RANDOM MATRIX MODEL FOR FREE MEIXNER LAWS

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Abstract. Applying the concept of matricial freeness which generalizes freeness in free probability we have recently studied asymptotic joint distributions of symmetric blocks of Gaussian random matrices (Gaussian Symmetric Block Ensemble). This approach gives a block refinement of the fundamental result of Voiculescu on asymptotic freeness of independent Gaussian random matrices. In this paper, we show that this framework is natural for constructing a random matrix model for free Meixner laws. We also demonstrate that the ensemble of independent matrices of this type is asymptotically conditionally free with respect to the pair of partial traces.

1. Introduction

It is well-known that free probability is an effective tool in the study of random matrices and their asymptotics. This approach was originated by Voiculescu in his fundamental paper [17], where he showed that independent Gaussian random matrices are asymptotically free (generalized to non-Gaussian entries by Dykema [9]). His result showed that the semicircle law obtained by Wigner [19] as the limit distribution of certain symmetric random matrices can now be viewed as an element of a much more general probability theory involving operator algebras [16].

If the complex-valued Gaussian variables which are entries of the considered random matrices are not identically distributed, one has to apply a more general scheme to study their asymptotics. One approach is to use operator-valued states and the associated notion of freeness with amalgamation, as in the paper of Shlakhtyenko on Gaussian band matrices [15]. This approach was further developed by Benayach-Georges [5] who described the asymptotics of blocks of random matrices and introduced a related additive convolution. Recently, we studied asymptotic joint distributions of symmetric blocks of random matrices by means of operatorial methods on Hilbert spaces. For this purpose, we employed a scheme based on arrays of scalar-valued states and the associated concept of matricial freeness introduced in [10].

In particular, we showed in [11,12] that the symmetric blocks of an ensemble of $n \times n$ complex Hermitian Gaussian random matrices $Y(u,n)$ with block-identical variances converge in moments under normalized partial traces to the mixed moments of symmetrized Gaussian operators, namely

\[ T_{p,q}(u,n) \to \tilde{\omega}_{p,q}(u) \]

where $u \in \mathcal{U}$ and $1 \leq p \leq q \leq r$, with $\mathcal{U}$ being an index set enumerating independent matrices. The operators $\tilde{\omega}_{p,q}(u)$ are natural symmetrizations of square arrays $(\omega_{p,q}(u))$ of matricially free Gaussian operators playing the role of basic Gaussian operators. By

\begin{itemize}
  \item \textbf{2010 Mathematics Subject Classification:} 46L53, 46L54, 15B52
  \item \textbf{Key words and phrases:} free probability, freeness, matricial freeness, random matrix, free Meixner law
\end{itemize}
a partial trace we understand a normalized trace over the subset of basis vectors related
to diagonal blocks.

In the random matrix context, the corresponding framework is thus a block refine-
ment of that used by Voiculescu and is closely related to his idea of decomposition of
Gaussian random matrices leading to semicircular and circular systems [18]. We studied
a deformation of this decomposition based on allowing the Gaussian variables to have
*block-identical variances* rather than identical and then computing their mixed moments
under (normalized) partial traces rather than under the (normalized) complete trace.
We would also like to remark that some results obtained by our methods can perhaps
be suitably reformulated in terms of freeness with amalgamation.

The key parameters of the block refinement are given by $r \times r$ symmetric variance
matrices $V(u) = (v_{p,q}(u))$ associated with symmetric blocks of the matrices $Y(u,n)$,
which, in turn are defined by the partition of the set

$$[n] = N_1 \cup N_2 \cup \ldots \cup N_r$$

into $r$ disjoint intervals (they depend on $n$, but this is suppressed in the notation), and
by the dimension matrix

$$D = \text{diag}(d_1, d_2, \ldots, d_r),$$

whose entries are given by non-negative numbers

$$d_j = \lim_{n \to \infty} \frac{|N_j|}{n}$$

called *asymptotic dimensions*. An important assumption is that we allow some of these
dimensions to vanish. Note that in our first paper, where we presented the block model
[11], we assumed that all asymptotic dimensions are positive.

It follows from the asymptotics of random symmetric blocks that the parameters of
random matrices are encoded in the products of the dimension matrix and the variance
matrices, namely

$$B(u) = DV(u)$$

and these matrices provide constants associated with blocks of colored non-crossing pair
partitions underlying the combinatorics of mixed moments of symmetrized Gaussian
operators. Let us add that we take the same dimension matrix for all random matrices.

In comparison with freeness of free probability, matricial freeness gives more flexibility
in treating such problems of random matrix theory as

1. evaluating limit distributions of random matrices,
2. studying asymptotic properties of random matrix ensembles,
3. constructing random matrix models for given probability measures,

and in that respect it reminds freeness with amalgamation. Some advantage of our
approach is that we rely on operators living in Hilbert spaces. This seems quite intu-
itive especially since computations involve operators which remind free creation and
annihilation operators and therefore their moments can be easily expressed in terms
of non-crossing (pair) partitions. A sample of such computations is contained in this
paper.

In particular, this flexibility allows us to treat sums and products of rectangular ran-
dom matrices in a unified manner, including *Wishart matrices* [20] as well as more gen-
eral products like those leading to *free Bessel laws* [4] and free products of Marchenko-
Pastur [14] distributions with arbitrary shape parameters. In fact, we were able to
compute the moments of the latter in the explicit form (known only in very special cases before) and introduce polynomials which can be viewed as multivariate Narayana polynomials [13]. A number of other new applications to the random matrix theory can be given. In particular, the matricially free Gaussian operators turned out to be effective in the construction of random matrix models for boolean independence, monotone independence and s-freeness [12]. In this paper, we also use these operators to construct a simple random matrix model for an important class of probability measures on the real line called free Meixner laws and prove the asymptotic conditional independence of the associated ensembles of random matrices.

Free Meixner systems of polynomials and the associated family of functionals were introduced and studied by Anshelevich [2,3]. Let us remark that free Meixner laws are free analogs of classical Meixner laws. In particular, up to affine transformations, they belong to one of the following six classes: free Gaussian (Wigner semicircle), free Poisson (Marchenko-Pastur), free negative binomial (free Pascal), free Gamma, free binomial and free hyperbolic secant, following the terminology of Anshelevich. Free Meixner laws turn out to display similar properties with respect to free independence as do the classical Meixner laws with respect to classical independence as Bryc and Bożejko showed in their study of the regression problem [6].

Random matrix models for certain special free Meixner laws are well-known, like the Gaussian Unitary Ensemble for the semicircle law, the Wishart Ensemble for the Marchenko-Pastur law or the Jacobi Ensemble for the free binomial law (see, for instance, [8,17,20]). However, a natural model for the whole class of free Meixner laws has not been given in the literature.

The paper is organized as follows. In Section 2, we recall a combinatorial formula for the moments of free Meixner laws. An operatorial realization of their moments in terms of matricially free Gaussian operators is proved in Section 3. A random matrix model for free Meixner laws is constructed in Section 4. An ensemble of independent random matrices of this type, called the Free Meixner Ensemble, is considered in Section 5, where we prove its asymptotic conditional freeness.

2. Moments of free Meixner laws

It is well-known that every probability measure on the real line with finite moments of all orders is characterized by two sequences of Jacobi parameters

$$\alpha = (\alpha_1, \alpha_2, \ldots) \text{ and } \beta = (\beta_1, \beta_2, \ldots),$$

where $\alpha_n \in \mathbb{R}$ and $\beta_n \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$, with the condition that if $\beta_k = 0$ for some $k$, then $\beta_m = 0$ for all $m > k$. We will call them Jacobi sequences and we will use the notation $J(\mu) = (\alpha, \omega)$. The Cauchy transform of $\mu$ can then be expressed as a continued fraction of the form

$$G_\mu(z) = \frac{1}{z - \alpha_1 - \frac{\beta_1}{z - \alpha_2 - \frac{\beta_2}{z - \alpha_3 - \frac{\beta_3}{\ddots}}}$$

and it is understood that if $\beta_m = 0$ for some $m$, then the fraction terminates and, for convenience, we set $\beta_n = \alpha_n = 0$ for all $n > m$. 
This continued fraction representation of Cauchy transforms turns out useful in our approach. Thus, let us first remark that the family of free Meixner laws is the family of probability measures on the real line associated with the pair of Jacobi sequences of the form

$$\alpha = (\alpha_1, \alpha_2, \alpha_2, \ldots) \text{ and } \beta = (\beta_1, \beta_2, \beta_2, \ldots),$$

i.e. they are constant starting from the second level of the corresponding continued fractions. If a free Meixner law corresponds to the pair of Jacobi sequences of the above form, we will say that it corresponds to $$(\alpha_1, \alpha_2, \beta_1, \beta_2)$$. In particular, if $\alpha_1 = 0$ and $\beta_1 = 1$, we obtain the standard free Meixner laws with mean zero and variance one. In that case, the absolutely continuous part of the associated measure $\mu$ takes the form

$$d\mu(x) = \frac{\sqrt{4\beta_2 - (x - \alpha_2)^2}}{2\pi(\beta_2 - 1)x^2 + \alpha_2x + 1}$$
on $[\alpha_2 - 2\sqrt{\beta_2}, \alpha_2 + 2\sqrt{\beta_2}]$, the measure can also have one or two atoms.

There is a useful combinatorial formula which expresses moments of probability measures on the real line in terms of non-crossing partitions consisting of 1-blocks (singletons) and 2-blocks (pairs). Namely, let $\mathcal{NC}_m^{1,2}$ be the set of non-crossing partitions of the set $[m] = \{1, 2, \ldots, m\}$ consisting of singletons and pairs, namely

$$\pi = \{\pi_1, \pi_2, \ldots, \pi_k\} \in \mathcal{NC}_m^{1,2}$$

where each $\pi_j$ contains one or two elements, respectively, and it is not possible to have two different 2-blocks $\pi_i = \{p, q\}$ and $\pi_j = \{r, s\}$, for which $p < r < q < s$.

In any non-crossing partition $\pi \in \mathcal{NC}_m$, if we put all numbers from the set $[m]$ in order and draw lines connecting all numbers which belong to the same block, the lines corresponding to different blocks cannot intersect each other. Further, its block $\pi_i$ is outer with respect to the block $\pi_j$ if there exist $r, s \in \pi_i$ such that for each $p \in \pi_j$ it holds that $r < p < s$. If $\pi$ consists of singletons and pairs, it is clear that any outer block must be a pair. We say that the block $\pi_i$ of $\pi \in \mathcal{NC}$ has depth $d(\pi_i) = d(i)$ if it has $d(i) - 1$ outer blocks. Thus, blocks which do not have outer blocks are assumed to have depth one. Note that if a block $\pi_i$ has at least one outer block, we can choose among them the one which lies immediately above $\pi_i$ and we will call it its nearest outer block.

If $\mu$ is a probability measure on the real line with all moments finite and the pair of Jacobi sequences $J(\mu) = (\alpha, \beta)$, its $n$-th moment is given by the combinatorial formula

$$M_n(\mu) = \sum_{\pi \in \mathcal{NC}_m^{1,2}} \prod_{i:|\pi_i|=1} \alpha_d(i) \prod_{j:|\pi_j|=2} \beta_d(j),$$

i.e. each block of depth $d$ of every $\pi \in \mathcal{NC}_m^{1,2}$ contributes $\alpha_d$ or $\beta_d$ if it is a singleton or a pair, respectively. This formula was first discovered by Cabanal-Duvillard and Ionesco for symmetric measures [7]. In that case, the first Jacobi sequence $\alpha$ vanishes and only pair partitions appear in the formula. The general version is due to Accardi and Bożejko [1].

3. Operatorial realization

We will use matricially free Gaussian operators living in the matricially free Fock space of tracial type introduced in [12] to find a realization of moments of free Meixner
laws. This Fock space is a generalization of the matricially free Fock space $\mathcal{M}$ introduced in [11].

For the purposes of this article, it suffices to consider the special case when

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2,$$

where both $\mathcal{M}_1$ and $\mathcal{M}_2$ are Hilbert space direct sums

$$\mathcal{M}_1 = \mathbb{C}\Omega_1 \oplus \bigoplus_{k=0}^{\infty}(\mathcal{H}^{\otimes k}_2 \otimes \mathcal{H}_1),$$

$$\mathcal{M}_2 = \mathbb{C}\Omega_2 \oplus \bigoplus_{k=1}^{\infty}\mathcal{H}^{\otimes k}_2,$$

where $\Omega_1, \Omega_2$ are unit vectors, $\mathcal{H}_j = \mathbb{C}e_j$ for $j \in \{1, 2\}$, where $e_1, e_2$ are unit vectors, and $\mathcal{H}^{\otimes 0} \otimes \mathcal{H}_1 = \mathcal{H}_1$. The space $\mathcal{M}$ is endowed with the canonical inner product.

Using the canonical basis of this Fock space,

$$\mathcal{B} = \{\Omega_1, \Omega_2, e_2^{\otimes k} \otimes e_1, \ e_2^{\otimes l} : k \in \mathbb{N} \cup \{0\}, l \in \mathbb{N}\},$$

we define creation operators $\varphi_1, \varphi_2 \in B(\mathcal{M})$ as follows. Let $(\beta_1, \beta_2)$ be a pair of non-negative numbers. We set

$$\varphi_1 \Omega_1 = \sqrt{\beta_1} e_1,$$

and we assume that $\varphi_1$ sends the remaining basis vectors to zero. In turn, $\varphi_2$ sends $\Omega_1$ to zero and otherwise,

$$\varphi_2 \Omega_2 = \sqrt{\beta_2} e_2,$$

$$\varphi_2(e_2^{\otimes k}) = \sqrt{\beta_2} e_2^{\otimes (k+1)}$$

$$\varphi_2(e_2^{\otimes l} \otimes e_1) = \sqrt{\beta_2} (e_2^{\otimes (l+1)} \otimes e_1)$$

for any $k \in \mathbb{N}, l \in \mathbb{N} \cup \{0\}$. By $\varphi_1^*$ and $\varphi_2^*$ we denote the adjoints of $\varphi_1$ and $\varphi_2$, respectively, and sums of the form

$$\omega_1 = \varphi_1 + \varphi_1^* \quad \text{and} \quad \omega_2 = \varphi_2 + \varphi_2^*$$

will be the corresponding Gaussian operators. Note that $\mathcal{M}_j$ is invariant with respect to $\varphi_i, \varphi_i^*, \omega_i$ for any $i, j \in \{1, 2\}$.

In particular, if we set $\beta_1 = \beta_2 = 1$, then the restrictions

$$(\varphi_1 + \varphi_2)|\mathcal{M}_1 \quad \text{and} \quad \varphi_2|\mathcal{M}_2$$

can be identified with the standard free creation operators living in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, and both spaces are isomorphic to the free Fock space over the one-dimensional Hilbert space.

**Remark 3.1.** Our Fock space $\mathcal{M}$ is a special case of the *matricially free Fock space of tracial type* associated with an array $(\mathcal{H}_{p,q})$ of Hilbert spaces, by which we understand the Hilbert space direct sum

$$\mathcal{M} = \bigoplus_{q=1}^{r} \mathcal{M}_q,$$
where each summand is of the form

\[ M_q = \bigoplus_{m=1}^{\infty} \bigoplus \bigoplus_{(p_1,p_2)\neq(p_3,p_4)\neq\ldots\neq(p_m,q)} F_{p_1,p_2}^0 \otimes F_{p_2,p_3}^0 \otimes \ldots \otimes F_{p_m,q}^0 \]

with tensor products built from free and boolean Fock spaces

\[ F_{p,q}^0 = \begin{cases} \bigoplus_{k=1}^{\infty} H_{q,k}^\otimes & \text{if } p = q \\ H_{p,q} & \text{if } p \neq q \end{cases}, \]

with vacuum spaces subtracted. In this paper, we suppose the array \((H_{p,q})\) consists of only two one-dimensional Hilbert spaces \(H_1 = H_2\). Clearly, an asymmetry in \((H_{p,q})\) leads to an asymmetry in the definitions of \(M_1\) and \(M_2\).

**Remark 3.2.** We can identify the creation operators \(\varphi_1, \varphi_2\) with the matricially free creation operators

\[ \varphi_1 = \varphi_{2,1} \quad \text{and} \quad \varphi_2 = \varphi_{2,2}, \]

where we use the matricial two-index notation of \([10,11]\). This notation is often helpful (and will be used when we refer to the results of these papers) since the second index shows onto which basis vectors the operators act non-trivially (it must match the first index of the basis vector). Therefore, \(\varphi_{p,q}\) acts non-trivially only onto \(\Omega_q\) and tensor products which begin with \(e_{q,r}\) for any \(r\). Thus, for instance,

\[ \varphi_{2,1}\Omega_2 = 0, \quad \varphi_{2,1}e_{2,1} = 0, \quad \varphi_{2,1}(e_{2,2} \otimes e_{2,1}) = 0, \quad \varphi_{2,2}\Omega_1 = 0, \]

which stands behind the definition of \(\varphi_1, \varphi_2\) (the fact that \(\varphi_{1,2}\) and \(\varphi_{1,1}\) are not used makes the one-index notation feasible). We also have \(\varphi_1^* = \varphi_{2,1}^*, \ \varphi_2^* = \varphi_{2,2}^*\), with the corresponding scalars \(\beta_1 = b_{2,1}\) and \(\beta_2 = b_{2,2}\). In turn, \(\omega_1\) and \(\omega_2\) can be identified with the corresponding matricially free Gaussian operators \(\omega_{2,1}\) and \(\omega_{2,2}\), respectively.

Details on the arrays of such operators can be found in \([10,11]\).

Using these operators, we can define operators in \(B(M_1)\) whose distributions in the state \(\Psi_1\) defined by the vector \(\Omega_1\) are free Meixner laws. For that purpose, the subspace \(M_2\) is not needed yet.

**Theorem 3.1.** If \(\mu\) is the free Meixner law corresponding to \((\alpha_1, \alpha_2, \beta_1, \beta_2)\), where \(\beta_1 \neq 0\) and \(\beta_2 \neq 0\), then its \(m\)-th moment is given by

\[ M_m(\mu) = \Psi_1((\omega + \gamma)^m), \]

where

\[ \omega = \omega_1 + \omega_2 \]

and

\[ \gamma = (\alpha_2 - \alpha_1)(\beta_1^{-1}\varphi_1\varphi_1^* + \beta_2^{-1}\varphi_2\varphi_2^*) + \alpha_1, \]

and \(\Psi_1\) is the state defined by the vector \(\Omega_1\).

**Proof.** Let us first analyze the moments of \(\omega\) since these were studied in \([12]\) in the general case of matricially free Gaussian operators. The operator \(\omega\) can be identified with

\[ \omega = \omega_{2,1} + \omega_{2,2}, \]
by Remark 3.2. Of course, if we set \( \omega_{1,2} = \omega_{1,1} = 0 \), we can use the combinatorial formula for the moments of the Gaussian pseudomatrix,

\[
\omega = \sum_{1 \leq p,q \leq 2} \omega_{p,q}
\]

associated with a \( 2 \times 2 \) array \((\omega_{p,q})\), in which we express these moments in terms of colored non-crossing pair partitions [11, Lemma 4.1].

By a colored non-crossing pair partition we shall understand a pair \((\pi, f)\), where \( \pi = \{\pi_1, \pi_2, \ldots, \pi_s\} \) is a non-crossing pair partition and \( f \) is a function on the set of its blocks with values in the set \([r]\). If we draw an additional 2-block which is outer with respect to all blocks of \( \pi \), called the imaginary block, and we color it by \( q \), we obtain the set of colored non-crossing pair partitions \( NC^2_{m,q}[r] \) colored by \([r]\) under condition that the imaginary block is colored by \( q \). Then, we have

\[
\Psi_q(\omega^m) = \sum_{(\pi,f) \in NC^2_{m,q}[r]} b_q(\pi, f),
\]

where the summation is over the empty set if \( m \) is odd, and

\[
b_q(\pi, f) = b_q(\pi_1, f)b_q(\pi_2, f) \ldots b_q(\pi_s, f)
\]

if \( m = 2s \), where \( \pi = \{\pi_1, \pi_2, \ldots, \pi_s\} \) and

\[
b_q(\pi_k, f) = b_{i,j}
\]

whenever block \( \pi_k \) is colored by \( i \) and its nearest outer block is colored by \( j \). In this formulation, we set \( b_{1,1} = b_{1,2} = 0 \) since there is no \( \omega_{1,1} \) or \( \omega_{1,2} \), but formally it holds for all colorings.

If we set \( q = 1 \), which refers to our theorem, the imaginary block gets colored by 1. Moreover, since \( b_{1,2} = b_{1,1} = 0 \), the non-vanishing contribution to \( \Psi_1(\omega^m) \) comes only from those colored partitions \((\pi, f) \in NC^2_{m,q}[r]\) in which each block of \( \pi \) is colored by 2. In fact, if some block \( \pi_k \) was colored by 1 and its nearest outer block (including the imaginary block) was colored by 1 or 2, then the corresponding \( b(\pi_k, f) \) would have to be \( b_{1,1} \) or \( b_{1,2} \), but these vanish. This means that to each block of depth one we assign the number \( b_{2,1} = \beta_1 \) since the imaginary block is colored by 1 and it is its nearest outer block, whereas to each block of depth greater than one we assign the number \( b_{2,2} = \beta_2 \) since each block of \( \pi \) is colored by 2. Namely

\[
b_1(\pi_k, f) = \begin{cases} 
\beta_1 & \text{if } d(\pi_k) = 1 \\
\beta_2 & \text{if } d(\pi_k) > 1 
\end{cases}
\]

This gives

\[
\Psi_1((\omega_{2,2} + \omega_{2,1})^m) = \sum_{\pi \in NC^2_{m}} b_1(\pi, f)
\]

since in this case the set \( NC^2_{m,1}[2] \) of colored non-crossing pair partitions of \([m]\) with the imaginary block colored by 1 reduces to the set in which all blocks colored by 2, which is in bijection with \( NC^2_{m} \). Switching back to the notations of this paper, we thus have

\[
\Psi_1(\omega^m) = \sum_{\pi \in NC^2_{m}} \beta_1^{B_1(\pi)} \beta_2^{B_2(\pi)},
\]
where $B_1(\pi)$ and $B_2(\pi)$ are the sets of 2-blocks of $\pi$ of depth 1 and of depth greater than 1, respectively.

In fact, the above formula for the moments of $\omega$ can be proved directly without invoking the general statement of [11, Lemma 4.1]. It suffices to observe that $(\pi, f)$ is uniquely determined by the sequence $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$ which appears in nonvanishing mixed moments of creation and annihilation operators of type

$$\Psi_1(\psi_1^{\epsilon_1} \psi_2^{\epsilon_2} \cdots \psi_m^{\epsilon_m}),$$

where $\epsilon_k \in \{1, \ast\}$ since the choice of $\epsilon$ uniquely determines the tuple $(q_1, q_2, \ldots, q_m)$ due to the 0-1 action of $\varphi_1$ and $\varphi_2$ and their adjoints. Namely, only $\varphi_1$ acts non-trivially onto $\Omega_1$, giving $\epsilon_1$, which corresponds to the right leg of each block of depth 1 (its adjoint corresponds to its left leg since it sends $\epsilon_1$ into $\Omega_1$). In turn, $\varphi_2$ acts non-trivially onto each basis element of $\mathcal{B}$ except $\Omega$ and thus it corresponds to the right leg of each block of depth greater than 1 (its adjoint corresponds to its left leg). Therefore, each block of $\pi$ of depth 1 is associated with the pair $(\varphi_1^*, \varphi_1)$ producing $\beta_1$, whereas the remaining blocks are associated with the pair $(\varphi_2^*, \varphi_2)$ producing $\beta_2$.

It remains to check what happens when we replace $\omega$ by $\omega + \gamma$. Observe that

$$\varphi_1 \varphi_1^* = \beta_1 P_1 \quad \text{and} \quad \varphi_2 \varphi_2^* = \beta_2 P_2,$$

where $P_1$ is the canonical projection onto $\mathcal{H}_1$ and $P_2$ is the canonical projection onto the subspace

$$\mathcal{F}_2 = \bigoplus_{k=1}^{\infty} (\mathcal{H}_2^\otimes_k \otimes \mathcal{H}_1),$$

respectively. Therefore,

$$\gamma = \alpha_1 P + \alpha_2 (P_1 + P_2),$$

where $P$ is the canonical projection onto $\mathbb{C} \Omega_1$, which means that $\gamma$ is diagonal in the basis $\mathcal{B}$, namely it multiplies $\Omega_1$ and all vectors from $\mathcal{B}\setminus\Omega_1$ by $\alpha_1$ and $\alpha_2$, respectively. Therefore, if we are given a mixed moment

$$\Psi_1(\psi_1^{\epsilon_1} \psi_2^{\epsilon_2} \cdots \psi_m^{\epsilon_m}),$$

associated with a non-crossing pair partition $\pi$ of the set $[k]$, each mixed moment of the form

$$\Psi_1(\gamma^{n_0} \psi_1^{\epsilon_1} \gamma^{n_1} \psi_2^{\epsilon_2} \gamma^{n_2} \cdots \psi_m^{\epsilon_m}),$$

where $n_0, n_1, \ldots, n_k$ are non-negative integers such that

$$k + n_0 + n_1 + \ldots + n_k = m,$$

which appears when we compute the $m$-th moment of $\omega + \gamma$, is naturally associated with a non-crossing partition $\tilde{\pi}$ of the set $[m]$ obtained from $\pi$ by adding $m - k$ singletons in such a way that $n_j$ singletons are placed right after the number $j$, with $n_0$ singletons placed before the number 1 belonging to the first pair. In this fashion we obtain all non-crossing partitions of $[m]$ which have $m - k$ singletons and $k$ pairs. Further, each $\tilde{\pi} \in \mathcal{NC}_m^{1,2}$ is obtained exactly once in this fashion from some $\pi \in \mathcal{NC}_m^2$.

Moreover, to each singleton of depth 1 we assign $\alpha_1$ and to each singleton of depth greater than 1 we assign $\alpha_2$ in view of the diagonal form of $\gamma$ in the basis $\mathcal{B}$. Therefore, we obtain

$$\Psi_1((\omega + \gamma)^m) = \sum_{\pi \in \mathcal{NC}_m^{1,2}} \alpha_1^{|S_1(\pi)|} \alpha_2^{|S_2(\pi)|} \beta_1^{|B_1(\pi)|} \beta_2^{|B_2(\pi)|},$$
where $S_1(\pi)$ and $S_2(\pi)$ are the sets of singletons of depth 1 and of depth greater than 1 in $\pi$, respectively. As we know from the combinatorial formula for the moments given in the Introduction, this is the $m$-th moment of the free Meixner law. This completes the proof.

**Example 3.1.** Let us give some examples of non-crossing partitions and the associated mixed moments. The diagrams are given in Figure 1. The partition $\pi$ consists of 4 pairs, namely $\pi_1 = \{1,8\}$, $\pi_2 = \{2,5\}$, $\pi_3 = \{3,4\}$, $\pi_4 = \{6,7\}$, with the imaginary block marked with a dotted line. There exists exactly one mixed moment of creation and annihilation operators that corresponds to this partition, namely we must have $\epsilon = (\ast,\ast,\ast,1,1,\ast,1,1)$ and the corresponding moment (the only non-trivial one which corresponds to this $\epsilon$) is

$$
\Psi_1(\psi_1^* \psi_2^* \psi_2^* \psi_2^* \psi_2^* \psi_1) = \beta_1^3 \beta_2^3
$$

since $\psi_1$ is the only creation operator which acts non-trivially onto $\Omega_1$, giving $e_1$, and $\psi_2$ is the only creation operator which acts non-trivially onto $e_1$ and $e_2 \otimes e_1$, giving $e_2 \otimes e_1$ and $e_2^\otimes \otimes e_1$, respectively. Next, $\psi_1^*$ is the only annihilation operator which acts non-trivially onto $e_1$, whereas $\psi_2^*$ is the only annihilation operator which acts non-trivially onto $e_2 \otimes e_1$ and $e_2^\otimes \otimes e_1$.

The partition $\sigma$ contains 3 pairs and 3 singletons, namely $\sigma_1 = \{1,8\}$, $\sigma_2 = \{2,7\}$, $\sigma_3 = \{3\}$, $\sigma_4 = \{4,5\}$, $\sigma_5 = \{6\}$, $\sigma_6 = \{9\}$. We assign the color 1 to all singletons of depth 1 and the color 2 to all remaining singletons. The colors assigned to singletons are to some extent arbitrary (they did not appear in [11,12], where we considered pair partitions only), but it is convenient to color all singletons of depth 1 by 1 and the remaining ones by 2 since this corresponds to the right Jacobi coefficients. The associated mixed moment is

$$
\Psi_1(\psi_1^* \psi_2^* \gamma \psi_2^* \psi_2^* \psi_1 \gamma) = \alpha_1^2 \beta_1 \beta_2^2,
$$

where the 2-blocks are associated with the pairs $(\psi_1^*, \psi_1)$ and $(\psi_2^*, \psi_2)$, which produce $\beta_1$ and $\beta_2$, respectively (like in the case of $\pi$), whereas the singletons are associated with $\gamma$, which produces $\alpha_1$ in the case of $\{9\}$ (since in this case $\gamma$ acts onto $\Omega_1$), and $\alpha_2$ in the case of $\{3\}$ and $\{9\}$ (since in this case $\gamma$ acts onto $e_2 \otimes e_1$).

If $\beta_1 = \beta_2 = 0$, we set $\omega_1 = 0$ and $\gamma_1 = \alpha$ which leads to the Dirac measure at $\alpha_1$. In turn, the case $\beta_2 = 0$ is treated below.

**Corollary 3.1.** If $\mu$ is the free Meixner law corresponding to $(\alpha_1, \alpha_2, \beta_1, 0)$, then its $m$-th moment is given by

$$
M_m(\mu) = \Psi_1((\omega_1 + \gamma_1)^m),
$$

Figure 1. Examples of colored non-crossing partitions.
where
\[ \gamma_1 = (a_2 - a_1)b_1^{-1}\varphi_1 \varphi_1^* + a_1 \]
and \( \Psi_1 \) is the state defined by the vector \( \Omega_1 \).

**Proof.** It suffices to observe that if we disregard \( \wp_2 \) and \( \wp_2^* \) in all computations in the proof of Theorem 3.1, then \( \beta_2 \) disappears from the formula for the moments of \( \omega + \gamma \) under \( \Psi_1 \). \( \Box \)

Finally, we would like to compute the moments of \( \omega + \gamma \) in the state \( \Psi_2 \). Observe that \( \wp_2, \wp_2^* \) vanishes on \( M_2 \) and therefore this reduces to the computation of moments of a slightly simpler operator.

**Corollary 3.2.** If \( \mu \) is the free Meixner law corresponding to \( (a_1, a_2, b_2, b_2) \), where \( b_2 \geq 0 \), then its \( m \)-th moment is given by
\[ M_m(\mu) = \Psi_2((\omega_2 + \gamma_2)^m), \]
where
\[ \gamma_2 = (a_2 - a_1)b_2^{-1}\varphi_2 \varphi_2^* + a_1 \]
and \( \Psi_2 \) is the state defined by the vector \( \Omega_2 \).

**Proof.** Observe that the action of \( \varphi_2, \varphi_2^* \) on \( M_2 \) is exactly the same as that of the free creation and annihilation operators, respectively, on the free Fock space. This means that the moments of \( \omega_2 \) under \( \Psi_2 \) agree with the moments of the (centered) semicircle law with variance \( \beta \), i.e. each moment of even order \( m = 2s \) is equal to \( \beta^s \) times the Catalan number \( C_s \). Represent \( C_s \) as the sum over \( NC_2^m \) and observe that if we replace \( \omega_2 \) by \( \omega_2 + \gamma_2 \), the effect is that \( NC_2^m \) gets replaced by \( NC_2^{1,2}_m \) as in the proof of Theorem 3.1, with singletons of depth 1 and 2 contributing \( a_1 \) and \( a_2 \), respectively. This gives the combinatorial formula for the \( m \)-th moment of the free Meixner law corresponding to \( (a_1, a_2, b_2, b_2) \). \( \Box \)

### 4. Random matrix model

Using our results on asymptotic distributions of random symmetric blocks and Theorem 3.1, we can now construct a random matrix model for free Meixner laws.

Consider the sequence of Gaussian Hermitian random matrices \( Y(n) \), where \( n \in \mathbb{N} \), under the assumptions of [11, Theorem 5.1]. Namely, we assume that \( Y(n) \) is a complex Gaussian \( n \times n \) random matrix of the block form
\[ Y(n) = \begin{pmatrix} A(n) & B(n) \\ C(n) & D(n) \end{pmatrix} \]
where the off-diagonal blocks are adjoints of each other, whereas the diagonal blocks are Hermitian and the sizes of blocks are defined by the partition of the set \( [n] = \{1, 2, \ldots, n\} \),
\[ [n] = N_1 \cup N_2, \text{ where } N_1 \cap N_2 = \emptyset \]
and
\[ d_1 = \lim_{n \to \infty} \frac{N_1}{n} = 0 \text{ and } d_2 = \lim_{n \to \infty} \frac{N_2}{n} = 1, \]
which corresponds to the situation in which
1. the sequence \( (D(n)) \) is balanced,
(2) the sequence of symmetric blocks built from \((B(n))\) and \((C(n))\) is unbalanced,
(3) the sequence \((A(n))\) is evanescent,

according to the natural terminology introduced in [12]. Since \((A(n))\) is evanescent, we can equivalently assume that each block of this sequence vanishes.

Using the notation of [12], where blocks are equipped with indices, we have
\[ A(n) = S_{1,1}(n), B(n) = S_{1,2}(n), C(n) = S_{2,1}(n), D(n) = S_{2,2}(n). \]

It is convenient to identify all blocks \(S_{p,q}(n)\) as well as the symmetric blocks
\[ T_{p,q}(n) = \left\{ \begin{array}{ll} S_{q,q}(n) & \text{if } p = q \\ S_{p,q}(n) + S_{q,p}(n) & \text{if } p \neq q \end{array} \right. \]
with their embeddings in the algebra of \(n \times n\) matrices, so that we can decompose matrices in terms of their blocks, namely
\[ Y(n) = \sum_{p,q} S_{p,q}(n) = \sum_{p \leq q} T_{p,q}(n), \]

which allows us to write the mixed moments of blocks under any partial trace \(\tau_j(n)\) over basis vectors of \(\mathbb{C}^n\) indexed by the set \(N_j\).

Shortly speaking, we shall assume that the matrices \(Y(n)\) are Gaussian Hermitian random matrices with block-identically distributed entries. More explicitly, we assume that

(1) each entry \(Y_{i,j}(n)\) of \(Y(n)\) is a complex Gaussian random variable of the form
\[ Y_{i,j}(n) = \text{Re}Y_{i,j}(n) + i\text{Im}Y_{i,j}(n), \]
(2) the family
\[ \{\text{Re}Y_{i,j}(n), \text{Im}Y_{i,j}(n) : 1 \leq i \leq j \leq n\} \]
is independent for any \(n\),
(3) the real-valued Gaussian variables have mean zero and
\[ \mathbb{E}(Y_{i,j}(n)Y_{i,j}(n)) = \frac{v_{p,q}}{n} \]
whenever \((i, j) \in N_p \times N_q\) for \(p, q \in \{1, 2\}\), where the variance matrix \(V = (v_{p,q})\) is symmetric.

**Theorem 4.1.** Under the above assumptions, let \(\tau_1(n)\) be the partial normalized trace over the set of first \(N_1\) basis vectors and let \(\beta_1 = v_{2,1} > 0\) and \(\beta_2 = v_{2,2} > 0\). Then
\[ \lim_{n \to \infty} \tau_1(n) ((M(n))^m) = \Psi_1((\omega + \gamma)^m) \]
where
\[ M(n) = Y(n) + \alpha_1 I_1(n) + \alpha_2 I_2(n) \]
for any \(n \in \mathbb{N}\), where \(I(n) = I_1(n) + I_2(n)\) is the decomposition of the \(n \times n\) unit matrix induced by the partition \([n] = N_1 \cup N_2\) and \(\omega, \gamma\) are given by Theorem 3.1.

**Proof.** We decompose \(Y(n)\) in terms of symmetric random blocks as
\[ Y(n) = T_{1,2}(n) + T_{1,1}(n) + T_{2,2}(n). \]
and therefore, by [11, Theorem 5.1], the moments of $Y(n)$ under any partial trace, including $\tau_1(n)$, tend to the moments of the corresponding Gaussian pseudomatrix $\omega$, namely
\[
\lim_{n \to \infty} \tau_1(n) ((Y(n))^m) = \Psi_1((\omega)^m)
\]
where
\[
\omega = \omega_{2,1} + \omega_{2,2}
\]
since $\omega_{1,2} = \omega_{1,1} = 0$ and that is why they do not appear in the above formula (each $\omega_{p,q}$ is associated with the scalar $b_{p,q} = d_p v_{p,q}$ and we have $d_1 = 0$). In the random matrix context, this means that the sequence $(T_{1,1}(n))$ is evanescent and $(T_{1,2}(n))$ is unbalanced. Moreover,
\[
b_{2,1} = d_2 v_{2,1} := \beta_1 \quad \text{and} \quad b_{2,2} = d_2 v_{2,2} := \beta_2
\]
since $d_2 = 1$ and $v_{2,1} = v_{1,2}$. This proves the assertion in the case when $\alpha_1 = \alpha_2 = 0$ (this includes Kesten laws).

Before we prove the assertion for the general case, let us observe that the block refinement of the above asymptotics can be written in the form
\[
\lim_{n \to \infty} \tau_1(n)(T_{p_1,q_1} T_{p_2,q_2} \cdots T_{p_m,q_m}) = \Psi_1(\omega_{p_1,q_1} \omega_{p_2,q_2} \cdots \omega_{p_m,q_m})
\]
provided we denote by $T_{2,1}$ rather than by $T_{1,2}$ the off-diagonal symmetric block. Namely, by [11, Theorem 5.1], the mixed moments of symmetric blocks $T_{p,q}$ under partial traces converge to the corresponding mixed moments of symmetrized Gaussian operators $\hat{\omega}_{p,q}$, where $\hat{\omega}_{1,1} = \omega_{1,1}$ and $\hat{\omega}_{2,2} = \omega_{2,2}$ and, more importantly,
\[
\hat{\omega}_{1,2} = \omega_{1,2} + \omega_{2,1}.
\]
Since, in the case considered in this theorem, $\omega_{1,2} = d_1 v_{1,2} = 0$ and thus $\hat{\omega}_{1,2} = \omega_{2,1}$, we can replace each $\hat{\omega}_{p,q}$ by $\omega_{p,q}$, which leads to the above equation. Moreover, even more information about these moments can be obtained. For that purpose, decompose $\mathbb{C}^n = W_1 \oplus W_2$, where $W_j$ is the linear span of basis vectors indexed by $i \in N_j$ and observe that
\[
T_{2,1}(W_1) \subseteq W_2, \quad T_{2,1}(W_2) \subseteq W_1 \quad \text{and} \quad T_{j,j}(W_j) \subseteq W_j
\]
for $j \in \{1, 2\}$. Since $\tau_1(n)$ is the partial trace over basis vectors from $W_1$, the above mixed moments of symmetric blocks vanishes unless it takes the form in which even powers of $T_{2,2}$ alternate with $T_{2,1}$, namely
\[
\tau_1(n)(T_{2,1} T_{2,2}^m T_{2,1} \cdots T_{2,1} T_{2,2}^r T_{2,1}),
\]
where $m_1, \ldots, m_r \in 2\mathbb{N} \cup \{0\}$ and $m_1 + m_2 + \cdots + m_r + 2r = m$. Likewise, the corresponding mixed moments of matricially free Gaussian operators vanish unless they take the form
\[
\Psi_1(\omega_{2,1} \omega_{2,2}^{m_1} \omega_{2,1} \cdots \omega_{2,1} \omega_{2,2}^{m_r} \omega_{2,1})
\]
since $\omega_{2,1}$ acts non-trivially onto $\Omega_1$ giving $e_1$ and sends $e_1$ back to $\Omega_1$, whereas $\omega_{2,2}$ kills both $\Omega_1$ and $e_1$, leaving $\mathcal{F}_2$ invariant. An even more detailed inspection leads to the formula
\[
\lim_{n \to \infty} \tau_1(n)(S_{1,2} T_{2,2}^m S_{2,1} \cdots S_{1,2} T_{2,2}^r S_{2,1}) = \Psi_1(\varphi_{2,1}^* \omega_{2,2}^{m_1} \varphi_{2,1}^* \cdots \varphi_{2,1}^* \omega_{2,2}^{m_r} \varphi_{2,1})
\]
since $\varphi_{2,1} \Omega_1 = e_1$ and $\varphi_{2,1}^* e_1 = \Omega_1$. Note that the last formula is not obvious since it is not true in general that $S_{1,2} \to \varphi_{2,1}^*$ and $S_{2,1} \to \varphi_{2,1}$ under the partial traces. However,
it is very convenient because it allows us to study the effect of inserting the diagonal deterministic matrix

\[ B = \alpha_1 I_1(n) + \alpha_2 I_2(n) \]

between the symmetric blocks, where the dependence of \( B \) on \( n \) is suppressed. We will show that an insertion of \( B \) somewhere on the LHS of the above formula corresponds to an insertion of the operator \( \gamma \) at the corresponding place on the RHS. Namely, this local analysis gives:

1. at the left or right end of the above moment, the matrix \( B \) reduces to \( \alpha_1 I_1 \) and thus it produces \( \alpha_1 \) since it acts onto \( W_1 \); the corresponding \( \gamma \) can also be replaced by \( \alpha_1 \) since it acts onto \( \Omega \),
2. in products of type \( BS_{2,1} \) and \( BT_{2,2} \), the matrix \( B \) reduces to \( \alpha_2 I_2 \) and gives \( \alpha_2 \) since it acts onto \( W_2 \); the corresponding pairs \( \gamma \phi_{2,1} \) and \( \gamma \omega_{2,2} \) can be replaced by \( \alpha_2 \phi_{2,1} \) and \( \alpha_2 \omega_{2,2} \), respectively, since \( \gamma \) acts here onto vectors from \( F_2 \).

Consequently, for all non-trivial mixed moments of \( T_{p,q} \) and \( B \), we can write

\[
\lim_{n \to \infty} \tau_1(n)(B^{n_0}Y B^{n_1}Y \ldots Y B^{n_k}) = \Psi_1(\gamma^{n_0} \omega \gamma^{n_1} \omega \ldots \omega \gamma^{n_k})
\]

for any nonnegative integers \( n_0, n_1, \ldots, n_k \) and any \( \alpha_1 \) and \( \alpha_2 \). This implies that

\[
\lim_{n \to \infty} \tau_1(n)((M(n))^m) = \Psi_1((\omega + \gamma)^m)
\]

which completes the proof of our theorem.

\[\square\]

**Corollary 4.1.** If \( \beta_1 = \nu_{2,1} > 0 \) and \( \beta_2 = \nu_{2,2} = 0 \) and under the remaining assumptions as in Theorem 4.1, it holds that

\[
\lim_{n \to \infty} \tau_1(n)((M(n))^m) = \Psi_1((\omega_1 + \gamma_1)^m)
\]

where \( \omega_1, \gamma_1 \) are given by Corollary 3.1.

**Proof.** The proof is similar to that of Theorem 4.1. The only difference is that blocks \( T_{2,2}(n, n) \) disappear from the computations under the trace \( \tau_1(n) \) and thus non-trivial mixed moments take the special form

\[
\tau_1(n)(B^{n_0}T_{2,1}B^{n_1}T_{2,1} \ldots T_{2,1}B^{n_m}) = \tau_1(n)(B^{n_0}S_{1,2}B^{n_1}S_{2,1} \ldots S_{2,1}B^{n_m})
\]

where \( m \) is even and \( S_{1,2} \) alternates with \( S_{2,1} \). They tend to

\[
\Psi_1(\gamma^{n_0} \omega_1 \gamma^{n_1} \omega_1 \ldots \omega_1 \gamma^{n_m}) = \Psi_1(\gamma^{n_0} \phi_1^* \gamma^{n_1} \phi_1 \ldots \phi_1 \gamma^{n_m})
\]

as \( n \to \infty \), where \( \phi_1^* \) alternates with \( \phi_1 \), since each \( B^j S_{1,2} B^k \) can be replaced by \( \alpha_1^j \alpha_2^k S_{1,2} \) for any \( j, k \in \mathbb{N} \) by the definition of \( B \) and, similarly, each \( \gamma^j \phi_1^* \gamma^k \) can be replaced by \( \alpha_1^j \alpha_2^k \phi_1^* \) be the definition of \( \gamma \). It remains to observe that in the situation when we have mixed moments of \( \omega_1 \) and \( \gamma \) under \( \Psi_1 \), we remain within \( H_1 \oplus \mathbb{C} \Omega_1 \) and thus \( \gamma \) can be replaced by \( \gamma_1 \), which completes the proof.

\[\square\]

**Corollary 4.2.** Under the assumptions of Theorem 4.1, it holds that

\[
\lim_{n \to \infty} \tau_2(n)((M(n))^m) = \Psi_2((\omega_2 + \gamma_2)^m)
\]

where \( \omega_2, \gamma_2 \) are given by Corollary 3.2.
Proof. The proof is similar to that of Theorem 4.1. In this case, when we compute the moments of $M(n)$ under $\tau_2(n)$, the mixed moments of $T_{1,1}(n), T_{2,1}(n), T_{2,2}(n)$ and $B$ become zero as $n \to \infty$ if there is $T_{1,1}(n)$ or $T_{2,1}(n)$ among them. On the level of matrices, this can be explained as follows: the fact that $(T_{2,1}(n))$ is unbalanced and is forced to act onto 'many' (of order $n$) basis vectors from $W_2$ giving 'few' (of order smaller than $n$) basis vectors from $W_1$ makes the moment containing $T_{2,1}(n)$ vanish in the limit $n \to \infty$ (in other words, zero asymptotic dimensions cannot be associated with inner blocks). Of course, the case of $T_{1,1}(n)$ is clear since it is evanescent. On the operatorial level, the effect of this is that the moments involving $\omega_1$ do not contribute to the limit moments since all operators act within $\mathcal{M}_2$, where $\omega_1$ is trivial and thus these moments reduce to the moments of $\omega_2$ and $\gamma$ under $\Psi_2$. Moreover, it is not hard to see that in fact $\gamma$ can be replaced with $\gamma_2$, which is the restriction of $\gamma$ to $\mathcal{M}_2$. ■

5. Free Meixner Ensemble

Let us consider an ensemble of independent random matrices of type considered in Section 4 and study their limit joint distributions under the state $\Psi_1$ as $n \to \infty$. The situation parallels that for the case of independent Gaussian random matrices and their asymptotic freeness [17]. As in Section 4, we will rely on the result derived in [12].

Definition 5.1. By the Free Meixner Ensemble we will understand the family of independent $n \times n$ Hermitian Gaussian random matrices $\{M(u,n) : n \in \mathbb{N}, u \in \mathcal{U}\}$, where matrices

$$M(u,n) = Y(u,n) + \alpha_1(u)I_1(n) + \alpha_2(u)I_2(n)$$

satisfy the assumptions of Theorem 4.1 or Corollary 4.1 for any $u \in \mathcal{U}$, where $\mathcal{U}$ is an index set, with the constants $\alpha_1(u), \alpha_2(u)$ as well as variances $\beta_1(u) = v_{2,1}(u), \beta_2(u) = v_{2,2}(u)$ depending on $u \in \mathcal{U}$. In particular, we assume that all matrices are decomposed into blocks in the same fashion for any fixed $n$ and that their asymptotic dimensions are $d_1 = 0$ and $d_2 = 1$ for all $u$.

We already know from Theorem 4.1 that the asymptotic distribution of $M(u,n)$ under the partial trace $\tau_1(n)$ is the free Meixner distribution associated with

$$(\alpha_1(u), \alpha_2(u), \beta_1(u), \beta_2(u)),$$

but we would like to find an asymptotic relation between independent random matrices from this ensemble. This relation is expected to be of asymptotic freeness type. In fact, we will demonstrate that the Free Meixner Ensemble is asymptotically conditionally free. As in Section 4, we exclude the case when $\beta_1(u) = 0$ for some $u$ since in this case the corresponding matrix realization would be purely deterministic, but one can easily extend all results to include this case.

We also know from [12] that the Hermitian Symmetric Gaussian Block Ensemble

$$\{T_{p,q}(u,n) : u \in \mathcal{U}, n \in \mathbb{N}\}$$

is asymptotically symmetrically matricially free, where symmetric matricial freeness is a symmetrized version of matricial freeness. More precisely, its asymptotics is determined by operators of type $\hat{\omega}_{p,q}(u)$ which are limit realizations of the corresponding symmetric
blocks $T_{p,q}(u)$. We shall use the results of [12], where we also studied the family of their sums

$$Y(u, n) = \sum_{p \leq q} T_{p,q}(u, n),$$

in order to find the limit distributions of the Free Meixner Ensemble.

We used the multivariate matricially free Fock space of tracial type. The definition of $\mathcal{M}$ remains the same as in Section 3, but instead of one-dimensional Hilbert spaces, we take direct sums

$$\mathcal{H}_j = \bigoplus_{u \in \mathcal{U}} \mathcal{H}_j(u)$$

where $\mathcal{H}_j(u) = \mathbb{C} e_j(u)$ for any $j \in \{1, 2\}$ and $u \in \mathcal{U}$, where $\{e_j(u) : j \in \{1, 2\}, u \in \mathcal{U}\}$ is an orthonormal set. Let

$$\mathcal{B} = \{\Omega_1, \Omega_2, e_2(u_1, \ldots, u_n), e_2(u_1, \ldots, u_{n-1}) \otimes e_1(u_n) : u_1, \ldots, u_n \in \mathcal{U}, n \in \mathbb{N}\}$$

be the orthonormal basis of $\mathcal{M}$, where we use a shorthand notation

$$e_2(u_1, \ldots, u_n) = e_2(u_1) \otimes \ldots \otimes e_2(u_n).$$

Then we define the family of creation operators $\varphi_1(u), \varphi_2(u)$ by the following rules:

$$\varphi_1(u) \Omega_1 = \sqrt{\beta_1(u)} e_1(u)$$

$$\varphi_2(u) \Omega_2 = \sqrt{\beta_2(u)} e_2(u)$$

$$\varphi_2(u) e_2(u_1, \ldots, u_n) = \sqrt{\beta_2(u)} e_2(u, u_1, \ldots, u_n)$$

$$\varphi_2(u) e_2(u_1, \ldots, u_{n-1}) \otimes e_1(u_n) = \sqrt{\beta_2(u)} e_2(u, u_1, \ldots, u_{n-1}) \otimes e_1(u_n)$$

and we assume that $\varphi_1(u), \varphi_2(u)$ send the remaining basis vectors to zero. By $\varphi_1^*(u)$ and $\varphi_2^*(u)$ we denote their adjoints, respectively, and sums of the form

$$\omega_j(u) = \varphi_j(u) + \varphi_j^*(u)$$

are the corresponding Gaussian operators. We have shown in [12] that operators of this type give the limit realization of the mixed moments of symmetric blocks of independent Hermitian Gaussian random matrices with block-identical variances (Gaussian Symmetric Block Ensemble). In other words, we showed that we have convergence of mixed moments

$$\lim_{n \to \infty} \tau_q(n)(T_{p_1,q_1}(u_1, n) \ldots T_{p_m,q_m}(u_m, n)) = \Psi_q(\hat{\omega}_{p,q}(u_1) \ldots \hat{\omega}_{p_m,q_m}(u_m)).$$

where $\hat{\omega}_{p,q}(u)$ is the same symmetrization as in the case of $\hat{\omega}_{p,q}$ in Section 3.

Let us give a definition of conditional freeness which is very similar to that of freeness and that will be helpful for us. The family of unital subalgebras $\{\mathcal{A}(u) : u \in \mathcal{U}\}$ of a unital algebra $\mathcal{A}$ is conditionally free with respect to the pair of states $(\varphi, \psi)$ on $\mathcal{A}$ if

$$\varphi(a_1 a_2 \ldots a_m) = 0$$

whenever $a_i \in \mathcal{A}(u_i) \cap \text{Ker } \psi$ for any $1 \leq i \leq n - 1$ and $a_n \in \mathcal{A}(u_n) \cap \text{Ker } \varphi$, where $u_1 \neq u_2 \neq \ldots \neq u_n$. This definition is equivalent to other definitions and immediately shows that there is a relation between different levels of Hilbert spaces in their free product and the corresponding states assigned to these levels. Consequently, there is a relation with the depths of the blocks of noncrossing partitions which contribute to the moments of conditionally free random variables. In more generality, we obtain freeness with infinitely many states [7].
Theorem 5.1. Let $\tau_j(n)$ be the partial trace over the set of basis vectors indexed by $N_j$, where $j \in \{1, 2\}$. The family of matrices

$$\{M(u, n) : u \in \mathcal{U}, n \in \mathbb{N}\}$$

is asymptotically conditionally free with respect to the pair of partial traces $(\tau_1(n), \tau_2(n))$ as $n \to \infty$.

Proof. In particular, if we consider the $2 \times 2$ block random matrices with asymptotic dimensions $d_1 = 0$ and $d_2 = 1$, the sequence $(T_{1,1}(n, u))$ is evanescent and $(T_{1,2}(n, u))$ is unbalanced and thus the corresponding arrays of symmetrized Gaussian operators reduce to arrays containing only $\omega_2(u) = \omega_{2,2}(u)$ and $\omega_1(u) = \omega_{2,1}(u)$ simply because $\omega_{1,1}(u) = 0$ and $\omega_{1,2}(u) = 0$. Thus, in view of the above, we have

$$\lim_{n \to \infty} \tau_1(n)(Y(u_1, n) \ldots Y(u_m, n)) = \Psi_1(\omega(u_1) \ldots \omega(u_m)),$$

where

$$\omega(u) = \omega_1(u) + \omega_2(u)$$

for any $u \in \mathcal{U}$. As in the proofs of Theorems 3.1 and 4.1, this can be generalized to the moments of matrices $M(u, n)$ from the Free Meixner Ensemble since all computations presented there are based on the relations between matricial indices of the considered blocks and of the considered operators and they depend on $u$ only in the sense that the blocks associated with symmetric blocks and with the corresponding operators labelled by $u$ give rise to parameters $\alpha_j(u), \beta_j(u)$ labelled by $u$. Thus, we have

$$\lim_{n \to \infty} \tau_1(n)(M(u_1, n) \ldots M(u_m, n)) = \Psi_1(y(u_1) \ldots y(u_m)),$$

where

$$y(u) = \omega(u) + \gamma(u)$$

and

$$\gamma(u) = (\alpha_2(u) - \alpha_1(u))(\beta_2^{-1}(u)\varphi_1(u)\varphi_1^*(u) + \beta_2^{-1}(u)\varphi_2(u)\varphi_2^*(u)) + \alpha_1(u),$$

for any $u \in \mathcal{U}$, where $\alpha_q(u) \in \mathbb{R}$ and $\beta_q(u) > 0$ for $q \in \{1, 2\}$. In a similar way one shows that

$$\lim_{n \to \infty} \tau_2(n)(M(u_1, n) \ldots M(u_m, n)) = \Psi_2(y(u_1) \ldots y(u_m))$$

where Corollaries 3.2 and 4.2 are used. Therefore, in order to prove our assertion, we need to show that the family

$$\{y(u) : u \in \mathcal{U}\}$$

is conditionally free with respect to the pair of states $(\Psi_1, \Psi_2)$, where $\Psi_q$ is the vector state associated with $\Omega_q$. We will prove a slightly more general result, namely that the family of unital *-algebras $\{\mathcal{A}(u) : u \in \mathcal{U}\}$, each generated by $\varphi_{2,1}(u)$ and $\varphi_{2,2}(u)$ for fixed $u$, respectively, is conditionally free with respect to $(\Psi_1, \Psi_2)$. We need to show that

$$\Psi_1(a_1a_2 \ldots a_n) = 0$$

for any $a_i \in \mathcal{A}(u_i) \cap \text{Ker}\Psi_2$, where $1 \leq i \leq n-1$ and $a_n \in \mathcal{A}_{u_n} \cap \text{Ker}\Psi_1$.

We claim that the variable $a_n$ is a polynomial in noncommuting variables

$$\varphi_1(u_n), \varphi_1^*(u_n), \varphi_2(u_n), \varphi_2^*(u_n)$$

which can be written as a linear combination of $P^\perp = 1 - P$ and of monomials

$$\varphi_2^{m_2}(u_n)\varphi_1^{m_1}(u_n)(\varphi_1^*(u_n))^{k_2}(\varphi_2^*(u_n))^{k_2}$$
where $m_2, k_2 \in \mathbb{N} \cup \{0\}$, $k_1, k_2 \in \{0, 1\}$ are such that $m_1 + m_2 + k_1 + k_2 > 0$. In order to reduce all monomials from Ker$\Psi$ to this form, first observe that $\mathcal{M}_1$ is invariant under the action of $\varphi_1, \varphi_2$ and their adjoints. Therefore, it suffices to consider all operators as their restrictions to $\mathcal{M}_1$. Then we have the relations
\[
\varphi_1^*(u)\varphi_1(u) = \beta_1(u)P, \quad \varphi_2^*(u)\varphi_2(u) = \beta_2(u)P^\perp \quad \text{and} \quad \varphi_q^*(u)\varphi_q(u') = 0
\]
as well as
\[
\varphi_1(u)\varphi_2(u') = 0, \quad P\varphi_1(u) = 0, \quad P^\perp\varphi_1(u) = \varphi_1(u), \quad P = \varphi_1(u)P^\perp = 0
\]
for any $q \in \{1, 2\}$ and $u \neq u'$, as well as their adjoints. Clearly, $P^\perp \in \text{Ker}\Psi_1$. Therefore, we can pull all starred operators to the right of the unstarred ones in $a_n$ and our claim is proved. This implies that $a_n$ maps $\Omega_1$ into $\mathcal{M}_1 \otimes \mathbb{C}\Omega_1$.

Now, any vector from the image $a_n(\Omega_1)$ is a linear combination of vectors which begin with $e_2(u_n)$. Therefore, the action of $a_{n-1}$ onto these vectors is the same as its action onto $\Omega_1$. Therefore, if we take $a_{n-1} \in \text{Ker}\Psi_2$ and we apply a similar reasoning as above, we can write it as a linear combination of monomials
\[
\varphi_1^{m_2}(u_{n-1})(\varphi_2^*(u_{n-1}))^{k_2}
\]
where $m_2 + k_2 > 0$ since $\mathcal{A}(u_{n-1})$ leaves $\mathcal{M}_1 \otimes \mathbb{C}\Omega_1$ invariant (recall that $u_{n-1} \neq u_n$) and the action of $\varphi_1^*(u_{n-1})$ is trivial on this space. Moreover, the constant term vanishes since $a_{n-1} \in \text{Ker}\Psi_2$ and thus $a_{n-1}a_n(\Omega_1)$ is a linear combination of vectors which begin with $e_2(u_{n-1})$. Continuing in this fashion, we obtain $a_1a_2\ldots a_n(\Omega_1) \perp \Omega_1$, which completes the proof. \hfill \qed

**Remark 5.1.** It can be easily seen that the family of algebras $\{\mathcal{A}(u) : u \in \mathcal{U}\}$ is, in general, not free with respect to $\Psi_1$. For instance, in the simple case when the Jacobi parameters are $(0, 0, \beta_1, \beta_2)$ for all $u \in \mathcal{U}$, where $0 \neq \beta_1 \neq \beta_2 \neq 0$, then we can take two polynomials, say $w_1 = y(s) \in \mathcal{A}(s)$ and $w_2 = (y(u))^2 - \beta_1 \in \mathcal{A}(u)$, where $u \neq s$, which are in Ker$\Psi_1$, but
\[
\Psi_1(w_1w_2w_1) = \beta_1(\beta_2 - \beta_1) \neq 0
\]
since
\[
\varphi_1^*(u)\varphi_1(u)\Omega_1 = \beta_1\Omega_1 \quad \text{and} \quad \varphi_2^*(u)\varphi_2(u)e_1 = \beta_2e_1.
\]
Of course, if we replace $w_2$ by $w_3 = (y(u))^2 - \beta_2 \in \text{Ker}\Psi_2$, we get zero in the above equation, which in in agreement with the conditional freeness of $w_1, w_3$ with respect to $(\Psi_1, \Psi_2)$ stated in Theorem 5.1.

**References**

[1] L. Accardi, M. Bożejko, Interacting Fock spaces and Gaussianization of probability measures, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **4** (1998), 663-670.

[2] M. Anshelevich, Free martingale polynomials, *J. Funct. Anal.* **201** (2003), 228-261.

[3] M. Anshelevich, Orthogonal polynomials with a resolvent-type generating function, *Trans. Amer. Math. Soc.* **360** (2008), 4125-4143.

[4] T. Banica, S.T. Belinschi, M. Capitaine, B. Collins, Free Bessel laws, *Canad. J. Math.* **63** (2011), 3-37.

[5] F. Benaych-Georges, Rectangular random matrices, related convolution, *Probab. Theory Relat. Fields** **144** (2009), 471-515.

[6] M. Bożejko, W. Bryc, On a class of free Lévy laws related to a regression problem, *J. Funct. Anal.* **236** (2006), 59-77.
[7] Th. Cabanal-Duvillard, V. Ionescu, Un théorème central limite pour de variables aléatoires non-commutatives, *C.R.A.S.* **325** (1997), Serie I, 1117-1120.

[8] M. Capitaine, M. Casalis, Asymptotic freeness by generalized moments for Gaussian and Wishart matrices. Applications to beta random matrices, *Indiana Univ. Math. J.* **53** (2004), 397-431.

[9] K. Dykema, On certain free product factors via an extended matrix model, *J. Funct. Anal.* **112** (1993), 31-60.

[10] R. Lenczewski, Matricially free random variables, *J. Funct. Anal.* **258** (2010), 4075-4121.

[11] R. Lenczewski, Asymptotic properties of random matrices and pseudomatrices, *Adv. Math.* **228** (2011), 2403-2440.

[12] R. Lenczewski, Limit distributions of random matrices, [arXiv:1208.3586](https://arxiv.org/abs/1208.3586) [math.OA], 2012.

[13] R. Lenczewski, R. Salapata, Multivariate Fuss-Narayana polynomials with application to random matrices, [arXiv:1210.3063](https://arxiv.org/abs/1210.3063) [math.CO], 2012.

[14] V. Marchenko, L. Pastur, The distribution of eigenvalues in certain sets of random matrices, *Math. Sb.* **72** (1967), 507-536.

[15] D. Shlyakhtenko, Random Gaussian band matrices and freeness with amalgamation, *Int. Math. Res. Notices* **20** (1996), 1013-1025.

[16] D. Voiculescu, K. Dykema, A. Nica, *Free random variables*, CRM Monograph Series, No.1, A.M.S., Providence, 1992.

[17] D. Voiculescu, Limit laws for random matrices and free products, *Invent. Math.* **104** (1991), 201-220.

[18] D. Voiculescu, Circular and semicircular systems and free product factors, *Progress in Math.* **92**, Birkhauser, 1990.

[19] E. Wigner, On the distribution of the roots of certain symmetric matrices, *Ann. Math.* **67** (1958), 325-327.

[20] J. Wishart, The generalized product moment distribution in samples from a normal multivariate population, *Biometrika* **20A** (1928), 32-52.

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