Geometrical McKay Correspondence for Isolated Singularities

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Introduction

Crepant Resolutions of Singularities of Calabi-Yau orbifolds

A Calabi-Yau manifold is a complex Kähler manifold with trivial canonical bundle. In the attempt to construct such manifolds it is useful to take into consideration singular Calabi-Yaus. One of the simplest singularities which can arise is an orbifold singularity. An orbifold is the quotient of a smooth Calabi-Yau manifold by a discrete group action which generically has fixed points. Locally such an orbifold is modeled on $\mathbb{C}^n/G$, where $G$ is a finite subgroup of $SL(n,\mathbb{C})$.

From a geometrical perspective we can try to resolve the orbifold singularity. A resolution $(X, \pi)$ of $\mathbb{C}^n/G$ is a nonsingular complex manifold $X$ of dimension $n$ with a proper biholomorphic map $\pi: X \to \mathbb{C}^n/G$ that induces a biholomorphism between dense open sets. We call $X$ a crepant resolution if the canonical bundles are isomorphic, $K_X \cong \pi^*(K_{\mathbb{C}^n/G})$. Since Calabi-Yau manifolds have trivial canonical bundle, to obtain a Calabi-Yau structure on $X$ one must choose a crepant resolution of singularities.

It turns out that the amount of information we know about a crepant resolution of singularities of $\mathbb{C}^n/G$ depends dramatically on the dimension $n$ of the orbifold:

$n = 2$: A crepant resolution always exists and is unique. Its topology is entirely described in terms of the finite group $G$ (via the McKay Correspondence).

$n = 3$: A crepant resolution always exists but it is not unique; they are related by flops. However all the crepant resolutions have the same Euler and Betti numbers: the stringy Betti and Hodge numbers of the orbifold [DHVW].

$n \geq 4$: In this case very little is known; crepant resolutions exist in rather special cases. Many singularities are terminal, which implies that they admit no crepant resolution.

We would like to completely understand the topology of crepant resolutions in the case $n = 3$. In this paper we are concerned with the study of the ring structure in cohomology. This is related to the generalization of the McKay Correspondence. In what follows we give a description of the problem by moving back and forth between the case $n = 2$ and $n = 3$.

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1 Etymology: For a resolution of singularities we can define a notion of discrepancy [R1]. A crepant resolution is a resolution without discrepancy.
The case $n = 2$. The quotient singularities $\mathbb{C}^2/G$, for $G$ a finite subgroup of $SL(2, \mathbb{C})$, were first classified by Klein in 1884 and are called Kleinian singularities (they are also known as Du Val singularities or rational double points). There are five families of finite subgroups of $SL(2, \mathbb{C})$: the cyclic subgroups $C_k$, the binary dihedral groups $D_{4k}$ of order $4k$, the binary tetrahedral group $T$ of order 24, the binary octahedral group $O$ of order 48, and the binary icosahedral group $I$ of order 120. A crepant resolution exists for each family and is unique. Moreover the finite group completely describes the topology of the resolution. This is encoded in the McKay Correspondence [McK1], which establishes a bijection between the set of irreducible representations of $G$ and the set of vertices of an extended Dynkin diagram of type $ADE$ (the Dynkin diagrams corresponding to the simple Lie algebras of the following five types: $A_{k−1}, D_{k+2}, E_6, E_7$ and $E_8$).

Concretely, let $\{R_0, R_1, \ldots, R_r\}$ be the set of irreducible representations of $G$, where $R_0$ denotes the one-dimensional trivial representation. To $G$ and its irreducible representations we associate an $(r+1) \times (r+1)$ adjacency matrix $A = [a_{ij}]$ with $i, j = 0, \ldots, r$. The entries $a_{ij}$ are positive integers; they are defined by the tensor product decompositions

$$R_i \otimes Q = \sum_{i=0}^{r} a_{ij} R_j,$$

where $Q$ denotes the natural two-dimensional representation of $G$ induced from the embedding $G \subset SL(2, \mathbb{C})$. McKay’s insight was to realize that the matrix $A$ is related to the Cartan matrix $C$ of a Dynkin diagram of type $ADE$, via

$$A = 2I - \tilde{C}. \quad (0.1)$$

(Here $\tilde{C}$ is the Cartan matrix of the extended Dynkin diagram; the matrix $C$ is the $r \times r$-minor obtained by removing the first row and the first column from $\tilde{C}$.)

Using McKay’s correspondence it is easy now to describe the crepant resolution $\pi : X \to \mathbb{C}^2/G$. The exceptional divisor $\pi^{-1}(0)$ is the dual of the Dynkin diagram: the vertices of the Dynkin diagram correspond naturally to rational curves $C_i$ with self-intersection $-2$. Two curves intersect transversally at one point if and only if the corresponding vertices are joined by an edge in the Dynkin diagram, otherwise they do not intersect. The curves above form a basis for $H_2(X, \mathbb{Z})$. The intersection form with respect to this basis is the negative of the Cartan matrix.

The first geometrical interpretation of the McKay Correspondence was given by Gonzalez-Sprinberg and Verdier [GV]. To each of the irreducible representations $R_i$ they associated a locally free coherent sheaf $\mathcal{R}_i$. The set of all these coherent sheaves form a basis for $K(X)$, the $K$-theory of X. Moreover, the first Chern classes $c_1(\mathcal{R}_i)$ form a basis in $H^2(X, \mathbb{Q})$ and the product of two such classes in $H^*(X, \mathbb{Q})$ is given by the formula

$$\left[\int_X c_1(\mathcal{R}_i)c_1(\mathcal{R}_j)\right]_{i,j=1,\ldots,r} = -C^{-1}, \quad (0.2)$$

where $C^{-1}$ is the inverse of the Cartan matrix. The proof given by Gonzalez-Sprinberg and Verdier uses a case by case analysis and techniques from algebraic geometry. Kronheimer and Nakajima gave a proof of the formula using techniques from gauge theory [KroN].

To summarize, in the case of surface singularities, $\mathbb{C}^2/G$, the representation theory of the finite group $G$ completely determines the topology the crepant resolution. The Dynkin
The case $n = 3$. The finite subgroups of $SL(3, \mathbb{C})$ were classified by Blichfeldt in 1917 [Bl]: there are ten families of such finite subgroups. In the early 1990’s a case by case analysis was used to construct a crepant resolution of $\mathbb{C}^3/G$ with the given stringy Euler and Betti numbers (see [Ro] and the references therein). As a consequence of these constructions, we know that all the crepant resolutions of $\mathbb{C}^3/G$ have the Euler and Betti numbers given by the stringy Euler and Betti numbers of the orbifold (since these numbers are unchanged under flops). In 1995 Nakamura made the conjecture that $\text{Hilb}^G(\mathbb{C}^3)$ is a crepant resolution of $\mathbb{C}^3/G$. In general, for $G$ a finite subgroup of $SL(n, \mathbb{C})$, the algebraic variety $\text{Hilb}^G(\mathbb{C}^n)$ parametrizes the 0-dimensional $G$-invariant subschemes of $\mathbb{C}^n$ whose space of global sections is isomorphic to the regular representation of $G$. Nakamura made the conjecture based on his computations for the case $n = 2$ [INak]; then he proved it in dimension $n = 3$ for the case of abelian groups [Nak]. In 1999 Bridgeland, King and Reid gave a general proof of the conjecture in the case $n = 3$, relying heavily on derived category techniques [BKR].

The Scope and Main Result of the paper

In the case of surface singularities, an important feature of the McKay Correspondence is that it gives the ring structure in cohomology in terms of the finite group. For the case $n \geq 3$, nothing is known about the multiplicative structures in cohomology or $K$-theory.

We start with $X$ a crepant resolution of $\mathbb{C}^n/G$. We assume that this resolution is given by $\text{Hilb}^G(\mathbb{C}^n)$. On this resolution, Ito and Nakajima, [IN], gave a recipe of extending the Gonzalez-Sprinberg-Verdier sheaves, associating a locally free sheaf (therefore an algebraic vector bundle) $\mathcal{R}_i$ to every irreducible representation $R_i$ of the finite group. In the case $n \leq 3$, these bundles span the $K$-theory of $X$, and via the Chern character isomorphism, \{ch($\mathcal{R}_0$), ch($\mathcal{R}_1$), ..., ch($\mathcal{R}_r$)\} span $H^*(X; \mathbb{Q})$.

The idea is to use the Atiyah-Patodi-Singer (APS) index theorem for studying the ring structure in cohomology. We define a generalization of the Cartan matrix of the case $n = 2$, and show a that a generalization of Kronheimer and Nakajima’s formula (0.2) holds:

$$\left[ \int_X (\text{ch}(\mathcal{R}_i) - \text{rk}(\mathcal{R}_i)) \left( \text{ch}(\mathcal{R}_j^*) - \text{rk}(\mathcal{R}_j) \right) \right]_{i,j=1,...,r} = C^{-1}. \quad (0.3)$$

Overview

In section 1 we present a construction of Nakamura’s $G$-Hilbert scheme as a symplectic reduction of a bigger space related to the representation theory of $G$. Then we restrict ourselves to the case $n = 3$ and assume that the singularity $\mathbb{C}^3/G$ is isolated. We prove that the induced metric is ALE of order 6. The decay of the metric allows us to use Joyce’s proof of the Calabi Conjecture on ALE manifolds to find a unique Ricci-flat ALE metric in its Kähler class.

In section 2 we consider the Dirac operator and analyze its index when completed in a weighted Sobolev norm. We compute the index of this operator using Atiyah-Patodi-Singer
index theorem. In section 3 we prove a vanishing result for the index of the Dirac operator. This result allows us to derive our geometrical interpretation of McKay Correspondence (0.3). We conclude with remarks about future work and related interpretations of the McKay Correspondence.

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1. Nakamura’s $G$-Hilbert scheme as an ALE space

For $G$ a nontrivial finite subgroup of $SL(n, \mathbb{C})$ we consider Nakamura’s $G$-Hilbert scheme \( \text{Hilb}^G(\mathbb{C}^n) \). This is the scheme parameterizing 0-dimensional subschemes $Z$ of $\mathbb{C}^n$ satisfying the following three conditions:

1. the length of $Z$ is equal to the order of $G$, $|G|$;
2. $Z$ is invariant under the $G$-action;
3. $H^0(O_Z)$ is the regular representation of $G$.

Let $X$ be the component of $\text{Hilb}^G(X)$ which contains the set of $G$-orbits $G \cdot (\mathbb{C}^n \setminus \{0\})$. (Usually $\text{Hilb}^G(\mathbb{C}^n)$ is the union of components of fixed points under the $G$-action in the Hilbert scheme of $|G|$-points in $\mathbb{C}^n$.) It comes with a natural morphism

$$\pi : X \to \mathbb{C}^n/G,$$

the Hilbert-Chow morphism. The variety $X$ can be described as a symplectic/GIT quotient [IN, SI, N, Kro]. We review these descriptions now.

Let $G$ be a finite subgroup of $SL(n, \mathbb{C})$ and let $Q$ be its canonical $n$-dimensional representation. Let $R$ be the regular representation of $G$:

$$R = \bigoplus_{i=0}^{r} \mathbb{C}^{n_i} \otimes R_i,$$

where $\{R_0, R_1, \ldots, R_r\}$ is the set of irreducible representations of $G$, with $R_0$ being the one-dimensional trivial representation. The number $n_i$ denotes the dimension of $R_i$. We consider the complex vector space

$$\mathcal{P} = Q \otimes \text{End}(R) \oplus \text{Hom}(R_0, R) \oplus \text{Hom}(R, R_0).$$

We choose an orthonormal basis of $Q$. With this choice, an element of $Q \otimes \text{End}(R)$ is represented as an $n$-tuple of endomorphisms $B = (B_1, \ldots, B_n)$. Therefore the group $G$ acts naturally on $\mathcal{P}$ via

$$g \cdot (B, i, j) = (g^{-1}Bg, g^{-1}i, jg).$$
We take the $G$-invariant part of $\mathcal{P}$ and denote it by $\mathcal{M}$. Inside $\mathcal{M}$ we consider the variety

$$\mathcal{N} = \{(B, i, j) \in \mathcal{M} | B \wedge B + ij = 0\},$$

with $B \wedge B = \sum_{\alpha < \beta} [B_\alpha, B_\beta]$. This is an algebraic variety (obtained from cutting the complex vector space $\mathcal{M}$ by quadratic equations); it is actually a cone in $\mathcal{M}$.

Let $GL(R)$ be the linear automorphism of $R$, and let $\mathcal{F}^c \subset GL(R)$ be the subgroup consisting of those elements which commute with the action of $G$ on $R$. We take the maximal compact subgroup of $\mathcal{F}^c$ and denote it by $\mathcal{F}$; it consists of those elements of $U(R)$ — unitary automorphisms of $R$ — which commute with the action of $G$:

$$\mathcal{F} = U^G(R_0) \times U^G(R_1) \times \ldots \times U^G(R_r). \quad (1.1)$$

The natural action of $\mathcal{F}^c$ on $\mathcal{P}$ is given by

$$(B, i, j) \mapsto (fBf^{-1}, f^{-1}i, jf), \quad f \in \mathcal{F}^c.$$}

It preserves $\mathcal{M}$ and the variety $\mathcal{N}$, as well as the induced Kähler forms and complex structures.

For a given character $\chi : \mathcal{F} \to \mathbb{C}^*$ there are two quotients we can associate to $\mathcal{N}$: the GIT quotient and the symplectic quotient. For the GIT quotient we take the complexification of the character $\chi$ — also denoted by $\chi$ — and construct it as

$$\mathcal{N} \sslash_\chi \mathcal{F}^c. \quad (1.2)$$

For the symplectic quotient we need to consider the moment map $\mu : \mathcal{P} \to \mathfrak{f}^*$ determined by the action of $\mathcal{F}$ on $\mathcal{P}$. Here $\mathfrak{f}^*$ denotes the Lie algebra of $\mathcal{F}$. Concretely, this moment map is

$$\mu(B, i, j) = \frac{\sqrt{-1}}{2} ([B, B^*] + ii^* - j^*j),$$

where $[B, B^*] = \sum_{\alpha=1}^n [B_\alpha, B_\alpha^*]$. The restriction of $\mu$ to $\mathcal{M}$ and respectively to $\mathcal{N}$ gives moment maps for the action of $\mathcal{F}$ on each of these spaces. If $d\chi \in \text{Center}(\mathfrak{f}^*)$, the center of the dual Lie algebra to $\mathcal{F}$, we construct the symplectic quotient as

$$\mathcal{N} \cap \mu^{-1}(\sqrt{-1}d\chi) \frac{\mathcal{F}}{\mathcal{F}}. \quad (1.3)$$

By a result of Kempf and Ness [KN] there is a bijection between the spaces obtained via the two constructions:

$$\mathcal{N} \sslash_\chi \mathcal{F}^c \cong \frac{\mathcal{N} \cap \mu^{-1}(\sqrt{-1}d\chi)}{\mathcal{F}}. \quad (1.4)$$

The following theorem gives the description of these quotients in two special circumstances: when $\chi$ is the trivial character and when $\chi$ is the determinant character.

**Theorem 1.4.** (i) Assume that the finite group $G$ acts freely on $\mathbb{C}^n \setminus \{0\}$. Then for $\chi : \mathcal{F} \to \mathbb{C}^*$ the trivial character,

$$\mathcal{N} \sslash \mathcal{F}^c \cong \frac{\mathcal{N} \cap \mu^{-1}(\sqrt{-1}d\chi)}{\mathcal{F}} \cong \mathbb{C}^n / G. \quad (1.5)$$

as algebraic varieties.
(ii) For $\chi = \det : \mathcal{F} \to \mathbb{C}^*$ the determinant character, there exists a bijection between the sets

$$\mathcal{N} \cap \mu^{-1}(\zeta_{\text{Hilb}}) \cong X,$$

where $\zeta_{\text{Hilb}} = d\chi \in \mathfrak{f}^*$ and $X$ is the largest connected component of Nakamura’s $G$-Hilbert scheme.

Part (i) of the Theorem was proved by Kronheimer [Kro] for $n = 2$, and by Sardo-Infirri [SI] for $n \geq 3$. In Sardo-Infirri’s proof it is noted that in the case when $G$ does not act freely on $\mathbb{C}^n \setminus \{0\}$, we just have an inclusion of $\mathbb{C}^n/G$ into the GIT quotient; in fact the GIT quotient might have many other components.

Part (ii) of the Theorem is proved by Nakajima [N] for $n = 2$, and by Ito and Nakajima [IN] for $n \geq 3$.

The relation between the above construction and crepant resolutions of Calabi-Yau orbifold singularities is established by the following theorem:

**Theorem 1.7.** Assume $n \leq 3$. Then $X$ is nonsingular, and the Hilbert-Chow morphism

$$\pi : X \to \mathbb{C}^n/G$$

is a crepant resolution.

In the case $n = 2$ (Kleinian singularities) this theorem was proved by Ito and Nakamura [INak]. In the case $n = 3$ the theorem was proved by Bridgeland, King and Reid [BKR] using derived categories techniques.

**Corollary 1.8 (Ito and Nakajima [IN], Corollary 4.6.).** Assume $n \leq 3$. Then $\mathcal{N} \cap \mu^{-1}(\zeta_{\text{Hilb}})$ is nonsingular and $\mathcal{F}$ acts freely on it, making the quotient map

$$\mathcal{N} \cap \mu^{-1}(\zeta_{\text{Hilb}}) \to X,$$

into a $\mathcal{F}$-principal bundle. Moreover, the bijection in (1.6) is an isomorphism.

This corollary allows us to use properties of the symplectic quotients to obtain information about the geometry of the crepant resolution $X$.

The geometry of the Crepant resolution

In the case $n = 2$ the geometry of the crepant resolution $X$ was completely described by Kronheimer [Kro] using the hyper-Kähler properties of this space. In what follows we restrict to the case $n = 3$ and try to give a similar description of the geometry of $X$, building up on previous work of Sardo-Infirri [SI].

**Assumption 1.9.** We make the essential assumption that the finite group $G$ acts freely on $\mathbb{C}^3 \setminus \{0\}$.

As a consequence, $G$ is an abelian group; it implies that the Lie group $\mathcal{F}$ is also abelian and that $\text{Center}(\mathfrak{f}^*) = \mathfrak{f}^*$. Therefore we can perform the symplectic quotient construction (1.3) at any point $\zeta \in \mathfrak{f}^*$.  

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Remark 1.10. The symplectic quotient for $\zeta = 0$ is $\mathbb{C}^3/G$ endowed with the induced orbifold metric, symplectic form and complex structure.

Notation 1.11. To make the writing easier we denote the space $\mathcal{N} \cap \mu^{-1}(\zeta)$ by $Y_\zeta$ for any $\zeta \in \mathfrak{f}^*$. The corresponding symplectic quotient is $X_\zeta = Y_\zeta/F$. In particular, $Y_{\text{global}} = \mathcal{N} \cap \mu^{-1}(\zeta_{\text{Hilb}})$ and $X_{\text{global}} = X$.

Corollary 1.8 tells us that $Y_{\text{global}}$ is smooth and that $Y_{\text{global}} \to X_{\text{global}}$ is an $F$-principal bundle. Therefore the same statement is true for $\zeta$ in a small convex neighborhood around $\zeta_{\text{Hilb}}$. We take the cone over this convex neighborhood, and denote it by $C$. At each point $\zeta \in C \setminus \{0\}$ the symplectic quotient $X_\zeta = Y_\zeta/F$ is defined. This is because the action by dilatation of the positive scalars on $\mathcal{N}$ induces a map $\mu^{-1}(\zeta) \to \mu^{-1}(\ell^2 \zeta)$ (here we use the property of the moment map of being quadratic on $P$ and therefore on $\mathcal{N}$). The quotients $X_\zeta$ for $\zeta \in C \setminus \{0\}$ are smooth and diffeomorphic to $X$. However the induced symplectic form $\omega_\zeta$ varies with $\zeta$. This variation was studied by Duistermaat-Heckman [DH]:

Lemma 1.12. For each $\zeta \in C \setminus \{0\}$, the $F$-principal bundle $Y_\zeta \to X_\zeta$ has a natural connection $A_\zeta$ induced by the symplectic form $\omega$ on $\mathcal{N}$.

Moreover, if $R_\zeta$ is the curvature 2-form associated to the connection $A_\zeta$, the variation of the symplectic form $\omega_\zeta$ is given by
\[
\partial_\lambda \omega_\zeta = R_\zeta \lambda,
\]
where for $\lambda \in \mathfrak{f}^*$, $\partial_\lambda \omega_\zeta$ denotes the differentiation in the direction of the constant vector field $\lambda$.

Proof. Let $p_\zeta : Y_\zeta \to X_\zeta$ denote the quotient projection and $i_\zeta : Y_\zeta \to \mathcal{N}$ denote the inclusion map. The symplectic form on $X_\zeta$ satisfies $p_\zeta^* \omega_\zeta = i_\zeta^* \omega$.

For $y \in Y_\zeta$ the tangent space to $Y_\zeta$ has the following decomposition:
\[
T_y Y_\zeta = T_y(F \cdot y) \oplus H_y,
\]
where $F \cdot y$ is the $F$-orbit through $y$, and $H_y = T_y(F \cdot y)^\omega \cap T_y Y_\zeta$ with $T_y(F \cdot y)^\omega$ being the symplectic complement of $T_y(F \cdot y)$ in $T_y \mathcal{N}$. The spaces $H_y$ for $y \in Y_\zeta$ determine a connection on the principal bundle $Y_\zeta \to X_\zeta$. We want to write down the corresponding connection 1-form.

The group $F$ acts on $\mathcal{N}$. For $\lambda \in \mathfrak{f}^*$ let $V_\lambda$ denote the corresponding vector field induced on $\mathcal{N}$. We define a $\mathfrak{f}$-valued one-form $A_\zeta$ on $Y_\zeta$ as follows
\[
A_\zeta(\lambda) = -i_\zeta^* (\iota(V_\lambda) \omega).
\]
We need to verify that this is the connection form corresponding to the horizontal subspaces $H_y$.

Let $\xi \in \mathfrak{f}$ and let $\xi^* \in \mathfrak{f}^*$ be the corresponding vector field on $Y_\zeta$. We need that $A_\zeta(\xi^*) = \xi$. This follows since for each $\lambda \in \mathfrak{f}^*$ we have
\[
\langle A_\zeta(\xi^*), \lambda \rangle = -i_\zeta^* (\iota(V_\lambda) \omega)(\xi^*) = -\omega(\xi^*, V_\lambda) = \langle d\mu(V_\lambda), \xi \rangle = \langle \lambda, \xi \rangle.
\]
It remains to check that $A_\zeta(v) = 0$ for any $v \in H_y$. Since $v$ lies in the symplectic complement we have $A_\zeta(\lambda)(v) = -\omega(V_\lambda, v) = 0$. Therefore $A_\zeta(v) = 0$ and $A_\zeta$ is the connection 1-form corresponding to the horizontal distribution given by $H_y$.
The last thing which remains to be proved is the variational formula. We have
\[ p_\zeta^*(\partial_\lambda \omega_\zeta) = \partial_\lambda (p_\zeta^* \omega_\zeta) = \partial_\lambda (i_\zeta^* \omega) = i_\zeta^* (\mathcal{L}_{V_\lambda} \omega) = \\
= i_\zeta^* (d (V_\lambda + ud \omega (V_\lambda))) = i_\zeta^* (d (V_\lambda) \omega) = \\
= di_\zeta^* (iv (V_\lambda) \omega) = dA_\zeta (\lambda) = \\
= p_\zeta^* R_\zeta (\lambda). \]

For the last equality we use the fact that the structure group \( \mathcal{F} \) of the bundle \( Y_\zeta \to X_\zeta \) is abelian. Since \( p_\zeta \) is a Riemannian submersion it follows that
\[ \partial_\lambda \omega_\zeta = R_\zeta (\lambda). \]

The complex structure on \( \mathcal{N} \) induces a natural complex structure on \( X_\zeta \) which is compatible to the symplectic structure. We also obtain a Riemannian \( g_\zeta \) on \( X_\zeta \) which is compatible to the complex structure and the symplectic form.

**Lemma 1.17.** For \( \zeta \in \mathcal{C} \setminus \{0\} \), the symplectic quotient \( X_\zeta \) inherits a complex structure \( J \) from \( \mathcal{N} \) and a metric \( g_\zeta \) which are compatible with the symplectic form \( \omega_\zeta \).

**Proof.** We need to show that \( H_y \) is also the orthogonal complement of \( T_y (\mathcal{F} \cdot y) \) in \( T_y (Y_\zeta) \) and that it is left invariant by the action of the complex structure \( J \) on \( \mathcal{N} \).

From the proof of the previous Lemma, we have \( H_y = T_y (\mathcal{F} \cdot y)^\omega \cap T_y Y_\zeta \). Let \( H'_y \) denote the orthogonal complement of \( T_y (\mathcal{F} \cdot y) \) in \( T_y Y_\zeta \). If \( v \in T_y Y_\zeta \) and \( \xi \in \mathfrak{f} \) we have
\[ g(Jv, \xi^*) = \omega(Jv, J\xi^*) = \omega(v, \xi^*) = \langle d\mu (v), \xi \rangle = 0, \]

since \( d\mu(v) = 0 \). Therefore \( \xi^* \perp Jv \) and \( J\xi^* \in T_y Y_\zeta^\perp \).

We want to check that \( J \) leaves \( H'_y \) invariant. Let \( v \in H'_y \). We have
\[ \langle d\mu (Jv), \xi \rangle = \omega(Jv, \xi^*) = -g(Jv, I\xi^*) = -g(v, \xi^*) = 0, \]

and therefore \( Jv \in \text{Ker} (d\mu) \). Thus \( Jv \in T_y (Y_\zeta) \); since \( Jv \perp \xi^* \) it follows that \( Jv \in H'_y \).

From the above considerations it follows that
\[ T_y (\mathcal{N}) = H'_y \oplus T_y (\mathcal{F} \cdot y) \oplus JT_y (\mathcal{F} \cdot y). \quad (1.18) \]

Since the symplectic form \( \omega \) on \( \mathcal{N} \) is compatible with the complex structure it follows that \( H'_y = H_y \).

The induced metric \( g_\zeta \) on \( X_\zeta \) has a special property: it is an “ALE metric”.

**ALE metrics**

**Definition 1.19.** A Riemannian manifold \((X, g)\) of real dimension \(m\) is called *asymptotically locally euclidean* (ALE) of order \( \mu > 0 \) if there exists a compact set \( K \subset X \) and a finite subgroup \( G \) of \( SO(m) \) so that \( X \setminus K \) — the “end” of \( X \) — is diffeomorphic to \((\mathbb{R}^m \setminus B_R)/G\) for some \( R > 0 \). Under this diffeomorphism the metric on \( X \setminus K \) is of the form
\[ g_{ij} = \delta_{ij} + O(r^{-\mu}), \quad \nabla^k g_{ij} = O(r^{-\mu-k}) \text{ for } k \leq 1, \]
in the geodesic polar coordinates \( \{r, \Theta\} \) — the ALE coordinates — induced on \( X \setminus K \).
This definition apparently depends on the choice of ALE coordinates. However, it can be shown \([PL]\) that the ALE structure is determined by the metric alone.

**Remark 1.20.** The ALE condition translates into the fact that the group \(G\) acts freely on \(S^{m-1}\). We think of an ALE manifold as a manifold with boundary \(Y = S^{m-1}/G\) at infinity.

**Example 1.21.** A crepant resolution of singularities of a Calabi-Yau orbifold admits ALE metrics, if the dimension of the orbifold is \(\leq 3\).

Sardo-Infirri showed that a crepant resolution of the isolated singularity \(\mathbb{C}^3/G\) admits an ALE metric. His ALE metric has order \(\mu = 4\). For our purposes (see Theorem 3.1) we need a stronger decay on the metric: ALE of order \(\mu = 6\). We work with Sardo-Infirri’s approach and show that his estimate can be improved.

**Proposition 1.22.** For \(\zeta \in \mathbb{C} \setminus \{0\}\) the metric \(g_\zeta\) on \(X_\zeta\) induced via the symplectic quotient construction is ALE of order \(\mu = 6\).

**Proof.** Let \(\zeta \in \mathbb{C} \setminus \{0\}\). Via the map

\[
\mathbb{R}^6 \setminus \{0\} \to \mathbb{R}^6/G \to X_\zeta,
\]

we transfer all the structure (symplectic form, Riemannian metric, complex structure) on \(X_\zeta\) to \(\mathbb{R}^6 \setminus \{0\}\). Therefore we have

- \(\{g_\zeta\}_{\zeta \in \mathbb{C}}\) family of metrics on \(\mathbb{R}^6 \setminus \{0\}\), with \(g_0 = \delta\) the Euclidean metric.
- \(\{\omega_\zeta\}_{\zeta \in \mathbb{C}}\) family of symplectic forms on \(\mathbb{R}^6 \setminus \{0\}\), with \(\omega_0\) being the standard symplectic form on \(\mathbb{C}^3 \setminus \{0\}\).
- \(\{A_\zeta\}_{\zeta \in \mathbb{C}}\) family of connections on the principal \(\mathcal{F}\)-bundle \((\mathbb{R}^n \setminus \{0\}) \times \mathcal{F} \to \mathbb{R}^n \setminus \{0\}\), with \(A_0\) the trivial connection.

Let \((r, \Theta)\) be the polar coordinates on \(\mathbb{R}^6 \setminus \{0\}\). The dependence of the family \(\{g_\zeta\}\) on \(\zeta\) is analytic, and we have the following power series expansion around 0:

\[
g_\zeta|_{r=1} = \sum_{|\nu| \geq 0} f_\nu \zeta^\nu. \tag{1.23}
\]

The rescaling via the map \(\mu^{-1}(\zeta) = \mu^{-1}(t^2 \zeta)\) gives

\[
g_\zeta(r, \Theta) = g_{r^{-2}\zeta}(1, \Theta).
\]

Then

\[
g_\zeta(r, \Theta) = g_{r^{-2}\zeta}(1, \Theta) = \sum_{|\nu| \geq 0} f_\nu \zeta^\nu r^{-2|\nu|}
\]

\[
= \sum_{k \geq 0} h_k(\Theta) r^{-2k},
\]

where \(h_k(\Theta) = \sum_{|\nu| = k} f_\nu \zeta^\nu\).

In order to show that \(g_\zeta\) is ALE we need that \(h_0 = \delta\) and \(h_1 = h_2 = 0\). For \(h_0\) we have

\[
h_0(\theta) = g_0(1, \theta) = \delta(1, \theta).
\]
For $h_1 = 0$ we need to show that $\partial V g|_{\zeta = 0}(1, \theta) = 0$, or equivalently that $\partial V \omega|_{\zeta = 0}(1, \theta) = 0$. Duistermaat-Heckman’s formula (1.13) gives that

$$\partial V \omega|_{\zeta = 0}(1, \theta) = \langle V, R_{\zeta}(1, \theta) \rangle,$$

(1.24)

where $R_{\zeta}$ is the curvature of the connection form $A_{\zeta}$. Since $R_0 = 0$ we obtain that $h_1(\theta) = 0$. We have left to show that $h_2 = 0$. By (1.13) this statement is equivalent to $\partial \lambda R|_{\zeta = 0} = 0$. We have

$$\partial \lambda R|_{\zeta = 0} = \frac{d}{dt}|_{t=0} R_{\lambda t} = \frac{d}{dt}|_{t=0} (\sqrt{t})^3 R_{\lambda} = 0,$$

since we have the rescaling $\mu^{-1}(\lambda t) \to \mu^{-1}(\lambda \sqrt{t})$ and since the ambient space $X_\zeta$ has complex dimension 3.

Tautological Vector Bundles on the Crepant Resolution

From the definition of $X_{\text{Hilb}}$ there exists a natural vector bundle

$$\mathcal{R} \to X_{\text{Hilb}},$$

$$\mathcal{R} = Y_\zeta \times_{\mathcal{F}} R,$$

with $R$ the regular representation of $G$. Let $\{R_0, R_1, \ldots, R_r\}$ be the irreducible representations of $G$ with $R_0$ the trivial representation. The regular representation decomposes as a $G \times G$ module as

$$R = \bigoplus_i W_i \otimes R_i,$$

where $\dim W_i = \dim R_i$.

**Remark 1.25.** Since we are in the case of isolated singularities, all the $W_i$ are one-dimensional complex vector spaces.

The vector bundle $\mathcal{R}$ has a corresponding decomposition as a $G$-module:

$$\mathcal{R} = \bigoplus_i \mathcal{R}_i \otimes R_i,$$

where we define the vector bundles $\mathcal{R}_i \to X_{\text{Hilb}}$ by

$$\mathcal{R}_i = Y_\zeta \times_{\mathcal{F}} W_i,$$

and $R_i$ denotes the trivial vector bundle with fiber $R_i$. We refer to the bundles $\mathcal{R}_i$ as **tautological vector bundles** on $X$.

**Proposition 1.26 (Bridgeand-King-Reid [BKR]).** *The set of Chern characters \{ch(\mathcal{R}_0), ch(\mathcal{R}_1), \ldots, ch(\mathcal{R}_r)\} forms a basis of $H^*(X, \mathbb{Q})$.***

We want to figure out the ring structure in cohomology of $H^*(X, \mathbb{Q})$. For this we need first to introduce an analytical set-up for doing analysis on the crepant resolution $X$. 

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First we need to notice that the bundles \( R_i \) are endowed with natural “asymptotically flat” connections. We call a connection \( A \) asymptotically flat if there exists a flat connection \( A_0 \) defined on the ALE end \( X \setminus K \) such that, under a suitable trivialization, the two connections satisfy
\[
A - A_0 = O(r^{-1}), \quad \nabla^k A - \nabla^k A_0 = O(r^{-1-k}).
\]

Lemma 1.27. The connection \( A_\zeta \) on \( X_\zeta \) is asymptotically flat and the corresponding curvature behaves like \( O(r^{-4}) \) at infinity.

This was proved by Gocho and Nakajima [GN] in the case \( n = 2 \). Their proof generalizes straightforwardly to our case. For the sake of completion we include it here.

Proof. The curvature form of the connection \( A_\zeta \) is given by
\[
R_\zeta(V, W) = -A_\zeta([\tilde{V}, \tilde{W}]^v),
\]
where \( \tilde{V} \) denotes the horizontal lift of the tangent vector \( V \) to \( X_\zeta \). The vertical component of a tangent vector \( U \) on \( Y_\zeta \) is denoted by \( U^v \). The bundle map \( Y_\zeta \to X_\zeta \) is a Riemannian submersion. Therefore the Levi-Civita connections on the two spaces are related by
\[
\nabla^X_\zeta V \cdot W = \nabla^Y_\zeta \tilde{V} \cdot \tilde{W} - \frac{1}{2} ([\tilde{V}, \tilde{W}]^v).
\]
The space \( Y_\zeta \) is a smooth submanifold of \( N \) and let \( \Pi \) denote the second fundamental form. Then the above formula becomes
\[
\nabla^X_\zeta V \cdot W = \nabla^N_{\tilde{V}} \tilde{W} - \frac{1}{2} ([\tilde{V}, \tilde{W}]^v).
\]
The previous formula applied to \( I\tilde{W} \) gives
\[
\nabla^N_{\tilde{V}} I\tilde{W} = \nabla^N_{\tilde{V}} \tilde{W} + \Pi(\tilde{V}, \tilde{W}) + \frac{1}{2} ([\tilde{V}, I\tilde{W}]^v).
\]
Since the complex structure on \( N \) is parallel with respect to \( \nabla^N \) and given the decompo- sition (1.18) of the tangent to \( T_y N \) we have
\[
\nabla^N_{\tilde{V}} I\tilde{W} = I\nabla^N_{\tilde{V}} \tilde{W}, \quad I \Pi(\tilde{V}, \tilde{W}) = \frac{1}{2} [\tilde{V}, \tilde{W}]^v, \quad \Pi(\tilde{V}, I\tilde{W}) = \frac{1}{2} I [\tilde{V}, I\tilde{W}]^v.
\]
The first relation shows that the complex structure on \( X_\zeta \) is parallel to the Levi-Civita connection. From the second identity we deduce that
\[
\Pi(\tilde{V}, \tilde{W}) = -\frac{1}{2} [\tilde{V}, \tilde{W}]^v.
\]
Therefore
\[
\nabla^N_{\tilde{V}} I\tilde{W} = \nabla^N_{\tilde{V}} \tilde{W} + \frac{1}{2} I [\tilde{V}, I\tilde{W}]^v - \frac{1}{2} [\tilde{V}, \tilde{W}]^v.
\]
For \( \xi^* \) a vertical tangent vector to \( Y_\zeta \) we have
\[
g([I\tilde{V}, \tilde{V}]^v, \xi^*) = g(I [I\tilde{V}, \tilde{W}]^v, I \xi^*)
= g(\nabla^N_{\tilde{V}} I\tilde{W}, I \xi^*)
= 2 \langle \xi, \text{Hess} \mu(\tilde{V}, \tilde{W}) \rangle.
\]
The last identity follows since
\[ \Pi_{-\text{grad} \mu} (\bar{V}, \bar{W}) = \langle \Pi(\bar{V}, \bar{W}), \text{grad} \mu \rangle = \text{Hess} \mu (\bar{V}, \bar{W}). \]
Choosing \( \xi = R_{\zeta} (V, W) \) the above formula leads to
\[ |R_{\zeta} (IV, W)|^2 = A_{\zeta} (|I \bar{V}, \bar{W}|^\nu), A_{\zeta} (|I \bar{V}, A_{\zeta} (R_{\zeta} (IV, W)^\nu))) \]
\[ \leq |A_{\zeta}|^2 g ([I \bar{V}, \bar{W}]^\nu, R_{\zeta} (IV, W)^\nu). \]
Putting everything together we obtain the following inequality
\[ |R_{\zeta} (IV, W)| \leq 2 |A_{\zeta}|^2 |\text{Hess} \mu (\bar{V}, \bar{W})|. \quad (1.28) \]
Since the connection form scales like \( A_{\zeta} (r, \Theta) = r^{-1} A_{r^{-2} \zeta} (1, \Theta) \) and since \( \text{Hess} \mu \) behaves like \( r^{-2} \) (the metric is ALE) it follows that the curvature is of the form \( R_{\zeta} = O(r^{-4}). \) □

**Remark 1.29.** Since the vector bundles \( \mathcal{R}_i \) are associated to the principal \( \mathcal{F} \)-bundle \( Y_{\zeta} \to X_{\zeta} \) the corresponding natural connection is asymptotically flat.

## 2. Analysis on ALE manifolds

Let \( (X^m, g) \) be an ALE space of order \( \mu > 0 \). From the Definition 1.19 this describes a Riemannian manifold with one end which at infinity resembles the quotient \( \mathbb{R}^m / G \) of the Euclidean space \( \mathbb{R}^m \) by a finite subgroup \( G \) of \( SO(m) \). The Riemannian metric \( g \) is required to approximate the Euclidean metric up to \( O(r^{-\mu}). \) From the previous section, ALE spaces arrive as crepant resolutions of isolated Calabi-Yau orbifolds.

For our purposes, we consider an ALE manifold which admits a Spin-structure. (If \( X \) is a crepant resolution, then \( c_1 (X) = 0 \), which implies that \( w_2 (X) = 0 \), i.e. \( X \) is spin.) The choice of a Spin-structure on the ALE end, \( X \setminus K \), is equivalent to the choice of a \( G \)-invariant Spin-structure on \( \mathbb{R}^m \setminus B_R \). Since \( H^1 (\mathbb{R}^m \setminus B_R; \mathbb{Z}_2) \) is trivial, the Spin-structure over this space is trivial, but with a \( G \)-action induced from the natural inclusion of \( G \) into \( SO(m) \). A Spin-structure on \( X \) is an extension of the Spin-structure on \( X \setminus K \) to the entire \( X \).

Moreover, since the ALE spaces we are dealing with arrive as crepant resolutions of Calabi-Yau orbifolds, we restrict ourselves to Kähler ALE spaces of complex dimension \( n \) (\( m = 2n \)). The action of the finite group \( G \) must preserve the Kähler structure on the end \( (\mathbb{C}^n \setminus B_R)/G \), and therefore we consider \( G \subset SU(n) \subset SO(2n) \).

On this spaces we have the two spin bundles, \( S^+ \) and \( S^- \), associated to the two half-spin representations of \( \text{Spin}(2n) \). For \( E \) a Hermitian complex vector bundle on \( X \) we consider the corresponding twisted Dirac operator
\[ D^+: \Omega^0 (X; S^+ \otimes E) \to \Omega^0 (X; S^- \otimes E). \]
We are interested in studying Fredholm properties of this first order elliptic differential operator.

If the appropriate analytical setting over compact manifolds is that of Sobolev spaces, in the case of ALE manifolds we need to take into consideration the behavior of the functions at infinity. Therefore the need to work with “weighted Sobolev spaces”. Let \( (X, g) \) be an ALE manifold, with ALE coordinates \( (r, \Theta) \) on \( X \setminus K \). Let \( \rho : X \to [1, +\infty) \) so that
• outside the ball of radius $2R$ we have $\rho(r, \Theta) = r$;
• on the compact subset $K$ of $X$ we have $\rho = 1$;
• on the collar $B_{2R} \setminus B_R$ we have $1 \leq \rho \leq 2R$.

Using $\rho$ we define the $\alpha$-weighted $L^2$-norm for a smooth compactly supported section of the Hermitian vector bundle $E$:

$$||f||^2_{L^2_\alpha} = \int_X \rho^{-2n} |\rho^\alpha f|^2 \text{dvol}(g).$$

We denote the completion of $\mathcal{O}^0_{\text{comp}}(X, E)$ with respect to the $L^2_\alpha$-norm by $L^2_\alpha(X, E)$, or $L^2_\alpha(X)$ when the bundle is clear from the context. A section $f$ which is bounded in the $L^2_\alpha$-norm behaves like $O(\rho^{-\alpha})$ on the infinite end.

We assume now that the Hermitian complex bundle $E$ is equipped with a connection $\nabla$ which is asymptotically flat. Using the connection $\nabla$ we extend the previously defined inner product to an $L^2_{k,\alpha}$-norm:

$$||f||^2_{L^2_{k,\alpha}} = \sum_{j=0}^k \int_X \rho^{-2n} |\rho^\alpha \nabla f|^2 \text{dvol}(g).$$

We denote the corresponding completions by $L^2_{k,\alpha}(X, E)$ or $L^2_{k,\alpha}(X)$.

The Fredholm properties of the Dirac operator extended in a weighted Sobolev space depend on the spectrum of the restriction to the boundary at infinity. We recall the structure of this spectrum in the proposition below. This result appears in [Bä] and [D1].

The spin bundles $S^\pm$ restricted to the boundary at infinity $Y = S^{2n-1}/G$ can each be identified with the spin bundle $S$ of $Y$ associated to the spin representation of $\text{Spin}(2n-1)$.

**Proposition 2.1.** For the round metric on $S^{2n-1}$, the eigenspaces of the Dirac operator are

$$V_{\frac{k}{2}} \geq \frac{k}{2} \geq \ldots \geq \frac{k}{2}$$

with eigenvalue $(-1)^n \frac{2n-1+2a}{2}$

$$V_{\frac{k}{2}} \geq \ldots \geq -\frac{k}{2} \geq -b - \frac{k}{2}$$

with eigenvalue $\frac{2n-1+2b}{2}$

$$V_{a+\frac{k}{2}} \geq \frac{k}{2} \geq \frac{k}{2} \geq \frac{k}{2} \geq b + \frac{k}{2}$$

with eigenvalues $\frac{-1)^{n+r+2(n+a+b)}}{2}$

$$\frac{-1)^{n+r-2(n+a+b)}}{2},$$

where $a$ and $b$ range over the positive integers. Here we denote by $\tilde{U}(n)$ the double cover of the unitary group in order to think of the sphere $S^{2n-1}$ as the homogeneous space $\tilde{U}(n)/U(n-1)$. The vector space $V_{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n}$ represents the irreducible representation of $\tilde{U}(n)$ with highest weight $\mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_n x_n$ (we choose the fundamental Weyl chamber of $\tilde{U}(n)$ to be $x_1 \geq x_2 \geq \ldots \geq x_n$).

Observe that from the above proposition the smallest positive eigenvalue is $\lambda^+ = \frac{2n-1}{2}$ and the largest negative one is $\lambda^- = -\frac{2n-1}{2}$.

**Theorem 2.3.** For the weight $\alpha$ so that $0 < \alpha < 2n - 1$ the closure of the Dirac operator in the $L^2_\alpha$-norm

$$D^+_\alpha : L^2_{1,\alpha}(X, S^+ \otimes E) \rightarrow L^2_{\alpha+1}(X, S^- \otimes E)$$

(2.4)
is Fredholm with index given by

$$\text{index } D^\pm_\alpha = \int_X \text{ch}(E)\hat{A}(p) - \frac{\eta_E}{2}. \quad (2.5)$$

Here $\eta_E(s)$ is the $\eta$-function of the Dirac operator restricted to the boundary at infinity $Y = S^{2n-1}/G$, and $\hat{A}(p)$ is the Hirzebruch $\hat{A}$-polynomial applied to the Pontrjagin forms $p$, of the ALE metric on $X$.

The proof of this theorem is inspired by techniques from gauge theory [MMR]; it is a consequence of combining the Atiyah-Patodi-Singer index theorem for manifolds with cylindrical ends and results of Lockhart and McOwen, [LMc]. There exist some other methods to see for which weights $\alpha$ the Dirac operator is Fredholm, [NW, CC, B]. The approach we choose here is convenient for our purposes of expressing the index as a topological object.

**Proof of the Theorem 2.3**

We change the ALE metric conformally into $\tilde{g} = \rho^{-2} g$. It transforms the ALE manifold into a manifold with a cylindrical end. The two conformally equivalent metrics give rise to the same spin bundles, with the same Hermitian metric, but with different Clifford multiplication. If $\gamma : TX \to \text{Hom}(S^+, S^-)$ is the Clifford multiplication corresponding to the metric $g$, then for $\tilde{g}$ it changes into $\tilde{\gamma} = \rho \gamma$. The Dirac operators, $D^\pm$ and $\tilde{D}^\pm$, corresponding to the two metrics are related via

$$\tilde{D}^\pm = \rho^{\frac{2n-1}{2}} D^\pm \rho^{-\frac{2n-1}{2}}. \quad (2.6)$$

Under the conformal change the compact set $K$ does not encounter any change in its geometry, but the ALE end $\mathbb{R}^{2n} \setminus B_R$ changes into the cylindrical end $[T, +\infty) \times Y$, where $T = \log R$. We take $\tau = \log \rho : X \to [0, +\infty)$. After a reparametrization we can assume that $\tau$ has the following properties:

- on $[3T, +\infty) \times Y$, $\tau$ agrees with the natural projection onto $[3T, +\infty)$;
- on $[2T, 3T]$, $\tau$ is between 0 and 1;
- on $X \setminus [2T, +\infty)$, $\tau \equiv 0$.

A function which behaves as $O(\rho^{-\alpha})$ at infinity behaves also like $O(e^{-\alpha \tau})$. Then the corresponding weighted Sobolev norm for the metric $\tilde{g}$ is

$$||f||^2_{\tilde{L}^2_\alpha} = \int_X |e^{\alpha \tau} f|^2 d\text{vol}(\tilde{g}).$$

We denote by $\tilde{L}^2_\alpha(X)$ the corresponding weighted Sobolev space. The Dirac operator (2.6) on the Riemannian manifold $(X, \tilde{g})$ acts in the following manner

$$\tilde{D}^\pm_{\alpha-\frac{2n-1}{2}} : \tilde{L}^2_{\alpha-\frac{2n-1}{2}}(X; S^\pm \otimes E) \to \tilde{L}^2_{\alpha-\frac{2n-1}{2}}(X; S^\mp \otimes E). \quad (2.7)$$

Let $\beta = \alpha - \frac{2n-1}{2}$. We have replaced the problem of studying Fredholm properties of the Dirac operator on a ALE manifold with the similar problem on a manifold with cylindrical end. The study of the operator (2.4) is equivalent to the study of

$$\tilde{D}^\pm_{\beta} : \tilde{L}^2_{\beta}(X; S^\pm \otimes E) \to \tilde{L}^2_{\beta}(X; S^\mp \otimes E). \quad (2.8)$$
We can further reduce the problem just to the study of Fredholm properties on a manifold with the product cylindrical metric on the infinite end. The following lemma takes care of the weighted Sobolev spaces.

**Lemma 2.9.** A metric \( \tilde{\mathfrak{g}} \) which arrives as the conformal transform of an ALE metric, is almost the product metric \( \tilde{\delta} = dt^2 + d\Theta^2 \) on the cylinder. As a consequence the \( \beta \)-weighted Sobolev norms are equivalent and therefore the Hilbert space completions in these norms are the same.

**Proof.** An orthonormal frame for the product metric on the cylinder is \( \{ dt, d\Theta \} \). Since

\[
|\tilde{\mathfrak{g}} - \tilde{\delta}| \leq e^{-\mu \tau},
\]

an orthonormal frame for \( \tilde{\mathfrak{g}} \) has the form \( \{ dt, h_{ij} d\theta^i \} \) where \( h_{ij} = 1 + O(e^{-(\mu+2)\tau}) \), and the volume form is \( dvol(\tilde{\mathfrak{g}}) = dvol(\tilde{\delta}) + O(e^{-\mu \tau}) \). Then,

\[
\int_X |e^{-\beta \tau} f|^2 dvol(\tilde{\mathfrak{g}}) \leq \int_X |e^{-\beta \tau} f|^2 dvol(\tilde{\delta}) + \int_X O(e^{-\mu \tau}) |e^{-\beta \tau} f|^2 dvol(\tilde{\delta})
\]

\[
\leq C \int_X |e^{-\beta \tau} f|^2 dvol(\tilde{\delta}).
\]

The other way around goes the same. \( \square \)

In order to complete the reduction to a manifold with the product metric on the cylindrical end we use the following result of Lockhart and McOwen [LMc].

**Proposition 2.10 (Lockhart and McOwen).** Let \( P_\beta \) be the Dirac operator associated to the product metric on the cylindrical end of \( X \). Then \( P_\beta \) and \( \tilde{D}_\beta \) are Fredholm for exactly the same values \( \beta \). Moreover, their Fredholm indices are equal, \( \text{index } P_\beta = \text{index } \tilde{D}_\beta \).

**Proof.** Assume that \( P_\beta \) is Fredholm. Let \( t \) be a positive real number. The idea is to cut and paste the operator \( P_\beta \) to a new operator \( P'_\beta = P_\beta + (1 - \phi_t)(D_\beta - P_\beta) \). Here \( \phi_t \) is a smooth function which is 1 on \( X_t \) and has support in \( X_{2t} \) (notation: \( X_t = X \setminus [t, +\infty) \times Y \)). Since the space of Fredholm operators is open, for \( t \) very large the operator \( P'_\beta \) is Fredholm and has index \( P'_\beta = \text{index } P_\beta \). This new operator \( P'_\beta \) has the property that for \( t > 3T \) is \( \tilde{D}_\beta \). Performing again a cutting and pasting procedure we obtain a parametrix for \( \tilde{D}_\beta \), constructed out of a parametrix for \( D_\beta \) on \( X_{4t} \), for example, and a parametrix for \( P_\beta \). Therefore \( \tilde{D}_\beta \) is Fredholm.

The operator norm between \( \tilde{D}_\beta \) and \( P_\beta \) is exponentially decaying. Therefore the two operators belong to the same connected set in the space of Fredholm operators and thus their Fredholm indices are equal. \( \square \)

On the cylindrical end the operator \( \tilde{D} \) has the form

\[
\tilde{D} = \frac{\partial}{\partial t} + B + O(e^{-\mu t}),
\]

where \( B \) is the Dirac operator restricted to the boundary at infinity \( Y = S^{2n-1}/G \). Lockhart and McOwen proved that the operator \( P_\beta \) (and therefore \( D_\beta \)) is Fredholm for those weights \( \beta \) for which the operator \( B - \beta I \) is invertible. In other words, if \( \beta \) is not an eigenvalue for \( B \) then \( P_\beta \) is Fredholm. In the case when \( B - \beta I \) is not invertible problems arise since
In this case the operator $P_\beta$ does not have closed range, which is an essential condition for the operator to be Fredholm.

In order to exhibit the index of $P_\beta$ we employ the Atiyah-Patodi-Singer theorem. First we need to establish the set-up: we consider the manifold with boundary $X_{2T}$ with the metric which is the product metric on $[T,2T] \times Y$. The Dirac operator

$$P : \Omega^0(X_{2T}, S^+ \otimes E) \to \Omega^0(X_{2T}, S^- \otimes E).$$

has the following form on the cylinder $[T,2T] \times Y$:

$$P = \frac{\partial}{\partial t} + B,$$

where $B$ is the Dirac operator on the boundary $Y$. Its $L^2$-adjoint is

$$P^* = \frac{\partial}{\partial t} - B.$$

We consider the spectral projection

$$\Pi^+ : \Omega^0(Y, S \otimes E) \to \Omega^0(Y, S \otimes E)$$

onto the span of eigenvectors corresponding to the positive eigenvalues of $B$. The space of all smooth sections which satisfy $\Pi^+ f(2T, \cdot)$ we denote by $\Omega^0(X_{2T}, S^+ \otimes E; \Pi^+)$. Its completion in the $L^2$-norm we denote by $\Omega^0_k(X, S^+ \otimes E; \Pi^+)$. We take the closure of the Dirac operator $P$ on $L^2$ with domain given by the global boundary condition induced by $\Pi^+$:

$$P : \Omega^0(X_{2T}, S^+ \otimes E; \Pi^+) \to \Omega^0(X_{2T}, S^- \otimes E).$$

The Atiyah-Patodi-Singer index theorem states that the operator (2.11) is Fredholm; its index is

$$\text{index } P = \int_{X_{2T}} \text{ch}(E) \hat{A}(p) - \frac{\eta_E}{2}.$$ 

To conclude the proof of Theorem 2.3 we need the following proposition:

**Proposition 2.12.** For $-\frac{2n-1}{2} < \beta < \frac{2n-1}{2}$ the operator $P_\beta$ is Fredholm and its Fredholm index is

$$\text{index } P_\beta = \text{index } P.$$

**Proof.** We assume that the metric is a product metric on the cylinder $[T, +\infty) \times Y$. The pair of operators $P_\beta$ and $P$ can be put together into an excision data

$$(Z_1 = X, \quad U_1 = X_{2T-\epsilon}, \quad V_1 = (T, +\infty) \times Y, \quad Q_1 = P_\beta, \quad Q_1' = P),$$

where

$$P_\beta : \tilde{L}^2_{1,\beta}(X) \to \tilde{L}^2_{\beta}(X),$$

and

$$P : L^2_1(X_{2T}, \Pi^+) \to L^2_2(X_{2T}).$$

Since $\tau = 0$ on $X_{2T}$, it follows that

$$P_\beta|_{U_1} = P|_{U_1}.$$
The second excision data we need is
\[(Z_2 = [T, +\infty) \times Y, \quad U_2 = [T, 2T - \epsilon) \times Y, \quad V_2 = (T, +\infty) \times Y, \quad Q_2, \quad Q'_2),\]
where \(Q_2\) is the restriction of \(P_{\beta}\) to
\[Q_2 : \left\{ f \in \tilde{L}^2_{1,\beta}(Z_2) \mid (1 - \Pi^+)f(T, \cdot) = 0 \right\} \to \tilde{L}^2_{\beta}(U_2),\]
and \(Q'_2\) is the restriction of \(P\) to
\[Q'_2 : \left\{ f \in L^2([T, 2T] \times Y) \mid (1 - \Pi^+)f(T, \cdot) = 0, \quad \Pi^+f(2T, \cdot) = 0 \right\} \to L^2([T, 2T] \times Y).
Again since \(\tau\) is 0 on \([T, 2T] \times Y\), we have
\[Q_2|_{U_2} = Q'_2|_{U_2}.
The two pairs of excision data are related in the following way
\[V_1 = V_2, \quad Q_1|_{V_1} = Q_2|_{V_2} \quad Q'_1|_{[T, 2T] \times Y} = Q'_2|_{[T, 2T] \times Y}.
The excision principle implies that
\[\text{index } Q_1 - \text{index } Q'_1 = \text{index } Q_2 - \text{index } Q'_2,\]
i.e.
\[\text{index } P_{\beta} - \text{index } P = \text{index } Q_2 - \text{index } Q'_2.
\textbf{Claim.} For the operators \(Q_2\) and \(Q'_2\) the following are true
\[\dim \ker Q_2 = 0, \quad \dim \ker Q'_2 = 0\]
\[\dim \coker Q_2 = 0, \quad \dim \coker Q'_2 = 0.
We sketch the proof for \(\dim \ker Q_2 = 0\); the other statements follow in a similar way.
Functions on \([T, +\infty) \times Y\) can be decomposed according to the eigenspaces of the Dirac operator \(B\) on \(Y\): \(f(t, y) = \sum \lambda f_\lambda(t) \phi_\lambda(y)\), where \(\phi_\lambda\) are the eigenfunctions of \(B\). Assume \(f(t, y) \in \ker Q_2\). This is equivalent to
\[
\begin{align*}
\frac{df_\lambda}{dt} + \lambda f_\lambda &= 0, \\
\text{with the boundary conditions:} & \\
\lambda f_\lambda(T) &= 0, \text{for } \lambda < 0 \\
f_\lambda &= O(e^{-\beta t}) \text{as } t \to +\infty, \text{for any } \lambda.
\end{align*}
\]
For \(\lambda < 0\) the solution to this system is
\[f_\lambda(t) = C \int_T^t e^{-\lambda u} du,\]
which satisfies the required boundary condition if \(\beta > -\lambda\). Since we choose \(-\frac{2n-1}{2} < \beta\), then the previous inequality never happens, so the only solution is the trivial one. For \(\lambda > 0\) the solution to the ODE is
\[f_\lambda(t) = -\int_t^{+\infty} e^{-\lambda u} du\]
which satisfies the decay condition only for \(\beta < -\frac{2n-1}{2}\).
Therefore for \(-\frac{2n-1}{2} < \beta < \frac{2n+1}{2}\) the operator \(P_\beta\) is Fredholm. Since \(\beta = \alpha - \frac{2n-1}{2}\), it means that for \(0 < \alpha < 2n - 1\) the Dirac operator \(D_\alpha\) in (2.4) is Fredholm. \(\square\)
Remark 2.14. It can be proved that the Dirac operator (2.4) is Fredholm for all weights \( \alpha = \beta - \frac{2n-1}{2} \) so that \( \beta \) is not an eigenvalue of \( B \) (in which case the Dirac operator does not have closed range). The index is going to jump – up or down – every time we cross an eigenvalue; the jump is by the dimension of the eigenspace corresponding to that eigenvalue.

3. A Vanishing Results on the Crepant Resolution

The index formula (2.4) holds for any ALE manifold with a metric of order \( \mu \geq 1 \). We are now interested in studying the case of an ALE metric which is Ricci-flat.

In Section 1 an ALE metric on the crepant resolution of the isolated orbifold singularity \( \mathbb{C}^n/G \) was constructed. Computations of Sardo-Infirri [SI] show that this metric is not Ricci-flat (the metric induced on \( O_{\mathbb{C}P^2}(-3) \) the crepant resolution of \( \mathbb{C}^3/G \) is not Ricci-flat; however this space has a Ricci-flat ALE metric, the Eguchi-Hanson metric). In the situation when the ALE metric \( g \) has order \( \mu = 2n \) there is a result which allows us to overcome this problem: Joyce’s proof of Calabi’s Conjecture for ALE spaces.

Theorem 3.1 (Joyce [J], Theorem 8.2.3). Let \( G \) be a finite group of \( SU(n) \) acting freely on \( \mathbb{C}^n \setminus \{0\} \), and \( (X, \pi) \) a crepant resolution of \( \mathbb{C}^n/G \). Then in each Kähler class of ALE Kähler metrics of order \( 2n \) on \( X \) there is a unique Ricci-flat ALE Kähler metric \( g \).

Assume now that \( X \) is the crepant resolution of \( \mathbb{C}^3/G \) given by Nakamura’s G-Hilbert scheme. We proved in Proposition 1.22 that \( X \) comes endowed with a natural ALE Kähler metric of order 6. By Joyce’s result we have a unique Ricci-flat ALE metric in its Kähler class. It is for this metric that we want to apply the index formula (2.4). We obtain the following vanishing result:

Theorem 3.2. Assume \( G \) is a finite subgroup of \( SL(3, \mathbb{C}) \) which acts freely on \( \mathbb{C}^3 \setminus \{0\} \). Let \( (X, \pi) \) be the crepant resolution given by the Nakamura’s G-Hilbert scheme, and \( g \) be the natural Ricci-flat ALE metric on it. Let \( E \) be a self-dual holomorphic Hermitian bundle \( (E = E^*) \) endowed with a connection which is asymptotically flat. Then, for the weight \( \alpha \) so that \( 0 < \alpha < 5 \) the Dirac operator

\[
D^{+}_{\alpha} : L^{2}_{1,\alpha}(X, S^{+} \otimes E) \rightarrow L^{2}_{\alpha+1}(X, S^{-} \otimes E),
\]

has vanishing index.

Proof. The \( L^2 \)-adjoint of the Dirac operator is

\[
D^{n}_{-\alpha+5} : L^{2}_{1,-\alpha+5}(X, S^{-} \otimes E) \rightarrow L^{2}_{\alpha+6}(X, S^{+} \otimes E)
\]

On a complex Kähler manifold, the Dirac operator and its adjoint are deeply related to the Dolbeault operator: we have \( S^{+} = \Lambda^{0,0}(X) \oplus \Lambda^{0,2}(X) \) and \( S^{-} = \Lambda^{0,1}(X) \oplus \Lambda^{0,3}(X) \). The Dirac operator is

\[
D^{+}_{A} = \sqrt{2} (\bar{\partial} A - \bar{\partial} A^*)
\]

Here \( \bar{\partial} A \) denotes the formal adjoint of \( \bar{\partial} A \); it is given by \( \bar{\partial} A = -\ast_{E} \bar{\partial}_{A} \ast_{E} \), where \( \ast_{E} \) is the usual Hodge star operator associated to the metric \( g \): \( \ast_{E} : \Omega^{0}(X, \Lambda^{p,q} \otimes E) \rightarrow \Omega^{0}(X, \Lambda^{n-p,n-q} \otimes E^{*}) \). Moreover since the metric \( g \) is Ricci-flat, the bundle \( \Lambda^{n,0} \) is trivial. It gives a natural isomorphism \( \Lambda^{n,n-k} \cong \Lambda^{0,n-k} \). Everything is encapsulated in the following diagram:
Completing the diagram in the $L^2$-norm, it follows that
\[ D^+_{\frac{5}{2}} = D^-_{\frac{5}{2}}. \] (3.3)

On the other hand we have that the indices of the two operators are related by
\[ \text{index } D^+_{\frac{5}{2}} = -\text{index } D^-_{\frac{5}{2}} \]
since $D^-_{\frac{5}{2}}$ is the $L^2$-dual of $D^+_{\frac{5}{2}}$. Therefore we have the desired vanishing result
\[ \text{index } D^+_{\frac{5}{2}} = 0. \] (3.4)

Since the index of the Dirac operator $D_\alpha$ is constant for the weight between two critical values, it follows that the vanishing holds for all the weights between 0 and 5.

**Remarks 3.5.**
1. The proof is valid for all other crepant resolutions of $\mathbb{C}^3/G$. They are obtained from $X$ via a flop. The flop changes the geometry of the exceptional fiber but since the singularity is isolated it does not affect the geometry at infinity.
2. Our proof also works for $n$ any odd number, provided that we know that a crepant resolution exists and it has an ALE metric of order $2n$.

4. Geometrical McKay Correspondence

We want to apply the index formula (2.4) together with the vanishing result of the Theorem 3.2 to obtain a geometrical interpretation of the McKay Correspondence similar to the one obtained by Kronheimer and Nakajima [KroN] for the case of surface singularities. The remaining needed piece of the puzzle is the $\eta$-invariant term.

**Proposition 4.1.** Let $X$ be a crepant resolution of the isolated orbifold singularity $\mathbb{C}^3/G$. Consider $E$ a Hermitian bundle over $X$ with fiber at infinity $E$. Then the corresponding $\eta$-invariant is
\[ \eta_E = \frac{1}{|G|} \sum_{g \in G \setminus I} \frac{\chi_E(g)}{-\chi_Q(g) + \chi_{\Lambda^2 Q}(g)}. \] (4.2)

In this formula $Q$ represents the 3-dimensional representation of $G$ induced by its inclusion into $SL(3, \mathbb{C})$. Also for a representation $V$ of $G$ we denote by $\chi_V$ the character of $V$.

This proposition is a consequence of the Lefschetz fixed-point formula, in the sense that $\eta_E$ is the contribution from the fixed locus under the action of $G$ on $\mathbb{C}^3$. It can be also proved using the definition of the $\eta$-invariant as the analytic continuation at 0 of the $\eta$-series corresponding to the spectrum of the Dirac operator on the boundary at infinity of the orbifold $\mathbb{C}^3/G$. This last approach gives the generalization of the above formula to the case of non-isolated singularities [D1].
Definition 4.3. Let \( \{ R_0, R_1, \ldots, R_r \} \) be the irreducible representations of the finite group \( G \) (here \( R_0 \) denotes the 1-dimensional trivial representation). We consider the tensor products \( Q \otimes R_i \) and \( \Lambda^2 Q \otimes R_i \) and decompose them into irreducible representations:

\[
R_i \otimes Q = \sum_{j=0}^{r} a_{ij} R_j
\]

\[
R_i \otimes \Lambda^2 Q = \sum_{j=0}^{r} b_{ij} R_j.
\]

We call the matrix

\[
\tilde{C} = [a_{ij} - b_{ij}]_{i,j=0,\ldots,r},
\]

the extended Cartan matrix associated to the finite group \( G \). By removing the first row and the first column of \( \tilde{C} \) we obtain a new matrix, \( C \), which we call the Cartan matrix associated to the finite group \( G \).

Remark 4.5. This definition is a generalization of the classical McKay Correspondence (0.1).

Now we are ready to state our geometrical interpretation of the McKay Correspondence, which is a consequence of the Vanishing Theorem 3.2.

Corollary 4.6. Assume that the finite subgroup \( G \) of \( SL(3, \mathbb{C}) \) acts freely on \( \mathbb{C}^3 \setminus \{0\} \), and let \( (X, \pi) \) be the crepant resolution of \( \mathbb{C}^3/G \) constructed using Nakamura’s \( G \)-Hilbert scheme. Let \( \{ R_0, R_1, \ldots, R_r \} \) be the tautological bundles on \( X \) corresponding to the irreducible representations of \( G \).

Then, the elements \( \{ \text{ch}(R_1), \ldots, \text{ch}(R_r) \} \) satisfy the following multiplicative relations

\[
\left[ \int_X (\text{ch}(R_i) - \text{rk}(R_i))(\text{ch}(R_j^*) - \text{rk}(R_j)) \right]_{i,j=1,\ldots,r} = C^{-1}.
\]

Remark 4.8. It is straightforward to see that since \( G \subset SL(3, \mathbb{C}) \), the Cartan matrix \( C \) is invertible.

Proof. The proof of Kronheimer and Nakajima goes through without major problems: We consider the bundle \( R \) which has \( R_{\text{reg}} \), the regular representation of the finite group, as fiber at infinity. We apply our Index Theorem 2.3 to the bundle \( R \otimes R^* \) and the Dirac operator completed in the weighted Sobolev norm with \( 0 < \alpha < 5 \):

\[
\text{index } D_{R \otimes R^*}^+ = \int_X \text{ch}(R \otimes R^*) \hat{A}(p) - \frac{\eta_{R_{\text{reg}} \otimes R_{\text{reg}}^*}}{2}.
\]

Since \( R \otimes R^* \) is a self-dual bundle, it follows according to the vanishing result, Theorem 3.2, that the left-hand-side of the above formula is zero.

The bundle \( R \) comes with a \( G \)-action on it. We decompose it under the \( G \)-action:

\[
R = \bigoplus_{i=0}^{r} R_i \otimes R_i.
\]

With respect this \( G \)-action the index formula becomes

\[
0 = \int_X \text{ch}(R_i \otimes R_j^*) \hat{A}(p) - \frac{\eta_{R_i \otimes R_j^*}}{2},
\]

where \( \eta_{R_i \otimes R_j^*} \) is the \( G \)-equivariant eta invariant of the induced bundle.
or equivalently
\[ \int_X \text{ch}(\mathcal{R}_i \otimes \mathcal{R}_j^*) \hat{A}(p) = \frac{\eta_{\mathcal{R}_i \otimes \mathcal{R}_j^*}}{2} \]

Multiplying on the right by the Cartan matrix \( \tilde{C} = [c_{ij}]_{i,j=0,\ldots,r} \), the left-hand-side becomes
\[
\sum_{k=0}^{r} c_{ik} \int_X \text{ch}(\mathcal{R}_k \otimes \mathcal{R}_j^*) \hat{A}(p) = \\
= \sum_{k=0}^{r} c_{ik} \int_X \text{ch}(\mathcal{R}_k) \hat{A}(p) + \sum_{k=0}^{r} c_{ik} \int_X \text{ch}(\mathcal{R}_j^*) \hat{A}(p) + \sum_{k=0}^{r} c_{ik} \int_X (\text{ch}(\mathcal{R}_k) - 1)(\text{ch}(\mathcal{R}_j^*) - 1)
\]
\[
= \sum_{k=0}^{r} c_{ik} \frac{\eta_{\mathcal{R}_k}}{2} + \sum_{k=0}^{r} c_{ik} \frac{\eta_{\mathcal{R}_j^*}}{2} + \sum_{k=0}^{r} c_{ik} \int_X (\text{ch}(\mathcal{R}_k) - 1)(\text{ch}(\mathcal{R}_j^*) - 1).
\]

To figure out what happens to the right-hand-side, we see that in terms of the corresponding characters, the relations (4.4) give
\[
(\chi_{\mathcal{Q}}(g) - \chi_{\Lambda^2 \mathcal{Q}}(g)) \chi_{\mathcal{R}_i}(g) = \sum_{j=0}^{r} c_{ij} \chi_{\mathcal{R}_j}(g).
\]

We multiply by \( \chi_{\mathcal{R}_k^*}(g) \) on both sides and then we sum after all \( g \neq 1 \) to obtain
\[
\sum_{g \neq 1} \chi_{\mathcal{R}_i}(g) \chi_{\mathcal{R}_k^*}(g) = \sum_{g \neq 1} c_{ij} \frac{\chi_{\mathcal{R}_j}(g) \chi_{\mathcal{R}_k^*}(g)}{\chi_{\mathcal{Q}}(g) - \chi_{\Lambda^2 \mathcal{Q}}(g)}
\]
(4.11)

Therefore the right-hand-side of the equation (4.11) is
\[
\sum_{j=0}^{r} c_{ij} \eta_{\mathcal{R}_j \otimes \mathcal{R}_k^*} = \frac{1}{|\Gamma|} (\delta_{ik} - 1).
\]

Putting these formulae together, and using again the index theorem, we have
\[
\frac{1}{2|\Gamma|} (\delta_{ij} - 1) = \frac{1}{2|\Gamma|} (\delta_{00} - 1) + \sum_{k=0}^{r} c_{ik} \int_X (\text{ch}(\mathcal{R}_k) - 1)(\text{ch}(\mathcal{R}_j^*) - 1).
\]

For \( i \neq 0 \) it gives
\[
\sum_{k=0}^{r} c_{ik} \int_X (\text{ch}(\mathcal{R}_k) - 1)(\text{ch}(\mathcal{R}_j^*) - 1) = \delta_{ij}.
\]

Since \( \text{ch}(\mathcal{R}_0) - 1 = 0 \), after inverting the above relation we obtain
\[
\int_X (\text{ch}(\mathcal{R}_k) - 1)(\text{ch}(\mathcal{R}_j^*) - 1) = (C^{-1})_{kj},
\]
for \( k, j \neq 0 \).

Comments

By the result of Bridgeland, King and Reid (see Proposition 1.26) the elements \( \{\text{ch}(\mathcal{R}_0), \text{ch}(\mathcal{R}_1), \ldots, \text{ch}(\mathcal{R}_r)\} \) form a basis in \( H^*(X, \mathbb{Q}) \). The multiplication relations in \( H^*(X, \mathbb{Q}) \) given by (4.7) are the generalization of the Kronheimer and Nakajima’s ones. However, in
our case they do not describe the entire ring structure in cohomology. For example the formula does not give any information about terms like $\int_X c_1(R_i)^3$.

Recent work of Craw and Ishii [CI] can be used to show that the multiplication formula (4.7) holds for any crepant resolution of an isolated singularity $\mathbb{C}^3/G$. In their work, they provide a description of any (projective) crepant resolution of $\mathbb{C}^3/G$, for $G$ abelian, as a moduli space of $G$-constellations. Our approach from Section 1 applies to this moduli space: it can be described as the symplectic reduction $X_\zeta = (N \cap \mu^{-1}(\zeta))/F$ at some point $\zeta \in f^*$. Proposition 1.22 generalizes straightforwardly and we obtain an ALE metric of order $\mu = 6$. Moreover the irreducible representations of the finite group $G$ give rise to tautological sheaves $R_{\zeta,i}$ which form a basis for the $K$-theory of $X_\zeta$. Their Chern characters satisfy the multiplication formula (4.7).

In our view, the essence of the McKay Correspondence is: how much of a crepant resolution of $\mathbb{C}^3/G$ can be described in terms of the finite group $G$ and its embedding in $SL(3, \mathbb{C})$?

From the discussion above all the crepant resolutions of $\mathbb{C}^3/G$ have a part of the cohomology ring which is the same – it is given by the multiplication formula (4.7). The remaining part of the cohomology ring should depend on the crepant resolution in the sense that it should change under a flop. The complete description of the cohomology ring should be carried out in future work.

Relationship to other Results

We present the relation of our multiplicative formula to the result of Ito and Nakajima [IN], and Bridgeland, King and Reid [BKR]. The author’s understanding of the following is due to a discussion with Andrei Caldararu and Alastair King at the Isaac Newton Institute in the summer of 2002.

Let $\pi : X \to \mathbb{C}^3/G$ be the crepant resolution of $\mathbb{C}^3/G$ given by Nakamura’s $G$-Hilbert scheme. We denote by $K(X)$ the Grothendieck group of coherent $O_X$-sheaves over $X$. The coherent sheaves which are supported on the exceptional divisor $\pi^{-1}(0)$ generate another group which we denote by $K_c(X)$. This can be thought of as the Grothendieck group of bounded complexes of algebraic vector bundles on $X$ which are exact outside $\pi^{-1}(0)$.

The result of Bridgeland, King and Reid [BKR] implies that $\{R_0, R_1, \ldots, R_r\}$ form a basis of $K(X)$. A procedure of Ito and Nakajima [IN] gives a basis $\{S_0, S_1, \ldots, S_r\}$ of $K_c(X)$ which is dual to the first one. Basically, $S_k$ is the class corresponding to the complex

$$R_k^* \to \bigoplus_l b_{kl} R_l^* \to \bigoplus_l a_{kl} R_l^* \to R_k^*.$$

The Fourier-Mukai transformation induces the isomorphisms:

- $\Phi : K(X) \to K^G(\mathbb{C}^3)$, where $K^G(\mathbb{C}^3)$ is the Grothendieck group of $G$-equivariant sheaves on $\mathbb{C}^3$. It acts by $\Phi(R_k) = R_k \otimes O_{\mathbb{C}^3}$.
- $\Phi_c : K_c(X) \to K_c^G(\mathbb{C}^3)$, where $K_c^G(\mathbb{C}^3)$ is the Grothendieck group of $G$-equivariant sheaves supported at the origin. It acts by $\Phi_c(S_k) = R_k \otimes O_0$.

Moreover each of the groups $K^G(\mathbb{C}^3)$ and $K_c^G(\mathbb{C}^3)$ is isomorphic to $R(G)$, the representation ring of $G$. The following diagram illustrates all these isomorphisms:
The bottom morphism $\kappa$ is the multiplication by the element

$$\lambda = \sum_{i=0}^{3} (-1)^i \Lambda^i Q$$

of $R(G)$. Since $G \subset SL(3, \mathbb{C})$ we have $\lambda = -Q + \Lambda^2 Q$. If we consider the basis $\{R_{\text{reg}}, R_1, \ldots, R_r\}$ on the left-hand-side and the basis $\{R_0, R_1, \ldots, R_r\}$ on the right-hand-side, then

$$\kappa = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} . \quad (4.12)$$

At the level of $K$-theory the bases $\{S_k\}$ and $\{R_k\}$ are dual to each other. Therefore the corresponding $\{\text{ch}c(S_k)\} \subset H^*_c(X, \mathbb{Q})$ and $\{\text{ch}(R_k)\} \subset H^*(X, \mathbb{Q})$ are dual to each other with respect to the pairing

$$\langle \text{ch}c(S_k), \text{ch}(R_l) \rangle = \int_X \text{ch}c(S_k) \cup \text{ch}(R_l) \hat{A} (X). \quad (4.13)$$

At the level at the representation ring it should be that the above pairing descends to the multiplication of the corresponding virtual characters. However, the bases we gave for $R(G)$ are not dual to each other since $\langle R_{\text{reg}}, R_k \rangle = \dim R_k$ for all $k$’s. Inspired by the multiplication formula (4.7) we modify the basis on the right-hand-side of $\kappa$ to

$$\{R_0, R_1 - \dim (R_1) R_0, \ldots, R_r - \dim (R_r) R_0\}.$$ 

This new basis does not modify the form (4.12) for the matrix of $\kappa$. Also, its lifting in $K(X)$ is

$$\{\text{ch}(R_0), \text{ch}(R_1) - \text{rk}(R_1)), \ldots, \text{ch}(R_r) - \text{rk}(R_r)\}.$$ 

Because of the properties of the Cartan matrix, this basis is still dual to the basis of $K_*(X)$ given by $S_k$’s. Therefore we modified the basis of $K(X)$ so that the multiplicative pairing in $K$-theory is compatible to the natural pairing in the representation ring.

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