On the separation of the roots of the generalized Fibonacci polynomial

Jonathan García * Carlos A. Gómez † Florian Luca ‡

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Abstract

In this paper we prove some separation results for the roots of the generalized Fibonacci polynomials and their absolute values.

Key words and phrases: $k$–generalized Fibonacci polynomial; Polynomial root separation; Distribution of roots of polynomials.

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1 Introduction

A sequence $(u_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C}$ is a linear recurrence sequence of order $k \in \mathbb{Z}^+$ if it satisfies the recurrence relation

$$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \cdots + a_k u_n \quad \text{for all } n \geq 0$$

with coefficients $a_1, \ldots, a_k \in \mathbb{C}$ and $a_k \neq 0$. We assume that $k$ is minimal with the above property. Such a sequence $(u_n)_{n \in \mathbb{Z}}$ has an associated characteristic polynomial given by $f(X) = X^k - a_1 X^{k-1} - \cdots - a_k$. Let $\alpha_1, \ldots, \alpha_s \in \mathbb{C}$ be the distinct roots of $\Psi_k(X)$. From the theory of linear recurrence sequences (see [7, Theorem C.1]) there are complex single–variable polynomials $A_1, A_2, \ldots, A_s$, uniquely determined by the initial values $u_0, \ldots, u_{k-1}$, such that for all $n \in \mathbb{Z}$

$$u_n = A_1(n)\alpha_1^n + A_2(n)\alpha_2^n + \cdots + A_s(n)\alpha_s^n.$$
From here we see the importance of knowing some properties of the roots of \( f(X) \) if one wants to deduce arithmetic properties for the members of \((u_n)_{n \in \mathbb{Z}}\).

A special and recently studied case is related to the roots of the characteristic polynomial of the \( k \)-generalized Fibonacci sequence. Let \( f_k(X) \) be the generalized Fibonacci polynomial given by

\[
f_k(X) = X^k - X^{k-1} - \cdots - X - 1.
\]

This polynomial has exactly one root outside the unit disk. It is real and larger than 1 and we denote it by \( \alpha_k \) (see [8]). Let \( \rho_j e^{i\theta_j} \), \( \theta_j \in (0, \pi] \) for \( j = 1, \ldots, K \) be roots of \( f_k(X) \) which are inside the upper half \( \text{Im}(z) \geq 0 \) of the disk \( |z| \leq 1 \). Then all roots of \( f_k(X) \) are

\[
\rho_1 e^{i\theta_1}, \rho_1 e^{-i\theta_1}, \ldots, \rho_K e^{i\theta_K}, \rho_K e^{-i\theta_K}, \alpha_k, \quad \text{when} \quad k \text{ is odd}
\]

and

\[
\rho_1 e^{i\theta_1}, \rho_1 e^{-i\theta_1}, \ldots, -\rho_K, \alpha_k, \quad \text{when} \quad k \text{ is even}.
\]

The following bounds on \( \rho_j \) for \( j = 1, \ldots, K \) appear in [3].

**Lemma 1.** We have

\[
1 - \frac{\log 3}{k} < \rho_j < 1 - \frac{1}{2^8 k^3} \quad \text{for} \quad j = 1, \ldots, K.
\]

The distribution of the arguments \( \theta_j \) is also understood. The following appears in [11], responding to a conjecture proposed in [3].

**Lemma 2.** Let \( \alpha_j = \rho_j e^{i\theta_j} \) with \( \theta_j \in [0, 2\pi) \) for \( j = 1, \ldots, k \) be all the roots of \( f_k(X) \). Then for every \( h \in \{0, 1, \ldots, k-1\} \), there exists \( j \) such that

\[
\left| \theta_j - \frac{2\pi h}{k} \right| < \frac{\pi}{k}.
\]

To reconcile with our notation, the angles \( \theta_j \) for \( j = K + 1, \ldots, k \) are chosen so that \( \theta_{j+\ell} = \theta_j + \pi \) for \( \ell = 1, \ldots, K \) when \( k \) is odd. When \( k \) is even, then the same formula holds for \( \ell = 1, \ldots, K-1 \) and \( \theta_K = \pi \). Finally, \( \theta_k = 0 \) corresponds to the dominant root \( \alpha_k \).

**Remark 1.** Since the intervals

\[
\left( \frac{2h\pi}{k} - \frac{\pi}{k}, \frac{2h\pi}{k} + \frac{\pi}{k} \right)
\]

are disjoint modulo \( 2\pi \) as \( h \) ranges in \( \{0, \ldots, k-1\} \), and since \( f_k(X) \) has \( k \) roots, each interval above contains the argument of only one of the roots of \( f_k(X) \).

*For an integer \( k \geq 2 \), the \( k \)-Fibonacci sequence is defined by the recurrence relation \( F_n^{(k)} = F_{n-k}^{(k)} + \cdots + F_1^{(k)} \) for all \( n \geq 2 \) and initial values \( F_i^{(k)} = 0 \) for \( i = -(k-2), \ldots, 0 \), followed by \( F_1^{(k)} = 1 \). The Fibonacci numbers are obtained for \( k = 2 \).
Another important and useful property is related to the separation of the roots of $f_k(X)$. In the previous paper [4] we proved that
\[
\frac{\rho_i}{\rho_j} > 1 + 8^{-k^4} \quad \text{for all} \quad 1 \leq i < j \leq K,
\]
and Dubickas [2] improved the right–hand side above to $1 + 1.454^{-k^3}$.

Our first result is improving this bound.

**Theorem 1.** The inequality
\[
\frac{\rho_i}{\rho_j} > 1 + \frac{1}{10k^{9.6}(\pi/e)^k} \quad \text{holds for all} \quad 1 \leq i < j \leq K. \quad (2)
\]
and all $k \geq 4$.

## 2 An auxiliar separation result

The above Theorem 1 is a separation result concerning the absolute values of the differences of the roots of $f_k(X)$. Quite general separation results of this kind appear in [2] but they are much worse (the denominator of the analogous expression from the right–hand side in (2) in [2] is exponential in $k^2$). To prove Theorem 1 we need a good separation result on the $\alpha_i$’s themselves.

Before presenting our result on the root separation of $f_k(X)$, with which we will prove Theorem 1, we will show how we can obtain a better result than the one proved by Dubickas [2] in our particular case by using results of Mahler [5] and Mignotte [6]. Namely, let us show that the inequality
\[
|\alpha_i - \alpha_j| > \frac{1}{k^{3/2}3k^2} \quad \text{holds for} \quad 1 \leq i < j \leq k \quad (3)
\]
provided $k \geq 100$. This can be proved using an off–the-shelf result. Namely, let
\[
g_k(X) = (X-1)f_k(X) = X^{k+1} - 2X^k + 1 \quad \text{and} \quad h_k(X) = X^{k+1}g_k(1/X).
\]
Then the roots of $h_k(X)$ are 1 and $1/\alpha_i$ for $i = 1, \ldots, k$. Let them be $y_1, \ldots, y_{k+1}$ with $y_\ell = 1/\alpha_\ell$ and $\alpha_{k+1} = 1$. By [5] and [6], the inequality
\[
|y_i - y_j| > \frac{\sqrt{3|\text{disc}(h_k)|}}{d^{d/2+1}\|h_k\|_d^{-d-1}} \quad \text{holds for} \quad 1 \leq i < j \leq k+1,
\]
holds where $d = \deg(h_k)$ and $\text{disc}(h_k)$ are the degree and discriminant of $h_k(X)$, respectively. For us, $d = k + 1$, and
\[
|\text{disc}(h_k)| = |\text{disc}(g_k)| = (k - 1)^2|\text{disc}(f_k)|
= 2^{k+1}k^k - (k + 1)^{k+1} = k^k \left(2^{k+1} - (k + 1) \left(1 + \frac{1}{k}\right)^k\right)
> k^k(2^{k+1} - e(k + 1)) > 2^{k}k^k \quad (4)
\]
since $2^k > e(k + 1)$ holds for $k \geq 100$. Further, $\|h_k\|_2 = \sqrt{1 + 2^2 + 1} = \sqrt{6}$.

We thus get

$$|y_i - y_j| > \frac{\sqrt{3 \cdot 2^k k^k}}{(k + 1)^{(k+3)/2}} \sqrt{6} = \frac{3^{1/2}}{(k + 1)^{3/2}} \frac{1}{(1 + 1/k)^{k/2}} \left( \frac{1}{3^{k/2}} \right)$$

$$\geq \left( \frac{3}{e} \right)^{1/2} \frac{1}{(k + 1)^{3/2} \cdot 3^{k/2}} > \frac{1.05}{k^{3/2} 3^{k/2}},$$

where we used the fact that $(3/e)^{1/2}(1/(k + 1)^{3/2}) > 1.05/k^{3/2}$ for $k \geq 100$. Evaluating the above in $y_i = 1/\alpha_i$, $y_j = 1/\alpha_j$, we get that

$$|\alpha_i - \alpha_j| > \frac{1.05|\alpha_i \alpha_j|}{k^{3/2} 3^{k/2}} > \frac{1}{k^{3/2} 3^{k/2}},$$

where we used the fact that

$$1.05|\alpha_i \alpha_j| > 1.05 \left( 1 - \frac{\log 3}{k} \right)^2 > 1$$

(see Lemma 1) for $k \geq 100$. We next prove a better result.

**Theorem 2.** The inequality

$$|\alpha_i - \alpha_j| > \frac{1}{k^{6.6}(\pi/e)^k}$$

holds for all $1 \leq i < j \leq k$

and all $k \geq 4$.

### 3 Proof of the separation result Theorem 2

We assume $k \geq 100$. The proof of Mahler’s and Mignotte’s results from [5] and [6], respectively, are based on a discriminant calculation together with an upper bound for a determinant which follows from Hadamard’s inequality. Since we know quite a few things about the roots of $f_k(X)$, we visit that proof and use at the appropriate place the extra informations that we have about the roots of $f_k(X)$. We assume that $\alpha_i$ and $\alpha_j$ are small roots (i.e., $i, j \in \{1, \ldots, k - 1\}$), for otherwise one of $i, j$ is $k$, say $i = k$ and then

$$|\alpha_i - \alpha_j| \geq \alpha_k - \rho_j \geq \left( 2 - \frac{1}{2k-1} \right) - \left( 1 - \frac{1}{2k^3} \right) = 1 + \frac{1}{2k^3} - \frac{1}{2k-1} > 0.9,$$

a much better inequality. We may also assume that $\theta_i \in (0, \pi]$ for otherwise we replace the pair of roots $(\alpha_i, \alpha_j)$ by the pair of roots $(\overline{\alpha_i}, \overline{\alpha_j})$ whose separation is the same since $|\alpha_i - \alpha_j| = |\overline{\alpha_i} - \overline{\alpha_j}|$. As in the proof of (3), we pass to $h_k(X)$ and write

$$|\text{disc}(h_k)| = \prod_{\ell=1}^{k+1} |h'_k(y_\ell)|.$$  

(5)
The left-hand side is
\[ |\text{disc}(h_k)| = 2^{k+1}k^k - (k+1)^{k+1} > 2^k k^k \]  
by the calculation (4). In the right-hand side we have
\[ h_k(X)' = (k+1)X^k - 2. \]
When \( \ell = k + 1 \) (\( y_\ell = 1 \)), the corresponding factor is
\[ |h'_k(1)| \leq k + 3. \]  
(7)
When \( \ell = k \) (\( y_\ell = 1/\alpha_k \)), the corresponding factor is
\[ |h'_k(1/\alpha_k)| = 2 - k + \frac{1}{\alpha_k} < 2 - \frac{k + 1}{2^k} < 2 \text{ for } k \geq 100. \]  
(8)
When \( y_\ell = 1/\alpha_\ell \) for \( \ell \in \{1, \ldots, k-1\}\setminus\{i, j\} \), we have
\[ |h'_k(1/\alpha_\ell)| \leq 2 + \frac{k + 1}{|\alpha_\ell|^k} \leq (k + 3)|\alpha_\ell|^{-k}, \quad \ell \in \{1, \ldots, k-1\}\setminus\{i, j\}. \]  
(9)
Multiplying (7), (8) and (9) for \( \ell \in \{1, \ldots, k-1\}\setminus\{i, j\} \), we get
\[ \prod_{1 \leq \ell \leq k+1, \ell \neq i, j} |h'_k(y_\ell)| \leq 2(k + 3)^{k-2} \prod_{1 \leq \ell \leq k-1, \ell \neq i, j} |\alpha_\ell|^{-k}. \]
By the Vieta relations,
\[ \left| \prod_{1 \leq \ell \leq k-1, \ell \neq i, j} \alpha_\ell \right|^{-1} = \alpha_k |\alpha_i \alpha_j| < \alpha_k < 2. \]
Hence,
\[ \prod_{1 \leq \ell \leq k+1, \ell \neq i, j} |h'_k(y_\ell)| < 2(k + 3)^{k-2}2^k. \]  
(10)
Now (5) together with bounds (6) and (10) give
\[ 2^k k^k < |\text{disc}(h_k)| = \prod_{\ell = 1}^{k+1} |h'_k(y_\ell)| < 2|h'_k(y_i)||h'_k(y_j)|(k + 3)^{k-2}2^k, \]
which gives
\[ |h'_k(y_i)||h'_k(y_j)| > \frac{(k + 3)^2}{2(1 + 3/k)^k} > \frac{(k + 3)^2}{2e^3} > \frac{k^2}{2e^3}, \]  
(11)
where we used the fact that \((1 + 1/x)^x < e\) for \(x > 1\) with \(x = k/3\). We work on the left–hand side. We have

\[
|h_k'(y_i)||h_k'(y_j)| = |y_i - y_j|^2 \prod_{1 \leq \ell \leq k+1} |y_i - y_\ell||y_j - y_\ell|.
\]

When \(y_\ell = 1\) or \(y_\ell = 1/\alpha_k\), we have

\[
|y_i - y_\ell||y_j - y_\ell| \leq (1 + 1/\rho_i)(1 + 1/\rho_j) < 2.1^2 \quad y_\ell \in \{1, 1/\alpha_k\}
\]

since \(\min\{\rho_i, \rho_j\} \geq 1 - \log 3/k > 1/1.1\) for \(k \geq 100\) by Lemma (1). We thus get that

\[
|h_k'(y_i)||h_k'(y_j)| < |y_i - y_j|^2 (2.1)^4 \prod_{1 \leq \ell \leq k-1} |y_i - y_\ell||y_j - y_\ell|. \tag{12}
\]

In the right, we write \(y_i - y_\ell = 1/\alpha_i - 1/\alpha_\ell\) and do the same for \(y_j - y_\ell\) to get that the right–hand side is

\[
|\alpha_i - \alpha_j|^2 (2.1)^4 \left( \prod_{1 \leq \ell \leq k-1} |\alpha_i - \alpha_\ell||\alpha_j - \alpha_\ell| \right) |\alpha_\ell|^-(k-1) \prod_{1 \leq \ell \leq k-1} |\alpha_\ell|^{-2}.
\]

Again by the Vieta relations,

\[
|\alpha_i|^{-2} |\alpha_j|^{-2} \prod_{1 \leq \ell \leq k-1} |\alpha_\ell|^{-2} = \alpha_k^2 < 4.
\]

Thus, we get that the right–hand side in (12) is at most

\[
|\alpha_i - \alpha_j|^2 (2.1)^4 \cdot 4 \left( \prod_{1 \leq \ell \leq k-1} |\alpha_i - \alpha_\ell||\alpha_j - \alpha_\ell| \right) |\alpha_i\alpha_j|^{-(k-3)}).
\]

Since

\[
|\alpha_i| = \rho_i > 1 - \frac{\log 3}{k} = \exp \left( \log \left( 1 - \frac{\log 3}{k} \right) \right) > \exp \left( - \frac{2 \log 3}{k} \right)
\]

(where we used that \(\log(1 - x) > -2x\) for \(x \in (0, 1/2)\)), we get that

\[
|\alpha_i|^{-(k-3)} < |\alpha_i|^{-k} < \exp (2 \log 3) = 9,
\]

and the same inequality holds with \(i\) replaced by \(j\). Thus, the right–hand side of (12) is at most

\[
|\alpha_i - \alpha_j|^2 (2.1)^4 \cdot 9^2 \left( \prod_{1 \leq \ell \leq k-1} |\alpha_i - \alpha_\ell||\alpha_j - \alpha_\ell| \right).
\]
Since $2.1^4 \cdot 4 \cdot 9^2 < 6500$, we get that

$$|h_k'(y_i)||h_k'(y_j)| < 6500|\alpha_i - \alpha_j|^2 \left( \prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j} |\alpha_i - \alpha_{\ell}| |\alpha_j - \alpha_{\ell}| \right).$$

Combining the above with (11), we get

$$\frac{k^2}{13000e^3} < |\alpha_i - \alpha_j|^2 \left( \prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j} |\alpha_i - \alpha_{\ell}| |\alpha_j - \alpha_{\ell}| \right).$$

(13)

It remains to bound the product in the right–hand side.

Let $h_i, h_j \in \{0, \ldots, k\}$ be such that

$$\theta_i \in \left(\frac{(2h_i - 1)\pi}{k}, \frac{(2h_i + 1)\pi}{k}\right)$$

and

$$\theta_j \in \left(\frac{(2h_j - 1)\pi}{k}, \frac{(2h_j + 1)\pi}{k}\right)$$

(14)

according to Lemma 2. By Remark 1, we have $h_i \neq h_j$. By the same remark, we have that $h_i, h_j$ are both nonzero since the real root $\alpha_k$ corresponds to $h = 0$. Since $\theta_i \in (0, \pi]$, it follows that $h_i \leq (k + 1)/2$. We now justify that we may assume that $h_i$ and $h_j$ are consecutive modulo $k$ and since $1 \leq h_i \leq (k + 1)/2$, it follows that either $h_j = h_i + 1$, or $h_j = h_i - 1$ and in the second case we must have $h_i \geq 2$. To see why, let us look at $|\theta_i - \theta_j|$. First of all, if $|\theta_i - \theta_j| \geq \pi/2$, it then follows that

$$|\alpha_i - \alpha_j| = |\rho_i e^{i\theta_i} - \rho_j e^{i\theta_j}| = \left| (e^{i\theta_i} - e^{i\theta_j}) + (\rho_i - 1) e^{i\theta_i} - (\rho_j - 1) e^{i\theta_j} \right|
\geq |e^{i\theta_i} - e^{i\theta_j}| - |1 - \rho_i| - |1 - \rho_j|
= 2 |\sin((\theta_i - \theta_j)/2)| - \frac{2 \log 3}{k} \geq \sqrt{2} - \frac{2 \log 3}{k} > 1$$

a much better bound. Thus, we may assume that $\theta_i - \theta_j \in (-\pi/2, \pi/2)$. Now, if $h_i$ and $h_j$ are not consecutive, it then follows that $\theta_i - \theta_j \geq 2\pi/k$. The same calculation as before then gives

$$|\alpha_i - \alpha_j| \geq 2 |\sin((\theta_i - \theta_j)/2)| - \frac{2 \log 3}{k}
\geq (2/\pi)|\theta_i - \theta_j| - \frac{2 \log 3}{k} \geq \frac{(4 - 2 \log 3)}{k},$$

again a much better bound. In the above, we used $|\sin x| \geq (2/\pi)|x|$ valid for $x \in (-\pi/2, \pi/2)$ with $x = \theta_i - \theta_j$.

A bit more can be said. Namely, if $h_j = h_i + 1$, then we must have

$$\frac{(2h_i - 0.5)\pi}{k} < \theta_i \leq \frac{(2h_i + 1)\pi}{k} \quad \text{and} \quad \frac{(2h_j - 0.5)\pi}{k} < \theta_j < \frac{(2h_j + 0.5)\pi}{k}.$$  

(15)
Indeed, say if the first one fails, then $(2h_j - 1)\pi/k < \theta_i \leq (2h_i - 0.5)\pi/k$, while $(2h_j - 1)\pi/k = (2h_i + 1)\pi/k < \theta_j$, which shows that $|\theta_j - \theta_i| \geq 1.5\pi/k$.

As with the previous argument, we get

$$|\alpha_i - \alpha_j| \geq 2|\sin((\theta_j - \theta_i)/2)| - \frac{2\log 3}{k} \geq (2/\pi)|\theta_j - \theta_i| - \frac{2\log 3}{k} \geq \frac{3 - 2\log 3}{k},$$

again a much better inequality. A similar inequality holds if the second inequality in (15) fails. Similar inequalities hold if $h_j = h_i - 1$ (just invert the roles of $i$ and $j$ in the above argument to obtain the desired inequalities).

From now on we assume that $h_j = h_i + 1$ since the other case is obtained by swapping the roles of $i$ and $j$. Let $\ell \in \{1, \ldots, k - 1\}\setminus\{i, j\}$ and let $h_\ell$ its corresponding $h$ in $\{0, 1, \ldots, k - 1\}$ according to Lemma 2. If $\ell_{i,j}$ is such that $h_{\ell_{i,j}} = h_i - 1$, we just bound trivially

$$|\alpha_i - \alpha_\ell| \leq 2. \quad (16)$$

If $\ell \neq i, j, \ell_{i,j}$, it then follows that

$$|\theta_i - \theta_\ell| \geq \frac{2(|h_\ell - h_i| - 1)\pi}{k}, \quad (17)$$

and $|h_\ell - h_i| \geq 2$. As $h_\ell$ circulates in $\{1, \ldots, k - 1\}$ such that $h_\ell \not\in \{h_i - 1, h_i, h_i + 1(= h_j)\}$ the numbers $|h_\ell - h_i| - 1$ are positive integers $1, 2, 3, \ldots$ and each one of them is attained at most twice. Assume that $\ell$ is such that $\theta_\ell$ is “far” from $\theta_i$, namely $|\theta_i - \theta_\ell| \geq \pi/2$. Then

$$|e^{i\theta_\ell} - e^{i\theta_i}| = 2|\sin((\theta_\ell - \theta_i)/2)| \geq \sqrt{2},$$

and

$$|\alpha_i - \alpha_\ell| = |e^{i\theta_i} - e^{i\theta_\ell} + (\rho_i - 1)e^{i\theta_i} - (\rho_\ell - 1)e^{i\theta_\ell}| \leq |e^{i\theta_i} - e^{i\theta_\ell}| + \frac{2\log 3}{k} \leq \left|e^{i\theta_i} - e^{i\theta_\ell}\right| \left(1 + \frac{2\log 3}{k|e^{i\theta_i} - e^{i\theta_\ell}|}\right) < \left|e^{i\theta_i} - e^{i\theta_\ell}\right| \left(1 + \frac{\sqrt{2}\log 3}{k}\right). \quad (18)$$

Assume now that $\ell$ is such that $\theta_\ell$ is “close” to $\theta_i$, namely $|\theta_i - \theta_\ell| < \pi/2$. Let $w \geq 1$ be such that $|h_\ell - h_i| - 1 = w$. We then have by (17) that

$$\frac{\pi}{2} > |\theta_i - \theta_\ell| \geq \frac{2w\pi}{k},$$

so $w \leq k/4$. Since every $w$ corresponds to at most two possible $\ell$'s it follows that there are at most $2 \cdot (k/4) = k/2$ such $\ell$'s. For them,

$$|e^{i\theta_i} - e^{i\theta_\ell}| = 2|\sin((\theta_i - \theta_\ell)/2)| \geq (2/\pi)|\theta_i - \theta_\ell| \geq \frac{4w}{k},$$
so that

\[ |\alpha_i - \alpha_\ell| = |e^{i\theta_i} - e^{i\theta_\ell} + (\rho_i - 1)e^{i\theta_i} - (\rho_\ell - 1)e^{i\theta_\ell}| \leq |e^{i\theta_i} - e^{i\theta_\ell}| + \frac{2\log 3}{k} \]

\[ \leq |e^{i\theta_i} - e^{i\theta_\ell}| \left( 1 + \frac{2\log 3}{k|e^{i\theta_i} - e^{i\theta_\ell}|} \right) \]

\[ < |e^{i\theta_i} - e^{i\theta_\ell}| \left( 1 + \frac{\log 3}{4w} \right). \] (19)

As a consequence of (16), (18) and (19), letting \( I \leq k \) be the number of \( \ell \)'s for which \( h_\ell \) is far from \( h_i \), we get that

\[ \prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j} |\alpha_i - \alpha_\ell| \leq 2 \prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j} |e^{i\theta_i} - e^{i\theta_\ell}| \]

\[ \times \left( 1 + \frac{\sqrt{2}\log 3}{k} \right)^I \prod_{1 \leq w \leq k/4} \left( 1 + \frac{\log 3}{4w} \right)^2. \]

The first factor in the second line is

\[ \left( 1 + \frac{\sqrt{2}\log 3}{k} \right)^I < \left( 1 + \frac{\sqrt{2}\log 3}{k} \right)^k \]

\[ < \exp(\sqrt{2}\log 3) = 3^{\sqrt{2}} < 5. \]

The second factor is

\[ \prod_{1 \leq w \leq k/4} \left( 1 + \frac{\log 3}{4w} \right)^2 < \exp \left( \frac{\log 3}{2} \sum_{1 \leq w \leq k/4} \frac{1}{w} \right) \]

\[ < \exp \left( \frac{\log 3}{2} \left( 1 + \log \left( \frac{k}{4} \right) \right) \right) \]

\[ < k^{\log 3/2} < k^{0.6}. \]

Thus,

\[ \prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j} |\alpha_i - \alpha_\ell| \leq 10k^{0.6} \prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j} |e^{i\theta_i} - e^{i\theta_\ell}|. \] (20)

Finally,

\[ |e^{i\theta_i} - e^{i\theta_\ell}| = 2|\sin((\theta_i - \theta_\ell)/2)| < |\theta_i - \theta_\ell| < \frac{2(|h_i - h_\ell| + 1)\pi}{k} \]

where in the last inequality we have used (14) with \( \ell \) instead of \( j \). As in the previous case, when we were counting \( \ell \) such that \( \theta_\ell \) were close from \( \theta_i \), we have that \( w := |h_i - h_\ell| \) are now integers of size at most \( k/2 \), each one of them is counted at twice except that 0 and 1 are not counted. Thus, we get that

\[ \prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j} |e^{i\theta_i} - e^{i\theta_\ell}| \leq \left( \frac{2\pi}{k} |k/2| ([k/2] + 1)! \right)^2 F, \]

where \( F \) accounts for the missing factors (more about that below).
Now, by Stirling’s formula:

\[
(2\pi/k)^{[k/2]}([k/2] + 1)! \leq (2\pi/k)^{[k/2]}(k/2 + 1)[k/2]! \\
\leq 1.1(k/2 + 1)(2\pi/k)^{[k/2]} \left( \frac{k/2}{e} \right)^{[k/2]} \sqrt{2\pi k/2} \\
\leq 1.1\sqrt{\pi}(k/2 + 1)k^{0.5}(\pi/e)^k \\
< k^{1.5}(\pi/e)^{k/2},
\]

since \( 1.1 \cdot \sqrt{\pi}(k/2 + 1) < k \) for \( k > 100 \). Thus,

\[
\prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j,i,j} |e^{i\theta_\ell} - e^{i\theta_i}| < k^3(\pi/e)^k F.
\]

On the other hand, \( F \) accounts for the fact that when upper bounding the product of \(|e^{i\theta_\ell} - e^{i\theta_i}| \) by the product of \( 2w\pi/k \), where \( w \leq [k/2] \) is a positive integer and each \( w \) is counted at most twice, we must not count \( w = 1 \) and \( w = 2 \) corresponding to \( h \in \{h_i - 1, h_i, h_i + 1\} \). Thus,

\[
F \leq \left( \frac{k}{2\pi} \right)^2 \left( \frac{k}{4\pi} \right)^2 = \frac{k^4}{64\pi^4}.
\]

Hence,

\[
\prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j,i,j} |e^{i\theta_\ell} - e^{i\theta_i}| < k^3(\pi/e)^k \left( \frac{k^4}{64\pi^4} \right) = \frac{k^7(\pi/e)^k}{64\pi^4},
\]

which together with (20) gives

\[
\prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j,i,j} |\alpha_i - \alpha_\ell| < \frac{10k^{7.6}(\pi/e)^k}{64\pi^4}.
\]

A similar bound applies for \( i \) replaced by \( j \), so we get that

\[
\prod_{1 \leq \ell \leq k-1 \atop \ell \neq i,j,i,j} |\alpha_i - \alpha_\ell||\alpha_j - \alpha_\ell| < \frac{100k^{15.2}(\pi/e)^{2k}}{2^{12}\pi^8},
\]

which together with (13) gives

\[
\frac{k^2}{13000e^3} < |\alpha_i - \alpha_j|^2 \left( \frac{100k^{15.2}(\pi/e)^{2k}}{2^{12}\pi^8} \right),
\]

so

\[
|\alpha_i - \alpha_j| > \frac{1}{k^{6.6}(\pi/e)^k}.
\]

Note that the above inequality was obtained under assumption that \( k \leq 100 \). However, a simple calculation in Mathematica shows that the above inequality holds also for \( k < 100 \), even without the exponential term \((\pi/e)^k\).

This completes the proof of Theorem [2]

\[ ^{\dagger}m! = \sqrt{2\pi m} \left( \frac{m}{e} \right)^m e^{\lambda_m}, \text{ where } \frac{1}{12m+1} < \lambda_m < \frac{1}{12m}. \]
4 The proof of Theorem 1

We keep the notations from the previous proof. We may assume that $\theta_i, \theta_j$ are both in $(0, \pi]$, otherwise we replace $\alpha_i$ by $\bar{\alpha}_i$ and/or $\alpha_j$ by $\bar{\alpha}_j$ since this replacement does not change the absolute values $\rho_i$ and $\rho_j$ of the roots. Let us assume that the claimed inequality does not hold, namely, that for some $1 \leq i < j \leq K$, we have

$$\rho_i - \rho_j \leq \frac{\rho_j}{10k^{9.6}(\pi/e)^k} < \frac{1}{10k^{9.6}(\pi/e)^k}.$$  

Evaluating $h_k(z) = 0$ for $z = y_i$ and its conjugate, we get

$$y_i^{k+1} = 2y_i - 1 \quad \text{and} \quad \overline{y}_i^{k+1} = 2\overline{y}_i - 1$$

and multiplying them side by side we get

$$r_2^{k+2} = (y_i\overline{y}_i)^{k+1} = (2y_i - 1)(2\overline{y}_i - 1) = 4r_i^2 - 2(y_i + \overline{y}_i) + 1,$$

where we write $r_\ell = |y_\ell| = 1/\rho_\ell$ for $\ell = 1, 2, \ldots, K$. A similar relation holds with $i$ replaced by $j$. Subtracting the two relations we get

$$(r_j - r_i)(r_i^{2k+1} + r_i^{2k}r_j + \cdots + r_j^{2k+1} - 4(r_i + r_j)) = 4(\text{Re}(y_i) - \text{Re}(y_j)).$$

We discuss the large factor in parentheses. We have

$$1 < r_\ell < \frac{1}{1 - \log 3/k} < 1 + \frac{1.2}{k} \quad \text{for} \quad \ell \in \{i, j\}$$

by Lemma 1 since $k > 100$. Thus, the factor in parenthesis is at least

$$2k + 2 - 4(r_i + r_j) > 2k + 2 - 4(2 + 2.4/k) > 2k + 2 - 9 = 2k - 7$$

is positive. On the other hand, this factor in parenthesis is at most as large as

$$(2k + 2) \left(1 + \frac{1.2}{k}\right)^{2k+1}
\leq (2k + 2) \left(1 + \frac{1.2}{k}\right) \left(1 + \frac{1.2}{k}\right)^{k/1.2} \left(1 + \frac{1.2}{k}\right)^{1/2}
\leq \left(2k + 2 + \frac{2.4k + 2.4}{k}\right) e^{2/1.2} < 5.3(2k + 5) < 12k$$

since $k > 100$. Further,

$$0 < r_j - r_i = \frac{\rho_i - \rho_j}{\rho_i\rho_j} < \frac{r_i}{10k^{9.6}(\pi/e)^k} < \frac{1}{9k^{9.6}(\pi/e)^k},$$

which shows that

$$0 < \text{Re}(y_i) - \text{Re}(y_j) < \frac{12k}{4 \cdot 9k^{9.6}(\pi/e)^k} = \frac{1}{3k^{8.6}(\pi/e)^k}. \quad (21)$$
But
\[ \text{Re}(y_i) - \text{Re}(y_j) = r_i \cos \theta_i - r_j \cos \theta_j = r_i (\cos \theta_i - \cos \theta_j) + (r_j - r_i) \cos \theta_j. \]

Hence, we get from (21) and (22) that
\[ r_i | \cos \theta_i - \cos \theta_j | < (\text{Re}(y_i) - \text{Re}(y_j)) + (r_j - r_i) \]
\[ < \left( 3 + \frac{1}{k} \right) \frac{1}{9k^{8.6}(\pi/e)^k} \]
\[ < \frac{1}{2.8k^{8.6}(\pi/e)^k}. \]

Since \( r_i > 1 \), we get that
\[ | \cos \theta_i - \cos \theta_j | < \frac{1}{2.8k^{8.6}(\pi/e)^k}. \]

We thus get that
\[ | \sin((\theta_i - \theta_j)/2)| | \sin((\theta_i + \theta_j)/2)| < \frac{1}{2.8k^{8.6}(\pi/e)^k}. \quad (23) \]

Fix
\[ x := (\theta_i + \theta_j)/2, \quad y := (\theta_i - \theta_j)/2 \]
so \( \theta_i = x + y, \quad \theta_j = x - y. \)

Assume that both \( \sin(x), \sin(y) \) above are smaller than \( 1/k^2 \). Since \( x \in (0, \pi) \), \( y \in (-\pi/2, \pi/2) \) and
\[ \frac{2|y|}{\pi} \leq |\sin y| \leq \frac{1}{k^2}, \]
we get that \( |y| \leq ((\pi/2)/k^2) < 2/k^2 \). Further as in the case of \( y \), if \( x \in (0, \pi/2) \), then also \( x \leq 2/k^2 \), while if \( x \in (\pi/2, \pi) \), we get that \( \pi - x < 2/k^2 \).

In case \( x \in (0, \pi/2) \), so \( x \in (0, 2/k^2) \), we get that
\[ \max\{\theta_i, \theta_j\} \leq x + |y| \leq 4/k^2. \]

Since \( \theta_i, \theta_j \in (0, \pi] \), it follows that \( \theta_i, \theta_j \in (0, 4/k^2) \). In particular, \( h_i = h_j = 0 \), which contradicts Remark [1].

In the case when \( \pi - x < 2/k^2 \) (i.e. \( x \in (\pi/2, \pi) \)), we get that \( \pi - \theta_\ell \in (0, 4/k^2) \) holds for both \( \ell \in \{i, j\} \). In particular, for \( k \) odd the interval corresponding to \( h = (k - 1)/2 \) contains both \( \theta_i \) and \( \theta_j \), and this is false. When \( k \) is even, we get that the interval corresponding to \( h = k/2 \) contains both \( \theta_i \) and \( \theta_j \) and this is again false.

So, in (23), the maximum of the factors is \( > 1/k^2 \), while the minimal factor is \( < \frac{1}{2.8k^{8.6}(\pi/e)^k} \). Assume, for example, that
\[ |\sin(y)| < \frac{1}{2.8k^{8.6}(\pi/e)^k}. \]
We now compute, using (21),

\[
|\text{Im}(y_i) - \text{Im}(y_j)| = |r_i \sin \theta_i - r_j \sin \theta_j| \\
\leq r_i |\sin \theta_i - \sin \theta_j| + (r_j - r_i) \\
< 2r_i |\sin(y) \cos(x)| + (r_j - r_i) \\
\leq \left( 2 \cdot 1.1 + \frac{2.8}{9k^3}\right) \frac{1}{2.8k^{6.6}(\pi/e)^k} \\
< \left( \frac{2.3}{2.8}\right) \frac{1}{k^{6.6}(\pi/e)^k}
\]

(24)

for \(k > 100\). It now follows, using (22) and (24), that

\[
|y_i - y_j| = \sqrt{(\text{Re}(y_i) - \text{Re}(y_j))^2 + (\text{Im}(y_i) - \text{Im}(y_j))^2} \\
< \left( \frac{1}{9k^4} + \left( \frac{2.3}{2.8}\right)^2 \right)^{1/2} \frac{1}{k^{6.6}(\pi/e)^k} < \frac{1}{k^{6.6}(\pi/e)^k}
\]

for \(k > 100\). In turn, this gives

\[
|\alpha_i - \alpha_j| < \frac{\rho_i \rho_j}{k^{6.6}(\pi/e)^k} < \frac{1}{k^{6.6}(\pi/e)^k},
\]

contradicting Theorem 2. This was when \(|\sin((\theta_i - \theta_j)/2)|\) was small. In the case when \(|\sin((\theta_i + \theta_j)/2)|\) is small, the same argument shows that \(|\alpha_i - \overline{\alpha_j}|\) is too small again contradicting Theorem 2. As before, a quick calculation with Mathematica shows that inequality (2) also holds for \(k < 100\), even without the exponential term \((\pi/e)^k\).

This finishes the proof of Theorem 1.

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