NATURAL CONNECTIONS ON RIEMANNIAN PRODUCT MANIFOLDS

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Abstract
A Riemannian almost product manifold with integrable almost product structure is called a Riemannian product manifold. In the present paper the natural connections on such manifolds are studied, i.e. the linear connections preserving the almost product structure and the Riemannian metric.

Key words: Riemannian almost product manifold, Riemannian metric, integrable structure, almost product structure, linear connection, torsion.

2010 Mathematics Subject Classification: 53C15, 53C25, 53B05.

1. Introduction

The systematic development of the theory of Riemannian almost product manifolds was started by K. Yano in [1], where basic facts of the differential geometry of these manifolds are given. A Riemannian almost product manifold $(M, P, g)$ is a differentiable manifold $M$ for which almost product structure $P$ is compatible with the Riemannian metric $g$ such that an isometry is induced in any tangent space of $M$.

The geometry of a Riemannian almost product manifold $(M, P, g)$ is a geometry of both structures $g$ and $P$. There are important in this geometry the linear connections with respect to which the parallel transport determine an isomorphism of the tangent spaces with the structures $g$ and $P$. This is valid if and only if the structures $g$ and $P$ are parallel with respect to such a connection. In the general case on a Riemannian almost product manifold there exist a countless number of linear connections regarding which $g$ and $P$ are parallel. Such connections are called natural in [2].

In the present work we consider the natural connections on the Riemannian product manifolds $(M, P, g)$, i.e. on the Riemannian almost product manifolds $(M, P, g)$ with an integrable structure $P$. In our investigations we suppose the condition $\text{tr}P = 0$, which implies that $\dim M$ is an even number.

In [3] A. M. Naveira gave a classification of Riemannian almost product manifolds with respect to the covariant differentiation $\nabla P$, where $\nabla$ is the Levi-Civita connection of $g$. Having in mind the results in [3], M. Staikova and K. Gribachev gave in [4] a classification of the Riemannian almost product manifolds $(M, P, g)$ with $\text{tr}P = 0$.

In Section 2 we give some necessary facts about the Riemannian almost product manifolds. We recall the classification of Staikova-Gribachev for these...
manifolds which is made regarding the tensor $F$ determined by $F(x, y, z) = g((\nabla_x P) y, z)$. The basic classes are $W_1$, $W_2$ and $W_3$. The class of the Riemannian product manifolds is $W_1 \oplus W_2$.

In Section 3 we recall a decomposition of the space of the torsion tensors on a Riemannian almost product manifold to invariant orthogonal subspaces $\mathcal{T}_i$ $(i = 1, 2, 3, 4)$ given in [2]. We establish some properties of the torsion of a natural connection on a manifold $(M, P, g) \in W_1 \oplus W_2$ in terms of the mentioned decomposition.

In Section 4 we establish that the unique natural connection on $(M, P, g) \in W_1 \oplus W_2$ with torsion $T$, which can be expressed by the components of $F$, is the canonical connection. We prove that this is the unique natural connection for which $T \in T_1$.

In Section 5 we consider the natural connections for which the torsion $T$ can be expressed by the components of the tensor $g \otimes \theta$, where $\theta$ is the Lee 1-form associated with $F$. Such connections exist only on a manifold $(M, P, g) \in W_1$ and their torsions belong to a 2-parametric family. When the natural connection does not coincide with the canonical connection then we have $T \in T_1 \oplus T_4$, $T \notin T_1$ and $T \notin T_4$.

2. Preliminaries

Let $(M, P, g)$ be a Riemannian almost product manifold, i.e. a differentiable manifold $M$ with a tensor field $P$ of type $(1, 1)$ and a Riemannian metric $g$ such that

$$P^2 x = x, \quad g(Px, Py) = g(x, y)$$

for arbitrary $x, y$ of the algebra $\mathfrak{X}(M)$ of the smooth vector fields on $M$. Obviously $g(Px, y) = g(x, Py)$.

Further $x, y, z, w$ will stand for arbitrary elements of $\mathfrak{X}(M)$ or vectors in the tangent space $T_p M$ at $p \in M$.

In this work we consider Riemannian almost product manifolds with $\text{tr} P = 0$. In this case $(M, P, g)$ is an even-dimensional manifold. We denote $\dim M = 2n$.

The classification in [4] of Riemannian almost product manifolds is made with respect to the tensor field $F$ of type $(0,3)$, defined by

$$F(x, y, z) = g((\nabla_x P) y, z),$$

where $\nabla$ is the Levi-Civita connection of $g$. The tensor $F$ has the following properties:

$$F(x, y, z) = F(x, z, y) = -F(x, Py, Pz),$$

$$F(x, y, Pz) = -F(x, Py, z).$$

The associated 1-form $\theta$ for $F$ is determined by the equality

$$\theta(x) = g^{ij} F(e_i, e_j, x),$$
where $g^{ij}$ are the components of the inverse matrix of $g$ with respect to the basis $\{e_i\}$ of $T_pM$.

The basic classes of the classification in [4] are $W_1$, $W_2$ and $W_3$. Their intersection is the class $W_0$ of the Riemannian $P$-manifolds, determined by the condition $F(x, y, z) = 0$ or equivalently $\nabla P = 0$. In the classification there are include the classes $W_1 \oplus W_2$, $W_1 \oplus W_3$, $W_2 \oplus W_3$ and the class $W_1 \oplus W_2 \oplus W_3$ of all Riemannian almost product manifolds.

In the present work we consider only the Riemannian almost product manifolds $(M, P, g)$ with integrable almost product structure $P$, i.e. the manifolds with zero Nijenhuis tensor $N$ determined by

$$N(x, y) = (\nabla_x P) P y - (\nabla_y P) P x + (\nabla_{Px} P) y - (\nabla_{Py} P) x.$$ 

These manifolds is called Riemannian product manifolds and they form the class $W_1 \oplus W_2$. The characteristic conditions for the classes $W_1$, $W_2$ and $W_1 \oplus W_2$ are the following

$W_1 :$ $F(x, y, z) = \frac{1}{2n} \left\{ g(x, y) \theta(z) + g(x, z) \theta(y) - g(x, P y) \theta(P z) - g(x, P z) \theta(P y) \right\}$;

$W_2 :$ $F(x, y, P z) + F(y, z, P x) + F(z, x, P y) = 0$, $\theta = 0$;

$W_1 \oplus W_2 :$ $F(x, y, P z) + F(y, z, P x) + F(z, x, P y) = 0$.

3. Natural connections on Riemannian product manifolds

The linear connections in our investigations have a torsion.

Let $\nabla'$ be a linear connection with a tensor $Q$ of the transformation $\nabla \to \nabla'$ and a torsion $T$, i.e.

$$\nabla'_x y = \nabla_x y + Q(x, y), \quad T(x, y) = \nabla'_x y - \nabla'_y x - [x, y].$$

The corresponding $(0,3)$-tensors are defined by

$$Q(x, y, z) = g(Q(x, y), z), \quad T(x, y, z) = g(T(x, y), z).$$

The symmetry of the Levi-Civita connection implies

(3.1) $T(x, y) = Q(x, y) - Q(y, x)$,

$$T(x, y) = -T(y, x).$$

A partial decomposition of the space $T$ of the torsion tensors $T$ of type $(0,3)$ is valid on a Riemannian almost product manifold $(M, P, g)$: $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4$, where $T_i$ ($i = 1, 2, 3, 4$) are invariant orthogonal subspaces
4 DOBRINKA GRIBACHEVA

[2]. For the projection operators $p_i$ of $\mathcal{T}$ in $\mathcal{T}_i$ is established:

$$p_1(x, y, z) = \frac{1}{8} \left\{ 2T(x, y, z) - T(y, z, x) - T(z, x, y) - T(Pz, x, Py) 
+ T(Py, z, Px) + T(z, Px, Py) - 2T(Px, Py, z) 
+ T(Py, Pz, x) + T(Pz, Px, y) - T(y, Pz, Px) \right\},$$

$$p_2(x, y, z) = \frac{1}{8} \left\{ 2T(x, y, z) + T(y, z, x) + T(z, x, y) + T(Pz, x, Py) 
- T(Py, z, Px) - T(z, Px, Py) - 2T(Px, Py, z) 
- T(Py, Pz, x) - T(Pz, Px, y) + T(y, Pz, Px) \right\},$$

$$p_3(x, y, z) = \frac{1}{4} \left\{ T(x, y, z) + T(Px, Py, z) - T(Px, y, Pz) - T(x, Py, Pz) \right\},$$

$$p_4(x, y, z) = \frac{1}{4} \left\{ T(x, y, z) + T(Px, Py, z) + T(Px, y, Pz) + T(x, Py, Pz) \right\}.$$

**Definition 3.1** ([2]). A linear connection $\nabla'$ on a Riemannian almost product manifold $(M, P, g)$ is called a natural connection if $\nabla' P = \nabla' g = 0$.

If $\nabla'$ is a linear connection with a tensor $Q$ of the transformation $\nabla \rightarrow \nabla'$ on a Riemannian almost product manifold, then it is a natural connection if and only if the following conditions are valid [2]:

$$(3.2) \quad F(x, y, z) = Q(x, y, Pz) - Q(x, Py, z),$$

$$(3.3) \quad Q(x, y, z) = -Q(x, z, y).$$

Let $\Phi$ be the $(0,3)$-tensor determined by

$$\Phi(x, y, z) = g \left( \tilde{\nabla} xy - \nabla xy, z \right),$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the associated metric $\tilde{g}$ determined by $\tilde{g}(x, y) = g(x, Py)$.

**Theorem 3.1** ([2]). A linear connection with the torsion $T$ on a Riemannian almost product manifold $(M, P, g)$ is natural if and only if

$$4p_1(x, y, z) = -\Phi(x, y, z) + \Phi(y, z, x) - \Phi(x, Py, Pz) 
- \Phi(y, Pz, Px) + 2\Phi(z, Px, Py),$$

$$4p_3(x, y, z) = -g(N(x, y), z) = -2 \left\{ \Phi(z, Px, Py) + \Phi(z, x, y) \right\}.$$
Let \((M, P, g)\) be a Riemannian almost product manifold, i.e. \((M, P, g) \in \mathcal{W}_1 \oplus \mathcal{W}_2\). For such a manifold is valid the following equality \([4]\)

\[
(3.4) \quad \Phi(x, y, z) = \frac{1}{2} \left\{ F(y, x, Pz) - F(Py, x, z) \right\}.
\]

By virtue of \((3.4)\), the characteristic condition for the class \(\mathcal{W}_1 \oplus \mathcal{W}_2\) and Theorem \(3.1\) we obtain the following

**Theorem 3.2.** A linear connection with the torsion \(T\) on a Riemannian product manifold \((M, P, g)\) is natural if and only if

\[
(3.5) \quad p_1(x, y, z) = \frac{1}{2} F(z, y, Px), \quad p_3(x, y, z) = 0.
\]

According to Theorem \(3.2\) and the conditions for the projection operators \(p_2\) and \(p_4\), we get the following

**Corollary 3.3.** For the torsion \(T\) of a natural connection on a Riemannian product manifold \((M, P, g)\) are valid the following equalities

\[
(3.6) \quad p_2(x, y, z) = \frac{1}{2} \left\{ T(x, y, z) - T(Px, Py, z) + F(z, x, Py) \right\},
\]

\[
p_4(x, y, z) = \frac{1}{2} \left\{ T(x, y, z) + T(Px, Py, z) \right\}.
\]

Further, we suppose that the considered Riemannian product manifold \((M, P, g)\) is not a Riemannian \(P\)-manifold, i.e. \(F\) is not a zero tensor.

According to Theorem \(3.2\) for the torsion \(T\) of a natural connection on a Riemannian product manifold \((M, P, g)\), we have \(T \in \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_4\). If we suppose that \(T = p_2 + p_4\) then, having in mind Corollary \(3.3\), we obtain \(F = 0\), which is a contradiction. Therefore \(T \notin \mathcal{T}_2 \oplus \mathcal{T}_4\). Then we have to consider the cases:

A) \(T \in \mathcal{T}_1\);

B) \(T \in \mathcal{T}_1 \oplus \mathcal{T}_4, T \notin \mathcal{T}_1, T \notin \mathcal{T}_4\);

C) \(T \in \mathcal{T}_1 \oplus \mathcal{T}_2, T \notin \mathcal{T}_1, T \notin \mathcal{T}_2\).

**4. Case A**

In \([2]\) a natural connection on a Riemannian almost product manifold \((M, P, g)\) is called *canonical* if for its torsion the following equality is valid

\[
T(x, y, z) + T(y, z, x) + T(Px, y, Pz) + T(y, Pz, Px) = 0.
\]

This connection is an analogue of the Hermitian connection on the Hermitian manifolds \([5]\). In \([2]\) it is proved that on any Riemannian product manifold \((M, P, g)\) there exist a unique canonical connection and for the torsion \(T\)
of this connection the condition $T \in T_1 \oplus T_3$ is valid. Then, according to Theorem 3.2 it is valid the following

**Theorem 4.1.** The case A for the torsion $T$ of a natural connection on a Riemannian product manifold $(M, P, g)$ is valid if and only if this connection is the canonical one. In this case the following equality is satisfied

$$T(x, y, z) = \frac{1}{2} F(z, y, Px).$$

Let $T$ is the torsion $T$ of a natural connection on a Riemannian product manifold $(M, P, g)$. Having in mind the characteristic condition for the class $W_1 \oplus W_2$ and conditions (2.1), we obtain the following expression of $T$ by the independent components of $F$:

$$T(x, y, z) = \lambda_1 F(x, y, z) + \lambda_2 F(y, z, x) + \lambda_3 F(Px, y, z) + \lambda_4 F(Py, z, x) + \lambda_5 F(x, y, Pz) + \lambda_6 F(y, z, Px) + \lambda_7 F(Px, Py, z) + \lambda_8 F(Py, Pz, x),$$

where $\lambda_i \in \mathbb{R}$ ($i = 1, 2, \ldots, 8$). From (4.2), using (2.1), (3.1) and (3.3), we get (4.1) and therefore $T$ is the torsion of the canonical connection on $(M, P, g)$. Hence we establish that it is valid the following

**Proposition 4.2.** The canonical connection is the unique natural connection on a Riemannian product manifold $(M, P, g)$, which torsion can be expressed by the tensor $F$. 

The canonical connection on a Riemannian product manifold $(M, P, g) \in \mathcal{W}_1$ is studied in [4]. Having in mind the characteristic condition for the class $\mathcal{W}_1$ and condition (4.1) for the torsion of the canonical connection on $(M, P, g) \in \mathcal{W}_1 \oplus \mathcal{W}_2$, we obtain the following

**Proposition 4.3.** For the torsion $T$ of the canonical connection on a Riemannian product manifold $(M, P, g) \in \mathcal{W}_1$ the following equality is valid

$$T(x, y, z) = \frac{1}{4n} \{ g(y, z) \theta(Px) - g(y, Pz) \theta(x) - g(x, z) \theta(Py) + g(x, Pz) \theta(y) \}.$$

5. **Case B and Case C**

Having in mind the latter two propositions, in the present section for the cases B and C we consider the existence of a natural connection with torsion
of the mentioned class implies then the comparison of the latter equality with the characteristic condition. Since the tensor $F$ from (5.1), using (3.1), (3.2) and (3.3), we obtain

$$T(x, y, z) = \lambda_1 g(x, y) \theta(z) + \lambda_2 g(y, z) \theta(x) + \lambda_3 g(z, x) \theta(y) + \lambda_4 g(x, y) \theta(Pz) + \lambda_5 g(y, z) \theta(Px) + \lambda_6 g(z, x) \theta(Py)$$

(5.1)

$$+ \lambda_7 g(x, Py) \theta(z) + \lambda_8 g(y, Pz) \theta(x) + \lambda_9 g(z, Px) \theta(y)$$

$$+ \lambda_{10} g(x, Py) \theta(Pz) + \lambda_{11} g(y, Pz) \theta(Px)$$

$$+ \lambda_{12} g(z, Px) \theta(Py).$$

From (5.1), using (3.1), (3.2) and (3.3), we obtain

$$F(x, y, z) =$$

$$(\lambda_1 - \lambda_{10}) \{g(x, y) \theta(Pz) - g(x, Pz) \theta(y) - g(x, Py) \theta(z) + g(x, z) \theta(Py)\}$$

$$+ (\lambda_4 - \lambda_7) \{g(x, y) \theta(z) - g(x, Pz) \theta(Py) - g(x, Py) \theta(Pz) + g(x, z) \theta(y)\}.$$ 

Since the tensor $F$ is expressed by the tensor $g \otimes \theta$ only for the class $W_1$, then the comparison of the latter equality with the characteristic condition of the mentioned class implies

$$\lambda_1 = \lambda_{10}, \quad \lambda_4 - \lambda_7 = \frac{1}{2n}, \quad \lambda_2 = \lambda_5 = \lambda_6 = \lambda_8 = \lambda_9 = \lambda_{11} = \lambda_{12} = 0.$$ 

Then from (5.1), using the denotations $\lambda = \lambda_1 = \lambda_{10}$ and $\mu = \lambda_7$, we obtain the following

**Theorem 5.1.** Let the torsion $T$ of a natural connection on a Riemannian product manifold $(M, P, g) \in W_1$ is expressed by $g \otimes \theta$. Then Case B or Case C is valid for $T$ if and only if $(M, P, g) \in W_1$. In this case $T$ has the following representation

$$T(x, y, z) =$$

$$\lambda \{g(y, z) \theta(x) - g(x, z) \theta(y) + g(y, Pz) \theta(Px) - g(x, Pz) \theta(Py)\}$$

(5.2)

$$+ \mu \{g(y, Pz) \theta(x) - g(x, Pz) \theta(y) + g(y, z) \theta(Px) - g(x, z) \theta(Py)\}$$

$$+ \frac{1}{2n} \{g(y, z) \theta(Px) - g(x, z) \theta(Py)\}, \quad \lambda, \mu \in \mathbb{R}.$$ 

Let us consider Case B, i.e. $T = p_1 + p_4$. Then, according to (3.5) and (3.6), we have that this case is valid if and only if

$$F(z, x, Py) = T(Px, Py, z) - T(x, y, z).$$

(5.3)

We verify directly that condition (5.3) is satisfied for any torsion $T$ determined by (5.2), i.e. for arbitrary $\lambda$ and $\mu$. Let us remark that for $\lambda = 0$ and $\mu = -\frac{1}{4n}$ from (5.2) we get condition (4.3) for the torsion of the canonical connection on $(M, P, g) \in W_1$, i.e. Case A for the class $W_1$. Therefore, it is valid the following
Theorem 5.2. Let the torsion $T$ of a natural connection on a Riemannian product manifold $(M, P, g) \in W_1$ is expressed by $g \otimes \theta$. Then Case B is valid if and only if $T$ is determined by (5.2) for $(\lambda, \mu) \neq (0, -\frac{1}{4n})$. □

Let us consider Case C. According to Theorem 5.1, the torsion $T$ is determined by (5.2). Then, having in mind Theorem 5.2, we establish that $T$ satisfies the conditions of Case B. Therefore, we obtain the following

Proposition 5.3. If the torsion $T$ of a natural connection on a Riemannian product manifold $(M, P, g) \in W_1$ is expressed by $g \otimes \theta$, then Case C does not exist. □

6. Conclusion

The canonical connection is the unique natural connection on any Riemannian product manifold $(M, P, g)$, which torsion can be expressed by $F$. This is the unique natural connection with torsion in Case A.

If a natural connection on $(M, P, g) \in W_1$ has a torsion $T$ expressed by $g \otimes \theta$, then $T$ belongs to a 2-parametric family determined by (5.2). Case A and Case B for $T$ are valid when $(\lambda, \mu) = (0, -\frac{1}{4n})$ and $(\lambda, \mu) \neq (0, -\frac{1}{4n})$, respectively.

Since $\theta = 0$ for the class $W_2$, then there do not exist any natural connection on $(M, P, g) \in W_2$ with torsion expressed by $g \otimes \theta$. Only Case A is valid on such a manifold.

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