Factorization over interpolation: A fast continuous orthogonal matching pursuit

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Abstract— We propose a fast greedy algorithm to compute sparse representations of signals from continuous dictionaries that are factorizable, i.e., with atoms that can be separated as a product of sub-atoms. Existing algorithms strongly reduce the computational complexity of the sparse decomposition of signals in discrete factorizable dictionaries. On another flavour, existing greedy algorithms use interpolation strategies from a discretization of continuous dictionaries to perform off-the-grid decomposition. Our algorithm aims to combine the factorization and the interpolation concepts to enable low complexity computation of continuous sparse representations of signals. The efficiency of our algorithm is highlighted by simulations of its application to a radar system.

1 Introduction

Computation of sparse representations is beneficial in many applicable fields such as radar signal processing, communication or remote sensing\textsuperscript{[1][2][3]}. This computation is based on the assumption that a signal $y$ decomposes as a linear combination of a few atoms taken from a dictionary $D$. In this paper, we focus on continuous parametric dictionaries\textsuperscript{[4][5][6]} which associate each parameter $p$ from the continuous parameter set $P$ to an atom $a(p) \in \mathbb{C}^M$. Thereby, the decomposition of $y \in \mathbb{C}^M$ reads

$$y = \sum_{k=1}^{K} \alpha_k a(p_k),$$

where for all $k \in [K]$, $\alpha_k \in \mathbb{C}$ and $p_k \in P$.

In some applications\textsuperscript{[7][8][9]}, the atoms $a(p_k)$ factorize as a product of sub-atoms, each depending on a distinct set of components of $P_k$. In that case, greedy algorithms such as presented in\textsuperscript{[10][11]} can leverage this property to strongly reduce the computational complexity of the decomposition.

These approaches, however, capitalize a discretization of $P$ and assume that the parameters $\{p_k\}_{k=1}^{K}$ match the resulting grid\textsuperscript{[12]}. Yet, estimations of parameters from such discretized models are affected by grid errors\textsuperscript{[13]}. Although a denser grid reduces this effect, it tremendously increases the dimensionality of the problem to solve. Continuous reconstruction algorithms do not require such dense grids as they perform off-the-grid estimations of parameters\textsuperscript{[14][15][16]}. In\textsuperscript{[17]}, a continuous version of the Basis Pursuit is derived from the construction of an interpolated model that approximates the atoms. In\textsuperscript{[18]}, the authors similarly designed the Continuous OMP (COMP) from the same interpolation concept.

We propose a Factorized COMP (F-COMP) that combines the concepts of interpolation and factorization to enable a fast and accurate reconstruction of sparse signals. We applied our algorithm to the estimation of the ranges and speeds of targets using a Frequency Modulated Continuous Wave (FMCW) radar. Simulations validated the superiority of using low-density grids with off-the-grid algorithms instead of denser grids with on-the-grid algorithms.

Notations: Matrices and vectors are denoted by bold uppercase and lowercase symbols, respectively. The tensors are denoted with bold calligraphic uppercase letters. The outer product is $\otimes$, $\| \cdot \|_F$ is the Frobenius norm, $[N] := \{1, \cdots, N\}$, $j = \sqrt{-1}$, and $c$ is the speed of light.

2 Problem Statement

We consider the problem of estimating the values of $K$ parameters $\{p_k\}_{k=1}^{K} \subset P$ from a measurement vector $y \in \mathbb{C}^M$. This measurement is assumed to decomposes as $[1]$, with $K \ll M$. The parameters are known to lie in a separable parameter domain $P \subset \mathbb{R}^K$ such that $P := \mathcal{P}_1 \times \cdots \times \mathcal{P}_L$ with $\mathcal{P}_l \subset \mathbb{R}$ for each $l \in [L]$. For all $k \in [K]$, $p_k$ decomposes in

$$p_k := (p_{k,1}, \cdots, p_{k,L})^\top,$$

with $p_{k,l} \in \mathcal{P}_l$ for all $l \in [L]$. In $[1]$ the atoms $a(p_k)$ are taken from a continuous dictionary defined by $D := \{a(p) : p \in P\}$.

In this paper, we consider the particular case of dictionaries of atoms that factorize in $L$ sub-atoms. More precisely, introducing the tensor $A(p) \in \mathbb{C}^{M_1 \times \cdots \times M_L}$ reshaping $a(p) \in \mathbb{C}^M$

$$A_{m_1, m_2, \cdots, m_L}(p) := a_m(p),$$

with $m := m_L + \sum_{l=1}^{L-1} (m_l - 1) \prod_{l'=1}^{L} M_{l'}$ for all $l \in [L]$ and $M = M_1 M_2 \cdots M_L$, we assume that the atom $A(p_k) \in \mathbb{C}^M$ decomposes in

$$A(p_k) := \psi_1(p_{k,1}) \otimes \cdots \otimes \psi_L(p_{k,L}).$$

In $[1]$, each $\psi_l(\cdot)$ is a sub-atom taken from the continuous dictionary $\mathcal{D}_l := \{\psi_l(p) : p \in \mathcal{P}_l\}$. In the tensor reshaped domain, the decomposition $[1]$ becomes

$$Y = \sum_{k=1}^{K} \alpha_k A(p_k),$$

where $Y \in \mathbb{C}^{M_1 \times \cdots \times M_L}$ is the tensor-shaped measurement, i.e., $Y_{m_1, m_2, \cdots, m_L} := y_m$.

Recovering the parameters $\{p_{k,l}\}_{k=1}^{K}$ from the factorized model $[5]$ can be made fast. For instance, in work $[10][11]$, the authors consider an adaptation of OMP, that we coin Factorized OMP (F-OMP), which leverages the decomposition $[1]$ to reduce the dimensionality of the recovery problem. Yet, F-OMP only enables the estimation of on-the-grid parameters taken from a finite discrete set of parameters. In the next section, we build a model based on the same grid which enables the greedy estimation of off-the-grid parameters while similarly leveraging the factorization.

3 Factorization over Interpolation

From the general non-factorized model $[1]$, the algorithm Continuous OMP (COMP) $[18]$ extends OMP and succeeds to greedily estimate off-the-grid parameters. COMP operates with a parameter grid which results from the sampling of $P$. The atoms of the continuous dictionary $D$ are approximated by a linear combination a multiple atoms which are defined from the grid. This combination enables to interpolate (from the grid) the atoms of $D$ that are parameterized from off-the-grid parameters. Our algorithm F-COMP applies the same interpolation concept to the atoms $A(p)$, which are factorized by $[4]$. 
Let us define the separable grid \( \Omega_P \subset \mathcal{P} \) such that \( \Omega_P = \Omega_{P_1} \times \cdots \times \Omega_{P_L} \), with \( \Omega_{P_l} = \{ \omega_{n_l}^k \}_{k=1}^{N_l} \subset \mathcal{P}_l \) for all \( l \in [L] \). We propose a “factorization over interpolation” strategy where each off-the-grid atom \( \mathbf{A}(p_k) \) is interpolated by \( I \) on-the-grid atoms

\[
\mathbf{A}(p_k) \approx \sum_{i=1}^{I} \mathbf{A}^{(i)}(p_k) \quad (6)
\]

In [6], the indices \( n(k) := (n_1(k), \ldots, n_L(k)) \) depend on the interpolation scheme and on \( p_k \), and each \( \mathbf{A}^{(i)}(n(k)) \) is the \( i \)-th interpolation atom associated to the \( n(k) \)-th element of the grid \( \Omega_P \). The coefficients \( e^{(i)}_k \) are obtained from

\[
e^{(i)}_k = C_m(n(k), p_k),
\]

where \( C_m(p) \) is a function defined from the choice of interpolation pattern [17] [18]. In this scheme, for all \( i \in [I] \), we decompose the global interpolation atoms \( \mathbf{A}^{(i)}(n(k)) \) using interpolation sub-atoms denoted by \( \psi_i^{(n)} \), i.e.,

\[
\mathbf{A}^{(i)}(n(k)) = \psi_i^{(1)}(n_1(k)) \circ \cdots \circ \psi_i^{(L)}(n_L(k)).
\]

The factorization [8] is enabled by the properties of the interpolation polynomials. It is for this case that the dictionaries describing FMCW chirp-modulated radar signals we detail in Sec. 5. From such signals, we can efficiently estimate off-the-grid values of \( p_{k_1} \) using the Factorized Continuous OMP that we explain in the next section.

### 4 Factorized Continuous OMP

Alg. 1 formulates F-OMP for a generic interpolation scheme. F-OMP leverages the factorized interpolated model (6) to estimate off-the-grid parameters with a reduced complexity with respect to COMP [13]. It follows the same steps as COMP and greedily minimizes \( \| \mathbf{Y} - \sum_{k=1}^{K} \alpha_k \mathbf{A}^{(i)}(n(k)) \|^2_F \). In Alg. 1 we use \( N \) to denote \( [N_1] \times \cdots \times [N_L] \) and \( \beta_k := (\beta_1^k, \ldots, \beta_L^k) \), where \( \beta_k \) estimates \( \alpha_k \).

The decomposition of the atoms \( \mathbf{A}(p_k) \) enables to compute the step in (9) with a complexity \( O(1) \) instead of \( O(1 \times N_1 \times \cdots \times N_L) \) in COMP. This is achieved by extending the methodology of F-OMP [11] to our interpolation-based model.

**Algorithm 1: Factorized Continuous OMP (F-OMP)**

**Input**: \( K, J, \{ \mathbf{A}^{(i)}[n] \}_{i \in [I]} \times [N] \times \Omega_P \).

**Output**: \( \{ \hat{\alpha}_k \}_{k=1}^K, \{ \hat{r}_k, \hat{\beta}_k \}_{k=1}^K \).

**Initialization**:

\[
\hat{\mathbf{r}}^{(1)} = \mathbf{Y}, \hat{\Omega} = \emptyset;
\]

While \( k \leq K : \)

\[
\hat{n}(k) = \arg \min_{n \in \mathcal{N}} \frac{1}{\beta} \sum_{i=1}^{I} \beta_i \mathbf{A}^{(i)}[n] - \hat{\mathbf{r}}^{(k)} \|_F^2 \quad (9)
\]

\[
\hat{\Omega} \leftarrow \hat{\Omega} \cup \{ \hat{n}(k) \}
\]

\[
\{ \hat{\beta}_k \}_{k=1}^K = \arg \min_{\beta \in \mathcal{C}} \frac{1}{\beta} \sum_{i=1}^{I} \beta_i \mathbf{A}^{(i)}[\hat{n}(k)] - \hat{\mathbf{r}}^{(k)} \|_F^2
\]

\[
r^{(k+1)} = \mathbf{y} - \sum_{k'=1}^{k} \sum_{i=1}^{I} \beta_{i}^{(k')} \mathbf{A}^{(i)}[\hat{n}(k')]
\]

\[
k \leftarrow k + 1
\]

For all \( k \in [K] \),

\[
(\hat{\alpha}_k, \hat{p}_k) = \arg \min_{\alpha \in \mathcal{C}(r, v)} \alpha \mathbf{C}_m(n_k)(p) - \hat{\beta}_k.
\]
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