Parameterized Inapproximability of
Exact Cover and Nearest Codeword

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Abstract

The $k$-ExactCover problem is a parameterized version of the ExactCover problem, in which we are given a universe $U$, a collection $S$ of subsets of $U$, and an integer $k$, and the task is to determine whether $U$ can be partitioned into $k$ sets in $S$. This is a natural extension of the well-studied SetCover problem; though in the parameterized regime we know it to be $W[1]$-complete in the exact case, its parameterized complexity with respect to approximability is not well understood.

We prove that, assuming ETH, for some $\gamma > 0$ there is no time $f(k) \cdot N^{\gamma k}$ algorithm that can, given a $k$-ExactCover instance $I$, distinguish between the case where $I$ has an exact cover of size $k$ and the case where every set cover of $I$ has size at least $\frac{1}{4} \log N$. This rules out even more than FPT algorithms, and additionally rules out any algorithm whose approximation ratio depends only on the parameter $k$. By assuming SETH, we instead improve the lower bound to requiring time $f(k) \cdot N^{k-\varepsilon}$, for any $\varepsilon > 0$.

In this work we also extend the inapproximability result to the $k$-Nearest-Codeword ($k$-NCP) problem. Specifically, given a generator matrix $A \in \mathbb{F}_2^{m \times n}$, a vector $y \in \mathbb{F}_2^m$, and the parameter $k$, we show that it is hard to distinguish between the case where there exists a codeword with distance at most $k$ from $y$ and the case where every codeword has distance at least $\frac{1}{4} \log N$ from $y$. This improves the best known parameterized inapproximability result, which rules out approximations with a factor of poly$(\log k)$, but requires us to assume ETH instead of $W[1] \neq$ FPT.

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1 Introduction

In the exact cover (ExactCover) problem, we are given a universe $U$, a collection $S$ of subsets of $U$, and an integer $k$; the goal is to determine whether there exist $k$ sets in $S$ that form a partition of $U$. Equivalently, given a bipartite graph $I = (U, S, E)$ and an integer $k$, the goal is to determine whether there exists a subset $A \subseteq S$ of size $k$ such that every element of $U$ is adjacent to exactly one element of $A$. As it was presented as one of Karp's 21 NP-complete problems in [Kar09], it is natural to cope with its intractability by turning to either approximation or parameterization—or both.

In looking purely at approximations, it is well-known that ExactCover does not admit any non-trivial approximation algorithms, assuming $P \neq NP$; this is due to the fact that some simple reductions will produce an exact cover instance containing a partition of size $k$ in one case, and in the other case produce an instance containing no partition of any size. On the other hand, if we allow for non-exact covers in the soundness case, the greedy algorithm achieves a $\ln n$ approximation, and there is a matching hardness result [Fei98, Mos15].

In this paper, though, our focus is on the parameterized version ($k$-ExactCover), in which the parameter $k$ is the desired size of the partition. There are a few known upper bounds for simply deciding whether a partition into $k$ subsets exists: the brute-force algorithm solves this problem in time $O(n^k+1)$, which can be improved slightly to $O(n^{k+o(1)})$ via a matrix multiplication technique [EG04]. Since $k$-ExactCover is known to be W[1]-hard [DF13], however, it is unlikely that it admits much faster algorithms in the exact case.

What about parameterized approximations? For $k$-ExactCover, we know of a result showing constant-factor inapproximability using Gap-ETH as the starting point of hardness [Man19]; we do not know of any other published results. However, there have been a recent series of advances in examining the parameterized approximability of the more standard $k$-SetCover problem, parameterized likewise by the size of the cover [CL16, CCK+17, KLM18, Lin19]; the most recent one shows the W[1]-hardness of approximation within a $(1-\epsilon)\frac{\log N}{\log \log N}$ ratio for any $\epsilon > 0$, with finer-grained results under ETH and SETH. Unfortunately, none of these results generalize immediately to $k$-ExactCover, as in the completeness case they do not yield a cover with $k$ disjoint sets. Following the general technique of [Lin19], our main contribution is an (essentially) matching inapproximability ratio for $k$-ExactCover:

**Theorem 1.1** (See Theorem 3.6). Assuming ETH, there exists $\gamma > 0$ such that for all computable $f$, there is no $f(k) \cdot N^{\gamma k}$ time algorithm that, given a set cover instance $(U, S)$ of size $N$, finds a set cover of size at most $\frac{1}{4}\frac{\log N}{\log \log N}$ whenever $U$ can be partitioned into $k$ sets from $S$. If SETH holds, then for every $\epsilon > 0$ and every computable $f$, there is no time $f(k) \cdot N^{k-\epsilon}$ algorithm for the same task.

This result unfortunately requires ETH, which is stronger than W[1] $\neq$ FPT. We also
briefly examine the nearest codeword problem (NCP), in which we are given a generator matrix $A \in \mathbb{F}_2^{m \times n}$ of a binary linear code, a vector $y$, and an integer $k$, and the task is to decide whether there exists a codeword with Hamming distance at most $k$ from $y$. NCP was shown to be NP-hard by Berlekamp et al. [BMT78], and was further shown to admit no constant-factor approximations in [ABBSS97] via a reduction from ExactCover, which we will use as well; reducing instead from LabelCover, they were able to show stronger results.

The parameterized variant of NCP ($k$-NCP), parameterized by this distance variable $k$, is known to be W[1]-hard [DFVW99]. This was the best known result until recently when [BGKM18] showed that, assuming Gap-ETH, there is no FPT constant-factor approximation. This assumption was subsequently weakened to $W[1] \neq FPT$ by [BELM18] via a reduction from Biclique, building upon the work of [Lin14, BEM16]. This result for NCP also completed a dichotomy for the parameterized complexity of MinCSPs [BEM16, BELM18]. In fact, they also improve the inapproximability ratio, showing the $W[1]$-hardness of not only constant-factor approximations but also poly$(\log k)$ factor approximations as well. We remark that the parameterized inapproximability of $k$-NCP implies that of the minimum distance problem ($k$-MDP), which was considered one of the biggest open questions in parameterized complexity [DF13] until resolved by [BGKM18]; prior to this work, not even the exact version was known to be $W[1]$-hard.

Using the ideas of [ABSS97] and [Lin19], by a simple extension we are able to obtain some parameterized inapproximability results for $k$-NCP:

**Theorem 1.2** (See Theorem 3.8). Assuming ETH, there exists $\gamma > 0$ such that for all computable $f$, there is no $f(k) \cdot N^{\gamma k}$ time algorithm that, given a generator matrix $A \in \mathbb{F}_2^{m \times n}$ of size $N$ and a vector $y \in \mathbb{F}_2^n$, distinguishes between:

- there exists $x \in \mathbb{F}_2^n$ such that $Ax$ and $y$ differ in at most $k$ indices;
- for every $x \in \mathbb{F}_2^n$, $Ax$ and $y$ differ in at least $\frac{1}{8} \sqrt{\frac{k \log N}{\log \log N}}$ indices.

If SETH holds, then for every $\varepsilon > 0$ and every computable $f$, there is no time $f(k) \cdot N^{k-\varepsilon}$ algorithm for the same task.

This improves the inapproximability factor of [BELM18] quite significantly, at the expense of needing ETH rather than just $W[1] \neq FPT$. In particular, our result shows “total inapproximability” of $k$-NCP, in the sense that it rules out $f(k)$-approximations for any computable function $f$.

### 2 Preliminaries

We begin by defining the exact versions of some problems we will discuss.

- **CNF-SAT.** In the $k$-SAT problem, we are given an $n$-variable CNF $\varphi$ with exactly $k$ literals in each clause, and the goal is to determine there exists a truth assignment...
such that each clause has at least one literal that is true. In the $b$-in-$k$-SAT problem, a clause is satisfied if exactly $b$ of its literals are true.

- **Clique.** In Clique, we are given an $n$-vertex graph $G$ and an integer $k$; the goal is to determine whether $G$ contains a clique of size $k$. The parameterized version is $k$-Clique, parameterized by the same $k$.

- **Set-Cover.** In SetCover, we are given a bipartite graph $I = (U, S, E)$ and an integer $k$; the goal is to determine whether there exists a subset $A \subseteq S$ of size $k$ such that every vertex in $U$ is adjacent to some vertex in $A$. The parameterized version is $k$-SetCover.

- **Exact-Cover.** The ExactCover (resp. $k$-ExactCover) problem is the same as SetCover (resp. $k$-SetCover), except we would like that every $u \in U$ is adjacent to exactly one vertex in $A$.

- **Nearest-Codeword.** In the Nearest-Codeword Problem (NCP), we are given a generator matrix $A \in \mathbb{F}_2^{m \times n}$ of a binary linear code, a target vector $y \in \mathbb{F}_2^m$, and an integer $k$; the goal is to decide whether there exists a codeword with distance at most $k$ from $y$, i.e. whether there exists $x \in \mathbb{F}_2^n$ such that $\|Ax - y\|_0 \leq k$. We denote its parameterized version by $k$-NCP.

We remark that the exact parameterized versions of these problems are all known to be $\text{W}[1]$-hard (in fact, $k$-SetCover is hard even for $\text{W}[2]$) [DF13] and are therefore unlikely to admit FPT algorithms. However, we are interested in their parameterized approximation variants, which we will define below:

**Definition 2.1** (Parameterized Approximation for SetCover). In $(k, h)$-SetCover, for $h \geq k$ we are given a set cover instance $I = (U, S, E)$, and the parameter is $k$. The goal is to distinguish between the following cases:

- $I$ has a set cover of size $k$;
- every set cover of $I$ has size greater than $h$.

**Definition 2.2** (Parameterized Approximation for NCP). In $(k, h)$-NCP, for $h \geq k$, we are given a matrix $A \in \mathbb{F}_2^{m \times n}$, a vector $y \in \mathbb{F}_2^m$, and the parameter is $k$. The goal is to distinguish between the following cases:

- for some $x \in \mathbb{F}_2^n$, $\|Ax - y\|_0 \leq k$;
- for every $x \in \mathbb{F}_2^n$, $\|Ax - y\|_0 > h$.

We define $(k, h)$-ExactCover somewhat differently as follows:
**Definition 2.3.** Let $I = (U, S, E)$ be a set cover instance. An exact cover of size $k$ is a partition of $U$ into $k$ subsets in $S$.

**Definition 2.4 (Parameterized Approximation for ExactCover).** In $(k, h)$-ExactCover, for $h \geq k$, we are given a set cover instance $I = (U, S, E)$, parameterized by $k$. The goal is to distinguish between the following cases:

- $I$ has an exact cover of size $k$;
- every set cover of $I$ has size greater than $h$.

Crucially, we observe that in the second case, we are still quantifying over all set covers, rather than exact covers. This formulation is important for downstream reductions to approximations for NCP, and, as described earlier, the formulation where the second case quantifies over exact covers as well is rather uninteresting, as it may be the case that no exact cover of any size exists.

Next, we define some formulations of hardness assumptions that we will discuss. The first one we will mention is $W[1] \neq FPT$, one of the most common hardness assumptions made in parameterized complexity, since many problems happen to be $W[1]$-hard. Discussion of the $W$ hierarchy is omitted (refer to [DF13] for more); since $k$-Clique is a canonical problem that is hard for $W[1]$, we will simply use the convenient formulation that it admits no FPT algorithms.

**Hypothesis 1 ($W[1] \neq FPT$ [DF13]).** For any computable function $T : \mathbb{N} \rightarrow \mathbb{N}$, there is no algorithm that can solve $k$-Clique in time $T(k) \cdot n^{O(1)}$.

We will also discuss several versions of the Exponential Time Hypothesis (ETH), another popular hardness assumption that simply states that solving 3-SAT cannot be done in subexponential time.

**Hypothesis 2 (Exponential Time Hypothesis (ETH) [IP01, IPZ01]).** There exists some $\delta > 0$ such that no algorithm can solve 3-SAT on $n$ variables in time $O(2^{\delta n})$. Further, there exists $C < \infty$ such that this holds even when only considering formulae having at most $Cn$ many clauses.

Note that the second assumption can be made due to the Sparsification Lemma [IPZ01], which, loosely speaking, states that any $k$-SAT instance can be converted into an equivalent one having $O_k(n)$ clauses.

**Hypothesis 3 (Strong Exponential Time Hypothesis (SETH) [IP01, IPZ01]).** For every $\varepsilon > 0$, there exists $k = k(\varepsilon)$ such that $k$-SAT cannot be solved in time $O(2^{(1-\varepsilon)n})$. Moreover, there exists $C = C(\varepsilon)$ such that this holds even when only considering $k$-CNFs with at most $Cn$ clauses.
As with ETH, the original formulation of SETH does not impose an upper bound on the number of clauses. However, by applying the Sparsification Lemma we may assume that our input formulae only have linearly many clauses.

**Hypothesis 4** (Gap Exponential Time Hypothesis (Gap-ETH) [Din16]). There exist $\varepsilon, \delta > 0$ such that any randomized algorithm that, given a 3-SAT instance $\varphi$, can distinguish between the following cases with probability at least $\frac{2}{3}$ requires time at least $2^{\delta n}$:

- $\varphi$ is a satisfiable instance;
- $\varphi$ is not even $(1 - \varepsilon)$-satisfiable, i.e. every assignment to the variables of $\varphi$ falsifies at least an $\varepsilon$ fraction of the clauses.

Note that SETH and Gap-ETH are both stronger assumptions than ETH. Since ETH implies there is no FPT algorithm for Clique, we remark that W[1] ≠ FPT is in fact the weakest hypothesis out of the listed assumptions.

3 Results

3.1 Inapproximability for Exact Cover

Our approach to showing inapproximability for $k$-ExactCover mirrors the approach in [Lin19] closely; we begin with a hardness assumption, convert it into an SetCover instance, and then apply a gadget to create the desired gap. The principal differences are twofold. First, we observe that the gadget application preserves the exactness of a set cover: this means that if we can reduce our starting assumption into not just a $(k,k)$-SetCover instance but rather an $(k,k)$-ExactCover instance, then we will have inapproximability for $(k,h)$-ExactCover rather than $(k,h)$-SetCover. [Lin19] presents two reductions to show hardness of approximation for $k$-SetCover, one from $\ell$-SAT and one from $k$-Clique; our reduction, which is presented in Lemma 3.4, is more in the spirit of the former but is a little different.

We will begin by defining the necessary ingredients from [Lin19].

**Definition 3.1** ($(k,n,m,\ell,h)$-Gap-Gadget [Lin19]). A $(k,n,m,\ell,h)$-Gap-Gadget is a bipartite graph $T = (A,B,E_T)$ such that

- $A$ is partitioned into $(A_1, A_2, \ldots, A_m)$, and $|A_i| = \ell$ for all $i \in [m]$.
- $B$ is partitioned into $(B_1, B_2, \ldots, B_k)$, and $|B_j| = n$ for all $j \in [k]$.
- For all $b_1 \in B_1, b_2 \in B_2, \ldots, b_k \in B_k$, there exist $a_1 \in A_1, a_2 \in A_2, \ldots, a_m \in A_m$ such that $(a_i, b_j) \in E_T$ for all $i \in [m], j \in [k]$.
- For all $X \subseteq B$ and $a_1 \in A_1, a_2 \in A_2, \ldots, a_m \in A_m$, if $|N(a_i) \cap X| \geq k + 1$ for all $i \in [m]$, then $|X| > h$. 

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Using the concept of an \((m,K)\)-universal set, namely a collection of \(m\)-bit strings such that their restriction to any \(K\) indices contain all \(2^K\) binary strings, one can efficiently construct such a gadget:

**Proposition 3.2** (Construction of Gap-Gadgets [Lin19]). There is an algorithm that, for every \(k, h, n \in \mathbb{N}\) satisfying \(k \log \log n \leq \log n\) and \(h \leq \frac{\log n}{(2+\varepsilon) \log \log n}\), can compute a \((k,n,n \log h, h^k,h)\)-Gap-Gadget in \(O(n^k)\) time.

Given a well-behaved set cover instance \(I\) and a Gap-Gadget \(T\) as described above, we can obtain our desired gap, as follows.

**Lemma 3.3** ([Lin19]). There is an algorithm that, given an integer \(k\), a set cover instance \(I = (U,S,E)\) where \(S\) is partitioned into \(\{S_1,S_2,\ldots,S_k\}\) and \(|S_i| = n\) for all \(i \in [k]\), and a \((k,n,m,\ell,h)\)-Gap-Gadget \(T = (A,S,E')\), outputs a set cover instance \(I' = (U',S,E')\) with \(|U'| = m|U|^\ell\) in \(|U|^{\ell \cdot n^{O(1)}}\) time such that

- if there exist \(s_1 \in S_1, s_2 \in S_2,\ldots,s_k \in S_k\) such that \(\{s_1,s_2,\ldots,s_k\}\) form an exact cover of \(U\), then \(I'\) has an exact cover of size \(k\);
- if every set cover of \(I\) has size greater than \(k\), then every set cover of \(I'\) has size greater than \(h\).

**Proof.** We will describe the reduction and reprove the completeness case, which is slightly different; the proof of soundness is identical to the cited lemma, but we include it for completeness.

For each \(i \in [m]\), set \(U_i = \{f : A_i \rightarrow U\}\), and let \(U' = \bigcup U_i\). We define \(E'\) as follows: for each \(s \in S\) and \(f \in U_i\), \((s,f) \in E'\) if and only if there exists \(a \in A_i\) such that \((s,f(a)) \in E\) and \((a,s) \in E_T\).

**Completeness.** Suppose that \(s_1 \in S_1, s_2 \in S_2,\ldots,s_k \in S_k\) cover \(U\) exactly. Let \(i \in [m]\) and \(f \in U_i\) be arbitrary; it suffices to show that \(f\) is covered by exactly one \(s_i\). By the definition of Gap-Gadget, there exists some \(a_i \in A_i\) such that \((a_i,s_j) \in E_T\) for all \(j \in [k]\). Further, \(f(a_i)\) is covered by exactly one such \(s_j\), and so we deduce that there is exactly one element of \(\{s_1,s_2,\ldots,s_k\}\) covering \(f\).

**Soundness.** Let \(X \subseteq S'\) be a set cover of \(U\), and suppose that every set cover of \(I\) has size greater than \(k\). We claim that for every \(i \in [m]\) there exists \(a \in A_i\) such that \(|N_T(a) \cap X| \geq k+1\). Suppose not. Then there exists an \(i \in [m]\) such that for every \(a \in A_i\), there exists some \(u_a \in U\) that is not covered by the set \(N_T(a) \cap X\). Let \(f : A_i \rightarrow U\) be defined via \(f(a) = u_a\); we claim that \(f\) is not covered by \(X\). Indeed, if \(f\) is covered by some \(s \in X\), for every \(a \in A_i\), either \(s \in N_T(a)\), in which case \((s,f(a)) = (s,u_a) \notin E\), or \(s \notin N_T(a)\), in which case \((a,s) \notin E_T\); either way, \((s,f) \notin E'\), as desired. With this claim in hand, for each \(i \in [m]\) we may choose \(a_i \in A_i\) such that \(|N_T(a_i) \cap X| \geq k+1\), from which we deduce that \(|X| > h\).
Next, we describe our reduction from \( \ell \text{-SAT} \) to \( k \text{-ExactCover} \). Very loosely speaking, given a formula \( \varphi \), our output \texttt{ExactCover} instance \( I \) has a universe of nodes corresponding to the literals in the clauses in \( \varphi \). The subsets come in two flavors: one that represents the truth assignment to variables, and one that represents how clauses in \( \varphi \) are satisfied. This construction ensures that these two must “interlock” in any cover, ensuring the exactness of the cover and allowing us to recover a satisfying assignment to \( \varphi \).

**Lemma 3.4.** There exists an algorithm that, given an integer \( k \) and a \( \ell \text{-SAT} \) instance \( \varphi \) with \( n \) variables and \( m \leq Cn \) many clauses, outputs an exact cover instance \( I = (U, S, E) \) where \( S = S_1 \cup S_2 \cup \cdots \cup S_{k'} \) is partitioned into \( k' = (1 + C\ell)k \) equal parts in \( O(C\ell k \cdot 2^{n/k}) \) time such that

- \(|U| \leq 2C\ell n + k'\) and \(|S| \leq C\ell k \cdot 2^{n/k} \);
- if \( \varphi \) is satisfiable, then there exist \( s_1 \in S_1, s_2 \in S_2, \ldots, s_{k'} \in S_{k'} \) such that \( \{s_1, s_2, \ldots, s_{k'}\} \) form an exact cover for \( I \);
- if \( \varphi \) is not satisfiable, then every set cover of \( I \) is of size greater than \( k' \).

**Proof.** Let \( k_1 = k \) and \( k_2 = C\ell k \), noting that \( k' = k_1 + k_2 \). Given \( \varphi \), partition its variables into \( k_1 \) groups \( V_1, V_2, \ldots, V_{k_1} \) of equal size, and partition its clauses into \( k_2 \) many groups \( C_1, C_2, \ldots, C_{k_2} \) of equal size. For \( i \in [k_1] \), let \( A_i \) be the set of partial assignments to the variables in \( V_i \), i.e. \( A_i = \{\sigma_i : V_i \to \{0,1\}\} \). For each \( j \in [k_2] \), let \( H_j \) be the set of possible truth assignments to the literals in the clauses of \( C_j \), i.e. \( H_j = \{\tau_j : C_j \to \{0,1\}^\ell\} \); for any clause \( C \), we will view the output \( \tau_j(C) \) as a function \( \tau_j(C) : [\ell] \to \{0,1\} \). Note that \(|A_i| = |H_j| = 2^{n/k} \). Finally, we let \( S \) be the union of all the \( A_i \)'s and the \( H_j \)'s.

Let \( U \) be the set containing \( \{p_{r,s}^b : r \in [m], s \in [\ell], b \in \{T,F\}\} \) as well as \( k' \) dummy nodes \( a_1, a_2, \ldots, a_{k_1}, h_1, h_2, \ldots, h_{k_2} \); informally, we create two nodes per literal per clause, which will represent their truth value in a purported assignment. The purpose of the dummy nodes is to enforce that at least one node from each \( A_i \) and \( H_j \) is chosen in a cover, and exactly one when the cover is exact.

\( E' \) is composed of the following types of edges:

- **Dummy edges.** For each \( i \in [k_1] \) and \( \sigma_i \in A_i \), put \((\sigma_i, a_i)\) in \( E' \). For each \( j \in [k_2] \) and \( \tau_j \in H_j \), put \((\tau_j, h_j)\) in \( E' \).

- **Variable edges.** For all \( i \in [k_1] \), \( \sigma_i \in A_i \), \( r \in [m] \), and \( s \in [\ell] \), do the following. If the \( s \)-th literal in the \( r \)-th clause of \( \varphi \) belongs to \( A_i \), then put \((\sigma_i, p_{r,s}^T)\) in \( E' \) if \( \sigma_i \) assigns it to 0 and \((\sigma_i, p_{r,s}^F)\) in \( E' \) if \( \sigma_i \) assigns it to 1. This defines the set \( \sigma_i \) such that, for every literal \( p_{r,s} \) that \( \sigma_i \) assigns a value to, the set \( \sigma_i \) covers the node opposite to that value.
Figure 1: An overview of the reduction in Lemma 3.4. Subsets in $S$ represent either partial assignments to the variables or truth assignments to the clauses. The former covers nodes representing literals with value opposite to what the assignment sets them to, while the latter covers the corresponding literals directly.

- **Clause edges.** For every $j \in [k_2]$, $\tau_j \in H_j$, $r \in [m]$, do the following. If clause $r$ belongs to $C_j$ and $\pi_r = \tau_j(r)$ satisfies the $\ell$-SAT predicate (i.e. $\pi_r(s) = 1$ for some $s \in [\ell]$), then for each $s \in [\ell]$ put $(\tau_j, p_r^{\pi(r)}(s))$ in $E'$. Each $\tau_j$ should cover the nodes corresponding to the purported satisfying assignment to each of its associated clauses.

A depiction of the reduction is shown in Figure 1. Informally, since the variable edges cover the opposite of the truth assignment and clause edges cover exactly the satisfying assignments, their sets “interlock” with each other. A satisfying assignment to $\varphi$ indicates that some collection of subsets do agree with each other in this way, giving an exact cover; moreover, any set cover must interlock, showing that $\varphi$ is satisfiable. We give a more detailed argument below.

**Completeness.** Suppose that $\varphi$ has a satisfying assignment $\sigma$. Then under $\sigma$, each clause has at least one literal set to $T$. We claim choosing the corresponding partial assign-
ments and clause assignments in $S$ creates an exact cover. Clearly, the dummy $a_i$ and $h_j$ nodes are uniquely covered. Now, let $r \in [m]$ and $s \in [\ell]$ be arbitrary; of the nodes $p^T_{r,s}$ and $p^F_{r,s}$, the partial variable assignments cover one of these and the partial clause assignments cover one of these as well. Since $\sigma$ is a satisfying assignment, choosing the assignments in the natural way ensures that they cover different $p_{r,s}$ nodes, and so we deduce that each of them is covered by exactly one set.

Soundness. Suppose that our produced instance $I'$ has any set cover of size at most $k$; by construction, this means we must choose one set per $A_i$ and one set per $H_j$. Observe that each clause corresponds to $2\ell$ nodes in $U'$; further, the $A_i$ nodes together can cover at most $\ell$ of these, and the $H_j$ nodes can also only cover at most $\ell$ (either $p^T_{r,s}$ or $p^F_{r,s}$ for a fixed $r$ and $s \in [\ell]$). Hence, this is actually an exact cover and we may recover a satisfying truth assignment to $\varphi$ in the natural way.

We remark that this reduction does not depend on the CSP being an $\ell$-SAT instance; since $\ell$ is constant, this reduction will hold for any relation $R \subseteq \{T,F\}^\ell$. If $\ell$ has a dependency on $n$, we note that this reduction can be modified slightly to run in time $O((\log |R| \cdot Ck \cdot 2^{n/k})$).

Finally, we may combine the results above.

Lemma 3.5. There is an algorithm that, given sufficiently large $k \in \mathbb{N}$ and an $\ell$-SAT instance $\varphi$ with $n$ variables and at most $Cn$ clauses, for sufficiently large $n$, outputs an exact cover instance $I = (U,S,E)$ of size at most $N = 2^{n/k}(1+1/2^k-3)$ in time $2^{5n/k}$ such that

- if $\varphi$ is satisfiable, then $I$ has an exact cover of size $k$;
- if $\varphi$ is not satisfiable, then every set cover of $I$ has size greater than $\frac{1}{4} \sqrt[4]{\log N}$.

Proof. Let our $\ell$-SAT instance $\varphi$ with $n$ variables and at most $Cn$ clauses be given. We first apply Lemma 3.4 to obtain an exact cover instance $I' = (U',S',E')$ where $S'$ is partitioned into $k' = (1+C\ell)k$ parts of equal size, and $I'$ satisfies the other properties stated in the lemma. Set $M = 2^{n/k}$ and $h = \frac{1}{2} \sqrt{\log M / \log \log M}$. By Lemma 3.2, setting $(k,n,m,\ell,h) = (k',M,M \log h,h^k,h)$, we get such a Gap-Gadget $T$ in $O(M^4) < 2^{5n/k}$ time, for large enough $n$. Note that $k' \log M \leq k' \log n \leq n/k = \log M$ for sufficiently large $n$ and $h \leq \frac{1}{7} \log M$, since $x^{1/k} < \varepsilon x$ for any $\varepsilon > 0$, for sufficiently large $x$, so we are indeed allowed to use this construction.

Now, using this exact cover instance $I'$ and the Gap-Gadget $T$, we may apply Lemma 3.3 to obtain a new exact cover instance $I = (U,S,E)$ with $S = S'$. We will argue that $I$ satisfies all the desired properties. The case where $\varphi$ is satisfiable follows directly from
the argument in Lemma 3.3; to see the case in which \( \varphi \) is unsatisfiable, observe that
\[
\frac{\log M}{\log \log M} > \frac{1}{1 + 1/2^{k-3}} \frac{\log N}{\log \log N} > \frac{1}{2^k} \frac{\log N}{\log \log N} \quad \text{for } k \geq 2,
\]
from which the result follows.

Finally, to compute the size of the instance \( I \), observe that
\[
|S| = |S'| \leq C\ell k \cdot 2^{n/k}
\]
and
\[
|U| = (M \log h)|U'| \leq (M \log h)(2\ell n + k')2^k \log \log M.
\]
With some computation, we may deduce that
\[
|S| + |U| \leq C\ell k M + (2C\ell M \log h)(n + k)2^k \log \log M \quad \text{via } 2C\ell k \geq k'
\]
\[
\leq C\ell M\left(k + (2 \log h)(2k \log M)2^k \log \log M\right) \quad \text{via } n \geq k \text{ for large } n
\]
\[
\leq C\ell M\left(k + (2 \log M)(\log M)2^k \log \log M\right) \quad \text{via } \log M \geq 2k \text{ for large } n
\]
\[
\leq C\ell M\left(k + (2 \log M)M^{1/2k-1}\right) \quad \text{via } 2 \log \log M \ll M^{1/2k-1} \text{ for large } n
\]
\[
\leq M^{1+1/2k-3} \quad \text{for large enough } n,
\]
which is precisely \( N \). This step can also be done within time \( 2^{5n/k} \), so we are done. \( \Box \)

As shown in [Lin19], if we assume sufficiently large \( k \) we can actually achieve a bound of \( \frac{1}{1+\delta} \sqrt[k]{\log N \log \log N} \) for any \( \delta > 0 \) by being slightly more careful with the bounding arguments. Since the constant factor is not significant compared to the \( \sqrt[k]{\log N \log \log N} \) term, however, for sake of simplicity we bound things a little more loosely.

It remains to apply this result to our starting hardness assumptions. We obtain the following:

**Theorem 3.6.** Assuming that ETH holds, there exists \( \gamma > 0 \) such that for every computable function \( f : \mathbb{N} \to \mathbb{N} \) and large enough \( k \), no time \( f(k) \cdot N^{\gamma k} \) algorithm can decide \( \left(k, \frac{1}{4} \sqrt[k]{\log N \log \log N}\right)\)-ExactCover. If SETH holds, for every \( \varepsilon > 0 \), every computable \( f \), and sufficiently large \( k \), no time \( f(k) \cdot N^{k-\varepsilon} \) algorithm can decide \( \left(k, \frac{1}{4} \sqrt[k]{\log N \log \log N}\right)\)-ExactCover.

**Proof.** Suppose for contradiction that, for every \( \gamma' > 0 \), such an algorithm \( A_{\gamma'} \) exists. By ETH, there exists \( \gamma \) such that 3-SAT cannot be solved in time \( 2^{\gamma n} \). Then, given a 3-SAT formula \( \varphi \), we may apply Lemma 3.5 to obtain an exact cover instance \( I \) of size \( N = 2^{n/k} \cdot (1+1/2^{k-3}) \) in time \( 2^{5n/k} \). Let \( \gamma' > \frac{\gamma}{2} \), observing that for large enough \( k \) we have \( \frac{\gamma}{k} < \frac{\gamma}{2} \). Using \( A_{\gamma'} \), we are then able to solve 3-SAT in time \( 2^{5n/k} + f(k) \cdot N^{\gamma k} = 2^{5n/k} + f(k) \cdot 2^{\gamma n(1+1/2^{k-3})} < 2^{\gamma n} \), giving the desired contradiction.
For SETH, suppose for contradiction there exists such an algorithm $A$ running in time $f(k) \cdot N^{k-\varepsilon}$, for some $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{k} - \frac{1}{2^{k-3}}$, observing that this is positive for sufficiently large values of $k$. By SETH, there exists $\ell$ such that $\ell$-SAT cannot be solved in time $2^{(1-\delta)n}$. Then, given a $\ell$-SAT formula $\varphi$, we may apply Lemma 3.5 to obtain an exact cover instance $I$ of size $N = 2^{n/k}(1+1/2^k-3)$ in time $2^{n/k} + f(k) \cdot 2^{n(1-\varepsilon/k)(1+1/2^k-3)} < 2^{n(1+1/2^k-3-\varepsilon/k)}$, which gives a contradiction to SETH.

3.2 Application to Nearest Codeword

In proving the NP-hardness of approximating NCP, [ABSS97] constructed a poly-time reduction from ExactCover. Since we have just shown hardness of approximation for $k$-ExactCover, this suits our purposes quite well, and we can use it to show inapproximability for $k$-NCP directly.

**Proposition 3.7 ([ABSS97]).** There exists an algorithm that, given an exact cover instance $I$ of size $N$, outputs a matrix $A \in \mathbb{F}_2^{m \times n}$ and vector $y \in \mathbb{F}_2^m$ with $m, n \leq N$ in time poly($N$) such that

- if $I$ has an exact cover of size $k$, then there exists an $x$ such that $\|Ax - y\|_0 \leq k$;
- if every set cover for $I$ has size at least $\gamma \cdot k$, then for every $x$, $\|Ax - y\|_0 \geq \gamma \cdot k$.

This gives us a very straightforward means to show inapproximability for $k$-NCP:

**Theorem 3.8.** Assuming that ETH holds, there exists $\gamma > 0$ such that for every computable $f : \mathbb{N} \rightarrow \mathbb{N}$ and sufficiently large $k$, no time $f(k) \cdot N^{\gamma k}$ algorithm can solve $(k, 1/\delta \sqrt{\log \log N})$-NCP. If SETH holds, for every $\varepsilon > 0$, every computable $f$, and large enough $k$, no time $f(k) \cdot N^{k-\varepsilon}$ algorithm can solve $(k, 1/\delta \sqrt{\log \log N})$-NCP.

**Proof.** We note that the algorithm in Proposition 3.7 runs in polynomial time and preserves the parameter $k$, and so it is in fact an FPT reduction as well. Given a set cover instance of size $N$, the size of the produced NCP instance is at most $N^2$, so starting from Theorem 3.6, we get a lower bound of $1/\delta \left(\frac{\log \sqrt{N}}{\log \log \sqrt{N}}\right)^{1/k} > 1/\delta \sqrt{\frac{\log N}{\log \log N}}$ in the second case. Since the upper bound in the first case remains unchanged, this completes the proof.

As with our proof of inapproximability for $k$-ExactCover, we are slightly lax with our bounds above. We can still achieve a lower bound of $1/\delta \sqrt{\frac{\log N}{\log \log N}}$ for any $\delta > 0$, so long as we take $k$ to be sufficiently large.
4 Conclusion

Assuming ETH, we have shown new hardness results for approximating the $k$-ExactCover problem, which extend to improving the inapproximability ratio for $k$-NCP. Notably, we show that any reasonably efficient algorithm must have $\Omega\left(\sqrt[\log\log N]{\log N}\right)$ approximation ratio, giving a “total inapproximability”-type result, i.e. that the approximability must depend on $n$ and cannot just be a function in $k$. But, we fall short of proving $W[1]$-hardness, which raises the natural question:

**Question 4.1.** Assuming $W[1] \neq FPT$, can we rule out constant-factor FPT approximation algorithms for $k$-ExactCover, or even $f(k)$ approximations for any computable function $f$?

In particular, [Lin19] shows $W[1]$-hardness of approximating $k$-SetCover to any $f(k)$ factor, but his analog of Lemma 3.4 proceeds via a reduction from $k$-Clique, which our technique does not immediately generalize to. Such a result would show inapproximability for $k$-ExactCover under a weaker assumption. Additionally, ruling out anything better than constant-factor approximation would translate to a new result for $k$-NCP; this would improve the result of [BELM18] without needing a stronger starting point of hardness.

In analyzing how to strengthen our results, one more point stands out:

**Question 4.2.** Does there exist an algorithm that can compute a $(k, n, m, \ell, h)$-Gap-Gadget with $\ell \ll h^k$ in time polynomial in $n$?

Since, roughly speaking, we need $\log M^\ell \leq M^{1+o(1)}$ in order for Lemma 3.5 to run sufficiently quickly, the application of Proposition 3.2 to obtain a $(k, n, n \log h, h^k, h)$-Gap-Gadget enforces that $h \leq \sqrt[\log\log M]{\log M}$. Any reduction to the size of $\ell$ directly translates to a stronger lower bound on the inapproximability ratio for $k$-ExactCover as well as for $k$-SetCover. As the construction in Proposition 3.2 uses the universality condition only to guarantee the existence of a string whose entries in the specified indices are all distinct, it might be the case that a better construction exists.

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5 References

[ABSS97] Sanjeev Arora, László Babai, Jacques Stern, and Z. Sweedyk. The Hardness of Approximate Optima in Lattices, Codes, and Systems of Linear Equations. *Journal of Computer and System Sciences*, 54(2):317 – 331, 1997.

[BELM18] Édouard Bonnet, László Egrí, Bingkai Lin, and Dániel Marx. Fixed-parameter Approximability of Boolean MinCSPs. *CoRR*, abs/1601.04935, 2018.
[BEM16] Édouard Bonnet, László Egri, and Dániel Marx. Fixed-Parameter Approximability of Boolean MinCSPs. In 24th Annual European Symposium on Algorithms, ESA 2016, August 22-24, 2016, Aarhus, Denmark, pages 18:1–18:18, 2016.

[BGKM18] A. Bhattacharyya, S. Ghoshal, C. S. Karthik, and P. Manurangsi. Parameterized Intractability of Even Set and Shortest Vector Problem from Gap-ETH. CoRR, March 2018.

[BMT78] E. Berlekamp, R. McEliece, and H. Van Tilborg. On the inherent intractability of certain coding problems (corresp.). IEEE Transactions on Information Theory, 24(3):384–386, 1978.

[CCK+17] P. Chalermsook, M. Cygan, G. Kortsarz, B. Laekhanukit, P. Manurangsi, D. Nanongkai, and L. Trevisan. From Gap-ETH to FPT-Inapproximability: Clique, Dominating Set, and More. In FOCS, pages 743–754, 2017.

[CL16] Yijia Chen and Bingkai Lin. The Constant Inapproximability of the Parameterized Dominating Set Problem. 2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS), 2016.

[DF13] Rodney G. Downey and Michael R. Fellows. Fundamentals of Parameterized Complexity. Springer, 2013.

[DFVW99] Rod G. Downey, Michael R. Fellows, Alexander Vardy, and Geoff Whittle. The Parameterized Complexity of Some Fundamental Problems in Coding Theory. SIAM Journal on Computing, 29(2):545–570, 1999.

[Din16] Irit Dinur. Mildly exponential reduction from gap 3SAT to polynomial-gap label-cover. In Electronic Colloquium on Computational Complexity (ECCC), 2016.

[EG04] Friedrich Eisenbrand and Fabrizio Grandoni. On the Complexity of Fixed Parameter Clique and Dominating Set. Theoretical Computer Science, 326(1-3):57–67, 2004.

[Fei98] Uriel Feige. A threshold of ln n for approximating set cover. J. ACM, 45(4):634–652, July 1998.

[IP01] R. Impagliazzo and R. Paturi. Complexity of k-SAT. Proceedings. Fourteenth Annual IEEE Conference on Computational Complexity (Formerly: Structure in Complexity Theory Conference) (Cat.No.99CB36317), Jan 2001.

[IPZ01] R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? Proceedings 39th Annual Symposium on Foundations of Computer Science (Cat. No.98CB36280), Dec 2001.
[Kar09] Richard M. Karp. Reducibility among combinatorial problems. 50 Years of Integer Programming 1958–2008, pages 219–241, 2009.

[KLM18] C. S. Karthik, Bundit Laekhanukit, and Pasin Manurangsi. On the Parameterized Complexity of Approximating Dominating Set. Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing - STOC 2018, 2018.

[Lin14] Bingkai Lin. The Parameterized Complexity of k-Biclique. Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, 2014.

[Lin19] Bingkai Lin. A Simple Gap-producing Reduction for the Parameterized Set Cover Problem. CoRR, abs/1902.03702, 2019.

[Man19] Pasin Manurangsi. Approximation and Hardness: Beyond P and NP. PhD thesis, University of California, Berkeley, 2019.

[Mos15] Dana Moshkovitz. The Projection Games Conjecture and the NP-Hardness of ln n-Approximating Set-Cover. Theory of Computing, 11:221–235, 2015.