Disconnected character graphs and odd dominating sets

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ABSTRACT
Suppose $\Gamma$ is a finite simple graph. If $D$ is a dominating set of $\Gamma$ such that each $x \in D$ is contained in the set of vertices of an odd cycle of $\Gamma$, then we say that $D$ is an odd dominating set for $\Gamma$. For a finite group $G$, let $\Delta(G)$ denote the character graph built on the set of degrees of the irreducible complex characters of $G$. In this paper, we show that the complement of $\Delta(G)$ contains an odd dominating set, if and only if $\Delta(G)$ is a disconnected graph with non-bipartite complement.

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1. Introduction

Let $G$ be a finite group and $R(G)$ be the solvable radical of $G$. Also let $cd(G)$ be the set of all character degrees of $G$, that is, $cd(G) = \{\chi(1) | \chi \in \text{Irr}(G)\}$, where $\text{Irr}(G)$ is the set of all complex irreducible characters of $G$. The set of prime divisors of character degrees of $G$ is denoted by $\rho(G)$. In order to get information about a group $G$, we can associate a graph to $cd(G)$ (see [11]). Its vertex set is $\rho(G)$ and two vertices $p$ and $q$ are joined by an edge if the product $pq$ divides some character degree of $G$. We refer the readers to a survey by Lewis [7] for results concerning this graph and related topics.

If we know $\Delta(G)$ is disconnected, we can often say something about the structure of the group $G$. For instance, Lewis and White proved in [8] that $\Delta(G)$ has three connected components if and only if $G = S \times A$, where $S \cong \text{PSL}_2(2^n)$ for some integer $n \geq 2$ and $A$ is an abelian group. Also they showed in [8] that when $G$ is non-solvable and $\Delta(G)$ has two connected components, then $G$ has normal subgroups $N \subseteq K$ so that $K/N \cong \text{PSL}_2(p^n)$ where $p$ is a prime and $n$ is an integer so that $p^n \geq 4$. Furthermore, $G/K$ is abelian and $N$ is either abelian or metabelian. As another instance, all finite solvable groups $G$ whose character graph $\Delta(G)$ is disconnected have been completely classified by Lewis in [6]. In this paper, we wish to use the dominating sets of the complement of $\Delta(G)$ to describe a new characterization of disconnected character graphs with non-bipartite complement.

A dominating set for a graph $\Gamma$ with vertex set $V$ is a subset $D$ of $V$ such that every vertex not in $D$ is adjacent to at least one member of $D$. The domination number of $\Gamma$ is the number of vertices in a smallest dominating set for $\Gamma$. If $D$ is a dominating set of $\Gamma$ such that each $x \in D$ is contained in the set of vertices of an odd cycle of $\Gamma$, then we say that $D$ is an odd dominating set.
of $\Gamma$. The existence of an odd dominating set is closely related to the existence of odd cycles. Not all groups whose complement graph contains an odd cycle have an odd dominating set. For example, for the group $G = \text{PSL}_2(29) \times \text{PSL}_2(67)$, the graph $\Delta(G)^c$ has a triangle but does not have any odd dominating set, since the vertex 2 is an isolated vertex in $\Delta(G)^c$. Now we are ready to state our main result.

**Main theorem.** Let $G$ be a finite group. Then the following are equivalent.

(a) The complement of $\Delta(G)$ contains an odd dominating set $D$.

(b) The complement of $\Delta(G)$ is non-bipartite with domination number 1.

(c) $\Delta(G)$ is a disconnected graph with non-bipartite complement.

Also when one of the above conditions holds, then there exists a normal subgroup $R(G) < M \leq G$ so that $G/R(G)$ is an almost simple group with socle $S := M/R(G) \cong \text{PSL}_2(u^2)$, where $u$ is a prime, $\alpha$ is a positive integer and $u^2 \geq 5$. Also either $R(G)$ is abelian or $R(G)$ has an abelian normal 2-complement.

**2. Preliminaries**

In this paper, all groups are assumed to be finite and all graphs are simple and finite. The complement of a graph $\Gamma$ is denoted by $\Gamma^c$. For a finite group $G$, the set of prime divisors of $|G|$ is denoted by $\pi(G)$. Also note that for an integer $n \geq 1$, the set of prime divisors of $n$ is denoted by $\pi(n)$. If $H \leq G$ and $\theta \in \text{Irr}(H)$, we denote by $\text{Irr}(G|\theta)$ the set of irreducible characters of $G$ lying over $\theta$ and define $\text{cd}(G|\theta) := \{\chi(1) | \chi \in \text{Irr}(G|\theta)\}$. We begin by recalling Corollary 11.29 of [5].

**Lemma 2.1.** Let $N \triangleleft G$ and $\varphi \in \text{Irr}(N)$. Then for every $\chi \in \text{Irr}(G|\varphi)$, $\chi(1)/\varphi(1)$ divides $[G : N]$.

**Lemma 2.2** ([10, Lemma 4.5]). Let $p$ be a prime, $f \geq 2$ be an integer, $q = p^f \geq 5$ and $S \cong \text{PGL}_2(q)$. If $q \neq 9$ and $S \leq G \leq \text{Aut}(S)$, then $G$ has irreducible characters of degrees $(q + 1)[G : G \cap \text{PGL}_2(q)]$ and $(q - 1)[G : G \cap \text{PGL}_2(q)]$.

**Lemma 2.3** ([13, Lemma 4.2]). Let $N$ be a normal subgroup of a group $G$ such that $G/N \cong S$, where $S$ is a non-abelian simple group. Let $\theta \in \text{Irr}(N)$. Then either $\chi(1)/\theta(1)$ is divisible by two distinct primes in $\pi(G/N)$ for some $\chi \in \text{Irr}(G|\theta)$ or $\theta$ is extendible to $\theta_0 \in \text{Irr}(G)$ and $G/N \cong A_5$ or $\text{PSL}_2(8)$.

We now state some relevant results on character graphs needed in the next section.

**Lemma 2.4** ([4, Lemma 3.10]). Let $G$ be a finite group, $R(G) < M \leq G$, $S := M/R(G)$ be isomorphic to $\text{PSL}_2(q)$, where for some prime $p$ and positive integer $f \geq 1$, $q = p^f$, $|\pi(S)| \geq 4$ and $S \leq G/R(G) \leq \text{Aut}(S)$. Also let $\theta \in \text{Irr}(R(G))$. If $\Delta(G)^c$ is not bipartite, then $\theta$ is $M$-invariant.

Using [8, Theorem 4.1] and [9, Lemma 3.2], we can state the following result.

**Lemma 2.5.** Suppose the character graph $\Delta(G)$ of a non-solvable group $G$ is disconnected. Then $\Delta(G)$ has at least an isolated vertex.

**Lemma 2.6** ([1, Theorem A]). Let $G$ be a solvable group. Then $\Delta(G)^c$ is bipartite.

When $\Delta(G)^c$ is not a bipartite graph, then there exists a useful restriction on the structure of $G$ as follows:
Lemma 2.7 ([2, Theorem A]). Let $G$ be a finite group and $\pi$ be a subset of the vertex set of $\Delta(G)$ such that $|\pi| > 1$ is an odd number. Then $\pi$ is the set of vertices of a cycle in $\Delta(G)^c$ if and only if $O^c(G) = S \times A$, where $A$ is abelian, $S \cong \text{SL}_2(u^z)$ or $S \cong \text{PSL}_2(u^z)$ for a prime $u \in \pi$ and a positive integer $z$, and the primes in $\pi - \{u\}$ are alternately odd divisors of $u^z + 1$ and $u^z - 1$.

Lemma 2.8 ([3, Proposition 2.4]). Let $G$ be a finite group. Also let $K$ be any (nonempty) set of normal subgroups of $G$ isomorphic to $\text{PSL}_2(u^z)$ or $\text{SL}_2(u^z)$, where $u^z \geq 4$ is a prime power (possibly with different values of $u^z$). Define $K$ as the product of all the subgroups in $K$ and $C := C_{G}(K)$. Then every prime $t$ in $\rho(C)$ is adjacent in $\Delta(G)$ to all the primes $q$ (different from $t$) in $[G/C]$, with the possible exception of $(t,q) = (2,u)$ when $|K| = 1, K \cong \text{SL}_2(u^z)$ for some $u \neq 2$ and $Z(K) = P$, $P \in \text{Syl}_2(C)$. In any case, $\rho(G) = \rho(G/C) \cup \rho(C)$.

3. Proof of main theorem

Now we prove a key lemma in order to prove our main result.

Lemma 3.1. Let $G$ be a finite group. If $\Delta(G)^c$ contains an odd dominating set $D$, then:

(a) There exists a normal subgroup $R(G) < M \leq G$ so that $G/R(G)$ is an almost simple group with socle $S := M/R(G) \cong \text{PSL}_2(u^z)$, where $u$ is a prime, $z$ is a positive integer and $u^z \geq 5$. Also $\rho(R(G)) \subseteq \{2\}$;

(b) $D \subseteq \pi(S)$ and for every $x \in D$, there exists $\pi_x \subseteq \pi(S)$ such that $x, u \in \pi_x$ and $\pi_x$ is the set of vertices of an odd cycle in $\Delta(G)^c$.

Proof. Let $x \in D$. Then there exists $\pi_x \subseteq \rho(G)$ such that $x \in \pi_x$ and $\pi_x$ is the set of vertices of an odd cycle in $\Delta(G)^c$. Thus by Lemma 2.7, $N_x := O^c(G) = R_x \times A_x$, where $A_x$ is abelian, $R_x \cong \text{SL}_2(u^z)$ or $R_x \cong \text{PSL}_2(u^z)$ for a prime $u \in \pi_x$ and a positive integer $z$, and the primes in $\pi_x - \{u\}$ are alternately odd divisors of $u^z + 1$ and $u^z - 1$. Let $K := \{R_x|x \in D\}$. Define $K$ as the product of all subgroups in $K$. We fix a subset $\{R_{x_1}, \ldots, R_{x_i}\} \subseteq K$ such that $K/Z(K) \cong R_{x_1}/Z(R_{x_1}) \times \cdots \times R_{x_i}/Z(R_{x_i})$. Note that $D \subseteq \pi(K/Z(K))$ and for every $1 \leq i \leq l, S_i := R_{x_i}/Z(R_{x_i}) \cong \text{PSL}_2(u_{x_i}^z)$ and $2 \in \pi(S_i)$. If $l > 1$, then we can see that $2 \notin D$ and in $\Delta(G)$, 2 is adjacent to all vertices in $D$. It is a contradiction as $D$ is an odd dominating set for $\Delta(G)^c$. Hence $l = 1$ and $K = \{R_{x_1}\}$. We set $N := N_{x_1}, u := u_{x_1}$ and $z := z_{x_1}$. Let $M := N(R(G))$. Then $S := M/R(G) \cong N/R(N) \cong \text{PSL}_2(u^z)$ is a non-abelian minimal normal subgroup of $G/R(G)$. When $S \cong A_5$, we choose $u := 5$. Note that for every $x \in D, u \in \pi_x \subseteq \pi(S)$. Let $C/R(G) = C_{G/R(G)}(M/R(G))$. We claim that $C = R(G)$. Suppose on the contrary that $C \neq R(G)$ and let $L/R(G)$ be a chief factor of $G$ with $L \leq C$. Then $L/R(G) \cong T^k$, for some non-abelian simple group $T$ and some integer $k \geq 1$. As $L \leq C, L/M(R(G)) \cong L/R(G) \times M/R(G) \cong S \times T^k$. Since $2 \in \pi(S) \cap \pi(T)$ and $\Delta(S \times T^k) \subseteq \Delta(G)$, it is easy to see that $2 \notin D$ and in $\Delta(G)$, 2 is adjacent to all vertices in $D$ which is impossible. Therefore $G/R(G)$ is an almost simple group with socle $S = M/R(G)$. Now suppose $C' := C_{G}(R_{x_1})$. It is clear that $G/C'$ is an almost simple group with socle isomorphic to $\text{PSL}_2(u^z)$. Since $R(G)/C'/C'$ is a normal solvable subgroup of $G/C', R(G) \subseteq C'$. Let $p \in \rho(G) - \{2\}$. Then by Lemma 2.8, $p$ is adjacent to all vertices in $D \subseteq \pi(G/C')$ which is impossible. Hence $\rho(R(G)) \subseteq \{2\}$ that completes the proof.

Proof of main theorem. $\textbf{a} \Rightarrow \textbf{b}$ Using Lemma 3.1, there exists a normal subgroup $R(G) < M \leq G$ so that $G/R(G)$ is an almost simple group with socle $S := M/R(G) \cong \text{PSL}_2(u^z)$, where $u$ is a prime, $z$ is a positive integer and $u^z \geq 5$. Since $\Delta(G)^c$ is non-bipartite, it is enough to show that $u$ is an isolated vertex for $\Delta(G)$. Suppose on the contrary that there exists $v \in \rho(G) - \{u\}$ such
that $u$ and $v$ are adjacent vertices in $\Delta(G)$. Then for some $\chi \in \text{Irr}(G)$, $uv$ divides $\chi(1)$. Let $\varphi \in \text{Irr}(M)$ and $\theta \in \text{Irr}(R(G))$ be constituents of $\chi_M$ and $\varphi_{R(G)}$, respectively. We claim that $u, v \not\in \pi([G : M])$. On the contrary we assume that for some $y \in \{u, v\}$, $y$ divides $[G : M]$. By Lemma 2.2 in $\Delta(G)$, $y$ is adjacent to all vertices in $\pi(u^{2a} - 1)$. Note that using Lemma 3.1(b), $D \subseteq \pi(S)$. If $y = u$, then there is no $\pi \subseteq \pi(S)$ with this property that $u \in \pi$ and $\pi$ is the set of vertices of an odd cycle in $\Delta(G)^c$. It is a contradiction with Lemma 3.1(b). If $y = v$ and $v \in D$, then we again obtain a contradiction with Lemma 3.1(b). Also if $y = v$ and $v \not\in D$, then $v$ is adjacent to all vertices in $D$ which is a contradiction with this fact that $D$ is an odd dominating set form $\Delta(G)^c$. Hence $u, v \not\in \pi([G : M])$. Thus by Lemma 2.1, $uv$ divides $\varphi(1) \in \text{cd}(M|\theta)$. Hence, $u, v \in \pi(S)$. Now we claim that $v = 2$. On the contrary, we suppose $v$ is odd. Since $\Delta(G)^c$ is non-bipartite, using Lemmas 2.3 and 2.4, $\theta$ is $M$-invariant. Therefore as $\text{SL}_2(u^2)$ is the Schur representation of $S$, we deduce that $\theta(1)$ is divisible by $u$ or $v$. Hence for some $y \in \{u, v\}$, $y$ is adjacent to all vertices in $\pi(S) - \{y\}$. It is a contradiction with Lemma 2.7. Thus $v = 2$ and $u$ is odd. We know that 2 is adjacent to all odd primes in $\pi(S)$ (different from $u$). Therefore, in the complement graph $\Delta(G)^c$ we have that $2 \not\in D$ is not adjacent to any prime in $D$, a contradiction. Thus $u$ is an isolated vertex and it completes the proof of this part.

b$\Rightarrow$c) Since the domination number of $\Delta(G)^c$ is equal to 1, we have nothing to prove.

c$\Rightarrow$a) Since $\Delta(G)^c$ is non-bipartite, by Lemma 2.6, $G$ is non-solvable. Thus as $\Delta(G)$ is disconnected, using Lemma 2.5, $\Delta(G)$ has an isolated vertex $u$. Hence $D := \{u\}$ is an odd dominating set for $\Delta(G)^c$.

Finally, as parts (a), (b) and (c) are equivalent, when one of these parts occurs, then by Lemma 3.1 (a), there exists a normal subgroup $R(G) < M \leq G$ such that $G/R(G)$ is an almost simple group with socle $S := M/R(G) \cong \text{PSL}_2(u^2)$, where $u$ is a prime, $x$ is a positive integer and $u^2 \geq 5$. Also $\rho(R(G)) \subseteq \{2\}$. If $\rho(R(G)) = \emptyset$, then $R(G)$ is abelian and we are done. Thus $\rho(R(G)) = \{2\}$ and using Ito–Michler’s Theorem [12], $R(G)$ has an abelian normal 2-complement.

Example 3.2. (a) Let $f \geq 3$ be an integer. Also let $\text{PSL}_2(2^f) \leq G \leq \text{Aut}(\text{PSL}_2(2^f))$. If $\pi(2^f \pm 1) \neq \emptyset$, then using [14, Theorem A], we can see that $\Delta(G)$ is a disconnected graph with non-bipartite complement. Thus by Main Theorem, the complement of $\Delta(G)$ contains an odd dominating set $D$.

(b) Assume that $q$ is an odd prime power such that $q - 1$ or $q + 1$ is 2-power. If $G := \text{PSL}_2(q)$, then using Main Theorem, we deduce that $\Delta(G)^c$ does not contain any odd dominating set.

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