Stability of the train of \( N \) solitary waves for the two-component Camassa-Holm shallow water system

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Abstract

Considered herein is the integrable two-component Camassa-Holm shallow water system derived in the context of shallow water theory, which admits blow-up solutions and the solitary waves interacting like solitons. Using modulation theory, and combining the almost monotonicity of a local version of energy with the argument on the stability of a single solitary wave, we prove that the train of \( N \) solitary waves, which are sufficiently decoupled, is orbitally stable in the energy space \( H^1(\mathbb{R}) \times L^2(\mathbb{R}) \).

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1 Introduction

In this paper, we are concerned with the following two-component Camassa-Holm shallow water system [4, 12, 28, 37]

\[
\begin{aligned}
\rho_t + (u\rho)_x &= 0, \\
m_t + 2u_x m + um_x + \rho \rho_x &= 0, \\
m &= u - u_{xx},
\end{aligned}
\]

where the variables \( u(t, x) \), \( \rho(t, x) \) describe the horizontal velocity of the fluid and the horizontal deviation of the surface from equilibrium (or scalar density), respectively. The system (1.1) was originally introduced by Chen et al. [4] and Falqui in [21]. It is completely integrable [12, 21, 27] as it can be written as a compatibility condition of two linear systems (Lax pair). Compared with the other integrable multicomponent Camassa-Holm-type systems, the system (1.1) has caught a large amount of attention, after Constantin and Ivanov [12] derived it in the context.
of shallow water regime. It is noticed that the boundary assumptions $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$, at any instant $t$, are required in their hydrodynamical derivation.

For $\rho \equiv 0$, the system (1.1) becomes the classical Camassa-Holm equation, which was first derived as an abstract bi-Hamiltonian partial differential equation by Fokas and Fuchssteiner [22]. Then Camassa and Holm [2] independently rediscovered it modeling shallow water waves with $u(t, x)$ representing the free surface over a flat bottom. Moreover, it was found by Dai [16] as a model for nonlinear waves in cylindrical hyperelastic rods where $u(t, x)$ stands for the radial stretch relative to a pre-stressed state. In the past two decades, the reason for the Camassa-Holm equation as the master equation in shallow water theory is that it gives a positive response to the question "What mathematical models for shallow water waves could include both the phenomena of soliton interaction and wave breaking?", which was proposed by Whitham [39]. The appearance of breaking waves as one of the remarkable properties of the Camassa-Holm equation, however, can not be captured by the KdV and BBM equations [13]. Plenty of impressive known results on wave breaking for the Camassa-Holm equation have been obtained in [3, 6, 10, 31, 34, 35, 41]. Recently, we notice that Brandolese [1] unifies some of earlier results by a more natural blow-up condition, that is, local-in-space blow-up criterion which means the condition on the initial data is purely local in space variable.

On the other hand, it was shown that the Camassa-Holm equation has solitary waves interacting like solitons [2, 3], which capture the essential features of the extreme water waves [7, 8, 11, 38]. Hence many papers addressed another fundamental qualitative property of solutions for the Camassa-Holm equation, which is the stability of solitary wave solutions. As commented in [14], due to the fact that a small perturbation of a solitary wave can yield another one with a different speed and phase shift, we could only expect orbital stability for solitary waves. Constantin and Strauss [14] gave a very simple proof of the orbital stability of the peakons by using the conservation laws. Then they [15] applied the general approach developed by [23], to cope with the stability of the smooth solitary waves. A series of works by El Dika and Molinet [17–19] were devoted to the study of the stability of the train of $N$ solitary waves, multipeakons and multi antipeakon-peakons, respectively. Moreover, Lenells [30] presented a variational proof of the stability of the periodic peakons.

For $\rho \neq 0$, the system (1.1) has also attracted much attention owing the fact that it has both solutions which blow up in finite time and solitary wave solutions interacting like solitons. The Cauchy problem of the system (1.1) has been studied extensively. The local well-posedness for the system (1.1) with initial data $(u_0, \rho_0)^t \in H^s \times H^{s-1}$, $s \geq 2$, by Kato’s semigroup theory [29], was established in [20]. Then Gui and Liu [26] improved the well-posedness result with initial data in the Besov spaces (especially in $H^s \times H^{s-1}$, $s > \frac{3}{2}$). More interestingly, singularities of the solutions for the system (1.1) can occur only in the form of wave breaking, while blow-
up solutions with a different class of certain initial profiles were shown in \[12, 20, 24, 26, 40\]. Moreover, the system \((1.1)\) has also global strong solutions \[12, 24, 25\]. Here we recall the following global existence result needed in our developments.

**Proposition 1.1. (Global existence)** Assume \(\bar{u}_0^t(0, x) = (u_0(x), \eta_0(x))^t \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s \geq 2\), and \(T > 0\) be the maximal time of the existence of the solution \(\bar{u}^t(t, x) = (u(t, x), \eta(t, x))^t \in C([0, T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))\) to the system \((1.1)\) with the initial profile \(\bar{u}^0(0, x)\). If \(\eta_0(x) \neq -1\), then \(T = +\infty\), i.e., the solution \(\bar{u}^t\) of the system \((1.1)\) is global.

On the other hand, Mustafa \[36\] proved that the existence of the smooth solitary waves for the system \((1.1)\) with a single crest profile of maximum amplitude. For convenience, we briefly collect the properties of the solitary waves from \[36, 40\] in the proposition below.

**Proposition 1.2. (Existence of solitary waves)** The system \((1.1)\) admits the smooth solitary wave solution \(\varphi_c^t(t, x) \triangleq (\varphi_c(t, x), \xi_c(t, x))^t\) with the speed \(c > 1\). Moreover, as \(|x| \to \infty\), we have

\[
\varphi(x) = O(\exp(-\sqrt{\frac{c^2-1}{c^2}}|x|)), \tag{1.2}
\]

and thus \(\xi(x) = \frac{c}{c-\varphi}\) also holds true.

Recently, Zhang and Liu \[40\] obtained the orbital stability of a single solitary wave of the system \((1.1)\) by following the general spectral method developed by Grillakis et al. \[23\]. Thus, an interesting problem is to investigate whether or not the train of \(N\) solitary wave solutions of the system \((1.1)\) is orbitally stable as the scalar Camassa-Holm equation \[17\]. This is the question we shall discuss in our present paper. For this purpose, we firstly rewrite the system \((1.1)\) with \(\rho \equiv 1 + \eta\) (\(\eta \to 0\) as \(|x| \to \infty\)) as follows

\[
\begin{cases}
    u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx} - (1 + \eta)\eta_x, & t > 0, x \in \mathbb{R}, \\
    \eta_t + (u(1 + \eta))_x = 0, & t > 0, x \in \mathbb{R},
\end{cases} \tag{1.3}
\]

or equivalently,

\[
\begin{cases}
    (1 - \partial_x^2)u_t = -\frac{1}{2}(1 - \partial_x^2)\partial_x^2 u^2 - \partial_x(u^2 + \frac{1}{2}u_x^2 + \eta + \frac{1}{2}\eta^2), & t > 0, x \in \mathbb{R}, \\
    \eta_t = -\partial_x(u + \eta), & t > 0, x \in \mathbb{R}. \tag{1.4}
\end{cases}
\]

Secondly, we define the space \(X = H^1(\mathbb{R}) \times L^2(\mathbb{R})\) with inner product \((\cdot, \cdot)\) and its norm \(\|\cdot\|_X\), and thus the dual \(X^* = H^{-1}(\mathbb{R}) \times L^2(\mathbb{R})\). Denote by \(\langle \cdot, \cdot \rangle\) the pairing between \(X\) and \(X^*\), and the space \(X^{**}\) is identified with \(X\) in the natural way. We have the isomorphism \(I : X \to X^*\) defined by

\[
I = \begin{pmatrix}
1 - \partial_x^2 & 0 \\
0 & 1
\end{pmatrix}.
\]
Thus, for \( \vec{u}^i, \vec{r}^i \in X \), we obtain \( \langle I \vec{u}^i, \vec{r}^i \rangle = (\vec{u}^i, \vec{r}^i) \). Moreover, we also need the functionals on \( X \), which are two useful conservation laws

\[
E(\vec{u}^i) \triangleq \frac{1}{2} \int_{\mathbb{R}} (u^2 + u_x^2 + \eta^2) \, dx \quad \text{and} \quad F(\vec{u}^i) \triangleq \frac{1}{2} \int_{\mathbb{R}} (u^3 + uu_x^2 + 2\eta u + \eta^2) \, dx.
\]

Now we are in the position to state our main result.

**Theorem 1.1.** Given \( N \) velocities \( c_1, ..., c_N \) such that \( 1 < c_1 < c_2 < ... < c_N \). Denote the \( N \) solitary waves as \( \sum_{i=1}^{N} \varphi_{c_i}(t, x) = \sum_{i=1}^{N} (\varphi_{c_i}(x - c_i t), \xi_{c_i}(x - c_i t))^t \) by Proposition 1.2. There exist \( \gamma_0, A_0, L_0, \varepsilon_0 > 0 \) such that for any initial date \( \vec{u}_0^i = \vec{r}_i(0, x) = (u_0, \eta_0)^t \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \), with \( s \geq 2 \), \( \eta_0(x) \neq -1 \) and satisfying

\[
\| \vec{u}_0^i - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - x_i) \|_X \leq \varepsilon,
\]

for some \( 0 < \varepsilon < \varepsilon_0 \), and \( x_i - x_{i-1} \geq L \) with \( L > L_0 \) for \( i = 2, ..., N \). Then, for the global strong solution \( \vec{u}^i = (u, \eta)^t \) of the system (1.3) or (1.4) with \( \vec{u}_0^i \) guaranteed by Proposition 1.1, there exist \( x_1(t), ..., x_N(t) \) such that

\[
\sup_{t \in [0, \infty)} \| \vec{u}^i(t, \cdot) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - x_i(t)) \|_X \leq A_0(\sqrt{\varepsilon} + e^{-\gamma_0 L}).
\]

As commented by El Dika and Molinet [17, 18], the strategy initiated in [33] for a scalar equation indicates that there are principally two required ingredients to prove the stability of the sum of \( N \) solitary waves. One of them is a dynamical proof of the stability of the single solitary wave, and the other is a property of almost monotonicity, which says for a solution close to \( \varphi_{c}^i \), the part of the energy traveling at the right of \( \varphi_{c}^i(\cdot - ct) \) is almost time decreasing. Our approach to prove Theorem 1.1 is to try to follow this method. However, we consider here a coupled system with two component, instead of a scalar equation. Hence, the same argument as in [17] or [33] for a single equation is not directly applicable here. More precisely, we need to overcome a difficulty encountered by the coupled system (1.1) in comparison with the Camassa-Holm equation, which is the mutual effect between the two component \( u(t, x) \) and \( \eta(t, x) \). To solve this problem, here we require more elaborate analysis on the decomposition of the solution by using modulation theory, and a local coerciveness inequality related to a Hessian operator \( H_c \) of \( cE - F \) around the solitary wave \( \varphi_{c}^i \). Moreover, we know that the method of the proof of the stability relies heavily on a property of almost monotonicity. Therefore, the key issue to prove Theorem 1.1 is to estimate precisely these coupled terms, which appear in the energy at the right of the \( (j - 1) \)-th bump of the solution. To this end, we first apply Hölder inequality to prove \( (p \ast \eta)^2 \leq p \ast \eta^2 \) (\( p \triangleq \frac{1}{2} e^{-|x|} \)), and then deduce the desired result by means of the Minkowski
inequality (see Lemma 2.2 below). Actually, to the best of our knowledge, our theorem is the first result on the stability of the sum of $N$ solitary wave solutions for the coupled shallow water system. Hence we expect there are more applications of this method to handle the stability of $N$ solitary waves for the other two-component system, such as a generalized two-component Camassa-Holm system [3], two-component Dullin-Gottwald-Holm system [32], and so on.

The remainder of the paper is dedicated to the proof of Theorem 1.1. In Section 2, we present some useful lemmas which will be used in the sequel. First, we control the distance between the different bumps of the solution by using a modulation argument. Then, we prove a almost monotonicity property and local coercivity of the solitary wave. In Section 3, we complete the proof of Theorem 1.1 by three steps.

Notation. As above and henceforth, we denote by $\ast$ the convolution. Since our discussion is all on the line $\mathbb{R}$, for simplicity, we omit $\mathbb{R}$ in our notations of function spaces. All the transpose of a row vector $\vec{f} = (f_1, f_2)$ is presented as $\vec{f}^t = \left(\begin{array}{c} f_1 \\ f_2 \end{array}\right)$.

2 Preliminaries

In this section, we shall establish some useful lemmas which are crucial to pursue our goal. We break them into the following three subsections.

2.1 Modulation

In this subsection, we show that we can decompose the solution $\vec{u}^t(t) = (u, \eta)^t$ as the sum of $N$ modulated solitary waves and a vector function $\vec{v}^t(t) \triangleq (v, \zeta)^t$, which remains small in the space $X$:

$$\vec{u}^t(t,x) = \sum_{i=1}^{N} (\varphi_{c_i}(x - x_i(t)), \xi_{c_i}(x - x_i(t)))^t + \vec{v}^t(t,x),$$

with $\vec{v}^t(t,x)$ is orthogonal to $((1 - \partial_x^2)\partial_x \varphi_{c_i}(x - x_i(t)), \partial_x \xi_{c_i}(x - x_i(t)))^t$ in $L^2$, for $i = 1, \ldots, N$. Moreover, we prove that the different bumps of $\vec{u}^t(t)$ that are individually close to a solitary wave get away from each other as time is increasing.

Let $1 < c_1 < c_2 \ldots < c_N$, $\sigma_0 = \frac{1}{4} \min(c_1, c_2 - c_1, \ldots, c_N - c_{N-1})$. For $\alpha, L > 0$, we define the neighborhood of size $\alpha$ of the superposition of $N$ solitary waves of speed $c_i$, located at a distance larger than $L$,

$$U(\alpha, L) = \{ \vec{u}^t \in X; \inf_{x_i - x_{i-1} > L} \| \vec{u}^t - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - x_i) \|_X < \alpha \}. $$

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Lemma 2.1. Suppose \( \bar{u}_0^i \) satisfy the assumptions (1.6) given in Theorem 1.1. There exist \( \alpha_0, L_0 \) such that for all \( 0 < \alpha < \alpha_0 \) and \( L > L_0 \), if the corresponding solution \( \bar{u}(t) \in U(\alpha, \frac{L}{2}) \) on \([0, t_0]\) for some \( 0 < t_0 \leq +\infty \), then there exist unique \( C^1 \)-functions \( x_i : [0, t_0] \to \mathbb{R}, i = 1, \ldots, N \), such that

\[
\bar{u}^i(t, x) = (v(t, x), \zeta(t, x))^t = \bar{u}^i(t, x) - \sum_{i=1}^N \vec{R}_i(t, x),
\]

where \( \vec{R}_i(t, x) = (R_i(t, x), S_i(t, x))^t = (\varphi_i(x - x_i(t)), \xi_i(x - x_i(t)))^t \), satisfies the following orthogonality conditions

\[
\int_{\mathbb{R}} v(t)(1 - \partial_x^2)\partial_x R_i(t)dx + \int_{\mathbb{R}} \zeta(t)\partial_x S_i(t)dx = 0, \quad i = 1, \ldots, N. \tag{2.1}
\]

Moreover, the following statements hold true:

\[
\|\bar{u}^i(t)\|_X = \|\bar{u}^i(t) - \sum_{i=1}^N \vec{R}_i(t)\|_X = O(\alpha), \tag{2.2}
\]

\[
\sup_{t \in [0, t_0]} |x_i(t) - c_i| \leq O(\alpha) + O(e^{-\sigma_0 L}), \quad i = 1, \ldots, N, \tag{2.3}
\]

and

\[
x_i(t) - x_{i-1}(t) \geq \frac{3L}{4} + 2\sigma_0 t, \quad i = 2, \ldots, N. \tag{2.4}
\]

Proof. For \( Z = (z_1, z_2, \ldots, z_N) \in \mathbb{R}^N \), such that \( z_i - z_{i-1} > \frac{L}{2} \), we set \( \vec{R}_Z(\cdot) = (R_Z(\cdot), S_Z(\cdot))^t = \left( \sum_{i=1}^N \varphi_i(\cdot - z_i), \sum_{i=1}^N \xi_i(\cdot - z_i) \right)^t \), and denote \( B_{H^1}(R_Z, \alpha), B_{L^2}(S_Z, \alpha) \) as the ball in \( H^1, L^2 \) of center \( R_Z, S_Z \) with radius \( \alpha \), respectively. For \( 0 < \alpha < \alpha_0 \), we define the following mapping

\[
Y : (-\alpha, \alpha)^N \times B_{H^1}(R_Z, \alpha) \times B_{L^2}(S_Z, \alpha) \to \mathbb{R}^N,
\]

\[
(y_1, \ldots, y_N, u, \eta) \mapsto (Y^1(y_1, \ldots, y_N, u, \eta), \ldots, Y^N(y_1, \ldots, y_N, u, \eta)),
\]

with

\[
Y^i(y_1, \ldots, y_N, u, \eta) = \int_{\mathbb{R}} \left( u - \sum_{i=1}^N \varphi_i(\cdot - y_i) \right) (1 - \partial_x^2)\partial_x \varphi_i(\cdot - y_i) dx
\]

\[
+ \int_{\mathbb{R}} \left( \eta - \sum_{i=1}^N \xi_i(\cdot - y_i) \right) \partial_x \xi_i(\cdot - y_i) dx. \tag{2.5}
\]

In the following, we verify that the function \( Y \) satisfies the properties:

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(i) $Y(0,\ldots,0,R_Z,S_Z) = (0,\ldots,0)$.

(ii) For $i,j = 1,\ldots,N$, by the dominated convergence theorem and the smoothness of $\varphi_c^x$, the partial derivatives $\frac{\partial Y^i}{\partial y_j}$, $\frac{\partial Y^i}{\partial x_i}$, and $\frac{\partial Y^i}{\partial \eta}$ are continuous. Indeed, for $i = 1,\ldots,N$,

$$\frac{\partial Y^i}{\partial y_j}(y_1,\ldots,y_N,u,\eta) = \int_{\mathbb{R}} (u_x - \sum_{j=1,j\neq i}^N \partial_x \varphi_{c_j}(\cdot - z_j - y_j))(1 - \partial_x^2)\partial_x \varphi_{c_i}(\cdot - z_j - y_j)dx$$

$$+ \int_{\mathbb{R}} (\eta_x - \sum_{j=1,j\neq i}^N \partial_y \xi_{c_j}(\cdot - z_j - y_j))\partial_y \xi_{c_i}(\cdot - z_j - y_j)dx,$$

$$\frac{\partial Y^i}{\partial y_j}(y_1,\ldots,y_N,u,\eta) = \int_{\mathbb{R}} \partial_x \varphi_{c_j}(\cdot - z_j - y_j)(1 - \partial_x^2)\partial_x \varphi_{c_i}(\cdot - z_j - y_j)dx$$

$$+ \int_{\mathbb{R}} \partial_y \xi_{c_j}(\cdot - z_j - y_j)\partial_y \xi_{c_i}(\cdot - z_j - y_j)dx,$$

for all $j \neq i$,

$$\frac{\partial Y^i}{\partial u}(y_1,\ldots,y_N,u,\eta) = \int_{\mathbb{R}} (1 - \partial_x^2)\partial_x \varphi_{c_i}(\cdot - z_j - y_j)dx,$$

and

$$\frac{\partial Y^i}{\partial \eta}(y_1,\ldots,y_N,u,\eta) = \int_{\mathbb{R}} \partial_y \xi_{c_i}(\cdot - z_j - y_j)dx.$$

Thus, the function $Y$ is of class $C^1$.

(iii) The determinant of the matrix $Y^t_{(y_1,\ldots,y_N)}(0,\ldots,0,R_Z,S_Z)$ is not equal to zero. In fact, from the above (ii), we obtain

$$\frac{\partial Y^i}{\partial y_i}(0,\ldots,0,R_Z,S_Z) = \|\partial_x \varphi_{c_i}\|_{H^1}^2 + \|\partial_y \xi_{c_i}\|_{L^2}^2 \geq \frac{1}{2} \min_{i=1,\ldots,N} \left(\|\partial_x \varphi_{c_i}\|_{H^1}^2 + \|\partial_y \xi_{c_i}\|_{L^2}^2\right),$$

and for $j \neq i$, by the exponential decay (12) of $\varphi_c^x$, and $z_i - z_{i-1} > \frac{L}{2}$, for $L_0$ large enough,

$$\frac{\partial Y^i}{\partial y_j}(0,\ldots,0,R_Z,S_Z) = (\partial_x \varphi_{c_i}(x - z_j), \partial_x \varphi_{c_i}(x - z_i))_{H^1} + (\partial_y \xi_{c_j}(x - z_j), \partial_y \xi_{c_i}(x - z_i))_{L^2} \leq O(e^{-\sigma_0 L}).$$

Thus, we deduce that for $L_0$ large enough, $\frac{\partial Y^i}{\partial y_j}(0,\ldots,0,R_Z,S_Z) \gg \frac{\partial Y^i}{\partial y_j}(0,\ldots,0,R_Z,S_Z)$ whenever $j \neq i$. So, $Y^t_{(y_1,\ldots,y_N)}(0,\ldots,0,R_Z,S_Z)$ is a diagonally dominant matrix.

Therefore, from the implicit function theorem, we find that there exists $\alpha_0 > 0$ and the uniquely determined $C^1$-functions $(y_1,\ldots,y_N)$ from $B_{H^1}(R_Z,\alpha_0) \times B_{L^2}(S_Z,\alpha_0)$ to a neighborhood of $(0,\ldots,0)$, such that

$$Y(y_1,\ldots,y_N,u,\eta) = (0,\ldots,0), \quad \text{for all } \vec{u} \in B_{H^1}(R_Z,\alpha_0) \times B_{L^2}(S_Z,\alpha_0).$$
In particular, if \( \vec{u}^t \in B_{H^1}(R_Z, \alpha) \times B_{L^2}(S_Z, \alpha) \) with \( 0 < \alpha \leq \alpha_0 \), there exists a constant \( C_0 > 0 \) such that

\[
\sum_{i=1}^{N} |y_i(\vec{u}^t)| \leq C_0 \alpha. \tag{2.6}
\]

Note that \( \alpha_0 \) and \( C_0 \) depend only on the velocity \( c_1 \) and \( L_0 \) and not on the point \((z_1, \ldots, z_N)\). For \( \vec{u}^t \in B_{H^1}(R_Z, \alpha_0) \times B_{L^2}(S_Z, \alpha_0) \), we set \( x_i(\vec{u}^t) = z_i + y_i(\vec{u}^t) \). If we take \( \alpha_0 \leq \frac{L_0}{8c_1} \), then \((x_1, \ldots, x_N)\) are \( C^1\)-functions on \( \vec{u}^t \in B_{H^1}(R_Z, \alpha) \times B_{L^2}(S_Z, \alpha) \), satisfying

\[
x_i(\vec{u}^t) - x_{i-1}(\vec{u}^t) > \frac{L}{2} - 2C_0 \alpha > \frac{L}{4}. \tag{2.7}
\]

Then by a modulation argument and the construction \((2.5)\) of the functions \( \vec{y}^i \), we can define \( N \) \( C^1\)-functions \( t \to x_i(t) = x_i(\vec{u}^i(t)) \) satisfying the orthogonality conditions \((2.1)\) for \( i = 1, \ldots, N \). Furthermore, from \( \vec{u}^i(t) \in U(\alpha, \frac{L}{2}) \), \((2.6)\) and the triangular inequality, there exists \( C_0 > 0 \), such that for all \( t \in [0, t_0] \)

\[
||\vec{u}^i(t)||_X \leq C_0 \alpha, \tag{2.8}
\]

hence, \((2.2)\) holds true.

Differentiating the orthogonality conditions \((2.1)\) with respect to time \( t \), we obtain

\[
\int_R v_t(1 - \partial_x^2) \partial_x R_i dx + \int_R \zeta_t \partial_x S_i dx = \dot{x}_i \left( \int_R v(1 - \partial_x^2) \partial_x^2 R_i dx + \int_R \zeta \partial_x^2 S_i dx \right),
\]

and thus, we have

\[
|\int_R v_t(1 - \partial_x^2) \partial_x R_i dx + \int_R \zeta_t \partial_x S_i dx| \leq |\dot{x}_i| O(||\vec{u}^i||_X) \leq |\dot{x}_i - c_i| O(||\vec{u}^i||_X) + O(||\vec{u}^i||_X). \tag{2.9}
\]

Substituting \( u = v + \sum_{i=1}^{N} R_i \) and \( \eta = \zeta + \sum_{i=1}^{N} S_i \) into the system \((1.4)\), it follows

\[
(1 - \partial_x^2)v_t + \sum_{i=1}^{N} (1 - \partial_x^2) \partial_t R_i = -\frac{1}{2} (1 - \partial_x^2) \partial_x \left( (v + \sum_{i=1}^{N} R_i)^2 \right) - \partial_x \left( (v + \sum_{i=1}^{N} R_i)^2 \right) - \frac{1}{2} (v_x + \sum_{i=1}^{N} \partial_x R_i)^2 + (\eta + \sum_{i=1}^{N} S_i) + \frac{1}{2} (\eta + \sum_{i=1}^{N} S_i)^2, \tag{2.10}
\]

and

\[
\zeta_t + \sum_{i=1}^{N} \partial_t S_i = -\partial_x \left( (v + \sum_{i=1}^{N} R_i) + (v + \sum_{i=1}^{N} R_i)(\eta + \sum_{i=1}^{N} S_i) \right). \tag{2.11}
\]
From the definition of $R_i$ and $S_i$, we obtain

$$(1 - \partial_x^2)\partial_t R_i + (\dot{x}_i - c_i)(1 - \partial_x^2)\partial_x R_i + 3R_i\partial_x R_i = 2\partial_x R_i\partial_x^2 R_i + R_i\partial_x^3 R_i - (1 + S_i)\partial_x S_i.$$ (2.12)

and

$$\partial_t S_i + (\dot{x}_i - c_i)\partial_x S_i + \partial_x R_i + \partial_x (R_i S_i) = 0.$$ (2.13)

Combining (2.10)-(2.13), we infer that $v$ and $\xi$ exist for all $0 < L \leq \alpha < \alpha_0$. This completes the proof of Lemma 2.1.

Taking the exponential decay of $\bar{R}_t$ and for $\zeta$, we get

$$(1 - \partial_x^2)\partial_t v_t - \sum_{i=1}^{N}(\dot{x}_i - c_i)(1 - \partial_x^2)\partial_x R_i = -\frac{1}{2}(1 - \partial_x^2)\partial_x ((v + \sum_{i=1}^{N} R_i)^2 - \partial_x ((v + \sum_{i=1}^{N} R_i)^2 - \sum_{i=1}^{N} R_i^2 + \frac{1}{2}(v_x + \sum_{i=1}^{N} \partial_x R_i)^2)$$

$-\frac{1}{2} \sum_{i=1}^{N}(\partial_x R_i)^2 - \partial_x (\eta + \sum_{i=1}^{N} S_i)^2 - \frac{1}{2} \sum_{i=1}^{N} S_i^2),$$

and for $\zeta(t)$

$$\zeta_t - \sum_{i=1}^{N}(\dot{x}_i - c_i)\partial_x S_i = -\partial_x (v + (v + \sum_{i=1}^{N} R_i)(\eta + \sum_{i=1}^{N} S_i) - \sum_{i=1}^{N} R_i S_i).$$ (2.15)

Taking the $L^2$-scalar product (2.14) with $\partial_x R_j$ and (2.15) with $\partial_x S_j$, integrating by parts, using the exponential decay of $\bar{R}_t$ and its derivatives, by (2.8), (2.9) and (2.7), then plugging the two obtained results, we get

$$|\dot{x}_j - c_j|([\|\partial_x R_j\|_{L^1}^2 + \|\partial_x S_j\|_{L^2}^2 + O(\alpha)) \leq O(\alpha) + O(e^{-\sigma_0 L}).$$

Taking $\alpha_0$ small enough and $L_0$ large enough depending only on $\{c_i\}_{i=1}^{N}$, we obtain (2.3).

To prove (2.4), for $\alpha_0$ sufficient small and $L_0$ large enough, we have $|\dot{x}_i - c_i| \leq \frac{\sigma_0^2}{4}$. Thus for all $0 < \alpha < \alpha_0$ and $L \geq L_0 > 4C_0\varepsilon$, by the mean value theorem, (1.6), (2.6) and (2.3), there exist $\xi \in [0, t]$ such that

$$x_i(t) - x_{i-1}(t) = x_i(t) - x_i(0) + x_i(0) - x_{i-1}(0) + x_i(0) - x_{i-1}(t)$$

$$= x_i(0) - x_{i-1}(0) + (\dot{x}_i(\xi) - x_{i-1}(\xi))t$$

$$> L - C_0\varepsilon + \frac{(c_i - c_{i-1})t}{2} \geq \frac{3L}{4} + 2\sigma_0 t, \quad \forall \ t \in [0, t_0].$$

This completes the proof of Lemma 2.1.
2.2 Monotonicity property

This subsection is devoted to proving the principal tool of our proof, which is the almost monotonicity of functionals that are very close to the energy at the right of the \((i-1)\)th bump of \(\vec{u}^t\), \(i=2,...,N\). Firstly, we define \(\Psi\) to be a \(C^\infty\) function such that

\[
\Psi(x) = \begin{cases} 
  e^{-|x|}, & x < -1, \\
  1 - e^{-|x|}, & x > 1,
\end{cases} \quad \text{and} \quad \begin{cases} 
  0 < \Psi \leq 1, \Psi' > 0, & x \in \mathbb{R}, \\
  |\Psi''| \leq 10\Psi', & x \in [-1, 1].
\end{cases}
\]

Consider \(\Psi_K = \Psi(\frac{x}{K})\), where the constant \(K > 0\) will be chosen later. Then, we introduce for \(j = 2,...,N\),

\[
I_{j,K}(t) = \int_{\mathbb{R}} (u^2(t) + u_x^2(t) + \eta^2(t)) \Psi_{j,K}(t,x) dx,
\]

where \(\Psi_{j,K}(t,x) = \Psi_K(x - y_j(t))\) for \(j = 2,...,N\). And for \(i = 1,...,N\), we define the following localized version of the conservation laws (1.5) of \(E\) and \(F\) as

\[
E'_{i}(\vec{u}^t) = \frac{1}{2} \int_{\mathbb{R}} \Phi_i(t)(u^2 + u_x^2 + \eta^2) dx \quad \text{and} \quad F'_{i}(\vec{u}^t) = \frac{1}{2} \int_{\mathbb{R}} \Phi_i(t)(u^3 + uw_x^2 + 2w\eta + w\eta^2) dx,\quad (2.16)
\]

where the weight functions \(\Phi_i = \Phi_i(t, x)\) are given by

\[
\Phi_1 = 1 - \Psi_{2,K}, \quad \Phi_N = \Psi_{N,K} \quad \text{and} \quad \Phi_i = \Psi_{i,K} - \Psi_{i+1,K}, \quad i = 2,...,N-1.
\]

Obviously, \(\sum_{i=1}^N \Phi_i \equiv 1\). Taking \(L > 0\) and \(\frac{L}{K} > 0\) large enough, we can deduce that

\[
|1 - \Phi_{i,K}| \leq 4e^{-\frac{L}{4}}, \quad \text{for} \quad x \in [x_i - \frac{L}{4}, x_i + \frac{L}{4}],
\]

and

\[
|\Phi_{i,K}| \leq 4e^{-\frac{L}{4}}, \quad \text{for} \quad x \in [x_j - \frac{L}{4}, x_j + \frac{L}{4}], \quad \text{whenever} \quad j \neq i.
\]

**Lemma 2.2.** Let \(\vec{u}^t(t,x)\) be the global strong solution of the system (1.1) such that \(\vec{u}^t(t) \in U(\alpha, \frac{L}{2})\) on \([0, +\infty)\), where \(x_i(t)\) are defined in Lemma 2.1. There exist \(\alpha_0 > 0\) and \(L_0 > 0\) only depending on \(\sigma_0\), such that if \(0 < \alpha < \alpha_0\) and \(L \geq L_0\) then for any \(5 \leq K = O(\sqrt{T})\),

\[
I_{j,K}(t) - I_{j,K}(0) \leq O(e^{-\sigma_0 L}), \quad t \in [0, +\infty).
\]

**Proof.** By the conversation law \(F(\vec{u}^t)\), we see that the system (1.1) can be written in Hamiltonian form as

\[
\frac{\partial \vec{u}^t}{\partial t} = J F'(\vec{u}^t),
\]
Integrating by parts and by (2.21), we can directly compute 

\[ J = \begin{pmatrix} -\partial_x (1 - \partial_x^2)^{-1} & 0 \\ 0 & -\partial_x \end{pmatrix}, \]

and \( F'(\vec{u}) \) is the Fréchet derivatives of \( F \) in \( X \) at \( \vec{u} \), which can be calculated as

\[ F'_u = \frac{3}{2} u^2 - \frac{1}{2} u_x^2 - uu_xx + \eta + \frac{1}{2} \eta^2 \quad \text{and} \quad F'_\eta = u + u\eta. \quad (2.21) \]

Differentiating \( I_j(t) \) with respect to time \( t \), we have

\[
\frac{d}{dt} I_j(t) = -\frac{\dot{y}_j(t)}{2} \int_R (u^2 + u_x^2 + \eta^2) \Psi_{j,k}^t dx + \int_R (uu_t + u_x u_xt + \eta \eta_t) \Psi_{j,k}^t dx
\]

\[ = -\frac{\dot{y}_j(t)}{2} \int_R (u^2 + u_x^2 + \eta^2) \Psi_{j,k}^t dx + J(t). \quad (2.22) \]

Integrating by parts, and by (2.20), (2.21), we deduce

\[ J(t) = \int_R uu_t \Psi_{j,k}^t dx - \int_R uu_{tx} \Psi_{j,k}^t dx - \int_R uu_{tz} \Psi_{j,k}^t dx + \int_R \eta \eta_t \Psi_{j,k}^t dx
\]

\[ = \int_R u \Psi_{j,k}^t (1 - \partial_x^2) u_t dx - \int_R uu_{tx} \Psi_{j,k}^t dx + \int_R \eta \eta_t \Psi_{j,k}^t dx
\]

\[ = -\int_R u \Psi_{j,k}^t \partial_x F'_u dx + \int_R u \Psi_{j,k}^t (1 - \partial_x^2)^{-1} \partial_x^2 F'_u dx - \int_R \eta \Psi_{j,k} \partial_x F'_t dx
\]

\[ = \int_R u_x \Psi_{j,k}^t F'_u dx + \int_R u \Psi_{j,k}^t (1 - \partial_x^2)^{-1} F'_t dx + \int_R \eta \Psi_{j,k} F'_t dx + \int_R \eta \Psi_{j,k}^t F'_\eta dx
\]

\[ \triangleq J_1 + J_2 + J_3 + J_4. \quad (2.23) \]

Integrating by parts and by (2.21), we can directly compute \( J_i \), \( 1 \leq j \leq 4 \) as follows

\[ J_1 = \int_R u_x \Psi_{j,k}^t (\frac{3}{2} u^2 - \frac{1}{2} u_x^2 - uu_xx + \eta + \frac{1}{2} \eta^2) dx
\]

\[ = -\frac{1}{2} \int_R \Psi_{j,k}^t u^3 dx + \frac{1}{2} \int_R \Psi_{j,k}^t uu^2 dx + \int_R u_x \Psi_{j,k} \eta dx + \frac{1}{2} \int_R u_x \Psi_{j,k} \eta^2 dx,
\]

\[ J_2 = \int_R u \Psi_{j,k}^t (1 - \partial_x^2)^{-1} (\frac{3}{2} u^2 - \frac{1}{2} u_x^2 - uu_xx + \eta + \frac{1}{2} \eta^2) dx
\]

\[ = \frac{1}{2} \int_R \Psi_{j,k}^t u^3 dx + \int_R u \Psi_{j,k}^t (1 - \partial_x^2)^{-1} (u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \eta^2) dx,
\]

\[ J_3 = \int_R \eta \Psi_{j,k}^t (u + u\eta) dx
\]

\[ = -\int R u_x \Psi_{j,k} \eta dx - \int R uu \Psi_{j,k}^t dx - \frac{1}{2} \int R u_x \Psi_{j,k} \eta^2 dx - \frac{1}{2} \int R u \Psi_{j,k} \eta^2 dx,
\]
and
\[ J_4 = \int_{\mathbb{R}} \eta \Psi_{j,k}'(u + u\eta)dx = \int_{\mathbb{R}} \eta \Psi_{j,k}'dx + \int_{\mathbb{R}} \eta^2 \Psi_{j,k}'dx. \]

Combining \( J_1 - J_4 \) with (2.22)-(2.23), we infer that \( \frac{d}{dt} I_j(t) \) can be written as
\[
\frac{d}{dt} I_j(t) = \frac{-\dot{y}_j(t)}{2} \int_{\mathbb{R}} (u^2 + u_x^2 + \eta^2) \Psi_{j,k}'dx + \frac{1}{2} \int_{\mathbb{R}} (uu_x^2 + u\eta^2) \Psi_{j,k}'dx \]
\[ + \int_{\mathbb{R}} \Psi_{j,k}'(1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\eta^2)dx + \int_{\mathbb{R}} \Psi_{j,k}'(1 - \partial_x^2)^{-1}\eta dx \]
\[ \leq \frac{-\dot{y}_j(t)}{2} \int_{\mathbb{R}} (u^2 + u_x^2 + \eta^2) \Psi_{j,k}'dx + J'_1 + J'_2 + J'_3. \] (2.24)

To estimate \( J'_1 \), we divide \( \mathbb{R} \) into two regions \( D_j \equiv [x_{j-1}(t) + \frac{L}{4}, x_j(t) - \frac{L}{4}] \) and its complement \( D'_j \). For \( x \in D_j \), from (2.22), we obtain
\[
\|u(t)\|_{L^\infty_{D_j}} \leq \sum_{i=1}^{N} \|\phi_{c_i}(\cdot - x_i(t))\|_{L^\infty_{D_j}} + \|u - \sum_{i=1}^{N} \phi_{c_i}(\cdot - x_i(t))\|_{L^\infty_{D_j}} \]
\[ \leq O(e^{-\sigma_0 L}) + O(\alpha). \] (2.25)

On the other hand, for \( x \in D'_j \) and by (2.4), we find
\[
|x - y_j(t)| \geq \frac{x_j(t) - x_{j-1}(t)}{2} - \frac{L}{4} \geq \frac{L}{8} + \sigma_0 t. \] (2.26)

Thus for \( \alpha_0 > 0 \) small enough and \( L_0 > 0 \) large enough, by (2.20) - (2.26), we obtain
\[
J'_1 = \frac{1}{2} \int_{\mathbb{R}} \eta \Psi_{j,k}'(u + u\eta)dx \]
\[ = \frac{1}{2} \int_{D'_j} (uu_x^2 + u\eta^2) \Psi_{j,k}'dx + \frac{1}{2} \int_{D_j} (uu_x^2 + u\eta^2) \Psi_{j,k}'dx \]
\[ \leq \frac{1}{2} \|u(t)\|_{L^\infty_{D'_j}} \sup_{x \in D'_j} \|\Psi_{j,K}(x - y_j(t))\|_{L^\infty_{D_j}} \int_{\mathbb{R}} (u_x^2 + \eta^2)dx + \frac{1}{2} \|u(t)\|_{L^\infty_{D_j}} \int_{D_j} (u_x^2 + \eta^2) \Psi_{j,k}'dx \]
\[ \leq C\|u_0\|_X^3 e^{-\frac{(\sigma_0 + L/8)}{K}} + \frac{\sigma_0}{8} \int_{\mathbb{R}} (u_x^2 + \eta^2) \Psi_{j,K}'dx. \] (2.27)

To bound \( J'_2 \), for \( x \in D'_j \), by the Young inequality, we get
\[
\int_{D'_j} \Psi_{j,K}'(1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\eta^2)dx \]
\[ \leq \|u(t)\|_{L^\infty_{D'_j}} \sup_{x \in D'_j} \|\Psi_{j,K}'(x - y_j(t))\|_{L^\infty_{D_j}} \int_{\mathbb{R}} p \ast (u^2 + u_x^2 + \eta^2)dx \]
\[ = \frac{1}{2} \|u(t)\|_{L^\infty_{D'_j}} \sup_{x \in D'_j} \|\Psi_{j,K}'(x - y_j(t))\|_{L^\infty_{D_j}} \int_{\mathbb{R}} e^{-|x|} \ast (u^2 + u_x^2 + \eta^2)dx \]
\[ \leq C\|u_0\|_X^3 e^{-\frac{(\sigma_0 + L/8)}{K}}. \] (2.28)
where \( p(x) \triangleq \frac{1}{2}e^{-|x|} \) is the Green function of \((1 - \partial^2_x)^{-1}\). On the other hand, since by the definition of \( \Psi \), we find

\[
(1 - \partial^2_x)\Psi'_{j,K} = \Psi'_{j,K} - \frac{1}{K^3} \Psi'''(\frac{x - y_j(t)}{K}) \geq (1 - \frac{10}{K^2})\Psi'_{j,K},
\]

hence, for \( K \geq 5 \),

\[
(1 - \partial^2_x)^{-1}\Psi'_{j,K} \leq (1 - \frac{10}{K^2})^{-1}\Psi'_{j,K}.
\tag{2.29}
\]

For \( x \in D_j \), noting that \( \Psi'_{j,K} \) and \( u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\eta^2 \) are non-negative, by (2.25) and (2.29), for \( \alpha_0 \) small enough and \( L_0 \) large enough, we have

\[
\int_{D_j} u\Psi'_{j,K}(1 - \partial^2_x)^{-1}(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\eta^2)dx \leq \|u(t)\|_{L^2_{D_j}} \int_{D_j} \Psi'_{j,K}(1 - \partial^2_x)^{-1}(u^2 + u_x^2 + \eta^2)dx
\]

\[
\leq \|u(t)\|_{L^2_{D_j}} \int_{\mathbb{R}} (u^2 + u_x^2 + \eta^2)(1 - \partial^2_x)^{-1}\Psi'_{j,K}dx
\]

\[
\leq \frac{\alpha_0}{8} \int_{\mathbb{R}} (u^2 + u_x^2 + \eta^2)\Psi'_{j,K}dx.
\tag{2.30}
\]

It thus remains to estimate \( J_3' \). For \( s > 0 \) to be chosen later, by the Cauchy-Schwarz inequality, we obtain

\[
J_3' = \int_{\mathbb{R}} u\Psi'_{j,k}(1 - \partial^2_x)^{-1}\eta dx = \int_{\mathbb{R}} u\Psi'_{j,k}p\ast \eta dx
\]

\[
\leq \frac{1}{2s} \int_{\mathbb{R}} u^2\Psi'_{j,k}dx + \frac{s}{2} \int_{\mathbb{R}} \Psi'_{j,k}(p\ast \eta)^2 dx \leq \frac{1}{2s} \int_{\mathbb{R}} u^2\Psi'_{j,k}dx + \frac{s}{2} J_{31}'.
\]

For \( J_{31}' \), by Hölder inequality, we firstly get

\[
(p\ast \eta)^2(x) = \left( \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|\eta(y)}dy \right)^2
\]

\[
\leq \frac{1}{4} \left( \int_{\mathbb{R}} e^{-|x-y|\eta(y)}dy \right) \cdot \left( \int_{\mathbb{R}} e^{-|x-y|\eta^2(y)}dy \right) = \frac{1}{2}(p\ast \eta^2)(x).
\]

Thus, we compute the term \( J_{31}' \) by the above inequality, the Minkowski inequality and (2.29) as

\[
J_{31}' = \int_{\mathbb{R}} \Psi'_{j,k}(p\ast \eta)^2 dx \leq \frac{1}{2} \int_{\mathbb{R}} \Psi'_{j,k}(p\ast \eta^2)dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}} \Psi'_{j,k}(t,x) \left( \int_{\mathbb{R}} e^{-|y-x|\eta^2(y)}dy \right) dx \leq \frac{1}{2} \int_{\mathbb{R}} \eta^2(y) \int_{\mathbb{R}} \frac{1}{2} e^{-|y-x|\eta^2(y)}\Psi'_{j,k}(t,x) dx dy
\]

\[
= \frac{1}{2} \int_{\mathbb{R}} \eta^2(y)(1 - \partial^2_x)^{-1}\Psi'_{j,K}dy \leq \frac{1}{2}(1 - \frac{10}{K^2})^{-1} \int_{\mathbb{R}} \eta^2(y)\Psi'_{j,K}dy.
\]

Hence, we have

\[
J_3' \leq \frac{1}{2s} \int_{\mathbb{R}} u^2\Psi'_{j,k}dx + \frac{s}{4}(1 - \frac{10}{K^2})^{-1} \int_{\mathbb{R}} \eta^2\Psi'_{j,K}dx.
\tag{2.31}
\]
For $\alpha_0 > 0$ small enough and $L_0 > 0$ large enough both depending only on $\sigma_0 > 0$, it follows from (2.3) that

$$-\frac{\dot{y}_j(t)}{2} = \frac{-\dot{x}_{j-1}(t) - c_{j-1}}{4} - \frac{-\dot{x}_j(t) - c_j}{4} - \frac{c_{j-1} + c_j}{4} \leq -\frac{c_1 + \sigma_0}{2}.$$ 

Therefore, plugging (2.27)-(2.28) and (2.30)-(2.31) into (2.24), and by the above inequality, we derive that

$$\frac{d}{dt}I_j(t) \leq C\|\vec{u}_t\|_X^3e^{-\frac{(\sigma_0 t + L/8)}{\kappa}} \sigma_0 \int_R (u^2 + u_x^2 + \eta^2)\Psi'_{j,K} dx
\leq \frac{1}{(1 - \frac{10}{K^2})^{-1}} \int_R \eta^2\Psi'_{j,K} dx.$$ 

Since $c_1 > 1$ and $K \geq 5$, we then take $s > 0$, such that

$$\frac{c_1}{2} + \frac{1}{2s} \leq 0 \text{ and } -\frac{c_1}{2} + \frac{s}{4}(1 - \frac{10}{K^2})^{-1} \leq 0.$$ 

In this way we obtain

$$\frac{d}{dt}I_j(t) \leq C\|\vec{u}_t\|_X^3e^{-\frac{(\sigma_0 t + L/8)}{\kappa}} \sigma_0 \int_R (u^2 + u_x^2 + \eta^2)\Psi'_{j,K} dx.$$ 

Then the almost monotonicity property (2.19) can be obtained by integrating the above inequality from 0 to $t$. This completes the proof of Lemma 2.2.

### 2.3 Local coercivity

In this subsection, we present a local coerciveness inequality which is crucial to our proof of the stability result. First, we recall that the Hessian operator $H_c$ of $cE - F$ around a solitary wave $\vec{\varphi}_c^t = (\varphi_c, \xi_c)^t$ is given by [40]

$$H_c = cE''(\vec{\varphi}_c^t) - F''(\vec{\varphi}_c^t) = \begin{pmatrix} L_c & -(1 + \xi_c) \\ -(1 + \xi_c) & \frac{c}{c - \varphi_c} \end{pmatrix},$$

where $L_c \triangleq -\partial_x((c - \varphi_c)\partial_x) - 3\varphi_c + \partial_x^2\varphi_c + c$. Using $\xi_c = \frac{\varphi_c}{c - \varphi_c}$, we have

$$H_c = \begin{pmatrix} L_c & -\frac{c}{c - \varphi_c} \\ -\frac{c}{c - \varphi_c} & \frac{c}{c - \varphi_c} \end{pmatrix}.$$ 

**Lemma 2.3.** There exist $\delta, C_\delta, C > 0$ depending only on $c_1 > 1$, such that for all $c \geq c_1$, $\Theta(x) \in C^2(\mathbb{R}) > 0$ and $\vec{\psi}^t = (\psi, \omega)^t \in X$, satisfying

$$|\langle \sqrt{\Theta}\vec{\psi}^t, ((1 - \partial_x^2)\varphi_c, \xi_c)^t \rangle_{L^2 \times L^2} + |\langle \sqrt{\Theta}\vec{\psi}^t, ((1 - \partial_x^2)\partial_x\varphi_c, \partial_x\xi_c)^t \rangle_{L^2 \times L^2}| \leq \delta \|\vec{\psi}^t\|_X, \quad (2.32)$$

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Then, we have

\[
\Lambda \triangleq \int_{\mathbb{R}} \Theta((c - \varphi_c)(\partial_x \psi)^2 + (-3\varphi_c + \partial_x^2 \varphi_c + c)\psi^2) + \partial_x \varphi_c \Theta' \psi^2 - 2\Theta \frac{c}{c - \varphi_c} \psi \omega \\
+ \Theta(c - \varphi_c)\omega^2 dx \geq C \int_{\mathbb{R}} \Theta(\psi^2 + (\partial_x \psi)^2 + \omega^2)dx. \tag{2.34}
\]

Proof. We directly calculate that

\[
\langle H_c \sqrt{\Theta} \psi^3, \sqrt{\Theta} \psi^3 \rangle_{L^2 \times L^2} = \langle H_c \sqrt{\Theta} \left( \psi \begin{pmatrix} \psi \\ \omega \end{pmatrix}, \sqrt{\Theta} \left( \psi \begin{pmatrix} \psi \\ \omega \end{pmatrix} \right) \right) \rangle_{L^2 \times L^2}
\]

\[
= \left\langle \left( - \partial_x ((c - \varphi_c)\partial_x \psi) - 3\varphi_c + \partial_x^2 \varphi_c + c \right) \sqrt{\Theta} \psi - \frac{c}{c - \varphi_c} \sqrt{\Theta} \omega \right), \left( \sqrt{\Theta} \psi \begin{pmatrix} \psi \\ \omega \end{pmatrix} \right) \right\rangle_{L^2 \times L^2} \\
= \int_{\mathbb{R}} \Theta((c - \varphi_c)(\partial_x \psi)^2 + (-3\varphi_c + \partial_x^2 \varphi_c + c)\psi^2) dx - 2\int_{\mathbb{R}} \Theta \frac{c}{c - \varphi_c} \psi \omega dx \\
+ \int_{\mathbb{R}} \Theta(c - \varphi_c)\omega^2 dx + \int_{\mathbb{R}} \frac{c - \varphi_c}{4} \Theta' \psi^2 dx + \int_{\mathbb{R}} (c - \varphi_c)\Theta' \psi dx \\
= \Lambda + \int_{\mathbb{R}} (c - \varphi_c) \left( \frac{(\Theta')^2}{4\Theta} - \frac{\Theta''}{2} - \frac{1}{2} \partial_x \varphi_c \Theta' \right) \psi dx, \tag{2.35}
\]

and

\[
\| \sqrt{\Theta} \psi^3 \|_{X}^2 = \int_{\mathbb{R}} (\sqrt{\Theta} \psi)^2 + (\partial_x (\sqrt{\Theta} \psi))^2 + (\sqrt{\Theta} \omega)^2 dx \\
= \int_{\mathbb{R}} \Theta(\psi^2 + (\partial_x \psi)^2 + \omega^2)dx + \int_{\mathbb{R}} \left( \frac{(\Theta')^2}{4\Theta} - \frac{\Theta''}{2} \right) \psi^2 dx. \tag{2.36}
\]

By the analysis on the spectrum of $H_c$ given in [41], we can easily deduce that there exist $\delta > 0$ and $C_6 > 0$, such that if $c \geq c_1$,

\[
| \langle \psi^3, ((1 - \partial_x^2)\varphi_c, \xi_c)^4 \rangle_{L^2 \times L^2} | + | \langle \psi^3, ((1 - \partial_x^2)\partial_x \varphi_c, \partial_x \xi_c)^4 \rangle_{L^2 \times L^2} | \leq \delta \| \psi^3 \|_X,
\]

then

\[
\langle H_c \psi^3, \psi^3 \rangle_{L^2 \times L^2} \geq C_6 \| \psi^3 \|_X^2.
\]
Therefore, under the hypotheses (2.32)-(2.33), we can derive (2.34) from (2.35)-(2.36) that
\[
A + c \cdot \frac{C_δ}{4c} \int R \Theta \psi^2 dx \geq (H_c \sqrt{\Theta \beta^3}, \sqrt{\Theta \beta^3}) \geq C_δ \|\sqrt{\Theta \beta^3}\|_X^2
\]
\[
\geq C_δ \int R \Theta (\psi^2 + (\partial_x \psi)^2 + \omega^2) dx - \min \left\{ \frac{1}{4}, \frac{C_δ}{4c} \right\} \int R \Theta \psi^2 dx,
\]
where we used the fact that \( \varphi_c \in [0, c - 1] \) and \( \partial_x \varphi_c \in [-c + 1, c - 1] \). This completes the proof of Lemma 2.3.

3 Proof of the orbital stability

Based on the series of lemmas in Section 2, we will complete the proof of the orbital stability of the train of \( N \) solitary waves for the system (1.1). By the continuity of \( \vec{w}(t) \) in \( H^s \times H^{s-1} \rightarrow X \), with \( s \geq 2 \), to prove Theorem 1.1, it is sufficient to show that there exist \( A_0, \gamma_0, L_0, \epsilon_0 > 0 \) such that for all \( L > L_0 \) and \( 0 < \epsilon < \epsilon_0 \), if \( \vec{w}_0 \) satisfies (1.6) and for some \( 0 < t_0 < T \), with \( 0 < T \leq +\infty \),

\[
\vec{w}(t) \in U(A_0(\sqrt{\epsilon} + e^{-\gamma_0 L}), L_0), \quad \text{for all} \quad t \in [0, t_0],
\]
then

\[
\vec{w}(t_0) \in U(A_0(\frac{1}{2} \sqrt{\epsilon} + e^{-\gamma_0 L}), \frac{2L_0}{3}).
\]

Therefore, we will conclude the proof of Theorem 1.1 if we prove the result (3.2) under the assumption (5.1) for some \( L > L_0 \) and \( 0 < \epsilon < \epsilon_0 \) with \( A_0, \gamma_0, L_0, \epsilon_0 > 0 \) to be specified later.

For simplicity, we set \( \vec{w} = \vec{w}(t_0) \), \( X = (x_1, ..., x_N) = (x_1(t_0), ..., x_N(t_0)) \). For \( i = 1, ..., N \), we define \( \psi_i^t = (\psi_i, \omega_i)^t \in X \) by

\[
\vec{w}^t = (1 + a_i) \vec{R}_X^t + \psi_i^t, \quad \langle (E_i'(\vec{R}_X^t), \psi_i^t) \rangle_{L^2 \times L^2} = 0,
\]
where \( \vec{R}_X^t = (R_X, S_X)^t = \sum_{i=1}^{N} R_i^t = (\sum_{i=1}^{N} R_i(\cdot), \sum_{i=1}^{N} S_i(\cdot))^t = (\sum_{i=1}^{N} \varphi_c(x_i(\cdot)), \sum_{i=1}^{N} \xi_c(x_i(\cdot)))^t \). Since, by (2.17) and (2.18), we have

\[
\langle (E_i'(\vec{R}_X^t), \vec{R}_X^t) \rangle_{L^2 \times L^2} = \langle E'(\varphi_c^t), \varphi_c^t \rangle_{L^2 \times L^2} + O(e^{-\sigma_0 L})
\]
\[
= \| \varphi_c^t \|_X^2 + O(e^{-\sigma_0 L}) > \frac{1}{2} \| \varphi_c^t \|_X^2.
\]

Thus, the functions \( \vec{w}_i^t \) are well defined. Then, we set \( \vec{v}^t = (v(t, x), \zeta(t, x))^t = \vec{w}^t - \vec{R}_X^t \) and in the sequel suppose that

\[
\| \vec{v}^t \|_X \geq \sqrt{\epsilon} + e^{-\gamma_0 L},
\]
otherwise we complete our proof with \( A_0 = 2 \) and \( \gamma_0 = \frac{\alpha}{2} \). We break our proof into three steps.
Proof. Step 1: In the first step, for all \( i = 1, \ldots, N \), we claim that the following estimates on \( a_i \) hold true:
\[
|a_i| \leq O(||v^i||_X^2),
\]
(3.6)

Indeed, according to the definitions (2.16) of \( I_i, E_i \) and \( F_i \), we have
\[
I_i(t, u) = \sum_{j=1}^{N} E_j^i(u^i), \quad \text{for } i = 2, \ldots, N, \quad E(u^i) = \sum_{j=1}^{N} E_j^i(u^i) \quad \text{and} \quad F(u^i) = \sum_{j=1}^{N} F_j^i(u^i). \quad (3.7)
\]

Using the exponentially asymptotic behavior of \( \varphi_i^c \), and by (2.17)-(2.18), one can easily find that
\[
E_j^i((R_j, S_j)^t) = E(\varphi_j^c) + O(e^{-\sigma_0 L}) \quad \text{and} \quad E_j^i((R_k, S_k)^t) \leq O(e^{-\sigma_0 L}) \quad \text{for } j \neq k. \quad (3.8)
\]

Hence, by Taylor formula, (3.3), (3.5) and (3.8), we obtain
\[
\sum_{j=1}^{N} E_j^i(u^i) = \sum_{j=1}^{N} E_j^i(\varphi_j^c) + \sum_{j=1}^{N} \langle (E_j^i)'(\varphi_j^c), \varphi_j^c \rangle_{L^2 \times L^2} + O(||\varphi_j^c||_X^2)
\]
\[
= \sum_{j=1}^{N} E_j^i(\varphi_j^c) + \sum_{j=1}^{N} a_j \langle (E_j^i)'(\varphi_j^c), \varphi_j^c \rangle_{L^2 \times L^2} + O(||\varphi_j^c||_X^2). \quad (3.9)
\]

Since \( u_0^i \) satisfies (1.6), on account of the conservation laws (1.5) of \( E \) and \( F \), we get
\[
E(u^i) = E(u_0^i) = \sum_{j=1}^{N} E(\varphi_j^c) + O(e^{-\sigma_0 L}) + O(\varepsilon), \quad (3.10)
\]

and
\[
F(u^i) = F(u_0^i) = \sum_{j=1}^{N} F(\varphi_j^c) + O(e^{-\sigma_0 L}) + O(\varepsilon). \quad (3.11)
\]

Thus, for all \( i = 1, \ldots, N \), we deduce from (3.7), (3.9)-(3.10) and (3.5) that
\[
\sum_{j=i}^{N} a_j \langle (E_j^i)'(\varphi_j^c), \varphi_j^c \rangle_{L^2 \times L^2} \leq O(||\varphi_j^c||_X^2). \quad (3.12)
\]

On the other hand, in a similar way, using Taylor formula, and by (2.17)-(2.18), we have
\[
F(u^i) = \sum_{i=1}^{N} F_i^i(u^i) = \sum_{i=1}^{N} F_i^i(\varphi_j^c) + \sum_{i=1}^{N} \langle (F_i^i)'(\varphi_j^c), \varphi_j^c \rangle_{L^2 \times L^2} + O(||\varphi_j^c||_X^2),
\]

and
\[
F(u^i) = \sum_{i=1}^{N} F(\varphi_j^c) + \sum_{i=1}^{N} \langle (F_i^i)'(\varphi_j^c), \varphi_j^c \rangle_{L^2 \times L^2} + O(||\varphi_j^c||_X^2) + O(e^{-\sigma_0 L}). \quad (3.13)
\]

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Hence, by (3.11), (3.13), (3.5) and (3.3), we get

\[ O(\|\tilde{v}\|^2_X) \]

\[ = \sum_{i=1}^{N} \langle (E_i')(\tilde{R}_X), \tilde{v}\rangle_{L^2 \times L^2} \]

\[ = \sum_{i=1}^{N} \langle (E_i')(\tilde{R}_X) - c_i(E_i')(\tilde{R}_X), \tilde{v}\rangle_{L^2 \times L^2} + \sum_{i=1}^{N} \langle c_i(E_i')(\tilde{R}_X), \tilde{v}\rangle_{L^2 \times L^2} \]

\[ = \sum_{i=1}^{N} \langle (E_i')(\tilde{R}_X) - c_i(E_i')(\tilde{R}_X), \tilde{v}\rangle_{L^2 \times L^2} + \sum_{i=1}^{N} c_i a_i \langle (E_i')(\tilde{R}_X), \tilde{R}_X \rangle_{L^2 \times L^2}. \quad (3.14) \]

Since the solitary waves satisfy the identity \( cE'(\tilde{\varphi}_i^d) - F'(\tilde{\varphi}_i^d) = 0 \), by (2.17)-(2.18), then we obtain

\[ \| (F_i')(\tilde{R}_X) - c_i(E_i')(\tilde{R}_X) \|_{X^*} = \| F'(\tilde{\varphi}_i^d) - c_iE'(\tilde{\varphi}_i^d) \|_{X^*} + O(e^{-\sigma_0L}) \leq O(e^{-\sigma_0L}). \quad (3.15) \]

Thus, by (3.14)-(3.15), we infer that

\[ \sum_{i=1}^{N} c_i a_i \langle (E_i')(\tilde{R}_X), \tilde{R}_X \rangle_{L^2 \times L^2} = O(\|\tilde{v}\|^2_X). \]

Then, using Abel transformation, we find that

\[ \sum_{i=1}^{N} (c_i - c_{i-1}) \sum_{j=1}^{N} a_j \langle (E_j')(\tilde{R}_X), \tilde{R}_X \rangle_{L^2 \times L^2} + c_1 \sum_{j=1}^{N} a_j \langle (E_j')(\tilde{R}_X), \tilde{R}_X \rangle_{L^2 \times L^2} = O(\|\tilde{v}\|^3_X). \]

\[ \quad \text{(3.16)} \]

Combining (3.12) with (3.10), for all \( i = 1, \ldots, N \), it follows that

\[ |a_i \langle (E_i')(\tilde{R}_X), \tilde{R}_X \rangle_{L^2 \times L^2}| \leq O(\|\tilde{v}\|^3_X). \]

Therefore, by (3.4), we prove our claim (3.10). Moreover, for all \( i = 1, \ldots, N \), we derive from our claim (3.3) and (3.3) that

\[ \|\tilde{v}\|_X \sim \|\tilde{\psi}_i^d\|_X. \]

\[ \quad \text{(3.17)} \]

Step 2: In the second step, we will apply the local coerciveness inequality (2.34) in Lemma 2.3 to prove

\[ \langle H_i'(\tilde{R}_i^d)\tilde{\psi}_i^d, \tilde{\psi}_i^d \rangle_{L^2 \times L^2} \geq CE_i^d(\tilde{\psi}_i^d), \]

\[ \quad \text{(3.18)} \]

where \( \tilde{R}_i^d(\cdot) = (R_i(\cdot), S_i(\cdot))^t = (\tilde{\varphi}_i(\cdot - x_i), \xi_i(\cdot - x_i))^t \) and \( \tilde{\psi}_i^d = (\psi_i, \omega_i)^t \in X \). And the operator \( H_i^d \) is given in the following form (3.19).
From the definitions of $E_i^t$ and $F_i^t$, we can explicitly compute the variational derivatives as in \[32\]

\[
\begin{cases}
(E_i^t)_\varphi = \varphi \Phi_i - \partial_x^2 \varphi \Phi_i - \partial_x \varphi \partial_x \Phi_i, \\
(E_i^t)_\xi = \xi \Phi_i,
\end{cases}
\]

and

\[
\begin{cases}
(F_i^t)_\varphi = \frac{3}{2} \Phi_i - \frac{1}{2} (\partial_x \varphi)^2 \Phi_i - \varphi \partial_x^2 \varphi \Phi_i + \xi \Phi_i + \frac{1}{2} \xi^2 \Phi_i - \varphi \partial_x \varphi \partial_x \Phi_i, \\
(F_i^t)_\xi = \varphi \Phi_i + \varphi \xi \Phi_i.
\end{cases}
\]

Hence, we have

\[
\begin{cases}
(E_i^t)''_\varphi = \Phi_i (1 - \partial_x^2) - \partial_x \Phi_i \partial_x, \\
(E_i^t)''_\xi = (E_i^t)''_\varphi = 0, \\
(E_i^t)''_{\xi \xi} = \Phi_i,
\end{cases}
\]

and

\[
\begin{cases}
(F_i^t)''_\varphi = (3 \varphi - \partial_x \varphi \partial_x - \varphi \partial_x^2 - \partial_x^2 \varphi) \Phi_i + (-\varphi \partial_x \Phi_i \partial_x - \partial_x \varphi \partial_x \Phi_i), \\
(F_i^t)''_\xi = (F_i^t)''_\varphi = \Phi_i + \Phi_i \xi, \\
(F_i^t)''_{\xi \xi} = \varphi \Phi_i.
\end{cases}
\]

Therefore, the linearized operator $H_i^t$ of $c(E_i^t)' - (F_i^t)'$ (or Hessian of $cE_i^t - F_i^t$) at $\varphi_i^t$ can be given as

\[
H_i^t(\varphi_i^t) = c(E_i^t)''(\varphi_i^t) - (F_i^t)''(\varphi_i^t)
\]

\[
= \left( \begin{array}{c}
\frac{L_i^t}{c \varphi_i - \Phi_i} \\
- \Phi_i (1 + \xi) \\
\Phi_i (c - \varphi_i)
\end{array} \right),
\]

where $L_i^t \triangleq -\partial_x (\Phi_i (c - \varphi_i) \partial_x) + \Phi_i (-3 \varphi_i + \partial_x^2 \varphi_i + c) + \partial_x \varphi \partial_x \Phi_i$. By $\xi = \frac{\varphi_i}{c - \varphi_i}$, we have

\[
H_i^t(\varphi_i^t) = \left( \begin{array}{c}
\frac{L_i^t}{c \varphi_i - \Phi_i} \\
\frac{- \frac{\Phi_i}{c \varphi_i - \Phi_i}}{\Phi_i (c - \varphi_i)}
\end{array} \right).
\]

According to Lemma \[243\] to prove \[32\], we only need to verify that $\tilde{\psi}_i^t$ satisfies the condition \[2.32\] with $\Theta = \Phi_i$ and $\varphi_i^t = \tilde{R}_i^t$. Indeed, using the fact $\tilde{\psi}_i^t = (\psi_i, \omega_i)^t = \bar{v}_i - a_i \bar{R}_X = (v - a_i R_X, \zeta - a_i S_X)^t$, by \[2.1\], we get

\[
\langle \sqrt{\Phi_i} \tilde{\psi}_i^t, (1 - \partial_x^2) \partial_x R_i, \partial_x S_i \rangle_{L^2 \times L^2}
\]

\[
= \langle \tilde{\psi}_i^t, (1 - \partial_x^2) \partial_x R_i, \partial_x S_i \rangle_{L^2 \times L^2} + \langle (\sqrt{\Phi_i} - 1) \tilde{\psi}_i^t, (1 - \partial_x^2) \partial_x R_i, \partial_x S_i \rangle_{L^2 \times L^2}
\]

\[
= \langle \psi_i, (1 - \partial_x^2) \partial_x R_i \rangle_{L^2} + \langle \omega_i, \partial_x S_i \rangle_{L^2} + \langle (\sqrt{\Phi_i} - 1) \psi_i, (1 - \partial_x^2) \partial_x R_i \rangle_{L^2}
\]

\[
+ \langle (\sqrt{\Phi_i} - 1) \omega_i, \partial_x S_i \rangle_{L^2}
\]

\[
= -a_i (R_X, (1 - \partial_x^2) \partial_x R_i \rangle_{L^2} + a_i (S_X, \partial_x S_i \rangle_{L^2} + \langle (\sqrt{\Phi_i} - 1) \psi_i, (1 - \partial_x^2) \partial_x R_i \rangle_{L^2}
\]

\[
+ \langle (\sqrt{\Phi_i} - 1) \omega_i, \partial_x S_i \rangle_{L^2}.
\]
Thus, gathering (2.17), (3.1), (3.6) and (3.17), we deduce that

\[ |\langle \sqrt{\Phi_1} \psi_1, ((1 - \partial_x^2)\partial_x R_i, \partial_x S_i) \rangle_{L^2}^4| \leq (O(\sqrt{\varepsilon}) + O(e^{-\sigma_0 L}) + O(e^{-\gamma_0 L}))(\psi_1^2)|_{X}. \]

In a similar manner as above and by (3.3), we then obtain

\[
\langle \sqrt{\Phi_1} \psi_1^2, ((1 - \partial_x^2)R_i, S_i) \rangle_{L^2}^4 = \langle \psi_1, ((1 - \partial_x^2)R_i, S_i) \rangle_{L^2}^4 + \langle (\sqrt{\Phi_1} - 1) \psi_1^2, ((1 - \partial_x^2)R_i, S_i) \rangle_{L^2}^4 \\
= \langle \psi_1, (1 - \partial_x^2)R_i \rangle_{L^2}^2 + \langle \omega_i, S_i \rangle_{L^2}^2 \leq \langle (\sqrt{\Phi_1} - 1) \psi_1, (1 - \partial_x^2)R_i \rangle_{L^2}^2 + \langle (\sqrt{\Phi_1} - 1) \omega_i, S_i \rangle_{L^2}^2 \\
\leq (O(\sqrt{\varepsilon}) + O(e^{-\sigma_0 L}) + O(e^{-\gamma_0 L}))(\psi_1^2)|_{X}. \]

**Step 3:** In the last step, we prove that there exists \( C > 0 \) independent of \( A_0 \) and \( \gamma_0 \) such that for \( L \geq L_0 \), with \( L_0 \) large enough,

\[
\| \tilde{v} \|^2_X = \| \tilde{u} \| - \sum_{i=1}^{N} \tilde{v}_{c_i}^2 (\cdot - x_i) \|_{X}^2 \leq C(\varepsilon + e^{-\sigma_0 L}).
\]

Then, we can conclude the proof of Theorem 1.1 with \( \gamma_0 = \frac{\sigma_0}{2} \) and \( A_0 = 2\sqrt{C} \).

By the monotonicity estimates (2.19) and (1.6), we have

\[
I_i(t_0, \tilde{u}^2) \leq I_i(0, \tilde{u}_0^2) + O(e^{-\sigma_0 L}) \leq \sum_{j=1}^{N} E_i^0(\varphi_{e_j}^2 (\cdot - x_j(0))) + O(e^{-\sigma_0 L}) + O(\varepsilon) \leq \sum_{j=1}^{N} E(\varphi_{e_j}^2) + O(e^{-\sigma_0 L}) + O(\varepsilon). \quad (3.20)
\]

Using Abel transformation, in view of the conservation law \( E \), then by (3.7), (3.20) and (3.9),
we deduce that

\[
\sum_{i=1}^{N} c_i E_i^l(\vec{u}^t) = \sum_{i=2}^{N} (c_i - c_{i-1}) \sum_{j=i}^{N} E_j^l(\vec{u}^t) + c_1 \sum_{j=1}^{N} E_j^l(\vec{u}^t)
\]

\[
= \sum_{i=2}^{N} (c_i - c_{i-1}) I_i(t_0, \vec{u}^t(t_0)) + c_1 E(\vec{u}^t(t_0))
\]

\[
\leq \sum_{i=2}^{N} (c_i - c_{i-1}) I_i(0, \vec{u}_0^t) + c_1 E(\vec{u}_0^t) + O(e^{-\sigma_0 L})
\]

\[
\leq \sum_{i=2}^{N} (c_i - c_{i-1}) \sum_{j=i}^{N} E(\varphi_{c_j}) + c_1 \sum_{j=1}^{N} E(\varphi_{c_j}) + O(e^{-\sigma_0 L}) + O(\varepsilon)
\]

\[
\leq \sum_{i=1}^{N} c_i E_i^l(\vec{R}_{3X}^t) + O(e^{-\sigma_0 L}) + O(\varepsilon).
\]

(3.21)

On the other hand, using Taylor formula, we derive from (3.3) and (3.15) that

\[
\sum_{i=1}^{N} \left( c_i E_i^l(\vec{u}^t) - F_i^l(\vec{u}^t) \right)
\]

\[
= \sum_{i=1}^{N} \left( c_i E_i^l(\vec{R}_{3X}^t) - F_i^l(\vec{R}_{3X}^t) \right) + \frac{1}{2} \sum_{i=1}^{N} \langle H_i^l(\vec{R}_{3X}^t) \psi_i^\gamma, \psi_i^\gamma \rangle_{L^2 \times L^2} + \sum_{i=1}^{N} a_i \langle H_i^l(\vec{R}_{3X}^t) \vec{R}_{3X}^t, \vec{R}_{3X}^t \rangle_{L^2 \times L^2}
\]

\[
+ \sum_{i=1}^{N} \frac{a_i^2}{2} \langle H_i^l(\vec{R}_{3X}^t) \vec{R}_{3X}^t, \vec{R}_{3X}^t \rangle_{L^2 \times L^2} + o(||\vec{\varphi}||_{X}^2) + O(e^{-\sigma_0 L}).
\]

(3.22)

Combining (3.21) with (3.22), we obtain

\[
F(\vec{R}_{3X}^t) - F(\vec{u}^t)
\]

\[
\geq \frac{1}{2} \sum_{i=1}^{N} \langle H_i^l(\vec{R}_{3X}^t) \psi_i^\gamma, \psi_i^\gamma \rangle_{L^2 \times L^2} + \sum_{i=1}^{N} a_i \langle H_i^l(\vec{R}_{3X}^t) \vec{R}_{3X}^t, \vec{R}_{3X}^t \rangle_{L^2 \times L^2}
\]

\[
+ \sum_{i=1}^{N} \frac{a_i^2}{2} \langle H_i^l(\vec{R}_{3X}^t) \vec{R}_{3X}^t, \vec{R}_{3X}^t \rangle_{L^2 \times L^2} + o(||\vec{\varphi}||_{X}^2) + O(e^{-\sigma_0 L}) + O(\varepsilon)
\]

(3.23)

Since, from (3.17), (3.19) and (3.6), we get

\[
\sum_{i=1}^{N} a_i \langle H_i^l(\vec{R}_{3X}^t) \vec{R}_{3X}^t, \vec{R}_{3X}^t \rangle_{L^2 \times L^2} = O(||\vec{\varphi}||_{X}^3) + O(e^{-\sigma_0 L}) + O(\varepsilon),
\]

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and we can easily infer that
\[
\langle H_t^i (\vec{R}_t^i \psi^i_t, \vec{\psi}^i_t) \rangle_{L^2 \times L^2} = \langle H_t^i (\vec{R}_t^i \psi^i_t, \vec{\psi}^i_t) \rangle_{L^2 \times L^2} = O(e^{-\sigma_0 L}) + O(\|v^i\|^3_X).
\]

Hence, by (3.18), (3.23) and the above two equalities,
\[
\sum_{i=1}^{N} E_t^i (\psi^i_t) \leq o(\|v^i\|^3_X) + O(e^{-\sigma_0 L}) + O(\varepsilon). \tag{3.24}
\]

Then, by (3.3) and (3.6), we get
\[
\sum_{i=1}^{N} E_t^i (\psi^i_t) = \sum_{i=1}^{N} E_t^i (\vec{\psi}^i_t) + O(\|v^i\|^3_X) = E (\vec{v}^i) + O(\|v^i\|^3_X) \geq \frac{1}{2} E (\vec{v}^i). \tag{3.25}
\]

Therefore, gathering (3.23)-(3.25) and (3.11), we complete the proof of Theorem 1.1.

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