An inverse kinematic problem with internal sources

Leonid Pestov\textsuperscript{1}, Gunther Uhlmann\textsuperscript{2} and Hanming Zhou\textsuperscript{3,4}

\textsuperscript{1}Immanuel Kant Baltic Federal University, Russia
\textsuperscript{2}Department of Mathematics, University of Washington, USA and IAS, Hong Kong University of Science and Technology, Hong Kong
\textsuperscript{3}Department of Mathematics, University of Washington, USA

E-mail: LPestov@kantiana.ru, gunther@math.washington.edu and hzhou@math.washington.edu

Received 27 September 2014, revised 22 February 2015
Accepted for publication 30 March 2015
Published 24 April 2015

Abstract

Given a bounded domain $M$ in $\mathbb{R}^n$ with a conformally Euclidean metric $g = \rho \, dx^2$, we consider the inverse problem of recovering a semigeodesic neighborhood of a domain $\Gamma \subset \partial M$ and the conformal factor $\rho$ in the neighborhood from the travel time data (defined below) and the Cartesian coordinates of $\Gamma$. We develop an explicit reconstruction procedure for this problem. The key ingredient is the relation between the reconstruction procedure and a Cauchy problem of the conformal Killing equation.

Keywords: inverse kinematic, internal data, conformal Killing vector fields

1. Introduction

The inverse kinematic problem arises in geophysics in an attempt to determine the substructure of the Earth by measuring at the surface of the Earth the travel times of seismic waves. The Earth is generally isotropic; thus one can assume that the geometry of the Earth is conformally Euclidean. In applications the conformal factor corresponds to $1/c^2(x)$, where $c(x)$ is the sound speed (index of refraction). This problem goes back to [6, 19], who considered the case of a radial metric conformal to the Euclidean metric. The geometric version of the problem is related to the boundary rigidity problem and its linearization, namely the geodesic ray transform (see [15, 18] for recent surveys).

The usual inverse kinematic problem takes into account only the travel times between boundary points. In the current paper we also make use of the internal data (travel times from internal sources to the boundary) so that we can reconstruct the geometry explicitly. The
problem of determining a sound speed of a medium from the corresponding Dirichlet-to-
Neumann map has been solved using the boundary control method (see [7]). An ingredient in
the method is to recover the boundary distance function between interior points and the
boundary. Our paper gives an explicit reconstruction in the case conformal to the Euclidean
metric. We describe now the mathematical setting of the problem.

Let \( (M, g) \) be a bounded domain in \( \mathbb{R}^n, n \geq 2 \) with smooth boundary \( \partial M \). We assume \( g \) is conformal to the Euclidean metric i.e., \( g = \rho \, dx^2 \), where \( \rho \) is a positive smooth function on \( M \) and \( dx^2 = (dx^1)^2 + \cdots + (dx^n)^2 \) is the Euclidean metric. Let \( \Gamma \) be a domain in \( \partial M \) (in particular, \( \Gamma \) could be \( \partial M \)). From each \( x' \in \Gamma \), there is a unique geodesic \( \gamma_{c}(t) \) with \( \gamma_{c}(0) = x', \gamma_{c}(0) = v(x'), \) where \( v(x') \) is the inward unit normal vector to \( \partial M \) at \( x' \) w.r.t. the metric \( g \). Moreover, since \( \gamma \) is a geodesic of unit speed w.r.t. the metric \( g \), we have \( \rho(\gamma_{c}) \, |\gamma_{c}'|^2 = 1 \), where \( |\cdot| \) is the Euclidean norm.

There is a positive smooth function \( T(x') \) on \( \Gamma \) such that for each \( x' \in \Gamma \), the geodesic \( \gamma_{c}(\cdot) \), which is orthogonal to \( \partial M \) at \( x' \), is defined on the interval \([0, T(x')]\). Let

\[ D := \{ (x', t) : x' \in \Gamma, \ 0 \leq t < T(x') \}. \]

We consider the map \( \gamma : D \to \gamma(D), \gamma(x', t) := \gamma_{c}(t) = x \). Generally such a map is not a diffeomorphism; for example, given the Euclidean disk with radius \( r \), let \( \Gamma \) be a domain of the boundary; then \( \gamma \) is not a diffeomorphism if \( T(x') > r \). Thus we modify \( T(x') \), i.e., \( D \), so that \( \gamma : D \to \gamma(D) \) is a diffeomorphism. Under this assumption, \( D \) actually provides a semigeodesic coordinate system (or boundary normal coordinates) for \( \gamma(D) \).

Now given a point \( x \in \gamma(D) \), if \( x' \in \Gamma \) such that \( x = \gamma(x', t) \) for some \( t \in [0, T(x')] \), let \( U(x') \subset \Gamma \) be a neighborhood of \( x' \). Moreover, we fix \( U(x') \) for all \( x \in \gamma_{c}([0, T(x')) \), and \( U(x') \) can be arbitrarily small. Notice that the choice of \( U(x') \) depends on a priori knowledge of \( x \); i.e., we need to know the geodesic projection \( x'(x) \in \Gamma \) of any \( x \in \gamma(D) \). We define the travel time data with respect to \( D \) by

\[ \Omega(D) := \{ (\tau(x, x'), x'(x)) : x \in \gamma(D), x' \in U(x') \}, \]

where \( \tau(x, x') := \text{dist}_{g}(x, x') \). In this paper, we consider the problem of recovering the neighborhood \( \gamma(D) \) and the conformal factor \( \rho \) in \( \gamma(D) \) from the travel time data \( \Omega(D) \). To solve the problem, we need some extra inverse data; namely, we assume the Cartesian coordinates of \( \Gamma \) are known. This is a reasonable assumption, since any rigid transformation of the domain \( M \) does not change the travel time data. We call the problem the inverse kinematic problem with internal sources. It is worth pointing out that we do not make any assumption on the convexity of the boundary (or \( \Gamma \)).

**Theorem 1.1.** Let \( M \) be a bounded domain in \( \mathbb{R}^n, n \geq 2 \), and \( g = \rho \, dx^2 \) be a conformally Euclidean metric on \( M \). Let \( \Gamma \) be a domain in \( \partial M \); then there exists a semigeodesic coordinate system \( D \) such that \( \gamma : D \to \gamma(D) \subset \mathbb{R}^n \) is a diffeomorphism, and \( \Gamma = \gamma([t = 0]) \). From the travel time data \( \Omega(D) \) and the Cartesian coordinates of \( \Gamma \), one can recover the diffeomorphism \( \gamma \) and the conformal factor \( \rho \) in \( \gamma(D) \).

Uniqueness for this inverse problem was proved by Anikonov [1]. In this paper we give a reconstruction procedure based on conformal Killing vector fields.

Generally one cannot expect to reconstruct \( \rho \) on the whole manifold, as the necessary assumption that \( \gamma \) is a diffeomorphism. However, if \( M \setminus \gamma(D) \) has an empty interior, we reconstruct the domain \( M \) and the conformal factor \( \rho \) globally, since points in \( M \setminus \gamma(D) \) are
limit points of $\gamma(D)$. The example mentioned above satisfies the assumption, and it is easy to see that $M^1_\gamma(D)$ is the center of the disk if $\Gamma = \partial M$, $T(x') \equiv r$.

In [7, 8] an isometric copy of a compact Riemannian manifold was recovered from the set of boundary distance functions $\{d(y) = (x, y) : x \in M, y \in \partial M\}$, which is also internal data. Such internal data is also related to the broken geodesic flow, which consists of two geodesic segments sharing a common end point inside the manifold (see, e.g., [7, 9]). A related reconstruction problem with different assumptions was considered in [5] by reducing the travel time data to measurements of the shape operators of the wave fronts of waves diffracted from interior points. Different from our method, their approach treated the case of two dimensions and the case of three and higher dimensions separately.

Notice that the statement of theorem 1.1 actually shows that this is a local problem; i.e., for a point $x = \gamma(x', t)$, we only need the travel time data from $x$ to an arbitrarily small neighborhood $U(x') \subset \Gamma$. If the function $T(x')$ is also uniformly small, the problem can be formulated just near one boundary point. Similar to the local version of the problem, the local boundary rigidity problem and local geodesic ray transform were considered in [16, 17], and a generalization to local ray transforms along arbitrary smooth curves was studied in [20].

As mentioned above, our arguments give a reconstruction procedure for the diffeomorphism $\gamma$ and the conformal factor $\rho$. The reconstruction procedure consists of two steps: step 1 (section 2) is devoted to the recovery of a semigeodesic (isometric) copy of the metric $g$ and the boundary restriction of the conformal factor from the inverse data. In step 2 (section 3), we reconstruct $\gamma$ and $\rho$ by studying the relation between $\gamma$ and conformal Killing vector fields on the semigeodesic copy $D$.

As noticed, the inverse dynamical problem for the wave equations (with boundary data) may be reduced by the boundary control method to the inverse kinematic problem with internal sources and then to the Yamabe problem [3]. It gives in our case the Cauchy problem for the Laplace operator. We use another approach (using conformal Killing vector fields, which are determined by a linear system of first-order partial differential equations) that gives stability for dimensions $n > 2$ [12].

### 2. Recovery of the semigeodesic copy of the metric

Given $x \in \gamma(D)$, there is a unique geodesic $\gamma_x$ (normal to $\partial M$) and $0 \leq t < T(x')$ such that $x = \gamma(x', t)$. Here $x'(x)$ is the geodesic projection of $x$ on $\Gamma$, so $t(x') = \tau(x, x'(x))$ is the distance from $x$ to the boundary. Thus the travel time data $\Omega(D)$ uniquely determines the semigeodesic coordinates of points in $\gamma(D)$.

Let the pair $(x'(x), t(x))$ be the semigeodesic coordinate of the point $x$. Thus for any point $y = (x', t) \in D$, the function $\tau(\gamma(x', t), x')$ is known. Define

$$\lambda(\gamma(x', t), x') = \tau(\gamma(x', t), x'),$$

then $\lambda(y, x')$, $y = (x', t)$ is the distance between points $y \in D$ and $x' \in U(x')$ in the metric $\tilde{g} := \gamma^* g$. Note that we identify $\Gamma$ with $\Gamma \times \{0\}$. We call $\tilde{g}$ the semigeodesic copy of the metric $g$.

We first recover the metric $\tilde{g}$. Notice that $\gamma$, now as an isometry, sends geodesics to geodesics. This implies

$$\tilde{g}_{k\ell}(x', 0) = \delta_{k\ell}, \quad x' \in \Gamma, \quad 1 \leq k, \ell \leq n,$$

where $\delta_{ij}$ is the Kronecker delta. In local coordinates, $y = (y^1, \ldots, y^n)$, where $y^n = t$. One has the following eikonal equation

$$3.$$
1 \leq \frac{1}{2} \left( \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - 2 R_{ij}^{k} u_k \right) - \frac{1}{n} \tilde{g}^{ij} \frac{\partial u_i}{\partial x^j} - g^{ij} \mu_k G_{ijkl} = 0.

Thus the conformal Killing equation is equivalent to a linear system of first-order partial differential equations.

However, not all of the metrics admit conformal Killing vector fields. Actually for \( n \geq 3 \), a ‘generic’ metric does not possess any non-trivial conformal Killing vector fields (see, e.g., [2, 10]). On the other hand, note that the metric \( \tilde{g} \) is conformal to the Euclidean metric; thus they share the same set of conformal Killing vector fields. When \( n = 2 \), in the Cartesian coordinates \( (x^1, x^2) \) all conformal Killing vector fields of the Euclidean metric have the form \( u = (u^1, u^2) \), where \( u^1 \) and \( u^2 \) are conjugate harmonic functions. In the case \( n > 2 \), the contravariant components of \( u \) in the Cartesian coordinates \( (x^1, \ldots, x^n) \) are given by

\[ u^i(x) = a_0 x^i + (Ax)^i - b^i \langle x \rangle^2 + 2x^j(b, x) + c^i, \]
where \( a_0 \) is a real constant, \( A \) is an \( n \times n \) skew-symmetric constant matrix, and \( b \) and \( c \) are vectors in \( \mathbb{R}^n \).

Now we are in a position to recover the map \( \gamma \) and the conformal factor \( \rho \). Let \( e_{(j)} = \frac{\partial}{\partial x^j}, j = 1, \ldots, n \) be the standard basis vectors in \( \mathbb{R}^n \). It is easy to see that they are conformal Killing vector fields in the Euclidean metric; thus also conformal Killing vector fields for \( g \). Then \( u_{(j)} = \gamma^ae_{(j)}, j = 1, \ldots, n \) are conformal Killing vector fields for the metric \( \tilde{g} = \gamma^ag \). This implies \( u_{(j)}, j = 1, \ldots, n \) satisfies the conformal Killing equation

\[
Ku_{(j)}(y) = 0, \ y \in D, j = 1, \ldots, n.
\]

It is known that \( u_{(j)} \) is uniquely determined by the Cauchy data \( \{u_{(j)}(x', 0): x' \in \Gamma \} \) (see, e.g., [4, 11]). Thus we calculate the Cauchy data of \( u_{(j)} \) first.

Since we have already recovered the semigeodesic copy \( \tilde{g} \), we denote the dual vector of \( u_{(j)} \) by \( v_{(j)} \). In the meantime, we denote the dual vector of \( e_{(j)} \) under the metric \( \rho = gdx^2 \) by \( w_{(j)} \); then \( w_{(j)} = \rho dx^j \). In local coordinates, the equality \( u_{(j)} = \gamma^ae_{(j)} \) means (for covariant components)

\[
u_{(j)}(y) = w_{(j)} \frac{\partial}{\partial y^j} = \rho \frac{\partial \gamma^j(y)}{\partial y^i}.
\]

This observation is crucial in our reconstruction procedure. It relates the isometry \( \gamma \) to the conformal Killing vector fields on the semigeodesic copy \( D \). Thus at \( y^\alpha = t = 0 \),

\[
u_{(j)}(x', 0) = \rho(x') \frac{\partial \gamma^j}{\partial y^\alpha}(x', 0) = \rho(x') \frac{\partial x^j}{\partial y^\alpha}, \quad \alpha = 1, \ldots, n - 1.
\]

Since \( \rho(x') \) and \( \gamma(x', 0) \) are known for \( x' \in \Gamma \), \( u_{(j)}(x', 0) \) are determined. To determine the value of \( u_{(j)} \) at \( t = 0 \), notice that \( \tilde{\gamma}'(0) = \nu(x') \), so

\[
u_{(j)}(x', 0) = \rho(x') \frac{\partial \gamma^j}{\partial t}(x', 0) = \rho(x') \nu^j(x').
\]

However, \( \nu \) is a unit vector w.r.t. metric \( g = \rho dx^2 \), if we denote the inward unit normal vector on \( \partial M \) w.r.t. the Euclidean metric by \( \nu_0 \). Then \( \nu = \frac{1}{\sqrt{\rho}} \nu_0 \); i.e.,

\[
u_{(j)}(x', 0) = \sqrt{\rho(x')} \nu_0^j(x').
\]

Fortunately, the Cartesian coordinates of the hypersurface \( \Gamma \) are given; thus \( \nu_0 \) as the normal vector to \( \Gamma \) is known. Together with the knowledge of \( \rho|_\Gamma \), we recover \( u_{(j)}|_{t=0} \) too.

From the Cauchy data, we uniquely recover the conformal Killing vector fields \( u_{(j)} = \gamma^ae_{(j)} \) and equivalently the dual vector \( v_{(j)} \). Now by defining \( v = (v^1, \ldots, v^n) \),

\[
u^j(x', t) := u_{(j)}(x', t) = \rho(\gamma(x', t)) \frac{\partial \gamma^j(x', t)}{\partial t},
\]

we have \( v = \rho(\gamma) \tilde{\gamma}' \). In the meantime, notice that

\[
|v|^2 = \rho^2(\gamma) |\tilde{\gamma}'|^2 = \rho(\gamma) \quad \text{(since} |\gamma'|_k = 1 \text{)};
\]

we obtain

\[
\tilde{\gamma}' = \frac{v}{|v|^2}.
\]
This implies
\[ \gamma(x', t) = \int_0^t \dot{\gamma}_x(t) \, dt + x' = \int_0^t \frac{v(x', t)}{|v|} \, dt + x'; \]
i.e., we recover the geodesics \( \gamma_x(t) \) and therefore the diffeomorphism \( \gamma : D \to \gamma(D) \), namely, the range \( \gamma(D) \). Moreover, by (3)
\[ \rho(\gamma(x', t)) = |v(x', t)|^2, \]
the conformal factor \( \rho|_{\gamma(D)} \) is determined, and this finishes the proof of the main theorem.

Acknowledgments

L Pestov was partly supported by grant 12-01-00260-a, RFBR; G Uhlmann was partly supported by NSF and a Simons Fellowship; H Zhou was partly supported by NSF.

References

[1] Anikonov Yu E 1978 Some methods of investigations of multidimensional inverse problems for differential equations Nauka Sibirsk. Otdel. Novosibirsk (in Russian)
[2] Beig R, Chruściel P and Schoen R 2005 KIDs are non-generic Annales Henri Poincaré 6 155–94
[3] Belishev M I and Glasman A K 2001 Dynamical inverse problem for the Maxwell system: recovering the velocity in a regular zone (the BC-method) St.-Petersburg Math. Journal 12 279–316
[4] Dairbekov N and Sharafutdinov V 2011 On conformal Killing symmetric tensor fields on Riemannian manifolds Siberian Ad. Math. 21 1–41
[5] De Hoop M V, Holman S F, Iversen E, Lassas M and Ursin B 2014 Reconstruction of a conformally Euclidean metric from local boundary diffraction travel times SIAM J. Math. Anal. 46 3705–26
[6] Herglotz G 1905 Über die Elastizität der Erde bei Berücksichtigung ihrer variablen Dichte Zeitschr. für Math. Phys. 52 275–99
[7] Kachalov A, Kurylev Y and Lassas M 2001 Inverse Boundary Spectral Problems (Monographs and Surveys in Pure and Applied Mathematics) (Boca Raton: Chapman and Hall/CRC)
[8] Kurylev Y 1997 Multidimensional Gel’fand inverse problem and boundary distance map Inverse Problems Related with Geometry ed H Soga pp 1–15
[9] Kurylev Y, Lassas M and Uhlmann G 2010 Rigidity of Broken Geodesic Flow and Inverse Problems Am. J. Math. 132 529–62
[10] Linnurinen T and Salo M 2012 Nowhere conformally homogeneous manifolds and limitig Carleman weights Inverse Problems and Imaging 6 523–30
[11] Lionheart W 1997 Conformal uniqueness results in anisotropic electrical impedance imaging Inverse Problems 13 125–34
[12] Reshetnyak Yu 1994 Stability Theorems in Geometry and Analysis Mathematics and its Applications 304 (Dordrecht: Kluwer) p 394
[13] Sharafutdinov V 1994 Integral Geometry of Tensor Fields (Utrecht: VSP)
[14] Stefanov P and Uhlmann G 1998 Rigidity for metrics with the same lengths of geodesics Math. Res. Lett. 5 83–96
[15] Stefanov P and Uhlmann G 2008 Boundary and lens rigidity, tensor tomography and analytic microlocal analysis Algebraic Analysis of Differential Equations (Berlin: Springer) pp 275–93
[16] Stefanov P, Uhlmann G and Vasy A Boundary rigidity with partial data (arXiv:1306.2995v4.)
[17] Uhlmann G and Vasy A The inverse problem for the local geodesic ray transform Adv. Math. to appear
[18] Uhlmann G 2014 Inverse problems: seeing the unseen Bulletin Math. Sci. 4 209–79
[19] Wiebert H and Zoeppritz K 1907 Über Erdbebenwellen Nachr. Koenigl Gesellschaft Wiss, Goettingen 4 415–549
[20] Zhou H The inverse problem for the local ray transform (arXiv:1304.7023v2).