FAKE DEGREES FOR REFLECTION ACTIONS ON ROOTS

VICTOR REINER AND ZHIWEI YUN

ABSTRACT. A finite irreducible real reflection group of rank \(\ell\) and Coxeter number \(h\) has root system of cardinality \(h \cdot \ell\). It is shown that the fake degree for the permutation action on its roots is divisible by \([h]_q = 1 + q + q^2 + \cdots + q^{h-1}\), and that in simply-laced types it equals \([h]_q \cdot \sum_{i=1}^{\ell} q^{d_i^*}\) where \(d_i^* = e_i - 1\) are the codegrees and \(e_i\) are the exponents.

1. Introduction

Consider a complex reflection group \(W \subset GL(V)\) with \(V = \mathbb{C}^\ell\), acting by linear substitutions on the polynomial algebra \(S = \text{Sym}(V^*) \cong \mathbb{C}[x_1, \ldots, x_n]\). Both Shephard and Todd [9] and Chevalley [2] proved that the invariant subalgebra is again a polynomial algebra \(S^W = \mathbb{C}[f_1, \ldots, f_\ell]\) for some homogeneous polynomials \(f_i\), and that the coinvariant algebra \(S/I\) where \(I = (f_1, \ldots, f_\ell)\) carries a graded version of the regular representation. Thus for any finite-dimensional \(\mathbb{C}W\)-module \(U\), the intertwiner space \(\text{Hom}_W(U, S/I) \cong (U^* \otimes S/I)^W\) is a graded \(\mathbb{C}\)-vector space, whose \(q\)-dimension or Hilbert series has been called its fake degree

\[ f_U(q) = \text{Hilb}(\text{Hom}_W(U, S/I), q). \]

Since \(f_U(1) = \dim_{\mathbb{C}} \text{Hom}(U, \mathbb{C}W) = \dim_{\mathbb{C}} U\), one may regard \(f_U(q)\) as a \(q\)-analogue of the degree \(\dim_{\mathbb{C}} U\). For example, the fake degree \(f^{V^*}(q)\) of the dual reflection representation \(V^*\) is determined by the degrees \(d_1 \leq \cdots \leq d_\ell\) of the invariants \(f_1, \ldots, f_\ell\) via \(f^{V^*}(q) = \sum_{i=1}^\ell q^{d_i-1}\). One also defines the codegrees \(d_i^* \leq \cdots \leq d_\ell^*\) via the fake degree polynomial \(f(q) = \sum_{i=1}^\ell q^{d_i^*+1}\) of the representation \(V\) itself.

We focus here on the case where \(W\) acts on \(V = \mathbb{C}^\ell\) as the complexification of an irreducible real reflection group, so that one has \(V \cong V^*\) and \(f(q) = f^{V^*}(q)\). In this setting, one defines the exponents \((e_1, \ldots, e_\ell)\) by \(e_i = d_i - 1 = d_i^* + 1\), and the Coxeter number \(h = d_\ell\). Choose a root system \(\Phi\), containing one opposite pair \(\{\pm \alpha\}\) of normals to each reflecting hyperplane, stable under the \(W\)-action. Given any \(W\)-stable subset \(\Phi'\) of \(\Phi\), we will consider the fake degree polynomial \(f_{\Phi'}(q) := f_U(q)\) for the \(W\)-permutation action \(U = \mathbb{C}\Phi'\). Recall [10, Chap. VI, §1, no. 2], [3] S 3.18] that the cardinality of \(\Phi\) has formula \(|\Phi| = h \cdot \ell\).

Theorem 1. Let \(W\) be an irreducible finite real reflection group, with root system \(\Phi\), and Coxeter number \(h\). Then for any \(W\)-stable subset of \(\Phi'\) of \(\Phi\),

1. \(f_{\Phi'}(q)\) is divisible by \([h]_q = 1 + q + q^2 + \cdots + q^{h-1}\), and
2. when \(W\) is simply-laced, \(f_{\Phi'}(q) = [h]_q \cdot (q^{d_{1*}} + \cdots + q^{d_{\ell*}})\).

Key words and phrases. Reflection group, Weyl group, fake degree, codegree, simply-laced.

First, second authors partially supported by the NSF grants DMS-0601010, DMS-0969470.

1 This follows as a consequence of Solomon’s result [10] that the \(W\)-invariant differential forms with polynomial coefficients \(S \otimes \wedge^* V\) form a free \(S^W\)-module with basis elements \(df_1 \wedge \cdots \wedge df_{\ell}\).
After posting this article to the arXiv, the authors learned that assertion (ii) of Theorem 1 appears in work of Stembridge [13, Lemma 4.3(c,d)], where it is proven via essentially the same method as in Section 3 below. Furthermore, Stembridge gives an explicit factorization of \( f_{\Phi'}(q)/[h]_q \) in the general crystallographic case, where \( \Phi' \) can be either of the two \( W \)-orbits of roots, short or long, using a notion of short exponents for \( W \).

2. Proof of assertion (i)

In the proof, one may assume without loss of generality that \( W \) acts transitively on the subset \( \Phi' \) of \( \Phi \). The desired divisibility will then be deduced from Lemma 2 below, applied to a Coxeter element of \( W \). The statement of the lemma involves Springer’s notion [11] of a regular element \( c \) in \( W \), with a regular eigenvalue \( \zeta \), meaning \( c(v) = \zeta v \) for an eigenvector \( v \) lying on none of the reflecting hyperplanes for \( W \). Then \( c \) and \( \zeta \) have the same multiplicative order \( n \). Denote by \( C \) the cyclic subgroup \( \langle c \rangle \) generated by \( c \).

Lemma 2. [8, Thm. 8.2] Let \( W \) be a complex reflection group acting transitively on a finite set \( X \), and \( c \) in \( W \) a regular element of order \( n \), with a regular eigenvalue \( \zeta \). Then for all \( m \), the fake degree \( f^X(q) := f^U(q) \) for the \( W \)-permutation action \( U = CX \) satisfies

\[
f^X(c^m) = \# \{ x \in X : c^m(x) = x \}.
\]

In particular, \( f^X(q) \) is divisible by \([n]_q\) if and only if \( C \) acts freely on \( X \).

Proof. For the sake of completeness, we recall the proof from [8]. Springer [11] extended the work of Shephard-Todd and Chevalley by proving one has an isomorphism \( W \times C \)-representations

\[
S/I \cong CW
\]

where \( W \) acts as before, and where \( C \) acts on \( CW \) via right-translation, and on \( S/I \) via scalar substitutions \( c(x_i) = \zeta^{-1} \cdot x_i \). Equivalently, \( c \) scales the \( d \)-th homogeneous component \((S/I)_d\) by the scalar \( \zeta^{-d} \).

Now identify the transitive \( W \)-permutation representation \( CX \) with a coset action \( \mathbb{C}[W/W'] \) for some subgroup \( W' \) of \( W \). Then one has an isomorphism \( \text{Hom}_{W'}(\mathbb{C}[W/W'], S/I) \cong (S/I)^{W'} \), and one can reformulate the fake degree:

\[
f^X(q) = \text{Hilb}((S/I)^{W'}, q).
\]

Taking \( W' \)-fixed spaces in (2.1) give an isomorphism of \( C \)-representations

\[
(S/I)^{W'} \cong (CW)^{W'} \cong CX
\]

and the result now follows by comparing the trace of \( c^m \) on the two ends of (2.3). □

To finish the proof of assertion (i), one applies Lemma 2 to a finite real reflection group \( W \), with Coxeter generators \( S = \{s_1, \ldots, s_\ell\} \), and with \( c = s_1s_2 \cdots s_\ell \) a Coxeter element. It is known that all Coxeter elements lie in a single \( W \)-conjugacy class, that they have multiplicative order \( h = d_\ell \), and that they are regular elements having \( \zeta = e^{\frac{2\pi i}{h}} \) as a regular eigenvalue; see [3, §3.16, 3.17]. Furthermore, it is known [4, Chap. V, §1, no. 11] that the cyclic group \( C \) generated by a Coxeter element \( c \) acts freely on the roots \( \Phi \). Assertion (i) now follows from Lemma 2.
3. Proof of assertion (ii)

We first recall a bit more of the root geometry for finite real reflection groups, in order to further reformulate the fake degree \( f_{\Phi'}(q) \); see e.g. [3] Chapters 1, 5.

Assume \( W \) is the complexification of a real reflection group acting on \( V_{\mathbb{R}} \cong \mathbb{R}^r \), that preserves a positive definite inner product \((-,-)\) on \( V_{\mathbb{R}} \). The reflecting hyperplanes dissect \( V_{\mathbb{R}} \) into open simplicial cones called chambers, which are permuted simply-transitively by \( W \). Choosing one such chamber \( C \) to be the dominant chamber, every \( W \)-orbit contains exactly one point in its closure \( \overline{C} \). The root system decomposes as \( \Phi = \Phi_+ \cup -\Phi_+ \), where the positive roots \( \Phi_+ \) are those having positive inner product with the points of \( C \). This also distinguishes the subset of simple roots \( \{\alpha_1, \ldots, \alpha_t\} \) inside \( \Phi_+ \), whose nonnegative linear combinations contain \( \Phi_+ \), and whose corresponding simple reflections \( S = \{s_1, \ldots, s_t\} \) gives rise to a Coxeter presentation \((W, S)\) for \( W \). The above discussion implies that every \( W \)-orbit of roots contains a unique dominant representative \( \alpha_0 \) lying in \( \overline{C} \), whose isotropy subgroup \( W_{\alpha_0} \) is a standard parabolic subgroup generated by some subset \( S \).

**Proposition 3.** Let \( W \) be a finite real reflection group \( W \) with root system \( \Phi \) and positive roots \( \Phi_+ \). Let \( \Phi' \) be a \( W \)-orbit of roots, with unique dominant representative \( \alpha_0 \). Then the fake degree for the \( W \)-permutation action on \( \Phi' \) can be expressed as

\[
f_{\Phi'}(q) = \sum_{\alpha \in \Phi'} q^{d(\alpha_0, \alpha)}
\]

where \( d(\alpha_0, \alpha) \) is the Coxeter group length \( \ell_S(w) \) of the minimum length representative \( w \) for the coset \( wW_{\alpha_0} = \{u \in W : u(\alpha_0) = \alpha \} \).

**Proof.** Note that \( S \) is a free \( S^W \)-module, because \( S^W = \mathbb{C}[f_1, \ldots, f_\ell] \) is a polynomial ring. One obtains \( S^W \)-module splittings for the ring inclusions \( S^{W_{\alpha_0}} \subset S^W \) and \( S^{W_{\alpha_0}} \subset S^{W_{\alpha_0}} \) by averaging over \( W_{\alpha_0} \) and over coset representatives for \( W/W_{\alpha_0} \), respectively. Hence \( S^{W_{\alpha_0}} \) is also a free \( S^W \)-module, with

\[
f_{\Phi'}(q) = \frac{\text{Hilb}((S/I)^{W_{\alpha_0}}, q)}{\text{Hilb}(S^W, q)} = \frac{\text{Hilb}(S^{W_{\alpha_0}}, q)}{\text{Hilb}(S^W, q)}.
\]

For any standard parabolic subgroup \( W' \) of \( W \), such as \( W' = W_{\alpha_0} \) or \( W' = W \) itself, one has [3] §3.15 that \( \text{Hilb}(S^{W'}, q)^{-1} = (1 - q)^{\ell} \sum_{w \in W'} q^{\ell_S(w)} \). Therefore

\[
f_{\Phi'}(q) = \frac{\sum_{w \in W} q^{\ell_S(w)}}{\sum_{w \in W_{\alpha_0}} q^{\ell_S(w)}} = \sum_{w} q^{\ell_S(w)}
\]

where in this last sum, \( w \) runs over the minimum-length coset representatives for the cosets \( wW_{\alpha_0} \) in \( W/W_{\alpha_0} \). \( \square \)

The crux of the proof of assertion (ii) will be the following lemma\(^2\). It relates, for simply-laced root systems with highest root \( \alpha_0 \), the quantity \( d(\alpha_0, \alpha) \) to the root height of \( \alpha \), which we recall here; see [11] Chap. VI, §8], [9] §3.20], [12] §3] for further discussion. When \( W \) is a crystallographic root system \( \Phi \), with simple roots\(^3\)

\(^2\) Although we will not need this information here, the table at the beginning of Section 8 lists the type for these standard parabolic subgroups \( W_{\alpha_0} \). When \( W \) is crystallographic and \( \alpha_0 \) is the highest root, \( W_{\alpha_0} \) is generated by the simple reflections of \( W \) not adjacent to the extra node \( s_0 \) in the extended Dynkin diagram for the affine Weyl group \( \tilde{W} \).

\(^3\) This lemma is similar in spirit to results of Stembridge [12] §2.3] on a quantity that he calls the depth \( d(\alpha) \) of the root \( \alpha \), closely related to the quantity \( d(\alpha_0, \alpha) \) defined here.
Lemma 4. Let $W$ be a simply-laced root Weyl group with root system $\Phi$, positive roots $\Phi_+$, and highest root $\alpha_0$. Then any root $\alpha$ in $\Phi$ has

$$d(\alpha_0, \alpha) = \begin{cases} 
\max(0, \ell_\alpha - \ell_0) & \text{if } \alpha \in \Phi_+, \\
\ell_\alpha - \ell_0 - 1 & \text{if } \alpha \in -\Phi_+.
\end{cases}$$

Proof. Rescale all roots $\alpha$ so that $(\alpha, \alpha) = 2$, and consequently $(\alpha, \beta)$ lies in $\{0, \pm 1, \pm 2\}$ for all pairs of roots $\alpha, \beta$. For any simple root, the formula

$$s_i(\beta) = \beta - (\beta, \alpha_i)\alpha_i$$

shows that applying the simple reflection $s_i$ to a root $\beta \neq \pm \alpha_i$ has the following effect on its height:

$$\ell(s_i \beta) = \begin{cases} 
\ell(\beta) & \text{if } \beta, \alpha_i = 0 \\
\ell(\beta) + 1 & \text{if } \beta, \alpha_i = -1 \\
\ell(\beta) - 1 & \text{if } \beta, \alpha_i = +1.
\end{cases}$$

When $\beta = \pm \alpha_i$, one has $\ell(s_i \beta) = \ell(\beta) = 0$, and $\ell(s_i \beta) = -\ell(\beta) = \pm 1$.

Consequently, when starting with the highest root $\alpha_0$, and applying a sequence of simple reflections $s_i$, the height can drop by at most one at each stage, except when one crosses from a simple root to its negative. This implies that the expression on the right side in the lemma (call it $b(\alpha)$) gives a lower bound on the length $\ell_S(w)$ for any $w$ sending $\alpha_0$ to $\alpha$. Thus $d(\alpha_0, \alpha) \ge b(\alpha)$.

To show $d(\alpha_0, \alpha) \le b(\alpha)$, induct on $b(\alpha)$. In the base case $b(\alpha) = 0$, so $\alpha = \alpha_0$ and $d(\alpha_0, \alpha) = 0$ also. In the inductive step, $b(\alpha) \neq 0$ implies $\alpha \neq \alpha_0$, so (as we are in the simply-laced case) $\alpha$ is not dominant, and there exists some simple root $\alpha_i$ with $(\alpha, \alpha_i) < 0$. It suffices to show that $b(s_i \alpha) = b(\alpha) - 1$.

If $(\alpha, \alpha_i) = -1$ then $\ell(s_i \alpha) = \ell(\alpha) + 1$, and either both $\alpha, s_i(\alpha)$ lie in $\Phi_+$ or both lie in $-\Phi_+$, so $b(s_i \alpha) = b(\alpha) - 1$.

If $(\alpha, \alpha_i) = -2$ then $\alpha = -\alpha_i$, so that $s_i \alpha = +\alpha_i$, and again $b(s_i \alpha) = b(\alpha) - 1$. 

The proof of assertion (ii) requires one more well-known fact [3 §3.20], relating the distribution of root heights to the exponents $e_i = d_i^* + 1$:

$$\sum_{\alpha \in \Phi_+} q^{\ell(\alpha)} = \sum_{i=1}^\ell (q^{1} + q^{2} + \cdots + q^{e_i}).$$
For $W$ simply-laced, there is only one orbit $\Phi$, whose dominant root $\alpha_0$ is the highest root, with $\text{ht}(\alpha_0) = h - 1$. Combining Proposition 3, Lemma 4, (3.2) gives

$$f^\Phi(q) = \sum_{\alpha \in \Phi_+} q^{h-\text{ht}(\alpha)} + \sum_{\alpha \in -\Phi_+} q^{h-2-\text{ht}(\alpha)}$$

$$= \ell (q^{h-e_1} + q^{h-e_2} + \ldots + q^{h-2}) + (q^{h-1} + q^h + \ldots + q^{h+e_i-2})$$

$$= (1-q)^{-1} \sum_{i=1}^\ell (q^{h-e_i-1} - q^{h+e_i-1})$$

where the last equality used the fact [3, §3.16] that $h - e_i = e_{I+1-i}$. Therefore

$$f^\Phi(q) = \frac{1-q^h}{1-q} \sum_{i=1}^\ell q^{e_i-1} = [h]_q \cdot \sum_{i=1}^\ell q^{d_i^*}$$

as desired.

4. Remarks and questions

4.1. Further divisibilities. The table below tabulates the polynomial $f^{\Phi'}(q)/[h]_q$ for root orbits $\Phi'$ in all real reflection groups. In the crystallographic types $A - E$, this can also be deduced from Stembridge’s exponent data [13, Table 4.1] together with his factorization [13, Lemma 4.2(c,d)]. The last column tabulates the additional data $\text{gcd}([h]_q, \sum_{i=1}^\ell q^{d_i^*})$, relevant for Proposition 5 below.

| $W$ | $h$ | $\Phi' = W.\alpha_0$ | $W_{\alpha_0}$ type | $f^{\Phi'}(q)/[h]_q$ | $\text{gcd}([h]_q, \sum_{i=1}^\ell q^{d_i^*})$ |
|-----|----|------------------|-------------------|-----------------|--------------------------------|
| $A_{n-1}$ | $n$ | $\Phi$ | $A_{n-3}$ | $[n-1]_q$ | $1$ |
| $B_n$ | $2n$ | $\{\pm e_i \pm e_j\}$ | $A_1 \times B_{n-2}$ | $[n-1]_q^2$ | $[n]_q^2$ |
| $B_n$ | $2n$ | $\{e_i\}$ | $B_{n-1}$ | $1$ | $1$ |
| $D_n$ | $2(n-1)$ | $\Phi$ | $A_1 \times D_{n-2}$ | $[n-2]_q^2 [n]_q [2]_{q^2}^2$ | $[2]_{q^2}^2 [4]_{q^6}^2$ |
| $E_6$ | $12$ | $\Phi$ | $A_5$ | $[2]_{q^2} [3]_{q^6}$ | $1$ |
| $E_7$ | $18$ | $\Phi$ | $D_6$ | $[2]_{q^2} [7]_{q^2}$ | $1$ |
| $E_8$ | $30$ | $\Phi$ | $E_7$ | $[2]_{q^6} [4]_{q^6}$ | $1$ |
| $F_4$ | $12$ | either orbit | $B_3$ | $[2]_{q^2}$ | $[2]_{q^6}$ |
| $H_3$ | $10$ | $\Phi$ | $A_1 \times A_1$ | $[3]_{q^2}$ | $1$ |
| $H_4$ | $30$ | $\Phi$ | $H_3$ | $[2]_{q^6} [2]_{q^{10}}$ | $1$ |
| $I_2(m)$ | $m$ | even | $A_1$ | $1$ | $1$ if $\frac{m}{2}$ odd |
| $I_2(m)$ | $m$ | odd | $A_1$ | $1$ if $\frac{m}{2}$ even |
| $I_2(m)$ | $m$ | $\Phi$ | $A_1$ | $1$ |

The table exhibits case-by-case two facts for which we lack uniform proofs.

**Proposition 5.** For finite real $W$ with one root orbit, $\text{gcd}([h]_q, \sum_{i=1}^\ell q^{d_i^*}) = 1$. 

Using (2.11), Proposition 5 is equivalent to the assertion that, when $W$ has only one orbit of roots, every power $c^m$ of a Coxeter element $c$ acts on $V$ with nonzero trace.

**Proposition 6.** For finite real $W$ which are at most doubly-laced, meaning that its Coxeter presentation relations $(s_is_j)^{m_{ij}} = e$ all have $m_{ij} \leq 4$, every $W$-stable root subset $\Phi'$ has fake degree $f_{\Phi'}(q)$ divisible by $\sum_{i=1}^\ell q^{\ell_i}$.

4.2. **Original motivation.** We originally observed Theorem 1 case-by-case while computing the fake degree of a certain irreducible representation of simply-laced $W$, arising naturally in [7, Chapter 3]. One can decompose the $W$-permutation representation $\mathbb{C}[\Phi']$ of any real reflection group $W$ on a root orbit $\Phi'$ into two direct summands, namely its symmetric and antisymmetric components $\mathbb{C}[\Phi']^+, \mathbb{C}[\Phi']^-$ with respect to the $W$-equivariant involution that simultaneously swaps each $+\alpha, -\alpha$. A straightforward calculation then shows the following.

**Proposition 7.** Let $W$ be a finite real reflection group $W$ and $\Phi'$ an orbit of its roots. Then any one of the three fake degrees for $\mathbb{C}[\Phi']$, $\mathbb{C}[\Phi']^+$, $\mathbb{C}[\Phi']^-$ determines the others via the relations $f_{\Phi'}(q) = f_{\Phi'}^+(q) + f_{\Phi'}^-(q)$ and $f_{\Phi'}^-(q) = q \cdot f_{\Phi'}^+(q)$.

It was further shown in [7, Chapter 3] that, for irreducible real reflection groups $W$, and any root orbit $\Phi'$, the antisymmetric component $\mathbb{C}[\Phi']^-$ has $W$-irreducible decomposition which is multiplicity-free. In the simply-laced case, it has only two irreducible constituents: $\mathbb{C}[\Phi']^- = V \oplus U$ where $V$ is the reflection representation $V$ of degree $\ell$, and $U$ is another $W$-irreducible, of degree $|\Phi^+| - \ell = \frac{\ell}{2} \cdot \ell$. Using Proposition 7 one can check that Theorem 1(ii) is equivalent to the assertion that this $W$-irreducible $U$ has fake degree $f_U(q) = q^2 \cdot \sum_{i=1}^\ell q^{\ell_i}$.

4.3. **M-V cycles.** Lemma 4 has a geometric interpretation. It is well-known that for a standard parabolic subgroup $W'$ of a Weyl group $W$ associated to simple complex algebraic group $G$ and Borel subgroup $B$, one can identify the invariant subalgebra $(S/I)^W$ with the cohomology $H^*(G/P)$ of $G/P$ where $P = \langle B, W' \rangle$. The Schubert cell decomposition of $G/P$ lets one express its Poincaré polynomial in terms of lengths of minimal coset representatives for $W/W'$. The expression (3.1) then arises in this way when $W' = W_{\ell_0}$ for a dominant root $\ell_0$.

When $\ell_0$ happens to be the highest root of a simply-laced root system, the cone over the variety $G/P$ also arises as a Schubert variety in the affine Grassmannian. The cell decomposition of $G/P$ as above can be used to give a decomposition of this cone into Mirković-Vilonen cycles introduced in [5]. In this picture, the dimension formula for the Mirković-Vilonen cycles is equivalent to Lemma 4 see Mirković and Vilonen [5, Theorem 3.2] with $\lambda = \ell_0$, and also Ngô and Polo [9, Lemme 7.4].

4.4. **A-D-E quivers?** For simply-laced $W$, the $W$-action permuting the roots can be modeled by reflection functors acting on the the bounded derived category of quiver representations, with a Coxeter element $c$ corresponding to the Auslander-Reiten translation. Here the $W$-equivariant map from an object to its dimension vector factors through the quotient category that mods out by the square of the shift map; see the discussion of the periodic Auslander-Reiten quiver by Kirillov and Thind [4]. Does Theorem 1(ii) reflect something lurking in this quiver picture?

**References**

[1] N. Bourbaki, Lie groups and Lie algebras, Chapters 4–6. *Elements of Mathematics*, Springer-Verlag, Berlin, 2002.
[2] C. Chevalley, Invariants of finite groups generated by reflections. Amer. J. Math. 77 (1955), 778–782.
[3] J.E. Humphreys, Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics 29. Cambridge University Press, Cambridge, 1990.
[4] A.A. Kirillov and J. Thind, Coxeter elements and periodic Auslander-Reiten quiver. J. Algebra 323 (2010), 1241–1265.
[5] I. Mirkovic and K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings. Ann. of Math. (2) 166 (2007), 95–143.
[6] B.C. Ngô and P. Polo, Résolutions de Demazure affines et formule de Casselman-Shalika géométrique. J. Algebraic Geom. 10 (2001), no. 3, 515–547
[7] V. Reiner, F. Saliola and V. Welker, Spectra of symmetrized shuffling operators, arxiv:1102.2460.
[8] V. Reiner, D. Stanton and D. White, The cyclic sieving phenomenon. J. Combin. Theory Ser. A 108 (2004), 17–50.
[9] G.C. Shephard and J.A. Todd, Finite unitary reflection groups. Canadian J. Math. 6, (1954), 274–304.
[10] L. Solomon, Invariants of finite reflection groups. Nagoya Math. J. 22 (1963), 57–64.
[11] T.A. Springer, Regular elements of finite reflection groups. Invent. Math. 25 (1974) 159–198.
[12] J.R. Stembridge, Quasi-minuscule quotients and reduced words for reflections. J. Algebraic Combin. 13 (2001) 275–293.
[13] J.R. Stembridge, Graded multiplicities in the Macdonald kernel. I. Int. Math. Res. Pap. (2005), no. 4, 183–236.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: reiner@math.umn.edu

Dept. of Mathematics, MIT, Cambridge, MA 02139
E-mail address: zyun@math.mit.edu