Entanglement entropy in long-range harmonic oscillators

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Abstract – We study the Von Neumann and Rényi entanglement entropy of long-range harmonic oscillators (LRHO) by both theoretical and numerical means. We show that the entanglement entropy in massless harmonic oscillators increases logarithmically with the sub-system size as $S = c \log l$. Although the entanglement entropy of LRHO’s shares some similarities with the entanglement entropy at conformal critical points we show that the Rényi entanglement entropy presents some deviations from the expected conformal behaviour. In the massive case we demonstrate that the behaviour of the entanglement entropy with respect to the correlation length is also logarithmic as the short range case.

Introduction. – While the entanglement entropy of short-range interacting quantum systems in one dimension is well studied with many different techniques [1], such as hamiltonian techniques [2–5], euclidean methods [6,7] and conformal field theory [8,9] there are few results concerning the long-range interacting systems. The main difficulty is the lack of exact solution for these systems which makes the problem much harder than the short-range counterparts. Because of the presence of non-trivial dynamical exponents the common euclidean techniques are usually useless in calculating the entanglement entropy in these systems. It is worth mentioning that although having a non-trivial dynamical exponents is not restricted to just long-range interacting systems, see for example [10,11], they usually provide a very natural knob to change it arbitrarily. Because of the huge finite size effects in these systems even the numerical calculations are not easy as their short-range counterparts. Nevertheless, recently much progress has been made in different directions: In [12] the entanglement entropy in Lipkin-Meshkov-Glick (LMG) model is studied. This model has hamiltonian similar to the XY model but while in the latter model the interaction only takes place between nearest neighbors, in the LMG model, all spins interact among themselves. In [13], the interactions are restricted to the Ising-type without external magnetic field which allows to study both the static and the dynamical entanglement properties of the system. Eisert et. all [14] found a logarithmically divergent geometric entropy in free fermions with long-range unshielded Coulomb interaction. Plenio et. all [15] studied the general properties of the entanglement entropy for interacting harmonic oscillators. In particular they found that the area law should be valid in higher dimensions even for long-range interacting harmonic oscillators, and most recently the entanglement entropy for the long range anti-ferromagnetic Ising chain is calculated by numerical means [16]. Finally we should mention that in [17] based on the matrix product states it was argued that for those long-range systems that do not have any short-range counterparts, in other words one can not approximate the ground state of the long-range model with the ground state of another short-range model, the presence of the long-range interaction implies larger entanglement entropy or the volume scaling of the entropy.

Harmonic oscillators are the building blocks of many quantum mechanical and field theoretical systems and of course this is also true for most of the long-range interacting quantum systems and non-local field theories. In this work we study the entanglement entropy of long-range harmonic oscillators in the massless and massive cases. The main approach we have used is the so called hamiltonian technique, which was first introduced in [2] and then elaborated in [7], for review see [5,6]. The important advantage of this method with respect to the others in calculating the entanglement entropy in our problem is that, firstly it is possible, to a large extent, to carry out
the calculations exactly, and on top of that the numerical calculations are very easy to implement. We will consider the lattice and continuum calculations in parallel. We first introduce the method and then apply it directly to our problem. In the analytical side we will write the main eigenvalue problem exactly, then we will try to find approximate solutions by using our numerical calculations. Using the solutions of the eigenvalue problem we then also calculate the Rényi entanglement entropy and finally we will solve numerically the problem of the massive interacting long-range harmonic oscillators.

**Results.**—The entanglement entropy of a subsystem $A$ is defined by using reduced density matrix $\rho_A$ as

$$S_A = -\text{tr} \rho_A \log \rho_A.$$  \hfill (1)

In this work we consider three main kinds of subsystems. In the massless case we consider two kinds of systems: in the first one the system is large and $A$ is a small sub-system with length $l$ and in the second case the system is finite with length $L$ and the sub-system is just half of it with length $L/2$. In the massive case the system is large and the sub-system is just half of the system. We will provide approximate formula just for the first case and study the two other cases numerically.

We define the hamiltonian of one dimensional interacting harmonic oscillators as

$$H = \frac{1}{2} \sum_{i=1}^{N} \phi_i^2 + \frac{1}{2} \sum_{i,j=1}^{N} \phi_i K_{ij} \phi_j,$$  \hfill (2)

where on the lattice we take

$$K_{i,j} = \int_0^{2\pi} \frac{dq}{2\pi} e^{iq(i-j)} \{- [2(1 - \cos(q))]^{\alpha/2} + m^\alpha \}$$

$$= \frac{\Gamma(-\frac{\alpha}{2} + i - j) \Gamma(\alpha + 1)}{\pi \Gamma(1 + \frac{\alpha}{2} + i - j)} \sin(\frac{\alpha}{2} \pi) + m^\alpha \delta_{i,j}. \hfill (3)$$

In the special case when $\alpha/2$ is an integer, $K(n) = (-1)^{\alpha-n+1} C_{\alpha,\frac{\alpha}{2}+n}$, where $C_{\alpha,\frac{\alpha}{2}+n}$ are binomial coefficients. In this case we remark that $K(n) = 0$ for $n > \alpha/2$.

It is easy to see that in the special case $\alpha = 2$ the $K$ matrix is just a simple laplacian. For non-integer values for large distances we have $K(n) \sim \frac{1}{n^\alpha}$.

In Fig 1 we depicted the phase diagram of the model which is gapless for $m = 0$ and gapped for non-zero value of $m$ for any value of $\alpha$.

In the continuum limit the above hamiltonian can be written as

$$\frac{1}{2} \sum_{i,j=1}^{N} \phi_i K_{ij} \phi_j \rightarrow \int \left[ \frac{1}{2} \phi(x)(-\nabla)^{\alpha/2} \phi(x) + \frac{1}{2} m^\alpha \phi^2 \right] dx,$$ \hfill (4)

where $-(\nabla)^{\alpha/2}$ is the fractional laplacian defined by its Fourier transform $|q|^\alpha$.

To measure the entanglement entropy of a sub-system with length $l$ of an infinite system we follow the method explained in [7]. First we make the matrices $W = K^{1/2}$ and $W^{-1} = K^{-1/2}$. In the continuum limit they have the following forms

$$W^{\pm 1}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk |k|^{\alpha} + m^\alpha \pm 1/2 \delta(x-y)$$

$$= \frac{1}{2\Gamma(\alpha/2) \cos(\frac{\alpha}{2}\pi)} \frac{1}{(x-y)^{1+\alpha/2}} + O(m^\alpha).$$ \hfill (5)

Then we define the matrix $\Lambda$ by multiplying $W$ and $W^{-1}$ in the complement of the sub-region $l$

$$\Lambda(x, y) = \frac{1}{\Omega} \int dz \left( W^{-1}(x, z) W(z, y) \right)$$

$$= \frac{1}{\Omega} \left[ \frac{1}{(x-y)^{1+\alpha/2}} \right], \hfill (6)$$

where $\Omega \in (-\infty < z < 0) \cup (l < z < \infty)$ and $\Lambda = \frac{1}{\Omega} \frac{\Gamma(1 + \frac{\alpha}{2})}{|\alpha/2| \cos(\frac{\alpha}{2}\pi)}$. For $\alpha = 2$ the matrix has a different form $\pi^2 \Lambda(x, y) = \frac{1}{(x-y)^{2+\alpha/2}} = \frac{1}{(x-y)^2} \log((l-x)-y) \log((l-y)-x)$.
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\[ \frac{\log(\alpha - 2) \log(\alpha)}{\log(1 + 2)} \]

We compared the matrix \( \Lambda(x, y) \) coming from the above equation with numerical \( \Lambda \) and found a very good agreement when the distances are more than four lattice sizes. The agreement gets better by increasing the size of the system.

The entanglement entropy can be expressed directly in terms of the eigenvalues \( E_i \) of the matrix \( \Lambda \) as [2,7]

\[ S = \sum_{E_i} \left[ \log \left( \frac{\sqrt{E_i}}{2} + \sqrt{1 + E_i} \log \left( \frac{1}{\sqrt{E_i}} + \sqrt{1 + \frac{1}{E_i}} \right) \right) \right]. \quad (7) \]

In the continuum limit one needs to solve the following eigenvalue problem

\[ \int dy \Lambda(x, y) \psi(y) = E \psi(x). \quad (8) \]

At the numerical level we followed the above algorithm and calculated the entanglement entropy for large enough system sizes such as \( L = 5000 \) and \( L = 6000 \), to avoid any finite size effects [19]. For these sizes the data show stability even for very small \( \alpha \)'s as far as we take the sub-system size less than \( L/100 \). Interestingly the entanglement entropy grows logarithmically with the sub-system size as

\[ S = \frac{c_{eff}}{3} \log l, \quad (9) \]

where \( c_{eff} \) is \( \alpha \) dependent, see Fig 2. The \( c_{eff} \) is maximum at the conformal short-range point and it starts to decrease by decreasing \( \alpha \).

For \( \alpha = 2 \) Callan and Wilczek [7] found a good estimation for the eigenvalues of the \( \Lambda \) operator when the subsystem is half of a finite system. Motivated by their work we found \( E(\omega) = \frac{1}{\sinh^2(\pi \omega)} \), where \( \omega(E_i) \log(l) = E_i \) is a very good approximation for the eigenvalues of the short-range case. Based on numerical comparison we found that apart from a constant the behaviour of the logarithm of the small eigenvalues (large \( i \), i.e. \( i > 6 \)) is independent of \( \alpha \) and one can safely conjecture the following behaviour for the eigenvalues of the \( \Lambda \) operator

\[ E(\omega) = \frac{a(\alpha)}{\sinh^2(\pi \omega)} + b(\alpha), \quad (10) \]

where \( a(\alpha) \) and \( b(\alpha) \) are functions of \( \alpha \) and \( \omega \) and it has a maximum at \( \alpha = 1 \). In principle \( b(\alpha) \) might be not a simple polynomial. Using the above formula one can get the logarithmic behaviour for the entanglement entropy for free and the \( c_{eff} \) is

\[ c_{eff}(\alpha) = \frac{6}{\pi} \int_0^\infty d\omega \left[ \log \left( \frac{E(\omega)}{2} + \sqrt{1 + E(\omega)} \right) \right] \times \log \left( \frac{1}{\sqrt{E(\omega)}} + \sqrt{1 + \frac{1}{E(\omega)}} \right). \quad (11) \]

In Fig 3 we compare the result coming from the above formula and the numerical calculations, with excellent agreement. The logarithmic behaviour of the entanglement entropy is reminiscent of the presence of conformal symmetry. To check this point we calculated the entanglement Rényi entropy using the same technique. The Rényi entropy \( S_n \) is defined as

\[ S_n = \frac{1}{1 - n} \log \text{tr} \rho^n, \quad n > 1 \quad (12) \]

and can be calculated using the following formula [0]

\[ S_n = \frac{1}{n - 1} \sum \log(1 - \xi^n) - n \log(1 - \xi)). \quad (13) \]
The best fit is $S_n = \frac{c_{eff}}{6} \log L$. The prefactor $c_{eff}$ is in general different from $c_{eff}$ except at conformal point, see Fig 5. This fact indicates that in the long-range interacting systems the entanglement entropy of a subsystem is not equal to the summation of the contribution of each boundary of the sub-system with its complement.

Finally we report the entanglement entropy of massive long-range interacting harmonic oscillators. The entanglement entropy of the positive half of an infinite massive system for short-range models comes from the Cardy-Calabrese formula $S = -\frac{c}{6} \log m$ where $c$ is the central charge of the system and it is one for short-range harmonic oscillators \[9\]. We calculated the same quantity for the long-range interacting systems. We found that $S$ saturates in $L \to \infty$ limit, and changes logarithmically with respect to the mass as

$$S = -\frac{c_{eff}}{6} \log m,$$

where $c_{eff}$ is different from $c_{eff}$ except at the $\alpha = 2$. We should mention that the presence of the pseudo central charge in long-range systems was already reported in \[10\].

It is worth to mention that interestingly $c_{eff}$ is equal to the prefactor of the free boundary condition case, see Fig 5. This can be understood as follows: In the definition of the $K$ matrix for free boundary condition we assumed that we just throw away those elements of the infinite system which are not inside the corresponding finite system which makes the summation of the every row of the matrix nonzero. This can show itself as an effective mass. One can check this guess by looking to the entanglement entropy of a subsystem of a large but finite system. By throwing away some elements of the infinite $K$ matrix we make the system gapped then one can easily check that one can introduce the same amount of gap by putting some mass in the infinite system. The corresponding mass is equivalent to correlation length $\xi = \frac{1}{m \bar{z}}$. As far as one takes a subsystem smaller than this length one will get just $c_{eff}$ but if we take the subsystem bigger than this length then the boundary effect will show itself as the mass gap.

Finally it is worth mentioning that since in the long-range interacting systems the correlation length changes as $\xi \sim \frac{1}{\sqrt{m \bar{z}}}$. One might expect that the the entanglement entropy be proportional to $\log \xi$ and then expect that $c_{eff}^\alpha$ be proportional to $\alpha$. Although it seems that this is the case in the region $1 < \alpha < 2$ the slop that we find is not compatible with the natural expectation $c_{eff}^\alpha = 0$ at $\alpha = 0$. Moreover our primary numerical calculations show strong deviation from the linear behaviour close to $\alpha = 0$.

**Conclusion and outline.** : In this paper we have found the entanglement entropy of long-range harmonic oscillators using the Hamiltonian technique. When we consider a sub-system with length $l$ the entanglement entropy follows the logarithmic behaviour as the short range cases...
but the prefactor is dependent on the power of the interaction $\alpha$.

Although it is tempting to interpret the prefactor of the logarithm $c_{eff}$ as the effective central charge of the system, by calculating the Rényi entropy we showed that the nature of this prefactor is different from the central charge at least at the level of the Rényi entropy.

This result is in contradiction with the theorem proved in [17], the reason is that the ground state of our long-range hamiltonian does not necessarily related to the ground state of any short-range hamiltonian! This is simply because if it was possible to approximate the ground state of our hamiltonian with ground state of a short-range hamiltonian we would expect conformal symmetry which is not the case in our model. The nature of this discrepancy is not clear for us.

We also calculated entanglement entropy for two other interesting cases, the system with boundary and also the massive case. Interestingly we found that the prefactor is the same in these two different cases, however, it is different from the massless infinite system. Our study shows that the entanglement entropy of long-range interacting systems shows some similarities with the entanglement entropy of short-range systems, however, since they have very different symmetries they start to show strong differences when we study their replica behaviour. We believe that long-range interacting systems show interesting features which deserve more intense studies. Our work can be extended in many directions: calculating the entanglement entropy in higher dimensions, the dynamics of the entanglement entropy and the finite size effects are just few among many other interesting directions.

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REFERENCES

[1] For recent reviews see: AMICO L., FAZIO R., OSTERLOH A. AND Vedral V., Rev. Mod. Phys., 80 (2008) 517; EISERT J., CRAMER M. AND PLENIO M. B., Rev. Mod. Phys., 82 (2010) 277; and references therein.

[2] Bombelli L., Koul R. K., Lee J. AND Sorkin R. D., Phys. Rev. D, 34 (1986) 373;

[3] Vidal G., Latorre J. I., Rico E. AND Kitaev A., Phys. Rev. Lett., 90 (2003) 227902; Latorre J. I., Rico E. AND Vidal G., Quant. Inf. Comp., 4 (2004) 48

[4] Srednicki M., Phys. Rev. Lett., 71 (1993) 666

[5] Peschel I., J. Phys. A: Math. Gen., 36, 14 (2009) L205-L208; Peschel I. AND Eisler V., J. Phys. A: Math. Theor., 42 (2009) 504003

[6] Casini H. AND Huerta M., J. Stat. Mech., (2005) P12012; J. Phys. A, 42 (2009) 504007

[7] Callan C. AND Wilczek F., Phys. Lett. B, 333 (1994) 55

[8] Holzhey C., Larsen F. AND Wilczek F., Nucl. Phys. B, 424 (1994) 443-467

[9] Calabrese P. AND Cardy J., J. Stat. Mech., (2004) P06002

[10] Refael G. AND Moore J. E., Phys. Rev. Lett., 93 (2004) 260602; Refael G. AND Moore J. E., J. Phys. A: Math. and Theor., 42 (2009) 504010

[11] Castro-Alvaredo O.A. AND Doyon B., Phys. Rev. Lett., 108 (2012) 120401

[12] Latorre J. I., Orúš R., Rico E. AND Vidal J., Phys. Rev. A, 71 (2005) 064101; Barthel T., Dusuel S. AND Vidal J., Phys. Rev. Lett., 97 (2006) 220402; Vidal J., Dusuel S. AND Barthel T., J. Stat. Mech., (2007) P01015; Dusuel S. AND Vidal J., Phys. Rev. B, 71 (2005) 224420

[13] Dürr W., Hartmann L., Hein M., Lewenstein M. AND Briegel H. J., Phys. Rev. Lett., 94 (2005) 097203

[14] Eisert J. AND Osborne T. J., Phys. Rev. Lett., 97 (2006) 150404

[15] Plenio M. B., Eisert J., Dreißig J. AND CRAMER M., Phys. Rev. Lett., 94 (2005) 060503; CRAMER M., Eisert J., PLENIO M.B. AND Dreißig J., Phys. Rev. A, 73 (2006) 012309

[16] Koffel T., Lewenstein M. AND Tagliacozzo L., [arXiv:1207.3957], (2012)

[17] Cadarso A., Sánz M., Wolf M. M., Cirac J. I. AND Perez-Garcia D., [arXiv:1209.3898], (2012)

[18] Dutta A. AND BHATTACHARJEE J. K., Phys. Rev. B., 64 (2001) 184106, and references therein.

[19] We discuss the finite size effects in this problem in a separate work.