac{1}{3}(\ell^*(y)\tau(x) + \ell^*(x)\tau(y) - \ell^*(x \star y) - \ell^*(x)\ell^*(y)))u \rangle = 0,
\]
which implies that Eq. (3.1) holds for \((-\tau^*, -\ell^*, V^*)\). \(\square\)

**Remark 3.6.** In fact, the above conclusion can be obtained from Remark 3.3 as follows. Note that for a representation \((S_{[\ , ]}, S_\circ, V)\) of a Poisson algebra \((P, [\ , ]\), \(\circ)\), \((S_{[\ , ]}^*, -S_\circ^*, V)\) is a representation of \((P, [\ , ]\), \(\circ)\) which is the dual representation of \((S_{[\ , ]}, S_\circ, V)\). Therefore for a representation \((\ell, \tau, V)\) of an adm-Poisson algebra \((P, \star)\), \(\left(\frac{1}{3}(1 - \ell), \frac{1}{3}I + \tau\right), V)\) is a representation of the corresponding Poisson algebra \((P, [\ , ]\), \(\circ)\) whose dual representation is \(\left(\frac{1}{3}(1 - \ell^*), -\frac{1}{2}(\ell^* + \tau^*), V^*\right)\), which in turn gives a representation \((-\tau^*, -\ell^*, V^*)\) of the original adm-Poisson algebra \((P, \star)\).

**Remark 3.7.** Note that for a representation \((\ell, \tau, V)\) of an associative algebra in the sense of bimodules, the dual representation is also \((-\tau^*, -\ell^*, V^*)\). Therefore for both associative and adm-Poisson algebras, the dual representations in the above sense have the same form.
Example 3.8. Let \((P, \ast)\) be an adm-Poisson algebra. Let \(L(x)\) and \(R(x)\) denote the left and right multiplication operators, respectively, that is, \(L(x)y = R(y)x = x \ast y\) for all \(x, y \in P\). Let \(L, R : P \to \text{End}_F(P)\) be two linear maps with \(x \to L(x)\) and \(x \to R(x)\) respectively. Then \((L, R, P)\) is a representation of \((P, \ast)\), called the adjoint representation. Furthermore, \((-R^\ast, -L^\ast, P^\ast)\) is also a representation of \((P, \ast)\).

The relationship between the adjoint representation \((L, R, P)\) of an adm-Poisson algebra \((P, \ast)\) and the representation \((\text{ad}, L_0, P)\) of the corresponding Poisson algebra \((P, \{\ , \ \}, \circ)\) is given by

\[
L = L_0 + \text{ad}, \quad R = L_0 - \text{ad}. \tag{3.6}
\]

Letting \((l, r, V) = (L, R, P)\) in Lemma 3.2, we have the following known conclusion.

Corollary 3.9. (\cite{12}) Let \((P, \ast)\) be an adm-Poisson algebra. Then we have

\[
(x \ast y) \ast z - x \ast (y \ast z) = z \ast (y \ast x) - (z \ast y) \ast x, \quad \forall x, y, z \in P. \tag{3.7}
\]

Definition 3.10. Let \((P_1, \ast_1)\) and \((P_2, \ast_2)\) be two adm-Poisson algebras. Let \(l_1, r_1 : P_1 \to \text{End}_F(P_2), l_2, r_2 : P_2 \to \text{End}_F(P_1)\) be four linear maps. \((P_1, P_2, l_1, r_1, l_2, r_2)\) is called a matched pair of adm-Poisson algebras if \((l_1, r_1, P_2)\) and \((l_2, r_2, P_1)\) are representations of \((P_1, \ast_1)\) and \((P_2, \ast_2)\) respectively, and for all \(x, y \in P_1, a, b \in P_2\), the following equations hold.

\[
\begin{align*}
\tau_2(a)(x \ast_1 y) & = \tau_2(l_1(y)a)x + x \ast_1 (\tau_2(a)y) + \frac{1}{3}(\tau_2(r_1(y)a)x + x \ast_1 (l_2(a)y) \\
& \quad - l_2(a)(x \ast_1 y) - y \ast_1 (\tau_2(a)x) - \tau_2(l_1(x)a)y + y \ast_1 (l_2(a)x) + \tau_2(r_1(x)a)y, \tag{3.8}
\end{align*}
\]

\[
\begin{align*}
l_2(a)(x \ast_1 y) & = (l_2(a)x) \ast_1 y + l_2(r_1(x)a)y + \frac{1}{3}(-l_2(a)(y \ast_1 x) + y \ast_1 (l_2(a)x) \\
& \quad + \tau_2(r_1(x)a)y + x \ast_1 (l_2(a)y)) + \tau_2(r_1(y)a)x - x \ast_1 (\tau_2(a)y) - \tau_2(r_1(y)a)x, \tag{3.9}
\end{align*}
\]

\[
\begin{align*}
(\tau_2(a)x) \ast_1 y & = -l_2(l_1(x)a)y + x \ast_1 (l_2(a)y) + \tau_2(r_1(y)a)x + \frac{1}{3}(x \ast_1 (\tau_2(a)y) \\
& \quad + \tau_2(l_1(y)a)x - y \ast_1 (\tau_2(a)x) - \tau_2(l_1(x)a)y - l_2(a)(x \ast_1 y) + l_2(a)(y \ast_1 x)), \tag{3.10}
\end{align*}
\]

\[
\begin{align*}
l_1(x)(a \ast_2 b) & = l_1(l_2(b)x)a + a \ast_2 (l_1(x)b) + \frac{1}{3}(l_1(r_2(b)x)a + a \ast_2 (l_1(x)b) \\
& \quad - l_1(x)(a \ast_2 b) - b \ast_2 (l_1(x)a) - r_1(l_2(a)x)b + b \ast_2 (l_1(x)a) \\
& \quad + r_1(l_2(b)x)b), \tag{3.11}
\end{align*}
\]

\[
\begin{align*}
l_1(x)(a \ast_2 b) & = (l_1(x)a) \ast_2 b + l_1(l_2(a)x)b + \frac{1}{3}(-l_1(x)(b \ast_2 a) + b \ast_2 (l_1(x)a) \\
& \quad + r_1(l_2(b)x)a + a \ast_2 (l_1(x)b) + r_1(r_2(b)x)a - a \ast_2 (r_1(x)b) \\
& \quad - r_1(l_2(b)x)a), \tag{3.12}
\end{align*}
\]

\[
\begin{align*}
(\tau_1(x)a) \ast_2 b & = -(l_1(l_2(a)x)b + a \ast_2 (l_1(x)b) + r_1(l_2(b)x)a + \frac{1}{3}(a \ast_2 (l_1(x)b) \\
& \quad + r_1(l_2(b)x)a - b \ast_2 (l_1(x)a) - r_1(l_2(a)x)b - l_1(x)(a \ast_2 b) \\
& \quad + l_1(x)(b \ast_2 a)), \tag{3.13}
\end{align*}
\]

By a straightforward proof, we get the following conclusion.
Proposition 3.11. Let \((P_1, \star_1)\) and \((P_2, \star_2)\) be two adm-Poisson algebras, \(l_1, r_1 : P_1 \to \text{End}_P(P_2)\), \(l_2, r_2 : P_2 \to \text{End}_P(P_1)\) be four linear maps. Define a bilinear operation \(\star : (P_1 \oplus P_2) \otimes (P_1 \oplus P_2) \to P_1 \oplus P_2\) on \(P_1 \oplus P_2\) by
\[
(x + a) \star (y + b) = x \star_1 y + r_2(b)x + l_2(a)y + l_1(x)b + r_1(y)a + a \star_2 b,
\]
where \(x, y \in P_1, a, b \in P_2\). Then \((P_1 \oplus P_2, \star)\) is an adm-Poisson algebra if and only if \((P_1, \star_1), (P_2, \star_2)\) is a matched pair of adm-Poisson algebras. We denote this adm-Poisson algebra by \(P_1 \bowtie_{l_1, r_2} P_2\) or simply \(P_1 \bowtie P_2\).

Note that the semi-direct product of an adm-Poisson algebra \((P, \star)\) by a representation \((l, r, V)\) given in Proposition 3.3 is a special case of the matched pairs of adm-Poisson algebras in Proposition 3.11 when \(P_2 = V\) is equipped with the zero multiplication, that is, \((P, V, l, r, 0, 0)\) is a matched pair of adm-Poisson algebras.

4. Admissible Poisson bialgebras

In this section, we introduce the notions of Manin triples of adm-Poisson algebras and adm-Poisson bialgebras. The equivalence between them is interpreted in terms of matched pairs of adm-Poisson algebras.

Definition 4.1. A bilinear form \(\mathcal{B}\) on an adm-Poisson algebra \((P, \star)\) is called invariant if
\[
\mathcal{B}(x \star y, z) = \mathcal{B}(x, y \star z), \quad \forall x, y, z \in P.
\]

Proposition 4.2. Let \((P, \star)\) be an adm-Poisson algebra. If there is a nondegenerate symmetric invariant bilinear form \(\mathcal{B}\) on \(P\), then the two representations \((L, R, P)\) and \((-R^*, -L^*, P^*)\) of the adm-Poisson algebra \((P, \star)\) are equivalent. Conversely, if the two representations \((L, R, P)\) and \((-R^*, -L^*, P^*)\) of the adm-Poisson algebra \((P, \star)\) are equivalent, then there exists a nondegenerate invariant bilinear form \(\mathcal{B}\) on \(P\).

Proof. Since \(\mathcal{B}\) is nondegenerate, there exists a linear isomorphism \(\varphi : P \to P^*\) defined by
\[
\langle \varphi(x), y \rangle = \mathcal{B}(x, y), \quad \forall x, y \in P.
\]
For all \(x, y, z \in P\), we have
\[
\langle \varphi(L(x)y), z \rangle = \mathcal{B}(x \star y, z) = \mathcal{B}(z \star x, y) = \langle \varphi(y), z \star x \rangle = \langle -R^*(x), \varphi(y), z \rangle;
\]
\[
\langle \varphi(R(x)y), z \rangle = \mathcal{B}(y \star x, z) = \mathcal{B}(y, x \star z) = \langle \varphi(y), x \star z \rangle = \langle -L^*(x), \varphi(y), z \rangle.
\]
Thus the two representations \((L, R, P)\) and \((-R^*, -L^*, P^*)\) of the adm-Poisson algebra \((P, \star)\) are equivalent. The converse can be proved similarly. We omit the details. \(\square\)

Definition 4.3. Let \((P, \star_P)\) be an adm-Poisson algebra. Suppose that \((P^*, \star_{P^*})\) is an adm-Poisson algebra on its dual space \(P^*\). If there exists an adm-Poisson algebra structure on the direct sum \(P \oplus P^*\) of the underlying vector spaces of \(P\) and \(P^*\) such that \((P, \star_P)\) and \((P^*, \star_{P^*})\) are adm-Poisson subalgebras and the following symmetric bilinear form \(\mathcal{B}_d\) on \(P \oplus P^*\) given by
\[
\mathcal{B}_d(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \quad \forall x, y \in P, \ a^*, b^* \in P^*
\]
is invariant, then \((P \oplus P^*, P^*)\) is called a (standard) Manin triple of adm-Poisson algebras associated to \(\mathcal{B}_d\).

Proposition 4.4. Let \((P, \star_P)\) be an adm-Poisson algebra. Suppose \((P^*, \star_{P^*})\) is an adm-Poisson algebra structure on the dual space \(P^*\). Then \((P \oplus P^*, P^*)\) is a standard Manin triple of adm-Poisson algebras associated to \(\mathcal{B}_d\) defined by Eq. (4.2) if and only if \((P, P^*, -R^*_P, -L^*_P, -R^*_P, -L^*_P)\) is a matched pair of adm-Poisson algebras.
Proof. It follows from the same proof of \[1\] Theorem 2.2.1.

\[\text{Theorem 4.5.} \text{ Let } (P, \star_P) \text{ be an adm-Poisson algebra. Suppose that there is an adm-Poisson structure } \“\star_P\” \text{ on its dual space } P^* \text{ given by a linear map } \alpha^* : P^* \otimes P^* \rightarrow P^*. \text{ Then } (P, P^*, -R^*_P, -L^*_P, -R^*_p, -L^*_p) \text{ is a matched pair of adm-Poisson algebras if and only if } \alpha \text{ satisfies the following equations}
\]

\[\alpha(x \star_P y) - (R_P(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes L_P(x))\alpha(y) = \frac{1}{3}((L_P(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes L_P(y))\alpha(x) + (L_P(x) \otimes \text{id})\alpha(y) - (R_P(x) \otimes \text{id})\alpha(y) + \tau(-\alpha(x \star_P y) + (L_P(x) \otimes \text{id})\alpha(y) + (L_P(y) \otimes \text{id})\alpha(x)), \]

\[\text{(4.3)}\]

\[\alpha(x \star_P y) - (R_P(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes L_P(x))\alpha(y) = \frac{1}{3}((R_P(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes L_P(x))\alpha(x) + (\text{id} \otimes L_P(x))\alpha(y) - (R_P(x) \otimes \text{id})\alpha(y) + \tau(\alpha(y \star_P x) - \alpha(x \star_P y))) \]

\[\text{(4.4)}\]

for all } x, y \in P.

\[\text{Proof.} \text{ By a direct calculation, in the case that } l_1 = -R^*_P, \tau_1 = -L^*_P, l_2 = -R^*_P, \text{ and } \tau_2 = -L^*_P, \text{ we have}
\]

\[\text{Eq. (3.8)} \iff \text{Eq. (4.3)}, \quad \text{Eq. (3.9)} \iff \text{Eq. (4.4)}, \quad \text{Eq. (3.10)} \iff \text{Eq. (4.5)}.\]

Thus if \((P, P^*, -R^*_P, -L^*_P, -R^*_P, -L^*_P)\) is a matched pair of adm-Poisson algebras, then Eqs. (4.3)–(4.5) hold.

Conversely, let } x, y \in P. \text{ By Eq. (4.3), we have}

\[\tau(\alpha(y \star_P x) - (R_P(x) \otimes \text{id})\alpha(y) - (\text{id} \otimes L_P(y))\alpha(x)) = \frac{1}{3}((L_P(y) \otimes \text{id})\alpha(x) + (L_P(x) \otimes \text{id})\alpha(y) - (R_P(x) \otimes \text{id})\alpha(y) + \tau(-\alpha(x \star_P y) + (L_P(y) \otimes \text{id})\alpha(x) + (L_P(x) \otimes \text{id})\alpha(y))). \]

\[\text{(4.6)}\]

By Eqs. (4.3)–(4.4), we obtain

\[\alpha(x \star_P y) - (R_P(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes L_P(x))\alpha(y) = \tau(\alpha(y \star_P x) - (R_P(x) \otimes \text{id})\alpha(y) - (\text{id} \otimes L_P(y))\alpha(x)). \]

\[\text{(4.7)}\]

Interchanging } x \text{ with } y \text{ in Eq. (4.3), we have}

\[((\text{id} \otimes R_P(x))\alpha(y) - (L_P(x) \otimes \text{id})\alpha(y) + \tau((\text{id} \otimes R_P(y))\alpha(x) - (L_P(y) \otimes \text{id})\alpha(x)) = \frac{1}{3}((R_P(x) \otimes \text{id})\alpha(y) - (\text{id} \otimes L_P(x))\alpha(y) + (\text{id} \otimes L_P(y))\alpha(x) - (R_P(y) \otimes \text{id})\alpha(x) + \tau(\alpha(x \star_P y) - \alpha(y \star_P x))). \]

\[\text{(4.8)}\]

Adding Eq. (4.7) and Eq. (4.8) together, we obtain

\[(L_P(x) \otimes \text{id})\alpha(y) - (\text{id} \otimes R_P(x))\alpha(y) + \tau((L_P(x) \otimes \text{id})\alpha(y) + (L_P(y) \otimes \text{id})\alpha(x)) = (\text{id} \otimes R_P(y))\alpha(x) - (L_P(y) \otimes \text{id})\alpha(x) + \tau((\text{id} \otimes R_P(y))\alpha(x) + (\text{id} \otimes R_P(x) \text{id})\alpha(y))(4.9)\]


By Eqs. (4.7) and (4.9), Eq. (1.3) implies
\[ \alpha(x \star_P y) - (R_P(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes L_P(x))\alpha(y) \]
\[ = \frac{1}{3} \left( (\text{id} \otimes R_P(x))\alpha(y) - (\text{id} \otimes R_P(x))\alpha(x) + \tau((\text{id} \otimes L_P(x))\alpha(y) - (R_P(y) \otimes \text{id})\alpha(x)) \right), \]
which is equivalent to Eq. (3.11).

By Eqs. (4.7) and (4.9), Eq. (1.3) implies
\[ \alpha(y \star_P x) - (R_P(x) \otimes \text{id})\alpha(y) - (\text{id} \otimes L_P(y))\alpha(x) \]
\[ = \frac{1}{3} \left( (\text{id} \otimes R_P(y))\alpha(x) + (\text{id} \otimes L_P(x))\alpha(y) - (R_P(y) \otimes \text{id})\alpha(x) + \tau(-\alpha(y \star_P x) + (R_P(y) \otimes \text{id})\alpha(x)) \right), \]
which is equivalent to Eq. (3.12).

By Eq. (4.7), Eq. (4.5) implies
\[ (L_P(x) \otimes \text{id})\alpha(y) - (\text{id} \otimes R_P(x))\alpha(y) + \tau((L_P(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes R_P(y))\alpha(x)) \]
\[ = \frac{1}{3} \left( (\text{id} \otimes L_P(x))\alpha(y) + (R_P(y) \otimes \text{id})\alpha(x) - (R_P(y) \otimes \text{id})\alpha(x) + \tau(-\alpha(y \star_P x) - (R_P(y) \otimes \text{id})\alpha(x)) \right), \]
which is equivalent to Eq. (3.13).

Therefore if Eqs. (1.3)-(1.5) hold, then \((P, P^*, -R_P^*, -L_P^*, -R_P^*, -L_P^*)\) is a matched pair of adm-Poisson algebras. \(\square\)

**Lemma 4.6.** Let \(P\) be a vector space and \(\alpha : P \to P \otimes P\) be a linear map. Then the dual map \(\alpha^* : P^* \otimes P^* \to P^*\) defines an adm-Poisson algebra structure on \(P^*\) if and only if \(\alpha\) satisfies
\[ (\text{id} \otimes \alpha)\alpha(x) - (\alpha \otimes \text{id})\alpha(x) + \frac{1}{3} (\text{id} \otimes \tau)(\text{id} \otimes \alpha)\alpha(x) - (\tau \otimes \text{id})(\text{id} \otimes \alpha)\alpha(x) \]
\[ - (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \alpha)\alpha(x) + (\tau \otimes \text{id})(\text{id} \otimes \tau)(\text{id} \otimes \alpha)\alpha(x)) = 0, \quad \forall x \in P. \quad (4.10)\]

**Proof.** It is straightforward. \(\square\)

See [5] for more details on other Lie admissible coalgebras and their applications.

**Definition 4.7.** Let \((P, \star_P)\) be an adm-Poisson algebra. An **adm-Poisson bialgebra** structure on \(P\) is a linear map \(\alpha : P \to P \otimes P\) such that \(\alpha\) satisfies Eqs. (4.3)-(4.5) and (4.10).

Combining Proposition 4.4 and Theorem 1.5, we have the following conclusion.

**Theorem 4.8.** Let \((P, \star_P)\) be an adm-Poisson algebra. Let \(\alpha : P \to P \otimes P\) be a linear map such that \(\alpha^* : P^* \otimes P^* \to P^*\) defines an adm-Poisson algebra structure on \(P^*\). Then the following conditions are equivalent:

1. \((P, \star_P, \alpha)\) is an adm-Poisson bialgebra;
2. \((P, P^*, -R_P^*, -L_P^*, -R_P^*, -L_P^*)\) is a matched pair of adm-Poisson algebras;
3. \((P \oplus P^*, P, P^*)\) is a standard Manin triple of adm-Poisson algebras associated to \(\mathcal{B}_d\) defined by Eq. (4.3).

There is a one-to-one correspondence between adm-Poisson bialgebras and Poisson bialgebras.
Proposition 4.9. Let \((P, \ast, \alpha)\) be an adm-Poisson bialgebra. Define two linear maps \(\delta, \Delta : P \to P \otimes P\) by
\[
\delta = \frac{1}{2} (\alpha - \tau \alpha), \quad \Delta = \frac{1}{2} (\alpha + \tau \alpha).
\]
Then \((P, [ \cdot, \cdot ], \circ, \delta, \Delta)\) is a Poisson bialgebra. Conversely, let \((P, [ \cdot, \cdot ], \circ, \delta, \Delta)\) be a Poisson bialgebra. Define a linear map \(\alpha : P \to P \otimes P\) by
\[
\alpha = \delta + \Delta.
\]
Then \((P, \ast, \alpha)\) is an adm-Poisson bialgebra.

Proof. Let \(x, y \in P\). Set
\[
A(x, y) := -\delta([x, y]) + (\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))\delta(y) - (\text{ad}(y) \otimes \text{id} + \text{id} \otimes \text{ad}(y))\delta(x),
\]
\[
B(x, y) := -\Delta(x \circ y) + (\text{id} \otimes L_0(x))\Delta(y) + (L_0(y) \otimes \text{id})\Delta(x),
\]
\[
C(x, y) := -\delta(x \circ y) + (\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))\Delta(y) + (L_0(y) \otimes \text{id})\Delta(x),
\]
\[
D(x, y) := -\delta(x \circ y) + (L_0(y) \otimes \text{id})\delta(x) + (L_0(x) \otimes \text{id})\delta(y) + (\text{id} \otimes \text{ad}(x))\Delta(y) + (\text{id} \otimes \text{ad}(y))\Delta(x),
\]
\[
E(x, y) := -\alpha(x \ast y) + (R(y) \otimes \text{id})\alpha(x) + (\text{id} \otimes L(x))\alpha(y)
\]
\[
\quad + \frac{1}{3} ((L(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes L(y))\alpha(x) + (L(x) \otimes \text{id})\alpha(y) - (R(x) \otimes \text{id})\alpha(y)
\]
\[
\quad + \tau(-\alpha(x \ast y) + (L(x) \otimes \text{id})\alpha(y) + (L(y) \otimes \text{id})\alpha(x))),
\]
\[
F(x, y) := -\alpha(x \ast y) + (R(y) \otimes \text{id})\alpha(x) + (\text{id} \otimes L(x))\alpha(y)
\]
\[
\quad + \frac{1}{3} (-\alpha(y \ast x) + (L(x) \otimes \text{id})\alpha(y) + (L(y) \otimes \text{id})\alpha(x))
\]
\[
\quad + \tau((L(x) \otimes \text{id})\alpha(y) + (L(y) \otimes \text{id})\alpha(x) - (R(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes L(x))\alpha(y))),
\]
\[
G(x, y) := (L(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes R(y))\alpha(x) + \tau((L(x) \otimes \text{id})\alpha(y) - (\text{id} \otimes R(x))\alpha(y))
\]
\[
\quad + \frac{1}{3} ((R(y) \otimes \text{id})\alpha(x) - (\text{id} \otimes L(y))\alpha(x) + (\text{id} \otimes L(x))\alpha(y) - (R(x) \otimes \text{id})\alpha(y)
\]
\[
\quad + \tau(\alpha(y \ast x) - \alpha(x \ast y))).
\]
Note that \((P, [ \cdot, \cdot ], \circ, \delta, \Delta)\) is a Poisson bialgebra if and only if \((P^\ast, \delta^\ast, \Delta^\ast)\) is a Poisson algebra and
\[
A(x, y) = B(x, y) = C(x, y) = D(x, y) = 0 \quad (4.11)
\]
and \((P, \ast, \alpha)\) is an adm-Poisson bialgebra if and only if \((P^\ast, \alpha^\ast)\) is an adm-Poisson algebra and
\[
E(x, y) = F(x, y) = G(x, y) = 0. \quad (4.12)
\]
It is straightforward to show that \((P^\ast, \delta^\ast, \Delta^\ast)\) is a Poisson algebra if and only if \((P^\ast, \alpha^\ast)\) is an adm-Poisson algebra.

By a straightforward calculation, we have
\[
A(x, y) = -\frac{1}{4}E(x, y) + \frac{3}{4}E(y, x) + \frac{3}{4}F(x, y) - \frac{1}{4}F(y, x) - \frac{1}{4}G(x, y) + \frac{1}{4}G(y, x);
\]
\[
B(x, y) = \frac{7}{8}E(x, y) + \frac{5}{8}E(y, x) - \frac{3}{8}F(x, y) - \frac{3}{8}F(y, x) - \frac{3}{8}G(x, y) - \frac{1}{8}G(y, x) + \frac{3}{8}\tau G(y, x);
\]
\[
C(x, y) = \frac{1}{4}G(x, y) - \frac{1}{2}G(y, x) + \frac{3}{4}\tau G(x, y);
\]
\[
D(x, y) = -\frac{3}{4}E(x, y) - \frac{3}{4}E(y, x) + \frac{3}{4}F(x, y) + \frac{3}{4}F(y, x) + \frac{1}{4}G(x, y) + \frac{1}{4}G(y, x);
\]
\[ E(x, y) = \frac{2}{3} A(x, y) + \frac{4}{3} B(x, y) + \frac{1}{3} C(x, y) - C(y, x) + D(x, y) + \frac{1}{3} \tau D(x, y); \]
\[ F(x, y) = \frac{2}{3} A(x, y) + \frac{4}{3} B(x, y) - \frac{2}{3} C(y, x) + \frac{4}{3} D(x, y); \]
\[ G(x, y) = -\frac{2}{3} A(x, y) + \frac{4}{3} B(x, y) - \frac{4}{3} B(y, x) + \frac{4}{3} C(x, y) + \frac{2}{3} C(y, x). \]

Therefore Eq. (5.1) holds if and only if Eq. (5.12) holds. Hence the conclusion follows. \(\square\)

**Remark 4.10.** On the one hand, the above proof generalizes and illustrates explicitly the polarization-depolarization process ([22], [23], [24]) in the sense of bialgebras for Poisson bialgebras with one cocommutative comultiplication \(\Delta\) and one anti-cocommutative comultiplication \(\delta\) and adm-Poisson bialgebras with one comultiplication \(\alpha\). On the other hand, the above correspondence between adm-Poisson bialgebras and Poisson bialgebras also can be given equivalently in terms of Manin triples or matched pairs ([24]).

5. A special class of adm-Poisson bialgebras

In this section, we consider a special class of adm-Poisson bialgebras, that is, an adm-Poisson bialgebra \((P, \star, \alpha)\) with \(\alpha\) in the form
\[ \alpha(x) = (\text{id} \otimes L(x) - R(x) \otimes \text{id})r, \quad \forall x \in P. \] (5.1)

Such adm-Poisson bialgebras are quite similar as the “coboundary Lie bialgebras” for Lie algebras ([1] [11]) or the “coboundary infinitesimal bialgebras” for associative algebras ([1], [3]), although to our knowledge, a well-constructed cohomology theory for adm-Poisson algebras is not known yet.

**Proposition 5.1.** Let \((P, \star)\) be an adm-Poisson algebra and \(r \in P \otimes P\). Let \(\alpha : P \to P \otimes P\) be a linear map defined by Eq. (5.1). Then for all \(x, y \in P\), we have

(a) \(\alpha\) satisfies Eq. (5.3) if and only if
\[ (\text{id} \otimes L(x))(L(y) \otimes \text{id} - \text{id} \otimes R(y))(r + \tau(r)) + (\text{id} \otimes L(y))(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r)) - (L(x \star y) \otimes \text{id} - \text{id} \otimes R(x \star y))(r + \tau(r)) = 0; \] (5.2)

(b) \(\alpha\) satisfies Eq. (5.4) if and only if
\[ (\text{id} \otimes L(x))(L(y) \otimes \text{id} - \text{id} \otimes R(y))(r + \tau(r)) + (\text{id} \otimes L(y))(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r)) - (L(x) \otimes \text{id})(L(y) \otimes \text{id} - \text{id} \otimes R(y))(r + \tau(r)) - (\text{id} \otimes R(y))(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r)) = 0; \] (5.3)

(c) \(\alpha\) satisfies Eq. (5.5) if and only if
\[ (R(x) \otimes \text{id} - \text{id} \otimes L(x))(L(y) \otimes \text{id} - \text{id} \otimes R(y))(r + \tau(r)) + \frac{1}{3}((L(x \star y) - L(y \star x)) \otimes \text{id} - \text{id} \otimes (R(x \star y) - R(y \star x)))(r + \tau(r)) = 0. \] (5.4)

**Proof.** We only give an explicit proof for (a) as an example and omit the proof for (b) and (c) since it is similar. Let \(x, y \in P\). Substituting Eq. (5.3) into the Eq. (5.3), we have
\[ 0 = - (\text{id} \otimes L(x \star y))r + (R(x \star y) \otimes \text{id})r + (R(y) \otimes L(x))r - (R(y)R(x) \otimes \text{id})r + (\text{id} \otimes L(x)L(y)r - (R(y) \otimes L(x))r \]
\[ + \frac{1}{3}((L(y) \otimes L(x))r - (L(y)R(x) \otimes \text{id})r - (\text{id} \otimes L(y)L(x))r + (R(x) \otimes L(y))r \\
+ (L(x) \otimes L(y))r - (L(x)R(y) \otimes \text{id})r - (R(x) \otimes L(y))r + (R(x)R(y) \otimes \text{id})r \\
- (L(x \ast y) \otimes \text{id})\tau(r) + (\text{id} \otimes R(x \ast y))\tau(r) + (L(y) \otimes L(x))\tau(r) \\
- (\text{id} \otimes L(x)R(y))\tau(r) + (L(x) \otimes L(y))\tau(r) - (\text{id} \otimes L(y)R(x))\tau(r)) \\
= (A1) + (A2) + (A3) + (A4), \]

where

\[ (A1) = ((-R(y)R(x) + R(x \ast y) + \frac{1}{3}(-L(y)R(x) - L(x)R(y) + L(x)L(y) + R(y \ast x))) \otimes \text{id})r, \]

\[ (A2) = (\text{id} \otimes (-L(x \ast y) + L(x)L(y)) + \frac{1}{3}(-L(y)L(x) + L(y)R(x) + L(x)R(y) - R(x \ast y))) r, \]

\[ (A3) = \frac{1}{3}(-L(x)L(y) \otimes \text{id} - R(y \ast x) \otimes \text{id} + R(x)R(y) \otimes \text{id})r, \]

\[ (A4) = \frac{1}{3}((L(x \ast y) \otimes \text{id})\tau(r) + (\text{id} \otimes R(x \ast y))\tau(r) + (\text{id} \otimes R(x \ast y))r \\
+ (\text{id} \otimes L(x))(L(y) \otimes \text{id} - \text{id} \otimes R(y))(r + \tau(r)) \\
+ (\text{id} \otimes L(y))(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r)) + (\text{id} \otimes R(x \ast y))(r + \tau(r))) \]

Since \((L, R, P)\) is a representation of the adm-Poisson algebra \((P, \ast)\), we get

\[ (A1) = (A2) = 0, \quad (A3) = -\frac{1}{3}(L(x \ast y) \otimes \text{id})r. \]

Thus we have

\[ (A3) + (A4) = \frac{1}{3}((\text{id} \otimes L(x))(L(y) \otimes \text{id} - \text{id} \otimes R(y))(r + \tau(r)) \\
+ (\text{id} \otimes L(y))(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r)) \\
- (L(x \ast y) \otimes \text{id} - \text{id} \otimes R(x \ast y))(r + \tau(r))) = 0. \]

Therefore Eq. \((1.3)\) holds if and only if Eq. \((5.3)\) holds. \(\square\)

**Proposition 5.2.** Let \((P, \ast)\) be an adm-Poisson algebra and \(r = \sum a_i \otimes b_i \in P \otimes P\). Let \(\alpha : P \to P \otimes P\) be a linear map defined by Eq. \((5.4)\). Then \(\alpha\) satisfies Eq. \((5.10)\) if and only if \(r\) satisfies

\[ (R(x) \otimes \text{id} \otimes \text{id} - \text{id} \otimes \text{id} \otimes L(x))P(r) + \frac{1}{3}((\text{id} \otimes R(x) \otimes \text{id} - \text{id} \otimes \text{id} \otimes R(x))P(r) \\
+ (R(x) \otimes \text{id} \otimes \text{id} - \text{id} \otimes R(x) \otimes \text{id})Q(r) \\
- \sum_i (R(x) \otimes \text{id} \otimes \text{id})(a_i \otimes ((L(b_i) \otimes \text{id} - \text{id} \otimes R(b_i))(r + \tau(r)))) \\
- \sum_i ((\text{id} \otimes R(x) \otimes \text{id})(\tau \otimes \text{id}))(a_i \otimes ((L(b_i) \otimes \text{id} - \text{id} \otimes R(b_i))(r + \tau(r)))) \\
+ \sum_i (\text{id} \otimes \tau + \text{id})((L(a_i) \otimes \text{id})(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r)) \otimes b_i) \\
+ \sum_i (\tau \otimes \text{id} + \text{id})(a_i \otimes ((L(x \ast b_i) \otimes \text{id} - \text{id} \otimes R(x \ast b_i))(r + \tau(r)))) \\
- \sum_i a_i \otimes ((R(b_i) \otimes \text{id})(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r))) \\
- \sum_i ((R(a_i) \otimes \text{id})(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r))) \otimes b_i = 0, \quad (5.5) \]
for all \( x \in P \), where
\[
P(r) = r_{23} \ast r_{12} - r_{13} \ast r_{23} - r_{12} \ast r_{13}, \quad Q(r) = r_{12} \ast r_{23} - r_{23} \ast r_{13} - r_{13} \ast r_{12}.
\] (5.6)

**Proof.** Let \( x \in A \). Then we have
\[
(id \otimes \alpha)\alpha(x) - (\alpha \otimes id)\alpha(x) - (R(x) \otimes id)id - id \otimes id \otimes L(x))P(r)
= \sum_{i,j} \left( a_i \otimes a_j \otimes ((x \ast b_i) \ast b_j - x \ast (b_i \ast b_j)) + a_i \otimes ((a_j \ast x) \ast b_i - a_j \ast (x \ast b_i)) \otimes b_j
+ ((a_j \ast a_i) \ast x - a_j \ast (a_i \ast x)) \otimes b_j \right) + \frac{1}{3} \sum_{i,j} \left( a_i \otimes a_j \otimes (x \ast (b_j \ast b_i) - b_j \ast (x \ast b_i) - b_i \ast (x \ast b_j) + b_i \ast (b_j \ast x))
+ a_i \otimes (a_j \ast (b_j \ast x) - b_i \ast (a_j \ast x) - x \ast (a_j \ast b_i) + x \ast (b_i \ast a_j)) \otimes b_j
+ (a_j \ast (x \ast a_i) - x \ast (a_j \ast i) - a_i \ast (a_j \ast x) + a_i \ast (x \ast a_j)) \otimes b_j \otimes b_i \right).
\] (5.7)

Furthermore, by Eq. (3.3), we have
\[
a_i \otimes a_j \otimes (x \ast (b_j \ast b_i) + b_i \ast (b_j \ast x)) = a_i \otimes a_j \otimes ((x \ast b_j) \ast b_i + (b_i \ast b_j) \ast x)
= a_i \otimes a_j \otimes (x \ast b_j) \ast b_i + (id \otimes id \otimes R(x))(r_{13} \ast r_{12}).
\] (5.8)

Similarly, we have
\[
a_i \otimes (a_j \ast (b_j \ast x) + x \ast (b_i \ast a_j)) \otimes b_j = a_i \otimes (x \ast b_i) \ast a_j \ast b_j + (id \otimes R(x) \otimes id)(r_{23} \ast r_{12});
\] (5.9)
\[
a_i \otimes (b_i \ast (a_j \ast x) + x \ast (a_j \ast b_i)) \otimes b_i = a_i \otimes (x \ast a_j) \ast b_i \otimes b_j + (id \otimes R(x) \otimes id)(r_{12} \ast r_{23});
\] (5.10)
\[
(x \ast (a_j \ast a_i) + a_i \ast (a_j \ast x)) \otimes b_j \otimes b_i = (x \ast a_j) \ast a_i \otimes b_j \otimes b_i + (R_x \otimes id \otimes id)(r_{13} \ast r_{12}).
\] (5.11)

Substituting Eqs. (5.8)-(5.11) into Eq. (5.7), by a direct calculation, we have
\[
(id \otimes \alpha)\alpha(x) - (\alpha \otimes id)\alpha(x) - (R(x) \otimes id)id - id \otimes id \otimes L(x))P(r)
= \frac{1}{3} \left( \sum_{i,j} \left( a_i \otimes a_j \otimes (x \ast b_j) \ast b_i - a_i \otimes a_j \otimes b_j \ast (x \ast b_i) - a_i \otimes a_j \otimes b_i \ast (x \ast b_j)
+ a_i \otimes (x \ast b_i) \ast a_j \ast b_j - a_i \otimes (x \ast a_j) \ast b_i \otimes b_j - (x \ast a_j) \ast a_i \otimes b_j \otimes b_i
+ a_j \ast (x \ast a_i) \otimes b_i \otimes b_i + a_i \ast (x \ast a_j) \otimes b_j \otimes b_i - (id \otimes R(x) \otimes id)(r_{23} \ast r_{13})
- (id \otimes R(x) \otimes id)(r_{13} \ast r_{12}) - (id \otimes id \otimes R(x))(r_{12} \ast r_{23}) + (id \otimes R(x) \otimes id)(r_{12} \ast r_{13})
+ (id \otimes R(x) \otimes id)(r_{13} \ast r_{23}) + (id \otimes id \otimes R(x))(r_{23} \ast r_{12}) + (R_x \otimes id \otimes id)(r_{23} \ast r_{13})
- (R_x \otimes id \otimes id)(r_{12} \ast r_{23}) \right).
\]

On the other hand, we have
\[
(id \otimes \tau)(id \otimes \alpha)\alpha(x) - (\tau \otimes id)(id \otimes \alpha)\alpha(x) - (id \otimes \tau)(\tau \otimes id)(id \otimes \alpha)\alpha(x)
+ (\tau \otimes id)(id \otimes \tau)(id \otimes \alpha)\alpha(x)
= \sum_{i,j} \left( a_i \otimes (x \ast b_i) \ast b_j \otimes a_j - a_i \otimes b_j \otimes a_j \ast (x \ast b_i) - a_i \ast x \ast b_i \ast b_j \otimes a_j + a_i \ast x \otimes b_j \otimes a_j \ast b_i
- a_i \otimes a_j \otimes (x \ast b_j) \ast b_i + a_j \ast (x \ast b_i) \otimes a_i \otimes b_j + a_j \otimes a_i \ast x \otimes b_i \ast b_j - a_j \ast b_i \otimes a_i \ast x \otimes b_j
- a_j \otimes (x \ast b_i) \ast b_j \otimes a_j + a_j \ast (x \ast b_i) \otimes b_j \otimes a_i + a_j \otimes b_i \otimes a_i \ast x - a_j \ast b_i \otimes b_j \otimes a_i \ast x
+ (x \ast b_i) \ast b_j \otimes a_j - b_j \otimes a_i \otimes a_j \ast (x \ast b_i) - b_i \ast b_j \otimes a_i \ast x \otimes a_j + b_j \otimes a_i \ast x \otimes a_j \ast b_i \right).
\]
Thus we have
\[
(id \otimes a)\alpha(x) - (\alpha \otimes \text{id})\alpha(x) + \frac{1}{3}((id \otimes \tau)(id \otimes \alpha)\alpha(x) - (\tau \otimes \text{id})(id \otimes \alpha)\alpha(x)
\]
\[
- (id \otimes \tau)(\tau \otimes \text{id})(id \otimes \alpha)\alpha(x) + (\tau \otimes \text{id})(id \otimes \alpha)\alpha(x))
\]
\[
= (R(x) \otimes \text{id} \otimes \text{id} - \text{id} \otimes \text{id} \otimes L(x))P(r) + \frac{1}{3}((id \otimes R(x) \otimes \text{id} - \text{id} \otimes \text{id} \otimes R(x))P(r)
\]
\[
+ (R(x) \otimes \text{id} \otimes \text{id} - \text{id} \otimes R(x) \otimes \text{id})Q(r) \bigg) + \frac{1}{3}(-A - B + C + D - E - F),
\]
where
\[
A = \sum_{i,j} (a_i \star x \otimes b_j \ast a_j \otimes b_i) + a_i \star x \otimes b_i \ast b_j \otimes a_j - a_i \star x \otimes a_j \otimes b_i - a_i \star x \otimes b_j \ast a_j \ast b_i;
\]
\[
B = \sum_{i,j} (b_i \ast a_j \otimes a_i \ast x \otimes b_j + b_i \ast b_j \otimes a_i \ast x \otimes a_j - a_i \ast x \otimes b_i \ast b_j - b_i \ast a_i \ast x \otimes a_j \ast b_i);
\]
\[
C = \sum_{i,j} (a_i \ast (x \ast b_j) \otimes b_i \ast b_j + a_i \ast (x \ast b_j) \otimes b_j \ast b_i - a_i \ast a_j \otimes b_j \ast x \otimes b_i - a_i \ast b_j \otimes a_j \ast x \otimes b_i
\]
\[
+ a_i \ast (x \ast a_j) \otimes b_i \ast b_j + a_i \ast (x \ast b_j) \otimes b_i \ast a_j - a_i \ast a_j \ast b_i \ast b_j \ast x - a_i \ast b_j \otimes b_i \otimes a_j \ast x);
\]
\[
D = \sum_{i,j} (a_i \otimes (x \ast b_i) \ast a_j \otimes b_j + a_i \otimes (x \ast b_i) \ast b_j \ast a_j - a_i \otimes a_j \ast b_j \ast (x \ast b_i) - a_i \otimes b_j \otimes a_j \ast (x \ast b_i)
\]
\[
+ (x \ast b_i) \ast a_j \otimes a_i \ast b_j + (x \ast b_i) \ast b_j \otimes a_i \ast a_j - a_i \ast a_j \otimes b_i \ast (x \ast b_i) - b_j \otimes a_i \otimes a_j \ast (x \ast b_i));
\]
\[
E = \sum_{i,j} (a_i \otimes (x \ast a_i) \ast b_i \ast b_j + a_i \otimes (x \ast b_i) \ast b_i \ast a_j - a_i \otimes a_j \ast b_i \otimes b_j \ast x - a_i \otimes b_j \otimes b_i \otimes a_j \ast x);
\]
\[
F = \sum_{i,j} ((x \ast a_j) \ast a_i \otimes b_j \ast b_i + (x \ast b_j) \ast a_i \otimes a_j \ast b_i - a_j \ast a_i \otimes b_j \ast x \otimes b_i - b_j \ast a_i \otimes a_j \ast x \otimes b_i).
\]
Furthermore, it is straightforward to check that
\[
A = \sum_i (R(x) \otimes \text{id} \otimes \text{id})(a_i \otimes ((L(b_i) \otimes \text{id} - \text{id} \otimes R(b_i))(r + \tau(r))));
\]
\[
B = \sum_i ((id \otimes R(x) \otimes \text{id})(\tau \otimes \text{id}))(a_i \otimes ((L(b_i) \otimes \text{id} - \text{id} \otimes R(b_i))(r + \tau(r))));
\]
\[
C = \sum_i (id \otimes \tau \otimes \text{id})((L(a_i) \otimes \text{id})(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r)) \otimes b_i);
\]
\[
D = \sum_i (\tau \otimes \text{id} \otimes \text{id}((L(x \ast b_i) \otimes \text{id} - \text{id} \otimes R(x \ast b_i))(r + \tau(r))));
\]
\[
E = \sum_i a_i \otimes ((R(b_i) \otimes \text{id})(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r)));
\]
\[
F = \sum_i ((R(a_i) \otimes \text{id})(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r))) \otimes b_i.
\]
By Lemma 1.6, the conclusion follows.

Combining Propositions 5.1 and 5.2, we have the following result.

**Theorem 5.3.** Let \((P, \ast)\) be an adm-Poisson algebra and \(r \in P \otimes P\). Let \(\alpha : P \rightarrow P \otimes P\) be a linear map defined by Eq. (5.1). Then \((P, \ast, \alpha)\) is an adm-Poisson bialgebra if and only if \(r\) satisfies Eqs. (5.1)-(5.3).
Corollary 5.4. Let \((P, \ast)\) be an adm-Poisson algebra and \(r \in P \otimes P\). Let \(\alpha : P \to P \otimes P\) be a linear map defined by Eq. \((5.11)\). If \(r\) satisfies
\[
(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r)) = 0, \quad \forall x \in P,
\]
then \((P, \ast, \alpha)\) is an adm-Poisson bialgebra if and only if \(r\) satisfies
\[
(R(x) \otimes \text{id} \otimes \text{id} - \text{id} \otimes \text{id} \otimes L(x))P(r) + \frac{1}{3}((\text{id} \otimes R(x) \otimes \text{id} - \text{id} \otimes \text{id} \otimes R(x))P(r)
+ (R(x) \otimes \text{id} \otimes \text{id} - \text{id} \otimes R(x) \otimes \text{id})Q(r)) = 0, \quad \forall x \in P.
\]

Proof. Since \(r\) satisfies Eq. \((5.12)\), Eqs. \((5.2) - (5.4)\) hold naturally and Eq. \((5.3)\) is equivalent to Eq. \((5.13)\). The conclusion follows immediately. \(\square\)

Lemma 5.5. Let \((P, \ast)\) be an adm-Poisson algebra. If \(r \in P \otimes P\) satisfies Eq. \((5.12)\), then
\[
P(r) = r_{23} \ast r_{12} - r_{13} \ast r_{23} - r_{12} \ast r_{13} = 0
\]
if and only if
\[
Q(r) = r_{12} \ast r_{23} - r_{23} \ast r_{13} - r_{13} \ast r_{12} = 0.
\]

Proof. Let \(a\) and \(s\) be the skew-symmetric and symmetric parts of \(r\) respectively, that is,
\[
a = \frac{1}{2}(r - \sigma(r)), \quad s = \frac{1}{2}(r + \sigma(r)).
\]
Then \(r = a + s\). Let \(a^*, b^*, c^* \in P^*\). Then we have
\[
(r_{23} \ast r_{12} - r_{13} \ast r_{23} - r_{12} \ast r_{13})(a^*, b^*, c^*)
= -(a^*(c^*) \ast a^*(b^*) \ast c^*) - (a^*(b^*) \ast a^*(c^*) \ast a^*) - (a^*(c^*) \ast a^*(a^*) \ast b^*)
- (s^*(a^*) \ast s^*(b^*) \ast c^*) - (s^*(b^*) \ast s^*(c^*) \ast a^*) - (s^*(c^*) \ast s^*(a^*) \ast b^*)
- (a^*(a^*) \ast a^*(b^*) \ast c^*) + (a^*(b^*) \ast a^*(c^*) \ast a^*) + (a^*(b^*) \ast a^*(c^*) \ast a^*)
+ (a^*(c^*) \ast a^*(a^*) \ast b^*) - (a^*(a^*) \ast a^*(b^*) \ast c^*) - (a^*(b^*) \ast a^*(c^*) \ast a^*) - (a^*(c^*) \ast a^*(c^*) \ast a^*).
\]

Thus we have
\[
(r_{23} \ast r_{12} - r_{13} \ast r_{23} - r_{12} \ast r_{13})(a^*, b^*, c^*)
= 2(-a^*(c^*) \ast s^*(a^*) \ast b^*) + (s^*(c^*) \ast a^*(a^*) \ast b^*) - (a^*(a^*) \ast s^*(b^*) \ast c^*)
- (s^*(a^*) \ast a^*(b^*) \ast c^*) + (a^*(b^*) \ast s^*(c^*) \ast a^*) + (s^*(b^*) \ast a^*(c^*) \ast a^*).
\]

Furthermore, by Eq. \((5.12)\), we have
\[
\langle s^*(L(x)a^*), b^* \rangle = \langle a^*, s^*(R(x)b^*) \rangle, \quad \forall x \in P, a^*, b^* \in P^*.
\]

Then we have
\[
\langle a^*(c^*) \ast s^*(a^*), b^* \rangle = -\langle s^*(a^*), L^*(a^*(c^*))b^* \rangle = -\langle s^*(R^*(a^*(c^*))a^*), b^* \rangle = \langle s^*(b^*) \ast a^*(c^*) \rangle a^*.
\]

Similarly, we have
\[
\langle s^*(c^*) \ast a^*(a^*), b^* \rangle = \langle a^*(a^*) \ast s^*(b^*) \rangle c^*; \langle a^*(b^*) \ast s^*(c^*) \rangle a^* = \langle s^*(a^*) \ast a^*(b^*) \rangle c^*.
\]

Thus
\[
(r_{23} \ast r_{12} - r_{13} \ast r_{23} - r_{12} \ast r_{13})(a^*, b^*, c^*) = (r_{12} \ast r_{23} - r_{23} \ast r_{13} - r_{13} \ast r_{12})(c^*, b^*, a^*).
\]

The conclusion follows immediately. \(\square\)
Corollary 5.6. Let \((P, \star)\) be an adm-Poisson algebra and \(r \in P \otimes P\). Let \(\alpha : P \to P \otimes P\) be a linear map defined by Eq. (5.11). Then \((P, \star, \alpha)\) is an adm-Poisson bialgebra if \(r\) satisfies Eq. (5.11) and the following equation

\[
P(r) = r_{23} \star r_{12} - r_{13} \star r_{23} - r_{12} \star r_{13} = 0.
\]  

(5.16)

In particular, if \(r\) is skew-symmetric and \(r\) satisfies Eq. (5.10), then \((P, \star, \alpha)\) is an adm-Poisson bialgebra.

Proof. By Lemma 5.5, it follows as a direct consequence of Corollary 5.4.

Definition 5.7. Let \((P, \star)\) be an adm-Poisson algebra and \(r \in P \otimes P\). Eq. (5.10) is called the adm-Poisson Yang-Baxter equation (adm-PYBE) in \((P, \star)\).

Remark 5.8. The notion of adm-Poisson Yang-Baxter equation in an adm-Poisson algebra is due to the fact that it is an analogue of the classical Yang-Baxter equation in a Lie algebra \((\mathfrak{g}, [\, , \,])\) or the associative Yang-Baxter equation in an associative algebra \((A, \cdot)\).

It is an unexpected consequence that both the adm-Poisson Yang-Baxter equation in an adm-Poisson algebra and the associative Yang-Baxter equation \((\mathfrak{g}, [\, , \,])\) in an associative algebra have the same form (5.11) (also see Eq. (2.13)). Hence both these two equations have some common properties. Next we give two properties of the adm-Poisson Yang-Baxter equation whose proofs are omitted since the proofs are the same as in the case of the associative Yang-Baxter equation.

Proposition 5.9. Let \((P, \star)\) be an adm-Poisson algebra and \(r \in P \otimes P\) be skew-symmetric. Then \(r\) is a solution of adm-Poisson Yang-Baxter equation if and only if \(r\) satisfies

\[
r^2(a^\star) \star r^2(b^\star) = r(-r^*(r^2(a^\star))b^\star - L^*(r^2(b^\star))a^\star), \quad \forall a^\star, b^\star \in P^*.
\]  

(5.17)

Remark 5.10. Since the dual representations of both adm-Poisson algebras and associative algebras have the same form (see Remark 5.7), the interpretation of adm-Poisson Yang-Baxter equation in terms of the operator form (5.17) in the above Proposition 5.1 explains partly why the adm-Poisson Yang-Baxter equation has the same form as of the associative Yang-Baxter equation.

Theorem 5.11. Let \((P, \star)\) be an adm-Poisson algebra and \(r \in P \otimes P\). Suppose that \(r\) is skew-symmetric and nondegenerate. Then \(r\) is a solution of adm-Poisson Yang-Baxter equation in \((P, \star)\) if and only if the inverse of the isomorphism \(P^* \to P\) induced by \(r\), regarded as a bilinear form \(\omega\) on \(P\) (that is, \(\omega(x, y) = \langle r^2 \rangle^{-1}(x), y\rangle\) for all \(x, y \in P\)), satisfies

\[
\omega(x \star y, z) + \omega(y \star z, x) + \omega(z \star x, y) = 0, \quad \forall x, y, z \in P.
\]  

(5.18)

At the end of this section, we study the relationship between the adm-Poisson bialgebras \((P, \star, \alpha)\) with \(\alpha\) defined by Eq. (5.1) and the coboundary Poisson bialgebras given in Section 2.

Let \((P, [\, , \,], \circ, \delta, \Delta)\) be a coboundary Poisson bialgebra with \(\delta, \Delta\) defined by Eqs. (2.14) and (2.13) respectively. Define a linear map \(\alpha : P \to P \otimes P\) by

\[
\alpha(x) = \delta(x) + \Delta(x), \quad \forall x \in P.
\]  

(5.19)

Then we have

\[
\alpha(x) = \delta(x) + \Delta(x) = (id \otimes L(x) - R(x) \otimes id)r, \quad \forall x \in P.
\]

Therefore, by Proposition 5.13, every coboundary Poisson bialgebra naturally induces an adm-Poisson bialgebra structure \((P, \star, \alpha)\) with \(\alpha\) satisfying Eq. (5.1).
Conversely, let \((P, \star, \alpha)\) be an adm-Poisson bialgebra with \(\alpha\) defined by Eq. (5.1) through \(r \in P \otimes P\). Define two linear maps \(\delta, \Delta : P \rightarrow P \otimes P\) as follows:

\[
\delta(x) = \frac{1}{2}(\alpha(x) - \tau \alpha(x)), \quad \Delta(x) = \frac{1}{2}(\alpha(x) + \tau \alpha(x)), \quad \forall x \in P.
\]

(5.20)

By Proposition 4.9, \((P, [\; , ;], \circ, \delta, \Delta)\) is a Poisson bialgebra. Let \(x \in P\). Then we have

\[
\delta(x) = \frac{1}{2}[(\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))(r - \tau(r)) + (\text{id} \otimes L(x) - L(x) \otimes \text{id})(r + \tau(r))],
\]

(5.21)

\[
\Delta(x) = \frac{1}{2}[(\text{id} \otimes L(x) - L(x) \otimes \text{id})(r - \tau(r)) + (\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))(r + \tau(r))].
\]

(5.22)

Then \((P, [\; , ;], \alpha, r)\) is a coboundary Poisson bialgebra if and only if there exists \(r_1 \in P \otimes P\) satisfying

\[
(\text{id} \otimes L(x) - L(x) \otimes \text{id})(r + \tau(r)) = (\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))(r_1),
\]

(5.23)

\[
(\text{id} \otimes L(x) - L(x) \otimes \text{id})(r - \tau(r)) = (\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))(r_1).
\]

(5.24)

Moreover, Eqs. (5.23) and (5.24) hold if and only if the following equations hold:

\[
(\text{id} \otimes L(x) - R(x) \otimes \text{id})(r + \tau(r) - r_1) = 0,
\]

(5.25)

\[
(L(x) \otimes \text{id} - \text{id} \otimes R(x))(r + \tau(r) + r_1) = 0.
\]

(5.26)

Therefore for adm-Poisson bialgebras with \(\alpha\) defined by Eq. (5.1), the corresponding Poisson bialgebras given in Proposition 4.9 include but not limited to coboundary ones. Furthermore, if Eq. (5.12) holds (in particular, when \(r\) is symmetric), then Eqs. (5.23) and (5.26) hold with \(r_1 = r + \tau(r)\). Therefore in this case, an adm-Poisson bialgebra with \(\alpha\) defined by Eq. (5.1) induces a coboundary Poisson bialgebra.

Summarizing the above study, we have the following conclusion.

**Proposition 5.12.** Let \((P, [\; , ;], \circ, \delta, \Delta)\) be a coboundary Poisson bialgebra with \(\delta, \Delta\) defined by Eqs. (2.14) and (2.13) through \(r \in P \otimes P\) respectively. Then \((P, \star, \alpha)\) is an adm-Poisson bialgebra, where \((P, \star)\) is the corresponding adm-Poisson algebra and \(\alpha\) is defined by Eq (5.1) satisfying Eq. (5.1). Conversely, let \((P, \star, \alpha)\) be an adm-Poisson bialgebra with \(\alpha\) defined by Eq. (5.1) through \(r \in P \otimes P\). Then \((P, [\; , ;], \circ, r)\) is a coboundary Poisson bialgebra, where \((P, [\; , ;], \circ)\) is the corresponding Poisson algebra and \(\Delta, \delta\) are defined by Eq. (5.20), if and only if Eqs. (5.21) and (5.20) hold for some \(r_1 \in P \otimes P\). In particular, when \(r\) satisfies Eq. (5.13) or \(r\) is skew-symmetric, an adm-Poisson bialgebra \((P, \star, \alpha)\) with \(\alpha\) defined by Eq. (5.1) through \(r \in P \otimes P\) exactly corresponds to a coboundary Poisson bialgebra.

The relationship between the PYBE and the adm-PYBE is given as follows.

**Proposition 5.13.** Let \((P, \star)\) be an adm-Poisson algebra and \((P, [\; , ;], \circ)\) be the corresponding Poisson algebra. Let \(r \in P \otimes P\). If \(r\) is a solution of PYBE, then \(r\) is a solution of adm-PYBE. Conversely, if \(r\) satisfies Eq. (5.12) and \(r\) is a solution of adm-PYBE, then \(r\) is a solution of PYBE. In particular, if \(r\) is skew-symmetric, then \(r\) is a solution of PYBE if and only if \(r\) is a solution of adm-PYBE.

**Proof.** As the same as in the proof of Lemma 5.2, let \(r = a + s \in P \otimes P\) with the skew-symmetric part \(a\) and the symmetric part \(s\). Let \(a^*, b^*, c^* \in P^*\). Then we have

\[
(r_23 \circ r_13 - r_13 \circ r_23 - r_12 \circ r_13)(a^*, b^*, c^*)
\]

\[
= -\langle \text{ad}(c) \circ \text{id}(a^*), b^* \rangle - \langle \text{id}(c) \circ \text{ad}(a^*), b^* \rangle - \langle \text{ad}(b) \circ \text{id}(c^*), a^* \rangle + \langle \text{id}(b) \circ \text{ad}(c^*), a^* \rangle + \langle \text{id}(a) \circ \text{ad}(b^*), c^* \rangle
\]

\[
- \langle \text{ad}(a) \circ \text{id}(b^*), c^* \rangle - \langle \text{id}(b) \circ \text{ad}(c^*), a^* \rangle - \langle \text{id}(c) \circ \text{ad}(a^*), b^* \rangle
\]

\[
+ \langle \text{id}(c) \circ \text{ad}(a^*), b^* \rangle
\]
\[-\langle a^\ast(a^\ast) \circ s^\ast(b^\ast), c^\ast \rangle - \langle a^\ast(a^\ast) \circ a^\ast(b^\ast), c^\ast \rangle + \langle a^\ast(b^\ast) \circ a^\ast(c^\ast), a^\ast \rangle + \langle s^\ast(b^\ast) \circ a^\ast(c^\ast), a^\ast \rangle,\]
\[(r_{23}, r_{12}) + [r_{23}, r_{13}] + [r_{13}, r_{12}]) (a^\ast, b^\ast, c^\ast)\]
\[= - \langle [a^\ast(a^\ast), a^\ast(b^\ast)], c^\ast \rangle - \langle [a^\ast(b^\ast), a^\ast(c^\ast)], a^\ast \rangle + \langle s^\ast(c^\ast), s^\ast(a^\ast) \rangle, b^\ast \rangle\]
\[- \langle [s^\ast(a^\ast), s^\ast(b^\ast)], c^\ast \rangle - \langle [s^\ast(b^\ast), s^\ast(c^\ast)], a^\ast \rangle - \langle [a^\ast(c^\ast), s^\ast(a^\ast)], b^\ast \rangle + \langle [s^\ast(c^\ast), a^\ast(a^\ast)], b^\ast \rangle\]
\[- \langle [a^\ast(a^\ast), s^\ast(b^\ast)], c^\ast \rangle - \langle [s^\ast(b^\ast), a^\ast(c^\ast)], a^\ast \rangle + \langle [a^\ast(b^\ast), s^\ast(c^\ast)], a^\ast \rangle + \langle [s^\ast(b^\ast), a^\ast(c^\ast)], a^\ast \rangle.\]

By Eqs. (5.14) and (5.27), we have

\[(r_{23} \ast r_{12} - r_{13} \ast r_{23} - r_{12} \ast r_{13}) (a^\ast, b^\ast, c^\ast)\]
\[= (r_{23} \circ r_{12} - r_{13} \circ r_{23} - r_{12} \circ r_{13}) (a^\ast, b^\ast, c^\ast) + ([r_{23}, r_{12}] + [r_{23}, r_{13}] + [r_{13}, r_{12}]) (a^\ast, b^\ast, c^\ast).\]

Thus if \(r\) is a solution of PYBE, then \(r\) is a solution of adm-PYBE.

Conversely, suppose that \(r\) is a solution of adm-PYBE. By Eqs. (5.14) and (5.27), we have

\[2(r_{23} \circ r_{12} - r_{13} \circ r_{23} - r_{12} \circ r_{13}) (a^\ast, b^\ast, c^\ast)\]
\[= (r_{23} \ast r_{12} - r_{13} \ast r_{23} - r_{12} \ast r_{13}) (a^\ast, b^\ast, c^\ast) + (r_{23} \ast r_{12} - r_{13} \ast r_{23} - r_{12} \ast r_{13}) (b^\ast, a^\ast, c^\ast)\]
\[+ 2(\langle s^\ast(a^\ast) \ast s^\ast(c^\ast), b^\ast \rangle - \langle s^\ast(c^\ast) \ast s^\ast(b^\ast), a^\ast \rangle - \langle s^\ast(a^\ast) \ast a^\ast(c^\ast), b^\ast \rangle + \langle a^\ast(c^\ast) \ast s^\ast(b^\ast), a^\ast \rangle).\]

By Eq. (5.12), we have

\[\langle s^\ast(a^\ast) \ast s^\ast(c^\ast), b^\ast \rangle = \langle s^\ast(c^\ast) \ast s^\ast(b^\ast), a^\ast \rangle, \quad \langle s^\ast(a^\ast) \ast a^\ast(c^\ast), b^\ast \rangle = \langle a^\ast(c^\ast) \ast s^\ast(b^\ast), a^\ast \rangle.\]

Then \(r\) satisfies

\[r_{23} \circ r_{12} - r_{13} \circ r_{23} - r_{12} \circ r_{13} = 0,\]

that is, \(r\) is a solution of CYBE. Hence \(r\) is a solution of PYBE.

There is an equivalent expression of Theorem 2.3 as follows.

**Corollary 5.14.** Let \((P, [ , ], \circ)\) be a Poisson algebra and \(r \in P \otimes P\). Suppose that the linear maps \(\delta : P \rightarrow \wedge^2 P\) and \(\Delta : P \otimes P \rightarrow P\) are defined by Eqs. (2.14) and (2.15) respectively. Then \((P, [ , ], \circ, \delta, \Delta)\) is a Poisson bialgebra if and only if the following conditions are satisfied:

1. \((L_0(x) + \text{ad}(x)) \otimes \text{id} - \text{id} \otimes (L_0(x) - \text{ad}(x)))(r + \tau(r)) = 0,
2. \((L_0(x) - \text{ad}(x)) \otimes \text{id} - \text{id} \otimes \text{id} \otimes (L_0(x) + \text{ad}(x)))(A(r) + C(r))\]
\[+ \frac{1}{3} \left( (\text{id} \otimes (L_0(x) - \text{ad}(x)) \otimes \text{id} - \text{id} \otimes \text{id} \otimes (L_0(x) - \text{ad}(x))))(A(r) + C(r))\]
\[+ ((L_0(x) - \text{ad}(x)) \otimes \text{id} \otimes \text{id} - \text{id} \otimes (L_0(x) - \text{ad}(x)) \otimes \text{id}) (A(r) - C(r)) \right) = 0,

for all \(x \in P\).

**Proof.** Suppose that \((P, [ , ], \circ, \delta, \Delta)\) is a Poisson bialgebra. By Proposition 5.12, \((P, \ast, \alpha)\) is an adm-Poisson bialgebra, where \((P, \ast)\) is the corresponding adm-Poisson algebra and \(\alpha\) is defined by Eq. (5.13). Furthermore, by Conditions (1) and (2) in Theorem 2.3, we have

\[(L_0(x) + \text{ad}(x)) \otimes \text{id} - \text{id} \otimes (L_0(x) - \text{ad}(x)))(r + \tau(r)) = 0,

that is, Condition (a) holds. By Corollary 5.4, Eq. (5.13) holds, that is, for all \(x \in P\),

\[(R(x) \otimes \text{id} \otimes \text{id} - \text{id} \otimes \text{id} \otimes L(x))P(r) + \frac{1}{3} ((\text{id} \otimes R(x) \otimes \text{id} - \text{id} \otimes \text{id} \otimes R(x))P(r))\]
\[+ (R(x) \otimes \text{id} \otimes \text{id} - \text{id} \otimes R(x) \otimes \text{id}) Q(r) = 0,

where \(L = L_0 + \text{ad}\) and \(R = L_0 - \text{ad}\). Note that

\[P(r) = A(r) + C(r), \quad Q(r) = A(r) - C(r).\]
Thus Condition (b) holds. Conversely, let \((P, \star)\) be the corresponding adm-Poisson algebra and \(\alpha\) be the linear map defined by Eq. (7.19). If Condition (b) holds, then Eq. (5.13) holds with \(L = L_0 + \text{ad}\) and \(R = L_0 - \text{ad}\). Therefore by Condition (a) and Corollary 5.3, \((P, \star, \alpha)\) is an adm-Poisson bialgebra. By Proposition 5.12 and Condition (a) again, \((P, [\cdot, \cdot], \circ, \delta, \Delta)\) is a Poisson bialgebra.  

6. \(\mathcal{O}\)-operators of adm-Poisson algebras and pre-adm-Poisson algebras

In this section, we introduce the notions of \(\mathcal{O}\)-operators of adm-Poisson algebras and pre-adm-Poisson algebras to construct skew-symmetric solutions of adm-Poisson Yang-Baxter equation and hence to construct the induced adm-Poisson bialgebras. Note the notion of pre-adm-Poisson algebras given here is an equivalent presentation for the notion of pre-Poisson algebras given by Aguiar in [7], like the correspondence between the presentation with one operation and the usual presentation for Poisson algebras.

Definition 6.1. Let \((P, \star)\) be an adm-Poisson algebra and \((I, r, V)\) be a representation of \((P, \star)\). A linear map \(\theta : V \to P\) is called an \textbf{\(\mathcal{O}\)-operator of \((P, \star)\)} \textbf{associated to} \((I, r, V)\) if \(\theta\) satisfies

\[
\theta(u) \star \theta(v) = \theta((\theta(u))v + r(\theta(v))u), \quad \forall u, v \in V. \tag{6.1}
\]

Example 6.2. Let \((P, \star)\) be an adm-Poisson algebra. An \(\mathcal{O}\)-operator \(R\) associated to the representation \((L, R, P)\) is called a \textbf{Rota-Baxter operator of weight zero}, that is, \(R\) satisfies

\[
R(x) \star R(y) = R(R(x) \star y + x \star R(y)), \quad \forall x, y \in P. \tag{6.2}
\]

Example 6.3. Let \((P, \star)\) be an adm-Poisson algebra and \(r \in P \otimes P\). If \(r\) is skew-symmetric, then by Proposition 7.9 \(r\) is a solution of adm-Poisson Yang-Baxter equation in \((P, \star)\) if and only if \(r^2 : P^* \to P\) is an \(\mathcal{O}\)-operator associated to the representation \((-R^*, -L^*, P^*)\).

There is the following construction of (skew-symmetric) solutions of adm-Poisson Yang-Baxter equation in a semi-direct product adm-Poisson algebra from an \(\mathcal{O}\)-operator of an adm-Poisson algebra which is similar as for associative algebras ([4], Theorem 2.5.5), hence the proof is omitted.

Theorem 6.4. Let \((P, \star)\) be an adm-Poisson algebra and \((I, r, V)\) be a representation of the adm-Poisson algebra \((P, \star)\). Let \(\theta : V \to P\) be a linear map which is identified as an element in \(P \ltimes_{-\sigma, -\tau} V^* \otimes P \ltimes_{-\sigma, -\tau} V^*\). Then \(r = \theta - \tau(\theta)\) is a skew-symmetric solution of adm-PYBE in \(P \ltimes_{-\sigma, -\tau} V^*\) if and only if \(\theta\) is an \(\mathcal{O}\)-operator of \((P, \star)\) associated to the representation \((I, r, V)\).

Definition 6.5. A pre-adm-Poisson algebra is a triple \((A, \succ, \prec)\) such that \(A\) is a vector space, \(\succ, \prec : A \otimes A \to A\) are two bilinear operations satisfying the following conditions:

\[
A(x, y, z) := -(x \succ y) \succ z - (x \prec y) \succ z + x \succ (y \succ z) + \frac{1}{3}(x \succ (z \prec y)) + z \prec (x \succ y) - z \prec (x \prec y) - y \succ (x \succ z) + y \succ (z \prec x) = 0, \tag{6.3}
\]

\[
B(x, y, z) := -x \succ (z \prec y) + (x \succ z) \prec y + \frac{1}{3}(-(x \succ (y \succ z)) + y \succ (x \succ z)) + z \prec (x \prec y) - z \prec (y \prec x) - z \prec (y \prec x) = 0, \tag{6.4}
\]

\[
C(x, y, z) := -z \prec (x \succ y) - z \prec (x \prec y) + (z \prec x) \prec y + \frac{1}{3}(-z \prec (y \succ x)) - z \prec (y \prec x) + y \succ (z \prec x) + x \succ (z \prec y) - x \prec (y \succ z) = 0, \tag{6.5}
\]

for all \(x, y, z \in A\).
Proposition 6.6. Let \((A, \succ, \preceq)\) be a pre-adm-Poisson algebra. Define
\[
x \star y = x \succ y + x \preceq y, \quad \forall x, y \in A.
\] (6.6)

Then \((A, \star)\) is an adm-Poisson algebra, which is called the sub-adjacent adm-Poisson algebra of \((A, \succ, \preceq)\) and denoted by \(A^c\) and \((A, \succ, \preceq)\) is called the compatible pre-adm-Poisson algebra structure on the adm-Poisson algebra \(A^c\).

Proof. Let \((A, \succ, \preceq)\) be a pre-adm-Poisson algebra. For all \(x, y, z \in A\), we have
\[
- (x \star y) \star z + x \star (y \star z) + \frac{1}{3} ( - x \star (z \star y) + z \star (x \star y) + y \star (x \star z) - y \star (z \star x) ) = A(x, y, z) + B(x, y, z) + C(x, y, z),
\]
By Eqs. (6.3)-(6.5), we have \(A(x, y, z) = B(x, y, z) = C(x, y, z) = 0\). Hence \((A^c, \star)\) is an adm-Poisson algebra. \(\square\)

Remark 6.7. In fact, the operad \textit{PreadmPois} of pre-adm-Poisson algebras is the dissuccessor (splitting an operad into two pieces) of the operad \textit{admPois} of adm-Poisson algebras in the sense of [1]. Note that the operad \textit{Dend} of dendriform algebras introduced by Loday ([24]) is the dissuccessor of the operad \textit{Ass} of associative algebras. Hence pre-adm-Poisson algebras can be regarded as analogue structures of dendriform algebras with many similar properties ([1],[18]).

Proposition 6.8. Let \((A, \succ, \preceq)\) be a pre-adm-Poisson algebra. Then \((L_\succ, R_\preceq, A)\) is a representation of the sub-adjacent adm-Poisson algebra \((A^c, \star)\), where \(L_\succ, R_\preceq : A \to \text{End}_A(A)\) are defined by
\[
L_\succ(x) y = x \succ y, \quad R_\preceq(x) y = y \preceq x, \quad \forall x, y \in A. \quad (6.7)
\]

Conversely, if \((A, \star)\) is an adm-Poisson algebra together with two bilinear operations \(\succ, \preceq : A \otimes A \to A\) such that \((L_\succ, R_\preceq, A)\) is a representation of \((A, \star)\), then \((A, \succ, \preceq)\) is a pre-adm-Poisson algebra.

Proof. Eq. (6.3) implies that Eq. (6.1) holds, Eq. (6.4) implies that Eq. (6.2) holds with \(I = L_\succ\) and \(R = R_\preceq\). Thus \((L_\succ, R_\preceq, A)\) is a representation of the sub-adjacent adm-Poisson algebra \((A^c, \star)\). The converse can be proved similarly. \(\square\)

A direct consequence is given as follows.

Corollary 6.9. Let \((A, \preceq, \succ)\) be a pre-adm-Poisson algebra. Then the identity map \(\text{id}\) is an \(O\)-operator of the sub-adjacent adm-Poisson algebra \((A^c, \star)\) associated to the representation \((L_\succ, R_\preceq, A)\).

Definition 6.10. (a) ([15]) A left pre-Lie algebra is a vector space \(A\) together with a bilinear operation \(* : A \otimes A \to A\) such that
\[
x \star (y \star z) - (x \star y) \star z = y \star (x \star z) - (y \star x) \star z, \quad \forall x, y, z \in A. \quad (6.8)
\]
(b) ([16]) A left Zinbiel algebra is a vector space \(A\) together with a bilinear operation \(\cdot : A \otimes A \to A\) such that
\[
x \cdot (y \cdot z) = (y \cdot x) \cdot z + (x \cdot y) \cdot z, \quad \forall x, y, z \in A. \quad (6.9)
\]
(c) ([17]) A left pre-Poisson algebra is a triple \((A, \cdot, \star)\) such that \((A, \cdot)\) is a left Zinbiel algebra, \((A, \star)\) is a left pre-Lie algebra and the following conditions hold:
\[
(x \star y - y \star x) \cdot z = x \star (y \cdot z) - y \cdot (x \star z), \quad (x \cdot y + y \cdot x) \star z = x \cdot (y \star z) + y \cdot (x \star z), \forall x, y, z \in A. \quad (6.10)(6.11)
\]
Proposition 6.11. Let \((A, \cdot, \ast)\) be a left pre-Poisson algebra. Define
\[ x \circ y = x \cdot y + y \cdot x, \quad [x, y] = x \ast y - y \ast x, \quad \forall x, y \in A. \]
Then \((A, [\cdot, \cdot], \circ)\) is a Poisson algebra.

There exists a one-to-one correspondence between pre-adm-Poisson algebras and pre-Poisson algebras.

Proposition 6.12. If \((A, \succ, \prec)\) is a pre-adm-Poisson algebra, define
\[ x \cdot y = \frac{1}{2}(x \succ y + y \succ x), \quad x \ast y = \frac{1}{2}(x \succ y - y \succ x), \quad \forall x, y \in A, \]
then \((A, \cdot, \ast)\) is a Poisson algebra. Conversely, if \((A, \cdot, \ast)\) is a pre-Poisson algebra, define
\[ x \succ y = x \cdot y + x \ast y, \quad x \prec y = y \cdot x - y \ast x, \quad \forall x, y \in A, \]
then \((A, \succ, \prec)\) is a pre-adm-Poisson algebra.

Proof. It is straightforward. \(\square\)

Theorem 6.13. Let \((P, \ast)\) be an adm-Poisson algebra and \(\theta : V \rightarrow P\) be an \(\mathcal{O}\)-operator of the adm-Poisson algebra \(P\) associated to the representation \((I, \tau, V)\). Then there exists a pre-adm-Poisson algebra structure on \(V\) given by
\[ u \succ v = I(\theta(u))v, \quad u \prec v = \tau(\theta(v))u, \quad \forall u, v \in V. \quad (6.12) \]
So there is the sub-adjacent adm-Poisson algebra structure on \(V\) given by Eq. \((6.12)\) and \(\theta\) is a homomorphism of adm-Poisson algebras. Moreover, \(\theta(V) = \{\theta(v) | v \in V\} \subset P\) is an adm-Poisson subalgebra of \((P, \ast)\) and there is an induced pre-adm-Poisson algebra structure on \(\theta(V)\) given by
\[ \theta(u) \succ \theta(v) = \theta(u \succ v), \quad \theta(u) \prec \theta(v) = \theta(u \prec v), \quad \forall u, v \in V. \quad (6.13) \]
Its corresponding sub-adjacent adm-Poisson algebra structure on \(T(V)\) given by Eq. \((6.12)\) is just the adm-Poisson subalgebra structure of \((P, \ast)\) and \(\theta\) is a homomorphism of pre-adm-Poisson algebras.

Proof. By Eq. \((6.12)\), for all \(u, v, w \in V\), we have
\[
\begin{align*}
-&(u \succ v) \succ w - (u \prec v) \succ w + u \succ (v \succ w) \\
+&\frac{1}{3}(u \succ (w \prec v) - w \prec (u \succ v) - w \prec (u \prec v) - v \prec (u \succ v) + v \succ (w \prec u))
\end{align*}
\]
\[
= -l(\theta(I(\theta(u))v))w - l(\theta(\tau(\theta(v))u)w + l(\theta(\theta(u)))l(\theta(v))w + \frac{1}{3}(l(\theta(u))\tau(\theta(v))w) - \tau(\theta(I(\theta(v))v))w - \tau(\theta(\theta(v)))l(\theta(u))w + l(\theta(v))\tau(\theta(u))w)
\]
\[
= -l(\theta(u) \ast \theta(v)) + l(\theta(u))l(\theta(v)) + \frac{1}{3}(l(\theta(v))\tau(\theta(u)) - \tau(\theta(u) \ast \theta(v)) - l(\theta(v)))l(\theta(u))w = 0.
\]
Hence Eq. \((6.3)\) holds. Similarly, by Eq. \((6.3)\), Eq. \((6.4)\) holds and by Eq. \((6.3)\), Eq. \((6.7)\) holds. Therefore, \((V, \prec, \succ)\) is a pre-adm-Poisson algebra. The rest is straightforward. \(\square\)

Finally we give the following construction of skew-symmetric solutions of adm-Poisson Yang-Baxter equation (hence adm-Poisson bialgebras) from pre-adm-Poisson algebras.
Proposition 6.14. Let \((A, >, <)\) be a pre-adm-Poisson algebra. Then

\[ r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i) \]  

(6.14)

is a solution of adm-Poisson Yang-Baxter equation in the adm-Poisson algebra \(A \otimes_{L^*_1} A^*\), where \(\{e_1, \cdots, e_n\}\) is a basis of \(A\) and \(\{e_1^*, \cdots, e_n^*\}\) is its dual basis.

**Proof.** Note that id = \(\sum_{i=1}^{n} e_i \otimes e_i^*\). The conclusion follows from Theorem 6.4 and Corollary 6.9. \(\square\)

**Acknowledgements.** This work is supported by NSFC (11931009, 11901501). C. Bai is also supported by the Fundamental Research Funds for the Central Universities and Nankai Zhida Foundation.

**REFERENCES**

[1] M. Aguiar, On the associative analog of Lie bialgebras. *J. Algebra* 244 (2001), 492-532.
[2] M. Aguiar, Pre-Poisson algebras. *Lett. Math. Phys.* 54 (2000), 263-277.
[3] V.I. Arnol’d, Mathematical Methods of Classical Mechanics. *Graduate Texts Math.* 60, Springer-Verlag, New York-Heidelberg, 1978.
[4] C. Bai, A unified algebraic approach to the classical Yang-Baxter equation. *J. Phys. A: Math. Gen.* 40 (2007), 11073-11082.
[5] C. Bai, Double constructions of Frobenius algebras, Connes cocycles and their duality. *J. Noncommu. Geom.* 4 (2010), 475-530.
[6] C. Bai, O. Bellier, L. Guo and X. Ni, Splitting of operations, Manin products and Rota-Baxter operators. *Int. Math. Res. Not.* (2013), no. 3, 485-524.
[7] C. Bai, L. Guo and X. Ni, \(O\)-operators on associative algebras and associative Yang-Baxter equations. *Pac. J. Math.* 256 (2012), 257-289.
[8] C. Bai, L. Guo and X. Ni, Relative Rota-Baxter operators and tridendriform algebras. *J. Algebra Appl.* 12 (2013), 1350027.
[9] V. Chari and A. Pressley, A Guide to Quantum Groups. Cambridge University Press, Cambridge, 1994.
[10] P.A.M. Dirac, Lectures on Quantum Mechanics. *Belfer Grad. Sch. Sci. Monogr. Ser.*, Yeshive University, New York, 1964.
[11] V.G. Drinfeld, Hamiltonian structure on the Lie groups, Lie bialgebras and the geometric sense of the classical Yang-Baxter equations. *Soviet Math. Dokl.* 27 (1983), 68-71.
[12] V.G. Drinfeld, Quantum groups. *Proc. Internat. Congr. Math.* (Berkeley, 1986), Amer. Math. Soc., Providence, RI, 1987, 798-820.
[13] M. Gerstenhaber, The cohomology structure of an associative ring. *Ann. Math.* 78 (1963), 267-288.
[14] V. Ginzburg and D. Kaledin, Poisson deformations of symplectic quotient singularities. *Adv. Math.* 186 (2004), 1-57.
[15] M. Goze and E. Remm, Cogèbres Lie-admissibles - Lie-admissible coalgebras. arXiv:math/0502318.
[16] M. Goze and E. Remm, Poisson algebras in terms of non-associative algebras. *J. Algebra* 320 (2008), 294-317.
[17] J. Huebschmann, Poisson cohomology and quantization. *J. Reine Angew. Math.* 408 (1990), 57-113.
[18] M. Kontsevich, Deformation quantization of Poisson manifolds. *Lett. Math. Phys.* 66 (2003), 157-216.
[19] B.A. Kupershmidt, What a classical \(r\)-matrix really is. *J. Nonlinear Math. Phys.* 6 (1999), 448-488.
[20] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées (French). *J. Diff. Geom.* 12 (1977), 253-300.
[21] J. Liu, C. Bai and Y. Sheng, Noncommutative Poisson bialgebras. *J. Algebra* 556 (2020), 35-66.
[22] M. Livernet and J.-L. Loday, The Poisson operad as a limit of associative operads. Unpublished preprint, March 1998.
[23] J.-L. Loday, Cup product for Leibniz cohomology and dual Leibniz algebras. *Math. Scand.* 77 (1995), 189-196.
[24] J.-L. Loday, Dialgebras and related operads. *Lecture Notes in Math.* 1763, Springer-Verlag, Berlin, 2001.
[25] J.-L. Loday and B. Vallette, Algebraic Operads. *Grundlehren Math. Wiss.* 346, Springer, Heidelberg, 2012.
[26] M. Markl and E. Remm, Algebras with one operation including Poisson and other Lie-admissible algebras. *J. Algebra* 299 (2006), 171-189.
[27] X. Ni and C. Bai, Poisson bialgebras. *J. Math. Phys.* 54 (2013), 023515.
[28] A. Odzijewicz, Hamiltonian and quantum mechanics. *Geom. Topol. Monogr.* 17 (2011), 385-472.
[29] C. Ospel, F. Panaite and P. Vanhaecke, Polarization and deformations of generalized dendriform algebras. arXiv:19120911.

[30] A. Polishchuk, Algebraic geometry of Poisson brackets. J. Math. Sci. 84 (1997), 1413-1444.

[31] E. Remm, Poisson superalgebras as nonassociative algebras. arXiv:1205.2910.

[32] E. Remm, Weakly associative algebras, Poisson algebras and deformation quantization. Comm. Algebra 49 (2021), 3881-3904.

[33] R. Schafer, An Introduction to Nonassociative Algebras. Pure and Applied Mathematics 22, Academic Press, New York-London, 1966.

[34] A. Weinstein, Lecture on Symplectic Manifolds. CBMS Regional Conference Series in Mathematics 29, Amer. Math. Soc., Providence, R.I., 1979.

[35] I. Vaisman, Lectures on the Geometry of Poisson Manifolds. Progr. Math. 118, Birkhäuser Verlag, Basel, 1994.

Department of Mathematics, Michigan State University, East Lansing, MI, 48823, U.S.A
Email address: liangj26@msu.edu

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
Email address: liujf120126.com

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, China
Email address: baicm@nankai.edu.cn