On thermodynamic approaches to conformal field theory

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Abstract

We present the thermodynamic Bethe ansatz as a way to factorize the partition function of a 2d field theory, in particular, a conformal field theory and we compare it with another approach to factorization due to K. Schoutens which consists of diagonalizing matrix recursion relations between the partition functions at consecutive levels. We prove that both are equivalent, taking as examples the $SU(2)$ spinons and the 3-state Potts model. In the latter case we see that there are two different thermodynamic Bethe ansatz equation systems with the same physical content, of which the second is new and corresponds to a one-quasiparticle representation, as opposed to the usual two-quasiparticle representation. This new thermodynamic Bethe ansatz system leads to a new dilogarithmic formula for the central charge of that model.

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1 Introduction

The representation of conformal field theories (CFT’s) in terms of quasi-particles is gaining importance, especially in regard to their application in statistical or condensed matter physics [1, 2]. In this representation the Hilbert space is built up from free particle states, in a similar way to the description of the Ising model in terms of free fermions. It is remarkable that the rules for constructing the eigenvalue spectra can generally be interpreted as a realization of Haldane exclusion statistics [3, 4]. Thus this representation is usually called fermionic or quasi-fermionic. The partition function of a particular CFT is easily derived as a combinatorial sum, once the energy is expressed as a sum of single particle levels of quasi-fermionic type.

On the other hand, the powerful methods of CFT based on free field representations provide expressions for the conformal characters and hence for the partition function in the form of bosonic sums [1]. Here one may question whether the quasi-particle representation contributes something, apart from the connections it implies. The answer lies in the fact, sometimes not sufficiently appreciated, that even though the partition function contains all the information on the CFT, this information is usually very hard to extract. The reason is that the expression of the partition function as a combinatorial sum is of little utility to obtain thermodynamic quantities. For this one needs the free energy, that is, the logarithm of the partition function, and its derivatives. In a free theory (take for example the fermionic representation of the Ising model) the summations in the (grand) partition function can be made, obtaining the usual infinite product expression. Its logarithm has a nice expression as an infinite sum, from which one derives the relevant thermodynamic quantities. Consequently, a sensible way to derive the thermodynamics of a CFT is to express the partition function in a factorized form. However, this is seldom achievable exactly. In some cases one can achieve an approximate factorization nevertheless.

It is in this context where the quasi-particle representation shows its utility. These quasi-particles are free except for the constraints embodied by their statistics or, in other words, the rules for the filling of single-particle levels. They turn out to be non trivial, unlike those for bosons or pure fermions, and constitute what has been called statistical interaction. Nevertheless, in the thermodynamic limit the entropy can be expressed as a sum, and the partition function as well, in a way that generalizes the derivation of the Bose or Fermi distributions in the microcanonical ensemble [1, 2]. It turns out that the equations for the occupation numbers $n_i$ are deduced from conditions which adopt the form of thermodynamic Bethe ansatz (TBA) equations for massless particles. This should come as no surprise for the TBA is essentially the same kind of approach: Starting with a system which can be described by the elastic scattering of quasi-particles (Bethe ansatz) the thermodynamic approach [3] consists of finding an expression for their entropy and hence deriving the distribution of quasi-particles.

A new method for factorizing the partition function of a CFT has been proposed

*For free CFT, bosonic or fermionic with periodic or twisted boundary conditions, it is a standard exercise in CFT. It admits a mathematical formulation as the factorization of Jacobi elliptic theta functions as a consequence of the Gauss-Jacobi triple product identity.*
recently by K. Schoutens [7]. He calls it “transfer matrix for truncated chiral spectra” and consists of finding recursion relations between partition functions truncated at some level in the spectrum. Typically, he finds that only a small number of truncated partition functions enter in the recursion relations, which can be encoded in a matrix form. Therefore, the partition function at a generic level is given by the product of the successive recursion matrices. One can then diagonalize the matrix and substitute the partition function by the product of the successive largest eigenvalues, putting it in a factorized form. However, the eigenvalues take rather complicated expressions even in the simpler cases. A useful simplification is to consider the relevant levels sufficiently high and hence to take the limit \( l \to \infty \) while keeping \( l/L \) finite, with \( l \) and \( L \) being the level number and the system size, respectively. This is again the thermodynamic limit. Schoutens’ method is thus in spirit closely related with the TBA. However, he asserts that the distributions he obtains are different from those obtained from TBA equations and says that the relation between the two approaches is not clear. We shall show here that, in addition to being conceptually analogous, these two approaches lead to the same equations. We shall also show that their comparison reveals new TBA systems of equations for known models.

The factorization properties of CFT and their connection with the TBA approach have been considered in [8] from a different point of view. They consider characters which factorize exactly and show that one can take advantage of it to calculate the central charge. They further show that a saddle point approximation of the characters, which is correct in the thermodynamic limit, leads to the TBA equations (see also [9]). However, they make no direct connection between the TBA equations and factorization nor do they attempt to connect with Schoutens’ approach.

## 2 A simple case: \( SU(2)_1 \) and semions

The simplest recursion relation occurs for the trivial case, massless free fermions. The one-fermion energy levels are \( \epsilon_l = \left( l + \frac{1}{2} \right) \frac{\pi}{L} \). Let us denote by \( Z_l(q, z) \), with \( q = e^{-\beta \frac{\pi}{L}} \) and the fugacity \( z = e^{\beta \mu} \), the partition function at level \( l \). The addition of the level \( l + 1 \) provides a new state to place a fermion so

\[
Z_{l+1}(q, z) = Z_l(q, z) + q^{j+\frac{j}{2}} z Z_l(q, z) = (1 + q^{j+\frac{j}{2}} z) Z_l(q, z). \tag{1}
\]

This recursion relation immediately gives the factorized form \( Z_l(q, z) = \prod_{j=0}^{l} (1 + q^{j+\frac{j}{2}} z) \), which is exact. As is well known, the central charge can be computed as the leading finite-size correction to the free energy

\[
-\beta F(\beta, \mu) = \sum_{j=0}^{l} \log \left[ 1 + q^{j+\frac{j}{2}} z \right]
\]

for \( \mu = 0 \), namely,

\[
\frac{\pi}{6 \beta} c = -\beta \frac{F(\beta, 0)}{L} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \ln \left( 1 + e^{-\beta |p|} \right). \tag{2}
\]

\(^1\)We consider the chiral or holomorphic part of the partition function, being understood that the total partition function is a product of the holomorphic part and the antiholomorphic part.
It yields $c = \frac{1}{2}$.

Apart from this trivial case, the simplest case is that of semions, namely, particles with statistical parameter $g = 1/2$ and double maximum occupation per level. They appear as the elementary excitations of a spin chain with long range interactions [10, 11], whose conformal limit is described by the $SU(2)_k$ WZW model. In regard to that connection they are called spinons and exhibit a particular symmetry, the Yangian symmetry, of the WZW model [2, 12]. There are two types of spinons, according to the polarization of their spin, usually denoted with a $+$ or $-$ script. If we consider only one type, $+$ say, we have a system of pure semions. Their TBA equations are a particular case of those of the ideal gas of particles with fractional statistics,

$$w(\zeta)^g [1 + w(\zeta)]^{1-g} = \zeta^{-1} \equiv e^{\beta (\epsilon - \mu)},$$  \hspace{1cm} \text{(3)}

$$n_i = \frac{1}{w(\zeta_i) + g},$$  \hspace{1cm} \text{(4)}

for $g = 1/2$ [4]. They have the simple solution

$$n_i = \frac{1}{\sqrt{1/4 + \exp[2 \beta (\epsilon_i - \mu)]}}.$$  \hspace{1cm} \text{(5)}

Furthermore, the free energy is given by the sum over one-particle levels

$$F = -T \sum_i \log [1 + w(\zeta_i)^{-1}],$$  \hspace{1cm} \text{(6)}

with the corresponding $w(\zeta_i)$.

The spinon partition function is

$$Z(q; z) = \sum_{n^+, n^-} z_{n^+} n^- q^{(n^+ + n^-)^2/4} (q)^{n^+} (q)^{n^-}$$  \hspace{1cm} \text{(7)}

with $(q)_n = \prod_{j=1}^n (1 - q^j)$ [2]. There are two independent characters included in it, namely, the vacuum character and the character corresponding to the $SU(2)$ fundamental representation. They are obtained by restricting the sum to $n^+ + n^-$ even or odd, respectively. One immediately deduces from (7) that only integer powers of $q$ appear in the first one whereas the powers of $q$ in the second one are integer modulo $1/4$. It is convenient to conventionally split every level $l$ into an odd sublevel $l + 1/2$ and an even sublevel $l + 3/2$ [7]. Then one concludes that the vacuum character correspond to a highest occupied odd level whereas the other one correspond to a highest occupied even level. Let us denote them by $Z_0 = \chi^0(q, z^+, z^-)$ and $Z_1 = \chi^1(q, z^+, z^-)$, respectively.

The partition function for $+$ spinons is obtained by setting the fugacity $z^- = 0$ in the total partition function (7). By examining the filling of levels it is possible to find a recursion relation for the truncated even and odd partition functions:

$$\begin{pmatrix} (Z_1)_l \\ (Z_0)_l \end{pmatrix} = \begin{pmatrix} 1 & q^{l-\frac{3}{4}} z^+ \\ q^{l-\frac{3}{4}} z_+ & 1 + q^{2l-1} z_+^2 \end{pmatrix} \begin{pmatrix} (Z_1)_{l-1} \\ (Z_0)_{l-1} \end{pmatrix},$$  \hspace{1cm} \text{(8)}
with initial values \( Z_0 = 1 \) and \( Z_1 = 0 \). The partition function at level \( l \) is given by the product of the successive recursion matrices for \( j = 1, \ldots, l \). Taking after the transfer matrix method, Schoutens then proposes to approximate this exact expression by the product of the largest eigenvalues of the matrices. One can easily calculate the eigenvalues of the recursion matrix

\[
\begin{vmatrix}
1 - \lambda_j & q^{j-\frac{3}{2}} z_+ \\
q^{j-\frac{3}{2}} z_+ & 1 + q^{2j-1} z_+^2 - \lambda_j
\end{vmatrix} = 1 + \left( -2 - q^{-1+2j} z_+^2 \right) \lambda_j + \lambda_j^2 = 0. \tag{9}
\]

Under the substitution \( \lambda \to 1 + w^{-1} \)

\[
1 - q^{2j-1} z_+^2 w (1 + w) = 0. \tag{10}
\]

It coincides with (3) for \( g = 1/2 \) if we take \( \epsilon_j = (j + \frac{1}{2}) \frac{\pi}{L} \). Notice that these energies exactly coincide with one-fermion levels.

The factorized partition function given by the product of the largest eigenvalues

\[
\lambda_l = 1 + \frac{q^{-1+2l} z_+^2}{2} + \frac{q^{-1+l} z_+ \sqrt{4q + q^{2l} z_+^2}}{2}
\]

also yields a free energy identical with (3):

\[
\beta F = - \log \left[ \prod_j \lambda_j \right] = \sum_j \log \left[ 1 + w_j^{-1} \right]. \tag{11}
\]

The central charge is

\[
c = \frac{6}{\pi^2} \int_0^1 d\zeta \log \left[ \frac{2 + \sqrt{\zeta^2 + \zeta \sqrt{4 + \zeta^2}}}{2} \right] = \frac{6}{\pi^2} L \left( \frac{\sqrt{5} - 1}{2} \right) = \frac{3}{5}, \tag{12}
\]

with \( L(x) \) being the Rogers dilogarithm function [13].

The unpolarized case leads to a more complicated recursion relation

\[
\begin{pmatrix}
(Z_1)_l \\
(Z_0)_l
\end{pmatrix} = \begin{pmatrix}
1 - q^{2l-2} z_+ z_- & q^{l-\frac{3}{2}} (z_+ + z_-) \\
q^{l-\frac{3}{2}} (z_+ + z_-) (1 - q^{2l-2} z_+ z_-) & 1 + q^{2l-1} (z_+^2 + z_+ z_- + z_-^2)
\end{pmatrix} \begin{pmatrix}
(Z_1)_{l-1} \\
(Z_0)_{l-1}
\end{pmatrix}
\]

and hence to a more complicated equation

\[
1 - q^{-2+2l} z_+ z_- - q^{-1+2l} z_+ z_- + q^{-3+4l} z_+^2 z_- - 2 + q^{-1+2l} z_+^2 - q^{-2+2l} z_+ z_- + q^{-1+2l} z_+ z_- + q^{-1+2l} z_-^2 \right) \lambda + \lambda^2 = 0. \tag{14}
\]

We would like to identify this equation with some sort of TBA equation. Before delving into this question, let us notice that the solutions of Eq. (14) are quite unwieldy. However, when the chemical potentials of both polarizations are equal, the system behaves as two independent fermions with total central charge \( c = 1 \), as corresponds to the \( SU(2)_1 \) WZW model [7]. Nevertheless, there is no simplification in the solutions of (14) unless we take the thermodynamic limit \( l \gg 1 \). Then (14) adopts the form

\[
\lambda^2 - \lambda \left( 2 + \zeta_+^2 + \zeta_-^2 \right) + (1 - \zeta_+ \zeta_-)^2 = 0, \tag{15}
\]
with $\zeta_\pm = q^\ell z_{\pm}$. In this limit, the solutions drastically simplify for $z_\pm = z_-$, yielding

$$
\lambda = (1 \pm \zeta)^2.
$$

(16)

The larger solution (with the plus sign) leads to

$$
\beta F = -\log \left( \prod_j \lambda_j \right) = -2 \sum_j \log \left( 1 + q^j z \right),
$$

(17)

corresponding to two free fermions.

The TBA equations for unpolarized spinons can be written once the statistical matrix $g_{ij}$ is known [4]. This matrix can be determined from the expression of the exponent of $q$ in the partition function (7) to be

$$
G = \left( \begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{array} \right).
$$

We obtain the TBA equations

$$
\frac{(\lambda_+ - 1) \sqrt{\lambda_-}}{\sqrt{\lambda_+}} = \zeta_+,
$$

(18)

$$
\frac{(\lambda_- - 1) \sqrt{\lambda_+}}{\sqrt{\lambda_-}} = \zeta_-,
$$

(19)

with the definition $\lambda_a = 1 + w_a^{-1}$ for $a = +, -$. Let us remark that $\lambda_a$ are not eigenvalues and are not related to any recursion matrix. The free energy is given by the sum over polarizations and over one-particle levels for each polarization

$$
\beta F = -\sum_{a,j} \log \left[ 1 + w_a (\zeta_j)^{-1} \right] = -\sum_j \log \left[ \lambda_+ \lambda_- \right]_j.
$$

(20)

Therefore, we are actually interested in the product $\lambda_+ \lambda_-$ of the solutions of the TBA equations (19) for given $\zeta_+$ and $\zeta_-$ rather than in their independent value. We can add to these two equations a third one, $\lambda = \lambda_+ \lambda_-$ and eliminate the intermediate variables $\lambda_+ \lambda_-$ to get an equation for $\lambda$. It turns out to be precisely the eigenvalue equation (15). Furthermore, the TBA free energy (20) corresponds to the factorized partition function being the product of eigenvalues $\prod_j \lambda_j(\zeta_+, \zeta_-)$.

For $\mu_+ \neq \mu_-$ there is an interesting physical quantity which can be read out from the free energy, namely, the total magnetization, that is, the difference between the total number of up and down spinons $N_+ - N_-$ [7]:

$$
N_+ - N_- = -\frac{\partial F}{\partial \mu_+} + \frac{\partial F}{\partial \mu_-} = \int_0^1 \frac{d\zeta}{\zeta} \frac{2 \zeta (z - z^{-1})}{\sqrt{4 + \zeta^2 (z - z^{-1})^2}}
$$

$$
= 2 \text{ArcSinh} \left( \frac{z - z^{-1}}{2} \right) = 2 \log z = \beta (\mu_+ - \mu_-),
$$

(21)

where we have introduced the variables

$$
z = \sqrt{\frac{\zeta_+/\zeta_-} = e^{\beta (\mu_+ - \mu_-)/2}}$$

6
and
\[ \zeta = \sqrt{\zeta_+ \zeta_-} = e^{-\beta \left[ \tau - (\mu_+ + \mu_-)/2 \right]} . \]

Eq. (21) expresses that the magnetization is proportional to the external magnetic field.

3 The three-state Potts model and \( z_n \) parafermions

The trivial case of massless real free fermions corresponds to the CFT of the critical Ising model. The natural (and oldest) generalization of this model consists of taking a site variable which can take three values instead of two, constituting the three-state Potts model. This model has been long known to be integrable and it has been long (but not as long) known to be describable in terms of particles with fractional statistics, which are generalizations of the Ising fermions and are called parafermions. A quasi-fermionic expression for the characters of the corresponding CFT is given by the Lepowski-Primc branching functions \([1, 14]\). This quasi-fermionic representation in terms of two conjugate parafermions carrying \( \mathbb{Z}_3 \) charges +1 or −1 is not the only one possible \([15]\) but it is the correct one to describe the three-state Potts model at a non-critical temperature as the perturbation produced by giving mass to the parafermions \([16, 17]\).

The TBA equations for the three-state Potts model were obtained in \([17]\). They can also be obtained from the Lepowski-Primc branching function representation of the characters,

\[
q^{1/30} b_2^{Q-1}(q) = q^{(d-1)/30} \sum_{m_1, m_2=0}^{\infty} \frac{q^{m C^{-1}_{A_2} m + L_i(m)}}{(q)_{m_1}(q)_{m_2}},
\]

where
\[ C_{A_2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]
is the Cartan matrix of the Lie algebra \( A_2 \), and \( L_0(m) = 0, L_1(m) = (2 m_1 + m_2)/3, L_2(m) = (m_1 + 2 m_2)/3 \). It leads to the statistical matrix
\[ G = 2 C_{A_2}^{-1} = \begin{pmatrix} 1 & -1/2 & 1/2 \\ 1/3 & 1/2 & 1/3 \end{pmatrix} . \]

Hence, the TBA equations are:

\[
(\lambda_1 - 1) \lambda_1^{1/3} \lambda_1^{2/3} = \zeta_1 , \tag{23}
\]

\[
(\lambda_1 - 1) \lambda_1^{2/3} \lambda_1^{1/3} = \zeta_1 . \tag{24}
\]

Unlike the TBA equations in the previous section, these equations are very complicated. Their algebraic solution can only be obtained with the help of algebraic computing programs, such as Mathematica, and it is huge: It occupies about 600 Kbytes of memory, that is, about 300 written pages! Moreover, it took 117847 seconds of CPU
time to obtain it with a modern Pentium computer. Therefore, it is sensible to examine first particular cases.

The simplest particular case occurs for equal fugacities of positive and negative particles, \( z_1 = z_\bar{1} \). Then it is like if the two equations are the same and we actually have only one

\[
(\lambda - 1) \lambda = \zeta. \tag{25}
\]

The solution of this equation is elementary

\[
\lambda = \frac{1 \pm \sqrt{1 + 4 \zeta}}{2}. \tag{26}
\]

The central charge is

\[
c = 2 \frac{6}{\pi^2} \int_0^1 \frac{d\zeta}{\zeta} \log \left( \frac{1 + \sqrt{1 + 4 \zeta}}{2} \right) = 2 \frac{6}{\pi^2} L \left( \frac{3 - \sqrt{5}}{2} \right) = \frac{4}{5}. \tag{27}
\]

Before proceeding to the other important particular case, let us remark that the algebraic expression obtained for the general solution is singular for \( \zeta_1 = \zeta_\bar{1} \). The limit \( \zeta_1 \to \zeta_\bar{1} \) coincides with (26) if performed correctly, for the function is continuous there.

The other obvious singularity of the expression for the general solution occurs for \( \zeta_\bar{1} = \zeta_1^2 \). The limit can again be performed and coincides with the solution of the equations (23) and (24) for \( \zeta_1 = \zeta_\bar{1}^2 \). This solution is still quite hefty but far from the general solution: It takes about two pages. We may wonder to what physical situation this solution applies. According to the definition of \( \zeta_1 \), it corresponds to having \( \mu_\bar{1} = 2 \mu_1 \) and \( \epsilon_\bar{1} = 2 \epsilon_1 \). The first condition has a simple interpretation: The levels are filled asymmetrically for particles 1 and \( \bar{1} \), with an average number of the latter double than of the former. The second condition seems to imply an asymmetry in the one-particle levels, with those for particle \( \bar{1} \) having double energy than those of particle 1. This would break the symmetry between both types of particles. However, there is an interpretation that encompasses both conditions and does not break this symmetry: The one-particle levels can still be the same if every particle \( \bar{1} \) is considered as a composite of two particles 1. This interpretation actually leads to a different representation of the model in terms of only one type of particle which is interesting on its own. Moreover, it is in this representation that the recursion matrix has been obtained [4]. It reads

\[
R = \begin{pmatrix}
(1 - \zeta^3) & \zeta^2 & \zeta \\
\zeta (1 - \zeta^3) & 1 & 2 \zeta^2 \\
2 \zeta^2 (1 - \zeta^3) & \zeta (1 + \zeta^3) & 1 + 2 \zeta^3
\end{pmatrix}. \tag{28}
\]

Its eigenvalue equation is

\[
-(1 - \zeta^3)^3 + (3 - 3 \zeta^3 - \zeta^6) \lambda - (3 + \zeta^3) \lambda^2 + \lambda^3 = 0. \tag{29}
\]

\[\text{Something similar happens for spinons described by the TBA equations (19). In this case the single equation for } \zeta_+ = \zeta_- \text{ is trivial and immediately leads to the solution (16).}\]
The solution of this equation is straightforward with the use of the Cardano formula for cubic equations.

At this point, we would like to relate the restricted TBA equations for the representation by one type of particles,

\begin{equation}
(\lambda_1 - 1) \lambda_1^{1/3} \lambda_1^{2/3} = \zeta, \tag{30}
\end{equation}

\begin{equation}
(\lambda_1 - 1) \lambda_1^{2/3} \lambda_1^{1/3} = \zeta^2, \tag{31}
\end{equation}

with the eigenvalue equation. Since in the TBA equations (30) and (31) we still have two variables, \(\lambda_1\) and \(\lambda_{\bar{1}}\), the first step must be to reduce them to only one variable that represents their combined contribution to the free energy, like we did for unpolarized spinons. We have now that

\begin{equation}
- \beta F = \sum_j \log \left[ 1 + w_1(\zeta_j)^{-1} \right] + 2 \sum_j \log \left[ 1 + w_1(\zeta_j)^{-1} \right] = \sum_j \log \left[ \lambda_1 \lambda_{\bar{1}}^2 \right]_j, \tag{32}
\end{equation}

since the particle of type \(\bar{1}\) is a composite of two particles of type 1 and contributes double. Then we define a variable \(\lambda = \lambda_1 \lambda_{\bar{1}}^2\) to express the partition function in the factorized form \(\prod_j \lambda(\zeta_j)\), corresponding to only one particle type. Now we can similarly reduce the two TBA equations to one equation for \(\lambda\). It coincides with the eigenvalue equation (29).

The central charge \(c = 4/5\) can also be computed with the help of Eq. (29) but it involves the integral of its largest solution and has to be made numerically. However, the central charge can be evaluated exactly in this representation from the partition function (32) and the TBA system (30, 31) in terms of the Rogers dilogarithm. Unlike the expression in the two particle representation (27) the calculation and the final expression are very complicated and are left for an appendix.

We can try to generalize the results above for the three-state Potts model to \(\mathbb{Z}_n\) parafermions. For example, the TBA equations in the one-particle representation of the \(\mathbb{Z}_4\) CFT turn out to be

\begin{equation}
(\lambda_1 - 1) \sqrt[3]{\lambda_1 \lambda_2} \sqrt[3]{\lambda_3} = \zeta, \tag{33}
\end{equation}

\begin{equation}
(\lambda_2 - 1) \lambda_1 \lambda_2 \lambda_3 = \zeta^2, \tag{34}
\end{equation}

\begin{equation}
(\lambda_3 - 1) \sqrt[3]{\lambda_1 \lambda_2} \sqrt[3]{\lambda_3} = \zeta^3, \tag{35}
\end{equation}

and the free energy

\begin{equation}
- \beta F = \sum_j \log \left[ \lambda_1 \lambda_2^2 \lambda_3^3 \right]_j. \tag{36}
\end{equation}

However, the algebraic calculation of the solution of these TBA equations is exceedingly difficult, even with algebraic computing programs. Nonetheless, one can solve them numerically. With a calculation of the solution in the interval \(\zeta \in [0, 1]\) at points separated by 0.1 we have obtained the value \(c = 1.00008\) for the central charge, sufficiently accurate.
4 Discussion

We have established the equivalence of the TBA approach and Schoutens’ recursion matrix method in particular but representative examples. Since we have shown that both methods essentially consist of an approximate factorization of the partition function in the thermodynamic limit, we deem reasonable to conjecture that they are equivalent in general. However, in the examples in hand they seem to operate in slightly different ways. Both start with an exact expression of the partition function—say the CFT characters—in the form of an infinite sum but take different routes: The TBA approach uses the fractional statistics properties entailed by that expression, which directly lead to TBA equations and an approximate factorization of it. Schoutens’ method demands first to find recursion matrix relations. In the examples studied here it has been done by inspection and, in our opinion, there is probably no systematic way to find recursion relations. Consequently, Schoutens’ method appears more difficult to implement in practice.

It is noteworthy that the problem of factorization of a CFT partition function can have several solutions with different physical interpretation. We have shown that for $\mathbb{Z}_n$ parafermions there are two possible quasiparticle representations and two corresponding factorizations: The first is based on a set of $n-1$ fundamental quasi-particles whereas the second is based on only one, which generates the remaining $n-2$ as composites of two, three, etc., of it. The latter representation leads to a much more difficult TBA equation system and to new (and more difficult as well) dilogarithmic formulas for their central charges.

A Calculation of the three-state Potts model central charge in the one-particle representation

The central charge is $c = \frac{6}{\pi^2} (I_1 + 2I_2)$ with

$$I_1 = \int_0^1 \frac{d\zeta}{\zeta} \log(\lambda_1), \quad (37)$$

$$I_2 = \int_0^1 \frac{d\zeta}{\zeta} \log(\lambda_1). \quad (38)$$

To calculate $I_1$ we must first express $\zeta$ as function of only $\lambda_1$ eliminating $\lambda_2$ in the TBA system (30,31). We obtain

$$(1 - \lambda_1)^3 + \zeta^3 (2 - \lambda_1)^2 \lambda_1 = 0, \quad (39)$$

and hence

$$\frac{d\zeta}{\zeta} = \left( \frac{1}{\lambda_1 - 1} - \frac{1}{3 \lambda_1} - \frac{2}{3 (\lambda_1 - 2)} \right) d\lambda_1. \quad (40)$$

Then

$$I_1 = \int_0^\rho \frac{dx}{x} \log(1 + x) - \frac{1}{3} \int_1^{1+\rho} \frac{d\lambda_1}{\lambda_1} \log(\lambda_1) - \frac{2}{3} \int_0^\rho \frac{dx}{x - 1} \log(1 + x), \quad (41)$$
where the change of integration variable $\lambda_1 = 1 + x$ has been used for the non-trivial first and third integrals. The integration limits are obtained by solving Eq. (39) and are expressed in terms of $\rho = \frac{-1 + \sqrt{5}}{2}$.

The non-trivial integrals are of the type expressible as the Euler dilogarithm function:

$$\int_0^\rho \frac{dx}{x} \log(1 + x) = -Li_2(-\rho),$$

and, with the change of variable $x = 2u - 1$,

$$\int_0^\rho \frac{dx}{x - 1} \log(1 + x) = -\frac{\pi^2}{12} + \frac{\log(2)^2}{2} + \log\left(\frac{1 - \rho}{2}\right) \log(1 + \rho) + Li_2\left(2, \frac{1 + \rho}{2}\right).$$

So

$$I_1 = -Li_2(-\rho) - \frac{1}{3} \frac{\log(1 + \rho)^2}{2} + \frac{\pi^2}{18} - \frac{\log(2)^2}{3} - \frac{2}{3} \log\left(\frac{1 - \rho}{2}\right) \log(1 + \rho) - \frac{2}{3} Li_2\left(2, \frac{1 + \rho}{2}\right).$$

We proceed analogously to calculate $I_2$. From the TBA system we get

$$\zeta^6 + (1 - \lambda_1)^3 - 2 \zeta^3 (-1 + \lambda_1)^2 = 0.$$  

(43)

Solving for $\zeta^3$

$$\zeta^3 = (-1 + \lambda_2) \left( -1 \pm \sqrt{-1 + \lambda_2 \sqrt{\lambda_2 + \lambda_2}} \right)$$

and selecting the positive solution we obtain

$$\frac{d\zeta}{\zeta} = \frac{d\lambda_2}{3 (\lambda_2 - 1)} + \frac{du}{3u},$$

(44)

where we have introduced the new variable $u = -1 + \sqrt{-1 + \lambda_2 \sqrt{\lambda_2 + \lambda_2}}$ or

$$\lambda_1 = \frac{(1 + u)^2}{1 + 2u}.$$

Then

$$I_2 = \frac{1}{3} \int_0^\rho \frac{dx}{x} \log(1 + x) + \frac{2}{3} \int_0^{\rho+1} \frac{du}{u} \log(1 + u) - \frac{1}{3} \int_0^{\rho+1} \frac{du}{u} \log(1 + 2u).$$

(45)

For the third integral we use the change of variable $v = 2u$ and it becomes

$$\int_0^{2(\rho+1)} \frac{dv}{v} \log(1 + v).$$

Now we have two integrals equal to $Li_2(z)$ for $|z| > 1$. We can transform them into standard dilogarithms with $|z| < 1$ using the change of variable $u = \frac{y}{1 - y}$ and

$$\int_0^z \frac{du}{u} \log(1 + u) = -\int_0^{1+y} \frac{dy}{y (1 - y)} \log(1 - y).$$

(46)
They yield
\[ \int_0^{\rho+1} \frac{du}{u} \log(1 + u) = 2 \log(\rho)^2 + Li_2(\rho) \]
and
\[ \int_0^{2(\rho+1)} \frac{dv}{v} \log(1 + v) = \frac{9 \log(\rho)^2}{2} + Li_2(2 \rho^2) \]
after using the identity \( \rho + 1 = 1/\rho \) a number of times. Hence
\[ I_2 = \frac{-\log(\rho)^2}{6} - \frac{Li_2(-\rho)}{3} + \frac{2 Li_2(\rho)}{3} - \frac{Li_2(2 \rho^2)}{3}. \]  
(47)

The contributions to the central charge from \( \lambda_1 \) and \( \bar{\lambda}_1 \),
\[ \frac{6}{\pi^2} I_1 = 0.421993347092241, \]  
(48)
\[ \frac{12}{\pi^2} I_2 = 0.378006652907758, \]  
(49)
are transcendental numbers but their sum is
\[ c = \frac{6}{\pi^2}(I_1 + 2 I_2) = \frac{4}{5}, \]  
(50)
as expected. This can be proved as follows. In
\[ I_1 + 2 I_2 = \frac{\pi^2}{18} - \frac{\log(2)^2}{3} - \frac{\log(\rho)^2}{3} - \frac{2 \log(1-\rho)}{3} \log(1 + \rho) - \frac{\log(1 + \rho)^2}{6} \]
\[ - \frac{5 Li_2(-\rho)}{3} + \frac{4 Li_2(\rho)}{3} - \frac{2 Li_2(2 \rho^2)}{3} - \frac{2 Li_2(1+\rho)}{3}. \]  
(51)
we can use the Euler identity
\[ Li_2(-x) = \frac{Li_2(x^2)}{2} - Li_2(x) \]
to get
\[ I_1 + 2 I_2 = \frac{\pi^2}{18} - \frac{\log(2)^2}{3} - \frac{\log(\rho)^2}{3} - \frac{2 \log(1-\rho)}{3} \log(1 + \rho) - \frac{\log(1 + \rho)^2}{6} \]
\[ + \frac{3 Li_2(\rho)}{6} - \frac{5 Li_2(\rho^2)}{6} - \frac{2 Li_2(2 \rho^2)}{3} - \frac{2 Li_2(1+\rho)}{3}. \]  
(52)
Now it is convenient to express it in terms of Rogers dilogarithms
\[ L(x) = Li_2(x) + \frac{1}{2} \log(x) \log(1-x), \]
\[ I_1 + 2 I_2 = \frac{\pi^2}{18} - \frac{\log(2)^2}{3} - \frac{3 \log(1-\rho)}{2} \log(\rho) - \frac{\log(\rho)^2}{3} + \frac{\log(1-\rho)}{3} \log(1-2 \rho^2) \]
\[ - \frac{2 \log(1-\rho)}{3} \log(1 + \rho) - \frac{\log(1 + \rho)^2}{6} + \frac{\log(2 \rho^2)}{3} \log(1-2 \rho^2) \]
\[ + \frac{5 \log(\rho^2)}{12} \log(1-\rho^2) + \frac{3 L(\rho)}{6} - \frac{5 L(\rho^2)}{6} - \frac{2 L(2 \rho^2)}{3} - \frac{2 L(1+\rho)}{3}. \]  
(53)
The Rogers dilogarithms with arguments $\rho$ and $\rho^2$ can be easily calculated \[13\]. For the others, with the help of \[13\]

$$L(x) + L(1-x) = L(1) = \frac{\pi^2}{6},$$

and the identities $1 - 2\rho^2 = \rho^3$ and $1 - \frac{1+\rho}{2} = \frac{\pi^2}{6}$, we obtain

$$L(2\rho^2) + L\left(\frac{1+\rho}{2}\right) = 2L(1) - \left[L(\rho^3) + L\left(\frac{\rho^2}{2}\right)\right]. \quad (54)$$

The last bracket can be reduced to computable Rogers dilogarithms \[18\]:

$$L(\rho^3) + L\left(\frac{\rho^2}{2}\right) = L(\rho) + L(\rho^2) - L\left(\frac{1}{2}\right). \quad (55)$$

It is another exercise in algebra to check that the contribution of the logarithmic terms in \[53\] vanishes. Finally,

$$I_1 + 2I_2 = \frac{-\pi^2}{6} - \frac{2L(\frac{1}{2})}{3} + \frac{11L(\rho)}{3} - \frac{L(\rho^2)}{6}$$

$$= \frac{-\pi^2}{6} - \frac{2\pi^2}{3} + \frac{11\pi^2}{12} - \frac{1\pi^2}{6} + \frac{4\pi^2}{15} = \frac{4\pi^2}{15}. \quad (56)$$

using the known values \[13\]

$$L\left(\frac{1}{2}\right) = \frac{\pi^2}{12}, \quad L(\rho) = \frac{\pi^2}{10}, \quad L(\rho^2) = \frac{\pi^2}{15}.$$

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