Analysis of the Axial Anomaly on the Lattice with $O(a)$-improved Wilson Action

D. Guadagnoli$^1$ and S. Simula$^2$

$^1$Dipartimento di Fisica, Università di Roma “La Sapienza”
Piazzale Aldo Moro 2, I-00185 Roma, Italy
$^2$Istituto Nazionale di Fisica Nucleare, Sezione Roma III
Via della Vasca Navale 84, I-00146 Roma, Italy

Abstract
Flavor singlet and non-singlet axial Ward identities are investigated using the Wilson formulation of lattice QCD with Clover $O(a)$-improvement, which breaks explicitly chiral symmetry. The matching at one-loop order of all the relevant renormalization constants with the continuum $\overline{\text{MS}}$ scheme is presented. Our calculations include: 1) the contributions arising from the Clover term of the action; 2) the complete mixing of the gluon operator $\tilde{G}\tilde{G}$ with the divergence of the singlet axial current; 3) the use of both local and extended definitions of the fermionic bilinear operators. A definition of the gluon operator $\tilde{G}\tilde{G}$ on the lattice outside the chiral limit is proposed. Our definition takes into account the possible power-divergent mixing with the pseudoscalar density, generated by the breaking of chiral symmetry. A non-perturbative procedure for the evaluation of such mixing constant is worked out. Finally, the renormalization properties of the composite insertion of the topological charge operator $\int d^4x \, \tilde{G}\tilde{G}(x)$ relevant for the lattice calculation of the neutron electric dipole moment, induced by the strong $CP$-violating term of the QCD Lagrangian, are discussed.

PACS numbers: 11.40.Ha, 12.38.Gc, 13.40.Em, 14.20.Dh

Keywords: Ward identities; lattice QCD; topological charge; electric dipole moment.
1 Introduction

The investigation of the axial anomaly is a powerful tool to unravel the structure of the QCD vacuum. Since the latter is highly non-trivial, the conservation of classical currents may be spoiled by quantum fluctuations of the vacuum. When this happens an anomaly is formed as in case of the flavor-singlet axial vector current. The axial anomaly can have deep consequences on various physical observables, like the large mass of the $\eta'$ meson or the smallness of the flavor-singlet nucleon axial coupling constant, which is commonly known as the proton spin crisis.

The connection between the topological structure of the QCD vacuum and the axial anomaly has been elucidated by 't Hooft [1]. A very interesting case where such an interrelation shows up, is provided by the neutron electric dipole moment (EDM) generated by the so-called $\theta$-term of the QCD Lagrangian [2], which in the Euclidean space is given by

$$L_\theta = i\theta \frac{g^2}{32\pi^2} G \tilde{G} = i\theta \frac{g^2}{64\pi^2} \varepsilon_{\alpha\beta\mu\nu} G^c_{\alpha\beta} G^c_{\mu\nu},$$

(1)

where $G^c_{\mu\nu}$ is the gluon field strength, $c$ the color octet index ($c = 1, \ldots, 8$) and $\theta$ a dimensionless parameter. The $\theta$-term [1] breaks both parity and time reversal symmetries and therefore it can generate a non-vanishing value of the neutron EDM\textsuperscript{a}. Available estimates of the relevant matrix element however are based on phenomenological models, as the MIT bag model of Ref. [4] or as the effective $\pi N$ chiral Lagrangian of Ref. [5]. Estimates relying on non-perturbative methods based on the fundamental theory, like lattice QCD, are still missing to date.

In Ref. [6] a strategy for evaluating the neutron EDM on the lattice induced by the strong CP violating term [1] was presented. Such a strategy is based on the standard definition of the neutron EDM, involving the insertion of the topological charge ($g^2/32\pi^2$) $\int d^4 x G \tilde{G}(x)$ in the presence of the charge density operator $J_0$ (see Eq. (4) of Ref. [6]). In case of three flavors with non-degenerate masses a complete diagrammatic analysis was performed [6] showing how the axial anomaly governs the replacement of the topological charge operator with well-defined insertions of the flavor-singlet pseudoscalar density. The applicability of the method to the case of lattice formulations that break explicitly chiral symmetry, like the Wilson and Clover actions, was discussed in Ref. [6] using general arguments.

The aim of this work is twofold: i) to present a complete one-loop calculation of the renormalization constants appearing in both singlet and non-singlet axial Ward Identities (WI’s) using Wilson fermions with the Clover $O(a)$-improvement of the action; ii) to provide a definition of the gluon operator $G \tilde{G}$ on the lattice outside the chiral limit, taking into account its possible power-divergent mixing with the pseudoscalar density. As for the one-loop matching, our calculations reproduce all the known results and add: 1) the

\textsuperscript{a}The present experimental upper limit on the neutron EDM is $d_N \equiv |d_N| < 6.3 \cdot 10^{-26} (e \cdot cm)$ at 90% confidence level \textsuperscript{3}, which corresponds to a severe bound on the magnitude of $\theta$. Indeed, using the available theoretical estimates from Refs. [4, 5], one has $d_N \approx 3 \cdot 10^{-16} |\theta| (e \cdot cm)$ leading to $|\theta| \lesssim 2 \cdot 10^{-10}$. The smallness of the parameter $\theta$ is usually referred to as the strong CP problem.
complete mixing of $G\tilde{G}$ with the divergence of the singlet axial current; and 2) the use of both local and extended definitions of the fermionic bilinear operators. As for the calculation of the mixing between a lattice discretization of $G\tilde{G}$ and the pseudoscalar density in case of lattice formulations breaking chiral symmetry a non-perturbative procedure is presented. Finally, as a separate issue, the renormalization properties of the composite insertion of the topological charge operator $\int d^4x \, G\tilde{G}(x)$ relevant for the lattice calculation of the neutron EDM are discussed.

The plan of the paper is as follows. In Section 2 the structure of singlet and non-singlet axial $WI$’s using the Wilson and Clover lattice $QCD$ formulations is briefly recalled to fix notations and basic definitions. In Sections 3 and 4 the non-singlet and singlet channels are considered, respectively. All the matching coefficients with the continuum $\overline{MS}$ scheme are explicitly calculated at one-loop order both with and without the $O(a)$-improvement of the action. The complete one-loop mixing of the gluon operator $G\tilde{G}$ with the divergence of the singlet axial current is evaluated. The issue of the possible power-divergent mixing of $G\tilde{G}$ with the pseudoscalar density is addressed and a non-perturbative procedure for evaluating the mixing constant is proposed. Moreover, in a separate subsection, the composite insertion of the topological charge operator $\int d^4x \, G\tilde{G}(x)$ relevant for the lattice calculation of the neutron EDM is considered and its renormalization properties are discussed. Section 5 is devoted to our conclusions. Finally, all the Feynman rules relevant for our calculations are collected in the Appendix.

2 Axial Ward identities on the lattice

In the subsequent discussion we will make use of the following definitions and notations. The $QCD$ action on the lattice is defined (in Euclidean space) as

$$S_{LQCD} = S_F + S_U$$
(2)

with $S_U$ being the pure Yang-Mills component \cite{7} and $S_F$ the Wilson fermion action

$$S_F = a^4 \sum_x \left\{ -\frac{1}{2a} \sum_{\mu} \bar{\psi}(x)(r - \gamma_\mu)U_\mu(x)\psi(x + \mu) 
+ \bar{\psi}(x + \mu)(r + \gamma_\mu)U_\mu^\dagger(x)\psi(x) \right\} + \bar{\psi}(x)\left(m_0 + \frac{4r}{a}\right)\psi(x),$$
(3)

where color and flavor indices are omitted, $m_0$ is the (bare) mass matrix, diagonal in flavor, and the terms proportional to $r$ are necessary to avoid the fermion doubling. The improvement of the action is represented by the Clover term

$$S_C = -a^4 \sum_x \sum_{\mu,\nu} c_{SW} \frac{ig_0ar}{4} \bar{\psi}(x)\sigma_{\mu\nu}P_{\mu\nu}(x)\psi(x),$$
(4)

with $P_{\mu\nu}$ being the usual lattice definition of the field-strength tensor $G_{\mu\nu}$ \cite{8}

$$P_{\mu\nu}(x) = \frac{1}{4a^2} \sum_{i=1}^4 \frac{1}{2i g_0} (U_i - U_i^\dagger)$$
(5)
where the sum is over the four plaquettes in the \(\mu-\nu\) plane stemming from \(x\) and taken in the counterclockwise sense (see also Appendix [1]).

To obtain axial WIs on the lattice, one starts with the usual definition of the vacuum expectation value of an operator \(O(x_1,\ldots,x_n)\)

\[
\langle O(x_1,\ldots,x_n) \rangle = \frac{1}{Z_0} \int d[G]d[\psi]d[\bar{\psi}] O(x_1,\ldots,x_n) e^{-S}
\]

where the fields are defined only on the nodes of the lattice, \(Z_0\) is the partition function and \(S = S_{LQCD} + S_C\). Performing local non-singlet axial rotations over the fermionic fields, namely

\[
\psi(x) \rightarrow \left[ 1 + i\alpha^a(x)\frac{\lambda^a}{2}\gamma_5 \right] \psi(x),
\]

\[
\bar{\psi}(x) \rightarrow \bar{\psi}(x) \left[ 1 + i\alpha^a(x)\frac{\lambda^a}{2}\gamma_5 \right],
\]

where \(\lambda^a\) are the usual \(SU(3)\) flavor matrices, and taking into account the invariance of the measure of integration in Eq. (6), one gets

\[
\langle O \frac{\delta S}{\delta (i\alpha^a(x))} \rangle = \langle \frac{\delta O}{\delta (i\alpha^a(x))} \rangle
\]

with

\[
\langle O \frac{\delta S}{\delta (i\alpha^a(x))} \rangle = -\Delta^x_{\mu}\langle O A^a_{\mu}(x) \rangle + \langle O \bar{\psi}(x)\{\frac{\lambda^a}{2},m_0\}\gamma_5 \psi(x) \rangle + \langle O [X^a(x) + X^a_C(x)] \rangle .
\]

In Eq. (9) we have indicated with \(\Delta^x_{\mu}\) the backward derivative in the \(\mu\)-direction with respect to \(x\). The non-singlet axial current \(A^a_{\mu}\) is given by

\[
A^a_{\mu}(x) = \frac{1}{2} \left[ \bar{\psi}(x)U_{\mu}(x)\gamma_\mu\frac{\lambda^a}{2}\psi(x + \mu) + h.c. \right],
\]

and the operators \(X^a\) and \(X^a_C\) are the chiral variations of the Wilson and Clover term in the action, respectively,

\[
X^a(x) = -\frac{r}{2a} \sum_\mu \left[ \bar{\psi}(x)\frac{\lambda^a}{2}\gamma_\mu U_{\mu}(x)\psi(x + \mu) + \bar{\psi}(x + \mu)\frac{\lambda^a}{2}\gamma_\mu U_{\mu}^\dagger(x)\psi(x) \right.
\]

\[
+ (x \rightarrow x - \mu) - 4\bar{\psi}(x)\frac{\lambda^a}{2}\gamma_5 \psi(x) \right] ,
\]

\[
X^a_C(x) = -\frac{ig_0ar}{2} c_{SW} \sum_{\mu\nu} \bar{\psi}(x)\frac{\lambda^a}{2}\gamma_\mu \sigma_{\mu\nu} P_{\nu}(x)\psi(x) .
\]
Similarly, performing flavor-singlet rotations
\[ \psi(x) \rightarrow [1 + i\alpha^0(x)\gamma_5] \psi(x), \]
\[ \bar{\psi}(x) \rightarrow \bar{\psi}(x) [1 + i\alpha^0(x)\gamma_5], \] (13)
one obtains
\[ \langle \mathcal{O} \frac{\delta S}{\delta (i\alpha^0(x))} \rangle = \langle \frac{\delta \mathcal{O}}{\delta (i\alpha^0(x))} \rangle \] (14)
where
\[ \langle \mathcal{O} \frac{\delta S}{\delta (i\alpha^0(x))} \rangle = -\Delta^X_{\mu} \langle \mathcal{O} A_{\mu}(x) \rangle + 2 \langle \mathcal{O} \bar{\psi}(x) m_0 \gamma_5 \psi(x) \rangle + \langle \mathcal{O} \left[ X^0(x) + X^0_C(x) \right] \rangle. \] (15)

The singlet current \( A_{\mu} \) and the operators \( X^0 \) and \( X^0_C \) are obtained from the corresponding octet operators (10)-(12), respectively, with the naive substitution \( \lambda^a/2 \rightarrow 1 \).

Equations (8)-(9) and (14)-(15) are expressed in terms of unrenormalized quantities. The operators \( X^a, X^a_C, X^0 \) and \( X^0_C \) are dimension-5 operators multiplied by one power of the lattice spacing; therefore, in any tree-level calculation they vanish identically in the continuum limit.

With the inclusion of quantum corrections the situation changes drastically. In the non-singlet channel, the operators \( X^a \) and \( X^a_C \) mix with the axial current and with the pseudoscalar density: these mixings result in a finite (multiplicative) renormalization constant for the current \( A_{\mu} \), and in an additive renormalization constant for the bare mass \( m_0 \), which multiplies the pseudoscalar density. The knowledge of such constants allows to identify the correct renormalized mass and axial current on the lattice, recalling that Eqs. (8)-(9) should reproduce the corresponding continuum \( W_I \) in the limit \( a \rightarrow 0 \).

In the singlet channel, the mixings of the operator \( X^0 \) with the divergence of the (singlet) axial current and with the (singlet) pseudoscalar density are in general different from those corresponding to \( X^a \), due to the presence of diagrams involving closed fermion loops (besides the ones present also in the non-singlet channel). This could cause in principle the singlet renormalized mass to be different from the non-singlet one: if it were so, we would be in trouble to identify the continuum limit of the lattice singlet \( W_I \), since in the continuum the renormalized quark masses in both the octet and the singlet channel are the same. However, using general non-perturbative arguments, it has been shown in Ref. [9] that renormalized masses are the same also on the lattice, so that, in this respect, a simple correspondence between the lattice \( W_I \)'s and the continuum ones can be established.

The singlet \( W_I \) on the lattice must reproduce the anomaly, represented by the term \( 2N_Fg_0^2/32\pi^2G\tilde{G} \), where \( G\tilde{G} = 1/2 \varepsilon_{\mu\nu\rho\sigma}G_{\mu\nu}G_{\rho\sigma} \) and \( G_{\mu\nu} \) is the usual gluon-field strength tensor. In a continuum regularization the singlet anomaly is generated by the fact that the regulator introduced during the renormalization process is not chiral invariant. Removing the regulator leaves a residual contribution in the action proportional to \( G\tilde{G} \).
On the lattice, which is a regulator, the action (3) with \( m_0 = r = 0 \) is perfectly chiral invariant, but it reproduces no anomaly, since it describes 16 quark species, with opposite chiral charges. A possible way to eliminate the 15 spurious fermions in the continuum limit, is represented by the Wilson term, which is however a chiral breaking term. The Wilson term generates the anomaly. Indeed, in the WI (14)-(15), the chiral variation of the Wilson term, namely \( X^0(x) \), mixes with \( \tilde{G}G \) in such a way to correctly reproduce the anomalous term present in the continuum [10]. The mixing is independent of \( r \), as long as \( r \neq 0 \).

Finally, given the anomalous non-conservation of the singlet axial current, the latter suffers an infinite (logarithmically divergent) renormalization. Analogous divergences occur for the operator \( \tilde{G}G \), so that one needs a mixing between the two operators to end up with finite quantities. Furthermore, on the lattice with broken chiral symmetry, one has to take into account that the discretization adopted for \( \tilde{G}G \) may mix with the pseudoscalar density.

Since the WI’s (8)-(9) and (14)-(15) behave differently under renormalization, they will be treated separately in the next two Sections.

3 Non-singlet axial WI’s

In this Section we analyze in detail the one-loop structure of the WI (8)-(9). For ease of presentation we will consider the case of degenerate bare masses \( m_0 \), which means \( \{\lambda^a/2, m_0\} \rightarrow 2m_0\lambda^a/2 \). First we notice that \( X^a \) and \( X^a_C \), having the same dimension and the same quantum numbers, should have an analogous behavior in perturbation theory. An analysis identical to the one made in Refs. [11, 9] for \( X^a \) leads to the identification

\[
X^a(x) + X^a_C(x) = \bar{X}^a(x) - 2\bar{M}^{(NS)}\psi(x)\frac{\lambda^a}{2}\gamma_5\psi(x) + (1 - Z_A^{(NS)})\partial_\mu A^a_\mu(x),
\]

where the operator \( \bar{X}^a(x) \) collectively refers to dimension-5 operators that vanish in the continuum limit for all matrix elements involving elementary operator insertions. Note that the renormalization constant \( Z_A^{(NS)} \) refers to the extended definition of the non-singlet axial current, given in Eq. (10).

The mixing coefficients \( \bar{M}^{(NS)} \) and \( [1 - Z_A^{(NS)}] \) appearing in Eq. (16) can be computed at the one-loop level simply by considering the correlator \( \langle \mathcal{O}(x) \psi(y)\psi(z) \rangle \) with amputation on the external legs, and \( \mathcal{O} \) given by \( X^a \) or \( X^a_C \). Its Fourier transform can be pictorially represented by the following diagrams (the corresponding Feynman rules are reported in Appendix E)

\[
\text{FT} \langle X^a(x) \psi(y)\psi(z) \rangle_{\text{amp}}^{(2)} = \]

\[+\]

\[\text{Diagram} \]

6
One gets

\[ M^{(NS)} = \frac{g_0^2}{16\pi^2} C_F \left( \frac{\Sigma_0}{a} + m_0 C \right) , \]  

where

\[ C = 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left\{ \frac{2r^2}{\Delta_2^2} (\Delta_1 \Delta_5 - \Delta_3) + c_{SW} \frac{4r^2}{\Delta_2^2} (-\Delta_3 \Delta_5 + \Delta_4) + c_{SW}^2 \frac{2r^2 \Delta_3}{\Delta_1 \Delta_2^2} (\Delta_3 \Delta_5 - \Delta_4) \right\} \]  

and \( \Sigma_0 \) is the additive mass renormalization, which appear in the one-loop approximation of the quark propagator [see below Eq. (25)].

The one-loop expression for the (finite) multiplicative renormalization of the axial current is

\[ Z_A^{(NS)} - 1 = \frac{g_0^2}{16\pi^2} C_F \left\{ 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left[ \frac{1}{2\Delta_2} + \frac{1}{\Delta_2^2} (5\Delta_1 \Delta_3 + 4\Delta_1 - 5\Delta_1^2 + \Delta_3^2 - 13\Delta_3) 
+ \frac{r^2 \Delta_1}{\Delta_2} (\Delta_3 - 6\Delta_1 + \Delta_1^2) + c_{SW} \left( \frac{1}{2\Delta_1 \Delta_2} (4\Delta_3 - \Delta_4) 
+ \frac{1}{\Delta_2^2} (10\Delta_1 \Delta_3 - 2\Delta_1 \Delta_4 - 2\Delta_1^2 \Delta_3 - 8\Delta_3 - 4\Delta_3^2 + 6\Delta_4 - 4\Delta_7) - \frac{\Delta_3}{2\Delta_2} \right) 
+ c_{SW}^2 \left( \frac{1}{\Delta_1 \Delta_2^2} (9\Delta_3 \Delta_4 - 4\Delta_3^2 - 2\Delta_4^2) + \frac{1}{2\Delta_1 \Delta_2} (4\Delta_3 + \Delta_3^2 - 6\Delta_4 + 2\Delta_7) 
+ \frac{1}{4\Delta_2} (\Delta_1 \Delta_3 - 6\Delta_3 + 3\Delta_4) + \frac{\Delta_3}{\Delta_2^2} (\Delta_3 - 2\Delta_4) \right) + c_{SW} \frac{2r^2 \Delta_1}{\Delta_2^2} (\Delta_4 - 3\Delta_3 \Delta_5) 
+ c_{SW}^2 \frac{r^2}{\Delta_2^4} (\Delta_1 (-8\Delta_3 - 3\Delta_3^2 + 12\Delta_4 - 4\Delta_7 + 6\Delta_1 \Delta_3 - 3\Delta_1 \Delta_4 - \Delta_1^2 \Delta_3) 
- \Delta_3 \Delta_4 + 4\Delta_3^2) \right] \right\} . \]  

The symbols \( \Delta_i \) stand for the following functions

\[ \Delta_1 = \sum_{\mu} \sin^2 \left( \frac{q_\mu}{2} \right) , \quad \Delta_2 = \sum_{\mu} \sin^2 (q_\mu) + \left( 2r \sum_{\mu} \sin^2 \left( \frac{q_\mu}{2} \right) \right)^2 , \]

\[ \Delta_3 = \frac{1}{4} \sum_{\mu} \sin^2 (q_\mu) , \quad \Delta_4 = \sum_{\mu} \sin^2 \left( \frac{q_\mu}{2} \right) \cos^4 \left( \frac{q_\mu}{2} \right) , \]

\[ 7 \]
\[\begin{align*}
\Delta_5 &= \sum_{\mu} \cos^2\left(\frac{q_\mu}{2}\right), \\
\Delta_6 &= \sum_{\mu} \cos^4\left(\frac{q_\mu}{2}\right), \\
\Delta_7 &= \sum_{\mu} \sin^2\left(\frac{q_\mu}{2}\right)\cos^6\left(\frac{q_\mu}{2}\right), \\
\Delta_8 &= \sum_{\mu} \sin^2\left(\frac{q_\mu}{2}\right)\cos^8\left(\frac{q_\mu}{2}\right),
\end{align*}\]

which for \(\Delta_1, ..., \Delta_7\) are identical to the corresponding functions used in Ref. [8], while \(\Delta_8\) is included here for completeness but used only in Section 4. Numerical results for the constants \(C\) and \(Z_A^{(NS)}\) at various values of the Wilson parameter \(r\) both for \(c_{SW} = 0\) and for \(c_{SW} = 1\) are reported in Table 1.

| \(r\) | \(C\) \(c_{SW}=0\) | \(Z_A^{(NS)} - 1\) \(c_{SW}=0\) | \(C\) \(c_{SW}=1\) | \(Z_A^{(NS)} - 1\) \(c_{SW}=1\) |
|-------|--------|-----------------|--------|-----------------|
| 0.0   | 0.0    | 0.0             | 0.0    | 0.0             |
| 0.2   | 8.09   | -3.62           | 5.29   | -2.67           |
| 0.4   | 11.41  | -5.82           | 6.20   | -4.13           |
| 0.6   | 11.25  | -7.00           | 5.11   | -4.87           |
| 0.8   | 10.43  | -7.89           | 3.96   | -5.37           |
| 1.0   | 9.64   | -8.66           | 3.07   | -5.75           |

Table 1: Numerical results for \(C\) and \([Z_A^{(NS)} - 1]\) for different values of \(r\) and \(c_{SW}\). A factor \(g_0^2 C_F/16\pi^2\) is understood for \([Z_A^{(NS)} - 1]\) [see Eq. (20)].

Equations (19)-(20), evaluated at \(c_{SW} = 0\), coincide with the ones obtained in Ref. [11], where the Clover term was not included. An explicit check of the correctness of the Clover contributions to Eqs. (19) and (20) can be obtained from the requirement that the WI (8)-(9) with \(O = \bar{\psi}(y)\psi(z)\) should be satisfied. This condition reads as

\[
\Delta_\mu^x \langle Z_A^{(NS)} A^a_\mu(x) \psi(y)\bar{\psi}(z) \rangle = 2(m_0 - \bar{M}^{(NS)}) \langle \bar{\psi}(x) \Lambda^a_5 \gamma_5 \psi(y) \bar{\psi}(z) \rangle - 2 \delta(x-y) \langle \bar{\psi}(z) \rangle - 2 \delta(x-z) \langle \psi(y) \bar{\psi}(z) \rangle, \tag{22}\]

where we have taken into account Eq. (13). In momentum space, denoting the amputated Green's functions with the insertion of the operators \(\Delta_\mu A^a_\mu\) and \((\bar{\psi} \lambda^a \gamma_5 \psi/2)\) by \(\Lambda^a_{\Delta A_\mu}\) and \(\Lambda^a_{\gamma_5}\), respectively, one gets

\[
Z_A^{(NS)} \Lambda^a_{\Delta A_\mu}(p', p) = 2 \left(m_0 - \bar{M}^{(NS)}\right) \Lambda^a_{\gamma_5}(p', p) - \gamma_5 \frac{\lambda^a}{2} S^{-1}(p) - S^{-1}(p') \gamma_5 \frac{\lambda^a}{2}, \tag{23}\]

where \(p\) (\(p'\)) denotes momentum of the incoming (outcoming) quark, and \(S(p)\) represents the quark propagator. The diagrams corresponding to \(\Lambda^a_{\Delta A_\mu}\) are analogous to the ones considered
for $X^a$ and $X^a_0$, reported in Eq. (17). For $\Lambda^a_0$, one has to consider only the first diagram, being the pseudoscalar current a local operator. After calculating at one loop all the terms in Eq. (23), including the Clover contributions, one is able to check Eqs. (18)-(20). The explicit one-loop expression for the lattice inverse quark-propagator $S^{-1}(p)$ is
\begin{equation}
S^{-1}(p) = i\psi + m_0 - \frac{g_0^2}{16\pi^2}C_F \left( \frac{1}{a} \Sigma_0 + i\psi \Sigma_1(p) + m_0 \Sigma_2(p) \right)
\end{equation}
with $\Sigma_{0,1,2}$ given by
\begin{align}
\Sigma_0 &= 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} r \left\{ \frac{\Delta_3}{\Delta_1 \Delta_2} - \frac{1}{2\Delta_1} - \frac{\Delta_5}{2\Delta_2} + \frac{r^2}{2\Delta_2} \left( \Delta_3 \Delta_5 - \Delta_4 \right) \right\}, \\
\Sigma_1(p) &= \gamma_E - F_{0001} + 2 \int_0^1 dx x \ln \left( a^2 m_0^2 (1-x) + a^2 p^2 x (1-x) \right) \\
&+ 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left\{ \frac{\Delta_3}{16\Delta_1} + \frac{1}{8\Delta_1} - \frac{\Delta_3}{8\Delta_1 \Delta_2} + \frac{1}{8\Delta_1^2} \left( 2\Delta_4 - \Delta_3 \Delta_5 \right) \right\}, \\
&+ \frac{r^2}{4\Delta_2} (2 - \Delta_1) + r^2 c_{SW} \left( \frac{1}{2\Delta_1 \Delta_2} (\Delta_4 - 4\Delta_3) + \frac{1}{8\Delta_2} (-9\Delta_1 + 2\Delta_3) \right) \\
&+ 4\Delta_3 + 4\Delta_5 - 2\Delta_6 + 4) + r^2 c_{SW}^2 \left( \frac{1}{8\Delta_1^2 \Delta_2} (-\Delta_3 \Delta_4 + 4\Delta_3^2) \right) \\
&+ \frac{1}{8\Delta_1 \Delta_2} (-4\Delta_3 \Delta_5 + 2\Delta_3 \Delta_6 - 4\Delta_3 + 4\Delta_4 - \Delta_7) \\
&+ \frac{1}{8\Delta_2} (-2\Delta_1 \Delta_3 + 9\Delta_3 - 2\Delta_4) \right\}, \\
\Sigma_2(p) &= 4 \left[ \gamma_E - F_{0000} + \int_0^1 dx x \ln \left( a^2 m_0^2 (1-x) + a^2 p^2 x (1-x) \right) \right] \\
&+ 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left\{ -\frac{4\Delta_1 - \Delta_2}{4\Delta_1^2 \Delta_2} + \frac{1}{4\Delta_2} + \frac{r^2}{4\Delta_2^2} \left( 2\Delta_1 \Delta_5 + \Delta_2 - 4\Delta_3 \right) \right\}, \\
&+ r^4 \left\{ -\frac{\Delta_3}{\Delta_2^2} \right\} + c_{SW}^2 \left( \frac{4r^2}{\Delta_2} \left( -\Delta_3 \Delta_5 + \Delta_4 \right) + c_{SW}^2 \frac{r^2}{4\Delta_1 \Delta_2} \left( -\Delta_3 \Delta_5 - \Delta_4 \right) \right) \\
&+ c_{SW}^2 \frac{2r^4 \Delta_1}{\Delta_2^3} \left( -\Delta_3 \Delta_5 + \Delta_4 \right) \right\}, \\
\end{align}
with $F_{0000} = 4.36898$, $F_{0001} = 1.31096$ and $\gamma_E = 0.577216$. Equations (25) coincide with the analytical results of Refs. [11, 12] (at $c_{SW} = 0$) and Ref. [8] (at $c_{SW} = 1$).

The check of Eqs. (18)-(20) via the $WT$ (23) can proceed now through the projection of all the quantities onto the structures $(\bar{p}' - \bar{p})\gamma_5$ and $\gamma_5$ (orthogonal to each other) with the appropriate normalizations. In other words we calculate the quantities
\begin{equation}
\Pi_{\Gamma}(p', p) = \frac{1}{N_{\Gamma}} Tr (T(p', p) \Gamma),
\end{equation}

9
where the trace is taken over the Dirac indices, $T(p', p)$ stands for any of the terms appearing in Eq. (23), $\Gamma$ is equal to $(p' - p)\gamma_5$ or $\gamma_5$ and $N_\Gamma$ is a normalization constant. We have calculated all the relevant projections and checked positively the correctness of Eqs. (19)-(20).

Note that the projection of the amputated diagram of the (extended) axial current over $(p' - p)\gamma_5$ and of the pseudoscalar density over $\gamma_5$ can be compared with the corresponding quantities calculated in the continuum MS-scheme in order to extract the matching constants between $\overline{\text{MS}}$ and the lattice [see Subsect. (3.1)], because the dependence on the external momenta is cancelled out in the difference between the two schemes.

Finally, from Eq. (22) it can be noticed that the explicit contribution of the operators $X^a$ and $X^a_C$ has been traded with a redefinition of the lattice axial current and of the bare mass $m_0$, i.e.

$$
\Delta_\mu A^a_\mu \rightarrow Z_A^{(NS)} \Delta_\mu A^a_\mu ,
$$

$$
m_0 \rightarrow m_0 - M^{(NS)} \equiv m_L.
$$

### 3.1 Matching with the continuum in the non-singlet channel

The Green’s functions calculated with the (bare) fields and operators on the lattice can be matched to the corresponding ones renormalized in the $\overline{\text{MS}}$ scheme, by rescaling the fields and the operators on the lattice with appropriate constants. For instance the bare quark field on the lattice $\psi_L(x)$ can be rescaled through $\psi_L(x) = Z_\psi \psi_R(x)$, where the subscript $R$ indicates a renormalized ($\overline{\text{MS}}$) quantity in the continuum. The constant $Z_\psi$ can be evaluated via the relation

$$
(S^{-1}(p))^R = Z_\psi (S^{-1}(p))^L
$$

which simply follows from the definition of the quark propagator. At one-loop order the relations

$$
(S^{-1}(p))^L = i\slashed{p} \left[ 1 - \frac{g_0^2}{16\pi^2} C_F \Sigma_1^L(p) \right] + m_L \left[ 1 - \frac{g_0^2}{16\pi^2} C_F \Sigma_2^L(p) \right],
$$

$$
(S^{-1}(p))^R = i\slashed{p} \left[ 1 - \frac{g_0^2}{16\pi^2} C_F \Sigma_1^R(p) \right] + m_R \left[ 1 - \frac{g_0^2}{16\pi^2} C_F \Sigma_2^R(p) \right],
$$

imply

$$
Z_\psi = 1 - \frac{g_0^2}{16\pi^2} C_F \Delta \Sigma_1, \quad \Delta \Sigma_1 = \Sigma_1^R(p) - \Sigma_1^L(p).
$$

Analogously, one can define a matching for the mass on the lattice from the relation

$$
m_R = Z_m m_L.
$$
obtaining, thanks to the use of Eq. (31) and of the definition (32),

\[ Z_m = Z_\psi (1 - \frac{g_0^2}{16\pi^2} C_F \Delta_{\Sigma_2}) . \] (34)

A similar procedure leads to the definition of the Z’s for the matching of composite operators. In the non-singlet channel, we are interested in Green’s functions involving fermion bilinear operators \( O \) contracted with two external quark fields, with amputation on the external legs. The corresponding functions in momentum space will be indicated by \( \Lambda_O(p', p) \). One gets the relation

\[ \Lambda_R^O(p', p) = Z_\psi Z_O \Lambda_L^O(p', p) \] (35)

which defines \( Z_O \) as the constant of matching for the operator \( O \). In perturbation theory one can express the constants in Eq. (35) as

\[ Z_\psi Z_O = 1 + \frac{g_0^2}{16\pi^2} C_F \Delta_O \ , \quad \Delta_O = \hat{\Lambda}_O^R - \hat{\Lambda}_O^L \] (36)

where the expressions \( \hat{\Lambda} \) collect all the one-loop diagrams. The relation (36) implies

\[ Z_O = 1 + \frac{g_0^2}{16\pi^2} C_F (\Delta_O + \Delta_{\Sigma_1}) . \] (37)

Denoting the subtraction scale in the continuum by the symbol \( \mu \), our results for \( \Delta_{\Sigma_1} \) and \( \Delta_{\Sigma_2} \) are

\[ \Delta_{\Sigma_1} = 1 - \ln a^2 \mu^2 - \gamma_E + F_{0001} - 16\pi^2 \int_\pi^\infty \frac{d^4q}{(2\pi)^4} \left\{ \frac{\Delta_3}{16\Delta_1^2} + \frac{1}{8\Delta_1} - \frac{\Delta_3}{8\Delta_1 \Delta_2} \right\} \]

\[ + \frac{1}{8\Delta_1^2 \Delta_2} (2\Delta_4 - \Delta_2^2) + \frac{r^2}{4\Delta_2} (2 - \Delta_1) + r^2 c_{SW} \left( \frac{1}{2\Delta_1 \Delta_2} (\Delta_4 - 4\Delta_3) \right) \]

\[ + \frac{1}{8\Delta_2} (-9\Delta_1 + 2\Delta_2^2 + 4\Delta_3 + 4\Delta_5 - 2\Delta_6 + 4) \]

\[ + r^2 c_{SW}^2 \left( \frac{1}{8\Delta_1^2 \Delta_2} (-\Delta_3 + 4\Delta_3^2) + \frac{1}{8\Delta_1 \Delta_2} (-4\Delta_3 \Delta_5 + 2\Delta_3 \Delta_6) \right) \]

\[ - 4\Delta_3 + 4\Delta_4 - \Delta_7 + \frac{1}{8\Delta_2} (-2\Delta_1 \Delta_3 + 9\Delta_3 - 2\Delta_4) \left. \right\}. \] (38)

\[ \Delta_{\Sigma_2} = 4 \left( \frac{1}{2} - \ln a^2 \mu^2 - \gamma_E + F_{0000} \right) - 16\pi^2 \int_\pi^\infty \frac{d^4q}{(2\pi)^4} \left\{ \frac{4\Delta_1 - \Delta_2}{4\Delta_1^2 \Delta_2} + \frac{1}{4\Delta_2} \right\} \]

\[ + \frac{r^2}{\Delta_2} (2\Delta_1 \Delta_5 + \Delta_4) + 4r^2 \left( -\frac{\Delta_2^2}{\Delta_2} \right) \]

\[ + c_{SW}^2 4r^2 (-\Delta_3 \Delta_5 + \Delta_4) + c_{SW}^2 r^2 \left( \frac{4\Delta_1 \Delta_2}{\Delta_2} (\Delta_3 \Delta_5 - \Delta_4) \right) \]

\[ + c_{SW}^2 2r^4 \left( \frac{1}{\Delta_2^3} (-\Delta_3 \Delta_5 + \Delta_4) \right). \] (39)
The numerical results obtained for Eqs. (25), (38) and (39) at various values of \(r\), are reported in Tables 2 and 3 for the two cases \(c_{SW} = 0\) and \(c_{SW} = 1\), respectively.

| \(r\) | \(\Sigma_0\) | \(\Delta_{\Sigma_1}\) | \(\Delta_{\Sigma_2}\) |
|-------|--------------|-----------------|-----------------|
| 0.0   | 0            | -6.04           | 33.17           |
| 0.2   | -19.79       | -7.13           | 18.63           |
| 0.4   | -30.70       | -8.91           | 8.35            |
| 0.6   | -38.29       | -10.47          | 3.86            |
| 0.8   | -44.96       | -11.77          | 1.55            |
| 1.0   | -51.43       | -12.85          | 0.10            |

Table 2: Numerical results for \(\Sigma_0\) and \(\Delta_{\Sigma_{1,2}}\) [see Eqs. (25), (38) and (39)] for different values of \(r\) and with \(c_{SW} = 0\). The choice \(\mu = 1/a\) is understood.

| \(r\) | \(\Sigma_0\) | \(\Delta_{\Sigma_1}\) | \(\Delta_{\Sigma_2}\) |
|-------|--------------|-----------------|-----------------|
| 0.0   | 0            | -6.04           | 33.17           |
| 0.2   | -12.04       | -6.52           | 21.81           |
| 0.4   | -18.31       | -7.40           | 14.69           |
| 0.6   | -22.98       | -8.16           | 11.95           |
| 0.8   | -27.44       | -8.75           | 10.74           |
| 1.0   | -31.99       | -9.21           | 10.10           |

Table 3: The same as in Table 2 but with \(c_{SW} = 1\).

As for the quantities \(\Delta_I\), \(\Delta_{\gamma_5}\) and \(\Delta_{ext_{\gamma_5}}\) one obtains

\[
\Delta_I = 4\left\{-1 + \ln a^2 \mu^2 - F_{0001} + \gamma_E - 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left[ -\frac{\Delta_3}{16\Delta_1^2} + \frac{\Delta_4}{4\Delta_1\Delta_2} + \frac{\Delta_5}{4\Delta_1}\right] \right\},
\]

\[
\Delta_{\gamma_5} = 4\left\{-1 + \ln a^2 \mu^2 - F_{0001} + \gamma_E - 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left[ -\frac{\Delta_3}{16\Delta_1^2} + \frac{\Delta_4}{4\Delta_1\Delta_2} + \frac{\Delta_5}{4\Delta_1}\right] \right\},
\]

\[
\Delta_{ext_{\gamma_5}} = 4\left\{-1 + \ln a^2 \mu^2 - F_{0001} + \gamma_E - 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left[ -\frac{\Delta_3}{16\Delta_1^2} + \frac{\Delta_4}{4\Delta_1\Delta_2} + \frac{\Delta_5}{4\Delta_1}\right] \right\},
\]
\[
\Delta_{\mu\gamma_5}^{ext} = \Pi_{\mu}^{(2)R} - \Pi_{\mu}^{(2)}_{\mu,ext} = \\
= -2 + \ln a^2 \mu^2 - F_{0001} + \gamma_E - 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \left[ -\frac{\Delta_3}{16\Delta_1^3} + \frac{\Delta_3}{\Delta_1 \Delta_2^2} - \frac{1}{8\Delta_1} \right] \\
+ \frac{\Delta_3}{4\Delta_1 \Delta_2} + \frac{1}{\Delta_1 \Delta_2^2} (3\Delta_3 + \Delta_2^2 - 5\Delta_4 + 2\Delta_7) + \frac{1}{2\Delta_2^2} (\Delta_1 \Delta_3 - 6\Delta_3) \\
+ 2\Delta_4 + r^2 \frac{\Delta_3}{4\Delta_2} + \frac{r^2}{2\Delta_2^2} (-3\Delta_1 \Delta_3 - 4\Delta_1 \Delta_5 + 2\Delta_1 \Delta_6 + 4\Delta_1 + 3\Delta_1^2) \\
- \Delta_1^2 + 7\Delta_3 - 2\Delta_4 + \frac{r^2 \Delta_3}{2\Delta_2} (2 - \Delta_1) + c_{SW} \frac{r^2}{\Delta_2} (-7\Delta_1 \Delta_3 + 4\Delta_1 \Delta_4) \\
- 2\Delta_1^2 \Delta_3 + 8\Delta_3 \Delta_5 - 4\Delta_3 \Delta_6 - 4\Delta_3 - 15\Delta_4 + 6\Delta_7) \\
+ c_{SW}^2 \frac{r^2}{2\Delta_1 \Delta_2^2} (3\Delta_3 \Delta_4 - 4\Delta_3 \Delta_4 - 4\Delta_3 \Delta_5 + 2\Delta_3 \Delta_6 + 4\Delta_3^2 + 2\Delta_4^2) \\
+ \Delta_1 (3\Delta_3^2 - \Delta_3 \Delta_4 - \Delta_1 \Delta_3^2) + c_{SW}^2 \frac{r^4 \Delta_1}{2\Delta_2^2} (4\Delta_3 \Delta_5 - 2\Delta_3 \Delta_6) \\
- 12\Delta_4 + 4\Delta_7 - 4\Delta_1 \Delta_3 + 3\Delta_1 \Delta_4 + \Delta_1^3 \Delta_3) \right]. \tag{42}
\]

We notice that using Eqs. (38) and (42) one has

\[
\frac{g_0^2}{16\pi^2} C_F \left[ \Delta_{\mu\gamma_5}^{ext} + \Delta_{\Sigma_1} \right] = Z_A^{(NS)} - 1, \tag{43}
\]

where \(Z_A^{(NS)}\) is given by Eq. (20).

For completeness, we report also the matching constants for the operators \(\bar{\psi}(x)\gamma_\mu \psi(x)\), \(\bar{\psi}(x)\gamma_\mu \gamma_5 \psi(x)\) and \(\bar{\psi}(x)\gamma_{\mu\nu} \psi(x)\) [8]:

\[
\Delta_\gamma = -2 + \ln a^2 \mu^2 - F_{0001} + \gamma_E - 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \left[ -\frac{\Delta_3}{16\Delta_1^3} + \frac{\Delta_4}{\Delta_1 \Delta_2^2} \right] \\
+ \frac{r^2}{2\Delta_2^2} \left( \Delta_1 \Delta_5 + 3\Delta_3 \right) + \frac{r^4 \Delta_3}{\Delta_2^2} + c_{SW} \frac{r^2}{2\Delta_2^2} \left( -\Delta_3 \Delta_5 + \Delta_4 \right) \\
+ \frac{r^2 c_{SW}^2 \Delta_3}{2\Delta_1 \Delta_2} \left( \Delta_3 \Delta_5 - \Delta_4 \right) \tag{44}
\]

\[
\Delta_{\gamma\gamma_5} = -2 + \ln a^2 \mu^2 - F_{0001} + \gamma_E - 16\pi^2 \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \left[ -\frac{\Delta_3}{16\Delta_1^3} + \frac{\Delta_4}{\Delta_1 \Delta_2^2} \right] \\
+ \frac{r^2}{2\Delta_2^2} \left( -\Delta_1 \Delta_5 + 5\Delta_3 \right) + \frac{r^4 \Delta_3}{\Delta_2^2} + c_{SW} \frac{r^2}{2\Delta_2^2} \left( \Delta_3 \Delta_5 - \Delta_4 \right) \\
+ \frac{r^2 c_{SW}^2 \Delta_3}{2\Delta_1 \Delta_2} \left( \Delta_4 - \Delta_3 \Delta_5 \right) \tag{45}
\]

\[
\Delta_{\sigma_{\mu\nu}} = -16\pi^2 \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \left[ \frac{1}{3\Delta_1 \Delta_2^2} \left( \Delta_1 \Delta_3 - 4(\Delta_3 - \Delta_4) \right) + 2r^2 \frac{\Delta_3}{\Delta_2^2} + r^4 \frac{\Delta_3}{\Delta_2^2} \right] \\
+ \frac{2r^2 c_{SW}}{3\Delta_2} \left( \Delta_1 - \Delta_3 \Delta_5 \right) + \frac{r^4 c_{SW}^2 \Delta_1}{3\Delta_2} \left( \Delta_1 - \Delta_3 \Delta_5 \right) \tag{46}
\]

13
The numerical values of the matching constants $\Delta_I$, $\Delta_{\gamma_5}$, $\Delta_{\gamma_5\gamma_5}$, $\Delta_{\gamma_5\mu}$, $\Delta_{\sigma_{\mu\nu}}$ and $\Delta_{\gamma_5\gamma_5}^{ext}$, obtained for various values of $r$, are collected in Tables 4 and 5. Our results for the local operators have been explicitly checked against the corresponding ones of Refs. [8, 13, 14], while those for $\Delta_{\gamma_5\gamma_5}^{ext}$ at $c_{SW} = 0$ coincide with the ones of Ref. [15].

### Table 4: Numerical results for various $\Delta_O$ for different values of $r$ and with $c_{SW} = 0$.  

| $r$  | $\Delta_I$ | $\Delta_{\gamma_5}$ | $\Delta_{\gamma_5\gamma_5}$ | $\Delta_{\gamma_5\mu}$ | $\Delta_{\sigma_{\mu\nu}}$ | $\Delta_{\gamma_5\gamma_5}^{ext}$ |
|------|------------|--------------------|-----------------|-----------------|-----------------|-----------------|
| 0.0  | -33.16     | -33.16             | -8.74           | -8.74           | 0.74            | 6.04            |
| 0.2  | -18.63     | -26.72             | -8.97           | -4.92           | -0.37           | 3.52            |
| 0.4  | -8.34      | -19.75             | -8.76           | -3.05           | -1.86           | 3.09            |
| 0.6  | -3.85      | -15.10             | -8.36           | -2.73           | -2.90           | 3.47            |
| 0.8  | -1.55      | -11.98             | -8.02           | -2.80           | -3.63           | 3.88            |
| 1.0  | -0.10      | -9.74              | -7.76           | -2.94           | -4.17           | 4.19            |

### Table 5: The same as in Table 4 but with $c_{SW} = 1$.  

| $r$  | $\Delta_I$ | $\Delta_{\gamma_5}$ | $\Delta_{\gamma_5\gamma_5}$ | $\Delta_{\gamma_5\mu}$ | $\Delta_{\sigma_{\mu\nu}}$ | $\Delta_{\gamma_5\gamma_5}^{ext}$ |
|------|------------|--------------------|-----------------|-----------------|-----------------|-----------------|
| 0.0  | -33.16     | -33.16             | -8.74           | -8.74           | 0.74            | 6.04            |
| 0.2  | -21.80     | -27.09             | -8.27           | -5.62           | -0.23           | 3.85            |
| 0.4  | -14.69     | -20.89             | -7.45           | -4.35           | -0.61           | 3.27            |
| 0.6  | -11.94     | -17.05             | -6.82           | -4.27           | -1.23           | 3.29            |
| 0.8  | -10.73     | -14.70             | -6.40           | -4.42           | -1.64           | 3.38            |
| 1.0  | -10.10     | -13.17             | -6.12           | -4.59           | -1.93           | 3.46            |

In order to get a further independent check of the expression (19) for the constant $C$ one can consider the relation

$$m_R = Z_m \left( m_0 - \frac{g_0^2}{16\pi^2} C_F \Sigma_0 \right) = \frac{m_0 - \bar{M}(NS)}{Z_{\gamma_5}},$$

where the first equality follows from the definition (33), while the second is a consequence of Eq. (22) and of the definition (35). Now, using the one-loop expressions

$$Z_m = 1 - \frac{g_0^2}{16\pi^2} C_F \left( \Delta_{\Sigma_1} + \Delta_{\Sigma_2} \right),$$

$$\bar{M}(NS) = \frac{g_0^2}{16\pi^2} C_F \left( \Sigma_0 + m_0 C \right),$$

$$Z_{\gamma_5} = 1 + \frac{g_0^2}{16\pi^2} C_F \left( \Delta_{\gamma_5} + \Delta_{\Sigma_1} \right),$$

(47)
one easily finds

$$C = -\Delta \Sigma - \Delta \gamma_5 ,$$  \hspace{1cm} (49)$$

which can be checked analytically using Eqs. (19), (39) and (41).

4 Singlet axial WI’s

Let us now turn to the singlet $WI$ (14)-(15). Because of the anomaly, the operator $X^0 + X_C^0$, representing the chiral variation of the Wilson and Clover terms, can be written as [11]

$$X^0(x) + X_C^0(x) = X^0(x) + X_C^0(x) - 2M_S^S \bar{\psi}(x) \gamma_5 \psi(x) + \left(1 - Z_A^S\right) \partial_{\mu} A_{\mu}(x)$$

$$+ 2N_F g_0^2 \frac{32\pi^2}{Z_{GG}} \bar{G} \bar{G}_{sub}(x) ,$$  \hspace{1cm} (50)$$

where, again, $X^0$ and $X_C^0$ indicate dimension-5 operators vanishing in the continuum limit in all matrix elements involving elementary operator insertions, and the symbol $G\bar{G}_{sub}$ indicates a lattice discretization of the corresponding continuum operator $G\bar{G}$. In Eq. (50) $M_S^S$ is the additive mass-renormalization constant, while $Z_A^S$ and $Z_{GG}$ are multiplicative constants in the singlet channel.

Before addressing the calculation of the renormalization constants, we point out that in Eq. (50) $G\bar{G}_{sub}$ should be defined in such a way to avoid ambiguities with the pseudoscalar density. Indeed, let us consider the commonly used discretization $G\bar{G}_L$ obtained through the use of the symmetric plaquette, viz. [10]

$$G\bar{G}_L(x) \equiv \varepsilon_{\mu\nu\rho\sigma} Tr(P_{\mu\nu}(x)P_{\rho\sigma}(x)) ,$$  \hspace{1cm} (51)$$

where $P_{\mu\nu}$ is defined in Eq. (5) and the sum over the greek indices is understood. Expanding the definition (51) in powers of $a$, one gets

$$G\bar{G}_L(x) = G\bar{G}(x) + a^2 O_6(x)$$  \hspace{1cm} (52)$$

where $G\bar{G} = 1/(\varepsilon_{\mu\nu\rho\sigma} G^\mu_{\nu} G^\rho_{\sigma})$ and $O_6$ denotes a dimension-6 operator. Since with Wilson fermions chiral symmetry is broken, there is no guarantee that the operator $O_6(x)$ does not mix with the pseudoscalar density. In case of mixing one expects by dimensional reasons a (cubic) divergence, leading to a linear-divergent mixing of $G\bar{G}_L$ with the pseudoscalar density. Such a divergence has to be subtracted and therefore the operator $G\bar{G}_{sub}$, appearing in Eq. (50), should be related to $G\bar{G}_L$ by

$$G\bar{G}_{sub}(x) = G\bar{G}_L(x) - \Delta M_L \bar{\psi}(x) \gamma_5 \psi(x)$$  \hspace{1cm} (53)$$

where $\Delta M_L$ contains the $1/a$ divergence and is in general dependent on the particular discretization $G\bar{G}_L$. Up to now, a non-perturbative definition of $\Delta M_L$ is not known, and
we anticipate here that at one-loop order we find no mixing, i.e. $\Delta M_L = O(g_0^2)$. However a non-perturbative definition of $\Delta M_L$ is mandatory, particularly outside the chiral limit. To this end we require that $G\tilde{G}_{\text{sub}}$ should be a chiral singlet.

First of all our requirement on $G\tilde{G}_{\text{sub}}$ is needed to guarantee the proportionality between $(m_0 - \bar{M}^{(NS)})$ and $(m_o - \bar{M}^{(S)})$, as it is demonstrated in Ref. [9]. This implies that: i) the chiral point is the same in the singlet and non-singlet channels; and ii) the renormalized mass is the same in the two channels and it can be therefore identified with the continuum renormalized mass. The latter point follows from a suitable choice of the renormalization constants for the singlet and octet pseudoscalar densities, which in turn arises from the transformation properties of both scalar and pseudoscalar densities as members of the same chiral multiplet.

Let us now consider the non-singlet $WI$ and the case $O = S^a(y_1, y_2) G\tilde{G}_{\text{sub}}(z)$, where $S^a(y_1, y_2) = \bar{\psi}(y_1)\lambda^a\psi(y_2)/2$ with $y_1 \neq y_2$; one has

$$\Delta^x_\mu(A^a_\mu(x)S^a(y_1, y_2) G\tilde{G}_{\text{sub}}(z)) = 2m_L\langle P^a_5(x) S^a(y_1, y_2) G\tilde{G}_{\text{sub}}(z) \rangle - C_F \langle \delta(x - y_1) + \delta(x - y_2) \rangle \langle P_5(y_1, y_2) G\tilde{G}_{\text{sub}}(z) \rangle + \langle \left[ \bar{\tilde{X}}^a(x) + \bar{X}^a_C(x) \right] S^a(y_1, y_2) G\tilde{G}_{\text{sub}}(z) \rangle,$$

where $P^a_5(x) = \bar{\psi}(x)\lambda^a\gamma_5\psi(x)/2$, $P_5(y_1, y_2) = \bar{\psi}(y_1)\gamma_5\psi(y_2)$, $A^a_\mu(x) = Z^{(NS)}_A A^a_\mu(x)$, $m_L = m_0 - \bar{M}^{(NS)}$ and the sum over the flavor index $a$ ($a = 1, ..., 8$) is understood. In Eq. (54) we have considered that $\delta G\tilde{G}_{\text{sub}}(z)/\delta(i\alpha^a(x)) = 0$, because $G\tilde{G}_{\text{sub}}$ is assumed to be a chiral singlet. After integration over the whole space-time in $x$ and $z$ one gets

$$0 = m_L \int d^4x d^4z \ P^a_5(x) S^a(y_1, y_2) G\tilde{G}_{\text{sub}}(z) - C_F \langle P_5(y_1, y_2) \int d^4z G\tilde{G}_{\text{sub}}(z) \rangle + \frac{1}{2} \langle \int d^4x d^4z \ [\bar{\tilde{X}}^a(x) + \bar{X}^a_C(x)] S^a(y_1, y_2) G\tilde{G}_{\text{sub}}(z) \rangle.$$

We have now to discuss the possible presence of contact terms which may arise in the first and third terms of the r.h.s. of Eq. (55) when $x \approx z$. Note that the operator $S^a(y_1, y_2)$ is a string of elementary operators taken at different space-time points $y_1 \neq y_2$; therefore it cannot generate contact terms when inserted with composite operators. The general structure of the possible contact terms in Eq. (55) can be derived using the results of Ref. 14, where the method of functional integral with the generalized mass term, $\sigma^a \bar{\psi}\lambda^a\psi/2 + i\pi^a \bar{\psi}\lambda^a\gamma_5\psi/2$, was developed to derive $WI$’s. The only possible contact terms should be as follows

$$\int d^4x d^4z \ P^a_5(x) S^a(y_1, y_2) O'(z),$$

$$\int d^4x d^4z \ [\bar{\tilde{X}}^a(x) + \bar{X}^a_C(x)] S^a(y_1, y_2) O'(z),$$

1The integration over $x$ is used to cancel out the l.h.s. of Eq. (54), including possible contact terms between $A^a_\mu(x)$ and $G\tilde{G}_{\text{sub}}(z)$. The integration over $z$ allows to get rid of the mixing of $G\tilde{G}_{\text{sub}}(z)$ with the singlet $\partial_\mu A_\mu(x)$ [see below Eq. (50)]. Note that such a mixing does not change the property of the renormalized $G\tilde{G}_R$ to be a chiral singlet.
\[
\int d^4x \, d^4z \, S^a(y_1, y_2) \, \frac{\delta O'(z)}{\delta \pi^b(x)} \, \delta^a_5 \pi^b(x) ,
\]  

(56)

where \( \delta^a_5 \pi^b(x) \) is the chiral variation of the mass field \( \pi^b(x) \) and \( O'(z) \) is an operator with the same transformation properties and dimension of \( G \tilde{G}_{\text{sub}}(z) \), i.e. a pseudo-scalar, chiral-singlet dimension-4 operator, build up with the fields \( \bar{\psi}(z) \), \( \psi(z) \), \( \sigma^a(z) \) and \( \pi^a(z) \) \[17\]. Since \( O'(z) \) should be a chiral singlet, the third form of the contact terms \( (56) \) is vanishing. Moreover, the only dimension-4, chiral singlet operator allowed is just the generalized mass term, i.e.: \( O' = \sigma^a \bar{\psi} \lambda^a \psi / 2 + i \pi^a \bar{\psi} \lambda^a \gamma_5 \psi / 2 \), which however is not pseudoscalar. Thus, we conclude that no contact terms are present in Eq. (55). Finally, according to Refs. [11, 17] in the continuum limit the third term in the l.h.s. of Eq. (55) vanishes and therefore it will be disregarded.

Substituting \( G \tilde{G}_{\text{sub}}(z) \), defined in Eq. (53), into Eq. (55) one gets

\[
\Delta \mathcal{M}_L = -C_F \langle \int d^4z \, P_5(y_1, y_2) \, G \tilde{G}_L(z) \rangle + m_L \langle \int d^4x \, d^4z \, P_5^a(x) \, S^a(y_1, y_2) \, G \tilde{G}_L(z) \rangle - C_F \langle \int d^4z \, P_5(y_1, y_2) \, \bar{P}_5(z) \rangle + m_L \langle \int d^4x \, d^4z \, P_5^a(x) \, S^a(y_1, y_2) \, \bar{P}_5(z) \rangle .
\]

(57)

Note that our non-perturbative definition of \( \Delta \mathcal{M}_L \) is gauge-invariant because the loss of gauge invariance is given by a factor which is the same in the numerator and in the denominator of the r.h.s. of Eq. (57).

Let us now address the calculation of the renormalization constants \( \bar{M}^{(S)} \), \( Z_A^{(S)} \) and \( Z_{G \tilde{G}} \) relevant to the singlet \( W_1 \). Substituting Eq. (50) into the \( W_1 \) (161 - 165), one gets

\[
\Delta_{\mu} \langle Z_A^{(S)} A_{\mu}(x) O \rangle = 2(m_0 - \bar{M}^{(S)}) \langle \bar{\psi}(x) \gamma_5 \psi(x) O \rangle + 2N_F \frac{g_6^2}{32\pi^2} \langle Z_{G \tilde{G}} G \tilde{G}_{\text{sub}}(x) O \rangle - \langle \frac{\delta O}{\delta (i\alpha^a(x))} \rangle ,
\]

(58)

where we have dropped the term involving \( X^0 + \bar{X}_C^0 \). Because of its anomalous non-conservation the singlet current \( A_{\mu} \) suffers a logarithmically divergent renormalization \[18\]. A mixing between \( Z_A^{(S)} A_{\mu} \) and \( Z_{G \tilde{G}} G \tilde{G}_{\text{sub}} \) is needed to obtain finite operators \[9\]

\[
A_R^\mu(x) = Z_A^{(S)} A_{\mu}(x) - (g_6^2 Z_C) Z_A^{(S)} A_{\mu}(x) ,
\]

\[
2N_F \frac{1}{32\pi^2} G \tilde{G}_{R}(x) = 2N_F \frac{1}{32\pi^2} Z_{G \tilde{G}} G \tilde{G}_{\text{sub}}(x) - Z_C Z_A^{(S)} \partial_\mu A_{\mu}(x) ,
\]

(59)

where we stress that \( G \tilde{G}_{\text{sub}}(x) = G \tilde{G}_L(x) - \Delta \mathcal{M}_L \bar{\psi}(x) \gamma_5 \psi(x) \). The above redefinition can be performed into Eq. (50) by adding and subtracting the counterterm \( \left[ Z_C Z_A^{(S)} \partial_\mu A_{\mu}(x) \right] \) \[6\].

We have evaluated at one loop level all the constants appearing in Eq. (59). The constant \( Z_A^{(S)} \) can be obtained by evaluating \( \langle X^0(x) + X_C^0(x) \psi(y) \psi(z) \rangle \): to \( O(g_6^2) \) its value is identical to that of \( Z_A^{(NS)} \), Eq. (20), while beyond the one-loop order the two constants will differ because of the contributions (in the singlet channel) coming from fermion loops. The same is true for \( \bar{M}^{(S)} \), which therefore at one-loop is given by Eq. (18), as well as for the
renormalization constants of the pseudoscalar density operators in both the singlet and non-singlet channels. This implies that at one-loop the renormalized masses \((m_0 - \overline{M}^{(S)})/Z_{\gamma_5}^{(S)}\) and \((m_0 - \overline{M}^{(NS)})/Z_{\gamma_5}^{(NS)}\) coincide and match the corresponding renormalized mass in the continuum. We stress that the equality between the renormalized masses in the singlet and non-singlet channels is a more general result, valid at any order of the perturbation theory and resulting from the property of \(\tilde{G}_{\text{sub}}\) to be a chiral singlet, as illustrated in Ref. [9].

The mixing between \(X_0^0(x) + X_0^C(x)\) and \(\tilde{G}_{\text{sub}}(x)\) can be computed at one loop by evaluating the correlator \(\langle [X_0^0(x) + X_0^C(x)] G_0^b(y) G_0^c(z) \rangle\). This was carried out in Ref. [10], where it was shown that the apparent dependence of the correlator on \(r\) actually disappears as far as \(r \neq 0\), and that the tree level expression \(2N_F g_0^2/32\pi^2 \tilde{G}\) is reproduced exactly. Thus, from Eq. (50) one has \(Z_{\tilde{G}_G} = 1 + O(g_0^2)\), and, since the logarithmic divergence in the singlet axial current manifests itself at the two-loop level, it follows that \(g_0^2 Z_C = O(g_0^4)\) and consequently \(Z_C = O(g_0^2)\).

The same considerations hold as well for the matching constants between the continuum and the lattice. For \(Z_{\tilde{G}_G}\) and \(Z_C\) appearing in Eq. (59), we can write

\[
Z_{\tilde{G}_G} = 1 + g_0^2 z_{gg}^{(2)} + O(g_0^4),
\]
\[
Z_C = g_0^2 \frac{C_F}{16\pi^2} \left( z_{g\psi}^{(2)} \ln(a^2 \mu^2) + \tilde{z}_{g\psi}^{(2)} \right) + O(g_0^4). \tag{60}
\]

The coefficient \(z_{gg}^{(2)}\) can be computed at one-loop in a pure gauge theory (see Ref. [19] and references therein quoted), and at this order it is unaffected by the presence of the Clover term in the action\(^c\). One obtains [19]

\[
z_{gg}^{(2)} = N \left( -\frac{1}{4N^2} + Z_{0000} + \frac{1}{8} + \frac{1}{2\pi^2} \right) \tag{61}
\]

where \(N\) is the dimension of the gauge group and \(Z_{0000} = 0.15493\). The constants \(z_{g\psi}^{(2)}\) and \(\tilde{z}_{g\psi}^{(2)}\) can be computed by evaluating, in the continuum and on the lattice, the correlator \(\langle \tilde{G}\tilde{G}(x)\psi(y)\bar{\psi}(z) \rangle\), with amputated external quark propagators. On the lattice \(\tilde{G}\tilde{G}\) should be \(\tilde{G}_{\text{sub}}\) given by Eq. (53). However, we have explicitly checked that the operator \(\tilde{G}\tilde{G}_L\) does not mix at one-loop with the pseudoscalar density and, therefore, in our leading-order approximation there is no difference between \(G\tilde{G}_{\text{sub}}\) and \(G\tilde{G}_L\). Thus, on the lattice we have to compute the following diagrams:

\[
FT\langle \tilde{G}\tilde{G}_L(x) \psi(y)\bar{\psi}(z) \rangle_{\text{amp}}^{(2)} =
\]

\[
\text{Diagram 1} + \text{Diagram 2} \tag{62}
\]

\(^c\)Beyond \(O(g_0^2)\) the presence of the Clover term in the action will, in general, affect also \(Z_{\tilde{G}\tilde{G}}\).
where the $\psi\psi g$ and $\psi\psi gg$ vertices include the contributions of both the Wilson and Clover action. The second diagram gives a null contribution, since the insertion of $G\tilde{G}$ contracted with two gluon lines is antisymmetric in the Lorentz indices of the gluons (via the $\varepsilon$-tensor) and symmetric in their color indices, whereas the $\psi\psi gg$ is symmetric in the Lorentz indices for the Wilson case and antisymmetric in the color matrices for the Clover case (see the Feynman rules in the Appendix). The first diagram displays a log divergence as $a \to 0$, in analogy with the continuum calculation. The difference between the two schemes gives the following result

$$z_{g\psi}^{(2)} = -6,$$

$$z_{g\psi}^{(2)} = 22 - 6(\gamma_E - F_{0001}) - 16\pi^2 \int \frac{d^4 q}{(2\pi)^4} \left[ \frac{3\Delta_3}{2\Delta_1^2} \left( \frac{1}{4\Delta_1} - \frac{1}{\Delta_2} \right) \right.$$}

$$+ \frac{1}{4\Delta_1^2 \Delta_2} \left( 10\Delta_3 \Delta_4 - 30\Delta_3 - 19\Delta_3^2 + 64\Delta_4 - 46\Delta_7 + 12\Delta_8 \right)$$

$$+ \frac{1}{4\Delta_1 \Delta_2} \left( 45\Delta_3 + 6\Delta_3^2 - 40\Delta_4 + 12\Delta_7 \right) + \frac{1}{4\Delta_2} \left( 2\Delta_1 \Delta_3 - 17\Delta_3 + 6\Delta_4 \right)$$

$$+ r^2 c_{SW} \left[ \frac{1}{2\Delta_1 \Delta_2} \left( 8\Delta_3 \Delta_4 - 3\Delta_3 \Delta_5^2 + 3\Delta_3 \Delta_6 - 4\Delta_3^2 \Delta_5 + 10\Delta_4 \Delta_5 + 2\Delta_4 \Delta_5^2 \right.$$

$$- 2\Delta_4 \Delta_6 - 8\Delta_5 \Delta_7 - 14\Delta_7 + 12\Delta_8 \left) + \frac{1}{\Delta_2} \left( \Delta_3 \Delta_5^2 - \Delta_3 \Delta_6 - 2\Delta_4 \Delta_5 + 2\Delta_7 \right) \right]$$

$$+ r^2 c_{SW}^2 \left[ \frac{1}{4\Delta_1 \Delta_2} \left( -10\Delta_3 \Delta_4 \Delta_5 - 2\Delta_3 \Delta_4 \Delta_5^2 + 2\Delta_3 \Delta_4 \Delta_6 + 8\Delta_3 \Delta_5 \Delta_7 \right.$$

$$+ 14\Delta_3 \Delta_7 - 12\Delta_3 \Delta_8 - 8\Delta_3^2 \Delta_4 + 3\Delta_3^2 \Delta_5^2 - 3\Delta_3^2 \Delta_6 + 4\Delta_3^3 \Delta_5 \right.$$

$$+ \frac{1}{2\Delta_1 \Delta_2} \left( 2\Delta_3 \Delta_4 \Delta_5 - 2\Delta_3 \Delta_7 - \Delta_3 \Delta_5^2 + \Delta_3^2 \Delta_6 \right) \right]. \quad (63)$$

The numerical values of the constant $z_{g\psi}^{(2)}$ obtained for various values of $r$ at $c_{SW} = 0$ and $c_{SW} = 1$ are reported in Table 6.

### 4.1 Composite insertions of the topological charge and the neutron EDM

In this subsection we take the opportunity to briefly address a separate issue concerning the composite insertion of the topological charge relevant for a lattice evaluation of the neutron EDM induced by the strong $\theta$-term. Following Ref. [6] the standard definition of the neutron EDM, $\vec{d}_N$, involves the insertion of the topological charge $(g^2/32\pi^2) \int d^4x \ G\tilde{G}(x)$ in the presence of the charge density operator $J_0$. Treating the $\theta$-term as a perturbation at first order, one has

$$\vec{d}_N \equiv -i\theta \frac{g^2}{32\pi^2} \int d^3y \ \bar{y}_0(y) \langle N|J_0(y) \left[ \int d^4x \ G\tilde{G}(x) \right] |N\rangle_0, \quad (64)$$
Table 6: Numerical results for \( \tilde{z}^{(2)}_{g\psi} \) [see Eq. (63)] for different values of \( r \) and \( c_{SW} \).

| \( r \) | \( \tilde{z}^{(2)}_{g\psi} \) \( c_{SW}=0 \) | \( \tilde{z}^{(2)}_{g\psi} \) \( c_{SW}=1 \) |
|------|----------------|----------------|
| 0.0  | 19.82          | 19.82          |
| 0.2  | 19.72          | 19.93          |
| 0.4  | 19.11          | 19.98          |
| 0.6  | 18.11          | 19.94          |
| 0.8  | 16.95          | 19.87          |
| 1.0  | 15.77          | 19.81          |

where \( |N\rangle_0 \) is a shorthand for \( |N\rangle_{\theta=0} \). In case of three flavors with non-degenerate masses a complete diagrammatic analysis was performed in Ref. [9] showing how the axial anomaly governs the replacement of the topological charge operator with well-defined insertions of the flavor-singlet pseudoscalar density\(^d\).

Thus the question is the possible presence of contact terms in the composite insertion of the topological charge operator with the electromagnetic (\( e.m. \)) current operator, which would lead to ambiguities in the numerical evaluation of Eq. (64). The contact terms in Eq. (64) should be of the form \( O^P_{\mu=0}(x) \cdot \delta^{(4)}(x-y) \), where \( O^P_{\mu} \) is a local operator of dimension-3, which transforms as a non-singlet pseudo-vector and is conserved. The last property derives from the fact that the \( e.m. \) current is conserved both with and without the \( \theta \)-term in the QCD action at any value of the parameter \( \theta \). The non-singlet nature of \( O^P_{\mu} \) is related to the non-singlet nature of the \( e.m. \) current operator. The only candidate for \( O^P_{\mu} \) is the non-singlet axial current which however is not conserved (outside the chiral limit).

The absence of contact terms in Eq. (64) can be derived also using the generalized mass insertion method of Ref. [17] applied to a vector \( WI \) with the operator \( O \) given by \( GG \).

5 Conclusions

A complete one-loop calculation of the renormalization constants appearing in both singlet and non-singlet axial Ward identities using Wilson fermions with the Clover \( O(a) \)-improvement of the action has been performed. Our calculations include: 1) the contributions arising from the Clover term of the action; 2) the complete mixing of the gluon operator \( GG \) with the divergence of the singlet axial current; 3) the use of both local and extended definitions of the fermionic bilinear operators.

In the singlet channel a definition of the gluon operator \( GG \) on the lattice outside the chiral limit has been proposed. Our definition takes into account the possible power-divergent

\(^d\)We point out that the final result of Ref. [9] crucially depends on the equality between the renormalized masses in the singlet and non-singlet \( WI \)’s. We stress once more that such an equality follows from the property of \( GG_{sub} \) [see Eq. (58)] to be a chiral singlet (cf. Ref. [3]).
mixing with the pseudoscalar density, generated by the breaking of chiral symmetry. No mixing has been found at one-loop order and a non-perturbative definition of the mixing constant has been developed.

Finally, the renormalization properties of the composite insertion of the topological charge operator $\int d^4x \, G\tilde{G}(x)$ relevant for the lattice calculation of the neutron electric dipole moment have been discussed, showing that no contact terms arise when the topological charge is inserted with a conserved current.

Acknowledgments

The authors gratefully acknowledge M. Testa for many useful discussions, which have been essential for clarifying the non-perturbative definition of the mixing between the gluon operator $G\tilde{G}$ on the lattice and the singlet pseudoscalar density. We warmly thank V. Lubicz and G. Martinelli for many useful comments and a critical reading of the manuscript.

Appendix: Feynman rules

The Wilson action is given by

$$S_F = a^4 \sum_x \left\{ -\frac{1}{2a} \sum_\mu \bar{\psi}(x)(r - \gamma_\mu)U_\mu(x)\psi(x + \mu) \\
+ \bar{\psi}(x + \mu)(r + \gamma_\mu)U_\mu^\dagger(x)\psi(x) + \bar{\psi}(x)\left(m_0 + \frac{4r}{a}\right)\psi(x) \right\}, \quad (65)$$

with the following notation $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2$, $U_\mu(x) = \exp[i g_0 a G_\mu(x)]$, $G_\mu = G_\mu^a t^a$ and $Tr(t^a t^b) = \delta_{ab}/2$.

The action is improved to $O(a)$ via the Clover term \[20\]

$$S_C = -a^4 \sum_x \sum_{\mu,\nu} c_{SW} \frac{i g_0 ar}{4} \bar{\psi}(x)\sigma_{\mu\nu} P_{\mu\nu}(x)\psi(x), \quad (66)$$

with $P_{\mu\nu}$ being the usual lattice definition of the field-strength tensor $G_{\mu\nu}$. \[8\]

$$P_{\mu\nu}(x) = \frac{1}{4a^2} \sum_{i=1}^4 \frac{1}{2i g_0} (U_i - U_i^\dagger) \quad (67)$$

where the sum is over the four plaquettes in the $\mu$-$\nu$ plane stemming from $x$ and taken in the counterclockwise sense, i.e.

$$
\begin{align*}
U_1 &= U_\mu(x) \, U_\nu(x + \mu) \, U_\mu^\dagger(x + \nu) \, U_\nu^\dagger(x), \\
U_2 &= U_\nu(x) \, U_\mu^\dagger(x - \mu + \nu) \, U_\nu(x - \mu) \, U_\mu(x - \mu), \\
U_3 &= U_\mu^\dagger(x - \mu) \, U_\nu^\dagger(x - \mu - \nu) \, U_\mu(x - \mu - \nu) \, U_\nu(x - \nu), \\
U_4 &= U_\nu^\dagger(x - \nu) \, U_\mu(x - \nu) \, U_\nu(x + \mu - \nu) \, U_\mu^\dagger(x). \quad (68)
\end{align*}
$$
Propagators

\[ G_{\mu\nu}(k) = \frac{\delta_{\mu\nu} - (1 - \alpha) \frac{k_{\mu}k_{\nu}}{k^2}}{k^2} \quad (69) \]

with

\[ \hat{p}_\mu = \frac{2}{a} \sin\left(\frac{p_\mu a}{2}\right), \quad \hat{p}^2 = \sum_\mu \hat{p}_\mu^2. \]

\[ S_0(p) = \]

\[ = \left[ \frac{1}{a} \sum_\lambda \gamma_\lambda \sin(p_\lambda a) + \left( m_0 + \frac{2r}{a} \sum_\lambda \sin^2 \frac{p_\lambda a}{2} \right) \right]^{-1} \quad (70) \]

QCD vertices

The indices \( W \) and \( C \) refer to the Wilson and Clover action respectively [8]:

\[ V_{\mu a}^{W}(q, q^{'}) = \]

\[ = -g_0(t^a)_{ij} \left\{ i\gamma_\mu \cos\left( (q + q^{'})_\mu \frac{a}{2} \right) + r \sin\left( (q + q^{'})_\mu \frac{a}{2} \right) \right\} \quad (71) \]

\[ V_{\mu a}^{C}(q, q^{'}) = \]
\[-g_0(t^a)_{ij}\{\sum_{\nu} \sigma_{\mu\nu} \sin((q - q')_{\nu}a) \cos((q - q')_{\mu}a)\} \tag{72}\]

\[\begin{align*}
= V^W_{\rho a, \sigma b}(p, p') = \\
= \frac{a g_0^2}{2} \delta_{\rho\sigma} \{t^a, t^b\}_{ij} [i \gamma_\rho \sin((p + p')_{\rho}a) - r \cos((p + p')_{\rho}a)] \tag{73}\end{align*}\]

**Extended operators**

For simplicity we report only the Feynman rules for flavor-singlet operators.

**Operator \( \Delta_\mu A_\mu(x) \):**

\[a \Delta_\mu A_\mu(x) = A_\mu(x) - A_\mu(x - \mu),\]
\[A_\mu(x) = \frac{1}{2} [\bar{\psi}(x + \mu) \gamma_\mu \gamma_5 U^d_\mu(x) \psi(x) + \bar{\psi}(x) \gamma_\mu \gamma_5 U_\mu(x) \psi(x + \mu)] . \tag{74}\]

\[= -\frac{i}{a} \sum_{\mu} \gamma_\mu \gamma_5 (\sin(p'_\mu a) - \sin(p_\mu a)) , \tag{75}\]
\[
\frac{g_0}{2} \gamma_\rho \gamma_5 (t^a)_{ij} \sum_\mu \left[ e^{i q_\rho} \left( e^{i p_\mu} - e^{-i p_\mu} \right) + e^{-i q_\rho} \left( e^{i p_\mu} - e^{-i p_\mu} \right) \right],
\]
(76)
\[
\begin{align*}
\left( 82 \right) & = \frac{g_0^2 a r}{4} \gamma_5 \delta_{\rho \sigma} \{ t^a, t^b \}_{ij} \sum_{\mu} \left[ e^{i q_{\mu} \frac{a}{2}} e^{-i q'_{\rho} \frac{a}{2}} (e^{i p_{\mu} a} + e^{-i p'_{\mu} a}) \\
& \quad + e^{-i q_{\mu} \frac{a}{2}} e^{i q'_{\rho} \frac{a}{2}} (e^{i p'_{\mu} a} + e^{-i p_{\mu} a}) \right]. \\
\end{align*}
\]

Operator \(X_C^0\):
\[
X_C^0(x) = -\frac{i g_0 a r}{2} \sum_{\mu \nu} \bar{\psi}(x) \gamma_5 \sigma_{\mu \nu} P_{\mu \nu} \psi(x) . \quad (83)
\]

\[
\begin{align*}
\left( 84 \right) & = \emptyset , \\
\end{align*}
\]

\[
\begin{align*}
\left( 85 \right) & = -g_0 \gamma_5 (t^a)_{ij} \sum_{\rho} \left[ \sigma_{\mu \nu} \sin(q_{\rho} a) \cos(q_{\rho} a) \right] ,
\end{align*}
\]
\[ G\tilde{G}_L = \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(P_{\mu\nu}P_{\rho\sigma}) . \]
[9] M. Testa, JHEP 9804, 2 (1998) [e-print archive hep-th/9803147].

[10] L.H. Karsten and J. Smit, Nucl. Phys. B183, 103 (1981).

[11] M. Bochicchio, L. Maiani, G. Martinelli, G.C. Rossi and M. Testa, Nucl. Phys. B262, 331 (1985).

[12] A. Gonzalez Arroyo, F.J. Yndurain and G. Martinelli, Phys. Lett. B117, 437 (1982).

[13] G. Martinelli and Zhang Yi-Cheng, Phys. Lett. B123, 433 (1983).

[14] S. Capitani et al., Nucl. Phys. B593, 183 (2001).

[15] G. Martinelli and Zhang Yi-Cheng, Phys. Lett. B125, 77 (1983).

[16] J. Mandula, G. Zweig and J. Govaerts, Nucl. Phys. B228, 91 (1983).

[17] L. Maiani, G. Martinelli, G.C. Rossi and M. Testa, Nucl. Phys. B289, 505 (1987).

[18] W.A. Bardeen, Nucl. Phys. B75, 246 (1974).

[19] B. Allés, A. Di Giacomo, H. Panagopoulos and E. Vicari, Phys. Lett. B350, 70 (1995).

[20] G. Heatlie, G. Martinelli, C. Pittori, G.C. Rossi and C.T. Sachrajda, Nucl. Phys. B352, 266 (1991).