The Euclidian-hyperboidal foliation method
and the nonlinear stability of Minkowski spacetime

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Abstract

We introduce a new method for analyzing nonlinear wave-Klein-Gordon systems and establishing global-in-time existence results for the Cauchy problem when the initial data need not have compact support. This method, which we call the Euclidian-Hyperboidal Foliation Method (EHFM), relies on the construction of a spacetime foliation obtained by glueing together asymptotically Euclidian and asymptotically hyperboloidal hypersurfaces. Well-chosen frames of vector fields (null-semi-hyperboloidal, Euclidian-hyperboloidal) allow us to exhibit the structure of the equations under consideration and analyze the decay of solutions in timelike and in spacelike directions. New Sobolev inequalities for Euclidian-hyperboloidal foliations involving the Killing fields of Minkowski spacetime (but not the scaling field), as well as pointwise bounds for wave and Klein-Gordon equations on curved spacetimes are established. Our bootstrap argument involves a hierarchy of (almost sharp) energy and pointwise bounds and distinguishes between low- and high-order derivatives of the solutions. We apply this method to the Einstein equations when the matter model is a massive field and the methods by Christodoulou and Klainerman and by Lindblad and Rodnianski do not apply.

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1. Purpose of this work

1.1. Coupled wave-Klein-Gordon systems

Our main motivation in the present work is the global evolution problem for self-gravitating massive matter fields and, especially, the global nonlinear stability of Minkowski spacetime in this context. We introduce here a new method for the global analysis of nonlinear coupled wave-Klein-Gordon systems, which does not require Minkowski’s scaling field. This method, which we refer to as the Euclidian-Hyperboloidal Foliation Method (EHFM), generalizes the Hyperboloidal Foliation Method (HFM) in [14,15,16,17]. The later method extended a methodology first proposed in Klainerman [13] and Hormander [10] but also included several novel techniques. In the present Note, we are able to handle solutions that need not be compactly supported, which is essential for the application to the Einstein equations. Our method applies to a broad class of nonlinear wave-Klein-Gordon systems.

In [14], we relied on a foliation of the interior of the light cone in Minkowski spacetime by spacelike hyperboloids and we derived sharp pointwise bounds and high-order energy estimates in order to study the nonlinear coupling taking place between wave and Klein-Gordon equations. Our method had the advantage of relying on the Lorentz boosts and the translations only, rather than on the full family of (conformal) Killing fields of Minkowski spacetime, as was the case in earlier works such as [2,20].

With the Hyperboloidal Foliation Method, we solved the global nonlinear stability problem when the Einstein equations are coupled to a massive scalar field and are expressed in wave gauge; see [17]. However, only the restricted class of initial data sets coinciding with Schwarzschild data outside a spatially compact domain was treated therein.

We provide here an outline of our new method while a full presentation can be found in the Monograph [18].

1.2. The basic strategy: the interior, transition, and exterior domains, together with adapted frames

We work in spacetimes $\mathcal{M}$, that is, a 4-manifold endowed with a Lorentzian metric $g$, on which we assume a global foliation $(x^\alpha) = (t, x) = (t, x^a)$ with $\alpha = 0, 1, 2, 3$ and $a = 1, 2, 3$. Our approach distinguishes within the spacetime between interior and exterior spacetime domains, in which different foliations are required. A gluing technique involving a transition region concentrated near the light cone around which the geometry of the foliation changes drastically from begin hyperboloidal to being Euclidian in nature.

Several frames of vector fields play a role here: the Cartesian frame $\partial_\alpha$, the semi-hyperboloidal frame $\tilde{\partial}_a = (x^a/r)\partial_t + \partial_\alpha$, as well as the null frame $\bar{\partial}_a = (x^a/r)\partial_t + \partial_\alpha$ is used at various stages of our analysis:

- **Vector fields tangent to the foliation:** $\tilde{\partial}_a$ in the interior domain and $\bar{\partial}_a$ in the exterior domain, which are used in expressing the energy estimates.
— Vector fields relevant for decomposing the metric and the nonlinearities: $\partial_a$ in the interior domain and $\tilde{\partial}_a$ in the exterior domain.

1.3. Main contributions

A major challenge is to cope with the nonlinear coupling taking place between the geometry and the matter terms of various gravity field equations (including the Einstein equations), or equivalently between wave equations and Klein-Gordon equations on a curved spacetime, which potentially could lead to a blow-up phenomena of the energy norm under consideration and prevent the global existence of solutions.

We rely on basic high-order energy estimates obtained by applying the translations, boosts, and spatial rotations of Minkowski spacetime.

— Considering a simpler model first and then analyzing the full problem of interest, we provide a classification of the nonlinearities arising in wave-Klein-Gordon equations and systematically we compare them with the terms we control with our energy functional.

— By proposing a general and synthetic proof, we establish that our frames of vector fields enjoy favorable commutator estimates in order for high-order energy estimates to be derived, and, especially, enjoy good commutation properties with the Killing fields of Minkowski spacetime

— We derive new Sobolev inequalities for Euclidian-hyperboloidal foliations, which are established by studying cone-like domains first.

— We establish sharp sup-norm estimates for solutions to wave and Klein-Gordon equations on curved spacetime, after decomposing the solution operators in suitable frames and building on the following three approaches: an ordinary differential equation argument along rays, a characteristic integration argument, and Kirchhoff’s explicit formula.

For gravity field problems, the wave gauge conditions play a central role in the derivation of energy and pointwise bounds and especially provide a control on the components of the metric associated with a propagation equation that does not satisfy the null condition. Let us point out that in the case of $1 + 1$ dimensions discussed in [21], solutions to wave-Klein-Gordon equations enjoy much better global properties [3,22]. The use of hyperboloidal foliations for wave equations was suggested first by Klainerman [13] and Hormander [10]. Earlier on, in [6], Friedrich also studied hyperboloidal foliations of Einstein spacetimes and succeed to establish global existence results for the Cauchy problem for the conformal vacuum field equations.

2. A new approach: the Euclidian-Hyperboloidal Foliation Method (EFHM)

2.1. A decomposition of the spacetime

We thus decompose the future $\mathcal{M}$ of an initial hypersurface $\mathcal{M}_0$, and distinguish between three regions: $\mathcal{M} = \mathcal{M}^{\text{int}} \cup \mathcal{M}^{\text{tran}} \cup \mathcal{M}^{\text{ext}}$. Without loss of generality, we label the initial hypersurface as $\{ t = 1 \}$ in our global coordinate chart $(t, x^1, x^2, x^3)$ with $r^2 = \sum (x^a)^2$. We write

$$\mathcal{M} \simeq \{ t \geq 1 \} \subset \mathbb{R}^{3+1}. \quad (2.1)$$

Symmetries of Minkowski spacetime are viewed as approximate symmetries for our spacetime $\mathcal{M}$:

— The translations are generated by the vector fields $\partial_\alpha \alpha = 0, 1, 2, 3$, which, for instance, will be tangent to the time slices in the exterior domain.

— The Lorentz boosts are generated by the vector fields $L_a = x_a \partial t + t \partial_a$, $a = 1, 2, 3$, which, for instance, will be tangent to the time slices in the interior domain.
— The spatial rotations are generated by the vector fields \( \Omega_{ab} = x_a \partial_b - x_b \partial_a \), \( a = 1, 2, 3 \), which will be tangent to the time slices in, both, the exterior and the interior domains.

We refer to the above as the family of admissible vector fields. These fields commute with the wave and Klein-Gordon operators in Minkowski spacetime. On the other hand, importantly, throughout our analysis we avoid to rely on Minkowski’s scaling field \( S = t \partial_t + r \partial_r \), since it does not commute with the Klein-Gordon operator in Minkowski spacetime.

— Within an interior domain \( \mathcal{M}^{\text{int}} \), we rely on the foliation based on the hyperboloidal slices

\[
\{ t^2 - r^2 = s^2 \} \subset \mathbb{R}^{3+1}
\]

with hyperbolic radius \( s \geq 2 \). These slices are most convenient in order to analyze wave propagation issues and establish the decay of solutions in timelike directions.

— Within an exterior domain denoted by \( \mathcal{M}^{\text{ext}} \), we rely on Euclidian slices of constant time \( c \geq 1 \)

\[
\{ t = c \} \subset \mathbb{R}^{3+1}.
\]

These slices are most relevant in order to analyze the asymptotic behavior of solutions in spacelike directions and the properties of asymptotically flat spacetimes.

2.2. Time foliation of interest

We take advantage of both foliations above by glueing them together within a transition region \( \mathcal{M}^{\text{tran}} \), as follows. Consider a cut-off function \( \chi = \chi(y) \) (suitably defined to provide a smooth transition between the region) satisfying \( \chi(y) = 0 \) for \( y \leq 0 \), while \( \chi(y) = 1 \) for \( y \geq 1 \), and being increasing within the interval \((0, 1)\). Then, we introduce the transition function

\[
\xi(s, r) := 1 - \chi(r + 1 - s^2/2) \in [0, 1],
\]

which is globally smooth and is defined for all \( s \geq 1 \) and all \( r \geq 0 \). The function \( \xi \) is not constant precisely in a transition region around the light cone \( 2r \simeq s^2 = t^2 - r^2 \).

The Euclidian-hyperboloidal time function is the function \( T = T(s, r) \) defined by the following ordinary differential problem:

\[
\partial_t T(s, r) = \chi(s - 1) \xi(s, r) \frac{r}{\sqrt{r^2 + s^2}},
\]

\[
T(s, 0) = s \geq 1.
\]

By definition, the Euclidian-Hyperboloidal foliation is determined from this time function and consists of the following family of spacelike hypersurfaces

\[
\mathcal{M}_s := \{ (t, x) / t = T(s, |x|) \}.
\]

Our analyzing is performed in the following spacetime regions:

\[
\mathcal{M}_{[s_0, s_1]} : = \{ (t, x) / T(s_0, |x|) \leq t \leq T(s_1, |x|) \} \subset \mathbb{R}^{3+1},
\]

\[
\mathcal{M}_{[s_0, +\infty)} : = \{ (t, x) / T(s_0, |x|) \leq t \} \subset \mathbb{R}^{3+1},
\]

and the interior, transition, and exterior domains (with \( r = |x| \))

\[
\mathcal{M}_s^{\text{int}} : = \{ t^2 = s^2 + r^2, \quad r \leq -1 + s^2/2 \} \quad \text{hyperboloidal region},
\]

\[
\mathcal{M}_s^{\text{tran}} : = \{ t = T(s, r), \quad -1 + s^2/2 \leq r \leq s^2/2 \} \quad \text{transition region},
\]

\[
\mathcal{M}_s^{\text{ext}} : = \{ t = T(s), \quad r \geq s^2/2 \} \quad \text{Euclidian region}.
\]

By construction, in the interior, the relation \( T^2 = s^2 + r^2 \) holds and the slices consist of hyperboloids of Minkowski spacetime. On the other hand, in the exterior, one has \( T = T(s) \simeq s^2 \) which is independent of \( r \) and represents a “slow time”, and the slices consists of flat hyperplanes of Minkowski spacetime.
We choose our cut-off function $\chi$ as follows, by setting first

\[
\rho(y) := \begin{cases} 
    e^{\frac{1}{y^2-(\theta-1)^2}}, & 0 < y < 1, \\
    0, & \text{otherwise}.
\end{cases} 
\]  

(2.6)

We then set $\rho_0 := \int_{-\infty}^{+\infty} \rho(y) \, dy = \int_0^1 \rho(y) \, dy > 0$ and define

\[
\chi(y) := \rho_0^{-1} \int_{-\infty}^{y} \rho(y') \, dy', \quad y \in \mathbb{R}. 
\]  

(2.7)

2.3. Weighted energy norm

We use the energy norm associated to the wave and Klein-Gordon equations and induced on our Euclidian-hyperboloidal slices. In addition, in the exterior domain we introduce a weight function which provides us with the required control of the decay in spacelike directions. This weight depends upon the Euclidian-hyperboloidal slices. In addition, in the exterior domain we introduce a weight function which we obtain the alternative form

\[
\mathcal{E}_{\eta,c}(s,v) := \int_{\mathcal{M}_s} (1 + \omega_r)^2 \left( (1 - \chi(1-s)^\frac{r^2}{t^2})(\partial_t v)^2 + \sum_a \left( \frac{\xi^a}{t} \partial_t v + \partial_a v \right)^2 + c^2 v^2 \right) \, dx,
\]

in which, by definition, one has $t = T(s,r)$ and $r = |x|$ on $\mathcal{M}_s$. With the notation

\[
\partial_a := \partial_a T(s,r) \partial_t + \partial_a 
\]

\[
= \chi(1-s) \xi(s,r) \frac{x^a}{T(s,r)} \partial_t v + \partial_a,
\]  

(2.9)

we obtain the alternative form

\[
\mathcal{E}_{\eta,c}(s,v) = \int_{\mathcal{M}_s} (1 + \omega_r)^2 \left( \zeta(s,r) \partial_t v \right)^2 + \sum_a \left( \partial_a v \right)^2 + c^2 v^2 \right) \, dx,
\]  

(2.10)

in which the coefficient $\zeta = \zeta(s,r) \in [0,1]$ is defined as

\[
\zeta := \chi(1-s) \sqrt{\frac{s^2 + c^2(r-1 + s^2/2r^2)}{s^2 + r^2}}.
\]  

(2.11)

This energy (together with its generalization to a curved metric) leads us to a control of the weighted wave-Klein-Gordon energy associated with the operator $\Box v - c^2 v$ with $c \geq 0$.

2.4. A Sobolev inequality for the transition and exterior domains

**Proposition 2.1** For all sufficiently regular functions defined on $\mathcal{M}_{[s_0,s_1]}$ with $2 \leq s_0 \leq s \leq s_1$, one has:

\[
|u(x)| \leq C(1+r)^{-1} \sum_{|I| + |J| \leq 2} \| \overrightarrow{\partial}^I \Omega^J \|_{L^2(\mathcal{M}_{s_0}^{\text{tran}} \cup \mathcal{M}_{s_1}^{\text{ext}})}, \quad x \in \mathcal{M}_s^{\text{tran}} \cup \mathcal{M}_s^{\text{ext}},
\]

(2.12)

\[
|u(x)| \leq C(1+r)^{-1} \sum_{|I| + |J| \leq 2} \| \overrightarrow{\partial}^I \Omega^J \|_{L^2(\mathcal{M}_s^{\text{tran}})}, \quad x \in \mathcal{M}_s^{\text{ext}}.
\]  

(2.13)

Here, $\overrightarrow{\partial}^I$ denotes any $|I|$-order operator determined from the fields $\{\partial_a\}_{a=1,2,3}$, while $\partial_a^I$ denotes any a $|I|$-order operator determined from the fields $\{\partial_a\}_{a=1,2,3}$.
Proof. We only sketch the argument. We consider the parametrization \((s, r)\) of \(\mathcal{M}_{(s_0, +\infty)}\), and recall that on \(\mathcal{M}_{(s_0, +\infty)}\), \(s\) is constant and \(t = T(s, r)\), \(-1 + s^2/2 \leq r\). We consider the restriction of \(u\) on \(\mathcal{M}_{(s_0, +\infty)}\), that is, the function \(v_s(x) = u(T(s, r), x)\) and we remark the relations
\[
\partial_a v_s = \nabla_{T} u = (x^a/r) \frac{\partial t}{\partial r} \partial_t u + \frac{\xi_s(r)x^a}{t} \partial_t u + \partial_a u,
\]
\[
\partial_b \partial_a v_s = \partial_b \nabla_{T} u = \frac{\partial^2_t u}{\partial b^2} + \frac{\partial_t u}{\partial r} \frac{\partial_{rb}}{\partial r} u + \partial_{ab} u,
\]
\[
\Omega_{ab} v_s = (x^a \partial_b - x^b \partial_a) u = \Omega_{ab} u.
\]

Now we apply a Sobolev inequality for \(v_s\) adapted to a cone-like region (cf. [18] for details), and (2.12) is established, while (2.13) is established in the same manner. \(\square\)

2.5. A Sobolev inequality for the interior domain

Proposition 2.2 For all sufficiently regular functions defined in a neighborhood of the hypersurface \(\mathcal{M}'_{s_{int}}\), the following estimate holds
\[
t^{3/2} |u(x)| \lesssim \sum_{|J| \leq 2} \|L^J u\|_{L^2(\mathcal{M}'_{s_{int}})}, \quad x \in \mathcal{M}'_{s_{int}}.
\]

Proof. We only sketch the proof and refer to [18] for details. We consider the restriction \(v_s(x) := u(\sqrt{s^2 + r^2}, x)\) of the function \(u\) on the hyperboloid \(\mathcal{H}_s\) with \(|x| \leq -1 + s^2/2\). Then we have
\[
\partial_a v_s = \nabla_{s} u = t^{-1} L_{a} u = (s^2 + r^2)^{-1/2} L_{a} u.
\]

Take a \(x_0 \in \mathcal{H}_s\), with out loss of generality, we can suppose that \(x_0 = -3^{-1/2}(r_0, r_0, r_0)\). We consider the positive cone \(C_{s/2, x_0} \subset \{|x| \leq -1 + s^2/2\}\). In this cone we introduce the change of variable: \(y^a := s^{-1}(x^a - x_0^a)\) and we define \(w_{s, x_0}(y) := v_s(sy + x_0)\) for \(y \in C_{1/2, 0}\). Therefore, we obtain
\[
\partial_a w_{s, x_0} = s \partial_a v_s = \frac{s}{\sqrt{s^2 + r^2}} L_{a} u,
\]
\[
\partial_b \partial_a w_{s, x_0} = \frac{s^2}{s^2 + r^2} L_{b} L_{a} u - \frac{s^2 x^b}{s^2 + r^2} (s^2 + r^2)^{-3/2} L_{a} u.
\]

Thus, for \(|I| \leq 2\), we obtain \(|\partial^I w_{s, x_0}| \leq C \sum_{|J| \leq 2} |L^J u|\). Then by a Sobolev inequality adapted to cone-like region (cf [18]) we see that
\[
|w_{s, x_0}(0)|^2 \leq C \sum_{|I| \leq 2} \int_{C_{1/2, 0}} |\partial^I w_{s, x_0}|^2 dy = C s^{-3} \sum_{|J| \leq 2} \int_{C_{1/2, 0}} |L^J u|^2 dx,
\]
which leads to
\[
|u(x_0)| \leq C s^{-3/2} \|L^J u\|_{L^2(\mathcal{H}_s)}.
\]

On the other hand, when \(r_0 \geq 1\), we consider the cone \(C_{r_0/2, x_0}\) and we introduce the function
\[
\omega_{x_0}(y) := v_s(r_0 y + x_0), \quad y \in C_{1/2, 0}.
\]

It is clear that \(\partial_a \omega_{x_0} = \frac{r_0}{\sqrt{r_0^2 + s^2}} L_{a} u\) and
\[
\partial_b \partial_a \omega_{x_0} = \frac{r_0^2}{r_0^2 + s^2} L_{b} L_{a} u - r_0^2 x^b (s^2 + r^2)^{-3/2} L_{a} u.
\]

In the cube \(C_{r_0/2, x_0}\) one has \(r \geq \sqrt{3} r_0\). Thus for \(|I| \leq 2\) we find
\[
|\partial^I \omega_{x_0}| \leq \sum_{|J| \leq 2} |L^J u|.
\]
Then, by a Sobolev inequality adapted to a cone-like region [18], we have
\[
|u(x)|^2 = |w_{x_0}(0)|^2 \leq C \sum_{|j|\leq 2} \int_{C_{1/2,0}} |\partial^j w_{x_0}|^2 dy
= Cr_0^{-3} \sum_{|j|\leq 2} \int_{C_{1/2,0}} |L^j u|^2 dx,
\]
which leads us to
\[
|u(x)| \leq Cr_0^{-3/2} \sum_{|j|\leq 2} \|L^j u\|_{L^2(H_+)}.
\]
When \(r_0 \leq 1\), we have \(\sqrt{s^2 + r_0^2} \leq 2s\) and the desired result is proved. \(\square\)

3. Existence theory and application to the Einstein equations

The proposed method leads us to new results of global-in-time existence for coupled systems of nonlinear wave-Klein-Gordon equations. A particularly interesting model is provided by the coupling of a massive scalar field to the Einstein equations, that is,
\[
\square g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g) + 16\pi \left( -\partial_\alpha \phi \partial_\beta \phi + U(\phi) g_{\alpha\beta} \right),
\]
\[
\square \phi - U'(\phi) = 0,
\]
whose unknowns are a Lorentz metric \(g = (g_{\alpha\beta})\) (assumed to satisfy the wave gauge conditions) and a scalar field \(\phi\), while \(U(\phi) = c^2 \phi^2 / 2\). We refer to [18] for a precise statement of our asymptotic conditions in spatial directions which allow us to define the notion of \((\sigma, \epsilon, N)\)-asymptotically tame initial data set. Here the exponent \(\sigma \in (1/2, 1)\) measures the pointwise decay \(r^{-\delta}\) of the metric in spacelike directions, in comparison to the Minkowski metric denoted by \(g_M\). The following result is a generalization to self-gravitating massive matter of the nonlinear stability theory in [2,20].

**Theorem 3.1 (Nonlinear stability of Minkowski space for self-gravitating massive fields)** For all sufficiently small \(\epsilon > 0\) and all sufficiently large integer \(N\), one can find \(\sigma, \eta > 0\) depending upon \((\epsilon, N)\) so that the following property holds. Consider any \((\sigma, \epsilon, N)\)-asymptotically tame initial data set for the Einstein-massive field system (3.1), i.e.
\[
(M_0 \simeq \mathbb{R}^3, g_0, k_0, \phi_0, \phi_1),
\]
which is assumed to be sufficiently close to a flat and vacuum spacelike slice of Minkowski spacetime \((\mathbb{R}^{3+1}, g_M)\) in the sense that, in a global coordinate chart \(x = (x^0)\) with \(r = |x|\),
\[
\|(1 + r)^{\sigma + |I|/2} Z^I \partial (g_0 - g_{M,0})\|_{L^2(\mathbb{R}^3)} \leq \epsilon,
\]
\[
\|(1 + r)^{\sigma + |I|/2} Z^I k_0\|_{L^2(\mathbb{R}^3)} \leq \epsilon,
\]
\[
\|(1 + r)^{\sigma + |I|/2 + 1/2} Z^I \phi_0\|_{L^2(\mathbb{R}^3)} \leq \epsilon,
\]
\[
\|(1 + r)^{\sigma + |I|/2 + 1/2} Z^I \phi_1\|_{L^2(\mathbb{R}^3)} \leq \epsilon,
\]
for all \(Z^I = \partial^I \Omega^I\) with \(|I_1| + |I_2| \leq N\), in which \(g_{M,0} = (\delta_{ab})\). Then, the corresponding initial value problem associated with (3.1) admits a global solution and therefore determines a globally hyperbolic Cauchy development \((M, g, \phi)\), which is future causally geodesically complete and asymptotically approaches Minkowski metric.

We recall that a future causally geodesically complete spacetime, by definition, has the property that every affinely parameterized geodesic (of null or timelike type) can be extended toward the future (for all values of its affine parameter). In view of the theorem for compactly supported kinetic fields established in [5], it would be very natural to also extend our new method to the Einstein-Vlasov model.
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