ASYMPTOTIC PROPERTIES OF DELAYED MATRIX EXPONENTIAL FUNCTIONS VIA LAMBERT FUNCTION

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ABSTRACT. In the case of first-order linear systems with single constant delay and with constant matrix, the application of the well-known “step by step” method (when ordinary differential equations with delay are solved) has recently been formalized using a special type matrix, called delayed matrix exponential. This matrix function is defined on the intervals \((k-1)\tau \leq t < k\tau, k = 0, 1, \ldots\) (where \(\tau > 0\) is a delay) as different matrix polynomials, and is continuous at nodes \(t = k\tau\). In the paper, the asymptotic properties of delayed matrix exponential are studied for \(k \to \infty\) and it is, e.g., proved that the sequence of values of a delayed matrix exponential at nodes is approximately represented by a geometric progression. A constant matrix has been found such that its matrix exponential is the “quotient” factor that depends on the principal branch of the Lambert function. Applications of the results obtained are given as well.

1. Introduction. The well-known “step by step” method is one of the basic concepts for the investigation of linear differential equations and systems with delay. The application of this method to first-order linear systems with single constant delay and with constant matrix of linear terms was formalized by using the notion of delayed matrix exponential \(e^{Bt}\), where \(B\) is a square constant matrix and \(\tau > 0\) is a delay, in [5, 6]. A special delayed matrix function is defined on every interval \((k-1)\tau \leq t < k\tau, k = 0, 1, \ldots\) (where \(\tau > 0\) is a delay) as a matrix polynomial depending on \(B\) and is continuous at nodes \(t = k\tau\). Such a step by step definition complicates its asymptotic analysis. The paper deals with the asymptotic properties of delayed matrix exponential. Proofs of the results derived below make use of the properties of the matrix Lambert function [8]. Therefore, some basic notations and results related to this function are recalled in this part, too. Auxiliary results overviewed in Part 1.1 can be found in [5, 6] and [2]. The results given in Part 1.2 are taken from [3] (see also the original source [8]). New auxiliary results are proved in Part 1.3. In Part 2, auxiliary determinants are computed and the results applied in Part 3 to prove the main result of the asymptotic behavior of a sequence of the

2010 Mathematics Subject Classification. Primary: 34K06; Secondary: 34K25.
Key words and phrases. Lambert function, delayed matrix exponential, asymptotic behavior, principal part, instability.

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ratios of delayed matrix exponentials at adjacent nodes. The sequence of values of delayed matrix exponential at nodes is approximately represented by a geometric progression. A constant matrix is found such that its matrix exponential is the “quotient” factor that depends on the principal branch of the Lambert function. Moreover, some further results on asymptotic properties of delayed matrix exponential are proved. Applications of the results derived are collected in Part 4.

1.1. First-order linear systems. Let $B$ be an $n \times n$ constant matrix, $\Theta$ an $n \times n$ null matrix, $I$ an $n \times n$ unit matrix and let $\tau > 0$ be a constant. The delayed matrix exponential $e^{Bt}_\tau$ of the matrix $B$ is an $n \times n$ matrix function mapping $\mathbb{R}$ to $\mathbb{R}^{n \times n}$, continuous on $\mathbb{R} \setminus \{-\tau\}$ and defined as follows:

$$e^{Bt}_\tau := \begin{cases} \sum_{j=0}^{k} B^j \frac{(t-(j-1)\tau)^j}{j!}, & t \geq -\tau, \\ \Theta, & t < -\tau \end{cases}$$

where $k = \lceil t/\tau \rceil$ is the ceiling function, i.e. the smallest integer greater than or equal to $t/\tau$. The main property of the delayed matrix exponential $e^{Bt}_\tau$ is the following:

$$(e^{Bt}_\tau)' = Be^{B(t-\tau)}_\tau, \quad t \in \mathbb{R} \setminus \{0\}$$

and the matrix $Y(t) = e^{Bt}_\tau$ solves the initial problem for a matrix differential system with a single delay

$$Y'(t) = BY(t-\tau), \quad t \in [0, \infty),$$
$$Y(t) = I, \quad t \in [-\tau, 0].$$

If $\varphi: [-\tau, 0] \to \mathbb{R}^n$ is a continuously differentiable vector-function, then the solution of the initial-value problem

$$y'(t) = By(t-\tau), \quad t \in [0, \infty), \quad (1)$$
$$y(t) = \varphi(t), \quad t \in [-\tau, 0] \quad (2)$$

can be represented in the form

$$y(t) = e^{Bt}_\tau \varphi(-\tau) + \int_{-\tau}^{0} e^{B(t-\tau-s)} \varphi'(s) ds, \quad t \in [-\tau, \infty). \quad (3)$$

Let $A$ be a regular $n \times n$ constant matrix and $AB = BA$. Then, the solution of the initial-value problem

$$y'(t) = Ay(t) + By(t-\tau), \quad t \in [0, \infty), \quad (4)$$
$$y(t) = \varphi(t), \quad t \in [-\tau, 0] \quad (5)$$

is given by the formula

$$y(t) = e^{At}e^{Bt}_\tau \varphi(-\tau) + \int_{-\tau}^{0} e^{At(t-\tau-s)} e^{Bt}_\tau e^{B(t-\tau-s)} e^{At}[\varphi'(s) - A\varphi(s)] ds \quad (6)$$

where $t \in [-\tau, \infty)$ and $B_1 = e^{-At}B$. 

1.2. The Lambert $W$ function. As can easily be seen from the definition, the above delayed matrix function is defined on intervals $(k-1)\tau \leq t < k\tau$, $k = 0, 1, \ldots$ as different matrix polynomials. As mentioned in Introduction, this is the reason why its asymptotic analysis is complicated.

Therefore, it seems to be important to study the sequence \( \{e^{B_{k\tau}}\}_{k=0}^{\infty} \) of values of the delayed exponential of a matrix $B$ at nodes $k\tau$, connecting two different matrix polynomials, as $k \to \infty$. Later, we will prove that, for the special matrix considered, this sequence approximately equals a geometric progression and we will find a constant $n \times n$ matrix $C$ such that its ordinary exponential $e^{C\tau}$ is the “quotient”, i.e., that

\[
\lim_{k \to \infty} e^{B_{(k+1)\tau}} (e^{B_{k\tau}})^{-1} = e^{C\tau},
\]

where $(\cdot)^{-1}$ denotes the inverse matrix, whose existence we assume.

This will be done using the so-called Lambert function (named after Johann Heinrich Lambert, see [8]). Recall its definition and some basic results on the Lambert function (published in [3]).

Lambert defined the function as the inverse to the function $f(w) = we^w$. This means that the Lambert function, usually denoted by $W = W(z)$, is defined implicitly by the equation

\[
z = W(z)e^{W(z)}.
\]

Such a function is multi-valued (except for the point $z = 0$). For real arguments $z = x$ such that $x > -1/e$ and real $W(x)$ satisfying $W(x) > -1$, equation (8) defines a single-valued function $W = W_0(x)$ called the principal branch of the Lambert $W(z)$ function, i.e.,

\[
W_0(x)e^{W_0(x)} = x, \quad x > -1/e.
\]

We prove that the matrix $C$ in (7) is defined by the principal branch $W_0(z)$ of the Lambert $W(z)$ function (see Corollary 1 below).

The Maclaurin expansion of $W_0(x)$ can be found easily being given by the series

\[
W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n,
\]

having the radius of convergence $r = 1/e$. The point $x = 0$ is a point of removable singularity of the function $W_0(x)/x$. It follows from (10) that the Maclaurin expansion of the function

\[
E(x) := \begin{cases} 
W_0(x), & x \neq 0, \\
1, & x = 0,
\end{cases}
\]

i.e.,

\[
E(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^{n-1}
\]

has the same radius of convergence $r = 1/e$. The function $E(x)$ is smooth and infinitely many times differentiable. Moreover, applying the Lagrange inversion
theorem (Lagrange-Bürmann formula), we obtain

$$\left( \frac{W_0(x)}{x} \right)^r = \exp(-rW_0(x)) = \sum_{n=0}^{\infty} \frac{r(n+r)^{n-1}}{n!} (-x)^n. \quad \text{(13)}$$

Differentiating the defining equation (8), we conclude that all branches of \( W(z) \) satisfy the differential equation

$$z(1 + W) \frac{dW}{dz} = W, \quad z \neq 0. \quad \text{(14)}$$

Let \( \lambda \) be a complex number. In determining the asymptotic properties of the exponential function \( \exp(\lambda x) \), where \( x \in \mathbb{R} \), the real part of the complex number \( \lambda \) often plays the principal role because the asymptotic properties differ for \( \Re \lambda x > 0 \) and \( \Re \lambda x < 0 \) and the domains for the real part being positive or negative are in the complex plane for \( \lambda \) “separated” by the set of points where \( \Re \lambda = 0 \). In the definition of the Lambert function by (8), the behavior of the exponential function plays an important role as well.

Define the set of complex numbers such that \( \Re W(z) = 0 \). Assuming \( z = x + iy \) and \( W(z) = u + iv \), from (8), we get

$$x + iy = W(z) = W(z)e^{W(z)} = ive^iv = iv(\cos v + i\sin v) = -v \sin v + iv \cos v,$$

i.e.,

$$x = -v \sin v, \quad \text{(15)}$$

$$y = v \cos v \quad \text{(16)}$$

where \( v \in \mathbb{R} \). Analyzing the part of this curve corresponding to the principal branch \( W_0(x + iy) \), i.e.,

$$x = -v \sin v > -\frac{1}{e}$$

we conclude that (15), (16) is a simple closed curve for the admissible range \( v \in [-\pi/2, \pi/2] \). This curve is depicted in Figure 1. From (15), (16), it is easy to deduce that the real part of the principal branch of the Lambert function is negative for

$$|z| < -\arctan \left( \frac{\Re z}{\Im z} \right). \quad \text{(17)}$$

This domain is bounded by the above curve (see Figure 1). Note that a Lambert \( W \) function cannot be expressed in terms of elementary functions. For more details, see [3].

1.3. Limits with principal part \( W_0 \) of the Lambert function. Let \( k \) be a nonnegative integer. Define a polynomial

$$P_k(x) = \sum_{j=0}^{k} \frac{(k + 1 - j)^j}{j!} x^j. \quad \text{(18)}$$

Then, the formula

$$e^{B^k \tau} = \sum_{j=0}^{k} B^j \frac{(k + 1 - j)^j}{j!} = P_k(B \tau), \quad \text{(19)}$$

where \( B^0 = I \), expressing the values of a delayed matrix exponential at the nodes \( t = k\tau, \ k = 0, 1, 2, \ldots \) holds and can be simply verified using the definition of the delayed matrix exponential.
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\( \text{Re} \, W_0(z) > 0 \)

\( \text{Im} z \)

\( \text{Re} \, W_0(z) < 0 \)

\( -\frac{\pi}{2} \)

\( \text{Re} z \)

Figure 1. The curve \( \text{Re} W_0(z) = 0 \)

Let \( x, \alpha \) and \( \beta \) be real numbers and let \( n \) be a positive integer. The following is a well-known Abel’s extension of the binomial theorem (see, e.g. [1])

\[
(x + \alpha)^n = x^n + \binom{n}{1} \alpha (x + \beta)^{n-1} + \binom{n}{2} \alpha (\alpha - 2\beta) (x + 2\beta)^{n-2} + \cdots + \binom{n}{\ell} \alpha (\alpha - \ell\beta)^{\ell-1} (x + \ell\beta)^{n-\ell} + \cdots + \binom{n}{n-1} \alpha (\alpha -(n-1)\beta)^{n-2} (x + (n-1)\beta) + \alpha (\alpha - n\beta)^{n-1},
\]

which, for \( \alpha \neq 0 \), can be rewritten as

\[
(x + \alpha)^n = \sum_{\ell=0}^{n} \binom{n}{\ell} \alpha (\alpha - \ell\beta)^{\ell-1} (x + \ell\beta)^{n-\ell} \tag{20}
\]

and will be used in the computations below.

Lemma 1.1. Let \( x \in (-1/e, 1/e) \) be fixed. Then,

\[
\lim_{k \to \infty} \frac{P_k(x)}{P_{k+1}(x)} = E(x), \tag{21}
\]

\[
\lim_{k \to \infty} \frac{P_{k+1}(x)}{P_k(x)} = \exp(W_0(x)) \tag{22}
\]

and, for \( l \in \mathbb{N} \), we have

\[
\lim_{k \to \infty} \frac{1}{P_{k+1}(x)} \left( P_k^{(l)}(x) - \sum_{\ell=1}^{l} \binom{l}{\ell} P_k^{(\ell)}(x) E^{(l-\ell)}(x) \right) = E^{(l)}(x). \tag{23}
\]

Proof. We decompose the ratio

\[
\frac{P_k(x)}{P_{k+1}(x)}
\]

into the Maclaurin power series with respect to \( x \)

\[
\frac{P_k(x)}{P_{k+1}(x)} = \sum_{\ell=0}^{\infty} a_\ell x^\ell, \quad a_\ell \in \mathbb{R},
\]
and show that the sum of the first \((k + 1)\) terms of this expansion where \(k \geq 0\) equals a polynomial of \(k\)-th degree (compare (12))

\[
E_k(x) = \sum_{\ell=0}^{k} \frac{(-\ell - 1)^{\ell}}{(\ell + 1)!} x^\ell,
\]

i.e.,

\[
a_\ell = \frac{(-\ell - 1)^{\ell}}{(\ell + 1)!}, \quad \ell = 0, 1, \ldots, k,
\]

and

\[
\frac{P_k(x)}{P_{k+1}(x)} = E_k(x) + \sum_{\ell=k+1}^{\infty} a_\ell x^\ell
\]

or

\[
\frac{P_k(x)}{P_{k+1}(x)} = E_k(x) + O(x^{k+1})
\]

where \(O\) is the Landau order symbol “big” \(O\). We prove this by matching the coefficients at identical powers \(n\), \(n = 0, 1, \ldots, k\) of two polynomials \(E_k(x)\) and \(P_k(x)\). The coefficient at the power \(x^n\) \((0 \leq n \leq k)\) of the product

\[
E_k(x)P_{k+1}(x) = \left(\sum_{\ell=0}^{k} \frac{(-\ell - 1)^{\ell}}{(\ell + 1)!} x^\ell\right) \cdot \left(\sum_{j=0}^{k+1} \frac{(k+2-j)^j}{j!} x^j\right)
\]

can be expressed as

\[
\sum_{\ell=0}^{n} \frac{(-\ell - 1)^{\ell} (k+2-n+\ell)^{n-\ell}}{(\ell + 1)! (n-\ell)!}
\]

\[
= \sum_{\ell=0}^{n} (-1)^{\ell} \frac{(-\ell - 1)^{\ell-1} (k+2-n+\ell)^{n-\ell}}{\ell! (n-\ell)!}
\]

\[
= \frac{1}{n!} \sum_{\ell=0}^{n} (-1)^{n} \binom{n}{\ell} (-\ell - 1)^{\ell-1}(k+2-n+\ell)^{n-\ell}
\]

\[
= (\text{we use identity (20) with } \alpha = -1, \beta = 1, x = k+2-n)
\]

\[
= \frac{(k+1-n)^n}{n!}
\]

and is the same as the coefficient at the power \(x^n\) of the polynomial \(P_k(x)\). Therefore, formula (26) holds with the indicated accuracy. Formula (21) now follows from the property

\[
\lim_{k \to \infty} E_k(x) = E(x).
\]

Formula (22) is a consequence of (21), (11) and (9) since

\[
\lim_{k \to \infty} \frac{P_{k+1}(x)}{P_k(x)} = \frac{1}{\lim_{k \to \infty} \frac{P_k(x)}{E(x)}} = \frac{1}{\frac{P_k(x)}{E(x)}} = \exp(W_0(x)).
\]

Now we will show that (23) holds. Without loss of generality, we assume \(k > l\) in the sequel. Since power series are infinitely many times differentiable within their
interval of convergence, from (24), we have
\[ E_k(x) = E(x) - \sum_{\ell=k+1}^{\infty} \frac{(-\ell - 1)^\ell}{(\ell + 1)!} x^\ell \]
and
\[ E_k^{(l)}(x) = E^{(l)}(x) + O(x^{k-l+1}). \] (27)
Rewriting (25) as
\[ P_k(x) = P_{k+1}(x)E_k(x) + P_{k+1}(x) \sum_{\ell=k+1}^{\infty} a_\ell x^\ell, \] (28)
differentiating (28) \( l \)-times, and using (27) we get
\[ P_k^{(l)}(x) = (P_{k+1}(x)E_k(x))^{(l)} + \left(P_{k+1}(x) \sum_{\ell=k+1}^{\infty} a_\ell x^\ell\right)^{(l)} \]
\[ = \sum_{\ell=0}^{l} \binom{l}{\ell} P_{k+1}^{(l)}(x)E_k^{(l-\ell)}(x) + O(x^{k-l+1}) \]
\[ = \sum_{\ell=0}^{l} \binom{l}{\ell} P_{k+1}^{(l)}(x)E^{(l-\ell)}(x) + O(x^{k-l+1}) \]
or
\[ P_k^{(l)}(x) - \sum_{\ell=1}^{l} \binom{l}{\ell} P_{k+1}^{(l)}(x)E^{(l-\ell)}(x) = P_{k+1}(x)E^{(l)}(x) + O(x^{k-l+1}). \]
Then,
\[ \frac{1}{P_{k+1}(x)} \left(P_k^{(l)}(x) - \sum_{\ell=1}^{l} \binom{l}{\ell} P_{k+1}^{(l)}(x)E^{(l-\ell)}(x)\right) = E^{(l)}(x) + O(x^{k-l+1}) \]
and, taking limit as \( k \to \infty \), we get formula (23).
\[ \square \]
Lemma 1.2. Let \( x \in (-1/e, 1/e) \) be fixed. Then,
\[ \lim_{k \to \infty} P_k(x) \exp(-kW_0(x)) = \frac{1}{E(x)(1 + W_0(x))}. \] (29)
Proof. We can decompose \( \exp(-kW_0(x))P_k(x) \), using (13) and (18), into the Maclaurin power series. In the following decomposition, the first \( (k + 1) \) terms are written exactly.
\[ (\exp(-kW_0(x)))P_k(x) = \left(\sum_{\ell=0}^{\infty} \frac{-k(-k - \ell)^{\ell-1}}{\ell!} x^\ell\right) \left(\sum_{\ell=0}^{k} \frac{(k + 1 - \ell)^\ell}{\ell!} x^\ell\right) \]
\[ = \sum_{l=0}^{k} x^l \sum_{\ell=0}^{l} \frac{-k(-k - \ell)^{\ell-1}}{\ell!} \cdot \frac{(k + 1 - l + \ell)^{l-\ell}}{(l-\ell)!} + O(x^{k+1}) \]
\[ = (\text{we use (20) with } n = l, \alpha = -k, \beta = 1, x = k + 1 - l) \]
\[ = \sum_{l=0}^{k} \frac{(1 - l)^l}{l!} x^l + O(x^{k+1}). \]
For the limit of this product, we obtain
\[
\lim_{k \to \infty} \exp(-kW_0(x))P_k(x) = \sum_{l=0}^{\infty} \frac{(1 - l)^l}{l!} x^l. \tag{30}
\]

Now we put \( r = -1 \) in (13) and develop the Maclaurin power series of the expression (below, the values for \( x = 0 \) are understood as limits for \( x \to 0 \))
\[
-x^2 \frac{d}{dx} \left( \frac{e^{W_0(x)}}{x} \right).
\]

We get
\[
-x^2 \frac{d}{dx} \left( \frac{e^{W_0(x)}}{x} \right) = x^2 \left( \sum_{n=0}^{\infty} \frac{-(1-n)^{n-1}(-x)^n}{n!} \right) = x^2 \sum_{n=0}^{\infty} \frac{(1-n)^{n-1}x^n}{n!} = \sum_{n=0}^{\infty} \frac{(1-n)^{n-1}x^n}{n!}. \tag{31}
\]

Comparing (30) with (31), we conclude that
\[
\lim_{k \to \infty} \exp(-kW_0(x))P_k(x) = -x^2 \frac{d}{dx} \left( \frac{e^{W_0(x)}}{x} \right).
\]

Using (9) and (14), we get
\[
-x^2 \frac{d}{dx} \left( \frac{e^{W_0(x)}}{x} \right) = x^2 \left( \frac{1}{W_0(x)} \right) = x^2 \frac{W_0'(x)}{W_0^2(x)} = \frac{x}{x(1 + W_0(x))} = \frac{1}{E(x)(1 + W_0(x))}. \tag{32}
\]

Now, formula (29) is a consequence of (30)–(32).

Remark 1. As it follows from formula (29) in Lemma 1.2, for fixed \( x \in (-1/e, 1/e) \), we have
\[
P_k(x) \sim \frac{\exp(kW_0(x))}{E(x)(1 + W_0(x))} \tag{33}
\]
where \( k \to \infty \), and
\[
\lim_{k \to \infty} P_k(x)E(x) \frac{1 + W_0(x)}{\exp(kW_0(x))} = 1.
\]

2. Preliminaries. Let us recall that two \( n \times n \) matrices \( A \) and \( B \) are called similar if \( B = P^{-1}_sAP_s \) for some invertible \( n \times n \) matrix \( P_s \) (for properties of matrices used in this part, we refer, e.g. to [4, Chapter V]). Let \( s \) be a positive integer and \( s \leq n \). An \( s \times s \) matrix \( J_{\lambda,s} \)
\[
J_{\lambda,s} = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & \lambda
\end{pmatrix},
\]
where $\lambda$ is a complex number, is called a Jordan block. Any block diagonal matrix whose blocks are Jordan blocks is called a Jordan matrix and any matrix $A$ is similar to an $n \times n$ Jordan matrix

$$J = \text{diag}(J_{\lambda_1, m_1}, J_{\lambda_2, m_2}, \ldots, J_{\lambda_N, m_N}) \quad (34)$$

where, for positive integers $m_i$, $i = 1, 2, \ldots, m_N$, we have $m_1 + m_2 + \cdots + m_N = n$ and $\lambda_i$ are the eigenvalues of $A$ with multiplicities $m_i$. The Jordan matrix $J$ given by (34) is unique up to a permutation of its diagonal blocks. $J$ is called the Jordan normal form of $A$ and, for some suitable invertible $n \times n$ matrix $P$, we have

$$A = P^{-1}JP.$$

For an analytic function with a radius of convergence $r$ given by the series

$$f(z) = \sum_{h=0}^{\infty} a_h z^h$$

and for any matrix $A$ with spectral radius $\rho(A) = \max_{i=1,2,\ldots,m_N} |\lambda_i|$ satisfying $\rho(A) < r$, also the matrix

$$f(A) = \sum_{h=0}^{\infty} a_h A^h = P^{-1}\text{diag}(f(J_{\lambda_1, m_1}), f(J_{\lambda_2, m_2}), \ldots, f(J_{\lambda_N, m_N}))P$$

is defined where the series has the same radius of convergence and the matrices

$$f(J_{\lambda_i, m_i}) = \sum_{h=0}^{\infty} a_h (J_{\lambda_i, m_i})^h, \quad i = 1, 2, \ldots, N,$$

defined by the series with the same radius of convergence $r$ again, satisfy:

$$f(J_{\lambda_i, m_i}) = \begin{pmatrix}
 f(\lambda_i) & f'(\lambda_i) & \cdots & f^{(m_i-2)}(\lambda_i) & f^{(m_i-1)}(\lambda_i) \\
 0 & f(\lambda_i) & \cdots & f^{(m_i-3)}(\lambda_i) & f^{(m_i-2)}(\lambda_i) \\
 0 & 0 & f(\lambda_i) & \cdots & f^{(m_i-4)}(\lambda_i) & f^{(m_i-3)}(\lambda_i) \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & f(\lambda_i) & f'(\lambda_i) \\
 0 & 0 & 0 & \cdots & 0 & f(\lambda_i)
\end{pmatrix}$$

Now we develop matrix analogies of the statements formulated in Lemma 1.1. Let $k$ be a nonnegative integer, $\lambda \in \mathbb{C}$, and $s$ be a positive integer. For $k \geq s$, we define an $s \times s$ matrix

$$P_k(J_{\lambda,s}) = (p_{ij}(k, \lambda, s))^s_{i,j=1}$$
as

\[
P_k(J\lambda,s) = \begin{pmatrix}
P_k(\lambda) & \frac{P_k^{(1)}(\lambda)}{1!} & \frac{P_k^{(2)}(\lambda)}{2!} & \cdots & \frac{P_k^{(s-2)}(\lambda)}{(s-2)!} & \frac{P_k^{(s-1)}(\lambda)}{(s-1)!} \\
0 & P_k(\lambda) & \frac{P_k^{(1)}(\lambda)}{1!} & \cdots & \frac{P_k^{(s-3)}(\lambda)}{(s-3)!} & \frac{P_k^{(s-2)}(\lambda)}{(s-2)!} \\
0 & 0 & P_k(\lambda) & \cdots & \frac{P_k^{(s-4)}(\lambda)}{(s-4)!} & \frac{P_k^{(s-3)}(\lambda)}{(s-3)!} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & P_k(\lambda) & \frac{P_k^{(1)}(\lambda)}{1!} \\
0 & 0 & 0 & \cdots & 0 & P_k(\lambda)
\end{pmatrix}
\]

where the polynomial \( P_k \) is given by formula (18). To avoid possible ambiguities in the following computations, we also define

\[
P_k(J\lambda,0) := (1)
\]

where \( k \) and \( \lambda \) are as above. In what follows, we do not consider zero points of the polynomial \( P_k \), so we will assume \( P_k(\lambda) \neq 0 \). Thus, \( P_k(J\lambda,s) \) is an invertible matrix.

To describe the result of the matrix product

\[
P_k(J\lambda,s) = (p_{ij}^k(J\lambda,s))^s_{i,j=1} := P_k(J\lambda,s)(P_k(J\lambda,s))^{-1}, \tag{35}
\]

we need to define some auxiliary determinants \( M_k(\lambda,s) \). The meaning of \( k \) and \( \lambda \) remains the same. The integer \( s \) in the following definition satisfies \( s \in \mathbb{Z} \).

**Definition 2.1.** Determinants \( M_k(\lambda,s) \) are defined as follows.

1. If \( s < 0 \), then \( M_k(\lambda,s) := 0 \).
2. If \( s = 0 \), then \( M_k(\lambda,0) := 1 \).
3. If \( s > 0 \), then

\[
M_k(\lambda,s) := \begin{vmatrix}
\frac{P_k^{(1)}(\lambda)}{1!} & \frac{P_k^{(2)}(\lambda)}{2!} & \cdots & \frac{P_k^{(s-1)}(\lambda)}{(s-1)!} & \frac{P_k^{(s)}(\lambda)}{s!} \\
P_k(\lambda) & \frac{P_k^{(1)}(\lambda)}{1!} & \cdots & \frac{P_k^{(s-2)}(\lambda)}{(s-2)!} & \frac{P_k^{(s-1)}(\lambda)}{(s-1)!} \\
0 & P_k(\lambda) & \cdots & \frac{P_k^{(s-3)}(\lambda)}{(s-3)!} & \frac{P_k^{(s-2)}(\lambda)}{(s-2)!} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & P_k(\lambda) & \frac{P_k^{(1)}(\lambda)}{1!}
\end{vmatrix}
\]

**Lemma 2.2.** Let \( M_{ij}, i,j = 1, \ldots, s \) be minors of the matrix \( P_{k+1}(J\lambda,s) \). Then,

a) \( M_{ij} = 0 \) if \( i < j \),

b) \( M_{ij} = (P_{k+1}(\lambda))^{s-1} \) if \( i = j \),

c) \( M_{ij} = (P_{k+1}(\lambda))^{s-1+j-i}M_{k+1}(\lambda,i-j) \) if \( i > j \).
Proof. a) Let $i < j$. Then, $M_{ij}$ is the determinant of an upper triangular matrix with the main diagonal

$$\begin{bmatrix}
(P_{k+1}(\lambda), \ldots, P_{k+1}(\lambda))_{i-1}, \ldots, 0, & P_{k+1}(\lambda), \ldots, P_{k+1}(\lambda)_{s-j}
\end{bmatrix}$$

and, consequently, $M_{ij} = 0$.

b) Let $i = j$. Then, the minor $M_{ii}$ is the determinant of an upper triangular matrix with the main diagonal

$$\begin{bmatrix}
(P_{k+1}(\lambda))_{s-1}, \ldots, P_{k+1}(\lambda)
\end{bmatrix}$$

and $M_{ij} = (P_{k+1}(\lambda))^{s-1}$.

c) Let $i > j$. Then, the minor $M_{ij}$ is the determinant of a matrix with the following structure - its main diagonal equals

$$\begin{bmatrix}
(P_{k+1}(\lambda), \ldots, P_{k+1}(\lambda))_{j-1}, & P'_{k+1}(\lambda), \ldots, P'_{k+1}(\lambda), & P_{k+1}(\lambda), \ldots, P_{k+1}(\lambda)_{s-1}
\end{bmatrix}$$

the elements $m_{pq} = 0$ if

$\alpha)\ q = 1, \ldots, j - 1$ and $p > q$,

$\beta)\ p = i + 1, \ldots, s - 1$ and $p > q$,

and the elements $m_{pq}$ where $p, q = j, \ldots, i - 1$ generate a matrix with the determinant $M_{k+1}(\lambda, i - j)$. We get

$$M_{ij} =$$

$$\begin{array}{cccccccc}
\ldots & \frac{P''_{k+1}(\lambda)}{(j - 2)!} & \frac{P_{k+1}(\lambda)}{(j - 1)!} & \frac{P''_{k+1}(\lambda)}{(j - 2)!} & \frac{P_{k+1}(\lambda)}{(j - 1)!} & \frac{P_{k+1}(\lambda)}{(j - 1)!} & \frac{P_{k+1}(\lambda)}{(j - 1)!} & \ldots
\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\\
\ldots & P_{k+1}(\lambda) & \frac{P'_{k+1}(\lambda)}{2!} & P_{k+1}(\lambda) & \frac{P'_{k+1}(\lambda)}{2!} & P_{k+1}(\lambda) & \frac{P'_{k+1}(\lambda)}{2!} & \ldots
\\
\ldots & 0 & \frac{P'_{k+1}(\lambda)}{2!} & 0 & \frac{P'_{k+1}(\lambda)}{2!} & 0 & \frac{P'_{k+1}(\lambda)}{2!} & \ldots
\\
\ldots & 0 & P_{k+1}(\lambda) & \frac{P'_{k+1}(\lambda)}{2!} & P_{k+1}(\lambda) & \frac{P'_{k+1}(\lambda)}{2!} & P_{k+1}(\lambda) & \ldots
\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\\
\ldots & \ldots & \ldots & 0 & P_{k+1}(\lambda) & \frac{P'_{k+1}(\lambda)}{2!} & P_{k+1}(\lambda) & \ldots
\\
\ldots & \ldots & \ldots & 0 & 0 & P_{k+1}(\lambda) & \ldots & \ldots
\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

from which it follows that

$$M_{ij} = (P_{k+1}(\lambda))^{s-1} M_{k+1}(\lambda, i-j) (P_{k+1}(\lambda))^{s-1-i} = (P_{k+1}(\lambda))^{s-1+j-i} M_{k+1}(\lambda, i-j).$$
Remark 2. In the sequel, we will need to express any minor $M_{ij}$ of the matrix $P_{k+1}(J_{\lambda,s})$ in terms of the determinants $M_{k+1}(\lambda, i - j)$. For every minor $M_{ij}$, $i, j = 1, \ldots, s$, the same formula

$$M_{ij} = (P_{k+1}(\lambda))^{s-1+j-i}M_{k+1}(\lambda, i - j)$$

holds since, using Definition 2.1, we can write the statements of Lemma 2.2 as

a) $M_{ij} = 0 = (P_{k+1}(\lambda))^{s-1+j-i}M_{k+1}(\lambda, i - j)$ if $i < j$, 
b) $M_{ij} = (P_{k+1}(\lambda))^{s-1} = (P_{k+1}(\lambda))^{s-1+j-i}M_{k+1}(\lambda, i - j)$ if $i = j$, 
c) $M_{ij} = (P_{k+1}(\lambda))^{s-1+j-i}M_{k+1}(\lambda, i - j)$ if $i > j$.

Using Remark 2, we can express the cofactors $C_{ij}$, $i, j = 1, \ldots, s$ of the matrix $P_{k+1}(J_{\lambda,s})$ as:

$$C_{ij}(J_{\lambda,s}) = (-1)^{i+j}M_{ij} = (-1)^{i+j}(P_{k+1}(\lambda))^{s-1+j-i}M_{k+1}(\lambda, i - j).$$

Now we will continue the computation of the matrix product (35). We can find the inverse matrix $(P_{k+1}(J_{\lambda,s}))^{-1}$ by a well-known procedure using the adjoint matrix whose elements can be defined through the cofactors $C_{ij}(J_{\lambda,s})$, $i, j = 1, \ldots, s$ and using the obvious formula $\det P_{k+1}(J_{\lambda,s}) = (P_{k+1}(\lambda))^s$.

We get

$$p_{ij}^k(J_{\lambda,s}) = \sum_{s=1}^{s} p_{\ell k}(k, \lambda, s) \frac{C_{ij}(J_{\lambda,s})}{(P_{k+1}(\lambda))^s} = \sum_{s=1}^{s} \frac{P_{k}(\ell-i)(\lambda)}{(\ell-i)!} \frac{C_{ij}(J_{\lambda,s})}{(P_{k+1}(\lambda))^s}$$

$$= \sum_{s=0}^{s-1} \frac{P_{k}(\ell-i)(\lambda)}{\ell!} \frac{C_{ij}(J_{\lambda,s})}{(P_{k+1}(\lambda))^s} = \sum_{s=0}^{s-1} \frac{P_{k}(\ell-i)(\lambda)}{\ell!} \frac{(-1)^{i+j+i}M_{k+1}(\lambda, j-l-i)}{(P_{k+1}(\lambda))^{1+j-i}}.$$

Because of the properties of determinants $M_k$ (see Definition 2.1), we have

$$p_{ij}^k(J_{\lambda,s}) = 0 \text{ if } i > j,$$

and, for the rest of the elements $p_{i+1, j}^k(J_{\lambda,s})$ with $j = 0, 1, \ldots, s-i$, we get

$$p_{i+1, j}^k(J_{\lambda,s}) = \sum_{l=0}^{s-i} \frac{P_{k}(\ell-i)(\lambda)}{\ell!} \frac{(-1)^{j+l}M_{k+1}(\lambda, j-l)}{(P_{k+1}(\lambda))^{1+j-l}}$$

$$= \sum_{j=0}^{s-i} \frac{P_{k}(\ell-i)(\lambda)}{\ell!} \frac{(-1)^{j+l}M_{k+1}(\lambda, j-l)}{(P_{k+1}(\lambda))^{1+j-l}}. \quad (36)$$

Due to (36), where the index $i$ is not included in the final formula, we can define

$$\hat{p}_{j}^k(J_{\lambda,s}) := p_{i+1, j}^k(J_{\lambda,s})$$

for any $i = 1, \ldots, s$, $j = 0, 1, \ldots, s-i$. 
Compute now the \((1,l)\)-cofactor of \(M_{k+1}(\lambda, s)\). It has the form

\[
(-1)^{1+l} \times \begin{vmatrix}
\frac{P_{k+1}^{(l)}(\lambda)}{l!} & \frac{P_{k+1}^{(l+2)}(\lambda)}{(l+1)!} & \cdots & \frac{P_{k+1}^{(s-1)}(\lambda)}{(s-2)!} & \frac{P_{k+1}^{(s)}(\lambda)}{(s-1)!} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \frac{P_{k+1}^{(l+1)}(\lambda)}{2!} & \cdots & \frac{P_{k+1}^{(s-1)}(\lambda)}{(s-l)!} & \frac{P_{k+1}^{(s)}(\lambda)}{(s-l-1)!} \\
0 & 0 & \frac{P_{k+1}^{(l+2)}(\lambda)}{3!} & \cdots & \frac{P_{k+1}^{(s-1)}(\lambda)}{(s-l-2)!} \\
0 & 0 & 0 & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \frac{P_{k+1}^{(l+1)}(\lambda)}{2!} \\
0 & \cdots & \cdots & \cdots & \frac{P_{k+1}^{(l+2)}(\lambda)}{3!} \\
0 & \cdots & \cdots & \cdots & \frac{P_{k+1}^{(l+3)}(\lambda)}{4!} \\
0 & \cdots & \cdots & \cdots & \frac{P_{k+1}^{(l+4)}(\lambda)}{5!} \\
0 & \cdots & \cdots & \cdots & \frac{P_{k+1}^{(l+5)}(\lambda)}{6!} \\
0 & \cdots & \cdots & \cdots & \frac{P_{k+1}^{(l+6)}(\lambda)}{7!} \\
0 & \cdots & \cdots & \cdots & \frac{P_{k+1}^{(l+7)}(\lambda)}{8!} \\
0 & \cdots & \cdots & \cdots & \frac{P_{k+1}^{(l+8)}(\lambda)}{9!} \\
0 & \cdots & \cdots & \cdots & \frac{P_{k+1}^{(l+9)}(\lambda)}{10!} \\
\end{vmatrix}
\]

and equals

\((-1)^{1+l}(P_{k+1}(\lambda))^{l+1}M_{k+1}(\lambda, s - l)\).

Applying the Laplace expansion of the determinant \(M_{k+1}(\lambda, s)\) along the first row, we get

\[
M_{k+1}(\lambda, s) = \sum_{l=1}^{s} \frac{P_{k+1}^{(l)}(\lambda)}{l!} (-1)^{1+l}(P_{k+1}(\lambda))^{l-1}M_{k+1}(\lambda, s - l).
\]

This equation can be rewritten in the form

\[
0 = \sum_{l=0}^{s} \frac{P_{k+1}^{(l)}(\lambda)}{l!} (-1)^{1+l}(P_{k+1}(\lambda))^{l-1}M_{k+1}(\lambda, s - l).
\]  \tag{38}

Using (38) for \(s \geq 1\), we can prove a recurring equation between the elements of the matrix product \(P_{k}(J_{\lambda,s})(P_{k+1}(J_{\lambda,s}))^{-1}\):

**Lemma 2.3.** For the elements \(\hat{p}_{j}^{k}(J_{\lambda,s})\) of the product \(P_{k}(J_{\lambda,s})(P_{k+1}(J_{\lambda,s}))^{-1}\), defined by (37), and integer \(1 \leq l \leq s - 1\), we have:

\[
\frac{P_{k}^{(l)}(\lambda)}{l!} = \sum_{\ell=0}^{l} \frac{P_{k+1}^{(\ell)}(\lambda)}{\ell!} \hat{p}_{l-\ell}^{k}(J_{\lambda,s})
\]  \tag{39}

**Proof.** Substitute (36) for \(\hat{p}_{l-\ell}^{k}(J_{\lambda,s})\) in the right-hand side of (39) to obtain:

\[
\sum_{\ell=0}^{l} \frac{P_{k+1}^{(\ell)}(\lambda)}{\ell!} \hat{p}_{l-\ell}^{k}(J_{\lambda,n})
= \sum_{\ell=0}^{l} \frac{P_{k+1}^{(\ell)}(\lambda)}{\ell!} \sum_{i=0}^{l-\ell} \frac{P_{k}^{(i)}(\lambda)}{i!} \frac{(-1)^{l-\ell+i}M_{k+1}(\lambda, l-\ell+i)}{(P_{k+1}(\lambda))^{l-\ell-l+i}}
= \sum_{\ell=0}^{l} \sum_{i=0}^{l-\ell} \frac{P_{k+1}^{(\ell)}(\lambda)}{\ell!} \frac{P_{k}^{(i)}(\lambda)}{i!} \frac{(-1)^{l-\ell+i}M_{k+1}(\lambda, l-\ell+i)}{(P_{k+1}(\lambda))^{l-\ell-l+i}} = (s)
\]
Now we rearrange \( \sum_{i=0}^{l-\ell} \sum_{\ell=0}^{l-i} a_{i\ell} = \sum_{i=0}^{l-\ell} a_{i\ell} \) the last sum (*) and apply the identity (38) to get:

\[
(*) = \sum_{i=0}^{l-\ell} \sum_{\ell=0}^{l-i} \frac{P_k^{(\ell)}(\lambda)}{\ell!} \frac{P_k^{(i)}(\lambda)}{i!} \frac{(-1)^{l-\ell+i}M_{k+1}(\lambda, l-\ell-i)}{(P_{k+1}(\lambda))^{l-\ell+i}}
\]

\[
= \sum_{i=0}^{l-\ell} \sum_{\ell=0}^{l-i} \frac{P_k^{(\ell)}(\lambda)}{\ell!} \frac{P_k^{(i)}(\lambda)}{i!} \frac{(-1)^{l-2\ell+i+1}}{(P_{k+1}(\lambda))^{l-\ell}} \frac{(-1)^{1+\ell}(P_{k+1}(\lambda))^{\ell-1}M_{k+1}(\lambda, l-\ell-i)}{i!} \sum_{\ell=0}^{l-i} \frac{P_k^{(\ell)}(\lambda)}{\ell!} (-1)^{1+\ell}(P_{k+1}(\lambda))^{\ell-1}M_{k+1}(\lambda, l-\ell-\ell)
\]

\[
= \frac{P_k^{(i)}(\lambda)}{i!} + \frac{P_k^{(i)}(\lambda)}{i!} \frac{(-1)^{l+i+1}}{(P_{k+1}(\lambda))^{l-i}} \times \sum_{\ell=0}^{l-i} \frac{P_k^{(\ell)}(\lambda)}{\ell!} (-1)^{1+\ell}(P_{k+1}(\lambda))^{\ell-1}M_{k+1}(\lambda, l-\ell-\ell) = \frac{P_k^{(i)}(\lambda)}{i!}
\]

\[
= 0 \text{ due to (38) with } s = l+i \geq 1
\]

3. Main results. Based on the auxiliary results proved we can now prove the main results of the paper.

**Theorem 3.1.** Let \( \tau > 0 \) and let an \( n \times n \) constant matrix \( B \neq \Theta \) be given. If the eigenvalues \( \lambda_i, i = 1, \ldots, n \) of the matrix \( B \) satisfy the inequality \( |\lambda_i| \tau < 1/e \), then

\[
\lim_{k \to \infty} e^{B \tau} (e^{B(k+1)})^{-1} = E(B\tau) \quad (40)
\]

and

\[
\lim_{k \to \infty} e^{B(k+1)} (e^{B \tau})^{-1} = \exp(W_0(B\tau)) \quad (41)
\]

**Proof.** First we show that (40) holds if \( B \) is replaced by a Jordan block \( J_{\lambda,n} \).

The limits of the elements \( p_k^i(J_{\lambda,n}) = p_k^0(J_{\lambda,n}) \), \( i = 1, \ldots, n \) of the product

\[
P_k(J_{\lambda,n})(P_{k+1}(J_{\lambda,n}))^{-1},
\]

as it follows from formula (36) (where \( j = 0 \)) and from formula (21), are

\[
\lim_{k \to \infty} p_k^i(J_{\lambda,n}) = \lim_{k \to \infty} \frac{P_k(\lambda)}{P_{k+1}(\lambda)} = E(\lambda).
\]

Now, by induction, we prove that, for the limits of other elements \( p_k^i(J_{\lambda,n}) = p_{k,i+1}(J_{\lambda,n}) \), \( l = 1, \ldots, n-i \), we have

\[
\lim_{k \to \infty} p_k^i(J_{\lambda,n}) = \frac{E^{(l)}(\lambda)}{l!}, \quad (42)
\]

i.e., for \( k \to \infty \) we have

\[
\hat{p}_k^i(J_{\lambda,n}) = \frac{E^{(l)}(\lambda)}{l!} + o(1)
\]

where \( o \) is the Landau order symbol "small" \( o \). The assertion is proved for \( l = 0 \).

Now we assume that this assertion holds for \( i = 0, \ldots, l \) where \( l < n-i \). We use formula (39) to express the element \( p_k^i(J_{\lambda,n}) \):
\[ \hat{p}^k_{l+1}(J_{\lambda,n}) = \frac{1}{P_{k+1}(\lambda)} \left( P^{(l+1)}(\lambda) - \sum_{\ell=1}^{l+1} \frac{P_{k+1}(\lambda)}{\ell!} \hat{p}^k_{l+1-\ell}(J_{\lambda,n}) \right) \]

Consequently, formula (42) holds.

Applying (23), we obtain:

\[ \lim_{k \to \infty} \hat{p}^k_{l+1}(J_{\lambda,n}) = \frac{E^{(l+1)}(\lambda)}{(l+1)!}. \]

The remaining elements \( p_{ij}^k(J_{\lambda,n}) \) of the product \( P_k(J_{\lambda,n})P_{k+1}(J_{\lambda,n})^{-1} \) with \( i > j \) (under the main diagonal) are equal to zero.

The Jordan block \( J_{\lambda,n} \) has the spectral radius \( \rho(J_{\lambda,n}) = |\lambda| \) and, by the assumption, \( |\lambda| \tau < 1/e \). Substituting \( J_{\lambda,n} \tau \) for \( x \) into (12), we conclude that there is a matrix \( E(J_{\lambda,n} \tau) \) as the value of the analytic function defined by the series (12) with the radius of convergence \( r = 1/e \) such that

\[ \lim_{k \to \infty} e^{J_{\lambda,n} \tau} \left( e^{J_{\lambda,n} (k+1) \tau} \right)^{-1} = \left[ \text{by (19)} \right] = \lim_{k \to \infty} P_k(J_{\lambda,n} \tau)(P_{k+1}(J_{\lambda,n})^{-1} = E(J_{\lambda,n} \tau). \]

From the representation

\[ B = P^{-1} \text{diag}(J_{\lambda_1,m_1}, J_{\lambda_2,m_2}, \ldots, J_{\lambda_N,m_N})P, \] (43)

we directly get

\[ e^{B \tau} = P^{-1} \text{diag} \left( e^{J_{\lambda_1,m_1} \tau}, \ldots, e^{J_{\lambda_N,m_N} \tau} \right) P \]

and

\[ e^{B \tau} \left( e^{B (k+1) \tau} \right)^{-1} = P^{-1} \text{diag} \left( e^{J_{\lambda_1,m_1} \tau}, \ldots, e^{J_{\lambda_N,m_N} \tau} \right)^{-1} \left( e^{J_{\lambda_1,m_1} \tau} \right)^{-1} \]

as well. Now we can obtain easily

\[ \lim_{k \to \infty} e^{B \tau} \left( e^{B (k+1) \tau} \right)^{-1} = P^{-1} \text{diag} \left( \lim_{k \to \infty} e^{J_{\lambda_1,m_1} \tau}, \ldots, \lim_{k \to \infty} e^{J_{\lambda_N,m_N} \tau} \right) \]

and (40) is proved.

Note that, due to formulas (10), (12), (43) and

\[ W_0(B \tau) = P^{-1} \text{diag}(W_0(J_{\lambda_1,m_1}), W_0(J_{\lambda_2,m_2}), \ldots, W_0(J_{\lambda_N,m_N}))P, \]

matrices \( B, E(B \tau) \) and \( W_0(B \tau) \) mutually commute (the Jordan canonical forms for \( B \) and \( W_0(B \tau) \) have, for the same regular matrix \( P \), diagonal blocks of the same
Then, formula (41) is a consequence of (40) since, by using (11) and (8), we get

\[
\lim_{k \to \infty} e^{B(k+1)\tau} (e^{Bk\tau})^{-1} = \left( \lim_{k \to \infty} e^{Bk\tau} (e^{B(k+1)\tau})^{-1} \right)^{-1} = (E(B\tau))^{-1} = B\tau (W_0(B\tau))^{-1} = \exp(W_0(B\tau)).
\] (44)

The following corollary specifies the matrix \( C \) mentioned in formula (7).

**Corollary 1.** From Theorem 3.1 and formula (44), we have

\[
\lim_{k \to \infty} e^{B(k+1)\tau} (e^{Bk\tau})^{-1} = e^{C\tau}
\]

where

\[
C := \frac{1}{\tau} W_0(B\tau).
\]

**Theorem 3.2.** Let \( \tau > 0 \) and let an \( n \times n \) constant matrix \( B \neq \Theta \) be given. If the eigenvalues \( \lambda_i, i = 1, \ldots, n \) of the matrix \( B \) satisfy the inequality \( |\lambda_i|\tau < 1/e \), then

\[
\lim_{k \to \infty} e^{Bk\tau} \exp(-kW_0(B\tau)) = B\tau (W_0(B\tau)(I + W_0(B\tau)))^{-1}.
\] (45)

**Proof.** Let \( n = 1 \). In the scalar case, (45) is a simple consequence of (29) since, by (19) and (11),

\[
\lim_{k \to \infty} e^{Bk\tau} \exp(-kW_0(B\tau)) = \lim_{k \to \infty} P_k(B\tau) \exp(-kW_0(B\tau))
\]

\[
= (E(B\tau)(1 + W_0(B\tau)))^{-1} = B\tau (W_0(B\tau)(1 + W_0(B\tau)))^{-1}.
\]

Let \( n > 1 \). The radius of convergence of the Maclaurin series of the function

\[
x(W_0(x)(1 + W_0(x)))^{-1}
\]

is \( r = 1/e \) (see formulas (10)–(12)). Since inequalities \( |\lambda_i|\tau < 1/e, i = 1, \ldots, n \) imply \( \rho(B\tau) < 1/e \), we can substitute \( x \to B\tau \) into this Maclaurin decomposition to get convergent matrix series. Its sum equals

\[
B\tau (W_0(B\tau)(I + W_0(B\tau)))^{-1}.
\]

Then,

\[
\lim_{k \to \infty} e^{Bk\tau} \exp(-kW_0(B\tau)) = \lim_{k \to \infty} P_k(B\tau) \exp(-kW_0(B\tau))
\]

\[
= B\tau (W_0(B\tau)(I + W_0(B\tau)))^{-1}.
\]

\[\square\]

Let \( F(k) = \{f_{ij}(k)\}_{i,j=1}^n \) and \( G = \{f_{ij}(k)\}_{i,j=1}^n \) be matrices defined for all sufficiently large \( k \). We say that

\[
F(k) \sim G(k), \ k \to \infty
\] (46)

if

\[
f_{ij}(k) = g_{ij}(k)(1 + o(1)), \ k \to \infty
\] (47)

where \( o(1) \) is the Landau order symbol “small” o.
Remark 3. Let all assumptions of Theorem 3.2 be valid. From formula (45), we get the asymptotic relation
\[ e^{Bk\tau} \approx B\tau \exp(kW_0(B\tau))(W_0(B\tau)(I + W_0(B\tau)))^{-1}, \quad k \to \infty \quad (48) \]
This formula can be useful, e.g., in the investigation of the asymptotic behavior of solutions of problem (1), (2) or (4), (5) at nodes \( t = k\tau \), as can be seen from formulas (3), (6).

The following theorem gives results on the behavior of the spectral radius \( \rho(\cdot) \) and spectral norm \( \|\cdot\|_\rho \) (defined for a matrix \( A \) as \( \|A\|_\rho = (\rho(AA^T))^{1/2} \)) of the sequence of values of delayed exponential \( e^{Bk\tau} \) for (discrete) \( k \to \infty \) and of delayed exponential \( e^{Bt\tau} \) for (continuous) \( t \to \infty \).

**Theorem 3.3.** Let \( \tau > 0 \) and let an \( n \times n \) constant matrix \( B \neq \Theta \) be given. Assume that the eigenvalues \( \lambda_i, i = 1, \ldots, n \) of the matrix \( B \) satisfy inequality \( \tau|\lambda_i| < 1/e, \quad i = 1, \ldots, n \). The following three statements are true:

(i) If all the eigenvalues \( \lambda_i, i = 1, \ldots, n \) satisfy
\[ \tau|\lambda_i| < -\arctan \left( \frac{\text{Re} \lambda_i}{|\text{Im} \lambda_i|} \right), \quad (49) \]
then
\[ \lim_{k \to \infty} \rho \left( e^{Bk\tau} \right) = 0. \quad (50) \]

(ii) If there exist an index \( i_0 \in \{1, \ldots, n\} \) such that
\[ \tau|\lambda_{i_0}| > -\arctan \left( \frac{\text{Re} \lambda_{i_0}}{|\text{Im} \lambda_{i_0}|} \right), \quad (51) \]
then
\[ \limsup_{k \to \infty} \|e^{Bk\tau}\|_\rho = \infty. \quad (52) \]

(iii) If all the eigenvalues \( \lambda_i, i = 1, \ldots, n \) are real and satisfy
\[ \tau|\lambda_i| > -\arctan \left( \frac{\text{Re} \lambda_i}{|\text{Im} \lambda_i|} \right), \quad (53) \]
then
\[ \lim_{t \to \infty} \|e^{Bt\tau}\|_\rho = \infty. \quad (54) \]

**Proof.** To prove this theorem we use Remark 3. Figure 2 details the eigenvalue domain for each case considered.

(i) From (49), we conclude that, for all the eigenvalues \( \lambda_i, i = 1, \ldots, n \), by (17), \( \text{Re} W_0(\lambda_i \tau) < 0 \) is true, therefore,
\[ \lim_{k \to \infty} \rho \left( \exp(kW_0(\lambda_i \tau)) \right) = 0. \]
It is well-known that the \( n \) roots of a polynomial of degree \( n \) depend continuously on the coefficients and that the eigenvalues of a matrix depend continuously on the matrix (we refer, e.g. to [9]). Then, (48) implies
\[ \lim_{k \to \infty} \rho \left( e^{Bk\tau} \right) = 0, \]
so that (50) holds.
(ii) From assumption (51), by (17), the existence follows of at least one eigenvalue \( \lambda_{i_0} \) such that \( \text{Re} W_0(\lambda_{i_0} \tau) > 0 \). Therefore,
\[
\limsup_{k \to \infty} \rho(\exp(k W_0(\lambda_{i_0} \tau))) = \infty.
\]
In much the same way as in part (i), by (48), we also deduce
\[
\limsup_{k \to \infty} \rho(e^{Bk\tau}) = \infty.
\]
Then the conclusion of part (ii) follows from the relation between the spectral radius and the spectral norm:
\[
\rho(A) \leq \|A\|_\rho
\]
for any matrix \( A \).

(iii) Let \( n = 1 \). In the scalar case, the condition (53) implies
\[
0 < \lambda_1 < 1/(e\tau).
\]
The delayed exponential function \( e^{\lambda_1 t} \) is a solution of the equation
\[
y'(t) = \lambda y(t - \tau)
\]
satisfying the initial condition
\[
y(t) = 1, \quad t \in [-\tau, 0].
\]
Since the solution \( y = y(t) \) of problem (55), (56) satisfies \( y(t) > 0, t \geq -\tau \) and \( y'(t) \geq \lambda_1 > 0 \) for \( t > 0 \), we have
\[
\lim_{t \to \infty} e^{\lambda_1 t} = \infty.
\]
Let \( n > 1 \). Then, as above, we have
\[
0 < \lambda_i < 1/(e\tau), \quad i = 1, \ldots, n.
\]
Let \( J \) be the Jordan canonical form of square matrix \( B \). I.e., there is an invertible matrix \( P_* \) such that \( B = P_*^{-1}JP_* \). Note that the Jordan canonical form of the delayed exponential of matrix \( e^{Bt} \) has the form \( P_*^{-1} e^{Jt} P_* \) and, due to this fact, all the eigenvalues of \( e^{\lambda_i t} \) are \( e^{\lambda_i t} \), \( i = 1, \ldots, n \) where \( \lambda_i, \; i = 1, \ldots, n \) are all the
eigenvalues of $B$. Proceeding similarly to the scalar case, we conclude that (54) holds.

4. Applications. In this part we make some suggestions for possible applications of the above results.

4.1. Equation of a showering person. Systems (1) often describe mathematical models of real-world phenomena. The solution of the initial problem (1), (2) is given by formula (3). Investigate the long-time behavior of the solutions generated by constant initial functions, i.e., assume $\varphi(t) \equiv C_\varphi$ for every fixed $t \in [-\tau, 0]$ and $C_\varphi \in \mathbb{R}^n$. Then, $\varphi'(t) \equiv \theta$, $t \in [-\tau, 0]$ where $\theta$ is the null vector. Formula (3) becomes

$$y(t) = e^{Bt}\varphi(-\tau) = e^{Bt}C_\varphi.$$  

If all assumptions of Theorem 3.2 hold, by formula (48), we get the asymptotic relation for (57) at nodes $t = k\tau$ as $k \to \infty$

$$y(k\tau) = e^{Bk\tau}\varphi(-\tau) = B_\tau \exp(kW_0(B\tau))(W_0(B\tau)(I + W_0(B\tau)))^{-1}C_\varphi.$$  

Consider the equation modeling the behavior of a showering person (for details we refer, e.g., to [7, part 3.6.3])

$$T'(t) = -\gamma[T(t - \tau) - T_d], \quad t \in [0, \infty)$$  

where $T$ is the regulated temperature of water leaving the mixer, $\gamma > 0$ and $T_d$ is the desired temperature of water agreeable for a showering person. Setting $y(t) = T(t) - T_d$ in (59), we get

$$y'(t) = -\gamma y(t - \tau), \quad t \in [0, \infty).$$  

Assuming the water temperature before regulation is constant, i.e. the initial condition is given by the equation

$$y(t) = y_0, \quad t \in [-\tau, 0],$$  

the solution of (60), (61) is

$$y(t) = e^{-\gamma t}y_0, \quad t \in [-\tau, \infty)$$  

and if $\gamma \tau e < 1$ then, by (46)–(48) and (58),

$$y(k\tau) = e^{-\gamma k\tau}y_0 = -\gamma\tau \exp(kW_0(-\gamma\tau)) \frac{y_0(1 + o(1))}{W_0(-\gamma\tau)(1 + W_0(-\gamma\tau))}$$

as $k \to \infty$. By (9), the last formula can be simplified to

$$y(k\tau) = \frac{y_0(1 + o(1))}{1 + W_0(-\gamma\tau)} e^{(1 + k)W_0(-\gamma\tau)}, \quad k \to \infty.$$  

Since, by (10),

$$W_0(-\gamma\tau) = -\gamma\tau - (\gamma\tau)^2 - \frac{3}{2}(\gamma\tau)^3 + \cdots,$$

we have $y(k\tau) > 0$ and $\lim_{k \to \infty} y(k\tau) = 0$. It means that the regulated temperature $T(k\tau)$ will tend to the desired value $T_d$ as $k \to \infty$.

The above example can be generalized, e.g., for two showering persons. Suppose that hot and cold water is supplied in two separate pipes to a bathroom with two showers. Inside the bathroom, each pipe branches into two pipes leading to the shower mixers. A person taking a shower regulates the water temperature flowing from the mixer to the sprinkler. Due to the changes in the water pressure caused by water being regulated by two persons simultaneously, there is a mutual
dependence between the temperatures \( T_1 \) and \( T_2 \) of the water flowing from mixer one to sprinkler one and from mixer two to sprinkler two, respectively. Then, a simple model modeling the behavior of two showering persons is

\[
T'_1(t) = -\gamma_{11}[T_1(t) - T_d1] + \gamma_{12}[T_2(t) - T_d2],
\]

\[
T'_2(t) = \gamma_{21}[T_1(t) - T_d1] - \gamma_{22}[T_2(t) - T_d2]
\]

where \( \gamma_{ij} > 0 \), \( i, j = 1, 2 \) and \( T_{di}, \ i = 1, 2 \) are the desired temperatures of water agreeable for each of the two showering persons. Substituting \( y_i(t) = T_i(t) - T_{di} \) in (62), (63) we get

\[
y'_1(t) = -\gamma_{11}y_1(t - \tau) + \gamma_{12}y_2(t - \tau),
\]

\[
y'_2(t) = \gamma_{21}y_1(t - \tau) - \gamma_{22}y_2(t - \tau).
\]

Assuming the water temperature before regulation is constant, i.e. the initial condition is given by the relation

\[
y_1(t) = y_2(t) = y_0, \quad t \in [-\tau, 0],
\]

the solution of (64)–(66) is

\[
y(t) = (y_1(t), y_2(t))^T = e^{-\Gamma t} y^0, \quad t \in [-\tau, \infty)
\]

where \( y^0 = (y_0, y_0)^T \) and

\[
\Gamma = \begin{pmatrix} -\gamma_{11} & \gamma_{12} \\ \gamma_{21} & -\gamma_{22} \end{pmatrix}.
\]

Let the eigenvalues

\[
\lambda_i = \frac{1}{2} \left[ -(\gamma_{11} + \gamma_{22}) + (-1)^i \sqrt{(\gamma_{11} - \gamma_{22})^2 + 4\gamma_{12}\gamma_{21}} \right], \ i = 1, 2
\]

of the matrix \( \Gamma \) satisfy \( |\lambda_i| \tau < 1, \ i = 1, 2 \). Then, by formula (58), at nodes \( t = k\tau \), the solution (67) has the asymptotic behavior

\[
y(k\tau) \approx \Gamma \tau \exp(kW_0(\Gamma \tau))(W_0(\Gamma \tau)(I + W_0(\Gamma \tau)))^{-1}y^0
\]

as \( k \to \infty \).

### 4.2. Instability of solutions.

In this part we give sufficient conditions for the instability of the system (1). In general, the instability of systems (1) will be proved if, in every \( \delta \)-neighborhood of zero initial function, there exist an initial function generating a solution not remaining in a given \( \varepsilon \)-neighborhood of the zero solution. In the proof of the following theorem, it is sufficient to restrict the set of initial functions to constant initial functions only.

**Theorem 4.1.** Let \( \tau > 0 \) and let an \( n \times n \) constant matrix \( B \neq \Theta \) be given. Assume that the eigenvalues \( \lambda_i, \ i = 1, \ldots, n \) of the matrix \( B \) satisfy the inequality \( \tau|\lambda_i| < 1/e, \ i = 1, \ldots, n \). If, moreover, there exist an index \( i_0 \in \{1, \ldots, n\} \) such that

\[
\tau|\lambda_{i_0}| > -\arctan \left( \frac{\Re \lambda_{i_0}}{|\Im \lambda_{i_0}|} \right),
\]

then the system (1) is instable.
Proof. We will employ constant initial functions
\[ \varphi^i(t) = C^i := (0, \ldots, 0, 1, 0, \ldots, 0)^T, \quad t \in [-\tau, 0], \quad i = 1, \ldots, n. \]
Generated by \( \varphi^i(t) \), solution \( y^i = y^i(t) \) equals
\[ y^i(t) = e^{B^i t} C^i, \quad t \in [-\tau, \infty), \quad i = 1, \ldots, n. \]
Consider a matrix equation
\[ Y'(t) = BY(t - \tau), \quad t \in [0, \infty) \tag{68} \]
where \( Y(t) \) is an \( n \times n \) matrix. Clearly, the matrix
\[ Y(t) := (y^1(t), \ldots, y^n(t)) = e^{B^i t} (C^1, \ldots, C^n) = e^{B^i t}, \quad t \in [-\tau, \infty) \]
is a solution of the system (68) satisfying \( Y(t) = I, \ t \in [-\tau, 0] \). Obviously,
\[ \|Y(t)\|_\rho = \|e^{B^i t}\|_\rho \]
and by applying the well-known result on the equivalence of norms, there exists a constant \( M > 0 \) such that, for the element-wise max norm \( \| \cdot \|_{\max} \) of a matrix, we have
\[ M \max_{i,j=1,\ldots,n} |y^i_j(t)| = M\|Y(t)\|_{\max} \geq \|Y(t)\|_\rho = \|e^{B^i t}\|_\rho, \quad t \in [0, \infty) \tag{69} \]
where \( y^i_j(t), \ j = 1, \ldots, n \) are co-ordinates of the solution \( y^i(t) \). All assumptions of Theorem 3.3, part (ii) are satisfied and, therefore, for \( t = k\tau \) and \( k \to \infty \), by formula (52), we have
\[ \limsup_{k \to \infty} \|e^{B^i k\tau}\|_\rho = \infty. \]
Then, from (69), we derive
\[ \limsup_{k \to \infty} \max_{i,j=1,\ldots,n} |y^i_j(k\tau)| = \infty. \]
This property proves the instability of the system (1). \( \square \)

Remark 4. A similar result on instability can be derived for the system (4) if the following modifications are taken into account. Instead of constant initial functions used in the proof of Theorem 4.1, initial functions as solutions of the system
\[ \varphi'(t) = A\varphi(t), \quad t \in [-\tau, 0] \]
can be used. Then, the formula (6) becomes
\[ y(t) = e^{A(t+\tau)} e^{B^i t} \varphi(-\tau), \quad t \in [-\tau, \infty) \]
where \( B_1 = e^{-A\tau} B \). In addition to this, additional assumptions on the matrix \( A \) for the statement on instability must be included.

Acknowledgments. The first author was supported by the Czech Science Foundation under the project 16-08549S. The research of the second author was carried out under the project CEITEC 2020 (LQ1601) with financial support from the Ministry of Education, Youth and Sports of the Czech Republic under the National Sustainability Programme II. The work was realized in CEITEC - Central European Institute of Technology with research infrastructure supported by the project CZ.1.05/1.1.00/02.0068 financed from European Regional Development Fund.
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Received for publication September 2016.

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