JOINING MEASURES FOR HOROCYCLE FLOWS
ON ABELIAN COVERS

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ABSTRACT. A celebrated result of Ratner from the eighties says that two horo-
cycle flows on hyperbolic surfaces of finite area are either the same up to al-
gebraic change of coordinates, or they have no non-trivial joinings. Recently,
Mohammadi and Oh extended Ratner’s theorem to horocycle flows on hyper-
bolic surfaces of infinite area but finite genus. In this paper, we present the
first joining classification result of a horocycle flow on a hyperbolic surface of
infinite genus: a \( \mathbb{Z} \) or \( \mathbb{Z}^2 \)-cover of a general compact hyperbolic surface. We
also discuss several applications.

1. INTRODUCTION

The starting point of our discussion is Ratner’s joining theorem for horocycle
flows on a finite volume quotient of \( \text{PSL}_2(\mathbb{R}) \) \cite{30}, which is a particular case
of her general classification theorem of invariant measures for unipotent flows
on any finite volume homogeneous space of a connected Lie group \cite{31}. For
infinite volume homogeneous spaces, such classification theorems are known
only for some special cases (~\cite{9, 33, 39, 3, 35}, etc.)

Recently, Mohammadi-Oh \cite{25} extended Ratner’s joining theorem to geomet-
rically finite discrete subgroups in \( \text{PSL}_2(\mathbb{R}) \) or \( \text{PSL}_2(\mathbb{C}) \). Their work is built on
earlier works of Flaminio and Spatzier on the rigidity of horospherical foliations
for such discrete subgroups (~\cite{12, 13}). In this paper, we extend Ratner’s joining
theorem to the unit tangent bundle of a \( \mathbb{Z}^d \)-cover of a compact hyperbolic sur-
face. To the best of our knowledge, this is the first joining classification result
for hyperbolic surface of infinite genus.

To state our results more precisely, let \( G = \text{PSL}_2(\mathbb{R}) \) and \( \Gamma_1, \Gamma_2 \) be discrete sub-
groups of \( G \). In the whole paper, all discrete subgroups of \( G \) are assumed to be
torsion-free and non-elementary. Assume further that \( \Gamma_1 \) is a normal subgroup
of a cocompact lattice \( \Gamma_1' \) of \( G \) so that \( \Gamma_1 \backslash \Gamma_1' \cong \mathbb{Z}^d \) for some positive integer \( d \).
Then \( \Gamma_1 \backslash G \) is a \( \mathbb{Z}^d \)-cover of the unit tangent bundle of the compact hyperbolic
surface \( \Gamma_1' \backslash \mathbb{H}^2 \). For simplicity, discrete subgroups like \( \Gamma_1 \) will be called \( \mathbb{Z}^d \)-covers. Let
\[
Z = \Gamma_1 \backslash G \times \Gamma_2 \backslash G.
\]
Set
\begin{equation}
U = \left\{ u_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}
\end{equation}
and \( \Delta(U) = \{(u_t, u_t) : t \in \mathbb{R} \} \). As is well known, the right translation action of \( u_t \) on \( \Gamma_i \backslash G \) corresponds to the contracting horocycle flow when we identify \( \Gamma_i \backslash G \) with the unit tangent bundle of the hyperbolic surface \( \Gamma_i \backslash \mathbb{H}^2 \).

**Definition 1.1.** Let \( \mu_i \) be a locally finite \( U \)-invariant Borel measure on \( \Gamma_i \backslash G \) for \( i = 1, 2 \). A locally finite \( \Delta(U) \)-invariant measure \( \mu \) on \( Z \) is called a \( \mu \)-joining with respect to the pair \( (\mu_1, \mu_2) \) if the push-forward \( (\pi_i)^* \mu \) is proportional to \( \mu_i \) for each \( i = 1, 2 \); here \( \pi_i \) denotes the canonical projection of \( Z \) to \( \Gamma_i \backslash G \). If \( \mu \) is \( \Delta(U) \)-ergodic, then \( \mu \) is called an ergodic \( U \)-joining.

In this paper, we investigate the \( U \)-joinings with respect to the pair of Haar measures \( (\mathcal{H}^{\text{Haar}}_{\Gamma_1}, \mathcal{H}^{\text{Haar}}_{\Gamma_2}) \). In fact, Ledrappier and Sarig showed in [20] that the Haar measure is the unique \( U \)-ergodic measure for \( \mathbb{Z}^d \)-covers which admits a generalized law of large numbers.

Our definition of \( U \)-joinings rules out the product measure \( \mathcal{H}^{\text{Haar}}_{\Gamma_1} \times \mathcal{H}^{\text{Haar}}_{\Gamma_2} \) since its projection to \( \Gamma_2 \backslash G \) is an infinite multiple of \( \mathcal{H}^{\text{Haar}}_{\Gamma_2} \). Nevertheless, a finite cover self-joining provides an example of \( U \)-joining. Recall that two subgroups of \( G \) are said to be commensurable with each other if their intersection has finite index in each of them.

**Definition 1.2** (Finite cover self-joining). Suppose that for some \( g_0 \in G \), \( \Gamma_1 \) and \( g_0^{-1} \Gamma_2 g_0 \) are commensurable with each other. Using the map
\[ \Gamma_1 \cap g_0^{-1} \Gamma_2 g_0 \backslash G \to Z \]
defined by \( [g] \mapsto ([g], [g_0 g]) \), the pushforward of the Haar measure \( \mathcal{H}^{\text{Haar}}_{\Gamma_1 \cap g_0^{-1} \Gamma_2 g_0} \) to \( Z \) gives a \( U \)-joining, which will be called a finite cover self-joining. If \( \mu \) is a \( U \)-joining, then any translation of \( \mu \) by \( (e, u_t) \) is also a \( U \)-joining. Such a translation of a finite cover self-joining will also be called a finite cover self-joining.

Our main result is as follows:

**Theorem 1.3.** Let \( \Gamma_1 \) be a \( \mathbb{Z} \) or \( \mathbb{Z}^2 \)-cover and let \( \Gamma_2 \) be any discrete subgroup of \( G \). Then any locally finite ergodic \( U \)-joining on \( Z \) is a finite cover self-joining.

The reason we assume \( \Gamma_1 \) is a \( \mathbb{Z} \) or \( \mathbb{Z}^2 \)-cover is that only for \( \mathbb{Z} \) and \( \mathbb{Z}^2 \)-covers, the geodesic flow is ergodic with respect to the Haar measures [32] and this property is essentially used in the proof of the main theorem.

**Corollary 1.4.** Let \( \Gamma_1 \) be as in Theorem 1.3. Suppose \( \Gamma_2 \) is a discrete subgroup of \( G \) such that the \( U \)-action is ergodic on \( (\Gamma_2 \backslash G, \mathcal{H}^{\text{Haar}}_{\Gamma_2}) \). Then \( Z \) admits a \( U \)-joining if and only if \( \Gamma_1 \) and \( \Gamma_2 \) are commensurable with each other, up to a conjugation.

Under our assumption, any \( U \)-joining measure on \( Z \) can be disintegrated into an integral over a probability space of a family of \( U \)-ergodic joinings. Thus Corollary 1.4 is an immediate application of Theorem 1.3.
Similar to the finite joining case, we can deduce the classification of \( U \)-equivariant factor maps from the classification of joinings:

**Corollary 1.5.** Let \( \Gamma \) be a \( \mathbb{Z} \) or \( \mathbb{Z}^2 \)-cover. Let \((Y, \nu)\) be a measure space with a locally finite \( U \)-invariant measure \( \nu \). Suppose \( p : (\Gamma \setminus G, m^\text{Haar}_\Gamma) \to (Y, \nu) \) is a \( U \)-equivariant factor map, that is, \( p_* m^\Gamma = \nu \). Then \((Y, \nu)\) is isomorphic to \((\Gamma_0 \setminus G, m^\text{Haar}_{\Gamma_0})\) where \( \Gamma_0 \) is a discrete subgroup of \( G \) containing \( \Gamma \) as a finite index subgroup. Moreover, the map \( p \) can be conjugated to the canonical projection \( \Gamma \setminus G \to \Gamma_0 \setminus G \).

Let \( A \) be the diagonal group in \( G \). As another application of the joining classification theorem, we obtain a classification of \( \Delta(AU) \)-invariant measures similar to [26]:

**Corollary 1.6.** Let \( \Gamma_1 \) be a \( \mathbb{Z} \) or \( \mathbb{Z}^2 \)-cover and let \( \Gamma_2 \) be a cocompact lattice of \( G \). Any \( \Delta(AU) \)-invariant, ergodic, conservative, infinite Radon measure \( \mu \) on \( \Gamma_1 \setminus G \times \Gamma_2 \setminus G \) is one of the following:

1. \( \mu \) is the product measure \( m^\text{Haar}_{\Gamma_1} \times m^\text{Haar}_{\Gamma_2} \);
2. \( \mu \) is the pushforward of the Haar measure on \( \Gamma_1 \cap g_0^{-1} \Gamma_2 g_0 \setminus G \) through the map:

\[
\phi : \Gamma_1 \cap g_0^{-1} \Gamma_2 g_0 \setminus G \to \Gamma_1 \setminus G \times \Gamma_2 \setminus G, \quad [g] \mapsto ([g], [g_0 g]),
\]

where \( g_0 \) is some element of \( G \) so that \( |\Gamma_1 : \Gamma_1 \cap g_0^{-1} \Gamma_2 g_0| < \infty \).

**On the proof of Theorem 1.3.** Our proof is loosely modeled on Mohammadi-Oh’s proof of classification of infinite \( U \)-joining measures for geometrically finite discrete subgroups [25]. In their proof, they utilize a close relation between Burger-Roblin measures and Bowen-Margulis-Sullivan measures (which will be called BR measures and BMS measures respectively for short) and the finiteness of BMS measures is crucially used. However, in our setting, both BR measures and BMS measures are Haar measures and hence such a passage to finite measures is not available. Here we discuss some of the main steps and difficulties.

One of the key ideas in Ratner’s proof [30] as well as our proof is to use the polynomial like behavior of unipotent flows to construct new invariants of a \( U \)-joining in concern. This idea is also used in Margulis’ proof of Oppenheim’s conjecture [21] using topological argument. To utilize this property, we need to demonstrate that the return times of a typical orbit to a fixed compact set has enough self similarities. More precisely, we show

**Theorem 1.7.** Suppose \( \Gamma \) is a \( \mathbb{Z}^d \)-cover for some positive integer \( d \). For any small \( 0 < \eta < 1 \), there exists \( 0 < r = r(\eta) < 1 \) such that for any non-negative \( \psi \in C_c(\Gamma \setminus G) \) and for almost every \( x \in \Gamma \setminus G \), there exists \( T_0 = T_0(\psi, x) > 0 \) so that

\[
\int_0^{rT} \psi(xu_t) \, dt \leq \eta \int_0^T \psi(xu_t) \, dt \quad \text{for all } T \geq T_0.
\]

This is one of the difficulties in extending Ratner’s rigidity theorems to infinite volume setting. For geometrically finite discrete subgroup, Flaminio and
Spatzier ([12, 13]) as well as Mohammadi and Oh [25] overcome this difficulty by using the self similarities of the conditional measure of the BMS measure. In our setting, we use symbolic description of the geodesic flow over the unit tangent bundle of $\Gamma \setminus \mathbb{H}^2$ and some ideas in Ledrappier and Sarig’s proof about the rational ergodicity of the horocycle flows for $\mathbb{Z}^d$-covers ([20], see also [36]).

As an application of Theorem 1.7, we classify the orbit closures of $\mathbb{Z}$ or $\mathbb{Z}^2$-cover group in the unit tangent bundle of compact hyperbolic surfaces in the appendix (Theorem 7.3).

With Theorem 1.7 available, we establish the following two properties about an arbitrary ergodic $U$-joining $\mu$ on $\Gamma$:

1. almost all fibers of projection of $\mu$ on $\Gamma \setminus G$ are finite;
2. $\mu$ is invariant under the diagonal embedding of $A$ (up to conjugation).

Let

$$U^+ := \left\{ u_t^+ := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

be the expanding horocyclic group, opposite to the subgroup $U$. Parametrize the elements of $A$ by $a_s := \begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{pmatrix}$. Extending the invariance of $\mu$ under $\Delta(U^+)$ involves showing that a measurable $AU$-equivariant set-valued map $\mathcal{Y} : \Gamma_1 \setminus G \to \Gamma_2 \setminus G$ is also $U^+$-equivariant. The rough idea is to demonstrate that if both $x a_{-s}$ and $x u_t^+ a_{-s}$ lie in some good compact subset, then the $U$-orbits of $\mathcal{Y}(x u_t^+ u_t^+, a_{-s})$ and $\mathcal{Y}(x a_{-s})$ do not diverge on average. More precisely, we show that

$$\sup_{t \in [0, e^s]} d(\mathcal{Y}(x u_t^+ a_{-s}) u_t^+, \mathcal{Y}(x a_{-s}) u_t) = O(1). \quad (1.3)$$

Such an argument is used by Ratner [30] as well as by Flaminio and Spatzier ([12, 13]). For the case when $\Gamma_1$ and $\Gamma_2$ are lattices, Birkhoff’s ergodic theorem and polynomial divergence of horocycle flows are two key inputs to obtain this estimate.

Letting $\mathcal{Y}(x u_t^+ a_{-s}) u_{-e^{s+t}}^+, = \mathcal{Y}(x a_{-s}) g_s$, a simple matrix computation yields $a_{-s} g_s a_s = O(e^{-s})$. Therefore,

$$d(\mathcal{Y}(x u_t^+) u_{-e^{s+t}}^+, \mathcal{Y}(x)) = d(\mathcal{Y}(x) a_{-s} g_s a_s, \mathcal{Y}(x)) = O(e^{-s}).$$

The ergodicity (and hence the conservativity) of the geodesic flow gives us an increasing sequence of $\{ s_j \}$ so that $x a_{-s_j}$ and $x u_t^+ a_{-s_j}$ lie in some good compact subset, which eventually shows that $\mathcal{Y}$ is $U^+$-equivariant. Now the assumption that $\Gamma_1$ is a $\mathbb{Z}$ or $\mathbb{Z}^2$-cover ensures the ergodicity of the geodesic flow, providing us the necessary dynamics between geodesic flows and horocycle flows. As $\Gamma_1 \setminus G$ is of infinite measure, to achieve (1.3), we make the most of the Hopf’s ratio theorem for horocycle flows and geodesic flows with respect to a series of compact subsets chosen with calibration.
Notational convention.

1. For any positive number \( a, b \) and \( \epsilon \), we write \( a = e^{\epsilon} b \) to mean that \( e^{-\epsilon} b \leq a \leq e^{\epsilon} b \).

2. For any discrete subgroup \( \Gamma \) in \( G \), denote the Haar measure on \( \Gamma \setminus G \) by \( m_\Gamma \).

When there is no ambiguity about \( \Gamma \), we simply denote it by \( m \).

2. Symbolic dynamics

For the rest of the paper, fix \( \Gamma_0 \) a cocompact lattice of \( G = \text{PSL}_2(\mathbb{R}) \) and \( \Gamma \) a normal subgroup of \( \Gamma_0 \) with \( \Gamma \Gamma_0 \cong \mathbb{Z}^d \) for some positive integer \( d \). Recall that we set

\[ A = \left\{ a_s := \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix} : s \in \mathbb{R} \right\}. \]

The right translation action of \( a_s \) on \( \Gamma \setminus G \) corresponds to the geodesic flow on the unit tangent bundle of \( \Gamma \setminus \mathcal{H} \) which can be identified with \( \Gamma \setminus G \). Recall the groups \( U \) and \( U^+ \) defined in (1.1) and (1.2) respectively.

In this section, we describe the geodesic flow on \( \Gamma \setminus G \) as a suspension flow, whose base is a skew product over a subshift of finite type. First recall some basic notions of symbolic dynamics.

A subshift of finite type with set of states \( S \) and transition matrix \( A = (a_{ij})_{S \times S} \) \( (i, j \in \{0, 1\}) \) is the set

\[ \Sigma := \{ x = (x_i) \in S^\mathbb{Z} : t_{x_i} x_{i+1} = 1 \} \]

together with the action of the left shift map \( \sigma : \Sigma \to \Sigma \), \( \sigma(x)_k = x_{k+1} \) and the metric \( d(x, y) = \sum_{k \in \mathbb{Z}} \frac{1}{2^{1+|k|}} (1 - \delta_{x_k, y_k}) \). There is a one-sided version \( \sigma : \Sigma^+ \to \Sigma^+ \) obtained by replacing \( \mathbb{Z} \) by \( \mathbb{N} \cup \{0\} \).

Suppose \( F \) is a real-valued function on \( \Sigma \) or \( \Sigma^+ \). The Birkhoff sums of \( F \) are denoted by \( F_n \),

\[ F_n := F + F \circ \sigma + \cdots + F \circ \sigma^{n-1}. \]

Symbolic dynamics for the geodesic flow. Fix \( \Omega_0 \) to be a connected relatively compact fundamental domain in \( \Gamma \setminus G \) for the left action of \( \Gamma \Gamma_0 \). As \( \Gamma \Gamma_0 \cong \mathbb{Z}^d \), the group \( \mathbb{Z}^d \) acts on \( \Gamma \setminus G \). For every \( \xi \in \mathbb{Z}^d \), we denote the left action of \( \xi \) on \( \Gamma \setminus G \) by \( D_\xi \).

Definition 2.1. For every \( g \in \Gamma \setminus G \), we call the unique integer \( \xi(g) \in \mathbb{Z}^d \) satisfying \( g \in D_{\xi(g)} \Omega_0 \) the \( \mathbb{Z}^d \)-coordinate of \( g \).

By a lifting argument of Bowen-Series symbolic dynamics of the geodesic flow on \( \Gamma_0 \setminus G \) (see [8, 37, 38, 27]), we obtain the following characterization of the geodesic flow on \( \Gamma \setminus G \):

Lemma 2.2. There exist a topologically mixing two-sided subshift of finite type \( (\Sigma, \sigma) \), a Hölder continuous function \( \tau : \Sigma \to \mathbb{R} \) which depends only on the non-negative coordinates, a function \( f : \Sigma \to \mathbb{Z}^d \) such that \( f(x) = f(x_0, x_1) \), a Hölder function \( h : \Sigma \to \mathbb{R} \) and a Hölder continuous map \( \pi : \Sigma \times \mathbb{Z}^d \times \mathbb{R} \to \Gamma \setminus G \) satisfying the following properties:
1. \( \tau^* := \tau + h - h \circ \sigma \) is non-negative, and there exists a constant \( n_0 \) such that \( \inf_{x \in X} \tau^*_n(x) > 0 \).

2. Let

\[
(\Sigma \times \{0\})_\tau := \{(x, 0, t) : x \in \Sigma, 0 \leq t < \tau^*(x)\}.
\]

The restriction map \( \pi : (\Sigma \times \{0\})_\tau \rightarrow \Omega_0 \) is a surjective finite-to-one map. Moreover, there exists a countable sequence \( \{g_i\} \subseteq \Gamma \setminus G \), such that every \( g \in \Gamma \setminus G \) outside \( \bigcup_{i=1}^{\infty} g_i \cdot AU \) and \( \bigcup_{i=1}^{\infty} g_i \cdot AU^+ \) has exactly one preimage [37].

3. For any \((\xi_0, t_0) \in \mathbb{Z}^d \times \mathbb{R}\), define the map \( Q_{\xi_0, t_0} \) on \( \Sigma \times \mathbb{Z}^d \times \mathbb{R} \) by \( Q_{\xi_0, t_0}(x, \xi, t) = (x, \xi + \xi_0, t + t_0) \). Then \( \pi \circ Q_{\xi_0, t_0}(x, \xi, t) = D_{\xi_0}(\pi(x, \xi, t) a_{t_0}) \) for all \((x, \xi, t) \in \Sigma \times \mathbb{Z}^d \times \mathbb{R}\).

4. \( \pi \circ T_{f, -\tau} = \pi \), where \( T_{f, -\tau}(x, \xi, t) = (\sigma x, \xi + f(x), t - \tau^*(x)) \).

5. Suppose \( g = \pi(x, \xi, t) \), \( g' = \pi(x', \xi', t') \). If there exist \( p, q \geq 0 \) such that

\[
x_p^\infty = (x')_q^\infty \quad (i.e., \, x_{p+i} = x_{q+i}' \text{ for any } i \in \mathbb{N});
\]

\[
t - t' = h(x) - h(x') + \tau_p(x) - \tau_q(x');
\]

\[
\xi - \xi' = f_q(x') - f_p(x),
\]

then \( g' = g u_s \) for some \( s \in \mathbb{R} \).

6. Suppose \( g = \pi(x, \xi, t), 0 \leq t < \tau^*(x) \). For every \( s \in \mathbb{R} \), all but at most countably many points \( g' \in g U a_s \) have a unique representation \( g' = \pi(x', \xi', t') \) such that \( 0 \leq t' < \tau^*(x') \) and there exist \( p, q \) with \( (x')_p^\infty = x_q^\infty \).

**Symbolic coordinates.** For every \( g_i \in \Gamma \setminus G \), the point described in Lemma 2.2 (2), choose a representation \( g_i = \pi(x_i, \xi_i, t_i) \) such that \( 0 \leq t_i < \tau^*(x_i) \). We call \((x_i, \xi_i, t_i) \in \Sigma \times \mathbb{Z}^d \times \mathbb{R}\) a symbolic coordinate for \( g \in \Gamma \setminus G \), if

1. \( g \notin \bigcup_{i=1}^{\infty} g_i \cdot AU \), \( g = \pi(x, \xi, t) \), and \( 0 \leq t < \tau^*(x) \);

2. \( g \in g_i \cdot U a_s \), \( g = \pi(x, \xi, t) \), \( 0 \leq t < \tau^*(x) \), and \( x_p^\infty = (x_i)_q^\infty \) for some \( p, q \).

Some points in \( \Gamma \setminus G \) have more than one symbolic coordinates. But for every \( g \in \Gamma \setminus G \), the set of points in \( g U \) with more than one symbolic coordinates is at most countable by Lemma 2.2 (2) and (6). In particular, for every \( g \), the Birkhoff integral \( \int_0^T f(g u_t) dt \) is determined by the \( t \)'s for which \( g u_t \) has a unique symbolic coordinate. We may therefore safely ignore the points with more than one symbolic coordinates.

**Ruelle's transfer operator and the Haar measure.** Consider the Ruelle's operator \( L_{-\tau} : C(\Sigma^+) \rightarrow C(\Sigma^+) \) given by

\[
L_{-\tau}(\varphi)(x) = \sum_{\sigma f = x} e^{-\tau(y)} \varphi(y).
\]

By Ruelle-Perron-Frobenius theorem, there exist a probability measure \( \nu' \) on \( \Sigma^+ \) and a Hölder continuous function \( \psi : \Sigma^+ \rightarrow \mathbb{R}^+ \) such that

\[
L_{-\tau} \psi = \psi, \quad L_{-\tau}^* \nu' = \nu', \quad \text{and} \quad \int \psi d\nu' = 1.
\]

The measure \( \psi d\nu' \) is a shift invariant probability measure which can be extended to the two-sided shift \( \Sigma \). Denote this extension by \( \nu \).
Let
\[
(\Sigma \times \mathbb{Z}^d)_{\tau^*} := \{(x, \xi, t) : 0 \leq t \leq \tau^*(x)\}.
\]
The following lemma is essentially in [6] (see also [3]).

**Lemma 2.3.** The Haar measure on \(G \setminus \Gamma\), subject to the normalization \(m_\Gamma(\Omega_0) = 1\), is given by
\[
\frac{1}{\tau^*} (v \times dm_{\mathbb{Z}^d} \times dt)|_{(\Sigma \times \mathbb{Z}^d)_{\tau^*}} \circ \pi^{-1}.
\]

**Symbolic local manifolds.** Suppose \(g \in \Gamma \setminus G\) has a symbolic coordinate \((x, \xi, t)\) with \(0 \leq t < \tau^*(x)\). Write \(t = s + h(x)\). The symbolic local stable manifold of \(g = \pi(x, \xi, s + h(x))\) is defined to be
\[
W_{ssloc}(g) := \pi\{y, \xi, s + h(y) : y_0^\infty = x_0^\infty\}.
\]
It follows from Lemma 2.2 (5) that \(W_{ssloc}(g) \subset gU\). Lemma 2.2 also implies that if \(W_{ssloc}(g)\) intersects \(W_{ssloc}(g')\) with positive measure for another \(g' \in \Gamma \setminus G\), then they are equal up to a set of measure 0.

Let the measure \(l_g\) on \(gU\) be given by the length measure
\[
l_g([g u_t : a < t < b]) = b - a.
\]

**Lemma 2.4 (Proposition 4.5 in [3]).** Suppose \(g \in \Gamma \setminus G\) has a symbolic coordinate \((x, \xi, s + h(x))\). Then
\[
l_g[W_{ssloc}(g)] = e^{-s}\psi(x_0, x_1, \ldots)
\]
where \(\psi : \Sigma^+ \to \mathbb{R}_0^+\) is the eigenfunction of the Ruelle's transfer operator given as (2.1).

### 3. Window property

Recall that \(\Gamma\) is a normal subgroup of a cocompact lattice \(\Gamma_0\) with \(\Gamma \setminus \Gamma_0 \cong \mathbb{Z}^d\) for some positive integer \(d\).

We keep the notations from Section 2. For \(g \in \Gamma \setminus G\) and \(T \in \mathbb{R}\), define
\[
\xi_T(g) := \xi(g a_T),
\]
where \(\xi(g a_T)\) is the \(\mathbb{Z}^d\)-coordinate of \(g a_T\) given by Definition 2.1.

It follows from the work of Ratner [28] and Katsuda-Sunada [16] that the distribution \(\frac{\xi_T(g)}{\sqrt{T}}\) as \(g\) ranges over \(\Omega_0\) converges to the distribution of a multivariate Gaussian random variable \(N\) on \(\mathbb{R}^d\), with a positive definite covariance matrix \(\text{Cov}(N)\). Denote
\[
\sigma := \sqrt{|\det\text{Cov}(N)|}.
\]

Consider the set
\[
W := \left\{ g \in \Gamma \setminus G : \lim_{T \to \infty} \frac{\xi_T(g)}{T} = 0, \limsup_{T \to \infty} \left| \frac{\xi_T(g)}{\sqrt{T } \ln T} \right| = \sqrt{2} \sigma \right\}.
\]
Then \(W\) is a conull set by Corollary 6.1 in [3] and Corollary 2 in [10].

In this section, we aim to prove the window property for the horocycle flow on \(G \setminus G\):
Theorem 3.1 (Window property I). For any $0 < \eta < 1$, there exists $0 < r = r(\eta) < 1$ so that the following holds: for any $g \in W$, and for any non-negative $\psi \in C_c(\Gamma \setminus G)$, there exists $T_0 = T_0(\psi, g) > 1$ such that for every $T > T_0$ we have

$$\int_0^{r T} \psi(g u_t) dt \leq \eta \int_0^T \psi(g u_t) dt.$$  

The following is another version of window property we need in the proof of joining classification.

Theorem 3.2 (Window property II). For any sufficiently small $0 < \delta < 1$, there exists $0 < \epsilon = c(\delta) < 1/4$ so that the following holds: for any $g \in W$ and for any non-negative $\psi \in C_c(\Gamma \setminus G)$, there exists $T_0 = T_0(\psi, g) > 1$ such that for every $T > T_0$ we have

$$\int_T^{(1+\delta) T} \psi(g u_t) dt \leq c \int_0^T \psi(g u_t) dt.$$  

3.1. Key Lemma. We show a key lemma (Lemma 3.3) leading to Theorems 3.1 and 3.2, which elaborates on the work of Ledrappier and Sarig [20], see also [36].

For $\varphi \in C(\Sigma^+)$, the topological pressure $P_{\text{top}}(\varphi)$ is given by

$$P_{\text{top}}(\varphi) := \sup_{\mu} \left( h_{\mu}(\sigma) + \int \varphi d\mu \right),$$

where the supremum is taken over all $\sigma$-invariant Borel probability measures $\mu$ on $\Sigma^+$; here $h_{\mu}(\sigma)$ denotes the measure theoretic entropy of $\sigma$ with respect to $\mu$.

Let $\tau$ and $f$ be as in Lemma 2.2. Define $P : \mathbb{R}^d \to \mathbb{R}$ implicitly by $u \mapsto P(u)$, where $P(u)$ is the root satisfying $P_{\text{top}}(-P(u)\tau + \langle u, f \rangle) = 0$. It is shown in [2] and [3] that $P$ is a convex analytic function with $P(0) = 1$, $\nabla P(0) = 0$ and $P''(0) = \text{Cov}(N)$.

Set

$$H : \mathbb{R}^d \to \mathbb{R}$$

to be minus the Legendre transform of $P$. Then $H$ is a concave analytic function with $H(0) = 1$, $\nabla H(0) = 0$ and $H''(0) = -\text{Cov}(N)^{-1}$.

Lemma 3.3 (Key Lemma). For every small $0 < \epsilon < 1$, there exist a Borel set $E \subset \Gamma \setminus G$ of positive measure, some compact neighborhood $K = K(E, \epsilon)$ of $0$ in $\mathbb{R}^d$ and $T_0 = T_0(E, \epsilon) > 1$ so that for any $g \in \Gamma \setminus G$, if $T > T_0$ and $\frac{\xi_{T}(g)}{T^*} \in K$ with $T^* = \ln T$, then

$$\int_0^T \chi_E(g u_t) dt = \frac{e^{cT} m(E)}{(2\pi^d T^*)^{\frac{d}{2}}} \cdot T \cdot \exp \left( T^* \left( H \left( \frac{\xi_{T}(g)}{T^*} \right) - 1 \right) \right),$$

where $\sigma$ is given by (3.1).

Fix some small $\epsilon^* = \epsilon^*(\epsilon) > 0$, which will be determined later. Recall the symbolic coding introduced in Section 2, in particular the definition of the eigenfunction $\psi$ of the Ruelle’s operator (2.1). Denote by $d_{\max}$ the maximal diameter of a symbolic local stable manifold, measured in the intrinsic metric of the horocycle that contains it. The coding can be modified so that

- $\max \tau^* < \epsilon^*$, $\max |h| < \epsilon^*$, $d_{\max} < \epsilon^*$, $\max \psi < \epsilon^*$,
- $\text{diam}(\pi([x, \xi_0, s] : x_0 = a_0, 0 \leq s < T^*(x))) < \epsilon^*$ for all $a_0, \xi_0$. 


Moreover, the coding can be adjusted to satisfy the following property:
\[
\frac{\max \psi}{\min \psi} < C_0,
\]
where \( C_0 \) does not depend on \( \epsilon^* \) or \( \epsilon \) (see Section 4.1 in [36] for details).

**Proof of Lemma 3.3.** We divide the proof into four steps. The first three steps follow from [20], which we recall for readers’ convenience.

Fix some cylinder set \([a] = [a_0, \ldots, a_{n−1}]\) such that \( \inf_{[a]} T^* > 0 \). Also fix some \( \epsilon_0 \in (0, \inf_{[a]} T^*) \) and \( \xi_0 \in \mathbb{Z}^d \). Our set \( E \) is going to be
\[
E := \pi([x, \xi_0, t + h(x)) : x \in [a], 0 \leq t < \epsilon_0]).
\]

For any \( g \in \Gamma \setminus G \), denote \( g U_T := \{g u_t : t \in [0, T]\} \). Viewing the integral \( \int_0^T \chi_E(g u_t) dt \) as an integral on the horocyclic arc \( g U_T \) with respect to the measure \( \nu \), we can write
\[
\int_0^T \chi_E(g u_t) dt = l_g(E \cap g U_T) = l_g(E \cap g a T \cdot U_1 a_{−T^*}).
\]

**Step 1.** We approximate the horocyclic arc \( g a T \cdot U_1 \) by symbolic local stable manifolds. More precisely, we claim that there exist \( N^* \in \mathbb{N} \) and \( g_i \in g a T \cdot U_1 \) for \( i = 1, \ldots, N^* \) such that setting \( J_{T^*}(g_i, E) = l_g(E \cap W_{loc}^{loc}(g_i) a_{−T^*}) \), we have
\[
\sum_{i=1}^{N^-} J_{T^*}(g_i, E) \leq l_g(E \cap g U_T) \leq \sum_{i=1}^{N^*} J_{T^*}(g_i, E),
\]
(3.4)

\[
\left| \sum_{i=1}^{N^*} l(W_{loc}^{loc}(g_i)) - l(g a T \cdot U_1) \right| \leq 4 \epsilon^*.
\]
(3.5)

In fact, this can be achieved by choosing \( g_i \)'s for \( i = 1, \ldots, N^- \) so that \( W_{loc}^{loc}(g_i) \) is contained in \( g a T \cdot U_1 \). Choose \( g_i \)'s for \( i = N^- + 1, \ldots, N^* \) so that \( W_{loc}^{loc}(g_i) \) intersects \( g a T \cdot U_1 \) with positive measure without being contained in it. Note that any two symbolic local stable manifolds are either equal or disjoint up to sets of measure 0. Therefore \( l_g(E \cap g U_T) \) can be sandwiched between \( \sum_{i=1}^{N^*} J_{T^*}(g_i, E) \) as (3.4).

The inequality (3.5) follows from the observation that every \( g_i \) lies in the \( d_{\max} \)-neighborhood of \( g a T \cdot U_1 \) and \( d_{\max} < \epsilon^* \).

**Step 2.** Suppose that \( g \) and \( g_i \) have symbolic coordinates \((x, \xi, t + h(x))\) and \((x_i, \xi_i, t_i + h(x_i))\), respectively. Assume \( T > e^{4\epsilon^*} \). Putting \( T_1^* = T^* - t_i \), it is shown in step 2 of Lemma 1 in [20] that
\[
J_{T^*}(g_i, E) = e^{\epsilon_0} \sum_{k=0}^{\infty} \sum_{y=(x) \in \Lambda^\mathbb{Z}} \chi_{[0,\epsilon_0]}(r_k(y) - T_i^*) \delta_{\xi_i - \xi_0}(f_k(y)) \chi_{[a]}(y) \psi(y),
\]
(3.6)

where the \( y \)'s in this sum take values in the one-sided shift \( \Sigma^+ \).

We note for future reference that \( |T_i^* - T^*| = |t_i| < \max \tau^* + \max |h| < 2 \epsilon^* \).
Step 3. Using an elaboration of Lalley’s method [18], it is proved in the appendix of [20] that there exists a compact neighborhood $\hat{K}_0$ of 0 in $\mathbb{R}^d$ and $T_0 > 1$ depending on $E$ and $\epsilon^*$ so that for every $T > T_0$ and every $i$, if $\frac{\xi_i}{T_i^*} \in \hat{K}_0$, then

$$(3.7) \quad J_{T^*}(g_i, E) = \frac{e^{\pm 10\epsilon^*}}{(2\pi \sigma T^*)^{d/2}} \cdot \exp \left( T_i^* H \left( \frac{\xi_i}{T_i^*} \right) \right) \cdot m(E) \cdot \psi(x_i),$$

where $\sigma$ is defined as (3.1).

Step 4. Now (3.4), (3.6) and (3.7) together imply that

$$(10) \quad l_g(E \cap g U_T) \leq \frac{e^{10\epsilon^*}}{(2\pi \sigma T^*)^{d/2}} \cdot m(E) \sum_{i=1}^{N^*} \exp \left( T_i^* H \left( \frac{\xi_i}{T_i^*} \right) \right) \psi(x_i).$$

We compare $T^* H \left( \frac{\xi_{T^*}(g)_{T^*}}{T^*} \right)$ with $T_i^* H \left( \frac{\xi_i}{T_i} \right)$. Without loss of generality, assume $\hat{K}_0$ is sufficiently small so that for every $x \in \hat{K}_0$, we have $|H(x) - H(0)| \leq \epsilon^*$ and $\|\nabla H(x) - \nabla H(0)\| \leq \epsilon^*$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. Let $K_0$ be some smaller compact neighborhood of 0 inside $\hat{K}_0$ with $\text{diam}(K_0) < \frac{1}{4} \text{diam}(\hat{K}_0)$.

Suppose $\frac{\xi_{T^*}(g)_{T^*}}{T^*} \in K_0$. By construction, all the $g_i$’s belong to a $d_{\text{max}}$-neighborhood of $A_1(g a T^*)$, a horocyclic arc of length 1. Their $Z^d$-coordinates $\xi_i = \xi(g_i)$ must therefore be within a bounded distance $D$ from each other and that of $g a T^*$. As a result, if $T$ is large enough, then $\frac{\xi_{T^*}(g)_{T^*}}{T^*} \in K_0$ implies that $\frac{\xi_i}{T_i} \in \hat{K}_0$.

Estimate the difference

$$\left| T^* H \left( \frac{\xi_{T^*}(g)_{T^*}}{T^*} \right) - T_i^* H \left( \frac{\xi_i}{T_i} \right) \right| \leq |T^* - T_i^*| \cdot \left| H \left( \frac{\xi_{T^*}(g)_{T^*}}{T^*} \right) \right| + |T_i^*| \cdot \left| H \left( \frac{\xi_{T^*}(g)_{T^*}}{T^*} \right) - H \left( \frac{\xi_i}{T_i} \right) \right| \leq 2\epsilon^* \cdot |H(0) + \epsilon^*| + (1 + \epsilon^*) \cdot T^* \cdot (\|\nabla H(0)\| + \epsilon^*) \cdot \left| \frac{\xi_{T^*}(g)_{T^*}}{T^*} - \frac{\xi_i}{T_i} \right|.$$

Since $T$ is large and $\frac{\xi_i}{T_i} \in \hat{K}_0$, we have

$$T^* \cdot \left| \frac{\xi_{T^*}(g)_{T^*}}{T^*} - \frac{\xi_i}{T_i} \right| \leq T^* \cdot \left| \frac{\xi_{T^*}(g)_{T^*}}{T^*} - \frac{\xi_i}{T^*} \right| + T^* \cdot \left| \frac{\xi_i}{T^*} - \frac{\xi_i}{T_i} \right| \leq D + 2\epsilon^* \cdot \text{diam} \hat{K}_0.$$

Since $H(0) = 1$ and $\nabla H(0) = 0$, there exists a constant $k > 0$ independent of $\epsilon^*$ so that for any $T$ large enough, if $\frac{\xi_{T^*}(g)_{T^*}}{T^*} \in K_0$, then

$$\exp \left( T^* H \left( \frac{\xi_{T^*}(g)_{T^*}}{T^*} \right) - T_i^* H \left( \frac{\xi_i}{T_i} \right) \right) \leq \exp k\epsilon^*.$$

Consequently, we get an upper bound for $l_g(E \cap g U_T)$:

$$l_g(E \cap g U_T) \leq \frac{e^{(10 + k)\epsilon^*}}{(2\pi \sigma T^*)^{d/2}} \cdot m(E) \cdot \exp \left( T^* H \left( \frac{\xi_{T^*}(g)_{T^*}}{T^*} \right) \right) \cdot \left( \sum_{i=1}^{N^*} \psi(x_i) \right).$$
For the sum of $\psi(x_i)$'s, Lemma 2.4 yields
\[ \psi(x_i) = e^{t_i} l_{g_i}(W_{\text{loc}}^{\text{ss}}(g_i)) = e^{\pm 2\epsilon^*} l_{g_i}(W_{\text{loc}}^{\text{ss}}(g_i)). \]

It follows from (3.5) that
\[ \sum_{i=1}^{N^+} \psi(x_i) \leq e^{2\epsilon^*} \sum_{i=1}^{N^+} l_{g_i}(W_{\text{loc}}^{\text{ss}}(g_i)) \leq e^{6\epsilon^*}. \]

Letting $\epsilon^* = \epsilon/(16 + k)$, we show the upper bound for $l_g(E \cap gU_T)$.

The lower bound can be obtained in a similar way. The proof is complete. \(\square\)

3.2. Proof of the window property I, II. Recall the following result about generic points for the horocycle flow for $\mathbb{Z}^d$-covers.

**Definition 3.4.** Suppose $\phi^t : X \to X$ is a continuous flow on a second countable and locally compact metric space $X$. A point $x \in X$ is called generic for a $\phi^t$-invariant Radon measure $\mu$, if for all $f, g \in C_c(X)$ with nonzero integrals,
\[ \lim_{T \to \infty} \int_0^T f(\phi^t x) dt = \int f d\mu, \]
\[ \lim_{T \to \infty} \int_0^T g(\phi^t x) dt = \int g d\mu. \]

**Theorem 3.5** (Sarig-Shapira [36]). A point $g \in \Gamma \backslash G$ is generic for the horocycle flow with respect to the Haar measure $m_\Gamma$ if and only if $\lim_{T \to \infty} \xi_{T^*}(g) = 0$. In particular, every point in $W$ (given by (3.2)) is generic.

**Proof of Theorem 3.1.** Fix $0 < \eta < 1$ and some small $0 < \epsilon < 1$ (which will be determined later). Let $E$ be the set given by Lemma 3.3 for $\epsilon$. We claim that there exists $0 < r = r(\eta) < 1$ such that for every $g \in W$, there exists $T_0 = T_0(g, \psi)$ so that for every $T > T_0$, we have
\[ \int_0^T \chi_E(g u_t) dt \leq \eta \int_0^T \chi_E(g u_t) dt. \]

In view of Lemma 3.3, it suffices to show the existence of $r$ satisfying the inequality
\[ \frac{e^\epsilon m(E)}{(2\pi \sigma(r T^*)^{d/2})} \cdot \exp \left( (r T)^* H \left( \frac{\xi_{(r T)^*}(g)}{(r T)^*} \right) \right) \leq \eta \cdot \frac{e^{-\epsilon m(E)}}{(2\pi \sigma T^*)^{d/2}} \cdot \exp \left( T^* H \left( \frac{\xi_{T^*}(g)}{T^*} \right) \right), \]

or equivalently the inequality
\[ \exp \left( (r T)^* H \left( \frac{\xi_{(r T)^*}(g)}{(r T)^*} \right) - T^* H \left( \frac{\xi_{T^*}(g)}{T^*} \right) \right) \leq \eta \cdot e^{-2\epsilon} \cdot \left( \frac{(r T)^*}{T^*} \right)^{d/2}, \]

where $T^* = \ln T$.

The key to obtain such $r$ is to estimate the upper bound for the following difference. Since $\xi_{T^*}(g) \to 0$, using the Taylor expansion for $H$, we have for any
sufficiently large $T$

\[
(rT)^*H\left(\frac{\xi_{(rT)^*}(g)}{(rT)^*} \right) - T^*H\left(\frac{\xi_{T^*}(g)}{T^*} \right)
\]

\[
= (rT)^*\left(H(0) + \frac{1}{2} \left(\frac{\xi_{(rT)^*}(g)}{(rT)^*}\right)^T H''(0) \left(\frac{\xi_{(rT)^*}(g)}{(rT)^*}\right) + O\left(\frac{\|\xi_{(rT)^*}(g)\|^3}{(rT)^*}\right)\right)
\]

\[
- T^*\left(H(0) + \frac{1}{2} \left(\frac{\xi_{T^*}(g)}{T^*}\right)^T H''(0) \left(\frac{\xi_{T^*}(g)}{T^*}\right) + O\left(\frac{\|\xi_{T^*}(g)\|^3}{(T^*)^2}\right)\right)
\]

\[
= \ln r + \frac{1}{2} \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}}\right)^T H''(0) \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}}\right) + O\left(\frac{\|\xi_{(rT)^*}(g)\|^3}{(T^*)^2}\right)
\]

We analyze the above sum term by term. Noting that $H''(0) = -(\text{Cov}(N))^{-1}$ with Cov($N$) positive definite, we have

\[
\left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}}\right)^T H''(0) \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}}\right) = \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} + \xi_{T^*}(g)\right)^T H''(0) \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} + \xi_{T^*}(g)\right)
\]

\[
= \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} + \frac{\xi_{T^*}(g)}{\sqrt{T^*}}\right)^T H''(0) \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} + \frac{\xi_{T^*}(g)}{\sqrt{T^*}}\right)
\]

\[
\leq C \cdot \left\|\frac{\xi_{T^*}(g)}{T^*} - \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}}\right\| \cdot \left\|\frac{\xi_{T^*}(g)}{T^*} + \frac{1}{\sqrt{T^*}} \cdot \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}}\right\|
\]

where $C > 0$ is some constant only depending on Cov($N$).

Since $g_{\mathbf{a}_{(rT)^*}}$ is at most $-\ln r$ away from $g_{\mathbf{a}_{(T^*)}}$, we have

\[
\left\|\frac{\xi_{T^*}(g)}{T^*} - \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}}\right\| \leq \ln r / M + 2,
\]

where $M := \text{diam}(\Omega_0)$. Then utilizing the property that $\limsup_{T \to \infty} \left|\frac{\xi_{T}(g)}{\sqrt{T \ln T}}\right| = \sqrt{2}\sigma$, we have for any large $T$,

\[
\left\|\frac{\xi_{T^*}(g)}{T^*} - \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}}\right\| \leq \left\|\frac{\xi_{T^*}(g)}{T^*} - \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}}\right\| + \left\|\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \cdot \left(\sqrt{T^*} - \sqrt{(rT)^*}\right)\right\|
\]

\[
\leq \frac{\ln r}{M} + 2 + \left\|\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \cdot \ln((rT)^*)\right\| \cdot \left|\sqrt{T^*} - \sqrt{(rT)^*}\right| \cdot \ln((rT)^*)
\]

\[
\leq \frac{\ln r}{M} + 2 + 3\sigma \cdot \frac{\ln r \cdot \ln((rT)^*)}{\sqrt{T^*} + \sqrt{(rT)^*}}
\]

\[
\leq -\ln r \left(\frac{1}{M} + \epsilon\right) + 2.
\]
Meanwhile, applying the property that \( \lim_{T \to \infty} \frac{\xi_T^*(g)}{T} = 0 \), we obtain
\[
\left\| \frac{\xi_T^*(g)}{T} + \frac{1}{\sqrt{T^*}} \cdot \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \right\| \leq \left\| \frac{\xi_T^*(g)}{T} \right\| + \left\| \frac{\xi_{(rT)^*}(g)}{(rT)^*} \right\| \cdot \sqrt{rT^*} \leq 3e.
\]
For the higher degree terms, we have the estimate:
\[
\left\| \frac{\xi_T^*(g)}{(T^*)^2} \right\| = \left\| \frac{\xi_T^*(g)}{\sqrt{T^* \ln T^*}} \right\|^3 \cdot \sqrt{\ln \ln T^*} \to 0.
\]
As a result, when \( c \) is appropriately chosen, for any large \( T > 0 \), we obtain an upper bound:
\[
(rT)^* \cdot H\left( \frac{\xi_{(rT)^*}(g)}{(rT)^*} \right) - T^* \cdot H\left( \frac{\xi_T^*(g)}{T^*} \right) \leq \sqrt{T} \cdot e^{(3C+2)e}.
\]
Since \( \left( \frac{(rT)^*}{T^*} \right)^{d/2} \to 0 \) as \( T \to \infty \), if \( 0 < r < 1 \) satisfies
\[
\sqrt{T} e^{(3C+2)e} \leq \frac{1}{2} \cdot \eta \cdot e^{-2e},
\]
then such \( r \) satisfies (3.9).

Now recall that every point in \( W \) is generic for the horocycle flow (Theorem 3.5). For a general non-negative function \( \psi \in C_c((\Gamma \setminus G)) \) and for any \( g \in W \), we have
\[
\lim_{T \to \infty} \frac{\int_0^T \psi(g u(t)) dt}{\int_0^T \chi_E(g u(t)) dt} = \frac{\int \psi dm}{\int \chi_E dm}.
\]
This limit together with (3.8) yield (3.3).

Proof of Theorem 3.2. Fix \( 0 < \delta < 1 \) and some small \( 0 < c < 1 \) to be determined later. Let \( E \) be the set given by Lemma 3.3 for \( c \). We just need to show Theorem 3.2 holds for \( \chi_E \) and the general statement follows from Hopf's ratio theorem. In view of Lemma 3.3, it suffices to show the existence of \( c \) satisfying the following inequality:
\[
(1 + \delta) \cdot H\left( \frac{\xi_{(1+\delta)T}^*(g)}{(1+\delta)T} \right) \leq \frac{(1 + \delta) \cdot H\left( \frac{\xi_T^*(g)}{T} \right)}{2}.
\]

Using the same argument as the proof of Theorem 3.1, we obtain an upper bound for the following difference for \( T \) large enough:
\[
(1 + \delta) \cdot H\left( \frac{\xi_{(1+\delta)T}^*(g)}{(1+\delta)T} \right) - T^* \cdot H\left( \frac{\xi_T^*(g)}{T^*} \right) \leq \frac{3}{2} \ln(1 + \delta) + (3C + 2)e,
\]
where \( C \) is a constant just depending on \( \text{Cov}(N) \).

Since \( \left( \frac{(1+\delta)T}{T} \right)^{d/2} \to 1 \) as \( T \to \infty \), if \( 0 < c = c(r) < 1/4 \) satisfies
\[
(1 + \delta)^{3/2} \cdot e^{(3C+2)e} < (1 + c) e^{-2e},
\]
then such \( c \) makes (3.10) hold.
**Remark 3.6.** It can be deduced from the proof that given any non-negative \( \psi \in C_c(\Gamma \backslash G) \) and any compact set \( \Omega \subset \Gamma \backslash G \), Theorems 3.1 and 3.2 can be made uniform on \( \Omega \) if \( \frac{\zeta_{\psi}(\cdot)}{T} \), \( \sup_{t \geq T} \left| \frac{\zeta_{\psi}(\cdot)}{T \ln t} \right| \) and \( \frac{\int_0^T u_t \psi(\cdot)dt}{\int_0^T u_t \chi(\cdot)dt} \) converge uniformly on \( \Omega \).

4. **Weak \((C, \alpha)\)-Good Property for \( \mathbb{Z}^d\)-Covers**

Recall that \( \Gamma \) is a \( \mathbb{Z}^d \)-cover for some positive integer \( d \). The following terminology is introduced in [17].

**Definition 4.1.** Let \( C, \alpha > 0 \) and denote the Lebesgue measure on \( \mathbb{R} \) by \( | \cdot | \). A function \( f : \mathbb{R} \to \mathbb{R} \) is said to be \((C, \alpha)\)-good on \( \mathbb{R} \) if for any interval \( J \subset \mathbb{R} \) and \( \epsilon > 0 \) one has

\[
|x \in J : |f(x)| < \epsilon| \leq C \cdot \left( \frac{\epsilon}{\sup_{x \in J}|f(x)|} \right)^{\alpha} |J|.
\]

It follows from Lagrange’s interpolation that if \( f \) is a polynomial of degree not greater than \( k \), then \( f \) is \((k(k + 1)^{1/k}, 1/k)\)-good on \( \mathbb{R} \).

We prove a weak form of \((C, \alpha)\)-good property of polynomials which is related to the recurrence of the horocycle flow \( \Gamma \backslash G \). For any positive integer \( k \), denote by \( \mathcal{P}_k \) the set of polynomials \( \Theta : U \to \mathbb{R} \) of degree at most \( k \).

**Lemma 4.2.** Fix \( k \geq 1 \). For any compact set \( K \subset \Gamma \backslash G \) and any small \( 0 < \epsilon < 1 \), there exists a constant \( 0 < C < 1 \) (independent of \( K \) and \( \epsilon \)), a compact subset \( K_0 \subset K \) with \( m(K_0) \geq (1 - \epsilon)m(K) \) and \( T_0 = T_0(K_0) > 1 \) so that the following inequality holds for every \( g \in K_0 \), \( T > T_0 \) and \( \Theta \in \mathcal{P}_k \):

\[
(4.1) \quad \int_0^T \chi_K(g(\cdot)u_t)\Theta(t)dt \geq C \cdot \int_0^T \chi_K(g(\cdot)u_t)dt \cdot \sup_{t \in [0,T]} |\Theta(t)|.
\]

**Proof.** Fix \( K \) and \( \epsilon \). By Theorem 3.1 and Remark 3.6, there exist \( 0 < r_0 = r_0(1/(2k)) < 1 \), \( T_0 > 1 \), a compact set \( K' \subset K \) with \( m(K') > (1 - \epsilon/2)m(K) \) and \( T_0 = T_0(K') > 1 \) such that for every \( T > T_0 \) and every \( g \in K' \), we have

\[
(4.2) \quad \int_0^{r_0 T} \chi_K(g(\cdot)u_t)dt \leq \frac{1}{2k} \int_0^T \chi_K(g(\cdot)u_t)dt.
\]

Fix some sufficiently small \( \delta > 0 \). By Theorem 3.2 together with Remark 3.6, there exist \( 0 < c = c(\delta) < 1 \), a compact subset \( K_0 \subset K' \) with \( m(K_0) > (1 - \epsilon/2)m(K') \) and \( T_0 = T_0(K_0) > 1 \) so that the following inequalities hold for every \( g \in K_0 \) and every \( T > T_0' \):

\[
(4.3) \quad \int_T^{(1+1/\delta) T} \chi_K(g(\cdot)u_t)dt \leq \frac{c}{2k} \int_0^T \chi_K(g(\cdot)u_t)dt,
\]

\[
\int_T^{(1-1/\delta) T} \chi_K(g(\cdot)u_t)dt \leq \frac{c}{2k} \int_0^T \chi_K(g(\cdot)u_t)dt,
\]

where \( l(\delta) := r_0^{-1}k(k + 1)^{1/k} \delta^{1/k} \).
We show that every $g \in K_0$ satisfies (4.1). Fix any $T > \max\{t_0, T^0/r_0\}$. We claim that there exists a constant $C_T \in (0, 1)$ such that for every $g \in K_0$ and every $\Theta \in \mathcal{P}_k$, we have

$$\int_0^T \chi_K(gu_t)|\Theta(t)|dt \geq C_T \cdot \int_0^T \chi_K(gu_t)dt \cdot \sup_{t \in [0, T]} |\Theta(t)|.$$  

(4.4)

It can be seen from the process that $C_T$ can be chosen independently of $T$, $K$ and $\epsilon$.

By multiplying both sides of (4.4) by a scalar if necessary, it suffices to verify that (4.4) holds for every polynomial in $\mathcal{P}_k^1 = \{\Theta \in \mathcal{P}_k : \sup_{t \in [0, T]} |\Theta(t)| = 1\}$.

Let $\Theta \in \mathcal{P}_k^1$. The potential obstacle to obtain (4.4) is the following set

$$I_{\Theta} := \{t \in [0, T] : |\Theta(t)| < \delta\}.$$

The $(C, \alpha)$-good property of polynomials on $\mathbb{R}$ implies that

$$|I_{\Theta}| \leq k(k+1)^{1/k} \delta^{1/k} T.$$

Then

$$\int_0^T \chi_K(gu_t)|\Theta(t)|dt \geq \delta \cdot \int_{I_{\Theta} \setminus I_0} \chi_K(gu_t)dt.$$  

As a result, (4.4) follows if there exists $0 < C'_T < 1$ such that

$$\int_{I_0} \chi_K(gu_t)dt \leq C'_T \cdot \int_0^T \chi_K(gu_t)dt.$$  

(4.5)

Since $\Theta$ is a polynomial of degree at most $k$, $I_{\Theta}$ consists of at most $k$ intervals with the length of each interval less than $k(k+1)^{1/k} \delta^{1/k} T$. Let $I$ be one of these intervals. There are two cases to discuss.

**Case 1.** Suppose $I \subset [0, r_0 T]$. Then it follows from (4.2) that

$$\int_I \chi_K(gu_t)dt \leq \int_0^{r_0 T} \chi_K(gu_t)dt \leq \frac{1}{2k} \int_0^T \chi_K(gu_t)dt.$$  

**Case 2.** There exists $t_0 \in I \cap (r_0 T, T]$. Recalling that $l(\delta) = r_0^{-1} k(k+1)^{1/k} \delta^{1/k}$, we have

$$I \subset [t_0 - k(k+1)^{1/k} \delta^{1/k} T, t_0 + k(k+1)^{1/k} \delta^{1/k} T] \subset [(1-l(\delta))t_0, (1+l(\delta))t_0].$$

Applying (4.3), we have

$$\int_I \chi_K(gu_t)dt \leq \int_{(1-l(\delta))t_0}^{(1+l(\delta))t_0} \chi_K(gu_t)dt \leq \frac{e}{k} \int_0^{t_0} \chi_K(gu_t)dt.$$  

Therefore (4.5) holds for $C'_T = k \cdot \max\{\frac{e}{k}, \frac{1}{2k}\}$. Noting that $C'_T$ does not depend on $T$, $K$ and $\epsilon$, the proof of the lemma is completed.
5. RIGIDITY OF $AU$-EQUIVARIANT MAPS

For the rest of the paper, let $\Gamma_1$ and $\Gamma_2$ be discrete subgroups of $G$. Denote $\Gamma_i \backslash G$ by $X_i$. Assume $\Gamma_1$ is a $\mathbb{Z}$ or $\mathbb{Z}^2$-cover. Let

$$\varphi_1, \ldots, \varphi_k : X_1 \to X_2$$

be Borel measurable maps such that for any two distinct $i, j$, we have $\varphi_i \neq \varphi_j$ almost everywhere. Define the set-valued map:

$$\Phi(x) = \{\varphi_1(x), \ldots, \varphi_k(x)\}.$$  

This section is devoted to showing the rigidity of $AU$-equivariant maps.

**Theorem 5.1.** Suppose that there exists a conull set $X' \subset X_1$ such that for every $x \in X'$ and every $a, u \in AU$, we have

$$\Phi(xa, u) = \Phi(x) a, u.$$  

Then there exists a conull set $X'' \subset X'$ such that for all $x \in X''$ and for every $u^+ \in U^+$ with $xu^+ \in X''$, we have

$$\Phi(xu^+) = \Phi(x) u^+. \quad (5.1)$$

The proof is inspired by the previous works of Ratner [29], Flaminio-Spatzier [12, 13] and Mohammadi-Oh [25]. Different from their setting, we now need to deal with infinite measures and make use of Hopf’s ratio theorem instead of Birkhoff ergodic theorem.

5.1. Reduction of Theorem 5.1.

**Lemma 5.2.** Theorem 5.1 holds if there exists a conull set $\tilde{X} \subset X'$ and $r_0 > 0$ such that for every $x \in \tilde{X}$ and every $r \in (-r_0, r_0)$ with $xu^+_r \in \tilde{X}$,

$$\Phi(xu^+_r) = \Phi(x) u^+_r.$$  

**Proof.** Set

$$X'' := \left\{ x \in \tilde{X} : \int_0^\infty \chi_{\tilde{X}}(xa) ds = 0 \right\}.$$

Then $X''$ is a conull set of $\tilde{X}$. We show $X''$ satisfies Theorem 5.1.

Fix any $x \in X''$ and $u^+_r \in U^+$ with $xu^+_r \in X''$. We may assume that $r > 0$. The property of $X''$ implies there exists $s > 0$ large enough so that $e^{-s} r < r_0$ and $xa_{-s}, xu^+_r a_{-s} \in \tilde{X}$. Then Lemma 5.2 can be deduced from a series of equivalent relations:

$$\Phi(xu^+_r) = \Phi(x) u^+_r \iff \Phi(xu^+_r a_{-s}) = \Phi(x) a_{-s} u^+_{e^{-s} r}$$

$$\iff \Phi(xu^+_r a_{-s}) = \Phi(xa_{-s}) u^+_{e^{-s} r} \quad (\text{by the } A\text{-equivariance})$$

$$\iff \Phi(xa_{-s} u^+_{e^{-s} r}) = \Phi(xa_{-s}) u^+_{e^{-s} r} \quad (\text{by the property of } \tilde{X}). \quad \square$$
5.2. **Key proposition for Theorem 5.1.** Recall the polynomial divergence of horocycle flow. It is known (see, for example, [12]) that there are universal constants $\rho_0 \in (0, 1)$, $C_0 > 1$ and $n_0 \in \mathbb{N}$ so that for all $x, y \in G$ and any interval $I \subset \mathbb{R}$ on which

$$(d(xu_t, yu_t))^2 < \rho_0^2,$$

there exists a polynomial $P$ of degree at most $n_0$ such that

$$P(s)/C_0 \leq (d(xu_t, yu_t))^2 \leq C_0 P(s)$$

for all $s \in I$.

We introduce three compact sets $K \subset \Omega \subset Q$ in $X_1$.

**Construction of $Q$.** Fix some small $\epsilon_1 > 0$. Choose a compact set $Q$ in $X_1$ so that there exists a symmetric neighborhood $U$ of $e$ in $G$ satisfying:

$$m(\bigcup_{u \in U} Qu \setminus \bigcap_{u \in U} Qu) < \epsilon_1 m(Q).$$

Denote

$$Q^+ = \bigcup_{u \in U} Qu \quad \text{and} \quad Q^- = \bigcap_{u \in U} Qu.$$ (5.2)

**Construction of $\Omega \subset Q$.** Let $\Omega$ be a compact subset of $Q$ satisfying the following properties:

- $\Omega \subset X'$ ($X'$ is given as Theorem 5.1).
- $m(\Omega) > (1 - \epsilon_1) m(Q)$.
- If $i \neq j$, then $\varphi_i(x) \neq \varphi_j(x)$ for every $x \in \Omega$.
- For every $i \in \{1, \ldots, k\}$, we have $\varphi_i$ continuous on $\Omega$.

In view of the properties of $\Omega$, there exists $\rho \in (0, \min\{\epsilon_1, \rho_0\})$ such that for every $x \in \Omega$, if $i \neq j$, then

$$d(\varphi_i(x), \varphi_j(x)) > 2\rho.$$ (5.3)

Set

$$\mathcal{F}_1 := \{\chi_{\Omega}, \chi_Q, \chi_{Q^+}, \chi_{Q^-}\}.$$ (5.4)

**Construction of $K \subset \Omega$.** Let $K$ be a compact subset in $\Omega$ satisfying the following properties:

- $m(K) > (1 - \epsilon_1) m(\Omega)$.
- Lemma 4.2 holds for $\chi_{\Omega}$ on $K$ with constants $C_1$ (independent of $\Omega$, $K$ and $\epsilon_1$) and $T_0$.
- Hopf’s ratio theorem for the horocycle flow holds uniformly on $K$ for the family of functions in $\mathcal{F}_1$.

Let $T_1 > 0$ be the starting point such that for every $T > T_1$, every $x \in K$, and every $f_1, f_2 \in \mathcal{F}_1$, we have

$$\frac{\int_0^T f_1(xu_t) dt}{\int_0^T f_2(xu_t) dt} > (1 - \epsilon_1) \frac{m(f_1)}{m(f_2)}.$$ (5.4)
Since $c_0$ and $c_1$ are independent of $\Omega$, $K$ and $e_1$, we may assume
\begin{equation}
(1 - e_1)^5 > \max \left\{ \frac{3}{4}, 1 - \frac{c_1}{4c_0^2} \right\}.
\end{equation}

Set
\[ \mathcal{F}_2 := \{ \chi_K, \chi_Q, \chi_Q^+, \chi_Q^- \}. \]

**Construction of conull set** $\tilde{X} \subset X'$. Let $\tilde{X}$ be a conull subset in $X'$ satisfying the following properties:
- for every $x \in \tilde{X}$, we have
  \[ \int_0^\infty \chi(xa_\cdot) ds = \infty. \]
- Hopf’s ratio theorem for the geodesic flow holds for every point in $\tilde{X}$ for the family of functions in $\mathcal{F}_2$.

We will show that there exists $r_0 > 0$ such that for every $x \in \tilde{X}$ and every $r \in (0, r_0)$ with $xu_r^+ \in \tilde{X}$,
\[ \Phi(xu_r^+) = \Phi(x)u_r^+. \]

We first prove the following intermediate result:

**Proposition 5.3.** Under the hypothesis of Theorem 5.1, there exists $r_0 > 0$ such that for every $x \in \tilde{X}$, $r \in (0, r_0)$ with $xu_r^+ \in \tilde{X}$, and for every $s > \max\{T_0, T_1\}$, if $xa_{-s}, xu_r^+ a_{-s} \in K$, then
\[ \Phi(xu_r^+) u_r^{+}_{-s} < \Phi(x) \cdot \{ g \in G : d(g, e) \leq c \cdot e^{-s} \}, \]
where $c > 1$ is an absolute constant.

**Proof.** Fix $x \in \tilde{X}$. For every $r > 0$ and $s > \max\{T_0, T_1\}$, if $xu_r^+ \in \tilde{X}$ then
\[ \Phi(x)a_{-s} = \Phi(xa_{-s}) \]
and
\[ \Phi(xu_r^+) u_r^{+}_{-s} = \Phi(xu_r^+ a_{-s}) u_r^{+}_{-s} = \Phi(xa_{-s} u_r^{+}_{e^{-s}r}) u_r^{+}_{e^{-s}r}. \]

We compare the distance between the $U$-orbits of $\Phi(xa_{-s} u_r^{+}_{e^{-s}r}) u_r^{+}_{e^{-s}r}$ and $\Phi(xa_{-s})$ and show that they do not diverge on average.

**Step 1.** Let $\rho$ be given as (5.3). There exist $e_2 \in (0, \rho/2)$ and $e_3 \in (0, e_2)$ such that for every $r \in (0, e_3)$ and $s > \max\{T_0, T_1\}$, if $xa_{-s}, xu_r^+ a_{-s}(= xa_{-s} u_r^{+}_{e^{-s}r}) \in K$, then
\[ d(\varphi_i(xu_r^+ a_{-s}) u_r^{+}_{e^{-s}r}, \varphi_i(xa_{-s})) < 2e_2. \]
Moreover we have for every $t \in [0, \max\{T_0, T_1\}]$
\[ d(\varphi_i(xu_r^+ a_{-s}) u_r^{+}_{-r}, u_t, \varphi_i(xa_{-s}) u_t) < \rho, \]
where $\rho$ is the constant given as (5.3).

Since $\varphi_i$ is continuous on $\Omega$, there exists $e_2 \in (0, \rho/2)$ such that for each $i$ and for every $x, y \in \Omega$ if
\[ d(\varphi_i(x), \varphi_i(y)) < 2e_2, \]
then for all \( t \in [0, \max\{T_0, T_1\}] \),
\[
d(\varphi_t(x) u_t, \varphi_t(y) u_t) < \rho,
\]
where \( \rho \) is the constant given as (5.3).

Let \( \varepsilon_3 \in (0, \varepsilon_2) \) be a constant so that for every \( x, y \in \Omega \), if
\[
d(x, y) < \varepsilon_3,
\]
then
\[
d(\varphi_t(x), \varphi_t(y)) < \varepsilon_2.
\]

Consequently, for any \( r \in (0, \varepsilon_3) \) and \( s > \max\{T_0, T_1\} \), if \( xa_{-s} \) and \( xu^+_r a_{-s} \in K \), then
\[
d(\varphi_t(xu^+_r a_{-s}) u^+_{e^{-s} r}, \varphi_t(xa_{-s})) < 2\varepsilon_2,
\]
and the second inequality follows from the choice of \( \varepsilon_2 \).

In view of (5.5), we can let \( \varepsilon_2 \) small enough such that
\[
4\varepsilon_2^2 + 2 \rho^2 (1 - (1 - \varepsilon_1)^5 (1 - \varepsilon_2)^2) < \frac{C_1 \rho^2}{2C_0}.
\]

For the rest of the proof, we fix any \( s > \max\{T_0, T_1\} \) and any \( r \in (0, \varepsilon_3) \) such that \( xa_{-s}, xu^+_r a_{-s} \in K \).

Define for \( t \in [0, e^s] \)
\[
\beta(t) := \frac{t}{1 - e^{-s} r t}, \quad g_t := \begin{pmatrix} (1 - e^{-s} r t)^{-1} & 0 \\ -e^{-s} r & 1 - e^{-s} r t \end{pmatrix}.
\]

It is easy to see \( d(e, g_t) < \varepsilon_3 \). And we have for every \( t \in [0, e^s] \),
\[
u^+_{e^{-s} r} u_t = u_{g(t)} g_t.
\]

**Step 2.** For \( t \in [0, e^s] \), if \( xa_{-s} u_t g_t^{-1}, xa_{-s} u_t \in \Omega \), then for every \( i \in \{1, \ldots, k\} \)
\[
d(\varphi_t(xa_{-s} u^+_{e^{-s} r}, \varphi_t(xa_{-s} u_t) < 2\varepsilon_2.
\]

In fact, we can obtain this inequality by using (5.7)
\[
d(\varphi_t(xa_{-s} u^+_{e^{-s} r}, u^+_{e^{-s} r} u_t, \varphi_t(xa_{-s} u_t)
\]
\[
= d(\varphi_t(xa_{-s} u^+_{e^{-s} r}, u_{\beta(t)} g_t, \varphi_t(xa_{-s} u_t)
\]
\[
= d(\varphi_{\beta(t)}(xa_{-s} u^+_{e^{-s} r}, u_{\beta(t)} g_t, \varphi_t(xa_{-s} u_t) \quad \text{(by the } U\text{-equivariance)}
\]
\[
\leq d(\varphi_{\beta(t)}(xa_{-s} u_t g_t^{-1}) g_t, \varphi_t(xa_{-s} u_t))
\]
\[
< 2\varepsilon_2.
\]

**Step 3.** We claim the following inequality holds:
\[
\frac{|\{t \in [0, e^s] : xa_{-s} u_t, xa_{-s} u_t g_t^{-1} \in \Omega\}|}{|\{t \in [0, e^s] : xa_{-s} u_t \in \Omega\}|} \geq 2 \cdot (1 - \varepsilon_1)^5 (1 - \varepsilon_2)^2 - 1.
\]
Let $Q^+, Q^-$ be the sets defined as (5.2). We may assume that $\epsilon_3 < \text{diam}(U)$. For every $t \in [0, e^s]$, since $d(e, g_t) < \epsilon_3$, we have the following relations:

$$
\begin{align*}
& x_{a-s} u_t g_t^{-1} \in Q^+ \\
& x_{a-s} u_t g_t^{-1} \in Q \\
& x_{a-s} u_t g_t^{-1} \in \Omega \\
& x_{a-s} u_t g_t^{-1} \in Q^- \\
& x_{a-s} u_t \in Q \\
& x_{a-s} u_t \in \Omega
\end{align*}
$$

Then

$$
\frac{\int_0^{e^s} \chi_\Omega(x_{a-s} u_t g_t^{-1}) \, dt}{\int_0^{e^s} \chi_{Q^+}(x_{a-s} u_t g_t^{-1}) \, dt} = \frac{\int_0^{e^s} \chi_\Omega(xu^+_t a_{-s} u_{\beta(t)}) \, dt}{\int_0^{e^s} \chi_{Q^+}(xu^+_t a_{-s} u_{\beta(t)}) \, dt} \quad \text{(by (5.7))}
$$

$$
\begin{align*}
& \geq (1 - r)^2 \cdot \frac{\int_0^{e^s} \text{diam}(\mathcal{R}(xu^+_t a_{-s} u_{\beta(t)})) \, dl}{\int_0^{e^s} \text{diam}(\mathcal{R}(xu^+_t a_{-s} u_{\beta(t)})) \, dl} \\
& \geq (1 - \epsilon_2)^2 \cdot (1 - \epsilon_1) \cdot \frac{m(\Omega)}{m(Q^+)} \quad \text{(since } xu^+_t a_{-s} \in K \text{ and } s > T_1) \\
& \geq (1 - \epsilon_2)^2 \cdot (1 - \epsilon_1)^3.
\end{align*}
$$

And

$$
\begin{align*}
& \int_0^{e^s} \chi_\Omega(xa_{-s} u_t) \, dt \geq (1 - \epsilon_1) \cdot \frac{m(\Omega)}{m(Q)} \cdot \int_0^{e^s} \chi_\Omega(xa_{-s} u_t) \, dt \\
& \geq (1 - \epsilon_1)^2 \cdot \int_0^{e^s} \chi_{Q^+}(xa_{-s} u_t g_t^{-1}) \, dt \\
& \geq (1 - \epsilon_1)^3 \cdot (1 - \epsilon_2)^2 \cdot \frac{m(Q^-)}{m(Q^+)} \cdot \int_0^{e^s} \chi_{Q^+}(xa_{-s} u_t g_t^{-1}) \, dt \\
& \geq (1 - \epsilon_1)^5 \cdot (1 - \epsilon_2)^2 \cdot \int_0^{e^s} \chi_{Q^+}(xa_{-s} u_t g_t^{-1}) \, dt.
\end{align*}
$$

Consequently,

$$
\begin{align*}
|\{t \in [0, e^s] : xa_{-s} u_t, xa_{-s} u_t g_t^{-1} \in \Omega\}| \\
\geq 2 \cdot (1 - \epsilon_1)^5 \cdot (1 - \epsilon_2)^2 \cdot |\{t \in [0, e^s] : xa_{-s} u_t g_t^{-1} \in Q^+\}| \\
\geq (2 \cdot (1 - \epsilon_1)^5 \cdot (1 - \epsilon_2)^2 - 1) \cdot |\{t \in [0, e^s] : xa_{-s} u_t \in \Omega\}|.
\end{align*}
$$

The claim is justified.
Step 4. Let $\rho$ be the constant given as (5.3). For each $i$, we claim that

$$\sup_{t \in [0, \epsilon^3]} (d(\varphi_i(xa_{-s}u^+_{e^{-s}t}, u_t, \varphi_i(xa_{-s})u_t))^2 \leq \rho^2.$$  

Set

$$\tilde{T} = \inf\{T \in [0, \epsilon^4] : (d(\varphi_i(xa_{-s}u^+_{e^{-s}t}) u^+_{e^{-s}T}, u_T, \varphi_i(xa_{-s})u_T)^2 = \rho^2\}.$$  

It follows from the choice of $\epsilon_2$ and $\epsilon_3$ in Step 1 that $\tilde{T} > \max\{T_0, T_1\}$. The polynomial divergence of horocycle flow implies that there exists a polynomial $P$ of degree at most $n_0$ such that for every $t \in [0, \tilde{T}]

$$P(t)/C_0 \leq (d(\varphi_i(xa_{-s}u^+_{e^{-s}t}) u^+_{e^{-s}t}, u_t, \varphi_i(xa_{-s})u_t))^2 \leq C_0 P(t).$$  

Define

$$\Theta_{i,s}(t) := \min\{(d(\varphi_i(xa_{-s}u^+_{e^{-s}t}) u^+_{e^{-s}t}, \Phi(xa_{-s})u_t)^2, \rho^2\}.  

We have that for any $t \in [0, \tilde{T})$, if $xa_{-s}u_t \in \Omega$, then

$$\Theta_{i,s}(t) = (d(\varphi_i(xa_{-s}u^+_{e^{-s}t}) u^+_{e^{-s}t}, \varphi_i(xa_{-s})u_t))^2.$$  

In fact, if there is another $j \neq i$ satisfying

$$\Theta_{i,s}(t) = (d(\varphi_i(xa_{-s}u^+_{e^{-s}t}) u^+_{e^{-s}t}, \varphi_j(xa_{-s})u_t))^2,$$  

then

$$d(\varphi_i(xa_{-s})u_t, \varphi_j(xa_{-s})u_t) \leq 2\rho.$$  

However both $\varphi_i(xa_{-s})u_t$ and $\varphi_j(xa_{-s})u_t$ belong to the set $\Phi(xa_{-s}u_t)$. It follows from the property of $\Omega$ that this is a contradiction.

Since $\tilde{T} > T_0$, applying Lemma 4.2, we get

$$\begin{align*}
\int_0^{\tilde{T}} \frac{\chi_{\Omega}(xa_{-s}u_t)\Theta_{i,s}(t) dt}{\int_0^{\tilde{T}} \chi_{\Omega}(xa_{-s}u_t) dt} \\
\geq \frac{\int_0^{\tilde{T}} \chi_{\Omega}(xa_{-s}u_t) Q(t) dt}{C_0 \int_0^{\tilde{T}} \chi_{\Omega}(xa_{-s}u_t) dt} \\
\geq \frac{C_1}{C_0} \cdot \sup_{t \in [0, \tilde{T}]} Q(t) \\
\geq \frac{C_1}{C_0} \cdot \sup_{t \in [0, \tilde{T}]} (d(\varphi_i(xa_{-s}u^+_{e^{-s}t}) u^+_{e^{-s}t}, \varphi_i(xa_{-s})u_t))^2.
\end{align*}$$

(5.8)

Meanwhile by the same argument as Steps 2 and 3, we have

$$\begin{align*}
\int_0^{\tilde{T}} \frac{\chi_{\Omega}(xa_{-s}u_t)\Theta_{i,s}(t) dt}{\int_0^{\tilde{T}} \chi_{\Omega}(xa_{-s}u_t) dt} \leq 4\epsilon_2^2 + 2\rho^2 \cdot (1 - (1 - \epsilon_1)^5(1 - \epsilon_2)^2).
\end{align*}$$

(5.9)

If $\tilde{T} < \epsilon^4$, then

$$\sup_{t \in [0, \tilde{T}]} (d(\varphi_i(xa_{-s}u^+_{e^{-s}t}) u^+_{e^{-s}t}, u_t, \varphi_j(xa_{-s})u_t))^2 = \rho^2.$$
And (5.8) and (5.9) together imply that
\[
\frac{C_1 \rho^2}{C_2} \leq 4 \varepsilon_2^2 + 2 \rho^2 \cdot (1 - (1 - \varepsilon_1)^5 (1 - \varepsilon_2)^2),
\]
 contradicting (5.6). Therefore \( \tilde{T} = e^s \) and the proof of Step 4 is completed.

**Step 5.** Completion of the proof of Proposition 5.3. Let \( g_{s,i} \in G \) satisfying
\[
\varphi_i(x u_r^+ a_{-s}) u_{-s}^{-1} = \varphi_i(x a_{-s} g_{s,i}).
\]
Step 4 in particular implies that \( g_{s,i} \) is contained in an \( O(1) \) neighborhood of the identity.

Write \( g_{s,i} = \begin{pmatrix} x_s & y_s \\ z_s & w_s \end{pmatrix} \). Then
\[
u_{-1} g_{s,i} \nu_t = \begin{pmatrix} x_s - t z_s & y_s + t (x_s - w_s) - t^2 z_s \\ z_s & w_s + t z_s \end{pmatrix}.
\]
Therefore it follows from Step 4 and the fact that \( \det g_{s,i} = 1 \) that
\[
|z_s| = O(e^{-2s}), |1 - x_s| = O(e^{-s}), |1 - w_s| = O(e^{-s}), |y_s| = O(1).
\]
This implies
\[
d(e, a_{-s} g_{s,i} a_s) = O(e^{-s}).
\]

In consequence,
\[
\varphi_i(x u_r^+ a_{-s}) u_{-s}^{-1} = \varphi_i(x a_{-s} g_{s,i} a_s) a_{-s} \in \Phi(x a_{-s}) a_s (a_{-s} g_{s,i} a_s) a_{-s} \in \Phi(x) \cdot \{ g \in G : \det(g, e) = O(e^{-s}) \} \cdot a_{-s}.
\]
Noting that \( \varphi_i(x u_r^+ a_{-s}) u_{-s}^{-1} \in \Phi(x u_r^+ u_{-s}^{-1} a_{-s}, a_{-s}) \), we conclude that
\[
\Phi(x u_r^+ u_{-s}^{-1} a_{-s}) \subset \Phi(x) \cdot \{ g \in G : \det(g, e) = O(e^{-s}) \}.
\]
This proves Proposition 5.3 with \( r_0 = \varepsilon_3 \) (constructed in Step 1).

**Proof of Theorem 5.1.** Fix any \( x \in \hat{X} \) and \( r \in (0, \varepsilon_3) \) with \( x u_r^+ \in \hat{X} \). We show that there exists an increasing sequence \( \{ s_n \} \subset \mathbb{R} \geq 0 \) such that \( x a_{-s_n}, x u_r^+ a_{-s_n} \in K \).

For any \( s > 0 \), noting that \( d(e, u_{-r}^+ u_{-s}^{-1}) < \varepsilon_3 \), we have the following relations:

\[
\begin{array}{c}
\text{x} a_{-s} \in Q^+ \\
\text{x} u_r^+ a_{-s} \in Q \\
\text{x} u_r^+ a_{-s} \in K \\
\text{x} a_{-s} \in Q^- \\
x a_{-s} \in K
\end{array}
\]

By construction, every point in \( \hat{X} \) is generic for the Hopf’s ratio theorem for the geodesic flow with respect to the family of functions in \( \mathcal{F}_2 \). For any
sufficiently large $T$, there exists a constant $c = c(T)$ such that
\[
\int_0^T \chi_K(x u^*_r a_{-s}) d s \geq c \cdot \frac{m(K)}{m(Q)} \cdot \int_0^T \chi_Q(x u^*_r a_{-s}) d s \geq c \cdot (1 - \epsilon_1)^2 \cdot \int_0^T \chi_Q'(x a_{-s}) d s \geq c^2 \cdot (1 - \epsilon_1)^3 \cdot \int_0^T \chi_Q'(x a_{-s}) d s.
\]
At the same time, we have
\[
\int_0^T \chi_K(x a_{-s}) d s \geq c \cdot \frac{m(K)}{m(Q')}, \quad \int_0^T \chi_Q'(x a_{-s}) d s \geq c \cdot (1 - \epsilon_1)^4 \cdot \int_0^T \chi_Q'(x a_{-s}) d s.
\]
It can be deduced from the above two inequalities that
\[
|\{s \in [0, T] : x u^*_r a_{-s}, x a_{-s} \in K\}| \geq (2c^2(1 - \epsilon_1)^3 - 1) \cdot |\{s \in [0, T] : x a_{-s} \in K\}|.
\]
The right-hand side of the above inequality is greater than 0 because $c$ is close to 1 when $T$ is sufficiently large and
\[
\int_0^\infty \chi_K(x a_{-s}) d s = \infty
\]
by the property of $\bar{X}$. Therefore there exists an increasing sequence $\{s_n\} \subset \mathbb{R}_{>0}$ such that $x a_{-s_n}, x u^*_r a_{-s_n} \in K$. Applying Proposition 5.3, we have
\[
\Phi(x u^*_r) u^*_r \subset \Phi(x) \cdot \{g \in G : d(e, g) = O(e^{-s_n})\}.
\]
As $s_n \to \infty$, this implies that
\[
\Phi(x u^*_r) u^*_r = \Phi(x).
\]

6. JOINING CLASSIFICATION

In this section, we prove the classification theorem of ergodic $U$-joinings (Theorem 1.3). The proof is divided into several steps. Let $\mu$ be any ergodic $U$-joining measure on $Z := X_1 \times X_2$. First we show that $\mu$ is invariant under the action of $\Delta(A)$ up to conjugation (Corollary 6.4): this consists of showing that $\mu$ is invariant under the action of a nontrivial connected subgroup of $\Delta(A)(\{e\} \times U)$ (Theorem 6.1) and that $\mu$ cannot be invariant under $\{e\} \times U$ (Lemma 6.3).

Next we prove that there exist a conull set $\Omega \subset Z$ and a positive integer $l$ so that $#\pi_{i}^{-1}(x^l) \cap \Omega = l$ for $m_{\Gamma_1}$-a.e. $x^l \in X_1$, where $\pi_1 : Z \to X_1$ is the canonical projection (Theorem 6.5). This will yield an $AU$-equivariant set-valued map $\mathcal{V} : X_1 \to X_2$. Applying Theorem 5.1 to $\mathcal{V}$, we prove that there exists $q_0 \in G$ so that $\Gamma_2 q_0 \Gamma_1 = \cup_{j=1}^l \Gamma_2 q_0 \gamma_j$ with $\gamma_j \in \Gamma_1$ and
\[
\mathcal{V}(\Gamma_1 g) = \{\Gamma_2 q_0 \gamma_j g, \ldots, \Gamma_2 q_0 \gamma_j g\}.
\]
for \( m_{\Gamma_1} \)-a.e. \( \Gamma_1 g \) (Proposition 6.6). This will eventually imply that \( \mu \) is in fact a finite cover self-joining (Definition 1.2), completing the proof of Theorem 1.3.

6.1. \( \Delta(A) \)-invariance of \( \mu \). Fix the followings:

1. a non-negative function \( \psi \in C_c(X_1) \) with \( m_{\Gamma_1}(\psi) > 0 \) and set \( \Psi = \psi \circ \pi \in C(Z) \);
2. a compact subset \( \Omega \subset X_1 \) so that Theorems 3.1 and 3.2 hold uniformly for \( \psi \) for all \( x \in \Omega \);
3. a constant \( 0 < r := \frac{1}{4} r(\frac{1}{2}; \Omega) < 1 \),

where \( r(\frac{1}{2}; \Omega) \) is given as Theorem 3.1;
4. a compact subset \( Q \subset \Omega \times X_2 \) such that for every \( x \in Q \), every \( f \in C_c(Z) \) and \( g \in C_c(X_1) \), the following holds:

\[
\lim_{T \to \infty} \frac{\int_0^T f(x\Delta(u_t)) dt}{\int_0^T g \circ \pi(x\Delta(u_t)) dt} = \frac{\mu(f)}{\mu(g \circ \pi_1)}.
\]

Fix a small \( \epsilon > 0 \) and choose \( \eta > 0 \) small enough so that \( \mu(Q|g : |g| < \eta) \leq (1 + \epsilon)\mu(Q) \). We put \( Q_+ := Q|g : |g| \leq \eta/4 \) and \( \mathcal{F} = \{ \chi_Q, \chi_{Q_+} \} \).

As every point \( Q \) satisfies Theorem 3.1 as well as (6.1), a simple computation yields

\[
\lim_{T \to \infty} \frac{\int_0^T f(x\Delta(u_t)) dt}{\int_0^T \Psi(x\Delta(u_t)) dt} = \frac{\mu(f)}{\mu(\Psi)}
\]
holds for every \( f \in \mathcal{F} \) and for \( \mu \)-a.e. \( x \in Q \). Set \( Q_c \) to be a compact subset in \( Q \) with \( \mu(Q_c) > (1 - \epsilon)\mu(Q) \) so that (6.2) converges uniformly on \( Q_c \).

Denote by \( N_{G \times G}(\Delta(U)) \) the normalizer of \( \Delta(U) \) in \( G \times G \).

**Theorem 6.1.** Let \( h_k \in G \times G - N_{G \times G}(\Delta(U)) \) be a sequence tending to \( e \) as \( k \to \infty \).
If \( Q_c h_k \cap Q_c \neq \emptyset \) for every \( k \), then \( \mu \) is invariant under a nontrivial connected subgroup of \( \Delta(A)(e) \times U \). Moreover, if \( \{h_k\} \) contains a subsequence in \( \{e\} \times G \), then \( \mu \) is invariant under \( \{e\} \times U \).

Given Theorems 3.1 and 3.2 in our setting, the proof of Theorem 7.12 in [25] works here. For readers’ convenience, we sketch the proof.

**Lemma 6.2** (Lemma 7.7 in [25]). If \( h \in N_{G \times G}(\Delta(U)) \) satisfies \( Q_c h \cap Q_c \neq \emptyset \), then \( \mu \) is \( h \)-invariant.

**Proof of Theorem 6.1.** Letting \( h_k = (h^1_k, h^2_k) \) and \( h^i_k = \left( a^i_k, b^i_k \right) \) for \( i = 1, 2 \), define for \( t \neq -d^{i1}_k / c^{i1}_k \)

\[
\alpha_k(t) = \frac{b^{i1}_k + a^{i1}_k t}{d^{i1}_k + c^{i1}_k t}.
\]
Let $x_k$ be a point in $Q_e$ so that $y_k = x_kh_k \in Q_e$. We can write

$$y_k \Delta(u_t) = x_kh_k\Delta(u_t) = x_k\Delta(u_{\alpha_k(t)})\varphi_k(t)$$

for some $\varphi_k(t) \in AU^+ \times G$. Associated to $\varphi_k(t)'s$, we obtain a quasi-regular map $\varphi : \mathbb{R} \to \Delta(A)((e) \times U)$ satisfying

$$\varphi(t) := \lim_k \varphi_k(R_k t),$$

where $\{R_k\}$ is a sequence of positive numbers tending to $\infty$ as $k \to \infty$. We refer readers to Section 7.1 in [25] or Section 5 in [23] for details.

Fix some sufficiently small $\sigma > 0$. Since $h_k \to e$ as $k \to \infty$, we can find an increasing sequence $\{T_k\}$ such that for all large $k$, the derivative of $\alpha_k$ satisfies

$$1 - \sigma \leq \alpha_k'(t) \leq 1 + \sigma$$

for any $t \in [0, T_k]$.

We claim that there exist constants $c_1 > 1$ and $\tilde{T} = \tilde{T}(Q_e, \Psi) > 1$ so that for all large $k$ and for every $f \in \mathcal{F}$,

$$c_1^{-1} \int_{r T}^{T} f(x\Delta(u_t)) dt \leq \int_{r T}^{T} f(x\Delta(u_{\alpha_k(t)})) dt \leq c_1 \int_{r T}^{T} f(x\Delta(u_t)) dt$$

holds for all $x \in Q_e$ and $t \in (\tilde{T}, T_k)$.

As $\alpha_k(0) \to 0$ as $k \to \infty$ and $\alpha_k'(t)$ is close to 1 for $t \in [0, T_k]$ ((6.3)), replacing $t$ by $\alpha_k(t)$, there exists $T_0 > 1$ such that for all large $k$, any $T \in [T_0, T_k]$, any $f \in \mathcal{F} \cup \{\Psi\}$ and $x \in Q_e$,

$$(1 + \sigma)^{-1} \int_{(1 - 2\sigma)r T}^{(1 - 2\sigma)T} f(x\Delta(u_t)) dt \leq \int_{r T}^{T} f(x\Delta(u_{\alpha_k(t)})) dt \leq (1 - \sigma)^{-1} \int_{(1 - 2\sigma)r T}^{(1 - 2\sigma)T} f(x\Delta(u_t)) dt.$$  

Applying Theorems 3.1 and 3.2 to the first and third equations in above inequalities for $f = \Psi$, we can verify that the claim is valid for $\Psi$. Note that the limit (6.2) converges uniformly on $Q_e$, we conclude that there are constants $c_1 > 1$ and $\tilde{T} = \tilde{T}(Q_e, \Psi) > 1$ so that for all large $k$ and for every $f \in \mathcal{F}$, (6.4) holds for all $x \in Q_e$ and $T \in (\tilde{T}, T_k)$.

Set $\tau'_{k}$ to be the infimum of $\tau > 0$ such that

$$\sup_{t \in [0, T]} d(e, \varphi_k(t)) = \eta/4,$$

and put $\tau_k = \min(\tau'_{k}, T_k)$. Note that $\theta_k = \tau_k/R_k$ is bounded away from 0. Passing to a subsequence if necessary, we may assume $\theta_k$’s converge to some $\theta \neq 0$.

Let $T' > 1$ be a constant satisfying for all $T > T'$ and for every $z \in Q_e$,

$$\frac{\int_{r T}^{T} \chi_Q(z\Delta(u_t)) dt}{f r T^T \chi_{Q^c}(z\Delta(u_t)) dt} > 1 - \epsilon.$$  

Note that we have the following relations:

$$x_k \Delta(u_{\alpha_k(t)}) \in Q \Rightarrow y_k \Delta(u_t) \in Q \Leftrightarrow y_k \Delta(u_t) \in Q.$$


The lower bound for the amount of time when $x_k \Delta(u_{a_k(t)}) \in Q$ is given as follows:

$$
\int_{rT}^T \chi_Q(x_k \Delta(u_{a_k(t)})) dt \geq c_1^{-1} \int_{rT}^T \chi_Q(x_k \Delta(u_t)) dt \\
\geq c_1^{-1}(1-\epsilon) \int_{rT}^T \chi_Q(y_k \Delta(u_{a_k^{-1}(t)})) dt \\
\geq c_1^{-2}(1-\epsilon) \int_{rT}^T \chi_Q(y_k \Delta(u_t)) dt \\
\geq c_1^{-2}(1-\epsilon)^2 \int_{rT}^T \chi_Q(y_k \Delta(u_t)) dt.
$$

We can give a lower bound for $||t \in [rT, T] : y_k \Delta(u_t) \in Q||$ in terms of $||t \in [rT, T] : y_k \Delta(u_t) \in Q_+||$ using (6.5).

These relations together imply that for all large $k$ and all $T \in [T', T_k]$

$$(6.6) \quad \{t \in [rT, T] : x_k \Delta(u_{a_k(t)}), y_k \Delta(u_t) \in Q \} > 0.$$

Now for each $k$, let $m_k$ be the largest integer so that $r^{m_k} \tau_k > T_0$. Then for any $l \geq 0$, we have $l \leq m_k$ holds for all large $k$. Applying (6.6) for $T_{k,l} = r^l \tau_k$, we obtain $t \in [r^{l+1} \tau_k, r^l \tau_k]$ and $z_{k,l} \in Q$ with $z_{k,l} \varphi_k(t) \in Q$. Passing to a subsequence we get $z_l \in Q$ and $s \in [r^{l+1} \theta, r^l \theta]$ so that $z_l \varphi(s) \in Q$. Therefore $\mu$ is $\varphi(s)$-invariant by Lemma 6.2. If $l$ is large enough, then $\varphi(s) \neq e$ gets arbitrarily close to $e$. The first claim of the theorem is proved noticing that the image of $\varphi$ is contained in $\Delta(A)(|e| \times G)$.

As for the second claim, the construction of $\varphi$ (see Section 7.1 for details) indicates that the image of $\varphi$ is contained in $N_{G < G}(\Delta(U)) \cap (|e| \times G)$ if $|h_k| \subset |e| \times G$. Consequently, under this situation, the joining measure $\mu$ is invariant under $|e| \times U$. \hfill $\square$

The following lemma follows from the proof of Lemma 7.16 in [25]:

**Lemma 6.3.** The ergodic joining measure $\mu$ is not invariant under $|e| \times U$.

Now we draw the following corollary from Theorem 6.1 and Lemma 6.3:

**Corollary 6.4.** The ergodic joining measure $\mu$ is invariant under a non-trivial connected subgroup $A'$ of $\Delta(A)(|e| \times U)$ which is not contained in $|e| \times U$.

**Proof.** Keep the same notations as in Theorem 6.1. In particular, $Q$ is a compact subset with $\mu(Q) > 0$ and $Q_\epsilon \subset Q$ with $\mu(Q_\epsilon) \geq (1-\epsilon)\mu(Q)$.

Let $\pi_i : Z \to X_i$ be the canonical projection for $i = 1, 2$. Since $(\pi_i)_* \mu = m_{\tau_i}$ and $m_{\tau_i}$ does not support on proper Zariski subvarieties, we can choose sequences $\{x_k\}, \{y_k\} \subset Q_\epsilon$ so that $y_k = x_k h_k$ with $h_k \notin N_{G < G}(\Delta(U))$ and $h_k$ tends to $e$ as $k \to \infty$.

Applying Theorem 6.1 to $\{h_k\}$, we get a map

$$
\varphi : \lim \rightarrow N_{G < G}(\Delta(U)) \cap \mathcal{L} = \Delta(A) \cdot (|e| \times U)
$$
so that $\mu$ is invariant under a non-trivial connected subgroup $L$ in the image of $\varphi$. The corollary follows from Lemma 6.3.

By replacing $\mu$ by $(e, u) \cdot \mu$, we may assume that $\mu$ is $\Delta(AU)$-invariant in the rest of the section.

6.2. Finiteness of fiber measures. Let $\mathcal{P}(X_2)$ be the set of probability measures. By the standard disintegration theorem, there exists an $m_{\Gamma_1}$-conull set $X'_1 \subset X_1$ and a measurable function $X'_1 \to \mathcal{P}(X_2)$ given by $x^1 \mapsto \mu_{x^1}$, such that for any Borel subsets $Y \subset Z$ and $C \subset X_1$,

$$\mu(Y \cap \pi_1^{-1}(C)) = \int_C \mu_{x^1}(Y) \, dm_{\Gamma_1}(x^1).$$

(6.7)

The measure $\mu_{x^1}$ is called the fiber measure over $\pi_1^{-1}(x^1)$.

**Theorem 6.5.** There exist a positive integer $l$ and an $m_{\Gamma_1}$-conull subset $X'_1 \subset X_1$ so that $\text{supp}(\mu_{x^1})$ is a finite set with cardinality $l$ for all $x^1 \in X'_1$. Furthermore, $\mu_{x^1}(x^2) = 1/l$ for any $x^1 \in X'_1$ and $x^2 \in \text{supp} \mu_{x^1}$.

**Proof.** This theorem can be regarded as a corollary of Theorem 6.1. It follows from the proof of Theorem 7.17 in [25].

6.3. Reduction to the rigidity of measurable factors. By Theorem 6.5, there exists a conull set $\tilde{X} \subset X_1$ and a positive integer $l$ so that $\mu_{x^1}$ is supported on $l$ points for every $x^1 \in \tilde{X}$.

Define a set-valued map $\mathcal{Y} : \tilde{X} \to X_2$ by

$$\mathcal{Y}(x^1) = \text{supp} \mu_{x^1}.$$  

(6.8)

It follows from [34] that there are measurable maps

$$\varphi_1, \ldots, \varphi_l : \tilde{X} \to X_2$$

so that $\mathcal{Y}(x^1) = (\varphi_1(x^1), \ldots, \varphi_l(x^1))$ for $x^1 \in \tilde{X}$. Furthermore, noting that $\mu$ is $\Delta(AU)$-invariant, by possibly changing $\mu_{x^1}$ on a set of $m_{\Gamma_1}$-measure zero, we may assume that $\mathcal{Y}$ is defined on $X_1$ and it is $AU$-equivariant.

**Proposition 6.6.** Let $\mathcal{Y} : X_1 \to X_2$ be defined as (6.8). In particular, we have that $\mathcal{Y}$ is $AU$-equivariant. Then there exists $q_0 \in G$ so that $[\Gamma_1 : \Gamma_1 \cap q_0^{-1} \Gamma_2 q_0] = l$. Putting $\Gamma_2 q_0 \Gamma_1 = \bigcup_{j=1}^l \Gamma_2 q_0 \gamma_j$ with $\gamma_j \in \Gamma_1$, we have

$$\mathcal{Y}(\Gamma_1 g) = \{\Gamma_2 q_0 \gamma_1 g, \ldots, \Gamma_2 q_0 \gamma_l g\}$$

for $m_{\Gamma_1}$-a.e. $\Gamma_1 g$.

**Lemma 6.7.** There exists a set-valued map $\mathcal{Y}_0 : X_1 \to X_2$ so that $\mathcal{Y}_0$ is $G$-equivariant and it agrees with $\mathcal{Y}$ on a conull set of $X_1$.  


Proof. Applying Theorem 5.1 to $\mathcal{Y}$, we obtain a conull subset $\tilde{X} \subset X_1$ so that for every $x^1 \in \tilde{X}$ and every $u^+_r \in U^+$ with $x u^+_r \in \tilde{X}$,

$$\mathcal{Y}(x u^+_r) = \mathcal{Y}(x) u^+_r.$$ 

Using Fubini theorem, we know that for $m_{\Gamma_1}$-a.e. $x \in X_1$,

$$\int_{\tilde{X}} \chi_{\tilde{X}}(x u^+_r) dr = 0.$$ 

Fix $x_0 \in \tilde{X}$ so that $x_0$ satisfies (6.9). Denote $U^+(x_0) = \{ u^+_r : x_0 u^+_r \in \tilde{X} \}$. Identifying $U^+$ with $\mathbb{R}$, $U^+(x_0)$ is a conull set in $U^+$.

Define another set-valued map $\mathcal{Y}_0 : X_1 \to X_2$ by $\mathcal{Y}_0(x_0 g) = \mathcal{Y}(x_0)g$. We need to verify that $\mathcal{Y}_0$ is well-defined. We first show that $\mathcal{Y}_0$ is well-defined on $x_0 U^+ AU$. It suffices to show that for any two points $x_0 u^+_r a_i u_t$ and $x_0 u^+_r$, if $x_0 u^+_r a_i u_t = x_0 u^+_r$, then

$$\mathcal{Y}_0(x_0 u^+_r a_i u_t) = \mathcal{Y}_0(x_0 u^+_r).$$

Since $U^+(x_0)$ is a conull set in $U^+$, there exists $u^+_r \in U^+$ satisfying

- $x_0 u^+_r \in \tilde{X};$
- $u^+_r a_i u_t u^+_r = u^+_r a_i u_t$ with $u^+_r \in U^+(x_0)$.

We have

$$\mathcal{Y}(x_0) u^+_r a_i u_t u^+_r = \mathcal{Y}(x_0 u^+_r a_i u_t) = \mathcal{Y}(x_0 u^+_r),$$

which implies $\mathcal{Y}_0$ is well-defined on $x_0 U^+ AU$.

Next we show that $\mathcal{Y}_0$ is well-defined on $X_1$. Suppose $x_0 g = x_0$. We prove

$$\mathcal{Y}_0(x_0) g = \mathcal{Y}(x_0).$$

Let $\{g_n\}$ be a sequence in $U^+ AU$ tending $g$ as $n \to \infty$. For every $i \in \{1, \ldots, l\}$, we have

$$d(\varphi_i(x_0), \mathcal{Y}(x_0) g) = \min_{1 \leq j \leq l} d(\varphi_i(x_0), \varphi_j(x_0) g)$$

$$\leq \min_{1 \leq j \leq l} (d(\varphi_i(x_0), \varphi_j(x_0) g_n) + d(\varphi_j(x_0) g_n, \varphi_j(x_0) g))$$

$$\leq d(\varphi_i(x_0), \mathcal{Y}_0(x_0 g_n)) + d(g_n, g)$$

$$= d(\varphi_i(x_0), \mathcal{Y}_0(x_0 g_n)) + (g_n, g).$$

Observe that $U^+ AU$ contains a neighborhood $V$ of $e$ in $G$. There exists a sequence $\{h_n\}$ in $V$ so that $x_0 h_n = x_0 g_n$ and $h_n$ tends to $e$ as $n \to \infty$. This implies that

$$d(\varphi_i(x_0), \mathcal{Y}(x_0) g) \leq d(\varphi_i(x_0), \mathcal{Y}_0(x_0 h_n)) + d(g_n, g)$$

$$= d(\varphi_i(x_0), \mathcal{Y}(x_0 h_n)) + d(g_n, g)$$

$$\leq d(\varphi_i(x_0), \varphi_i(x_0) h_n) + d(g_n, g)$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

Therefore $\mathcal{Y}_0$ is well-defined and $\mathcal{Y}_0$ agrees with $\mathcal{Y}$ on $x_0 U^+(x_0) AU$. \qed
Proof of Proposition 6.6. By Lemma 6.7, we can show the proposition for \( \mathcal{Y}_0 \). Let \( x_0 \in X_1 \) be the point given in the proof of Lemma 6.7. Write \( x_0 = \Gamma_1 g_0 \) and \( \mathcal{Y}_0(\Gamma_1 g_0) = \{ \Gamma_2 h_1, \ldots, \Gamma_2 h_l \} \).

The \( G \)-equivariance of \( \mathcal{Y}_0 \) implies \( \mathcal{Y}_0(\Gamma_1 g_0) \) is \( g_0^{-1} \Gamma_1 g_0 \)-invariant. Putting \( q_i = h_i g_0^{-1} \) for \( i = 1, \ldots, l \), we have for every \( i \)\

\[ \Gamma_2 q_i \Gamma_1 \subset \{ \Gamma_2 q_1, \ldots, \Gamma_2 q_l \} \] (6.10)

This implies \( \Gamma_1 \cap q_i^{-1} \Gamma_2 q_i \) is a finite index subgroup of \( \Gamma_1 \).

Fixing \( i \), assume that \( |\Gamma_1 : \Gamma_1 \cap q_i^{-1} \Gamma_2 q_i| = l_i \leq l \). In view of (6.10), we have

\[ \Gamma_2 q_i \Gamma_1 = \{ \Gamma_2 q_i, \ldots, \Gamma_2 q_l \} \]

Consider the set

\[ X_i := \{ (x^1, x^2) : x^1 = \Gamma_1 g, x^2 \in \{ \Gamma_2 q_1 g, \ldots, \Gamma_2 q_l g \} \} \]

Observe the set

\[ X = \{ (x^1, x^2) : x^1 = \Gamma_1 g, x^2 \in \mathcal{Y}_0(x^1) = \{ \Gamma_2 q_1 g, \ldots, \Gamma_2 q_l g \} \} \]

is a conull set for the joining measure \( \mu \) since \( \mathcal{Y}_0 \) agrees with \( \mathcal{Y} \) almost everywhere. Then \( \mu \mathcal{L}_1^1 (X_i) = l_i / l \) for \( m_{\Gamma_1} \)-a.e. \( x^1 \in X_1 \). As \( X_i \) is \( \Delta(\Upsilon) \)-invariant set with positive measure, we conclude \( l_i = l \) and \( X_i \) agrees with \( X \) up to sets of measure zero. Therefore \( q_i \in G \) is an element satisfying Proposition 6.6. \( \square \)

Proof of Theorem 1.3. Keep the notations in Proposition 6.6. In particular, let \( q_0 \in G \) be an element satisfying Proposition 6.6 so that \( \Gamma_2 q_0 \Gamma_1 = \bigcup_{j=1}^l \Gamma_2 q_0 \gamma_j \) with \( \gamma_j \in \Gamma_1 \).

For the ergodic \( \Upsilon \)-joining measure \( \mu \), recall the disintegration of \( \mu \) in terms of \( \mu_{\Gamma_1} \) (6.7). It follows from Proposition 6.6 that \( \mu_{\Gamma_1} \) is a uniformly distributed on \( \{ \Gamma_2 q_0 \gamma_1 g, \ldots, \Gamma_2 q_0 \gamma_l g \} \) for \( m_{\Gamma_1} \)-a.e. \( \Gamma_1 g \). This implies that \( \mu \) is \( \Delta(G) \)-invariant.

Letting \( \Gamma_0 = \Gamma_1 \cap q_0^{-1} \Gamma_2 q_0 \), the map

\[ \psi : \Gamma_0 \backslash G \rightarrow \Gamma_1 \backslash G \times \Gamma_2 \backslash G \]

given by \( \Gamma_0 g \rightarrow (\Gamma_1 g, \Gamma_2 q_0 g) \) provides a homeomorphism between \( \Gamma_0 \backslash G \) and its image. Then the pullback of \( \mu \) through \( \psi \) provides a \( G \)-invariant measure on \( \Gamma_0 \backslash G \). Therefore \( \mu \) is a multiple of the pushforward of \( m_{\Gamma_0} \) through \( \psi \).

Now we show \( \Gamma_0 \) is also a finite index subgroup of \( q_0^{-1} \Gamma_2 q_0 \). Choose a neighborhood \( B \) of \( e \) in \( G \) so that \( B \cap q_0^{-1} \Gamma_2 q_0 = \{ e \} \). Up to scalars, we have

\[
\begin{align*}
m_{\Gamma_2}(\Gamma_2 q_0 B) &= \mu(\Gamma_1 \backslash G \times \Gamma_2 q_0 B) \\
&= m_{\Gamma_0}(\psi^{-1}(\Gamma_1 \backslash G \times \Gamma_2 q_0 B)) \\
&= \sum_{\alpha} m_{\Gamma}(\Gamma \gamma_\alpha B)
\end{align*}
\]

where \( \{ \Gamma \gamma_\alpha \}_\alpha \) are the cosets of \( \Gamma \) in \( q_0^{-1} \Gamma_2 q_0 \). Since \( m_{\Gamma_2}(\Gamma_2 q_0 B) < \infty \), this equality implies that \( \Gamma_0 \) is a finite index subgroup of \( q_0^{-1} \Gamma_2 q_0 \).

In conclusion, the ergodic \( \Upsilon \)-joining measure \( \mu \) is a finite cover self-joining (Definition 1.2). \( \square \)
Remark 6.8. We provide a proof here showing that the $U$-action on $\Gamma_0 \backslash G$ is ergodic with respect to $m_{\Gamma_0}$. To see this, note that $\Gamma_1$ is of divergent type by Rees [32]. Hence $\Gamma_0$, as a finite index subgroup of $\Gamma_1$, is also of divergent type. Any non-elementary discrete subgroup of $\text{PSL}_2(\mathbb{R})$ has non-arithmetic length spectrum. Therefore the ergodicity of $m_{\Gamma_0}$ with respect to $U$-action can be deduced from the works of Kaimanovich [15] and Roblin [33].

Now we deduce Corollary 1.6 as a corollary of Theorem 1.3.

Proof of Corollary 1.6. Denote by $\pi$ the projection from $\Gamma_1 \backslash G \times \Gamma_2 \backslash G$ to $\Gamma_1 \backslash G$. Let $\mu$ be any $\Delta(AU)$-invariant, ergodic, conservative, infinite Radon measure on the product space. Then the pushforward of $\mu$ through $\pi$, denoted by $(\pi)_* \mu$, is a $\Delta(AU)$-invariant, ergodic measure on $\Gamma_1 \backslash G$. It follows from the main theorem in [1] that $(\pi)_* \mu = m_{\Gamma_1}$. Applying the disintegration theorem to $\mu$, we have

$$\mu = \int_{x \in \Gamma_1 \backslash G} \mu_x dm_{\Gamma_1}(x),$$

where $\mu_x$ is a probability measure on $\{x\} \times \Gamma_2 \backslash G$ for $m_{\Gamma_1}$-a.e. $x$.

The discrepancy of $\mu$ is determined by whether $\mu$ is invariant under $\{e\} \times U$ or not. Suppose $\mu$ is not invariant under $\{e\} \times U$. Note that in the proof of Theorem 1.3, we require the ergodic $U$-joining is not invariant under $\{e\} \times U$ (Corollary 6.4). Now applying Theorem 1.3 to $\mu$, we conclude that $\mu$ is of the form described in case (2).

If $\mu$ is invariant under $\{e\} \times U$, then $\mu_x$ is a $\{e\} \times U$-invariant on $\{x\} \times \Gamma_2 \backslash G$ for $m_{\Gamma_1}$-a.e. $x$. By the unique ergodicity of $U$ on $\Gamma_2 \backslash G$ [11], we have $\mu_x = m_{\Gamma_2}$ for $m_{\Gamma_1}$-a.e. $x$. Hence $\mu = m_{\Gamma_1} \times m_{\Gamma_2}$.

Next we show that $m_{\Gamma_1} \times m_{\Gamma_2}$ is $\Delta(AU)$-ergodic. Suppose $m_{\Gamma_1} \times m_{\Gamma_2}$ is not $\Delta(AU)$-ergodic. Let $\tau$ be any ergodic component in the ergodic decomposition of $m_{\Gamma_1} \times m_{\Gamma_2}$. Then $\tau$ is conservative under the action of $\Delta(AU)$ for $(\pi)_* \tau = m_{\Gamma_1}$.

The above analysis implies $\tau$ should be of the form described in Corollary 1.6 (2). Now set

$$\text{Comm}(\Gamma_1; \Gamma_2) = \{g \in G : [\Gamma_1 : \Gamma_1 \cap g^{-1}\Gamma_2 g] < \infty\}. $$

Since $\Gamma_1$ and $\Gamma_2$ are countable, there exists a countable field $k$ so that $\Gamma_i \subset \text{SL}_2(k)$ for $i = 1, 2$. For every $g \in \text{Comm}(\Gamma_1; \Gamma_2)$, we have that $g \in \text{SL}_2(k)$ by Chapter VII, Lemma 6.2 in [22]. (In fact, the proof of the lemma is valid as long as $\Gamma_1$ and $\Gamma_2$ are Zariski dense.) This implies that the set $\text{Comm}(\Gamma_1; \Gamma_2)$ is countable.

Note that $m_{\Gamma_1} \times m_{\Gamma_2}$ gives measure zero to the sets of the form

$$([e],[g])\Delta(G)((\{e\} \times AU),$$

where $g \in \text{Comm}(\Gamma_1; \Gamma_2)$. Then $m_{\Gamma_1} \times m_{\Gamma_2}$ is a zero measure by the countability of $\text{Comm}(\Gamma_1; \Gamma_2)$, which is a contradiction. Therefore, we have that the action of $\Delta(AU)$ is ergodic with respect to $m_{\Gamma_1} \times m_{\Gamma_2}$. \qed
7. \( U \)-factor classification

Let \( \Gamma \) be a \( \mathbb{Z} \) or \( \mathbb{Z}^2 \)-cover. This section is devoted to proving Corollary 1.5. Given a \( U \)-equivariant factor map \( p : (\Gamma \backslash G, m_\Gamma) \to (Y, \nu) \), consider the following map

\[
\Gamma \backslash G \to Y \times \Gamma \backslash G, \quad [g] \mapsto (p([g]), [g]).
\]

The pushforward of \( m_\Gamma \) through this map, denoted by \( \mu \), is an ergodic \( U \)-joining measure with respect to the pair of measures \((\nu, m_\Gamma)\). And \( \mu \) can be disintegrated into the following form:

\[
\mu = \int_{y \in Y} \tau_y d\nu(y),
\]

where \( \tau_y \) is a probability measure supported on \([y] \times p^{-1}(y)\) for \( \nu \)-a.e. \( y \).

We first show that the measure \( \tau_y \) is fully atomic for \( \nu \)-a.e. \( y \).

**Proposition 7.1.** Under the assumption of Corollary 1.5, there exist a conull set \( \Omega \) in \( \Gamma \backslash G \) and a positive integer \( l_0 \) so that \( \# p^{-1}(y) \cap \Omega = l_0 \) for \( \nu \)-a.e. \( y \). Furthermore, the measure \( \tau_y \) is uniform distributed on \([y] \times (p^{-1}(y) \cap \Omega)\) for \( \nu \)-a.e. \( y \).

**Proof.** The proof is parallel to the proof of Theorem 6.5. The key lies in obtaining window property I (Theorem 3.1) for \( Y \) using the factor map \( p \). We claim that \( \tau_y \) is fully atomic for \( \nu \)-a.e. \( y \), or equivalently, the set \( B' = \{ y \in Y : \tau_y \) is not fully atomic\} is a null set.

Suppose the claim fails. Then \( \nu(B') > 0 \). For every \( y \in B' \), decompose \( \tau_y \) into the following form:

\[
\tau_y = (\tau_y)^a + (\tau_y)^c,
\]

where \((\tau_y)^a\) and \((\tau_y)^c\) are respectively the purely atomic part and the continuous part of \( \tau_y \). Let

\[
B = \{(y, [g]) \in Y \times \Gamma \backslash G : y \in B' \text{ and } [g] \in \text{supp}(\tau_y)^c\}.
\]

We will construct two compact subsets \( Q \) and \( Q_\varepsilon \) in \( B \) as Section 6.1. To be precise, fix a nonnegative function \( \psi \in C_c(Y) \) with \( \nu(\psi) > 0 \). Then \( \psi \circ p \in L^1(\Gamma \backslash G, m_\Gamma) \). Let \( \pi_1 \) be the canonical projection from \( Y \times \Gamma \backslash G \) to \( Y \). Set

\[
\Psi = \psi \circ \pi_1 \in C(Y \times \Gamma \backslash G).
\]

Choose a compact subset \( D \) in \( p^{-1}(B') \) so that \( p|_D \) is continuous and the window property I (Theorem 3.1) holds for \( \psi \circ p \) uniformly for all \([g] \in D\). Let

\[
0 < r := \frac{1}{4} r(1/2; D) < 1
\]

be the constant given as Theorem 3.1. As a result, there exists \( T_0 > 1 \) so that we have for every \( T > T_0 \) and for every \((y, [g]) \in p(D) \times \Gamma \backslash G \cap B\)

\[
\int_0^T \Psi((y, [g]) \Delta(u_t)) dt \leq \frac{1}{2} \int_0^T \Psi((y, [g]) \Delta(u_t)) dt.
\]
Set $Q$ to be a compact subset in $\rho(D) \times \Gamma \setminus \Gamma \cap B$ so that the following holds for every $(y, [g]) \in Q$ and for every $f \in C_c(\mathbb{G} \times \Gamma)$:

$$
\lim_{T \to \infty} \int_0^T f((y, [g])\Delta(u_t))\,dt = \frac{\mu(f)}{\mu(\mathbb{G})}.
$$

Fix a small $\epsilon > 0$ and choose $\eta > 0$ small enough so that $\mu(Q((e, g) : g \in G, |g| \leq \eta)) < (1 + \epsilon)\mu(Q)$. Set $Q_+ = Q((e, g) : g \in G, |g| \leq \eta/4)$.

In view of (7.2), we have for $\mu$-a.e. $(y, [g]) \in Q$

$$
\lim_{T \to \infty} \int_0^T \frac{\chi_y((y, [g])\Delta(u_t))\,dt}{\mu(\mathbb{G})} = \frac{\mu(Q)}{\mu(\mathbb{G})},
$$

Let $Q_c \subset Q$ be a compact subset so that $\mu(Q_c) > (1 - \epsilon)\mu(Q)$ and (7.3) converges uniformly on $Q_c$.

If the claim fails, there exists a sequence $\{(y, [g_k])\} \subset Q_c$ converging to some point $(y, [g]) \in Q_c$. This is because $Q_c$ is a subset of $B$ and applying Fubini’s theorem to $\mu(Q_c)$, we have

$$
\mu(Q_c) = \int_{y \in B'} (\tau_y)^c(Q_c)\,dv(y) > 0.
$$

Write $(y, [g_k]) = (y, [g])(e, h_k)$ where $h_k \neq e$ and $h_k \to e$ as $k \to \infty$. Then $Q_c \cap Q_e \neq \emptyset$. Applying the argument of Theorem 6.1 to $Q_e$ (Theorem 7.12 in [25]), we deduce that there exists a sequence $\{(e, u_k)\} \subset \{e\} \times U$ converging to $(e, e)$ so that $Q_e(e, u_k) \cap Q_e \neq \emptyset$. This implies $\mu$ is invariant under $\{e\} \times U$ (cf. Lemma 7.7 in [25]). However, it follows from the proof as Lemma 7.16 in [25] that $\mu$ cannot be invariant under $\{e\} \times U$. Therefore, the measure $\tau_y$ is fully atomic for $\nu$-a.e. $y$.

Now set

$$
\Omega' = \{(y, [g]) \in \mathbb{G} \times \Gamma \setminus \Gamma : \tau_y([g]) = \max_{[g^'] \in \rho^{-1}(y)} \tau_y([g^'])\}.
$$

This is a $\Delta(U)$-invariant set of positive $\mu$-measure. The ergodicity of $\mu$ yields that $\Omega'$ is a conull set. Moreover, there exists a positive integer $l_0$ so that $\tau_y$ is uniform distributed on $l_0$-points. Let $\pi_2$ be the canonical projection from $\mathbb{G} \times \Gamma \setminus \Gamma$ to $\Gamma \setminus \Gamma$. Then $\Omega := \pi_2(\Omega')$ is a conull set satisfying Proposition 7.1. □

Denote the Haar measure on $G$ by $\tilde{m}$. Let $\text{Comm}_G(\Gamma)$ be the commensurator subgroup of $\Gamma$ in $G$, that is, $g \in \text{Comm}_G(\Gamma)$ if and only if $G$ and $g^{-1}\Gamma g$ are commensurable with each other.

**Lemma 7.2.** For $i = 1, 2$, let $h_i \in \text{Comm}_G(\Gamma)$, $u_i \in U$, and $\varphi_i$ be the map

$$
\Gamma \cap h_i^{-1} \Gamma h_i \setminus \Gamma \to \Gamma \setminus \Gamma \times \Gamma \setminus \Gamma
$$

given by $[g] \mapsto ([g], [h_i g u_i])$. Set $\mu_i = (\varphi_i)_* m_{\Gamma \cap h_i^{-1} \Gamma h_i}$. If $\mu_1$ is not proportional to $\mu_2$, then

$$
\Gamma h_1 g u_1 \neq \Gamma h_2 g u_2
$$

for $\tilde{m}$-a.e. $g$. 
Proof. Set

\[ W = \{ g \in G : \Gamma h_1 g u_1 = \Gamma h_2 g u_2 \}. \]

We show that \( W \) is a null set in \( G \). Suppose \( W \) is of positive measure.

Let \( \Gamma_i = \Gamma \cap h_i^{-1} \Gamma h_i \) and \( \rho_i : G \to \Gamma_i \backslash G \) be the natural projection. Consider the following diagram:

\[ \begin{array}{ccc}
\Gamma \backslash G & \xrightarrow{\rho_1} & \Gamma_1 \backslash G \\
\phi_1 \downarrow & & \downarrow \phi_1 \\
\Gamma \backslash G \times \Gamma \backslash G & \xrightarrow{\rho_2} & \Gamma_2 \backslash G \\
\phi_2 \downarrow & & \downarrow \phi_2 \\
\end{array} \]

We have \( \phi_1 \circ \rho_1|_W = \phi_2 \circ \rho_2|_W \). Observe that \( W \) is a conull set because \( \rho_1(W) \) is \( U \)-invariant and \( m_{\Gamma_1} \) is \( U \)-ergodic (Remark 6.8). When restricting \( \mu_1 \) and \( \mu_2 \) to \( \phi_1 \circ \rho_1(W) \), any \( \mu_1 \)-measure zero set \( A \) is also \( \mu_2 \)-measure zero. Hence we can consider the Radon-Nikodym derivative \( d\mu_2/d\mu_1 \). Note that \( d\mu_2/d\mu_1 \) is \( \Delta(U) \)-invariant. Therefore, \( \mu_1 = c\mu_2 \) for some \( c > 0 \), which is a contradiction. \( \square \)

Proof of Corollary 1.5. Follow the notations in Proposition 7.1. Recall the measures \( \tau_y \)'s given as (7.1). Denote by \( \sigma_y \) the pushforward measure \( (\pi_2)_* \tau_y \), where \( \pi_2 \) is the canonical projection from \( Y \times \Gamma \backslash G \) to \( \Gamma \backslash G \). Consider the following measure on \( \Gamma \backslash G \times \Gamma \backslash G \):

\[ \tilde{\mu} = \int_{y \in Y} \sigma_y \otimes \sigma_y d\nu(y), \]

where \( \sigma_y \otimes \sigma_y \)'s are the product measures on \( \Gamma \backslash G \times \Gamma \backslash G \). The measure \( \tilde{\mu} \) is a \( U \)-joining measure with respect to the pair \( (m_{\Gamma}, m_{\Gamma}) \). Let \( \Omega \) be the conull subset given by Proposition 7.1. The set

\[ \Omega \times_p \Omega := \{(x_1, x_2) \in \Omega \times \Omega : p(x_1) = p(x_2)\} \]

is a \( \tilde{\mu} \)-conull set. We claim that there exist finitely many \( h_1, \ldots, h_k \in \text{Comm}_C(\Gamma) \) and \( u_1, \ldots, u_k \in U \) so that up to sets of measure zero

\[ \Omega \times_p \Omega = \cup_{1 \leq i, k} \{(e, h_i) \Delta(G)(e, u_i)\}. \]

Let \( \mu_\Delta \) be the \( U \)-ergodic measure on \( \Gamma \backslash G \times \Gamma \backslash G \) attained by pushing forward the Haar measure \( m_{\Gamma} \) on \( \Gamma \backslash G \) through the diagonal embedding:

\[ \Gamma \backslash G \to \Gamma \backslash G \times \Gamma \backslash G, \quad [g] \mapsto ([g], [g]). \]

If \( \tilde{\mu} \) equals a multiple of \( \mu_\Delta \), the claim is obvious. Now suppose \( \tilde{\mu} \) is not a multiple of \( \mu_\Delta \). Consider the \( \Delta(U) \)-ergodic decomposition of \( \tilde{\mu} \):

\[ \tilde{\mu} = \int_{z \in Z} \mu_z d\sigma(z), \]

where \( (Z, \sigma) \) is a probability space. For \( \sigma \)-a.e. \( z \), the measure \( \mu_z \) is an ergodic \( U \)-joining measure so that \( \Omega \times_p \Omega \) is a \( \mu_z \)-conull set. Choose any ergodic component \( \mu_1 \) that is not a multiple of \( \mu_\Delta \). Applying the joining classification theorem
Up where $\mu$ then (7.5) and (7.6) imply $\Omega \times p \Omega$ is a $\mu_1$-conull set, we have $p(\Gamma g) = p(\Gamma h_1 g u_1)$ and $\tau_{p(\Gamma g)}(\Gamma h_1 g u_1) = 1/l_0$ for $\tilde{m}$-a.e. $g$. Let $i_1 = [\Gamma : \Gamma \cap h_1^{-1} \Gamma h_1]$. By Lemma 7.2, for $\nu$-a.e. $y$,

$$\sigma_y \otimes \sigma_y([\{e, e\} \Delta(G) \cup [\{e, h\}] \Delta(G)(e, u_1)) = (i_1 + 1)/l_0.$$ 

If $i_1 + 1 < l_0$, choose another ergodic component $\mu_2$ of $\tilde{m}$ so that $\mu_2$ is a $U$-ergodic joining measure and $\Omega \times p \Omega$ is a $\mu_2$-conull set. The claim can be verified by repeating the above process finitely many times.

The sets $\{h_1, \ldots, h_k\} \subseteq \text{Comm}_G(\Gamma)$ and $\{u_1, \ldots, u_k\} \subseteq U$ yield a set $\{c_1 = e, \ldots, c_n\}$ and a set $\{u_{p_1} = e, \ldots, u_{p_n}\}$ satisfying:

(7.4) 

$$p^{-1}(p(\Gamma g)) \cap \Omega = [\Gamma c_1 g u_{p_1}, \ldots, \Gamma c_n g u_{p_n}]$$

for every $c_i$ and every $y, c_i y \in \Gamma c_j$ for some $j$.

We show that $p_1 = p_2 = \ldots = p_n = 0$.

Fix any $s \neq 0$. For $\tilde{m}$-a.e. $g$, we have

$$p^{-1}(\Gamma g a_s) \cap \Omega = [\Gamma c_1 g a_{p_1} u_{p_1}, \ldots, \Gamma c_n g a_{p_n} u_{p_n}]$$

where $b_1 = p_1(1 - e^{-s})$ for $1 \leq i \leq n$.

Set $B = \{b_1, \ldots, b_n\}$. For $m_{\Gamma}$-a.e. $x, y \in \Gamma \backslash G$, if $p(x) = p(y)$, then

$$p(xa_s) = p(y a_s u_{b(y, x)})$$

for some $b(y, x) \in B$ and $p(y a_s) = p(xa_s u_{b(x, y)})$ for some $b(x, y) \in B$. Since $p$ is $U$-equivariant, we get

(7.5) 

$$b(x, y) = -b(y, x).$$

This implies for $\tilde{m}$-a.e. $x, y, z \in \Gamma \backslash G$, if $p(x) = p(y) = p(z)$, then

(7.6) 

$$b(x, z) = b(y, z) - b(y, x).$$

Suppose there exists $p_i \neq 0$. Then $b_1 = p_1(1 - e^{-s}) \neq 0$. Denote $\bar{b} = \max\{b_1, \ldots, b_n\}$ and $\breve{b} = \min\{b_1, \ldots, b_n\}$. Let $x, y, z \in \Gamma \backslash G$ be such that

$$p(x) = p(y) = p(z) \quad \text{and} \quad b(y, z) = \bar{b}, b(y, x) = \bar{b}.$$ 

Then (7.5) and (7.6) imply

$$b(x, z) = b(y, z) - b(y, x) = \bar{b} - \breve{b} = 2\bar{b} > \bar{b},$$

which contradicts the maximality of $\bar{b}$. Hence $p_1 = \ldots = p_n = 0$.

Now for $\tilde{m}$-a.e. $g$ and for every $1 \leq i \leq n$, we have

$$p^{-1}(p(\Gamma g)) \cap \Omega = [\Gamma c_1 g \ldots, \Gamma c_n g],$$

$$p^{-1}(p(\Gamma c^{-1} g)) \cap \Omega = [\Gamma c_1 c^{-1} g \ldots, \Gamma c_n c^{-1} g],$$

$$p^{-1}(p(\Gamma c i g)) \cap \Omega = [\Gamma c_1 c_i g \ldots, \Gamma c_n c_i g].$$
So for every $i, j \in \{1, \ldots, n\}$, we have $c_i^{-1} \in \Gamma c_j$ and $c_i c_j \in \Gamma c_l$ for some $1 \leq l \leq n$. Let $\Gamma_0$ be the group generated by $\Gamma$ and $\{c_1, \ldots, c_n\}$. We deduce from the above relation between $\Gamma$ and $\{c_1, \ldots, c_n\}$ together with (7.4) that $\Gamma$ is a finite index subgroup of $\Gamma_0$. The proof is completed.

APPENDIX: $\mathbb{Z}^d$-COVER GROUP ORBITS IN COMPACT HYPERBOLIC SURFACES

Let $\Gamma_1$ be a $\mathbb{Z}$ or $\mathbb{Z}^2$-cover and let $\Gamma_2$ be a cocompact lattice in $\text{PSL}_2(\mathbb{R})$. We show the following theorem:

**Theorem 7.3.** Any $\Gamma_1$-orbit on $\Gamma_2 \backslash \text{PSL}_2(\mathbb{R})$ is either finite or dense.

When $\Gamma_1$ is a non-elementary finitely generated discrete subgroup, such an orbit classification theorem is shown by Benoist-Quint [4] using the classification of stationary measures. Later, Benoist and Oh provided an elementary and topological proof [5], inspired by the work of McMullen-Mohammadi-Oh [24]. Our proof of Theorem 7.3 is modeled on Benoist-Oh’s proof. In particular, Theorem 7.3 can be deduced from the following Theorem 7.4 (see [5] for the deduction). Let

$$G := \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}),$$
$$H := \{(h, h) : h \in \text{PSL}_2(\mathbb{R})\},$$
$$\Gamma := \Gamma_1 \times \Gamma_2.$$

**Theorem 7.4.** For any $x \in \Gamma \backslash G$, the orbit $xH$ is either closed or dense.

7.1. Dynamics of unipotent flows. A key input in the proof of Theorem 7.4 is the window property of the horocycle flow on $\Gamma_1 \backslash \text{PSL}_2(\mathbb{R})$ (Theorem 3.1). Set

- $N := \{u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\};$
- $D := \{a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R}\}, A = \{(a_t, a_t)\};$
- $U_1 = \{(u_t, e)\}, U_2 = \{(e, u_t)\}, U = \{(u_t, u_t)\}.$

For simplicity, we write $\tilde{u}_t$ for $(u_t, u_t)$ and $\tilde{a}_t$ for $(a_t, a_t).$

**Definition 7.5.** Let $K > 1$. A subset $T \subset \mathbb{R}$ is called $K$-thick if $T$ intersects $[-Kt, -t] \cup [t, Kt]$ for all $t > 0$. Denote the Haar measure on $\Gamma_1 \backslash \text{PSL}_2(\mathbb{R})$ by $m_{\Gamma_1}$. The following proposition can be easily deduced from Theorem 3.1.

**Proposition 7.6.** For any compact subset $Q_1$ in $\Gamma_1 \backslash \text{PSL}_2(\mathbb{R})$ with $m_{\Gamma_1}(Q_1) > 0$, there exist a compact subset $Q_2 \subset Q_1$ of positive measure and constants $K, T_0 > 1$ such that for $Q_1(T_0) = \bigcup_{-T_0 \leq t \leq T_0} Q_1 u_t$, the set

$$\{t \in \mathbb{R} : x u_t \in Q_1(T_0)\}$$

is $K$-thick for every $x \in Q_2$. 

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7.2. Proof of Theorem 7.4. Let $X = \Gamma \backslash G$. Our proof is modeled on [5] using the $U$-minimal sets relative to a fixed compact subset of $X$. In the construction of minimal sets, we need to find a compact subset $\Omega \subset X$ such that the $U$-orbit of every element of $\Omega$ returns to $\Omega$ for $K$-thick amount of time for some $K > 1$. When $\Gamma_1$ is finitely generated, there is a natural compact subset in $X$ to use, which is the non-wandering set of the geodesic flow. When it comes to our setting, such a non-wandering set is the whole $X$ and hence non-compact. In view of Proposition 7.6, instead of finding one such compact subset, we construct two compact subsets $\Omega_2 \subset \Omega_1$ in $X$ such that the $U$-orbit of every element of $\Omega_2$ returns to $\Omega_1$ for $K$-thick amount of time for some $K > 1$. This difference results in some modification in the statement. But with Proposition 7.6 available, the proof is essentially a verbatim repetition of Benoist-Oh’s proof. We will list the steps of the proof and point out the necessary modification.

Set $Q_1'$ to be a compact subset in $\Gamma_1 \backslash \text{PSL}_2(\mathbb{R})$ of positive measure such that for every point $x_1 \in Q_1'$, the orbit $x_1 N$ is dense in $\Gamma_1 \backslash \text{PSL}_2(\mathbb{R})$. It is shown in [19] that for $m_{\Gamma_1}$-a.e. $x_1$, the orbit $x_1 N$ is dense in $\Gamma_1 \backslash \text{PSL}_2(\mathbb{R})$. Hence such a compact set $Q_1'$ exists.

Let $Q_2$ be a compact subset in $Q_1'$ such that for every $x_1 \in Q_2$, the set $\{ t \in \mathbb{R}_{\geq 0} : x_1 a_t \in Q_1' \}$ is unbounded and the set $\{ t \in \mathbb{R} : x_1 u_t \in \bigcup_{|t| \leq T_0} Q_1' \}$ is $K$-thick for some constants $K, T_0 > 1$. The existence of such a compact set $Q_2$ follows from Proposition 7.6 and the fact that the $D$-action on $\Gamma_1 \backslash \text{PSL}_2(\mathbb{R})$ is conservative [32]. Let $Q_1 = \bigcup_{|t| \leq T_0} Q_1' u_t$. Set

$$
\Omega_1 := Q_1 \times \Gamma_2 \backslash \text{PSL}_2(\mathbb{R}) \text{ and } \Omega_2 := Q_2 \times \Gamma_2 \backslash \text{PSL}_2(\mathbb{R}).
$$

Note that for each $x \in \Omega_2$, the set

$$
T(x, \Omega_1) := \{ t \in \mathbb{R} : x a_t \in \Omega_1 \}
$$

is $K$-thick and the set $\{ t \in \mathbb{R}_{\geq 0} : x a_t \in \Omega_1 \}$ is unbounded.

Let $x = (x_1, x_2) \in X$ and consider the orbit $x H$. Let $Y$ be an $H$-minimal subset of the closure $x H$ with respect to $\Omega_1$, i.e., $Y$ is a closed $H$-invariant subset of $x H$ such that $Y \cap \Omega_1 \neq \emptyset$ and $y H$ is dense in $Y$ for every $y \in Y \cap \Omega_1$. Let $Z$ be a $U$-minimal subset of $x H$ with respect to $\Omega_1$. Such minimal sets $Y$ and $Z$ exist as $\Omega_1$ is compact and $x H$ intersects $\Omega_1$ non-trivially.

In the following, we assume that the orbit $x H$ is not closed and show that $x H$ is dense in $X$.

**Lemma 7.7.** The set $Z$ intersects $\Omega_2$ non-trivially.

**Proof.** Let $z = (z_1, z_2) \in Z \cap \Omega_1$ and $w_1 \in Q_2$. It follows from the construction of $Q_1$ that the orbit $z_1 N$ is dense in $\Gamma_1 \backslash \text{PSL}_2(\mathbb{R})$. As a result, there exists a sequence $\{ t_n \} \subset \mathbb{R}$ such that $z_1 u_{t_n}$ converges to $w_1$. Since $\Gamma_2 \backslash \text{PSL}_2(\mathbb{R})$ is compact, the sequence $\{ z_2 u_{t_n} \}$ has a limit point $w_2 \in \Gamma_2 \backslash \text{PSL}_2(\mathbb{R})$. Consequently, the point $(w_1, w_2) \in Z U \cap \Omega_2 = Z \cap \Omega_2$. 


Theorem 7.4 follows from the similar argument as in [5]. In particular, we apply the proofs of Lemmas 3.3, 3.4 and Propositions 3.5, 3.6 in [5] to a point $z \in Z \cap \Omega_2$.

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