Supergravity and the Poincaré group

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An action for (3+1)-dimensional supergravity genuinely invariant under the Poincaré supergroup is proposed. The construction of the action is carried out considering a bosonic lagrangian invariant under both local Lorentz rotations and local Poincaré translations as well as under diffeomorphism, and therefore the Poincaré algebra closes off-shell.

Since the lagrangian is invariant under the Poincaré supergroup, the supersymmetry algebra closes off-shell without the need of auxiliary fields.

I. INTRODUCTION

The construction of a supergravity theory without auxiliary fields in (3+1)-dimensions has remained as an interesting open problem. Recently it has been shown that the three and five dimensional supergravities studied by Achucarro-Townsend \cite{achucarro} and by Chamseddine \cite{chamseddine} respectively, as well as the higher dimensional theory studied by Bañados-Troncoso-Zanelli \cite{bananos} are Chern-Simons theories. Their supersymmetry transformations can be written in the form

\[ \delta A = \nabla \lambda \]  

and therefore the supersymmetry algebra closes off-shell without the need of auxiliary fields.

It is the purpose of this paper to show that it is also possible to construct a four dimensional supergravity without auxiliary fields provided one chooses the bosonic lagrangian in an appropriate way. In fact, the correct lagrangian for the bosonic sector is the Hilbert lagrangian constructed with help of the one-form vierbein defined by Stelle \cite{stelle}, and Grignani-Nardelli \cite{grignani}. This vierbein, also called solder form \cite{stelle}, \cite{grignani} was considered as a smooth map between the tangent space to the space-time manifold at \( x^\mu \), and the tangent space to the internal AdS space at the point whose AdS coordinates are \( \zeta (x) \), as the point \( P \) ranges over the whole manifold \( M \). The fig.1 of ref. \cite{grignani} illustrates that such vierbein \( V^a_\mu (x) \) is the matrix of the map between the tangent space \( T_x(M) \) to the space-time manifold at \( x^\mu \), and the tangent space \( T_{\zeta(x)} \{ G/H \} \) to the internal AdS space \( \{ G/H \} \) at the point \( \zeta^a(x) \), whose explicit form is given by the eq.(3.19) of ref. \cite{grignani}.

Taking the limit \( m \to 0 \) in such eq.(3.19) we obtain \( V^a_\mu (x) = D_\mu \xi^a + e^a_\mu \) which is the map between the tangent space \( T_x(M) \) to the space-time manifold at \( x^\mu \) and the tangent space \( T_{\zeta(x)} \{ ISO (3,1)/SO (3,1) \} \) to the internal Poincaré space \( \{ ISO (3,1)/SO (3,1) \} \) at the point \( \zeta^a(x) \). The same result was obtained in ref. \cite{grignani} by gauging the action of a free particle defined in the internal Minkowski space.

II. GRAVITY AND THE POINCARÉ GROUP

In this section we shall review some aspects of the torsion-free condition in gravity. The main point of this section is to display the differences in the invariances of the Hilbert action when different definitions of vierbein are used.

A. The torsion-free condition in general relativity

The generators of the Poincaré group \( P_a \) and \( J_{ab} \) satisfy the Lie algebra,

\[ [P_a, P_b] = 0; \]

\[ [J_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a; \]

\[ [J_{ab}, J_{cd}] = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc}. \]  

Here the operators carry Lorentz indices not related to coordinate transformations. The Yang-Mills connection for this group is given by

\[ A = A^A T_A = e^a_\mu P_a + \frac{1}{2} \omega^{ab} J_{ab}. \]  

Using the algebra \cite{grignani} and the general form for the gauge transformations on \( A \)

\[ \delta A = \nabla \lambda = d\lambda + [A, \lambda] \]
with
\[ \lambda = \rho^a P_a + \frac{1}{2} \kappa^{ab} J_{ab}, \quad (5) \]
we obtain that \( e^a \) and \( \omega^{ab} \), under Poincaré translations, transform as
\[ \delta e^a = D \rho^a; \quad \delta \omega^{ab} = 0, \quad (6) \]
and under Lorentz rotations, as
\[ \delta e^a = \kappa_b^a e^b; \quad \delta \omega^{ab} = -D \kappa^{ab}, \quad (7) \]
where \( D \) is the covariant derivative in the spin connection \( \omega^{ab} \). The corresponding curvature is
\[ F = F^A T_A = dA + AA \]
where
\[ T^a = D e^a = d e^a + \omega^a_b e^b \quad (9) \]
is the torsion 1-form, and
\[ R^{ab} = d \omega^{ab} + \omega^c_e \omega^{eb} \quad (10) \]
is the curvature 2-form.

The Hilbert action
\[ S_{EH} = \int \varepsilon_{abcd} R^{ab} e^c e^d, \quad (11) \]
is invariant under diffeomorphism and under Lorentz rotations, but is not invariant under Poincaré translations. In fact
\[ \delta S_{EH} = 2 \int \varepsilon_{abcd} R^{ab} e^c \delta e^d \]
\[ = 2 \int \varepsilon_{abcd} R^{ab} T^c \rho^d + \text{surf. term} \quad (12) \]
where we see that the invariance of the action requires imposing the torsion free condition
\[ T^a = D e^a = d e^a + \omega^a_b e^b = 0 \quad (13) \]
which has effects on the algebra of local Poincaré transformations. If we impose this condition, then the local Poincaré translations take the form of a local change of coordinates, as we can see from the respective transformation law
\[ \delta_{\text{loc}} e^a = D \rho^a + \kappa_b^a e^b \quad (14) \]
\[ \delta_{\text{loc}} e^a = D \rho^a + \rho \cdot \omega^a_b e^b + \rho \cdot T^a. \quad (15) \]

The condition \( T^a = 0 \) permits replacing local Poincaré translations by a local change of coordinates which acts together with the local Lorentz transformations on the gauge fields as:
\[ \delta e^a = D \rho^a + \kappa_b^a e^b \]
\[ \delta \omega^{ab} = -D \kappa^{ab} + \varepsilon \cdot R^{ab}. \quad (16) \]

The commutator of two local Poincaré translations can now be computed and gives
\[ [\delta (\rho_2), \delta (\rho_1)] = \delta (\kappa) \quad (17) \]
with \( \kappa^{ab} = \rho^1_b \rho^2_a R^{ab} \). Furthermore one finds
\[ [\delta (\kappa^{ab}), \delta (\rho^c)] = \delta (\rho^d) \]
with \( \rho^d = \rho_b \kappa^{ba} \quad (18) \)
and
\[ [\delta (\kappa_2), \delta (\kappa_1)] = \delta (\kappa_3) \] with \( \kappa_3 = [\kappa_1, \kappa_2] \). \quad (19)

This means that, for non-vanishing \( R^{ab} \), the local Poincaré translations no longer commute, but their commutator is a local Lorentz transformation proportional to the Riemann curvature. The rest of the algebra is unchanged. Thus an effect of the torsion-free condition is that the Poincaré algebra only closes on-shell, but does not close off-shell.

Another consequence of the torsion-free condition is that it is an equation of motion of the action, which implies that the invariance of the action under diffeomorphisms does not result from the transformation properties of the fields alone, but it is a property of their dynamics as well. The problem stems from the identification between diffeomorphism, which is a genuine invariance of the action, and local Poincaré translation which is not a genuine invariance.

The torsion-free condition breaks local translation invariance in Lorentz space, and uniquely identifies the origin of the local Lorentz frame with the space-time point at which it is constructed.

**B. Gravity invariant under the Poincaré group**

Now we show that the formalism proposed by Stelle and Grignani-Nardelli (SGN) leads to a formulation of general relativity where Hilbert’s action is invariant both under local Poincaré translations and under local Lorentz transformations as well as under diffeomorphism and therefore the Poincaré algebra closes off-shell.

The key ingredients of the (SGN) formalism are the so-called Poincare coordinates \( \xi^a (x) \) which behave as vectors under \( ISO (3, 1) \) and are involved in the definition of the 1-form vierbein \( V^a \), which is not identified with the component \( e^a \) of the gauge potential, but is given by
\[ V^a = D \xi^a + e^a = d \xi^a + \omega^a_b \xi^b + e^a \quad (20) \]
Since $\zeta^a$, $e^a$, $\omega^{ab}$ under local Poincare translations change as

$$\delta\zeta^a = -\rho^a, \quad \delta e^a = D\rho^a, \quad \delta\omega^{ab} = 0; \quad (21)$$

and under local Lorentz rotations change as

$$\delta\zeta^a = \kappa^a \zeta^b, \quad \delta e^a = \kappa^a e^b, \quad \delta\omega^{ab} = -D\kappa^{ab}; \quad (22)$$

we have that the vierbein $V^a$ is invariant under local Poincare translations

$$\delta V^a = 0 \quad (23)$$

and, under local Lorentz rotations, transforms as

$$\delta V^a = \kappa^a V^b. \quad (24)$$

The space-time metric is postulated to be

$$g_{\mu\nu} = \eta_{ab} V^a \eta_{\nu} \quad (25)$$

with $\eta_{ab} = (-1, 1, 1, 1)$. Thus the corresponding curvature is given by $\mathbf{[3]}$, but now $\mathbf{[3]}$ does not correspond to the space-time torsion because the vierbein is not given by $e^a$. The space-time torsion $T^a$ is given by

$$\mathbf{[3]} \quad (26)$$

The Hilbert action can be rewritten as

$$S_{EH} = \int \varepsilon_{abcd} V^a V^b R^{cd} \quad (27)$$

which is invariant under general coordinate transformations, under local Lorentz rotations, as well as under local Poincare translation. In fact

$$\delta S_{EH} = \int \varepsilon_{abcd} \delta (R^{ab} V^c V^d) \quad (28)$$

$$\delta S_{EH} = 2 \int \varepsilon_{abcd} R^{ab} V^c \delta V^d = 0. \quad (29)$$

Thus the action is genuinely invariant under the Poincaré group without imposing a torsion-free condition.

The variations of the action with respect to $\zeta^a$, $e^a$, $\omega^{ab}$ lead to the following equations:

$$\varepsilon_{abcd} T^b R^{cd} = 0 \quad (30)$$

$$\varepsilon_{abcd} V^b R^{cd} = 0 \quad (31)$$

$$\varepsilon_{[aece]} \zeta^b V^c R^{ed} + \varepsilon_{abcd} V^c T^d = 0 \quad (32)$$

which reproduce the correct Einstein equations:

$$T^a = DV^a = 0$$

$$\varepsilon_{abcd} V^b R^{cd} = 0.$$ 

The commutator of two local Poincaré translations is given by

$$[\delta(\rho_2), \delta(\rho_1)] = 0 \quad (33)$$

i.e. the local Poincaré translations now commute. The rest of the algebra is unchanged. Thus the Poincaré algebra closes off-shell. This fact has deep consequences in supergravity.

### III. Supergravity in 3 + 1 without auxiliary fields

In this section we shall review some aspects of the torsion-free condition in supergravity. The main point of this section is to show that the (SGN) formalism permits constructing a supergravity invariant under local Lorentz rotation and under local Poincare translation as well as under local supersymmetry transformations. This means that the superpoincare algebra closes off shell without the need of any auxiliary fields.

#### A. The torsion-free condition in $N = 1$ supergravity

$D = 3 + 1$, $N = 1$ supergravity is based on the superpoincare algebra

$$[P_a, P_b] = 0$$

$$[J_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a$$

$$[J_{ab}, J_{cd}] = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc}$$

$$[J_{ab}, Q^a] = -\frac{1}{2} (\gamma_{ab})^{[\alpha\beta} Q_{\beta]}$$

$$[P_a, Q_{\beta}] = 0$$

$$[Q^a, Q_{\beta}] = \frac{1}{2} (\gamma^a)^\alpha P_\alpha. \quad (34)$$

The connection for this group is given by

$$A = A^\lambda T_\lambda = e^a P_a + \frac{1}{2} \omega^{ab} J_{ab} + \overline{Q} \psi. \quad (35)$$

Using the algebra (34) and the general form for gauge transformations on $A$

$$\delta A = D\lambda = d\lambda + [A, \lambda] \quad (36)$$

with

$$\lambda = \rho^a P_a + \frac{1}{2} \kappa^{ab} J_{ab} + \overline{Q} \varepsilon \quad (37)$$

we obtain that $e^a$, $\omega^{ab}$, and $\psi$ under Poincaré translations, transform as

$$\delta e^a = D\rho^a; \quad \delta \omega^{ab} = 0; \quad \delta \psi = 0; \quad (38)$$

under Lorentz rotations, as

$$\delta e^a = \kappa^a e^b; \quad \delta \omega^{ab} = -D\kappa^{ab}; \quad \delta \psi = \frac{1}{4} \kappa^{ab} \kappa_{ab} \psi; \quad (39)$$

and under supersymmetry transformations, as
\[ \delta \varepsilon^a = \frac{1}{2} \gamma^a \psi; \quad \delta \omega^{ab} = 0; \quad \delta \psi = D \varepsilon. \] 

(40)

The consistency of the propagation of the massless \( \text{Rarita-Schwinger field} \) in a classical gravitational back-ground field is proved by contracting its field equation

\[ \gamma_5 \varepsilon^a \gamma_\alpha D \psi = 0 \] 

(41)

by the covariante derivative \( D \),

\[ D (\gamma_5 \varepsilon^a \gamma_\alpha D \psi) = 0 \]

\[ \gamma_5 \gamma_\alpha T^a D \psi + \gamma_5 \varepsilon^a \gamma_\alpha D D \psi = 0. \] 

(42)

The Einstein equation and the Bianchi identity reduce \[ (42) \] to an identity.

Equation \[ (11) \] does not take into account the back reaction of the spin 3/2 field on the gravitational field. It turns out that this back reaction of the spin-3/2 field on the gravitational and on itself can be taken into account by a generalizing of Weyl’s lemma \[ [5] \]:

\[ D e^a_\mu = \partial_\mu e^a_\nu - \omega^a_\mu e^b_\nu - \frac{1}{4} \bar{\psi}_\mu \gamma^a \psi_\nu - \Gamma^\lambda_\mu_\nu e^a_\lambda = 0, \] 

(43)

which implies that the corresponding torsion is given by

\[ \hat{T}^a = T^a - \frac{1}{2} \psi \gamma^a \psi. \] 

(44)

Supergravity is the theory of the gravitational field interacting with a spin 3/2 Rarita Schwinger field \[ \[3, 3\] \]. In the simplest case there is just one spin 3/2 Majorana fermion, usually called the gravitino. The corresponding action

\[ S = \int \varepsilon_{abcd} e^a e^b R^{cd} + 4 \psi \gamma_5 \varepsilon^a \gamma_\alpha D \psi \] 

(45)

is invariant under diffeomorphism, under local Lorentz rotations and local supersymmetry transformations, but it is not invariant under Poincaré translations. In fact, under local Poincaré translations

\[ \delta S = 2 \int \varepsilon_{abcd} R^{ab} \left( T^c - \frac{1}{2} \gamma^c \psi \right) \rho^d + \text{surf. term.} \] 

(46)

\[ \delta S = 2 \int \varepsilon_{abcd} R^{ab} \hat{T}^c \rho^d + \text{surf. term.} \] 

(47)

The invariance of the action requires the vanishing of the torsion

\[ \hat{T}^a = 0 \] 

(48)

An effect of the supertorsion-free condition on the local Poincaré superalgebra is that all commutators on \( e^a, \psi \) close except the commutator of two local supersymmetry transformations on the gravitino. For this commutator on the vierbein one finds

\[ \left[ \delta (\varepsilon_1), \delta (\varepsilon_2) \right] e^a = \frac{1}{2} \gamma_5 \varepsilon_1 D \varepsilon_2 - \frac{1}{2} \gamma_5 \varepsilon_2 D \varepsilon_1 = \frac{1}{2} D (\varepsilon_1 \gamma^a \varepsilon_2). \] 

(49)

with \( \rho^a = \frac{1}{2} \gamma_5 \varepsilon_1 \varepsilon_2 \), we can write

\[ \left[ \delta (\varepsilon_1), \delta (\varepsilon_2) \right] e^a = D \rho^a. \] 

(50)

This means that, in the absence of the torsion-free condition, the commutator of two local supersymmetry transformations on the vierbein is a local Poincaré translation. However, the action is invariant by construction under general coordinate transformations and not under local Poincaré translation. The general coordinate transformation and the local Poincaré translation can be identified if we impose the torsion-free condition: since \( \rho^a = \rho^e e^a \), we can write

\[ D \rho^a = (\partial_\mu \rho^e) e^a_\mu + \rho^e \left( \partial_\nu e^a_\mu \right) + \frac{1}{2} \rho^e \left( \bar{\psi}_\mu \gamma^a \psi_\nu \right) + \rho^e \left( \omega^a_{\mu e} \right) \] 

(51)

This means that, if \( T_{\mu \nu} = 0 \), then the following commutator is valid:

\[ \left[ \delta Q (\varepsilon_1), \delta Q (\varepsilon_2) \right] = \delta_{\text{GCT}} (\rho^e) + \delta_{\text{LLT}} \left( \rho^e \omega^a_{\mu e} \right) + \delta Q \left( \rho^e \bar{\psi}_\mu \right) \] 

(52)

where we can see that \( P \in \{ Q, Q \} = P \), i.e. local Poincaré translation, is replaced by general coordinate transformations besides two other gauge symmetries. The structure constants defined by this result are field-dependent \[ [10] \], which is a property of supergravity not present in Yang-Mills Theory.

The commutator of two local supersymmetry transformations on the gravitino is given by

\[ \left[ \delta (\varepsilon_1), \delta (\varepsilon_2) \right] \psi = \frac{1}{2} (\sigma_{ab} \varepsilon_2) \left[ \delta (\varepsilon_1) \omega^{ab} \right] + \frac{1}{2} (\sigma_{ab} \varepsilon_1) \left[ \delta (\varepsilon_2) \omega^{ab} \right]. \] 

(53)

The condition \( \hat{T}^a = 0 \) leads to \( \omega^{ab} = \omega^{ab} (e, \psi) \) which implies that the connection is no longer an independent variable and its variation \( \delta (\varepsilon) \omega^{ab} \) is given in terms of \( \delta (\varepsilon) e^a \) and \( \delta (\varepsilon) \psi \). Introducing \( \delta (\varepsilon) \omega^{ab} (e, \psi) \) into \[ (53) \] we see that, without the auxiliary fields, the gauge algebra does not close, as shows the eq. \[ (10) \] of ref. \[ [10] \].
Therefore the condition $\hat{T}^a = 0$ not only breaks local Poincaré invariance, but also the supersymmetry transformations. We show that, if we use the (SGN) formalism, the gauge algebra closes without the auxiliary fields because it is not necessary to impose the torsion free condition.

**B. Supergravity invariant under the Poincaré group**

Analogous to the pure gravity case, the action for supergravity in $3 + 1$ dimensions is not invariant under local Poincaré translations. The invariance of the action requires, in accord with 1.5 formalism, the vanishing of the torsion $\hat{T}^a$, which implies that the connection is no longer an independent variable. Rather, its variation is given in terms of $\delta e^a$ and $\delta \psi$, and differs from the one dictated by group theory. As a consequence the supersymmetry algebra, acting on the spinor field, closes off-shell only with auxiliary fields. How to construct supergravity in four dimensions without auxiliary fields is an interesting open problem. Now we will show that this construction is possible.

The massless Rarita-Schwinger field is a spin $3/2$ field that can be described by a Majorana vector-spinor $\psi_\mu$ that satisfies the equation

$$\gamma_5 V^a \gamma_a d\psi = 0$$

where now,

$$V^a = D\xi^a + e^a_\mu.$$  \hspace{1cm} (55)

The coupling of this field to a gravitational field satisfying the free Einstein equations is achieved by minimal coupling. According to this prescription, it is possible to generalize the free spin-3/2 equation consistently to include interaction with a gravitational background field, as

$$\gamma_5 V^a \gamma_a D\psi = 0.$$  \hspace{1cm} (56)

The consistency of the propagation of a spin $3/2$ particle in a classical gravitational background field is proved by contracting (56) with another derivative $D$:

$$D (\gamma_5 V^a \gamma_a D\psi) = 0$$

$$\gamma_5 \gamma_a D^a \psi + \gamma_5 V^a \gamma_a DD \psi = 0.$$  \hspace{1cm} (57)

The Einstein equation and the Bianchi identity reduce [54] to an identity. Of course, using the free Einstein equation implies that we have not taken into account the back reaction of the spin $3/2$ field on the gravitational field. In that sense the gravitational field here is just a fixed classical background field. The more general situation, in which both the gravitational field and the Rarita-Schwinger field are dynamical, is the situation encountered in supergravity.

Equation (54) does not take into account the back reaction of the spin $3/2$ field on the gravitational field; namely, the spin-3/2 field itself can act as a source for the gravitational field. Since the spin $3/2$ field is coupled to the gravitational field through the covariant derivative $D$, this also induces a coupling of the gravitino field with itself. It turns out that this back reaction of the spin-3/2 field on the gravitational and on itself can be taken into account by generalizing Weyl’s lemma [7]:

$$D_\nu V^a_\mu = \partial_\nu V^a_\mu - \omega^a_{\nu\sigma} V^\sigma_\mu - \frac{1}{4} \psi^a_\mu \gamma^\nu_\lambda V^\lambda_\sigma = 0.$$

This implies that the corresponding torsion is given by

$$\hat{T}^a = T^a - \frac{1}{2} \psi \gamma^a \psi.$$  \hspace{1cm} (59)

Within the (SGN) formalism the action for supergravity can be rewritten as

$$S = \int \varepsilon_{abcd} V^a V^b R^d + 4\sqrt{-g} \gamma_5 V^a \gamma_a D\psi$$

which is invariant under local Lorentz rotations

$$\delta V^a = \kappa^a b V^b; \hspace{1cm} \delta \psi = \frac{1}{4} \kappa^a c_\rho \gamma_\rho \psi;$$

i.e. under $\delta \omega_{ab} = -D\kappa_{ab}; \delta e^a = \kappa^a \epsilon^b; \delta \psi = \frac{1}{4} \kappa_{ab} \gamma_\rho \psi; \delta \xi^a = \kappa^a b \xi^b$; under local Poincaré translations

$$\delta V^a = 0; \hspace{1cm} \delta \psi = 0;$$

i.e. under $\delta \omega_{ab} = 0; \delta e^a = D\rho^a; \delta \psi = 0; \delta \xi^a = -\rho^a$; and under local supersymmetry transformations

$$\delta V^a = i\varepsilon^a \gamma^b \psi; \hspace{1cm} \delta \psi = D\varepsilon;$$

i.e. under $\delta \omega_{ab} = 0; \delta e^a = i\varepsilon^a \gamma^b \psi; \delta \psi = D\varepsilon; \delta \xi^a = 0$.

This means that the action (54) is invariant without the need to impose a torsion-free condition. We can see that, in the local Poincaré superalgebra, all commutators on $e^a, \psi$ close including the commutators of two local supersymmetry transformations on the vierbein and on the gravitino. In fact, for this commutator on the vierbein one finds:

$$[\delta (\varepsilon_1), \delta (\varepsilon_2)] e^a = D\rho^a$$

where $\rho^a = \varepsilon_2 \gamma^a \varepsilon_1$, i.e. the commutator of two local supersymmetry transformations on the vierbein is a local Poincaré translation.

The commutator of two local supersymmetry transformations on the gravitino is given by

$$[\delta (\varepsilon_1), \delta (\varepsilon_2)] \psi = \frac{1}{2} (\sigma_{ab} \varepsilon_2) [\delta (\varepsilon_1) \omega_{ab}]$$
\[-\frac{1}{2} (\sigma_{ab} \varepsilon_1) \left[ \delta (\varepsilon_2) \omega^{ab} \right] = 0 \]  

(65)

because now the connection is an independent variable i.e. $\delta (\varepsilon) \omega^{ab} = 0$ in accord with the group theory. This proves that, if we use the (SGN) formalism, the gauge algebra closes without the auxiliary fields, because it is not necessary to impose the torsion-free condition.

In the context of a genuinely first order formalism, i.e. where the spin connection $\omega^{ab}_\mu$ transforms independently of the graviton field $e^a_{\mu}$, of the gravitino field $\psi$, and of the Poincare field $\xi$, the field equations can be obtained by varying (60) with respect to $\psi_\mu$, $e^a_{\mu}$ and with respect to $\omega^{ab}_{\mu}$

$$\gamma_5 V^a \gamma_\alpha D\psi = 0,$$

(66)

$$\varepsilon_{abcd} V^b R^{cd} + 2 \overline{\psi} \gamma_5 \gamma_\alpha D\psi = 0,$$

(67)

$$\varepsilon_{abcd} V^c \delta^d + \varepsilon_{[acde] \xi_0} V^e R^{cd} + 2 \xi_5 \bar{\psi} \gamma_5 \gamma_\alpha D\psi = 0.$$  

(68)

IV. COMMENTS

The off-shell closure of the gauge algebra in (3 + 1)-dimensional supergravity is a problem that was studied long ago by Kaku, Townsend, van Nieuwenhuizen [11], [10]. They found that the action for conformal supergravity is invariant under local K-gauges if the $P$-curvature vanishes $T^a_{\mu\nu} = 0$ and if the $Q$-curvature is chiral dual $R_{\mu\nu}(Q) + 2 \varepsilon_\sigma^{\mu\nu} R_{\rho\sigma}(Q) \gamma_5 = 0$. Local Q-invariance follows if one imposes one more constraint $R_{\mu\nu}(Q) \sigma^{\mu\nu} = 0$. The torsion free condition leads to $\omega^{ab}_\mu = \omega^{ab}_\mu (e, \psi)$ and the duality constraint leads to $\phi_\mu = \phi_\mu (K)$. The transformation of $\omega^{ab}_\mu$ and $\phi_\mu$ obtained by application of the chain rule differs from the group transformation. Using the cyclic identity $\varepsilon^{\mu\nu\rho\sigma} \gamma_\rho R_{\rho\sigma}(Q) = 0$ one obtains the transformation of $\omega^{ab}_\mu$: $\delta \omega^{ab}_\mu = \frac{1}{2} [R^{ab}(Q) \gamma_\mu \varepsilon]$. This transformation permits showing that in conformal supergravity the gauge algebra closes without the need to use the field equations.

In conformal supergravity, as in usual supergravity, the Poincaré translation is not a symmetry of the action, but one must consider instead general coordinate transformations. The reason why the Poincaré algebra does not close on $\psi$ is that the cyclic identity $\varepsilon^{\mu\nu\rho\sigma} \gamma_\rho R_{\rho\sigma}(Q) = 0$ is not available to cast $\delta \omega^{ab}_\mu$ into the simple form $\delta \omega^{ab}_\mu$ that appears in conformal supergravity.

We have shown in this work that the successful formalism used by Stelle [4] and Grignani-Nardelli [5] to construct an action for (3 + 1)-dimensional gravity invariant under the Poincaré group can be generalized to supergravity in (3 + 1)-dimensions. The extension to other even dimensions remains an open problem. The main result of this paper is that we have shown that the (SGN) formalism permits constructing a supergravity invariant under local Lorentz rotation and under local Poincaré translation as well as under local supersymmetry transformations, which means that the gauge algebra closes off shell without the need of any auxiliary fields.

The action reduces to the usual action for supergravity if we choose $\xi^a = 0$, i.e. if we identify uniquely the origin of the local Lorentz frame with the space-time point at which it is constructed.

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