ASYMPTOTIC DYNAMICS OF A SYSTEM OF CONSERVATION LAWS FROM CHEMOTAXIS

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Abstract. This paper is devoted to the analytical study of the long-time asymptotic behavior of solutions to the Cauchy problem of a system of conservation laws in one space dimension, which is derived from a repulsive chemotaxis model with singular sensitivity and nonlinear chemical production rate. Assuming the $H^2$-norm of the initial perturbation around a constant ground state is finite and using energy methods, we show that there exists a unique global-in-time solution to the Cauchy problem, and the constant ground state is globally asymptotically stable. In addition, the explicit decay rates of the solutions to the chemically diffusive and non-diffusive models are identified under different exponent ranges of the nonlinear chemical production function.

1. Introduction. In this paper, we consider the following Cauchy problem:

\[
\begin{aligned}
& p_t - (pq)_x = p_{xx}, \quad x \in \mathbb{R}, \quad t > 0; \\
& q_t - (p^\gamma + \varepsilon q^2)_x = \varepsilon q_{xx}, \quad x \in \mathbb{R}, \quad t > 0; \\
& (p, q)(x, 0) = (p_0, q_0)(x), \quad x \in \mathbb{R}.
\end{aligned}
\]  

where $\gamma > 1$ and $\varepsilon \geq 0$. The purpose of this paper is to understand the underlying mechanisms of the global dynamics of large-data solutions to the Cauchy problem (1). In particular, we aim to identify the explicit decay rates of large-data solutions to the Cauchy problem (1) with $\varepsilon = 0$ and $\varepsilon > 0$ under different ranges of $\gamma$. 

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1.1. **Background.** Chemotaxis is the oriented movement of cells in response to a chemical stimulus in the environment. It is an indispensable means for cellular communication in immune response, wound healing, progression of diseases, bacteria foraging, cancer metastasis, embryonic development and tissue homeostasis [32]. The well-known chemotaxis model was proposed in [17, 18, 19], which in the general form reads

\[
\begin{align*}
    p_t &= Dp_{xx} - (\chi p \phi (v)_x)_x, \\
    v_t &= \varepsilon v_{xx} + g(p, v),
\end{align*}
\] (2)

where \( p(x, t) \) and \( v(x, t) \) denote the cell density and chemical concentration, respectively; the potential function \( \phi (v) \) is regarded as the chemotactic sensitivity function revealing the signal detection mechanism, and \( g(p, v) \) is the chemical kinetics including chemical production, consumption and degradation. The parameters \( D > 0 \) and \( \varepsilon \geq 0 \) are the cellular and chemical diffusion coefficients, respectively; and \( \chi \) is the chemotactic sensitivity coefficient. The chemotaxis is considered to be attractive if \( \chi > 0 \) and repulsive if \( \chi < 0 \) with \( |\chi| \) measuring the intensity of the chemotaxis.

The experimental and mathematical studies of chemotaxis date back to the pioneering works of Adler [1, 2] and Keller and Segel [17, 18, 19], respectively. Up to date, the extensive studies on the mathematical analysis of the Keller-Segel type chemotaxis model are mainly concerned with the classical attractive case \( (\chi > 0) \), and we refer the readers to the survey papers [3, 10, 11, 42] and the references therein. In contrast, the analytical results for the repulsive chemotaxis model \( (\chi < 0) \) are much less, see [38, 40] and the references therein. In this paper, we consider the following repulsive chemotaxis model with singular sensitivity and nonlinear production rate in one space dimension:

\[
\begin{align*}
    p_t &= Dp_{xx} - (\chi p (\ln v)_x)_x, \\
    v_t &= \varepsilon v_{xx} + f(p)v - \mu v,
\end{align*}
\] (3)

where \( \chi < 0 \), and the constant \( \mu > 0 \) is the natural degradation rate of the chemical signal. When \( f(p) = p \), the model (3) was introduced in [20, 33] to describe the motion of reinforced random walk. When \( f(p) = p^\gamma (\gamma > 1) \), the model (3) was proposed and analyzed in [27, 48, 49], as part of the efforts to understand the complicated interactions between cellular populations and chemical signals. Since the model (3) contains possible singularities stemming from the logarithmic sensitivity function, a commonly practiced preprocessing of the model is to take the Cole-Hopf transformation: \( q \equiv (\ln v)_x \) (cf. [20, 43]). In this paper, we consider the kinetics function \( f(p) = p^\gamma (\gamma > 1) \), and by using the Cole-Hopf transformation, we reformulate (3) into a non-singular system of conservation laws:

\[
\begin{align*}
    p_t - (pq)_x &= p_{xx}, \\
    q_t - (p^\gamma + \varepsilon q^2)_x &= \varepsilon q_{xx},
\end{align*}
\] (4)

where \( \varepsilon \geq 0 \), and we have assumed \( D = -\chi = 1 \) for brevity since their specific values are not of importance in our analysis.

1.2. **Literature review and motivation.** The system (4) has been studied in recent years. We now briefly review the relevant literatures on (4) with respect to \( \gamma \).

When \( \gamma = 1 \), the related results of the qualitative behavior of solutions to the non-diffusive \( (\varepsilon = 0) \) and diffusive \( (\varepsilon > 0) \) systems of (4) are as follows:
existence and nonlinear stability of one-dimensional traveling wave solutions on \( \mathbb{R} \) \([15, 24, 26, 27, 28, 29, 31, 34, 42, 43]\),

- global well-posedness and long-time behavior of one-dimensional and small amplitude strong solutions on finite intervals \([47]\),

- global well-posedness, long-time behavior, chemical diffusion limit and boundary layer formation of one-dimensional and large amplitude strong solutions on finite intervals \([6, 12, 14, 23, 25, 35, 39, 45]\),

- global well-posedness, long-time behavior, chemical diffusion limit and numerical analysis of multi-dimensional and small amplitude strong solutions on bounded domains in \( \mathbb{R}^n \) \([25, 37]\),

- local well-posedness and blowup criteria of multi-dimensional and large amplitude strong solutions on \( \mathbb{R}^n \) \([5, 21]\),

- global well-posedness, long-time behavior and chemical diffusion limit of one-dimensional and large amplitude strong solutions on \( \mathbb{R} \) \([8, 22, 30]\),

- global well-posedness, long-time behavior and chemical diffusion limit of multi-dimensional and small amplitude strong solutions on \( \mathbb{R}^n \) \([4, 9, 13, 21, 36, 41, 44]\).

In contrast, when \( \gamma > 1 \), the qualitative behavior of solutions to the system (4) has been investigated relatively less. The following results have been recently established:

- existence and local stability of one-dimensional traveling wave solutions on \( \mathbb{R} \) \([27]\),

- global well-posedness, long-time behavior of one-dimensional and small amplitude strong solutions on \( \mathbb{R} \) \([48]\),

- global well-posedness, long-time behavior and chemical diffusion limit of one-dimensional and large amplitude weak and strong solutions on \( \mathbb{R} \) \([49]\).

Next, we would like to point out the facts that motivate the current work and state the specific goals to be achieved in this paper.

First of all, we note that the analytical results reported in \([49]\) are under low regularity assumptions on the initial data (at most \( H^1 \)). The qualitative behavior of large amplitude solutions to (1) with higher order regularity has not been investigated. Mathematically, building up higher order regularity of solutions to (1) is challenging, due to the differentiation of \( p^\gamma \) with respect to \( x \) generates higher order nonlinearities when \( \gamma > 1 \). This is in contrast to the case when \( \gamma = 1 \), in which \( \partial_x p \) is always linear. The first goal of this paper is to establish higher order regularity of global and large amplitude solutions to the Cauchy problem (1) by assuming more regular initial data. We achieve the goal by developing novel energy estimates based on delicate applications of various inequalities.

Secondly, we observe that although the global stability of constant ground states associated with (1) has been established in \([49]\), the explicit decay rates of the perturbations have not been identified. Because of the physical importance and mathematical challenge of the problem, the second goal of this paper is set to identify the explicit decay rates of large amplitude solutions to (1) toward constant ground states. We achieve the goal by developing weighted (in time) energy methods. We stress again that due to the nonlinearities stemming from \( \partial_x (p^\gamma) \) when \( \gamma > 1 \), the weighted energy method developed in this paper is more complicated than the case when \( \gamma = 1 \) (cf. \([22, 30]\)).

Furthermore, we note that since \( \epsilon \) is a coefficient of both diffusion and “convection”, it is desirable to understand how such a parameter affects the long-time...
dynamics of the solution to (1). This is the third fact that motivates the current work. As a matter of fact, the readers will see from the proofs in Section 3 and Section 4 that there is an intrinsic distinction in the process of mathematical analysis between the non-diffusive and diffusive systems, for which \( \varepsilon \) takes the major responsibility. Roughly speaking, when \( \varepsilon > 0 \), the parabolic structure of the model presents a strong dissipation mechanism that suppresses the influence of the nonlinear convection term and enhances the long-time dynamics of the solution, comparing with the non-diffusive model (i.e. \( \varepsilon = 0 \)).

1.3. Statement of results. To present our main results, we first introduce some notations.

Notation. Throughout this paper, we use \( \| \cdot \|_{L^2} \), \( \| \cdot \|_{H^s} \), and \( \| \cdot \|_{L^\infty} \) to denote the norms of the usual Lebesgue space \( L^2(\mathbb{R}) \), the Hilbert space \( H^s(\mathbb{R}) \), and the Sobolev space \( L^\infty(\mathbb{R}) \), respectively. Unless otherwise specified, we use \( C \) and \( C_i \) to denote generic constants which are independent of the unknown functions. The values of the constants may vary line by line according to the context.

The first result of this paper on the global well-posedness and long-time behavior of large amplitude strong solutions to the Cauchy problem (1) is stated as follows.

**Theorem 1.1 (Global Dynamics for \( \varepsilon \geq 0 \)).** Consider the Cauchy problem (1). Let \( 3 \leq \gamma \in \mathbb{R} \) and \( \varepsilon \geq 0 \) be fixed. Assume that the initial data satisfy \( p_0 \geq 0 \), \( p_0 \not\equiv 0 \) and \( (p_0 - \bar{p}, q_0) \in H^2(\mathbb{R}) \) for some constant ground state \( \bar{p} > 0 \). Then there exists a unique global-in-time solution \( (p, q) \) to the Cauchy problem (1), such that for any \( t > 0 \), it holds that

\[
\| (p - \bar{p})(t) \|_{H^2}^2 + \| q(t) \|_{H^2}^2 + \int_0^t \left( \| p_x(\tau) \|_{H^2}^2 + \| q_x(\tau) \|_{H^1}^2 + \varepsilon \| q_x(\tau) \|_{H^2}^2 \right) d\tau \leq C,
\]

where the constant \( C > 0 \) is independent of \( t \) and \( \varepsilon \), and depends only on \( p_0, q_0, \bar{p} \) and \( \gamma \). Furthermore, it holds that

\[
\lim_{t \to \infty} \left( \| p(t) - \bar{p} \|_{L^\infty}^2 + \| p_x(t) \|_{L^\infty}^2 + \| q(t) \|_{L^\infty}^2 + \| q_x(t) \|_{L^\infty}^2 \right) = 0.
\]

**Remark 1.** Theorem 1.1 improves the regularity of the solutions established in [49], which serves as the foundation for identifying the explicit decay rates of large-data strong solutions to the Cauchy problem (1). Moreover, since the energy estimates recorded in Theorem 1.1 are \( \varepsilon \)-independent, one can upgrade the zero chemical diffusion limit and convergence rate results obtained in [49] to higher order spatial derivatives of solutions to (1). Since the proof is in the same spirit of [49], we omit the technical details in order to simplify the presentation.

The second result is concerned with the explicit decay rates of the non-diffusive problem, i.e., (1) with \( \varepsilon = 0 \). We remark that an essential difficulty encountered in the proof is the control of certain low frequency part of the solution, due to the lack of the Poincaré inequality in the whole space case. To resolve the issue, we invoke the ideas of [22, 30] to define anti-derivatives of the perturbations and perform time-weighted energy estimates. To state the result, we introduce the following anti-derivatives:

\[
\phi(x, t) = \int_{-\infty}^{x} (p(y, t) - \bar{p}) dy, \quad \psi(x, t) = \int_{-\infty}^{x} q(y, t) dy, \quad t \geq 0,
\]

(5)
where \( \tilde{p} > 0 \) is any given constant ground state. As in \([26, 27, 28] \), we assume that 
\[
\phi_0(\pm \infty) = \psi_0(\pm \infty) = 0, \quad \text{which implies } \phi(\pm \infty, t) = \psi(\pm \infty, t) = 0 \text{ for all } t > 0
\]
according to the conservative structures of the equations in (1). Then we have

**Theorem 1.2** (Decay Rates for \( \varepsilon = 0 \)). Consider the Cauchy problem (1) with (5) when \( 3 \leq \gamma \in \mathbb{R} \) and \( \varepsilon = 0 \). Assume that the initial data satisfy \( (\phi_0, \psi_0)(x) \in H^3(\mathbb{R}) \) and there exists a sufficiently small constant \( \zeta_1 > 0 \) such that 
\[
\|\phi_0\|_{L^2}^2 + \|\psi_0\|_{L^2}^2 \leq \zeta_1.
\]
Then there exists a unique solution to (1) satisfying \( p(\cdot - \tilde{p}) \in C([0, \infty); H^2(\mathbb{R})) \cap L^2((0, \infty); H^3(\mathbb{R})) \) and \( q \in C([0, \infty); H^2(\mathbb{R})) \cap L^2((0, \infty); H^3(\mathbb{R})) \).

Moreover, there exists a constant \( C > 0 \) which is independent of \( t \), and a finite time \( T > 0 \), such that for any \( t > T \) it holds that
\[
\begin{align*}
\sum_{k=1}^{3} \left[ (1 + t)^{k-1} \|p^{-1}(p(\cdot - \tilde{p}))\|_{L^2}^2 + \|q^{-1}(p(\cdot - \tilde{p}))\|_{L^2}^2 \right] + \int_{T}^{t} (1 + \tau)^{k} \|p^{-k}(p(\cdot - \tilde{p}))\|_{L^2}^2 d\tau \\
+ \sum_{m=1}^{2} \int_{T}^{t} (1 + \tau)^{m} \|p^{-m}(p(\cdot - \tilde{p}))\|_{L^2}^2 d\tau \leq C.
\end{align*}
\]

In particular, it holds that
\[
\|p(t) - \tilde{p}, q(t)\|_{L^\infty} \leq C(1 + t)^{-\frac{3}{4}}, \quad \text{and} \quad \|p_{\cdot x}(t)\|_{L^\infty} \leq C(1 + t)^{-\frac{5}{4}}.
\]

**Remark 2.** Although Theorem 1.2 requires the smallness of the \( L^2 \)-level energy of the initial perturbation, the initial oscillations of the anti-derivatives can be potentially large. This is because Theorem 1.1 holds for potentially large perturbations which correspond to large oscillations of the anti-derivatives. Moreover, the result recorded in Theorem 1.2 generalizes the previous one reported in [22] by extending the exponent \( \gamma \) of the chemical production rate function from a single value to the half real line.

**Remark 3.** The main reason for studying the higher order regularity of the solution to (1) is that the uniform (in time) energy estimate of \( \|p_x\|_{L^\infty} \) is necessary for proving the explicit decay rates of the perturbation when \( \varepsilon = 0 \), see (58). Since 
\[
\|p_x\|_{L^\infty} \leq 2 \|p_x\|_{L^2} \|p_{xx}\|_{L^2},
\]
one needs to get the uniform estimates of \( \|p_x\|_{L^2} \) and \( \|p_{xx}\|_{L^2} \) in order to establish the uniform estimate of \( \|p_x\|_{L^\infty} \). Note that the uniform estimate of \( \|p_x\|_{L^2} \) has been obtained in [49] for \( \gamma \geq 2 \) and \( H^1 \) initial data. For \( H^2 \) initial data, in order to gain the uniform estimate of \( \|p_{xx}\|_{L^2} \) one needs to estimate \( (\gamma)^2_{xx} \), which generates the nonlinear term: \( \gamma(\gamma - 1)(\gamma - 2)p^{\gamma-3}(p_x)^3 \). Based on our analysis, such a term can be controlled only when \( \gamma - 3 \geq 0 \), otherwise one loses control of \( p^{\gamma-3} \). This explains why we require \( \gamma \geq 3 \) in Theorem 1.2. However, note that when \( \gamma = 2 \), the aforementioned nonlinear term does not appear in \( (\gamma)^2_{xxx} \). In this case, the uniform estimate of \( \|p_{xx}\|_{L^2} \) can be established and the explicit decay rates can be identified. Hence, based on such an observation and the previous results for the case when \( \gamma = 1 \), the cases for which the explicit decay rates are not identified for the non-diffusive model (\( \varepsilon = 0 \)) are when \( 1 < \gamma < 2 \) and \( 2 < \gamma < 3 \). We leave the investigation in a future work.

The last theorem addresses the explicit decay rates of the diffusive problem, i.e., (1) with \( \varepsilon > 0 \).

**Theorem 1.3** (Decay Rate for \( \varepsilon > 0 \)). Consider the Cauchy problem (1) with (5) when \( 2 \leq \gamma \in \mathbb{R} \) and \( \varepsilon > 0 \). Assume that the initial data satisfy \( (\phi_0, \psi_0)(x) \in H^3(\mathbb{R}) \) and there exists a sufficiently small constant \( \zeta_2 > 0 \) such that 
\[
\|\phi_0\|_{L^2}^2 + \|\psi_0\|_{L^2}^2 \leq \zeta_2.
\]
Then there exists a unique solution to (1) satisfying \((p - \bar{p}, q) \in C([0, \infty) ; H^1(\mathbb{R})) \cap L^2([0, \infty) ; H^2(\mathbb{R}))\). Moreover, there exists a constant \(C > 0\) which is independent of \(t\), such that for any \(t > 0\) it holds that

\[
\sum_{k=1}^{2} \left[ (1 + t)^k \left( \| \partial_x^{k-1}(p(t) - \bar{p}) \|_{L^2}^2 + \| \partial_x^{k-1}q(t) \|_{L^2}^2 \right) + \int_0^t (1 + \tau)^k \left( \| \partial_x^k p(\tau) \|_{L^2}^2 + \| \partial_x^k q(\tau) \|_{L^2}^2 \right)d\tau \right] \leq C.
\]

Furthermore, it holds that

\[
\| (p(t) - \bar{p}, q(t)) \|_{L^\infty} \leq C(1 + t)^{-\frac{3}{4}}.
\]

**Remark 4.** Because of the parabolic structure of the diffusive model \((\varepsilon > 0)\), both the requirements on the exponent \(\gamma\) and the smallness of the initial perturbation in Theorem 1.3 are less demanding than the ones in Theorem 1.2. In addition, the result recorded in Theorem 1.3 improves the previous one reported in [30], again by extending the exponent \(\gamma\) of the chemical production rate function from a single value to the half real line.

**Remark 5.** The global regularity and explicit decay rates recorded in Theorems 1.1, 1.2 and 1.3 are among the first generation of analytical results for the system of conservation laws (4). The novelty of our results lies in the fact that despite the presence of strong nonlinearities, the results are valid for potentially large amplitude perturbations. This is rare among the existing results on the general frameworks for hyperbolic-parabolic conservation laws, where the full spectrum of the initial perturbations must be assumed to be small in order to obtain the global well-posedness and long-time behavior of strong solutions (see e.g. [46]). Moreover, unlike the case of bounded domains, in which the Poincaré inequality can be utilized to obtain exponential decay rates of large-data solutions (cf. [23, 25, 37, 45]), identifying explicit decay rates of large-data solutions to the Cauchy problem for systems of conservation laws is usually much more difficult, due to the non-compactness of the spatial domain.

We have mentioned that the major technical obstacle encountered in the analysis of (1) is the strong nonlinearities stemming from \(p^\gamma\) when building up the higher order regularity of the solution. We overcome the obstacle through deriving delicate energy estimates by utilizing \(L^p\)-based energy methods and applying various inequalities, such as Hölder, Young, Cauchy-Schwarz, Gagliardo-Nirenberg, and Grönwall inequalities. Comparing with the previous studies on the model when \(\gamma = 1\) [22, 30] and \(\gamma > 1\) for \(H^1\) initial data [49], the energy method developed in this paper is more involved, as the control of the second order spatial derivative of the solution contains the estimation of \((p^\gamma)^{xxx}\), which consists in the major novelty of this paper. As in the previous studies (cf. [22, 30, 49]), another ingredient of the proof is the derivation of an inhomogeneous damping equation for the spatial derivatives of the \(q\)-component, which allows us to obtain the global dynamics and zero chemical diffusion limit results in one stroke. Moreover, the explicit decay rates are obtained by using time-weighted energy method and the uniform estimate of the solution.

**Remark 6.** The explicit decay rates recorded in Theorems 1.2 and 1.3 are the best rates possible via time-weighted energy method. However, the parabolic structure
of the equations in (4) suggests that these rates are not optimal, in the sense that they are slower than the decay rates of the solution to the heat equation. Hence, it is desirable to know whether the rates in Theorems 1.2 and 1.3 can be improved or not. We anticipate that the combination of detailed spectral analysis and Duhamel’s principle will be an effective vehicle to carry the analysis for handling such an exquisite situation. We leave the investigation in a forthcoming paper.

Remark 7. In [34], the authors studied the nonlinear stability of traveling wave solutions to (1) when $\gamma = 1$. We note that the initial perturbation is assumed to be in $L^2$ only instead of $H^2$, and its amplitude can be arbitrarily large. However, the regularity of the solution can still be improved to $H^2$ due to the diffusive dissipation in the system. Hence, the results reported in this paper may be improved by prescribing lower regularity on initial data using the idea of [34]. We leave the investigation in a future work.

Remark 8. We further remark that our proofs of Theorems 1.2 and 1.3 rely heavily on the uniform-in-time energy estimates of the solutions. To extend our results to multi-space dimensions, one must overcome the difficulty in obtaining energy estimate on low frequency due to the lack of Poincaré inequality in the whole space. The main issue with $\varepsilon > 0$ is how to find an appropriate entropy function in higher space dimensions, while in the case of $\varepsilon = 0$ the issue is the lack of sufficient dissipation. This raises an interesting and challenging question for future studies in this area.

The rest of this paper is organized as follows. In Sections 2–4, we complete the proofs of Theorems 1.1–1.3, respectively. The paper finishes with concluding remarks.

2. Global dynamics for $\varepsilon \geq 0$. In this section, we prove Theorem 1.1. First of all, we note that under the conditions of Theorem 1.1 the local well-posedness of strong solutions to the Cauchy problem (1) can be established by applying Kawashima’s theory on a general system of conservation laws [16]. Moreover, it follows from the maximum principle (cf. [7]) that the local solution satisfies $p \geq 0$ within its lifespan. We collect the results in the following:

Proposition 1 (Local Existence). Consider the Cauchy problem (1). Let the conditions of Theorem 1.1 hold. Then there exists a finite $T_0 > 0$ and a unique solution $(p, q)$ to (1) such that $p \geq 0$ and

- $(p - \bar{p}, q) \in C([0, T_0); H^2(\mathbb{R}))$, $(p_x, q_x) \in L^2([0, T_0); H^2(\mathbb{R}))$, for $\varepsilon > 0$;
- $(p - \bar{p}, q) \in C([0, T_0); H^2(\mathbb{R}))$, $p_x \in L^2([0, T_0); H^2(\mathbb{R}))$, $q_x \in L^2([0, T_0); H^1(\mathbb{R}))$, for $\varepsilon = 0$.

Next, we derive a priori estimates of the local solution, in order to extend it to be a global one.

Proposition 2 (A priori Estimates). Let $(p, q)$ be the solution obtained in Proposition 1. Then it holds that

$$\|p(t) - \bar{p}\|^2_{H^2} + \|q(t)\|^2_{H^2} + \int_0^t \left( \|p_x(\tau)\|^2_{H^2} + \|q_x(\tau)\|^2_{H^1} + \varepsilon \|q_x(\tau)\|^2_{H^2} \right) d\tau \leq C,$$

where the constant $C > 0$ is independent of $t$ and $\varepsilon$. 

In what follows, we divide the proof of Proposition 2 into two parts which are separated by the lower order and higher order estimates of the solution. To proceed, we first quote on the lower order regularity results of the solution (cf. [49]).

**Proposition 3** ([49, Theorem 1.1]). Let \(2 \leq \gamma \in \mathbb{R} \) and \(\varepsilon \geq 0\) be fixed. Assume that the initial data satisfies \(p_0 \geq 0\) and \((p_0 - \bar{p}, q_0) \in H^1(\mathbb{R})\) for some constant ground state \(\bar{p} > 0\). Then there exists a unique global-in-time solution \((p, q)\) to the Cauchy problem (1), such that

\[
\|(p - \bar{p})(t)\|_{H^1}^2 + \|q(t)\|_{H^1}^2 + \int_0^t \left(\|p_x(\tau)\|_{H^1}^2 + \|q_x(\tau)\|_{L^2}^2 + \varepsilon \|q_{xx}(\tau)\|_{L^2}^2\right) \, d\tau \leq C,
\]

where the constant \(C > 0\) is independent of \(t\) and \(\varepsilon\), and depends only on \(p_0, q_0, \bar{p}\) and \(\gamma\). In addition, it holds that

\[
\lim_{t \to \infty} \left(\|(p - \bar{p})(t)\|_{L^\infty}^2 + \|q(t)\|_{L^\infty}^2\right) = 0.
\]

Next, we move on to the uniform-in-time estimate of higher order spatial derivatives of the solution. Since the proof of the following lemma is rather lengthy, we divide it into four steps.

**Lemma 2.1 (H^2-Estimate).** Under the conditions of Theorem 1.1, for any \(3 \leq \gamma \in \mathbb{R}, \varepsilon \geq 0\), and \(t > 0\), it holds that

\[
\|\bar{p}_{xx}(t)\|_{L^2}^2 + \|q_{xx}(t)\|_{L^2}^2 + \int_0^t \left(\|\bar{p}_{xxx}(\tau)\|_{L^2}^2 + \|q_{xxx}(\tau)\|_{L^2}^2 + \varepsilon \|q_{xxxx}(\tau)\|_{L^2}^2\right) \, d\tau \leq C,
\]

where \(\bar{p} = p - \bar{p}\) and the constant \(C > 0\) is independent of \(t\) and \(\varepsilon\).

**Proof. Step 1.** To carry out asymptotic analysis, we reformulate (1) by letting \(\bar{p} = p - \bar{p}\), and take the second order spatial derivatives of (4), then we have

\[
\begin{align*}
\bar{p}_{xxt} - [(\bar{p} + \bar{p})q]_{xxx} &= \bar{p}_{xxxxx}, \\
q_{xxt} - [(\bar{p} + \bar{p})q]_{xxx} &= \varepsilon q_{xxxxx} + \varepsilon (q^2)_{xxx}.
\end{align*}
\]

(6)

Taking the \(L^2\) inner product of the first equation of (6) with \(\gamma \bar{p}^{\gamma-2} \bar{p}_{xx}\) and the second one with \(q_{xx}\), and then adding the results, we obtain

\[
\begin{align*}
\frac{d}{dt} \left(\frac{\gamma}{2} \bar{p}^{\gamma-2} \|\bar{p}_{xx}\|_{L^2}^2 + \frac{1}{2} \|q_{xx}\|_{L^2}^2\right) + \gamma \bar{p}^{\gamma-2} \|\bar{p}_{xxx}\|_{L^2}^2 + \varepsilon \|q_{xxx}\|_{L^2}^2 \\
= -\gamma \bar{p}^{\gamma-2} \int_{\mathbb{R}} (\bar{p}q)_{xx} \bar{p}_{xx} \, dx + \gamma \int_{\mathbb{R}} [(\bar{p} + \bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}] \bar{p}_{xxx} q_{xx} \, dx \\
+ \gamma (\gamma - 1) (\gamma - 2) \int_{\mathbb{R}} (\bar{p} + \bar{p})^{\gamma-3} (\bar{p}_x)^3 q_{xx} \, dx \\
+ 3 \gamma (\gamma - 1) \int_{\mathbb{R}} (\bar{p} + \bar{p})^{\gamma-2} \bar{p}_x q_{xxx} \, dx - \varepsilon \int_{\mathbb{R}} q_{xxx} (q^2)_{xx} \, dx.
\end{align*}
\]

(7)

Before proceeding to bound the terms on the right-hand side of (7), we note that the energy estimation will inevitably depend on the temporal integral of \(\|q_{xx}\|_{L^2}^2\). However, we see from Proposition 3 that the temporal integral of \(\|q_{xx}\|_{L^2}^2\) depends reciprocally on \(\varepsilon\). To obtain the \(\varepsilon\)-independent estimate of the temporal integral of \(\|q_{xx}\|_{L^2}^2\), we first derive a damping equation for \(q_{xx}\). By using the first and second
equations of (6), we can show that
\[
q_{xx} = \gamma \gamma^{-1} \bar{p}_{xx} - (\bar{p} q)_{xx} - \bar{p} q_{xx} + \gamma [ (\bar{p} + \bar{p}) \gamma^{-1} - \bar{p} \gamma^{-1} ] \bar{p}_{xxx} \\
+ \varepsilon (q^2)_{xxx} + \gamma (\gamma - 1) (\gamma - 2) (\bar{p} + \bar{p}) \gamma^{-3} (\bar{p} x)^3 \\
+ 3 \gamma (\gamma - 1) (\bar{p} + \bar{p}) \gamma^{-2} \bar{p} \bar{p}_{xx} + \varepsilon q_{xxxx}.
\]

Multiplying (8) by \(q_{xx}\) and integrating by parts, we derive
\[
\frac{d}{dt} \left( \frac{1}{2} \| q_{xx} \|_{L^2}^2 \right) + \gamma \bar{p} \gamma^{-1} \int \int \bar{p} q_{xx} dx - \gamma \bar{p} \gamma^{-1} \int \int \bar{p} q_{xx} dx - \gamma \bar{p} \gamma^{-1} \int (\bar{p} q)_{xx} q_{xx} dx \\
+ \int \int [(\bar{p} + \bar{p}) \gamma^{-1} - \bar{p} \gamma^{-1}] \bar{p}_{xxx} q_{xx} dx \\
+ \gamma (\gamma - 1) (\gamma - 2) \int \int (\bar{p} + \bar{p}) \gamma^{-3} (\bar{p} x)^3 q_{xx} dx \\
+ 3 \gamma (\gamma - 1) \int (\bar{p} + \bar{p}) \gamma^{-2} \bar{p} \bar{p}_{xx} q_{xx} dx - \varepsilon \int \int q_{xxxx} (q^2)_{xx} dx.
\]

For the second term on the right-hand side of (9), by employing the second equation of (6) and integrating by parts, we have
\[
-\gamma \bar{p} \gamma^{-1} \int \int \bar{p} q_{xx} dx = \gamma^2 (\gamma - 1) \bar{p} \gamma^{-1} \int \int \bar{p} q_{xx} dx \\
+ \gamma^2 \bar{p} \gamma^{-1} \int \int (\bar{p} + \bar{p}) \gamma^{-2} (\bar{p} x)^2 dx \\
+ \varepsilon \gamma \bar{p} \gamma^{-1} \int \int \bar{p} q_{xx} dx + \varepsilon \gamma \bar{p} \gamma^{-1} \int \int \bar{p} q_{xx} (q^2)_{xx} dx,
\]

which, combined with (9), and then added to (7), yields
\[
\frac{d}{dt} \left( \frac{\gamma}{2} \bar{p} \gamma^{-2} \| \bar{p}_{xx} \|_{L^2}^2 + \| q_{xx} \|_{L^2}^2 - \gamma \bar{p} \gamma^{-1} \int \int \bar{p} q_{xx} dx \right) \\
\quad + \gamma \bar{p} \gamma^{-2} \| \bar{p}_{xx} \|_{L^2}^2 + \gamma \bar{p} \gamma^{-2} \| \bar{p} q_{xx} \|_{L^2}^2 + 2 \varepsilon \| q_{xx} \|_{L^2}^2 \\
= 2 \gamma (\gamma - 1) (\gamma - 2) \int \int (\bar{p} + \bar{p}) \gamma^{-3} (\bar{p} x)^3 q_{xx} dx \\
+ 6 \gamma (\gamma - 1) \int \int (\bar{p} + \bar{p}) \gamma^{-2} \bar{p} \bar{p}_{xx} q_{xx} dx \\
+ 2 \gamma \int \int [(\bar{p} + \bar{p}) \gamma^{-1} - \bar{p} \gamma^{-1}] \bar{p} q_{xx} dx - \gamma \bar{p} \gamma^{-2} \int \int (\bar{p} q)_{xx} \bar{p} q_{xx} dx \\
- \gamma \bar{p} \gamma^{-1} \int \int (\bar{p} q)_{xx} dx + \gamma^2 (\gamma - 1) \bar{p} \gamma^{-1} \int \int \bar{p} (\bar{p} + \bar{p}) \gamma^{-2} (\bar{p} x)^2 dx \\
+ \gamma^2 \bar{p} \gamma^{-1} \int \int (\bar{p} + \bar{p}) \gamma^{-1} (\bar{p} x)^2 dx + \varepsilon \gamma \bar{p} \gamma^{-1} \int \int \bar{p} q_{xx} dx \\
+ \varepsilon \gamma \bar{p} \gamma^{-1} \int \int \bar{p} q_{xx} (q^2)_{xx} dx - 2 \varepsilon \int \int q_{xxxx} (q^2)_{xx} dx \\
\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10}.
\]

Next, we proceed to estimate \(I_1, \ldots, I_{10}\). From Proposition 3, we can see that

\[ \| \bar{p} \|_{L^\infty}^2 \leq C_1 \] and \[ \| q \|_{L^\infty}^2 \leq C_2, \]
where the positive constants are independent of $t$ and $\varepsilon$, and we have used the inequality
\[
\|f\|_{L^2}^2 \leq 2 \|f\|_{L^1} \|f_x\|_{L^2}, \quad f \in H^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),
\] 
(12)
for $\tilde{p}$ and $q$. By using the Hölder and Young inequalities, since $\gamma \geq 3$, we can estimate $I_1$ as
\[
|I_1| = \left| 2\gamma(\gamma - 1)(\gamma - 2) \int_\mathbb{R} (\tilde{p} + \tilde{p})^{\gamma-3}(\tilde{p}_x)^3 q_{xx} dx \right|
\leq C_3 \int_\mathbb{R} (|\tilde{p}|^{\gamma-3} + \tilde{p}^{\gamma-3}) |\tilde{p}_x|^3 |q_{xx}| dx
\leq C_3 (|\tilde{p}|_{L^\infty}^{\gamma-3} + \tilde{p}^{\gamma-3}) |\tilde{p}_x|_{L^2}^2 \|q_{xx}\|_{L^2}
\leq C_4 |\tilde{p}_x|_{L^2} \|q_{xx}\|_{L^2}
\leq \delta |q_{xx}|_{L^2}^2 + C(\delta) \|\tilde{p}_x\|_{L^2}^2,
\]
(13)
where we have used (12) for $\tilde{p}_x$, $\delta > 0$ is a constant to be determined, and $C_4$ depends only on $\gamma, \tilde{p}$ and the uniform estimates of $\|\tilde{p}\|_{L^\infty}$ and $\|\tilde{p}_x\|_{L^2}$ from Proposition 3.

By utilizing the Young inequality, for $I_2$, one can show that
\[
|I_2| = \left| 6\gamma(\gamma - 1) \int_\mathbb{R} (\tilde{p} + \tilde{p})^{\gamma-2}\tilde{p}_x q_{xx} dx \right|
\leq C_5 (|\tilde{p}|_{L^\infty}^{\gamma-2} + \tilde{p}^{\gamma-2}) |\tilde{p}_x|_{L^\infty} \|\tilde{p}_x\|_{L^2} \|q_{xx}\|_{L^2}
\leq C_5 |\tilde{p}_x|_{L^2}^2 \|q_{xx}\|_{L^2}
\leq \delta |q_{xx}|_{L^2}^2 + C(\delta) |\tilde{p}_x|_{L^2}^2,
\]
(14)
where we have used (12) for $\tilde{p}_x$, and $C_5$ depends only on $\gamma, \tilde{p}$ and the uniform estimates of $\|\tilde{p}\|_{L^\infty}$ and $\|\tilde{p}_x\|_{L^2}$.

For $I_3$, by the Mean Value Theorem,
\[
(\tilde{p} + \tilde{p})^{\gamma-1} - \tilde{p}^{\gamma-1} = (\gamma - 1)(p^*)^{\gamma-2} \tilde{p},
\]
where $p^*$ is between $\tilde{p} + \tilde{p}$ and $\tilde{p}$, satisfying $|p^*| \leq |\tilde{p}| + |\tilde{p}|$. Then we have
\[
|I_3| = \left| 2\gamma \int_\mathbb{R} [(\tilde{p} + \tilde{p})^{\gamma-1} - \tilde{p}^{\gamma-1}] \tilde{p}_x q_{xx} dx \right|
\leq 2\gamma(\gamma - 1) \int_\mathbb{R} (|\tilde{p}| + |\tilde{p}|)^{\gamma-2}|\tilde{p}| |\tilde{p}_x| \|q_{xx}\| dx
\leq C_7 (|\tilde{p}|_{L^\infty}^{\gamma-2} + \tilde{p}^{\gamma-2}) |\tilde{p}| \|\tilde{p}_x\|_{L^2} \|q_{xx}\|_{L^2}
\leq C_8 |\tilde{p}_x|_{L^2}^2 \|q_{xx}\|_{L^2}
\leq \theta |\tilde{p}_x|_{L^2}^2 + C(\theta) \|\tilde{p}_x\|_{L^2} \|q_{xx}\|_{L^2},
\]
(15)
where we have used (12) for $\tilde{p}$, and $\theta > 0$ is a constant to be determined, and $C_8$ depends only on $\gamma, \tilde{p}$ and the uniform estimates of $\|\tilde{p}\|_{L^\infty}$ and $\|\tilde{p}_x\|_{L^2}$. By employing the Young inequality, we have
\[
C(\theta) \|\tilde{p}_x\|_{L^2} \leq \delta + C(\delta, \theta) |\tilde{p}_x|_{L^2}^2.
\]
(16)
Plugging (16) into (15), we obtain
\[
|I_3| \leq \theta |\tilde{p}_x|_{L^2}^2 + \delta |q_{xx}|_{L^2}^2 + C(\delta, \theta) |\tilde{p}_x|_{L^2}^2 |q_{xx}|_{L^2}^2.
\]
(17)
By using (12) for $\hat{p}$ and $\tilde{p}_x$, and the Young inequality, we infer that
\[
|I_4| = \left| \gamma \hat{p}^{\gamma-2} \int_R (\tilde{p} q)_{xx} \tilde{p} x x x dx \right|
\leq \theta \|\tilde{p} x x x\|_{L^2}^2 + C(\theta)\|(\tilde{p} q)_{xx}\|_{L^2}^2
\leq \theta \|\tilde{p} x x x\|_{L^2}^2 + C(\theta)\|q\|_{L^\infty}^2 \|\tilde{p} x x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^\infty}^2 \|q_x\|_{L^2}^2 + \|\tilde{p}\|_{L^\infty}^2 \|q_{xx}\|_{L^2}^2 \]  
(18)
\leq \theta \|\tilde{p} x x x\|_{L^2}^2 + C(\theta)(\|\tilde{p} x x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2)
\leq \theta \|\tilde{p} x x x\|_{L^2}^2 + \delta \|q x x\|_{L^2}^2 + C(\theta)(\|\tilde{p} x x\|_{L^2}^2 + \|q_x\|_{L^2}^2)
+ C(\delta, \theta)\|\tilde{p}_x\|_{L^2}^2 \|q_{xx}\|_{L^2}^2,
\]
where we have used (16) and the uniform estimates of $\|q\|_{L^\infty}$, $\|\tilde{p}_x\|_{L^2}$, $\|\tilde{p}\|_{L^2}$ and $\|q_x\|_{L^2}$.

Similar to (18), by using the Young inequality, we can show that
\[
|I_5| = \left| \gamma \hat{p}^{\gamma-1} \int_R (\tilde{p} q)_{x} q_{x x} dx \right|
\leq \delta \|q_{x x}\|_{L^2}^2 + C(\delta)\|q\|_{L^\infty}^2 \|\tilde{p} x x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^\infty}^2 \|q_x\|_{L^2}^2 + \|\tilde{p}\|_{L^\infty}^2 \|q_{xx}\|_{L^2}^2 \]  
(19)
\leq \delta \|q_{x x}\|_{L^2}^2 + C(\delta)(\|\tilde{p} x x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2 \|q_{xx}\|_{L^2}^2).
\]

By employing the Young inequality again, for $I_6$, we derive
\[
|I_6| = \left| \gamma^2 (\gamma - 1) \hat{p}^{\gamma-2} \int_R \tilde{p} x x (\hat{p} + \tilde{p}) \gamma - 2 (\tilde{p}_x)^2 dx \right|
\leq C_9 \|\tilde{p}\|_{L^\infty}^2 + \tilde{p}^{\gamma-2} \|\tilde{p}_x\|_{L^\infty} \|\tilde{p} x x\|_{L^2} \|\tilde{p}_x\|_{L^2} \]  
(20)
\leq C_10 \|\tilde{p}_x\|_{L^2}^3 \|\tilde{p} x x\|_{L^2}^3
\leq \|\tilde{p} x x\|_{L^2}^2 + C_{11} \|\tilde{p}_x\|_{L^2}^2,
\]
where we have used (12) for $\tilde{p}_x$, and $C_{11}$ depends only on $\gamma, \tilde{p}$ and the uniform estimates of $\|\tilde{p}\|_{L^\infty}$ and $\|\tilde{p}_x\|_{L^2}$.

For $I_7$, by using the uniform estimate of $\|\tilde{p}\|_{L^\infty}$ again, we obtain
\[
|I_7| = \left| \gamma^2 \hat{p}^{\gamma-1} \int_R (\hat{p} + \tilde{p}) \gamma - 1 (\tilde{p}_x)^2 dx \right|
\leq C_{12} \|\tilde{p}\|_{L^\infty}^{-1} \tilde{p}^{\gamma-1} \|\tilde{p}_x\|_{L^2}^2
\leq C_{13} \|\tilde{p}_x\|_{L^2}^2,
\]
where $C_{13}$ is independent of $t$ and $\varepsilon$.

The combination of (13), (14), (17), (18), (19), (20) and (21) gives
\[
\sum_{k=1}^7 |I_k| \leq 2\theta \|\tilde{p} x x x\|_{L^2}^2 + 5\delta \|q_{x x}\|_{L^2}^2
+ C(\delta, \theta)(\|\tilde{p} x x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)(\|\tilde{p} x x\|_{L^2}^2 + \|q_{x x}\|_{L^2}^2)
+ \|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2 + \|q_{x x}\|_{L^2}^2). \]  
(22)
For the $I_k$'s involving $\varepsilon$, we have the following estimates. For $I_8$, by using the Cauchy-Schwarz inequality, we derive

$$\begin{align*}
|I_8| &= \left| \varepsilon \gamma \bar{p}^{-1} \int_R \bar{p}_{xx} q_{xxx} \, dx \right| \\
&\leq \frac{\varepsilon}{4} \|q_{xxx}\|_2^2 + \varepsilon (\gamma \bar{p}^{-1})^2 \|\bar{p}_{xx}\|_2^2. 
\end{align*} \tag{23}$$

For $I_9$, by using (12) for $q_x$ and the uniform estimates of $\|q\|_{L^\infty}$ and $\|q_x\|_{L^2}$, we obtain

$$\begin{align*}
|I_9| &= \left| \varepsilon \gamma \bar{p}^{-1} \int_R \bar{p}_{xx} (q^2)_{xxx} \, dx \right| \\
&\leq \varepsilon (\gamma \bar{p}^{-1})^2 \|\bar{p}_{xx}\|_2^2 + \frac{\varepsilon}{4} \|q_x\|_{L^2}^2 \\
&\leq \varepsilon (\gamma \bar{p}^{-1})^2 \|\bar{p}_{xx}\|_2^2 + C_{14} \varepsilon (\|q_x\|_{L^\infty}^2 \|q_x\|_{L^2}^2 + \|q_x\|_{L^2}^\infty \|q_{xxx}\|_{L^2}^2) \\
&\leq \varepsilon (\gamma \bar{p}^{-1})^2 \|\bar{p}_{xx}\|_2^2 + C_{15} (\varepsilon \|q_{xx}\|_{L^2}^2 + \varepsilon \|q_x\|_{L^2}^2),
\end{align*} \tag{24}$$

where the positive constants are independent of $t$ and $\varepsilon$.

Similarly, for $I_{10}$, we can show that

$$\begin{align*}
|I_{10}| &= \left| 2\varepsilon \int_R q_{xxx} (q^2)_{xxx} \, dx \right| \\
&\leq \frac{\varepsilon}{4} \|q_{xxx}\|_2^2 + C_{16} (\varepsilon \|q_{xx}\|_{L^2}^2 + \varepsilon \|q_x\|_{L^2}^2), 
\end{align*} \tag{25}$$

where $C_{16}$ is independent of $t$ and $\varepsilon$. By combing (23), (24) and (25), we have

$$|I_8| + |I_9| + |I_{10}| \leq \frac{\varepsilon}{2} \|q_{xxx}\|_{L^2}^2 + C_{17} (\varepsilon \|\bar{p}_{xx}\|_{L^2}^2 + \varepsilon \|q_{xx}\|_{L^2}^2 + \varepsilon \|q_x\|_{L^2}^2), \tag{26}$$

where $C_{17}$ is independent of $t$ and $\varepsilon$.

Substituting (22) and (26) into (11), we infer that

$$\begin{align*}
\frac{d}{dt} \left[ \gamma \frac{1}{2} \bar{p}^{-2} \|\bar{p}_{xx}\|_{L^2}^2 + \|q_{xx}\|_{L^2}^2 - \gamma \bar{p}^{-1} \int_R \bar{p}_{xx} q_{xxx} \, dx \right] \\
&+ \gamma \bar{p}^{-2} \|\bar{p}_{xxx}\|_{L^2}^2 + \gamma \bar{p}^{-1} \|q_{xxx}\|_{L^2}^2 + \frac{3\varepsilon}{2} \|q_{xxx}\|_{L^2}^2 \\
&\leq 2\theta \|\bar{p}_{xxx}\|_{L^2}^2 + 5\delta \|q_{xx}\|_{L^2}^2 + C_{17} (\varepsilon \|\bar{p}_{xx}\|_{L^2}^2 + \varepsilon \|q_{xx}\|_{L^2}^2 + \varepsilon \|q_x\|_{L^2}^2) \\
&+ C_{17} \overline{\left[ \|\bar{p}_{xx}\|_{L^2}^2 + \|\bar{p}_x\|_{L^2}^2 \right]} \left[ \|\bar{p}_{xx}\|_{L^2}^2 + \|q_{xx}\|_{L^2}^2 + \|q_x\|_{L^2}^2 \right]. 
\end{align*} \tag{27}$$

By choosing

$$\theta = \gamma \bar{p}^{-2} \quad \text{and} \quad \delta = \gamma \bar{p}^r \quad \frac{1}{10},$$

we update (27) as

$$\begin{align*}
\frac{d}{dt} \left[ \gamma \frac{1}{2} \bar{p}^{-2} \|\bar{p}_{xx}\|_{L^2}^2 + \|q_{xx}\|_{L^2}^2 - \gamma \bar{p}^{-1} \int_R \bar{p}_{xx} q_{xxx} \, dx \right] \\
&+ \gamma \frac{1}{2} \bar{p}^{-2} \|\bar{p}_{xxx}\|_{L^2}^2 + \gamma \bar{p}^r \|q_{xxx}\|_{L^2}^2 + \frac{3\varepsilon}{2} \|q_{xxx}\|_{L^2}^2 \\
&\leq C_{18} \left[ \|\bar{p}_{xx}\|_{L^2}^2 + \|\bar{p}_x\|_{L^2}^2 \right] \left[ \|\bar{p}_{xx}\|_{L^2}^2 + \|q_{xx}\|_{L^2}^2 \right] \\
&+ \|\bar{p}_{xx}\|_{L^2}^2 + \|\bar{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \varepsilon \|q_{xx}\|_{L^2}^2 + \varepsilon \|q_x\|_{L^2}^2, 
\end{align*} \tag{28}$$
We observe that the quantity inside of the temporal derivative in (28) is not necessarily non-negative. Next, we need to make a coupling of the above estimate with the following similar estimate, in order to apply the Grönwall inequality to close the energy estimate.

**Step 3.** Similar to (7), we can show that

\[
\frac{d}{dt} \left( \frac{\gamma}{2} \bar{p}^{\gamma-2} \| \bar{p}_x \|_{L^2}^2 + \frac{1}{2} \| q_x \|_{L^2}^2 \right) + \gamma \bar{p}^{\gamma-2} \| \bar{p}_{xx} \|_{L^2}^2 + \varepsilon \| q_{xx} \|_{L^2}^2 \\
= - \gamma \bar{p}^{\gamma-2} \int R (\bar{p}q)_x \bar{p}_{xx} dx + \gamma (\gamma - 1) \int R (\bar{p} + \bar{p})^{\gamma-2} (\bar{p}_x)^2 q_x dx \\
+ \gamma \int R [(\bar{p} + \bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}] \bar{p}_{xx} q_x dx - 2 \varepsilon \int R q_x q_{xx} dx \\
\equiv I_{11} + I_{12} + I_{13} + I_{14}.
\]

Let us estimate $I_{11}, ..., I_{14}$. By using the uniform estimates of $\| \bar{p} \|_{L^\infty}$ and $\| q \|_{L^\infty}$, and the Cauchy-Schwarz inequality, we can show that

\[
|I_{11}| = \left| \gamma \bar{p}^{\gamma-2} \int R (\bar{p}q)_x \bar{p}_{xx} dx \right| \\
\leq \frac{1}{4} \gamma \bar{p}^{\gamma-2} \| \bar{p}_{xx} \|_{L^2}^2 + C_{19} (\| \bar{p} \|_{L^\infty} \| q_x \|_{L^2} + \| q \|_{L^\infty} \| \bar{p}_x \|_{L^2}) \\
\leq \frac{1}{4} \gamma \bar{p}^{\gamma-2} \| \bar{p}_{xx} \|_{L^2}^2 + C_{20} (\| \bar{p}_x \|_{L^2}^2 + \| q_x \|_{L^2}) \tag{30}
\]

By using the Young inequality, we obtain

\[
|I_{12}| = \left| \gamma (\gamma - 1) \int R (\bar{p} + \bar{p})^{\gamma-2} (\bar{p}_x)^2 q_x dx \right| \\
\leq C_{21} (\| \bar{p} \|_{L^\infty}^2 + \bar{p}^{\gamma-2}) \| \bar{p}_x \|_{L^\infty} \| \bar{p}_x \|_{L^2} \| q_x \|_{L^2} \\
\leq C_{22} \| \bar{p}_x \|_{L^2} \| \bar{p}_{xx} \|_{L^2} \| q_x \|_{L^2} \tag{31}
\]

\[
\leq \frac{1}{4} \gamma \bar{p}^{\gamma-2} \| \bar{p}_{xx} \|_{L^2}^2 + C_{23} \| \bar{p}_x \|_{L^2}^2 \| q_x \|_{L^2} \frac{3}{4} \\
\leq \frac{1}{4} \gamma \bar{p}^{\gamma-2} \| \bar{p}_{xx} \|_{L^2}^2 + C_{24} (\| \bar{p}_x \|_{L^2}^2 + \| q_x \|_{L^2})^2,
\]

where we have used (12) for $\bar{p}_x$ and the uniform estimates of $\| \bar{p} \|_{L^\infty}$ and $\| \bar{p}_x \|_{L^2}$.

Similar to (15), we have

\[
|I_{13}| = \left| \gamma \int R [(\bar{p} + \bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}] \bar{p}_{xx} q_x dx \right| \\
\leq \gamma (\gamma - 1) \int R [(\| \bar{p} \|_{L^\infty}^2 + \bar{p}^{\gamma-2}) \| \bar{p}_x \|_{L^\infty} \| \bar{p}_{xx} \|_{L^2} \| q_x \|_{L^2} \\
\leq C_{25} (\| \bar{p} \|_{L^\infty}^2 + \bar{p}^{\gamma-2}) \| \bar{p} \|_{L^\infty} \| \bar{p}_{xx} \|_{L^2} \| q_x \|_{L^2} \tag{32}
\]

\[
\leq C_{26} \| \bar{p}_{xx} \|_{L^2} \| q_x \|_{L^2} \\
\leq \frac{1}{4} \gamma \bar{p}^{\gamma-2} \| \bar{p}_{xx} \|_{L^2}^2 + C_{27} \| q_x \|_{L^2}^2,
\]
where we have used the Young inequality and the uniform estimate of $\|\tilde{p}\|_{L^\infty}$. By using the uniform estimate of $\|q\|_{L^\infty}$ again, we get
\[
|I_{14}| = \left| \frac{\epsilon}{2} \int_\mathbb{R} q_x q_{xx} dx \right|
\leq \frac{\epsilon}{4} \|q_{xx}\|_{L^2}^2 + 4\epsilon \|q\|_{L^\infty} \|q_x\|_{L^2}^2
\leq \frac{\epsilon}{4} \|q_{xx}\|_{L^2}^2 + C_{29\epsilon} \|q_x\|_{L^2}^2.
\tag{33}
\]
Plugging (30)-(33) into (29), we end up with
\[
\frac{d}{dt} \left( \frac{\gamma}{2} \|\tilde{p}_x\|_{L^2}^2 + \frac{1}{2} \|\tilde{p}_x\|_{L^2}^2 \right) + \frac{\gamma}{4} \|\tilde{p}_{xx}\|_{L^2}^2 + \frac{3\epsilon}{4} \|q_{xx}\|_{L^2}^2
\leq C_{29}(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \epsilon \|\tilde{q}_x\|_{L^2}^2),
\tag{34}
\]
where $C_{29}$ is independent of $t$ and $\epsilon$.

**Step 4.** Let
\[
K = \frac{2 + (\gamma \tilde{p}^{-1})^2}{\gamma \tilde{p}^{-2}}.
\]
Multiplying (34) by $K$, then adding the result to (28), we obtain
\[
\frac{d}{dt} X(t) + Y(t) \leq C_{18}(\|\tilde{p}_{xx}\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)(\|\tilde{p}_{xx}\|_{L^2}^2 + \|q_{xx}\|_{L^2}^2)
+ C_{30}(\|\tilde{p}_{xx}\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \epsilon \|q_{xx}\|_{L^2}^2),
\tag{35}
\]
where
\[
X(t) = \frac{\gamma}{2} \|\tilde{p}_{xx}\|_{L^2}^2 + \frac{1}{2} \|q_{xx}\|_{L^2}^2 + \frac{1}{2} \|q_{xx} - \gamma \tilde{p}^{-1} \tilde{p}_x\|_{L^2}^2
+ \|\tilde{p}_x\|_{L^2}^2 + \frac{K}{2} \|q_x\|_{L^2}^2,
\]
\[
Y(t) = \frac{\gamma}{2} \|\tilde{p}_{xx}\|_{L^2}^2 + \frac{\gamma}{2} \|q_{xx}\|_{L^2}^2 + \frac{3\epsilon}{2} \|q_{xx}\|_{L^2}^2
+ \frac{K\gamma}{4} \tilde{p}^{-2} \|\tilde{p}_{xx}\|_{L^2}^2 + \frac{3K}{4} \epsilon \|q_{xx}\|_{L^2}^2.
\]
By the definition of $X(t)$, we can show that
\[
C_{18}(\|\tilde{p}_{xx}\|_{L^2}^2 + \|q_{xx}\|_{L^2}^2) \leq C_{31} X(t)
\tag{36}
\]
for some constant $C_{31}$ which is independent of $t$ and $\epsilon$. Plugging (36) into (35), we have
\[
\frac{d}{dt} X(t) + Y(t) \leq C_{31}(\|\tilde{p}_{xx}\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2) X(t)
+ C_{30}(\|\tilde{p}_{xx}\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \epsilon \|q_{xx}\|_{L^2}^2 + \epsilon \|q_x\|_{L^2}^2).
\tag{37}
\]
Applying the Grönwall inequality to (37) and using Proposition 3, we can show that
\[
X(t) \leq C_{32}
\tag{38}
\]
for some constant which is independent of $t$ and $\epsilon$. Substituting (38) into (37) and integrating the result with respect to $t$, we can show that
\[
\int_0^t Y(\tau) d\tau \leq C_{33}
\tag{39}
\]
for some constant which is independent of $t$ and $\varepsilon$. From (38), (39) and the definition of $X(t)$ and $Y(t)$, we can derive that

$$\|\tilde{p}_{xx}\|_{L^2}^2 + \|q_{xx}\|_{L^2}^2 + \int_0^t (\|\tilde{p}_{xxx}\|_{L^2}^2 + \|q_{xxx}\|_{L^2}^2 + \varepsilon \|q_{xxxx}\|_{L^2}^2)\,dt \leq C_{34}$$

for some constant which is independent of $t$ and $\varepsilon$. Thus this completes the proof of Lemma 2.1.

Proposition 2 then follows from the combination of Proposition 3 and Lemma 2.1, which along with the local existence result in Proposition 1, yields a global-in-time solution to (1). The uniqueness of the solution can be proved by utilizing standard arguments (cf. [25]). Next, we prove the asymptotic stability of the solution.

**Lemma 2.2 (Asymptotic Stability).** Under the conditions of Theorem 1.1, for any $3 \leq \gamma \in \mathbb{R}$ and $\varepsilon \geq 0$, it holds that

$$\lim_{t \to \infty} (\|p(t) - \bar{p}\|_{L^\infty}^2 + \|\tilde{p}_x(t)\|_{L^2}^2 + \|q(t)\|_{L^\infty}^2 + \|q_x(t)\|_{L^\infty}^2) = 0.$$  

**Proof.** In view of Proposition 3 we see that one needs only to prove $\|\tilde{p}_x(t)\|_{L^\infty}^2 + \|q_x(t)\|_{L^\infty}^2 \to 0$ as $t \to \infty$. First of all, we would like to remark that a function of $t$, belonging to $W^{1,1}(0, \infty)$, converges to zero as time goes to infinity (cf. [49]). In what follows, we use such a fact, together with the energy estimates obtained in the previous subsections, to establish the asymptotic decay estimate stated in Theorem 1.1. In view of (7), for any $3 \leq \gamma \in \mathbb{R}$ and $\varepsilon \geq 0$, we can show that

$$\left| \frac{d}{dt} \left( \frac{\gamma}{2} \tilde{p}^{\gamma-2}\|\tilde{p}_{xx}\|_{L^2}^2 + \frac{1}{2} \|q_{xx}\|_{L^2}^2 \right) \right| \leq \gamma \tilde{p}^{\gamma-2}\|\tilde{p}_{xxx}\|_{L^2}^2 + \varepsilon \|q_{xxx}\|_{L^2}^2 + \gamma \tilde{p}^{\gamma-2}\int_{\mathbb{R}} (\tilde{p}q)_{xx}\tilde{p}_{xxx} \,dx$$

$$+ \gamma \int_{\mathbb{R}} [(\tilde{p} + \bar{p})^{\gamma-1} - p^{\gamma-1}] \tilde{p}_{xxx}q_{xx} \,dx$$

$$+ \gamma(\gamma - 1)(\gamma - 2) \int_{\mathbb{R}} (\tilde{p} + \bar{p})^{\gamma-3}(\tilde{p}_x)^3 q_{xx} \,dx$$

$$+ 3\gamma(\gamma - 1) \int_{\mathbb{R}} (\tilde{p} + \bar{p})^{\gamma-2}\tilde{p}_x\tilde{p}_{xx}q_{xx} \,dx + \varepsilon \int_{\mathbb{R}} q_{xxx}(q^2)_{xx} \,dx$$

$$\equiv \gamma \tilde{p}^{\gamma-2}\|\tilde{p}_{xxx}\|_{L^2}^2 + \varepsilon \|q_{xxx}\|_{L^2}^2 + |J_1| + |J_2| + |J_3| + |J_4| + |J_5|.$$  

Let us estimate $|J_i|$ $(i = 1, \ldots, 5)$. By using (12), we deduce from Proposition 3 and Lemma 2.1 that

$$\|\tilde{p}_x\|_{L^\infty}^2 \leq C_{35} \text{ and } \|q_x\|_{L^\infty}^2 \leq C_{36},$$

where the positive constants are independent of $t$ and $\varepsilon$. By the Cauchy-Schwarz inequality, we obtain

$$|J_1| = \gamma \tilde{p}^{\gamma-2}\int_{\mathbb{R}} (\tilde{p}q)_{xx}\tilde{p}_{xxx} \,dx$$

$$\leq \gamma \tilde{p}^{\gamma-2}\|\tilde{p}_{xxx}\|_{L^2}^2 + C_{37}(\|\tilde{p}\|_{L^\infty}^2 \|q_{xx}\|_{L^2}^2 + \|\tilde{p}_x\|_{L^\infty}^2 \|q_x\|_{L^\infty}^2 + \|q\|_{L^\infty}^2 \|\tilde{p}_{xx}\|_{L^2}^2)$$

$$\leq \gamma \tilde{p}^{\gamma-2}\|\tilde{p}_{xxx}\|_{L^2}^2 + C_{38}(\|\tilde{p}_{xx}\|_{L^2}^2 + \|q_{xx}\|_{L^2}^2 + \|q_x\|_{L^2}^2),$
where we have used the uniform estimates of $\|\tilde{p}\|_{L^\infty}$, $\|\tilde{p}_x\|_{L^\infty}$ and $\|q\|_{L^\infty}$. Similar to (15), we can estimate $|J_2|$ as

$$|J_2| = \left| \gamma \int_\mathbb{R} (\tilde{p} + \tilde{p})^{\gamma-1} \tilde{p}_x q_x dx \right|$$
\[ \leq \gamma (\gamma - 1) \int_\mathbb{R} (\tilde{p} + \tilde{p})^{\gamma-2} |\tilde{p}_x| q_{xx} dx \]
\[ \leq C_39 (\|\tilde{p}\|_{L^\infty}^{\gamma-2} + \|\tilde{p}\|_{L^\infty}^{\gamma-2}) \|\tilde{p}_x\|_{L^\infty} \|\tilde{p}_x\|_{L^2} \|q_{xx}\|_{L^2} \quad (42) \]

where we have used the uniform estimate of $\|\tilde{p}\|_{L^\infty}$. In a similar fashion to (13), since $\gamma \geq 3$, we have

$$|J_3| = \left| \gamma (\gamma - 1)(\gamma - 2) \int_\mathbb{R} (\tilde{p} + \tilde{p})^{\gamma-3} (\tilde{p}_x)^3 q_{xx} dx \right|$$
\[ \leq C_41 (\|\tilde{p}\|_{L^\infty}^{\gamma-3} + \|\tilde{p}\|_{L^\infty}^{\gamma-3}) \|\tilde{p}_x\|_{L^\infty} \|\tilde{p}_x\|_{L^2} \|q_{xx}\|_{L^2} \]
\[ \leq C_44 \|\tilde{p}_x\|_{L^2} \|q_{xx}\|_{L^2} \]
\[ \leq C_44 \left( \|\tilde{p}_x\|_{L^2}^{2} + \|q_{xx}\|_{L^2}^{2} \right) \quad (43) \]

where we have used the uniform estimates of $\|\tilde{p}\|_{L^\infty}$ and $\|\tilde{p}_x\|_{L^\infty}$. Similarly, we can show that

$$|J_4| = \left| 3\gamma (\gamma - 1) \int_\mathbb{R} (\tilde{p} + \tilde{p})^{\gamma-2} \tilde{p}_x \tilde{p}_x q_{xx} dx \right|$$
\[ \leq C_43 (\|\tilde{p}\|_{L^\infty}^{\gamma-2} + \|\tilde{p}\|_{L^\infty}^{\gamma-2}) \|\tilde{p}_x\|_{L^\infty} \|\tilde{p}_x\|_{L^2} \|q_{xx}\|_{L^2} \]
\[ \leq C_44 \|\tilde{p}_x\|_{L^2} \|q_{xx}\|_{L^2} \]
\[ \leq C_44 \left( \|\tilde{p}_x\|_{L^2}^{2} + \|q_{xx}\|_{L^2}^{2} \right) \quad (44) \]

Again, by using the uniform estimates of $\|q\|_{L^\infty}$ and $\|q_x\|_{L^\infty}$, we derive

$$|J_5| = \left| \varepsilon \int_\mathbb{R} q_{xxx} (q_x^2)_{xx} dx \right|$$
\[ \leq \varepsilon \|q_{xxx}\|_{L^2}^2 + C_{45} \varepsilon (\|q_x\|_{L^\infty}^2 \|q_x\|_{L^2}^2 + \|q\|_{L^\infty}^2 \|q_{xx}\|_{L^2}^2) \]
\[ \leq \varepsilon \|q_{xxx}\|_{L^2}^2 + C_{46} (\varepsilon \|q_x\|_{L^2}^2 + \varepsilon \|q_{xx}\|_{L^2}^2) \quad (45) \]

Plugging (41)-(45) into (40), we end up with

$$\left| \frac{d}{dt} \left( \frac{\gamma}{2} \tilde{p}_x \tilde{p}_x^2 + \frac{1}{2} \|q_{xx}\|_{L^2}^2 \right) \right|$$
\[ \leq C_47 (\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2 + \|q_{xx}\|_{L^2}^2 + \|q_x\|_{L^2}^2 ) \]
\[ + \varepsilon \|q_x\|_{L^2}^2 + \varepsilon \|q_{xx}\|_{L^2}^2 + \varepsilon \|q_{xxx}\|_{L^2}^2 \quad (46) \]

Integrating (46) over $[0, t]$, and using Proposition 3 and Lemma 2.1, we infer that

$$\int_0^t \left| \frac{d}{dt} \left( \frac{\gamma}{2} \tilde{p}_x \tilde{p}_x^2 + \frac{1}{2} \|q_{xx}\|_{L^2}^2 \right) \right| dt \leq C_{48}.$$
Cauchy problem:

of (1) with respect to conditions on the initial data. For this purpose, by integrating the perturbed system

Our next goal is to identify the algebraic decay rates of the solution under mild decay rates of the perturbations, which is mathematically challenging.

mentioned before, in view of (1), the present literatures have no related report on behavior of the solution obtained in Theorem 1.1, namely, prove Theorem 1.2. As

Decay rate for \( \varepsilon = 0 \). In this section, we shall investigate the asymptotic decay behavior of the solution obtained in Theorem 1.1, namely, prove Theorem 1.2. As mentioned before, in view of (1), the present literatures have no related report on the explicit decay rates of the perturbations, which is mathematically challenging. Our next goal is to identify the algebraic decay rates of the solution under mild conditions on the initial data. For this purpose, by integrating the perturbed system of (1) with respect to \( x \) over \((-\infty, x]\) and using (5), we can derive the following Cauchy problem:

\[
\begin{align*}
\phi_t - \phi_x \psi_x - \bar{p} \psi_x &= \phi_{xx}, \\
\psi_t - [(\phi_x + \bar{p})^\gamma - \bar{p}] &= \varepsilon \psi_{xx} + \varepsilon (\psi_x)^2, \\
(\phi, \psi)(x, 0) &= (\phi_0, \psi_0)(x) = \left( \int_{-\infty}^{x} (p_0(y) - \bar{p}) dy, \int_{-\infty}^{x} g_0(y) dy \right),
\end{align*}
\]

\( x \in \mathbb{R}, \ t > 0 \). Following standard continuation arguments, we shall establish energy estimates for the solution to (47) under the \textit{a priori} assumption:

\[
\sup_{0 \leq t \leq T} \left( \|\phi(t)\|_{L^2}^2 + \|\psi(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2 \right) \leq \xi_1,
\]

where \( \xi_1 \) is a positive constant to be determined. Firstly, we establish the uniform-in-time estimates of \( \|\phi(t)\|_{L^2} \) and \( \|\psi(t)\|_{L^2} \). In what follows, \( D_i \) will denote a generic constant which is independent of time and the unknown functions.

**Lemma 3.1.** Under the conditions of Theorem 1.2, for any \( \gamma \geq 3 \) and \( \varepsilon = 0 \), it holds that

\[
\|\phi\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \|\phi_x\|_{L^2}^2 + \|q\|_{L^2}^2 \leq D_1,
\]

where \( D_1 \) is a positive constant independent of \( t \).

**Proof.** **Step 1.** Multiplying the first equation of (47) by \( \gamma \bar{p}^{\gamma-2} \phi \) and the second equation of (47) with \( \varepsilon = 0 \) by \( \psi \), adding the results and integrating by parts with respect to \( x \), we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \gamma \bar{p}^{\gamma-2} \|\phi\|_{L^2}^2 + \|\psi\|_{L^2}^2 \right) + \gamma \bar{p}^{\gamma-2} \|\phi_x\|_{L^2}^2 \\
= \gamma \bar{p}^{\gamma-2} \int_{\mathbb{R}} \phi_x \psi_x \phi dx + \int_{\mathbb{R}} \left[ (\phi_x + \bar{p})^\gamma - \bar{p}^\gamma - \gamma \bar{p}^{\gamma-1} \phi_x \right] \psi dx \\
\equiv R_1 + R_2.
\end{align*}
\]
Next, we proceed to estimate $R_1$ and $R_2$. By the definition of $\phi$ and Theorem 1.1, we have $\|\phi_x\|_{L^2} = \|p - \bar{p}\|_{L^2} \leq D_2$. By using (48), we obtain

$$|R_1| = \left| \gamma \bar{p}^{\gamma-2} \int_R \phi_x \psi_x \phi dx \right|$$

$$\leq \gamma \bar{p}^{\gamma-2} \|\phi\|_{L^\infty} \|\phi_x\|_{L^2} \|\psi_x\|_{L^2}$$

$$\leq \gamma \bar{p}^{\gamma-2} \sqrt{2} \|\phi\|_{L^2}^{\frac{1}{2}} \|\phi_x\|_{L^2}^{\frac{1}{2}} \|\psi_x\|_{L^2}$$

$$\leq D_3 \xi_1^2 \|\phi_x\|_{L^2} \|\psi_x\|_{L^2}$$

$$\leq \frac{D_3}{2} \xi_1^2 (\|\phi_x\|_{L^2}^2 + \|\psi_x\|_{L^2}^2),$$

where $D_3 = \gamma \bar{p}^{\gamma-2} \sqrt{2D_2}$.

For $R_2$, by the Taylor Theorem,

$$(\phi_x + \bar{p})^\gamma - \bar{p}^\gamma - \gamma \bar{p}^{\gamma-1} \phi_x = \frac{\gamma (\gamma - 1)}{2} (p^*)^{\gamma-2} (\phi_x)^2,$$

where $p^*$ is between $\phi_x + \bar{p}$ and $\bar{p}$, satisfying $|p^*| \leq |\phi_x| + |\bar{p}| = |\phi| + |\bar{p}|$, then we have

$$|R_2| = \left| \int_R [(\phi_x + \bar{p})^\gamma - \bar{p}^\gamma - \gamma \bar{p}^{\gamma-1} \phi_x] \psi dx \right|$$

$$\leq \frac{\gamma (\gamma - 1)}{2} \int_R (|\bar{p}| + |\bar{p}|)^{\gamma-2} (\phi_x)^2 |\psi| dx$$

$$\leq D_4 \int_R (\|\bar{p}\|_{L^\infty}^{\gamma-2} + \|\bar{p}\|_{L^\infty}^{\gamma-2}) \|\phi_x\|_{L^2}^2$$

$$\leq \sqrt{2} D_4 (\|\bar{p}\|_{L^\infty}^{\gamma-2} + \|\bar{p}\|_{L^\infty}^{\gamma-2}) \|\phi_x\|_{L^2} \|\psi_x\|_{L^2} \|\phi_x\|_{L^2}^2$$

$$\leq D_5 \xi_1^4 \|\phi_x\|_{L^2}^2,$$

where we have used (12) and (48) for $\psi$, $D_5$ depends only on $\gamma, \bar{p}$ and the uniform estimates of $\|\bar{p}\|_{L^\infty}$ and $\|\psi_x\|_{L^2} = \|q\|_{L^2}$ from Theorem 1.1. With the above estimates of $R_1$ and $R_2$, we update (49) as

$$\frac{1}{2} \frac{d}{dt} (\gamma \bar{p}^{\gamma-2} \|\phi\|_{L^2}^2 + \|\psi\|_{L^2}^2) + \gamma \bar{p}^{\gamma-2} \|\phi_x\|_{L^2}^2 \leq D_6 \xi_1^4 (\|\phi_x\|_{L^2}^2 + \|\psi_x\|_{L^2}^2).$$

**Step 2.** For the second term on the right-hand side of (52), we invoke the spirit of [22] to derive a damping equation for $\psi_x$, and then obtain the control of the dissipation $\|\psi_x\|_{L^2}$ by coupling with (52). To this end, by taking the spatial derivatives to the second equation of (47) with $\varepsilon = 0$ and combing with the first equation, we have

$$\psi_{xt} = \gamma [(\phi_x + \bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}] \phi_{xx} + \gamma \bar{p}^{\gamma-1} (\phi_t - \phi_x \psi_x - \bar{p} \psi_x).$$

Taking the $L^2$ inner product of (53) with $\psi_x$, we get

$$\frac{1}{2} \frac{d}{dt} \|\psi_x\|_{L^2}^2 + \gamma \bar{p} \|\psi_x\|_{L^2}^2 = \gamma \bar{p}^{\gamma-1} \int_R \psi_x \phi_t dx - \gamma \bar{p}^{\gamma-1} \int_R \phi_x (\psi_x)^2 dx$$

$$+ \gamma \int_R [(\phi_x + \bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}] \phi_{xx} \psi_x dx.$$
By using the second equation of (47) with $\varepsilon = 0$ again, we have
\[
\gamma \bar{p}^{\gamma - 1} \int_R \psi_x \phi_t \, dx = \frac{d}{dt} \left( \gamma \bar{p}^{\gamma - 1} \int_R \psi_x \phi \, dx \right) - \gamma \bar{p}^{\gamma - 1} \int_R \phi \psi_x \, dx
\]
\[
= \frac{d}{dt} \left( \gamma \bar{p}^{\gamma - 1} \int_R \psi_x \phi \, dx \right) + \gamma \bar{p}^{\gamma - 1} \int_R [(\phi_x + \bar{p})^{\gamma - 1} - \bar{p}^{\gamma - 1}] \phi_x \psi_x \, dx \tag{55}
\]
Plugging (55) into (54), we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} \|\psi_x\|_{L^2}^2 - \gamma \bar{p}^{\gamma - 1} \int_R \psi_x \phi \, dx \right) + \gamma \bar{p}^{\gamma} \|\psi_x\|_{L^2}^2 = - \gamma \bar{p}^{\gamma - 1} \int_R \phi_x (\psi_x)^2 \, dx + \gamma \int_R [(\phi_x + \bar{p})^{\gamma - 1} - \bar{p}^{\gamma - 1}] \phi_x \psi_x \, dx
\]
\[
+ \gamma \bar{p}^{\gamma - 1} \int_R [(\phi_x + \bar{p})^{\gamma} - \bar{p}^{\gamma}] \phi_x \, dx \equiv R_3 + R_4 + R_5.
\]
Let us estimate $R_3, R_4$ and $R_5$. By using the uniform estimate of $\|\psi_x\|_{L^\infty} = \|q\|_{L^\infty}$ from Theorem 1.1, we have
\[
|R_3| = \left| \gamma \bar{p}^{\gamma - 1} \int_R \phi_x (\psi_x)^2 \, dx \right|
\]
\[
\leq \frac{\gamma \bar{p}^{\gamma}}{4} \|\psi_x\|_{L^2}^2 + \gamma \bar{p}^{\gamma - 2} \|\phi_x\|_{L^\infty}^2 \|\phi_x\|_{L^2}^2 \tag{57}
\]
\[
\leq \frac{\gamma \bar{p}^{\gamma}}{4} \|\psi_x\|_{L^2}^2 + D_7 \|\phi_x\|_{L^2}^2.
\]
Similar to (15), by using the Young inequality, we can show that
\[
|R_4| = \left| \gamma \int_R [(\phi_x + \bar{p})^{\gamma - 1} - \bar{p}^{\gamma - 1}] \phi_x \psi_x \, dx \right|
\]
\[
\leq \gamma (\gamma - 1) \int_R [(\bar{p} + \bar{p})^{\gamma - 2}] |\phi_x| \|\phi_x\| \|\psi_x\| \, dx
\]
\[
\leq D_8 \int_R [(\bar{p})^{\gamma - 2} + (\bar{p})^{\gamma - 2}] |\phi_x| \|\phi_x\| \|\psi_x\| \, dx
\]
\[
\leq D_9 \|\phi_x\|_{L^2} \|\psi_x\|_{L^2} \leq D_9 \|\phi_x\|_{L^2} \|\psi_x\|_{L^2}
\]
\[
\leq \frac{\gamma \bar{p}^{\gamma}}{4} \|\psi_x\|_{L^2}^2 + D_{10} \|\phi_x\|_{L^2}^2,
\]
where we have used the uniform estimates of $\|\bar{p}\|_{L^\infty}$ and $\|\phi_{xx}\|_{L^\infty} = \|\bar{p}_x\|_{L^\infty}$ obtained from Theorem (1.1) when $\gamma \geq 3$. We remark that this is the main reason why we need to establish the higher order spatial regularity of the solution recorded in Theorem 1.1.

For $R_5$, by using the uniform estimate of $\|\bar{p}\|_{L^\infty}$ again, we can show that
\[
|R_5| = \left| \gamma \bar{p}^{\gamma - 1} \int_R [(\phi_x + \bar{p})^{\gamma} - \bar{p}^{\gamma}] \phi_x \, dx \right|
\]
\[
\leq \gamma^2 \bar{p}^{\gamma - 1} \int_R [(\bar{p} + \bar{p})^{\gamma - 1}] (\phi_x)^2 \, dx
\]
\[
\leq D_{11} \|\bar{p}\|_{L^\infty}^{\gamma - 1} \|\bar{p}_x\|_{L^2}^2 \|\phi_x\|_{L^2}^2
\]
\[
\leq D_{12} \|\phi_x\|_{L^2}^2.
\]
Substituting (57), (58) and (59) into (56), we have
\[
\frac{d}{dt} \left( \frac{1}{2} \| \psi_x \|_{L^2}^2 - \gamma \bar{p} \gamma^{-1} \int \psi_x \phi dx \right) + \frac{\gamma \bar{p} \gamma^{-1}}{2} \| \psi_x \|_{L^2}^2 \leq D_{13} \| \phi_x \|_{L^2}^2. \tag{60}
\]

**Step 3.** By choosing
\[
D_{14} = \frac{2(\gamma \bar{p} \gamma^{-1})^2 + D_{13}}{\gamma \bar{p} \gamma^{-2}},
\]
and adding \(D_{14} \times (52)\) to (60), we obtain
\[
\frac{d}{dt} \left( \frac{D_{13}}{2} \| \phi \|_{L^2}^2 + \frac{D_{14}}{2} \| \psi \|_{L^2}^2 + \frac{1}{4} \| \psi_x - 2 \gamma \bar{p} \gamma^{-1} \phi \|_{L^2}^2 + \frac{1}{4} \| \psi_x \|_{L^2}^2 \right) + 2(\gamma \bar{p} \gamma^{-1})^2 \| \phi_x \|_{L^2}^2 + \frac{\gamma \bar{p}}{2} \| \psi_x \|_{L^2}^2 \leq D_8 D_{14} \xi_1^2 (\| \phi_x \|_{L^2}^2 + \| \psi_x \|_{L^2}^2).
\]
From (61) we see that, by choosing \(\xi_1\) sufficiently small, it holds that
\[
\frac{d}{dt} \left( \frac{D_{13}}{2} \| \phi \|_{L^2}^2 + \frac{D_{14}}{2} \| \psi \|_{L^2}^2 + \frac{1}{4} \| \psi_x - 2 \gamma \bar{p} \gamma^{-1} \phi \|_{L^2}^2 + \frac{1}{4} \| \psi_x \|_{L^2}^2 \right) + D_{15} (\| \phi_x \|_{L^2}^2 + \| \psi_x \|_{L^2}^2) \leq 0.
\]
Integrating (62) over \([0,t]\), we obtain
\[
\| \phi(t) \|_{L^2}^2 + \| \psi(t) \|_{L^2}^2 + \| \psi_x(t) \|_{L^2}^2 + \int_0^t (\| \phi_x(\tau) \|_{L^2}^2 + \| \psi_x(\tau) \|_{L^2}^2) d\tau \leq D_{16} (\| \phi(0) \|_{L^2}^2 + \| \psi(0) \|_{L^2}^2 + \| \psi_x(0) \|_{L^2}^2).
\]
From standard continuation argument, we know that (63) holds true for all time provided that \(\| \phi(0) \|_{L^2}^2 + \| \psi(0) \|_{L^2}^2 + \| \psi_x(0) \|_{L^2}^2\) is sufficiently small. Hence the *a priori* assumption (48) is closed. This proves Lemma 3.1.

Next, we shall carry out the time-weighted energy estimates and then obtain the explicit decay rates of the perturbation.

**Lemma 3.2.** Under the conditions of Theorem 1.2, for any \(\gamma \geq 3\) and \(\varepsilon = 0\), then there exists a finite time \(T_1 > 0\), such that for any \(t > T_1\) it follows that
\[
(1 + t)(\| \phi_x \|_{L^2}^2 + \| \psi_x \|_{L^2}^2) + \int_{T_1}^t (1 + \tau)(\| \phi_{xx}(\tau) \|_{L^2}^2 + \| \psi_{xx}(\tau) \|_{L^2}^2) d\tau \leq D_{17},
\]
where \(D_{17}\) is a positive constant independent of \(t\).

**Proof.** **Step 1.** Taking spatial derivatives of the first and second equations of (47) with \(\varepsilon = 0\), we have
\[
\begin{cases}
\phi_{xt} - (\phi_x \psi_x)_x - \bar{p} \psi_{xx} = \phi_{xxx}, \\
\psi_{xt} - [(\phi_x + \bar{p}) \gamma - \bar{p} \gamma]_x = 0.
\end{cases}
\tag{64}
\]
Taking the $L^2$ inner product of the first equation of (64) with $\gamma \bar{p}^{-2}(1 + t)\phi_x$ and the second equation with $(1 + t)\psi_x$, we have

$$\frac{1}{2} \frac{d}{dt} (1 + t)(\gamma \bar{p}^{-2}\|\phi_x\|^2_{L^2} + \|\psi_x\|^2_{L^2}) + \gamma \bar{p}^{-2}(1 + t)\|\phi_{xx}\|^2_{L^2}$$

$$= \frac{1}{2}(\gamma \bar{p}^{-2}\|\phi_x\|^2_{L^2} + \|\psi_x\|^2_{L^2}) + \gamma \bar{p}^{-2}(1 + t)\int_{\mathbb{R}} (\phi_x \psi_x)_{xx} dx$$

$$- (1 + t)\int_{\mathbb{R}} [(\phi_x + \bar{p})^\gamma - \bar{p}^\gamma - \gamma \bar{p}^{-1}\phi_x] \psi_{xx} dx$$

$$\equiv \frac{1}{2}(\gamma \bar{p}^{-2}\|\phi_x\|^2_{L^2} + \|\psi_x\|^2_{L^2}) + R_6 + R_7. \tag{65}$$

Let us estimate $R_6$ and $R_7$. By using the Gagliardo-Nirenberg type interpolation inequality:

$$\|\nabla f\|_{L^4} \leq D_{18}\|f\|^{1/2}_{L^\infty} \|\nabla^2 f\|^{1/2}_{L^2}, \quad \forall f \in W^{2,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \tag{66}$$

we obtain

$$|R_6| = \left| \frac{\gamma \bar{p}^{-2}}{2} (1 + t) \int_{\mathbb{R}} (\phi_x \psi_x)_{xx} \phi_x dx \right|$$

$$\leq \frac{\sigma}{4} (1 + t)\|\psi_{xx}\|^2_{L^2} + \frac{(\gamma \bar{p}^{-2})^2}{4\sigma} (1 + t)\|\phi_x\|^4_{L^4}$$

$$\leq \frac{\sigma}{4} (1 + t)\|\psi_{xx}\|^2_{L^2} + D(\sigma)(1 + t)\|\phi\|^2_{L^\infty}\|\phi_{xx}\|^2_{L^2}, \tag{67}$$

where $\sigma > 0$ is a constant to be determined. Similar to (51), we can show that

$$|R_7| = \left| (1 + t)\int_{\mathbb{R}} [(\phi_x + \bar{p})^\gamma - \bar{p}^\gamma - \gamma \bar{p}^{-1}\phi_x] \psi_{xx} dx \right|$$

$$\leq \frac{\gamma(\gamma - 1)}{2} (1 + t)\int_{\mathbb{R}} (\|\bar{p} + \bar{p}\|^\gamma - \gamma \bar{p}^{-2}\phi_x))^2 \psi_{xx} dx$$

$$\leq D_{19}(1 + t)(\|\bar{p}\|^\gamma_{L^\infty} + \|\bar{p}^{-2}\|_{L^\infty})\|\phi_x\|^2_{L^4}\|\psi_{xx}\|_{L^2}$$

$$\leq D_{20}(1 + t)\|\phi_x\|^2_{L^4}\|\psi_{xx}\|_{L^2}$$

$$\leq \frac{\sigma}{4} (1 + t)\|\psi_{xx}\|^2_{L^2} + \frac{(D_{20})^2}{\sigma} (1 + t)\|\phi_x\|^4_{L^4}$$

$$\leq \frac{\sigma}{4} (1 + t)\|\psi_{xx}\|^2_{L^2} + D(\sigma)(1 + t)\|\phi\|^2_{L^\infty}\|\phi_{xx}\|^2_{L^2}, \tag{68}$$

where we have used (66) for $\phi$. Substituting (67) and (68) into (65), we rewrite (65) as

$$\frac{1}{2} \frac{d}{dt} (1 + t)(\gamma \bar{p}^{-2}\|\phi_x\|^2_{L^2} + \|\psi_x\|^2_{L^2}) + \gamma \bar{p}^{-2}(1 + t)\|\phi_{xx}\|^2_{L^2}$$

$$\leq \frac{1}{2}(\gamma \bar{p}^{-2}\|\phi_x\|^2_{L^2} + \|\psi_x\|^2_{L^2}) + \frac{\sigma}{4} (1 + t)\|\psi_{xx}\|^2_{L^2}$$

$$+ D(\sigma)(1 + t)\|\phi\|^2_{L^\infty}\|\phi_{xx}\|^2_{L^2}. \tag{69}$$

In order to estimate the right-hand-side terms of (69), we need to further investigate the higher-order spatial derivatives of (64).
Step 2. Similar to (53), we derive

\[
\psi_{xx} = \gamma (\gamma - 1)(\phi_x + \bar{p})^{\gamma - 2}(\phi_{xx})^2 + \gamma \left[(\phi_x + \bar{p})^{\gamma - 1} - \bar{p}^{\gamma - 1}\right] \phi_{xx} + \gamma \bar{p}^{\gamma - 1}(\phi_{xt} - (\phi_x \psi_x) - \bar{p} \psi_{xx}).
\]

(70)

Multiplying (70) by \((1 + t)\psi_{xx}\), integrating by parts with respect to \(x\), and utilizing similar arguments as those in (54)-(56), we can show that

\[
\frac{d}{dt} (1 + t) \left( \frac{1}{2} \| \psi_{xx} \|_{L^2}^2 - \gamma \bar{p}^{\gamma - 1} \int_{\mathbb{R}} \psi_{xx} \phi_x \, dx \right) + \gamma \bar{p}^{\gamma - 1} (1 + t) \| \psi_{xx} \|_{L^2}^2 = \frac{1}{2} \| \psi_{xx} \|_{L^2}^2 + \gamma (\gamma - 1)(1 + t) \int_{\mathbb{R}} (\phi_x + \bar{p})^{\gamma - 2}(\phi_{xx})^2 \psi_{xx} \, dx
\]

\[
+ \gamma (1 + t) \int_{\mathbb{R}} [((\phi_x + \bar{p})^{\gamma - 1} - \bar{p}^{\gamma - 1}] \phi_{xxx} \psi_{xx} \, dx - \gamma \bar{p}^{\gamma - 1} \int_{\mathbb{R}} \psi_{xx} \phi_x \, dx
\]

\[
+ \gamma^2 \bar{p}^{\gamma - 1} (1 + t) \int_{\mathbb{R}} (\phi_x + \bar{p})^{\gamma - 1}(\phi_{xx})^2 \, dx - \gamma \bar{p}^{\gamma - 1} (1 + t) \int_{\mathbb{R}} (\phi_x \psi_x)_x \psi_{xx} \, dx.
\]

(71)

Similar to (65), we can show that

\[
\frac{1}{2} \frac{d}{dt} (1 + t) (\gamma \bar{p}^{\gamma - 2} \| \phi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2) + \gamma \bar{p}^{\gamma - 2} (1 + t) \| \phi_{xxx} \|_{L^2}^2
\]

\[
= \frac{1}{2} (\gamma \bar{p}^{\gamma - 2} \| \phi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2) - \gamma \bar{p}^{\gamma - 2} (1 + t) \int_{\mathbb{R}} (\phi_x \psi_x)_x \phi_{xx} \, dx
\]

\[
+ \gamma (\gamma - 1)(1 + t) \int_{\mathbb{R}} (\phi_x + \bar{p})^{\gamma - 2}(\phi_{xx})^2 \psi_{xx} \, dx
\]

\[
+ \gamma (1 + t) \int_{\mathbb{R}} [((\phi_x + \bar{p})^{\gamma - 1} - \bar{p}^{\gamma - 1}] \phi_{xxx} \psi_{xx} \, dx,
\]

(72)

which, combined with (71), yields

\[
\frac{d}{dt} (1 + t) \left( \frac{1}{2} \gamma \bar{p}^{\gamma - 2} \| \phi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2 - \gamma \bar{p}^{\gamma - 1} \int_{\mathbb{R}} \psi_{xx} \phi_x \, dx \right)
\]

\[
+ \gamma \bar{p}^{\gamma - 2} (1 + t) \| \phi_{xxx} \|_{L^2}^2 + \gamma \bar{p}^{\gamma - 1} (1 + t) \| \psi_{xx} \|_{L^2}^2
\]

\[
= \frac{1}{2} \gamma \bar{p}^{\gamma - 2} \| \phi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2 + 2 \gamma (\gamma - 1)(1 + t) \int_{\mathbb{R}} (\phi_x + \bar{p})^{\gamma - 2}(\phi_{xx})^2 \psi_{xx} \, dx
\]

\[
+ 2 \gamma (1 + t) \int_{\mathbb{R}} [((\phi_x + \bar{p})^{\gamma - 1} - \bar{p}^{\gamma - 1}] \phi_{xxx} \psi_{xx} \, dx
\]

\[
- \gamma \bar{p}^{\gamma - 2} (1 + t) \int_{\mathbb{R}} (\phi_x \psi_x)_x \phi_{xxx} \, dx + \gamma^2 \bar{p}^{\gamma - 1} (1 + t) \int_{\mathbb{R}} (\phi_x + \bar{p})^{\gamma - 1}(\phi_{xx})^2 \, dx
\]

\[
- \gamma \bar{p}^{\gamma - 1} (1 + t) \int_{\mathbb{R}} (\phi_x \psi_x)_x \psi_{xx} \, dx - \gamma \bar{p}^{\gamma - 1} \int_{\mathbb{R}} \psi_{xx} \phi_x \, dx
\]

\[
= \frac{1}{2} \gamma \bar{p}^{\gamma - 2} \| \phi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2 + R_8 + R_9 + R_{10} + R_{11} + R_{12} + R_{13}.
\]

(73)
We proceed to estimate $R_8, \ldots, R_{13}$. By using the uniform estimate of $\|\phi_x\|_{L^\infty} = \|\bar{\rho}\|_{L^\infty}$ from Theorem 1.1, we can show that

\[
|R_8| = \left| 2\gamma(\gamma - 1)(1 + t) \int_{\mathbb{R}} (\phi_x + \bar{\rho})^{\gamma - 2} (\phi_{xx})^2 \psi_{xx} \, dx \right|
\leq D_{21}(1 + t)(\|\bar{\rho}\|_{L^\infty}^{\gamma - 2} + \bar{\rho}^{\gamma - 2})(\phi_{xx})^2 \|\psi_{xx}\|_{L^2}
\leq D_{22}(1 + t)(\phi_{xx})^2 \|\psi_{xx}\|_{L^2}
\leq \frac{\gamma \bar{\rho}^{\gamma - 1}}{4} (1 + t)\|\psi_{xx}\|_{L^2}^2 + \frac{(D_{22})^2}{\gamma \bar{\rho}^{\gamma - 1}} (1 + t)\|\phi_{xx}\|_{L^4}^4
\leq \frac{\gamma \bar{\rho}^{\gamma - 1}}{4} (1 + t)\|\psi_{xx}\|_{L^2}^2 + D_{23}(1 + t)\|\phi_x\|_{L^\infty}^2 \|\phi_{xx}\|_{L^2}^2, \tag{74}
\]

where we have used (66) for $\phi_x$. As a similar fashion in (58), we obtain

\[
|R_9| = \left| 2\gamma(\gamma - 1)(1 + t) \int_{\mathbb{R}} [(\phi_x + \bar{\rho})^{\gamma - 1} - \bar{\rho}^{\gamma - 1}] \phi_{xx}\psi_{xx} \, dx \right|
\leq 2\gamma(\gamma - 1)(1 + t) \int_{\mathbb{R}} (|\bar{\rho}| + \bar{\rho})^{\gamma - 2}|\phi_x|\|\phi_{xx}\|\|\psi_{xx}\| \, dx
\leq D_{24}(1 + t)\|\phi_x\|_{L^\infty} \|\phi_{xx}\|_{L^2} \|\psi_{xx}\|_{L^2}
\leq \frac{\gamma \bar{\rho}^{\gamma - 1}}{4} (1 + t)\|\psi_{xx}\|_{L^2}^2 + \frac{(D_{24})^2}{\gamma \bar{\rho}^{\gamma - 1}} (1 + t)\|\phi_x\|_{L^\infty}^2 \|\phi_{xx}\|_{L^2}^2. \tag{75}
\]

By using the Cauchy-Schwarz inequality, we get

\[
|R_{10}| = \left| \gamma \bar{\rho}^{\gamma - 2}(1 + t) \int_{\mathbb{R}} (\phi_x \psi_x)_x \phi_{xx} \, dx \right|
\leq \frac{\gamma \bar{\rho}^{\gamma - 2}}{4} (1 + t)\|\phi_{xx}\|_{L^2}^2
+ 2\gamma \bar{\rho}^{\gamma - 2}(1 + t)(\|\psi_x\|_{L^\infty}^2 \|\phi_{xx}\|_{L^2}^2 + \|\phi_x\|_{L^\infty}^2 \|\psi_{xx}\|_{L^2}^2). \tag{76}
\]

By using the uniform estimate of $\|\phi_x\|_{L^\infty} = \|\bar{\rho}\|_{L^\infty}$ again, we have

\[
|R_{11}| = \left| \gamma^{2} \bar{\rho}^{\gamma - 1}(1 + t) \int_{\mathbb{R}} (\phi_x + \bar{\rho})^{\gamma - 1}(\phi_{xx})^2 \, dx \right|
\leq D_{25}(1 + t)(\|\bar{\rho}\|_{L^\infty}^{\gamma - 1} + \bar{\rho}^{\gamma - 1})\|\phi_{xx}\|_{L^2}^2
\leq D_{26}(1 + t)\|\phi_{xx}\|_{L^2}^2. \tag{77}
\]

With the Cauchy-Schwarz inequality again, we can show that

\[
|R_{12}| = \left| \gamma \bar{\rho}^{\gamma - 1}(1 + t) \int_{\mathbb{R}} (\phi_x \psi_x)_x \psi_{xx} \, dx \right|
\leq \frac{\gamma \bar{\rho}^{\gamma}}{4} (1 + t)\|\psi_{xx}\|_{L^2}^2
+ 2\gamma \bar{\rho}^{\gamma - 2}(1 + t)(\|\psi_x\|_{L^\infty}^2 \|\phi_{xx}\|_{L^2}^2 + \|\phi_x\|_{L^\infty}^2 \|\psi_{xx}\|_{L^2}^2), \tag{78}
\]

and

\[
|R_{13}| = \left| \gamma \bar{\rho}^{\gamma - 1} \int_{\mathbb{R}} \psi_{xx} \phi_x \, dx \right| \leq \|\psi_{xx}\|_{L^2}^2 + \frac{(\gamma \bar{\rho}^{\gamma - 1})^2}{4} \|\phi_x\|_{L^2}^2. \tag{79}
\]
With the above estimates of \( R_i \) \((i = 8, \ldots, 13)\), we update (73) as
\[
\frac{d}{dt}(1 + t) \left( \frac{1}{2} \gamma \bar{p}^{-2} \| \phi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2 - \gamma \bar{p}^{-1} \int_R \psi_{xx} \phi_{xx} dx \right)
+ \frac{3}{4} \gamma \bar{p}^{-2} (1 + t) \| \phi_{xxx} \|_{L^2}^2 + \frac{1}{4} \gamma \bar{p}^{-1} (1 + t) \| \psi_{xx} \|_{L^2}^2
\leq \frac{1}{2} \gamma \bar{p}^{-2} \| \phi_{xx} \|_{L^2}^2 + 2 \| \psi_{xx} \|_{L^2}^2 + \frac{1}{4} (\gamma \bar{p}^{-1})^2 \| \phi_{xx} \|_{L^2}^2 + D_{28} (1 + t) \| \phi_{xx} \|_{L^2}^2
+ D_{27} (1 + t) (\| \phi_{x} \|_{L^\infty}^2 + \| \psi_{x} \|_{L^\infty}^2)(\| \phi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2 + \| \phi_{xxx} \|_{L^2}^2).
\]

**Step 3.** By choosing
\[
D_{28} = \frac{(\gamma \bar{p}^{-1})^2 + D_{26}}{\gamma \bar{p}^{-2}} \quad \text{and} \quad \sigma = \frac{\gamma \bar{p}^{-1}}{4D_{28}},
\]
and adding \( D_{28} \times (69) \) to (80), we get
\[
\frac{1}{2} \frac{d}{dt} (1 + t) \left( D_{26} \| \phi_{xx} \|_{L^2}^2 + D_{28} \| \psi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2 + \gamma \bar{p}^{-1} \| \phi_{xx} \|_{L^2}^2 \right)
+ \| \psi_{xx} \|_{L^2}^2 + \gamma \bar{p}^{-2} \| \phi_{xx} \|_{L^2}^2
+ (\gamma \bar{p}^{-1})^2 (1 + t) \| \phi_{xx} \|_{L^2}^2 + \frac{1}{8} \gamma \bar{p}^{-1} (1 + t) \| \psi_{xx} \|_{L^2}^2
+ \frac{3}{4} \gamma \bar{p}^{-2} (1 + t) \| \phi_{xxx} \|_{L^2}^2
\leq D_{29} (\| \phi_{x} \|_{L^2}^2 + \| \psi_{x} \|_{L^2}^2 + \| \phi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2)
+ D_{30} (1 + t) (\| \phi_{x} \|_{L^\infty}^2 + \| \phi_{xx} \|_{L^\infty}^2 + \| \psi_{x} \|_{L^\infty}^2)
\times (\| \phi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2 + \| \phi_{xxx} \|_{L^2}^2).
\]

Taking the \( L^2 \) inner product of the first equation of (64) with \( \phi_{x} \), we can show that
\[
\frac{1}{2} \frac{d}{dt} \| \phi_{x} \|_{L^2}^2 = - \| \phi_{xx} \|_{L^2}^2 - \int_R \phi_{x} \psi_{x} \phi_{xx} dx + \bar{p} \int_R \phi_{x} \psi_{xx} dx
= - \| p_{x} \|_{L^2}^2 - \int_R \phi_{x} q p_{x} dx + \bar{p} \int_R \phi_{x} q_{x} dx.
\]

By employing the uniform estimate of \( \| q \|_{L^\infty} \), we can show that
\[
\int_{0}^{t} \left| \frac{d}{dt} \| \phi_{x} \|_{L^2}^2 \right| d\tau \lesssim \int_{0}^{t} \left( \| p_{x} \|_{L^2}^2 + \| \phi_{x} \|_{L^2}^2 + \| q_{x} \|_{L^2}^2 \right) d\tau.
\]

It follows from Proposition 3 and Lemma 3.1 that \( \frac{d}{dt} \| \phi_{x} \|_{L^2}^2 \in L^{1}(0, \infty) \). By employing Lemma 3.1 again we infer that \( \| \phi_{x}(t) \|_{L^2}^2 \in W^{1,1}(0, \infty) \), and hence \( \| \phi_{x}(t) \|_{L^2} \to 0 \) as \( t \to \infty \). Since \( \| \phi \|_{L^\infty} \lesssim \| \phi \|_{L^2} \| \phi_{x} \|_{L^2} \), from Lemma 3.1 we know that \( \| \phi(t) \|_{L^\infty} \to 0 \) as \( t \to \infty \). From (5) and Theorem 1.1, we then know that there exists a finite time \( T_{1} > 0 \) such that \( \| \phi \|_{L^\infty} + \| \phi_{x} \|_{L^\infty} + \| \psi_{x} \|_{L^2} \) is sufficiently small for all \( t \geq T_{1} \). Then we derive from (81) that
\[
\frac{1}{2} \frac{d}{dt} (1 + t) \left( D_{26} \| \phi_{x} \|_{L^2}^2 + D_{28} \| \psi_{x} \|_{L^2}^2 + \| \psi_{x} \|_{L^2}^2 - \gamma \bar{p}^{-1} \| \phi_{x} \|_{L^2}^2 \right)
+ \| \psi_{xx} \|_{L^2}^2 + \gamma \bar{p}^{-2} \| \phi_{xx} \|_{L^2}^2
+ D_{31} (1 + t) (\| \phi_{xx} \|_{L^2}^2 + \| \psi_{xx} \|_{L^2}^2 + \| \phi_{xxx} \|_{L^2}^2)
\leq D_{32} (\| \phi_{x} \|_{L^2}^2 + \| \psi_{x} \|_{L^2}^2 + \| \phi_{xx} \|_{L^2}^2 + \| \phi_{xxx} \|_{L^2}^2), \quad \forall \ t > T_{1}.
\]
Integrating (82) over time and using (5) and Theorem 1.1 and Lemma 3.1, we end up with

\[(1 + t)(\|\phi_x\|^2_L^2 + \|\psi_x\|^2_L^2 + \|\phi_{xx}\|^2_L^2 + \|\phi_{xxx}\|^2_L^2)
\]

\[+ \int_{\bar{T}_1}^t (1 + \tau)(\|\phi_{xx}\|^2_L^2 + \|\psi_{xx}\|^2_L^2 + \|\phi_{xxx}\|^2_L^2)d\tau \leq D_{33}, \quad \forall \ t > \bar{T}_1. \tag{83}\]

Thus this completes the proof of Lemma 3.2.

**Lemma 3.3.** Under the conditions of Theorem 1.2, for any \(\gamma \geq 3\) and \(\varepsilon = 0\), then there exists a finite time \(\bar{T}_2 \geq \bar{T}_1\), such that for any \(t > \bar{T}_2\), it holds that

\[(1 + t)^2(\|\phi_{xx}\|^2_L^2 + \|\psi_{xx}\|^2_L^2 + \|\phi_{xxx}\|^2_L^2 + \|\phi_{xxxx}\|^2_L^2)
\]

\[+ \int_{\bar{T}_2}^t (1 + \tau)^2(\|\phi_{xxx}(\tau)\|^2_L^2 + \|\psi_{xxx}(\tau)\|^2_L^2)d\tau \leq D_{34}, \quad \forall \ t > \bar{T}_2. \tag{84}\]

where \(D_{34}\) is a positive constant independent of \(t > \bar{T}_2\).

**Proof.** **Step 1.** By performing in a completely similar fashion to the proof of Lemma 3.2, we can show that

\[(1 + t)(\|\phi_{xx}\|^2_L^2 + \|\psi_{xx}\|^2_L^2 + \|\phi_{xxx}\|^2_L^2 + \|\phi_{xxxx}\|^2_L^2)
\]

\[+ \int_{\bar{T}_1}^t (1 + \tau)(\|\phi_{xx}\|^2_L^2 + \|\psi_{xx}\|^2_L^2 + \|\phi_{xxx}\|^2_L^2)d\tau \leq D_{35}, \quad \forall \ t > \bar{T}_1. \tag{85}\]

Here we omitted the technical details for brevity. Next, we move on to the decay estimates of the second order spatial derivatives of \(\phi\) and \(\psi\).

**Step 2.** From (83), by using (12) for \(\phi_x\) and \(\psi_x\), we obtain

\[\|\phi_{xx}\|^2_L^2 \leq D_{36}(1 + t)^{-1} \quad \text{and} \quad \|\psi_{xx}\|^2_L^2 \leq D_{37}(1 + t)^{-1}, \quad \forall \ t > \bar{T}_1. \tag{86}\]

As a similar fashion in (72), we get

\[\frac{1}{2} \frac{d}{dt} (1 + t)^2(\gamma \bar{p}^{\gamma - 2}\|\phi_{xx}\|^2_L^2 + \|\psi_{xx}\|^2_L^2) + \gamma \bar{p}^{\gamma - 2}(1 + t)^2\|\phi_{xxx}\|^2_L^2
\]

\[\equiv (1 + t)(\gamma \bar{p}^{\gamma - 2}\|\phi_{xx}\|^2_L^2 + \|\psi_{xx}\|^2_L^2) - \gamma \bar{p}^{\gamma - 2}(1 + t)^2\int_{\mathbb{R}} (\phi_x \psi_x) x \phi_{xxx} dx
\]

\[+ \gamma (\gamma - 1)(1 + t)^2\int_{\mathbb{R}} (\phi_x + \bar{p})^{\gamma - 2}(\phi_{xx})^2 \psi_x dx
\]

\[+ \gamma (1 + t)^2\int_{\mathbb{R}} [(\phi_x + \bar{p})^{\gamma - 1} - \bar{p}^{\gamma - 1}] \phi_{xxx} \psi_{xx} dx
\]

\[\equiv (1 + t)(\gamma \bar{p}^{\gamma - 2}\|\phi_{xx}\|^2_L^2 + \|\psi_{xx}\|^2_L^2) + R_{14} + R_{15} + R_{16}. \tag{87}\]

Let us estimate \(R_{14}, R_{15}\) and \(R_{16}\). By using (85), we can show that

\[|R_{14}| = \left| \gamma \bar{p}^{\gamma - 2}(1 + t)^2 \int_{\mathbb{R}} (\phi_x \psi_x) x \phi_{xxx} dx \right|
\]

\[\leq \frac{1}{4} \gamma \bar{p}^{\gamma - 2}(1 + t)^2\|\phi_{xxx}\|^2_L^2
\]

\[+ 2\gamma \bar{p}^{\gamma - 2}(1 + t)^2(\|\psi_x\|^2_L^2 \|\phi_{xx}\|^2_L^2 + \|\phi_x\|^2_L^2 \|\psi_{xx}\|^2_L^2)
\]

\[\leq \frac{1}{4} \gamma \bar{p}^{\gamma - 2}(1 + t)^2\|\phi_{xxx}\|^2_L^2 + D_{38}(1 + t)(\|\phi_{xx}\|^2_L^2 + \|\psi_{xx}\|^2_L^2). \tag{88}\]
By using (85) and the uniform estimate of $\|\phi_x\|_{L^\infty} = \|\hat{p}\|_{L^\infty}$, we can show that
\[
|R_{15}| = \left| \gamma(\gamma - 1)(1 + t)^2 \int_{\mathbb{R}} (\phi_x + \hat{p})^{\gamma - 1} (\phi_{xx})^2 \psi_{xx} dx \right|
\leq D_{39}(1 + t)^2 (\|\hat{p}\|_{L^\infty}^{\gamma - 2} + \hat{p}^{\gamma - 2})(\|\phi_{xx}\|_{L^2}^2 \|\psi_{xx}\|_{L^2})
\leq D_{40}(1 + t)^2 \|\phi_x\|_{L^\infty} \|\phi_{xxx}\|_{L^2} \|\psi_{xx}\|_{L^2}
\leq \frac{1}{4} \gamma \hat{p}^{\gamma - 2}(1 + t)^2 \|\phi_{xxx}\|_{L^2}^2 + \frac{(D_{40})^2}{\gamma \hat{p}^{\gamma - 2}} (1 + t)^2 \|\phi_x\|_{L^\infty}^2 \|\psi_{xx}\|_{L^2}^2
\leq \frac{1}{4} \gamma \hat{p}^{\gamma - 2}(1 + t)^2 \|\phi_{xxx}\|_{L^2}^2 + D_{41}(1 + t) \|\psi_{xx}\|_{L^2}^2,
\] (88)

where we have used (66) for $\phi_x$. As a similar fashion in (75), by using (85) again, we have
\[
|R_{16}| = \left| \gamma(\gamma - 1)(1 + t)^2 \int_{\mathbb{R}} ((\phi_x + \hat{p})^{\gamma - 1} - \hat{p}^{\gamma - 1}) \phi_{xxx} \psi_{xx} dx \right|
\leq \gamma(\gamma - 1)(1 + t)^2 \int_{\mathbb{R}} (\|\hat{p}\| + \hat{p})^{\gamma - 2} |\phi_x| \|\phi_{xxx}\| \|\psi_{xx}\| dx
\leq D_{42}(1 + t)^2 \|\phi_x\|_{L^\infty} \|\phi_{xxx}\|_{L^2} \|\psi_{xx}\|_{L^2}
\leq \frac{1}{4} \gamma \hat{p}^{\gamma - 2}(1 + t)^2 \|\phi_{xxx}\|_{L^2}^2 + \frac{(D_{42})^2}{\gamma \hat{p}^{\gamma - 2}} (1 + t)^2 \|\phi_x\|_{L^\infty}^2 \|\psi_{xx}\|_{L^2}^2
\leq \frac{1}{4} \gamma \hat{p}^{\gamma - 2}(1 + t)^2 \|\phi_{xxx}\|_{L^2}^2 + D_{43}(1 + t) \|\psi_{xx}\|_{L^2}^2.
\] (89)

Plugging (87)-(89) into (86), we obtain
\[
\frac{1}{2} \frac{d}{dt} (1 + t)^2 (\gamma \hat{p}^{\gamma - 2}\|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2) + \frac{1}{4} \gamma \hat{p}^{\gamma - 2}(1 + t)^2 \|\phi_{xxx}\|_{L^2}^2
\leq D_{44}(1 + t)(\|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2).
\] (90)

Integrating (90) over time and using (83), we end up with
\[
(1 + t)^2 (\|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2) + \int_{T_1}^t (1 + \tau)^2 \|\phi_{xxx}(\tau)\|_{L^2}^2 d\tau \leq D_{45}, \quad \forall t > T_1. \quad (91)
\]

**Step 3.** Next, we proceed to the uniform estimate of the temporal integral of $(1 + t)^2 \|\psi_{xxx}\|_{L^2}^2$. As a similar fashion to the derivation of (80), we can show that
\[
\frac{d}{dt} (1 + t)^2 \left( \frac{1}{2} \gamma \hat{p}^{\gamma - 2}\|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 - \gamma \hat{p}^{\gamma - 1} \int_{\mathbb{R}} \psi_{xxx} \phi_{xx} dx \right)
+ D_{46}(1 + t)^2 (\|\phi_{xxx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2)
\leq D_{47}(1 + t)(\|\phi_{xxx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 + \|\phi_{xx}\|_{L^2}^2) + D_{48}(1 + t)^2 \|\phi_{xxx}\|_{L^2}^2
+ D_{49}(1 + t)^2 (\|\phi_x\|_{L^\infty} + \|\psi_x\|_{L^\infty})(\|\phi_{xxx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 + \|\phi_{xxx}\|_{L^2}^2),
\] (92)

where we have omitted the technical details for brevity. Let
\[
D_{50} = \frac{(\gamma \hat{p}^{\gamma - 1})^2 + 4D_{48}}{\gamma \hat{p}^{\gamma - 2}}.
\]
Multiplying (90) by $D_{50}$, and adding the result to (92), we have

$$
\frac{1}{2} \frac{d}{dt} (1 + t)^2 (4D_{48} \| \phi_{xx} \|^2_{L^2} + D_{50} \| \psi_{xx} \|^2_{L^2} + \| \psi_{xxx} - \gamma \bar{p}^{-1} \phi_{xx} \|^2_{L^2} \\
+ \| \psi_{xxx} \|^2_{L^2} + \bar{p}^{-2} \| \phi_{xxx} \|^2_{L^2} \\
+ D_{51} (1 + t)^2 (\| \phi_{xx} \|^2_{L^2} + \| \psi_{xx} \|^2_{L^2} + \| \phi_{xxx} \|^2_{L^2} + \| \psi_{xxx} \|^2_{L^2}) \\
\leq D_{52} (1 + t) (\| \phi_{xx} \|^2_{L^2} + \| \psi_{xx} \|^2_{L^2} + \| \phi_{xxx} \|^2_{L^2} + \| \psi_{xxx} \|^2_{L^2}) \\
+ D_{49} (1 + t)^2 (\| \phi_{x} \|^2_{L^\infty} + \| \psi_{x} \|^2_{L^\infty}) (\| \phi_{xxx} \|^2_{L^2} + \| \psi_{xxx} \|^2_{L^2} + \| \phi_{xxxx} \|^2_{L^2}).
$$

(93)

By using the smallness of $\| \phi_{x} \|^2_{L^\infty}$ and $\| \psi_{x} \|^2_{L^\infty}$ after a finite time $\bar{T}_2 \geq \bar{T}_1$, we obtain

$$
\frac{1}{2} \frac{d}{dt} (1 + t)^2 (4D_{48} \| \phi_{xx} \|^2_{L^2} + D_{50} \| \psi_{xx} \|^2_{L^2} + \| \psi_{xxx} - \gamma \bar{p}^{-1} \phi_{xx} \|^2_{L^2} \\
+ \| \psi_{xxx} \|^2_{L^2} + \bar{p}^{-2} \| \phi_{xxx} \|^2_{L^2} \\
+ D_{53} (1 + t)^2 (\| \phi_{xx} \|^2_{L^2} + \| \psi_{xx} \|^2_{L^2} + \| \phi_{xxx} \|^2_{L^2} + \| \psi_{xxx} \|^2_{L^2}) \\
\leq D_{54} (1 + t) (\| \phi_{xx} \|^2_{L^2} + \| \psi_{xx} \|^2_{L^2} + \| \phi_{xxx} \|^2_{L^2} + \| \psi_{xxx} \|^2_{L^2}).
$$

(94)

Integrating (94) over time and using (83) and (84), we end up with

$$
(1 + t)^2 (\| \phi_{xx} \|^2_{L^2} + \| \psi_{xx} \|^2_{L^2} + \| \phi_{xxx} \|^2_{L^2} + \| \psi_{xxx} \|^2_{L^2}) \\
+ \int_{\bar{T}_2}^t (1 + \tau)^2 (\| \phi_{xx} \|^2_{L^2} + \| \psi_{xx} \|^2_{L^2} + \| \phi_{xxx} \|^2_{L^2} + \| \psi_{xxx} \|^2_{L^2}) d\tau \leq D_{55}, \quad \forall \ t > \bar{T}_2.
$$

(95)

Thus the proof of Lemma 3.3 is completed.

By utilizing an iterative argument as in the proof of Lemma 3.3, we can identify the decay rate of the third order spatial derivatives of the perturbation. Since the proof is in a completely similar fashion to the proof of Lemma 3.3, we only state the result and omit the technical details for brevity.

**Lemma 3.4.** Under the conditions of Theorem 1.2, for any $\gamma \geq 3$ and $\varepsilon = 0$, then there exists a finite time $\bar{T}_3 \geq \bar{T}_2$, such that for any $t > \bar{T}_3$, it holds that

$$
(1 + t)^3 (\| \phi_{xxx} \|^2_{L^2} + \| \psi_{xxx} \|^2_{L^2}) + \int_{\bar{T}_3}^t (1 + \tau)^3 (\| \phi_{xxx} (\tau) \|^2_{L^2} + \| \psi_{xxx} (\tau) \|^2_{L^2} d\tau \leq D_{56},
$$

where $D_{56}$ is a positive constant independent of $t > \bar{T}_3$.

3.1. **Proof of Theorem 1.2.** According to the definition of anti-derivatives from (5), by combining Lemma 3.2, Lemma 3.3 and Lemma 3.4, we can obtain the explicit decay rate of the solution to the Cauchy problem of (1) for any $\gamma \geq 3$ and $\varepsilon = 0$. In addition, there exists a finite time $\bar{T} > 0$, such that for all $t > \bar{T}$,

$$
\|(p(t) - \bar{p}, q(t))\|_{L^2} \leq D_{57} (1 + t)^{-\frac{1}{2}},
$$

$$
\|(p_x, q_x)(t)\|_{L^2} \leq D_{58} (1 + t)^{-1},
$$

$$
\|(p_{xx}, q_{xx})(t)\|_{L^2} \leq D_{59} (1 + t)^{-\frac{1}{2}}.
$$

By using Sobolev embedding, we have

$$
\|p(t) - \bar{p}\|_{L^\infty} \leq \sqrt{2} \|p(t) - \bar{p}\|^\frac{1}{2}_{L^2} \|p(t)\|_{L^2}^{\frac{1}{2}} \|ar{p}\|_{L^2}^{\frac{1}{2}} \leq D_{60} (1 + t)^{-\frac{1}{2}}.
$$
Similarly, we can show that
\[ \|q(t)\|_{L^\infty} \leq C_6(1 + t)^{-\frac{3}{2}}, \quad \text{and} \quad \|(p_x, q_x)(t)\|_{L^\infty} \leq C_6(1 + t)^{-\frac{3}{2}}. \]
Thus the proof of Theorem 1.2 is completed.

4. Decay Rate for \( \varepsilon > 0 \). In this section, we prove Theorem 1.3. We present energy estimates under the \textit{a priori} assumption:
\[
\sup_{0 \leq t \leq T} (\|\phi(t)\|_{L^2}^2 + \|\psi(t)\|_{L^2}^2) \leq \xi_2, \tag{96}
\]
where \( \xi_2 \) is a positive constant to be determined. First of all, we shall establish the uniform-in-time \( L^2 \) estimates of \( \phi \) and \( \psi \).

**Lemma 4.1.** Under the conditions of Theorem 1.3, for any \( \gamma \geq 2 \) and \( \varepsilon > 0 \), it holds that
\[
\|\phi\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \int_0^t (\|\phi_x(\tau)\|_{L^2}^2 + \|\psi_x(\tau)\|_{L^2}^2) d\tau \leq D_63,
\]
where \( D_63 \) is a positive constant independent of \( t \).

**Proof.** Taking the \( L^2 \) inner products of the first equation of (47) by \( \gamma \bar{p}^{-2}\phi \) and the second equation of (47) with \( \varepsilon > 0 \) by \( \psi \), and adding the results and integrating by parts with respect to \( x \), we have
\[
\frac{1}{2} \frac{d}{dt} (\gamma \bar{p}^{-2}\|\phi\|_{L^2}^2 + \|\psi\|_{L^2}^2) + \gamma \bar{p}^{-2}\|\phi_x\|_{L^2}^2 + \varepsilon \|\psi_x\|_{L^2}^2
\]
\[
= \gamma \bar{p}^{-2} \int \phi_x \psi_x dx + \int \left[ (\phi_x + \bar{p}) \gamma - \bar{p}^{-1} p \right] \psi dx + \varepsilon \int \psi_x^2 dx \tag{97}
\]
\[
\equiv R_1 + R_2 + R_{17},
\]
where \( R_1 \) and \( R_2 \) are defined in (49). In view of (50), (51) and (96), we see that the estimates of \( R_1 \) and \( R_2 \) are still valid for \( \gamma \geq 2 \) and \( \varepsilon > 0 \). Then we can show that
\[
|R_1| + |R_2| \leq D_{64}\xi_2^2 (\|\phi_x\|_{L^2}^2 + \|\psi_x\|_{L^2}^2). \tag{98}
\]
By using (96) and the definition of \( \psi \), and the uniform estimate of \( \|\psi_x\|_{L^2} = \|q\|_{L^2} \) from Theorem 1.1, we can show that
\[
|R_{17}| = \left| \varepsilon \int \psi_x^2 dx \right|
\]
\[
\leq \varepsilon \|\psi\|_{L^\infty} \|\psi_x\|_{L^2}^2
\]
\[
\leq \varepsilon \|\psi\|_{L^\infty} \|\psi_x\|_{L^2}^2 \|\psi_x\|_{L^2}^2
\]
\[
\leq D_{65}\varepsilon \xi_2^2 \|\psi_x\|_{L^2}^2. \tag{99}
\]
Plugging (98) and (99) into (97), we derive
\[
\frac{1}{2} \frac{d}{dt} (\gamma \bar{p}^{-2}\|\phi\|_{L^2}^2 + \|\psi\|_{L^2}^2) + \gamma \bar{p}^{-2}\|\phi_x\|_{L^2}^2 + \varepsilon \|\psi_x\|_{L^2}^2
\]
\[
\leq D_{66}\xi_2^2 (\|\phi_x\|_{L^2}^2 + \|\psi_x\|_{L^2}^2), \tag{100}
\]
where the constant \( D_{66} \) depends on \( \varepsilon \), but is independent of \( t \). By choosing \( \xi_2 \) sufficiently small, we have
\[
\frac{1}{2} \frac{d}{dt} (\gamma \bar{p}^{-2}\|\phi\|_{L^2}^2 + \|\psi\|_{L^2}^2) + D_{67}(\|\phi_x\|_{L^2}^2 + \|\psi_x\|_{L^2}^2) \leq 0, \tag{101}
\]
which implies that
\[
\| \phi \|^2_{L^2} + \| \psi \|^2_{L^2} + \int_0^t (\| \phi_x (\tau) \|^2_{L^2} + \| \psi_x (\tau) \|^2_{L^2}) d\tau \leq D_{69} (\| \phi_0 \|^2_{L^2} + \| \psi_0 \|^2_{L^2}).
\] (102)

Based on standard continuation argument we know that (102) holds true for all time provided that \( \| \phi_0 \|^2_{L^2} + \| \psi_0 \|^2_{L^2} \) is sufficiently small. Then the \textit{a priori} assumption (96) is closed. This completes the proof of Lemma 4.1.

\[\square\]

\textbf{Lemma 4.2.} \textit{Under the conditions of Theorem 1.3, for any} \( \gamma \geq 2 \) \textit{and} \( \varepsilon > 0 \), \textit{it holds that}
\[
(1 + t) (\| \phi_x \|^2_{L^2} + \| \psi_x \|^2_{L^2}) + \int_0^t (1 + \tau) (\| \phi_{xx} (\tau) \|^2_{L^2} + \| \psi_{xx} (\tau) \|^2_{L^2}) d\tau \leq D_{69}, \quad \forall t > 0,
\]
\textit{where} \( D_{69} \) \textit{is a positive constant independent of} \( t \).

\textbf{Proof.} As a similar fashion in (65), we obtain
\[
\frac{1}{2} \frac{d}{dt} (1 + t) (\gamma \bar{p} \gamma^{-2} \| \phi_x \|^2_{L^2} + \| \psi_x \|^2_{L^2}) \\
+ \gamma \bar{p} \gamma^{-2} (1 + t) \| \phi_{xx} \|^2_{L^2} + \varepsilon (1 + t) \| \psi_{xx} \|^2_{L^2} \\
= \frac{1}{2} (\gamma \bar{p} \gamma^{-2} \| \phi_x \|^2_{L^2} + \| \psi_x \|^2_{L^2}) + \gamma \bar{p} \gamma^{-2} (1 + t) \int_{\mathbb{R}} (\phi_x \psi_x + \psi_x \phi_x) dx \\
- (1 + t) \int_{\mathbb{R}} [(\phi_x + \tilde{p}) \gamma - \bar{p} \gamma - \gamma \bar{p} \gamma^{-1} \phi_x] \psi_x dx \\
\equiv \frac{1}{2} (\gamma \bar{p} \gamma^{-2} \| \phi_x \|^2_{L^2} + \| \psi_x \|^2_{L^2}) + R_6 + R_7,
\]
where \( R_6 \) and \( R_7 \) are the same as those defined in (65). In a similar fashion, we can show that (67) and (68) are still valid in this case for any \( \gamma \geq 2 \) and \( \varepsilon > 0 \). Choosing \( \sigma = \varepsilon \), and substituting (67) and (68) into (103), we have
\[
\frac{1}{2} \frac{d}{dt} (1 + t) (\gamma \bar{p} \gamma^{-2} \| \phi_x \|^2_{L^2} + \| \psi_x \|^2_{L^2}) \\
+ \gamma \bar{p} \gamma^{-2} (1 + t) \| \phi_{xx} \|^2_{L^2} + \frac{\varepsilon}{2} (1 + t) \| \psi_{xx} \|^2_{L^2} \\
\leq \frac{1}{2} (\gamma \bar{p} \gamma^{-2} \| \phi_x \|^2_{L^2} + \| \psi_x \|^2_{L^2}) + D_{70} (1 + t) \| \phi \|^2_{L^\infty} \| \phi_{xx} \|^2_{L^2} \\
\leq \frac{1}{2} (\gamma \bar{p} \gamma^{-2} \| \phi_x \|^2_{L^2} + \| \psi_x \|^2_{L^2}) + 2D_{70} (1 + t) \| \phi \|_{L^2} \| \phi_x \|_{L^2} \| \phi_{xx} \|^2_{L^2} \\
\leq \frac{1}{2} (\gamma \bar{p} \gamma^{-2} \| \phi_x \|^2_{L^2} + \| \psi_x \|^2_{L^2}) + D_{71 \xi_2} (1 + t) \| \phi_{xx} \|^2_{L^2},
\]
where we have used (96) and the uniform estimate of \( \| \phi_x \|_{L^2} = \| \tilde{p} \|_{L^2} \) obtained from Theorem 1.1. By choosing \( \xi_2 \) sufficiently small, we update (104) as
\[
\frac{1}{2} \frac{d}{dt} (1 + t) (\gamma \bar{p} \gamma^{-2} \| \phi_x \|^2_{L^2} + \| \psi_x \|^2_{L^2}) \\
+ D_{72} (1 + t) \| \phi_{xx} \|^2_{L^2} + \frac{\varepsilon}{2} (1 + t) \| \psi_{xx} \|^2_{L^2} \\
\leq D_{73} (\| \phi_x \|^2_{L^2} + \| \psi_x \|^2_{L^2}).
\]
Integrating (105) over \([0, t]\) and using Lemma 4.1, we obtain Lemma 4.2.

\[\square\]
Lemma 4.3. Under the conditions of Theorem 1.3, for any $\gamma \geq 2$, $\varepsilon > 0$, and $t > 0$, it holds that

$$(1 + t)^2(\|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2) + \int_0^t (1 + \tau)^2(\|\phi_{xxx}(\tau)\|_{L^2}^2 + \|\psi_{xxx}(\tau)\|_{L^2}^2)d\tau \leq D_{T4},$$

where $D_{T4}$ is a positive constant independent of $t$.

Proof. **Step 1.** As a similar fashion in (72), we obtain

$$
\frac{1}{2} \frac{d}{dt} (1 + t) (\gamma \bar{p}^{\gamma - 2}\|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2)
+ \gamma \bar{p}^{\gamma - 2}(1 + t)\|\phi_{xxx}\|_{L^2}^2 + \varepsilon (1 + t)\|\psi_{xxx}\|_{L^2}^2
= \frac{1}{2} (\gamma \bar{p}^{\gamma - 2}\|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2) - \gamma \bar{p}^{\gamma - 2}(1 + t) \int_\mathbb{R} (\phi_x \psi_x) \phi_{xxx} dx
+ \gamma (\gamma - 1)(1 + t) \int_\mathbb{R} (\phi_x + \bar{p})^{\gamma - 2}(\phi_{xx})^2 \psi_{xx} dx
- 2 \varepsilon (1 + t) \int_\mathbb{R} \psi_x \psi_{xxx} \psi_{xx} dx
+ \gamma (1 + t) \int_\mathbb{R} [(\phi_x + \bar{p})^{\gamma - 1} - \bar{p}^{-1}] \phi_{xxx} \psi_{xx} dx
\equiv \frac{1}{2} (\gamma \bar{p}^{\gamma - 2}\|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2) + R_{18} + R_{19} + R_{20} + R_{21}.
$$

Let us estimate $R_{18}, \ldots, R_{21}$. By using the uniform estimates of $\|\phi_x\|_{L^\infty} = \|\bar{p}\|_{L^\infty}$ and $\|\psi_x\|_{L^\infty} = \|q\|_{L^\infty}$ obtained from Theorem 1.1, we can show that

$$|R_{18}| = \left| \gamma \bar{p}^{\gamma - 2}(1 + t) \int_\mathbb{R} (\phi_x \psi_x) \phi_{xxx} dx \right|
\leq \frac{1}{4} \gamma \bar{p}^{\gamma - 2}(1 + t)\|\phi_{xxx}\|_{L^2}^2
+ 2 \gamma \bar{p}^{\gamma - 2}(1 + t)(\|\psi_{xx}\|_{L^\infty} \|\phi_{xx}\|_{L^2}^2 + \|\phi_x\|_{L^\infty} \|\psi_{xx}\|_{L^2}^2)
\leq \frac{1}{4} \gamma \bar{p}^{\gamma - 2}(1 + t)\|\phi_{xxx}\|_{L^2}^2 + D_{T5}(1 + t)(\|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2).$$

By using the Young inequality and the uniform estimate of $\|\phi_x\|_{L^\infty} = \|\bar{p}\|_{L^\infty}$ again, we derive

$$|R_{19}| = \left| \gamma (\gamma - 1)(1 + t) \int_\mathbb{R} (\phi_x + \bar{p})^{\gamma - 2}(\phi_{xx})^2 \psi_{xx} dx \right|
\leq D_{T6}(1 + t)(\|\bar{p}\|_{L^\infty} \gamma^{\gamma - 2} + \bar{p}^{\gamma - 2})\|\phi_{xx}\|_{L^2} \|\psi_{xx}\|_{L^2}
\leq D_{T7}(1 + t)\|\phi_x\|_{L^\infty} \|\phi_{xxx}\|_{L^2} \|\psi_{xx}\|_{L^2}
\leq D_{T8}(1 + t)\|\phi_{xxx}\|_{L^2} \|\psi_{xx}\|_{L^2}
\leq \frac{1}{4} \gamma \bar{p}^{\gamma - 2}(1 + t)\|\phi_{xxx}\|_{L^2}^2 + D_{T9}(1 + t)\|\psi_{xx}\|_{L^2}^2,$$
where we have used (66) for $\phi_x$. By using the uniform estimate of $\|\psi_x\|_{L^\infty} = \|q\|_{L^\infty}$ again, we get

$$\left|R_{20}\right| = \left|2\varepsilon (1+t) \int_R \psi_x \psi_{xx} \psi_{xxx} dx \right|$$

$$\leq \frac{1}{4} \varepsilon (1+t) \|\psi_{xxx}\|_{L^2}^2 + 4\varepsilon (1+t) \|\psi_x\|_{L^\infty} \|\psi_{xx}\|_{L^2}$$

(109)

$$\leq \frac{1}{4} \varepsilon (1+t) \|\psi_{xxx}\|_{L^2}^2 + D_{80} (1+t) \|\psi_{xx}\|_{L^2}.$$  

As a similar fashion in (75), and using the Young inequality again, we obtain

$$\left|R_{21}\right| = \left|\gamma (1+t) \int_R \left[ (\phi_x + \bar{p})^{\gamma-1} - \bar{p}^{\gamma-1} \right] \phi_{xxx} \psi_{xx} dx \right|$$

$$\leq \gamma \left( \gamma - 1 \right) (1+t) \int_R \left( |\bar{p}| + |\bar{p}| \right) \gamma^{-2} |\phi_x||\phi_{xxx}| |\psi_{xx}| dx$$

(110)

$$\leq D_{81} (1+t) \left( \|\bar{p}\|_{L^\infty}^{-2} + \|\bar{p}\|_{L^2}^{-2} \right) \|\phi_x\|_{L^\infty} \|\phi_{xxx}\|_{L^2} \|\psi_{xx}\|_{L^2}$$

$$\leq D_{82} (1+t) \|\phi_{xxx}\|_{L^2} \|\psi_{xx}\|_{L^2}$$

$$\leq \frac{1}{4} \gamma \bar{p}^{-2} (1+t) \|\phi_{xxx}\|_{L^2}^2 + D_{83} (1+t) \|\psi_{xx}\|_{L^2}^2.$$  

Substituting (107)-(110) into (106), we end up with

$$\frac{1}{2} \frac{d}{dt} (1+t) \left( \gamma \bar{p}^{-2} \|\phi_x\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 \right)$$

$$+ \frac{1}{4} \gamma \bar{p}^{-2} (1+t) \|\phi_{xxx}\|_{L^2}^2 + \frac{3}{4} \varepsilon (1+t) \|\psi_{xxx}\|_{L^2}^2$$

(111)

$$\leq \frac{1}{2} \gamma \bar{p}^{-2} \|\phi_{xxx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 + D_{84} (1+t) \left( \|\phi_{xxx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 \right).$$

Integrating (111) over time, and using Theorem 1.1 and Lemma 4.2, we deduce

$$(1+t) \left( \|\phi_x\|_{L^\infty}^2 + \|\psi_{xx}\|_{L^2}^2 \right) + \int_0^t (1+t) \left( \|\phi_{xxx}(\tau)\|_{L^2}^2 + \|\psi_{xx}(\tau)\|_{L^2}^2 \right) d\tau \leq D_{85},$$

(112)

which, combined with Lemma 4.2 and Sobolev inequality, yields

$$\|\phi_x\|_{L^\infty}^2 \leq D_{86} (1+t)^{-1} \quad \text{and} \quad \|\psi_{xx}\|_{L^2}^2 \leq D_{87} (1+t)^{-1}, \quad \forall t > 0.$$  

(113)

**Step 2.** As a similar fashion in (86), we obtain

$$\frac{1}{2} \frac{d}{dt} (1+t)^2 \left( \gamma \bar{p}^{-2} \|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 \right)$$

$$+ \gamma \bar{p}^{-2} (1+t)^2 \|\phi_{xxx}\|_{L^2}^2 + \varepsilon (1+t)^2 \|\psi_{xxx}\|_{L^2}^2$$

$$= (1+t) \left( \gamma \bar{p}^{-2} \|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 \right) - \bar{p}^{-2} (1+t)^2 \int_R (\phi_x \psi_x)x \phi_{xx} dx$$

$$+ \gamma (\gamma - 1) (1+t)^2 \int_R (\phi_x + \bar{p})^{\gamma-1} - \bar{p}^{\gamma-1} \phi_{xxx} \psi_{xx} dx$$

(114)

$$+ \gamma (1+t)^2 \int_R ((\phi_x + \bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}) \phi_{xxx} \psi_{xx} dx$$

$$- 2\varepsilon (1+t)^2 \int_R \psi_x \psi_{xx} \psi_{xxx} dx$$

$$\equiv (1+t) \left( \gamma \bar{p}^{-2} \|\phi_{xx}\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 \right) + R_{14} + R_{15} + R_{16} + R_{22},$$

where $R_{14}, R_{15}, R_{16}, R_{22}$ are nonnegative.
where $R_{14}$, $R_{15}$ and $R_{16}$ are the same as those defined in (86). Similarly, by using (113), we can show that estimates of $R_{14}$, $R_{15}$ and $R_{16}$ still valid for any $γ ≥ 2$ and $ε > 0$ (see (87), (88) and (89)). Thus we have

$$|R_{14}| + |R_{15}| + |R_{16}| \leq \frac{3}{4} γ \bar{p}^{γ-2}(1+t)^2 \||φ_{xxx}\|_{L^2}^2 + D_{88}(1+t)(\|φ_{xx}\|_{L^2}^2 + \|ψ_{xx}\|_{L^2}^2).$$

(115)

Similar to (109), and using (113), we can show that

$$\|R_{22}\| = \left| 2ε(1+t)^2 \int_{\mathbb{R}} φ_x φ_xx ψ_xxxx dx \right|$$

$$\leq \frac{1}{4} ε(1+t)^2 \|φ_{xxx}\|_{L^2}^2 + 4ε(1+t)^2 \|ψ_{xx}\|_{L^∞}^2 \|ψ_{xxx}\|_{L^2}^2$$

$$≤ \frac{1}{4} ε(1+t)^2 \|φ_{xxx}\|_{L^2}^2 + D_{88}(1+t) \|ψ_{xx}\|_{L^2}^2.$$  

(116)

Substituting (115) and (116) into (114), we derive

$$\frac{1}{2} \frac{d}{dt}(1+t)^2(γ \bar{p}^{γ-2} \|φ_{xx}\|_{L^2}^2 + \|ψ_{xx}\|_{L^2}^2)$$

$$+ \frac{3}{4} γ \bar{p}^{γ-2}(1+t)^2 \|φ_{xxx}\|_{L^2}^2 + \frac{3}{4} ε(1+t)^2 \|ψ_{xxx}\|_{L^2}^2$$

$$\leq D_{90}(1+t)(\|φ_{xx}\|_{L^2}^2 + \|ψ_{xx}\|_{L^2}^2).$$

(117)

Integrating (117) over $[0,t]$ and using Lemma 4.2, we can obtain Lemma 4.3. □

4.1. Proof of Theorem 1.3. Finally, the combination of Lemma 4.2 and Lemma 4.3 gives the explicit decay rates of the solution to the Cauchy problem of (1) for any $γ ≥ 2$ and $ε > 0$. Moreover, the explicit decay rates of $\|p(t) - \bar{p}\|_{L^∞}$ and $\|q(t)\|_{L^∞}$ can be identified by a completely similar fashion as in the proof of Theorem 1.2. Here we omit the further details for brevity. Thus the proof of Theorem 1.3 is completed.

5. Conclusion. We have investigated the long-time asymptotic behavior of the system of hyperbolic-parabolic conservation laws (4) with strong nonlinearities in one space dimension. When the exponent of the chemical production rate, $γ$, satisfies $γ ≥ 3$, by utilizing energy methods, we proved that for general initial data satisfying $(p_0 - \bar{p}, q_0) ∈ H^2(\mathbb{R})$ and $p_0(x) ≥ 0$, where $\bar{p} > 0$ is a constant, there exists a unique global-in-time solution to the Cauchy problem of the model. Our result improved the regularity of the solution previously obtained in [49]. Despite the complexity of the problem, we managed to show that the energy estimates of the solution, obtained in Theorem 1.1, are entirely independent of time and the diffusion coefficient of the $q$-component. Thus we revealed that constant ground states are globally asymptotically stable in a stronger topology than that obtained in [49]. Moreover, by utilizing time-weighted energy methods and combining with the energy estimates established in Theorem 1.1, we identified the explicit decay rates of the solutions to the Cauchy problem of the non-diffusive and diffusive models under mild conditions on the initial data. Our results generalized the ones previously reported in [22, 30, 48] by extending the exponent of the chemical production rate from a single value to the half real line.

Furthermore, the transformed system (4) itself has deep mathematical interest as it serves as a prototype of general hyperbolic-parabolic conservation laws. The nature of non-uniform parabolicity coupled with nonlinear flux functions in this type
of equations presents significant challenge in mathematical analysis. Besides being an effective vehicle in the study of the chemotaxis model (3), the analysis of (4) helps to shed light on how to advance fundamental research of hyperbolic-parabolic conservation laws in related topics by carefully developing a sophisticated approach to meet the specific challenge. This is to start with energy estimate, continued with weighted energy estimate in its full capacity for explicit decay rates. We expect the approach or its varieties to offer future opportunities in the rigorous studies of chemotaxis models and to be hopeful to other hyperbolic-parabolic conservation laws.

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