On Schur Complements In Bicomplex Representation Of q EP

S. SRIDEVI1, K. GUNASEKARAN2

PG and Research Department of Mathematics,
Government Arts College (Autonomous),
Kumbakonam - 612 002, Tamil Nadu (India)

Corresponding author Email: sridevimahendran21@gmail.com.
http://dx.doi.org/10.22147/jusps-A/300303

Acceptance Date 3rd February, 2018, Online Publication Date 2nd March, 2018

Abstract

It is established that under certain conditions a Schur Complement in bicomplex representation of q-EP is as well a q-EP matrix. As an application a decomposition of a partitioned matrix into a sum of bicomplex representation of q-EP matrices is given.

Key words : q-EP matrix, Schur Complements in q-EP, Schur Complements in Bicomplex representation of q-EP.

AMS Classification : 15A57, 15A15, 15A09

Introduction

Through we shall deal with \( n \times n \) quaternion matrices: Let \( A^* \) denote the conjugate transpose of \( A \). Any matrix \( A \in H_{n \times n} \) is called q-EP. If \( R(A) = R(A^*) \) and is called q-EP, if is q-EP \( ^4(Q^*_{E_{n \times n}}) \) and \( rk(A) = r \), where \( N(A), R(A) \) and \( rk(A) \) denote the null space, range space and rank of \( A \) respectively. It is well known that sum and product of q-EP, Generalized Inverse Group Inverse and Reverse order law for q-EP and Bicomplex representation methods and application of q-EP matrices.

For any q-EP matrix \( A \), \( A \) can be uniquely represented as

\[
A = A_h + A_i j
\]  

[by 8]

\[
R(A) = R(A_h) + R(A_i j)
\]  

[by 5, Theorem 1]
Where \( A_j \in C_{n \times n} \) (\( s=0,1 \)), \( A_j \) means to multiply each entries of \( A_j \) by \( j \) from right hand side and \( \text{rk}(A_0) = \text{rk}(A_1) \).

In this section, Schur complements in bicomplex representation of q-EP matrices.

**Lemma 1.1:**

If \( X \) and \( Y \) are generalized inverse of \( A = A_0 + A_1 j \), then
\[
(C_0 + C_1 j) + (B_0 + B_1 j) = (C_0 + C_1 j)Y(B_0 + B_1 j) \quad \text{if and only if} \quad N(A_0 + A_1 j) \subseteq M(C_0 + C_1 j) \quad \text{and} \quad N(A_0 + A_1 j^+) \subseteq N(B_0 + B_1 j^+) \quad \text{or, equivalently if and only if}
\]
\[
C = (C_0 + C_1 j)(A_0 + A_1 j)^-(A_0 + A_1 j) \quad \text{and} \quad B = (A_0 + A_1 j)(A_0 + A_1 j)^- B \quad \text{for every} \quad (A_0 + A_1 j)^-
\]

Throughout this paper, we are concerned with \( n \times n \) quaternion matrices \( M \) partitioned in the form
\[
M = M_0 + M_1 j \quad \text{where,}
\]
\[
M_0 + M_1 j = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_0 + A_1 j & B_0 + B_1 j \\ C_0 + C_1 j & D_0 + D_1 j \end{bmatrix}
\]

Where \( A_0 + A_1 j \) and \( D_0 + D_1 j \) are square matrices with respect to this partitioning a Schur complements of \( A \) in \( M \) is a matrix at the form
\[
\left( (M_0 + M_1 j) / (A_0 + A_1 j) \right) = \left( D_0 + D_1 j \right) - \left( C_0 + C_1 j \right)(A_0 + A_1 j)^-(B_0 + B_1 j). \quad \text{For entries of Schur complements one may refer to}^{23,5}
\]

On account of lemma 1 it is obvious that under certain conditions \( (M_0 + M_1 j) / (A_0 + A_1 j) \) is independent of the choice of \( (A_0 + A_1 j)^- \). However in the sequel we shall always assume that \( (M_0 + M_1 j) / (A_0 + A_1 j) \) is given in terms of specific choice of \( (A_0 + A_1 j)^- \).

In\(^9\) necessary and sufficient conditions are derived for a matrix of the (2) with \( C_0 + C_1 j = 0 \) to be q-EP. The results are here extended for general matrices of the form (2). If a partitioned matrix of the form (2) is q-EP, then in general \( (M_0 + M_1 j) / (A_0 + A_1 j) \) is not q-EP. Here we determine necessary and sufficient conditions for \( (M_0 + M_1 j) / (A_0 + A_1 j) \) to be q-EP. In particular, when \( \text{rk}(M_0 + M_1 j) = \text{rk}(A_0 + A_1 j) \) our results include as special cases the results of paper\(^{14}\). In\(^5\) we have given conditions for a sum of q-EP matrices to be q-EP.

**Theorem 1.2:**

Let \( M \) be a matrix of the form (2) with
\[
N(A_0 + A_1 j) \subseteq N(C_0 + C_1 j) \quad \text{and} \quad N(M_0 + M_1 j) / (A_0 + A_1 j) \subseteq N(B_0 + B_1 j), \quad \text{then the following are equivalent.}
\]
i. \( M_0 + M_1j \) is a q-EP matrix

ii. \( A_0 + A_1j \) and \( (M_0 + M_1j)/(A_0 + A_1j) \) are q-EP, \( N(A_0 + A_1j)^* \subseteq N(B_0 + B_1j)^* \) and \( N((M_0 + M_1j)/(A_0 + A_1j)^*) \subseteq M(C_0 + C_1j)^* \);

iii. Both the matrices

\[
\begin{pmatrix} A_0 + A_1j & 0 \\ C_0 + C_1j & (M_0 + M_1j)/(A_0 + A_1j) \end{pmatrix}
\]

and

\[
\begin{pmatrix} A_0 + A_1j & B_0 + B_1j \\ 0 & (M_0 + M_1j)/(A_0 + A_1j) \end{pmatrix}
\]

are q-EP.

**Proof:**

(i) \( \Rightarrow \) (ii)

Let us consider the matrices

\[
P = \begin{pmatrix} I_0 + I_1j & 0 \\ (C_0 + C_1j)(A_0 + A_1j)^* & I_0 + I_1j \end{pmatrix}, \quad Q = \begin{pmatrix} I_0 + I_1j & B_0 + B_1j(M_0 + M_1j)/(A_0 + A_1j)^* \\ 0 & I_0 + I_1j \end{pmatrix}.
\]

\[
L = \begin{pmatrix} A_0 + A_1j & 0 \\ 0 & (M_0 + M_1j)/(A_0 + A_1j) \end{pmatrix}
\]

Clearly \( P \) and \( Q \) are non-singular. By assumption \( N(A_0 + A_1j) \subseteq N(C_0 + C_1j) \) and \( N(M_0 + M_1j)/(A_0 + A_1j) \subseteq N(B_0 + B_1j) \) and by using Lemma 1.1 it is obvious that \( M_0 + M_1j \) can be factorized as \( (M_0 + M_1j) = PQL \). Hence \( rk(M_0 + M_1j) = rk(L_0 + L_1j) \) and \( N(M_0 + M_1j) = N(L_0 + L_1j) \).

But \( M_0 + M_1j \) is q-EP, e.g. \( N(M_0 + M_1j)^* = N(M_0 + M_1j) = N(L_0 + L_1j) \). Therefore by using Lemma 1.1 again \( (M_0 + M_1j)^* = (M_0 + M_1j)^*(L_0 + L_1j)^*(L_0 + L_1j) \) holds for every \( (L_0 + L_1j)^* \).

One choice of \( (L_0 + L_1j)^* \) is

\[
(L_0 + L_1j)^* = \begin{pmatrix} A_0 + A_1j^* & 0 \\ 0 & ((M_0 + M_1j)/(A_0 + A_1j))^* \end{pmatrix}, \quad \text{which gives}
\]

\[
(M_0 + M_1j)^* = \begin{pmatrix} (A_0 + A_1j)^* & (C_0 + C_1j)^* \\ (B_0 + B_1j)^* & (D_0 + D_1j)^* \end{pmatrix}
\]

\[
= \begin{pmatrix} (A_0 + A_1j)^* & (C_0 + C_1j)^* \\ (B_0 + B_1j)^* & (D_0 + D_1j)^* \end{pmatrix} \begin{pmatrix} A_0 + A_1j^* & 0 \\ 0 & ((M_0 + M_1j)/(A_0 + A_1j))^*((M_0 + M_1j)/(A_0 + A_1j))^* \end{pmatrix}.
\]
\[(A_0 + A_j)^* = (A_0 + A_j)\] implies \(N(A_0 + A_j) \supseteq N(A_0 + A_j)^*\), and since \(rk(A_0 + A_j)^* = rk(A_0 + A_j)\) these imply \(N(A_0 + A_j) = N(A_0 + A_j)^*\). Hence \(A_0 + A_j\) is q-EP. From \((B_0 + B_j)^* = (B_0 + B_j)(A_0 + A_j)^*(A_0 + A_j)^\top\) it follows that \(N(B_0 + B_j) \supseteq N(A_0 + A_j) = N(A_0 + A_j)^*\).

After substituting \(D_0 + D_j = (M_0 + M_j)/\langle A_0 + A_j \rangle + ((B_0 + B_j)(A_0 + A_j)^*(C_0 + C_j))\) and using

\[(C_0 + C_j)^* = (C_0 + C_j)((M_0 + M_j)/\langle A_0 + A_j \rangle)^*(M_0 + M_j)/\langle A_0 + A_j \rangle)\] in

\[(D_0 + D_j)^* = (D_0 + D_j)^*((M_0 + M_j)/\langle A_0 + A_j \rangle)^*(M_0 + M_j)/\langle A_0 + A_j \rangle)\] we get

\[(M_0 + M_j)/\langle A_0 + A_j \rangle)^* = (M_0 + M_j)/\langle A_0 + A_j \rangle + (A_0 + A_j)^*(M_0 + M_j)/\langle A_0 + A_j \rangle)^*(M_0 + M_j)/\langle A_0 + A_j \rangle)\]

This implies that \(N((M_0 + M_j)/\langle A_0 + A_j \rangle)^*) \supseteq N((M_0 + M_j)/\langle A_0 + A_j \rangle)\) and since

\[rk((M_0 + M_j)/\langle A_0 + A_j \rangle)^* = rk((M_0 + M_j)/\langle A_0 + A_j \rangle)\]

we get \(N((M_0 + M_j)/\langle A_0 + A_j \rangle)^* = N((M_0 + M_j)/\langle A_0 + A_j \rangle)\)

Thus \((M_0 + M_j)/\langle A_0 + A_j \rangle)\) is q-EP. Further

\[N(C_0 + C_j)^* \supseteq N((M_0 + M_j)/\langle A_0 + A_j \rangle) = N((M_0 + M_j)/\langle A_0 + A_j \rangle)^*\]

Hence (ii) holds.

(i) \(\Rightarrow\) (ii). Since \(N(A_0 + A_j) \subseteq N(C_0 + C_j)\) and \(N(A_0 + A_j)^* \subseteq N(B_0 + B_j)^*\).

\[N((M_0 + M_j)/\langle A_0 + A_j \rangle) \subseteq N(B_0 + B_j)\] and \(N((M_0 + M_j)/\langle A_0 + A_j \rangle)^* \subseteq N(C_0 + C_j)^*\) hold according to the assumption. So \((M_0 + M_j)^*\) is given buy the formula

\[
(M_0 + M_j)^* = \left(\begin{array}{c}
(A_0 + A_j)^\top + (A_0 + A_j)^\top (B_0 + B_j)(M_0 + M_j)/\langle A_0 + A_j \rangle)^\top (C_0 + C_j)(A_0 + A_j)^\top
\end{array}
\left(\begin{array}{c}
-(A_0 + A_j)^\top (B_0 + B_j)(M_0 + M_j)/\langle A_0 + A_j \rangle
\end{array}
\left(\begin{array}{c}
-(M_0 + M_j)/\langle A_0 + A_j \rangle
\end{array}
\right)
\right)
\right)
\]

According to lemma 1.1 the assumptions \(N(A_0 + A_j) \subseteq N(C_0 + C_j)\) and

\[N(A_0 + A_j)^* \subseteq N(B_0 + B_j)^*\] imply that \((M_0 + M_j)/\langle A_0 + A_j \rangle)\) is invariant for every choice of \((A_0 + A_j)^\top\). Hence \((M_0 + M_j)/\langle A_0 + A_j \rangle = (D_0 + D_j) - ((C_0 + C_j)(A_0 + A_j)^\top (B_0 + B_j)\).

Further, using \((C_0 + C_j) = ((M_0 + M_j)/\langle A_0 + A_j \rangle)(M_0 + M_j)/\langle A_0 + A_j \rangle)^\top (C_0 + C_j)\) and \((B_0 + B_j) = (A_0 + A_j)(A_0 + A_j)^\top (B_0 + B_j)\), \((M_0 + M_j)(M_0 + M_j)^\top\) is reduced to the form
\[(M_0 + M_j)^{(M_0 + M_j)} = \begin{cases} (A_0 + A_j)(A_0 + A_j)^\dagger & \text{if} \quad ((M_0 + M_j)/(A_0 + A_j))(A_0 + A_j) = ((M_0 + M_j)/(A_0 + A_j))^\dagger \\ 0 & \text{otherwise} \end{cases} \]

The relations \((A_0 + A_j)(A_0 + A_j)^\dagger = (A_0 + A_j)^\dagger (A_0 + A_j)\) and 
\[((M_0 + M_j)/(A_0 + A_j))(A_0 + A_j)^\dagger = ((M_0 + M_j)/(A_0 + A_j))^\dagger ((M_0 + M_j)/(A_0 + A_j))\]
result \((M_0 + M_j)(M_0 + M_j)^\dagger = (M_0 + M_j)^\dagger (M_0 + M_j)\), e.g., \((M_0 + M_j)\) is q-EP.

Thus (i) holds.

(iii) \iff (iiii) By corollary 8 in \(^10\)

\[
\begin{pmatrix} A_0 + A_j & 0 \\ C_0 + C_j & (M_0 + M_j)/(A_0 + A_j) \end{pmatrix}
\]
is q-EP iff \(A_0 + A_j\) and 
\([(M_0 + M_j)/(A_0 + A_j)]\) are q-EP,

Further \(N(A_0 + A_j) \subseteq N(C_0 + C_j)\) and \(N((M_0 + M_j)/(A_0 + A_j))^\dagger \subseteq N(C_0 + C_j)^\dagger\)

\[
\begin{pmatrix} A_0 + A_j & B_0 + B_j \\ 0 & (M_0 + M_j)/(A_0 + A_j) \end{pmatrix}
\]
is q-EP iff \(A\) and \((M_0 + M_j)/(A_0 + A_j)\) are q-EP, further \(N(A_0 + A_j)^\dagger \subseteq N(B_0 + B_j)^\dagger\) and 
\(N((M_0 + M_j)/(A_0 + A_j)) \subseteq N(B_0 + B_j)\). This proves the equivalence of (ii) and (iii).

The proof is complete.

\[
(M_0 + M_j) = \begin{bmatrix} 1 & 1 & 1 & j \\ 1 & 1 & 1 & -j \\ 1 & 1 & 1 & 1 \\ j & 0 & j & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

**Theorem 1.3:**

Let \((M_0 + M_j)\) be a matrix of the form(2) with \(N(A_0 + A_j)^\dagger \subseteq N(B_0 + B_j)^\dagger\) and 
\(N((M_0 + M_j)/(A_0 + A_j))^\dagger \subseteq N(C_0 + C_j)^\dagger\), then the following are equivalent.

i. \((M_0 + M_j)\) is an q-EP matrix

ii. \((A_0 + A_j)\) and \((M_0 + M_j)/(A_0 + A_j)\) are q-EP, further \(N(A_0 + A_j) \subseteq N(C_0 + A_j)\)

and \(N((M_0 + M_j)/(A_0 + A_j)) \subseteq N(B_0 + B_j)\):
iii. Both the matrices
\[
\begin{bmatrix}
(A_0 + A_1) & 0 \\
(C_0 + C_1) & (M_0 + M_1)/j/(A_0 + A_1)
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
(A_0 + A_1) & (B_0 + B_1) \\
(C_0 + C_1) & (M_0 + M_1)/j/(A_0 + A_1)
\end{bmatrix}
\]
are q-EP.

Proof:

Theorem 1.3 follows immediately from theorem 1.2 and from the fact that \((M_0 + M_1)\) is q-EP iff
\((M_0 + M_1)^*\) is q-EP iff \((M_0 + M_1)\) is q-EP.

In this special case when \((B_0 + B_1) = (C_0 + C_1)^*\) we get the following

Corollary 1.4:

Let \((M_0 + M_1) = \begin{bmatrix}
(A_0 + A_1) & (C_0 + C_1)^* \\
(C_0 + C_1) & (D_0 + D_1)
\end{bmatrix}
\)

with \(N(A_0 + A_1) \subseteq N(C_0 + C_1)\) and

\(N((M_0 + M_1)/(A_0 + A_1)) \subseteq (C_0 + C_1)^*\), then the following are equivalent.

i. \((M_0 + M_1)\) is an q-EP matrix

ii. \((A_0 + A_1)\) and \(((M_0 + M_1)/(A_0 + A_1))\) are q-EP matrices.

iii. the matrix \(\begin{bmatrix}
(A_0 + A_1) & 0 \\
(C_0 + C_1) & (M_0 + M_1)/(A_0 + A_1)
\end{bmatrix}\) is q-EP.

Remark 1.5:

The conditions that taken on \(M = M_0 + M_1\) in the previous theorems are essential. This is illustrated in the following example. Let

\[
M = \begin{bmatrix}
1 & 1 & 1 & 1+ j \\
1 & 1 & 1− j & 1 \\
1 & 1 & 1 & 1 \\
1+ j & 1 & 1 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
−1 & 0 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]

\(M\) is symmetric and

\((B_0 + B_1) = (C_0 + C_1) = \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}\)

\(((M_0 + M_1)/(A_0 + A_1)) = (D_0 + D_1) = (C_0 + C_1) (A_0 + A_1)^*(B_0 + B_1) = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}\)
Clearly \((A_0 + A_j)\) and \((M_0 + M_j)/(A_0 + A_j)\) are q-EP, \(N(A_0 + A_j) \subseteq N(C_0 + C_j)\) and 
\(N(A_0 + A_j)^* \subseteq N(B_0 + B_j)^*\), but \(N((M_0 + M_j)/(A_0 + A_j)) \subseteq N(B_0 + B_j)\) and 
\(N((M_0 + M_j)/(A_0 + A_j))^* \not\subseteq N(C_0 + C_j)^*\), further
\[
\begin{pmatrix}
(A_0 + A_j) & 0 \\
(C_0 + C_j) & (M_0 + M_j)/(A_0 + A_j)
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
(A_0 + A_j) & (B_0 + B_j) \\
0 & (M_0 + M_j)/(A_0 + A_j)
\end{pmatrix}
\]
or not q-EP. Thus theorem 1.2 and 1.3 as well as corollary 1.4 fail.

**Remark 1.6:**

We conclude from Theorem 1.2 and Theorem 1.3 that for a q-EP matrix \(M\) of the form equation (2) the following are equivalent
\[
N(A_0 + A_j) \subseteq N(C_0 + C_j), \quad N((M_0 + M_j)/(A_0 + A_j)) \subseteq N(B_0 + B_j)
\quad \text{(4)}
\]
\[
N(A_0 + A_j)^* \subseteq N(B_0 + B_j)^*, \quad N((M_0 + M_j)/(A_0 + A_j))^* \subseteq N(C_0 + C_j)^*
\quad \text{(5)}
\]
However this fails if we omit the condition that \((M_0 + M_j)\) is q-EP. For example Let
\[
(M_0 + M_j) = \begin{bmatrix}
1 & 1 & 1 & j \\
1 & 1 & 1 & -j \\
1 & 1 & 1 & 1 \\
1 & 0 & j & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\((M_0 + M_j)\) is not q-EP. Here
\[
(A_0 + A_j) = \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}, \quad (B_0 + B_j) = (C_0 + C_j)^* = \begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix}
\]
\((A_0 + A_j)\) is q-EP, \(N(A_0 + A_j) \subseteq N(C_0 + C_j)\) and \(N(A_0 + A_j)^* \subseteq N(B_0 + B_j)^*\).

Hence \((M_0 + M_j)/(A_0 + A_j)\) is independent of the choice of \((A_0 + A_j)^-\) and so
\[
((M_0 + M_j)/(A_0 + A_j)) = (D_0 + D_1)/(C_0 + C_1)/(A_0 + A_j)^+ (B_0 + B_j) = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]
\((M_0 + M_j)/(A_0 + A_j)\) is not q-EP, \(N((M_0 + M_j)/(A_0 + A_j))^* \subseteq N(C_0 + C_j)^*\), but 
\(N(A_0 + A_j) \not\subseteq N(B_0 + B_j)\). Thus equation(5) holds, while equation(5) fails.

**Remark 1.7:**

It has been proved is that for any matrix \(A\) its Moore-Penrose inverse, \((M_0 + M_j)^+\) is given by the
formula equation(3) iff both equation (4) and equation (5) holds. However it is clear by the previous remark 1.6 that for an q-EP matrix formula (3) gives $(M_0 + M_1)^{\dagger}$ iff either (4) or (5) holds.

**Theorem 1.8 :**

Let $(M_0 + M_1)$ be of the form equ(2) with \( rk(M_0 + M_1) = rk(A_0 + A_1) = r \). Then $(M_0 + M_1)$ is an q-EP, matrix if and only if $A$ is q-EP, and

\[
(C_0 + C_1)(A_0 + A_1)^{\dagger} = ((A_0 + A_1)^{\dagger}(B_0 + B_1))^*. 
\]

**Proof :**

Since \( rk(M_0 + M_1) = rk(A_0 + A_1) = r \), we have by reason of the corollary of theorem (1) in \(^7\) that,

\[
N(A_0 + A_1) \subseteq N(C_0 + C_1), \quad N(A_0 + A_1)^* \subseteq N(B_0 + B_1)^* \quad \text{and} 
\]

\[
(M_0 + M_1)(A_0 + A_1)^{\dagger} = (D_0 + D_1) - (C_0 + C_1)(A_0 + A_1)^{\dagger}(B_0 + B_1) = 0. 
\]

According to Theorem 1.1 these relation are equivalent $(C_0 + C_1)^{\dagger} = (C_0 + C_1)(A_0 + A_1)^{\dagger}(A_0 + A_1)$.

$(B_0 + B_1) = (A_0 + A_1)(A_0 + A_1)^{\dagger}(B_0 + B_1)$ and $(D_0 + D_1) = (C_0 + C_1)(A_0 + A_1)^{\dagger}(B_0 + B_1)$. Let us consider the matrices

\[
P = \begin{pmatrix} (I_0 + I_1) & 0 \\ (C_0 + C_1)(A_0 + A_1)^{\dagger} & (I_0 + I_1) \end{pmatrix} \quad Q = \begin{pmatrix} (I_0 + I_1) & (A_0 + A_1)^{\dagger}(B_0 + B_1) \\ 0 & (I_0 + I_1) \end{pmatrix},
\]

\[
L = \begin{pmatrix} (A_0 + A_1)^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}. 
\]

$P$ and $Q$ are non-singular and by assumption

\[
(C_0 + C_1)(A_0 + A_1)^{\dagger} = ((A_0 + A_1)^{\dagger}(B_0 + B_1))^* \quad \text{it holds} \quad P = (Q_0 + Q_1)^g. 
\]

Therefore $(M_0 + M_1)$ can be factorized as $M = (P_0 + P_1)(L_0 + L_1)(P_0 + P_1)^g$. Since $(A_0 + A_1)$ is q-EP, consequently $(L_0 + L_1)$ is as well q-EP.

Hence $N(L_0 + L_1) = N(L_0 + L_1)$ and so we have according to Lemma 3 of paper\(^1\) that

\[
N(M_0 + M_1) = N(P_0 + P_1)(L_0 + L_1)(P_0 + P_1)^g = N((P_0 + P_1)(L_0 + L_1)(P_0 + P_1)^g) = N(M_0 + M_1)^g 
\]

This shows that is q-EP.

Conversely, Let us assume that $(M_0 + M_1)$ is q-EP. Since

\[
(M_0 + M_1) = (P_0 + P_1)(L_0 + L_1)(Q_0 + Q_1), \quad \text{one choice of} \quad (A_0 + A_1)^g 
\]

is
\[(M_0 + M_1)^\dagger = (Q_0 + Q_1)^\dagger \begin{pmatrix} (A_0 + A_1)^\dagger & 0 \\ 0 & (P_0 + P_1)^\dagger \end{pmatrix}\]

We know that \(N(M_0 + M_1) = N((M_0 + M_1)^\dagger)\), therefore by Lemma 1.1 \((M_0 + M_1)^\dagger = (M_0 + M_1)^\dagger (M_0 + M_1)^\dagger (M_0 + M_1)\) holds, e.g.

\[(M_0 + M_1)^\dagger = \begin{pmatrix} (A_0 + A_1)^\dagger (C_0 + C_1)^\dagger \\ (B_0 + B_1)^\dagger (D_0 + D_1)^\dagger \end{pmatrix} \begin{pmatrix} (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger (B_0 + B_1)^\dagger \\ 0 \end{pmatrix} \begin{pmatrix} (A_0 + A_1)^\dagger (C_0 + C_1)^\dagger (D_0 + D_1)^\dagger \\ 0 \end{pmatrix} \]

or equivalently, \((A_0 + A_1)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger\) and

\[(C_0 + C_1)^\dagger = (C_0 + C_1)^\dagger (A_0 + A_1)^\dagger (B_0 + B_1)^\dagger, \quad (A_0 + A_1)^\dagger = (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger\]

it follows \(N(A_0 + A_1)^\dagger = N(A_0 + A_1)^\dagger\), i.e., \(A\) is q-EP, and therefore

\[(A_0 + A_1)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger\) taking into account

\[(C_0 + C_1)^\dagger = (C_0 + C_1)^\dagger (A_0 + A_1)^\dagger (B_0 + B_1)^\dagger, \quad we\ have

\[(C_0 + C_1)(A_0 + A_1)^\dagger = (B_0 + B_1)^\dagger ((A_0 + A_1)^\dagger)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger)

\[= (B_0 + B_1)^\dagger ((A_0 + A_1)^\dagger)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger (A_0 + A_1)^\dagger)

\[= (B_0 + B_1)^\dagger ((A_0 + A_1)^\dagger)^\dagger = ((A_0 + A_1)^\dagger (B_0 + B_1)^\dagger)^\dagger\]

The theorem is proved.

**Corollary 1.9:**

Let \((M_0 + M_1)\) of the form (2) with \((A_0 + A_1)^\dagger\) non-singular matrix and

\[rk(M_0 + M_1) = rk(A_0 + A_1)^\dagger\]. Then \(M\) is q-EP if and only if

\[(C_0 + C_1)(A_0 + A_1)^\dagger = ((A_0 + A_1)^\dagger (B_0 + B_1)^\dagger)^\dagger\].

**Corollary 1.10:**

Let \(M = (M_0 + M_1)\) be an \(n \times n\) matrix f rank \(r\). Then \((M_0 + M_1)\) is q-EP, if and only if every principal sub matrix of rank \(r\) is q-EP.
Proof:

Suppose $M=(M_0 + M_1)$ is an q-EP$_r$ matrix. Let $(A_0 + A_j)$ be any principal submatrix of $(M_0 + M_1)$ such that $rk(M_0 + M_1) = rk (A_0 + A_j) = r$. Then there exists a permutation matrix such that

$$
\overline{M} = PMP^T = \begin{pmatrix} (A_0 + A_j) & (B_0 + B_j) \\ (C_0 + C_j) & (D_0 + D_j) \end{pmatrix} \text{ and } rk (A_0 + A_j) = r
$$

According to Lemma (3) in $^1$, $M$ is q-EP$_r$. Now, we conclude from theorem (1.3) that $A$ q-EP$_r$ as well. Since $A$ was arbitrary, it follows that very principal submatrix of rank $r$ is q-EP$_r$. The converse is obvious.

Remark 1.11:

Theorem 1.8 fails if we relax the condition on rank of $M = (M_0 + M_1)$.

2. Application:

We give conditions under which a partitioned matrix is decomposed into complementary summands of q-EP matrices. $M_1$ and $M_2$ are called complementary summand of $(M_0 + M_1)$ if $M = M_1 + M_2$ and $rk(M) = rk(M_1) + rk(M_2)$.

Theorem 2.1

Let $(M_0 + M_1)$ of the form (2) with

$$
k(M_0 + M_1) = rk((M_0 + M_1)/(A_0 + A_j)), \text{ where}
$$

$$
((M_0 + M_1)/(A_0 + A_j)) = (D_0 + D_j) - (C_0 + C_j)(A_0 + A_j)^*(B_0 + B_j). \text{ If } (A_0 + A_j) \text{ and } (C_0 + C_j)(A_0 + A_j)^* \text{ are q-EP matrices such that}
$$

$$
(C_0 + C_j)(A_0 + A_j)^* = ((A_0 + A_j) + (B_0 + B_j))^* \text{ and}
$$

$$
B((M_0 + M_1)/(A_0 + A_j))^* = (((M_0 + M_1)/(A_0 + A_j))^*(C_0 + C_j)^*) \text{ then } (M_0 + M_1) \text{ can be decomposed into complementary summands of q-EP matrices.}
$$

Proof:

Let us consider the matrices

$$
M_1 = \begin{pmatrix} (A_0 + A_j) & (A_0 + A_j)(A_0 + A_j)^*(B_0 + B_j) \\ (C_0 + C_j)(A_0 + A_j)^*(A_0 + A_j) & (C_0 + C_j)(A_0 + A_j)^*(B_0 + B_j) \end{pmatrix}
$$

$$
M_2 = \begin{pmatrix} 0 & ((I_0 + I_j) - (A_0 + A_j)(A_0 + A_j)^*)(B_0 + B_j) \\ (C_0 + C_j)((I_0 + I_j) - (A_0 + A_j)(A_0 + A_j)^*)(A_0 + A_j)) & (M_0 + M_1)/(A_0 + A_j) \end{pmatrix}
$$

Taking into account that $N(A_0 + A_j) \subseteq N((C_0 + C_j)(A_0 + A_j)^*(A_0 + A_j))$,

$$
N((A_0 + A_j)^*) \subseteq N((A_0 + A_j)(A_0 + A_j)^*(B_0 + B_j))^*
$$

and
we obtain by the corollary after Theorem 1 in\textsuperscript{5}, that $\text{rk}(M) = \text{rk}(A_0 + A_1j)$.

Since $(A_0 + A_1j)$ is q-EP and ($(C_0 + C_1j)(A_0 + A_1j)(A_0 + A_1j)^\dagger) = (C_0 + C_1j)(A_0 + A_1j)^\dagger$

$= ((A_0 + A_1j)^\dagger(B_0 + B_1j))^* = ((A_0 + A_1j)^\dagger(A_0 + A_1j)(A_0 + A_1j)^\dagger(B_0 + B_1j))^*$

We have from Theorem 1.8 that $M_1$ is q-EP. Since

$\text{rk}(M) = \text{rk}(A_0 + A_1j) + \text{rk}((M_0 + M_1j)/(A_0 + A_1j))$, Theorem 1. of paper \textsuperscript{5}

gives $N((M_0 + M_1j)/(A_0 + A_1j)) \subseteq \mathbb{N}((I_0 + I_1j) - (A_0 + A_1j)(A_0 + A_1j)^\dagger) = (B_0 + B_1j).

$N((M_0 + M_1j)/(A_0 + A_1j)) \subseteq \mathbb{N}((I_0 + I_1j) - (A_0 + A_1j)^\dagger)(C_0 + C_1j)^*$ and

$((I_0 + I_1j) - (A_0 + A_1j)(A_0 + A_1j)^\dagger)M((M_0 + M_1j)/(A_0 + A_1j)^\dagger)(C_0 + C_1j) = (B_0 + B_1j)(C_0 + C_1j) - (A_0 + A_1j)^\dagger(A_0 + A_1j) = 0

Thus by the corollary of the just applied Theorem 1.1 in\textsuperscript{5}, we have $\text{rk}(M_2) = \text{rk}(M_0 + M_1j)/(A_0 + A_1j))$.

Further, using $(A_0 + A_1j)(A_0 + A_1j)^\dagger = (A_0 + A_1j)^\dagger(A_0 + A_1j)$, we obtain

$((I_0 + I_1j) - (A_0 + A_1j)(A_0 + A_1j)^\dagger)(B_0 + B_1j)(M_0 + M_1j)/(A_0 + A_1j))$

$= ((I_0 + I_1j)(A_0 + A_1j)^\dagger)((M_0 + M_1j)/(A_0 + A_1j)^\dagger)(C_0 + C_1j)^*$

$= (((M_0 + M_1j)/(A_0 + A_1j)^\dagger)(C_0 + C_1j)(I_0 + I_1j) - (A_0 + A_1j)(A_0 + A_1j))^*$

Thus by Theorem 1.8 $M_2$ is also q-EP. Clearly $M = M_1 + M_2$, where both $M_1$ and $M_2$ are q-EP matrices and $\text{rk}(M_0 + M_1j) = \text{rk}(A) + \text{rk}((M_0 + M_1j)/(A_0 + A_1j)) = \text{rk}(M_1) + \text{rk}(M_2)$. Hence $M_1$ and $M_2$ are complementary summands of q-EP matrices.

References

1. T.S. Baskett and I.J. Katz, Theorems on products of EP, matrices, Linear Algebra and its Appl. 2 (1969), 87-103. MR 40: 4280
2. F. Burns, D. Carlson, E. Haynsworth and TH. Markham, Generalized inverse formulas using the Schur Complement, SIAM J. Appl. Math. 26, 254-259. MR 48, 8519 (1974).
3. D. Carlson, E. Haynsworth and TH. Markham, A generalization of the Schur complement by means of the Moore-Penrose inverse, SIAM J. Appl. Math. 26, 169-175. MR 50: 344 (1974).
4. Gunasekaran K. and Sridevi S., On Range Quaternion Hermitian Matrices; Inter; J; Math., Archive-6(8), 159-163 (2015).
5. Gunasekaran. K. and Sridevi S., On Sums of Range Quaternion Hermitian Matrices; Inter; J. Modern Engineering Research -.5, ISSN11. 44 – 49 (2015).
6. Gunasekaran. K. and Sridevi. S., On Product of Range Quaternion Hermitian Matrices; Inter., Journal of pure algebra – 6(6), 318-321 (2016).
7. Gunasekaran. K. and Sridevi. S., Generalized inverses, Group inverses and Revers order law of range quaternion Hermitian matrices., IOSR-JM, Vol.,12, Iss.4 ver.II (Jul – Aug 2016) , 51- 55.
8. Gunasekaran. K. and Sridevi. S., On Bicomplex Representation Methods and Application of Range Quaternion Hermitian Matrices (q-Ep)., IJMTT- Vol. 48, No. 4 August, 250 – 259 (2017).
9. Gunasekaran. K. and Sridevi. S., On Schur Complements in Range Quaternion Hermitian Matrices., IJMAA- Vol. 5, Issue 4-F, 985-991 (2017).
10. M.H. Pearl, On generalized inverses of matrices, Proc. Cambridge Phil. Soc. 62, 673-677. MR 33: 5650 (1966).
11. C.D. Meyer, Generalized inverses of block triangular matrices, SIAM J. Appl. Math. 19, 741- 750. Mr 42: 7676 (1970).
12. R. Penrose, A Generalized inverse for matrices, Proc. Cambridge Philos. Soc. 51, 406-415. MR 16, 1082 (1955).
13. C.R. Rao and S.K. Mitra, Generalized inverse of matrices and its applications, Wiley, New York, MR 49: 2780 (1971).
14. P. Robert, On the group inverse of a linear transformation, J. Math. Anal.Appl. 22, 658-669. MR 37:5232(1968).
15. H. Schwerdtfeger, Introduction to linear algebra and the theory of matrices, Noordhoff, Groningen, 1950. Mr12, 470