Characterization of multipartite entanglement

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In this paper, we provide a characterization of multipartite entanglement in terms of equivalence classes of states under local unitary transformation (LU) by demonstrating a simple method to construct all homogenous polynomials that are invariant under local unitary group (LU-IPs) for any fixed degree. We give an upper bound on the degree of the LU-IP such that the ring of LU-IPs can be generated by LU-IPs with degree lower than the bound. Our characterization can directly generate an algorithm whose output is a generating set of LU-IPs. By employing the concept of LU-IPs, we prove that multipartite entanglement is additive in the sense that two multipartite states are LU equivalent if and only if \( n \)-copies of these two states are LU equivalent for some \( n \). The method for studying LU equivalence is also used to classify the different types of multipartite mixed entanglement according to equivalence classes of states under stochastic local operations and classical communication (SLOCC), where the pure states case was previously studied by Gour and Wallach using another approach.

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Introduction—Multipartite entanglement is considered as an essential asset to information processing and computational tasks, which include measurement-based quantum computation\textsuperscript{[1,2]}, quantum error correction schemes\textsuperscript{[3]}, and quantum secret sharing\textsuperscript{[4]}. The intriguing properties and potential applications of entanglement spark many literature dedicated to quantify it as a resource, and numerous results have been obtained. In bipartite case, it is well known that the vector of Schmidt coefficients, instead of any scalar, is a proper entanglement measure when exact transformations are considered. However, it is much more complicated for multipartite entanglement even in the asymptotic manner\textsuperscript{[5–7]}. Even though great efforts and lots of progresses has been made, we are still we are far from completely understanding the non-local properties of multipartite entanglement.

Entanglement transformation can often be utilized to study the equivalence of entanglement. The following entanglement transformation problem is fundamental: whether two pure \( N \)-partite states \( |\psi\rangle \) and \( |\phi\rangle \) could be transformed into each other, assuming that each party may use only local operations on their respective systems with the help of unlimited two-way classical communication (LOCC). Theoretically, this LOCC equivalence class is defined such that within this class any two quantum states are interconvertible by local unitary (LU) operators\textsuperscript{[8]}. This problem is significant for characterizing entanglement by noticing that LOCC could never increase entanglement, then two quantum states are LOCC(LU) equivalent indicates that their entanglement are exactly the same in any perspective of view. Therefore, multipartite entanglement are characterized according to their inter-convertibility under local unitary transformation. Often, one may release the local unitary operator restriction to invertible local transformation, the LOCC equivalence then turns to the widely studied stochastic local operations and classical communication (SLOCC) equivalence\textsuperscript{[8–15]}

In order to study the LU equivalent problem, the concept of the local polynomial invariants is presented in \textsuperscript{[16–18]}. A complete classification has been obtained only for very few simple case, \( 2 \times 2 \times n \) system and \( 2 \times 2 \times 2 \times 2 \) system\textsuperscript{[19–21]}. Beyond that, and despite the extensive literature, very little is known, even for pure tripartite states, about the set of all such LU-invariant polynomials (LU-IPs) except for few techniques that were used to construct some of the LU-IPs for multi-partite states, see\textsuperscript{[22]} as a very incomplete list. Very recently, another approach of studying the LU equivalence of \( n \)-qubit states is presented in\textsuperscript{[23]}. Such method creatively deduces this problem into solving finite set of non linear equations whose variables are two-qubit unitaries. When it is unavailable to solve such equations, this method is no longer operational. Thus, a characterization of LU-IPs for multipartite system is highly desirable.

In this paper, we give a characterization of multipartite entanglement by exploiting a systematic method to describe the ring of all LU-IPs. In order to do so, we first employ some techniques and concepts of invariant theory to demonstrate an explicit upper bound of the degree of LU-IPs for which any LUIP can be generated by LU-IPs with degree less than the bound. We then provide an algorithm to construct all LU-IPs with fixed degree. Our main idea here is to look at the ring of homogenous polynomials that are invariant under local unitary group applied on one fixed party, then the ring of LU-IPs is the intersection of these rings, which can be computed by computing the intersection of finite dimensional subspaces. By using the concept of LU-IPs, we are able to show that multipartite entanglement is additive in the sense that
two multipartite states are LU equivalent if and only if 
n-copies of these two states are LU equivalent for some n.
Interestingly, this idea can be used to study the multipar-
tite mixed entanglement in terms of equivalence classes
of states under SLOCC. For pure state, we provide an al-
gorithm to compute the whole set of SL-invariant poly-
nomials(SLIPs) which was recently obtained by Gour and
Wallach in [13] using another approach. The problem of
the rank one SLOCC equivalence relation between
mixed state is transformed into the problem of equiva-
ience relation for pure states under the group of ten-
sor product of SL group and unitary group(SLU), then the whole set
of SLU-invariant polynomials(SLUIPs) is constructed by
employing our previous approach. It would be worth to
notice that our algorithms are dramatically simple by
only computing finite intersections between subspaces,
which is of course feasible in all dimensions.

Preliminaries—In order to demonstrate our main re-
sult, we need to employ the result in [24] by Derksen: Let
group G over an algebraically closed field K of character-
istic 0, acting on an s-dimensional vector space V as fol-
ows: there exist polynomials h1, ..., hs ∈ K[z1, ..., zs]
ai,j ∈ K[z1, ..., zs] such that G is the the zero set of
these polynomials: On the other hand, there exist poly-
nomials ai,j for i, j ≤ s such that g : G → GL(V) is given
by g → (ai,j(g))i,j≤s, where GL(V) is the general linear
group of V, i.e., the group of invertible matrices.

The coordinate ring of V can be identified as R =
K[x1, ..., xs], and G induces an action on R. The in-
variant ring of G on V, denoted as R_G, consists of those
polynomials in R invariant under G, i.e.,

\[ R_G = \{ r : r(g \cdot v) = r(v), \forall g \in G, v \in V \} \]

It is known that R_G is finitely generated according to
Hilbert’s famous result [25]. The question here is to de-
rive an explicit degree bound for this finite generation.

Let \( A = \max \{ \deg(a_{i,j}) \mid i, j \leq s \} \), \( H = \max \deg(h_i) \), and
d = \( \dim(G) \) where \( \dim(G) \) denotes the dimension of the
algebraic variety \( G \) [26]. Derksen shows that [24]

\[ \sigma(V, G) \leq H^{1-d}A^d, \quad \beta(V, G) \leq \max(2, \frac{3}{8} s^2) \].

Main Results—In this section, we demonstrate an ex-
plicit upper bound for rings of invariant, then provide an
algorithm to compute the rings of invariant.

During this paper, we consider the Hilbert space
\( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \),
with \( d_i \) being the dimension of each Hilbert space \( \mathcal{H}_i \).

Let \( U(d_i) \) be the group of \( d_i \times d_j \) unitary group, local
unitary group is defined as
\[ \text{LU} = U(d_1) \otimes U(d_2) \otimes \cdots \otimes U(d_n). \]

For \( |\Psi\rangle \in \mathcal{H} \), the orbit \( \text{LU}|\Psi\rangle := \{ g|\Psi\rangle : g \in \text{LU} \} \) is
consisting of quantum states in the LU equivalent class
of \( |\Psi\rangle \). The entanglement of \( |\Psi\rangle \) is the same as that of
any \( |\Phi\rangle \in \text{LU}|\Psi\rangle \) from any point of view.

We are interested here characterizing multipartite en-
tanglement by classifying classes of LU equivalent states.
It is direct to see that the whole state space is classified
into the union of distinct orbits. In order to separate dis-
tinct orbits, one may use a series of functions \( f_m : \mathcal{H} \to \mathbb{C} \)
for \( 1 \leq m \in \mathbb{N} \). To maintain the definition is also well
defined on orbits, a natural requirement is that \( f_m \) is
LU-invariants, i.e., invariant under local unitary trans-
formations. As the simplest and the most well studied
functions, \( f_m \) is chosen to be polynomial. Those poly-
nomials which is invariant under local unitaries is called
LUIP. More precisely, for the local unitary group, a func-
tion \( f : \mathcal{H} \to \mathbb{C} \) is an LUIP, if \( f(|\Psi\rangle|\Psi\rangle) \) is the homoge-
nous polynomial on entries of \( |\Psi\rangle|\Psi\rangle \) and
\[ f(g|\Psi\rangle|g^\dagger) = f(|\Psi\rangle|\Psi\rangle), \quad \forall g \in \text{LU} \quad \text{and} \quad \forall |\Psi\rangle \in \mathcal{H}. \]

It is known that the LUIPs can be used to determine
whether two states in \( \mathcal{H} \) belong to the same LU class
[18, 19], i.e., two pure state are LU equivalent if and
only if \( f_i(|\Psi\rangle|\Psi\rangle) = f_i(|\Phi\rangle|\Phi\rangle) \) holds for all LUIPs \( f_i \).
Since the degree of LUIP can be arbitrary large, it is not
feasible to calculate \( f_i(|\Psi\rangle|\Psi\rangle) \) for all LUIP \( f_i \).

Our first result is to show that the ring of LUIPs is
finitely generated. Moreover, we demonstrate a degree
bounds for a generator for the ring of LUIPs.

**Theorem 1.** The set of all LUIPs is generated by
the LUIPs with degree less than \( N(d_1, d_2, \cdots, d_n) = \frac{3}{8} \prod_i d_i^{3/2}, 2d^2 + 2 \).

**Proof:** For a linear operation \( \rho \in \mathcal{H} \), \( g \in \text{LU} \subseteq
\text{GL}(\mathcal{H}) \) acts on \( \rho \) by sending \( \rho \) to \( g\rho g^{-1} \). Let
\( R \) be the polynomial ring in \( \prod_i d_i^2 \) variables, identified
as the coordinate ring of \( L(\mathcal{H}, \mathcal{H}) \), and \( R_{\text{LU}} \) be LUIPs,
\( i.e., \) the invariant ring of local unitary.
It is not feasible to apply Derksen’s bound directly, as \( U \) cannot be viewed as zero set of polynomials over algebraically closed field \( \mathbb{C} \). This can be got around by considering the complexification of \( U \), which yields \( G = GL(d_1, \mathbb{C}) \times \cdots \times GL(d_n, \mathbb{C}) \subseteq GL(\mathcal{H}) \). Recall that we can view \( R \) as the space of representations of \( LU \) and \( G \), and note that invariant polynomials are just dimension-1 irreducible representations. Thus by the correspondence between irreducible representations of \( LU \) and \( G \), \( R^{G \mid R} = R^G \). Thus it is enough to get a degree bound for the action of \( G \).

To get a degree bound for the action of \( G \) on \( R \), we compute as follows. First \( s := \dim(R^G) = (\prod_i d_i)^2 \). To bound \( \sigma(\mathcal{H}, G) \), we observe that \( t = \sum_i d_i^2 \), \( H = 1 \), \( d = \sum_i d_i^2 \), and \( A = \prod_i d_i \). Thus

\[
\sigma(\mathcal{H}, G) \leq (\prod_i d_i) \Sigma d_i^2,
\]

\[
\Rightarrow \beta(\mathcal{H}, LU) = \beta(\mathcal{H}, G) \leq \frac{3}{8}(\prod_i d_i) \Sigma, 2d_i^2 + 2.
\]

In other words, two quantum states \( |\Psi\rangle, |\Phi\rangle \in \mathcal{H} \) are LU equivalent if and only if \( f_i(|\Psi\rangle |\Phi\rangle) = f_i(|\Phi\rangle |\Phi\rangle) \) holds for any LUIPs \( f_i \) with degree less than \( N(d_1, d_2, \cdots, d_n) \).

Now we provide a new view of LUIPs which leads to an algorithm for finding all of them. Let \( I_j \) be the identity operator of system \( \mathcal{H}_j \), we can define group \( U_i \) as follows,

\[
U_i = I_1 \otimes \cdots \otimes I_{i-1} \otimes U(d_i) \otimes I_{i+1} \otimes \cdots \otimes I_n.
\]

A direct but useful observation is as follows

\[
LU = U_1U_2 \cdots U_n, \text{and } U_i \subset LU.
\]

The advantage of this observation on studying the polynomial invariants is based on the following relation between the polynomial invariants of \( LU \), says \( P \), and those polynomial invariants of \( U_i \)'s, says \( P_i \),

\[
P = \bigcap_{i=1}^n P_i.
\]

To see the validity of the above relation \( \square \), we notice that \( U_i \subset LU \) leads us to the fact that \( P \subset P_i \), thus, \( P \subset \bigcap_{i=1}^n P_i \). On the other hand, one can verify that for any \( p \in \bigcap_{i=1}^n P_i \), \( g = g_1g_2 \cdots g_n \in LU \) with \( g_i \in U_i \) and \( |\Psi\rangle \in \mathcal{H} \), we have \( p \in P \) by observing

\[
p(|\Psi\rangle |\Psi\rangle g^{-1}) = p(g_1 \cdots g_n |\Psi\rangle |\Psi\rangle g_n^{-1} \cdots g_1^{-1})
\]

\[
= p(g_2 \cdots g_n |\Psi\rangle |\Psi\rangle g_n^{-1} \cdots g_2^{-1})
\]

\[= \cdots
\]

\[= p(|\Psi\rangle |\Psi\rangle).
\]

Therefore, \( P \subset \bigcap_{i=1}^n P_i \Rightarrow P = \bigcap_{i=1}^n P_i \).

\( P_i \)'s are very easy to compute in the following way. Suppose \( |\alpha\rangle = \sum_{j_1,j_2,\cdots,j_n} x_{j_1,j_2,\cdots,j_n} |1_{j_1}2_{j_2} \cdots n_{j_n}\rangle \) with variables \( x_{j_1,j_2,\cdots,j_n} \in \mathbb{C} \). Notice that any homogenous polynomial \( q \in P_i \) is a polynomial on the elements of the reduced density matrix \( \rho_i = Tr_{\mathcal{V}}|\alpha\rangle \langle \alpha| = [u_{kl}^{(i)}(x)] \). We know that \( u_{kl}^{(i)}(x) \) are quadratic polynomial of \( x \). On the other hand, any polynomial on \( u_{kl}^{(i)}(x) \) is an element of \( P_i \). Therefore, \( P_i \) is the ring with generators \( u_{kl}^{(i)}(x) \).

More precisely,

\[
P_i = \mathbb{C}[u_{kl}^{(i)}(x)].
\]

Notice that \( P \) is fully characterized by the intersection of polynomial rings \( P_i \), we can conclude that the degree of LU equivalent must be even. The homogeneous LUIPs with degree \( 2m \in \mathbb{N} \) is

\[
Q_{(2m)} : = \bigcap_{i=1}^n \text{span} \{u_{kl}^{(i)}(x)\}^m,
\]

where the product of two set of polynomials is defined as \( A \times B = \{a \times b : a \in A, b \in B\} \).

We know that the set of all LUIPs with fixed degree form a vector subspace over \( \mathbb{C} \). \( Q_{(2m)} \) can be viewed as a linear subspace. Therefore, Eq. (2) indicates that the subspace \( Q_{(2m)} \) is the intersection of \( n \)– subspaces, which are easily computed.

In order to characterize the LU equivalence class, according to Theorem 1, we only need to find all LUIPs with degree less than \( N(d_1, d_2, \cdots, d_n) \). Moreover, a basis of such LUIPs is sufficient to decide the LU equivalent class instead of the set of all such LUIPs.

\[
R := \{q_{(2m),j} : 0 \leq j \leq r(2m), 0 \leq 2m \leq N(d_1, d_2, \cdots, d_n)\},
\]

where \( q_{(2m),1}, q_{(2m),2}, \cdots, q_{(2m),r(2m)} \) is a basis of \( Q_{(2m)} \).

The above equation indeed provides an algorithm to separate local unitary equivalent orbits by noticing that computing a basis of \( Q_{(2m)} \) is simple after obtaining the subspace description of \( Q_{(2m)} \).

On the other hand, the characterization of \( P \) can be given as

\[
P = \bigcap_{i=1}^n P_i = \text{span} \bigcup_{m=1}^\infty Q_{(2m)} = \text{span} \bigcup_{j=1}^\infty R^j.
\]

We have already provided a method to separate the LU orbits of pure states, interestingly, this can help us to separate the LU orbits of mixed states by noticing that two \( n \)-partite mixed states are LU equivalent if and only if their purifications, two \( n+1 \)-partite mixed states, are LU equivalent \([22]\).

The LU equivalence of quantum states has the application on the equivalence between quantum channel, where two quantum channel \( E \) and \( F \) are said to be equivalent if there are unitary channels \( U \) and \( V \) such that

\[
F = V \circ E \circ U.
\]

Here, unitary channels \( U \) and \( V \) can be regarded as encoding channel and decoding channel, respectively. It is direct to verify that \( E \) and \( F \) have the same ability on transmit information.
One can observe that $E, F : L(H_A) \rightarrow L(H_B)$ are equivalent if and only if their Choi-matrices are LU equivalent, where the Choi-matrix of quantum channel $E(\cdot) = \sum_i E_i \otimes E_i^t$ is defined as bipartite mixed state (non-normalized) $\rho_{AA'} = (I_{AA'} \otimes E(|\Psi\rangle\langle\Psi|))$ with $|\Psi\rangle = \sum_{i=1}^d |i\rangle|i\rangle$ and noiseless channel $I_{AA'}$ on quantum system $H_{AA'}$ which has the same dimension $d$ with system $H_A$. Invoking our previous argument, this problem can be reduced to the LU equivalence between tripartite pure states, which can be solved using our algorithm on searching the LUIPs of this tripartite system.

In the following, we demonstrate that the concept of LUIPs is a powerful tool on study the LU equivalence between quantum states by showing the following result.

**Theorem 2.** There is some $r \in \mathbb{N}$ such that $|\Psi\rangle^\otimes r, |\Phi\rangle^\otimes r$ are LU equivalent, then $|\Psi\rangle, |\Phi\rangle$ are LU equivalent.

**Remark:**— The bipartite case is obvious since the structure of bipartite pure entanglement is simply determined by it Schmidt coefficients. **Proof:**— By applying the above procedure to study the LUIPs for $r$ copies of states, says $|\Psi\rangle^\otimes r$, $f_1f_2 \cdots f_r$ is an LUIP for any LUIPs for the original system $f_1f_2 \cdots f_k$ of degree $l_1,l_2, \ldots , l_k$ with $\sum l_k$ divisible by $2r$. More precisely, we can conclude $\prod_{i=1}^k f_i(|\Psi\rangle\langle\Psi|) = \prod_{i=1}^k f_i(|\Phi\rangle\langle\Phi|)$. Then according to the LU equivalence of $|\Psi\rangle^\otimes r$ and $|\Phi\rangle^\otimes r$, we know the following equation is valid for any LUIP $f'$ of degree $2l$ with $i+l$ divisible by $r$,

$$f'_0(|\Psi\rangle\langle\Psi|) f'(|\Psi\rangle\langle\Psi|) = f'_0(|\Phi\rangle\langle\Phi|) f'(|\Phi\rangle\langle\Phi|)$$

Therefore, $f'(|\Psi\rangle\langle\Psi|) = f'(|\Phi\rangle\langle\Phi|)$ is valid for any LUIP $g$. By invoking the result that LUIPs are sufficient to separate any two distinct orbits under local unitary, we can conclude that $|\Psi\rangle, |\Phi\rangle$ are LU equivalent.

It is direct to see that similar statement is true for mixed states, and for quantum channels by recalling our previous arguments.

To see the power of our method on constructing LUIPs, we apply it on studying the SLOCC equivalence, where the pure state case was recently solved by Gour and WALLACH using Schur-Weyl duality [12].

We apply Derksen’s bound to the following action related to the SLOCC equivalence of pure state and obtain an explicit upper bound for the degree of some generating set of SLUIPs.

Let $G = GL(d_1, \mathbb{C}) \times \cdots \times GL(d_n, \mathbb{C})$ acts on $H$ in the natural way. Let $R$ be a polynomial ring over $\mathbb{C}$ in $\prod d_i$ variables, identified as the coordinate ring of $H$, and $\mathbb{R}_G^n$ denote the invariant ring.

Note that $G$ is the zero locus of $\det(z_{i,j}^{(k)})_{i,j \in [d_k]}$ for $k \leq n$, we see that $t = \sum d_i^2$, $H = \max\{d_i | i \leq n\}$, and $d := \dim(G) = t - n$, $A = n$, and get

$$\sigma(H, G) \leq \max(d_i)^n \cdot n \sum d_i^2 - n.$$

As $s := \dim(R_G^n) \leq \prod d_i$, we get

$$\beta(H, G) \leq \frac{3}{8} \sum d_i \max(d_i)^2 n \cdot n \sum d_i^2 - 2n.$$

In [12], an algorithm of construct the SLUIPs for fixed degree is given, here we provide an alternative algorithm.

According to the argument of the local unitary case, we only need to compute the invariant polynomials of group $SLOCC$, with $SLOCC = I_1 \otimes \cdots \otimes I_{n-1} \otimes SL(d_1) \otimes I_{n+1} \otimes \cdots \otimes I_n$ and $SL(d_i)$ standing for $d_i \times d_i$ matrix with determinant $1$. Therefore, we can regard the multipartite state as bipartite pure state which is just a matrix, says $X$, and the action of the group is just the left matrix multiplication, $i.e., \ X \rightarrow LX$ with $\det(L) = 1$. Fortunately, the invariant polynomials of such map are fully characterized by the determinant of all square matrix with columns consisting of $X$. Therefore, these argument simple implies an algorithm whose output is the SLUIPs for the multipartite system $H$.

Using similar proof technique as that of Theorem 2, we can know that the statement is also true for stable pure states under SLOCC by employing the result from [12] that the orbits of stable states can be distinct by SLUIPs.

**Discussion**— In order to study the equivalence relation of mixed states under the acting group $SL = SL(d_1) \otimes \cdots \otimes SL(d_n)$, i.e., two mixed states $\rho$ and $\sigma$ are said equivalent under rank one SLOCC if there is some $\rho = A_1 \otimes \cdots \otimes A_n$ such that $\rho$ is proportional to $A_1 A_1^t$, $\{A_j\}_{j=1}^n$ are invertible $m_j \times m_j$ matrices. One can derive the following observation,

**Proposition 1.** $\rho$ and $\sigma$ are equivalent under rank one SLOCC if and only if there is some $s \in SLU$ such that $|\Psi\rangle$ is proportional to $|s\rangle$, where $|\Psi\rangle, |\Phi\rangle \in H$ are some purification of $\rho, \sigma$ with $d_{n+1}$ being the dimension of $H_{n+1}$, and

$$SLU \equiv SL(d_1) \otimes SL(d_2) \otimes \cdots \otimes SL(d_n) \otimes U(d_{n+1}).$$

It is showed that SLUIPs can be used to determine whether two stable states in $H$ belong to the same SLOCC class, where a state $|\Psi\rangle \in H$ is said to be stable if its orbit $S|\Psi\rangle$ is closed [13]. Proposition 1 motivates us to generalize the definition of stable pure state [13] to mixed state case. The following definition coincides with that of [13] for pure states.

**Definition 1.** Mixed state $\rho$ of system $H$ with some purification $|\Psi\rangle \in H \otimes H_{n+1}$ is said to be stable under SLOCC if its orbit closed, where the orbit of $\rho$ under $SLU$ is defined as $\{A \otimes u|\Psi\rangle, A \in S, u \in U(d_{n+1})\}$ with $d_{n+1}$ being the dimension of $H_{n+1}$.

In order to study the SLOCC equivalence of mixed state, we only need to classify the distinct orbits of pure states under group $SLU$. Similarly, we employ the concept of invariant polynomial under group $SLU$. 

An $SLU$-invariant polynomial ($SLUIP$) is a polynomial $f : \mathcal{H}_n \rightarrow \mathbb{C}$ that satisfies
\[
f(s\ket{\Psi}\bra{\Psi}s^+) = f(\ket{\Psi}\bra{\Psi}) \quad \forall \, s \in \mathbb{S} \text{ and } \forall \ket{\Psi} \in \mathcal{H} \otimes \mathcal{H}_{n+1},
\]
Our previous method can be used to construct the whole set of SLUIPs. Now, it is desirable to know that whether $SLU$-invariant polynomial is finitely generated. Moreover, we have the following conjecture,

**Conjecture 1.** There is a finite set of generators of invariant polynomial under group $SLU$, where each generator has degree less than $M(d_1, d_2, \cdots, d_{n+1}).$

If the above conjecture is valid, we are able to generalize the result of [13], we show that the invariant polynomial of group $SLU$ can be used to determine whether two stable mixed states in $\mathcal{H}$ belong to the same orbit. More precisely, Let $\ket{\Psi}, \ket{\Phi} \in \mathcal{H} \otimes \mathcal{H}_{n+1}$ be two stable states (i.e. states $\ket{\Psi}$ whose orbits $SLU(\ket{\Psi})$ are closed). Then, they share the same orbit if and only if for all homogeneous invariant polynomial of group $SLU$ of degree $k$, $f_k, g_k$, with $h_k(\ket{\Psi}) \neq 0$,
\[
f_k(\ket{\Psi}\bra{\Psi})/h_k(\ket{\Psi}\bra{\Psi}) = f_k(\ket{\Phi}\bra{\Phi})/h_k(\ket{\Phi}\bra{\Phi}).
\]

**Conclusion—**In this paper, we give a characterization of multipartite entanglement by exploiting a systematic method to describe the ring of all LUIPs. More precisely, we then provide an algorithm to construct a set of generators of the ring of LUIPs. By using the concept of LUIPs, we are able to show that multipartite entanglement is additive in the sense that two multipartite states are LU equivalent if and only if $n$-copies of these two states are LU equivalent for some $n$. Interestingly, this idea can be used to study the multipartite mixed entanglement in terms of equivalence classes of states under SLOCC. We construct the whole set of SL-invariant polynomials (SLIPs) for the pure state case. Then the problem of the equivalence between pure states under the SLU group is studied, and the set of invariant polynomials are demonstrated.

There are still several unsolved problems left in our paper. First, it is interesting to understand the computational complexity of studying the LU (or SLOCC) equivalence of given quantum states, lower bound and upper bound, especially, does polynomial degree bound of invariant polynomials exist, for the group LU (or SLOCC)? The second problem is to provide an explicit upper bound of the group of SLU.

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sion of $W$ is: The maximal length $d$ of the chains $V_0 \subset V_1 \subset \ldots \subset V_d$ of distinct nonempty subvarieties.