A mathematical model for apportionment and reapportionment using F26A graph and the magic labeling

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Abstract
The Allotment of a set of integers to vertices and edges of a graph satisfying certain conditions termed as labelling of a graph. It was introduced in the late 1960s by (5). The concept of magic labelling which is one of the many labellings itself has a variety. By fixing some natural number (here some atomic numbers) to the edges of F26A graph, it is proved to be a magic graph.

Keywords
F26A graph, Magic Labeling, Apportionment, Reapportionment, Atomic Number of Elements with their symbols.

1. Introduction
The applications of concepts of graph theory keep increasing day by day. In particular, the concept of Graph labeling has a noteworthy contribution in such applications. The Authors of this paper were inspired to work in graph labelings, the source being “A Dynamic survey of graph labeling”, the Electronic Journal of combinatorics, 19(2009), 1-219 by J.A.Gallian [1]. It is natural to be attracted towards Magic and the authors decide to use the magic labeling and present an application and hence this paper. By choosing the Graph F26A and the Magic Labeling they have proposed a Mathematical model for Apportionment and Reapportionment. In order to maintain secrecy with respect to the details of the apportionment, Atomic Numbers and symbols of certain elements are used.

2. Pre-requisites
Definition 2.1 (F26A [3]). The F26A graph is a cubic graph with 26 vertices and 39 edges.

Definition 2.2 (Magic Graph [5]). By assigning natural numbers to the edges such that the sum total of the edge values...
Theorem 3.1. The F26A graph admits the magic labeling, where the edge labels are assigned positive integers.

Proof. There are 26 vertices and 39 edges in the F26A. Three edges pass through each \( v_i \).

The magic total, that is the sum of distinct edge labels incident with the vertex \( v_i \) is denoted by \( S(v_i) \) should be the same for \( i = 1, 2, \ldots, 26 \) for the graph to be a magic graph.

Let take the magic sum as \( 200 \). Take the condition, the magic total, that is the sum of distinct edge labels incident with the vertex \( v_i \) is denoted by \( S(v_i) \) should be the same for \( i = 1, 2, \ldots, 26 \) for the graph to be a magic graph.

Adding,

\[
S(v_1) = f(e_1) + f(e_2) + f(e_3) = x_1 + x_2 + x - (x_1 + x_2) = x
\]

By repeating the same procedure, avoiding repetition for edge rules, the graph is found to admit the magic labeling and so becomes a magic graph. \( \square \)

### 3.1 Verification

The edge labels assume any positive integer. To be specific take the magic sum as 200. Take

\[
f(e_i) = (2q + 3) - i \quad \text{for} \ i = 1, 2, \ldots, 8 \quad (3.1)
\]

Therefore,

\[
f(e_1) = 80,
f(e_2) = 79,
\]

\[
\ldots
f(e_8) = 73
\]

As the vertex \( v_9 \) is connected to \( v_2 \) by the edge \( e_2 \) (forming a cycle), The formula 3.1 fails for \( i = 9 \) onwards. \( f(e_9) = 72 \) according to 3.1. According to the condition,

\[
f(e_1) + f(e_2) + f(e_28) = 200
\]

\[
\Rightarrow 80 + 79 + f(e_28) = 200
\]

\[
\Rightarrow f(e_28) = 41
\]

The edges \( e_{28}, e_8, e_9 \) are incident at \( v_9 \) so,

\[
f(e_{28}) + f(e_8) + f(e_9) = 200
\]

\[
\Rightarrow 41 + 73 + f(e_9) = 200
\]

So, \( f(e_9) \neq 72 \) but \( f(e_9) = 86 \). From this stage, the edge labels are to be allotted vertex by vertex.

\[
f(e_{8} + i) = 2q + 7 + i \ (i = 1, 2, e_{10})
\]

\[
f(e_{10} + i) = 2q - 11 + i \ (i = 1, 2)
\]

\[
f(e_{12} + i) = 2q + 3 + i \ (i = 1, 2)
\]

\[
f(e_{15} + i) = 2q - 14 + 2i \ (i = 0, 1)
\]

\[
f(e_{17} + 2i) = 3q - 10 - i \ (i = 0, 1)
\]

\[
f(e_{18} + 2i) = q - 7 + 2i \ (i = 0, 1)
\]

The rest of the edges \( e_{21} \) to \( e_{26} \) cannot be combined even two by two

\[
f(e_{21}) = 132 = 3q + 15
\]

\[
f(e_{22}) = 25 = q - 14
\]

\[
f(e_{23}) = 105 = 3q - 12
\]

\[
f(e_{24}) = 48 = q + 9
\]

\[
f(e_{25}) = 90 = 2q + 12
\]

\[
f(e_{26}) = 59 = 2q - 19
\]

The edges are labelled 27 to 34

\[
f(e_{27}) = 61 = 2q - 17
\]

\[
f(e_{35}) = 27 = q - 11
\]

\[
f(e_{36}) = 63 = 2q - 15
\]

\[
f(e_{37}) = 35 = q - 3
\]

\[
f(e_{38}) = 70 = 2q - 8
\]

\[
f(e_{39}) = 62 = 2q - 16
\]

\[
f(e_{28} + i) = q + 2 + 2i \ (i = 0, 1, \ldots, 6)
\]

\[
f(e_{35}) = 27 = q - 11
\]

\[
f(e_{36}) = 63 = 2q - 15
\]

\[
f(e_{37}) = 35 = q - 3
\]

\[
f(e_{38}) = 70 = 2q - 8
\]

\[
f(e_{39}) = 62 = 2q - 16
\]

Here the vertex sum is checked just for two vertices

1. \( s(v_4) = f(e_3) + f(e_4) + f(e_{30}) = 78 + 77 + 45 = 200 \) and

2. \( s(v_{23}) = 25 + 105 + 70 = 200 \)
So the F26A graph with respect to the suggested Magic labeling is found to satisfy the required conditions. Hence it is a Magic graph.

4. Description of Mathematical Model

Description of Mathematical Model is given below:

A mathematic model using the magic labeling on F26A graph. The great business magnet of South India BMB wants to distribute a very huge sum to 25 people. He contacts the Veeru-Kanchan (VKA) and describes the mode of distribution with certain condition, allowing VKA to have a certain degree of freedom.

1) To divide the huge sum into 200 parts and to apportion it to three persons (named say $P_1, P_2, P_3$) in the ratio 80:61:59

The first two have to reapportion to two different persons while the third need not reapportion to anyone. He enjoys the apportionment from BMB and reapportionment through two persons (of course it is again the fortune from BMB).

The person who receives a portion as reapportionment needs to divide it into 200 parts, keeping with him the same number of parts (which will be specified to him, not the same amount) and reapportion it to different persons. A total of 14 persons have to reapportion to two persons, 10 of them to one person (not the same person) and one to none.

The VKA works at the problem and recognizes it in the form of F26A graph and the magic labeling and submits the solution to BMB.

The above diagram is the solution proposed by VKA (It is not unique; it is one among many, according to freedom given to VKA, they find the solution). The great BMB, scrutinizes and accepts and decides the persons to receive fixing people for the vertices starting V1 with VKA.

According to the request from BMB, the VKA makes the solution secretive by allotting the names of the elements instead of numbers which are atomic numbers of such elements. The diagram is given below.

5. Conclusion

The Authors have made an effort and have come out with a Mathematical model for Apportionment, using a graph and a graph labeling. They believe that this opens an avenue for such and many more problems with different graphs and different labeling.

References

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