THE TEICHMÜLLER–RANDERS METRIC

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Abstract. In this paper, we introduce a new asymmetric weak metric on the Teichmüller space of a closed orientable surface with (possibly empty) punctures. This new metric, which we call the Teichmüller–Randers metric, is an asymmetric deformation of the Teichmüller metric, and is obtained by adding to the infinitesimal form of the Teichmüller metric a differential 1-form. We study basic properties of the Teichmüller–Randers metric. In the case when the 1-form is exact, any Teichmüller geodesic between two points is a unique Teichmüller–Randers geodesic between them. A particularly interesting case is when the differential 1-form is (up to a factor) the differential of the logarithm of the extremal length function associated with a measured foliation. We show that in this case the Teichmüller–Randers metric is incomplete in any Teichmüller disc, and we give a characterisation of geodesic rays with bounded length in this disc in terms of their directing measured foliations.

Résumé. Dans cet article, nous introduisons une nouvelle métrique asymétrique sur l’espace de Teichmüller d’une surface fermée orientable avec ou sans perforations. Cette nouvelle métrique, que nous appelons métrique de Teichmüller–Randers, est une déformation asymétrique de la métrique de Teichmüller obtenue en ajoutant à la forme infinitésimale de cette dernière une forme différentielle de degré 1. Nous étudions les propriétés de base de la métrique de Teichmüller–Randers. Nous démontrons que dans le cas où la forme différentielle est exacte, toute géodésique entre deux points pour la métrique de Teichmüller est aussi une géodésique pour la métrique de Teichmüller–Randers, et que c’est l’unique géodésique joignant ces deux points. Un cas particulièrement intéressant est celui où la forme différentielle est (à un multiple près) la différentielle du logarithme de la longueur extrême associée à un feuilletage mesuré. Nous montrons que dans ce cas la métrique de Teichmüller–Randers restreinte à un disque de Teichmüller quelconque n’est pas complète et nous caractérisons les rayons géodésiques de longueur bornée dans ce disque en fonction des feuilletages mesurés qui les dirigent.

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1. Introduction

A Randers metric is a deformation of a Riemannian or Finsler metric obtained by adding to its infinitesimal form a differential 1-form. In [15], in the case where the surface is a torus, we exhibited a natural family of Randers metrics which connects the Teichmüller metric on the Teichmüller space of that surface to its Thurston asymmetric metric. It is natural to study now the same kind of deformation of the Teichmüller metric on the Teichmüller space \( \mathcal{T}_{g,m} \) of a general closed orientable surface \( \Sigma_{g,m} \) of genus \( g \) with \( m \) punctures, and this is what we do in the present paper. It turns out that the metrics in this family are interesting to study in this general setting and this is what we propose to show in this paper.

In its original form given in [17], a Randers metric is associated with an \( n \)-dimensional Riemannian manifold \((M, g)\) and a 1-form \( \omega \) on \( M \) satisfying \( \|\omega\|_g < 1 \) at every point of \( M \). In this situation, the associated Randers metric is a Finsler asymmetric metric on \( M \) defined infinitesimally by \( F(v) = g(v, v)^{1/2} + \omega(v) \). Randers metrics have applications in the physical world, and they been widely studied since their appearance. The same construction also works when the original metric is not Riemannian, but Finsler, like in the case we study here.

In this paper we study a Randers deformation of the Teichmüller metric \( \kappa \) on \( \mathcal{T}_{g,m} \), which we call the Teichmüller–Randers metric associated with a real 1-form \( \omega \), defined using

\[
(1.1) \quad \kappa^\omega(x; v) = \kappa(x; v) + \omega(v)
\]

In a natural way, the lengths of differentiable arcs on \( \mathcal{T}_{g,m} \) can be defined using this metric, and the distance between two points is set to be the infimum of the lengths of arcs connecting them. The Teichmüller–Randers distance may take negative values for a general 1-form \( \omega \), but it gives a Finsler metric when the Teichmüller norm \( \|\omega\|_T(x) \) of \( \omega \) at \( x \), (i.e., the supremum of the value of \( \omega \) on the tangent vectors at \( x \) with Teichmüller norm \( \leq 1 \)) is less than 1 at every point \( x \) of \( \mathcal{T}_{g,m} \). The Teichmüller–Randers metric becomes a weak metric when the Teichmüller norm of \( \omega \) is 1 (see Section 2.1).

We have already introduced and studied the Teichmüller–Randers metric in the case of torus. In [2], Belkhirat, Papadopoulos and Troyanov showed that Thurston’s asymmetric metric coincides with the weak distance on the upper-half plane \( \mathbb{H} \) defined by

\[
(1.2) \quad \delta(\zeta_1, \zeta_2) = \log \sup_{x \in \mathbb{R}} \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right|
\]

for \( \zeta_1, \zeta_2 \in \mathbb{H} \) if we identify the Teichmüller space of a torus \( T \) with the hyperbolic plane by choosing a generator system \( a, b \) of \( \pi_1(T) \), and consider normalized flat structures on \( T \) such that \( a \) has length 1. In [15], we showed that this weak distance is indeed a Finsler metric and that it is also given
by the formula
\[ ds_{hyp} + \frac{1}{2}d\log \text{Im}(\zeta), \]
where \( ds_{hyp} \) is the hyperbolic metric on \( \mathbb{H} \) of constant curvature \(-4\). Since the Teichmüller metric coincides with the hyperbolic metric in this setting, the weak distance in Eq. (1.2) is nothing but the Teichmüller–Randers metric given by Eq. (1.3), that is, associated with the 1-form \( \omega = -(1/2)d\log \text{Im}(\zeta)^{-1} \). For \( 0 \leq t \leq 1 \), we define \( \delta_t \) be the weak metric defined by the Finsler norm \( ds_{hyp} + \frac{t}{2}d\log \text{Im}(\zeta) \). We also note that the 1-form \( \omega = -(1/2)d\log \text{Im}(\zeta)^{-1} \) is exact and \( \text{Im}(\zeta)^{-1} \) coincides with the extremal length of the isotopy class of simple closed curves corresponding to \( a \).

We now turn to stating our main theorems. Before that, we recall that the Teichmüller distance is a uniquely geodesic metric, and that any geodesic extends to a holomorphic disc called a Teichmüller disc. Namely, for any two points in \( \mathcal{T}_{g,m} \), there is a holomorphic (or anti-holomorphic) isometry \((\mathbb{H}, ds_{hyp}) \to (\mathcal{T}_{g,m}, d_T)\) whose image contains the two points, and this image is called a Teichmüller disc. A Teichmüller disc is determined by a holomorphic quadratic differential \( q \), hence we denote it by \( \mathbb{D}_q \) (see Section 4.2).

For a measured foliation \( F \) on \( \Sigma_{g,m} \), we denote by \( \text{Ext}_x(F) \) the function on \( \mathcal{T}_{g,m} \) taking a point \( x \) to the extremal length of a measured foliation \( F \) at \( x \), and by \( q_{F,x} \) the Hubbard–Masur differential on \( x \) for \( F \) (see Section 2.2). We shall show the following three main theorems.

**Theorem 1.1** (Geodesics of the Teichmüller–Randers metric). Let \( F \) be a measured foliation on \( \Sigma_{g,m} \), and set \( \omega = -\frac{1}{2}d\log \text{Ext}_i(F) \). Then the following hold.

(i) The (asymmetric) metric space \((\mathcal{T}_{g,m}, \delta_T^\omega)\) is a uniquely geodesic space such that the Teichmüller geodesics are the geodesics.

(ii) For any \( x \in \mathcal{T}_{g,m} \) and for any \( 0 \leq t \leq 1 \), the Teichmüller disc defined by \( q_{F,x} \) coincides with the image of an isometric embedding of \((\mathbb{H}, \delta_t)\) into \((\mathcal{T}_{g,m}, \delta_T^\omega)\).

**Theorem 1.2** (Isometric discs). Suppose that there is an isometry \( \phi: (\mathbb{H}, \delta) \to (\mathcal{T}_{g,m}, \delta_T^\omega) \) where \( \omega \) is exact and satisfies \( \|\omega\|_T \leq 1 \) in a neighbourhood of the image of \( \phi \). Then, there is a measured foliation \( F \) on \( \Sigma_{g,m} \) such that \( \phi \) is a holomorphic or anti-holomorphic isometry onto the Teichmüller disc associated with \( q_{F,x} \) with \( x = \phi(i) \), and such that \( \omega = -(1/2)d\log \text{Ext}_i(F) \) holds on the image of that isometry.

From Theorem 1.1 and Theorem 1.2 for a fixed measured foliation \( F \), we have a characteristic property of the geometry of the weak distance \( \delta_T^\omega \) with \( \omega = -(1/2)d\log \text{Ext}_i(F) \) on the Teichmüller disc defined by the Hubbard–Masur differential for \( F \). Since, by Theorem 1.1, any Teichmüller disc is
totally geodesic with respect to $\delta_T^\omega$, it is natural to ask how the weak distance $\delta_T^\omega$ behaves on Teichmüller discs other than the one associated with $q_{F,x}$.

**Theorem 1.3.** Let $F$ be a measured foliation on $\Sigma_{g,m}$, and set $\omega = -(1/2) d \log \text{Ext}(\cdot)(F)$. For any $x \in T_{g,m}$ and for any measured foliation $G$ on $\Sigma_{g,m}$, we have the following.

1. If $q_{G,x}$ is not a complex constant multiple of $q_{F,x}$, then the restriction of $\delta_T^\omega$ to the Teichmüller disc $D_{q_{G,x}}$ is a weak non-negative distance function which separates any two points.

2. The following two conditions are equivalent:
   - $\langle F, G \rangle \neq 0$.
   - The Teichmüller geodesic ray directed by $q_{G,x}$ has bounded length with respect to $\delta_T^\omega$.

In particular, the restriction of $\delta_T^\omega$ to every Teichmüller disc is incomplete.

We note that the weak distance $\delta$ on $\mathbb{H}$, which corresponds to the Teichmüller space of a torus, does not separate points. Theorem 1.3 gives a generalisation of this torus case. See Section 2.1 for more details.

As can been seen in the definition, our metric depends on the choice of the (projective class) of a measured foliation $F$. Because of this, our metric is not invariant under the entire mapping class group. Still if we consider the family of metrics making $x$ and $F$ vary, then the family is invariant under the action of the mapping class group.

Besides the theorems stated above, we shall also discuss the extension of the Hamilton-Krushkal condition (see Theorem 3.1), and the Teichmüller–Randers cometric on the cotangent space (see Theorem 3.3).

2. Preliminaries

2.1. Weak metric. A weak metric $\delta$ on a set $X$ is a map $\delta : X \times X \to \mathbb{R}$ satisfying the following.

1. $\delta(x, x) = 0$ for every $x$ in $X$;
2. $\delta(x, y) \geq 0$ for every $x$ and $y$ in $X$;
3. $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$ for every $x$, $y$ and $z$ in $X$.

A weak metric $\delta$ is said to separate points if $\delta(x_1, x_2) = 0$ for $x_1, x_2 \in X$ implies $x_1 = x_2$, and to be complete if for any sequence $(x_n)$ in $X$ satisfying $\delta(x_n, x_{n+m}) \to 0$ as $n, m \to \infty$, the sequence $(x_n)$ converges in $X$ (see [3, I.1]). (Notice that since the metric is not symmetric, the order of the arguments in $\delta(x_1, x_2)$ is important.)

In [2], the following weak metric was introduced on $\mathbb{H}$. First, for $\zeta_1 \neq \zeta_2 \in \mathbb{H}$, we set

$$M(\zeta_1, \zeta_2) = \sup_{x \in \mathbb{R}} \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right|.$$  

(2.1)

For $\zeta_1 = \zeta_2$, we set $M(\zeta_1, \zeta_2) = 1$. We set

$$\delta(\zeta_1, \zeta_2) = \log M(\zeta_1, \zeta_2) \quad (\zeta_1, \zeta_2 \in \mathbb{H}).$$  


Then, $\delta$ is an asymmetric weak metric on $\mathbb{H}$. Furthermore, $\delta$ does not separate points of $\mathbb{H}$. Indeed, when $\zeta_1 = y_1 i$, $\zeta_2 = y_2 i \in \mathbb{H}$ with $y_1 < y_2$, $\delta(\zeta_1, \zeta_2) = 0$. In particular, $\delta$ is not complete (see [2, Proposition 1]). The distance between $\zeta_1$ and $\zeta_2 \in \mathbb{H}$ is explicitly given by

$$\delta(\zeta_1, \zeta_2) = \log \left( \frac{|\zeta_2 - \overline{\zeta_1}| + |\zeta_2 - \zeta_1|}{|\zeta_1 - \overline{\zeta_1}|} \right)$$

(see [2]). Hence, any geodesic ray tending to $\mathbb{R} \subset \partial \mathbb{H}$ has bounded length, and the length of a geodesic ray is infinite only if it goes upward in the vertical direction.

We note that when we identify $\mathbb{H}$ with the Teichmüller space of a torus, the ideal boundary $\partial \mathbb{H}$ is naturally thought of as the Thurston boundary, which is the space of projective measured foliations on the torus (see [4]). Using this identification, we see that the intersection number $i(F_x, F_\infty)$ is zero if and only if $x = \infty$, where $[F_x]$ is the projective measured foliation corresponding to an arbitrary $x \in \partial \mathbb{H}$. This corresponds to the condition of Theorem 1.3 in the case of torus.

### 2.2. Teichmüller theory

We review some of Teichmüller theory. We refer the reader to [5] for more details.

#### 2.2.1. Teichmüller space.

Let $\Sigma_{g,m}$ be a closed orientable surface of type $(g,m)$, that is, of genus $g$ with $m$ points deleted. The integers $g$ and $m$ may take all nonnegative values except that if $g = 0$ we assume $m \geq 4$ and if $g = 1$ we assume that $m \geq 1$. A marked Riemann surface $(X, f)$ of type $(g, m)$ is a pair of an analytically finite Riemann surface $X$ of type $(g, m)$ and an orientation-preserving homeomorphism $f : \Sigma_{g,m} \to X$. Two marked Riemann surfaces $(X_1, f_1)$ and $(X_2, f_2)$ are said to be Teichmüller equivalent if there is a conformal mapping $h : X_1 \to X_2$ such that $h \circ f_1$ is homotopic to $f_2$. The set $T_{g,m}$ of Teichmüller equivalence classes of marked Riemann surfaces of type $(g, m)$ is called the Teichmüller space of analytically finite Riemann surfaces of type $(g, m)$. The Teichmüller distance $d_T$ on $T_{g,m}$ is defined by

$$d_T(x, y) = \frac{1}{2} \log \inf_h K(h)$$

where $h$ ranges over all quasi-conformal maps $h : X_1 \to X_2$ homotopic to $f_2 \circ f_1^{-1}$ and where $K(h)$ denotes the maximal quasiconformal dilatation of $h$. The Teichmüller space is known to be a complex manifold which is biholomorphically equivalent to a bounded domain in $\mathbb{C}^{3g-3+m}$. Furthermore, the Teichmüller distance is complete, uniquely geodesic, and coincides with the Kobayashi distance.

#### 2.2.2. Infinitesimal theory.

For a Riemann surface $X$, let $L^\infty(X)$ be the complex Banach space of bounded measurable $(-1,1)$-forms $\mu = \mu(z)(d\overline{z}/dz)$ on $X$ with the norm

$$\|\mu\|_\infty = \text{ess.sup}\{ |\mu(z)| \mid z \in X \}.$$
A form in $L^\infty(X)$ is called a Beltrami differential.

Let $A^2(X)$ be the Banach space of holomorphic quadratic differentials $\varphi = \varphi(z)dz^2$ with the norm

$$\|\varphi\|_1 = \int_X |\varphi(z)|dxdy.$$ 

There is a natural pairing between Beltrami differentials and holomorphic quadratic differentials as follows.

$$L^\infty(X) \times A^2(X) \ni (\mu, \varphi) \mapsto \int_X \mu \varphi.$$ 

Let $N^\infty(X) \subset L^\infty(X)$ be the subspace orthogonal to $A^2(X)$ with respect to the pairing. Namely,

$$N^\infty(X) := \left\{ \mu \in L^\infty(X) \mid \int_X \mu \varphi = 0, \forall \varphi \in A^2(X) \right\}.$$ 

Two Beltrami differentials $\mu$ and $\nu$ are infinitesimally Teichmüller equivalent if $\mu - \nu \in N^\infty(X)$.

Fix a basepoint $x_0 = (X_0, f_0)$ in $\mathcal{T}_{g,m}$. For $t \geq 0$ and $\varphi \in A^2(X_0)$, we denote by $F_t: X_0 \to X_t = F_t(X_0)$ a quasi-conformal map with the property that

$$\overline{\partial}F_t = \tanh(t)(\overline{\varphi}/|\varphi|)\partial F_t.$$ 

Then, a path

$$r_\varphi: [0, \infty) \ni t \mapsto (X_t, F_t \circ f_0) \in \mathcal{T}_{g,m}$$

is called a Teichmüller differential when it has a form $\mu = c\overline{\varphi}/|\varphi|$ for some $\varphi \in A^2(X) - \{0\}$. It is known that when a Beltrami differential $\mu$ is infinitesimally extremal in the sense that $\|\mu\|_\infty = \text{Re}(\varphi, \varphi)$, then $\mu$ must be a Teichmüller differential. Fix a basepoint $x_0 = (X_0, f_0)$ in $\mathcal{T}_{g,m}$. For $t \geq 0$ and $\varphi \in A^2(X_0)$, we denote by $F_t: X_0 \to X_t = F_t(X_0)$ a quasi-conformal map with the property that

$$\overline{\partial}F_t = \tanh(t)(\overline{\varphi}/|\varphi|)\partial F_t.$$ 

Then, a path

$$r_\varphi: [0, \infty) \ni t \mapsto (X_t, F_t \circ f_0) \in \mathcal{T}_{g,m}$$

is called a Teichmüller differential.
constitutes a geodesic ray with respect to $d_T$. We call such a geodesic the Teichmüller geodesic ray emanating from $x_0$. It is known that for any $x \in \mathcal{T}_{g,m} - \{x_0\}$, there is a unique Teichmüller geodesic ray passing $x$ and emanating from $x_0$. Furthermore,

$$(0, \infty) \times \{ \varphi \in A^2(X_0) \mid \| \varphi \| = 1 \} \ni (t, \varphi) \mapsto r_\varphi(t) \in \mathcal{T}_{g,m} \setminus \{x_0\}$$

is a homeomorphism.

Unless $(g, m)$ is either $(1, 1)$ or $(0, 4)$, the Teichmüller metric is not Riemannian. This was known to Teichmüller, but we can prove it just by using the fact that the group of linear isometries of a tangent (or cotangent) space of any Riemannian metric is an orthogonal group, whereas by [4], the linear isometry group of a tangent/cotangent space with respect to the Teichmüller metric is a finite union of $S^1$.

The following might be well known and follows from the discussion in the proof of Lemma 3 in [6, p. 173]. For completeness, we give a brief proof.

**Lemma 2.1** (Derivative of the Teichmüller norm). Take $x = (X, f) \in \mathcal{T}_{g,m}$ and $v_0 \in T_x \mathcal{T}_{g,m} - \{0\}$. Suppose that $v_0$ is represented by the Teichmüller differential $|\varphi_0|/|\alpha_0|$, $\|\alpha_0\| = 1$, $\beta > 0$). Then

$$\frac{d}{dt} \bigg|_{t=0} \kappa(x; v_0 + tv) = \text{Re}(v_0, \alpha_0)$$

for any $v \in T_x \mathcal{T}_{g,m}$.

**Proof.** For $t \in \mathbb{R}$, we take $\alpha_t \in A^2(X)$ with $\|\alpha_t\|_1 = 1$ such that

$$\kappa(x; v_0 + tv) = \text{Re}(v_0 + tv, \alpha_t).$$

We claim that $\mathbb{R} \ni t \mapsto \alpha_t \in A^2(X)$ is well defined and continuous. Indeed, by the Lebesgue dominated convergence theorem, the map

$$L_1 : A^2(X) \not\ni \alpha \mapsto \ell_\alpha = \left[ A^2(X) \not\ni \varphi \mapsto \|\alpha\| \int_X \frac{\varphi}{|\alpha|} \right] \in A^2(X)^*$$

is continuous with respect to the weak topology on $A^2(X)^*$. Since $A^2(X)$ is finite-dimensional, the weak topology on $A^2(X)^*$ coincides with the topology derived from the dual norm (the operator norm). One can see that $\text{Re}(\ell_\alpha(\psi)) \leq \|\psi\|_1$ for all $\psi \in A^2(X)$, and $\text{Re}(\ell_\alpha(\alpha)) = \|\alpha\|_1 \|\alpha\|_1$ if and only if $\varphi = \alpha$. Hence, $\alpha_t$ is well defined for $t \in \mathbb{R}$, and the map defined in Eq. (2.3) is injective and proper. Since $A^2(X)$ and $A^2(X)^*$ are homeomorphic to the Euclidean space $\mathbb{R}^{6g-6+2m}$, from the invariance of domains, Eq. (2.3) is homomorphic. Since

$$L_2 : T_x \mathcal{T}_{g,m} \not\ni v \mapsto [\varphi \mapsto \langle v, \varphi \rangle] \in A^2(X)^*$$

is a complex linear isomorphism, and continuous with respect to the Teichmüller metric and the dual norm, we see that the map $\mathbb{R} \ni t \mapsto \alpha_t = L_1^{-1} \circ L_2(v_0 + tv) \in A^2(X)$ is continuous.
Then,
\[\kappa(x; v_0 + tv) - \kappa(x; v_0) = \text{Re}\langle v_0 + tv, \alpha_t \rangle - \text{Re}\langle v_0, \alpha_0 \rangle \]
\[\geq \text{Re}\langle v_0 + tv, \alpha_0 \rangle - \text{Re}\langle v_0, \alpha_0 \rangle = t \text{Re}\langle v, \alpha_0 \rangle\]
and
\[\kappa(x; v_0 + tv) - \kappa(x; v_0) \leq \text{Re}\langle v_0 + tv, \alpha_t \rangle - \text{Re}\langle v_0, \alpha_t \rangle \]
\[= t \text{Re}\langle v, \alpha_0 \rangle + t \text{Re}\langle v, \alpha_t - \alpha_0 \rangle \]
as \(t \to 0\). Therefore,
\[|\kappa(x; v_0 + tv) - \kappa(x; v_0) - t \text{Re}\langle v, \alpha_0 \rangle| = o(t)\]
as \(t \to 0\). \hfill \Box

2.2.3. Measured foliations. Let \(S\) be the set of homotopy classes of non-trivial and non-peripheral simple closed curves on \(\Sigma_{g,m}\). Let \(\mathcal{W}S\) be the set of formal scalar products \(\{t\alpha \mid t \geq 0, \alpha \in S\}\), which we call the set of weighted simple closed curves on \(\Sigma_{g,m}\). Consider the embedding
\[\mathcal{W}S \ni t\alpha \mapsto \left[\mathcal{S} \ni \beta \mapsto t_i(\alpha, \beta)\right] \in \mathbb{R}_{\geq 0}.\]
We equip the function space \(\mathbb{R}_{\geq 0}^S\) with the pointwise convergence topology. The closure \(\mathcal{M}F\) of the image of \(\mathcal{W}S\) in \(\mathbb{R}_{\geq 0}^S\) is called the space of measured foliations on \(\Sigma_{g,m}\). For \(F \in \mathcal{M}F\), we call the value \(F(\alpha)\) the intersection number of \(F\) with \(\alpha\), and denote it by \(i(F, \alpha)\). Set \(i(F, t\alpha) = ti(F, \alpha)\) for \(t\alpha \in \mathcal{W}S\). It is known that the intersection number \(i(\cdot, \cdot)\) on \(\mathcal{M}F \times \mathcal{W}S\) extends continuously to a function \(i(\cdot, \cdot)\) on \(\mathcal{M}F \times \mathcal{M}F\) which satisfies \(i(F, G) = i(G, F)\) for \(F, G \in \mathcal{M}F\).

2.2.4. Hubbard–Masur differentials and Extremal length. Let \(x = (X, f)\) be a point in \(\mathcal{T}_{g,m}\). For \(q = q(z)dz^2 \in A^2(X)\), we set
\[v(q)(\alpha) = \inf_{\alpha' \in f(\alpha)} \int_{\alpha'} |\text{Re}(\sqrt{q(z)}dz)|\]
for \(\alpha \in S\). Regarding \(v(q)\) as contained in \(\mathbb{R}_{\geq 0}^S\), we call it the vertical foliation of \(q\). It is known that \(v(q) \in \mathcal{M}F\).

For \(x = (X, f) \in \mathcal{T}_{g,m}\) and \(F \in \mathcal{M}F\), there is a unique quadratic differential \(q_{F,x} \in A^2(X)\) such that \(i(F, \alpha) = v(q)(\alpha)\) for all \(\alpha \in S\). We call the differential \(q_{F,x}\) the Hubbard–Masur differential for \(F\) on \(x\). The norm
\[\text{Ext}_x(F) = \int_X |q_{F,x}(z)|dxdy\]
is called the extremal length of \(F\) on \(x\). The extremal length function
\[\mathcal{T}_{g,m} \times \mathcal{M}F \ni (x, F) \mapsto \text{Ext}_x(F)\]
The Teichmüller–Randers metric is continuous. When $F \in \mathcal{MF}$ is fixed, the extremal length function is of class $C^1$. The following formula, called the Gardiner formula, is known:

$$d\text{Ext.}(F) = -2\text{Re} \int_X \mu q_{F,x}$$

for $v = \mu \in TX \mathcal{T}_{g,m} \cong L^\infty(X)/N^\infty(X)$ (cf. [3]). Notice that the minus sign in the right-hand side of Eq. (2.5) comes from the fact that $q_{F,x}$ has $F$ as the vertical foliation, while Gardiner considers the horizontal foliations when he concludes the formula Eq. (2.5).

### 2.3. Teichmüller–Randers metric

For a given $n$-dimensional Riemannian manifold $(M, g)$ and a 1-form $\omega$ on $M$ with $||\omega||_g < 1$ at every point of $M$, the associated Randers metric is a Finsler metric on $M$ defined by $F(v) = g(v,v)^{1/2} + \omega(v)$. Although in general literatures, Randers metrics refer to deformations of Riemannian metrics by 1-forms, we can think of such deformations for Finsler (symmetric) metrics in the same way. Furthermore, even in the case when $||\omega||_g = 1$, the Randers metric makes sense as a weak Finsler metric. In this paper, we study Randers-type deformations of the Teichmüller metric $\kappa$ on $\mathcal{T}_{g,m}$ which we explained in the previous subsection, by a 1-form $\omega$ as we presented in Introduction;

$$\kappa^\omega(x;v) = \kappa(x;v) + \omega(v).$$

As a 1-form $\omega$, we shall consider in particular the form expressed as $-\frac{1}{2} d\log\text{Ext}_{\ell}(F)$ for a measured foliation $F$ on $\Sigma_{g,m}$. This metric does depends on the choice of $F$, but only on the projective class of $F$ since we are taking log in the second term. This metric can be regarded as a generalisation of the weak Finsler metric which we studied in [15].

### 2.4. References to background materials

We now give some references for background materials which we briefly explained in this section. The Teichmüller metric was introduced and studied thoroughly by Teichmüller in his paper [19] (see its English translation [20]). In this paper there is a long section (§25) on the Finsler nature of this metric. In the same section, Teichmüller introduced and studied what are now called Teichmüller discs (isometric images of the hyperbolic plane, defined by quadratic differentials), which he calls complex geodesics. As modern introductions to Teichmüller theory, we refer the to [6] and [8]. For the theory of measured foliations and measured foliation spaces, we refer the reader to [4], and for a comprehensive introduction to extremal length [14] for instance. For Randers’ metric, we refer the reader to Rander’s original paper [16].

### 3. Extension of the Hamilton–Krushkal condition

In this section, we discuss the infinitesimal extremal property for our Teichmüller–Randers metric.
Let $X$ be a Riemann surface, and fix $\varphi_0 \in A^2(X)$. We consider the following functional on the space $L^\infty(X)$ of Beltrami differentials:

\begin{equation}
\beta(\mu, \varphi_0) = \sup \left\{ \left| \int_X \mu \varphi \right| + \text{Re} \int_X \mu \varphi_0 \mid \varphi \in A^2(X), \|\varphi\|_1 = 1 \right\}
\end{equation}

for $\mu \in L^\infty(S)$. It immediately follows from the definition that

\begin{equation}
\beta(\mu, \varphi_0) \leq \|\mu\|_\infty + \text{Re} \int_X \mu \varphi_0
\end{equation}

for all $\mu \in L^\infty(X)$. We say that a Beltrami differential $\mu$ is infinitesimally $\varphi_0$-extremal if

\begin{equation}
\beta(\mu, \varphi_0) = \|\mu\|_\infty + \text{Re} \int_X \mu \varphi_0,
\end{equation}

and that $\mu$ satisfies the Hamilton condition if

\begin{equation}
\sup \left\{ \left| \int_X \mu \varphi \right| \mid \varphi \in A^2(X), \|\varphi\|_1 \leq 1 \right\} = \|\mu\|_\infty.
\end{equation}

It is known that $\mu \in L^\infty(X)$ is infinitesimally Teichmüller extremal in the sense that $\|\mu - \nu\|_\infty \geq \|\mu\|_\infty$ for all $\nu \in N^\infty(X)$ if and only if it satisfies the Hamilton condition ([7] and [10]).

**Theorem 3.1 (Extension of the Hamilton–Krushkal condition).** Let $X$ be a Riemann surface and $\varphi_0$ a holomorphic quadratic differential on $X$.

1. If two Beltrami differentials $\mu, \nu \in L^\infty(X)$ are infinitesimally Teichmüller equivalent, then $\beta(\mu, \varphi_0) = \beta(\nu, \varphi_0)$.

2. For a Beltrami differential $\mu \in L^\infty(X)$, the following three conditions are equivalent:
   
   (a) $\mu$ is infinitesimally $\varphi_0$-extremal;
   
   (b) $\mu$ is infinitesimally Teichmüller extremal; and
   
   (c) $\mu$ satisfies the Hamilton condition.

**Proof.**

1. The assumption that $\mu$ and $\nu$ are Teichmüller equivalent means by definition that $\int_X \mu \varphi = \int_X \nu \varphi$ for all $\varphi \in A^2(X)$. Hence, we have

   \[
   \left| \int_X \mu \varphi \right| + \text{Re} \int_X \mu \varphi_0 = \left| \int_X \nu \varphi \right| + \text{Re} \int_X \nu \varphi_0
   \]

   for all $\varphi \in A^2(X)$, which implies $\beta(\mu, \varphi_0) = \beta(\nu, \varphi_0)$.

2. We only need to show the equivalence between conditions (a) and (b), for the equivalence of the condition (c) with (a) and (b) follows immediately then. Suppose that $\mu$ is infinitesimally $\varphi_0$-extremal. Then for $\nu \in N^\infty(X)$, we have

   \[
   \|\mu\|_\infty + \text{Re} \int_X \mu \varphi_0 = \beta(\mu, \varphi_0) = \beta(\mu - \nu, \varphi_0)
   \]

   \[
   \leq \|\mu - \nu\|_\infty + \text{Re} \int_X (\mu - \nu) \varphi_0 = \|\mu - \nu\|_\infty + \text{Re} \int_X \mu \varphi_0,
   \]

   where $\text{Re} \int_X (\mu - \nu) \varphi_0$ is taken as the integral of a real-valued function.
and hence $\|\mu\|_\infty \leq \|\mu - \nu\|_\infty$. This means that $\mu$ is infinitesimally Teichmüller extremal.

Conversely, suppose that $\mu$ is infinitesimally Teichmüller extremal. Then, by definition, there exists a sequence $(\varphi_n)$ in $A^2(X)$ such that $\|\int_X \mu \varphi_n\| \to \|\mu\|_\infty$. Therefore,

$$\|\mu\|_\infty + \Re \int_X \mu \varphi_0 = \lim_{n \to \infty} \left| \int_X \mu \varphi_n \right| + \Re \int_X \mu \varphi_0 \leq \beta(\mu, \varphi_0).$$

Combining this with Eq. (3.2), we see that $\mu$ is infinitesimally $\varphi_0$-extremal.

$$\square$$

In the case of analytically finite Riemann surfaces, an infinitesimally Teichmüller extremal Beltrami differential is a Teichmüller Beltrami differential, and vice versa. Hence we have the following.

**Corollary 3.2** (Analytically finite case). Let $X$ be an analytically finite Riemann surface and $\varphi_0$ a holomorphic quadratic differential on $X$. Then, for $\mu \in L^\infty(X)$, the following two conditions are equivalent.

1. $\mu$ is infinitesimally $\varphi_0$-extremal; and
2. $\mu$ is a Teichmüller Beltrami differential. Namely, there are $\psi \in A^2(X) - \{0\}$ and $c \geq 0$ such that $\mu = c\overline{\psi}/|\psi|$.

### 3.1. Teichmüller–Randers cometric

Let $x = (X, f)$ be a point in $\mathcal{T}_{g,m}$, and $\omega$ a 1-form on $\mathcal{T}_{g,m}$ with $\|\omega\|_T(x) \leq 1$. We define the **Teichmüller–Randers cometric** on the space of holomorphic quadratic differentials, which identified with the cotangent space as $A^2(X) \cong T^*_x \mathcal{T}_{g,m}$, by

$$G_\omega(\varphi) = \sup_{\kappa(x,v) = 1} |\langle v, \varphi \rangle| = \sup_{\kappa(x,v) = 1} \Re \langle v, \varphi \rangle$$

for $\varphi \in A^2(X)$. This is dual to the Randers–Teichmüller metric. When $\omega = 0$, it is known that

$$G_0(\varphi) = \|\varphi\|_1.$$

Even if $\|\omega\|_T(x) < 1$, as we have seen above, $G_\omega$ defines an (asymmetric) norm on $A^2(X)$, whereas $G_\omega$ is not a norm then. Indeed, take a tangent vector $v \in T_x \mathcal{T}_{g,m}$ with $\kappa(x,v) = 1$ and $\omega(v) = -1$. Then, $\kappa^\omega(x,v) = 0$ by definition of $\kappa^\omega$. Take $\alpha \in A^2(X)$ such that $\Re \langle v, \alpha \rangle = \|\alpha\|_1 = 1$, and $\{v_n\} \subset T_x \mathcal{T}_{g,m}$ converging to $v$. Then, we have

$$\frac{\Re \langle v_n, \alpha \rangle}{\kappa^\omega(x,v_n)} \to \infty$$

as $n \to \infty$, which shows that $G_\omega$ is not a norm. For this reason, when we discuss the dual $G_\omega$, we always assume that $\|\omega\|_T(x) < 1$.

**Theorem 3.3** (Teichmüller–Randers cometric). Let $x = (X, f)$ be a point in $\mathcal{T}_{g,m}$. Suppose that $\omega$ is represented by $\psi \in A^2(X) \cong T^*_x \mathcal{T}_{g,m}$ at $x$ and that $\|\omega\|_T(x) = \|\psi\|_1 < 1$. Then,

$$G_\omega(\varphi) = \inf \left\{ t > 0 \mid \frac{\varphi}{t} - \psi \right\} \leq 1 \right\}.$$
In particular, if \( \varphi \neq 0 \), then we have
\[
\left\| \frac{\varphi}{G_\omega(\varphi)} - \psi \right\|_1 = 1.
\]

Proof. By Corollary 3.2 for a Teichmüller Beltrami differential \( \mu = \overline{\alpha}/|\alpha| \) \((\alpha \in A^2(X))\), we have
\[
(3.4) \quad \kappa^\omega(x; [\mu]) = 1 + \Re \int_X \overline{\alpha}/|\alpha| \psi.
\]
We may assume that neither \( \varphi \) nor \( \psi \) is 0, for our claim evidently holds if one of them is 0. From the definition of \( G_\omega \) and Eq. (3.4), we have
\[
(3.5) \quad G_\omega(\varphi) = \sup_{\kappa(x; v) = 1} \Re \langle v, \varphi \rangle / (1 + \Re \langle v, \psi \rangle).
\]
If \( v \) is represented by the Teichmüller Beltrami differential \( \alpha/|\alpha| \), the function in the supremum in the right-hand side of Eq. (3.5) is positive. Hence we have \( G_\omega(\varphi) > 0 \).
Let \( v_0 \) be a tangent vector represented by a Teichmüller Beltrami differential \( \alpha_0/|\alpha_0| \) \((\|\alpha_0\|_1 = 1)\) which attains the supremum in the right-hand side of Eq. (3.5). Then
\[
G_\omega(\varphi) = \frac{\Re \langle v_0, \varphi \rangle}{1 + \Re \langle v_0, \psi \rangle}.
\]
We note that \( \Re \langle v, \varphi \rangle > 0 \) and \( |\langle v_0, \psi \rangle| < 1 \) since \( G_\omega(\varphi) > 0 \) and \( \|\psi\|_1 < 1 \).
For \( v \in T_x \mathcal{T}_{g,m} \), we can compute
\[
\frac{\Re \langle v_0 + tv, \varphi \rangle}{1 + \Re \langle v_0 + tv, \psi \rangle} = G_\omega(\varphi) + tG_\omega(\varphi) \Re \left\langle v, \frac{\varphi}{\Re \langle v_0, \varphi \rangle} - \frac{\psi}{1 + \Re \langle v_0, \psi \rangle} \right\rangle + o(t)
\]
as \( t \to 0 \). As remarked above, \( G_\omega(\varphi) > 0 \). Since the left hand side of the above equality attains the supremum in \( \{ v \mid \kappa(x; v) = 1 \} \) at \( v_0 \), by Lemma 2.1 we have
\[
\Re \left\langle v, \frac{\varphi}{\Re \langle v_0, \varphi \rangle} - \frac{\psi}{1 + \Re \langle v_0, \psi \rangle} \right\rangle = 0
\]
for all \( v \in T_x \mathcal{T}_{g,m} \) with \( \Re \langle v, \alpha_0 \rangle = 0 \). This means that there exists \( t \in \mathbb{R} \) such that
\[
\frac{\varphi}{\Re \langle v_0, \varphi \rangle} - \frac{\psi}{1 + \Re \langle v_0, \psi \rangle} = t\alpha_0.
\]
By taking pairing with \( v_0 \) on both sides, we get
\[
\frac{\Re \langle v_0, \varphi \rangle}{\Re \langle v_0, \varphi \rangle} - \frac{\Re \langle v_0, \psi \rangle}{1 + \Re \langle v_0, \psi \rangle} = t\|\alpha_0\|_1 = t,
\]
which means \( t = (1 + \Re \langle v_0, \psi \rangle)^{-1} \). Thus we obtain
\[
\frac{\varphi}{G_\omega(\varphi)} - \psi = \frac{1 + \Re \langle v_0, \psi \rangle}{\Re \langle v_0, \varphi \rangle} \varphi - \psi = \alpha_0,
\]
which implies the desired equalities. \( \square \)
3.2. The case when \( \omega \) is exact. Assume that \( \omega \) is a continuous exact form, that is, \( \omega = dF_\omega \) for some \( C^1 \)-function \( F_\omega \) on \( \mathcal{T}_{g,m} \). Then, the length of any \( C^1 \)-path \( \gamma: [a, b] \to \mathcal{T}_{g,m} \) is expressed as

\[
\int_a^b \kappa_\omega(\gamma(t); \dot{\gamma}(t)) \, dt = \int_a^b (\kappa(\gamma(t); \dot{\gamma}(t)) + \omega(\dot{\gamma}(t))) \, dt
\]

\[
= \int_a^b \kappa(\gamma(t); \dot{\gamma}(t)) \, dt + F_\omega(\gamma(b)) - F_\omega(\gamma(a))
\]

with respect to the Teichmüller–Randers metric \( \kappa_\omega \). Therefore, taking the infimum on the lengths of paths connecting \( x_1 \in \mathcal{T}_{g,m} \) to \( x_2 \in \mathcal{T}_{g,m} \), the weak metric \( \delta_T^\omega \) associated with the Teichmüller–Randers metric satisfies

\[
\delta_T^\omega(x_1, x_2) = d_T(x_1, x_2) + F_\omega(x_2) - F_\omega(x_1).
\]

This inequality implies the following proposition.

**Proposition 3.4.** For any continuous exact form \( \omega \) on \( \mathcal{T}_{g,m} \), any Teichmüller geodesic is a unique geodesic with respect to the Teichmüller–Randers distance \( \delta_T^\omega \).

This gives a generalisation of Theorem 2.1 in [15]. We also note that in the case when \( \omega \) is exact, the symmetrisation of the weak metric associated with the Teichmüller–Randers metric coincides with the Teichmüller distance. Indeed, we have

\[
S(\delta_T^\omega)(x_1, x_2) = \frac{1}{2} (\delta_T^\omega(x_1, x_2) + \delta_T^\omega(x_2, x_1))
\]

\[
= \frac{1}{2} (d_T(x_1, x_2) + (F_\omega(x_2) - F_\omega(x_1)) + d_T(x_2, x_1) + (F_\omega(x_1) - F_\omega(x_2)))
\]

\[
= \frac{1}{2} (d_T(x_1, x_2) + d_T(x_2, x_1)) = d_T(x_1, x_2).
\]

**Proof of Proposition 3.4.** By Eq. (3.6), we see immediately that every Teichmüller geodesic is also a geodesic with respect to \( \delta_T^\omega \). It remains to check the uniqueness of geodesics. Let \( x_1, x_2 \in \mathcal{T}_{g,m} \) and \( \gamma: [a, b] \to \mathcal{T}_{g,m} \) be a \( C^1 \)-path connecting \( x_1 \) to \( x_2 \). If \( \gamma \) is not a Teichmüller geodesic, by the uniqueness of Teichmüller geodesics, we have

\[
d_T(x_1, x_2) + F_\omega(x_2) - F_\omega(x_1)
\]

\[
< \int_a^b (\kappa(\gamma(t); \dot{\gamma}(t)) \, dt + F_\omega(x_2) - F_\omega(x_1) = \int_a^b \kappa_\omega(\gamma(t); \dot{\gamma}(t)) \, dt,
\]

which implies that \( \gamma \) is not a geodesic with respect to \( \delta_T^\omega \) either. \( \square \)

4. Proof of theorems

4.1. Teichmüller discs. Let \( x = (X, f) \) be a point in \( \mathcal{T}_{g,m} \). For \( q \in A^2(X) \) and \( \lambda \in \mathbb{H} \), we define

\[
\mu_{\lambda, q} := \frac{\lambda - i \, q}{\overline{\lambda} + i \, |q|}.
\]
Let $f_{\lambda,q}$ be a quasi-conformal map on $X$ with $\bar{\partial}f_{\lambda,q} = \mu_{\lambda,q} \partial f_{\lambda,q}$, and set $X_{\lambda,q}$ to be the image of $f_{\lambda,q}$. The Teichmüller disc associated with $q$, which is denoted by $\mathbb{D}_q$, is a holomorphic disc in $T_{g,m}$ defined by

$$\phi_q : \mathbb{H} \ni \lambda \mapsto x(\lambda, q) := (X_{\lambda,q}, f_{\lambda,q}) \in T_{g,m}.$$  

The following lemma shows basic properties of Teichmüller discs.

**Lemma 4.1.** For $x = (X, f) \in T_{g,m}$ and a measured foliation $F$ on $S$, we have the following.

(a) The extremal length function satisfies

$$\text{Ext}_{x(\lambda, q_F,x)}(F) = \frac{1}{\text{Im}(\lambda)} \text{Ext}_x(F).$$

(b) For the Teichmüller disc defined as in Eq. (4.2) for $q = q_{F,x}$, the image of any vertical geodesic line in $\mathbb{H}$ is the Teichmüller geodesic defined by holomorphic quadratic differentials whose vertical foliations are $F$.

(c) For any measured foliation $F$ on $S$ and any $\lambda \in \mathbb{H}$, the unit tangent vector $v_\lambda = (\phi_{q_F,x})_* (2i \text{Im}(\lambda) \partial / \partial \lambda)$ is represented by $\frac{q_{F,\phi(\lambda)}}{|q_{F,\phi(\lambda)}|}$.

**Proof.** The assertions follow from the discussion by Marden and Masur in [11, §1.3]. We review some details for the convenience of the reader.

(a) We shall only show Eq. (4.3) for $\alpha \in S$. Since the weighted simple closed curves are dense in $\mathcal{MF}$ and $\mathcal{MF} \ni F \mapsto q_{F,x} \in A^2(X)$ is continuous, we can then conclude Eq. (4.3) for general measured foliations by taking limits.

One of the characterisations of the extremal length of $\alpha$ is that it is the reciprocal of the modulus of the ‘characteristic annulus’ of $q_\alpha$, that is, the maximal (open) annulus formed by closed leaves of the vertical trajectories of $q_\alpha$. (See also [18, §20.3]). By the discussion by Marden and Masur in [11, §1.3], the extremal length $\text{Ext}_{x(\lambda, q_{F,x})}(\alpha)$ satisfies

$$\text{Ext}_{x(\lambda, q_{F,x})}(\alpha) = \frac{1}{1 + \text{Re}(\lambda')} \text{Ext}_x(\alpha)$$

where $\lambda'$ is a complex number satisfying $\text{Re}(\lambda') > -1$ and

$$\frac{\lambda - i}{\lambda + i} = \frac{\lambda'}{2 + \lambda'}.$$ 

Since $\text{Re}(\lambda') = \text{Re}(-1 - i\lambda) = -1 + \text{Im}(\lambda)$, we obtain Eq. (4.3) from Eq. (4.4) in the case when $F = \alpha \in S$.

(b) Let $A$ be the characteristic annulus of $q_{\alpha,x}$. The Teichmüller map $f_{\lambda,q}$ defined by $\mu_{\lambda,q_{\alpha,x}}$ is expressed as a map $h_\lambda$ defined by

$$h_\lambda(z) = z |z|^{-i\lambda - 1} = z |z|^\text{Im}(\lambda) - 1 - i\text{Re}(\lambda)$$
on the characteristic annulus \( A \cong \{ 1 < |z| < r \} \) with \( r = \exp(2\pi/\text{Ext}_x(\alpha)) \). The image \( h_\lambda(\{ 1 < |z| < r \}) = \{ 1 < |z| < r^{\text{Im}(\lambda)} \} \) corresponds to the characteristic annulus of the terminal quadratic differential \( q_{x,F,x} \). Therefore, the deformation along the vertical line in \( \mathbb{H} \) passing through \( \lambda \in \mathbb{H} \) is the Teichmüller geodesic associated with the differential \( q_{x,F,x} \).

(c) Let \( v_\lambda \in T_{x,F,x} \mathcal{T}_{g,m} \) be the unit tangent vector in \( \mathcal{D}_{q_{F,x}} \) at \( x(\lambda, q_{F,x}) \) as given in the statement (c). Then \( v_\lambda \) is represented by a Teichmüller Beltrami differential \( \overline{\psi}/\psi \) with \( \psi \in A^2(X_{x,F,x}) \). From (a) above and the Gardiner formula Eq. (2.3),

\[
\begin{align*}
-\text{Re} \int_{X_{x,F,x}} \frac{\psi}{|\psi|} q_{F,x}(\lambda, q_{F,x}) &= \frac{1}{2} d \log \text{Ext}_{x,F}(F)[v_\lambda] \\
&= \text{Re} \left( 2i \text{Im}(\lambda) \frac{d}{d\lambda} \log \text{Ext}_{x,F}(F) \right) \\
&= \text{Re} \left( 2i \text{Im}(\lambda) \cdot \left( -\frac{1}{2i \text{Im}(\lambda)} \right) \right) \\
&= -1,
\end{align*}
\]

which means that \( \psi = q_{F,x}(\lambda, q_{F,x}) \).

4.2. Proof of Theorem 1.1. The part (i) follows from Proposition 3.4. Let \( x \) be a point in \( \mathcal{T}_{g,m} \) and let \( \phi: \mathbb{H} \to \mathcal{T}_{g,m} \) be the Teichmüller disc defined by \( q_{F,x} \) with \( \phi(i) = x \). Since \( \omega \) is exact, by Proposition 3.4 for any two points \( \zeta_1, \zeta_2 \in \mathbb{H} \), the hyperbolic geodesic \( \gamma: [a, b] \to \mathbb{H} \) connecting \( \zeta_1 \) to \( \zeta_2 \) is mapped to a geodesic with respect to \( \delta^\omega \) connecting \( x_1 = \phi(\zeta_1) \) to \( x_2 = \phi(\zeta_2) \). From Eq. (1.3) and Lemma 4.4 we have

\[
\delta^\omega(x_1, x_2) = \int_a^b (\kappa(\phi(\gamma(s)); \phi_* \gamma(t)) + \omega(\phi_* \gamma(t))) \, dt
\]

\[
= d_{\text{hyp}}(\zeta_1, \zeta_2) - \frac{1}{2} \int_{\phi(\gamma)} d \log \text{Ext}_x(F)
\]

\[
= d_{\text{hyp}}(\zeta_1, \zeta_2) + \frac{1}{2} \log \text{Ext}_{x_1}(F) - \frac{1}{2} \log \text{Ext}_{x_2}(F)
\]

\[
= d_{\text{hyp}}(\zeta_1, \zeta_2) + \frac{1}{2} \log \frac{1}{\text{Im}(\zeta_1)} - \frac{1}{2} \log \frac{1}{\text{Im}(\zeta_2)}
\]

\[
= \int_{\gamma} \left( ds_{\text{hyp}} + \frac{1}{2} d \log \text{Im}(\zeta) \right) = \delta(\zeta_1, \zeta_2),
\]

which implies the part (ii) of Theorem 1.1.

4.3. Proof of Theorem 1.2. Let \( \phi: \mathbb{H} \to \mathcal{T}_{g,m} \) be an isometry as in the statement. We may assume that \( \omega \) is exact on \( \mathcal{T}_{g,m} \) by changing it outside a neighbourhood of \( \phi(\mathbb{H}) \). Then, there is a \( C^1 \)-function \( F_\omega \) on \( \mathcal{T}_{g,m} \) such that \( dF_\omega = \omega \). We set \( f_\omega = F_\omega \circ \phi \).
Take two points \( \zeta_1 = \xi_1 + i\eta_1 \) and \( \zeta_2 = \xi_2 + i\eta_2 \in \mathbb{H} \). Let \( \gamma: [0, s_0] \to \mathbb{H} \) be a hyperbolic geodesic connecting \( \zeta_1 \) to \( \zeta_2 \). By Proposition \ref{prop:TeichmullerMap}, \( \phi\gamma: [0, s_0] \to \mathcal{T}_{g,m} \) is a Teichmüller geodesic, and since \( \phi \) is an isometry, we obtain

\[
\delta(\zeta_1, \zeta_2) = \frac{1}{2} \log \frac{\eta_2}{\eta_1} = \int_0^{s_0} (\kappa(\phi(\gamma(t))) ; \phi_* \circ \dot{\gamma}(t)) + \omega(\phi_* \circ \dot{\gamma}(t)) \, dt.
\]

(4.5)

**Case 1.** (horizontal lines) Suppose that \( \eta_1 = \eta_2 \). Since both \( d_{hyp} \) and \( d_T \) are symmetric, from Eq. (4.5), we obtain

\[
f_\omega(\zeta_1) = f_\omega(\zeta_2), \quad \text{and hence} \quad d_T(\phi(\zeta_1), \phi(\zeta_2)) = d_{hyp}(\zeta_1, \zeta_2).
\]

(4.6)

**Case 2.** (vertical lines) Suppose \( \xi_1 = \xi_2 \) and \( \eta_1 > \eta_2 \). In this case, the geodesic \( \gamma \) is a vertical segment from \( \zeta_1 \) to \( \zeta_2 \). Since \( \delta(\zeta_1, \zeta_2) = 0 \) in this case, from Eq. (4.5), we have

\[
f_\omega(\zeta_1) - f_\omega(\zeta_2) = d_T(\phi(\zeta_1), \phi(\zeta_2)).
\]

(4.7)

For \( x \in \mathbb{R} \), let \( L_\xi = \{ \xi + \eta i \mid \eta > 0 \} \). Then by Eq. (4.7), we see that \( f(L_\xi) \) is a geodesic with respect to \( d_T \). Take a measured foliation \( F_\xi \) on \( S \) such that \( \phi(L_\xi) \) is the Teichmüller geodesic defined by the Hubbard–Masur differential for \( F_\xi \). To describe this more precisely, fix \( \eta_0 > 0 \) and set \( x(\xi) = \phi(\xi + i\eta_0) \in \mathcal{T}_{g,m} \). Let \( x(\xi + i\eta) \) be the image of the Teichmüller map from \( x(\xi) \) with the Betrami differential

\[
\tanh(t) \frac{dF_\xi \cdot x(\xi)}{dF_\xi \cdot x(\xi)},
\]

(4.8)

where \( t = t(\eta) \) satisfies \( |t| = d_T(x(\xi + i\eta), x(\xi)), \), \( t > 0 \) if \( \eta > \eta_0 \), and \( t \leq 0 \) otherwise. Then we have \( \phi(L_\xi) = \{ x(\xi + i\eta) \mid \eta > 0 \} \). By the Gardiner formula Eq. (2.5), \( \text{Ext}_{x(\xi + i\eta)}(F_\xi) \) decreases as \( \eta \) increases. Hence, from the Kerckhoff formula, we have

\[
\text{Ext}_{x(\xi + i\eta)}(F_\xi) = \begin{cases} 
  e^{-2d_T(x(\xi + i\eta), x(\xi))} \text{Ext}_{x(\xi)}(F_\xi) & (\eta \geq \eta_0) \\
  e^{2d_T(x(\xi + i\eta), x(\xi))} \text{Ext}_{x(\xi)}(F_\xi) & (\eta \leq \eta_0).
\end{cases}
\]
Now, for any \( \eta, \eta' \), take \( \eta_3 > 0 \) smaller than \( \min\{\eta, \eta'\} \). From Eq. (4.17), we can compute as follows:

\[
\int_{\gamma'} \omega = f_\omega(\xi + i\eta') - f_\omega(\xi + i\eta)
\]

\[
= (f_\omega(\xi + i\eta_2) - f_\omega(\xi + i\eta_3)) - (f_\omega(\xi + i\eta_1) - f_\omega(\xi + i\eta_3))
\]

\[
= \frac{1}{2} \log \frac{\text{Ext}_x(\xi + i\eta_2)(F_\xi)}{\text{Ext}_x(\xi + i\eta_1)(F_\xi)} - \frac{1}{2} \log \frac{\text{Ext}_x(\xi + i\eta_3)(F_\xi)}{\text{Ext}_x(\xi + i\eta_1)(F_\xi)}
\]

\[
= \frac{1}{2} \int_{\gamma'} d\log \text{Ext}(F_\xi),
\]

where \( \gamma' \) is the image under \( \phi \) of the vertical segment from \( \xi + i\eta \) to \( \xi + i\eta' \) in \( \mathbb{H} \). We note that by Eq. (4.8) or (c) of Lemma 4.31, the tangent vector \( v_y \in T_y \mathcal{T}_{g,m} \) along \( \phi(L_\xi) \) at \( y = \phi(L_\xi) \) has unit length with respect to the Teichmüller metric, and is given by the Beltrami differential

\[
\frac{\eta \partial}{|\eta \partial|}.
\]

Hence we obtain

\[
\left| \frac{1}{2} d\log \text{Ext}(F_\xi)[v_y] \right| = \frac{1}{2} \frac{2}{\|\eta \partial\|} \text{Re} \int_X \frac{\eta \partial}{|\eta \partial|} \eta \partial = 1.
\]

Since \( \|\omega\|_T \leq 1 \) on the image \( \phi(\mathbb{H}) \), from Eq. (4.9), we conclude that we have

\[
\omega = -\frac{1}{2} d\log \text{Ext}(F_\xi).
\]

on \( L_\xi \).

**Case 3.** (general case) We take \( \zeta_1 = \xi_1 + i\eta_1 \) and \( \zeta_2 = \xi_2 + i\eta_2 \in \mathbb{H} \) to be arbitrary. Set \( \zeta_3 = \xi_2 + i\eta_1 \). By Eq. (4.12) we have

\[
\text{Ext}_{\phi(\xi + i\eta_1)}(F_\xi) = \frac{\eta_2}{\eta_1} \text{Ext}_{\phi(\xi + i\eta_2)}(F_\xi)
\]

for all \( \xi \in \mathbb{R} \) and \( \eta_1, \eta_2 > 0 \), and \( \text{Ext}_{\phi(\zeta_1)}(F_{\xi_2}) = \text{Ext}_{\phi(\zeta_1)}(F_{\xi_2}) \). Combining this with the argument in Case 2, we have

\[
f_\omega(\zeta_2) - f_\omega(\zeta_1) = (f_\omega(\zeta_2) - f_\omega(\zeta_3)) + (f_\omega(\zeta_3) - f_\omega(\zeta_1))
\]

\[
= -\frac{1}{2} \log \text{Ext}_{\phi(\zeta_2)}(F_{\xi_2}) + \frac{1}{2} \log \text{Ext}_{\phi(\zeta_1)}(F_{\xi_2})
\]

\[
= -\frac{1}{2} \log \frac{\eta_1}{\eta_2}.
\]

Then, from Eq. (4.15), we conclude that \( d_T(\phi(\zeta_1), \phi(\zeta_2)) = d_{h\text{hyp}}(\zeta_1, \zeta_2) \) for any \( \zeta_1, \zeta_2 \in \mathbb{H} \). Hence, \( \phi: (\mathbb{H}, d_{h\text{hyp}}) \rightarrow (\mathcal{T}_{g,m}, d_T) \) is an isometry. [1] Theorem 1.1] shows that in this situation, \( \phi \) is either holomorphic or anti-holomorphic,
and the image is the Teichmüller disc. As shown in Lemma 4.1, $F_{\xi_1} = F_{\xi_2}$ for all $\xi_1, \xi_2 \in \mathbb{R}$. Setting $F = F_{\xi} (\xi \in \mathbb{R})$, we see that the image $\phi(\mathbb{H})$ is the Teichmüller disc defined by the Hubbard–Masur differential for $F$.

Consider $\zeta = \xi + i\eta \in \mathbb{H}$ and $L_{\xi}$ defined above. By Eq. (4.6), the derivative of $f_\omega$ at $\zeta$ in the horizontal direction is constantly zero. As shown in (c) of Lemma 4.1, the image $v \in T_{\phi(\zeta)} T_{q,m}$ of the unit tangent vector $2i \text{Im}(\zeta)(\partial/\partial \zeta) \in T_{\zeta}\mathbb{H}$ to $L_{\xi}$ at $\zeta$ is represented by the Teichmüller Beltrami differential $q_{F,\phi(\zeta)} / |q_{F,\phi(\zeta)}|$. Hence, by the Gardiner formula Eq. (2.5), the derivative of the function

$$\mathbb{H} \ni \zeta \mapsto -\frac{1}{2} \log \text{Ext}_{\phi(\zeta)}(F)$$

is also zero in the horizontal direction in $\mathbb{H}$. As a consequence, by Eq. (4.11),

$$\omega = -\frac{1}{2} d\log \text{Ext}(F)$$

on the image $\phi(\mathbb{H})$.

### 4.4. Proof of Theorem 1.3

Let $x$ be a point in $T_{g,m}$, and $G$ a measured foliation on $S$.

1) Suppose that $\alpha q_{F,x} \neq q_{G,x}$ for any complex number $\alpha$, and hence $D_{q_{F,x}} \cap D_{q_{G,x}} = \{x\}$.

Then we claim the following.

**Claim 1.** For any $y \in D_{q_{G,x}}$, we have $D_{q_{F,y}} \cap D_{q_{G,y}} = \{y\}$.

**Proof.** Otherwise, there is $y \in D_{q_{G,x}}$ such that $D_{q_{F,y}}$ and $D_{q_{G,y}}$ share at least two points. By the uniqueness of the Teichmüller geodesic, $D_{q_{F,y}}$ and $D_{q_{G,y}}$ share a common Teichmüller geodesic line passing through these two points. Since $D_{q_{F,y}}$ and $D_{q_{G,y}}$ are holomorphic discs, by the identity theorem, $D_{q_{F,y}} = D_{q_{G,y}}$. Since both $x$ and $y$ lie in $D_{q_{G,x}} = D_{q_{F,y}}$, from the discussion in Lemma 4.1 (or the discussion in [11, §1.3]), we have $D_{q_{F,y}} = D_{q_{F,x}}$ and $D_{q_{G,x}} = D_{q_{G,y}}$. Therefore, we obtain

$$D_{q_{G,x}} = D_{q_{G,y}} = D_{q_{F,y}} = D_{q_{F,x}},$$

which contradicts our assumption. \qed

Let $y$ be a point in $D_{q_{G,x}}$, and $v_y$ the unit tangent vector to $D_{q_{G,x}}$ at $y$ represented by $q_{G,y} / |q_{G,y}|$ (cf. (c) of Lemma 4.1). We note that by Claim 1, $q_{G,y}$ is not a complex scalar multiple of $q_{F,y}$. Hence,

$$\left| -\frac{1}{2} d\log \text{Ext}(F)[v_y] \right| = \left| \Re \int_{X_{q_{G,y}}} q_{G,y} q_{F,x}(\lambda, q_{G,y}) / q_{G,y} / \|q_{F,y}\|_1 \right| < 1.$$
Therefore, for any compact set $K$ in $\mathbb{D}_{q_G,x}$, there is a constant $C_K < 1$ such that

$$\left| -\frac{1}{2} d \log \text{Ext}(F)[v_y] \right| \leq C_K$$

for all $y \in K$.

Let $x_1$ and $x_2$ be distinct points on $\mathbb{D}_{q_G,x}$, and $\gamma \subset \mathbb{D}_{q_G,x}$ the Teichmüller geodesic containing $x_1$ and $x_2$. From the above discussion, we have

$$\left| \frac{1}{2} \log \text{Ext}_{x_1}(F) - \frac{1}{2} \log \text{Ext}_{x_2}(F) \right| < d_T(x_1, x_2)$$

and

$$\delta_T^\omega(x_1, x_2) = d_T(x_1, x_2) + \frac{1}{2} \log \text{Ext}_{x_1}(F) - \frac{1}{2} \log \text{Ext}_{x_2}(F) > 0,$$

which implies that $\delta_T^\omega$ separates two points in $\mathbb{D}_{q_G,x}$.

(2) Let $r_T = r_{q_G,x} : [0, \infty) \to \mathcal{F}_{g,m}$ be the Teichmüller geodesic ray defined by $q_{G,x}$ with arclength parameterisation. By [13, Lemma 1], the function

$$[0, \infty) \ni t \mapsto e^{-\delta_T^\omega(x, r_{q_G}(t))} = e^{-t \left( \frac{\text{Ext}_{r_G(t)}(F)}{\text{Ext}_{x_0}(F)} \right)^{1/2}}$$

is non-increasing and tends to $\frac{\mathcal{E}(F)}{\text{Ext}_{x_0}(F)^{1/2}}$ as $t \to \infty$ where $\mathcal{E}$ is some continuous function defined on $\mathcal{M}\mathcal{F}$ (see also [12 Theorem 1.1]). Let $G = G_1 + \cdots + G_m$ be the decomposition of $G$ into indecomposable components (for detail, see [12]). In [21 Corollary 1], Walsh showed that the limit function $\mathcal{E}$ is expressed as

$$\mathcal{E}(H) = \sqrt{\sum_{i=1}^{m} \frac{i(G_i, H)^2}{i(G_i, \mathcal{H}(q_{G,x}))}}$$

for $H \in \mathcal{M}\mathcal{F}$, where $\mathcal{H}(q_{G,x})$ is the horizontal foliation of $q_{G,x}$. Therefore, $\mathcal{E}(H) = 0$ if and only if $i(G, H) = 0$. This means that $\delta_T^\omega(x, r_{q_G}(t))$ is uniformly bounded in terms of $t \geq 0$ if and only if $i(F, G) \neq 0$.

Finally, we prove the incompleteness of the restriction of $\delta_T^\omega$ to any Teichmüller disc. Let $x$ be a point in $\mathcal{F}_{g,m}$ and $G$ a measured foliation on $S$. By [9 Theorem 2], the vertical foliation $G_0$ of $e^{i\theta} q_{G,x}$ is uniquely ergodic for almost every $\theta$. Therefore, $i(F, G_0) \neq 0$ for almost every $\theta$. It follows that almost all Teichmüller geodesic rays emanating from $x$ in $\mathbb{D}_{q_G,x}$ have bounded length with respect to the distance $\delta_T^\omega$, and in particular, the restriction of $\delta_T^\omega$ to $\mathbb{D}_{q_G,x}$ is incomplete.

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