Computational algorithms for modeling systems with piecewise constant parameters

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Abstract. Modeling systems with piecewise constant parameters imposes special requirements on the computational algorithms used. In places where the parameters are discontinuous, the model variables may experience discontinuities or smoothness disturbances. In this case, the conditions for matching the values of the variables on both sides of the discontinuity boundary, reflecting the physical conditions at this boundary, must be satisfied. The paper proposes computational algorithms for modeling these systems, based on the formulation of the initial initial-boundary value problem in the form of a system of partial differential equations for generalized functions. In this case, the initial conditions, conditions on the external and internal boundaries are included in these equations in a weak form. The latter circumstance makes it possible to avoid the need to subordinate the basis functions, according to which the desired solution is expanded, to the conditions on the outer and inner boundaries, which is an important circumstance, especially for problems with many spatial variables. The paper presents a solution to the generalized Riemann problem with conditions on the outer or inner boundary for a differential equation with second-order derivatives with respect to spatial variables. The solution to the Riemann problem is based on the formulation of the problem in the form of a partial differential equation for generalized functions and the construction of a fundamental solution to the problem operator. When constructing a computational algorithm, the solution of the generalized Riemann problem is used on the inner and outer boundaries. The proposed technique allows, by using high-degree polynomials as basis functions, to build computational algorithms of a high order of approximation.

1. Introduction

Numerical modelling of systems in which the parameters characterizing the physical properties of the medium are piecewise constant and, therefore, abruptly change when crossing the interface between the parts of the system, is undoubtedly an urgent problem and occurs in a variety of applied areas. For example, when simulating the propagation of elastic or electromagnetic disturbances in layered media or media consisting of parts with different physical properties, the behaviour of elastically supported bearing structures consisting of parts with different properties of the materials used, heat propagation in composite materials, and many others. Also, it is a very common technique when, in mathematical modelling, the system is divided into subdomains, within each such subdomain, the system parameters...
are assumed to be constant and abruptly change when crossing the boundary separating the parts. We would especially like to highlight the case when the system is under the action of external forces, which are transmitted through a small in comparison with the size of the system "contact patch", such as in the case of a bearing structure, supported on elastic supports. This system also belongs to the considered ones – the elastic properties of the supports on which the structure is located change abruptly.

At the boundaries separating the parts of the system, at which the parameters of the system change abruptly, the smoothness conditions for the variables of the mathematical model may be violated. These jumps of variables or their derivatives cannot be arbitrary but must obey conditions reflecting the physical conditions at the interface between the media. For example, there must be a continuous flow of heat through the boundary of the media, or Newton's third law must be fulfilled: the force with which one of the contacting media acts on the other is equal and oppositely directed to the force with which the second acts on the first. These physical conditions must also be met for the model variables. Otherwise, the conservatism of the model will be violated, and the numerical model will not be adequate to the investigated physical situation.

One of the most frequently used algorithms in the numerical simulation of systems with piecewise constant parameters is the semi-discrete Galerkin method and its various modifications, in which the physical conditions at the interface between media are included in the model equations in a "weak" form [1]-[7]. This approach makes it possible to approximately satisfy the conditions at the interface between the media with an accuracy corresponding to the accuracy of the numerical model.

A feature of the Galerkin method is that it leads to implicit computational algorithms at each time step, it is necessary to solve a system of algebraic equations, which significantly reduces the speed.

In this work, on the example of the problem of vibrations of an elastic string supported on intermediate supports, we propose a unified approach to the numerical modeling of systems with piecewise constant parameters, which allows one to construct explicit numerical algorithms. This approach is based on the formulation of the initial-boundary value problem in the form of a system of partial differential equations for generalized functions, in the construction of the fundamental solution of the operator of the problem and the representation of the solution of the problem as a compression of the fundamental solution with the right-hand side of the system of equations for generalized functions.

2. Problem statement
The methods for constructing numerical models outlined below will be demonstrated by the example of a model of vibration of an elastic string with free ends, supported on several intermediate elastic supports, each of which is in contact with the string through a small "contact patch".

Let us denote \( u(t,x) \) the vertical coordinates of the position of the string points, \( \rho(x) \) piecewise constant string density, \( \kappa(x) \) piecewise constant string elasticity, \( \eta_i \) coefficient of elasticity of the \( s \)-th intermediate support. Then the mathematical model has the form of the initial-boundary value problem

\[
\rho \frac{\partial^2 u}{\partial t^2} - \kappa \frac{\partial^4 u}{\partial x^4} = -\rho g - \eta u; \quad x \in \Omega = (-l,l) \\
\frac{\partial u}{\partial x}(t,x = -l) = \frac{\partial u}{\partial x}(t,x = l) = 0; u(t = 0,x) = u_0(x); \quad \frac{\partial u}{\partial t}(t = 0,x) = u_1(x)
\]

(1)

\( \eta(x) \) – piecewise constant coefficient of elasticity of intermediate supports. In the area of the "contact patch" of the \( s \)-th intermediate support, it is equal to \( \eta_i \), and outside these areas it is zero. The equations are satisfied everywhere, except for the points of discontinuity of the coefficients of the equation. At the points \( x' \) of discontinuity of the parameters of the equation, the conjugation conditions must be satisfied:
\[ u(t, x^- - 0) = u(t, x^+ + 0), \]  
reflecting the condition of string continuity, and the condition reflecting the fulfillment of Newton's third law (the forces acting on the boundary from both sides are equal in magnitude and oppositely directed).

\[ \kappa(x^+ - 0) \frac{\partial u}{\partial x}(t, x^- - 0) = \kappa(x^- + 0) \frac{\partial u}{\partial x}(t, x^+ + 0) \]  

3. Generalized Riemann problem with additional conditions on the boundary

In the further presentation, we will use the concepts and statements of the theory of generalized functions of slow growth, a presentation of which can be found, for example, in [5] or [11]. Let us define the space of basic vector-functions \( S(R^N) \). The elements of this space will be the \( M \)-dimensional vector-functions \( \varphi = (\varphi_1, ..., \varphi_M) \), whose components \( \varphi_i(x), ..., \varphi_M(x) \) belong to the space \( S(R^N) \), which consists of functions \( C^\infty(R^N) \), decreasing at \( |x| \to \infty \) together with all derivatives faster than any degree \( |x|^{-1} \).

Generalized vector-functions \( f = (f_1, ..., f_M) \in S^* \) are linear continuous functionals on the vector base space \( S(R^N) \), and the functional \( f \) acts on the base vector-function \( \varphi = (\varphi_1, ..., \varphi_M) \) by the formula

\[ (f, \varphi) = (f_1, \varphi_1) + ... + (f_M, \varphi_M). \]

The algorithms proposed in this work are based on the solution of the generalized Riemann problem with additional conditions on the boundary. In papers [4, 5, 8], the statement and solution of the generalized Riemann problem with additional conditions on the boundary are given as applied to systems of partial differential equations of the first order. Below is a solution to the generalized Riemann problem for equations with second-order derivatives.

In the application to the problem under consideration, the generalized Riemann problem with additional conditions on the boundary is formulated as follows. To find a solution to the Cauchy problem for the equation

\[ \rho(x) \frac{\partial^2 u}{\partial t^2} - \kappa(x) \frac{\partial^2 u}{\partial x^2} = 0 \]

\[ u(t = 0, x) = \sigma(x); \quad \frac{\partial u}{\partial t}(t = 0, x) = \gamma(x) \]  

Equation coefficients are constant in the left and right half-planes \( \rho(x < 0) = \rho^-, \rho(x > 0) = \rho^+ \), \( \kappa(x < 0) = \kappa^- \), \( \kappa(x > 0) = \kappa^+ \). The initial data \( \sigma(x) \) and \( \gamma(x) \) are everywhere twice continuously differentiable, except for the point \( x = 0 \). A solution of the Cauchy problem at every moment of time \( t \geq 0 \) at the point \( x^+ = 0 \) must satisfy conjugation conditions (2) and (3).

Let the function \( u(t, x) \) is a solution to the Cauchy problem (4). Let us plot the functions \( u^-(t, x), \sigma^-(x), \gamma^-(x) \), which coincide respectively with \( u(t, x), \sigma(x), \gamma(x) \) at \( x < 0, t \geq 0 \) and are equal to zero otherwise. Then let us construct functions \( u^+(t, x), \sigma^+(x), \gamma^+(x) \), which coincide respectively with \( u(t, x), \sigma(x), \gamma(x) \) at \( x > 0, t \geq 0 \) and equal to zero otherwise. We also denote by \( v^-_0(t) = u(t, x = 0 - 0), \quad v^-_1(t) = u(t, x = 0 + 0) \) and \( v^+_0(t) = \frac{\partial u}{\partial x}(t, x = 0 - 0), \quad v^+_1(t) = \frac{\partial u}{\partial x}(t, x = 0 + 0) \) the value of the solution and its derivative, respectively, on the right and left sides of the boundary.
The function $u^- (t, x)$ considered as a generalized function from $\mathcal{S}'$ satisfies the equation
\[
\rho \frac{\partial^2 u^-}{\partial t^2} - k \frac{\partial^2 u^-}{\partial x^2} = \rho \sigma^- (x) \delta'(t) + \rho \gamma^- (x) \delta(t) + k \delta'(x) + k \delta(x)
\]  
(5)

The function $u^+ (t, x)$ considered as a generalized function from $\mathcal{S}'$ satisfies the equation
\[
\rho \frac{\partial^2 u^+}{\partial t^2} - k \frac{\partial^2 u^+}{\partial x^2} = \rho \sigma^+ (x) \delta'(t) + \rho \gamma^+ (x) \delta(t) - k \delta'(x) - k \delta(x)
\]  
(6)

Let us introduce the notation $a^- = \left(\frac{\kappa}{\rho}\right)^{1/2}$. We make sure by direct verification that the function
\[
E^-(t, x) = \frac{1}{2a} \theta(a^- t - |x|)
\]  
(7)

is a fundamental solution to the operator of problem (5). That is
\[
\rho \frac{\partial^2 E^-}{\partial t^2} - k \frac{\partial^2 E^-}{\partial x^2} = \delta(t, x)
\]  
(8)

Then the solution to equation (5) can be written as a compression of the fundamental solution with the right-hand side
\[
u^-(t, x) = E^-(t, x) \ast \left(\rho \sigma^- (x) \delta'(t) + \rho \gamma^- (x) \delta(t) + k \delta'(x) + k \delta(x)\right)
\]  
(9)

Considering that
\[
E^-(t, x) \ast \rho \sigma^- (x) \delta'(t) = \frac{1}{2} \sigma^- \left(x - a^- t\right)
\]
\[
E^-(t, x) \ast \rho \gamma^- (x) \delta(t) = \frac{1}{2a} \int_{x-a^- t}^0 \gamma^- (\xi) d\xi
\]
\[
E^-(t, x) \ast k \delta'(x) = \frac{1}{2} v_0^\prime \left(t - \frac{|x|}{a^-}\right)
\]
\[
E^-(t, x) \ast k \delta(x) = \frac{a^-}{2} \int_0^{\frac{t}{a^-}} v_1^\prime (\tau) d\tau
\]  
(10)

the solution $u^- (t, x < 0)$ can be written as
\[
u^-(t, x) = \frac{1}{2} \sigma^- \left(x - a^- t\right) + \frac{1}{2a} \int_{x-a^- t}^0 \gamma^- (\xi) d\xi + \frac{1}{2} v_0^\prime \left(t + \frac{x}{a^-}\right) + \frac{a^-}{2} \int_0^{\frac{t-x}{a^-}} v_1^\prime (\tau) d\tau
\]  
(11)

Passing in equalities (11) to the limit at $x \to 0$ from the left, we obtain...
\( v'_0(t) = \sigma^- (-a^- t) + \left( \frac{\rho^-}{\kappa^-} \right)^{1/2} \int_{-a^- t}^0 \gamma^- (\xi) d\xi + \left( \frac{\kappa^-}{\rho^-} \right)^{1/2} \int_0^t v'_i(\tau) d\tau \)

\( v'_i(t) = \frac{\partial \sigma^+}{\partial x} (-a^+ t) - \left( \frac{\rho^+}{\kappa^+} \right)^{1/2} \gamma^+ (-a^+ t) + \left( \frac{\rho^+}{\kappa^+} \right)^{1/2} \frac{\partial v'_0}{\partial t} \)

Similarly, the solution \( u^+ (t, x > 0) \) to equation (6) has the form

\( u^+(t, x) = \frac{1}{2} \sigma^+(x + a^+ t) + \frac{1}{2a^+} \int_{x - a^+ t}^x \gamma^+(\xi) d\xi + \frac{1}{2} v'_0 \left( \frac{t - x}{a^+} \right) - \frac{1}{2a^+} \int_0^t \nu'_i(\tau) d\tau \)

\( \frac{\partial u^+}{\partial x}(t, x) = \frac{1}{2} \frac{\partial \sigma^+}{\partial x} (x + a^+ t) + \frac{1}{2a^+} \gamma^+(x + a^+ t) - \frac{1}{2a^+} \frac{\partial v'_0}{\partial t} \left( \frac{t - x}{a^+} \right) + \frac{1}{2} v'_i \left( \frac{t - x}{a^+} \right) \)

Passing in equalities (13) to the limit at \( x \to 0 \) from the right, we obtain

\( v'_0(t) = \sigma^+ (+a^+ t) + \left( \frac{\rho^+}{\kappa^+} \right)^{1/2} \int_{a^+ t}^0 \gamma^+(\xi) d\xi - \left( \frac{\kappa^+}{\rho^+} \right)^{1/2} \int_0^t v'_i(\tau) d\tau \)

\( v'_i(t) = \frac{\partial \sigma^+}{\partial x} (+a^+ t) + \left( \frac{\rho^+}{\kappa^+} \right)^{1/2} \gamma^+ (+a^+ t) - \left( \frac{\rho^+}{\kappa^+} \right)^{1/2} \frac{\partial v'_0}{\partial t} \)

Let us multiply the first equation in (12) by \( \sqrt{\rho^+ \kappa^+} \) and add to it the first equation from (14) multiplied by \( \sqrt{\rho^- \kappa^-} \). Let us take into account the conjugation conditions (2) and (3)

\( v'_0 = v'_0 = \frac{\rho^+ a^+ \sigma^+ (-a^- t) + \rho^- a^- \sigma^+ (+a^+ t) + \rho^+ \int_{a^- t}^0 \gamma^- (\xi) d\xi + \rho^- \int_0^t \gamma^+ (\xi) d\xi}{\rho^+ a^+ \sigma^+ + \rho^- a^- \sigma^+} \)

Let us multiply the second equation in (12) by \( \kappa^- \) and add to it the second equation from (14) multiplied by \( \kappa^+ \). Let us take into account the conjugation conditions (2) and (3)

\( v'_0 = \kappa^- \frac{a^- \frac{\partial \sigma^-}{\partial x} (-a^- t) + a^+ \frac{\partial \sigma^+}{\partial x} (+a^+ t) + a^- a^+ \gamma^- (-a^- t) + \gamma^+ (+a^+ t)}{\kappa^- a^- + \kappa^+ a^+} \)

\( v'_i = \kappa^- \frac{a^- \frac{\partial \sigma^-}{\partial x} (-a^- t) + a^+ \frac{\partial \sigma^+}{\partial x} (+a^+ t) + a^- a^+ \gamma^- (-a^- t) + \gamma^+ (+a^+ t)}{\kappa^- a^- + \kappa^+ a^+} \)

Expressions (15) and (16) give a solution to the generalized Riemann problem and its derivative on the boundary \( x = 0 \) on both sides. Knowing \( v'_0 (t), v'_i (t) \) and \( v'_0 (t), v'_i (t) \), using formulas (11) and (13), we can calculate the solution of the generalized Riemann problem with additional conditions on the boundary at an arbitrary point \((t, x)\).

In the limit \( t \to 0^+ \), relations (15) and (16) take the form

\( v'_0 = v'_0 = \frac{\rho^+ a^+}{\rho^+ a^- + \rho^- a^+} u^+(0, -0) + \frac{\rho^- a^-}{\rho^+ a^- + \rho^- a^+} u^+(0, +0) \)
\( \kappa^{-1} \psi_1 = \kappa^{-1} \psi_1 = \frac{\kappa \kappa' a^+}{\kappa' a^+ + \kappa a^-} \frac{\partial u^-}{\partial x} (0, -0) + \frac{\kappa \kappa' a^-}{\kappa' a^+ + \kappa a^-} \frac{\partial u^+}{\partial x} (0, +0) \) \tag{18}

Relations (17) and (18) will be used in what follows when constructing computational algorithms.

Also, when constructing a computational algorithm, the solution of a somewhat different problem will be used, which we will call the generalized Riemann problem with conditions on the outer boundary, and which is formulated as follows. Find a solution in the left-hand side-plane \( x \leq 0 \) of the initial-boundary value problem for a partial differential equation with constant coefficients that satisfies the free boundary conditions

\[
\begin{align*}
0 & \leq t \leq T, \quad x \leq 0 \\
\kappa \frac{\partial u}{\partial x} (t, x = 0) & = 0 \\
u(t = 0, x) & = \sigma(x); \quad \frac{\partial u}{\partial t} (t = 0, x) = \gamma(x)
\end{align*}
\tag{19}
\]

We obtain a solution to this problem similarly to the above solution to the generalized Riemann problem with conditions on the internal boundaries.

Let the function \( u(t, x) \) be a solution to problem (19). Let us construct functions \( u^-(t, x), \sigma^-(x), \gamma^-(x) \), which coincide, respectively, with \( u(t, x), \sigma(x), \gamma(x) \), at \( x < 0, t \geq 0 \) and are equal to zero otherwise. We also denote by \( v^-_0(t) = u(t, x = 0 - 0) \) and \( v^-_1(t) = \frac{\partial u}{\partial x}(t, x = 0 - 0) \) the value of the solution and its derivative on the left side of the boundary.

A function \( u^-(t, x) \), considered as a generalized function from \( S^+ \), satisfies the equation

\[
\rho \frac{\partial^2 u^-}{\partial t^2} - \kappa \frac{\partial^2 u^-}{\partial x^2} = \rho \sigma(x) \delta(t) + \rho \gamma(x) \delta(t) + k v^-_0(t) \delta'(x) + k v^-_1(t) \delta(x)
\tag{20}
\]

The solution to this equation \( u^-(t, x < 0) \) can be written in the form \( a = \left( \frac{\kappa}{\rho} \right)^{1/2} \)

\[
u^-(t, x) = \frac{1}{2} \sigma^- (x- at) + \frac{1}{2a} \int_{x- at}^0 \gamma^- (\xi) d\xi + \frac{1}{2} v^-_0(t + \frac{x}{a}) + \frac{a}{2} \int_0^1 \int_0^1 \psi^-_1(t, \tau) d\tau
\tag{21}
\]

Passing in equality (21) to the limit \( x \to 0 \) at the left, we obtain

\[
v^-_0(t) = \sigma^- (- at) + \left( \frac{\rho}{\kappa} \right)^{1/2} \int_{- at}^0 \gamma^- (\xi) d\xi + \left( \frac{\kappa}{\rho} \right)^{1/2} \int_0^1 \psi^-_1(t, \tau) d\tau
\tag{22}
\]

Taking into account the boundary conditions on the outer boundary \( \Gamma: x = 0 \), which can be rewritten in the form \( \kappa v^-_1(t) = 0 \), we obtain

\[
v^-_0(t) = \sigma^- (- at) + \left( \frac{\rho}{\kappa} \right)^{1/2} \int_{- at}^0 \gamma^- (\xi) d\xi
\tag{23}
\]

\[v^-_1(t) = 0\]
The obtained expressions (23) give a solution to the generalized Riemann problem and its derivative on the boundary $\Gamma : x=0$. Knowing $v^+_{\sigma_i}(t), v^-_{\sigma_i}(t)$, by formula (21) we can calculate the solution of the generalized Riemann problem with conditions on the outer boundary at an arbitrary point $(t,x)$.

In the limit $t \to 0$, relations (23) take the form

$$v^+_{\sigma_i}(t = +0) = \sigma^-(x = -0) = u^-(t = +0, x = -0)$$

$$v^-_{\sigma_i}(t = +0) = 0$$

(24)

Relations (24), along with relations (17) and (18), will be used in the further presentation when constructing computational algorithms.

4. Formulation of the problem in generalized functions

Let us divide the interval $[-l,l]$ into subdomains $\Omega_i = [x_i, x_{i+1}]$ by nodes $x_i, i = 1:I + 1$. In this case, we require that all points at which some of the parameters of equation (1) is discontinuous coincide with some of the nodes $x_i$. Then, inside each of the subdomains $\Omega_i$, the parameters of the equation are constant and equal, respectively, $\rho_i, k_i, \eta_i$.

Let the function $u(t,x)$ be a solution to the initial boundary value problem (1). Let us construct functions $u_i(t,x)$ that coincide with $u(t,x)$ at $x \in \Omega_i, t \geq 0$ and are equal to zero otherwise. We also construct functions $\sigma_i(x)$ and $\gamma_i(x)$, which for $x \in \Omega_i$ coincide with $\sigma(x)$ or $\gamma(x)$, respectively, and are equal to zero otherwise. Let us define vector-functions $u(t,x) = u_i(t,x), \sigma(x) = \sigma_i(x)$ and $\gamma(x) = \gamma_i(x)$.

Let us show that vector-functions $u_i(t,x)$, considered as generalized functions from $S'$, satisfy the equations

$$i = 1$$

$$\rho_i \frac{\partial^2 u_i}{\partial t^2} - k_i \frac{\partial^2 u_i}{\partial x^2} - f_i - \rho_i u_{0,i} \delta'(t) - \rho_i u_{i,0} \delta(t)$$

$$= k_i v^+_{0,i} \delta'(x-x_i) - k_i v^-_{0,i+1} \delta'(x-x_{i+1}) - k_i v^+_{i+1} \delta(x-x_{i+1}),$$

(25)

$$i = 2 : I - 1$$

$$\rho_i \frac{\partial^2 u_i}{\partial t^2} - k_i \frac{\partial^2 u_i}{\partial x^2} - f_i - \rho_i u_{0,i} \delta'(t) - \rho_i u_{i,0} \delta(t)$$

$$= k_i v^+_{0,i} \delta'(x-x_i) + k_i v^-_{0,i} \delta(x-x_i) - k_i v^+_{i+1} \delta'(x-x_{i+1}) - k_i v^-_{i+1} \delta(x-x_{i+1}),$$

(26)

$$i = I$$

$$\rho_i \frac{\partial^2 u_i}{\partial t^2} - k_i \frac{\partial^2 u_i}{\partial x^2} - f_i - \rho_i u_{0,i} \delta'(t) - \rho_i u_{i,0} \delta(t)$$

$$= k_i v^+_{0,i} \delta'(x-x_i) + k_i v^-_{i} \delta(x-x_i) - k_i v^+_{0,i+1} \delta'(x-x_{i+1}) - k_i v^-_{i+1} \delta(x-x_{i+1}).$$

(27)

Here, the quantities $v^+_{\sigma_i}(t), v^-_{\sigma_i}(t)$ and $v^+_{\gamma_i}(t), v^-_{\gamma_i}(t)$ for $i = 2 : I - 1$ are determined by expressions (17) and (18), as a solution to the generalized Riemann problem with conditions on the inner boundary. And the same values for $i = 1, I$ are determined by expressions (24), as a solution to the generalized Riemann problem with conditions on the outer boundary.

These equations, using matrix notation, can be rewritten
\[
D \frac{\partial^3 u}{\partial t^3} + K \frac{\partial^2 u}{\partial x^2} = \sum_{j=1}^{l+1} KR_j \frac{\partial u}{\partial x}(x - x_j) + \sum_{j=1}^{l+1} KQ_j u \delta'(x - x_j) + Eu + b + D \sigma \delta'(t) + D\gamma \delta(t) \quad (28)
\]

Here \(D, K, E\) are the diagonal matrices whose elements in the \(i\)-th row are equal, respectively \(\rho_i, -\kappa_i, -\eta_i, b\) the vector whose elements in the \(i\)-th line are equal \(-\rho_i\). The matrix elements \(R_j, Q\), are determined by the coefficients of the equation of the original problem (1).

Indeed, by definition, for an arbitrary base function \(\varphi(t,x)\in \mathcal{S}'\)
\[
\left(D \frac{\partial^2 u}{\partial t^2} + K \frac{\partial^2 u}{\partial x^2}, \varphi \right) = \int_{-\infty}^{\infty} \left( -\frac{\partial^2 \varphi}{\partial t^2} D + \frac{\partial^2 \varphi}{\partial x^2} K \right) u dt dx \quad (29)
\]

At all \(\varphi(t,x)\in \mathcal{S}'\) the equality is true
\[
\int_{-\infty}^{\infty} \varphi(t) \frac{\partial^2 u}{\partial t^2} dt dx + \int_{-\infty}^{\infty} \varphi(t) \frac{\partial^2 u}{\partial t^2} dt dx = \int_{-\infty}^{\infty} \varphi(t) D \frac{\partial u}{\partial t} \bigg|_{0}^{\infty} dx - \int_{-\infty}^{\infty} \varphi(t) D \frac{\partial u}{\partial t} \bigg|_{0}^{\infty} dx.
\]

Taking into account the initial conditions (1), we get
\[
\int_{-\infty}^{\infty} \varphi(t) \frac{\partial^2 u}{\partial t^2} dt dx = \int_{-\infty}^{\infty} \varphi(t) D \frac{\partial u}{\partial t} \bigg|_{0}^{\infty} dx + \int_{-\infty}^{\infty} \varphi(t) D \gamma \bigg|_{0}^{\infty} dx. \quad (30)
\]

Similarly,
\[
\int_{-\infty}^{\infty} \varphi(t) \frac{\partial^2 u}{\partial t^2} dt dx = \int_{-\infty}^{\infty} \varphi(t) D \frac{\partial u}{\partial t} \bigg|_{0}^{\infty} dx - \sum_{j=1}^{l+1} \int_{-\infty}^{\infty} \varphi(t) K u + \varphi(t) K u \bigg|_{j}^{\infty} dt.
\]

Here \(u = u(t,x) + 0 - u(t,x) - 0\) and \(\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} t,x_j + 0 - \frac{\partial u}{\partial x} t,x_j - 0\) the jump of the vectors \(u\) and \(\frac{\partial u}{\partial x}\) when crossing the point \(x_j\).

Let us express the jump at the boundary nodes through the values of the vectors \(u\) and \(\frac{\partial u}{\partial x}\) at these nodes, taking into account the boundary conditions (1): \(u_j = Q_j u(t,x), \quad \frac{\partial u}{\partial x} \bigg|_{j}^{\infty} = R_j \frac{\partial u}{\partial x} t,x_j, \quad j=1,I+1\). The jump \(u_j\) and \(\frac{\partial u}{\partial x} \bigg|_{j}^{\infty}\) at the internal nodes, we express through the values of the vectors \(u\) and \(\frac{\partial u}{\partial x}\) in the solution of the generalized Riemann problem with additional conditions on the boundary \(x=x_j\) (17)-(18): \(u_j = Q_j u(t,x), \quad \frac{\partial u}{\partial x} \bigg|_{j}^{\infty} = R_j \frac{\partial u}{\partial x} t,x_j, \quad j=2: I\). As a result, we get
\[
\int_{-\infty}^{\infty} \varphi(t) \frac{\partial^2 u}{\partial x^2} dt dx = \int_{-\infty}^{\infty} \varphi(t) K u \frac{\partial^2 u}{\partial x^2} dt dx - \sum_{j=1}^{l+1} \int_{-\infty}^{\infty} \varphi(t) KQ_j u \bigg|_{t,x_j}^{\infty} - \varphi(t) K u \frac{\partial u}{\partial x} \bigg|_{t,x_j}^{\infty} \bigg|_{j}^{\infty} dt. \quad (31)
\]
Adding equations (21), (22) and taking into account that \( u_i(t,x) \) the components of the vector \( \mathbf{u}(t,x) \) satisfy equation (1) almost everywhere, equality (19) follows.

Again, if the generalized vector functions \( \mathbf{u}, t, x \) satisfy the system of equations (19), then these generalized functions are regular and the function \( u(t,x) = \sum_{i=1}^{I} u_i(t,x) \), as an ordinary function, satisfies the equation, the initial and boundary conditions of problem (1) and the conjugation conditions (2) (3).

5. Computational algorithm

Equations (19) are used as the basis for a computational algorithm for finding an approximate solution to the initial-boundary value problem (1).

Let the functions \( H_{i,m}(x) \) be basic interpolation polynomials on an interval \([x_i, x_{i+1}]\) constructed at the nodes located at the zeros of a Chebyshev polynomial of degree \( M \) and equal to zero outside the interval \([x_i, x_{i+1}]\). Also let \( \tilde{H}_{i,m}(x) \) be the polynomials conjugate to them. That is

\[
\int_{x_i}^{x_{i+1}} \tilde{H}_{i,m}(x) dx = \delta_i^m \delta_m^m.
\]

Let us extend the polynomials \( \tilde{H}_{i,m}(x) \) smoothly outside the interval \([x_i, x_{i+1}]\) so as to ultimately obtain a finite infinitely differentiable function. For the function thus obtained, we will leave the former name \( \tilde{H}_{i,m}(x) \).

We will seek an approximate solution to equations (19) in the form of a vector-function

\[
u_i(t,x) = \sum_{m=1}^{M} H_{i,m}(x) u_{i,m}(t) = H_{i,m}(x) u_{i,m}^\tau(t)
\]

In the previous expression, the agreement is used that if any index is repeated twice in the expression, then summation over all values of the index is implied. If there is a dot ("\( i \)") to the right of the repeated indices, then summation is not implied. We will use this agreement in further presentation. Superscripts and subscripts are not distinguished.

As test functions from the space of basic functions, we take finite vector-functions \( \mathbf{\phi}_{i,m}(t,x) \in \mathbf{S}' \), all elements of which are equal to zero, except for the element in the \( i \)-th row, equal to \( \phi_i^m(t) \tilde{H}_{i,m}(x) \in \mathbf{S} \), where \( \phi_i^m(t) \) are arbitrary finite infinitely differentiable functions.

Substituting into equation (19), and taking into account that \( \phi_i^m(t) \) arbitrary finite infinitely differentiable functions, we obtain that the coefficients \( u_{i,m}^\tau(t) \) must be a solution to the Cauchy problem for the system of ordinary differential equations

\[
\begin{align*}
\frac{d^2 u_{i,m}^\tau}{dt^2} + A_{i,m}^\tau u_{i,m}^\tau &= f_{i,m}^\tau, \\
u_{i,m}^\tau(t = 0) &= \mu(t), \\
\frac{du_{i,m}^\tau}{dt}(t = 0) &= \nu(t)
\end{align*}
\]

Here

\[
A_{i,m}^\tau = \frac{\kappa_{i,m}^\tau}{\rho_{i,m}^\tau} \left( \int_{x_i}^{x_{i+1}} \frac{\partial \tilde{H}_{i,m}^\tau}{\partial x} \frac{\partial H_{i,m}}{\partial x} dx + \frac{\partial \tilde{H}_{i,m}^\tau}{\partial x} (x_{i+1}) H_{i,m}(x_{i+1}) - \frac{\partial \tilde{H}_{i,m}^\tau}{\partial x} (x_i) H_{i,m}(x_i) + \ldots \right) + \ldots
\]

\[
+ \frac{\kappa_{i,m}^\tau}{\rho_{i,m}^\tau} \left( \sum_{j=1}^{I} Q_{i,j}^\tau \tilde{H}_{i,m}^\tau(x_j) H_{i,m}(x_j) - \sum_{j=1}^{I} R_{j,m}^\tau \tilde{H}_{i,m}^\tau(x_j) \frac{\partial H_{i,m}}{\partial x}(x_j) + \eta_{i,m}^\tau \delta_i^m \delta_m^m \right)
\]

(32)
\[ f^{i,m} = -g \int_{x_i}^{x_{i+1}} H^{i,m} \, dx; \quad \mu(t) = \int_{x_i}^{x_{i+1}} \rho^{i} H^{i,m} \, dx; \quad \nu(t) = \int_{x_i}^{x_{i+1}} \gamma^{i} H^{i,m} \, dx \]  
\tag{34}

We solve this ODE system by one or another numerical method and by the formulas 
\[ u_i(t,x) = \sum_{m=1}^{M} H_{i,m}(x) u^{i,m}(t) \]  
we determine an approximate solution to the original problem.

From the above scheme for constructing a computational algorithm, we see that the original problem is reduced to solving the Cauchy problem for a system of ODEs solved with respect to derivatives. If an explicit difference scheme is used for the numerical solution of the Cauchy problem, then the entire algorithm is also explicit, in contrast to the algorithms obtained on the basis of the Galerkin method. In the latter, due to the fact that the resulting system of ODEs is not resolved with respect to derivatives, it is necessary to solve a system of linear algebraic equations at each time level.

The described algorithm is applicable not only for problems with second-order derivatives but also for problems in which the processes under study are described by derivatives of an arbitrary order.

Using a basis of high degree polynomials, we can construct computational algorithms of an arbitrarily high order of approximation.

Probably one of the main features of the proposed algorithm is that the conjugation conditions, which are a reflection of the physical conditions, are met exactly everywhere. This property excludes the possibility of the appearance of artificial sources of disturbances leading to computational instability.

The algorithm is based on the use of a basis of discontinuous piecewise-smooth functions, this basis does not need to be subordinated to the boundary conditions, which is a difficult task, especially in the case of many spatial variables (boundary conditions are included in the "weak" form in the equations for generalized functions) and, as a result, it leads to a system of linear ODEs with highly sparse matrices, which allows efficient implementation on parallel computing systems.

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