A NOTE ON THE MULTIPLICITY OF $SL(n)$ OVER FUNCTION FIELDS

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Abstract. In [Laf12], Vicent Lafforgue attaches a semisimple Langlands parameter (or, what amounts to the same thing, a $\hat{G}$-pseudocharacter) to every cuspidal automorphic representation of a reductive group $G$ over the field of functions of a smooth projective algebraic curve $X$ over a finite field. Hence, gets a decomposition of the space of cusp forms. In this note, we show that in the case of $G = SL(n)$, Lafforgue’s decomposition coincides with the classical decomposition using $L$-packets, and moreover, the number of ($G$-equivalence classes of) extensions of an unramified Hecke character of $G$ to $\hat{G}$-pseudocharacters serves as a natural upper bound on the multiplicity of $SL(n)$.

1. Overview

Let $k = \mathbb{F}_q$, where $q = p^m$ is a prime power, and $X/k$ be a smooth irreducible projective curve and $F = k(X)$ its field of rational functions. The set of places of $F$ is denoted as $|X|$, which is the same as the set of closed points of $X$. For each $v \in |X|$, we have $F_v$ its completion, and $\mathcal{O}_v$ its ring of integer in $F_v$. Let $G$ be a connected reductive group over $F$, and $N = \sum_{v \mid X} n_v v$ be an effective divisor on $X$, and $K_N = \{ k \in \Pi_{v \mid X}(G(\mathcal{O}_v)) : k \equiv 1(\text{mod } m_v^n) \}$ be the open compact subgroup of level $N$.

Fix some prime number $l \neq p$, let $A_0(G, \mathbb{Q}_l)$ denote the space of cusp forms, and $A_0(G, K_N, \mathbb{Q}_l) = A_0(G, \mathbb{Q}_l)_{K_N}$. In [Laf12], V. Lafforgue construct a commutative algebra $B_N$, containing the normal Hecke algebra $T_N = \bigotimes_{v \mid N} T_v$, called the excursion algebra(of level $N$). Moreover, for each excursion character $\nu: B_N \to \mathbb{Q}_l$, one can associate a unique Langlands parameter $\sigma_{\nu}: \Gamma \to \hat{G}(\mathbb{Q}_l)$, up to conjugation, where $\Gamma$ is the Galois group of the maximal separable extension of $F$ unramified outside of the support of $N$, and $\hat{G}$ is the Langlands dual group of $G$. More precisely, $B_N$ is generated by excursion operators $S_{m, f, \gamma} \in \text{End}(A_0(G, K_N, \mathbb{Q}_l))$, where $f \in \mathcal{O}(\hat{G}^n)$, the conjugate invariant functions on $\hat{G}^n$, and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma^n$. When $V \in \text{Rep}(\hat{G})$, the Hecke operator $h_{V, \nu}$ is $S_{1, Tr V, Fr_{obo}}$. Moreover, if $\nu$ is a character of $B_N$, then the function $\Theta(m, f, \gamma) = \nu(S_{m, f, \gamma})$ is called a pseudocharacter of $G$. It is the pseudocharacter that gives a unique Langlands parameter $\sigma$, up to conjugacy, they are related by:

$$\Theta(m, f, \gamma) = \nu(S_{m, f, \gamma}) = f(\sigma(\gamma)).$$

Then V. Lafforgue proved a decomposition [Laf12]:

$$A_0(G, K_N, \mathbb{Q}_l) = \bigoplus_{\nu} A_{0, \nu} = \bigoplus_{\sigma} A_{0, \sigma}.$$  

Moreover, by the identification of excursion operators $S_{1, Tr V, Fr_{obo}}$ with the Hecke operators, it is easy to see this decomposition is also compatible with the Satake isomorphism, more
precisely, for any $\nu : B_N \to \mathbb{Q}_l$, and $\nu \nmid N$, then $\sigma_{\nu}$ is unramified at $v$, and the semisimple conjugacy class $\sigma_{\nu}(\text{Frob}_v)$ corresponds to the Hecke character $\sigma_{\nu}|_{\mathcal{T}_v}$ under the Satake isomorphism.

On the other hand, we know that $A_0(G, \mathbb{Q}_l)$ is a discrete $G(\mathbb{A})$ module, and has a decomposition:

$$A_0(G, \mathbb{Q}_l) = \bigoplus_{\pi} m(\pi)\pi,$$

with pairwise inequivalent irreducible admissible $G(\mathbb{A})$ representations, and $m(\pi)$ is a finite nonnegative integer. After taking the $K_N$ fixed part, decomposition (1.2) just becomes:

$$A_0(G, K_N, \mathbb{Q}_l) = \bigoplus_{\pi} m(\pi^{K_N})\pi^{K_N},$$

While it is well-known that $m(\pi) = 1$ or 0 in the case of $G = GL(n)$, in [Bla94], Blasius constructed infinitely many families of automorphic cuspidal representations that are isomorphic, but not coincide in the case of $G = SL(n)$ over number fields, namely $m(\pi) > 1$ for infinitely many $\pi$. This raises the question of higher multiplicities of $SL(n)$. We studied Lafforgue’s excursion character in the case of $G = SL(n)$ over function fields, and see how the Lafforgue’s decomposition (1.1), and the idea of pseudocharacters account for the multiplicities of $SL(n)$.

After showing that Lafforgue’s decomposition (1.1) coincides with the classical decomposition by $L$-packets induced from $GL(n)$ in Subsection 5.1, we come to Proposition 5.1, which gives an upper bound of multiplicities of $SL(n)$ in terms of the number of extensions:

**Proposition 1.4.** The number of isomorphic irreducible components of $A_0(G, \mathbb{Q}_l)$ (which corresponds to a character $\lambda$ of some unramified Hecke algebra of $SL(n)$), is bounded above by the number of $G$-equivalent classes of pseudocharacters $\hat{\Theta}(m, f, (\gamma_i))$ of $GL(n)$, such that $\hat{\Theta}(1, Tr_V, \gamma)$ (where $V$ is any representation of $GL(n)$ that factors through $PGL(n)$) is given by $\lambda$, and $\hat{\Theta}(GL(n))(1, Det, \gamma)$ is given by $\hat{\mu}$.

And we will see that the $\lambda$ and $\hat{\mu}$ in Proposition 1.4 determine pseudocharacters $\hat{\Theta}$ up to $n$-th roots of unity.

In the rest of this paper, we will fix the following notations, $n$ is a positive integer coprime to $p$, and $\tilde{G} = GL(n)$, $G = SL(n)$. All notations with $\sim$ will refer to the corresponding notion of $GL(n)$, and those without $\sim$ will refer to those of $SL(n)$. For example, $\mathbb{T}_N$ will indicate the Hecke algebra (of level $N$) of $GL(n)$, and $\pi$ will indicate a cuspidal representation of $SL(n)$, i.e. $\pi \subset A_0(SL(n), \mathbb{Q}_l)$, unless otherwise specified.

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2. **Restriction of Cuspidal Forms**

In this section, we review classical theory about the restriction of cuspidal forms from $GL(n)$ to $SL(n)$, our main reference is [HS12]. Although the theorems proved in [HS12] is for number fields, but the proofs work verbatim in the function field case, provided our standing
assumption that \((n, \text{Char}(F)) = 1\). Denote \(\tilde{Z}\) to be the center for \(\tilde{G}\), and \(Z = \tilde{Z} \cap G\). Let \(\tilde{\mu} \in (\tilde{Z}(\mathbb{A})/\tilde{Z}(F))^D\), the Pontryagin dual of \(\tilde{Z}(\mathbb{A})/\tilde{Z}(F)\), and \(\mu \in (Z(\mathbb{A})/Z(F))^D\). We will suppress the coefficient \(\mathbb{Q}_l\), and write \(A_0(\tilde{G}, \tilde{\mu})\) to indicate the cusp forms with central character \(\tilde{\mu}\). We denote \(\Pi_{\text{cusp}}(\tilde{G}, \tilde{\mu})\) to be the set of irreducible representations of \(\tilde{G}(\mathbb{A})\) appearing in \(A_0(\tilde{G}, \tilde{\mu})\). Let \(\mathcal{L}(\tilde{\pi})\) be the maximal \(\tilde{\pi}\)-isotypic subspace of \(A_0(\tilde{G}, \tilde{\mu})\). Then \(\mathcal{L}(\tilde{\pi}) \cong m(\tilde{\pi})\tilde{\pi}\), where \(m(\tilde{\pi}) = 1\) by Multiplicity One Theorem for \(GL_n\). Similarly, we define \(A_0(G, \mu), \Pi_{\text{cusp}}(G, \mu),\) and \(\mathcal{L}(\pi)\) for \(\pi \in \Pi_{\text{cusp}}(G, \mu)\) for \(SL(n)\).

Before we go into the global theory of restriction of cusp forms, we need to make a little preparations in the local case.

### 2.1. Restriction in the local case.

We first make a little general assumption in the local case, suppose \(\tilde{G}\) is a locally compact, totally disconnected group, i.e. it has a fundamental basis of neighbourhood around \(e\) by open compact subgroups. Let \(G \subset \tilde{G}\) to be an open normal subgroup of finite index.

The following lemmas are carefully proved in [GK82].

**Lemma 2.1.** If \(\tilde{\pi}\) is an irreducible admissible representation of \(\tilde{G}\), then \(\text{Res}_{\tilde{G}}^G(\tilde{\pi})\) is a direct sum of finite number of irreducible admissible representations of \(G\) with the same multiplicity.

We denote by \(\Pi(\tilde{\pi}) = \Pi_{\tilde{G}}^G(\tilde{\pi})\) the set of equivalence classes of irreducible admissible representations of \(G\) appearing in the composition series of \(\text{Res}_{\tilde{G}}^G(\tilde{\pi})\). Then the above lemma asserts that

\[
\text{Res}_{\tilde{G}}^G(\tilde{\pi}) = \bigoplus_{\pi \in \Pi(\tilde{\pi})} m \cdot \pi,
\]

where \(m\) is the common multiplicity of \(\pi \in \Pi(\tilde{\pi})\).

**Lemma 2.3.** Let \(\tilde{\pi}\) and \(\tilde{\pi}'\) be irreducible admissible representations of \(\tilde{G}\), then the following are equivalent:

- \(\Pi(\tilde{\pi}) \cap \Pi(\tilde{\pi}') = \emptyset\),
- \(\Pi(\tilde{\pi}) = \Pi(\tilde{\pi}')\),
- \(\tilde{\pi}' \cong \tilde{\pi} \otimes \omega\) for some \(\omega \in (\tilde{G}/G)^D\).

Now, we specialize to our interesting case. For any \(v \in |X|\), let \(\tilde{G}_v = GL(n, F_v)\), and similarly \(G\) be \(G_v = SL(n, F_v)\), and \(\tilde{\pi}_v\) is a component of an irreducible cuspidal representation \(\pi_v\), with \((n, \text{Char}(F)) = 1\). Suppose \(\tilde{\pi}\) has a central character, i.e. there exists a character \(\tilde{\mu}\) of \(F^*\) such that \(\tilde{\pi}(x) = \tilde{\mu}(x) \cdot \text{Id}\), where \(x\) is identified with its diagonal embedding of \(F^*\) into \(\tilde{G}_v\). Let \(H = \tilde{Z} \cdot G\) be a subgroup of \(\tilde{G}\), we then have an isomorphism (as a topological group) \(\tilde{G}/G \cong F^*\) through determinant map, where \(H/G\) is the inverse image of \(F^{*n}\), the group of \(n\)-th power. Since \((n, \text{Char}(F)) = 1\), we know that \(F^{*n}\) is an open normal subgroup of \(F^*\) of finite index. Hence, the same holds for \(H \subset \tilde{G}\).

Since \(H = \tilde{Z}G\), where \(\tilde{\pi}_v\) is just the scalar when restricted to \(\tilde{Z}\). We know that a representation of \(\tilde{Z}G\) is irreducible if and only if its restriction on \(G\) is irreducible, hence Lemma 2.1 and Lemma 2.3 still holds for our \(\tilde{G}_v\) and \(G_v\). Finally, we note the following theorem of Tadic [Tad92] (Theorem 1.2):
Theorem 2.4. For an irreducible smooth representation $(\tilde{\pi}_v, V)$ of $G_v$, $\tilde{\pi}_v|_{G_v}$ is multiplicity free.

2.2. (Locally) $G$-equivalence. For $\mu \in (Z(\tilde{A})/Z(F))^D$, we put

$$[\mu] = \{\hat{\mu} \in (\tilde{Z}(A)/\tilde{Z}(F))^D | \text{Res}^{\tilde{Z}(A)}_Z \hat{\mu} = \mu\}.$$ 

For $\hat{\mu}, \hat{\mu}' \in (\tilde{Z}(A)/\tilde{Z}(F))^D$, we say that $\hat{\mu}$ and $\hat{\mu}'$ are $G$-equivalent if there exists $\omega \in (G(A)/G(F)G(A))^D$ such that

$$\hat{\mu}' \cong \hat{\mu} \otimes \text{Res}^{\tilde{Z}(A)}_Z \omega.$$ 

It can be shown that $\hat{\mu}, \hat{\mu}'$ are $G$-equivalent, if and only if they are in the same $[\mu]$.

For $\bar{f} \in A_0(\tilde{G}, \tilde{\mu})$, we define $\text{res}^{\tilde{G}}_G(\bar{f})$ to be the restriction of $\bar{f}$ to a function on $G(\tilde{A})$. For $\hat{\mu} \in [\mu]$, we clearly have $\text{res}^{\tilde{G}}_G A_0(\tilde{G}, \tilde{\mu}) \subset A_0(G, \mu)$.

Remark 2.5. This is different from Res, which we reserved for restriction as abstract representation.

Let $\tilde{\pi} \in \Pi_{\text{cusp}}(\tilde{G}, \tilde{\mu})$, and $\tilde{\pi}' \in \Pi_{\text{cusp}}(\tilde{G}, \tilde{\mu}')$. We say that $\tilde{\pi}'$ and $\tilde{\pi}$ are locally $G$-equivalent if, for any place $v$ of $F$, there exists $\omega_v \in (G(F_v)/G(F_v))^D$ such that $\bar{\pi}_v' \cong \bar{\pi}_v \otimes \omega_v$ (or equivalently, there exists $\omega \in (G(A)/G(A))^D$ such that $\tilde{\pi}' \cong \tilde{\pi} \otimes \omega$). Furthermore, $\tilde{\pi}' \in \Pi_{\text{cusp}}(\tilde{G}, \tilde{\mu}')$ and $\tilde{\pi} \in \Pi_{\text{cusp}}(\tilde{G}, \tilde{\mu})$ are $G$-equivalent, if there exists $\omega \in (G(A)/G(\tilde{A})G(F))^D$, such that $\tilde{\pi}' \cong \tilde{\pi} \otimes \omega$. If $\tilde{\pi}'$ and $\tilde{\pi}$ are $G$-equivalent, then $\hat{\mu}$ and $\hat{\mu}'$ are $G$-equivalent. If $\tilde{\pi}, \tilde{\pi}' \in \Pi_{\text{cusp}}(\tilde{G}, \tilde{\mu})$, then they are $G$-equivalent, if and only if the chosen $\omega \in (G(\tilde{A})/G(\tilde{A})\tilde{Z}(A)G(F))^D$. We denote the (resp. locally) $G$-equivalent class of $\tilde{\pi}$ in $\Pi_{\text{cusp}}(\tilde{G}, \tilde{\mu})$ to be $\{\tilde{\pi}\}_G$ (resp. $\{\tilde{\pi}\}_G^{\text{loc}}$), and we write $\Pi_{\text{cusp}}(\tilde{G}, \tilde{\mu})_G$ to be the set of $G$-equivalent classes in $\Pi_{\text{cusp}}(\tilde{G}, \tilde{\mu})$.

We put

$$X(\tilde{\pi}) = \{\omega \in (G(\tilde{A})/G(F)G(\tilde{A})\tilde{Z}(A))^D | \tilde{\pi} \otimes \omega \cong \tilde{\pi}\},$$

$$X_{\text{loc}}(\tilde{\pi}) = \{\omega \in (G(\tilde{A})/G(\tilde{A})\tilde{Z}(A))^D | \tilde{\pi} \otimes \omega \cong \tilde{\pi}\}.$$

For $\omega \in (G(\tilde{A})/G(F)G(\tilde{A}))^D$, we define the twisting operator

$$I_\omega : A_0(\tilde{G}, \tilde{\mu}) \to A_0(\tilde{G}, \tilde{\mu} \otimes \text{Res}^{\tilde{Z}(A)}_Z \omega)$$

by $I_\omega \bar{f}(x) = \omega(x) \bar{f}(x)$. Then

$$I_\omega (\mathcal{L}(\tilde{\pi})) = \mathcal{L}(\tilde{\pi} \otimes \omega).$$

This implies

$$\text{res}^{\tilde{G}}_G \mathcal{L}(\tilde{\pi}) = \text{res}^{\tilde{G}}_G \mathcal{L}(\tilde{\pi} \otimes \omega).$$

Note that we need $\omega \in (G(\tilde{A})/G(F)G(\tilde{A}))^D$ rather than simply $\omega \in (G(\tilde{A})/G(A))^D$. Otherwise $\omega(x) \bar{f}(x)$ is not necessarily $G(F)$-invariant.

Let $S(\tilde{\pi}) = \{I_\omega | \omega \in X(\tilde{\pi})\}$, then $S(\tilde{\pi})$ is commutative and acts on $\mathcal{L}(\tilde{\pi}) \cong \tilde{\pi}$. For $\eta \in X(\tilde{\pi})^D$, we put

$$\mathcal{L}(\tilde{\pi})^\eta = \{\bar{f} \in \mathcal{L}(\tilde{\pi}) | I_\omega \bar{f} = \eta(\omega) \bar{f}, \text{for all } \omega \in X(\tilde{\pi})\}.$$

Then

$$\mathcal{L}(\tilde{\pi}) = \bigoplus_{\eta \in X(\tilde{\pi})^D} \mathcal{L}(\tilde{\pi})^\eta.$$
The subspace $L(\tilde{\pi})^1$ is where $S(\tilde{\pi})$ acts trivially. Since $\text{supp}(f) \subset \{g \in \tilde{G}(A) | \omega(g) = \eta(\omega)\}$ for all $\omega \in X(\tilde{\pi})$. We see that

$$\text{res}_G^\tilde{G} L(\tilde{\pi})^n = 0, \quad \text{unless } \eta = 1.$$ 

Thus, $\text{res}_G^\tilde{G} L(\tilde{\pi}) = \text{res}_G^\tilde{G} L(\tilde{\pi})^1$.

The main theorem about the restriction of the cusp forms is the following:

**Theorem 2.6.** For any $\tilde{\mu} \in [\mu]$, the morphism

$$(2.7) \quad \Theta_{\{\tilde{\pi} \in \Pi_{\text{cus}}(\tilde{G}, \tilde{\mu}): L(\tilde{\pi})^1 \overset{\text{res}_G^{\tilde{G}}}{\rightarrow} A_0(G, \mu)}$$

is a bijection.

On the other hand, by Theorem 2.4, we will get the following lemma about the restriction of irreducible cuspidal representations of $\tilde{G}$ to $G$ as abstract representations.

**Lemma 2.8.** Let $\tilde{\pi} \in \Pi_{\text{cus}}(\tilde{G}, \tilde{\mu})$, then $\text{Res}_{G}(\tilde{\pi})$ is a direct sum of irreducible admissible representations of $G$, and is multiplicity free.

We will summarize what we know about restrictions of cusp forms. For any cuspidal representation $\pi$ of $G(A)$, we put

$$[\pi] = \bigcup_{\tilde{\mu} \in [\mu]}\{\tilde{\pi} \in \Pi_{\text{cus}}(\tilde{G}, \tilde{\mu}) | \pi_v \in \Pi_{G_v}(\tilde{\pi}_v)\}, \text{ for all places } v.$$ 

By Lemma 2.3, we know $\tilde{\pi}_1, \tilde{\pi}_2 \in [\pi]$ is the same thing as $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are locally $G$-equivalent, and their restrictions to $G$ (as an abstract representation) has a constituent $\pi$.

For any $\tilde{\pi} \in \Pi_{\text{cus}}(\tilde{G}, \tilde{\mu})$, we have $\text{res}_G^{\tilde{G}} \tilde{\pi}$ is a subrepresentation of $A_0(G, \mu)$, and hence a direct sum of irreducible cuspidal representations of $G$. For each component $\pi$ in $\text{res}_G^{\tilde{G}} \tilde{\pi}$, Theorem 2.6 implies that $\text{res}_G^{\tilde{G}}$ induces a $G$-isomorphism between $\pi$ and its preimage in $\tilde{\pi}$. By Lemma 2.8 we know that $\text{res}_G^{\tilde{G}} \tilde{\pi}$ is multiplicity free as well.

By Theorem 2.6, we have $A_0(G, \mu) = \Theta_{\{\tilde{\pi} \in \Pi_{\text{cus}}(\tilde{G}, \tilde{\mu}): \text{res}_G^{\tilde{G}} \tilde{\pi}\}}$. Moreover, if $\pi \in \Pi_{\text{cus}}(G, \mu)$ appears in both $\text{res}_G^{\tilde{G}} \tilde{\pi}_1$, and $\text{res}_G^{\tilde{G}} \tilde{\pi}_2$, then $\tilde{\pi}_1, \tilde{\pi}_2 \in [\pi]$. However, we don’t know if the converse is true, i.e., if $\tilde{\pi} \in [\pi]$, we can’t say $\pi$ appears in $\text{res}_G^{\tilde{G}} \tilde{\pi}$ a priori.

In summary, there is a decomposition of $A_0(G, \mu)$ into a direct sum of irreducible cuspidal representations, and each component $\pi$ lift to an irreducible representation $\tilde{\pi}$ of $\tilde{G}$ as functions, i.e. the restriction of functions in $V_\pi$ contains $V_{\tilde{\pi}}$.

### 3. Excursion characters for $GL(n)$

As a warm-up, we will first treat the excursion characters in the case of $G = GL(n)$. We will show that a excursion character $\tilde{\nu}$ is determined by its restriction to unramified Hecke subalgebra $\tilde{T}_N = \bigotimes_{i=1}^n H_v$, where $N$ is the fixed effective divisor. As a consequence, Lafforgue’s decomposition (1.1) coincides with the classical decomposition (1.3).

Suppose $\tilde{\nu} : B_N \rightarrow \tilde{Q}_l$ is a character, then by studying the pseudocharacter $\tilde{\nu}(S_{n,f}(\gamma_1, \gamma_2, ..., \gamma_n))$, where $\gamma_i \in \Gamma$, and $f \in O(\hat{G}^n/\hat{G})$, we know that $\tilde{\nu}$ is determined by $\tilde{\nu}(S_{1, \text{Tr}_0})$, where $\gamma \in \Gamma$. Since $\gamma \mapsto \tilde{\nu}(S_{1, \text{Tr}_\gamma})$ is continuous, $\tilde{\nu}$ is determined by $\tilde{\nu}(S_{1, \text{Tr}_0})$, where $v$ runs through
places of $F$ outside $N$. This is because $\{\text{Frob}_v | v + N\}$ is a dense subset of $\Gamma$ by Chebotorev’s Theorem.

However, by compatibility with Hecke operators, we know that $S_{1, \text{Tr,Frob}_v}$ is the Hecke operator of $h_{\nu,v}$ corresponds to the standard representation $V$ of $\hat{G} = GL(n, \hat{Q}_l)$. Thus, $\tilde{\nu}$ is determined by $\tilde{\nu}|_{\tilde{T}_N}$.

We have the decomposition

$$A_0(G, K_N, \hat{Q}_l) = \bigoplus \pi^K_N = \bigoplus \lambda A_{0,\lambda},$$

where each $\pi^K_N$ becomes the eigenspace $A_{0,\lambda}$, for some character $\lambda$ of $T_N$. By Strong Multiplicity One theorem, we know that different $\pi^K_N$ corresponds to different $\lambda$.

On the other hand, we also have the decomposition

$$A_0(G, K_N, \hat{Q}_l) = \bigoplus \nu A_{0,\nu} = \bigoplus \lambda A_{0,\lambda},$$

where $A_{0,\nu}$ is the eigenspace of character $\tilde{\nu}$ of $B_N$. When restricting to $\tilde{T}_N$, $A_{0,\nu}$ becomes the eigenspace $A_{0,\lambda}$, where $\lambda = \tilde{\nu}|_{T_N}$, we just showed that different $\tilde{\nu}$ correspond to different $\lambda$. Hence, the two decompositions (1.1) and (1.3) coincide with each other.

4. Decomposition induced from $\mathcal{H}(GL(n))$

Following Section 2 we will first study the restriction of cusp forms from $\hat{G}(\mathbb{A}) = GL(n, \mathbb{A})$ to $G(\mathbb{A}) = SL(n, \mathbb{A})$ more carefully, and define an action on $A_0(G, K, \hat{Q}_l)$ via this restriction. Recall that we have a bijection

$$\bigoplus_{(\pi) \in \Pi_{\text{cusp}}(\hat{G}, \hat{\rho})_G} \mathcal{L}(\tilde{\pi})^1 \xrightarrow{\text{res}_G^\hat{G}} A_0(G, \mu),$$

where $\tilde{\mu} \in [\mu]$, from Theorem 2.6. Note that we can always find $\tilde{\mu} \in [\mu]$ of finite order. Since $\mathbb{I}/F^* \cong \mathbb{I}/F^* \times \mathbb{Z}$ as a topological group, and $Z(\mathbb{A})/Z(F)$ is isomorphic to the closed subgroup of $n$-th roots of unity in $\mathbb{I}/F^*$, one can lift $\mu$ to a finite order character of $\mathbb{I}/F^*$, and define $\tilde{\mu}$ to be that lift tensoring with the trivial character on $\mathbb{Z}$. We will henceforth fix such a finite order $\tilde{\mu}$ that lifts $\mu$.

In the following discussion, we need to fix a specific lifting, i.e., we need to specify a cuspidal representation $\tilde{\pi}$ from each $G$-equivalence class $\{\tilde{\pi}\}_G$. The actions induced from $GL(n, \mathbb{A})$ and its Hecke algebra do depend on this choice, however, all choices of $\tilde{\pi}$ will result in the same conclusion of Proposition 4.3, and the same decomposition in Proposition 4.4 only parameterized by different characters.

4.1. Embedding Hecke Algebra. We first study how to embed Hecke algebra on $SL(n)$ to that of $GL(n)$. We fix the ground field $k = F_v$ to be a local field, with $\pi$ a uniformizer. For $G = SL(n)$, we fix the standard maximal torus in the Borel group, $T \subset B \subset G$, and $X^* = \text{res} X_*$ be the (resp. co)character group of $G$. Let $\Phi = \Phi^+ \cup -\Phi^+$ be the roots, $\rho = \frac{1}{2} \sum_\alpha \alpha$, and $P^+$ be the positive coroots, $W$ be the Weyl group. Let $G$ also denotes the $k$-points $G(k)$, and $K = G(O)$ be the maximal compact open subgroup of $G$. Let $\mathcal{H}(G, K)$ be the Hecke algebra of locally constant compactly supported function, bi-invariant under $K$, we normalize the measure $dx$, so that volume of $K$ is 1. And we define everything with $^\sim$ to
indicate the corresponding notions of $GL(n)$, with the exceptions that $\Phi^+$ and $\rho$ can indicate both for $GL(n)$ and $SL(n)$, since they have the identical root system.

We have the Cartan Decomposition:

**Proposition 4.1.** The group $G$(resp. $\tilde{G}$) is disjoint union of double coset $K\lambda(\pi)K$(resp. $\tilde{K}\lambda(\pi)\tilde{K}$), where $\lambda$ runs over $P^+$.

Hence a basis of Hecke algebra $\mathcal{H}(\tilde{G}, \tilde{K})$(resp. $\mathcal{H}(G, K)$) is the set of characteristic functions $\tilde{\chi}_{\lambda} = \text{Char}(\tilde{K}\lambda(\pi)\tilde{K})$(resp. $c_{\lambda} = \text{Char}(K\lambda(\pi)K)$), where $\lambda$ runs over $P^+$. We define the embedding to be:

$$\iota: \mathcal{H}(G, K) \to \mathcal{H}(\tilde{G}, \tilde{K})$$

$$c_{\lambda} \mapsto \tilde{\chi}_{\lambda},$$

for all $\lambda \in P^+(SL(n))$. Since $\{c_{\lambda} | \lambda \in P^+(SL(n))\}$ form a $\mathbb{Q}_l$-basis of Hecke algebra, we can extend this linearly to $\mathcal{H}(G, K)$. This is clearly injective, and additive, it suffices to show that it is also multiplicative.

**Lemma 4.2.** If $K\lambda(\pi)K = \bigsqcup x_iK$, then $\tilde{K}\lambda(\pi)\tilde{K} = \bigsqcup x_i\tilde{K}$, where $\lambda \in P^+(SL(n))$.

**Proof.** It is clear that $\bigcup x_i\tilde{K} \subset \tilde{K}\lambda(\pi)\tilde{K}$, and it is a disjoint union. Moreover, any element

$$\tilde{k}\lambda(\pi)\tilde{k}' = (kd)\lambda(\pi)\tilde{k}'$$

$$= kd\lambda(\pi)\tilde{k}'$$

$$= k\lambda(\pi)d\tilde{k}'$$

$$= x_i k' d \tilde{k}' \in x_i \tilde{K}, \text{ for some } x_i$$

where $\tilde{k}, \tilde{k}' \in \tilde{K}, k, k' \in K$, and $d \in \tilde{K}$ is some diagonal matrix, with the first entry equals $\det(k)$, and 1 elsewhere. Since $\lambda(\pi)$ is diagonal too, it commutes with $d$. \qed

Standard computations [Gro98] show that

$$\tilde{\chi}_{\lambda} \cdot \tilde{\chi}_{\mu} = n_{\lambda, \mu}(\nu)\tilde{\chi}_{\nu},$$

where $n_{\lambda, \mu}(\nu) = \|\{(i, j) : \nu(\pi) \in x_i y_j \tilde{K}\}$. For $\nu \in P^+$, Lemma 4.2 implies that $n_{\lambda, \mu}(\nu) = n_{\lambda, \mu}(\nu)$, hence $\iota$ is a $\mathbb{Q}_l$-algebra map. The embedding $\iota$ also preserves Satake isomorphism:

$$S(f)(t) = \delta(t)^{1/2} \cdot \int_N f(tn)dn, \quad f \in \mathcal{H}(G, K),$$

where $dn$ is the unique haar measure on $N$, such that $N \cap K$ has volume 1, and $\delta(t)^{1/2} = q^{-<\mu, \rho>}$ for $t = \mu(\pi)$. This is because $\tilde{G}$ and $G$ has the same coroots, and hence $\rho$. It can also be checked for $f = c_{\lambda}$ that $S(\iota(f))$ is supported on $X_*(T)$, and

$$S(\iota(f))|_{X_*(T)} = S(f), \quad f \in \mathcal{H}(\tilde{G}, \tilde{K}).$$

Thus, $S(\iota(f)) = \iota'(S(f))$, where $\iota'$ is the natural inclusion of $\mathbb{Q}_l[X^\bullet(\hat{T})]^W \to \mathbb{Q}_l[X^\bullet(\hat{T})]^W$. Hence, if $V$ is a representation $\text{PGL}(n, \mathbb{Q}_l) \to \text{GL}(m, \mathbb{C})$, which lifts to a representation $\hat{V}$ of $\text{GL}(n, \mathbb{Q}_l)$, then $\iota(h_{V, \nu}) = h_{\hat{V}, \nu}$, where $h_{V, \nu}$ is the Hecke operator corresponds to $V$. 
4.2. Action induced from Hecke algebra. We can get an action of Hecke algebra of GL(n) to \( A_0(G, \mu) \), via the restriction in Theorem 2.6.

Fix \( N_0 \) an effective divisor, and \( K_{N_0} \subset G(\mathcal{O}_F) \) the corresponding open compact subgroup of \( G(\A) \), where \( K_v = G(\mathcal{O}_{F_v}) \) for \( v \notin N_0 \). Hence we have the decomposition of cusp forms of level \( K_{N_0} \):

\[
A_0(G, K_{N_0}, \mu) = \bigoplus \pi_{K_{N_0}},
\]

which is finite dimensional. Choose a finite basis \( \{ f_i \} \) of \( A_0(G, K_{N_0}, \mu) \), we get a lifting \( \{ \tilde{f}_i \} \subset A_0(\tilde{G}, \tilde{\mu}) \), where \( \tilde{\mu} \in \mu \), and of finite order. Since \( \tilde{f}_i \) is locally constant, there exists a compact open subgroup \( \tilde{K}_i \) such that \( \tilde{f}_i \) is right-invariant under \( \tilde{K}_i \). By taking finite intersection, we can assume that \( \{ \tilde{f}_i \} \subset A_0(\tilde{G}, \tilde{\mu}, \tilde{\mu}) \), for some open compact subgroup \( \tilde{K} \subset \prod_v \tilde{G}(\mathcal{O}_v) \) of \( \tilde{G}(\A) \).

We can then choose \( N \), such that \( N_0 \subset N \), and \( \tilde{K}_v = \tilde{G}(\mathcal{O}_v) \), for \( v \notin N \). Let \( T_N = \bigotimes_{v \notin N} H_v(G(F_v), K_v) \), and \( \tilde{T}_N = \bigotimes_{v \notin N} H_v(\tilde{G}(F_v), \tilde{K}_v) \) be the unramified commutative Hecke algebra of \( G \), and \( \tilde{G} \) respectively. Therefore, we can extend the action of \( T_N \) on \( A_0(G, K_{N_0}, \mu) \) to \( \tilde{T}_N \) via \( \text{res}_G^{\tilde{G}} \). More precisely, \( h \cdot f = \text{res}_G^{\tilde{G}}(\tilde{h} \cdot \tilde{f}) \). Note that \( \tilde{f} \in \tilde{\pi}^K \), hence for all places \( v \notin N \), we have \( \tilde{h}_v \cdot f = \tilde{\lambda}_v(\tilde{h}_v) f \), where \( \tilde{\lambda}_v \) is the Hecke character of \( \tilde{\pi}_v \). If \( \tilde{h}_v = \iota(h_v) \), then \( \tilde{h}_v \cdot f = h_v \cdot f \). This can be checked similarly by Lemma 4.2. Let \( \tilde{\lambda} = \bigotimes_{v \notin N} \tilde{\lambda}_v \), then \( \lambda = \tilde{\lambda}|_T \) is the normal character for \( T \).

Suppose \( \tilde{\pi} = \bigotimes_{v \notin N} \tilde{\pi}_v \), by Lemma 2.8 we know that

\[
\text{Res}^{\tilde{G}}_{G} \tilde{\pi} = \text{Res}^{\tilde{G}}_{G} \bigotimes_{v \notin N} \tilde{\pi}_v \cong \bigoplus_{\text{each } v, i_v = 1 \ldots r_v} \bigotimes_{v,i=1}^{v,i} \pi_{v,i},
\]

where \( \text{Res}^{\tilde{G}}_{G} \tilde{\pi}_v = \bigotimes_{i=1}^{r_v} \pi_{v,i} \) is multiplicity free, and \( \pi_{v,1} \) is unramified , \( i_v = 1 \) for almost all \( v \).

Similarly we have

\[
\text{Res}^{\tilde{G}}_{G} \tilde{\pi} = \bigotimes_{v \notin N} \tilde{\pi}_v = \bigotimes_{i=1}^{r_v} \pi_{v,i},
\]

is the restriction of cusp forms, where \( \pi_v \)'s are irreducible cuspidal representations of \( G \), and it is multiplicity free. We call \( \{ \pi_v \mid \text{Res}^{\tilde{G}}_{G} \tilde{\pi} = \bigotimes \pi_{v,i} \} \) an L-packet of \( \tilde{\pi} \), denoted as \( L(\tilde{\pi}) \) (we’ll sometimes abuse this term by referring \( \text{Res}^{\tilde{G}}_{G} \tilde{\pi} \) as L-packet too). Also, let \( \overline{L(\tilde{\pi})} \) be the set of isomorphism classes of \( G(\A) \) representations in \( L(\tilde{\pi}) \), it is clearly a subset of the set of (classes of) irreducible components of \( \text{Res}^{\tilde{G}}_{G} \tilde{\pi} \).

The following proposition is essentially showed in [GK82]

**Proposition 4.3.** If \( \overline{L(\tilde{\pi}_1)} \cap \overline{L(\tilde{\pi}_2)} = \emptyset \), then \( \overline{L(\tilde{\pi}_1)} = \overline{L(\tilde{\pi}_2)} \). In particular, if two irreducible cuspidal representations of \( G(\A) \) are in the same L-packet, then they have the same multiplicity in \( A_0(G, \mu) \).

Thus, it is easy to see that the L-packet \( \text{Res}^{\tilde{G}}_{G} \tilde{\pi}^{K_{N_0}} = \bigotimes \pi_{K_{N_0}} \) is contained in \( \tilde{\lambda} \)-eigenspace \( A_0(G, K_{N_0}, \mu) \), where \( \tilde{\lambda} \) is the unramified Hecke character for \( \tilde{\pi} \). By Multiplicity One, they are actually equal. If two L-packets are isomorphic (contains isomorphic irreducible components), then they will be contained in the same \( \lambda \)-eigenspace \( A_0(G, K_{N_0}, \mu) \). We conclude in the following proposition:

**Proposition 4.4.** The action of \( \tilde{T}_N \) extends the action of \( T_N \), and we have the decomposition \( A_0(G, K_{N_0}, \mu) = \bigoplus_{\lambda} A_0(G, K_{N_0}, \mu) \), where \( A_0(G, K_{N_0}, \mu) = \bigoplus_{\lambda} A_0(G, K_{N_0}, \mu) \), where
\( \lambda \) and \( \bar{\lambda} \) are characters of \( T_N \) and \( \bar{T}_N \). Moreover, \( A_0(G, K_{N_0}, \mu)_{\lambda} = \text{res}_G^\pi(\bar{\pi})^{K_{N_0}} \) is an L-packet. For each \( A_0(G, K_{N_0}, \mu)_{\lambda} \), we can associate a Langlands parameter \( \sigma \) to it, which is compatible with Satake Isomorphism.

For \( A_0(G, K_{N}, \mu)_{\lambda} = \text{res}_G^\pi(\bar{\pi})^{K_{N_0}} \), we have a Langlands parameter \( \bar{\sigma} : \Gamma \rightarrow \text{GL}(n, \mathbb{Q}_l) \) corresponding to \( \bar{\pi} \), compose it with the projection to \( \text{PGL}(n, \mathbb{Q}_l) \), we get the desired Langlands parameter \( \sigma \). Let \( V \) be an irreducible representation of \( \text{PGL}(n, \mathbb{Q}_l) \), it lifts to an irreducible representation \( \check{V} \) of \( \text{GL}(n, \mathbb{Q}_l) \). By compatibility with Satake Isomorphism for \( \text{GL}(n) \), we know that \( h_{\check{V}, \nu} \) acts on \( A_0(G, K, \mu)_{\sigma} = \text{res}_G^\pi(\bar{\pi})^{K} \), by multiplication by the scalar \( \chi_{\check{V}}(\bar{\sigma}(\text{Frob}_v)) \), where \( \chi_{\check{V}} \) is the character of \( \check{V} \). Since \( \iota \) is compatible with Satake Isomorphism, \( h_{V, \nu} \) acts on \( A_0(G, K, \mu)_{\sigma} = \text{res}_G^\pi(\bar{\pi})^{K} \), also by multiplication by the scalar \( \chi_{V}(\sigma(\text{Frob}_v)) = \chi_{\check{V}}(\bar{\sigma}(\text{Frob}_v)) \).

Remark 4.5. The definition of \( T_N \) and \( \bar{T}_N \) involves \( N \). Hence, in Proposition 4.4, the decomposition under the action of \( T_N \) does depend on \( N \). However, the decomposition under the action of \( \bar{T}_N \) does not depend on \( N \), thanks to Multiplicity One.

Remark 4.6. The action induced from \( \bar{T}_N \) does depend on the choice of lifting. If we choose \( \bar{\pi}_1 = \bar{\pi}_2 \otimes \omega \), where \( \omega \in (\hat{G}(\mathbb{A})/\hat{G}(F)G(\mathbb{A})Z(\mathbb{A}))^{\mathbb{A}} \), then \( \text{res}_G^\pi(\bar{\pi}_1) = \text{res}_G^\pi(\bar{\pi}_2) \), but \( \bar{T}_N \) acts on it as \( \bar{\lambda}_i \) if we lift it to \( \bar{\pi}_i \), where \( \bar{\lambda}_i \) are the normal Hecke character for \( \bar{\pi}_i \). By Strong Multiplicity One, \( \bar{\lambda}_1 \neq \bar{\lambda}_2 \). Again, the decomposition in Proposition 4.4 doesn’t depend on lifting, different liftings will only result in different characters \( \bar{\lambda} \) parameterizing the same components.

5. Excursion character for \( \text{SL}(n) \)

5.1. Decomposition coincide. In Subsection 4.2, we defined an induced action of unramified Hecke algebra \( \bar{T}_N \) of \( \text{GL}(n) \) to \( A_0(G, K_{N_0}, \mu) \). By the discussion in Section 3, we know that the action of \( \bar{B}_N \) on \( A_0(G, K_{N_0}, \mu) \) uniquely extends that of \( \bar{T}_N \). We can similarly define the induced action of \( \bar{B}_N \) on \( A_0(G, K_{N_0}, \mu) \), this will result in the same decomposition in Proposition 4.4 but parameterized by characters \( \nu \) of \( \bar{B}_N \).

Then the decomposition in Proposition 4.4 can be written as:

\[
A_0(G, K_{N_0}, \mu) = \bigoplus_{\bar{\nu}} A_0(G, K_{N_0}, \mu)_{\bar{\nu}},
\]

where \( A_0(G, K_{N_0}, \mu)_{\bar{\nu}} = A_0(G, K_{N_0}, \mu)_{\bar{\lambda}} = \text{res}_G^\pi(\bar{\pi})^{K_{N_0}} \), for the unique character \( \bar{\nu} \) of \( \bar{B}_N \), such that \( \bar{\lambda} = \bar{\nu}\mid_{\bar{T}_N} \), and \( \bar{\pi} \) corresponds to \( \bar{\lambda} \). On the other hand, we have the actual decomposition given in Lafforgue’s paper for \( \text{SL}(n) \).

\[
A_0(G, K_{N_0}, \mu) = \bigoplus_{\nu} A_0(G, K_{N_0}, \mu)_{\nu},
\]

where \( \nu \) is the character of \( B_N = B_N(\text{SL}(n)) \). There is an obvious way that \( B_N \) can be embedded into \( \bar{B}_N \), if we get \( \nu \) by restricting \( \bar{\nu} \) to \( B_N \), then the corresponding Langlands parameter \( \sigma \) is the composition of \( \bar{\sigma} \) with the projection \( \text{GL}(n) \rightarrow \text{PGL}(n) \).

By Proposition 12.5 of [Laf12], we have \( A_0(G, K_{N_0}, \mu)_{\bar{\nu}} \subset A_0(G, K_{N_0}, \mu)_{\nu} \). We know that the equality holds \( A_0(G, K_{N_0}, \mu)_{\bar{\nu}} = A_0(G, K_{N_0}, \mu)_{\nu} \), because if \( \bar{\nu}_1|_{B_N} = \bar{\nu}_2|_{B_N} \), then the Langlands parameter \( \bar{\sigma}_1 \) and \( \bar{\sigma}_2 \) are equal after composing with the projection \( \text{GL}(n) \rightarrow \text{PGL}(n) \), it is easy to verify that \( \bar{\pi}_1 \) and \( \bar{\pi}_2 \) are hence \( G \)-equivalent, which implies \( \bar{\pi}_1 = \bar{\pi}_2 \), and \( \nu_1 = \nu_2 \), since we only choose one representative from a \( G \)-equivalence class in the decomposition of
Theorem 2.6: Meanwhile, since $\text{res}_l^G(\tilde{\pi})$ is multiplicity free. The multiplicity of $SL(n)$ can be bounded by the number of $L$-packets $\text{res}_l^G(\tilde{\pi})^{K_{N_0}}$, in the $\lambda$-eigenspace, this will be the subject of the next subsection.

5.2. Higher Multiplicities of $SL(n)$. Using the decomposition in Proposition 4.4 and its coincidence with the Lafforgue’s decomposition, we may find an upper bound for the multiplicities of $SL(n)$, $M(\pi_0) = \#\{\pi \in A_0(G, \mathbb{Q}_l) | \pi \cong \pi_0\}$. Since $\pi_0$ is countable dimension(countable restricted tensor product of countable dimension representations), and $\mathbb{Q}_l$ is algebraically closed and uncountable, by Schur’s Lemma, $\pi_0$ has a central character $\mu$. Let $\pi_1 \cong \pi_2$ be two irreducible cuspidal representations in $A_0(G, \mu)$ for that $\mu$. In addition, there exists $N_0$, such that $\pi_{i, N_0} \not\cong 0$, and both $\pi_{i, N_0}$ appear in $A_0(G, K_{N_0}, \mu_{\lambda})$, for some character $\lambda$ of an unramified Hecke algebra $T_N$, hence the multiplicity of $SL(n)$ is bounded by the multiplicity in $A_0(G, K_{N}, \mu_{\lambda})$. By Proposition 4.3, this is bounded by the cardinality of $\{\lambda \in \text{Hom}(\mathcal{F}_N, \mathbb{Q}_l) | \lambda|_{T_N} = \lambda\}$, and is realized as a cuspidal representation in $A_0(\tilde{G}, \tilde{\mu})$ by Langlands Correspondance for $GL(n)$. We summarize:

Proposition 5.1. The number of isomorphic irreducible components of $A_0(G, \mathbb{Q}_l)$ (which corresponds to a character $\lambda$ of some unramified Hecke algebra of $SL(n)$), is bounded above by the number of $G$-equivalent classes of pseudocharacters $\tilde{\Theta}(m, f, (\gamma_i))$ of $GL(n)$, such that $\tilde{\Theta}(1, Tr_V, \gamma)$ (where $V$ is any representation of $GL(n)$ that factors through $\text{PGL}(n)$) is given by $\lambda$, and $\tilde{\Theta}(\text{GL}(n))(1, Det, \gamma)$ is given by $\tilde{\mu}$.

By abuse of the notation, we say two pseudocharacters are $G$-equivalent, if their corresponding cuspidal representations are. We know that the pseudocharacters of $GL(n)$ is determined by $\tilde{\Theta}(1, Tr, \gamma)$, where $\gamma$ runs over $\{\text{Frob}_v | v + N\}$.

In Proposition 5.1 we can take $V = (\text{Std})^{\otimes n} \otimes \wedge^n \text{Std}$, and we see that $Tr_V = \frac{T^n}{\text{Det}}$, hence $\tilde{\Theta}(1, Tr, \gamma)^n$, where $\gamma$ runs over $\{\text{Frob}_v | v + N\}$, are given by $\lambda$ and $\tilde{\mu}$. In another word, $\lambda$ and $\tilde{\mu}$ in Proposition 5.1 determine such extensions up to $n$-th roots of unity.

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