Spinning cosmic strings: a general class of solutions

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Abstract

In this work, we give a general class of solutions of the spinning cosmic string in Einstein’s theory of gravity. After treating same problem in Einstein Cartan (EC) theory of gravity, the exact solution satisfying both exterior and interior space-times representing a spin fluid moving along the symmetry axis is presented in the EC theory. The existence of closed timelike curves in this spacetime are also examined.
1 Introduction

Cosmic strings are line-like objects. Like other topological defects [1], they might have been formed during the phase transitions in the very early universe. The cosmic string space-time is locally flat. Globally it has a conical structure that may result in some non-trivial global effects. The current experimental results exclude the predictions of the cosmic strings about the large scale structure and the anisotropy in the cosmic microwave background radiation. Their gravitational effects, however, make it worthwhile to study cosmic strings. These effects may be listed as: vacuum fluctuations, gravitational lensing, bending of light. They may also give an explanation to the most energetic events in the universe such as ultrahigh energy neutrinos, cosmological gamma-ray bursts. Furthermore, a mixture of topological effects and inflation still gives a theory consistent with the current cosmic microwave background data [2].

A straight cosmic string can be characterized by its linear mass density $\mu$ which depends on the formation energy scale. For the cosmic strings in the grand-unified theories, the linear mass density is $\mu = 10^{22}\text{gr/cm}$. Here we are interested in a special class of cosmic strings, namely the spinning cosmic strings which can be determined by their linear energy density $\mu$ and intrinsic spin $J$. Geometrically the spinning cosmic string space-time can be obtained by cutting a wedge from four dimensional Minkowski space-time and gluing it back after making a boost in one of the faces. This boost is different from the one made in the "straight string" case. Since there is no Lorentz boost invariance anymore, an observer moving along the $z$-axis will see a twisted metric with the spacelike helical structure [3-6]. In the spinning cosmic string spacetime, some physical properties are studied in the literature.
like the quantization of the energy [7], semi-classical gravitational effects [8], vacuum fluctuations [9, 10] and closed timelike curves [11].

In the following section, we present a general class of solutions for the field equations of the Einstein’s gravitational theory in the vacuum for the spinning string. We start our calculation using the metric proposed by [3], where the strings with helical structure in the time coordinate as well as the space coordinate along the string is presented. It is also shown that these solutions can be proposed by considering thick cylindrical sources which do not have δ-type singularity in the curvature and energy-momentum tensor.

In either method, other singular and non-singular sources can also be considered such as electric current carrying strings on which small structures (wiggles) exist. But here spinning fluid is considered as the reason for the singularities in the curvature and torsion.

In the continuum mechanical point of view, a cosmic string can be represented by disclinations in the space-time and the helical structure of the spinning cosmic string is associated with the timelike and spacelike dislocations. In [4], by introducing torsion along the string or treating the problem in the EC (Einstein-Cartan) theory, some ill-defined quantities are removed, or the missing part of the spacetime defects are recovered. Since torsion is coupled to the theory by means of spin, it is seen that, for the spacetime presented, the torsion tensor has the same δ-type singularities as the curvature and the energy-momentum tensor [12].

In the framework of the distorted spacetimes some exact solutions of the EC theory, which can be interpreted as spin-polarized cosmic strings and cosmic dislocations, are given in [6]. One may also interpret these solutions as those that exist in the presence of sources which have non-zero thickness such as spinning fluids. In section 3, we suppose that a spin fluid exists
in a cylinder with radius $r_0$ moving along the $z$-axis and fulfilling the exact
solution conditions for the EC theory. We consider the interior (the spin fluid
with matter and torsion distribution) and exterior (vacuum of the spin fluid)
space-times induced by the spinning fluid. In this example the interior and
the exterior space-times are subject to the continuity and the Arkuszewski-
Kopczynski-Ponomariev (AKP) junction conditions [13], i.e., continuity in
the metric components and discontinuity in the first derivatives of the metric
components.

We also study closed timelike curve (CTC) properties of the metrics cor-
responding to the spinning cosmic strings.

2 Spinning cosmic string in Einstein theory of gravity

Spinning cosmic string spacetime line element can be given by

$$ds^2 = -(dt + 4GJ^t d\phi)^2 + dr^2 + b^2 r^2 d\phi^2 + (dz + 4GJ^z d\phi)^2$$ (1)

where $b$ is the string parameter and defined in terms of string energy density
$\mu$ and Newton’s constant $G$ as $b = 1 - 4G\mu$. The constants $J^t$ and $J^z$ are
spin of the string and dislocations of the space respectively [3, 4]. In these
references they examine the physical structures of the space-times depending
on the three different values of $j^2 = (J^t)^2 - (J^z)^2$:

i) $j^2 = 0$ (or $|J^z| = |J^t|$),

$$ds^2 = -(dt + 4GJ^t d\phi)^2 + dr^2 + b^2 r^2 d\phi^2 + (dz + 4GJ^z d\phi)^2$$ (2)

represents a string interacting with a circularly polarized plane-fronted grav-
itational wave,
ii) $|J^z| = 0,$

$$ds^2 = -(dt + 4GJ^t d\phi)^2 + dr^2 + b^2 r^2 d\phi^2 + dz^2$$  \hspace{1cm} (3)

represents a spinning cosmic string with no cosmic dislocations and

iii) $|J^t| = 0,$

$$ds^2 = -dt^2 + dr^2 + b^2 r^2 d\phi^2 + (dz + 4GJ^z d\phi)^2$$  \hspace{1cm} (4)

here the spacetime represents screw dislocations; the superposition of screw dislocation ($2GJ^z / \pi$ is analogous to Burgers vector) and disclination.

Now we consider a general vacuum solution of the Einstein field equations corresponding to the spinning cosmic string lying along the $z$-axis which has the line element

$$ds^2 = -[dt + F(z) d\phi]^2 + dr^2 + b^2 r^2 d\phi^2 + [dz + H(t) d\phi]^2$$  \hspace{1cm} (5)

where $F(z)$ is spin of the string and $H(t)$ is the analogue of the Burgers vector of dislocations. Vacuum field equations require that $F$ and $H$ must be linear in their variables such that $F(z) = az + J^t$ and $H(t) = at + J^z$ with $a$, $J^t$ and $J^z$ are constants. The space-time given by (1) is a special form of (5), if we take $a = 0$ the previous case is obtained [3, 4]. Since the nature of the problem will not change, for the sake of the simplicity we make transformations $z \to (z - J^t / a)$ and $t \to (t - J^z / a)$ and take $a = 4GJ$, then we get

$$ds^2 = -[dt + 4GJ z d\phi]^2 + dr^2 + b^2 r^2 d\phi^2 + [dz + 4GJ t d\phi]^2.$$  \hspace{1cm} (6)

Since the space-time is locally flat except along the $z$-axis, to examine the structure of the singularity we also need to make the transformation $r \to r^b / b$
and write (6) in Cartesian coordinates
\[ ds^2 = -[dt+4 G J z(Xdy-Ydx)]^2 + r^{-8G\mu}(dx^2+dy^2)+[dz+4 G J t(Xdy-Ydx)]^2, \]
where \( X = x/r^2, \ Y = y/r^2 \) and \( r = \sqrt{x^2+y^2} \). In this spacetime one forms are defined as
\begin{align*}
\theta^0 &= \ dx^0 + 4 G J x \left( Xdx^2 - Ydx^1 \right) \\
\theta^3 &= \ dx^3 + 4 G J x \left( Xdx^2 - Ydx^1 \right) \\
\theta^1 &= \ r^{-4G\mu}dx^1 \\
\theta^2 &= \ r^{-4G\mu}dx^2
\end{align*}
(0, 3, 1, 2) represents \((t, z, x, y)\) and metric is defined as \( \eta_{ij} = diag(-1, 1, 1, 1) \) and we use the definition \( \partial X/\partial x^1 + \partial Y/\partial x^2 = 2\pi\delta(x^1)\delta(x^2) \), same notation given in [3, 4].

During the calculations of Einstein’s tensors, in addition to \( \delta \)-type distributions we obtain \((\delta(x^1)\delta(x^2))^2\) type singularities. We can approach this problem in two ways: First, \([\delta(x^1)\delta(x^2)]^2\) term is taken to be zero or lower order than \( \delta(x^1)\delta(x^2) \) as examined in [3, 4] and second, the metric (5) represents the exterior spacetime of the thick non-zero source and the interior solution and matching conditions should be satisfied [14].

In the first approach the non-zero components of the Einstein tensor are
\begin{align*}
G_{00} &= -G_{33} = 8\pi G\mu r^{8G\mu}\delta(x^1)\delta(x^2) \\
G_{01} &= -G_{31} = 4\pi G J x^3 r^{12G\mu}\partial[\delta(x^1)\delta(x^2)]/\partial x^2 \\
G_{02} &= -G_{32} = -4\pi G J x^0 r^{12G\mu}\partial[(\delta(x^1)\delta(x^2))]/\partial x^1
\end{align*}
and the scalar curvature is
\[ R = 16\pi G\mu r^{8G\mu}\delta(x^1)\delta(x^2). \]
We see that the spacetime given by (7) represents a cosmic string since tension
of the string is equal to its linear energy density i.e., \( G_{00} = G_{33} \) and it is valid
both for exterior spacetime and along the singular line source [15, 16].

In the second approach, we consider thick string, a non-singular source,
as an interior solution which is cylindrically symmetric. The exterior metric
is (7) and satisfies matching conditions; continuity in the metric components
and junction condition in the exterior curvatures of exterior and interior
spacetimes on the cylinder with radius \( r_0 \) [?]. Now we introduce interior
metric as

\[
ds^2 = -(dt + A d\phi)^2 + dr^2 + W^2 d\phi^2 + (dz + B d\phi)^2
\]

(11)

where \( A = cz + q(r) \) is function of \( z \) and \( r \), \( B = ct + k(r) \) is function of \( t \) and
\( r \) and \( W \) is function of \( r \) only.

The interior spacetime can be defined by the anisotropic fluid moving
along the string (\( z \)-axis) and the energy momentum tensor of an anisotropic
fluid in the Riemannian (torsionless) space is given by

\[
t_{ij} = (\epsilon + p)u_i u_j - p \eta_{ij} + (P - p)v_i v_j
\]

(12)

Here, \( u \) is the fluid’s velocity, \( p \) is the isotropic pressure in the \( r \) and \( \phi \)
directions and \( P \) is the pressure in the \( z \)-direction. \( v \) is a spacelike vector
orthogonal to \( u \) and \( \epsilon \) is the linear density of the string. Velocities obey the
normalization conditions \( u^i u_i = -1, v^i v_i = 1 \) and the orthogonality condition
\( u^i v_i = 0 \). Then, corresponding equations become

\[
G_{00} = -(\frac{W'}{W} - \frac{q'^2}{2W^2}) = (\epsilon + p)u_0 u_0 + p + (P - p)v_0 v_0
\]

\[
G_{33} = (\frac{W'}{W} + \frac{k'^2}{2W^2}) = (\epsilon + p)u_3 u_3 - p + (P - p)v_3 v_3
\]
\[ G_{11} = \left( \frac{q'^2}{4W^2} - \frac{k'^2}{4W^2} \right) = -p + (P - p)v_1v_1 \]

\[ G_{22} = \left( \frac{q'^2}{4W^2} - \frac{k'^2}{4W^2} \right) = -p + (P - p)v_2v_2 \]

\[ G_{01} = -\frac{k'c}{2W^2} = (P - p)v_0v_1 \]

\[ G_{02} = \left( \frac{q'}{2W} \right)_r = (P - p)v_0v_2 \]

\[ G_{31} = \frac{q'c}{2W^2} = (P - p)v_3v_1 \]

\[ G_{32} = -\left( \frac{k'}{2W} \right)_r = (P - p)v_3v_2 \]

\[ G_{03} = -\frac{q'k'}{2W^2} = (\epsilon + p)u_0u_3 + (P - p)v_0v_3 \] (13)

If we take \( q(r) = \text{constant} \) and \( k(r) = \text{constant} \) the interior solution corresponds to spinning cosmic string where \( G_0^0 = G_3^3 \). In general, it represents spinning string with the Einstein’s tensors given by (13). And matching conditions on the cylinder with radius \( r_0 \) \((t = t_0 \text{ and } z = z_0)\) are

\[ g_{\mu\nu}|_+ = g_{\mu\nu}|_- \]

\[ K_{\mu\nu}^+ = K_{\mu\nu}^- \] (14)

where \( +(-) \) represents exterior (interior) space-time and \( K_{\mu\nu}^\pm \) represents the corresponding exterior derivative, \( K_{ij}^\pm = e_i^\alpha e_j^\beta n_{\alpha \beta}^\pm \). \( e_i^\alpha \) is the orthonormal triad lying in the junction surface and \( n_{\alpha}^\pm \) is the unit normal vector outward (inward) for the interior (exterior) space-times. These conditions give the following set of equations

\[-az_0^2 + a^2t_0^2 + b^2r_0^2 = [cz_0 + q(r_0)]^2 + [ct_0 + k(r_0)]^2 + W^2(r_0)\]

\[ az_0 = cz_0 + q(r_0) \]
\[ at_0 = ct_0 + k(r_0) \]
\[ b^2r_0 = qq' + kk' + WW' \]
\[ 0 = k' \]
\[ 0 = q' \]  \hspace{1cm} (15)

From last two equations it is seen that \( q \) and \( k \) must be functions of \((r - r_0)\) or equal to constants. We can make assumptions \( W(r) = \sin r, \ q = \sin(r + \pi/2) \) and \( k = r + \text{const.} \) such that our solution will represent fluid moving in a cylinder with radius \( r_0 \) which is similar to the straight string solution but a little complicated. Corresponding \( r \) dependent velocities are as follows

\[ u_0 = \frac{1}{\sqrt{\epsilon + p}} \left( -p + \frac{q^2}{2W^2} - \frac{c^2k'^2}{2W^2(2pW^2 - k'^2 + q'^2)} \right)^{1/2} \]
\[ u_3 = \frac{1}{\sqrt{\epsilon + p}} \left( p + \frac{k'^2}{2W^2} - \frac{c^2q'^2}{2W^2(2pW^2 - k'^2 + q'^2)} \right)^{1/2} \]
\[ v_0 = -\frac{ck'}{W\sqrt{2(P - p)}} \left( 2pW^2 - k'^2 + q'^2 \right)^{1/2} \]
\[ v_3 = \frac{cq'}{W\sqrt{2(P - p)}} \left( 2pW^2 - k'^2 + q'^2 \right)^{1/2} \]
\[ v_1 = \frac{1}{\sqrt{P - p}} \left( p - \frac{k'^2}{2W^2} + \frac{q'^2}{2W^2} \right)^{1/2} \]
\[ v_2 = \frac{1}{\sqrt{P - p}} \left( p - \frac{k'^2}{2W^2} + \frac{q'^2}{2W^2} \right)^{1/2} . \]  \hspace{1cm} (16)

With the constraints coming from last two equations of (13) are

\[ \frac{cq' - k'W' + Wk''}{2W^2} = 0 \]
\[ \frac{ck' + q'W' + Wq''}{2W^2} = 0, \]  \hspace{1cm} (17)
and from the orthogonality of \( \mathbf{u} \cdot \mathbf{v} = 0 \) we get

\[
k' \sqrt{-2pW^2 + q^2 + \frac{c^2k'^2}{-2pW^2 + k'^2 - q'^2}} + q' \sqrt{2pW^2 + k'^2 + \frac{c^2q'^2}{-2pW^2 + k'^2 - q'^2}} = 0. \tag{18}
\]

Pressures can be obtained from the normalization conditions of \( \mathbf{u} \cdot \mathbf{u} = -1 \) and \( \mathbf{v} \cdot \mathbf{v} = 1 \)

\[
\frac{8p^2W^4 + 2pW^2(-k'^2 + q'^2) + (k'^2 - q'^2)(c^2 - k'^2 + q'^2)}{2(\epsilon + p)W^2(2pW^2 - k'^2 + q'^2)} = -1
\]

\[
\frac{-8p^2W^4 + 8pW^2(k'^2 - q'^2) + (k'^2 - q'^2)(c^2 - 2k'^2 + 2q'^2)}{2(p - P)W^2(2pW^2 - k'^2 + q'^2)} = 1 \tag{19}
\]

and they should be equal to zero on the boundary surface \( p = P = 0 \) at \( r = r_0 \). If we choose \( W(r) = br \) or \( W(r) = b \sin(r/b) \) as in the literature, we meet non-simple choice of \( q(r) \) and \( k(r) \) therefore non-simple pressures but, one can always find solutions desired functions for \( W, q, k \) and/or pressures.

### 3 Spinning cosmic string in Einstein-Cartan theory

We can treat the same problem; the general form of the spinning cosmic string in the framework of EC-theory of gravity in which the torsion is different from zero and coupled to the theory through the spin momentum density of the matter. The field equations to be satisfied in EC theory are

\[
R_{ij} - \frac{1}{2} g_{ij} R = \kappa t_{ij} \tag{20}
\]

\[
T^i_{jk} - \delta^i_j T^l_{lk} - \delta^i_k T^l_{jl} = \kappa s^i_{jk} \tag{21}
\]

where \( \kappa \) is gravitational constant and we take \(-8\pi G\). Here, \( t_{ij} \) energy-momentum, \( T^i_{jk} \) torsion and \( s^i_{jk} \) spin tensors of the matter. From, (21) it
is seen that the torsion is coupled to the system by means of spin distribution of the matter. Here, $s_{jk}^i = u^i S_{jk}$, $u^i$ is 4-velocity and $S_{jk}$ intrinsic-spin angular momentum of the matter which are subject to the Frenkel condition $u^i S_{ji} = 0$.

In this section we try to find a spacetime of EC theory which satisfies (20),(21). Let us suppose that the spacetime (5) represents a spin fluid flowing along the $z$-axis with velocity $u = (u^0, u^3, 0, 0)$ then spin fluid moving along the $z$-axis generates torsion distribution as $T^0 = T_{12}^0 \theta^1 \wedge \theta^2$, $T^3 = T_{12}^3 \theta^1 \wedge \theta^2$. Now we consider two different torsions

i)

\[
\begin{align*}
T_{12}^0 &= 8\pi G J r^8 G^\mu (x^3 - 1) \delta(x^1) \delta(x^2) \\
T_{12}^3 &= 8\pi G J r^8 G^\mu (x^0 - 1) \delta(x^1) \delta(x^2)
\end{align*}
\]

(22)

which has the same functional form with (13) and

ii)

\[
\begin{align*}
T_{12}^0 &= 8\pi G J r^8 G^\mu x^3 \delta(x^1) \delta(x^2) \\
T_{12}^3 &= 8\pi G J r^8 G^\mu x^0 \delta(x^1) \delta(x^2)
\end{align*}
\]

(23)

which gives the same energy-momentum with the straight cosmic string with non-zero torsion,

\[
G_{00} = -G_{33} = 8\pi G \mu r^8 G^\mu \delta(x^1) \delta(x^2).
\]

(24)

The non-zero components of the spin tensor which are the solutions of (21) are $s_{12}^0 = -T_{12}^0/8\pi G$ and $s_{12}^3 = -T_{12}^3/8\pi G$, and non-singular spin angular momentum is $S_{12} = s_{12}^0/u^0 = s_{12}^3/u^3$.

In Einstein theory of gravity, the space-time is torsionless and the curvature and non-zero components of the energy momentum tensor have $\delta$-type
singularities. In EC theory in addition to these singularities we see that the torsion tensor has the same type of singularity.

Torsion distributions given by (22) and (23), and corresponding energy-momentum densities given by (13) and (24) defines a spinning cosmic string both exterior and along the singular source with torsion.

Our aim is to find the exact solution of the field equations both satisfied by vacuum solution (5) as exterior, and the spin fluid as interior. Now we introduce cylindrically symmetric interior metric as

$$ds^2 = -[dt + A(t, z, r)d\phi]^2 + dr^2 + W^2(r)d\phi^2 + [dz + B(t, z, r)d\phi]^2$$  \hspace{1cm} (25)

with $A = f(t, z) + q(r)$, $B = h(t, z) + k(r)$ and the spin fluid moving along the $z$-axis produces the torsion

$$T_{12}^0 = A'/W = q'/W$$
$$T_{12}^3 = B'/W = k'/W.$$  \hspace{1cm} (26)

In this spacetime 1-forms are

$$\theta^0 = dx^0 + A dx^2$$
$$\theta^3 = dx^3 + B dx^2$$
$$\theta^1 = dx^1$$
$$\theta^2 = W dx^2$$  \hspace{1cm} (27)

where $(x^0, x^1, x^2, x^3)$ corresponds to $(t, z, r, \phi)$ and the connection 1-forms are

$$\omega_3^0 = \frac{f_0}{W} \theta^2 = \frac{f_3}{W} \theta^2$$
$$\omega_0^0 = 0$$
$$\omega_0^3 = -\frac{f_0}{W} \theta^0$$

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\[
\begin{align*}
\omega_1^3 &= 0 \\
\omega_2^3 &= -\frac{h_3}{W} \theta^3 \\
\omega_2^1 &= -\frac{W'}{W} \theta^2.
\end{align*}
\] (28)

Therefore, corresponding energy momentum components become

\[
\begin{align*}
G_{00} &= -\left(\frac{h_3^2}{W} + \frac{A h_{03}}{W} + \frac{W''}{W}\right) \\
G_{33} &= \frac{f_0^2}{W} + B \frac{f_{03}}{W} + \frac{W''}{W} \\
G_{11} &= -\left[\frac{f_0 h_3}{W^2} + \frac{f_0^2}{W} + \frac{h_3^2}{W} + \frac{f_{03} B}{W} + \frac{h_{03} A}{W}\right] \\
G_{22} &= -\frac{f_0 h_3}{W^2} \\
G_{01} &= -\frac{f_0 q'}{W^2} \\
G_{02} &= \frac{f_{33}}{W} \\
G_{31} &= -\frac{h_3 k'}{W^2} \\
G_{32} &= \frac{h_{00}}{W} \\
G_{03} &= (f_0 - h_3) \frac{h_0}{W^2} \\
G_{12} &= -(f_0 + h_3) \frac{W'}{W^2}
\end{align*}
\] (29)

with the condition \( f_3 = h_0 \). \( f_0 = \partial f/\partial x^0 \), etc., and a prime denotes a partial derivative with respect to \( x^1 \). If we take \( h_3 = f_0 = 0 \) in (29), i.e., \( f \) and \( h \) are functions of \( z \) and \( t \) respectively, we conclude that only non-zero components
of the $G_{ij}$ are

$$G_{00} = G_{33} = -\frac{W'}{W}$$

which represents a spinning cosmic string with radius $r_0$ and its tension is equal to its linear rest energy.

For the general form of the $f$ and $h$ space-time represents an anisotropic spin fluid with the velocity $u = (u^0, u^3, 0, 0)$, pressures $p$ (in the $r$ and $\phi$ directions) and $P$ (in the $z$ direction) and energy momentum components are defined as

$$t_{ij} = (\epsilon + p) u_i u_j - p \eta_{ij} + (P - p) v_i v_j + 2 u_j u^k \dot{S}_{ki}$$

where $\epsilon$ is the rest energy of the spin fluid, $v$ is a spacelike vector orthogonal to timelike velocity vector $u$ and antisymmetric part of the hypermomentum density is equal to the spin angular momentum $S_{ij}$ [17]. Since only $S_{12}$, $u^0$ and $u^3$ are different from zero the Frenkel conditions automatically are satisfied and the last term in (31) does not contribute to the energy momentum tensor. Combined energy-momentum equations are given as

$$G_{00} = -\left(\frac{h_3^2}{W} + \frac{A}{W} \frac{h_{03}}{W} + \frac{W''}{W}\right) = (\epsilon + p)u_0 u_0 + p + (P - p)v_0 v_0$$

$$G_{33} = \frac{f_0^2}{W} + \frac{B f_{03}}{W} + \frac{W''}{W} = (\epsilon + p)u_3 u_3 - p + (P - p)v_3 v_3$$

$$G_{11} = -\left[\frac{f_0 h_3}{W^2} + \frac{f_0^2}{W} + \frac{h_3^2}{W^2} + \frac{f_{03}}{W} \frac{B}{W} + \frac{h_{03}}{W} \frac{A}{W}\right] = -p + (P - p)v_1 v_1$$

$$G_{22} = -\frac{f_0 h_3}{W^2} = -p + (P - p)v_2 v_2$$

$$G_{01} = -\frac{f_0 q'}{W^2} = (P - p)v_0 v_1$$

$$G_{02} = \frac{f_3}{W} = (P - p)v_0 v_2$$
\[ G_{31} = -\frac{h_3 k'}{W^2} = (P-p)v_3 v_1 \]
\[ G_{32} = \frac{h_{00}}{W} = (P-p)v_3 v_2 \]
\[ G_{03} = (f_0 - h_3) \frac{h_0}{W^2} = (P-p)v_0 v_3 \]
\[ G_{12} = -(f_0 + h_3) \frac{W'}{W} = (P-p)v_2 v_1 \]  
(32)

We have also normalization \(-u_0^2 + u_3^2 = -1, -v_0^2 + v_1^2 + v_2^2 + v_3^2 = 1\) and orthogonality conditions \(u_0 v_0 = u_3 v_3\). It is seen that since \(u^0\) and \(u^3\) are different from zero, fluid’s velocities \(u_t, u_z\) and \(u_{\phi}\) are different from zero and there is no movement in \(r\) direction as expected.

In the space-time with torsion junction conditions are different from 14 and here they obey the the matching conditions given by [13]

\[ g_{\mu\nu}|_- = g_{\mu\nu}|_- \]
\[ \frac{\partial g_{\mu\nu}}{\partial x^\alpha}|_- + 2g_{\alpha\rho}K^\rho_{(\mu\nu)}|_- = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}|_- + 2g_{\alpha\rho}K^\rho_{(\mu\nu)}|_- \]  
(33)

on the cylinder with radius \(r = r_0\) and constant \(t = t_0, z = z_0\). For our example, these conditions reduce to

\[ g_{\phi\phi}|_+ = g_{\phi\phi}|_- \]
\[ g_{\phi z}|_+ = g_{\phi z}|_- \]
\[ g_{\phi t}|_+ = g_{\phi t}|_- \]
\[ g_{\phi\phi, r}|_+ = g_{\phi\phi, r}|_- + 2K^r_{\phi\phi} \]
\[ g_{\phi z, r}|_+ = g_{\phi z, r}|_- + 2K^r_{\phi z} \]
\[ g_{\phi t, r}|_+ = g_{\phi t, r}|_- + 2K^r_{\phi t} \]

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where \((\_\_\_\_)\) represents torsionless exterior space-time and \((\_\_\_\_)\) represents the interior space-time with \(K_{\mu\nu}^\rho\) which is the contorsion part of the connection \(\Gamma_{ij}^k = \Gamma_{ij}^{\{}k - K_{ij}^k\). Symmetric part of the connection \(\Gamma_{ij}^{\{}k\) represents the Riemannian contribution and the other quantities can be defined as

\[
\begin{align*}
g_{ij} &= e^\alpha_i e^\beta_j \eta_{\alpha\beta} \\
K_{kij} &= e^\alpha_i e^\beta_j K_{k\alpha\beta} \\
T^i &= K^i_j \wedge \theta^j \\
K^i_j &= K^\alpha_\beta dx^k
\end{align*}
\]

(35)

And if we rewrite conditions (34) explicitly

\[
\begin{align*}
-a^2 z_0^2 + a^2 t_0^2 + b^2 r_0^2 &= -A^2 + B^2 + W^2 \\
 a z_0 &= A \\
 a t_0 &= B \\
 2b^2 r_0 - (-AA_r + BB_r + WW') &= AA_r - BB_r \\
 A_r &= A_r \\
 B_r &= B_r
\end{align*}
\]

(36) (37) (38) (39) (40) (41)

Some exact solutions of Einstein field equations have CTCs. It means that a timelike observer instead of going from past to future intersects itself. In [11] Bonnor suggests that instead of spacetime, a closed timelike curve may exist in the laboratory by considering two spinning massive particles having
angular momenta and parallel spins. It is shown that a CTC can be created in the lab depending on the particles’ unit angular momenta.

The spacetime given with (5) has CTCs in the region

\[ r > \sqrt{F^2 - B^2 / b} \]

which changes the sign of \( g_{\phi\phi} \) for \( F^2 > B^2 \) and spacetime has global hyperbolic structure for \( F^2 < B^2 \).

If we consider \( z_0 = \text{const.} \) hypersurface, spin and dislocation parameters \( J^t \) and \( J^z \) play role in formation of the CTCs.

4 Conclusion

In this work, a general form of the spinning cosmic string in the Einstein theory of gravity is presented and the same problem is considered in the Einstein Cartan theory of gravity. Our metric is studied in the EC theory and as pointed out in [6] and [5] besides the curvature and energy-momentum singularity space-time has the torsion singularity characteristic. The exact solution of this space-time is considered as a fluid which is non-zero source in Einstein theory and spin fluid with anisotropic pressure moving along the symmetry axis in EC theory.

Since it is important to understand CTC’s role in the general relativity, CTCs are calculated in the spacetime suggested. Our solution can also be expanded multi-string solutions by redefining coordinate parameters[3, 4].

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References

[1] Vilenkin A. and Shellard E.P.S. 2000 *Cosmics and Other Topological Defects* (Cambridge: Cambridge Univ. Press)

[2] Pogosian L., Tye H.S.H., Wasserman I., Wyman M., 2003 *Phys. Rev. D*68* 023506; Contaldi C. R., 2000, preprint ”Cosmic String in the age of the Boomerang”, astro-ph/0005115; Simatos N., Perivolaropoulos L. 2001 *Phys. Rev. D*63* 025018 Landriau M., Shellard E.P.S. 2003 *Phys. Rev. D*67* 103512; Bouchet F.R., Peter P., Riazuelo P.A., Sakellariadou M. 2002 *Phys. Rev. D*65* 021301.

[3] Gal’tsov D.V., Letelier P.S., 1993 *Phys. Rev. D*47* 4273

[4] Letelier P.S., 1995 *Class. Quant. Grav. 12* 471; Letelier P.S., 2001 *Class. Quant. Grav. 17* 3639

[5] Tod K.P., 1994 *Class. Quant. Grav. 11* 1331-1339; A.N.Aliev, Y.Nutku unpublished work.

[6] Puntigam R.A., Soleng H.H., 1997 *Class. Quant. Grav. 11* 1331

[7] Mazur P.O, 1986 *Phys. Rev. Let. 59* 929 Samuel J., Iyer B.R., 1987 *Phys. Rev. Let. 59* 2379

[8] Mastas G.E.A., 1990 *Phys. Rev. D*42* 2927
[9] Lorenci V.A., Moreira E.S., 2000 *Phys. Rev.* **D63** 027501

[10] Lorenci V.A., De Paola R.D.M. and Svaiter N.F., 1999 *Class. Quant. Grav.* **16** 3047-3055

[11] Bonnor W.B., 2003, *Int.J.Mod.Phys.* **D12** 1705-1708; De Lorenci V.A., Moreira E.S., *Spinnin strings, cosmic dislocations and chronology protection*, gr-qc/0309122

[12] Hehl F.W., von der Heyde P., Kerlick G.D., 1974 *Phys. Rev.* **D10** 1066; Hehl F.W., von der Heyde P., Kerlick G.D., 1976 *Rev. Mod. Phys.* **48** 393

[13] Arkuszewski W., Kopczynski W., Ponomariev V.N., 1975 *Commun. Math. Phys.* **45** 183

[14] Bekenstein J.D., 1992 *Phys. Rev.* **D45** 2794; Jensen B., Soleng H., 1992 *Phys. Rev.* **D45** 3528

[15] Hiscock W.A., 1985 *Phys. Rev. D.* **31** 3288

[16] Vilenkin A.V., 1981 *Phys. Rev. D.* **23** 852

[17] Obukhov Y.N., Korotky V.A., 1987 *Class. Quant. Grav.* **4** 1633; Obukhov Y.N., Tresguerres R., 1993 *Phys. Lett.* **A184** 17