Equivalent T-Q relations and exact results for the open TASEP

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Abstract. Starting from the Bethe ansatz solution for the open totally asymmetric simple exclusion process (TASEP), we compute the largest eigenvalue of the deformed Markovian matrix, in exact agreement with results obtained by the matrix ansatz. We also compute the eigenvalues of the higher conserved charges. The key step is to find a simpler equivalent T-Q relation, which is similar to the one for the TASEP with periodic boundary conditions.

Keywords: exact results, exclusion processes, integrable spin chains and vertex models, quantum integrability (Bethe ansatz)
1. Introduction

Numerous analytical results have been obtained in non-equilibrium statistical mechanics by using methods that were originally developed for quantum integrable systems. Indeed, a connection between the Markovian matrix of the asymmetric simple exclusion process (ASEP) and a Hamiltonian describing an integrable quantum spin chain has been pointed out in [23]. The Bethe ansatz, which was invented to compute the spectrum of quantum models, can therefore also be used in the context of exclusion processes. For example, the Bethe ansatz was used to compute the current fluctuations [11, 19, 21, 22] and their spectral gaps [12, 20] for various exclusion processes.

Another method, called the matrix ansatz or DEHP method [10], was specially developed to compute the stationary states as well as various correlation functions. This approach was subsequently used to compute the fluctuations of the current for the totally ASEP (TASEP) with open boundaries [15] and for the ASEP [13]. The Bethe ansatz was not used for such computations since the necessary generalization of the Bethe ansatz for generic boundary conditions had not been developed until recently [1, 2, 4, 5, 7, 14, 18, 25, 27]. Based on these generalizations, the Bethe ansatz for the TASEP with open boundary conditions was formulated in [6], and for the ASEP in [26]. The main purpose of this letter is to use the recent Bethe ansatz results to recover the analytical results obtained by matrix ansatz. To this end, we map the so-called T-Q relation obtained in [6] for the open TASEP to a simpler equivalent T-Q relation, which is similar to the one for the TASEP with periodic boundary conditions. Then, using the same approach developed in [11], we recover the matrix ansatz results of [15]. Remarkably, the two distinct T-Q relations describe exactly the same transfer-matrix
eigenvalue (‘T’), for any value of length \( L \). It would be very interesting to find more such examples.

The plan of the paper is as follows. In section 2.1 we recall the definition of the open TASEP model, and its one-parameter (\( \mu \)) deformation (deformed Markovian matrix), which allows one to compute the fluctuations of the current. The transfer matrix associated to this model as well as its eigenvalues expressed in terms of Bethe roots (i.e. the T-Q relation) are briefly reviewed in section 2.2. A perturbative approach for computing the eigenvalue for small values of \( \mu \) is presented in section 2.3. The main result of this paper is presented in section 3.1, where a second T-Q relation is introduced, which reproduces the largest eigenvalue of the deformed Markovian matrix. In section 3.2, we show how the matrix ansatz results of [15] can be recovered and generalized from these new Bethe equations.

2. Bethe ansatz for the open TASEP

2.1. The open TASEP model

The TASEP on a segment of \( L \) sites in contact with particle reservoirs at the two ends is a stochastic model, which evolves in time according to the following rules: during any time interval \( dt \), each particle jumps with probability \( dt \) to the neighboring site on its right if that site is empty, enters at site 1 (if no particle is present there) with probability \( \alpha dt \), or exits from site \( L \) with probability \( \beta dt \). This process is ‘totally asymmetric’, since particles cannot jump to the left; ‘simple’, since particles can jump no more than 1 site in the time \( dt \); and ‘exclusive’, since a site cannot be occupied by more than one particle. A schematic representation of the dynamical rules is displayed in figure 1.

The probabilities \( P_t(C) \) of finding the system in the configuration \( C \) at time \( t \) satisfy

\[
\frac{d}{dt} P_t(C) = \sum_{C'} M(C, C') P_t(C'),
\]

(2.1)

where, for \( C' \neq C \), \( M(C, C')dt \) is the probability to go from \( C' \) to \( C \) during the interval \( dt \) and \( M(C, C) = -\sum_{C' \neq C} M(C', C) \). In a suitable basis, the Markovian matrix for the TASEP is a matrix acting on \((\mathbb{C}^2)^\otimes L\)

\[
M = B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L,
\]

(2.2)

where the subscripts indicate on which space the following matrices \( w, B \) and \( \overline{B} \) act non trivially.

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\[ B = \begin{pmatrix} -\alpha & 0 \\ \alpha & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \overline{B} = \begin{pmatrix} 0 & \beta \\ 0 & -\beta \end{pmatrix}. \quad (2.3) \]

We are in fact more interested in a deformation of the Markovian matrix (2.2)

\[ M(\mu) = B_1(\mu) + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L \quad \text{where} \quad B(\mu) = \begin{pmatrix} -\alpha & 0 \\ \alpha e^\mu & 0 \end{pmatrix}, \quad (2.4) \]

and \( \mu \) is a real parameter. Evidently, \( M(\mu) \to M \) as \( \mu \to 0 \). This deformed Markovian matrix is important, since its largest eigenvalue \( \lambda(\mu) \) can be used to determine the fluctuations of the current entering in the system [17]. The latter has been computed previously in [15] by using a generalization of the matrix ansatz [10]. We shall recover this result (and generalize it) by using the Bethe ansatz solution discovered recently in [6] following the lines of [11], where similar computations have been done for a periodic TASEP.

2.2. The T-Q relation for the open TASEP

As usual in the context of the algebraic Bethe ansatz, instead of diagonalizing only the Markovian matrix, we diagonalize a transfer matrix depending on a parameter \( x \) (called the spectral parameter) from which we can recover the (deformed) Markovian matrix. The transfer matrix associated to the deformed Markovian matrix (2.4) is given by [6]

\[ t(x) = \text{tr}_0 \left( \tilde{\mathbf{K}}_0(x) \ R_{0L}(x) \ldots R_{01}(x) \ K_0(x) \ R_{10}(x) \ldots R_{L0}(x) \right), \quad (2.5) \]

where

\[ \tilde{\mathbf{K}}(x) = \frac{1}{1 + xb} \begin{pmatrix} 1 & 1 \\ 0 & xb \end{pmatrix}, \quad R(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 1 & 1 - x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad K(x) = \begin{pmatrix} \frac{(a+x)x}{xa+1} & 0 \\ 0 & \frac{e^\mu (1-x^2)}{xa+1} \end{pmatrix}, \quad (2.6) \]

and the parameters \( a \) and \( b \) above are related to the boundary rates \( \alpha \) and \( \beta \) as follows

\[ a = \frac{1}{\alpha} - 1 \quad \text{and} \quad b = \frac{1}{\beta} - 1. \quad (2.7) \]

A key feature of the transfer matrix is its commutativity property

\[ [t(x), t(y)] = 0, \quad (2.8) \]

which implies that the corresponding eigenvalues \( \Lambda(x) \) have the form

\[ \Lambda(x) = \frac{S(x)}{(1 + ax)(1 + bx)}, \quad (2.9) \]

where \( S(x) \) is a polynomial in \( x \) of order \( L + 2 \). (The factors in the denominator in (2.9) are due to corresponding factors in the definitions of \( \tilde{\mathbf{K}}(x) \) and \( K(x) \) (2.6).) The proof of (2.8) given in [24] does not apply here since \( R^\dagger \) is not invertible; nevertheless, an alternative proof is available [9].

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The deformed Markovian matrix (2.4) is recovered from the transfer matrix (2.5) as follows

\[ M(\mu) = -\frac{1}{2} \frac{d}{dx} t(x) \bigg|_{x=1}. \tag{2.10} \]

Therefore, once we know the eigenvalues of the transfer matrix, we readily obtain the eigenvalues of the deformed Markovian matrix,

\[ \lambda(\mu) = -\frac{1}{2} \frac{d}{dx} \Lambda(x) \bigg|_{x=1}. \tag{2.11} \]

The higher charges defined by

\[ H^r = (-1)^r \frac{d^r}{dx^r} \ln(t(x)) \bigg|_{x=1}, \quad r = 1, 2, \ldots \tag{2.12} \]

commute among themselves as a further consequence of the commutativity property (2.8). Since \( t(1) = 1 \), we see that \( H^1 = M(\mu) \).

In [6], the eigenvalues of the transfer matrix \( t(x) \) have been obtained by using the modified algebraic ansatz [1]:

\[ \Lambda(x) = x^{L+1} \frac{b + x}{bx + 1} \prod_{k=1}^{L} \frac{x u_k - 1}{u_k - x} - \frac{(x - 1)^2 (x^2 - 1)}{(ax + 1)(bx + 1)} \prod_{k=1}^{L} \frac{u_k}{u_k - x}, \tag{2.13} \]

where the parameters \( \{ u_k \mid k = 1, 2, \ldots, L \} \), called Bethe roots, must satisfy the following Bethe equations

\[ (a u_j - e^\mu)(u_j - 1)^{2L} (u_j^2 - 1) = u_j^{L+1}(au_j + 1)(u_j + b) \prod_{k=1}^{L} \left( u_j - \frac{1}{u_k} \right) \quad \text{for} \quad j = 1, 2, \ldots, L. \tag{2.14} \]

The eigenvectors are also computed in [6], but in this letter we focus only on the eigenvalues.

Let us introduce the following polynomial w.r.t. the spectral parameter \( Q(x) \)

\[ Q(x) = \prod_{k=1}^{L} \left( 1 - \frac{x}{u_k} \right), \tag{2.15} \]

whose the zeros are the Bethe roots. This polynomial is usually called the \( Q \)-polynomial and allows us to rewrite relation (2.13) for the eigenvalues as follows

\[ \Lambda(x) Q(x) = x^{2L+1} \frac{b + x}{bx + 1} Q(1/x) - \frac{(x - 1)^2 (ax + e^\mu)(x^2 - 1)}{(ax + 1)(bx + 1)}. \tag{2.16} \]

This relation is called the (functional) T-Q relation.

Although the transfer matrix (2.5) has \( 2^L \) eigenvalues, we shall henceforth restrict our attention to only one of them, namely, the (unique) eigenvalue that tends to 1 as \( \mu \) tends to 0

\[ \Lambda(x) \bigg|_{\mu=0} = 1. \tag{2.17} \]
We are interested in this eigenvalue because it corresponds to the eigenvalue \( \lambda(\mu) \) of the deformed Markovian matrix \( M(\mu) \) with the largest real part, which is the only eigenvalue of \( M(\mu) \) that tends to 0 as \( \mu \) tends to 0, see (2.11).

2.3. Expansion for small \( \mu \)

The fact (2.17) that \( \Lambda(x) \) has a simple limit for \( \mu = 0 \) suggests to solve the T-Q relation perturbatively in terms of \( \mu \). This idea has been exploited previously for the periodic exclusion processes [19, 21].

Let us introduce the following expansions, for small \( \mu \),

\[
\Lambda(x) = 1 + \mu \Lambda^{(1)}(x) + \mu^2 \Lambda^{(2)}(x) + \ldots \tag{2.18}
\]

\[
Q(x) = Q^{(0)}(x) + \mu Q^{(1)}(x) + \mu^2 Q^{(2)}(x) + \ldots \tag{2.19}
\]

where

\[
Q^{(0)}(x) = \sum_{k=0}^{L} q_k^{(0)} x^k, \quad Q^{(j)}(x) = \sum_{k=1}^{L} q_k^{(j)} x^k, \tag{2.20}
\]

\[
\Lambda^{(j)}(x) = \frac{1}{(ax+1)(bx+1)} \sum_{k=0}^{L+2} \ell_k^{(j)} x^k. \tag{2.21}
\]

We can easily show from the T-Q relation that \( \Lambda(0) = e^\mu \), which implies \( \ell_0^{(j)} = \frac{1}{j!} \). The T-Q relation (2.16) implies

\[
Q^{(0)}(x) = x^{2L+1} \frac{b+x}{bx+1} Q^{(0)}(1/x) - \frac{(x-1)^{2L}(x^2-1)}{bx+1}, \tag{2.22}
\]

\[
Q^{(1)}(x) + Q^{(0)}(x)\Lambda^{(1)}(x) = x^{2L+1} \frac{b+x}{bx+1} Q^{(1)}(1/x) - \frac{(x-1)^{2L}(x^2-1)}{(ax+1)(bx+1)}, \ldots \tag{2.23}
\]

and so on. With these choices of expansions, these equations have a unique solution, which corresponds to the largest eigenvalue \( \lambda(\mu) \) since one gets \( \Lambda(x) \to 1 \) as \( \mu \to 0 \). In particular, we can find a closed formula for \( Q^{(0)}(x) \):

\[
Q^{(0)}(x) = 2 \sum_{k=0}^{L} (-x)^k \sum_{p=0}^{k} b^{k-p} \frac{L-p+1}{2L-p+2} \binom{2L+1}{p}. \tag{2.24}
\]

Let us emphasize that \( Q^{(0)}(x) \) corresponds to the \( Q \)-polynomial for the non-deformed (\( \mu = 0 \)) Markovian model, and its \( L \) roots are the Bethe roots solution of the Bethe equation (2.14) for \( \mu = 0 \), corresponding to the eigenvalue \( \Lambda(x) = 1 \). Using this explicit formula for the \( Q \)-polynomial, we can easily find the Bethe roots for very large systems. We display an example in figure 2. Let us recall that a surprising connection between the \( Q \)-polynomial \( Q^{(0)}(x) \) and the normalization of the stationary state has been discovered in [8], which relates the roots of the \( Q \)-polynomial \( Q^{(0)}(x) \) and the Lee–Yang zeros studied in [3].

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3. Equivalent T-Q relation and exact eigenvalues of the higher charges

3.1. Equivalent T-Q relation

In the previous section, we reviewed the T-Q relation for the open TASEP. We then explained how to expand the eigenvalue with the property (2.17) for small $\mu$. Remarkably, there exists another T-Q relation which gives exactly the same eigenvalue $\Lambda(x)$, for any value of $L$. Indeed, the solution $\Lambda(x)$ of the T-Q relation (2.16) with the expansions (2.18)–(2.21), is also a solution of the following T-Q relation

$$\Lambda(x) Q(x) = xL(x+b)(x+a)e^{-\mu} + \frac{Q(0)(1-x)^{2L+2}(x+1)^2}{(ax+1)(bx+1)}e^{\mu},$$

(3.1)

where the expansion of $Q(x)$ is given by

$$Q(x) = Q^{(0)}(x) + \mu Q^{(1)}(x) + \mu^2 Q^{(2)}(x) + \ldots$$

(3.2)

$$Q^{(0)}(x) = \sum_{k=0}^{L+2} q^{(0)}_k x^k , \quad Q^{(j)}(x) = \sum_{k=0}^{L+1} q^{(j)}_k x^k .$$

(3.3)

If we add the additional requirement $\ell^{(j)}_0 = \frac{1}{j!}$ (see below (2.21)), then the solution of the T-Q relation (3.1) is unique. We did not succeed in proving the equivalence between the T-Q relations (2.16) and (3.1); however, we have strong evidence that this equivalence holds. For $a = b = 0$, we get the same expansions of $\Lambda(x)$ up to the order $\mu^{30}$ and for $L = 1, 2, \ldots, 35$ from both T-Q relations. Similar positive results have been obtained.
for $a \neq 0$ and $b \neq 0$, up to the order $\mu^5$ and $L = 1, \ldots, 5$. Moreover, both T-Q relations with $a = b = 0$ have the same exact solution $\Lambda(x)$ for $L = 0$ and $L = 1$

$$L = 0 : \Lambda(x) = 1 - (\mu^2 - 1)(x^2 - 1), \quad Q(x) = 1, \quad \overline{Q}(x) = x^2 + e^{-\mu} - 1, \quad (3.4)$$

$$L = 1 : \quad \Lambda(x) = 1 + (\mu^2 - 1)(\mu^2 - 1)(x^2 - 1), \quad Q(x) = 1 - x - e^{-\mu}x, \quad \overline{Q}(x) = (x^2 - 1)(x - 1) + e^{-\mu}x + e^{-\mu/2}(x^2 - 1). \quad (3.5)$$

Let us emphasize that, although $\Lambda(x)$ is the same for both T-Q relations, the $Q$-polynomials $Q(x)$ and $\overline{Q}(x)$ are different. For example, one can see that

$$\overline{Q}^{(0)}(x) = x^L(x + a)(x + b), \quad (3.6)$$

which is different from $Q^{(0)}(x)$ given by (2.24). See also (3.4) and (3.5).

As usual, we call the roots of $\overline{Q}(x)$ Bethe roots, and we denote them by $\overline{\nu}_j$. Due to the expansion (3.3) and of the explicit form (3.6) of $\overline{Q}^{(0)}(x)$, we know that

$$\overline{Q}(x) = \prod_{k=1}^{L+2}(x - \overline{\nu}_k). \quad (3.7)$$

By knowing the explicit expression (3.6) of $\overline{Q}^{(0)}(x)$, we deduce that $L$ Bethe roots $\overline{\nu}_k$ tends to 0 when $\mu \to 0$, one tends to $-a$, and the last one tends to $-b$.

From the T-Q relation (3.1), we deduce that these Bethe roots satisfy the following Bethe equations, for $j = 1, 2, \ldots, L + 2$,

$$\overline{\nu}_j^L(\overline{\nu}_j + b)(\overline{\nu}_j + a)(a\overline{\nu}_j + 1)(b\overline{\nu}_j + 1) = (-1)^{L+1}e^{2\mu}(1 - \overline{\nu}_j)^{2L+2}(\overline{\nu}_j + 1)^2 \prod_{k=1}^{L+2}\overline{\nu}_k. \quad (3.8)$$

At this point, the new T-Q relation (3.1) seems artificial. However, for $a = b = 0$, it becomes

$$\Lambda(x) \overline{Q}(x) = x^{L+2}e^{-\mu} + \overline{Q}(0)(1 - x)^{2L+2}(x + 1)^2 e^\mu, \quad (3.9)$$

and we recognize the T-Q relation associated to the diagonalization of the following transfer matrix

$$t(x) = \text{tr}_0\left( Z_0 R_{0,2L+4}(-x) R_{0,2L+3}(-x) R_{0,2L+2}(x) R_{0,2L+1}(x) \ldots R_{01}(x) \right), \quad (3.10)$$

where $Z = \text{diag}(e^{-\mu}, e^{\mu})$. The matrix $\overline{t}(x)$ is evidently a transfer matrix for a TASEP with $2L + 4$ sites, quasi-periodic boundary conditions (with twist matrix $Z$), and inhomogeneities at sites $2L + 4$ and $2L + 3$ (responsible for the minus signs in the corresponding $R$-matrices). The T-Q relation (3.9) corresponds to the sector with $L + 2$ particles.

We note that the T-Q relation (3.1) may be obtained as a limit of the T-Q relation obtained in [16] by functional Bethe ansatz for the ASEP.

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3.2. Exact expansions for the eigenvalues of the higher charges

Since the T-Q relation (3.1) is closely related to the one for the TASEP with periodic boundary conditions, the computations done previously for the periodic case can be generalized to obtain results for the T-Q relation (3.1) and, by consequence, for the TASEP with open boundaries. In particular, we want to generalize the results of [11] to obtain exact expansions for the eigenvalues of the higher conserved charges (2.12).

3.2.1. Case with \( a = b = 0 \). By setting \( a = b = 0 \) and by performing the change of variables \( \overline{u}_j = 1 - \frac{z_j}{\Delta} \) with \( \Delta L + 2 = e^{\epsilon} \), the Bethe equation (3.8) become, for \( j = 1, 2, \ldots, L + 2 \),

\[
(\Delta - z_j)^{L+2} + z_j^{2L+2}(2\Delta - z_j)^2 \prod_{k=1}^{L+2} (z_k - \Delta) = 0. \tag{3.11}
\]

In this case, all the Bethe roots \( z_j \) tend to 1 when \( \mu \to 0 \).

From the T-Q relation (2.16), we deduce also that \( \Lambda(1) = 1 \). By setting this result in the T-Q relation (3.9), and by using the explicit expression (3.7) of \( Q(x) \), one obtains

\[
\prod_{k=1}^{L+2} z_k = 1. \tag{3.12}
\]

In [11], the authors succeeded in computing a development for the largest eigenvalue for the periodic TASEP starting from relations similar to (3.11) and (3.12). We shall adapt their approach to compute the eigenvalues of the first \( 2L + 1 \) conserved charges (2.12)

\[
I^{(r)} = \frac{(-1)^r}{2(r - 1)!} \frac{d^r}{dx^r} \ln \left( \Lambda(x) \right) \bigg|_{x=1} \quad \text{for} \quad r = 1, 2, \ldots, 2L + 1 \tag{3.13}
\]

\[
= \sum_{j=1}^{L+2} h^{(r)}(z_j) \quad \text{where} \quad h^{(r)}(z) = \frac{1}{2} \left( \frac{\Delta^r - z^r}{z^r} \right). \tag{3.14}
\]

The equality (3.14) has been obtained by using the T-Q relation (3.9). Note that \( I^{(1)} \) corresponds to the largest eigenvalue for the deformed Markovian matrix.

Let us introduce the following polynomial

\[
P(z) = (\Delta - z)^{L+2} - A z^{2L+2} (2\Delta - z)^2. \tag{3.15}
\]

Then, by Cauchy’s theorem, we obtain the following expression for the eigenvalues

\[
I^{(r)} = \sum_{j=1}^{L+2} h^{(r)}(z_j) = \oint_{|z-\Delta|=\epsilon} \frac{dz}{2i\pi} h^{(r)}(z) \frac{P'(z)}{P(z)}, \tag{3.16}
\]

where \( z_j \) are the \( L + 2 \) solutions of \( P(z) = 0 \) satisfying \( z_j \xrightarrow{A \to 0} \Delta \). By using the explicit form of \( P(z) \), this expression becomes

\[
I^{(r)} = -\oint_{|z-\Delta|=\epsilon} \frac{dz}{2i\pi} \frac{(\Delta^r - z^r)(\Delta - z)^{L+1}}{2z^{r+1}(2\Delta - z)} \frac{(L + 2)z^2 + 4\Delta(L + 1)(\Delta - z)}{(\Delta - z)^{L+2} - A z^{2L+2}(2\Delta - z)^2}. \tag{3.17}
\]

We have used the fact that \( P(z_j) = 0 \) to simplify the \( A \) in the numerator.

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By expanding w.r.t. $A$ in the integrand of (3.17) and performing the integrals, we eventually obtain, for $r = 1, 2, \ldots, 2L + 1$,

$$I^{(r)} = -\frac{r}{2} \sum_{k=1}^{\infty} B^k \frac{(2k)!((2k(L + 1) - r - 1)!}{k(k(L + 2) - 1)!} \sum_{p=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} \binom{r-1}{2p} \frac{(-1)^p}{(k-p)!(k(L + 1) - r + p)!}$$  \hspace{1cm} (3.18)

with $B = -A(-\Delta)^{L+2}$. In particular, for $r = 1$,

$$I^{(1)} = -\frac{1}{2} \sum_{k=1}^{\infty} B^k \frac{(2k)!(2k(L + 1) - 2)!}{k(k(L + 2) - 1)!} \frac{1}{k!(k(L + 1) - 1)!}.$$  \hspace{1cm} (3.19)

Similarly, with the help of (3.12), one gets

$$0 = \sum_{j=1}^{L+2} \ln(z_j) = -\oint_{|z-\Delta|=\epsilon} \frac{dz}{2i\pi} \ln(z) \frac{(\Delta - z)^{L+1}}{z(2\Delta - z)} \frac{(L + 2)z^2 + 4\Delta(L + 1)(\Delta - z)}{(\Delta - z)^{L+2} - Az^{2L+2}(2\Delta - z)^2}$$  \hspace{1cm} (3.20)

and, by expanding w.r.t. $A$ as before, we obtain

$$\mu = -\sum_{k=1}^{\infty} B^k \frac{(2k)!(2k(L + 1))!}{2k(k(L + 1))!k!}.$$  \hspace{1cm} (3.21)

Expansions (3.19) and (3.21) of $I^{(1)}$ and $\mu$ in terms of $B$ reproduce exactly the matrix ansatz result of [15]. We have generalized their result by computing also expansions for the eigenvalues $I^{(r)}$ of the higher conserved charges.

3.2.2. Case with $a \neq 0$ and $b \neq 0$. This case can be treated as previously. The main difference is that $L$ Bethe roots $z_j$ tend to 1, while the last two Bethe roots tend to $1 + a$ and $1 + b$. We must take in account this difference when we define the contour of integration. Then, the previous computation can be done for this more general case by using instead the following polynomial

$$P(z) = (\Delta - z)^{L}(\Delta(1 + a) - z)(\Delta(1 + b) - z)(\Delta(1 + a) - az)(\Delta(1 + b) - bz) - A\Delta^2z^{2L+2}(2\Delta - z)^2,$$  \hspace{1cm} (3.22)

in view of the Bethe equation (3.8). However, the integrals are significantly more complicated, and we have not managed to obtain comparably simple results for general values of $r$.

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