On the Involution Generators of the Mapping Class Group of a Punctured Surface

Tülin Altunöz, Mehmetcik Pamuk and Oğuz Yildiz

Abstract. Let $\text{Mod}(\Sigma_{g,p})$ denote the mapping class group of a connected orientable surface of genus $g$ with $p$ punctures. For $g \geq 14$ and even $p \geq 10$, we prove that $\text{Mod}(\Sigma_{g,p})$ can be generated by three involutions. For $g \geq 13$ and even $p \geq 9$, we prove that $\text{Mod}(\Sigma_{g,p})$ can be generated by four involutions. Moreover, for even $p \geq 4$ and $3 \leq g \leq 6$, $\text{Mod}(\Sigma_{g,p})$ can be generated by four involutions.

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1. Introduction

Let $\Sigma_{g,p}$ denote a connected orientable surface of genus $g$ with $p$ punctures; when $p = 0$ we write $\Sigma_g$. The mapping class group of $\Sigma_{g,p}$ is the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,p}$ preserving the set of punctures.

Here is a brief history of generating sets for $\text{Mod}(\Sigma_{g,p})$: Dehn [4] showed that $\text{Mod}(\Sigma_g)$ can be generated by $2g(g-1)$ Dehn twists. Later, Lickorish [14] gave a generating set consisting of $3g-1$ Dehn twists and Humphries [8] reduced the number of Dehn twist generators to $2g+1$. Humphries also proved that the number $2g+1$ is minimal for $g \geq 2$. Johnson [10] proved that the same set of Dehn twists also generates $\text{Mod}(\Sigma_{g,1})$. In the presence of multiple punctures, Gervais [7] proved that $\text{Mod}(\Sigma_{g,p})$ can be generated by $2g+p$ Dehn twists for $p \geq 1$.

If it is not required that the generators are Dehn twists, then it is possible to obtain smaller generating sets for $\text{Mod}(\Sigma_{g,p})$: for $g \geq 3$ and $p = 0$, Lu [15, Theorem 1.3] proved that $\text{Mod}(\Sigma_g)$ can be generated by three elements. For $g \geq 1$ and $p = 0$ or 1, a minimal (since the group is not abelian) generating set of two elements, a product of two Dehn twists and
a product of $2g$ Dehn twists, was first given by Wajnryb [21]. Korkmaz [12, Theorem 5] improved this result by showing that one of these two generators can be taken as a Dehn twist. He also showed that this group is generated by two elements of finite order [12, Theorem 14].

Generating the mapping class group by involutions, which is the main theme of this paper, was first considered by McCarthy and Papadopoulos [17]. They showed that $\text{Mod}(\Sigma_g)$ can be generated by infinitely many conjugates of a single involution for $g \geq 3$. In terms of generating by finitely many involutions, Luo [16] showed that any Dehn twist about a nonseparating simple closed curve can be written as a product of six involutions, which in turn implies that $\text{Mod}(\Sigma_g)$ can be generated by $12g + 6$ involutions. Brendle and Farb [3] obtained a generating set of six involutions for $g \geq 3$. Following their work, Kassabov [11] showed that $\text{Mod}(\Sigma_g)$ can be generated by four involutions if $g \geq 7$. Korkmaz [13] showed that $\text{Mod}(\Sigma_g)$ is generated by three involutions if $g \geq 8$ and four involutions if $g \geq 3$. Also, the third author improved his result showing that it is generated by three involutions if $g \geq 6$ [22]. For surfaces with multiple punctures, if $g \geq 3$, Kassabov obtained a generating set of involution elements where the number of generators depends on $g$ and the parity of $p$ (see [11, Theorem 1]). Later, Monden [18] removed the parity condition on $p$ for $g = 7$ and $g = 5$. For $g \geq 1$ and $p \geq 2$, Monden [19] also gave a generating set for $\text{Mod}(\Sigma_{g,p})$ consisting of three elements. Recently, he [20] gave a minimal generating set for $\text{Mod}(\Sigma_{g,p})$ containing two elements for $g \geq 3$.

Note that any infinite group generated by two involutions must be isomorphic to the infinite dihedral group whose subgroups are either cyclic or dihedral. Since $\text{Mod}(\Sigma_{g,p})$ contains nonabelian free groups, it cannot be generated by two involutions. In this paper, we obtain the following result:

**Theorem A.** For every even integer $p \geq 10$ and $g \geq 14$, $\text{Mod}(\Sigma_{g,p})$ can be generated by three involutions. Moreover, for every even integer $p \geq 4$ and for $g = 3, 4, 5$ or 6, $\text{Mod}(\Sigma_{g,p})$ can be generated by four involutions.

At the end of the paper, we also show that Theorem A also holds for the cases $p = 2$ or $p = 3$. For surfaces with odd number of punctures, we have the following result:

**Theorem B.** For every odd integer $p \geq 9$ and $g \geq 13$, $\text{Mod}(\Sigma_{g,p})$ is generated by four involutions. Moreover, for every odd integer $p \geq 5$ and for $g = 3, 4, 5$ or 6, $\text{Mod}(\Sigma_{g,p})$ can be generated by five involutions.

The paper is organized as follows: in Sect. 2, we quickly provide the necessary background on mapping class groups. In Sect. 3, we first provide subsets of $\text{Mod}(\Sigma_{g,p})$, consisting of involutions, containing generators of $\text{Mod}_0(\Sigma_{g,p})$ (for definition, see Sect. 2). Then, using a well-known result from algebra we present proofs of Theorem A and Theorem B.

### 2. Background and Results on Mapping Class Groups

Let $\Sigma_{g,p}$ denote a connected orientable surface of genus $g$ with $p$ punctures specified by the set $P = \{z_1, z_2, \ldots, z_p\}$ of $p$ distinguished points. If $p$ is
zero then we omit it from the notation and write \( \Sigma_g \). The mapping class group \( \text{Mod}(\Sigma_{g,p}) \) of the surface \( \Sigma_{g,p} \) is defined to be the group of the isotopy classes of orientation-preserving self-diffeomorphisms of \( \Sigma_{g,p} \) which fix the set \( P \). Let \( \text{Mod}_0(\Sigma_{g,p}) \) denote the subgroup of \( \text{Mod}(\Sigma_{g,p}) \) consisting of elements which fix the set \( P \) pointwise. It is obvious that we have the following exact sequence:

\[
1 \longrightarrow \text{Mod}_0(\Sigma_{g,p}) \longrightarrow \text{Mod}(\Sigma_{g,p}) \longrightarrow \text{Sym}_p \longrightarrow 1,
\]

where \( \text{Sym}_p \) denotes the symmetric group on the set \( \{1, 2, \ldots, p\} \) and the last projection is given by the restriction of the isotopy class of a diffeomorphism to its action on the punctures.

Let \( \beta_{i,j} \) be an embedded arc that joins two punctures \( z_i \) and \( z_j \) and does not intersect \( \delta \) on \( \Sigma_{g,p} \). Let \( D_{i,j} \) denote a closed regular neighborhood of \( \beta_{i,j} \), which is a disk with two punctures. There exists a diffeomorphism \( H_{i,j} : D_{i,j} \rightarrow D_{i,j} \), which interchanges the punctures such that \( H_{i,j}^2 \) is equal to the right-handed Dehn twist about \( \partial D_{i,j} \) and is the identity on the complement of the interior of \( D_{i,j} \). Such a diffeomorphism is said to be the (right handed) half twist about \( \beta_{i,j} \). It can be extended to a diffeomorphism of \( \text{Mod}(\Sigma_{g,p}) \). Throughout the paper we do not distinguish a diffeomorphism from its isotopy class. For the composition of two diffeomorphisms, we use the functional notation; if \( f \) and \( g \) are two diffeomorphisms, then the composition \( fg \) means that \( g \) is applied first and then \( f \).

For a simple closed curve \( a \) on \( \Sigma_{g,p} \), following [1, 13], we denote the right-handed Dehn twist \( t_a \) about \( a \) by the corresponding capital letter \( A \). Let us also remind the following basic facts of Dehn twists that we use frequently throughout the paper: let \( a \) and \( b \) be simple closed curves on \( \Sigma_{g,p} \) and \( f \in \text{Mod}(\Sigma_{g,p}) \).

- If \( a \) and \( b \) are disjoint, then \( AB = BA \) (Commutativity).
- If \( f(a) = b \), then \( fAf^{-1} = B \) (Conjugation).

Let us finish this section by noting that we denote \( gf \) by \( f \) for any \( f, g \in \text{Mod}(\Sigma_{g,p}) \).

3. Involution Generators for \( \text{Mod}(\Sigma_{g,p}) \)

Let us start this section by recalling the following set of generators given by Korkmaz [13, Theorem 5].

**Theorem 3.1.** If \( g \geq 3 \), then the mapping class group \( \text{Mod}(\Sigma_g) \) can be generated by the four elements \( R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1} \).

Let us also recall the following basic fact in group theory.

**Lemma 3.2.** Let \( G \) and \( K \) be groups. Suppose that the following short exact sequence holds:

\[
1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} K \longrightarrow 1.
\]

Then a subgroup \( \Gamma \) of \( G \) satisfies \( i(N) \subseteq \Gamma \) and \( \pi(\Gamma) = K \) if and only if \( \Gamma = G \).
In our case where $G = \text{Mod}(\Sigma_{g,p})$ and $N = \text{Mod}_0(\Sigma_{g,p})$, we have the following short exact sequence:

$$1 \longrightarrow \text{Mod}_0(\Sigma_{g,p}) \longrightarrow \text{Mod}(\Sigma_{g,p}) \longrightarrow \text{Sym}_p \longrightarrow 1.$$ 

Therefore, we obtain the following useful result which follows immediately from Lemma 3.2. If $\Gamma$ is a subgroup of $\text{Mod}(\Sigma_{g,p})$ with $\text{Mod}_0(\Sigma_{g,p}) \subseteq \Gamma$ and $\pi(\Gamma) = \text{Sym}_p$, then $\Gamma = \text{Mod}(\Sigma_{g,p})$.

Throughout the paper, we consider the embeddings of $\Sigma_{g,p}$ into $\mathbb{R}^3$ in such a way that it is invariant under the rotations $\rho_1$ and $\rho_2$. Here, $\rho_1$ and $\rho_2$ are the rotations by $\pi$ about the $z$-axis (see Figs. 1, 2). Note that $\text{Mod}(\Sigma_{g,p})$ contains the element $R = \rho_1 \rho_2$ which satisfies the following:

(i) $R(a_i) = a_{i+1}$, $R(b_i) = b_{i+1}$ for $i = 1, \ldots, g - 1$ and $R(b_g) = b_1$,
(ii) $R(c_i) = c_{i+1}$ for $i = 1, \ldots, g - 2$,
(iii) $R(z_1) = z_p$ and $R(z_i) = z_{i-1}$ for $i = 2, \ldots, p$.

We want to note here that, in the following lemmata, where we present generating sets for surfaces with even number of punctures, we mainly follow the proof of [13, Theorem 5]. We use them in the proof of Theorem A and then for surfaces with odd number of punctures we explain how our arguments are modified.

\textbf{Lemma 3.3.} For every even integer $g = 2k \geq 14$ and every even integer $p = 2b \geq 10$, the subgroup of $\text{Mod}(\Sigma_{g,p})$ generated by the elements

$$\rho_1, \rho_2 \text{ and } \rho_1 H_{b-1,b} H_{b+1,b}^{-1} C_{k-3} B_{k-1} A_k A_k^{-1} B_{k+1}^{-1} C_{k+4}^{-1}$$

contains the Dehn twists $A_i$, $B_i$ and $C_i$ for $i = 1, \ldots, g$. 
Figure 2. The involutions $\rho_1$ and $\rho_2$ for $g = 2k + 1$ and $p = 2b$

Proof. Consider the models of $\Sigma_{g,p}$ depicted in Fig. 1. Let $F_1 := H_{b-1,b}H_{b+1,b}^{-1}C_{k-3}B_{k-1}A_{k}A_{k+2}^{-1}B_{k+3}^{-1}C_{k+4}^{-1}$ and let $\Gamma$ be the subgroup of $\text{Mod}(\Sigma_{g,p})$ generated by the elements $\rho_1$, $\rho_2$ and $\rho_1 F_1$. One can see that $R = \rho_1 \rho_2$ and $F_1 = \rho_1 \rho_1 F_2$ are contained in $\Gamma$. Let $F_2$ be the element obtained by the conjugation of $F_1$ by $R^{-3}$. Since

$$R^{-3}(c_{k-3}, b_{k-1}, a_k, a_{k+2}, b_{k+3}, c_{k+4}) = (c_{k-6}, b_{k-4}, a_{k-3}, a_{k-1}, b_k, c_{k+1})$$

and

$$R^{-3}(z_{b-1}, z_b, z_{b+1}) = (z_{b+2}, z_{b+3}, z_{b+4}),$$

we have

$$F_2 = F_1^{R^{-3}} = H_{b+2,b+3}H_{b+4,b+3}^{-1}C_{k-6}B_{k-4}A_{k-3}A_{k-1}B_{k}^{-1}C_{k+1}^{-1} \in \Gamma.$$  

Let $F_3$ be the element $F_1 F_2^{-1}$, that is $F_3 = H_{b-1,b}H_{b+1,b}^{-1}C_{k-3}A_{k-1}B_{k}$ $A_{k+2}^{-1}B_{k+3}^{-1}C_{k+4}^{-1} \in \Gamma$.

Since we use repeatedly similar calculations in the remaining parts of the paper, let us provide some details here. It can be shown that the diffeomorphism $F_1 F_2^{-1}$ maps the curves $c_{k-3}, b_{k-1}, a_k, a_{k+2}, b_{k+3}, c_{k+4}$ to $c_{k-3}, a_{k-1}, b_k, a_{k+2}, b_{k+3}, c_{k+4}$, respectively. Also it follows from the composites of half twists $H_{b-1,b}H_{b+1,b}$ and $H_{b-4,b-3}H_{b-2,b-3}$ commute that we get

$$F_3 = F_1 F_2^{-1} = (F_1 F_2^{-1})(H_{b-1,b}H_{b+1,b}^{-1}C_{k-3}B_{k-1}A_{k}A_{k+2}^{-1}B_{k+3}^{-1}C_{k+4}^{-1})(F_1 F_2^{-1})^{-1}$$

$$= H_{b-1,b}H_{b+1,b}^{-1}C_{k-3}A_{k-1}B_{k}A_{k+2}^{-1}B_{k+3}^{-1}C_{k+4}^{-1}.$$
The subgroup $\Gamma$ contains
\[ F_4 = F_3^{R^{-3}} = H_{b+2,b+3}H_{b+4,b+3}^{-1}C_{k-6}A_{k-4}B_{k-3}A_{k-1}^{-1}B_k^{-1}C_{k+1}^{-1}. \]
One can verify that $F_3F_4$ sends the curves $c_{k-3}, a_{k-1}, b_k, a_{k+2}, b_{k+3}, c_{k+4}$ to the curves $b_{k-3}, a_{k-1}, b_k, a_{k+2}, b_{k+3}, c_{k+4}$, respectively. This implies that
\[ F_5 = F_3^{-1}F_4 = H_{b-1,b}^{-1}B_{k-3}A_{k-1}B_kA_{k+2}^{-1}B_{k+3}^{-1}C_{k+4}^{-1} \in \Gamma. \]
Thus, we obtain $F_5F_3^{-1} = B_{k-3}C_{k-3}^{-1}$, which is contained in $\Gamma$. By conjugating this element with powers of $R$, we conclude that
\[ B_iC_i^{-1} \in \Gamma \text{ for } i = 1, \ldots, g-1. \]
The subgroup $\Gamma$ also contains $F_1F_3^{-1} = B_{k-1}A_kB_{k-1}^{-1}A_{k-1}^{-1}$. After conjugating with $R^3$ and considering the inverse, we have $A_{k+2}B_{k+3}A_{k+3}B_{k+2}^{-1} \in \Gamma$. This in turn implies that for $i = 1, \ldots, g-1$,
\[ A_iB_{i+1}A_{i+1}^{-1}B_i^{-1} \in \Gamma. \]
We also have the following elements in $\Gamma$:
\[ F_6 = F_1(A_{k+2}B_{k+3}A_{k+3}^{-1}B_{k+2}^{-1}) = H_{b-1,b}H_{b+1,b}^{-1}C_{k-3}B_{k-1}A_kB_{k+2}A_{k+3}^{-1}C_{k+4}^{-1} \]
and
\[ F_7 = F_6^{R^{-3}} = H_{b+2,b+3}H_{b+4,b+3}^{-1}C_{k-6}B_{k-4}A_{k-3}B_{k-1}^{-1}A_k^{-1}C_{k+1}^{-1}. \]
It can be checked that $F_6F_7$ maps $c_{k-3}, b_{k-1}, a_k, b_{k+2}, a_{k+3}, c_{k+4}$ to the curves $c_{k-3}, b_{k-1}, a_k, c_{k+1}, a_{k+3}, c_{k+4}$, respectively. Then we get
\[ F_8 = F_6^{-1}F_7 = H_{b-1,b}H_{b+1,b}^{-1}C_{k-3}B_{k-1}A_kC_{k+1}A_{k+1}^{-1}C_{k+4}^{-1} \in \Gamma. \]
Hence, we can conclude that $F_8^{-1}F_6 = C_{k+1}B_{k+2}^{-1} \in \Gamma$. Again conjugating with $R$ implies that
\[ C_iB_{i+1}^{-1} \in \Gamma, \text{ for all } i = 1, \ldots, g-1. \]
Furthermore, we can see that $\Gamma$ contains
\[ F_9 = (B_{k-3}C_{k-3}^{-1}F_3(C_{k+4}B_{k+5}^{-1})) = H_{b-1,b}H_{b+1,b}^{-1}B_kA_{k-1}B_kA_{k+2}^{-1}B_{k+3}^{-1}B_{k+5}^{-1}, \]
and
\[ F_{10} = F_9^{R^{-3}} = H_{b+2,b+3}H_{b+4,b+3}^{-1}B_{k-6}A_{k-3}B_{k-3}A_{k-1}^{-1}B_k^{-1}B_{k+2}^{-1}. \]
It is easy to see that $F_9F_{10}$ sends $b_{k-3}, a_{k-1}, b_k, a_{k+2}, b_{k+3}, b_{k+5}$ to $b_{k-3}, a_{k-1}, b_k, b_{k+2}, b_{k+3}, b_{k+5}$, respectively, so that
\[ F_{11} = F_9^{-1}F_{10} = H_{b-1,b}H_{b+1,b}^{-1}B_kB_{k-3}A_{k-1}B_kB_{k+2}^{-1}B_{k+3}^{-1}B_{k+5}^{-1} \in \Gamma. \]
From this, we obtain $F_{11}^{-1}F_9^{-1} = B_{k+2}^{-1}A_{k+2}^{-1} \in \Gamma$. By the action of $R$, we can conclude that
\[ A_iB_i^{-1} \in \Gamma \text{ for all } i = 1, \ldots, g. \]
The subgroup $\Gamma$ contains
\[ A_1A_2^{-1} = (A_1B_1^{-1})(B_1C_1^{-1})(C_1B_2^{-1})(B_2A_2^{-1}), \]
\[ B_1B_2^{-1} = (B_1C_1^{-1})(C_1B_2^{-1}) \text{ and } \]
\[ C_1C_2^{-1} = (C_1B_2^{-1})(B_2C_2^{-1}). \]
It follows from Theorem 3.1 that $A_i, B_i, C_i$ all belong to the subgroup generated by $R, A_1A_2^{-1}, B_1B_2^{-1}$ and $C_1C_2^{-1}$, which completes the proof.

If $g$ is odd and $p$ is even, we have the following result:

Lemma 3.4. For every odd integer $g = 2k+1 \geq 15$ and even integer $p = 2b \geq 10$, the subgroup of $\text{Mod}(\Sigma_{g,p})$ generated by the elements

$$\rho_1, \rho_2 \text{ and } \rho_1H_{b-1,b}\rho_1H_{b+1,b}^{-1}C_{k-3}B_kA_{k+1}A_{k+2}^{-1}B_{k+3}^{-1}C_{k+5}^{-1}$$

contains the Dehn twists $A_i, B_i$ and $C_i$ for $i = 1, \ldots, g$.

Proof. Consider the models for $\Sigma_{g,p}$ as shown in Fig. 2. Let $\Gamma$ denote the subgroup of $\text{Mod}(\Sigma_{g,p})$ generated by the elements $\rho_1, \rho_2$ and $\rho_1G_1$, where $G_1 = H_{b-1,b}H_{b+1,b}^{-1}C_{k-3}B_kA_{k+1}A_{k+2}^{-1}B_{k+3}^{-1}C_{k+5}^{-1}$. The elements $R = \rho_1\rho_2$ and $G_1 = \rho_1\rho_1G_1$ belong to $\Gamma$. Let $G_2$ denote the conjugation of $G_1$ by $R^{-3}$,

$$G_2 = G_1R^{-3} = H_{b+2,b+3}H_{b+4,b+3}^{-1}C_{k-6}B_kA_{k-2}A_{k-1}^{-1}B_{k-1}^{-1}C_{k+2}^{-1}.$$ 

It is easy to verify that

$$G_3 = G_1G_2^{-1} = H_{b-1,b}H_{b+1,b}^{-1}B_kA_{k+1}A_{k+2}^{-1}C_{k+5}^{-1}$$

is contained in $\Gamma$. Let

$$G_4 = G_3R^{-3} = H_{b+2,b+3}H_{b+4,b+3}^{-1}B_kA_{k-2}A_{k-1}^{-1}C_{k-1}^{-1}C_{k+2}^{-1}.$$ 

Thus, we get

$$G_5 = G_3G_4^{-1} = H_{b-1,b}H_{b+1,b}^{-1}B_kC_{k-1}A_{k+1}A_{k+2}^{-1}C_{k+5}^{-1},$$

which is contained in $\Gamma$. This implies that $G_3G_5^{-1} = B_kC_{k-1}^{-1} \in \Gamma$. By conjugating $B_kC_{k-1}^{-1}$ with powers of $R$, we see that

$$B_{i+1}C_{i-1}^{-1} \in \Gamma,$$

for all $i = 1, \ldots, g-1$. In particular, $C_{k+5}B_{k+6}^{-1} \in \Gamma$. Hence, $\Gamma$ contains

$$G_6 = G_1(C_{k+5}B_{k+6}^{-1}) = H_{b-1,b}H_{b+1,b}^{-1}C_{k-3}B_kA_{k+1}A_{k+2}^{-1}B_{k+3}^{-1}B_{k+6}^{-1}.$$ 

Then, we see that

$$G_7 = G_6R^{-3} = H_{b+2,b+3}H_{b+4,b+3}^{-1}C_{k-6}B_kA_{k-2}A_{k-1}^{-1}B_{k+3}^{-1}B_{k+6}^{-1}$$

and

$$G_8 = G_6G_7 = H_{b-1,b}H_{b+1,b}^{-1}B_kA_{k+1}A_{k+2}^{-1}B_{k+3}^{-1}B_{k+6}^{-1}$$

are contained in $\Gamma$, which implies that $\Gamma$ contains $G_6G_8^{-1} = C_{k-3}B_{k-3}^{-1}$. By the action of $R$, we see that

$$C_iB_{i-1}^{-1} \in \Gamma$$

for all $i = 1, \ldots, g-1$. Moreover, we get

$$G_9 = (B_{k-2}C_{k-3}^{-1})G_6 = H_{b-1,b}H_{b+1,b}^{-1}B_kA_{k+1}A_{k+2}^{-1}B_{k+3}^{-1}B_{k+6}^{-1} \in \Gamma,$$

$$G_{10} = G_9R^{-3} = H_{b+2,b+3}H_{b+4,b+3}^{-1}B_{k-5}B_kA_{k-2}A_{k-1}^{-1}B_{k-1}^{-1}B_{k+3}^{-1} \in \Gamma$$

and

$$G_{11} = G_9G_{10} = H_{b-1,b}H_{b+1,b}^{-1}A_{k-2}B_kA_{k+1}A_{k+2}^{-1}B_{k+3}^{-1}B_{k+6}^{-1} \in \Gamma.$$
From these, we have $G_9 G_{11}^{-1} = B_{k-2} A_{k-2}^{-1} \in \Gamma$ so that

$$B_i A_i^{-1} \in \Gamma,$$

for $i = 1, \ldots, g$, by the action of $R$. The remaining part of the proof can be completed as in the proof of Lemma 3.3.

In the following four lemmata, we give generating sets for smaller genera.

**Lemma 3.5.** For $g = 6$ and every even integer $p \geq 4$, the group generated by the elements

$$\rho_1, \rho_2, \rho_2 B_2 A_3 A_4^{-1} B_5^{-1} \text{ and } \rho_1 H_{b-1, b} H_{b+1, b}^{-1} C_3 C_4^{-1}$$

contains the Dehn twists $A_i, B_i$ and $C_i$ for $i = 1, \ldots, g$.

**Proof.** Consider the models for $\Sigma_{g,p}$ as shown in Fig. 1. Let $\Gamma$ be the subgroup of $\text{Mod}(\Sigma_{g,p})$ generated by the elements $\rho_1, \rho_2, \rho_2 F_1$ and $\rho_1 E_1$ where $F_1 = B_2 A_3 A_4^{-1} B_5^{-1}$ and $E_1 = H_{b-1, b} H_{b+1, b}^{-1} C_3 C_4^{-1}$. Then the elements $R = \rho_1 \rho_2, F_1 = \rho_2 \rho_2 F_1$ and $E_1 = \rho_1 \rho_1 E_1$ are contained in $\Gamma$.

The subgroup $\Gamma$ contains the following elements:

$$F_2 = F_1^R = B_3 A_4 A_5^{-1} B_6^{-1},$$
$$F_3 = F_1 F_2 = B_2 B_3 A_4^{-1} A_5^{-1},$$
$$F_4 = F_3^R = B_3 B_4 A_5^{-1} A_6^{-1} \text{ and}$$
$$F_5 = F_3 F_4^{-1} = B_2 B_3 B_4^{-1} A_5^{-1}.$$

Hence, we get $F_5^{-1} F_3 = B_4 A_4^{-1} \in \Gamma$. By the action of $R$, for all $i = 1, \ldots, 6$, $A_i B_i^{-1} \in \Gamma$. Moreover, we have

$$F_6 = E_1 E_1 F_3 = H_{b-1, b} H_{b+1, b}^{-1} B_3 C_4^{-1} \in \Gamma \text{ and}$$
$$F_7 = E_1 E_1 F_1 = H_{b-1, b} H_{b+1, b}^{-1} C_3 B_5^{-1} \in \Gamma.$$

This implies that $F_6 E_1^{-1} = B_3 C_4^{-1} \in \Gamma$ and $F_7^{-1} E_1 = B_5 C_4^{-1} \in \Gamma$ and so $B_i C_i^{-1} \in \Gamma$ and $B_i B_{i+1} C_i^{-1} \in \Gamma$, for all $i = 1, \ldots, 5$, by conjugating these elements with powers of $R$. The proof can be completed as in the proof of Lemma 3.3.

**Lemma 3.6.** For $g = 5$ and every even integer $p \geq 4$, the group generated by the elements

$$\rho_1, \rho_2, \rho_1 H_{b-1, b} H_{b+1, b}^{-1} A_3 A_4^{-1} \text{ and } \rho_2 A_2 B_2 C_2 C_3^{-1} B_4^{-1} A_4^{-1}$$

contains the Dehn twists $A_i, B_i$ and $C_i$ for $i = 1, \ldots, g$.

**Proof.** Consider the models for $\Sigma_{5,p}$ as shown in Fig. 2. Let $\Gamma$ denote the subgroup of $\text{Mod}(\Sigma_{5,p})$ generated by the elements $\rho_1, \rho_2, \rho_1 F_1$ and $\rho_2 E_1$ where $F_1 = H_{b-1, b} H_{b+1, b}^{-1} A_3 A_4^{-1}$ and $E_1 = A_2 B_2 C_2 C_3^{-1} B_4^{-1} A_4^{-1}$. Thus, $R = \rho_1 \rho_2, F_1 = \rho_1 \rho_1 F_1$ and $E_1 = \rho_2 \rho_2 E_1$ are in $\Gamma$.

One can obtain the following elements:

$$F_2 = F_1^{-1} = H_{b, b+1} H_{b+2, b+1}^{-1} A_2 A_3^{-1}.$$
\[ F_3 = F_2^E_1 = H_{b,b+1}H_{b+2,b+1}^{-1}B_2A_3^{-1} \] and
\[ F_4 = F_3^E_1 = H_{b,b+1}H_{b+2,b+1}^{-1}C_2A_3^{-1}, \]
which are contained in \( \Gamma \). Thus, we get that \( F_2F_3^{-1} = A_2B_2^{-1} \in \Gamma \) and \( F_3F_4^{-1} = B_2C_2^{-1} \in \Gamma \). By conjugating these elements with powers of \( R \), we see that
\[ A_iB_i^{-1} \in \Gamma \text{ and } B_jC_j^{-1} \in \Gamma, \]
which also implies that \( A_iC_i^{-1} \in \Gamma \) for all \( i = 1, \ldots, 5 \) and \( j = 1, \ldots, 4 \).

Finally, it can be verified that
\[ E_1(a_3, c_3) = (a_3, b_4) \]
so that the group \( \Gamma \) contains
\[ (A_3C_3^{-1})E_1 = A_3B_4^{-1}. \]

Hence, \( A_iB_i^{-1} \in \Gamma \) for all \( i = 1, \ldots, 5 \) by the action of \( R \). The rest of the proof is similar to that of Lemma 3.3.

Lemma 3.7. For \( g = 4 \) and every even integer \( p \geq 4 \), the group generated by the elements
\[ \rho_1, \rho_2, \rho_2B_1A_2A_3^{-1}B_4^{-1} \text{ and } \rho_1H_{b-1,b}H_{b+1,b}^{-1}C_2C_3^{-1} \]
contains the Dehn twists \( A_i, B_i \text{ and } C_i \) for \( i = 1, \ldots, g \).

Proof. Let us consider the models for \( \Sigma_{4,p} \) as shown in Fig. 1 and let \( \Gamma \) be the subgroup of \( \text{Mod}(\Sigma_{4,p}) \) generated by the elements \( \rho_1, \rho_2, \rho_2F_1 \) and \( \rho_1E_1 \) where \( F_1 = B_1A_2A_3^{-1}B_4^{-1} \) and \( E_1 = H_{b-1,b}H_{b+1,b}^{-1}C_2C_3^{-1} \). Thus, it is clear that \( R = \rho_1\rho_2, F_1 = \rho_2\rho_2F_1 \) and \( E_1 = \rho_1\rho_1E_1 \) belong to the subgroup \( \Gamma \). We have
\[ F_2 = E_1^E_1F_1 = H_{b-1,b}H_{b+1,b}^{-1}C_2B_4^{-1} \in \Gamma. \]

Thus, \( \Gamma \) contains \( F_2^{-1}E_1 = B_4C_3^{-1} \) and \( \rho_1(B_4C_3^{-1}) = B_2C_2^{-1} \). By conjugating these elements with powers of \( R \), we get
\[ B_{i+1}C_i^{-1} \in \Gamma \text{ and } B_iC_i^{-1} \in \Gamma \]
for all \( i = 1, 2, 3 \). One can also obtain that \( \Gamma \) contains the following elements:
\[ F_3 = (C_1B_1^{-1})F_1 = C_1A_2A_3^{-1}B_4^{-1}, \]
\[ F_4 = F_3^R(B_1C_1^{-1}) = C_2A_3A_4^{-1}B_1^{-1}(B_1C_1^{-1}) = C_2A_3A_4^{-1}C_1^{-1} \text{ and } \]
\[ F_5 = F_3^F_3F_4 = C_1A_2A_3^{-1}A_4^{-1}. \]

Thus, we obtain that \( F_5F_3^{-1} = A_4B_4^{-1} \in \Gamma \). By the action of \( R \), \( A_iB_i^{-1} \in \Gamma \) for all \( i = 1, 2, 3, 4 \). The remaining part of the proof is very similar to that of Lemma 3.3.

Lemma 3.8. For \( g = 3 \) and every even \( p \geq 4 \), the group generated by the elements
\[ \rho_1, \rho_2, \rho_1H_{b-1,b}H_{b+1,b}^{-1}A_2A_3^{-1} \text{ and } \rho_2A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1} \]
contains the Dehn twists \( A_i, B_i \text{ and } C_i \) for \( i = 1, 2, 3 \).
Proof. Consider the models for $\Sigma_{3,p}$ as shown in Fig. 2. Let $\Gamma$ be the subgroup of $\text{Mod}(\Sigma_{3,p})$ generated by the elements $\rho_1$, $\rho_2$, $\rho_1F_1$ and $\rho_2E_1$ where $F_1 = H_{b-1,b}^{-1}H_{b+1,b}^{-1}A_2A_3^{-1}$ and $E_1 = A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}$. Thus, the elements $R = \rho_1\rho_2$, $F_1 = \rho_1\rho_1F_1$ and $E_1 = \rho_2\rho_2E_1$ are contained in $\Gamma$. We get

$$F_2 = F_1^{R^{-1}} = H_{b,b}^{-1}H_{b+1,b}^{-1}A_1A_2^{-1} \in \Gamma,$$

$$F_3 = F_2^{E_1i} = H_{b,b}^{-1}H_{b+1,b}^{-1}B_1A_2^{-1} \in \Gamma$$

and

$$F_4 = F_3^{E_1i} = H_{b,b}^{-1}H_{b+1,b}^{-1}C_1A_2^{-1} \in \Gamma.$$ 

From these, $\Gamma$ contains $F_2F_3^{-1} = A_1B_1^{-1}$ and $F_3F_4^{-1} = B_1C_1^{-1}$, which implies that $A_1C_1^{-1} \in \Gamma$. Hence,

$$A_iB_i^{-1} \in \Gamma, B_jC_j^{-1} \in \Gamma$$

for all $i = 1, 2, 3$ and $j = 1, 2$, by the action of $R$. We also have

$$(A_2C_2^{-1})E_1 = A_2B_3^{-1},$$

which is contained in $\Gamma$. This implies that

$$A_iB_{i+1}^{-1} \in \Gamma$$

for $i = 1, 2$ by the action of $R$. One can complete the proof as in the proof of Lemma 3.3. \qed

**Remark 3.9.** To see that our results in lemmata 3.3–3.8 work for surfaces with odd number of punctures, see Figures 3 and 5 in [2].

Now, in the remainder of the paper, let $\Gamma$ be the subgroup of $\text{Mod}(\Sigma_{g,p})$ generated by the elements given explicitly in lemmata 3.3–3.8 with the conditions mentioned in these lemmata. The proof of the following lemma is similar to that of [2, Lemma 4.6]; nevertheless, we give a proof for the sake of completeness of the paper.

**Lemma 3.10.** The group $\text{Mod}_0(\Sigma_{g,p})$ is contained in the group $\Gamma$.

Proof. The group $\text{Mod}_0(\Sigma_{g,p})$ is generated by the Dehn twists $A_i$, $B_i$, $C_j$ for $i = 1, \ldots, g$ and $j = 1, \ldots, g-1$ and also $E_{k,l}$ for some fixed $k$ and $l = 1, 2, \ldots, p-1$. It follows from $\Gamma$ contains $A_i$, $B_i$ and $C_j$ for all $i = 1, \ldots, g$ and $j = 1, \ldots, g-1$ by lemmata 3.3–3.8 that it is sufficient to prove that $\Gamma$ contains the Dehn twists $E_{i,j}$ for some fixed $i$ ($j = 1, 2, \ldots, p-1$). Let us first note that $\Gamma$ contains $A_g$ and $R = \rho_1\rho_2$. Consider the models for $\Sigma_{g,p}$ as shown in Figs. 1 and 2. By the fact that the diffeomorphism $R$ maps $a_g$ to $e_{1,p-1}$, we get

$$RA_gR^{-1} = E_{1,p-1} \in \Gamma.$$ 

The diffeomorphism $\phi_i = B_{i+1}^{-1}C_iB_i$ which maps each $e_{i,j}$ to $e_{i+1,j}$ for $j = 1, 2, \ldots, p-1$ (see Fig. 3). By the proof of [2, Lemma 4.5], the group $\Gamma$ contains $\phi_{g}$. Thus, we have

$$\phi_{g-1} \ldots \phi_2 \phi_1 E_{1,p-1} \phi_2 \phi_1^{-1} = E_{g,p-1} \in H.$$ 

Likewise, the diffeomorphism $R$ sends $e_{g,p-1}$ to $e_{1,p-2}$. Then we obtain

$$RE_{g,p-1}R^{-1} = E_{1,p-2} \in \Gamma.$$
Figure 3. The curves $e_{i,j}$ and $\gamma_i$ on the surface $\Sigma_{g,p}$

It follows from

$$\phi_{g-1} \ldots \phi_2 \phi_1 E_{1,p-2}(\phi_{g-1} \ldots \phi_2 \phi_1)^{-1} = E_{g,p-2} \in \Gamma$$

that

$$R(E_{g,p-2})R^{-1} = E_{1,p-3} \in \Gamma$$

Continuing in this way, we conclude that $E_{1,1}, E_{1,2}, \ldots, E_{1,p-1}$ belong to $\Gamma$, which completes the proof. $\square$

Proof of Theorem A. Consider the surface $\Sigma_{g,p}$ as in Figs. 1 and 2.

For $g = 2k \geq 14$ and $p = 2b \geq 10$: In this case, consider the surface $\Sigma_{g,p}$ as in Fig. 1. Since

$$\rho_1(c_{k-3}) = c_{k+4}, \rho_1(b_{k-1}) = b_{k+3} \text{ and } \rho_1(a_k) = a_{k+2},$$

we get

$$\rho_1 C_{k-3} \rho_1 = C_{k+4}, \rho_1 B_{k-1} \rho_1 = B_{k+3} \text{ and } \rho_1 A_k \rho_1 = A_{k+2}.$$ 

Also, since $\rho_1 H_{b-1,b} \rho_1 = H_{b+1, b}$, it is easy to see that $\rho_1 H_{b-1,b} H_{b+1, b}^{-1} C_{k-3}$ $B_{k-1} A_k A_{k+2} B_{k+3}^{-1} C_{k+4}^{-1}$ is an involution. Therefore, the generators of $\Gamma$ given in Lemma 3.3 are involutions.

For $g = 2k + 1 \geq 13$ and $p = 2b \geq 10$: In this case, consider the surface $\Sigma_{g,p}$ as in Fig. 2. It follows from

$$\rho_1(c_{k-3}) = c_{k+5}, \rho_1(b_k) = b_{k+3} \text{ and } \rho_1(a_{k+1}) = a_{k+2},$$

we get

$$\rho_1 C_{k-3} \rho_1 = C_{k+5}, \rho_1 B_{k-1} \rho_1 = B_{k+3} \text{ and } \rho_1 A_k \rho_1 = A_{k+2}.$$ 

Also, since $\rho_1 H_{b-1,b} \rho_1 = H_{b+1, b}$, it is easy to see that $\rho_1 H_{b-1,b} H_{b+1, b}^{-1} C_{k-3}$ $B_{k-1} A_k A_{k+2} B_{k+3}^{-1} C_{k+4}^{-1}$ is an involution. Therefore, the generators of $\Gamma$ given in Lemma 3.3 are involutions.
that we have

\[ \rho_1 C_{k-3} \rho_1 = C_{k+5}, \rho_1 B_k \rho_1 = B_{k+3} \text{ and } \rho_1 A_{k+1} \rho_1 = A_{k+2}. \]

Also, by the fact that \( \rho_1 H_{b-1,b} \rho_1 = H_{b+1,b} \), it is easy to see that \( \rho_1 H_{b-1,b} H_{b+1,b}^{-1} C_{k-3}^{-1} A_k A_{k+1}^{-1} B_{k+3}^{-1} C_{k+5}^{-1} \) is an involution. We conclude that the generators of \( \Gamma \) given in Lemma 3.4 are involutions.

For \( g = 3, 4, 5 \) or \( g = 6 \) and \( p = 2b \geq 4 \): it follows from

- \( \rho_2(b_2) = b_5, \rho_2(a_3) = a_4 \) and \( \rho_1(c_3) = c_4 \) if \( g = 6 \),
- \( \rho_1(a_3) = a_4, \rho_2(a_2) = a_4, \rho_2(b_2) = b_4 \) and \( \rho_2(c_2) = c_3 \) if \( g = 5 \),
- \( \rho_2(b_1) = b_4, \rho_2(a_2) = a_3 \) and \( \rho_1(c_2) = c_3 \) if \( g = 4 \),
- \( \rho_1(a_2) = a_3, \rho_2(a_1) = a_3, \rho_2(b_1) = b_3 \) and \( \rho_2(c_1) = c_2 \) if \( g = 3 \) and
- \( \rho_1 H_{b-1,b} \rho_1 = H_{b+1,b} \) if \( g = 3, 4, 5 \) or \( g = 6 \)

that the following elements:

- \( \rho_2 B_2 A_3 A_1^{-1} B_5^{-1} \) and \( \rho_1 H_{b-1,b} H_{b+1,b}^{-1} C_3 C_4^{-1} \) if \( g = 6 \),
- \( \rho_1 H_{b-1,b} H_{b+1,b}^{-1} A_3 A_4^{-1} \) and \( \rho_2 A_2 B_2 C_2 C_3^{-1} B_4^{-1} A_4^{-1} \) if \( g = 5 \),
- \( \rho_2 B_1 A_2 A_3^{-1} B_4^{-1} \) and \( \rho_1 H_{b-1,b} H_{b+1,b}^{-1} C_3^{-1} \) if \( g = 4 \) and
- \( \rho_1 H_{b-1,b} H_{b+1,b}^{-1} A_2 A_3^{-1} \) and \( \rho_2 A_1 B_1 C_1 C_2^{-1} B_3^{-1} A_3^{-1} \) if \( g = 3 \)

given in lemmata 3.5–3.8 are involutions.

Next, we show that \( \Gamma \) is equal to the mapping class group \( \text{Mod}(\Sigma_{g,p}) \). By Lemma 3.10, the group \( \text{Mod}_0(\Sigma_{g,p}) \) is contained in the group \( \Gamma \). Hence, by Lemma 3.2, we need to prove that \( \Gamma \) is mapped surjectively onto \( \text{Sym}_p \).

The element \( \rho_1 \rho_2 \in \Gamma \) has the image \((1,2,\ldots,p) \in \text{Sym}_p \).

As proven above, \( A_i, B_i \) and \( C_i \) belong to the subgroup \( \Gamma \). Thus, it can be easily observed that the composite of half twists \( H_{b-1,b} H_{b+1,b}^{-1} \) is contained in \( \Gamma \). Therefore, the group \( \Gamma \) also contains the following element:

\[
R^{b-2} \left( H_{b-1,b} H_{b+1,b}^{-1} \right) R^{2-b} = H_{1,2} H_{3,2}^{-1},
\]

which has the image \((1,2,3) \in \text{Sym}_p \). This completes the proof since \((1,2,\ldots,p)\) and \((1,2,3)\) generate the whole group \( \text{Sym}_p \) if \( p \) is even [9, Theorem B].

When the number of punctures is odd, we introduce an additional involution \( \rho_3 \) (depicted in Fig. 4) to our generating set. The main reason behind adding an extra involution is for generating the symmetric group \( \text{Sym}_p \). We want to point out that aside from generating \( \text{Sym}_p \), all of our proofs in the case of even number of punctures work for odd number of punctures. For \( \rho_1 \) and \( \rho_2 \), we distribute punctures as in Figures 3 and 5 in [2] (see also Remark 3.9).

**Proof of Theorem B.** For the first part of the proof, we show that

(i) For every even integer \( g = 2k \geq 14 \) and every odd integer \( p = 2b+1 \geq 9 \), the subgroup \( \text{Mod}_0(\Sigma_{g,p}) \) of \( \text{Mod}(\Sigma_{g,p}) \) is generated by

\[
\rho_1, \rho_2 \text{ and } \rho_1 H_{b-1,b} H_{b+1,b}^{-1} C_{k-3} B_{k-1} A_k A_{k+1}^{-1} B_{k+3}^{-1} C_{k+4}^{-1}, \rho_3.
\]
(ii) For every odd integer $g = 2k + 1 \geq 15$ and odd integer $p = 2b + 1 \geq 9$, the subgroup Mod$_0(\Sigma_{g,p})$ of Mod$(\Sigma_{g,p})$ is generated by

$$\rho_1, \rho_2 \text{ and } \rho_1 H_{b-1,b} H_{b+1,b}^{-1} C_{k-3} B_k A_{k+1}^{-1} B_{k+2}^{-1} C_{k+5}^{-1}, \rho_3.$$ 

Note that, it is enough to prove that the subgroup generated by the elements above is mapped surjectively onto $Sym_p$. For this, consider the images of $\rho_1$, $\rho_2$ and $\rho_3$

$$(1, p - 1)(2, p - 2) \ldots (b, b + 1),$$

$$(1, p)(2, p - 1) \ldots (b, b + 2),$$

$$(2, p - 1)(3, p - 2) \ldots (b, b + 2).$$

This finishes the proof for the first part, since these elements generate $Sym_p$, see [18, Lemma 6]. For the second part of the theorem, note that adding $\rho_3$ to the corresponding generating set given in Theorem A, finishes the proof.

As a last observation, one can prove that Theorem A also holds for the cases $p = 2$ or $p = 3$. In these cases, the generating set of $\Gamma$ can be chosen as

$$\{\rho_1, \rho_2, \rho_1 C_{k-3} B_{k-1} A_{k-1} B_{k+2}^{-1} C_{k+4}^{-1}\} \text{ if } g = 2k \geq 14,$$

$$\{\rho_1, \rho_2, \rho_1 C_{k-3} B_k A_{k+1}^{-1} B_{k+2}^{-1} C_{k+5}^{-1}\} \text{ if } g = 2k + 1 \geq 13,$$

$$\{\rho_1, \rho_2, \rho_2 B_2 A_3 A_4^{-1} B_5^{-1}, \rho_1 C_3 C_4^{-1}\} \text{ if } g = 6,$$

$$\{\rho_1, \rho_2, \rho_1 A_3 A_4^{-1}, \rho_2 A_2 B_2 C_3 C_4^{-1} B_4^{-1} A_4^{-1}\} \text{ if } g = 5,$$

$$\{\rho_1, \rho_2, \rho_2 B_1 A_2 A_3^{-1} B_4^{-1}, \rho_1 C_2 C_3^{-1}\} \text{ if } g = 4,$$

$$\{\rho_1, \rho_2, \rho_1 A_2 A_3^{-1}, \rho_2 A_1 B_1 C_1 C_2^{-1} B_3^{-1} A_3^{-1}\} \text{ if } g = 3.$$
It can be easily proven that $\Gamma$ contains $\text{Mod}_0(\Sigma_{g,p})$ by the similar arguments in the proofs of lemmata 3.3–3.8. The element $\rho_1\rho_2 \in \Gamma$ has the image $(1,2,\ldots,p) \in \text{Sym}_p$. Hence, this element generates $\text{Sym}_p$ for $p = 2$. If $p = 3$, we distribute the punctures as in [11, Figure 1]. Then $\rho_1$ has image $(1,3)$, which together with $(1,2,3)$ generate $\text{Sym}_p$. Therefore, the group $\Gamma$ is mapped surjectively onto $\text{Sym}_p$ for $p = 2,3$. One can conclude that the group $\Gamma$ is equal to $\text{Mod}(\Sigma_{g,p})$.

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Tülin Altunöz  
Faculty of Engineering  
Başkent University  
Ankara  
Turkey  

e-mail: tulinaltunoz@baskent.edu.tr
Mehmetcik Pamuk and Oğuz Yıldız
Department of Mathematics
Middle East Technical University
Ankara
Turkey
e-mail: mpamuk@metu.edu.tr

Oğuz Yıldız
e-mail: oguzyildiz16@gmail.com

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