Improvement and Renormalization Constants in $O(a)$ Improved Lattice QCD

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We present results at $\beta = 6.0$ and 6.2 for the $O(a)$ improvement and renormalization constants for bilinear operators using axial and vector Ward identities. We discuss the extraction of the mass dependence of the renormalization constants and the coefficients of the equation of motion operators.

1. INTRODUCTION

In quenched Lattice QCD, axial and vector Ward identities can be used to determine, at $O(a)$, all the scale independent renormalization constants for bilinear currents ($Z_A$, $Z_V$, and $Z_S/Z_P$), the improvement constants ($c_A$, $c_V$, and $c_T$), the quark mass dependence of all five $Z_O$, and the coefficients of the equation of motion operators \( \langle \delta S^{(12)} \rangle \). Here we summarize results at $\beta = 6.0$ and 6.2 and discuss the highlights of our calculations.

We start with the general axial Ward identity involving operators improved on and off-shell

\[
\langle \delta S^{(12)} \rangle = \langle \delta S_{\text{on-shell}}^{(12)} \rangle = \langle \delta S_{\text{off-shell}}^{(12)} \rangle,
\]

where $\delta S$ is the result of the axial variation of $O$ ($A_\mu \leftrightarrow V_\mu$, $S \leftrightarrow P$, and $T_{\mu\nu} \rightarrow \epsilon_{\mu\rho\sigma\tau} T_{\rho\sigma}$), and $\delta S$ is the variation in the action.

At $O(a)$ there exists only one dimension 4 off-shell operator (which vanishes by the equations of motion) for each bilinear that has the appropriate symmetries $\mathbb{T}$. Consequently, we define

\[
\langle \delta S_{\text{on-shell}}^{(12)} \rangle = \langle \delta S_{\text{on-shell}}^{(12)} \rangle = \langle \delta S_{\text{on-shell}}^{(12)} \rangle \equiv 0,
\]

\[
\langle \delta S_{\text{off-shell}}^{(12)} \rangle = \langle \delta S_{\text{off-shell}}^{(12)} \rangle = \langle \delta S_{\text{off-shell}}^{(12)} \rangle = 0,
\]

\[
\langle \delta S_{\text{on-shell}}^{(12)} \rangle = \langle \delta S_{\text{on-shell}}^{(12)} \rangle = \langle \delta S_{\text{on-shell}}^{(12)} \rangle = 0,
\]

\[
\langle \delta S_{\text{off-shell}}^{(12)} \rangle = \langle \delta S_{\text{off-shell}}^{(12)} \rangle = \langle \delta S_{\text{off-shell}}^{(12)} \rangle = 0.
\]

This ensures that the equation-of-motion operator $E_{O}$ gives rise only to contact terms, and does not change the overall normalization $Z_O$. The $O(a)$ on-shell improved renormalized operators $O_{R}^{(ij)}$ are

\[
O_{R}^{(ij)} = Z_{O}^{0}(1 + b_{O} m_{ij}) O_{I}^{(ij)},
\]

\[
(4)
\]

\[
O_{R}^{(ij)} = Z_{O}^{0}(1 + b_{O} m_{ij}) O_{I}^{(ij)}.
\]

\[
(5)
\]

\[
(A_{I})_{\mu} = A_{\mu} + a c_{A} \partial_{\mu} P_{I},
\]

\[
(6)
\]

\[
(V_{I})_{\mu} = V_{\mu} + a c_{V} \partial_{\mu} T_{I} P_{I},
\]

\[
(7)
\]

\[
(T_{I})_{\mu\nu} = T_{\mu\nu} + a c_{T} \epsilon_{\mu\nu\rho\sigma} V_{\rho} - \partial_{\sigma} V_{\mu},
\]

\[
(8)
\]

\[
P_{I} = P_{I}, \quad S_{I} \equiv S_{I},
\]

\[
(9)
\]

The $Z_{O}$ are renormalization constants in the chiral limit, $m_{ij} \equiv (m_{i} + m_{j})/2$ is the average bare quark mass, $m_{ij} = 1/2 k_{i} - 1/2 k_{j}$, $k_{i}$ is the value of the hopping parameter in the chiral limit, and $m_{ij}$ is the quark mass defined by the axial Ward identity (AWI) in Eq. (12). Note that $m$ and $\tilde{m}$ are identical in a discretized theory with chiral symmetry, like staggered fermions. With these definitions, $b_{O} = 1$, $c_{O} = 0$, $c_{T} = 1$ at tree level $\mathbb{T}$.

Since the equation-of-motion operators contribute only contact terms, Eq. (13) can be rewritten in terms of just on-shell improved operators:

\[
\langle \delta S_{I}^{(13)} \rangle = \langle \delta S_{I}^{(13)} \rangle / \langle \delta S_{I}^{(12)} \rangle
\]

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\]

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\]

where

\[
\delta S_{I}(x) \equiv 2 \tilde{m}_{12} P_{I}^{(12)}(x) - \partial_{\mu} (A_{I})_{\mu}^{(12)}(x).
\]
Our calculation is limited to the case $\tilde{m}_1 = \tilde{m}_2$ (this simplification was used due to limited computer resources), in which case the r.h.s. of Eq. 10 reduces to

$$\frac{Z^0_{A \Sigma}}{Z^0_{A \Sigma}} \left[ 1 + (\tilde{b}_A - \tilde{b}_O) \frac{a \tilde{m}_3}{2} \right] +$$

$$\left[ \frac{Z^0_{A \Sigma}}{Z^0_{A \Sigma}} \left( \tilde{b}_A - \tilde{b}_O \right) - \tilde{b}_A \right] + c_{\rho}' + c_{\sigma}' \right] a \tilde{m}_1 \right(11)$$

where $\tilde{m}_i \equiv \tilde{m}_{ij} |_{m_i = m_i}$. Using Eqs. 10 and 11, all the $b_{ij}$ (except $b_T$ which requires $m_1 \neq m_2$), $c_{ij}$, $c_{\rho}'$, and $c_{\sigma}'$, and the scale independent normalization constants are determined by making suitable choices for $J$, $O$, and $y$ in Eq. 10 and studying it as a function of $\tilde{m}_1$ and $\tilde{m}_3$ (Eq. 11).

2. RESULTS

The lattice parameters used in our calculation are given in Tab. 1 and 2. Our final results, which supercede those in Ref. 1, are given in Tab. 2.

In many cases a given on-shell improvement and normalization constants can be determined in a number of ways as discussed in 1. Results in Tab. 2 are based on the AWI with the best signal and smallest error. Table 2 also includes results by the ALPHA collaboration 3, 4, 5 and the one-loop tadpole improved perturbative results. To simplify comparison with previous results, we quote both $b_V$, $b_A$ and $b_V$, $b_A$.

One of the goals of our calculation is to quantify the residual $O(a^2)$ errors and to understand the shortcomings of 1-loop perturbation theory. For $O(a^2)$ errors we use two estimates: (i) the difference between our results and those by the ALPHA collaboration 3, 4, 5, and (ii) the difference between using 2-point and 3-point discretization of the derivatives 6 in the extraction of $c_A$ from

$$\frac{\sum_{\vec{x}} \langle \bar{A}_\mu | A_\mu + ac_{\rho} \bar{J} (\vec{x}) \rangle}{\sum_{\vec{x}} \langle \bar{P} | J^2 (\vec{x}) \rangle}$$

$$= 2 \tilde{m}_{ij}, \quad (12)$$

and the subsequent effect of the difference in $c_A$ on other constants. This latter variation is quoted as the second error in Tab. 2.

These differences are compared to the expected size of the residual discretization errors: $|a_{\Lambda_{QCD}}| \approx 0.15$ and 0.1 for the improvement constants and $|a_{\Lambda_{QCD}}|^2 \approx 0.02$ and 0.01 for the normalization constants at $\beta = 6.0$ and 6.2 respectively.

A comparison, at $\beta = 6.0$, between simulation at $c_{SW} = 1.4755$ (tadpole improved theory) and $c_{SW} = 1.769$ (non-perturbatively $O(a)$ improved theory) shows that all the constants are sensitive to the choice of $c_{SW}$. It is therefore important to use $c_{SW}$ determined non-perturbatively.

The most significant comparison is between our results and those of the ALPHA collaboration. The only results which do not agree within 2-$\sigma$ statistical errors are those for $Z^0_V$, $c_A$ and $c_V$ at $\beta = 6$, and for $Z^0_V$ at $\beta = 6.2$. The differences for $Z^0_V$ are of size 0.01 and 0.005 at $\beta = 6$ and 6.2 respectively, and are thus consistent with the expected differences of $O(a^2)$. The differences for $c_A$ and $c_V$ are also consistent with the size expected of $O(a)$ differences, but are more notable because they correspond to very large fractional differences (e.g. our $c_A$ at $\beta = 6$ has less than half the magnitude of that found by the ALPHA collaboration). What we learn is that (i) $c_O$, which vanish at tree level and are numerically small, depend

Table 1

Simulation parameters, statistics, and the time interval in $x_4$ defining the volume $V$ over which the chiral rotation is performed in the AWI. The lattice spacing is fixed using $r_0 = 0.5$ fermi, and is thus independent of the fermion action. The source $J$ is placed at $t = 0.$

| Label | $\beta$ | $c_{SW}$ | $a^{-1}$ (GeV) | Volume (fm$^3$) | Conf. | $x_4$ |
|-------|---------|----------|---------------|----------------|-------|-------|
| 60TI  | 6.0     | 1.4755   | 2.12          | $16^3 \times 48$ | 1.5   | 83    | 4–18 |
| 60NPF | 6.0     | 1.769    | 2.12          | $16^3 \times 48$ | 1.5   | 125   | 4–18 |
| 60NPb | 6.2     | 1.614    | 2.91          | $24^3 \times 64$ | 1.65  | 70    | 6–25 |
| 62NP  | 6.2     | 1.014    | 1.61          | $24^3 \times 64$ | 1.65  | 70    | 39–58|
Table 2
Values of $\kappa$ used in the three simulations, and the corresponding values of $aM_\pi$ and the quark mass $a\hat{m}$ extracted. $\hat{m}$ is defined by the AWI in Eq. [13]. $\kappa_c$ is the zero of $\hat{m}$ obtained from quadratic fits in $1/\kappa$. The non-zero value of $aM_\pi$ at $\kappa_c$ is indicative of the inadequacy of quadratic fits, $a^2M_\pi^2$ as a function of $1/2\kappa$, used to extract it, and discretization errors. Of these, the first is the dominant cause and points to the need for including quenched chiral logs in the fits [13].

| Label | $\kappa$ | $a\hat{m}$ | $aM_\pi$ | $\kappa$ | $a\hat{m}$ | $aM_\pi$ | $\kappa$ | $a\hat{m}$ | $aM_\pi$ |
|-------|---------|---------|--------|---------|---------|--------|---------|---------|--------|
| $\kappa_1$ | 0.11900 | 0.443(8) | 1.530(1) | 0.1300 | 0.144(1) | 0.711(2) | 0.1310 | 0.1345(6) | 0.609(1) |
| $\kappa_2$ | 0.13524 | 0.105(1) | 0.571(2) | 0.1310 | 0.118(1) | 0.630(2) | 0.1321 | 0.1054(4) | 0.522(1) |
| $\kappa_3$ | 0.13606 | 0.084(1) | 0.504(2) | 0.1320 | 0.092(1) | 0.544(2) | 0.1333 | 0.0727(3) | 0.418(1) |
| $\kappa_4$ | 0.13688 | 0.063(1) | 0.431(2) | 0.1326 | 0.075(1) | 0.488(2) | 0.1339 | 0.0560(2) | 0.360(2) |
| $\kappa_5$ | 0.13770 | 0.042(1) | 0.348(3) | 0.1333 | 0.056(1) | 0.416(2) | 0.1344 | 0.0419(2) | 0.307(2) |
| $\kappa_6$ | 0.13851 | 0.020(1) | 0.244(4) | 0.1342 | 0.032(1) | 0.308(3) | 0.1348 | 0.0306(2) | 0.261(2) |
| $\kappa_7$ | 0.13878 | 0.013(1) | 0.195(8) | 0.1345 | 0.025(4) | 0.262(12) | 0.1350 | 0.0248(1) | 0.235(2) |
| $\kappa_c$ | 0.13926(2) | 0 | 0.082(15) | 0.13532(3) | 0 | 0.083(20) | 0 | 0.135861(5) | 0 | 0.066(10) |

substantially, at $\beta = 6$, on the method/definition used to extract them; (ii) the variation between 2-pt and 3-pt derivatives significantly smaller than the difference between our results and those of the ALPHA collaboration; and (iii) these differences in $c_V$, and even more so in $c_A$, are substantially reduced at $\beta = 6.2$. The change appears too rapid to be an $O(a)$ effect.

Both $c_V$ and $c_T$ are obtained as a small difference between two large terms. Nevertheless, we are able to design Ward identities that yield these quantities with reasonable precision. In particular, the significant improvement we obtain in determining $c_V$ using methods described in [3] reduces the error in $Z_{\lambda}^0$, $Z_{P}^0/Z_{S}^0$, $c_T$ and $c_A'\pi$ as the uncertainty in $c_V$ feeds into these quantities.

When comparing against perturbative estimates, the yardstick we use for the missing higher order terms is $\sim \alpha_s^2 \approx 0.02$ and 0.016, respectively. We find that tadpole-improved 1-loop perturbation theory underestimates the deviations of renormalization and improvement constants from their tree level values. In all but one case, however, these discrepancies can be understood as a combination of a 2-loop correction of size $(1 - 2) \times \alpha_s^2$ [for $Z_{\lambda}^0$, $Z_{A}^0$, and $c_A$], higher order discretization errors of size $(1 - 2) \times aA_{\text{QCD}}$ [for $c_V$, $c_T$ and $b_V$], and statistical errors [for $b_A$, $b_P$, and $b_S$]. The only exception is $Z_{P}^0/Z_{S}^0$, for which a very large higher order perturbative con-

Table 4
Results for off-shell mixing coefficients.

|       | 60TI | 60NP | 60NPb | 62NP |
|-------|------|------|-------|------|
| $c_V'$ | +3.72(73) | +2.38(50) | +3.00(37) | +1.72(16) |
| $c_A'$ | +3.28(94) | +1.99(56) | +2.45(46) | +1.53(20) |
| $c_P'$ | -0.98(76) | +0.44(49) | -0.33(29) | +0.91(12) |
| $c_S'$ | +3.00(73) | +2.00(48) | +2.72(33) | +1.49(14) |
| $c_T'$ | +3.24(75) | +1.96(49) | +2.60(38) | +1.51(15) |

tribution of size $4 \times \alpha_s^2$ is needed to reconcile our non-perturbative results with 1-loop perturbation theory.

In Tab. 4, we present, first results for the equation-of-motion improvement constants $c_X'$. The combination $c_P' + c_O'$ is extracted by studying the dependence of Eq. [14] on $\hat{m}_1$ once the other constants defined in Eq. [11] have been determined. The errors in the determination of the $c_O'$ are dominated by two quantities: (i) The uncertainty in $c_A$ feeds into the extraction of $c_A'$, and (ii) the correlation function from which $c_P' + c_P$ is determined has a poor signal (the intermediate state is a scalar for $J = S$, $O = P$ and $\delta O = S$ in Eq. [14]). The uncertainty in $c_P'$ then feeds into $c_V'$, $c_S'$, and $c_T'$. Overall, we find a very significant improvement in the quality of the results with increasing $\beta$, i.e., between $\beta = 6.0$ and $\beta = 6.2$.

Finally, we comment on results presented in
two recent papers. Using the Schrödinger functional, Ref. [8] calculates $b_A - b_P$, $b_S$ and $Z_P^0/(Z_S^0 Z_A^0)$ for a range of $\beta \geq 6$. Their most striking result is that different discretizations of derivatives lead to very different results for $b_A - b_P$. For example, at $\beta = 6$, this quantity varies roughly from 0.17 to $-0.17$. While our number lies within this range, our estimate of $O(a)$ uncertainties is clearly a substantial underestimate.

Reference [9] has determined $c_A$ using the same method and similar lattice parameters as here but with significantly more configurations. They study, at one $\kappa$ (\sim $\kappa_5$ at both $\beta = 6.0$ and $6.2$), the effect of using derivatives that are tree-level improved through $O(a^2)$ (our 3-pt), $O(a^3)$ and $O(a^6)$. They find a larger dependence than what we get between 2-pt and 3-pt discretizations at $\kappa_5$. The $O(a^2)$ errors in the two calculations are, however, different due to the choice of source and the fit range in time. Also, we find that after chiral extrapolation these discretization effects are significantly reduced. Nevertheless, once again the large variation should serve as a warning that the $O(a)$ errors in $c_0$ can be substantial.

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