LOGARITHMIC MODELS AND MEROMORPHIC FUNCTIONS IN DIMENSION TWO

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Abstract. In this article we describe the construction of logarithmic models for germs of plane singular analytic foliations, both real and complex. A logarithmic model is a germ of closed meromorphic 1-form with simple poles produced upon some specified geometric data: the structure of dicritical (non-invariant) components in the exceptional divisor of its reduction of singularities, a prescribed finite set of separatrices — invariant analytic branches at the origin — and Camacho-Sad indices with respect to these separatrices. As an application, we use logarithmic models in order to construct real and complex germs of meromorphic functions with a given indeterminacy structure and prescribed sets of zeroes and poles. Also, in the real case, logarithmic models are used in order to build germs of analytic vector fields with a given Bendixson’s sectorial decomposition of a neighborhood of $0 \in \mathbb{R}^2$ into hyperbolic, parabolic and elliptic sectors. This is carried out in the specific case where all trajectories accumulating to the origin are contained in analytic curves. As a consequence, we can produce real meromorphic functions with a prescribed sectorial decomposition.

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1. Introduction

The fundamental motivation for this article is the following question:

Are there germs of meromorphic functions, in the real or in the complex plane, with a given indeterminacy structure and prescribed zeroes and poles?

By the indeterminacy structure of a meromorphic function we mean the information given by the dicritical components of its resolution by a sequence of quadratic blow-ups, i.e. the irreducible components of the exceptional divisor not contained in a level set of the lifted function. We prescribe zeroes and poles by providing a finite set $\mathcal{S}$ of analytic branches, i.e. irreducible analytic curves at the origin. In the complex case, we can give a positive answer to this question once we complete $\mathcal{S}$ into a finite set of branches satisfying some conditions of axiomatic nature, so that it forms a set of “separatrices” (Theorem B). In the real case, the answer is always positive, whoever $\mathcal{S}$ is (Theorem E).

We handle the above question in the broader context of the local theory of singular holomorphic foliations in the complex plane, by addressing the specific problem of building the so-called logarithmic models. They consist in the construction of germs of singular foliations of logarithmic type — those defined by closed meromorphic 1-forms with simple poles — with a prescribed set of geometric data: dicritical components (non-invariant components of the exceptional divisor in the reduction of singularities), separatrices (invariant complex analytic branches), and Camacho-Sad indices. The latter are local residue-type invariants, associated with pairs singularity/separatrix, that play a significant role in the local topological characterization of the foliation and also impose combinatorial restrictions along its reduction of singularities [5, 12, 17]. These data may originate, for instance, from a germ of singular complex analytic foliation $\mathcal{F}$ of generalized curve type [4], which means that, if $\pi$ is a reduction of singularities for $\mathcal{F}$ by a sequence of quadratic blow-ups, then the lifted foliation $\pi^*\mathcal{F}$ has no saddle-node singularities, i.e. simple singularities with one zero eigenvalue (see Examples 2.2 and 2.4). In this case, a logarithmic model for $\mathcal{F}$ is a germ of logarithmic foliation $\mathcal{L}$ that, roughly speaking, has the same reduction of singularities of $\mathcal{F}$, with dicritical components positioned in the same places, having the same isolated separatrices (those whose lift by $\pi$ touch invariant components of the exceptional divisor) and the same Camacho-Sad indices. All of this is considered up to the existence of some extra points, placed in the smooth part of dicritical components, where $\mathcal{L}$ has a holomorphic first integral and demands additional blow-ups in order to complete its reduction of singularities. These are called escape points. They give rise to some isolated separatrices for $\mathcal{L}$, called accordingly escape separatrices. The logarithmic model is said to be strict if there are no escape points, thus $\mathcal{F}$ and $\mathcal{L}$ have the same reduction of singularities and, hence, coincident sets of isolated separatrices. Logarithmic models of foliations in dimension two are studied in the articles [8], [6] and [9], where the cases considered are, respectively, complex non-dicritical, complex dicritical and real non-dicritical. Several ideas and arguments we present here owe much to these three papers.
Our purpose in this article is to make a construction of general nature that encompasses the three aforementioned cases and, further, deals with the remaining real dicritical case. Our approach is somewhat different from that of [8] and [6], bearing some resemblance to the philosophy of [9]. The core of our method is the consideration of blocks of information, with an axiomatic structure inspired in the properties of foliations of generalized curve type, having, in principle, no connection with a concrete foliation. They are built upon the various steps of a sequence of quadratic blow-ups \( \pi \), that simulates the reduction of singularities of some hypothetical foliation. The first of these blocks is formed by the exceptional divisors of \( \pi \) and of its intermediate factors, having some distinguished irreducible components that play the role of dicritical components of the reduction divisor of a singular foliation. The second is a finite set \( S_0 \) of complex analytic branches at \((\mathbb{C}^2, 0)\), along with their strict transforms by all steps of \( \pi \), that impersonate separatrices of a foliation and are thus named this way. These two blocks are called, respectively, dicritical structure (Definition 2.1) and configuration of separatrices (Definition 2.3). Together they form a dicritical duplet. Then, we have a set of local complex invariants assigned to separatrices and non-dicritical components, which emulate Camacho-Sad indices, called system of indices (Definition 3.1). Its attachment to a dicritical duplet forms a dicritical triplet. Our goal is then to assign, to separatrices and non-dicritical components, invariants in \( \mathbb{C}^* \) called residues, to be assembled in a fourth block called system of residues (Definition 3.4), in order to form a dicritical quadruplet. Simultaneously, we wish that a logarithmic 1-form with poles in \( S_0 \) and the given residues defines a germ of singular foliation \( L \) that is a logarithmic model for all blocks of data considered (Definition 4.2). In a nutshell, up to the existence of escape points and escape separatrices, this means that the reduction of singularities of \( L \), its isolated separatrices, Camacho-Sad indices and residues are precisely those provided by the dicritical quadruplet. The existence of consistent data of residues and of a logarithmic 1-form with these properties is given by Theorem A, which will be the main source for all other results in this paper. Its proof relies mostly on arguments of [6], properly adapted to our objects. The axiomatic construction of our dicritical multiplets is carried out in Section 2 and in Section 3, while the proof of Theorem A is done in Section 4.

Next, in Section 5, we state and prove Theorem B, which gives an answer to the complex version of the question that opens this article. We start with a dicritical duplet containing the prescribed data of indeterminacy and branches of zeroes and poles. Then, using combinatorial tools of [3], we produce a system of rational negative indices at the final level — the one corresponding to the whole sequence of blow-ups \( \pi \). This condition, necessary for them to be actual Camacho-Sad indices of reduced models of a meromorphic function, also happens to be sufficient. Indeed, Theorem A provides a system of rational residues, so that the wished meromorphic function arises in a straightforward way.

Then, in Section 6 and in Section 7, we turn our attention to the real case, proving the existence of real logarithmic models in Theorem C. Our point of departure is a sort of real dicritical triplet containing only real information (sequences of quadratic blow-ups, analytic branches and indices, all of them real), with a weakened axiomatic structure (Definition 6.4). Its complexification can be completed into a dicritical triplet, symmetric with respect to the involution induced by the complex conjugation. The application of Theorem A then provides a system of real residues and a logarithmic 1-form, both symmetric with
respect to the conjugation. The restriction of this complex logarithmic 1-form to the real trace provides the logarithmic model we seek. It turns out that, in the real case, by adding some dicritical separatrices with appropriate residues, real escape points can be eliminated (Proposition 7.4). This, combined with Theorem C can be used in order to produce strict real logarithmic models, as we do in Theorem D where they are obtained for germs of real analytic foliations of real generalized curve type, i.e. whose real reductions of singularities lead only to non-degenerate final models. Also, by the application of the ideas converging to Theorem C we obtain Theorem E which gives an affirmative answer to the real version of our opening question.

Finally, in the last part of the article, Section 8 we study sectorial decompositions of germs of real analytic vector fields. A classical result by I. Bendixson [1] asserts that a real analytic vector field, with an isolated singularity at \(0 \in \mathbb{R}^2\), of non-monodromic type, induces a decomposition of a small neighborhood of the origin in a finite number of sectors. The term non-monodromic refers to the fact that there exists at least one characteristic orbit, i.e. one accumulating to the origin with a well defined tangent at the limit point. Then, each sector is limited by a pair of characteristic orbits and bears a classification into hyperbolic, parabolic or elliptic, according to the topological behavior of the orbits inside of it (see Section 8 and also [10, 15] for a detailed description). The essential information concerning a sectorial decomposition — a finite number of real analytic semicurves, with the mentioned accumulation properties, defining sectors classified into hyperbolic, parabolic or elliptic — would then define a sectorial model. A second guiding question, also a motivation for the development of real logarithmic models described above, is posed in the following terms:

*Given a sectorial model at \((\mathbb{R}^2, 0)\), is there a germ of real analytic vector field that realizes it?*

We answer this question in a particular case. Within the family of non-monodromic vector fields, we consider those whose orbits accumulating to \(0 \in \mathbb{R}^2\) are contained in the trace of real analytic branches and call them \(\ell\)-analytic (Definition 8.1). With them, we associate more refined sectorial models, also called \(\ell\)-analytic, that aggregate infinitesimal information of dicritical components, which in turn account for the existence of parabolic and elliptic sectors (Definition 8.2). An almost immediate application of the existence of real logarithmic models gives Theorem F which asserts that any such a model is realized by a germ of \(\ell\)-analytic vector field, and Theorem G which assures that it can be actually realized by a germ of real meromorphic function.

The authors are grateful to Felipe Cano for asking us the above question on the existence of meromorphic functions.

### 2. Dicritical structure and configuration of separatrices

The problems we deal with concern the construction of singular holomorphic foliations with prescribed dicritical components in their reductions of singularities and prescribed sets of invariant analytic curves. As we commented in the introduction, we approach this by means of abstract structures that handle these information in an axiomatic manner. In this section we introduce two of them: dicritical structure and configuration of separatrices.
2.1. **Infinitesimal classes.** We start by establishing an intrinsic way to identify an irreducible component of the exceptional divisor of a sequence of quadratic blow-ups. Let \( \pi : (\tilde{M}, D) \to (\mathbb{C}^2, 0) \) be a sequence of quadratic blow-ups. The set \( D = \pi^{-1}(0) \), the exceptional divisor of \( \pi \), is a normal crossings divisor with finitely many irreducible components — to which we refer simply as components — biholomorphic to projective lines. We call the regular points of \( D \) trace points and its singular points corners.

We denote by \( \mathcal{B}_0 \) the family of all complex analytic branches — i.e. germs of irreducible complex analytic curves — at \((\mathbb{C}^2, 0)\). More generally, if \( M \) is a complex surface, represent by \( \mathcal{B}_p \) the family of complex analytic branches at \( p \in M \) and, if \( C \subset M \) is a complex analytic curve, by \( \mathcal{B}_p(C) \subset \mathcal{B}_p \) the set of irreducible local components of \( C \) at \( p \). If \( B \in \mathcal{B}_0 \), we say that the strict transform \( \pi^*B \) is transversal to \( D = \pi^{-1}(0) \) if \( \pi^*B \) is smooth and touches \( D \) transversally at a trace point. In this case, we denote by \( D^\pi(B) \) the unique component of \( D \) touched by \( \pi^*B \).

Let \( D \subset D \) be a component of the exceptional divisor of the above \( \pi \). We associate with \( D \) an infinitesimal class \( \kappa(D) \), which is, essentially, the family of all sequences of blow-ups that “generate \( D \):” sequences of blow-ups or blow-downs over \((\tilde{M}, D)\), the latter not collapsing \( D \), considered up to isomorphism. For the sake of formality, we give a more precise definition. Let us consider, along with \((\pi, D)\), all pairs \((\pi', D')\), where \( \pi' : (\tilde{M}', D') \to (\mathbb{C}^2, 0) \) is a sequence of blow-ups and \( D' \subset D' \) is a component, such that:

- either \( \pi \) factors \( \pi' \), that is, there exists a sequence of blow-ups \( \varsigma \) such that \( \pi' = \pi \circ \varsigma \), and \( D' = \varsigma^*D \);
- or \( \pi' \) factors \( \pi \), that is, there exists a sequence of blow-ups \( \varsigma' \) such that \( \pi = \pi' \circ \varsigma' \), and \( D = \varsigma'^*D' \).

Next, extend this construction putting, in place of \( \pi \) and \( D \), any sequence of blow-ups isomorphic to \( \pi \) and the isomorphic image of \( D \). The family of all these mappings is the infinitesimal class \( \kappa(D) \) of \( D \subset D \). Any of its elements is a a realization of \( \kappa(D) \). Recall that two sequences of blow-ups \( \pi^{(1)} : (\tilde{M}^{(1)}, D^{(1)}) \to (\mathbb{C}^2, 0) \) and \( \pi^{(2)} : (\tilde{M}^{(2)}, D^{(2)}) \to (\mathbb{C}^2, 0) \) are isomorphic if there is a germ of biholomorphism \( \Phi : (\tilde{M}^{(1)}, D^{(1)}) \to (\tilde{M}^{(2)}, D^{(2)}) \) such that \( \pi^{(1)} = \pi^{(2)} \circ \Phi \). A realization of \( \kappa(D) \) is minimal if it is factored by no other realization of \( D \). A minimal realization is unique up isomorphism.

If \( D^* \) is a union of components of \( D \), then we define the infinitesimal class \( \kappa(D^*) \) as the family of sequences of blow-ups that simultaneously realize all infinitesimal classes \( \kappa(D) \), for \( D \subset D^* \), and also separate these components. The latter means that, if \( \pi' : (\tilde{M}', D') \to (\mathbb{C}^2, 0) \) is a realization of \( \kappa(D^*) \), then no two components of the divisor \( D' = \pi'^{-1}(0) \) corresponding to components in \( D^* \) intersect. We can also define, in an evident manner, a minimal realization for \( \kappa(D^*) \), which is unique up to isomorphism.

In the same line of the above definitions, if \( S \subset \mathcal{B}_0 \) is a finite set, we define its equisingularity class \( \varepsilon(S) \) as the family of all sequences of blow-ups \( \pi : (\tilde{M}, D) \to (\mathbb{C}^2, 0) \) such that \( \pi \) desingularizes \( S \). By this we mean that the strict transforms \( \pi^*B \), for \( B \in S \), are disjoint and transversal to \( D \). Each such a sequence \( \pi \) is a realization of \( \varepsilon(S) \), and we can also talk about minimal realizations, all of them isomorphic.

2.2. **Dicritical structures.** Let \( \pi : (\tilde{M}, D) \to (\mathbb{C}^2, 0) \) be a composition of quadratic blow-ups as above. We write \( \pi = \sigma_1 \circ \cdots \circ \sigma_n \) its factorization into individual blow-ups \( \sigma_j = (\tilde{M}_j, D_j) \to (\tilde{M}_{j-1}, D_{j-1}) \), for \( j = 1, \ldots, n \), where \( D_j = (\sigma_1 \circ \cdots \circ \sigma_j)^{-1}(0) \), \( \sigma_j \) is
a punctual blow-up at \( q_{j-1} \in D_{j-1} \), with the convention that \( q_0 = 0 \), \((\tilde{M}_0, D_0) = (\mathbb{C}^2, 0)\) and \((\tilde{M}_n, D_n) = (\tilde{M}, D)\). All objects and invariants pertaining to \( \tilde{M}_j \) are said to be at level \( j \), with \( j = 0 \) and \( j = n \) being, respectively, the initial and final levels. For each \( j = 1, \ldots, n \), write the factorization \( \pi = \pi_j \circ \varsigma_j \) into maps \( \varsigma_j : (\tilde{M}_n, D_n) \to (\tilde{M}_j, D_j) \) and \( \pi_j : (\tilde{M}_j, D_j) \to (\tilde{M}_0, D_0) \), where \( \varsigma_j = \sigma_{j+1} \circ \cdots \circ \sigma_n \) and \( \pi_j = \sigma_1 \circ \cdots \circ \sigma_j \). This is depicted in the diagram:

\[
(1) \quad (\tilde{M}_n, D_n) \xrightarrow{\sigma_n} \cdots \xrightarrow{\sigma_{j+1}} (\tilde{M}_j, D_j) \xrightarrow{\sigma_j} (\tilde{M}_{j-1}, D_{j-1}) \xrightarrow{\sigma_{j-1}} \cdots \xrightarrow{\sigma_1} (\tilde{M}_0, D_0) \simeq (\mathbb{C}^2, 0).
\]

**Definition 2.1.** A dicritical structure at \((\mathbb{C}^2, 0)\) with underlying sequence of blow-ups \( \pi : (M, D) \to (\mathbb{C}^2, 0) \) is the set of data \( \Delta \) given, for each \( j = 1, \ldots, n \), by a decomposition \( D_j = D^j_\delta \cup D^j_\epsilon \), where \( D^j_\delta \) and \( D^j_\epsilon \) are unions of components of the exceptional divisor \( D_j = \pi_j^{-1}(0) \) satisfying:

\begin{enumerate}[(D.1)]
  \item \( D^j_\delta \) and \( D^j_\epsilon \) have no common components for \( j = 1, \ldots, n \);
  \item \( D \) is in \( D^j_{\delta-1} \) if and only if the strict transform \( \sigma_j^*D \) is in \( D^j_\delta \), for \( j = 2, \ldots, n \);
  \item two components in \( D^j_\delta \) do not intersect.
\end{enumerate}

Components in \( D^j_\delta \) are called dicritical or non-invariant, whereas the ones in \( D^j_\epsilon \) are called non-dicritical or invariant. If \( D = \sigma_j^{-1}(q_{j-1}) \subset D^j_\delta \), we will also call the blow-up \( \sigma_j \) dicritical, the same happening in the non-dicritical case. We will say that \( n \) is the height of \( \Delta \) and we denote \( n = h(\Delta) \). Note that, if we choose a point \( p \in D_j \), for \( j = 1, \ldots, n \), the dicritical structure \( \Delta \) can be localized at \( p \), by considering the decomposition into dicritical and non-dicritical components induced by the sequence of blow-ups \( \varsigma_j \) over \( p \). We denote this localized dicritical structure by \( \Delta_p \). Evidently, if \( \varsigma_j \) is trivial over \( p \), then this dicritical structure is also trivial.

Let \( \Delta \) and \( \Delta' \) be two dicritical structures with underlying sequences of blow-ups \( \pi = \sigma_1 \circ \cdots \circ \sigma_n \) and \( \pi' = \sigma'_1 \circ \cdots \circ \sigma'_n \). We say that \( \Delta' \) dominates \( \Delta \), and we denote \( \Delta' \geq \Delta \), if:

\begin{itemize}
  \item \( n' = h(\Delta') \geq h(\Delta) = n \) and \( \sigma'_j = \sigma_j \) for \( j = 1, \ldots, n \);
  \item the decomposition \( D_j = D^j_\delta \cup D^j_\epsilon \) is the same for \( \Delta \) and \( \Delta' \), for \( j = 1, \ldots, n \);
  \item if \( n' > n \), then \( \varsigma'_n = \sigma'_{n+1} \circ \cdots \circ \sigma'_{n'} \) is a composition of non-dicritical blow-ups which is non-trivial only over finitely many trace points of components in \( D^j_\delta \).
\end{itemize}

**Example 2.2.** Let \( F \) be a germ of singular holomorphic foliation at \((\mathbb{C}^2, 0)\), induced by a germ of holomorphic 1-form \( \omega \) with isolated singularity at the origin. We take \( \pi : (M, D) \to (\mathbb{C}^2, 0) \), a minimal reduction of singularities for \( F \) (see [16, 4]). This means that we obtain a strict transform foliation \( \tilde{F} = \pi^*F \) around the exceptional divisor \( D = \pi^{-1}(0) \) with the following properties:

\begin{itemize}
  \item \( \tilde{F} \) has a finite number of singularities over \( D \), which are all simple, meaning that, at each of them, \( \tilde{F} \) is defined by a germ of holomorphic vector field with non-nilpotent linear part with eigenvalues with ratio outside \( \mathbb{Q}_+ \);
  \item \( \tilde{F} \) has no singularities and no points of tangency in the non-invariant components of \( D \);
\end{itemize}
If we write the decomposition of $\pi$ as in (1), at each step we have a foliation $\mathcal{F}_j$, the strict transform of $\mathcal{F}$ by $\pi_j$ (with the convention that $\mathcal{F}_n = \mathcal{F}$), which is a germ of singular holomorphic foliation around $D_j$. We define a decomposition $D_j = D^\delta_j \cup D^\iota_j$ by setting $D \subset D^\delta_j$ if and only if $D$ is non-invariant by $\mathcal{F}_j$. This establishes a dicritical structure that we denote by $\Delta(\mathcal{F})$ or by $\Delta(\omega)$.

2.3. Configuration of separatrices. Considering a dicritical structure $\Delta$ at $(\mathbb{C}^2, 0)$, as defined in the previous subsection, we write the decomposition

\[ D_n^\iota = \cup_{k=1}^\ell A_{n,k}^\iota \]

into disjoint topologically connected components. These components, which appear in the following definition, also play an important role in some of the arguments in this text. For the next definition, recall that the valence of a component $D \subset D_n$, denoted $\text{Val}(D)$, is the number of intersections of $D$ with other components of $D_n$.

**Definition 2.3.** Let $\Delta$ be a dicritical structure at $(\mathbb{C}^2, 0)$. A configuration of separatrices framed on $\Delta$ is the collection of information $\Sigma$ given by a finite set $S_0 \subset B_0$, having a decomposition into disjoint subsets $S_0 = S_0^\delta \cup S_0^\iota$, together with its propagation along the sequence of blow-ups $\pi$, $S_j = S_j^\delta \cup S_j^\iota$, where $S_j^\delta = \{ \pi_j^* S; S \in S_0^\delta \}$ and $S_j^\iota = \{ \pi_j^* S; S \in S_0^\iota \}$, satisfying the following conditions:

- (S.1) $\pi$ is a realization of $\varepsilon(S_0)$ and $S \in S_0^\delta$ if and only if $D^\pi(S) \subset D_n^\delta$;
- (S.2) for every $A_{n,k}^\iota$, there is $S \in S_0^\iota$ such that $D^\pi(S) \subset A_{n,k}^\iota$;
- (S.3) if $D \subset D_n^\iota$ is such that $\text{Val}(D) = 1$, then there exists $S \in S_0^\iota$ such that $D^\pi(S) = D$;
- (S.4) $\pi$ is a minimal sequence of blow-ups that simultaneously realizes the infinitesimal class $\kappa(D_n^\delta)$ and the equisingularity class $\varepsilon(S_0)$.

Taking into account Definitions 2.1 and 2.3, we say that the pair $D = (\Delta, \Sigma)$ is a dicritical duplet at $(\mathbb{C}^2, 0)$. Branches in $S_0^\delta$ and in $S_0^\iota$ are called, respectively, dicritical and isolated separatrices. The support of $D$ is defined, at each level $j = 1, \ldots, n$, as the analytic curve $Z_j = S_j \cup D_j^\iota$, seen as germ around $D_j$. Define, at level $j$, the singular set of $D$ as the singular set of its support, $\text{Sing}(Z_j)$. At the initial level, we have $Z_0 = S_0$. Note that $S_0^\delta$ may be empty. On the other hand, $S_0^\iota$ is empty if and only if $h(\Delta) = 1$ and $D_1 = D_0^\iota$. This is the radial case, which is trivial for our purposes and will not be taken as our starting point. Nevertheless, in some inductive constructions, the radial case may appear at intermediate levels of the dicritical structure, as a result of the localization of our objects. Anyhow, if we wish to include the initial radial case, we should modify condition (S.3) and also ask that, if $D \in D_n^\iota$ is such that $\text{Val}(D) = 0$, then there are at least two separatrices in $S_0^\delta$.

Given two dicritical duplets $D = (\Delta, \Sigma)$ and $D' = (\Delta', \Sigma')$ at $(\mathbb{C}^2, 0)$, we say that $D'$ dominates $D$, and denote $D' \geq D$, if:

- $\Delta' \geq \Delta$;
- $S_0' \supset S_0$, where $S_0'$ denotes the set of separatrices of $\Sigma'$ at level 0, and $S_0' \setminus S_0$ contains only isolated separatrices for $\Sigma'$;
- separatrices in $S_0$ are qualified in the same way, as dicritical or isolated, for both $\Sigma$ and $\Sigma'$;
• if \( S \in S_0 \setminus S_0 \), then \( D = D^\pi(S) \subset D_n^\delta \) and \( \pi^*S \cap D \) does not belong to \( Z_n \), the support of \( D \) at level \( n \).

We make some comments on these conditions. For \( S \in S_0 \setminus S_0 \), let \( q = \pi^*S \cap D \), where \( D = D^\pi(S) \). Also denote by \( \zeta_n = \sigma_{n+1} \circ \ldots \circ \sigma_{n'} \) the blow-ups of \( \pi' \) after level \( n = h(\Delta) \), where \( \pi' \) is the sequence of blow-ups subjacent to \( \Delta' \) and \( n' = h(\Delta') \). Since \( S \) is isolated for \( \Sigma' \) and \( D \) is dicritical for \( \Delta' \), we have that \( \zeta_n \) is non-trivial over \( q \). We call \( q \) a *escape point* and \( S \) a *escape separatrix*. A detailed discussion of these objects will be carried out later in Section 7. We also refer the reader to Section 5 of [9].

As we did for a dicritical structure \( \Delta \), we can localize a configuration of separatrices \( \Sigma \), framed on \( \Delta \), at a point \( p \in D_j \), for some \( j = 1, \ldots, n \), obtaining a configuration of separatrices \( \Sigma_p \) framed on the dicritical structure \( \Delta_p \). The separatrices at level 0 for \( \Sigma_p \) will be precisely the branches in \( B_p(Z_j) \). Branches in \( B_p(Z_j) \) will inherit their classification as dicritical or isolated. On the other hand, a local component of \( D \subset D_j \) at \( p \) will be dicritical if and only if \( \zeta_j \) touches \( \zeta_j^{-1}(p) \) at a component of \( D_n^\delta \) contained in \( \zeta_j^{-1}(p) \). In particular, if \( \zeta_j \) is trivial over \( p \), these local components will be isolated separatrices. The fact that \( \Sigma_p \) thus defined is framed on \( \Delta_p \) is easy to check and will be left to the reader. We will say that \( D_p = (\Delta_p, \Sigma_p) \) is the *localization* of \( D = (\Delta, \Sigma) \) at the point \( p \).

**Example 2.4.** We revisit Example 2.2. As we have seen, there is a dicritical structure \( \Delta(\mathcal{F}) \) associated with the minimal reduction of singularities of a germ of singular holomorphic foliation \( \mathcal{F} \) at \((\mathbb{C}^2,0)\). Recall that a *separatrix* for \( \mathcal{F} \) is an analytic branch at \((\mathbb{C}^2,0)\) invariant by \( \mathcal{F} \). Denote their set by \( \text{Sep}_0(\mathcal{F}) \). As proven in [5] (or in [3]), \( \text{Sep}_0(\mathcal{F}) \neq \emptyset \). From this point on, we suppose that \( \mathcal{F} \) is of *generalized curve* type (see [3]), meaning that there are no saddle-node singularities — i.e. simple singularities with one zero eigenvalue — in the final models of its reduction of singularities. For a generalized curve type foliation, a desingularization for \( \text{Sep}_0(\mathcal{F}) \) is a reduction of singularities for \( \mathcal{F} \) [4, Th. 2]. A separatrix \( S \in \text{Sep}_0(\mathcal{F}) \) is classified as dicritical or isolated, following the classification of the component \( D = D^\pi(S) \) in the reduction divisor. Denote their sets by, respectively \( \text{Dic}_0(\mathcal{F}) \) and \( \text{Iso}_0(\mathcal{F}) \). Isolated separatrices are finite in number. However, there are infinitely many dicritical separatrices, provided there is some dicritical component in \( D_n \). In order to define a configuration of separatrices framed on \( \Delta(\mathcal{F}) \) we proceed as follows:

- \( S_0(\mathcal{F}) = \text{Iso}_0(\mathcal{F}) \) is the set of all isolated separatrices;
- \( S_0^\delta(\mathcal{F}) \) includes finitely many separatrices in \( \text{Dic}_0(\mathcal{F}) \), with at least one separatrix attached to each component of valence one in \( D_n^\delta \).

The configuration of separatrices \( \Sigma(\mathcal{F}) \) constructed upon \( S_0(\mathcal{F}) = S_0^\delta(\mathcal{F}) \cup S_0(\mathcal{F}) \) is framed on \( \Delta(\mathcal{F}) \). Indeed, (S.1) and (S.3) are true by construction, while (S.2) follows, for instance, from [13, Prop. 4]. Finally, (S.4) is a consequence of the minimality property of the reduction of singularities \( \pi \) along with the above mentioned fact that the desingularization of the separatrices and of the foliation are equivalent in the generalized curve type context. We then have a dicritical duplet by \( D(\mathcal{F}) = (\Delta(\mathcal{F}), \Sigma(\mathcal{F})) \), also denoted by \( D(\omega) = (\Delta(\omega), \Sigma(\omega)) \) when \( \mathcal{F} \) is defined by the germ of holomorphic 1-form \( \omega \). This duplet is evidently non-unique provided the reduction of singularities of \( \mathcal{F} \) contains a dicritical blow-up.
3. Systems of indices and systems of residues

In the preceding section we introduced the object $\mathbf{D} = (\Delta, \Sigma)$, called dicritical duplet. It joins together two nested blocks of abstract information related to singular holomorphic foliations at $(\mathbb{C}^2, 0)$: a dicritical configuration $\Delta$, which works as a chart of dicritical components, and a configuration of separatrices $\Sigma$, which is a list of mandatory invariant branches. In this section, we will introduce other two abstract blocks, systems of indices and systems of residues, that assemble information corresponding to Camacho-Sad indices and residues of logarithmic 1-forms. These four blocks altogether will mould the construction of logarithmic foliations.

For the following definition, suppose that the underlying sequence of blow-ups of $\Delta$ is $\pi : (\tilde{M}, D) \to (\mathbb{C}^2, 0)$ and that $n = h(\Delta)$. Recall that the support of $\mathbf{D}$ at level $j$ is $Z_j = S_j \cup D^j_j$, for $j = 0, \ldots, n$.

**Definition 3.1.** A system of indices associated with the dicritical duplet $\mathbf{D} = (\Delta, \Sigma)$, or $\mathbf{D}$-system of indices for short, is the set of data $\Upsilon$ obtained by assigning, for each $j = 0, \ldots, n$, to every $p \in Z_j$ and each local branch $S \in B_p(Z_j)$, a number $I_p(S) \in \mathbb{C}$ in a way that the following conditions are respected:

(I.1) Suppose that $\sigma_j$ is a blow-up at $q \in D_{j-1}$, for some $1 \leq j \leq n$. Let $p \in Z_{j-1}$, $S \in B_p(Z_{j-1})$, $\tilde{S} = \sigma_j^*S$ and $\tilde{p} = D_j \cap \tilde{S}$. Then the following transformation law is satisfied:

$$I_{\tilde{p}}(\tilde{S}) = \begin{cases} I_p(S) - \nu_p(S)^2 & \text{if } p = q \\ I_p(S) & \text{if } p \neq q. \end{cases}$$

(I.2) If $p$ is a regular point of $Z_j$ and $S$ is the only branch of $Z_j$ at $p$, then $I_p(S) = 0$ (regular point condition).

(I.3) For $1 \leq j \leq n$ and every component $D \subset D^j_j$, the Camacho-Sad formula is valid:

$$\sum_{p \in D} I_p(D) = D \cdot D,$$

where in the right side stands the self-intersection number of $D$ in $M_j$, which coincides with $c(D)$, the first Chern class of the normal bundle of $D$ in $M_j$.

(I.4) If $p \in Z_n$ is a singular point and $S_1, S_2$ are the two smooth branches of $Z_n$ at $p$ (one of them necessarily in $D^j_n$) then we have the simple singularity rule:

$$I_p(S_1), I_p(S_2) \in \mathbb{C}^* \setminus \mathbb{Q}_+, \quad \text{and} \quad I_p(S_2) = 1/I_p(S_1).$$

In this situation, we say that $\mathbf{T} = (\mathbf{D}, \Upsilon) = (\Delta, \Sigma, \Upsilon)$ is a dicritical triplet at $(\mathbb{C}^2, 0)$.

**Example 3.2.** We resume Examples 2.2 and 2.4. Let $\mathbf{D}(\mathcal{F}) = (\Delta(\mathcal{F}), \Sigma(\mathcal{F}))$ be a dicritical duplet of a germ of singular holomorphic foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$ of generalized curve type. We obtain a system of indices $\Upsilon(\mathcal{F})$ by setting, for each $p \in Z_j$ and each $S \in B_p(Z_j)$, $I_p(S)$ as the Camacho-Sad index of $\tilde{\mathcal{F}}_j = \pi_j^*\mathcal{F}$ at $p$ (see [5, 12, 17] and also [2, 18]). Conditions (I.1) to (I.4) are known properties of this index. In particular, condition (I.3) follows from the theorem asserting that the sum of Camacho-Sad indices along a compact analytic curve invariant by a foliation in an ambient complex surface is the curve’s self intersection. We say that $\mathbf{T}(\mathcal{F}) = (\mathbf{D}(\mathcal{F}), \Upsilon(\mathcal{F})) = (\Delta(\mathcal{F}), \Sigma(\mathcal{F}), \Upsilon(\mathcal{F}))$ is a dicritical triplet associated with $\mathcal{F}$.
A system of indices is determined in a unique way by its prescription at the final level, as shown next.

**Proposition 3.3.** Let $D = (\Delta, \Sigma)$ be a dicritical duplet at $(\mathbb{C}^2, 0)$. If $n = h(\Delta)$, then any set of data given by complex numbers $I_p(S) \in \mathbb{C}^*$, for every $p \in \mathbb{Z}_n$ and every $S \in B_p(\mathbb{Z}_n)$, satisfying (I.2), (I.3) and (I.4) of Definition 3.1 for $j = n$, extends in a unique way to a $D$-system of indices $\Upsilon$.

**Proof.** The transformation law (I.1) ought to be used in descending the indices assigned at the final level and, thus, it will be automatically fulfilled. Besides, since (I.4) concerns only level $n$, it is enough to verify (I.2) and (I.3) for lower levels. The proof goes by a reverse induction argument. The initialization, at level $n$, works by hypothesis. Suppose then that $0 \leq j < n$ and that conditions (I.1) to (I.4) are valid at all levels above $j$.

We first prove that (I.2) is true at level $j$. Let $p$ be a smooth point of $\mathcal{Z}_j$. If $p \in \mathcal{D}_j$, then $p$ is a trace point of some component $D \subset \mathcal{D}_j$ and there are no components of $S_j$ at $p$. By the minimality condition (S.4), $\varsigma_j$ would be non-trivial at $p$ if and only if $\varsigma_j^{-1}(p)$ contained some dicritical component, say $D' \subset \mathcal{D}_n$. But we would have that either $D'$ disconnects $\mathcal{D}_n$ or $\text{Val}(D') = 1$. In both cases, by (S.2) and (S.3), we would find some element of $S_j$ at $p$, which is impossible. Hence, $\varsigma_j$ is a local isomorphism over $p$. Thus, using (I.2) at level $n$ and successively applying (I.1), we conclude that $I_p(D) = 0$.

Suppose now that $p \notin \mathcal{D}_j$. Then, $p$ is a trace point of a component $D \subset \mathcal{D}_j$ and there is a unique $S \in S_j$ at $p$, which is smooth. Suppose first that $S \in S_j^0$. If $\varsigma_j$ were non-trivial over $p$, then, by (D.3), $\varsigma_j^{-1}(p)$ would contain some non-dicritical component, which in turn would imply the existence of a least one element of $S_j$ at $p$, which is not allowed. Thus $\varsigma_j$ is trivial over $p$ and, as in the previous paragraph, we find that $I_p(S) = 0$. On the other hand, if $S \in S_j^1$, then $S$ is tangent to $D$, $\varsigma_j$ is non-trivial over $p$ and, as a consequence of (S.2), $\varsigma_j^{-1}(p)$ contains only non-dicritical components. By the minimality property (S.4), it is easy to see that $\varsigma_j^{-1}(p)$ is a linear chain of components $D_1 \cup \cdots \cup D_r \subset \mathcal{D}_n$ such that $D_i \cap D_{i+1} \neq \emptyset$ for $i = 1, \ldots, r - 1$, $\tilde{S} = \varsigma_j^{-1}(p)$ touches $D_i$ at a point $\tilde{p}$ and no other components of $S_j^1$ touches $\varsigma_j^{-1}(p)$. Besides, $D_1 \cdot D_i = -1$ and $D_1 \cdot D_i = -2$ for $i = 2, \ldots, r$.

By the application, at level $n$, of (I.3) and (I.4) along this linear chain, we find that $I_p(\tilde{S}) = -r$. Finally, the successive application of (I.1) gives that $I_p(S) = 0$. This proves (I.2) at level $j$.

In order to prove (I.3) at level $j$, suppose that $D \subset \mathcal{D}_j$ is an irreducible component. Suppose that $\sigma_{j+1}$ is a blow-up at $p \in \mathcal{D}_j$. If $p \notin D$, the Camacho-Sad formula (I.3) will evidently hold for $D$, since it holds for $\tilde{D} = \sigma_{j+1}^*D$ and all invariants involved are preserved by $\sigma_{j+1}$. Now, in the case $p \in D$, we have, on the one hand, $\tilde{D} \cdot \tilde{D} = D \cdot D - 1$. On the other hand, if $\tilde{p} = \tilde{D} \cap \sigma_{j+1}^{-1}(p)$, then the transformation law (I.1) gives $I_{\tilde{p}}(\tilde{D}) = I_p(D) - 1$. The Camacho-Sad formula then holds for $D$, since the indices of $D$ at points other than $p$ are preserved by $\sigma_{j+1}$. This proves (I.3) at level $j$, concluding the proof of the proposition. □

Recall that a meromorphic 1-form $\eta$ — and also the germ of singular holomorphic foliation $\mathcal{L}$ that it defines — is logarithmic if it can be written as

\[
\eta = \sum_{S \in S} \lambda_S \frac{df_S}{f_S} + \alpha,
\]
where $\alpha$ is some closed germ of holomorphic 1-form, $S \subset B_0$ is a finite set and, for every $S \in S$, $f_S \in \mathcal{O}_0$ is a local irreducible equation for $S$ and $\lambda_S \in \mathbb{C}^*$. Note that, since $\eta$ is closed, all branches in $S$ are separatrices of the foliation $\mathcal{L}$.

We define a dicritical duplet $\mathbf{D}(\eta) = (\Delta(\eta), \sigma(\eta))$ intrinsically associated with the logarithmic 1-form $\eta$. The dicritical structure is $\Delta(\eta) = \Delta(\mathcal{L})$, the one defined by the reduction of singularities of the foliation $\mathcal{L}$, as in Example 2.2. As for the configuration of separatrices, in $\Sigma(\eta)$ we include all branches in $S$ and also all isolated separatrices of $\mathcal{L}$. For $\Sigma(\eta)$ to be a configuration of separatrices, it remains to assure that condition (S.3) is true, in other words, that there is $S \in S$ such that $D^\tau(S) = D$ for every component $D \subset D^\delta_n$ with $\text{Val}(D) = 1$, where $n = h(\Delta(\eta))$ and $\pi$ is the sequence of blow-ups beneath $\Delta(\eta)$. Nevertheless, this happens in the cases considered in this text, more specifically, if we assume that $\eta$ is a faithful logarithmic 1-form (see Definition 4.1 and Section 7). Thus, henceforth we can consider that $\mathbf{D}(\eta) = (\Delta(\eta), \Sigma(\eta))$ is a dicritical duplet. We obtain a dicritical triplet $\mathbf{T}(\eta) = (\mathbf{D}(\eta), \mathbf{Y}(\eta)) = (\Delta(\eta), \sigma(\eta), \mathbf{Y}(\eta))$ by prescribing as indices those provided by the Camacho-Sad indices of $\mathcal{L}$.

In an attempt to handle residues of logarithmic 1-forms, we add another element to our construction:

**Definition 3.4.** Let $\mathbf{D} = (\Delta, \Sigma)$ be a dicritical duplet at $(\mathbb{C}^2, 0)$. A system of residues for $\mathbf{D} = (\Delta, \Sigma)$, or simply a $\mathbf{D}$-system of residues, is the set of data $\Lambda$ obtained by assigning numbers $\lambda_S \in \mathbb{C}^*$ to each irreducible component $S \subset Z_j$ ($S$ is either a component of $\mathcal{D}_j^\delta$ or a germ of separatrix of $S_j$), for $j = 0, \ldots, n$, obeying the following rules:

(R.1) Let $S \subset Z_j$ and $\sigma_{j+1}$ be the factor of $\pi$ corresponding to a blow-up at a point $q \in Z_j$, for some $j < n$. If $S = \sigma_{j+1}^* S$, then $\lambda_S = \lambda_S$ (invariance under blow-ups).

(R.2) If $\sigma_{j+1}$ is a non-dicritical blow-up at $q \in Z_j$, for some $0 \leq j < n$, and $D = \sigma_{j+1}^{-1}(q)$, then
\[
\lambda_D = \sum_{S \subset Z_j} \nu_q(S) \lambda_S \neq 0.
\]

(R.3) If $\sigma_{j+1}$ is a dicritical blow-up at $q \in Z_j$, for some $0 \leq j < n$, then
\[
\sum_{S \subset Z_j} \nu_q(S) \lambda_S = 0 \quad \text{(main resonance)}.
\]

The sum in items (R.2) and (R.3) has finitely many non-zero terms, only those corresponding to components $S$ containing $q$. We call it weighted balance of residues. It is worth pointing out that the assignment of residues at level 0 determines, by the successive application of rules (R.1) and (R.2), residues for all components of the support at all levels. Evidently, rules (R.2) and (R.3) impose compatibility conditions on a set of residues at level 0 that actually engender a $\mathbf{D}$-system of residues. Two $\mathbf{D}$-systems of residues $\Lambda$ and $\Lambda'$ are equivalent if there exists a common ratio $\alpha \in \mathbb{C}^*$ for corresponding residues. We denote, in this case, $\Lambda = \alpha \Lambda'$. If $\Lambda$ is a $\mathbf{D}$-system of residues, then the principal part of the logarithmic 1-form shown in equation (3) for $S = S_0$ is denoted by $\eta_\Lambda$. If $\Lambda$ and $\Lambda'$ are equivalent $\mathbf{D}$-systems of residues, with a proportionality ratio $\alpha \in \mathbb{C}^*$ as above, then $\eta_\Lambda = \alpha \eta_{\Lambda'}$, the two 1-forms defining the same germ of singular holomorphic foliation at $(\mathbb{C}^2, 0)$.

We establish a bond between systems of indices and systems of residues with the following:
Definition 3.5. Let $\Upsilon$ and $\Lambda$ be $\mathbf{D}$-systems of indices and residues, where $\mathbf{D} = (\Delta, \Sigma)$ is a dicritical duplet at $(\mathbb{C}^{2}, 0)$ of height $n = h(\Delta)$. We say that $\Upsilon$ and $\Lambda$ are consistent if, for every $p \in \text{Sing}(\mathbb{C}^{n})$,

$$I_p(S_1) = -\frac{\lambda S_2}{\lambda S_1} \quad \text{and} \quad I_p(S_2) = -\frac{\lambda S_1}{\lambda S_2},$$

where $S_1, S_2$ are the two local components of $\mathcal{B}_p(\mathbb{C}^{n})$. In this situation, we say that $Q = (\Delta, \Sigma, \Upsilon, \Lambda)$ forms a dicritical quadruplet.

Note that, by (I.4), the two equalities in the definition are equivalent. Note also that there are no conditions on the separatrices in $S_n^{0}$, and this gives some degree of freedom in their assignment of residues. This remark will be particularly useful in Section 7. Another important point is that, if $\Lambda$ and $\Lambda'$ are equivalent $\mathbf{D}$-systems of residues, then $\Lambda$ is consistent with a $\mathbf{D}$-system of indices $\Upsilon$ if and only if $\Lambda'$ is. It is apparent that, given $p \in D_j$, both a system of indices $\Upsilon$ and a system of residues $\Lambda$ can be localized at $p$, keeping their consistency relation, if it is the case. Thus, denoting these localizations by $\Upsilon_p$ and $\Lambda_p$, we can consider the localizations $T_p$ and $Q_p$ at $p$ of, respectively, a dicritical triplet $T = (\Delta, \Sigma, \Upsilon)$ and a dicritical quadruplet $Q = (\Delta, \Sigma, \Upsilon, \Lambda)$.

It is always possible to assign residues at the final level in a consistent way with a system of indices. More precisely:

Proposition 3.6. Suppose that $T = (\Delta, \Sigma, \Upsilon)$ is a dicritical triplet at $(\mathbb{C}^{2}, 0)$. Then, it is possible to assign numbers $\lambda_S \in \mathbb{C}^*$ for the irreducible components $S \subset \mathbb{C}^{n}$, where $n = h(\Delta)$, in such a way that the consistency condition of Definition 3.5 is satisfied.

Proof. We can work separately in each topologically connected component defined in $\mathbb{C}^{n}$ and we fix some $A = A_0^{n,k}$. Our task is to assign residues to each component of $\mathbb{C}^{n}$ intersecting $A$. To that end, after fixing an arbitrary component $D \subset A$, define a partial order in the components of $A$ in the following inductive way: $D$ is the maximal element, its immediate predecessors are the components $D'$ such that $D \cap D' \neq \emptyset$ and so forth. This is possible since the dual graph of $A$ is a tree (see the paragraph preceding Proposition 3.5). Choose an arbitrary value $\lambda_D \in \mathbb{C}^*$. Then, by the consistency condition in Definition 3.5, $\lambda_D$ determines all residues $\lambda_S$ for components $S \subset \mathbb{C}^{n}$ intersecting $D$: we must set $\lambda_S = -\lambda_D I_p(D)$, where $p = S \cap D$. In particular, we find all residues $\lambda_{D'}$ for components $D'$ preceding $D$. Again, for each such $D'$, all residues of components of $\mathbb{C}^{n}$ intersecting $D'$ are calculated by the consistency condition. This clearly provides an inductive process that will end once we reach the minimal elements with respect to this order. □

Proposition 3.6 will provide the initial step, in a reverse induction process, of the construction of a $\mathbf{D}$-system of residues $\Lambda$ consistent with a $\mathbf{D}$-system of indices $\Upsilon$. The general step of the induction is the amalgamation procedure, described in the next section. For it, we need the proposition below. All elements of its proof are in [9 Prop. 3] and [6 Lem. 15]. We include a proof here adapted to our setting.

Proposition 3.7. Let $Q = (\Delta, \Sigma, \Upsilon, \Lambda)$ be a dicritical quadruplet at $(\mathbb{C}^{2}, 0)$. If $S \in \mathcal{B}_0(\mathbb{C}^{n})$, then

$$\lambda_S I_0(S) = -\sum_{S' \in \mathcal{B}_0(\mathbb{C}^{n}) \setminus S} \lambda_{S'}(S', S)_0,$$

where $(\cdot, \cdot)_0$ denotes the intersection number at $0 \in \mathbb{C}^{2}$.
Proof. We will use the following induction assertion at level \(j\), for \(0 \leq j \leq n\): if \(p \in Z_j\) and \(S\) is a local component of \(Z_j\) at \(p\), then

\[
\lambda_S I_p(S) = - \sum_{S' \in B_p(Z_j) \mid S} \lambda_{S'}(S', S)_p.
\]

We are evidently employing here data from the localized quadruplet \(Q_p = (\Delta_p, \Sigma_p, \Upsilon_p, \Lambda_p)\).

At level \(n\), the assertion is true: it is trivial for each regular point of \(Z_n\), and, for singular points, it is the very definition of consistency. Let \(1 \leq j \leq n\) and suppose that the result is true, as stated, for all points in \(Z_j\). We will prove that it is also true for points at level \(j - 1\). Suppose that \(p \in Z_{j-1}\). If the blow-up \(\sigma_j\) is trivial over \(p\), there is nothing to prove and, thus, we are reduced to the case where \(\sigma_j\) is a blow-up at \(p\).

Let us denote \(\tilde{S}\) and \(\tilde{S}'\) the strict transforms by \(\sigma_j\) of the components \(S\) and \(S'\), \(D = \sigma_j^{-1}(p)\) and \(\tilde{p} = \tilde{S} \cap D\). We have

\[
\sum_{S' \in B_p(Z_{j-1}) \mid S} \lambda_{S'}(S', S)_p = \sum_{S' \in B_p(Z_{j-1}) \mid S} \lambda_{S'} \left( \nu_p(S')\nu_p(S) + (\tilde{S}', \tilde{S})_{\tilde{p}} \right)
\]

\[
= \nu_p(S) \sum_{S' \in B_p(Z_{j-1}) \mid S} \lambda_{S'} \nu_p(S') + \sum_{S' \in B_p(Z_{j-1}) \mid S} \lambda_{S'}(\tilde{S}', \tilde{S})_{\tilde{p}}
\]

\[
= -\lambda_S \nu_p(S)^2 + \nu_p(S) \sum_{S' \in B_p(Z_{j-1}) \mid S} \lambda_{S'} \nu_p(S') + \sum_{S' \in B_p(Z_{j-1}) \mid S} \lambda_{S'}(\tilde{S}', \tilde{S})_{\tilde{p}}
\]

\[
\sum_{S' \in B_p(Z_{j-1}) \mid S} \lambda_{S'}(S', S)_p = -\lambda_S \nu_p(S)^2 - \lambda_S I_p(S)
\]

proving thereby the result in this case.

On the other hand, if \(\sigma_j\) is a non-dicritical blow-up, the summation \((*)\) equals \(\lambda_D\) by (R.2) and, by applying the induction assertion for \(\tilde{p}\) followed by (R.1) and (I.1), we get

\[
\sum_{S' \in B_p(Z_{j-1}) \mid S} \lambda_{S'}(S', S)_p = -\lambda_S \nu_p(S)^2 + \lambda_D(D, \tilde{S})_{\tilde{p}} + \sum_{S' \in B_p(Z_{j-1}) \mid S} \lambda_{\tilde{S}'}(\tilde{S}', \tilde{S})_{\tilde{p}}
\]

\[
= -\lambda_S \nu_p(S)^2 - \lambda_{\tilde{S}} I_{\tilde{p}}(\tilde{S}) = -\lambda_S (\nu_p(S)^2 + I_{\tilde{p}}(\tilde{S})) = -\lambda_S I_p(S).
\]

This concludes the proof of the proposition. \(\square\)

4. Existence of dicritical quadruplets and logarithmic 1-forms

We start by recalling a definition from [6] that will be crucial to the development of the results of this section. Let \(\eta\) be a germ of logarithmic 1-form at \((C^2, 0)\), written as

\[
(4) \quad \eta = \lambda_1 \frac{df_1}{f_1} + \cdots + \lambda_r \frac{df_r}{f_r} + \alpha,
\]

where \(\lambda_1, \ldots, \lambda_r\) are (not necessarily rational) functions on \(V\) with \(\lambda_j = 0\) when \(|z| \to 0\) and \(\alpha\) is a nowhere vanishing analytic function on \(V\) with \(|\alpha| \to +\infty\) when \(|z| \to 0\).
where \( \alpha \) is a germ of closed holomorphic 1-form, \( r \geq 2 \), and, for \( i = 1, \ldots, r \), \( f_i \in \mathcal{O}_0 \) is irreducible with order \( \nu_i = \nu_0(f_i) \) and \( \lambda_i \in \mathbb{C}^* \). Denote by \( \mathcal{L} \) the singular holomorphic foliation defined by \( \eta \).

**Definition 4.1.** The 1-form \( \eta \) is 1-\textit{faithful} if

\[
\nu_0(f \eta) = m - 1,
\]

where \( f = f_1 \cdots f_r \) and \( m = \nu_0(f) = \nu_1 + \cdots + \nu_r \). We plainly say that \( \eta \) is \textit{faithful} if its strict transforms are 1-faithful along the reduction of singularities of \( \mathcal{L} \).

In the definition, the closed holomorphic 1-form \( \alpha \) in (4) is inert, since \( \nu_0(f \alpha) \geq m \). Also, the definition does not depend on the choice of irreducible equations \( f_1, \ldots, f_r \) for the components of the polar set of \( \eta \). This gives, in particular, the following remark, that will be used a couple of times in the proof of Theorem \( \mathbf{A} \) below: if \( \eta \) and \( \eta' \) are germs of logarithmic 1-forms at \((\mathbb{C}^2, 0)\) differing from each other by a germ of closed holomorphic 1-form, then \( \eta \) is faithful if and only if \( \eta' \) is. In Section \( \mathbf{7} \) we make a brief discussion on the notion of faithfulness. Notably, the following result, from [6, Lem. 7], will be explained: if \( \lambda_0 = \sum_{i=1}^r \nu_i \lambda_i \) is the weighted balance of residues of \( \eta \), then

- \( \lambda_0 \neq 0 \) implies that \( \eta \) is non-dicritical at \( 0 \in \mathbb{C}^2 \) and 1-faithful;
- \( \lambda_0 = 0 \) and \( \eta \) 1-faithful implies that \( \eta \) is dicritical \( 0 \in \mathbb{C}^2 \) (main resonant case).

In other words, for a faithful logarithmic 1-form, the dicritical or non-dicritical character of a blow-up in its reduction of singularities is determined by, respectively, the vanishing or non-vanishing of the weighted balance of residues. Another important point is that, if \( \eta \) is faithful logarithmic and \( \pi : (\tilde{M}, \mathcal{D}) \to (\mathbb{C}^2, 0) \) is its reduction of singularities, then \( \eta \) is faithful if and only if \( \eta' \) is. A necessary condition in order that \( D(\eta) \) be a dicritical duplet, follows from the Residue Theorem applied to the restriction of \( \pi^* \eta \) to \( D \). Bearing in mind these observations, we make the following definition:

**Definition 4.2.** Let \( Q = (\Delta, \Sigma, \Upsilon, \Lambda) \) be a dicritical quadruplet at \((\mathbb{C}^2, 0)\). We say that a germ of closed meromorphic 1-form \( \eta \) is \textit{Q-logarithmic} if:

1. there is a writing

\[
\eta = \sum_{S \in S_0} \lambda_S \frac{df_S}{f_S} + \alpha,
\]

where \( S_0 \) is the level 0 of \( \Sigma \), \( f_S \in \mathcal{O}_0 \) are local equations for the components \( S \in S_0 \), \( \lambda_S \) are the residues assigned by \( \Lambda \) and \( \alpha \) is a germ of closed holomorphic 1-form;
2. \( D(\eta) \geq D \), where \( D(\eta) = (\Delta(\eta), \Sigma(\eta)) \) and \( D = (\Delta, \Sigma) \);
3. \( \eta \) is faithful.

We say that \( \eta \) is strictly \textit{Q-logarithmic} if \( D(\eta) = D \).

Above, we also say that \( \eta \) is a \textit{logarithmic model} for \( Q \). Definition \( \mathbf{4.2} \) deserves a few comments. First, the poles of \( \eta \) are prescribed by \( S_0 \), the level 0 of the configuration of separatrices \( \Sigma \), and their residues are those defined by the system of residues \( \Lambda \). Second, the dominance condition \( D(\eta) \geq D \) says, aside from the fact that the reduction of singularities
of the foliation \( \mathcal{L} \) is longer than the underlying sequence of blow-ups of \( \Delta \), that the isolated separatrices of \( \mathcal{L} \) that are not poles of \( \eta \) are escape separatrices with respect to \( \mathbf{D} = (\Delta, \Sigma) \). Finally, the axiomatics defining systems of residues and systems of indices, along with the notion of consistency in Definition 3.5, gives that the Camacho-Sad indices of \( \Delta(\eta) \) and \( \Delta \) correspond, which is part of the information given by condition (2). Actually, we are demanding that whenever the weighted balance of residues \( \lambda_0 \) vanishes, the corresponding blow-up is dicritical, but, additionally, that it is dicritical in a “1-faithful” way.

The following theorem is a cornerstone to all subsequent results of this article. The remainder of this section is devoted to its proof.

**Theorem A.** Let \( \mathbf{T} = (\Delta, \Sigma, \Upsilon) \) be a dicritical triplet. Then there exists:

1. a system of residues \( \Lambda \) such that \( \mathbf{Q} = (\mathbf{T}, \Lambda) = (\Delta, \Sigma, \Upsilon, \Lambda) \) is a dicritical quadruplet;
2. a \( \mathbf{Q} \)-logarithmic 1-form \( \eta \).

**Proof.** The two assertions above are proved simultaneously. We essentially adapt the arguments in [6] to the combinatorics involved in the definition of a dicritical quadruplet. The idea is to descend the dicritical structure, starting at the final level \( n = h(\Delta) \). For a fixed level \( j \), we will furnish the following objects:

(i) a partial system of residues \( \Lambda^{(j)} \), that is to say, residue data for all components of the support for levels \( j \) and higher, satisfying (R.1), (R.2) and (R.3), consistent with the indices of \( \Upsilon \) at level \( n \).

(ii) for each \( p \in \mathbb{Z}_j \), a \( \mathbf{Q}^{(j)}_p \)-logarithmic 1-form \( \eta^{(j)}_p \), where \( \mathbf{Q}^{(j)}_p = (\mathbf{T}_p, \Lambda^{(j)}_p) \) is the dicritical quadruplet at \((\tilde{\mathbf{M}}_j, p)\) defined by the localizations of \( \mathbf{T} \) and of \( \Lambda^{(j)} \) at \( p \).

At level \( n \), residues of \( \Lambda^{(n)} \) are provided by Proposition 3.6. As for the \( \mathbf{Q}^{(n)}_p \)-logarithmic 1-forms, at a singular point \( p \in \mathbb{Z}_n \), we take local coordinates \((x, y)\) at \( p \) such that \( x = 0 \) and \( y = 0 \) are equations of the two components of \( \mathbb{Z}_n \) at \( p \), having residues \( \lambda_1 \) and \( \lambda_2 \), and then we set \( \eta^{(n)}_p = \lambda_1 dx/x + \lambda_2 dy/y \). At a regular point \( p \in \mathbb{Z}_n \), in local coordinates \((x, y)\) at \( p \) such that the component of \( \mathbb{Z}_n \) at \( p \) has equation \( x = 0 \), we set \( \eta^{(n)}_p = \lambda dx/x \), where \( \lambda \in \mathbb{C}^* \) is the corresponding residue.

Suppose that, for a fixed \( j \) with \( 1 \leq j \leq n \), residues and logarithmic 1-forms have been provided as described in items (i) and (ii) above. We will amalgamate data from level \( j \) in order to construct, at level \( j - 1 \), a partial system of residues \( \Lambda^{(j-1)} \), as well as a \( \mathbf{Q}^{(j-1)}_p \)-logarithmic 1-form for every \( p \in \mathbb{Z}_{j-1} \), where \( \mathbf{Q}^{(j-1)}_p = (\mathbf{T}_p, \Lambda^{(j-1)}_p) \). Actually, we only need to do this for the point \( p \in \mathbb{Z}_{j-1} \) that is the center of the blow-up \( \sigma_j \). Denote by \( D = \sigma_j^{-1}(p) \) and let \( p_1, \ldots, p_\ell \) be the points of intersection of \( D \) with \( \mathbb{Z}_{j-1} \). In our inductive construction, in addition to the partial system of residues \( \Lambda^{(j)} \), for each \( k = 1, \ldots, \ell \), we have a germ of \( \mathbf{Q}^{(j)}_{p_k} \)-logarithmic 1-form \( \eta^{(j)}_{p_k} \) at \( p_k \), where \( \mathbf{Q}^{(j)}_{p_k} = (\mathbf{T}_{p_k}, \Lambda^{(j)}_{p_k}) \).

The non-dicritical and dicritical cases are then treated separately.

**Case 1:** \( \sigma_j \) is a non-dicritical blow-up. The amalgamation of data from level \( j \) is quite instantaneous here and will be a consequence of Proposition 3.7. Indeed, for each \( p_k \in D \),
we have
\[
(5) \quad \lambda_D I_{p_k}(D) = - \sum_{\tilde{S} \in B_{p_k}(Z_j) \setminus D} \lambda_{\tilde{S}}(\tilde{S}, D)_{p_k},
\]
where residues are those of \( \Lambda^{(j)} \). Applying the Camacho-Sad formula (I.3) followed by equation \((5)\), we get
\[
(6) \quad \lambda_D = \lambda_D \left( - \sum_{k=1}^{\ell} I_{p_k}(D) \right) = \sum_{k=1}^{\ell} (-\lambda_D I_{p_k}(D))
\]
\[
= \sum_{k=1}^{\ell} \left( \sum_{\tilde{S} \in B_{p_k}(Z_j) \setminus D} \lambda_{\tilde{S}}(\tilde{S}, D)_{p_k} \right)
\]
\[
= \sum_{\tilde{S} \in B_p(Z_{j-1})} \nu_p(S) \lambda_{\tilde{S}},
\]
where \( \tilde{S} = \sigma_{j}^{*}S \) is the strict transform of \( S \in B_p(Z_{j-1}) \), which belongs to some \( B_{p_k}(Z_j) \).

Equation \((6)\) allows us to define \( \lambda_S = \lambda_{\tilde{S}} \), in such a way that condition (R.2) is satisfied by the blow-up \( \sigma_j \). Therefore, we can define residues at level \( j - 1 \) by simply importing then from level \( j \): for every component \( S \subset Z_{j-1} \), set \( \lambda_S = \lambda_{\tilde{S}} \), where \( \tilde{S} = \sigma_{j}^{*}S \). In this way, we have a partial system of residues \( \Lambda^{(j-1)} \).

In our downwards construction, write the \( Q_{p_k}^{(j)} \)-logarithmic 1-form at \( p_k \in D = \sigma_j^{-1}(p) \) as
\[
\eta_{p_k}^{(j)} = \lambda_{D} \frac{d f_D}{f_D} + \sum_{\tilde{S} \in B_{p_k}(Z_j) \setminus D} \lambda_{\tilde{S}} \frac{d \tilde{S}}{f_{\tilde{S}}} + \alpha_{p_k},
\]
where \( f_{\tilde{S}} \in \mathcal{O}_{p_k} \) are local equations for \( \tilde{S} \in B_{p_k}(Z_j) \setminus D \), \( f_D \in \mathcal{O}_{p_k} \) is a local equation for \( D \) and \( \alpha_{p_k} \) is a germ of closed holomorphic 1-form at \( p_k \). For each \( S \in B_p(Z_{j-1}) \), take \( f_S \in \mathcal{O}_p \) an equation of \( S \) such that \( f_S \circ \sigma_j = f_D^{(j)}(S)\tilde{f}_{\tilde{S}} \), where \( \tilde{S} = \sigma_{j}^{*}S \) and \( f_D \) is the local equation of \( D \) at \( p_k = \tilde{S} \cap D \). We then define
\[
\eta_{p}^{(j-1)} = \sum_{S \in B_p(Z_{j-1})} \lambda_S \frac{d f_S}{f_S}.
\]
Setting \( Q_{p}^{(j-1)} = (T_p, \Lambda_p^{(j-1)}) \), then \( \eta_{p}^{(j-1)} \) is \( Q_{p}^{(j-1)} \)-logarithmic. Indeed, by equation \((6)\), the residue of \( \sigma_{j}^{*}\eta_{p}^{(j-1)} \) with respect to \( D = \sigma_j^{-1}(p) \) is precisely \( \lambda_D \neq 0 \), which simultaneously implies that \( \eta_{p}^{(j-1)} \) is 1-faithful and that \( D \) is invariant by \( \sigma_{j}^{*}\eta_{p}^{(j-1)} \) [6]

Lem. 7]. On the other hand, faithfulness of \( \eta_{p}^{(j-1)} \) at higher levels is assured by the inductive construction, since, at each point \( p_k \in D \), the logarithmic 1-forms \( \sigma_{j}^{*}\eta_{p}^{(j-1)} \) and \( \eta_{p_k}^{(j)} \) differ from each other by a germ of closed holomorphic 1-form.

Case 2: \( \sigma_j \) is a dicritical blow-up. We consider again \( p_1, \ldots, p_{\ell} \), which now are precisely the points of \( Z_j \) on \( D = \sigma_j^{-1}(p) \). We start by proving the following:

Assertion 1. \( \ell \geq 2 \).
Proof. Indeed, if \( j = n \), so that \( D \subset D_n^\delta \), the result is clear: recalling that, by condition (D.3), \( \text{Val}(D) \) is actually the number of intersections of \( D \) with \( D_n^\delta \), the result is obvious if \( \text{Val}(D) \geq 2 \) and, if \( \text{Val}(D) = 1 \), it is a consequence of condition (S.3). Suppose now that \( 1 \leq j < n \) and \( D \subset D_j^\delta \). If \( \varsigma_j \) is trivial over \( D \), then the result follows from the case \( j = n \). When this is not the case, we remark that, for each point \( q \in D \) over which \( \varsigma_j \) is non-trivial, we find, as a consequence of (D.3) and (S.2), at least one separatrix in \( S_j^p \) at \( q \).

Besides, each component of \( D_j \), other than \( D \), intersecting \( D \) at a point \( q' \) over which \( \varsigma_j \) is trivial is necessarily in \( D^\delta \) by (D.3). We are then reduced to the situation where there is only one such point \( q \in D \) and none of those points \( q' \in D \). But, in this case, denoting \( \bar{D} = \varsigma_j^* D \), then \( \text{Val}(D) = 1 \) in \( D_n \). By (S.3), we must have a separatrix of \( S_j^p \) touching \( D \) at a point other than \( q \), proving thereby the assertion. \( \square \)

Now, at \( p_k \in D = \sigma_j^{-1}(p) \), the given \( Q_p^j \)-logarithmic 1-form is written as

\[
\eta_{p_k}^{(j)} = \sum_{S \in B_{p_k}(Z_j)} \lambda_{\bar{S}} \frac{df_{\bar{S}}}{f_{\bar{S}}} + \alpha_{p_k},
\]

where \( f_{\bar{S}} \in \mathcal{O}_{p_k} \) is a local equation for \( \bar{S} \) and \( \alpha_{p_k} \) is a germ of closed holomorphic 1-form at \( p_k \).

For each \( k = 1, \ldots, \ell \), denote by \( B_{p_k}^{\sigma_j}(Z_{j-1}) \subset B_p(Z_{j-1}) \) the subset of all branches \( S \) such that \( \bar{S} = \sigma_j^* S \in B_{p_k}(Z_j) \). Also define

\[
\rho_{p_k} = \sum_{S \in B_{p_k}^{\sigma_j}(Z_{j-1})} \nu_p(S) \lambda_{\bar{S}}.
\]

Another important point is the following:

**Assertion 2.** \( \rho_{p_k} \neq 0 \) for every \( k = 1, \ldots, \ell \).

Proof. Suppose, to the contrary, that \( \rho_{p_k} = 0 \) for some \( k \). Consider the germ of logarithmic 1-form at \( p \in Z_{j-1} \) defined as

\[
\zeta_p^{(j-1)} = \sum_{S \in B_{p_k}^{\sigma_j}(Z_{j-1})} \lambda_{\bar{S}} \frac{df_{\bar{S}}}{f_{\bar{S}}},
\]

where \( f_S \in \mathcal{O}_p \) is the equation of \( S \) corresponding to the equation \( f_{\bar{S}} \in \mathcal{O}_{p_k} \) of \( \bar{S} = \sigma_j^* S \). Since \( \zeta_p^{(j-1)} \) has a main resonance at \( p \) and all its poles share the same tangent cone, by [6] Prop. 10], \( \sigma_j \) is a non-dicritical blow-up for \( \zeta_p^{(j-1)} \). Denote the germ of \( \sigma_j^* \zeta_p^{(j-1)} \) at \( p_k \) by \( \zeta_{p_k}^{(j)} \). Since \( \rho_{p_k} = 0 \), the logarithmic 1-forms \( \zeta_{p_k}^{(j)} \) and \( \eta_{p_k}^{(j)} \) differ from each other by the germ of closed holomorphic 1-form \( \alpha_{p_k} \). Thereafter, \( \zeta_{p_k}^{(j)} \) is also \( Q_{p_k}^{(j)} \)-logarithmic. We thus have that \( D_{p_k} \), the germ of \( D \) at \( p_k \), is not a pole for \( \zeta_{p_k}^{(j)} \), although it is invariant. For this to occur, \( D_{p_k} \) should be a escape separatrix for the logarithmic 1-form \( \zeta_{p_k}^{(j)} \) and, as a consequence, \( \zeta_{j}^* D_{p_k} \) should touch \( \varsigma_{j}^{-1}(p_k) \subset D_n \) at a dicritical component. However, this would mean that two components in \( D_n^\delta \) meet each other, which is not allowed by (D.3). This contradiction proves the assertion. \( \square \)
In order to amalgamate data at level \( j - 1 \), we chose \( c_1, \ldots, c_\ell \in \mathbb{C}^* \) such that
\[
c_1 \rho_{p_1} + \cdots + c_\ell \rho_{p_\ell} = 0,
\]
which is possible by the two assertions just proven. Next, we modify the data of \( \Lambda^{(j)} \) in the following way: at each \( p_k \in D \), for each branch \( S \in B_{p_k}(Z_j) \) we replace \( \lambda_S \) by \( c_k \lambda_S \) and perform the same correction of residues at all points in the connected component of \( D^p_j \) containing \( p_k \). Once accomplished this adjustment at level \( j \), we accordingly modify residues at higher levels, obtaining new partial system of residues, that we denote by \( \tilde{\Lambda}^{(j)} \).

Next, we define a partial system of residues \( \Lambda^{(j-1)} \) by using information from \( \tilde{\Lambda}^{(j)} \) at levels \( j \) and higher and also descending it to level \( j - 1 \): for a component \( S \subset Z_{j-1} \), we take \( \lambda_S \) as the residue of \( \tilde{S} = \sigma_S^* S \) in \( \tilde{\Lambda}^{(j)} \). We clearly have a partial system of residues. In particular, condition (R.3) for \( \sigma_j \) is a consequence of \( \square \).

As for the \( Q_p^{(j-1)} \)-logarithmic 1-form, where \( Q_p^{(j-1)} = (T_p, \Lambda_p^{(j-1)}) \), in the same way as in the non-dicritical case, we define
\[
\eta_p^{(j-1)} = \sum_{S \in B_p(Z_{j-1})} \lambda_S \frac{df_S}{f_S}.
\]
It is a consequence of equation \( \square \) that the germ of \( \sigma_p^* \eta_p^{(j-1)} \) at \( p_k \) differs from \( c_k \eta_p^{(j-1)} \) by a germ of closed holomorphic 1-form. This implies that \( \sigma_p^* \eta_p^{(j-1)} \) is faithful and, thus, \( Q_p^{(j)} \)-logarithmic. On the other hand, \( \eta_p^{(j-1)} \) is 1-faithful as a consequence of Proposition \( 7.1 \) to be proven below in Section \( 7 \). In particular, it is dicritical for the blow-up \( \sigma_j \). We can thus conclude that \( \eta_p^{(j-1)} \) is faithful, and, therefore, that it is \( Q_p^{(j-1)} \)-logarithmic. \( \square \)

**Example 4.3.** In Examples 2.2, 2.4 and 3.2 we built \( T(F) = (\Delta(F), \Sigma(F), \Upsilon(F)) \), a dicritical triplet associated with a singular holomorphic foliation \( F \) at \((\mathbb{C}^2,0)\). As a consequence of Theorem 4A we can complete \( T(F) \) into a dicritical quadruplet \( Q(F) = (T(F), \Lambda(F)) = (\Delta(F), \Sigma(F), \Upsilon(F), \Lambda(F)) \) and find a \( Q(F) \)-logarithmic 1-form \( \eta_F \). We say that \( \eta_F \) is a logarithmic model for \( F \). The existence of logarithmic models for germs of singular holomorphic foliations in the plane has been established in \( \mathbb{S} \) and \( \mathbb{R} \).

5. **Complex meromorphic functions**

Let \( h \) be a germ of meromorphic function at \((\mathbb{C}^2,0)\). Let \( \pi : (\tilde{M}, D) \to (\mathbb{C}^2,0) \) be the reduction of singularities of the foliation \( H \) defined by the levels of \( h \). In particular, \( \pi \) raises the indeterminacy of \( h \): there exists a germ of holomorphic map \( \tilde{h} : \tilde{M} \to \mathbb{P}^{1}_C \) in a neighborhood of \( D \) such that, outside \( D \), \( \tilde{h} = h \circ \pi \). We associate with \( h \) the dicritical duplet of the logarithmic 1-form \( \eta_h = dh/h : D(h) = D(\eta_h) \), where \( D(h) = (\Delta(h), \Sigma(h)) \). Actually, we have to assume that (S.3) is satisfied, as it will be the case below. The typical fibers of \( h \) have irreducible components whose equisingularity types are determined by its indeterminacy structure, i.e. the dicritical components of \( D^h_n \). On the other hand, the isolated separatrices of \( S^h_0 \) are irreducible components of non-typical fibers.

**Example 5.1.** Let \( h = x^p/y^q \), with \( p, q \in \mathbb{Z}_+ \) relatively prime. The combinatorics involved in \( \Delta(h) \) is the same of that of Euclid’s algorithm of the pair \((p, q)\). The sole dicritical blow-up will be the last one, \( \sigma_n \). In other words, the dicritical structure depends on the residues of the 1-form \( \eta_h = dh/h = p dx/x - q dy/y \). If \( p, q > 1 \), then \( S_1 : x = 0 \) and \( S_2 : y = 0 \).
are isolated separatrices. On the other hand, if, for instance, \( p > 1 \) and \( q = 1 \), then \( S_1 \) is isolated and \( S_2 \) is dicritical.

The central result of this section concerns the construction of meromorphic functions with prescribed indeterminacy structure, zeros and poles. The indeterminacy structure of a meromorphic function \( h \) can be specified by an infinitesimal class \( \kappa \) of a union of components of some blow-up divisor, meant to identify the dicritical components of the reduction of singularities of the foliation \( \mathcal{H} \) given by the levels of \( h \). We can then state:

**Theorem B.** Let \( \kappa \) be an infinitesimal class and \( \mathcal{S} \) be a finite set of branches of analytic curves at \((\mathbb{C}^2,0)\). Then there exists a germ of meromorphic function \( h \) at \((\mathbb{C}^2,0)\) whose indeterminacy structure is given by \( \kappa \) and such that the set of all branches of \( h = 0 \) and \( h = \infty \) contains \( \mathcal{S} \).

**Proof.** Let \( \pi : (\tilde{M}, D) \to (\mathbb{C}^2,0) \) be a sequence of blow-ups that is a minimal realization for the infinitesimal class \( \kappa \) and for the equisingularity class \( \varepsilon(\mathcal{S}) \). Build a dicritical structure \( \Delta \) over \( \pi \) with dicritical components determined by \( \kappa \). Next, complete \( \mathcal{S} \) into a finite subset \( \mathcal{S}_0 \subset B_0 \) so that conditions (S.2) and (S.3) are respected, taking care that also (S.4) is satisfied. Then \( \mathcal{S}_0 \), with the decomposition given by \( \Delta \), is the level 0 of a configuration of separatrices \( \Sigma \), and thus we have a dicritical duplet \( D \) at \((\mathbb{C}^2,0)\). We obtain the theorem’s statement by proving the following: there exists a germ of meromorphic function \( h \) such that \( D(h) \geq D \). This result will follow from Theorem A once we prove that we can associate a system of indices \( \Upsilon \) having only negative rational indices at the singular points of the support of \( D \) at the final level. We prove this next, in Proposition 5.2 and assume it for now. Hence, applying Theorem A we can find a system of residues \( \Lambda \) that completes a dicritical quadruplet \( \mathcal{Q} = (\Delta, \Sigma, \Upsilon, \Lambda) \) as well as a \( \mathcal{Q} \)-logarithmic 1-form \( \eta \). An attentive look at the induction in its proof shows that \( \Lambda \) can be obtained having only rational residues. Indeed, its initialization works by the application of Proposition 3.6, where residues are obtained in \( \mathbb{Q}^* \) once we depart from \( \lambda_D \in \mathbb{Q}^* \) for the maximal element \( D \) of each connected component \( A = A'_{n,k} \). It is also easy to see that the amalgamation of data belonging to the general step \( j \) can be done in such a way that all residues are generated in \( \mathbb{Q}^* \). Notably, in the dicritical case, in equation (7), if all coefficients \( \rho_{p,k} \) are in \( \mathbb{Q}^* \), then we can get a solution vector with coordinates in \( \mathbb{Q}^* \).

With the writing of Definition 4.2 after replacing some \( f_S \) by \( u f_S \), where \( u \in \mathcal{O}_0 \) is a unit such that \( \lambda_S du/u = \alpha \), and after multiplying by the least common multiple of the denominators of the residues, we can write

\[
\eta = \sum_{S \in \mathcal{S}_0} n_S \frac{df_S}{f_S},
\]

where \( n_S \in \mathbb{Z}^* \). By setting

\[
\eta = \prod_{S \in \mathcal{S}_0} n_S^{f_S},
\]

we have that \( \eta = \eta_h = dh/h \), proving the theorem. \( \square \)

In order to complete the proof of Theorem B, we still have to show the possibility of assigning negative rational indices at the final level. For this purpose, we shall explore the combinatorics dictated by rules (I.2), (I.3) and (I.4). Let us then associate with \( D = \pi^{-1}(0) \) a weighted graph \( \text{Gr}(D) \), in which each irreducible component \( D \) corresponds to a vertex,
More precisely, there is $j$ is proved once induction reaches level $\text{Gr}(A)$ gives that

Remark. Following this order, denote by $s$ be the size of the tree $\text{Gr}(A)$ and stratify its vertices in levels $j = 1, \ldots, s$, where level $s$ corresponds to the maximal vertex $D_0$, level $s - 1$ to its immediate predecessors and so on. Next, for each component $D \subset A$ and for each trace singularity $q \in D$, with $q \neq p$, choose for $I_q(D)$ a sufficiently small value in $\mathbb{Q}_-$ in such a way that their sum over $D$ satisfies $|\sum I_q(D)| < \epsilon$. Define $c(D) = c(D) - \sum I_q(D)$. In this way, $\text{Gr}(A)$ has a negative definite intersection matrix. The idea now is to use \cite{prop. 3.3} in a context where the sum of indices over a component $D$ is $c(D)$, in order to show that all indices, for all corners and for the point $p$, are in $\mathbb{Q}_-$. The argument goes by induction, with the initialization and the general step treated with the same argument. Let $1 \leq j \leq s$ and suppose that all corners in components at levels lower than $j$ have indices in $\mathbb{Q}_-$ (an empty condition, if $j = 1$). If $D$ is a component at level $j$, denote by $\text{Gr}(A_D)$ the subtree of $\text{Gr}(A)$ formed by $D$ and all its predecessors. $\text{Gr}(A_D)$ also has a negative definite intersection matrix. If $j < s$, let $D'$ be the successor of $D$ and denote $p' = D \cap D'$. If $j = s$, set $D' = S$ and $p' = p$. Denote by $q_1, \ldots, q_r$ the corners of $D$ with its predecessors, if there are any. Then the induction hypothesis and (I.4) gives that $I_{q_i}(D) \in \mathbb{Q}_-$ for $i = 1, \ldots, r$. The sum of indices over $D$ equaling $\tilde{c}(D)$ gives that $I_{p'}(D) \in \mathbb{Q}_-$. Finally, regarding $\text{Gr}(A_D)$ as a combinatorial portrait of an invariant divisor with only one trace singularity at $p'$, with sums of indices given by $\tilde{c}$ in place of $c$, an application of \cite{prop. 3.3} gives that $I_{p'}(D) \notin \mathbb{R}_{\geq 0}$. Thus, $I_{p'}(D) \in \mathbb{Q}_-$ and also $I_{p'}(D') \in \mathbb{Q}_-$. The proposition is proved once induction reaches level $j = s$. \hfill $\square$

Remark. In Theorem \cite{prop. 3.3} the set of branches of zeroes and poles of $h$ contains $S$ as a proper subset, in general. As it is clear in the beginning of its proof, we have to include in $S$ some branches in order to obtain a set where conditions (S.2) and (S.3) are valid. The necessity of condition (S.3) is apparent: if $D \subset \mathcal{D}$ is a dicritical component of valence one, then the lift $\tilde{h}$ of $h$ to $\tilde{M}$, restricted $D \simeq \mathbb{P}^1_{\mathbb{C}}$, must have a zero and a pole. One of
them being given by the component of $\overline{D \setminus D}$ intersecting $D$, the other must be given by a separatrix touching $D$.

6. Real logarithmic models

6.1. Symmetric dicritical quadruplets. Let $J : (x, y) \in \mathbb{C}^2 \mapsto (\bar{x}, \bar{y}) \in \mathbb{C}^2$ be the canonical anti-holomorphic involution defined by the complex conjugation. The real trace of $\mathbb{C}^2$, the fixed set of $J$, is identified with $\mathbb{R}^2$. Points are said to be real when they belong to the real trace, and non-real when they do not. For $f \in \mathcal{O}_0$, let $f^\vee \in \mathcal{O}_0$ be such that $f^\vee(x, y) = \overline{f(J(x, y))}$ for every $(x, y)$ near $0 \in \mathbb{C}^2$. We say that $f \in \mathcal{O}_0$ is real if $f = f^\vee$. This is equivalent to $f$ having a Taylor series expansion at $0 \in \mathbb{C}^2$ with only real coefficients or to $f$ assuming only real values over the real trace. For a branch $S \in \mathcal{B}_0$ having $f \in \mathcal{O}_0$ as equation, we denote by $S^\vee \in \mathcal{B}_0$ the branch with equation $f^\vee$, whose trace is the $J$-image of the trace of $S$. We say that a branch $S \in \mathcal{B}_0$ is real if $S = S^\vee$.

This happens if and only if $S$ has a real $f \in \mathcal{O}_0$ as equation.

A germ of holomorphic 1-form $\omega = Pdx + Qdy$ at $(\mathbb{C}^2, 0)$ is symmetric if $P, Q \in \mathcal{O}_0$ are real. In this case, the germ of holomorphic foliation $\mathcal{F}$ defined by $\omega$, also said to be symmetric, is invariant by $J$: if $U$ is a $J$-invariant neighborhood of $0 \in \mathbb{C}^2$ where $\omega$ is realized, and $p \in U$ is a regular point, denoting by $L_p$ the leaf through $p$, then $J(L_p)$ is the leaf through $p^\vee = J(p)$. In particular, if $p \in \mathbb{R}^2$, then $L_p = J(L_p)$. Thus, $L_p|\mathbb{R}^2$ is a local curve in $\mathbb{R}^2$ (of topological dimension one), which is the local leaf of the foliation $\mathcal{F}_\mathbb{R}$ defined by $\omega_\mathbb{R}$, the restriction of $\omega$ to the real trace. Note also that, if $\mathcal{F}$ is symmetric, $S$ is a separatrix at a real point $p$ if and only if $S^\vee$ is.

If $\sigma : (\mathbb{C}^2, D) \to (\mathbb{C}^2, 0)$ is a punctual blow-up, then there is a unique continuous involution $J' : (\mathbb{C}^2, D) \to (\mathbb{C}^2, D)$ such that $\sigma(J'(p)) = J(\sigma(p))$ for every $p \in \mathbb{C}^2$. Its fixed set, $\mathbb{R}^2$, is the real trace of $\mathbb{C}^2$. Evidently, the involution $J$ will be lifted by successive blow-ups provided their centers lie in the real trace. More generally, we say that a sequence $\pi : (\mathbb{C}^2, D) \to (\mathbb{C}^2, 0)$ of quadratic blow-ups is symmetric if the anti-holomorphic involution $J$ has a continuous lift $\bar{J}$ to $\mathbb{C}^2$. Loosely speaking, this happens if, among the individual blow-ups factoring $\pi$, those with non-real centers are even in number and paired by conjugation.

A dicritical structure $\Delta$ of height $n = b(\Delta)$ at $(\mathbb{C}^2, 0)$ is symmetric if its underlying sequence of blow-ups is symmetric and, for every component $D \subset \mathcal{D}_n$, we have that $D \subset \mathcal{D}_n^0$ if and only if $D^\vee = J(D) \subset \mathcal{D}_n^0$. A configuration of separatrices $\Sigma$ at $(\mathbb{C}^2, 0)$ is symmetric if it is framed on a symmetric dicritical structure $\Delta$ and if $S_\Sigma$ is invariant by $J$. Clearly, this also gives that $S_n$ is invariant by $J$ and, as a consequence, it holds that $S \in S_\Sigma^0$ if and only if $S^\vee \in S_\Sigma^0$. By (S.4), if $p \in \mathcal{D}_n$ is a real point and $S \in S_n$ is a separatrix through $p$, then $S$ is real. In particular, if $p \in \mathcal{Z}_n$ is a real singular point, then both branches of $\mathcal{Z}_n$ at $p$ are real. We say that $\mathcal{D} = (\Delta, \Sigma)$ is a symmetric dicritical duplet.

A system of indices $\Upsilon$ associated with a symmetric dicritical duplet $\mathcal{D}$ is symmetric if whenever $p \in \mathcal{Z}_n$ and $S \in \mathcal{B}_p(\mathcal{Z}_n)$, then $I_p^\vee(S^\vee) = I_p(S)$. In particular, if $p \in \mathcal{Z}_n$ is real singular point, then $I_p(S) \in \mathbb{R}^* \setminus \mathbb{Q}_+$ for each branch $S \in \mathcal{B}_p(\mathcal{Z}_n)$. We say that $\Upsilon = (\Delta, \Sigma, \Upsilon)$ is a symmetric dicritical triplet. Finally, a symmetric system of residues $\Lambda$ for the symmetric dicritical duplet $\mathcal{D} = (\Delta, \Sigma)$ is one for which $\lambda S^\vee = \lambda S$ whenever $S \in S_0$. Thus, $\lambda S \in \mathbb{R}^*$ whenever $S$ is a real separatrix. The application of (R.1), (R.2) and (R.3) gives that $\lambda S^\vee = \overline{\lambda S}$ for every component $S \subset \mathcal{Z}_n$. If $\Upsilon$ and $\Lambda$ are consistent.
symmetric $D$-systems of indices and residues, then we say that $Q = (\Delta, \Sigma, Y, \Lambda)$ is a symmetric dicritical quadruplet.

We say that a logarithmic 1-form as in (3) is symmetric if its set of poles $S \subset B_0$ is symmetric, as well as the corresponding residues and the holomorphic 1-form $\alpha$. We can restate Theorem A in the context of symmetric multiplets. The very same proof presented in Section § works here and we leave to the reader its step-by-step verification.

**Theorem 6.1.** Let $T = (\Delta, \Sigma, Y)$ be a symmetric dicritical triplet. Then there exists:

1. a symmetric system of residues $\Lambda$ such that $Q = (T, \Lambda) = (\Delta, \Sigma, Y, \Lambda)$ is a symmetric dicritical quadruplet;
2. a Q-logarithmic symmetric 1-form $\eta$.

In the next subsection, we will handle abstract multiplets formed by real objects that will model the desingularization, separatrices and indices of germs of real analytic vector fields at $(\mathbb{R}^2, 0)$. The complexification of these real objects will eventually give rise to symmetric multiplets. In order to achieve this, we first have to consider the following:

**Definition 6.2.** A quasi dicritical triplet at $(\mathbb{C}^2, 0)$, or q-dicritical triplet for short, is the object $T^R = (\Delta^R, \Sigma^R, Y^R)$ composed by:

- a dicritical structure $\Delta^R$;
- a configuration of separatrices $\Sigma^R$ with conditions (S.2) and (S.3) removed;
- a system of indices $Y^R$ with condition (I.3) deleted.

The q-dicritical triplet $T^R$ is symmetric if all its elements are invariant by conjugation. In this case, we say that a symmetric dicritical triplet $T = (\Delta, \Sigma, Y)$ extends $T^R$ if:

- The sequence of blow-ups subjacent to $T$ is obtained from that of $T^R$ by additional blow-ups at non-real points; $\Delta$ and $\Delta^R$ provide the same classification as dicritical or non-dicritical for the common components.
- If $S_0^R$ and $S_0$ denote the set of separatrices at level 0 of $\Sigma^R$ and $\Sigma$, respectively, then $S_0^R \subset S_0$ and $S_0 \setminus S_0^R$ contains only non-real separatrices.
- For separatrices in $S_0^R$, indices assigned by $T$ and $T^R$ are the same.

We can always extend a symmetric q-dicritical triplet, as shown in the following result.

**Proposition 6.3.** Let $T^R = (\Delta^R, \Sigma^R, Y^R)$ be a symmetric q-dicritical triplet at $(\mathbb{C}^2, 0)$. Then there exists a symmetric dicritical triplet $T = (\Delta, \Sigma, Y)$ that extends $T^R$. All indices in $\Sigma$ can be obtained in $\mathbb{R}$.

**Proof.** We take $\Delta = \Delta^R$. Our task is to add non-real separatrices to $\Sigma^R$ and assign them indices in such a way that rules (S.2), (S.3) and (I.3) are observed. Let $A = A_{n,k}$ be a connected component of $D_n$, the invariant part of the final level of $\Delta$. Let $D \subset A$ be a component. Define $c^R(D) = \sum I_p(D)$, where $p$ runs over the singular points of $D$ with respect to $T^R$, i.e. its corners and points where separatrices in $S^R_n$ meet $D$. If $c^R(D) = c(D)$, we do nothing. If $c(D) - c^R(D) \in \mathbb{R} \setminus \mathbb{Q}_{\geq 0}$, we choose a non-real point $p \in D$, take $S$ any smooth branch at $p$ transversal to $D$, add it to $S^R_n$, with index $I_p(S) = 2/(c(D) - c^R(D))$ getting, as a consequence, $I_p(D) = (c(D) - c^R(D))/2$. We also include $S^R$ in $S^R_n$, with $I_p(S^R) = I_p(S)$. Now, if $c(D) - c^R(D) \in \mathbb{Q}_{+}$, then we choose $b_1, b_2 \in \mathbb{R} \setminus \mathbb{Q}$ such that $b_1 + b_2 = (c(D) - c^R(D))/2$ and two distinct, non-conjugate, non-real points $p_1, p_2 \in D$. Take $S_1$ and $S_2$ two smooth branches at, respectively, $p_1$ and
According to the definition in the previous section, \( \omega \) is an isolated singularity at \( 0 \in F \). 

Then, \( I_{p_1}(D) + I_{p_2}(D) = b_1 + b_2 = (c(D) - c^{\mathcal{K}}(D))/2 \). Next, also include \( S_1^\mathcal{K} \) and \( S_2^\mathcal{K} \) to \( S_n^\mathcal{K} \) with indices \( I_{p_1}(S_1^\mathcal{K}) = 1/b_1 \) and \( I_{p_2}(S_2^\mathcal{K}) = 1/b_2 \).

The object \( T = (\Delta, \Sigma, \Upsilon) \) constructed so far satisfies (I.3) for all components \( D \in A \). It does not comply with (S.2) if \( S_n^\mathcal{K} \) has no separatrices touching \( A \) and if \( c^{\mathcal{K}}(D) = c(D) \) for every \( D \subset A \). If this is so, fix \( D \subset A \), pick two distinct, non-conjugate, non-real points \( p_1, p_2 \in D \), then choose \( b_1, b_2 \in \mathbb{R} \setminus \mathbb{Q} \) such that \( b_1 + b_2 = 0 \) and repeat the construction that closes the previous paragraph. The above procedure, applied to all connected components of \( D_n^\mathcal{K} \), results in the validity of (I.3) and (S.2). To finish the proof, if for some \( D \subset D_n^\mathcal{K} \), the foliation \( T \) has no separatrices touching \( D \) transversally at pair of different trace points, and give them both zero as indices, doing the corresponding symmetric intervention in \( D^\mathcal{V} \) if \( D \) is non-real.

\[ \square \]

6.2. \textbf{Real analytic 1-forms.} Suppose that \((x, y)\) are now coordinates in \( \mathbb{R}^2 \). Let \( C^\omega_0 \) denote the ring of germs of real analytic functions with real values at \((\mathbb{R}^2, 0)\). Let \( \omega_\mathbb{R} = P(x, y)dx + Q(x, y)dy \) be a germ of real analytic 1-form at \((\mathbb{R}^2, 0)\), where \( P, Q \in C^\omega_0 \) are relatively prime non-units. Then, \( \omega_\mathbb{R} \) defines, near \( 0 \in \mathbb{R}^2 \), a real analytic foliation \( F_\mathbb{R} \) with isolated singularity at the origin. We denote by \( \omega \) the complexification of \( \omega_\mathbb{R} \), which amounts to regarding \((x, y)\) as coordinates of \( \mathbb{C}^2 \) and \( P, Q \) as elements of \( \mathcal{O}_0 \). Then, according to the definition in the previous section, \( \omega \) is a symmetric 1-form, with an isolated singularity at \( 0 \in \mathbb{C}^2 \), defining a singular holomorphic foliation \( F \) invariant by the involution \( J \). Clearly, viewing \( \mathbb{R}^2 \) as the real trace of \( \mathbb{C}^2 \), we have that \( F_\mathbb{R} = F|_{\mathbb{R}^2} \).

The foliation \( F_\mathbb{R} \) has a \textit{reduction of singularities} by real quadratic blow-ups, say \( \pi_\mathbb{R} : (M_\mathbb{R}, D_\mathbb{R}) \to (\mathbb{R}^2, 0) \). This means that, for the strict transform foliation \( \hat{F}_\mathbb{R} = \pi_\mathbb{R}^*F_\mathbb{R} \), we can obtain the real equivalent of the four items listed in Example 2.2 with the additional condition that real eigenvalues are associated with simple singularities.

In order to obtain \( \pi_\mathbb{R} \), we can consider, for instance, the minimal reduction of singularities for \( F \), say \( \pi : (M, D) \to (\mathbb{C}^2, 0) \), and take \( M_\mathbb{R} \) as the real trace of \( M \), determined by the lift of \( J \), getting \( D_\mathbb{R} = D \cap M_\mathbb{R} \) and \( \pi_\mathbb{R} = \pi|_{\mathbb{M}_\mathbb{R}} \).

The following definition associates with a sequence of real quadratic blow-ups an abstract triplet composed only by real objects.

\textbf{Definition 6.4.} A \textit{real quasi dicritical triplet}, or \textit{real q-dicritical triplet}, framed on a sequence of real quadratic blow-ups \( \pi_\mathbb{R} \) is the object \( \mathbf{T}^\mathbb{R} = (\Delta^\mathbb{R}, \Sigma^\mathbb{R}, \Upsilon^\mathbb{R}) \) whose components are:

- \( \Delta^\mathbb{R} \) is the structure of real divisors produced by the sequence of blow-ups \( \pi_\mathbb{R} \);
- \( \Sigma^\mathbb{R} \) is formed by real analytic branches;
- \( \Upsilon^\mathbb{R} \) includes only real indices.

They satisfy the set of axioms that defines a dicritical triplet, except for (S.2), (S.3) and (I.3).

Evidently, the complexification of the entries of \( \mathbf{T}^\mathbb{R} \) gives readily a symmetric q-dicritical triplet \( \mathbf{T}^\mathbb{C} \), framed on the sequence of complex quadratic blow-ups \( \pi \), the complexification of \( \pi_\mathbb{R} \).
Following the definition in [14], $\mathcal{F}_R$ is of real generalized curve type if $\tilde{\mathcal{F}}_R$ has no real algebraic saddle-node singularities, i.e. simple singularities with one zero eigenvalue. This is equivalent to asking that $\tilde{\mathcal{F}}$ has no saddle-node singularities over the real trace of $\tilde{M}$. With a foliation of this kind, we can associate a real q-dicritical triplet $T^N_R(\mathcal{F}_R) = (\Delta^N_R(\mathcal{F}_R), \Sigma^N_R(\mathcal{F}_R), \Upsilon^N_R(\mathcal{F}_R))$, including in $\Sigma^N_R(\mathcal{F}_R)$ all isolated separatrices and a finite number of dicritical separatrices. It gives rise, by complexification, to a symmetric q-dicritical triplet $T^N(\mathcal{F}_R) = (\Delta^N(\mathcal{F}_R), \Sigma^N(\mathcal{F}_R), \Upsilon^N(\mathcal{F}_R))$.

We say that a germ of 1-form $\eta_R$ at $(\mathbb{R}^2, 0)$ is real logarithmic if it has a writing as in [4], where each $f_i \in C^\infty_0$ is irreducible, the residues $\lambda_i$ are real and $\alpha$ is a germ of real analytic 1-form. Note that the zero set of some of the $f_i$ may degenerate to the origin. When $f_i = 0$ is one dimensional, we say that it defines a real pole $S_i$ of $\eta_R$. The notion of 1-faithful logarithmic 1-form, set in Definition 3.4, we can build a real q-dicritical triplet $\eta_R$ of real logarithmic 1-form, preceding Definition 6.3, we may degenerate to the origin. We then say that a logarithmic 1-form is real faithful if it is 1-faithful along the reduction of singularities of $L_R$, the germ of real analytic foliation at $(\mathbb{R}^2, 0)$ defined by $\eta_R$.

Following the same steps of the construction of a dicritical triplet for a complex logarithmic 1-form, preceding Definition 3.4 we can build a real q-dicritical triplet $T^N_R(\eta_R) = (\Delta^N_R(\eta_R), \Sigma^N_R(\eta_R), \Upsilon^N_R(\eta_R))$ for a real logarithmic $\eta_R$. Briefly, $\Delta^N_R(\eta_R)$ is based on the real reduction of singularities of $L_R$, $\Sigma^N_R(\eta_R)$ includes all real poles of $\eta_R$ along with all isolated separatrices of $L_R$, and $\Upsilon^N_R(\eta_R)$ collects all Camacho-Sad indices of $L_R$. Next, the notion of dominance for dicritical duplets can be transposed in a straightforward way to real q-dicritical duplets, and, with it, the concepts of escape point and escape separatrix. Keeping our notation, the writing $D^N_R \geq D^N_R$ means that $D^N_R = (\Delta^N_R, \Sigma^N_R, \Upsilon^N_R)$ dominates $D^N_R = (\Delta^N_R, \Sigma^N_R, \Upsilon^N_R)$.

**Definition 6.5.** Let $T^N_R = (\Delta^N_R, \Sigma^N_R, \Upsilon^N_R)$ be a real q-dicritical triplet at $(\mathbb{R}^2, 0)$. A germ of real logarithmic 1-form $\eta_R$ is a real logarithmic model for $T^N_R$ if:

1. the set of real poles of $\eta_R$ is $S^N_0$, the set of separatrices of $\Sigma^N_R$ at level 0;
2. $D^N_R(\eta_R) \geq D^N_R$;
3. $\eta_R$ is real faithful;
4. indices for separatrices in $S^N_0$ are the same, for $\Upsilon^N_R(\eta_R)$ and for $\Upsilon^N_R(\eta_R)$.

The real logarithmic model is strict if $D^N_R(\eta_R) = D^N_R$. The following result generalizes, to the dicritical case, the main theorem in [9]:

**Theorem C.** Let $T^N_R = (\Delta^N_R, \Sigma^N_R, \Upsilon^N_R)$ be a real q-dicritical triplet at $(\mathbb{R}^2, 0)$. Then there exists a germ of real logarithmic 1-form $\eta_R$ that is a real logarithmic model for $T^N_R$.

**Proof.** We complexify $T^N_R$, obtaining the symmetric q-dicritical triplet $T^N$. By Proposition 6.3, $T^N$ can be completed into a symmetric dicritical triplet $T$ with real indices. As a consequence of Theorem E.1, there are a symmetric system of residues $\Lambda$, with real residues, such that $Q = (T, \Lambda)$ is a symmetric dicritical quadruplet, and a symmetric Q-logarithmic 1-form $\eta$, whose residues are real. If $\eta_R$ is the restriction of $\eta$ to the real trace of $\mathbb{C}^2$, then $\eta_R$ is a real logarithmic model for $T^N_R$. \(\square\)

In Definition 6.5, when $T^N_R = T^N_R(\mathcal{F}_R)$ for some germ of singular real analytic foliation $\mathcal{F}_R$ of real generalized curve type, we say that $\eta_R$ is a real logarithmic model for $\mathcal{F}_R$. The logarithmic model is strict if $\eta_R$ is a strict real logarithmic model for some real q-dicritical
triplet $T^\infty_{\mathcal{F}_R}(\mathcal{F}_R)$ associated with $\mathcal{F}_R$. For germs of real analytic foliations, we can assure the existence of strict real logarithmic models:

**Theorem D.** Let $\mathcal{F}_R$ be a germ of singular real analytic foliation at $(\mathbb{R}^2, 0)$ of real generalized curve type. Then there exists a strict real logarithmic model for $\mathcal{F}_R$.

**Proof.** The existence of a logarithmic model is a consequence of Theorem C. The possibility of obtaining a strict real logarithmic model is justified in the next section, by Proposition 7.4 and the comments following it.

As we did for complex sequences of blow-ups in Section 2, we can define real infinitesimal classes for sequences of real quadratic blow-ups. In this case, they are denoted by $\kappa_R$. We can then apply the machinery of this section to the construction of real meromorphic functions, yielding the following real version of Theorem B:

**Theorem E.** Let $\kappa_R$ be a real infinitesimal class and $S$ be a finite set of branches of real analytic curves at $(\mathbb{R}^2, 0)$ whose indeterminacy structure is given by $\kappa_R$ and such that the set of all branches of $h = 0$ and $h = \infty$ equals $S$.

**Proof.** Information from $\kappa_R$ and $S$ define a real q-dicritical duplet $D^\infty_R = (\Delta^\infty_R, \Sigma^\infty_R)$. We complexify it and add a symmetric set of complex separatrices in order that conditions (S.2) and (S.3) are accomplished, obtaining a dicritical duplet $D = (\Delta, \Sigma)$. Now we proceed as in Section 5. By Proposition 5.2, we can produce a system of indices $\Upsilon$ associated with $D$ having indices in $\mathbb{Q}^-$, and we do it in a symmetric way with respect to the involution $J$. As in Theorem B, having as reference the symmetric dicritical triplet $T = (\Delta, \Sigma, \Upsilon)$, we build a germ of meromorphic function $h$ at $(\mathbb{C}^2, 0)$, which is also symmetric with respect to $J$. The restriction of $h$ to the real trace $\mathbb{R}^2$ is the desired germ of real meromorphic function.

7. **Escape set and escape separatrices**

In the construction of logarithmic models, as byproducts, some separatrices not originally modelled by the dicritical quadruplet appear. If $Q$ is a dicritical quadruplet and $\eta$ is a $Q$-logarithmic 1-form, it turns out that the reduction of singularities of the singular foliation $L$ defined by $\eta$ may be longer than the underlying sequence of blow-ups of $Q$. The additional blow-ups start at centers called escape points, which are dicritical points for $Q$, outside its support, where the strict transform of $\eta$ is closed holomorphic, having thus a holomorphic first integral. Escape points give rise to some isolated separatrices for $L$, accordingly called escape separatrices. In this section we shall present a description of these objects. In the real case, escape points can be eliminated in a process called logarithmic modification, by which additional dicritical separatrices are included, without affecting the data conveyed by the original dicritical quadruplet.

7.1. **Escape function and faithful 1-forms.** Let $\eta$ be a logarithmic 1–form at $(\mathbb{C}^2, 0)$, written as in [3]. We can suppose that the holomorphic closed part $\alpha$ has been incorporated into one of the equations of the polar set, in such a way that

$$\eta = \lambda_1 \frac{df_1}{f_1} + \cdots + \lambda_r \frac{df_r}{f_r},$$
where \( r \geq 2 \) and, for \( i = 1, \ldots, r \), \( f_i \in \mathcal{O}_0 \) is irreducible with order \( \nu_i = \nu_0(f_i) \) and \( \lambda_i \in \mathbb{C}^* \). Set \( \lambda_0 = \sum_{i=1}^r \nu_i \lambda_i \), the weighted balance of residues of \( \eta \). Denote by \( \mathcal{L} \) the germ of holomorphic foliation at \((\mathbb{C}^2,0)\) defined by \( \eta \). This foliation is also induced by the holomorphic 1-form \( \omega = f \eta \), where \( f = f_1 \cdots f_r \), which may have a one dimensional singular set. For the quadratic blow-up \( \sigma : (\mathbb{C}^2, D) \to (\mathbb{C}^2, 0) \), we consider coordinates \((x, t) \in \mathbb{C}^2 \) such that \( \sigma(x, t) = (x, xt) \), in which \( D = \sigma^{-1}(0) \) has \( x = 0 \) as an equation. Then the divided blow-up of \( \omega \) (even in the non-isolated singularity case) is

\[
\text{(8) (i) non-dicritical: } \tilde{\omega} = \sigma^* \omega/x^{m_0} \text{ or (ii) dicritical: } \tilde{\omega} = \sigma^* \omega/x^{m_0+1},
\]

where \( m_0 = \nu_0(\omega) = \nu_0(f \eta) \) (see, for instance, [4]). The 1-form \( \tilde{\omega} \), which induces \( \tilde{\mathcal{L}} = \sigma^* \mathcal{L} \) in the coordinates \((x, t) \), is holomorphic and does not contain \( D \) in its singular set.

Recall, from Definition 4.1 that \( \eta \) is 1-faithful if and only if \( m_0 = \nu_0(\omega) = \nu_0(f \eta) = m-1 \), where \( m = \nu_0(f) = \nu_1 + \cdots + \nu_r \). This is equivalent to the fact that, in

\[
\omega = f \eta = \sum_{i=1}^r \lambda_i f_1 \cdots \tilde{f}_i \cdots f_r df_i,
\]

the initial term is precisely that obtained by operating with the initial terms of the equations \( f_i \). In other words, it is expressed exclusively in terms of the residues \( \lambda_i \) and of the equations of the tangent cones of the functions \( f_i \).

By a linear change of coordinates, we can assume that none of the functions \( f_i \) contains the \( y \)-axis in its tangent cone. We write, after possibly multiplying by non-zero constants,

\[
f_i(x, y) = (y - \alpha_i x)^{\nu_i} + g_i(x, y),
\]

where \( \alpha_i \in \mathbb{C} \) and \( g_i(x, y) \in \mathcal{O}_0 \) is such that \( \nu_0(g_i) \geq \nu_i + 1 \).

We have

\[
f_i(x, xt) = x^{\nu_i} \tilde{f}_i(x, t) = x^{\nu_i} ((t - \alpha_i)^{\nu_i} + x \tilde{g}_i(x, t)),
\]

where \( \tilde{g}_i(x, t) = g_i(x, xt)/x^{\nu_i+1} \). It follows readily that

\[
\tilde{\eta} = \sigma^* \eta = \lambda_0 \frac{dx}{x} + \lambda_1 \frac{df_1}{f_1} + \cdots + \lambda_r \frac{df_r}{f_r}.
\]

If we set \( \tilde{f} = \tilde{f}_1 \cdots \tilde{f}_r \), we have

\[
\text{(10) } \sigma^* \omega = \sigma^*(f \eta) = x^m \tilde{f} \tilde{\eta}.
\]

If \( \lambda_0 \neq 0 \), by cancelling the poles in (9), we get

\[
\tilde{\omega} = x \tilde{f} \tilde{\eta} = \lambda_0 \tilde{f} dx + x \sum_{i=1}^r \lambda_i \tilde{f}_1 \cdots \tilde{f}_i \cdots \tilde{f}_r df_i,
\]

where, as usual, “\( \tilde{\cdots} \)” indicates the absence of the corresponding factor. It is clear that \( D = \sigma^{-1}(0) \) is invariant by \( \tilde{\omega} \), that is, the blow-up \( \sigma \) is non-dicritical. Comparing with (8) and (10), we conclude that \( m_0 = m - 1 \), and, thus, \( \eta \) is 1-faithful.

On the other hand, if \( \lambda_0 = 0 \), then

\[
\text{(11) } \tilde{\omega} = \tilde{f} \tilde{\eta} = \sum_{i=1}^r \lambda_i \tilde{f}_1 \cdots \tilde{f}_i \cdots \tilde{f}_r df_i.
\]

If we suppose \( \eta \) to be 1-faithful then, by definition, \( m - 1 = m_0 = \nu_0(f \eta) \). Looking again at (8) and (10), the equality \( m = m_0 + 1 \) implies that \( \sigma \) is dicritical. That is, \( D \) is
non-invariant, which is equivalent to the non-vanishing of the term independent of \( x \) in the coefficient of \( dt \) in \( \tilde{\omega} \).

Let us thus assume \( \eta \) to be 1-faithful and main resonant, i.e. \( \lambda_0 = 0 \). Our goal is to track the tangency points between \( \tilde{\mathcal{L}} = \sigma^*\mathcal{L} \) and \( D \). The term independent of \( x \) in the coefficient of \( dt \) in \( \tilde{\omega} \) is

\[
\sum_{i=1}^{r} \lambda_i \nu_i (t - \alpha_1)^{\nu_1} \cdots (t - \alpha_i)^{\nu_i - 1} \cdots (t - \alpha_r)^{\nu_r} = (t - \alpha_1)^{\nu_1 - 1} \cdots (t - \alpha_r)^{\nu_r - 1} \sum_{i=1}^{r} \lambda_i \nu_i (t - \alpha_1) \cdots (\tilde{t} - \tilde{\alpha}_i) \cdots (t - \alpha_r).
\]

Let \( a_1, \ldots, a_\ell \) denote the distinct points of the set \( \{\alpha_1, \ldots, \alpha_r\} \). For \( k = 1, \ldots, \ell \), let \( I_k = \{ 1 \leq i \leq r; \alpha_i = a_k \} \). For each \( k \), set

\[
m_k = \sum_{i \in I_k} \nu_i \quad \text{and} \quad \rho_k = \sum_{i \in I_k} \lambda_i \nu_i.
\]

With this notation, we have that (12) is the polynomial

\[
P_\eta(t) = \sum_{k=1}^{\ell} \rho_k (t - a_1) \cdots (t - a_k) \cdots (t - a_\ell)
\]
times \((t - a_1)^{m_1 - 1} \cdots (t - a_\ell)^{m_\ell - 1}\). The preceding discussion allows us to register the following conclusion:

**Fact 1.** A logarithmic 1-form \( \eta \), main resonant at \( 0 \in \mathbb{C}^2 \), is 1-faithful if and only if \( P_\eta \neq 0 \).

The points of tangency between \( \tilde{\mathcal{L}} \) and \( D \) out of the set \( \{a_1, \ldots, a_\ell\} \) are the roots of \( P_\eta(t) \). At each one of them, \( \tilde{\eta} \) is closed and holomorphic and, thus, \( \tilde{\mathcal{L}} \) has a holomorphic first integral. The roots of \( P_\eta(t) \) determine points of two kinds, depending on \( \tilde{\mathcal{L}} \) being transversal to \( D \) or not. At point \( p \in D \) of the first kind, the leaf through \( p \) is contained in the singular set of \( \tilde{\eta} \). However, if \( p \in D \) is of the second kind, the reduction of singularities of \( \mathcal{L} \) will be non trivial over \( p \), formed by non-dicritical blow-ups, which give rise to some isolated separatrices for \( \mathcal{L} \). It is a escape point, with respect to the branches in the polar set of \( \eta \) or with respect to any dicritical quadruplet \( Q = (\Delta, \Sigma, \Upsilon, \Lambda) \) for which \( \eta \) is \( Q \)-logarithmic. Observe that, in this latter case, \( \eta \) is faithful by hypothesis and, from Assertion 2 in the proof of Theorem A, we have that, in (13), \( \rho_k \neq 0 \), for \( k = 1, \ldots, \ell \).

The vanishing of the weighted balance of residues, that is \( \sum_{i=1}^{r} \nu_i \lambda_i = \sum_{k=1}^{\ell} \rho_k = 0 \), gives that \( P_\eta(t) \) has degree at most \( \ell - 2 \). The coefficient of the term of degree \( \ell - 2 \) is, up to changing sign,

\[
-\sum_{k=1}^{\ell} \left( \sum_{1 \leq i \leq \ell, i \neq k} \rho_k \right) a_k = \sum_{k=1}^{\ell} \rho_k a_k.
\]
Equivalent calculations in the coordinates \((u,y)\) of the blow-up, where \(x = uy\), lead to the polynomial
\[
\hat{P}_η(u) = - \sum_{k=1}^\ell \rho_k a_k (1 - a_1 u) \cdots (1 - a_k u) \cdots (1 - a_\ell u),
\]
whose constant term is \(- \sum_{k=1}^\ell \rho_k a_k\). Thus, \(P_η(t)\) has degree \(\ell - 2\) if and only if \(\hat{P}_η(0) \neq 0\), which means that \(L\) and \(D\) are transversal at \((u,y) = (0,0)\). Thus, by choosing appropriately the blow-up coordinates, we can always assume that \(P_η(t)\) has degree \(\ell - 2\) and, thus, the search for escape points of \(η\) on \(D\) can be entirely done in the coordinates \((x,t)\). In particular, if \(\ell = 2\), \(P_η\) is constant and there are no escape points.

Let us define
\[
R_η(t) = \sum_{k=1}^{\ell} \frac{\rho_k}{t - a_k} = \frac{P_η(t)}{Q_η(t)},
\]
where \(Q_η(t) = (t - a_1) \cdots (t - a_\ell)\). We name \(R_η\) escape function. The affine zeroes of this rational function are the points of tangency considered, among which we find the escape points of \(η\) on \(D\). As a consequence of Fact \(\Pi\) we have:

**Fact 2.** A logarithmic 1-form \(η\), main resonant at \(0 \in \mathbb{C}^2\), is 1-faithful if and only if \(R_η \neq 0\).

The above is the key to prove the following result, which was used in the dicritical amalgamation in the proof of Theorem \(A\).

**Proposition 7.1.** Let \(η = \eta_1 + \cdots + \eta_\ell\), where each \(\eta_j\) is a germ of logarithmic 1-form at \((\mathbb{C}^2,0)\) and \(\ell \geq 2\). Suppose that:
- the tangent cone of the polar set of each \(\eta_k\) is a singleton, say \(p_k \in D = \sigma^{-1}(0)\);
- \(p_1, \ldots, p_\ell \in D\) are distinct points;
- if \(\rho_k\) denotes the weighted balance of residues of \(\eta_k\), then \(\rho_1 + \cdots + \rho_\ell = 0\);
- \(\rho_k \neq 0\) for some \(k\).

Then, \(η\) is 1-faithfull.

**Proof.** We take advantage of the above notation and write the blow-up \(σ\) at \(0 \in \mathbb{C}^2\) in coordinates \((x,t)\) such that \(σ(x,t) = (x,tx)\), in which the points \(p_k\) have coordinates \((x,t) = (0,a_k)\), for \(k = 1, \ldots, \ell\). On account of Fact \(\Pi\) it is enough to see that \(R_η \neq 0\). This is however evident from formula \(\lfloor i \rfloor\), since some \(\rho_k\) is non-zero. \(\Box\)

### 7.2. Logarithmic modifications

We remain in the setting of the previous subsection, keeping the notation. Let \(a'_1, \ldots, a'_s \in \mathbb{C}\) be distinct numbers, none of them in \(\{a_1, \ldots, a_\ell\}\). Let \(λ'_1, \ldots, λ'_s \in \mathbb{C}^*\). A 1-logarithmic modification of \(η\) with parameters \(τ = \{(a'_1, λ'_1), \ldots, (a'_s, λ'_s)\}\) is a logarithmic 1-form of the kind
\[
η^τ = η + λ'_1 df'_1/f'_1 + \cdots + λ'_s df'_s/f'_s,
\]
where, for \(i = 1, \ldots, s\), the function \(f'_i \in \mathcal{O}_0\) is an equation of a smooth branch at \((\mathbb{C}^2,0)\) whose tangent cone is \(t = a'_i\). If \(λ'_1 + \cdots + λ'_s = 0\) then the polynomial \(P_{η^τ}\) has degree at
most \( \ell + s - 2 \). In analogy with [14], the coefficient of its term of degree \( \ell + s - 2 \) is, up to sign,

\[
(16) \quad \sum_{k=1}^{\ell} \rho_k a_k + \sum_{i=1}^{s} \lambda'_i a'_i.
\]

Supposing that \( P_\eta \) has degree \( \ell - 2 \), then, for a generic choice of \((a'_1, \ldots, a'_s, \lambda'_1, \ldots, \lambda'_s)\) in the hyperplane of equation \( \lambda'_1 + \cdots + \lambda'_s = 0 \) in \( \mathbb{C}^s \), the degree of \( P_{\eta^*} \) is precisely \( \ell + s - 2 \), so that the escape points of \( \eta^* \) are found among the affine roots of \( R_{\eta^*} \). In this case, we say that both \( \tau \) and the corresponding \( \eta^* \) are \textit{balanced}.

Now, let \( Q = (\Delta, \Sigma, \Upsilon, \Lambda) \) be a dicritical quadruplet at \((\mathbb{C}^2, 0)\), main resonant at \( 0 \in \mathbb{C}^2 \), and \( \pi \) be its underlying sequence of blow-ups. Suppose that \( \{a_1, \ldots, a_\ell\} \) lists all points of the tangent cone of \( S_0 \) — which coincides with the intersection of the support \( Z_1 \) with \( \Delta = \sigma^{-1}(0) \), where \( \sigma \) is the initial blow up of \( \pi \) — as values of \( t \) of coordinates \((x, t)\) for \( \sigma \). As in the above paragraph, let \( \tau \) be a balanced set of parameters, accompanied by a choice of smooth branches \( S'_i \in B_0 \) of equations \( f'_i = 0 \), for \( i = 1, \ldots, s \). We define the \textit{1-logarithmic modification} of \( Q \) with parameters \( \tau \) as the dicritical quadruplet \( Q^\tau = (\Delta^\tau, \Sigma^\tau, \Upsilon^\tau, \Lambda^\tau) \) obtained from \( Q \) in the following manner:

- \( \Delta^\tau = \Delta \);
- \( \Sigma^\tau \) is built upon the decomposition at level 0 obtained by attaching to \( S^\delta_0 \) the curves \( S'_i \) for \( i = 1, \ldots, s \);
- \( \Upsilon^\tau \) preserves the indices of \( \Upsilon \) and assigns to the new elements of \( S^\delta_0 \) indices \( I_0(S'_i) = 1 \);
- \( \Lambda^\tau \) preserves the residues of \( \Lambda \) and assigns residues \( \lambda(S'_i) = \lambda'_i \).

In this framework, we have:

**Proposition 7.2.** Suppose that \( \eta \) is a \( Q \)-logarithmic 1-form for some dicritical quadruplet \( Q = (\Delta, \Sigma, \Upsilon, \Lambda) \) at \((\mathbb{C}^2, 0)\), main resonant at \( 0 \in \mathbb{C}^2 \). Let \( \eta^\tau \) be a balanced 1-logarithmic modification of \( \eta \) with parameters \( \tau \). Then \( \eta^\tau \) is \( Q^\tau \)-logarithmic.

**Proof.** First, the initial blow-up \( \sigma \) is dicritical for \( \eta^\tau \). Indeed, \( \eta^\tau \) is main resonant and \( 1 \)-faithful at \( 0 \in \mathbb{C}^2 \), the latter being a consequence of \( R_{\eta^\tau} \neq 0 \). Next, we examine the points of the support of the dicritical duplet \( D^\tau = (\Delta^\tau, \Sigma^\tau) \) over \( D = \sigma^{-1}(0) \). The points \( p_k \), corresponding to \( t = a_k \), for \( k = 1, \ldots, \ell \), are simultaneously in the supports of \( D^\tau \) and \( D = (\Delta, \Sigma) \). At each of these points, \( \sigma^* \eta^\tau \) differs from \( \sigma^* \eta \) by a closed holomorphic 1-form. Hence also \( \sigma^* \eta^\tau \) is \( Q_{p_k}^\tau \)-logarithmic, which is the same of being \( Q_{p_k}^\tau \)-logarithmic, since \( Q_{p_k}^\tau \) and \( Q_{p_k}^\tau \) coincide. The other points of the support of \( D^\tau \) are \( p'_1, \ldots, p'_s \), corresponding to \( t = a'_1, \ldots, t = a'_s \). The fact that \( \sigma^* \eta^\tau \) at each \( p'_i \) is \( Q_{p'_i}^\tau \)-logarithmic is obvious. \( \square \)

Suppose that \( Q = (\Delta, \Sigma, \Upsilon, \Lambda) \) has height \( n = h(\Delta) \). Let \( d \) be the number of dicritical blow-ups of \( \Delta \) and denote by \( q_{j_k} \in D_{j_k} \) their centers, where \( 0 \leq j_1 < \cdots < j_d < n \). Starting with \( q = q_{j_d} \), we can consider the localization \( Q_q \) and, for a balanced set of parameters \( \tau_q \), perform the corresponding 1-logarithmic modification, obtaining a dicritical quadruplet \( Q_q^{\tau_q} \) at \((M_{j_d}, q) = (M_{j_d}, q_{j_d}) \). The incorporation of new dicritical separatrices and their residues at levels of \( Q \) lower than \( j_d \) gives rise to a new dicritical quadruplet \( Q_q^{\tau_q} = (\Delta^{\tau_q}, \Sigma^{\tau_q}, \Upsilon^{\tau_q}, \Lambda^{\tau_q}) \) at \((\mathbb{C}^2, 0)\). Since \( \tau_q \) is balanced and all new dicritical separatrices introduced are equisingular, their weighted balances of residues, at levels lower than \( j_d \),
will always vanish. Therefore, at any level, these new separatrices give no contribution to the weighted balances of residues prescribed by $Q$. Descending the dicritical structure, we carry out successive 1-logarithmic modifications at each $q_{j_k}$ with balanced set of parameters $\tau_{j_k}$. The outcome is a dicritical quadruplet $Q^1 = (\Delta^T, \Sigma^T, Y^T, \Lambda^T)$, where the superscript $T$ makes reference to the set of information given by the pairs $(q_{j_k}, \tau_{j_k})$, for $k = 1, \ldots, d$, along with the new separatrices introduced. Now, if $\eta$ is a $Q$-logarithmic 1-form at $(C^2, 0)$, the successive 1-logarithmic modifications just described can be applied to $\eta$. The result is a germ logarithmic 1-form $\eta^T$, which, by a careful application of Proposition 7.2, can be shown to be $Q^1$-logarithmic. It is noteworthy that a 1-logarithmic modification at a point $q_{j_k}$ modifies the escape function on $D_{j_k+1} = \sigma_{j_k+1}^{-1}(q_{j_k})$ but does not change escape functions on dicritical components at higher or lower levels.

7.3. **Symmetric logarithmic modifications.** Suppose now that $\eta$ is a symmetric logarithmic 1-form at $(C^2, 0)$, defining a germ of singular holomorphic foliation $L$. A set of parameters $\tau = \{ (a_1', \lambda_1'), \ldots, (a_d', \lambda_d') \}$ is symmetric if its invariant by conjugation. In this case we say that $\tau$ parametrizes a symmetric 1-logarithmic modification of $\eta$ provided the set of equations $f_1', \ldots, f_s' \in \mathcal{O}_0$ is also invariant by the involution $J$. In particular, if $a_i' \in \mathbb{R}$, then also $\lambda_i' \in \mathbb{R}$ and $f_i' \in \mathcal{O}_0$ defines a real curve. As a consequence, $\eta^T$ is also a symmetric logarithmic 1-form. Remark that, if $\eta$ is main resonant and symmetric, then the set of parameters involved in formula (15) is symmetric under conjugation. Thus, $P_\eta$ can be obtained with real coefficients (the same for $P_{\eta^T}$, for a balanced symmetric 1-logarithmic modification $\eta^T$). In particular, the number of real roots of $P_\eta$ and its degree have the same parity.

Balanced symmetric 1-logarithmic modifications can be used to eliminate real escape points of symmetric logarithmic 1-forms, as described in the next proposition.

**Proposition 7.3.** Suppose that $\eta$ is a main resonant 1-faithful symmetric logarithmic 1-form at $(C^2, 0)$. Then there is a balanced symmetric set of real parameters $\tau$ such that the corresponding symmetric 1-logarithmic modification $\eta^\tau$ has no escape points in the real trace of the divisor of the quadratic blow-up at $0 \in C^2$.

**Proof.** As above, we fix $(x, t)$ coordinates for the blow-up $\sigma$ at $0 \in C^2$ so that the affine zeroes of the escape function $R_\eta(t)$ give all tangency points between $\tilde{\eta} = \sigma^* \eta$ and $D = \sigma^{-1}(0)$. We can suppose that $P_\eta(t)$ has even degree. If this is not so, we make a balanced symmetric 1-logarithmic modification with parameters $\{ (a_1', \lambda_1'), (a_2', \lambda_2'), (a_3', \lambda_3') \}$, where $a_i', \lambda_i' \in \mathbb{R}^*$ for $i = 1, 2, 3$, with $\lambda_1', \lambda_2', \lambda_3'$ sufficiently small. Let $t_0$ be a real zero of $R_\eta(t)$, which can be supposed to be $0 \in C$ after a real translation in the coordinate $t$. We write, locally,

$$R_\eta(t) = \frac{P_\eta(t)}{Q_\eta(t)} = t^n g(t),$$

for some $n > 0$, where $g(t)$ is a holomorphic function in a neighborhood of $0 \in C$ such that $\alpha = g(0) \in \mathbb{R}^*$. Choose $r > 0$ sufficiently small such that $|\alpha|/2 < |g(t)| < 2|\alpha|$ over $\overline{D}_r$. This implies that $g(t)$ has no zeroes over $\overline{D}_r$ and, for $t \in \mathbb{R}$ with $|t| < r$, $g(t)$ and $\alpha = g(0)$ have the same sign. Note also that $|t^n g(t)| > r^n |\alpha|/2$ over $\partial \overline{D}_r$.

**Case 1: n even.** We perform a balanced 1-logarithmic modification with parameters $\tau_0 = \{ (a, \lambda), (-a, -\lambda) \}$, where the sufficiently small $a, \lambda \in \mathbb{R}^*$, with $a > 0$, are to be specified
later, getting a logarithmic 1-form \( \eta^0 \) for which
\[
R_{\eta^0}(t) = t^n g(t) + \frac{\lambda}{t-a} + \frac{-\lambda}{t+a} = \frac{t^n(t^2 - a^2)g(t) + 2a\lambda}{t^2 - a^2}.
\]
Choose \( |a| < r/2 \), so that \( |\lambda/(t \pm a)| < 2|\lambda|/r \) over \( \partial D_r \). If we also choose \( \lambda \in \mathbb{R}^* \) such that \( |\lambda| < r^{n+1}|\alpha|/8 \), we assure that
\[
\left| \frac{\lambda}{t-a} + \frac{-\lambda}{t+a} \right| < 4|\lambda|/r < \frac{r^n|\alpha|}{2} < |t^n g(t)|
\]
over \( \partial D_r \). Hence, applying Rouché’s theorem, \( t^n g(t) + \lambda/(t-a) - \lambda/(t+a) \) has exactly \( n + 2 \) zeroes in \( D_r \). They are roots of the function \( t^n(t^2 - a^2)g(t) + 2a\lambda \). If \( t_0 \in \mathbb{R} \) is one of these roots, then
\[
(17)\quad t_0^n(t_0^2 - a^2) = -\frac{2a\lambda}{g(t_0)}.
\]
Observe that \( t^n(t^2 - a^2) \geq -2Ka^{n+2} \) for every \( t \in \mathbb{R} \), where \( K = \frac{1}{n+2}\left(\frac{n}{n+2}\right)^n \). Choose first a sufficiently small \( a \in \mathbb{R}^* \), with \( a > 0 \), and then pick \( \lambda \in \mathbb{R}^* \), also small, but satisfying \( |\lambda| > 2|\alpha|/K a^{n+1} \). We then have
\[
\frac{2a|\lambda|}{|g(t_0)|} > (2a)2|\alpha|/K a^{n+1} = 2Ka^{n+2}.
\]
Taking \( \lambda \) with the same sign as \( \alpha = g(0) \), we have
\[
-\frac{2a\lambda}{g(t_0)} < -2Ka^{n+2} \leq t^n(t^2 - a^2) \quad \text{for every } t \in \mathbb{R}.
\]
Comparing with \((17)\), we conclude that the real root \( t_0 \) cannot exist.

**Case 2:** \( n \) odd. Now we do a non-balanced 1-logarithmic modification with parameter \( \tau_0 = \{(a, \lambda)\} \), with sufficiently small \( a, \lambda \in \mathbb{R}^* \). The logarithmic 1-form \( \eta^0 \) obtained is such that
\[
R_{\eta^0}(t) = t^n g(t) + \frac{\lambda}{t-a} = \frac{t^n(t-a)g(t) + \lambda}{t-a}.
\]
If \( |a| < r/2 \), then \( |\lambda/(t-a)| < 2|\lambda|/r \) over \( \partial D_r \). Choosing \( \lambda \in \mathbb{R}^* \) such that \( |\lambda| < r^{n+1}|\alpha|/4 \), we assure that \( |\lambda/(t-a)| < r^n|\alpha|/2 < |t^n g(t)| \) over \( \partial D_r \). Hence, applying again Rouché’s theorem, \( t^n g(t) + \lambda/(t-a) \) has exactly \( n + 1 \) zeroes in \( D_r \). These zeroes are roots of the function \( t^n(t-a)g(t) + \lambda \). If \( t_0 \in \mathbb{R} \) is one of these roots, then
\[
(18)\quad t_0^n(t_0-a) = -\frac{\lambda}{g(t_0)}.
\]
We have that \( t^n(t-a) \geq -Ka^{n+1} \) for every \( t \in \mathbb{R} \), where \( K = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n \). Choose first a sufficiently small \( a \in \mathbb{R}^* \) and then pick \( \lambda \in \mathbb{R}^* \), also small, but satisfying \( |\lambda| > 2|\alpha|/K a^{n+1} \). We then have
\[
\frac{|\lambda|}{|g(t_0)|} > 2|\alpha|/K a^{n+1} = Ka^{n+1}.
\]
Taking \( \lambda \) with the same sign as \( \alpha = g(0) \), we have
\[
-\frac{\lambda}{g(t_0)} < -Ka^{n+1} \leq t^n(t-a) \quad \text{for every } t \in \mathbb{R}.
\]
Comparing with (15), this precludes the existence of such a real root.

We complete the proof by performing successive individual 1-logarithmic modifications as above, whenever we find a real root for the escape function. Once a root has been worked out, generating new roots in a small disc around it, subsequent 1-logarithmic modifications will not perturb any of these new roots back to the real axis, provided all parameters are sufficiently small. Finally, since \( P_\eta(t) \) has real coefficients and even degree, there is an even number of real roots with \( n \) odd. Thus, if any such a root exists, there are at least two of them, allowing us to choose the residues \( \lambda \) in a way that the overall 1-logarithmic modification is balanced.

Remark that in the proof of Proposition 7.3, all residues can be obtained in \( \mathbb{Q}^* \). This is particularly important if our goal is to use logarithmic models in order to produce real meromorphic functions, as in Theorem G below. The iterated application of Proposition 7.3 in a symmetric dicritical structure gives promptly the following result, which was used in Theorem D in order to obtain a strict real logarithmic model.

**Proposition 7.4.** Let \( Q = (\Delta, \Sigma, \Upsilon, \Lambda) \) be a symmetric dicritical quadruplet at \((\mathbb{C}^2, 0)\) and \( \eta \) be a symmetric \( Q \)-logarithmic 1-form. Then there is a finite set of balanced symmetric logarithmic modifications, with parameters \( T \), such that \( \eta^T \) is a symmetric \( Q^T \)-logarithmic 1-form without real escape points. Besides, all individual 1-logarithmic modifications can be taken with real parameters, with residues in \( \mathbb{Q}^* \).

Proposition 7.4 completes the proof of Theorem D on the existence of strict logarithmic models for germs of singular real analytic foliations. Indeed, having a germ of real analytic foliation \( F_R \) of real generalized curve type, we provide a q-dicritical triplet \( T_\mathbb{R}^\# = (T, \Lambda) \) and Theorem C gives it a logarithmic model \( \eta_\mathbb{R} \). In this process, \( T_\mathbb{R}^\# \) is complexified, then completed into a symmetric dicritical triplet \( T \) with real indices, which turns into a symmetric dicritical quadruplet \( Q = (T, \Lambda) \) with real residues, to finally obtain a \( Q \)-logarithmic 1-form \( \eta \), whose decomplexification is \( \eta_\mathbb{R} \). If \( \eta_\mathbb{R} \) fails to be a strict logarithmic model for \( F_\mathbb{R} \), that is to say, if escape points exist, we apply Proposition 7.4 to \( \eta \). In this case, the set of parameters \( T \) will be chosen in such a way that all separatrices involved are actual real dicritical separatrices of \( F \), the germ of singular holomorphic foliation defined by \( \eta \). The corresponding logarithmic modifications then produce a new q-dicritical triplet \( (T_\mathbb{R}^\#)^\dagger \), the real trace of \( T^\dagger \), which in practice only incorporates the data of the real separatrices introduced, being also a q-dicritical triplet associated with \( F_\mathbb{R} \). Thus, \( \eta_\mathbb{R}^\dagger \), the decomplexification of \( \eta^\dagger \), is a strict logarithmic model for \( F_\mathbb{R} \).

8. **Bendixson’s sectorial decomposition**

The existence of a sectorial decomposition for a planar real analytic vector field with isolated singularity appeared in the seminal paper of I. Bendixson [1], where, following the ideas of H. Poincaré, he developed a qualitative study of the orbits of a planar vector field. Bendixson described and proved the finiteness of such a decomposition in a sufficiently small neighborhood of the singularity, say \( 0 \in \mathbb{R}^2 \), when some orbit approaches the singularity with a “determined tangent”. In modern terminology, this is a characteristic orbit and the situation considered is called non-monodromic. In this case, it turns out that all orbits approaching the origin have tangents, defined as limits of secants. FINITELY many of these orbits will be the boundary of sectors, which have a topological classification...
as hyperbolic, parabolic or elliptic. In a hyperbolic sector no trajectory has the origin in its limit set, in a parabolic one all trajectories have the origin either in the $\alpha$ or in the $\omega$-limit set and, finally, in an elliptic sector all trajectories have the origin both in their $\alpha$ and $\omega$-limit sets. We refer the reader to [10] for a proof of the existence of a sectorial decomposition based on the technique of reduction of singularities, which unveils other analytic aspects that will be relevant to the forthcoming discussion.

Let $\pi_R : (M_R, D_R) \to (\mathbb{R}^2, 0)$ be a sequence of real quadratic blow-ups and let $\rho_R : (\tilde{N}_R, E_R) \to (\mathbb{R}^2, 0)$ be the corresponding sequence of trigonometric blow-ups, where $\tilde{N}_R$ is a real analytic surface with boundaries and corners (see definitions in [10]). Sequences of trigonometric blow-ups to be considered henceforth are of this kind. There is a canonical analytic map $\psi : (\tilde{N}_R, E_R) \to (M_R, D_R)$ satisfying $\pi_R \circ \psi = \rho_R$, obtained by the consideration of the double covering $\mathbb{S}^1 \to \mathbb{P}_R^1$ for each single blow-up. We associate with each component $D \subset D_R$ its real infinitesimal class $\kappa_D$. If $D$ has valence $m$, then the subset of trace points of $D$ has $m$ connected components, whose pre-images by $\psi$ are $2m$ connected components of the regular part of $E_R$. They determine $2m$ distinct infinitesimal semiclasses and all these semiclasses are said to be correlate to the infinitesimal class $\kappa_D$ and also correlate to each other, and the same is said of the corresponding irreducible components of $E_R$ and $D_R$. The infinitesimal semiclass of $E \subset E_R$ will be denoted by $\hat{k}_E$.

Let $E_x \subset E$ be the component corresponding to the positive $x$-semiaxis. If $E, E' \subset E$ are two distinct components, the notation $\hat{k}_E < \hat{k}_{E'}$ means that either $E = E_x$ or $E$ lies between $E_x$ and $E'$, considering the counterclockwise orientation around $0 \in \mathbb{R}^2$.

Let $U$ be a small neighborhood of $0 \in \mathbb{R}^2$. An analytic semicurve in $U \setminus \{0\}$ is defined by a real analytic parametrization of the kind $\gamma : [t_0, \infty) \to U \setminus \{0\}$ or $\gamma : (-\infty, t_0] \to U \setminus \{0\}$, for some $t_0 \in \mathbb{R}$. We shall only consider analytic semicurves $\gamma$ in $U \setminus \{0\}$ whose limit set is either $\ell(\gamma) = \{0\}$ or $\ell(\gamma) = \emptyset$. Let $X_R$ be a germ of real analytic vector field, with isolated singularity at $(\mathbb{R}^2, 0)$, of non-monodromic type. Any integral semicurve $\gamma$ of $X_R$ such that $\ell(\gamma) = 0$ has the iterated tangents property, meaning that $\gamma$, as well as all its lifts by successive blow-ups, have tangents at their limit points (we refer to [7], Sec. 2.2] for a proof). In a small neighborhood of $0 \in \mathbb{R}^2$, an integral semicurve does not intercept a real analytic branch (unless the semicurve is contained in the branch) and two integral semicurves do not intercept. Thus, a pair $\gamma, \gamma'$ of integral solutions of $X_R$, both accumulating to the origin, can be counterclockwise ordered having the positive $x$-semiaxis $\gamma_x$ as reference: $\gamma \prec \gamma'$ means that either $\gamma = \gamma_x$ or $\gamma$ lies between $\gamma_x$ and $\gamma'$ in a sufficiently small neighborhood of $0 \in \mathbb{R}^2$. We also say that $\gamma$ has type $\hat{k}_E$, for some $E \subset E$, if the $\rho_R$-lifting of $\gamma$ touches $E$ in a trace point. In this case, if $E', E'' \subset E$ are such that $\hat{k}_{E'} \prec \hat{k}_E \prec \hat{k}_{E''}$, we denote $\hat{k}_E \prec \gamma \prec \hat{k}_{E''}$.

A sectorial model corresponds to the prescription, in a sufficiently small neighborhood $U$ of $0 \in \mathbb{R}^2$, of the following information:

- a finite number of real analytic semicurves $\gamma_1, \ldots, \gamma_n$ in $U \setminus \{0\}$, with limit set $0 \in \mathbb{R}^2$, not intercepting each other and having the property of iterated tangents, such that $\gamma_1 \prec \ldots \prec \gamma_n$;
- a classification of the region $\chi = \chi(\gamma_i, \gamma_{i+1})$, named sector, formed by points of $U$ between $\gamma_i$ and $\gamma_{i+1}$, where $i = 1, \ldots, n$ and $\gamma_{n+1} = \gamma_1$, as hyperbolic, parabolic or elliptic, with no parabolic sector neighboring other parabolic sectors.
Evidently, in the first of these items, we could also ask other necessary conditions for the semicurves $\gamma_i$ to be integral curves of a non-monodromic real analytic vector field. Taking into account the machinery developed throughout this text, our proposal is to produce examples of real analytic vector fields whose sectorial decomposition fits a prescribed sectorial model. Our techniques allow us to do this in a variety of situations, having as output a vector field of logarithmic nature, i.e. whose dual 1-form is a multiple of a real logarithmic 1-form by a unity in $C_0^\omega$. We will exemplify this procedure within a specific category of non-monodromic real analytic vector fields that we call $\ell$-analytic, delimited in the following paragraph.

Denote the set of irreducible germs of real analytic curves at $(\mathbb{R}^2,0)$ by $B^R_0$. If $S \in B^R_0$, then $S \setminus \{0\}$ splits into two semibranches, $\gamma$ and $\gamma^*$, said to be adjoint to each other. Evidently, a semibranch $\gamma$ is an integral curve of a germ of real analytic vector field $X_R$ if and only if $\gamma^*$ is. An analytic semicurve $\gamma$ in $U \setminus \{0\}$ is analytic in the limit, or $\ell$-analytic for short, if either $\ell(\gamma) = 0$ or $\ell(\gamma) = \{0\}$ and there is $S \in B^R_0$ such that $\gamma(t) \in S$ whenever $|t|$ is sufficiently large. In an abuse of language, we say that $\gamma$ itself is an analytic semibranch. An analytic curve $\Gamma : (-\infty, \infty) \to U \setminus \{0\}$ is $\ell$-analytic if it can be partitioned in two analytic semi-curves, both $\ell$-analytic.

**Definition 8.1.** We say that a germ of real analytic vector field $X_R$ is $\ell$-analytic if all its orbits in $U \setminus 0$ are $\ell$-analytic, for some small neighborhood $U$ of the origin.

Clearly, a vector field whose underlying singular foliation is defined by the levels of a real meromorphic function is $\ell$-analytic. Simple saddle-type vector fields — whose linear part has eigenvalues $\lambda_1, \lambda_2$ satisfying $\lambda_2/\lambda_1 \in \mathbb{R}_- —$ are $\ell$-analytic. On the other hand, simple node-type vector fields — with eigenvalues satisfying $\lambda_2/\lambda_1 \in \mathbb{R}_+ \setminus \mathbb{Q}_+$ — are not $\ell$-analytic. A simple vector field with one zero eigenvalue — an algebraic saddle-node — is $\ell$-analytic if and only if it is a topological saddle. In particular, its weak separatrix — the one corresponding to the zero eigenvalue — is convergent. Indeed, otherwise, by the center manifold theorem, there would exist an orbit asymptotic to the non-convergent weak separatrix, which would not be $\ell$-analytic. A non-monodromic germ of vector field is $\ell$-analytic if and only if it is of center type. The concept of $\ell$-analytic vector field is evidently invariant under quadratic blow-ups. Thus, if $\pi_R : (M_R, \mathcal{D}_R) \to (\mathbb{R}^2,0)$ is a reduction of singularities of $\mathcal{F}_R$ by a sequence of real quadratic blow-ups, then $X_R$ is $\ell$-analytic if and only if all simple singularities of $\tilde{\mathcal{F}}_R = \pi^*_R \mathcal{F}_R$ over $\mathcal{D}_R$ are topological saddles.

We will try to present a more refined description of the sectorial decomposition of an $\ell$-analytic vector field $X_R$. Let $\rho_R : (\tilde{N}_R, \mathcal{E}) \to (\mathbb{R}^2,0)$ be the reduction of singularities of the associated singular foliation $\mathcal{F}_R$ by a sequence of trigonometric blow-ups and denote $\tilde{\mathcal{F}}_R = \rho^*_R \mathcal{F}_R$. Let $E, E' \subset \mathcal{E}$ be consecutive dicritical components with $\kappa_E < \kappa_{E'}$. Let $A$ denote the union of invariant components of $\mathcal{E}$ between $E$ and $E'$, which is non-empty by the properties of the reduction of singularities. Let $\tau$ denote the number of separatrices — i.e. invariant semibranches — of $\tilde{\mathcal{F}}_R$ touching $A$. The figure gives a portrait for the cases $\tau = 0, 1$ and 2. Define $p = A \cap E$ and $p' = A \cap E'$. The dicritical components $E$ and $E'$ determine two pieces of parabolic sectors, $\chi_E$ and $\chi_{E'}$ — here we say “piece” since they may be proper subsectors of parabolic sectors. We attach to them the infinitesimal semiclasses $\tilde{\kappa}_E$ and $\tilde{\kappa}_{E'}$, on the grounds that their interior orbits are of types, respectively, $E$ and $E'$. Suppose that $\tau = 0$. Then, leaves of $\tilde{\mathcal{F}}_R$ touching $E$ in points near $p$ reach $E'$.
in points near $p'$. Choosing $\xi$ and $\xi'$ orbits whose $p_\mathbb{R}$-lifts touch, respectively, $E$ in a point sufficiently near $p$ and $E'$ in a point sufficiently near $p'$, then $\chi(\xi, \xi')$ is an elliptic sector. The semibranch $\xi$ will be the upper boundary of $\chi_E$ and $\xi'$ will be the lower boundary of $\chi_{E'}$. On the other hand, if $\tau > 0$, then the first of the separatrices reaching $\mathcal{A}$, say $\gamma$, will be the upper boundary of $\chi_E$ and the last one, say $\gamma'$, will be the lower boundary of $\chi_{E'}$. If $\tau = 1$, then $\gamma = \gamma'$ and $\chi_E$ neighbor $\chi_{E'}$. Hence, these pieces of parabolic sectors are merged, as part of the formation of a maximal parabolic sector. On the other hand, if $\tau > 1$, there are $\tau - 1$ hyperbolic sectors between $\chi_E$ and $\chi_{E'}$, each one bounded by a consecutive pair of separatrices of $\mathcal{A}$. In other words, hyperbolic sectors are bounded by isolated invariant semibranches of $\mathcal{F}_\mathbb{R}$. Observe that elliptic sectors always neighbor parabolic sectors and their boundaries are always dicritical invariant semibranches of $\mathcal{F}_\mathbb{R}$. By possibly reducing $U$ and changing the choice of some of these semibranches, we can also suppose that a semibranch $\gamma$ is the boundary of a sector or of a piece of sector if and only if $\gamma^*$ is. Finally, it is also clear that if $E \subset \mathcal{E}$ is a dicritical component defining a piece of parabolic sector, then all components of $\mathcal{E}$ correlate to $E$ also define pieces of parabolic sectors.

The above description matches the model we describe next. An \textit{infinitesimal sentence} is an object $W = W_1W_2\cdots W_r$, formed by \textit{words} $W_i$ of one of two \textit{types}: either $W_i = \gamma_i$ is a semibranch or $W_i = \hat{\kappa}_i$ is an infinitesimal semiclass. The following syntax must be obeyed:

- $W_1 < W_2 < \cdots < W_r$, where "<" denotes the counterclockwise order of semibranches and semiclasss described above;
- if $W_i = \gamma_i$ is a semibranch, then the adjoint $\gamma_i^*$ is also a word of $W$;
- if $W_i = \hat{\kappa}_i$ is an infinitesimal semiclass, then all semiclasss correlate to $\hat{\kappa}_i$ are words of $W$.

For convenience, consider $W_0 = W_r$ and $W_{r+1} = W_1$. We then establish the following:

\textbf{Definition 8.2.} Let $W$ be an infinitesimal sentence as above. An \textit{\ell-analytic sectorial model} of pattern $W$ is the prescription of a sectorial decomposition of a small neighborhood of $0 \in \mathbb{R}^2$ complying with the following conditions:

- if $W_i = \gamma_i$ is a semibranch, then it is the boundary of a hyperbolic sector or of a piece of parabolic sector;
• if \( W_i = \hat{\kappa}_i \) is an infinitesimal semiclass, then it defines a piece of parabolic sector;
  if \( W_{i-1} \) or \( W_{i+1} \) is a semibranch, say \( \gamma \), then \( \gamma \) is in its boundary;
• if \( W_i = \gamma \) and \( W_{i+1} = \gamma' \) are semibranches, then \( \chi(\gamma, \gamma') \) is a hyperbolic sector;
• if \( W_i = \hat{\kappa} \) and \( W_{i+1} = \hat{\kappa}' \) are infinitesimal semiclasses, then there is an elliptic sector \( \chi(\xi, \xi') \), bounded by semibranches \( \xi \) and \( \xi' \) of types, respectively, \( \hat{\kappa} \) and \( \hat{\kappa}' \); these semibranches are also boundaries of pieces of parabolic sectors;
• neighboring pieces of parabolic sectors are merged into a single parabolic sector;

From the definition, we infer that parabolic sectors correspond, in the sentence \( \mathcal{W} \), to maximal sequences of words of alternating types. Aiming at giving them a more accurate description, we could say, for instance, that each such a sequence determines an \textit{infinitesimal multitype} for the corresponding parabolic sector.

From our discussion, every \( \ell \)-analytic vector field has a sectorial decomposition of the above type. Reciprocally, we have:

**Theorem F.** There are \( \ell \)-analytic vector fields fitting any preassigned \( \ell \)-analytic sectorial model.

**Proof.** Let \( \mathcal{W} \) be the infinitesimal sentence subjacent to the given \( \ell \)-analytic sectorial model. Let \( \mathcal{S} \subset \mathcal{B}_0^R \) be the set of branches whose semibranches are words of \( \mathcal{W} \) and \( \varepsilon = \varepsilon(\mathcal{S}) \) be its real equisingularity class. Let \( \kappa \) be the real infinitesimal class that aggregates all semiclasss that are words of \( \mathcal{W} \). Let \( \pi_R \) be a sequence of real quadratic blow-ups that is a minimal simultaneous realization for \( \kappa \) and \( \varepsilon \). Consider a real \( q \)-dicritical triplet \( \mathcal{T}_R^q \) built upon \( \pi_R \), with the dicritical structure defined by \( \kappa \) and separatrices given by \( \mathcal{S} \). As for the indices, for each singular point \( p \) at the final level, choose, for the two local branches of the support at \( p \), negative indices respecting (I.4). By Theorem C, there is a real logarithmic 1-form \( \eta_R^1 \) with real residues, which is a real logarithmic model for \( \mathcal{T}_R^q \). Using Proposition 7.4, after a finite number of logarithmic modifications, we obtain a real logarithmic 1-form \( \eta^1 \) without escape points. We cancel the poles of \( \eta^1 \) and take the dual vector field. This vector field has the preassigned \( \ell \)-analytic sectorial model. 

We can also obtain a version of the above theorem concerning the existence of real meromorphic functions:

**Theorem G.** There are real analytic meromorphic functions fitting any preassigned \( \ell \)-analytic sectorial model.

**Proof.** We follow the steps of the proof of Theorem F associating with the given sectorial model a dicritical class \( \kappa \) and an equisingularity class \( \varepsilon \). Then, as in Theorem B, we produce a real meromorphic function upon these data, using, in this process, logarithmic modifications with residues in \( \mathbb{Q}^* \) in order to get rid of all real escape points, as proposed in Proposition 7.4.

**9. Examples**

As an illustration of methods and concepts introduced in the text, we present below a pair of examples. In order to lighten notation, we denote by the same symbol a curve and its strict transform by blow-up maps. Ambiguity will be avoided by making precise the level we are working at. Also, all pictures presented correspond to the real trace.
Example 9.1. Consider, at $(\mathbb{C}^2, 0)$, the branches

$$S_1 : f_1 = x - y^2 = 0 \quad \text{and} \quad S_2 : f_2 = x + y^2 = 0.$$  

Let $\pi = \sigma_1 \circ \sigma_2$ be the sequence of blow-ups desingularizing $\mathbb{C}^2 = \{S_1, S_2\}$. We will produce a symmetric logarithmic model, with rational residues, in which $\mathbb{C}^2$ are separatrices at level 0 of a q-dicritical duplet, corresponding to a dicritical structure determined by $\pi$, where $\sigma_1$ is a non-dicritical blow-up and $\sigma_2$ is a dicritical one.

Denote by $D_1$ and $D_2$ the components of the exceptional divisor introduced by, respectively, $\sigma_1$ and $\sigma_2$. In order that (S.2) is satisfied by $D_1$ at the final level, we introduce the pair non-real conjugate branches (represented in the picture by dotted lines)

$$S_3 : f_3 = y - ix = 0 \quad \text{and} \quad S_4 : f_4 = y + ix = 0,$$

so that $\mathbb{C}^2 = \{S_1, S_2, S_3, S_4\}$ are separatrices at level 0 of a dicritical duplet. We shall now work at level 1. Let $p \in D_1$ be the center of $\sigma_2$. If we set $\lambda_{D_1} = 1$, in order to have a zero weighted balance of residues at $p$, we can put, for instance $\lambda_{S_1} = \lambda_{S_2} = -1/2$. Let $q_3$ and $q_4$ denote the points of intersection of $S_3$ and $S_4$ with $D_1$. Since $i_p(D_1) = 1$ and $c(D_1) = -1$, in order to have (I.3) satisfied on $D_1$, we can put $i_{q_3}(D_1) = i_{q_4}(D_1) = -1$. Since $\sigma_2$ is trivial over $q_3$ and $q_4$, we have, by (I.4), $i_{q_3}(S_3) = i_{q_4}(S_4) = -1$. By the consistency condition of Definition 3.5 we then find $\lambda_{S_3} = \lambda_{S_4} = 1$. The residues obtained are such that

$$\eta = -\frac{1}{2} f_1 \frac{df_1}{f_1} - \frac{1}{2} f_2 \frac{df_2}{f_2} + \frac{df_3}{f_3} + \frac{df_4}{f_4} = \frac{1}{2} \frac{dh}{h}, \quad \text{where} \quad h = \frac{f_3^2 f_4^2}{f_1 f_2},$$

is a logarithmic model for the data described above. Let us write $\sigma_1$ in coordinates $(u, y)$ such that $x = uy$ and, then, the blow-up $\sigma_2$ at $p$ in coordinates $(u, t)$ such that $y = tu$. In these coordinates, at level 2, we have $S_1 : t - 1 = 0$, $S_2 : t + 1 = 0$ and $D_1 : t = 0$. The escape function corresponding to $\eta_1 = \sigma_1^* \eta$ at $p$ is

$$R_{\eta_1} = \frac{1}{t} + \frac{-1/2}{t - 1} + \frac{-1/2}{t + 1} = \frac{-1}{t(t + 1)(t - 1)} = \frac{P_m}{Q_m}.$$  

Since $\deg(P_m) < 1$, $\eta_2 = \sigma_2^* \eta_1 = \pi^* \eta$ has a point of tangency with $D_2$ at the point corresponding to $t = \infty$. The level $-1$ of $h$ has equation

$$f_3^2 f_4^2 + f_1 f_2 = x^2(x^2 + 2y^2 + 1).$$

That is, the $y$-axis is a fiber of multiplicity two, thereby contained in $\text{Sing}(\eta)$. This one-dimensional component accounts for the point of tangency detected. It is not a escape separatrix, since it does not force the introduction of additional blow-ups in the desingularization process.
The real figure of this example corresponds to an $\ell$-analytic sectorial model patterned on the infinitesimal sentence $W = W_1W_2$, whose words are the infinitesimal semiclasses $W_1 = \hat{\kappa}E_1$ and $W_2 = \hat{\kappa}E_2$. The semiclasses $\hat{\kappa}E_i$, $i = 1, 2$, yield two parabolic sector, while the transitions $\hat{\kappa}E_i \leftrightarrow \hat{\kappa}E_j$, $i \neq j$, two elliptic sectors.

Example 9.2. Take, for $i = 1, 2, 3, 4$, the germs of cuspids at $(C^2, 0)$ defined by the equations

$$S_i : f_i(x, y) = x^3 - (y - a_i x)^2 = 0,$$

where $a_1 = 1$, $a_2 = -1$, $a_3 = 4$, $a_4 = -4$. We consider $S_0 = \{S_1, S_2, S_3, S_4\}$ as the initial level of a configuration of separatrices framed on a dicritical structure determined by the sequence of blow-ups that desingularizes $S_0$, in which only the first blow-up is dicritical. We produce a logarithmic model with these data having rational residues.

In order to zero the weighted balance of residues at $0 \in C^2$, we can take, for instance, $\lambda_1 = \lambda_3 = 1/10$ and $\lambda_2 = \lambda_4 = -1/10$. We then set

$$\eta = \sum_{i=1}^{4} \lambda_i \frac{df_i}{f_i} = \frac{1}{10} \frac{dh}{h}, \quad \text{where} \quad h = \frac{f_1f_3}{f_2f_4}.$$

Denoting by $\sigma_1 : (\tilde{C}^2, D_1) \rightarrow (C^2, 0)$ the dicritical blow-up at $0 \in C^2$ and considering coordinates $(x, t)$ such that $y = tx$, at level 1, each $S_i$ has $x - (t - a_i)^2 = 0$ as equation. Hence, in these coordinates, we get the escape function

$$R_\eta = \frac{1}{10} \left( \frac{1}{t - 1} + \frac{-1}{t + 1} + \frac{1}{t - 4} + \frac{-1}{t + 4} \right) = \frac{t^2 - 4}{(t^2 - 1)(t^2 - 16)},$$

which has two affine zeroes, $t = 2$ and $t = -2$. The curves of equations

$$4(t - 2)^4 - 9(t - 2)^3 - 8(t - 2)^2 x - 18(t - 2)^2 + 9(t - 2)x + 4x^2 - 18x = 0$$
and
\[-4(t + 2)^4 - 9(t + 2)^3 + 8(t + 2)^2 x + 18(t + 2)^2 + 9(t + 2)x - 4x^2 + 18x = 0\]
are invariant by $\eta_1 = \sigma^* \eta$ and are tangent to the divisor $D_1$ at, respectively, $t = 2$ and $t = -2$ (they correspond to the level curves $h = 1/81$ and $h = 81$). Their local branches at these points are, thus, escape separatrices.

Now, let us consider a set of parameters of the form $\tau = (2 + a, \lambda), (-2 - a, -\lambda)$, where $a, \lambda \in \mathbb{R}^*$ are small, to be determined following the guidelines of the proof of Proposition 7.3. We consider the balanced symmetric logarithmic modification with parameters $\tau$ given, for instance, by
\[
\eta^* = \eta + \lambda \frac{dg_1}{g_1} - \lambda \frac{dg_2}{g_2},
\]
where $g_1 = y - (2 + a)x$ and $g_2 = y + (2 + a)x$. Its escape function is
\[
R_{\eta^*} = R_\eta + \frac{\lambda}{t - (2 + a)} - \frac{\lambda}{t + (2 + a)} = \frac{t^2 - 4}{(t^2 - 1)(t^2 - 16)} + \frac{2\lambda(2 + a)}{t^2 - (2 + a)^2} = \frac{P_{\eta^*}}{Q_{\eta^*}},
\]
whose numerator is the biquadratic polynomial
\[
P_{\eta^*}(t) = (1 + 2\lambda(2 + a))t^4 + (-2 + a)^2 - 4 - 34(2 + a)\lambda t^2 + (4(2 + a)^2 + 32(2 + a)\lambda).
\]
The discriminant of the corresponding quadratic polynomial obtained by setting $t^2 = z$ is
\[
\Delta = (2 + a)^4 + 36(2 + a)^3\lambda + 900(2 + a)^2\lambda^2 - 8(2 + a)^2 - 144(2 + a)\lambda + 16.
\]
Taking, for instance, $a = 1/10$ and $\lambda = -1/10$ we get $\Delta = -237215/10000 < 0$. This means that $P_{\eta^*}(t)$ has no real roots and, thus, $\eta^*$ has no real escape points. Using these values, we have $g_1 = y - (2 + 1/10)x$ and $g_2 = y + (2 + 1/10)x$, so that
\[
\eta^* = \eta - \frac{1}{10} \frac{dg_1}{g_1} + \frac{1}{10} \frac{dg_2}{g_2} = \frac{1}{10} \frac{dh_1}{h_1}, \quad \text{where} \quad h_1 = h \frac{g_2}{g_1} = h = \frac{f_1 f_2 g_2}{f_2 f_4 g_1}.
\]
We finish by inviting the reader to draw the figure corresponding to the sectorial decomposition of this example and also to try to devise other examples by himself.

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