VOLUMES, TRACES AND ZETA FUNCTIONS

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ABSTRACT. Let $Q(x)$ be a quadratic form over $\mathbb{R}^n$. The Epstein zeta function associated to $Q(x)$ is a well known function in number theory. We generalize the construction of the Epstein zeta function to a class of function $\phi(x)$ defined in $\mathbb{R}^n$ that we call $A-$homogeneous, where $A$ is a real square matrix of order $n$ having each eigenvalue in the left half space $\text{Re}\lambda > 0$. Such a class includes all the homogeneous polynomials (positive outside the origin) and all the norms on $\mathbb{R}^n$ which are smooth outside the origin. As in the classical (i.e. quadratic) case we prove that such zeta functions are obtained from the Mellin transforms of theta function of Jacobi type associated to the $A-$homogeneous function $\phi(x)$. We prove that the zeta function associated to a $A-$homogeneous function $\phi(x)$ which is positive and smooth outside the origin is an entire meromorphic function having a unique simple pole at $s = \alpha$ the trace of the matrix $A$ with residue given by the product of the trace $\alpha$ and the Lebesgue volume of the unit ball associated to $\phi(x)$, that is the volume of the set $x \in \mathbb{R}^n$ satisfying $\phi(x) < 1$. We also prove that the theta function associated to $\phi(x)$ has an asymptotic expansion near the origin. We find that the coefficients of such expansion depend on the values that the zeta function associated to $\phi(x)$ assumes at the negative integers.

1. INTRODUCTION

For $s \in \mathbb{C}$ we denote by $\text{Re} s$ and $\text{Im} s$ respectively the real part and the imaginary part of $s$; $\arg s$ is the principal determination of the argument of $s$, defined when $s$ is not a negative real number.

Let $n$ be a positive integer, $A$ a real square matrix of order $n$, and $t > 0$ a positive real number; we put

$$t^A := e^{(\log t)A}$$

We denote by $M^+(n, \mathbb{R})$ the set of real square matrices of order $n$ with $\text{Re}\lambda > 0$ for each eigenvalue $\lambda$; $I_n$ will denote the identity matrix of order $n$.

Given a fixed $A \in M^+(n, \mathbb{R})$ we say that a function $\phi : \mathbb{R}^n \to \mathbb{R}$ is $A-$homogeneous, if for each $t > 0$ and each $x \in \mathbb{R}^n$

$$\phi(t^A x) = t\phi(x).$$

Then the unit ball associated to $\phi$ is

$$B_\phi = \{ x \in \mathbb{R}^n \mid \phi(x) < 1 \}.$$ 

For $r > 0$ we also set

$$B_\phi(r) = \{ x \in \mathbb{R}^n \mid \phi(x) < r \}.$$
Let $A \in M^+(n, \mathbb{R})$ and let $\varphi : \mathbb{R}^n \to [0, +\infty]$ be an $A$–homogeneous function. Setting $t = 2$ and $x = 0$ in (1) we obtain $\varphi(0) = 2\varphi(0)$ and hence $\varphi(0) = 0$. We say that $\varphi$ is positive if $\varphi(x) > 0$ when $x \neq 0$.

Let us point out some examples of $A$–homogeneous functions.

Any homogeneous polynomial of degree $d$ is $d^{-1}I_n$–homogeneous.

If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is the Minkowsky functional associated to an open star-like domain $D \subset \mathbb{R}^n$ with respect to the origin then $\varphi$ is the unique $I_n$–homogeneous function such that $B_\varphi = D$.

Let now $A \in M^+(n, \mathbb{R})$ and let $\varphi : \mathbb{R}^n \to [0, +\infty]$ be a continuous positive $A$–homogeneous function. Given a complex variable $s \in \mathbb{C}$ we define

$$
\zeta(\varphi, s) = \sum_{\omega \in \mathbb{Z}^n \setminus \{0\}} \varphi(\omega)^{-s},
$$

when the series on the right hand side converges.

We say that $\zeta(\varphi, s)$ is the $\zeta$-function associated to $\varphi$.

The $\theta$–functions associated to $\varphi$ is defined when $\tau$ is a complex number in the upper half plane $\text{Im} \tau > 0$ by the series

$$
\theta(\varphi, \tau) = \sum_{\omega \in \mathbb{Z}^n} e^{i\tau \varphi(\omega)},
$$

and

$$
\theta^*(\varphi, \tau) = \theta(\varphi, \tau) - 1 = \sum_{\omega \in \mathbb{Z}^n \setminus \{0\}} e^{i\tau \varphi(\omega)}.
$$

The purpose of this paper is to study convergence and analytic continuation of such $\zeta$-functions and to give asymptotic expansion for the corresponding $\theta$–functions.

Let us fix now some further notations.

For each (finite) set $X$ we denote by $\#(X)$ the cardinality of $X$.

For each (measurable) subset $E \subset \mathbb{R}^n$ we denote by $|E|$ the Lebesgue measure of $E$.

When $D \subset \mathbb{R}^n$ is an open domain we denote (as usual) by $C^0(D)$ (resp. $C^k(D)$ and $C^\infty(D)$) the space of the real continuous (resp. differentiable of order $k$ and indefinitely differentiable) functions on $D$.

The main results of this paper are the following:

**Theorem 1.1.** $A \in M^+(n, \mathbb{R})$ be a square matrix and let $\alpha$ be the trace of the matrix $A$.

Let $\varphi \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ be a continuous positive $A$–homogeneous function. Then the series on the right hand side of (5) converges to a holomorphic function on the half space $\text{Re} \ s > \alpha$ and extends to a holomorphic function on $\mathbb{C} \setminus \{\alpha\}$ having a simple pole at $s = \alpha$ with residue

$$
\text{Res}_{s=\alpha} \frac{\zeta(\varphi, s)}{s-\alpha} = \alpha |B_\varphi|.
$$

We also have $\zeta(\varphi, 0) = -1$.

When $n = 1$ and $\varphi(x) = |x|$ then $\zeta(\varphi, s) = 2\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function; in this case the result is a classical one.

When $n > 1$ and $\varphi$ is a positive definite quadratic form then $\varphi$ is $A$–homogeneous for $A = 2^{-1}I_n$; in this case $\zeta(\varphi, s)$ is the Epstein zeta function associated to $\varphi$ and the result also is well known.
When \( \varphi \) is a homogeneous polynomial of degree \( d \) such that \( \varphi(x) > 0 \) when \( x \neq 0 \) then \( \varphi \) is \( A \)-homogeneous for \( A = d^{-1}I_n \); in this case the meromorphic extension of \( \zeta(\varphi, s) \) has been established in [1] but without an explicit computation of the residue at the (unique) pole \( n/d \) of \( \zeta(\varphi, s) \).

**Theorem 1.2.** Let \( A \in M^+(n, \mathbb{R}) \) be a square matrix with trace \( \alpha \).
Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be a continuous positive \( A \)-homogeneous function Then:

1. the series on the right hand side of (5) converges when \( \Re s > \alpha \);
2. if \( \sigma \in \mathbb{R} \) we have
   \[
   \lim_{\sigma \to \alpha^+} (\sigma - \alpha) \zeta(\varphi, \sigma) = \alpha |B_\varphi|;
   \]
3. if \( r > 0 \) we have
   \[
   \lim_{r \to +\infty} \frac{\#(B_\varphi(r) \cap \mathbb{Z}^n)}{r^\alpha} = |B_\varphi|;
   \]

Our next general result on the \( \zeta \) functions says that when \( \varphi \) is not smooth throughout \( \mathbb{R}^n \setminus \{0\} \) any result concerning the analytic continuation of the series \( \zeta(\varphi, s) \) cannot be given simply by approximation with respect to the \( C^0 \) topology (and so we are compelled to a tricky approach).

**Theorem 1.3.** Let \( A \in M^+(n, \mathbb{R}) \) be a square matrix with trace \( \alpha \).
Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be a continuous positive \( A \)-homogeneous function Given any \( \delta > 0, \varepsilon > 0 \) (arbitrarily small) and \( M > 0 \) (arbitrarily large) there exists \( \psi \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\}) \) such that for each \( x \in \mathbb{R}^n \)

\[
\varphi(x) \leq \psi(x) \leq (1 + \varepsilon)\varphi(x)
\]

but

\[
\sup_{|s - \alpha| = \delta} |\zeta(\psi, s)| \geq M.
\]

The following theorem describes the behaviour of the \( \theta \)-function associated to a continuous positive \( A \)-homogeneous function smooth on \( \mathbb{R}^n \setminus \{0\} \).

If \( \sigma \in \mathbb{R} \) we adopt the following notation for the Cauchy integral of a holomorphic function over a vertical line

\[
\int_{(\sigma)} f(s)ds = \int_{\sigma - i\infty}^{\sigma + i\infty} f(s)ds = \lim_{T \to +\infty} \int_{\sigma - iT}^{\sigma + iT} f(s)ds.
\]

**Theorem 1.4.** Let \( \varphi \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\}) \) be a positive \( A \)-homogeneous function.
Let \( w \) be a complex number, and assume \( \Re w > 0 \). Let \( N \) be a positive integer and let \( 0 < \varepsilon < 1 \). Then

\[
\theta(\varphi, iw) = \Gamma(\alpha + 1) |B_\varphi| w^{-\alpha} + \sum_{k=1}^{N} \frac{(-1)^k \zeta(\varphi, -k)}{k!} w^k
\]

\[
+ \frac{1}{2\pi i} \int_{(-N-1+\varepsilon)}^{\varepsilon} \Gamma(s) \zeta(\varphi, s) w^{-s}ds.
\]
Moreover, given $0 < \delta < \pi/2$, there exists $c > 0$ which depends on $N$, $\varepsilon$ and $\delta$ only such that when $\text{Re} w > 0$ and $|\arg w| \leq \pi/2 - \delta$ then

\begin{equation}
\left| \int_{(-N-1+\varepsilon)} \Gamma(s) \zeta(\varphi, s)w^{-s}ds \right| \leq c |w|^{N+1-\varepsilon}.
\end{equation}

The paper is organized as follows.

In section 2 we fix some basic notations and recall various known results that we need in the sequel of the paper.

In section 3 we study the Mellin transform of a large class of theta function associated to any function $g(x)$ which satisfies $g(x) = O(||x||^\sigma)$ as $||x|| \to +\infty$ and $\hat{g}(y) = O(||y||^\tau)$ as $||y|| \to +\infty$, where $\hat{g}(y)$ is the Fourier transform of $g(x)$ and $\sigma, \tau > n$.

In section 4 we give a detailed description of the behaviour of an $A$–homogeneous functions $\phi(x)$ near the origin and for $||x|| \to \infty$.

In the remaining sections we give the proofs of the main results of this paper.

2. NOTATIONS AND SOME BASIC RESULTS

We will denote by $c_1, c_2, \ldots$ suitable real positive constants.

If $f \in C^1(\mathbb{R}^n)$ and $1 \leq i \leq n$ we denote by $D_i f$ the derivative of $f$ with respect to the variable $x_i$ and also set $D f = (D_1 f, \ldots, D_n f)$.

When $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$ we set $x^p = x_1^{p_1} \cdots x_n^{p_n}$ and $|p| = p_1 + \cdots + p_n$. We also set $D^p = D_1^{p_1} \cdots D_n^{p_n}$, as usual.

When $x, y \in \mathbb{R}^n$ we denote by $\langle x, y \rangle$ the Euclidean inner product of the vectors $x$ and $y$.

$GL(n, \mathbb{R})$ is the group of all invertible real square matrices of order $n$.

The transpose of the matrix $A$ is denoted by $A^t$.

When $s$ is a complex variable $\Gamma(s)$ denotes the Euler Gamma function.

Let us recall that $L^1(\mathbb{R}^n)$ is the space of absolutely integrable functions on $\mathbb{R}^n$.

For $f \in L^1(\mathbb{R}^n)$

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e^{-2\pi i (x,y)}dx$$

is the Fourier transform of $f$.

It is well known that if $A$ is any square real matrix then, for some positive constant $\beta$,

$$\left| t^A x \right| \leq c_1 t^\beta \|x\| \quad (x \in \mathbb{R}^n, 1 \leq t < +\infty).$$

Let $L$ be a positive definite symmetric matrix. We say that $L$ is a Ljapunov matrix for the matrix $A$ if

\begin{equation}
\langle Ax, Lx \rangle > 0
\end{equation}

for each $x \in \mathbb{R}^n \setminus \{0\}$.

The following proposition give some characterizations of the space $M^+(n, \mathbb{R})$.

**Proposition 2.1.** Let $n$ be a positive integer and let $A$ be a real square matrix of order $n$. Then the following conditions are equivalent.

1. $A \in M^+(n, \mathbb{R})$;
2. there exists a Ljapunov matrix for the matrix $A$;
(3) there exists a positive real constant $\gamma$ such that
\[ x \in \mathbb{R}^n, \ 0 < t \leq 1 \implies \left\| t^A x \right\| \leq c_1 t \gamma \left\| x \right\|. \]

For a proof of the proposition see e.g. [3], section 22.

The proof of the following proposition is straightforward.

**Proposition 2.2.** Let $A \in M^+(n, \mathbb{R})$. Then there exist real constants $0 < \gamma \leq \beta < +\infty$ such that for each $x \in \mathbb{R}^n$

\[ t \geq 1 \implies c_1 t^\gamma \left\| x \right\| \leq \left\| t^A x \right\| \leq c_2 t^\beta \left\| x \right\|, \]

\[ 0 < t \leq 1 \implies c_1 t^\beta \left\| x \right\| \leq \left\| t^A x \right\| \leq c_2 t^\gamma \left\| x \right\|. \]

The following proposition describes the “A—polar coordinates” when $A \in M^+(n, \mathbb{R})$.

**Proposition 2.3.** Let $A \in M^+(n, \mathbb{R})$. Let $0 < \gamma \leq \beta < +\infty$ be as in Proposition 2.2. Let $L$ be a Ljapunov matrix for $A$ and set
\[ S_L = \{ x \in \mathbb{R}^n \mid \langle Lx, x \rangle = 1 \}. \]

Then the map
\[ S_L \times [0, +\infty[ \ni (x, t) \mapsto x = t^A x \in \mathbb{R}^n \setminus \{0\} \]

is a diffeomorphism and when $x \in \mathbb{R}^n \setminus \{0\}, \ x \in S_L$ and $t \in [0, +\infty[$ then

\[ t \geq 1, \ x = t^A x \implies c_1 \left\| x \right\|^{\frac{\gamma}{\beta}} \leq t \leq c_2 \left\| x \right\|^{\frac{\beta}{\gamma}}, \]

\[ 0 < t \leq 1, \ x = t^A x \implies c_3 \left\| x \right\|^{\frac{\beta}{\gamma}} \leq t \leq c_4 \left\| x \right\|^{\frac{\beta}{\gamma}}. \]

For a proof of the proposition see e.g. [3], section 22.

Let $\mu$ be a Radon measure on the half line $]0, +\infty[$ and let $s \in \mathbb{C}, \ a, b \in \mathbb{R}$, with $a < b$.

The Mellin transform of the measure $\mu$ is defined formally by the Mellin integral
\[ \hat{\mu}(s) = \int_{]0, +\infty[} t^{s-1} d\mu(t). \]

We also set
\[ \hat{\mu}^-(s) = \int_{]0,1[} t^{s-1} d\mu(t), \]

\[ \hat{\mu}^+(s) = \int_{[1, +\infty[} t^{s-1} d\mu(t) \]

and say that $\hat{\mu}^-(s)$ (resp. $\hat{\mu}^+(s)$) converges if the integral on the right hand side of (12) (resp. (13)) is absolutely convergent.

When $a \in \mathbb{R}$ we say that the function $\hat{\mu}^-(s)$ (resp. $\hat{\mu}^+(s)$) is defined on the half plane $\text{Re} \ s > a$ (resp. $\text{Re} \ s < a$) if there exists $a' \geq a$ (resp. $a' \leq a$) such that the integral defining $\hat{\mu}^-(s)$ (resp. $\hat{\mu}^+(s)$) converges absolutely when $\text{Re} \ s > a'$ (resp. $\text{Re} \ s < a'$) and defines a holomorphic function which extends meromorphically on the larger half plane $\text{Re} \ s > a$ (resp. $\text{Re} \ s < a$).
When \( a, b \in \mathbb{R} \) satisfies \( a < b \) we say that the Mellin transform \( \hat{\mu}(s) \) is defined on the strip \( a < \text{Re} \, s < b \) if the function \( \hat{\mu}^{-}(s) \) is defined when \( \text{Re} \, s > a \), and the function \( \hat{\mu}^{+}(s) \) is defined when \( \text{Re} \, s < b \); in this case we define

\[
\hat{\mu}(s) = \hat{\mu}^{-}(s) + \hat{\mu}^{+}(s).
\]

We recall now Phragmén-Landau’s theorem on Dirichlet integrals (see e.g. [10], Theorem 6, pag. 111)

**Theorem 2.1.** Let \( \mu \) be a positive Radon measure on \( \mathbb{R} \) having support on \( [\varepsilon, +\infty[ \) for some \( \varepsilon > 0 \). Let \( a, a' \in \mathbb{R}, \ a \leq a' \), and suppose that

\[
D_{\mu}(s) = \int_{0}^{+\infty} t^{-s} \, d\mu(t)
\]

converges absolutely when \( \text{Re} \, s > a' \) and extends holomorphically on the larger half space \( \text{Re} \, s > a \). Then the integral \( D_{\mu}(s) \) also is absolutely convergent when \( \text{Re} \, s > a \).

Lastly we recall the Ikehara-Wiener theorem (see e.g. [6], pag. 305).

**Theorem 2.2.** Let \( \mu \) be a positive Radon measure on \( \mathbb{R} \) having support on \( [\varepsilon, +\infty[ \) for some \( \varepsilon > 0 \) and for \( x > 0 \) set

\[
F(x) = \int_{[0,x]} d\mu(t) = \mu([0,x])
\]

Let \( a \in \mathbb{R}, \ a \leq 0 \), and suppose that

\[
D_{\mu}(s) = \int_{0}^{+\infty} t^{-s} \, d\mu(t)
\]

converges absolutely when \( \text{Re} \, s > a \) and extends to a meromorphic function in neighbourhood of the region \( \text{Re} \, s \geq a \) having no pole except for a simple pole at \( s = a \) with residue \( R \). Then

\[
\lim_{x \to +\infty} \frac{F(x)}{x^{a}} = \frac{R}{a}.
\]

3. FUNCTIONAL EQUATIONS

This section is reminiscent of the classical arguments related to the functional equation of the Riemann zeta function.

Although in a totally different setting, our approach looks formally like the Tate’s thesis treatment of the global functional equation for zeta function associated to functions defined on adeles of a number field: compare Theorem 12 and Theorem 13 of e.g. [6], pag. 205-206) with our theorem (3.2).

From now to the end of the paper \( n \) is a positive integer, \( A \in M^{+}(n, \mathbb{R}), \ \alpha > 0 \) is the trace of the matrix \( A \) and \( 0 < \beta \leq \gamma < +\infty \) are constants satisfying (8) and (9) of proposition [2.2].

Let \( \sigma \) and \( \tau \) two positive constants.
Given \( g \in C^0(\mathbb{R}^n) \) and \( t > 0 \) we put
\[
\theta_A(g, it) = \sum_{\omega \in \mathbb{Z}^n} g(t^A \omega),
\]
\[
\theta_A^*(g, it) = \sum_{\omega \in \mathbb{Z}^n \setminus \{0\}} g(t^A \omega) = \theta_A(g, t) - g(0)
\]
when the series on the right hand side converge absolutely.

Observe that such theta function are defined on a (possibly empty) subset of the half line of the complex plane having zero real part and positive imaginary part.

When \( \theta_A^*(g, it) \) is defined for each \( t > 0 \) we denote the “Mellin transform” of the Radon measure \( \theta_A^*(g, it) \, dt \) by
\[
\xi_A(g, s) = \int_0^{+\infty} \theta_A^*(g, it) \frac{dt}{t},
\]
\[
\xi_A^+(g, s) = \int_1^{+\infty} \theta_A^*(g, it) \frac{dt}{t},
\]
\[
\xi_A^-(g, s) = \int_0^1 \theta_A^*(g, it) \frac{dt}{t},
\]
where \( s \) is a complex variable (see the discussion at the end of the previous section).

**Definition 3.1.** Let \( \sigma > 0 \) be a positive real constant. We denote by
\[
\mathcal{S}_\sigma(\mathbb{R}^n)
\]
the space of all continuous function \( g : \mathbb{R}^n \to \mathbb{C} \) such that
\[
\|g\|_{\sigma} = \sup_{x \in \mathbb{R}^n} |g(x)| (1 + ||x||^\sigma) < +\infty,
\]

**Definition 3.2.** Let \( \sigma > 0 \) and \( \tau > 0 \) be two positive real constants. We denote by
\[
\mathcal{S}_\tau^\sigma(\mathbb{R}^n)
\]
the space of all continuous function \( g : \mathbb{R}^n \to \mathbb{C} \) such that \( \|g\|_{\sigma} < +\infty \) and \( \|\hat{g}\|_\tau < +\infty \), where \( \hat{g} \) is the Fourier transform of \( g \).

Observe that \( \mathcal{S}_\tau^\sigma(\mathbb{R}^n) \) endowed with the norm \( \|g\|_{\sigma, \tau} = \|g\|_\sigma + \|\hat{g}\|_\tau \) is a Banach space and the Fourier transform is an isometry between the two spaces \( \mathcal{S}_\sigma^\sigma(\mathbb{R}^n) \) and \( \mathcal{S}_\tau^\tau(\mathbb{R}^n) \). Moreover,
\[
\bigcap_{\sigma > 0, \tau > 0} \mathcal{S}_\tau^\sigma(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)
\]
is the usual Schwartz space of smooth function \( g \in C^\infty(\mathbb{R}^n) \) such that
\[
\sup_{x \in \mathbb{R}^n} |x^p D^q g(x)| < +\infty
\]
for each \( p, q \in \mathbb{N}^n \).

**Lemma 3.1.** Let \( g \in \mathcal{S}_\sigma(\mathbb{R}^n) \). Then, for each \( t > 0 \) and each \( x \in \mathbb{R}^n \setminus \{0\} \),
\[
t \geq 1 \quad \Rightarrow \quad |g(t^A x)| \leq c_1 \|g\|_\sigma t^{-\gamma \sigma} ||x||^{-\sigma},
\]
\[
0 < t \leq 1 \quad \Rightarrow \quad |g(t^A x)| \leq c_2 \|g\|_\sigma t^{-\beta \sigma} ||x||^{-\sigma}.
\]
where the constants $c_1$ and $c_2$ depend only on $A$.

**Proof.** Since $g \in \mathcal{S}_\sigma(\mathbb{R}^n)$ then

$$|g(t^Ax)| \leq \frac{\|g\|_\sigma}{1 + \|t^Ax\|^\sigma} \leq \frac{\|g\|_\sigma}{\|t^Ax\|^\sigma}.$$  

By (8) and (9) we have $|t^Ax| \geq c_1 t^\gamma \|x\|$ when $t \geq 1$ and $|t^Ax| \geq c_2 t^\beta \|x\|$ when $0 < t \leq 1$ and the assertion follows. 

**Lemma 3.2.** Let $g \in \mathcal{S}_\sigma(\mathbb{R}^n)$ and $\sigma > n$. Then the theta functions $\theta_A(g, it)$ and $\theta_A^*(g, it)$ are defined for each $t > 0$ and

(14) \hspace{1cm} t \geq 1 \implies |\theta_A^*(g, t)| \leq c_1 \|g\|_\sigma t^{-\gamma \sigma},

(15) \hspace{1cm} 0 < t \leq 1 \implies |\theta_A^*(g, t)| \leq c_2 \|g\|_\sigma t^{-\beta \sigma},

where the constant $c_1$ and $c_2$ depend only on $A$ and $\sigma$.

**Proof.** The assertion follows immediately from the previous lemma, observing that if $\sigma > n$ then

$$\sum_{\omega \in \mathbb{Z}^n \setminus \{0\}} \|\omega\|^{-\sigma} < +\infty.$$  

**Theorem 3.1.** Let $g \in \mathcal{S}_\sigma(\mathbb{R}^n)$ with $\sigma > n$ and $\tau > n$. Then the theta functions $\theta_A(g, t)$ and $\theta_A^*(\hat{g}, it)$ are defined for each $t > 0$ and satisfies the identity

(16) \hspace{1cm} \theta_A(g, \frac{i}{t}) = t^\alpha \theta_A^*(\hat{g}, it).$$

**Proof.** It suffices to apply the Poisson summation formula (see e.g. [10], Theorem 1, pag. 91, or [9], Corollary 2.6, pag. 252) to the function $g_t(x) = g(t^{-A}x)$, observing that $\hat{g}_t(x) = t^\alpha \hat{g}(t^{A'}x)$. 

**Lemma 3.3.** Let $g \in \mathcal{S}_\sigma(\mathbb{R}^n)$ and $\sigma > n$.

If $\text{Re}_s < \gamma \sigma$ then $\xi_A^+(g, s)$ converges and satisfies

(17) \hspace{1cm} |\xi_A^+(g, s)| \leq \frac{c_1 \|g\|_\sigma}{\gamma \sigma - \text{Re}_s}.

If $\text{Re}_s > \beta \sigma$ then $\xi_A^-(g, s)$ converges and satisfies

(18) \hspace{1cm} |\xi_A^-(g, s)| \leq \frac{c_2 \|g\|_\sigma}{\text{Re}_s - \beta \sigma}.

The constant $c_1$ and $c_2$ depend only on $A$ and $\sigma$. 


Proposition 3.1. Let \( g \in \mathcal{S}_\sigma^\tau (\mathbb{R}^n) \) with \( \sigma > n \) and \( \tau > n \). Then \( \xi^-_A (g, s) \) converges when \( \text{Re} s > \alpha \) and is defined when \( \text{Re} s > \alpha - \gamma \tau \). Moreover, \( \xi^+_{A'} (\hat{g}, \alpha - s) \) converges when \( \text{Re} s > \alpha - \gamma \tau \) and satisfies the identities

\[
(19) \quad \xi^-_A (g, s) = -\frac{g(0)}{s} - \frac{\hat{g}(0)}{\alpha - s} + \xi^+_{A'} (\hat{g}, \alpha - s)
\]

**Proof.** After inserting \( t^{-1} \) in (16) we easily obtain

\[
\theta^*_A (g, it) = -g(0) + \hat{g}(0)t^{-\alpha} - t^{-\alpha}\theta^*_{A'} (\hat{g}, \frac{i}{t}).
\]

By (14), when \( 0 < t \leq 1 \)

\[
\left| t^{-\alpha}\theta^*_{A'} (\hat{g}, \frac{i}{t}) \right| \leq c_1 \| \hat{g} \|_\tau t^{\gamma \tau - \alpha},
\]

and hence, when \( \text{Re} s > \alpha \),

\[
\xi^-_A (g, s) = -\frac{g(0)}{s} - \frac{\hat{g}(0)}{\alpha - s} + \int_0^1 \theta^*_{A'} (\hat{g}, \frac{i}{t}) t^{s-\alpha} \frac{dt}{t},
\]

where all the integrals are absolutely convergent if \( \text{Re} s > \alpha \).

Making the change of variable \( t \to 1/t \) in the last integral we obtain

\[
\int_0^1 \theta^*_{A'} (\hat{g}, \frac{i}{t}) t^{s-\alpha} \frac{dt}{t} = \int_1^{\infty} \theta^*_{A'} (\hat{g}, it) t^{\alpha-\gamma s} \frac{dt}{t} = \xi^+_{A'} (\hat{g}, \alpha - s).
\]

To end the proof it is enough to note that, by the previous lemma, \( \xi^+_{A'} (\hat{g}, \alpha - s) \) converges when \( \text{Re} s > \alpha - \gamma \tau \). //

**Theorem 3.2.** Let \( g \in \mathcal{S}_\sigma^\tau (\mathbb{R}^n) \) with \( \sigma > n \) and \( \tau > n \). Assume also that

\[
\sigma + \tau > \frac{\alpha}{\gamma}.
\]

Then the functions \( \xi_A (g, s) \) and \( \xi_{A'} (\hat{g}, s) \) are defined in the strip

\[
\alpha - \gamma \tau < \text{Re} s < \gamma \sigma
\]

and satisfy the identities

\[
(20) \quad \xi_A (g, s) = -\frac{g(0)}{s} - \frac{\hat{g}(0)}{\alpha - s} + \xi^+_A (g, s) + \xi^+_{A'} (\hat{g}, \alpha - s)
\]

and

\[
(21) \quad \xi_A (g, s) = \xi_{A'} (\hat{g}, \alpha - s).
\]
Proof. Condition (20) ensures that the strip defined in (21) is not empty.

By proposition (3.1), \( \xi_A^-(g, s) \) is defined when \( \text{Re } s > \alpha - \gamma \tau \) and satisfies

\[
\xi_A^-(g, s) = -\frac{g(0)}{s} - \frac{\hat{g}(0)}{\alpha - s} + \xi_A^+(\hat{g}, \alpha - s)
\]

Adding \( \xi_A^+(g, s) \), which by lemma (3.3) converges when \( \text{Re } s < \gamma \sigma \), we obtain immediately (22).

Replacing \( g \) with its Fourier transform \( \hat{g} \), \( A \) with its transpose and \( s \) with \( \alpha - s \) in equation (22) we easily obtain

\[
(24) \quad \xi_A^+(\hat{g}, \alpha - s) = -\frac{\hat{g}(0)}{s} - \frac{\hat{g}(0)}{\alpha - s} + \xi_A^+(\hat{g}, \alpha - s),
\]

where \( \hat{g} \) is the function defined for each \( x \in \mathbb{R}^n \) as \( \hat{g}(x) = g(-x) \).

Of course \( \hat{g}(0) = g(0) \), and by the symmetry with respect to the origin of \( \mathbb{Z}^n \setminus \{0\} \) obviously

\[
\theta_A^*(\hat{g}, it) = \theta_A^*(g, it)
\]

which implies

\[
\xi_A^+(\hat{g}, s) = \xi_A^+(g, s)
\]

and so obtaining

\[
(25) \quad \xi_A^+(\hat{g}, \alpha - s) = -\frac{\hat{g}(0)}{s} - \frac{\hat{g}(0)}{\alpha - s} + \xi_A^+(\hat{g}, \alpha - s).
\]

The identity (23) follows now by comparing (22) and (25).

4. \( A \)-HOMOGENEOUS FUNCTIONS

The purpose of this section is to prove that if \( \varphi \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\}) \) is \( A \)-homogeneous and positive then the functions of the form \( e^{-\varphi^A} \) and \( \varphi^\mu e^{-\varphi} \) are in \( \mathcal{A}_\sigma^T(\mathbb{R}^n) \) for each \( \sigma > 0 \) and for suitable \( \tau > n \) when the exponents \( \lambda \) and \( \mu \) are large enough.

From elementary calculus we have the following:

**Lemma 4.1.** Let \( \Omega \in \mathbb{R}^n \) be an open domain and let \( f \in C^1(\Omega) \). Let \( \Lambda = (\lambda_{ij}) \in GL(n, \mathbb{R}) \) and set \( g = f \circ \Lambda^{-1} \). Then, for each \( x \in \Omega \),

\[
Df(\Lambda^{-1} x) = Dg(x) \Lambda,
\]

that is, for \( j = 1, \ldots, n \),

\[
(27) \quad D_{ij} f(\Lambda^{-1} x) = \sum_{i=1}^n D_i g(x) \lambda_{ij}.
\]

**Proposition 4.1.** Let \( \lambda \in \mathbb{R} \) and \( \Lambda \in GL(n, \mathbb{R}) \). Let \( k > 0 \) be an integer and let \( f \in C^k(\mathbb{R}^n \setminus \{0\}) \). Suppose that for each \( x \in \mathbb{R}^n \setminus \{0\} \)

\[
\lambda f(x) = f(\Lambda^{-1} x).
\]
Then, given \(k\) integers \(1 \leq j_1, \ldots, j_k \leq n\),

\[
D_{j_1} \cdots D_{j_k} f(\Lambda^{-1}x) = \sum_{1 \leq i_1, \ldots, i_k \leq n} D_{i_1} \cdots D_{i_k} f(x) \lambda \prod_{l=1}^{k} \lambda_{i_l j_l}.
\]

Proof. The proof is by induction on \(k\). Using (28) and (27),

\[
D_{j_1} f(\Lambda^{-1}x) = \sum_{i_1=1}^{n} D_{i_1} (f \circ \Lambda^{-1})(x) \lambda_{i_1 j_1}
\]

which is just (29) when \(k = 1\).

Assume that (29) holds for \(k - 1\), that is

\[
D_{j_1} \cdots D_{j_{k-1}} f(\Lambda^{-1}x) = \sum_{1 \leq i_1, \ldots, j_{k-1} \leq n} D_{i_1} \cdots D_{i_{k-1}} f(x) \lambda \prod_{l=1}^{k-1} \lambda_{i_l j_l}.
\]

Then, applying \(D_{j_k}\) and using (27) again, we obtain

\[
D_{j_1} \cdots D_{j_k} f(\Lambda^{-1}x) = D_{j_k} \left(D_{j_1} \cdots D_{j_{k-1}} f(\Lambda^{-1}x)\right) \lambda_{i_k j_k}
\]

as desired. \(\Box\)

**Proposition 4.2.** Let \(k\) and \(m\), \(0 \leq k \leq m\), be two non negative integers and let \(\varphi \in C^0(\mathbb{R}^n) \cap C^m(\mathbb{R}^n \setminus \{0\})\) be a \(A\)-homogeneous function. Then, if \(1 \leq j_1, \ldots, j_k \leq n\),

\[
\|x\| \leq 1 \quad \implies \quad |D_{j_1} \cdots D_{j_k} \varphi(x)| \leq c_1 \|x\|^{\beta - k}
\]

\[
\|x\| \geq 1 \quad \implies \quad |D_{j_1} \cdots D_{j_k} \varphi(x)| \leq c_2 \|x\|^{\beta - k}
\]

Proof. Let us prove (30). Let \(L\) be a Ljapunov matrix for \(A\) and set

\[
S_L = \{x \in \mathbb{R}^n | \langle Lx, x \rangle = 1\}.
\]

Replacing \(L\) with a \(aL\) for a suitable \(a\), if necessary, we may assume that for each \(x \in \mathbb{R}^n\)

\[
\langle Lx, x \rangle < 1 \quad \text{if} \quad \|x\| \leq 1.
\]

Since \(S_L\) is compact the quantity

\[
c_3 = \max \{|D^p \varphi(\overline{x})| | \overline{x} \in S_L, 0 \leq |p| \leq m\}
\]

is finite.

Let \(x \in \mathbb{R}^n\) and suppose that \(\|x\| \leq 1\). Choose \(\overline{x} \in S_L\) and \(t \in [0, 1]\) such that

\[
x = tA\overline{x}.
\]

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Denoting by \( e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1) \) the canonical basis of \( \mathbb{R}^n \), from (29) it follows that
\[
D_{j_1} \cdots D_{j_k} \varphi(x) = D_{j_1} \cdots D_{j_k} \varphi(t^A x)
\]
\[
= \sum_{1 \leq i_1, \ldots, i_k \leq n} D_{i_1} \cdots D_{i_k} \varphi(x) t^\sum_{l=1}^{k} \langle e_{i_l}, t^{-A} e_{i_l} \rangle,
\]
and hence, by (9) and (11),
\[
|D_{j_1} \cdots D_{j_k} \varphi(x)| \leq c_4 t^\sum_{l=1}^{k} |t^{-A} e_l| \leq c_5 t^{1-\beta k} \leq c_6 \|x\|^\frac{1}{\beta} (1-\beta k) = c_6 \|x\|^\frac{1}{\beta} - k.
\]
This proves (30); an analogous argumentation establishes (31). \( \| \)

In the same way we still have:

**Proposition 4.3.** Let \( \varphi \in C^0(\mathbb{R}^n) \) be a positive \( A \)-homogeneous function. Then, for each \( x \in \mathbb{R}^n \),
\[
\|x\| \leq 1 \implies c_1 \|x\|^\frac{1}{\beta} \leq \varphi(x) \leq c_2 \|x\|^\frac{1}{\beta}
\]
\[
\|x\| \geq 1 \implies c_3 \|x\|^\frac{1}{\beta} \leq \varphi(x) \leq c_4 \|x\|^\frac{1}{\beta}
\]

**Corollary 4.1.** Let \( m > 0 \) be a positive integer and let \( \lambda > 0 \) be a positive real number. Let \( \varphi \in C^0(\mathbb{R}^n) \cap C^m(\mathbb{R}^n \setminus \{0\}) \) be a positive \( A \)-homogeneous function. Assume that
\[
\lambda > \beta m.
\]
Then
\[
\varphi^\lambda \in C^m(\mathbb{R}^n)
\]
and for each \( p \in \mathbb{N}^n \) such that \( |p| \leq m \)
\[
D^p (\varphi^\lambda)(0) = 0.
\]

**Proof.** Observe that \( \varphi^\lambda \in C^0(\mathbb{R}^n) \cap C^m(\mathbb{R}^n \setminus \{0\}) \) and also it is a \( \lambda^{-1} A \)-homogeneous positive function; the matrix \( B = \lambda^{-1} A \) satisfies
\[
t \geq 1 \implies c_1 t^\frac{\lambda}{\beta} \|x\| \leq \|t^B x\| \leq c_2 t^\frac{\beta}{\lambda} \|x\|,
\]
\[
0 < t \leq 1 \implies c_1 t^\frac{\beta}{\lambda} \|x\| \leq \|t^B x\| \leq c_2 t^\frac{\lambda}{\beta} \|x\|.
\]

**Proposition 4.2** implies that if \( 1 \leq j_1, \ldots, j_k \leq n \), \( k \leq m \), then
\[
\|x\| \leq 1 \implies \left| D_{j_1} \cdots D_{j_k} (\varphi^\lambda)(x) \right| \leq c_3 \|x\|^\frac{1}{\beta} - k
\]
and hence, being by hypotesis,
\[
\frac{\lambda}{\beta} - k \geq \frac{\lambda}{\beta} - m > 0,
\]
we obtain
\[
\lim_{x \to 0} D_{j_1} \cdots D_{j_k} (\varphi^\lambda)(x) = 0.
\]
Since \( 1 \leq j_1, \ldots, j_k \leq n \) and \( k \leq m \) are arbitrary the assertion easily follows. \( \| \)
Theorem 4.1. Let \( m > 0 \) be a positive integer and let \( \lambda > 0 \) and \( \sigma > 0 \) be positive real numbers. Let \( \varphi \in C^0(\mathbb{R}^n) \cap C^m(\mathbb{R}^n \setminus \{0\}) \) be a positive \( A \)-homogeneous function. Assume that
\[
\lambda > \beta m.
\]
Then
\[
e^{-\varphi^\lambda}, \varphi^\lambda e^{-\varphi} \in \mathcal{S}_\sigma(\mathbb{R}^n).
\]

Proof. The functions \( e^{-\varphi^\lambda} \) and \( \varphi^\lambda e^{-\varphi} \) are obviously continuous on \( \mathbb{R}^n \).

By the estimates of proposition 4.3 we easily obtain that for \( x \in \mathbb{R}^n \) and \( ||x|| \geq 1 \),
\[
|e^{-\varphi(x)^\lambda}| \leq e^{-c_1||x||^{\frac{1}{\beta}}},
\]
and
\[
|\varphi(x)^\lambda e^{-\varphi(x)}| \leq c_2 ||x||^{\frac{1}{\beta}} e^{-c_3||x||^{\frac{1}{\sigma}}}.
\]
Since for each \( a > 0, M > 0 \)
\[
\lim_{t \to +\infty} t^a e^{-Mt} = 0
\]
it follows that for each \( \sigma > 0 \)
\[
e^{-\varphi^\lambda}, \varphi^\lambda e^{-\varphi} \in \mathcal{S}_\sigma(\mathbb{R}^n).
\]
To complete the proof it suffices to prove that the functions \( e^{-\varphi^\lambda} \) and \( \varphi^\lambda e^{-\varphi} \) are of class \( C^m \) and all their derivatives of order \( k \leq m \) are absolutely integrable on \( \mathbb{R}^n \).

Set \( \psi = \varphi^\lambda \).

Let \( k \leq m \). By induction on \( k \) it easy to prove that the derivatives of the function \( e^{-\psi} = e^{-\varphi} \) are linear combinations (with real coefficients) of functions of the form
\[
(D^{p_1} \psi \cdots D^{p_s} \psi)e^{-\psi}
\]
where
\[
|p_1| + \cdots + |p_s| = k
\]
and the derivatives of the function \( \varphi^\lambda e^{-\varphi} = \psi e^{-\varphi} \) are linear combinations (with real coefficients) of functions of the form
\[
(D^q \psi D^{p_1} \varphi \cdots D^{p_s} \varphi)e^{-\varphi}
\]
where
\[
|q| + |p_1| + \cdots + |p_s| = k.
\]

It is now easy to show that each function of the form either or is \( O(||x||^a) \) for some \( a > 0 \) as \( x \to 0 \) and \( O(||x||^b e^{-M||x||^c}) \) for some \( b, c, M > 0 \) as \( ||x|| \to +\infty \).

The proof of the theorem is therefore completed.

\( \Box \)

We end this section with some approximation results.
Proposition 4.4. Let $\varphi_1, \varphi_2 : \mathbb{R}^n \to [0, +\infty]$ be two $A$–homogeneous function. Assume that $\varphi_1$ is upper semicontinuous, $\varphi_2$ is lower semicontinuous and for each $x \in \mathbb{R}^n \setminus \{0\}$ $\varphi_1(x) < \varphi_2(x)$.

Then there exists a sequence $\psi_v \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ of positive $A$–homogeneous functions such that for each $x \in \mathbb{R}^n$

$$
\psi_1(x) \leq \varphi_2(x),
$$
$$
\psi_{v+1}(x) \leq \psi_v(x), \quad v = 1, 2, \ldots
$$

and

$$
\lim_{v \to +\infty} \psi_v(x) = \varphi_1(x).
$$

Proof. Let $L$ be a Ljapunov matrix for the matrix $A$ and set

$$
S_L = \{ x \in \mathbb{R}^n | \langle Lx, x \rangle = 1 \}.
$$

Then $S_L$ is a compact (sub)manifold. By standard approximation arguments there exists a non increasing sequence of smooth positive functions $f_v$ on $S_L$ which converges pointwise to the restriction of the function $\varphi_1$ to $S_L$ and $f_1(x) \leq \varphi_2(x)$ for each $x \in S_L$. For each $v > 0$ let $\psi_v : \mathbb{R}^n \to [0, +\infty]$ be the unique $A$-homegeneous functions which extends the function $f_v$. Then the sequence $\psi_v$ has the required properties. //

A similar argument yields:

Proposition 4.5. Let $\varphi : \mathbb{R}^n \to [0, +\infty]$ be a continuous $A$–homogeneous function and let $0 < \varepsilon < 1$. Then there exists two positive $A$–homogeneous functions $\psi_1, \psi_2 \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ such that for each $x \in \mathbb{R}^n$

$$
(1 - \varepsilon)\varphi(x) \leq \psi_1(x) \leq \varphi(x) \leq \psi_2(x) \leq (1 + \varepsilon)\varphi(x).
$$

5. Proof of Theorem 1.1

We begin with the following lemma.

Lemma 5.1. The serie (5) defining $\zeta(\varphi, s)$ converges absolutely when $\text{Re } s > \beta n$.

Proof. If $\omega \in \mathbb{Z}^n \setminus \{0\}$ then $|\omega| \geq 1$, and hence, by Proposition (4.3)

$$
|\varphi(\omega)| \geq c_1 ||\omega||^\frac{1}{\beta}.
$$

If $s \in \mathcal{C}$ then

$$
|\zeta(\varphi, s)| \leq \sum_{\omega \in \mathbb{Z}^n \setminus \{0\}} \varphi(\omega)^{-\text{Re } s} \leq c_1^{-1} \sum_{\omega \in \mathbb{Z}^n \setminus \{0\}} ||\omega||^{-\frac{\text{Re } s}{\beta}}
$$

and the latter series converges (absolutely) when $\frac{\text{Re } s}{\beta} > n$, that is, if $\text{Re } s > \beta n$. //
Proposition 5.1. Let $\phi : \mathbb{R}^n \to [0, +\infty]$ be a continuous positive $A$–homogeneous function. Let $a > 0$, $b > 0$ and $c$ be real constants. Then

(36) $g = e^{-\phi}$, $\Re s > \beta n \implies \xi_A(g, s) = \Gamma(s) \zeta(\phi, s)$,
(37) $g = e^{-a\phi}$, $\Re s > \beta n \implies \xi_A(g, s) = a^{-s} \Gamma(s) \zeta(\phi, s)$,
(38) $g = e^{-\phi^b}$, $\Re s > \beta n \implies \xi_{b^{-1}A}(g, b^{-1}s) = \Gamma\left(\frac{s}{b}\right) \zeta(\phi, s)$,
(39) $g = \phi^c e^{-\phi}$, $\Re s > \max\{\beta n, -c\}, \implies \xi_A(g, s) = \Gamma(s + c) \zeta(\phi, s)$.

Proof. Let $c \in \mathbb{R}$ and set $g = \phi^c e^{-\phi}$ for $x \neq 0$. If $\omega \in \mathbb{Z}^n \setminus \{0\}$ and $\Re s > -c$ then

$$
\int_0^{+\infty} g \left( t^A \omega \right) t^s dt = \int_0^{+\infty} \left( t \phi(\omega) \right)^c e^{-t \phi(\omega)} t^s dt
= \phi(\omega)^c \int_0^{+\infty} e^{-u} \left( \frac{u}{\phi(\omega)} \right)^{c+s} dt
= \phi(\omega)^{-s} \Gamma(c+s).
$$

By Lemma 5.1 if $\Re s > \max\{\beta n, -c\}$ then, summing on $\omega \in \mathbb{Z}^n \setminus \{0\}$, we obtain

$$
\xi_A(g, s) = \Gamma(s + c) \zeta(\phi, s)
$$

and this proves (39).

Setting $c = 0$ in (39) we obtain (36).

Let $a > 0$. Then the assertion (37) follows from (36) applied to the $A$–homogeneous function $\psi = a\phi$, observing that when $\Re s > \beta n$ we trivially have

$$
\zeta(\psi, s) = a^{-s} \zeta(\phi, s).
$$

Let now $b > 0$. Then $\phi^b$ is a positive $b^{-1}A$–homogeneous function and the assertion (38) follows replacing in (36) the matrix $A$ with $b^{-1}A$, $\phi$ with $\phi^b$, and $s$ with $\frac{s}{b}$, observing that when $\Re s > \beta n$ we have

$$
\zeta(\phi^b, \frac{s}{b}) = \zeta(\phi, s).
$$

Proposition 5.2. Let $\phi \in C^0(\mathbb{R}^n)$ be a positive $A$–homogeneous function. If $a > 0$ is a positive constant then

(40) $\int_{\mathbb{R}^n} e^{-a\phi(x)} dx = a^{-\alpha} \Gamma(\alpha + 1)|B_\phi|$. 

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Proof. Let \( x \in \mathbb{R}^n \) and let \( r > 0 \). By \( A \)–homogeneity we have
\[
\varphi(x) < r \iff r^{-1} \varphi(x) < 1 \iff \varphi(r^{-A} x) < 1,
\]
that is
\[
x \in B_\varphi(r) \iff r^{-A} x \in B_\varphi
\]
and hence
\[
(41) \quad |B_\varphi(r)| = r^\alpha |B_\varphi|.
\]

Given \( t > 0 \) we set
\[
E_t = \{ x \in \mathbb{R}^n \mid e^{-\alpha \varphi(x)} > t \}.
\]

If \( 0 < t < 1 \) then
\[
E_t = B_\varphi \left( \frac{1}{\alpha} \log \frac{1}{t} \right)
\]
and so, using (41),
\[
\int_{\mathbb{R}^n} e^{-\alpha \varphi(x)} dx = \int_0^1 |E_t| dt = \int_0^1 |B_\varphi \left( \frac{1}{\alpha} \log \frac{1}{t} \right)| dt
\]
\[
= a^{-\alpha} |B_\varphi| \int_0^1 \left( \log \frac{1}{t} \right)^\alpha dt
\]
\[
= a^{-\alpha} \Gamma (\alpha + 1) |B_\varphi|.
\]

\[
II
\]

Theorem 1.1 will be now an immediate consequence of the following:

**Theorem 5.1.** Let \( m \) be a positive integer and assume that \( m > n \). Let \( \varphi \in C^1(\mathbb{R}^n) \cap C^m(\mathbb{R}^n \setminus \{0\}) \) be a positive \( A \)–homogeneous function. Then the series on the right hand side of (5) converges to a holomorphic function on the half space \( \Re s > \alpha \) and extends to a meromorphic function on the half space \( \Re s > \alpha - \gamma m \) having only a simple pole at \( s = \alpha \) with residue \( \alpha |B_\varphi| \). If \( \alpha - \gamma m < 0 \) then we also have \( \zeta(\varphi, 0) = -1 \).

**Proof.** Let \( \lambda > 0 \) be a positive real constant such that \( \lambda > \beta m \).

Choose \( \sigma \) satisfying \( \sigma > n, \sigma > \beta n \) and \( \sigma + m > \frac{\alpha}{\gamma} \).

If \( g = \varphi^\lambda e^{-\varphi} \) then, by (39)
\[
\beta n < \Re s < \sigma \implies \xi_\lambda (g, s) = \Gamma(s + \lambda) \zeta(\varphi, s).
\]

By Theorem 4.1 we also have \( g \in \mathcal{S}_\sigma^m(\mathbb{R}^n) \).

Since \( g(0) = 0 \), by (22) of Theorem 3.2, when \( \alpha - \gamma m < \Re s < \sigma \)
\[
\xi_\lambda (g, s) = -\frac{\hat{\varphi}(0)}{\alpha - s} + \xi_0^+ (g, s) + \xi_\lambda^+ (g, \alpha - s),
\]
the functions \( \xi_0^+ (g, s) \) and \( \xi_\lambda^+ (g, \alpha - s) \) being holomorphic when \( \alpha - \gamma m < \Re s < \sigma \).

Since \( \lambda > \beta n \) and \( \gamma < \beta \) the function \( \Gamma(s + \lambda) \) is holomorphic and not zero on the strip \( \alpha - \gamma m < \Re s < \sigma \).

It follows that the function \( \Gamma(s + \lambda)^{-1} \xi_\lambda (g, s) \) is meromorphic on the strip \( \alpha - \gamma m < \Re s < \sigma \) having only a simple pole at \( s = \alpha \) and it coincide with \( \zeta(\varphi, s) \) when \( \beta n < \Re s < \sigma \); this shows that the function \( \zeta(\varphi, s) \) has a meromorphic extension to the half
plane \( \text{Re} \, s > \alpha - \gamma m \) which is holomorphic when \( s \neq \alpha \) and has a simple pole at \( s = \alpha \) with residue

\[
R = \Gamma(\alpha + \lambda)^{-1} \tilde{g}(0) = \Gamma(\alpha + \lambda)^{-1} \int_{\mathbb{R}^n} \phi(x)^\lambda e^{-\phi(x)} \, dx.
\]

Considering \( \lambda \) as a complex variable, we observe that the function

\[
\lambda \mapsto G(\lambda) = \Gamma(\alpha + \lambda)^{-1} \int_{\mathbb{R}^n} \phi(x)^\lambda e^{-\phi(x)} \, dx
\]

is holomorphic with respect to \( \lambda \) when \( \text{Re} \, \lambda > 0 \). Since \( G(\lambda) = R \) when \( \lambda \) is real and \( \lambda > \beta n \) then, by the identity principle, \( G(\lambda) = R \) when \( \text{Re} \, \lambda > 0 \).

Using the Lebesgue theorem on dominated convergence and (40) we obtain

\[
R = \lim_{\lambda \to 0^+} G(\lambda) = \Gamma(\alpha)^{-1} \int_{\mathbb{R}^n} e^{-\phi(x)} \, dx = \Gamma(\alpha)^{-1} \Gamma(\alpha + 1) |B_\phi|.
\]

Recalling that the functional equation for the Euler Gamma function is

\[
\Gamma(\alpha + 1) = \alpha \Gamma(\alpha),
\]

it follows that

\[
R = \alpha |B_\phi|.
\]

Observe now that the series (5) defining \( \zeta(\phi, s) \) converges when \( \text{Re} \, s > \beta n \) and extends to a holomorphic function on the bigger half plane \( \text{Re} \, s > \alpha \). The convergence of the series (5) when \( \text{Re} \, s > \alpha \) follows then from Landau’s Theorem 2.1.

Assume now that \( \alpha - \gamma m < 0 \). If we set \( g_\lambda = e^{\phi(x)} \) then, by Theorem 4.1 we have \( g_\lambda \in \mathcal{S}_\sigma^m(\mathbb{R}^n) \), so that by (38) and analytic continuation, if \( \alpha - \gamma m < \text{Re} \, s < \sigma \),

\[
\zeta(\phi, s) = \Gamma\left(\frac{s}{\lambda}\right)^{-1} \xi_{\lambda^{-1} A}(g, \lambda^{-1} s).
\]

Again by (22) of Theorem 3.2 since \( g_\lambda(0) = 1 \), in a neighbourhood of \( s = 0 \) we have

\[
\xi_{\lambda^{-1} A}(g, \lambda^{-1} s) = \frac{1}{s} + O(1).
\]

Since

\[
\lim_{s \to 0} s \Gamma(s) = 1
\]

it follows that

\[
\zeta(\phi, 0) = \lim_{s \to 0} \Gamma\left(\frac{s}{\lambda}\right)^{-1} \xi_{\lambda^{-1} A}(g, \lambda^{-1} s) = \lim_{s \to 0} \Gamma\left(\frac{s}{\lambda}\right)^{-1} \left( -\frac{\lambda}{s} + O(1) \right) = -1.
\]



\[
\text{Theorem 5.2. Let } m \text{ be a positive integer and suppose that } m > n. \text{ Let } \phi \in C^0(\mathbb{R}^n) \cap C^m(\mathbb{R}^n \setminus \{0\}) \text{ be a positive } A \text{--homogeneous function. Let } b \text{ and } c \text{ be positive real constants. If } \text{Re} \, s > \alpha - m \gamma, s \neq 0, \alpha, \text{ then}
\]

\[
(42) \quad g = e^{-\phi^c}, b > m \beta, \quad \Rightarrow \quad \xi_{b^{-1} A}(g, b^{-1} s) = \Gamma\left(\frac{s}{b}\right) \zeta(\phi, s);
\]

\[
(43) \quad g = \phi^c e^{-\phi}, c > m \beta, \quad \Rightarrow \quad \xi_A(g, s) = \Gamma(s + c) \zeta(\phi, s).
\]

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6. Proof of Theorem 1.2

Let $A$ and $\varphi$ be as in the hypothesis of Theorem 1.2.
When $r > 0$ set
\[ F_\varphi(r) = #(B_{\varphi_1}(r) \cap \mathbb{Z}^n) \]
and let $\mu_\varphi$ be the unique Radon measure such that for each $a < b \in \mathbb{R}$
\[ \mu_\varphi([a,b]) = \lim_{r \to a^+} F_\varphi(r) - \lim_{r \to b^-} F_\varphi(r). \]

When $\Re s > \beta n$ we have
\[ \zeta(\varphi, s) = \int_0^{+\infty} t^{-s} d\mu_\varphi(t). \]

Assume that $\varphi \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$. Then Theorem 1.2 follows immediately from Theorem 1.1 and the Ikehara-Wiener Theorem 2.2.
Assume now that $\varphi \in C^0(\mathbb{R}^n)$. Then by theorem 4.5 there exist a positive $A$–homogeneous function $\psi \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ such that for every $x \in \mathbb{R}^n$
\[ \varphi(x) \geq \psi(x). \]

Let $\Re s > \alpha$. Then
\[ |\zeta(\varphi, s)| \leq \sum_{\omega \in \mathbb{Z}^n \setminus \{0\}} \varphi(\omega)^{-\Re s} \leq \sum_{\omega \in \mathbb{Z}^n \setminus \{0\}} \psi(\omega)^{-\Re s} < +\infty. \]

This completes the proof of assertion 1 of Theorem 1.2.

Fix now $0 < \varepsilon < 1$. By Theorem 4.3 there exist two positive $A$–homogeneous functions $\psi_1, \psi_2 \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ such that for every $x \in \mathbb{R}^n$
\[ (1 - \varepsilon)\varphi(x) \leq \psi_1(x) \leq \varphi(x) \leq \psi_2(x) \leq (1 + \varepsilon)\varphi(x). \]

Then we have respectively
\[ (1 + \varepsilon)^{-n} \alpha |B_\varphi| \leq \alpha |B_{\psi_1}| \leq \liminf_{\sigma \to +\alpha^+} (\sigma - \alpha) \zeta(\psi_2, \sigma) \]
\[ \leq \liminf_{\sigma \to +\alpha^+} (\sigma - \alpha) \zeta(\varphi, \sigma) \leq \limsup_{\sigma \to +\alpha^+} (\sigma - \alpha) \zeta(\varphi, \sigma) \]
\[ \leq \limsup_{\sigma \to +\alpha^+} (\sigma - \alpha) \zeta(\psi_1, \sigma) \]
\[ \leq \alpha |B_{\psi_1}| \leq (1 - \varepsilon)^{-n} \alpha |B_\varphi| \]
and
\[ (1 + \varepsilon)^{-n}|B_\varphi| \leq |B_{\psi_2}| \leq \liminf_{r \to +\infty} \frac{\#(B_{\psi_2}(r) \cap \mathbb{Z}^n)}{r^\alpha} \]
\[ \leq \liminf_{r \to +\infty} \frac{\#(B_\varphi(r) \cap \mathbb{Z}^n)}{r^\alpha} \leq \limsup_{r \to +\infty} \frac{\#(B_{\psi_1}(r) \cap \mathbb{Z}^n)}{r^\alpha} \]
\[ \leq |B_{\psi_1}| \leq (1 - \varepsilon)^{-n}|B_\varphi|. \]

Since $\varepsilon > 0$ can be made arbitrarily small the proof of assertions 2 and 3 of Theorem 1.2 is so completed.
7. Proof of Theorem 1.3

Let $A, \varphi, \varepsilon$ and $\delta$ be as in Theorem 1.3.
Let $L$ be a Lyapunov matrix for the matrix $A$ and set (again)
\[ S_L = \{ x \in \mathbb{R}^n \mid \langle Lx, x \rangle = 1 \} . \]

For each (Borel) subset $E \subset S_L$ we denote by $|E|_{n-1}$ the $(n-1)$-dimensional Euclidean measure of the set $E$.

Let $D$ be the set of the points $x \in S_L$ of the form $tA(\omega)$, where $t > 0$ and $\omega \in \mathbb{Z}^n$.
As $D$ is countable, it follows that $|D|_{n-1} = 0$. By standard measure theory approximation arguments there exists a compact set $K \subset S_L$ such that $K \cap D = \emptyset$ but $|K|_{n-1} > 0$.

We now define $\varphi_1 : \mathbb{R}^n \to [0, +\infty]$ as the unique $A$-homogeneous function such that when $x \in S_L$
\[ \varphi_1(x) = \begin{cases} \varphi(t), & x \in S_L \setminus K, \\ (1 + \varepsilon/2)\varphi(t), & x \in K. \end{cases} \]

We also set $\varphi_2 = (1 + \varepsilon)\varphi$.
Then $\varphi_1$ is upper semicontinuous, $\varphi_2$ is lower semicontinuous and $\varphi_1(x) < \varphi_2(x)$ for each $x \in \mathbb{R}^n \setminus \{0\}$.

By Proposition 4.4 there exists a sequence
\[ \psi_\nu \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\}) \]
of positive $A$-homogeneous functions such that for each $x \in \mathbb{R}^n$,
\[ \psi_\nu(x) \leq \varphi_2(x) = (1 + \varepsilon)\varphi(x), \]
\[ \psi_{\nu+1}(x) \leq \psi_\nu(x), \quad \nu = 1, 2, \ldots \]
and
\[ \lim_{\nu \to +\infty} \psi_\nu(x) = \varphi_1(x) \geq \varphi(x). \]

It follows that
\[ B_{\varphi_2} \subset B_{\psi_1}, \]
\[ B_{\psi_\nu} \subset B_{\psi_{\nu+1}}, \quad \nu = 1, 2, \ldots \]
\[ \bigcup_{\nu} B_{\psi_\nu} = B_{\psi_1}, \]
and hence
\[ (1 + \varepsilon)^{-n} |B_{\varphi}| \leq |B_{\psi_\nu}| \leq |B_{\varphi}|, \quad \nu = 1, 2, \ldots \]
\[ \lim_{\nu \to +\infty} |B_{\psi_\nu}| = |B_{\psi_1}|. \]

Since $|K|_{n-1} > 0$ and $|S_L \setminus K|_{n-1} > 0$ it follows that
\[ |B_{\psi_1}| < |B_{\varphi}| \]
and hence
\[ \lim_{\nu \to +\infty} |B_{\psi_\nu}| < |B_{\varphi}|. \]
Observe also that if $\omega \in \mathbb{Z}^n$ then, by construction,

\begin{equation}
\lim_{\nu \to +\infty} \psi_\nu(\omega) = \varphi(\omega).
\end{equation}

Set now

$$M_\nu = \sup_{|s - \alpha| = \delta} |\zeta(\psi_\nu, s)|.$$

We complete the proof of Theorem 1.3 showing that

$$\lim_{\nu \to +\infty} M_\nu = +\infty.$$

If not, taking a subsequence if necessary, we have

$$\sup_{|s - \alpha| = \delta} |\zeta(\psi_\nu, s)| \leq M$$

for some real constant $M$.

Consider the sequence of functions defined as

\begin{equation}
g_\nu(s) = \zeta(\psi_\nu, s) - \frac{\alpha|B_{\psi_\nu}|}{s - \alpha}.
\end{equation}

By Theorem 1.1 the functions $g_\nu(s)$ are holomorphic throughout over all the complex plane. When $|s - \alpha| = \delta$ we have

\begin{equation}
|g_\nu(s)| \leq M + \frac{\alpha|B_{\varphi}|}{\delta}.
\end{equation}

By the maximum principle the same inequality holds when $|s - \alpha| \leq \delta$.

By Vitali-Montel theorem, taking again a subsequence if necessary, the sequence $g_\nu(s)$ converges uniformly on the compact subsets of the disk $U = \{s \mid |s - \alpha| < \delta\}$ to a holomorphic function $g : U \to \mathbb{C}$.

Set

$$f(s) = g(s) + \frac{\alpha|B_{\varphi}|}{s - \alpha}.$$

Since

$$\lim_{\nu \to +\infty} \frac{\alpha|B_{\psi_\nu}|}{s - \alpha} = \frac{\alpha|B_{\varphi}|}{s - \alpha}$$

uniformly on the compact sets of $\mathbb{C} \setminus \{\alpha\}$ it follows that

\begin{equation}
\lim_{\nu \to +\infty} \zeta(\psi_\nu, s) = f(s)
\end{equation}

uniformly on the compact sets of $U \setminus \{\alpha\}$.

Assume now that $s = \sigma \in \mathbb{R}$ and $\alpha < \sigma < \alpha + \delta$. Then, by assertion 1 of Theorem 1.2 using (45) and the Beppo-Levi monotone convergence theorem we obtain

$$f(\sigma) = \lim_{\nu \to +\infty} \sum_{\omega \in \mathbb{Z}^n \setminus \{0\}} \psi_\nu(\omega)^{-\sigma} = \sum_{\omega \in \mathbb{Z}^n \setminus \{0\}} \varphi(\omega)^{-\sigma} = \zeta(\varphi, \sigma).$$

By assertion 2 of Theorem 1.2 and using (48) we would obtain

$$\alpha|B_{\varphi}| = \lim_{\sigma \to +\alpha^+} (\sigma - \alpha) \zeta(\varphi, \sigma) = \operatorname{Res} f(s) = \lim_{\nu \to +\infty} \operatorname{Res} \zeta(\psi_\nu, s) = \lim_{\nu \to +\infty} \alpha|B_{\psi_\nu}|$$

and this contradicts (44).
8. ASYMPTOTIC EXPANSIONS AND PROOF OF THEOREM 1.4

We begin with an estimate of the growing of the zeta functions $\zeta(\varphi, s)$ on the imaginary directions.

**Proposition 8.1.** Let $\varphi \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ be a positive $A$–homogeneous function.

Let $a < b \in \mathbb{R}$ be given, and let $\varepsilon > 0$. If $a \leq \text{Re} s \leq b$ and $|\text{Im} s| \geq 1$ then

\begin{align*}
|\zeta(\varphi, s)| &\leq c_1e^{\varepsilon \text{Im} s}, \\
|\Gamma(s)\zeta(\varphi, s)| &\leq c_2e^{-(\pi/2-\varepsilon)|\text{Im} s|}.
\end{align*}

**Proof.** The Stirling formula for the Euler Gamma function implies that if $a \leq \text{Re} s \leq b$ and $|\text{Im} s| \geq 1$, then for each $\varepsilon > 0$,

\begin{equation}
|\Gamma(s)| \leq c_3e^{-(\pi/2+\delta)|\text{Im} s|}
\end{equation}

and hence the estimates (49) and (50) are equivalent. So it suffices to prove (50).

Let $\sigma > n$ and $\tau > n$ be chosen in such a way that respectively $\gamma\sigma > b$ and $\alpha - \gamma\tau < a$. Let $m$ be any positive integer which satisfies $m > \gamma\sigma$ which will be chosen later on.

From Gauss’s multiplication formula

\[ \prod_{k=0}^{m-1} \Gamma \left( s + \frac{k}{m} \right) = m^{\frac{1}{2} - ms} (2\pi)^{\frac{1}{2}(m-1)} \Gamma(ms) \]

it follows that

\[ \Gamma(ms) = (2\pi)^{\frac{1}{2}(1-m)m^{ms-\frac{1}{2}}} \Gamma(s) \prod_{k=1}^{m-1} \Gamma \left( s + \frac{k}{m} \right) ; \]

then

\[ \Gamma(s) = (2\pi)^{\frac{1}{2}(1-m)m^{s-\frac{1}{2}}} \Gamma \left( \frac{s}{m} \right) \prod_{k=1}^{m-1} \Gamma \left( \frac{s+k}{m} \right) . \]

Setting $g_m(x) = e^{-\varphi(x)m}$ and using the identity (42) of Theorem 5.2 we obtain

\[ \Gamma(s)\zeta(\varphi, s) = \xi_{m^{-1}A} \left( g_m, \frac{s}{m} \right) (2\pi)^{\frac{1}{2}(1-m)m^{s-\frac{1}{2}}} \prod_{k=1}^{m-1} \Gamma \left( \frac{s+k}{m} \right) . \]

The identity (22) of Theorem 5.2 yields therefore

\[ \xi_{m^{-1}A} \left( g_m, \frac{s}{m} \right) = -\frac{m}{s} - \frac{m\hat{g}_m(0)}{\alpha - s} + \xi_{m^{-1}A} \left( g_m, \frac{s}{m} \right) + \xi_{m^{-1}A'} \left( \hat{g}_m, \frac{\alpha - s}{m} \right) \]

Assume now that $s$ lies in the region $S$ defined by the conditions $a \leq \text{Re} s \leq b$ and $|\text{Im} s| \geq 1$. Then the estimate (17) of Lemma 3.3 implies

\[ \left| \xi_{m^{-1}A} \left( g_m, \frac{s}{m} \right) \right| \leq \frac{mc_5 \|\hat{g}_m\|_{\sigma}}{\gamma\sigma - \text{Re} s} \leq \frac{mc_5 \|\hat{g}_m\|_{\sigma}}{\gamma\sigma - \gamma\tau} \]

and

\[ \left| \xi_{m^{-1}A'} \left( \hat{g}_m, \frac{\alpha - s}{m} \right) \right| \leq \frac{mc_6 \|\hat{g}_m\|_{\tau}}{\text{Re} s - \alpha + \beta}\frac{1}{21} \leq \frac{mc_6 \|\hat{g}_m\|_{\tau}}{a - \alpha + \beta} . \]
It follows easily now that the function $\xi_{m,A}(s, \frac{\lambda}{m})$ is bounded on the region $S$ and hence
\[
|\Gamma(s)\zeta(\varphi, s)| \leq c_7(2\pi)^{\frac{1}{2}(1-m)}m^{-\frac{1}{2}}\prod_{k=1}^{m-1} \left| \Gamma \left( \frac{s+k}{m} \right) \right| = c_8 \prod_{k=1}^{m-1} \left| \Gamma \left( \frac{s+k}{m} \right) \right|.
\]

Setting $\delta = \varepsilon/2$ in (51) we easily obtain
\[
|\Gamma(s)\zeta(\varphi, s)| \leq c_9 e^{-\left(\frac{\pi-\varepsilon}{2}\right)(\frac{m-1}{m})}\operatorname{Im}s.
\]

We now chose $m$ large enough to satisfy
\[
\left(\frac{\pi-\varepsilon}{2}\right)\left(\frac{m-1}{m}\right) > \frac{\pi}{2} - \varepsilon
\]
and we are done. //

**Proposition 8.2.** Let $\varphi \in C^0(\mathbb{R}^n)$ be a positive $A$–homogeneous function. If $t > 0$ is a positive real number
\[
\theta(\varphi, it) = \theta_A\left(e^{-\varphi}, it\right)
\]
and if $\operatorname{Im} \tau > 0$,
\[
|\theta(\varphi, \tau)| \leq \theta(\varphi, \operatorname{Im} \tau).
\]

**Proof.** When $t > 0$, by $A$–homogeneity, we have
\[
\theta(\varphi, it) = \sum_{\omega \in \mathbb{Z}^n} e^{-i\varphi(\omega)} = \sum_{\omega \in \mathbb{Z}^n} e^{-\varphi(\omega)} = \theta_A\left(e^{-\varphi}, it\right).
\]

When $\operatorname{Im} \tau > 0$
\[
|\theta(\varphi, \tau)| \leq \sum_{\omega \in \mathbb{Z}^n} \left| e^{i\tau \varphi(\omega)} \right| \leq \sum_{\omega \in \mathbb{Z}^n} e^{-\operatorname{Im} \tau \varphi(\omega)} = \theta(\varphi, \operatorname{Im} \tau).
\]

//

We are now ready to prove Theorem 1.4.
Let $\operatorname{Re} w > 0$ and $\operatorname{Re} s > \beta n$. Then
\[
w^{-s}\Gamma(s)\zeta(\varphi, s) = \int_{0}^{+\infty} \theta^* (\varphi, it) w \frac{dt}{t}.
\]

Indeed both sides are holomorphic with respect to the variable $w$ on the half plane $\operatorname{Re} w > 0$ and coincide when $w = a > 0$ by (37) of proposition 5.1.

Using the estimates for $|\Gamma(s)\zeta(\varphi, s)|$ given in proposition 8.1 we easily obtain that for each $\varepsilon > 0$
\[
(52) \quad |w^{-s}\Gamma(s)\zeta(\varphi, s)| = O(e^{-(\pi/2-|\arg w|)\varepsilon})
\]
as $|\operatorname{Im} s| \to +\infty$ uniformly with respect to $\operatorname{Re} s$. The Mellin inversion formula implies therefore that, when $b > \beta n$,
\[
(53) \quad \theta(\varphi, iw) = 1 + \theta^* (\varphi, iw) = 1 + \frac{1}{2\pi i} \int_{(b)} w^{-s} \Gamma(s)\zeta(\varphi, s) ds.
\]

Let $N$ be a positive integer and let $0 < \varepsilon < 1$. Given a positive real number $T > 0$ consider the rectangle $R(T)$ with vertices $-N - 1 + \varepsilon \pm iT$ and $b \pm iT$. The poles of the function $w^{-s}\Gamma(s)\zeta(\varphi, s)$ inside $R(T)$ are $s = \alpha$ with residue $R = \Gamma(\alpha + 1) |B_\varphi| w^{-\alpha}$,
This case Theorem 1.4 yields that the equations \( \Re \zeta(s) = \sum_{n=1}^{\infty} \frac{\arg n}{n^s} = -\frac{1}{s} \) for each integer \( k = 1, \ldots, N, s = -k \) with residue \( R = ((-1)^k \zeta(\varphi, -k)/k!)w^k \).

The residue theorem implies that

\[
\frac{1}{2\pi i} \int_{\partial R(T)} w^{-s} \Gamma(s) \zeta(\varphi, s) ds = \Gamma(\alpha + 1) |B_{\varphi}| w^{-\alpha} - 1 + \sum_{k=1}^{N} \frac{(-1)^k \zeta(\varphi, -k)}{k!} w^k.
\]

Letting \( T \to +\infty \), the estimate \( (52) \) implies that

\[
\frac{1}{2\pi i} \int_{(b)} w^{-s} \Gamma(s) \zeta(\varphi, s) ds = \frac{1}{2\pi i} \int_{(-N+1+\epsilon)} w^{-s} \Gamma(s) \zeta(\varphi, s) ds + \Gamma(\alpha + 1) |B_{\varphi}| w^{-\alpha} - 1 + \sum_{k=1}^{N} \frac{(-1)^k \zeta(\varphi, -k)}{k!} w^k.
\]

Inserting such expression in \( (53) \) we obtain \( (5) \). The estimate \( (5) \) is an easy consequence of \( (52) \) and the proof is complete.

**Classical Example.** Let \( 0 < \delta < \pi/2 \). Let denote by \( E(\delta) \) the open angle defined by the equations \( \Re w > 0 \) and \( |\arg w| < \pi/2 - \delta \). Let \( p > 0 \). For \( n = 1, \varphi(x) = |x|^p \) we have \( \zeta(\varphi, s) = 2\zeta(ps) \), where \( \zeta(s) \) is the Riemann zeta function and \( |B_{\varphi}| = |[-1, 1]| = 2 \); in this case Theorem 1.4 yields

\[
\sum_{\omega=-\infty}^{+\infty} e^{-w|\omega|^p} = 2\Gamma \left( \frac{1}{p} + 1 \right) w^{-1/p} + 2 \sum_{k=1}^{N} \frac{(-1)^k \zeta(-kp)}{k!} w^k + O(|w|^{N+1-\epsilon}), \quad w \in E(\delta).
\]

Since the Riemann zeta function has its real zeroes at the even negative integers, when \( p = 2m \) is an even positive integer we obtain that for each \( \sigma > 0 \)

\[
\frac{1}{2} \sum_{\omega=-\infty}^{+\infty} e^{-w|\omega|^{2m}} = \Gamma \left( \frac{1}{2m} + 1 \right) w^{-1/2m} + O(|w|^\sigma), \quad w \in E(\delta).
\]

When \( p = 1 \) we obtain easily

\[
\frac{1}{2} \frac{1 + e^{-w}}{1 - e^{-w}} = \frac{1}{w} + \sum_{k=1}^{N} \frac{(-1)^k \zeta(-k)}{k!} w^k + O(|w|^{N+1-\epsilon}),
\]

according to the fact that when \( k = 1, 2, \ldots, -(k+1) \zeta(-k) = B_{k+1} \), where \( B_{k+1} \) is the \((k + 1)\)th Bernoulli number.
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