Framed Knots at Large $N$

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We study the framing dependence of the Wilson loop observable of $U(N)$ Chern-Simons gauge theory at large $N$. Using proposed geometrical large $N$ dual, this leads to a direct computation of certain topological string amplitudes in a closed form. This yields new formulae for intersection numbers of cohomology classes on moduli of Riemann surfaces with punctures (including all the amplitudes of pure topological gravity in two dimensions). The reinterpretation of these computations in terms of BPS degeneracies of domain walls leads to novel integrality predictions for these amplitudes. Moreover we find evidence that large $N$ dualities are more naturally formulated in the context of $U(N)$ gauge theories rather than $SU(N)$.
1. Introduction

Topological strings have been studied for about a decade now. Most of the progress in their study has been in the context of closed strings (i.e. Riemann surfaces without boundaries). In the type II superstrings these amplitudes correspond to certain computation for the effective 4 dimensional theory with $\mathcal{N} = 2$ supersymmetry (or more generally for theories with 8 supercharges). More recently progress has been made also in understanding topological strings involving open strings, i.e. with D-branes in target space geometry. These translate in the context of superstrings to F-term computations for the underlying 4 dimensional theory with $\mathcal{N} = 1$ (or more generally for theories with 4 supercharges).

This progress has come from three different directions: On the one hand large $N$ duality of Chern-Simons gauge theory on $S^3$ with closed topological strings on $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ [1] in the context of Wilson loop observables at large $N$ formulated in [2] has led to predictions for a special class of open topological string amplitudes for all genera and arbitrary number of holes [2][3][4][5][6]. On the other hand techniques from mirror symmetry have led to computations of disc amplitudes for a large class of open string topological amplitudes [7][8]. Finally, more recently, from a direct mathematical computation of Gromov-Witten invariants using localization ideas a number of results have emerged [9][10][11] that lead in principle to computation of the amplitudes at all genera and arbitrary number of holes for a large class of open topological amplitudes, in terms of intersection theory on moduli of Riemann surfaces with punctures, for which there are known computational algorithms.

However, it was discovered in [8], that there is an inherent ambiguity in the open topological string amplitude related to the IR geometry of the D-brane. Moreover it was shown there that in the context of duality with large $N$ Chern-Simons theory, this gets mapped to the well known framing ambiguity for knot invariants [12]. The existence of this choice, labeled by an integer, was also verified in [9][11].

A major goal of this paper is to extend the computation of [8] for the framing dependence of the amplitudes, which was carried out for disc amplitudes, to arbitrary genus Riemann surfaces with holes, for D-branes which are large $N$ duals of knots. For the case of the unknot in a special limit (the limit of large $\mathbb{P}^1$ volume) the framing dependence of Gromov-Witten invariants was computed for all Riemann surfaces with boundaries, where it was reduced to certain intersection classes on moduli of Riemann surfaces with punctures, which can be computed using known algorithms [13][14]. Given our result we find a
closed expression for all such intersections in terms of framing dependence of the unknot. This also turns out to exhibit certain novel integrality properties.

The organization of this paper is as follows: In section 2 we review aspects of topological strings in the context of Riemann surfaces with boundaries. This includes the discussion about integrality predictions for these amplitudes. In section 3 we review the framing dependence of knots for Chern-Simons theory. In section 4 we check that the large $N$ expansion for knots for Chern-Simons theory has the expansion and integrality properties expected for open topological strings for arbitrary framing. In section 5 we compare the special case of our results to the results of [1] which leads to a closed formula for intersection theory of Mumford classes with insertion of up to three Chern classes of the Hodge bundle on moduli of punctured Riemann surfaces. In section 6 we compute using techniques from mirror symmetry [7][8] the framing dependence of the unknot for arbitrary volume for $\mathbb{P}^1$ and find agreement with the framing dependence of the large $N$ limit of unknot.

2. Open topological string

Open topological string computes certain invariants related to the space of holomorphic maps from Riemann surfaces with boundaries to Calabi-Yau manifolds where the boundary lies on a Lagrangian submanifold, identified with a topological D-brane. These invariants are called Gromov-Witten invariants. For simplicity, and in view of the application to the large $N$ dual of Chern-Simons theory, consider the case where the Calabi-Yau manifold $X$ has one Kähler moduli denoted by $t$ and assume that the Lagrangian submanifold $C$ has one non-trivial $H_1$. Let us assume we wrap $M$ D-branes around $C$ and denote the holonomy around the non-trivial $H_1$ by the matrix $V$.

Let us consider the topological string theory associated to the maps of an open Riemann surface $\Sigma_{g,h}$ (with genus $g$ and $h$ holes) to $X$ with holes mapped to $C$. To each boundary we can associate an integer related to how many times it wraps the corresponding element of $C$: if the $i$-th hole winds around the one-cycle $n_i$ times, the homotopy class of the boundary can be labeled by $h$ integers $\vec{n} = (n_1, \ldots, n_h)$. The free energy of topological string theory in the topological sector labeled by $\vec{n}$ can be regarded as a generating functional of open Gromov-Witten invariants:

$$F_{g,\vec{n}}(t) = \sum_Q F^{Q}_{\vec{n},g} e^{-Qt}. \quad (2.1)$$
In this equation, \( t \) is the complexified Kähler parameter of the Calabi-Yau manifold, and \( Q \) labels the relative homology class of the embedded Riemann surface. The quantities \( F_{g,\vec{n}}^Q \) are the open string analog of the Gromov-Witten invariants and they “count” in an appropriate sense the number of holomorphically embedded Riemann surfaces of genus \( g \) in \( X \) with Lagrangian boundary conditions specified by \( \mathcal{C} \) with the class represented by \( Q,\vec{n} \). These are in general rational numbers. Mathematical aspects of defining these quantities have been considered recently in [9][10][11].

We can now consider the total free energy, which is the generating functional for all topological sectors:

\[
F(V) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \sum \frac{i^h}{h!} g_s^{2g-2+h} F_{g,\vec{n}}(t) \text{Tr} V^{n_1} \cdots \text{Tr} V^{n_h},
\]

(2.2)

where \( g_s \) is the string coupling constant. The factor \( i^h \) is very convenient in order to compare to the Chern-Simons free energy, as we will see in a moment. The factor \( h! \) is a symmetry factor which takes into account that the holes are indistinguishable (or one could have absorbed them into the definition of \( F_{g,\vec{n}} \)). We take all \( n_i > 0 \) (as discussed in [2] this can be achieved if necessary by analytic continuation of the amplitude).

It is convenient to rewrite (2.2) in terms of a vector \( \vec{k} \). Given a vector \( \vec{n} = (n_1, \cdots, n_h) \), we define a vector \( \vec{k} \) as follows: the \( i \)-th entry of \( \vec{k} \) is the number of \( n_j \)'s which take the value \( i \). For example, if \( n_1 = n_2 = 1 \) and \( n_3 = 2 \), then this corresponds to \( \vec{k} = (2, 1, 0, \cdots) \). In terms of \( \vec{k} \), the number of holes and the total winding number are given by

\[
h = |\vec{k}| = \sum_j k_j, \quad \ell = \sum_i n_i = \sum_j jk_j.
\]

(2.3)

Note that a given \( \vec{k} \) will correspond to many \( \vec{n} \)'s which differ by permutation of entries. In fact there are \( h!/\prod_j k_j! \) vectors \( \vec{n} \) which give the same vector \( \vec{k} \) (and the same amplitude). We can then write the total free energy as:

\[
F(V) = \sum_{g=0}^{\infty} \sum_{\vec{k}} \frac{i^{|\vec{k}|}}{\prod_j k_j} g_s^{2g-2+h} F_{g,\vec{n}}(t) \Upsilon_{\vec{k}}(V)
\]

(2.4)

where

\[
\Upsilon_{\vec{k}}(V) = \prod_{j=1}^{\infty} (\text{Tr} V^j)^{k_j}
\]
2.1. Integrality properties

Let us define

\[ q = e^{i g_s}, \quad \lambda = e^{t}. \]

We now define the generating functions \( f_R(q, \lambda) \) through the following equation:

\[ F(V) = - \sum_{n=1}^{\infty} \sum_{R} \frac{1}{n} f_R(q^n, \lambda^n) \text{Tr}_R V^n \tag{2.5} \]

where \( R \) denotes a representation of \( U(M) \) and we are considering the limit \( M \to \infty \). In this limit we can exchange the basis consisting of product of traces of powers in the fundamental representation, with the trace in arbitrary representations. It was shown in [2], following similar ideas in the closed string case [15], that the open topological strings compute the partition function of BPS domain walls in a related superstring theory. This led to the result that \( F(V) \) has an integral expansion structure. This result was further refined in [3] where it was shown that the corresponding integral expansion leads to the following formula for \( f_R(q, \lambda) \):

\[ f_R(q, \lambda) = \sum_{g \geq 0} \sum_{Q} \sum_{R', R''} C_{R, R', R''} S_{R'}(q) \hat{N}_{R', g, Q}(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{2g-1} \lambda^Q. \tag{2.6} \]

In this formula \( R, R', R'' \) label representations of the symmetric group \( S_\ell \), which can be labeled by a Young tableau with a total of \( \ell \) boxes. In this equation, \( C_{R, R', R''} \) are the Clebsch-Gordon coefficients of the symmetric group, and the monomials \( S_R(q) \) are defined as follows. If \( R \) is a hook representation

\[ \begin{array}{c}
\cdot & \cdot & \cdot \\
\cdot & \cdot \\
\cdot
\end{array} \tag{2.7} \]

with \( \ell \) boxes in total, and with \( \ell - d \) boxes in the first row, then

\[ S_R(q) = (-1)^d q^{-\frac{d-1}{2}+d}, \tag{2.8} \]

and it is zero otherwise. The \( \hat{N}_{R, g, Q} \) are integers and compute the net number of BPS domain walls of charge \( Q \) and spin \( g \) transforming in the representation \( R \) of \( U(M) \), where we are using the fact that representations of \( U(M) \) can also be labeled by Young tableaux.
2.2. Multicovering formulae

In order to exhibit the multicovering aspects of \( F(V) \), it is convenient to use the related invariants \( n_{\vec{k},g,Q} = \sum_R \chi_R(C(\vec{k})) \hat{N}_{R,g,Q} \),

\[
(2.9)
\]

where \( \chi_R(C(\vec{k})) \) are the characters of the symmetric group, and \( C(\vec{k}) \) denotes the conjugacy class of the symmetric group with \( k_i \) cycles of size \( i \). Notice that these invariants are not as fundamental as \( \hat{N}_{R,g,Q} \). For example, integrality of \( n_{\vec{k},g,Q} \) follows from integrality of \( \hat{N}_{R,g,Q} \) (since the characters are integers), but not the other way around, and there are some further integrality constraints on \( n_{\vec{k},g,Q} \).

The multicovering formula derived in [5] states that the free energies of open topological string theory with a fixed homotopy class \( \vec{k} \) can be written in terms of the integer invariants \( n_{\vec{k},g,Q} \) as follows:

\[
\sum_{g=0}^{\infty} g_s^{2g-2+|\vec{k}|} F_{g,\vec{k}}(t) = \frac{1}{\prod_j j^{k_j}} \sum_{d|\vec{k}} (-1)^{d|\vec{k}|+g} n_{\vec{k_1/d},g,Q} d^{d|\vec{k}|} \left( 2 \sin \frac{dg_s}{2} \right)^{2g-2} \prod_j \left( 2 \sin \frac{jg_s}{2} \right)^{k_j} \lambda^{Qd}. \tag{2.10}
\]

Notice there is one such identity for each \( \vec{k} \). In this expression, the sum is over all integers \( d \) which satisfy the following condition: \( d|j \) for every \( j \) with \( k_j \neq 0 \). When this is the case, we define the vector \( \vec{k}_{1/d} \) whose components are \( (\vec{k}_{1/d})_i = \vec{k}_{di} \). Remember that \( |\vec{k}| = h \) is the number of holes.

As shown in [5], the formula (2.10) is in fact a consequence of (2.6), and gives a natural generalization of the expression derived in [15] for the closed string case. Just like the multicovering formula of [15] expresses the usual Gromov-Witten invariants in terms of other integer invariants, the formula (2.10) implies that the open Gromov-Witten invariants \( F_{Q \vec{k},g} \) can be also written in terms of the integer invariants (2.9). Up to genus 2
one finds,

\[ F_{k,g=0}^Q = (-1)^{|k|} \sum_{d|k} d^{|k|} n_{k_1/d,0,Q/d}, \]

\[ F_{k,g=1}^Q = - (-1)^{|k|} \sum_{d|k} \left( d^{|k|-1} n_{k_1/d,1,Q/d} - \frac{d^{|k|-3}}{24} (2d^2 - \sum_j j^2 k_j) n_{k_1/d,0,Q/d} \right), \]

\[ F_{k,g=2}^Q = (-1)^{|k|} \sum_{d|k} \left( d^{|k|+1} n_{k_1/d,2,Q/d} + \frac{d^{|k|-1}}{24} n_{k_1/d,1,Q/d} \sum_j j^2 k_j \right) \]

\[ + \frac{d^{|k|-3}}{5760} (24d^4 - 20d^2 \sum_j j^2 k_j - 2 \sum_j j^4 k_j + 5 \sum_{j_1,j_2} j_1^2 j_2^2 k_{j_1} k_{j_2}) n_{k_1/d,0,Q/d}. \]

(2.11)

In these equations, the integer \( d \) has to divide the vector \( k \) (in the sense explained above) and it is understood that \( n_{k_1/d,g,Q/d} \) is zero if \( Q/d \) is not a relative homology class.

3. **U(N) Chern-Simons theory and the framed knots**

The duality between Chern-Simons theory on \( S^3 \) and topological string theory on the resolved conifold was formulated in [1] for the \( SU(N) \) gauge theory. At the level of partition function, which was checked in [1], the difference between the \( U(N) \) and \( SU(N) \) Chern-Simons gauge theories is an additive constant which is not unambiguously defined in the context of topological strings. However, as we will see in this and the next section, consideration of Wilson loop observables indicates that the duality is in fact far more natural for the \( U(N) \) theory, especially when one takes into account the framing dependence, as we will discuss below. This also suggests that perhaps also in other large \( N \) superstring duals, it is the \( U(N) \) gauge theory which is dual to the string theory. In fact \( U(N) \) gauge theory is the more natural version of the large \( N \) duality that ‘t Hooft proposed.

3.1. **Wilson loops in U(N) Chern-Simons**

Chern-Simons theory has an action given by

\[ S(A) = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \]

(3.1)

where \( A \) is a gauge connection for a gauge group \( G \). The generators of the Lie algebra of \( G, T^a \), are normalized as \( \text{Tr}(T^a T^b) = -\delta^{ab} \). The gauge-invariant observables of this theory
are the Wilson loop operators, which are defined as follows. Let $U$ be the holonomy of the gauge connection $A$ around a knot $K$ described by a loop $\gamma$

$$U_\gamma = P \exp \oint_\gamma A.$$  \hfill (3.2)

The Wilson loop operator in the representation $R$ is

$$\text{Tr}_RU_\gamma.$$ \hfill (3.3)

We will denote the vevs of products of Wilson loops by

$$W_{(R_1,\ldots,R_L)} = \langle \prod_i \text{Tr}_{R_i} U_{\gamma_i} \rangle,$$ \hfill (3.4)

where the vev is a normalized one (i.e. we divide by $Z(S^3)$).

If we take $G = U(1)$, Chern-Simons theory turns out to be extremely simple, since it is essentially a Gaussian theory \[16\]. The different representations are labeled by integers, and in particular the vevs of Wilson loop operators

$$\langle \prod_i \exp(n_i \int_{\gamma_i} A) \rangle$$ \hfill (3.5)

can be computed exactly. In order to compute them, however, one has to choose a framing for each of the knots $\gamma_i$. This arises as follows: in evaluating the vev, contractions of the holonomies corresponding to different $\gamma_i$ produce the following integral:

$$\text{lk}(K_i, K_j) = \frac{1}{4\pi} \oint_{\gamma_i} dx^\mu \oint_{\gamma_j} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x - y)^\rho}{|x - y|^3}.$$ \hfill (3.6)

This is in fact a topological invariant, i.e. it is invariant under deformations of the contours $\gamma_i$, $\gamma_j$ and it is in fact the linking number of the knots $K_i$ and $K_j$. On the other hand, contractions of the holonomies corresponding to the same knot $\gamma$ involve the integral

$$\phi(K) = \frac{1}{4\pi} \oint_{\gamma} dx^\mu \oint_{\gamma} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x - y)^\rho}{|x - y|^3}.$$ \hfill (3.7)

This integral is well-defined and finite (see, for example, \[17\],[18]), and it is called the cotorsion of $\gamma$. The problem is that the cotorsion is not invariant under deformations of the knot. In order to preserve topological invariance one has to choose another definition of the composite operator $(\int_\gamma A)^2$ by means of a framing. A framing of the knot consists
of choosing a contour $\gamma^f$ for $\gamma$, specified by a normal vector field $n$. The cotorsion $\phi(K)$ becomes then

$$\phi_f(K) = \frac{1}{4\pi} \oint dx^\mu \oint_{\gamma^f} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} = \text{lk}(K, K^f).$$

(3.8)

The correlation function that we obtain in this way is a topological invariant (a linking number) but the price that we have to pay is that our regularization depends on a set of integers $p_i = \text{lk}(K_i, K^f_i)$ (one for each knot). The vev (3.3) can now be computed, after choosing the framings, as follows:

$$\langle \prod_i \exp(n_i \int_{\gamma_i} A) \rangle = \exp\left(\frac{\pi i}{k} \sum_i \ell_i^2 p_i + \frac{\pi i}{k} \sum_{i \neq j} \ell_i \ell_j \text{lk}(K_i, K_j)\right).$$

(3.9)

This regularization is nothing but the ‘point-splitting’ method familiar in the context of QFT’s.

Let us now consider a $U(N)$ Chern-Simons theory. The $U(1)$ factor decouples from the $SU(N)$ theory, and all the vevs factorize into an $U(1)$ and an $SU(N)$ piece. A representation of $U(N)$ is labeled by a Young tableau, and it decomposes into a representation of $SU(N)$ corresponding to that tableau, and a representation of $U(1)$ with charge:

$$n = \ell \sqrt{N},$$

(3.10)

where $\ell$ is the number of boxes in the Young tableau. In order to compute the vevs associated to the $U(1)$ of $U(N)$, one has to take also into account that the coupling constant $k$ is shifted as $k \to k + N$. We then find that the vev of a product of $U(N)$ Wilson loops in representations $R_i$ is given by:

$$W^{U(N)}_{(R_1, \ldots, R_L)} = \exp\left(\frac{\pi i}{N(k + N)} \sum_i \ell_i^2 p_i + \frac{\pi i}{N(k + N)} \sum_{i \neq j} \ell_i \ell_j \text{lk}(K_i, K_j)\right) W^{SU(N)}_{(R_1, \ldots, R_L)},$$

(3.11)

where the $SU(N)$ vev is computed in the framing specified by $p_i$. Notice that, in the case of knots, the $SU(N)$ and $U(N)$ computations differ in a factor which only depends on the choice of framing, while for links the answers also differ in a topological piece involving the linking numbers. For knots and links in $S^3$, there is a standard or canonical framing, defined by asking that the self-linking number is zero. This corresponds to $p_i = 0$, and in that case we find:

$$W^{U(N), sf}_{(R_1, \ldots, R_L)} = \exp\left(\frac{\pi i}{N(k + N)} \sum_{i \neq j} \ell_i \ell_j \text{lk}(K_i, K_j)\right) W^{SU(N), sf}_{(R_1, \ldots, R_L)}.$$  

(3.12)
This is precisely the corrected vev that was introduced in [4], eq. (4.40), in order to match the Chern-Simons vevs with the topological string answer. This indicates that the geometric duality advocated in [1] is in fact a duality between $U(N)$ Chern-Simons gauge theory in $S^3$ and topological string theory in the resolved conifold. We will find more evidence for this when we analyze the framing dependence.

3.2. Framing dependence

One can now study the effect of a change of framing on the vacuum expectation values of Wilson loops, in a general Chern-Simons theory with gauge group $G$. Consider $L$ Wilson loops $W_{R_i}$ in representations $R_i$ of $G$, with $i = 1, \cdots, L$. It was shown in [12] that, under a change of framing of $K_i$ by $p_i$ units, the vev of the product of Wilson loops changes as follows:

$$W_{(R_1, \cdots, R_L)} \rightarrow \exp \left[ 2\pi i \sum_i p_i h_{R_i} \right] W_{(R_1, \cdots, R_L)},$$

(3.13)

In this equation, $h_R$ is the conformal weight of the WZW primary field corresponding to the representation $R$, and it is given by:

$$h_R = \frac{C_R}{2(k + N)},$$

(3.14)

where $C_R$ is the quadratic Casimir of the group $G$ in the representation $R$. When $G = U(N)$, the representations $R$ can be labeled by the lengths of rows in a Young tableau, $l_i$, where $l_1 \geq l_2 \geq \cdots$. In terms of these, the quadratic Casimir for $U(N)$ reads,

$$C_R = N\ell + \kappa_R,$$

(3.15)

where $\ell$ is the total number of boxes in the tableau, and

$$\kappa_R = \ell + \sum_i \left( l_i^2 - 2il_i \right).$$

(3.16)

For $SU(N)$, one has

$$C_R^{SU(N)} = N\ell + \kappa_R - \frac{\ell^2}{N},$$

(3.17)

Notice that the difference between the change of $SU(N)$ and $U(N)$ vevs under the change of framing is consistent with (3.11). If we now introduce the variables (in anticipation of the large $N$ duality [1])

$$q = \exp \left( \frac{2\pi i}{k + N} \right), \quad \lambda = q^N,$$

(3.18)
we see that $U(N)$ vevs change, under the change of framing, as

$$W_{(R_1,\ldots,R_L)} \rightarrow q^{\frac{1}{2}} \sum_i \kappa_{R_i} p_i \lambda^{\frac{1}{2}} \sum_i \ell_i p_i W_{(R_1,\ldots,R_L)}.$$  \hspace{1cm} (3.19)

4. Framing dependence and the large $N$ dual

It was proposed in [1] that large $N$ limit of Chern-Simons gauge theory is given by closed topological strings on $O(-1) \oplus O(-1) \to \mathbb{P}^1$, with the identification of parameters $q, \lambda$ as given in sections 2 and 3. This was formulated in the context of the string theory realization of Chern-Simons theory on $T^*S^3$ with $N$ Dbranes wrapping $S^3$ [19]. In the large $N$ duality proposed in [1] the conifold undergoes a geometric transition to the resolved conifold $O(-1) \oplus O(-1) \to \mathbb{P}^1$ where the branes wrapping $S^3$ have disappeared. In the context of the large $N$ duality the formulation of Wilson loop observables was studied in [2] which we now review.

4.1. Review of the large $N$ duality for Wilson loops

For simplicity let us consider the case of knots, as the extension to links is straightforward. The formulation of Wilson loop observables was achieved in [2] by considering $M$ branes in the $T^*S^3$ geometry where $M$ branes wrap a Lagrangian submanifold intersecting $S^3$ on the knot. Moreover it was conjectured that for each knot in the large $N$ dual, the $M$ branes deform to a Lagrangian submanifold in the resolved conifold geometry. This was shown for algebraic knots in [3]. More recently it has been extended to all knots [20]. Moreover, the partition function of open topological string $F(V)$ with $M$ branes in the resolved conifold geometry is related to the Wilson loop observable as follows: Consider the operator in Chern-Simons theory

$$Z(U, V) = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} U^n \text{Tr} V^n \right],$$  \hspace{1cm} (4.1)

where $U$ is the holonomy of the $U(N)$ Chern-Simons gauge field around a given knot, and $V$ is a $U(M)$ matrix. It is convenient to expand the exponential in (4.1), and the result can be written as follows. Let $\vec{k}$ be a vector with an infinite number of entries, almost all zero, and whose nonzero entries are positive integers. Given such a vector, we define, as in [23]:

$$\ell = \sum j \ k_j, \quad |\vec{k}| = \sum k_j.$$  \hspace{1cm} (4.2)
We can associate to any vector $\vec{k}$ a conjugacy class $C(\vec{k})$ of the permutation group $S_\ell$. This class has $k_1$ cycles of length 1, $k_2$ cycles of length 2, and so on. The number of elements of the permutation group in such a class is given by $\ell!/z_\vec{k}$, where

$$z_\vec{k} = \prod_j k_j! j^{k_j}. \quad (4.3)$$

We now introduce the following operators, labeled by the vectors $\vec{k}$:

$$\Upsilon_\vec{k}(U) = \prod_{j=1}^\infty (\text{Tr} U^j)^{k_j}. \quad (4.4)$$

It is easy to see that:

$$Z(U, V) = 1 + \sum_{\vec{k}} \frac{1}{z_\vec{k}^2} \Upsilon_\vec{k}(U) \Upsilon_\vec{k}(V), \quad (4.5)$$

since we are assuming $\ell > 0$. The basis of operators (4.4), labeled by conjugacy classes of the permutation group, is related to the operators labeled by representations $R$ of $U(N)$ by Frobenius formula,

$$\Upsilon_\vec{k}(U) = \sum_R \chi_R(C(\vec{k})) \text{Tr}_R U, \quad (4.6)$$

where $\chi_R(C(\vec{k}))$ is the character of the conjugacy class $C(\vec{k})$ in the representation labeled by the Young tableau of $R$. In particular, one has

$$W_\vec{k} \equiv \langle \Upsilon_\vec{k}(U) \rangle = \sum_R \chi_R(C(\vec{k})) W_R. \quad (4.7)$$

Using (4.6), one can also write (4.1) as

$$Z(U, V) = 1 + \sum_R \text{Tr}_R U \text{Tr}_R V, \quad (4.8)$$

where the sum over $R$ starts with the fundamental representation.

The main statement of [2] is

$$\langle Z(U, V) \rangle = \exp(-F(V)) = \exp\left(\sum_{n=1}^\infty \sum_R \frac{1}{n} f_R(q^n, \lambda^n) \text{Tr}_R V^n\right). \quad (4.9)$$
To check this prediction, given that the Chern-Simons amplitudes are computable, one can find \( f_R \) and see if it has the predicted integrality properties. In fact one can show that this equation defines the \( f_R(q, \lambda) \) in terms of \( W_R \). The explicit equation is

\[
f_R(q, \lambda) = \sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{dm} \sum_{\vec{k}_1, \ldots, \vec{k}_m} \sum_{R_1, \ldots, R_m} \chi_R(\sum_{j=1}^l \vec{k}_j) \frac{W_R(q^d, \lambda^d)}{z_{\vec{k}_j}},
\]

(4.10)

where \( \vec{k}_d \) is defined as follows: \( (\vec{k}_d)_d = k_1 \) and has zero entries for the other components. Therefore, if \( \vec{k} = (k_1, k_2, \ldots) \), then

\[
\vec{k}_d = (0, \ldots, 0, k_1, 0, \ldots, 0, k_2, 0, \ldots)
\]

where \( k_1 \) is in the \( d \)-th entry, \( k_2 \) is in the \( 2d \)-th entry, and so on. The sum over \( \vec{k}_1, \ldots, \vec{k}_m \) is over all vectors with \( |\vec{k}_j| > 0 \). In (4.10), \( \mu(d) \) denotes the Moebius function. Recall that the Moebius function is defined as follows: if \( d \) has the prime decomposition \( d = \prod_{i=1}^a p_i^{m_i} \), then \( \mu(d) = 0 \) if any of the \( m_i \) is greater than one. If all \( m_i = 1 \) (i.e. \( d \) is square-free) then \( \mu(d) = (-1)^a \). Some examples of (4.10) are

\[
\begin{align*}
f_\Box(q, \lambda) &= W_\Box(q, \lambda), \\
f_\Box(q, \lambda) &= W_\Box(q, \lambda) - \frac{1}{2}(W_\Box(q, \lambda)^2 + W_\Box(q^2, \lambda^2)), \\
f_\Box(q, \lambda) &= W_\Box(q, \lambda) - \frac{1}{2}(W_\Box(q, \lambda)^2 - W_\Box(q^2, \lambda^2)).
\end{align*}
\]

(4.11)

These results have been generalized to links in [6][7]. Moreover it has been checked that they satisfy the integrality constraint predicted in (2.6) in many examples. In the above equations, we have assumed that all the knot invariants have been computed in the standard framing.

Notice that the logarithm of \( \langle Z(U, V) \rangle \) is the generating function of connected vevs \( W^{(c)}_\vec{k} \),

\[
\log\langle Z(U, V) \rangle = \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} W^{(c)}_\vec{k} \Upsilon_\vec{k}(V).
\]

(4.12)

Therefore, the free energies of open strings are given by:

\[
i^{i|\vec{k}|} \sum_{g=0}^{\infty} F_{g, \vec{k}}(t) g_s^{2g-2+|\vec{k}|} = -\frac{1}{\prod_j j^{k_j}} W^{(c)}_\vec{k}.
\]

(4.13)
4.2. Framing dependence of the integer invariants

Since the Chern-Simons vevs change under a change of framing in the way prescribed by (3.19), it is natural to ask what is the effect of this change on the invariants $\hat{N}_{R,g,Q}$. In fact as noted in [8] for any choice of framing we should get integer values for $\hat{N}_{R,g,Q}$ related to the BPS degeneracies of domain walls in a geometry with different IR behavior. Notice that, from the point of view of Chern-Simons theory, the fact that $\hat{N}_{R,g,Q}$ are integers in the standard framing is already highly nontrivial, and it provides one of the major evidences we have for the duality advocated in [1]. The integrality predictions for any $p$ are even more surprising.

In principle, one would think that the change of framing for the integer invariants can be determined by using the change of framing (3.19), and then by plugging the new vevs (which depend on $p$) in (4.10). This should give $p$-dependent functions $f_R(q,\lambda)$ from which one can extract $\hat{N}_{R,g,Q}(p)$. It turns out that there is a subtlety here [8]: we should expand the partition function $F$ in terms of flat coordinates which in this case does depend on the choice of framing in a simple form. In fact, the appropriate redefinition of $V$ can be read from the results in [7]: the $V$ corresponds to $e^{\hat{u}}$ of [8]. Since under a change of framing one has to redefine $\hat{u} \to \hat{u}_p = \hat{u} + ip\pi$, the natural redefinition of $V$ turns out to be

$$V \to (-1)^{\ell p}V, \quad (4.14)$$

and this means that

$$\text{Tr}_RV \to (-1)^{\ell p} \text{Tr}_RV, \quad (4.15)$$

where $\ell$ is the number of boxes in $R$. By looking at the generating functional (4.8), it is clear that the effect of the redefinition is to change the sign of $W_R$ by $(-1)^{\ell p}$. This gives a relative sign $f_R$ which is crucial for integrality. To compare to topological string theory, it is also useful to reabsorb the $\lambda^{\frac{\ell}{2}}$ factor of (3.19) in $V$ (notice that the $\lambda^{\frac{\ell}{2}}$ gives a global factor in $f_R$, since all the “lower order” terms in (4.10) change by the same factor).

The main conclusion is that, for a framing labeled by the integer $p$, the integer invariants $\hat{N}_{R,g,Q}(p)$ are obtained from (4.10) but with the vevs

$$W_R^{(p)}(q,\lambda) = (-1)^{\ell p} q^{\frac{1}{2} p w_R} W_R(q,\lambda), \quad (4.16)$$
where $\kappa_R$ is defined in (3.16). One has, for example,

\[
\begin{align*}
    f^{(p)}(q, \lambda) &= (-1)^p W(q, \lambda), \\
    f_{\Box}^{(p)}(q, \lambda) &= q^p W_{\Box}(q, \lambda) - \frac{1}{2} (W(q, \lambda)^2 + (-1)^p W_{\Box}(q^2, \lambda^2)), \\
    f_{\square}^{(p)}(q, \lambda) &= q^{-p} W_{\square}(q, \lambda) - \frac{1}{2} (W(q, \lambda)^2 - (-1)^p W_{\Box}(q^2, \lambda^2)),
\end{align*}
\]

and so on. Notice that the right framing factor in order to match the topological string theory prediction is (3.15), and not (3.17). This is yet another indication that the duality of [1] involves the $U(N)$ gauge group, not the $SU(N)$ group. The extension to links is straightforward: one has just to include the factor in (4.16) for each component.

### 4.3. Examples

Using now the explicit expressions for the $U(N)$ Wilson loops, together with (4.10), (2.6) and (4.16), one can check that the invariants $\hat{N}_{R,g,Q}$ are in fact integer for any $p$. We present in the following tables some of the results for the unknot and for the trefoil knot (some of the invariants of the trefoil knot were computed in [5] in the standard framing). One sees that, for given $R$ and $p$, there is only a finite number of $g, Q$ for which $\hat{N}_{R,g,Q} \neq 0$. As we increase $p$ in absolute value, the $g$’s which have a nonzero invariant also increase.

#### Table 1: The integers $\hat{N}_{\Box,g,Q}(p)$ for the framed unknot.

| $Q$ | $g = 0$ | $g = 1$ |
|-----|---------|---------|
| $-1$ | $\frac{1}{8} (1 - (-1)^p - 4p + 2p^2)$ | $\frac{1}{96} (-3 + 3(-1)^p + 8p + 4p^2 - 8p^3 + 2p^4)$ |
| $0$ | $-\frac{1}{2} p(p - 1)$ | $-\frac{1}{24} p(p + 1)(p - 1)(p - 2)$ |
| $1$ | $\frac{1}{8} (-1 + (-1)^p + 2p^2)$ | $\frac{1}{96} (3 - 3(-1)^p - 8p^2 + 2p^4)$ |

#### Table 2: The integers $\hat{N}_{\square,g,Q}(p)$ for the framed unknot (continuation).

| $Q$ | $g = 2$ |
|-----|---------|
| $-1$ | $\frac{1}{5760} (45 - 45(-1)^p - 96p - 104p^2 + 120p^3 + 10p^4 - 24p^5 + 4p^6)$ |
| $0$ | $-\frac{1}{720} (p - 3)(p - 2)(p - 1)p(p + 1)(p + 2)$ |
| $1$ | $\frac{1}{5760} (-45 + 45(-1)^p + 136p^2 - 50p^4 + 4p^6)$ |
### Table 3: The integers $\hat{N}_{g,Q}(p)$ for the framed unknot.

| $Q$  | $g = 0$                                                                 | $g = 1$                                                                 |
|------|-------------------------------------------------------------------------|-------------------------------------------------------------------------|
| $-1$ | $\frac{1}{6}(-1 + (-1)^p + 2p^2)$                                       | $\frac{1}{36}(3 - 3(-1)^p - 8p^2 + 2p^4)$                               |
| $0$  | $-\frac{1}{p}(p+1)$                                                    | $-\frac{1}{24}p(p+1)(p-1)(p+2)$                                        |
| $1$  | $\frac{1}{8}(1 - (-1)^p + 4p + 2p^2)$                                   | $\frac{1}{36}(-3 + 3(-1)^p - 8p + 4p^2 + 8p^3 + 2p^4)$                  |

### Table 4: The integers $\hat{N}_{g,Q}(p)$ for the framed unknot (continuation).

| $Q$  | $g = 2$                                                                 |
|------|-------------------------------------------------------------------------|
| $-1$ | $\frac{1}{5760}(-45 + 45(-1)^p + 136p^2 - 50p^4 + 4p^6)$                  |
| $0$  | $-\frac{1}{240}(p + 3)(p + 2)(p + 1)p(p - 1)(p - 2)$                      |
| $1$  | $\frac{1}{5760}(45 - 45(-1)^p + 96p - 104p^2 - 120p^3 + 10p^4 + 24p^5 + 4p^6)$ |

### Table 5: The integers $\hat{N}_{g,Q}(p)$ for the framed unknot.

| $Q$  | $g = 0$                                                                 | $g = 1$                                                                 |
|------|-------------------------------------------------------------------------|-------------------------------------------------------------------------|
| $-3/2$ | $\frac{1}{6}(-1)^p p(p-2)(p-1)^2$                                       | $\frac{1}{36}(-1)^p p(p-2)(p-1)^2(-3 - 4p + 2p^2)$                       |
| $-1/2$ | $-\frac{1}{6}(-1)^p p(p-1)(-1 + 5p + 3p^2)$                              | $-\frac{1}{24}(-1)^p p(p-2)(p-1)(1 - 3p - 4p^2 + 4p^3)$                  |
| $1/2$  | $\frac{1}{6}(-1)^p p(p-1)(1 - p + 3p^2)$                                 | $\frac{1}{24}(-1)^p (p+1)(p-1)(2 + p - 8p^2 + 4p^3)$                    |
| $3/2$  | $-\frac{1}{6}(-1)^p p^3(p-1)(p+1)$                                      | $-\frac{1}{36}(-1)^p p^3(p-1)(p+1)(-5 + 2p^2)$                          |

### Table 6: The integers $\hat{N}_{g,Q}(p)$ for the framed unknot.

| $Q$  | $g = 0$                                                                 | $g = 1$                                                                 |
|------|-------------------------------------------------------------------------|-------------------------------------------------------------------------|
| $-3/2$ | $\frac{1}{6}(-1)^p p(p-1)(-1 - 2p + 2p^2)$                              | $\frac{1}{36}(-1)^p p(p-2)(p-1)(p+1)(-3 - 8p + 8p^2)$                    |
| $-1/2$ | $-\frac{1}{6}(-1)^p p(p-1)(-1 + 2p + 6p^2)$                              | $-\frac{1}{24}(-1)^p p(p-1)(p+1)(2 - 5p - 8p^2 + 8p^3)$                  |
| $1/2$  | $\frac{1}{6}(-1)^p p(p+1)(-1 - 2p + 6p^2)$                               | $\frac{1}{24}(-1)^p (p+1)(p-1)(-2 - 5p + 8p^2 + 8p^3)$                   |
| $3/2$  | $-\frac{1}{6}(-1)^p p(p+1)(-1 + 2p + 2p^2)$                              | $-\frac{1}{36}(-1)^p p(p+2)(p-1)(p+1)(-3 + 8p + 8p^2)$                   |
The above integer invariants for the unknot satisfy the relation

\[ \hat{N}_{R,g,Q}(p) = (-1)^{\ell} \hat{N}_{R',g,-Q}(p), \]  

(4.18)

where \( R' \) denotes the representation whose Young tableau is transposed to the Young tableau of \( R \). This symmetry is easy to explain. Given a knot \( K \) and its mirror image \( K^* \), the vevs of the corresponding Wilson loops are related as follows (see for example [18]):

\[ W_R(q, \lambda)(K^*) = W_R(q^{-1}, \lambda^{-1})(K). \]  

(4.19)

Using (2.6), it is easy to see that (4.19) implies the following relation for the integer invariants:

\[ \hat{N}_{R,g,Q}(K^*) = (-1)^{\ell} \hat{N}_{R',g,-Q}(K). \]  

(4.20)

Since the mirror image of the unknot with framing \( p \) is the unknot with framing \(-p\), (4.18) follows from (4.20).

| \( Q \) | \( g = 0 \) | \( g = 1 \) |
|-------|----------------|----------------|
| \(-3/2\) | \(-\frac{1}{4}(-1)^pp^2(p-1)(p+1)\) | \(\frac{1}{36}(-1)^pp^2(p-1)(p+1)(-5+2p^2)\) |
| \(-1/2\) | \(-\frac{1}{6}(-1)^pp(p+1)(-1+p+3p^2)\) | \(-\frac{1}{72}(-1)^pp(p+1)(p-1)(-2+p+8p^2+4p^3)\) |
| \(1/2\) | \(\frac{1}{6}(-1)^pp(p+1)(1+5p+3p^2)\) | \(\frac{1}{72}(-1)^pp+2)(p+1)(-1-3p+4p^2+4p^3)\) |
| \(3/2\) | \(-\frac{1}{4}(-1)^pp(p+2)(p+1)^2\) | \(-\frac{1}{36}(-1)^pp(p+2)(p+1)^2(-3+4p+2p^2)\) |

| \( Q \) | \( g = 0 \) | \( g = 1 \) |
|-------|----------------|----------------|
| \(1\) | \(\frac{1}{3}(9-(-1)^p+2p+4p^2)\) | \(\frac{1}{36}(69-21(-1)^p-4p+98p^2+4p^3+4p^4)\) |
| \(2\) | \(-8-5p-p^2\) | \(-\frac{1}{12}(72+80p+75p^2+10p^3+3p^4)\) |
| \(3\) | \(\frac{1}{6}(93+3(-1)^p+76p+26p^2)\) | \(\frac{1}{54}(921+39(-1)^p+1288p+724p^2+152p^3+26p^4)\) |
| \(4\) | \(-\frac{1}{2}(16+13p+3p^2)\) | \(-\frac{1}{44}(2+p)(72+65p+20p^2+3p^3)\) |
| \(5\) | \(\frac{1}{5}(17-(-1)^p+12p+2p^2)\) | \(\frac{1}{54}(93+3(-1)^p+168p+100p^2+24p^3+2p^4)\) |

**Table 7:** The integers \( \hat{N}_{R,g,Q}(p) \) for the framed unknot.

**Table 8:** The integers \( \hat{N}_{R,g,Q}(p) \) for the framed trefoil knot.
Table 9: The integers $\hat{N}_{g,\theta}(p)$ for the framed trefoil knot.

5. Comparison with the direct A-model computation

In the case of the unknot with an arbitrary framing given by $p$, one can find a rather explicit expression for the connected vevs. The dual geometry for the unknot is known \[3\], and some explicit computations of open Gromov-Witten invariants for this geometry have been done using localization techniques \[9\] \[10\] \[11\]. In particular, Katz and Liu \[9\] were able to give an explicit expression for some of these invariants in an arbitrary framing, and therefore comparing the Chern-Simons answer with their computation gives a very powerful check of the duality. Some checks (for genus 0) have been done already in \[9\] in comparison with disk amplitudes of large $N$ Chern-Simons dual \[8\]. An interesting corollary of the comparison, as we will discuss, is that all the correlation functions of two-dimensional topological gravity, and all Hodge integrals involving up to three $\lambda$ classes (Chern classes of the Hodge bundle) can be computed from the quantum dimensions of a Wess-Zumino-Witten model! In particular, the multicovering formula (2.10) predicts that some combinations of Hodge integrals are integers.

The computation of the connected vevs for the unknot in an arbitrary framing is in principle straightforward. A well-known result in Chern-Simons theory \[12\] is that the vev of the unknot in the representation $R$ (in the standard framing) is given by:

$$W_R = \frac{S_{\rho,\rho+\Lambda}}{S_{\rho,\rho}},$$

(5.1)

where $S_{ij}$ are the entries of the $S$-matrix of the $SU(N)$ WZW model, $\Lambda$ is the highest weight associated to the representation $R$, and $\rho$ is the Weyl vector that represents the
compute the vacuum state. The right-hand side of (5.1) can be written in terms of a character of $SU(N)$, and it is also called the quantum dimension of $R$:

$$\dim_q R = \text{ch}_A \left[ -\frac{2\pi i}{k + N} \rho \right]. \quad (5.2)$$

The quantum dimension can be explicitly written in terms of $q$-numbers as follows. Define:

$$[x] = q^{\frac{x}{2}} - q^{-\frac{x}{2}}, \quad [x]_\lambda = \lambda^x q^{\frac{x}{2}} - \lambda^{-\frac{x}{2}} q^{-\frac{x}{2}}. \quad (5.3)$$

If $R$ has a Young tableau with $c_R$ rows of lengths $l_i, i = 1, \ldots, c_R$, then the quantum dimension can be explicitly written as:

$$\dim_q R = \prod_{1 \leq i < j \leq c_R} \frac{|l_i - l_j + j - i|}{|j - i|} \prod_{i=1}^{c_R} \frac{\prod_{v=i+1}^{l_i-i}[v]_\lambda}{\prod_{v=1}^{l_i}[v - i + c_R]}. \quad (5.4)$$

Now we have all the ingredients to compute $F_{g,\bar{k}}(t)$ in an arbitrary framing given by $p$. According to (1.13), the generating functional for $F_{g,\bar{k}}(t)$ is determined by the connected vev $W_{\bar{k}}^{(c)}$. Therefore, we just have to correct the $W_R$ given in (5.4) by the framing factor, compute the $W_{\bar{k}}$ with Frobenius formula, and then extract the connected piece by using:

$$W_{\bar{k}}^{(c)} = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \sum_{\bar{k}_1, \cdots, \bar{k}_n} \delta \sum_{n=1}^{\bar{n}} \bar{k}_{i,k} \prod_{i=1}^{n} \frac{W_{\bar{k}_i}}{z_{\bar{k}_i}}. \quad (5.5)$$

In this equation, the second sum is over $n$ vectors $\bar{k}_i, \cdots, \bar{k}_n$ such that $\sum_{i=1}^{n} \bar{k}_i = \bar{k}$ (as indicated by the Kronecker delta), and therefore the right hand side of (5.5) involves a finite number of terms. The generating functional for the open Gromov-Witten invariants is then explicitly given by:

$$\sum_{Q} \sum_{g=0}^\infty \frac{F_{k_g}^Q}{k^g} g_{2g-2+|\bar{k}|} e^{2\pi i Q t} = \prod_{i=1}^{\ell} \prod_{j,k} \sum_{1 \leq i < j \leq c_{R_s}} \frac{(-1)^n}{n} \sum_{\bar{k}_1, \cdots, \bar{k}_n} \delta \sum_{\sigma=1}^{n} \bar{k}_{i,k} \prod_{\sigma=1}^{R_s} \frac{\chi_{R_s}(C(\bar{k}_\sigma))}{z_{\bar{k}_\sigma}} \cdot e^{i\pi R_s g_{i,j}} \prod_{1 \leq i < j \leq c_{R_s}} \frac{\sin \left( (l^g_i - l^g_j + j - i) g_{s/2} / 2 \right)}{\sin \left( (j - i) g_{s/2} / 2 \right)} \prod_{i=1}^{c_{R_s}} \prod_{v=i+1}^{l^g_i-i} \left( e^{2\pi i \frac{v}{g_{s}}} - e^{-2\pi i \frac{v}{g_{s}}} \right) \prod_{v=1}^{2\pi i \frac{g_{s}}{2}} \frac{\sin \left( (v - i + c_{R_s}) g_{s/2} / 2 \right)}{\sin \left( (v - i) g_{s/2} / 2 \right)}. \quad (5.6)$$

The open Gromov-Witten invariants $F_{k_g}^Q$ have been computed in the A-model by Katz and Liu in [3] for $Q = \ell/2$, with $\ell = \sum j k_j$. This corresponds to the leading power of $e^{Qt}$ in (5.6). Their formula is written in terms of the vector $\bar{n} = (n_1, \cdots, n_h)$ and reads:

$$F_{\bar{n},g}^{\ell/2} = (-1)^{\ell} (p(p + 1))^{h-1} \left( \prod_{i=1}^{h} \prod_{j=1}^{n_i-1} (j + ni p) \right)^{h-1} \frac{1}{(n_i - 1)!} \cdot \text{Res}_{u=0} \int \prod_{j=1}^{h} \prod_{i=1}^{n_i-1} \left( c_g(\bar{\Pi}_{\psi_i}^+ u) c_g(\bar{\Pi}_{\psi_i}^- (u - p - 1) u) c_g(\bar{\Pi}_{\psi_i}^+ (pu)) u^{2h-4} \right) \prod_{i=1}^{h} (u - n_i \psi_i). \quad (5.7)$$
In this formula, \( \overline{M}_{g,h} \) is the Deligne-Mumford moduli space of genus \( g \) stable curves with \( h \) marked points, and has complex dimension \( 3g - 3 + h \) (see for example [21] for a survey of these moduli spaces and their properties). \( \mathcal{E} \) is the Hodge bundle over \( \overline{M}_{g,h} \). It is a complex vector bundle of rank \( g \) whose fiber at a point \( \Sigma \) is \( H^0(\Sigma, K_\Sigma) \). Its dual is denoted by \( \mathcal{E}^\vee \), and its Chern classes are denoted by:

\[
\lambda_j = c_j(\mathcal{E}).
\]

(5.8)

In (5.7), we have written

\[
c_g(\mathcal{E}^\vee(u)) = \sum_{i=0}^{g} c_{g-i}(\mathcal{E}^\vee)u^i,
\]

(5.9)

and similarly for the other two factors. The integral in (5.7) also involves the \( \psi_i \) classes of two-dimensional topological gravity, which are defined as follows. We first define the line bundle \( \mathcal{L}_i \) over \( \overline{M}_{g,h} \) to be the line bundle whose fiber over each stable curve \( \Sigma \) is the cotangent space of \( \Sigma \) at \( x_i \) (where \( x_i \) is the \( i \)-th marked point). We then have,

\[
\psi_i = c_1(\mathcal{L}_i), \quad i = 1, \cdots, h.
\]

(5.10)

The integrals of the \( \psi \) classes can be obtained by the results of Witten and Kontsevich on 2d topological gravity [22][23], while the integrals involving \( \psi \) and \( \lambda \) classes (the so-called Hodge integrals) can be in principle computed by reducing them to pure \( \psi \) integrals [13]. Explicit formulae for some Hodge integrals have been recently found in [14]. In writing (5.7), we have taken into account that the variable \( a \) used in [9] corresponds to \(-p\) here, and we have included the global factor \((-1)^p\ell\) which is crucial in order to extract integer invariants by means of the multicovering formulae.

The Chern-Simons computation (5.6) gives an explicit generating functional for all the open Gromov-Witten invariants, including those in (5.7) with \( Q = \ell/2 \). In particular, it is in principle possible to compute all the integrals over \( \overline{M}_{g,h} \) that appear in (5.7) from the explicit expression (5.6)! These Hodge integrals include an arbitrary number of \( \psi \) classes and up to three \( \lambda \) classes. Therefore, all correlation functions of two-dimensional topological gravity can in principle be extracted from (5.6). It should be noted, however, that some of the simple structural properties of (5.7) are not at all obvious from (5.6). For example, for \( g = 0, h = 1 \), (5.7) gives a fairly compact expression for the open Gromov-Witten invariant, and the fact that this equals the Chern-Simons answer amounts to a rather nontrivial combinatorial identity, as it was already observed in [8].
Let us compare the two expressions, (5.7) and (5.6), in some simple examples with $h = 1$. Since for Riemann surfaces with one hole the homotopy class of the map is given by the winding number $j$, we will denote the invariants by $F_{j,g}^Q$ and $n_{j,g,Q}$ (and we are going to take $Q = j/2$ to compare with \[8\]|\[9\]). For $g = 1$, one finds:

$$F_{j,1}^{j/2} = \frac{(-1)^p}{j-1} \sum_{l=1}^{j-1} (l + jp) \left( \left( \int_{M_{1,1}} \lambda_1 - j \psi_1 \right) p(p+1) + \int_{M_{1,1}} \lambda_1 \right), \quad (5.11)$$

and for $g = 2$,

$$F_{j,2}^{j/2} = \frac{(-1)^p}{j-1} \sum_{l=1}^{j-1} (l + jp) \left( \left( \int_{M_{2,1}} j^2 \psi_1^4 - j \psi_1^3 \psi_2 \right) j^2 p^3 (p+2) \right. $$

$$+ \left. \left( \int_{M_{2,1}} j^3 \psi_1^4 - 2j^2 \psi_1^3 \psi_2 - j \psi_1 \psi_2 \right) j^2 \right)$$

$$+ \left. \left( \int_{M_{2,1}} -j^2 \psi_1^3 \psi_2 - j \psi_1 \psi_2 \right) j^2 \int_{M_{2,1}} \psi_1^2 \psi_2 \right). \quad (5.12)$$

To obtain this expression, we have used the Mumford relations $\lambda_2^2 = 0$ and $\lambda_1^2 = 2 \lambda_2$ [24]. All the integrals involved here can be extracted from the generating functional (5.6), computed up to order $g_s^4$, and for two values of $j$, say $j = 1, 2$. These are easily computed to be:

$$i W_1^{(c)}(g_s) = \frac{(-1)^p}{g_s} \left( 1 + \frac{1}{24} g_s^2 + \frac{7}{5760} g_s^4 + O(g_s^6) \right),$$

$$\frac{i}{2} W_2^{(c)}(g_s) = \frac{1 + 2p}{g_s} \left( \frac{1}{4} - \frac{1}{24} (p^2 + p - 1) g_s^2 + \frac{1}{1440} (7 - 11p - 8p^2 + 6p^3 + 3p^4) g_s^4 + O(g_s^6) \right). \quad (5.13)$$

After some simple algebra, one finds for $g = 1$:

$$\int_{M_{1,1}} \psi_1 = \int_{M_{1,1}} \lambda_1 = \frac{1}{24} \quad (5.14)$$

and for $g = 2$

$$\int_{M_{2,1}} \psi_1^4 = \frac{1}{1152}, \quad \int_{M_{2,1}} \psi_1^2 \lambda_1 = \frac{1}{480},$$

$$\int_{M_{2,1}} \psi_1^2 \lambda_2 = \frac{7}{5760}, \quad \int_{M_{2,1}} \psi_1 \lambda_1 \lambda_2 = \frac{1}{2880}, \quad (5.15)$$

in agreement with known results (see for example [14]).
In general, for arbitrary $h$, the coefficient of the leading power of $p$ is a sum of Hodge integrals which includes

$$
\sum_{k_1, \ldots, k_h} n_{i_1}^{k_1} \cdots n_{i_h}^{k_h} \int_{M_{g,h}} \psi_1^{k_1} \cdots \psi_h^{k_h},
$$

so in principle one can extract the correlation functions of 2d topological gravity from (5.16). This in turn suggests that the Kontsevich matrix integral \cite{23}, which can be viewed as a large $N$ duality for a 0-dimensional gauge theory, may be obtained in an appropriate limit from the large $N$ duality conjecture of \cite{1} for three dimensional Chern-Simons gauge theory with an insertion of observables like (4.1). This would be very interesting to directly establish.

One can obtain integer invariants from the Gromov-Witten invariants by using the multicovering formulae. The most basic integer invariants are the integers $\hat{N}_{R,g,Q}(p)$, and one can compute the integers $n_{k,g,Q}$ that enter the multicovering formula (2.10) from (2.9). The relevant formulae simplify in some particular cases. For example, if we want to compute $n_{j,g,Q}$ (corresponding to one hole with wrapping number $j$), then the generating functional

$$
f_j(q, \lambda) = (q^{-\frac{j}{2}} - q^{\frac{j}{2}}) \sum_{g,Q} n_{j,g,Q}(q^{-\frac{j}{2}} - q^{\frac{j}{2}})^{2g-2} \lambda^Q
$$

is given by

$$
f_j(q, \lambda) = \sum_{d|j} \mu(d) W_{j/d}^{(c)}(q^d, \lambda^d),
$$

and we recall that $\mu(d)$ is the Moebius function. This expression can be derived from (2.4) and (4.10) (see \cite{3} for more details). In the case $h = 1$, the integer invariants that correspond to the open Gromov-Witten invariants $F_{j/g}^{1/2}$ computed by Katz and Liu are $n_{j,g,j}$. The relation between them is precisely the one written in (2.11) for lower genera. In the following tables we list some of these integer invariants. For $g = 0$, these results were obtained in \cite{8} in the context of the B-model.

| $j$  | $g = 0$                          | $g = 1$                          |
|------|----------------------------------|----------------------------------|
| 1    | $-(-1)^p$                        | 0                                |
| 2    | $-\frac{1}{4}(2p + 1 - (-1)^p)$ | $-\frac{1}{48}(-3 + 3(-1)^p - 4p + 6p^2 + 4p^3)$ |
| 3    | $-\frac{1}{2}(-1)^pp(p + 1)$    | $-\frac{1}{8}(-1)^p(p + 1)(-2 + 3p + 3p^2)$ |
| 4    | $-\frac{1}{7}p(p + 1)(2p + 1)$  | $-\frac{1}{7}p(p + 1)(2p + 1)(-1 + 2p + 2p^2)$ |
| 5    | $-\frac{5}{22}(-1)^p(p + 1)(2 + 5p + 5p^2)$ | $-\frac{1}{144}(-1)^p(p + 1)(-96 - 50p + 575p^2 + 1250p^3 + 625p^4)$ |

**Table 10**: The integers $n_{j,g,j}(p)$ for the framed unknot.
The open Gromov-Witten invariants for $g = 0, h = 1$ (disk amplitudes) can also be computed in the B-model by using the techniques of [7][8]. In [8], the mirror geometry to the framed unknot was studied in detail in the large volume limit, and this allowed to obtain an explicit expression for the invariants $F_{j/2}^{Q_j, g=0}$. In fact, one can extend the computation in [8] and obtain the explicit expression of $F_{j, g=0}^Q$ for $Q = -j/2, \ldots, j/2$, which we will now do, in order to compare with the results we have obtained from Chern-Simons theory.

The A-model geometry for the unknot in $S^3$ was found in [2], and it is a Lagrangian submanifold in the resolved conifold $O(-1) \oplus O(-1) \rightarrow P^1$. Actually, more precisely it is the same Lagrangian submanifold discussed in [2], but in the “flopped phase” of the Lagrangian submanifold (Phase II, p. 25 of [4][8]). This also avoids the problem of having the $S^1$ of the original Lagrangian submanifold getting flopped in the blow up geometry, which was noticed by the authors of [2] (but not written in the paper). The mirror geometry for the brane is characterized by the Riemann surface,

$$e^t e^{v+u} + e^u + e^v + 1 = 0. \quad (6.1)$$

The right variables to use are, according to the analysis in [8], $\hat{u} = u + i\pi$, $\hat{v} = v + i\pi$. This finally gives the equation,

$$e^{\hat{u}} + e^{\hat{v}} - e^{t} e^{\hat{u} + \hat{v}} = 1. \quad (6.2)$$

An arbitrary framing specified by $p$ corresponds to a shift $\hat{u} \rightarrow \hat{u} + p\hat{v}$, and the equation to be solved reads:

$$xy^p + y - e^t xy^{p+1} = 1, \quad (6.3)$$

---

1 We are grateful to Mina Aganagic for discussions leading to this clarification.
where we have denoted
\[ x = e^{\hat{u}}, \quad y = e^{\hat{v}}. \] (6.4)

The algebraic equation (6.3) can be solved with the ansatz
\[ y = \sum_{m=0}^{\infty} a_m x^m, \] as in [3].

One gets a recursive relation for the coefficients with the following explicit solution:
\[ a_m = \sum_{l=0}^{m} a_{m,l} e^{lt}, \] (6.5)

where
\[ a_{m,l} = \frac{(-1)^{m+l}}{m!} \binom{m}{l} \prod_{j=-l}^{m-l-2} (mp - j). \] (6.6)

The open Gromov-Witten invariants are the coefficients of the superpotential,
\[ W = \sum_{m=1}^{\infty} \sum_{l=0}^{m} W_{m,l} e^{m\hat{u}+lt}, \] (6.7)

which can be obtained from the equation \( \hat{v} = \partial_{\hat{u}} W \), or equivalently \( x \partial_x W = \log y \). One then finds,
\[ W_{m,l} = (-1)^{m+l} \binom{m}{l} \prod_{j=-l+1}^{m-l-1} (mp - j), \quad l = 0, \ldots, m. \] (6.8)

The result of [3] for the “almost \( \mathbb{C}^3 \)” geometry is a particular case of (6.6) and (6.8) when \( l = 0 \). Notice the symmetry
\[ W_{m,l}(-p) = -W_{m,m-l}(p). \] (6.9)

We have checked in many cases that the above result agrees with the Chern-Simons result for the framed unknot (5.6). More precisely, one has that
\[ F_{m,g=0}^Q = (-1)^{pm} W_{m,Q+m/2}, \quad Q = -m/2, \ldots, m/2. \] (6.10)

The multicovering formulae (2.11) predict that, if we write the superpotential as
\[ W = -\sum_{m,l} \sum_{k>0} \frac{n_{m,l}}{k^2} e^{k(m\hat{u}_p+lt)}, \] (6.11)

where \( \hat{u}_p = \hat{u} + \pi ip \), then the \( n_{m,l} \) are integers. In fact, they are the integer invariants \( n_{m,g=0,l-m/2} \) that appear in (2.11) and that can be computed from Chern-Simons theory. The case \( l = 0 \) was obtained in [3]. For \( l > 0 \) one finds, for example:
\[ n_{2,1} = p, \quad n_{2,2} = \frac{p}{2} + \frac{(-1)^p - 1}{4}, \]
\[ n_{3,1} = -\frac{(-1)^p}{2} p(3p-1), \quad n_{3,2} = \frac{(-1)^p}{2} p(3p+1), \quad n_{3,3} = -\frac{(-1)^p}{2} p(p+1), \]

\[ n_{4,1} = \frac{p}{3} (2p-1)(4p-1), \quad n_{4,2} = -4p^3, \quad n_{4,3} = \frac{p}{3} (2p+1)(4p+1), \quad n_{4,4} = -\frac{p}{3} (p+1)(2p+1), \]

and so on. For \( m = 2, 3 \), one can immediately check that the above expressions agree with the invariants \( (2.9) \) obtained from tables 1-7. It follows from \( (6.9) \) that

\[ n_{m,l}(-p) = -n_{m,m-l}(p), \quad (6.12) \]

which from the Chern-Simons point of view is a consequence of \( (4.18) \). Notice that, when \( p = 1 \), the integers \( n_{m,l} \) are precisely the \( d_{l,m} \) computed in \[7\] for phase II of the Lagrangian submanifold in \( O(-1) \oplus O(-1) \to \mathbb{P}^1 \).

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