Asymptotic behaviour of the $S$-stopped branching processes with countable state space

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Abstract: The starting process with countable number of types $\mu(t)$ generates a stopped branching process $\xi(t)$. The starting process stops, by falling into the nonempty set $S$. It is assumed, that the starting process is subcritical, indecomposable and noncyclic. It is proved, that the extinction probability converges to the cyclic function with period 1.

Keywords: branching processes; Markov chain; extinction probability; asymptotic behavior.

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1. Let us consider a measured state space $(X, \mathcal{A})$, where $\mathcal{A}$ is the $\sigma$-algebra on $X$. On this space we consider unbreakable homogenous Markov process with transition probability $P(t, x, A)$, where $t$ denotes time, $x \in X$ and $A \in \mathcal{A}$. Considering every trajectory of the given process as an evolution of the motion of a particle, $P(t, x, A)$ can be interpreted as a probability that a particle, which starts its motion from $x \in X$, falls into the set $A \in \mathcal{A}$ till the time $t$. It is assumed, that the time is discrete and the lifetime of a particle is equal to 1. At the end of its life the particle promptly gives rise to a number of offsprings, starting position of which are randomly distributed on the space $X$. The number and the position of these offsprings depends only on

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the position of the particle-ancestor at the transformation time point. Further every
offspring evolutes analogously and independently of other particles.

Let $\mu_{xt}(A)$ be a random measure, which for every $A \in \mathcal{A}$ is equal to the number of the
particles at time point $t$, types of which fall into set $A$, under condition that the process
started with one particle $x \in X$. $\mu_t(A)$ is a random measure equal to the number of
particles at the time $t$, which types are from the set $A$, but without any knowledge
about starting group of particles.

Further we assume, that the space $X$ consists of a countable number of elements
$x_1, \ldots, x_n, \ldots$. This means that the set of types of particles $\{T_1, \ldots, T_n, \ldots\}$ is count-
able.

Based on the measure $\mu_{xt}(A)$ we introduce the multivariate measure $\mu_{xt}(A)$

$$
\mu_{xt}(A) = \begin{cases} 
\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \mu_{x_{ij}}(x_m), & \text{if } x_m \in A \\
0, & \text{else}
\end{cases}
$$

where $x = \{x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2}, \ldots\}$, $x_{ij} \in X$ is the $j$-th element $i$-th type.

Let us denote $\mathcal{N}_0 = \{0, 1, 2, \ldots\}$, and respectively $\mathcal{N}_0^\infty$ is an infinite dimensional mea-
sured space with elements $x_i \in \mathcal{N}_0$.

Having $P(t, x, A)$ let us introduce $\hat{P}(t, x, y)$, $(x, y \in \mathcal{N}_0^\infty)$, where $\hat{P}$ is a probability
that we obtain vector $y$ till time $t$, assuming that we started from $x$. $\hat{P}$ could be rewriten in terms of $\mu_{xt}$ as

$$
\hat{P}(t, x, y) = P\{\mu_{xt}(X) = y\}.
$$

Let $E(i) = (\delta_{i1}, \ldots, \delta_{im}, \ldots)$, where $\delta_{ij}$ is the Kronecker symbol, $\delta_{xy} = \prod_{i=1}^{\infty} \delta_{x_{ij}}$, $E(i)$
is the particle of the $i$-th type. We also assume, that $a^b = a_1^b a_2^b \cdots a_n^b \cdots$, $a! = a_1 a_2 \cdots a_n ! \cdots$, $\overline{a} = a_1 + \cdots + a_n + \cdots$, $a_i^{[b]} = a_i(a_i - 1) \cdots (a_i - b_i + 1)$.

**Definition 1. Functional**

$$
F(s(\cdot)) = F(s) = \mathbb{E} \exp \left\{ \int \ln s(x) \mu(dx) \right\}
$$
is called a generated functional of the random measure $\mu$, where $s(x)$ is a measured
bounded function.

Generated functional $F(s)$ is always defined, when $0 < |s(x)| \leq 1$ and integral $\int \ln s(x) \mu(dx)$
exists.

For our process the generated functional is given as

$$
h(t, s(\cdot)) = \mathbb{E} \exp \left\{ \int_X \ln s(z) \mu_t(dz) \right\}.
$$
where $\mu_t$ is the same multivariate measure as $\mu_{xt}$ but not taking into account any combination of the starting position of the process. Further we will consider the case $s(\cdot) = \text{const} = s = (s_1, s_2, \ldots)$. It is easy to check whether the introduced generated functional is generated, as in the case of finite number of types (in that case it is not a functional but a function).

Let us denote

$$
\begin{align*}
  h^i(t, s) &= h^{E(i)}(t, s), \\
  h^{\beta}(t, s) &= ((h^{E(1)}(t, s))^\beta_1, (h^{E(2)}(t, s))^\beta_2, \ldots), \\
  h(t, s) &= (h^{E(1)}(t, s), h^{E(2)}(t, s), \ldots).
\end{align*}
$$

It is proved in [3], that the introduced generated function follows the main functional equation ($\forall t, \tau = 0, 1, 2, \ldots$)

$$
  h(t + \tau, s) = h(t, h(\tau, s)).
$$

Let us fix the finite subset $S \subset \mathbb{N}_0^\infty$, $0 \notin S$. *Stopped* or $S$-stopped multitype branching process is the process $\xi_{xt}(X)$, defined for $t = 1, 2, \ldots$ and $x \in \mathbb{N}_0^\infty$ by equations

$$
\xi_{xt}(X) = \begin{cases} 
  \mu_{xt}(X), & \text{if } \forall v, 0 \leq v < t, \mu_{xe}(X) \notin S \\
  \mu_{xu}(X), & \text{if } \forall v, 0 \leq v < u, \mu_{xe}(X) \notin S, \mu_{xu}(X) \in S, u < t.
\end{cases}
$$

From this, for the $S$-stopped process $\xi_{xt}(X)$, points of the set $S$ are additional states of absorption compared to the process $\mu_{xt}(X)$. The latter had only one point of absorption 0. In contrast to the process $\mu_{xt}(X)$, in the $S$-stopped branching process $\xi_{xt}(X)$ single particles in generation $t$ multiplies independently following probability law defined by the generated functional $h(\cdot)$, only if $\xi_{xt}(X) \notin S$. If the random vector $\xi_{xt}(X)$ falls into the set $S$, the evolution of the process stops.

Since the process $\mu_{xt}(X)$ is a Markov chain, then

$$
\tilde{P}(t_1 + t_2, \alpha, \beta) = \sum_{\gamma \in \mathbb{N}_0^\infty} \tilde{P}(t_1, \alpha, \gamma) \tilde{P}(t_2, \gamma, \beta).
$$

For further needs, we also consider probabilities $\tilde{P}(t, \alpha, r)$, defined as

$$
\tilde{P}(t, \alpha, r) = \begin{cases} 
  \tilde{P}(1, \alpha, r), & t = 1; \\
  \sum_{\beta \notin S} \tilde{P}(1, \alpha, \beta) \tilde{P}(t - 1, \beta, r), & t \geq 2.
\end{cases}
$$

(1)

It is easy to see, that $\tilde{P}(l, \alpha, r)$ is a conditional probability of the event

$$
\{\mu_{\alpha l}(X) = r\} \cap \left( \bigcap_{l' = 1}^{l-1} \{\mu_{\alpha l'}(X) \notin S\} \right).
$$
Let

\[ q^n_r(t) = P\{ \xi_{nt}(X) = r \} \]

be the probability of an extinction of the \( S \)-stopped branching process \( \xi_{xt}(X) \) into state \( r \in S \) till time \( t \), starting from state \( n \in \mathbb{N}_0 \).

2. Main facts.

**Theorem 1.** For any \( n \not\in S, n \neq 0, r \in S, t \geq 1 \) holds

\[ q^n_r(t) = \sum_{\alpha \in S} \sum_{l=1}^t c_{\alpha r}(t, l) \tilde{P}(l, n, \alpha), \tag{2} \]

where coefficients \( c_{\alpha r}(t, l) \) can be found from

\[ c_{\alpha r}(t + 1, l + 1) = c_{\alpha r}(t, l), \tag{3} \]
\[ c_{\alpha r}(t + 1, 1) = \delta_{\alpha r} - \sum_{l=1}^{t-1} \tilde{P}(l, \alpha, r), \tag{4} \]
\[ c_{\alpha r}(1, 1) = \delta_{\alpha r}. \tag{5} \]

**Proof.** Let

\[ \tau = \min \{ t : \mu_{nt}(X) \in S \} \]

be the moment of the first fall into \( S \), then for \( t \geq l \)

\[ P\{ \xi_{nt}(X) = r, \tau = l \} = P\{ \xi_{nt}(X) = r \} = \tilde{P}(l, n, r). \]

Applying (1) to \( \tilde{P}(l, n, r) \), \( l \geq 2 \), we get

\[ \tilde{P}(l, n, r) = \sum_{\alpha \in S} \tilde{P}(1, n, \alpha) \tilde{P}(l - 1, \alpha, r) = \sum_{\alpha \in S} \tilde{P}(2, n, \alpha) \tilde{P}(l - 2, \alpha, r) - \sum_{\alpha \in S} \tilde{P}(1, n, \alpha) \tilde{P}(l - 1, \alpha, r). \]

The first sum on the right hand side of this formula can be transformed similarly

\[ \sum_{\alpha \in S} \tilde{P}(2, n, \alpha) \tilde{P}(l - 2, \alpha, r) = \sum_{\alpha \in S} \tilde{P}(3, n, \alpha) \tilde{P}(l - 3, \alpha, r) - \sum_{\alpha \in S} \tilde{P}(2, n, \alpha) \tilde{P}(l - 2, \alpha, r). \]

Making the same transformations in the sum \( \sum_{\alpha \in S} \tilde{P}(i, n, \alpha) \tilde{P}(l - i, \alpha, r) \), we get

\[ \tilde{P}(l, n, r) = \tilde{P}(l, n, r) - \sum_{\alpha \in S} \sum_{i=1}^{l-1} \tilde{P}(l - i, n, \alpha) \tilde{P}(i, \alpha, r), \tag{6} \]
\[ l = 2, \ldots, t, \quad \tilde{P}(l, n, r) = \tilde{P}(1, n, r). \tag{7} \]

As \( q^n_r(t) = \sum_{l=1}^t \tilde{P}(l, n, r) \), from formulas (6), (7) we get (3), (4), (5).

\[ \square \]
Late on we will consider the process similarly to [1]. Let

\[ A_1(x, D) = E\{\xi_{x1}(D)\} \]

be the first factorial moment, where \( \xi_{x1}(D) \) is such a random measure, which for each \( D \in \mathcal{A} \) is equal to the number of particles at time point 1, which types belong to set \( D \), conditional on \( S \)-stopped process. It also taken into account that at the beginning there was only one particle of the type \( x \in X \), what means \( \xi_{x1}(D) = \sum_{i=1}^{\infty} \xi_{x1}(D) \). From the linearity of \( E \) we have \( A_1(x, D) = E\{\xi_{x1}(D)\} = \sum_{i=1}^{\infty} A_1(x_i, D) \). It is important that \( D \) could be a vector or a set.

**Definition 2.** Let \( A_1(x, D) = A(x, D) \) and

\[ A_{n+1}(x, D) = \int_X A_n(y, D) dA(x, y) = \int_X A(y, D) dA_n(x, y). \]

It is assumed, that \( A_0(x, D) = 1 \), if \( x \in D \) and \( A_0(x, D) = 0 \) else.

In [4] it is proved, that iterations of the operator \( A \) coincide with the first moments of \( \xi \). This means, that for matrix of the linear operator \( A(t) \), with \( A_{ij}(t) = A_t(x_i, x_j) \), it holds that \( A(t) = A^t \) will take place, where \( A = A(1) \).

Let

\[ B_t(x, D_1, D_2) = E\{\xi_{xt}(D_1) \cdot \xi_{xt}(D_2) - \xi_{xt}(D_1 \cap D_2)\} \]

be the second factorial moment.

For further work we have to introduce some definitions, describing classes of branching processes (see [3]).

**Definition 3.** Branching process in which all types form a single class of equivalent types is called indecomposable. All other processes are called decomposable. Branching process is called fully indecomposable if the set of types could be split-up into two nonempty closed sets.

**Definition 4.** An indecomposable discrete time branching process is called cyclic with period \( d \), if the greatest divisor for all \( t \), such that \( \langle A_t(x_i, x_i) \rangle > 0 \), is equal to \( d \). If \( d = 1 \) then the process is called noncyclic.

**Definition 5.** An indecomposable discrete time branching process is called subcritical, if the largest eigenvalue (Perron’s root) \( \delta \) of the matrix \( A \) is smaller than 1, supercritical, if \( \delta > 1 \) and critical if \( \delta = 1 \) and \( f(x_i)B^i_{jk} \nu(x_j)\nu(x_k) > 0 \), where \( B^i_{jk} \) is the matrix of the operator \( B \), and \( f \) and \( \nu \) eigenfunction and invariant measure respectively which correspond to the Perron’s root \( \delta \).

**Assumption 1.** The kernel \( E\xi_{xt}(S) \) is assumed to be indecomposable, noncyclic and subcritical.
Correspondingly to the assumption \( \text{1} \) the operator \( A \), which is defined by the kernel \( E \{ \xi \} \) in the space of measurable functions and in the space of measures, has the eigenfunction \( f(\cdot) \) and the invariant measure \( \nu(\cdot) \), such that

\[
\int_X f(y) A_t(x, dy) = f(x) = \sum_{i=1}^{\infty} f(y_i) A_t(x, y_i),
\]

\[
\int_X A_t(x, Y) \nu(dx) = \nu(Y) = \sum_{i=1}^{\infty} A_t(x_i, Y) \nu(x_i).
\]

Further we assume, that \( 0 < x_1 < f(x) < x_2 < \infty, \nu(X) < \infty \) and

\[
\int_X f(y) \nu(dy) = 1 = \sum_{i=1}^{\infty} f(y_i) \nu(y_i). \tag{8}
\]

The operator induced by the above defined kernel in the space of bounded functions has \( \{1\} \) as an isolated point of the spectrum.

**Assumption 2.** We assume \( E\{\mu_{E(j)}(x_i) \log \mu_{E(j)}(x_i)\} \) is finite for \( \forall i, j = 1, 2, \ldots \).

**Assumption 3.** The expansion \( A_t(x, y) = \sum_k f(x_k) \delta^k_t \nu(y_k) \) exists.

As in indecomposable, noncyclic, subcritical processes with discrete time all absolute values of eigenvalues are less than one, then based on the assumption \( \text{3} \) we can conclude, that when \( t \to \infty \)

\[
A_t(x_i, y_j) = f(x_i) \delta^t \nu(y_j) + o(\delta^t),
\]

where \( \delta \) is the largest eigenvalue. Thus

\[
\lim_{t \to \infty} A_t(x_i, y_j) \delta^{-t} = f(x_i) \nu(y_j). \tag{9}
\]

Let us denote

\[
R^i(t, s) = 1 - h^i(t, s),
\]

\[
R(t, s) = (R^1(t, s), \ldots, R^n(t, s), \ldots),
\]

\[
R(t, 0) = Q(t) = (Q^1(t), \ldots, Q^n(t), \ldots) = \lim_{s \to 0} R(t, s).
\]

As in the case with the finite number of types, the following inequalities could be easily proved (see [3])

\[
0 \leq R^i(t, s) \leq Q^i(t) \quad 0 < |s| \leq 1,
\]

\[
|R^i(t, s)| \leq 2Q^i(t) \quad 0 < |s| \leq 1. \tag{11}
\]

(11) implies that for the degenerating branching processes \( R^i(t, s) \) converges uniformly to zero on \( 0 < |s| \leq 1 \).

We need following technical assumption on the process
Assumption 4. Let $A^t > 0$ for some $t > 0$ in the sense $\forall i, j \ a_{ij} > 0$ and $h^t(t, s) \neq A_{ij}(t)$.

Hereafter the notation $A = \{a_{ij}\} > 0$, means that $a_{ij} > 0 \ \forall i, j$, and the notation $A > B$, where $A = \{a_{ij}\}, B = \{b_{ij}\}$ are matrices, means that $a_{ij} > b_{ij} \ \forall i, j$.

Let $h(s) = h(1, s)$.

Assumption 5. Following the above defined assumptions for this process, it holds that

$$1 - h(s) = [A - E(s)](1 - s), \quad (12)$$

where matrix $E(s)$ with $0 \leq s \leq s' \leq 1$ satisfies conditions $0 \leq E(s') \leq E(s) \leq A$ and $\lim_{s \to 1} E(s) = 0$.

Theorem 2. With Assumptions 3-5

$$\lim_{t \to \infty} \frac{R^i(t, s)}{f(x_k)R^k(t, s)} = \nu(x_i)$$

uniformly on all $s \neq 1, 0 \leq s \leq 1$.

This theorem is proved analogically to theorem 1 on page 192 in [3], by replacing the right and left eigenvectors by eigenfunction and invariant measure respectively. Matrices are from the class of matrices of infinite measurable linear operator.

Theorem 3. By assumptions 1-5 for any $i, j = 1, 2, \ldots$ and for $l \to \infty$ probability that the process extinct to 0 from one particle of type $j$ over $l$ is

$$1 - \hat{P}(l, \mathcal{E}(j), 0) = K(S_j)\delta(1 + o(1)), \quad K(S_j) > 0; \quad (13)$$

a) the limit of the conditional probabilities exists

$$\lim_{l \to \infty} P\{\mu_n(X) = k| n \neq 0\} = p^*_k, \quad (14)$$

and the generating function $h^*(s) = \sum_{k \in \mathbb{N}^\infty} p^*_k s^k$ is not depending on $n$ and satisfies the relationships

$$1 - h^*(h(\cdot)) = \delta(1 - h^*(s)),$$

$$h^*(0, \ldots, 0, \ldots) = 0, \quad h^*(1, \ldots, 1, \ldots) = 1; \quad (15)$$

b) distribution $p^*_k$ has positive expectation

$$h^*_j(1) = \lim_{s \to 1} h^*_i(s) = \sum_{k \in \mathbb{N}^\infty} k_j p^*_k,$$

where $h^*_j(s) = \frac{\partial h^*_i(s)}{\partial s_j}.$
It is proved by mimicking the theorem 3 on page 198 from [3] with the use of theorem 2 for the representation of the limit of the generating function of the conditional distribution by getting result similar to one in [2].

Let us fix one more assumption

**Assumption 6.** Let \( h_{ij}(s) = \frac{\partial h_i(s)}{\partial s_j} \), then for all \( j, \ 1 \leq j < \infty \) there exists such \( i, \ 1 \leq i < \infty \), that \( h_{ij}(0) \) are positive.

From the equality
\[
h_{ij}(0) = \hat{P}(0, \mathcal{E}(i), \mathcal{E}(j)) = P\{ \mu_{\mathcal{E}(i)}(X) = \mathcal{E}(j) \}
\]
this means, that the corresponding probabilities \( \hat{P}(0, \mathcal{E}(i), \mathcal{E}(j)) \) are positive.

To proceed further we need following lemma

**Lemma 1.** Under the assumptions 1-6, the limit of conditional probabilities is positive, for all \( i = 1, 2, \ldots \)
\[
\lim_{t \to \infty} P\{ \mu_n(X) = \mathcal{E}(i) | n \neq 0 \} = p_{\mathcal{E}(i)}^* > 0,
\]

**Proof.** The generating function \( h^*(s) = \sum_k p_k^* s^k \) in Theorem 3 satisfies the equation (15). If we replace in this equation \( s \) by \( h(s) \), and repeat this replacement \( t \) times, we get the equality
\[
1 - h^*(h(t, s)) = \delta^t(1 - h^*(s)),
\]
where \( h(t, s) \) is \( t \)-th iteration of the function implied by the main differential equation. By differentiating (16) with respect to \( s_j \) at \( s = 0 \), we obtain
\[
\sum_{i=1}^{\infty} h_i^*(h(t, 0)) h_{ij}(t, 0) = \delta^t h_j^*(0) = \delta^t p_{\mathcal{E}(j)}^*.
\]

As all coordinates of \( h(t, 0) \) converge to 1, for \( t \to \infty \), then by the theorem 8 we can find such \( T \) and \( C_1 \), that \( h_i^*(h(t, 0)) \geq C_1 > 0 \) for \( t > T \). According to the assumption 6 this implies that for all \( 1 \leq j \leq \infty \) we can found such \( i \), that \( h_{ij}(t, 0) > 0 \). For all \( i_1, i_2, \ldots, i_{t+1} \) holds
\[
h_{i_1 i_2 + 1}(t, 0) \geq \prod_{l=1}^{t} h_{i_1 i_{l+1}}(0).
\]

Thus (17) implies
\[
\delta^t p_{\mathcal{E}(j)}^* \geq C_1 \sum_{i=1}^{t} h_{ij}(t, 0) > 0, \forall 1 \leq j \leq \infty.
\]

\[ \square \]
Theorem 4. By the assumption the limiting extinction probabilities \( q^n_r(t) = \lim_{t \to \infty} q^n_r(t), \forall n \not\in S, r \in S \), can be written in the series representation

\[
q^n_r = \sum_{l=1}^{\infty} \sum_{\alpha \in S} c_{\alpha r} \tilde{P}(l, n, \alpha),
\]

where \( c_{\alpha r} = \lim_{t \to \infty} c_{\alpha r}(t, l) = \delta_{\alpha r} - \sum_{u=1}^{\infty} \tilde{P}(u, \alpha r). \)

Proof. Probabilities \( q^n_r(t) \) increase with \( t \) and are bounded above by 1. Then the limit \( q^n_r = \lim_{t \to \infty} q^n_r(t) \) exists.

We can pass to the limits on the left and on the right hand sides of the formula (2), when \( t \to \infty \), as for all \( \alpha, r \in S \) holds that \( \tilde{P}(l, \alpha, r) \leq \hat{P}(l, \alpha, r) \) and Chebyshev inequality and assumption 3 imply that

\[
\hat{P}(l, \alpha, r) \leq P\left\{ \sum_{j=1}^{\infty} \mu_{\alpha l}(\mathcal{E}(j)) \geq 1 \right\} \\
\leq \sum_{j=1}^{\infty} E\{\mu_{\alpha l}(\mathcal{E}(j))\} \\
= \sum_{i=1}^{\infty} \alpha_i \sum_{j=1}^{\infty} d_{ij} \delta^i (1 + o(1)).
\]

This means that series \( \sum_l \tilde{P}(l, \alpha, r) \) and \( \sum_l \hat{P}(l, \alpha, r) \) converge to each other. This implies (18).

As in [2] let us consider the asymptotic behavior of \( q^n_r \) for \( \pi \to \infty \).

Theorem 5. Let assumptions are fulfilled and \( \lim_{\pi \to \infty}(n_i/\pi) = a_i \), where \( a = (a_1, a_2, \ldots) \). In this case for \( r \in S \) and \( \pi \to \infty \)

\[
q^n_r - H(\log_\delta \pi) \to 0,
\]

where \( H(x) \) is a cyclic function with period 1, defined through the following equalities

\[
H(x) = \sum_{j=1}^{r_0} c_j H_j(x),
\]

\[
H_j(x) = \sum_{L=-\infty}^{\infty} \delta^{j(L+x)} e^{-\langle a, K \rangle \delta^{L+x}},
\]

where \( \langle a, K \rangle = \sum_{i=1}^{\infty} a_i K_i, K_i \) as in [12], \( r_0 = \max\{r \mid r = r_1 + r_2 + \ldots : r \in S\} \). Constants \( c_j = c_j(r, a, p^*) \) depend on \( r, a \) and the limit distribution \( p^* = \{p^*_r\} \) which is defined in lemma [7].
Proof. Let \( \theta(l) = (\theta_1(l), \theta_2(l), \ldots) \) be a random vector, which components \( \theta_i(l) \) are equal to the number of particles of type \( i \) which give an offspring to the \( l \)-th generation. Thus we can write, that for all \( \alpha \in S, \ l \geq 1 \ \ \ n \notin S \), we have

\[
\hat{P}(l, n, \alpha) = \sum_{\{\beta: 1 \leq \beta \leq \alpha\}} P\{\mu_{n,l}(X) = \alpha, \ \theta(0, l) = \beta\} = \sum_{\{\beta: 1 \leq \beta \leq \alpha\}} P\{\theta(0, l) = \beta\} P\{\mu_{\beta l}(X) = \alpha | \theta(0, l) = \beta\}.
\]

Under the assumptions of the theorem 5

\[
P\{\theta(0, l) = \beta\} = \prod_{i=1}^{\infty} \left( \frac{n_i}{\beta_i} \right) \left( \hat{P}(l, \mathcal{E}(i), 0) \right)^{n_i - \beta_i} (1 - \hat{P}(l, \mathcal{E}(i), 0))^{\beta_i} = \frac{\alpha^{l\beta}}{\beta!} K^\beta \delta(l\beta) e^{-(a,K)\overline{m}^{l+1/o(1)}} (1 + o(1)),
\]

and the probability, not depending on \( n \)

\[
P\{\mu_{\beta l}(X) = \alpha | \theta(0, l) = \beta\}
\]

\[
= \sum_{\{\alpha^{(j)}\}} \prod_{k=1}^{\infty} \prod_{j=1}^{\beta_k} P\{\mu_{\mathcal{E}(k), l}(X) = \alpha^{(j)} | \mathcal{E}(k) \neq 0\}
\]

\[
\rightarrow \sum_{\{\alpha^{(j)}\}} \prod_{k=1}^{\infty} \prod_{j=1}^{\beta_k} p^{*}_{\alpha^{(j)}} \ l \rightarrow \infty,
\]

where \( \mu_{\mathcal{E}(k), l}(X) \) are branching processes, whith the same distribution as \( \mu_{\mathcal{E}(k), l}(X) \).

The summation in \( \sum_{\{\alpha^{(j)}\}} \) is done over all such \( \alpha^{(j)} \), which \( \sum_{k=1}^{\infty} \sum_{j=1}^{\beta_k} \alpha^{(j)} = \alpha \).

The statements in (20)-(22) imply, that the general component of the series (18) for \( \overline{\pi} \rightarrow \infty, \ l \rightarrow \infty \) can be written in the form

\[
(1 + o(1)) \sum_{\alpha \in S} \sum_{\{\beta: 1 \leq \beta \leq \alpha\}} g(\alpha, \beta) \sum_{l=0}^{r_0} \delta(l + \log_2 \overline{\pi}) \beta \times \exp \left\{ -(a, K)\delta^{l+\log_2 \overline{\pi}} (1 + o(1)) \right\},
\]

where \( g(\alpha, \beta) \) is an independent of \( n \) and \( l \) function. It is easy to see that in formula (18) for \( \overline{\pi} \rightarrow \infty \) each component of series with any \( l \geq 1 \) converges to zero.

Let us choose \( L_1 < L_2 \) in such way that sums

\[
\sum_{L_1}^{L_2} \delta L e^{-(a,K)\delta x} \quad \text{and} \quad \sum_{L=-\infty}^{L_2} \delta L e^{-(a,K)\delta x}
\]

are small. We set \( l_i + \log_2 \overline{\pi} = l_i + x_i \overline{\pi} \), for \( i = 1, 2 \), where \( 0 \leq x_i \overline{\pi} \leq 1 \). (23) and (24) imply, that we can choose such \( L_1, L_2 \) and \( n_0 \), that tails of the sum in (18), bounded
from 1 to \( t_1 \) and from \( t_2 \) to infinity, are less then \( \varepsilon/2 \), where \( \varepsilon > 0 \) is small. Elements of the series \( \sum_{k=1}^{l_2} \) with \( l_1 < l < l_2 \) can be replaced by a limited expressions \( \sum_{k=1}^{l_2-1} \) for \( \alpha \rightarrow \infty \) as well as for \( l \rightarrow \infty \). The number of summands in the sum \( \sum_{k=1}^{l_2-1} \) in expression \( \sum_{k=1}^{l_2} \) is finite \( l_2 - l_1 - 1 = L_2 - L_1 - 1 \). This means that \( n_0 \) can be chosen in such a way, that for all \( n > n_0 \) the approximation error will be also less than \( \varepsilon/2 \). This implies the statement of the theorem, while \( \varepsilon > 0 \) is any real number.  

From the theorem it cannot be concluded directly, whether the coefficients \( c_j \) in the formula \( \sum_{k=1}^{l_2} \) are such, that \( H(x) > 0 \). For this we introduce the next lemma.

**Lemma 2.** Under assumptions \( \sum_{k=1}^{l_2} \), there exists such a constant \( \Theta > 0 \), that for some number \( n_0 \)

\[
q^n_t > \Theta, \text{ for } \forall n \text{ with } n \geq n_0 \text{ and } \forall r \in S.
\]

**Proof.** As for any \( t, q^n(t) = \lim_{t \rightarrow \infty} q^n(t) \geq q^n_t(t) \), it is enough to prove, that the inequality \( q^n(t) \geq \Theta > 0 \) holds for any large enough \( t \), for all \( r \in S \) and \( n \) from \( \alpha \geq n_0 \). Let us use the upper defined random vector \( \theta(0, t) \) and introduce one more random vector \( \theta'_1(t-1) = (\theta'_1(0), \theta'_2(0), \ldots) \), where \( \theta'_1(t) \) is the number of starting particles of \( i \)-th type, from \( \alpha \)-th generation. For \( n = (n_1, n_2, \ldots) \), \( n_1 \geq r_0 + 1 \), where \( r_0 = \max_{r \in S} r \), we use the inequality

\[
q^n_t \geq P\{\mu_n(X) = r, \theta'(t) = (r_0 + 1 - r), \theta(0, t) = rE(1)\}. \tag{25}
\]

The right side of \( \sum_{k=1}^{l_2} \) we write as a product of \( P_1(n, t)P_2(t) \), where \( P_1(n, t) = \sum_{k=1}^{l_2} \) and depends on \( n \) and \( t \), but \( P_2(t) = \sum_{k=1}^{l_2} \) depends only on \( t \). From the definition of the random vectors \( \theta(0, t) \) and \( \theta'(t-1) \) we have, that

\[
P_1(n, t) = \left(\frac{n_1}{n_1 - r_0 - 1}\right)^{n_1 - r_0 - 1}\left(1 - \hat{P}(t-1, \mathcal{E}(1), 0)\right)^r \times \prod_{i=1}^{\infty} \hat{P}(1, \mathcal{E}(1), 0)^{n_1} \sum_{k=1}^{l_2} P\{\mu'_r(X) = 0 \mid r_0 + 1 - r \neq 0\}; \tag{26}
\]

\[
P_2(t) = \prod_{k=1}^{l_2} \prod_{j=1}^{r_k} P\{\mu^{(jk)}_{\mathcal{E}(1), t}(X) = \mathcal{E}(k) \mid \mathcal{E}(k) \neq 0\}. \tag{27}
\]

Here \( \mu, \mu', \mu^{(jk)} \) are branching processes, whose evolution is defined by a generating function \( h(s) = (h_1(s), h_2(s), \ldots) \). Setting \( t \rightarrow \infty \), in such a way, that \( \alpha \rightarrow 0 \), for \( \alpha \rightarrow 0 \), we get in the right side of the equality \( \sum_{k=1}^{l_2} \) a positive constant multiplied
by a conditional probability, which stays at the end of formula. Using the limiting relationship $P\{\mu'(X) = k | k \neq 0\} \to p^*_k$, of the theorem 3 and the equality

$$\sum_{k \in \mathbb{N}_0} p^*_k(\hat{P}^{k_1}(1, \mathcal{E}(1), 0)\hat{P}^{k_2}(1, \mathcal{E}(2), 0) \cdots) = h^*(h(0))$$

we have that this conditional probability is equal in limit to $h^*(h(0))$. Expression (27) does not depend on $n$ and is equal to the product $\prod_{i=1}^{\infty}[p^*_{\mathcal{E}(i)}]^{r_i}$, for $t \to \infty$. From lemma 1 this product is positive. That completes the proof.

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