Soliton Scattering in Noncommutative Spaces

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Abstract

We discuss exact multi-soliton solutions to integrable hierarchies on noncommutative space-times in diverse dimension. The solutions are represented by quasi-determinants in compact forms. We study soliton scattering processes in the asymptotic region where the configurations could be real-valued. We find that the asymptotic configurations in the soliton scatterings can be all the same as commutative ones, that is, the configuration of $N$-soliton solution has $N$ isolated localized lump of energy and each solitary wave-packet lump preserves its shape and velocity in the scattering process. The phase shifts are also the same as commutative ones. As new results, we present multi-soliton solutions to noncommutative anti-self-dual Yang-Mills hierarchy and discuss 2-soliton scattering in detail.

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1 Introduction

Noncommutative (NC) extension of integrable systems has attracted many researchers in both mathematics and physics for long time. It would be a recent breakthrough that exact multi-soliton solutions to the noncommutative KP hierarchy are constructed in terms of quasi-determinants [11]. The quasideterminants are first introduced in 1991 by Gelfand and Retakh [14] in the context of noncommutative generalization of theory of determinants of matrices. It has been found that the quasideterminants play important roles in construction of exact solutions to noncommutative integrable systems. (See e.g. [10, 12, 15, 16, 17, 18, 19, 20, 21, 27, 28, 37, 48, 50] and references therein.) It is interesting that quasideterminants simplify proofs in commutative theories.

For the last several years, extension of integrable systems to noncommutative space-times has been studied intensively. This can be realized by using the star-product. From now on, the word “noncommutative” is assumed to refer to generalization to noncommutative spaces, not to non-abelian and so on. For surveys on integrable systems in noncommutative space-times, see e.g. [9, 22, 24, 31, 35, 41, 54].

N-soliton solutions are stable in the sense that the configuration has $N$ localized lump of energy where shape and velocity of each localized lump keep intact in the scattering process. Existence of them closely relates to existence of infinite conserved quantities or infinite dimensional symmetry. Hence it is worth studying the stability of the noncommutative soliton dynamics. In the star-product formalism, space-time coordinates and functions take c-number values and scattering dynamics can be clarified explicitly. However, there are few studies on it [8, 39, 47].

In this paper, we study exact multi-soliton solutions to noncommutative integrable hierarchies and the asymptotic behavior of them where the asymptotic configurations are real-valued. We focus on the noncommutative Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP) and anti-self-dual Yang-Mills (ASDYM) equations in $(1+1)$, $(2+1)$ and 4 dimension, respectively. We find that the asymptotic behavior in soliton scatterings is all the same as commutative ones, that is, the $N$-soliton solution has $N$ isolated localized lump of energy and each wave-packet lump preserves its shape and velocity in the scattering process. The phase shift is also the same as commutative one. The analysis of 2-soliton scattering in the noncommutative ASDYM equation is new. Property of the quasideterminants and the star products plays crucial roles.

This paper is organized as follows. In section 2, we give a brief introduction to noncommutative field theory in the star-product formalism. In section 3, we make a brief review of the quasi-determinants. In section 4, we define noncommutative integrable hierarchy and construct exact multi-soliton solutions by using quasideterminants. Asymptotic
behaviors of noncommutative KdV and KP solitons are discussed in the star-product formalism [27]. In section 5, we define noncommutative ASDYM hierarchy and give exact solutions to it in terms of quasideterminants. Asymptotic behavior of 2-soliton solutions are discussed. This section gives new results.

2 Integrable Equations in Noncommutative Spaces

Noncommutative spaces are defined by the noncommutativity of the coordinates:

\[ [x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{2.1} \]

where the constant \( \theta^{\mu\nu} \) is called the noncommutative parameter. If the coordinates are real, noncommutative parameters should be real because of hermicity of the coordinates. We note that the noncommutative parameter \( \theta^{\mu\nu} \) is anti-symmetric with respect to \( \mu \) and \( \nu \) which implies that the rank of it is even. In \((1+1)\)-dimension with the coordinate \((t, x)\), there is unique choice of noncommutativity : \([x, t] = i\theta\) which is space-time noncommutativity. In \((2+1)\)-dimension with the coordinate \((t, x, y)\), there are essentially two kind of choices of noncommutativity, that is, space-space noncommutativity: \([x, y] = i\theta\) and space-time noncommutativity: \([x, t] = i\theta\) or \([y, t] = i\theta\).

Noncommutative field theories are given by the replacement of ordinary products in the commutative field theories with the star-products. The star-product is defined for ordinary fields. On flat spaces, it is represented explicitly by

\[
\begin{align*}
    f \star g(x) & := \exp \left( \frac{i}{2} \theta^{\mu\nu} \partial^{(x_1)}_{\mu} \partial^{(x_2)}_{\nu} \right) f(x_1)g(x_2) \bigg|_{x_1=x_2=x} \\
    &= f(x)g(x) + \frac{i}{2} \theta^{\mu\nu} \partial_{\mu} f(x) \partial_{\nu} g(x) + \mathcal{O}(\theta^2),
\end{align*}
\tag{2.2}
\]

where \( \partial^{(x)}_{\mu} := \partial/\partial x^{\mu} \). This is known as the Moyal product [42]. The ordering of fields in nonlinear terms are determined so that some structures such as gauge symmetries should be preserved.

The star-product has associativity: \( f \star (g \star h) = (f \star g) \star h \). It reduces to the ordinary product in the commutative limit: \( \theta^{\mu\nu} \to 0 \). In this sense, the noncommutative field theories are deformed theories from the commutative ones. The replacement of the product makes the ordinary spatial coordinates “noncommutative,” that is, \([x^\mu, x^\nu]_{\star} := x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu} \).

We note that the fields themselves take c-number values and the differentiation and the integration for them are the same as commutative ones.

Here is a gallery of noncommutative integrable equations. Time and spatial coordinates are denoted by \( t \) and \( x, y \), respectively.
• In (1 + 1) dimension:
  
  – Noncommutative KdV equation
  \[ \dot{u} = \frac{1}{4}u'''' + \frac{3}{4} (u' * u + u * u'), \]  
  \( (2.3) \)
  
  where \( \dot{u} := \partial f / \partial t, \ u' := \partial f / \partial x, \ u'' := \partial^2 f / \partial x^2 \) and so on. 2-soliton dynamics is discussed \[8\].

  – Noncommutative Boussinesq equation
  \[ 3\dot{u} + u'''' + 2(u * u)'' - 2[u, \partial^{-1} \dot{u}]_{\times} = 0. \]  
  \( (2.4) \)

  – Noncommutative Non-Linear Schrödinger equation
  \[ i \dot{\psi} = \psi'' - 2\varepsilon \psi * \bar{\psi} * \psi. \]  
  \( (2.5) \)

  – Noncommutative modified KdV equation
  \[ \dot{v} = \frac{1}{4}v'''' - \frac{3}{4} (v * v * v' + v' * v * v). \]  
  \( (2.6) \)

  This is connected with the noncommutative KdV equation via the noncommutative Miura map: \( u = v' - v^2 \) \[8\].

  – Noncommutative Burgers equation
  \[ \dot{u} = u'' + 2u * u' \quad \text{or} \quad \dot{u} = u'' - 2u' * u. \]  
  \( (2.7) \)

  These can be linearized by the noncommutative Cole-Hopf transformations: \( u = \psi^{-1} * \psi' \) or \( u = -\psi' * \psi^{-1} \), respectively. (See e.g. \[30\].) 2 shock-wave dynamics is discussed \[39\].

• In (2 + 1) dimension:
  
  – Noncommutative KP equation
  \[ u_t = \frac{1}{4}u_{xxx} + \frac{3}{4} (u_x * u + u * u_x) + \partial^{-1} u_{yy} - [u, \partial^{-1} u_y]_{\times}. \]  
  \( (2.8) \)

  where the subscripts denote partial derivatives and \( \partial^{-1} f(x) = \int^x dx' f(x') \). This reduces to the KdV equation in (1 + 1)-dimension by \( \partial_y = 0 \). 2-soliton dynamics is discussed in detail \[47\].
Noncommutative Zakharov system \[26\]

\[
i\psi_t = \psi_{xy} - \varepsilon \psi \star \partial_x^{-1} \partial_y (\bar{\psi} \star \psi) - \varepsilon \partial_x^{-1} \partial_y (\psi \star \bar{\psi}) \star \psi,
\]

where \(\varepsilon = \pm 1\). This reduces to the noncommutative Non-Linear Schrödinger (NLS) equation in (1 + 1)-dimension by \(x = y\).

Noncommutative Bogoyavlenskii-Calogero-Schiff equation \[55\]

\[
4 u_t = u_{xxy} + 2(u \star u)_y + (u_y \star \partial_x^{-1} u + \partial_x^{-1} u \star u_y) + \partial_x^{-1}[u, \partial_x^{-1}[u, \partial_x^{-1} u_y]]_*. 
\]

This reduces to the noncommutative KdV equation by \(x = y\).

Noncommutative Davey-Stewartson equation \[25\]

\[
\begin{aligned}
2i q_t &= (\partial_x^2 - \partial_y^2)q + R_1 \star q - q \star R_2, \\
2i r_t &= -(\partial_x^2 - \partial_y^2)r + R_2 \star r - r \star R_1,
\end{aligned}
\]

where \((\partial_x - i\partial_y)R_1 = -(\partial_x + i\partial_y)(q \star r), (\partial_x + i\partial_y)R_2 = (\partial_x - i\partial_y)(r \star q)\). This reduces to the noncommutative NLS equation in (1 + 1)-dimension by \(\partial_y = 0\) and \(R_1 = -q \star r, R_2 = r \star q, q = \psi, r = \bar{\psi}\).

In 4-dimension, there is an important integrable equation: noncommutative anti-self-dual Yang-Mills (ASDYM) equation

\[
F^w_{z\bar{w}} = 0, \ F^w_{\bar{z}w} = 0, \ F^w_{zw} + F^w_{w\bar{z}} = 0, \quad (2.9)
\]

where \(z\) and \(w\) denote local coordinates of the 4-dimensional Euclidean plane \(\mathbb{C}^2\), and \(F^w_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_\star\) denotes the field strength. There are two choices of rank 2 and 4 with respect to noncommutativity. We note that the noncommutative ASDYM equation gives rise by reduction to various noncommutative lower-dimensional integrable equations including all equations above except for noncommutative Burgers equation. More examples are summarized in \[25, 26\], which would be evidence for noncommutative version \[29\] of the Ward conjecture \[58\]. (See also \[1, 40\].)

3 Review of Quasi-determinants

In this section, we briefly review quasi-determinants introduced by Gelfand and Retakh \[14\] and present a few properties of them which play important roles in the following sections. The detailed discussion is seen in e.g. \[13\].
Quasi-determinants are not just a generalization of usual commutative determinants but rather related to inverse matrices. From now on, we assume existence of the inverses in any case.

Let $A = (a_{ij})$ be a $N \times N$ matrix and $B = (b_{ij})$ be the inverse matrix of $A$, that is, $A \star B = B \star A = 1$. In this paper, all products of matrix elements are assumed to be star-products.

Quasi-determinants of $A$ are defined formally as the inverse of the elements of $B = A^{-1}$:

$$|A|_{ij} := b_{ji}^{-1}.$$  \hspace{1cm} (3.1)

In the commutative limit, this reduces to

$$|A|_{ij} \xrightarrow{\theta \to 0} (-1)^{i+j} \frac{\det A}{\det A^i_j},$$  \hspace{1cm} (3.2)

where $A^i_j$ is the matrix obtained from $A$ deleting the $i$-th row and the $j$-th column.

We can write down more explicit form of quasi-determinants. In order to see it, let us recall the following formula for the inverse $2 \times 2$ block matrix:

$$\begin{bmatrix} A & B \\ C & d \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1} \star B \star S^{-1} \\ -S^{-1} \star C \star A^{-1} & S^{-1} \end{bmatrix},$$

where $A$ is a square matrix and $d$ is a single element and $S := d - C \star A^{-1} \star B$ is called the Schur complement. We note that any matrix can be decomposed as a $2 \times 2$ matrix by block decomposition where one of the diagonal parts is $1 \times 1$. We note that by choosing an appropriate partitioning, any element in the inverse of a square matrix can be expressed as the inverse of the Schur complement. Hence quasi-determinants can be defined iteratively by:

$$|A|_{ij} = a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} \star (|A^{i'}|_j^{j'})^{-1} \star a_{j'j}$$

$$= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} \star (|A^i|_j^{j'})^{-1} \star a_{j'j}. \hspace{1cm} (3.3)$$

It is convenient to represent the quasi-determinant as follows:

$$|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & & a_{ij} & & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}. \hspace{1cm} (3.4)$$
Examples of quasi-determinants are, for a $1 \times 1$ matrix $A = a$

$$|A| = a,$$

and for a $2 \times 2$ matrix $A = (a_{ij})$

$$|A|_{11} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12} \star a_{22}^{-1} \star a_{21}, \quad |A|_{12} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11} \star a_{22}^{-1} \star a_{22},$$

$$|A|_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{21} - a_{22} \star a_{12}^{-1} \star a_{11}, \quad |A|_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{22} - a_{21} \star a_{11}^{-1} \star a_{12},$$

and for a $3 \times 3$ matrix $A = (a_{ij})$

$$|A|_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} - (a_{12}, a_{13}) \star \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \star \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix}$$

$$= a_{11} - a_{12} \star \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} \star a_{21} - a_{12} \star \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} \star a_{31}$$

$$- a_{13} \star \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} \star a_{21} - a_{13} \star \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} \star a_{31},$$

and so on.

Quasideterminants have various interesting properties similar to those of determinants. The following ones are relevant to the discussion on soliton scattering.

**Proposition 3.1** [14] Let $A = (a_{ij})$ be a square matrix of order $n$.

(i) Permutation of Rows and Columns.

The quasi-determinant $|A|_{ij}$ does not depend on permutations of rows and columns in the matrix $A$ that do not involve the $i$-th row and $j$-th column.

(ii) The multiplication of rows and columns.

Let the matrix $M = (m_{ij})$ be obtained from the matrix $A$ by multiplying the $i$-th row by $f(x)$ from the left, that is, $m_{ij} = f \star a_{ij}$ and $m_{kj} = a_{kj}$ for $k \neq i$. Then

$$|M|_{kj} = \begin{cases} 
    f \star |A|_{ij} & \text{for } k = i \\
    |A|_{kj} & \text{for } k \neq i \text{ and } f \text{ is invertible}
\end{cases} \quad (3.5)$$

Let the matrix $N = (n_{ij})$ be obtained from the matrix $A$ by multiplying the $j$-th column by $f(x)$ from the right, that is, $n_{ij} = a_{ij} \star f$ and $n_{il} = a_{il}$ for $l \neq j$. Then

$$|N|_{il} = \begin{cases} 
    |A|_{ij} \star f & \text{for } l = j \\
    |A|_{il} & \text{for } l \neq j \text{ and } f \text{ is invertible}
\end{cases} \quad (3.6)$$
4 NC Integrable Hierarchy and Soliton Solutions

In this section, we give exact multi-soliton solutions to noncommutative integrable hierarchies in terms of quasi-determinants. In the commutative case, determinants of Wronski matrices play crucial roles. In the noncommutative case, quasi-determinants give a better formulation. We review foundation of the noncommutative KP hierarchy and the reduced hierarchies, so called noncommutative Gelfand-Dickey (GD) hierarchies, and present the exact multi-soliton solutions to them developed by Etingof, Gelfand and Retakh [11]. (See also [18])

An $N$-th order pseudo-differential operator $A$ is represented as follows

$$A = a_N \partial^N_x + a_{N-1} \partial^{N-1}_x + \cdots + a_0 + a_{-1} \partial^{-1}_x + a_{-2} \partial^{-2}_x + \cdots,$$

where $a_i$ is a function of $x$ associated with noncommutative associative products (here, the star products). When the coefficient of the highest order $a_N$ equals to 1, we call it monic. Here we introduce the following symbols:

$$A_{\geq r} := \partial^N_x + a_{N-1} \partial^{N-1}_x + \cdots + a_r \partial^r_x,$n
$$A_{\leq r} := A - A_{\geq r+1} = a_r \partial^r_x + a_{r-1} \partial^{r-1}_x + \cdots.$$

The action of a differential operator $\partial^n_x$ on a multiplicity operator $f$ is formally defined as the following generalized Leibniz rule:

$$\partial^n_x \cdot f := \sum_{i\geq 0} \binom{n}{i} (\partial^i_x f) \partial^{n-i},$$

where the binomial coefficient is given by

$$\binom{n}{i} := \frac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots1}.$$

We note that the definition of the binomial coefficient (4.5) is applicable to the case of negative $n$, which implies that the action of negative power of differential operators is defined.

The composition of pseudo-differential operators is also well-defined and the total set of pseudo-differential operators forms an operator algebra. For a monic pseudo-differential operator $A$, there exist the unique inverse $A^{-1}$ and the unique $m$-th root $A^{1/m}$ which commute with $A$. (These proofs are all the same as commutative ones as far as the commutative limit exists.) For more on pseudo-differential operators and Sato’s theory, see e.g. [3, 4, 7, 34].
4.1 Noncommutative KP and KdV hierarchies

In order to define the noncommutative KP hierarchy, let us introduce a monic pseudo-differential operator:

\[ L = \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots, \]

where the coefficients \( u_k (k = 2, 3, \ldots) \) are functions of infinite coordinates \( \vec{x} := (x_1, x_2, \ldots) \) with \( x_1 \equiv x \):

\[ u_k = u_k(x_1, x_2, \ldots). \]  

The noncommutativity is introduced into the coordinates \((x_1, x_2, \ldots)\) as Eq. (2.1) here.

In order to define the noncommutative KP hierarchy, let us introduce a differential operator \( B_m \) as follows:

\[ B_m := (L \ast \cdots \ast L) \geq 0 =: (L^m) \geq 0. \]  

The noncommutative KP hierarchy is defined as follows:

\[ \partial_m L = [B_m, L], \quad m = 1, 2, \ldots, \]

where the action of \( \partial_m := \partial/\partial x_m \) on the pseudo-differential operator \( L \) is defined by \( \partial_m L := [\partial_m, L] \), or \( \partial_m \partial^k_x = 0 \). The KP hierarchy gives rise to a set of infinite differential equations with respect to infinite kind of fields from the coefficients in Eq. (4.9) for a fixed \( m \). Hence it contains huge amount of differential (evolution) equations for all \( m \). The LHS of Eq. (4.9) becomes \( \partial_m u_k \) which shows a kind of flow in the \( x_m \) direction.

If we put the constraint \( (L^l) \leq 1 = 0 \) or equivalently \( L^l = B_l \) \( l = 2, 3, \cdots \) on the noncommutative KP hierarchy (4.9), we get a reduced noncommutative KP hierarchy which is called the \( l \)-reduction of the noncommutative KP hierarchy, or the noncommutative \( l \)KdV hierarchy, or the \( l \)-th noncommutative GD hierarchy. In particular, the 2-reduction of noncommutative KP hierarchy is just the noncommutative KdV hierarchy. We can easily show

\[ \frac{\partial u_k}{\partial x_{nl}} = 0, \]  

for all \( n, k \) because \( \partial L^l/\partial x_{nl} = [B_{nl}, L^l] = [(L^l)^n, L^l] = 0 \). This time, the constraint \( L^l = B_l \) gives simple relationships which make it possible to represent infinite kind of fields \( u_{l+1}, u_{l+2}, u_{l+3}, \ldots \) in terms of \( (l - 1) \) kind of fields \( u_2, u_3, \ldots, u_l \). (cf. Appendix A in [23].)

Let us see explicit examples.
• Noncommutative KP hierarchy

The coefficients of each powers of (pseudo-)differential operators in the noncommutative KP hierarchy (4.9) yield a series of infinite noncommutative “evolution equations,” that is, for \( m = 1 \)

\[
\partial_x^{1-k} \partial_1 u_k = u_k', \quad k = 2, 3, \ldots \quad \Rightarrow \quad x^1 \equiv x, \quad (4.11)
\]

for \( m = 2 \)

\[
\begin{align*}
\partial_x^{-1} \partial_2 u_2 & = u_2'' + 2u_3', \\
\partial_x^{-2} \partial_2 u_3 & = u_3'' + 2u_4' + 2u_2 \ast u_2' + 2[u_2, u_3]_*, \\
\partial_x^{-3} \partial_2 u_4 & = u_4'' + 2u_5' + 4u_3 \ast u_2' - 2u_2 \ast u_2' + 2[u_2, u_4]_*, \\
\partial_x^{-4} \partial_2 u_5 & = \cdots \quad ,
\end{align*}
\]

and for \( m = 3 \)

\[
\begin{align*}
\partial_x^{-1} \partial_3 u_2 & = u_2''' + 3u_3'' + 3u_4' + 3u_2' \ast u_2 + 3u_2' \ast u_2', \\
\partial_x^{-2} \partial_3 u_3 & = u_3''' + 3u_4'' + 3u_5' + 6u_2 \ast u_3' + 3u_2' \ast u_3 + 3u_3 \ast u_2 + 3[u_2, u_4]_*, \\
\partial_x^{-3} \partial_3 u_4 & = u_4''' + 3u_5'' + 3u_6' + 3u_2' \ast u_4 + 3u_2 \ast u_4' + 6u_4 \ast u_2 \\
& \quad - 3u_2 \ast u_3' - 3u_3 \ast u_2' + 6u_3 \ast u_3' + 3[u_2, u_5]_* + 3[u_3, u_4]_* , \\
\partial_x^{-4} \partial_3 u_5 & = \cdots \quad .
\end{align*}
\]

These contain the \((2 + 1)\)-dimensional noncommutative KP equation (2.8) with \( 2u_2 \equiv u, x_2 \equiv y, x_3 \equiv t \) and \( \partial_x^{-1} f(x) = \int x dx f(x') \). We note that infinite kind of fields \( u_3, u_4, u_5, \ldots \) are represented in terms of one kind of field \( 2u_2 \equiv u \) as is seen in Eq. (4.12).

• Noncommutative KdV Hierarchy (2-reduction of the noncommutative KP hierarchy)

Putting the constraint \( L^2 = B_2 = \partial_x^2 + u \) on the noncommutative KP hierarchy, we get the noncommutative KdV hierarchy. We note that the even-th flows are trivial. The following noncommutative hierarchy

\[
\frac{\partial u}{\partial x^m} = [B_m, L^2]_* , \quad (4.14)
\]

has neither positive nor negative power of (pseudo-)differential operators for the same reason as commutative case and gives rise to the \( m \)-th KdV equation for each \( m = 1, 3, 5, \cdots \). The noncommutative KdV hierarchy (4.14) coincides with the \((1 + 1)\)-dimensional noncommutative KdV equation for \( m = 3 \) with \( x_3 \equiv t \), (2.3)
and with the \((1 + 1)\)-dimensional 5-th noncommutative KdV equation \[55\] for \(m = 5\) with \(x_5 \equiv t\):

\[
\dot{u} = \frac{1}{16} u'''''' + \frac{5}{16} (u*u'' + u''*u) + \frac{5}{8} (u'*u' + u*u*u').
\] (4.15)

In this way, we can generate infinite set of the \(l\)-reduced noncommutative KP hierarchies. Explicit examples are seen in e.g. \[23\]. (See also \[5, 33, 38, 46, 56, 57\].)

### 4.2 Multi-soliton Solutions to NC KP and KdV hierarchies

Now we construct multi-soliton solutions of the noncommutative KP hierarchy. Let us introduce the following functions,

\[
f_s(\vec{x}) = e^{\xi(\vec{x}; k_s)} + a_s e^{\xi(\vec{x}; k'_s)}, \quad \text{where} \quad \xi(\vec{x}; k) = x_1 k + x_2 k^2 + x_3 k^3 + \cdots,
\] (4.16)

where \(k_s, k'_s\) and \(a_s\) are constants. Star exponential functions are defined by

\[
e^f(x) := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} f(x)^n.
\] (4.17)

An \(N\)-soliton solution to the noncommutative KP hierarchy (4.9) is given by \[11\],

\[
L = \Phi_N \star \partial_\vec{x} \Phi^{-1}_N,
\] (4.18)

where

\[
\Phi_N \star f = |W(f_1, \ldots, f_N, f)|_{N+1,N+1},
\]

\[
= \left| \begin{array}{cccc}
  f_1 & f_2 & \cdots & f_N & f \\
  f'_1 & f'_2 & \cdots & f'_N & f' \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  f^{(N-1)}_1 & f^{(N-1)}_2 & \cdots & f^{(N-1)}_N & f^{(N-1)} \\
  f^{(N)}_1 & f^{(N)}_2 & \cdots & f^{(N)}_N & f^{(N)} \\
\end{array} \right|.
\] (4.19)

The Wronski matrix \(W(f_1, f_2, \ldots, f_m)\) is given by

\[
W(f_1, f_2, \ldots, f_m) := \left[ \begin{array}{cccc}
  f_1 & f_2 & \cdots & f_m \\
  f'_1 & f'_2 & \cdots & f'_m \\
  \vdots & \vdots & \ddots & \vdots \\
  f^{(m-1)}_1 & f^{(m-1)}_2 & \cdots & f^{(m-1)}_m \\
\end{array} \right],
\] (4.20)

where \(f_1, f_2, \ldots, f_m\) are functions of \(x\) and \(f' := \partial f/\partial x, f'' := \partial^2 f/\partial x^2, f^{(m)} := \partial^m f/\partial x^m\) and so on.
In the commutative limit, $\Phi_N \ast f$ is reduced to
\[
\Phi_N \ast f \to \frac{\det W(f_1, f_2, \ldots, f_N, f)}{\det W(f_1, f_2, \ldots, f_N)},
\]
which just coincides with the commutative $N$-soliton solution \[7\]. In this respect, quasi-determinants are fit to this framework of the Wronskian solutions, however, give a new formulation of it.

From Eq. (4.18), we have a more explicit form as
\[
u_2 = \partial_x \left( \sum_{s=1}^N W_s' W_s^{-1} \right), \quad \text{where} \quad W_s := |W(f_1, \ldots, f_s)|_{ss}.
\]

In the soliton solutions, The $l$-reduction condition $(L^l)_{\leq -1} = 0$ or $L^l = B_l$ is equivalent to the constraint $k_s = \epsilon k_s' (s = 1, 2, \ldots, N)$, where $\epsilon$ is the $l$-th root of unity.

### 4.3 Asymptotic Behavior of the Exact Soliton Solutions

In this subsection, we discuss asymptotic behavior of the $N$-soliton solutions in asymptotic region of infinitely past and future. In the star-product formalism, all coordinates are regarded as c-number functions. We can as usual plot the configurations and interpret the positions of localized wave packet lump, and read phase shifts. Here we restrict ourselves to noncommutative KdV and KP equations (the third flow of the hierarchies) with space-time noncommutativity $[x,t]_s = i\theta$ where $(x,t) \equiv (x_1, x_3)$. Discussion to other noncommutative hierarchies is similarly made \[27\].

First, let us comment on an important formula which is relevant to one-soliton solutions. Let $x, t$ be noncommutative space-time coordinates. Introducing new noncommutative coordinates as $z := x + vt, \bar{z} := x - vt$, we can easily find
\[
f(z) \ast g(z) = f(z)g(z)
\]
because the star-product (2.2) is rewritten in terms of $(z, \bar{z})$ as
\[
f(z, \bar{z}) \ast g(z, \bar{z}) = e^{i\theta(\partial_{z_1} \partial_{\bar{z}_2} - \partial_{z_2} \partial_{\bar{z}_1})} f(z_1, \bar{z}_1)g(z_2, \bar{z}_2) \bigg|_{\begin{array}{c} z_1 = z_2 = z \\ \bar{z}_1 = \bar{z}_2 = \bar{z}\end{array}} \tag{4.24}
\]

Hence noncommutative one soliton-solutions can be the same as commutative ones.

When $f(x)$ is a linear function, the star exponential function $e^f(x)$ is tractable because it satisfies
\[
(e^{f(x)})^{-1} = e^{-f(x)}, \quad \partial_x e^{f(x)} = ke^{f(x)}. \tag{4.25}
\]
These formula play crucial roles in discussion on asymptotic behavior of the \( N \)-soliton solutions.

### 4.3.1 Asymptotic behavior of noncommutative KdV solitons

First, let us discuss the asymptotic behavior of the \( N \)-soliton solutions to the noncommutative KdV equation. The noncommutative KdV hierarchy is the 2-reduction of the noncommutative KP hierarchy and realized by putting \( k'_s = -k_s \) on the \( N \)-soliton solutions to the noncommutative KP hierarchy. Here the constants \( k_s \) and \( a_s \) are non-zero real numbers and \( a_s \) is positive. Because of the permutation property of the columns of quasi-determinants in Proposition 3.1 (i), we can assume \( k_1 < k_2 < \cdots < k_N \).

Let us discuss the soliton solutions to the noncommutative KdV equation where the coordinates are specified as \( (x, t) \equiv (x_1, x_3) \). Let us define a new coordinate \( X := x + k^2 t \) comoving with the \( I \)-th soliton and take \( t \to \pm \infty \) limit. We note that \( X \) is finite at any time. Then, because of \( x + k^2 t = x + k^2 t + (k^2 - k^2 t)I, \) either \( e_s e_k(x+k^2 t) \) or \( e_s e_k(x+k^2 t) \) goes to zero for \( s \neq I \). Hence the behavior of \( f_s \) becomes at \( t \to +\infty \):

\[
f_s(x) \to \begin{cases} a_s e_s^{-k_s(x+k^2 t)} & s < I \\ e_s^{k_s(x+k^2 t)} - a_t e_s^{-k_s(x+k^2 t)} & s = I \\ e_s^{k_s(x+k^2 t)} & s > I, \end{cases} \quad (4.27)
\]

and at \( t \to -\infty \):

\[
f_s(x) \to \begin{cases} e_s^{k_s(x+k^2 t)} & s < I \\ e_s^{k_s(x+k^2 t)} - a_t e_s^{-k_s(x+k^2 t)} & s = I \\ a_s e_s^{-k_s(x+k^2 t)} & s > I. \end{cases} \quad (4.28)
\]

We note that the \( s \)-th \( (s \neq I) \) column is proportional to a single exponential function \( e_s^{\pm \kappa_s(x+k^2 t)} \) due to Eq. (4.26). Because of the multiplication property of columns of quasi-determinants in Proposition 3.1 (ii), we can eliminate a common invertible factor from the \( s \)-th column in \( |A|_{ij} \) where \( s \neq j \). (Note that this exponential function is actually invertible as is shown in Eq. (4.29).) Hence the \( N \)-soliton solution becomes the following simple form where only the \( I \)-th column is non-trivial, at \( t \to +\infty \):

\[
\Phi_N \ast f \to \begin{bmatrix} 1 & \cdots & 1 \\ -k_1 & \cdots & -k_{I-1} \\ \vdots & \ddots & \vdots \\ (-k_1)^{N-1} & \cdots & (-k_{I-1})^{N-1} \\ (-k_1)^N & \cdots & (k_{I-1})^N \end{bmatrix} \begin{bmatrix} e_s^{\xi(x;k_1)} + a_t e_s^{-\xi(x;k_1)} & 1 \\ k_1 e_s^{\xi(x;k_1)} - a_t e_s^{-\xi(x;k_1)} & k_{I+1} \\ \vdots & \vdots \\ k_{I-1}^{N-1} e_s^{\xi(x;k_1)} + (-1)^{N-1} a_t e_s^{-\xi(x;k_1)} & k_{N-1} \\ k_1 e_s^{\xi(x;k_1)} + (-1)^N a_t e_s^{-\xi(x;k_1)} & k_{I+1} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} f \\ f' \\ \vdots \\ f^{(N-1)} \\ f^{(N)} \end{bmatrix},
\]
and at \( t \to -\infty \):

\[
\Phi_N \star f \to \begin{pmatrix}
1 & \ldots & 1 & e^\xi_{\mathbf{k},1} + a_1 e^{-\xi_{\mathbf{k},1}} & 1 & \ldots & 1 & f \\
k_1 & \ldots & k_{I-1} & k_I (e^\xi_{\mathbf{k},1} - a_I e^{-\xi_{\mathbf{k},1}}) & -k_{I+1} & \ldots & -k_N & f'
n \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
k_1^{N-1} & \ldots & k_{I-1}^{N-1} & k_I^{N-1} (e^\xi_{\mathbf{k},1} + (-1)^{N-1} a_I e^{-\xi_{\mathbf{k},1}}) & (-k_{I+1})^{N-1} & \ldots & (-k_N)^{N-1} & f^{(N-1)} \\
k_1^N & \ldots & k_{I-1}^N & k_I^N (e^\xi_{\mathbf{k},1} + (-1)^N a_I e^{-\xi_{\mathbf{k},1}}) & (-k_{I+1})^N & \ldots & (-k_N)^N & f^{(N)}
\end{pmatrix}
\]

Here we can see that all elements in between the first column and the \( N \)-th column commute and depend only on \( x + k_I^2 t \) in \( \xi(\mathbf{x}; k_I) \), which implies that the corresponding asymptotic configuration coincides with the commutative one, that is, the \( I \)-th one-soliton configuration with some coordinate shift, so called the phase shift. (We note that because \( f \) is arbitrary, there is no need to consider the products between a column and the \( (N+1) \)-th column. This observation for asymptotic behavior can be made from Eq. (4.22) also.)

The commutative discussion has been studied in this way by many authors, and therefore, we conclude that for the noncommutative KdV hierarchy, asymptotic behavior of the multi-soliton solutions is all the same as commutative one, and as the results, the \( N \)-soliton solutions have \( N \) isolated localized lump of energy and in the scattering process, they never decay and preserve their shapes and velocities. The phase shifts also appear by the same degree as commutative ones.

### 4.3.2 Asymptotic behavior of noncommutative KP solitons

Next, let us focus on the asymptotic behavior of the \( N \)-soliton solutions to the noncommutative KP equation where the space and time coordinates are \((x, y, t) \equiv (x_1, x_2, x_3)\) with the space-time noncommutativity \([x, t]_s = i\theta\). Here the constants \( k_s \) and \( k'_s \) are non-zero real numbers and the constant \( a_s \) will be redefined later.

As we mentioned at the beginning of the present section, one-soliton solutions are all the same as commutative ones. However, we have to treat carefully for the noncommutative KP hierarchy.

First we comment on the the Baker-Campbell-Hausdorff (BCH) formula for the the star exponential function \( \xi(\xi; \mathbf{k}) \) in the solution. Let’s focus on the noncommutative part of the star exponential function. The relevant part in the linear function is \( \xi(\xi; \mathbf{k}) = k(x + k^2 t) \). We note that we sometimes meet the following calculation:

\[
e^\xi_{\mathbf{k},1} \star e^\xi_{\mathbf{k}',1} = e^{(i/2)\theta (kk' - k_3 k'_3)} e^\xi_{\mathbf{k},1} + e^{(i/2)\theta (k'^2 - k^2)} e^\xi_{\mathbf{k}',1} \star e^\xi_{\mathbf{k},1}.
\]

Let us see how we should eliminate the complex factor \( \Delta := (i/2)\theta kk'(k'^2 - k^2) \) in the asymptotic region under the condition that the configurations take real values.
From Eq. (4.22), naive one-soliton solution can be expressed as follows:

\[ u_2 = \partial_x \left( \partial_x (\xi^\alpha(x,k) + ae_{\xi}(x,k')) * (e^{\xi(x,k)} + ae^{\xi(x,k')})^{-1} \right) \]

\[ = \partial_x \left( (k^2 + ak^b \Delta \eta(x,k,k')) * (1 + a\Delta e_{\xi}(x,k,k'))^{-1} \right), \tag{4.30} \]

where \( \eta(x,k,k') := x(k' - k) + y(k'^2 - k^2) + t(k'^3 - k^3) \). We note that the complex factor \( \Delta \) cannot be absorbed by redefining a coordinate such as \( x \to x + (k' - k)^{-1} \Delta \) because the space-time coordinates are real. Instead of this, we redefine a positive real constant \( \tilde{a} := a\Delta \) in order to absorb the complex factor \( \Delta \) so that \( f_1 = e^{\xi(x,k)} + ae^{\xi(x,k')} = (1 + \tilde{a}e_{\xi}(x,k')) * e^{\xi(x,k)} \). This avoids the coordinate shift by a complex number. The configuration in asymptotic region is real.

This point becomes important for scattering process of the multi-soliton solutions. We will soon see that the constants \( a_s \) in the \( N \)-soliton solution to the noncommutative KP equation should be replaced with a positive real number \( \tilde{a}_s \) which satisfies \( a_s = \tilde{a}_s \Delta_s^{-1} \) where \( \Delta_s := e^{(i/2)\theta k_s k_s'}(k_s^2 - k_s'^2) \).

Let us define new coordinates comoving with the \( I \)-th soliton as follows:

\[ X := x + k_I y + k_I^2 t, \quad Y := x + k_I' y + k_I'^2 t, \tag{4.31} \]

so that \( X, Y \) are finite in the asymptotic region. Then the function \( \xi(x,y,t;k_s) \) can be rewritten in terms of the new coordinates as \( \xi(X,Y,t;k_s) = A(k_s)X + B(k_s)Y + C(k_s)t \) where \( A(k_s), B(k_s) \) and \( C(k_s) \) are real constants depending on \( k_I, k_I' \) and \( k_s \). We can get from Eq. (4.31)

\[ \left( \begin{array}{c} x \\ y \end{array} \right) = \frac{1}{k_I' - k_I} \left( \begin{array}{c} k_I' X - k_I Y + k_I k_I' (k_I' - k_I)t \\ -X + Y + (k_I^2 - k_I'^2)t \end{array} \right), \tag{4.32} \]

and find

\[ \xi = x + k_s y + k_s^{-1} t = \frac{k_I' - k_s}{k_I' - k_I} X + \frac{k_s - k_I}{k_I' - k_I} Y + (k_s - k_I)(k_s - k_I)t. \]

Here we assume that \( C(k_s) \neq C(k_s') \) which corresponds to pure soliton scatterings. (The condition \( C(k_s) = C(k_s') \) could lead to soliton resonances. For commutative discussion, see e.g. [35], and [32] as well.)

Now let us take \( t \to \pm \infty \) limit, then, for the same reason as in the noncommutative KdV equation, we can see that the asymptotic behavior of \( f_s \) becomes:

\[ f_s(x) \to \begin{cases} A_s e^{\xi(x,k_s)} & s \neq I \\ e^{\xi(x,k_I)} + a_I e^{\xi(x,k_I')} & s = I \end{cases} \tag{4.33} \]

\[ ^3 \text{In [39], similar observations are made of the noncommutative Burgers equation where the noncommutative parameter is not real but pure imaginary. This implies that the factor } \Delta \text{ is real and can be absorbed by a coordinate shift which affects the phase shift.} \]
where $A_s$ is some real constant whose value is 1 or $a_s$, and $\tilde{k}_s$ is a real constant taking a value of $k_s$ or $k_s'$. As in the case of the noncommutative KdV equation, the $s$-th ($s \neq I$) column is proportional to a single exponential function and we can eliminate this factor from the $s$-th column. Hence in the asymptotic region $t \to \pm \infty$, the $N$-soliton solution becomes the following simple form where only the $I$-th column is non-trivial:

\[
\Phi_N \ast f \rightarrow \begin{pmatrix}
1 & \cdots & 1 & e_\ast^{\xi(\bar{x};k_I)} + a_I e_\ast^{\xi(\bar{x};k'_I)} & 1 & \cdots & 1 & f \\
\tilde{k}_1 & \cdots & k_{I-1} & k_I e_\ast^{\xi(\bar{x};k_I)} + a_I e_\ast^{\xi(\bar{x};k'_I)} & \tilde{k}_{I+1} & \cdots & \tilde{k}_N & f' \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{k}_1^{N-1} & \cdots & \tilde{k}_1^{N-1} & k_I^{N-1} e_\ast^{\xi(\bar{x};k_I)} + a_I k_I^{N-1} e_\ast^{\xi(\bar{x};k'_I)} & \tilde{k}_{I+1}^{N-1} & \cdots & \tilde{k}_N^{N-1} & f^{(N-1)} \\
\tilde{k}_1^N & \cdots & \tilde{k}_1^N & k_I^N e_\ast^{\xi(\bar{x};k_I)} + a_I k_I^N e_\ast^{\xi(\bar{x};k'_I)} & \tilde{k}_{I+1}^N & \cdots & \tilde{k}_N^N & f^{(N)}
\end{pmatrix}
\]

Here we can see that all elements between the first column and the $N$-th are real and depend only on $x(k'_I - k_I) + t(k_I^3 - k'_I^3)$ for noncommutative coordinates. This implies that the corresponding asymptotic configuration coincides with the commutative one. Hence, we can also conclude that for the noncommutative KP equation, asymptotic behavior of the multi-soliton solutions is all the same as commutative one in the process of pure soliton scatterings. As the results, the $N$-soliton solutions possess $N$ isolated localized lump of energy and in the pure scattering process, they never decay and preserve their shapes and velocities of the localized solitary waves. This coincides with the result on 2-soliton scattering studied by Paniak \[47\].

As is suggested by the stability of the $N$-soliton solution, there actually exist infinite conserved densities of the noncommutative KP equation with space-time noncommutativity:

\[
\sigma_n = \text{coef}_{-1} L^n - 3\theta ((\text{coef}_{-1} L^n) \ast u'_3 + (\text{coef}_{-2} L^n) \ast u'_2), \quad n = 1, 2, \cdots \quad (4.34)
\]

where $\text{coef}_{-1} L^n$ denotes the coefficient of $\partial^{-1}$ in $L^n$. (In particular, $\text{coef}_{-1} L^n$ is the residue of $L^n$.) The product “$\ast$” is called the Strachan’s product \[51\] and defined by

\[
f(x) \ast g(x) := \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p + 1)!} \left( \frac{1}{2} \delta_{ij} \partial^{(x')}_{i} \partial^{(x'')}_{j} \right)^{2p} f(x') g(x'') \bigg|_{x' = x'' = x}.
\]

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This is a commutative and non-associative product. Conserved densities for one-soliton configuration are not deformed in the noncommutative extension because one soliton solutions can be always reduced to commutative ones.

5 NC ASDYM Hierarchy and Soliton Solutions

Finally, we present noncommutative anti-self-dual Yang-Mills (ASDYM) equations in 4 dimension and its hierarchy generalization. Let \((x_0, x_1, x_2, \ldots ; y_0, y_1, y_2, \ldots)\) be complex coordinates and define covariant derivatives \(D_{x_k} := \partial_{x_k} + A_{x_k}, \ D_{y_l} := \partial_{y_l} + A_{y_l} \) \((k, l = 0, 1, 2, \ldots)\) where \(A_{x_k}\) and \(A_{y_l}\) are \(n \times n\) complex matrices. Noncommutativity is introduced into the coordinates.

Let us consider the following linear systems:

\[
L_k \psi := (D_{x_k} - \zeta D_{x_{k-1}}) \psi, \quad (5.1)
\]
\[
M_l \psi := (D_{y_l} - \zeta D_{y_{l-1}}) \psi, \quad (5.2)
\]

where \(\zeta \in \mathbb{C}P^1\) is a spectral parameter which commutes with all spatial coordinates. The compatibility condition of the linear systems is \([L_k, M_l]_* = 0\) for any \(k, l\). This yields an infinite systems of partial differential equations, which is called the noncommutative anti-self-dual Yang-Mills hierarchy. By identification of \(x_0 = \bar{z}, x_1 = w, y_0 = -\bar{w}, y_1 = z\), the compatibility condition for \(k = l = 1\) coincides with the noncommutative anti-self-dual Yang-Mills equation \((2.9)\). For the commutative anti-self-dual Yang-Mills hierarchies, see e.g. \([2, 40, 43, 52, 53]\).

The noncommutative anti-self-dual Yang-Mills hierarchy equations can be rewritten as the following form:

\[
\partial_{y_l}(J^{-1} \ast \partial_{x_{k-1}}J) - \partial_{x_k}(J^{-1} \ast \partial_{y_{l-1}}J) = 0, \quad (5.3)
\]

where \(J\) is an \(n \times n\) matrix. The equation \((5.3)\) and the matrix \(J\) is called the noncommutative Yang’s hierarchy equation and the Yang’s \(J\)-matrix, respectively. For \(k = l = 1\), this coincides with the noncommutative Yang’s equation.

Anti-self-dual gauge fields can be reproduced from the solution \(J\) to the equation \((5.3)\) by decomposition \(J = \tilde{h}^{-1} \ast h\) as

\[
A_{y_l} = -(\partial_{y_l} h) \ast h^{-1}, \quad A_{x_k} = -(\partial_{x_k} h) \ast h^{-1}, \quad A_{x_{k-1}} = -(\partial_{x_{k-1}} \tilde{h}) \ast \tilde{h}^{-1}, \quad A_{y_{l-1}} = -(\partial_{y_{l-1}} \tilde{h}) \ast \tilde{h}^{-1}.
\]

The proof is the same as the noncommutative anti-self-dual Yang-Mills equation. (See e.g. \([15]\).)
From now on, we focus on the \( n = 2 \) case. The \( J \)-matrix can be reparametrized without loss of generality as follows:

\[
J = \begin{bmatrix}
p - r \ast q^{-1} \ast s & -r \ast q^{-1} \\
q^{-1} \ast s & q^{-1}
\end{bmatrix}.
\] (5.4)

Exact solutions of the \( m \)-th Atiyah-Ward ansatz are represented in terms of quasideterminants (\( m = 0, 1, 2, \cdots \)):

\[
p_m = \begin{vmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-m} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{1-m} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_m & \varphi_{m-1} & \cdots & \varphi_0 \
\end{vmatrix}^{-1} \\
q_m = \begin{vmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-m} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{1-m} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_m & \varphi_{m-1} & \cdots & \varphi_0 \
\end{vmatrix}^{-1} \\
r_m = \begin{vmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-m} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{1-m} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_m & \varphi_{m-1} & \cdots & \varphi_0 \
\end{vmatrix}^{-1} \\
s_m = \begin{vmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-m} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{1-m} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_m & \varphi_{m-1} & \cdots & \varphi_0 \
\end{vmatrix}^{-1}
\] (5.5)

where the scalar functions \( \varphi_i(x; y) \) can be determined from a scalar function \( \varphi_0 \) recursively by the chasing relation:

\[
\frac{\partial \varphi_i}{\partial y_l} = -\frac{\partial \varphi_{i+1}}{\partial y_{l-1}}, \quad \frac{\partial \varphi_i}{\partial x_k} = -\frac{\partial \varphi_{i+1}}{\partial x_{k-1}}, \quad -m \leq i \leq m - 1 \ (m \geq 2).
\] (5.6)

The scalar function \( \varphi_0 \) is a solution to the linear equation \((\partial x_k \partial y_{k-1} - \partial y_k \partial x_{k-1}) \varphi_0 = 0\).

### 5.1 Asymptotic behavior of noncommutative ASDYM solitons

Here let us discuss \( N \) soliton solutions to the anti-self-dual Yang-Mills hierarchy (5.3) and asymptotic behaviors of the 2-soliton case. In order to discuss it, we have to pick a specified coordinate up in order to identify it with time coordinate.

By the identification of \( x_{k-1} \equiv z = t_1 - it_2, x_k \equiv w = t_3 + it_4, y_{l-1} \equiv -\bar{w} = -t_3 - it_4, y_l \equiv z = t_1 + it_2 \) where \( t_\mu (\mu = 1, 2, 3, 4) \) is a real coordinate, the linear equation becomes the Laplace equation in 4-dimension:

\[
\partial^2 \varphi_0(t) = 0,
\] (5.7)

where \( \partial^2 := \partial_\mu \partial^\mu = \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2 \). The Yang’s equation (5.3) becomes

\[
\partial_z (J^{-1} \ast \partial_z J) + \partial_w (J^{-1} \ast \partial_w J) = 0.
\] (5.8)
The following solution to the Laplace equation (5.7) leads to an \( N \) soliton solution to the anti-self-dual Yang-Mills hierarchy equation:

\[
\varphi_0(t) = 1 + \sum_{s=1}^{N} a_s e^{\xi(k_s; t)} + \sum_{i_1=1, i_2=1, i_1 < i_2} a_{i_1 i_2} e^{\xi(k_{i_1}; t) + \xi(k_{i_2}; t)} + \ldots
\]

\[
+ a_{12 \ldots N} e^{\xi(k_{1}; t) + \xi(k_{2}; t) + \ldots + \xi(k_{N}; t)}, \quad \xi(k; t) := k_{\mu} t^\mu
\]

where the coefficients \( a_{i_1 \ldots i_s} \) \((s = 1, \ldots, N)\) are complex constants and \( k_{(s)\mu} \) \((\mu = 1, 2, 3, 4)\) are real parameters which satisfy \( k_{(s)\mu} k_{(s)\mu} = 0 \). The commutative limit of this \( N \)-soliton solution reduces to the \( N \) non-linear plane wave solution [6]. We note that other scalar functions can be the same representation as (5.9) because of the chasing relation (5.6).

We note that the coefficients \( a_{i_1 \ldots i_s} \) \((s = 1, \ldots, N)\) are in general not real but complex, because there is a gauge freedom: \( A_{\mu} \mapsto g^{-1} * A_{\mu} * g + g^{-1} * \partial_{\mu} g \). Here we focus, however, on real-valued configurations in order to compare with the previous discussion. We finally need to check the asymptotic behavior of gauge invariant quantities such as \( \int d^4 t \text{Tr} F_{\mu \nu}^* \) and \( \int d^4 t \text{Tr} F_{\mu \nu}^* * F_{\mu \nu}^* \).

Let us discuss the asymptotic behavior of the \( N \) soliton solutions of the anti-self-dual Yang-Mills equation, which is called the ASDYM solitons. Now noncommutativity is assumed to be introduced into a spatial coordinate \( x \equiv t_1 \) and time coordinate \( t \equiv t_3 \) such that \([x, t]_s = i \theta \). We consider the \( t \to \pm \infty \) limit.

One soliton solution is given by

\[
\varphi_0 = 1 + a e^{\xi(k; t)}. \quad (5.9)
\]

Dependence of noncommutative coordinates is \( \varphi_0(x + vt) \) \((v := k_3/k_1)\) and hence the configuration reduces to the commutative one. This can be interpreted as a domain wall in 4-dimension. D-brane interpretation of this solution is worth studying.

Two soliton solution is given by

\[
\varphi_0 = 1 + a_1 e^{\xi(k; t)} + a_2 e^{\xi(k'; t)} + a_{12} e^{\xi(k; t) + \xi(k'; t)}. \quad (5.10)
\]

Let us ride on the comoving frame with the first soliton so that \( k_{\mu} t^\mu \) and \( e^{\xi(k; t)} \) are finite. In the limit of \( t \to \pm \infty \), the term \( e^{\xi(k'; t)} \) goes to 0 or infinity. Hence in the asymptotic region, the 2-soliton solution becomes the following case (i) or (ii):

\[
\varphi_0 \rightarrow \begin{cases} 
(i) & 1 + a_1 e^{\xi(k; t)} \\
(ii) & a_2 e^{\xi(k'; t)} + a_{12} e^{\xi(k; t) + \xi(k'; t)} = (a_2 + a_{12} e^{\xi(k; t)}) * e^{\xi(k'; t)}
\end{cases}
\]

where \( a_{12} = \tilde{a}_{12} \Delta \) and \( \Delta := e^{(i/2)\theta(k_1 k_5 - k_3 k'_1)} \). We assume that \( \tilde{a}_{12} \) is real.
In the case of (i), we can find that \( \varphi_0(t, x) = \varphi_0(x + vt) \) and hence the configuration coincides with commutative one. In the case of (ii), we have to proceed the calculation. As is commented, other scalar functions in the Atiyah-Ward ansatz solution (5.5) have the form: \( \varphi_i = b_1 + c_i e^{i(k_1 t)} + d_i e^{i(k_1 t) + \xi(k_i t)} \) where \( b_1, c_i, d_i, r_i \) are constants, and \( b_1, c_i, d_i \) are real. Asymptotic behavior of (ii) is \( \varphi_i \to (c_i + \tilde{r}_i e^{i(k_i t)}) * e^{i(k_i t)} \). where \( r_i = \tilde{r}_i \Delta \) so that \( \tilde{r}_i \) is real.

Because of the multiplication property of columns of quasideterminants, the Atiyah-Ward ansatz solution (5.5) have the common asymptotic form: \( p, q, r, s \to f(t+ux) * e^{i(k_i t)} \).

The gauge fields can be recovered from the matrices \( h \) and \( \tilde{h} \) as in (5.11). Let us decompose the matrix \( J \) into \( h \) and \( \tilde{h} \) as follows:

\[
J = \begin{bmatrix} p & -r & q^{-1} s & -r & q^{-1} \end{bmatrix} * \begin{bmatrix} 1 & r & 0 & q & s \end{bmatrix} = \tilde{h}^{-1} * h.
\]

The gauge fields are calculated as

\[
A_z = - (\partial_z h) * h^{-1} = \begin{bmatrix} - (\partial_z p) * p^{-1} & 0 \\
- (\partial_z s) * p^{-1} & 0 \end{bmatrix}, \quad A_w = - (\partial_w h) * h^{-1} = \begin{bmatrix} - (\partial_w p) * p^{-1} & 0 \\
- (\partial_w s) * p^{-1} & 0 \end{bmatrix}, \quad A_{\tilde{z}} = - (\partial_{\tilde{z}} \tilde{h}) * \tilde{h}^{-1} = \begin{bmatrix} 0 & - (\partial_{\tilde{z}} r) * q^{-1} \\
0 & - (\partial_{\tilde{z}} q) * q^{-1} \end{bmatrix}, \quad A_{\tilde{w}} = - (\partial_{\tilde{w}} \tilde{h}) * \tilde{h}^{-1} = \begin{bmatrix} 0 & - (\partial_{\tilde{w}} r) * q^{-1} \\
0 & - (\partial_{\tilde{w}} q) * q^{-1} \end{bmatrix}.
\]

We can see that the common factor \( e^{i(k_i t)} \) in \( p, q, r, s \) is canceled out here, and the coordinate dependence in the gauge fields becomes \( A_\mu(t, x) = A_\mu(x + vt) \). Note that there is no difference between commutative case and noncommutative case in the derivation from (5.11). We can therefore conclude the gauge invariant quantities consist of \( F^*_{\mu\nu} \) are the same as commutative ones.

Let us consider the comoving frame with the second soliton where \( k_1 t^\mu \) and \( e^{i(k_i t)} \) are finite. In the limit of \( t \to \pm \infty \), the factor \( e^{i(k_i t)} \) goes to (i) 0 or (ii) infinity. The case (i) reduces to one-soliton configuration. The case (ii) leads to, in similar way, the following asymptotic behaviors of the scalar functions: \( \varphi_i \to e^{i(k_i t)} * (a_1 + \tilde{a}_1 e^{i(k_i t)}) \), \( \varphi_i \to e^{i(k_i t)} * (d_i + \tilde{r}_i e^{i(k_i t)}) \), and \( p, q, r, s \to e^{i(k_i t)} * f(x + vt) \). We note that the coefficients \( a_1, \tilde{a}_1, d_i, \tilde{r}_i \) are the same as those in case (i). We can see that the common factor \( e^{i(k_i t)} \) appears in the gauge fields as \( A_\mu(x, t) \to e^{i(k_i t)} * A_\mu(x + vt) * e^{-i(k_i t)} \). This is essentially gauge equivalent to \( A(x + vt) \) up to constants which do not contribute the field strength. Hence the gauge invariant quantities consist of \( F^*_{\mu\nu} \) are the same as commutative ones in this case as well.

Therefore we can conclude that the asymptotic behavior of the 2-soliton solutions is the same as commutative one [6] and as the results, the 2-soliton solutions has 2 isolated localized lump of energy and in the scattering process, they never decay and preserve their shapes and velocities of the localized solitary waves.
Higher-charge soliton scattering is worth studying. For this purpose, Wronskian-type solutions [44] would be suitable. This will be reported elsewhere.

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