An exact nilpotent non-perturbative BRST symmetry for the Gribov-Zwanziger action in the linear covariant gauge

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We point out the existence of a non-perturbative exact nilpotent BRST symmetry for the Gribov-Zwanziger action in the Landau gauge. We then put forward a manifestly BRST invariant resolution of the Gribov gauge fixing ambiguity in the linear covariant gauge.

I. INTRODUCTION

The Gribov-Zwanziger framework [1, 2] is a non-perturbative approach to face the hard problem of understanding the behavior of Yang-Mills theories in the infrared region, where standard perturbation theory cannot be applied. It takes into account the existence of Gribov copies [1], resulting in a modification of the Euclidean functional integral. Gribov copies are present whenever the gauge fixing condition allows multiple solutions, a very generic feature as shown by [5]. So far, a non-trivial set of results has been obtained from this approach, ranging from the gluon and ghost two-point functions [6–8], to the glueball spectrum [9, 10], to thermodynamic quantities and phase transitions [11–17], to supersymmetric theories [18, 19] and to the case where Higgs matter fields are present [20]. Nevertheless, the important issue of the BRST symmetry still lacks a simple answer, see [21–37] for an overview of the on-going discussion. In the present paper we propose a manifestly BRST invariant formulation of the Gribov-Zwanziger framework, resulting in the existence of a non-perturbative exact BRST symmetry. We limit ourselves here to outline the main steps of our reasoning, postponing all details to a longer and complete work.

II. THE ORIGINAL GRIBOV-ZWANZIGER ACTION IN THE LANDAU GAUGE

The framework [1, 2], applied to SU(N) gauge theories in Euclidean space-time, implements the restriction of the path integral to the Gribov region Ω in the Landau gauge, δμAμa = 0, namely

Ω = { Aμa | δμAμa = 0, Mab(A) > 0 },

(1)

where Mab is the Faddeev-Popov operator

Mab = −δab d2 + gf abc Aμγδμ, with δμAμa = 0.

(2)

According to [1, 2], for the partition function of quantized Yang-Mills theory we write

Z = \int Ω [DA] δ(∂Aμa) det(M) e−SYM.

(3)

The restriction of the domain of integration to the region Ω can be effectively implemented by adding to the starting action an additional non-local term H(A), known as the horizon function. More precisely [1, 2]

\int Ω [DA] δ(∂Aμa) det(M) e−SYM

= \int Ω [DA] δ(∂Aμa) det(M) e−(SYM+γH(A)−4γH(A)(N^2−1))

(4)
where

\[ H(A) = g^2 \int d^4 x d^4 y \, f^{abc} A^a_\mu(x) \left[ \mathcal{M}^{-1}(x,y) \right]^{ad} f^{dec} A^e_\mu(y), \]  

with \([\mathcal{M}^{-1}]\) denoting the inverse of the Faddeev-Popov operator, see eq. (3). The mass parameter \(\gamma^2\) appearing in expression (3) is known as the Gribov parameter. It is determined in a self-consistent way by the gap equation (2)

\[ \langle H \rangle = 4V(N^2-1), \]  

where the vacuum expectation value \(\langle H \rangle\) has to be evaluated with the measure defined in eq. (4). \(V\) denotes the space-time volume. Expression (4) can be cast in a more suitable form with the measure defined in eq. (4); namely

\[ \int_{\Omega} [\mathcal{D}A] \delta(\partial A^a) \, \det(M) \, e^{-S_{GZ}} = \int_{\Omega} [\mathcal{D}\Phi] \, e^{-\left(S_{GZ} - 4V\gamma^2(N^2-1)\right)}, \]

where \(\Phi\) refers to all fields present and \(S_{GZ}\) stands for the Gribov-Zwanziger action²

\[ S_{GZ} = S_{FP} + \int d^4 x \left( \bar{\phi} \mathcal{M}(A) \phi - \bar{\phi} \mathcal{M}(A) \phi + \gamma^2 A(\bar{\phi} + \phi) \right), \]

with \(S_{FP}\) being the Faddeev-Popov action in the Landau gauge

\[ S_{FP} = S_{YM} + \int d^4 x \left( b^a \partial_\mu A^a_\mu + c^a \partial_\mu D^a_\mu + \epsilon^b \right) \]  

Notice that the gap equation (6) can be rewritten as

\[ \frac{\partial E_\nu}{\partial x^\nu} = 0, \quad e^{-V\epsilon} = \int_{\Omega} [\mathcal{D}\Phi] \, e^{-\left(S_{GZ} - 4V\gamma^2(N^2-1)\right)}, \]

where \(\epsilon\) denotes the vacuum energy. As already mentioned, till now, a simple resolution of the issue of the BRST symmetry for the action (3) is still lacking.

One important property which should be underlined here is that, as observed in (3), the Gribov region \(\Omega\) does not support anymore infinitesimal gauge transformations. If one performs an infinitesimal gauge transformation of a generic field \(A_\delta\) belonging to \(\Omega\), the resulting transformed field lies outside the region \(\Omega\). From this simple argument, one easily understands that the restriction of the functional integral to the region \(\Omega\) might give rise to possible incompatibilities with the standard BRST symmetry.

### III. WARMING UP: A NON-PERTURBATIVE EXACT BRST SYMMETRY FOR THE GRIBOV-ZWANZIGER ACTION IN THE LANDAU GAUGE

The previous observation has led us to consider a non-local gauge invariant transverse field \(A^h_\mu\), \(\partial_\mu A^h_\mu = 0\), obtained by

\[ A^h_\mu = P_{p\nu}\left( A_\nu - ig \left( \frac{\partial A_\nu}{\partial x^{\nu}} \phi \right) + ig_2 \phi \frac{\partial A_\nu}{\partial x^{\nu}} + O(A^3) \right) = A_\mu - \frac{1}{\gamma^2} \partial_\mu \phi + \frac{1}{\gamma^2} \partial_\mu \phi + O(A^3), \]

minimizing the auxiliary functional \(Tr \int d^4 x A^a_\mu A^{\mu}_a\) along the gauge orbit of \(A_\mu\), cf. (38) [40] and Appendix A

\[ A^h_\mu = \frac{\partial}{\partial x^\mu} \left( \frac{\partial A^a_\mu}{\partial x^a} \phi + \frac{\partial A^a_\mu}{\partial x^a} \phi \right) + O(A^3), \]

with \(P_{p\nu}\) being the transverse projector. Expression (11) is left invariant by infinitesimal gauge transformations order by order. Moreover, looking at eq. (11), one realizes that a divergence \(\partial A\) is present in all higher order terms. As a consequence, we can rewrite Zwanziger’s horizon function \(H(A)\) in terms of the invariant field \(A^h\) as

\[ H(A) = H(A^h) = H(A) - R(A) (\partial A) \]

where \(R(A)(\partial A)\) is a short-hand notation, \(R(A)(\partial A) = \int d^4 x d^4 y (R^a(x,y)(\partial A))^a\), \(R(A)\) being an infinite non-local power series of \(A_\mu\). Therefore, for the Gribov-Zwanziger action, we may write, omitting color indices for brevity,

\[ S_{GZ} = S_{YM} + \int d^4 x \left( b^a \partial_\mu A^a_\mu + c^a \partial_\mu D^a_\mu + \gamma^2 H(A^h) \right) \]

\[ = S_{YM} + \int d^4 x \left( b^a \partial_\mu A^a_\mu + c^a \partial_\mu D^a_\mu + \gamma^2 H(A^h) \right) - \gamma^4 R(A)(\partial A) \]

where the new field \(b^h\) stands for

\[ b^h = \beta^4 R(A) \].

The use of the field \(b^h\) enables us to write down an exact nilpotent non-perturbative BRST transformation. Rewriting the Gribov-Zwanziger action by using the auxiliary fields \((\bar{\phi}, \phi, \omega, \phi)\), i.e.

\[ S_{GZ} = S_{YM} + \int d^4 x \left( b^h \partial_\mu A^a_\mu + c^a \partial_\mu D^a_\mu + \gamma^2 H(A^h) \right) \]

\[ + \int d^4 x \left( \bar{\phi} \mathcal{M}(A^h) \phi - \bar{\phi} \mathcal{M}(A^h) \phi + \gamma^2 A(\bar{\phi} + \phi) \right). \]

It becomes clear that expression (15) is left invariant by the nilpotent non-perturbative BRST transformation

\[ s_F^2 = s + \delta s_F^2, \quad s_F^2 = 0, \quad s_F S_{GZ} = 0. \]

In eqs. (16), the operator \(s\) stands for the usual BRST operator

\[ s A^a_\mu = -D^a_\mu b^h, \quad s c^a = \frac{g}{2} f^{abc} c^b c^c, \quad s b^a = b^a, \quad s a^a = 0, \quad s w^ab_\mu = \omega^{ab}_\mu, \quad s \omega^{ab} = 0, \quad s \omega^{ab} = \partial^{ab}_\mu, \quad s \partial^{ab}_\mu = 0, \]

while

\[ \delta s_F^2 = -\gamma^4 R^a(A), \quad \delta s_F b^a = \gamma^4 s R^a(A), \]

\[ \delta s_F \omega^{ab} = \gamma^2 f^{abc} A^k_\mu \left[ \mathcal{M}^{-1}(A^h) \right]^{ba}_k, \quad \delta s_F (\text{rest}) = 0 \]
The operators \( (s, \delta \gamma) \) obey the nice algebra
\[
\{ s, \delta \gamma \} = s^2 = \delta \gamma^2 = s^2 = 0 \quad (19)
\]
and clearly, for \( \gamma^2 \to 0 \) we have \( s \gamma \to s \).

The operator \( s \gamma \) is a genuine non-perturbative BRST operator, as it depends explicitly on the non-perturbative Gribov parameter \( \gamma^2 \).

Thanks to \( s \gamma \), we can write down non-perturbative Ward identities which clarify the origin of the breaking of the standard BRST symmetry. Moreover, we can formulate a non-perturbative BRST method which is immediately seen to belong to the cohomology of the new BRST operator \( s \gamma \).

expression (23) naturally generalizes the Gribov-Zwanziger action of the Landau gauge to an arbitrary linear covariant gauge in a manifestly non-perturbative BRST invariant way, namely
\[
S^l_{\text{GZ}} = 0 . \quad (25)
\]

The action (23) reduces precisely to the Gribov-Zwanziger action in the limit \( \alpha \to 0 \)
\[
S^l_{\text{GZ}} |_{\alpha = 0} = S_{\text{GZ}} , \quad (26)
\]
while yielding the usual action of the linear covariant gauge when \( \gamma^2 = 0 \), i.e.
\[
s^l_{\text{GZ}} |_{\gamma^2 = 0} = S_{FP} = S_{YM} + \int d^4x \left( b^b \partial_\mu A_\mu - \frac{\alpha}{2} b^b b^h + \bar{c} \partial_\mu D_\mu c \right) , \quad (27)
\]
Expression (27) is nothing but the Faddeev-Popov action of the linear covariant gauges
\[
\partial_\mu A_\mu = \alpha b , \quad (28)
\]
where \( \alpha \) stands for the gauge parameter and \( b \) for the Lagrange multiplier.

Since in expression (23) the gauge parameter \( \alpha \) is coupled to a \( s \gamma \)-exact quantity, expectation values of \( s \gamma \)-invariant quantities will not depend on \( \alpha \). In particular, this will be the case for the dynamical mass scale \( \gamma^2 \). As we shall see at the end of this section, the independence of \( \gamma^2 \) from \( \alpha \) is a consequence of the fact that \( \gamma^2 \) is now determined by the gauge invariant horizon condition
\[
\frac{\partial \langle H(A^b) \rangle}{\partial \gamma^2} = 0 \Rightarrow \langle H(A^b) \rangle = 4V(N^2 - 1) \quad (29)
\]
valid for an arbitrary functional \( \langle \rangle \).

It is also interesting to note that, integrating out the field \( b^h \) in expression (23), one gets the nice equation
\[
\int d^4x \left( b^h \partial_\mu A_\mu - \frac{\alpha}{2} b^h b^h \right) \Rightarrow \int d^4x \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 . \quad (31)
\]

We point out that, recently, the linear covariant gauges have been studied in lattice numerical simulations by \cite{41, 42} or with functional methods by \cite{43, 44}. It is worth underlining that the tree level gluon propagator \cite{47} stemming from
expression turns out to be in qualitative agreement with the available lattice numerical simulations, exhibiting an infrared suppression in the gluon sector. A more detailed analysis will involve taking into account additional $d = 2$ condensates, following. Let us provide a geometrical understanding of the action by showing that it enables one to eliminate infinitesimal gauge copies.

The Faddeev-Popov operator for general $\alpha$ reads

$$
\mathcal{M}^{ab}(A) = -\partial_\mu D^{h\mu}_b = -\partial_\mu (\delta^{ab} \partial_\mu - g f^{abc} A^c_\mu) = -\delta^{ab} \partial_\mu^2 + \alpha_\mu g f^{abc} b^c + g f^{abc} A^c_\mu \partial_\mu.
$$

(32)

Infinitesimal Gribov copies will appear whenever

$$
\mathcal{M}^{ab}(A)^{\zeta^a} = 0,
$$

with $\zeta^a$ a normalizable zero mode, in which case $A^a_\mu - D^{h\mu\nu}_b \zeta^b$ also fulfills condition if $A^a_\mu$ does.

Unlike the case of the Landau gauge, we notice that, when $\alpha \neq 0$, the partial derivative $\partial_\mu$ and the covariant one $D_\mu$ do not commute. As a consequence, the Faddeev-Popov operator in eq. (32) is not Hermitian. The Hermiticity of $\mathcal{M}^{ab}$ plays an important role in the original Gribov-Zwanziger analysis. Let us therefore consider

$$
\mathcal{M}^{ab}(A^h) = -\partial_\mu (\delta^{ab} \partial_\mu - g f^{abc} A^c_\mu),
$$

(34)

with $A^h$ the gauge invariant field defined in eq. (11). By construction, the operator $\mathcal{M}(A^h)$ in eq. (35) is gauge invariant order by order and Hermitian, thanks to the transversality of $A^h$. It thus makes sense to define the region

$$
\Omega^h = \{ A_\mu | \partial_\mu A^a_\mu = \alpha b^a, \partial_\mu A^h_\mu = 0, \mathcal{M}^{ab}(A^h) > 0 \}.
$$

(35)

The region $\Omega^h$ shares the important properties of the Gribov region $\Omega$ of the Landau gauge of being convex and bounded in all directions. Those properties follow from the linearity of the operator $\mathcal{M}^{ab}(A^h)$ in the field $A^h$.

Let us recall that the Landau gauge is, as far as we know, the only gauge for which it has been proven that every gauge orbit crosses at least once the Gribov region $\Omega$, i.e. a gauge field configuration located outside of the region $\Omega$ is a copy of some configuration located within $\Omega$. The essential ingredient in the proof is that the functional $\text{Tr} \int d^4 x A_\mu A_\mu$ achieves its absolute minimum along the gauge orbit of $A$, and this for an arbitrary starting gauge configuration $A$. Said otherwise, the search for the minima along the gauge orbit can be regarded as a pure mathematical problem for the functional $\text{Tr} \int d^4 x A_\mu A_\mu$, not related to the particular gauge condition obeyed by the configuration $A$. Actually, it turns out that the functional $\text{Tr} \int d^4 x A_\mu A_\mu$ has many relative minima along the gauge orbit before attaining its absolute minimum. The set of the relative minima of $\text{Tr} \int d^4 x A_\mu A_\mu$ is precisely the Gribov region $\Omega$. The proof of shows that, given an arbitrary gauge configuration $A$, it is always possible to introduce a related transverse field $A^h$ through the process of minimization of the functional $\text{Tr} \int d^4 x A_\mu A_\mu$ along the gauge orbit of $A$. Any configuration $A^h$ can be identified with a local minimum of the functional $\text{Tr} \int d^4 x A_\mu A_\mu$, while any such minimum is left invariant by infinitesimal gauge transformations.

Our construction of a non-perturbative BRST operator is possible with any $A^h$, but for our purposes we use the unique order by order representation given in eq. (11). These considerations make the region $\Omega^h$ a suitable candidate to integrate over.

Let us proceed by showing that the use of the region enables us to eliminate a large class of infinitesimal gauge copies from the partition function. This proposition borrows from an earlier insight of some of us in [47, 50], where only the transverse component $A^h_\mu$, $A^h_\mu = (\delta_\mu^\nu - \partial_\mu \partial_\nu)A_\nu$, was considered instead of the complete invariant gauge field $A^h$.

Following [47, 50], let us assume that $\zeta^a$ is a zero mode of the Faddeev-Popov operator having a Taylor expansion in $\alpha$,

$$
\zeta^a = \sum_{n=0}^{\infty} \alpha^n \zeta^a_n.
$$

(36)

Let us decompose the gauge field $A^a_\mu$ according to

$$
A_\mu = A^h_\mu + \tau_\mu, \quad \partial_\mu \tau_\mu = \alpha b,
$$

(37)

so that, in view of eq. (37), we can write

$$
\tau_\mu = \sum_{n=0}^{\infty} \alpha^{n+1} \zeta^a_n = \alpha \tau_\mu,
$$

(38)

since $\tau_\mu$ has to vanish in the limit $\alpha \to 0$. If $A_\mu \in \Omega^h$, we can write

$$
\zeta^a = \frac{-g}{\mathcal{M}^{(A^h)^{-1}}(\zeta^a)} f^{abc} \partial_\mu \left( \zeta^b \zeta^c \right)
$$

$$
= -g \alpha \left[ (M^{A^h})^{-1} \right] f^{abc} \partial_\mu \left( \zeta^b \zeta^c \right),
$$

(39)

or, expanding in powers of $\alpha$,

$$
\sum_n \alpha^n \zeta^a_n = -\sum_n \alpha^{n+1} \left[ (M^{A^h})^{-1} \right] f^{abc} \partial_\mu \left( \zeta^a_n \zeta^a_n \right).
$$

(40)

Matching orders of $\alpha$ shows that the $n^{th}$ order coefficient $\zeta^a_n$ is proportional to the $(n-1)^{th}$. Since for the first coefficient we find $\zeta^a_0 = 0$, we immediately find $\zeta^a_n = 0$, and thus $\zeta^a = 0$. Said otherwise, all zero modes that possess a Taylor expansion around $\alpha = 0$, are automatically vanishing. As such, the restriction to $\Omega^h$ excludes at least the set of infinitesimally connected gauge copies related to the aforementioned zero modes.

We proceed by implementing $\mathcal{M}^{h} \equiv \mathcal{M}^{ab}(A^h) > 0$ into the path integral. We rely on the so-called Gribov no-pole condition, whose all order implementation can be found in [27]. For any external field $A^h$, we can use Wick’s theorem to invert the operator $\mathcal{M}^{ab}(A^h)$ in any dimension $d$. Denoting by $G^{ab}(A^h, p^2) = \langle p | \frac{1}{\mathcal{M}^{ab}(A^h)} | p \rangle$ the Fourier-transform of the inverse of $\mathcal{M}^{ab}(A^h)$, one introduces the so-called Gribov form...
factor \( \sigma(A^h, p^2) \) through

\[
G^{ab}(A^h, p^2) = \frac{8^{ab}}{N^2-1} G^{cc}(A^h, p^2) = \frac{8^{ab}}{N^2-1} \frac{1 + \sigma(A^h, p^2)}{p^2}.
\]

(41)

Repeating the procedure outlined in \([27]\), it follows that at zero momentum

\[
\sigma(A^h, 0) = - \frac{g^2}{V d(N^2-1)} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} A_{\mu,ab}^{sp}(k) \left[ (M^h)^{-1} \right]_{bc}^{k-q} A^{h,ca}_\mu(q).
\]

(42)

Comparison of eqns. (3) and (42) learns that \( \sigma(A^h, 0) = \frac{H(A^h)}{V d(N^2-1)} \). We will concentrate on the zero momentum limit, since it is expected on general grounds\(^3\) that the smallest eigenvalue of \( M^{ab}(A^h) \) will carry no momentum, so it would be sufficient to avoid this eigenvalue becoming negative. At the level of expectation values, we can rewrite eq. (42) as

\[
G^h(p^2) = \langle G^{aur}(A^h, p^2) \rangle_{\text{comm}} = \frac{1}{p^2 (1 - \langle \sigma(A^h, p^2) \rangle_{\text{PI}}^2)},
\]

(43)

so that we must impose at the level of the path integral \( \langle \sigma(A^h, 0) \rangle_{\text{PI}} \leq 1 \), or

\[
\langle H(A^h) \rangle_{\text{PI}} \leq V d(N^2-1).
\]

(44)

We can add this constraint to the path integral measure with a step function. Via a saddle point evaluation in the thermodynamic limit \([1, 26]\), one then finds

\[
\begin{align*}
[D\Phi] \int \frac{d\eta}{2\pi\eta} e^{-S_{P}^{p} + \eta \left[Vd(N^2-1) - H(A^h)\right]} & = \left[D\Phi\right] \int \frac{d\eta}{2\pi\eta} e^{-S_{P}^{p} + \eta \left[Vd(N^2-1) - H(A^h)\right]} \\
& \rightarrow \left[D\Phi\right] e^{-S_{P}^{p} + \eta \left[Vd(N^2-1) - H(A^h)\right]},
\end{align*}
\]

(45)

where \( S_{P}^{p} \) stands for the expression given in eq. (24). The saddle point equation precisely amounts to eq. (29), i.e. the horizon condition with identification \( \eta^\alpha = \gamma^\alpha \). As the horizon condition is writable in terms of the vacuum energy and since the only contributing diagrams to the latter are \( 1\text{PI} \) (see also \([27]\)), it indeed follows that condition (44) is met. As such, we do have excluded a large set of zero modes by effectively having imposed that \( \mathcal{M}(A^h) > 0 \) via the action (23). Upon introduction of the auxiliary fields \( (\phi, \phi, \omega, \bar{\omega}) \), the latter is equivalent to the action appearing in eq. (23), given that eq. (29) holds.

\[\text{V. CONCLUSION}\]

For the first time, we have identified a non-perturbative nilpotent BRST symmetry for gauge theories quantized à la Gribov-Zwanziger, that is by further restricting the domain of integration in the path integral. This eliminates a large set of gauge copies and deeply affects the infrared low-momentum regime of the gauge theory. The new BRST operator \( s_{2} \) depends explicitly on the gauge invariant mass parameter \( m^2 \) that is linked to the aforementioned restriction. As such, the operator \( s_{2} \) itself is intertwined with this geometric restriction.

The introduction of \( s_{2} \) opens up whole new strata of applications. We have already discussed a first one in this paper, namely a non-perturbative extension of the usual linear covariant gauge to a setting where the Gribov gauge fixing ambiguity is also faced in this gauge. Our setup generalizes to the Refined Gribov-Zwanziger approach \([7]\), in which case we can make contact with the gauge invariant \( d = 2 \) condensate \( \langle A_{\mu}^{2} \rangle \), of important phenomenological interest \([51, 52]\). A renormalization analysis of the proposed framework is already in preparation, of relevance to explicit studies of propagators, spectrum and thermodynamics. Generalizations, compatible with the new non-perturbative BRST, to the matter sector are also possible. Moreover, it would also be interesting to make contact with lattice studies of the linear covariant gauge, e.g. to find out if a practical numerical implementation of our proposal exists. We are already studying a functional depending on the original gauge field \( A_{\mu} \) and an auxiliary field \( B_{\mu} \), with the property that the minimum occurs for \( \partial_{\mu} A_{\mu} = q \) (thus effectively implementing the linear covariant gauge) and for \( B_{\mu} = A_{\mu}^{h} \) with \( \mathcal{M}(B) \geq 0 \). This could circumvent potential issues with the convergence of the series expression used in eq. (11) to define \( A^{h} \) in case of “large” gauge fields, while it would also open the road to simulation of our proposed non-perturbative linear covariant gauge. We will report on this in future work.

As a final but most crucial remark, we stress that no sacrifices have to be made w.r.t. gauge invariance, even when the Gribov problem is taken into account. The physical content of the theory is described by the \( s_{2} - \text{cohomology} \), which can be studied along the lines of \([53, 54]\) upon localization of our approach, another matter of current investigation.

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\[\footnote{\text{We can consider} \( \mathcal{M}^{ab}(A^h) \) as a perturbed system around \( -\hat{\sigma}^2 \) which reaches its lowest eigenvalue at zero momentum. A few comments regarding this were made in \([3]\). One can also check, a posteriori but explicitly, that the expectation value \( \langle \sigma(A^h, 0) \rangle \) is maximal.}\]
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Appendix A: A gauge invariant transversal gauge field

As it will turn out, the construction of the transverse gauge field $A^h_\mu$ follows from the minimization of the functional $f_A[u]$

$$f_A[u] \equiv \text{Tr} \int d^4x A^a_\mu A^a_\mu = \text{Tr} \int d^4x \left( u^\dagger A_\mu u + \frac{i}{g} u^\dagger \partial_\mu u \right) \left( u^\dagger A_\mu u + \frac{i}{g} u^\dagger \partial_\mu u \right) \quad (A1)$$

along the gauge orbit of a given configuration $A_\mu$. To give a well defined mathematical meaning to expression (A1), we shall require that both $A^a_\mu$ and the local gauge transformations, $u \in \mathcal{U}$, are square-integrable, i.e.

$$||A||^2 = \text{Tr} \int d^4x A_\mu A^\dagger_\mu < +\infty, \quad ||u^\dagger \partial_\mu u||^2 = \text{Tr} \int d^4x \left( u^\dagger \partial_\mu u \right) \left( u^\dagger \partial_\mu u \right) < +\infty. \quad (A2)$$

Then, it has been shown [48, 49] that $f_A[u]$ reaches its absolute minimum along the gauge orbit of $A_\mu$, i.e. there exists a certain $h$ such that

$$\delta f_A[h] = 0, \quad (A3)$$
$$\delta^2 f_A[h] \geq 0, \quad (A4)$$
$$f_A[h] \leq f_A[u], \quad \forall u \in \mathcal{U}. \quad (A5)$$

Following [38–40], we can work out the conditions (A3) and (A4) in a series expansion. We set

$$v = he^{i\omega} = he^{i\omega^a T^a}, \quad (A6)$$

with

$$[T^a, T^b] = i f^{abc}, \quad \text{Tr} \left( T^a T^b \right) = \frac{1}{2} \delta^{ab}, \quad (A7)$$

We first obtain

$$A^h_\mu = A^A_\mu + ig[A^h_\mu, \omega] + \frac{g^2}{2} \left[ \left( \omega, A^h_\mu \right), \omega \right] - \partial_\mu \omega + \frac{i g}{2} \left[ \omega, \partial_\mu \omega \right] + O(\omega^3), \quad (A8)$$

One subsequently finds

$$f_A[v] = f_A[h] + 2 \text{Tr} \int d^4x \left( \omega \partial_\mu A^h_\mu \right) - \text{Tr} \int d^4x \omega \partial_\mu D_\mu (A^h) + O(\omega^3), \quad (A9)$$

Armed with this expression, one simply realizes that

$$\delta f_A[h] = 0 \iff \partial_\mu A^h_\mu = 0, \quad (A10)$$
$$\delta^2 f_A[h] \geq 0 \iff -\partial_\mu D_\mu (A^h) > 0 \quad (A10)$$

are the conditions for a local minimum. Clearly, this is the a priori reason why the Gribov region $\Omega$, eq. (1), is introduced as it is.

The transversality condition, $\partial_\mu A^h_\mu = 0$, can be solved for $h = \tilde{h}(A)$ as a power series in $A_\mu$. Setting

$$A^h_\mu = h^A_\mu h + \frac{i}{g} h^A_\mu \partial_\mu h, \quad h = e^{i\xi} = e^{ig^a T^a}, \quad (A11)$$

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4 We refer to [40] for technical details.
we expand the gauge transformation matrix $h$ in powers of $\xi$

$$h = 1 + ig\xi - \frac{g^2}{2} \xi^2 + O(\xi^3) \, .$$  \hspace{1cm} (A12)

As such,

$$A^h_\mu = A_\mu - \partial_\mu \xi + ig[A_\mu, \xi] + \frac{g}{2} [\xi, \partial_\mu \xi] + g^2 \xi A_\mu \xi - \frac{g^2}{2} A_\mu \xi^2 - \frac{g^2}{2} \xi^2 A_\mu + O(\xi^3) \, .$$  \hspace{1cm} (A13)

Imposing $\partial_\mu A^h_\mu = 0$ yields

$$\partial^2 \xi = \partial_\nu A + ig[\partial_\nu A, \xi] + ig[A_\nu, \partial_\nu \xi] + g^2 \partial_\nu \xi A_\nu \xi + g^2 \xi A_\nu \partial_\nu \xi + \frac{g^2}{2} \xi A_\nu \partial_\nu \xi - \frac{g^2}{2} \partial_\nu \xi A_\nu - \frac{g^2}{2} \xi^2 \partial_\nu A_\nu + i \frac{g}{2} [\xi, \partial_\nu \xi] + O(\xi^3) \, .$$  \hspace{1cm} (A14)

Solving iteratively, we arrive at

$$\xi = \frac{1}{\partial^2} \partial_\mu A_\mu + i \frac{g}{\partial^2} \left[A_\nu, \frac{\partial A_\nu}{\partial^2} \right] + i \frac{g}{2} \left[A_\nu, \frac{\partial A_\nu}{\partial^2} \right] + i \frac{g}{2} \frac{\partial A_\nu}{\partial^2} \frac{\partial A_\nu}{\partial^2} + O(\xi^3) \, ,$$  \hspace{1cm} (A15)

and thus

$$A^h_\mu = A_\mu - \frac{1}{\partial^2} \partial_\mu A_\mu - ig \frac{\partial A_\mu}{\partial^2} \left[ A_\nu, \frac{\partial A_\nu}{\partial^2} \right] + ig \left[ A_\nu, \frac{1}{\partial^2} \frac{\partial A_\nu}{\partial^2} \right] + O(\xi^3) \, .$$  \hspace{1cm} (A16)

It is interesting to rewrite $A^h_\mu$ as

$$A^h_\mu = \left( \delta_\mu^\nu - \frac{\partial_\mu \partial_\nu}{2 \partial^2} \right) \left( A_\nu - ig \frac{1}{2} \frac{\partial A_\nu}{\partial^2} \right) + i \frac{g}{2} \left[ \frac{\partial A_\nu}{\partial^2}, \frac{1}{\partial^2} \frac{\partial A_\nu}{\partial^2} \right] + O(\xi^3) \, .$$  \hspace{1cm} (A17)

Under an infinitesimal gauge transformation

$$\delta A_\mu = -\partial_\mu \lambda + ig[A_\mu, \lambda] \, .$$  \hspace{1cm} (A18)

it can be checked that

$$\delta \Psi_\nu = -\partial_\nu \left( \lambda - \frac{g}{2} \frac{\partial A_\nu}{\partial^2} \right) + O(\lambda^2) \, .$$  \hspace{1cm} (A19)

The combined knowledge of (A17) and (A19) nicely displays that $A^h_\mu$ is indeed transverse, while it is also gauge invariant, order by order. It is perhaps interesting to notice here that in [53], the one loop renormalizability of the non-local operator $\frac{1}{2} \int d^4 \lambda A^h A^h$, i.e. the local minimum of eq. (A2), was explicitly checked.

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