Simulating local fields in carbon nanotube-reinforced composites for infinite strip with voids

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Received: 30 January 2022 / Accepted: 28 April 2022 / Published online: 1 June 2022 © The Author(s) 2022

Abstract A numerical simulation of the thermal properties is conducted for an isotropic and homogeneous infinite strip composite reinforced by carbon nanotubes (CNTs) and containing voids. The CNTs can be uniformly or randomly distributed but are non-overlapping. We model the CNTs as thin perfectly conducting elliptic inclusions and assume the voids to be of circular shape and act as barriers to heat flow. We also impose isothermal conditions on the external boundaries by assuming the lower infinite wall to be a heater under a given temperature, and the upper wall to be a cooler that can be held at a lower fixed temperature. The mathematical model, which takes the form of a mixed Dirichlet–Neumann problem, is solved by applying the boundary integral equation with the generalized Neumann kernel. We illustrate the performance of the proposed method through several numerical examples including the case of the presence of a large number of CNTs and voids.

Keywords Boundary integral equation · CNTs-reinforced composites · Thermal conductivity

1 Introduction

Nanofibers embedded in polymer matrices have attracted attention as one of the reinforcements for composite materials. Carbon nanotubes (CNTs)-reinforced polymer nanocomposites are considered as conventional micro- and macro-composites [1]. Their thermal, mechanical, and electric properties are determined by experimental and theoretical investigations [2–4]. CNTs are considered as perfectly conducting inclusions, which suggests imposing
Dirichlet boundary conditions on the boundary of CNTs. On the other hand, the classical problems for materials with holes in porous media and materials with voids and insulating inclusions are modeled by the Neumann boundary condition [5,6].

The present work is devoted to the thermal conductivity in a 2D (two-dimensional) isotropic and homogeneous nanocomposite, which takes the form of an infinite strip, when it is reinforced by non-overlapping CNTs and contains defects or voids. In particular, we are interested in studying the effect of CNTs as well as the presence of voids on the macroscopic conductive and mechanical properties of this composite. Physical measurement shows that CNTs act as superconductors and voids act as insulators with an extremely low conductivity. Henceforth, one can assume that the conductivity of CNTs is infinity and that of the voids is equal to zero. The polymer host has certainly lower conductivity $\lambda$ than CNTs but higher than voids. In the sequel, the polymer conductivity is normalized to unity, i.e., $\lambda = 1$. The governing equations can be then cast into a mixed boundary value problem (BVP).

Theoretical investigation of mixed BVPs by integral equations can be found in [7–11]. In the same time, implementation of numerical methods for large number of inclusions and holes is still a challenging problem of applied and computational mathematics. We propose in this work a fast and effective algorithm for the numerical solution of the formulated mixed BVP. The method employs the boundary integral equation (BIE) with the generalized Neumann kernel (GNK), which has been used in [10] to solve a similar mixed BVP related to the capacity of generalized condensers. An appealing property of the proposed method is that it can be employed even when the number of perfectly conducting inclusions and holes is very large.

As a result of simulations, we first study the 2D local fields for three types of media. In the first type, we consider the case of pure $m$ void cracks with $m = 5, 30,$ and 50. The second type consists of pure $\ell$ CNT inclusions with $\ell = 5$ and 200. Finally, we treat the case of a large number of combined inclusions and holes by considering either 2000 of one of the two or 1000 of each. Afterward, we take up the systematic investigation of the effective conductivity of the considered composites. It is important in applications to predict the macroscopic properties of composites which depend on the concentration of perfectly conducting CNTs as well as on the concentration of insulating voids. It is worth noting that the notation of concentration is different for slit shapes of CNTs and circular shapes of holes. The performed simulations of local fields and computation of their averaged conductivity for various concentrations allow to establish the dependence of the macroscopic conductivity on the main geometrical parameters.

We point out that for definiteness, we use in the present paper the terminology of heat conduction. However, other physical phenomena can be discussed by the same method due to the universality of the mathematical model; see Table 1 [12].

## 2 Problem formulation

Let us consider a channel medium embedding $m$ inhomogeneities in the form of $\ell$ nanofillers and $p = m - \ell$ holes (voids). As many nanofillers (e.g., carbon nanotubes [13]) have cross sections of elliptical shapes, we model them as ellipses $C_1, \ldots, C_\ell$. Furthermore, we represent the non-conducting holes by inner circles $C_{\ell+1}, \ldots, C_m$. The top

| Table 1 | Weber’s table [12] |
| --- | --- |
| Heat conduction | Heat flux, $\mathbf{q}$ |
| Electrical conduction | Electrical current, $\mathbf{j}$ |
| Dielectrics | Displacement field, $\mathbf{d}$ |
| Magnetism | Magnetic induction, $\mathbf{B}$ |
| Diffusion | Particle current, $\mathbf{j}$ |
| Flow in porous media | Velocity, $\mathbf{v}$ |
| Antiplane elasticity | Stresses, $(\tau_{13}, \tau_{23})$ |
| | Temperature gradient, $-\nabla T$ |
| | Electric field, $\mathbf{e}$ |
| | Electric field, $\mathbf{e}$ |
| | Magnetic field, $\mathbf{H}$ |
| | Concentration gradient, $-\nabla c$ |
| | Pressure gradient, $\nabla P$ |
| | Vertical displacement gradient, $\nabla u_3$ |
| | Thermal conductivity, $\lambda$ |
| | Electrical conductivity, $\sigma$ |
| | Electric permittivity, $\epsilon$ |
| | Magnetic permeability, $\mu$ |
| | Diffusivity, $D$ |
| | Fluid permeability, $K$ |
| | Shear modulus, $G$ |
and bottom infinite walls of the channel are denoted, respectively, by $C'_0$ and $C''_0$, which yields a multiply connected domain $\Omega$ with a boundary set $C = \bigcup_{j=0}^m C_j$, where $C_0 = C'_0 \cup C''_0$. An example of this domain for the case of $\ell = 4$ and $m = 7$ is illustrated in Fig. 1.

The medium matrix without inhomogeneities is supposed to be homogeneous and isotropic with a constant thermal conductivity $\lambda = 1$. We also assume that conduction is the only dominating mechanism of heat transfer in the medium. Except being non-overlapping, no other restriction is imposed on the inhomogeneities as they can be placed at random orientation and position.

The nanofillers are treated as heat superconductors with an almost uniform temperature distribution within each one. Therefore, the temperature $T$ is considered to be equal to an indeterminate real constant $\delta_j$ on each ellipse $C_j$ for $j = 1, \ldots, \ell$. This assumption is consistent with the numerical results reported in [14] for CNT-reinforced polymer composites. Furthermore, by the law of energy conservation in steady-state heat conduction, there should be no net thermal flow through each nanofiller. This constraint is written by means of the net heat flux boundary condition (1e).

On the other hand, the curves $C_{\ell+1}, \ldots, C_m$ are assumed to be perfect insulators, and therefore, they act as barriers to heat flow. Henceforth, the Neumann boundary condition (1f) is imposed along the holes contours. Finally, isothermal conditions are imposed on the external boundaries by assuming that the lower infinite wall is a heater of temperature $T_1$, and the upper wall acts as a heat sink, which can be held at a fixed temperature $T_0 < T_1$. These two values $T_0$ and $T_1$ of the temperature on the external boundaries are normalized to 0 and 1, respectively.

Under steady-state conditions, Fourier’s law of heat conduction and the above specified heat boundaries conditions yield that the temperature distribution $T$ is governed by the mixed Dirichlet–Neumann BVP given by

$$\Delta T = 0 \text{ in } \Omega, \quad (1a)$$
$$T = 0 \text{ on } C'_0, \quad (1b)$$
$$T = 1 \text{ on } C''_0, \quad (1c)$$
$$T = \delta_j \text{ on } C_j, \quad j = 1, 2, \ldots, \ell, \quad (1d)$$
$$\int_{C_j} \frac{\partial T}{\partial n} \, ds = 0, \quad j = 1, 2, \ldots, \ell, \quad (1e)$$
$$\frac{\partial T}{\partial n} = 0 \text{ on } C_j, \quad j = \ell + 1, \ell + 2, \ldots, m, \quad (1f)$$

where $\partial T/\partial n$ stands for the normal derivative of $T$. Besides computing the distribution temperature $T$, we need to determine the values of the constants $\delta_1, \ldots, \delta_m$ alongside.

### 3 The integral equation method

The BIE method with GNK is not directly applicable to (1) because of the external boundary component. However, the above BVP is conformal mapping invariant. The function

$$...$$
Fig. 2 The computational domain \( G \) corresponding to the physical domain in Fig. 1

\[ z = \Psi(\zeta) = \frac{1}{\pi} \log \frac{1 + \zeta}{1 - \zeta} + \frac{i}{2} \]

conformally maps the unit disk \( |\zeta| < 1 \) onto the infinite strip \( 0 < \text{Im} \, z < 1 \). Thus, the inverse mapping

\[ \zeta = \Psi^{-1}(z) = \tanh \left( \frac{\pi z}{2} - \frac{\pi i}{4} \right) \]

conformally maps the infinite strip \( 0 < \text{Im} \, z < 1 \) onto the unit disk \( |\zeta| < 1 \), the real axis onto the lower half of the unit circle, the line \( \text{Im} \, z = 1 \) onto the upper half of the unit circle, and satisfies \( \Psi^{-1}(\pm\infty + 0i) = \pm1 \). Hence, the function \( \Psi^{-1} \) maps the multiply connected domain \( \Omega \) in the \( z \)-plane (the physical domain) onto a multiply connected domain \( G \) in the \( \zeta \)-plane interior of the unit circle and exterior of \( m \) smooth Jordan curves (the computational domain). In Fig. 2, we display the result of the conformal mapping of the example shown in Fig. 1.

In this way, the harmonic function \( T \) can be obtained through

\[ T(z) = U(\Psi^{-1}(z)) \]

after computing \( U \) solution to the BVP in the \( \zeta \)-plane given by

\begin{align*}
\Delta U &= 0 \quad \text{in} \ G, \\
U &= 0 \quad \text{on} \ \Gamma'_0, \\
U &= 1 \quad \text{on} \ \Gamma''_0, \\
U &= \delta_j \quad \text{on} \ \Gamma_j, \quad j = 1, 2, \ldots, \ell, \\
\int_{\Gamma_j} \frac{\partial U}{\partial n} \, ds &= 0, \quad j = 1, 2, \ldots, \ell, \\
\frac{\partial U}{\partial n} &= 0 \quad \text{on} \ \Gamma_j, \quad j = \ell + 1, \ell + 2, \ldots, m,
\end{align*}

where \( \Gamma'_0 = \Psi^{-1}(C'_0) \), \( \Gamma''_0 = \Psi^{-1}(C''_0) \), and \( \Gamma_j = \Psi^{-1}(C_j) \) for \( j = 1, 2, \ldots, m \). Notice that the restriction of the function \( U(\zeta) \) on the external boundary is discontinuous at \( \zeta = \pm1 \). However, the function \( U \) can be cast into the form

\[ U(\zeta) = u_0(\zeta) + u(\zeta), \]

where \( u(\zeta) \) is a harmonic function in \( G \), and

\[ u_0(\zeta) = \frac{1}{\pi} \text{Im} \log \frac{1 - \zeta}{1 + \zeta} + \frac{1}{2}. \]

The function \( u_0(\zeta) \) is harmonic in \( G \) with \( u_0(\zeta) = 1 \) on the lower half of the unit circle and \( u_0(\zeta) = 0 \) on the upper part. The function \( u(\zeta) \) is the solution of the BVP:

\begin{align*}
\Delta u(\zeta) &= 0 \quad \text{if} \ \zeta \in G, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \ \Gamma_j, \quad j = \ell + 1, \ell + 2, \ldots, m.
\end{align*}
\[ u(\zeta) = 0 \quad \text{if } \zeta \in \Gamma_0, \]  
\[ u(\zeta) = \delta_j \frac{1}{\pi} \text{Im} \log \frac{1-\zeta}{1+\zeta} - \frac{1}{2} \quad \text{if } \zeta \in \Gamma_j, \quad j = 1, 2, \ldots, \ell, \]  
\[ \int_{\Gamma_j} \frac{\partial u}{\partial n} ds = 0 \quad j = 1, 2, \ldots, \ell, \]  
\[ \frac{\partial u}{\partial n} \bigg|_{\zeta} = -\frac{\partial u_0}{\partial n} \bigg|_{\zeta} \quad \text{if } \zeta \in \Gamma_j, \quad j = \ell + 1, \ell + 2, \ldots, m, \]

where \( \Gamma_0 \) is the unit circle.

For the orientation of the boundary components of \( G \), we assume that \( \Gamma_0 \) is oriented counterclockwise and the other curves \( \Gamma_1, \ldots, \Gamma_m \) are oriented clockwise. We assume that the boundaries \( \Gamma_j, \quad j = 0, 1, \ldots, m \) are parametrized by \( 2\pi \)-periodic functions \( \eta_j(t) \) for \( t \in J_j := [0, 2\pi] \) such that \( \eta_j'(t) \neq 0 \). The whole boundary \( \Gamma \) is then parametrized by the complex function \( \eta \) defined on the disjoint union \( J \) of \( J_0, J_1, \ldots, J_m \) by \([15,16]\)

\[ \eta(t) = \begin{cases} 
\eta_0(t), & t \in J_0, \\
\eta_1(t), & t \in J_1, \\
\vdots \\
\eta_m(t), & t \in J_m.
\end{cases} \]

Note that the unit circle \( \Gamma_0 \) is parametrized by \( \eta_0(t) = e^{it}, \quad t \in J_0 = [0, 2\pi] \).

Let \( n(\zeta) \) be the unit outward normal vector at \( \zeta \in \Gamma \) and let \( \nu(\zeta) \) be the angle between the normal vector \( n(\zeta) \) and the positive real axis. Then, for \( \zeta = \eta(t) \in \Gamma \),

\[ n(\zeta) = e^{i\nu(\zeta)} = e^{-i \frac{\eta'(t)}{||\eta'(t)||}}, \quad (4) \]

Thus,

\[ \frac{\partial u_0}{\partial n} = \nabla u_0 \cdot n = \cos \nu \frac{\partial u_0}{\partial x} + \sin \nu \frac{\partial u_0}{\partial y} = \text{Re} \left[ e^{i\nu} \left( \frac{\partial u_0}{\partial x} - i \frac{\partial u_0}{\partial y} \right) \right]. \quad (5) \]

The harmonic function \( u_0(\zeta) \) is the real part of a single-valued analytic function \( f_0(\zeta) \), i.e., \( u_0(\zeta) = \text{Re} [f_0(\zeta)] \), where

\[ f_0(\zeta) = \frac{1}{\pi i} \log \frac{1-\zeta}{1+\zeta} + \frac{1}{2}, \quad (6) \]

and the branch of the logarithm function is chosen such that \( \log 1 = 0 \). On the basis of the Cauchy–Riemann equations, it follows that

\[ f_0'(\zeta) = \frac{\partial u_0(\zeta)}{\partial x} - i \frac{\partial u_0(\zeta)}{\partial y}, \]

which, in view of (4) and (5), implies that

\[ |\eta'(t)| \frac{\partial u_0}{\partial n} \bigg|_{\eta(t)} = \text{Re} \left[ -i \eta'(t) f_0'(\eta(t)) \right], \quad \eta(t) \in \Gamma_j, \quad j = \ell + 1, \ldots, m. \quad (7) \]

Since

\[ f_0'(\zeta) = \frac{i}{\pi} \left( \frac{1}{1-\zeta} + \frac{1}{1+\zeta} \right), \]
it follows that for \( \eta(t) \in \Gamma_j \) and \( j = \ell + 1, \ldots, m \),

\[
|\eta'(t)| \frac{\partial u_0}{\partial n}_{\zeta = \eta(t)} = \frac{1}{\pi} \text{Re} \left[ \frac{\eta'(t)}{1 - \eta(t)} + \frac{\eta'(t)}{1 + \eta(t)} \right].
\]  

(8)

The harmonic function \( u \) can be assumed to be a real part of an analytic function \( f(\zeta), \zeta \in G \). The boundary conditions (3b) and (3c) give the real parts of the function \( f(\zeta) \) on \( \Gamma_j \) for \( j = 0, 1, \ldots, \ell \). Specifically, we have

\[
\text{Re} \left[ f(\eta(t)) \right] = 0, \quad \eta(t) \in \Gamma_0
\]  

(9)

and

\[
\text{Re} \left[ f(\eta(t)) \right] = \delta_j - \frac{1}{\pi} \text{Im} \log \frac{1 - \eta(t)}{1 + \eta(t)} - \frac{1}{2} \quad \text{if } \eta(t) \in \Gamma_j, \quad j = 1, \ldots, \ell.
\]  

(10)

For the remaining boundary components, we use the condition (3e) to determine the boundary condition on \( f(\eta) \).

By the Cauchy–Riemann equations, we can show using the same arguments as in (7) that

\[
|\eta'(t)| \frac{\partial u}{\partial n}_{\eta(t)} = \text{Re} \left[ -i\eta'(t) f'(\eta(t)) \right].
\]  

(11)

Thus, for \( \eta(t) \in \Gamma_j \) for \( j = \ell + 1, \ldots, m \), it follows from (3e), (8), and (11) that

\[
\text{Re} \left[ -i\eta'(t) f'(\eta(t)) \right] = -\frac{1}{\pi} \text{Re} \left[ \frac{\eta'(t)}{1 - \eta(t)} + \frac{\eta'(t)}{1 + \eta(t)} \right].
\]  

We integrate both sides with respect to the parameter \( t \) for \( t \in J_j \) to obtain

\[
\text{Re} \left[ -if(\eta(t)) \right] = \frac{1}{\pi} \log \left| \frac{1 - \eta(t)}{1 + \eta(t)} \right| + \delta_j, \quad j = \ell + 1, \ldots, m,
\]  

(12)

where the integration constants \( \delta_j \) are undetermined. The constants \( \delta_j, j = 1, \ldots, m \) in (10) and (12) are determined so that \( f(z) \) is a single-valued analytic function.

Since we aim to compute the real function \( u = \text{Re} [f] \), without loss of generality, we assume \( c = f(\alpha) \in \mathbb{R} \) for some given point \( \alpha \in G \). Then, the function \( g(\zeta) \) defined for \( \zeta \in G \) through

\[
f(\zeta) = (\zeta - \alpha)g(\zeta) + c
\]  

(13)

is analytic on \( G \). Define

\[
A(t) = e^{-i\theta(t)}(\eta(t) - \alpha),
\]  

(14)

where

\[
\theta(t) = \begin{cases} 
0, & t \in J_0, \\
\vdots \\
0, & t \in J_\ell, \\
\pi/2, & t \in J_{\ell+1}, \\
\vdots \\
\pi/2, & t \in J_m.
\end{cases}
\]  

(15)
Thus, 
\[ e^{-i\theta(t)} f(\eta(t)) = A(t)g(\eta(t)) + e^{-i\theta(t)} c, \]
which implies that 
\[ \text{Re}[A(t)g(\eta(t))] = \text{Re}[e^{-i\theta(t)} f(\eta(t))] - c \cos \theta(t). \]

On the basis of the conditions (9), (10), and (12), \( g(z) \) fulfills the Riemann–Hilbert boundary value problem 
\[ \text{Re}[A(t)g(\eta(t))] = \gamma(t) + h(t), \quad (16) \]

where 
\[
h(t) = \begin{cases} 
-c, & t \in J_0, \\
\delta_1 - \frac{1}{2} - c, & t \in J_1, \\
\vdots \\
\delta_\ell - \frac{1}{2} - c, & t \in J_\ell, \\
\delta_{\ell+1}, & t \in J_{\ell+1}, \\
\vdots \\
\delta_m, & t \in J_{m}. 
\end{cases}
\]

\[
\gamma(t) = \begin{cases} 
0, & t \in J_0, \\
-\frac{1}{\pi} \text{Im} \log \frac{1-\eta(t)}{1+\eta(t)}, & t \in J_1, \\
\vdots \\
-\frac{1}{\pi} \text{Im} \log \frac{1-\eta(t)}{1+\eta(t)}, & t \in J_\ell, \\
\frac{1}{\pi} \log \frac{1-\eta(t)}{1+\eta(t)}, & t \in J_{\ell+1}, \\
\vdots \\
\frac{1}{\pi} \log \frac{1-\eta(t)}{1+\eta(t)}, & t \in J_m. 
\end{cases}
\quad (17)
\]

Clearly, \( \gamma \) is known but the piecewise constant function \( h \) is not and should be determined. Let \( \mu(t) = \text{Im}[A(t)g(\eta(t))] \), i.e., the boundary values of an analytic function \( g \) are given by 
\[ g(\eta(t)) = \frac{\gamma(t) + h(t) + i\mu(t)}{A(t)}, \quad t \in J. \quad (18) \]

Thus, to find the boundary values of \( g \), one should determine the two unknown functions \( \mu \) and \( h \). These two functions can be computed using the BIE with GNK [15–17].

We define two integral operators \( N \) and \( M \) on the space \( H \) of all real Hölder continuous functions on the boundary \( \Gamma \) by 
\[
N\gamma(s) = \int_J \frac{1}{\pi} \text{Im} \left( \frac{A(s)}{A(t)} \frac{\eta'(t)}{\eta(t) - \eta(s)} \right) \gamma(t) dt, \quad s \in J, \\
M\gamma(s) = \int_J \frac{1}{\pi} \text{Re} \left( \frac{A(s)}{A(t)} \frac{\eta'(t)}{\eta(t) - \eta(s)} \right) \gamma(t) dt, \quad s \in J.
\]

On account of [17], we have \( \mu \) is the unique solution of the integral equation 
\[ (I - N)\mu = -M\gamma, \quad (19) \]

where \( I \) denotes the identity operator. Additionally, \( h \) can be computed using the operators \( M \) and \( N \) by 
\[ h = [M\mu - (I - N)\gamma]/2. \quad (20) \]

The kernel of the operator \( N \) is known as the generalized Neumann kernel which is a generalization of the well-known Neumann kernel that corresponds to \( A = 1 \); see [15–17] for more details. The Neumann kernel appears in the double layer integral equation for Laplace’s equation in planar domains [18, p. 282]. See also [19–23].

In employing (19) and (20), respectively, we approximate \( \mu \) and \( h \) by the MATLAB function \( \text{fbie} \) from [15]. This function utilizes a discretization of (19) by the Nyström method associated with the trapezoidal rule [19] to
obtain an algebraic linear system of size \((m+1)n \times (m+1)n\), where \(n\) is the number of discretization points in each boundary component. The solution of the resulting system is obtained by applying the generalized minimal residual method through the MATLAB function \texttt{gmres}. What will drastically reduce the computational effort in case of a large number of CNTs or voids is the use of the fast multipole method to compute the matrix-vector multiplication when required in \texttt{gmres}. In this context, we take advantage of the MATLAB function \texttt{zfmm2dpart} from the MATLAB toolbox \texttt{FMMLIB2D} [24]. The values of the other parameters in the function \texttt{fbie} are chosen as in [25].

For the number of discretization points \(n\), when the boundaries of the domain are well separated from each other, accurate results can be obtained for moderate values of \(n\), say \(n = 2^7\) or \(n = 2^8\). If the boundaries are close to each other, then large values of \(n\) are required to obtain accurate results. Further, in the MATLAB function \texttt{fbie}, singularity subtraction [15,26] has been used in solving the integral equation which improves the accuracy of the numerical solution of the integral equation for domains with close to touching boundaries.

The MATLAB function \texttt{zfmm2dpart} has been employed also in [15] to write a MATLAB function \texttt{fcau} for fast and accurate computation of the Cauchy integral formula. Although the integral in the Cauchy integral formula becomes nearly singular when the formula is used to compute the values of an analytic function for points near to the boundary, applying singularity subtraction can overcome this singularity [15,27,28]. This technique of singularity subtraction has been implemented in the function \texttt{fcau}. For more details, we refer the reader to [15].

### 4 Computing the temperature distribution and the heat flux

By obtaining \(\mu\) and \(h\), we can compute the boundary values of \(g\) through (18). The values of \(g(\xi)\) for \(\xi \in G\) can be computed by the Cauchy integral formula. For this, we use the MATLAB function \texttt{fcau} from [15]. Then, the values of \(f(\xi)\) can be computed by (13), and hence, the values of the solution of the BVP (2) are given for \(\xi \in G\) by

\[
U(\xi) = \text{Re} \left[ f(\xi) + f_0(\xi) \right].
\]

We deduce the values of the temperature \(T(z)\) for any \(z \in \Omega\) by

\[
T(z) = \text{Re} \left[ f(\Psi^{-1}(z)) + f_0(\Psi^{-1}(z)) \right].
\]

Moreover, by computing the piecewise constant function \(h\), we can compute as well the values of \(c, \delta_1, \ldots, \delta_m\) from (16).

The function \(T(z)\) is the real part of the function

\[
W(z) = f(\Psi^{-1}(z)) + f_0(\Psi^{-1}(z)), \quad z \in \Omega.
\]

According to the Cauchy–Riemann equations, we get the derivative of the complex potential \(W(z)\) on \(\Omega\) by

\[
W'(z) = \frac{\partial T}{\partial x} - i \frac{\partial T}{\partial y}.
\]

Also,

\[
W'(z) = \frac{f'(\Psi^{-1}(z))}{\Psi'(\Psi^{-1}(z))} + \frac{f_0'(\Psi^{-1}(z))}{\Psi'(\Psi^{-1}(z))}, \quad z \in \Omega.
\] (21)

The denominator in (21) is nonzero on \(\Omega\) because \(\Psi\) is a conformal mapping. Consequently, the heat flux \(q(z)\) can be written for \(z \in \Omega\) as

\[
q(z) = - \left( \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right) \bigg|_z = -W'(z).
\] (22)
Hence,

\[
\frac{\partial T}{\partial y} = -\text{Im} \ W'(z).
\]  

The two derivatives \( f'_0(\Psi^{-1}(z)) \) and \( \Psi'(\Psi^{-1}(z)) \) in (21) can be computed analytically. So, \( q \) can be approximated on \( \Omega \) by approximating the derivatives of the boundary values of the function \( f \) on each boundary component. This can be done by approximating the function \( f(\eta(t)) \) using trigonometric polynomial interpolation and then differentiation. Finally, the values of \( f'(\Psi^{-1}(z)) \) in (21) can be then calculated for \( z \in \Omega \) through the Cauchy integral formula.

5 Computing the effective thermal conductivity

The medium matrix without inhomogeneities is assumed to be homogeneous and isotropic. We will assume that the CNTs and the circular voids are in the part of the domain between \( x = -1 \) and \( x = 1 \). Thus, the effective conductivity of a layer in the \( \lambda \) direction \( \lambda_y \) is calculated by the formula (3.2.33) from the book [6, p. 53], which in our case on account of (23) becomes

\[
\lambda_y = -\frac{1}{2} \int_{-1}^{1} \frac{\partial T}{\partial y}(x, 0) \, dx = \frac{1}{2} \text{Im} \left[ \int_{-1}^{1} W'(x) \, dx \right].
\]  

Since

\[
\xi_0(t) = \Psi(\eta_0(t)) = \Psi(e^{it}) = \frac{1}{\pi} \log \frac{1 + e^{it}}{1 - e^{it}} + i \frac{1}{2}, \quad 0 \leq t \leq 2\pi,
\]

where for \( 0 < t < \pi \), \( \xi_0(t) \) is on the line \( y = 1 \) and for \( \pi < t < 2\pi \), \( \xi_0(t) \) is on the real line. Thus, for \( \pi < t < 2\pi \), we have

\[
\xi_0(t) = \frac{1}{\pi} \log \left| \cot \frac{t}{2} \right| = -\frac{1}{\pi} \log \left| \tan \frac{t}{2} \right|,
\]

and hence, for \( \pi < t < 2\pi \),

\[
t = 2\pi - 2 \tan^{-1} \left( e^{-\pi \xi_0(t)} \right).
\]

Let \( t_1, t_2 \in (\pi, 2\pi) \) be such that \( \xi_0(t_1) = -1 \) and \( \xi_0(t_2) = 1 \). Then,

\[
\pi < t_1 = 2\pi - 2 \tan^{-1} \left( e^{\pi} \right) < t_2 = 2\pi - 2 \tan^{-1} \left( e^{-\pi} \right) < 2\pi.
\]  

Consequently, (24) can be written as

\[
\lambda_y = \frac{1}{2} \text{Im} \left[ \int_{t_1}^{t_2} W'(\xi_0(t)) \xi_0'(t) \, dt \right].
\]  

In combining (21) with the fact that \( \xi_0'(t) = ie^{it} \Psi'(e^{it}) \), we can see that

\[
W'(\xi_0(t)) = \frac{f'(\Psi^{-1}(\xi_0(t)))}{\Psi'(\Psi^{-1}(\xi_0(t)))} + \frac{f'_0(\Psi^{-1}(\xi_0(t)))}{\Psi'(\Psi^{-1}(\xi_0(t)))} = \frac{f'(e^{it})}{\Psi'(e^{it})} + \frac{f'_0(e^{it})}{\Psi'(e^{it})}.
\]
Hence,
\[
\lambda_y = \frac{1}{2} \Im \left[ \int_{t_1}^{t_2} \left[ i e^{it} \left( f'(e^{it}) + f'_0(e^{it}) \right) \right] dt \right],
\]  
(27)
which implies that
\[
\lambda_y = \frac{1}{2} \Im \left[ f(e^{it_2}) - f(e^{it_1}) \right] + \frac{1}{2} \Im \left[ f_0(e^{it_2}) - f_0(e^{it_1}) \right].
\]  
(28)
The second term in the last sum does not depend on neither the CNTs nor the voids. In view of (6) and (25), we have
\[
\frac{1}{2} \Im \left[ f_0(e^{it_2}) - f_0(e^{it_1}) \right] = 1,
\]  
and hence
\[
\lambda_y = 1 + \frac{1}{2} \Im \left[ f(e^{it_2}) - f(e^{it_1}) \right].
\]  
(29)
Since \( e^{it_1} \) and \( e^{it_2} \) are on the unit circle \( \Gamma_0 \), the external boundary of \( G \), and taking into account (13), (14), (15), and (18), Eq. (29) can be written as
\[
\lambda_y = 1 + \frac{1}{2} [\mu(t_2) - \mu(t_1)].
\]  
(30)

By solving (19), we obtain approximate values of \( \mu \) at the discretization points. These values are employed to interpolate the approximate solution \( \mu \) on \( J_0 \) by a trigonometric interpolation polynomial, which is then used to approximate the values of \( \mu(t_1) \) and \( \mu(t_2) \).

6 Numerical results

The above-presented method is applied to approximate the temperature distribution \( T \) as well as the heat flux \( q \) for several examples. Most of these examples include nearly touching boundaries, and therefore, we use \( n = 2^{11} \) discretization points on each boundary component. We will choose the CNTs and the circular voids within the part of the domain between \( x = -1 \) and \( x = 1 \). For some of the presented examples, we also discretize part of the domain \( \Omega \), namely for \(-1.5 \leq x \leq 1.5 \) and \( 0.0001 \leq y \leq 0.9999 \), and compute the values of \( T \) and \( q \) at these discrete points as described in Section 4.

6.1 The domain \( \Omega \) with only circular voids

In this subsection, we consider the domain \( \Omega \) with \( m \) non-overlapping circular holes and without any CNT (i.e., \( \ell = 0 \)). We also assume that all circular holes have the same radius \( r \) with the parametrization
\[
\eta_j(t) = z_j + re^{-it}, \quad 0 \leq t \leq 2\pi, \quad j = 1, 2, \ldots, m,
\]  
in which \( z_1, z_2, \ldots, z_m \) are the centers of the circular holes. As these circular holes are chosen in the part of the domain \( \Omega \) between \( x = -1 \) and \( x = 1 \), we define the concentration \( c(m, r) \) of these voids to be the area of these circular holes over the area of the rectangle \( \{(x, y) : -1 \leq x \leq 1, \ 0 \leq y \leq 1\} \), i.e.,
\[
c(m, r) = \frac{mr^2\pi}{2}.
\]  
(31)
The Clausius–Mossotti approximation (CMA) also known as Maxwell’s formula can be applied for dilute composites when the concentration (31) is sufficiently small. Below, we write this formula for a macroscopically isotropic media with insulators of identical circular holes within the precisely established precision in [29]

\[ \lambda_e = \frac{1 - c}{1 + c} + O(c^3). \]  

(32)

Example 1 We consider \( m = 5 \) circular holes with the radius \( r \) for \( 0 < r < 0.2 \). For Case I, we assume the centers of the holes to be set to \(-0.8 + 0.5i, -0.4 + 0.5i, 0.5i, 0.4 + 0.5i, \) and \( 0.8 + 0.5i \). The contour plot of \( T \) and \( |q| \) for \( r = 0.1 \) is shown in Fig. 3 (first row). The approximate value of the effective thermal conductivity for \( r = 0.1 \) is \( \lambda_y = 0.8533491 \).

When \( r \) is close to 0.2, the circular holes become adjacent to each other. To show the effects of the radius \( r \) on the effective thermal conductivity \( \lambda_y \), we compute the values of \( \lambda_y \) for several values of \( r, 0.00001 \leq r \leq 0.19999 \). The results are presented in Fig. 4 where, by (31), the concentration of these 5 holes is \( c = c(5, r) = 5r^2\pi/2 \approx 7.854r^2 \) for \( 0 < c < \pi/10 \) and \( 0 < r < 0.2 \). The values of the estimated effective conductivity \( \lambda_e \) are given also in Fig. 4.

As one can expect, there is a good agreement between \( \lambda_y \) and \( \lambda_e \) for small values of \( c \). In the same time, the divergence of \( \lambda_y \) and \( \lambda_e \) is observed for the concentrations greater than 0.1.

For Case II, the centers of the holes become \(-0.8 + 0.5i, -0.4 + 0.3i, 0.5i, 0.4 + 0.7i, \) and \( 0.8 + 0.5i \), which means the centers are not anymore horizontally aligned as the second and fourth centers are now shifted by 0.2 up and down, respectively. This is displayed in Fig. 3 (second row). The curve showing the obtained values of \( \lambda_y \) as a function of the concentration is depicted in Fig. 4.

Figure 4 illustrates that the values of \( \lambda_y \) depend on the position of the circular holes’ centers while the values of \( \lambda_e \) are the same for both cases since it depends only on the concentration of the circular holes and not on their positions. We notice a better agreement between \( \lambda_y \) and \( \lambda_e \) in Case II when comparing to Case I.

Example 2 We consider \( m = 30 \) circular holes with centers \( x_j + 0.25i, x_j + 0.5i, \) and \( x_j + 0.75i \), where \( x_j = -0.9 + 0.2(k - 1) \) for \( j = 1, 2, \ldots, 10 \), and with radius \( r \) for \( 0 < r < 0.1 \). The contour plot of \( T \) and \( |q| \) for \( r = 0.099 \) is shown in Fig. 5. The approximate value of the effective thermal conductivity for \( r = 0.099 \) is \( \lambda_y = 0.1519156 \).
The effective thermal conductivity $\lambda_y$ and the estimated effective conductivity $\lambda_e$ in (32) vs. the concentration $c(m, r) = 5r^2\pi/2$ for the domain $\Omega$ with $m = 5$ circular holes for $0.00001 \leq r \leq 0.19999$. The vertical dotted line is $c = \pi/10$.

When $r$ is close to 0.1, the circular holes become adjacent to each other. We compute the values of $\lambda_y$ for several values of $r$, $0.00001 \leq r \leq 0.09999$. The obtained results are depicted in Fig. 6 where, by (31), the concentration of these 30 holes is $c = c(30, r) = 30r^2\pi/2 \approx 47.124r^2$. Note that $0 < c < 3\pi/20$ for $0 < r < 0.1$.

Example 3 We take up here the case of $m = 50$ circular holes with centers $x_j + 0.1i, x_j + 0.3i, x_j + 0.5i, x_j + 0.7i,$ and $x_j + 0.9i$, where $x_j = -0.9 + 0.2(k - 1)$ for $j = 1, 2, \ldots, 10$, and with radius $r$ for $0 < r < 0.1$. On the basis of (31), the concentration of these 50 holes is $c = c(50, r) = 50r^2\pi/2 \approx 78.54r^2$. For $0 < r < 0.1$, we have $0 < c < \pi/4$. When $r$ is close to 0.1, the circular holes become adjacent to each other, and the concentration is almost equal to $\pi/4$. The obtained results showing the behavior of $\lambda_y$ as a function of the radius $r$, for $0.001 \leq r \leq 0.099$, are presented in Fig. 7.
The effective thermal conductivity $\lambda_y$ vs. the concentration $c(m, r) = mr^2\pi/2$ for $m = 50$ and $0.001 \leq r \leq 0.099$ (Example 3). The vertical dotted line is $c = \pi/4$.

6.2 The domain $\Omega$ with only CNTs

In this subsection, we consider the domain $\Omega$ with $m$ non-overlapping elliptic CNTs without any circular holes (i.e., $m = \ell$). We assume that all CNTs have equal sizes and are of elliptic shape where the ellipses have the parametrization

$$\eta_j(t) = z_j + a \cos t - ib \sin t, \quad 0 \leq t \leq 2\pi, \quad j = 1, 2, \ldots, m,$$

where $z_j$ is the center of the ellipse, $2a$ and $2b$ are the length of the ellipses axes in the $x$ and $y$-directions, respectively. If $a/b > 1$, the major axis of the ellipses is horizontal, if $a/b < 1$, the major axis of the ellipses is vertical, and if $a/b = 1$, the ellipses reduce to circles. Here, we choose $a$ and $b$ such that their ratio satisfies $0.1 \leq a/b \leq 10$. These elliptic shape CNTs are chosen in the part of the domain $\Omega$ between $x = -1$ and $x = 1$. So, we define the concentration $c(m, a, b)$ of these CNTs to be

$$c(m, a, b) = \frac{mab\pi}{2}.$$

(34)

If $\frac{a}{b} \ll 1$, instead of (34) the plane slits density is considered in the theory of composites and porous media

$$\phi = \frac{mb^2}{|\Omega|} = \frac{mb^2}{2}.$$

(35)

For a macroscopically isotropic media with only perfectly conducting identical circular inclusions (CNTs), an approximation of the effective conductivity $\lambda_e$ is given by the reciprocal to (32) value (see [29])

$$\lambda_e = \frac{1 + c}{1 - c} + O(c^3).$$

(36)

Example 4 We consider $m = 5$ elliptic CNTs with centers $-0.8 + 0.5i, -0.4 + 0.5i, 0.5i, 0.4 + 0.5i,$ and $0.8 + 0.5i$, and where $0 < a < 0.2$ and $0 < b < 0.5$. Figure 8 (first row) presents the contour plot of $T$ and $|q|$ for $a = 0.19$ and $b = 0.019$ (the ellipses are horizontal). For these values of $a$ and $b$, the approximate value of the effective thermal conductivity is

$$\lambda_y = 1.0272480.$$
For \( a = 0.019 \) and \( b = 0.19 \), the contour plot of \( T \) and \( |q| \) is shown in Fig. 8 (second row). The approximate value of the effective thermal conductivity for these values of \( a \) and \( b \) is

\[
\lambda_y = 1.2804116.
\]

The CNTs in Fig. 8 have the same concentration. However, the value of \( \lambda_y \) is larger for the vertical ellipses case.

When \( a \) approaches 0.2, the ellipses get adjacent to each other. On the other hand, they come close to the upper and lower walls when \( b \) approaches 0.5. We compute the values of \( \lambda_y \) for several values of \( a, 0.0001 \leq a \leq 0.1999 \), with \( b = 0.1a \). The obtained results are presented in Fig. 9 (left) where, by (34), the concentration of these 5 ellipses is

\[
c = c(5, a, b) = 5ab\pi/2 = a^{2}\pi/4 \approx 0.7854a^2.
\]

Note that, for \( 0 \leq a < 0.2 \) and \( b = 0.1a \), we have

\[
0 < c < \pi/100.
\]

The values of \( \lambda_y \) are also computed for several values of \( b \) for \( 0.001 \leq b \leq 0.499 \) with \( a = 0.1b \). Since \( a/b = 0.1 \) is small, the obtained values of \( \lambda_y \) are plotted versus the values of \( \phi = 2.5b^2 \), given by (35), where

\[
0 < \phi < 5/8 \text{ for } 0 < b < 0.5.
\]

The obtained results are presented in the right side of Fig. 9.

**Example 5** We consider \( m = 200 \) elliptic CNTs with centers \( x_k + iy_j \) for \( k = 1, 2, \ldots, 20 \) and \( j = 1, 2, \ldots, 10 \) where \( x_k = -0.95 + (k - 1)/10 \) and \( y_j = 0.05 + (j - 1)/10 \), and with \( 0 < a < 0.05 \) and \( 0 < b < 0.05 \).

We compute the values of \( \lambda_y \) for several values of \( a, 0.0002 \leq a \leq 0.0498 \), and \( a/b = 10 \) (i.e., the ellipses are horizontal) where the ellipses become close to each other when \( a \) approaches 0.05. The obtained results are presented in Fig. 10 (left) where the concentration of these 200 ellipses is

\[
c = 10a^2\pi \approx 31.416a^2.
\]

For \( 0 < a < 0.05 \) and \( a/b = 10 \), we have \( 0 < c < \pi/40 \). Then, we compute the values of \( \lambda_y \) for several values of \( b, 0.0002 \leq b \leq 0.0498 \), and \( a/b = 0.1 \). The obtained results for \( \lambda_y \) versus the plane slits density \( \phi = mb^2/2 = 100b^2 \) are presented in Fig. 10 (right), where

\[
0 < \phi < 1/4 \text{ for } 0 < b < 0.05 \text{ and } a/b = 0.1.
\]

When \( a/b = 1 \), the ellipses reduce to circles. We compute the values of \( \lambda_y \) for several values of \( a, 0.0002 \leq a \leq 0.0498 \). The obtained results are presented in Fig. 11 where the concentration of these 200 ellipses is

\[
c = 100a^2\pi \approx 314.16a^2.
\]

For \( 0 < a < 0.05 \) and \( a/b = 1 \), we have \( 0 < c < \pi/4 \). Figure 11 presents also the values of the estimated effective conductivity \( \lambda_e \) given by (36).
Fig. 9 The effective thermal conductivity $\lambda_y$ for the domain $\Omega$ with $m = 5$ elliptic CNTs (Example 4). On the left, the effective thermal conductivity $\lambda_y$ vs. the concentration $c = a^2 \pi / 4$ for $0.0001 \leq a \leq 0.1999$ and $a/b = 10$. The vertical dotted line is $c = \pi/100$. On the right, the effective thermal conductivity $\lambda_y$ vs. the plane slits density $\phi = 2.5b^2$ for $0.001 \leq b \leq 0.499$ and $a/b = 0.1$. The vertical dotted line is $\phi = 0.625$.

Fig. 10 The effective thermal conductivity $\lambda_y$ for the domain $\Omega$ with $m = 200$ elliptic CNTs (Example 5). On the left, the effective thermal conductivity $\lambda_y$ vs. the concentration $c = 10a^2 \pi$ for $0.0002 \leq a \leq 0.0498$ with $a/b = 10$. The vertical dotted line is $c = \pi/40$. On the right, the effective thermal conductivity $\lambda_y$ vs. the plane slits density $\phi = 100b^2$ for $0.0002 \leq b \leq 0.0498$, with $a/b = 0.1$. The vertical dotted line is $\phi = 1/4$. The vertical dotted line is $c = 1/4$.

Fig. 11 The effective thermal conductivity $\lambda_y$ (for the domain $\Omega$ with $m = 200$ circular CNTs obtained by setting $b = a$ in Example 5) and the estimated effective conductivity $\lambda_e$ in (36) vs. the concentration $c = 100a^2 \pi$ for $0.0002 \leq a \leq 0.0498$. The vertical dotted line is $c = \pi/4$. The vertical dotted line is $c = 1/4$. 
Fig. 12 The domain \( \Omega \) with \( m = 2000 \) circular holes. Case I: We have \( p = 1000 \) voids (blue circles) and \( \ell = 1000 \) CNTs (red circles).

Fig. 13 The values of the effective thermal conductivity \( \lambda_y \) and the estimated effective conductivity \( \lambda_e \) in (37) (for the domain \( \Omega \) with \( m = 2000 \) circular holes in Example 6) vs. the number of the experiment for Case I (first row), Case II (second row, left), and Case III (second row, right).

6.3 The domain \( \Omega \) with 2000 CNTs/voids

We are concerned in this section with the study of a large number of inclusions/holes. We consider two examples where in the first both perfectly conducting inclusions and insulating holes have the same circular shape, while in the second, conductors have an elliptic shape and insulators have a circular shape. The present investigation is useful when studying the impact of geometric shapes on the macroscopic properties of three-phases high-contrast media.
Table 2 The value of $\lambda_e$, average, and standard deviation of the values of the effective thermal conductivity $\lambda_y$ for the 20 different locations for the domain $\Omega$ in Example 6

|                | Case I | Case II | Case III |
|----------------|--------|---------|----------|
| $\lambda_e$    | 1      | 0.6996  | 1.4293   |
| The average of the values of $\lambda_y$ | 0.9995 | 0.6991  | 1.4289   |
| The standard deviation of the values of $\lambda_y$ | 0.0034 | 0.0015  | 0.0020   |

Example 6 We take $m = 2000$ circular holes of equal size with radius $r = 0.0075$. In this example, the concentration $c = c(m, r) = 1000r^2\pi \approx 0.1767$ is constant and the locations of these holes are chosen randomly. In this case, the following extension of CMA may be used

$$\lambda_e = \frac{1 + c_1 - c_2}{1 - c_1 + c_2} + O(c^3),$$  (37)

where $c_1$ denotes the conductor concentration, $c_2$ the insulator concentration, and $c = c_1 + c_2$. Three cases are considered:

- Case I: We assume that half of the holes are CNTs and the other half are voids (see Fig. 12). For this case, $c_1$ and $c_2$ are given by $c_1 = c_2 = 500r^2\pi \approx 0.0884$.
- Case II: All holes are voids, and hence $c_1 = 0$ while $c_2 = 1000r^2\pi \approx 0.1767$.
- Case III: All holes are CNTs, and hence $c_1 = 1000r^2\pi \approx 0.1767$ while $c_2 = 0$.

For each case, we run the code for 20 times, so that to get 20 different locations for these circular holes. In each of these 20 experiments, we compute the value of the effective thermal conductivity $\lambda_y$ by the presented method and the values of the estimated effective conductivity $\lambda_e$ by (32). As we can see from Fig. 13, $\lambda_e$ is a constant and the values of $\lambda_y$ depend on the locations of the holes. For the three cases, the value of $\lambda_e$ as well as the average and standard deviation of the values of the effective thermal conductivity $\lambda_y$ for the 20 different locations are presented in Table 2.

Example 7 We consider $m = 2000$ with $\ell = 1000$ elliptic perfect conductors and $p = 1000$ circular insulators of equal area $\pi r^2$ (see Fig. 14). The radius $r$ is chosen to be the same as in the previous example, i.e., $r = 0.0075$. The locations of both elliptic and circular holes are chosen randomly. For the ellipses, we assume that the ratio between the length of the major axis and the minor axis is 4, and the angles between the major axis and the $x$-axis are chosen randomly. As in the previous example, we run the code for 20 times. In each of these 20 experiments, we compute the value of the effective thermal conductivity $\lambda_y$ by the presented method. The computed values are shown in Fig. 14 (left).

Since we have the same number of elliptic perfect conductors and circular insulators of equal area $\pi r^2$, the conductor concentration $c_1$ and the insulator concentration $c_2$ are equal and given by $c_1 = c_2 = 500r^2\pi \approx 0.0884$. Although $c_1$ and $c_2$ here are the same as in Case I of the previous example, it is clear from Figs. 13 (first row) and 14 (right) that the values $\lambda_y$ in this example (elliptic conductors) are larger than those in the previous example (circular conductors).

6.4 The dependence of $\lambda_y$ on $\phi$ and $c$

We consider now a domain $\Omega$ with $m = \ell = 276$ non-overlapping elliptic CNTs without any void. We assume that all CNTs are of equal size and elliptic shape. The ellipses are parametrized by (33) with $a < b$, which means they are taken to be vertical.
First we assume that the concentration \( c = c(m, a, b) \) is constant and we choose the values of the parameters \( a \) and \( b \) such that the plane slits density \( \phi = \phi(m, b) \in [0.4, 1.3] \). The domain \( \Omega \) for \( c = 0.5 \) and \( \phi = 1.3 \) is shown in Fig. 15. The contour plots of \( T \) and \( |q| \) for \( c = 0.3 \) and \( \phi = 0.8 \) are presented in Fig. 16.

We also consider five distinct values of the concentration, \( c = 0.1, 0.2, 0.3, 0.4, 0.5 \). Then, for each of these values, we compute and show in Fig. 17 (left) the values of \( \lambda_y = \lambda_y(\phi) \). Afterward, we take up the plane slits density \( \phi = \phi(m, b) \) to be constant and we choose the values of the parameters \( a \) and \( b \) such that the concentration \( c = c(m, a, b) \in [0.1, 0.5] \). We consider four values of \( \phi, \phi = 0.4, 0.7, 1, 1.3 \), and compute again \( \lambda_y = \lambda_y(c) \) for each case. The obtained results are presented in Fig. 17 (right).

The dependence of the conductivity \( \lambda_y \) on \( \phi \) and \( c \) is numerically illustrated in Fig. 17, which reveals that \( \lambda_y \) is more sensitive to the density parameter \( \phi \) than to the concentration \( c \). Therefore, the percolation effect for the superconductivity is dominating due to the elliptical shapes of the inclusions contained in the definition of the parameter \( \phi \). The impact of circular inclusions of concentration \( c \) onto the percolation is negligible. This observation
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Fig. 17 The effective thermal conductivity $\lambda_y$ for the domain $\Omega$ with $m = 276$ elliptic CNTs. On the left, the values of $\lambda_y = \lambda_y(\phi)$ for $\phi \in [0.4, 1.3]$ and for several values of $c$. On the right, the values of $\lambda_y = \lambda_y(c)$ for $c \in [0.1, 0.5]$ and for several values of $\phi$.

is inline with the analytical formulas from [30] concerning the impact of circular and elliptic inclusions onto the effective conductivity.

7 Conclusion

A systematic estimation of the local fields and the effective conductivity properties of 2D composites reinforced by uniformly and randomly distributed CNTs is carried out. It is assumed that the medium may contain voids as well. The CNTs are considered as perfectly conducting elliptic inclusions and the voids as circular insulators. For definiteness, a composite strip is considered with the given constant external field passing through the strip. The local field is governed by the Laplace equation in the multiply connected domain formed by the strip without two types of holes, CNTs, and voids. The Dirichlet boundary condition is imposed on the CNTs boundary, and the Neumann boundary condition governs the void boundary. A numerical method is developed to solve the mixed problem for a large number of CNTs and voids. The method makes use of the boundary integral equation with the generalized Neumann kernel [15,16]. One key feature of this method is that it can be employed for domains with complex geometry as it provides accurate results even when the boundaries are close together. To solve the integral equation, the Fast Multipole Method has been employed, which enables to treat the case of thousands of CNTs and voids. With the help of conformal mappings, the presented method can be extended to treat the case when CNTs and voids are rectilinear slits as done in [25] for example.

The computational study has shown a dependence of the local fields and the effective conductivity $\lambda_y$ on the concentration of voids $c$ given by (34) as well as on the density $\phi$ of CNTs given by (35). Besides the opposite conductive properties, voids and CNTs have also different types of the geometric parameters $c$ and $\phi$ that are not reduced to each other. Hence, the present study is concerned additionally with the case of three-phase composites of high-contrast conductivity. It is demonstrated that our simulations are covered with the classical lower order approximations (Clausius–Mossotti, Maxwell) for dilute composites. The high order concentrations and densities led to different results from the classical ones. It is worth noting that the huge number of numerical experiments for uniformly distributed inclusions yield the graphical dependencies of $\lambda_y$ on $c$ and $\phi$, which can be used in practical applications.

All MATLAB codes for the computations in this article are available in GitHub at github.com/mmsnasser/localfields.
Acknowledgements The authors would like to thank two anonymous reviewers for their valuable comments and suggestions which improved the presentation of this paper.

Funding Open Access funding provided by the Qatar National Library.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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