Equivalence of the Chern-Simons state and the Hartle-Hawking and Vilenkin wave-functions

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We show that the Chern-Simons (CS) state when reduced to mini-superspace is the Fourier dual of the Hartle-Hawking (HH) and Vilenkin (V) wave-functions of the Universe. This is to be expected, given that the former and latter solve the same constraint equation, written in terms of conjugate variables (loosely the expansion factor and the Hubble parameter). A number of subtleties in the mapping, related to the contour of integration of the connection, shed light on the issue of boundary conditions in quantum cosmology. If we insist on a real Hubble parameter, then only the HH wave-function can be represented by the CS state, with the Hubble parameter covering the whole real line. For the V (or tunneling) wave-function the Hubble parameter is restricted to the positive real line (which makes sense, since the state only admits outgoing waves), but the contour also covers the whole negative imaginary axis. Hence the state is not admissible if reality conditions are imposed upon the connection. Modifications of the V state, requiring the addition of source terms to the Hamiltonian constraint, are examined and found to be more palatable. In the dual picture the HH state predicts a uniform distribution for the Hubble parameter over the whole real line; the modified V state a uniform distribution over the positive real line.

It is well known that the Chern-Simons (CS) state (also called the Kodama state) solves the full, non-perturbative Hamiltonian constraint in the self-dual, or Ashtekar formulation [1-5]. The CS state is given by:

$$\psi(A) = N \exp \left( -\frac{3}{2\kappa b^2} Y_{\text{CS}} \right)$$

(1)

where

$$Y_{\text{CS}} = \int L_{\text{CS}} = \int A^I dA^I + \frac{2}{3} \epsilon_{IJK} A^I A^J A^K$$

(2)

is the CS functional, $A^I$ is the $SU(2)$ Ashtekar self-dual connection (with $I$ its $SU(2)$ indices), $\Lambda$ is the cosmological constant, and $\kappa b^2 = 16\pi G N_b$. A number of fair criticisms have been levelled against this state (e.g. [2]), namely regarding its non-normalizability, CPT violating properties (and consequent impossibility of a positive energy property), and lack of gauge invariance under large gauge transformations. All of these criticisms hinge on the fact that state’s phase is not purely imaginary, for example proportional to $i3Y_{\text{CS}}$. If that were the case, then the Lorentzian theory would resemble the Euclidean theory, for which these problems evaporate [6, 7]. We will comment on this issue later in this paper. Suffice it to say at this stage that in the minisuperspace (MSS) approximation the state’s phase is always purely imaginary.

At this point, we could simply evaluate (1) in MSS without further ado. However, in order to facilitate comparison with the work of Hartle and Hawking and Vilenkin, we choose an alternative derivation. The basis for the Ashtekar formalism is the Einstein-Cartan (EC) formulation, upon which a canonical transformation is applied [8]. The reduction of the EC action to MSS leads to a very simple Hamiltonian system (see e.g. [9, 10]), resulting from the action:

$$S = 3\kappa V_c \int dt \left( 2a^2 \dot{b} + 2Na \left( b^2 + k - \frac{\Lambda}{3a^2} \right) \right).$$

(3)

Here $\kappa = 1/(16\pi G N)$, $a$ is the expansion factor, $b \approx \dot{a}$ (i.e., on-shell, if there is no torsion), $k = 0, \pm 1$ is the spatial curvature and $V_c$ is the comoving volume of the region under study (in most quantum cosmology work $k = 1$ and $V_c = 2\pi^2$). Hence the Poisson bracket is:

$$\{b, a^2\} = \frac{1}{6\kappa V_c}$$

(4)

and the system reduces to a single constraint (the Hamiltonian constraint) multiplying Lagrange multiplier $N$. Upon quantization [4] implies:

$$\{\hat{b}, \hat{a}^2\} = \frac{i\kappa b^2}{3V_c}$$

(5)

so that in the $b$ representation:

$$\hat{a}^2 = -\frac{i\kappa b^2}{3V_c} \frac{d}{db}.$$  

(6)

Assuming the ordering implied in [3], the quantum Hamiltonian constraint equation therefore is:

$$\hat{H}\psi = \left( i\kappa^2 \frac{d}{db} + k + b^2 \right) \psi = 0.$$  

(7)

Its most general solution has the form:

$$\psi_{\text{CS}} = N \exp \left[ i \left( \frac{9V_c}{\Delta P} \left( \frac{b^3}{3} + kb \right) + \phi_0 \right) \right]$$

(8)
where the only ambiguity is in the constant phase $\phi_0$ (which we will set to zero, as it does not affect our considerations). The real constant $\mathcal{N}$ is fixed by the normalization condition: with delta function normalization, as suggested in [6], one has $\mathcal{N} = 1/\sqrt{2\pi}$. There is no ± ambiguity in the phase: the plus sign is fixed and will play an important role.

We note that (8) is nothing but the CS state (1) reduced to MSS (as explained, this could have been derived directly, right at the start of this paper). Indeed for $k = 0$ we have $A^i_j = ib\delta^i_j$, leading to to (8) trivially. The calculation is more involved for $k \neq 0$ (see [10, 11]), but the conclusion remains true. This is hardly surprising, since equation (7) is nothing but a MSS reduction of the full Hamiltonian constraint with an appropriate ordering. Note that the fluxes conjugate to $A^i_j$ are the denitized inverse triads $E^i_j$ which in MSS become $E^i_j = a^2\delta^i_j$, in agreement with (1). We will say more about this later, but we stress that from this perspective it is clear that the base variable for discussing quantum cosmology in the metric representation should be not the expansion factor, $a$, but its square, $a^2$. This innocent remark has many a radical implication.

We now move on to the main point of this paper, and the reason for our alternative derivation of (6). Our MSS Hamiltonian constraint equation (7) is nothing but the standard Wheeler-deWitt (WdW) equation in the complementary representation implied by commutator (5). Had we chosen the metric (or rather, the $a^2$) representation, then:

$$\dot{b} = \frac{i\hbar^2}{3V_c} \frac{d}{d(a^2)}$$

and the Hamiltonian constraint equation would have read:

$$\left[ \frac{d^2}{da} - \frac{1}{a} \frac{d}{da} - U(a) \right] \psi = 0.$$  

with

$$U(a) = 4\left(\frac{3V_c}{\Lambda^2}\right)^2 a^2 \left( k - \frac{\Lambda a^2}{3} \right).$$

This is just the usual WdW equation with a specific ordering. We use the excellent review by Vilenkin [12] as the gold standard. Setting $k = 1$, $V_c = 2\pi^2$, and choosing the ordering parameter (as defined in [12]) $\alpha = -1$, we find that indeed there is agreement . This is not surprising, since the EC action reduces to the Einstein-Hilbert if there is no torsion. The solutions of this equation include the Hartle and Hawking (HH) [12, 13] and the Vilenkin (V) or tunnelling wave-functions [12, 14], depending on which boundary conditions (BC) one adds to this equation.

What can, therefore, be the relation between the HH and V wave-functions, on the one hand, and the CS state, on the other? Obviously, in some sense, the two have to be related by a Fourier transform (FT), since they solve the same quantum equation in terms of complementary variables. The FT inferred from (5) is:

$$\psi_{a^2}(a^2) = \frac{3V_c}{i\hbar^2} \int \frac{db}{\sqrt{2\pi}e} e^{-i\frac{V_c}{\hbar^2}a^2b} \psi_b(b).$$

But at once we notice an oddity. The WdW equation in the metric representation is second order (allowing two linearly independent solutions: HH and V), whereas in the $b$ representation it is first order, so that the CS wave-function is essentially unique up to an irrelevant phase and normalization constant. This points to an ambiguity in the FT, capable of incorporating this disparity in degrees of freedom. Resolving the matter will explain how the CS state can be dual to both the HH and V proposals.

The simplest way to unveil the detailed map is to examine concrete solutions. In the $a^2$ representation these are Airy-type functions [14], specifically:

$$\psi_V \propto \text{Ai}(-z) + i\text{Bi}(-z) \quad (13)$$

for Vilenkin BC, and

$$\psi_H \propto \text{Ai}(-z) \quad (14)$$

for HH BC, with:

$$z = \left( \frac{9V_c}{\Lambda^2} \right)^{2/3} \left( k - \frac{\Lambda a^2}{3} \right).$$

We can now appeal to well-known results in the theory of Airy functions [15, 16] familiar in optics and quantum optics. These special functions have integral representation

$$\phi(z) = \frac{1}{2\pi} \int e^{i\left(\frac{z^2}{3} + zt\right)} dt \quad (16)$$

where $\phi$ can be Airy, Bi or a combination thereof depending on the choice of contour over which the $t$ integration is undertaken. It is a central result of this paper that inserting (8) (the CS state) into (12) (the proposed FT) leads precisely to integral (16) with replacements (15) and:

$$t = \left( \frac{9V_c}{\Lambda^2} \right)^{1/3} b. \quad (17)$$

Hence the CS wave-function is indeed the FT dual of HH and V, with the choice of range for the connection $b$ (or of contour for the integral (16)) dictating which of the two functions is represented.

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1 To bridge notation notice that $6\epsilon V_c = 1/2a^2$ as defined in [12]. Also note that $\Lambda$ is defined with an extra factor of $1/3$ there.
This choice can be identified from standard results \[13, 16\]. The integration contour must start and finish at complex infinity within one of the 3 sectors:

\[
\begin{align*}
S_1 : & \quad 0 < \arg(t) < \frac{\pi}{3} \\
S_2 : & \quad \frac{2\pi}{3} < \arg(t) < \pi \\
S_3 : & \quad \frac{5\pi}{3} < \arg(t) < \frac{7\pi}{3},
\end{align*}
\]

(18)
as depicted in Fig. 1. This is because (16) only solves the Airy equation following an integration by parts, producing a boundary term that requires the integrand to vanish at the endpoints. Any contour starting (at infinity) in sector S2 and finishing (at infinity) in sector S1 produces the HH function. Instead, any contour starting in S3 and finishing in S1 produces the V wave-function. These are the only independent possibilities\(^2\). Examples of such contours are drawn in Fig. 1.

Two further qualifications are in order. Firstly, the inequalities defining sectors \[18\] may be non-strict (i.e. include equalities) if we accept an extended sense of convergence \[15, 17\]. This will include delta-function normalization of the CS state, as we shall see. Secondly, even though the contours just described are completely generic, two particular choices stand out. If we insist on \(b\) being real, then only the HH wave-function can be dual to the CS state. The integral in (16) is then to be seen as over the real line, containing both expanding and contracting Universes (see Fig. 2). Should we required strict convergence (non-delta function normalization) we can shift the contour by:

\[
b \to b + i\epsilon
\]

(19)
and then let \(\epsilon \to 0\), as is common in some QFT integrations. In contrast, the V wave-function requires any integration over real \(b\) to extend at most over the positive axis only. This makes sense, since the wave-function is restricted to representing outgoing waves after tunnelling. However, the integral cannot start at zero and then follow the positive real axis. Such integral represents Scorer's functions \(G_i\) and \(H_i\) and they solve a different differential equation \[16\] (i.e. not the Airy/WdW equation):

\[
\psi'' + z\psi = \frac{1}{\pi}.
\]

(20)
Hence, to obtain the V wave-function, we must allow for imaginary values of \( b \), for example, starting along the negative imaginary axis, then swerving into the positive real axis (see Fig. 2). The implications will be studied below.

Although all of this is standard mathematics, alternative derivations may be found which make contact with familiar results in quantum cosmology. For example, the WKB approximations (often used in quantum cosmology [12]) both in the non-oscillatory regime, under the “barrier” of \( U(a) \), and in the oscillatory regime at large \( a \) can be recovered from a stationary phase approximation to \( \Psi \). The derivation is instructive. The integrand in (16) may be written as \( e^{iS} \), not to be confused with the Euclidean path integral [13]. Unwrapping the integral in its full glory it reads:

\[
\psi_{a^2}(a^2) \propto \int \frac{db}{2\pi} \exp \left[ \frac{9iV_c}{A^2_H} \left( \frac{b^3}{3} + kb - \frac{\Lambda a^2}{3} \right) \right] \tag{21}
\]

so that:

\[
\frac{\partial S}{\partial t} \propto \frac{\partial S}{\partial b} \propto H = b^2 + k - \frac{\Lambda a^2}{3}. \tag{22}
\]

Hence the stationary points of phase \( S \) (containing the CS functional, not the Hamiltonian) are the solutions to the classical Hamiltonian constraint \( H \approx 0 \), given by:

\[
b_{\pm} = \pm \sqrt{\frac{\Lambda a^2}{3} - k} \tag{23}
\]

(or the equivalent expression in terms of \( t \) and \( z \), according to (17) and (15)). By taking the Taylor expansion to second order around these points:

\[
S_{\pm} = -\frac{2}{3}a^3_{\pm} + t_{\pm}(t - t_{\pm})^2 \tag{24}
\]

we find that the integral (16) can be carried out (see the relevant Appendix in [15] for details). It leads to the WKB expressions for the HH wave-function if both \( S_{\pm} \) are included; to the V wave-function if only \( S_{+} \) is selected. We found this to be the simplest way to make contact with these well-trodden territories.

So far our equivalence is purely formal, but what can all of this mean? We feel that a deep connection between these two hitherto separate fields must exist. In the final part of this paper we content ourselves with picking the lowest-hanging fruit, hoping to motivate further work.

First of all, we learn an important lesson about the Kodama/Chern-Simons wave-function: that this function, by itself, does not fix a quantum state. To turn it into a quantum state one must specify the range (or contour) of the connection, in lieu of what are standard boundary conditions in the dual metric representation. Thus, we should distinguish between the CS wave-function defined for \( b \in D_1 = \mathbb{R} = (-\infty, \infty) \) and for \( b \in D_2 = (-\infty, 0) \cup (0, \infty) \).

Once this is recognized, however, it makes little sense to distinguish between the CS state (i.e. function and its domain) on the one hand, and the HH or the V wave-functions, on the other. These two “hands” are the same quantum state expressed in different representations. Only \( |\psi_V\rangle \) exists in Hilbert space, with:

\[
\psi_V(a^2) = \langle a^2 | \psi_V \rangle \tag{25}
\]

\[
\psi_{CS}(b; b \in D_2) = \langle b | \psi_V \rangle. \tag{26}
\]

and likewise for \( |\psi_{HH}\rangle \). Having understood this simple but important fact we may now benefit from the cross-pollination resulting from examining the same quantum state from the complementary perspectives of conjugate variables.

Foremost, we find the issue of reality conditions in the Ashtekar formulation [8]. Face value these imply a re-identification of the Vilenkin state. The reality conditions require the reality/hermiticity of \( E_I^T \) and that the anti-self dual connection \( A_I^a \) be the complex/hermitian conjugate of the self-dual connection \( A_I^a \). Thus, in MSS the reality conditions imply that \( a^2 \) and \( b \) must be real. This disqualifies the Vilenkin wave-function, due to its compulsory foray into the negative imaginary axis of \( b \), but it is possible that a less strict interpretation of the reality conditions might change this conclusion. We stress that the V state’s forced inclusion of the imaginary axis of \( b \), in particular, and likewise for \( |\psi_{HH}\rangle \). Having understood this simple but important fact we may now benefit from the cross-pollination resulting from examining the same quantum state from the complementary perspectives of conjugate variables.

Curiously, in the reverse direction, the reality conditions only require that \( a^2 \) be real: they do not require that it be positive. As already pointed out, the fluxes of \( A_I^a \) are the densitized inverse triads \( E_I^T \) and these are proportional to \( a^2 \) in MSS. This is also the canonical variable conjugate to \( b \), and it explains why all the wave-functions are functions of \( a^2 \) alone. Therefore \( a^2 \in (-\infty, \infty) \) is natural coming from the connection perspective; indeed this is needed to render [12] invertible and the basis in \( a^2 \) complete. Strangely, the reality conditions imply that we need to consider Euclidean regions for the FRW metric. This has consequence for the normalization of the various wave-functions, a matter we now turn to.

The CS state, under the guise of the Kodama state, has been much maligned on the grounds of its
non-normalizability, among other perceived deficiencies (e.g. [2]). As already pointed out at the start of this paper, these crimes vanish for the state’s Euclidean formulation [6, 7], where the state becomes a pure phase (i.e. its exponent is imaginary). As we have seen here the same happens in the Lorentzian theory in MSS. Elsewhere [11, 18] we will show that it is possible to mimic the MSS treatment starting from the Einstein-Cartan action (3) in the full theory. This leads to the modified state:

\[ \psi_{CS} = N^2 \exp \left( -\frac{3i}{\hbar \Lambda} 3Y_{CS} \right). \]  

(27)

With the imaginary phase property of the wave-function assured, the state is as normalizable as a plane wave extending over the whole state, i.e. it is delta-function normalizable, belonging to a rigged Hilbert space. This assumes, of course, the reality of \( b \).

A probabilistic interpretation may now be attempted. For the HH state, the wave-function in \( b \) space is a pure phase over the whole of its domain, so the prediction is a uniform distribution in \( b \) over the real line. This is to be interpreted in the same way as the probability distribution in space for a plane wave extending over the whole space. It can be regulated with a UV cut-off in \( |b| \), for example, or else with prescription [19]. Such a uniform distribution is the flip-side of the distribution of \( a^2 \) implied by the HH wave-function (i.e. \( P_{HH}(a^2) \propto A\pi \delta(-z) \)), see Fig. 3. This is of course not uniform, indeed in the classically allowed region the wave-function is a standing wave, so the probability is modulated by oscillations. The fact that this is not strictly convergent as \( a \to \infty \) reflects the same issues found in the \( b \) representation: that the state is only delta-function normalizable. In terms of the \( a^2 \) representation this means that:

\[ \int_{-\infty}^{\infty} dz \psi^*_H(z+x)\psi_H(z+y) = \delta(x-y). \]  

(28)

We can still make sense of relative probabilities over the whole \( a^2 \in (-\infty, \infty) \). As \( a^2 \to -\infty \) the probability dies down exponentially.

The probability distribution for the \( V \) state in \( b \) space is more difficult to interpret, given that \( b \) must abandon the real line. Naively, the state predicts a uniform distribution in \( b \) over the positive real line. Over the negative imaginary line, written as \( b = i\delta(b) \), the prediction is:

\[ P_V(3b) = \frac{1}{2\pi} \exp \left[ \frac{18V_c}{\Lambda b} \left( \frac{3b^3}{3} - k3b \right) \right]. \]  

(29)

rising to a peak at \( b = -i \) (for \( k = 1 \), then falling off exponentially to zero, as \( b \to -i\infty \) (see Fig. 3). How these conclusions map into \( a^2 \) space is less obvious. Note that in \( a^2 \) space the \( V \) wave-function does not exhibit the same modulations as the HH state in the classically allowed regime, since it is a travelling wave. More importantly, given that \( a^2 \) (seen as a dual to \( b \)) should extend to minus infinity, the state appears problematic. As \( a^2 \to -\infty \) the \( V \) state diverges exponentially (due to the Bi function). Hence the regulating procedure analogous to that proposed for HH should not exist.

Naturally the tunneling state can be retouched to make it more palatable. It was suggested (e.g. [12]) that the state is only non-zero for \( a > 0 \), in which case it solves a modified WdW equation, with a delta-Dirac source term. \( V \)'s state is then a Green’s function of the WdW operator. This is a different wave-function and quantum state, which we will label V1. Although it was obtained with more sophisticated methods (e.g. the path integral formalism) a pedestrian derivation follows from writing:

\[ \psi_{V1}(a^2) = \langle a^2 | \psi_{V1} \rangle = \psi_V(a^2)\Theta(a^2). \]  

(30)

Insertion into (10) generates a source term in \( \delta(a^2) \). Had we dressed \( \psi_V(a^2) \) with \( \Theta(a) \) a source term proportional to \( \delta(a^2) \) would also have been obtained.

We stress that this wave-function represents a state different from \( \psi_V \). It solves a different equation. As in the case of the range of \( b \) and the CS function, the range of \( a^2 \) now becomes as relevant in defining the state as the function itself. Its dual representation \( \psi_{V1}(b) = \langle b | \psi_{V2} \rangle \) no longer is the CS wave-function, subject to whatever contour. A source term proportional to \( \delta(a^2) \) in (10) translates into a constant source term in the \( b \) dual representation, Eq. (7). The CS wave-function is not a solution. Elsewhere we will study the modified wave-function in the connection representation associated with this state.

Here, instead, we will do something simpler. Once we accept the introduction of delta-function sources in the WdW equation as justified means to an end (that of imposing desirable domain truncations) there is no reason not to do it in the \( b \) representation. We therefore backtrack to the point in this paper (around Eq. (20)) where we dismissed the possibility of starting the \( b \) contour at the origin, following the positive real axis only. Such a
The state $|\psi_{V_2}\rangle$ is very interesting. It shares with other V proposals the feature that it contains only outgoing waves (in the sense that its transform only contains $b > 0$, i.e., expanding Universes, with $\dot{a} > 0$). Wave-function \( |\psi_2\rangle \) is the dual of the CS wave-function defined over a contour that complies with the reality conditions. Indeed the offensive contribution (the integral \( \phi_{IM} \) over the imaginary negative axis) has form:

$$\phi_{IM}(z) = \frac{i}{\pi} \int_{-\infty}^{0} e^{\frac{i}{\pi} (z - \bar{z})} d\bar{t} = i\text{Hi}(-z),$$

(34)

with $\bar{t} = \Im(t)$. Since $B_i = G_i + H_i$ we have:

$$\psi_{V_2}(a^2) = \psi_V(a^2) - \psi_{IM}(a^2)$$

(35)

The state is also well behaved as $a^2 \to -\infty$, just like the HH state. Is this the best of both worlds?

To conclude, perhaps the most radical implication of the exploration of dual pictures pursued in this paper is the damning of “creation of the Universe out of nothing”. “Nothing” here is $a = 0$, but coming from the canonical perspective which gives primacy to the connection, the natural dual variable is $a^2$, the densitized metric. The question then is not nucleation from nothing ($a = 0$, excluding $a < 0$), but whether or not to include the Euclidean section ($a^2 < 0$). From the connection perspective there is no reason not to consider $a^2 \in (-\infty, \infty)$. The relevant issue is therefore, what is the probability for a Lorentzian Universe, $P_L$? For the HH, $V_1$ and $V_2$ states it is 1. For the unexpurgated V state it is zero.

The point $a = 0$ is unexceptional. Also, all our results are functions of $z$ alone (defined in \( |\psi_{1}\rangle \)), so they apply equally well to non-spherical Universes ($k = 0, -1$). For $k = 0, -1$ we can consider topologically non-trivial versions with finite $V_c$ and integrate over the whole space; or we can consider the quantum mechanics of a given finite comoving region. Whatever the case, the results are essentially the same. Different choices of $k$ (as well as $\Lambda > 0$ and $V_c$) merely shift the value of $z$ where Euclidean gives way to Lorentzian spaces, but the results found are generic. For $\Lambda < 0$ the relation between the sign of $a^2$ and that of $z$ reverses, so our conclusions reverse. Negative Lambda seems to favour Euclidean Universes.

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