DISPERSE EFFECTS IN A PERIDYNAMIC MODEL

GIUSEPPE MARIA COCLITE, SERENA DIPIERRO, GIUSEPPE FANIZZA, FRANCESCO MADDALENA, AND ENRICO VALDINOCI

ABSTRACT. We study the dispersive properties of a linear peridynamic equation in one spatial dimension. The interplay between nonlocality and dispersion is analyzed in detail through the study of the asymptotics at low and high frequencies, revealing new features ruling the wave propagation in continua where nonlocal characteristics must be taken into account. Global dispersive estimates and existence of conserved functionals are proved. A comparison between these new effects and the classical local scenario is deepen also through a numerical analysis.

INTRODUCTION

A fundamental trait in the mathematical modeling of continuum physics relies in capturing the essential phenomena of a complex problem still keeping the technical difficulties as manageable as possible. A typical example of this strategy is provided by classical linear elasticity ([10]) which, in spite of its very long-lasting tradition, represents yet an unavoidable comparison term even for the more recent mechanical theories aimed to describe old and new material behaviors that were not contemplated in the original theory. In particular, the spontaneous creation of singularities like cracks or damages and their evolution process, as well as the dispersive characteristics affecting wave propagation, have originated a great effort in exploiting new and subtle mathematical formulations, leading to a general consensus on the fact that macroscopic manifestations as plasticity or failure are governed by intricate mechanisms acting at different scales. With respect to these considerations, the analysis of the nonlocal features of a model has become a consolidated strategy towards a better rationale understanding of “how nature works”.

While the insertion of nonlocal descriptors in classical differential formulations of continuum mechanics (as strain gradient, convolution kernels, etc.) has a long tradition ([15, 14, 8, 9, 13, 12]), the acceptance ab initio of a material intrinsic length-scale in a genuinely nonlocal theory is a more recent achievement and peridynamics, as initiated by S.A. Silling (see [18, 19, 20, 21, 17]), seems to have good chances to shed some new light on these problems.

In the present paper we deal with possibly the simplest evolution equation arising in linear peridynamics, in one spatial dimension, with the aim of investigating the dispersive features of wave propagation and detect a number of original features due to nonlocality.

In particular, it is known that material dispersion manifests through propagating pulses with frequency components traveling at a different speed [3, 24]: as the distance increases, the pulse becomes...
broader, hence the mathematical analysis mainly concerns the study of the properties of the dispersion relation and the determination of decay properties of various functional norms of the solutions of the initial value problem (1.2). In this paper we pursue this program (namely, understanding the dispersion relation of propagating pulses and establishing regularity and decay estimates) through a detailed study on how nonlocality influences the dispersive behavior.

The article is organized as follows: In Section 1 we introduce the initial value problem given by (1.3) following the general framework studied in [5] and deduce the corresponding dispersion relation. In Section 2 the dispersion relation is studied in detail and the asymptotics at low and high frequencies clearly exhibit the scale effects ruled by nonlocality (see Theorems 2.1 and 2.2). In particular one sees that at low frequencies, hence at large physical scales, the propagation is quite similar to that governed by the classical wave equation, while at high frequencies, hence at small physical scales, the propagation is remarkably different, due to nonlocality. Furthermore, the analysis of the derivatives of the dispersion reveals new and somehow unexpected features. Indeed (see Theorem 2.3), due to nonlocality these derivatives present exotic decay with highly oscillatory behavior at high frequencies. Since these quantities are related with the velocity of energy transport, this suggests that some pieces of information could be hidden at small scales because of the strange behavior of their propagation velocity. In Section 3 we prove some decay properties of the solution of the peridynamic problem by establishing properly dispersive estimates. Those results play a key role in the subsequent Section 4 where we prove the conservation of energy, momentum and angular momentum (Theorem 4.2). Section 5 is devoted to a numerical study on the comparison between classical wave equation and the present nonlocal problem and the outcome of this analysis essentially shows that the peridynamic case seems to produce additional oscillations. The paper ends with three Appendix sections in which some facts previously mentioned in the paper are specifically proved.

1. THE LINEAR MODEL

In [5] the Cauchy problem related to a very general model of nonlocal continuum mechanics, inspired by the seminal work by S.A. Silling (see [15], was studied and the analytical aspects concerning global solutions in energy space were exploited in the framework of nonlinear hyperelastic constitutive assumptions. More precisely, the governing equation of the motion of an infinite body is modeled in [5] by the initial-value problem

\[\begin{align*}
&\partial_t u(x, t) = (K u(\cdot, t))(x), \quad x \in \mathbb{R}^N, \ t > 0, \\
&u(x, 0) = u_0(x), \ \partial_t u(x, 0) = v_0(x), \quad x \in \mathbb{R}^N,
\end{align*}\]

where

\[(K u)(x) := \int_{B_{3\delta}(x)} f(x' - x, u(x') - u(x)) \, dx', \quad \text{for every} \ x \in \mathbb{R}^N,\]

for a given \(\delta > 0\). Of course, the physically meaningful cases correspond to the dimensions \(N = 1, 2, 3\). From the point of view of peridynamics, the parameter \(\delta\) takes into account the finite horizon of the nonlocal bond which is governed by the long-range interaction integral \(K\). The \(\mathbb{R}^N\)-valued function \(f\) is defined on the set \(\Omega := (\mathbb{R}^N \setminus \{0\}) \times \mathbb{R}^N\) and is supposed to satisfy the following general constitutive assumptions:

(H.1) \(f \in C^1(\Omega; \mathbb{R}^N);\)

(H.2) \(f(\mathbf{y}, -\mathbf{u}) = -f(\mathbf{y}, \mathbf{u}), \ \text{for every} \ (\mathbf{y}, \mathbf{u}) \in \Omega \times \mathbb{R}^N;\)

(H.3) there exists a function \(\Phi \in C^2(\Omega)\) such that

\[f = \nabla_u \Phi, \quad \Phi(\mathbf{y}, \mathbf{u}) = \kappa \frac{|u|^p}{|y|^{N+\alpha p}} + \Psi(\mathbf{y}, \mathbf{u}), \quad \text{for every} \ (\mathbf{y}, \mathbf{u}) \in \Omega,\]

where \(\kappa, \ p, \ \alpha\) are constants such that

\[\kappa > 0, \quad 0 < \alpha < 1, \quad p \geq 2,\]
and
\[
\Psi(y, 0) = 0 \leq \Psi(y, u), \quad |\nabla u \Psi(y, u)|, |D^2 u \Psi(y, u)| \leq g(y), \quad \text{for every } (y, u) \in \Omega,
\]
for some nonnegative function \(g \in L^2_{\text{loc}}(\mathbb{R}^N)\).

We recall that in the peridynamics model the \(\mathbb{R}^N\)-valued function \(u\) models the displacement vector field. With this respect, assumption \(\text{(H.2)}\) can be seen as a counterpart of Newton’s Third Law of Motion (the Action-Reaction Law). Also, assumption \(\text{(H.3)}\) states that the material is hyperelastic (the linear elastic case corresponding to \(p = 2\) and \(\Psi = 0\), and the hyperelasticity taking into account nonlinear elastic responses of the material).

In the present paper we focus on the simplest case planned by the previous theory, namely we will deal with \(p = 2\) (quadratic elastic energy) and \(N = 1\) which represents a linear one-dimensional nonlocal mechanical model (higher dimensional cases can be taken into account as well, but the analysis is obviously more transparent when \(N = 1\)). We intend to exploit all the relevant analytical aspects encoded in this problem and compare them with their physical counterparts.

In this perspective (1.1) reduces to the study of the following Cauchy problem
\[
\begin{cases}
\rho u_{tt} = -2 \kappa \int_{-\delta}^{\delta} \frac{u(t, x) - u(t, x - y)}{|y|^{1+2 \alpha}} dy =: K(u), & t > 0, x \in \mathbb{R}, \\
u(0, x) = v_0(x), & x \in \mathbb{R}, \\
u_t(0, x) = v_1(x), & x \in \mathbb{R},
\end{cases}
\]
where \(\delta, \kappa\) and \(\rho\) are positive real constants and \(0 < \alpha < 1\).

As customary, the integral in (1.3) is interpreted in the “principal value” sense to “average out” the singularity, namely
\[
\int_{-\delta}^{\delta} \frac{u(t, x) - u(t, x - y)}{|y|^{1+2 \alpha}} dy := \lim_{\epsilon \to 0^+} \int_{(-\delta, \delta) \setminus (-\epsilon, \epsilon)} \frac{u(t, x) - u(t, x - y)}{|y|^{1+2 \alpha}} dy
\]
\[
= \frac{1}{2} \lim_{\epsilon \to 0^+} \int_{(-\delta, \delta) \setminus (-\epsilon, \epsilon)} \frac{2u(t, x) - u(t, x + y) - u(t, x - y)}{|y|^{1+2 \alpha}} dy
\]
\[
= \frac{1}{2} \int_{-\delta}^{\delta} \frac{2u(t, x) - u(t, x + y) - u(t, x - y)}{|y|^{1+2 \alpha}} dy.
\]

The evolution problem in (1.3) is explicitly solvable, according to the following result:

**Theorem 1.1.** Let \(v_0, v_1 \in \mathcal{S}(\mathbb{R})\) and \(0 < \alpha < 1\). Then problem (1.3) has the unique solution \(u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}\) given by
\[
u(t, x) = \int_{\mathbb{R}} e^{-i\xi x} \left[ \hat{v}_0(\xi) \cos (\omega(\xi) t) + \frac{\hat{v}_1(\xi)}{\omega(\xi)} \sin (\omega(\xi) t) \right] d\xi,
\]

\(^1\)In this paper, for simplicity, unless differently specified, we will take the initial data \(v_0\) and \(v_1\) in the Schwartz Space of smooth and rapidly decreasing functions (more general settings can be treated similarly with technical modifications).
where \( \hat{v}_0(\xi) \) and \( \hat{v}_1(\xi) \) represent the Fourier transform of \( v_0(x) \) and \( v_1(x) \), and \( \omega: \mathbb{R} \to \mathbb{R}^+ \) is the dispersion relation defined by

\[
\omega(\xi) = \left( \frac{2\kappa}{\rho \delta^{2\alpha}} \int_{-1}^{1} \frac{1 - \cos(\xi \delta z)}{|z|^{1+2\alpha}} dz \right)^{1/2}.
\]

Additionally,

\[
\omega^2(\xi) \leq \frac{C\kappa}{\rho \delta^{2\alpha}} \left[ \frac{|\xi|^2 \delta^2}{1 - \alpha} \min \left\{ \frac{1}{|\xi|^{2-2\alpha \delta^2-2\alpha}}, 1 \right\} + \frac{1}{\alpha} \chi_{(0,1)} \left( \frac{2\kappa}{|\xi|^\delta} \right) \left( \frac{|\xi|^{2\alpha \delta^{2\alpha}}}{2^{2\alpha}} - 1 \right) \right],
\]

for some constant \( C > 0 \) (independent of all the parameters involved in problem (1.3)).

**Proof.** The existence and uniqueness of the solution of (1.3) follow from [5]. Thus, to check (1.5), up to a superposition, we seek a solution of the form

\[
u(t, x) = e^{\pm (i\omega t + i\xi x)}.
\]

After substituting this expression into (1.3), we obtain the equation

\[
\omega^2 = \frac{2\kappa}{\rho} \int_{-\delta}^{\delta} \frac{1 - e^{\pm i\xi y}}{|y|^{1+2\alpha}} dy
= \frac{2\kappa}{\rho} \int_{-\delta}^{\delta} \frac{1 - \cos(\xi y)}{|y|^{1+2\alpha}} dy + \frac{2i\kappa}{\rho} \int_{-\delta}^{\delta} \sin(\xi y) \frac{\delta^{-2\alpha}}{|y|^{1+2\alpha}} dy
= \frac{2\kappa}{\rho} \int_{-\delta}^{\delta} \frac{1 - \cos(\xi y)}{|y|^{1+2\alpha}} dy - \frac{2\kappa}{\rho} \delta^{-2\alpha} \int_{-1}^{1} \frac{1 - \cos(\xi \delta z)}{|z|^{1+2\alpha}} dz,
\]

which leads to the dispersion relation (1.6).

Introducing the functions

\[
\alpha(\xi) := \frac{1}{2} \left( \hat{v}_0(\xi) + i \hat{v}_1(\xi) \right) \quad \text{and} \quad \beta(\xi) := \frac{1}{2} \left( \hat{v}_0(\xi) - i \hat{v}_1(\xi) \right),
\]

we have that \( \alpha + \beta = \hat{v}_0 \) and \( -i\omega(\alpha - \beta) = \hat{v}_1 \). As a result, if

\[
u(t, x) := \int_{\mathbb{R}} \left\{ \alpha(\xi) e^{-i[\xi x + \omega(\xi) t]} + \beta(\xi) e^{-i[\xi x - \omega(\xi) t]} \right\} d\xi,
\]

we see that

\[
u(0, x) = \int_{\mathbb{R}} \left\{ \alpha(\xi) e^{-i\xi x} + \beta(\xi) e^{-i\xi x} \right\} d\xi = \int_{\mathbb{R}} \hat{v}_0(\xi) e^{-i\xi x} d\xi = v_0(x)
\]

and

\[
u_t(0, x) = \int_{\mathbb{R}} \left\{ -i\alpha(\xi) \omega(\xi) e^{-i\xi x} + i\beta(\xi) \omega(\xi) e^{-i\xi x} \right\} d\xi = \int_{\mathbb{R}} \hat{v}_1(\xi) e^{-i\xi x} d\xi = v_1(x),
\]

hence (1.8) provides a solution of (1.3). We can also rewrite (1.8) in the form given by (1.5). Additionally, we observe that our assumptions \( v_0, v_1 \in S(\mathbb{R}) \) allow us to state that \( \hat{v}_0, \hat{v}_1 \in S(\mathbb{R}) \).

Moreover, for every \( t \in \mathbb{R} \), we have that

\[
1 - \cos t \leq \min \left\{ \frac{t^2}{2}, 2 \right\},
\]

\[\text{We use here the nonunitary convention that}
\hat{v}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} v(x) e^{ix\xi} dx.
\]

In this way, the inversion formula reads

\[
v(x) = \int_{\mathbb{R}} \hat{v}(\xi) e^{-i\xi x} d\xi.
\]
hence it follows from (1.6) that
\[
\omega^2(\xi) = \frac{2\kappa}{\rho \delta^{2\alpha}} \int_{-1}^{1} \frac{1 - \cos(\xi \delta z)}{|z|^{1+2\alpha}} dz
\]
\[
\leq \frac{2\kappa}{\rho \delta^{2\alpha}} \int_{-1}^{1} \min \left\{ \frac{\xi^2 \delta^2 z^2}{2}, 2 \right\} \frac{dz}{|z|^{1+2\alpha}}
\]
\[
\leq \frac{2\kappa}{\rho \delta^{2\alpha}} \left[ \int_{\{z| \leq \min\{\frac{\pi}{\delta}, 1\}\}} \frac{\xi^2 \delta^2 z^2}{2|z|^{1+2\alpha}} dz + \int_{\{\pi \delta < |z| \leq 1\}} \frac{2}{|z|^{1+2\alpha}} dz \right]
\]
\[
= \frac{2\kappa}{\rho \delta^{2\alpha}} \left[ \frac{\xi^2 \delta^2}{2 - 2\alpha} \min \left\{ \frac{2^{2-2\alpha}}{|\xi|^{2-2\alpha} \delta^2 - 2\alpha}, 1 \right\} + \frac{2}{\alpha} \chi(0,1) \left( \frac{2}{|\xi| \delta} \left( \frac{|\xi|^2 \delta^2 - 2\alpha}{2\alpha} - 1 \right) \right) \right],
\]
that gives (1.7).

We point out that problem (1.3) reduces to the classical wave equation as \(\alpha \to 1^-\), in a sense which is made precise in Lemma A.1.

Similarly, the explicit solution provided in (1.5) and (1.6) approaches the one obtained by Fourier methods for the classical wave equation, as specified in Lemma A.2.

2. Dispersion relation

We now deepen our analysis of the dispersion relation introduced in (1.6) by supporting the estimate in (1.7) with some precise asymptotics:

**Theorem 2.1.** For \(\delta > 0\) and \(0 < \alpha < 1\), we have that

\[
\lim_{\xi \to 0} \xi^{-2} \omega^2(\xi) = \frac{\kappa \delta^{2(1-\alpha)}}{(1-\alpha) \rho}
\]
and

\[
\lim_{\xi \to \pm \infty} |\xi|^{-2\alpha} \omega^2(\xi) = \frac{4\kappa}{\rho} \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau.
\]

We observe that the asymptotics in (2.1) and (2.2) show a different power law behavior of the dispersion relation \(\omega\) at zero and at infinity. This different behavior is also confirmed numerically in Figure 1, where \(\omega\) is plotted in logarithmic scale: as usual, in this setting, the two different power laws correspond to straight lines with different slopes. Furthermore, we point out that a suitable constitutive restriction on the elastic material parameter \(\kappa\) should take into account the asymptotic scaling stated in (2.1), namely

\(\kappa \sim \frac{1}{\delta^{2(1-\alpha)}}\).

**Proof of Theorem 2.1** Let \(\xi_j\) be an infinitesimal sequence and

\[
F_j(z) := \frac{1 - \cos(\xi_j \delta z)}{\xi_j^2 \delta^2 |z|^{1+2\alpha}} = \frac{\delta^2 |z|^{1-2\alpha}(1 - \cos(\xi_j \delta z))}{(\xi_j \delta z)^2}.
\]

We point out that, for each \(z \in [-1, 1]\),

\[
\lim_{j \to +\infty} F_j(z) = \frac{\delta^2 |z|^{1-2\alpha}}{2}.
\]

Additionally, recalling (1.9),

\[
|F_j(z)| \leq \frac{\xi_j^2 \delta^2 z^2}{2 \xi_j^2 \delta^2 |z|^{1+2\alpha}} = \frac{\delta^2 |z|^{1-2\alpha}}{2} =: G(z).
\]
Since $G \in L^1([-1, 1])$, we can use the Dominated Convergence Theorem and infer that

$$\lim_{j \to +\infty} \int_{-1}^{1} \frac{1 - \cos(\xi_j \delta z)}{\xi_j^2 |z|^{1+2\alpha}} \, dz = \lim_{j \to +\infty} \int_{-1}^{1} F_j(z) \, dz = \int_{-1}^{1} \frac{\delta^2 |z|^{1-2\alpha}}{2} \, dz = \frac{\delta^2}{2 - 2\alpha}.$$  

From this equation and the definition of $\omega$ in (1.6) we plainly obtain the desired result in (2.1).

Moreover, using the substitution $\tau := |\xi| \delta z$,

$$\lim_{\xi \to \pm \infty} |\xi|^{-2\alpha} \omega^2(\xi) = \lim_{\xi \to \pm \infty} \frac{4\kappa}{\rho \delta^2} \int_{0}^{1} \frac{1 - \cos(\xi \delta z)}{z^{1+2\alpha}} \, dz = \frac{4\kappa}{\rho} \int_{0}^{1} \frac{|\xi|^{1-2\alpha}}{\tau^{1+2\alpha}} \, d\tau,$$

from which (2.2) plainly follows.

Now we present a sharpening of Theorem 2.1 in a logarithmic scale, in view of an asymptotic as $\alpha \to 1^-$. 

![Figure 1. Numerical plots of $\omega$ in logarithmic scale and $\kappa = 1/2$ and $\rho = \delta = 1$.](image)
Theorem 2.2. Let \( \delta > 0 \) and \( \frac{1}{2} \leq \alpha < 1 \). Then, given \( b > 0 \) there exists \( C > 0 \), that depends only on \( b \) and \( \delta \), such that for all \( \xi \in \mathbb{R} \setminus (-b, b) \)

\[
(2.3) \quad \left| \frac{\sqrt{(1-\alpha)\rho \omega(\xi)}}{\sqrt{R} |\xi|^{\alpha}} - 1 \right| \leq \sqrt{C} (1-\alpha).
\]

Also, there exists \( c \in (0, \frac{1}{2}) \), that depends only on \( b \), \( \delta \) and \( \kappa \), such that if \( 1-\alpha \leq c \) then, for all \( \xi \in \mathbb{R} \setminus (-b, b) \),

\[
(2.4) \quad \left| \log \frac{\sqrt{(1-\alpha)\rho \omega(\xi)}}{\sqrt{R}} - \alpha \log |\xi| \right| \leq \sqrt{C} (1-\alpha).
\]

We observe that (2.4) gives a convergence of the dispersion relation to a straight line in logarithmic scale (with an explicit error bound). Also (2.4) states that this convergence is uniform\(^3\) outside the origin. For a numerical evidence of the convergence of the dispersion relation to a straight line in logarithmic scale see Figure 2.

![Figure 2. Numerical plots of \( \omega \) (left) and \( \sqrt{1-\alpha} \omega \) (right) in logarithmic scale for \( \alpha = 1 - 10^{-6}, 1 - 10^{-10}, 1 - 10^{-20} \) and \( \kappa = 1/2 \) and \( \rho = \delta = 1 \).](image)

Proof of Theorem 2.2. Let \( a_0 \in (0, 1) \). First of all, we claim that there exists \( C > 0 \) depending only on \( a_0 \) such that for every \( t \geq a_0 \)

\[
(2.5) \quad \left| \int_0^t \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau - \frac{1}{4(1-\alpha)} \right| \leq C.
\]

To check this, we distinguish two cases, according to whether \( t \in [a_0, 1] \) or \( t > 1 \). If \( t \in [a_0, 1] \), we use that

\[
1 - \cos \tau - \frac{\tau^2}{2} + \frac{\tau^3}{6} \geq 0 \quad \text{for all} \quad \tau \in \mathbb{R}
\]

and we see that

\[
(2.6) \quad \int_0^t \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \geq \int_0^t \frac{\tau^2}{2} - \frac{\tau^3}{6} d\tau = \frac{t^2 - 2\alpha}{4(1-\alpha)} + \frac{t^{3-2\alpha}}{6(2\alpha - 3)}.
\]

Similarly, since

\[
1 - \cos \tau - \frac{\tau^2}{2} \leq 0 \quad \text{for all} \quad \tau \in \mathbb{R},
\]

\(^3\)We cannot expect uniform convergence up to the origin. Indeed, as we will see in (2.19), near the origin

\[
\omega(\xi) = \frac{\sqrt{R} |\xi|^{1-\alpha}}{\sqrt{(1-\alpha)\rho}} (1 + O(\xi^2))
\]

and therefore

\[
\left| \log \frac{\sqrt{(1-\alpha)\rho \omega(\xi)}}{\sqrt{R}} - \alpha \log |\xi| \right| = (1-\alpha) |\log \delta + \log |\xi||
\]

which diverges as \( \xi \to 0 \).
we obtain that
\[
\int_0^t \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \leq \int_0^t \frac{\tau^2}{\tau^{1+2\alpha}} d\tau = \frac{t^{2-2\alpha}}{4(1 - \alpha)}.
\]
By combining this and (2.6), we obtain that, for all \( t \in [a_0, 1] \),
\[
\left| \int_0^t \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau - \frac{1}{4(1 - \alpha)} \right| \leq \frac{1 - t^{2-2\alpha}}{4(1 - \alpha)} + \frac{t^{3-2\alpha}}{6(3 - 2\alpha)} \leq 1 - a_0 \frac{2(1-\alpha)}{4(1 - \alpha)} + \frac{1}{6}.
\]
Thus, since
\[
a_0 \frac{2(1-\alpha)}{4(1 - \alpha)} = \exp \left( (1 - \alpha) \log(a_0^2) \right) \geq 1 + (1 - \alpha) \log(a_0^2),
\]
thanks to the convexity of the exponential function, we conclude that
\[
\left| \int_0^t \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau - \frac{1}{4(1 - \alpha)} \right| \leq -\log(a_0^2) + \frac{1}{6}.
\]
This proves (2.3) when \( t \in [a_0, 1] \). If instead \( t > 1 \), we use (2.5) with \( t := 1 \) to see that
\[
\left| \int_0^t \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau - \frac{1}{4(1 - \alpha)} \right| \leq \left| \int_0^1 \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau - \frac{1}{4(1 - \alpha)} \right| + \int_1^t \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \leq C + 2 \int_1^{+\infty} \frac{d\tau}{\tau^2},
\]
from which we obtain (2.5) in this case as well.

Hence, combining (1.6) and (2.5), for all \( \xi \geq a_0/\delta \),
\[
\frac{\sqrt{\kappa}^{\alpha}}{(1 - \alpha)\rho} \left| \omega(\xi) - \frac{\sqrt{\kappa}^{\alpha}}{(1 - \alpha)\rho} \right| \leq \left( \omega(\xi) + \frac{\sqrt{\kappa}^{\alpha}}{(1 - \alpha)\rho} \right) \left| \omega(\xi) - \frac{\sqrt{\kappa}^{\alpha}}{(1 - \alpha)\rho} \right|
= \omega^2(\xi) - \frac{\kappa^{2\alpha}}{(1 - \alpha)\rho}
= \frac{4\kappa}{\rho \delta^{2\alpha}} \int_0^1 \frac{1 - \cos(\xi \delta z)}{z^{1+2\alpha}} dz - \frac{\kappa^{2\alpha}}{(1 - \alpha)\rho}
= \frac{4\kappa^{2\alpha}}{\rho} \left| \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau - \frac{1}{4(1 - \alpha)} \right|
\leq C \frac{\kappa^{2\alpha}}{\rho}
\]
and therefore the claim in (2.3) plainly follows by taking \( a_0 := b\delta \) and recalling that \( \omega \) is an even function.

Moreover, we claim that
\[
(2.7) \quad |\log(1 + r)| \leq 4|r| \quad \text{for every } r \in \left[ -\frac{1}{2}, \frac{1}{2} \right].
\]
Indeed, suppose not, namely
\[
\min \left\{ |\psi|, 0 \right\} < 0,
\]
where \( \psi(r) := 4|r| - |\log(1 + r)| \). Let \( r_0 \) be the point attaining the above minimum. Thus, since \( \psi(0) = 0 \), \( \psi \left( -\frac{1}{2} \right) = 2 - |\log \frac{1}{2}| > 0 \) and \( \psi \left( \frac{1}{2} \right) = 2 - \log \frac{3}{2} > 0 \), necessarily \( \psi'(r_0) = 0 \) and \( r_0 \neq 0 \). This gives that
\[
0 = \psi'(r_0) = \begin{cases} 
4 - \frac{1}{1 + r_0}, & \text{if } r_0 \in (0, \frac{1}{2}), \\
-4 + \frac{1}{1 + r_0}, & \text{if } r_0 \in (-\frac{1}{2}, 0).
\end{cases}
\]
The above cases readily produce a contradiction, hence (2.7) is proved.

Using together (2.3) and (2.7) with \( r := \frac{\sqrt{(1-\alpha)\rho} \omega(\xi)}{\sqrt{\kappa}^{\alpha}} - 1 \), we obtain that
\[
\left| \log \frac{\sqrt{(1-\alpha)\rho} \omega(\xi)}{\sqrt{\kappa}^{\alpha}} - \alpha \log |\xi| \right| = \left| \log \frac{\sqrt{(1-\alpha)\rho} \omega(\xi)}{\sqrt{\kappa}^{\alpha}} \right| \leq 4 \left| \frac{(1-\alpha)\rho \omega(\xi)}{\sqrt{\kappa}^{\alpha}} - 1 \right| \leq 4C (1 - \alpha)
\]
as long as $\sqrt{C(1-\alpha)} \leq \frac{1}{2}$. This establishes (2.4), up to renaming $C$. \hfill \Box

As it is well known, in wave propagation in dispersive medium, initiated by Lord Rayleigh (3), a crucial role is played by the notion of group velocity which is given by the derivative of the dispersion with respect to the frequency variable. Indeed, this remarkable role stays in the property that the group velocity corresponds to the velocity of energy transport (2) in a large class of so called nondissipative media. Therefore, now we are going to extend the asymptotics of Theorem 2.1 to the derivatives of the dispersion, to the aim of obtaining quantitative estimates on the behavior of the group velocity.

**Theorem 2.3.** For $\delta > 0$ and $0 \leq \alpha < 1$, we have that

\begin{equation}
\lim_{\xi \to 0^\pm} \omega'(|\xi|) = \pm \frac{\sqrt{\kappa \delta}^{1-\alpha}}{\sqrt{(1-\alpha)\rho}},
\end{equation}

\begin{equation}
\lim_{\xi \to \pm \infty} |\xi|^{-\alpha} \omega'(\xi) = \pm 2\alpha \sqrt{\frac{\kappa}{\rho}} \int_0^{\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau,
\end{equation}

\begin{equation}
\lim_{\xi \to 0^\pm} |\xi|^{-\omega''(\xi)} = -\sqrt{\frac{\kappa(1-\alpha)}{4(2-\alpha)\sqrt{\rho}}},
\end{equation}

\begin{equation}
\lim_{\xi \to 0^\pm} \omega''(\xi) = 0,
\end{equation}

\begin{equation}
\begin{split}
\text{if } \alpha \in \left(\frac{1}{2}, 1\right) & \text{ then } \lim_{\xi \to \pm \infty} |\xi|^{-\alpha} \omega''(\xi) = -2\alpha(1-\alpha)\sqrt{\kappa} \left(\int_0^{\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau\right)^{1/2}, \\
\text{if } \alpha \in \left[0, \frac{1}{2}\right) & \text{ then } \lim_{\xi \to \pm \infty} |\xi|^{-\alpha} \omega''(\xi) = -\sqrt{\frac{\kappa \delta}{\rho}} \int_0^{\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \left(\int_0^{\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau\right)^{-1/2}, \\
\text{if } \alpha = \frac{1}{2} & \text{ then } \lim_{\xi \to \pm \infty} |\xi|^{-\alpha} \omega''(\xi) = \left[\frac{\kappa}{\sqrt{\rho}} \left(\int_0^{\infty} \frac{1 - \cos \tau}{\tau^{2}} d\tau\right)^{-1/2} - \frac{1}{2} \left(\frac{\kappa}{\rho} \int_0^{\infty} \frac{1 - \cos \tau}{\tau^{2}} d\tau\right)^{1/2}\right] \left[\frac{\kappa}{\sqrt{\rho}} \left(\int_0^{\infty} \frac{1 - \cos \tau}{\tau^{2}} d\tau\right)^{-1/2} - \frac{1}{2} \left(\frac{\kappa}{\rho} \int_0^{\infty} \frac{1 - \cos \tau}{\tau^{2}} d\tau\right)^{1/2}\right]^{-1/2},
\end{split}
\end{equation}

\begin{equation}
\lim_{\xi \to \pm \infty} |\xi|^{-\alpha} \omega''(\xi),
\end{equation}

---

We stress that Theorem 2.3 highlights a number of special features of the dispersion relation. Indeed, it follows from the asymptotics in (2.8) and (2.11) of Theorem 2.3 that $\omega'$ has a jump discontinuity at the origin (hence $\omega$ presents a corner), but $\omega''$ (as a function defined in $\mathbb{R} \setminus \{0\}$) can be extended continuously through the origin.

Moreover, the convexity properties of the dispersion relation present an interesting dependence on $\alpha$. Specifically, when $\alpha \in \left(\frac{1}{2}, 1\right)$, formula (2.12) in Theorem 2.3 gives that $\omega''$ is negative at infinity and therefore $\omega$ is concave at infinity. Instead, when $\alpha \in \left(0, \frac{1}{2}\right]$, the asymptotics in formulas (2.13) and (2.14) of Theorem 2.3 state that $\omega''$ changes sign infinitely many times at infinity and consequently in this range the dispersion relation $\omega$ switches from convex to concave infinitely often. Besides detailed analytic proofs, we also provide numerical confirmations of these phenomena. In particular, the function $\omega''$ is plotted in Figure 3, notice that $\omega''$ is shown to intersect the horizontal axis infinitely many
times when $\alpha \in \left(0, \frac{1}{2}\right]$ in agreement with (2.13) and (2.14) (and differently from the case $\alpha \in \left(\frac{1}{2}, 1\right)$ which instead is in agreement with (2.12)).

An additional interesting feature showcased by Theorem 2.3 is that the derivatives of the dispersion relation do not inherit the “natural decay at infinity” from the original function. In particular, while $\omega$ at infinity behaves like $|\xi|^\alpha$ in light of (2.2), contrary to the usual situations it is not always true in this setting that $\omega''$ behaves at infinity like $|\xi|^{\alpha-2}$ (that is, like $\frac{\omega}{\xi^2}$). More precisely, while this is true when $\alpha \in \left(\frac{1}{2}, 1\right)$, thanks to formula (2.12) in Theorem 2.3, in the range $\alpha \in \left(0, \frac{1}{2}\right]$ the behavior is completely different and the leading order happens to be $|\xi|^{1+\alpha}$ (surprisingly corresponding to $\frac{\omega}{\xi}$, and also presenting oscillatory behaviors).

The different power law behaviors of $\omega''$ at infinity stated in (2.12), (2.13) and (2.14) are also numerically confirmed by Figures 4 and 5. In particular, Figure 4 showcases a numerical plot of $|\xi|^{2-\alpha}\omega''$ that confirms the convergence at infinity if $\alpha \in \left(\frac{1}{2}, 1\right)$, in agreement with (2.12), its divergence if $\alpha \in \left(0, \frac{1}{2}\right)$, in agreement with (2.13), its oscillatory boundedness $\alpha = \frac{1}{2}$, in agreement with (2.14). Instead, Figure 5 showcases a numerical plot of $|\xi|^{1+\alpha}\omega''$ that confirms the divergence at infinity if $\alpha \in \left(\frac{1}{2}, 1\right)$, in agreement with (2.12), and its bounded oscillatory behavior if $\alpha \in \left(0, \frac{1}{2}\right]$, in agreement with (2.13) and (2.14). We also notice that when $\alpha = \frac{1}{2}$ the plots in Figures 4 and 5 agree, consistently with the fact that $2 - \alpha = 1 + \alpha$ in this specific case.

The unusual phenomena detected in (2.12), (2.13) and (2.14) are deeply related to the nonlocal nature of the problem and to the appearance of divergent singular integrals in the formal expansions.

**Figure 3.** Numerical plot for $\omega''$ when $\kappa = 1/2$ and $\rho = \delta = 1$. 
of the dispersion relation. This important technical details prevent us to use lightly formal expansions and soft arguments of general flavor since, roughly speaking, terms that are usually “negligible” in a standard expansion may become “dominant” in our setting since they may end up being multiplied by a “divergent” coefficient induced by a singular integral and, quite interestingly, as emphasized by the asymptotics in formulas (2.13) and (2.14) of Theorem 2.3, these new significant terms may even be of oscillatory type.

The “numerology” of Theorem 2.3 is also somewhat interesting since all the coefficients appearing in the asymptotics are determined explicitly (and finding an explicit representation of a coefficient is often the most direct way to prove that it is finite as well). As a matter of fact, though we do not make use of this fact, we mention that the trigonometric integral appearing in Theorem 2.3 (as well as in (2.2)) can be computed in terms of the Euler Gamma Function, since

\[
\int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} \, d\tau = - \cos(\pi\alpha) \Gamma(-2\alpha),
\]

with the left hand side continuously extended to the value \(\frac{\pi}{2}\) when \(\alpha = \frac{1}{2}\), see Appendix B for an elementary (or Appendix C for a shorter, but more sophisticated) proof of (2.15). See also Figure 6 for a numerical confirmation of (2.15): indeed, in Figure 6 the plots of the functions \(\alpha \mapsto \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} \, d\tau \) and \(\alpha \mapsto - \cos(\pi\alpha) \Gamma(-2\alpha)\) are given and one can observe that the two graphs are the same, up to a sign change, in full agreement with (2.15).
Figure 5. Numerical plot for $|\xi|^{1+\alpha} \omega''$ when $\kappa = 1/2$ and $\rho = \delta = 1$.

Figure 6. Numerical evidence for (2.15).

It is also interesting to recall that the threshold $\alpha = \frac{1}{2}$ that emerges in formulas (2.12), (2.13) and (2.14) of Theorem 2.3 is also an important threshold for several other nonlocal problems, see e.g. [4, 16, 7].
**Proof of Theorem 2.3** Differentiating (1.6) and using the change of variable \( w := \xi \delta z \) we see that

\[
2\omega(\xi)\omega'(\xi) = \frac{d}{d\xi} \omega^2(\xi) = \frac{d}{d\xi} \left( \frac{2\kappa}{\rho \delta^{2\alpha}} \int_{-1}^{1} \frac{1 - \cos(\xi \delta z)}{|z|^{1+2\alpha}} dz \right) = \frac{2\kappa}{\rho \delta^{2\alpha-1}} \int_{-1}^{1} \frac{z \sin(\xi \delta z)}{|z|^{1+2\alpha}} dz
\]

(2.16)

From this and (2.1), using l'Hôpital’s Rule we find that

\[
\frac{2\sqrt{\kappa} \delta^{1-\alpha}}{\sqrt{(1-\alpha) \rho}} \lim_{\xi \to 0^\pm} \omega'(\xi) = \lim_{\xi \to 0^\pm} \frac{2\omega(\xi)\omega'(\xi)}{|\xi|} = \lim_{\xi \to 0^\pm} \frac{4\kappa}{\rho |\xi|^{2-2\alpha}} \int_{0}^{\xi \delta \rho} \sin |w| |w|^{2\alpha} dw
\]

(2.17)

\[
= \lim_{\xi \to 0^\pm} \frac{4\kappa}{(2-2\alpha)\rho |\xi|^{2-2\alpha}} \sin |\xi \delta| = \lim_{\xi \to 0^\pm} \frac{2\kappa \delta^{1-2\alpha}}{(1-\alpha)\rho} \sin |\xi \delta|
\]

\[
= \pm \frac{2\kappa \delta^{2-2\alpha}}{(1-\alpha)\rho}.
\]

From this, we obtain formula (2.8) in Theorem 2.3

Similarly, using (2.2) and (2.16),

\[
\frac{4\sqrt{\kappa}}{\sqrt{\rho}} \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \lim_{\xi \to \pm\infty} |\xi|^{1-\alpha} \omega'(\xi) = \lim_{\xi \to \pm\infty} 2|\xi|^{1-2\alpha} \omega(\xi)\omega'(\xi)
\]

\[
= \lim_{\xi \to \pm\infty} \frac{4\kappa}{\rho} \int_{0}^{\xi \delta} w \sin w |w|^{1+2\alpha} dw = \frac{4\kappa}{\rho} \int_{0}^{+\infty} \sin w |w|^{2\alpha} dw = \pm \frac{4\kappa}{\rho} \int_{0}^{+\infty} \sin \frac{\tau}{\tau^{2\alpha}} d\tau.
\]

We stress that when \( \alpha \in (0, \frac{1}{2}] \) the function \( \sin \frac{\tau}{\tau^{2\alpha}} \) is not Lebesgue summable in \((0, +\infty)\); nevertheless, for every \( \alpha \in (0, 1) \) one can write the improper Riemann integral

\[
\int_{0}^{+\infty} \frac{\sin \tau}{\tau^{2\alpha}} d\tau := \lim_{R \to +\infty} \int_{0}^{R} \frac{\sin \tau}{\tau^{2\alpha}} d\tau = \lim_{R \to +\infty} \int_{0}^{R} \left[ \frac{1 - \cos \tau}{\tau^{2\alpha}} \right]' + 2\alpha \left( \frac{1 - \cos \tau}{\tau^{1+2\alpha}} \right) d\tau
\]

\[
(2.17)
\]

\[
= \lim_{R \to +\infty} \int_{0}^{R} \frac{1 - \cos \tau}{R^{2\alpha}} - \lim_{R \to +\infty} \frac{1 - \cos t}{t^{2\alpha}} + 2\alpha \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau
\]

\[
= 0 - \lim_{t \to 0} \frac{t^{2-2\alpha}}{t^{2\alpha}} t^{1-2\alpha} + 2\alpha \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau = 2\alpha \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau.
\]

These observations lead to

\[
\frac{4\sqrt{\kappa}}{\sqrt{\rho}} \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \lim_{\xi \to \pm\infty} |\xi|^{1-\alpha} \omega'(\xi) = \pm \frac{8\kappa}{\rho} \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau,
\]

which produces formula (2.9) in Theorem 2.3

Also, differentiating once more in (2.16), for all \( \xi \neq 0 \),

\[
\omega(\xi)\omega''(\xi) + (\omega'(\xi))^2 = \frac{d}{d\xi} \left( \omega(\xi)\omega'(\xi) \right) = \frac{d}{d\xi} \left( \frac{2\kappa}{\rho \delta^{2\alpha-1}} \int_{-1}^{1} \frac{\sin(\xi \delta z)}{z^{2\alpha}} dz \right)
\]

\[
= \frac{2\kappa \delta^{2-2\alpha}}{\rho} \int_{0}^{1} \cos(\xi \delta z) \frac{z^{2-2\alpha}}{z^{2\alpha-1}} dz.
\]
Thus, using again (2.16),
\[
\omega''(\xi) = \frac{2 \kappa \delta^{2-2\alpha}}{\rho} \int_0^1 \frac{\cos(\xi \delta z)}{z^{2\alpha-1}} dz - (\omega'(\xi))^2 \\
\tag{2.18}
\]
\[
= \frac{2 \kappa \delta^{2-2\alpha}}{\rho} \int_0^1 \frac{\cos(\xi \delta z)}{z^{2\alpha-1}} dz - \left( \frac{2 \kappa}{\rho \delta^{2\alpha-1}} \omega(\xi) \int_0^1 \frac{\sin(\xi \delta z)}{z^{2\alpha}} dz \right)^2.
\]

Also, in light of (1.6), as \( \xi \to 0^+ \),
\[
\omega^2(\xi) = \frac{2 \kappa}{\rho \delta^{2\alpha}} \int_{-1}^1 \frac{1 - \cos(\xi \delta z)}{|z|^{1+2\alpha}} dz = \frac{4 \kappa}{\rho \delta^{2\alpha}} \int_0^1 \frac{1}{z^{1+2\alpha}} dz - \frac{(\xi \delta z)^2}{24} + O(\xi^6) dz
\]
\[
\tag{2.19}
= \frac{4 \kappa}{\rho \delta^{2\alpha}} \left( \frac{\xi^2 \delta^2}{4(1 - \alpha)} - \frac{\xi^4 \delta^4}{48(2 - \alpha)} + O(\xi^6) \right) = \frac{\kappa \xi^2 \delta^2(1 - \alpha)}{(1 - \alpha) \rho} - \frac{\kappa \xi^4 \delta^2(2 - \alpha)}{12(2 - \alpha) \rho} + O(\xi^6)
\]
which can be seen as an enhanced version of (2.1).

As a result,
\[
\left( \frac{2 \kappa}{\rho \delta^{2\alpha-1}} \omega(\xi) \right)^2 = \frac{4(1 - \alpha) \kappa}{\xi^2 \delta^2 \rho} \left( 1 - \frac{(1 - \alpha) \xi^2 \delta^2}{12(2 - \alpha)} + O(\xi^4) \right) = \frac{4(1 - \alpha) \kappa}{\xi^2 \delta^2 \rho} \left( 1 + \frac{(1 - \alpha) \xi^2 \delta^2}{12(2 - \alpha)} + O(\xi^4) \right).
\]

Hence, by (2.18), as \( \xi \to 0^+ \),
\[
\frac{\sqrt{\kappa \xi^{1-\alpha}}}{\sqrt{(1 - \alpha) \rho}} \sqrt{1 + O(\xi^2)} \omega''(\xi)
\]
\[
= \frac{2 \kappa \delta^{2-2\alpha}}{\rho} \int_0^1 \frac{\cos(\xi \delta z)}{z^{2\alpha-1}} dz - \frac{4(1 - \alpha) \kappa}{\xi^2 \delta^2 \rho} \left( 1 + \frac{(1 - \alpha) \xi^2 \delta^2}{12(2 - \alpha)} + O(\xi^4) \right) \left( \int_0^1 \frac{\sin(\xi \delta z)}{z^{2\alpha}} dz \right)^2.
\]

Thus, noticing that, as \( \xi \to 0^+ \),
\[
\int_0^1 \frac{\cos(\xi \delta z)}{z^{2\alpha-1}} dz = \int_0^1 \frac{1 - (\xi \delta z)^2}{2} + O(\xi^4) \frac{dz}{z^{2\alpha}} dz = \frac{1}{2(1 - \alpha)} - \frac{\xi^2 \delta^2}{4(2 - \alpha)} + O(\xi^4)
\]
and
\[
\int_0^1 \frac{\sin(\xi \delta z)}{z^{2\alpha}} dz = \int_0^1 \xi \delta z - \frac{(\xi \delta z)^3}{6} + O(\xi^5) z^5 \frac{dz}{z^{2\alpha}} dz = \frac{\xi \delta}{2(1 - \alpha)} - \frac{\xi^3 \delta^3}{12(2 - \alpha)} + O(\xi^5),
\]
we conclude that
\[
\frac{\sqrt{\kappa \xi^{1-\alpha}}}{\sqrt{(1 - \alpha) \rho}} \sqrt{1 + O(\xi^2)} \omega''(\xi)
\]
\[
= \frac{2 \kappa \delta^{2-2\alpha}}{\rho} \left( \frac{1}{2(1 - \alpha)} - \frac{\xi^2 \delta^2}{4(2 - \alpha)} + O(\xi^4) \right) - \frac{4(1 - \alpha) \kappa \delta^{2-2\alpha}}{\rho} \left( 1 + \frac{(1 - \alpha) \xi^2 \delta^2}{12(2 - \alpha)} + O(\xi^4) \right) \left( \frac{1}{2(1 - \alpha)} - \frac{\xi^2 \delta^2}{12(2 - \alpha)} + O(\xi^4) \right)^2
\]
\[
= \frac{2 \kappa \delta^{2-2\alpha}}{\rho} \left( \frac{1}{2(1 - \alpha)} - \frac{\xi^2 \delta^2}{4(2 - \alpha)} + O(\xi^4) \right) - \frac{4(1 - \alpha) \kappa \delta^{2-2\alpha}}{\rho} \left( 1 + \frac{(1 - \alpha) \xi^2 \delta^2}{12(2 - \alpha)} + O(\xi^4) \right) \left( \frac{1}{4(1 - \alpha)^2} - \frac{\xi^2 \delta^2}{12(1 - \alpha)(2 - \alpha)} + O(\xi^4) \right)
\]
\[
= \frac{\kappa \delta^{2-2\alpha}}{\rho} \left( \frac{1}{1 - \alpha} - \frac{\xi^2 \delta^2}{2(2 - \alpha)} + O(\xi^4) \right) - \frac{\kappa \delta^{2-2\alpha}}{\rho} \left( \frac{1}{1 - \alpha} + \frac{\xi^2 \delta^2}{12(2 - \alpha)} - \frac{\xi^2 \delta^2}{3(2 - \alpha)} + O(\xi^4) \right)\]
This entails that
\[ \frac{\sqrt{\kappa\delta^{1-\alpha}}}{\sqrt{(1-\alpha)\rho}} \lim_{\xi \to 0^+} \xi^{-1} \omega''(\xi) = -\frac{\kappa\delta^{1-2\alpha}}{4(2-\alpha)\rho} \]
and therefore
\[ (2.20) \quad \lim_{\xi \to 0^+} |\xi|^{-1} \omega''(\xi) = -\frac{\sqrt{\kappa(1-\alpha)\delta^{3-\alpha}}}{4(2-\alpha)\sqrt{\rho}}. \]
Since \( \omega \) (and thus \( \omega'' \)) is an even function, we also have that
\[ \lim_{\xi \to 0^-} |\xi|^{-1} \omega''(\xi) = -\frac{\sqrt{\kappa(1-\alpha)\delta^{3-\alpha}}}{4(2-\alpha)\sqrt{\rho}}. \]
This and \( (2.20) \) give formula \( (2.10) \) in Theorem 2.3 (from which formula \( (2.11) \) in Theorem 2.3 follows at once).
Let now \( \xi > 0 \). From \( (1.6) \),
\[ \omega(\xi) = \left( \frac{4\kappa}{\rho \delta^{2\alpha}} \int_0^{\xi\delta} \frac{1 - \cos(\xi\delta z)}{z^{1+2\alpha}} dz \right)^{1/2} = \xi^\alpha \left( \frac{4\kappa}{\rho} \int_0^{\xi\delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{1/2} \]
and therefore\(^4\)
\[ (2.21) \quad \omega'(\xi) = \alpha \xi^{\alpha-1} \left( \frac{4\kappa}{\rho} \int_0^{\xi\delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{1/2} + \frac{2\kappa \xi^{-\alpha-1}(1 - \cos(\xi\delta))}{\rho \delta^{2\alpha}} \left( \frac{4\kappa}{\rho} \int_0^{\xi\delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{-1/2}. \]
Taking one more derivative,
\[ \omega''(\xi) = \alpha(\alpha - 1) \xi^{\alpha-2} \left( \frac{4\kappa}{\rho} \int_0^{\xi\delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{1/2} + \frac{2\alpha \kappa \xi^{-\alpha-2}(1 - \cos(\xi\delta))}{\rho \delta^{2\alpha}} \left( \frac{4\kappa}{\rho} \int_0^{\xi\delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{-1/2} - \frac{2(\alpha + 1) \kappa \xi^{\alpha-2}(1 - \cos(\xi\delta))}{\rho \delta^{2\alpha}} \left( \frac{4\kappa}{\rho} \int_0^{\xi\delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{-1/2} + \frac{2\kappa \xi^{-\alpha-1} \sin(\xi\delta)}{\rho \delta^{2\alpha-1}} \left( \frac{4\kappa}{\rho} \int_0^{\xi\delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{-1/2} - \frac{4\kappa^2 \xi^{-3\alpha-2}(1 - \cos(\xi\delta))}{\rho^2 \delta^{4\alpha}} \left( \frac{4\kappa}{\rho} \int_0^{\xi\delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{-3/2}. \]
\(^4\)We observe that an alternative proof of formula \( (2.9) \) in Theorem 2.3 follows from \( (2.21) \), namely
\[ \lim_{\xi \to +\infty} \xi^{1-\alpha} \omega'(\xi) = \lim_{\xi \to +\infty} \left[ \alpha \left( \frac{4\kappa}{\rho} \int_0^{\xi\delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{1/2} + \frac{2\kappa \xi^{-2\alpha}(1 - \cos(\xi\delta))}{\rho \delta^{2\alpha}} \left( \frac{4\kappa}{\rho} \int_0^{\xi\delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{-1/2} \right] = \alpha \left( \frac{4\kappa}{\rho} \int_0^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{1/2} + \frac{2\kappa \xi^{-2\alpha}(1 - \cos(\xi\delta))}{\rho \delta^{2\alpha}} \left( \frac{4\kappa}{\rho} \int_0^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right)^{-1/2} \]
and formula \( (2.9) \) of Theorem 2.3 follows from this and the fact that \( \omega' \) is odd.
That is, as \( \xi \to +\infty \),
\[
\omega''(\xi) = \alpha(\alpha - 1)\xi^{\alpha - 2} \left( \frac{4\kappa}{\rho} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} \ d\tau \right)^{1/2} 
+ \frac{2\kappa \xi^{-\alpha - 1} \sin(\xi \delta)}{\rho \delta^{2\alpha - 1}} \left( \frac{4\kappa}{\rho} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} \ d\tau \right)^{-1/2} + o(\xi^{\alpha - 2}) + o(\xi^{-\alpha - 1}).
\]

(2.23)

Consequently, if \( \alpha \in \left( \frac{1}{2}, 1 \right) \),
\[
|\xi|^{-2-\alpha} \omega''(\xi) = \alpha(\alpha - 1) \left( \frac{4\kappa}{\rho} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} \ d\tau \right)^{1/2} + o(1)
\]
as \( \xi \to +\infty \).

Similarly, if \( \alpha \in \left( 0, \frac{1}{2} \right) \), one deduces from (2.23) that
\[
|\xi|^{|+\alpha} \omega''(\xi) = \frac{2\kappa \sin(\xi \delta)}{\rho \delta^{2\alpha - 1}} \left( \frac{4\kappa}{\rho} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} \ d\tau \right)^{-1/2} + o(1)
\]
as \( \xi \to +\infty \).

And also, if \( \alpha = \frac{1}{2} \), we get from (2.23) that
\[
|\xi|^{3/2} \omega''(\xi) = -\frac{1}{4} \left( \frac{4\kappa}{\rho} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{2}} \ d\tau \right)^{1/2} + \frac{2\kappa \sin(\xi \delta)}{\rho} \left( \frac{4\kappa}{\rho} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{2}} \ d\tau \right)^{-1/2} + o(1)
\]
as \( \xi \to +\infty \).

These observations (and the fact that \( \omega'' \) is an even function) lead to formulas (2.12), (2.13) and (2.14) of Theorem 2.3.

For completeness, we now provide alternative proofs for the statements in (2.12), (2.13) and (2.14) of Theorem 2.3. To this end, using (2.17), we observe that, when \( \alpha \in \left[ \frac{1}{2}, 1 \right) \),
\[
\lim \inf_{R \to +\infty} \int_0^{R} \frac{\cos \tau}{\tau^{2\alpha - 1}} \ d\tau = \lim \inf_{R \to +\infty} \int_0^{R} \left( \frac{\sin \tau}{\tau^{2\alpha - 1}} \right)' + (2\alpha - 1) \frac{\sin \tau}{\tau^{2\alpha}} \ d\tau 
= \lim \inf_{R \to +\infty} \frac{\sin R}{R^{2\alpha - 1}} + (2\alpha - 1) \int_0^{+\infty} \frac{\sin \tau}{\tau^{2\alpha}} \ d\tau 
= \begin{cases} 
-1 & \text{if } \alpha = \frac{1}{2}, \\
2\alpha(2\alpha - 1) \int_0^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} \ d\tau & \text{if } \alpha \in \left( \frac{1}{2}, 1 \right) \).
\end{cases}
\]

Similarly,
\[
\lim \sup_{R \to +\infty} \int_0^{R} \frac{\cos \tau}{\tau^{2\alpha - 1}} \ d\tau = \begin{cases} 
1 & \text{if } \alpha = \frac{1}{2}, \\
2\alpha(2\alpha - 1) \int_0^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} \ d\tau & \text{if } \alpha \in \left( \frac{1}{2}, 1 \right) \).
\end{cases}
\]

Accordingly, recalling (2.2) and (2.18), exploiting (2.17) once again, and using the substitution \( \tau := |\xi|\delta z \), if \( \alpha \in \left[ \frac{1}{2}, 1 \right) \),
\[
\sqrt{\frac{4\kappa}{\rho} \int_0^{+\infty} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} \ d\tau} \lim \inf_{\xi \to +\infty} |\xi|^{2-\alpha} \omega''(\xi) 
= \lim \inf_{\xi \to +\infty} |\xi|^{2-2\alpha} \omega(\xi) \omega''(\xi) 
= \lim \inf_{\xi \to +\infty} |\xi|^{2-2\alpha} \left( \frac{2\kappa \delta^{2-2\alpha}}{\rho} \int_0^{1} \frac{\cos(\xi \delta z)}{z^{2\alpha - 1}} \ dz - \left( \frac{2\kappa}{\rho \delta^{2\alpha - 1}} \omega(\xi) \int_0^{1} \frac{\sin(\xi \delta z)}{z^{2\alpha}} \ dz \right)^{2} \right)
\]
These observations give formulas (2.12) and (2.14) in Theorem 2.3.

Similarly, when

$$\alpha = \frac{1}{2},$$

$$\left\{ \begin{array}{l}
\frac{2\kappa}{\rho} \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{2 \alpha - 1}} d\tau \\
\frac{4\alpha(\alpha - 1)\kappa}{\rho} \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1 + 2\alpha}} d\tau
\end{array} \right.\quad \text{if } \alpha \in \left(\frac{1}{2}, 1\right).$$

These observation give formulas (2.12) and (2.14) in Theorem 2.3.

Besides, when $\alpha \in \left(0, \frac{1}{2}\right)$, we can infer from (2.2), (2.17) and (2.18), that

$$\sqrt{\frac{4\kappa}{\rho}} \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1 + 2\alpha}} d\tau \limsup_{\xi \to \pm\infty} |\xi|^{2-\alpha} \omega''(\xi) = \left\{ \begin{array}{l}
\frac{2\kappa}{\rho} \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{2\alpha}} d\tau \\
\frac{4\alpha(\alpha - 1)\kappa}{\rho} \int_{0}^{+\infty} \frac{1 - \cos \tau}{\tau^{1 + 2\alpha}} d\tau
\end{array} \right.\quad \text{if } \alpha = \frac{1}{2},$$

These observations give formula (2.13) in Theorem 2.3.

3. Decay estimates

In this section we prove the relevant spatial decay properties of the solution of (1.3) which also will be useful in the next section to prove the existence of conserved quantities for the peridynamic problem.
Theorem 3.1. For every given $t \geq 0$, the function $\mathbb{R} \ni x \mapsto u(t, x)$ in (1.5) belongs to the Schwartz Space.

Proof. Let
\begin{equation}
(3.1) \quad \varpi(\xi) := \omega^2(\xi)
\end{equation}
and $\Psi$ be an even real analytic function such that
\begin{equation}
(3.2) \quad \text{the derivatives of } \Psi \text{ of any order are bounded in } [1, +\infty).
\end{equation}
Thus, by the analyticity of $\Psi$, for suitable coefficients $\Psi_j \in \mathbb{R}$, with $|\Psi_j| \leq \frac{C_j}{(2j)!}$ for some $C > 0$, we write
\begin{equation}
(3.3) \quad \Psi(r) := \sum_{j=0}^{+\infty} \Psi_j r^{2j}.
\end{equation}
Therefore, setting
\begin{equation}
(3.4) \quad \psi(r) := \sum_{j=0}^{+\infty} \Psi_j r^j,
\end{equation}
we find that
\begin{equation}
(3.5) \quad \psi \text{ is also a real analytic function}.
\end{equation}
By (1.6), we know that
\begin{equation}
(3.6) \quad \varpi(\xi) = \frac{4\kappa}{\rho \delta 2\alpha} \int_0^1 \frac{1 - \cos(\xi \delta z)}{z^{1+2\alpha}} \, dz = \frac{4\kappa}{\rho \delta 2\alpha} \int_0^{+\infty} \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}(\xi \delta)^{2k} z^{2k-1-2\alpha}}{(2k)!} \, dz
\end{equation}
In particular, we have that
\begin{equation}
(3.7) \quad \varpi \text{ is also a real analytic function}.
\end{equation}
By composition, it follows that
\begin{equation}
(3.8) \quad \phi(\xi) := \psi(\varpi(\xi)) \text{ is also real analytic}.
\end{equation}
Notice that, by construction
\begin{equation}
(3.9) \quad \phi(\xi) = \sum_{j=0}^{+\infty} \Psi_j (\varpi(\xi))^j = \sum_{j=0}^{+\infty} \Psi_j (\omega(\xi))^{2j} = \Psi(\omega(\xi)).
\end{equation}
We claim that
\begin{equation}
(3.10) \quad \text{the derivatives of } \varpi \text{ of any order divided by } (1 + |\xi|^{2\alpha}) \text{ are bounded}.
\end{equation}
To this aim, in view of (3.7), it suffices to show that, for all $\ell \in \mathbb{N}$ and $|\xi| \geq 1$,
\begin{equation}
(3.11) \quad |\varpi^{(\ell)}(\xi)| \leq C_\ell (1 + |\xi|^{2\alpha-\ell}).
\end{equation}

\textit{Statements like (3.2) mean that for all } \ell \in \mathbb{N},
\begin{equation}
\sup_{r \in [1, +\infty)} |\Psi^{(\ell)}(r)| < +\infty,
\end{equation}
where $\Psi^{(\ell)}$ denotes the derivative of $\Psi$ of order $\ell$. 
For this, we first show that for every \( \ell \) there exist \( C_\ell \in \mathbb{R} \) and a function \( F_\ell \in C^\infty((1/2, +\infty)) \) whose derivatives of any order are bounded in \([1, +\infty)\) such that, for every \( \xi \in [1, +\infty) \),

\[
\varpi^{(\ell)}(\xi) = F_\ell(\xi) + C_\ell \xi^{2\alpha - \ell} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau.
\]

(3.12)

In this notation, \( C_\ell \) and \( F_\ell \) may also depend on the structural parameters \( \kappa, \rho \) and \( \delta \), which are supposed to be fixed quantities. Thus, we argue by induction over \( \ell \). When \( \ell = 0 \), changing variable \( \tau := \xi \delta z \) in the first integral of (3.6) we find that

\[
\varpi(\xi) = \frac{4\kappa \xi^{2\alpha}}{\rho} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau,
\]

which is (3.12) with \( C_0 := \frac{4\kappa}{\rho} \) and \( F_0 := 0 \). We now suppose recursively that (3.12) holds true for the index \( \ell \) and we establish it for the index \( \ell + 1 \). For this, taking one further derivative, the inductive assumption leads to

\[
\varpi^{(\ell+1)}(\xi) = \frac{d}{d\xi} \left[ F_\ell(\xi) + C_\ell \xi^{2\alpha - \ell} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \right]
\]

\[
= F'_\ell(\xi) + C_\ell \delta \xi^{2\alpha - \ell} \frac{1 - \cos(\xi \delta)}{\xi \delta} + (2\alpha - \ell) C_\ell \xi^{2\alpha - \ell - 1} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau
\]

\[
= F'_\ell(\xi) + C_\ell \delta^{-2\alpha} \xi^{-(\ell+1)} (1 - \cos(\xi \delta)) + (2\alpha - \ell) C_\ell \xi^{2\alpha - (\ell+1)} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau.
\]

Hence, we define \( C_{\ell+1} := (2\alpha - \ell) C_\ell \) and

\[
F_{\ell+1}(\xi) := F_\ell(\xi) + C_\ell \delta^{-2\alpha} \xi^{-(\ell+1)} (1 - \cos(\xi \delta)),
\]

and we stress that all the derivatives of \( F_{\ell+1} \) are bounded in \([1, +\infty)\), since so are the ones of \( F_\ell \). The inductive step is thereby complete and we have thus established (3.12).

We also notice that, if \( \xi \geq 1 \), then

\[
\xi^{2\alpha} \int_0^{\xi \delta} \frac{1 - \cos \tau}{\tau^{1+2\alpha}} d\tau \leq \xi^{2\alpha} \left[ \int_0^{\delta} \frac{\tau^2}{\tau^{1+2\alpha}} d\tau + \int_{\delta}^{\xi \delta} \frac{2}{\tau^{1+2\alpha}} d\tau \right] \leq \xi^{2\alpha} \left[ \frac{\delta^{2-2\alpha}}{2-2\alpha} + \frac{\delta^{-2\alpha}}{2} \right] \leq C \xi^{2\alpha},
\]

for some \( C > 0 \) depending on \( \delta \). This and (3.12) yield that

\[
|\varpi^{(\ell)}(\xi)| \leq |F_\ell(\xi)| + C C_\ell \xi^{2\alpha - \ell}.
\]

From this and the parity of \( \varpi \) we obtain (3.11) (up to renaming \( C_\ell \)) and thus (3.10).

Now we show that

\[
(3.13)
\]

the derivatives of \( \psi \) of any order are bounded in \([0, +\infty)\).

From (3.5), it suffices to check this claim in \([1, +\infty)\). For this, using (3.3) and (3.4), we observe that \( \psi(r) = \Psi(\sqrt{r}) \) for all \( r > 0 \). For this reason, by the Faà di Bruno’s Formula,

\[
\psi^{(\ell)}(r) = \frac{d^\ell}{d\xi^\ell} (\Psi(\sqrt{r})) = \sum_{m_1 + \cdots + m_\ell = \ell} \frac{\ell!}{m!} \Psi^{(m_1 + \cdots + m_\ell)}(\sqrt{r}) \prod_{j=1}^\ell \left( \frac{d^j}{d\xi^j} \sqrt{r} \right)^{m_j}
\]

with the sum above ranging over all \( m \in \mathbb{N}^\ell \) satisfying the constraint

\[
\sum_{j=1}^\ell j m_j = \ell
\]

(3.15)

and the standard multiindex notation \( m! = m_1! m_2! \cdots m_\ell! \) has been used. We also observe the recursive fact that, for every \( j \geq 1 \),

\[
\frac{d^j}{dr^j} \sqrt{r} = \frac{(-1)^{j+1} r^{1/2-j}}{2^j} \prod_{k=1}^{j-2} (2k + 1).
\]

(3.16)
As a consequence, for all $r \geq 1$ and $j \geq 1$,  
\[
\left| \frac{d^j}{dr^j} \sqrt{r} \right| \leq \frac{r^{\frac{1-2j}{2}}}{2^j} \prod_{k=1}^{j-2} (2k+1) \leq \frac{r^{\frac{1-2j}{2}}}{2^j} \prod_{k=1}^{j-2} (k+1) = \frac{r^{\frac{1-2j}{2}} (j-1)!}{4} \leq \frac{(j-1)!}{4}
\]
and therefore
\[
\prod_{j=1}^{\ell} \left( \frac{d^j}{dr^j} \sqrt{r} \right)^{m_j} \leq \prod_{j=1}^{\ell} \left( \frac{1}{4j} \right)^{m_j} \leq \prod_{j=1}^{\ell} \left( \frac{1}{4} \right)^{m_j} \leq C(\ell),
\]
for some $C(\ell) > 0$. Hence, exploiting (3.2) and (3.14), and denoting by $C_\xi^\#$ a bound in $[1, +\infty)$ for the derivatives of $\Psi$ up to order $\ell$ (that takes into account the previous $C(\ell)$ too),
\[
|\psi^{(\ell)}(r)| \leq C(\ell) \sum \frac{\ell!}{m!},
\]
from which the desired result in (3.13) plainly follows.

We now claim that for every $\ell \in \mathbb{N}$ there exists $C_\ell^\# \geq 1$ such that, for every $\xi \in \mathbb{R}$,
\[
(3.17) \quad |\phi^{(\ell)}(\xi)| \leq C_\ell^\# (1 + |\xi|)^{C_\ell^\#}.
\]
To this end, we exploit the Faà di Bruno’s Formula to see that, for every $\ell \in \mathbb{N}$,
\[
\phi^{(\ell)}(\xi) = \frac{d^\ell}{d\xi^\ell} (\psi(\varphi(\xi))) = \sum \frac{\ell!}{m!} \psi^{(m_1 + \cdots + m_\ell)}(\varphi(\xi)) \prod_{j=1}^{\ell} \left( \frac{\varphi(j)(\xi)}{j!} \right)^{m_j},
\]
with the sum above ranging over all $m \in \mathbb{N}^\ell$ satisfying the constraint in (3.15). Thus, recalling (3.10) and (3.13), we pick $C_\ell^\# \geq 1$ sufficiently large such that, for all $j \leq \ell$,
\[
|\varphi(j)(\xi)| \leq C_\ell^* (1 + |\xi|^{2\alpha}) \quad \text{for every } \xi \in \mathbb{R}
\]
and
\[
|\psi^{(j)}(r)| \leq C_\ell^* \quad \text{for every } r \in [0, +\infty).
\]
In this way, we find that
\[
|\phi^{(\ell)}(\xi)| \leq C_\ell^* \sum \frac{\ell!}{m!} \prod_{j=1}^{\ell} \left( C_\ell^* \frac{(1 + |\xi|^{2\alpha})}{j!} \right)^{m_j}
\]
\[
\leq C_\ell^* \sum \frac{\ell!}{m!} \prod_{j=1}^{\ell} \left( C_\ell^* \left( 1 + |\xi|^{2\alpha} \right) \right)^{\ell} = (C_\ell^*)^{1+\ell^2} (1 + |\xi|^{2\alpha})^{\ell^2} \sum \frac{\ell!}{m!},
\]
which leads to (3.17).

Thus, from (3.8), (3.9) and (3.17), we obtain that if $v$ is a smooth and rapidly decreasing function in the Schwartz Space,
\[
(3.18) \quad \text{then so is the function } \xi \mapsto \hat{v}(\xi) \phi(\xi) = \hat{v}(\xi) \Psi(\omega(\xi)),
\]
and so is its Fourier antitransform.

Applying this with $\Psi(r) := \cos r$ and with $\Psi(r) := \frac{\sin r}{r}$, and recalling the representation formula in (1.5), we obtain the desired result in Theorem 3.1. However, to perform this last step, we need to check that the functions $\Psi(r) := \cos r$ and $\Psi(r) := \frac{\sin r}{r}$, satisfy the hypothesis in (3.2). This is obvious if $\Psi(r) := \cos r$. If instead $\Psi(r) := \frac{\sin r}{r}$, we use that
\[
\Psi(r) = \sum_{j=1}^{+\infty} \frac{(-1)^j r^{2j}}{(2j)!},
\]
to see that $\Psi$ is analytic, hence all its derivatives are bounded in $(-1, 1)$. Thus, it only remains to check that all its derivatives are bounded in $\mathbb{R} \setminus (-1, 1)$. By even parity, it suffices to focus on $[1, +\infty)$. For this, we observe that $\Psi = \Psi_1 \Psi_2$, where $\Psi_1(r) := \sin r$ and $\Psi_2(r) := 1/r$. Notice that the derivatives
of \( \Psi_1 \) and \( \Psi_2 \) of any order are bounded in \([1, +\infty)\). This fact and the General Leibniz Rule yield that the derivatives of \( \Psi \) of all orders are bounded in \([1, +\infty)\), and the proof of Theorem 3.1 is thereby complete.

As a byproduct of the previous results, we now point out some integrability estimates that will be used in Section 4 to introduce some useful conserved quantities for solutions of equation (1.3).

**Corollary 3.2.** Let \( v_0, v_1 \in S(\mathbb{R}) \) and \( 0 < \alpha < 1 \). Let \( u \) be a solution of problem (1.3). Let also \( w(t,x) := (1 + |x|)u(t,x) \) and \( W(t,x) := (1 + |x|)u_t(t,x) \). Then, for all \( t > 0 \),

\[
\begin{align*}
(3.19) & \quad u_t(t,\cdot) \in L^2(\mathbb{R}), \\
(3.20) & \quad w(t,\cdot) \in L^1(\mathbb{R}), \\
(3.21) & \quad \text{and} \quad W(t,\cdot) \in L^1(\mathbb{R}).
\end{align*}
\]

Moreover,

\[
\begin{align*}
(3.22) & \quad \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{|u(t,x) - u(t,x - y)|^2}{|y|^{1+2\alpha}} \ dx \ dy < +\infty, \\
(3.23) & \quad \lim_{t \to 0^+} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{|u(t,x) - u(t,x - y)|^2}{|y|^{1+2\alpha}} \ dx \ dy = \frac{\rho}{\kappa} \int_{\mathbb{R}} \omega^2(\xi)|\tilde{v}_0(\xi)|^2 \ d\xi, \\
(3.24) & \quad \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{|2u(t,x) - u(t,x + y) - u(t,x - y)|}{|y|^{1+2\alpha}} \ dx \ dy < +\infty
\end{align*}
\]

and \( ^6 \)

\[
(3.25) \quad \lim_{t \to 0^+} \int_{\mathbb{R}} x u_t(t,x) \ dx = \int_{\mathbb{R}} x v_1(x) \ dx.
\]

**Proof.** From equation (1.5), we have that

\[
(3.26) \quad u_t(t,x) = \int_{\mathbb{R}} e^{-i\xi x} \left[ -\omega(\xi)\tilde{v}_0(\xi) \sin (\omega(\xi) t) + \tilde{v}_1(\xi) \cos (\omega(\xi) t) \right] d\xi.
\]

\( ^6 \)Here and in the rest of this paper, the notation \( \int_A \int_B f(x,y) \ dx \ dy \) means that we are integrating over \( x \in A \) and \( y \in B \) (not vice-versa).
Then, using \(\ast\) to denote complex conjugation and \(\delta(\cdot)\) to denote the Dirac Delta Function, it follows that

\[
\|u_t(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(\xi-q)x} \left[ -\omega(\xi)\tilde{v}_0(\xi) \sin(\omega(\xi)t) + \tilde{v}_1(\xi) \cos(\omega(\xi)t) \right] \\
\times \left[ -\omega(q)\tilde{v}_0(q) \sin(\omega(q)t) + \tilde{v}_1^*(q) \cos(\omega(q)t) \right] dx \, d\xi \, dq
\]

\[
= 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \delta(\xi-q) \left[ -\omega(\xi)\tilde{v}_0(\xi) \sin(\omega(\xi)t) + \tilde{v}_1(\xi) \cos(\omega(\xi)t) \right] \\
\times \left[ -\omega(q)\tilde{v}_0(q) \sin(\omega(q)t) + \tilde{v}_1^*(q) \cos(\omega(q)t) \right] dx \, d\xi \, dq
\]

\[
= 2\pi \int_{\mathbb{R}} \left\{ \omega^2(\xi)|\tilde{v}_0(\xi)|^2 \sin^2(\omega(\xi)t) + |\tilde{v}_1(\xi)|^2 \cos^2(\omega(\xi)t) \right\} dx
\]

\[
- \omega(\xi) \sin(\omega(\xi)t) \cos(\omega(\xi)t) \left[ \tilde{v}_0(\xi)\tilde{v}_1^*(\xi) + \tilde{v}_1(\xi)\tilde{v}_0^*(\xi) \right] \right\} d\xi.
\]

From this and the bound on \(\omega\) in (1.7), we obtain the desired result in (3.19).

Also, the claim in (3.20) follows directly from Theorem 3.1.

Additionally, we have that

\[
r \sin r = \sum_{j=0}^{+\infty} \frac{(-1)^j r^{2j+2}}{(2j+1)!}
\]

and therefore, given \(t > 0\), recalling the notation in (3.1) and the result in (3.7),

\[
Z(t, \xi) := \omega(\xi) \sin(\omega(\xi)t) = \frac{\omega(\xi)t \sin(\omega(\xi)t)}{t} = \sum_{j=0}^{+\infty} \frac{(-1)^j t^j (\omega(\xi))^j+1}{(2j+1)!},
\]

which is a real analytic function in the variable \(\xi\). Also, in view of (1.7), we know that \(Z\) grows at most polynomially at infinity in \(\xi\), whence, if \(v\) belongs to the Schwartz Space, then also the function

\[
\xi \mapsto \tilde{v}(\xi)Z(t, \xi)
\]

belongs to the Schwartz Space.

Moreover, using (3.18), we have that if \(v\) belongs to the Schwartz Space, then also the function

\[
\xi \mapsto \tilde{v}(\xi)\cos(\omega(\xi)t)
\]

belongs to the Schwartz Space.

\footnote{It is useful to recall that}

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\xi y} \psi(x) \hat{\phi}(\xi) \, dx \, d\xi = 2\pi \int_{\mathbb{R}} \hat{\psi}(\xi) \hat{\phi}(\xi) \, d\xi.
\]

Also, the Fourier transform of the convolution between \(\hat{\psi}\) and \(\hat{\phi}\) is

\[
\hat{\psi} \ast \hat{\phi}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} (\psi \ast \phi)(x) e^{ix\xi} \, dx = \frac{1}{2\pi} \int_{\mathbb{R}} \psi(x-y) \phi(y) e^{i\xi y} \, dy
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(t) \phi(y) e^{i\xi(t-y)} \, dt \, dy = 2\pi \hat{\psi}(\xi) \hat{\phi}(\xi).
\]

From these observations and the inversion formula (recall footnote 2), we find that

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\xi x} \psi(x) \hat{\phi}(\xi) \, dx \, d\xi = \int_{\mathbb{R}} \psi \ast \hat{\phi}(\xi) \, d\xi = \int_{\mathbb{R}} e^{-i\xi0} \psi \ast \hat{\phi}(\xi) \, d\xi = \psi \ast \phi(0) = \int_{\mathbb{R}} \hat{\psi}(\xi) \phi(x) \, dx.
\]

Hence, taking \(\psi := 1\) and arguing in the sense of distributions,

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\xi x} \hat{\phi}(\xi) \, dx \, d\xi = \int_{\mathbb{R}} \phi(x) \, dx = \int_{\mathbb{R}} e^{i\xi0} \phi(x) \, dx = 2\pi \hat{\phi}(0) = 2\pi \int_{\mathbb{R}} \delta(\xi) \hat{\phi}(\xi) \, d\xi,
\]

that is, distributionally,

\[
\int_{\mathbb{R}} e^{i\xi x} \, dx = 2\pi \delta(\xi).
\]
Combining \((3.26), (3.29)\) and \((3.30)\) we obtain \((3.21)\), as desired.

Furthermore, from equation \((1.5)\), we have that
\[
R_{t,x} - R_{t,x} = 4 = 4 \delta v_0(\xi) \cos(\omega(\xi) t) + \frac{\delta_1(\xi)}{\omega(\xi)} \sin(\omega(\xi) t)
\]
\[
\text{Furthermore, from equation (1.5), we have that}
\]
\[
R_{t,x} - R_{t,x} = 4 = 4 \delta v_0(\xi) \cos(\omega(\xi) t) + \frac{\delta_1(\xi)}{\omega(\xi)} \sin(\omega(\xi) t)
\]
\[
(3.31)
\]
\[
\text{Hence, we obtain}
\]
\[
\int \int_\delta |u(t,x) - u(t,x-y)|^2 dx dy = \int \int_\delta \frac{1}{|y|^{1+2\alpha}} \int \int_\delta e^{-i(\xi-q)x}
\]
\[
\times \{1 - \cos(\omega(\xi) t) - i \sin(\omega(\xi) t)\}
\]
\[
\times \left[ R_{t,x} - R_{t,x} - \frac{\delta_1(\xi)}{\omega(\xi)} \sin(\omega(\xi) t) \right]
\]
\[
\int \int_\delta \frac{1}{|y|^{1+2\alpha}} \int \int_\delta \left[ R_{t,x} - R_{t,x} - \frac{\delta_1(\xi)}{\omega(\xi)} \sin(\omega(\xi) t) \right] dx dy d\xi dq
\]
\[
= 2\pi \int \int_\delta \frac{1}{|y|^{1+2\alpha}} \int \int_\delta \left[ R_{t,x} - R_{t,x} - \frac{\delta_1(\xi)}{\omega(\xi)} \sin(\omega(\xi) t) \right] dx dy d\xi dq
\]
\[
(3.32)
\]
\[
\text{where we have performed the change of variable } z := \delta y \text{ and used } (1.6). \text{ Thus, recalling the bound on } \omega \text{ in (1.7), we obtain (3.22), as desired.}
\]

By taking the limit as } t \to 0^+ in (3.22), we obtain the first identity in (3.23). Also, the second identity in (3.23) follows by (1.6), the translation invariance of the norm and Plancherel Theorem; more precisely
\[
\int \int_\delta |v_0(x) - v_0(x-y)|^2 dx dy = \int \int_\delta \frac{v_0^2(x) + v_0^2(x-y) - 2v_0(x)v_0(x-y)}{|y|^{1+2\alpha}} dx dy
\]
for some $C > 0$ independent of $x$. This and (3.33) lead to
\[
\int_{\mathbb{R}} \int_{-\delta}^{\delta} \frac{|2u(t, x) - u(t, x + y) - u(t, x - y)|}{|y|^{1+2\alpha}} dy dx \leq C (1 + \delta)^3 \int_{\mathbb{R}} \int_{-\delta}^{\delta} \frac{|x|}{(1 + |x|)^3} dx dy
\]
which is finite, thus proving (3.24).

Now we use again (3.26), combined with the bound on $\omega$ obtained in (1.7), to see that
\[
\left| \int_{\mathbb{R}} x u(t, x) dx - \int_{\mathbb{R}} x v_1(x) dx \right|
= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} x e^{-i\xi x} \left[ -\omega(\xi) \hat{v}_0(\xi) \sin(\omega(\xi) t) + \hat{v}_1(\xi) \cos(\omega(\xi) t) \right] d\xi dx - \int_{\mathbb{R}} x v_1(x) dx \right|
\leq \left| \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi \omega(\xi) \hat{v}_0(\xi) \sin(\omega(\xi) t) d\xi dx \right|
+ \left| \int_{\mathbb{R}} \int_{\mathbb{R}} x e^{-i\xi x} \hat{v}_1(\xi) \cos(\omega(\xi) t) d\xi dx - \int_{\mathbb{R}} \int_{\mathbb{R}} x e^{-i\xi x} \hat{v}_1(\xi) d\xi dx \right|
= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi x} \frac{d\Theta_1}{d\xi}(t, \xi) d\xi dx \right|
+ \left| \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi x} \frac{d\Theta_2}{d\xi}(t, \xi) d\xi dx \right|
= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi x} \frac{d\Theta_1}{d\xi}(t, \xi) d\xi dx,
\]
where
\[
\Theta_1(t, \xi) := \omega(\xi) \hat{v}_0(\xi) \sin(\omega(\xi) t) \quad \text{and} \quad \Theta_2(t, \xi) := \hat{v}_1(\xi) \left( 1 - \cos(\omega(\xi) t) \right).
\]
Recalling (3.28), it is also convenient to consider the function
\[
(0, +\infty) \ni r \mapsto S(r) := \sqrt{r} \sin(\sqrt{r}) = \sum_{j=0}^{+\infty} \frac{(-1)^j r^{j+1}}{(2j + 1)!}
\]
We notice that $S$ can be extended to an analytic function defined for all $r \in \mathbb{R}$ by using the series expansion in the right hand side of (3.35).
Also, for every $\ell \in \mathbb{N}$ and $r > 0$,
\[ |S^{(\ell)}(r)| \leq C_\ell (1 + \sqrt{r}), \]
for suitable $C_\ell > 0$. Indeed, if $r \in [0, 1]$ this claim follows by taking derivatives in the series expansion in (3.35), and if $r \geq 1$ it follows from (3.16) and the General Leibniz Rule.

As a result, using (3.1), (3.10) and the Faà di Bruno’s Formula, for every $m \in \mathbb{N}$,
\[ \left| \frac{d^m}{d\xi^m} [\varphi_0(\xi) \sin (\omega(\xi) t)] \right| = \left| \frac{d^m}{d\xi^m} S(\varphi_0(\xi) t^2) \right| \leq C^*_m t^2 (1 + |\xi|)^{C^*_m}, \]
for a suitable $C^*_m > 0$, and consequently, using again the General Leibniz Rule and the fact that $\hat{\varphi}_0$ belongs to the Schwartz Space,
\[ \left| \frac{d^m}{d\xi^m} \Theta_1(t, \xi) \right| \leq \frac{C^*_m t}{1 + \xi^2}, \]
for some $C^*_m > 0$.

Integrating twice by parts in the variable $\xi$ when $|x| > 1$, we thus find that
\[
\begin{align*}
\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi x} \frac{d\Theta_1}{d\xi}(t, \xi) \, d\xi \, dx & \leq \int_{\mathbb{R}\setminus[-1,1]} \int_{\mathbb{R}} e^{-i\xi x} \frac{d\Theta_1}{d\xi}(t, \xi) \, dx \, d\xi + \int_{[-1,1]} \int_{\mathbb{R}} e^{-i\xi x} \frac{d\Theta_1}{d\xi}(t, \xi) \, dx \, d\xi \\
& \leq \int_{\mathbb{R}\setminus[-1,1]} \int_{\mathbb{R}} C^*_1 \frac{t}{1 + \xi^2} \, dx \, d\xi + \int_{[-1,1]} \int_{\mathbb{R}} e^{-i\xi x} \frac{d^3\Theta_1}{d\xi^3}(t, \xi) \, dx \, d\xi \\
& \leq O(t) + \int_{[-1,1]} \int_{\mathbb{R}} C^*_1 \frac{t}{x^2 (1 + \xi^2)} \, dx \, d\xi = O(t).
\end{align*}
\]
(3.36)

Similarly,
\[
\begin{align*}
\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi x} \frac{d\Theta_2}{d\xi}(t, \xi) \, dx \, d\xi & = O(t).
\end{align*}
\]
Plugging this and (3.36) into (3.34) we obtain (3.25) as desired. \(\square\)

We conclude this section with the following decay properties of the solutions of (1.3).

**Theorem 3.3.** Let $u(t, x)$ be a solution of (1.3) according to (1.5). Then for all $t > 0$ the following inequality holds true.

\[ \|u(t, \cdot)\|_{L^2(\mathbb{R})} \leq \|\hat{\varphi}_0\|_{L^2(\mathbb{R})} + \sqrt{2\pi} \left\| \tilde{\varphi}_1 \min \left\{ t, \frac{1}{\omega} \right\} \right\|_{L^2(\mathbb{R})}. \]
(3.37)

**Proof.** By (1.5) and Plancherel Theorem,
\[
\frac{\|u(t, \cdot)\|_{L^2(\mathbb{R})}^2}{2\pi} = \int_{\mathbb{R}} \left| \hat{\varphi}_0(\xi) \cos (\omega(\xi) t) + \frac{\hat{\varphi}_1(\xi)}{\omega(\xi)} \sin (\omega(\xi) t) \right|^2 \, d\xi \\
\leq \int_{\mathbb{R}} \left( |\hat{\varphi}_0(\xi)| + \left| \frac{\hat{\varphi}_1(\xi)}{\omega(\xi)} \right| |\sin (\omega(\xi) t) | \right)^2 \, d\xi.
\]
(3.38)

Thus, since, for all $r \in \mathbb{R}$,
\[
\frac{\sin r}{r} \leq \min \left\{ 1, \frac{1}{|r|} \right\},
\]
we deduce from (3.38) that
\[
\frac{\|u(t, \cdot)\|_{L^2(\mathbb{R})}^2}{2\pi} \leq \int_{\mathbb{R}} \left( |\hat{\varphi}_0(\xi)| + t |\hat{\varphi}_1(\xi)| \min \left\{ 1, \frac{1}{\omega(\xi) t} \right\} \right)^2 \, d\xi = \left\| \hat{\varphi}_0 + t |\hat{\varphi}_1| \min \left\{ t, \frac{1}{\omega} \right\} \right\|_{L^2(\mathbb{R})}^2.
\]
and therefore
\[ \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq \left\| \mathfrak{v}_0 + |\mathfrak{v}_1| \min \left\{ t, \frac{1}{\omega} \right\} \right\|_{L^2(\mathbb{R})} \leq \| \mathfrak{v}_0 \|_{L^2(\mathbb{R})} + \| \mathfrak{v}_1 \|_{L^2(\mathbb{R})}, \]
from which we obtain (3.37) using again Plancherel Theorem.

\[ \Box \]

**Theorem 3.4.** Let \( u(t, x) \) be a solution of the (1.3) according to (1.5). Then, for every \((t, x) \in \mathbb{R}_+ \times \mathbb{R},\)
\[ |u(t, x)| \leq \min \left\{ \| \mathfrak{v}_0 \|_{L^1(\mathbb{R})}, \| \mathfrak{v}_1 \|_{L^1(\mathbb{R})} \right\}, \]
and therefore
\[ \left(1 + \frac{|x|}{t} \right)^{\frac{1}{2}} \left( \left\| \mathfrak{v}_0 \right\|_{W^{1,1}((-\infty, 0)_t)} + \left\| \mathfrak{v}_1 \right\|_{W^{1,1}((0, +\infty)_t)} \right). \]

**Proof.** From
\[ (3.41) \quad |u(t, x)| = \left\| \int_\mathbb{R} e^{-ix\xi} \left[ \mathfrak{v}_0(\xi) \cos(\omega(\xi) t) + \mathfrak{v}_1(\xi) \sin(\omega(\xi) t) \right] d\xi \right\|. \]
and (3.39) we deduce that
\[ (3.42) \quad |u(t, x)| \leq \int_\mathbb{R} |\mathfrak{v}_0(\xi)| + |\mathfrak{v}_1(\xi)| \min \left\{ t, \frac{1}{\omega(\xi)} \right\} d\xi. \]
Another consequence of (3.41) is that
\[ (3.43) \quad |u(t, x)| = \left\| \int_\mathbb{R} e^{-ix\xi} \left[ \frac{\mathfrak{v}_0(\xi)}{\omega'()} t \partial_\xi \sin(\omega(\xi) t) - \frac{\mathfrak{v}_1(\xi)}{\omega(\xi) \omega'(\xi)} t \partial_\xi \cos(\omega(\xi) t) \right] d\xi \right\|. \]
Moreover, recalling (2.8) and (2.9), we see that
\[ \left(1 + \frac{|x|}{t} \right)^{\frac{1}{2}} \left( \left\| \mathfrak{v}_0 \right\|_{W^{1,1}((-\infty, 0)_t)} + \left\| \mathfrak{v}_1 \right\|_{W^{1,1}((0, +\infty)_t)} \right). \]
Additionally,
\[ \left(1 + \frac{|x|}{t} \right)^{\frac{1}{2}} \left( \left\| \mathfrak{v}_0 \right\|_{W^{1,1}((-\infty, 0)_t)} + \left\| \mathfrak{v}_1 \right\|_{W^{1,1}((0, +\infty)_t)} \right). \]
and, as a consequence,

$$\left| \int_{\mathbb{R}} e^{-i\xi x} \frac{\hat{v}_1(\xi)}{\omega(\xi) \omega'(\xi)t} \partial_\xi \cos(\omega(\xi)t) d\xi \right| \leq \frac{1 + |x|}{t} \left( \left\| \frac{\hat{v}_1}{\omega'} \right\|_{W^{1,1}((-\infty,0))} + \left\| \frac{\hat{v}_1}{\omega'} \right\|_{W^{1,1}(0,\infty)} \right).$$

Owing to (3.43), the latter estimate and (3.44) entail that

$$|u(t, x)| \leq \frac{1 + |x|}{t} \left( \left\| \frac{\hat{v}_0}{\omega'} \right\|_{W^{1,1}((-\infty,0))} + \left\| \frac{\hat{v}_0}{\omega'} \right\|_{W^{1,1}(0,\infty)} + \left\| \frac{\hat{v}_1}{\omega'} \right\|_{W^{1,1}((-\infty,0))} + \left\| \frac{\hat{v}_1}{\omega'} \right\|_{W^{1,1}(0,\infty)} \right).$$

Thus, recalling (3.40), we obtain the desired result in (3.40).

**Remark 3.5.** We stress that if $v_0$ and $v_1$ belong to the Schwartz Space, then in particular

$$\left\| \frac{\hat{v}_0}{\omega} \right\|_{L^1(\mathbb{R})} + \left\| \frac{\hat{v}_1}{\omega} \right\|_{L^1(\mathbb{R})} < +\infty$$

and consequently the right hand side of (3.40) is finite.

Furthermore, in light of (2.1), (2.2), (2.8), (2.9), (2.11), (2.12) and (2.13),

$$\frac{1}{\omega|\omega'|} + \frac{1}{\omega^2|\omega'|} + \frac{|\omega''|}{\omega(\omega')^2} = O \left( \frac{1}{|\xi|} + \frac{1}{\xi^2} + 1 \right) = O \left( \frac{1}{\xi^2} \right)$$

as $\xi \to 0^\pm$ and

$$\frac{1}{\omega|\omega'|} + \frac{1}{\omega^2|\omega'|} + \frac{|\omega''|}{\omega(\omega')^2} = O \left( \frac{1}{|\xi|^{2\alpha - 1}} + \frac{1}{|\xi|^{3\alpha - 1}} + \frac{|\xi|^{\max\{2\alpha - 1, 1 - \alpha\}}}{|\xi|^{3\alpha - 2}} \right)
= O \left( \frac{1}{|\xi|^{2\alpha - 1}} + \frac{1}{|\xi|^{3\alpha - 1}} \right) = O \left( \frac{1}{|\xi|^{2\alpha - 1}} \right)$$

as $\xi \to \pm\infty$.

By (3.45), it follows that additional assumptions (beside being in the Schwartz Space) must be taken on $\hat{v}_1$ if one wishes that

$$\left\| \frac{\hat{v}_1}{\omega'} \right\|_{W^{1,1}((-\infty,0))} + \left\| \frac{\hat{v}_1}{\omega'} \right\|_{W^{1,1}(0,\infty)} < +\infty.$$

**4. Conserved Quantities**

In this section, we investigate the conservation properties of equation (1.3). For this, we introduce the following definition:

**Definition 4.1.** Let $u(t, x)$ be a solution of (1.3). We define the following functionals:

(4.1) **Energy**

$$E[u(t, \cdot)] := \frac{\rho}{2} \| u_t(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{\kappa}{2} \int_{-\delta}^{\delta} \int_{\mathbb{R}} \frac{|u(t, x) - u(t, x - y)|^2}{|y|^{1 + 2\alpha}} dx dy,$$

(4.2) **Momentum**

$$P[u(t, \cdot)] := \rho \int_{\mathbb{R}} u_t(t, x) dx,$$

(4.3) **Angular momentum**

$$L[u(t, \cdot)] := \rho \int_{\mathbb{R}} x u_t(t, x) dx.$$

We stress that the definition in (4.1) is well posed thanks to (3.19) and (3.22). Similarly, the definitions in (4.2) and (4.3) are well posed thanks to (3.21).

All the quantities introduced in Definition 4.1 are conserved by the equation, according to the following result:

**Theorem 4.2.** If $u$ is a solution of problem (1.3), then

i) $u$ preserves Energy according to (4.1) in Definition 4.1,

ii) $u$ preserves Momentum according to (4.2) in Definition 4.1.
iii) $u$ preserves Angular Momentum according to (4.3) in Definition 4.1.

Theorem 4.2 is actually the byproduct of the forthcoming Theorems 4.4 and 4.5.

**Theorem 4.3 (Angular Momentum conservation).** Let $u(t, x)$ be a solution of (1.3). Then (4.4)

$$L[u(t, \cdot)] = \rho \int_x v_1(x) \, dx.$$  

**Proof.** Let $u(t, x)$ be a solution of (1.3). It follows that

$$\rho \frac{d}{dt} \int_R x \, u_t \, dx = \int_R \rho \, x \, u_t \, dx = -2 \kappa \int_R \int_{-\delta}^{\delta} \frac{u(t, x) - u(t, x - y)}{|y|^{1+2\alpha}} \, dx \, dy$$

$$= -2 \kappa \int_{-\delta}^{\delta} \left[ x \, u(t, x) - (x - y) \, u(t, x - y) - y \, u(t, x - y) \right] \, dx \, \frac{dy}{|y|^{1+2\alpha}}$$

$$= -2 \kappa \lim_{\epsilon \to 0, \delta \to 0} \int_{-\delta - \epsilon}^{\delta + \epsilon} \left[ \int_R x \, u(t, x) \, dx - \int_R (x - y) \, u(t, x - y) \, dx \right] \, \frac{dy}{|y|^{1+2\alpha}}$$

$$= 2 \kappa \lim_{\epsilon \to 0} \left\{ \int_{\epsilon}^{\delta} \frac{1}{y^{2\alpha}} \left[ \int_R u(t, x - y) \, dx \right] \, dy - \int_{-\delta}^{-\epsilon} \frac{1}{y^{2\alpha}} \left[ \int_R u(t, x - y) \, dx \right] \, dy \right\}$$

$$= 2 \kappa \lim_{\epsilon \to 0} \left\{ \int_{\epsilon}^{\delta} \frac{1}{y^{2\alpha}} \left[ \int_R u(t, x - y) \, dx \right] \, dy - \int_{\epsilon}^{\delta} \frac{1}{y^{2\alpha}} \left[ \int_R u(t, x + y) \, dx \right] \, dy \right\}$$

$$= 2 \lim_{\epsilon \to 0} \left\{ \int_{\epsilon}^{\delta} \frac{1}{y^{2\alpha}} \left[ \int_R u(t, x - y) \, dx - \int_R u(t, x + y) \, dx \right] \, dy \right\},$$

where the two integrals in the third and last lines cancel due to the translational invariance – and we stress that the integrals involved in the computations are finite, thanks to (3.20), (3.21) and (3.24) (recall also (1.4)).

We have thus shown that the angular momentum is constant in time, whence (4.4) follows from (3.25) and (4.3).

**Theorem 4.4 (Energy conservation).** Let $u(t, x)$ be a solution of the (1.3) according to (1.5). Then (4.5)

$$E[u(t, \cdot)] = \rho \pi \int_R \left\{ \omega^2(\xi) |\hat{v}_0(\xi)|^2 + |\hat{v}_1(\xi)|^2 \right\} \, d\xi$$

$$= \rho \pi \int_R \left\{ \omega^2(\xi) |\hat{v}_0(\xi)|^2 \sin^2 (\omega(\xi) \, t) + |\hat{v}_1(\xi)|^2 \cos^2 (\omega(\xi) \, t) \right\} \frac{2}{2} \frac{\int_R \int_{-\delta}^{\delta} \frac{|v_0(x) - v_0(x - y)|^2}{|y|^{1+2\alpha}} \, dx \, dy}{dx \, dy}.$$  

**Proof.** From (4.1) in Definition 4.1 and equations (3.27) and (3.32), we get

$$E[u(t, \cdot)] = \rho \pi \int_R \left\{ \omega^2(\xi) |\hat{v}_0(\xi)|^2 \sin^2 (\omega(\xi) \, t) + |\hat{v}_1(\xi)|^2 \cos^2 (\omega(\xi) \, t) \right\} \frac{2}{2} \frac{\int_R \int_{-\delta}^{\delta} \frac{|v_0(x) - v_0(x - y)|^2}{|y|^{1+2\alpha}} \, dx \, dy}{dx \, dy}.$$
This establishes the first identity in (4.5). In particular, the energy is constant in time and recalling (3.23) we obtain the second identity in (4.5).

**Theorem 4.5 (Momentum conservation).** Let \( u(t, x) \) be a solution of the (1.3) according to (1.5). Then

\[
P[u(t, \cdot)] = \rho \int_{\mathbb{R}} v_1(x) \, dx.
\]

**Proof.** From (2.1) and (3.26) it follows that

\[
P[u(t, \cdot)] = 2\rho \pi \int_{\mathbb{R}} \delta(0) \left[ -\omega(\xi) \tilde{v}_0(\xi) \sin (\omega(\xi) t) + \tilde{v}_1(\xi) \cos (\omega(\xi) t) \right] \, d\xi
\]

\[
= 2\rho \pi \left[ -\omega(0) \tilde{v}_0(0) \sin (\omega(0) t) + \tilde{v}_1(0) \cos (\omega(0) t) \right] = 2\rho \pi \tilde{v}_1(0).
\]

\[
\]

5. Numerics

From now on, let us consider the numerical integration for the case \( \alpha = 1/10, \rho = 1 \) and \( \kappa = 1/2 \), whereas \( \delta \) will be fixed case by case. Moreover, we fix initial conditions such that

\[
v_0(x) = \sqrt{2\pi} e^{-x^2/2} \quad \text{and} \quad v_1(x) = 4 \, x \, \sqrt{2\pi} \, e^{-x^2/2},
\]

i.e. the initial deformation is Gaussian, with square root of the variance \( \sigma = 1/2 \) and the initial velocity is given by the initial condition of a traveling wave \( v_1(x) = v \, v'_0(x) \). The value of \( v \) will be specified case by case later as well. Thus, in Fourier Space, we have

\[
\tilde{v}_0(\xi) = \frac{1}{2} e^{-\xi^2/8} \quad \text{and} \quad \tilde{v}_1(\xi) = i \, \xi \, \tilde{v}_0(\xi).
\]

Notice that \( \tilde{v}_0(\xi) \) is a Gaussian with \( \tilde{\sigma} = 2 \). The numerical evolution of this Gaussian according to (1.3) and (1.5) is depicted in Figure 7, where \( \delta = 1 \) and \( v = 0 \).

Let us emphasize some important differences exhibited in Figure 7 with respect to the classical case of the wave equation (in which the solution is simply the sum of two traveling positive Gaussians, as shown in Figure 8). First of all, the pattern in Figure 7 is that of a sign-changing solutions. Also, multiple critical points happen to arise as time goes. Overall, in this situation, with respect to the classical wave equation, the peridynamic case seems to produce additional oscillations. Certainly, it is desirable to carry on further analytical and numerical investigations of these possible phenomena.

Moreover, in Figure 9, we report numerical solutions for the Cauchy problem given by initial conditions (5.1) with \( v = 1 \) and \( \delta = 1 \).

We notice the presence of secondary oscillations left behind the wavefront, whose amplitude slowly decreases as time evolves. These two features are not present in the solution of the classical equation \( u_{tt} = u_{xx} \). Indeed, in this case only a single Gaussian is expected to travel without neither deformation or damping of the amplitude.

Finally, in Figure 10, we show numerical solutions of the peridynamic model for three different values of \( \delta \) and initial velocity given by \( v = \delta^{1-\alpha} / \sqrt{2(1-\alpha)} \). The latter choice is inspired by the limit of the dispersion relation on large scales, proven in Theorem 2.1.

Figure 10 refers to \( \delta = 5/2 \) (red), \( \delta = 1 \) (green) and \( \delta = 1/10 \) (blue). To properly interpret this numerical solution, we compare the values of \( \delta \) with the dispersion of the Gaussian in the chosen initial condition \( (\sigma = 1/2) \). For \( \delta = 5/2 \), we have that \( \delta = 5\sigma \). This means that the deformation introduced by the initial conditions involves scales which are small when compared with \( \delta \), i.e. the characteristic range of the peridynamic model. This leads to a naive expectations that most of the modes \( \xi \) propagates with dispersion relation of order \( \xi^0 \) and hence the evolution is highly dispersive. This expectation is in qualitative agreement with what shown in the red plots of Figure 10.

---

\(^8\)This numerical solution holds for the wave equation \( u_{tt} = u_{xx} \), with initial conditions given by equation (5.1) with \( v = 0 \). This leads to the analogous of (1.5) where \( \omega(\xi) \) is replaced by \( \xi \).
Figure 7. Numerical solution at different times, from \( t = 0 \) until \( t = 8 \), with unitary time-step. We imposed Gaussian initial conditions with \( \sigma = \delta/2 \) and initial velocity 0. This case refers to the parameters \( \alpha = 1/10, \rho = 1, \kappa = 1/2 \) and \( \delta = 1 \).

On the opposite case, when \( \delta = 1/10 \), we have that \( \delta = \sigma/5 \) and then the deformation induced by the initial conditions are on large scales when compared with \( \delta \). In this case, most of the involved scales propagates with dispersion relation \( \approx \delta^{1-\alpha}/\sqrt{2(1-\alpha)}\xi \), whose group velocity is the same as the one given in the initial conditions. Hence we expect that this case is nondispersive. Bottom panels of Figure 10 are in line with this expectation.

Finally, middle panels in Figure 10 exploit the case when \( \delta = 1 \) and \( \sigma = 1/2 \) are of the same order. We notice an evolution which is still dispersive, just as in Figure 9. However, in Figure 10 the secondary oscillations have smaller amplitude that in Figure 9. We address this behavior to the fact that, in Figure 10, deformation of large scales travel with velocity \( v = \delta^{1-\alpha}/\sqrt{2(1-\alpha)} \) which is just the one emerging from the dispersion relation when \( \xi \to 0 \).

Appendix A. Recovering the classical wave equation as \( \alpha \to 1^- \)

In this appendix we discuss how problem (1.3) recovers the classical wave equation as \( \alpha \to 1^- \), and how the explicit solution provided in (1.5) and (1.6) recovers in the limit the one obtained for the wave equation via Fourier methods.

Lemma A.1. If \( u \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), then

\[
\lim_{\alpha \to 1^-} (1-\alpha)K(u) = C\kappa\Delta u,
\]

for a suitable constant \( C > 0 \).
Using this, (A.1), (A.2) and [6, Proposition 4.4(ii)], we conclude that and consequently

\[ \lim_{t,x} \text{ for some } C_\alpha \text{ for all } x \]

By the definition of \( K(u) \) in [1.3], and exploiting [6] equations (3.1) and (4.3), we have that for all \( \alpha \in (0, 1) \),

\[ (A.1) \quad (-\Delta)^\alpha u = C_\alpha \int_{\mathbb{R}} \frac{u(t, x-y) - u(t, x)}{|y|^{1+2\alpha}} dy = C_\alpha \left[ \frac{K(u)}{2\kappa} + \int_{\mathbb{R}\setminus(-\delta, \delta)} \frac{u(t, x-y) - u(t, x)}{|y|^{1+2\alpha}} dy \right], \]

for some \( C_\alpha \) such that

\[ (A.2) \quad \lim_{\alpha \to 1^-} \frac{C_\alpha}{1-\alpha} = C_*, \]

for a suitable constant \( C_* > 0 \).

Furthermore,

\[ \int_{\mathbb{R}\setminus(-\delta, \delta)} \frac{|u(t, x-y) - u(t, x)|}{|y|^{1+2\alpha}} dy \leq \int_{\mathbb{R}\setminus(-\delta, \delta)} \frac{2||u||_{L^\infty(\mathbb{R})}}{|y|^{1+2\alpha}} dy = \int_{\delta}^{+\infty} \frac{4||u||_{L^\infty(\mathbb{R})}}{|y|^{1+2\alpha}} dy = \frac{2||u||_{L^\infty(\mathbb{R})}}{\alpha\delta^{2\alpha}} \]

and consequently

\[ \lim_{\alpha \to 1^-} \left(1-\alpha \right) \int_{\mathbb{R}\setminus(-\delta, \delta)} \frac{|u(t, x-y) - u(t, x)|}{|y|^{1+2\alpha}} dy = 0. \]

Using this, (A.1), (A.2) and [6] Proposition 4.4(ii)], we conclude that

\[ \Delta u = \lim_{\alpha \to 1^-} (-\Delta)^\alpha u = \lim_{\alpha \to 1^-} C_\alpha (1-\alpha) \left[ \frac{K(u)}{2\kappa} + \int_{\mathbb{R}\setminus(-\delta, \delta)} \frac{u(t, x-y) - u(t, x)}{|y|^{1+2\alpha}} dy \right] \]

\[ = \frac{C_*}{2\kappa} \lim_{\alpha \to 1^-} (1-\alpha)K(u), \]
as desired. □

Lemma A.2. We have that
\[
\lim_{\alpha \to 1^-} \sqrt{1 - \alpha} \omega(\xi) = \frac{C \sqrt{\kappa} |\xi|}{\sqrt{\rho}},
\]
for a suitable constant \(C > 0\).

Proof. In view of the parity of \(\omega\), we can suppose that \(\xi > 0\). Also, by [6, equation (4.3)], we know that
\[
\lim_{\alpha \to 1^-} (1 - \alpha) \int_{\mathbb{R}} \frac{1 - \cos \tau}{|\tau|^{1 + 2\alpha}} \, d\tau = C_*,
\]
for some \(C_* > 0\).

Thus, from (1.6)
\[
\lim_{\alpha \to 1^-} (1 - \alpha)\omega^2(\xi) = \lim_{\alpha \to 1^-} \frac{2(1 - \alpha)\kappa}{\rho \delta^{2\alpha}} \int_{-1}^{1} \frac{1 - \cos(\xi \delta z)}{|z|^{1 + 2\alpha}} \, dz,
\]
\[
= \lim_{\alpha \to 1^-} \frac{2(1 - \alpha)\kappa |\xi|^{2\alpha}}{\rho} \int_{-\xi \delta}^{\xi \delta} \frac{1 - \cos(\tau)}{|\tau|^{1 + 2\alpha}} \, d\tau,
\]
\[
= \frac{2\kappa C_* |\xi|^2}{\rho} + \lim_{\alpha \to 1^-} \frac{4(1 - \alpha)\kappa |\xi|^{2\alpha}}{\rho} \left( \int_{\xi \delta}^{+\infty} \frac{1 - \cos(\tau)}{\tau^{1 + 2\alpha}} \, d\tau \right).
\]

Additionally,
\[
\int_{\xi \delta}^{+\infty} \frac{1 - \cos(\tau)}{\tau^{1 + 2\alpha}} \, d\tau \leq \int_{\xi \delta}^{+\infty} \frac{2}{\tau^{1 + 2\alpha}} \, d\tau = \frac{(\xi \delta)^{2\alpha}}{\alpha}.
\]
This and (A.3) entail that
\[
\lim_{\alpha \to 1^-} (1 - \alpha) \omega^2(\xi) = \frac{2\kappa C^* |\xi|^2}{\rho},
\]
from which the desired result follows.

\[\square\]

**Appendix B. An elementary proof of (2.15)**

We use an ad-hoc and rather delicate modification of a classical argument used in complex analysis (see e.g. [22, page 44]). For this we use a contour integration as in Figure 11, with \(\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4\), oriented counterclockwise.

We observe that, by Cauchy’s Theorem,
\[
\int_{\gamma} \frac{e^{iz} - 1}{z^{1+2\alpha}} \, dz = 0
\]
and therefore
\[
\lim_{R \to +\infty} \Re \left( \int_{\gamma} \frac{e^{iz} - 1}{z^{1+2\alpha}} \, dz \right) = 0.
\]
Thus, since
\[
\lim_{\epsilon \to 0^+} \lim_{R \to +\infty} \Re \left( \int_{\gamma_1} e^{iz} - \frac{1}{z^{1+2\alpha}} \, dz \right) = \lim_{\epsilon \to 0^+} \lim_{R \to +\infty} \Re \left( \int_{\epsilon}^{R} e^{it} - \frac{1}{t^{1+2\alpha}} \, dt \right) = \lim_{\epsilon \to 0^+} \lim_{R \to +\infty} \int_{\epsilon}^{R} \cos t - \frac{1}{t^{1+2\alpha}} \, dt = \int_{0}^{+\infty} \cos t - \frac{1}{t^{1+2\alpha}} \, dt,
\]
we deduce from (B.1) that
\[
(B.2) \quad \int_{0}^{+\infty} \frac{1 - \cos t}{t^{1+2\alpha}} \, dt = \lim_{\epsilon \to 0^+} \lim_{R \to +\infty} \Re \left( \int_{\gamma_2 \cup \gamma_3 \cup \gamma_4} e^{iz} - \frac{1}{z^{1+2\alpha}} \, dz \right).
\]
We also point out that if \( z = x + iy \) with \( x \in \mathbb{R} \) and \( y \geq 0 \) then
\[
|e^{iz}| = |e^{-y}| \leq 1.
\]
Hence, since \( \gamma_2 \) is the quarter of circle (traveled anticlockwise) of the form \( \{ z = Re^{it}, t \in (0, \pi/2) \} \), we have that if \( z \in \gamma_2 \) then
\[
\left| \frac{e^{iz} - 1}{z^{1+2\alpha}} \right| = \left| \frac{e^{iz} - 1}{|z|^{1+2\alpha}} \right| \leq \frac{2}{R^{1+2\alpha}}.
\]
Accordingly,
\[
(B.3) \quad \left| \lim_{\epsilon \to 0^+} \lim_{R \to +\infty} \Re \left( \int_{\gamma_2} e^{iz} - \frac{1}{z^{1+2\alpha}} \, dz \right) \right| \leq \lim_{R \to +\infty} \frac{\pi R}{R^{1+2\alpha}} = 0.
\]
Now we note that \( \gamma_4 \) is the quarter of circle (traveled clockwise) of the form \( \{ z = e^{it}, t \in (0, \pi/2) \} \). Also, if \( z \in \gamma_4 \), for small \( \epsilon \) we have that
\[
\frac{e^{iz} - 1}{z^{1+2\alpha}} = \frac{i}{z^{2\alpha}} + O(\epsilon^{1-2\alpha})
\]
and therefore
\[
\Re \left( \int_{\gamma_4} e^{iz} - \frac{1}{z^{1+2\alpha}} \, dz \right) = \Re \left( \int_{\gamma_4} \frac{i}{z^{2\alpha}} \, dz \right) + O(\epsilon^{2-2\alpha}) = \epsilon^{1-2\alpha} \Re \left( \int_{0}^{\pi/2} \frac{e^{it}}{e^{2i\alpha t}} \, dt \right) + O(\epsilon^{2-2\alpha})
\]
\[
(B.4) \quad = \epsilon^{1-2\alpha} \int_{0}^{\pi/2} \cos((2\alpha - 1)t) \, dt + O(\epsilon^{2-2\alpha}) = \frac{\epsilon^{1-2\alpha}}{2\alpha - 1} \sin \left( \frac{2\alpha - 1}{2} \pi \right) + O(\epsilon^{2-2\alpha})
\]
\[
= - \frac{\epsilon^{1-2\alpha}}{2\alpha - 1} \cos(\pi \alpha) + O(\epsilon^{2-2\alpha}).
\]
As for the integral on \( \gamma_3 = \{ z = it, t \in (\epsilon, R) \} \) (oriented downwards), using that
\[
(B.5) \quad i^{2\alpha} = (e^{i\pi})^{2\alpha} = e^{i\pi \alpha}
\]
we have
\[
\Re \left( \int_{\gamma_3} \frac{e^{iz} - 1}{z^{1+2\alpha}} \, dz \right) = -\Re \left( i \int_{\epsilon}^{R} \frac{e^{-t} - 1}{t^{1+2\alpha}} \, dt \right) = -\Re \left( e^{-i\pi \alpha} \int_{\epsilon}^{R} \frac{e^{-t} - 1}{t^{1+2\alpha}} \, dt \right)
\]
\[
= \frac{1}{2\alpha} \cos(\pi \alpha) \int_{\epsilon}^{R} \left[ \frac{d}{dt} \left( (e^{-t} - 1)t^{-2\alpha} \right) + e^{-t}t^{-2\alpha} \right] \, dt
\]
\[
= \frac{1}{2\alpha} \cos(\pi \alpha) \left[ (e^{-R} - 1)R^{-2\alpha} - (e^{-\epsilon} - 1)e^{-2\alpha} + \int_{\epsilon}^{R} e^{-t}t^{-2\alpha} \, dt \right]
\]
\[
= \frac{1}{2\alpha} \cos(\pi \alpha) \left[ e^{1-2\alpha} + \int_{\epsilon}^{R} e^{-t}t^{-2\alpha} \, dt \right] + O(R^{-2\alpha}) + O(e^{2-2\alpha})
\]
\[
= \frac{1}{2\alpha} \cos(\pi \alpha) \left[ e^{1-2\alpha} + \frac{1}{1-2\alpha} \int_{\epsilon}^{R} \left( \frac{d}{dt} \left( e^{-t}t^{1-2\alpha} \right) + e^{-t}t^{1-2\alpha} \right) \, dt \right] + O(R^{-2\alpha}) + O(e^{2-2\alpha})
\]
\[
= \frac{1}{2\alpha} \cos(\pi \alpha) \left[ e^{1-2\alpha} + \frac{1}{1-2\alpha} \left( e^{-R}R^{1-2\alpha} - e^{-\epsilon}e^{1-2\alpha} + \int_{\epsilon}^{R} e^{-t}t^{1-2\alpha} \, dt \right) \right] + O(R^{-2\alpha}) + O(e^{2-2\alpha})
\]
\[
= \frac{1}{2\alpha(1-2\alpha)} \cos(\pi \alpha) \left[ -2\alpha e^{1-2\alpha} + \int_{\epsilon}^{R} e^{-t}t^{1-2\alpha} \, dt \right] + O(R^{-2\alpha}) + O(e^{2-2\alpha}).
\]

As a result, recalling (B.4),
\[
\lim_{\epsilon \to 0^+} \lim_{R \to +\infty} \Re \left( \int_{\gamma_3 \cup \gamma_4} \frac{e^{iz} - 1}{z^{1+2\alpha}} \, dz \right) = \lim_{\epsilon \to 0^+} \lim_{R \to +\infty} \left[ \frac{\cos(\pi \alpha)}{2\alpha(1-2\alpha)} \int_{\epsilon}^{R} e^{-t}t^{1-2\alpha} \, dt + O(R^{-2\alpha}) + O(e^{2-2\alpha}) \right]
\]
\[
= \frac{\cos(\pi \alpha)}{2\alpha(1-2\alpha)} \Gamma(2-2\alpha) = -\cos(\pi \alpha) \Gamma(-2\alpha).
\]

By inserting this information and (B.3) into (B.2), we thereby obtain the desired result in (2.15).

**APPENDIX C. A SHORTER (BUT LESS ELEMENTARY) PROOF OF (2.15)**

From Ramanujan’s Master Theorem (see e.g. Formula (B) in Section 11.2 on page 186 of [1] or Theorem 3.2 in [2]), if a complex-valued function \( f \) has an expansion of the form
\[
f(x) = \sum_{k=0}^{+\infty} \frac{\ell(k)}{k!} (-x)^k,
\]
then
\[
(C.1) \int_{0}^{+\infty} x^{s-1} f(x) \, dx = \Gamma(s) \ell(-s).
\]

We take \( s := 1 - 2\alpha \) and \( \ell(s) := \sin\left(\frac{\pi s}{2}\right) \). In this way,
\[
\ell(k) = \begin{cases} 0 & \text{if } k \in 2\mathbb{N}, \\ 1 & \text{if } k \in 4\mathbb{N} + 1, \\ -1 & \text{if } k \in 4\mathbb{N} + 3, \end{cases}
\]
thus
\[
f(x) = -\sum_{j=0}^{+\infty} \frac{1}{(4j+1)!} x^{4j+1} + \sum_{j=0}^{+\infty} \frac{1}{(4j+3)!} x^{4j+3} = -\sum_{m=0}^{+\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} = -\sin x.
\]
Then, we deduce from (C.1) that

\[- \int_0^{+\infty} x^{-2\alpha} \sin x\, dx = \Gamma(1-2\alpha) \sin \left( \frac{\pi}{2} (1-2\alpha) \right) = -2\alpha \Gamma(-2\alpha) \cos(\pi\alpha). \]

This and (2.17) entail (2.15).

References

[1] T. Amdeberhan, O. Espinosa, I. Gonzalez, M. Harrison, V. H. Moll, and A. Straub. Ramanujan’s master theorem. *Ramanujan J.*, 29(1-3):103–120, 2012.
[2] M. A. Biot. General theorems on the equivalence of group velocity and energy transport. *Phys. Rev.*, 105:1129–1137, Feb 1957.
[3] L. Brillouin. *Wave propagation and group velocity / Leon Brillouin*. Pure and applied physics ; v.8. Academic Press, New York, 1960.
[4] L. A. Caffarelli and P. E. Souganidis. Convergence of nonlocal threshold dynamics approximations to front propagation. *Arch. Ration. Mech. Anal.*, 195(1):1–23, 2010.
[5] G. M. Coclite, S. Dipierro, F. Maddalena, and E. Valdinoci. Wellposedness of a nonlinear peridynamic model. *Nonlinearity*, 32(1):1–21, Nov 2018.
[6] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.
[7] S. Dipierro, E. Proietti Lippi, and E. Valdinoci. (Non)local logistic equations with Neumann conditions. *arXiv e-prints*, page arXiv:2101.02315, Jan. 2021.
[8] A. C. Eringen. *Nonlocal continuum field theories*. Springer-Verlag, New York, 2002.
[9] A. C. Eringen and D. G. B. Edelen. On nonlocal elasticity. *Internat. J. Engrg. Sci.*, 10:233–248, 1972.
[10] M. E. Gurtin. The linear theory of elasticity. In *Linear theories of elasticity and thermoelasticity*, pages 1–295. Springer, 1973.
[11] G. H. Hardy. *Ramanujan. Twelve lectures on subjects suggested by his life and work*. Cambridge University Press, Cambridge, England; Macmillan Company, New York, 1940.
[12] E. Kröner. Elasticity theory of materials with long range cohesive forces. *Internat. J. Solids Structures*, 3(5):731–742, 1967.
[13] I. A. Kunin. *Elastic media with microstructure. I*, volume 26 of *Springer Series in Solid-State Sciences*. Springer-Verlag, Berlin-New York, 1982. One-dimensional models, Translated from the Russian.
[14] C. Lim, G. Zhang, and J. Reddy. A higher-order nonlocal elasticity and strain gradient theory and its applications in wave propagation. *J. Mech. Phys. Solids*, 78:298–313, 2015.
[15] S. Papargyri-Beskou, D. Polyzos, and D. Beskos. Wave dispersion in gradient elastic solids and structures: A unified treatment. *Internat. J. Solids Structures*, 46(21):3751–3759, 2009.
[16] O. Savin and E. Valdinoci. Γ-convergence for nonlocal phase transitions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(4):479–500, 2012.
[17] S. Silling and R. Lehoucq. Peridynamic theory of solid mechanics. In H. Aref and E. van der Giessen, editors, *Advances in Applied Mechanics*, volume 44 of *Advances in Applied Mechanics*, pages 73–168. Elsevier, 2010.
[18] S. A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *J. Mech. Phys. Solids*, 48(1):175–209, 2000.
[19] S. A. Silling. Linearized theory of peridynamic states. *J. Elasticity*, 99(1):85–111, 2010.
[20] S. A. Silling, M. Epton, O. Weckner, J. Xu, and E. Askari. Peridynamic states and constitutive modeling. *J. Elasticity*, 88(2):151–184, 2007.
[21] S. A. Silling and R. B. Lehoucq. Convergence of peridynamics to classical elasticity theory. *J. Elasticity*, 93(1):13–37, 2008.
[22] E. M. Stein and R. Shakarchi. *Complex analysis*, volume 2 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2003.
[23] T. Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.
[24] G. B. Whitham. *Linear and nonlinear waves*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.
