ON THE ROAD TOWARDS
THE QUANTUM GEOMETER’S UNIVERSE:
AN INTRODUCTION TO FOUR-DIMENSIONAL
SUPERSYMMETRIC QUANTUM FIELD THEORY

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This brief set of notes presents a modest introduction to the basic features entering the construction of supersymmetric quantum field theories in four-dimensional Minkowski spacetime, building a bridge from similar lectures presented at a previous Workshop of this series, and reaching only at the doorstep of the full edifice of such theories.

1. Introduction

The organisers of the third edition of the COPROMAPH Workshops had thought it worthwhile to have the second series of lectures during the week-long meeting dedicated to an introduction to supersymmetric quantum field theories. An internationally renowned expert in the field had been invited, and was to deliver the course. Unfortunately, at the last minute fate had it decided otherwise, depriving the participants of what would have been an introduction to the subject of outstanding quality. The present author was finally found to be on hand, without being able to do full justice to the wide relevance of the topic, ranging from pure mathematics and topology to particle physics phenomenology at its utmost best in anticipation of the running of the LHC at CERN by 2007.

Fate had it also that the same author had already delivered a similar series of lectures at the previous edition of the COPROMAPH Workshops,\(^1\) which in broad brush strokes attempted to paint with vivid colours the fundamental principles of XX\(^{th}\) century physics, underlying all the basic conceptual progresses having led to the relativistic quantum gauge field theory and classical general relativity frameworks for the present description of all known forms of elementary matter constituents and their fundamental

\(^1\)
interactions, as inscribed in the Standard Model of particle physics and Einstein’s classical theory of general relativity. At the same time, a few of the doors onto the roads winding deep into the unchartered territories of the physics that must lie well beyond were also opened. It is thus all too fitting that we get the opportunity to trace together a few steps onto one of these roads, in the embodiment of a Minkowski spacetime structure extended into a superspace including now also anticommuting coordinates in addition to the usual commuting spacetime ones. We are truly embarking on a journey onto the roads leading towards the quantum geometer’s universe! Even if only by marking the path by a few white and precious pebbles to guide us into the unknown territory when the time will have come for more solitary explorations of one’s own in the composition, with a definite African beat, of the music scores of the unfinished symphony of XXI\textsuperscript{st} century physics.\textsuperscript{1}

Even though none are based on actual experimental facts, there exist a series of theoretical and conceptual motivations for considering supersymmetric extensions of ordinary Yang–Mills theories in the quest for a fundamental unification. Spacetime supersymmetry is a symmetry that exchanges particles of integer — bosons — and half-integer — fermions — spin,\textsuperscript{a} enforcing specific relations between the properties and couplings of each class of particles, when supersymmetry remains manifest in the spectrum of the system. In particular, since for what concerns ultra-violet (UV) short-distance divergences of quantum field theories in four-dimensional Minkowski spacetime fermionic fields are less ill-behaved than bosonic fields (namely, in terms of a cut-off in energy, divergences in fermionic loop amplitudes are usually only logarithmically divergent whereas those of bosonic loops are quadratically divergent), one should expect that in the presence of manifest supersymmetry, UV divergences should be better tamed for bosonic fields, being reduced to a logarithmic behaviour only as in the fermionic sector (this has important consequences which we shall not delve into here). Another aspect is that within the context of superstring and M-theory\textsuperscript{2} with bosonic and fermionic states, quantum consistency is ensured provided supersymmetries are restricting the dynamics. In this sense, the existence of supersymmetry at some stage of unification beyond the Standard Model is often considered to be a natural prediction of M-theory.

\textsuperscript{a}The lectures delivered at COPROMAPH\textsuperscript{2} did not deal with field theories associated to fermionic degrees of freedom described using Grassmann odd variables, and considered only bosonic theories.\textsuperscript{1} Quantised fermionic field theories are briefly dealt with in Sec. 3.
Besides such physics motivations just hinted at, supersymmetry has also proved to be of great value in mathematical physics, in the understanding of nonperturbative phenomena in quantum field theories and M-theory,\textsuperscript{3,4} and for uncovering deep connections between different fields of pure mathematics. The algebraic structures associated to Grassmann graded algebras are powerful tools with which to explore new limits in the concepts of geometry, topology and algebra.\textsuperscript{4} One cannot help but feel that a great opportunity would be missed if tomorrow’s quantum geometry would not make any use of supersymmetric algebraic structures.

Since its discovery in the early 1970’s,\textsuperscript{5,6} applications of supersymmetry have been developed in such a diversity of directions and in so large a variety of fields of physics and mathematics, that it is impossible to do any justice to all that work in the span of any set of lectures, let alone only a few. Our aim here will thus be very modest. Namely, starting from the contents of the previous lecture notes,\textsuperscript{1} build a bridge reaching the entry roads and the shores towards supersymmetric field theories and the fundamental concepts entering their construction. Not that the lectures delivered at the Workshop did not discuss the general superfield approach over superspace as the most efficient and transparent techniques for such constructions in the case of $\mathcal{N} = 1$ supersymmetry, but the latter material being so widely and in such detailed form available from the literature, it is felt that rather a detailed introduction to the topics missing from Ref. 1 but necessary to understand supersymmetric field theories is of greater use and interest to most readers of this Proceedings volume. With these notes, our aim is thus to equip any interested reader with a few handy concepts and tools to be added to the backpack to be carried on his/her explorer’s journey towards the quantum geometers universe of XXI\textsuperscript{st} century physics, in search of the new principle beyond the symmetry principle of XX\textsuperscript{th} century physics.\textsuperscript{1}

Also by lack of space and time, even of the anticommuting type if the world happens to be supersymmetric indeed, we shall thus stop short of discussing explicitly any supersymmetric field theory in 4-dimensional Minkowski spacetime, even the simplest example of the $\mathcal{N} = 1$ Wess-Zumino model\textsuperscript{6} that may be constructed using the hand-made tools of an amateur artist-composer in the art of supersymmetries. From where we shall leave the subject in these notes, further study could branch off into a variety of directions of wide ranging applications, beginning with general supersymmetric quantum mechanics and the general superspace and superfield techniques for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric field theories with Yang–Mills internal gauge symmetries and the associated Higgs
mechanism of gauge symmetry breaking, to further encompass the search for new physics at the LHC through the construction of supersymmetric extensions of the Standard Model, or also reaching towards the duality properties of supersymmetric Yang–Mills and M-theory, mirror geometry, topological string and quantum field theories, etc., to name just a few examples.2,3,4

Let us thus point out a few standard textbooks and lectures for large and diversified accounts of these classes of theories and more complete references to the original literature. Some important such material is listed in Refs. 8 and 9. In particular, the lectures delivered at the Workshop were to a significant degree inspired by the contents of Ref. 10. Any further search through the SPIRES database (http://www.slac.stanford.edu/spires/hep/; UK mirror: http://www-spires.dur.ac.uk/spires/hep/) will quickly uncover many more useful reviews.

In Sec. 2, we briefly recall the basic facts of relativistic quantum field theory for bosonic degrees of freedom, discussed at greater length in Ref. 1, in order to explain why such systems are the natural framework for describing relativistic quantum point-particles. The same considerations are then developed in Sec. 3 in the case of fermionic degrees of freedom associated to particles of half-integer spin, based on a discussion of the theory of finite dimensional representations of the Lorentz group, leading in particular to the free Dirac equation for the description of spin 1/2 particles without interactions. Section 4 then considers, as a simple introductory illustration of some facts essential and generic to supersymmetric field theories, and much in the same spirit as that of the discussion in Sec. 2, the \( \mathcal{N} = 1 \) supersymmetric harmonic oscillator which already displays quite a number of interesting properties. Section 5 then concludes with a series of final remarks related to the actual construction of supersymmetric field theories based on the general concepts of the Lie symmetry algebraic structures inherent to such relativistic invariant quantum field theories and their manifest realisations through specific choices of field content, indeed the underlying theme to both these lectures and the previous ones.1
2. Basics of Quantum Field Theory: A Compendium for Scalar Fields

Within a relativistic classical framework, material reality consists, on the one hand, of dynamical fields, and on the other hand, of point-particles. Fields act on particles through forces that they develop, such as the Lorentz force of the electromagnetic field for charged particles, while particles react back onto the fields being sources for the latter, for instance through the charge and current density sources of the electromagnetic field in Maxwell’s equations (the same characterisation applies to the gravitational field equations of general relativity). This dichotomic distinction between matter and radiation is unified in a dual form when considering a relativistic quantum framework. Indeed, it then turns out that particles are nothing else than the quanta, i.e., the quantum states of definite energy, momentum and spin, of quantum fields. Particles and fields are just the two complementary aspects of the quantum relativistic world of point-particles. All electrons, for example, are identical, being quanta of a single electron field filling all of spacetime. To each distinct species of particle corresponds a field, and vice-versa. This, in a word, is the essence of quantum field theory: the natural framework for a description of relativistic quantum point-particles, explaining their corpuscular properties when detected in energy-momentum eigenstates and their wave behaviour when considering their spacetime propagation. Let us briefly express these points in a somewhat more mathematical setting.

2.1. Particles and Fields

A free relativistic field may be seen to correspond to an infinite collection of harmonic oscillators sitting at each point in space, and coupled to one another through a nearest-neighbour term in the action of the field’s dynamics. Let us first recall a few basic facts about the one-dimensional harmonic oscillator. Its dynamics derives through the variational principle from the action

\[ S[q] = \int dt \left[ \frac{1}{2} \left( \frac{dq}{dt} \right)^2 - \frac{1}{2} \omega^2 q^2 \right], \tag{1} \]

where \( q(t) \in \mathbb{R} \) is the configuration space of the system. How to perform the standard canonical operator quantisation of this system is well known.\(^1\)

\(^b\)Throughout most of these notes, units such that \( c = 1 = \hbar \) are used.
leading, in the Heisenberg picture, to the following quantum operator representation,

\[
\hat{q}(t) = \sqrt{\frac{\hbar}{2m\omega}} \left[ a e^{-i\omega t} + a^\dagger e^{i\omega t} \right],
\]

(2)
obeying the operator equation of motion

\[
\left[ \frac{d^2}{dt^2} + \omega^2 \right] \hat{q}(t) = 0.
\]

(3)

Here, \( a \) and \( a^\dagger \) are the annihilation and creation operators for the quanta of the system (they are complex conjugate integration constants at the classical level), obeying the Fock space algebra

\[
[a, a^\dagger] = 1.
\]

(4)

The quantum Hamiltonian \( \hat{H} = \hbar\omega (a^\dagger a + 1/2) \) is diagonal in the Fock state basis, constructed as follows for all natural numbers \( n = 0, 1, 2, \ldots \),

\[
|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle, \quad \langle n|m\rangle = \delta_{nm}, \quad a|0\rangle = 0, \quad \hat{H}|n\rangle = \hbar\omega (n + \frac{1}{2})|n\rangle.
\]

(5)

The physical interpretation is that the state \( |0\rangle \) defines the ground state or vacuum of the quantum oscillator, with the discrete set of states \( |n\rangle \) \( (n = 1, 2, \ldots) \) corresponding to excitations of the oscillator with \( n \) quanta each contributing an energy \( \hbar\omega \) on top of the vacuum quantum energy \( \hbar\omega/2 \) due to so-called vacuum quantum fluctuations. In particular, the operators \( a \) and \( a^\dagger \) are ladder operators between the Fock states,

\[
a|n\rangle = \sqrt{n}|n\rangle, \quad a^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle, \quad a^\dagger a|n\rangle = n|n\rangle.
\]

(6)

Thus, here we have a mathematical framework in which the quantisation of a configuration space \( q(t) \in \mathbb{R} \) leads to an algebra of quantum operators representing the creation and annihilation of energy eigenstates. In order to describe the dynamics of relativistic quantum point-particles which likewise, as observed in experiments, may be created and annihilated, we shall borrow a similar mathematical framework. Since the harmonic oscillator is a system invariant under translations in time, according to Noether’s theorem\(^{11}\) there must exist a conserved quantity associated to this continuous symmetry whose value coincides with the energy of the system, namely its Hamiltonian. In the case of relativistic particles defined over Minkowski spacetime,\(^c\) invariance under spacetime translations implies the existence

\(^{11}\)Our choice of Minkowski spacetime metric signature is such that \( \eta_{\mu\nu} = \text{diag}(+ - - -) \) with \( \mu, \nu = 0, 1, 2, D - 1 \) with \( D = 4 \).
of conserved quantities associated to these symmetries, namely the particle’s total energy and momentum, which we shall denote \( k^\mu = (k^0, \vec{k}) \) with 
\[ k^0 = \omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}, \] 
m being the particle’s mass. Consequently, let us introduce the annihilation and creation operators, \( a(\vec{k}) \) and \( a^\dagger(\vec{k}) \), respectively, for particles of given momentum \( \vec{k} \) and energy \( \omega(\vec{k}) \), and obeying the commutation relations
\[
\left[ a(\vec{k}), a^\dagger(\vec{\ell}) \right] = (2\pi)^3 2\omega(\vec{k}) \delta(3)(\vec{k} - \vec{\ell}).
\] (7)

For instance, 1-particle states are thus constructed as
\[
|\vec{k}\rangle = a^\dagger(\vec{k}) |0\rangle, \quad \langle \vec{k}|\vec{\ell}\rangle = (2\pi)^3 2\omega(\vec{k}) \delta(3)(\vec{k} - \vec{\ell}),
\] (8)

|0\rangle being the Fock vacuum of the system.

In order to identify the actual configuration space of the system that is being considered, by analogy with (2), let us construct the following quantum operator in the Heisenberg picture,
\[
\hat{\phi}(x^\mu) = \int \frac{d\vec{k}}{(2\pi)^3 2\omega(\vec{k})} \left[ a(\vec{k}) e^{-ik \cdot x} + a^\dagger(\vec{k}) e^{ik \cdot x} \right],
\] (9)

where the inner product in the plane wave factors is defined to be given by \( k \cdot x = \omega(\vec{k}) x^0 - \vec{k} \cdot \vec{x} \), thus with the on-shell energy value \( k^0 = \omega(\vec{k}) \).

Note that in comparison to (2), the plane wave factors, corresponding to positive- and negative-frequency components of the wave equation and involving only a time dependence in the case of the harmonic oscillator, have now been extended to the spacetime dependent Lorentz invariant quantity \( k \cdot x \). The requirement of a relativistic covariant description of quantum point-particles being created and annihilated requires such an extension of the plane wave contributions. Furthermore, the operator \( \hat{\phi}(x^\mu) \) obeys the quantum equation of motion
\[
\left[ \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2 \right] \hat{\phi}(x^\mu) = 0,
\] (10)

in which one recognises of course the Klein–Gordon equation, deriving also from the action
\[
S[\phi] = \int dt \int_{(\infty)} d^3 \vec{x} \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \vec{\nabla} \phi \right)^2 - \frac{1}{2} m^2 \phi^2 \right].
\] (11)

\(^d\)Only the nonvanishing commutators are given. The choice of normalisation is made such that the momentum integration measure in the mode decomposition of the fields later on is Lorentz invariant. This choice also implies that particle states are normalised in a Lorentz covariant manner.
In other words, such a framework for the description of relativistic quantum point-particles and their creation and annihilation naturally leads to a relativistic quantum field theory. The configuration space of such a system is that of the relativistic real scalar field $\phi(x^\mu)$. In the quantum world, configurations of this field are observed through its energy and momentum eigenstates — thanks to the invariance under spacetime translations of the Klein–Gordon action — which are nothing else than the particle quanta of the field. To any relativistic quantum field one associates relativistic quantum point-particles, and to any ensemble of indistinguishable relativistic quantum point-particles one associates a relativistic quantum field. Fields and particles are only two dual aspects in a relativistic quantum universe whose basic “constituents” are dynamical fields. Note that relativistic covariance, which forced the extension to the Lorentz invariant $k \cdot x$ in the plane wave contributions, also explains the appearance of the gradient terms $\nabla \phi$ in the Klein–Gordon wave equation and action. Had this term been absent, indeed the field $\phi(x^\mu)$ would have described simply an infinite collection of harmonic oscillators $q_\xi(t) = \phi(t, \vec{x})$ each fixed at each of the points in space, and oscillating independently of one another. However, the gradient term introduces a specific nearest-neighbour coupling between these oscillators, such that any disturbance set-up in any one of them will quickly spread throughout space in a wave-like manner because of the gradient coupling between adjacent oscillators. These linear waves are characterised by their wave-number vector $\vec{k}$ and frequency $\omega(\vec{k})$, which, at the quantum level, are identified with the quanta’s or particles’ conserved momentum and energy. Indeed, because of Noether’s theorem associated to the invariance under spacetime translations, the total energy and momentum of the field take these values for the 1-particle quantum states $|\vec{k}\rangle$. The system is also invariant under the full Lorentz group of spacetime (pseudo)rotations, hence leading also to further conserved quantum numbers of the field and its quanta associated to their spin. In the present instance, since the field $\phi(x^\mu)$ transforms as a scalar under the Lorentz group (namely, it is left invariant), the quanta of such a real scalar field carry zero spin.

The above argument thus explains why relativistic quantum field theory provides the natural framework for the description of relativistic quantum point-particles that may be annihilated and created. An explicit canonical quantisation starting from the classical Klein–Gordon action (11) of course recovers all the above results, simply by applying the usual rules of quantum mechanics to this system of degrees of freedom $q_\xi(t) = \phi(t, \vec{x})$.\footnote{1}
Furthermore, it is also possible to set up a perturbation expansion for the introduction of spacetime local interactions between such fields or with themselves (simply by adding to the Klein–Gordon Lagrangian density higher order products of the fields at each point in spacetime, thus preserving spacetime locality and causality), and the systematic calculation, through Feynman diagrams and the corresponding Feynman rules, of matrix elements of the scattering S-matrix. Hence finally, decay rates and cross-sections for processes occurring between the quanta associated to such fields may be evaluated, at least through perturbation theory, starting from any given quantum field theory extended to include also interactions.

As discussed in Ref. 1, it is at this stage that the short-distance UV divergences appear in loop amplitudes, for which the renormalisation programme has been designed. Large classes of renormalisable, i.e., theories for which specific predictions may be made, have thereby been identified, and they all fall within the class of Yang–Mills theories of local gauge interactions extended in different manners and involving particles of spin 0 and 1/2 for the matter fields, and of spin 1 for the gauge fields associated to the gauge interactions. These are the basic concepts going into the construction of the Standard Model of the strong and electroweak interactions, based on the gauge symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$. A brief discussion of Yang–Mills theories, the Higgs mechanism of spontaneous symmetry breaking and the generation of mass is available in Ref. 1.

All the above may readily be extended to collections of real scalar fields. When further internal symmetries appear, additional internal quantum numbers exist, by virtue of Noether’s theorem, and particle quanta may then be classified according to specific linear representations of that internal symmetry group when realised in the Wigner mode. It thus proves very efficient to base the consideration of the construction of relativistic

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Footnotes:

1For example, taking two real scalar fields of identical mass defines a system with a global $SO(2) = U(1)$ symmetry. The associated conserved quantum number thus distinguishes quanta according to their $U(1)$ quantum number, taking a value either (+1) for certain quanta — particles — or (−1) for other quanta — antiparticles —, all sharing otherwise the same kinematical and spacetime properties such as mass and spin. The existence of matter and antimatter is thus a natural outcome of relativistic quantum field theory, extended to complex valued fields. Indeed, the two mass-degenerate real scalar fields combine into a single complex scalar field, invariant under any global, namely spacetime independent transformation of its phase.

1This is no longer the case if the symmetry is realised in the Goldstone mode, namely when it is spontaneously broken by the vacuum which is then not invariant under the action of the symmetry.
quantum field theories, of which the particle quanta carry collections of conserved quantum numbers and specific interactions governed by the associated symmetries, on a Lagrangian formulation, since symmetries of the dynamics are then made manifest, readily leading to the identification of the conserved quantities through the Noether theorem. Thus when turning to the construction of field theories possessing the invariance under supersymmetry transformations, the analysis will be performed directly in terms of the Lagrangian density once the field content is specified.

2.2. Spacetime Symmetries

Since supersymmetry relates particles of integer and half-integer spin, it is a symmetry that intertwines with the spacetime symmetry of the full Poincaré group in Minkowski spacetime. It is thus important that we first understand the basics of the Poincaré group algebra, involved in the construction of any relativistic quantum field theory over Minkowski spacetime.

Acting on the spacetime coordinates, the ISO(1,3) Poincaré group (in a 4-dimensional Minkowski spacetime) is defined by the transformations

\[ x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu , \]

where the 4-vector \( a^\mu \) represents a constant spacetime translation, and \( \Lambda^\mu_\nu \) a constant SO(1,3) Lorentz (pseudo)rotation leaving invariant the Minkowski metric,

\[ \eta_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu = \eta_{\mu\nu} . \]

These transformations also act on the field content of a given theory. In the case of scalar fields, one has simply

\[ \phi'(x') = \phi(x) , \]

and more generally for a collection of real or complex scalar fields, a similar relation holds component by component. Note that at the quantum level for the quantum field operators, such transformations are generated by a representation \( U(a, \Lambda) \) of the Poincaré group acting through the adjoint action on the field operator in the Heisenberg picture,

\[ \hat{\phi}(x') = U(a, \Lambda) \hat{\phi}(x) U^\dagger(a, \Lambda) . \]

The Poincaré group being an abstract Lie group, possesses a collection of generators each of which is associated to each independent type of transformation. Thus for spacetime translations with the parameters \( a^\mu \) one has the generators \( P^\mu \), while for spacetime Lorentz (pseudo)rotations
with the parameters $\Lambda^\mu$. One has the generators $M^{\mu\nu} = -M^{\nu\mu}$. At the abstract level, the corresponding Lie algebra is given by the nonvanishing Lie brackets,

\[
[P^\mu, P^\nu] = 0 \quad , \quad [P^\mu, M^{\nu\rho}] = i \left[ \eta^{\mu\nu} P^\rho - \eta^{\mu\rho} P^\nu \right] , \quad (16)
\]

\[
[M^{\mu\nu}, M^{\rho\sigma}] = -i \left[ \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} \right] ,
\]

where the last set of brackets determines the Lorentz algebra. These generators induce finite Poincaré transformations through their exponentiated action in the abstract realisation of the group,

\[
U(a, \Lambda) = e^{ia_\mu P^\mu + i\frac{1}{2} \omega_{\mu\nu} M^{\mu\nu}} . \quad (17)
\]

According to the Noether theorem, any dynamics of which the Lagrange function is invariant (possibly up to a total divergence) under Poincaré transformations possesses conserved quantities — the Noether charges — for solutions to the classical equations of motion, which, in the Hamiltonian formulation, generate through Poisson brackets these symmetry transformations on phase space, and possess among themselves Poisson brackets which coincide with the above Lie algebra brackets. Hence, the Noether charges provide the explicit realisation within a given system of the associated symmetry generators in terms of the relevant degrees of freedom.

Thus for instance, in the case of the relativistic quantum scalar particle in the configuration space wave function representation, the Poincaré algebra is realised by the operators

\[
P_\mu = -i\hbar \frac{\partial}{\partial x^\mu} = -i\hbar \partial_\mu \quad , \quad M_{\mu\nu} = P_\mu x_\nu - P_\nu x_\mu , \quad (18)
\]

where in the last expression the Lorentz covariant extension of the usual orbital angular-momentum definition is recognised.

Likewise given any relativistic invariant local field theory, Noether’s theorem guarantees the existence of conserved charges given by explicit functionals of the fields which generate Poincaré transformations through Poisson brackets at the classical level, and through commutation relations

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8The notion of a dynamics invariant under a set of symmetry transformations requires in fact that the action of the system, rather than its Lagrange function, be invariant up to a surface term, since the latter does not affect the equations of motion. If indeed the Lagrange function is invariant only up to a surface term, central extensions of the symmetry Lie brackets are also possible, already at the classical level. Nonetheless, Noether’s theorem then remains valid, though with a contribution of the induced surface terms to the conserved charges.\(^{11}\)
at the quantum level. However, due to Lorentz covariance and spacetime locality of the Lagrange function given as a space integral of a Lagrangian density, it follows that the conservation condition is expressed through a divergenceless condition on a conserved current density, of which the conserved charge is given by the integral over space of its time component. In general terms,

$$\partial_\mu J^\mu = 0 \quad \text{(on-shell)} \quad , \quad Q = \int_{(\infty)} d^3 \vec{x} J^\mu = 0 ,$$

(19)

where $J^\mu$ denotes the Noether current density and $Q$ the associated Noether charge, while "on-shell" stands for the fact that the conservation property holds only for solutions to the classical equations of motion.

In the case of a single real scalar field, the detailed analysis of the Noether identities\textsuperscript{11,12} associated to the invariance of the Lagrangian density under Poincaré transformations establishes that the Noether density is given by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L} ,$$

(20)

a quantity which defines the energy-momentum density of the system. In particular, the total energy-momentum content $P^\mu$ of the field is then given as

$$P^\mu = \int_{(\infty)} d^3 \vec{x} T^{0\mu} : \quad P^0 = H_0 = \int_{(\infty)} d^3 \vec{x} \mathcal{H}_0 \quad , \quad \vec{P} = -\int_{(\infty)} d^3 \vec{x} \pi_\phi \vec{\nabla} \phi ,$$

(21)

where $\mathcal{H}_0$ stands for the canonical Hamiltonian density of the system, and $\pi_\phi = \partial_0 \phi$ for the momentum conjugate to the scalar field. For the total angular-momentum content, one also has

$$M^{\mu\nu} = \int_{(\infty)} d^3 \vec{x} \left[ T^{0\mu}_\nu x^\nu - T^{0\nu}_\mu x^\nu \right] .$$

(22)

Once given such expressions as well as the mode expansions of the scalar field and its conjugate momentum in terms of the creation and annihilation operators of its quanta, it is possible to also determine the representations of these Poincaré charges in terms of the particle content of the field, whether at the classical or the quantum level, as operators acting on Hilbert space. Thus, once a normal ordering prescription is applied onto composite operators — whereby creation operators are always brought to the left of annihilation operators\textsuperscript{1,12} —, one finds for the energy-momentum content
of the field,

\[ \hat{P}^\mu = \int \frac{d^3 \vec{k}}{(2\pi)^3 2\omega(k)} k^\mu \ a^\dagger(\vec{k}) a(\vec{k}) , \]  

(23)

while its angular-momentum \( \hat{M}^{\mu\nu} \) decomposes according to,\(^{12}\)

\[ \hat{M}_{0j} = i \int \frac{d^3 \vec{k}}{(2\pi)^3 2\omega(k)} a^\dagger(\vec{k}) \left[ \omega(k) \frac{\partial}{\partial k^j} \right] a(\vec{k}) , \]

\[ \hat{M}_{j\ell} = i \int \frac{d^3 \vec{k}}{(2\pi)^3 2\omega(k)} a^\dagger(\vec{k}) \left[ k_j \frac{\partial}{\partial k^\ell} - k_\ell \frac{\partial}{\partial k^j} \right] a(\vec{k}) . \]

(24)

The expectation values of these quantities may thus be determined for whatever quantum state the quantum field finds itself in. In particular, 1-particle states define specific eigenstates of these Poincaré generators (see below).

As is well known, representations of the Poincaré algebra ISO(1,3) are characterised by the eigenstates of its two Casimir operators, namely the invariant energy \( \hat{P}^2 = \hat{P}^\mu \hat{P}_\mu \) which measures the invariant mass of field configurations, and the relativistic invariant \( \hat{W}^2 \) of the Pauli-Lubanski 4-vector \( \hat{W}^\mu = \frac{1}{2} \epsilon^{\mu\rho\sigma\tau} \hat{P}_\rho \hat{M}_{\sigma\tau} \), which commutes with \( \hat{P}^\mu \).

A massive representation of the Poincaré group is thus characterised by the eigenvalues \( \hat{P}^2 = m^2 \) and \( \hat{W}^2 = -m^2 s(s+1) \), where \( m > 0 \) stands for its mass and \( s \) for its spin, an integer or half-integer valued quantity defining an irreducible representation of the group SU(2), the universal covering group the 3-dimensional rotation group SO(3) sharing the same Lie algebra of infinitesimal rotations in space. For a massless representation, one has \( \hat{P}^2 = 0 \) and \( \hat{W}^2 = 0 \). Such representations are characterised by the helicity \( s \) of the state, namely a specific representation of the helicity group SO(2), the rotation subgroup of the Wigner little group for a massless particle.\(^1\)

\(^{1}\)Note that for a massive particle in its rest-frame, the Pauli-Lubanski 4-vector does indeed reduce to its total angular-momentum, i.e., its spin.

\(^{1}\)By definition, the Wigner little group of a particle is the subgroup of the full Lorentz group leaving invariant the particle’s energy-momentum 4-vector. For a massive particle, by going to its rest-frame, it is immediate to establish that its little group is isomorphic to the space rotation group SO(3) (at least for its component connected to the identity transformation, namely homotopic to the identity) or SO(D-1) in a D-dimensional Minkowski spacetime. For a massless particle whose energy-momentum 4-vector is light-like, a detailed analysis, based on the Lorentz algebra, shows that the little group is isomorphic to the euclidean group E(D-2) for a Minkowski spacetime of dimension D which combines the rotations SO(D-2) in the space directions transverse to the particle momentum with specific combinations of Lorentz boosts in the momentum direction with space rotations around that momentum direction. At the quantum level, the notion
For instance, for a light-like energy-momentum 4-vector $P^\mu = E(1,0,0,1)$, one has $W^\mu = M_{12} P^\mu$, so that $M_{12}$ takes the possible eigenvalues $\pm s$.

In the case of the scalar field, it is then straightforward to identify the particle content of its Hilbert space. A 1-particle state $|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle$ is characterised by the eigenvalues

$$\hat{P}^0|\vec{k}\rangle = \omega(\vec{k})|\vec{k}\rangle, \quad \hat{\vec{P}}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle, \quad \hat{W}^2|\vec{k}\rangle = 0,$$

thus showing that indeed, the quanta of such a quantum field may be identified with particles of definite energy-momentum and mass $m$, carrying a vanishing spin (in the massive case) or helicity (in the massless case). Relativistic quantum field theories are thus the natural framework in which to describe all the relativistic quantum properties, including the processes of their annihilation and creation in interactions, of relativistic quantum point-particles. It is the Poincaré invariance properties, namely the relativistic covariance of such systems, that also justifies, on account of Noether’s theorem, this physical interpretation.

One has to learn how to extend the above description to more general field theories whose quanta are particles of nonvanishing spin or helicity. Clearly, one then has to consider collections of fields whose components also mix under Lorentz transformations, namely nontrivial representations of the Lorentz group.

3. Spinor Representations of the Lorentz Group and Spin 1/2 Particles

3.1. The Lorentz Group and Its Covering Algebra

Let us now consider the possibility that a collection of fields $\phi_\alpha(x)$ (whether real or complex), distinguished by a component index $\alpha$, provide a linear representation space of the Poincaré group, whose action is defined accord-
The sought for collection of fields is to provide a representation space of the associated Lorentz \( so(1, 3) \) algebra,
\[
[M^{\mu\nu}, M^{\rho\sigma}] = -i [\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\nu\rho}] ,
\]
where the Lorentz boost generators \( M^{0i} \) \((i = 1, 2, 3)\) must be taken to be anti-hermitian, and the generators of rotations in space \( M^{ij} \) hermitian,\(^1\)
\[
(M^{0i})^\dagger = -M^{0i} , \quad (M^{ij})^\dagger = M^{ji} , i, j = 1, 2, 3 .
\]
\(^2\)

In order to exploit now a feature unique to 4-dimensional Minkowski spacetime, let us introduce the following change of basis in the complexified Lorentz Lie algebra,
\[
L^i_\pm = \frac{1}{2} [L^i \pm iK^i] , \quad K^i = M^{0i} , \quad L^i = \frac{1}{2} \epsilon^{ijk} M^{jk} ,
\]
\[
L^i_\pm \dagger = L^i_\pm , \quad K^{i\dagger} = -K^i , \quad L^{i\dagger} = L^i .
\]
\(^3\)
Note that the generators \( L^i_\pm \) combine a Lorentz boost in the direction \( i \) with a rotation around that direction in opposite directions, hence in effect defining chiral rotations in spacetime, and leading to hermitian generators for the complexified Lorentz algebra. A direct calculation then readily finds that in terms of these chiral generators \( L^i_\pm \), the Lorentz algebra factorises into the direct sum of two \( su(2) \) algebras,
\[
\left[ L^i_\pm , L^j_\pm \right] = i \epsilon^{ijk} L^k_\pm \quad , \quad \left[ L^i_\pm , L^j_\mp \right] = 0 .
\]
\(^4\)
In other words, the complexification of the \( so(1, 3) \) Lorentz algebra is isomorphic to the algebra \( su(2)_+ \oplus su(2)_- = sl(2, \mathbb{C}) \).\(^m\) Consequently, the universal covering algebra (over \( \mathbb{C} \)) of the Lorentz group algebra \( so(1, 3) \) is that of the group \( SU(2)_+ \times SU(2)_- \).

\(^1\)Note well that the fields are taken to transform under the trivial representation of the spacetime translation subgroup of the full Poincaré group. Hence it is only for Lorentz transformations that we need to understand the representation theory to be discussed in the present section.

\(^2\)A finite dimensional representation of a noncompact Lie algebra as is that of the Lorentz group is necessarily nonunitary.

\(^3\)The relation to \( SL(2, \mathbb{C}) \) is discussed hereafter.
The obvious advantage of this result is that the representation theory of the Lorentz group in 4-dimensional Minkowski spacetime may be understood in terms of representations of the SU(2) group, which are well known from the notion of spin and angular-momentum in nonrelativistic quantum mechanics. To each of the factors $su(2)_{\pm}$ one must associate an integer or half-integer value $j_{\pm}$ which determines a specific irreducible representation of SU(2), namely that of “spin” $j_{\pm}$. Thus finite dimensional irreducible representations of the Lorentz group SO(1,3) are characterised by a pair of integer or half-integer values $(j_{+},j_{-})$. The trivial representation is that characterised by $(j_{+},j_{-}) = (0,0)$. Next one has the two inequivalent representations $(j_{+},j_{-}) = (1/2,0)$ and $(j_{+},j_{-}) = (0,1/2)$, which will be seen to play a fundamental role hereafter. One may also have for instance $(j_{+},j_{-}) = (1/2,1/2), (1,0), (0,1)$, etc. In fact, since in SU(2), all representations may be obtained through tensor products of the fundamental $j = 1/2$ spinor representation, likewise for the Lorentz group, all its finite dimensional irreducible representations may be obtained through tensor products of the two inequivalent spinor representations $(j_{+},j_{-}) = (1/2,0)$ and $(j_{+},j_{-}) = (0,1/2)$, which are thus the two fundamental representations of the Lorentz group, known as the Weyl spinors of opposite right- or left-handed chiralities, respectively.

Given any such $(j_{+},j_{-})$ Lorentz representation, its spin content may also easily be identified. Indeed, in terms of the chiral generators $L_{\pm}^{i}$, the SO(3) angular-momentum generators $L^{i}$ are obtained simply as the direct sum $L^{i} = L_{+}^{i} + L_{-}^{i}$. Thus, the spin content of a given $(j_{+},j_{-})$ representation is simply obtained through the usual rules for spin reduction of tensor products of SU(2) representations. Consequently, a $(j_{+},j_{-})$ representation contains spin representations of $so(3) = su(2)$ of values spanning the range

$$|j_{+} - j_{-}|, |j_{+} - j_{-}| + 1, \cdots, j_{+} + j_{-}.$$  \hspace{1cm} (32)

Finally, a given $(j_{+},j_{-})$ Lorentz representation is not invariant under parity. Indeed, under this transformation in space, the Lorentz boost generators $K^{i}$ change sign whereas the angular-momentum ones $L^{i}$ do not. Hence under parity, the two classes of chiral operators $L_{\pm}^{i}$ are simply exchanged, inducing the correspondence under parity of the representations $(j_{+},j_{-})$ and $(j_{-},j_{+})$. Consequently, when the Lorentz group SO(1,3) is extended to also include the parity transformation, its irreducible representations are to be combined into the direct sums $(j_{+},j_{-}) \oplus (j_{-},j_{+})$ in the case of distinct values for $j_{+}$ and $j_{-}$.
Given all these considerations, one may list the representations which are invariant under parity and correspond to the lowest spin or helicity content possible,

\begin{align*}
(0,0) & : \text{ scalar field } \phi; \\
(1/2,0) \oplus (0,1/2) & : \text{ Dirac spinor } \psi; \\
(1/2,1/2) & : \text{ vector field } A_\mu; \\
(1,0) \oplus (0,1) & : \text{ electromagnetic field strength tensor } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \simeq (\vec{E}, \vec{B}).
\end{align*}

The simplest \( \mathcal{N} = 1 \) supersymmetry realisation in 4-dimensional Minkowski spacetime in fact relates scalar and spinor fields, as well as spinor and vector fields. The fundamental Lorentz spinors correspond to the right- and left-handed Weyl spinors \((1/2, 0)\) and \((0, 1/2)\), respectively, which are exchanged under the parity transformation. In terms of the quanta of such fields, Weyl spinors describe massless particles of fixed helicity \( s = \pm 1/2 \) equal to the chirality \( \pm 1/2 \) of the Weyl spinor, and antiparticles of the opposite helicity \( s = \mp 1/2 \). Weyl spinors must be combined in order to describe massive spin 1/2 particles, one possibility being the celebrated Dirac spinor and its Dirac equation, describing massive spin 1/2 particles and antiparticles invariant under parity, which is to be discussed hereafter.

3.2. An Interlude on SU(2) Representations

Let us pause for a moment to recall a few well known facts concerning SU(2) representations, that will become relevant in the next section. The \( \text{su}(2) \) Lie algebra is spanned by three generators \( T^i \) \((i = 1, 2, 3)\) with the Lie bracket algebra

\[
[T^i, T^j] = i \epsilon^{ijk} T^k , \quad \epsilon^{123} = +1.
\]

As is the case for any SU(N) algebra, \textit{a priori}, SU(2) possesses two fundamental representations of dimension two, complex conjugates of one another, namely the spinor representations of SU(2) or SO(3). There is the “covariant” 2-dimensional representation \( \mathbf{2} \), a vector space spanned by covariant complex valued doublet vectors \( a_\alpha \) \((\alpha = 1, 2)\) transforming under a SU(2) group element \( U_\alpha^\beta \), with \( U^\dagger = U^{-1} \) and \( \det U = 1 \), as

\[
a'^\alpha = U_\alpha^\beta a_\beta .
\]

This representation is also associated to the generators

\[
T^i = \frac{1}{2} \sigma_i , \quad i = 1, 2, 3 ,
\]
the $\sigma_i$ being the usual Pauli matrices,\(^n\)
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
(37)

Correspondingly, the “contravariant” complex conjugate 2-dimensional representation $\overline{\mathbf{2}}$, spanned by vectors $a^\alpha (\alpha = 1, 2)$, consists of complex valued vectors transforming under SU(2) group elements as
\[
a'^\alpha = a^\beta U^\dagger_\beta a^\gamma U_\gamma^{-1} \alpha ,
\]
(38)
and associated to the generators $T^i = \sigma^i / 2$.

Similar considerations apply to the SU(N) case. The fact that in this general case these are the two fundamental representations is related to the existence of two SU(N)-invariant tensors, namely the Kronecker symbols $\delta_{\alpha\beta}$ and $\delta_{\beta\alpha}$ and the totally antisymmetric symbols $\epsilon_{\alpha_1 \cdots \alpha_N}$ and $\epsilon_{\alpha_1 \cdots \alpha_N}$, which themselves are directly connected to the defining properties of SU(N) matrices, namely the fact that they are unitary, $U^\dagger = U^{-1}$, and of unit determinant, $\det U = 1$. Particularised to the SU(2) case, these simple properties may easily be checked. Indeed, using the transformation rules recalled above for co- and contra-variant indices under the SU(2) action, one has, for instance,
\[
\delta'^{\alpha} = U^\alpha_{\alpha_1} \delta_{\alpha_1 \beta_1} U^\dagger_{\beta_1} = \delta^\beta ,
\]
(39)
a result which readily follows from the unitarity property of SU(2) elements, $U^\dagger = U^{-1}$. Likewise for the $\epsilon_{\alpha\beta}$ tensor, for instance,
\[
\epsilon'^{\alpha \beta} = U^\alpha_{\alpha_1} U^\beta_{\beta_1} \epsilon_{\alpha_1 \beta_1} = \epsilon_{\alpha \beta} ,
\]
(40)
a result which follows from the unit determinant value, $\det U = 1$.

In the general SU(N) case, these considerations imply that the $N$-dimensional contravariant representation $\overline{\mathbf{N}}$, the complex conjugate of the $N$-dimensional covariant one $\mathbf{N}$, is also equivalent to the totally antisymmetry representation obtained through the $(N - 1)$-times totally antisymmetrised tensor product of the latter representation with itself,
\[
a_{\alpha_1 \cdots \alpha_{N-1}} = \epsilon_{\alpha_1 \cdots \alpha_{N-1} \beta} a^\beta .
\]
(41)
However, the SU(2) case is distinguished in this regard by the fact that this transformation also defines a unitary transformation on representation

\(^n\)The position of the index $i$ is important in these relations, for reasons to become clear later on.
space. In other words, the relations
\[ a^\alpha = \epsilon^{\alpha \beta} a_\beta, \quad a_\alpha = \epsilon_{\alpha \beta} a^\beta, \tag{42} \]
establish the unitary equivalence of the two 2-dimensional SU(2) representations \( \mathbf{2} \) and \( \mathbf{\bar{2}} \). For example, one may check that these quantities do indeed transform under SU(2) according to the rules associated to the position of the index \( \alpha \), using the invariant properties of the two available SU(2) invariant tensors, for instance,
\[ a'^\alpha = \epsilon^{\alpha \beta} U_\beta \gamma a_\gamma = (U^{-1})_\beta^\alpha \epsilon^{\beta \gamma} a_\gamma = a^\beta U^{-1\beta} = a^\beta U^{\dagger \beta}. \tag{43} \]

The unitary equivalence between the two 2-dimensional SU(2) representations is thus determined by the unitary matrix
\[ \epsilon^{\alpha \beta} = (i\sigma_2)^{\alpha \beta}, \quad \epsilon_{\alpha \beta} = (-i\sigma_2)_{\alpha \beta}. \tag{44} \]

This matrix being also antisymmetric, means that in fact the 2-dimensional SU(2) representation \( \mathbf{2} \) or \( \mathbf{\bar{2}} \) is a pseudoreal representation. Contrary to SU(N) with \( N > 2 \) for which the \( \mathbf{N} \) and \( \mathbf{\bar{N}} \) representations are the two inequivalent fundamental complex representations, in the SU(2) case there is only a single fundamental representation which is also pseudoreal. This is the SU(2) spinor representation. Consequently, it is also clear that all higher spin SU(2) representations are either real or pseudoreal, namely are unitarily equivalent to their complex conjugate representations with a unitary matrix defining this equivalence which is either symmetric or antisymmetric, respectively, according to whether they are obtained with an even or an odd number of tensor product factors of the fundamental spinor representation. In other words, all integer spin SU(2) representations are strictly real, whereas all half-integer spinor representations are pseudoreal representations. In fact, all integer spin representations are actual representations of \( \text{SO}(3) = \text{SU}(2)/\mathbb{Z}_2 \) corresponding to all tensor representations of arbitrary rank, whereas all half-integer spin representations are representations of SU(2), the universal covering group of SO(3), but not of SO(3) itself because of the \( \mathbb{Z}_2 \) factor, the center of SU(2) (a rotation of angle \( 2\pi \) in a half-integer spin or spinor representation is given by \( -1 \)) but by \( 1 \) in an integer spin or tensor representation). This distinction between tensor and spinor representations of the rotation group SO(3) is related to the fact that SO(3) is a doubly-connected Lie group, a 3-dimensional manifold equivalent to the solid 2-sphere of radius \( \pi \) and with opposite points identified, obtained as the quotient of SU(2) by its center \( \mathbb{Z}_2 \) taking on the value \(-1\) (resp. \(+1\)) for any rotation by \( 2\pi \) (resp. \( 4\pi \)) around any given axis.
In contradistinction the SU(2) manifold is that of the 3-sphere, which is simply connected.

This detailed characterisation of SU(2) representations enables the direct construction of quantities which are SU(2) invariants. For instance, consider two covariant spinors $a_\alpha$ and $b_\beta$. Since the tensor product of the spin 1/2 representation with itself includes the trivial representation of zero spin, the associated SU(2) invariant must exist, and is given by the explicit SU(2) invariant contraction of the different indices in a manner involving the two invariant tensors available,

$$\epsilon^{\alpha\beta} a_\alpha b_\beta = a_\alpha b_\beta = a_\alpha \delta^\alpha_\beta b^\beta, \quad (45)$$

showing how the singlet component may be identified within the tensor products $\mathbf{2} \otimes \mathbf{2}$ or $\mathbf{2} \otimes \bar{\mathbf{2}}$. This simple rule for the construction of SU(2) invariants for SU(2) tensor products will thus readily extend to the construction of Lorentz invariant quantities, since the SO(1,3) Lorentz group shares the same algebra as the SU(2)$^+ \times$SU(2)$^-$ group, of which the two independent spinor representations define the two fundamental Weyl spinor representations of the Lorentz group.

3.3. The Fundamental Lorentz Representations: Weyl Spinors

We have seen how, on the basis of the chiral SU(2)$^+ \times$SU(2)$^-$ group, it is possible to readily identify the finite dimensional representation theory of the Lorentz group SO(1,3). Let us now discuss yet another construction of its two fundamental Weyl spinor representations, which is also of importance in the construction of supersymmetric field theories. The present discussion shall also make explicit why the universal covering group of the Lorentz group SO(1,3) is the group SL(2,\mathbb{C}) of complex 2×2 matrices of unit determinant. We shall thus establish the relation, at the level of the corresponding Lie algebras,

$$so(1,3)_{\mathbb{C}} = su(2)^+ \oplus su(2)^- = sl(2,\mathbb{C}). \quad (46)$$

Let us introduce the notation

$$\sigma_\mu = (1, \sigma_i) \quad , \quad \sigma^\mu = (1, \sigma^i) = (1, -\sigma_i), \quad (47)$$

This is readily established by considering a real parametrisation of 2×2 complex matrices, and imposing the constraints of unitarity and unit determinant defining the SU(2) matrix group.
where the space index \( i \) carried by the usual Pauli matrices is raised and lowered according to our choice of signature for the Minkowski spacetime metric, namely \( \eta_{\mu\nu} = \text{diag} (+ - - -) \). Consider now an arbitrary spacetime 4-vector \( x^\mu \), and construct the 2×2 hermitian matrix

\[
X = x^\mu \sigma_\mu = \begin{pmatrix}
x^0 + x^3 & x^1 - ix^2 \\
x^1 + ix^2 & x^0 - x^3
\end{pmatrix}
\]

(48)

Note that conversely, any 2×2 hermitian matrix \( X = X^\dagger \) possesses such a decomposition, and may thus be associated to some spacetime 4-vector \( x^\mu \) through the above relation. In particular, the determinant of any such matrix is equal to the Lorentz invariant inner product of the associated 4-vector with itself,

\[
\det X = x^2 = \eta_{\mu\nu} x^\mu x^\nu
\]

(49)

Consider now an arbitrary SL(2,\( \mathbb{Z} \)) group element \( M \), thus of unit determinant, \( \det M = 1 \), and its adjoint action on any hermitian matrix \( X \) as

\[
X' = MXM^\dagger
\]

(50)

It should be clear that the transformed matrix itself is hermitian, \( X'^\dagger = X' \), hence possesses a decomposition in terms of a 4-vector \( x'^\mu \), \( X' = x'^\mu \sigma_\mu \), of which the Lorentz invariant takes the value

\[
x'^2 = \det X' = \det MXM^\dagger = \det X = x^2
\]

(51)

In other words, any SL(2,\( \mathbb{C} \)) transformation induces a Lorentz transformation on the 4-vector \( x^\mu \). The group SL(2,\( \mathbb{C} \)) determines a covering group of the Lorentz group SO(1,3). In fact, it is the universal covering of the latter, as the discussion hereafter in terms of its fundamental representations establishes. This conclusion is thus analogous to that which states that SU(2) is the universal covering group of the group SO(3) of spatial rotations. Indeed, the above discussion may also be developed in the latter case, simply by ignoring the time component of the matrices \( \sigma_\mu \) and then restricting further the matrices \( X \) to be both hermitian and traceless.

This conclusion having been reached, the next question is: how does one construct the two fundamental Weyl spinor representations of SO(1,3) in terms of SL(2,\( \mathbb{C} \)) representations? An arbitrary SL(2,\( \mathbb{C} \)) matrix \( M \) with \( \det M = 1 \) may be decomposed according to

\[
M = e^{(a_j + ib_j)\sigma_j}, \quad M^\dagger = e^{(a_j - ib_j)\sigma_j}
\]

(52)
where $a_j$ and $b_j$ ($j = 1, 2, 3$) are triplets of real numbers. In these terms, the SU($2_+ \times SU(2)_-$ structure of the $sl(2, \mathbb{C})$ algebra should again be obvious, with in particular the hermitian (related to space rotations) and antihermitian (related to Lorentz boosts) components of the Lie algebra. In the case of SU(2), the additional property is that the matrices defining the group are also unitary, $U^\dagger = U^{-1}$. As a consequence, we have seen that the two fundamental 2-dimensional SU(2) representations, complex conjugates of one another, are unitarily equivalent. In the SL(2,$\mathbb{C}$) case, these two representations are no longer equivalent. However, because of the property $\det M = 1$, $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ still define SL(2,$\mathbb{C}$) invariant tensors, that may be used to raise and lower indices. Because of this latter fact, there exist only two independent fundamental 2-dimensional representations of SL(2,$\mathbb{C}$), in direct correspondence with the two chiral Weyl spinors considered previously.

Another way of arguing the same conclusion is as follows. Given a matrix $M \in$ SL(2,$\mathbb{C}$), each of the matrices $M$, $M^{-1}$, $M^*$ and $(M^*)^{-1}$ defines a priori another 2-dimensional representation of the same group. As pointed out above, $M$ and $M^*$ are necessarily not unitarily equivalent. However, $M$ and $M^{-1}$ on the one hand, and $M^*$ and $(M^*)^{-1}$ on the other hand, are each unitarily equivalent in pairs, using the invariant tensors $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ because of the property $\det M = 1$ for these 2 $\times$ 2 matrices.

In conclusion, first we have the right-handed Weyl spinor representation $(1/2, 0)$, $\psi_\alpha$ or $\psi^\alpha$, such that

$$\psi'^\alpha = \epsilon^{\alpha\beta} \psi_\beta , \quad \psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta , \quad \epsilon^{12} = +1 , \quad \epsilon_{12} = -1 , \quad (53)$$

and transforming under SL(2,$\mathbb{C}$) according to

$$\psi'^\alpha = M_\alpha^\beta \psi_\beta , \quad \psi'^\alpha = \psi_\beta (M^{-1})^\alpha_\beta . \quad (54)$$

Likewise, the left-handed Weyl spinor representation $(0, 1/2)$, $\bar{\psi}_\dot{\alpha}$ or $\bar{\psi}^{\dot{\alpha}}$, is such that

$$\bar{\psi}^{\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_\dot{\alpha} , \quad \bar{\psi}_\dot{\alpha} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}} , \quad \epsilon^{\dot{1}\dot{2}} = +1 , \quad \epsilon_{\dot{1}\dot{2}} = -1 , \quad (55)$$

each of these spinors transforming according to

$$\bar{\psi}'_{\dot{\alpha}} = M_{\dot{\alpha}}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} , \quad \bar{\psi}'_{\dot{\alpha}} = \bar{\psi}^{\dot{\beta}} (M^*)^{-1}_{\dot{\beta}} . \quad (56)$$

Note that the counting of independent parameters of the different SO(1,3), SU(2)$_+ \times$SU(2)$_-$ and SL(2,$\mathbb{C}$) groups also matches this correspondence. These three Lie groups are all 6-dimensional.
Here, in order to distinguish these two SL(2, \mathbb{C}) representations, or equivalently the two Weyl spinors, the van der Waerden dotted and undotted index notation has been introduced. This notation proves particularly valuable for the construction of manifestly supersymmetric invariant Lagrangian densities.

The undotted indices \( \alpha, \beta \), on the one hand, and dotted indices \( \dot{\alpha}, \dot{\beta} \), on the other hand, have the same meaning as the \( \alpha, \beta \) indices for the SU(2) spinor representations. Consequently, Lorentz invariant quantities are readily constructed in terms of the Weyl spinors \( \psi_{\alpha} \) and \( \overline{\psi}_{\dot{\alpha}} \), through simple contraction of the indices using the invariant tensors available. Furthermore, given that we have

\[
x^\mu \sigma_\mu = X' = MXM^\dagger = M (x^\mu \sigma_\mu) M^\dagger, \quad (57)
\]

it follows that the SL(2, \mathbb{C}) or SO(1,3) Lorentz transformation properties of the matrices \( \sigma_\mu \) are those characterised by the index structure,

\[
\sigma^{\mu} : \ (\sigma^{\mu})_{\alpha\dot{\alpha}}, \quad \sigma_\mu = (1, \sigma_i), \quad \sigma^{\mu} = (1, -\sigma_i) = (1, \sigma^i). \quad (58)
\]

By raising the indices, one introduces the quantities

\[
\sigma^\mu : \ (\sigma^\mu)^{\alpha\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} (\sigma_\mu)_{\beta\dot{\beta}}, \quad \sigma^\mu = (1, \sigma_i), \quad \sigma_{\mu} = (1, -\sigma_i). \quad (59)
\]

Note that these properties also justify why indeed a 4-vector \( A_\mu \) is equivalent to the \((1/2, 1/2) = (1/2, 0) \oplus (0, 1/2)\) Lorentz representation, \( A^\mu \sigma_{\mu\alpha\dot{\alpha}} = A_{\alpha\dot{\alpha}}, \quad A^\mu \sigma_{\mu}^{\alpha\dot{\alpha}} = \overline{A}^{\dot{\alpha}\alpha} \).

Let us now consider different Weyl spinors \( \psi, \chi, \overline{\psi}, \overline{\chi}, \ldots \) and the Lorentz invariant spinor bilinears that may constructed out of these quantities. For this purpose, it is important to realise that such field degrees of freedom, at the classical level, need to be described in terms of Grassmann odd variables, namely variables \( \theta_1, \theta_2, \ldots \) which anticommute with one another, \( \theta_2 \theta_1 = -\theta_1 \theta_2 \), in contradistinction to commuting variables used for fields describing particles of integer spin and obeying Bose–Einstein statistics. The reasons for this necessary choice will be discussed somewhat further later on, but at this stage, it suffices to say that spinorial fields are associated to particles of half-integer spin which should thus obey Fermi–Dirac statistics with the consequent Pauli exclusion principle, a result which is readily achieved provided Grassmann odd degrees of freedom are used even at the classical level. The associated Grassmann graded Poisson brackets\(^{11}\) then correspond, at the quantum level, to anticommutation rather than commutation relations for the degrees of freedom, ensuring the Fermi–Dirac statistics. The anticommuting character of the Weyl spinors
hereafter is an important fact to always keep in mind when performing explicit calculations.

Since dotted and undotted indices cannot be contracted with one another in a Lorentz invariant way, there are only two types of Lorentz invariant spinor bilinears that may be considered. By definition, those associated to undotted spinors write as,

$$\psi \chi = \psi^\alpha \chi_\alpha = \epsilon^{\alpha \beta} \psi_\beta \chi_\alpha = -\epsilon^{\alpha \beta} \psi_\alpha \chi_\beta$$

$$= -\psi_\alpha \chi^\alpha = \chi^\alpha \psi_\alpha = \chi \psi . \quad (60)$$

The convention here, implicit throughout the supersymmetry literature, is that for undotted spinors, the Lorentz invariant contraction denoted $\psi \chi$ without displaying the indices, is that in which the undotted indices are contracted from top-left to bottom-right. Note that the Grassmann odd property of the Weyl spinors has been used to derive the above identity, $\psi \chi = \chi \psi$.

In contradistinction for dotted spinors, the convention is that the contraction is taken from bottom-left to top-right, namely

$$\overline{\psi} \overline{\chi} = \overline{\psi}_\dot{\alpha} \overline{\chi}^{\dot{\beta}} = \epsilon_{\dot{\alpha} \dot{\beta}} \overline{\psi}^{\dot{\beta}} \overline{\chi}^{\dot{\alpha}} = -\epsilon_{\dot{\alpha} \dot{\beta}} \overline{\psi}^{\dot{\alpha}} \overline{\chi}^{\dot{\beta}}$$

$$= -\overline{\psi}^{\dot{\alpha}} \overline{\chi}_{\dot{\alpha}} \overline{\psi}^{\dot{\beta}} = \overline{\chi} \overline{\psi} . \quad (61)$$

Further identities that may be established in a likewise manner are,

$$(\psi \chi) = \overline{\chi} \overline{\psi} = \overline{\psi} \overline{\chi} , \quad (\overline{\psi} \overline{\chi}) = \chi \psi = \psi \chi . \quad (62)$$

For the construction of Lorentz covariant spinor bilinears, one has to also involve the matrices $\sigma_\mu$ and $\overline{\sigma}_\mu$. Thus for instance, we have the quantities transforming as 4-vectors under Lorentz transformations,

$$\psi \sigma^\mu \overline{\chi} = \psi^\alpha \sigma^\mu_{\alpha \beta} \overline{\chi}^{\beta} , \quad \overline{\psi} \overline{\sigma}^\mu \chi = \overline{\psi}_{\dot{\alpha}} \overline{\sigma}^{\dot{\alpha} \dot{\beta}} \chi^{\beta} . \quad (63)$$

Such quantities also obey a series of identities, for instance,

$$\chi \sigma^\mu \overline{\psi} = -\overline{\psi} \overline{\sigma}^\mu \chi , \quad \chi \sigma^\mu \overline{\sigma}^{\mu'} \psi = \psi \sigma^\nu \overline{\sigma}^{\nu} \chi , \quad (64)$$

$$(\chi \sigma^\mu \overline{\psi}) = \psi \sigma^\mu \overline{\chi} , \quad (\chi \sigma^\mu \overline{\sigma}^{\mu'} \psi)^\dagger = \overline{\psi} \overline{\sigma}^{\nu} \sigma^\mu \overline{\chi} \ . \quad (65)$$

Identities of this type enter the explicit construction of supersymmetric invariant field theories.
3.4. The Dirac Spinor

As mentioned earlier, Weyl spinors are not parity invariant representations of the Lorentz group. The fundamental parity invariant representation is obtained as the direct sum of a right- and a left-handed Weyl spinor, leading to the Dirac spinor, a 4-dimensional spinor representation of the Lorentz group, which is irreducible for the Lorentz group SO(1,3) extended to also include the parity transformation. Furthermore, since the dotted and undotted notation is not as familiar as the Dirac spinor construction, the latter will now be considered in detail through its relation to the previous discussion.

Given the 4-dimensional Dirac representation, it is useful to combine the $\sigma^\mu$ and $\tau^\mu$ matrices into a collection of $4 \times 4$ matrices,

\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tau^\mu & 0 \end{pmatrix}, \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

known as the Dirac matrices. As a matter of fact, the above definition provides a specific representation of the Dirac-Clifford algebra that these matrices obey,

\[ \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \{\gamma^\mu, \gamma_5\} = 0, \]

\[ \gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0, \quad \gamma_5^\dagger = \gamma_5, \quad \gamma_5^2 = 1. \]

Other matrix representations of this algebra exist (among which that originally constructed by Dirac himself\textsuperscript{12} when he discovered the celebrated Dirac equation). However in a Minkowski spacetime of even dimension, all these representations are unitarily equivalent.\textsuperscript{15} The above representation of the Dirac-Clifford algebra is known as the chiral or Weyl representation, since the chiral projection operator $\gamma_5$ is then diagonal.

Being the direct sum of a right- and a left-handed Weyl spinor, within the chiral representation a Dirac spinor decomposes according to

\[ \psi_{(4)}^{\text{Dirac}} = \begin{pmatrix} \psi^\alpha \\ \chi^\dot{\alpha} \end{pmatrix}, \]

where $\psi^\alpha$ is a $(1/2,0)$ right-handed Weyl spinor, and $\chi^\dot{\alpha}$ a $(0,1/2)$ left-handed one. These two chiral components are indeed projected from the Dirac spinor through the projectors

\[ P_R = \frac{1}{2} (1 + \gamma_5), \quad P_L = \frac{1}{2} (1 - \gamma_5), \]

where $\gamma_5$ is the chiral projection operator for the Dirac spinor.
with the properties,
\[ P^2_R = P_R, \quad P^2_L = P_L, \quad P_LP_R = 0 = P_RP_L. \] (70)

A priori, the two Weyl spinors \( \psi_\alpha \) and \( \chi^{\dot\alpha} \) are independent spinors, leading to the construction of an actual Dirac spinor with these many independent degrees of freedom. However, it could be that these two Weyl spinors are complex conjugates of one another, in which case the above construction defines what is known as a Majorana spinor,
\[ \psi_{\text{Majorana}}^{(4)} = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot\alpha} \end{pmatrix}. \] (71)

A Majorana spinor is to a Dirac spinor what a real scalar field is to a complex scalar field. Namely, whereas the quanta of a real scalar field are particles that cannot be distinguished from their antiparticles (they do not carry a conserved quantum number that could distinguish them, such as the electric charge), the quanta of a complex scalar field are classified in terms of particles and antiparticles, which may be distinguished according to a conserved quantum number, for instance their electric charge, associated to the global symmetry invariance under arbitrary spacetime constant variations in the complex phase of the complex scalar field.\(^1\) Likewise for the above spinors, since the Majorana spinor obeys some sort of restriction under complex conjugation (its Weyl components of opposite chiralities are related through complex conjugation), a Majorana spinor describes spin or helicity \(1/2\) particles which are their own antiparticles, and thus cannot carry a conserved quantum number such as the electric charge.\(^3\) In contradistinction, the quanta associated to a Dirac spinor may be distinguished in terms of particles and their antiparticles carrying opposite values of a conserved quantum number, such as for instance the electric charge (or baryon or lepton number), associated to a symmetry under arbitrary global phase transformations of the Dirac spinor. As the above construction clearly shows, in a 4-dimensional Minkowski spacetime, one cannot have both a Weyl and a Majorana condition imposed on a Dirac spinor. In such a case, one has either only Dirac spinors, Majorana spinors, or Weyl spinors of definite chirality, while the fundamental constructs of Lorentz covariant spinors are the two fundamental right- and left-handed Weyl spinors. In fact, it

\(^{3}\)Consequently, among quarks and leptons, only neutrinos could possibly be Majorana particles. The experimental verdict is still out, and is an important issue in the quest for the fundamental unification of all interactions and particles.
may be shown,\textsuperscript{15} using the properties of the Dirac-Clifford algebra, that Majorana-Weyl spinors exist only in a Minkowski spacetime of dimension $D = 2 \pmod{8}$, which includes the dimension $D = 10$ in which superstrings may be constructed, which is not an accident.

Given that the Dirac $\gamma^\mu$ matrices provide a representation space of the Lorentz group, it should be possible to display explicitly the associated generators. Indeed, it may be shown that the latter are obtained as

$$\Sigma^{\mu\nu} = \frac{1}{2} i \gamma^{\mu\nu}, \quad \gamma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu],$$

(72)

with

$$\gamma^{\mu\nu} = \frac{1}{2} \begin{pmatrix} \sigma^{\mu\nu} - \sigma^{\nu\mu} & 0 \\ 0 & \sigma^{\mu\nu} - \sigma^{\nu\mu} \end{pmatrix}.$$  

(73)

Thus a right-handed spinor $\psi_\alpha$ transforms according to the generators,

$$\Sigma^{\mu\nu}_{R} : (\Sigma^{\mu\nu}_{R})_{\alpha} = \frac{1}{4} i \left[ \sigma^{\mu\nu}_{\alpha\beta} - \sigma^{\nu\mu}_{\alpha\beta} \right],$$

(74)

while a left-handed Weyl spinor $\chi^{\dot{\alpha}}$ according to

$$\Sigma^{\mu\nu}_{L} : (\Sigma^{\mu\nu}_{L})^{\dot{\alpha}} = \frac{1}{4} i \left[ \sigma^{\mu\nu}_{\dot{\alpha}\dot{\beta}} - \sigma^{\nu\mu}_{\dot{\alpha}\dot{\beta}} \right].$$

(75)

Given these different considerations, it should not come as a surprise that once a free quantum field theory dynamics is constructed, it turns out that such fundamental spinor representations of the Lorentz group describe quanta which are massive or massless particles whose spin or helicity is $1/2$.

Extending the above considerations to an arbitrary representation of the Dirac-Clifford algebra, any Dirac spinor may be decomposed into its chiral components,

$$\psi = \psi_L + \psi_R, \quad \psi_L = P_L \psi = \frac{1}{2} (1 - \gamma_5) \psi, \quad \psi_R = P_R \psi = \frac{1}{2} (1 + \gamma_5) \psi.$$  

(76)

The $\text{SL}(2,\mathbb{C})$ invariant tensors that enable the raising and lowering of dotted and undotted indices provide for a transformation which, given a Dirac spinor $\psi$ and its complex conjugate, constructs another Dirac spinor also transforming according to the correct rules under Lorentz transformations. This operation, known as charge conjugation since it exchanges the roles played by particles and their antiparticles, is represented through a matrix $C$ such that

$$C \gamma^\mu C^{-1} = -\gamma^\mu, \quad C = i \gamma^2 \gamma^0, \quad C^\dagger = C^T = -C, \quad C^2 = -\mathbf{1},$$

(77)
where, except for the very first identity, the last series of properties is valid, for instance, in the Dirac and chiral representations of the $\gamma^\mu$ matrices, but not necessarily in just any other representation of the Dirac-Clifford algebra. The charge conjugate Dirac spinor $\psi_C$ associated to a given Dirac spinor $\psi$ is given by,

$$\psi_C = C\psi^\dagger,$$

up to an arbitrary phase factor. Consequently, a Majorana spinor $\psi$ obeys the Majorana condition,

$$\psi = \psi_C = C\psi^\dagger,$$

thus extending to the Dirac spinor representation of the Lorentz group in a manner consistent with Lorentz transformation, the reality condition under complex conjugation for such fields, in a way similar to the simple reality condition $\phi = \phi^\dagger$ for a scalar field real under complex conjugation describing spin 0 particles which are their own antiparticles.

Given all the above, different properties may be established. For instance, one has

$$\overline{\psi} L = \overline{\psi} R, \quad \overline{\psi R} = \overline{\psi} L, \quad (\psi L)_C = (\psi R)_C, \quad (\psi R)_C = (\psi L)_R.$$

Lorentz invariant spinor bilinears decompose as

$$\overline{\psi} \chi = \overline{\psi} L \chi_R + \overline{\psi} R \chi_L, \quad \overline{\psi} \gamma^5 \chi = \overline{\psi} L \gamma^5 \chi_R + \overline{\psi} R \gamma^5 \chi_L = \overline{\psi} L \chi_R - \overline{\psi} R \chi_L,$$

where, under parity, the first quantity is a pure scalar, and the second a pseudoscalar. Likewise, one has the Lorentz covariants,

$$\overline{\psi} \gamma^\mu \chi = \overline{\psi} L \gamma^\mu \chi_R + \overline{\psi} R \gamma^\mu \chi_L,$$

$$\overline{\psi} \gamma^\mu \gamma^5 \chi = -\overline{\psi} L \gamma^\mu \chi_R + \overline{\psi} R \gamma^\mu \chi_L,$$

$$\overline{\psi} \sigma^{\mu\nu} \chi = \overline{\psi} L \sigma^{\mu\nu} \chi_R + \overline{\psi} R \sigma^{\mu\nu} \chi_L,$$

where in the last relation one defines $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Note that the bilinears $\overline{\psi} \gamma^\mu \chi, \overline{\psi} \gamma^\mu \gamma^5 \chi$ and $\overline{\psi} \sigma^{\mu\nu} \chi$ transform as a 4-vector, an axial 4-vector, and a $(1,0) \oplus (0,1)$ tensor, respectively. In fact, the whole $2^4 = 16$ dimensional Dirac-Clifford algebra, generated by the $2^2 \times 2^2$ matrices $\mathbf{1}$ and $\gamma^\mu$, is spanned by the $2^4 = 16$ independent quantities $\mathbf{1}, \gamma_5, \gamma^\mu, \gamma^\mu \gamma_5$ and $\sigma^{\mu\nu}$ (one has indeed $\sigma^{\mu\nu} \gamma_5 = i\epsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma}/2$ where $\epsilon^{0123} = +1$).

Further identities involving four Dirac spinors are also important to establish supersymmetry invariance. These involve the celebrated Fierz
identities, the simplest of which is of the form,
\[ \psi_1 \bar{\psi}_2 \bar{\psi}_3 \bar{\psi}_4 = -\frac{1}{4} \left\{ \psi_1 \bar{\psi}_4 \bar{\psi}_3 \bar{\psi}_2 + \psi_1 \gamma^\mu \psi_4 \bar{\psi}_3 \gamma_\mu \psi_2 + \bar{\psi}_1 \sigma^{\mu\nu} \psi_4 \bar{\psi}_3 \sigma_{\mu\nu} \psi_2 - \bar{\psi}_1 \gamma^\mu \gamma_5 \psi_4 \bar{\psi}_3 \gamma_5 \psi_2 + \bar{\psi}_1 \gamma_5 \psi_4 \bar{\psi}_3 \gamma_5 \psi_2 \right\}, \]
where \( \psi_1, \psi_2, \psi_3 \) and \( \psi_4 \) are arbitrary Grassmann odd Dirac spinors. An application of this identity leads, for instance, to the relation
\[ \epsilon_{1R} \partial_\mu \psi_L \gamma^\mu \epsilon_{2R} = -\frac{1}{2} \epsilon_{1R} \gamma^\nu \epsilon_{2R} \gamma^{\mu\nu} \partial_\mu \psi_L, \]
where \( \epsilon_{1R}, \epsilon_{2R} \) and \( \psi_L \) are Grassmann odd Dirac spinors of definite chirality as indicated by their lower label. This relation is central in establishing the supersymmetry invariance property of the simplest example of a supersymmetric field theory, the so-called Wess-Zumino model involving a scalar and a Weyl or Majorana spinor.\(^{6,10}\)

In the case of Grassmann odd Majorana spinors \( \epsilon \) and \( \lambda \), one also has,
\[ \tau \lambda = \bar{\lambda} \epsilon = (\tau \lambda)^\dagger, \]
\[ \tau \gamma_5 \lambda = \bar{\lambda} \gamma_5 \epsilon = -(\tau \gamma_5 \lambda)^\dagger, \]
\[ \tau \gamma^\mu \lambda = -\bar{\lambda} \gamma^\mu \epsilon = -(\tau \gamma^\mu \lambda)^\dagger, \]
\[ \tau \gamma^\mu \gamma_5 \lambda = \bar{\lambda} \gamma^\mu \gamma_5 \epsilon = (\tau \gamma^\mu \gamma_5 \lambda)^\dagger, \]
\[ \tau \gamma^{\mu\nu} \lambda = -\bar{\lambda} \gamma^{\mu\nu} \epsilon = (\tau \gamma^{\mu\nu} \lambda)^\dagger. \]

It is a useful exercise to establish any of these identities.

### 3.5. The Dirac Equation

Let us now consider the dynamics of a single free Dirac spinor field, thus described, at the classical level, by complex valued Grassmann odd variables forming a 4-component Dirac spinor \( \psi(x^\mu) \). The action principle for a such a system is given by the Lorentz invariant quantity
\[ S[\psi, \bar{\psi}] = \int d^4x^\mu \mathcal{L}(\psi, \partial_\mu \psi), \]

\(^{1}\)As a matter of fact, all other Fierz identities follow from the present one, by appropriate choices of the spinors involved.
with the Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \left[ \overline{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \overline{\psi} \gamma^\mu \psi \right] - m \overline{\psi} \psi . \]  

(87)

Through the variational principle, the associated equation of motion is the celebrated Dirac equation,

\[ [i \gamma_\mu \partial_\mu - m] \psi(x^\mu) = 0 . \]  

(88)

A few remarks are in order. Given the relations in (81) and (82), it is clear that the kinetic term \( \overline{\psi} \gamma^\mu \partial_\mu \psi \) couples the chiral components of the Dirac spinor by preserving their chirality, while the coupling \( m \overline{\psi} \psi \) switches between the two chirality components. As will become clear hereafter, since the real parameter \( m \geq 0 \) in fact determines the mass of the particle quanta associated to such a field, a massless Dirac particle propagates without flipping its chirality, whereas a massive particle sees both its chiral components contribute to its spacetime dynamics.

The term \( m \overline{\psi} \psi \) is known as the Dirac mass term. In particular, it preserves the symmetry of the kinetic term under global phase transformations of the Dirac spinor,

\[ U_V(1) : \quad \psi'(x) = e^{i\alpha} \psi(x) , \]  

(89)

leading to a conserved \( U_V(1) \) quantum number which, effectively, counts the difference between the numbers of fermions and antifermions present in the system. This \( U_V(1) \) phase symmetry is thus that of the fermion number, which may coincide with the electric charge quantum number when coupled to the electromagnetic interaction. The corresponding conserved Noether current is simply the vector bilinear \( J^\mu = \overline{\psi} \gamma^\mu \psi \), thus obeying the divergenceless condition \( \partial^\mu J_\mu = 0 \) for solutions to the Dirac equation (88).

Furthermore, since under the transformation

\[ \psi'(x) = \gamma_5 \psi(x) , \]  

(90)

the mass term \( m \overline{\psi} \psi \) changes sign, \( \overline{\psi'} \psi' = -\overline{\psi} \psi \), it may always be assumed that the parameter \( m \) is not negative, \( m \geq 0 \).

One may also consider \( U(1)_A \) axial transformations,

\[ U_A(1) : \quad \psi'(x) = e^{i\alpha \gamma_5} \psi(x) , \]  

(91)

\(^{\text{often, this Lagrangian density is given as } \mathcal{L} = i \overline{\psi} \gamma^\mu \partial_\mu \psi - m \overline{\psi} \psi \text{, which differs from the one given here by a total divergence with no consequence for a choice of boundary conditions at infinity such that fields vanish asymptotically. Note however that the form chosen in (87) is manifestly real under complex conjugation, as befits any Lagrangian density.}}\)
leaving the kinetic term of the Lagrangian density invariant, but not the Dirac mass term. When \( m = 0 \), the associated conserved Noether current density is the axial vector spinor bilinear, \( J_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi \), which is indeed conserved for solutions to Dirac’s equation (88) only provided \( m = 0 \), as may explicitly be checked through direct calculation. These vector and axial symmetries of the Dirac Lagrangian density are important aspects for the theory of the strong interactions, quantum chromodynamics (QCD).

Rather than considering a Dirac mass term, one may also use the charge conjugate spinor \( \psi_C \) to define another type of mass term,

\[
m_{M} \bar{\psi} \psi_C + \text{hermitian conjugate},
\]

known as a Majorana mass term. However, it should be clear that such a term breaks not only the axial symmetry as does a Dirac mass term, but also the above vector symmetry under phase transformations. Hence, a Majorana mass term leads to a violation of the fermion number, again a reason why such a possibility may be contemplated for neutrinos only within the Standard Model of the quarks and leptons and their strong and electroweak interactions.

A detailed analysis, similar to that applied to the Klein–Gordon equation,\(^1\) considering the plane wave solutions\(^4\) to the Dirac equation (88), reveals that the general solution may be expressed through the following mode expansion

\[
\psi(x^\mu) = \int \frac{d^3 \vec{k}}{(2\pi)^3 2\omega(\vec{k})} \sum_{s=\pm} \left\{ e^{-i\vec{k} \cdot \vec{x}} u(\vec{k}, s) b(\vec{k}, s) + e^{i\vec{k} \cdot \vec{x}} v(\vec{k}, s) d^\dagger(\vec{k}, s) \right\},
\]

where the plane wave spinors \( u(\vec{k}, s) \) and \( v(\vec{k}, s) \) are positive- and negative-frequency solutions to the Dirac equation in energy-momentum space,

\[
[\gamma^\mu k_\mu - m] u(\vec{k}, s) = 0 \quad \text{and} \quad [\gamma^\mu k_\mu + m] v(\vec{k}, s) = 0.
\]

The normalisation of these spinors is such that

\[
\sum_{s=\pm} u(\vec{k}, s) \bar{u}(\vec{k}, s) = (\gamma^\mu k_\mu + m) \quad \text{and} \quad \sum_{s=\pm} v(\vec{k}, s) \bar{v}(\vec{k}, s) = (\gamma^\mu k_\mu - m).
\]

The index \( s = \pm \) taking two values is related to a spin or a helicity projection degree of freedom, specifying the polarisation state of the solution. The

\(^1\)Such solutions must exist since the Dirac equation is invariant under spacetime translations and is linear in the field.
general solution has to include a summation over the two possible polarisation states of the field. The spinors \( u(\vec{k}, s) \) and \( v(\vec{k}, s) \) thus also correspond to polarisation spinors characterising the polarisation state of the field (in the same way that a polarisation vector characterises the polarisation state of a vector field \( A_\mu(x^\mu) \), such as the electromagnetic vector field).

Finally, in exactly the same manner as for the scalar field, the quantities \( b(\vec{k}, s) \) and \( d^\dagger(\vec{k}, s) \) are, at the classical level, Grassmann odd integration constants specifying a unique solution to the Dirac equation, which, at the quantum level, correspond to quantum operators for which the quantum algebraic structure is given by the following anticommutation relations

\[
\{ b(\vec{k}, s), b^\dagger(\vec{\ell}, r) \} = (2\pi)^3 2\omega(\vec{k}) \delta_{sr} \delta^{(3)}(\vec{k} - \vec{\ell}) = \{ d(\vec{k}, s), d^\dagger(\vec{\ell}, r) \} \quad . (96)
\]

Note that the normalisation of these relations is the same as that of the creation and annihilation operators for a scalar field. As explained in Ref. 1, this choice leads a Lorentz covariant normalisation of 1-particle states and mode decomposition of fields.

One very important point should be emphasised here. By giving the above anticommutation relations, it is understood, as it is also in the bosonic case, that only the nonvanishing anticommutators are displayed. Thus the following anticommutation relations are implicit,

\[
\{ b(\vec{k}, s), b(\vec{\ell}, r) \} = 0 = \{ d(\vec{k}, s), d(\vec{\ell}, r) \} ,
\]

\[
\{ b^\dagger(\vec{k}, s), b^\dagger(\vec{\ell}, r) \} = 0 = \{ d^\dagger(\vec{k}, s), d^\dagger(\vec{\ell}, r) \} .
\]

(97)

Given that \( b(\vec{k}, s) \) and \( d(\vec{k}, s) \) are to be interpreted as annihilation operators for particles and antiparticles, and \( b^\dagger(\vec{k}, s) \) and \( d^\dagger(\vec{k}, s) \) as creation operators for particles and antiparticles, respectively, the anticommutators in (97) have as consequence that no two identical particles may occupy the same quantum state specified by the quantum numbers \( \vec{k} \) and \( s \). In other words, in contradistinction to commutation relations for bosonic degrees of freedom as is the case for a scalar field, anticommutation relations provide a manifest realisation of the Pauli exclusion principle at the operator level. Subsequent action with the same creation operator on a 1-particle state, \( \hat{b}(\vec{k}, s; -) = b^\dagger(\vec{k}, s)|0\rangle \) or \( \hat{b}(\vec{k}, s; +) = d^\dagger(\vec{k}, s)|0\rangle \), leads to the null vector in Hilbert space, since \( b^{\dagger 2}(\vec{k}, s) = 0 = d^{\dagger 2}(\vec{k}, s) \). It thus appears that half-integer spin fields, namely fermionic degrees of freedom, must be quantised according to anticommutation relations, whereas integer spin fields, namely...
bosonic degrees of freedom, must be quantised with commutation relations. This is the realisation of the spin-statistics connection.

The justification of this choice may be seen from a series of arguments. The one often invoked goes as follows. Given the different mode expansions of the bosonic and fermionic fields in terms of creation and annihilation operators in a Fock space representation of their Fock algebra, it is necessary to specify an ordering prescription for composite operators, such as for instance the Hamiltonian operator measuring the total energy content of the field. Within the perturbative Fock space representation, it is customary and natural to choose normal ordering, whereby all creation operators are brought to sit to the left of all annihilation operators. In the case of the Dirac spinor though, when using commutation relations rather than anticommutation ones, this prescription leads to an energy spectrum which is not bounded below: the contribution of the $d^\dagger d$ type (antiparticles) is negative-definite! On the other hand, using anticommutation relations brings in the required minus sign, rendering the energy spectrum of the system positive-definite both for particles and antiparticles. Half-integer spin fields must be quantised according to anticommutation relations.

For that reason, it is also necessary to use at the classical level Grassmann odd degrees of freedom to describe half-integer spin systems. Consequently, the usual Hamiltonian formulation of such systems involves now Grassmann graded Poisson brackets,$^{11}$ extending the properties of the usual bosonic Poisson brackets based on commuting degrees of freedom, as is the case for the scalar field for instance. Through the correspondence principle, such Grassmann graded Poisson brackets must then correspond to Grassmann graded (anti)commutation relations for the quantised system, in particular anticommutation relations for fermionic degrees of freedom of half-integer spin and commutation relations for bosonic degrees of freedom of integer spin. The algebraic properties shared by Grassmann graded Poisson brackets and Grassmann graded (anti)commutation relations are indeed identical, hence the necessity of such a coherent prescription for their correspondence.

From yet another point of view, the necessity of Grassmann odd degrees of freedom for spinor fields may be seen as follows. Note that the Lagrangian function for the Dirac field is linear in the spacetime gradient $\partial_\mu \psi$, whereas that for the scalar field is quadratic in $\partial_\mu \phi$. This is a crucial fact, when considered in relation to the possibility of adding total deriv-
tives to Lagrange functions. Indeed, for the sake of the argument, consider a one degree of freedom system of configuration space coordinate $\theta(t)$, for which the Lagrange function is first-order in the time derivative,

$$L = N\theta \frac{d\theta}{dt} - V(\theta),$$

(98)

$N$ being some normalisation constant with properties under complex conjugation such that $L$ be real ($\theta$ could be complex valued). However, one may also write

$$\theta \frac{d\theta}{dt} = \frac{d}{dt} (\theta^2) - \frac{d\theta}{dt}\theta.$$

(99)

Thus, if the variable $\theta$ is Grassmann even, namely implying that $\theta$ and $\dot{\theta}$ commute, one has

$$\theta \frac{d\theta}{dt} = \frac{d}{dt} \left( \frac{1}{2} \theta^2 \right),$$

(100)

showing that such a first-order contribution to such an action for a bosonic degree of freedom reduces purely to a total time derivative, hence leads to an equation of motion which is not a dynamical equation but rather a constraint condition, $\partial_\theta V(\theta) = 0$, involving only the $\theta$-derivative of the potential contribution $V(\theta)$ to the Lagrange function. On the other hand, if the variable $\theta$ is Grassmann odd, namely such that $\theta^2 = 0$ and $\dot{\theta}\theta = -\theta\dot{\theta}$ (since $\theta_1\theta_2 = -\theta_2\theta_1$ for Grassmann odd variables $\theta_1$ and $\theta_2$), the first-order contribution $\theta \dot{\theta}$ to the Lagrange function does indeed lead to an equation of motion describing dynamics, namely

$$\dot{\theta} = \frac{1}{2N} \frac{\partial V}{\partial \theta},$$

(101)

where in the r.h.s. a left-derivative is implicitly understood. Hence, first-order actions of the above type, which generically apply for spinor field representations of the Lorentz group, need to be defined in terms of Grassmann odd variables in order to lead to nontrivial dynamics. Consequently, at the quantum level, they need to be quantised using anticommutation, rather than commutation relations.

The whole mathematical framework is thus consistent, both at the classical as well as the quantum level, provided integer spin degrees of freedom are described in terms of bosonic or commuting Grassmann even variables, hence commutation relations at the quantum level, and half-integer spin degrees of freedom are described in terms of fermionic or anticommuting Grassmann odd variables, hence anticommutation relations at the quantum level.
Having understood how to quantise the Dirac spinor field, let us conclude with a few more remarks. First, consider the Majorana condition

\[ \psi^C(x) = \psi(x) \]

imposed on such a spinor. The associated Lagrangian density then reads,

\[
\mathcal{L} = \frac{1}{4} i \left[ \overline{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \overline{\psi} \gamma^\mu \psi \right] - \frac{1}{2} m \overline{\psi} \psi
\]

\[ (102) \]

where the choice of factor 1/2 in comparison to the Dirac Lagrangian density is made in order to have a convenient normalisation of the field, leading to the usual normalisation of the anticommutation relations for the creation and annihilation operators of its quanta. This factor is also related to the avoidance of double counting of degrees of freedom. In fact, it is the same factor that appears in the Lagrangian density for a real scalar field, as compared to that for a complex scalar field \( \phi(x) \), namely related to the factor \( 1/\sqrt{2} \) in the real and imaginary components of the complex field in terms of real fields, \( \phi(x) = (\phi_1(x) + i\phi_2(x))/\sqrt{2} \).

Solving the Dirac equation following from the above Majorana field Lagrangian density, subject to the Majorana condition, leads to the mode decomposition,

\[
\psi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} \sum_{s = \pm} \left\{ e^{-ik\cdot x} u(k, s) b(k, s) + e^{ik\cdot x} v(k, s) b^\dagger(k, s) \right\},
\]

\[ (103) \]

with the same quantities as those that appear in the solution (93) for the Dirac spinor. Note well that indeed there no longer appears the creation operator \( d^\dagger(k, s) \) for antiparticles, but that only the annihilation, \( b(k, s) \), and creation, \( b^\dagger(k, s) \), operators of particles of a single type contribute to the Majorana spinor field operator.\(^\text{7}\) A Majorana spinor describes quanta which are their own antiparticles. Hence, they cannot carry a conserved quantum number, such as fermion number, as was already observed previously. A Majorana spinor describes neutral spin 1/2 particles, whereas a Dirac spinor describes charged (for some symmetry, for instance the U(1) symmetry of electric charge or fermion number) spin 1/2 particles.

The fermion number of the Dirac spinor is determined, through Noether’s theorem, from the time component of the conserved vector cur-
rent $J^\mu = \bar{\psi}\gamma^\mu\psi$. In terms of the mode expansion, one has

$$F = \int\frac{d^3k}{(2\pi)^3\omega(k)} \sum_{s=\pm} \left\{ b^\dagger(k, s)b(k, s) - d^\dagger(k, s)d(k, s) \right\},$$

(104)

where the normal ordering prescription has been applied. Clearly, this expression shows that states created by $b^\dagger(k, s)$ carry an $F$ value opposite to that carried by states created by $d^\dagger(k, s)$. The conserved $F$ quantum number, related to the invariance of the Dirac Lagrangian density under arbitrary global phase transformations of the Dirac spinor field, is what distinguishes particles from antiparticles of spin 1/2 in this system. If this quantum number is also identified to the electric charge of the electromagnetic interaction for electrons, it is thus seen that the Dirac spinor describes both electrons and their antiparticles, positrons, of identical mass and spin, but opposite electric charge, which remains a conserved quantum number. Gauging the associated $U(1)$ vector symmetry then leads to a complete description of the quantum electromagnetic interactions between electrons, positrons and photons, namely quantum electrodynamics (QED). When this is extended to nonabelian internal symmetries, one obtains Yang–Mills theories which, for the choice of gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, enter the construction of the Standard Model of quarks and leptons and their interactions. The sector of the strong interactions among quarks is thus based on the colour symmetry $SU(3)_C$ and the associated Yang–Mills gauge theory of quantum chromodynamics (QCD).

For what concerns spacetime symmetries, the Poincaré generators are now given by the expressions,

$$P^\mu = \int\frac{d^3k}{(2\pi)^3\omega(k)} k^\mu \sum_{s=\pm} \left\{ b^\dagger(k, s)b(k, s) + d^\dagger(k, s)d(k, s) \right\},$$

(105)

$$M^{\mu\nu} = \int\frac{d^3k}{(2\pi)^3\omega(k)} \left( \Theta^{0\mu}x^\nu - \Theta^{0\nu}x^\mu - \frac{1}{4}\bar{\psi}\gamma^0\sigma^{\mu\nu}\psi \right),$$

(106)

where

$$\Theta^{\mu\nu} = \frac{1}{2i} \left[ \bar{\psi}\gamma^\mu\partial^\nu\psi - \partial^\mu\bar{\psi}\gamma^\nu\psi \right],$$

(107)

while fermion normal ordering is implicit of course. It then follows that the 1-particle states obtained by acting with the creation operators $b^\dagger(k, s)$ and $d^\dagger(k, s)$ on the Fock vacuum $|0\rangle$ are energy-momentum eigenstates of momentum $k$ and mass $m$, possessing spin or helicity 1/2.
In the same way as for the scalar field, it is possible to compute the Feynman propagator of the Dirac field, namely the causal probability amplitude for seeing a particle created at a given point in spacetime and annihilated at some other such point. This time-ordered amplitude is thus defined by the 2-point correlation function

\[
\langle 0| T\psi_{\alpha}(x)\bar{\psi}_{\beta}(y)|0\rangle = \theta(x^0 - y^0)\langle 0| \psi_{\alpha}(x)\bar{\psi}_{\beta}(y)|0\rangle - 
\theta(y^0 - x^0)\langle 0| \bar{\psi}_{\beta}(y)\psi_{\alpha}(x)|0\rangle ,
\]

where the anticommuting nature of the spinor is accounted for through the negative sign in the second contribution in the r.h.s. (\(\theta(x)\) denotes the usual step function, \(\theta(x > 0) = 1\) and \(\theta(x < 0) = 0\)). In the case of the free Dirac field, a direct substitution of the mode expansion (93) leads to the integral representation,

\[
\langle 0| T\psi_{\alpha}(x)\bar{\psi}_{\beta}(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \left( \frac{i}{\gamma^\mu k_\mu - m + i\epsilon} \right)_{\alpha\beta} ,
\]

where, as usual, \(\epsilon > 0\) corresponds to an infinitesimal imaginary part in the denominator of the momentum-space propagator introduced to specify the contour integration in the complex \(k^0\) energy plane in order to pick up the correct pole contributions associated to the positive- and negative-frequency components of the Dirac spinor mode expansion. This Dirac propagator is the basis for perturbation theory involving Dirac spinors, in the same way that the Feynman propagator for scalar fields enables the evaluation of the perturbation theory corrections stemming from interactions between scalar particles.

4. On the Road Towards Supersymmetry: A Simple Quantum Mechanical Model

The previous sections have reviewed how, by enforcing at all steps the consequences of spacetime Lorentz and Poincaré covariance, relativistic quantum field theories lead to a conceptual framework deeply rooted in basic physical principles which naturally describes the relativistic and quantum properties of point-particles of given mass and spin, and the possibility of their creation and annihilation in a variety of processes for which the fundamental interactions are responsible. The Poincaré symmetry invariance properties of Minkowski spacetime allow for the particle interpretation of definite energy-momentum and spin values for the quanta of such fields. Any further internal symmetries then also account for further conserved
quantum numbers that particles carry. When gauged, such internal symmetries lead to specific interactions of the Yang–Mills type, which are at the basis of the construction of the successful Standard Model for quarks and leptons and their fundamental interactions.

We have also made clear how bosonic particles of integer spin need to be described in terms of commuting degrees of freedom and quantum commutation relations for the tensor field representations of the Lorentz group, whereas fermionic particles of half-integer spin need to be described in terms of anticommuting degrees of freedom and quantum anticommutation relations for the spinor field representations of the Lorentz group.

As briefly discussed in Ref. 1, this widely encompassing framework aiming towards a fundamental unification has now come to a cross-roads at which an irreconcilable clash has arisen between the principles of general relativity, the relativistic invariant classical field theory for the gravitational interaction described through the dynamics of spacetime geometry, and the principles of relativistic quantum field theory, the natural framework for all of matter and the other three fundamental interactions. Many extensions beyond the Standard Model aiming at a resolution of this conflict have been contemplated, most of which involve in one way or another algebraic structures relating fermionic and bosonic degrees of freedom, so-called supersymmetry algebras. Indeed, the distinct separation between boson and fermionic fields at the same time attracts the suggestion of a possible unification within a larger framework in which such degrees of freedom could appear on an equal footing, a specific type of a fundamental unification of matter (half-integer spin particles, namely the quarks and leptons) and interactions (integer spin particles, namely the Yang–Mills gauge bosons of the strong and electroweak interactions, the higgs particle yet to be discovered and the graviton). One should expect that assuming this to be achievable, such a unification should also extend the usual commuting coordinates of Minkowski spacetime into a superspace including both commuting and anticommuting coordinates, truly a first embodiment of an eventual fundamental quantum geometry.

The stage has thus been set to embark onto a journey on the roads towards the construction of supersymmetric quantum field theories. These notes shall stop short of such a discussion, which is widely available in the literature, and conclude in this section with a series of remarks pointing towards the generic features of such systems, as a way of opening the reader’s mind for whom this is unknown territory of theoretical physics, to what he/she may expect from a study on his/her own of supersymmetry.
We shall do this starting again from ordinary quantum mechanics. Hopefully, it should have been made abundantly clear\(^1\) that the “essence” of relativistic quantum fields is their harmonic oscillator characteristics, extended in such a manner as to make their spacetime dynamics also consistent with the Poincaré invariance of Minkowski spacetime. This is true whether for bosonic or fermionic quantum fields, the simplest examples of which are the fields describing particles of spin or helicity 0 and 1/2. Let us thus reduce to the extreme again these field situations, by restricting the discussion to simple harmonic oscillator degrees of freedom finite in number. The generalisation to field degrees of freedom will then be restricted and guided by the constraints stemming from Poincaré invariance, leading in fine to supersymmetric relativistic quantum field theories.

To begin with, let us consider a single bosonic harmonic oscillator.\(^1\) Once quantised, to such a system is associated a representation space of its quantum states, its physical Hilbert space, on which act the annihilation, \(a\), and creation, \(a^\dagger\), operators of energy quanta subjected to the commutation relation \([a, a^\dagger] = 1\) (the other commutators vanish identically, \([a, a] = 0 = [a^\dagger, a^\dagger]\)). A canonical basis of the Fock algebra is the Fock basis, constructed from a vacuum state \(|0\rangle\) annihilated by \(a\), \(a|0\rangle = 0\), on which acts the creation operator \(a^\dagger\), leading to the discrete set of states \(|n\rangle = (a^\dagger)^n|0\rangle/\sqrt{n!}(n = 0, 1, 2, \cdots)\) obeying the properties,

\[
\langle n|m \rangle = \delta_{nm}, \quad a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a^\dagger a|n\rangle = n|n\rangle.
\]

(110)

The quantum Hamiltonian of the system, generating also its dynamical evolution in time, is diagonal in the Fock basis, and is given by

\[
H_B = \frac{1}{2} \hbar \omega \{a^\dagger, a\} = \frac{1}{2} \hbar \omega \left( a^\dagger a + aa^\dagger \right) = \hbar \omega \left[ a^\dagger a + \frac{1}{2} \right],
\]

(111)

where the vacuum quantum energy contribution \(\hbar \omega / 2\) has been retained, while \(\omega\) denotes the angular frequency of the system, setting its energy scale in combination with Planck’s constant \(\hbar\). The energy spectrum is thus equally spaced in steps of \(\hbar \omega\), with the eigenvalues \(E_B(n) = \hbar \omega(n + 1/2)\), \(H_B|n\rangle = E_B(n)|n\rangle\), starting with the vacuum state at \(E_B(n = 0) = \hbar \omega / 2\).

Let us now consider likewise the quantum fermionic oscillator of same angular frequency \(\omega\) (the reason being that later on we shall introduce a symmetry relating the bosonic and fermionic systems). The space of states provides a representation for the fermionic anticommutator algebra

\[
\{b, b\} = 0 = \{b^\dagger, b^\dagger\}, \quad \{b, b^\dagger\} = 1,
\]

(112)
where \( b \) and \( b^\dagger \) are the fermionic annihilation and creation operators, respectively. Note that by having replaced commutation relations with anti-commutation ones, the vanishing anticommutators in fact imply the properties \( b^2 = 0 = b^\dagger b^\dagger \), the manifest realisation of the Pauli exclusion principle for fermions. As a consequence, the Fock space representation of this fermionic Fock algebra is 2-dimensional (to be contrasted with the discrete infinite dimension of the bosonic Hilbert space), and is spanned by a vacuum state \( |0\rangle \) and its first excitation \( |1\rangle = b^\dagger |0\rangle \), with the properties,

\[
   b|0\rangle = 0 \quad , \quad b^\dagger |0\rangle = |1\rangle \quad , \quad b|1\rangle = |0\rangle \quad , \quad b^\dagger |1\rangle = 0 ,
\]

(113)

For the quantum Hamiltonian, we shall also choose

\[
   H_F = \frac{1}{2} \hbar \omega \left[ b^\dagger b \right] = \frac{1}{2} \hbar \omega \left[ b^\dagger b - b b^\dagger \right] = \hbar \omega \left[ b^\dagger b - \frac{1}{2} \right] ,
\]

(114)

where this time the vacuum quantum energy is negative because of the fermionic character of the degree of freedom. The Fock state basis diagonalises this operator, with the energy spectrum,

\[
   H_F |0\rangle = -\frac{1}{2} \hbar \omega \quad , \quad H_F |1\rangle = \frac{1}{2} \hbar \omega ,
\]

(115)

thus describing a 2-level quantum system split by an energy \( \hbar \omega \).

Let us now combine these two systems, and consider the tensor product of their operator algebras and representation spaces. Hence, the complete Hilbert space is spanned by the states \( |n, 0\rangle \) and \( |n, 1\rangle \), where the first entry stands for the bosonic excitation level, and the second entry for that of the fermionic sector. The total Hamiltonian of the system then reads,

\[
   H = H_B + H_F = \frac{1}{2} \hbar \omega \left[ a^\dagger a + aa^\dagger + b^\dagger b - bb^\dagger \right] = \hbar \omega \left[ a^\dagger a + b^\dagger b \right] ,
\]

(116)

in which the vacuum quantum energies of the bosonic and fermionic sectors have cancelled one another. Consequently, the energy eigenspectrum is still equally spaced in steps of \( \hbar \omega \), is doubly degenerate at each level with the states \( |n, 0\rangle \) and \( |n - 1, 1\rangle \) at level \( n \) of energy \( \hbar \omega n \), except for the single ground state or vacuum state \( |n = 0, 0\rangle \) at level \( n = 0 \) whose energy vanishes identically,

\[
   H|n = 0, 0\rangle = 0 \quad , \quad H|n, 0\rangle = \hbar \omega n|n, 0\rangle \quad , \quad H|n - 1, 1\rangle = \hbar \omega n|n - 1, 1\rangle .
\]

(117)
With these simple remarks, in fact we already encounter a series of features quite unique to supersymmetry. If a system possesses a symmetry that relates fermionic and bosonic degrees of freedom, there are general classes of cancellations between quantum fluctuations and corrections stemming from the two sectors, leading to better behaved short-distance UV divergences generic of 4-dimensional quantum field theories. Indeed, there are even certain classes of quantum operators which, in supersymmetric field theories, are not at all renormalised by perturbative quantum corrections, leading to very powerful so-called no-renormalisation theorems. In addition, the cancellation between bosonic and fermionic vacuum quantum energy contributions implies that in field theories in which supersymmetry is not spontaneously broken, the vacuum state possesses an exactly vanishing energy, suggesting a possible connection with the famous problem of the extremely small (in comparison to the Planck energy scale relevant to quantum gravity, $10^{19}$ GeV) and yet not exactly vanishing cosmological constant of our universe.\textsuperscript{17} If only from that perspective, dynamical spontaneous symmetry breaking of supersymmetry is thus an extremely fascinating issue in the quest for a fundamental unification.\textsuperscript{18}

The degeneracy between the bosonic states $|n, 0\rangle$ and the fermionic ones $|n, 1\rangle$ suggests that there exists a symmetry — a supersymmetry — relating these two sectors of the system. We need to construct the operators generating such transformations, by creating a fermion and annihilating a boson, or vice-versa, thus mapping between bosonic and fermionic states degenerate in energy. Clearly these operators are given by

$$Q = \sqrt{\hbar \omega} a^\dagger b , \quad Q^\dagger = \sqrt{\hbar \omega} a b^\dagger ,$$

(118)

acting as

$$Q|n, 0\rangle = 0 , \quad Q|n, 1\rangle = \sqrt{\hbar \omega} \sqrt{n + 1}|n + 1, 0\rangle ,$$

$$Q^\dagger |n, 0\rangle = \sqrt{\hbar \omega} \sqrt{n}|n - 1, 1\rangle , \quad Q^\dagger |n, 1\rangle = 0 .$$

(119)

Note that the vacuum $|n = 0, 0\rangle$ is the single state which is annihilated by both $Q$ and $Q^\dagger$, as it must since it is not degenerate in energy with any other state. The operators $Q$ and $Q^\dagger$ are thus the generators of a supersymmetry present in this system. Their algebra is given by

$$\{Q, Q\} = 0 = \{Q^\dagger, Q^\dagger\} , \quad \{Q, Q^\dagger\} = H , \quad [Q, H] = 0 = [Q^\dagger, H] .$$

(120)

The fact that they define a symmetry is confirmed by their vanishing commutation relations with the Hamiltonian $H$. 
Once again, we uncover here a general feature of supersymmetry algebras, namely the fact that acting twice with a supersymmetry generator, in fact one gets an identically vanishing result, \( Q^2 = 0 = Q^\dagger_2 \), a property directly reminiscent of cohomology classes of differential forms in differential geometry.\(^\text{19}\) In addition, the anticommutator of a supersymmetry generator with its adjoint gives the Hamiltonian of the system. In a certain sense thus, making a system supersymmetric amounts to taking a square-root of its Hamiltonian. Put differently, the square-root of the Klein–Gordon equation is the Dirac equation, when this correspondence is extended to field theories. From these simply remarks it already transpires that supersymmetry algebras provide powerful new tools with which to explore mathematics questions within a context which may draw on a lot of insight and intuition from quantum physics.\(^\text{4,19}\) Results have indeed been very rewarding already, and many more are still to be established along such lines.

To complete the algebraic relations in (120), it is also useful to display the supersymmetry action on the creation and annihilation operators,

\[
\begin{align*}
\{Q, a\} &= -\sqrt{\hbar \omega} b , \quad \{Q, a^\dagger\} = 0 , \quad \{Q^\dagger, a\} = 0 , \quad \{Q^\dagger, a^\dagger\} = \sqrt{\hbar \omega} b^\dagger , \\
\{Q, b\} &= 0 , \quad \{Q, b^\dagger\} = \sqrt{\hbar \omega} a^\dagger , \quad \{Q^\dagger, b\} = \sqrt{\hbar \omega} a , \quad \{Q^\dagger, b^\dagger\} = 0 .
\end{align*}
\]

The properties \( Q^2 = 0 = Q^\dagger_2 \) also suggest that it should be possible to obtain wave function representations of the fermionic and supersymmetry algebras using complex valued Grassmann odd variables \( \theta \), such that \( \theta_1 \theta_2 = -\theta_2 \theta_1 \) and thus \( \theta_2 = 0 = \theta_2^\dagger \), in the same way that the bosonic Fock algebra possesses wave function representations in terms of commuting coordinates, a configuration space coordinate \( x \) and its conjugate momentum \( p \), obeying the Heisenberg algebra.\(^\text{1}\) In the latter case, these two variables may be combined into a single complex commuting variable \( z \), leading for instance to the usual holomorphic representation in the bosonic sector,

\[
a = \frac{\partial}{\partial z} , \quad a^\dagger = z .
\]

Thus likewise for the fermionic algebra, let us take

\[
b = \frac{\partial}{\partial \theta} , \quad b^\dagger = \theta ,
\]

where it is understood that all derivatives with respect to Grassmann odd variables are taken from the left (left-derivatives). Consequently the supersymmetry generators are represented by

\[
Q = \sqrt{\hbar \omega} z \frac{\partial}{\partial \theta} , \quad Q^\dagger = \sqrt{\hbar \omega} \theta \frac{\partial}{\partial z} ,
\]
leading to the representation for the Hamiltonian,

\[ H = Q^\dagger Q + QQ^\dagger = \hbar \omega \left[ a^\dagger a + b^\dagger b \right] = \hbar \omega \left[ z \frac{\partial}{\partial z} + \theta \frac{\partial}{\partial \theta} \right]. \]  

(125)

These operators thus act on wave functions \( \psi(z, \theta) \). Because of the Grassmann property \( \theta^2 = 0 \), a power series expansion of such a function terminates at a finite order, in the present case at first order since only one \( \theta \) variable is involved,

\[ \psi(z, \theta) = \psi_B(z) + \theta \psi_F(z), \quad \psi_F(z) = \frac{\partial}{\partial \theta} \psi(z, \theta), \]  

(126)

where, assuming that \( \psi(z, \theta) \) itself is Grassmann even, the bosonic component \( \psi_B(z) \) is Grassmann even while the fermionic one \( \psi_F(z) \) is Grassmann odd, as it should considering the analogous structure of the space of quantum states. In particular, the general wave function representing the energy eigenstates \( |n, 0\rangle \) and \( |n - 1, 1\rangle \) with value \( E(n) = \hbar \omega n \) is given as

\[ \psi_n(z, \theta) = B_n \frac{z^n}{\sqrt{n!}} + F_n \theta \frac{z^{n-1}}{\sqrt{(n-1)!}}, \]  

(127)

where \( B_n \) and \( F_n \) are arbitrary phase factors associated to the bosonic and fermionic components of this wave function.

The supersymmetry charges \( Q \) and \( Q^\dagger \) act on such general wave functions as

\[ Q \psi(z, \theta) = \sqrt{\hbar \omega} z \psi_F(z), \quad Q^\dagger \psi(z, \theta) = \sqrt{\hbar \omega} \theta \partial_z \psi_B(z). \]  

(128)

Thus introducing a complex valued Grassmann odd constant parameter \( \epsilon \) associated to the symmetries generated by the supercharges \( Q \) and \( Q^\dagger \), one has for the general self-adjoint combination of supercharges

\[ Q_\epsilon = \epsilon Q + \epsilon^\dagger Q^\dagger = \epsilon Q - \epsilon^\dagger Q^\dagger, \]  

(129)

the action

\[ Q_\epsilon \psi(z, \theta) = \sqrt{\hbar \omega} \left[ (z \epsilon \psi_F(z)) + \theta (\epsilon^\dagger \partial_z \psi_B(z)) \right]. \]  

(130)

Consequently, given the variations \( \delta_\epsilon \psi(z, \theta) = iQ_\epsilon \psi(z, \theta) \), the bosonic and fermionic components of such wave functions are transformed according to the rules

\[ \delta_\epsilon \psi_B(z) = i\sqrt{\hbar \omega} z \epsilon \psi_F(z), \quad \delta_\epsilon \psi_F(z) = i\sqrt{\hbar \omega} \epsilon^\dagger \partial_z \psi_B(x). \]  

(131)

These expressions thus provide the infinitesimal supersymmetry transformations of the wave functions of the system. We shall come back to these relations hereafter.
In order to identify which type of classical system corresponds to the present situation, let us now introduce the configuration and momentum space degrees of freedom through the usual relations,

\[ a = \sqrt{\frac{m\omega}{2\hbar}} \left[ x + \frac{i}{m\omega}p \right] , \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left[ x - \frac{i}{m\omega}p \right] , \]

\[ b = \sqrt{\frac{m\omega}{2\hbar}} \left[ \theta_1 + i\theta_2 \right] , \quad b^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left[ \theta_1 - i\theta_2 \right] . \]

Note well that the variables \( x, p, \theta_1 \) and \( \theta_2 \), which are assumed to be self-adjoint, \( x^\dagger = x, p^\dagger = p, \theta_1^\dagger = \theta_1, \theta_2^\dagger = \theta_2 \), are still operators at this stage.

The decomposition of the fermionic operators \( b \) and \( b^\dagger \) in these terms is of course to maintain as manifest as possible the parallel between the bosonic and fermionic sectors of the system, which are exchanged under supersymmetry transformations. Given these operator redefinitions, it follows that the only nonvanishing (anti)commutators are (note that the operators \( \theta_1 \) and \( \theta_2 \) thus anticommute with one another, \( \{ \theta_1, \theta_2 \} = 0 \))

\[ [x, p] = i\hbar , \quad \{ \theta_1, \theta_1 \} = \frac{\hbar}{m\omega} = \{ \theta_2, \theta_2 \} . \]

Furthermore, the Hamiltonian operators is then expressed as

\[ H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + im\omega^2\theta_1\theta_2 , \]

leading to the operator equations of motion in the Heisenberg picture,

\[ i\hbar \dot{x} = [x, H] = i\hbar \frac{p}{m} , \quad i\hbar \dot{p} = [p, H] = -i\hbar m\omega^2 x , \]

\[ i\hbar \dot{\theta}_1 = [\theta_1, H] = i\hbar \omega \theta_2 , \quad i\hbar \dot{\theta}_2 = [\theta_2, H] = -i\hbar \omega \theta_1 . \]

It is also possible to determine how the supercharges \( Q \) and \( Q^\dagger \) act on the operators \( x, p, \theta_1 \) and \( \theta_2 \), an exercise left to the reader (of which the results are used hereafter).

Through the correspondence principle, the (anti)commutator relations (133) are required to translate into the following classical Grassmann graded Poisson brackets for the associated degrees of freedom,

\[ \{ x, p \} = 1 , \quad \{ \theta_1, \theta_1 \} = -\frac{i}{m\omega} = \{ \theta_2, \theta_2 \} , \]

with now all the variables \( x, p, \theta_1 \) and \( \theta_2 \) real under complex conjugation, \( x \) and \( p \) being ordinary commuting Grassmann even degrees of freedom, but \( \theta_1 \) and \( \theta_2 \) being anticommuting Grassmann odd degrees of freedom associated to the fermionic sector of the system. At the classical level, the Hamiltonian is given by the same expression as in (134). In particular,
using these Grassmann graded Poisson brackets, at the classical level the same Hamiltonian equations of motion are recovered as those in (135) for the quantum operators. These classical equations of motion follow through the variational principle from the first-order Hamiltonian action

$$S[x, p, \theta_1, \theta_2] = \int dt \left\{ \frac{1}{2} [\dot{x}p - \dot{p}x] - \frac{1}{2} im\omega \left[ \dot{\theta}_1 \dot{\theta}_1 + \dot{\theta}_2 \dot{\theta}_2 \right] - H \right\}.$$  \hspace{1cm} (137)

Using the Hamiltonian equation of motion for $x$ in order to reduce its conjugate momentum $p$, namely $p = m\dot{x}$, and also introducing the complex valued Grassmann odd variable $\theta = \theta_1 + i\theta_2$, it then follows finally that the Lagrange function of the system is given by,\(^\text{w}\)

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} im\omega \dot{\theta} \dot{\theta}^\dagger - \frac{1}{2} m\omega^2 \theta \theta^\dagger.$$  \hspace{1cm} (138)

From the above considerations, it should then follow that the transformations associated to the supercharges $Q$ and $Q^\dagger$ generate global symmetries of this action. These transformations are given by\(^\text{x}\)

$$\delta Q x = i\epsilon \theta + i\epsilon^\dagger \theta^\dagger , \quad \delta Q \theta = 2i\epsilon^\dagger \left( x + \frac{i}{\omega} \dot{x} \right), \quad \delta Q \theta^\dagger = -2i\epsilon \left( x - \frac{i}{\omega} \dot{x} \right).$$  \hspace{1cm} (139)

And indeed, it may readily be checked that the infinitesimal variation of the Lagrange function (138) then reduces to a simple total time derivative, thus establishing the supersymmetry invariance of this system also at the classical level. Applying Noether’s general analysis to this new type of symmetry for which the parameters are Grassmann odd quantities, leads back to the conserved supercharges generating these transformations.\(^\text{y}\)

Once again, a few general lessons may be drawn from the above considerations, which remain valid in the case also of supersymmetric field theories. The supercharges $Q$ and $Q^\dagger$ define transformations between the

\(^{\text{w}}\)Note that up to a total time derivative term this function is indeed real under complex conjugation, because of the Grassmann odd character of the fermionic degree of freedom $\theta(t)$. Some total derivative terms in time have been ignored to reach this expression, and to bring it into such a form that no time derivatives of order strictly larger than unity appear in the action.

\(^{\text{x}}\)Compared to the previous parametrisation, a factor $(-\sqrt{\hbar\omega}/2)$ has been absorbed into the normalisation of the supersymmetry constant parameters $\epsilon$ and $\epsilon^\dagger$ or supercharges $Q$ and $Q^\dagger$. Note also that these expressions are consistent with the properties under complex conjugation of the different degrees of freedom as well as their Grassmann parity.

\(^{\text{y}}\)In such an analysis, one should beware of the surface terms induced by the supersymmetry transformation applied to the action, which also contribute to the definition of the Noether charges.\(^\text{11}\)
states $|n,0\rangle$ and $|n-1,1\rangle$, except for the vacuum state $|n=0,0\rangle$ which remains invariant under supersymmetry. Hence, all these pairs of states for $n \geq 1$ define 2-dimensional supermultiplets, namely irreducible representations of the supersymmetry algebra, combining a bosonic and a fermionic state degenerate in energy. In the holomorphic wave function representation, the bosonic component is given by $\psi(z,\theta) = \psi_B(z) = z^n/\sqrt{n!}$, and the fermionic one by $\psi(z,\theta) = \theta \psi_F(z) = \theta z^{n-1}/\sqrt{(n-1)!}$. From a certain point of view, the bosonic phase space $z$ of the system has been extended into a super-phase space of degrees of freedom $(z,\theta)$, on which supersymmetry transformations act through $Q = \sqrt{\hbar \omega} z \partial_\theta$ and $Q^\dagger = \sqrt{\hbar \omega} \theta \partial_z$, thereby inducing a map between the bosonic and fermionic components of a general super-phase space wave function $\psi(z,\theta)$, according to the rules in (131). In particular, note that the lowest component $\psi_B(z)$ of such a super-wave function is mapped into its fermionic component, while its highest component $\psi_F(z)$ is mapped into the $z$-derivative of its lowest component. If one recalls that through a process of second-quantisation, quantum fields may be seen, in a certain sense, to correspond to quantum mechanical wave functions of 1-particle states, this remark suggests that by extension, supersymmetric quantum field theories should be constructed in terms of superfields depending not only on the usual spacetime coordinates $x^\mu$, but also on some collection of Grassmann odd variables, in order to extend usual Minkowski spacetime into some form of a superspace\textsuperscript{20} Minkowski spacetime, all in a manner consistent with the Poincaré covariance properties required of such theories. Consequently, these Grassmann odd variables must be chosen to determine specific spinor representations of the Lorentz group, namely right- and left-handed Weyl spinors $\theta_\alpha$ and $\bar{\theta}^{\dot{\alpha}}$.

Indeed, when developing this point of view, it appears that such superfields decompose into specific bosonic and fermionic components, defining a supermultiplet, with supersymmetry transformations mapping these components into one another. In particular, the transformation of the highest component always includes the spacetime derivative of some of the lower components.

From the field theory point of view however, it would be more appropriate to develop the same considerations rather in terms of the system degrees of freedom $x(t)$ and $\theta(t)$, which are transformed into one another as shown in (139). Indeed, as argued in Sec. 2, fields may be viewed as collections of oscillators fixed at all points in space, namely $\phi(t,\vec{x}) = x_{\vec{x}}(t)$ and $\psi(t,\vec{x}) = \theta_{\vec{x}}(t)$ for scalar and spinor fields, respectively, and coupled to one another through their spatial gradients in order to ensure space-
time Poincaré and Lorentz invariance. In the case of the present simple supersymmetric mechanical model, these bosonic and fermionic degrees of freedom $x(t)$ and $\theta(t)$ thus define a certain type of “field” supermultiplet (rather than a supermultiplet of quantum states, as in the discussion above), of which the variation of its highest component includes again the time derivative of some of its lower components. Upon quantisation, these transformation properties also apply to the quantum operators, and translate into the specific transformation rules for the quantum states described above. When extended to field theories, these features survive, this time in terms of bosonic and fermionic field degrees of freedom. Note that in (138), time derivatives of bosonic degrees of freedom contribute in quadratic form to the Lagrange function, whereas for fermionic ones they contribute to linear order. This fact, when extended to a relativistic framework, remains valid as well. The Klein–Gordon Lagrangian density is quadratic in space-time gradients of the scalar field, but the Dirac Lagrangian density is linear in such derivatives of the Dirac spinor. For reasons explained previously, these features are necessary for a consistent dynamics of Grassmann even and odd, or integer and half-integer spin degrees of freedom.

For the sake of completing the discussion of the present simple supersymmetric quantum mechanical model of harmonic oscillators, let us indeed show how a superfield calculus may be developed already in this case. Again, the general lessons following from such an approach readily extend to the superfield constructions of supersymmetric field theories in which the constraints of spacetime Poincaré covariance are then also accounted for through the knowledge of the representation theory of the Lorentz group.

The Hamiltonian of any given system is the generator of translations in time. Given that the anticommutator of supercharges produces the Hamiltonian, as shown for example in (120), means that supersymmetry transformations correspond to taking some sort of square-root of translations in (space)time, in a new “dimension” of (space)time which must be parametrised by a Grassmann odd coordinate this time, since supercharges map bosonic and fermionic states into one another. In addition to the bosonic time coordinate $t$, let us thus extend time into a “supertime” by also introducing a complex valued Grassmann odd coordinate, which shall be denoted $\eta$ (the customary notation $\theta$ for Grassmann odd superspace coordinates being already used for the fermionic degrees of freedom of the system, $\theta(t)$), and its complex conjugate $\eta^\dagger$. Thus we now have the superspace (or rather for this mechanical model simply “supertime”) spanned by the coordinates $(t, \eta, \eta^\dagger)$. Supersymmetry transformations generated by $Q$
and $Q^\dagger$ should then correspond to translations in the Grassmann odd directions in superspace, in the same way that transformations in time generated by the Hamiltonian correspond to translations in the Grassmann even direction of superspace. By analogy with the operator $i\partial_t$ generating the latter translations, and representing the action of the Hamiltonian on degrees of freedom, the naive choice for the supercharges would be $Q = -i\partial_\eta$ and $Q^\dagger = i\partial_{\eta^\dagger}$.

However, a quick check then finds that all anticommutators $\{Q, Q\}$, $\{Q^\dagger, Q^\dagger\}$ and $\{Q, Q^\dagger\}$ vanish, thus not reproducing the supersymmetry algebra in (120). Hence, in order that the anticommutator of $Q$ and $Q^\dagger$ also reproduces the Hamiltonian, it is necessary that while a translation is performed in $\eta$ and $\eta^\dagger$, a translation in $t$ be also included in an amount proportional to the Grassmann odd coordinates in superspace.

It turns out that an appropriate choice is given by

$$Q = -i\partial_\eta + \frac{2}{\omega} \eta^\dagger \partial_t , \quad Q^\dagger = i\partial_{\eta^\dagger} - \frac{2}{\omega} \eta \partial_t .$$

A direct calculation finds that these operators obey the supersymmetry algebra

$$\{Q, Q\} = 0 = \{Q^\dagger, Q^\dagger\} , \quad \{Q, Q^\dagger\} = \left( -\frac{2}{\sqrt{\hbar}\omega} \right)^2 (i\hbar \partial_t) ,$$

in perfect correspondence with the abstract algebra in (120) (one should recall that a rescaling by a factor $(-\sqrt{\hbar}\omega/2)$ of the supersymmetry parameters $\epsilon$ and $\epsilon^\dagger$ or the supercharges $Q$ and $Q^\dagger$ has been applied in the intervening discussion).

In order to readily construct manifestly supersymmetric invariant Lagrange functions, it proves necessary to also use another pair of superspace differential operators, that anticommute with the supercharges, and define so-called superspace covariant derivatives. These supercovariant derivatives thus enable one to take derivatives of superfields in a manner consistent with supersymmetry transformations. Again, a convenient choice turns out to be

$$D = \partial_\eta - \frac{2i}{\omega} \eta^\dagger \partial_t , \quad D^\dagger = -\partial_{\eta^\dagger} + \frac{2i}{\omega} \eta \partial_t .$$

---

Some properties have to be met in the whole construction, such as preserving under supersymmetry transformations the real character under complex conjugation of the superfield considered hereafter. This leaves open a series of possible choices, essentially related to possible phase factors in the combinations defining the superspace differential operators introduced hereafter.
leading to the algebra
\[ \{D, D\} = 0 = \{D^\dagger, D^\dagger\}, \quad \{D, D^\dagger\} = \left(-\frac{2}{\sqrt{\hbar \omega}}\right)^2 (i\hbar \partial_t), \tag{143} \]
as well as the required properties
\[ \{Q, D\} = 0, \quad \{Q, D^\dagger\} = 0, \quad \{Q^\dagger, D\} = 0, \quad \{Q^\dagger, D^\dagger\} = 0. \tag{144} \]

Consider now an arbitrary Grassmann even superfield on superspace, namely a function \(X(t, \eta, \eta^\dagger)\). Without loss of generality (by distinguishing its real and imaginary parts), it is always possible to assume that such a superfield obeys a reality condition,
\[ \eta^\dagger(t, \eta, \eta^\dagger) = X(t, \eta, \eta^\dagger). \tag{145} \]

On account of the Grassmann odd character of the coordinate \(\eta\), namely the fact that \(\eta^2 = 0 = \eta^\dagger^2\), the general form of such a real superfield is given by
\[ X(t, \eta, \eta^\dagger) = x(t) + i\eta \theta(t) + i\eta^\dagger \theta^\dagger(t) + \eta^\dagger \eta f(t), \tag{146} \]
where \(x(t)\) and \(f(t)\) are real bosonic degrees of freedom, whereas \(\theta(t)\) and \(\theta^\dagger(t)\) are complex valued fermionic ones, complex conjugates of one another. Indeed, it will turn out that \(x(t)\) and \(\theta(t)\) correspond to the degrees of freedom considered above, while \(f(t)\) will be seen to be simply an auxiliary degree of freedom without dynamics, whose equation of motion is purely algebraic and such that upon its reduction the system described in (138) is recovered. This is a generic feature of superfields in supersymmetric field theories: they include auxiliary fields which are reduced through their algebraic equations of motion. However, in the superspace formulation, there are required for a supersymmetric covariant superspace calculus.

These choices having been specified, it is now straightforward to establish how the different components \((x, \theta, \theta^\dagger, f)\) (namely, the components of the terms in 1, \(i\eta \theta\), \(i\eta^\dagger \theta^\dagger\) in the \(\eta\)-expansion of superfields) of real superfields transform under supersymmetry transformations. By considering the explicit evaluation of
\[ \delta_Q X = i \left[ \epsilon Q - \epsilon^\dagger Q^\dagger \right] X, \tag{147} \]
\(\epsilon\) and \(\epsilon^\dagger\) being the arbitrary complex valued Grassmann odd constant supersymmetry parameters, complex conjugates of one another, it readily follows
that the components vary according to
\[
\delta Qx = i\epsilon \theta + i\epsilon^\dagger \theta^\dagger, \quad \delta Q\theta = i\epsilon^\dagger \left[ f + \frac{2i}{\omega} \dot{x} \right],
\]
\[
\delta Q\theta^\dagger = -i\epsilon \left[ f - \frac{2i}{\omega} \dot{x} \right], \quad \delta Qf = -\frac{2}{\omega} \left[ \epsilon \dot{\theta} - \epsilon^\dagger \dot{\theta}^\dagger \right].
\]
(148)

It is of interest to compare these transformation rules to those given in (139).

Here appears yet another generic feature of the superfield technique. One notices that the highest component \( f(t) \) in \( \eta^\dagger \eta \) of the superfield \( X(t, \eta, \eta^\dagger) \) transforms under supersymmetry as a total derivative in time. In the context of supersymmetric field theories, the highest component of superfields transforms as a total spacetime divergence. Thus, if one chooses for the Lagrange function or Lagrangian density the highest component of any relevant superfield, under any supersymmetry transformation the action of the system is invariant up to a total derivative, thus indeed defining an invariance of its equations of motion. In superspace, supersymmetric invariant actions are given by the highest component of superfields, in our case the \( \eta^\dagger \eta \) component,
\[
S[X] = \int dt \, d\eta \, d\eta^\dagger \, F(X), \quad (149)
\]
where \( F(X) \) is any real valued superfield constructed out of the basic superfield \( X \) and its derivatives obtained through the action of the supercovariant derivatives \( D \) and \( D^\dagger \). In this expression, the definition of Grassmann integration is such that
\[
\int d\eta \, d\eta^\dagger \, 1 = 0, \quad \int d\eta \, d\eta^\dagger \, \eta = 0, \quad \int d\eta \, d\eta^\dagger \, \eta^\dagger = 0, \quad \int d\eta \, d\eta^\dagger \, \eta^\dagger \eta = 1,
\]
(150)
while the result for any linear combination of these \( \eta \)-monomials is given by the appropriate linear combination of the resulting integrations (the usual integral over Grassmann even variables being also linear for polynomials).

It turns out that the choice corresponding to the supersymmetric harmonic oscillator in (138) is given by (one has, by construction of the supercovariant derivatives, \( (DX)^\dagger = D^\dagger X \) for the real superfield \( X \))
\[
S[X] = \int dt \, d\eta \, d\eta^\dagger \left[ -\frac{1}{8} m\omega^2 (D^\dagger X) (DX) - \frac{1}{4} m\omega^2 X^2 \right]. \quad (151)
\]
Working out the superspace components of this expression, it reduces to
\[ S[x, \theta, \theta^\dagger, f] = \int dt \left\{ \frac{1}{8} m \omega^2 \left[ f^2 + \frac{1}{x^2} \dot{x}^2 + 2 \left( \dot{\theta}^\dagger \dot{\theta} + \theta \dot{\theta} \right) \right] - \right. \\
\left. \frac{1}{2} m \omega^2 \left( f x + \theta \dot{\theta} \right) \right\}. \tag{152} \]

Since no time derivatives of the highest superfield component \( f(t) \) contribute to this action, this degree of freedom is indeed auxiliary with a purely algebraic equation of motion given by
\[ f(t) = 2 \dot{x}(t). \tag{153} \]

Upon reduction of this auxiliary degree of freedom, one recovers precisely the Lagrange function in (138), up to a total derivative in time \( d/dt(-i m \omega \theta \dot{\theta}/4) \), while for the remaining dynamical degrees of freedom \( x(t), \theta(t) \) and \( \theta^\dagger(t) \), the supersymmetry transformations (148) coincide then exactly with those in (139).

Having achieved the construction of the harmonic oscillator with a single supersymmetry generator \( \mathcal{N} = 1 \) from these different but complementary points of view, one may wonder whether generalisations to types of potentials other than the quadratic one in \( X^2 \), to more general dynamics, and for a larger number \( \mathcal{N} \) of supersymmetries, are possible. The interested reader is invited to explore such issues further, which have been addressed in the literature already to a certain extent.

We have thus shown how a superspace extension of the time coordinate into superspace coordinates \((t, \eta, \eta^\dagger)\) over which a superspace calculus is defined for superfields, readily allows for a systematic approach to the construction of supersymmetric quantum mechanical models. This superspace calculus displays already the features generic to the superspace techniques of superfields for the construction of supersymmetric invariant field theories in Minkowski spacetime. In the latter case, it is spacetime itself which is extended into a “superspacetime” \((x^\mu, \theta_\alpha, \theta^\dagger_\alpha)\) of bosonic and Weyl spinor coordinates, the latter appearing with multiplicities depending on the number \( \mathcal{N} \) of supersymmetries acting on the theory.

5. An Invitation to Superspace Exploration

As recalled in Sec. 2, by enforcing Poincaré invariance, the ordinary bosonic harmonic oscillator extends naturally into the quantum field theory of a scalar field describing relativistic quantum point-particles of zero spin. One
could attempt pursuing the same road starting from the fermionic harmonic oscillator described above and reach again the Dirac or the Majorana equation for spin 1/2 charged or neutral particles, but the task would be quite much more involved, since the answer is known to require a 4-component complex valued field, the parity invariant spinor representation of the Lorentz algebra. Rather, it is by considering the detailed representation theory of the Lorentz group that the correct answer is readily identified in simple algebraic terms.

Likewise, one could attempt to extend the simple $\mathcal{N} = 1$ supersymmetric harmonic oscillator model into a relativistic invariant quantum field theory, which should thus include both a scalar field and a Dirac–Majorana spinor. It is indeed possible to construct by hand such a field theory, known as the Wess-Zumino model, the simplest example of a $\mathcal{N} = 1$ supersymmetric quantum field theory in 4-dimensional Minkowski spacetime. However, the approach is very much streamlined by expressing everything in terms of superfields defined over some superspace which extends Minkowski spacetime by including some further Grassmann odd coordinates corresponding to specific Weyl spinors. This is the superspace construction of $\mathcal{N} = 1$ supersymmetric quantum field theories. Truly a quantum geometer’s approach to a possible quantum geometry of spacetime.

However once again, in order to classify and identify the realm of possible supersymmetric quantum field theories, for whatever number $\mathcal{N}$ of supersymmetries, and whatever dimension of Minkowski spacetime, and even whatever type of interactions consistent with the requirements of perturbative renormalisability, a discussion based on the possible algebraic structures merging and intertwining together the Poincaré algebra with Grassmann odd generators mapping bosons to fermions and vice-versa, is the most efficient approach. It is no small feat that such a complete and finite dimensional classification has been achieved. It is a nontrivial fact that such solutions exist, and also that they are only a small finite number of possibilities consistent with the rules of quantum field theory, in particular unitarity and causality. Clearly, such a situation gives credence to the suggestion that such a combination of Poincaré covariance and supersymmetry invariance brings us onto the right track towards the quest for a final unification.

The usefulness, relevance, and even meaning, of these different remarks should find a nice simple illustration with the previous quantum mechanical model. These different avenues towards the construction and classification of supersymmetric quantum field theories have been developed and dis-
discussed during the actual lectures delivered at the Workshop. However, since this material is widely available in the literature, and in much detailed form, while the lectures themselves were to a large extent based on those of Ref. 10, we shall stop short here from pursuing any further the discussion of such field theories and their particle content, except for just one last remark.

The supersymmetric field theories simplest to construct in 4-dimensional Minkowski spacetime involve a single supersymmetry generator, \( \mathcal{N} = 1 \), represented by a right-handed Weyl spinor supercharge \( Q_\alpha \) and its complex conjugate left-handed Weyl spinor supercharge \( \overline{Q}^\alpha \), using the dotted and undotted index notation (thus the single supercharge combines into a single Majorana spinor). In this case, the supersymmetry algebra is defined by the anticommutation relations,

\[
\{Q_\alpha, Q_\beta\} = 0 = \{\overline{Q}^\alpha, \overline{Q}^\beta\}, \quad \{Q_\alpha, \overline{Q}^\beta\} = 2P_\mu (\sigma^\mu)^{\alpha\beta}.
\]

Clearly, these relations are the natural extension to a Poincaré covariant setting of the supersymmetry algebra in (120) relevant to the quantum mechanical oscillator model. Indeed, the Hamiltonian is the time component of the energy-momentum 4-vector \( P^\mu \), while the components of the Weyl supercharges \( Q \) and \( \overline{Q} \) all square to a vanishing operator, implying again important cohomology properties in supersymmetric quantum field theories. The Noether charge \( P^\mu \) being also the generator for spacetime translations, implies that in a certain sense supersymmetry transformations correspond to taking the square-root of spacetime translations, requiring spinor degrees of freedom for consistency with Lorentz covariance.

As mentioned in Ref. 1, often one prefers, if only for aesthetical reasons having to do with spacetime locality and causality, to have a local or gauged symmetry as compared to a global symmetry acting identically and instantaneously throughout all of spacetime. Supersymmetry transformations as described in these notes correspond to global symmetries. Indeed, their infinitesimal action generated by the supercharges and mapping bosonic and fermionic degrees of freedom into one another, involves arbitrary Grassmann odd parameters which are spacetime independent constants. This situation suggests that one should gauge supersymmetry transformations, namely consider the possibility of constructing quantum field theories invariant under the same types of transformations between their degrees of freedom for which though, this time, the parameters are local functions of spacetime. Since the anticommutator of supercharges induces spacetime translations, it is clear that by gauging supersymmetry one has to introduce
new field degrees of freedom\(^1\) — the associated gauge fields, possessing both bosonic and fermionic degrees of freedom — which are in direct relation to spacetime reparametrisations and local Lorentz transformations, the latter being precisely the local gauge symmetries of general relativity for instance. In other words, in exactly the same way that gauged internal symmetries lead to Yang–Mills interactions, gauged supersymmetry implies the gravitational interaction through a dynamical spacetime metric field of helicity \(\pm 2\) and its supersymmetric partner field, in fact a Majorana Rarita–Schwinger of helicity \(\pm 3/2\). For this reason, gauged supersymmetric field theories are known as supergravity theories.\(^8\)\(^,\)\(^22\) Such theories exist for spacetime dimensions ranging from \(D = 2\) to \(D = 11\). Again, it is no accident that M-theory, the modern nonperturbative extension of superstring theory and a possible candidate for a final unification yet to be constructed, exists only in a spacetime of dimension \(D = 11\).\(^2\)

It is hoped that through the above analysis of a simple supersymmetric quantum mechanical model, the reader will have understood enough of the general concepts and generic features entering the formulation and the construction of supersymmetric field theories, as well as of their potential to address from novel and powerful points of view large fields of pure mathematics itself, that he/she may feel sufficiently secure in following the lead of such little white and precious pebbles along the path, to embark on a journey of one’s own onto the roads of the quantum geometer’s superspaces, deep into the uncharted territories of supersymmetries and their yet to be discovered treasure troves in the eternally fascinating worlds of physics and mathematics, thereby fulfilling ever a little more, this time with a definite African beat in the symphony, humanity’s unswaying and yet never ending quest for a complete understanding of our destiny in the physical Universe, the eternal yearning of man’s soul.\(^1\)

**Acknowledgements**

The author acknowledges the Workshop participants for their many constructive discussions and contributions on the matter of these lectures. This work is partially supported by the Federal Office for Scientific, Technical and Cultural Affairs (Belgium) through the Interuniversity Attraction Pole P5/27.
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