DESINGULARIZATIONS OF CALABI–YAU 3-FOLDS WITH A CONICAL SINGULARITY

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Abstract

We study Calabi–Yau 3-folds $M_0$ with a conical singularity $x$ modelled on a Calabi–Yau cone $V$. We construct desingularizations of $M_0$, obtaining a 1-parameter family of compact, nonsingular Calabi–Yau 3-folds which has $M_0$ as the limit. The way we do is to choose an Asymptotically Conical Calabi–Yau 3-fold $Y$ modelled on the same cone $V$, and then glue into $M_0$ at $x$ after applying a homothety to $Y$. We then get a 1-parameter family of nearly Calabi–Yau 3-folds $M_t$ depending on a small real variable $t$. For sufficiently small $t$, we show that the nearly Calabi–Yau structures on $M_t$ can be deformed to genuine Calabi–Yau structures, and therefore obtaining the desingularizations of $M_0$. Our result can be applied to resolving orbifold singularities and hence provides a quantitative description of the Calabi–Yau metrics on the crepant resolutions.

1. Introduction

A Calabi–Yau 3-fold is a Kähler manifold $(M, J, g)$ of complex dimension 3 with a covariant constant holomorphic volume form $\Omega$ satisfying $\omega^3 = \frac{3}{4} \Omega \wedge \bar{\Omega}$ where $\omega$ is the Kähler form of $g$. Equivalently it is a Riemannian 6-fold with a torsion-free SU(3)-structure.

Our starting point is to introduce the notion of a nearly Calabi–Yau structure on some 6-fold $M$. It is basically an SU(3)-structure with small torsion. We then show that if the torsion is small enough, the nearly Calabi–Yau structure on $M$ can be deformed to a torsion-free SU(3)-structure, that is, a genuine Calabi–Yau structure on $M$. This is achieved by applying Joyce’s existence result [7, Thm. 11.6.1] for torsion-free $G_2$-structures on compact 7-folds.

We shall then study a kind of singular Calabi–Yau 3-fold with isolated singularities, known as Calabi–Yau 3-folds with conical singularities. Roughly speaking, they are compact Calabi–Yau 3-folds with finitely many isolated singularities such that they approach some Calabi–Yau cones near the singular points in some sense. For simplicity we shall consider instead Calabi–Yau 3-folds with only one conical singularity modelled on a Calabi–Yau cone, and no other singularities.

The goal of this paper is to desingularize the compact Calabi–Yau 3-fold $M_0$ with a conical singularity modelled on a Calabi–Yau cone $V$ by an analytic technique called gluing. The construction proceeds as follows. Suppose $Y$ is an Asymptotically Conical Calabi–Yau 3-fold modelled on the same cone $V$. It has a similar definition to $M_0$, but it is asymptotic to $V$ at infinity. We apply a homothety to $Y$ and then glue into $M_0$ to get a 1-parameter family of compact, nonsingular smooth 6-folds $M_t$. We will then construct nearly Calabi–Yau structures on $M_t$ for small $t > 0$, and when $t$ is sufficiently small, the nearly Calabi–Yau
structures can be deformed to genuine Calabi–Yau structures by applying our result before.

An application of our result involves desingularizing Calabi–Yau 3-orbifolds with isolated singularities. We can then describe what the Calabi–Yau metrics on the crepant resolution of the orbifold locally look like.

We begin in §2 by giving some background material for this paper. Section 3 introduces nearly Calabi–Yau structures on 6-dimensional manifolds and the induced $G_2$-structures on 7-dimensional manifolds. We also prove the existence result for genuine Calabi–Yau structures, using Joyce’s result with some modifications. Then we define in §4 the main objects of this paper, namely the Calabi–Yau cones, Calabi–Yau m-folds with a conical singularity and Asymptotically Conical Calabi–Yau m-folds, and provide some examples. Finally we show in §5 the construction of the desingularization in the simplest case where there are no obstructions. We shall then give an application of our result and have a short discussion on the obstructed case.

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2. Background material

In this section we provide some background on Calabi–Yau 3-folds, SU(3)-structures on 6-folds and $G_2$-structures on 7-folds. They will play an essential role in our construction of desingularizations of compact Calabi–Yau 3-folds with a conical singularity. Let us begin with studying Calabi–Yau 3-folds. Some useful introductory references on Calabi–Yau manifolds are [6] and [7, Chapter 6].

Definition 2.1. A Calabi–Yau 3-fold is a Kähler manifold $(M, J, g)$ of complex dimension 3 with a covariant constant holomorphic volume form $\Omega$ such that it satisfies $\omega^3 = \frac{1}{4} \Omega \wedge \bar{\Omega}$ where $\omega$ is the Kähler form of $g$. We say that $(J, \omega, \Omega)$ constitutes a Calabi–Yau structure on $M$ and write a Calabi–Yau manifold as a quadruple $(M, J, \omega, \Omega)$.

Thus for each $x \in M$, there is an isomorphism between $T_xM$ and $\mathbb{C}^3$ that identifies $g_x, \omega_x$ and $\Omega_x$ with the flat metric $g_0$, the real 2-form $\omega_0$ and the complex 3-form $\Omega_0$ on $\mathbb{C}^3$ given by

$$g_0 = |dz_1|^2 + |dz_2|^2 + |dz_3|^2, \quad \omega_0 = \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$$

and

$$\Omega_0 = dz_1 \wedge dz_2 \wedge dz_3,$$

where $(z_1, z_2, z_3)$ are coordinates on $\mathbb{C}^3$. Calabi–Yau manifolds are automatically Ricci-flat, and one can use Yau’s solution of the Calabi conjecture ([12] and [7, Chapter 5]) to show the existence of families of Calabi–Yau manifolds. Another equivalent way of defining a Calabi–Yau 3-fold is to require that the Riemannian 6-fold $(M, g)$ has holonomy group $\text{Hol}(g)$ contained in SU(3). One can then show that $M$ admits a holomorphic volume form satisfying the normalization formula.
We shall now consider SU(3)-structures on 6-folds and $G_2$-structures on 7-folds and relate them to Calabi–Yau structures. An SU(3)-structure on a real 6-fold $M$ is a principal subbundle of the frame bundle of $M$, with fibre SU(3). Each SU(3)-structure gives rise to an almost complex structure $J$, a real 2-form $\omega$ and a complex 3-form $\Omega$ with the properties that

1. $\omega$ is of type (1,1) w.r.t. $J$ and is positive,
2. $\Omega$ is of type (3,0) w.r.t. $J$ and is nonvanishing, and
3. $\omega^3 = \frac{1}{3} \Omega \wedge \bar{\Omega}$.

We will refer to $(J, \omega, \Omega)$ as an SU(3)-structure. If in addition $d\omega = 0$ and $d\Omega = 0$, then $J$ is integrable and $\Omega$ is a holomorphic (3,0)-form, and the closedness of $\omega$ implies the associated Hermitian metric $g$ is Kähler. In this case $d\omega$ and $d\Omega$ can be thought of as the torsion of the SU(3)-structure, and when they both vanish the SU(3)-structure is torsion-free. Note that property (3) implies that the holomorphic (3,0)-form $\Omega$ has constant length, so it is covariant constant. Therefore we have

**Proposition 2.2.** Let $M$ be a real 6-fold and $(J, \omega, \Omega)$ an SU(3)-structure on $M$. Let $g$ be the Hermitian metric with Hermitian form $\omega$. Then the followings are equivalent:

(i) $d\omega = 0$ and $d\Omega = 0$ on $M$,
(ii) $(J, \omega, \Omega)$ is torsion-free,
(iii) $(J, \omega, \Omega)$ gives a Calabi–Yau structure on $M$, and
(iv) $\text{Hol}(g) \subseteq \text{SU}(3)$.

Now we discuss $G_2$-structures in 7-folds. The books by Salamon [9 §11–§12] and Joyce [7, Chapter 10] are good introductions to $G_2$. In the theory of Riemannian holonomy groups, one of the exceptional cases in Berger’s classification [1] is given by $G_2$ in 7 dimensions. Bryant and Salamon [3] found explicit, complete metrics with holonomy $G_2$ on noncompact manifolds, and Joyce [7] constructed examples of compact 7-folds with holonomy $G_2$. The exceptional Lie group $G_2$ is the subgroup of GL(7,$\mathbb{R}$) preserving the 3-form

$$\varphi_0 = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_4 \wedge dx_5 + dx_1 \wedge dx_6 \wedge dx_7 + dx_2 \wedge dx_4 \wedge dx_6 - dx_2 \wedge dx_5 \wedge dx_7 - dx_3 \wedge dx_4 \wedge dx_5 - dx_3 \wedge dx_4 \wedge dx_6$$

on $\mathbb{R}^7$ with coordinates $(x_1, \ldots, x_7)$. This group also preserves the metric

$$g_0 = dx_1^2 + \cdots + dx_7^2,$$

the 4-form

$$\ast \varphi_0 = dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7 + dx_2 \wedge dx_3 \wedge dx_6 \wedge dx_7 + dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5$$

$$+ dx_1 \wedge dx_3 \wedge dx_5 \wedge dx_7 - dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_6 - dx_1 \wedge dx_2 \wedge dx_5 \wedge dx_6 - dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_7,$$

and the orientation on $\mathbb{R}^7$. Let $X$ be an oriented 7-fold. We say that a 3-form $\varphi$ (a 4-form $\psi$) on $X$ is positive if for each $p \in X$, there exists an oriented isomorphism between $T_p X$ and $\mathbb{R}^7$ identifying $\varphi$ and the 3-form $\varphi_0$ (the 4-form $\ast \varphi_0$).

A $G_2$-structure on a 7-fold $X$ is a principal subbundle of the oriented frame bundle of $X$, with fibre $G_2$. Thus there is a 1-1 correspondence between positive 3-forms and $G_2$-structures on $X$. Moreover, to any positive 3-form on $X$ one can associate a unique positive
4-form $\ast \varphi$ and metric $g$, such that $\varphi$, $\ast \varphi$ and $g$ are identified with $\varphi_0$, $\ast \varphi_0$ and $g_0$ under an isomorphism between $T_pX$ and $\mathbb{R}^7$, for each $p \in M$. We shall refer to $(\varphi, g)$ as a $G_2$-structure. Suppose $(\varphi, g)$ is a $G_2$-structure on $X$, and $\nabla$ is the Levi-Civita connection of $g$. We call $\nabla \varphi$ the torsion of the $G_2$-structure $(\varphi, g)$, and if $\nabla \varphi = 0$, then the $G_2$-structure is torsion-free. Here is a result from [7, Prop. 10.1.3]:

**Proposition 2.3.** Let $X$ be a real 7-fold and $(\varphi, g)$ a $G_2$-structure on $X$. Then the followings are equivalent:

(i) $(\varphi, g)$ is torsion-free,

(ii) $\nabla \varphi = 0$ on $X$, where $\nabla$ is the Levi-Civita connection of $g$,

(iii) $d \varphi = d^* \varphi = 0$ on $X$, and

(iv) $\text{Hol}(g) \subseteq G_2$, and $\varphi$ is the induced 3-form.

Now if $(M, J, \omega, \Omega)$ is a Calabi–Yau 3-fold with the Calabi–Yau metric $g_M$, then by Proposition 2.2, $(J, \omega, \Omega)$ gives a torsion-free SU(3)-structure on $M$, and $\text{Hol}(g_M) \subseteq \text{SU}(3)$. By considering SU(3) as a subgroup of $G_2$, the 7-fold $S^1 \times M$ has a torsion-free $G_2$-structure, which is constructed by the following result [7, Prop. 11.1.2]:

**Proposition 2.4.** Suppose $(J, \omega, \Omega)$ is a torsion-free SU(3)-structure on a 6-fold $M$. Let $s$ be a coordinate on $S^1$. Define a metric $g$ and a 3-form $\varphi$ on $S^1 \times M$ by

$$g = ds^2 + g_M \quad \text{and} \quad \varphi = ds \wedge \omega + \text{Re} \omega \Omega.$$

Then $(\varphi, g)$ is a torsion-free $G_2$-structure on $S^1 \times M$, and

$$\ast \varphi = \frac{1}{2} \omega \wedge \omega - ds \wedge \text{Im} \omega \Omega.$$

### 3. Nearly Calabi–Yau structures

This section introduces the notion of a nearly Calabi–Yau structure on an oriented 6-fold $M$. We begin in §3.1 by giving the definition of a nearly Calabi–Yau structure $(\omega, \Omega)$ on $M$, and showing that if $M$ admits a genuine Calabi–Yau structure, then any $(\omega, \Omega)$ which is sufficiently close to it gives a nearly Calabi–Yau structure. Section 3.2 constructs $G_2$-structures on the 7-fold $S^1 \times M$. Finally, we give the main result of the section, the existence of genuine Calabi–Yau structures on $M$, in §3.3. It is based on the existence result for torsion-free $G_2$-structures on compact 7-folds by Joyce [M Thm. 11.6.1].

#### 3.1. Introduction to nearly Calabi–Yau structures

Let $M$ be an oriented 6-fold. A nearly Calabi–Yau structure on $M$ consists of a real closed 2-form $\omega$, and a complex closed 3-form $\Omega$ on $M$. Basically, the idea of a nearly Calabi–Yau structure $(\omega, \Omega)$ is that it corresponds to an SU(3)-structure with “small torsion”, and hence approximates a genuine Calabi–Yau structure, which is equivalent to a torsion-free SU(3)-structure. So let us start with generating an SU(3)-structure on $M$ from $(\omega, \Omega)$. 
First we write \( \Omega = \theta_1 + i \theta_2 \), so \( \theta_1 \) and \( \theta_2 \) are both real closed 3-forms. Suppose \( \theta_1 \) has stabilizer \( \text{SL}(3, \mathbb{C}) \subset \text{GL}_+(6, \mathbb{R}) \) at each \( p \in M \), then the orbit of \( \theta_1 \) in \( \bigwedge^3 T^*_p M \) under the action of \( \text{GL}_+(6, \mathbb{R}) \) is \( \text{GL}_+(6, \mathbb{R})/\text{SL}(3, \mathbb{C}) \). For each \( p \in M \), define \( \bigwedge^3 T^*_p M \) to be the subset of 3-forms \( \theta \in \bigwedge^3 T^*_p M \) for which there exists an oriented isomorphism between \( T^*_p M \) and \( \mathbb{R}^6 \cong \mathbb{C}^3 \) identifying \( \theta \) and the 3-form \( \text{Re}(dz_1 \wedge dz_2 \wedge dz_3) \) where \((z_1, z_2, z_3)\) are coordinates on \( \mathbb{C}^3 \). Then \( \bigwedge^3 T^*_p M \cong \text{GL}_+(6, \mathbb{R})/\text{SL}(3, \mathbb{C}) \), as \( \text{Re}(dz_1 \wedge dz_2 \wedge dz_3) \) has stabilizer \( \text{SL}(3, \mathbb{C}) \). Then \( \theta_{1|p} \) lies in \( \bigwedge^3 T^*_p M \) for each \( p \in M \). It is easy to see that \( \dim \bigwedge^3 T^*_p M = \dim \text{GL}_+(6, \mathbb{R})/\text{SL}(3, \mathbb{C}) = \dim \bigwedge^3 T^*_p M = 20 \), so \( \bigwedge^3 T^*_p M \) is an open subset of \( \bigwedge^3 T^*_p M \) for each \( p \in M \). Therefore any 3-form on \( M \) which is sufficiently close to a 3-form in \( \bigwedge^3 T^*_p M \) still lies in \( \bigwedge^3 T^*_p M \), or equivalently, has stabilizer \( \text{SL}(3, \mathbb{C}) \) at each point on \( M \).

The oriented frame bundle \( F_+ \) of \( M \) is the bundle over \( M \) whose fibre at \( p \in M \) is the set of oriented isomorphisms between \( T^*_p M \) and \( \mathbb{R}^6 \). Let \( P \) be the subset of \( F_+ \) consisting of oriented isomorphisms between \( T^*_p M \) and \( \mathbb{R}^6 \) which identify \( \theta_1 \) at \( p \) and \( \text{Re}(dz_1 \wedge dz_2 \wedge dz_3) \). It is well-defined as we have assumed that \( \theta_1 \) and \( \theta_2 \) are \( \mathbb{R} \)-linearly independent. Thus \( \theta_1 \) defines a principal sub-bundle \( P \) of the oriented frame bundle \( F_+ \), with fibre \( \text{SL}(3, \mathbb{C}) \), that is, an \( \text{SL}(3, \mathbb{C}) \)-structure on \( M \). As \( \text{SL}(3, \mathbb{C}) \) acts on \( \mathbb{R}^6 \cong \mathbb{C}^3 \) preserving the complex structure \( J_0 \) on \( \mathbb{C}^3 \), we obtain a unique almost complex structure \( J' \) on \( M \).

Note that the forms \( \omega, \Omega \) are not necessarily of type \((1,1) \) and \((3,0) \) with respect to \( J' \) respectively. We then denote \( \omega^{(1,1)} \) by the \((1,1)\)-component of \( \omega \) with respect to \( J' \) and define a 3-form \( \theta'_2 \) on \( M \) by \( \theta'_2(u,v,w) := \theta_2(J'u,v,w) \) for all \( u,v,w \in TM \), or in index notation, \( (\theta'_2)_{abc} = (J')^a_{\phantom{a}d} (\theta_2)_{dcb} \). Suppose that \( \omega^{(1,1)} \) is a positive \((1,1)\)-form, that is, \( \omega^{(1,1)}(v,J'v) > 0 \) for any nonzero \( v \in TM \). Write \( \Omega' = \theta_1 + i\theta'_2 \), then \( \Omega' \) is a \((3,0)\)-form with respect to \( J' \). In general, \( \theta'_2 \) will not be a closed 3-form, unless \( J' \) is integrable.

We want \((J',\omega^{(1,1)},\Omega')\) to be an \( \text{SU}(3) \)-structure, but the problem with this is the usual normalization formula defining a Calabi–Yau manifold may not hold for \( \omega^{(1,1)} \) and \( \Omega' \), that is, \( (\omega^{(1,1)})^3 \neq \frac{3}{2} \Omega' \wedge \Omega' \) in general. We then define a smooth function \( f : M \to (0, \infty) \) by

\[
(\omega^{(1,1)})^3 = f \cdot \frac{3}{2} \theta_1 \wedge \theta'_2.
\]

Consequently, if we rescale \( \omega^{(1,1)} \) by setting \( \omega' = f^{-\frac{3}{4}} \omega^{(1,1)} \), we have \( \omega'^3 = \frac{3}{2} \theta_1 \wedge \theta'_2 \). Given that \( \omega^{(1,1)} \), and hence \( \omega' \), is positive, then one can determine a Hermitian metric \( g_M \) on \( M \) from \( \omega' \) and \( J' \) by \( g_M(u,v) = \omega'(u,J'v) \) for all \( u,v \in TM \).

Now we are ready to give the definition of a nearly Calabi–Yau structure on \( M \):

**Definition 3.1.** Let \( M \) be an oriented 6-fold, and let \( \omega \) be a real closed 2-form, and \( \Omega = \theta_1 + i \theta_2 \) a complex closed 3-form on \( M \). Let \( \epsilon_0 \in (0, 1] \) be a fixed small constant, to be chosen later in Lemma 3.3. Then \((\omega, \Omega)\) constitutes a nearly Calabi–Yau structure on \( M \) if

(i) the real closed 3-form \( \theta_1 \) has stabilizer \( \text{SL}(3, \mathbb{C}) \) at each point of \( M \), or equivalently, it lies in \( \bigwedge^3 T^*_p M \) for each \( p \in M \).

Then we can associate a unique almost complex structure \( J' \) and a unique real 3-form \( \theta'_2 \) such that \( \Omega' = \theta_1 + i \theta'_2 \) is a \((3,0)\)-form with respect to \( J' \).

(ii) the \((1,1)\)-component \( \omega^{(1,1)} \) of \( \omega \) with respect to \( J' \) is positive.
Then we can associate a Hermitian metric $g_M$ on $M$ from $\omega'$ and $J'$, where $\omega' = f^{-\frac{i}{2}} \omega^{(1,1)}$ is the rescaled $(1,1)$-part of $\omega$ and $f$ is defined by (3.1).

(iii) the following inequalities hold for some $\epsilon$ with $0 < \epsilon \leq \epsilon_0$:

$$|\theta_2 - \tilde{\theta}_2|_{g_M} < \epsilon,$$

$$|\omega^{(2,0)}|_{g_M} < \epsilon,$$

$$|\omega^3 - \frac{3}{2} \theta_1 \wedge \theta_2|_{g_M} < \epsilon$$

where the norms $|\cdot|_{g_M}$ are measured by the metric $g_M$.

If $(\omega, \Omega)$ is a nearly Calabi–Yau structure on $M$, one can show that the function $f$ defined in (3.1) satisfies

$$|f - 1| < C_0 \epsilon,$$

for some constant $C_0 > 0$, i.e. $f$ is approximately equal to 1 for sufficiently small $\epsilon$, as we would expect.

The next result shows that if $M$ admits a genuine Calabi–Yau structure, then any $(\omega, \Omega)$ which is sufficiently close to it gives a nearly Calabi–Yau structure on $M$.

**Proposition 3.2.** There exist constants $\epsilon_1, C, C' > 0$ such that whenever $0 < \epsilon \leq \epsilon_1$, the following is true.

Let $M$ be an oriented 6-fold. Suppose $(\tilde{J}, \tilde{\omega}, \tilde{\Omega})$ is a Calabi–Yau structure with Calabi–Yau metric $\tilde{g}$, $\omega$ a real closed 2-form, and $\Omega = \theta_1 + i\theta_2$ a complex closed 3-form on $M$, satisfying

$$|\tilde{\omega} - \omega|_{\tilde{g}} < \epsilon \quad \text{and} \quad |\tilde{\Omega} - \Omega|_{\tilde{g}} < \epsilon,$$

then $(\omega, \Omega)$ is a nearly Calabi–Yau structure on $M$ with metric $g_M$ satisfying

$$|\tilde{g} - g_M|_{\tilde{g}} < C \epsilon \quad \text{and} \quad |\tilde{g}^{-1} - g_M^{-1}|_{\tilde{g}} < C' \epsilon.$$

**Proof.** From (3.6) we have $|\text{Re}(\tilde{\Omega}) - \theta_1|_{\tilde{g}} < \epsilon$, which means that if we choose $\epsilon_1$ to be sufficiently small, then $\theta_1$ has stabilizer $\text{SL}(3, \mathbb{C})$ since the stabilizer condition is an open condition as we mentioned before. So we can associate a unique almost complex structure $J'$, with $|\tilde{J} - J'|_{\tilde{g}} < C_1 \epsilon$ for some constant $C_1 > 0$, and a unique real 3-form $\tilde{\theta}'_2$ such that $\tilde{\Omega}' = \theta_1 + i\tilde{\theta}'_2$ is a $(3,0)$-form with respect to $J'$.

One can deduce from (3.6) and $|\tilde{J} - J'|_{\tilde{g}} < C_1 \epsilon$ that $|\tilde{\omega} - \omega^{(1,1)}|_{\tilde{g}} < C_2 \epsilon$ for some $C_2 > 0$. Make $\epsilon_1$ smaller if necessary, then $\omega^{(1,1)}$ is a positive $(1,1)$-form with respect to $J'$ since the positivity is also an open condition. Then we can define a metric $g_M$ by $g_M(u, v) = \omega'(u, J'v)$ for all $u, v \in TM$, where $\omega' = f^{-\frac{i}{2}} \omega^{(1,1)}$ and $f$ is defined by (3.1). Now we show that $f$ is close to 1. In fact,

$$|(f - 1)\omega'^3|_{g_M} = |(\omega^{(1,1)})^3 - \tilde{\omega}^3|_{\tilde{g}}$$

$$\leq |(\omega^{(1,1)})^3 - \tilde{\omega}^3|_{\tilde{g}} + \left| \tilde{\omega}^3 - \frac{3}{2} \text{Re}(\tilde{\Omega}) \wedge \text{Im}(\tilde{\Omega}) \right|_{\tilde{g}}$$

$$+ \left| \frac{3}{2} \text{Re}(\tilde{\Omega}) \wedge \text{Im}(\tilde{\Omega}) - \frac{3}{2} \theta_1 \wedge \tilde{\theta}'_2 \right|_{\tilde{g}}.$$
Since $|	ilde{\omega} - \omega^{(1,1)}|_{\tilde{g}} < C_2\epsilon$, the first term of right hand side of (3.8) is of size $O(\epsilon)$. The second term vanishes as $(\tilde{\omega}, \tilde{\Omega})$ is a Calabi-Yau structure. From $|\text{Re}(\Omega) - \theta_1|_{\tilde{g}} < \epsilon$, and $|\tilde{J} - J'|_{\tilde{g}} < C_1\epsilon$, we have $|\text{Im}(\Omega) - \theta_2|_{\tilde{g}} < C_3\epsilon$ for some $C_3 > 0$ and hence the third term also has size $O(\epsilon)$. Summing up, we have $f - 1$ is of size $O(\epsilon)$.

It can then be shown that $|\tilde{\omega} - \omega'|_{\tilde{g}} < C_4\epsilon$ for some $C_4 > 0$. Together with $|\tilde{J} - J'|_{\tilde{g}} < C_1\epsilon$, we obtain first part of (3.7), that is, $|\tilde{g} - g_M|_{\tilde{g}} < C\epsilon$ for some $C > 0$. If $\epsilon$ is small enough such that $C\epsilon < \frac{1}{2}$, then one can deduce that $|\tilde{g}^{-1} - g_M^{-1}|_{\tilde{g}} < C'\epsilon$ for some $C' > 0$. This implies that $\tilde{g}$ and $g_M$ are uniformly equivalent metrics, and hence norms of any tensor on $M$ taken with respect to $\tilde{g}$ and with respect to $g_M$ differ by a bounded factor.

It remains to check (3.2)-(3.4) of Definition 3.1. But it is not hard to get bounds for (3.2)-(3.4) in terms of $\epsilon$ with respect to the metric $\tilde{g}$, and so by making $\epsilon$ smaller and using the equivalence of the metrics, we obtain (3.2)-(3.4). \hfill \Box

### 3.2. $G_2$-structures on $S^1 \times M$

Let $(\omega, \Omega)$ be a nearly Calabi-Yau structure on $M$. From §3.1 we know that $(J', \omega', \Omega')$ gives an SU(3)-structure with metric $g_M$ on $M$. In this section, we would like to discuss $G_2$-structures on the 7-fold $S^1 \times M$, which is essential for the main result in next section. Let $s$ be a coordinate on $S^1$. Now define a 3-form $\varphi'$ and a metric $g'$ on $S^1 \times M$ by

\begin{equation}
\varphi' = ds \wedge \omega' + \theta_1 \quad \text{and} \quad g' = ds^2 + g_M.
\end{equation}

It turns out that $(\varphi', g')$ defines a $G_2$-structure (with torsion) on $S^1 \times M$. The associated 4-form $\ast_{g'} \varphi'$ on $S^1 \times M$ is then given by

\begin{equation}
\ast_{g'} \varphi' = \frac{1}{2} \omega' \wedge \omega' - ds \wedge \theta_2.
\end{equation}

Also, we can construct another 3-form $\varphi$ and 4-form $\chi$ on $S^1 \times M$ by

\begin{equation}
\varphi = ds \wedge \omega + \theta_1 \quad \text{and} \quad \chi = \frac{1}{2} \omega \wedge \omega - ds \wedge \theta_2.
\end{equation}

The next lemma shows that the forms in (3.11) are close to the $G_2$-forms $\varphi'$ and $\ast_{g'} \varphi'$ if we make $\epsilon_0$ in the definition of nearly Calabi-Yau manifolds to be sufficiently small.

**Lemma 3.3.** There exist constants $C_1, C_2, C_3$ and $C_4 > 0$ such that if $\epsilon_0$ in Definition 3.1 is chosen sufficiently small, then the following is true.

Let $(\varphi', g')$ be the $G_2$-structure given by (3.9), $\ast_{g'} \varphi'$ the associated 4-form given by (3.10), $\varphi$ the 3-form and $\chi$ the 4-form given by (3.11) on $S^1 \times M$. Then

\begin{equation}
|\varphi - \varphi'|_{g'} < C_1\epsilon
\end{equation}

where $\epsilon \in (0, \epsilon_0]$ is the small constant in (iii) of Definition 3.1. Hence $\varphi$ is also a positive 3-form on $S^1 \times M$, and it defines another $G_2$-structure $(\varphi, g)$. Moreover, the associated
metric $g$ and the 4-form $\ast_g \varphi$ satisfy
\begin{align}
|g - g'|_{g'} &< C_2 \epsilon, \quad |g^{-1} - g'^{-1}|_{g'} < C_3 \epsilon \text{ and} \\
|\ast_g \varphi - \chi|_{g'} &< C_4 \epsilon.
\end{align}

Proof. From (3.9) and (3.11), we have $|\varphi - \varphi'|_{g'} = |ds \wedge (\omega - \omega')|_{g'} = |\omega - \omega'|_{g_M} < C_1 \epsilon$ for some constant $C_1 > 0$, where we used (3.3) and (3.5). Now we choose $\epsilon_0$ in Definition 3.1 such that $C_1 \epsilon_0$ is small enough and $\varphi$ is a positive 3-form on $S^3 \times M$.

It follows from general facts about $G_2$-forms that the associated metric $g$ satisfies $|g - g'|_{g'} < C_2 \epsilon$ for some $C_2 > 0$. Also, by using the same argument as in Proposition 3.2, we obtain $|g^{-1} - g'^{-1}|_{g'} < C_3 \epsilon$ for some $C_3 > 0$. Now,
\begin{align}
|\ast_g \varphi - \chi|_{g'} &\leq |\ast_g \varphi - \ast_{g'} \varphi'|_{g'} + |\ast_{g'} \varphi' - \chi|_{g'} \\
&\leq |(\ast_g - \ast_{g'}) \varphi|_{g'} + |\ast_{g'} (\varphi - \varphi')|_{g'} + |\ast_{g'} \varphi' - \chi|_{g'}.
\end{align}

The first term of right hand side has the same size as $|g - g'|_{g'}$, which is bounded in (3.13). The second term is just $|\varphi - \varphi'|_{g'}$ since $\ast_{g'}$ is an isometry with respect to $g'$, and so it is bounded in (3.12). For the last term, we have
\begin{align}
|\ast_{g'} \varphi' - \chi|_{g'} &= \left| \frac{1}{2} (\omega' \wedge \omega' - \omega \wedge \omega) - ds \wedge (\theta_2' - \theta_2) \right|_{g'}
\end{align}
from (3.10) and (3.11), which can be shown by using $|\omega - \omega'|_{g_M} < C_1 \epsilon$ and (3.2) that has size $O(\epsilon)$. Summing up together, we get $|\ast_g \varphi - \chi|_{g'} < C_4 \epsilon$ for some constant $C_4 > 0$. \hfill \Box

### 3.3. An existence result for Calabi–Yau structures on $M$

In the last part of this section we present our main result which shows that when $\epsilon_0$ is sufficiently small, the nearly Calabi–Yau structure on $M$ can be deformed to a genuine Calabi–Yau structure. The proof of it is based on an existence result for torsion-free $G_2$-structures given by Joyce [7, Thm. 11.6.1], which shows using analysis that any $G_2$-structure on a compact 7-fold with sufficiently small torsion can be deformed to a nearby torsion-free $G_2$-structure. We shall adopt a slightly modified version of this result, which improves the bounds of various norms to fit into our situation. We refer to the spaces $L^q$, $L^q_k$, $C^k$ and $C^{k,\alpha}$ as the Banach spaces defined in [7, §1.2]. We begin by stating Joyce’s result, with improvements to powers of $t$:

**Theorem 3.4.** Let $\kappa > 0$ and $D_1, D_2, D_3 > 0$ be constants. Then there exist constants $\epsilon \in (0, 1]$ and $K > 0$ such that whenever $0 < t \leq \epsilon$, the following is true.

Let $X$ be compact 7-fold, and $(\varphi, g)$ a $G_2$-structure on $X$ with $d \varphi = 0$. Suppose $\psi$ is a smooth 3-form on $X$ with $d^* \psi = d \varphi$, and

- (i) $\|\psi\|_{L^2} \leq D_1 t^{2+\kappa}$, $\|\psi\|_{C^0} \leq D_1 t^{\kappa}$ and $\|d^* \psi\|_{L^1} \leq D_1 t^{-\frac{3}{2}+\kappa}$,
- (ii) the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq D_2 t$, and
- (iii) the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq D_2 t^{-2}$.

Then there exists a smooth, torsion-free $G_2$-structure $(\hat{\varphi}, \hat{g})$ on $X$ such that $\|\hat{\varphi} - \varphi\|_{C^0} \leq K t^\kappa$ and $[\hat{\varphi}] = [\varphi]$ in $H^3(X, \mathbb{R})$. 


The proof of it depends upon the following two results. We state them here and then we will improve the powers of $t$ so that Theorem 3.4 can be modified to fit into our situation for the 7-fold $S^1 \times M$.

**Theorem 3.5.** Let $D_2, D_3, t > 0$ be constants, and suppose $(X, g)$ is a complete Riemannian 7-fold, whose injectivity radius $\delta(g)$ and Riemann curvature $R(g)$ satisfy $\delta(g) \geq D_2 t$ and $\|R(g)\|_{C^0} \leq D_3 t^{-2}$. Then there exist $K_1, K_2 > 0$ depending only on $D_2$ and $D_3$, such that if $\chi \in L^{14}_1(\Lambda^3 T^*X) \cap L^2(\Lambda^3 T^*X)$ then

$$\|\nabla \chi\|_{L^{14}} \leq K_1 (\|d \chi\|_{L^{14}} + \|d^* \chi\|_{L^{14}} + t^{-4} \|\chi\|_{L^2})$$

and

$$\|\chi\|_{C^0} \leq K_2 (t^{\frac{2}{7}} \|\nabla \chi\|_{L^{14}} + t^{-\frac{3}{7}} \|\chi\|_{L^2}).$$

The second result is:

**Theorem 3.6.** Let $\kappa > 0$ and $D_1, K_1, K_2 > 0$ be constants. Then there exist constants $\epsilon \in [0, 1]$, $K_3$ and $K > 0$ such that whenever $0 < t \leq \epsilon$, the following is true.

Let $X$ be a compact 7-fold, and $(\varphi, g)$ a $G_2$-structure on $X$ with $d \varphi = 0$. Suppose $\psi$ is a smooth 3-form on $X$ with $d^* \psi = d^* \varphi$, and

1. $\|\psi\|_{L^2} \leq D_1 t^{\frac{2}{7} + \kappa}$, $\|d^* \psi\|_{C^0} \leq D_1 t^{\kappa}$ and $\|d^* \psi\|_{L^{14}} \leq D_1 t^{-\frac{2}{7} + \kappa}$,
2. if $\chi \in L^{14}_1(\Lambda^3 T^*X)$ then $\|\nabla \chi\|_{L^{14}} \leq K_1 (\|d \chi\|_{L^{14}} + \|d^* \chi\|_{L^{14}} + t^{-4} \|\chi\|_{L^2})$, and
3. if $\chi \in L^{14}_1(\Lambda^3 T^*X)$ then $\|\chi\|_{C^0} \leq K_2 (t^{\frac{2}{7}} \|\nabla \chi\|_{L^{14}} + t^{-\frac{3}{7}} \|\chi\|_{L^2}).$

Let $\epsilon_1$ be as in Definition 10.3.3, and $F$ as in Proposition 10.3.5 of Joyce [2]. Denote $\pi_1$ by the orthogonal projection from $\Lambda^3 T^*X$ to the 1-dimensional component of the irreducible representation of $G_2$. Then there exist sequences $\{\eta_j\}_{j=0}^\infty$ in $L^{14}_2(\Lambda^2 T^*X)$ and $\{f_j\}_{j=0}^\infty$ in $L^{14}_1(X)$ with $\eta_0 = f_0 = 0$, satisfying the equations

$$(dd^* + d^* d) \eta_j = d^* \psi + d^* (f_{j-1} \psi) + * dF(d \eta_{j-1})$$

and $f_j \varphi = \frac{7}{3} \pi_1 (d \eta_j)$ for each $j > 0$, and the inequalities

(a) $\|d \eta_j\|_{L^2} \leq 2D_1 t^{\frac{2}{7} + \kappa}$, (d) $\|d \eta_j - d \eta_{j-1}\|_{L^2} \leq 2D_1 2^{-j} t^{\frac{2}{7} + \kappa}$,

(b) $\|\nabla d \eta_j\|_{L^{14}} \leq K_3 t^{-\frac{2}{7} + \kappa}$, (e) $\|\nabla (d \eta_j - d \eta_{j-1})\|_{L^{14}} \leq K_3 2^{-j} t^{-\frac{2}{7} + \kappa}$,

(c) $\|d \eta_j\|_{C^0} \leq K t^\kappa \leq \epsilon_1$ and (f) $\|d \eta_j - d \eta_{j-1}\|_{C^0} \leq K 2^{-j} t^\kappa$.

We shall first modify Theorem 3.5 by considering the 6-dimensional version of those analytic estimates. In Theorem 3.5, the first inequality is derived from an elliptic regularity estimate for the operator $d + d^*$ on 3-forms on $X$. The second inequality follows from the Sobolev Embedding Theorem, which states that $L^r_k$ embeds in $C^{l,\alpha}$ if $\frac{1}{r} \leq \frac{k-l-\alpha}{n}$ where $n$ is the dimension of the underlying Riemannian manifold. For the 7-dimensional case, we have $L^1_1$ embeds in $C^{0,1/2}$ which then embeds in $C^0$, whereas in 6 dimensions, we have $L^1_1$ embeds in $C^{0,1/2}$. We can use this to show

**Theorem 3.7.** Let $D_2, D_3, t > 0$ be constants, and suppose $(M, g)$ is a complete Riemannian 6-fold, whose injectivity radius $\delta(g)$ and Riemann curvature $R(g)$ satisfy $\delta(g) \geq D_2 t$ and
\[\|R(g)\|_{C^0} \leq D_3 t^{-2}.\] Then there exist \(K_1, K_2 > 0\) depending only on \(D_2\) and \(D_3\), such that if \(\chi \in L^2_1(M^3 T^* M) \cap L^2(\Lambda^3 T^* M)\) then
\[\|\nabla \chi\|_{L^2} \leq K_1 (\|d\chi\|_{L^2} + \|d^* \chi\|_{L^2} + t^{-\frac{5}{2}} \|\chi\|_{L^2})\]
and
\[\|\chi\|_{C^0} \leq K_2 (t^{\frac{7}{2}} \|\nabla \chi\|_{L^2} + t^{-3} \|\chi\|_{L^2}).\]

The proof of it is similar to [4, Thm. G1, p. 298]. We can first prove the case for \(t = 1\), and the case for general \(t > 0\) follows by conformal rescaling: apply the \(t = 1\) case to the metric \(t^{-2} g\). The factors of \(t\) compensate for powers of \(t\) which the norms scaled by in replacing \(g\) by \(t^{-2} g\).

By considering \(S^1\)-invariant forms and \(S^1\)-invariant \(G_2\)-structures on the 7-fold \(S^1 \times M\), we can use the Sobolev Embedding Theorem in 6 dimensions, rather than in 7 dimensions. This has an advantage that the powers of \(t\) and the inequalities are calculated in 6 dimensions, though the norms are computed on the 7-fold \(S^1 \times M\). Here is the modified version of Theorem 3.6:

**Theorem 3.8.** Let \(\kappa > 0\) and \(D_1, K_1, K_2 > 0\) be constants. Then there exist constants \(\epsilon \in (0, 1]\), \(K_3\) and \(K > 0\) such that whenever \(0 < t \leq \epsilon\), the following is true.

Let \(M\) be a compact 6-fold, and \((\varphi, g)\) an \(S^1\)-invariant \(G_2\)-structure on \(S^1 \times M\) with \(d \varphi = 0\). Suppose \(\psi\) is an \(S^1\)-invariant smooth 3-form on the 7-fold \(S^1 \times M\) with \(d^* \psi = d^* \varphi\), and

\[(d^* + d^* d) \eta_j = d^* \psi + d^* (f_j \psi) + (\star dF(d \eta_{j-1}))\]

and

\[f_j \varphi = \frac{7}{3} \pi_1(d \eta_j)\]

for each \(j > 0\), and the inequalities

\[(a) \|d \eta_j\|_{L^2} \leq 2 D_1 t^{3+j\kappa}, \quad (d) \|d \eta_j - d \eta_{j-1}\|_{L^2} \leq 2 D_1 2^{-j} t^{3+j\kappa},\]

\[(b) \|\nabla d \eta_j\|_{L^2} \leq K_3 t^{-\frac{5}{2} + \kappa}, \quad (c) \|\nabla (d \eta_j - d \eta_{j-1})\|_{L^2} \leq K_3 2^{-j} t^{-\frac{5}{2} + \kappa},\]

\[(c) \|d \eta_j\|_{C^0} \leq K t^\kappa \leq \epsilon_1\] and

\[(f) \|d \eta_j - d \eta_{j-1}\|_{C^0} \leq K 2^{-j} t^\kappa.\]

Here \(\nabla\) and \(\| \cdot \|\) are computed using \(g\) on \(S^1 \times M\).

Thus Theorem 3.8 is essentially an \(S^1\)-invariant version of Theorem 3.6. We are now ready to state the modified version of Theorem 3.4, to be used to obtain our main result.

**Theorem 3.9.** Let \(\kappa > 0\) and \(D_1, D_2, D_3 > 0\) be constants. Then there exist constants \(\epsilon \in (0, 1]\) and \(K > 0\) such that whenever \(0 < t \leq \epsilon\), the following is true.
Let $M$ be a compact 6-fold, and $(\varphi, g)$ an $S^1$-invariant $G_2$-structure on $S^1 \times M$ with $d\varphi = 0$. Suppose $\psi$ is an $S^1$-invariant smooth 3-form on $S^1 \times M$ with $d^*\psi = d^*\varphi$, and

(i) $\|\psi\|_{L^2} \leq D_1 t^{3+\kappa}$, $\|\psi\|_{C^0} \leq D_1 t^\kappa$ and $\|d^*\psi\|_{L^{1,2}} \leq D_1 t^{-\frac{3}{2}+\kappa}$,

(ii) the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq D_2 t$, and

(iii) the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq D_3 t^{-2}$.

Then there exists a smooth, torsion-free $G_2$-structure $(\tilde{\varphi}, \tilde{g})$ on $S^1 \times M$ such that $\|\tilde{\varphi} - \varphi\|_{C^0} \leq K t^\kappa$ and $[\tilde{\varphi}] = [\varphi]$ in $H^3(S^1 \times M, \mathbb{R})$.

Theorem 3.9 follows from Theorems 3.7 and 3.8 as on [7, p.296-297].

In the remaining part of this section, we shall derive an existence result for genuine Calabi–Yau structures. Our strategy is the following. We start with the nearly Calabi–Yau structure $(\omega, \Omega)$ on $M$, and then be shown in the following theorem that, under appropriate hypotheses on the nearly Calabi–Yau structure $(\omega, \Omega)$, the induced $G_2$-structure satisfies all the conditions in Theorem 3.9, and therefore can be deformed to have zero torsion. Finally, we pull back this torsion-free $G_2$-structure to obtain a genuine Calabi–Yau structure on $M$.

**Theorem 3.10.** Let $\kappa > 0$ and $E_1, E_2, E_3, E_4 > 0$ be constants. Then there exist constants $\epsilon \in (0, 1)$ and $K > 0$ such that whenever $0 < t \leq \epsilon$, the following is true.

Let $M$ be a compact 6-fold, and $(\omega, \Omega)$ a nearly Calabi–Yau structure on $M$. Let $\omega'$, $g_M$ and $\theta_2'$ be as in §3.1. Suppose $\omega - \omega'$ $L^2 \leq E_1 t^{3+\kappa}$, $\|\omega - \omega'\|_{C^0} \leq E_1 t^\kappa$, $\|\theta_2 - \theta_2'\|_{L^2} \leq E_1 t^{3+\kappa}$ and $\|\nabla (\omega - \omega')\|_{L^{1,2}} \leq E_1 t^{-\frac{3}{2}+\kappa}$;

(ii) $\|\nabla (\omega - \omega')\|_{L^{1,2}} \leq E_1 t^{-\frac{3}{2}+\kappa}$,

(iii) $\|\nabla^2 (\omega - \omega')\|_{C^0} \leq E_2 t^{\kappa-1}$, and $\|\nabla^2 (\omega - \omega')\|_{C^0} \leq E_2 t^{\kappa-2}$,

(iv) the injectivity radius $\delta(g_M)$ satisfies $\delta(g_M) \geq E_3 t$, and

(v) the Riemann curvature $R(g_M)$ satisfies $\|R(g_M)\|_{C^0} \leq E_4 t^{-2}$.

Then there exists a Calabi–Yau structure $(J, \tilde{\omega}, \tilde{\Omega})$ on $M$ such that $\|\tilde{\omega} - \omega\|_{C^0} \leq K t^\kappa$ and $\|\tilde{\Omega} - \Omega\|_{C^0} \leq K t^\kappa$. Moreover, if $H^3(M, \mathbb{R}) = 0$, then the cohomology classes satisfy $[\text{Re}(\tilde{\Omega})] = [\text{Re}(\Omega)] \in H^3(M, \mathbb{R})$ and $[\omega] = c [\tilde{\omega}] \in H^2(M, \mathbb{R})$ for some $c > 0$. Here the connection $\nabla$ and all norms are computed with respect to $g_M$.

**Proof.** Let $\varphi$ be the 3-form on $S^1 \times M$ given by (3.11). Then Lemma 3.3 shows that $(\varphi, g)$ is a $G_2$-structure on $S^1 \times M$, with $d\varphi = 0$ as $d\varphi = d\theta_1 = 0$. Define a 3-form $\psi = \varphi - g \chi$ on $S^1 \times M$, where $\chi$ is the 4-form given by (3.11). Then $d^*\psi = d^*\varphi - d^*(g \chi) = d^*\varphi + g d\chi = d^*\varphi$, since $d^*g = -g d$ on 3-forms and $d\chi = 0$.

Now $|\psi|_g = |g \varphi - \chi|_g \leq C |g \varphi - \chi|_{g'}$ where $C > 0$ is some constant relating norms w.r.t. the uniformly equivalent metrics $g$ and $g' = ds^2 + g_M$. From (3.15), one can show that $|g \varphi - \chi|_{g'} \leq C_1 (|\omega - \omega'|_{g_M} + |\omega - \omega'|_{g_M} + |\theta_2 - \theta_2'|_{g_M})$ for some $C_1 > 0$, and hence

(3.16) $|\psi|_g \leq C_2 (|\omega - \omega'|_{g_M} + |\omega - \omega'|_{g_M} + |\theta_2 - \theta_2'|_{g_M})$.
for some $C_2 > 0$. Consequently, we have

$$\|\psi\|_{C^0} \leq C_2 \left( \|\omega - \omega'\|_{C^0}^2 + \|\omega - \omega'\|_{C^0} + \|\theta_2 - \theta'_2\|_{C^0} \right),$$

and

$$\|\psi\|_{L^2} \leq C_2 \left( \|\omega - \omega'\|_{C^0} \cdot \|\omega - \omega'\|_{L^2} + \|\omega - \omega'\|_{L^2} + \|\theta_2 - \theta'_2\|_{L^2} \right)$$

which then imply

$$\|\psi\|_{C^0} \leq C_3 t^\kappa \quad \text{and} \quad \|\psi\|_{L^2} \leq C_3 t^{3+\kappa} \quad \text{for some} \quad C_3 > 0,$$

where we have used condition (i) and $t \leq \epsilon \leq 1$. This verifies the first two inequalities of (i) in Theorem 3.9, and we now proceed to the last one. Denote by $\nabla^g$ and $\nabla'^{g'}$ the connections computed using $g$ and $g'$ respectively. Consider now the term $\Box_d \psi = \partial^d \varphi$, we get

$$|\Box_d \psi|_g = |\Box^g \varphi|_g \leq C |\nabla^g \varphi|_{g'}.$$

Denote by $\Delta$ the difference of the two torsion-free connections $\nabla^g$ and $\nabla'^{g'}$. Thus $\Delta$ transforms as a tensor and it satisfies $A^{g'}_{ij} = A^g_{ij}$. We need the following proposition to obtain the bound for $|\Box^g \varphi|_g$.

**Proposition 3.11.** *In the situation above, we have*

\begin{align*}
\|\nabla^g \varphi|_{g'} &\leq |\nabla \omega|_{gM} + |\nabla \theta_1|_{gM}, \quad (3.19) \\
|\nabla^g \varphi'|_{g'} &\leq |\nabla (\omega - \omega')|_{gM} + |\nabla \omega|_{gM} + |\nabla \theta_1|_{gM}, \quad (3.20) \\
|A_{g'}| &\leq \frac{3}{2} |g^{-1}|_{g'} \cdot |\nabla^g g|_{g'}, \quad (3.21) \\
|\nabla^g g|_{g'} &\leq B_1 |\nabla^g (\varphi - \varphi')|_{g'} + B_2 |\varphi - \varphi'|_{g'} \cdot (|\nabla^g \varphi|_{g'} + |\nabla^g \varphi'|_{g'}) \quad (3.22)
\end{align*}

*for some $B_1, B_2 > 0$ depending on a small upper bound for $|\varphi - \varphi'|_{g'}$.*

**Proof.** For the first one, note that

$$|\nabla^g \varphi|_{g'} = |\nabla^g (ds \wedge \omega + \theta_1)|_{g'}$$

$$\leq |\nabla^g ds|_{g'} \cdot |\omega|_{g'} + |ds|_{g'} \cdot |\nabla^g \omega|_{g'} + |\nabla^g \theta_1|_{g'}. $$

Since $ds$ is a constant 1-form with length 1 w.r.t. the metric $g'$, equation (3.19) follows. The second inequality follows easily from the first one. For (3.21), we have from the definition of the tensor $A$,

$$A^{g'}_{ij} = \Gamma^{g'}_{ij} - \Gamma^g_{ij}$$

where $\Gamma^{g'}_{ij}$ and $\Gamma^g_{ij}$ are the Christoffel symbols of the Levi-Civita connections $\nabla^g$ and $\nabla'^{g'}$ respectively. Consider now the term $\nabla^g g$, and expressing it in index notation,

$$\nabla^g_a g_{bc} = \partial_a g_{bc} - \Gamma^d_{ab} g_{dc} - \Gamma^d_{ac} g_{bd}. $$
Then
\[ \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}) \]
\[ = \frac{1}{2} g^{kl} \left[ (\nabla_l g_{ij} + \Gamma^m g_{lj} + \Gamma^m g_{jm}) + (\nabla_l g_{ij} + \Gamma^m g_{il} + \Gamma^m g_{im}) \right. \]
\[ \left. - (\nabla_l g_{ij} + \Gamma^m g_{im} + \Gamma^m g_{jm}) \right] \]
\[ = \frac{1}{2} g^{kl} (\nabla_l g_{ij} + \nabla_l g_{ij} - \nabla_l g_{ij} + 2 \Gamma^m g_{ml}) \]

by the fact that \( \nabla^g \) is torsion-free. Hence,
\[ A^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ij} \]
\[ = \frac{1}{2} g^{kl} (\nabla_l g_{ij} + \nabla_l g_{ij} - \nabla_l g_{ij} + 2 \Gamma^m g_{ml}) \]

which gives rise to (3.21).

To verify the last inequality, first let \( F \) be the smooth function that maps each positive 3-form to its associated metric. Then \( F(\varphi) = g \) and \( F(\varphi') = g' \). As \( F(\varphi) \) depends pointwise on \( \varphi \), we can write \( F(\varphi)(x) = F(x, \varphi(x)) \) for all \( x \in S^1 \times M \) and \( \varphi(x) \) in the vector space \( \wedge^3 T^*_x(S^1 \times M) \). We may then take partial derivative in the \( \varphi(x) \) direction without using a connection, and write \( \partial \) for the partial derivative in this direction. Now,
\[ |\nabla^g g|_{g'} = |\nabla^g (g - g')|_{g'} \]
\[ = |\nabla^g (F(\varphi) - F(\varphi'))|_{g'} \]
\[ = \left| \int_0^1 \frac{d}{dr} \nabla^g (F(x, \varphi(x) + r(\varphi(x) - \varphi'(x))) \, dr \right|_{g'} \]
\[ = \left| \int_0^1 \nabla^g \left( \frac{d}{dr} F(x, \varphi(x) + r(\varphi(x) - \varphi'(x))) \right) \, dr \right|_{g'} \]
\[ = \left| \int_0^1 \nabla^g \left( \partial F(x, \varphi(x) + r(\varphi(x) - \varphi'(x))) \cdot (\varphi(x) - \varphi'(x)) \right) \, dr \right|_{g'} \]
\[ = \left| \int_0^1 \left[ (\nabla^g \partial F)(x, \varphi(x) + r(\varphi(x) - \varphi'(x))) \right. \]
\[ + \partial^2 F(x, \varphi'(x) + r(\varphi(x) - \varphi'(x))) \cdot \nabla^g (\varphi'(x) + r(\varphi(x) - \varphi'(x))) \]
\[ \left. \cdot (\varphi(x) - \varphi'(x)) + \partial F(x, \varphi'(x) + r(\varphi(x) - \varphi'(x))) \right) \, dr \right|_{g'} \]

It can be shown, by using the fact that continuous functions are bounded over compact spaces, that for any \( \phi \) which is close enough to \( \varphi' \), we have
\[ |\partial F(x, \phi(x))|_{g'} \leq B_1 \quad \text{and} \quad |\partial^2 F(x, \phi(x))|_{g'} \leq 2B_2 \]
for some constants \( B_1, B_2 > 0 \), and as this is a calculation at a point, \( B_1, B_2 \) are constants depend only on a small upper bound for \( |\phi - \varphi'|_{g'} \). Moreover, if we choose geodesic normal coordinates at \( x \), then the Christoffel symbols \( \Gamma^k_{ij} \) of \( \nabla^g \) vanish at \( x \), so \( \nabla^g \) reduces to the usual partial differentiation at \( x \) and it follows that
\[ \nabla^g \partial F(x, \varphi'(x) + r(\varphi(x) - \varphi'(x))) = 0 \]
since $F$, and hence $\partial F$ is invariant under translation along the directions of coordinate vectors. Consequently,
\[
|\nabla^g \varphi|_{g'} \leq B_1 |\nabla^g \Phi (\varphi(x) - \varphi'(x))|_{g'} + 2 B_2 |\varphi(x) - \varphi'(x)|_{g'} \int_0^1 |\nabla^g \Phi (\varphi(x) + r(\varphi(x) - \varphi'(x)))|_{g'} dr \\
\leq B_1 |\nabla^g (\varphi(x) - \varphi'(x))|_{g'} + B_2 |\varphi(x) - \varphi'(x)|_{g'} \cdot (|\nabla^g \varphi|_{g'} + |\nabla^g \varphi'|_{g'})
\]
and this finishes the proof of Proposition 3.11.

Applying the above estimates to (3.18) shows that
\[
|d^* \psi|_g \leq C |\nabla^g \Phi|_{g'} \leq C |\nabla^g \Phi|_{g'} + C |A|_{g'} \cdot |\varphi|_{g'} \\
\leq C |\nabla \omega|_{g_M} + C |\nabla \theta_1|_{g_M} + C |\nabla \theta_2|_{g_M} + 3 C |\nabla^g \Phi|_{g'} \cdot |\nabla^g \varphi|_{g'} \\
\leq C |\nabla \omega|_{g_M} + C |\nabla \theta_1|_{g_M} + C_4 (B_1 |\nabla^g \Phi (\varphi - \varphi')|_{g'} + B_2 |\varphi - \varphi'|_{g'} \cdot (|\nabla^g \Phi|_{g'} + |\nabla^g \varphi'|_{g'})) \\
\leq C |\nabla \omega|_{g_M} + C |\nabla \theta_1|_{g_M} + C_5 |\nabla (\omega - \omega')|_{g_M} + C_6 |\omega - \omega'|_{g_M} \\
(\|\nabla (\omega - \omega')\|_{g_M} + 2 \|\nabla \omega\|_{g_M} + 2 \|\nabla \theta_1\|_{g_M}) \text{ by (3.20)},
\]
where $C_4, C_5, C_6 > 0$ are some constants. From the second inequality of (i), we have $\|\omega - \omega'\|_{C^0} \leq \lambda t^{1/2} \leq \lambda$ as $t \leq \kappa \leq 1$. Combining with condition (ii) shows that
\[
\|d^* \psi\|_{L^1} \leq C \|\nabla \omega\|_{L^1} + C \|\nabla \theta_1\|_{L^1} + C_5 \|\nabla (\omega - \omega')\|_{L^1} \\
+ C_6 \|\omega - \omega'\|_{C^0} \cdot (\|\nabla (\omega - \omega')\|_{L^1} + 2 \|\nabla \omega\|_{L^1} + 2 \|\nabla \theta_1\|_{L^1}) \\
\leq (C \lambda + C \lambda + C_5 \lambda + C_6 \lambda (\lambda + 2 \lambda + 2 \lambda)) t^{-\frac{3}{2} + \kappa}.
\]
Thus, together with (3.17), we have verified condition (i) of Theorem 3.9.

Given (iv) and (v), the injectivity radius and the Riemann curvature of $g' = ds^2 + g_M$ satisfy $\delta(g') \geq \min(\mu t, \pi) = \mu t$ for small $t$, and $\|R(g')\|_{C^0} \leq \nu t^{-2}$. Basically, the role of the estimates on injectivity radius and the $C^0$-norm of the Riemann curvature in the proof of Theorem 3.9 is to show that there exist coordinate systems on small balls such that the metric in these coordinate systems is $C^{1,\alpha}$-close to the Euclidean metric on $\mathbb{R}^7$ for $\alpha \in (0, 1)$. Therefore, all we have to show is that the metric $g$ is $C^{1,\alpha}$-close to the Euclidean metric. Now we know that $g'$ is $C^{1,\alpha}$-close to the Euclidean metric on $\mathbb{R}^7$ by Joyce [11 Prop. 11.7.2]. But the second inequality of condition (i) and condition (iii) of the hypotheses ensure that $g$ and $g'$ are $C^2$-close, and this implies that $g$ is also $C^{1,\alpha}$-close to the Euclidean metric on $\mathbb{R}^7$, which is what we need.

Therefore Theorem 3.9 gives a torsion-free $G_2$-structure $(\tilde{\varphi}, \tilde{g})$ on $S^1 \times M$. It remains to construct a Calabi–Yau structure on $M$ from $(\tilde{\varphi}, \tilde{g})$. Denote $\partial_{s\alpha}$ by the Killing vector w.r.t. $g$ such that $\iota (\partial_{s\alpha}) ds = 1$. Then $\partial_{s\alpha}$ is also a Killing vector w.r.t. $\tilde{g}$ since $(\tilde{\varphi}, \tilde{g})$ is $S^1$-invariant.
It follows from a general fact about Killing vectors on a torsion-free, compact $G_2$-manifold that $\nabla_\vartheta \frac{\partial}{\partial s} = 0$, and hence $\left| \frac{\partial}{\partial s} \right|_g$ equals to some constant $c$. Define a 1-form $d\tilde{s}$ on $S^1 \times M$ by $(d\tilde{s})_a = \frac{1}{c} \tilde{g}_{ab} \left( \frac{\partial}{\partial s} \right)^b$. Then $d\tilde{s}$ is closed, of unit length w.r.t. $\tilde{g}$, and $\iota \left( \frac{\partial}{\partial s} \right) d\tilde{s} = c$, and we may write $d\tilde{s} = c \, ds + \alpha'$ for some closed 1-form $\alpha'$ on $S^1 \times M$ with $\iota \left( \frac{\partial}{\partial s} \right) \alpha' = 0$. The 1-form $\alpha'$ is thus the pullback of some closed 1-form $\alpha$ on $M$ via the projection map $\pi : S^1 \times M \to M$, i.e. $\alpha' = \pi^* (\alpha)$. Since by assumption $H^1(M, \mathbb{R}) = 0$, $\alpha$, and hence $\alpha'$, is exact. Therefore we have $[d\tilde{s}] = c \, [ds]$.

Using the fact that $\frac{\partial}{\partial s}$ is a Killing vector and $\tilde{\varphi}$ is a closed 3-form, we have

$$\iota \left( \frac{\partial}{\partial s} \right) \tilde{\varphi} + \iota \left( \frac{\partial}{\partial s} \right) (\ast \tilde{\varphi}) = \mathcal{L}_\frac{\partial}{\partial s} \tilde{\varphi} = 0,$$

so $\iota \left( \frac{\partial}{\partial s} \right) \tilde{\varphi}$, and similarly $\iota \left( \frac{\partial}{\partial s} \right) (\ast \tilde{\varphi})$ are both $S^1$-invariant closed forms on $S^1 \times M$. Then we can define a closed 2-form $\tilde{\omega}$ and closed 3-forms $\tilde{\theta}_1$ and $\tilde{\theta}_2$ on $M$ by

$$\tilde{\omega}_x = \frac{1}{c} \left( \iota \left( \frac{\partial}{\partial s} \right) \tilde{\varphi} \right)_{(s,x) \mid T_x M}, \quad (\tilde{\theta}_1)_x = \tilde{\varphi}_{(s,x)} - (d\tilde{s} \wedge \tilde{\omega})_{(s,x)} |_{T_x M},$$

and

$$\tilde{\theta}_2_x = - \frac{1}{c} \left( \iota \left( \frac{\partial}{\partial s} \right) (\ast \tilde{\varphi}) \right)_{(s,x) \mid T_x M}$$

for each $x \in M$ and any $s \in S^1$. Identify $T_{(s,x)} (S^1 \times M)$ with $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ such that $T_x M$ is identified with $\mathbb{C}^3$, $(\tilde{\varphi}, \tilde{\omega})$ with the flat $G_2$-structure $(\varphi_0, g_0)$, $\frac{1}{c} \left( \frac{\partial}{\partial s} \right)$ with $\frac{\partial}{\partial x_1}$, and $d\tilde{s}$ with $dx_1$ where $x_1$ is the coordinate on $\mathbb{R}$. Then calculation shows that at each $x \in M$, $(\tilde{J}, \tilde{\omega}, \tilde{\Omega})$ can be identified with the standard Calabi–Yau structure $(J_0, \omega_0, \Omega_0)$ on $\mathbb{C}^3$, where $\tilde{\Omega} = \tilde{\theta}_1 + i \tilde{\theta}_2$, and $\tilde{J}$ is the associated complex structure. It follows that $(\tilde{J}, \tilde{\omega}, \tilde{\Omega})$ gives a Calabi–Yau structure on $M$ with $\tilde{\varphi} = d\tilde{s} \wedge \tilde{\omega} + \tilde{\theta}_1$ and $\ast \tilde{\varphi} = \frac{1}{2} \tilde{\omega} \wedge \tilde{\omega} - d\tilde{s} \wedge \tilde{\theta}_2$ on $S^1 \times M$.

It is not hard to show $\| \tilde{\omega} - \omega \|_{C^0} \leq K t^e$ and, by making $K$ larger if necessary, $\| \tilde{\Omega} - \Omega \|_{C^0} \leq K t^e$ provided that $\| \tilde{\varphi} - \varphi \|_{C^0} \leq K t^e$, which is a consequence from Theorem 3.9. Moreover, as $\| \varphi \| = \| \tilde{\varphi} \|$ and $[d\tilde{s}] = c \, [ds]$, it follows that $[\omega] = c \, [\tilde{\omega}]$ and $[\tilde{\theta}_1] = [\tilde{\theta}_1]$. This completes the proof of Theorem 3.10.

Remarks

(1) In general we can’t guarantee $[\text{Im}(\tilde{\Omega})] = [\text{Im}(\Omega)]$ since, roughly speaking, $[\text{Im}(\tilde{\Omega})]$ is locally determined by $[\text{Im}(\Omega)]$, whereas $[\text{Im}(\Omega)]$ is free to change slightly, as long as the inequality (3.2) is satisfied. Hence $[\text{Im}(\tilde{\Omega})]$ is independent of $[\text{Re}(\Omega)]$ and it follows that $[\text{Im}(\tilde{\Omega})]$ can’t possibly be determined by $[\text{Im}(\Omega)]$.

(2) If $H^1(M, \mathbb{R}) \neq 0$, then $\alpha'$ may not be exact, and we have to modify the cohomological formula for $[\text{Re}(\tilde{\Omega})]$ to

$$[\text{Re}(\tilde{\Omega})] = [\text{Re}(\Omega)] - [\alpha] \cup [\tilde{\omega}].$$

(3) There is an alternative way of obtaining the Calabi–Yau structure on $M$ from the holonomy point of view. Since $(\tilde{\varphi}, \tilde{\omega})$ is torsion-free, $\text{Hol}(\tilde{g}) \subseteq G_2$. Moreover, $\text{Hol}(\tilde{g})$ fixes the vector $\frac{\partial}{\partial s}$ as $\nabla_\vartheta \frac{\partial}{\partial s} = 0$. It turns out that $\text{Hol}(\tilde{g})$ actually lies in $\text{SU}(3)$ and hence the torsion-free $G_2$-structure $(\tilde{\varphi}, \tilde{\omega})$ must be coming from a Calabi–Yau structure on $M$. 

\[ \square \]
4. Calabi–Yau cones, Calabi–Yau manifolds with a conical singularity and Asymptotically Conical Calabi–Yau manifolds

In this section we define Calabi–Yau cones, Calabi–Yau manifolds with a conical singularity and Asymptotically Conical Calabi–Yau manifolds. We will give some examples and provide results analogous to the usual Darboux Theorem on symplectic manifolds for the Calabi–Yau manifolds with a conical singularity and Asymptotically Conical Calabi–Yau manifolds. The conical singularities in Calabi–Yau 3-folds will be desingularized in section 5 by using the existence result obtained in section 3.

4.1. Preliminaries on Calabi–Yau cones

We will give our definition of Calabi–Yau cones and provide several examples in this section. Let us first consider the $\mathbb{C}^m$ case. Write $\mathbb{C}^m$ as $S^{2m-1} \times (0, \infty) \cup \{0\}$, a cone over the $(2m-1)$-dimensional sphere. Let $r$ be a coordinate on $(0, \infty)$. Then the standard metric $g_0$, Kähler form $\omega_0$ and holomorphic volume form $\Omega_0$ on $\mathbb{C}^m$ can be written as

$$g_0 = r^2 g_0|_{S^{2m-1}} + dr^2, \quad \omega_0 = r^2 \omega_0|_{S^{2m-1}} + r dr \wedge \alpha$$

and

$$\Omega_0 = r^m \Omega_0|_{S^{2m-1}} + r^{m-1} dr \wedge \beta,$$

where $\alpha$ is a real 1-form and $\beta$ a complex $(m-1)$-form on $S^{2m-1}$. Hence they scale as

$$g_0|_{S^{2m-1} \times \{r\}} = r^2 g_0|_{S^{2m-1}}, \quad \omega_0|_{S^{2m-1} \times \{r\}} = r^2 \omega_0|_{S^{2m-1}}$$

and

$$\Omega_0|_{S^{2m-1} \times \{r\}} = r^m \Omega_0|_{S^{2m-1}}.$$

Motivated by this standard case, we give our definition of a Calabi–Yau cone:

**Definition 4.1.** Let $\Gamma$ be a compact $(2m-1)$-dimensional smooth manifold, and let $V = \{0\} \cup V'$ where $V' = \Gamma \times (0, \infty)$. Write points on $V'$ as $(\gamma, r)$. $V$ is called a Calabi–Yau cone if $V$ is a Calabi–Yau $m$-fold with a Calabi–Yau structure $(J_V, \omega_V, \Omega_V)$ and its associated Calabi–Yau metric $g_V$ satisfying

$$g_V = r^2 g_V|_{\Gamma \times \{1\}} + dr^2, \quad \omega_V = r^2 \omega_V|_{\Gamma \times \{1\}} + r dr \wedge \alpha$$

(4.1)

and

$$\Omega_V = r^m \Omega_V|_{\Gamma \times \{1\}} + r^{m-1} dr \wedge \beta.$$

Here we identify $\Gamma$ with $\Gamma \times \{1\}$, and $\alpha$ is a real 1-form and $\beta$ a complex $(m-1)$-form on $\Gamma$.

Let $X$ be the radial vector field on $V$ such that $X_{(\gamma, r)} = r \frac{\partial}{\partial r}$ for any $(\gamma, r) \in \Gamma \times (0, \infty)$. Then $r^2 \alpha = \iota(X) \omega_V$ and $r^m \beta = \iota(X) \Omega_V$. Moreover,

$$\mathcal{L}_X \omega_V = d(\iota(X) \omega_V) \quad \text{as} \quad d\omega_V = 0 = d(r^2 \alpha)$$

$$= r^2 d\alpha + 2r dr \wedge \alpha.$$

It can be shown that $d\alpha = 2 \omega_V|_{\Gamma}$ by using $d\omega_V = 0$ and the formula for $\omega_V$ in (4.1). Therefore we have $\mathcal{L}_X \omega_V = 2 \omega_V$. In a similar way, we can show $\mathcal{L}_X \Omega_V = m \Omega_V$ and $\mathcal{L}_X g_V = 2 g_V$. It follows that $\mathcal{L}_X J_V = 0$, and hence $X$ is a holomorphic vector field. The flow of $X$ thus expands the Calabi–Yau metric $g_V$, the Kähler form $\omega_V$ and the holomorphic volume form
$\Omega_V$ exponentially and $X$ is then a Liouville vector field. In particular, the 1-form $\alpha$ defines a contact form on $\Gamma$, which makes $\Gamma$ a contact $(2m - 1)$-fold.

The tangent space $T_{(\gamma, r)} V$ decomposes as $T_{(\gamma, r)} V = T_\gamma \Gamma \oplus X_{(\gamma, r)} \mathbb{R}$ for any $(\gamma, r) \in \Gamma \times (0, \infty)$. Note that $Z := J_V X$ is a vector field on $\Gamma$, and it is complete as $\Gamma$ is compact. Now $\iota(Z) \omega_V$ is a 1-form such that $\iota(X) (\iota(Z) \omega_V) = g_V (X, X) = r^2$ and $\iota(Z) \omega_V \big|_{\Gamma \times (r)} = 0$, hence we can write $\iota(Z) \omega_V = rdr$. It follows that

$$L_Z \omega_V = d(\iota(Z) \omega_V) = d rdr = 0.$$ 

For the holomorphic volume form, we use the fact that if $\Omega$ is a holomorphic $(m, 0)$-form and $\nu$ a holomorphic vector field, then $L_{J_V \nu} \Omega = i L_{\nu} \Omega$ where $J$ is the complex structure. Now $Z = J_V X$ is a holomorphic vector field, this gives $L_Z \Omega_V = i m \Omega_V$.

Now we define a complex dilation on the Calabi–Yau cone $V$. The flow of $Z$ generates the diffeomorphism $\exp(\theta Z)$ on $\Gamma$ for each $\theta \in \mathbb{R}$. Thus for each $\theta \in \mathbb{R}$ and $t > 0$, we can define a complex dilation $\lambda$ on $V$ which is given by $\lambda(0) = 0$ and $\lambda(\gamma, r) = (\exp(\theta Z)(\gamma), tr)$.

**Lemma 4.2.** Let $\lambda: V \to V$ be the complex dilation defined above. Then $\lambda^* (g_V) = t^2 g_V$, $\lambda^* (\omega_V) = t^2 \omega_V$ and $\lambda^* (\Omega_V) = t^m e^{im \theta} \Omega_V$.

**Proof.** It follows from $L_Z \omega_V = 0$ that $\exp(\theta Z)^* (\omega_V) = \omega_V$ and hence $\lambda^* (\omega_V) = t^2 \omega_V$ by the scaling of $t$. The formula for the metric $g_V$ follows similarly. For the holomorphic $(m, 0)$-form $\Omega_V$, observe that

$$\frac{d}{d\theta} \exp(\theta Z)^* (\Omega_V) \big|_{\theta = 0} = L_Z \Omega_V = i m \Omega_V,$$

and this means $\exp(\theta Z)^* (\Omega_V) = e^{im \theta} \Omega_V$. Thus together with the scaling of $t$, we have $\lambda^* (\Omega_V) = t^m e^{im \theta} \Omega_V$. $\square$

In the situation of our standard example $\mathbb{C}^m$, the complex dilation is given by complex multiplication $\lambda: \mathbb{C}^m \to \mathbb{C}^m$ sending $z$ to $\lambda z$, where $\lambda = te^{i \theta} \in \mathbb{C}$. It is easy to see the above properties for the standard structures $g_0$, $\omega_0$ and $\Omega_0$.

**Example 4.3.** A trivial example is given by $\mathbb{C}^m$, a cone on $S^{2m-1}$. Some nontrivial examples can be constructed as follows: Let $G$ be a finite subgroup of $SU(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$, then the quotient singularity $\mathbb{C}^m / G$ is a Calabi–Yau cone. An example of this type is given by the $\mathbb{Z}_m$-action on $\mathbb{C}^m$:

$$\zeta^k \cdot (z_1, \ldots, z_m) = (\zeta^k z_1, \ldots, \zeta^k z_m)$$

where $\zeta = e^{2\pi i / m}$ and $1 \leq k \leq m$. Note that $\zeta^m = 1$, so $\mathbb{Z}_m$ is a subgroup of $SU(m)$ and acts freely on $\mathbb{C}^m \setminus \{0\}$. Then $\mathbb{C}^m / \mathbb{Z}_m$ is a Calabi–Yau cone.

**Example 4.4.** Consider the cone $V$ defined by the quadric $\sum_{j=1}^{m+1} z_j^2 = 0$ on $\mathbb{C}^{m+1}$. The singularity at the origin is known as an ordinary double point, or a node. It can be shown that $\Gamma$ is an $S^{m-1}$-bundle over $S^m$. Stenzel [11] constructed a Calabi–Yau metric on $V$, thus making it a Calabi–Yau cone. We are particularly interested in the case $m = 3$. Then
of $\Gamma$ has the topology of $S^2 \times S^3$, and hence $V$ is topologically a cone on $S^2 \times S^3$ since any $S^2$-bundle over $S^3$ is trivial. One can also describe $V$ as follows. Consider the blow-up $\mathbb{C}^4$ of $\mathbb{C}^4$ at origin. It introduces an exceptional divisor $\mathbb{CP}^3$, and the blow-up $\mathbb{V}$ of the cone $V$ inside $\mathbb{C}^4$ meets this $\mathbb{CP}^3$ at $S \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. The exceptional divisor $\mathbb{CP}^3$ corresponds to the zero section of the line bundle $L$ given by $\mathbb{C}^4 \rightarrow \mathbb{CP}^3$, and so its normal bundle is isomorphic to $L$. Hence the normal bundle $\mathcal{O}(-1, -1)$ over $\mathbb{CP}^1 \times \mathbb{CP}^1$ is isomorphic to the line bundle $\mathbb{V} \rightarrow S$. This gives us the following isomorphisms:

$$V \setminus \{0\} \cong \mathbb{V} \setminus S \cong \mathcal{O}(-1, -1) \setminus (\mathbb{CP}^1 \times \mathbb{CP}^1).$$

Example 4.5. Suppose $S$ is Kähler-Einstein with positive scalar curvature, Calabi [4] p.284-5] constructed a 1-parameter family of Calabi–Yau metrics $g_t$ for $t \geq 0$ on the canonical line bundle $K_S$. When $t > 0$, $g_t$ is a nonsingular complete metric on $K_S$ and when $t = 0$, $g_0$ degenerates on $S$ and thus gives a cone metric on $K_S \setminus S$, which then makes $K_S \setminus S$ a Calabi–Yau cone with $S$ “collapsed” to the vertex of the cone.

(i) One of the standard examples of Kähler-Einstein manifolds with positive scalar curvature is given by the complex surface $S \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. Calabi’s construction thus applies to it and yields a Calabi–Yau metric on $K_S = \mathcal{O}(-2, -2) \rightarrow S$. Note that $\mathcal{O}(-1, -1)$ is a double cover of $\mathcal{O}(-2, -2)$ away from the zero section $S$, so we have the following relation between the cone $K_S \setminus S$ and the cone $V$ described in Example 4.4:

$$K_S \setminus S \cong (V \setminus \{0\})/\mathbb{Z}_2.$$

(ii) Boyer, Galicki and Kollár [2] constructed Kähler-Einstein metrics on some compact orbifolds, particularly on orbifolds of the form given by the quotient $(L \setminus \{0\})/\mathbb{C}^*$, where $L = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : \sum_{j=1}^m z_j^{a_j} = 0\}$ for some positive integers $a_j$ satisfying certain conditions. The set $L$ is a hypersurface in $\mathbb{C}^m$, and $\mathbb{C}^*$ acts naturally on $\mathbb{C}^m$ by $\lambda : (z_1, \ldots, z_m) \mapsto (\lambda^{a_1} z_1, \ldots, \lambda^{a_m} z_m)$. It follows from Calabi’s construction, in the category of orbifolds, that $L \setminus \{0\}$ is a Calabi–Yau cone.

4.2. Calabi–Yau $m$-folds with a conical singularity

We define Calabi–Yau $m$-folds with a conical singularity in this section. Moreover, we show that there exist coordinate systems that can trivialize the symplectic forms of Calabi–Yau $m$-folds with a conical singularity.

Definition 4.6. Let $(M_0, J_0, \omega_0, \Omega_0)$ be a singular Calabi–Yau $m$-fold with a singularity at $x \in M_0$, and no other singularities. We say that $M_0$ is a Calabi–Yau $m$-fold with a conical singularity at $x$ with rate $\nu > 0$ modelled on a Calabi–Yau cone $(V, J_V, \omega_V, \Omega_V)$ if there exist a small $\epsilon > 0$, a small open neighbourhood $S$ of $x$ in $M_0$, and a diffeomorphism
\( \Phi : \Gamma \times (0, \epsilon) \to S \setminus \{x\} \) such that

\[
\begin{align*}
|\nabla^k (\Phi^*(\omega_0) - \omega_V)|_{g_V} &= O(r^{\nu-k}), \\
|\nabla^k (\Phi^*(\Omega_0) - \Omega_V)|_{g_V} &= O(r^{\nu-k}) \quad \text{as } r \to 0 \text{ and for all } k \geq 0.
\end{align*}
\]

Here \( \nabla \) and \( | \cdot | \) are computed using the cone metric \( g_V \).

Note that the asymptotic conditions on \( g_0 \) and \( J_0 \) follow from those on \( \omega_0 \) and \( \Omega_0 \), namely,

\[
\begin{align*}
|\nabla^k (\Phi^*(g_0) - g_V)|_{g_V} &= O(r^{\nu-k}), \\
|\nabla^k (\Phi^*(J_0) - J_V)|_{g_V} &= O(r^{\nu-k}) \quad \text{as } r \to 0 \text{ and for all } k \geq 0,
\end{align*}
\]

and so it is enough to just assume asymptotic conditions on \( \omega_0 \) and \( \Omega_0 \).

We will usually assume that \( M_0 \) is compact. The point of the definition is that \( M_0 \) has one end modelled on \( \Gamma \times (0, \epsilon) \) and as \( r \to 0 \), all the structures \( g_0, J_0, \omega_0 \) and \( \Omega_0 \) on \( M_0 \) converge to the cone structures \( g_V, J_V, \omega_V \) and \( \Omega_V \) with rate \( \nu \) and with all their derivatives.

The 2-forms \( \Phi^*(\omega_0) \) and \( \omega_V \) are closed on \( \Gamma \times (0, \epsilon) \) and so \( \Phi^*(\omega_0) - \omega_V \) represents a cohomology class in \( H^2(\Gamma \times (0, \epsilon), \mathbb{R}) \cong H^2(\Gamma, \mathbb{R}) \). Similarly, \( \Phi^*(\Omega_0) - \Omega_V \) represents a cohomology class in \( H^m(\Gamma \times (0, \epsilon), \mathbb{C}) \cong H^m(\Gamma, \mathbb{C}) \). It turns out that in the conical singularity case, these two classes \( [\Phi^*(\omega_0) - \omega_V] \) and \( [\Phi^*(\Omega_0) - \Omega_V] \) are automatically zero.

**Lemma 4.7.** Let \( (M_0, J_0, \omega_0, \Omega_0) \) be a compact Calabi–Yau \( m \)-fold with a conical singularity at \( x \) with rate \( \nu > 0 \) modelled on a Calabi–Yau cone \( (V, J_V, \omega_V, \Omega_V) \). Then \( [\Phi^*(\omega_0) - \omega_V] = 0 \) in \( H^2(\Gamma, \mathbb{R}) \) and \( [\Phi^*(\Omega_0) - \Omega_V] = 0 \) in \( H^m(\Gamma, \mathbb{C}) \).

**Proof.** Suppose \( \Sigma \) is a 2-cycle in \( \Gamma \). Then

\[
\begin{align*}
\int_{\Sigma \times \{r\}} (\Phi^*(\omega_0) - \omega_V) &= \text{vol}(\Sigma) \cdot O(r^{\nu}) \quad \text{by (4.2)} \\
&= O(r^{\nu+2})
\end{align*}
\]

Hence \( \nu > 0 \) implies the above integral approaches 0 as \( r \to 0 \). Then

\[
[\Phi^*(\omega_0) - \omega_V] \cdot [\Sigma] = 0
\]

for any 2-cycle \( \Sigma \), and hence \( [\Phi^*(\omega_0) - \omega_V] = 0 \) in \( H^2(\Gamma \times (0, \epsilon), \mathbb{R}) \cong H^2(\Gamma, \mathbb{R}) \). The case for \( [\Phi^*(\Omega_0) - \Omega_V] \) follows similarly by considering 3-cycles in \( \Gamma \).

One may ask whether the symplectic form \( \omega_0 \) on \( S \) near \( x \) in \( M_0 \) can actually be symplectomorphic to the cone form \( \omega_V \) near the origin, rather than just having an asymptotic relation in (4.2). Our next result shows that this is indeed the case for Calabi–Yau \( m \)-folds with conical singularities, and can be regarded as an analogue of the usual Darboux theorem on symplectic manifolds.

**Theorem 4.8.** Let \( (M_0, J_0, \omega_0, \Omega_0) \) be a compact Calabi–Yau \( m \)-fold with a conical singularity at \( x \) with rate \( \nu > 0 \) modelled on a Calabi–Yau cone \( (V, J_V, \omega_V, \Omega_V) \). Then there exist an \( \epsilon' > 0 \), an open neighbourhood \( S' \) of \( x \) and a diffeomorphism \( \Phi : \Gamma \times (0, \epsilon') \to S' \setminus \{x\} \) such that \( (\Phi')^*(\omega_0) = \omega_V \) and \( |\nabla^k ((\Phi')^*(\Omega_0) - \Omega_V)|_{g_V} = O(r^{\nu-k}) \) for all \( k \geq 0 \).
For each fixed \( r \), write \( \omega \) as \( (4.7) \) and \( \omega^t \) as \( (4.8) \), \( \eta \) is an exact 2-form such that \( (4.11) \). Make \( \epsilon \) smaller if necessary so that \( \omega^t \) is symplectic for all \( t \in [0,1] \). Then
\[
\frac{d}{dt} \omega^t = \omega - \omega_V.
\]

By Lemma 4.7, \(|\omega - \omega_V| = 0\) in \( H^2(\Gamma, \mathbb{R}) \) and hence \( \omega - \omega_V \) is exact. Suppose \( \eta = \omega - \omega_V \), and write \( \eta \) as \( \eta_0(\gamma, r) + \eta_1(\gamma, r) \wedge dr \), where \( \eta_0(\gamma, r) \in \Lambda^2 T_{r}^* \Gamma \) and \( \eta_1(\gamma, r) \in \Lambda^1 T_{r}^* \Gamma \). Then \( \eta \) is an exact 2-form such that
\[
\frac{d}{dt} \omega^t = \eta.
\]

Now we want to choose a 1-form \( \sigma \) such that \( \eta = d\sigma \). Define
\[
\sigma(\gamma, r) = -\int_0^r \eta_1(\gamma, s)ds.
\]

By the fact that \(|\eta|_{g_V} = |\omega - \omega_V|_{g_V} = O(r^\nu)\) as \( r \to 0 \), we have \(|\eta_0|_{g_V} = |\eta_1|_{g_V} = O(r^{\nu})\) as \( r \to 0 \) and using \( \nu > 0 \), we deduce that \( \sigma \) is well-defined and it satisfies \(|\sigma|_{g_V} = O(r^{\nu+1})\) as \( r \to 0 \). Since \( d\eta = 0 \), we have \( d_T \eta_0 = 0 \) and
\[
\frac{\partial \eta_0}{\partial r} + d_T \eta_1 = 0,
\]
where \( d_T \) denotes the exterior differentiation in the \( \gamma \) direction. Therefore,
\[
\begin{align*}
  d\sigma(\gamma, r) &= -\int_0^r d_T(\eta_1(\gamma, s))ds - dr \wedge \frac{\partial}{\partial r} \left( \int_0^r \eta_1(\gamma, s)ds \right) \\
  &= \int_0^r \frac{\partial}{\partial s}(\gamma, s))ds - dr \wedge \eta_1(\gamma, r) \quad \text{by (4.9).}
\end{align*}
\]

For each fixed \( r \), \( \eta_0^r(\gamma) := \eta_0(\gamma, r) \) is a 2-form on \( \Gamma \cong \Gamma \times \{1\} \), and so \(|\eta_0^r(\gamma)|_{g_V|\Gamma \times \{1\}} = r^2|\eta_0(\gamma, r)|_{g_V} = O(r^{\nu+2})\) since \(|\eta_0|_{g_V} = O(r^{\nu})\). It follows that \(|\eta_0^r(\gamma)|_{g_V|\Gamma \times \{1\}} \to 0\) as \( r \to 0 \) and hence \( \eta_0(\gamma, r) \to 0 \) as \( r \to 0 \). We then deduce that \( d\sigma(\gamma, r) = \eta_0(\gamma, r) + \eta_1(\gamma, r) \wedge dr = \eta(\gamma, r) \). Therefore, we obtain a 1-form \( \sigma \) on \( \Gamma \times (0, \epsilon) \) such that
\[
\frac{d}{dt} \omega^t = d\sigma.
\]

Also, \(|\nabla^k \sigma|_{g_V} = |\nabla^{k-1} d\sigma|_{g_V} = |\nabla^{k-1} \eta|_{g_V} = O(r^{\nu-k+1})\) as \( r \to 0 \) and for all \( k \geq 1 \). Together with \(|\sigma|_{g_V} = O(r^{\nu+1})\), we get a 1-form \( \sigma \) satisfying
\[
|\nabla^k \sigma|_{g_V} = O(r^{\nu-k+1}) \quad \text{as } r \to 0 \text{ and for all } k \geq 0.
\]

Now define a family of vector fields \( X_t \) via
\[
\sigma + \epsilon(X_t) \omega^t = 0,
\]
The flow of this family of vector fields yields a family of diffeomorphisms $\psi_t$ on $V$ such that $\psi_t^*(\omega^t) = \omega^0$. In particular, we have constructed $\psi_1 : \Gamma \times (0, \epsilon') \to \Gamma \times (0, \epsilon)$ for some $\epsilon' \in (0, \epsilon)$ which is a diffeomorphism with its image satisfying

$$\psi_1^*(\omega) = \omega_1 = \omega_\Gamma.$$  

Write $\Phi' = \Phi \circ \psi_1$ and $S' = \Phi \circ \psi_1(\Gamma \times (0, \epsilon'))$, then $\Phi' : \Gamma \times (0, \epsilon') \to S'$ is a diffeomorphism such that

$$(\Phi')^*(\omega_0) = \psi_1^*(\Phi^*(\omega_0)) = \psi_1^*(\omega) = \omega_\Gamma,$$

as required.

From (4.11) we have $|\nabla^k X_t|_{g_V} = O(r^{-k+1})$ for all $k \geq 0$. Roughly speaking, $\psi_1 = \text{Id} + \int_0^1 X_t ds$ to the first order, and hence $|\nabla^k \psi_t|_{g_V} = O(r^{-k+1})$ for all $k \geq 0$. It doesn’t exactly make sense as $\psi_1$ and Id map to different points on $V$. But we could express them in terms of local coordinates $(x_1, \ldots, x_{2m-1}, r)$ on $\Gamma \times (0, \epsilon')$. Let $\psi_1^t(x_1, \ldots, x_{2m-1}, r)$ be the $j$-th component function of $\psi_1$ for $j = 1, \ldots, 2m$. Then $\partial^k(\psi_1^t(x_1, \ldots, x_{2m-1}, r) - x_j) = O(r^{-k+1})$ for $j = 1, \ldots, 2m - 1$ and for all $k \geq 0$, and $\partial^k(\psi_1^2(x_1, \ldots, x_{2m-1}, r) - r) = O(r^{-k+1})$ for all $k \geq 0$ where $\partial$ denotes the usual partial differentiation at the point $(x_1, \ldots, x_{2m-1}, r)$. It follows that $\partial^k(\psi_1 - \text{Id})(x_1, \ldots, x_{2m-1}, r) = O(r^{-k+1})$ for all $k \geq 0$. Consequently we have

$$\partial^k(\psi_1 - \text{Id})^*(\Phi^*(\Omega_0))_{g_V} = O(r^{-k})$$

at the point $(x_1, \ldots, x_{2m-1}, r)$. As a result, we have at each point on $\Gamma \times (0, \epsilon')$

$$|\nabla^k((\Phi')^*(\Omega_0) - \Omega_\Gamma)|_{g_V} = |\nabla^k(\psi_1^*(\Phi^*(\Omega_0)) - \Omega_\Gamma)|_{g_V} \\
\quad \leq |\partial^k(\psi_1^*(\Phi^*(\Omega_0)) - \Phi^*(\Omega_0))|_{g_V} \\
\quad + |\partial^k(\Phi^*(\Omega_0) - \Omega_\Gamma)|_{g_V} \\
= O(r^{-k}) + O(r^{-k}) = O(r^{-k})$$

for all $k \geq 0$. This completes the proof.

\[\square\]

### 4.3. Asymptotically Conical Calabi–Yau m-folds

In the last part we study Asymptotically Conical (AC) Calabi–Yau m-folds. We shall provide some examples and give an analogue of Theorem 4.8 for AC Calabi–Yau m-folds.

**Definition 4.9.** Let $(Y, J_Y, \omega_Y, \Omega_Y)$ be a complete, nonsingular Calabi–Yau m-fold. Then $Y$ is an Asymptotically Conical (AC) Calabi–Yau m-fold with rate $\lambda < 0$ modelled on a Calabi–Yau cone $(V, J_V, \omega_V, \Omega_V)$ if there exist a compact subset $K \subset Y$, and a diffeomorphism $\Upsilon : \Gamma \times (R, \infty) \to Y \setminus K$ for some $R > 0$ such that

\begin{align}
|\nabla^k(\Upsilon^*(\omega_Y) - \omega_Y)|_{g_Y} & = O(r^{\lambda-k}), \quad \text{and} \\
|\nabla^k(\Upsilon^*(\Omega_Y) - \Omega_Y)|_{g_Y} & = O(r^{\lambda-k}) \quad \text{as} \quad r \to \infty \quad \text{and for all} \quad k \geq 0.
\end{align}

Here $\nabla$ and $|\cdot|$ are computed using the cone metric $g_Y$. 

Similar asymptotic conditions on $g_Y$ and $J_Y$ can be deduced from (4.12) and (4.13). Unlike the conical singularity case, $[\Omega^*(\omega_Y) - \omega_Y]$ and $[\Omega^*(\Omega_Y) - \Omega_Y]$ need not be zero cohomology classes. Here are some conditions:

**Lemma 4.10.** Let $(V, J_Y, \omega_Y, \Omega_Y)$ be an AC Calabi–Yau m-fold with rate $\lambda < 0$ modelled on a Calabi–Yau cone $(V, J_Y, \omega_Y, \Omega_Y)$. If $\lambda < -2$ or $H^2(\Gamma, \mathbb{R}) = 0$, then $[\Omega^*(\omega_Y) - \omega_Y] = 0$. If $\lambda < -m$ or $H^m(\Gamma, \mathbb{C}) = 0$, then $[\Omega^*(\Omega_Y) - \Omega_Y] = 0$.

The proof of it is similar to that of Lemma 4.7, except we now have $O(r^{\lambda+2})$ for the integral of the difference of symplectic forms and $O(r^{\lambda+m})$ for the holomorphic $(m,0)$-forms. Hence if $\lambda < -2$, the integral approaches 0 as $r \to \infty$, which implies $[\Omega^*(\omega_Y) - \omega_Y] = 0$. The same argument applies to the holomorphic $(m,0)$-forms.

We shall normally consider the case $\lambda < -2$, so that $[\Omega^*(\omega_Y) - \omega_Y]$ is always exact. Moreover, when $\lambda < -2$, the proof for the analogue of Darboux Theorem works similarly as that for the conical singularities case. It is not clear whether the theorem holds for $\lambda \geq -2$ and $[\Omega^*(\omega_Y) - \omega_Y] = 0$ or not.

**Theorem 4.11.** Let $(V, J_Y, \omega_Y, \Omega_Y)$ be an AC Calabi–Yau m-fold with rate $\lambda < -2$ modelled on a Calabi–Yau cone $(V, J_Y, \omega_Y, \Omega_Y)$. Then there exist a $R' > 0$ and a diffeomorphism $\Upsilon': \Gamma \times (R', \infty) \to V \setminus K$ such that $(\Upsilon')^*(\omega_Y) = \omega_Y$ and $|\nabla^k((\Upsilon')^*(\Omega_Y) - \Omega_Y)|_{g_Y} = O(r^{\lambda-k})$ for all $k \geq 0$.

One can prove it in the same way as the proof of Theorem 4.8. The condition on the rate $\lambda$ is essential for this proof to work. Since $|\eta_1|_{g_Y} = O(r^{\lambda})$, we need $\lambda < -2$ to construct the 1-form $\sigma$. Moreover, in proving Theorem 4.8, we encountered the norm of $\eta_0$: $|\eta_0(\gamma)|_{g_Y}|_{\Gamma \setminus \{1\}} = r^2|\eta_0(\gamma, r)|_{g_Y}$, which is equal to $O(r^{\lambda+2})$ in this case. Therefore we need $\lambda < -2$ in order to have $\eta_0 \to 0$ as $r \to \infty$.

**Example 4.12.** Let $G$ be a finite subgroup of $SU(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$, and $(X, \pi)$ a crepant resolution of the Calabi–Yau cone $V = \mathbb{C}^m / G$ given in Example 4.3. Then in each Kähler class of ALE Kähler metrics on $X$ there is a unique ALE Ricci-flat Kähler metric (see Joyce [4], Chapter 8) and $X$ is then an AC Calabi–Yau m-fold asymptotic to the cone $\mathbb{C}^m / G$. In this case, it follows from [4, Thm. 8.2.3] that the rate $\lambda = -2m$.

If we take $G = \mathbb{Z}_m$ acting on $\mathbb{C}^m$ as in Example 4.3, then a crepant resolution is given by the blow-up of $\mathbb{C}^m / \mathbb{Z}_m$ at 0, which is also the total space of the canonical line bundle over $\mathbb{CP}^{m-1}$. An explicit ALE Ricci-flat Kähler metric is given in [4] p.284-5 and also in [7] Example 8.2.5).

**Example 4.13.** Consider the Calabi–Yau cone $V = \{ z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \}$ described in Example 4.4. One can construct two kinds of AC Calabi–Yau manifolds. The first one is called the small resolution of $V$, given by

$$\tilde{V} = \{((z_1, \ldots, z_4), [w_1, w_2]) \in \mathbb{C}^4 \times \mathbb{CP}^1 : z_1w_2 = z_4w_1, \ z_3w_2 = z_2w_1 \}.$$
It is essentially isomorphic to the normal bundle \(\mathcal{O}(-1) \oplus \mathcal{O}(-1)\) over \(\mathbb{CP}^1\) with fibre \(\mathbb{C}^2\), and is also isomorphic to \(V\) away from the origin where it is replaced by the whole \(\mathbb{CP}^1\). Note that one can obtain a second small resolution by swapping \(z_3\) and \(z_4\) in \(\tilde{V}\). Candelas et al \[5, p.258\] constructed Calabi–Yau metrics on \(\tilde{V}\), and it is an AC Calabi–Yau 3-fold with rate \(-2\).

The other is known as the \textit{deformation} or \textit{smoothing}, where \(V\) is deformed to \(Q_{\epsilon} = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = \epsilon^2\}\) with \(\epsilon\) a nonzero constant. This has the effect of replacing the node by an \(S^3\). An important fact is that there is a symplectomorphism which identifies the standard symplectic form on \(\mathbb{C}^4\) restricted to \(Q_{\epsilon}\) and the canonical symplectic form on the cotangent bundle \(T^*S^3\) of \(S^3\). Stenzel \[11, p.161\] constructed a Calabi–Yau metric on \(T^*S^3\), which makes \(Q_{\epsilon}\) into an AC Calabi–Yau 3-fold with rate \(\lambda = -3\).

**Example 4.14.** Calabi \[4, p.284-5\] constructed a 1-parameter family of AC Calabi–Yau metrics on the canonical bundle \(K_S\) of any Kähler-Einstein \((m-1)\)-fold \(S\) with positive scalar curvature, so that \(K_S\) is an AC Calabi–Yau \(m\)-fold modelled on the Calabi–Yau cone \(K_S \setminus S\) with rate \(\lambda = -2m\).

For the case in Example 4.5 (i), the \(\mathcal{O}(-2,-2)\)-bundle is AC Calabi–Yau asymptotic to the cone \(\mathcal{O}(-2,-2) \setminus (\mathbb{CP}^1 \times \mathbb{CP}^1)\) with rate \(-6\).

Note that if we take \(S = \mathbb{CP}^{m-1}\), then we recover the case in Example 4.12 with \(G = \mathbb{Z}_m\).

### 5. Calabi–Yau desingularizations

This section studies desingularizations of a compact Calabi–Yau 3-fold \(M_0\) with a conical singularity using an AC Calabi–Yau 3-fold \(Y\) with rate \(\lambda\). We shall only treat the simplest case here, in which \(\lambda < -3\) so that \(Y^*(t^3 \Omega_Y) - \Omega_Y\) is exact by Lemma 4.10. We explicitly construct a 1-parameter family of diffeomorphic, nonsingular compact 6-folds \(M_t\) for small \(t\) in §5.1. Then in §5.2 we construct a real closed 2-form \(\omega_t\) and a complex closed 3-form \(\Omega_t\) on \(M_t\) and show that they give nearly Calabi–Yau structures on \(M_t\) for small enough \(t\). Section 5.3 contains the main result of this paper, in which we show that the nearly Calabi–Yau structure \((\omega_t, \Omega_t)\) on \(M_t\) can be deformed to a genuine Calabi–Yau structure \((\tilde{\omega}_t, \tilde{\Omega}_t)\) for small \(t\) by applying Theorem 3.10. Finally in §5.4, we apply our result to some examples studied before. We shall also discuss the case when \(\lambda = -3\).

#### 5.1. Construction of \(M_t\)

Let \((M_0, J_0, \omega_0, \Omega_0)\) be a compact Calabi–Yau 3-fold with a conical singularity at \(x\) with rate \(\nu\) modelled on a Calabi–Yau cone \((V, J_V, \omega_V, \Omega_V)\). By Theorem 4.8, there exists an \(\epsilon > 0\), a small open neighbourhood \(S\) of \(x\) in \(M_0\) and a diffeomorphism \(\Phi : \Gamma \times (0, \epsilon) \to S \setminus \{x\}\) such that \(\Phi^*(\omega_0) = \omega_V\).

Let \((Y, J_Y, \omega_Y, \Omega_Y)\) be an AC Calabi–Yau 3-fold with rate \(\lambda < -3\) modelled on the same Calabi–Yau cone \(V\). Theorem 4.11 shows that there is a diffeomorphism \(\Upsilon : \Gamma \times (R, \infty) \to \)
Our goal is to desingularize $(M \setminus K)$ for some $R > 0$ such that
\[ \Upsilon^*(\omega_Y) = \omega_V \quad \text{and} \quad |\nabla^k(\Upsilon^*(\Omega_Y) - \Omega_V)|_{g_V} = O(r^{\lambda-k}) \]
as $r \to \infty$ for all $k \geq 0$. We then apply a homothety to $Y$ such that
\[ (Y, J_Y, t^2 \omega_Y, t^3 \Omega_Y) \to (Y, J_Y, t^2 \omega_Y, t^3 \Omega_Y). \]
Then $(Y, J_Y, t^2 \omega_Y, t^3 \Omega_Y)$ is also an AC Calabi–Yau 3-fold, with the diffeomorphism $\Upsilon_t : \Gamma \times (tR, \infty) \to Y \setminus K$ given by
\[ \Upsilon_t(\gamma, r) = \Upsilon(\gamma, t^{-1}r). \]
Our goal is to desingularize $(M_0, J_0, \omega_0, \Omega_0)$ by gluing $(Y, J_Y, t^2 \omega_Y, t^3 \Omega_Y)$ at $x$ to produce a family of compact nonsingular Calabi–Yau 3-folds.

Fix $\alpha \in (0, 1)$ and let $t > 0$ be small enough that $tR < t^\alpha < 2t^\alpha < \epsilon$. Define
\[ P_t = K \cup \Upsilon_t(\Gamma \times (tR, 2t^\alpha)) \subset Y \quad \text{and} \quad Q_t = (M_0 \setminus S) \cup \Phi(\Gamma \times (t^\alpha, \epsilon)) \subset M_0. \]
The diffeomorphism $\Phi \circ \Upsilon_t^{-1}$ identifies $\Upsilon_t(\Gamma \times (t^\alpha, 2t^\alpha)) \subset P_t$ and $\Phi(\Gamma \times (t^\alpha, 2t^\alpha)) \subset Q_t$, and we define the intersection $P_t \cap Q_t$ to be the region $\Upsilon_t(\Gamma \times (t^\alpha, 2t^\alpha)) \cong \Phi(\Gamma \times (t^\alpha, 2t^\alpha)) \cong \Gamma \times (t^\alpha, 2t^\alpha)$. Define $M_t$ to be the quotient space of the union $P_t \cup Q_t$ under the equivalence relation identifying the two annuli $\Upsilon_t(\Gamma \times (t^\alpha, 2t^\alpha))$ and $\Phi(\Gamma \times (t^\alpha, 2t^\alpha))$. Then $M_t$ is a smooth nonsingular compact 6-fold for each $t$.

### 5.2. Nearly Calabi–Yau structures $(\omega_t, \Omega_t)$: the case $\lambda < -3$

In this section we construct on $M_t$ a real closed 2-form $\omega_t$ and a complex closed 3-form $\Omega_t$, and show they together give nearly Calabi–Yau structures on $M_t$ for small enough $t$. Define
\[ \omega_t = \begin{cases} \omega_0 & \text{on } Q_t, \\ t^2 \omega_Y & \text{on } P_t. \end{cases} \]
This is well-defined as $\Phi^*(\omega_0) = \omega_V = \Upsilon_t^*(t^2 \omega_Y)$ on the intersection $P_t \cap Q_t$ by Theorem 4.8 and 4.11. Thus $\omega_t$ gives a symplectic form on $M_t$.

Let $F : \mathbb{R} \to [0, 1]$ be a smooth, increasing function with $F(s) = 0$ for $s \leq 1$ and $F(s) = 1$ for $s \geq 2$. Then for $r \in (tR, \epsilon)$, $F(t^{-\alpha}r) = 0$ for $tR < r \leq t^\alpha$ and $F(t^{-\alpha}r) = 1$ for $2t^\alpha \leq r < \epsilon$. We now define a complex 3-form on $M_t$. From (4.3), we have $|\Phi^*(\Omega_0) - \Omega_V|_{g_V} = O(r^\nu)$. As $\nu > 0$, it follows that $\Phi^*(\Omega_0) - \Omega_V$ is exact, and we can write
\begin{equation}
\Phi^*(\Omega_0) = \Omega_V + dA 
\end{equation}
for some complex 2-form $A(\gamma, r)$ on $\Gamma \times (0, \epsilon)$ satisfying
\begin{equation}
|\nabla^k A(\gamma, r)|_{g_V} = O(r^{\nu+1-k}) \quad \text{as } r \to 0 \quad \text{for all } k \geq 0.
\end{equation}
The case $k = 0$ follows by defining $A$ by integration as in Theorem 4.8. Similarly, as we have assumed $\lambda < -3$ to simplify the problem, the 3-form $\Upsilon^*(\Omega_V) - \Omega_V$ is exact by Lemma 4.10 and we can write
\[ \Upsilon^*(\Omega_Y) = \Omega_V + dB \]
for some complex 2-form \( B(\gamma, r) \) on \( \Gamma \times (R, \infty) \) satisfying
\[
|\nabla^k B(\gamma, r)|_{g_r} = O(r^{\lambda+1-k}) \quad \text{as } r \to \infty \text{ and for all } k \geq 0.
\]

Then we apply a homothety to \( Y \) and rescale the forms to get \( B(\gamma, t^{-1}r) \) on \( \Gamma \times (tR, \infty) \) such that
\[
(5.3) \quad \nabla^k (t^3 \Omega_Y) = \Omega_Y + t^3 dB(\gamma, t^{-1}r)
\]
and
\[
(5.4) \quad |\nabla^k B(\gamma, t^{-1}r)|_{g_r} = O(t^{-\lambda-3} r^{\lambda+1-k}) \quad \text{for } r > tR \text{ and for all } k \geq 0.
\]

Define a smooth, complex closed 3-form \( \Omega_t \) on \( M_t \) by
\[
(5.5) \quad \Omega_t = \begin{cases} 
\Omega_0 & \text{on } Q_t \setminus (P_t \cap Q_t), \\
\Omega_Y + d[\nabla t^{-\alpha} A(\gamma, r) + t^{3}(1 - F(t^{-\alpha} r))B(\gamma, t^{-1}r)] & \text{on } P_t \cap Q_t, \\
\lambda t^3 \Omega_Y & \text{on } P_t \setminus (P_t \cap Q_t).
\end{cases}
\]

Recall that when \( 2t^\alpha \leq r < \epsilon \) we have \( F(t^{-\alpha} r) = 1 \) so that \( \Omega_{t} = \Phi^*(\Omega_0) \) by (5.1), and when \( tR < r \leq t^\alpha \) we have \( F(t^{-\alpha} r) = 0 \), so that \( \Omega_{t} = \nabla^k t^3 \Omega_Y \) by (5.3). Therefore, \( \Omega_{t} \) interpolates between \( \Phi^*(\Omega_0) \) near \( r = \epsilon \) and \( \nabla^k t^3 \Omega_Y \) near \( r = tR \).

Proposition 5.1. Let \( M_t, \omega_t \) and \( \Omega_t \) be defined as above. Then \( (\omega_t, \Omega_t) \) gives a nearly Calabi–Yau structure on \( M_t \) for sufficiently small \( t \).

Proof. We only have to prove the statement on \( P_t \cap Q_t \), as \( (M_t, \omega_t, \Omega_t) \) is Calabi–Yau on \( P_t \setminus P_t \cap Q_t \) and on \( Q_t \setminus P_t \cap Q_t \), and hence is nearly Calabi-Yau. We prove it by applying Proposition 3.2, that is, we show on \( P_t \cap Q_t \) that \( (\omega_t, \Omega_t) \) is sufficiently close to the genuine Calabi–Yau structure \( (\omega_Y, \Omega_Y) \) coming from the Calabi–Yau cone \( V \) for small \( t \). We choose to compare with \( (\omega_Y, \Omega_Y) \) rather than either of the Calabi–Yau structures \( (\omega_0, \Omega_0) \) and \( (\omega, \Omega) \) on \( P_t \cap Q_t \) since we have already got bounds on norms for various forms w.r.t. the cone metric \( g_Y \). Now \( \omega_t = \omega_V \) on \( P_t \cap Q_t \), while
\[
\Omega_t - \Omega_Y = d[F(t^{-\alpha} r) A(\gamma, r) + t^3(1 - F(t^{-\alpha} r))B(\gamma, t^{-1}r)] \quad \text{on } P_t \cap Q_t
\]
by (5.5). Calculation shows that
\[
(5.6) \quad |\Omega_t - \Omega_Y|_{g_Y} = O(t^{-\lambda(1-\alpha)}) + O(t^{3\alpha}) \quad \text{for } r \in (t^\alpha, 2t^\alpha),
\]
and hence \( |\Omega_t - \Omega_Y|_{g_Y} \leq C_0 t^\gamma \) where \( C_0 > 0 \) is some constant and \( \gamma = \min(-\lambda(1-\alpha), 3\alpha) \).

Hence Proposition 3.2 applies with \( \epsilon = C_0 t^\gamma \) if \( t \) is small enough such that \( C_0 t^\gamma \leq \epsilon_1 \), and so \( (\omega_t, \Omega_t) \) gives a nearly Calabi–Yau structure on \( P_t \cap Q_t \). This completes the proof. \( \square \)

Therefore we can associate an almost complex structure \( J_t \) and a real 3-form \( \theta'_{2,1} \), such that \( \Omega'_t := \text{Re}(\Omega_t) + i\theta'_{2,1} \) is a \((3,0)\)-form w.r.t. \( J_t \). Moreover, we have the 2-form \( \omega'_t \), which is the rescaled \((1,1)\)-part of \( \omega_t \) w.r.t. \( J_t \), and the associated metric \( g_t \) on \( M_t \). Following similar arguments to Proposition 3.2, we conclude that \( |g_t - g_Y|_{g_Y} = O(t^{-\lambda(1-\alpha)}) + O(t^{3\alpha}) = \ldots \)
Furthermore, $|g_t^{-1} - g_V^{-1}|_{g_V}$.

5.3. The main result

We are now ready to prove our main result on desingularization of compact Calabi–Yau 3-folds $M_0$ with a conical singularity in the simplest case when $\lambda < -3$. The proof of it uses Theorem 3.10.

**Theorem 5.2.** Suppose $(M_0, J_0, \omega_0, \Omega_0)$ is a compact Calabi–Yau 3-fold with a conical singularity at $x$ with rate $\nu > 0$ modeled on a Calabi–Yau cone $(V, J_V, \omega_V, \Omega_V)$. Let $(Y, J_Y, \omega_Y, \Omega_Y)$ be an AC Calabi–Yau 3-fold with rate $\lambda < -3$ modeled on the same Calabi–Yau cone $V$. Define a family $(M_t, \omega_t, \Omega_t)$ of nonsingular compact nearly Calabi–Yau 3-folds, with Calabi–Yau metrics $g_t$, as in §5.1 and §5.2.

Then $M_t$ admits a Calabi–Yau structure $(\tilde{J}_t, \tilde{\omega}_t, \tilde{\Omega}_t)$ such that $\Vert \tilde{\omega}_t - \omega_t \Vert_{C^0} \leq K t^\kappa$ and $\Vert \tilde{\Omega}_t - \Omega_t \Vert_{C^0} \leq K t^\kappa$ for some $\kappa, K > 0$ and for sufficiently small $t$. The cohomology classes satisfy $\vert \text{Re}(\Omega_t) \vert = \vert \text{Re}(\tilde{\Omega}_t) \vert \in H^3(M_t, \mathbb{R})$ and $\vert \omega_t \vert = c_t \vert \tilde{\omega}_t \vert \in H^2(M_t, \mathbb{R})$ for some $c_t > 0$. Here all norms are computed with respect to $g_t$.

**Proof.** First we estimate the norms of $\omega_t - \omega_t'$ and $\text{Im}(\Omega_t) - \theta_{2,t}' = \text{Im}(\Omega_t) - \text{Im}(\Omega_t')$ on $P_t \cap Q_t$, as in part (i) of Theorem 3.10. Since $\omega_t'$ depends on $\text{Re}(\Omega_t)$ and $\omega_t = \omega_V$ on $P_t \cap Q_t$ on $\text{Re}(\Omega_V)$, it follows that

$$\vert \omega_t - \omega_t' \vert_{g_t} \leq C_1 \vert \omega_t - \omega_t' \vert_{g_V} \leq C_2 \vert \text{Re}(\Omega_t) - \text{Re}(\Omega_V) \vert_{g_V} \leq C_2 \vert \Omega_t - \Omega_V \vert_{g_V} = O(t^{-\lambda (1-\alpha)}) + O(t^{\alpha \nu})$$

for some constants $C_1, C_2 > 0$ and hence

$$\vert \omega_t - \omega_t' \vert_{C^0} = O(t^{-\lambda (1-\alpha)}) + O(t^{\alpha \nu}).$$

From the fact that $\text{vol}(P_t \cap Q_t) = O(t^{6\alpha})$, we have

$$\vert \omega_t - \omega_t' \vert_{L^2} = O(t^{\frac{3\alpha}{2}}) \cdot \vert \omega_t - \omega_t' \vert_{C^0} = O(t^{3\alpha - \lambda (1-\alpha)}) + O(t^{3\alpha + \alpha \nu}).$$

Furthermore,

$$\vert \text{Im}(\Omega_t) - \text{Im}(\Omega_t') \vert_{g_t} \leq C_3 \vert \text{Im}(\Omega_t) - \text{Im}(\Omega_t') \vert_{g_V} \leq C_3 \vert \text{Im}(\Omega_t) - \text{Im}(\Omega_V) \vert_{g_V} + C_3 \vert \text{Im}(\Omega_V) - \text{Im}(\Omega_t') \vert_{g_V} \leq C_3 \vert \Omega_t - \Omega_V \vert_{g_V} + C_4 \vert \text{Re}(\Omega_V) - \text{Re}(\Omega_t) \vert_{g_V} = O(t^{-\lambda (1-\alpha)}) + O(t^{\alpha \nu})$$

for some constants $C_3, C_4 > 0$, as $\text{Im}(\Omega_V)$ is determined by $\text{Re}(\Omega_V)$ and $\text{Im}(\Omega_t)'$ by $\text{Re}(\Omega_t)$. Therefore,

$$\vert \text{Im}(\Omega_t) - \text{Im}(\Omega_t') \vert_{C^0} = O(t^{-\lambda (1-\alpha)}) + O(t^{\alpha \nu})$$

and

$$\vert \text{Im}(\Omega_t) - \text{Im}(\Omega_t') \vert_{L^2} = O(t^{3\alpha - \lambda (1-\alpha)}) + O(t^{3\alpha + \alpha \nu}).$$
It can be deduced from (5.5) and (5.6) that $|\nabla^{g_t}(\Omega_t - \Omega_V)|_{g_t} = O(t^{-\lambda(1-\alpha)-\alpha}) + O(t^{\alpha\nu-\alpha})$ and $|\nabla^{g_t}{\nu}(\Omega_t - \Omega_V)|_{g_t} = O(t^{-\lambda(1-\alpha) - 2\alpha}) + O(t^{\alpha\nu-2\alpha})$, which imply the equalities

\begin{equation}
(5.11) \quad \|\nabla^{g_t}(\omega_t - \omega^t)\|_{C^0} = O(t^{-\lambda(1-\alpha)-\alpha}) + O(t^{\alpha\nu-\alpha})
\end{equation}

and

\begin{equation}
(5.12) \quad \|\nabla^{g_t}{\nu}(\omega_t - \omega^t)\|_{C^0} = O(t^{-\lambda(1-\alpha) - 2\alpha}) + O(t^{\alpha\nu-2\alpha}).
\end{equation}

Then the $L^{12}$-norm satisfies

\begin{equation}
(5.13) \quad \|\nabla^{g_t}(\omega_t - \omega^t)\|_{L^{12}} = O(t^{-\frac{\lambda(1-\alpha)}{2} + \alpha\nu}) + O(t^{-\frac{\lambda(1-\alpha)}{2}}).
\end{equation}

Finally, we estimate the $L^{12}$-norms of $\nabla^{g_t}\omega_t$ and $\nabla^{g_t}\text{Re}(\Omega_t)$. Note that

\begin{equation}
|\nabla^{g_t}\omega_t|_{g_t} \leq C_5 |(\nabla^{g_t} - \nabla^{g_V})\omega_t|_{g_V} + C_5 |\nabla^{g_V}\omega_t|_{g_V} = C_5 |(\nabla^{g_t} - \nabla^{g_V})\omega_t|_{g_V}
\end{equation}

for some constant $C_5 > 0$, as $\omega_t = \omega_V$ on $P_t \cap Q_t$ and $\nabla^{g_V}\omega_V = 0$. Then

\begin{align*}
|\nabla^{g_t}\omega_t|_{g_t} &\leq C_5 |(\nabla^{g_t} - \nabla^{g_V})|_{g_V} \cdot |\omega_t|_{g_V} \\
&\leq \frac{3}{2} C_5 |g_t|^{-1} \cdot |\nabla^{g_V}g_t| \cdot |\omega_t|_{g_V} \quad \text{by (3.21)} \\
&= C_6 |\nabla^{g_V}(g_t - g_V)|_{g_V} \quad \text{as } \nabla^{g_V}g_V = 0.
\end{align*}

Here $C_6$ is an upper bound for $\frac{3}{2} C_5 |g_t|^{-1} \cdot |\omega_t|_{g_V}$ which is independent of $t$. It follows that

\begin{equation}
|\nabla^{g_t}\omega_t|_{g_t} = O(t^{-\lambda(1-\alpha) - \alpha}) + O(t^{\alpha\nu-\alpha}),
\end{equation}

and consequently

\begin{equation}
(5.14) \quad \|\nabla^{g_t}\omega_t\|_{L^{12}} = O(t^{-\frac{\lambda(1-\alpha)}{2} + \alpha\nu}) + O(t^{-\frac{\lambda(1-\alpha)}{2}}).
\end{equation}

A similar argument shows

\begin{equation}
(5.15) \quad \|\nabla^{g_t}\text{Re}(\Omega_t)\|_{L^{12}} = O(t^{-\frac{\lambda(1-\alpha)}{2} + \alpha\nu}) + O(t^{-\frac{\lambda(1-\alpha)}{2}}).
\end{equation}

Now for parts (i) to (iii) of Theorem 3.10 to hold, we need:

\begin{align*}
-\lambda(1-\alpha) &\geq \kappa, \quad \alpha\nu \geq \kappa \quad \text{from (5.7) and (5.9),} \\
3\alpha - \lambda(1-\alpha) &\geq 3 + \kappa, \quad 3\alpha + \alpha\nu \geq 3 + \kappa \quad \text{from (5.8) and (5.10),} \\
-\frac{\lambda}{2} - \lambda(1-\alpha) &\geq -\frac{3}{2} + \kappa, \quad -\frac{\lambda}{2} + \alpha\nu \geq -\frac{3}{2} + \kappa \quad \text{from (5.13), (5.14) and (5.15),} \\
-\lambda(1-\alpha) - \alpha &\geq \kappa - 1, \quad \alpha\nu - \alpha \geq \kappa - 1 \quad \text{from (5.11),} \\
-\lambda(1-\alpha) - 2\alpha &\geq \kappa - 2, \quad \alpha\nu - 2\alpha \geq \kappa - 2 \quad \text{from (5.12).}
\end{align*}

Observe that the second set of inequalities imply all the others, as $\alpha \leq 1$. Therefore, calculations using these two inequalities show that there exist solutions $\alpha \in (0,1)$ and $\kappa > 0$ for any $\nu > 0$ and $\lambda < -3$. For example, we could take

\[ \alpha = \frac{1}{2} \left( \frac{6 + \nu}{3 + \nu} \right) \in (0,1) \quad \text{and} \quad \kappa = \min \left( (1 - \alpha)(-3 - \lambda), \frac{\nu}{2} \right) > 0. \]

For parts (iv) and (v) of Theorem 3.10, note that under the homothety $g_Y \mapsto t^2 g_Y$ on the AC Calabi–Yau 3-fold $Y$ we have $\delta(t^2 g_Y) = t \delta(g_Y)$ and $\|R(t^2 g_Y)\|_{C^0} = t^{-2} \|R(g_Y)\|_{C^0}$. Moreover, the dominant contributions to $\delta(g_Y)$ and $\|R(g_Y)\|_{C^0}$ for small $t$ come from $\delta(g_Y)$.
and \( \| R(g_Y) \|_{C^0} \) which are proportional to \( t \) and \( t^{-2} \). Thus there exist constants \( E_3, E_4 > 0 \) such that (iv), (v) of Theorem 3.10 hold for sufficiently small \( t \). Hence by Theorem 3.10, \( M_t \) admits a Calabi–Yau structure \( (\tilde{J}_t, \tilde{\omega}_t, \tilde{\Omega}_t) \) such that \( \| \tilde{\omega}_t - \omega_t \|_{C^0} \leq K t^\kappa \) and \( \| \tilde{\Omega}_t - \Omega_t \|_{C^0} \leq K t^\kappa \) for some \( \kappa, K > 0 \) and for sufficiently small \( t \).

Finally, the cohomology condition in Theorem 3.10 holds automatically here. This is because if \( Y \) is an AC Calabi–Yau 3-fold, then it can be shown that \( \text{Hol}(g_Y) = \text{SU}(3) \) except for the trivial case where \( Y = V = \mathbb{C}^3 \). Moreover, \( \text{Hol}(g_Y) \) lies inside the holonomy group \( \text{Hol}(\tilde{g}_t) \) of the Calabi–Yau metric \( \tilde{g}_t \) on \( M_t \), and hence \( \text{Hol}(\tilde{g}_t) \) must be the whole \( \text{SU}(3) \). The first cohomology group \( H^1(M_t, \mathbb{R}) \) therefore vanishes for each sufficiently small \( t \), and the theorem now follows from Theorem 3.10.

Theorem 5.2 can be extended to the case when the Calabi–Yau 3-fold \( (M_0, J_0, \omega_0, \Omega_0) \) has finitely many conical singularities at \( x_1, \ldots, x_n \) with rates \( \nu_1, \ldots, \nu_n > 0 \) modelled on Calabi–Yau cones \( V_1, \ldots, V_n \). Let \( (Y_i, J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i}) \) be AC Calabi–Yau 3-folds with rates \( \lambda_i < -3 \) modelled on cones \( V_i \) for \( i = 1, \ldots, n \). We then glue in at the singular points AC Calabi–Yau 3-folds \( (Y_i, J_{Y_i}, t^2 \omega_{Y_i}, t^3 \Omega_{Y_i}) \) and produce a 1-parameter family of nonsingular Calabi–Yau 3-folds.

### 5.4. Conclusions

We conclude by applying the above result to some examples given in §3 and discussing briefly the case \( \lambda = -3 \). First consider the situation in Example 4.12 and take \( m = 3 \). Then the crepant resolution \( X \) of the Calabi–Yau cone \( V = \mathbb{C}^3/G \) is an AC Calabi–Yau 3-fold with rate \( \lambda = -6 \). Thus Theorem 5.2 applies and we can desingularize any compact Calabi–Yau 3-fold \( M_0 \) with conical singularities modelled on \( V \), or equivalently, any Calabi–Yau 3-orbifold with isolated singularities, by the gluing process. In particular when \( G = \mathbb{Z}_3 \), a standard example of compact Calabi–Yau 3-orbifold with isolated singularities is given in [2, Example 6.6.3]. Let \( \Lambda = \langle 1, e^{2\pi i/3} \rangle \) be a lattice in \( \mathbb{C} \), and take \( T^6 = T^2 \times T^2 \times T^2 \) where we regard each \( T^2 \) as the quotient \( \mathbb{C}/\Lambda \). The group \( \mathbb{Z}_3 \) acts on \( T^6 \) by multiplication by \( e^{2\pi i/3} \) on each \( T^2 \)-component. Then \( M_0 = T^6/\mathbb{Z}_3 \) is a Calabi–Yau 3-orbifold and it is not hard to see that \( M_0 \) has 27 isolated singular points modelled on \( \mathbb{C}^3/\mathbb{Z}_3 \). Thus by gluing in the total space \( K_{\mathbb{P}^2} = O(-3) \rightarrow \mathbb{P}^2 \) of the canonical bundle over \( \mathbb{P}^2 \) in each of the singular points, we obtain a desingularization of \( M_0 = T^6/\mathbb{Z}_3 \).

Note that if we desingularize a Calabi–Yau 3-orbifold with isolated singularities modelled on \( \mathbb{C}^3/G \) by gluing, the Schlessinger Rigidity Theorem [10] tells us that \( \mathbb{C}^3/G \) admits no nontrivial deformations, and hence what we will get will be a crepant resolution of the original orbifold. Potentially we will get a crepant resolution of a deformation of the orbifold, but it will be the crepant resolution of the original orbifold if \( [\text{Re}(\Omega_t)] = [\text{Re}(\Omega)] \) for all sufficiently small \( t \), or equivalently if \( [\Omega_Y] = 0 \) in \( H^3(Y, \mathbb{C}) \).

Now Yau’s solution to the Calabi conjecture [12] gives the existence of Calabi–Yau metrics on the crepant resolution. However, it does not provide a way to write down the Calabi–Yau metrics explicitly, and so in general we do not know much about what the Calabi–Yau metrics are like. But in the orbifold case, our result tells a bit more by giving a quantitative description of these Calabi–Yau metrics, showing that these metrics locally look like...
the metrics obtained by gluing the orbifold metrics and the ALE metrics on the crepant resolution of $\mathbb{C}^3/G$.

Our result can also be applied to desingularize compact Calabi–Yau 3-folds with conical singularities modelled on the Calabi–Yau cone $\mathcal{O}(-2, -2) \setminus (\mathbb{C}P^1 \times \mathbb{C}P^1)$ by gluing in the AC Calabi–Yau 3-fold $\mathcal{O}(-2, -2)$-bundle with rate $-6$. Thus we could resolve a kind of singularity which is not of orbifold type.

Finally we would like to discuss the case when $\lambda = -3$. In Theorem 5.2 we desingularize a Calabi–Yau 3-fold with a conical singularity using an AC Calabi–Yau 3-fold with rate $\lambda$ where we assumed that $\lambda < -3$ and hence $[\Upsilon_Y^*(t^3\Omega_Y) - \Omega_V] = 0$ by Lemma 4.10. If we relax this to allow $\lambda = -3$, then the cohomology class $[\Upsilon_Y^*(t^3\Omega_Y) - \Omega_V]$ may be nonzero in $H^3(\Gamma, \mathbb{C})$. But the cohomology class $[\Phi^*(\Omega_0) - \Omega_V]$ is always zero by Lemma 4.7, so there can be topological obstructions to defining a closed 3-form which interpolates between $\Phi^*(\Omega_0)$ and $\Upsilon_Y^*(t^3\Omega_Y)$. Thus allowing $\lambda = -3$ introduces global problems for our gluing method. Here is a very short sketch of a method the author hopes to use to tackle the problem. We replace the holomorphic $(3,0)$-form $\Omega_0$ on $M_0$ by $\Omega_0 + t^3\chi$, where $\chi$ is some closed and coclosed $(2,1)$-form with appropriate asymptotic behaviour, and $[\Phi^*(\chi)] = [\Upsilon_Y^*(\Omega_Y)]$ on $H^3(\Gamma, \mathbb{C})$. Calculations by the author suggest that such $\chi$ exists and it cancels out the $O(t^3r^{-3})$ terms such that Theorem 3.10 can handle the size of the error introduced. The advantage of extending our result to the case $\lambda = -3$ is that it can be applicable to a larger class of AC Calabi–Yau 3-folds such as the deformation $Q_\varepsilon$ of the cone $V = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}$ in Example 4.13. Hence by extending our result, we could smooth ordinary double points by analytic rather than algebro-geometric methods.

References

[1] M. Berger, Sur les groupes d’holonomie homogène des variétés à connexion affine et des variétés riemanniennes, Bull. Soc. Math. France 83 (1955), 225-238.
[2] C.P. Boyer, K. Galicki and J. Kollár, Einstein Metrics on Spheres, math.DG/0309408.
[3] R.L. Bryant and S.M. Salamon, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. 58 (1989) 829-850.
[4] E. Calabi, Métriques kählériennes et fibrés holomorphes, Ann. Sci. École Norm. Sup. 12 (1979), 269-294.
[5] P. Candelas and X.C. de la Ossa, Comments on Conifolds, Nucl. Phys. B342 (1990), 246-268.
[6] M. Gross, D. Huybrechts and D.D. Joyce, Calabi–Yau Manifolds and Related Geometries, Springer-Verlag, 2003.
[7] D.D. Joyce, Compact Manifolds with Special Holonomy, Oxford University Press, Oxford, 2000.
[8] D. McDuff and D. Salamon, Introduction to Symplectic Topology, second edition, Oxford University Press, Oxford, 1998.
[9] S.M. Salamon, Riemannian Geometry and Holonomy Groups, Pitman Research Notes in Mathematics, volume 201, Longman, Harlow, 1989.
[10] M. Schlessinger, Rigidity of quotient singularities, Inventiones mathematicae 14 (1971), 17-26.
[11] M.B. Stenzel, Ricci-flat metrics on the complexification of a compact rank one symmetric space, Manuscripta math. 80 (1993), 151-163.
[12] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equations. I, Communications on pure and applied mathematics 31 (1978), 339-411.