NOTE ON MULTIPLE $q$-ZETA FUNCTIONS

T. Kim

Abstract. In this paper we consider the analytic continuation of the multiple Euler $q$-zeta function in the complex number field as follows:

$$
\zeta^E_{r,q}(s, x) = [2]^r_q \sum_{m_1, \cdots, m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}}{[x+m_1+\cdots+m_r]_q^s},
$$

where $q \in \mathbb{C}$ with $|q| < 1$, $\Re(x) > 0$, and $r \in \mathbb{N}$. Thus, we investigate their behavior near the poles and give the corresponding functional equations.

§1. Introduction/ Preliminaries

Let $\mathbb{C}$ be the complex number field. For $s \in \mathbb{C}$, the Hurwitz’s type Euler zeta function is defined by

$$
(1) \quad \zeta^E(s, x) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+z)^s}, \text{ where } s \in \mathbb{C}, z \neq 0, -1, -2, \cdots, \text{ (see [11]).}
$$

Thus, we note that $\zeta^E(s, x)$ is a meromorphic function in whole complex $s$-plane. It is well known that the Euler polynomials are defined as

$$
(2) \quad \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ for } |t| < \pi,
$$

and $E_n = E_n(0)$ are called the $n$-th Euler numbers (see [7, 8, 9, 11]). By (1) and (2), we note that $\zeta^E(-n, x) = E_n(x)$, for $n \in \mathbb{Z}_+$. Throughout this paper we assume

Key words and phrases. : multiple $q$-zeta function, $q$-Euler numbers, Laurent series, Cauchy integral.

2000 AMS Subject Classification: 11B68, 11S80

The present Research has been conducted by the research Grant of Kwangwoon University in 2010

Typeset by \LaTeX
that \( q \in \mathbb{C} \) with \( |q| < 1 \) and we use the notation of \( q \)-numbers as \([x]_q = \frac{1-q^x}{1-q}\). The \( q \)-Euler numbers are defined as

\[
E_{0,q} = \frac{2}{[2]_q}, \quad \text{and} \quad (qE + 1)^n + E_{n,q} = 0 \quad \text{if} \quad n \geq 1,
\]

where we use the standard convention about replacing \( E^k \) by \( E_{k,q} \) (see [7]). Thus, we define the \( q \)-Euler polynomials as follows:

\[
E_{n,q}(x) = \sum_{l=0}^{n} \left( \begin{array}{l} n \\ l \end{array} \right) q^l[x]_q^{n-l} q^lx E_{l,q}, \quad \text{(see [7, 8, 15]).}
\]

For \( s \in \mathbb{C} \), the \( q \)-extension of Hurwitz’s type \( q \)-Euler zeta function is defined by

\[
\zeta^E(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n}{[n+x]_q^s}, \quad \text{where} \quad x \neq 0, -1, -2, \ldots.
\]

For \( n \in \mathbb{Z}_+ \), we have \( \zeta^E(-n, x) = E_{n,q}(x) \) (see [6, 7, 15]). Let \( \chi \) be a Dirichlet’s character with conductor \( f \in \mathbb{N} \) with \( f \equiv 1 \pmod{2} \). It is known that the generalized \( q \)-Euler polynomials attached to \( \chi \) are defined by

\[
F_{q,\chi}(t, x) = [2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m)e^{[m+x]_q^t} = \sum_{m=0}^{\infty} E_{m,\chi,q}(x) \frac{t^m}{m!}, \quad \text{see [7].}
\]

Note that

\[
\lim_{q \to 1} F_{q,\chi}(t, x) = \sum_{m=0}^{\infty} E_{m,\chi}(x) \frac{t^m}{m!},
\]

where \( E_{m,\chi}(x) \) are called the \( m \)-th generalized Euler polynomials attached to \( \chi \). From (6), we can derive the following equation.

\[
E_{n,\chi,q}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m)[m+x]_q^n = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \left( \begin{array}{l} n \\ l \end{array} \right) (-q^n)^l \sum_{a=0}^{f-1} \frac{(-1)^a \chi(a)q^{la}t^a}{1+q^f}.
\]

Now, we consider the Dirichlet’s type Euler \( q \)-\( l \)-function which interpolate \( E_{n,\chi,q}(x) \) at negative integer. For \( s \in \mathbb{C} \), define

\[
l_q(s, x|\chi) = \sum_{n=0}^{\infty} \frac{\chi(n)(-1)^n}{[n+x]_q^s}, \quad x \neq 0, -1, -2, \ldots, \quad \text{(see [6, 7, 8, 15]).}
\]

Note that \( l_q(-n, x|\chi) = E_{n,\chi,q}(x) \) for \( n \in \mathbb{Z}_+ \). In the special case \( x = 0 \), \( E_{n,\chi,q}(= E_{n,x,\chi,q}(0)) \) are called the \( n \)-th generalized Euler numbers attached to \( \chi \). The theory of
quantum groups has been quite successful in producing identities for \(q\)-special function. Recently, several mathematicians have studied \(q\)-theory in the several areas (see [1-23]). In this paper we approach the \(q\)-theory in the area of special function. That is, we first consider the analytic continuation of multiple \(q\)-Euler zeta function in the complex plane as follows:

\[
\zeta^E_{r,q}(s, x) = [2]_q^r \sum_{m_1, \ldots, m_r = 0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r}}{[x + m_1 + \cdots + m_r]_q^s}, \quad s \in \mathbb{C}, \; x \neq 0, -1, \ldots.
\]

From (8), we investigate some identities for the multiple \(q\)-Euler numbers and polynomials. Finally, we give interesting functional equation related to the multiple \(q\)-Euler polynomials, gamma functions and multiple \(q\)-Euler zeta function.

§2. Multiple \(q\)-Euler polynomials and multiple \(q\)-Euler zeta functions

From (3), we note that

\[
E_{n,q} = \frac{[2]_q^n}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-q^x)^l}{(1 + q^t)} = [2]_q \sum_{m=0}^{\infty} (-1)^m [m + x]_q^n.
\]

Let \(F_q(t, x) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}\). Then we see that

\[
F_q(t, x) = [2]_q \sum_{m=0}^{\infty} (-1)^m [m + x]_q^n t.
\]

From (10), we note that \(\lim_{q \to 1} F_q(t, x) = \frac{2}{e^x + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}\), where \(E_n(x)\) are called the \(n\)-th Euler polynomials. For \(s \in \mathbb{C}\), we have

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F_q(-t, x) dt = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n}{[n + x]_q^n}, \quad \text{where} \; x \neq 0, -1, -2, \ldots.
\]

By Cauchy residue theorem and Laurent series, we see that \(\zeta^E_q(-n, x) = E_{n,q}(x)\) for \(n \in \mathbb{Z}_+\). Let \(\chi\) be the Dirichlet’s character with conductor \(f(=\text{odd}) \in \mathbb{N}\). From (6), we can derive

\[
F_{q,\chi}(t, x) = [2]_q \sum_{a=0}^{f-1} (-1)^a \chi(a) \sum_{n=0}^{\infty} (-1)^n e^{[a + x + nf]_q t}
\]

\[
= [2]_q \sum_{a=0}^{f-1} (-1)^a \chi(a) \sum_{n=0}^{\infty} (-1)^n e^{[f]_q [\frac{x + a}{f} + n] t}
\]

Let us define the Dirichlet’s type \(q\)-Euler \(l\)-function as follows:

\[
l_q(s, x|\chi) = [2]_q \sum_{m=0}^{\infty} \chi(m) (-1)^m \frac{1}{[m + x]_q^s}, \quad \text{where} \; s \in \mathbb{C}, \; x \neq 0, -1, -2, \ldots.
\]
From the Mellin transformation of $F_{q,\chi}(t, x)$, we note that
\begin{equation}
\frac{1}{\Gamma(s)} \int_0^\infty F_{q,\chi}(-t, x) t^{s-1} dt = [2]_q \sum_{n=0}^\infty \frac{(-1)^n \chi(n)}{(n+x)_q^s}, \text{ where } s \in \mathbb{C}, x \neq 0, -1, -2, \ldots.
\end{equation}

By Laurent series and Cauchy residue theorem, we see that $l_q(-n, x|\chi) = E_{n,\chi,q}(x)$ for $n \in \mathbb{Z}_+$. Let us consider the following $q$-Euler polynomials of order $r(\in \mathbb{N})$.
\begin{equation}
F^{(r)}_q(t, x) = [2]_q^r \sum_{m_1, \ldots, m_r = 0}^\infty (-1)^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r+x]q} t^{m_1+\cdots+m_r} \frac{t^n}{n!}.
\end{equation}

In the special case $x = 0$, $E^{(r)}_{n,q}(= E^{(r)}_{n,q}(0))$ are called the $n$-th $q$-Euler numbers of order $r$. It is easy to show that $\lim_{q \to 1} F^{(r)}_q(t, x) = \left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^\infty E^{(r)}_n(x) \frac{t^n}{n!}$, where $E^{(r)}_n(x)$ are called the $n$-th Euler polynomials of order $r$. From (15), we note that
\begin{equation}
\sum_{n=0}^\infty E^{(r)}_{n,q}(x) \frac{t^n}{n!} = [2]_q^r \sum_{m_1, \ldots, m_r = 0}^\infty (-1)^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r+x]q} t^{m_1+\cdots+m_r}
\end{equation}
\begin{equation}
= [2]_q^r \sum_{m=0}^\infty \binom{m+r-1}{m} (-1)^m e^{[m+x]q} t^m.
\end{equation}

Thus, we have
\begin{equation*}
E^{(r)}_{n,q}(x) = \frac{[2]_q^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^lx \left(\frac{1}{1+q^l}\right)^r.
\end{equation*}

Therefore, we obtain the following proposition.

Proposition 1. For $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have
\begin{equation}
E^{(r)}_{n,q}(x) = [2]_q^r \sum_{m_1, \ldots, m_r = 0}^\infty (-1)^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r+x]q} t^{m_1+\cdots+m_r}
\end{equation}
\begin{equation}
= [2]_q^r \sum_{m=0}^\infty \binom{m+r-1}{m} (-1)^m e^{[m+x]q} t^m
\end{equation}
\begin{equation}
= \frac{[2]_q^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^lx \left(\frac{1}{1+q^l}\right)^r.
\end{equation}

By Mellin transformation of $F^{(r)}_q(t, x)$, we see that
\begin{equation}
\frac{1}{\Gamma(s)} \int_0^\infty F^{(r)}_q(-t, x) t^{s-1} dt = [2]_q^r \sum_{m=0}^\infty \frac{(m+r-1)(-1)^m}{[m+x]_q^s}\frac{(m+r-1)(-1)^m}{[m+x]_q^s}
\end{equation}
\begin{equation}
= [2]_q^r \sum_{m_1, \ldots, m_r = 0}^\infty \frac{(-1)^{m_1+\cdots+m_r}}{[m_1+\cdots+m_r+x]_q^s}, \text{ where } s \in \mathbb{C}, x \neq 0, -1, -2, \ldots.
\end{equation}

From (17), we can consider the following multiple $q$-Euler zeta function.
Definition 2. For $s \in \mathbb{C}$, $x \in \mathbb{R}$ with $x \neq 0, -1, -2, \cdots$, we define the multiple $q$-Euler zeta function as follows:

$$
\zeta_{r,q}^{E}(s, x) = [2]_{q}^{r} \sum_{m_1, \cdots, m_r = 0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r}}{[m_1 + \cdots + m_r + x]_{q}^{s}}.
$$

Note that $\zeta_{r,q}^{E}$ is meromorphic function in whole complex $s$-plane. By using Cauchy residue theorem and Laurent series in (15) and (17), we obtain the following theorem.

Theorem 3. For $n \in \mathbb{Z}_{+}$, $r \in \mathbb{N}$, we have

$$
\zeta_{r,q}^{E}(-n, x) = E_{n,q}^{(r)}(x).
$$

In (15), we have

(18)

$$
F_{q}^{(r)}(t, x) = [2]_{q}^{r} \sum_{a_1, \cdots, a_r = 0}^{f-1} (-1)^{a_1 + \cdots + a_r} \sum_{m_1, \cdots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} e^{[\sum_{i=1}^{r}(a_i + f m_i) + x]q} t.
$$

By (17) and (18), we obtain the following theorem.

Theorem 4. (Distribution relation for $E_{m,q}^{(r)}(x)$)

For $n \in \mathbb{Z}_{+}$, $f \in \mathbb{N}$ with $f \equiv 1 \pmod{2}$, we have

$$
E_{n,q}^{(r)}(x) = \left( [2]_{q}^{n} \left( [f]_{q}^{n} \sum_{a_1, \cdots, a_r = 0}^{f-1} (-1)^{a_1 + \cdots + a_r} E_{n,q}^{(r)} \left( \frac{a_1 + \cdots + a_r + x}{f} \right) \right) \right).
$$

Moreover,

$$
E_{n,q}^{(r)}(x) = \frac{[2]_{q}^{n}}{(1 - q)^{n}} \sum_{l=0}^{n} \binom{n}{l} (-q^{x})^{l} \sum_{a_1, \cdots, a_r = 0}^{f-1} (-1)^{a_1 + \cdots + a_r} q^{l(a_1 + \cdots + a_r)} \frac{1}{(1 + q^{l})^{r}}.
$$

Let $\chi$ be the Dirichlet’s character with conductor $f (= \text{odd}) \in \mathbb{N}$. Then we define the generalized $q$-Euler polynomials of order $r$ attached to $\chi$ as follows:

$$
F_{q,\chi}^{(r)}(t, x) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x) \frac{t^{n}}{n!}
$$

(19)

$$
= \sum_{m_1, \cdots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} \frac{\prod_{j=1}^{r} \chi(m_j)}{[m_1 + \cdots + m_r]_{q}} e^{[x + m_1 + \cdots + m_r]_{q}t}.
$$
In the special case \( x = 0 \), \( E_{n, \chi, q}^{(r)}(= E_{n, \chi, q}^{(r)}(0)) \) are called the \( n \)-th generalized \( q \)-Euler numbers of order \( r \) attached to \( \chi \). From (19), we can derive (20)

\[
F_{q, \chi}^{(r)}(t, x) = [2]^r_q \sum_{m_1, \ldots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} \left( \prod_{j=1}^{r} \chi(m_j) \right) e^{[x + m_1 + \cdots + m_r]_q t} \]

\[
= [2]^r_q \sum_{m=0}^{\infty} \left( \frac{m + r - 1}{m} \right) (-1)^m \sum_{a_1, \ldots, a_r = 0}^{f-1} \left( \prod_{j=1}^{r} \chi(a_j) \right) (-1) \Sigma_{j=1}^{r} a_j e^{[x + mf + \Sigma_{j=1}^{r} a_j]_q t}. \]

By (16) and (20), we obtain the following theorem.

**Theorem 5.** For \( f \in \mathbb{N} \) with \( f \equiv 1 \pmod{2} \), we have

\[
E_{n, \chi, q}^{(r)}(x) = [f]^n_q \left[ \frac{[2]^r_q}{[2]^r_q} \right] \sum_{a_1, \ldots, a_r = 0}^{f-1} \left( \prod_{j=1}^{r} \chi(a_j) \right) (-1) \Sigma_{j=1}^{r} a_j E_{n, q^j}^{(r)} \left( \frac{x + \Sigma_{j=1}^{r} a_j}{f} \right),
\]

and

\[
E_{n, \chi, q}^{(r)}(x) = \frac{[2]^r_q}{(1 - q)^n} \sum_{l=0}^{\infty} \binom{n}{l} (-q)^l \sum_{a_1, \ldots, a_r = 0}^{f-1} \left( \prod_{j=1}^{r} \chi(a_j) \right) \left( \frac{1}{1 + q^{lf}} \right)^r.
\]

From the Mellin transformation of \( F_{q, \chi}^{(r)}(t, x) \), we note that

\[
(21) \quad \frac{1}{\Gamma(s)} \int_{0}^{\infty} F_{q, \chi}^{(r)}(-t, x) t^{s-1} dt = [2]^r_q \sum_{m_1, \ldots, m_r = 0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r} \left( \prod_{j=1}^{r} \chi(m_j) \right)}{[m_1 + \cdots + m_r + x]_q^s},
\]

where \( s \in \mathbb{C} \), \( \Re(s) > 0 \). From (21) we can also consider the following Dirichlet’s type multiple \( q \)-Euler \( l \)-function.

**Definition 6.** For \( s \in \mathbb{C}, x \in \mathbb{R} \) with \( x \neq 0, -1, -2, \ldots \), we define Dirichlet’s type \( q \)-Euler \( l \)-function as follows:

\[
l^{(r)}_q(s, x | \chi) = [2]^r_q \sum_{m_1, \ldots, m_r = 0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r} \left( \prod_{j=1}^{r} \chi(m_j) \right)}{[m_1 + \cdots + m_r + x]_q^s}.
\]

Note that \( l^{(r)}_q(s, x | \chi) \) is also holomorphic function in whole complex \( s \)-plane. By (20) and (21), we see that

\[
l^{(r)}_q(s, x | \chi) = \frac{1}{[f]^s_q} \left[ \frac{[2]^r_q}{[2]^r_q} \right] \sum_{a_1, \ldots, a_r = 0}^{f-1} \left( \frac{\prod_{j=1}^{r} \chi(a_j)}{(1 - q)^r} \right) (-1) \Sigma_{i=1}^{r} a_i E_{r, q^j}^{(r)}(s, \frac{a_1 + \cdots + a_r + x}{f}).
\]
Theorem 7. For $n \in \mathbb{Z}_+$, we have

$$l_q^{(r)}(-n, x|\chi) = E_{n,\chi,q}^{(r)}(x).$$

For $q = 1$, Theorem 7 seems to be similar type of Dirichlet’s $L$-function in complex analysis. That is, let $\chi$ be the Dirichlet’s character with conductor $d \in \mathbb{N}$. Then Dirichlet $L$-function is defined as

$$L(s, x|\chi) = \sum_{n=0}^{\infty} \frac{\chi(n)}{(n+x)^s}, \text{ where } s \in \mathbb{C}, x \neq 0, -1, -2, \ldots.$$

Let $n$ be positive integer. Then we have $L(-n, x|\chi) = -\frac{B_{n,\chi}(x)}{n}$, where $B_{n,\chi}(x)$ are called the $n$-th generalized Bernoulli polynomials attached to $\chi$ (see [13, 14, 16, 18, 2, 3, 20-23]).

References

[1] I. N. Cangul, V. Kurt, H. Ozden, Y. Simsek, On the higher-order $w$-$q$-Genocchi numbers, Adv. Stud. Contemp. Math. 19 (2009), 39–57.
[2] L. Comtet, Advanced combinatories, Reidel, Dordrecht, 1974.
[3] E. Deeba, D. Rodriguez, Stirling’s series and Bernoulli numbers, Amer. Math. Monthly 98 (1991), 423–426.
[4] N. K. Govil, V. Gupta, Convergence of $q$-Meyer-Konig-Zeller-Durrmeyer operators, Adv. Stud. Contemp. Math. 19 (2009), 97–108.
[5] L.-C. Jang, A study on the distribution of twisted $q$-Genocchi polynomials, Adv. Stud. Contemp. Math. 18 (2009), 181–189.
[6] T. Kim, $q$-Euler numbers and polynomials associated with p-adic $q$-integrals, J. Nonlinear Math. Phys. 14 (2007), 15–27.
[7] T. Kim, The modified $q$-Euler numbers and polynomials, Adv. Stud. Contemp. Math. 16 (2008), 161–170.
[8] T. Kim, Some identities on the $q$-Euler polynomials of higher order and $q$-stirling numbers by the fermionic $p$-adic integrals on $\mathbb{Z}_p$, Russ. J. Math. Phys. 16 (2009), 484–491.
[9] T. Kim, $q$–Volkenborn integration, Russ. J. Math. Phys. 9 (2002), 288–299.
[10] T. Kim, A Note on $p$-Adic $q$-integral on $\mathbb{Z}_p$ Associated with $q$-Euler Numbers, Adv. Stud. Contemp. Math. 15 (2003), 133–138.
[11] T. Kim, On Euler-Barnes multiple zeta functions, Russ. J. Math. Phys. 10 (2003), 261–267.
[12] T. Kim, $q$-Extension of the Euler formula and trigonometric functions, Russ. J. Math. Phys. 14 (2007), 275–278.
[13] T. Kim, Power series and asymptotic series associated with the $q$-analogue of the two-variable $p$-adic $L$-function, Russ. J. Math. Phys. 12 (2005), 186–196.
[14] T. Kim, Non-Archimedean $q$-integrals associated with multiple Changhee $q$-Bernoulli polynomials, Russ. J. Math. Phys. 10 (2003), 91–98.
[15] T. Kim, Note on the Euler $q$-zeta functions, J. Number Theory 129 (2009), 1798–1804.
[16] Y. H. Kim, K. W. Hwang, Symmetry of power sum and twisted Bernoulli polynomials, Adv. Stud. Contemp. Math. 18 (2009), 127–133.
[17] Y. H. Kim, W. Kim, C. S. Ryoo, On the twisted $q$-Euler zeta function associated with twisted $q$-Euler numbers, Proc. Jangjeon Math. Soc. 12 (2009), 93-100.
[18] H. Ozden, I. N. Cangul, Y. Simsek, Remarks on $q$-Bernoulli numbers associated with Daehee numbers, Adv. Stud. Contemp. Math. 18 (2009), 41-48.
[19] H. Ozden, Y. Simsek, S.-H. Rim, I.N. Cangul, *A note on p-adic q-Euler measure*, Adv. Stud. Contemp. Math. 14 (2007), 233–239.

[20] K. Shiratani, S. Yamamoto, *On a p-adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci., Kyushu University Ser. A 39 (1985), 113-125.

[21] Y. Simsek, *Theorems on twisted L-function and twisted Bernoulli numbers*, Advan. Stud. Contemp. Math. 11 (2005), 205–218.

[22] H.J.H. Tuenter, *A Symmetry of power sum polynomials and Bernoulli numbers*, Amer. Math. Monthly 108 (2001), 258-261.

[23] Z. Zhang, Y. Zhang, *Summation formulas of q-series by modified Abel’s lemma*, Adv. Stud. Contemp. Math. 17 (2008), 119–129.

Taekyun Kim
Division of General Education-Mathematics,
Kwangwoon University,
Seoul 139-701, S. Korea e-mail: tkkim@kw.ac.kr