Dynamical analysis of Lorenz System on traffic problem in Yogyakarta, Indonesia

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Abstract. The traffic congestion becomes a routine problem which occurs in Yogyakarta. This study was to develop a mathematical model of traffic congestion using Lorenz System. This system was used to analyze the traffic condition in Yogyakarta road. The data was taken from the observation of the road and it had been validated. Routh Hourwith used to analyse the stability of free jam equilibrium. The dynamic analysis showed that the system was stabil. Numerical solution showed that the bigger value of ratio of the distance relaxation time to velocity relaxation time resulting the more time to reach the stabil condition. It showed that more and more time to reach optimal velocity, needed more and more time as well to reach equilibrium. Traffic condition at the time of observation was ideal and there was no transition of traffic jam.

1. Introduction
Traffic congestion is a problem that occurs almost in all cities in Indonesia, in which Special District of Yogyakarta as one of the examples. Yogyakarta as the city center of education and culture has a road along 619.34 km with around 2.2 million of vehicles in 2015. Road traffic congestion produces undesirable impacts. Increases the fuel consumption, delays, and air pollution are the examples of congestion impacts, [1,2]. Therefore, an analysis for traffic congestion is necessary to be studied to overcome traffic jam.

Traffic congestion analysis can be obtained from the result of traffic flow analysis. The mathematical models of traffic flow from the scale of detail are divided into three models. Those are microscopic, mesoscopic, and macroscopic. Flow characteristic of different types of different vehicles need to be described differently, [3]. The microscopic model focuses on the behaviour and interaction between two vehicles in a traffic state, [4]. Real time congestion prediction can be modeled based on microscopic model, [5].

There are numerous models developed using microscopic. One of the microscopic models was firstly proposed by Pipes in 1953, [6]. Various mathematical models based on the theory of microscopic car following model have been widely researched and continue to grow. Based on Zhu in [7], among of the microscopic models, car following model is the most suitable one. To describe the dynamical characteristics of the traffic condition, there are numerous factors to be considered. In this paper, some factors are neglected. To evaluate the model performance, the key issue is by understanding the driving behavior [8]. However, drivers always react to the stimuli generated by surrounding vehicles.
Traffic flow models are based on the assumption that there is some relation between the distance between vehicles and velocity. [9]. The problem of traffic congestion transition was represented as a matter of first order non-equilibrium transition phase and analogous to self-organization process. The term self-organization means the process in which the elements interact in order to achieve global function or behaviour. One of the schemes that describe the self-organization process is Lorenz system [10].

Traffic jamming transition problem within the framework of Lorenz system had been studied by several researchers [10,11]. These studies are seem applicable to be implemented in Yogyakarta’s road since the divergence properties of the drivers behavior. The result of the research can be used to determine the safe distance for one vehicle to the one in front. This paper discusses the construction of traffic congestion model with Lorenz system and its application on a road in Yogyakarta. This paper also discusses the stability of the equilibrium point to analyze traffic conditions and simulate numerically using the Runge Kutta method using Octave software.

2. Mathematical model of traffic jam with lorenz system

In this section, we determine the modeling process of the Lorenz system in traffic congestion situations which are depending on the distance deviation, velocity and time acceleration of observed vehicles as a function of time [11]. The several parameters and variables that need to be considered can be seen in Table 1.

| Notation | Means |
|----------|-------|
| \( h \) | The optimum vehicle distance in a traffic condition, in this case, the safe distance or the ideal distance between vehicles is taken. |
| \( x_k \) | Position of the front of the \( k \)-th observed vehicle. |
| \( \mu \) | The distance deviation between the observed vehicle distance and the optimal vehicle distance. |
| \( t_0 \) | Ideal time to get the optimum distance. |
| \( v_0 \) | Optimum speed vehicles |
| \( v_t \) | Velocity of observed vehicles |
| \( \tau \) | Time acceleration of observed vehicles |
| \( \tau_0 \) | Initial acceleration of the observed vehicle |
| \( t_v \) | Relaxation time it takes a vehicle that was observed by his speed to get the speed deviation equal to 0. |
| \( t_\mu \) | The required relaxation time of the vehicle being observed at the speed it has to obtain a distance deviation equal to 0. |
| \( t_\tau \) | Relaxation time it takes a vehicle that was observed with the speed he has to get acceleration deviation equal to 0. |

Figure 1 is a traffic illustration of two cars on the road. Suppose that \( \Delta x \) is the headway or distance rate of the front of two respectively observed vehicles. Therefore, the distance deviation, the difference between the observed vehicle distance rate and the optimal vehicle distance, in mathematical notation can be written as Equation 1.

\[
\mu = |\Delta x - h|
\]  

((1))
Next, we modeled the velocity deviation as a result of the variation of the distance deviation from the traffic condition to its optimum speed, as in Equation 2.

\[ v \equiv \Delta x - v_x - v_o \]  

(2)

Having explained about the parameters related to the mathematical model of traffic congestion, we formed a derived mathematical model of the three parameters. The first derivative of is the rate of change of the optimal vehicle distance to its stationary position at the moment. It is defined that the rate of parameter change over time will be proportional to the velocity deviation of the vehicle.

\[ \dot{\mu} = \frac{0 - \mu}{t_\mu} + v \]  

(3)

The first derivative of \( v \) is the rate of change of vehicle velocity observed against its stationary velocity at the moment. The parameter change of time will be proportional to the distance deviation and its acceleration. When written in mathematical notation, it is obtained Equation 4.

\[ \dot{v} = \frac{0 - v}{t_v} + g_x \tau_\mu \]  

(4)

The first derivative of \( \tau \) is the rate of change of acceleration of the stationary acceleration at the moment. If the value increases, then it will cause a reduction in the distance deviation and the velocity deviation. So obtained the model as in Equation 5.

\[ \dot{\tau} = \frac{\tau_0 - \tau}{t_x} - g_x \mu \]  

(5)

Equations 4 and 5 consist constant values \( g_x \) and \( g_x \). Therefore, to simplify the application, those have been eliminated through the process of scaling parameters related to [10]. We obtain as in Equation 6.

\[ \bar{\eta} = \frac{\eta}{\eta_m}, \bar{\eta} = \frac{\eta}{\tau}, \bar{y} = \frac{y}{v_m}, \bar{\tau} = \frac{\tau}{\tau_m}, \bar{t} = \frac{t}{\tau}, \bar{g}_x = g_x \]  

\[ \eta_m = \left(t_x g_x t_{g_x}\right)^{-1/2}, v_m = \frac{\left(t_x g_x t_{g_x}\right)^{1/2}}{t_x}, \tau_m = \left(t_x g_x\right)^{-1} \]  

(6)

Here, the Equation 3, 4 and 5 can be written in equation system 7.

\[
\begin{align*}
\dot{\mu} & = -\mu + v \\
\dot{v} & = \frac{-\left(v - \mu \tau\right)}{\alpha} \\
\dot{\tau} & = \frac{\left(\tau_0 - \tau\right) - \mu \nu}{\delta}
\end{align*}
\]  

(7)
Where $\alpha = \frac{t_v}{t_\mu}, \delta = \frac{t_\tau}{t_\mu}$. This system (7) is then called as the Lorenz System of traffic congestion transition problems.

3. Dynamic analysis of traffic situation in Yogyakarta

To analyze the traffic condition is needed observation data. Data can be found by observing velocity in certain position and after certain time. Time relaxation can be found using proposition method. After the data examined being found that the Cronbach's Alpha value Based on Standardized Items is 0.992. Showing that the data is reliable. Table 2 is a result of observations made on traffic conditions on the northern ring road.

| Information | Value |
|-------------|-------|
| $t_\mu$     | 10    |
| $t_\tau$    | 2     |
| $\tau_0$    | 7     |
| $\delta$    | 1     |
| $\mu$       | 5     |

Based on the table has been analysed the situation of road traffic in ring road north, Yogyakarta. By substituting the parameter values into system 7, the mathematical model for traffic congestion problem in ring road can be written in system 8.

$$
\begin{align*}
\dot{\mu} &= -\mu + \nu \\
\dot{\nu} &= -\left(\nu - \mu \tau\right) / \alpha \\
\dot{\tau} &= \frac{35}{4} - 5\tau - 5\mu \nu
\end{align*}
$$

(8)

Next will be analyzed the stability of the System 8 by finding the equilibrium point(s).

Lemma 1

System 8 has an equilibrium point, ie $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1\right)$.

Proof:
An equilibrium point $(\dot{\mu}, \dot{\nu}, \dot{\tau})$ will occur if the rate of change of the system over time will be equal to zero. Therefore from the system 8, we obtained

$$
\begin{align*}
\dot{\mu} &= \dot{\nu} \quad \text{and} \quad \dot{\nu} = \dot{\mu} \dot{\tau}
\end{align*}
$$

(9)

So, here we obtained that

$$
\dot{\mu} = 0, \quad \text{and} \quad \dot{\tau} = 1.
$$

Next, for the rate of change of acceleration deviation, the equilibrium point should satisfies

$$
0 = \frac{35}{4} - 5\dot{\tau} - 5\dot{\nu} \dot{\mu}
$$

(10)

Because of the value of $\dot{\mu} = \dot{\nu}$ and $\dot{\tau} = 1$ from the equation 10 we obtained that

$$
\dot{\mu} = \pm \frac{\sqrt{3}}{2}
$$
Since \( \mu \) is the distance deviation between the observed vehicle distance and the optimal vehicle distance, the value should not be negative. Therefore, \( \hat{\mu} = \frac{\sqrt{3}}{2} \). Thus, the equilibrium point of System 8 is \( (\hat{\mu}, \hat{\nu}, \hat{\tau}) = \left( \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1 \right) \).

**Lemma 2**

System (2) is said to be stable in the interval \( \alpha = (0, \infty) \).

**Proof:**

The system resulted in nonlinear systems equations so that the linearization process is required to obtain a solution. Suppose the linear equation using Jacobian matric be

\[
\begin{pmatrix}
\dot{\mu} \\
\dot{\nu} \\
\dot{\tau}
\end{pmatrix} = 
\begin{pmatrix}
-1 & 1 & 0 \\
\frac{\tau}{\alpha} & -\frac{1}{\alpha} & \frac{\mu}{\alpha} \\
-5\nu & -5\mu & -5
\end{pmatrix}
\begin{pmatrix}
\mu \\
\nu \\
\tau
\end{pmatrix}
\]

The above linear system will be stable if the real part of its eigen value is negative. Whatever the above eigen system values are obtained from

\[
\lambda^3 + \left( 6 + \frac{1}{\alpha} \right) \lambda^2 + \left( \frac{5\mu^2 + 5\alpha - \tau + 6}{\alpha} \right) \lambda + \left( \frac{5\mu^2 + 5\mu\nu - 5\tau + 5}{\alpha} \right) = 0
\]

Substitution point of equilibrium \( (\hat{\mu}, \hat{\nu}, \hat{\tau}) = \left( \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1 \right) \), so that Equation (12) becomes

\[
\lambda^3 + \left( 6 + \frac{1}{\alpha} \right) \lambda^2 + \left( \frac{35}{4\alpha} + 5 \right) \lambda + \left( \frac{30}{4\alpha} \right) = 0
\]

Furthermore, to obtain the eigen value with the real negative part used Routh-Hurwitz criteria that shown in Table 3. If the value in the second column of the table in the Routh-Hurwitz criterion has the same mark (all positive or all negative) then it will satisfy all of its eigen values having a negative real part.

| \( \lambda \) | \( a_0 \) | \( a_1 \) | \( a_2 \) | \( b_1 \) | \( c_0 \) |
|---|---|---|---|---|---|
| \( \lambda^3 \) | 1 | \( 6 + \frac{1}{\alpha} \) | \( \frac{35}{4\alpha} + 5 \) | \( \frac{24(\alpha + 20)}{24} - \frac{393}{24} \) | \( \frac{35}{4\alpha} + 5 \) |
| \( \lambda^2 \) | \( \frac{35}{4\alpha} \) | \( 0 \) | \( \frac{30}{4\alpha} \) |
| \( \lambda^1 \) | \( b_1 \) | \( 0 \) |
| \( \lambda^0 \) | \( c_1 \) | \( 0 \) |

Due to a positive value, all values are in the second column should be positive in this case must be greater than zero. Therefore, we obtained

\[
a_1 = 6 + \frac{1}{\alpha} > 0 \rightarrow \alpha > -\frac{1}{6} = -0.166, \alpha \neq 0
\]
\[ a_2 = \frac{35}{4\alpha} + 5 > 0 \rightarrow \alpha > 0 \]
\[ a_3 = \frac{30}{4\alpha} > 0 \rightarrow \alpha > 0 \]
\[ b_1 = \frac{5}{4\alpha(6\alpha + 1)} \left( 24\left(\alpha + \frac{20}{24}\right)^2 - \frac{393}{24} \right) > 0 \rightarrow \alpha < -\frac{20 - \sqrt{393}}{24} = -1.6591 \text{ or } \alpha > -\frac{20 + \sqrt{393}}{24} = -0.0075 \]

The equilibrium point is \( \left( \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1 \right) \) means that the deviation between the vehicle distance and the optimal distance is \( \frac{\sqrt{3}}{2} \). The deviation between the vehicle velocity and the optimal velocity is \( \frac{\sqrt{3}}{2} \). The time acceleration is 1. It has been analyzed that the equilibrium point is stable. It means that longer time remaining on the equilibrium state. When the data was taken the condition of the traffic was dense but smooth.

4. Analysis traffic conditions in Yogyakarta

It was more appropriate to build a model using microscopic because of the divergence drivers. This model had considered to accommodate the velocity difference of drivers in the same traffic condition [4]. After analysing dynamical behavior, numerical solution analysing to know the changing of the \( \alpha \) value was done. Figures 2 and 3 show numerical solution with a different \( \alpha \) value.

![Figure 2. Traffic Flow model at \( \alpha = 2 \)

![Figure 3. Traffic Flow model at \( \alpha = 8 \)]

Figures 2 and 3 show that the numerical solution leading to equilibrium position that is \( \left( \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1 \right) \). It has been in accordance to the dynamical behavior. The traffic will reach the optimum condition and the stability will not change. Using the same initial value that is \( (1.3, 1.3, 0.99999) \), it is analyzed the difference of \( \alpha \) value. From figures 3 and 4, showing that the bigger \( \alpha \) value the longer to reach stability. At \( \alpha = 2 \), after fifteen minutes stabilization just can be reached. At \( \alpha = 8 \), after twenty minutes stabilization just can be reached. It showed that more and more time to reach optimal velocity, needed more and more time as well to reach equilibrium.

After analysing Lorenz model solution, the distance was analysed.
Vehicle road distance can be analyzed using distance deviation transformation. Figures 4 and 5 show the road distances at $\alpha = 2$ and at $\alpha = 8$. From Figures 4 and 5 show that the ratio between distance relaxation time and velocity relaxation time is bigger, the road distance will be farther. Figures 4 showing that the road distance is 166,7660 during 10 minutes unity at $\alpha = 2$. Figures 5 showing that the road distance is 167,8080 during 10 minutes at $\alpha = 8$. Smaller value of $\alpha$ caused more relaxation time to reach optimal distance and resulted shorter road distance.

5. Conclusion
Mathematical model of traffic jam considering divergence driver has been modeled into Lorentz system. Stable equilibrium point has been found in this system. It indicated that traffic jam remained stable. Equilibrium point resulted was equilibrium free of jam. Numerical simulation showed that bigger value of $\alpha$ caused more time to reach stability. Parameter resulted could be used as a reference to get traffic situation without jam. The model and reality had similar traffic behavior.

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