Free-Fermion Multiply Excited Eigenstates and Their Experimental Signatures in 1D Arrays of Two-Level Atoms

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One-dimensional (1D) subwavelength atom arrays display multiply excited subradiant eigenstates which are reminiscent of free fermions. So far, these states have been associated with subradiant states with decay rates \( \propto N^{-3} \), with \( N \) the number of atoms, which fundamentally prevents detection of their fermionic features by optical means. In this Letter, we show that free-fermion states generally appear whenever the band of singly excited states has a quadratic dispersion relation at the band edge and, hence, may also be obtained with radiant and even superradiant states. 1D arrays have free-fermion multiply excited eigenstates that are typically either subradiant or (super)radiant, and we show that a simple transformation acts between the two families. Based on this correspondence, we propose different means for their preparation and analyze their experimental signature in optical detection.

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Subwavelength atom arrays support extremely subradiant states and significant optical nonlinearity [1], which bring promises for photon storage [2,3], lossless mirrors [4], and quantum metasurfaces [5], etc. Collective effects in subwavelength atom arrays can be described by the resonant dipole-dipole interaction (RDDI) mediated by the quantized radiation field. The RDDI Hamiltonians are long-range flip-flop spin models, and their multiply excited eigenstates display strong spatial correlations. Such states have been obtained through studies of 1D atom arrays coupled to waveguides [3,6–22] and to the free space vacuum [3,23–30]. For the waveguide case where the RDDI Hamiltonian has a simple form [31], a variety of spatial correlations of the two-excitation eigenstates have been obtained, see a summary of results in Ref. [32]. Slater determinants formed by singly excited eigenstates, so-called free-fermion states [3], are regarded as a generic family of subradiant states in 1D atom arrays which is also expected to appear for RDDI mediated by other fields, e.g., the free space vacuum field [4], and fields supported by hyperbolic metamaterials [33] or photonic crystal waveguides [34], etc. Within the ideal 1D waveguide model [7], we have shown a mapping of the RDDI Hamiltonian to the Lieb-Liniger model [35] and interpreted the free-fermion subradiant states as a Tonks-Girardeau gas of hard-core bosons [36,37].

However, two questions about the free-fermion states have remained unresolved. First, a conclusive proof has not been given for their general existence. Second, there has been a lack of methods for their detection, because the optical emission from the free-fermion subradiant states is suppressed by a factor of \( N^{-3} \) [3], with \( N \) the number of atoms.

In this Letter, the first question is addressed by a system-independent approach. We find that, if the band of singly excited states induced by RDDI has a quadratic extremum point \( k_{ex} \), i.e., its dispersion relation can be expanded as \( \omega_{\text{eff}}(k) \approx \omega_{\text{eff}}(k_{ex}) + a_{2}(k - k_{ex})^{2} \) for \( k \approx k_{ex} \) with \( a_{2} \) an expansion coefficient, there is a family of free-fermion multiply excited eigenstates defined in the vicinity of \( k_{ex} \). This result provides a sufficient condition for the generic existence of the free-fermion eigenstates, and it dismisses the notion that they must be extremely subradiant: There exist free-fermion states with finite decay rates, and this paves new ways for their experimental detection. Thus, we propose two schemes to prepare and detect radiant (and even superradiant) free-fermion multiply excited eigenstates of 1D atom arrays.

**Preliminaries.**—We consider atoms with two levels, the ground state \( |g\rangle \) and an excited state \( |e\rangle \), between which the energy gap is \( \omega_{0} (\hbar = 1) \). In a regular 1D array, the atoms are equally spaced with coordinates \( z_{m} = md \). The light field can be specified by its dyadic Green’s tensor \( G \). Assuming the Born-Markov approximation and translation symmetry of the light field in the direction along the array, the effective RDDI Hamiltonian is expressed as [38–40]

\[
H_{\text{eff}} = -\mu_{0} \omega_{0}^{2} \sum_{m,n=1}^{N} \mathbf{d}_{m} \cdot G(z_{m} - z_{n}, \omega_{0}) \cdot \mathbf{d}_{n} \sigma_{m}^{\dagger} \sigma_{n},
\]

where \( \mu_{0} \) denotes the vacuum permeability, \( \mathbf{d} \) is the transition dipole moment, and \( \sigma^{\dagger} = |e\rangle \langle g| \). The Hamiltonian (1) can be rewritten in Fourier space as

\[
H_{\text{eff}} = -\mu_{0} \omega_{0}^{2} \sum_{m,n=1}^{N} \mathbf{d}_{m} \cdot G_{\text{eff}}(\mathbf{k}, \omega_{0}) \cdot \mathbf{d}_{n} \sigma_{m}^{\dagger} \sigma_{n},
\]
\[ H_{\text{eff}} = Nd \int_{-\pi/d}^{\pi/d} \frac{dk}{2\pi} \omega_{\text{eff}}(k) \sigma^\dagger_k \sigma_k, \quad (2) \]

where the spin-wave operator reads
\[ \sigma^\dagger_k = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{i k \varpi_m} \sigma^\dagger_m, \quad (3) \]

and \( \omega_{\text{eff}}(k) \) is the complex dispersion relation of the band of singly excited eigenstates. For an infinite array, the state \( |k\rangle \equiv \sigma^\dagger_k |G\rangle \), with \( |G\rangle \) the atomic ground state, has the energy \( \Re \omega_{\text{eff}}(k) \) and decay rate \( \gamma(k) = -2 \Im \omega_{\text{eff}}(k) \), where \( \Re \) and \( \Im \) denote the real and imaginary parts, respectively.

According to Eq. (2), \( H_{\text{eff}} \) is fully specified by the single excitation dispersion relation \( \omega_{\text{eff}}(k) \). Thus, any generic feature of its eigenstates must reflect a common mathematical property of \( \omega_{\text{eff}}(k) \). We note that, for finite \( N \), Eq. (2) does not diagonalize the Hamiltonian (1) because of nontrivial spin commutator relations [\( \sigma_i, \sigma^\dagger_{i'} \neq \delta(k-k') \)]. This leads to a rich variety of multiply excited states and the free-fermion state is only one of them, see, e.g., recent work on atom arrays coupled to a 1D waveguide [32]. The free-fermion states, however, are special as they exist for any Hamiltonian (1) as long as its \( \omega_{\text{eff}}(k) \) has a quadratic band edge \( k_{\text{ex}} \) where \( \omega_{\text{eff}}(k) \approx \omega_{\text{eff}}(k_{\text{ex}}) + a_2(k-k_{\text{ex}})^2 \).

**Free-fermion states.**—We shall show that \( H_{\text{eff}} \) can be approximated (near the band edge) by a simpler Hamiltonian
\[ H_1 = c_1 \tilde{N}_e - \frac{a_2}{d^2} \sum_{j=1}^{N-1} (e^{-ik_{\text{ex}}d} \sigma^\dagger_j \sigma_{j+1} + e^{ik_{\text{ex}}d} \sigma^\dagger_{j+1} \sigma_j), \quad (4) \]

where \( c_1 = \omega_{\text{eff}}(k_{\text{ex}}) + 2a_2/d^2 \) and \( \tilde{N}_e = \sum_{j=1}^{N} \sigma^\dagger_j \sigma_j \).

The dispersion relation of \( H_1 \) is \( \omega_1(k) = c_1 - 2a_2/d^2 \cos[(k-k_{\text{ex}})d] \), which equals \( \omega_{\text{eff}}(k) \) of \( H_{\text{eff}} \) near \( k_{\text{ex}} \). \( H_1 \) can be exactly diagonalized by the Jordan-Wigner transformation \[ \sigma^\dagger = e^{i\varpi \sum_{m=1}^{N} f^\dagger_m f_m} \] and its Hermitian conjugate, where \( f_m \) and \( f^\dagger_m \) are fermionic operators satisfying the anticommutation relations \( \{f_i, f^\dagger_{i'}\} = \delta_{i,i'} \) and \( \{f_i, f_{i'}\} = 0 \). The transformation leads to
\[ H_1 = \sum_{\bar{\xi}=1}^{N} \omega_1(k_{\text{ex}} + q_{\bar{\xi}}) f^\dagger_{\bar{\xi}} f_{\bar{\xi}}, \quad (5) \]

where \( f_{\bar{\xi}} = \sum_{j=1}^{N} (\psi^\dagger_{\bar{\xi}} \sigma^\dagger_j |G\rangle f_j \) is the annihilation operator for the single-excitation orthonormal mode
\[ \psi^\dagger_{\bar{\xi}} = \frac{1}{\sqrt{2}} (\sigma^\dagger_{k_{\text{ex}}+q_{\bar{\xi}}} - \sigma^\dagger_{k_{\text{ex}}-q_{\bar{\xi}}})|G\rangle, \quad (6a) \]

with
\[ q_{\bar{\xi}} = \frac{\pi/d}{N+1}, \quad 1 \leq \bar{\xi} \leq N. \quad (6b) \]

An eigenstate of \( H_1 \) with \( n_e \) excitations has the form
\[ |F^e_{\bar{\xi}}\rangle = f^e_{\bar{\xi}_1} f^e_{\bar{\xi}_2} \cdots f^e_{\bar{\xi}_{n_e}} |G\rangle, \quad (7) \]

where \( g \) denotes the string \( \bar{\xi}_1 \leq \bar{\xi}_2 \leq \cdots \leq \bar{\xi}_{n_e} \). The state lives in the vicinity of \( k_{\text{ex}} \) if \( \frac{\pi}{N} \ll \Delta \).

The idea of studying \( H_{\text{eff}} \) using a simpler Hamiltonian was recently used to prove a power-law scaling of the decay rates of the singly excited subradiant states [42]. As in [42], to prove that \( H_1 \) approximates \( H_{\text{eff}} \), the idea is to write \( H_{\text{eff}} = H_1 + \Delta H \) and show that \( \Delta H \) can be treated as a perturbation to \( H_1 \). To proceed, we write \( \Delta H \) in the form of Eq. (2) with a dispersion relation \( \delta \omega(k) = \omega_{\text{eff}}(k) - \omega_1(k) \) and evaluate the perturbative expression
\[ \langle F_{\bar{\xi}} | \Delta H | F_{\bar{\xi}} \rangle = Nd \int_{-\pi/d}^{\pi/d} \frac{dk}{2\pi} \delta \omega(k) \langle F_{\bar{\xi}} | \sigma^\dagger_k \sigma_k | F_{\bar{\xi}} \rangle. \quad (8) \]

By definition, \( \delta \omega(k) \) scales as \( N^{-3} \) for \( |k - k_{\text{ex}}| \sim N^{-1} \), but generally, as \( O(1) \) outside the neighborhood of \( k_{\text{ex}} \). We assume that \( \omega_{\text{eff}}(k_{\text{ex}}) \) is not degenerate with \( \omega_{\text{eff}}(k) \) at other wave numbers, hybridization of these states may require further treatment.) We derive in the Supplemental Material [43] that the occupation \( \langle F_{\bar{\xi}} | \sigma^\dagger_k \sigma_k | F_{\bar{\xi}} \rangle \) scales as \( O(1) \) for \( |k - k_{\text{ex}}| \sim N^{-1} \) and as \( N^{-4} \) elsewhere. Thus, Eq. (8) yields the scaling \( \langle F_{\bar{\xi}} | \Delta H | F_{\bar{\xi}} \rangle \sim N^{-3} \), which is a factor \( N \) smaller than the separation of the eigenvalues of \( H_1 \). The same scaling also holds for off-diagonal terms \( \langle F_{\bar{\xi}} | \Delta H | F_{\bar{\xi'}} \rangle \) where \( \bar{\xi} \neq \bar{\xi}' \). Therefore, \( \Delta H \) can be consistently viewed as a perturbation to \( H_1 \), and \( \langle F_{\bar{\xi}} \rangle \) are the leading order eigenstates of \( H_{\text{eff}} \).

Our result solidifies the following physical argument: A quadratic \( \omega_{\text{eff}}(k) \) corresponds to a kinetic energy that can be represented by \( \alpha (\partial f)^2 \). Discrete versions of this operator reduce to nearest-neighbor tunneling. Therefore, although displaying long-range hopping terms, \( H_{\text{eff}} \) can be approximated by \( H_1 \) and give rise to Jordan-Wigner fermions.

**Experimental signatures.**—No assumption about subradiance is applied above. Radiant and even superradiant free-fermion states are obtained if \( |k_{\text{ex}}| \) is smaller than \( k_0 = \omega_0/c \) (\( c \) is the speed of light). Within the Markov approximation, the electric field (positive frequency part) of the emission from the atoms reads
\[ \hat{E}^{(+)}(r) = \mu_0 \omega_0^2 \sum_{j=1}^{N} \mathbf{G}(r - r_j, \omega_0) \cdot \mathbf{d}_j \sigma_j(t). \quad (9) \]

In the far field, \( \mathbf{G}(r, \omega_0) \propto r^{-1} e^{i k_0 r} \mathbf{f}(\theta, \phi) \), where \( \mathbf{f}(\theta, \phi) \) is the radiation pattern at polar angle \( \theta \) and azimuthal
angle $\phi$ [44]. Thus, $\hat{E}^+ (\theta) \propto \sqrt{N}\sigma_{\text{eff}} \cos \theta$ and quantities in the form of $\langle \sigma_k^+ \sigma_q \rangle$ and $(\sigma_k^+ \sigma_q \sigma_k \sigma_q)$, etc., can be efficiently measured if $k, q \in [k_0, k_0]$. Thus, we propose two experimental schemes for the study of signatures of the free-fermion states.

Detection scheme 1.—First, we note that 1D atom arrays usually have two band edges, which are $k_{\text{ex,0}} = 0$ and $k_{\text{ex,1}} = \pi / d$ if the system satisfies parity symmetry $\omega_{\text{eff}}(k) = \omega_{\text{eff}}(-k)$. The implied free-fermion states are denoted by $|F^0_{\xi}\rangle$ and $|F^\pi_{\xi}\rangle$, respectively. The states $|F^\pi_{\xi}\rangle$ have been previously recognized as subradiant states when $d < \pi / k_0$, while states $|F^0_{\xi}\rangle$ are radiant and even super-radiant. A one-to-one correspondence between members of these two families is established by a single unitary $U_x = \mathcal{O}_{m=1}^N |\psi_m\rangle \langle \psi_m| + (-1)^m |\psi_m\rangle \langle \psi_m|$, so that $U_x |F^0_{\xi}\rangle = |F^\pi_{\xi}\rangle$ and $U_x |F^\pi_{\xi}\rangle = |F^0_{\xi}\rangle$ for any $\xi$, $U_x$ factorizes and can be realized, e.g., by geometric phase control [45].

Subradiant states are difficult to excite directly by external lasers. Instead, we can initialize the array in the symmetric state, e.g., by geometric phase control [45]. Subsequent atom dynamics, the atomic state follows the trajectory $\Psi_{\text{eff}}(\theta) \propto e^{-iH_{\text{eff}} t} U_x |\psi_0\rangle$ and gradually converges to the most long-lived component in the eigenstate expansion of $U_x |B_{0,0,0}\rangle$. This state, in turn, is dominated by the desired free-fermion state, $|F^\pi_{1,2,3}\rangle$ (note that atom arrays coupled to a 1D waveguide may result in bound states with even longer lifetime [8,9]).

We can arrest the “cooling” at any time and apply $U_x$ to convert the state to a symmetric state around $k_{\text{ex,0}}$. Importantly, the fact that $U_x$ maps uniformly between $|F^0_{\xi}\rangle$ and $|F^\pi_{\xi}\rangle$ for any $\xi$ implies little deformation of the state during preparation. Then, the time evolution is governed by the master equation master equation

$$i \frac{d}{dt} \rho = H_{\text{eff}} \rho - \rho H_{\text{eff}}^\dagger + i \sum_{\xi = 1}^N \gamma_\xi \sigma_{\phi_\xi} \rho \sigma_{\phi_\xi}^\dagger,$$

where $\sigma_{\phi_\xi} = \sum_{j=1}^N \langle \phi_\xi | \sigma_j^\dagger | G \rangle \sigma_j \gamma_\xi$ and $|\phi_\xi\rangle$ are eigenvalues and eigenstates of $2H_{\text{eff}}^\dagger$, the dissipative part of $H_{\text{eff}}$ defined through $H_{\text{eff}} = H_{\text{eff}}^\dagger - i H_{\text{eff}}^\dagger$ [46], and the wave number subscript $\xi$ is counted with respect to $k_{\text{ex,0}}$ in the manner of Eq. (6a). Since $k_{\text{ex,0}}$ is also a quadratic band edge of $\omega_{\text{eff}}(k)$, we have $|\phi_\xi\rangle \approx |\psi_\xi\rangle$ for $\xi \approx N$.

We consider a 1D atom array in vacuum in 3D space with $d = \lambda_0 / 4$ ($\lambda_0 = 2\pi / k_0$) and $N = 20$, where the atomic transition dipoles $d$ are aligned parallel to the array [43]. Such systems can be realized with subwavelength optical lattices [47–51]. We refer the reader to Ref. [48] for a thorough discussion of the influence of atomic motion, which is ignored here. In Fig. 1(a), we simulate the evaporative cooling process and plot the fidelities $F_k = \langle |\psi_{\text{eff}}| U_x |B_{0,0,0}\rangle \rangle^2$ (blue line) and $F_{\text{eff}} = \langle |\psi_{\text{eff}}| F_{\text{eff}} \rangle$ (red line), respectively. Along the no-jump trajectory, the atomic state coincides with the symmetrically excited state $|F^\pi_{1,2,3}\rangle$ by a lower peak and quantities in $k_0$. These states are acted upon by $U_x$ and then used as the initial state for the simulation of radiative emission governed by (10). In Fig. 1(b), we plot the renormalized photon axial momentum distribution $P_k$ for $|B_{0,0,0}\rangle$, $U_x |\psi_{\text{eff}}\rangle$ and $|F^\pi_{1,2,3}\rangle$, respectively. The log scale inset shows the central parts of $P_k$.

FIG. 1. Detection scheme 1. (a) Dissipative state evolution through the triply excited states $U_x |B_{0,0,0}\rangle$ and $|F^\pi_{1,2,3}\rangle$ of 20 atoms. Fidelities between $|\psi_{\text{eff}}\rangle$ and $U_x |B_{0,0,0}\rangle$ ($F_k$, blue) and $|F^\pi_{1,2,3}\rangle$ ($F_{\text{eff}}$, red) as a function of time, in units of $1 / \gamma_0$ where $\gamma_0$ is the single atom spontaneous emission rate in 3D free space. The intermediate state $|\psi_{\text{eff}}\rangle$ has $F_k = F_{\text{eff}}$. (b) Renormalized photon axial momentum distribution $P_k$ for $|B_{0,0,0}\rangle$, $U_x |\psi_{\text{eff}}\rangle$ and $|F^\pi_{1,2,3}\rangle$, respectively. The log scale inset shows the central parts of $P_k$. States $|F^\pi_{1,2,3}\rangle$ and $|B_{0,0,0}\rangle$ are not orthogonal. Fully populating one implies a 0.53 population of the other. However, as shown in Fig. 1(b), the emission profile $P_k$ of $|F^\pi_{1,2,3}\rangle$ is different from that of $|B_{0,0,0}\rangle$ by a lower peak value and wider shoulders. The inset in Fig. 1(b) emphasizes the destructive interference in $P_k$ for $|B_{0,0,0}\rangle$ at axial photon momentum $k \approx \pm 0.1 \pi / d$. The intermediate state
$U_x|\Psi_{\text{inter}}\rangle$ overlaps equally with $|B_{0,0}\rangle$ and $|F_{1,2,3}\rangle$, and shares emission features of both states.

**Detection scheme 2.—**An alternative scheme may employ continuous laser excitation with a constant spatial phase, driving excitations with $k \approx k_{\text{ex,0}}$ of the atoms. Each driving is also studied in Refs. [28,52,53]. The collective driving is modeled by

$$H_L = \Omega (e^{-i\delta_L} \sigma^+_{k=0} + e^{i\delta_L} \sigma^-_{k=0}), \quad (11)$$

and we assume the detuning $\delta_L = \omega_0 + \eta \omega_{\text{eff}}(k_{\text{ex,0}})$ so that $|F_{1,2}\rangle$ is the doubly excited state closest to resonance. The amplitude $\Omega$ is assumed to be weak so that excited state components with $n_e \geq 3$ are neglected. The emitted radiation signal may be dominated by the most populated singly excited components of the steady state, but we can extract the properties of the doubly excited components by observation of photon coincidences, described by the second-order equal-time correlation function $G(k_1,k_2) \equiv N^2 \langle \sigma^+_{k_1} \sigma^+_{k_2} \sigma^-_{k_1} \sigma^-_{k_2} \rangle$.

If the RDDI is negligible, the two-excitation component of the steady state will be $|B_{0,0}\rangle \propto (\sigma^-_{k=0})^2 |G\rangle$, and only for sufficiently strong interactions will the steady state be dominated by the optical emission show features of $|F_{1,2}\rangle$. To focus on the essential physical mechanisms rather than system-dependent details, we propose a minimal model

$$i \frac{d}{dt} \rho = \mathcal{H}_1 \rho - \rho \mathcal{H}_1^\dagger - [\rho, H_L] + i\beta \gamma \sum_{\xi=1}^{N} \sigma_{\psi_\xi} \rho \sigma_{\psi_\xi}^\dagger, \quad (12)$$

where $\mathcal{H}_1$ is derived from $H_1$,

$$\mathcal{H}_1 = \beta \gamma \frac{N_e}{2i} \sum_{j=1}^{N-1} (\sigma^-_j \sigma^+_j + \sigma_j^+ \sigma^-_j). \quad (13)$$

In this model, $\Delta \omega = \eta \rho_{2}/(Nd)^2$ characterizes the energy gaps between the free-fermion states, and $\beta \gamma$ characterizes their linewidths, where $\beta$ is introduced to explicitly control the role of the dimensionless ratio $r_\beta = \Delta \omega / (\beta \gamma)$. When $r_\beta$ is large, eigenstates other than $|F_{1,2}\rangle$ far from resonance and the doubly excited states predominantly occupy $|F_{1,2}\rangle$. Moreover, the quantum jump operators in Eq. (12) appear with the same magnitude. Thus, they are equivalent to individual atomic decays. Hence, Eq. (12) describes decaying independently with decay rate $\beta \gamma$ while being coherently coupled to the nearest neighbors with “renormalized” tunneling strength $\eta \rho_{2}/d^2$.

A simulation of Eq. (12) is compared with a simulation based on Eq. (10), including $H_1$, for the same system and parameters of $H_{\text{eff}}$ as used in scheme 1 but allowing variation of the dissipative part through the factor $\beta$, i.e., $H_{\text{eff}} \rightarrow H_{\text{eff}}^\beta - i\beta H_{\text{eff}}^{\text{nl}}$. This choice of $H_{\text{eff}}$ yields

$$\eta \omega_{\text{eff}}(k_{\text{ex}}) \approx -1.03 \gamma_0, \quad \gamma_{ex} \approx 3 \gamma_0, \quad \text{and} \quad \eta \alpha_2/d^2 \approx 0.17 \gamma_0,$$

that we apply in Eq. (12). In Fig. 2, we show the results of the simulation for $N = 20$ atoms with paired parameters $(\beta = 1/25, \Omega = 0.017 \gamma_0)$ and $(\beta = 1/150, \Omega = 0.008 \gamma_0)$. In the Supplemental Material [43], we show results for a larger value of $N = 30$ which blurs some of the features of $|F_{1,2}\rangle$. The results are obtained by averaging over 1000 quantum trajectories [46].

In Fig. 2(a), we extract the two-excitation component of each quantum state trajectory, renormalize it, and plot its fidelity with $|F_{1,2}\rangle$ and $|B_{0,0}\rangle \propto (\sigma^-_{k=0})^2 |G\rangle$, for both Eqs. (12) and (10). We select two values of $\beta$, 1/25 and 1/150. For $\beta = 1/25$, the steady state of Eq. (10) (left panel, dotted lines, $r_\beta \approx 0.004$) is at an intermediate stage with equal overlaps with $|F_{1,2}\rangle$ and $|B_{0,0}\rangle$ while for Eq. (12) (solid line) the dominant overlap is with $|B_{0,0}\rangle$. For $\beta = 150$ (right panel, $r_\beta \approx 0.02$), both models yield dominant overlap with the free-fermion state $|F_{1,2}\rangle$.

In Fig. 2(b), we plot the two-photon coincidences, $\log_{10} G(k_1,k_2)$ with $0 \leq k_1, k_2 \leq 0.2 d / d$ for the steady states of Eq. (12) (the first row, labeled by “toy”) and

![FIG. 2. Scheme 2. (a) Fidelities between the two-excitation components of the steady states (normalized) and the bosonic state $|B_{0,0}\rangle$ (blue) and $|F_{1,2}\rangle$ (red). Solid lines are for Eq. (12) and dotted lines are for Eq. (10). (b) Two-photon correlation function $\log_{10} G(k_1,k_2)$ evaluated in the steady states, with the first row for the steady states of Eq. (12), the second row for steady states of Eq. (10), and the third row for $|B_{0,0}\rangle$ and $|F_{1,2}\rangle$. The color map on the right applies to the first and second rows. Color map of the third row is not shown. (c) $\log_{10} G(k,k)$, i.e., values along the diagonal in the plots in (b).](image-url)
Eq. (10) (the second row). In the plots, the patterns colored by dark blue represent suppression of two-photon coincidences. In each row, plots for $\beta = 1/25$ and $\beta = 1/150$ are shown in the left and right columns, respectively. In the third row, we show results evaluated as expectation values in the states $|B_{0,0}\rangle$ (left) and $|F_{1,2}\rangle$ (right).

The almost identical patterns obtained for the same $\beta$ in Fig. 2(b) show that Eq. (12) approximates Eq. (10) well. For $\beta = 1/25$ the patterns display an upright cross as a signature of the state $|B_{0,0}\rangle$. While for $\beta = 1/150$, we see two sloping lines characterizing $|F_{1,2}\rangle$. To distinguish free-fermion and free-boson ansatz states beyond nearest neighbor tunneling processes. After the Jordan-Wigner transformation, this corresponds to a strongly interacting fermionic model.

Discussions and conclusions.—To conclude, by approximating the RDDI Hamiltonian $H_{\text{eff}}$ with the solvable model $H_I$, the free-fermion states are found to be a generic consequence of the quadratic dispersion relation in the band edge of the singly excited states. We propose to observe the free-fermion states by their optical emission in two basic schemes. Scheme 1 combines unitary and dissipative evolution to prepare a subradiant free-fermion state and exploits a transfer of the quantum system between the sub- and superradiant states to observe the directional distribution of radiation. Scheme 2 observes the second-order correlation function of the steady state emission by the atoms subject to constant laser driving.

The free-fermion state is not a precise ansatz if the extremum point of $\omega_{\text{eff}}(k)$ is not quadratic. Quartic extremum points exist in atom arrays in 3D free space [42]. In this case, $H_I$ (4) should be replaced by one with beyond nearest neighbor tunneling processes. After the Jordan-Wigner transformation, this corresponds to a strongly interacting fermionic model.

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