Note on the holographic $c$-function

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Abstract
We discuss the holographic $c$-function and describe an algorithm for the practical computation of the changing central charge in arbitrary RG-flows. The conclusions are drawn from studying a particular example, which is worked out in detail. The renormalisation procedure of hep-th/0112150 necessary to obtain the central charge is reviewed.
Since the work of Zamolodchikov [1], the proof of a c-theorem for quantum field theories in more than two dimensions has remained an open problem. In the last years the idea of holography, namely the extension of the AdS/CFT correspondence to non conformal theories, has provided new tools for addressing this issue. In [2, 3] a holographic c-theorem has been established. It is defined for quantum field theories, which allow a holographic description in terms of gravity. This is quite a restriction. In fact the existence of this construction seems to hinge on a special identity fulfilled for the coefficients $a$ and $c$ of the two contributions to the conformal anomaly, i.e. the Euler density and the Weyl tensor [4]. For a holographic renormalisation group flow (RG-flow) these coefficients coincide. Within field theory there are counterexamples for c-theorem conjectures based on $c$, while for all known field theory examples $a_{UV} \geq a_{IR}$ for the coefficient of the Euler density. So the c-theorem should be actually dubbed a-theorem.

Inspired by [5] we constructed explicit examples of holographic RG-flows between two conformal field theories (CFT) by glueing asymptotically AdS-spacetimes together and studied the corresponding c-functions [6, 7]. A crucial feature of the examples was the necessity to introduce artificially a “renormalised” c-function in order to obtain the expected values for the central charges. The renormalisation must be performed in order to get rid of a divergence caused by the presence of a “massless pole”, which lacks a physical interpretation. We argued that the appearance of the divergence was an artefact of the construction, i.e. induced by the glueing procedure, which required only continuity at the gluing point. We speculated that for a smooth RG-flow no renormalisation must be performed. Here we want to push the topic a little bit further by analysing a smooth example. We first shortly summarise the construction of the c-function and then focus on the lesson taught by the smooth model.

In [5] a natural definition of a holographic c-function was proposed, the so-called “canonical” c-function, which is related to the OPE of the stress-energy tensor [8] as (here in $d = 4$)

$$
\langle T_{\mu\nu}(x)T_{\rho\sigma}(0) \rangle = -\frac{1}{48\pi^3} \Pi^{(2)}_{\mu\nu,\rho\sigma} \left\{ \frac{c(x)}{x^4} \right\} + \pi_{\mu\nu} \pi_{\rho\sigma} \left\{ \frac{f(x)}{x^4} \right\}
$$

(1)

where $\pi_{\mu\nu} = \partial_\mu \partial_\nu - \eta_{\mu\nu} \Box$ and $\Pi^{(2)}_{\mu\nu,\rho\sigma} = 2\pi_{\mu\nu} \pi_{\rho\sigma} - 3(\pi_{\mu\rho} \pi_{\nu\sigma} + \pi_{\mu\sigma} \pi_{\nu\rho})$. For a 4-d conformal QFT the function $f(x)$ vanishes and $c(x)$ becomes a constant, the so called central charge $c$. We want to compute this 4-dimensional correlator by applying the AdS/CFT correspondence to the 5-dimensional gravity action below:

$$
S = \int d^{d+1}x \sqrt{g} \left[ R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right] - 2 \int d^d x \sqrt{g} K
$$

(2)

This is done as follows. First one has to choose a classical solution

$$
ds^2 = e^{2A(z)} \left( dz^2 + \eta_{ij} dx^i dx^j \right), \quad \phi = \phi(z)
$$

(3)
of the equation of motion of the action \[ \frac{dA}{dz} = \frac{e^A}{2(d-1)} W , \quad \frac{d\phi}{dz} = -e^A W_\phi \] (4)

where as \( z \) moves from zero to infinity, \( \phi(z) \) interpolates between two nearby extrema of the potential \( V(\phi) \). Needless to mention that we have to choose an appropriate \( V(\phi) \), of course. If \( \phi(z) \) approaches one of these extrema the geometry of the metric asymptotes to AdS. The function \( W(\phi) \) appearing above is related to the potential by

\[ V(\phi) = \frac{1}{2} W'^2 - \frac{d}{4(d-1)} W^2 . \] (5)

The gravity calculation of \( < TT > \)-correlator for a specified background flow was done in [9, 10, 11] and boils down to the solution of the fluctuation equation for the linearised transverse traceless graviton (TT)

\[ h^{TT}_{ij} = e^{ikx} \chi(z) \xi_{ij}(x) , \] (6)

which satisfies the equation

\[ \left[ -\frac{d^2}{dz^2} + \left( V_{QM}(z) - k^2 \right) \right] e^{\frac{d-1}{2}A(z)} \chi(z) = 0 \] (7)

\[ V_{QM} = \left( \frac{d-1}{2} \right)^2 A'^2 + \frac{d-1}{2} A'' . \] (8)

The \( c \)-function in eq. (11) can be obtained via the four dimensional Fourier transform of the so called flux factor \( \mathcal{F}(q) \) \( ( q = i |k| ) \), which can be constructed out of the fluctuation \( \chi(z) \):

\[ \frac{c(z)}{z^4} = \mathfrak{F} \left[ \frac{\mathcal{F}(q)}{q^4} \right] (z) \] (9)

with

\[ \mathcal{F}(q) = \frac{1}{\varepsilon^{2A-1}} \lim_{z \to \varepsilon} \frac{d}{dz} \frac{\chi(z)}{\chi(\varepsilon)} . \] (10)

The evaluation of eq. (10) for a generic potential is the one part of the problem. We solved it by showing that the potentials which arise from a given background are those to which scattering theory applies. The solution \( \chi(z) \) of the fluctuation equation one has to choose is fixed by regularity in the IR. This is due to an finite action argument. Having chosen this
solution, which is unique up to a scaling, one simply considers its decomposition into two solutions with prescribed power law close to the origin. The solution reads

$$\chi(q, z) = e^{-\frac{d-1}{2} A(z)} \left[ J(\lambda, -q) \varphi_{\text{irr}}(z) - J(-\lambda, -q) \varphi_{\text{reg}}(z) \right]$$

(11)

where $J(\pm \lambda, -q)$ are the so called Jost functions. Here the Jost functions contain the necessary informations about the potential. The flux computed in [6] reads

$$\mathcal{F}(q) = \mathcal{F}_{\log}(q) + \mathcal{F}_{\text{int}}(q)$$

(12)

with

$$\mathcal{F}_{\log}(q) = \frac{(-1)^{\lambda+1}}{2^{2\lambda-2}[\Gamma(\lambda)]^2} q^{2\lambda} \log q$$

(13)

$$\mathcal{F}_{\text{int}}(q) = -2 \lambda R_{\text{UV}}^{2\lambda-1} \frac{J(-\lambda, -q)}{J(\lambda, -q)}.$$  

(14)

In [6,7] we observed the necessity to accomplish the definition of the $c$-function by a subtraction procedure in order to obtain the expected values for the central charges. The reason for this “renormalisation” could not be explained. Relying on a heuristic argument, we conjectured that the subtraction should be absent for a continuous flow.

The Jost functions $J(-\lambda, -q)$ vanishes for large $q$. So the term $\mathcal{F}_{\log}(q)$ is the only one, which survives in the large $q$-limit. It determines the central charge $c_{\text{UV}}$ in the UV, which can be conveniently normalised by multiplying the whole flux factor with an appropriate constant. A practical choice is to set $c_{\text{UV}} = 1$. The exact computation of $\mathcal{F}_{\text{int}}(q)$ turns out to be rather difficult. The problem is easy to understand. We are interested in the small $q$-expansion of $\mathcal{F}_{\text{int}}(q)$ and in particular in the logarithmic terms, which vanish close to zero. The $q$-expansion contains terms divergent for small $q$, too. So we have to extract a subleading order numerically.

To make the task tractable, we decided to map the problem into an 2-point boundary value problem, which is numerically accessible. This is achieved by first rescaling the solution according to $\tilde{y}(z) = z^{3/2} \cdot y(z)$ so that close to $z = 0$ the irregular solution approaches the value 1, while the regular solution tends to zero as before. Any solution must be decomposable into a sum of the irregular and regular solutions. Taking this and the non uniqueness of $\chi(z)$ due to a rescaling ( see eq. (10) ) into account one may impose at the left boundary the condition $y(0) = 1$. Mapping now the interval $I_z = [0, \infty]$ by means of the transformation $z = -1 + \sqrt{x}$ to the interval $I_x = [0, 1]$ one can single out the exponentially decaying

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*For technical details see appendix A.*
solution by the additional boundary condition $\bar{y}(0) = 0$. Thus we obtain \(\dagger\),
\[
-4x^3 \frac{d^2 \bar{y}(x)}{dx^2} - 6x^2 \left(1 + \frac{1}{1 - \sqrt{x}}\right) \frac{d\bar{y}(x)}{dx} + \left(U(x) + q^2\right) \bar{y}(x) = 0
\]
the numerical solution of which is performed via a finite difference algorithm [13]. Please note that the contraction of the interval provides us with a substantial benefit. The equal spacing of the contracted interval means an unequal spacing on the positive line, i.e. close to zero we collect more and more mesh points. The two alternatives, the shooting algorithm and the solution via an integral equation, did not work. The first due to the peculiar boundary conditions we have to impose and the other since the numerical data lack precision we must obtain.

Now we are going to investigate a particular example of a smooth RG-flow defined by the quantum mechanical potential
\[
V_{QM}(z) = \frac{15}{4} \frac{1}{z^2} + U(z) = \frac{15}{4} \frac{1}{z^2} + \frac{z}{(1 + z^2)^2},
\]
which meets all the requirements imposed in [6], i.e. it is dominated at zero and infinity by the angular momentum barrier and smooth everywhere else in between. So it corresponds to an interpolating RG-flow. The result of the computation of $F_{int}(q)/q^2$ is shown in Fig. 1 below.

The only cosmetic correction we have done here is to subtract the constant appearing in the $q$ expansion of $F_{int}(q)^{\dagger \dagger}$. It is important to notice that for very small $q$ the remaining function

\(\dagger\)For the definition of $U$ see eq. [19].
\(\dagger \dagger\)The value of the constant is $-0.598164897814$
becomes a constant. Again this signals the presence of a $q^2$-term in the expansion of $F_{\text{int}}(q)$, dubbed “massless pole” in the introduction. In order to obtain the proper behaviour of a $c$-function, we have to perform the renormalisation procedure introduced in [6] again, i.e. we must subtract the divergence introduced by the “massless pole”\textsuperscript{*}. The Fourier transformation of eq. (3) is performed using a code provided in [12]. In Fig. 2 we show the result of the Fourier transformation after applying the renormalisation. The actual computation of Fig. 2 involves an additional trick because the numerical Fourier transformation is not stable. Instead the plot is generated by transforming $F_{\text{int}}(q)$ multiplied by a gaussian distribution. This is merely a sort of cutoff and does not spoil the validity of the result as long as we are only interested in the behaviour at large $q$. This is due to the fact that the Fourier transformation just intertwines the region close to zero in $q$ with far from zero in $x$ and vice versa. The Gaussian does not affect the behaviour of $F_{\text{int}}(q)$ for small $q$.

As a cross check for the validity of the content of Fig. 2 for large $x$ we just performed the inverse Fourier transformation on the unrenormalised data underlying Fig. 2 in order to see if we can reconstruct Fig. 1 correctly. The ratio of the original data shown in Fig. 1 to the reconstructed ones is shown in Fig. 3.

![Fig. 3. Ratio $r = \frac{F_{\text{int}}(q)}{F_{\text{int}}(q)}$](image)

The ratio is close to one nearly everywhere with exception of $q$ very close to zero. This is not a blow for our procedure because the deviation is related to the fact that we have not tabulated the transformed function on the whole real line but just on a finite interval.

The sharp peak in Fig. 2 is a complete artefact of the multiplication by the Gaussian hence completely meaningless. The straight line in the same figure denotes the value which must be approached for $x$ very large in order to represent a decline in the central charge to $2/3$ of the original value. With a different sampling of the data it is possible to find the right behaviour of the Fourier transform for small $x$, too. The result is shown in Fig. 4. Here we

\textsuperscript{*}One must subtract $0.09428 \cdot x^2$
have summed up the two contributions to the c-function coming from $\mathcal{F}_{\text{log}}(q)$ and $\mathcal{F}_{\text{int}}(q)$ and rescaled to make the UV-charge becoming 1.

\[
\begin{align*}
\text{Fig.4. } c_{\text{ren}}(z)
\end{align*}
\]

As a comparison we computed the gravitational $c$-function of [2, 3]. In the last two papers a holographic $c$-theorem was established using the warp factor $A(z)$ appearing in the metric eq. (3). It is used to introduce a function,

\[
c(z) = -\frac{e^{(d-1)A(z)}}{A'(z)d^{d-1}},
\]

which satisfies all the properties of a $c$-function, i.e. $c(z)$ is a monotonous function, which correctly interpolates between the UV and IR central charges. The factors in the definition eq. (15) are chosen in order to ensure $c_{\text{UV}} = 1$. So by construction the two $c$-functions, eq. (9) and eq. (15), can be made to become equal at $z = 0$ (cf. second paragraph after eq. (14) on page 4). By a straightforward calculation we express the suitable normalised $c$-function entirely in terms of the zero mode $\psi_0(z) = e^{\frac{d-1}{2}A(z)}$ of eq. (7). It reads

\[
c_{\text{grav}}(z) = -\left(\frac{d-1}{2}\right)^{d-1} \frac{\psi_0(z)^{d+1}}{\psi_0'(z)^{d-1}}.
\]

For $d = 4$ we can compute the gravitational $c$-function by constructing the zero mode of eq. (7) applying the finite difference algorithm once more. The corresponding plot is shown in Fig. 5 below. The horizontal line again represents the value 2/3. The both $c$-functions coincide in their predictions for the central charge.
Conclusions

We considered the holographic c-function connected with a RG-flow, which can be computed from a supersymmetric quantum mechanical scattering problem. The purpose of this work was to show that for a smooth scattering potential the c-function does not require the renormalisation procedure introduced in [6]. Instead we found that even here a “massless pole” appears and renormalisation must be applied again.

The problem of the “massless pole” is related to the following fact. For $\frac{c(x)}{x^4}$ to have a well-defined Fourier transform, it has to be of the form

$$c(x) = \frac{\partial^2}{x^2} h(t), \quad t = \ln \mu^2 x^2,$$

with some suitable function $h(t)$. The massless pole is of a peculiar type with respect to the structure of $h(t)$. This is due to the fact that the flux $\mathcal{F}(q)$ can be expressed as

$$\mathcal{F}(q) = q^5 \cdot \int_0^\infty h(t) J_1(qx) dx$$

and each $t^k$ transforms into terms of the type $q^4 \cdot P_k(\ln q)$ with $P_k$ a polynomial of degree $k$. The highest power in $P_k(\ln(q))$ is $(-1)^k 2^k (\ln q)^k$. Unless there is an astonishing conspiracy between all the powers $t^k$ no isolated $q^2$ can be obtained.

The final goal is a proof of a holographic c-theorem based entirely on the field theoretic definition given by eq. (1). The example points out that the definition of the flux-factor must be improved or accompanied by the renormalisation procedure in order to get rid of the divergence, which apparently has no physical meaning. The possibility to express the
gravitational $c$-function completely in terms of the zero mode is interesting for itself and constitute the link of both definitions. This will be investigated more carefully in the future.

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A Notation

The two independent solutions $\varphi_{\text{reg}}(z)$ and $\varphi_{\text{irr}}(z)$ of

$$\left[-\frac{h d^2}{dz^2} + (V_{\text{QM}} - k^2)\right] \varphi(z) = 0$$

with

$$V_{\text{QM}}(z) = \frac{\lambda^2 - 1/4}{z^2} + U(z)$$

(19)

and $\lambda = \frac{d}{2}$ are distinguished by their power behaviour at $z = 0$. The regular solution is uniquely defined by the asymptotic behaviour at zero.

$$\varphi_{\text{reg}}(z) = z^{\lambda+1/2} \cdot (1 + \mathcal{O}(1))$$

Instead the irregular solution is not uniquely defined by the asymptotic behaviour at zero. One can add an arbitrary multiple of $\varphi_{\text{reg}}(z)$ without changing the asymptotics. We choose the following combination:

$$\varphi_{\text{irr}}(z) = z^{-\lambda+1/2} \cdot (1 + \mathcal{O}(1)) + \frac{(-1)^{\lambda+1}}{2^{2\lambda-1} \Gamma(\lambda) \Gamma(\lambda+1)} \cdot q^{2\lambda} \cdot \left[ \ln(qx) + \text{const} \right] \cdot \varphi_{\text{reg}}(z)$$

The value of the dummy “const” depends on $\lambda$ and for $\lambda = 2$ it becomes $\ln 2 + \gamma - 3/4$. It is chosen in order to reproduce the $K_2(qx)$-Bessel function in the case $U(z) = 0$. The term $\ln q$ is responsible for the splitting of $\mathcal{F}(q)$ in the sum of two terms (see eq. 12). The dummy “const” can’t be used to kill the “massless pole”.

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