On the average value of the least common multiple of $k$ positive integers

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On the average value of the least common multiple of $k$ positive integers

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Abstract

We deduce an asymptotic formula with error term for the sum
\[ \sum_{\substack{n_1,\ldots,n_k \leq x}} f([n_1,\ldots,n_k]), \]
where $[n_1,\ldots,n_k]$ stands for the least common multiple of the positive integers $n_1,\ldots,n_k$ ($k \geq 2$) and $f$ belongs to a large class of multiplicative arithmetic functions, including, among others, the functions $f(n) = n^r$, $\varphi(n)^r$, $\sigma(n)^r$ ($r > -1$ real), where $\varphi$ is Euler’s totient function and $\sigma$ is the sum-of-divisors function. The proof is by elementary arguments, using the extension of the convolution method for arithmetic functions of several variables, starting with the observation that given a multiplicative function $f$, the function of $k$ variables $f([n_1,\ldots,n_k])$ is multiplicative.

Keywords: greatest common divisor, least common multiple, arithmetic function of several variables, multiplicative function, Dirichlet series, asymptotic formula

2000 MSC: 11A05, 11A25, 11N37

1. Introduction

We use the following notation: $\mathbb{N} = \{1,2,\ldots\}$, $\ast$ is the Dirichlet convolution of arithmetic functions, $\text{id}_r$ ($r \in \mathbb{R}$) is the function $\text{id}_r(n) = n^r$ ($n \in \mathbb{N}$), $1 = \text{id}_0$, $\text{id} = \text{id}_1$, $\mu$ denotes the Möbius function, $\lambda$ is the Liouville function,
\[ \sigma_r = 1 \ast \text{id}_r, \quad \sigma = \sigma_1 \text{ is the sum-of-divisors function, \( \tau = \sigma_0 \) is the divisor function,} \]
\[ \beta_r = \lambda \ast \text{id}_r, \quad \beta = \beta_1 \text{ is the alternating sum-of-divisors function (cf. [19]),} \]
\[ \varphi_r = \mu \ast \text{id}_r \text{ is the generalized Euler function,} \]
\[ \varphi = \varphi_1 \text{ is Euler’s totient function,} \]
\[ \psi_r = \mu^2 \ast \text{id}_r \text{ is the generalized Dedekind function,} \]
\[ \psi = \psi_1 \text{ is the classical Dedekind function.} \]

If \( n \in \mathbb{N} \), then \( n = \prod_p p^{\nu_p(n)} \) is its prime power factorization, the product being over the primes \( p \), where all but a finite number of the exponents \( \nu_p(n) \) are zero.

Furthermore, let \((n_1, \ldots, n_k)\) and \([n_1, \ldots, n_k]\) denote the greatest common divisor (gcd) and the least common multiple (lcm) of \( n_1, \ldots, n_k \in \mathbb{N} \) (\( k \geq 2 \)), respectively.

It is easy to see that for any arithmetic function \( f \) we have the identity
\[ \sum_{n_1, \ldots, n_k \leq x} f((n_1, \ldots, n_k)) = \sum_{d \leq x} (\mu \ast f)(d) \left\lfloor \frac{x}{d} \right\rfloor^k, \] which leads to asymptotic formulas for this sum. For example, if \( f = \text{id} \) and \( k \geq 3 \), then we have
\[ \sum_{n_1, \ldots, n_k \leq x} (n_1, \ldots, n_k) = \frac{\zeta(k-1)}{\zeta(k)} x^k + O(R_k(x)), \]
where \( R_3(x) = x^2 \log x \) and \( R_k(x) = x^{k-1} \) for \( k \geq 4 \). The case \( f = \text{id}, \ k = 2 \) can be treated separately by writing
\[ \sum_{m,n \leq x} (m,n) = 2 \sum_{m \leq n \leq x} (m,n) - \sum_{n \leq x} n \]
\[ = 2 \sum_{n \leq x} (\mu \ast \text{id} \tau)(n) - \frac{x^2}{2} + O(x), \]
giving, by using elementary arguments, the formula
\[ \sum_{m,n \leq x} (m,n) = \frac{x^2}{\zeta(2)} \left( \log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O(x^{1+\theta+\varepsilon}), \]
valid for every \( \varepsilon > 0 \), where \( \gamma \) is Euler’s constant and \( \theta \) is the exponent appearing in Dirichlet’s divisor problem.

For the lcm of \( k \) positive integers there is no formula similar to (1). However, in the case \( k = 2 \), the lcm of the integers \( m, n \in \mathbb{N} \) can be written using
their gcd as $[m,n] = mn/(m,n)$, which enables to establish the following asymptotic formula, valid for any positive real number $r$:

$$
\sum_{m,n \leq x} [m,n]^r = \frac{\zeta(r+2)}{\zeta(2)} \cdot \frac{x^{2(r+1)}}{(r+1)^2} + O(x^{2r+1} \log x). \tag{4}
$$

If $r \in \mathbb{N}$, then the error term in (4) can be improved into $O(x^{2r+1} (\log x)^2 (\log \log x)^{4/3})$, which is a consequence of the result of Walfisz [23, Satz 1, p. 144] for $\sum_{n \leq x} \varphi(n)$.

For $k = 2$ the asymptotic formulas concerning $\sum_{m,n \leq x} (m,n)^r$ and $\sum_{m,n \leq x} [m,n]^r$ are equivalent to those for $\sum_{n \leq x} g_r(n)$ and $\sum_{n \leq x} \ell_r(n)$, respectively, where $g_r(n) = \sum_{1 \leq j \leq n} (j,n)^r$ is the gcd-sum function and $\ell_r(n) = \sum_{1 \leq j \leq n} [j,n]^r$ is the lcm-sum function. The function $g_1(n) = \sum_{1 \leq j \leq n} (j,n)$, investigated by S. S. Pillai [16], is also called Pillai’s function in the literature.

The above and related results go back, in chronological order, to the work of E. Cesàro [6], E. Cohen [9, 10, 11], K. Alladi [1], P. Diaconis and P. Erdős [12], J. Chidambaramswamy and R. Sitaramachandrarao [7], K. A. Broughan [5], O. Bordellès [2, 3, 4], Y. Tanigawa and W. Zhai [17], S. Ikeda and K. Matsuoka [15], and others.

For example, formula (3) with the weaker error $O(x^{3/2} \log x)$ was given in [12, Th. 2, Eq. (1.4)] and was recovered in [5, Th. 4.7]. Formula (3) with the above error term was established in [7, Th. 3.1] and recovered in [2, Th. 1.1] (in both papers for Pillai’s function). Formula (4) was established in [12, Th. 2, Eq. (1.6)]. The better error term for (4) in the case $r \in \mathbb{N}$ was obtained in [15, Th. 2]. Asymptotic formulas for (1) in the case $k = 2$ and for various choices of the function $f$, including $f = \sigma$ and $f = \varphi$ were deduced in [4, 9, 10, 11]. See also the survey paper [18].

The result

$$
\sum_{m,n,q \leq x} [m,n,q]^r \sim c_r \frac{x^{3(r+1)}}{(r+1)^3} \quad (x \to \infty),
$$

valid for $r \in \mathbb{N}$, without any error term and with a computable constant $c_r$ given in an implicit form, was obtained by J. L. Fernández and P. Fernández [13, Th. 3(b)]. Their proof is by an ingenious method based on the identity $[m,n,q](m,n)(m,q)(n,q) = mnq(m,n,q)$ ($m,n,q \in \mathbb{N}$) and using the dominated convergence theorem. As far as we know, there are no other asymptotic results in the literature for the sum

$$
\sum_{n_1, \ldots, n_k \leq x} f([n_1, \ldots, n_k]), \tag{5}
$$
in the case \( k \geq 3 \), where \( f \) is an arithmetic function. It seems that the method of [13] can not be extended for \( k \geq 3 \), even in the case \( f = \text{id} \). Also, it is not possible to reduce the estimation of the sum (5) to sums of a single variable, like in (1).

In this paper we deduce an asymptotic formula with remainder term for the sum (5), where \( k \geq 2 \) and \( f \) belongs to a large class of multiplicative arithmetic functions, including the functions \( \text{id}_r \) with \( r > -1 \) real and \( \sigma_r \), \( \beta_r \), \( \varphi_r \), \( \psi_r \) with \( r \geq 1/2 \) real. The proof is by elementary arguments, using the extension of the convolution method for arithmetic functions of several variables starting with the observation that given a multiplicative function \( f \), the function of \( k \) variables \( f([n_1, \ldots, n_k]) \) is multiplicative and the associated multiple Dirichlet series factorizes as an Euler product. The same method was used by the second author [21] for a different problem. See the survey paper [20] of the second author for basic properties of multiplicative functions of several variables and related convolutions.

We also extend to the \( k \) dimensional case the formula

\[
\sum_{m,n \leq x} \frac{[m,n]}{(m,n)} = \frac{\pi^2}{60} x^4 + O(x^3 \log x),
\]

which can be obtained in a similar manner to the results (2) and (4). Properties of the operation \( m \circ n = [m,n]/(m,n) \) were investigated by the first author [14].

Note that the following recent result of different type, concerning the lcm of several positive integers, was obtained by J. Cilleruelo, J. Rué, P. Šarka and A. Zumalacárregui [8]:

\[
\text{lcm}\{a : a \in A\} = 2^n(1+o(1)) \quad \text{for almost all subsets } A \subset \{1, \ldots, n\}.
\]

2. Main results

Let \( r \in \mathbb{R} \) be a fixed number. Let \( \mathcal{A}_r \) denote the class of complex valued multiplicative arithmetic functions satisfying the following properties: there exist real constants \( C_1, C_2 \) such that

\[
|f(p) - p^r| \leq C_1 p^{r-1/2} \quad \text{for every prime } p,
\]

and

\[
|f(p^\nu)| \leq C_2 p^{\nu r} \quad \text{for every prime power } p^\nu \text{ with } \nu \geq 2.
\]
Note that conditions (i) and (ii) imply that
\[ |f(p^\nu)| \leq C_3 p^{\nu r} \]
for every prime power \( p^\nu \) with \( \nu \geq 1 \),
\[ \text{(iii)} \]
where \( C_3 = \max(C_1 + 1, C_2) \).

Observe that \( \text{id}_r \in \mathcal{A}_r \) for every \( r \in \mathbb{R} \), while \( \sigma_r, \beta_r, \varphi_r, \psi_r \in \mathcal{A}_r \) for every \( r \in \mathbb{R} \) with \( r \geq 1/2 \). The functions \( f(n) = \sigma(n)^r, \beta(n)^r, \varphi(n)^r, \psi(n)^r \) also belong to the class \( \mathcal{A}_r \) for every \( r \in \mathbb{R} \). As other examples of functions in the class \( \mathcal{A}_r \), with \( r \in \mathbb{R} \), we mention \( \varphi^*(n)r, \sigma^*(n)r \) and \( \sigma^{(e)}(n)r \), where \( \varphi^*(n) = \prod_{p|n} (p^\nu(n) - 1) \) is the unitary Euler totient, \( \sigma^*(n) = \prod_{p|n} (p^\nu(n) + 1) \) is the sum-of-unitary-divisors function and \( \sigma^{(e)}(n) = \prod_{p|n} \sum_{d|\nu_p(n)} p^d \) denotes the sum of exponential divisors of \( n \). Furthermore, if \( f \) is a bounded multiplicative function such that \( f(p^\nu) = 1 \) for every prime \( p \), then \( f \in \mathcal{A}_0 \). In particular, \( \mu^2 \in \mathcal{A}_0 \).

We prove the following results.

**Theorem 2.1.** Let \( k \geq 2 \) be a fixed integer and let \( f \in \mathcal{A}_r \) be a function, where \( r > -1 \) is real. Then for every \( \epsilon > 0 \),
\[ \sum_{n_1, \ldots, n_k \leq x} f([n_1, \ldots, n_k]) = C_{f,k} \frac{x^{k(r+1)}}{(r + 1)^k} + O\left(x^{k(r+1)-\frac{1}{2} \min(r+1,1)+\epsilon}\right), \]
\[ \text{(7)} \]
and
\[ \sum_{n_1, \ldots, n_k \leq x} \frac{f([n_1, \ldots, n_k])}{(n_1 \cdots n_k)^r} = C_{f,k} x^k + O\left(x^{k-\frac{1}{2} \min(r+1,1)+\epsilon}\right), \]
\[ \text{(8)} \]
where
\[ C_{f,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \ldots, \nu_k = 0}^{\infty} \frac{f(p^\max(\nu_1, \ldots, \nu_k))}{p^{(r+1)(\nu_1 + \cdots + \nu_k)}}. \]

Formula (7) shows that the average order of \( f([n_1, \ldots, n_k]) \) is \( C_{f,k}(n_1 \cdots n_k)^r \), in the sense that
\[ \sum_{n_1, \ldots, n_k \leq x} f([n_1, \ldots, n_k]) \sim \sum_{n_1, \ldots, n_k \leq x} C_{f,k}(n_1 \cdots n_k)^r \quad (x \to \infty). \]

From (8) we deduce that
\[ \lim_{x \to \infty} \frac{1}{x^k} \sum_{n_1, \ldots, n_k \leq x} \frac{f([n_1, \ldots, n_k])}{(n_1 \cdots n_k)^r} = C_{f,k}. \]
representing the mean value of the function \( f([n_1, \ldots, n_k])/(n_1 \cdots n_k)^r \). See N. Ushiroya [22, Th. 4] and the second author [20, Prop. 19] for general results on mean values of multiplicative arithmetic functions of several variables.

**Theorem 2.2.** Let \( k \geq 2 \) be a fixed integer and let \( f \in \mathcal{A}_r \) be a function, where \( r \geq 0 \) is real. Then for every \( \varepsilon > 0 \),

\[
\sum_{n_1, \ldots, n_k \leq x} f\left(\frac{[n_1, \ldots, n_k]}{n_1 \cdots n_k}\right) = D_{f,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}+\varepsilon}\right),
\]

(9)

where

\[
D_{f,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \ldots, \nu_k=0}^{\infty} f\left(p^{\max(\nu_1, \ldots, \nu_k)}-\min(\nu_1, \ldots, \nu_k)\right)\frac{p^{\nu_1+\cdots+\nu_k}}{p^{(r+1)(\nu_1+\cdots+\nu_k)}}.
\]

In the case \( f = \text{id}_r \) we obtain from Theorem 2.1 the next result:

**Corollary 1.** Let \( k \geq 3 \) and \( r > -1 \) be a real number. Then for every \( \varepsilon > 0 \),

\[
\sum_{n_1, \ldots, n_k \leq x} [n_1, \ldots, n_k]^r = C_{r,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2} \min(r+1,1)+\varepsilon}\right),
\]

(10)

and

\[
\sum_{n_1, \ldots, n_k \leq x} \left(\frac{[n_1, \ldots, n_k]}{n_1 \cdots n_k}\right)^r = C_{r,k} x^k + O\left(x^{k-\frac{1}{2} \min(r+1,1)+\varepsilon}\right),
\]

where

\[
C_{r,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \ldots, \nu_k=0}^{\infty} \frac{p^{r \max(\nu_1, \ldots, \nu_k)}}{p^{(r+1)(\nu_1+\cdots+\nu_k)}}.
\]

In particular,

\[
C_{r,3} = \zeta(r+2)\zeta(2r+3) \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3} + \frac{2}{p^{r+2}} - \frac{3}{p^{r+3}} + \frac{1}{p^{r+5}}\right),
\]

(11)

\[
C_{r,4} = \zeta(r+2)\zeta(2r+3)\zeta(3r+4) \prod_p \left(1 - \frac{6}{p^2} + \frac{8}{p^3} - \frac{3}{p^4} + \frac{5}{p^{r+2}} - \frac{12}{p^{r+3}} + \frac{6}{p^{r+4}} + \frac{4}{p^{r+5}}
\]

\[- \frac{3}{p^{r+6}} + \frac{3}{p^{2r+3}} - \frac{4}{p^{2r+4}} - \frac{6}{p^{2r+5}} + \frac{12}{p^{2r+6}} - \frac{5}{p^{2r+7}} + \frac{3}{p^{3r+5}} - \frac{8}{p^{3r+6}} + \frac{6}{p^{3r+7}} - \frac{1}{p^{3r+9}}\right).
\]

(12)
In the case \( f = \text{id}_r \) we deduce from Theorem 2.2:

**Corollary 2.** Let \( k \geq 3 \) and \( r > 0 \) be a real number. Then for every \( \varepsilon > 0 \),

\[
\sum_{n_1, \ldots, n_k \leq x} \left( \frac{[n_1, \ldots, n_k]}{(n_1, \ldots, n_k)} \right)^r = D_{r,k} \frac{x^{k(r+1)}}{(r+1)^k} + O \left( x^{k(r+1) - \frac{1}{2} + \varepsilon} \right), \tag{13}
\]

where

\[
D_{r,k} = \prod_p \left( 1 - \frac{1}{p} \right)^k \sum_{\nu_1, \ldots, \nu_k = 0}^{\infty} \frac{p^{r(\max(\nu_1, \ldots, \nu_k) - \min(\nu_1, \ldots, \nu_k))}}{p^{r+1}(\nu_1 + \ldots + \nu_k)}.
\]

In particular,

\[
D_{r,3} = C_{r,3} \frac{\zeta(3r+3)}{\zeta(2r+3)}, \quad D_{r,4} = C_{r,4} \frac{\zeta(4r+4)}{\zeta(3r+4)}.
\]

We remark that in the case \( k = 2 \) asymptotic formulas (10) and (13) reduce to (4) and (6) (case \( r = 1 \)), respectively, but the latter ones have better error terms. Note that \( D_{r,2} = \zeta(2r+2)/\zeta(2) \).

Among other special cases we consider the functions \( \sigma, \varphi \in A_1 \) and \( \mu^2 \in A_0 \).

**Corollary 3.** Let \( k \geq 2 \). Then for every \( \varepsilon > 0 \),

\[
\sum_{n_1, \ldots, n_k \leq x} \sigma([n_1, \ldots, n_k]) = C_{\sigma,k} x^{2k} + O \left( x^{2k - 1/2 + \varepsilon} \right),
\]

and

\[
\sum_{n_1, \ldots, n_k \leq x} \frac{\sigma([n_1, \ldots, n_k])}{n_1 \cdots n_k} = C_{\sigma,k} x^k + O \left( x^{k - 1/2 + \varepsilon} \right),
\]

where

\[
C_{\sigma,k} = \prod_p \left( 1 - \frac{1}{p} \right)^k \sum_{\nu_1, \ldots, \nu_k = 0}^{\infty} \frac{\sigma(p^{\max(\nu_1, \ldots, \nu_k)})}{p^{2(\nu_1 + \ldots + \nu_k)}}.
\]

In particular,

\[
C_{\sigma,2} = \zeta(3)\zeta(4) \prod_p \left( 1 + \frac{1}{p^2} - \frac{2}{p^3} - \frac{2}{p^5} + \frac{2}{p^6} \right).
\]
Corollary 4. Let $k \geq 2$. Then for every $\varepsilon > 0$,

$$\sum_{n_1, \ldots, n_k \leq x} \varphi([n_1, \ldots, n_k]) = C_{\varphi,k} \frac{x^{2k}}{2^k} + O\left(x^{2k-1/2+\varepsilon}\right),$$

and

$$\sum_{n_1, \ldots, n_k \leq x} \frac{\varphi([n_1, \ldots, n_k])}{n_1 \cdots n_k} = C_{\varphi,k} x^k + O\left(x^{k-1/2+\varepsilon}\right),$$

where

$$C_{\varphi,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \ldots, \nu_k = 0}^{\infty} \frac{\varphi(p^{\max(\nu_1, \ldots, \nu_k)})}{p^{2(\nu_1 + \ldots + \nu_k)}}.$$ 

In particular,

$$C_{\varphi,2} = \zeta(3) \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3} - \frac{1}{p^4} + \frac{2}{p^5} - \frac{1}{p^6}\right).$$

Corollary 5. Let $k \geq 2$. Then for every $\varepsilon > 0$,

$$\sum_{n_1, \ldots, n_k \leq x} \mu^2([n_1, \ldots, n_k]) = \frac{x^k}{\zeta(2)^k} + O\left(x^{k-1/2+\varepsilon}\right).$$

Remark 1. It would be interesting to find the best possible error, especially in particular cases. For example, for $r = 1$ in Corollary 1, the relative error is $O(x^{-1/2+\varepsilon})$. Can we improve the exponent further and if so, by how much?

3. Proofs

An arithmetic function $g$ of $k$ variables is called multiplicative if

$$g(m_1 n_1, \ldots, m_k n_k) = g(m_1, \ldots, m_k) g(n_1, \ldots, n_k),$$

provided that $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. Hence

$$g(n_1, \ldots, n_k) = \prod_p g(p^{\nu_p(n_1)}, \ldots, p^{\nu_p(n_k)})$$

for every $n_1, \ldots, n_k \in \mathbb{N}$. In this case the multiple Dirichlet series of the function $g$ can be expanded into an Euler product:

$$\sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{g(n_1, \ldots, n_k)}{n_1^{z_1} \cdots n_k^{z_k}} = \prod_p \sum_{\nu_1, \ldots, \nu_k = 0}^{\infty} \frac{g(p^{\nu_1}, \ldots, p^{\nu_k})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}}.$$

We need the following lemmas.
Lemma 3.1. If \( k \geq 2 \) and \( f \in \mathcal{A}_r \) with \( r > -1 \) real, then

\[
L_{f,k}(z_1, \ldots, z_k) := \sum_{n_1, \ldots, n_k=1}^{\infty} \frac{f([n_1, \ldots, n_k])}{n_1^{z_1} \cdots n_k^{z_k}} = \zeta(z_1-r) \cdots \zeta(z_k-r) H_{f,k}(z_1, \ldots, z_k),
\]

where the multiple Dirichlet series \( H_{f,k}(z_1, \ldots, z_k) \) is absolutely convergent for

\[
\Re z_1, \ldots, \Re z_k > A := \begin{cases} 
& \frac{r+\frac{1}{2}}{2}, \quad \text{if } r \geq 0, \\
& \frac{r+1}{2}, \quad \text{if } -1 < r < 0.
\end{cases}
\]

Proof. If \( f \) is a multiplicative function of a single variable, then the arithmetic function of \( k \) variables \( f([n_1, \ldots, n_k]) \) is multiplicative. It follows that

\[
L_{f,k}(z_1, \ldots, z_k) = \prod_p \sum_{\nu_1, \ldots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \ldots, \nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}}
\]

\[\tag{15}\]

Case I. Assume that \( r \geq 0 \). Grouping the terms of the sum in (15) according to the values \( \nu_1 + \cdots + \nu_k \) we have

\[
L_{f,k}(z_1, \ldots, z_k) = \prod_p \left(1 + \frac{f(p)}{p^{z_1}} + \cdots + \frac{f(p)}{p^{z_k}} + \sum_{\nu_1+\cdots+\nu_k \geq 2} \frac{f(p^{\max(\nu_1, \ldots, \nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}}\right).
\]

\[\tag{16}\]

Let \( \Re z_1, \ldots, \Re z_k \geq \delta > r \). By using condition (i) from the definition of the class \( \mathcal{A}_r \),

\[
\frac{f(p)}{p^{z_j}} = \frac{1}{p^{z_j-r}} + O \left( \frac{1}{p^{\delta-r+1/2}} \right) \quad (1 \leq j \leq k).
\]

Also, by condition (iii) following the definition of the class \( \mathcal{A}_r \) and by using that \( r \geq 0 \) we deduce that

\[
\left| \frac{f(p^{\max(\nu_1, \ldots, \nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}} \right| \leq C_3 \frac{p^{\max(\nu_1, \ldots, \nu_k)}}{p^{\delta(\nu_1+\cdots+\nu_k)}} \leq C_3 \frac{1}{p^{(\delta-r)(\nu_1+\cdots+\nu_k)}}.
\]

Thus the sum in (16) over \( \nu_1 + \cdots + \nu_k \geq 2 \) is \( O \left( p^{-2(\delta-r)} \right) \). We obtain

\[
L_{f,k}(z_1, \ldots, z_k) \zeta^{-1}(z_1-r) \cdots \zeta^{-1}(z_k-r)
\]

\[= \prod_p \left(1 - \frac{1}{p^{z_1-r}}\right) \cdots \left(1 - \frac{1}{p^{z_k-r}}\right) \left(1 + \frac{1}{p^{z_1-r}} + \cdots + \frac{1}{p^{z_k-r}} + O \left( \frac{1}{p^{\delta-r+1/2}} \right) \right) \]


\[ +O \left( \frac{1}{p^{2(\delta-r)}} \right) = \prod_p \left( 1 + O \left( \frac{1}{p^{\delta-r+1/2}} \right) + O \left( \frac{1}{p^{2(\delta-r)}} \right) \right), \]

since \( \Re z_j \geq \delta \) (1 \( \leq j \leq k \)), where the terms \( \pm \frac{1}{p^{j-r}} \) (1 \( \leq j \leq k \)) cancel out. Here the latter product converges absolutely when \( \delta - r + 1/2 > 1 \) and \( 2(\delta - r) > 1 \), that is, for \( \delta > r + 1/2 \).

Case II. Assume that \( -1 < r < 0 \). Now we group the terms of the sum in (15) according to the values \( \max(\nu_1, \ldots, \nu_k) \):

\[ L_{f,k}(z_1, \ldots, z_k) = \prod_p \left( 1 + \sum_{\max(\nu_1, \ldots, \nu_k) = 1} f(p) \frac{p^{\nu_1 z_1 + \cdots + \nu_k z_k}}{p^{\nu_1 + \cdots + \nu_k}} + \sum_{\max(\nu_1, \ldots, \nu_k) \geq 2} \frac{f(p_{\max(\nu_1, \ldots, \nu_k)})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}} \right). \]  

(17)

Let \( \Re z_1, \ldots, \Re z_k \geq \delta \geq 0 \). Consider the sum in (17) over \( \max(\nu_1, \ldots, \nu_k) = 1 \) and suppose that \( \nu_i = 1 \) for \( m \) (1 \( \leq m \leq k \)) distinct values of \( i \). If \( m = 1 \), then by condition (i) from the definition of the class \( A_r \) we have

\[ \frac{f(p)}{p^{\nu_i z_i + \cdots + \nu_k z_k}} = 1 + O \left( \frac{1}{p^{\delta-r+1/2}} \right) \quad (1 \leq j \leq k). \]

If \( m \geq 2 \), then

\[ \left| \frac{f(p)}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}} \right| \leq (C_1 + 1)p^r \frac{1}{p^{m\delta}} = O \left( \frac{1}{p^{2(\delta-r)}} \right). \]

This shows that the sum in (17) over \( \max(\nu_1, \ldots, \nu_k) = 1 \) is

\[ \frac{1}{p^{z_1-r}} + \cdots + \frac{1}{p^{z_k-r}} + O \left( \frac{1}{p^{\delta-r+1/2}} \right) + O \left( \frac{1}{p^{2(\delta-r)}} \right). \]

Furthermore, by condition (ii) we deduce that for \( \max(\nu_1, \ldots, \nu_k) \geq 2 \),

\[ \left| \frac{f(p_{\max(\nu_1, \ldots, \nu_k)})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}} \right| \leq C_2 p^r \frac{p^{\max(\nu_1, \ldots, \nu_k)}}{p^{\delta(\nu_1 + \cdots + \nu_k)}} \leq C_2 \frac{1}{p^r} \frac{1}{p^{(\delta-r)\max(\nu_1, \ldots, \nu_k)}} \]

(\( \delta \geq 0 \)) and it follows that the sum in (17) over \( \max(\nu_1, \ldots, \nu_k) \geq 2 \) is

\[ O \left( p^{-2(\delta-r)} \right) = O \left( p^{-2(\delta-r)} \right), \]

since \( r < 0 \).

We obtain that

\[ L_{f,k}(z_1, \ldots, z_k) = \prod_p \left( 1 + \frac{1}{p^{z_1-r}} + \cdots + \frac{1}{p^{z_k-r}} + O \left( \frac{1}{p^{\delta-r+1/2}} \right) + O \left( \frac{1}{p^{2(\delta-r)}} \right) \right) \]
Thus the sum in (18) over out, similar to Case I. Here the latter product converges absolutely when use the same arguments as in the previous proof.

where the multiple Dirichlet series

\[
L_{f,k}(z_1, \ldots, z_k) = \prod_p \left(1 - \frac{1}{p^{z_1-r}}\right) \cdots \left(1 - \frac{1}{p^{z_k-r}}\right) \prod_p \left(1 + \frac{1}{p^{z_1-r}} + \cdots + \frac{1}{p^{z_k-r}}\right)
\]

\[+ O\left(\frac{1}{p^{\delta-r+1/2}}\right) + O\left(\frac{1}{p^{2\delta-r}}\right)\]

\[= \prod_p \left(1 + O\left(\frac{1}{p^{\delta-r+1/2}}\right) + O\left(\frac{1}{p^{2\delta-r}}\right)\right),\]

since \(\Re z_j \geq \delta (1 \leq j \leq k)\), where the terms \(\pm \frac{1}{p_j^{\nu_j}}(1 \leq j \leq k)\) cancel out, similar to Case I. Here the latter product converges absolutely when \(\delta - r + 1/2 > 1\) and \(2\delta - r > 1\), that is, for \(\delta > (r+1)/2 > 0\). \(\square\)

**Lemma 3.2.** If \(k \geq 2\) and \(f \in A_r\) with \(r \geq 0\), then

\[
\mathcal{L}_{f,k}(z_1, \ldots, z_k) := \sum_{n_1, \ldots, n_k = 1}^{\infty} f\left(\frac{[n_1, \ldots, n_k]}{(n_1, \ldots, n_k)}\right) = \zeta(z_1-r) \cdots \zeta(z_k-r) \mathcal{H}_{f,k}(z_1, \ldots, z_k),
\]

where the multiple Dirichlet series \(\mathcal{H}_{f,k}(z_1, \ldots, z_k)\) is absolutely convergent for \(\Re z_1, \ldots, \Re z_k > r + 1/2\).

**Proof.** Similar to the proof of Lemma 3.1, Case I. If \(f\) is multiplicative, then the function \(f([n_1, \ldots, n_k]/(n_1, \ldots, n_k))\) is also multiplicative and we have

\[
\mathcal{L}_{f,k}(z_1, \ldots, z_k) = \prod_p \sum_{\nu_1, \ldots, \nu_k = 0}^{\infty} f\left(p^{\max(\nu_1, \ldots, \nu_k) - \min(\nu_1, \ldots, \nu_k)}\right) / p^{\nu_1 z_1 + \cdots + \nu_k z_k}
\]

\[= \prod_p \left(1 + \frac{f(p)}{p^{z_1}} + \cdots + \frac{f(p)}{p^{z_k}} + \sum_{\nu_1 + \cdots + \nu_k \geq 2} \frac{f\left(p^{\max(\nu_1, \ldots, \nu_k) - \min(\nu_1, \ldots, \nu_k)}\right)}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}}\right).\]

(18)

If \(\Re z_1, \ldots, \Re z_k \geq \delta > r\), then it follows that

\[
\left|\frac{f\left(p^{\max(\nu_1, \ldots, \nu_k) - \min(\nu_1, \ldots, \nu_k)}\right)}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}}\right| \leq C p^{\nu_1 z_1 + \cdots + \nu_k z_k} / p^\delta (\nu_1 + \cdots + \nu_k) \leq C \frac{1}{p^{(\delta-r)(\nu_1 + \cdots + \nu_k)}},
\]

thus the sum in (18) over \(\nu_1 + \cdots + \nu_k \geq 2\) is \(O\left(p^{-2(\delta-r)}\right)\). Furthermore, we use the same arguments as in the previous proof. \(\square\)
Proof of Theorem 2.1. From Lemma 3.1 we deduce the convolutional identity
\[ f([n_1, \ldots, n_k]) = \sum_{j_1d_1 = n_1, \ldots, j_kd_k = n_k} j_1^r \cdots j_k^r h_{f,k}(d_1, \ldots, d_k), \]
where
\[ \sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{h_{f,k}(n_1, \ldots, n_k)}{n_1^2 \cdots n_k^2} = H_{f,k}(z_1, \ldots, z_k). \]
Therefore
\[ \sum_{n_1, \ldots, n_k \leq x} f([n_1, \ldots, n_k]) = \sum_{j_1d_1 \leq x, \ldots, j_kd_k \leq x} j_1^r \cdots j_k^r h_{f,k}(d_1, \ldots, d_k) \]
\[ = \sum_{d_1, \ldots, d_k \leq x} h_{f,k}(d_1, \ldots, d_k) \sum_{j_1 \leq x/d_1} j_1^r \cdots \sum_{j_k \leq x/d_k} j_k^r \]
\[ = \sum_{d_1, \ldots, d_k \leq x} h_{f,k}(d_1, \ldots, d_k) \left( \frac{x^{r+1}}{(r+1)d_1^{r+1}} + O\left( \frac{x^R}{d_1^R} \right) \right) \cdots \left( \frac{x^{r+1}}{(r+1)d_k^{r+1}} + O\left( \frac{x^R}{d_k^R} \right) \right), \]
where \( R := \max(r, 0) \). We deduce that
\[ \sum_{n_1, \ldots, n_k \leq x} f([n_1, \ldots, n_k]) = \frac{x^{k(r+1)}}{(r+1)^k} \sum_{d_1, \ldots, d_k \leq x} \frac{h_{f,k}(d_1, \ldots, d_k)}{d_1^{r+1} \cdots d_k^{r+1}} + S_{k,r}(x), \tag{19} \]
with
\[ S_{k,r}(x) \ll \sum_{u_1, \ldots, u_k} x^{u_1 + \cdots + u_k} \sum_{d_1, \ldots, d_k \leq x} \frac{|h_{f,k}(d_1, \ldots, d_k)|}{d_1^{u_1} \cdots d_k^{u_k}}, \tag{20} \]
where the first sum is over \( u_1, \ldots, u_k \in \{r+1, R\} \) such that at least one \( u_i \) is \( R \). Let \( u_1, \ldots, u_k \) be fixed and assume that \( u_i = R \) for \( t \) (1 \( \leq t \leq k \)) values of \( i \), we take the first \( t \) values of \( i \). Then \( x^{u_1 + \cdots + u_k} \) times the inner sum of (20) is, using the notation \( A \) given by (14),
\[ \ll x^{(k-t)(r+1)+tR} \sum_{d_1, \ldots, d_k \leq x} \frac{|h_{f,k}(d_1, \ldots, d_k)|}{d_1^R \cdots d_t^R d_{t+1}^{R+1} \cdots d_k^{R+1}} \]
\[ = x^{(k-t)(r+1)+tR} \sum_{d_1, \ldots, d_k \leq x} \frac{|h_{f,k}(d_1, \ldots, d_k)|d_1^{A-R+\varepsilon} \cdots d_t^{A-R+\varepsilon} \cdots d_k^{A-R+\varepsilon}}{d_1^{A+\varepsilon} \cdots d_t^{A+\varepsilon} d_{t+1}^{R+1} \cdots d_k^{R+1}} \]
\[
\leq x^{k(t)(r+1)+tR_x t(A-R+\varepsilon)} \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{|h_{f,k}(d_1,\ldots,d_k)|}{d_1^{\varepsilon} \cdots d_t^{\varepsilon} d_{t+1}^{\varepsilon} \cdots d_k^{\varepsilon}} \\
= x^{k(r+1)-t(r+1-A)+te} H_{f,k}(A+\varepsilon,\ldots,A+\varepsilon,r+1,\ldots,r+1) \\
\ll x^{k(r+1)-t(r+1-A)+te},
\]
since the latter series is convergent by Lemma 3.1. Using that \(r+1-A = \frac{1}{2} \min(r+1,1) > 0\), the obtained error is maximal for \(t=1\) giving
\[
O \left( x^{k(r+1)-\frac{1}{2} \min(r+1,1)+\varepsilon} \right).
\]

Furthermore, for the sum in the main term of (19) we have
\[
\sum_{d_1,\ldots,d_k \leq x} \frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^{\varepsilon} \cdots d_k^{\varepsilon}} \\
= \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^{\varepsilon} \cdots d_k^{\varepsilon}} \\
- \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} \sum_{d_i \geq x, i \in I} \sum_{d_j \leq x, j \notin I} \frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^{\varepsilon} \cdots d_k^{\varepsilon}},
\]
where the series is convergent by Lemma 3.1, and its sum is \(H_{f,k}(r+1,\ldots,r+1)\).

Let \(I\) be fixed and assume that \(I = \{1,2,\ldots,s\}\), that is \(d_1,\ldots,d_s > x\) and \(d_{s+1},\ldots,d_k \leq x\), where \(s \geq 1\). We deduce, by noting that \(A-(r+1) = -\frac{1}{2} \min(r+1,1) < 0\,
\[
\sum_{d_1,\ldots,d_s > x} \sum_{d_{s+1},\ldots,d_k \leq x} \frac{|h_{f,k}(d_1,\ldots,d_k)|}{d_1^{\varepsilon} \cdots d_k^{\varepsilon}} \\
= \sum_{d_1,\ldots,d_s > x} \sum_{d_{s+1},\ldots,d_k \leq x} \frac{|h_{f,k}(d_1,\ldots,d_k)|}{d_1^{\varepsilon} \cdots d_k^{\varepsilon}} \\
\leq x^{s(A-(r+1)+\varepsilon)} \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{|h_{f,k}(d_1,\ldots,d_k)|}{d_1^{\varepsilon} \cdots d_k^{\varepsilon}} \\
= x^{s(A-(r+1)+\varepsilon)} H_{f,k}(A+\varepsilon,\ldots,A+\varepsilon,r+1,\ldots,r+1) \\
\ll x^{-\frac{1}{2} \min(r+1,1)+\varepsilon},
\]
the latter series (the same as before) being convergent, and the obtained error is maximal for \( s = 1 \) giving, according to (19) and (21), the same error
\[
O \left( x^{k(r+1) - \frac{1}{2} \min(r+1,1)+\varepsilon} \right).
\]

This proves asymptotic formula (7) with the constant \( C_{f,k} = H_{f,k}(r+1,\ldots,r+1) \). Here, according to Lemma 3.1,
\[
C_{f,k} = \prod_p \left( 1 - \frac{1}{p} \right)^k \sum_{\nu_1,\ldots,\nu_k = 0}^\infty \frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{r(\nu_1+\ldots+\nu_k)}}.
\]

The proof of (8) is similar, based on Lemma 3.1 and the convolutional identity
\[
\frac{f([n_1,\ldots,n_k])}{(n_1\ldots n_k)^r} = \sum_{j_1d_1=n_1,\ldots,j_kd_k=n_k} \frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^r \cdots d_k^r},
\]
which implies that
\[
\sum_{n_1,\ldots,n_k \leq x} \frac{f([n_1,\ldots,n_k])}{(n_1\ldots n_k)^r} = \sum_{d_1,\ldots,d_k \leq x} \frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^r \cdots d_k^r} \sum_{j_1 \leq x/d_1} 1 \cdots \sum_{j_k \leq x/d_k} 1.
\]

Proof of Theorem 2.2. Formula (9) is obtained by using Lemma 3.2, in exactly the same way as (7) (here \( r \geq 0 \) and \( R = \max(r,0) = r \)), with the constant \( D_{f,k} = \overline{H}_{f,k}(r+1,\ldots,r+1) \).

Proof of Corollary 1. Apply Theorem 2.1 for \( f = \text{id}_r \). Here
\[
C_{r,3} = \prod_p \left( 1 - \frac{1}{p} \right)^3 \sum_{a,b,c=0}^\infty \frac{p^{r\max(a,b,c)}}{p^{r(1+a+b+c)}}
\]
\[
= \prod_p \left( 1 - \frac{1}{p} \right)^3 \left( 6S_1 + 3S_2 + 3S_3 + S_4 \right),
\]
with
\[
S_1 = \sum_{0 \leq a < b < c} \frac{p^{re}}{p^{r+1}(a+b+c)}, \quad S_2 = \sum_{0 \leq a = b < c} \frac{p^{re}}{p^{r+1}(2a+c)},
\]

14
\[ S_3 = \sum_{0 \leq a < b < c} \frac{p^{rc}}{p^{(r+1)(a+2c)}}, \quad S_4 = \sum_{0 \leq a = b < c} \frac{p^{rc}}{p^{(r+1)3c}}, \]

which gives (11). Formula (12) for the constant \( C_{r,4} \) can be computed in a similar manner. \( \square \)

**Proof of Corollary 2.** Apply Theorem 2.2 for \( f = \text{id}_r \). The constants \( D_{r,3} \) and \( D_{r,4} \) can be evaluated like above. \( \square \)

**Proof of Corollaries 3, 4, 5.** Apply Theorem 2.1 for \( f = \sigma \), \( f = \varphi \) with \( r = 1 \), resp. \( f = \mu^2 \) with \( r = 0 \). \( \square \)

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