SUSY Associated Vector Coherent States and Generalized Landau Levels Arising From 2-dimensional SUSY

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Abstract

We describe a method for constructing vector coherent states for quantum supersymmetric partner Hamiltonians. The method is then applied to such partner Hamiltonians arising from a generalization of the fractional quantum Hall effect. Explicit examples are worked out.

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I Introduction

Two quantum mechanical problems are addressed in this paper. The first is the construction of vector coherent states, associated to supersymmetric pairs of Hamiltonians and the second is a generalization of the concept of Landau levels in the fractional quantum Hall effect, via supersymmetric pairs of Hamiltonians. Coherent states, in the context of supersymmetric quantum mechanics, have been studied before (see for example, [14, 15, 18]). These attempts were mainly centered around building such states from the eigenvectors of the Hamiltonian for the fermionic sector, exploiting the trilinear lowering operator that can be constructed using these vectors. In this paper we adopt a different strategy, in that we build vector coherent states using the eigenvectors and eigenvalues of the pair of supersymmetric Hamiltonians. This gives us coherent states which represent both the bosonic and fermionic sectors. Our construction also makes contact with another suggestion, that has recently been made in the literature, which explicitly uses anti-commuting Grassmann variables [12, 19], to introduce a quantization using super Toeplitz operators.

The so-called Landau levels appear in the analysis of the quantum motion of an electron in a uniform magnetic field. This, in turn, is the building block of a fascinating problem in many-body theory, the quantum Hall effect (QHE), (see [3] and references therein). We will not discuss here the role of these Landau levels in the context of the QHE, which have been analyzed in many papers and textbooks. Rather, we shall show how the use of two-dimensional supersymmetry (2d-SUSY), as discussed in [13], can be useful to construct different super-partner Hamiltonians which, in many ways, behave analogously to the Hamiltonian of the electron in the magnetic field. Finally, as already mentioned, we shall construct vector coherent states using these pairs partner Hamiltonians.
II VCS for SUSY quantum models

In this section we outline a method for building vector coherent states (VCS) for supersymmetric (SUSY) quantum models. A SUSY model (see, for example, [17]) consists of two Hamiltonians, $H_b$ and $H^f$, acting on a Hilbert space $\mathcal{H}$ and factorizable in the manner,

$$H_b = A^\dagger A, \quad H^f = AA^\dagger.$$  (2.1)

Each Hamiltonian has a purely discrete spectrum and the two spectra coincide, except possibly, for the lowest eigenvalue. Let us denote the normalized eigenvectors of $H_b$ by $\phi_n^b$, $n = 0, 1, 2, \ldots \infty$, and those of $H^f$ by $\phi_n^f$, $n = 0, 1, 2, \ldots \infty$. We shall assume the lowest eigenvalue of $H_b$ to be zero and that of $H^f$ to coincide with the first non-zero eigenvalue of $H_b$. Thus, we write $\varepsilon_n$, with $\varepsilon_0 = 0$, for the eigenvalues corresponding to the eigenvectors $\phi_n^b$, $n = 0, 1, 2, \ldots$, and $\varepsilon_{n+1}$ for the eigenvalues corresponding to the eigenvectors $\phi_n^f$, $n = 0, 1, 2, \ldots$. The operators $A$ and $A^\dagger$ act on the eigenvectors in the manner,

$$A\phi_n^b = \sqrt{\varepsilon_n} \phi_{n-1}^f, \quad A\phi_n^b = 0, \quad A^\dagger\phi_n^f = \sqrt{\varepsilon_{n+1}} \phi_{n+1}^b, \quad n = 0, 1, 2, \ldots,$$  (2.2)

and each set of eigenvectors forms an orthonormal basis for $\mathcal{H}$. The full SUSY Hamiltonian, $H^{\text{SUSY}}$ is then defined as

$$H^{\text{SUSY}} = \begin{pmatrix} H_b & 0 \\ 0 & H^f \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix},$$  (2.3)

on the Hilbert space $\mathcal{H}^{\text{SUSY}} = \mathbb{C}^2 \otimes \mathcal{H}$. The Hamiltonian can also be written as $H^{\text{SUSY}} = \{ Q^\dagger, Q \}$, where $Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$ and $Q^\dagger$ are the supercharges. On $\mathcal{H}^{\text{SUSY}}$ we define the vectors

$$\Phi_n^b = \begin{pmatrix} \phi_n^b \\ 0 \end{pmatrix}, \quad \Phi_n^f = \begin{pmatrix} 0 \\ \phi_n^f \end{pmatrix}, \quad n = 0, 1, 2, \ldots,$$  (2.4)

which together form an orthonormal basis for this Hilbert space.
II.1 Construction of the VCS

Vector coherent states (VCS), of the type we are about to construct here, have been introduced in [1, 22] and we shall follow the method outlined there to build vector coherent states for SUSY systems. We start by defining the vectors,

$$\Psi_0 = \begin{pmatrix} \phi^b_0 \\ 0 \end{pmatrix}, \quad \Psi_n = \Phi^b_n \oplus \Phi^f_{n-1} = \begin{pmatrix} \phi^b_n \\ \phi^f_{n-1} \end{pmatrix}, \quad n = 1, 2, 3, \ldots \quad (2.5)$$

These vectors are mutually orthogonal but not all normalized:

$$\|\Psi_0\|^2 = 1, \quad \|\Psi_n\|^2 = 2, \quad n = 1, 2, 3, \ldots.$$ 

However, they are eigenvectors of the SUSY Hamiltonian:

$$H_{\text{SUSY}} \Psi_n = \varepsilon_n \Psi_n, \quad n = 0, 1, 2, \ldots, \quad (2.6)$$

but they do not span all of $\mathfrak{H}_{\text{SUSY}}$ since, for instance, the vector $\begin{pmatrix} 0 \\ \phi^f_0 \end{pmatrix}$ belongs to $\mathfrak{H}_{\text{SUSY}}$ but cannot be written as a linear combination of the $\Psi_n$'s.

Next let $\lim_{n \to \infty} \varepsilon_n = L$, which could be infinity, and define the domain $\mathcal{D} = \{ z \in \mathbb{C} \mid |z| < \sqrt{L} \} \subseteq \mathbb{C}$. We also assume that the sequence $\{\varepsilon_n!\}_{n=0}^{\infty}$, where, by definition $\varepsilon_0! = 1$ and $\varepsilon_n! = \varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots \varepsilon_n$, $n = 0, 1, 2, \ldots$, is a moment sequence. This means that we assume that there exists a measure $d\lambda$ on $(0, \sqrt{L})$ such that

$$2\pi \int_0^{\sqrt{L}} r^{2n} d\lambda(r) = \varepsilon_n!, \quad n = 0, 1, 2, \ldots \quad (2.7)$$

Vector coherent states $|z, \bar{z}\rangle \in \mathfrak{H}_{\text{SUSY}}$ are now defined, for each $z \in \mathcal{D}$, as

$$|z, \bar{z}\rangle = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\varepsilon_n!}} \Psi_n,$$

$$= \mathcal{N}(|z|^2)^{-\frac{1}{2}} \left[ \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\varepsilon_n!}} \Phi^b_n + \sum_{n=0}^{\infty} \frac{\bar{z}^{n+1}}{\sqrt{\varepsilon_{n+1}!}} \Phi^f_n \right], \quad (2.8)$$

$$= \begin{pmatrix} z \\ 0 \end{pmatrix}.$$
where the normalization constant,

\[ \mathcal{N}(|z|^2) = 1 + 2 \sum_{n=1}^{\infty} \frac{|z|^{2n}}{\varepsilon_n!}, \]  

(2.9)
is chosen so that \( \langle z, \overline{z} \mid z, \overline{z} \rangle = 1 \), independently of \( z \in \mathcal{D} \). Notice that this series converges for all \( z \in \mathcal{D} \). Defining the measure

\[ d\mu(z, \overline{z}) = d\lambda(r) \, d\theta, \quad \text{where} \quad z = re^{i\theta}, \]

it is easy to verify that these VCS satisfy the resolution of the identity,

\[ \int_{\mathcal{D}} |z, \overline{z}\rangle \langle z, \overline{z}| \, \mathcal{N}(|z|^2) \, d\mu(z, \overline{z}) = I_{\mathcal{H}_{\text{SUSY}}}, \]  

(2.10)
on \( \mathcal{H}_{\text{SUSY}} \). We shall call the vectors (2.8) \textit{SUSY associated VCS}. The term vector coherent state reflects the fact that they can also be written as the two-component vectors:

\[ |z, \overline{z}\rangle = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \begin{pmatrix} \frac{z^n}{\sqrt{\varepsilon_n}} \phi_n^b \\ \frac{\overline{z}^{n+1}}{\sqrt{\varepsilon_{n+1}}} \phi_n^f \end{pmatrix}. \]  

(2.11)

\section*{II.2 Holomorphic representation}

Let us re-emphasize that the VCS (2.8) are built using eigenvectors of the SUSY Hamiltonian, with the degeneracy of the levels \( \varepsilon_n \), \( n = 1, 2, 3, \ldots \), reflected in the choice of the vectors \( \Psi_n \), \( n = 1, 2, 3, \ldots \). We proceed to study some analytic features of these VCS. Consider the Hilbert space \( L^2(\mathcal{D}, d\mu(z, \overline{z})) \), in which we identify the two subspaces, \( \mathcal{H}_{\text{hol}}^b \), consisting of all functions analytic in \( z \), including the constant function and \( \mathcal{H}_{\text{hol}}^f \), consisting of all functions analytic in \( \overline{z} \), excluding the constant function. Clearly, the two subspaces are mutually orthogonal. We write \( \mathcal{H}_{\text{hol}} = \mathcal{H}_{\text{hol}}^b \oplus \mathcal{H}_{\text{hol}}^f \), for the subspace consisting of all functions either analytic or anti-analytic in \( z \). Let \( P^b_{\text{hol}} \) and \( P^f_{\text{hol}} \) be the corresponding projection operators:

\[ P^b_{\text{hol}} \mathcal{H}_{\text{hol}} = \mathcal{H}_{\text{hol}}^b, \quad P^f_{\text{hol}} \mathcal{H}_{\text{hol}} = \mathcal{H}_{\text{hol}}^f. \]  

(2.12)
Note that the Hilbert space $\mathcal{H}_{\text{SUSY}} = \mathbb{C}^2 \otimes \mathcal{H}$ can also be written as the direct sum
\[
\mathcal{H}_{\text{SUSY}} = \mathcal{H}_{\text{SUSY}}^b \oplus \mathcal{H}_{\text{SUSY}}^f,
\]
(2.13)
of a bosonic subspace $\mathcal{H}_{\text{SUSY}}^b$, spanned by the vectors $\Phi_n^b$ and a fermionic subspace $\mathcal{H}_{\text{SUSY}}^f$, spanned by the vectors $\Phi_n^f$.

In view of the resolution of the identity (2.10), the mapping
\[
W : \mathcal{H}_{\text{SUSY}} \longrightarrow \mathcal{H}_{\text{hol}}, \quad (W \Phi)(z, \bar{z}) = \mathcal{N}(|z|^2)^{1/2} \langle \bar{z}, z | \Phi \rangle,
\]
(2.14)
where the order of $z$ and $\bar{z}$ is important, is unitary, and maps the bosonic sector $\mathcal{H}_{\text{SUSY}}^b$ onto the subspace $\mathcal{H}_{\text{hol}}^b$ of analytic functions in $z$ (including the constant function) and the fermionic sector $\mathcal{H}_{\text{SUSY}}^f$ onto the subspace $\mathcal{H}_{\text{hol}}^f$ of analytic functions in $\bar{z}$ (excluding the constant function). It is easy to see that under this mapping the basis vectors $\Phi_n^b$ and $\Phi_n^f$ transform into the monomials,
\[
(W \Phi_n^b)(z, \bar{z}) = \frac{z^n}{\sqrt{\varepsilon_n!}} := \xi_n(z),
\]
\[
(W \Phi_n^f)(z, \bar{z}) = \frac{\bar{z}^{n+1}}{\sqrt{\varepsilon_{n+1}!}} = \xi_{n+1}(z), \quad n = 0, 1, 2, \ldots
\]
(2.15)
so that the vectors $\Psi_n$, used to construct the VCS, transform to
\[
(W \Psi_0)(z, \bar{z}) = \xi_0(z) = 1,
\]
\[
(W \Psi_n)(z, \bar{z}) = \xi_n(z) + \frac{\bar{z}^{n-1}}{\sqrt{\varepsilon_{n-1}!}}, \quad n = 1, 2, \ldots
\]
(2.16)
We shall then write,
\[
|z, \bar{z}\rangle_{\text{hol}} := W |z, \bar{z}\rangle = \mathcal{N}(|z|^2)^{-1/2} \left[ \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\varepsilon_n!}} \xi_n + \sum_{n=1}^{\infty} \frac{\bar{z}^{n-1}}{\sqrt{\varepsilon_{n-1}!}} \bar{\xi}_n \right],
\]
(2.17)
Also writing,
\[
Q_{\text{hol}} = W Q W^{-1}, \quad Q_{\text{hol}}^\dagger = W Q^\dagger W^{-1},
\]
(2.18)
for the ‘holomorphic supercharges’, we see that they act on the vectors $\xi_n$ as follows:
\[
Q_{\text{hol}} \left( \frac{z^n}{\sqrt{\varepsilon_n!}} \right) = \frac{\bar{z}^{n-1}}{\sqrt{\varepsilon_{n-1}!}}, \quad Q_{\text{hol}}^\dagger \left( \frac{\bar{z}^{n}}{\sqrt{\varepsilon_n!}} \right) = \frac{z^n}{\sqrt{\varepsilon_{n-1}!}}, \quad n = 1, 2, 3, \ldots
\]
and

\[ Q_{\text{hol}} \xi_0(z) = Q_{\text{hol}}^\dagger \xi_0(z) = Q_{\text{hol}} \xi_n(\bar{z}) = Q_{\text{hol}}^\dagger \xi_n(z) = 0, \quad n = 1, 2, 3, \ldots, \]

so that, apart from the constant function, they basically interchange the holomorphic and antiholomorphic sectors. Clearly,

\[ \{Q_{\text{hol}}^\dagger, Q_{\text{hol}}\} = Q_{\text{hol}}^\dagger Q_{\text{hol}} + Q_{\text{hol}} Q_{\text{hol}}^\dagger = WH^{\text{SUSY}} W^{-1} =: H_{\text{hol}}^{\text{SUSY}}. \tag{2.19} \]

Note that the ground state wave function, \( \xi_0 \), used in constructing the VCS in (2.17), satisfies

\[ Q_{\text{hol}} \xi_0 = Q_{\text{hol}}^\dagger \xi_0 = 0, \]

which is reflective of the fact that we are using a model where SUSY is unbroken.

### II.3 Creation and annihilation operators

Suppose we define the formal annihilation operator, \( \mathcal{A} \), by its action on the VCS (2.8),

\[ \mathcal{A}|z, \bar{z}\rangle = 3|z, \bar{z}\rangle, \tag{2.20} \]

where, on the right hand side, multiplication of the vector \( |z, \bar{z}\rangle \), considered as an element in \( \mathbb{C}^2 \), by the matrix \( 3 \) is implied. It is easily seen that the above equation is recovered by the following action of \( \mathcal{A} \) on the vectors \( \Psi_n \):

\[ \mathcal{A} \Psi_0 = 0, \quad \mathcal{A} \Psi_n = \sqrt{\varepsilon_n} \Psi_{n-1}, \quad n = 1, 2, \ldots, \tag{2.21} \]

which has the familiar form of shift operators. We would like to define an adjoint operator, \( \mathcal{A}^\dagger \), such that \( \mathcal{A}^\dagger \mathcal{A} \) would coincide with \( H^{\text{SUSY}} \). However, since the vectors \( \Psi_n \) do not span the whole of \( \mathcal{F}^{\text{SUSY}} \) and since they are not all normalized, the usual relations, \( \mathcal{A}^\dagger \Psi_n = \sqrt{\varepsilon_{n+1}} \Psi_{n+1} \), will not define the adjoint. In fact, if we compute the adjoint of \( \mathcal{A} \) on the subspace generated by the orthonormal set of vectors \( \Psi_0, \frac{1}{\sqrt{2}} \Psi_n, n = 1, 2, 3, \ldots, \)

we easily obtain,

\[ \mathcal{A}^\dagger \Psi_0 = \sqrt{\frac{\varepsilon_1}{2}} \frac{\Psi_1}{\sqrt{2}}, \quad \mathcal{A}^\dagger \frac{\Psi_n}{\sqrt{2}} = \sqrt{\frac{\varepsilon_{n+1}}{2}} \frac{\Psi_{n+1}}{\sqrt{2}}, \quad n = 1, 2, 3, \ldots. \]
Also, it is easily checked that $A^\dagger A\Psi_n = \varepsilon_n \Psi_n$, $n = 0, 2, 3, \ldots$, but $A^\dagger A\Psi_1 = \frac{\varepsilon_1}{2} \Psi_1$, so that $A^\dagger A$ does coincide with $H_{\text{SUSY}}$ on this subspace.

To proceed further, we first extend $A$ and $A^\dagger$ to the entire set of basis vectors $\Phi_n^b, \Phi_n^f, n = 0, 1, 2, \ldots$, (see (2.4)), spanning $H_{\text{SUSY}}$. Let $a_b, a_b^\dagger$ denote the usual shift operators in $H_{\text{SUSY}}$ acting on the normalized eigenvectors, $\phi_n^b, n = 0, 1, 2, \ldots$, of the bosonic Hamiltonian $H^b = A^\dagger A$ (see (2.1)-(2.2)):

$$a_b \phi_0^b = 0, \quad a_b^\dagger \phi_n^b = \sqrt{\varepsilon_n} \phi_{n-1}^b, \quad n = 1, 2, 3, \ldots, \quad a_b^\dagger \phi_0^b = \sqrt{\varepsilon_1} \phi_1^b, \quad n = 0, 1, 2, \ldots,$$

(2.22)

Then $a_b^\dagger a_b = A^\dagger A = H^b$. We now want to define similar operators $a_f, a_f^\dagger$, acting on the normalized eigenvectors $\phi_n^f, n = 0, 1, 2, \ldots$, of the fermionic Hamiltonian $H^f = A A^\dagger$, such that $a_f^\dagger a_f = AA^\dagger$. Note, however, that the lowest eigenvalue of $H^f$ is $\varepsilon_1 \neq 0$.

Let us start by defining

$$a_f \phi_n^f = \sqrt{\varepsilon_{n+1}} \phi_{n-1}^f, \quad n = 1, 2, 3, \ldots, \quad a_f^\dagger \phi_0^f = 0,$$

$$a_f^\dagger \phi_n^f = \sqrt{\varepsilon_{n+2}} \phi_{n+1}^f, \quad n = 0, 1, 2, \ldots$$

(2.23)

However, this gives $a_f^\dagger a_f \phi_0^f = 0$ and $a_f^\dagger a_f \phi_n^f = \varepsilon_n \phi_n^f, n = 1, 2, 3, \ldots$. In order to correct for the appearance of 0 and not $\varepsilon_1$ as the lowest eigenvalue, it is convenient to extend the operators $a_f^\dagger, a_f$ to a larger Hilbert space. To do this, we adjoin an abstract vector $\tilde{\chi}$ to the Hilbert space $H_{\text{SUSY}}$ and extend its scalar product so that $\tilde{\chi}$ has unit norm and is orthogonal to $H_{\text{SUSY}}$ in this product. Let $\tilde{H}_{\text{SUSY}}$ and $\langle \cdot \mid \cdot \rangle^\sim$ denote this extended space and scalar product, respectively, so that,

$$\langle \chi \mid \chi \rangle^\sim = 1, \quad \langle \chi \mid \phi \rangle^\sim = 0, \quad \langle \psi \mid \phi \rangle^\sim = \langle \psi \mid \phi \rangle_{\text{SUSY}}, \quad \forall \psi, \phi \in \tilde{H}_{\text{SUSY}}. \quad (2.24)$$

An arbitrary vector $\tilde{\phi} \in \tilde{H}_{\text{SUSY}}$ has the form $\tilde{\phi} = u\chi + v\phi$, for some $u, v \in \mathbb{C}$ and $\phi \in H_{\text{SUSY}}$. On $\tilde{H}_{\text{SUSY}}$ we define the operators $\tilde{a}_f, \tilde{a}_f^\dagger$ as

$$\tilde{a}_f \chi = 0, \quad \tilde{a}_f \phi_0^f = \sqrt{\varepsilon_1} \chi, \quad \tilde{a}_f^\dagger \phi_0^f = \sqrt{\varepsilon_1} \phi_0^f, \quad n = 1, 2, 3, \ldots,$$

$$\tilde{a}_f^\dagger \chi = \sqrt{\varepsilon_1} \phi_0^f, \quad \tilde{a}_f^\dagger \phi_n^f = \sqrt{\varepsilon_{n+1}} \phi_{n+1}^f, \quad n = 0, 1, 2, \ldots$$

(2.25)

Clearly, on $H_{\text{SUSY}}$ we have

$$a_f = P_{\text{SUSY}} \tilde{a}_f P_{\text{SUSY}} = \tilde{a}_f P_{\text{SUSY}}, \quad a_f^\dagger = P_{\text{SUSY}} \tilde{a}_f^\dagger P_{\text{SUSY}} = \tilde{a}_f^\dagger P_{\text{SUSY}}, \quad \tilde{a}_f^\dagger \tilde{a}_f P_{\text{SUSY}} = AA^\dagger,$$

(2.26)
\( \mathbb{P}_\Phi \) being the projector from \( \tilde{\mathcal{H}} \) to \( \mathcal{H} \) which acts as \( \mathbb{P}_\Phi \tilde{\Phi} = \tilde{\Phi} - \langle \chi, \tilde{\Phi} \rangle \chi \). We similarly extend the fermionic subspace \( \tilde{\mathcal{H}}_{\text{SUSY}}^f \) of \( \mathcal{H}_{\text{SUSY}} \) (see (2.13)), by adding the vector

\[
\tilde{\Phi}_{00} = \begin{pmatrix} 0 \\ \chi \end{pmatrix},
\]

(2.27)

and extending the scalar product, as before, so that \( \tilde{\Phi}_{00} \) has unit norm and is orthogonal to \( \tilde{\mathcal{H}}_{\text{SUSY}}^f \). We denote the extended space by \( \tilde{\mathcal{H}}_{\text{SUSY}}^f \) and write

\[
\tilde{\mathcal{H}}_{\text{SUSY}} = \mathcal{H}_{\text{SUSY}}^b \oplus \tilde{\mathcal{H}}_{\text{SUSY}}^f.
\]

On this extended space \( \tilde{\mathcal{H}}_{\text{SUSY}}^f \), we now define the two two operators,

\[
\tilde{A} = \begin{pmatrix} a_b & 0 \\ 0 & \tilde{a}_f \end{pmatrix}, \quad \tilde{A}^\dagger = \begin{pmatrix} a_b^\dagger & 0 \\ 0 & \tilde{a}_f^\dagger \end{pmatrix},
\]

(2.28)

so that, denoting the projector from \( \tilde{\mathcal{H}}_{\text{SUSY}}^f \) to \( \mathcal{H}_{\text{SUSY}} \) by \( \tilde{\mathbb{P}} \), we set

\[
\mathcal{A}_{\text{SUSY}} = \tilde{\mathbb{P}} \tilde{A} \tilde{\mathbb{P}}, \quad \mathcal{A}^\dagger_{\text{SUSY}} = \tilde{\mathbb{P}} \tilde{A}^\dagger \tilde{\mathbb{P}}.
\]

(2.29)

Clearly, \( \mathcal{A}_{\text{SUSY}} \) is the extension to \( \tilde{\mathcal{H}}_{\text{SUSY}}^f \) of the operator \( \mathcal{A} \) defined in (2.20)-(2.21). Also, these operators act on the vectors \( \tilde{\Phi}_{b}^n, \tilde{\Phi}_{f}^n, \) \( n = 0, 1, 2, 3, \ldots \), in the expected manner:

\[
\mathcal{A}_{\text{SUSY}} \tilde{\Phi}_{b}^n = 0, \quad \mathcal{A}_{\text{SUSY}} \tilde{\Phi}_{b}^n = \sqrt{\varepsilon_n} \tilde{\Phi}_{b}^{n-1}, \quad n = 1, 2, \ldots,
\]

\[
\mathcal{A}_{\text{SUSY}}^\dagger \tilde{\Phi}_{b}^n = \sqrt{\varepsilon_{n+1}} \tilde{\Phi}_{b}^{n+1}, \quad n = 1, 2, \ldots
\]

\[
\mathcal{A}_{\text{SUSY}} \tilde{\Phi}_{f}^n = 0, \quad \mathcal{A}_{\text{SUSY}} \tilde{\Phi}_{f}^n = \sqrt{\varepsilon_{n+1}} \tilde{\Phi}_{f}^{n-1}, \quad n = 1, 2, 3, \ldots,
\]

\[
\mathcal{A}_{\text{SUSY}}^\dagger \tilde{\Phi}_{f}^n = \sqrt{\varepsilon_{n+2}} \tilde{\Phi}_{f}^{n+1}, \quad n = 0, 1, 2, \ldots.
\]

(2.30)

The SUSY Hamiltonian can now be written as (see (2.3)):

\[
H_{\text{SUSY}} = \tilde{\mathbb{P}} \begin{pmatrix} a_b^\dagger a_b & 0 \\ 0 & \tilde{a}_f^\dagger \tilde{a}_f \end{pmatrix} \tilde{\mathbb{P}} = \tilde{\mathbb{P}} \tilde{A}^\dagger \tilde{A} \tilde{\mathbb{P}}.
\]

(2.31)

Note that while this Hamiltonian now appears in the form \( B^\dagger B \), with \( B = \tilde{A} \tilde{\mathbb{P}} \), the range of the operator \( B \) includes the additional vector \( \tilde{\Phi}_{00} \) and the domain of \( B^\dagger \) is the extended space \( \tilde{\mathcal{H}}_{\text{SUSY}}^f \).
II.4 VCS on the extended space

It is interesting to define now VCS on the enlarged Hilbert space $\tilde{H}^{\text{SUSY}}$, which extend the SUSY associated VCS introduced in (2.8). We define the vectors (see (2.5)),

$$\tilde{\Psi}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_0^b \\ \chi \end{pmatrix}, \quad \tilde{\Psi}_n = \frac{1}{\sqrt{2}} \Psi_n, \quad n = 1, 2, 3, \ldots,$$

and set

$$|z, \bar{z}\rangle \sim = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{3}{\varepsilon_n!} \tilde{\Psi}_n, \quad \mathcal{Z} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, \quad (2.32)$$

with $\mathcal{N}$ defined as before (see (2.9)). These vectors are normalized. Indeed,

$$\sim\langle z, \bar{z} | z, \bar{z}\rangle \sim = 1,$$

and we still have a resolution of the identity,

$$\int_{\mathcal{D}} |z, \bar{z}\rangle \sim \sim\langle z, \bar{z} | \mathcal{N}(|z|^2) \, d\mu(z, \bar{z}) = I_{\tilde{H}^{\text{SUSY}}}, \quad (2.33)$$

on the enlarged space $\tilde{H}^{\text{SUSY}}$. The physical or SUSY associated VCS (2.8) are now obtained by simple projection,

$$|z, \bar{z}\rangle = \tilde{\mathcal{P}}|z, \bar{z}\rangle \sim, \quad z \in \mathcal{D}. \quad (2.34)$$

Furthermore, we easily verify the relations,

$$\tilde{A}|z, \bar{z}\rangle \sim = \mathcal{Z}|z, \bar{z}\rangle \sim, \quad \tilde{A}\tilde{\Psi}_n = \sqrt{\varepsilon_n} \tilde{\Psi}_{n-1}, \quad \tilde{A}^\dagger \tilde{\Psi}_n = \sqrt{\varepsilon_{n+1}} \tilde{\Psi}_{n+1}.$$

Finally let us note that the appearance of the vectors $\chi$ and $\Phi_{00}$ in the discussion (see (2.27)) above is not entirely spurious. Indeed, the existence of such a vector is guaranteed when SUSY is not broken. In a generic SUSY model, the two operators, $A$ and $A^\dagger$ act on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dx)$ and have the form:

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad (2.35)$$
where $W(x)$ is a real ‘superpotential’. Since we are assuming that the bosonic ground state $\phi_0^b$ is an eigenstate of $H^b = A^\dagger A$ with eigenvalue $\varepsilon_0 = 0$, this wave function satisfies

$$A\phi_0^b = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \phi_0^b + W(x)\phi_0^b = 0,$$

from which we get

$$\phi_0^b(x) = \exp \left[ -\frac{\sqrt{2m}}{\hbar} \int_0^x W(x') \, dx' \right]. \quad (2.36)$$

Next, if we try to find a vector $\chi$ which would correspond to the zero eigenvalue of $AA^\dagger$, we need to solve

$$A^\dagger \chi = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \chi + W(x)\chi = 0.$$

We thus find

$$\chi(x) = \exp \left[ \frac{\sqrt{2m}}{\hbar} \int_0^x W(x') \, dx' \right], \quad (2.37)$$

which will generally not be square-integrable, if the solution in (2.36) is square-integrable. It is this vector that we adjoined to the Hilbert space $\mathcal{H}$ to obtain the space $\tilde{\mathcal{H}}$ above, but of course, we had to extend the scalar product of $\mathcal{H} = L^2(\mathbb{R}, dx)$ to accommodate it (see (2.24)). Thus, the extended VCS in (2.32) include this “unphysical” vector which is not $L^2$-normalizable.

### II.5 An alternative realization

Before ending this discussion on the general construction of SUSY associated VCS, let us note that the vectors (2.17) can also be written in the standard SUSY formalism, using anticommuting variables. We start by introducing the complex Grassmann variables $\zeta, \bar{\zeta}$ which satisfy

$$\zeta^2 = \bar{\zeta}^2 = 0, \quad \zeta \bar{\zeta} = -\bar{\zeta} \zeta, \quad (2.38)$$

and with respect to the formal measure $d\zeta$ have the “fermionic (Berezin) integration” properties:

$$\int_{\mathbb{C}^{1|1}} \zeta \, d\zeta = \int_{\mathbb{C}^{1|1}} \bar{\zeta} \, d\zeta = \int_{\mathbb{C}^{1|1}} d\zeta = 0, \quad \int_{\mathbb{C}^{1|1}} \bar{\zeta} \zeta \, d\zeta = 1, \quad (2.39)$$

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\( \mathbb{C}^{1|1} \) denoting the formal domain of the Grassmann variable \( \zeta \). We consider now the Hilbert space \( \mathcal{H}_{\text{hol}}^b \) of holomorphic functions, defined earlier, and its subspace \( \mathcal{H}_{\text{hol}}^{1|1} \) which consists of all functions in \( \mathcal{H}_{\text{hol}}^b \) except for the constant function. Consider next functions in the two variables \( z, \zeta \), of the type \( \xi(z, \zeta) = \xi^b(z) + \zeta \psi(z) \), with \( \xi^b \in \mathcal{H}_{\text{hol}}^b \) and \( \psi \in \mathcal{H}_{\text{hol}}^{1|1} \). These functions form a Hilbert space with respect to the scalar product

\[
\langle \xi_1 | \xi_2 \rangle = \int_{D^{1|1}} \xi_1(z, \zeta) \xi_2(z, \zeta) [1 + \zeta \zeta] d\zeta \, d\mu(z, \overline{z})
= \int_D \xi_1^b(z) \xi_2^b(z) d\mu(z, \overline{z}) + \int_D \psi_1(z) \psi_2(z) d\mu(z, \overline{z}) , \tag{2.40}
\]

where \( D^{1|1} \) now denotes the joint domain of the variables \( \zeta \) and \( z \). We denote this Hilbert space by \( \mathcal{K}^{\text{SUSY}} \) and note that (2.40) implies the formal orthogonal decomposition, \( \mathcal{K}^{\text{SUSY}} \simeq \mathcal{H}_{\text{hol}}^b \oplus \mathcal{H}_{\text{hol}}^{1|1} \). The coherent states (2.17), expressed in this alternative notation now appear as

\[
|z, \zeta \rangle = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \left[ \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\varepsilon_n!}} \xi_n + \zeta \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{\varepsilon_n!}} \xi_n \right]
= \mathcal{N}(|z|^2)^{-\frac{1}{2}} \left[ \xi_0 + (1 + \zeta) \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{\varepsilon_n!}} \xi_n \right] , \tag{2.41}
\]

and they satisfy the formal resolution of the identity,

\[
\int_{D^{1|1}} |z, \zeta \rangle \langle z, \zeta | \mathcal{N}(|z|^2) \zeta \zeta - 1 \rangle d\zeta \, d\mu(z, \overline{z}) = \mathcal{I}_{\mathcal{H}_{\text{hol}}^b} \oplus \mathcal{I}_{\mathcal{H}_{\text{hol}}^{1|1}} \simeq \mathcal{I}_{\mathcal{K}^{\text{SUSY}}} , \tag{2.42}
\]

which is to be compared to (2.10).

### III Landau levels

We proceed to apply the theory of supersymmetric coherent states just developed, to certain concrete physical models related to the quantum Hall effect and some of its generalizations.
III.1 Standard Landau levels

The Hamiltonian of a single electron, moving on a two-dimensional plane and subject to a uniform magnetic field along the $z$-direction, is given by

$$H_0 = \frac{1}{2} \left( \mathbf{p} + \mathbf{A}(r) \right)^2 = \frac{1}{2} \left( p_x - \frac{y}{2} \right)^2 + \frac{1}{2} \left( p_y + \frac{x}{2} \right)^2,$$

(3.1)

where we have used minimal coupling and the symmetric gauge $\mathbf{A} = \frac{1}{2}(-y, x, 0)$.

The spectrum of this Hamiltonian is easily obtained by first introducing the new variables

$$P' = p_x - y/2, \quad Q' = p_y + x/2.$$  

(3.2)

In terms of $P'$ and $Q'$ the single electron Hamiltonian, $H_0$, can be rewritten as

$$H_0 = \frac{1}{2}(Q'^2 + P'^2).$$

(3.3)

The transformation (3.2) is part of a canonical map from the phase space variables $(x, y, p_x, p_y)$ to $(Q, P, Q', P')$, where

$$P = p_y - x/2, \quad Q = p_x + y/2.$$  

(3.4)

Indeed, we easily see that

$$\begin{pmatrix} Q \\ Q' \\ P \\ P' \end{pmatrix} = S \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}, \quad \text{where} \quad S = \begin{pmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ -\frac{1}{2} & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 1 & 0 \end{pmatrix},$$

and $S$ is a symplectic matrix:

$$SJST = J, \quad \text{with} \quad J = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix}, \quad \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, at the classical level one also verifies the invariance of the associated two-form:

$$dx \wedge dp_x + dy \wedge dp_y = dQ \wedge dP + dQ' \wedge dP'.$$
under this transformation.

The corresponding quantized operators satisfy the commutation relations:

\[ [x, p_x] = [y, p_y] = i, \quad [x, p_y] = [y, p_x] = [p_x, p_y] = 0, \]

and

\[ [Q, P] = [Q', P'] = i, \quad [Q, P] = [Q', P] = [Q, Q'] = [P, P'] = 0. \quad (3.5) \]

As discussed extensively in the literature (see, for example, \[3\] and references therein), a wave function in the \((x, y)\)-space is related to its \(PP'\)-counterpart by the formula

\[
\Psi(x, y) = \frac{e^{ixy/2}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(xP' + yP + PP')} \Psi(P, P') dP dP',
\]

which can be easily inverted:

\[
\Psi(P, P') = \frac{e^{-iPP'}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(xP' + yP + xy/2)} \Psi(x, y) dxdy.
\]

The usefulness of the \(PP'\)-representation has been widely analyzed in several papers over the years, in particular in connection with the problem of finding the ground state for the fractional quantum Hall effect (QHE), using techniques of multi-resolution analysis (see \[8, 6, 7, 5\] and references therein).

It is clear that, introducing the ladder operators \(B, B^\dagger\) as follows

\[ B = \frac{Q' + iP'}{\sqrt{2}}, \quad B^\dagger = \frac{Q' - iP'}{\sqrt{2}} \Rightarrow [B, B^\dagger] = I, \quad (3.8) \]

and the hamiltonian can be written as \(H_0 = B^\dagger B + \frac{1}{2}\). It is well known that for the standard harmonic oscillator there is not much to be gained by introducing the supersymmetric partner Hamiltonians \(H^b\) and \(H^f\): indeed they are simply the same hamiltonian apart from an additive constant. If we define \(H^b = H_0 - \frac{1}{2}I = B^\dagger B\) and \(H^f = H_0 + \frac{1}{2}I = BB^\dagger\) then the eigenvalues of \(H^b\) are \(E_n^{(b)} = n, n \in \mathbb{N} \cup \{0\}\), and its eigenstates are \(\Psi^{(b)}_n = (\sqrt{E_n^{(b)}}\Psi^{(b)}_0, \) where \(B\Psi^{(b)}_0 = 0, H^b\Psi^{(b)}_n = E_n^{(b)}\Psi^{(b)}_n, \) while for \(H^f\) we have \(E_n^{(f)} = E_{n+1}^{(b)} = n + 1, n \in \mathbb{N} \cup \{0\}, \) and \(\Psi^{(f)}_n = \frac{1}{\sqrt{E_n^{(f)}}} B \Psi^{(b)}_{n+1} = \Psi^{(b)}_n.\)
This illustrates what we can call the triviality of the SUSY approach for the Hamiltonian of the standard Landau levels: $H^b$ and $H^f$ are essentially the same operator, and they are both very closely related to the original quantum mechanical hamiltonian, $H_0$. Nevertheless, the formalism of one-dimensional supersymmetry has been employed in the study of Landau levels in a recent paper, [20]. This was done in a rather complicated way, viz by defining a family of radial Hamiltonians, depending on the orbital angular momentum eigenvalue $\ell$ of the original two-dimensional system. In this way a family of $\ell$-dependent supersymmetric partner hamiltonians were constructed. In other words, a two-dimensional physical system was mapped into an infinite family of one-dimensional systems.

In this paper we adopt a different point of view, using a truly two-dimensional SUSY [13], which we slightly adapt to our purposes.

It is clear that, because of the commutation rules (3.5), each Landau level is infinitely degenerate (see, for example, [5]). It is instructive to construct the vector coherent states associated to this system, since this will also serve as a model for the other cases, discussed below.

Since the energy levels of $H_0$ are infinitely degenerate, we denote the corresponding normalized eigenstates by $|n,k\rangle$, $n,k = 0,1,2,3,\ldots,\infty$, with $H_0 |n,k\rangle = (n+\frac{1}{2}) |n,k\rangle$ and $k$ denoting the degeneracy parameter. These vectors form an orthonormal basis for the Hilbert space $\mathcal{H}$ of the system. Vector coherent states, for the SUSY pair of Hamiltonians $H^b,H^f$ are now defined in $C^2 \otimes \mathcal{H}$ for each degeneracy level $k$, following (2.8), as

$$|z,\bar{z} ; k\rangle = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \left( \frac{z^n}{\sqrt{n!}} \frac{\sqrt{n+1}}{\sqrt{(n+1)!}} \right) \otimes |n,k\rangle , \quad k = 0,1,2,\ldots,\infty . \quad (3.9)$$

Here $\mathcal{N}(|z|^2) = 2e^{|z|^2} - 1$. These vectors then satisfy the resolution of the identity,

$$\sum_{k=0}^{\infty} \int_C |z,\bar{z} ; k\rangle \langle z,\bar{z} ; k| \mathcal{N}(|z|^2) e^{-|z|^2} \frac{dx\,dy}{\pi} = \begin{pmatrix} I_\mathcal{H} & 0 \\ 0 & I_\mathcal{H} \end{pmatrix} , \quad z = x + iy \quad (3.10)$$
III.2 Generalized Landau levels

This section is devoted to the analysis of some quantum mechanical models naturally arising from $H_0$ when SUSY is taken into account.

Introducing the function $\vec{W}_0 = -\frac{1}{2}(x, y, 0) = (W_{0,1}, W_{0,2}, 0)$ we may rewrite the operators in (3.2) and (3.4) as

$$
P' = p_x + W_{0,2}, \quad Q' = p_y - W_{0,1}, \quad P = p_y + W_{0,1}, \quad Q = p_x - W_{0,2}.
$$

This definition can be extended as follows

$$
p' = p_x + W_2, \quad q' = p_y - W_1, \quad p = p_y + W_1, \quad q = p_x - W_2,
$$

introducing a vector superpotential $\vec{W} = (W_1, W_2, 0)$. Our notation is the following: small letters (like $q, p, q'$ and $p'$) refer to a generic superpotential $\vec{W}$, while capital letters (like $Q, P, Q'$ and $P'$) refer to the particular choice of superpotential $\vec{W}_0$, i.e. when we consider the standard Landau levels.

We now put

$$
e = -\frac{1}{\sqrt{2}}(q' + ip''), \quad e^\dagger = -\frac{1}{\sqrt{2}}(q' - ip''), \quad k = -\frac{1}{\sqrt{2}}(q + ip), \quad k^\dagger = -\frac{1}{\sqrt{2}}(q - ip),
$$

where the overall minus sign has been introduced everywhere in order to preserve the same notation as in [13]. Thus, $E = -\frac{1}{\sqrt{2}}(Q' + iP') = -B$, $E^\dagger = -\frac{1}{\sqrt{2}}(Q' - iP') = -B^\dagger$, $K = -\frac{1}{\sqrt{2}}(Q + iP)$, and $K^\dagger = -\frac{1}{\sqrt{2}}(Q - iP)$. The following commutation rules can be easily obtained:

$$
\begin{align*}
[q, p] &= [q', p'] = -i\vec{\nabla} \cdot \vec{W}, \\
[p', p] &= [q', q] = -i(\partial_x W_1) + i(\partial_y W_2), \\
[q', p] &= -2i(\partial_y W_1), \quad [p', q] = 2i(\partial_x W_2),
\end{align*}
$$

which immediately imply

$$
\begin{align*}
[e, e^\dagger] &= [k, k^\dagger] = -\vec{\nabla} \cdot \vec{W}, \\
[k, e] &= (\partial_x W_2) - (\partial_y W_1), \\
[k, e^\dagger] &= -(\partial_x W_2) - (\partial_y W_1).
\end{align*}
$$

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It is easy to check that if we take $\vec{W} = \vec{W}_0$, these commutation relations yield those of the previous subsection. Note also, that classically the transformation (3.12) is canonical, i.e., $dx \wedge dp_x + dy \wedge dp_y = dQ \wedge dP + dQ' \wedge dP'$, if and only if $\vec{W} = \vec{W}_0$, so that $\vec{\nabla} \cdot \vec{W} = -1$.

We now introduce two pairs of supersymmetric partner Hamiltonians

$$h^b = e^\dagger e, \quad h^f = ee^\dagger, \quad h^b = k^\dagger k, \quad h^f = k k^\dagger,$$

which are related to each other by

$$h^b - h^f = h^b - h^f = -\vec{\nabla} \cdot \vec{W} \quad (3.17)$$

Let us focus our attention on $h^b$ and $h^f$ which can also be written as

$$\begin{cases}
    h^b = e^\dagger e = \frac{1}{2}(p_x + W_2)^2 + \frac{1}{2}(p_y - W_1)^2 + \frac{1}{2} \vec{\nabla} \cdot \vec{W}, \\
    h^f = ee^\dagger = \frac{1}{2}(p_x + W_2)^2 + \frac{1}{2}(p_y - W_1)^2 - \frac{1}{2} \vec{\nabla} \cdot \vec{W}.
\end{cases} \quad (3.18)$$

The capital counterparts of these relations turn out to be $H^b = E^\dagger E = \frac{1}{2}(p_x - y/2)^2 + \frac{1}{2}(p_y + x/2)^2 - \frac{1}{2} I = H_0 - \frac{1}{2} I$ and $H^f = E^\dagger E = \frac{1}{2}(p_x - y/2)^2 + \frac{1}{2}(p_y + x/2)^2 + \frac{1}{2} I = H_0 + \frac{1}{2} I$, which we have already discussed. The analysis of $h^b$ and $h^f$ is not significantly different from that of $h^b$ and $h^f$, and will be omitted here.

If we now compare the expression of $H_0$ in (3.1) with those of $h^b - \frac{1}{2} \vec{\nabla} \cdot \vec{W}$ and $h^f + \frac{1}{2} \vec{\nabla} \cdot \vec{W}$ in (3.18), it is easy to see that the superpotential $\vec{W}$ is related to the vector potential and, therefore, to the magnetic field, as follows:

$$A_1 = W_2, \quad A_2 = -W_1, \quad \Rightarrow \vec{B} = \vec{\nabla} \wedge \vec{A} = -\hat{k}(\vec{\nabla} \cdot \vec{W}), \quad (3.19)$$

where $\hat{k} = (0, 0, 1)$. Needless to say that, when $\vec{W} = \vec{W}_0$, the situation reverts to the one discussed in the previous section. However, for different choices of $\vec{W}$, the supersymmetry produces inequivalent conjugate Hamiltonians which, in some sense, extend the original operator $H_0$. Our goal is to find explicit examples of such partner Hamiltonians, whose spectra are completely discrete, with each energy level being infinitely degenerate, and which therefore come under the purview of both a generalized quantum Hall effect and a proper supersymmetric theory.
III.2.1 Case 1: $\vec{\nabla} \cdot \vec{W} = 0$.

At first sight this choice may seem rather trivial since, because of (3.19), it corresponds to a zero magnetic field: $\vec{B} = \vec{0}$. However, as we show below, some non trivial mathematics and physics do nevertheless appear.

Since $\vec{\nabla} \cdot \vec{W} = 0$ we have:

$$
\vec{B} = \vec{0}, \quad h^b = h^t = \frac{1}{2} (p_x + W_2)^2 + \frac{1}{2} (p_y - W_1)^2, \quad [e, e^\dagger] = [k, k^\dagger] = 0,
$$

(3.20)

while, on the other hand, $[k, e]$ and $[k, e^\dagger]$ need not to be zero. To be concrete, let us fix $\vec{W} = \frac{1}{2} (-y, x, 0)$, as an example. With this choice we have that

$$
[\varphi^{(k)}_0] = 0,
$$

(3.21)

and

$$
[\varphi^{(k)}_n] = \frac{1}{2} (-y, x, 0),
$$

(3.22)

as well as

$$
[\varphi^{(e)}_n] = \frac{1}{2} (-y, x, 0),
$$

(3.23)

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• these two operators are related to each other: $X_+ - X_- = I$;

• if the vectors $\varphi^{(e)}_n = k^n \varphi^{(e)}_0$ and $\varphi^{(k)}_n = e^n \varphi^{(k)}_0$ are different from zero, then they are eigenstates of, respectively, $X_-$ and $X_+$:

$$
\begin{align*}
X_- \varphi^{(e)}_n &= -(n + 1) \varphi^{(e)}_n, \\
X_+ \varphi^{(k)}_n &= (n + 1) \varphi^{(k)}_n,
\end{align*}
$$

(3.21)

for all $n = 0, 1, 2, 3, \ldots$

• It is clear from their definition that $X_\pm$ are not expected to be positive or negative operators, even though (3.21) might suggest something different. Indeed this first impression is correct, since it is also easy to continue the analysis of the spectra of $X_\pm$ getting the following result:

$$
\begin{align*}
X_+ \varphi^{(k)}_n &= (n + 1) \varphi^{(k)}_n, \quad n = 0, 1, 2, \ldots \\
X_+ \varphi^{(e)}_n &= -n \varphi^{(e)}_n, \quad n = 0, 1, 2, \ldots
\end{align*}
$$

(3.22)

as well as

$$
\begin{align*}
X_- \varphi^{(e)}_n &= -(n + 1) \varphi^{(e)}_n, \quad n = 0, 1, 2, \ldots \\
X_- \varphi^{(k)}_n &= n \varphi^{(k)}_n, \quad n = 0, 1, 2, \ldots
\end{align*}
$$

(3.23)
It is clear that, since \((X_{\pm})^\dagger \neq X_{\pm}\), there is no reason for all these different eigenstates to be mutually orthogonal, and in fact they are not. For the same reason, we can only conclude that \(Z \subseteq \sigma(X_{\pm})\), where \(\sigma(X_{\pm})\) are the spectra of the operators \(X_{+}\) and \(X_{-}\).

- Using the explicit expressions for \(e\) and \(k\) we find that
  \[
  X_{+} - \frac{1}{2} I = X_{-} + \frac{1}{2} I = \frac{i}{2} \left\{ (p_x - iy/2)^2 + (p_y + ix/2)^2 \right\},
  \]
  which shows that \(-i(\frac{1}{2} I)\) and \(-i(\frac{1}{2} I)\) may be interpreted as a sort of non-self adjoint Hamiltonian of a purely imaginary magnetic field \(\vec{B}_c\) arising from the following complex vector potential \(\vec{A}_c = \frac{i}{2}(-y, x, 0)\). This is amazing, because we started with a Landau Hamiltonian with no magnetic field at all and we have eventually recovered an imaginary and uniform \(\vec{B}_c\). The reason for this is related to the fact that the system in question has a non-trivial geometry. In effect we are quantizing a classical system living on the two dimensional plane with the origin removed. The introduction of a vector potential with zero magnetic field implies a gauge change which is reflected in the quantum theory. The situation is reminiscent of the Bohm-Aharonov effect.

This is not yet the end of the story: other interesting operators can still be defined starting from the ones we have considered above. In particular, let us define

\[
a = \frac{k + e^\dagger}{\sqrt{2}}, \quad a^\dagger = \frac{k^\dagger + e}{\sqrt{2}}, \quad \Rightarrow [a, a^\dagger] = I.
\]

It is a simple exercise to check that \(a^\dagger a = H^\dagger_0 + \frac{1}{2} I\), where \(H^\dagger_0 = \frac{1}{2} \left( (p_x + \frac{y}{2})^2 + \frac{1}{2} (p_y - \frac{x}{2})^2 \right)\) differs from \(H_0\) only through the change of sign \(\vec{A} \rightarrow -\vec{A}\), implying that \(\vec{B} \rightarrow -\vec{B}\). Again, this result looks rather interesting: although we started with a Hamiltonian for a free electron, the introduction of a two-dimensional SUSY naturally produced several operators, some self-adjoint, others not, and describing real or imaginary magnetic fields, yet whose spectra are analyzable in great detail.

Of course the natural question, at this stage, is the following: is it really SUSY that is responsible for the appearance of \(-\vec{B}\) in \(H^\dagger_0\)?
III.2.2 Case 2: $\partial_x W_2 = \partial_y W_1 = 0$.

Let us consider again the commutation rules in (3.15). What we want to do now is to mimic, as far as possible, the standard Landau level situation. This means, in particular, that we want $e, e^\dagger$ to commute with $k, k^\dagger$. Therefore, because of (3.15), we need to have $\partial_x W_2 = \partial_y W_1 = 0$ or, in other words, the superpotential must have the following general expression: $\vec{W} = (W_1(x), W_2(y), 0)$. Needless to say, $\vec{W}_0$ satisfies this property, but it is also clear that this is not the only possibility. Different choices produce, in general, superpartner Hamiltonians which are really different, since $\vec{\nabla} \cdot \vec{W} \neq 0$. The following results can be easily deduced:

- if $\xi$ is an eigenstate of $h^b$ in (3.16) with eigenvalue $\epsilon$, then $e \xi$ is an eigenstate of $h^f$ with the same eigenvalue. This is a standard result for partner Hamiltonians;

- more interestingly, if we define the unitary operator $T = e^{\alpha k - ak^\dagger}$, and we put $\xi_n := T^n \xi$, $n \in \mathbb{Z}$, it is also clear that $\xi_n$ is an eigenstate of $h^b$ with eigenvalue $\epsilon$ while $a \xi_n$ is an eigenstate of $h^f$ again with the same eigenvalue. This situation extends the analogous result valid for standard Landau levels: once again, each generalized Landau level is infinitely degenerate!

Thus, if we are able to generate superpotentials for which the spectrum of $h^b$ is completely discrete we would be in the standard SUSY situation and could build coherent states, using the formalism presented above and generalizing (3.9).

III.2.3 Examples

Our first choice of a superpotential $\vec{W}$ which is different from the standard one, $\vec{W}_0$, is the following: $\vec{W} = -\left(\frac{x+y}{2}, \frac{x+y}{2}, 0\right)$. Note that with this choice, even though $\partial_x W_2$ and $\partial_y W_1$ are different from zero, in view of (3.15) we still have $[e, e^\dagger] = [k, k^\dagger] = I$, $[k, e] = 0$ and $[k, e^\dagger] = I$. Therefore, $e$ and $k$ behave as a pair of coupled annihilation operators. However, using (3.19), the magnetic field associated to this $\vec{W}$ coincides with the one arising from $\vec{W}_0$. So they describe the same physical situation.
A perhaps more interesting choice is the superpotential, \( \vec{W} = \kappa \left( \frac{1}{x}, 0, 0 \right) \), where \( \kappa \) is a constant to be conveniently determined later. Clearly, for this potential \( \partial_x W_2 = \partial_y W_1 = 0 \), so that we are within the framework of Case 2 of the previous subsection. With this choice, let us introduce a slight change of notation, the reasons for which will become clear shortly:

\[
H^f = h^b, \quad H^b = h^f, \quad A = e^\dagger, \quad A^\dagger = e. \tag{3.26}
\]

Then,

\[
H^f = AA^\dagger = \frac{1}{2} \left[ p_x^2 + \left( p_y - \frac{\kappa}{x} \right)^2 - \frac{\kappa}{x^2} \right] = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\kappa(\kappa - 1)}{2x^2} + \frac{i \kappa}{x} \frac{\partial}{\partial y} - \frac{1}{2} \frac{\partial^2}{\partial y^2}. \tag{3.27}
\]

It is clear that \([H^f, p_y] = 0\), so that the eigenstates of \( H^f \) can be found among the eigenstates of the operator \( p_y \). Consider the function

\[
\Psi_{jm}(x, y) = \psi(x) \chi_{jm}(x, y), \quad x, y \in \mathbb{R}, \ j = 0, \pm 1, \pm 2, \ldots, \pm \infty, \ m = 0, 1, 2, 3, \ldots, \infty, \tag{3.28}
\]

where,

\[
\chi_{jm}(x, y) = \begin{cases} 
\frac{1}{\sqrt{2\pi}} \exp \left( -i \frac{x}{|x|} my \right), & \text{for } y \in [2j\pi, 2(j + 1)\pi] \\
0, & \text{otherwise}
\end{cases} \tag{3.29}
\]

We then see that in order to obtain a solution to the eigenvalue problem \( H^f \Psi_{jm} = \varepsilon_{jm} \Psi_{jm} \), the function \( \psi(x) \) has to satisfy,

\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\kappa m}{|x|} + \frac{\kappa(\kappa - 1)}{2x^2} \right] \psi(x) = \left( \varepsilon_{jm} - \frac{m^2}{2} \right) \psi(x). \tag{3.30}
\]

Comparing this equation with the well-known radial equation for the hydrogen atom:

\[
\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} - \frac{Ze^2}{r} + \frac{\ell(\ell + 1)\hbar^2}{2\mu r^2} \right] u = Eu, \tag{3.31}
\]

we find that for \( m \neq 0 \) and \( \kappa = -1 \) (3.30) reduces to (3.31), with the choice \( \ell = 1, \ \frac{\hbar^2}{\mu} = 1, \ Ze^2 = m \) and \( E = \varepsilon_{jm} - \frac{m^2}{2} \) and if we restrict \( x \) to either \( 0 \leq x < \infty \) or \( -\infty < x \leq 0 \).
 Explicitly, we then get

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} - \frac{m}{|x|} + \frac{1}{x^2}\right] \psi(x) = \left(\varepsilon_{jm}^f - \frac{m^2}{2}\right) \psi(x). \quad (3.32)$$

The solutions to (3.31) come out in terms of the Laguerre polynomials and the corresponding eigenvalues are

$$E = -\frac{Z^2\epsilon^4\mu}{2(n + \ell + 1)^2\hbar^2}, \quad n = 0, 1, 2, 3, \ldots \infty.$$ 

Hence the eigenvalues of $H^f$ satisfy,

$$-\frac{m^2}{2(n+2)^2} = \varepsilon_{jm}^f - \frac{m^2}{2}$$

whence, for $j = 0, \pm 1, \pm 2, \pm \infty$,

$$\varepsilon_{njm}^f := \varepsilon_{jm}^f = \frac{m^2}{2} \left[1 - \frac{1}{(n+2)^2}\right], \quad m = 1, 2, 3, \ldots \infty, \quad n = 0, 1, 2, \ldots \infty. \quad (3.33)$$

Note that the eigenvalues $\varepsilon_{njm}^f$ do not depend on $j$ and hence each level, corresponding to fixed values of $n$ and $m$, is infinitely degenerate. Moreover, the lowest eigenvalue $\varepsilon_{01m}^f$ is not zero: $\varepsilon_{01m}^f = \frac{3}{8}$.

Let us next look at the other Hamiltonian, $H^b$. From (3.18) we easily get,

$$H^b = \mathbb{A}^\dagger \mathbb{A} = \frac{1}{2} \left[p_x^2 + \left(p_y - \frac{\kappa}{x}\right)^2 - \frac{\kappa}{x^2}\right] = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\kappa(\kappa+1)}{2x^2} + \frac{i\kappa}{x} \frac{\partial}{\partial y} - \frac{1}{2} \frac{\partial^2}{\partial y^2}. \quad (3.34)$$

Thus, taking $\kappa = -1$ and again assuming a solution of the type (3.29) and taking note of (3.32), we are lead to the eigenvalue problem:

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} - \frac{m}{|x|}\right] \psi(x) = \left(\varepsilon_{jm}^b - \frac{m^2}{2}\right) \psi(x). \quad (3.35)$$

Comparing with the equation for the hydrogen atom, (3.31), we see that we are in the case where $\ell = 0$. Thus, analogously to (3.33) we get, for $j = 0, \pm 1, \pm 2, \pm \infty$,

$$\varepsilon_{njm}^b := \varepsilon_{jm}^b = \frac{m^2}{2} \left[1 - \frac{1}{(n+1)^2}\right], \quad m = 1, 2, 3, \ldots \infty, \quad n = 0, 1, 2, \ldots \infty. \quad (3.36)$$
Note that, as expected,  
\[ \varepsilon_{njm}^f = \varepsilon_{(n+1)jm}^b. \]  
(3.37)

This time, the lowest eigenvalue, coming at \( n = 0 \), is in fact zero, which justifies the change in the identification of the bosonic and fermionic sectors in (3.26). Moreover, this eigenvalue is doubly degenerate, i.e.,  
\[ \varepsilon_{0jm}^b = 0, \quad m = 1, 2, 3, \ldots, \infty, \quad j = 0, \pm 1, \pm 2, \ldots, \pm \infty. \]  
(3.38)

Finally, a straightforward computation (or an inspection of the well-known ground state radial wave functions for the hydrogen atom) yields for the eigenstate, \( \Psi_{0jm}(x, y) \) corresponding to the eigenvalue \( \varepsilon_{0jm}^b = 0 \), the function,  
\[ \Psi_{0jm}(x, y) = N|x|e^{-m|x|} \chi_{jm}(x, y), \]  
(3.39)

with \( \chi_{jm} \) as in (3.29) and \( N \) being a normalization constant. Finally, it is easily checked that \( A \Psi_{0jm} = 0 \).

It is now possible to construct VCS for the supersymmetric pair \( \{ H^b, H^f \} \), for fixed values of \( m \) and \( j \). Let \( \Psi_{njm}^b \) be the eigenvectors of the Hamiltonian \( H^b \), corresponding to the eigenvalues \( \varepsilon_{njm}^b \) and let \( \mathcal{H}_{jm}^b \) be the Hilbert space generated by the vectors \( \Psi_{njm}^b, \ n = 0, 1, 2, \ldots, \infty \). Similarly, let \( \Psi_{njm}^f \) be the eigenvectors of the Hamiltonian \( H^f \), corresponding to the eigenvalues \( \varepsilon_{njm}^f \) and \( \mathcal{H}_{jm}^f \) the Hilbert space generated by the vectors \( \Psi_{njm}^f, \ n = 0, 1, 2, \ldots, \infty \). Set \( \mathcal{H}_{jm}^{SUSY} = \mathcal{H}_{jm}^b \oplus \mathcal{H}_{jm}^f \). Then, following (2.11), we define vector coherent states on \( \mathcal{H}_{jm}^{SUSY} \) as  
\[ |z, \overline{z}; jm\rangle = \mathcal{N}(|z|^2)^{-\frac{1}{4}} \sum_{n=0}^{\infty} \left( \begin{array}{c} \frac{z^n}{\varepsilon_{njm}^b!} \Psi_{njm}^b \\ \frac{\overline{z}^{n+1}}{\varepsilon_{(n+1)jm}^b!} \Psi_{njm}^f \end{array} \right), \quad \mathcal{N}(|z|^2) = 1 + 2 \sum_{n=1}^{\infty} \frac{|z|^2}{\varepsilon_{njm}^b}. \]  
(3.40)

In order to get a resolution of the identity, we note that the radius of convergence of the series representing \( \mathcal{N}(|z|^2) \) is \( \frac{m^2}{2} \) and furthermore,  
\[ \varepsilon_{njm}^b = \frac{m^{2n}}{2n+1} \left[ 1 + \frac{1}{n+1} \right]. \]  
(3.41)
Thus following (2.7), we need to find a measure $d\lambda$ such that
\[
2\pi \int_0^{\frac{m}{\sqrt{2}}} r^{2n} d\lambda = \frac{m^{2n}}{2^{n+1}} \left[ 1 + \frac{1}{n+1} \right].
\]
(3.42)
The measure in question is easily found to be (see also [16] for a similar computation)
\[
d\lambda(r) = \frac{1}{4\pi} \delta(r - \frac{m}{\sqrt{2}}) \, dr + \frac{1}{\pi m^2 r} \, dr.
\]
(3.43)
Thus, there follows the resolution of the identity,
\[
\int_0^{2\pi} \int_0^{\frac{m}{\sqrt{2}}} |z, \bar{z}; jm\rangle \langle z, \bar{z}; jm| \mathcal{N}(|z|^2) \, d\lambda(r) \, d\theta = \begin{pmatrix} I_{g^b_{jm}} & 0 \\ 0 & I_{g^f_{jm}} \end{pmatrix} = I_{\text{SUSY}}.
\]
(3.44)
(\text{where } z = re^{i\theta}). Also, in view of the fact that
\[
\epsilon^b_{n,m} = \frac{2^{n+1}}{m^{2n}} \left[ 1 - \frac{1}{n+2} \right],
\]
the normalization factor $\mathcal{N}(|z|^2)$ is computed to be,
\[
\mathcal{N}(|z|^2) = \frac{4u}{1-u} - \frac{u^2}{4} \log(1-u) - 3, \quad u = \frac{2|z|^2}{m^2}.
\]
(3.45)
Of course there could be other choices of $\vec{W}$ as well, producing other families of VCS. For example, $\vec{W} = \left( -\frac{x^2}{2}, 0, 0 \right)$ could be one such interesting choice. For this superpotential the operator $h^b$ looks like
\[
h^b = \frac{1}{2} \left( p_x^2 + p_y^2 + x^2 p_y + \frac{x^4}{4} - x \right),
\]
which again commutes with $p_y$, and with the choice of a trial solution similar to (3.28), it is easy to deduce the following one-dimensional eigenvalue equation:
\[
\frac{1}{2} \left( -\frac{d^2}{dx^2} + \frac{x^4}{4} + x^2 k - x \right) \psi(x) = \left( E - \frac{k^2}{2} \right) \psi(x),
\]
(3.46)
where as before $k \in \mathbb{Z}$. Solving this equation is harder than the previous one. However, it is an easy exercise to find the ground state $\psi_0$ which, as before, turns out to be
infinitely degenerate. The equation for $\psi_0$ is $e\psi_0 = 0$, whose solution is $\psi_0(x, y) = N \sqrt{2\pi} e^{-kx + iky - x^3/6}$. Again, it is easy to check that for each $k \in \mathbb{Z}$, this satisfies the equation $h^b \psi_0 = 0$.

Finding the excited eigenstates is a more difficult problem this time, and one can look for solutions of (3.46) in power series. In this way one can get a (formal) solution, which, however, does not appear to be in closed form. Next, these (formal) eigenstates, can be used to find the eigenstates of the hamiltonian $h^f$, using (2.2). Subsequently, using (2.8), one can again construct VCS.

IV Conclusions

As mentioned in the Introduction, we have presented in this paper a method for constructing vector coherent states for supersymmetric Hamiltonian pairs and then applied it to constructing such states to pairs of Hamiltonians arising from a generalization of the fractional quantum Hall effect. While the general scheme adopted here for constructing VCS has been developed elsewhere, the application to supersymmetric Hamiltonians is new. Two interesting facts ought to be reiterated here. The first is the appearance of both analytic and anti-analytic functions in the complex representation of the underlying Hilbert space, in which the bosonic part occupies the analytic and the fermionic part the anti-analytic sectors. The second is the fact that this complex representation is naturally equivalent to a representation using the standard anti-commuting Grassmann variables, common to treatments of supersymmetry.

As a concrete example of our construction we have considered the Hamiltonian of the Landau levels and some natural generalizations of it, suggested by SUSY: this in turn, produced several SUSY partner Hamiltonians and, as a consequence, their related VCS.
Acknowledgements

This work was partially supported by the Ministero Affari Esteri, Italy, through its program of financial support for international cooperations, Bando CORI 2003, cap. B.U. 9.3.0001.0001.0001, and through grants from the Natural Sciences and Engineering Research Council (NSERC), Canada and the Fonds québécois de la recherche sur la nature et les technologies (FQRNT), Québec. One of us (STA) would like to thank H. Upmeier for useful suggestions.

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