Study of Convexo-Symmetric Networks via Fractional Dimensions

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The work of Muhammad Kamran Aslam and Muhammad Javaid was supported by the Higher Education Commission of Pakistan through the National Research Program for Universities under Grant 20-16188/NRPU/R&D/HEC/2021 2021. The work of Samuel Asefa Fufa was supported by the Department of Mathematics, Addis Ababa University.

**ABSTRACT** For having an in-depth study and analysis of various network’s structural properties such as interconnection, extensibility, availability, centralization, vulnerability and reliability, we require distance based graph theoretic parameters. Numerous parameters like distance based dimensions help in designing queuing models in restaurants, public health and service facilities and production lines. Likewise, allocations of robots in several production units are indebted to these. Moreover, chemists and druggists use these parameters in finding out new drug type and structural formula of a chemical compound. In this article, we are finding out the extremal values of local fractional metric dimension of a class of network bearing convex as well as symmetric properties known by the name of convex polytopes. Also we have related the significance of our findings towards the end of the manuscript in the form of fire exit plan. The same will serve as a guide for future architects in planning a floor with a conducive fire exit plan.

**INDEX TERMS** Metric dimension, fractional metric dimension, convex polytopes, resolving sets.

**I. INTRODUCTION**

A network \( \mathbb{G} \) comprises of two sets that are \( V(\mathbb{G}) \) set of vertices and \( E(\mathbb{G}) \) the set of objects that forms connection among the nodes/vertices called edges where \( E(\mathbb{G}) \subseteq V(\mathbb{G}) \times V(\mathbb{G}) \). The cardinality of \( V(\mathbb{G}) \) identifies the order of \( \mathbb{G} \) denoted by \( p \) and size of \( \mathbb{G} \) is identified by the cardinality of \( E(\mathbb{G}) \) denoted by \( q \). For any \( s, t \in \mathbb{G} \), \( d(s, t) \) represents the shortest distance of path between \( s \) and \( t \). For further studies of graph theoretic concepts, we refer to [2], [11], and [1].

The advancements in technology is intervening our lives in the form of replacing manpower with AI(artificial intelligence) enabled and automated machines and robots. Meanwhile, effective allocation of these are required keeping in view cost effectiveness and utilizing them in minimum numbers. Both these concerns are answered by a distance based parameter namely metric dimension (MD).

Suppose that \( \mathbb{N} = \{n_1, n_2, \ldots, n_k\} \subseteq V(\mathbb{G}) \) then \( \mathbb{N} \) after the imposition of some ordering forms a set of vertices bearing that ordering. For any \( s \in V(\mathbb{G}) \), the \( k \)-tuple represented by \( r(s[\mathbb{N}]) = (d(s, n_1), d(s, n_2), d(s, n_3), \ldots, d(s, n_k)) \) denotes the distance of \( s \) from the elements of \( \mathbb{N} \). The set \( \mathbb{N} \) changes into resolving set if for each \( s, t \in V(\mathbb{G}) - \mathbb{M} \), \( r(s[\mathbb{N}]) \neq r(t[\mathbb{N}]) \). The resolving set having fewer number of nodes in \( \mathbb{G} \) forms MD of \( \mathbb{G} \) denoted by \( \lambda(\mathbb{G}) \) [4], [22].

Slater established the notion of resolving set for any connected network by using locating set as its title, [23], [24]. After revisiting and analysing the aforesaid notions by themselves, Harary and Melter proclaimed them as MD [12]. The other types of MD includes fault tolerant, edge and strong. Afterwards, numerous researchers investigated the MD of different networks. Chartrand et al. on the other hand, proved the MD of path and cycle [4]. Similarly, one can find numerous results regarding the MD of connected networks in, [4] and [16]. Similarly, for fault tolerant metric dimension of circulant and wheel related networks see, [33], [34], and [35].
Chartrand et al. put MD at work for solving integer programming problem (IPP) [4]. Afterwards, for obtaining the results of IPP with maximum possible accuracy, Currie and Oellermann [3] pioneered the notion of fractional metric dimension (FMD). Similarly, one can find the parsimonious solution of a certain linear programming relaxation problem employing the FMD in [21]. Arumugam et al. [29] unveiled the overlooked properties of FMD. After its inception, numerous results on this topic related to various networks appeared for instance the networks formed out of the products of networks namely cartesian, comb, corona, hierarchial, and lexicographic products, see [18], [19], [20], [29], and [28]. In the same manner, the FMD of the generalized Jahangir network is investigated in [9]. The FMD of metal organic networks are identified in [15].

Aisyah et al. introduced the notion of local fractional metric dimension (LFMD) and for putting same tot the test the corona product of two networks was chosen [31]. Similarly, one can find results regarding the LFMD of circular networks in [10]. In the same manner, one can find the general criteria for the interval of LFMD for all the connected networks in [13] and improved lower bound criteria in [14]. In this paper, the family of networks (convex polytopes) is chosen that bears the symmetry with respect to rotation. Mainly, for the convex polytopes networks, we found the local resolving neighbourhood sets with their cardinalities and computed the local fractional metric dimensions in the form of lower and upper bounds. The bounded and unboundedness is checked at the larger order of the networks as possible. The significance of the present study is also highlighted by presenting the fire exit plan for the floor of a mall. The flux of this paper is in the wind ups the paper with hand full of concluding remarks.

II. APPLICATIONS

A. CHEMISTRY

The structural formula of a chemical compound after its turned into a network theoretic form is termed as a molecular graph with nodes as atoms and links between them as bonds [4]. The compound in its graph-theoretic version and distance-based parameters together help chemists to remove the discrepancies in some chemical structures but also enable them to reach out to sites exhibiting the same properties in them. The aforesaid techniques are the topic of [23], [26], and [25].

B. ROBOTICS

The propulsion of strata of networking is nourishing the topics of distance based dimensions. The tools like these assist an interpolar to be allocated in a suitable region for employing its effective usage [12], [23]. Similarly, the allocation of robots in some production unit as well as in public health facility have been given attention in [27]. Also, numerous techniques that involve the rectification of example and picture handling, information handling, see [26]. The aforementioned parameters have made it possible for engineers and scientists to employ robots to serve a production line, public health facilities and restaurants with the least number of these but also minimized their responding time.

III. PRELIMINARIES

A vertex $c$ is said to resolve a pair $\{s, t\} \subseteq V(G)$ if $d(s, c) \neq d(t, c)$. The resolving neighborhood (RN) set of $\{s, t\}$ is defined as $R(s, t) = \{c \in V(G) | d(s, c) \neq d(t, c)\}$. A mapping $\kappa : V(G) \rightarrow [0, 1]$ is called local resolving function (LRF) if $\kappa(LRN(st)) \geq 1$ for LRN of each $st \in G$, where $\kappa(LRN(st)) = \sum_{c \in LRN(st)} \kappa(c)$. An LRF $\kappa$ of $G$ is known as local minimal resolving function if any function $\psi : V(G) \rightarrow [0, 1]$ such that $\psi \leq \kappa$ and $\psi(c) \neq \kappa(c)$ for at least one $c \in V(G)$, that is not a resolving function of $G$.

For a connected network $G$ its LFMD is given by $dim_{LR}(G) = min(|\kappa| : \kappa$ is the minimal LRF of $G)$ [31].

A. CONVEX POLYTOPES

The network $G \cong B_4$ bearing $n$ 4-sided faces, $3n$ 3-sided faces, $n$ 5-sided faces and a pair of $n$-sided faces that is the resultant of the combination of convex polytope $Q_n$ and prism network $D_n$ [17] is known as type I convex polytope. Its vertex and edge sets are:

$V(B_n) = \{s_j, u_j, v_j, w_j|1 \leq j \leq n\}$ and $E(B_n) = \{s_js_{j+1}, t_ju_{j+1}, u_jw_{j+1}, v_jv_{j+1}|1 \leq j \leq n\}$ \cup \{s_jt_j, t_{j+1}u_j, u_jw_j, v_jv_j|1 \leq j \leq n\}$ respectively. The induced cycles namely inner, internal, external and outer cycle are $\{w_j|1 \leq j \leq n\}, \{t_j|1 \leq j \leq n\}, \{u_j|1 \leq j \leq n\}, \{v_j|1 \leq j \leq n\}$ and $\{w_j|1 \leq j \leq n\}$ respectively. The Figure 1 illustrates $B_n$.

Likewise, by adding new edges $v_{j+1}w_j$ in $B_n$ we obtain the Type II Convex Polytope network $G \cong C_n$. It consists of $n$ 4-sided faces, $n$ 5-sided faces, $3n$ 3-sided faces, and a pair of
n-sided faces. In the same manner, the cycles \{s_j|1 \leq j \leq n\}, \{t_j|1 \leq j \leq n\}, \{u_j|1 \leq j \leq n\}, \{v_j|1 \leq j \leq n\} and \{\gamma_j|1 \leq j \leq n\} are known as inner, internal, external and outer cycle respectively. The Figure 2 illustrates \(C_n\).

The sets \(V(C_n)\) and \(E(C_n)\) are given by
\[
V(C_n) = \{s_j, t_j, u_j, v_j|1 \leq j \leq n\}
\]
and
\[
E(C_n) = \{s_jt_j+1, t_ju_j+1, u_jv_j+1, v_jt_j+1|1 \leq j \leq n\} \cup \{s_jt_j, t_j+1u_j, u_jv_j, v_j+1t_j|1 \leq j \leq n\}.
\]

We close this section by presenting 2 important results that will be required to prove our findings. We are presenting here their precise points in the proofs, for those details reader may see [13] and [14].

**Theorem A [13]:** Suppose that \(G(V(G), E(G))\) is a connected network. Let \(LRN(ab)\) be the local resolving neighborhood of \(ab \in E(G)\) in \(G\) with order \(p\). If \(|LR(ab)| \cap W| \geq \beta\), \(\forall ab \in E(G)\), then
\[
1 \leq dim_f(G) \leq \frac{|W|}{\beta},
\]
where \(W = \bigcup_{ab \in E(G)} LR(ab)\), \(\beta = \min\{|LR(ab)|) \cap W| \geq \beta\), and \(2 \leq \beta \leq p\).

**Proof:** The proof will proceed followed by two main cases (Case I and Case II) subject to the values of \(\beta\).

**Case I:** Let \(\beta \neq p\) and \(W\) as defined in theorem. Define a mapping \(f : V(G) \rightarrow [0, 1]\) as
\[
f(w) = \begin{cases}
\frac{1}{\beta}, & w \in W; \\
0, & v \in V(G) \cap W.
\end{cases}
\]

Since \(\forall e \in E(G)\),
\[
f(LR(e)) = \sum_{w \in LR(e)} f(w) = \sum_{w \in LR(e) \cap W} \frac{1}{\beta} = |LR(e) \cap W| \frac{1}{\beta} \geq 1,
\]
which provides the evident of the fact that \(f\) is LRF. To find, \(f\) is a minimal resolving function, suppose that there is another resolving function \(g\) such that \(g \leq f\). By definition, \(g(w) < f(w)\) for some \(w \in W\). Assume \(w \in LR(e)\) and for some \(e \in E(G)\) with \(|LR(e)| = \beta\). Then
\[
g(LR(e)) = \sum_{w \in LR(e)} g(w) < \sum_{w \in LR(e)} f(w) < 1
\]
which shows that our supposition was wrong about \(g\) being the LRF. Hence, \(f\) is the minimal LRF. Suppose that \(f\) as another LRF of \(G\). Consider,
\[
|\hat{f}| = \sum_{w \in LR(e)} \hat{f}(w) = \sum_{w \in W} \hat{f}(w) + \sum_{w \in V(G) - W} \hat{f}(w)
\]
In this way, the following 3 subcases comes into play:

i. \(\hat{f}(w) < \frac{1}{\beta}, \forall w \in W\),

ii. \(\hat{f}(w) \geq \frac{1}{\beta}, \forall w \in W\),

iii. for some \(w \in W\), \(\hat{f}(w) < \frac{1}{\beta}\).

In the light of aforementioned subcases, equation 1 takes the form:
\[
dim_f = |\hat{f}| = \sum_{w \in LR(e)} \leq |\hat{f}| = \frac{|W|}{\beta}.
\]

**Case II:** Let \(\beta = p\) then \(LR(e) = V(G)\) and \(W = V(G) \forall e \in E(G)\). For any \(f : V(G) \rightarrow [0, 1]\) such that \(|LR(e)| = 1\) is minimal LRF with the smallest cardinality, where \(e \in E(G)\). Hence, \(dim_f(G) = 1\).

Finally, from both **Case I** and **Case II**, we get
\[
1 \leq dim_f(G) \leq \frac{|W|}{\beta}.
\]

**Theorem B [14]:** Suppose that \(G(V(G), E(G))\) is a connected network with order \(p\). Let \(LRN(ab)\) be the local resolving neighborhood of \(ab \in E(G)\) in \(G\). Then
\[
\frac{p}{\gamma} \leq dim_f(G),
\]
where \(\gamma = \max\{|LR(ab)|: \forall ab \in E(G)\}\) and \(2 \leq \gamma \leq p\).

**Proof:** The proof of this theorem follows the same pursuit as of Theorem A.

**Corollary C [13]:** Suppose that \(G(V(G), E(G))\) is a connected bipartite network. Then \(dim_f = 1\).

**IV. MAIN RESULTS**

The current section contains the major results related to the LFMD of the networks under consideration. The results related to the LRFs of \(B_n\) and \(C_n\) are given by Lemma 4.1 and Lemma 4.3 respectively. Likewise, Theorem 4.2 and 4.4 are the results related to the bounds of LFMD of the aforementioned networks.

**Lemma 4.1:** Suppose that \(G \cong B_n\) is a convex polytope type I network with \(n \geq 6\) and \(n \equiv 0(\mod 2)\). Then

i. \(\frac{|G|}{2} + 1 \leq |LR(e)| \forall e \in E(G)\),

ii. \(|LR(e) \cap X| \geq \frac{7n}{2} + 1 \forall e \in E(G)\) and \(X = \bigcup_{e \in E(G)} |LR(e)| = \frac{7n}{2} + 1\).

**Proof:**

i. From FIGURE 1, we note that there are 6 types of edges in \(G\) that are \(t_ju_j, t_ju_{j-1}, s_{j}t_{j+1}, t_{j}t_{j+1}, v_{j}v_{j+1}
\]
and \( u_jv_j \), where \( 1 \leq l, j \leq n \). The LRNs of these edges are
\[
LR\{t_lu_l\} = V(\mathbb{B}_n)
\]
\[
\{ s_h | h \equiv l + 1, l + 2, \ldots, l + \frac{n}{2} (\text{mod } n) \}
\]
\[
\cup \{ t_h | h \equiv l + 1, l + 2, \ldots, l + \frac{n}{2} (\text{mod } n) \}
\]
\[
\cup \{ w_h | h \equiv l + 1, l + 2, \ldots, l + \frac{n}{2} (\text{mod } n) \}
\]
\[
LR\{t_ju_{j-1}\} = V(\mathbb{B}_n)
\]
\[
\{ s_h | h \equiv j - 1, j - 2, \ldots, j - \frac{n}{2} (\text{mod } n) \}
\]
\[
\cup \{ t_h | h \equiv j - 1, j - 2, \ldots, j - \frac{n}{2} (\text{mod } n) \}
\]
\[
\cup \{ w_h | h \equiv j - 1, j - 2, \ldots, j - \frac{n}{2} (\text{mod } n) \}
\]
\[
LR\{s_js_{j+1}\} = V(\mathbb{B}_n) - \{ u_h | h \equiv r, j + \frac{n}{2} (\text{mod } n) \}
\]
\[
\cup \{ v_h | h \equiv r, j + \frac{n}{2} (\text{mod } n) \}
\]
\[
\cup \{ w_h | h \equiv r, j + \frac{n}{2} (\text{mod } n) \} = LR\{t_{j+1}\},
\]
\[
LR\{v_jv_{j+1}\} = V(\mathbb{B}_n) - \{ s_h | h \equiv j + 1, j + \frac{n + 2}{2} (\text{mod } n) \}
\]
\[
\cup \{ t_h | h \equiv j + 1, j + \frac{n + 2}{2} (\text{mod } n) \}
\]
\[
\cup \{ u_h | v_j, v_{j+1} \} = V(\mathbb{B}_n) - \{ s_h | h \equiv j - 1, j + 1 (\text{mod } n) \}
\]
and
\[
LR\{s_ft_f\} = V(\mathbb{B}_n) = LR\{v_ju_j\} \text{ respectively. Their cardinalities are}
\]
\[
|LR\{t_lu_l\}| = |LR\{t_ju_{j-1}\}| = \frac{7n}{2} + 1, |LR\{s_js_{j+1}\}| = |LR\{t_{j+1}\}| = 5n - 6, |LR\{v_jv_{j+1}\}| = 5n - 4 \text{ and } |LR\{u_jv_j\}| = |LR\{s_j|t_f|\}| = 5n \text{ respectively. From all the aforementioned cardinalities we have seen that } \frac{7n}{2} + 1 \leq |LR(e)| \forall e \in G.
\]
(b) From proof of part (i), we have
\[
|LR\{t_lu_l\}| = |LR\{t_ju_{j-1}\}| = \frac{7n}{2} + 1 \leq |LR(e)| \forall e \in E(G).
\]
Now consider,
\[
\bigcup_{l=1}^{n} LR\{t_lu_l\} = V(G)
\]
and
\[
\bigcup_{l=1}^{n} LR\{t_ju_{j-1}\} = V(G).
\]
Take \( X = \bigcup_{l=1}^{n} LR\{t_lu_l\} \cup \bigcup_{l=1}^{n} LR\{t_ju_{j-1}\} = V(G) \). Since for any \( e \in E(G), LR(e) \subseteq V(G) \Rightarrow |LR(e) \cap X| = |LR(e) \cap V(G)| = |LR(e)| \geq \frac{7n}{2} + 1 \) (By part (i)). This completes the proof.

**Theorem 4.2:** If \( G \cong \mathbb{B}_n \) with \( n \geq 6 \) and \( n \equiv 0(\text{mod } 2) \). Then, \( 1 < \dim_G(\mathbb{B}_n) \leq \frac{10n}{7} + 2 \).

**Proof:**

**Case I:** When \( n = 6 \).

The LRNs are given by:

| \( n \) | \( LR_j \) |
|---|---|
| 12 | \( V(\mathbb{B}_6) \) |
| 20 | \( LR\{s_1t_1\} \) |
| 21 | \( LR\{s_2t_2\} \) |
| 22 | \( LR\{s_3t_3\} \) |
| 23 | \( LR\{s_4t_4\} \) |
| 24 | \( LR\{s_5t_5\} \) |
| 25 | \( LR\{s_6t_6\} \) |

Now, we define a function \( \kappa : V(\mathbb{B}_6) \rightarrow \{0, 1\} \) such that \( \kappa(s_j) = \kappa(t_j) = \kappa(u_j) = \kappa(v_j) = \frac{n}{7} \). It is observed in the LRNs of minimum cardinality show the following behaviour \( \cap_{j=1}^{12} LR_j = \emptyset \) but \( LR_j \cap LR_i \neq \emptyset \) for \( 1 \leq i, j \leq 12 \) of \( \mathbb{B}_6 \), that means they are not pairwise disjoint. Thus by
Theorem A

$$\text{dim}_f(\mathcal{B}_6) \leq \sum_{j=1}^{30} \frac{1}{\gamma} = \frac{30}{22}. \quad (2)$$

Likewise, in Table 5 LRNs with maximum cardinality of $30 = \gamma$ are shown thus by Theorem B and Corollary C, if $\text{G}$ is not bipartite, we have

$$\frac{|V(\mathcal{B}_6)|}{\gamma} = \frac{30}{30} = 1 < \text{dim}_f(\mathcal{B}_6) \quad (3)$$

Therefore, from 2 and 3, we have

$$1 < \text{dim}_f(\mathcal{B}_6) \leq \frac{30}{22}$$

**Case II:** $n \geq 8$.

From Lemma 4.1 we have found that the LRNs with minimum cardinality of $\frac{2n}{3} + 1$ are $\text{LR}(t_ju_j)$ and $\text{LR}(t_ju_j-1)$ respectively. Also $\bigcup_{j=1}^{n} \text{LR}_j = V(\mathcal{B}_n)$, $\bigcup_{j=1}^{n} \text{LR}_l = V(\mathcal{B}_n)$ and $|\text{LR}_j \cap \bigcup_{j=1}^{n} \text{LR}_j = V(\mathcal{B}_n)| = |\text{LR}|$, where $1 \leq j \leq 7$. Let

$$\beta = \frac{2n}{3} + 1$$

$$\delta = |\bigcup_{j=1}^{n} \text{LR}_j| = 5n$$

Now we define a mapping $\kappa : V(\mathcal{B}_n) \rightarrow [0, 1]$ such that

$$\kappa(a) = \sum_{j=1}^{\beta} \frac{1}{\beta} \frac{1}{\frac{2n}{3} + 1}.$$ 

Also for $1 \leq j \leq n$ LR$_j$ are not pairwise disjoint, hence by Theorem A, we have

$$\text{dim}_f(\mathcal{B}_n) \leq \sum_{j=1}^{\beta} \frac{1}{\beta} \frac{1}{\frac{2n}{3} + 1} = \frac{10n}{7n + 2}. \quad (4)$$

Likewise, the LRNs with maximum cardinality of $|V(\mathcal{B}_n)|$ are $\gamma = |\bigcup_{j=1}^{n} \text{LR}(s_jt_j)| = |\bigcup_{j=1}^{n} \text{LR}(t_ju_j)|$, thus by Theorem B and Corollary C, we have

$$\frac{|V(\mathcal{B}_n)|}{\gamma} = \frac{|V(\mathcal{B}_n)| \text{dim}_f(\mathcal{B}_n)}{\text{dim}_f(\mathcal{B}_n)} = \frac{10n}{7n + 2}. \quad (5)$$

Therefore, from 4 and 5, we have

$$1 < \text{dim}_f(\mathcal{B}_n) \leq \frac{10n}{7n + 2}$$

**Lemma 4.3** Suppose that $G \cong C_n$ is a convex polytope type II network with $n \geq 6$ and $n \equiv 0 \text{(mod 2)}$. Then

(i) $3(n+1) \leq |\text{LR}(e)| \forall e \in E(G)$,

(ii) $|\text{LR}(e) \cap X| \geq 3(n+1) \forall e \in E(G)$ and $X = \bigcup_{e \in E(G)} |\text{LR}(e)| = 3(n+1)$.

**Proof:**

(i) From FIGURE 2, we note that there are 8 types of edges in $G$ that are $v_jw_j, v_jw_{j-1}, t_ju_j, t_ju_{j-1}, s_jt_{j+1}, t_jf_{j+1}, v_jf_{j+1}, u_jv_j$, and $s_jt_j$, where $1 \leq i, j \leq n$. The LRNs of these edges are

$$\text{LR}(v_jw_j) = V(C_n)$$

$$\text{LR}(v_jw_{j-1}) = V(C_n)$$

$$\text{LR}(t_ju_j) = V(C_n)$$

$$\text{LR}(t_ju_{j-1}) = V(C_n)$$

$$\text{LR}(v_jf_{j+1}) = V(C_n)$$

$$\text{LR}(v_jf_{j+1}) = V(C_n)$$

$$\text{LR}(u_jv_j) = V(C_n)$$

$$\text{LR}(u_jv_j) = V(C_n)$$

respectively. Their cardinalities are

$$|\text{LR}(v_jw_j)| = |\text{LR}(v_jw_{j-1})| = 3(n+1),$$

$$|\text{LR}(t_ju_j)| = |\text{LR}(t_ju_{j-1})| = \frac{7}{2}n + 1,$$

$$|\text{LR}(s_jt_{j+1})| = |\text{LR}(t_jf_{j+1})| = 5n - 6,$$

and $|\text{LR}(s_jt_j)| = |\text{LR}(u_jv_j)| = 5n$ respectively. From all the aforementioned cardinalities we have seen that $3(n+1) \leq |\text{LR}(e)| \forall e \in G$.

(b) From proof of part (i), we have

$$|\text{LR}(v_jw_j)| = |\text{LR}(v_jw_{j-1})| = 3(n+1) \leq |\text{LR}(e)| \forall e \in E(G).$$

Now consider,

$$\bigcup_{i=1}^{n} |\text{LR}(v_jw_j)| = V(G)$$
Take $X = \bigcup_{i=1}^{n} LR(v_i) = V(G)$. Since for any $e \in E(G)$, $LR(e) \subseteq V(G)$, $|LR(e)| \subseteq V(G)|$, $|LR(e)| \geq 3(n+1)$ (By part (ii)). This completes the proof.

Theorem 4.4: If $G \cong C_{n}$ with $n \geq 6$ and $n \equiv 0$(mod 2). Then, $1 < dimy(C_{n}) \leq \frac{5n}{3n+1}$. 

Proof: Case I: When $n = 6$.

The LRNs are given by:

It can be seen that Table 7, Table 8 and Table 9 show the LRNs with cardinalities of $22, 24$ and $26$ respectively. Table 6 bears LRNs with the minimum cardinality of $21$ and Table 10 represents the LRNs with maximum cardinality of $30$. Also $\bigcup_{j=1}^{12} LR_{j} = V(C_{6})$ this implies $|\bigcup_{j=1}^{12} LR_{j}| = 30$ and $|LR_{j} \cap \bigcup_{j=1}^{12} LR_{j}| \geq |LR_{j}|$. 

Now, we define a function $\kappa : V(C_{6}) \to [0, 1]$ such that $\kappa(s_{j}) = \kappa(t_{j}) = \kappa(u_{j}) = \kappa(v_{j}) = \kappa(v_{j}) = \frac{1}{27}$. As $LR_{j}$ for $1 \leq j \leq 12$ of $C_{6}$ are not pairwise disjoint. Therefore, by Theorem A, we have

$$dimy(C_{6}) \leq \frac{30}{22} = \frac{15}{11} \leq \frac{30}{21}$$

Likewise, in Table 10 shows the LRNs with maximum cardinality of $30 = \gamma$ hence by Theorem B and Corollary C ($G$ is not bipartite), we have

$$\frac{|V(C_{6})|}{\gamma} = \frac{30}{30} = 1 < dimy(C_{6})$$

Therefore, from 6 and 7, we have

$$1 < dimy(C_{6}) \leq \frac{30}{21}$$

Case II: $n \geq 8$.

It is evident from Lemma 4.3 that the LRNs with minimum cardinality of $3(n+1)$ are $LR_{j} = V(C_{n})$, $LR_{j} = V(C_{n})$ and respectively. Also $\bigcup_{j=1}^{n} LR_{j} = V(C_{n})$, $\bigcup_{j=1}^{n} LR_{j} = V(C_{n})$ and $|LR_{j} \cap \bigcup_{j=1}^{n} LR_{j}| \geq |LR_{j}|$, where $1 \leq j \leq 7$. Let $\beta = 3(n+1)$ and $\delta = |\bigcup_{j=1}^{n} LR_{j}| = 5n$. Now we define a mapping $\kappa : V(C_{n}) \to [0, 1]$ such that

$$dimy(C_{n}) \leq \frac{30}{22} = \frac{15}{11} \leq \frac{30}{21}$$

Likewise, in Table 10 shows the LRNs with maximum cardinality of $30 = \gamma$ hence by Theorem B and Corollary C ($G$ is not bipartite), we have

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$$1 < dimy(C_{n}) \leq \frac{30}{21}$$

Case II: $n \geq 8$.

It is evident from Lemma 4.3 that the LRNs with minimum cardinality of $3(n+1)$ are $LR_{j} = V(C_{n})$, $LR_{j} = V(C_{n})$ and respectively. Also $\bigcup_{j=1}^{n} LR_{j} = V(C_{n})$, $\bigcup_{j=1}^{n} LR_{j} = V(C_{n})$ and $|LR_{j} \cap \bigcup_{j=1}^{n} LR_{j}| \geq |LR_{j}|$, where $1 \leq j \leq 7$. Let $\beta = 3(n+1)$ and $\delta = |\bigcup_{j=1}^{n} LR_{j}| = 5n$. Now we define a mapping $\kappa : V(C_{n}) \to [0, 1]$ such that

$$dimy(C_{n}) \leq \frac{30}{22} = \frac{15}{11} \leq \frac{30}{21}$$

Likewise, in Table 10 shows the LRNs with maximum cardinality of $30 = \gamma$ hence by Theorem B and Corollary C ($G$ is not bipartite), we have

$$\frac{|V(C_{n})|}{\gamma} = \frac{30}{30} = 1 < dimy(C_{n})$$

Therefore, from 6 and 7, we have

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V. CONCLUSION

In this article, we have found the local resolving neighbourhood sets of convex polytopes networks with their cardinalities and computed the extremal values of their local fractional metric dimensions under certain conditions are as follows:

- The obtained values are within the interval as prescribed by [13] and [14].
- Table 11 shows the summary of main results and Table 12 gives the values of LFMDs as they tends to ∞.
- The graphical analysis of results as shown in Figure 3 reveals that $B_n$ bears the dominant LFMD.
- Based on the findings of our results, the fire exit plan for the floor of building can be designed. For instance, consider a floor of the mall comprising of shops, food court, washrooms, lifts and escalators as shown on the top left of Figure 4. As a first step its graph theoretic version is obtained as shown on the top right of Figure 4. Whereas at the center of Figure 4 we obtain the underlying graph as $C_6$. From Theorem 4.4 the LFMD of $C_n$ is $30/21 \approx 1.43$ which means that the complete evacuation for this floor in case of fire will be done in 1 hr and 43 min. Similarly, the path for the fire exit plan can be found from Table 6.
- To find the extremal values of fractional metric dimensions of asymmetric networks is still an open problem.

ACKNOWLEDGMENT

The authors would like to thank the anonymous referees for their valuable thoughts and comments which improved the original version of this paper.

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