On the Capacity of Wireless Networks with Random Transmission Delay

Niv Voskoboynik, Haim H. Permuter and Asaf Cohen

Abstract—In this paper, we introduce novel coding schemes for wireless networks with random transmission delays. These coding schemes obviate the need for synchronicity, reduce the number of transmissions and achieve the optimal rate region in the corresponding wired model for both multiple unicast and multicast cases with up to three users under the equal rate constraint. The coding schemes are presented in two phases; first, coding schemes for line, star and line-star topologies with random transmission delays are provided. Second, any general topology with multiple bidirectional unicast and multicast sessions is shown to be decomposable into these canonical topologies to reduce the number of transmissions without rate redundancy. As a result, the coding schemes developed for the line, star and line-star topologies can improve performance and power efficiency.

Index Terms—Multiple unicast, Network coding, Random delay, Wireless networks.

I. INTRODUCTION

Network Coding (NC) [1], [2] is a networking technique used to better exploit the available bandwidth, use energy efficiently and increase the network’s security. Unlike the traditional approach of routing, NC enables messages to be encoded and decoded at the intermediate nodes.

In wireless networks, NC allows the broadcast nature of the channel to be exploited and significant throughput gains to be achieved compared to traditional routing. Yet, in many practical scenarios, random transmission delays may render existing NC solutions impractical. In this paper, we focus on energy efficient coding schemes for wireless networks that, on the one hand, exploit the benefits of NC in wireless networks and achieve the fundamental limits in several practical configurations but, on the other hand, obviate the need for synchronicity, which is a key impediment in such networks.

This research was supported in part by the Israel Science Foundation (grant no. 684/11) and an ERC starting grant. This paper was presented in part at the 2014 IEEE International Symposium on Network Coding, Aalborg, Denmark. N. Voskoboinik and H. H. Permuter are with the Department of Electrical and Computer Engineering, Ben-Gurion University of the Negev, 84105, Beer-Sheva, Israel (email: voskoboy@post.bgu.ac.il; haimp@bgu.ac.il). A. Cohen is with the Department of Communication Systems Engineering, Ben-Gurion University of the Negev, 84105, Beer-Sheva, Israel (email: coasaf@bgu.ac.il).

Fig. 1: Schematic illustrations of (a) wireless and (b) wired setups, where nodes 1 and 2 exchange their information through node r.

The following example demonstrates the difficulty of implementing a NC scheme for a network with random transmission delay. Consider two users, 1 and 2, that exchange packets \{W_1^{(0)}, ..., W_1^{(t)}\} and \{W_2^{(0)}, ..., W_2^{(t)}\} through a relay, r, in a packet-based communication scheme (Fig. 1a). The links can carry one packet per time unit, and each transmission has a discrete random delay. We assume that at time instant t’ the relay, r, receives a packet-pair \(W_1^{(p)}, W_2^{(q)}\), where p and q are arbitrary integers that are smaller than t. Using simple routing, the relay transmits each packet separately, and therefore, the minimum number of transmissions required by node r is two for each packet-pair \(W_1^{(p)}, W_2^{(q)}\). In contrast, the NC approach suggests that node r could encode two packets and broadcast the encoded packet by exploiting the wireless medium, e.g., \(W_1^{(p)} \oplus W_2^{(q)}\), where \(\oplus\) is a bitwise XOR operation. However, to decode the encoded packet at both nodes 1 and 2, the values of p and q are required. A straightforward NC implementation, therefore, cannot be decoded successfully. A feasible solution, which is to code only within a single generation of packets and decode each generation separately, may incur an unacceptable delay. Of course, the problem becomes even more involved with the inclusion of multiple sources and terminals, a scenario that introduces several independent, bi-directional sessions. A key goal in our work, therefore, is to find coding schemes that not only exploit NC to minimize the number of transmissions, but that also have a feasible decoding process with minimum decoding delay and a small overhead.

We first define the wireless network model to be used throughout this work. Following [3], we transform a given...
wireless network into a corresponding wired model (Fig. 1). Namely, we define a bidirectional graph with random transmission delays, translating the criterion for optimality from minimum transmissions (in a wireless model) into maximum rates (in a wired model). For example, in the wired model of Fig. 1b, using simple routing yields a rate of 
\[
R \mid i = \min \{R_i + R_j \} 
\]
where 
\[
R_i \leq R \leq C, \quad i \in \{1, 2\}. 
\]
Our objective is thus to derive coding schemes for the wireless model that achieve the capacity rate region in the corresponding wired model with minimum delay.

A key network characteristic that significantly impacts our ability to give tight results and optimal NC schemes is the demand structure. For example, in a multicast scenario [4], [5], all terminal nodes wish to decode all sources. For this case, practical and rate-optimal solutions exist under several network models. However, in a multiple unicast scenario, where independent source-destination pairs wish to communicate, the problem is much less tractable, and the general case is still unsolved. In fact, the authors of [6] showed that any acyclic directed network (with general demand structure) has an equivalent multiple unicast network, conferring significant interest on the study of such networks. Specifically, unlike the multicast case, linear NC [7] fails to achieve the capacity region in this setting [8]. A few special cases are the capacity region when only XOR operations are permitted, given in [9], and a coding scheme for three unicast sessions that achieves a rate of half the minimum cut using the interference alignment approach, which was presented in [10]. Our setup, on the other hand, presents three bidirectional sessions. The case of three unicast sessions was also studied in [11], where lower bounds on the connectivity of the network, which allows a unity rate, were introduced. Here, however, we give a coding scheme that achieves capacity without assuming anything about the connectivity of the network.

Thus, we continue to explore the case of three unicast sessions, but in our case, the three sessions are bidirectional and experience random delays. Additionally, similar to the opportunistic coding approach that was presented in [12], [13], we decrease the number of transmissions. Motivated by the conjecture in [14], which claimed that in undirected graphs there is no advantage to using NC, we show that despite the fact that the use of NC does not result in rate improvement, we introduce other advantages, such as a reduction in the number of transmissions in the corresponding bidirectional wired model. Specifically, in that model, which is a directed graph, we achieve the capacity rate region where simple routing schemes failed. We can now summarize our main contributions.

Main Contributions:

We study in detail the two demand structures above under the suggested wireless model with random transmission delays. Specifically, we consider bidirectional multicast and multiple unicast [15]–[17], both for up to three users. In the first model (Fig. 2a), three users 1, 2 and 3 exchange messages in a bidirectional multicast manner through a wireless network. Multicast transmission is a widely used networking technique in which a message is sent to a set of receivers, an example of which is a video conference between three users. In the second model, each user generates two different messages, one for each of the remaining users, i.e., it exchanges independent messages with two different users in a bidirectional manner through a wireless network (Fig. 2b). For an example of the multiple unicast case, consider a messaging application with several users, where each user communicates with its partners, but each message is addressed to a specific partner. The two paradigms, multicast and multiple unicast, are used as subnetworks in many different communication networks, such as wireless ad-hoc networks, server-client communication in cloud computing and networks of optical fibers.

For both demand structures, we achieve the capacity rate regions in corresponding wired models, where each two-way communication is carried out at the same rate. Additionally, our coding schemes are shown to be Real Time (RT) NC, which we define as a NC scheme that allows decoding with minimum delay. Such a scheme is suitable for several applications, such as video conferencing and instant messaging.

In the multicast scenario, the benefits of the coding schemes are in their practicality and applicability to networks with random delays. In the multiple unicast scenario, which is...
generally open, we also extend the current state of the art as to when network coding is optimal and what are the achievable rates.

Our coding schemes are based on a modular approach. We begin by providing simple constructions for line and star topologies (e.g., [13], [19]), after which we use graph-theoretic tools to show how to decompose general networks into the above building blocks. Specifically, constructive algorithms are given to show that wireless networks with random transmission delays can be decomposed into line and star topologies as building blocks without rate redundancy and RT coding schemes can be deployed with a small overhead and minimum transmissions.

The rest of the paper is organized as follows. In Section II we present the network model. In Section III we outline the preliminaries of the coding schemes for the line, star and line-star topologies and then use these coding schemes in Sections IV and V as building blocks to derive a coding scheme for a general wireless network with multicast and multiple unicast sessions, respectively. Simulation results are given in Section VI and then some extensions are presented in Section VII. Finally, in Section VIII we summarize the paper with our conclusions.

II. Notation and Problem Setup

A bidirectional wireless network with delays is defined as a directed graph, $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, M\}$ is a set of nodes and $\mathcal{E} \subseteq \{1, \ldots, M\} \times \{1, \ldots, M\}$ is a set of bidirectional edges. Each edge $(i, j) \in \mathcal{E}$ represents a directed link from node $i$ to node $j$ with a capacity of $C$ bits per time unit. Since the edges are bidirectional, each edge $(i, j) \in \mathcal{E}$ induces a corresponding edge $(j, i) \in \mathcal{E}$ with the same capacity. Additionally, we consider a set of source nodes $\mathcal{S} \subseteq \mathcal{V}$.

Next, we present a model that allows us to explore the broadcast ability of the wireless medium. Therefore, we introduce an equivalent directed graph, $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$, with the same set of source nodes $\mathcal{S} \subseteq \mathcal{V}'$, by splitting each relay node $i \in \mathcal{V}' \setminus \mathcal{S}$ into two nodes $\{i, i'\} \subset \mathcal{V}'$ (Fig. 3). Each pair of directed edges $(i, j)$ and $(j, i)$ in $\mathcal{E}$ corresponds to a pair of new directed edges, one entering $i$, $(j', i) \in \mathcal{E}'$, with capacity $C$ and another leaving $i'$, $(i', j) \in \mathcal{E}'$, with the same capacity. In addition, there is an edge directed from $i$ to $i'$ with capacity $C$ that models the broadcast constraint at relay node $i$.

In addition, we introduce the following notations:

- $W^{(i)}_i$ is the multicast message of source $i$ at time $t$. We assume each message is distributed uniformly over $\{1, \ldots, 2^R_i\}$.
- $W^{(i)}_{i \rightarrow j}$ is the unicast message of source $i$ intended for node $j$ at time $t$. Again, we assume each message is distributed uniformly over $\{1, \ldots, 2^R_{i \rightarrow j}\}$.
- $W^{(i)}_i = \{W^{(i)}_0, W^{(i)}_1, \ldots, W^{(i)}_t\}$ represents the set of messages that was produced by source $i$ up to time $t$. Similarly, $W^{(i)}_{i \rightarrow j} = \{W^{(i)}_{i \rightarrow j}^{(0)}, W^{(i)}_{i \rightarrow j}^{(1)}, \ldots, W^{(i)}_{i \rightarrow j}^{(t)}\}$.
- $X^{(i)}_i$ represents the binary vector transmitted on all the edges leaving node $i \{\{i, j\} : j = 1, \ldots, M, (i, j) \in \mathcal{E}'\}$ at discrete time $t$.

We assume messages at negative times equal zero, i.e., $W^{(i)}_{i \rightarrow j}^{(t)} = 0$ and $W^{(i)}_i^{(t)} = 0$, $\forall t < 0$ and $\forall i, j \in \mathcal{S}$. Our model consists of random transmission delays, i.e., $X^{(i)}_i$ is sent from node $i$ to node $j$ in time slot $t$ and yet received by node $j$ after an arbitrary discrete delay, $d^{(i)}_{i \rightarrow j} = d^{(j)}_{i \rightarrow j} \in \mathcal{D}$. The delay is assumed to be bounded by $D$, i.e., $d^{(i)}_{i \rightarrow j} \leq D, \forall (i, j) \in \mathcal{E}'$ and $\forall t$. We assume there is no delay in the node processors, i.e., the transmission over the edge $(i, i') \in \mathcal{E}'$ has no delay, for all $i \in \mathcal{V}' \setminus \mathcal{S}$. The outgoing transmission from every node at any particular time instant $t$ is a function of the incoming transmissions to that node at earlier time instants and of its own messages. Throughout the paper, we use the operators floor $[\cdot]$ and ceiling $\lceil \cdot \rceil$.

We denote by $C_{i,j}$ the value of the minimal cut between nodes $i$ and $j$ in $\mathcal{G}'$. Symmetry between source nodes $i$ and $j$ induces that $C_{i,j} = C_{j,i}$. Similarly, we denote by $P_{i,j}$ a subset of the set of disjoint paths from node $i$ to node $j$ in $\mathcal{G}'$, where it follows that $|P_{i,j}| \leq \frac{C_{i,j}}{C_{i,j} - 1}$. In addition, we define a maximum distance in a graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ as $L = \max P_{i,j}$, where the maximum is taken with respect to all $P_{i,j}, i, j \in \mathcal{S}$. The maximum distance excludes the edges $(i, i') \in \mathcal{E}'$, $\forall i \in \mathcal{V}' \setminus \mathcal{S}$ since they have no delay. Furthermore, the diameter of the graph is the maximum distance in a network with $\mathcal{S} = \mathcal{V}$.

Additionally, we will use the following definitions throughout the paper.

Definition 1 (Achievable Rate) A coding scheme of rates $(R_{i \rightarrow j}, R_i, R_j)$ is said to be achievable if every node $j \in \mathcal{S}$ receives messages that are destined for it, i.e., $W^{(i)}_{i \rightarrow j}$ and $W^{(i)}_i$ for all $t$, where $c$ is some constant that is independent of $D$.

Definition 2 (Equal Rate Capacity Region) The capacity region under the equal rate constraint is defined as the closure of the set of all achievable rate tuples $(R_{i \rightarrow j}, R_{j \rightarrow i}, R_i, R_j)$, $i, j \in \mathcal{S}$, with the demands $R_{i \rightarrow j} = R_{j \rightarrow i} = R_i = R_j$.

Definition 3 (Real Time) A coding scheme is said to be RT for a graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ with a maximum distance $L$ and an arbitrary delay bounded by $D$ if every node $j \in \mathcal{S}$ decodes all messages up to time $t$, $W^{(i)}_i, W^{(i)}_{i \rightarrow j}$, with a maximum delay.

Fig. 3: Diagram showing the conversion of a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ in (a) into a graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ in (b) $i, j \in \mathcal{V} \setminus \mathcal{S}$. 
of \(LD + t + c, \forall t\), where \(c\) is some constant that is independent of \(D\).

The minimum delay of any coding scheme on a path with length \(L\) is bounded by \(LD\), where a minimum delay is the upper bound of the worst case decoding delay of a coding scheme. An RT coding scheme decodes a new message in each time slot after an initialization duration of \(LD + c\), i.e., a RT coding scheme achieves the minimum decoding delay. For comparison, a scheme based on random linear network coding (RLNC), e.g., [20], is not RT since it has to accumulate multiple transmissions before decoding is possible. Additionally, in a RT coding scheme the decoding delay is independent of the minimum cut between the source and the sink, where in RLNC this parameter has a huge impact on the decoding delay.

The goal is to find a RT coding scheme for a graph \(G'(V', E')\) that achieves the capacity region under an equal rate assumption. Furthermore, the coding scheme presented here requires only an overhead (as a header) of the order of \(O(\log_2 h)\) bits to operate the network into the building blocks, while a RLNC scheme requires \(O(h)\) bits to transmit the global encoding vector, where \(h\) is the minimal cut of separating one source from the network.

### III. Preliminaries

In this section, we describe the key concepts of the coding scheme for the line, star and line-star topologies. We later use these schemes as building blocks for more complex networks.

#### A. Line Topology

A line topology of \(M\) nodes is defined as a graph \(G'(V', E')\) of two source nodes \(S = \{1, M\}\) that exchange messages \(\mathcal{W}_i^t\) and \(\mathcal{W}_M^t\) through a line of nodes (Fig. 4). The coding scheme for this topology was first derived in [21]. For completeness and since we use it extensively later, we now present a sketch of the scheme. The main result for a line topology is summarized in the following theorem.

**Theorem 1** For any line topology with an arbitrary delay bounded by \(D\), there exists a RT coding scheme that achieves the equal rate capacity, which is \(C\). Furthermore, the coding scheme has a decoding delay of at most \(LD\) and it includes a fixed header per transmission of \(2\lceil\log_2 2D\rceil\) bits, independent of \(C\).

![Graph G'(V', E') of a line topology, where S = {1, M}.](image)

Since the minimal cut \(C_{1:M} = C\), we obtain that the equal rate upper bound is \(R \leq C\). Next, we prove Theorem 1 for the case where the delay per each transmission \(d_{i,j}^t\) at time instant \(t\) is fixed and equal to one, \(\forall (i, j) \in E'\) and later we provide a proof for the case where \(d_{i,j}^t\) is an arbitrary integer, yet it is bounded by \(D\). Note that for a case of unbounded delay, a retransmission policy is required, i.e., messages that were lost have to be retransmitted, however, in our case no retransmission is necessary.

**Proof:** We first note that Equation (2) is no longer applicable for the unsynchronized case, and therefore, we introduce a new coding scheme for this case.
Coding scheme: The transmission from each node $r \in \mathcal{V}'$ is a linear combination in $\mathbb{F}_{2^c}$, as follows
\[ X_i^{(t)} = k_i W_i^{(p)} + k_M W_M^{(q)}, \]
where $k_i \neq 0$ is a coefficient over the field $\mathbb{F}_{2^c}$, and $W_i^{(p)}$ and $W_M^{(q)}$ are some arbitrary messages from the sets $\mathcal{W}_i$ and $\mathcal{W}_M$, respectively.

Decoding: For each incoming transmission, each decoder $i \in S$ subtracts the message $k_i W_i^{(p)}$ to decode the information $W_j^{(q)}$, $j \in S \setminus \{i\}$, for some arbitrary integers $p$ and $q$.

We verify that each node is able to produce a transmission in the form of (4) at any particular time instant $t$. First, we add two indices, each of length $\lceil \log_2 2D \rceil$, as metadata to each transmission (Fig. 5). This header represents the messages from the sets $\mathcal{W}_i$ and $\mathcal{W}_M$ that were encoded. Second, we demand that each node will decode and store the messages. We argue that the set of messages $\mathcal{W}_i$ that node $r$ holds at time instant $t$ is a subset of the set of messages $\mathcal{W}_i^{(q)}$ held by node $r - 1$. Similarly, node $r$ holds $\mathcal{W}_M^{(q)}$, which is a subset of the messages $\mathcal{W}_M^{(q)}$ that node $r + 1$ holds. We verify this claim using induction. At time $t = 0$, the claim is trivial, and thus, we inductively assume that up to time $t$ the claim is true. We demonstrate the claim subsequently, where node $r$ obtains a transmission $X_{r-1}^{(t)}$, e.g., $k_i W_i^{(p)} + k_M W_M^{(q)}$ from node $r - 1$. Since by the inductive assumption node $r$ has message $W_M^{(q)}$, it is able to decode message $W_i^{(p)}$, likewise, from the transmission from node $r + 1$, e.g., $X_{r+1}^{(t)} = k_i W_i^{(p)} + k_M W_M^{(q)}$, node $r$ is able to decode message $W_M^{(q)}$, and hence, the claim is established. We summarize with the inference that node $r \in \mathcal{V}'$ is able to produce transmissions in the form of (4) at any particular time instant $t$ from messages it holds, independent of $d_i^{(t)}$, $\forall (i, j) \in \mathcal{E}'$. The maximum delay in this case is also bounded by $DL + t$ for all $t$, where $L = M - 1$, and therefore, this coding scheme is a RT coding scheme. 

![Fig. 5: Packet header of an unsynchronized coding scheme for a line topology.](image)

In the corresponding wireless model, i.e., a line network $\mathcal{G}(\mathcal{V}, \mathcal{E})$, applying this coding scheme requires $|\mathcal{V}| - 2$ transmissions by the relay nodes for each pair of messages $W_i^{(t)}$, $W_M^{(t)}$, where using a simple routing scheme yields $2(|\mathcal{V}| - 2)$ transmissions. Hence, by exploiting the broadcast ability of the wireless medium, we obtain an energy efficient coding scheme for the wireless setting.

B. Star Topology

A star topology of three source nodes is defined by a network $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$, where $\mathcal{S} = \{1, 2, 3\}$ and all the source nodes in $\mathcal{S}$ try to communicate through a single relay node, 4 (Fig. 6). Our main result for the star topology is summarized in the following theorem.

**Theorem 2** For a star topology with an arbitrary delay bounded by $D$, there exists a RT coding scheme that achieves the equal rate capacity, which is $\frac{C}{2}$. Furthermore, the coding scheme includes a fixed header per transmission of $3\lfloor \log_2 2D \rfloor + 1$ bits.

![Fig. 6: Graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ of a star topology, where $\mathcal{S} = \{1, 2, 3\}$.](image)

Since the minimal cut $C_{i,j|i} = C$, we obtain that $R_i + R_j \leq C$, $\forall i, j, l \in \mathcal{S}$ and the upper bound of the equal rate is $R \leq \frac{C}{2}$. Next, we prove Theorem 2 for the case where $d_i^{(t)}$ is fixed and equal to one, $\forall (i, j) \in \mathcal{E}'$ and $\forall t$.

Proof: Coding scheme: We choose two non-zero triplets $a = [a_1, a_2, a_3]$ and $b = [b_1, b_2, b_3]$ over the field $\mathbb{F}_{2^c}$ that satisfy
\[ a_i a_j \neq b_i b_j, \quad \forall (i, j) \in \{1, 2, 3\}, i \neq j. \]

Each source $i \in \mathcal{S}$ generates a transmission in $\mathbb{F}_{2^c}$ as follows
\[ X_i^{(t)} = k_i W_i^{(2)} + k_j W_j^{(2)} + k_k W_k^{(2)}, \]
where $k_i = a_i$ or $k_i = b_i$, $\forall i \in \{1, 2, 3\}$, i.e., for each message $W_i^{(t)}$ that node $i$ generates, it transmits two transmissions (one with coefficients $k_i = a_i$ and the other with coefficients $k_i = b_i$). As such, we set the rate of each source node $i \in \mathcal{S}$ to $R_i = \frac{C}{2}$. The relay node, 4, generates a transmission
\[ X_4^{(t)} = X_1^{(t-1)} + X_2^{(t-1)} + X_3^{(t-1)}. \]

We now show that the coding scheme defined in (6) and (7) implies that
\[ X_4^{(t)} = X_1^{(t-1)} + X_2^{(t-1)} + X_3^{(t-1)} = k_1 W_1^{(2)} + k_2 W_2^{(2)} + k_3 W_3^{(2)} \]
\[ + k_1 W_1^{(2)} + k_2 W_2^{(2)} + k_3 W_3^{(2)} \]
\[ + k_1 W_1^{(2)} + k_2 W_2^{(2)} + k_3 W_3^{(2)} \]
\[ = k_1 W_1^{(2)} + k_2 W_2^{(2)} + k_3 W_3^{(2)} \] (8)

Decoding: Each decoder $i \in \mathcal{S}$ obtains two independent equations for each two time instants $t + 1$ and $t + 2$ and is able to decode two messages $W_j^{(2)}$, $\forall j \in \mathcal{S} \setminus \{i\}$.

We verify that this NC scheme enables all source nodes to recover all the messages with a delay of $L + 1$, where $L = 2$, using induction over $t$. At time $t = 0$ the claim is trivial and, by the inductive assumption, up to time $t$ node $i$ can recover $W_i^{(2)}$, $\forall j \in \mathcal{S} \setminus \{i\}$. At time $t + 1$ and $t + 2$, node $i$ obtains two transmissions, $X_i^{(t)}$ and $X_i^{(t+1)}$, respectively, and since the coefficients $a$ and $b$ were chosen according to (5), it is
able to decode two messages, $W_{j}^{(i)} \mod \left( \frac{L+L-1}{L} \right)$, $j \in S \setminus \{i\}$, thereby proving the claim. Moreover, this claim also proves that this NC scheme is a RT coding scheme since the maximum delay is $L + t + 1$ for all $t$, where $L = 2$.

Now we present the proof of Theorem 2 for the case where $d_{i,j}^{(t)}$ is an arbitrary integer that can be different for each transmission over the edge $(i, j) \in E'$ but is bounded by $D$.

Proof: We first note that Equation (7) is no longer applicable for the unsynchronized case, and therefore, we introduce a new coding scheme for this case.

Coding scheme: The relay node, 4, transmits a linear combination in $\mathbb{F}_2^c$, as follows

\begin{equation}
X^{(t)}_4 = k_1W_1^{(p)} + k_2W_2^{(q)} + k_3W_3^{(u)},
\end{equation}

where $k_i = a_i$ or $k_i = b_i$, $\forall i \in \{1, 2, 3\}$, and $W_1^{(p)}, W_2^{(q)}, W_3^{(u)}$ are some arbitrary messages from the sets $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$, respectively. Namely, relay node 4 transmits two independent transmissions (one with coefficients $k_i = a_i$ and the other with coefficients $k_i = b_i$). Similarly, each source node $i \in S$ transmits

\begin{equation}
X^{(t)}_i = k_1W_1^{(t)} + k_2W_2^{(p)} + k_3W_3^{(q)},
\end{equation}

where $j, l \in S \setminus \{i\}$.

Decoding: Each decoder $i \in S$ obtains two independent equations and is able to decode two messages $W_1^{(p)}, W_l^{(q)}$, $j, l \in S \setminus \{i\}$, and $W_1^{(p)}, W_i^{(q)}$ are some arbitrary messages from the sets $\mathcal{W}_1, \mathcal{W}_l, \mathcal{W}_i$, respectively.

We verify that the relay node, 4, is able to produce a transmission in the form of (9) at any particular time instant $t$. First, we add a header to each transmission to indicate which messages from the set $\mathcal{W}_i$ were encoded, $\forall i \in S$ and another bit to indicate the coefficients ($k_i = a_i$ or $k_i = b_i$), as illustrated in Fig. 2. Second, we demand that relay node 4 will decode and store the messages. We argue that the set of messages $\mathcal{W}_i$ held by source node $j$ is a subset of the messages $\mathcal{W}_j$ held by node 4 for all $t$, $\forall i \in S \setminus \{j\}$. We verify the claim using induction. At time $t = 0$ the claim is trivial, and therefore, we inductively assume that up to time $t$ the claim is true. We demonstrate the claim subsequently, where source node $j$ receives two transmissions, $k_1W_1^{(p)} + k_2W_2^{(q)} + k_3W_3^{(u)}$, one with coefficients $a_i$ and the other with coefficients $b_i$, $\forall i \in \{1, 2, 3\}$. Since by the inductive assumption node $j$ has messages $\mathcal{W}_i^{(\frac{j}{2})}$ and the coefficients were chosen according to (5), the node is able to decode two messages and the claim is established. To conclude, each source node $i \in S$ is able to decode the messages sent by $S \setminus \{i\}$ at any particular time instant $t$ independent of $d_{i,j}^{(t)}$, $\forall (i, j) \in E'$. Furthermore, the maximum delay is bounded by $DL + t + 1$ for all $t$, where $L = 2$, and therefore this NC scheme is a RT coding scheme.

![Fig. 7: Packet header of an unsynchronized coding scheme for a star topology.](image)

In the corresponding wireless model, i.e., a star network $\mathcal{G}(V, E)$, applying our coding scheme requires two transmissions by the relay node for each triplet of messages, $W_1^{(t)}, W_2^{(t)}, W_3^{(t)}$, where using a simple routing scheme yields three transmissions. Hence, by exploiting the broadcast ability of the wireless medium, we obtain an energy efficient coding scheme for the wireless setting.

C. Line-Star Topology

The line-star topology is generally defined as a combination of line and star topologies (Fig. 8), in which a line of nodes connects a source node to the star topology structure. The upper bound of the equal rate, which is identical to that for the star topology in Section III-B, is proved in the same manner. We demonstrate the coding scheme for this topology in the following section.

![Fig. 8: Graph $\mathcal{G}'(V', E')$ of a line-star topology, where $S = \{1, 2, 3\}$. A line topology is formed between nodes 1 and 5, and a star topology is formed around node 4.](image)

First, we show a case where $d_{i,j}^{(t)}$ is fixed and equal to one, $\forall (i, j) \in E'$ and $\forall t$ and we consider another single relay node, 5, between nodes 1 and 4 (Fig. 8).

Coding scheme: Relay node 4 and each source node $i \in S$ perform coding in a manner identical to that described for the star topology in Section III-B. Node 5 generates transmissions in the form of (3) in the same manner as the relays of the line topology in Section III-A i.e.,

\begin{equation}
X_5^{(t)} = X_1^{(t-1)} + X_4^{(t-1)} + X_5^{(t-2)}. \tag{11}
\end{equation}

This coding implies that

\begin{align*}
X_5^{(t)} &= X_1^{(t-1)} + X_4^{(t-1)} + X_5^{(t-2)} \\
&= k_1W_1^{(2\frac{t}{L-2})} + k_2W_2^{(2\frac{t}{L-2})} + k_3W_3^{(2\frac{t}{L-2})} + k_4W_4^{(2\frac{t}{L-2})} + k_5W_5^{(2\frac{t}{L-2})} + k_6W_6^{(2\frac{t}{L-2})} \\
&= k_1W_1^{(2\frac{t}{L-2})} + k_2W_2^{(2\frac{t}{L-2})} + k_3W_3^{(2\frac{t}{L-2})} + k_4W_4^{(2\frac{t}{L-2})} + k_5W_5^{(2\frac{t}{L-2})} + k_6W_6^{(2\frac{t}{L-2})}.
\end{align*}

Decoding: The decoding process is the same as the coding scheme of the star topology in Section III-B.

We verify that this NC scheme enables all source nodes to recover all the messages with a delay of $L + 1$, where $L = 3$, using induction over $t$. At time $t = 0$ the claim is trivial and, by the inductive assumption, up to time $t$ node $i$ can recover $\mathcal{W}_1^{(\frac{L}{2})}, \mathcal{W}_j^{(\frac{L}{2})}, \forall j \in S \setminus \{i\}$. At time $t + 1$ and $t + 2$, node $i$ obtains two transmissions, $X_i^{(t)}$ and $X_i^{(t+1)}$, respectively,
where \( r \in V \setminus S \), and since the coefficients \( a \) and \( b \) were chosen according to \([5]\), it is able to decode two messages, \( W_j^{(1)}(t_{\frac{t}{t+1}}) \), \( j \in S \setminus \{t\} \), thereby proving the claim. Moreover, this claim also proves that this NC scheme is a RT coding scheme since the maximum delay is \( L + t + 1 \) for all \( t \), where \( L = 3 \).

Now we present the case where \( d_{i,j}^{(t)} \) is an arbitrary integer that can be different for each transmission over the edge \((i, j) \in E'\) but that is bounded by \( D \).

We first note that Equation \([11]\) is no longer applicable for the unsynchronized case, and therefore, we introduce a new coding scheme for this case.

**Coding scheme:** Relay node 4 and each source node \( i \in S \) perform coding in a manner identical to that described for the unsynchronized case of the star topology in Section III-B.

Node 5 transmits
\[
X_{5}^{(t)} = k_1W_{1}^{(p)} + k_2W_{2}^{(q)} + k_3W_{3}^{(u)},
\]
where \( k_1 = a_i \) or \( k_1 = b_i \), \( \forall i \in \{1, 2, 3\} \), \( W_{1}^{(p)} \), \( W_{2}^{(q)} \) and \( W_{3}^{(u)} \) are some arbitrary messages from the sets \( W_1 \), \( W_2 \) and \( W_3 \), respectively. Namely, it transmits two independent transmissions (one with coefficients \( k_i = a_i \) and the other with coefficients \( k_i = b_i \)) with the same header as described in Fig. 1. Node 5 needs to decode all messages, \( W_1 \), \( W_2 \) and \( W_3 \), and then to store them for this coding scheme to work. We argue that the set of messages \( W_{1}^{(p)} \) held by source node \( j \) is a subset of the messages \( W_{1}^{(p)} \) held by nodes 4 and 5 for all \( t \), \( \forall i \in S \setminus \{j\} \). We verify the claim using induction. At time \( t = 0 \) the claim is trivial, and therefore, we inductively assume that up to time \( t \) the claim is true. We demonstrate the claim subsequently, where source node \( j \) receives two transmissions, \( k_1W_{1}^{(p)} + k_2W_{2}^{(q)} + k_3W_{3}^{(u)} \), one with coefficients \( a_i \) and the other with coefficients \( b_i \), \( \forall i \in \{1, 2, 3\} \). Since the inductive assumption node \( j \) has messages \( W_{j}^{(2,t)} \) and the coefficients were chosen according to \([5]\), the node is able to decode two messages and the claim is established. To conclude, each source node \( l \in S \) is able to decode the messages sent by \( S \setminus \{t\} \) at any particular time instant \( t \) independent of \( d_{i,j}^{(t)} \), \( \forall (i, j) \in E' \). Furthermore, the maximum delay is bounded by \( DL + t + 1 \) for all \( t \), and therefore, this NC scheme is a RT coding scheme.

This process of adding a relay node to the star topology can now be extended to any given number of nodes connecting each source node as a line topology to the star topology structure. In the corresponding wireless model, i.e., a line-star network \( G(V, E) \), applying our coding scheme requires \( 2(|V|−3) \) transmissions for each triplet of messages, \( W_1^{(1)}, W_2^{(1)}, W_3^{(1)} \), i.e., two transmissions by each relay node, \( V \setminus S \), where using a simple routing scheme yields \( 3(|V|−3) \) transmissions. Hence, by exploiting the broadcast ability of the wireless medium, we obtain an energy efficient coding scheme for the wireless setting.

The topologies of line, star, and line-star are all from the same family in the sense that they all represent networks with a minimum cut of one between each two source nodes. Additionally, by applying a simple routing scheme to the wireless model of each of the topologies, we can achieve the maximum rate. However, by using NC, we achieve the minimum number of transmissions, i.e., an efficient energy consumption, without rate redundancy. The coding schemes presented in Sections III-A, III-B and III-C hold the properties of RT and are innovative \([22]\). A coding scheme is called innovative if each incoming transmission to source node \( i \in S \) is not contained in the span of messages previously received by \( i \). Furthermore, the line topology coding scheme of Section III-A also holds a property of instantly decodable, a coding scheme in which a new message is decoded for each incoming transmission, i.e., a RT coding scheme with \( c = 0 \). In the rest of the paper, we use these coding schemes as building blocks and exploit their properties to construct a coding scheme for a general network.

**IV. Multicast Network**

In this section, we combine the line topology from Section III-A and the line-star topology from Section III-C and use them as building blocks to present a new coding scheme for a general multicast network. A general network is defined by a graph \( G(V', E') \) with a set of source nodes \( S = \{1, 2, 3\} \) (Fig. 2a). Every network is shown to be decomposable into line-star and ring topologies, where a ring is defined as three special line topologies. The special line topology is valid for a ring if deleting that line will not decrease the minimal cut of the remaining source node, e.g., if deleting \( P_{1,2} \) will not decrease \( C_{3,1,2} \). Each network may have many decompositions of those building blocks. However, we prove that there exists at least one decomposition that, using the coding schemes of the building blocks, achieves the equal rate capacity for a general wireless network, in which three source nodes communicate bidirectionally in a multicast manner.

**A. Capacity and Coding for a Multicast Network Based on Line and Star Topologies**

We denote the minimum of all the cuts that separate one source from the network by
\[
h = \min_{i \in S} C_{i; S \setminus \{i\}}.
\]
(14)

Our main result is summarized in the following theorem.

**Theorem 3** For any network \( G(V', E') \) with an arbitrary delay bounded by \( D \), there exists a RT coding scheme that achieves the equal rate capacity, which is \( \frac{3}{2}C \). Furthermore, the coding scheme includes a fixed header per transmission of \( 3\lceil \log_2 2D \rceil + 1 + \lceil \log_2 h \rceil \) bits.

The upper bound is obtained by the standard minimum-cut arguments \([23]\). We assume, without loss of generality, that \( h = \frac{C_{i; S \setminus \{i\}}}{C} \) for some \( i \in S \). Therefore, we get an upper bound of \( R_i + R_l \leq hC, j, l \in S \setminus \{i\} \) and the equal rate upper bound is \( h \leq \frac{C}{3} \). This means that the three users can exchange information that was generated up to time \( t \) at rate \( R \leq \frac{hC}{3} \) and with delay at most \( DL+t+1, \forall t \).

To prove the achievability of Theorem 3 we first introduce Lemma 11 which shows how to partition each network \( G' \) into sub-topologies of line and line-star networks. The line
topologies are formed by a set of rings $\mathcal{R}$, where each $r \in \mathcal{R}$ is defined in the following definition.

**Definition 4 (Ring)** A ring, $r$, is defined for a graph, $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$, with three source nodes, $\mathcal{S} = \{1, 2, 3\}$, as a set of edges, i.e., $r \subseteq \mathcal{E}'$, which forms three bidirectional paths between each two source nodes under the condition that removing a path between $i$ and $j$ is not lessening the minimum cut of the remaining source node $l$, i.e., $C_{l,i,j}$. Each ring in a graph contributes only a rate of $\frac{1}{3}$ to each source node by using the line topology coding scheme from Section III-C, i.e., $q \in \mathcal{Q}$ to each source node by using the line topology coding scheme of Section III-A at each bidirectional path, i.e., $r_1 \cap r_2 = \emptyset$, $\forall r_1, r_2 \in \mathcal{R}$. For example, consider the ring in Fig. 10a. The line-star topologies are defined by a set $\mathcal{Q}$, where each $q \in \mathcal{Q}$ is defined by a union two bidirectional paths, where both paths leave the same source node, but each path is destined for another node, as illustrated in Fig. 10b.

![Fig. 10: Schematic illustrations of (a) an element in $\mathcal{R}$ (a ring) and (b) an element in $\mathcal{Q}$ (a line-star topology).](image)

Each ring in also an element in $\mathcal{Q}$. However, each $q \in \mathcal{Q}$ contributes only a rate of $\frac{1}{3}$ to each source node by using the line-star topology coding scheme from Section III-C, i.e., $q_1 \cap q_2 = \emptyset$, $\forall q_1, q_2 \in \mathcal{Q}$. Although there exist many decompositions of $\mathcal{R}$ and $\mathcal{Q}$ in a network $\mathcal{G}'$, in the following lemma we show that by first finding the maximum number of rings, we can assure that there exist enough $\mathcal{R}$ and $\mathcal{Q}$ to achieve the equal rate upper bound.

**Lemma 1** For a network $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$, there exist $\mathcal{R}$ and $\mathcal{Q}$ such that $|\mathcal{R}| + \frac{|\mathcal{Q}|}{3} \geq \frac{1}{2}$, where $\mathcal{R} \cap \mathcal{Q} = \emptyset$. Namely, $\mathcal{R}$ and $\mathcal{Q}$ have no mutual edges.

**Proof of Lemma 1** The proof is by construction. First, we search for the maximum number of rings, $|\mathcal{R}|$. Then, we construct a network $\mathcal{G}''$ that is the network $\mathcal{G}'$ without $\mathcal{R}$, i.e., we remove the edges in $\mathcal{R}$ from $\mathcal{G}'$. Without loss of generality, we assume that between source nodes 1 and 2 there are no more paths, such that deleting them from $\mathcal{G}''$ will not decrease $C_{1,2}$. We can always find a pair of source nodes that satisfies this condition, because otherwise, we could increase $\mathcal{R}$. Finally, we find a set of new paths $\mathcal{Q}$ in $\mathcal{G}''$, where $|\mathcal{Q}| = \frac{C_{1,2}}{C}$. Each path, $P_{1:2} \in \mathcal{P}_{1:2}$, from node 1 to node 2 shares at least one common node with a special path, $P_{1:3} \in \mathcal{P}_{1:3}$ from node 1 to node 3. This path, $P_{1:3}$, shares no common nodes with all the other paths, $\mathcal{P}_{1:2} \setminus \{P_{1:2}\}$, between nodes 1 and 2. At least one special path $P_{1:3}$ exists that corresponds to each selection of $P_{1:2}$, since otherwise deleting $P_{1:2}$ will not decrease $C_{3,1,2}$. As a result, finding the path $P_{1:2}$ and one of the special paths $P_{1:3}$ that corresponds to it is equivalent to finding a line-star topology in the network. The line-star topology consists of the union between $P_{1:2}$ and $P_{1:3}$, i.e., the union of $P_{1:2}$ and $P_{1:3}$ is a $q \in \mathcal{Q}$. Therefore,

$$|\mathcal{Q}| \geq \frac{C''_{1,2}}{C} \quad (a)$$

$$\leq \frac{\min\{C''_{1,2,3}, C''_{2,1,3}\}}{C} \quad (b)$$

$$\leq \frac{\min\{C_{1,2,3}, C_{2,1,3}\}}{C} - 2|\mathcal{R}| \quad (c)$$

$$\geq h - 2|\mathcal{R}| \quad (d)$$

where $(a)$ follows from the fact that there are $\frac{3C''_{1,2}}{C}$ paths $P_{1:2}$ that has a corresponding path $P_{1:3}$ that creates a line-star topology in the network. $(b)$ is true since node 3 can only be in one of the cuts separating nodes 1 and 2, and $C_{1,2}$ is the minimization of all the cuts separating them. $(c)$ is the transition to the network $\mathcal{G}'$ and $d$ follows from $(14)$. ■

![Fig. 9: A network $\mathcal{G}$ is shown in (a) where all the source nodes 1, 2 and 3 communicate in a bidirectional multicast manner. By using the ring and line-star building blocks approach from Section IV, we get the set $\mathcal{R}$, which represents the ring topologies in the network, illustrated in (b) and the set $\mathcal{Q}$, which represents the line-star topologies, as illustrated in (c).](image)
Next, we provide the proof for Theorem 3.

Proof of Theorem 3: Using the line topology coding scheme from Section III-A and another bits according to the coding scheme of the star topology we obtain a RT NC scheme that achieves the equal rate is an upper bound, we have an equality. Moreover, we obtain yields a rate of \( R \). Since there is only one path from node \( j \) and then we use the coding schemes of those two canonical topologies to form disjoint line and line-star topologies in the network. Therefore, we obtain a RT NC scheme that achieves the equal rate capacity.

B. Example

In this subsection, we show a multicast network in Fig. 9. This network is decomposed into ring and line-star topologies, and then we use the coding schemes of those two canonical topologies to obtain a new coding scheme that achieves the equal rate capacity. In Fig. 9(a) we visualize the ring topologies in the network. In this example, only one element in \( R \) exists, since there is only one path from node 1 to node 3 that is deleted from the network will not reduce the minimum cut of node 2, i.e., \( C_{2,1,3} \). Note that this is not the only choice for a ring in the network, e.g., a different choice, \( R' \), include nodes 1, 2, 3, 6, 7 and 11 and all the direct links that connect them. As a negative example, consider a path from node 1 to node 3 through nodes 9 and 10. This path cannot be a part of a ring topology since deleting it will decrease the minimum cut of node 2, i.e., \( C_{2,1,3} \).

In Fig. 9(b) we present the line-star topologies in the network. In case \( R' \) is chosen a different choice of a line-star topology include nodes 1, 2, 3, 8, 9 and 10 and all the direct links that connect them. By using those topologies as building blocks, we achieve a rate of \( R = 1.5C \), which is also the equal rate upper bound of this network, i.e., \( \frac{3}{2} \), where \( h = 3C \).

C. Algorithm to Find the Building Blocks

The difficulty of finding \( R \) and \( Q \), i.e., the sets of edges that serve as building blocks of ring and line-star topologies in the network, is shown to be equivalent to solving a binary multicommodity flow problem. This class of optimization problem was shown to be a NP-complete problem [24]. However, there exist many approximation algorithms for the integral multicommodity flow problem [25, 26]. Furthermore, finding \( R \) and \( Q \) can be facilitated by introducing the upper bound \( |R| + |Q| \leq \min_{i \in S} \text{deg}(i) \), where \( \text{deg}(i) \) is defined to be the degree of node \( i \) in \( V' \), i.e., the number of edges that are initiated at node \( i \). The two sets \( R \) and \( Q \) represent disjoint edges and each element in these sets represents a path that passes through each source node and therefore, we obtain an upper bound of the size of the sets.

To find the set \( R \), we introduce an optimization problem. We denote by \( O_j \) and \( I_j \) the output and input flows from node \( j \), respectively. Furthermore, the flow, denoted by \( f \), is a binary vector that represents the flow in each edge, where an element \( f_{i,j} \in f \) represents a binary flow in edge \( (i, j) \in E' \). In this problem, we would like to find the maximum number of disjoint paths from node \( i \) to node \( j \) that deleting them from the network will not reduce the maximum number of disjoint paths from nodes \( i \) and \( j \) to node \( l \), \( C_{l,i,j} \), where \( i, j, l \in S \). Therefore, we demand maximal input flow to node \( l \), i.e.,

\[
\mathcal{I}_l = \frac{\mathcal{C}_{l,i,j}}{C}.
\] (19)

Additionally, we demand that \( O_l = 0 \), i.e., the entire flow will originate at nodes \( i \) and \( j \) and that \( I_l = 0 \) since we are interested in finding a flow that terminates at nodes \( j \) and \( l \). With condition (19) satisfied, we maximize the flow that originates at node \( l \) and terminates at node \( j \), i.e., the flow that is consumed by node \( j \), \( \mathcal{I}_j - O_j \). Since \( f \) is a binary flow and each relay, \( m \in V' \setminus S \), has equal input and output flows, we get two sets of disjoint paths. First, paths that terminate at node \( l \) and satisfy condition (19), and second, paths from \( i \) to \( j \). Because the two sets are disjoint, even if we remove the edges associated with the second set, i.e., paths from \( i \) to \( j \), condition (19) will still be satisfied.

\[
\begin{align*}
\text{maximize} & \quad \mathcal{I}_j - O_j \\
\text{subject to} & \quad O_m = \sum_{n: (m,n) \in E'} f_{m,n}, \quad m = 1 \ldots |V| \\
& \quad I_m = \sum_{n: (n,m) \in E'} f_{n,m}, \quad m = 1 \ldots |V| \\
& \quad I_l = \frac{\mathcal{C}_{l,i,j}}{C} \\
& \quad I_l = 0 \\
& \quad O_l = 0 \\
& \quad O_l + O_j = \mathcal{I}_j + \mathcal{I}_l \\
& \quad O_k = \mathcal{I}_k, \quad k = 4, \ldots, |V|, \\
& \forall i, j, l \in S.
\end{align*}
\] (20)

Algorithm 1 Find \( R \)

1: \( R \leftarrow \emptyset \)
2: while True do
3: solve problem (20) for each pair \( i, j \in S \).
4: find \( T' \) which is the input flow to node \( j \) after removing the cyclic flow.
5: if \( T' = 0 \) for any \( j \in S \) then
6: break.
7: end if
8: for each pair \( i, j \in S \), choose randomly one path, \( P_{i,j} \), out of \( f \) between nodes \( i \) and \( j \).
9: add the three paths \( P_{i,j} \) for each pair \( i, j \in S \) to \( R \) and remove them from the network.
10: end while

The result of applying this optimization problem is a binary flow that satisfies the conditions of the optimization problem. From this flow, we subtract the cyclic flow, which is defined as the flow that originates and terminates at the same node, i.e.,
paths from node \( j \) to node \( j \) that do not pass through nodes \( i \) and \( l \), \( i, j, l \in S \). After the substraction, we obtain a new input flow to node \( j \), \( T'_j \). This flow, \( T'_j \), represents the maximum number of paths between nodes \( i \) and \( j \), where condition (19) is satisfied even if we delete those paths from the network, i.e., we remove their edges from the graph. Therefore, they are applicable to the set \( R \). Algorithm 1 describes how to find all of the paths in \( R \).

**Algorithm 2 Find \( Q \)**

1. Remove the edges associated with \( R \) from the graph.
2. \( Q \leftarrow \emptyset \)
3. \textbf{while} True \textbf{do}
4. \textbf{if} \(|Q| \geq h - 2|R|\) \textbf{then}
5. \hspace{1em} Break.
6. \textbf{end if}
7. Find two source nodes, \( i, j \in S \), which have no path between them that deleting it from the network will not decrease \( C_{i:j} \), \( l \in S \setminus \{i, j\} \).
8. Find a path between nodes \( i \) and \( j \), \( P_{i:j} \), which deleting it from the network will minimally decrease \( C_{i:j} \).
9. Remove the edges associated with \( P_{i:j} \) from the graph.
10. Find all the paths from node \( l \) to node \( i \) in the new network and delete them from the network, i.e., remove their edges from the graph.
11. Recover the path \( P_{i:j} \) to the network and find a path \( P_{i:l} \) in the new network.
12. Add the union of \( P_{i:l} \) and \( P_{i:j} \) to \( Q \) and remove it from the network, i.e., remove the edges associated with \( Q \) from \( L \).
13. \textbf{end while}

Next, we present Algorithm 2 which describes how to find the line-star topologies, \( Q \). This algorithm is based on the assumption that we already deleted the set \( R \) from the network, i.e., we removed the edges associated with \( R \) from the graph. Therefore, there exists at least one pair of source nodes \( i \) and \( j \) that have no path between them that deleting it will not decrease \( C_{i:j} \). After finding \( i \) and \( j \), we search for a path \( P_{i:j} \in P_{i:j} \) that deleting it from the network will cause the smallest possible decrement to \( C_{i:j} \), because we would like to avoid crossing other paths. Since deleting \( P_{i:j} \) results in a decrement to \( C_{i:j} \), there exists at least one special path, \( P_{i:l} \in P_{i:l} \), from node \( i \) to node \( l \) that does not intersect with any of the other paths, \( P_{i:j} \setminus \{P_{i:l}\} \) from \( i \) to \( j \), i.e., it only shares a common node with \( P_{i:j} \). To find this special path, we delete all the remaining paths from node \( i \) to node \( l \). By now, we have deleted from the network the path \( P_{i:j} \) and all the remaining paths from node \( i \) to node \( l \). Next, by returning \( P_{i:l} \) to the network, we assure that there exists a path between \( i \) and \( l \), \( P_{i:l} \), which is a special path. A line-star topology structure, which is the union of \( P_{i:j} \) and \( P_{i:l} \), is then found.

To demonstrate Algorithm 2 we apply it in an example. Graph \( \mathcal{G}(V, E) \), which is illustrated in Fig. 11 contains three source nodes 1, 2, and 3. We note that there is no path from node 1 to node 2 under the condition that deleting that path will not change the minimum cut of node 3, i.e., \( C_{3:1,2} \). Therefore, there are no rings in the graph. In the first step, we find a path between 1 and 2, \( P_{1:2} \), that deleting it from the network decreases the minimum cut of node 3, \( C_{3:1,2} \), by \( C \), which is minimal, e.g., \( P_{1:2} = \{(1, 4), (4, 2)\} \). Next, we delete \( P_{1:2} \) from the network and then find all the remaining paths from node 3 to node 1 and delete them as well, e.g., \( \{(3, 5), (5, 1)\} \). In the third phase, we return the path \( P_{1:2} \) to the network and search for a path from node 1 to node 3, e.g., \( P_{1:3} = \{(1, 4), (4, 3)\} \). Finally, we conclude that the union of \( P_{1:2} \) and \( P_{1:3} \) is an element in \( Q \), i.e., \( q = \{(1, 4), (2, 4), (3, 4)\}, q \in Q \).

**V. MULTIPLE UNICAST NETWORK**

In this section, we present the coding scheme for the multiple unicast network, depicted in Fig. 2b. This network is defined by a graph \( \mathcal{G}(V', E') \) with a set of source nodes \( S = \{1, 2, 3\} \) (Fig. 2b). In the multiple unicast network, each source node \( i \in S \) produces two different messages \( W^{(i)} \) and \( W^{(i)} \) that are intended for the two remaining source nodes \( j, l \in S \setminus \{i\} \), i.e., each two source nodes communicate bidirectionally in a unicast manner.

**A. Capacity and Coding for a Multiple Unicast Network Based on Line Topology**

Our coding scheme achieves the capacity region under an equal rate demand. Specifically, each two-way communication is carried out at the same rate, \( R_{i\rightarrow j} = R_{j\rightarrow i}, \forall i, j \in S \). To show the RT coding scheme for this network, we use the line topology coding scheme from Section IIIA. Since each unicast session represents a flow from one source node to another, we show that no inter-flow coding is needed to achieve the capacity region. Under the equal rate assumption, we show that the minimum cut upper bound is achievable. However, in the multiple unicast general case the minimum cut upper bound is not tight [3], i.e., our coding scheme is optimal under an equal rate assumption, but sub-optimal in the general case. Our main result is summarized in the following theorem.

**Theorem 4** For any network \( \mathcal{G}(V', E') \) with an arbitrary delay bounded by \( D \), there exists a RT coding scheme with
any set of rates within the equal rate capacity region, which is for all \(i,j,l \in S\),

\[
R_{i \rightarrow j} + R_{i \rightarrow l} \leq C_{i;j,l},
\]

\[
R_{i \rightarrow j} \leq C_{i;j},
\]

(21)

where \(R_{i \rightarrow j} = R_{j \rightarrow i}\) and \(R_{i \rightarrow l} = R_{l \rightarrow i}\). Furthermore, the coding scheme includes a fixed header per transmission of \(2[\log_2 2D] + \lceil \log_2 h \rceil\) bits.

The capacity region is upper bounded by standard minimum cut arguments. It has corner points of \((R_{1 \rightarrow 2}, R_{1 \rightarrow 3}, R_{2 \rightarrow 3})\) that can be found by setting the rate \(R_{1 \rightarrow 2} = C_{1;2}\), which yields that \(R_{1 \rightarrow 3} = C_{1;2} - C_{1;2}\) and \(R_{2 \rightarrow 3} = C_{1;3} - C_{1;2}\). Hence, either \(R_{2 \rightarrow 3}\) or \(R_{1 \rightarrow 3}\) equals zero, since node 3 can only be in one of the cuts between nodes 1 and 2 and \(C_{1;2}\) is the minimization of all the cuts separating them. Therefore,

\[
C_{1;2} = \min\{C_{1;2}, C_{2;1,3}\}.
\]

(22)

A general expression for all the corner points includes \(R_{i \rightarrow j} = C_{i;j}, i,j \in S\). Then, in case \(C_{i;j} < C_{i;j,l}\), we get \(R_{i \rightarrow l} = 0\) and \(R_{j \rightarrow l} = \min\{C_{j;1,l} - C_{i;j}, l \in S \setminus \{i,j\}\}\). Furthermore, the fact that \(C_{i;j,l} \leq C_{i;j,l} + C_{i;l,j}\) implies that the corner point \((C_{i;j} = C_{i;j,l})\) can be expressed as

\[
R_{i \rightarrow j} = C_{i;j},
\]

\[
R_{j \rightarrow l} = C_{j;1,l} - C_{i;j},
\]

\[
R_{i \rightarrow l} = 0.
\]

(23)

To show an achievable coding scheme for the region in (21), it is sufficient to prove that we achieve all corner points in the form of (23). Therefore, we present a lemma that shows how to partition each network \(G'\) into sub-topologies of line networks. The decomposition includes a set of disjoint paths \(P_{i;j}\), where \(|P_{i;j}| = \frac{C_{i;j}}{C}\), and a set of disjoint paths \(P_{j;l}\), where \(|P_{j;l}| = \frac{C_{j;1,l} - C_{i;j}}{C}\) and \(P_{i;j} \cap P_{j;l} = \emptyset\). By using this lemma, we can achieve the corner points in the capacity region.

**Lemma 2** For a network \(G'(V', E')\) with \(C_{i;j} = C_{i;j,l}\), there exist sets of disjoint paths \(P_{i;j}\) and \(P_{j;l}\) such that \(|P_{i;j}| = \frac{C_{i;j}}{C}\) and \(|P_{j;l}| = \frac{C_{j;1,l} - C_{i;j}}{C}\), where \(P_{i;j} \cap P_{j;l} = \emptyset\), \(i,j,l \in S\). Namely, \(P_{i;j}\) and \(P_{j;l}\) have no mutual edges.

**Proof of Lemma 2** The maximum number of disjoint paths between nodes \(i\) and \(j\), \(|P_{i;j}|\), is \(\frac{C_{i;j}}{C}\) (the Max-flow Min-cut theorem [27], Theorem 1). Since \(C_{i;j} = C_{i;j,l}\), this is also the maximum number of paths between \(i\) and \(j\). However, in case \(C_{i;j,l} < C_{i;j,l}\), there are more paths between \(j\) and \(i\) and then \(\frac{C_{i;j,l}}{C}\). These paths, \(P_{i;j}\), are between nodes \(i\) and \(j\).

To prove that \(P_{i;j}\) and \(P_{j;l}\) are disjoint, we add \(\frac{C_{j;1,l} - C_{i;j}}{C}\) direct paths between \(i\) and \(l\), \(P_{i;l}\), which yields a new network in which \(C'_{i;j,l} = C_{i;j,l}\). As a result of (22), we conclude that there are \(\frac{C_{i;j,l} - C_{i;j,l}}{C}\) new paths between nodes \(i\) and \(j\), \(P'_{i;j}\). Paths \(P'_{i;j}\) are a union of \(P_{i;l}\) and \(P_{j;l}\), i.e., \(|P_{j;l}| = \frac{C_{j;1,l} - C_{i;j}}{C}\). Hence, since \(P_{i;j}\) and \(P'_{i;j}\) are disjoint according to (27), we conclude that \(P_{j;l}\) and \(P_{i;j}\) are also disjoint.

Next, we provide the proof for Theorem 3.

**Proof of Theorem 3** Using the line topology coding scheme from Section III-A and Lemma 2, we can achieve all the corner points in the capacity region. Specifically, we use the line topology coding scheme from Section III-A at each path from Lemma 2 to achieve a rate of \(R_{i \rightarrow j} = C_{i;j}\) and \(R_{j \rightarrow l} = C_{j;1,l} - C_{i;j}\), \(i,j,l \in S\). Using time sharing arguments, it is straightforward to see that we achieve the capacity region under the equal rate constraint. Moreover, we obtain an asynchronous coding scheme with a maximum delay of \(LD + t\) for all \(t\) by using a header of \(2[\log_2 2D] + \lceil \log_2 h \rceil\) bits according to the coding scheme of the line topology from Section III-A. Another \(\lceil \log_2 h \rceil\) bits are used to form disjoint paths in the network. Therefore, by exploiting the broadcast ability of the wireless medium, we obtain a RT NC scheme.

**B. Algorithm to Find the Building Blocks**

We introduce an optimization problem that we use to find the set of paths from Lemma 2, \(P_{i;j}\) and \(P_{j;l}\), \(i,j,l \in S\). In this problem, we would like to find the minimum flow, \(f\), which maintains the condition that \(|P_{i;j}| = \frac{C_{i;j}}{C}\) paths from \(j\) to \(l\) and another \(|P_{i;j}| = \frac{C_{j;1,l} - C_{i;j}}{C}\) paths from \(i\) to \(j\) are disjoint. Since we would like to find \(\frac{C_{i;j}}{C}\) paths between nodes \(i\) and \(j\), we demand that \(I_{i} = \frac{C_{i;j}}{C}\) and \(O_{j} = \frac{C_{j;1,l} - C_{i;j}}{C}\). However, for the case in which \(C_{i;j} < C_{j;1,l}\), there are more paths emerging from node \(j\). Those paths are consumed by node \(l\), i.e., \(I_{i} - O_{l} = \frac{C_{j;1,l} - C_{i;j}}{C}\). We guarantee that all of the paths are disjoint since \(f\) is a binary vector and each relay node in the network, \(m \in V' \setminus S\), has an equal input and output flows. Hence, the outcome of this optimization problem is a binary flow that includes two sets of disjoint paths \(P_{i;j}\) and \(P_{j;l}\).

\[
\begin{align*}
\text{minimize} & \quad \sum_{f(u,v) \in \mathcal{F}} f(u,v) \\
\text{subject to} & \quad \mathcal{O}_{m} = \sum_{n:(m,n) \in \mathcal{E}'} f(m,n), m = 1, \ldots, |V| \\
& \quad \mathcal{I}_{m} = \sum_{n:(n,m) \in \mathcal{E}'} f(m,n), m = 1, \ldots, |V| \\
& \quad \mathcal{I}_{j} = 0 \\
& \quad \mathcal{I}_{i} = \frac{C_{i;j}}{C} \\
& \quad \mathcal{I}_{l} - \mathcal{O}_{l} = \frac{C_{j;1,l} - C_{i;j}}{C} \\
& \quad \mathcal{O}_{j} = \frac{C_{j;1,l}}{C} \\
& \quad \mathcal{O}_{l} = 0 \\
& \quad \mathcal{O}_{k} = \mathcal{I}_{k}, k = 4, \ldots, |V|,
\end{align*}
\]

(24)

**C. Example** In this subsection, we show an example of a network \(G\) with three source nodes that communicate in a bidirectional unicast manner (Fig. 12). In fact, we include an example in which we apply the algorithm from the previous Subsection V.B i.e., the algorithm for finding the disjoint paths between the three source nodes that, by using the line topology coding
scheme from Section II-A will produce a corner point in the equal rate capacity region. The corner point that is shown in Fig. 12(b) and (c) is \((R_{1\rightarrow2}, R_{1\rightarrow3}, R_{2\rightarrow3}) = (C_{1;2,3} - C_{1;3}, C_{1;3}, 0)\). Specifically, in this network \(C_{1;3} = 3C\) and \(C_{1;2,3} = 5C\), and therefore, we show three disjoint paths from node 1 to node 3 (Fig. 12b) and another two disjoint paths from node 1 to node 2 (Fig. 12c). The paths are disjoint since they do not pass through a common relay node or, similarly, in the equivalent graph \(G'\) they have no mutual edges.

VI. SIMULATION

We simulated our results, i.e., we used a fixed number of nodes but generated a random number of edges, on an Erdős-Rényi random graph model \([28]\). Each edge is a bidirectional link that is drawn independently of the other links with the same probability \(p\) to be with capacity \(C\) or \((1 - p)\) to be 0. Then, we constructed an equivalent graph \(G'(V', E')\), as illustrated in Fig. 3 and searched for the line and star topologies in the graphs.

![Fig. 13: Simulation results of the average size of the sets \(P_{1;2}\) and \(P_{1;3}\). This simulation shows the sum rate increment that was attained using the coding scheme from Section V.](image)

In Fig. 13, we show a simulation result of a corner point in the equal rate capacity region of a multiple unicast network as described in Section V. This corner point is the maximization on the rate between nodes 1 and 2, i.e., \(R_{1\rightarrow2} = C_{1;2}\). Then, we search for the maximum number of disjoint paths between nodes 1 and 3, i.e., \(R_{1\rightarrow3} = C_{1;2,3} - C_{1;2}\). Therefore, the number of paths between nodes 1 and 2 are likely to be greater than the number of paths between nodes 1 and 3. For each simulation, we compared 8, 16, 32, 64 and 128 nodes and created ten different graphs by varying the value of \(p\). We show the average number of \(|P_{1;2}|\) and \(|P_{1;3}|\) over the number of nodes in a graph \(G\). As expected, our result shows that each of the examined variables \(|P_{1;2}|\) or \(|P_{1;3}|\) increases with the scale of the network, where \(|P_{1;2}| > |P_{1;3}|\). Furthermore, this simulation shows the sum rate increment that was achieved using the coding scheme from Section V.

By using the line topology coding scheme of Section III-A on each of the paths in \(P_{1;2}\) and \(P_{1;3}\), we can construct an optimal practical asynchronous coding scheme for a general network under the equal rate demand.

We further simulated our results on a network with the multicast demands of Section IV illustrated in Fig. 14. In this network, we searched for the maximum number of ring (Algorithm 1) and line-star (Algorithm 2) topologies. The simulation shows a comparison between 8, 16, 32, 64 and 128 nodes. For each number of nodes, we created ten different graphs by varying the value of \(p\). We show the average number of \(|R|\) and \(|Q|\) over the number of nodes in a graph \(G\). This simulation shows the decomposing opportunities in a graph. We recall that each ring topology contributes a rate of \(C\) to each source, where each line-star topology only contributes \(\frac{C}{2}\). However, we see from Fig. 14 that the line-star topology makes a more significant contribution than the ring topology in the multicast problem of three users under the equal rate constraint.

![Fig. 12: A network \(G\) is shown in (a) where all the source nodes 1, 2 and 3 communicate in a bidirectional unicast manner. In (b) and (c) we show a corner point in the equal rate capacity region by applying the algorithm from Section V-B where \(R_{1\rightarrow3} = C_{1;3}\), i.e., \(3C\), and \(R_{1\rightarrow2} = C_{1;2,3} - C_{1;3}\), i.e., \(2C\).](image)
allow at least two adjacent nodes to gain new information from
the proposed coding schemes we employ NC to
utilization of the wireless medium. Unlike a simple routing
each network. The simulation shows a marked improvement
in the coding schemes that achieve the maximum rates for
sessions, we show that such a combination is feasible, i.e.,

\[ 2R + R_{i\rightarrow j} + R_{l\rightarrow i} \leq C_{i,j,l} \]

where \( R_{i\rightarrow j} = R_{j\rightarrow i} \) and \( R_{l\rightarrow i} = R_{i\rightarrow l} \), \( \forall i,j,l \in S \). Each source node \( i \in S \) produces three different
messages, \( W_i^{(t)}, W_{i\rightarrow j}^{(t)} \) and \( W_{l\rightarrow i}^{(t)} \), which are intended for the
two remaining source nodes \( j,l \in S \setminus \{i\} \).

Corollary 5 For any network \( G'(V', E') \) with an arbitrary
delay bounded by \( D \), there exists a RT coding scheme with
any set of rates within the equal rate capacity region, which is

\[ 2R + R_{i\rightarrow j} + R_{l\rightarrow i} \leq C_{i,j,l} \]

\[ R_{i\rightarrow l} \leq C_{i,j} \]  

Proof: The capacity region can be upper bounded by stan-
dard minimum cut arguments. For example, \( C_{1:2,3} \) represents
the cut set bound of all the information that node 1 receives.
Nodes 2 and 3 transmit multicast and unicast messages to
node 1, e.g., \( W_2^{(t)}, W_3^{(t)}, W_{2\rightarrow 1}^{(t)} \) and \( W_{3\rightarrow 1}^{(t)} \). Therefore, we
obtain the first upper bound in the region (25). The second
upper bound is obtained straightforwardly by the minimum-
cut maximum-flow theorem.

To show an achievable coding scheme for the region in (25), it is sufficient to prove that we achieve all corner points.

VII. EXTENSIONS

In this section, we show that the building block approach
also yields a coding scheme that achieves capacity for the
combined problem, namely, a network of multicast and mul-
tiple unicast demands. By using the coding schemes for the
multicast (Section IV) and the multiple unicast (Section V)

In the next simulation, Figs. 15 and 16 we compared the
performance of our coding schemes to simple routing schemes in wireless settings, \( G'(V, E) \). Specifically, we measured the
number of transmissions required by the relay nodes, \( V \setminus S \),
in the coding schemes that achieve the maximum rates for
each network. The simulation shows a marked improvement
in the number of required transmissions due to the efficient
utilization of the wireless medium. Unlike a simple routing technique, in the proposed coding schemes we employ NC to
allow at least two adjacent nodes to gain new information from
each transmission. The differences between the multicast (Fig.
15) and the multiple unicast (Fig. 16) scenarios are due to the
differences between the star topology (Section III-B), which makes a greater contribution in the multicast network, and
the line topology (Section III-A), which is the only topology
that is used in the multiple unicast network. Therefore, the
improvement for the multiple unicast network is 100%, and
for the multicast network is about 30%. Lastly, from these
simulations we learned that for the same amount of informa-
tion, the number of transmissions decreases significantly by
using the proposed coding schemes, or equivalently, for the
same number of transmissions, we achieve a higher rate.

Fig. 16: Simulation results of the average number of
transmissions required by the relay nodes in a multiple
unicast network. As shown, we achieve significantly better
performance using the proposed coding scheme.
There exist two different corner points: the first is the case 
\[ R_{i\to j} = C_{i,j}, \] which yields that \( R = 0 \) since
\[ C_{i,j} = \min\{C_{i,j,l} \mid C_{i,j,l} \leq \}, \tag{26} \]
where \( i, j, l \in S \). Therefore, this corner point is strictly the multiple unicast case, which we already discussed in Section VI. The second corner point is the case where \( R = \frac{1}{2} \). In this case, there are two bidirectional unicasts sessions that equal zero. For example, assuming \( h = C_{1,2,3} \) yields that \( R_{1\to 2} \) and \( R_{3\to 4} \) both equal zero. The remaining unicast rate, e.g., \( R_{2\to 3} \), can be larger than zero in the general case, i.e., we can find a set of disjoint paths between nodes 2 and 3 such that deleting them from the network will not affect the maximum rate of the multicast session. For example, consider the network depicted in Fig. [18] In this network, there exists a path between nodes 1 and 2 that does not intersect with the star topology structure, which is required to obtain the maximum multicast rate, in this case \( \frac{1}{2} C \).

VIII. CONCLUSION

We conclude that a wireless network with three source nodes under any communication demands can be decomposed into line and line-star topologies using the building block approach. Then, by exploiting the broadcast ability of the wireless medium, we can achieve the capacity region under an equal rate constraint in the corresponding wired model. Furthermore, the coding schemes, which are based on those two canonical topologies, include many advantages, such as RT decoding delay and a small overhead. In practice, these coding schemes can be implemented on wireless networks with random transmission delays to gain better performance and power efficiency.

REFERENCES

[1] R. Ahlswede, N. Cai, S. R. Li, and R. W. Yeung, “Network information flow,” IEEE Trans. Inf. Theory, vol. 46, no. 4, pp. 1204–1216, 2000.
[2] R. Koetter and M. Médard, “An algebraic approach to network coding,” IEEE/ACM Trans. Netw., vol. 11, no. 5, pp. 782–795, 2003.
[3] Y. Wu, P. A. Chou, Q. Zhang, K. Jain, Z. Wenwu, and S. Y. Kung, “Network planning in wireless ad hoc networks: a cross-layer approach,” IEEE J. Sel. Areas Commun., vol. 23, no. 1, pp. 136–150, 2011.
[4] S. Jaggi, P. Sanders, P. Chou, M. Effros, S. Egner, K. Jain, and L. M. G. M. Tolhuizen, “Polynomial time algorithms for multicast network code construction,” IEEE Trans. Inf. Theory, vol. 51, no. 6, pp. 1973–1982, 2005.
[5] T. Ho, M. Médard, R. Koetter, D. R. Karger, M. Effros, J. Shi, and B. Leong, “A random linear network coding approach to multicast,” IEEE Trans. Inf. Theory, vol. 52, no. 10, pp. 4413–4430, 2006.
[6] R. Dougherty and K. Zeger, “Nonreversibility and equivalent constructions of multiple-unicast networks,” IEEE Trans. Inf. Theory, vol. 52, no. 11, pp. 5067–5077, 2006.
[7] S. R. Li, R. W. Yeung, and N. Cai, “Linear network coding,” IEEE Trans. Inf. Theory, vol. 49, no. 2, pp. 371–381, 2003.
[8] R. Dougherty, C. Freiling, and K. Zeger, “Insufficiency of linear coding in network information flow,” IEEE Trans. Inf. Theory, vol. 51, no. 8, pp. 2745–2759, 2005.
[9] D. Traskov, N. Ratnakar, D. Lun, R. Koetter, and M. Médard, “Network coding for multidimensional unicast: An approach based on linear optimization,” in International Symposium on Information Theory (ISIT), 2006, pp. 1758–1762.
[10] A. Ramakrishnan, A. Das, H. Maleki, A. Markopoulos, S. Jafar, and S. Vishwanath, “Network coding for three unicast sessions: Interference alignment approaches,” in Proc. Allerton Conference Communication, Control, and Computing, 2010, pp. 1054–1061.
[11] S. Huang and A. Ramamoorthy, “On the multiple-unicast capacity of 3-source, 3-terminal directed acyclic networks,” IEEE/ACM Trans. Netw., vol. 22, no. 1, pp. 285–299, 2014.
[12] S. Katti, H. Rahul, W. Hu, D. Katabi, M. Médard, and J. Crowcroft, “Xors in the air: Practical wireless network coding,” IEEE/ACM Trans. Netw., vol. 16, no. 3, pp. 497–510, 2008.
[13] S. Sengupta, S. Rayanchu, and S. Banerjee, “Network coding-aware routing in wireless networks,” IEEE/ACM Trans. Netw., vol. 18, no. 4, pp. 1158–1170, 2010.
[14] Z. Li and B. Li, “Network coding in undirected networks,” in Conference on Information Sciences and Systems (CISS), Princeton, New Jersey, USA, 2004.
[15] C. Wang, T. Gou, and S. Jafar, “Multiple unicast capacity of 2-source 2-sink networks,” in Global Telecommunications Conference (GLOBECOM), Houston, Texas, USA, 2011.
[16] S. Sengupta, S. Rayanchu, and S. Banerjee, “An analysis of wireless network coding for unicast sessions: The case for coding-aware routing,” in International Conference on Computer Communications (INFCOM), Anchorage, Alaska, USA, 2007.
[17] K. Jain, V. Vazirani, and G. Yuval, “On the capacity of multiple unicast sessions in undirected graphs,” IEEE Trans. Inf. Theory, vol. 52, no. 6, pp. 2805–2809, 2006.
[18] M. Bakshi, M. Effros, W. Gu, and R. Koetter, “On network coding of independent and dependent sources in line networks,” in International Symposium on Information Theory (ISIT), Nice, France, 2007.
[19] S. M. S. Tabatabaei Yazdi, S. A. Savari, and G. Kramer, “Network coding in node-constrained line and star networks,” IEEE Trans. Inf. Theory, vol. 57, no. 7, pp. 4452–4468, 2011.
[20] P. A. Chou, Y. Wu, and K. Jain, “Practical network coding,” in Proc. Allerton Conference Communication, Control, and Computing, 2003.
[21] Y. Wu, P. A. Chou, and S. Y. Kung, “Information exchange in wireless networks with network coding and physical-layer broadcast,” Microsoft Research, Tech. Rep. MSR-TR-2004-78, 2004.
[22] P. Sadeghi, D. Traskov, and R. Koetter, “Adaptive network coding for broadcast channels,” in Network Coding, Theory, and Applications (NetCod), Lausanne, Switzerland, 2009.
[23] L. R. Ford and D. R. Fulkerson, Flows in Networks. Princeton University Press, 1962.
[24] R. Karp, “Reducibility among combinatorial problems,” in Complexity of Computer Computations, ser. The IBM Research Symposia Series, R. Miller, J. Thatcher, and J. Bohlinger, Eds., 1972, pp. 85–103.
[25] N. Garg and J. Koenemann, “Faster and simpler algorithms for multi-commodity flow and other fractional packing problems,” SIAM Journal on Computing, vol. 37, no. 2, pp. 630–652, 2007.
[26] N. Garg, V. V. Vazirani, and M. Yannakakis, “Primal-dual approximation algorithms for integral flow and multicut in trees,” Algorithmica, vol. 18, no. 1, pp. 3–20, 1997.
[27] L. R. Ford and D. R. Fulkerson, “Maximal flow through a network,” Canadian Math., vol. 8, pp. 399–404, 1956.
[28] P. Erdös and A. Rényi, “On the evolution of random graphs,” Publ. Math. Inst. Hung. Acad. Sci, vol. 5, pp. 17–61, 1960.