I. INTRODUCTION

Hardy’s paradox is the simplest demonstration of Bell nonlocality, the impossibility of describing all quantum correlations in terms of local hidden variables [1–3]. In the original form, the paradox occurs when some two-particle entangled states are measured by two observers, each with two von Neumann measurements, each with two outcomes. However, if one wants to study nonlocality from the point of paradoxes in a systematic way, one must investigate a general paradox for any given number of parties, settings, and outcomes, and check the tightness of Bell’s inequalities induced by the paradox.

For two spin-half particles, increasing the number of settings at each end is an efficient method to improve the success probability of demonstration of “nonlocality without inequalities” [4]. Subsequently, a similar proof of nonlocality is extended to two spin-1 particles [5], and using 5-settings at each end the proof worked for 0.171 of pairs. But, for a long time, the methods of extending Hardy’s paradox to high-dimensional systems can not improve the maximal success probability without inequalities [5, 6–8].

Until 2013, from an paradox, equivalent to Hardy’s paradox for spin-1/2 systems, for spin-s (s ≥ 1/2) system [9], the maximal probability of the nonlocal events can grow with the dimension of the local systems. Up to now, a number of experiments have been carried out to confirm the nonlocality without inequalities [4, 10–16] in two-particle systems. Hardy’s paradox has also been extended to the case of more than two particles [17, 18].

Hardy’s paradox for multi-settings and high-dimensional systems

Recently, Chen et al introduced an alternative form of Hardy’s paradox for 2-settings and high-dimensional systems [Phy. Rev. A 88, 062116 (2013)], in which there is a great progress in improving the maximum probability of the nonlocal event. Here, we construct a general Hardy’s paradox for multi-settings and high-dimensional systems, which (i) includes the paradox in [Phy. Rev. A 88, 062116 (2013)] as a special case, (ii) for spin-1/2 systems, is equivalent to the ladder proof of nonlocality without inequalities in [Phy. Rev. Lett. 13, 2755 (1997)], (iii) for spin-1 systems, increases the maximum probability of the nonlocal event by adding the number of settings, specially, with only 5-settings it can be improved to 0.40184, which is more than two times higher than 0.171, the maximal success probability to prove nonlocality in Adan’s paradox [Phy. Rev. A 58, 1687 (1998)].

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II. HARDY’S PARADOX FOR k-SETTINGS AND d-DIMENSIONAL SYSTEMS

A bipartite d-dimensional system can be described by a quantum pure state

$$|\psi\rangle = \sum_{i,j} h_{ij} |i\rangle \otimes |j\rangle, \quad i, j = 0, 1, ..., d - 1,$$

where |i⟩’s are a set of orthonomalous bases, and h_{ij}’s denote the coefficients that satisfy the normalization requirement: $$\sum_{i,j} |h_{ij}|^2 = 1$$. Below, for simplicity, it is tacitly assumed that these coefficients are all real-valued. The state can thus be represented uniquely by a coeﬃ-
cient matrix

\[ H = \begin{bmatrix}
  \vdots & \vdots \\
  h_{ij} & \vdots \\
  \vdots & \vdots
\end{bmatrix}. \tag{2} \]

In this paper, we consider two observers: Alice, who can make only one measurement \( A_i \) from a set of \( \{ A_i : i = 1, 2, \ldots, k \} \) on her subsystem, and Bob, who can also make only one measurement \( B_j \) from a set of \( \{ B_j : j = 1, 2, \ldots, k \} \) on his. Suppose that each of these measurements has \( d \) outcomes that we will number as \( 0, 1, 2, \ldots, d - 1 \). In the following, we assume that the von Neumann measurements \( \text{VNM}s \) form as

\[ A_i = \{ |A_{i,0}\rangle, |A_{i,1}\rangle, \ldots, |A_{i,d-1}\rangle \}, \]

\[ B_i = \{ |B_{i,0}\rangle, |B_{i,1}\rangle, \ldots, |B_{i,d-1}\rangle \}, \]

where

\[ \{|A_{i,s}\rangle : s = 0, 1, \ldots, d - 1\} \]

and

\[ \{|B_{i,t}\rangle : t = 0, 1, \ldots, d - 1\} \]

are orthonormal bases of \( d \)-dimensional system \( \mathbb{C}^d \). Following the symbols used in [9], \( P(A_i < B_j) \) denoted the joint conditional probability that the result of \( A_i \) is strictly smaller than the result of \( B_j \). Then

\[ P(A_i < B_j) = \sum_{s=0}^{d-2} \sum_{t=s+1}^{d-1} |\langle \psi | A_{i,s} \rangle B_j t |^2. \tag{3} \]

Following from the fact that, according to quantum theory, there exist two-qudit entangled states and local measurements satisfying, simultaneously,

\[ P(A_k < B_k) > 0, \]

\[ P(A_i < B_{i-1}) = 0, \]

\[ P(B_{i-1} < A_i) = 0, \]

\[ P(A_i < B_k) = 0, \tag{4} \]

for any \( i = 2, 3, \ldots, k \). However, if events \( A_i \), \( B_i, A_{i-1}, B_{i-1} < A_i, (i = 2, 3, \ldots, k) \), \( A_1 < B_k \) never happen, then, in any local theory, the event \( A_k < B_k \) never happen. For \( k = 2 \), it is just the Hardy’s paradox for \( d \)-dimensional systems presented by Chen in [9]. Therefore, (4) is a general Hardy’s paradox for \( k \)-settings and \( d \)-dimensional systems.

Let us define

\[ SP_{k,d} = \max P(A_k < B_k) \tag{5} \]

satisfying conditions in (4). Then \( SP_{k,d} \) denotes the maximal successful probability to prove nonlocality in Hardy’s paradox (4) for \( k \)-settings and \( d \)-dimensional systems.

### III. HARDY’S PARADOX (4) FOR TWO-QUBIT SYSTEMS

For two-qubit systems, each of both Alice’s and Bob’s measurements has only 2 outcomes, 0 or 1. By the constrain conditions \( P(A_k < B_{k-1}) = 0 \) and \( P(B_{k-1} < A_{k-1}) = 0 \) in (4), we obtain

\[ \langle \psi | A_{k,0} \rangle B_{k-1,1} = 0, \text{i.e.,} |B_{k-1,1} \rangle \perp H^T |A_{k,0} \rangle \]

and

\[ \langle \psi | A_{k-1,1} \rangle B_{k-1,0} = 0, \text{i.e.,} |A_{k-1,1} \rangle \perp H |B_{k-1,0} \rangle \]

respectively, which imply

\[ |B_{k-1,0} \rangle \propto H^T |A_{k,0} \rangle \]

by the orthogonality of \( |B_{k-1,1} \rangle \) and \( |B_{k-1,0} \rangle \), and

\[ |A_{k-1,0} \rangle \propto H |B_{k-1,0} \rangle \propto H H^T |A_{k,0} \rangle, \]

by the orthogonality of \( |A_{k-1,0} \rangle \) and \( |A_{k-1,1} \rangle \). Similarly, for any \( i = 1, 2, \ldots, k \), we can represent \( |A_{i,0} \rangle, |A_{i,1} \rangle, |B_{i,0} \rangle, |B_{i,1} \rangle \) with \( |A_{0,0} \rangle \) and \( H \). In particular,

\[ |B_{k,0} \rangle \propto H^T (H H^T)^{k-1} |A_{k,0} \rangle. \]

To calculate \( SP_{k,2} \), it is sufficient to take

\[ H = \begin{pmatrix} \cos \theta & 0 \\ 0 & \sin \theta \end{pmatrix}, |A_{k,0} \rangle = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \]

and then

\[ |B_{k,0} \rangle \propto \begin{pmatrix} \cos^2 k-1 \theta \cos \phi \\ \sin^2 k-1 \theta \sin \phi \end{pmatrix}, \]

\[ |B_{k,1} \rangle \propto \begin{pmatrix} \sin^2 k-1 \theta \sin \phi \\ -\cos^2 k-1 \theta \cos \phi \end{pmatrix}. \]

Therefore,

\[ P(A_k < B_k) = |\langle \psi | A_{k,0} \rangle B_{k,1} |^2 \]

\[ = \sin^2 \phi \cos^2 \theta \cos^2 \phi \cos^2 \theta \sin^2 \theta (\sin^2 k-2 \theta - \cos^2 k-2 \theta)^2 \]

\[ \frac{\sin^2 k-1 \theta \sin \phi)^2 + (\cos^2 k-1 \theta \cos \phi)^2}, \]

which implies that the paradox in (4) is equivalent to the ladder proof of nonlocality in [4]. In table I, we list \( SP_{k,d} \) for \( k = 2, 3, 4, 5, 6 \) and \( d = 2 \). As \( k \to \infty \), \( SP_{2,2} \to 0.5 \) and \( P(A_k < B_k) = 0 \) for the maximally entangled state.

| \( k \) | \( SP_{k,2} \) |
|---|---|
| 2 | 0.09017 |
| 3 | 0.17455 |
| 4 | 0.23126 |
| 5 | 0.27088 |
| 6 | 0.29995 |
IV. HARDY’S PARADOX (4) FOR TWO-QUTRIT SYSTEMS

Two-qutrit system is much richer than two-qubit system. For 2-setting scenarios, [9] introduced alternative formulation of Hardy’s paradox, which is just the paradox (4) with \( k = 2 \), and numerically proved that \( SP_{2,3} = P_{\text{Hardy}} = 0.1413 \) the maximal probability of nonlocal events can be higher than \( SP_{2,2} = P_{\text{Hardy}} = 0.0917 \). Next, we investigate \( SP_{k,3} \) for Hardy’s paradox (4) as follows:

\( k=3 \): It is sufficient to let Alice and Bob choose VNMs:

\[
A_1 = B_1 = \{ |0\rangle \langle 0|, |1\rangle \langle 1|, |2\rangle \langle 2| \},
\]

then condition \( P(B_1 < A_1) = 0 \) leads to \( h_{ij} = 0 \) for \( i > j \). This implies that the matrix \( H \) is an upper-triangular matrix. Condition \( P(A_1 < B_3) = 0 \) leads to

\[
\langle B_{3,2} | \perp \langle 0 | H, \langle B_{3,2} | \perp \langle 1 | H, \langle B_{3,1} | \perp \langle 0 | H,
\]

because of mutual orthogonality of \( \langle B_{3,2} | \), \( \langle B_{3,1} | \) and \( \langle B_{3,0} | \), we can use entries of \( H \) to denote the VNM \( B_3 \). Condition \( P(A_2 < B_1) = 0 \) implies

\[
\langle A_{2,0} | \perp H | 1 |, \langle A_{2,0} | \perp H | 2 |, \langle A_{2,1} | \perp H | 2 |
\]

because of mutual orthogonality of \( \langle A_{2,0} | \), \( \langle A_{2,1} | \) and \( \langle A_{2,2} | \), we can use entries of \( H \) to denote the VNM \( A_2 \). Similarly, by the constraint conditions in Hardy’s paradox (4), we can also use the elements of \( H \) to denote VNMs \( B_2 \) and \( A_3 \). Thus,

\[
P(A_3 < B_3) = \langle \phi | A_{3,0} | B_{3,1} |^2 + \langle \phi | A_{3,1} | B_{3,2} |^2 + \langle \phi | A_{3,1} | B_{3,2} |^2
\]

is a function over entries of \( H \). Even though the analytic expression of the function is complex, we can compute its maximal value 0.267769 numerically, which is obtained when the system is in the state, written in the representation of \( H \),

\[
H = \begin{pmatrix} 0.636671 & 0.289003 & 0.197472 \\ 0 & 0.473914 & 0.164544 \\ 0 & 0 & 0.469534 \end{pmatrix}.
\]

To help people reproduce the above results, we list the optimal local measurements in the appendix.

In conclusion, there exist two-qutrit and local measurements satisfying the constraint conditions \( P(A_3 < B_2) = P(B_2 < A_2) = P(A_2 < B_1) = P(B_1 < A_1) = P(A_1 < B_3) = 0 \) such that the maximal probability of nonlocal event is \( P(A_3 < B_3) \approx 0.267769 \).

\( k=4 \): Without loss of generality, let Alice and Bob choose VNMs:

\[
A_2 = B_2 = \{ |0\rangle \langle 0|, |1\rangle \langle 1|, |2\rangle \langle 2| \},
\]

for paradox (4), we are capable of only using entries of \( H \) to denote the probability of the nonlocal event \( A_4 < B_4 \), and can compute its maximal value 0.348158 numerically, which is obtained when the system is in the state

\[
H = \begin{pmatrix} 0.551527 & -0.209186 & 0.184342 \\ 0 & 0.519748 & -0.209186 \\ 0 & 0 & 0.551527 \end{pmatrix},
\]

and the optimal local measurements are listed in the appendix.

In short, there exist two-qutrit and local measurements satisfying the constraint conditions \( P(A_4 < B_4) = P(B_4 < A_4) = P(A_4 < B_4) = P(B_2 < A_2) = P(A_2 < B_3) = P(B_1 < A_1) = P(A_1 < B_3) \approx 0.348158 \).

\( k=5 \): Let Alice and Bob choose VNMs:

\[
A_3 = B_3 = \{ |0\rangle \langle 0|, |1\rangle \langle 1|, |2\rangle \langle 2| \},
\]

we also have the ability to obtain the maximal probability of nonlocal event \( P(A_5 < B_5) \approx 0.40184 \), which is attained when the system’s state

\[
H = \begin{pmatrix} 0.560108 & 0 & 0 \\ 0.17891 & 0.534063 & 0 \\ 0.152703 & 0.17891 & 0.560108 \end{pmatrix},
\]

and the optimal local measurements, listed in the appendix, satisfy the constraint conditions \( P(A_5 < B_4) = P(B_4 < A_4) = P(A_4 < B_4) = P(B_3 < A_3) = P(A_3 < B_3) = P(B_2 < A_2) = P(A_2 < B_1) = P(B_1 < A_1) = P(A_1 < B_5) = 0 \).

The calculations for \( k > 5 \) are beyond our computer’s capability. In table II, we list \( SP_{k,3} \) for \( k = 2, 3, 4, 5 \).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\( k \) & 2 & 3 & 4 & 5 \\
\hline
\( SP_{k,3} \) & 0.141327 & 0.267769 & 0.348158 & 0.40184 \\
\hline
\end{tabular}
\caption{For two-qutrit systems, the maximal successful probability to prove nonlocality in Hardy’s paradox (4) for \( k = 2, 3, 4, 5 \).}
\end{table}

Remark 1 For the maximally entangled state (MES), the corresponding matrix \( H \) can be written as \( \frac{1}{2} \sqrt{3} I_3 \), and then from the above discussion one can obtain that the constrain conditions in (4) imply \( A_k = B_{k-1} = A_{k-1} = B_{k-2} = \cdots = A_1 = B_k \), which means that MES does not violate the paradox (4). Even though the paradox introduced by [5] holds for MES, paradox (4) has more than two times successful probability 0.40184 than 0.171 that of the paradox in [5].
that improving the success probability to prove nonlocality makes the paradox more adequate for experimental observation of Hardy-like nonlocality and for applications based on this type of nonlocality. It is worth to

V. GENERALIZED HARDY’S INEQUALITIES
FOR k-SETTINGS AND d-DIMENSIONAL SYSTEMS

Based on the paradox (4) with k-settings and two d-dimensional systems, we can have the corresponding generalized Hardy’s inequality as

\[
GH_{k,d}(x, y, z) = \min\{x, y, z\} P(A_k < B_k) - x \sum_{i=2}^{k} P(A_i < B_{i-1}) - y \sum_{i=2}^{k} P(B_{i-1} < A_{i-1}) - z P(A_1 < B_k) \leq 0. \tag{6}
\]

with \(x > 0, y > 0, z > 0\). Usually for convenience, one can choose \(x, y, z\) as positive integers, and the coefficient \(\min\{x, y, z\}\) is used to make the inequality satisfied by local theory.

Remark 2 In table III, we list the maximally values allowed by quantum theory and the maximal entangled states for some general Hardy’s inequalities (6). Then we find that the optimal states are the nonmaximally entangled states.

Remark 3 In table IV, we list the tightness of inequalities \(GH_{k,d}(x, y, z) \leq 0\), where “\(\leq\)” means that we have not found suitable \(x, y, z\) such that the inequality \(GH_{k,d}(x, y, z) \leq 0\) is tight. Then we conjecture that \(GH_{k,2}(1, 1, 1) \leq 0\) is tight if and only if \(k = 2\) and for \(d = 3\) there always exist tight inequalities (6) for any \(k > 2\).

Remark 4 In table V, we list the nonlocal threshold value (NTV) of the general Hardy’s inequalities (6) with \(x = y = z = 1\) and \(d = 2\), the Hardy’s inequalities in [4] and the chained Bell inequalities [20, 21]. This means that, based on the visibility criterion, they are equivalent.

VI. CONCLUSIONS

In this paper, we have presented a general Hardy’s paradox for \(k\)-settings and spin-\(s\) systems which generalizes a ladder proof of nonlocality without inequalities and Chen’s alternative form of Hardy’s paradox. It is well known that improving the success probability to prove nonlocality makes the paradox more adequate for experimental observation of Hardy-like nonlocality and for applications based on this type of nonlocality. It is worth to

TABLE V. The NTV of the inequalities \(GH_{k,2}(1, 1, 1)\), Hardy inequalities in [4] and the chained inequalities.

| \(k = 3\) | \(k = 4\) | \(k = 5\) |
|---|---|---|
| NTV of \(GH_{k,2}(1, 1, 1)\) | 0.7698 | 0.811794 | 0.8411697 |
| NTV of Hardy inequalities in [4] | 0.7698 | 0.811794 | 0.8411697 |
| NTV of the chained inequalities | 0.7698 | 0.811794 | 0.8411697 |

| \(|A_{1,0}\)\rangle = |0\rangle, \(|A_{1,1}\)\rangle = |1\rangle, \(|A_{1,2}\)\rangle = |2\rangle, |
| \(|A_{2,0}\)\rangle = \begin{pmatrix} 0.840759 \\ -0.512713 \\ -0.173922 \end{pmatrix}, \(|A_{2,1}\)\rangle = \begin{pmatrix} 0.39627 \\ 0.801645 \\ -0.447589 \end{pmatrix}, |
| \(|A_{2,2}\)\rangle = \begin{pmatrix} 0.368908 \\ 0.307394 \\ 0.877163 \end{pmatrix}, \(|A_{3,0}\)\rangle = \begin{pmatrix} 0.604857 \\ -0.783492 \\ -0.142437 \end{pmatrix}, |
| \(|A_{3,1}\)\rangle = \begin{pmatrix} -0.361768 \\ -0.429694 \\ 0.827337 \end{pmatrix}, \(|A_{3,2}\)\rangle = \begin{pmatrix} 0.709416 \\ 0.448892 \\ 0.543346 \end{pmatrix}, |

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APPENDIX

A. Optimal local measurements in paradox (4) for 3-settings and spin-1 systems
TABLE III. Maximal values (MV) allowed by the quantum theory (QT) and the maximally entangled states (MES) for (6).

|                         | $k = 2$   | $k = 3$   | $k = 4$   | $k = 5$   |
|-------------------------|-----------|-----------|-----------|-----------|
| MV of $GH_{k,3}(1, 1, 1)$ allowed by QT | 0.304951  | 0.429015  | 0.491126  | 0.527868  |
| MV of $GH_{k,3}(1, 1, 1)$ allowed by MES | 0.290078  | 0.414408  | 0.47795   | 0.516216  |
| MV of $GH_{k,3}(2, 1, 1)$ allowed by QT | 0.268075  | 0.393554  | 0.460468  | 0.501445  |
| MV of $GH_{k,3}(2, 1, 1)$ allowed by MES | 0.240655  | 0.364543  | 0.434436  | 0.478489  |

TABLE IV. Tightness of general Hardy’s inequality for $k$-settings and two $d$-dimensional systems.

| Tightness of (6) | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ |
|------------------|---------|---------|---------|---------|---------|
| $d = 2$ | $x = y = z = 1$, Yes | - | - | - | - |
| $d = 3$ | - | $x = 2, y = z = 1$, Yes | $x = y = z = 1$, Yes | $x = y = z = 1$, Yes | $x = y = z = 1$, Yes | - |
| $d = 4$ | - | $x = 2, y = z = 1$, Yes | $x = y = z = 1$, Yes | $x = y = z = 1$, Yes | - | - |
| $d = 5$ | - | - | - | - | - | - |

$$|B_{1,0}⟩ = |0⟩, |B_{1,1}⟩ = |1⟩, |B_{1,2}⟩ = |2⟩,$$

$$|B_{2,0}⟩ = \begin{pmatrix} -0.0700471 \\ 0.0357429 \\ 0.0138888 \end{pmatrix}, |B_{2,1}⟩ = \begin{pmatrix} 0.00836581 \\ 0.021817 \\ -0.0139536 \end{pmatrix},$$

$$|B_{2,2}⟩ = \begin{pmatrix} 0.125725 \\ 0.13505 \\ 0.286533 \end{pmatrix}, |B_{3,0}⟩ = \begin{pmatrix} 0.876299 \\ 0.397777 \\ 0.271796 \end{pmatrix},$$

$$|B_{3,1}⟩ = \begin{pmatrix} -0.460165 \\ 0.858127 \\ 0.227742 \end{pmatrix}, |B_{3,2}⟩ = \begin{pmatrix} 0.142645 \\ 0.324641 \\ -0.935019 \end{pmatrix}.$$

$$|A_{1,0}⟩ = \begin{pmatrix} 0.905512 \\ -0.349204 \\ 0.241051 \end{pmatrix}, |A_{1,1}⟩ = \begin{pmatrix} -0.399076 \\ -0.893898 \\ 0.204168 \end{pmatrix},$$

$$|A_{1,2}⟩ = \begin{pmatrix} 0.144179 \\ -0.281074 \\ -0.948794 \end{pmatrix},$$

$$|A_{2,0}⟩ = |0⟩, |A_{2,1}⟩ = |1⟩, |A_{2,2}⟩ = |2⟩,$$

$$|A_{3,0}⟩ = \begin{pmatrix} 0.914794 \\ 0.368182 \\ -0.166114 \end{pmatrix}, |A_{3,1}⟩ = \begin{pmatrix} -0.272352 \\ 0.865949 \\ 0.419472 \end{pmatrix},$$

$$|A_{4,0}⟩ = \begin{pmatrix} 0.298288 \\ -0.338489 \\ 0.89244 \end{pmatrix}, |A_{4,1}⟩ = \begin{pmatrix} 0.307453 \\ -0.639559 \\ -0.704583 \end{pmatrix},$$

$$|A_{4,2}⟩ = \begin{pmatrix} 0.573965 \\ -0.514156 \\ 0.68856 \end{pmatrix},$$

$$|B_{1,0}⟩ = \begin{pmatrix} 0.89244 \\ -0.338489 \\ 0.298288 \end{pmatrix}, |B_{1,1}⟩ = \begin{pmatrix} 0.166114 \\ -0.368182 \\ -0.914794 \end{pmatrix},$$

$$|B_{1,2}⟩ = |0⟩, |B_{2,1}⟩ = |1⟩, |B_{2,2}⟩ = |2⟩.$$
C. Optimal local measurements in paradox (4) for 5-settings and spin-1 systems

\[ |A_{1,0}⟩ = \begin{pmatrix} 0.749824 \\ 0.477546 \\ 0.457945 \end{pmatrix}, \quad |A_{1,1}⟩ = \begin{pmatrix} 0.634354 \\ -0.715587 \\ -0.292456 \end{pmatrix}, \quad |B_{2,0}⟩ = |0⟩, |B_{2,1}⟩ = |1⟩, |B_{2,2}⟩ = |2⟩, \]

\[ |A_{1,2}⟩ = \begin{pmatrix} 0.188038 \\ 0.50979 \\ -0.839497 \end{pmatrix}, \quad |A_{2,0}⟩ = \begin{pmatrix} 0.921999 \\ 0.294506 \\ 0.251365 \end{pmatrix}, \quad |B_{3,0}⟩ = |0⟩, |B_{3,1}⟩ = |1⟩, |B_{3,2}⟩ = |2⟩, \]

\[ |A_{2,1}⟩ = \begin{pmatrix} -0.354751 \\ 0.902659 \\ 0.243634 \end{pmatrix}, \quad |A_{2,2}⟩ = \begin{pmatrix} 0.155145 \\ 0.313803 \\ -0.936727 \end{pmatrix}, \quad |B_{3,2}⟩ = |0⟩, |B_{4,0}⟩ = |1⟩, |B_{4,1}⟩ = |2⟩, \]

\[ |A_{3,0}⟩ = |0⟩, |A_{3,1}⟩ = |1⟩, |A_{3,2}⟩ = |2⟩, \quad |B_{4,2}⟩ = |0⟩, |B_{4,1}⟩ = |1⟩, |B_{4,0}⟩ = |2⟩, \]

\[ |A_{4,0}⟩ = \begin{pmatrix} -0.958481 \\ 0.248819 \\ 0.139295 \end{pmatrix}, \quad |A_{4,1}⟩ = \begin{pmatrix} 0.187595 \\ 0.918099 \\ -0.349146 \end{pmatrix}, \quad |B_{5,0}⟩ = |0⟩, |B_{5,1}⟩ = |1⟩, |B_{5,2}⟩ = |2⟩, \]

\[ |A_{4,2}⟩ = \begin{pmatrix} 0.214761 \\ 0.308518 \\ 0.926658 \end{pmatrix}, \quad |A_{5,0}⟩ = \begin{pmatrix} 0.898853 \\ -0.40709 \\ -0.162297 \end{pmatrix}, \quad |B_{5,0}⟩ = |0⟩, |B_{5,1}⟩ = |1⟩, |B_{5,2}⟩ = |2⟩, \]

\[ |A_{5,1}⟩ = \begin{pmatrix} -0.251302 \\ -0.782172 \\ 0.570136 \end{pmatrix}, \quad |A_{5,2}⟩ = \begin{pmatrix} 0.35904 \\ 0.471683 \\ 0.80536 \end{pmatrix}, \quad |B_{5,2}⟩ = |0⟩, |B_{5,0}⟩ = |1⟩, |B_{5,1}⟩ = |2⟩, \]

\[ |B_{1,0}⟩ = \begin{pmatrix} 0.926658 \\ 0.308518 \\ 0.214761 \end{pmatrix}, \quad |B_{1,1}⟩ = \begin{pmatrix} 0.349146 \\ -0.918099 \end{pmatrix}, \quad |B_{1,2}⟩ = \begin{pmatrix} 0.139295 \\ 0.248819 \\ -0.958481 \end{pmatrix}, \]

\[ |B_{2,0}⟩ = |0⟩, |B_{2,1}⟩ = |1⟩, |B_{2,2}⟩ = |2⟩, \quad |B_{3,0}⟩ = |0⟩, |B_{3,1}⟩ = |1⟩, |B_{3,2}⟩ = |2⟩, \]

\[ |B_{4,0}⟩ = |0⟩, |B_{4,1}⟩ = |1⟩, |B_{4,2}⟩ = |2⟩, \quad |B_{5,0}⟩ = |0⟩, |B_{5,1}⟩ = |1⟩, |B_{5,2}⟩ = |2⟩. \]

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