ZERO-SUM MULTISETS MOD $p$ WITH AN APPLICATION TO SURFACE AUTOMORPHISMS

ANTHONY WEAVER

Abstract. We solve a problem in enumerative combinatorics which is equivalent to counting topological types of certain group actions on compact Riemann surfaces. Let $V_2(F_p)$ be the two-dimensional vector space over $F_p$, the field with $p$ elements, $p$ an odd prime. We count orbits of the general linear group $GL_2(F_p)$ on certain multisets consisting of $R \geq 3$ non-zero columns from $V_2(F_p)$. The $R$-multisets are 'zero-sum;' that is, the sum (mod $p$) over the columns in the multiset is $\left[\frac{0}{0}\right]$. The orbit count yields the number of topological types of fully ramified actions of the elementary abelian $p$-group of rank 2 on compact Riemann surfaces of genus $1 + Rp(p-1)/2 - p^2$.

1. Introduction

Let $p$ be an odd prime, $F_p$ the field with $p$ elements, and $V_2(F_p)$ the two-dimensional vector space over $F_p$. The one-dimensional subspaces are the points $\infty$, 0, 1, 2, ..., $p-1$ of the projective line $PG(1,F_p)$. A standard set of representatives in $V_2(F_p)$ is given by the correspondence

$$ (1) \quad \infty \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad 0 \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad i \leftrightarrow \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad i = 1, \ldots, p-1. $$

A multiset consisting of $R \geq 3$ non-zero columns of $V_2(F_p)$ is conveniently presented as a $2 \times R$ matrix in block form

$$ (2) \quad M = \begin{bmatrix} S_\infty & S_0 & S_1 & \cdots & S_{p-1} \end{bmatrix}, $$

where $S_i$ consists of $\mu_i \geq 0$ non-zero scalar multiples of the column $i$. The zero-sum condition on $M$ is the requirement that both row sums $\equiv 0$ (mod $p$). The tuple

$$ (3) \quad \mathcal{P} = (\mu_\infty, \mu_0, \mu_1, \ldots, \mu_{p-1}) $$

is the projective marking induced by $M$. Since we also require that $M$ have rank 2, at least two of the $\mu$'s must be non-zero. The general linear group $GL_2(F_p)$ acts on zero-sum multisets by column-wise left-multiplication. Dependence relations are preserved, hence, also, the zero-sum and rank conditions.

The projective marking induced by $M$ is not necessarily preserved. The $R$-partition,

$$ R = \mu_\infty + \mu_0 + \cdots + \mu_{p-1}, $$

however, remains unchanged (though the subscripts may be permuted). Thus we may speak of the partition type $\mathcal{P}$ of the $GL_2(F_p)$-orbit of $M$. ‘Forgetting’ the subscripts, we may write the parts of the partition, with multiplicities, in decreasing order of size,

$$ (4) \quad \mathcal{P} = (a^{[n]}, b^{[m]}, \ldots, z^{[l]}), \quad a > b > \cdots > z \geq 1. $$
It is also convenient to represent $\mathcal{P}$ by its Ferrers diagram, in which the parts are exhibited as rows of the corresponding number of dots. The total multiplicity, $s = n + m + \cdots + t$, which is also the number of rows of the Ferrers diagram, must satisfy the inequality
\[ 2 \leq s \leq p + 1. \]

$s < 2$ is inadmissible by the rank condition on $M$; $s > p + 1$ is inadmissible since each part of the partition is associated with a distinct subspace, of which there are exactly $p + 1$. Additionally, if $s = 2$, $\mathcal{P} = (R - 1, 1)$ is inadmissible since the zero-sum condition cannot be satisfied by a multiset with all but one column belonging to a single subspace.

Invariance of the partition type under the $GL_2(F_p)$-action reduces the problem of counting orbits on zero-sum multisets of size $R$ to the sub-problem of counting orbits on multisets of a given partition type. Let $\mathcal{M}(\mathcal{P})$ be the collection of multisets of partition type $\mathcal{P}$.

Burnside’s Lemma (see, e.g., [10], Theorem 10.5), states that the number of orbits of a finite group acting on a finite set is the average number of points fixed by an element of the group. For $\sigma \in GL_2(F_p)$ and $M \in \mathcal{M}(\mathcal{P})$, let
\[ \text{Fix}_\mathcal{P}(\sigma) = \{ M \in \mathcal{M}(\mathcal{P}) \mid \sigma M = M \}. \]

Then by Burnside’s Lemma the number of orbits of $GL_2(F_p)$ on $\mathcal{M}(\mathcal{P})$ is
\[ \frac{1}{|GL_2(F_p)|} \sum_{\sigma \in GL_2(F_p)} |\text{Fix}_\mathcal{P}(\sigma)|. \]

The summation can be reduced to a sum over a set of representatives of the conjugacy classes in $GL_2(F_p)$ since conjugate elements fix the same number of points. If $\sigma$ is the identity element, $|\text{Fix}_\mathcal{P}(\sigma)| = |\mathcal{M}(\mathcal{P})|$. This alone is a non-trivial computation. See Section 3 and in particular, Table 3.1 which gives $|\mathcal{M}(\mathcal{P})|$ for each admissible partition of $3 \leq R \leq 6$. For non-identity $\sigma$, $|\text{Fix}_\mathcal{P}(\sigma)| > 0$ only if $\mathcal{P}$ satisfies certain constraints involving the particular prime $p$ under investigation, and certain number-theoretical parameters associated with the geometric type of the conjugacy class of $\sigma$. See Section 4 and in particular, Table 4.2 which gives a complete enumeration of the conjugacy classes, by class type, in $GL_2(F_p)$, $p = 3, 5, 7$. A parameter of particular importance is the central indicator, which is the smallest power of $\sigma$ contained in the center of $GL_2(F_p)$.

Constraints imposed on $\mathcal{P}$ are most easily described with reference to the Ferrers diagram. When the block form (2) of $M$ is ‘stacked’ in the form
\[ S_\infty \]
\[ S_0 \]
\[ \vdots \]
\[ S_{p-1} \]

the resulting array of columns, after possible reordering and deleting of empty blocks, has the same form as the Ferrers diagram of $\mathcal{P}$. If $\sigma M = M$, $M$ is a concatenation of $\sigma$-orbits which can be exhibited as rectangular arrays overlaying or ‘tiling’ the Ferrers diagram. Certain rows are distinguished as corresponding to eigenspaces (if $\sigma$ has any). Enumeration of feasible orbit-tilings yields formulae for $|\text{Fix}_\mathcal{P}(\sigma)|$. See Section 5 Theorems 4, 5, and 6.
Theorem 1 in Section 2 is a number-theoretical result, possibly of independent interest, which we use at several crucial points in the paper. In Section 6 we gather together all the tools developed in the previous sections to apply Burnside’s Lemma. In Section 6.1 we give a complete enumeration of \( GL_2(F_p) \)-orbits on zero-sum \( R \)-multisets in the smallest cases \( p = 3, 5, 7, \ R = 3, 4, 5, 6 \).

1.1. Surface automorphisms. We briefly describe the relevance of our enumeration problem to the theory of surface automorphisms. Let \( \mathbb{Z}_p^2 \) be the elementary abelian \( p \)-group of rank 2. As an additive group it is isomorphic to \( V_2(F_p) \) with vector addition mod \( p \). Let the group act by automorphisms on a closed surface \( X \) of genus \( g > 1 \). In a fully ramified action, the quotient surface \( X/\mathbb{Z}_p^2 \) has genus 0, and the quotient map \( X \to X/\mathbb{Z}_p^2 \) branches over \( R \geq 3 \) points. \( R \) is related to \( g \) by the Riemann-Hurwitz relation

\[
\begin{align*}
g &= 1 + \frac{Rp(p-1)}{2} - p^2.
\end{align*}
\]

The existence of such an action is equivalent to the existence of a short exact sequence of group homomorphisms

\[
1 \to \Lambda_g \xrightarrow{i} \Gamma \xrightarrow{\eta} \mathbb{Z}_p^2 \to 1,
\]

where 1 is the trivial group, and \( \Gamma \) is a Fuchsian group with signature \((0; R)\) and presentation

\[
\Gamma = \Gamma(0; R) = \langle \gamma_1, \ldots, \gamma_R \mid \gamma_1^p = \cdots = \gamma_R^p = \prod_{i=1}^{R} \gamma_i = \text{id} \rangle.
\]

\( \Lambda_g \) is a torsion-free (surface) group with signature \((g; -)\) and presentation

\[
\Lambda_g = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \prod_{i=1}^{g} \alpha_i^{-1} \beta_i^{-1} = \text{id} \rangle.
\]

Two short exact sequences determine topologically equivalent actions if there are group automorphisms \( \phi : \Gamma \to \Gamma \) and \( \psi : \mathbb{Z}_p^2 \to \mathbb{Z}_p^2 \) such that the diagram

\[
\begin{array}{ccc}
1 & \to & \Lambda_g \\
\| & & \downarrow \phi \\
1 & \to & \Lambda_g
\end{array}
\begin{array}{ccc}
\Gamma & \xrightarrow{\eta_1} & \mathbb{Z}_p^2 \\
\phi & & \psi \\
\Gamma & \xrightarrow{\eta_2} & \mathbb{Z}_p^2
\end{array}
\to 1
\]

commutes. Each epimorphism \( \eta_i \) specifies a generating vector

\[
(\eta_i(\gamma_1), \ldots, \eta_i(\gamma_R)) \in \mathbb{Z}_p^2 \times \cdots \times \mathbb{Z}_p^2,
\]

which is an ordered \( R \)-tuple of elements generating the group, whose product is the identity (this is a consequence of the final relation in (9)). In an abelian group, of course, the ‘product’ of a set of elements is independent of the order in which they are taken, so a generating vector is just a generating multiset which sums to the identity. A pair

\[
(\phi, \psi) \in \text{Aut}(\Gamma) \times \text{Aut}(\mathbb{Z}_p^2)
\]

acts on generating vectors by sending

\[
(\eta(\gamma_1), \ldots, \eta(\gamma_R)) \mapsto (\psi \circ \eta \circ \phi^{-1}(\gamma_1), \ldots, \psi \circ \eta \circ \phi^{-1}(\gamma_R)).
\]
Aut(\(\Gamma\)) merely permutes the generators \(\gamma_1, \ldots, \gamma_R \in \Gamma\) in all possible ways (see [2] Prop. 4.2 and Remark 4.2; or [3], Sec. 2.3). Hence the effective action on (unordered) generating sets is by Aut(\(\mathbb{Z}_p^2\)) = GL_2(\(F_p\)) alone. The topological types of the \(\mathbb{Z}_p^2\)-action are thus in bijection with GL_2(\(F_p\))-orbits on zero-sum \(R\)-multisets of columns in \(V_2(F_p)\). Table 6.1 summarizes the results of Section 6.1 from this point of view.

The notion of topological equivalence of group actions goes back at least to Nielsen [9], who studied the problem for cyclic groups of prime order. This early work was extended by Harvey [7] and others in the early 1970’s. Since then there have been many attempts to count topological types of finite group actions in a given genus. The main motivation is the bijection between topological types and conjugacy classes of finite subgroups of the mapping class group. Lloyd [8] gave a generating function for the number of topologically inequivalent \(\mathbb{Z}_p\)-actions in genus \(g\). Gilman [6] gave an explicit count in this cyclic case, using combinatorial techniques similar to ours. Costa and Natanzon [4] gave a bijective classification of topological types of elementary abelian \(p\)-group actions which specifies (without solving) the crucial counting problem. Broughton and Wootton [3] describe a technique using representation theory and Möbius inversion to count topological types of abelian actions, and provide explicit results in some low genus cases, e.g., actions of \(\mathbb{Z}_p^2\) with four branch points (cf. their Section 4). Their method depends on enumeration of the finite subgroups of the symmetric groups.

2. Weighted multi-partitions

The theorem in this section is purely number-theoretical. We cite it on several occasions throughout the rest of the paper. The proof, which is somewhat involved, is given in an Appendix.

Let \((\mu_1, \mu_2, \ldots, \mu_m)\) be an \(m\)-tuple of positive integers, and \((\omega_1, \omega_2, \ldots, \omega_m)\) an \(m\)-tuple of weights, which are non-zero residue classes mod \(p\). Let \(\alpha\) be an arbitrary residue class mod \(p\). We seek \((\mu_1 + \cdots + \mu_m)\)-tuples

\[
(r_1, \ldots, r_{\mu_1}, s_1, \ldots, s_{\mu_2}, \ldots, z_1, \ldots, z_{\mu_m})
\]

of non-zero residue classes mod \(p\) which satisfy the congruence

\[
\omega_1(r_1 + \cdots + r_{\mu_1}) + \omega_2(s_1 + \cdots + s_{\mu_2}) + \cdots + \omega_m(z_1 + \cdots + z_{\mu_m}) \equiv \alpha \pmod{p}.
\]

The sums in parentheses are to be interpreted as unordered sums, or partitions, so we may assume

\[
p - 1 \geq r_1 \geq \cdots \geq r_{\mu_1} \geq 1,
\]

\[
p - 1 \geq s_1 \geq \cdots \geq s_{\mu_2} \geq 1,
\]

\[
\vdots
\]

\[
p - 1 \geq z_1 \geq \cdots \geq z_{\mu_m} \geq 1.
\]

We call (11), subject to (12), a weighted multi-partition congruence and a solution (10) a weighted multi-partition. The number of solutions is independent of the weights. Indeed (10) is a solution to (11) if and only if

\[
(\omega_1^{-1}r_1, \ldots, \omega_1^{-1}r_{\mu_1}, \omega_2^{-1}s_1, \ldots, \omega_2^{-1}s_{\mu_2}, \ldots, \omega_m^{-1}z_1, \ldots, \omega_m^{-1}z_{\mu_m})
\]
is a solution to
\[(r_1 + \cdots + r_{\mu_1}) + (s_1 + \cdots + s_{\mu_2}) + \cdots + (z_1 + \cdots + z_{\mu_m}) \equiv \alpha \pmod{p},\]
with all weights equal to 1. Inverses are taken mod $p$, and sub-tuples such as $\omega_1^{-1} r_1, \ldots, \omega_1^{-1} r_{\mu_1}$ are reordered, if necessary, to satisfy (12). Perhaps surprisingly, the number of solutions of (11) is not always independent of the residue class $\alpha$. Theorem 1 shows that solutions are equidistributed across the residue classes if and only if the $m$-tuple $(\mu_1, \ldots, \mu_m)$ contains an integer $\not\equiv 0, 1 \pmod{p}$.

If the $m$-tuple is a singleton $(\mu_1)$, the multi-partition reduces to an ordinary partition. On the other hand, if the $m$-tuple consists of $m$ 1’s, the multi-partition is an ordered additive composition with $m$ positive summands. Thus, for example, if $p = 7$ and $(\mu_1) = (3)$, we count 8 ordinary 3-part partitions of $\alpha \equiv 0 \pmod{p}$, namely
\[
\begin{align*}
1 + 2 + 4 &\quad 1 + 1 + 5 &\quad 2 + 2 + 3 &\quad 3 + 3 + 1 \\
3 + 5 + 6 &\quad 4 + 4 + 6 &\quad 5 + 5 + 4 &\quad 6 + 6 + 2,
\end{align*}
\]
while if $(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$, we count 30 3-part compositions, since each distinguishable reordering of the eight partitions contributes to the count. For the intermediate 3-part case,
\[
(r_1 + r_2) + s \equiv 0 \pmod{7},
\]
where $(\mu_1, \mu_2) = (2, 1)$, we assume $r_1 \geq r_2$, so the six sums in (13) of the form $a + a + b$ count twice as $(a + a) + b$ and $(a + b) + a$, while the two sums of the form $a + b + c$ count three times each as $(a + b) + c$, $(a + c) + b$, and $(b + c) + a$, for a total of 18.

It is convenient, here, and elsewhere in the paper, to interpret the binomial coefficient
\[
\binom{N + M - 1}{N}
\]
as the number of ways of distributing $N$ identical balls in $M$ distinguishable boxes. The contents of the individual boxes are separated by the insertion of $M - 1$ barriers, arbitrarily, in a linear array of $N$ balls.

**Theorem 1.** Let $(\mu_1, \mu_2, \ldots, \mu_m) = \mathcal{P}$ be an $m$-tuple of positive integers, $(\omega_1, \omega_2, \ldots, \omega_m)$ an $m$-tuple of non-zero residue classes mod $p$, and $\alpha$ an arbitrary residue class mod $p$. Let $n_{\mathcal{P}, \alpha}$ be the number of $(\mu_1 + \cdots + \mu_m)$-tuples (10) satisfying the weighted multi-partition congruence (11), subject to (12). Let
\[
B_{\mathcal{P}} = \prod_{i=1}^{m} \binom{\mu_i + p - 2}{\mu_i}.
\]
Then
\[
(n_{\mathcal{P}, 0}, n_{\mathcal{P}, 1}, \ldots, n_{\mathcal{P}, p-1}) = (W_{\mathcal{P}}, Z_{\mathcal{P}}, \ldots, Z_{\mathcal{P}})
\]
where

\[
W_P = \begin{cases} \frac{B_P}{p} & \text{if } \exists i, \mu_i \not\equiv 0, 1 \pmod{p}; \\ \frac{B_P + (-1)^{t+1}}{p} + (-1)^t & \text{otherwise}; \end{cases}
\]

\[
Z_P = \begin{cases} \frac{B_P}{p} & \text{if } \exists i, \mu_i \not\equiv 0, 1 \pmod{p}; \\ \frac{B_P + (-1)^{t+1}}{p} & \text{otherwise}; \end{cases}
\]

where \( t \geq 0 \) is the number of elements \( \mu_i \in P \) such that \( \mu_i \equiv 1 \pmod{p} \).

Proof. See Appendix. \( \square \)

3. The cardinality of \( \mathcal{M}(P) \)

To determine \( |\mathcal{M}(P)| \) for a general partition \( P \), we begin with types

(15) \( P = (1^\lfloor \frac{s}{p} \rfloor) = (1, \ldots, 1), \quad 2 \leq s \leq p + 1. \)

Let \( A \) be a \( 2 \times s \) matrix consisting of \( s \) distinct columns chosen from the set of standard representatives of \( PG(1, F_p) \),

(16) \( \left\{ \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ \end{bmatrix}, \ldots, \begin{bmatrix} p-1 \\ 1 \\ \end{bmatrix} \right\}. \)

A zero-sum multiset \( M = M(A) \in \mathcal{M}(P) \) corresponds to a solution \( \bar{x} = (x_1, x_2, \ldots, x_s) \) in positive integers \( 1 \leq x_i \leq p - 1 \), of the homogeneous system

(17) \( A\bar{x} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{p}. \)

There are no such solutions if \( s = 2 \) or \( 1 \). If \( s \geq 3 \), let \( d_s \) be the number of solutions. We could arbitrarily choose \( s - 2 \) of the \( x_i \)'s in any of \((p - 1)^{s-2}\) ways, and obtain solutions \( (x_1, x_2, \ldots, x_s) \) mod \( p \) by solving for the remaining two \( x_i \)'s. If \( s = 3 \), all the \( x_i \)'s obtained are non-zero, and this yields \( d_3 = p - 1 \). If \( s > 3 \), there is no guarantee that the two coordinates found in this way are both positive (i.e., \( \not\equiv 0 \pmod{p} \)). To get around this obstacle, note that if \( s \leq p - 1 \) we may assume, by a change of basis, that \( A \) does not contain the columns \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) or \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Then \( A \) has the form

\[
\begin{bmatrix}
    a_1 & a_2 & \cdots & a_s \\
    1 & 1 & \cdots & 1
\end{bmatrix}, \quad 1 \leq a_1 < a_2 < \cdots < a_s \leq p - 1.
\]

If \( *, ** \) denote arbitrary residue classes mod \( p \), not necessarily distinct, the number of positive solutions of

\( A\bar{x} \equiv \begin{bmatrix} * \\ 0 \end{bmatrix} \pmod{p} \)

is equal to the number of positive solutions of

(18) \( A\bar{x} \equiv \begin{bmatrix} 0 \\ ** \end{bmatrix} \pmod{p}, \)
since \((x_1, x_2, \ldots, x_s)\) is a positive solution of the former if and only if \((a_1^{-1}x_1, a_2^{-1}x_2, \ldots, a_s^{-1}x_s)\) is a positive solution of the latter. By Theorem 1 the number of positive solutions of (18) is

\[
W_{(1^{|s|})} = \frac{(p - 1)^s + (-1)^{s+1} + (-1)^s}{p}, \quad s \geq 1.
\]

With (19) and the principle of inclusion and exclusion, we derive a recursion for \(d_s\) as follows. Let \(A'\) consist of any \(s - 2\) columns of \(A\), say, for definiteness, the last \(s - 2\) columns. Choose positive \(\overline{\gamma} = (x_3, \ldots, x_s)\) in all possible ways, obtaining a set of cardinality \((p - 1)^{s-2}\). We count those \(\overline{\gamma}\)'s for which \(A'\overline{\gamma} \equiv \left[ \frac{s}{3} \right], \alpha, \beta \neq 0 \mod p\). Each such \(\overline{\gamma}\) can be uniquely completed to a positive solution \((x_1, x_2, x_3, \ldots, x_s)\), \(1 \leq x_i \leq p - 1\), of the original system (17), by assigning \(x_1 = -(a_{13}x_3 + a_{14}x_4 + \cdots + a_{1s}x_s) \equiv \alpha \mod p\) and \(x_2 = -(x_3 + x_4 + \cdots + x_s) \equiv \beta \mod p\). From the set of \((p - 1)^{s-2}\) \(\overline{\gamma}\)'s, we first discard the \(W_{(1^{s-2})}\) \(\overline{\gamma}\)'s making the first row \(\equiv 0 \mod p\). We then separately discard those \(\overline{\gamma}\)'s making the second row \(\equiv 0 \mod p\). There are \(W_{(1^{s-2})}\) of these \(\overline{\gamma}\)'s as well. To the remaining set of \((p - 1)^2 - 2W_{(1^{s-2})}\) \(\overline{\gamma}\)'s, we restore the \(d_{s-2}\) \(\overline{\gamma}\)'s making both rows \(simultaneously\) \(\equiv 0 \mod p\). It follows that \(d_s = (p - 1)^{s-2} - 2W_{(1^{s-2})} + d_{s-2}, \quad s \geq 3\). From this recursion we obtain the explicit formula

\[
d_s = (p - 1) \sum_{k=0}^{s-3} (-1)^k \binom{s-1}{k} p^{s-3-k}, \quad s \geq 3.
\]

It is convenient to define \(d_0 = 1\) and extend (20) to

\[
d_s = \begin{cases} 
1 & \text{if } s = 0 \\
0 & \text{if } s = 1, 2 \\
(p - 1) \sum_{k=0}^{s-3} (-1)^k \binom{s-1}{k} p^{s-3-k} & \text{if } 3 \leq s \leq p + 1.
\end{cases}
\]

The first few cases are

\[
\begin{align*}
d_0 &= 1; \\
d_1 &= 0; \\
d_2 &= 0; \\
d_3 &= (p - 1); \\
d_4 &= (p - 1)(p - 3); \\
d_5 &= (p - 1)(p^2 - 4p + 6); \\
d_6 &= (p - 1)(p^3 - 5p^2 + 10p - 10); \\
d_7 &= (p - 1)(p^4 - 6p^3 + 15p^2 - 20p + 15); \\
d_8 &= (p - 1)(p^5 - 7p^4 + 21p^3 - 35p^2 + 35p - 21).
\end{align*}
\]

**Theorem 2.** Let \(\mathcal{P} = (1^{|s|}), s \leq p + 1\).

\[
|\mathcal{M}(\mathcal{P})| = \binom{p + 1}{s} d_s.
\]
Proof. We need only note that the $s$ columns comprising $A$ can be selected from $PG(1, F_p)$ in $\binom{p^2 + 1}{s}$ ways.

We now consider more general partition types of the form $\mathcal{P} = (a^{[n]}, b^{[m]}, 1^{[s]})$, $a > b > 1$, $n, m, s \geq 0$. This will suffice for our computations, but Theorem 3 below, which treats this case, can be extended in a straightforward way to the general case. The theorem reduces to Theorem 2 in the case $n = m = 0$.

**Theorem 3.** Let $\mathcal{P} = (a^{[n]}, b^{[m]}, 1^{[s]})$, $a > b > 1$, $n, m, s \geq 0$, $n + m + s \leq p + 1$.

$$|\mathcal{M}(\mathcal{P})| = \frac{(p + 1)!}{s!(p + 1 - s - n - m)!} \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{W_{(a)}^{n-i} Z_{(a)}^{i} W_{(b)}^{m-j} Z_{(b)}^{j} d_{s+i+j}'}{i! j!(n-i)!(m-j)!},$$

where $d_s$ is defined at (21) and $W_s$, $Z_s$ are defined in Theorem 1.

Proof. We construct zero-sum multisets by a process of expansion starting from solutions of (17), where the coefficient matrix consists of $s'$ distinct columns chosen from (16), $s \leq s' \leq s + n + m \leq p + 1$. By Theorem 2 there are $d_{s'}$ such solutions. Associate entries from $\mathcal{P}$ with the elements of a solution $\mathbf{x} = (x_1, x_2, \ldots, x_s, \ldots, x_{s'})$ as follows. Partition the set $\{x_1, \ldots, x_{s'}\}$ into three disjoint subsets: $X_a$, $X_b$ and $X_1$, consisting of $i$, $j$ and $s$ elements, respectively, where $i \leq n$, $j \leq m$, and $i + j = s' - s$. Associate $i$ entries equal to $a$ with the elements of $X_a$, $j$ entries equal to $b$ with the elements of $X_b$, and all $s$ entries equal to 1 with the elements of $X_1$. The remaining $n - i$ entries equal to $a$ and $m - j$ entries equal to $b$, if any, are associated to columns in (16) outside of $A$. For each $a$ assigned to an $x_k \neq 0$ to any of the $Z_{(a)}$ a-part partitions of $x_k$ with parts of size at most $p - 1$. Similarly, for each $b$ assigned to an $x_l$, expand $x_l$ by any of the $Z_{(b)}$ b-part partitions of $x_l \neq 0$ with parts of size at most $p - 1$. The remaining $n - i$ a’s (resp. $m - j$ b’s) are expanded to any of the $W_{(a)}$ a-part (resp., any of the $W_{(b)}$ b-part partitions) of an integer $\equiv 0 \mod p$.

We count the independent choices at each step of the foregoing process. First choose $s + n + m$ of the $p + 1$ columns of (16),

$$\binom{p + 1}{n + m + s};$$

choose $s'$ columns from this set,

$$\binom{n + m + s}{s'};$$

distribute a-, b- and 1- parts of $\mathcal{P}$ over these $s'$ columns in

$$\frac{s'}{s! i! j!}$$

ways; distribute any leftover a- and b-parts in

$$\frac{(n + m + s - s')!}{(n - i)!(m - j)!}$$

ways. The product of these numbers is

$$\frac{(p + 1)!}{s!(p + 1 - s - n - m)! i! j!(n - i)!(m - j)!}.$$
Each distribution of parts is associated with \(d_{s+i+j}\) solutions of (17), each of which can be expanded in \(W_{(a)}^{n-i}Z_{(a)}^{i}W_{(b)}^{m-j}Z_{(b)}^{j}\) ways. Summing over all possible \(i\) and \(j\) yields the formula.

Results of Theorem 3 are given in Table 3.1 for all admissible partitions of \(3 \leq R \leq 6\). Formulae given for \(p > 3\) are valid for \(p = 3\) as well, except in the cases where the partition contains a part 3 or 4 (\(\equiv 0, 1 \mod 3\)). Recall that the inadmissible partition types are the singletons \((R[1])\) and the two-part partitions of the form \(((R − 1)[1], 1[1])\).

| \(R\)-partition | \(\mathcal{P}\) | \(p = 3\) | \(p > 3\) |
|-----------------|----------------|---------|---------|
| 3               | \(\{1^{3}\}\) | 8       | \((p+1)p(p-1)^2/6\) |
| 4               | \(\{1^{4}\}\) | 0       | \((p+1)p(p-1)^2(p-2)(p-3)/24\) |
|                 | \(\{2, 1^{3}\}\) | 24     | \((p+1)p(p-1)^3/4\) |
|                 | \(\{2^{2}\}\) | 6       | \((p+1)p(p-1)^2/8\) |
| 5               | \(\{1^{5}\}\) | 0       | \((p+1)p(p-1)^2(p-2)(p-3)(p^2-4p+6)/120\) |
|                 | \(\{2, 1^{3}\}\) | 8       | \((p+1)p(p-1)^3(p-2)^2/12\) |
|                 | \(\{2^{2}, 1\}\) | 24     | \((p+1)p(p-1)^4/8\) |
|                 | \(\{3, 1^{2}\}\) | 24     | \((p+1)^2p(p-1)^3/12\) |
|                 | \(\{3^{2}, 2^{1}\}\) | 48     | \((p+1)^2p(p-1)^2/12\) |
| 6               | \(\{1^{6}\}\) | 0       | \((p+1)p(p-1)^2(p-2)(p-3)(p-4)(p^3-5p^2+10p-10)/720\) |
|                 | \(\{2, 1^{4}\}\) | 0       | \((p+1)p(p-1)^3(p-2)(p-3)(p^2-3p+3)/48\) |
|                 | \(\{2^{2}, 1^{2}\}\) | 24     | \((p+1)p(p-1)^5(p-2)/16\) |
|                 | \(\{3, 1^{3}\}\) | 16     | \((p+1)^2p(p-1)^3(p-2)^2/36\) |
|                 | \(\{4, 1^{2}\}\) | 48     | \((p+2)(p+1)^2p(p-1)^3/48\) |
|                 | \(\{3^{2}, 2^{1}, 1\}\) | 48     | \((p+1)^2p(p-1)^2/12\) |
|                 | \(\{2^{3}\}\) | 12     | \((p+1)p^2(p-1)^4/48\) |
|                 | \(\{3^{2}\}\) | 24     | \((p+1)^3p(p-1)^2/72\) |
|                 | \(\{4^{2}, 2^{1}\}\) | 12     | \((p+2)(p+1)^2p(p-1)^2/48\) |

Table 3.1. \(|\mathcal{M}(\mathcal{P})|\) for admissible partitions \(\mathcal{P}\) of \(3 \leq R \leq 6\).

4. Conjugacy classes in \(GL_2(F_p)\)

\(GL_2(F_p)\) has order \((p^2 - 1)(p^2 - p)\). Let \(\sigma \in GL_2(F_p)\) be a non-central element, and let \(\tilde{\sigma}\) denote its image in the projective group \(PGL_2(F_p) = GL_2(F_p)/\{[x, y] : x \in F_p^\ast\}. \sigma\) is elliptic, parabolic, hyperbolic if \(\tilde{\sigma}\) has, respectively, zero, one, or two fixed points on \(PG(1, F_p)\). Equivalently, \(\sigma\) has, respectively, zero, one or two eigenspaces in \(V_2(F_p)\). Representatives of each type are given in Table 4.1 (see [5], §5.2). In the Table, \(\epsilon\) denotes a primitive generator of the multiplicative group \(F_p^\ast\) of non-zero elements in \(F_p\). Each geometric type is a union of conjugacy classes; the number of classes is the number of distinct representatives of the given type. The factor of \(\frac{1}{2}\) for elliptic elements comes from the fact that representatives in
| Type      | Class representatives | No. of Classes | No. of elements in a Class |
|-----------|-----------------------|----------------|---------------------------|
| Central   | $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$, $x \neq 0$ | $p - 1$ | 1 |
| Elliptic  | $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$, $y \neq 0$ | $p(p-1)/2$ | $p^2 - p$ |
| Parabolic | $\begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}$, $x \neq 0$ | $p - 1$ | $p^2 - 1$ |
| Hyperbolic| $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$, $x \neq y$ | $(p-1)(p-2)/2$ | $p^2 + p$ |

Table 4.1. Elements in $GL_2(F_p)$. $x, y \in F_p$, $\epsilon \in F_p^*$ a non-square

which $y$ is replaced by $-y$ are conjugate. Similarly the factor of $\frac{1}{2}$ for hyperbolic elements comes from the fact that representatives with $x$ and $y$ interchanged are conjugate.

The central indicator $d'$ of an element $\sigma \in GL_2(F_p)$ is the smallest positive power of $\sigma$ which is contained in the center of $GL_2(F_p)$. Equivalently, $d'$ is the order of $\tilde{\sigma}$ in $PGL_2(F_p)$. Thus $d'$ is a divisor of $d$, the order of $\sigma$, and we put $d = d'd''$, where $d''$ is the central quotient.

An eigen-order of an element is the multiplicative order of an eigenvalue (if there is one) in $F_p^*$.

Let

$$pC(d',d''), \quad pE(d',d''), \quad pP^e(d',d''), \quad pH^f(d',d'')$$

 denote the sets of central, elliptic, parabolic, hyperbolic elements, respectively, with central indicator $d'$, central quotient $d'' = d/d'$, order $d'd'' = d$, and eigen-orders (if any) $e, f$. We omit the initial $p$ when it is understood or clear from the context. We call the sets (22) **class types** since they comprise smaller collections of conjugacy classes within the larger geometric types. For hyperbolic elements, up to conjugacy, we may assume $e \leq f$. For central elements, $d' = 1$, and for parabolic elements, $d' = p$. In the following lemmata, we give necessary and sufficient conditions (on the various parameters) for these sets to be non-empty, and, when non-empty, the number of conjugacy classes they contain. Complete results for $p = 3, 5, 7$ are given in Table 4.2. $\phi(n)$ denotes Euler’s totient function, defined, for a positive integer $n$, as the number of positive integers less than or equal to $n$ and relatively prime to $n$. In a cyclic group of order $n$, $\phi(d)$ is the number of elements of order $d \mid n$.

4.1. **Central and parabolic elements.**

**Lemma 1.** Class types $pC(1,d'')$ and $pP^d''(p,d'')$ are non-empty if and only if $d'' \mid p - 1$, in which case, each contains $\phi(d'')$ conjugacy classes.

**Proof.** A parabolic element is conjugate to $\begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}$. Its $p$th power is the central element $\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$. Hence there is a conjugacy class in $\sigma \in pP^d''(p,d'')$ if and only if $x \in F_p^*$ has multiplicative order $d'' \mid p - 1$. There are $\phi(d'')$ such elements $x \in F_p^*$.

\[ \square \]
4.2. **Elliptic elements.** The $p^2 - 1$ elements

$$\left\{ \begin{bmatrix} x & ye \\ y & x \end{bmatrix}, \ x, y \text{ not both } = 0 \right\}$$

form a cyclic subgroup of order $p^2 - 1$ isomorphic to $F_{p^2}^*$. By analogy with complex numbers, one can think of these elements as $x + y\sqrt{\epsilon} \in F_{p^2}$ ($\epsilon$ being a non-square in $F_p$.) Each element with $y \neq 0$ is conjugate to its $p$th power,

$$\begin{bmatrix} x & ye \\ y & x \end{bmatrix}^p = \begin{bmatrix} x & -y\epsilon \\ -y & x \end{bmatrix}.$$ 

Moreover,

$$\begin{bmatrix} x & ye \\ y & x \end{bmatrix}^{p+1} = \begin{bmatrix} x^2 - y^2\epsilon & 0 \\ 0 & x^2 - y^2\epsilon \end{bmatrix}.$$ 

It follows that the central indicator of an elliptic element is a divisor $d' > 1$ of $p + 1$. Thus the general elliptic element has order $d = d'd''$, with $d' \mid p + 1$, $d' > 1$, and $d'' \mid p - 1$.

**Lemma 2.** For $d' > 1$, $d' \mid p + 1$, and $d'' \mid p - 1$, the class type $\_E(d', d'')$ is non-empty if and only if

$$\gcd((p - 1)/d'', d') = 1,$$

in which case $\_E(d', d'')$ contains $\phi(d'd'')/2$ conjugacy classes.

**Proof.** $\gcd((p - 1)/d'', d') \leq \gcd(p - 1, p + 1) = 2$. Suppose $\gcd((p - 1)/d'', d') = 2$. Then $d = d'd''$ is even. If $\sigma$ is an elliptic element of order $d = 2k$, then $\sigma^k$ is the central element

$$\begin{bmatrix} \epsilon^{(p-1)/2} & 0 \\ 0 & \epsilon^{(p-1)/2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

This implies that $d''$ is even. With both $d'$ and $d''$ even, $d = 4l$, and $\sigma^l$ is conjugate to one of the order 4 elements

$$\begin{bmatrix} 0 & \pm \epsilon^{(p+1)/4} \\ \pm \epsilon^{(p-3)/4} & 0 \end{bmatrix},$$

which implies $p \equiv -1 \pmod{4}$. On the other hand, the assumption $\gcd((p - 1)/d'', d') = 2$ implies $(p - 1)/d''$ is even. But if $d''$ and $(p - 1)/d''$ are both even, $p \equiv 1 \pmod{4}$, a contradiction. Thus $\gcd((p - 1)/d'', d') = 1$ is a necessary condition.

To prove sufficiency, note that every elliptic element is a power of an elliptic element $\sigma$ of maximum order $p^2 - 1$, with central indicator $p + 1$. Let $d'$ and $d''$ satisfy the conditions of the lemma. Let

$$n = \frac{p + 1}{d'} \cdot \frac{p - 1}{d''}.$$ 

Then $\sigma^n$ has order $d = d'd''$ and $\sigma^{nd'}$ is a central element of order $d''$. If $d'$ is not the central indicator, there exists $d'_1 \mid d'$, $1 < d'_1 < d'$, such that $\sigma^{nd'_1}$ is a central element of order

$$\frac{p - 1}{\gcd(\frac{p - 1}{d'_1}, p - 1)} = d'_1d''.$$
Since \( d'_1 \mid \gcd(p+1, p-1) \), the only possibility is \( d'_1 = 2 \). Then \( d' \) is even, and condition (24) implies that \( \frac{p-1}{d'} \) is odd, so that \( \frac{p-1}{d'} \) is not an integer. Thus \( d'_1 = d' \) and \( d' \) is the central indicator. \( \square \)

4.3. Hyperbolic elements. Let \( x, y, z \geq 1 \) be three positive integers with \( \text{lcm}(x, y, z) = d \). We say that \( \{x, y, z\} \) satisfies the strong lcm condition if

\[
\text{lcm}(x, y, z) = \text{lcm}(x, y) = \text{lcm}(x, z) = \text{lcm}(y, z).
\]

An equivalent condition is that the prime divisors \( q \mid d \) separate into two disjoint sets:

- \( U = \) primes \( q \mid d \) having uniform multiplicity \( m_q \) in the prime factorizations of \( x, y, z \);
- \( \overline{U} = \) primes \( q \mid d \) whose multiplicity \( j_q \) is non-maximal \((j_q < m_q)\) in the prime factorization of exactly one of \( x, y, z \).

**Lemma 3.** For \( e, f, d' \mid p - 1, e \leq f \), and \( \text{lcm}(e, f, d') = d \), the class type \( \phi_H^e(d', d/d') \) is non-empty if and only if \( \{e, f, d'\} \) satisfies the strong lcm condition and the additional condition \( 2 \in \overline{U} \) if \( 2 \mid d \). In this case, the number of conjugacy classes in the class type is

\[
(25) \quad \prod_{q \in U} \phi(q^{m_q})q^{m_q-1}(q-2) \prod_{q \in \overline{U}} \phi(q^{m_q})\phi(q)\]

if \( e < f \), or half this number, if \( e = f \).

**Proof.** First suppose \( d = q^m \), a power of single prime. A hyperbolic element of order \( q^m \) is conjugate to a diagonal matrix \( \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \), where \( a, b \in F_p^* \) have multiplicative orders \( e = q^n \), \( f = q^m \), and we assume \( n \leq m \). Let the central indicator \( d' = q^j \), for some \( j \leq m \). Let \( D_q(n, m, j) \) be the set of diagonal matrices with \( \{e, f, d'\} = (q^n, q^m, q^j) \). We prove that \( n < m \) implies \( j = m \), and conversely, \( j < m \) implies \( n = m \) (i.e., that \( q \in U \) or \( q \in \overline{U} \)).

For computational purposes, we associate diagonal matrices in \( D_q(n, m, j) \) with pairs \((u, v) \in N \times N\),

\[
N = \{0, 1, 2, \ldots, q^m - 1\},
\]

by choosing a generator \( \gamma \) of the multiplicative group \( F_{q^m}^* \), and defining the correspondence

\[
(u, v) \leftrightarrow \begin{bmatrix} \gamma^u & 0 \\ 0 & \gamma^v \end{bmatrix}.
\]

Membership in \( D_q(n, m, j) \) is equivalent to the conditions

\[
(26) \quad \gcd(u, q^m) = q^{m-n}, \quad \gcd(v, q^m) = 1,
\]

and, since \( d' \) is the smallest solution of the congruence \( u d' \equiv v d' \mod q^m \),

\[
(27) \quad \gcd(u - v, q^m) = q^{m-j}.
\]

The set \( \{u \in N \mid \gcd(u, q^m) = q^{m-n}\} \) can be parametrized as

\[
\{q^{m-n}(Aq + B) \mid A = 0, 1, \ldots, q^n - 1, \quad B = 1, \ldots, q - 1\},
\]

and the set \( \{v \in N \mid \gcd(v, q^m) = 1\} \) as

\[
\{aq + b \mid a = 0, 1, \ldots, q^{m-1} - 1, \quad b = 1, \ldots, q - 1\}.
\]
The differences $u - v$ between elements of the first and second sets can be laid out in an array of dimensions $\phi(q^n) \times \phi(q^m)$,

$$\Delta = \{q^{m-n}(Aq + B) - (aq + b)\} = \{q^{m-n}[q(A - a) + B] - b\}.$$ 

If $n < m$, $\Delta$ consists of elements $\equiv -b \pmod{q}$, that is, $\not\equiv 0 \pmod{q}$. Hence for $n < m$ it follows from condition (27) that $j = m$ is the only possibility.

$$|D_q(n, m, j)| = \begin{cases} 0 & \text{if } 0 \leq j \leq m - 1 \\ \phi(q^n)\phi(q^m) & \text{if } j = m. \end{cases} \quad (n < m)$$

If $n = m$, the $\phi(q^n) \times \phi(q^m)$ array of differences is

$$\Delta = \{q(A - a) + B - b\}.$$ 

In this array, $u - v \not\equiv 0 \pmod{q}$ if and only if $B - b \not\equiv 0 \pmod{q}$, and this occurs in

$$\frac{q - 2}{q - 1}(\phi(q^m))^2 = \phi(q^m)q^{m-1}(q - 2)$$

positions. The remaining $\frac{1}{q - 1}(\phi(m))^2$ positions have $u - v = q(A - a) \equiv q^{m-j} \pmod{q^m}$, $1 \leq j \leq m$, or, equivalently,

$$A - a \equiv q^{m-1-j} \pmod{q^{m-1}}, \quad 0 \leq j \leq m - 1.$$ 

For each $j = 0, 1, \ldots, m - 1$ there are $\phi(q^j)$ differences $\equiv q^{m-1-j}$ in each of the $\phi(q^m)$ rows of the array. Thus if $n = m$,

$$|D_q(m, m, j)| = \begin{cases} \phi(q^j)\phi(q^m) & \text{if } 0 \leq j \leq m - 1 \\ \phi(q^m)q^{m-1}(q - 2) & \text{if } j = m. \end{cases}$$

We assumed $n \leq m$ in the triples $(n, m, j)$, but when $n \not\equiv m$,

$$|D_q(n, m, m)| = |D_q(m, n, m)|,$$

since conjugacy gives a one-to-one correspondence between the two sets. Hence we have shown that

$$(28) \quad |D_q(j, m, m)| = |D_q(m, j, m)| = |D_q(m, m, j)| = \begin{cases} \phi(q^j)\phi(q^m) & \text{if } j < m \\ \phi(q^m)q^{m-1}(q - 2) & \text{if } j = m, \end{cases}$$

while, for all other triples $(i, j, k)$, $0 \leq i, j, k \leq m$, $|D_q(i, j, k)| = 0$.

The general hyperbolic element of order $d = \text{lcm}(x, y, z)$ is conjugate to a product of diagonal matrices of prime power order. The formula (25) is the product of (28) over the set $U \cup U'$ of prime divisors of $d$. The necessity of the $U \cup U'$ bipartition of the prime divisors of $d$, equivalently, of the strong lcm condition on $\{e, f, d\}$, follows immediately. The formula would yield 0 if $2 \in U$, due to the factor $q - 2$, which explains the extra condition.

If $e = f$ (and $d' \neq 1$), the diagonal matrices $[a \: 0 \: 0]$ and $[b \: 0 \: 0]$ are distinct but conjugate. This explains the necessity of reducing the formula by a factor of $\frac{1}{2}$ in this case. \hfill \square
Table 4.2. Class types in $GL_2(F_p)$, $p = 3, 5, 7$.

5. The cardinality of $\text{Fix}_P(\sigma)$

The cardinality of $\text{Fix}_P(\sigma) = \{M \in \mathcal{M}(P) \mid \sigma M = M, \sigma \in GL_2(F_p)\}$ depends only on the conjugacy class of $\sigma$. This follows from: (i) $M \in \mathcal{M}(P)$ if and only if $\gamma M \in \mathcal{M}(P)$, $\gamma \in GL_2(F_p)$; and (ii) $\sigma M = M$ if and only if $\gamma \sigma \gamma^{-1}(\gamma M) = \gamma M$. If $M \in \text{Fix}_P(\sigma)$ then $M$ is a concatenation of $\sigma$-orbits on $V_2(F_p)$. We do not say ‘disjoint union’ because $M$, being a multiset, may contain the same orbit more than once.

5.1. The zero-sum condition.

Lemma 4. If $\sigma \in GL_2(F_p)$ has no eigenvalue equal to 1, $\sigma M = M$ implies the zero-sum condition on $M$.

Proof. Let $\sigma$ have order $d > 1$. $\sigma^d - I_2 = [0 \ 0; 0 \ 0]$, and if there is no eigenvalue equal to 1, $\sigma - I_2$ is invertible. Therefore,

$$(\sigma^d - I_2)(\sigma - I_2)^{-1} \equiv I_2 + \sigma + \sigma^2 + \cdots + \sigma^{d-1} \equiv [0 \ 0; 0 \ 0] \pmod{p}. \quad (29)$$

Hence the $\sigma$-orbit of any column $[\begin{array}{c} a \\ b \end{array}]$ of $M$ sums to

$$[\begin{array}{c} a \\ b \end{array}] (I_2 + \sigma + \sigma^2 + \cdots + \sigma^{d-1}) \equiv [0 \ 0] \pmod{p}. \quad \square$$
If $\sigma$ has an eigenvalue equal to 1, it belongs to one of the conjugacy classes in $pP^1(p,1)$ or $pH^1_1(f,1)$. By suitable change of basis, we may take $\sigma$ to be

$$
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} \in pP^1(p,1) \text{ or } \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \in pH^1_1(f,1).
$$

$\infty$ is the 1-eigenspace in both of these cases, and $\overline{0}$ is the $f$-eigenspace in the hyperbolic case. Let $M$ be a multiset in the form (2). If $\sigma M = M$, we show that the zero-sum condition, in both cases, is a weighted multi-partition congruence involving only the top row of $M$.

For $\sigma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in pP^1(p,1)$, a column $\begin{bmatrix} a \\ b \end{bmatrix} \in M$ with $b \neq 0$ (a non-eigenvector) belongs to a $\sigma$-orbit

$$
\begin{bmatrix}
a \\
b
\end{bmatrix}, \begin{bmatrix}
a + b \\
b
\end{bmatrix}, \begin{bmatrix}
a + 2b \\
b
\end{bmatrix}, \ldots, \begin{bmatrix}
a + (p-1)b \\
b
\end{bmatrix}
$$

of length $p$, which spans all subspaces of $PG(1,F_p)$ except $\infty$ and sums to

$$
\begin{bmatrix}
ap + bp(p-1)/2 \\
bp
\end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{p}.
$$

Hence if $\sigma M = M$, the zero-sum condition is simply

$$
\sum S_\infty \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{p}.
$$

For $\sigma = \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} \in pH^1_1(f,1)$, a column $\begin{bmatrix} a \\ b \end{bmatrix} \in M$ belongs to the $\sigma$-orbit

$$
\begin{bmatrix}
a \\
b
\end{bmatrix}, \begin{bmatrix}
a \\
yb
\end{bmatrix}, \begin{bmatrix}
a \\
y^2b
\end{bmatrix}, \ldots, \begin{bmatrix}
a \\
y^{f-1}b
\end{bmatrix}
$$

which sums to

$$
\begin{bmatrix}
fa \\
b(1 + y + y^2 + \cdots + y^{f-1})
\end{bmatrix} \equiv \begin{bmatrix} fa \\ 0 \end{bmatrix} \pmod{p}.
$$

If $\sigma M = M$, the zero-sum condition is

$$
\sum S_\infty \equiv -\sum [S_1 | \cdots | S_{p-1}] \pmod{p}.
$$

Note that (33) implies

$$
\sum S_0 \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{p}
$$

automatically. It is easily verified that zero sum conditions (32) and (34) involve only the entries in the top row of $M$.

5.2. The shape of a $\sigma$-orbit. Let $\sigma$ be an arbitrary element of $GL_2(F_p)$, and let $\bar{\sigma}$ denote the image of $\sigma$ in the projective group $PGL_2(F_p)$. $\bar{\sigma}$ permutes the $p+1$ points of $PG(1,F_p)$. A proper orbit of $\bar{\sigma}$ consists of $d'$ points, where $d'$ is the central indicator. The number of proper orbits of $\bar{\sigma}$ is

$$
Q = Q(\sigma) = \begin{cases}
\frac{p+1}{d'} & \text{if } \sigma \text{ is elliptic or central;}
\frac{p}{d'} & \text{if } \sigma \text{ is hyperbolic;}
1 & \text{if } \sigma \text{ is parabolic.}
\end{cases}
$$
We define a *generic* orbit of $\sigma$ to be an orbit that projects to a proper orbit of $\tilde{\sigma}$ under the projection $V_2(F_p) \to PG(1, F_p)$.

Let $v \in V_2(F_p)$ belong to a generic orbit of $\sigma$, and let $\overline{v}$ be its image in $PG(1, F_p)$. Then $v, \sigma(v), \ldots, \sigma^{d-1}(v)$ project onto the proper $\tilde{\sigma}$-orbit $\overline{v}, \tilde{\sigma}(\overline{v}), \ldots, \tilde{\sigma}^{d'-1}(\overline{v})$ on $PG(1, F_p)$. $\sigma^d$ is a central element $[y \ 0 \ 0], y \in F_p^*$, of multiplicative order $d''$. The $\sigma$-orbit of $v$ can be displayed in the shape of a $d' \times d''$ array

$$X(v, y) = \begin{bmatrix}
v & yv & \ldots & y^{d''-1}v \\
\sigma(v) & y\sigma(v) & \ldots & y^{d''-1}\sigma(v) \\
\vdots & & & \\
\sigma^{d-1}(v) & y\sigma^{d-1}(v) & \ldots & y^{d''-1}\sigma^{d-1}(v)
\end{bmatrix}.$$ 

$X(v, y)$ is marked in the vertical (projective) dimension by the $\tilde{\sigma}$-orbit

$$\langle \tilde{\sigma} \rangle(\overline{v}) = \{\overline{v}, \tilde{\sigma}(\overline{v}), \ldots, \tilde{\sigma}^{d'-1}(\overline{v})\},$$

and in the horizontal (central) dimension by the $y$-orbit

$$\langle y \rangle = \{1, y, y^2, \ldots, y^{d''-1}\} \subseteq F_p^*.$$

$X(v, y)$ is generated by any one of its entries, e.g., starting with $y^jv$ rather than $v$ yields the same entries, with the columns shifted cyclically. However, starting with $zv, z \in F_p^*$, $z \not\in \langle y \rangle$, produces a different array, $X(zv, y)$, disjoint from $X(v, y)$, with the same vertical marking but a different horizontal marking, namely

$$z\langle y \rangle = \{z, zy, zy^2, \ldots, zy^{d''-1}\} \subseteq F_p^*.$$

The number of distinct horizontal markings is the number of distinct orbits of $\langle y \rangle$ on $F_p^*$, namely, $\frac{d''-1}{e}$.

If $v$ were an eigenvector of $\sigma$, the array would be an *eigen-orbit* of shape $1 \times e$ (where $e$ is the order of the eigenvalue $y$), marked in the vertical dimension by the singleton orbit $\{\tilde{\sigma}(\overline{v})\} = \{\overline{v}\}$ and in the horizontal dimension by the $y$-orbit $\langle y \rangle \subseteq F_p^*$. If $v$ were replaced by $zv, z \in F_p^*$, $z \not\in \langle y \rangle$, we would obtain a different linear array $X(zv, y)$, disjoint from $X(v, y)$ with the same vertical marking $\{\overline{v}\}$ and a different horizontal marking. The number of distinct horizontal markings is the number of orbits of $\langle y \rangle$ on $F_p^*$, namely $\frac{d''-1}{e}$.

It follows that, for each $\sigma \in GL_2(F_p)$, the set $V_2(F_p) - \{[0\ 0]\}$ is partitioned into marked $\sigma$-orbits of generic shape $d' \times d''$, together, possibly, with marked eigen-orbits of shape $1 \times e$ and/or $1 \times f$.

### 5.3. Tiling the Ferrers diagram

If $\sigma M = M, M \in M(\mathcal{P})$, the Ferrers diagram of $\mathcal{P}$ can be overlaid or *tiled* by arrays (of dots) of shape $d' \times d''$, together (possibly) with some arrays of shape $1 \times e$ and/or $1 \times f$ confined to one or two rows. This follows from the fact that $M$ is a concatenation of $\sigma$-orbits. The $d' \times d''$ arrays (generic tiles) are realized in $M$ by marked generic orbits, and the $1 \times e$ or $1 \times f$ tiles by marked eigen-orbits, if any. There are at most $Q = Q(\sigma)$ rows of generic tiles, spanning $d'$ rows each, there being only $Q$ distinct proper $\tilde{\sigma}$-orbits with which to mark them. There is no limit on the number of tiles in a particular row of tiles, since $M$ is a multiset with repeats allowed. If there are $1 \leq s \leq Q$ rows of generic tiles, the tiled Ferrers diagram has the form
Horizontal tiles $(1 \times e$ or $1 \times f$) are confined to the first two rows, corresponding to eigenspaces, if any. (This may entail a violation of the usual convention that rows in a Ferrers diagram decrease weakly in length from top to bottom.) The integer tuple $(l,n,m_1,\ldots,m_s)$ on the left indicates the number of tiles in the corresponding row. We assume $m_1 \geq m_2 \geq \cdots \geq m_s$. It is convenient for counting purposes to express $(l,n,m_1,\ldots,m_s)$ with multiplicities as

\begin{equation}
(l,n,[m_1]_1,\ldots,[m_s]_u), \quad k_1 + k_2 + \cdots + k_u = s, \quad m_{i_{\mu}} > m_{i_{\nu}} \text{ if } \mu > \nu.
\end{equation}

If there are no eigenspaces (i.e., if $\sigma$ is elliptic or central), $l = n = 0$. Of course $l$ or $n$ may be zero even when eigenspaces exist. Similarly, $s$ may be zero (if there are no marked generic orbits). The tiling (37) is feasible for $\sigma$ if a zero-sum multiset $M \in \text{Fix}_P(\sigma)$ can be realized by associating marked $\sigma$-orbits with each tile. Feasibility conditions are just those which ensure that the rank and zero-sum conditions on $M$ can be satisfied.

**Lemma 5.** The tiled Ferrers diagram $T$ given by (37) is feasible for $\sigma$ if the tuple $(l,n,m_1,\ldots,m_s)$ satisfies $s \leq Q(\sigma)$ and

| Type of $\sigma$ | Conditions |
|-----------------|------------|
| Central         | $l = n = 0$, $s > 1$ |
| Elliptic        | $l = n = 0$, $s > 0$ |
| Parabolic       | $n = 0$, $s = 1$, $(l \neq 1)^*$ |
| Hyperbolic      | If $s = 0$, then $l, n > 0$, $(l \neq 1)^*$. |

* If $\sigma$ has an eigenvalue equal to 1.

**Proof.** Central and Elliptic elements have no eigenspaces, so $l = n = 0$. For central elements, generic orbits have shape $1 \times d''$, so $s > 1$ is necessary to satisfy the rank condition on $M$. For parabolic elements, $s \leq Q = 1$, and $s = 1$ is necessary to satisfy the rank condition. For hyperbolic elements, if $s = 0$, then both $l$ and $n$ must be non-zero to satisfy the rank condition. If $\sigma$ has an eigenvalue equal to 1, the extra condition on $l$ comes from the non-trivial zero-sum condition. This is clarified at the end of the proofs of Theorems 5 and 6 below.

**Theorem 4.** Let $\sigma \in \text{GL}_2(F_p)$ have central indicator $d'$, central quotient $d''$, and, if $\sigma$ has eigenspaces, eigenorders $e, f > 1$, i.e., no eigenvalue equal to 1. Let the tiled Ferrers diagram $T$ given by (37) be feasible for $\sigma$. Then

$$|\text{Fix}_T(\sigma)| = \frac{Q(\sigma)!}{(Q(\sigma) - s)!k_1!k_2!\cdots k_u!} \left(\frac{l + \frac{e}{e} - 1}{l}\right)^* \left(\frac{n + \frac{f}{f} - 1}{n}\right)^* \prod_{\mu=1}^{u} \left(\frac{m_{i_{\mu}} + \frac{e_{\mu}}{e_{\mu}} - 1}{m_{i_{\mu}}}\right)^{k_{i_{\mu}}}. $$

* These factors are set to 1 if the corresponding eigenspace does not exist.
Theorem 5. Let \( \sigma \) have no eigenvalue equal to 1, the zero-sum condition is automatic on any concatenation of \( \sigma \)-orbits. It follows that a multiset \( M \) can be realized by assigning the vertical and horizontal markings of \( [37] \) arbitrarily. A vertical (projective) marking of \( [37] \) is obtained by marking the \( s \) rows of generic tiles by a selection of distinct proper 3-orbits \( \langle \tilde{\sigma} \rangle \{ \overline{\pi}_i \} \), \( i = 1, \ldots, s \) in some order. The particular choices of orbit representatives \( \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_s \) need not be given explicitly. The number of possible projective markings is

\[
Q(\sigma)! \\
\left( Q(\sigma) - s! \right) k_1! k_2! \ldots k_u!.
\]

This accounts for the first factor in the formula. Each of the \( m_i \) tiles in the \( i \)th row of generic tiles can have any one of \( \frac{e-1}{f} \) or \( \frac{e-1}{f} \) horizontal markings. Similarly, each tile in an eigenspace can have any one of \( \frac{e-1}{f} \) horizontal markings. Within a row, tiles are unordered. For counting purposes, we view them as ‘balls’ placed arbitrarily into ‘boxes’ labelled with the available markings. The binomial coefficient \( [14] \) accounts for the remaining factors in the formula. The factors involving \( l \) and \( n \) reduce to 1 if the corresponding eigenspace exists but is unmarked \((l \text{ or } n = 0)\).

We now determine \( |\text{Fix}_T(\sigma)| \) when \( \sigma \) has an eigenvalue equal to 1.

Theorem 5. Let \( \sigma \in \pi H_f^1(f, 1) \). Let the tiled Ferrers diagram \( T \) given by \( [37] \), \( e = 1 \), \( d' = f \), \( d'' = 1 \), be feasible for \( \sigma \). Then

\[
|\text{Fix}_T(\sigma)| = \frac{\left( \frac{e-1}{f} \right)!}{\left( \frac{e-1}{f} - s! \right) k_1! k_2! \ldots k_u!} \left( n + \frac{e-1}{f} - 1 \right) W_{(l, m_1, \ldots, m_s)},
\]

where \( W_{(\_\_\_)} \) has the definition given in Theorem 7.

Proof. \( [37] \) takes the form

\[
\begin{array}{c|c|c}
    l & \cdot & \cdot \\
    n & \vdots & \vdots \\
    m_1 & \vdots & \vdots \\
    m_2 & \vdots & \vdots \\
    m_s & \vdots & \vdots \\
\end{array}
\]

\[
\begin{array}{c|c|c}
    & \overline{w}_1 \\
    \overline{\pi}_1 & \langle \tilde{\sigma} \rangle \{ \overline{\pi}_1 \} \\
    \overline{\pi}_2 & \langle \tilde{\sigma} \rangle \{ \overline{\pi}_2 \} \\
    \overline{\pi}_s & \langle \tilde{\sigma} \rangle \{ \overline{\pi}_s \} \\
\end{array}
\]

We include a projective (vertical) marking on the right. Thus \( \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_s \) is set of representatives of \( s \) distinct proper 3-orbits on \( PG(1, F_p) \). In the 1-eigenspace \( (\overline{w}_1) \), tiles are single dots (shape \( 1 \times 1 \)). In the \( f \)-eigenspace \( (\overline{w}_f) \) tiles are horizontal segments of shape \( 1 \times f \). Generic tiles are vertical segments of shape \( f \times 1 \). The number of possible projective markings, as before, is given by \( [39] \), with \( Q(\sigma) = \frac{e-1}{f} \). Without loss of generality, we may assume (by a possible change of basis) that \( \sigma = [1/0 \ 0] \), so the eigenspaces are \( \overline{w}_e = \overline{\pi} \) and \( \overline{w}_f = \overline{0} \). Then the set \( \{ \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_s \} \) can be given more explicitly as \( \overline{v}_k = [a_k] \), \( k = 1, \ldots, s \), \( 1 \leq a_k \leq p-1 \), \( a_i \neq a_j \) if \( i \neq j \). The horizontal markings of the \( 1 \times 1 \) tiles in the 1-eigenspace and the generic \( f \times 1 \) tiles must be chosen so that the zero-sum condition \( [34] \) is satisfied. Because these tiles have horizontal dimension \( 1 = e = d'' \), a horizontal marking is just a choice of scalar from \( F_p^* \) for each tile. Let \( r_1, \ldots, r_l \) be the scalar markings of the \( 1 \times 1 \) tiles,
and \( z_{i1}, \ldots, z_{im_i}, i = 1, \ldots, s \), be the scalar markings of tiles in the \( i \)th row of generic tiles. The zero-sum condition (34) is a weighted multi-partition congruence, with tuple \((l, m_1, \ldots, m_s)\) and weights \((1, f_{a_1}, \ldots, f_{a_s})\).

\[
(42) \quad \sum_{j=1}^{l} r_j + \sum_{k=1}^{s} (f_{a_k}) \sum_{i=1}^{m_k} z_{ki} \equiv 0 \pmod{p}.
\]

The number of solutions of (42), by Theorem 1, is \( W_{(l,m_1,\ldots,m_s)} \). By the proof of the theorem, this number is independent of the weights. The horizontal marking of the \( f \)-eigenspace can be chosen arbitrarily in

\[
\binom{n + \frac{p-1}{n} - 1}{n}
\]

ways, in view of (35). If \( s = 0 \), \( W_{(l,m_1,\ldots,m_s)} = W_{(l)} = 0 \) if \( l = 1 \). Thus we have the additional feasibility condition \( l \neq 1 \) in this case.

**Theorem 6.** Let \( \sigma \in \mathcal{P}_1(p,1) \). A feasible tiled Ferrers diagram \( \mathcal{T} \) for \( \sigma \) has the form

\[
(43) \quad \begin{array}{cccccc}
\cdot & \cdots & \cdots & \cdot \\
\mid & \mid & \cdots & \mid \\
\end{array}
\]

with eigenspace tiles of shape \( 1 \times 1 \) and \( m > 0 \) generic tiles of shape \( p \times 1 \). Then

\[
(44) \quad |\text{Fix}_{\mathcal{T}}(\sigma)| = \binom{m + p - 2}{m} W_{(l)},
\]

where \( W_{(-)} \) has the definition given in Theorem 1.

**Proof.** Without loss of generality we may assume \( \sigma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) with eigenspace \( \infty \). The single row of \( m > 0 \) generic tiles of shape \( p \times 1 \) can be projectively marked in only one way by the single proper orbit \( \langle \tilde{\sigma} \rangle(\overline{0}) = \{ \overline{0}, \overline{1}, \ldots, \overline{p-1} \} \). Generic \( \sigma \)-orbits are zero-sum (cf. (31)), so the generic tiles may be horizontally marked by scalars from \( F_p^* \) arbitrarily, in

\[
\binom{m + p - 2}{m}
\]

ways, by the balls-in-boxes analogy. Dot tiles in the top (eigenspace) row are also marked by scalar choices from \( F_p^* \). The zero-sum condition (32) is simply that these \( l \geq 0 \) scalars sum to 0 mod \( p \). By Theorem 1, the number of scalar choices satisfying this condition is \( W_{(l)} \). Since \( W_{(1)} = 0 \), we have the additional feasibility condition \( l \neq 1 \) in this case.

Let \( \mathcal{P} \) be the partition underlying a tiling \( \mathcal{T} \). We write \( \mathcal{T} \supset \mathcal{P} \) in this situation. It is possible that there are two or more tilings with the same underlying partition feasible for the same \( \sigma \). Then

\[
|\text{Fix}_{\mathcal{P}}(\sigma)| = \sum_{\mathcal{T} \supset \mathcal{P}} |\text{Fix}_{\mathcal{T}}(\sigma)|.
\]
For example, the partition $\mathcal{P} = (2^3)$ has two feasible tilings for $\sigma \in \mathcal{P}H^1_2(2,1)$ (for any odd $p$):

(45) \quad T_1 = \text{\textbullet\textbullet\textbullet}, \quad T_2 = \text{\textbullet\textbullet\textbullet} \supset \mathcal{P} = \text{\textbullet\textbullet\textbullet\textbullet}\text{\textbullet\textbullet\textbullet}

From Theorem 5 and Theorem 1,

\[ |\text{Fix}_{T_1}(\sigma)| = \frac{p-1}{2} W(2,2) = \frac{p(p-1)^3}{8}, \]
\[ |\text{Fix}_{T_2}(\sigma)| = \left(\frac{p-1}{2}\right)^2 W(2) = \frac{(p-1)^3}{8}, \]
\[ |\text{Fix}_{\mathcal{P}}(\sigma)| = \frac{p(p-1)^3}{8} + \frac{(p-1)^3}{8} = \frac{(p+1)(p-1)^3}{8}. \]

Similarly, $\mathcal{P} = (3^1, 1^3)$ has two feasible tilings for $\sigma \in \mathcal{P}H^3_3(3,1)$ (for $p \equiv 1 \pmod{3}$), depending on which of the two 3-eigenspaces is marked:

(46) \quad T_1 = \text{\textbullet\textbullet\textbullet}, \quad T_2 = \text{\textbullet\textbullet\textbullet} \supset \mathcal{P} = \text{\textbullet\textbullet\textbullet\textbullet}

(The unmarked eigenspace is indicated by a light horizontal segment.) From Theorem 4,

\[ |\text{Fix}_{T_1}(\sigma)| = |\text{Fix}_{T_2}(\sigma)| = \left(\frac{p-1}{3}\right)^2 (p-1), \]
\[ |\text{Fix}_{\mathcal{P}}(\sigma)| = \frac{2(p-1)^3}{9}. \]

6. **Burnside’s Lemma**

For given $\mathcal{P}$, the summation over the elements of $GL_2(F_p)$ in Burnside’s Lemma (6) can be reduced to a sum over a subset consisting of the identity element and one representative of each class type for which $\mathcal{P}$ has a feasible tiling. The identity contributes the summand $|\mathcal{M}(\mathcal{P})|$, obtained from Theorem 3 or its generalization. Each remaining summand is a three-fold product of the form

(47) \quad t(\sigma)c(\sigma) \sum_{T \supset \mathcal{P}} |\text{Fix}_{T}(\sigma)|,

where $\sigma$ is a representative of a feasible class type; $t = t(\sigma)$ is the number of elements in the class (obtained from Table 4.1); and $c = c(\sigma)$ is the number of classes in the class type (obtained from Lemmas 1, 2, or 3). The factors in (47) are all independent of the choice of $\sigma$ within the class type. Let $F(\mathcal{P}) \subset GL_2(F_p)$ be a set containing one representative of each class type for which $\mathcal{P}$ has a feasible tiling. Let $O(\mathcal{P})$ be the set of orbits of $GL_2(F_p)$ on $\mathcal{M}(\mathcal{P})$. Burnside’s Lemma is then

(48) \quad |O(\mathcal{P})| = (p^2 - 1)^{-1}(p^2 - p)^{-1}\{|\mathcal{M}(\mathcal{P})| + \sum_{\sigma \in F(\mathcal{P})} t(\sigma)c(\sigma) \sum_{T \supset \mathcal{P}} |\text{Fix}_{T}(\sigma)|\}.\]
The computations below for the 6-partition $\mathcal{P} = (2^{[3]})$ yield $|O(\mathcal{P})| = 2, 6$ for the primes $p = 3, 5$ respectively.

| $T$ | Class Type | Summand |
|-----|-------------|---------|
| $\text{Id}$ | 12 |
| $3C(1, 2)$ | 4 |
| $3H_2^1(2, 1)$ | 36 |
| $3H_2^2(2, 1)$ | 12 |
| $3P^2(3, 2)$ | 8 |
| $3P^1(3, 1)$ | 24 |

Sum/48 = 96/48 = 2

$p = 3$

| $T$ | Class Type | Summand |
|-----|-------------|---------|
| $\text{Id}$ | 800 |
| $5C(1, 2)$ | 160 |
| $5H_2^1(2, 1)$ | 1200 |
| $5H_2^2(2, 1)$ | 240 |
| $5E(3, 2)$ | 80 |
| $5E(3, 1)$ | 400 |

Sum/480 = 2880/480 = 6

$p = 5$

Computations for all admissible $R$-partitions, $R = 3, 4, 5, 6,$ and $p = 3, 5, 7,$ are given in Section 6.1. Tables 3.1 and 4.2 contain the explicit numerical data needed for the computations.

Given $R$ and $p,$ the sum $\sum_{\mathcal{P}} |O(\mathcal{P})|$ over all admissible $R$-partitions yields the number of topological types of $\mathbb{Z}_p^2$ actions in genus $g = 1 + Rp(p - 1)/2 - p^2.$ This information, gleaned from the computations in Section 6.1, is summarized in Table 6.1 below.

| $R$ | Genus | # Top. Types | Genus | # Top. Types | Genus | # Top. Types |
|-----|-------|--------------|-------|--------------|-------|--------------|
| 3   | 1     | 1            | 6     | 1            | 15    | 1            |
| 4   | 4     | 2            | 16    | 4            | 36    | 6            |
| 5   | 7     | 5            | 26    | 14           | 57    | 39           |
| 6   | 10    | 10           | 36    | 57           | 78    | 282          |

Table 6.1. Topological types of $\mathbb{Z}_p^2$ actions fully ramified over $R$ points, $p = 3, 5, 7.$

6.1. Computations. For each $R$-partition $\mathcal{P}$ and prime $p,$ the summands in (48) are given in the form $\mathcal{P} : |\mathcal{M}(\mathcal{P})|$; followed by triples consisting of a class type, a tiling $T,$ and the integer $t(\sigma)c(\sigma)|\text{Fix}_T(\sigma)|$ for a representative element $\sigma$ in the class type.

$R = 3$:

$p = 3, g = 1$: 1 orbit

$1^{[3]}$: 8; $P^1(3, 1) \mid 16; H_2^1(2, 1) \hat{1} 24; (8 + 16 + 24)/48 = 1.$

$p = 5, g = 6$: 1 orbit

$1^{[3]}$: 80; $E(3, 1) \mid 160; H_2^1(2, 1) \hat{1} 240; (80 + 160 + 240)/480 = 1.$

$p = 7, g = 15$: 1 orbit

$1^{[3]}$: 336; $H_2^1(2, 1) \hat{1} 1008; H_3^3(3, 1) \mid 672; (336 + 1008 + 672)/2016 = 1.$

$R = 4$:
\[ p = 3, q = 4: \text{2 orbits} \]
\[ 1^{[4]}: 0; \]
\[ 2^{[1]}1^{[2]}: 24; H_2^1(2,1) \square 24; \frac{(24+24)}{48} = 1. \]
\[ 2^{[2]}: 6; C(1,2) \equiv 6; E(2,2) \square 12; H_2^1(2,1) \square 12; H_2^1(1,2) \equiv 12; \]
\[ \frac{(6+6+12+12+12)}{48} = 1. \]

\[ p = 5, q = 16: \text{4 orbits} \]
\[ 1^{[4]}: 120; H_2^1(2,1) \square 120; H_2^4(4,1) \square 240; \frac{(120+120+240)}{480} = 1. \]
\[ 2^{[1]}1^{[2]}: 480; H_2^2(2,1) \square 480; \frac{(480+480)}{480} = 2. \]
\[ 2^{[2]}: 60; C(1,2) \equiv 60; H_2^1(2,1) \square 120; H_1^3(1,2) \equiv 120; H_1^4(2,2) \square 120; \]
\[ \frac{(60+60+120+120+120)}{480} = 1. \]

\[ p = 7, q = 36: \text{6 orbits} \]
\[ 1^{[4]}: 1680; H_2^2(2,1) \square 1008; H_2^3(3,1) \square 1344; \frac{(1680+1008+1344)}{2016} = 2. \]
\[ 2^{[1]}1^{[2]}: 3024; H_2^1(2,1) \square 3024; \frac{(3024+3024)}{2016} = 3. \]
\[ 2^{[2]}: 252; C(1,2) \equiv 252; E(2,2) \square 504; H_2^2(2,1) \square 504; H_2^4(2,1) \equiv 504; \]
\[ \frac{(252+252+504+504+504)}{2016} = 1. \]

\( R = 5: \)

\[ p = 3, q = 7: \text{5 orbits} \]
\[ 1^{[5]}: 0; \]
\[ 2^{[1]}1^{[3]}: 8; P^1(3,1) \square 16; H_2^1(2,1) \square 24; \frac{(8+16+24)}{48} = 1. \]
\[ 2^{[2]}1^{[1]}: 24; H_2^1(2,1) \square 24; \frac{(24+24)}{48} = 1. \]
\[ 3^{[1]}1^{[2]}: 24; H_2^1(2,1) \square 24; \frac{(24+24)}{48} = 1. \]
\[ 3^{[1]}2^{[1]}: 48; H_2^3(2,1) \equiv 48; \frac{(48+48)}{48} = 2. \]

\[ p = 5, q = 26: \text{14 orbits} \]
\[ 1^{[5]}: 264; P^1(5,1) \square 96; H_2^1(2,1) \square 360; H_2^4(4,1) \square 240; \]
\[ \frac{(264+96+360+240)}{480} = 2. \]
\[ 2^{[1]}1^{[3]}: 1440; H_2^2(2,1) \square 480; \frac{(1440+480)}{480} = 4. \]
\[ 2^{[2]}1^{[1]}: 960; H_2^1(2,1) \square 480; \frac{(960+480)}{480} = 3. \]
\[ 3^{[1]}1^{[2]}: 960; H_2^1(2,1) \equiv 960; \frac{(960+960)}{480} = 4. \]
\[ 3^{[1]}2^{[1]}: 240; H_2^3(2,1) \equiv 240; \frac{(240+240)}{480} = 1. \]

\[ p = 7, q = 57: \text{39 orbits} \]
\[ 1^{[5]}: 9072; H_2^2(2,1) \square 5040; \frac{(9072+5040)}{2016} = 7. \]
\[ 2^{[1]}1^{[3]}: 25200; H_2^2(2,1) \square 3024; H_2^3(3,1) \equiv 4032; \]
\[ \frac{(25200+3024+4032)}{2016} = 16. \]
\[ 2^{[2]}1^{[1]}: 9072; H_2^2(2,1) \square 3024; \frac{(9072+3024)}{2016} = 6. \]
\[ 3^{[1]}1^{[2]}: 8064; H_2^1(2,1) \equiv 8064; \frac{(8064+8064)}{2016} = 8. \]
\[ 3^{[1]}2^{[1]}: 1344; H_2^3(2,1) \equiv 1344; H_2^4(3,1) \equiv 672; H_2^2(6,1) \equiv 672; \]
\[ \frac{(1344+1344+672+672)}{2016} = 2. \]

\( R = 6: \)

\[ p = 3, q = 10: \text{10 orbits} \]
\[1^6; 0;\]
\[2^{[1]} 1^{[4]}; 0;\]
\[2^2 1^2; 24; \ H_2^1(2, 1) \ \vdash \ 72; \ (24 + 72)/48 = 2.\]
\[3^1 1^{[3]}; 16; \ P^1(3, 1) \ \vdash \ 32; \ (16 + 32)/48 = 1.\]
\[4^1 1^2; 48 \ H_2^1(2, 1) \ \vdash \ 48; \ (48 + 48)/48 = 2.\]
\[3^2 1^1; 48; \ 48/48 = 1.\]
\[2^3; 12; \ C(1, 2) \ \vdash 4; \ P^1(3, 1) \ \vdash 24; \ P^2(3, 2) \ \vdash 8; \ H_2^2(2, 1) \ \vdash 36; \ H_2^2(2, 1) \ \vdash 12; \]
\[(12 + 4 + 24 + 8 + 36 + 12)/48 = 2.\]
\[3^2; 24; \ H_2^2(2, 1) \ \vdash 24; \ (24 + 24)/48 = 1.\]
\[4^1 1^2; 12; \ C(1, 2) \ \vdash 12; \ H_2^2(2, 1) \ \vdash 12; \ H_2^2(2, 1) \ \vdash 12; \]
\[(12 + 12 + 12 + 12)/48 = 1.\]

\[p = 5, g = 36: \ 57 \text{ orbits}\]
\[1^6; 160; \ E(3, 1) \ \vdash 320; \ (160 + 320)/480 = 1.\]
\[2^1 1^{[4]}; 3120; \ H_2^2(2, 1) \ \vdash 960; \ H_2^2(2, 1) \ \vdash 240; \ H_2^4(4, 1) \ \vdash 480; \]
\[H_2^4(4, 1) \ \vdash 480; \ (3120 + 960 + 240 + 480 + 480)/480 = 11.\]
\[2^2 1^2; 5760; \ H_2^2(2, 1) \ \vdash 480; \ H_2^2(2, 1) \ \vdash 480; \ (5760 + 480 + 480)/480 = 14.\]
\[3^1 1^{[3]}; \ 2880; \ 2880/480 = 6.\]
\[4^1 1^2; 1680; \ H_2^2(2, 1) \ \vdash 1680; \ (1680 + 1680)/480 = 7.\]
\[3^2 1^1; 3840; \ 3840/480 = 8.\]
\[3^1 2^1 1^{[1]}; 48; \ 48/48 = 1.\]

\[p = 7, g = 78: \ 282 \text{ orbits}\]
\[1^6; 26544; \ H_2^1(2, 1) \ \vdash 1680; \ H_3^1(3, 1) \ \vdash 672; \ H_3^2(3, 1) \ \vdash 2016; \ H_6^6(6, 1) \ \vdash 672; \]
\[H_2^6(6, 1) \ \vdash 672; \ (26544 + 1680 + 672 + 2016 + 672 + 672)/2016 = 16.\]
\[2^1 1^{[4]}; 156240; \ H_2^2(2, 1) \ \vdash 18144; \ H_2^4(4, 1) \ \vdash 3024; \]
\[(156240 + 18144 + 3024)/2016 = 88.\]
\[2^2 1^2; 136080; \ H_2^2(2, 1) \ \vdash 9072; \ H_2^4(4, 1) \ \vdash 6048; \]
\[(136080 + 9072 + 6048)/2016 = 75.\]
\[3^1 1^{[3]}; 67200; \ H_3^1(3, 1) \ \vdash 10752; \ H_3^3(3, 1) \ \vdash 1344; \ H_3^3(3, 1) \ \vdash 1344; \]
\[(67200 + 10752 + 1344 + 1344)/2016 = 40.\]
\[4^1 1^2; 18144; \ H_2^2(2, 1) \ \vdash 18144; \ (18144 + 18144)/2016 = 18.\]
\[3^{[1]}2^{[1]}1^{[1]}: \quad 48384; \quad (48384)/2016 = 24.\]

\[2^{[3]}: \quad 10584; \quad C(1, 2) \equiv 1512; \quad H_2^1(2, 1) \equiv 10584; \quad H_3^1(2, 1) \equiv 1512; \quad H_3^1(3, 1) \equiv 672; \quad H_3^2(3, 2) \equiv 2352; \quad H_6^0(3, 2) \equiv 336;\]

\[(10584 + 1512 + 10584 + 1512 + 672 + 2352 + 672 + 336)/2016 = 14.\]

\[3^{[2]}: \quad 1792; \quad C(1, 3) \equiv 224; \quad H_1^2(2, 1) \equiv 1344; \quad H_3^1(3, 1) \equiv 1792; \quad H_3^2(3, 1) \equiv 224; \quad H_3^3(2, 3) \equiv 1008;\]

\[(1792 + 224 + 1344 + 1792 + 224 + 672)/2016 = 3.\]

\[4^{[1]}2^{[1]}: \quad 3024; \quad C(1, 2) \equiv 1008; \quad H_1^2(2, 1) \equiv 3024; \quad H_2^1(2, 1) \equiv 1008;\]

\[(3024 + 1008 + 3024 + 1008)/2016 = 4.\]

**Appendix: Proof of Theorem \(\text{[1]}\)**

The \(q\)-binomial coefficients are

\[
\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}
\]

where \(n \geq k\) are positive integers. If \(q\) is a prime power, \(\binom{n}{k}_q\) is the number of \(k\)-dimensional subspaces of an \(n\)-dimensional vector space over the field with \(q\) elements. It is easily verified that

\[
\binom{n}{k}_q = \binom{n}{n-k}_q.
\]

Though defined as a rational functions, the \(q\)-binomial coefficients turn out to be polynomials with integer coefficients,

\[
\binom{n}{k}_q = \sum_{l=0}^{k(n-k)} t_l q^l,
\]

where \(t_l\) is equal to the number of partitions of \(1 \leq l \leq k(n-k)\) into at most \(k\) parts of size at most \(n-k\). A proof of this statement, and further details on the \(q\) binomial coefficients, can be found in, e.g., [10], Ch. 24.

In accordance with the remarks preceding the statement of the theorem, we may assume the weights \(\omega_1 \equiv \omega_2 \equiv \cdots \equiv \omega_m \equiv 1 \mod p.\)

Let \(a_{ij} \geq 0\) be the number of parts equal to \(j\), \(1 \leq j \leq p-1\), in the first partition \(r_1 + r_2 + \cdots + r_{\mu_1}\) in \(\text{[1]}\). Similarly let \(a_{2j}, \ldots, a_{mj}\) be the numbers of parts equal to \(j\) in the remaining \(m-1\) partitions comprising \(\text{[1]}\). Then the \(m \times (p-1)\) matrix \([a_{ij}]\) satisfies the conditions

\[
\sum_{j=1}^{p-1} a_{ij} = \mu_i, \quad i = 1, \ldots, m,
\]

and

\[
\sum_{i=1}^{m} \sum_{j=1}^{p-1} ja_{ij} \equiv \alpha \quad (\mod p),
\]
and specifies the multi-partition \((\square)\) uniquely. Thus determination of \(n_\alpha\) is equivalent to counting \(m \times (p - 1)\) matrices \([a_{ij}]\) of nonnegative integers satisfying (51) and (52).

First suppose the matrices have an extra initial column \(a_{i0}, i = 1, \ldots, m\) and that the expanded matrices satisfy the conditions analogous to (51) and (52), namely,

\[
\sum_{j=0}^{p-1} a_{ij} = \mu_i, \quad i = 1, \ldots, m, \tag{53}
\]

and

\[
\sum_{i=1}^{m} \sum_{j=0}^{p-1} j a_{ij} \equiv \alpha \pmod{p}, \tag{54}
\]

the latter being, in fact, identical to (52). Let \(\tilde{n}_\alpha\) be the number of expanded matrices satisfying (53) and (54). As a first step, we show that

\[
(\tilde{n}_0, \tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_{p-1}) = \begin{cases} \left(\frac{E}{p}, \frac{E}{p}, \ldots, \frac{E}{p}\right) & \text{if } \exists i, \mu_i \equiv 0 \pmod{p}; \\ \left(\frac{E-1}{p} + 1, \frac{E-1}{p}, \frac{E-1}{p}, \ldots, \frac{E-1}{p}\right) & \text{otherwise}, \end{cases} \tag{55}
\]

where \(E = \prod_{i=1}^{m} e_{\mu_i}\), and \(e_{\mu_i} = (\mu_i + p - 1)\). \(e_{\mu_i}\) is the number of ways of populating the \(i\)th row of the expanded matrix. (\(\mu_i\) ‘balls’ distribute themselves over \(p\) ‘boxes’ labelled 0, 1, \ldots, \(p - 1\).) Hence the total number of matrices is \(\prod_i e_{\mu_i} = E\). Let \(Y_\alpha\) be the subset of matrices with weighted sum \(\sum_{i=1}^{m} \sum_{j=0}^{p-1} j a_{ij}\) in the specified residue class \(\alpha\). To prove the first case of (55), apply the right shift map \(\sigma\) to a row, say, the \(i\)th row, having \(\mu_i \equiv 0 \pmod{p}\).

\[
\sigma : a_{ij} \mapsto a_{i, j+1}, \quad j = 0, \ldots, p - 2, \\
\sigma : a_{i, p-1} \mapsto a_{i, 0}.
\]

Extend it by the identity map on the other rows. Then

\[
\sum_{i=1}^{m} \sum_{j=0}^{p-1} j \sigma(a_{ij}) \equiv \sum_{i=1}^{m} \sum_{j=0}^{p-1} j a_{ij} + \mu_i \pmod{p}.
\]

Construct the inverse map in the obvious way using a left shift map. We obtain a bijection between \(Y_\alpha\) and \(Y_{\alpha + k\mu_i}\). Composing this bijection with itself \(k\) times gives a bijection between \(Y_\alpha\) and \(Y_{\alpha + k\mu_i}\). The numbers \(\alpha + k\mu_i, k = 0, 1, \ldots, p - 1\) determine the complete set of residue classes \(\pmod{p}\); this yields the first case of (55). To prove the second case of (55), apply the following map, for \(\beta \not\equiv 0 \pmod{p},
\]

\[
\phi_\beta : (a_{i0}, a_{i1}, \ldots, a_{ip-1}) \mapsto (a_{i(\beta 0)}, a_{i(\beta 1)}, a_{i(\beta 2)}, \ldots, a_{i(\beta(p-1))}) \tag{57}
\]

to all rows of the matrix. This defines a bijection \(\phi_\beta : Y_\alpha \to Y_{\beta \alpha}\) with inverse \(\phi_{\beta^{-1}} : Y_{\beta \alpha} \to Y_\alpha\), as can be seen from

\[
\sum_{i=1}^{m} \sum_{j=0}^{p-1} \beta j a_{ij} = \beta \sum_{i=1}^{m} \sum_{j=0}^{p-1} j a_{ij}.
\]
From this we conclude again that $\tilde{n}_1 = \tilde{n}_\alpha$ for all $\alpha \neq 0 \mod p$, now including the case in which all $\mu_i \equiv 0 \mod p$. The second case of (55) will follow if we show that $\tilde{n}_0 = \frac{E - 1}{p} + 1$.

The presence of an initial column $(a_{ij}, i = 1, \ldots, m)$, with weight $j_0 = 0$, allows the $i$th row of $[a_{ij}]$ to be interpreted as a partition of $0 \leq l = \sum_{j_0}^{p-1} j a_{ij} \leq \mu_i(p - 1)$ into at most $\mu_i$ positive parts of size at most $p - 1$. Hence the number of ways of populating the $i$th row with weighted sum $l$ is the coefficient of $q^l$ in the polynomial form of the $q$-binomial coefficient

$$\left[ \begin{array}{c} \mu_i + p - 1 \\ \mu_i \end{array} \right]_q = \sum_{l=0}^{\mu_i(p-1)} t_{l,i} q^l.$$

It follows that the number of matrices $[a_{ij}]$ with total weighted sum

$$L = \sum_{i=1}^{m} \sum_{j=0}^{p-1} j a_{ij}, \quad 0 \leq L \leq R(p - 1), \quad R = \sum_{i} \mu_i,$$

is the coefficient of $q^L$ in the polynomial product

$$\prod_{i=1}^{m} \left[ \begin{array}{c} \mu_i + p - 1 \\ \mu_i \end{array} \right]_q = \sum_{L=0}^{R(p-1)} T_L q^L,$$

where

$$T_L = \sum_{l_1 + l_2 + \cdots + l_m = L} t_{l,i}.$$

The number we seek is

$$\tilde{n}_{p,0} = \sum_{p|L} T_L.$$

To pick out the desired coefficients, we use

$$\sum_{j=0}^{p-1} (e^{2\pi i/p})^{jm} = \begin{cases} p & \text{if } p \mid m, \\ 0 & \text{if } p \nmid m, \end{cases}$$

where $e$ is the base of natural logarithms and $i$ is the imaginary unit (see [1], Theorem 8.1). It follows that

$$p \tilde{n}_0 = p \sum_{p|L} T_L = \sum_{j=0}^{p-1} \sum_{L=0}^{R(p-1)} T_L (e^{2\pi i/p})^{jL}.$$

The first summand in the outer summation ($j = 0$) is

$$\sum_{L=0}^{R(p-1)} T_L = E,$$

and the remaining summands ($j = 1, 2, \ldots, p - 1$) are all reorderings of the summation

$$\sum_{L=0}^{R(p-1)} T_L (e^{2\pi i/p})^L = \left[ R + p - 1 \right]_{e^{2\pi i/p}}.$$
Putting $q = e^{2\pi i/p}$, and using (49) and (50), we have
\[
\begin{bmatrix} R+p-1 \\ R \end{bmatrix}_q = \frac{(q^{R+p-1} - 1)(q^{R+p-2} - 1) \ldots (q^{R+1} - 1)}{(q^{p-1} - 1)(q^{p-2} - 1) \ldots (q - 1)},
\]
which is equal to 1, since $R \equiv 0 \mod p$. Thus
\[p\tilde{n}_0 = E + p - 1,
\]
which completes the proof of second case of (55).

We now return to the conditions in the statement of the theorem, by assuming the extra initial column of the expanded matrix $[a_{ij}]$ consists entirely of 0’s, i.e., $a_{i0} = 0, i = 1, \ldots, m$. The number of ways of choosing the $i$th row such a matrix is
\[b_{\mu_i} = e_{\mu_i} - e_{\mu_i-1} = \left(\frac{\mu_i + p - 2}{\mu_i}\right).
\]
If both $\mu_i$ and $\mu_i - 1 \not\equiv 0 \mod p$ (i.e., $\mu_i \not\equiv 0, 1 \mod p$), (55) applied to the singletons $\mathcal{P} = (\mu_i)$ and $\mathcal{P} = (\mu_i - 1)$, implies that there are
\[\frac{e_{\mu_i} - e_{\mu_i-1}}{p} = \frac{b_{\mu_i}}{p}
\]
ways of populating the $i$th row $[0, a_{i1} \ldots, a_{ij} \ldots a_{ip-1}]$ such that the weighted sum $\sum_{j=1}^{p-1} ja_{ij}$ belongs to any given residue class $\alpha \mod p$. Then as long as $\mathcal{P}$ contains an element $\mu_i \not\equiv 0, 1 \mod p,$
\[(n_{\mathcal{P},0}, n_{\mathcal{P},1}, \ldots, n_{\mathcal{P},p-1}) = \left(\frac{B_{\mathcal{P}}}{p}, \frac{B_{\mathcal{P}}}{p}, \ldots, \frac{B_{\mathcal{P}}}{p}\right).
\]
Even without this restriction on $\mathcal{P}$, we have $n_{\mathcal{P},\alpha} = n_{\mathcal{P},\beta}$ if $\alpha, \beta \not\equiv 0$, because the map (57) fixes the initial column (of 0’s). Thus in the remaining case, in which $\mathcal{P}$ consists entirely of elements $\mu_i \equiv 0$ or $\mu_i \equiv 1 \mod p$, there exist nonnegative integers $\Omega_{\mathcal{P}}$ and $\zeta_{\mathcal{P}}$, satisfying $\Omega_{\mathcal{P}} + (p - 1)\zeta_{\mathcal{P}} = B_{\mathcal{P}}$, such that
\[(n_{\mathcal{P},0}, n_{\mathcal{P},1}, \ldots, n_{\mathcal{P},p-1}) = (\Omega_{\mathcal{P}}, \zeta_{\mathcal{P}}, \ldots, \zeta_{\mathcal{P}}).
\]
To obtain expressions for $\Omega_{\mathcal{P}}$ and $\zeta_{\mathcal{P}}$, relabel
\begin{equation}
(58) \quad \mathcal{P} = (\mu_1, \mu_2, \ldots, \mu_s, \nu_1, \nu_2, \ldots, \nu_t), \quad s + t = m, \quad s, t \geq 0,
\end{equation}
so that $\mu_i \equiv 0 \mod p$ and $\nu_j \equiv 1 \mod p$. Assume all the $\mu_i$ are nonzero. Consider the $m$-tuples $\tilde{\mathcal{P}}$ that can be formed from (58) by reducing some, none, or all elements by 1. All but one of these tuples contain an element $\not\equiv 0 \mod p$. The exception is
\[\mathcal{P}_0 = (\mu_1, \mu_2, \ldots, \mu_s, \nu_1 - 1, \ldots, \nu_t - 1).
\]
Applying (55) to each $\tilde{\mathcal{P}} \neq \mathcal{P}_0$, we obtain
\[(n_{\mathcal{P},0}, n_{\mathcal{P},1}, \ldots, n_{\mathcal{P},p-1}) = \left(\frac{E}{p}, \frac{E}{p}, \ldots, \frac{E}{p}\right),
\]
for some $E$. On the other hand, applying (55) to $\tilde{\mathcal{P}}_0$ we obtain
\begin{equation}
(59) \quad (n_{\mathcal{P},0}, n_{\mathcal{P},1}, \ldots, n_{\mathcal{P},p-1}) = \left(\frac{E_0 - 1}{p} + 1, \frac{E_0 - 1}{p}, \frac{E_0 - 1}{p}, \ldots, \frac{E_0 - 1}{p}\right),
\end{equation}
where $E_0 = e_{\mu_1}e_{\mu_2} \cdots e_{\mu_t}e_{\nu_2} \cdots e_{\nu_s}$. The numbers $\tilde{E}$, $E_0$ are (up to sign) the sum-mands in the expansion of

$$B_{\mathcal{P}} = \prod_{i=1}^{s} (e_{\mu_i} - e_{\mu_i-1}) \prod_{j=1}^{t} (e_{\nu_j} - e_{\nu_j-1}).$$

$E_0$ appears with sign $(-1)^t$, so $B_{\mathcal{P}} + (-1)^{t+1}E_0$ is the sum of all the other (signed) terms, each of which counts a set of matrices whose weighted sums, by (55), are equidistributed mod $p$. $E_0$ counts a set of matrices whose weighted sums are distributed according to (59). From this and the fact that $b_{\mu_i} = e_{\mu_i} - e_{\mu_i-1}$, we deduce that

$$\zeta_{\mathcal{P}} = n_{\mathcal{P},1} = n_{\mathcal{P},2} = \cdots = n_{\mathcal{P},p-1} = \frac{B_{\mathcal{P}} + (-1)^{t+1}E_0}{p} + (-1)^{t} \frac{E_0 - 1}{p}$$

while

$$\Omega_{\mathcal{P}} = n_{\mathcal{P},0} = \frac{B_{\mathcal{P}} + (-1)^{t+1}E_0}{p} + (-1)^{t+1} \left( \frac{E_0 - 1}{p} + 1 \right).$$

Terms involving $E_0$ drop out of both expressions, and we find $\Omega_{\mathcal{P}} = W_{\mathcal{P}}$ and $\zeta_{\mathcal{P}} = Z_{\mathcal{P}}$ as in the statement of the theorem.

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