LOCALITY AND RENORMALISATION:
UNIVERSAL PROPERTIES AND INTEGRALS ON TREES

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Abstract. The purpose of this paper is to build an algebraic framework suited to
regularise branched structures emanating from rooted forests and which encodes the
locality principle. This is achieved by means of the universal properties in the locality
framework of properly decorated rooted forests. These universal properties are then
applied to derive the multivariate regularisation of integrals indexed by rooted forests.
We study their renormalisation, along the lines of Kreimer’s toy model for Feynman
integrals.

Contents

1. Introduction 2
2. Locality operated sets and algebras 3
2.1. Operated structures and free objects 3
2.2. Locality sets and algebras 7
2.3. Locality operated structures 9
3. Universal properties of decorated rooted forests: the locality version 10
3.1. Properly decorated forests 10
3.2. The universal property of properly decorated rooted forests 12
3.3. Branching of locality maps 14
4. Multivariate regularisation of branched integrals 16
4.1. Linear complex powers: the locality algebra $M[L]$ 16
4.2. $M[L]$ as an operated locality algebra 18
4.3. Locality morphisms on rooted forests 19
5. Multivariate renormalisation of branched integrals 20
5.1. The minimal substraction scheme on fractions 20
5.2. Locality morphisms on properly decorated rooted forests 22
5.3. Renormalised values on similar properly decorated rooted forests 23
References 24
1. Introduction

The purpose of this paper is to apply to the study of concrete models an algebraic formulation of the locality principle developed in [CGPZ1] in the context of the (co)algebraic approach to perturbative quantum field theory initiated by Connes and Kreimer [CK, K]. The concrete model we focus on is Kreimer’s rooted forests toy model, which serves as an experimental model to study renormalisation and related renormalisation groups. Rooted forests are useful to generate branched structures, such as branched integrals and branched sums, but also geometric rough paths.

To apply the algebraic formulation of [CGPZ1], we need to investigate locality structures for rooted forests and to build locality algebra homomorphisms which encode the regularisation. We then study the renormalisation of branched integrals, and show an invariance property of renormalised values under a similarity transform of decorated rooted forests. The case of branched sums was studied in [CGPZ2], using some of the results of the present paper.

The paper starts with an abstract algebraic part (Sections 2 and 3), which establishes the universal property of properly decorated rooted forests, and gives the general algebraic framework to regularize branched structures. This is followed by an application (Sections 4 and 5) to branched integrals in the context of Kreimer’s toy model for Feynman integrals, which we revisit using a multivariate renormalisation approach.

So in order to renormalise these a priori divergent integrals, we use a multivariate locality regularisation derived from the universal property of properly decorated rooted forests instead of the usual univariate regularisation procedure [CK] and the multivariate regularisation we used before [GPZ4], with the locality setting playing a key role. This multivariate approach gives rise to renormalised branched integrals which inherit the locality property from properly decorated rooted forests.

We next review the contents of the paper in some detail. The tools underlying the algebraic constructions in this paper are operated locality sets and algebras discussed in Section 2. They are obtained as a combination of the notions of operated sets (resp. algebras) and locality sets (resp. algebras). Paragraph 2.1 is dedicated to operated structures, leaving locality aside. Theorem 2.3 builds from rooted trees decorated by a set Ω, the free object in the category of Ω-operated monoids. Alongside these results, in Corollary 2.4 we derive anew the fact that the space spanned by rooted forests decorated by Ω is the initial object in the category of Ω-operated monoids [G1, Md].

The universal property is then implemented to lift maps from the decoration set Ω to maps defined on rooted forests \( \mathcal{F}_\Omega \) decorated by Ω when Ω is equipped with additional structures. In Proposition 2.6 a map \( \phi : \Omega \rightarrow \Omega \) on the decorating monoid (resp. algebra) \( \Omega \) is lifted to a morphism of monoids (resp. algebras) \( \hat{\phi} : \mathcal{F}_\Omega \rightarrow \Omega \) (resp. \( \hat{\phi} : K\mathcal{F}_\Omega \rightarrow \Omega \) for the monoid algebra \( K\mathcal{F}_\Omega \) with coefficients in a field \( K \)) from rooted forests decorated by \( \Omega \). The universal property is further discussed in a relative context in Proposition 2.7 to lift a morphism \( \phi : \Omega_1 \rightarrow \Omega_2 \) of monoids (algebras) to a morphism \( \hat{\phi} : \mathcal{F}_{\Omega_1} \rightarrow \mathcal{F}_{\Omega_2} \) (\( \hat{\phi} : K\mathcal{F}_{\Omega_1} \rightarrow K\mathcal{F}_{\Omega_2} \)) of monoids (algebras).

In Paragraph 2.2, we recall the concept of locality and in Paragraph 2.3 we introduce the notions of operated locality structures, preceded by the notions of operated locality sets in Definition 2.13, of operated locality semigroups, monoids and algebras.

Section 3 deals with the universal property of properly decorated rooted forests. In Paragraph 3.2 we consider properly decorated rooted forests, to which we extend the
universal properties of ordinary decorated rooted forests incorporating locality. Theorem 3.6 is the locality version of Corollary 2.4, while Proposition 3.7 and Corollary 3.9 are the locality version of Propositions 2.7 and 2.6 respectively. Corollary 3.9 was used in [CGPZ2] to renormalise discrete sums attached to trees.

In Section 4 we implement the aforementioned universal locality properties on forests to the study of branched integrals in the context of Kreimer’s toy model [K], to construct a regularisation of branched integrals in the locality setup.

For this purpose, we build a locality morphism in several steps. In Paragraph 4.1, we first introduce the group ring $M[\mathcal{L}]$ over the algebra $M$ of meromorphic germs with linear poles generated by the additive monoid $\mathcal{L}$ of multivariate linear forms, and equip it with a locality algebra structure $\perp$. In Paragraph 4.2, we view $(M[\mathcal{L}], \perp)$ as a locality algebra operated by $\mathcal{L}$ (Lemma 4.2). Paragraph 4.3 is dedicated to locality morphisms defined on forests decorated by $(\mathcal{L}, \perp)$. The universal property discussed in Theorem 3.6 yields a $M[\mathcal{L}]$-valued locality algebra homomorphism $R$ (Lemma 4.3, Lemma 4.3), defined on $\mathcal{L}$-decorated forests, from which we build an $\mathcal{L}$-valued locality morphism $R_1$ on $\mathcal{L}$-decorated rooted forests. We moreover provide an explicit formula of its evaluation on properly decorated forests and show that it is a locality algebra homomorphism in Proposition 5.4.

In Section 5, following the renormalisation scheme by locality morphisms presented in Paragraph 5.2, we build a renormalised map $\pi_+ R_1$. It takes values in holomorphic germs at zero which we evaluate at zero to build the renormalised map as a locality character $R_{\text{ren}}$ on the locality algebra of properly decorated rooted forests (Proposition-Definition 5.3). An interesting feature of this renormalisation process is that similar properly decorated rooted forests (Definition 5.3) have the same renormalised values, as shown in Paragraph 5.8. This results from Theorem 5.1 which yields an algorithm to evaluate the renormalised value of any given branched integral and whose proof uses computational techniques for multivariate meromorphic germs with linear poles developed in [GPZ3].

In conclusion, this paper aims at the application of our locality principle, provides new algebraic tools for multivariate regularisation and renormalisation associated with rooted tree structures, some of which were used in [CGPZ2]. This multivariate renormalisation scheme is then implemented on a non-trivial example, namely Kreimer’s toy model.

2. Locality operated sets and algebras

In this section we introduce the concepts of locality operated set, semigroup and algebra, and construct the free objects in the corresponding categories. For this purpose we first revisit these concepts and constructions without the locality conditions, and later adapt to the locality setting.

2.1. Operated structures and free objects. After recalling the concepts of operated structures, we give the free construction for operated sets, operated semigroups and monoids, and operated algebras, successively.

2.1.1. The notion of operated structures. Let $\Omega$ be a set. Recall that an $\Omega$-operated set (resp. semigroup, monoid, vector space, unital algebra) $(U, \beta)$ (resp. $(U, m_U, \beta)$, $(U, m_U, 1_U, \beta)$, $(U, +, \beta)$, $(U, m_U, 1_U, +, \beta)$) (see [G1, G2]) is a set (resp. semigroup, monoid, vector space, unital algebra) $U$ together with a set of operators

$$\beta := \beta^\Omega_U := \{\beta^\omega := \beta^\omega_U : U \to U \mid \omega \in \Omega\}$$
parameterised by $\Omega$. More precisely, this means that there is a map

$$\beta_U = \beta^\omega_U : \Omega \times U \to U, \quad (\omega, u) \mapsto \beta^\omega_U(u).$$

The maps $\beta^\omega_U$ are often called grafting operators. In the case of a vector space or a unital algebra, we also assume that the operators $\beta^\omega_U$ are linear.

A homomorphism from an $\Omega$-operated object $(U, \beta_U)$ to an $\Omega$-operated object $(V, \beta_V)$ is a morphism $f : U \to V$ in the corresponding category without the $\Omega$-actions with the property

$$f(\beta^\omega_U(u)) = \beta^\omega_V(f(u)) \quad \text{for all } u \in U, \omega \in \Omega.$$

We therefore have the category $\text{OS}_\Omega$ (resp. $\text{OSG}_\Omega$, resp. $\text{OM}_\Omega$, resp. $\text{OA}_\Omega$) of $\Omega$-operated sets (resp. semigroups, resp. monoids, resp. algebras).

2.1.2. Free operated monoids and algebras. Let us consider free objects in the categories of various operated algebraic structures. For “classical” algebraic structures without operators, such as associative and Lie algebras, the free objects have a generating set $X$. For operated algebraic structures, we already have a set of operators. So we need to be careful in distinguishing the two sets: the set $\Omega$ of operators and the set $X$ of generators for a free object. Even though for the applications in this paper, the generating set $X$ will be taken to be the empty set, we give a uniform approach with arbitrary generating sets which might be applied to broader contexts.

We next construct free objects in the category of $\Omega$-operated monoids and algebras, with the latter following naturally from the former.

Let us introduce some terminology. A rooted tree (resp. planar rooted tree) is a connected loopless graph (resp. planar graph), whose edges are oriented, thereby equipping the tree with a partial order on its set of vertices with a unique minimal element, called the root. A rooted forest (resp. planar rooted forest) is a concatenation of (resp. planar) rooted trees. Any maximal vertex (i.e. one that has no element above it) for this partial order is called a leaf. The set of leaves of a forest $F$ is denoted by $l(F)$. For a rooted forest or planar rooted forest $F$, let $V(F)$ denote the set of vertices of $F$. A vertex of $F$ is called non-leaf or interior if it is not a leaf vertex. For the tree with unique vertex, the vertex is taken to be a leaf vertex.

The regularised integrals parametrised by trees which arise in the renormalisation procedure considered later in this paper obey the commutativity property. Consequently, we will focus on (non planar) trees and take the forest concatenation to be commutative.

The following concept is a natural generalisation of notions from [G1, ZGG]. See also [G2] for the planar case.

**Definition 2.1.** Let $\Omega$ and $X$ be disjoint sets. An ($\Omega, X$)-decorated rooted forest is a pair $(F, d)$, where $F$ is a rooted forest and $d : V(F) \to \Omega \cup X$ from the set $V(F)$ of vertices of $F$ is such that $d(V(F) \setminus l(F)) \subseteq \Omega$, i.e. whose restriction to the non-leaf vertices is in $\Omega$ (but whose restriction to the leaf vertices is in $\Omega \cup X$). Let $\mathcal{F}_{\Omega, X}$ denote the set of ($\Omega, X$)-decorated rooted forests together with the “empty tree” denoted $1$.

For $\omega \in \Omega$, we define the grafting operator

$$B^\omega_+ : \mathcal{F}_{\Omega, X} \to \mathcal{F}_{\Omega, X}$$
which sends any rooted forest \((F,d)\) to a rooted tree by adding to \((F,d)\) a new root decorated by \(\omega\), and sends the empty tree 1 to the tree \(\bullet_\omega\) with a single leaf decorated by \(\omega\). We set \(B := \{B_\omega^+ \mid \omega \in \Omega\}\).

The number of vertices of a rooted forest, which we call the degree of the rooted forest, provides a grading on forests and decorated forests. The following simple result further follows from the analog statement in the undecorated case. See [G1] for example.

**Lemma 2.2.** Let \(i : X \to \mathcal{F}_{\Omega,X}, x \mapsto \bullet_x, x \in X\), be the canonical embedding of \(X\) into \(\mathcal{F}_{\Omega,X}\). An \((\Omega, X)\)-decorated rooted forest \((F,d)\) is either 1 or can be written in an unique way as follows:

(i) \(F \in \text{Im}(i)\), that is \(F = \bullet_x\) for some \(x \in X\).

(ii) If \(F\) is a non-empty tree not in the image of \(i\), then \((F,d) = B_\omega^+(\mathcal{T}, \bar{d})\) for some \((\Omega, X)\)-decorated rooted forest \((\mathcal{T}, \bar{d})\) (which might be 1) with \(\deg(F,d) = \deg(\mathcal{T}, \bar{d}) + 1\);

(iii) If \(F\) is not a tree, then \((F,d) = (F_1, d_1) \cdots (F_k, d_k)\), \(k \geq 2\), with \((\Omega, X)\)-rooted trees \(F_i \neq 1\) and with \(\deg(F,d) = \deg(F_1, d_1) + \cdots + \deg(F_n, d_n)\).

The following results for planar rooted forests have been obtained in [ZGG] when \(|X| = 1\) and in [GZ] for general \(X\). The results here, for rooted forests, follows from the same argument.

**Theorem 2.3.** Let \(\Omega\) and \(X\) be sets with \(\Omega\) non-empty. Let \(K\) be a field.

(i) The operated set \((\mathcal{F}_{\Omega,X}, B)\), with the forest concatenation, is an \(\Omega\)-operated commutative monoid with unit 1. The linear span \((K\mathcal{F}_{\Omega,X}, B)\) is an \(\Omega\)-operated unital commutative \(K\)-algebra.

(ii) The operated monoid \(\mathcal{F}_{\Omega,X}\) together with the map \(i : X \to \mathcal{F}_{\Omega,X}, x \mapsto \bullet_x, x \in X\), is the free object on \(X\) in the category of \(\Omega\)-operated commutative monoids. More precisely, for any \(\Omega\)-operated commutative monoid \((U, \beta)\) and map \(f : X \to U\), there is unique morphism \(\bar{f} : \mathcal{F}_{\Omega,X} \to U\) of \(\Omega\)-operated commutative monoids such that \(\bar{f} \circ i = f\).

(iii) The operated algebra \(K\mathcal{F}_{\Omega,X}\) is the free object on \(X\) in the category of \(\Omega\)-operated commutative algebras, characterised by a universal property similar to the previous one.

### 2.1.3. Initial objects and the relative extensions.

We will be most interested in a special case of \((\Omega, X)\)-decorated forests, namely when \(X = \emptyset\) is the empty set. Since there is unique map (known as the empty map) from \(\emptyset\) to any set, the subsequent statement follows directly from Theorem 2.3.

**Corollary 2.4.** Let a set \(\Omega\) be given.

(i) The set \(\mathcal{F}_\Omega := \mathcal{F}_{\Omega,\emptyset}\) of \(\Omega\)-decorated rooted forests with all vertices decorated by \(\Omega\) is the initial object in the category of \(\Omega\)-operated commutative monoids.

(ii) The space \(K\mathcal{F}_\Omega\) of \(\Omega\)-decorated rooted forests is the initial object in the category of \(\Omega\)-operated commutative algebras.

**Remark 2.5.** This result was already shown in aforementioned papers [G2], [M2] by recursively constructing the morphism of \(\Omega\)-operated algebras. In contrast, our proof is a simple application of Theorem 2.3.
Let $\Omega$ be a monoid (resp. an algebra). Let $\phi : \Omega \to \Omega$ be a map (resp. linear map). Then $\Omega$ becomes an $\Omega$-operated structure with the operators

$$\beta^\omega : \Omega \to \Omega, \quad \omega' \mapsto \phi(\omega \omega'), \quad \omega, \omega' \in \Omega.$$ (3)

It then follows from Corollary 2.4 that

Proposition 2.6. Let $\Omega$ be a commutative monoid (resp. algebra). A (resp. linear) map $\phi : \Omega \to \Omega$ induces a unique homomorphism

$$\hat{\phi} : \mathcal{F}_\Omega \to \Omega \quad (\text{resp. } \mathbf{K}\mathcal{F}_\Omega \to \Omega)$$

of commutative monoids (resp. algebras) such that

$$\hat{\phi}(B^\omega(F)) = \phi(\omega(\hat{\phi}(F))).$$ (4)

Proof. As we observed before the proposition, $(\Omega, \beta_\phi := \{\beta^\omega | \omega \in \Omega\})$ is an $\Omega$-operated commutative monoid (resp. algebra). The existence and uniqueness of a homomorphism $\hat{\phi} : \mathcal{F}_\Omega \to \Omega$ of $\Omega$-operated monoids (resp. algebras) then follows from Corollary 2.4, and Eq. (4) boils down to the compatibility condition for the $\Omega$-operations.

We next extend the universal property of $\mathcal{F}_\Omega$ to the relative context.

Proposition 2.7. Let $\phi : \Omega_1 \to \Omega_2$ be a map and let $(U, \beta)$ be an $\Omega_2$-operated commutative monoid. Then $U$ has an $\Omega_1$-action induced from $\phi$. Further $\phi$ lifts uniquely to a homomorphism of $\Omega_1$-operated monoids as defined in Eq. (2):

$$\phi^\phi : \mathcal{F}_{\Omega_1} \to U.$$ (5)

More precisely $\phi^\phi$ is characterised by the properties

$$\phi^\phi(1) = 1_U,$$ (6)

$$\phi^\phi((F_1, d_1) \cdots (F_n, d_n)) = \phi^\phi(F_1, d_1) \cdots \phi^\phi(F_n, d_n),$$ (7)

for $\Omega_1$-decorated rooted forests $(F, d), (F_1, d_1), \cdots, (F_n, d_n)$ and $\omega \in \Omega_1$.

The same applies when $\Omega_1, \Omega_2$ are monoids (resp. algebras), $U$ is an $\Omega_1$-operated monoid (resp. algebra) and $\phi : \Omega_1 \to \Omega_2$ is a map (resp. linear map).

Note that the proposition applies when $\Omega_1 = \Omega_2$, whether or not $\phi$ is the identity map. The case $\Omega_1 = \Omega_2$ and $\phi = \text{Id}$ will be of interest later (Corollary 3.9).

Proof. The map $\phi : \Omega_1 \to \Omega_2$ induces an $\Omega_1$-operated monoid structure $\tilde{\beta} := \{\tilde{\beta}^\omega | \omega \in \Omega_1\}$ on $U$ by pull-back

$$\tilde{\beta}^\omega(u) := \beta^\phi(\omega)(u), \quad \text{for all } \omega \in \Omega_1, u \in U.$$ (8)

The universal property of $\mathcal{F}_{\Omega_1}$ in Corollary 2.4 then yields a unique homomorphism $\phi^\phi : \mathcal{F}_{\Omega_1} \to U$ of $\Omega_1$-operated commutative monoids as stated in the proposition.

□
2.2. Locality sets and algebras. We first recall the concept of a locality set introduced in [CGPZ1].

**Definition 2.8.** A locality set is a couple \((X, \top)\) where \(X\) is a set and \(\top \subseteq X \times X\) is a binary symmetric relation on \(X\). For \(x_1, x_2 \in X\), denote \(x_1 \top x_2\) if \((x_1, x_2) \in \top\). We also use the alternative notations \(X \times \top X\) and \(X^\top\) for \(\top\).

In general, for any subset \(U \subseteq X\), let

\[ U^\top := \{x \in X \mid (x, U) \subseteq \top\} \]

denote the polar subset of \(U\). For integers \(k \geq 2\), denote

\[ X^{\top k} := X \times \top \cdots \top X := \{(x_1, \ldots, x_k) \in X^k \mid x_i \top x_j \text{ for all } 1 \leq i < j \leq k\}. \]

We call two subsets \(A\) and \(B\) of a locality set \((X, \top)\) independent if \(A \times B \subseteq \top\). Thus a locality relation \(\top\) on a set \(X\) induces a relation on the power set \(\mathcal{P}(X)\), which we denote by the same symbol \(\top\).

Let \((X, \top)\) be a locality set. Then the relation locality induced on \(\mathcal{P}(X)\) is symmetric, so that \((\mathcal{P}(X), \top)\) is a locality set [CGPZ1]. Furthermore, \(\mathcal{P}(X)^\top = \mathcal{P}(X^\top)\), as can be checked directly.

Recall that two maps \(\Phi, \Psi : (X, \top_X) \to (Y, \top_Y)\) are called independent and we write \(\Phi \top \Psi\) if \((\Phi \times \Psi)(\top_X) \subseteq \top_Y\), that is, \(x_1 \top_X x_2\) implies \(\Phi(x_1) \top_Y \Psi(x_2)\) for \(x_1, x_2 \in X\). A map \(\Phi : (X, \top_X) \to (Y, \top_Y)\) is called a locality map if \(\Phi \top \Phi\). Given two locality sets \((X, \top_X)\) and \((Y, \top_Y)\), let \(\text{Mor}_{\top}(X, Y)\) denote the set of locality maps from \(X\) to \(Y\).

Here are some examples of locality sets used later on.

**Example 2.9.**

(i) The power set \(\mathcal{P}(S)\) of any set \(S\) can be equipped with the independence relation:

\[ A \top B \iff A \cap B = \emptyset, \]

so that \((\mathcal{P}(S), \top)\) is a locality set with \(\mathcal{P}(S)^\top = \{\emptyset\}\).

(ii) A locality structure on decorated forests can be deduced from a locality structure on the set of decorations. Indeed, given a locality set \((\Omega, \top_\Omega)\), the set \(\mathcal{F}_\Omega\) of \(\Omega\)-decorated rooted forests can be equipped with the following independence relation induced by that of \(\mathcal{P}(\Omega)\):

\[ (F_1, d_1) \top_{\mathcal{F}_\Omega} (F_2, d_2) \iff d_1(V(F_1)) \top_{\Omega} d_2(V(F_2)) \]

Then \((\mathcal{F}_\Omega, \top_{\mathcal{F}_\Omega})\) is a locality set. Let \(K\mathcal{F}_\Omega\) be its linear span, with the induced locality relation denoted by \(\top_{\mathcal{K}\mathcal{F}_\Omega}\).

We also recall the concepts of locality monoids and locality algebras.

**Definition 2.10.**

(i) A locality semigroup is a locality set \((G, \top)\) together with a product law defined on \(\top\):

\[ m_G : G \times \top G \to G, (x, y) \mapsto x \cdot y = m_G(x, y), \quad \text{for all } (x, y) \in \top_G \]

for which the product is compatible with the locality relation on \(G\), more precisely

\[ m_G((U^\top \times U^\top) \cap \top G) \]

and such that

\[ (x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \text{for all } (x, y, z) \in G \times \top G \times \top G. \]
Note that, because of the condition (4), both sides of Eq. (1) are well-defined for any triple in the given subset.

(ii) A locality semigroup is commutative if \( m_G(x, y) = m_G(y, x) \) for \((x, y) \in \Gamma\).

(iii) A locality monoid is a locality semigroup \((G, \Gamma, m_G)\) together with a unit element \(1_G \in G\) given by the defining property

\[
\{1_G\}^\Gamma = G \quad \text{and} \quad m_G(x, 1_G) = m_G(1_G, x) = x \quad \text{for all} \ x \in G.
\]

We denote the locality monoid by \((G, \Gamma, m_G, 1_G)\).

(iv) A locality group is a locality monoid \((G, \Gamma, m_G, 1_G)\) equipped with a locality map

\[
s : G \rightarrow G, \quad g \mapsto s(g), \ \text{for all} \ g \in G,
\]

such that \((g, s(g)) \in \Gamma\) and \(m_G(g, s(g)) = m_G(s(g), g) = 1_G\) for any \(g \in G\).

(v) A locality vector space is a vector space \(V\) equipped with a locality relation \(\Gamma\) which is compatible with the linear structure on \(V\) in the sense that, for any subset \(X\) of \(V\), \(X^\Gamma\) is a linear subspace of \(V\).

(vi) Given two vector spaces \(V\) and \(W\), we equip their cartesian product \(V \times W\) with a locality structure \(\Gamma := \Gamma \times \Gamma \subseteq V \times W\) compatible with the vector space structure on the cartesian product, that is, for \(X \subseteq V\) and \(Y \subseteq W\), the subsets \(X^\Gamma\) and \(Y^\Gamma\) are subspaces of \(V\) and \(W\) respectively. A map \(f : V \times W \rightarrow U\) to a vector space \(U\) is called locality bilinear if

\[
f(v_1 + v_2, w_1) = f(v_1, w_1) + f(v_2, w_1), \quad f(v_1, w_1 + w_2) = f(v_1, w_1) + f(v_1, w_2),
\]

\[
f(kv_1, w_1) = kf(v_1, w_1), \quad f(v_1, kw_1) = kf(v_1, w_1)
\]

for all \(v_1, v_2 \in V\), \(w_1, w_2 \in W\) and \(k \in K\) for which all the pairs arising in the above expressions are in \(V \times W\).

(vii) A nonunitary locality algebra over \(K\) is a locality vector space \((A, \Gamma)\) over \(K\) together with a locality bilinear map

\[
m_A : A \times^\Gamma A \rightarrow A
\]

such that \((A, \Gamma, m_A)\) is a locality semigroup.

(viii) A locality algebra is a nonunitary locality algebra \((A, \Gamma, m_A)\) together with a unit \(1_A : K \rightarrow A\) in the sense that \((A, \Gamma, m_A, 1_A)\) is a locality monoid. We shall omit explicitly mentioning the unit \(1_A\) and the product \(m_A\) unless doing so generates ambiguity.

Here is a straightforward consequence of the above definition.

**Lemma 2.11.** Let \((G, m_G, \Gamma_G)\) be a locality semigroup. Let \(k \geq 2\) and \(1 \leq i \leq k\). For \((x_1, \ldots, x_k) \in G^\Gamma_k\) we have

(i) \((id_G^{k-1} \times m_G \times id_G^{k-1})(x_1, \ldots, x_k) \in G^\Gamma_{(k-1)}\),

(ii) \((x_1, \ldots, x_i, x_{i+1}, \ldots, x_k) \in G \times G\).

**Example 2.12.** A central example of a locality monoid in this paper is that of trees. Given a locality set \((\Omega, \Gamma_\Omega)\), the set \(\mathcal{F}_\Omega\) of \(\Omega\)-decorated rooted forests can be equipped with the independence relation defined in Example 2.11. The concatenation product of forests induces a disjoint product \((F_1, d_1) \cdot (F_2, d_2)\) of \(\Omega\)-decorated forests defined as \((F_1 \cdot F_2, d_{F_1, F_2})\) with \(d_{F_1, F_2}(V(F_1)) = d_1\) and \(d_{F_1, F_2}(V(F_2)) = d_2\). Then \((\mathcal{F}_\Omega, \Gamma_\mathcal{F}, \cdot, 1)\) is a locality monoid which induces a locality algebra structure on \(\mathbf{K}\mathcal{F}_\Omega\).
2.3. Locality operated structures. We now combine the locality and operated structures.

Definition 2.13. Let \((\Omega, \mathbb{T})\) be a locality set. An \((\Omega, \mathbb{T})\)-operated locality set or simply a locality operated set is a locality set \((U, \mathbb{T}_U)\) together with a partial action \(\beta\) of \(\Omega\) on \(U\) on a subset \(\mathbb{T}_{\Omega,U} := \Omega \times \mathbb{T} U \subseteq \Omega \times U\)

\[ \beta : \Omega \times \mathbb{T} U \rightarrow U, (\omega, x) \mapsto \beta^\omega(x), \]

satisfying the following compatibility conditions.

(i) For

\[ \Omega \times \mathbb{T} U \times \mathbb{T} U := \{ (\omega, u, u') \in \Omega \times \mathbb{T} U \times \mathbb{T} U \mid (u, u') \in \mathbb{T}_U, (\omega, u), (\omega, u') \in \Omega \times \mathbb{T} U \}, \]

the map \(\beta \times \text{Id}_U : (\Omega \times \mathbb{T} U) \times \mathbb{T} U \rightarrow \mathbb{T} U \times \mathbb{T} U\) restricts to

\[ \beta \times \text{Id}_U : \Omega \times \mathbb{T} U \times \mathbb{T} U \rightarrow U \times \mathbb{T} U. \]

In other words, if \((\omega, u, u')\) lies in \(\Omega \times \mathbb{T} U \times \mathbb{T} U\), then \((\beta^\omega u, u')\) lies in \(\mathbb{T}_U\).

(ii) For

\[ \Omega \times \mathbb{T} \Omega \times \mathbb{T} U := \{ (\omega, \omega', u) \in \Omega \times \Omega \times U \mid (\omega, \omega') \in \mathbb{T}_{\Omega}, (\omega, u), (\omega', u) \in \Omega \times \mathbb{T} U \}, \]

the map \(\text{Id}_{\Omega} \times \beta : \Omega \times (U \times \mathbb{T} U) \rightarrow \Omega \times U\) restricts to

\[ \text{Id}_{\Omega} \times \beta : \Omega \times \mathbb{T} \Omega \times \mathbb{T} U \rightarrow \Omega \times \mathbb{T} U. \]

In other words, if \((\omega, \omega', u)\) lies in \(\Omega \times \mathbb{T} \Omega \times \mathbb{T} U\), then \((\omega, \beta^{\omega'} u)\) lies in \(\Omega \times \mathbb{T} U\).

There are variations and generalisations of the compatibility conditions, such as the subsequent direct consequences of the axioms.

Lemma 2.14. Let \((U, \mathbb{T}_U)\) be an \((\Omega, \mathbb{T}_\Omega)\)-operated locality set. For \(m, n \geq 1\), denote

\[ \Omega^{m} \times \mathbb{T} U \times \mathbb{T}^n := \left\{ (\omega_1, \cdots, \omega_m, x_1, \cdots, x_n) \in \Omega^m \times \mathbb{T} U \times \mathbb{T}^n \mid \begin{array}{c} (\omega_1, \cdots, \omega_m) \in \Omega^m \\ (x_1, \cdots, x_n) \in \mathbb{T}^n \\ \omega_i \泰国 U x_j \forall (i, j) \in [m] \times [n] \end{array} \right\}. \]

\((\text{Id}_{\Omega}^{m-1} \times \beta^{\omega_m} \times \text{Id}_{\Omega}^{n-1})(\Omega^{m} \times \mathbb{T} U \times \mathbb{T}^n)\) is contained in \(\Omega^{(m-1)} \times \mathbb{T} U \times \mathbb{T}^n\).

With a similar notation, we have \(\beta (\times \beta)(\Omega \times \mathbb{T} U \times \mathbb{T} \Omega \times \mathbb{T} U) \subseteq U \times \mathbb{T} U\).

Definition 2.15. Let \((\Omega, \mathbb{T})\) be a locality set.

(i) A locality \((\Omega, \mathbb{T})\)-operated semigroup is a quadruple \((U, \mathbb{T}_U, \beta, m_U)\), where \((U, \mathbb{T}_U, m_U)\) is a locality semigroup and \((U, \mathbb{T}_U, \beta)\) is a \((\Omega, \mathbb{T})\)-operated locality set such that

\[ (\omega, u, u') \in \Omega \times \mathbb{T} U \times \mathbb{T} U \implies (\omega, uu') \in \Omega \times \mathbb{T} U; \]

(ii) A locality \((\Omega, \mathbb{T})\)-operated monoid is a quintuple \((U, \mathbb{T}_U, \beta, m_U, 1_U)\), where \((U, \mathbb{T}_U, m_U, 1_U)\) is an locality monoid and \((U, \mathbb{T}_U, \beta, m_U)\) is a \((\Omega, \mathbb{T})\)-operated locality semigroup such that \(\Omega \times 1_U \subseteq U \times \mathbb{T} U\).
(iii) A \((\Omega, \top)\)-operated locality nonunitary algebra (resp. \((\Omega, \top)\)-operated locality unitary algebra) is a quadruple \((U, \top_U, \beta, m_U)\) (resp. quintuple \((U, \top_U, \beta, m_U, 1_U)\)) which is a locality algebra (resp. unitary algebra) and a locality \((\Omega, \top)\)-operated semigroup (resp. monoid), satisfying the additional condition that for any \(\omega \in \Omega\), the set \(\{\omega\}^{\top_{\Omega, U}} := \{u \in U \mid \omega \top_{\Omega, U} u\}\) is a subspace of \(U\) on which the action of \(\omega\) is linear. More precisely, the last condition means that for any \(u_1, u_2 \in \{\omega\}^{\top_{\Omega, U}}\) and for any \(k_1, k_2 \in K\), we have \(k_1u_1 + k_2u_2 \in \{\omega\}^{\top_{\Omega, U}}\) and \(\beta^\omega(k_1u_1 + k_2u_2) = k_1\beta^\omega(u_1) + k_2\beta^\omega(u_2)\) (resp. this condition and \(\Omega \times 1_U \subset \Omega \times \top U\)).

In each case, the \((\Omega, \top)\)-operated structure is called commutative if the corresponding locality structure is.

Example 2.16. A locality semigroup \((G, \top, \cdot)\) is a locality \((G, \top)\)-operated commutative semigroup for the action \(\alpha : G \times \top G \to G\) given by the product on \(G\).

Directly from the definition, we have

Lemma 2.17. Let \((U, \top_U)\) be an \((\Omega, \top_{\Omega})\)-operated locality semigroup. For \(i, j \geq 1\), the subset \((\text{Id}_{\Omega} \times m_U \times \text{Id}_{U^j}^{-1})((\Omega^{\top i} \times \top U) \times (\top U^{-j}))\) is contained in \(\Omega^{\top i} \times \top U^{-j}\).

Lemma 2.18. A locality operated semigroup \((U, \top_U, \beta, m_U)\) expands to a locality operated nonunitary algebra \((KU, \top_{KU}, \beta, m_{KU})\) by linearity. The same holds for a locality operated monoid and unitary algebra.

The proof follows from that of the case without the \((\Omega, \top_{\Omega})\)-action.

Definition 2.19. Given \((\Omega, \top_{\Omega})\)-operated locality structures (sets, semigroups, monoids, nonunitary algebras, algebras) \((U_i, \top_{U_i}, \beta_i), i = 1, 2\), a morphism of locality operated locality structures is a locality morphism (of sets, semigroups, monoids, nonunitary algebras, algebras) \(f : U_1 \to U_2\) such that

- \((\text{Id}_{\Omega} \times f)(\Omega \times \top U_1) \subseteq \Omega \times \top U_2\) and
- \(f \circ \beta^\omega_i = \beta^\omega_f \circ f\) for all \(\omega \in \Omega\).

We therefore have the categories of \(\text{OS}_{\Omega, \top_{\Omega}}\) (resp \(\text{OSG}_{\Omega, \top_{\Omega}}\), resp. \(\text{OM}_{\Omega, \top_{\Omega}}\), resp. \(\text{OA}_{\Omega, \top_{\Omega}}\)) of \((\Omega, \top_{\Omega})\)-operated sets (resp. semigroups, resp. monoids, resp. algebras).

3. Universal properties of decorated rooted forests: The locality version

3.1. Properly decorated forests. We first equip properly decorated rooted forests and the resulting linear space with the structures of a locality \(\Omega\)-operated commutative monoid and algebra. We then prove their universal properties in the category of locality operated commutative monoids and algebras.

Definition 3.1. Let \((\Omega, \top_{\Omega})\) be a locality set. An \(\Omega\)-properly decorated rooted forest is a decorated rooted forest \(f = (F, d_F)\) whose vertices are decorated by mutually independent elements of \(\Omega\). When \(\Omega\) is clear from context, we call them properly decorated forests.

Let \(\mathcal{F}_{\Omega, \top_{\Omega}}\) denote the set of \(\Omega\)-properly decorated rooted forests, and \(K\mathcal{F}_{\Omega, \top_{\Omega}}\) be its linear span. The set \(\mathcal{F}_{\Omega, \top_{\Omega}}\) inherits the independence relation \(\top_{\mathcal{F}_{\Omega}}\) of \(\mathcal{F}_{\Omega}\), denoted by \(\top_{\mathcal{F}_{\Omega, \top_{\Omega}}}\), and \(K\mathcal{F}_{\Omega, \top_{\Omega}}\) inherits the independence relation \(\top_{K\mathcal{F}_{\Omega}}\) of \(K\mathcal{F}_{\Omega}\) denoted by \(\top_{K\mathcal{F}_{\Omega, \top_{\Omega}}}\).
It is easy to see that the disjoint union of forests in \( \mathcal{F}_{\Omega} \) defines a locality monoid structure on \( \mathcal{F}_{\Omega, \top_{\Omega}} \), and thus a locality algebra structure on \( K \mathcal{F}_{\Omega, \top_{\Omega}} \). This leads to the following straightforward yet fundamental result.

**Proposition 3.2.** Let \((\Omega, \top_{\Omega})\) be a locality set. Then

(i) \((\mathcal{F}_{\Omega, \top_{\Omega}}, \top_{\mathcal{F}_{\Omega, \top_{\Omega}}}, B_{\top}, \cdot, 1)\) is a locality \((\Omega, \top_{\Omega})\)-operated commutative monoid;

(ii) \((K \mathcal{F}_{\Omega, \top_{\Omega}}, \top_{K \mathcal{F}_{\Omega, \top_{\Omega}}}, B_{\top}, \cdot, 1)\) is a locality \((\Omega, \top_{\Omega})\)-operated commutative algebra.

**Proof.** (i) It follows from the definition that \((\mathcal{F}_{\Omega, \top_{\Omega}}, \top_{\mathcal{F}_{\Omega, \top_{\Omega}}}, 1)\) is a locality monoid. The grafting operators on this locality monoid further satisfy the conditions of a locality \((\Omega, \top_{\Omega})\)-operated monoid;

(ii) follows from (i) by linear extension. \( \square \)

**Lemma 3.3.** Let \((\Omega, \top_{\Omega})\) be a locality set. An \(\Omega\)-properly decorated rooted forest in \( \mathcal{F}_{\Omega, \top_{\Omega}} \)

(i) \((F_1, d_1) \cdots (F_n, d_n), \ n \geq 2, \) with rooted trees \(F_i \neq 1\) such that

\[
(F_i, d_i) \top_{\mathcal{F}_{\Omega, \top_{\Omega}}} (F_j, d_j), \ 1 \leq i \neq j \leq n.
\]

Furthermore \(\deg(F, d) = \deg(F_1, d_1) + \cdots + \deg(F_n, d_n)\);

(ii) \(B^\omega_{\top}(F, d)\) for some \((F, d) \in \mathcal{F}_{\Omega, \top_{\Omega}}\) which is independent of \(\bullet_{\omega}\). Furthermore,

\[
\deg(B^\omega_{\top}(F, d)) = \deg(F, d) + 1.
\]

**Proof.** The statements hold for any decorated forest without any independence requirement \([\square]\). If a decorated forest has independent decorations, then the independence conditions in the statements automatically hold. \( \square \)

The following results are also easy to verify.

**Lemma 3.4.**

(i) For rooted forests \((F_1, d_1), \ldots, (F_n, d_n) \in \mathcal{F}_{\Omega, \top_{\Omega}}\), the product forest \((F_1, d_1) \cdots (F_n, d_n)\) lies in \( \mathcal{F}_{\Omega, \top_{\Omega}} \) if and only if \((F_i, d_i) \top_{\mathcal{F}_{\Omega, \top_{\Omega}}} (F_j, d_j)\) for \(1 \leq i \neq j \leq n\);

(ii) For a decorated rooted forest \((F, d) \in \mathcal{F}_{\Omega, \top_{\Omega}}, \) the rooted tree \(B^\omega_{\top}(F, d)\) lies in \( \mathcal{F}_{\Omega, \top_{\Omega}} \) if and only if \((F, d)\) is independent of \(\bullet_{\omega}\).

The following characterisation of a morphism of operated commutative monoids will later be useful.

**Lemma 3.5.** Let \(U\) be an \((\Omega, \top_{\Omega})\)-operated commutative monoid. A map \(\eta : \mathcal{F}_{\Omega, \top_{\Omega}} \to U\) is a morphism of \((\Omega, \top_{\Omega})\)-operated commutative monoids if and only if

(i) \(\eta(1) = 1_U\);

(ii) for any \((F, d), (F', d') \in \mathcal{F}_{\Omega, \top_{\Omega}}, \) if \((F, d) \top_{\mathcal{F}_{\Omega, \top_{\Omega}}} (F', d')\), then \(\eta(F, d) \top_U \eta(F', d')\);

(iii) For any \((T_1, d_1), \ldots, (T_n, d_n) \in \mathcal{F}_{\Omega, \top_{\Omega}}\) where \((T_1, d_1), \ldots, (T_n, d_n)\) are decorated rooted trees, the equation

\[
\eta(T_1, d_1) \cdots (T_n, d_n) = \eta(T_1, d_1) \cdots \eta(T_n, d_n)
\]

holds;

(iv) If \((\omega, (F, d))\) is in \(\Omega \times_{\top} \mathcal{F}_{\Omega, \top_{\Omega}}, \) then \((\omega, \eta(F, d))\) is in \(\Omega \times_{\top} U\);

(v) For any \(B^\omega_{\top}(F, d) \in \mathcal{F}_{\Omega, \top_{\Omega}}, \) the equation \(\eta(B^\omega_{\top}(F, d)) = B^\omega_{\top}(\eta(F, d))\) holds.
Proof. (⇒). Suppose that \( \eta : F_{\Omega,T} \to U \) is a morphism of operated commutative monoids. Then conditions (i), (ii) and (iii) hold by the locality of the map \( \eta \) and its compatibility with the actions of \( (\Omega, T_{\Omega}) \). Condition (iv) is the unitary condition and Condition (v) follows since the concatenation is the product in \( F_{\Omega,T} \).

(⇐). Now suppose that all the conditions are satisfied, so that we only need to verify that \( \eta \) is multiplicative for any \( (F_1, d_1), (F_2, d_2) \in F_{\Omega,T} \) with \( (F_1, d_1)T_{\Omega}(F_2, d_2) \).

By Lemma 3.3, we have decompositions
\[
(F_i, d_i) = (T_{i,1}, d_{i,1}) \cdots (T_{i,n_i}, d_{i,n_i}), i = 1, 2,
\]
of \( (F_i, d_i) \) into rooted trees. Furthermore, by Lemma 3.4, the concatenation \( (F_1, d_1)(F_2, d_2) \) is well-defined in \( F_{\Omega,T} \) and then the decomposition of \( (F_1, d_1)(F_2, d_2) \) in Lemma 3.3 is
\[
(F_1, d_1)(F_2, d_2) = (T_{1,1}, d_{1,1}) \cdots (T_{1,n_1}, d_{1,n_1})(T_{2,1}, d_{1,1}) \cdots (T_{2,n_2}, d_{2,n_2}).
\]
This gives the multiplicativity of \( \eta \):
\[
\eta((F_1, d_1)(F_2, d_2)) = \eta((T_{1,1}, d_{1,1}) \cdots (T_{1,n_1}, d_{1,n_1})(T_{2,1}, d_{1,1}) \cdots (T_{2,n_2}, d_{2,n_2}))
\]
\[
= \eta(T_{1,1}, d_{1,1}) \cdots \eta(T_{1,n_1}, d_{1,n_1})\eta(T_{2,1}, d_{1,1}) \cdots \eta(T_{2,n_2}, d_{2,n_2})
\]
\[
= \eta(F_1, d_1)\eta(F_2, d_2),
\]
as required. \( \square \)

3.2. The universal property of properly decorated rooted forests. In this part, we study the universal properties of properly decorated rooted forests. It is well known that the number of vertices in a forest defines a grading which makes \( F \) into a graded monoid, and further \( KF \) into a graded algebra. Restricting the grading to properly decorated forests, we obtain locality graded operated monoids and algebras.

**Theorem 3.6.** Let a locality set \( (\Omega, T_{\Omega}) \) be given.

(i) The quintuple \( (F_{\Omega,T_{\Omega}}, \mathcal{T}_{\Omega,T_{\Omega}}, B_+, \cdot, 1) \) is the initial object in the category of \( (\Omega, T_{\Omega}) \)-operated commutative locality monoids. More precisely, for any \( (\Omega, T_{\Omega}) \)-operated commutative locality monoid \( U := (U, T_U, \beta_U, m_U, 1_U) \), there is a unique morphism \( \eta_U : F_{\Omega,T_{\Omega}} \to U \) of \( (\Omega, T_{\Omega}) \)-operated commutative locality monoids;

(ii) \( (K F_{\Omega,T_{\Omega}}, \mathcal{T}_{K F_{\Omega,T_{\Omega}}, B_+, \cdot, 1}) \) is the initial object in the category of \( (\Omega, T_{\Omega}) \)-operated commutative locality algebras;

**Proof.** (i) Let an \( (\Omega, T_{\Omega}) \)-operated commutative locality monoid \( (U, T_U, \beta_U, m_U, 1_U) \) be given. We only need to prove that there is a unique morphism of \( (\Omega, T_{\Omega}) \)-operated commutative locality algebras
\[
\eta = \eta_U : F_{\Omega,T_{\Omega}} \to U
\]
satisfying the conditions (i) — (v). For \( k \geq 0 \), we set
\[
F_k := \{(F, d) \in F_{\Omega,T_{\Omega}} \mid |F| \leq k\}.
\]
We will prove by induction on \( k \geq 0 \) that there is a unique map \( \eta_k : F_k \to U \) fulfilling the five conditions in Lemma 3.3 when \( F_{\Omega,T_{\Omega}} \) is replaced by \( F_k \), in which case, we call the corresponding conditions Condition \( (j)_k \) for \( j \in \{1, 2, 3, 4, 5\} \).
When \( k = 0 \), we have \( \mathcal{F}_k = \{1\} \). Then only Condition \( (\Box) \) and Condition \( (\square) \) apply when \( (F, d) = (F', d') = 1 \), giving the unique map
\[
\eta_0 : \mathcal{F}_0 \to U, \quad 1 \mapsto 1_U.
\]
Since \((1_U, 1_U) \in U \times_T U\), Condition \( (\square) \) is satisfied.

Another instructive example to study before the inductive step is the case \( k = 1 \), so \( \mathcal{F}_1 = \{1\} \cup \{\omega \mid \omega \in \Omega\} \). Since \( \omega = B^\omega(1) \), the only map \( \eta : \mathcal{F}_1 \to U \) satisfying Conditions \( (\Box) - (\square) \) is given by
\[
\eta(1) = 1_U, \quad \eta(\omega) = \eta(\beta^\omega(1)) = \beta^\omega(\eta(1)) = \beta^\omega(1_U).
\]

Now let \( k \geq 0 \) be given and assume that there is a unique map \( \eta_k : \mathcal{F}_k \to U \) satisfying Conditions \( (\Box)_k - (\square)_k \). Consider \( f = (F, d) \in \mathcal{F}_{k+1} \). If \( f \) is already in \( \mathcal{F}_k \), then \( \eta(f) \) is uniquely defined by the induction hypothesis. If \( f \in \mathcal{F}_{k+1} \) is not in \( \mathcal{F}_k \), then the degree of \( f \) is at least 1. So Lemma \( 2.11 \) shows that either there is a factorisation \( f = f_1 \cdots f_n, n \geq 2 \), into independent properly decorated rooted trees, or \( f = B^\omega_{\pm}(\overline{f}) \) for \( \overline{f} \) independent of \( \omega \) and necessarily in \( \mathcal{F}_k \). The assignment
\[
\eta_{k+1}(f) = \begin{cases} \eta_k(f_1) \cdots \eta_k(f_n), & \text{if } f = f_1 \cdots f_n; \\ \beta^\omega(\eta_k(\overline{f})), & \text{if } f = B^\omega_{\pm}(\overline{f}). \end{cases}
\]
is then well-defined since \( \eta_k \) satisfies Conditions \( (\Box)_k - (\square)_k \) by assumption.

Note that this is in fact the only way to define \( \eta_{k+1} \) satisfying the conditions in Lemma \( 2.13 \), proving the uniqueness of \( \eta_{k+1}(f) \).

Next we verify that the map \( \eta_{k+1} \) obtained this way indeed satisfies Conditions \( (\Box)_{k+1} - (\square)_{k+1} \). By the above equation and the inductive hypothesis, \( \eta_{k+1} \) satisfies Conditions \( (\Box)_{k+1}, (\square)_{k+1} \) and \( (\square)_{k+1} \).

To verify Condition \( (\square)_{k+1} \), consider \( f, f' \in \mathcal{F}_{k+1} \) with \( f^\top \mathcal{F}_{\Omega, \gamma} f' \). Depending on whether \( f \) or \( f' \) lies or not in \( \mathcal{F}_k \), there are four cases to consider. In the case when both \( f \) and \( f' \) are in \( \mathcal{F}_k \), the condition is satisfied by the induction hypothesis. For the remaining three cases, the verifications are similar, the most complicated one being when neither \( f \) nor \( f' \) lies in \( \mathcal{F}_k \). So we will only verify this case. For this case, we further have four subcases depending on which of the two forms in Lemma \( 2.13 \) that \( f \) or \( f' \) takes.

**Subcase 1.** \( f = f_1 \cdots f_n, n \geq 1 \), for independent properly decorated trees \( f_1, \ldots, f_n \) and \( f' = f'_1 \cdots f'_{n'}, f' \) for independent properly decorated trees \( f'_1, \ldots, f'_{n'} \). Then all the factor trees are pairwise independent. Since all the factor trees are in \( \mathcal{F}_k \), by the inductive assumption of Condition \( (\square)_k \), their images
\[
\eta_k(f_i), 1 \leq i \leq n, \quad \eta_k(f'_j), 1 \leq j \leq n',
\]
are pairwise independent in \( U \). By Lemma \( 2.11 \), the products \( \eta_1(f_1) \cdots \eta_n(f_n) \) and \( \eta_{n'}(f'_1) \cdots \eta_{n'}(f'_{n'}) \) are independent. But by the construction of \( \eta_{k+1} \) in Eq. \( (12) \), the last two products equal to \( \eta_{k+1}(f_1) \cdots \eta_n(f_n) \) and \( \eta_{k+1}(f'_1) \cdots \eta_{n'}(f'_{n'}) \). This gives Condition \( (\Box)_{k+1} \) in this subcase.

**Subcase 2.** \( f = B^\omega_{\pm}(\overline{f}) \) for \( (\omega, \overline{f}) \in \Omega \times_T \mathcal{F}_{\Omega, \gamma} \) and \( f' = f'_1 \cdots f'_{n'} \), \( n \geq 2 \), for independent properly decorated trees \( f'_1, \ldots, f'_{n'} \). Since \( n \geq 2 \), we have \( f' = f'_A f'_B \) with independent \( f'_A \) and \( f'_B \), both in \( \mathcal{F}_k \). Since \( f \) and \( f' \) are independent, we have
\[
(\omega, \overline{f}, f'_A, f'_B) \in \Omega \times_T \mathcal{F}_k \times_T \mathcal{F}_k \times_T \mathcal{F}_k.
\]
Then Conditions \( (\Box)_{k} \) and \( (\square)_{k} \) lead to
\[
(\omega, \eta_k(\overline{f}), \eta_k(f'_A), \eta_k(f'_B)) \in \Omega \times_T U \times_T U \times_T U.
\]
Applying Lemma 2.14 gives \((\beta^\omega(\eta_k(\bar{f})), \eta_k(\bar{f}_A)\eta_k(\bar{f}_B)) \in U \times T U\), that is, \((\eta_k(\beta^\omega(\bar{f})), \eta_k(\bar{f}_A)\bar{f}_B))\) is in \(U \times T U\), by Eq. (12). This gives Condition (14)_{k+1} in this subcase.

**Subcase 3.** \(\bar{f} = f_1 \cdots f_n, n \geq 2\), for independent properly decorated trees \(f_1, \cdots, f_n\) and \(\bar{f} = B^\omega_+ (\bar{f})\) for \((\omega, \bar{f}) \in \Omega \times T F_{\Omega, \tau_\Omega}\). This subcase follows from the previous subcase by the commutativity of the concatenation of the forest product and the locality relation.

**Subcase 4.** \(\bar{f} = B^\omega_+ (\bar{f})\) and \(\bar{f}' = B^{\omega'}_+ (\bar{f}')\) for \((\omega, \bar{f}), (\omega', \bar{f}') \in \Omega \times T F_{\Omega, \tau_\Omega}\). Since the two forests are independent, we have
\[
(\omega, \bar{f}, \omega', \bar{f}') \in \Omega \times T F_k \times \Omega \times T F_k.
\]
Then the locality of \(\eta_k\), in particular Condition (14)\(_k\), guaranteed by the induction hypothesis, gives
\[
(\omega, \eta(\bar{f}), \omega', \eta(\bar{f}')) \in \Omega \times T F_k \times \Omega \times T F_k,
\]
which yields, by Lemma 2.14 and Eq. (12),
\[
(\eta(B^\omega_+ (\bar{f})), \eta(B^{\omega'}_+ (\bar{f}'))) = (\beta^\omega(\eta(\bar{f})), \beta^{\omega'}(\eta(\bar{f}))) \in U \times T U.
\]
This gives Condition (14)\(_{k+1}\) in this subcase.

We have therefore completed the verification of Condition (14)\(_{k+1}\).

Let us finally check Condition (14)\(_{k+1}\) assuming the induction hypothesis, distinguishing two cases: \(\bar{f} \in F_{k+1}\) is of the form \(\bar{f} = f_1 \cdots f_n, n \geq 2\), for independent properly decorated trees \(f_1, \cdots, f_n \in F_k\) or \(\bar{f} = B^\omega_+ (\bar{f})\) for \((\omega, \bar{f}) \in \Omega \times T F_k\).

In the first case, we write \(\bar{f} = f_A f_B\) with \(f_A, f_B \in F_k\). Then from \((\omega, \bar{f}) \in \Omega \times T F_{k+1}\) we have \((\omega, f_A, f_B) \in \Omega \times T F_k \times F_k\) which implies \((\omega, \eta_k(f_A), \eta_k(f_B)) \in \Omega \times T U \times T U\). This gives
\[
(\omega, \eta_{k+1}(f_A f_B)) = (\omega, \eta(f_A)\eta(f_B)) \in \Omega \times T U,
\]
as needed.

In the second case, similarly we have \((\omega, \omega', \bar{f}) \in \Omega \times T \Omega \times T F_k\) which implies \((\omega, \omega', \eta(\bar{f})) \in \Omega \times T \Omega \times T U\). Therefore, \((\omega, \eta(B^\omega_+ (\bar{f}))) = (\omega, \beta^\omega(\eta(\bar{f}))) \) is in \(\Omega \times T U\) by the assumption on \(\beta\).

This completes the verification of Condition (14)\(_{k+1}\). Together with the verification of the other conditions for the existence of \(\eta_{k+1}\) above, as well as that of the uniqueness of \(\eta_{k+1}\) after Eq. (12), the inductive step is completed.

### 3.3. Branching of locality maps

We next move to the relative case. Given two locality sets \((\Omega_i, T_{\Omega_i}), i = 1, 2\), let

- \(L_{\tau} (\Omega_1, \Omega_2)\) denote the set of locality maps \(\phi : \Omega_1 \rightarrow \Omega_2\);
- \(L_{\Omega_1, \Omega_2} (U_1, U_2)\) denote the set of morphisms from a \((\Omega_1, T_{\Omega_1})\)-operated locality structure \((U_1, T_{U_1}, \beta_1)\) of certain type (i.e., semigroup, monoid, nonunitary algebra, algebra) to a \((\Omega_2, T_{\Omega_2})\)-operated locality structures \((U_2, T_{U_2}, \beta_2)\) of the same type.

Recall that all these sets are equipped with the independence relation of maps: \(\phi, \psi : (A, T_A) \rightarrow (B, T_B)\)
\[
\phi \subset \psi \iff (a_1 T_A a_2 \iff \phi(a_1) T_B \psi(a_2)).
\]

By the same proof as the one of Proposition 2.7, we obtain

**Proposition 3.7.** Let \(\phi : (\Omega_1, T_{\Omega_1}) \rightarrow (\Omega_2, T_{\Omega_2})\) be a locality map between locality sets \((\Omega_1, T_{\Omega_1})\) and \((\Omega_2, T_{\Omega_2})\).
Corollary 3.9. Let \((\Omega, \top)\) be a commutative locality monoid (resp. a unital commutative locality algebra). A map
\[
\phi : (\Omega, \top) \longrightarrow (\Omega, \top)
\]
such that \(\phi \top \text{Id}_\Omega\) induces a unique morphism of commutative locality monoids (resp. unital commutative locality algebras)
\[
\hat{\phi} : \mathcal{F}_{\Omega, \top} \longrightarrow (\Omega, \top),
\]

We call \(\phi^\sharp\) the \(\phi\)-lifted map, which by construction is characterised by the following properties:

(i) For any commutative \((\Omega_2, \top_{\Omega_2})\)-operated locality monoid \((U, \top_U, \beta_{U,+}, m_U, 1_U)\) (with its \((\Omega_1, \top_{\Omega_1})\)-operated locality monoid structure induced by \(\phi\)), there is a unique lift of \(\phi\) to a morphism of operated commutative locality monoids \(\phi^\sharp : \mathcal{F}_{\Omega_1, \top_{\Omega_1}} \longrightarrow U\), which gives rise to a map
\[
\# : (\mathcal{L}_\top(\Omega_1, \Omega_2), \top) \longrightarrow (\mathcal{L}_{\Omega_1, \Omega_2}(\mathcal{F}_{\Omega_1, \top_{\Omega_1}}, U), \top), \quad \phi \mapsto \phi^\sharp.
\]

(ii) For any commutative locality algebra \((U, \top_U, \beta_{U,+}, m_U, 1_U)\), there is a unique lift of \(\phi\) to a morphism of operated commutative locality algebras \(\phi^\sharp : \mathcal{F}_{\Omega_1, \top_{\Omega_1}} \longrightarrow U\), which gives rise to a map
\[
\# : (\mathcal{L}_\top(\Omega_1, \Omega_2), \top) \longrightarrow (\mathcal{L}_{\Omega_1, \Omega_2}(\mathcal{K}\mathcal{F}_{\Omega_1, \top_{\Omega_1}}, U), \top), \quad \phi \mapsto \phi^\sharp.
\]

The following example of locality operated monoids will be useful in the sequel.

Lemma 3.8. Let \((\Omega, \top_{\Omega}, m_{\Omega}, 1_{\Omega})\) be a locality monoid and let \(\phi : \Omega \longrightarrow \Omega\) be a map such that \(\phi \top \text{Id}_\Omega\). Define
\[
\beta_\phi : \top_{\Omega} \longrightarrow \Omega, \quad (\omega, \omega') \longmapsto \beta_\phi(\omega) := \phi(m_{\Omega}(\omega, \omega')).
\]

Then \((\Omega, \top_{\Omega}, \beta_\phi, m_{\Omega}, 1_{\Omega})\) is an \((\Omega, \top_{\Omega})\)-operated monoid.

Proof. Let \((\Omega, \top_{\Omega})\) and \(\phi : \Omega \longrightarrow \Omega\) be as in the statement of the lemma. By symmetry of \(\top_{\Omega}\), \(\phi \top \text{Id}_\Omega\) implies \(\phi \top \text{Id}_\Omega\), i.e. that \(\phi\) is a locality map.

The axioms for an \((\Omega, \top_{\Omega})\)-locality operated monoid on \(U = \Omega\) are checked as follows.

Let \((\omega, u, u')\) be in \(\Omega^{\top_3}\). Then \(m_{\Omega}(\omega, u) \top_{\Omega} u'\) holds. Since \(\phi \top \text{Id}_\Omega\), we obtain \((\beta_\phi(\omega), u') = (\phi(m_{\Omega}(\omega, u)), u')\) which is in \(\top_{\Omega}\).

Let \((\omega, \omega', u)\) be in \(\Omega^{\top_3}\). Then \(\omega \top_{\Omega} m_{\Omega}(\omega', u)\) and hence \(\omega \top_{\Omega} \phi(m_{\Omega}(\omega', u))\). This means that \((\omega, \beta_\phi(\omega') = m_{\Omega}(\omega, u'))\) is in \(\top_{\Omega}\).

Finally \(\Omega \times 1_{\Omega} \subset \top_{\Omega}\) since \((\Omega, \top_{\Omega}, m_{\Omega}, 1_{\Omega})\) is a locality monoid.

Applying Proposition 3.7, we obtain

Corollary 3.9. Let \((\Omega, \top_{\Omega})\) be a commutative locality monoid (resp. a unital commutative locality algebra). A map
\[
\phi : (\Omega, \top_{\Omega}) \longrightarrow (\Omega, \top_{\Omega})
\]
such that \(\phi \top \text{Id}_\Omega\) induces a unique morphism of commutative locality monoids (resp. unital commutative locality algebras)
\[
\hat{\phi} : \mathcal{F}_{\Omega, \top_{\Omega}} \longrightarrow (\Omega, \top_{\Omega}),
\]
(resp. $\hat{\phi} : \mathcal{F}_{\Omega, \Omega} \rightarrow (\Omega, \Omega))$. $\hat{\phi}$ is called the $\phi$-branched map. By construction it is characterised by the following properties:

\begin{align}
\hat{\phi}(\emptyset) &= 1_{\Omega} \\
\hat{\phi}((F_1, d_1) \cdots (F_n, d_n)) &= \hat{\phi}(F_1, d_1) \cdots \hat{\phi}(F_n, d_n) \\
\hat{\phi}(B_{\omega}^F(F, d)) &= \phi(\omega(\hat{\phi}(F, d))),
\end{align}

for any mutually independent properly decorated forests $(F_1, d_1), \ldots, (F_n, d_n) \in \mathcal{F}_{\Omega, \Omega}$, and any $\omega \in \Omega$ which is independent of $(F, d)$.

Proof. As before, $\phi \triangleright Id$ implies $\phi \triangleright \phi$, i.e. that $\phi$ is a locality map. Then by Lemma 3.8 we can apply Proposition 3.7 in the special case $\Omega_1 = \Omega_2 = \Omega$ with the map $Id_{\Omega} : (\Omega, \Omega) \rightarrow (\Omega, \beta_{\phi})$. Then we have $\hat{\phi} := Id_{\phi}$, which has the stated properties by the definition of $\phi$. □

Remark 3.10. • Writing $\hat{\phi} := Id_{\phi}$ might seem confusing since no $\phi$ appears on the right hand side. Of course, this is a notation artifact: $\phi$ does play a role in the right hand side, since the lift $\phi$ is made with respect to the operation $\beta_{\phi}$. A more rigorous notation would be $Id_{\phi, \beta_{\phi}}$ which we have not opted for in order to lighten the notations.
• One could also prove the Corollary as a consequence of Theorem 3.6, in the same way that Proposition 2.6 is derived from Corollary 2.4.

4. Multivariate regularisation of branched integrals

We apply the framework previously developed to study Kreimer’s toy model in building a regularisation by means of locality algebra homomorphisms. So our first goal is to define a suitable decoration set. There are several possible decoration sets; in this paper we use a very simple one similar to that in [GPZ1]. An alternative choice can be found in [CGPZ1].

4.1. Linear complex powers: the locality algebra $\mathcal{M}[L]$. We adapt the terminology from [GPZ1] to the algebraic locality framework developed in [CGPZ1].

Consider a sequence $(\mathbb{R}^\infty, Q) := (\mathbb{R}^n, Q_n)_{n \geq 1}$ of Euclidean space with inner product $Q_n$ on $\mathbb{R}^n$ such that, under the direct system $i_n : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 1$, of standard embeddings, we have $Q_{n+1}|_{\mathbb{R}^n} = Q_n, n \geq 1$.

Since $Q_n$ on $\mathbb{R}^n$ induces an isomorphism

$$Q_n^* : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^*,$$

the maps

$$j_n = (Q_n^*)^{-1} \circ i_n^* \circ Q_{n+1}^* : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n,$$

give rise to the direct systems

$$j_n^* : (\mathbb{R}^n)^* \rightarrow (\mathbb{R}^{n+1})^*,
\quad j_n^* : \mathcal{M}(\mathbb{R}^n \otimes \mathbb{C}) \rightarrow \mathcal{M}(\mathbb{R}^{n+1} \otimes \mathbb{C}).$$

Here $\mathcal{M}(\mathbb{R}^n \otimes \mathbb{C})$ is the algebra of multivariate meromorphic germs at 0 with linear poles and real coefficients [GPZ1, GPZ2].

We set

$$L := \lim_{n \rightarrow} (\mathbb{R}^n)^*, \quad \mathcal{M} := \lim_{n \rightarrow} \mathcal{M}(\mathbb{R}^n \otimes \mathbb{C}).$$
By \cite{GPZ2,GPZ3}, the algebra $\mathcal{M}(\mathbb{R}^n \otimes \mathbb{C})$ is equipped with the linear decomposition
\[
\mathcal{M}(\mathbb{R}^n \otimes \mathbb{C}) = \mathcal{M}_+ (\mathbb{R}^n \otimes \mathbb{C}) \oplus \mathcal{M}_- (\mathbb{R}^n \otimes \mathbb{C})
\]
as the direct sum of the subalgebra $\mathcal{M}_+ (\mathbb{R}^n \otimes \mathbb{C})$ of holomorphic germs and the space $\mathcal{M}_- (\mathbb{R}^n \otimes \mathbb{C})$ generated by polar germs, defined to be fractions of the form
\[
\frac{h(\ell_1, \cdots, \ell_k)}{L_1 \cdots L_n},
\]
where $\ell_1, \cdots, \ell_k, L_1, \cdots, L_n$ are linear form such that $\{\ell_1, \cdots, \ell_k\} \perp \{L_1, \cdots, L_n\}$. These decompositions are compatible with the maps $j_n^\star$, thus we have
\[
\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-, \quad \mathcal{M}_\pm := \lim_{n \to \infty} \mathcal{M}_\pm (\mathbb{R}^n \otimes \mathbb{C}).
\]

Let
\[
\pi_+: \mathcal{M} \to \mathcal{M}_+
\]
denote the projection onto $\mathcal{M}_+$ along $\mathcal{M}_-$. In the algebra of functions with complex variables in $\mathcal{C}^\infty$ and with one real variable $x$, we consider the $\mathcal{M}$-module of linear combinations
\[
\left\{ \sum_{i=1}^k f_i x^{L_i} \big| f_i \in \mathcal{M}, L_i \in \mathcal{L}, 1 \leq i \leq k, k \geq 1 \right\}.
\]
It is an $\mathcal{M}$-subalgebra since $x^{L}, L \in \mathcal{L}$ is closed under multiplication. It can further be checked that $x^{L}, L \in \mathcal{L}$, are linearly independent over $\mathcal{M}$. Thus it is isomorphic to the group ring $\mathcal{M}[\mathcal{L}]$ over $\mathcal{M}$ generated by the additive monoid $\mathcal{L}$. We will henceforth make this identification.

**Remark 4.1.** Elements $x^{L} \in \mathcal{M}[\mathcal{L}]$ are particular instances of the more general (multivariate) holomorphic families of classical symbols considered in \cite{CGPZ2} – with the difference that in \cite{CGPZ2} they are symbols on $[0, +\infty)$ whereas here they are defined on $(0, +\infty)$.

We equip the algebra $\mathcal{M}$ with the independence relation $\perp$ introduced in \cite{CGPZ1}, Definition 2.14
\[
(f \perp g) \iff (\text{Dep}(f) \perp \text{Dep}(g)).
\]
The pointwise product gives rise to a locality algebra $(\mathcal{M}, \perp)$ and $(\mathcal{L}, \perp)$ is viewed as a locality subspace of the locality linear space $(\mathcal{M}, \perp)$. See \cite{CGPZ3} for another locality structure on $\mathcal{M}$.

The locality relation $\perp$ on $\mathcal{M}$ induces one on the space $\mathcal{M}[\mathcal{L}]$:
\[
\left( \sum_i f_i x^{L_i} \perp \sum_j g_j x^{L_j} \right) \iff (\{f_i, L_i\}_i \perp \{g_j, \ell_j\}_j \text{ in } \mathcal{M})
\]
and $(\mathcal{M}[\mathcal{L}], \perp)$ is a locality algebra.

Using $\mathcal{L}$ as the set of decorations for rooted forests in the previous sections, we have the $\mathcal{L}$-operated monoid $\mathcal{F}_\mathcal{L}$, the $(\mathcal{L}, \perp)$-operated locality monoid $\mathcal{F}_{\mathcal{L}, \perp}$ and the corresponding locality algebra $\mathbb{R}[\mathcal{F}_{\mathcal{L}, \perp}]$. 

**LOCALITY AND RENORMALISATION 17**
4.2. \( M[\mathcal{L}] \) as an operated locality algebra. We revisit a map defined in \([GPZ4]\), viewed here as an operating map on \( M[\mathcal{L}] \):

**Lemma 4.2.** For any \( L \in \mathcal{L} \), the operator

\[
\mathcal{J}^L(f)(x) := \int_0^\infty \frac{f(y)y^{-L}}{y + x} \, dy,
\]

defines a linear map from \( M[\mathcal{L}] \) to \( M[\mathcal{L}] \). With

\[
\mathcal{J} : \mathcal{L} \times M[\mathcal{L}] \longrightarrow M[\mathcal{L}], \quad (L, f) \longmapsto \mathcal{J}^L(f),
\]
denoting the resulting action of \( \mathcal{L} \) on \( M[\mathcal{L}] \), the triple \((M[\mathcal{L}], \perp, \mathcal{J})\) is an \((\mathcal{L}, \perp)\)-operated algebra.

**Proof.** We first prove that \( \mathcal{J}^L \) defines a map from \( M[\mathcal{L}] \) to \( M[\mathcal{L}] \). By the \( M \)-linearity, we only need to prove \( \mathcal{J}^L(x^\ell) \in M[\mathcal{L}] \).

For fixed \( x > 0 \) and \( z \in \mathbb{C} \), the map \( y \longmapsto \frac{y^{-z}}{(y+x)^{k}} \) is locally integrable on \((0, +\infty)\). Since \( \frac{y^{-z}}{(y+x)^{k}} \sim C y^{-z} \) and \( \frac{y^{-z}}{(y+x)^{k}} \sim y^{-z-k} \), the integral \( \int_0^\infty \frac{y^{-z}}{(y+x)^{k}} \, dy \) converges absolutely on the strip \( \Re(z) \in (1-k, 1) \) and the map \( x \longmapsto \frac{y^{-z}}{y+x} \) is smooth with \( k \)-th derivative \( x \longmapsto (-1)^k k! \int_0^\infty \frac{y^{-z}}{(y+x)^{k}} \, dy \).

For \( \alpha > 0 \), we have

\[
\lim_{x \rightarrow 0^+} x^\alpha \ln(x) = 0, \quad \lim_{x \rightarrow \infty} x^{-\alpha} \ln(x) = 0,
\]
so that for fixed \( x > 0 \), the map \( z \longmapsto \int_0^\infty \frac{y^{-z}}{y+x} \, dy \) is holomorphic in \( z \) on the strip \( \Re(z) \in (0, 1) \) with derivative \( z \longmapsto -z \int_0^\infty \frac{\ln y y^{-z}}{y+x} \, dy \). For any real number \( a \in (0, 1) \), an explicit computation \([GPZ4]\) Lemma 4.5] gives:

\[
\int_0^\infty \frac{x^{-a}}{x+1} \, dx = \frac{\pi}{\sin(\pi a)},
\]

and hence

\[
\int_0^\infty \frac{y^{-a}}{y+x} \, dy = x^{-a} \int_0^\infty \frac{y^{-a}}{y+1} \, dy = \frac{\pi x^{-a}}{\sin(\pi a)}.
\]

It follows that for any complex number in the strip \( \Re(z) \in (0, 1) \), we have

\[
\int_0^\infty \frac{y^{-z}}{y+x} \, dy = \frac{\pi}{\sin(\pi z)} x^{-z}.
\]

As a consequence of this explicit formula, the map \( z \longmapsto \int_0^\infty \frac{y^{-z}}{y+x} \, dy \) extends to a meromorphic family (in the parameter \( z \)) of smooth functions in \( x \). Thus, when restricted to a neighborhood of 0 in the parameter space, \( \mathcal{J}^L \) can be viewed as a \( M \)-linear map from \( M[\mathcal{L}] \) to \( M[\mathcal{L}] \). More precisely,

\[
\mathcal{J}^L(f x^{-\ell}) = \left( x \mapsto f(x) \frac{\pi}{\sin(\pi (L + \ell))} x^{-L-\ell} \right), \quad f \in M, L \in \mathcal{L}
\]

(on the l.h.s. we write \( x^{-\ell} \) for \( x \rightarrow x^{-\ell} \in M[\mathcal{L}] \)). The conditions for \((M[\mathcal{L}], \mathcal{J})\) to be an \((\mathcal{L}, \perp)\)-operated locality algebra are then easy to check. \(\square\)
4.3. Locality morphisms on rooted forests. Recall from Eq. (23) that $\mathbb{R}\mathcal{F}_{\mathcal{L},\bot}$ comes equipped with a locality relation induced by the locality relation $\bot$ on $\mathcal{L}$, for which $(\mathbb{R}\mathcal{F}_{\mathcal{L},\bot}, \bot)$ is a locality algebra. The universal property of the $(\mathcal{L}, \bot)$-operated algebra $\mathbb{R}\mathcal{F}_{\mathcal{L},\bot}$ discussed in Theorem 4.3 yields a unique locality algebra homomorphism

$$\mathcal{R} : (\mathbb{R}\mathcal{F}_{\mathcal{L},\bot}, \mathcal{T}_{\mathcal{F}_{\mathcal{L},\bot}, B^\bot}) \to (\mathcal{M}[\mathcal{L}], \mathcal{T}, \mathcal{I}),$$

(with $\mathcal{I} := \{\mathcal{I}^L, L \in \mathcal{L}\}$) which is characterized by the following conditions:

\begin{align*}
(25) & \quad \mathcal{R}(\bullet_L) = \int_0^\infty \frac{y^{-L}}{y + x} dy = \frac{\pi}{\sin(\pi L)} x^{-L}, \\
(26) & \quad \mathcal{R}((F_1, d_1)(F_2, d_2)) = \mathcal{R}(F_1, d_1)\mathcal{R}(F_1, d_2) \quad \text{for all } (F_1, d_1)(F_2, d_2) \in \mathcal{F}_{\mathcal{L},\bot}, \\
(27) & \quad \mathcal{R}(B^L_+(F, d)) = \mathcal{I}^L(\mathcal{R}(F, d)) \quad \text{for all } B^L_+(F, d) \in \mathcal{F}_{\mathcal{L},\bot}.
\end{align*}

The subsequent statement follows from [GPZ4].

**Lemma 4.3.** For any properly decorated rooted forest $(F, d) \in \mathcal{F}_{\mathcal{L},\bot}$,

$$\mathcal{R}(F, d)(x) = x^{-\sum_{v \in V(F)} d(v)} \prod_{v \in V(F)} \frac{\pi}{\sin(\pi L_v)},$$

where, for any $v \in V(F)$, $L_v$ is the sum of decorations associated to vertices of the subtrees with root $v$: $L_v := \sum_{w \geq v} d(w)$.

**Proof.** Because of the uniqueness of the operated algebra homomorphism $\mathcal{R}$ satisfying the conditions in Eqs. (23)–(27), we only need to verify that the desired equation in the lemma satisfies the same three properties. Taking $(F, d) = \bullet_L$ gives Eq. (23). The second and third properties are easy to verify noting that the right hand side of the desired equation is

$$\prod_{v \in V(F)} x^{-d(v)} \frac{\pi}{\sin(\pi L_v)}$$

and using the explicit expression (23) of $\mathcal{I}^L$. \hfill \Box

We also prove the following property for later use.

**Lemma 4.4.** With the notation in Lemma 4.3, the set $\{L_v\}_{v \in V(F)}$ is linearly independent.

**Proof.** Suppose not, then there is a linear combination $\sum_{v \in V(F)} c_v L_v = 0$ in which not all coefficients are zero. Let $v_0$ be maximal such that $c_{v_0} \neq 0$ with respect to the order $v \leq w$ if $w \in F_v$. By the definition of $L_v$, we have

$$\sum_{v \in V(F)} c_v L_v = \sum_{w \in V(F)} b_w d(w),$$

where $b_w = \sum_{v \leq w} c_v$. Since $F$ is properly decorated, from $\sum_{v \in V(F)} c_v L_v = 0$ we obtain $b_w = 0$ for all $w \in V(F)$. Now by the maximality of $v_0$ with nonzero coefficients, we have $0 = b_{v_0} = \sum_{v \leq v_0} c_v = c_{v_0}$. This gives the desired contradiction. \hfill \Box
5. Multivariate renormalisation of branched integrals

5.1. The minimal subtraction scheme on fractions. In preparation for the actual multivariate renormalisation, in this paragraph, we investigate the holomorphic projections of a special family of meromorphic germs, that is \( \pi_+ \left( \frac{g(L_w, w \in W)}{\prod_{v \in V} L_v} \right) \) of a fraction \( \frac{g(L_w, w \in W)}{\prod_{v \in V} L_v} \) built from a holomorphic germ \( g \), together with their evaluation at zero \( \text{ev}_0 \circ \pi_+ \left( \frac{g(L_w, w \in W)}{\prod_{v \in V} L_v} \right) \), for a fixed set of independent linear forms \( \{L_w, w \in W\} \) indexed by \( W \) and a subset \( V \) of \( W \). The latter expression amounts to implementing a minimal substraction scheme on the fraction \( \frac{g(L_w, w \in W)}{\prod_{v \in V} L_v} \).

**Theorem 5.1.** Given \( V \subset W \) and \( g = g(z_w, w \in W) \) a holomorphic germ at zero,

(i) there are holomorphic germs at zero \( f_v(L_w, w \in W) \) for \( v \in V \) such that

\[
\pi_+ \left( \frac{g(L_w, w \in W)}{\prod_{v \in V} L_v} \right) = \sum_{v \in V} \pi_+ \left( \frac{f_v(L_w, w \in W)}{\prod_{u \in V \setminus \{v\}} L_u} \right).
\]

(ii) for two sets of linear forms \( L_{i v}, v \in W, i = 1, 2 \) related by

\[
Q(L_{1 v}, L_{1 w}) = c Q(L_{2 v}, L_{2 w})\quad \text{for all } v, w \in W,
\]

with \( c \) some constant, we have

\[
\text{ev}_0 \circ \pi_+ \left( \frac{g(L_{1 v}, w \in W)}{\prod_{v \in V} L_{1 v}} \right) = \text{ev}_0 \circ \pi_+ \left( \frac{g(L_{2 v}, w \in W)}{\prod_{v \in V} L_{2 v}} \right).
\]

**Proof.** The proof uses techniques similar to those of the proof of [GPPZ Theorem 2.11].

When \( |V| = 0 \), the fraction is already holomorphic and hence the result is the holomorphic function. In general, for a given \( V \subset W \), let

\[ (\mathbb{R}^n)^* = \text{span}(V) \bigoplus \text{span}(V)^\perp \]

be the decomposition of \((\mathbb{R}^n)^*\) as the direct sum of the linear span of \( L_v, v \in V \) (which, without loss of generality, can be assumed to be linearly independent) and its orthogonal complement. Then \( w \in W \) has the decomposition

\[
L_w = L''_w + L' = \sum_{v \in V} a_{w v} L_v + L'_{w}.
\]

The projections \( L'_{w} \) and \( L''_{w} \) are unique, thus the choice of the coefficients \( a_{w v} \) with respect to the linearly independent set \( \{L_v, v \in V\} \) is unique.

We therefore get the decomposition

\[
\frac{g(L_w, w \in W)}{\prod_{v \in V} L_v} = \frac{g(L'_{w}, w \in W)}{\prod_{v \in V} L_v} + \frac{g(L_w, w \in W) - g(L'_{w}, w \in W)}{\prod_{v \in V} L_v},
\]

where

\[
\frac{g(L'_{w}, w \in W)}{\prod_{v \in V} L_v}
\]
is a polar germ as defined in Eq. (13) and hence is annihilated by $\pi_+$. Thus we only need to consider the second fraction

$$\frac{g(L_w, w \in W) - g(L'_w, w \in W)}{\prod_{v \in V} L_v} \times \prod_{v \in V} L_v$$

(32)

Using the expression of $L'_w = L_w - \sum a_{uv} L_v$ from Eq. (31), we note that the fraction is a function in $L_v, v \in W$.

Before proceeding further, we introduce some notations to simplify the presentation, setting $\vec{L} := (L_w, w \in W), \vec{L}' := (L'_w, w \in W)$. For $v \in V, u \in W$, we set $\vec{L}_{uv} := (\delta_{uw} L_v, w \in W)$. Then we have

$$\vec{L} = \vec{L}' + \sum_{j=1}^{N} c_j \vec{M}_j, \quad \text{where } \vec{M}_j = \vec{L}_{uv} \text{ and } c_j = a_{uv} \text{ for some } v \in V, u \in W.$$ (33)

With these notations, we have the telescopic expansion

$$g(L'_w + \sum_{v \in V} c_{uv} L_v, w \in W) - g(L'_w, w \in W)$$

$$= g(\vec{L}' + \sum_{j=1}^{N} c_j \vec{M}_j) - g(\vec{L}')$$

$$= \sum_{\ell=1}^{N} \left( g(\vec{L}' + \sum_{j=1}^{\ell} c_j \vec{M}_j) - g(\vec{L}' + \sum_{j=1}^{\ell-1} c_j \vec{M}_j) \right)$$

with the convention that the sum over the empty set is zero.

If $\vec{M}_\ell = \vec{L}_{uv}$, then the fraction

$$h_\ell(L_w, w \in W) := \frac{g(\vec{L}' + \sum_{j=1}^{\ell} c_j \vec{M}_j) - g(\vec{L}' + \sum_{j=1}^{\ell-1} c_j \vec{M}_j)}{L_{uv}}$$

is holomorphic as a function of $L_w, w \in W$. So

$$\frac{g(\vec{L}' + \sum_{j=1}^{\ell} c_j \vec{M}_j) - g(\vec{L}' + \sum_{j=1}^{\ell-1} c_j \vec{M}_j)}{\prod_{v \in V} L_v} = \frac{h_\ell(L_w, w \in W)}{\prod_{v \in V \setminus \{v_{\ell} \}} L_v}, 1 \leq \ell \leq N.$$ 

Now taking

$$f_v(L_w, w \in W) := \sum h_\ell(L_w, w \in W)$$

where the sum is over all $\ell$ with $\vec{M}_\ell = \vec{L}_{uv}$ for some $u \in W$, we have Eq. (34).

To prove Eq. (34) we proceed by induction on $|V|$ along the lines of the first part of the proof. The identity clearly holds when $|V| = 0$ since both sides equal to $g(z_w, w \in W)|_{z_w=0}$. 

[1]
Assume that the equation holds when \(|V| = k\) for \(k \geq 0\) and consider the case when \(|V| = k + 1\). For \(w \in W\) and \(i = 1, 2\), the decomposition in Eq. (31) yields

\[
L_{iw} = L_{iw}^{n} + L_{iw}^{t} = \sum_{v \in V} a_{iwv}L_{iv} + L_{iw}^{t}.
\]  

By Eq. (24), we have

**Claim 5.2.** For any \(w \in W\) and \(v \in V\), the equations \(a_{1wv} = a_{2wv}\) holds.

**Proof.** Taking inner product with \(L_{iu}, u \in V\) in Eq. (34), we have a linear system for \(a_{iwv}\):

\[
\sum_{v \in V} Q(L_{iv}, L_{iu})a_{iwv} = Q(L_{iw}, L_{iu}).
\]

Since \(\{L_{iu}, u \in V\}\) is linearly independent by assumption, the matrix

\[
(Q(L_{iv}, L_{iu})_{u,v \in V})
\]

is invertible and this system has a unique solution, as already shown before. Further, by Eq. (29), the coefficients of the linear system for \(i = 2\) are \(c\) times those for \(i = 1\). Hence the two systems have the same solution, proving the equation. \(\square\)

With notations similar to those of the first part of the proof,

\[
ev_0 \circ \pi_+ \left( \frac{g(L_{iw}, w \in W)}{\prod_{v \in V} L_{iv}} \right) = \sum_{\ell=1}^{N} \left( \frac{h_{\ell}(L_{iw}, v \in W)}{\prod_{v \in V \backslash \{v_{\ell}\}} L_{iv}} \right) \]

By Claim 5.2, if \(L_{1v}\) is replaced by \(L_{2v}, v \in W\), then \(L_{1v}', L_{1j}', c_{1j}, M_{1j}, \hat{M}_{1j}\) are replaced by \(L_{2v}', L_{2j}', c_{2j}, M_{2j}, \hat{M}_{2j}\) respectively. Thus the fraction

\[
\frac{h_{\ell}(L_{iw}, v \in W)}{\prod_{v \in V \backslash \{v_{\ell}\}} L_{iw}}
\]

when \(i = 2\) is obtained from the fraction when \(i = 1\) upon replacing its variables \(L_{iv}\) by \(L_{2v}, v \in W\). Thus the induction assumption yields the conclusion. \(\square\)

### 5.2. Locality morphisms on properly decorated rooted forests

In order to renormalise branched integrals by means of a locality multivariate regularisation, we need to introduce some locality morphisms. We state a straightforward yet useful preliminary result without proof.

**Lemma 5.3.** The evaluation map at \(x = 1\):

\[
ev_{1} := ev_{x=1} : (\mathcal{M}[\mathcal{L}], \perp) \rightarrow (\mathcal{M}, \perp), \quad f x^{L} \mapsto f, \quad f \in \mathcal{M}, L \in \mathcal{L},
\]

is a homomorphism of locality algebras.

Combining the locality properties of \(ev_{1}\) and \(\mathcal{R}\) leads to the following properties of their composition:
Proposition 5.4. The map
\[ R_1 := \text{ev}_1 \circ R : \mathcal{RF}_{\mathcal{L}_\perp} \to \mathcal{M} \]
sends properly decorated forests to
\[ R_1(F,d) = \prod_{v \in V(F)} \frac{\pi}{\sin(\pi L_v)} \quad (\text{with } L_v \text{ as in Lemma 4.3}) \]
and defines a homomorphism of locality algebras
\[ R_1 : (\mathcal{RF}_{\mathcal{L}_\perp}, \perp, \perp) \to (\mathcal{M}, \perp). \]
In particular, for independent properly decorated forests \( (F,d) \) and \( (F',d') \), we have
\[ R_1(F,d) \perp R_1(F',d') \quad \text{and} \quad R_1((F,d) \bullet (F',d')) = R_1(F,d) R_1(F',d'). \]

Recall from Eq. (21) that \( \mathcal{M}_- \) is a locality ideal of \( \mathcal{M} \), implying that \( \pi_+ : \mathcal{M} \to \mathcal{M}_+ \) is a locality morphism as proved in [CGPZ1]. Composing the locality algebra homomorphism \( R_1 : \mathcal{RF}_{\mathcal{L}_\perp} \to \mathcal{M} \) with the locality morphism \( \pi_+ \) and the evaluation
\[ \text{ev}_0 := \text{ev}_{z=0} : \mathcal{M}_+ \to \mathbb{C} \]
of a holomorphic germ at zero yields the subsequent statement.

Proposition-Definition 5.5. The map \( R_{\text{ren}} \) on \( \mathcal{RF}_{\mathcal{L}_\perp} \) defined by
\[ R_{\text{ren}} := \text{ev}_0 \circ \pi_+ \circ R_1 \]
is a locality character on the locality algebra \( \mathcal{RF}_{\mathcal{L}_\perp} \), called the renormalised character.

5.3. Renormalised values on similar properly decorated rooted forests. Theorem 5.1 provides a useful recursive procedure for \( R_{\text{ren}} \) by means of an algorithm to evaluate the renormalised value of any given branched integral.

Inserting in Eq. (38), the Laurent expansion at \( x = 0 \),
\[ \frac{\pi}{\sin(\pi x)} = \frac{1}{x} + h(x), \]
where \( h(x) \) is holomorphic, yields
\[ R_1(F,d) = \prod_{v \in V(F)} \left( \frac{1}{L_v} + h(L_v) \right) \]
\[ = \sum_{V \subseteq V(F)} \frac{1}{\prod_{v \in V} L_v} \left( \prod_{v \in V(F) \setminus V} h(L_v) \right) \]
\[ = \sum_{V \subseteq V(F)} g(L_w, w \in V(F) \setminus V), \]
where \( g(z_w, w \in V(F) \setminus V) := \prod_{w \in V(F) \setminus V} h(z_w) \) is holomorphic. In view of Lemma 4.4, the linear forms \( \{ L_w, w \in V(F) \} \) are linearly independent, thus the fraction can be regarded as a function with variables in \( L_w, w \in V(F) \).

The renormalised locality character \( R_{\text{ren}} \) has very special properties besides functorial ones.
Definition 5.6. Two properly decorated rooted forests \((F_1,d_1)\) and \((F_2,d_2)\) are called similar if \(F_1 = F_2\) and if there exists a constant \(c \in \mathbb{R}_{>0}\) such that \(Q(d_1(v),d_1(v)) = cQ(d_2(v),d_2(v))\) for any \(v \in V(F_1) = V(F_2)\).

Corollary 5.7. For similar properly decorated rooted forests \((F_1,d_1)\) and \((F_2,d_2)\), we have 
\[
\mathcal{R}^{\text{ren}}(F_1,d_1) = \mathcal{R}^{\text{ren}}(F_2,d_2).
\]

Proof. Let \(F_v\) denote the (maximal) subtree of \(F = F_1 = F_2\) with root \(v\). As before let 
\[
L_{iv} = \sum_{w \in V(F_v)} d_i(w) \quad \text{for} \quad i = 1,2,
\]
and \(V(F) = V(F_1) = V(F_2)\).

Note that for any \(v,w \in V(F)\), the intersection \(F_v \cap F_w\) is either \(\emptyset\) or \(F_v\) (when \(v \in V(F_w)\)) or \(F_w\) (when \(w \in V(F_v)\)). Thus 
\[
Q(L_{iv}, L_{iw}) = \begin{cases} 
0, & F_v \cap F_w = \emptyset, \\
\sum_{u \in F_v} Q(d_i(u),d_i(u)), & F_v \cap F_w = F_v, \\
\sum_{u \in F_w} Q(d_i(u),d_i(u)), & F_v \cap F_w = F_w.
\end{cases}
\]

Thus by the similarity of \((F_1,d_1)\) and \((F_2,d_2)\), we have Eq. (29).

Let \(F = F_i, i = 1,2\), in Eq. (29), we have 
\[
\mathcal{R}_1(F_i,d_i) = \sum_{V \subset V(F)} \frac{g(L_{iw}, w \in V(F))}{\prod_{v \in V} L_{iv}},
\]
where \(g(z_v, v \in V(F) \setminus V) := \prod_{v \in V(F \setminus V)} h_{i,w}(z_v)\) is holomorphic in variables \(\{L_{iw}, w \in V(F)\}\). Theorem 29 and 30 with \(W = V(F)\) yields the statement.

This concludes the renormalisation of branched integrals via locality morphisms by means of the multivariate renormalisation scheme developed in [GPZ1] in the framework of Kreimer’s toy model.

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