Eigenstate thermalization hypothesis in conformal field theory

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Received 3 November 2017
Accepted for publication 6 February 2018
Published 8 March 2018

Online at stacks.iop.org/JSTAT/2018/033101
https://doi.org/10.1088/1742-5468/aab020

Abstract. We investigate the eigenstate thermalization hypothesis (ETH) in $d+1$ dimensional conformal field theories by studying the reduced density matrices in energy eigenstates. We show that if the local probes of the finitely excited primary eigenstates satisfy ETH, then any finite energy observable with support on a subsystem of finite size satisfies ETH. In two dimensions, we discover that if ETH holds locally, the finite size reduced density matrix of states created by heavy primary operators is well-approximated by a projection to the Virasoro identity block.

Keywords: conformal field theory, entanglement in extended quantum systems, quantum thermalization
1. Introduction

In the past decade, information-theoretic insights have proved helpful in the study of certain aspects of out-of-equilibrium physics such as quantum quenches (e.g. see [1–4]). However, there is still a limited set of analytic tools to address the fundamental question of how quantum statistical physics emerges from an isolated non-integrable quantum many-body system in a pure state with sufficiently large number of degrees of freedom. There is a powerful conjecture for non-integrable systems—the eigenstate thermalization hypothesis (ETH)—which states that the expectation values of generic few-body observables in a finitely excited energy eigenstate should coincide with those in the micro-canonical ensemble up to corrections that are exponentially small in entropy [5, 6]. Assuming ETH, one can deduce that in the thermodynamic limit: (i) the expectation values of few-body observables in a generic state coincide with those in a thermal ensemble; (ii) the expectation values of few-body observables in a non-equilibrium initial state evolve towards those of a thermal ensemble.

By now, there is good numerical evidence for ETH in a variety of quantum systems [7]. While ETH is often stated for few-body observables, recently numerical support was found for the convergence of the full reduced density matrices of finite subsystems to those in the microcanonical ensemble [8, 13].

The power of ETH is in replacing the hard problem of following the time evolution of a general non-equilibrium state with an in-principle simpler problem of understanding the properties of finitely excited energy eigenstates. In contrast to the significant recent progress in numerical methods, the energy eigenstates of a general interacting
Eigenstate thermalization hypothesis in conformal field theory

many-body system are still, unfortunately, too complex to be tractable by analytic methods. A proof of ETH for general non-integrable systems appears out of reach at this stage.

In this paper, we show that analytic progress can be made in conformal field theories (CFT), which is a natural starting point to obtain insights into the quantum chaotic behavior of finitely exited states of general quantum field theories. CFTs capture the essential aspects of ETH for a general continuum system, but exhibit more symmetries which make them more accessible to study. We are able to prove that for CFTs, the weakest form of ETH which applies to local primary operators (to which we refer as local ETH) leads to the strongest form of ETH which applies to the reduced density matrix of a general subsystem of finite size. Thus to prove ETH for CFTs it is enough to prove that it applies to the local primary operators, which in turn reduces to a statement regarding the coefficients in the operator product expansions (OPE). Clearly, not all CFTs satisfy ETH, thus proving local ETH should amount to understanding how the corresponding statement regarding the OPE coefficients connects with non-integrability.

We believe that local ETH provides a powerful technical handle for obtaining many dynamical properties of a non-integrable CFT. As a simple application, we use it to compute the Renyi entropies for an interval in an energy eigenstate in a $(1+1)$-dimensional CFT. Interestingly, we find that even though the reduced density matrix for the region approaches that of the grand canonical ensemble with the same charges in the thermodynamic limit, the Renyi entropies differ by a finite amount. Previous work on the entanglement of excited states in $(1+1)$-dimensional CFTs was focused on light states and integrable models (e.g. [11, 12]).

2. Setup

Consider a finitely excited energy eigenstate $|E\rangle$ of a conformal field theory on a $d$-dimensional sphere of radius $L$ (i.e. the total spacetime is $\mathbb{R}_t \times S^d$). A key simplification of conformal symmetry is that in the Euclidean signature any state $|\psi\rangle$ on $S^d$ can be mapped to a local operator $\Psi$ inserted at origin ($r=0$) in $\mathbb{R}^{d+1}$ via the conformal transformation

$$d_{\text{cylinder}}^2 = dr^2 + L^2 d\Omega_d^2 = \Lambda^2 (dr^2 + r^2 d\Omega_d^2)$$

$$\Lambda = \frac{L}{r}, \quad \tau = L \log r.$$ (1)

Similarly, the conjugate state $\langle \psi|$ is mapped to a local operator $\Psi^\dagger$ inserted at $r=\infty$ in $\mathbb{R}^{d+1}$. The energy eigenstates are in one-to-one correspondence to the operators $P_{\mu_1} \cdots P_{\mu_m} \Psi_a$, where $P_\mu$ is a momentum operator and $\Psi_a$ is a primary operator. Below, we use the state and the operator languages interchangeably.

The kinematics of conformal symmetry fix the correlators in descendant states $P_{\mu_1} \cdots P_{\mu_m} \Psi_a$, in terms of those of their corresponding primaries $\Psi_a$. The dynamical content of a CFT is the spectrum of its primaries and the OPE coefficients. ETH is a dynamical statement about the energy eigenstates which are not related by symmetries, therefore we restrict our analysis to the primary energy eigenstates. Moreover, we
restrict our attention to the homogeneous energy eigenstates $S^d$ which carry no angular momentum under the $SO(d+1)$ rotations. Such an energy eigenstate with energy $E_a$ corresponds to a spinless primary $\Psi_a$ of dimension $h_a = E_a L$. The energy density of the system is $\epsilon_a = \frac{E_a}{L^d \omega_d} = \frac{h_a}{L^{d+1} \omega_d}$, where $\omega_d$ is the volume of a unit $S^d$. For a CFT in a thermal state of temperature $T$, $\epsilon_a \sim C T^{d+1}$ where $C$ is the central charge. We define the ‘thermal’ length scale associated with $|E_a\rangle$ as

$$\lambda_T = \left(\frac{\epsilon_a}{C}\right)^{-\frac{1}{d+1}} \sim T^{-1}. \quad (2)$$

We are interested in the thermodynamic limit with $L \to \infty$, while keeping a finite energy density $\epsilon_a$, and hence a finite $\lambda_T$. In this limit, the scaling dimension $h_a$ should scale with $L$ as

$$h_a = C \omega_d \left(\frac{L}{\lambda_T}\right)^{d+1}. \quad (3)$$

The local ETH condition is defined as

$$\langle E_a | \mathcal{O}_p | E_b \rangle = \mathcal{O}_p(E) \delta_{ab} + \Delta^p_{ab}, \quad (4)$$

where $\mathcal{O}_p$ is a local primary operator (with $p$ labeling different operators), the diagonal element $\mathcal{O}_p(E)$ is a smooth function of $E = \frac{E_a + E_b}{2}$, and $\Delta^p_{ab} \sim e^{-O(S(E))}$ where $e^{S(E)}$ is the density of states at energy $E$. Since the state is homogeneous it does not matter where $\mathcal{O}_p$ is inserted on $S^d$, and we keep the location of $\mathcal{O}_p$ implicit.

In experiments what we observe are the wave-packets of local fields rather than operators inserted at a point. Furthermore, any physically meaningful observable has small fluctuations measured by the composite operator $\mathcal{O}^2$. Thus, any formulation of ETH in quantum field theory has to restrict to the set of bounded smeared operators. In equation (4), $\mathcal{O}_p$ should be understood as a smeared operator $\mathcal{O}_p(g) = \int g \mathcal{O}_p$ with a smearing function $g$. The set of observables $\{\mathcal{O}_p(g)\}$ in region $B$ form a subalgebra. If $g$ has support only inside a region $B$, then the restriction of the wavefunction to this subalgebra defines the reduced density matrix on $B^4$. The operator $\mathcal{O}_p(g)$ is formally unbounded, however it is always possible to replace it with a bounded operator on the dense set of physical states with no loss of physics $[9, 10]^5$. The resulting smoothed operator has finite moments, in particular the operator $\mathcal{O}_p^2$ is well-defined and has finite expectation values in physical states. Below for notational simplicity we continue to denote the bounded smooth operator by $\mathcal{O}_p$.

We now introduce subsystem ETH which states that for a subregion $B$ [13]

1. There exists a ‘universal’ density matrix $\rho_B(E)$ (which depends only on $B$ and energy $E$) such that for any energy eigenstate $|E_a\rangle$

$$||\rho_B^a - \rho_B(E = E_a)|| \sim e^{-O(S(E))} \quad (5)$$

$^4$ A state is a map from the algebra of observables to expectation values. Restricting this map to a subalgebra defines a ‘reduced’ state.

$^5$ The idea is that projecting out the part of an operator that is in the vector space spanned by energy eigenstates of very high energies does not affect the low-energy physics.
where \( \rho_B \equiv \text{Tr}_{B'} |E_a\rangle \langle E_a| \). In (5), \( \| \rho - \sigma \| = \frac{1}{2} \text{Tr} \sqrt{\left( \rho - \sigma \right)^2} = \frac{1}{2} \sum_i |\lambda_i| \) is the trace distance of two density matrices \( \rho \) and \( \sigma \), and \( \lambda_i \) are the eigenvalues of the Hermitian operator \( \rho - \sigma \).

2. Introducing \( \sigma_{ab} \equiv \text{Tr}_{B'} |E_a\rangle \langle E_b| \), then

\[
\| e^{ia} \sigma_{ab} + e^{-ia} \sigma_{ba} \| \sim e^{-O(S(E))},
\]

for all \( \alpha \).

Note that the subsystem ETH (5) and (6) implies local ETH, that is, for an operator \( \mathcal{O} \) supported inside region \( B \) we have (see appendix C)

\[
\text{Tr}((\rho_B^a - \rho_B(E)) \mathcal{O}) \leq \| \rho_B^a - \rho_B(E) \| \frac{1}{2} \text{Tr}((\rho_B^a + \rho_B(E)) \mathcal{O}^2) \frac{1}{2}
\]

and thus is exponentially small. Similarly, \( \text{Tr}(\mathcal{O} \sigma_{ab}) = e^{-O(S(E))} \).

To close our setup, let us briefly comment on the descendant eigenstates. As an example consider \( |E⟩ \) corresponding to a spinless operator \((P^a_l)\Psi_a\) where \( l \) is an integer. For such a state \( E = E_{a,l} = E_a + \frac{2l}{L} \). The matrix element \( \langle E|\mathcal{O}_p|E⟩ \) is controlled by that in the primary state \( \Psi_a \), i.e. by \( \mathcal{O}_p(E - \frac{2l}{L}) \) rather than \( \mathcal{O}_p(E) \). Thus as stated earlier, we should not include the descendant states in either (4) or (5) and (6). Of course if one assumes (4), for \( h_a \gg l \), from the smoothness of function \( \rho_B(E) \), \( \mathcal{O}_p(E - \frac{2l}{L}) \) is related to \( \mathcal{O}_p(E) \) only by corrections of order \( 1/h_a \) and similarly the corresponding reduced density matrix is close to \( \rho_B(E) \) in trace distance with corrections that are polynomially suppressed in \( 1/h_a \). But such statements do not contain any new dynamical information beyond (4)–(6). For completeness, in appendix B we discuss descendant states in more detail.

3. From local ETH to subsystem ETH

Let us now look at the implications of (4). Using the conformal map in (1) the matrix element \( \langle E_b|\mathcal{O}_p|E_a⟩ \) on \( S^d \) is mapped to the Euclidean three-point function

\[
\langle E_b|\mathcal{O}_p|E_a⟩ = \frac{L^{-h_p} \langle \Psi_b^1(\infty)\mathcal{O}_p(1)\Psi_a(0) \rangle}{\langle \Psi_b^1(\infty)\Psi_a(0) \rangle \langle \Psi_b^1(\infty)\Psi_b(0) \rangle} = C^p_{ab} L^{-h_p},
\]

where \( C^p_{ab} \) is the OPE coefficient for \( \mathcal{O}_p \) appearing in the operator product of two primaries \( \Psi_a \) and \( \Psi_b^1 \). We assume that the thermodynamic limit in (4) exists for any operator \( \mathcal{O}_p \) whose dimension \( h_p \) does not scale with \( L \) (we refer to such observables as finite energy in the thermodynamic limit). From (8) and expressing \( L \) in term of \( h_a \) using (3) we conclude that the OPE coefficient \( C^p_{ab} \) must scale with \( h_a \rightarrow \infty \) as

\[6\]

Expectation values depend on the normalization of the observable and could be arbitrarily large for unbounded operators. It is important that density matrices with small trace distance are physically indistinguishable, even though corresponding expectation values of certain operators might not be close [14].

https://doi.org/10.1088/1742-5468/aab020
where \( f_p(E) = \lambda h^p \omega(\omega_{a}^{-})^{-\frac{h}{\pi}} O_p(E) \) is a smooth function of \( E \) independent of label \( a \), and \( R_{ab}^p = L^a R \Delta_{ab}^p \sim e^{-O(h_{a}^{-})} + \frac{h_{a}^{-}}{\pi} \log h_{a} \) as on general ground we expect that \( S(E) \propto L^d \) in the thermodynamic limit. Note that equation (9) implies the following: operators \( O_p \) whose \( C_{aa}^p \) grow slower than \( h_{a}^{-} \) with \( h_{a} \) cannot have a non-vanishing expectation value in the thermodynamic limit, while it is impossible for an operator with \( C_{ab}^p \) to grow faster than \( h_{a}^{-} \) as that would imply the thermodynamic limit does not exist.

We are interested in the reduced density matrix of a region \( B \) of a finite size in the thermodynamic limit. Consider \( B \) to be a ball around the north pole of \( S^d \) with the angular radius \( \theta_0 = \tan^{-1}(\frac{R}{L}) \). In the the thermodynamic limit of the angular size \( R/L \to 0 \) keeping \( R \) finite, \( B \) becomes a ball of radius \( R \) in \( \mathbb{R}^d \). We will discuss other shapes or regions with disconnected components towards the end of this letter.

The distance is invariant under unitary rotation of states:

\[
\|\rho - \sigma\| = \|\tilde{\rho} - \tilde{\sigma}\|, \quad \tilde{A} = UAU^\dagger.
\]

A conformal transformation acts on the density matrix as a unitary operator. Therefore, we are free to compare density matrices in any conformal frame. A convenient conformal frame is the Rindler frame

\[
ds_{\text{cylinder}}^2 = \Lambda^2(X) dX^i dX^i, \quad i = 0, \ldots, d,
\]

where the ball-shaped region \( B \) on \( S^d \) is mapped to negative half-space, \( X^i \leq 0; \) see figure 1 and appendix A for details. Under this map \( \Psi_a \) and \( \Psi_{a}^\dagger \) that create the state vector and its dual are mapped to \( X_{\pm}^i = (\pm \sin \theta_0, \cos \theta_0, X^{i>1} = 0) \), respectively. Thus, in the thermodynamic limit \( \theta_0 \to 0 \), the two operators are colliding in this frame, and we can use their operator product expansions to represent them.

Denote the reduced density matrix for region \( B \) in the energy eigenstate \( |E_a\rangle \) in the Rindler frame with \( \tilde{\rho}_B \). The expectation values in this state can be written as

\[
\Psi^\dagger(\infty) \\
S^d \times R \_t \\
\Psi(-\infty)
\]

\[
\Psi^\dagger(\infty) \\
B \\
\Psi(0)
\]

\[
\Psi^\dagger(\sin \theta_0) \\
\Psi(-\sin \theta_0)
\]

**Figure 1.** (a) The cylinder \( S^d \times R \_t \) conformal frame (b) Radial quantization \( \mathbb{R}^{d+1} \) conformal frame. (c) The conformal frame convenient for the study of the density matrix on subsystem \( B \).
Formally, the reduced density matrix is as a map from the algebra of observables on the subsystem to expectation values: \( \rho(O \in \{O\}) = \text{Tr}(\rho O) \). For a fixed subregion \( B \), we restrict the list of observables of interest to operators \( O \) whose dimensions \( h_p \) remain finite in the thermodynamic limit (i.e. do not scale \( L \)). The operators of weight \( h_p \geq O(L) \) do contribute to the OPE; however, they are unimportant in the thermodynamic limit as they would only affect the value of observables that produce a large amount of energy locally, while the observables with the fixed \( h_p \) will be unaffected. Hence, we restrict the sum in (11) to operators with \( h_p \) parametrically smaller than \( O(L) \). It should be understood that in the sum in (11) the descendants of \( O \) are included implicitly (similarly below).

Using (8) and (4) we can write (11) in an operator form

\[
\tilde{\rho}_B^a = \tilde{\rho}_B(E) + \mathcal{R}_B^a
\]

where

\[
\tilde{\rho}_B(E) = \sum_{h_p} O_p(E) R^{h_p} O_p(X_-)
\]

\[
\mathcal{R}_B^a = \sum_{h_p} \Delta_a^{rb} R^{h_p} O_p(X_-)
\]

Equation (13) (and similarly for \( \mathcal{R}_B^a \)) should be understood as follows: \( \tilde{\rho}_B(E) \) is prepared in the Rindler frame via a Euclidean path-integral over \( \mathbb{R}^{d+1} \) with boundary conditions above and below the negative half-space and a local operator inserted at \( X_\mu \) (see figure 2). In (13) we denote the density matrix by the specific operator inserted at \( X_- \). Similarly, we can write \( \sigma_{ab} \) introduced before (6) as (with \( a \neq b \))

\[
\tilde{\sigma}_{ab} = \sum_{h_p} \Delta_a^{rb} R^{h_p} O_p(X_-)
\]

The equations above are to be understood as operator equalities inside the Euclidean path-integral in the Rindler frame. Mapping back to the radial quantization frame the universal density matrix \( \rho_B(E) \) is

\[
\rho_B(E) = \sum_{h_p} O_p(E) R^{h_p} \hat{O}_p(0)
\]

\[
\hat{O}_p = U^a O_p U^a,
\]

and \( U \) is the unitarity corresponding to the conformal transformation from the Rindler frame to radial quantization frame; see figure 2(c).

Note that since we have already taken \( L \to \infty \) we dropped the upper bound on the dimension of operators \( h_p \). In the infinite volume limit, it is not a physical question to
ask the response of the state to a local operator that has energy scaling with volume. More formally, one can say that the density matrix above is the restriction of the density matrix to finite energy observables, which is sufficient for all experimental tests and the problem of distinguishability.

From (12) and the finite radius of convergence of OPE we immediately find that for the observables in region $B$,

$$\frac{\text{Tr}(\hat{\rho}_B^a - \hat{\rho}_B \cdots)}{\text{Tr}(\hat{\rho}_B \cdots)} = \frac{\sum h_p \Delta_{aa}^p \rho^{hp}_p \langle O_p \cdots \rangle}{\sum h_p \rho_p(E) R^{hp}_p \langle O_p \cdots \rangle} \sim e^{-O(S(E))}.$$

In fact, we can prove that density matrices $\rho_B^a$ and $\rho_B(E)$ are close in trace distance. Trace distance is hard to compute in continuum theories. Instead, we use another measure of distance called the relative entropy that is a measure of distinguishability of two density matrices $\rho$ and $\sigma$, and is defined to be

$$S(\rho \| \sigma) = \text{tr}(\rho \log \rho) - \text{tr}(\rho \log \sigma).$$

It is non-negative, and vanishes if and only if $\rho = \sigma$. The Pinsker inequality [15, 16] provides an upper bound on trace distance of states in terms of relative entropy:

$$\|\rho - \sigma\|^2 \leq 2S(\rho \| \sigma).$$

Applied to our case of interest, we find

$$\|\rho_B^a - \rho_B(E)\|^2 = \|\rho_B^a - \hat{\rho}_B(E)\|^2 \leq 2S(\rho_B^a \| \hat{\rho}_B(E)).$$

Since the two density matrices are close, we only need to compute the relative entropy $S(\rho_B^a \| \hat{\rho}_B)$ perturbatively in small $\mathcal{R}_B$. To the second order, we find that the relative entropy is given by the quantum Fisher information

$$S(\rho_B^a \| \hat{\rho}_B) \simeq \int_0^\infty \frac{ds}{2} \text{Tr} \left( (\hat{\rho}_B + s)^{-1} \mathcal{R}_B^a (\hat{\rho}_B + s)^{-1} \mathcal{R}_B^a \right)$$

$$= O(\eta_a^2),$$

$$\eta_a = \sup_p \Delta_{aa}^p = e^{-O(S(E))},$$

where we have used the integral representation of the logarithm of a positive operator. In appendix D, we expand relative entropy to all orders in $\eta_a$ using the replica trick in

\[https://doi.org/10.1088/1742-5468/aab020\]
[18], and show (19) assuming that the $n$-point correlators on an $n$-sheeted manifold are finite. Invoking (19) it is evident that
\[
\|\rho_B^a - \rho_B(E)\|_2^2 \leq 2S(\tilde{\rho}_B \| \tilde{\rho}_B(E)) = O(\eta_a^2),
\]
which demonstrates part one of the subsystem ETH, (5). Note that subsystem ETH holds for any finite ball-shaped subsystem of finite radius $R$. The monotonicity of trace distance under partial trace implies that subsystem ETH holds for any subsystem of arbitrary shape and disconnected regions that can be encircled in a ball of finite size:
\[
\forall A \text{ such that } \exists B : A \subset B, \quad \|\rho_A^a - \rho(E)_A\| \leq \|\rho_B^a - \rho(E)_B\| = O(\eta_a),
\]
where $\rho_A(E) = \text{Tr}_{B-A}(\rho_B(E))$; see figure 2(c). Repeating the argument above for the state $|E_{a,b,a}\rangle = \frac{1}{\sqrt{2}}|E_a\rangle + e^{i\alpha}|E_b\rangle$ with $|E_a\rangle$ and $|E_b\rangle$ two energy eigenstates proves part two of the subsystem ETH (6) (see the appendix C equation (C.5)).

The expression in (12) is an operator equality. To obtain an intuition about how the eigenvalues of $\rho_B^a$ and $\rho_B(E)$ compare, we compute their Renyi entropies. The Renyi entropy of a density matrix $\rho$ is defined by
\[
S_n(\rho) = \frac{1}{1-n} \log \text{tr}(\rho^n).
\]
Expanding the Renyi entropies of the density matrix in the energy eigenstate it is easy to see that
\[
S_n(\rho_B^a) - S_n(\rho_B(E)) = e^{-O(S(E))}.
\]
That is to say that the Renyi entropies of reduced density matrices in energy eigenstates are universal.

4. Further applications of local ETH

CFTs in $1+1$-dimensions are special in that the expectation value of local operators that are not Virasoro descendants of identity vanish in the thermal state. That is due to the fact that the thermal state on a line is conformally flat. Then, the local ETH implies that in the thermodynamic limit $C_{\rho a}^p \to 0$ for all quasi-primaries, except for those made of stress tensors $T$ and $\bar{T}$. In other words, the density matrices in eigenstates are well-approximated by their projection to the Virasoro identity block. Assuming local ETH one can compute the universal density matrix $\tilde{\rho}_B(E)$ defined by (13). Since the quasi-primaries that appear in $\rho_B(E)$ are all only made of stress tensor one can directly compute the Renyi entropies of subsystem $B$ order by order in $R/\lambda T$. In [19] we compute the vacuum subtracted Renyi entropies, $\Delta S_n$, using the replica trick in our universal density matrix and find
\[
\Delta S_n(\rho_B(E)) = \left(\frac{1+n}{12n}\right)(R/\lambda T)^2 - \left(\frac{1+n}{120n}\right)(R/\lambda T)^4(\frac{n^2 + 11}{12n^2})
+ \left(\frac{1+n}{630n}\right)(R/\lambda T)^6(\frac{4-n^2}{144n^4})(n^2 + 47) + \cdots
\]
https://doi.org/10.1088/1742-5468/aab020
which is to be compared with the Renyi entropies in the thermal reduced density matrix $\rho_B(T) = \text{Tr}_{B^c}(e^{-\beta H}/Z)$ computed in [17]:

$$\Delta S_n(\rho_B(T)) = \frac{(1 + n)c}{12n} (R/\lambda_T)^2 - \frac{(1 + n)c}{120n} (R/\lambda_T)^4$$

$$+ \frac{(1 + n)c}{630n} (R/\lambda_T)^6 + \cdots. \quad (24)$$

We observe that while for $n > 1$ the Renyi entropies do not match, the entanglement entropies ($n = 1$) match perfectly. The discrepancy between the Renyi entropies is a consequence of the infinite dimensionality of the Hilbert space. The Fannes–Audenaert inequality and its generalizations for $n > 1$ restrict the difference between entropies to be bound by the trace distance multiplied by a factor proportional to the dimension of the Hilbert space. As the latter diverges, the Renyi entropies could be different for arbitrarily close $\rho_B(E)$ and $\rho_B(T)$. However, the equivalence of entanglement entropies demonstrates that up to the order $(R/\lambda_T)^4$

$$\|\rho_B(E) - \rho_B(T)\|^2 \leq S(\rho_B(E)) \|\rho_B(T)\|$$

$$\Delta \langle H(\rho_B(T)) \rangle = \Delta S = e^{-O(S(E))}. \quad (25)$$

Here $H(\rho_B(T)) = -\log \rho_B(T)$ is a local integral over $T_{00}$ in $1 + 1$-dimensions, and we have tuned the two states to have the same energy density. Therefore, $\Delta \langle H(\rho_B(T)) \rangle = 0$ by construction.

An alternative ‘thermodynamic’ limit can be defined at large central charge and finite volume. Then, it has been argued that in $1 + 1$-dimensional CFTs with a sparse spectrum of low dimension operators the density matrix in eigenstates of energy order $c$ is well-approximated by its projection to the Virasoro identity block [20]. This is the analogue of the local ETH assumption at large central charge. For a discussion of ETH in $1 + 1$-dimensional CFT at large central charge see [21, 23]. It is important to keep in mind that the universal terms in the density matrix in the large central charge and finite volume are different from those in the finite central charge and infinite volume limit. For instance, the derivative operator is suppressed in the infinite volume limit, but not in the infinite central charge limit.

5. Conclusions

In this paper, we studied ETH in chaotic CFTs as a first step towards understanding ETH in the continuum limit. We showed that the reduced density matrix on any finite subsystem of arbitrary shape in finitely excited energy eigenstates are well-approximated by a universal density matrix. In order to prove this we assumed local ETH. All integrable models we checked (free theories and minimal models in $1 + 1$-dimensions) failed to satisfy our local ETH assumption. Therefore, we can interpret the local ETH assumption as our working definition of chaos in CFTs. It would be interesting to connect the local ETH to the more conventional definitions of quantum chaos in the literature. In particular, one might hope to use the exponential decay of out-of-time order correlators and bootstrap equations to prove a statement similar to local ETH. Finally,
holographic CFTs are a large class of theories that are believed to be chaotic [24]. It is interesting to study the validity of local ETH in such theories.

Acknowledgments

We would like to thank Ahmed Almheiri, John Cardy, Thomas Faulkner, Liam Fitzpatrick, Daniel Harlow, Thomas Hartman, Tarun Grover, Mark Srednicki and Sasha Zhiboedov for valuable discussions. The research of NL is supported in part by funds provided by MIT-Skoltech Initiative. This paper has the preprint number Technical Report MIT-CTP/4841. This work is supported by the Office of High Energy Physics of US Department of Energy under grant Contract Number DE-SC0012567.

Appendix A. Rindler space: convenient conformal frame

Consider a \((d+1)\)-dimensional CFT in radial quantization with a ball-shaped subsystem of angular size \(\theta_0\) on \(S^d\) at \(r = 1\). According to the operator/state correspondence the density matrix in the subsystem is given by a path-integral over the \((d+1)\)-dimensional space with two operators inserted, \(\Psi\) at \(r = \epsilon\) and \(\Psi^\dagger\) at \(r = 1/\epsilon\) with \(\epsilon \to 0\), and a cut open at the location of the subsystem. The initial metric in the radial quantization is

\[
ds^2 = dr^2 + r^2 d\Omega_d^2
\]

with \((\theta_1, \cdots \theta_d)\) the coordinates on \(S^d\). We perform the following conformal transformation

\[
L(r^2 - 1) = \frac{X^0}{1 - 2X^1 + X \cdot X} \quad \text{and} \quad \frac{2Lr \sin \theta_1 \cos \theta_2}{1 + r^2 + 2r \cos \theta_1} = \frac{1 - 2X^1 + X \cdot X}{(1 - X \cdot X)/2}
\]

that maps the subsystem at \(r = 0\) and \(\theta_1 \leq \theta_0\) to the negative half-space, i.e. \((0, X^1 < 0, 0 \cdots 0)\). Here \(L\) is the radius of \(S^d\) in units where \(R\) is set to one. The new metric in the \(X\)-coordinates that we call Rindler frame is given by

\[
ds^2 = \Lambda(X)^2 dX^i dX^i
\]

\[
\Lambda(X) = \left( X^0 - \frac{LV_-}{2} - \frac{V_+}{8L} \right)^{-1}
\]

\[
V_\pm = (1 \pm 2X^1 + X \cdot X).
\]

In these coordinates the path-integral without operator insertions prepares the Rindler density matrix in vacuum. The operators \(\Psi\) and \(\Psi^\dagger\) are now inserted at \(X_-\) and \(X_+\) respectively.
Eigenstate thermalization hypothesis in conformal field theory

\[ X_\pm = (\pm \sin \theta_0, \cos \theta_0, 0 \cdots, 0), \]
\[ \Lambda(X_-) = (2 \sin \theta_0)^{-1}, \]
\[ \Lambda(X_+) = \varepsilon^{-2}(2 \sin \theta_0)^{-1}. \]  

Under this map a conformal primary transforms according to
\[ \langle \Psi(r = 0) \cdots \rangle_{\Lambda(X)\delta_{ij}} = \Lambda(X(r = 0))^{-h}(\Psi(X(r = 0)) \cdots)_{\delta_{ij}}. \]

Therefore,
\[ \langle \Psi(1/\varepsilon)\Psi(\varepsilon) \cdots \rangle_{\text{radial}} = (2\sin \theta_0)^{2h}\langle \Psi(X_+)\Psi(X_-) \cdots \rangle_{\text{Rind}} \]

In the thermodynamic limit \( \theta_0 \ll 1 \) the distance between \( \Psi \) and \( \Psi^\dagger \) goes to zero: \( |X_+ - X_-| = 2 \sin \theta_0 \ll 1 \), and we use the OPE to obtain
\[ \langle \Psi(1/\varepsilon)\Psi(\varepsilon) \cdots \rangle_{\text{radial}} = \varepsilon^{2h} \sum_P C_{\Psi\Psi}^P (2 \sin \theta_0)^{h_P} \langle O_P(\varepsilon) \cdots \rangle. \]

Appendix B. Spinless descendant eigenstates

B.1. Local probes

An arbitrary descendant energy eigenstate in conformal field is created by the operator \( P_{l_1} \cdots P_{l_m} \Psi_{i_1 \cdots i_m} \). In order to simplify the presentation and avoid unnecessary manipulation of indices we focus on a particular class of spinless primaries: \( (P^2)\Psi_a \). The argument generalizes to arbitrary descendants. Our eigenstates of interest are labelled by \((a,l)\). In a conformal theory the matrix element of a scaling operator \( O_p \) in these states is given by
\[ \langle E_{(b,l)}|O_p|E_{(a,m)}\rangle = \frac{1}{L^b_p} \frac{\langle \Psi_b(\infty) (K_\mu K^\mu)^l O_p(1)(P_\nu P^\nu)^m \Psi_a(0) \rangle}{\sqrt{\langle X_{a,m}\rangle \langle X_{b,l}\rangle}}, \]  

where \( \langle X_{a,m}\rangle = \langle \Psi_a(\infty) (K_\mu K^\mu)^m (P_\nu P^\nu)^m \Psi_a(0) \rangle \). Note that in radial quantization \( (P_\mu|\Psi)^l = \langle \Psi|K_\mu \rangle \). Here, we assume that \( l \) and \( m \) are much smaller than \( h_a \) and \( h_b \).

First, consider the term \( \langle X_{a,m}\rangle \) in the denominator. Primary field \( \Psi_a \) is killed by \( K_\mu \), so we only need to compute \( [ (K_\mu K^\mu)^m, (P_\nu P^\nu)^m ] \Psi_a \). This operator can be simplified by the successive application of the following commutation relations of the generators of the conformal group:
\[ [K_\mu, P_\nu] = 2(\delta_{\mu\nu}D - M_{\mu\nu}) \]
\[ [M_{\mu\nu}, P_\rho] = P_\rho \delta_{\mu\nu} \]
\[ [M_{\mu\nu}, K_\rho] = K_\rho \delta_{\mu\nu} \]
\[ [M_{\mu\nu}, M_{\rho\sigma}] = M_{\mu\rho} \delta_{\nu\sigma} \] \hspace{1cm} (B.2)

Every time the operator \( D \) appears it acts on its eigenstate and we get a factor \( h_a + i \), where \( 0 \leq i \leq m \). All other terms carry no powers of \( h_a \). At large \( h_a \) we find
\[ \langle X_{a,m}\rangle = (d + 1)^m (2h_a)^{2m}(1 + O(h_a^{-1})). \] \hspace{1cm} (B.3)

https://doi.org/10.1088/1742-5468/aab020
The numerator in (B.1) has two terms
\[ \langle \Psi_q \mathcal{O}_p (K_\mu K^\mu)^l (P_\nu P^\nu)^m \Psi_s(0) \rangle \\
+ \langle \Psi_q (K_\mu K^\mu)^l \mathcal{O}_p (P_\nu P^\nu)^m \Psi_s(0) \rangle. \] (B.4)

With no loss of generality we assume \( m \geq l \). Here we argue that the first term wins over the second term in the large \( h_a \) limit. As we saw above, the first term at large \( h_a \) scales as
\[ (d + 1)^l (2h_a)^{2l} \langle \Psi_q(\infty)\mathcal{O}_p(1)(P_\nu P^\nu)^{m-l} \Psi_s(0) \rangle \]
\[ = (d + 1)^l (2h_a)^{2l} \lim_{z \to 0} (\partial^2)^{m-l} \langle \Psi_q(\infty)\mathcal{O}_p(1)\Psi_s(z) \rangle \]
\[ = (d + 1)^l (2h_a)^{2l} C_q^p f_{m-l}(h_p) \] (B.5)
where \( f_{m-l}(h_p) \) is a polynomial of degree \( 2(m - l) \) in \( h_p \). To compute the second term in (B.4) we work out the commutator
\[ [(K_\mu K^\mu)^l, \mathcal{O}_p] = l [K_\mu, \mathcal{O}_p] K^\mu (K_\alpha K^\alpha)^{l-1} \]
\[ + l(l - 1) [K_\mu, [K_\nu, \mathcal{O}_p]] K^\mu K^\nu (K_\alpha K^\alpha)^{l-2} \]
\[ + \cdots + [K_{\mu_1}, \cdots [K_{\mu_n}, \mathcal{O}_p] \cdots ] K^{\mu_1} \cdots K^{\mu_n}. \]

The special conformal transformation generated by vector field \( \epsilon \xi^\mu \) sends
\[ x'^\mu = \frac{x^\mu - \epsilon \xi^\mu x^2}{(1 - 2\epsilon \xi \mu x^\mu + \epsilon^2 \xi^2 x^2)} \] (B.6)
which transforms the scaling operator \( \mathcal{O}_p \) according to
\[ (1 - 2\epsilon \xi_\mu x^\mu + \epsilon^2 \xi^2 x^2)^{h_p} \mathcal{O}(x'^\mu) = e^{-i\epsilon \xi_\mu K^\mu} \mathcal{O}(x)e^{i\epsilon \xi_\mu K^\mu}. \]

Matching the coefficients of \( \frac{\xi^\mu}{n!} \) in a series expansion on both sides gives
\[ [K_{\mu_1}, \cdots [K_{\mu_n}, \mathcal{O}_p] \cdots ] = \partial_{n}^{\mu} \left( (1 - 2\epsilon \xi_\mu x^\mu + \epsilon^2 \xi^2 x^2)^{h_p} \mathcal{O}(x'^\mu) \right). \]

Terms that appear on the right-hand side of the equation above have the form
\[ f(x^\mu, \xi^\mu, h_p)(\partial \cdots \partial \mathcal{O}_p)K_{\mu_1} \cdots K_{\mu_n}. \] (B.7)

Putting this back in the second term in (B.4) we obtain terms that are
\[ g(h_p)(d + 1)^j (2h_a)^2 \langle \Psi_\ell(\partial \cdots \partial \mathcal{O}_p)(1)\Psi_a \rangle \] (B.8)
for \( j < l \). Note that extra derivatives on \( \mathcal{O}_p \) do not lead to any extra powers of \( h_a \). As a result, the first term in (B.4) dominates. Putting all these terms back in equation (B.1) we obtain the matrix elements of \( \mathcal{O}_p \) in energy eigenbasis
\[ \langle E_{(b, l)} | \mathcal{O}_p | E_{(a, m)} \rangle \]
\[ = \frac{C_p^b}{L^{bp}} (d + 1)^{l-m}/2 (2h_a)^{l-m} f_{m-l}(h_p)(1 + O(h_a^{-1})). \]

In the case \( l = m \) the above expression becomes
The conformal algebra fixes their value in terms of $C_{a b}^p$, $h_a$, $h_b$ and $h_p$. In appendix B, we work out these matrix elements and argue that at large $h_a$ and $h_b$ they are given by

$$\langle E_{(a,b)}|O_p|E_{(a,m)}\rangle = \frac{C_{a b}^p}{h_p^{\frac{g(m-l)}{2}}} (h_p)^{l-m} L^{h_p} (1 + O(h_p/h_a))$$

where without loss of generality, we have assumed $m > l$, and $g_k(h_p)$ is a polynomial of order $2k$ in $h_p$ with $g_0(h_p) = 1$. Then, from equation (9) we find that

$$\langle E_{(a,m)}|O_p|E_{(a,m)}\rangle = C_{a a}^p (1 + O(h_a^{-1})),$$

which together with the assumption of local ETH for primary energy eigenstates implies

$$\langle E_{(a,b)}|O_p|E_{(a,m)}\rangle = (O_p(E)\delta_{a b} + \Delta_{a b}^p) (\delta_m + O(h_a^{-1})).$$

### B.2. Density matrix

Now consider the spinless energy eigenstate $|E_{a,l}\rangle$ created with a path-integral over the unit ball with $(P^m P_\mu) |\Psi_a(0)\rangle = \Box^{1/2} |\Psi_a(0)\rangle$ in the center of radial quantization. In the Rindler frame the Laplacian is

$$\Box = \frac{1}{\Lambda(X)^{d+1}} \frac{1}{\partial_l} \left( \Lambda(X)^{d+1} \partial_l \right)$$

and the unit ball is mapped to the lower half-plane $X^0 < 0$. In these coordinates, the operators that create the state and its conjugate are

$$\hat{\Psi}(X_-) = \left( \Lambda(X)_- \right)^{d-1} \partial_l \left( \Lambda(X)_-^{d-1} \partial_l \right)^l \Lambda(X)_-^{-h} \hat{\Psi}(X_-)$$

$$\hat{\Psi}^\dagger(X_+) = \lim_{\epsilon \to 0} \epsilon^{-2(h+2l)} \left( \Lambda(X_{1/\epsilon})^{-d-1} \partial_l \left( \Lambda(X_{1/\epsilon})^{d-1} \partial_l \right)^l \Lambda(X_{1/\epsilon})^{-h} \hat{\Psi}(X_{1/\epsilon})

$$where we have used the fact that under conjugation $X^0 \to -X^0$. Note that

$$\Lambda(X_{1/\epsilon}) \sim \epsilon^{-2}, \quad \partial_l^a \Lambda(X_{1/\epsilon}) \sim \epsilon^{-(2+n)}$$

and since $\partial_l \Psi$ will not carry any powers of $\epsilon$ the conjugate operator has the form

$$\hat{\Psi}(X_+) = f(L) \hat{\Psi}(X_+).$$

Therefore, the density matrix in these coordinates is

$$tr(\rho \cdots) = \frac{\langle \hat{\Psi}(X_-) \hat{\Psi}(X_+) \cdots \rangle}{\langle \langle \hat{\Psi}(X_-) \hat{\Psi}(X_+) \rangle \rangle}.$$
The operator $\tilde{\Psi}\Psi$ simplifies further in the thermodynamic limit $L \gg 1$. To see this, apply $\partial_i$ to the OPE

$$\frac{\partial}{\partial X_i} \Psi(X)\Psi(X_+) = \sum_p \frac{\partial}{\partial X_i} ((X - X_+)^{h_p - 2h}\mathcal{O}(X_+))$$

$$= (-2h)(X - X_+)^{h_p - 2h - 1}\mathcal{O}(X_+) + O(1/h), \quad (B.16)$$

where we have used $h \gg h_p$. Now, notice that

$$\Lambda(X_\epsilon) \sim L,$n

$$\partial^n\Lambda(X_\epsilon) \sim L^{n+1}.$$n

However when $\partial_i$ acts on $\Lambda(X_\epsilon)$ we get a factor of $hL$. Therefore,

$$\tilde{\Psi}(X_-)\Psi(X_+) = \Lambda(X_-)^{-(h+2)} ((\partial_i\partial_i)^2\Psi)(X_-)\Psi(X_+).$$n

Finally, the expression for the density matrix in these coordinates becomes

$$\sum_p (2/L)^{h_p} C_p^\psi (\partial_i\partial_i)^l (z - X_+)^{-2h}\big|_{z \to X_-} \mathcal{O}_p \cdots$$

$$\sum_p (2/L)^{h_p} C_p^\psi (\partial_i\partial_i)^l (z - X_+)^{-2h}\big|_{z \to X_-}$$

which is the same as the density matrix in the primary state from which $|E_a\rangle$ descends. As a result, we find

$$\|\rho^{(a,m)}_B - \rho^a_B\| \sim O \left( (E_a L)^{-1} \right). \quad (B.17)$$

**Appendix C. Subsystem ETH implies local ETH**

Consider an observable $A = \sum_a a|a\rangle\langle a|$ and the operator $\rho - \rho_T$ in this basis:

$$\rho - \rho_T = \sum_{ab} c_{ab}|a\rangle\langle b| \quad (C.1)$$

The expectation value of $A$ is

$$\text{Tr}((\rho - \rho_T)A) = \sum_a c_{aa} a \leq \sum_a |c_{aa}|$$

$$\leq \left( \sum_a |c_{aa}| \right)^{1/2} \left( \sum_a |c_{aa}|a^2 \right)^{1/2}$$

$$\leq \|\Phi[\rho - \rho_T]\|^1/2 \left( \sum_a |c_{aa}|a^2 \right)^{1/2}$$

$$\leq \|\rho - \rho_T\|^{1/2} \left( \sum_a |c_{aa}|a^2 \right)^{1/2}. \quad (C.2)$$
where $\Phi[\rho] = \sum_a |\rho_{aa}| a\langle a|$ is the map that decoheres $\rho$ in the basis of $A$, and we have used the fact that this map decreases the trace distance of operators. Note that $c_{aa}$ are real, but could have either sign. Denote by $V$ and $W$ the projectors that project to $c_{aa}$ that are, respectively, positive and negative. Then,

$$\sum_a |c_{aa}|^2 \leq \mathrm{Tr}((\rho + \rho_T)A^2), \quad (C.3)$$

where we used the fact that for a projector $V$: $\mathrm{Tr}(\rho V A^2) \leq \mathrm{Tr}(\rho A^2)$. Putting this back into (C.2) we find

$$\mathrm{Tr}((\rho - \rho_T)A) \leq ||\rho - \rho_T||^{1/2} \mathrm{Tr}((\rho + \rho_T)A^2)^{1/2}. \quad (C.4)$$

Repeating the argument above for $\tilde{\rho}_B^{a,b,\alpha} = \frac{1}{2}(\tilde{\rho}_B(E_a) + \tilde{\rho}_B(E_a))$ we find

$$\mathrm{Tr}((e^{i\alpha}\sigma_{ab} + e^{-i\alpha}\sigma_{ba})A) \leq \|e^{i\alpha}\sigma_{ab} + e^{-i\alpha}\sigma_{ba}\|^{1/2}$$

$$\times \mathrm{Tr}\left(\tilde{\rho}_B^{a,b,\alpha} + \frac{1}{2}(\tilde{\rho}_B(E_a) + \tilde{\rho}_B(E_a))A^2\right)^{1/2}, \quad (C.5)$$

for all $\alpha$. By adding and subtracting the inequality above for values of $\alpha = 0$ and $\alpha = \pi$ we find $\mathrm{Tr}(A\sigma_{ab}) = e^{-O(S(E))}$.

**Appendix D. Relative entropy replica trick**

Similar to entanglement entropy there is a replica trick that computes the relative entropy of arbitrary states in quantum field theory [18, 22]. The relative entropy is found from the analytic continuation in $n$ of

$$S(\tilde{\rho}_B^n \| \tilde{\rho}_B) = \lim_{n \to 1} \frac{1}{n-1} \log \frac{\mathrm{Tr}((\tilde{\rho}_B^n)\mathrm{Tr}(\tilde{\rho}_B)^{n-1})}{\mathrm{Tr}(\tilde{\rho}_B^n \tilde{\rho}_B^{n-1}) \mathrm{Tr}(\tilde{\rho}_B)^{n-1}}.$$

See figure C1. Now inserting (12) into the above expression we find that
\[ S(\hat{\rho}_B^a \parallel \hat{\rho}_B) = \partial_n \log \left[ \frac{1 + nA_n^{(1)} + nA_n^{(2)} + \cdots}{1 + A_n^{(1)}} \right] \]

\[ A_n^{(1)} = \frac{\text{Tr}(R_B^a \hat{\rho}_B^a \cdots \hat{\rho}_B^a)}{\text{Tr}(\hat{\rho}_B^a)} \]

\[ A_n^{(2)} = \sum_{i=1}^{n-2} \frac{\text{Tr} \left( R_B^a \hat{\rho}_B^a R_B^a \hat{\rho}_B^a \cdots R_B^a \hat{\rho}_B^a \right)}{\text{Tr}(\hat{\rho}_B^a)} \]

\[ A_n^{(k)} = \sum_{m_1 + \cdots + m_k = n-k} \frac{\text{Tr} \left( R_B^a \hat{\rho}_B^a \cdots R_B^a \hat{\rho}_B^a \right)}{\text{Tr}(\hat{\rho}_B^a)} \]

Consider the term \( A^{(k)} \). From the definition (13) we find

\[ A_n^{(k)} = \sum_{m_1 + \cdots + m_k = n-k} \sum (\Delta_{a_1} \cdots \Delta_{a_k} f_{p_1 \cdots p_k} R_{p_1 \cdots p_k}^a) \]

\[ f_{p_1 \cdots p_k}^{m_1 \cdots m_k} = \frac{\text{Tr} \left( O_{p_1} \hat{\rho}_B^a \cdots O_{p_k} \hat{\rho}_B^a \right)}{\text{Tr}(\hat{\rho}_B^a)} \]

It is clear that \( \Delta_{a_1} \cdots \Delta_{a_k} = 0(E) \) where \( \eta_a = \sup_p \Delta_{a} = e^{-O(S(E))} \). Therefore, if we argue that \( f_{p_1 \cdots p_k}^{m_1 \cdots m_k} \) is not entropically suppressed we have argued \( A^{(k)} = O(\eta_a^k) \). Using (13) we have

\[ f_{p_1 \cdots p_k}^{m_1 \cdots m_k} = \frac{\sum_{q_1 \cdots q_k} O_{q_1}(E) \cdots O_{q_k}(E) \langle O_{p_1} \hat{\rho}_B^a \cdots O_{p_k} \hat{\rho}_B^a \rangle R_{q_1 \cdots q_k}^a}{\sum_{q_1 \cdots q_k} O_{q_1}(E) \cdots O_{q_k}(E) \langle O_{p_1} \cdots O_{p_k} \rangle R_{q_1 \cdots q_k}^a} \]

Neither the correlators, nor \( O_p(\rho) \) have any entropic suppressions. Therefore, as long as the correlators on the \( n \)-sheeted manifold are finite, the sums are convergent and \( A^{(k)} = O(\eta_a^k) \). Therefore, expanding (D.1) in \( \eta_a \) we have

\[ S(\hat{\rho}_B^a \parallel \hat{\rho}_B) = \partial_n \left( n + A_n^{(1)} \right) n \rightarrow 1 + O(\eta_a^2) \]

\[ = O(\eta_a^2) = e^{-O(S(E))}. \]

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Eigenstate thermalization hypothesis in conformal field theory

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