FUNDAMENTAL THEOREMS OF SUMMABILITY THEORY FOR A
NEW TYPE OF SUBSEQUENCES OF DOUBLE SEQUENCES

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Abstract. In 2000, the notion of a subsequence of a double sequence was introduced [3]. Using this definition, a multidimensional analogue to a result from H. Steinhaus, that states that for any regular matrix $A$ there exists a sequence of zeros and ones that is not $A$-summable, was proved. Additionally, an analogue of a result of R. C. Buck that states that a sequence $x$ is convergent if and only if there exists a regular matrix $A$ that sums every subsequence of $x$ was presented. However, this definition imposes a restrictive condition on the entries of the double sequence that can be considered for the subsequence. In this article, we introduce a less restrictive new definition of a subsequence. We denote them by $\beta$-subsequences of a double sequence and show that analogues to these two fundamental theorems of summability still hold for these new subsequences.

1. Introduction

In a seminal article, Patterson introduced the definition of a subsequence of a double sequence [3]. He, then, established two fundamental theorems of summability theory for these subsequences. Namely, the author showed that for any regular 4-dimensional matrix transformation, in the sense of Robison and Hamilton [2, 6], $A$, there exists a double sequence of 0’s and 1’s that is not $A$-summable. Additionally, he showed that the following characterization holds for these subsequences: “A double sequence $x$ is convergent in the Pringsheim sense if and only if there exists a regular 4-dimensional matrix transformation, $A$, such that $A$ sums every subsequence of $x$.”

However, the construction of these subsequences requires that one imposes a very stringent condition on the subindices eligible to form them. It is the goal of this article to introduce a family of sequences, to be denoted $\beta$-subsequences ($\beta > 1$), of double sequences that still satisfy the stated summability theorems but that do not impose such stringent condition.

Therefore, in Section 2, we use an idea similar to that used for $\beta$-rearrangements [4], to introduce the concept of a $\beta$-subsequence of a double sequence. In Section 3, we start by establishing the following basic notions of analysis of sequences for $\beta$-subsequences, that is, we show that if a double sequence is convergent, all of its $\beta$-subsequences are convergent and converge to the same limit (see Proposition 1). Following that, we show that for any $\beta$-regular 4-dimensional matrix transformation,
A, there exists a double sequence of 0’s and 1’s that is not A-summable (see Definition 7 and Theorem 2). We conclude by showing that a double sequence $x$ is convergent in the Pringsheim sense if and only if there exists a $\beta$-regular 4-dimensional matrix transformation, $A$, such that $A$ sums every $\beta$-subsequence of $x$.

2. Definitions and notation

Let $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given in the following way

$$(1,1) \mapsto 1 \quad (1,2) \mapsto 2$$
$$(2,2) \mapsto 3 \quad (2,1) \mapsto 4$$
$$(1,3) \mapsto 5 \quad (2,3) \mapsto 6$$
$$(3,3) \mapsto 7 \quad (3,2) \mapsto 8$$

In matrix form, this can be encoded as

$$\begin{pmatrix}
\psi(1,1) & \psi(1,2) & \psi(1,3) & \psi(1,4) & \cdots \\
\psi(2,1) & \psi(2,2) & \psi(2,3) & \psi(2,4) & \cdots \\
\psi(3,1) & \psi(3,2) & \psi(3,3) & \psi(3,4) & \cdots \\
\psi(4,1) & \psi(4,2) & \psi(4,3) & \psi(4,4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} =
\begin{pmatrix}
1 & 2 & 5 & 10 & \cdots \\
4 & 3 & 6 & 11 & \cdots \\
9 & 8 & 7 & 12 & \cdots \\
16 & 15 & 14 & 13 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Clearly, $\psi$ is a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. Thus, it is invertible. This map $\psi$ should be though of as a flattening function of the double sequence. We use this flattening function to introduce the definition of a $\beta$-subsequence of a double sequence. Before that, we start by defining a $\beta$-section $S_{\beta} \subset \mathbb{N} \times \mathbb{N}$ by

$$S_{\beta} := \left\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid \frac{1}{\beta} \leq \frac{m}{n} \leq \beta \right\}.$$

**DEFINITION 1.** ($\beta$-subsequence) Let $x = [x_{k,l}]$ be a double sequence and let $\beta > 1$ be an extended real. The double sequence $y_{\psi(p,q)}^{(\pi,\beta)}$ is called a $\beta$-subsequence of the double sequence $x$ if and only if there exists a strictly increasing function $\pi : \psi(S_{\beta}) \to \psi(S_{\beta})$ such that

$$y_{\psi(p,q)}^{(\pi,\beta)}(z_{\psi(p,q)}) = \begin{cases} 
1_{\psi(p,q)}, & \text{if } \frac{1}{\beta} > \frac{p}{q} \text{ or } \frac{p}{q} > \beta, \\
1_{\pi(\psi(p,q))}, & \text{if } \frac{1}{\beta} \leq \frac{p}{q} \leq \beta,
\end{cases}$$

where $z_i = x_{\psi^{-1}(i)}$. If $\beta = +\infty$, the inequalities are understood in the limit sense.

Some remarks are in order.

**REMARK 1.** Firstly, it must be noted that a double subsequence in the sense of [3] of $x$ can be realized as a $+\infty$-subsequence of $x$. However, an arbitrary $\beta$-sequence, cannot be realized as a double subsequence in the sense of [3]. Thus, the previous definition provides a generalization of the concept of subsequence of a double sequence.
REMARK 2. Second, a double subsequence in the sense of [3] is not a subsequence of itself. However, every double subsequence is a $\beta$-subsequence of itself where the map $\pi$ is the identity map on $S_{\beta}$.

For convenience, we consider the compatible decomposition of the double sequence $x$ as

$$x = \Pi(x) + B(x) + \Upsilon(x),$$

where

$$B(x)_{m,n} = \begin{cases} x_{m,n}, & \text{if } \frac{1}{\pi} \leq \frac{p}{q} \leq \beta, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Pi(x)_{m,n} = \begin{cases} x_{m,n}, & \text{if } \frac{p}{q} > \beta, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Upsilon(x)_{m,n} = \begin{cases} x_{m,n}, & \text{if } \frac{p}{q} < \frac{1}{\pi}, \\ 0, & \text{otherwise}. \end{cases}$$

For computational convenience, we assume the convention $\psi^{-1}(i) = (m_i, n_i)$.

DEFINITION 2. (Summability method [6]) Let $A$ be a four dimensional summability method that maps the complex double sequences $x$ into the double sequence $Ax$ where the $m,n$-th term of $Ax$ is given by

$$(Ax)_{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}.$$  

DEFINITION 3. (P-convergence [5]) A double sequence $x = [x_{k,l}]$ has a Pringsheim limit $L$ if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_{k,l} - L| < \epsilon,$$

whenever $k,l > N$. In this case, we say $x$ is $P$-convergent and we denote it by

$$L = \lim_{k,l \to \infty} x_{k,l}.$$  

Unless otherwise specified, the notation $\lim_{k,l \to \infty}$ is reserved in this article to limits in the Pringsheim sense.

For our purposes, we need to give an equivalent definition of a P-limit point as the one given in [3]. This is due to the fact stated in Remark 3. The advantage of the following definition is its independence from the definition of subsequence.

DEFINITION 4. (P-limit points) A double sequence $x = [x_{k,l}]$ has a Pringsheim limit point $L$ if and only if for every $\epsilon > 0$ and $N \in \mathbb{N}$, there exist $k,l \geq N$ such that

$$|x_{k,l} - L| < \epsilon.$$
Remark 3. Subsequences in the sense of [3] satisfy the following statement: “If \(L\) is a P-limit point of \(x\), then there exists a subsequence of \(x\) whose P-limit is \(L\).” This is not the case for \(\beta\)-subsequences. Indeed, consider the double sequence \(x\) such that

\[
x_{m,n} = \begin{cases} 
1, & \text{if } m = n, \\
0, & \text{if } m \neq n,
\end{cases}
\]

and a finite \(\beta > 1\). Clearly, 0 and 1 are P-limit points of \(x\). While there are \(\beta\)-subsequences of \(x\) converging to 0, there are no \(\beta\)-subsequences converging to 1.

Pringsheim also introduces a stronger notion of divergence.

Definition 5. (Definite divergence [5]) A double sequence \(x = [x_{k,l}]\) is said to be definite divergent if for every \(G > 0\), there exist naturals \(n, m\) such that \(|x_{k,l}| > G\) for all \(k > n, l > m\).

Definition 6. (RH-regular [6]) Let \(A\) be a four dimensional matrix. \(A\) is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Theorem 1. (Hamilton [2], Robison [6]) A 4-dimensional matrix \(A\) is RH-regular if and only if:

(RH1) \(\lim_{m,n \to \infty} a_{m,n,k,l} = 0\), for each \((k,l) \in \mathbb{N}^2\);

(RH2) \(\lim_{m,n \to \infty} \sum_{k,l=0}^{\infty} a_{m,n,k,l} = 1\);

(RH3) \(\lim_{m,n \to \infty} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0\), for each \(l \in \mathbb{N}\);

(RH4) \(\lim_{m,n \to \infty} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0\), for each \(k \in \mathbb{N}\);

(RH5) \(\lim_{m,n \to \infty} \sum_{k,l=0}^{\infty} |a_{m,n,k,l}|\) is P-convergent;

(RH6) there exist finite positive integers \(A\) and \(B\) such that

\[
\sum_{\substack{k>B \\ l>B}} |a_{m,n,k,l}| < A,
\]

for each \((m,n) \in \mathbb{N}^2\).

Definition 7. (\(\beta\)-regular) A 4-dimensional matrix \(A\) is said to be \(\beta\)-regular if and only if \(A\) is RH-regular and \(\lim_{m,n \to \infty} \sum_{(k,l) \in S_\beta} a_{m,n,k,l} = 1\).
3. Results

**Proposition 1.** Let $\beta > 1$. Then, the double sequence $x$ is $P$-convergent to $L$ if and only if every $\beta$-subsequence of $x$ is $P$-convergent to $L$.

**Proof.** Assume $x$ is $P$-convergent to $L$ and let $\varepsilon > 0$. Assume, $y^{(\pi, \beta)}$ is a subsequence of $x$. By the $P$-convergence of $x$, there exists $N \in \mathbb{N}$ such that

$$|x_{k,l} - L| < \varepsilon,$$

whenever $k, l > N$.

We consider two cases, when $(k, l) \not\in S_\beta$ and when $(k, l) \in S_\beta$. In the former case, $x_{k,l} = y^{(\pi, \beta)}_{k,l}$ so

$$|y^{(\pi, \beta)}_{k,l} - L| < \varepsilon,$$

in this case.

The latter case, when $(k, l) \in S_\beta$ is a little more delicate. Consider $y^{(\pi, \beta)}_{k,l}$ where $k > N, l > N$. Then,

$$y^{(\pi, \beta)}_{k,l} = x_{\psi^{-1}(\pi(\psi(k,l)))}.$$

If $\pi(\psi(k,l)) = \psi(p, q)$, there is no guarantee that $p > N$ or $q > N$, thus $|y^{(\pi, \beta)}_{k,l} - L| < \varepsilon$ may not be satisfied. See Figure 1.

To circumvent this situation, define $M \in \mathbb{N}$ by

$$M = \max\{p \in \mathbb{N} \mid 1/\beta < p/N \leq \beta\}$$

and consider $y^{(\pi, \beta)}_{p,q} = x_{p,q}$, where $k > M, l > M$ where $(k, l) \in S_\beta$. Notice that since $\pi$ is strictly increasing and $\pi(1) \geq 1$ we have that $\psi(p, q) = \pi(\psi(k,l)) \geq \psi(k,l)$. If equality holds, there is nothing to show, so assume that $\psi(p, q) > \psi(k,l)$. By the construction of $\psi$, this implies $p > k$ or $q > l$. We claim it is not possible for $p \leq N$ or $q \leq N$. For a contradiction, assume $q \leq N$. By the definition of $N$, it is clear that $N \leq M$. Thus, $q \leq N \leq M < k < p$ as it is not possible for $q > l$ to hold.

This implies that $1 < p/N \leq \beta$. Since $(p, q) \in S_\beta$, we have

$$\frac{1}{\beta} < 1 \leq \frac{p}{N} \leq \frac{p}{q} < \beta.$$
Figure 1: Pictorial representation of how an element in the subsequence (circled) may fail to belong to \( \{(k,l) \mid k,l > N\} \).

**Theorem 2.** Let \( \beta > 1 \) and let \( A \) be a \( \beta \)-regular four dimensional matrix. Then, there exists a sequence \( x \) with support on \( S_\beta \) whose entries are only equal to \( 1 \) or \( 0 \) such that \( x \) is not \( A \)-summable.

**Proof.** As in [3, Theorem 3.1] for each \( i \in \mathbb{N} \), we pick coefficients

\[
\begin{align*}
m_0 &< \cdots < m_i, & k_0 &< \cdots < k_i, \\
n_0 &< \cdots < n_i, & l_0 &< \cdots < l_i,
\end{align*}
\]

inductively with such that by (RH1),

\[
\sum_{k \leq k_i} \sum_{l \leq l_i} |a_{m_i,n_i,k,l}| \leq \sum_{k \leq k_i} \sum_{l \leq l_i} |a_{m_i,n_i,k,l}| < \frac{1}{(i+2)^2},
\]

and by (RH3), (RH4)

\[
\begin{align*}
\sum_{k \leq k_i} \sum_{l > l_i} |a_{m_i,n_i,k,l}| &\leq \sum_{k \leq k_i} \sum_{l > l_i} |a_{m_i,n_i,k,l}| < \frac{1}{(i+2)^2}, \\
\sum_{k > k_i} \sum_{l \leq l_i} |a_{m_i,n_i,k,l}| &\leq \sum_{k > k_i} \sum_{l \leq l_i} |a_{m_i,n_i,k,l}| < \frac{1}{(i+2)^2}.
\end{align*}
\]
In addition, by the $\beta$-regularity of $A$, pick $m_i$ and $n_i$ so that

$$\left| \sum_{(k,l) \in S_{\beta}} a_{m_i,n_i,k,l} \right| > 1 - \frac{1}{(i+2)^2}.$$ 

So that

$$\sum_{k > k_i \atop l > l_i \atop (k,l) \in S_{\beta}} |a_{m_i,n_i,k,l}| \geq \sum_{(k,l) \in S_{\beta}} |a_{m_i,n_i,k,l}| - \sum_{k \leq k_i \atop l \leq l_i \atop (k,l) \in S_{\beta}} |a_{m_i,n_i,k,l}| - \sum_{k \leq k_i \atop l > l_i \atop (k,l) \in S_{\beta}} |a_{m_i,n_i,k,l}|$$

$$- \sum_{k > k_i \atop l \leq l_i \atop (k,l) \in S_{\beta}} |a_{m_i,n_i,k,l}| > 1 - \frac{4}{(i+2)^2}.$$ 

With these coefficients chosen, we proceed to choose $k_{i+1} > k_i$ and $l_{i+1} > l_i$ such that

$$\left| \sum_{k_i < k < k_{i+1} \atop l_i < l < l_{i+1} \atop (k,l) \in S_{\beta}} a_{m_i,n_i,k,l} \right| > 1 - \frac{4}{(i+2)^2}, \quad \sum_{k_i \leq k < k_{i+1} \atop l_l < l < l_{i+1} \atop (k,l) \in S_{\beta}} |a_{m_i,n_i,k,l}| \leq \frac{1}{(i+2)^2},$$

$$\sum_{k_i < k < k_{i+1} \atop l \geq l_{i+1}} |a_{m_i,n_i,k,l}| \leq \frac{1}{(i+2)^2}, \quad \sum_{k_i \leq k \leq k_{i+1} \atop l_i < l \leq l_{i+1} \atop (k,l) \in S_{\beta}} |a_{m_i,n_i,k,l}| \leq \frac{1}{(i+2)^2}.$$ \hspace{1cm} (2)

Now, we define the double sequence $x$ by

$$x_{k,l} = \begin{cases} 1, & \text{if } (k,l) \in S_{\beta}, k_{2p} < k < k_{2p+1} \text{ and } l_{2p} < l < l_{2p+1}, \text{ for } p \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \hspace{1cm} (3)$$

Noting that $a_{m,n,k,l}x_{k,l} = 0$ whenever $(k,l) \not\in S_{\beta}$, we have that the $m_i,n_i$’th term of the double sequence $Ax$ is given by

$$(Ax)_{m_i,n_i} = \sum_{(k,l) \in S_{\beta}} a_{m_i,n_i,k,l}$$

$$= \sum_{k \leq k_i \atop l \leq l_i \atop (k,l) \in S_{\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{k \leq k_i \atop l \geq l_i \atop (k,l) \in S_{\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{k > k_i \atop l \leq l_i \atop (k,l) \in S_{\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{k > k_i \atop l > l_i \atop (k,l) \in S_{\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{k_i < k < k_{i+1} \atop l_i < l < l_{i+1} \atop (k,l) \in S_{\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{k_i < k < k_{i+1} \atop l \geq l_{i+1} \atop (k,l) \in S_{\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{k \geq k_{i+1} \atop l_i < l < l_{i+1} \atop (k,l) \in S_{\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{k \geq k_{i+1} \atop l \geq l_{i+1} \atop (k,l) \in S_{\beta}} a_{m_i,n_i,k,l}x_{k,l}$$ \hspace{1cm} (4)
We index each of the sums by $I_j$ for $j = 1, \ldots, 7$. Now based on (3), we note that $I_4 = 0$ or $I_4 > 1 - \frac{4}{(i+2)^2}$ depending on whether $i$ is odd or even, respectively. So, whenever $i$ is odd, we have that
\[
| (Ax)_{m_i, n_i} | \leq \sum_{j \neq 4} |I_j| \leq \frac{6}{(i+2)^2},
\]
which has P-limit equal to zero. On the other hand, however, when $i$ is even, by the reverse triangle inequality and (4) we obtain
\[
| (Ax)_{m_i, n_i} | \geq |I_4| - \sum_{j \neq 4} |I_j|
\]
and by (1) and (2), we have
\[
| (Ax)_{m_i, n_i} | > 1 - \frac{4}{(i+2)^2} - \sum_{j \neq 4} \frac{1}{(i+2)^2}.
\]
The latter expression has P-limit equal to 1. Thus, $A$ cannot sum $x$. □

**Lemma 1.** Suppose that $y$ and $z$ are two convergent $\beta$-subsequences of $x$. If
\[
\lim_{m,n \to \infty} y_{m,n} = \lim_{m,n \to \infty} z_{m,n},
\]
then
\[
\lim_{m,n \to \infty} (B(y)_{m,n} - B(z)_{m,n}) = 0.
\]
In particular, if $B(x)$ is not summable, then $x$ is not $A$ summable.

**Proof.** Notice that $\Pi(y) = \Pi(z)$ and $\Upsilon(y) = \Upsilon(z)$ as $y, z$ are subsequences of $x$. Therefore,
\[
(y - z)_{m,n} = (B(y) - B(z))_{m,n},
\]
for all $m, n \in \mathbb{N}$. Therefore, by assumption
\[
\lim_{m,n \to \infty} (B(y)_{m,n} - B(z)_{m,n}) = 0.
\]
For the second statement, suppose that for some bounded subsequences $y, z$ of $x$,
\[
\lim_{m,n \to \infty} (B(Ay)_{m,n} - B(Az)_{m,n}) \neq 0.
\]
Then, by what we just showed we have that
\[
\lim_{m,n \to \infty} (Ay)_{m,n} \neq \lim_{m,n \to \infty} (Az)_{m,n},
\]
thus implying that $x$ is not summable. □
REMARK 4. The converse of the lemma is not true. Consider the double sequence \( z_{m,n} \) such that \( z_{m,n} = 1 \) if \( \frac{m}{n} < 1/\beta \) and \( z_{m,n} = 0 \) otherwise. Further, let \( y \) be the null-double-sequence. In that case,

\[
\lim_{m,n \to \infty} (B(y)_{m,n} - B(z)_{m,n}) = 0,
\]

but clearly \( \lim_{m,n \to \infty} z_{m,n} \) is undefined, while \( \lim_{m,n \to \infty} y_{m,n} = 0 \).

In [3], Patterson showed that for a special type of \( \beta \)-subsequence a “Buck-type” result (see [1, 3]) holds for this special case of double subsequence. As it turns out, this happens to be true for the more general \( \beta \)-subsequences.

THEOREM 3. Let \( \beta > 1 \). A bounded double sequence \( x \) is P-convergent if and only if there exists a \( \beta \)-regular matrix \( A \) such that \( A \) sums every \( \beta \)-subsequence of \( x \).

Proof. The implication follows from Proposition 1, as any \( \beta \)-subsequence of a bounded convergent double sequence is bounded and convergent. Thus, any RH-regular matrix \( A \) sums it. In particular, any \( \beta \)-regular matrix sums it.

For the converse, we shall show that for a bounded but not P-convergent \( x \) and any \( \beta \)-regular matrix \( A \) there exists a \( \beta \)-subsequence of \( x \) that is not summed by \( A \). By Lemma 1, it suffices to consider subsequences of \( B(x) \). Therefore, assume \( x \) is supported on \( S_\beta \).

If \( x \) is bounded but not P-convergent, it must have more than one limit point. Consider the flattened sequence corresponding to \( x \), namely the sequence defined by \( (x_{yi}^{-1}(i))_{i=1}^{\infty} = (x_{mi,ni})_{i=1}^{\infty} \) and define

\[
\alpha = \limsup_{i \to \infty} x_{mi,ni} \quad \text{and} \quad \beta = \liminf_{i \to \infty} x_{mi,ni}.
\]

Since, the P-limit is not unique, we necessarily have that \( \alpha \neq \beta \).

As in [3], we define the double sequence \([y_{m,n}]\) by

\[
y_{m,n} = \frac{x_{m,n} - \beta}{\alpha - \beta}, \quad \text{for all } n,m \in \mathbb{N}.
\]

Note that \([y_{m,n}]\) is supported on \( S_\beta \), as is \( x \). It is also clear that

\[
\limsup_{i \to \infty} y_{mi,ni} = 1 \quad \text{and} \quad \liminf_{i \to \infty} y_{mi,ni} = 0.
\]

Then, there exists a subsequences \((y_{mi_j,n_{i_j}})_{j=1}^{\infty}\) and \((y_{mi_k,n_{i_k}})_{k=1}^{\infty}\) of the flattened sequence \((y_{mi,n_i})_{i=1}^{\infty}\) such that

\[
\frac{1}{\beta} < \frac{m_{i_k}}{n_{i_k}} < \beta \quad \text{and} \quad \frac{1}{\beta} < \frac{m_{i_j}}{n_{i_j}} < \beta
\]

and

\[
\lim_{j \to \infty} y_{mi_j,n_{i_j}} = 0 \quad \text{and} \quad \lim_{k \to \infty} y_{mi_k,n_{i_k}} = 1.
\]
Notice that by Remark 3, there are no such $\beta$-subsequences. Indeed, the Pringsheim limit of the $\bar{\beta}$-subsequence corresponding to $(y_{m_k,n_{k}})_{k=1}^{\infty}$ is undefined. However, this flattened subsequence shall suffice for our purposes.

Define

$$y^*_{m,n} = \begin{cases} 
1, & \text{if } (m,n) = (m_{i_k}, n_{i_k}) \text{ for some } k \in \mathbb{N}, \\
0, & \text{if } (m,n) = (m_{i_j}, n_{i_j}) \text{ for some } j \in \mathbb{N}, \\
y_{m,n}, & \text{otherwise.}
\end{cases}$$

Since $y^*$ has infinitely many 0’s and 1’s in its $S_\beta$ component, by Theorem 2, there exists a $\beta$-subsequence $z^{(\pi,\beta)}$ of $y^*$ that is not $A$-summable. Let $y^{(\pi,\beta)}$ denote the $\beta$-subsequence of $y$ induced by the same injection $\pi$ that defines $z^{(\pi,\beta)}$. It is easy to see that

$$\lim_{m,n \to \infty} (y^{(\pi,\beta)}_{m,n} - z^{(\pi,\beta)}_{m,n}) = 0.$$

Thus by the linearity and regularity of $A$, we have

$$\lim_{m,n \to \infty} (Ay^{(\pi,\beta)}_{m,n} - Az^{(\pi,\beta)}_{m,n}) = 0.$$

This, in turn, implies that the $\beta$-subsequence $y^{(\pi,\beta)}$ is not $A$-summable. By the definition of $y_{m,n}$, this implies that the corresponding subsequence $x^{(\pi,\beta)}$ of $x$ is not $A$-summable. □

**Remark 5.** In the particular case when $\beta = +\infty$, Theorem 3 implies Theorem 3.2 in [3], as the set of all subsequences in their sense is contained in the set of all $+\infty$-subsequences. Thus, this theorem presents a generalization of the results therein.

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