NON-COMMUTATIVE INTEGRABLE SYSTEMS ON 
b-SYMPLECTIC MANIFOLDS

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Abstract. In this paper we study non-commutative integrable systems on $b$-Poisson manifolds. One important source of examples (and motivation) of such systems comes from considering non-commutative systems on manifolds with boundary having the right asymptotics on the boundary. In this paper we describe this and other examples and we prove an action-angle theorem for non-commutative integrable systems on a $b$-symplectic manifold in a neighbourhood of a Liouville torus inside the critical set of the Poisson structure associated to the $b$-symplectic structure.

1. Introduction

A non-commutative integrable system on a symplectic manifold with boundary yields a non-commutative system on a class of Poisson manifolds called $b$-Poisson manifolds, as long as the asymptotics of the system satisfy certain conditions near the boundary. $b$-Poisson manifolds constitute a class of Poisson manifolds which recently has been studied extensively (see for instance [GMP11], [GMP12], [GMPS13] and [GLPR14]) and integrable systems on such manifolds have been the object of study in [KMS15], [KM16] and [DKM15].

In [LMV11] an action-angle coordinate for Poisson manifolds is proved on a neighbourhood of a regular Liouville torus. This theorem cannot be applied to a neighborhood of a Liouville torus contained inside the critical set of the Poisson structure where the rank of the bivector field is no longer maximal. In this paper we extend the techniques in [LMV11] to consider a neighbourhood of a Liouville torus inside the critical set of a $b$-Poisson manifolds thus proving an action-angle theorem for non-commutative systems on $b$-Poisson manifolds.

The action-angle theorem for non-commutative integrable systems for symplectic manifolds was proved by Nehoroshev in [N72]. Our proof follows a combination of techniques from [LMV11] with techniques native to $b$-symplectic geometry. As in [LMV11] the key point of the proof is to find

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a torus action attached to a non-commutative integrable system and extend
the Darboux-Carathéodory coordinates in a neighbourhood of the invariant
subset. The upshot is the use of \( b \)-symplectic techniques and toric actions
on these manifolds [GMPS13, GMPS2] as we did in [KMS15] and [KM16]
for commutative systems on \( b \)-manifolds. The proof is a combination of the
theory of torus actions with a refinement of the commutative proof by con-
sidering Cas-basic forms and working with them as a subcomplex of the \( b \)-De
Rham complex. The action-angle theorem for commutative integrable sys-
tems on \( b \)-symplectic manifolds yields semilocal models as twisted cotangent
lifts (see [KM16]). It is also possible to visualize the action-angle theorem
for non-commutative systems using twisted cotangent lifts.

The organization of this paper is as follows: In Section 2 we introduce the
basic tools that will be needed in this paper. In Section 3 we provide a list of
examples which includes non-commutative systems on symplectic manifolds
with boundary and examples obtained from group actions including twisted
\( b \)-cotangent lifts. We end this section exploring the Galilean group as a
source of non-commutative examples in \( b \)-symplectic manifolds. In Section
4 we state and prove the action-angle coordinate theorem for \( b \)-symplectic
manifolds.

2. Preliminaries

2.1. Integrable systems and action-angle coordinates on Poisson
manifolds. A Poisson manifold is a pair \((M, \Pi)\) where \( \Pi \) is a bivector field
such that the associated bracket on functions

\[
\{f, g\} := \Pi(df, dg), \quad f, g : M \to \mathbb{R}
\]

satisfies the Jacobi identity. The Hamiltonian vector field of a function \( f \) is
defined as \( X_f := \Pi(df, \cdot) \). This allows us to formulate equations of motion
just as in the symplectic setting, i.e. given a Hamiltonian function \( H \) we
consider the flow of the vector field \( X_H \). The concept of integrable systems
is well understood in the symplectic context. A similar definition is possible
in the Poisson setting and the famous Arnold-Liouville-Mineur theorem on
the semilocal structure of integrable systems has its analogue in the Poisson
context. Both commutative and non-commutative integrable systems on
Poisson manifolds were studied in [LMV11].

**Definition 1** (Non-commutative integrable system on a Poisson manifold).
Let \((M, \Pi)\) be a Poisson manifold of (maximal) rank \( 2r \). An \( s \)-tuple of
functions \( F = (f_1, \ldots, f_s) \) on \( M \) is a non-commutative (Liouville) in-
tegrable system of rank \( r \) on \((M, \Pi)\) if

1. \( f_1, \ldots, f_s \) are independent (i.e. their differentials are independent
   on a dense open subset of \( M \));
2. The functions \( f_1, \ldots, f_r \) are in involution with the functions \( f_1, \ldots, f_s \);
3. \( r + s = \dim M \);
(4) The Hamiltonian vector fields of the functions $f_1, \ldots, f_r$ are linearly independent at some point of $M$.

Viewed as a map, $F : M \to \mathbb{R}^s$ is called the **momentum map** of $(M, \Pi, F)$.

When all the integrals commute, i.e. $r = s$, then we are dealing with the conventional case of a commutative integrable system.

**Example 2** (A generic example). Consider the manifold $\mathbb{T}^r \times \mathbb{R}^s$ with coordinates $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ equipped with the Poisson structure

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \pi'$$

where $\pi'$ is any Poisson structure on $\mathbb{R}^{s-r}$. Then the functions $(p_1, \ldots, p_r, z_1, \ldots, z_s)$ define a non-commutative integrable system of rank $r$.

As we will see in Theorem 3 below, any non-commutative integrable system semilocally takes this form, more precisely in the neighborhood of a regular compact connected level set of its integrals $(f_1, \ldots, f_s)$.

2.1.1. **Standard Liouville tori.** Let $(M, \Pi, F)$ be a non-commutative integrable system of rank $r$. We denote the non-empty subset of $M$ where the differentials $df_1, \ldots, df_s$ (resp. the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_r}$) are independent by $U_F$ (resp. $M_{F,r}$).

On the non-empty open subset $M_{F,r} \cap U_F$ of $M$, the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_r}$ define an integrable distribution of rank $r$ and hence a foliation $\mathcal{F}$ with $r$-dimensional leaves, see [LMV11].

We will only deal with the case where $\mathcal{F}_m$ is compact. Under this assumption, $\mathcal{F}_m$ is a compact $r$-dimensional manifold, equipped with $r$ independent commuting vector fields, hence it is diffeomorphic to an $r$-dimensional torus $\mathbb{T}^r$. The set $\mathcal{F}_m$ is called a **standard Liouville torus** of $F$.

The action-angle coordinate theorem proved in [LMV11] (Theorem 1.1) gives a semilocal description of the Poisson structure around a standard Liouville torus of a non-commutative integrable system:

**Theorem 3** (Action-angle coordinate theorem for non-commutative integrable systems on Poisson manifolds). Let $(M, \Pi, F)$ be a non-commutative integrable system of rank $r$, where $F = (f_1, \ldots, f_s)$ and suppose that $\mathcal{F}_m$ is a standard Liouville torus, where $m \in M_{F,r} \cap U_F$. Then there exist $\mathbb{R}$-valued smooth functions $(p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$, defined in a neighborhood $U$ of $\mathcal{F}_m$, and functions $\phi_{kl} = -\phi_{lk}$, which are independent of $\theta_1, \ldots, \theta_r, p_1, \ldots, p_r$, such that

1. The functions $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ define a diffeomorphism $U \simeq \mathbb{T}^r \times B^s$. 
(2) The Poisson structure can be written in terms of these coordinates as,
\[\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{k,l=1}^{s-r} \phi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l};\]

(3) The leaves of the surjective submersion \(F = (f_1,\ldots,f_s)\) are given by the projection onto the second component \(\mathbb{T}^r \times B^s\), in particular, the functions \(f_1,\ldots,f_s\) depend on \(p_1,\ldots,p_r, z_1,\ldots, z_{s-r}\) only.

The functions \(\theta_1,\ldots,\theta_r\) are called angle coordinates, the functions \(p_1,\ldots,p_r\) are called action coordinates and the remaining coordinates \(z_1,\ldots, z_{s-r}\) are called transverse coordinates.

2.2. \(b\)-Poisson and \(b\)-symplectic manifolds. A symplectic form \(\omega\) induces a Poisson structure \(\Pi\) defined via
\[\Pi(df, dg) = \omega(X_f, X_g)\]
where \(X_f, X_g\) are the Hamiltonian vector fields defined with respect to \(\omega\).

On the other hand, a Poisson structure which does not have full rank everywhere, i.e. the set of Hamiltonian vector fields spans the tangent space at every point, does not induce a symplectic structure. However, if the Poisson structure drops rank in a controlled way as defined below, it is possible to associate a so-called \(b\)-symplectic structure.

**Definition 4** (\(b\)-Poisson structure). Let \((M^{2n}, \Pi)\) be an oriented Poisson manifold. If the map
\[p \in M \mapsto (\Pi(p))^n \in \bigwedge^{2n}(TM)\]
is transverse to the zero section, then \(\Pi\) is called a **\(b\)-Poisson structure** on \(M\). The hypersurface \(Z = \{p \in M | (\Pi(p))^n = 0\}\) is the **critical hypersurface** of \(\Pi\). The pair \((M, \Pi)\) is called a **\(b\)-Poisson manifold**.

It is possible and convenient to work in the “dual” language of forms instead of bivector fields. The object equivalent to a \(b\)-Poisson structure will be a \(b\)-symplectic structure. To define \(b\)-symplectic structures and, in general, \(b\)-forms we introduce the concept of \(b\)-manifolds and the \(b\)-tangent bundle associated to the critical set \(Z\):

**Definition 5.** A **\(b\)-manifold** is a pair \((M, Z)\) of an oriented manifold \(M\) and an oriented hypersurface \(Z \subset M\). A **\(b\)-vector field** on a \(b\)-manifold \((M, Z)\) is a vector field which is tangent to \(Z\) at every point \(p \in Z\).

The set of \(b\)-vector fields is a Lie subalgebra of the algebra of all vector fields on \(M\). Moreover, if \(x\) is a local defining function for \(Z\) on some open set \(U \subset M\) and \((x,y_1,\ldots,y_{N-1})\) is a chart on \(U\), then the set of \(b\)-vector fields on \(U\) is a free \(C^\infty(M)\)-module with basis \((x \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{N-1}})\).

A locally \(C^\infty(M)\)-module has a vector bundle associated to it. We call the vector bundle associated to the sheaf of \(b\)-vector fields the **\(b\)-tangent**
bundle denoted $\mathcal{B}TM$. The $\mathcal{B}$-cotangent bundle $\mathcal{B}T^*M$ is, by definition, the vector bundle dual to $\mathcal{B}TM$.

Given a defining function $f$ for $Z$, let $\mu \in \Omega^1(M \setminus Z)$ be the one-form $\frac{df}{f}$. If $v$ is a $\mathcal{B}$-vector field then the pairing $\mu(v) \in C^\infty(M \setminus Z)$ extends smoothly over $Z$ and hence $\mu$ itself extends smoothly over $Z$ as a section of $\mathcal{B}T^*M$. We will write $\mu = \frac{df}{f}$, keeping in mind that on $Z$ the expression only makes sense when evaluated on $\mathcal{B}$-tangent vectors.

**Definition 6 ($\mathcal{B}$-de Rham-$k$-forms).** The sections of the vector bundle $\Lambda^k(\mathcal{B}T^*M)$ are called $\mathcal{B}$-$k$-forms ($\mathcal{B}$-de Rham-$k$-forms) and the sheaf of these forms is denoted $\mathcal{B}\Omega^k(M)$.

For $f$ a defining function of $Z$ every $\mathcal{B}$-$k$-form can be written as

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M).$$

(1)

The decomposition (1) enables us to extend the exterior $d$ operator to $\mathcal{B}\Omega^k(M)$ by setting

$$d\omega = d\alpha \wedge \frac{df}{f} + d\beta.$$

The right hand side is well defined and agrees with the usual exterior $d$ operator on $M \setminus Z$ and also extends smoothly over $M$ as a section of $\Lambda^{k+1}(\mathcal{B}T^*M)$. Since we have $d^2 = 0$, we can define the differential complex of $\mathcal{B}$-forms, the $\mathcal{B}$-de Rham complex.

**Definition 7.** Let $(M^{2n}, Z)$ be a $\mathcal{B}$-manifold and $\omega \in \mathcal{B}\Omega^2(M)$ a closed $\mathcal{B}$-form. We say that $\omega$ is $\mathcal{B}$-symplectic if $\omega_p$ is of maximal rank as an element of $\Lambda^2(\mathcal{B}T^*_pM)$ for all $p \in M$.

It was shown in [GMP12] that $\mathcal{B}$-symplectic and $\mathcal{B}$-Poisson manifolds are in one-to-one correspondence.

The classical Darboux theorem for symplectic manifolds has its analogue in the $\mathcal{B}$-symplectic case:

**Theorem 8 ($\mathcal{B}$-Darboux theorem [GMP12]).** Let $(M, Z, \omega)$ be a $\mathcal{B}$-symplectic manifold. Let $p \in Z$ be a point and $z$ a local defining function for $Z$. Then, on a neighborhood of $p$ there exist coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z, t)$ such that

$$\omega = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z} dz \wedge dt.$$
Under the Mazzeo-Melrose isomorphism, a $b$-form of degree $p$ has two parts: its first summand, the smooth part, is determined (by Poincaré duality) by integrating the form along any $p$-dimensional cycle transverse to $Z$ (such an integral is improper due to the singularity along $Z$, but the principal value of this integral is well-defined). The second summand, the singular part, is the residue of the form along $Z$.

2.3. $b$-functions. It is convenient to enlarge the set of smooth functions to the set of $b$-functions $bC^\infty(M)$, so that the $b$-form $\frac{df}{f}$ is exact, where $f$ is a defining function for $Z$. We define a $b$-function to be a function on $M$ with values in $\mathbb{R} \cup \{\infty\}$ of the form $c \log |f| + g$, where $c \in \mathbb{R}$ and $g$ is a smooth function. For ease of notation, from now on we identify $\mathbb{R}$ with the completion $\mathbb{R} \cup \{\infty\}$.

We define the differential operator $d$ on this space in the obvious way:

$$d(c \log |f| + g) := \frac{c df}{f} + dg \in b\Omega^1(M),$$

where $dg$ is the standard de Rham derivative.

As in the smooth case, we define the ($b$-)Hamiltonian vector field of a $b$-function $f \in bC^\infty(M)$ as the (smooth) vector field $X_f$ satisfying

$$\iota_{X_f} \omega = -df.$$

Obviously, the flow of a $b$-Hamiltonian vector field preserves the $b$-symplectic form and hence the Poisson structure, so $b$-Hamiltonian vector fields are in particular Poisson vector fields.

2.4. Twisted $b$-cotangent lift. Given a Lie group action on a smooth manifold $M$,

$$\rho : G \times M \to M : (g, m) \mapsto \rho_g(m),$$

we define the cotangent lift of the action to $T^*M$ via the pullback:

$$\hat{\rho} : G \times^{b} T^*M \to^{b} T^*M : (g, p) \mapsto \rho_g^*(p).$$

It is well-known that the lifted action $\hat{\rho}$ is Hamiltonian with respect to the canonical symplectic structure on $T^*M$ (see [GS90]).

We want to view the lifted action as a $b$-Hamiltonian action by means of a construction first described in [KM16].

Consider $T^*S^1$ with standard coordinates $(\theta, a)$. We endow it with the following one-form defined for $a \neq 0$, which we call the logarithmic Liouville one-form in analogy to the construction in the symplectic case: $\lambda_{tw,c} = \log |a| d\theta$ for $a \neq 0$.

Now for any $(n-1)$-dimensional manifold $N$, let $\lambda_N$ be the classical Liouville one-form on $T^*N$. We endow the product $T^*(S^1 \times N) \cong T^*S^1 \times T^*N$ with the product structure $\lambda := (\lambda_{tw,c}, \lambda_N)$ (defined for $a \neq 0$). Its negative differential $\omega = -d\lambda$ extends to a $b$-symplectic structure on the whole manifold and the critical hypersurface is given by $a = 0$. 
Let $K$ be a Lie group acting on $N$ and consider the component-wise action of $G := S^1 \times K$ on $M := S^1 \times N$ where $S^1$ acts on itself by rotations. We lift this action to $T^*M$ as described above. This construction, where $T^*M$ is endowed with the $b$-symplectic form $\omega$, is called the twisted $b$-contangent lift.

If $(x_1, \ldots, x_{n-1})$ is a chart on $N$ and $(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})$ the corresponding chart on $T^*N$ we have the following local expression for $\lambda$

$$\lambda = \log |a| d\theta + \sum_{i=1}^{n-1} y_i dx_i.$$ 

Just as in the symplectic case, this action is Hamiltonian with moment map given by contracting the fundamental vector fields with the Liouville one-form $\lambda$.

3. Non-commutative $b$-integrable systems

In [KMS15] we introduced a definition of integrable systems for $b$-symplectic manifolds, where we allow the integrals to be $b$-functions. Such a “$b$-integrable system” on a $2n$-dimensional manifold consists of $n$ integrals, just as in the symplectic case. Here we introduce the definition for the more general non-commutative case:

**Definition 10** (Non-commutative $b$-integrable system). A non-commutative $b$-integrable system of rank $r$ on a $2n$-dimensional $b$-symplectic manifold $(M^{2n}, \omega)$ is an $s$-tuple of functions $F = (f_1, \ldots, f_r, f_{r+1}, \ldots, f_s)$ where $f_1, \ldots, f_r$ are $b$-functions and $f_{r+1}, \ldots, f_s$ are smooth such that the following conditions are satisfied:

1. The differentials $df_1, \ldots, df_s$ are linearly independent as $b$-cotangent vectors on a dense open subset of $M$ and on a dense open subset of $Z$;
2. The functions $f_1, \ldots, f_r$ are in involution with the functions $f_1, \ldots, f_s$;
3. $r + s = 2n$;
4. The Hamiltonian vector fields of the functions $f_1, \ldots, f_r$ are linearly independent as smooth vector fields at some point of $Z$.

We call the first $r$ functions $(f_1, \ldots, f_r)$ the commuting part of the system and the last $s - r$ functions the non-commuting part.

The case $r = s = n$ where we are dealing with a commutative system was studied in [KMS15].

We denote the non-empty subsets of $M$ where condition (1) resp. (4) are satisfied by $U_F$ resp. $M_{F,r}$. The points of the intersection $M_{F,r} \cap U_F$ are called regular. As in the general Poisson case, the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_r}$ fields define an integrable distribution of rank $r$ on this set and we denote the corresponding foliation by $\mathcal{F}$. If the leaf through a point $m \in M$ is compact, then it is an $r$-torus (“Liouville torus”), denoted $\mathcal{F}_m$. 
Remark 11. In the symplectic case, if the differentials $df_i(i = 1, \ldots, r)$ are linearly independent at a point $p$, then also the corresponding Hamiltonian vector fields $X_{f_i}$ are independent at $p$. However, the situation is more delicate in the $b$-symplectic case. The differentials $df_i$ are $b$-one-forms. At a point $p$ where the $df_i$ are independent as $b$-cotangent vectors, the corresponding Hamiltonian vector fields $X_{f_i}$ are independent at $p$ as $b$-tangent vectors. However, for $p \in Z$ the natural map $bTM|_p \to TZ|_p$ is not injective and therefore we cannot guarantee independence of the $X_{f_i}$ as smooth vector fields. This is why the condition (4) is needed. As an example, consider $\mathbb{R}^2$ with standard coordinates $(t, z)$ and $b$-symplectic structure $\frac{1}{t} dt \wedge dz$.

Then the function $z$ has a differential $dz$ which is non-zero at all points of $\mathbb{R}^2$, but the Hamiltonian vector field of $z$ is $t \frac{\partial}{\partial t}$ and vanishes along $Z = \{t = 0\}$. We do not allow this kind of systems in our definition, since we are interested precisely in the dynamics on $Z$ and the existence of $r$-dimensional Liouville tori there. We remark that the definition has already been given in an analogous way for general Poisson manifolds in [LMVT1].

4. Examples of (non-commutative) $b$-integrable systems

4.1. Non-commutative integrable systems on manifolds with boundary. In [KMS15] we introduced new examples of integrable systems using existing examples on manifolds with boundary. We can reproduce a similar scheme in the non-commutative case. As a concrete example, let the manifold with boundary be $M = N \times H_+$, where $(N, \omega_N)$ is any symplectic manifold and $H_+$ is the upper hemisphere including the equator. We endow the interior of $H_+$ with the symplectic form $\frac{1}{h} dh \wedge d\theta$, where $(h, \theta)$ are the standard height and angle coordinates and the interior of $M$ with the corresponding product structure. Now let $(f_1, \ldots, f_s)$ be a non-commutative integrable system of rank $r$ on $N$. Then on the interior of $M$ we can, for instance, define the following (smooth) non-commutative integrable system:

$$(\log|h|, f_1, \ldots, f_s)$$

Taking the double of $M$ we obtain a non-commutative $b$-integrable system on $N \times S^2$.

4.2. Examples coming from $b$-Hamiltonian $\mathbb{T}^r$-actions. In [Bo03] it is shown how to construct integrable systems from the Hamiltonian action of a Lie group $G$ on a symplectic manifold $M$: Let $\mu : M \to \mathfrak{g}^*$ be the moment map of the action and consider the algebra of functions on $M$ generated by $\mu$-basic functions and $G$-invariant functions. Then under certain assumptions, this algebra is complete in the sense of [Bo03], Definition 1.1 therein. This result is the content of Theorem 2.1 in [Bo03]. In our terminology, this means that the algebra of functions admits a basis of functions $f_1, \ldots, f_s$ which form a non-commutative integrable system on $M$. The assumptions
needed for this to hold are satisfied in particular when the action is proper, which is the case for any compact Lie group $G$.

This result can be used in the $b$-symplectic case to semilocally construct a non-commutative $b$-integrable system on a $b$-symplectic manifold $M^{2n}$ with an effective Hamiltonian $T^r$-action as follows: Let us denote the critical hypersurface of $M$ by $Z$ and assume $Z$ is connected. Let $t$ be a defining function for $Z$ and assume $Z$ is connected. Let $t$ be a defining function for $Z$. A Hamiltonian $T^r$-action on a $b$-symplectic manifold, by definition, satisfies that the $b$-one-form $\iota_X \# \omega$ is exact for all $X \in t$. We consider an action with the property that, moreover, for some $X \in t$ the $b$-one-form $\iota_X \# \omega$ is a genuine $b$-one-form, i.e. not smooth. Then the following proposition proved in [GMPS13] about the “splitting” of the action holds: The critical hypersurface $Z$ is a product $L \times S^1$, where $L$ is a symplectic leaf inside $Z$ and in a neighborhood of $Z$ there is a splitting of the Lie algebra $t \simeq t_Z \times \langle X \rangle$ such that the $T^r_Z^{-1}$-action on $Z$ induces a Hamiltonian $T^r_{Z^{-1}}$-action on $L$. Let $\mu_L : L \to t^*_Z$ be the moment map of the latter. Then on a neighborhood $L \times S^1 \times ((-\varepsilon, \varepsilon) \setminus \{0\}) \simeq U \subset M$ of $Z$ the $T^r$-action has moment map

$$
\mu_{U \setminus Z} : L \times S^1 \times ((-\varepsilon, \varepsilon) \setminus \{0\}) \to t^* \simeq t^*_Z \times \mathbb{R}
$$

$$
(\ell, p, t) \mapsto (\mu_L(\ell), c \log |t|).
$$

Let $(f_1, \ldots, f_s)$ be the non-commutative integrable system induced on $L$ by applying the theorem in [Bo03] to the $T^r$-action on $L$. This system has rank $r-1$. On a neighborhood $L \times \{\delta < \theta < \delta\} \times \{-\varepsilon < t < \varepsilon\}$ it extends to a non-commutative $b$-integrable system $(\log |t|, f_1, \ldots, f_s)$ of rank $r$. The Liouville tori of the system are the orbits of the action.

4.3. The geodesic flow. A special case of a $T^r$-action is obtained in the case of a Riemannian manifold $M$ which is assumed to have the property that all its geodesics are closed. These manifolds are called P-manifolds. In this case the geodesics admit a common period (see e.g. [Be12], Lemma 7.11); hence their flow induces an $S^1$-action on $M$. In the same way the standard cotangent lift induces a system on $T^*M$ we can use the twisted $b$-cotangent lift (see subsection 2.4) to obtain a Hamiltonian $S^1$-action on $T^*M$ and hence a non-commutative $b$-integrable system on $T^*M$. In dimension two, examples of P-manifolds are Zoll and Tannery surfaces (see Chapter 4 in [Be12]).

4.4. The Galilean group. The Galilean group has its physical origin in the (non-relativistic) transformations between two reference frames which differ by relative motion at a constant velocity $b$. Together with spatial rotations and translations in time and space, this is the so-called (in)homogeneous Galilean group $G$. We now present in detail this example as a non-commutative integrable system, see also [MM16].

We consider the evolution space

$$
V = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (t, x, y),
$$

\begin{aligned}
\mu \circ j : L \times S^1 \times ((-\varepsilon, \varepsilon) \setminus \{0\}) &
\end{aligned}
where \( t \in \mathbb{R} \) is time and \( x, y \in \mathbb{R}^3 \) are the position and velocity respectively.

The Galilean group can be viewed as a Lie subgroup of \( \text{GL}(\mathbb{R}, 5) \) consisting of matrices of the form
\[
\begin{pmatrix}
A & b & c \\
0 & 1 & e \\
0 & 0 & 1
\end{pmatrix}, \quad A \in \text{SO}(3), b \in \mathbb{R}^3, c \in \mathbb{R}^3, e \in \mathbb{R}.
\tag{2}
\]

If we denote the matrix above by \( a \) then the action \( a_V \) of the Galilean group on \( V \) is defined as follows:
\[ a_V(t, x, v) = (t^*, x^*, y^*) \]
where \( t^* = t + e, \quad x^* = Ax + bt + c, \quad y^* = Ay + b. \)

The Lie algebra \( \mathfrak{g} \) of \( G \) is given by the set of matrices [S70]:
\[
\begin{pmatrix}
\omega & \beta & \gamma \\
0 & 0 & \epsilon \\
0 & 0 & 0
\end{pmatrix}, \quad \epsilon \in \mathbb{R}, \omega \in \mathbb{R}^3, \beta \in \mathbb{R}^3, \gamma \in \mathbb{R}^3.
\]

Here, \( j \) is the map that identifies \( \mathbb{R}^3 \) with \( \mathfrak{so}(3) \). Now instead of letting \( G \) act on the evolution space \( \mathbb{R}^7 \), we consider the action on the “space of motions” \( \mathbb{R}^3 \times \mathbb{R}^3 \), which is obtained by fixing time, \( t = t_0 \). This space is symplectic with the canonical symplectic form and the action of \( G \) on it is Hamiltonian.

In the literature the following integrals of the action are considered [S70]:
Consider the basis of \( \mathfrak{g} \) given by the union of the standard basis on each of its components \( \mathfrak{so}(3), \mathbb{R}^3 \) (corresponding to spatial translation \( \gamma \)), \( \mathbb{R} \) (corresponding to time translation \( \epsilon \)) and the Galilei boost Lie algebra \( \mathbb{R}^3 \) (corresponding to the shift in velocity \( \beta \)). The corresponding integrals are, respectively, the components of the angular momentum \( J = x \times y \), velocity vector \( y \) and position vector \( x \) and the energy \( E \). This system is non-commutative.

We want to investigate the action of certain subgroups of \( G \) and construct \( b \)-versions of the integrable systems. We will consider the space of motions \( \mathbb{R}^6 \) with coordinates \( (x, y) \) as described above and time \( t = 0 \).

**Subgroup given by** \( A = \text{Id} \). First, consider the subgroup of matrices of the form (2) where \( A \) is the identity matrix \( \text{Id} \in \text{SO}(3) \). Then we have an action of \( \mathbb{R}^6 \) on itself; in coordinates \( (x, y) \) as above the action consists of shifts in the \( x \) and \( y \) directions. This action is Hamiltonian with moment map and given by the full set of coordinates \( (x_1, x_2, x_3, y_1, y_2, y_3) \). Clearly, this defines a non-commutative integrable system (of rank zero).

**Subgroup \( \text{SO}(3) \times \mathbb{R}^3 \).** Now let \( c, e \) be constant; for the sake of simplicity we assume they are equal to zero. Consider the subgroup of \( G \) where only \( A \in \text{SO}(3) \) and \( b \in \mathbb{R}^3 \) vary. Then the action on \( \mathbb{R}^6 \) is given by
\[ A \cdot (x, y) = (Ax, Ay + b). \tag{3} \]

First we want to see that the \( \text{SO}(3) \)-action is Hamiltonian. Consider the standard basis of the Lie algebra \( \mathfrak{so}(3) \) corresponding under \( j \) to the unit
vectors in $\mathbb{R}^3$:

\[
e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

On $\mathbb{R}^3$ they describe rotations around the $x_1$, $x_2$- and $x_3$-axis respectively. The corresponding fundamental vector fields on $\mathbb{R}^6$ are

\[
e_1^\# = x_3 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_3} - x_2 \frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial y_2},
\]

\[
e_2^\# = x_1 \frac{\partial}{\partial x_3} - y_3 \frac{\partial}{\partial y_1} - x_3 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_3},
\]

\[
e_3^\# = x_2 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_2} - x_1 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_1}.
\]

One checks that these vector fields are Hamiltonian with respect to the following functions:

\[
f_1 = x_2 y_3 - x_3 y_2, \quad f_2 = x_3 y_1 - x_1 y_3, \quad f_3 = x_1 y_2 - x_2 y_1.
\]

Note that the $f_i$ are the components of angular momentum $J = x \times y$. Hence we have seen that the SO(3)-action is Hamiltonian. The commutators are:

\[
\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2}) = x_1 y_2 - x_2 y_1 = f_3,
\]

and similarly $\{f_2, f_3\} = f_1$ and $\{f_3, f_1\} = f_2$.

Since the $f_i$ do not commute we need additional functions to define an integrable system on $\mathbb{R}^6$. This is where the $\mathbb{R}^3$ action, given by the parameter $b$ in Equation (3) comes into play. It has fundamental vector fields $\frac{\partial}{\partial y_i}$ and the corresponding Hamiltonian functions are the coordinates $x_i$. Together with the integrals $f_i$ they form a non-commutative integrable system $(f_1, f_2, f_3, x_1, x_2, x_3)$ of rank zero.

**Subgroup $S^1 \times \mathbb{R}^3 \times \mathbb{R}^3$.** Above we have studied the SO(3) action on $\mathbb{R}^6$. Now we restrict to the $S^1$-subgroup of SO(3) given by rotations around the $x_1$- and $y_1$-axis. The associated integral is $f_1 = x_2 y_3 - x_3 y_2$. To obtain a non-commutative integrable system of non-zero rank, we can e.g. add the functions $x_2, x_3, y_2$, which do not commute with $f_1$, and the function $y_1$, which commutes with all the other functions. Hence we have obtained a non-commutative integrable system $(y_1, f_1, x_2, x_3, y_2)$ of rank one.

**Some $b$-versions of these constructions.** We view $\mathbb{R}^6$ as a $b$-symplectic manifold with critical hypersurface given by $Z = \{ y_1 = 0 \}$ and canonical $b$-symplectic structure

\[
\frac{dy_1}{y_1} \wedge dx_1 + \sum_{i=2}^{r} dy_i \wedge dx_i.
\]

We want to see if the actions of the subgroups above can be seen as Hamiltonian actions on the $b$-symplectic manifold $\mathbb{R}^6$ (i.e. their fundamental vector fields are Hamiltonian with respect to the $b$-symplectic structure). We treat the above cases one by one:
• The system \((x_1, x_2, x_3, y_1, y_2, y_3)\) translates into the non-commutative \(b\)-integrable system \((x_1, x_2, x_3, \log |y_1|, y_2, y_3)\), i.e. the Hamiltonian vector fields with respect to the \(b\)-symplectic structure are the same and the system fulfils the required independence and commutativity properties.

• The \(\text{SO}(3) \times \mathbb{R}^3\) action with moment map \((f_1, f_2, f_3, x_1, x_2, x_3)\) is \textit{not} Hamiltonian with respect to the \(b\)-symplectic structure. Indeed, away from \(Z\), the fundamental vector field of the \(\text{SO}(3)\)-action above associated to the Lie algebra element \(e_2\) has Hamiltonian function

\[
x_3 \log |y_1| - x_1 y_3,
\]

but this does not extend to a \(b\)-function on \(\mathbb{R}^6\).

• The system \((y_1, f_1, x_2, x_3, y_2)\) translates into the non-commutative \(b\)-integrable system \((\log |y_1|, f_1, x_2, x_3, y_2)\); the induced action is the same as in the smooth case. On the other hand, the smooth system where we replace \(y_1\) by \(x_1\), i.e. \((x_1, f_1, x_2, x_3, y_2)\), does not have such an analogue in the \(b\)-setting. Indeed, with respect to the \(b\)-symplectic form, the Hamiltonian vector field of the first function \(x_1\) is \(y_1\) and vanishes on \(Z\), so the Hamiltonian vector fields of these functions are nowhere independent on \(Z\).

5. Action-angle coordinates for non-commutative \(b\)-integrable systems

In Theorem 8 we recalled the action-angle coordinate theorem for non-commutative integrable systems on Poisson manifolds, which was proved in [LMV11]. For \(b\)-symplectic manifolds and the commutative \(b\)-integrable systems defined there, we have proved an action-angle coordinate theorem [KMS15], which is similar to the symplectic case in the sense that even on the hypersurface \(Z\) where the Poisson structure drops rank there is a foliation by Liouville tori (with dimension equal to the rank of the system) and a semi-local neighborhood with “action-angle coordinates” around them. The main goal of this paper is to establish a similar result in the non-commutative case, proving the existence of \(r\)-dimensional invariant tori on \(Z\) and action-angle coordinates around them.

5.1. Cas-basic functions. Consider a non-commutative \(b\)-integrable system \(F\) on any Poisson manifold \((M, \Pi)\), where we denote the Poisson bracket by \(\{\cdot, \cdot\}\). Let \(V := F(M) \cap \mathbb{R}^s\) be the “finite” target space of the integrals \(F\). If we want to emphasize the functions \(F\) we are referring to, we will also write \(V_F\). The space \(V\) inherits a Poisson structure \(\{\cdot, \cdot\}_V\) satisfying the following property:

\[
\{g, h\}_V \circ F = \{g \circ F, h \circ F\},
\]

where \(g, h\) are functions on \(V\). Note that the values of the brackets \(\{f_i, f_j\}\) on \(M\) uniquely define the Poisson bracket \(\{\cdot, \cdot\}_V\).
An $F$-basic function on $M$ is a function of the form $g \circ F$. The Poisson structure $\{\cdot, \cdot\}_V$ allows us to define the following important class of functions:

**Definition 12 (Cas-basic function).** An $F$-basic function $g \circ F$ is called **Cas-basic** if $g$ is a Casimir function with respect to $\{\cdot, \cdot\}_V$, i.e. the Hamiltonian vector field of $g$ on $V$ is zero.

We recall the following characterisation of Cas-basic functions proved in [LMV11] in the setting of integrable systems on Poisson manifolds. The proof in the $b$-case is the same.

**Proposition 13.** A function is Cas-basic if and only if it commutes with all $F$-basic functions.

### 5.2. Normal forms for non-commutative $b$-integrable systems.

**Definition 14 (Equivalence of non-commutative $b$-integrable systems).** Two non-commutative $b$-integrable systems $F$ and $F'$ are equivalent if there exists a Poisson map $\mu : V_F \rightarrow V_{F'}$ taking one to the other: $F' = \mu \circ F$. Here, $\mu$ is a Poisson map with respect to the Poisson structures induced on $V_F$ and $V_{F'}$ as defined in the previous section.

We will not distinguish between equivalent systems: if the action-angle coordinate theorem that we will prove holds for one system then it holds for all equivalent systems too.

We prove a first “normal form” result for non-commutative $b$-integrable systems:

**Proposition 15.** Let $(M, \omega)$ be a $b$-symplectic manifold of dimension $2n$ with critical hypersurface $Z$. Given a non-commutative $b$-integrable system $F = (f_1, \ldots, f_s)$ of rank $r$ there exists an equivalent non-commutative $b$-integrable system of the form $(\log |t|, f_2, \ldots, f_s)$ where $t$ is a defining function of $Z$ and the functions $f_2, \ldots, f_s$ are smooth.

**Proof.** First, assume that one of the functions $f_1, \ldots, f_r$ is a genuine $b$-function, without loss of generality $f_1 = g + c \log |t'|$ where $c \neq 0$ and $t'$ a defining function of $Z$. Dividing $f_1$ by the constant $c$ and replacing the defining function $t'$ by $t := e^c t'$, we can restrict to the case $f_1 = \log |t|$. We subtract an appropriate multiple of $f_1$ from the other functions $f_2, \ldots, f_s$ so that they become smooth. Note that this does not affect their independence nor the commutativity condition for $f_1, \ldots, f_r$, since $f_1$ commutes with all the integrals. Also, since these operations do not affect the non-commutative part of the system, the induced Poisson bracket on the target space (cf. Section 5.1) remains unchanged. Hence we have obtained an equivalent $b$-integrable system of the desired form.

If all the functions $f_1, \ldots, f_s$ are smooth then from the independence of $df_i$ ($i = 1, \ldots, s$) as $b$-one-forms on the set of regular points $U_F \cap M_{F,r}$ it
follows that
\[ df_1 \wedge \ldots \wedge df_s \wedge dt \neq 0 \in \Omega^*_p \quad \text{for } p \in \mathcal{U}_F \cap M_{F,r}, \]  
where \( t \) is a defining function of \( Z \). Therefore the functions \( f_1, \ldots, f_s, t \) define a submersion on \( \mathcal{U}_F \cap M_{F,r} \) whose level sets are \((r - 1)\)-dimensional.

On the other hand, the Hamiltonian vector fields \( X_{f_1}, \ldots, X_{f_r} \) are linearly independent (on \( \mathcal{U}_F \cap M_{F,r} \)) and tangent to the leaves of this submersion, because \( f_1, \ldots, f_r \) commute with all \( f_j, j = 1, \ldots, s \) and also with \( t \), since any Hamiltonian vector field is tangent to \( Z \). Contradiction. \( \square \)

**Remark 16.** Recall that the Liouville tori of a non-commutative \( b \)-integrable system \( F \) are, by definition, the leaves of the foliation induced by \( X_{f_i}, i = 1, \ldots, r \) on \( \mathcal{U}_F \cap M_{F,r} \). A Liouville torus that intersects \( Z \) lies inside \( Z \), since the Hamiltonian vector fields are Poisson vector fields and therefore tangent to \( Z \). Moreover, since at least one of the first \( r \) integrals \( f_1, \ldots, f_r \) has non-vanishing “log” part, the Liouville tori inside \( Z \) are transverse to the symplectic leaves.

We now prove a normal form result which holds semilocally around a Liouville torus. It describes the topology of the system: we will see that semilocally the foliation of Liouville tori is a product \( \mathbb{T}^r \times B^s \), but the result does not yet give information about the Poisson structure.

**Proposition 17.** Let \( m \in Z \) be a regular point of a non-commutative \( b \)-integrable system \( (M, \omega, F) \). Assume that the integral manifold \( F_m \) through \( m \) is compact (i.e. a torus \( \mathbb{T}^r \)). Then there exist a neighborhood \( U \subset \mathcal{U}_F \cap M_{F,r} \) of \( F_m \) and a diffeomorphism
\[ \phi : U \simeq \mathbb{T}^r \times B^s, \]
which takes the foliation \( F \) induced by the system to the trivial foliation \( \{ \mathbb{T}^r \times \{ b \} \}_{b \in B^s} \).

**Proof.** As described in the previous proposition, we can assume that our system has the form \((\log |t|, f_2, \ldots, f_s)\) where \( f_2, \ldots, f_s \) are smooth. Consider the submersion
\[ \tilde{F} := (t, f_2, \ldots, f_s) : \mathcal{U}_F \to \mathbb{R}^s \]
which has \( r \)-dimensional level sets. The Hamiltonian vector fields \( X_{f_1}, \ldots, X_{f_r} \) are tangent to the level sets. By comparing dimensions we see that the level sets of \( \tilde{F} \) are precisely the Liouville tori spanned by \( X_{f_1}, \ldots, X_{f_r} \).

Now, as described in [LMV11](Prop. 3.2) for classical non-commutative integrable systems, choosing an arbitrary Riemannian metric on \( M \) defines a canonical projection \( \psi : U \to F_m \). Setting \( \phi := \psi \times \tilde{F} \) we have a commuting
The change does not affect the Poisson structure on the target space. The commuting diagram (5) implies that
\[ F = (\log |\pi_1|, \pi_2, \ldots, \pi_s) \circ \phi =: \pi' \]
so the Poisson structure on the target space \( V = F(U) = \pi'(\mathbb{T}^r \times B^s) \)
induced by \( F \) and \( \pi' \) is the same.

The upshot is that for the semi-local study of non-commutative \( b \)-integrable systems around a Liouville torus we can restrict our attention to systems on \( (\mathbb{T}^r \times B^s, \omega) \) where \( \omega \) is the \( b \)-symplectic structure induced by the diffeomorphism \( \phi \) in the proof above and where the integrals \( F = (f_1, \ldots, f_s) \) are given by
\[ f_1 = \log |\pi_1|, f_2 = \pi_2, \ldots, f_s = \pi_s, \]
where \( \pi_1, \ldots, \pi_s \) are the projections on to the components of \( B^s \) and where we assume that the \( b \)-symplectic structure has exceptional hypersurface \( \{\pi_1 = 0\} \). Also, we can assume that the system is regular on the whole manifold \( M = \mathbb{T}^r \times B^s \). We refer to this system as the standard non-commutative \( b \)-integrable system on \( \mathbb{T}^r \times B^s \).

**Remark 18.** The previous result gives a semilocal description of the manifold and the integrals. However, no information is given about the symplectic structure. In contrast, the action-angle coordinate theorem will specify the integrable system with respect to the canonical \( b \)-symplectic form (\( b \)-Darboux form) on \( \mathbb{T}^r \times B^s \).

### 5.3. Darboux-Carathéodory theorem

The following is a key ingredient for the proof of the action-angle coordinate theorem. It tells us that we can locally extend a set of independent commuting functions to a \( b \)-Darboux chart.

**Lemma 19 (Darboux-Carathéodory theorem for \( b \)-integrable systems).** Let \( m \) be a point lying inside the exceptional hypersurface \( Z \) of a \( b \)-symplectic manifold \( (M^{2n}, \omega) \). Let \( t \) be a local defining function of \( Z \) around \( m \). Let \( f_1, \ldots, f_k \) be a set of commuting \( C^\infty \) functions with differentials that are linearly independent at \( m \) as elements of \( {}^bT^*_m(M) \). Then there exist,
on a neighborhood $U$ of $m$, functions $g_1, \ldots, g_k, t, p_2, \ldots, p_{n-k}, q_1, \ldots, q_{n-k}$, such that

(a) The $2n$ functions $(f_1, g_1, \ldots, f_k, g_k, t, q_1, p_1, q_2, \ldots, p_{n-k}, q_{n-k})$ form a system of coordinates on $U$ centered at $m$.

(b) The $b$-symplectic form $\omega$ is given on $U$ by

$$\omega = \sum_{i=1}^{k} df_i \wedge dg_i + \frac{1}{t} dt \wedge dq_1 + \sum_{i=2}^{n-k} dp_i \wedge dq_i.$$

Proof. Let us denote the $b$-Poisson structure dual to $\omega$ by $\Pi$. From the Darboux-Carathéodory Theorem for non-commutative integrable systems on Poisson manifolds it follows that on a neighborhood $U$ of $m$ we can complete the functions $f_1, \ldots, f_k$ to a coordinate system $(f_1, g_1, \ldots, f_k, g_k, z_1, \ldots, z_{2n-2r+2})$ centred at $m$ such that the $b$-Poisson structure reads

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial f_i} \wedge \frac{\partial}{\partial g_i} + \sum_{i,j=1}^{2n-2k} \phi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

for some functions $\phi_{ij}$. The image of the coordinate functions is an open subset of $\mathbb{R}^{2n}$; we can assume that it is a product $U_1 \times U_2$ where $U_2$ corresponds to the image of $z_1, \ldots, z_{2n-2r}$. Then

$$\Pi_2 = \sum_{i,j=1}^{2n-2r+2} \phi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

is a $b$-Poisson structure on $U_2$ and hence by the $b$-Darboux theorem (Theorem 8), there exist coordinates on $U_2$

$$(t, q_1, p_2, q_2, \ldots, p_{n-k}, q_{n-k}),$$

where $t$ is the local defining function for $Z$ that we fixed in the beginning, such that

$$\Pi_2 = t \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial q_1} + \sum_{i=2}^{n-r} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}.$$

The result follows immediately. \qed

Remark 20. A different proof can be given using the tools of [KMS15].

5.4. Action-angle coordinates. Let $(M^{2n}, \omega, F)$ be a non-commutative $b$-integrable system of rank $r$. Let $p \in M_{F,r} \cap \mathcal{U}_F$ be a regular point of the system lying inside the critical hypersurface and let $\mathcal{F}_p$ be the Liouville torus passing through $p$. For a semilocal description of the system around $\mathcal{F}_p$, by Proposition 17 we can assume that we are dealing with the “standard model” of a non-commutative $b$-integrable system, i.e. the manifold is the cylinder $\mathbb{T}^r \times B^s$ with some $b$-symplectic form $\omega$ whose critical hypersurface is $Z = \{ \pi_1 = 0 \} = \mathbb{T}^r \times \{ 0 \} \times B^{s-1}$ and the integrals are $f_1 = \log |\pi_1|, f_i = \pi_i, i = 2, \ldots, r$. Let $c$ be the modular period of $Z$. 
Theorem 21. Then on a neighborhood $W$ of $\mathcal{F}_m$ there exist $\mathbb{R}/\mathbb{Z}$-valued smooth functions
\[ \theta_1, \ldots, \theta_r \]
and $\mathbb{R}$-valued smooth functions
\[ t, a_2, \ldots, a_r, p_1, \ldots, p_\ell, q_1, \ldots, q_\ell \]
where $\ell = n - r = \frac{s - r}{2}$ and $t$ is a defining function of $Z$, such that
1. The functions $(\theta_1, \ldots, \theta_r, t, a_2, \ldots, a_r, p_1, \ldots, p_{n-r}, q_1, \ldots, q_{n-r})$ define a diffeomorphism $W \simeq \mathbb{T}^r \times B^s$.
2. The $b$-symplectic structure can be written in terms of these coordinates as
\[ \omega = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^r d\theta_i \wedge da_i + \sum_{k=1}^\ell dp_k \wedge dq_k. \]
3. The leaves of the surjective submersion $F = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbb{T}^r \times B^s$, in particular, the functions $f_1, \ldots, f_s$ depend on $t, a_2, \ldots, a_r, p_1, \ldots, p_\ell, q_1, \ldots, q_\ell$ only.

The functions $\theta_1, \ldots, \theta_r$ are called angle coordinates, the functions $t, a_2, \ldots, a_r$ are called action coordinates and the remaining coordinates $p_1, \ldots, p_{n-r}, q_1, \ldots, q_{n-r}$ are called transverse coordinates.

We will need the following two lemmas for the proof of this theorem:

Lemma 22. Let $F : M \to \overline{\mathbb{R}}^s$ be an $s$-tuple of $b$-functions on the $b$-symplectic manifold $M = \mathbb{T}^r \times B^s$. If the coefficients of a vector field of the form $Z = \sum_{j=1}^r \psi_j X_{f_j}$ are $F$-basic and the vector field has period one, then the coefficients are Cas-basic.

Proof. The proof is exactly the same as in [LMV11] replacing Hamiltonian by $b$-Hamiltonian vector field. □

The following lemma was proved in [LMV11] (see Claim 2),

Lemma 23. If $Y$ is a complete vector field of period one and $P$ is a bivector field for which $L_Y^2 P = 0$, then $L_Y P = 0$.

We can now proceed with the proof of Theorem 21.

Proof. (of Theorem 21) In the first step we perform “uniformization of periods” similar to [LMV11] and [KMS15]. The joint flow of the vector fields $X_{f_1}, \ldots, X_{f_r}$ defines an $\mathbb{R}^r$-action on $M$, but in general not a $\mathbb{T}^r$-action, although it is periodic on each of its orbits $\mathbb{T}^r \times \{\text{const}\}$.
Denoting the time-$s$ flow of the Hamiltonian vector field $X_f$ by $\Phi^s_{X_f}$, the joint flow of the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_r}$ is

$$\Phi : \mathbb{R}^r \times (\mathbb{T}^r \times B^s) \to \mathbb{T}^r \times B^s$$

$$((s_1, \ldots, s_r), (x, b)) \mapsto \Phi^s_{X_{f_1}} \circ \cdots \circ \Phi^s_{X_{f_r}}(x, b).$$

Because the $X_{f_i}$ are complete and commute with one another, this defines an $\mathbb{R}^r$-action on $\mathbb{T}^r \times B^s$. When restricted to a single orbit $\mathbb{T}^r \times \{b\}$ for some $b \in B^s$, the kernel of this action is a discrete subgroup of $\mathbb{R}^r$, hence a lattice $\Lambda_b$, called the \textit{period lattice} of the orbit $\mathbb{T}^r \times \{b\}$. Since the orbit is compact, the rank of $\Lambda_b$ is $r$. We can find smooth functions (after shrinking the ball $B^s$ if necessary)

$$\lambda_i : B^s \to \mathbb{R}^r, \ i = 1, \ldots, r$$

such that

- $(\lambda_1(b), \lambda_2(b), \ldots, \lambda_r(b))$ is a basis for the period lattice $\Lambda_b$ for all $b \in B^s$
- $\lambda^1_i$ vanishes along $\{0\} \times B^{s-1}$ for $i > 1$, and $\lambda^1_1$ equals the modular period $c$ along $\{0\} \times B^{s-1}$. Here, $\lambda^j_i$ denotes the $j$th component of $\lambda_i$.

Using these functions $\lambda_i$ we define the “uniformized” flow

$$\tilde{\Phi} : \mathbb{R}^r \times (\mathbb{T}^r \times B^s) \to (\mathbb{T}^r \times B^s)$$

$$((s_1, \ldots, s_r), (x, b)) \mapsto \Phi\left( \sum_{i=1}^r s_i \lambda_i(b), (x, b) \right).$$

The period lattice of this $\mathbb{R}^r$-action is constant now (namely $\mathbb{Z}^r$) and hence the action naturally defines a $\mathbb{T}^r$ action. In the following we will interpret the functions $\lambda_i$ as functions on $\mathbb{T}^r \times B^s$ (instead of $B^s$) which are constant on the tori $\mathbb{T}^r \times \{b\}$.

We denote by $Y_1, \ldots, Y_r$ the fundamental vector fields of this action. Note that $Y_i = \sum_{j=1}^r \lambda^j_i X_{f_j}$. We now use the Cartan formula for $b$-symplectic forms (where the differential is the one of the complex of $b$-forms \cite{GMP12}) to compute the following expression:

$$\mathcal{L}_{Y_i} \mathcal{L}_{Y_i} \omega = \mathcal{L}_{Y_i} (d(i_{Y_i} \omega) + i_{Y_i} d\omega)$$

$$= \mathcal{L}_{Y_i} (d(- \sum_{j=1}^n \lambda^j_i df_j))$$

$$= -\mathcal{L}_{Y_i} \left( \sum_{j=1}^n d\lambda^j_i \wedge df_j \right) = 0$$

\footnote{The decomposition of a $b$-form of degree $k$ as $\omega = \mathcal{F} \wedge \alpha + \beta$ for $\alpha, \beta$ De Rham forms proved in \cite{GMP12} allows to extend the Cartan formula valid for smooth De Rham forms to $b$-forms.}
where in the last equality we used the fact that $\lambda_i^j$ are constant on the level sets of $F$. By applying Lemma 23 this yields $\mathcal{L}_{Y_i}\omega = 0$, so the vector fields $Y_i$ are Poisson vector fields, i.e. they preserve the $b$-symplectic form.

We now show that the $Y_i$ are Hamiltonian, i.e. the ($b$-)one-forms

$$\alpha_i := \iota_{Y_i}\omega = -\sum_{j=1}^r \lambda_i^j df_j, \quad i = 1, \ldots, r,$$

(9)

which are closed (because $Y_i$ are Poisson) have a ($bC^\infty$-)primitive $a_i$. Since $\lambda_i^1$ vanishes along $T^r \times \{0\} \times B^{s-1}$ for $i > 1$, the one-forms $\alpha_i$ defined in Equation (9) and hence the functions $a_i$ are smooth for $i > 1$. On the other hand, $\lambda_i^1$ equals the modular period $c$ along $T^r \times \{0\} \times B^{s-1}$ and therefore $a_1 = c \log |t|$ for some defining function $t$.

We compute the functions $a_2, \ldots, a_r$ explicitly by applying a homotopy formula to the smooth one-forms $\alpha_2, \ldots, \alpha_r$. This not only yields that these one-forms are exact but moreover that their $C^\infty$-primitives $a_2, \ldots, a_r$ are Cas-basic. (For the $b$-function $a_1 = c \log |t|$ this is clear.) This is equivalent to proving that these closed forms are exact for the corresponding subcomplex of Cas-basic $b$-forms. We do this by means of adapted homotopy operators.

Consider the following homotopy formula (see for instance [MS12]):

$$\alpha_i - \phi_0^*(\alpha_i) = I(d(\alpha_i)) + d(I(\alpha_i)), \quad i = 2, \ldots, r$$

where the functional $I$ will be defined below and $\phi_\tau$ is the retraction from $T^r \times B^s$ to $T^r \times \{0\} \times B^{s-r}$:

$$\phi_\tau(x_1, \ldots, x_r, b_1, \ldots, b_r, b_{r+1}, \ldots, b_s) = (x, \tau b_1, \ldots, \tau b_r, b_{r+1}, \ldots, b_s).$$

Note that $\phi_0^*(\alpha_i) = 0$ since for any vector field $X \in \mathcal{X}(T^r \times \{0\} \times B^{s-r})$ we have $\alpha_i(X) = 0$. Recall that $\alpha_i$ is a linear combination of $d\pi_2, \ldots, d\pi_r$ and therefore evaluates to zero for $X$ a linear combination of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}, \frac{\partial}{\partial \pi_{r+1}}, \ldots, \frac{\partial}{\partial \pi_s}$. Therefore the homotopy formula tells us that the Hamiltonian function of $\alpha_i$ ($i = 2, \ldots, r$) is explicitly given by $I(\alpha_i)$, which is defined as follows:

$$I(\alpha_i) = \int_0^1 \phi_\tau^*(\iota_{\xi_\tau}(\alpha_i)).$$

Here $\xi_\tau$ is the vector field associated with the retraction:

$$\xi_\tau = \frac{d\phi_\tau}{d\tau} \circ \phi_\tau^{-1} = \frac{1}{\tau} \sum_{k=1}^s \pi_k \frac{\partial}{\partial \pi_k}.$$

Therefore we have

$$\iota_{\xi_\tau}(\alpha_i) = \frac{1}{\tau} \sum_{j=2}^r \lambda_i^j d\pi_j(\xi_\tau) = \frac{1}{\tau} \sum_{j=2}^r \sum_{k=1}^s \lambda_i^j \pi_k d\pi_j \left(\frac{\partial}{\partial \pi_k}\right) = \frac{1}{\tau} \sum_{j=2}^r \lambda_i^j \pi_j.$$

In the last equality we have used $d\pi_j(\frac{\partial}{\partial \pi_k}) = \delta_{jk}$ for $j > 2$. 


The projections $\pi_j, j = 1, \ldots, r$, are obviously Cas-basic. The functions $\lambda^j_i$ are Cas-basic by Lemma 22. The pullback $\phi^*_r$ does not affect the Cas-basic property since it leaves the non-commutative part of the system invariant. We conclude that the functions $\phi^*_r(\xi_r(\alpha_i))$ and hence $a_1, \ldots, a_r$ are Cas-basic.

We apply the Darboux-Carathéodory theorem for $b$-integrable systems to a point $p \in T^r \times \{0\}$ and the independent commuting smooth functions $a_2, \ldots, a_n$. Then on a neighborhood $U$ of $p$ we obtain a set of coordinates $(t, g_1, a_2, g_2, \ldots, a_r, g_r, q_1, p_1, q_2, p_2, \ldots, q_\ell, p_\ell)$, where $\ell = (s - 2r)/2$, such that

$$\omega|_U = \frac{c}{t} dt \wedge dg_1 + \sum_{i=2}^k da_i \wedge dg_i + \sum_{i=1}^\ell dp_i \wedge dq_i. \quad (10)$$

The idea of the next steps is to extend this local expression to a neighborhood of the Liouville torus using the $T^r$-action given by the vector fields $X_{a_k}$. First, note that the functions $(q_1, p_1, q_2, p_2, \ldots, q_\ell, p_\ell)$ do not depend on $f_i$ and therefore can be extended to the saturated neighborhood $W := \pi^{-1}(\pi(U))$. Note that $Y_i = \frac{\partial}{\partial g_i}$ and therefore the flow of the fundamental vector fields of the $Y_i$-action corresponds to translations in the $g_i$-coordinates. In particular, we can naturally extend the functions $g_i$ to the whole set $W$ as well.

We want to see that the functions

$$t, g_1, a_2, g_2, \ldots, a_r, g_r, q_1, p_1, q_2, p_2, \ldots, q_\ell, p_\ell \quad (11)$$

which are defined on $W$, indeed define a chart there (i.e. they are independent) and that $\omega$ still has the form given in Equation (10).

It is clear that $\{a_i, g_j\} = \delta_{ij}$ on $W$. To show that $\{g_i, g_j\} = 0$, we note that this relation holds on $U$ and flowing with the vector fields $X_{a_k}$ we see that it holds on the whole set $W$:

$$X_{a_k}(\{g_i, g_j\}) = \{\{g_i, g_j\}, a_k\} = \{g_i, \delta_{ij}\} - \{g_j, \delta_{ik}\} = 0.$$ 

This verifies that $\omega$ has the form (10) above and in particular, we conclude that the derivatives of the functions (11) are independent on $W$, hence these functions define a coordinate system.

Since the vector fields $\frac{\partial}{\partial g_i}$ have period one, we can view $g_1, \ldots, g_r$ as $\mathbb{R}\setminus\mathbb{Z}$-valued functions (“angles”) and therefore use the letter $\theta_i$ instead of $g_i$.

Remark 24. In the language of cotangent models introduced in [KM16], this theorem can be expressed as saying that a non-commutative $b$-integrable system is semilocally equivalent given by the the twisted $b$-cotangent lift of the $T^r$-action on itself by translations.

References

[Be12] A. Besse, Manifolds all of whose geodesics are closed., Vol. 93. Springer Science and Business Media, 2012.
[Bo03] Alexey V. Bolsinov, Bozidar Jovanovic, *Non-commutative integrability, moment map and geodesic flows*, Annals of Global Analysis and Geometry 23 (2003), no. 4, 305–322.

[DKM15] A. Delshams, A. Kiesenhofer, E. Miranda, *Examples of integrable and non-integrable systems on singular symplectic manifolds*, arXiv:1512.08293 (2015).

[RLV15] R. L. Fernandes, C. Laurent-Gengoux, and P. Vanhaecke, *Global action-angle variables for non-commutative integrable systems*, arXiv:1503.00084 (2015).

[GLPR14] M. Gualtieri, S. Li, A. Pelayo, T. Ratiu, *Tropical moment maps for toric log symplectic manifolds*, arXiv:1407.3300 (2014).

[GMP11] V. Guillemin, E. Miranda and A.R. Pires, *Codimension one symplectic foliations and regular Poisson structures*, Bulletin of the Brazilian Mathematical Society, New Series 42 (4), 2011, pp. 1–17.

[GMP12] V. Guillemin, E. Miranda, and A. Pires, *Symplectic and Poisson geometry on b-manifolds*. Adv. Math. 264 (2014), 864–896.

[GMPS13] V. Guillemin, E. Miranda, A. Pires, and G. Scott, *Toric actions on b-symplectic manifolds*. Int Math Res Notices 2014:rnu108v1-31.

[GMPS2] V. Guillemin, E. Miranda, A. R. Pires and G. Scott, *Convexity for Hamiltonian torus actions on b-symplectic manifolds*, arXiv:1412.2488, Mathematical Research Letters, to appear, 2016.

[GS90] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*. Second edition. Cambridge University Press, Cambridge, 1990. xii+468 pp. ISBN: 0-521-38990-9.

[KM16] A. Kiesenhofer and E. Miranda, *Cotangent models for integrable systems*, arXiv:1601.05041 to appear at Communications in Mathematical Physics.

[KMS15] A. Kiesenhofer, E. Miranda and G. Scott, *Action-angle variables and a KAM theorem for b-Poisson manifolds*. J. Math. Pures Appl. (9) 105 (2016), no. 1, 66–85.

[LMV11] C. Laurent-Gengoux, E. Miranda, and P. Vanhaecke. *Action-angle coordinates for integrable systems on Poisson manifolds*. Int. Math. Res. Not. IMRN 2011, no. 8, 1839–1869.

[MS12] E. Miranda, R. Solha, *On a Poincaré Lemma for Foliations*. Foliations 2012, World Scientific Publishing, pp. 115.

[MM16] D. Martinez-Torres and E. Miranda, *Weakly Hamiltonian actions*, Journal of Geometry and Physics, published online 2016, doi:10.1016/j.geomphys.2016.04.022.

[N72] N. Nekhoroshev, *Action-angle variables and their generalization*, Trudy Moskov Mat. Obsch., 26 (1972), 181-198 (in Russian). English translation: Trans. Moscow Math. Soc. 26 (1972), 180-198.

[S70] J.-M. Souriau. *Structure des systèmes dynamiques*. (French) Maîtrises de mathématiques Dunod, Paris 1970 xxxii+414 pp. 69.00 (70.00)

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