INTEGRABILITY VIA REVERSIBILITY

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Abstract. A class of left-invariant second order reversible systems with functional parameter is introduced which exhibits the phenomenon of robust integrability: an open and dense subset of the phase space is filled with invariant tori carrying quasi-periodic motions, and this behavior persists under perturbations within the class.

Real-analytic volume preserving systems are found in this class which have positive Lyapunov exponents on an open subset, and the complement filled with invariant tori.

1. Introduction

We study a family of second order dynamical systems on a locally homogeneous Riemannian space $M$, modeled on a special solvable Lie group. The simplest example is the geodesic flow of a left-invariant metric. Our class generalizes the examples discovered by Butler [Bu1], and Bolsinov and Taimanov [B-T]. In these examples the complete integrability of the geodesic flow in the tangent bundle $TM$ is accompanied by highly non-integrable behavior on an invariant submanifold of codimension $n = \dim M$. The dynamics there is the suspension of a toral automorphism, and in [B-T] the hyperbolic automorphism is chosen, which leads to an Anosov flow. The presence of an Anosov flow as a subsystem guarantees the positivity of topological entropy.

We show that such a behavior extends to a larger class of left-invariant second order systems. This class is parametrized by a matrix $L$ and a functional parameter $H$: a smooth vector field in the unit ball of the Euclidean space. We call them $L$-$H$ systems. The crucial property of $L$-$H$ systems is their $J$-reversibility, where $J$ is an appropriate involution of the tangent bundle $TM$. Let us recall that a system is $J$-reversible if the involution $J$ conjugates the forward in time dynamics with the backward in time dynamics.

There is a vast literature devoted to $J$-reversible systems. The survey paper of Lamb and Roberts [L-R] contains an extensive bibliography.

In particular there is a version of the KAM theory for the $J$-reversible systems. It goes back to Moser [M], and Sevryuk [S]. In our case we establish robust integrability: it persists under any small perturbation.

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as long as we stay in the family of L-H systems. Note that this family is parametrized by an infinite dimensional Banach space of vector fields $H$. What is notable is that we do not assume volume preservation, the symmetries imposed on the system force the integrability, and the occurrence of a finite absolutely continuous invariant measure. This measure has a density with respect to the Liouville volume which is only $C^\infty$, and typically no real-analytic invariant density exists (Section 8).

The $J$-reversible KAM theory would give us large subsets of quasi-periodic motions for perturbations which are not left-invariant, as long as they are $J$-reversible. We were unable to check the non-degeneracy of the unperturbed system required for the application of the KAM theory. However we conjecture that the non-degeneracy does hold for most systems under consideration.

Butler, [Bu2], [Bu3], used the mechanism discovered in [B-T] to obtain $C^\infty$ examples of integrable volume preserving systems with positive metric entropy. In our class we find whole families of real-analytic systems with positive metric entropy and a subset filled with quasi-periodic motions, open but not dense (Section 9).

The phenomenon of robust integrability, accompanied by positive topological entropy occurs already for geodesic flows of linear connections. The generalization of the geodesic flow of the Levi-Civita connection to more general linear connections was discussed in [P-W]. Such a generalization appears naturally in the study of Gaussian thermostats, a class of systems introduced by Hoover [H]. The paper of Gallavotti and Ruelle [G-R] introduces the Gaussian thermostats in the physical context. In particular in our class of systems we find a Gaussian thermostat with the following paradoxical behavior (Section 10). For small kinetic energy the system is asymptotic to an Anosov flow, which has a large codimension in the whole phase space. For larger values of the kinetic energy the system undergoes a drastic change, it becomes integrable: an open and dense subset in the phase space is filled with quasi-periodic motions. The Anosov subsystem is still present, however for a subset of initial conditions of full Lebesgue measure the solutions stay away from that chaotic subsystem, and fill densely invariant tori.

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2. THE CONFIGURATION SPACE

Our configuration space is a locally homogeneous Riemannian space modeled on a special Lie group $G$. We start its description with the Lie algebra $\mathfrak{g}$. We assume that $\text{dim } \mathfrak{g} = n+1$ and that $\mathfrak{g}$ contains an $n$
dimensional abelian ideal \( g_0 \). We choose an arbitrary scalar product in \( g \), and let \( b \) denote a unit vector orthogonal to \( g_0 \). Since the ideal \( g_0 \) is assumed to be abelian the Jacobi identity imposes no conditions on the operator \( L : g_0 \rightarrow g_0, L = ad_b \). At this stage we place no restrictions on the operator \( L \). Later on we will consider various special cases. We endow the Lie group with the left invariant metric determined by our choice of the scalar product in \( g \).

The Levi-Civita connection \( \nabla \) on the Riemannian manifold \( G \) can be expressed as a tensor on \( g \). It can be calculated directly, which is done in the fundamental paper of Milnor [Mi], where extensive explanations can be found. The formulas read

\[
\begin{align*}
\nabla_b b &= 0, \\
\nabla_b u &= Au, \quad \text{for } u \in g_0, \\
\nabla_u b &= -Su, \\
\nabla_u v &= \langle Su, v \rangle b, \quad \text{for } u, v \in g_0
\end{align*}
\]

where \( S = \frac{1}{2}(L + L^*) \) and \( A = \frac{1}{2}(L - L^*) \) denote the symmetric and skew-symmetric parts of the operator \( L \).

The Lie algebra has an important automorphism \( K : g \rightarrow g \), where \(-K\) is equal to the euclidean reflection in \( g_0 \). There are only few Lie algebras with this kind of additional symmetry. It is not difficult to enumerate all of them. We did not find the other algebras useful for our purposes. If a Lie algebra has an automorphism \( K \) such that \(-K\) is the reflection in a codimension 1 subspace then the subspace must be an abelian subalgebra. Our class of Lie algebras is singled out by the additional property that the invariant subspace of \( K \) not only is an abelian subalgebra but it is also an ideal.

Since the automorphism \( K \) is orthogonal it generates the automorphism \( K \) of the Lie group \( G \) which is an isometry. This isometry will play crucial role in our discussion. Let us note that both \( K \) and \( K \) are involutive, i.e., \( K = K^{-1}, K = K^{-1} \).

The Lie group has the following matrix representation, which we will also denote by \( G \). It consists of matrices with the block form

\[
\begin{bmatrix}
1 & 0 \\
w & e^{u L}
\end{bmatrix}, \quad w \in \mathbb{R}^n, u \in \mathbb{R}.
\]

The abelian ideal \( g_0 \) corresponds to the normal abelian subgroup \( G_0 \) consisting of the matrices with \( u = 0 \). We obtain convenient coordinates \((w, u) \in \mathbb{R}^n \times \mathbb{R}\) in the group \( G \), which is the semi-direct product \( G_0 \rtimes G_1 \), of the abelian additive groups \( G_0 = \mathbb{R}^n \) and \( G_1 = \mathbb{R} \).

The Lie algebra of our matrix group consists of matrices of the form

\[
\begin{bmatrix}
0 & 0 \\
\xi & \eta L
\end{bmatrix}, \quad \xi \in \mathbb{R}^n, \eta \in \mathbb{R}.
\]

We will be using the linear coordinates \((\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}\) in the Lie algebra \( g \).

Let us choose a lattice \( \Gamma_0 \) in \( G_0 \) of rank \( n \), so that \( \Gamma_0 \backslash G_0 \) is an \( n \)-dimensional torus. Our basic configuration space \( M \) is the Riemannian
The manifold $M = \Gamma_0 \backslash G$. The manifold $M = \Gamma_0 \backslash (G_0 \times G_1)$ is canonically diffeomorphic to $( \Gamma_0 \backslash G_0 ) \times G_1 = \mathbb{T}^n \times \mathbb{R}$. In particular each torus in this product has a canonical affine structure inherited from the abelian subgroup $G_0$. This foliation into tori will play an important role. We will refer to the leaves as toral leaves.

In general the Riemannian manifold $M$ is only locally homogeneous, since the left translations do not factor to the coset space $\Gamma_0 \backslash G$. However the left translations by elements from the subgroup $G_0$ do factor onto $M$, and these isometries preserve all the toral leaves, acting on them as translations.

The configuration space $M = \Gamma_0 \backslash G$ is non-compact. In some cases the discrete subgroup $\Gamma_0$ can be enlarged to a discrete subgroup $\Gamma$ so that the resulting quotient space $N = \Gamma \backslash M$ is compact. It is so when the one parameter subgroup $e^{ul}$, $u \in \mathbb{R}$, contains an automorphism of $\Gamma_0 \subset \mathbb{R}^n$. In general the necessary condition for the existence of a quotient with finite volume is the unimodularity of the group $G$. [M]. In our case the group $G$ is unimodular if and only if $tr L = 0$.

Let us assume for simplicity that $A = e^L$ is an automorphism of the discrete subgroup $\Gamma_0$ (and the torus $\Gamma_0 \backslash G_0$). The discrete subgroup $\Gamma$ is generated by $\Gamma_0$ and $A \in G$ ($A = e^L$ can be considered as an element of the subgroup $G_1$: $w = 0$, $u = 1$.) We will consider such subgroups and the resulting compact locally homogeneous spaces $N = \Gamma \backslash G$.

3. SECOND ORDER LEFT INVARIANT EQUATIONS ON LIE GROUPS

We will need some general properties of second order equations on Lie groups, which are invariant under left translations.

A second order equation on a manifold $M$ is a dynamical system, i.e., a continuous flow $\Psi^t$, $t \in \mathbb{R}$, on the tangent bundle $TM$, such that for a trajectory $(x(t), v(t)) = \Psi^t(x(0), v(0)), t \in \mathbb{R}$, where $x(t) \in M$ and $v(t) \in T_{x(t)}M$, we have $\frac{dx}{dt} = v(t)$.

Let further $G$ be a Lie group. The tangent bundle $TG$ of the group is canonically diffeomorphic to the cartesian product $G \times \mathfrak{g}$ by the use of left translations. With this identification the derivative of a left translation on $G$, as a mapping of $TG = G \times \mathfrak{g}$, is equal to the identity in the second factor.

A second order equation on a Lie group is left-invariant if left translations on the tangent bundle $TG = G \times \mathfrak{g}$ commute with the flow. In other words the trajectories of the flow are taken to trajectories by left translations: for any $g \in G$ and any trajectory $(x(t), v(t)) = \Psi^t(x(0), v(0)), t \in \mathbb{R}$, the parametrized curve $(gx(t), v(t))$ in $G \times \mathfrak{g}$ is also a trajectory of the flow.

The structure of a left-invariant second order equation is described in the following
Proposition 3.1. If a continuous flow \( \Psi^t \) on the tangent bundle \( TG = G \times g \) of a Lie group is a left-invariant second order equation then it is a group extension of a flow on \( g \), i.e., there is a continuous flow \( \psi^t : g \to g \) and a continuous cocycle over \( \psi^t \) with values in \( G \), \( h : \mathbb{R} \times g \to G \), such that

\[
\Psi^t(g, v) = (gh(t, v), \psi^t(v)), \quad \text{for} \quad (g, v) \in G \times g.
\]

Proof. Let \( \pi_G : TG \to G \) and \( \pi_g : TG \to g \) be the projections associated with the identification of \( TG \) with \( G \times g \). We define the flow \( \psi^t \) in the Lie algebra by

\[
\psi^t(v) = \pi_g(\Psi^t(g, v)) \quad \text{for any} \quad g \in G.
\]

The mapping \( \psi^t : g \to g \) is well defined because of left invariance, and hence it must be a flow.

Further let

\[
h(t, v) = g^{-1} \pi_G(\Psi^t(g, v)) \quad \text{for any} \quad g \in G.
\]

Again since it is well defined the cocyle property can be easily checked. \( \square \)

The flow on \( g \) described in Proposition 3.1 will be referred to as the Euler flow. This choice of terminology comes from the Euler equation of the rigid body dynamics, see \( [A] \), Appendix 2.

It follows from this Proposition 3.1 that if the Euler flow has a periodic trajectory through \( v_0 \in g \) of period \( T \) then the map \( \Psi^T \) preserves the set \( G \times \{v_0\} \) and it is equal there to a right translation on \( G \).

4. The dynamical systems: the \( L - H \) systems

The simplest dynamical system that is of interest to us is the geodesic flow, which we consider in the tangent bundle \( TM \), rather than the cotangent bundle. Using the Riemannian metric we can identify the tangent and cotangent bundles, and the tangent bundle acquires the natural symplectic form \( \omega \).

The isometry \( K : G \to G \) projects to \( M = \Gamma_0 \backslash G \), and we denote it again as \( \mathcal{K} \). This isometry is involutive, i.e., \( \mathcal{K} = \mathcal{K}^{-1} \).

The derivative \( D\mathcal{K} : TM \to TM \) is also involutive and it commutes with the geodesic flow \( \Psi^t : TM \to TM, \ t \in \mathbb{R} \),

\[
D\mathcal{K} \circ \Psi^t = \Psi^t \circ D\mathcal{K}, \ t \in \mathbb{R}.
\]

Let \( \tilde{J} : TM \to TM \) be the involution which is identity in the base \( M \) (it does not move points in \( M \) and it is equal to minus identity in every tangent space. It is well known that the geodesic flow is \( \tilde{J} \)-reversible, i.e.,

\[
\tilde{J} \circ \Psi^t = \Psi^{-t} \circ \tilde{J}, \ t \in \mathbb{R}.
\]
The involutions $\tilde{J}$ and $D\mathcal{K}$ commute, hence their composition is also an involution, and we denote it by $J = \tilde{J} \circ D\mathcal{K} = D\mathcal{K} \circ \tilde{J}$. Since the geodesic flow commutes with $D\mathcal{K}$ it must be also $J$-reversible.

The $J$-reversibility is the fundamental self symmetry that is present in our family of dynamical systems on $SM$.

The first generalization is from the Levi-Civita connection of a left invariant metric on $G$ to a left invariant linear connection $\nabla^*\mathcal{K}$ on $G$. Such a connection differs from the Levi-Civita connection by a tensor $B$ in $\mathfrak{g}$

\[ \nabla^*_{X} Y - \nabla_{X} Y = B(X, Y), \quad X, Y \in \mathfrak{g}. \]

We assume that the connection $\nabla^*$ has two additional properties. Firstly we require that the parametrization of its geodesics is proportional to the arc length. We will call such connections para-metric, since they generalize the concept of metric connections, which have isometric parallel transport. A connection is para-metric if and only if $\langle B(X, X), X \rangle = 0$ for every $X \in \mathfrak{g}$. The geodesic flow of a para-metric connection $\nabla^*$ preserves the unit sphere bundle, and it will be denoted again by $\Psi_t : SM \to SM$. A discussion of geodesic flows of linear connections can be found in [P-W].

Secondly we assume that the symmetric part of the tensor $B = \nabla^* - \nabla$ is invariant under the isometric involution $K : \mathfrak{g} \to \mathfrak{g}$, i.e., for every $X \in \mathfrak{g}$

\[ B(KX, KX) = KB(X, X). \]

If such a property holds we say that the connection is weakly $\mathcal{K}$-invariant. It follows that for a left-invariant and weakly $\mathcal{K}$-invariant connection the geodesic flow $\Psi_t$ is again $J$-reversible.

The equations of the geodesic flow can be written as

\[ \frac{dx}{dt} = v, \quad \nabla_x v = 0, \]

where $x(t) \in G$ is a parametrized geodesic.

With the identification of the tangent bundle $TG$ with $G \times \mathfrak{g}$ by left translations we get there the coordinates $(w, u; \xi, \eta) \in \mathbb{R}^{2n+2}$, which where introduced in Section 1. In these coordinates the above involutions are given by

\[ D\mathcal{K}(w, u; \xi, \eta) = (-w, u; -\xi, -\eta), \quad \tilde{J}(w, u; \xi, \eta) = (w, u; -\xi, -\eta), \]

\[ J(w, u; \xi, \eta) = (-w, u; \xi, -\eta) \]

We need to establish the form of a left-invariant, para-metric, weakly $\mathcal{K}$-invariant connection on $G$. Recall that $b \in \mathfrak{g}$ is a unit vector orthogonal to $\mathfrak{g}_0$. An arbitrary element $X \in \mathfrak{g}$ can be written as $X = \xi + \eta b, \xi, \eta \in \mathfrak{g}_0$.

**Proposition 4.1.** A connection $\nabla^* = \nabla + B$ on the group $G$ is left-invariant, para-metric and weakly $\mathcal{K}$-invariant if and only if there is a
linear operator $C : g_0 \to g_0$ such that for any $X = \xi + \eta b, \xi \in g_0$.

$$B(X, X) = \eta C \xi - \langle C \xi, \xi \rangle b$$

Proof. Without loss of generality we can assume that $B$ is symmetric.

We have for every $X = \xi + \eta b, \xi \in g_0$,

$$0 \equiv \langle B(X, X), X \rangle = \langle B(\xi, \xi), \xi \rangle + \eta \langle B(\xi, b), b \rangle + 2\eta \langle B(b, \xi), \xi \rangle + \eta^2 \langle B(b, b), \xi \rangle + \eta^3 \langle B(b, b), b \rangle.$$

Since this cubic polynomial in $\eta$ must vanish we get for every $\xi \in g_0$

$$\langle B(\xi, \xi), b \rangle + 2\eta \langle B(b, \xi), \xi \rangle = 0, \qquad 2\eta \langle B(b, b), \xi \rangle + \langle B(b, b), b \rangle = 0.$$

Since the connection is weakly $\mathcal{K}$-invariant we obtain further that for every $\xi$

$$B(b, b) = 0 \implies \langle B(\xi, b), b \rangle = 0, \quad B(\xi, \xi) \perp g_0 \implies B(\xi, \xi) = -2\langle B(\xi, b), \xi \rangle b.$$ 

Putting $C \xi = 2B(\xi, b)$ we get the desired formula. $\Box$

Using Proposition 4.1 and the formulas (1) we obtain by direct calculations the equations of the geodesic flow for our special connections.

**Theorem 4.2.** For any para-metric left-invariant and weakly $\mathcal{K}$-invariant connection, the geodesic equations in the coordinates $(w, u; \xi, \eta)$ in the tangent bundle $TG$ are

$$\begin{align*}
\frac{d\xi}{dt} &= \eta H(\xi), & \frac{d\eta}{dt} &= -\langle H(\xi), \xi \rangle \\
\frac{dw}{dt} &= e^{uL_\xi} \eta, & \frac{du}{dt} &= \eta.
\end{align*}$$

where $H(\xi) = L^* \xi - C \xi$, and the matrix $C$ depends only on the tensor $B = \hat{\nabla} - \nabla$, namely for $X = \xi + \eta b, \xi \in g_0$ we have $B(X, X) = \eta C \xi - \langle C \xi, \xi \rangle b$. $\Box$

Note that in the last Theorem any matrix $C$ can occur with an appropriate choice of the connection $\hat{\nabla}$.

The equations (2) factor to the Lie algebra $g$ as the Euler equations, the first line of (2). Let us recall that the Euler equations are obtained by left translations of velocities along geodesics. It follows from the left invariance of the connection that the resulting curves in $g$ must satisfy the Euler equation, see [A], Appendix 2.

Our final generalization is to replace the linear vector field $H(\xi) = H^* \xi, \xi \in g_0 = \mathbb{R}^n$ in (2) by a general (non-linear) vector field $H : g_0 \to g_0$. Such a dynamical system will be called an $L - H$ system. In the special case of a left invariant para-metric connection, when the vector field $H$ is linear, we will call it a quadratic $L - H$ system.
Let us recall that the operator $L : \mathfrak{g}_0 \to \mathfrak{g}_0$ in the euclidean space $\mathfrak{g}_0$ determines the Lie group $G$ with a chosen left invariant metric. The vector field $H$ determines then the second order equations (2) in $TG$ which are preserved by any left translation of $G$, and hence project naturally to $TM$, or $SM$. Note that to define an $L-H$ system on $SM$ it is enough to have the vector field $H$ defined in the closed unit ball of $\mathfrak{g}_0$.

The crucial property of general $L-H$ systems on $SM$ (or $TM$) is their $J$-reversibility.

Actually the $L-H$ systems can be characterized as smooth second order left invariant equations on $SM$ which are $J$-reversible.

**Proposition 4.3.** Any second order left-invariant equation on $SM$ which is also $J$-reversible is an $L-H$ system.

**Proof.** We have the Lie group $G$ with the chosen left-invariant metric. We write the equations using the left-invariant Levi-Civita connection. These equations on $SM$ factor to the unit sphere $S^n = \{ (\xi, \eta) \in \mathfrak{g} | \xi^2 + \eta^2 = 1 \}$. Using the $(\xi, \eta)$ coordinates we obtain from the assumption of $J$-reversibility that the function

$$U(\xi, \eta) = \frac{d\xi}{dt}$$

is odd in the $\eta$ variable, i.e., $U(\xi, -\eta) = -U(\xi, \eta)$. It follows that if $U$ is a smooth function on the unit sphere then the function

$$H(\xi) = \frac{1}{\sqrt{1-\xi^2}} U(\xi, \sqrt{1-\xi^2})$$

is well defined and smooth in the closed unit ball in $\mathfrak{g}_0$, and $U(\xi, \eta) = \eta H(\xi)$. $\square$

### 5. Periodic solutions in $J$-reversible systems

For $J$-reversible systems there is a very convenient way of searching for periodic solutions. It goes so far back that it is by now a part of the mathematical folklore. It was formulated explicitly by DeVogelaere [DeV], and Devaney [D]. We do it again for the convenience of the reader.

Let us consider the subset $N$ of fixed points of the involution $J$, $N = \{ p | J(p) = p \}$.

**Theorem 5.1.** Any trajectory of a $J$-reversible flow $\Psi^t$ which visits $N$ twice must be periodic. Moreover such a trajectory is invariant under $J$ with the reversal of time.

**Proof.** Let $p_0 \in N$ be such that there is $t_0 > 0$ with $\Psi^{t_0}(p_0) \in N$. We have

$$\Psi^{t_0}(p_0) = J (\Psi^{t_0}(p_0)) = \Psi^{-t_0} J(p_0) = \Psi^{-t_0}(p_0).$$
Hence the trajectory \( p(t) = \Psi^t(p_0) \) is periodic with the period \( T = 2t_0 \). Moreover
\[
Jp(t) = J\Psi^t(p_0) = \Psi^{-t}(Jp_0) = \Psi^{-t}(p_0) = p(-t).
\]

The minimal period of a trajectory of \( p_0 \) in the above proof is \( T = 2t_0 \) if and only if \( t_0 > 0 \) is the time of the first return of \( p_0 \) to the set \( N \) of fixed points of \( J \).

We apply this principle not to the full \( L - H \) system but only to its factor, the Euler equations
\[
\frac{d\xi}{dt} = \eta H(\xi), \quad \frac{d\eta}{dt} = -\langle H(\xi), \xi \rangle,
\]
(3)

The involution \( J \) descends naturally to the phase space of (3) \( (\xi, \eta) \in g \), and we denote it again by \( J \). Clearly the Euler equation is \( J \)-reversible. The set of fixed points of \( J \) is equal to \( N = g_0 = \{ (\xi, \eta) | \eta = 0 \} \).

Let us consider the open unit ball \( B \subset g_0 \), with the boundary, the unit sphere, \( S = \partial B \).

**Definition 5.1.** For a smooth vector field \( H = H(\zeta) \) defined on the closed unit ball in \( g_0 \), we say that a point \( \zeta_0 \) in the open unit ball \( B \) is **escaping** if the integral curve \( \zeta = \zeta(s) \) of \( H \) through \( \zeta_0 \) is defined in a finite closed interval \( [s_-, s_+] \ni 0 \), \( \zeta(0) = \zeta_0 \), \( \zeta(s) \in B \) for \( s \in (s_-, s_+) \), the endpoints \( \zeta(s_{\pm}) \) belong to the unit sphere \( S = \partial B \), and the vector field \( H \) is transversal to the unit sphere \( S \) at the endpoints of the integral curve, i.e.,
\[
\langle H(\zeta(s_-)), \zeta(s_-) \rangle < 0, \quad \langle H(\zeta(s_+)), \zeta(s_+) \rangle > 0.
\]
(4)

The integral curve through an escaping point is called an **escaping trajectory**.

Let us note that for a given smooth vector field \( H \) the set of escaping points in the unit ball is open.

We have the following crucial

**Theorem 5.2.** For any escaping trajectory \( \zeta(s), s \in [s_-, s_+] \) of the vector field \( H = H(\zeta), \zeta \in g_0 \), there are periodic functions \( \eta(t) \) and \( u(t) \) with the period \( T = 2t_0 \) such that \( \eta(0) = \eta(t_0) = 0, u(0) = s_-, u(t_0) = s_+, s_- \leq u(t) \leq s_+ \), and \( (\xi(t), \eta(t)) \) is a \( T \)-periodic solution of the Euler equation (3), where \( \xi(t) = \zeta(u(t)) \). Moreover \( \eta(t) \) is an odd function \( \eta(-t) = -\eta(t), t \in \mathbb{R} \), and \( u(t) \) is an even function
\[
u(t) = s_- + \int_0^t \eta(s)ds.
\]

**Proof.** We restrict the Euler equation (3) to the upper half of the unit sphere in \( g \), i.e., to the subset \( \{ (\zeta, \eta) | \zeta^2 + \eta^2 = 1, \eta > 0 \} \). In this
submanifold we can use \( \zeta \) as coordinates, and we introduce there the
time change \( \frac{ds}{dt} = \eta \). After this time change the Euler equation becomes
the following system

\[
\frac{d\zeta}{ds} = \mathcal{H}(\zeta).
\]

Hence the trajectory \( \zeta(s), s_- < u < s_+ \), of this vector field in the
open ball \( B \subset g_0 \) gives rise to the trajectory \( (\xi(t), \eta(t)), t_- < t < t_+ \),
of the Euler equation in the upper half of the unit sphere in \( g \), with
\( \lim_{t \to t_-} \eta(t) = \lim_{t \to t_+} \eta(t) = 0 \). It may happen (and it does) that
\( t_\pm = \pm \infty \). This is excluded in our case by the condition (4) which gives
us \( \lim_{t \to t_-} \frac{d\eta}{dt} > 0, \lim_{t \to t_+} \frac{d\eta}{dt} < 0 \). Hence we have a finite time interval
\([t_-, t_+]\) and we can shift it to the time interval \([0, t_0], t_0 = t_+ - t_- \).

By Theorem 5.1 this trajectory extends to the \( T = 2t_0 \) periodic
solution of the Euler equation \( (\xi(t), \eta(t)), t \in \mathbb{R} \). Putting \( u(t) = s_- + \int_{t_-}^t \eta(s)ds \) we have \( \xi(t) = \zeta(u(t)) \) for \( 0 \leq t \leq t_0 \). Since the periodic
solution is invariant under \( J \) with the reversal of time, we obtain that
\( \xi(-t) = \xi(t), \eta(-t) = -\eta(t) \). It follows that \( \int_{t_0}^{t_0} \eta(s)ds = 0 \), and
consequently \( u(t) \) is a \( T \) periodic function \( 0 \leq u(t) \leq u_0 \). Hence we
have also \( \xi(t) = \zeta(u(t)) \) for any \( t \in \mathbb{R} \). \( \square \)

In simple terms what this proof reveals is that an integral curve of
\( H \) is up to a time change also a trajectory of the respective Euler flow
in the upper semi-sphere. For escaping trajectories of \( H \) the solution
of the Euler equation crosses transversally the “equator” \( \{ \eta = 0 \} \) and
in the lower semi-sphere it follows the same integral curve of \( H \) but in
the reversed direction. Hence it must be a periodic trajectory of the
Euler flow.

In the case of a quadratic \( L - H \) system we get the following

**Corollary 5.3.** If the linear vector field \( H(\xi) = H\xi \) has eigenvalues
both with positive and negative real parts, then for the Euler equation
(3) the periodic trajectories fill an open and dense subset of \( S^n \subset g \).

If all the eigenvalues of \( H \) are on the imaginary axis then for the
Euler equation (3) an open and dense subset of \( S^n \subset g \) is filled with
trajectories which are either periodic or quasi-periodic after an appro-
priate smooth time change.

**Proof.** Let us first note that for linear vector field \( H \) the condition (4)
is satisfied on an open and dense subset of the unit sphere \( S \subset g_0 \),
unless \( H^* = -H \).

If the matrix \( H \) has eigenvalues with both positive and negative real
parts then the set of escaping points in the unit ball \( B \subset g_0 \) is not only
open but also dense in the unit ball. That is so because the instability
as \( t \to \pm \infty \) forces typical solutions to reach the boundary unit sphere
at some finite time both in the future and in the past. By Theorem
5.2 they give rise to the periodic solutions of the Euler equations.
If the matrix $H$ has only purely imaginary eigenvalues, and there is no resonance, then escaping trajectories of the linear system (5) are not dense in the unit ball. There is an open set of trajectories which stay in the unit ball for all times. However these bounded trajectories are typically quasi-periodic, and their closure is an invariant torus of the linear system (5). Note that the solution of the Euler equation need not be quasi-periodic since a time change is involved.

In the skew symmetric case $H^* = -H$, the unit ball contains no escaping trajectories. An open and dense subset of the unit ball is filled with quasi-periodic bounded solutions of the linear system (5).

In the resonant case we have instability for $t \to \pm \infty$, and hence there is a dense subset of escaping trajectories, both in the case of purely imaginary nonzero eigenvalues and the zero eigenvalue. By Theorem 5.2 we get an open and dense subset of the unit sphere $S^n \subset g$ filled with periodic solutions of the Euler equations.

6. ROBUST INTEGRABILITY OF $L - H$ SYSTEMS

The notion of integrability of a dynamical system has a long history and several different versions. In Hamiltonian dynamics the main concept of integrability is associated with the Liouville-Arnold theorem, where families of invariant tori carrying quasi-periodic motions appear. The issues involved in the general definition of integrability in Hamiltonian dynamics were explored by Bogoyavlensky, [Bo1],[Bo2], and Fasso [F].

We study a family of reversible, non-Hamiltonian systems in which quasi-periodic motions occur robustly. We introduce the following working definition

**Definition 6.1.** We call a dynamical system semi-integrable if an open subset $U$ of the phase space is filled with invariant tori carrying the quasi-periodic motions. If $U$ is also dense in the phase space then the dynamical system will be called integrable.

It turns out that any periodic solution of the Euler equation constructed in Theorem 5.2 gives rise to quasi-periodic solutions of the $L - H$ system. This is the contents of the following theorems.

**Theorem 6.1.** If a vector field $H$ in $g_0$ has escaping points in the unit ball then the $L - H$ system on $SM$ is semi-integrable.

If the vector field $H$ has a dense set of escaping points in the unit ball then the $L - H$ system on $SM$ is integrable.

For quadratic $L - H$ system we can formulate an effective criterion of integrability. Moreover the integrability persists under small perturbations in the space of $L - H$ systems.

**Theorem 6.2.** If for a linear vector field $H = H\xi, \xi \in g_0$ the matrix $H$ has eigenvalues with both positive and negative real parts then the
quadratic $L - H$ system on $TM$ is integrable. Moreover the invariant tori are common level sets of real-analytic first integrals.

Any small perturbation of such a quadratic integrable $L - H$ system in the space of all $L - H$ systems must be integrable.

The $L - H$ system is not in general hamiltonian with respect to the natural symplectic form $\omega$ on $TM$. However the geodesic flow of the Levi-Cita connection is hamiltonian. In this case we have the following refinement of the previous theorem.

**Theorem 6.3.** If the operator $L : \mathfrak{g}_0 \to \mathfrak{g}_0$ has eigenvalues with both positive and negative real parts then for the left-invariant metric on $M$ the geodesic flow on $TM$ is integrable.

Moreover the $(n + 1)$-dimensional invariant tori are lagrangian, and they are common level sets of real-analytic first integrals.

The last theorem recovers in particular the examples of Butler, [Bu1], and Bolsinov and Taimanov, [BT]. It follows from the hamiltonian formalism that if the common level sets of first integrals are all lagrangian then the first integrals are by necessity in involution. It is our approach in these systems the first integrals are secondary to the langrangian foliation.

It seems that within this hamiltonian class of systems, for an open dense subset of operators $L$ the system is iso-energetically non-degenerate, i.e., it has a rich family of frequencies on $SM$, [A], Appendix 8. In particular the KAM theory would be applicable. The calculations required to establish the non-degeneracy are cumbersome, and we did not find a satisfactory way to do it.

More generally, for the quadratic integrable systems, the reversible version of the KAM theory, developed by Moser, [M], and Sevryuk, [S], would be applicable upon the establishment of the non-degeneracy. The allowed perturbations would be $J$-reversible second order equations, but not necessarily left-invariant.

We will give a joint proof of Theorems 6.1, 6.2 and 6.3.

**Proof.** We will establish that the periodic solution of the Euler equation in $\mathfrak{g}$ constructed in Theorem 5.2 is covered by invariant tori in $TM$ carrying quasi-periodic solutions. To that end, given the $T$-periodic solution $v(t) \in \mathfrak{g}, v(t) = (\xi(t), \eta(t))$ of Theorem 5.2, $T = 2t_0$, let us integrate the equations (2) of the $L - H$ system. We get

\[
\begin{align*}
u(t) &= a + a_0(t), \quad a_0(t) = \int_0^t \eta(s)ds, \\
w(t) &= b + b_0(t), \quad b_0(t) = \int_0^t e^{u(s)L} \xi(s)ds.
\end{align*}
\]

Since the function $\eta$ is odd and periodic we obtain that $a_0(t)$ is also a periodic even function and assumes values between $-u_0$ and $u_0$, where
\[ u_0 = \int_0^t \eta(s)ds. \] Note that although \( b_0(t) \) is not periodic, its time derivative is \( T \)-periodic.

Fixing the value \( a = u(0) \) we obtain an invariant torus. Indeed consider the flow \( \Psi^t \) defined by the \( L - H \) system (3), which is a group extension flow by Proposition 3.1. The mapping \( \Psi^T \) takes \( M \times \{v(0)\} \) into itself and it acts there as a right translation. Since \( u(t) \) is periodic the translation is by an element from \( G_0 \), and hence it preserves all the toral leaves. For \( a \in \mathbb{R} \) and \( v \in \mathfrak{g} \) denote by \( \mathcal{L}(a, v) \) the toral leaf \( \mathcal{L}(a, v) = \{(w, u; \xi, \eta) \in SM \mid w \mod \Gamma_0, u = a, (\xi, \eta) = v \} \). Since \( \Psi^T \) takes \( \mathcal{L}(a, v(0)) \) into itself, and it acts there by the translation by \( c_0 = \int_0^{2\pi} e^{u(s)L\xi(s)}ds \), then the union of the \( n \)-dimensional tori \( \Psi^t \mathcal{L}(a, v(0)) \), \( t \in \mathbb{R} \), is an \( (n + 1) \)-dimensional torus carrying the suspension flow of the toral translation, i.e., a quasi-periodic flow.

In the case of a linear vector field \( H(\xi) = H\xi \), for some constant matrix \( H \), the real analytic first integrals are given by the vector valued function

\[ F(w, u; \xi, \eta) = e^{-uH}\xi. \]

These first integrals are functionally independent, hence their common level sets must be in general unions of submanifolds of dimension \( n + 1 \). Clearly the toral leaves \( \mathcal{L}(a, v) \) belong to the level sets, and so do the \( n + 1 \)-dimensional tori constructed above. Hence in general the level sets must be unions of the \( (n + 1) \)-dimensional invariant tori.

In the Levi-Civita case we still need to establish that the tori are lagrangian. To that effect we will use some hamiltonian formalism, in a fairly geometric way. The toral leaves \( \mathcal{L}(a, v) \) are orbits of the action of the torus \( G_0 \) on \( SM \) by left translations. Since these left translations are isometries, the action on \( SM \) is by symplectomorphisms. Since the group \( G_0 \) is abelian the orbits of its action must be isotropic, i.e., the symplectic form vanishes on any of the orbits. The orbits of this action are exactly the toral leaves \( L(a, v) \). The torus constructed above is the image of this \( n \)-dimensional toral orbit by the hamiltonian flow \( \Psi^t \).

Since the toral leaves lie in the level set of the hamiltonian the image is automatically isotropic, and being \( n + 1 \)-dimensional it is a lagrangian submanifold.

Note that the flow \( \Psi^t \) commutes with the action of \( G_0 \) on \( SM \).

Since in the hamiltonian case the level sets of the first integrals are unions of lagrangian tori then the first integrals are in involution.

7. Compact configuration spaces

In this section we consider the additional properties of an \( L - H \) system on the compact phase space, namely the unit sphere bundle \( SN \) where \( \Gamma \) is a cocompact lattice in \( G \) and \( N = \Gamma \backslash G \), as discussed in Section 1.
Let us note first that in this case the group must be unimodular, and the operator $L$ has zero trace. Hence either $L$ has eigenvalues with both positive and negative real parts, or all of its eigenvalues are on the imaginary axis.

The integrability and semi-integrability are passed from the non-compact phase space $SM$ to the compact phase space $SN$ without any limitations. What is new is the appearance of hyperbolic behavior made possible by the recurrence in the compact phase space. For any quadratic $L - H$ system the submanifold $A \subset SM$ given by the equations $\xi = 0, \eta = 1$ is diffeomorphic to $M$ and it carries trajectories escaping to infinity both in the future and in the past. The projection of $A$ to the compact phase space $SN$ is diffeomorphic to $N$ and the $L - H$ flow is diffeomorphic to the suspension of the toral automorphism $e^L$. If the toral automorphism has no eigenvalues on the unit circle then it is an Anosov diffeomorphism, and its suspension is an Anosov flow. If it has only some eigenvalues outside the unit circle then we get partially hyperbolic flows as suspensions, [K-H]. Such flows in the compact phase space $SN$ while integrable have positive topological entropy. The existence of such flows was the discovery of Bolsinov and Taimanov, [B-T].

More generally let us assume that the automorphism $e^L$ has eigenvalues outside of the unit circle. Then for an arbitrary left-invariant second order equations on $SN$ (not necessarily an $L - H$ system) we have the following

**Proposition 7.1.** If the variable $\eta$ has positive lower (upper) time average on some trajectory then this trajectory has negative and positive lower (upper) Lyapunov exponents. The dimension of the respective stable and unstable subspaces is equal to the dimension of the stable and unstable subspaces of the automorphism $e^L$.

**Proof.** The flow $\Psi^t$ has the structure of a group extension flow over the Euler flow as described in Proposition 3.1. We fix a trajectory $v(t) = (\xi_0(t), \eta_0(t))$ of the Euler flow. There is a unique parametrized curve $(w_0(t), u_0(t)) \in G, w_0(0) = 0, u_0(0) = 0$, such that for every $(w, u) \in G$

$$\Psi^t(w, u; v(0)) = (w + e^{uL}w_0(t), u + u_0(t); v(t)).$$

Hence the flow takes the $n$-dimensional toral leaf $\mathcal{L}(u, v(0))$ onto the toral leaf $\mathcal{L}(u + u_0, v(t))$. We have the coordinates $w$ for both toral leaves. In these coordinates the restriction of the flow is defined by $w \mapsto w + e^{uL}w_0(t)$. Differentiating this mapping with respect to $w$ we obtain that $D\Psi^t$ restricted to the $n$-dimensional tangent subspace of the toral leaf is given by the identity operator in the coordinates $w$. However we measure the tangent vectors using the left-invariant metric and that means

$$||dw||^2 = (e^{-(u+u_0)L}dw)^2.$$
Hence we get the exponential growth (decay) as long as \( u_0 \) has linear growth (decay). Since \( \frac{du_0}{dt} = \eta_0 \), the assumption of the positive time average of \( \eta_0 \) leads to the exponential growth (decay) of \( ||dw||^2 \) in the unstable (stable) subspace of the operator \( L \).

We have not chosen any invariant measure so the Lyapunov exponents need to be understood as the upper (or lower) limits. It follows from the proof that assuming negative lower (upper) time average we will arrive at the same conclusion, with the reversal of the number of positive and negative Lyapunov exponents.

The phenomenon described in Proposition 7.1 appears explicitly in the work of Butler and Paternain on special magnetic flows on \( SOL \), [Bu-P].

Let us consider first integrals of a quadratic \( L - H \) flow \( \Psi^t \) on the compact phase space \( SN \). The real-analytic first integrals of Theorem 6.1 can be considered as real analytic multivalued first integrals. We will establish that for a residual subset of integrable quadratic \( L - H \) systems on \( SN \) there are no single valued real-analytic first integrals constant on the toral leaves. If an integrable system is non-degenerate (i.e., it has a rich family of frequencies for the quasi-periodic motions on the tori) then by necessity any continuous first integral must be constant on the toral leaves. Since we have not established the non-degeneracy we need to include it into the assumptions.

Let us recall that by Theorem 6.1 quadratic \( L - H \) flows on \( SN \) for which the matrix \( H \) has eigenvalues with positive and negative real parts are integrable, and they form an open subset in the space of all quadratic \( L - H \) flows. The complement of this subset has an interior filled with flows asymptotic to the suspension of the toral automorphism \( e^t \).

**Proposition 7.2.** For a residual subset of integrable quadratic \( L - H \) flows on \( SN \) there are no single valued real-analytic first integrals constant on the \( n \)-dimensional toral leaves, i.e., functions of \((u; \xi, \eta)\).

**Proof.** We consider only diagonalizable operators \( H \), with all different eigenvalues \((\lambda_1, \lambda_2, \ldots, \lambda_n)\), at least some of them with positive and some of them with negative real parts. This gives us an open and dense subset of integrable quadratic \( L - H \) systems. If there is a real-analytic first integral \( R \) for such a system on \( SN \) then it can be lifted to the first integral of the system on \( SM \), which we denote also by \( R \). Now \( R \) can be considered as a function of \((u; \xi, \eta)\). We restrict our attention to the unit ball \( B \) in \( g_0 \) (the “\( \xi \)-space”), and we put \( \eta = \sqrt{1 - \xi^2} \). This gives us a real-analytic function

\[
\tilde{R}(u; \xi), (u; \xi) \in \mathbb{R} \times B, \text{ which is also a first integral of the system.}
\]
We further consider the time change 
\[
\frac{du}{dt} = \eta \quad \text{in our domain } \mathbb{R} \times B.
\]
The equations of the system in the variables \((u; \xi)\) are
\[
(6) \quad \frac{du}{ds} = 1, \quad \frac{d\xi}{ds} = H\xi.
\]
The function \(\tilde{R}\) is a first integral of (6) so that its time derivative is identically zero
\[
\frac{\partial \tilde{R}}{\partial u} + \frac{\partial \tilde{R}}{\partial \xi} H\xi \equiv 0.
\]
It follows that for the real analytic function \(P(u; \xi) = \tilde{R}(u; e^{uH} \xi)\) defined in the vicinity of the origin we have
\[
\frac{\partial P}{\partial u} = \frac{\partial \tilde{R}}{\partial u} + \frac{\partial \tilde{R}}{\partial \xi} e^{uH} \xi \equiv 0.
\]
Hence \(P\) is the function of \(\xi\) alone defined in the vicinity of the origin in \(\mathbb{R}^n\) such that \(\tilde{R}(u; \xi) = P(e^{-uH} \xi)\). Now we use the fact that the function \(\tilde{R}(u; \xi) = P(e^{-uH} \xi)\) must be actually a periodic function of \(u\) with the period 1, since it is lifted from a function on \(SN\). Hence
\[
(7) \quad P(e^{-H} \xi) = P(\xi), \quad \text{for all } \xi \in \mathbb{R}^n.
\]
It goes back to Poincare that such a relation is impossible if the eigenvalues of \(H\) are non-resonant. More precisely, if there is no integer vector \((k_1, k_2, \ldots, k_n)\), with non-negative entries, such that
\[
k_1 \lambda_1 + \cdots + k_n \lambda_n = 0
\]
then the relation (7) can hold only for a constant function.

The family of \(L - H\) systems with non-resonant eigenvalues form a residual subset of the open set of integrable quadratic \(L - H\) systems introduced at the beginning of the proof. \(\square\)

8. Invariant Measures

In this section we discuss smooth invariant measures for the \(L-H\) systems, for the non-compact \(SM\) and compact \(SN\) phase spaces. First of all we consider the Lebesgue measure \(dw \, du\) in \(G\), which is the right-invariant Haar measure. It is also invariant under the left translations by elements from \(\Gamma_0\). So it projects into a \(\sigma\)-finite measure \(\mu\) on \(M = \Gamma_0 \backslash G\). The product of \(\mu\) by the standard Lebesgue measure in \(S^n \subset g\) (or \(g\)) will be referred to as the Lebesgue measure in \(SM = M \times S^n\) (or \(TM\)) and denoted by \(\nu\). Once the Lebesgue measure \(\nu\) is chosen we call a smooth function \(\rho \geq 0\) an invariant density of a dynamical system if the measure \(\rho \nu\) is preserved by the dynamical system.

**Proposition 8.1.** If the vector field \(H\) has constant divergence in the unit ball of \(g_0\) then \(\rho = \exp(-u \text{ div } H)\) is an invariant density for the \(L - H\) system on \(SM\).
Proof. Since the $L - H$ system defined in $TM = M \times g$ preserves all the sphere bundles it suffices to check the invariance of the measure in $TM$. It can be easily seen that the divergence of the $L - H$ system in $TM$ is equal to $\eta \ div H$. Since

$$\frac{d}{dt}\rho = -\eta \ div H,$$

the claim is proven. \hfill \Box

In the case of a compact phase space $SN$ the group $G$ must be unimodular, i.e., $tr L = 0$. For unimodular Lie groups the left Haar measure is equal to the right Haar measure, and the Lebesgue measures $\mu$ and $\nu$ project as finite measures to $N$ and $SN$, respectively. We will denote the resulting measures again as $\mu$ and $\nu$.

It follows from the above Proposition that if $div H = 0$ then the $L - H$ system preserves the Lebesgue measure $\nu$ in $SN$. Moreover the Euler flow of the $L - H$ system preserves the Lebesgue measure on $S^n \subset g$.

Conversely, if the $L - H$ system has a (real-analytic) invariant density $\rho$ in $SN$ then the Euler flow in $S^n \subset g$ has a (real-analytic) invariant density. Indeed, we obtain the density of the projected measure by integrating out the $w, u$ coordinates. It preserves the smoothness, and the real-analyticity of the density.

For the quadratic $L - H$ systems the Euler flow cannot have a smooth invariant density which is positive at the fixed point $\xi = 0$, unless $div H = 0$. However there may be smooth, or even real-analytic, invariant densities which vanish at the fixed point, even if $div H \neq 0$.

**Proposition 8.2.** For a residual subset in the space of quadratic $L - H$ systems in $SN$ there are no real-analytic invariant densities.

For the open family of integrable quadratic $L - H$ systems in $SN$ with all different real eigenvalues of $H$, there are $C^\infty$ invariant densities. Moreover there is a dense subset in this family with real-analytic invariant densities.

**Proof.** It follows from the above considerations that if an $L - H$ system in $SN$ has a real analytic invariant density then the field $H$ has a real analytic invariant density $\rho = \rho(\xi)$ in the unit ball in $g_0$. Hence we have

$$div (\rho H) = 0.$$

For a diagonalizable matrix $H$ we can analyze the last equation in the coordinates in which $H$ is diagonal. This may require a linear complex change of coordinates, but that is not a problem in calculations involving Taylor expansions. Assuming that $H$ is already diagonal with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ we get

$$\sum_{i=1}^{n} \lambda_i \xi_i \frac{\partial \rho}{\partial \xi_i} = -tr H \rho$$

\hfill (8)
This relation can hold for a function real-analytic in the neighborhood of the origin $\xi = 0$ if and only if there are monomials which satisfy it. For $\rho(\xi) = \prod_{i=1}^{n} \xi^{r_i}$ we get substituting into (8)

$$\sum_{i=1}^{n} \lambda_i (r_i + 1) = 0.$$  \hspace{1cm} (9)

The set of matrices $H$ for which (9) does not hold for any vector $(r_1, r_2, \ldots, r_n)$ with natural entries is a residual subset in the space of all matrices. The first part of the Theorem is proven.

To prove the second part consider the set of matrices with all different real eigenvalues, at least one positive and one negative. It is an open set of matrices. Its subset where the eigenvalues satisfy (9) for some vector $(r_1, r_2, \ldots, r_n)$ with natural even entries is dense. For such systems the function $\rho(\xi) = \prod_{i=1}^{n} \xi^{r_i} \geq 0$ is an invariant density of the Euler flow, and also the full $L - H$ system in $SN$.

To get a $C^\infty$ invariant density we choose a vector $(r_1, r_2, \ldots, r_n)$ with positive entries satisfying (9) and consider the function $\tilde{\rho}(\xi) = \prod_{i=1}^{n} |\xi|^{r_i} \geq 0$. It satisfies (9) in the open dense subset where it is positive, but it is not in general a smooth function. We consider the first integral of the Euler flow $f(\xi) = \prod_{i=1}^{n} |\xi|^{r_i+1}$ and we get a $C^\infty$ invariant density $\rho(\xi) = exp(-f(\xi)^{-1})\tilde{\rho}(\xi)$. \hfill $\Box$

9. Coexistence of integrability and positive metric entropy

The results of Sections 6 and 7 may be used to find real-analytic volume preserving $L - H$ flows with both quasi-periodic motions in an open subset and positive metric entropy. Such examples were constructed by Butler in the $C^\infty$ class, \cite{Bu2},\cite{Bu3}. The recent survey of Chen, Hu and Pesin, \cite{C-H-P} describes other kinds of coexistence phenomenon.

Let $H$ have purely imaginary all different eigenvalues, but $H^* \neq -H$. For such a linear vector field the open unit ball is not invariant, hence there is an open subset of escaping points. More precisely the whole open unit ball is the closure of the union of two open subsets $V_e$ and $V_b$. The open subset $V_e$ contains only escaping points, and the open subset $V_b$ contains points whose trajectories have compact closures in the unit ball. Actually these compact closures are invariant tori of the linear system defined by the matrix $H$. Note also that the linear vector field has zero divergence, so the $L - H$ flow preserves the Lebesgue measure.

These considerations apply already for $n = 2$. If $H$ has purely imaginary eigenvalues, but $H^* \neq -H$, then the open unit disk contains an open ellipse filled with elliptical integral curves, The rest of the open unit disk is filled with escaping trajectories.
The open set of escaping trajectories $V_e$ gives rise by Theorem 6.1 to semi-integrability of the $L - H$ flow. If the matrix $L$ has eigenvalues with positive (and negative) real parts then the open set of bounded trajectories $V_b$ leads by Proposition 7.1 to positive Lyapunov exponents. By Pesin formula, [KH], we obtain also positive metric entropy with respect to the invariant Lebesgue measure in $SN$. We get

**Theorem 9.1.** There are volume-preserving quadratic $L - H$ systems on $SN$ which are semi-integrable and have positive metric entropy.

Let us note that these properties will hold also for small divergence free perturbations of $H$ as long as we guarantee that the perturbed vector field in $g$ still has a positive Lebesgue measure of quasi-periodic motions near the origin $\xi = 0$. This can be achieved for instance when the linear vector field $H$ and the perturbation are hamiltonian, and $H$ satisfies the non-degeneracy conditions of the KAM theory, [A], Appendix 8.

Similarly by using a modified linear vector field $H_1 = H - \epsilon I$, we obtain for $\epsilon > 0$ the asymptotic stability of the origin $\xi = 0$. At the same time for small $\epsilon$ the restriction of the vector field $H_1$ to the unit ball still has an open set of escaping trajectories.

**Theorem 9.2.** There is an open set of quadratic $L - H$ systems on $SN$ which are semi-integrable and the interior of the complement to the set of invariant tori is filled with orbits which are asymptotic as $t \to +\infty (-\infty)$ to the attractor (repellor) carrying the suspension of the toral automorphism $e^k (e^{-L})$.

We conjecture that the following is true.

**Conjecture.** There are real-analytic divergence free vector fields in the closed unit ball with an open and dense subset filled with escaping trajectories, and with the complement of positive Lebesgue measure, filled with trajectories defined for all times and which are contained in the interior of the unit ball.

In the Arnold diffusion scenario a generic hamiltonian perturbation of an integrable hamiltonian system has a nowhere dense subset of invariant tori of positive Lebesgue measure. In the complement almost all orbits move away unboundedly both in the future and in the past. However this scenario was not so far rigorously established in any real-analytic examples.

If this conjecture holds then the respective real-analytic $L - H$ system would be integrable and of positive metric entropy.

10. **Gaussian thermostats**

A Gaussian thermostat is defined by a Riemannian metric on a manifold $M$ and a vector field $E$. The trajectories of the Gaussian thermostat satisfy the following ordinary differential equations in the tangent
bundle $TM$, [P-W].

\begin{equation}
\frac{d}{dt} x = v, \nabla_v v = E - \frac{\langle E, v \rangle}{\langle v, v \rangle} v,
\end{equation}

where $x = x(t) \in M$ is a parametrized curve in $M$.

By the force of these equations the “kinetic energy” $v^2$ is constant. Fixing the value of this constant $k = v^2$, and introducing the auxiliary vector field $F_k = \frac{1}{k} E$ we can rewrite the equations (10) as

\begin{equation}
\frac{d}{dt} x = v, \nabla_v v = v^2 F_k - \langle F_k, v \rangle v.
\end{equation}

It was observed in [P-W] that the equations (11) describe geodesics of a special metric connection defined by the field $F_k$

\[ \tilde{\nabla}_X Y = \nabla_X Y - \langle X, Y \rangle F_k + \langle Y, F_k \rangle X, \]

where $X, Y$ are arbitrary smooth vector fields on $M$. This connection is not symmetric, it is the unique metric connection with the torsion $T(X,Y) = \langle Y, F_k \rangle X - \langle X, F_k \rangle Y$, [P-W].

Note that the equations of a Gaussian thermostat define significantly different flows for different values of $k = v^2$, while the equations (11) scale so that for different values of $v^2$ we get the same flow up to a change of time by a constant factor. To get complete understanding of the dynamics of (10) we need to consider the dynamics of the geodesic flow of the metric connection (11) for the whole family of vector fields $F_k = \frac{1}{k} E, k > 0$.

If the manifold $M$ is the homogeneous space of Section 1 and the vector field $E$ is left invariant, then we obtain a family of left-invariant para-metric connections. To get a weakly $X$-invariant connection we have to have $KE = E$, i.e., $E$ must be orthogonal to $g_0$. The equations of geodesics for these connections give us a family of quadratic $L - H$ systems. Assuming that $E^2 = 1$, we get that in this case the vector field

\[ H(\xi) = L^* \xi - \frac{1}{k} \xi. \]

We obtain the following corollary of Theorem 6.2 and the considerations leading to Theorem 9.2.

**Corollary 10.1.** Let $r_{\min}$ and $r_{\max}$ be the smallest and the largest real parts of the eigenvalues of $L$. For $k$ such that $\frac{1}{k} > r_{\max}$ the trajectories of the Gaussian thermostat (10) are asymptotic to a subsystem, which is the suspension of the automorphism $e^L$.

If $r_{\min} < r_{\max}$ then for $k$ such that $r_{\min} < \frac{1}{k} < r_{\max}$ the Gaussian thermostat (10) is integrable.

In particular if $r_{\min} < 0 < r_{\max}$ then for $0 < k < \frac{1}{r_{\max}}$ the Gaussian thermostat (10), for the field $E$ orthogonal to $g_0$, has a global attractor carrying the suspension of $e^L$ (which can be an Anosov flow). However
for $k > \frac{1}{r_{max}}$ it is integrable. This is a somewhat paradoxical behavior: strong mixing properties at low kinetic energy, and quasi-periodic motions for high kinetic energy. Admittedly the strong mixing properties occur at the asymptotic submanifold of half the dimension of the phase space. And the subsystem is present for all values of the kinetic energy, but it is not an attractor or a repellor in the integrable case.

11. A FAMILY OF RIEMANNIAN METRICS WITH INTEGRABLE GEODESIC FLOWS

The Gaussian thermostats can be associated with Weyl connections, [W1, W2]. The left invariant vector field $E$ of Section 10 is a gradient of a function on the group $G$, which factors to $M = \Gamma_0 \setminus G$. Indeed we have $E = \nabla e^u$. The Weyl connection defined by gradient vector fields are Levi-Civita connections of modified metrics, namely for the vector field $F_k = \frac{1}{k} E$ we need to multiply the initial left invariant metric by the function $e^{2u}$.

Let for simplicity $L$ be diagonal with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. The left invariant metric on $G$ is given by

$$ds^2 = e^{-2u\lambda_1} dw_1^2 + e^{-2u\lambda_2} dw_2^2 + \cdots + e^{-2u\lambda_n} dw_n^2 + du^2.$$  

Introducing the new variables

$$x_0 = e^{2u}, x_i = \frac{1}{k} w_i, i = 1, \ldots, n,$$

we obtain

$$e^{2u} ds^2 = k^2 \left( x_{0}^{2\tau_1} dx_1^2 + x_{0}^{2\tau_2} dx_2^2 + \cdots + x_{0}^{2\tau_n} dx_n^2 + dx_0^2 \right),$$

where $\tau_i = 1 - k\lambda_i, i = 1, \ldots, n$. It was calculated in [T-W] that the Weyl sectional curvatures are non-positive in this case if and only if $\tau_i \geq 1$ or $\tau_i = 0$ for $i = 1, \ldots, n$. The sign of sectional curvatures of the last metric must be the same as for the Weyl connection ([P-W]).

Considering the geodesic flow of the Riemannian metric on $M = \mathbb{T}^n \times \mathbb{R}_+ = \{(x; x_0) | x \mod \Gamma_0, x_0 > 0\}$, where $\Gamma_0$ is a lattice in $\mathbb{R}^n$, we conclude that it is integrable if and only if there are exponents $\tau_i$ of opposite signs. It follows from the considerations of Section 10, but this phenomenon can be greatly generalized. Note that the metric is not complete since the variable $x_0$ is assumed positive. Nevertheless in the integrable case geodesics from an open and dense subset can be extended indefinitely since they do not leave a compact subset of $M$. If the conditions of integrability are not satisfied then almost all geodesics leave every compact subset of $M$.

For an open interval $(a, b) \subset \mathbb{R}$, finite or infinite, let us consider a Riemannian metric on

$$M = \mathbb{T}^n \times (a, b) = \{(x; x_0) | x \mod \Gamma_0, a < x_0 < b\},$$

$$ds^2 = \alpha_1^2 dx_1^2 + \alpha_2^2 dx_2^2 + \cdots + \alpha_n^2 dx_n^2 + dx_0^2,$$
where all the functions $\alpha_i$ are positive functions of the variable $x_0$ alone, defined on the interval $(a, b)$.

**Proposition 11.1.** If $\lim_{x_0 \to a} \alpha_i(x_0) = 0$ and $\lim_{x_0 \to b} \alpha_l(x_0) = 0$ for some $i$ and $l$ then the geodesic flow is integrable.

**Proof.** Let us pass to the Hamiltonian formulation. We get the momenta

$$p_0 = \dot{x}_0, \quad p_j = \alpha_j^2 \dot{x}_j, \quad j = 1, \ldots, n,$$

and the Hamiltonian

$$H = \frac{1}{2} \left( p_0^2 + \sum_{j=1}^{n} \alpha_j^{-2} p_j^2 \right).$$

We have the obvious $n$ first integrals $p_1, \ldots, p_n$. Together with $H$ we get $n+1$ first integrals in involution. Under the assumption that $p_i \neq 0$ and $p_l \neq 0$ the common level set of the first integrals must be compact in the variable $x_0$, hence altogether compact. Indeed, we have

$$H \geq \alpha_i^{-2} p_i^2 + \alpha_l^{-2} p_l^2 \to +\infty \quad \text{as} \quad x_0 \to a, b.$$

By the Arnold-Liouville Theorem all regular compact level sets are tori carrying the quasi-periodic motions. □

This last Proposition takes us away from homogeneous systems, now we need only the $\mathbb{T}^n$ symmetry.

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