Tomographic probability representation for quantum fermion fields

V. A. Andreev\textsuperscript{1}, M. A. Man’ko\textsuperscript{1}, V. I. Man’ko\textsuperscript{1}, Nguyen Hung Son\textsuperscript{2}, Nguyen Cong Thanh\textsuperscript{2}, Yu.P.Timofeev\textsuperscript{1} and S.D.Zakharov\textsuperscript{1}

\textsuperscript{1} P. N. Lebedev Physical Institute
Leninskii Prospect 53, Moscow 119991, Russia

\textsuperscript{2} Institute of Physics
10 Dao Tan Street, Ha Noi, Viet Nam

Abstract

Tomographic probability representation is introduced for fermion fields. The states of the fermions are mapped onto probability distribution of discrete random variables (spin projections). The operators acting on the fermion states are described by fermionic tomographic symbols. The product of the operators acting on the fermion states is mapped onto star-product of the fermionic symbols. The kernel of the star-product is obtained. The antisymmetry of the fermion states is formulated as the specific symmetry property of the tomographic joint probability distribution associated with the states.

1 Introduction

In quantum mechanics the states are described (in standard formulation) by vectors in Hilbert space \cite{1} (wave function \cite{2}) or density operators \cite{3}, \cite{4}. In the quantum field theory one considers not only states of the fixed numbers of particles but also the process of creation and annihilation of the particles (see \cite{5}, \cite{6}). The formulation of quantum field theory is based on the second quantization procedure (see, e.g \cite{7}). The
fields are divided into two classes: boson fields and fermion fields. The states of the boson fields must be symmetric with respect to permutation of the particles (bosons). But the states of the fermion fields must be antisymmetric ones. Recently probability representation of quantum states (tomographic probability representation) in quantum mechanics was introduced [8], (see also recent review [9]). In quantum field theory the tomographic probability representation was studied for bosons still now in few works [10], [11], [12]. The more systematic tomographic approach to study quantum boson fields (scalar fields) was suggested in our previous work [13]. In this work the star-product formalism for tomographic probabilities [14] describing the boson field states was presented. The aim of our work is to suggest the tomographic probability representation for fermion fields. We consider the identical spin-1/2 particles and their states in framework of free fields. To do this we review and apply spin-tomographic description of fermion states in quantum mechanics [15], [16], [17]. Then we extend the spin-tomographic approach to the fermion fields and their states description. It is worthy to note that the explicit matrix representation of fermion anticommutation relation was discussed in [18]. In our construction we use and develop the approach given in this book.

The paper is organized as follows. In Sec. 2 we present the spin-tomographic scheme for one and two particles states. In Sec. 3 the star-product formalism is developed for the spin-tomographic symbols of the operators acting in the space of fermion states. In Sec. 4 matrix representation of fermionic operators is given. In Sec.5 the important operators like creation and annihilation operators as well as vacuum state density matrix will be studied in the tomographic framework. The conclusion and perspective are given in Sec. 6.
2 State of spin-1/2 particle

First we study the pure state of spin-1/2 particle in quantum mechanics. The state is described by a spinor

$$|\Psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix},$$

(1)

where $a, b$ are complex numbers and normalization condition holds

$$|a|^2 + |b|^2 = 1.$$  (2)

The tomogram of the state (1) is determined by the probability vector $\vec{\omega}$ with the two component $\omega(+\frac{1}{2}, u), \omega(-\frac{1}{2}, u)$ where

$$\omega(m, u) = |\langle m | u | \Psi \rangle|^2,$$

$$m = \pm \frac{1}{2},$$

(3)

and the unitary matrix $u$ rotating the spinor (1) in terms of Euler angels $\varphi, \theta$ and $\psi$ is

$$u = \left( \begin{array}{cc} \cos \frac{\varphi}{2} e^{i(\varphi+\psi)/2} & \sin \frac{\varphi}{2} e^{i(\varphi-\psi)/2} \\ -\sin \frac{\varphi}{2} e^{-i(\varphi-\psi)/2} & \cos \frac{\varphi}{2} e^{-i(\varphi+\psi)/2} \end{array} \right).$$

(4)

For mixed state with density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix},$$

(5)

the tomogram is given by diagonal elements of rotated density matrix

$$\omega(m, u) = (u \rho u^\dagger)_{mm}.$$  (6)

As was sown in \[15],\[16\] the tomogram determines the density matrix. Also the tomogram satisfies the normalization condition

$$\sum_{m=-1/2}^{1/2} \omega(m, u) = 1,$$

(7)
for arbitrary values of Euler angles. For two particles with spin 1/2 the tomogram is introduced as joint probability distribution of two random spin projection $m_1, m_2$

$$\omega(m_1, m_2, u) = (u\rho(1, 2)u^+)_{m_1 m_2, m_1 m_2}. \quad (8)$$

Here $\rho(1, 2)$ is density 4×4 matrix of the two particle spin state. The indices $m_1, m_2 = \pm \frac{1}{2}$ and correspond to pure spin state $| m_1 m_2 \rangle$ basis. Thus

$$\rho(1, 2)_{m_1 m_2, n_1 n_2} = \langle m_1 m_2 | \hat{\rho}(1, 2) | n_1 n_2 \rangle. \quad (9)$$

For pure state of two particles $| \Psi \rangle$ the tomogram according to formulaes (8), (9) reads

$$\omega(m_1, m_2, u) = |\langle \Psi | u | m_1 m_2 \rangle|^2. \quad (10)$$

The unitary 4×4-matrix $u$ is tensor product of two 2×2-matrices (11) each depending on their Euler angles

$$u = u_1 \otimes u_2. \quad (11)$$

It is clear that for multiparticle state construction of tomographic probability distribution (join probability distribution) is straightforward. The tomograms (8), (10) are nonnegative and satisfy the normalization condition

$$\sum_{m_1 = -1/2}^{1/2} \sum_{m_2 = -1/2}^{1/2} \omega(m_1, m_2, u) = 1, \quad (12)$$

for arbitrary Euler angles determining matrices $u_1$ and $u_2$.

## 3 Star-product for spin-1/2 operators

The quantum operators can be mapped onto the functions [7], [19], [20]. The product of the functions called symbols of the operators is the star-product which is associative but not commutative one. The product is determined by a kernel. For standard pointwise product of functions the kernel is local one. In general case the product
The kernel is non-local. The scheme of the star-product construction is the following \[14\]. If one has an operator \( \hat{A} \) and specific pair of operators \( \hat{U}(\vec{x}) \) called dequantizer and \( \hat{D}(\vec{x}) \) called quantizer satisfying relation

\[
Tr \hat{D}(\vec{x})\hat{U}(\vec{x}') = \delta(\vec{x} - \vec{x}'),
\]  

the symbol of operator \( \hat{A} \) is defined as

\[
f_{A}(\vec{x}) = Tr \hat{A} \hat{U}(\vec{x}).
\]

The operator is reconstructed from its symbol by means of the quantizer

\[
\hat{A} = \int f_{A}(\vec{x}) \hat{D}(\vec{x}) d\vec{x}.
\]

In (13)-(15) the coordinates \( \vec{x} \) can be any set of continuous or discrete variables \( \vec{x} = (x_1, x_2, \ldots, x_n) \) and integration in (15) is understood as integration over continuous variables and summation over the discrete variables. The symbols of the operators \( f_{A}(\vec{x}) \) and \( f_{B}(\vec{x}) \) provide the symbol of operator \( f_{C}(\vec{x}) \) if

\[
\hat{C} = \hat{A} \hat{B},
\]

using the star-product definition

\[
f_{C}(\vec{x}) := f_{AB}(\vec{x}) = Tr \hat{A} \hat{B} \hat{U}(\vec{x}).
\]

The symbol \( f_{C}(\vec{x}) \) can be presented in the integral form

\[
f_{C}(\vec{x}) = \int f_{A}(\vec{y}) f_{B}(\vec{z}) K(\vec{y}, \vec{z}, \vec{x}) d\vec{y} d\vec{z}.
\]

Here non-local kernel determining the star-product of symbols reads

\[
K(\vec{y}, \vec{z}, \vec{x}) = Tr (\hat{D}(\vec{y}) \hat{D}(\vec{z}) \hat{U}(\vec{x})).
\]

For spin-1/2 particle we present the tomographic star-product of the spin-tomograms of the operators. The coordinate \( \vec{x} = (m, \theta, \psi) \equiv (m, \vec{n}) \) contains three variables. The
discrete variable \( m = \pm \frac{1}{2} \) is the spin projection on z-axis. The continuous variables \( \theta \) and \( \varphi \) are coordinates of a point on the sphere \( 0 \leq \psi \leq 2\pi, 0 \leq \theta \leq \pi \). These coordinates can be described by unit vector \( \vec{n} = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta) \), \( \vec{n}^2 = 1 \).

The dequantizer \( \hat{U}(\vec{x}) = \hat{U}(m, \vec{n}) \) which is 2×2 matrix reads [21]

\[
\hat{U}(m, \vec{n}) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m \begin{pmatrix} \cos \theta & -e^{i\psi} \sin \theta \\ -e^{-i\psi} \sin \theta & -\cos \theta \end{pmatrix},
\]

and the quantizer \( \hat{D}(\vec{x}) = \hat{D}(m, \vec{n}) \) is

\[
\hat{D}(m, \vec{n}) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3m \begin{pmatrix} \cos \theta & -e^{i\psi} \sin \theta \\ -e^{-i\psi} \sin \theta & -\cos \theta \end{pmatrix}.
\]

The symbol of arbitrary operator is defined by equality

\[
f_A(m, \vec{n}) = Tr \hat{A} \hat{U}(m, \vec{n}),
\]

where the matrix of the operator is

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

i.e. the symbol reads

\[
f_A(m, \vec{n}) = \frac{1}{2} + m[A_{11} \cos \theta - e^{-i\psi} \sin \theta A_{12} + A_{21}(-e^{i\psi} \sin \theta) - A_{22} \cos \theta].
\]

One can check that condition (13) is given by symbolic formula

\[
\delta(\vec{x} - \vec{x}') = Tr \hat{D}(m_1, \vec{n}_1)\hat{U}(m_2, \vec{n}_2) = \frac{1}{2} + 6m_1m_2(\vec{n}_1\vec{n}_2).
\]

The symbol with delta-function in Eq.(25) is not usual Dirac delta function (or Kronecker one). It only means that the right-hand side of this equation (25) is equivalent to matrix element of identity operator acting in the set of tomograms.

The tomogram (3) of pure quantum state \( |\Psi\rangle \) is defined in framework of presented star-product scheme using the density operator \( \hat{\rho} = |\Psi\rangle\langle\Psi| \) and corresponding density matrix in (6).
The kernel of star-product reads [21]

\[ K(m_1 \vec{n}_1, m_2 \vec{n}_2, m_3 \vec{n}_3) = Tr(\hat{D}(m_1, \vec{n}_1) \hat{D}(m_2, \vec{n}_2) \hat{U}(m_3, \vec{n}_3)) \]

\[ = \frac{1}{4} + 3m_1 m_2 (\vec{n}_1 \vec{n}_2) + 9m_1 m_3 (\vec{n}_1 \vec{n}_3) + 9m_2 m_3 (\vec{n}_2 \vec{n}_3) \]  

(26)

\[ + 18im_1 m_2 m_3 (\vec{n}_1 \vec{n}_2 \vec{n}_3). \]

4 Matrix representation of fermion operators.

Let us construct matrix representation of the creation and annihilation fermion operators for \( N \) fermions. If \( N = 1 \) one can construct the 2×2 matrices

\[ a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_1^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]

(27)

which satisfy the relations

\[ a_1 a_1 + a_1 a_1 = 0, \]

\[ a_1^+ a_1^+ + a_1^+ a_1^+ = 0, \]

(28)

\[ a_1 a_1^+ + a_1^+ a_1 = 1, \]

which can be rewritten as

\[ a_i a_j + a_j a_i = \{a_i, a_j\} = 0, \]

\[ a_i^+ a_j^+ + a_j^+ a_i^+ = \{a_i^+, a_j^+\} = 0, \]

(29)

\[ a_i a_j^+ + a_j a_i^+ = \{a_i, a_j^+\} = \delta_{ij}, \]

where \( i, j = 1 \). If \( i, j = 1, 2 \) the relation (29) can be satisfied by 4×4 matrices

\[ a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_1^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \]

\[ a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a_2^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

(30)

If \( i, j = 1, 2, 3 \) the relations (29) are satisfied by the 8×8-matrices which we present in the form of tensor products

\[ a_1 = 1 \otimes 1 \otimes \sigma_+, \]

\[ a_2 = 1 \otimes \sigma_+ \otimes \sigma_z, \]

\[ a_3 = \sigma_+ \otimes \sigma_z \otimes \sigma_z. \]
and corresponding matrices $a_j^+$ associated with fermion creation operators.

For $N = 4$ i.e. $i,j = 1,2,3,4$ in (29) the construction of the matrices $a_j$ looks as follows

$$a_1 = 1 \otimes 1 \otimes 1 \otimes \sigma_+, \quad (32)$$

$$a_2 = 1 \otimes 1 \otimes \sigma_+ \otimes \sigma_z,$$

$$a_3 = 1 \otimes \sigma_+ \otimes \sigma_z \otimes \sigma_z,$$

$$a_4 = \sigma_+ \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z.$$

We introduced standard notations for $2 \times 2$ matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_- = (\sigma_+)^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (33)$$

The generalization of the construction for arbitrary number $N$ is straightforward, i.e.

$$a_i = 1^{(1)} \otimes \cdots \otimes 1^{(N-i)} \otimes \sigma_+^{(N-i+1)} \otimes \sigma_z^{(N-i+2)} \cdots \otimes \sigma_z^{(N)}. \quad (34)$$

Corresponding matrices $a_i^+$ are obtained by changing in (32) all $\sigma_+$ by $\sigma_-$ matrices.

The vector $| \text{vac} \rangle$ which satisfies the condition

$$a_j | \text{vac} \rangle = 0, j = 1,2,\ldots N, \quad (35)$$

has the form of tensor product

$$| \text{vac} \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (36)$$

For $N = 2$

$$| \text{vac} \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (37)$$

The density matrix of this state reads

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
The states
\[ |\ldots i \ldots j \ldots \rangle = a_i^+ a_j^+ |\text{vac}\rangle, \]  
(38)
are antisymmetric
\[ |\ldots i \ldots j \ldots \rangle = - |\ldots j \ldots i \ldots \rangle, \]  
(39)
by construction

5 Tomographic representation of Fermi-operators.

Let us express the creation and annihilation Fermi-operators in tomographic form. To do this we first express the separate factors ingredient in the tomographic form. There are three such factors, namely factors given by (33).

According to general construction the spin-tomographic symbols of the matrix factors are calculated using the diagonal matrix elements of the matrices \( u\sigma_u u^+, \; u\sigma_z u^+, \) \( u1u^+ \) as well as \( u\sigma_u u^+ \). We denote three symbols \( \Omega_+(m, \vec{n}) \), \( \Omega_z(m, \vec{n}) \), \( \Omega_1(m, \vec{n}) \) and \( \Omega_-(m, \vec{n}) \) respectively. By construction

\[ \Omega_+(m, \vec{n}) = \Omega_-(m, \vec{n})^*. \]  
(40)

We remind that \( m \) takes two values \( m = \pm 1/2 \) and the unit vector \( \vec{n} = (\sin\theta \cos\psi, \sin\theta \sin\psi, \cos\theta) \).

The symbol of identity operator reads

\[ \Omega_1(\pm 1/2, \vec{n}) = \Omega_1(-\pm 1/2, \vec{n}) = 1. \]  
(41)

One can check that the symbol of the operators \( \sigma_z \) has the following values

\[ \Omega_z(\pm 1/2, \vec{n}) = \cos\theta, \; \Omega_z(-\pm 1/2, \vec{n}) = -\cos\theta. \]  
(42)

The tomographic symbols of the operator \( \sigma_+ \) is obtained from the product of three matrices

\[ \Omega_+(\pm 1/2, \vec{n}) = (u\sigma_+ u^+)_1, \; \Omega_+(\pm 1/2, \vec{n}) = (u\sigma_+ u^+)_2. \]  
(43)
Straightforward calculation yields
\[
\Omega_+\left(\pm \frac{1}{2}, \vec{n}\right) = \frac{1}{2}\sin \theta e^{i\psi}, \quad \Omega_+\left(-\frac{1}{2}, \vec{n}\right) = -\frac{1}{2}\sin \theta e^{i\psi}.
\]

Using the obtained expression we can construct the tomographic symbols for all the cases of one, two, three, etc. fermions. For one fermion the tomographic symbols \(\Omega_a(m, \vec{n}), \Omega_a^+(m, \vec{n})\) and \(\Omega_1(m, \vec{n})\) read
\[
\Omega_a(m, \vec{n}) = \Omega_+(m, \vec{n}) = \begin{cases} 
\frac{1}{2}\sin \theta e^{i\psi}, m = \frac{1}{2} \\
-\frac{1}{2}\sin \theta e^{i\psi}, m = -\frac{1}{2}
\end{cases}
\]
\[
\Omega_a^+(m, \vec{n}) = (\Omega_a)^*(m, \vec{n}) = \begin{cases} 
\frac{1}{2}\sin \theta e^{-i\psi}, m = \frac{1}{2} \\
-\frac{1}{2}\sin \theta e^{-i\psi}, m = -\frac{1}{2}
\end{cases}
\]
\[
\Omega_1(m, \vec{n}) = \begin{cases} 
1, m = +\frac{1}{2} \\
1, m = -\frac{1}{2}
\end{cases}
\]

For two fermions we introduce the notations \(\vec{n}_j = (\sin \theta_j \cos \psi_j, \sin \theta_j \sin \psi_j, \cos \theta_j), a_j, a_j^+; j = 1, 2\) as well as \(m_j = \pm \frac{1}{2}\).

Thus we get the tomographic symbols \((j = 1, 2)\)
\[
\Omega_+^{(j)}(m_j, \vec{n}_j) = \begin{cases} 
\frac{1}{2}\sin \theta_j e^{i\psi_j}, m_j = \frac{1}{2} \\
-\frac{1}{2}\sin \theta_j e^{i\psi_j}, m_j = -\frac{1}{2}
\end{cases}
\]
\[
\Omega_-^{(j)}(m_j, \vec{n}_j) = \Omega_+^{(j)*}(m_j, \vec{n}_j).
\]
\[
\Omega_1^{(j)}(m_j, \vec{n}_j) = 1, m_j = \pm \frac{1}{2}.
\]
\[
\Omega_z^{(j)}(m_j, \vec{n}_j) = \begin{cases} 
\cos \theta_j, m_j = +\frac{1}{2} \\
-\cos \theta_j, m_j = -\frac{1}{2}
\end{cases}
\]

The tomographic symbols of creation and annihilation operators for two fermions read
\[
\Omega_{a_1}(m_1, \vec{n}_1, m_2, \vec{n}_2) = \Omega_1^{(1)}(m_1, \vec{n}_1)\Omega_+^{(2)}(m_2, \vec{n}_2) = \begin{cases} 
\frac{1}{2}\sin \theta_2 e^{i\psi_2}, m_1 = +\frac{1}{2}, m_2 = +\frac{1}{2} \\
-\frac{1}{2}\sin \theta_2 e^{i\psi_2}, m_1 = +\frac{1}{2}, m_2 = -\frac{1}{2} \\
\frac{1}{2}\sin \theta_2 e^{i\psi_2}, m_1 = -\frac{1}{2}, m_2 = +\frac{1}{2} \\
-\frac{1}{2}\sin \theta_2 e^{i\psi_2}, m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2}
\end{cases}
\]
\( \Omega_{a_2}(m_1, \bar{n}_1, m_2, \bar{n}_2) = \Omega_+(1)(m_1, \bar{n}_1)\Omega_1(2)(m_2, \bar{n}_2) = \begin{cases} 
 \frac{1}{2}\sin\theta_1 e^{i\psi_1}, m_1 = +\frac{1}{2}, m_2 = +\frac{1}{2} \\
 -\frac{1}{2}\sin\theta_1 e^{i\psi_1}, m_1 = +\frac{1}{2}, m_2 = -\frac{1}{2} \\
 \frac{1}{2}\sin\theta_1 e^{i\psi_1}, m_1 = -\frac{1}{2}, m_2 = +\frac{1}{2} \\
 -\frac{1}{2}\sin\theta_1 e^{i\psi_1}, m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2} \end{cases} \) (53)

For creation operators one has

\( \Omega_{a_1^+}(m_1, \bar{n}_1, m_2, \bar{n}_2) = \Omega_{a_1^+}(m_1, \bar{n}_1, m_2, \bar{n}_2), \) (54)

\( \Omega_{a_2^+}(m_1, \bar{n}_1, m_2, \bar{n}_2) = \Omega_{a_2^+}(m_1, \bar{n}_1, m_2, \bar{n}_2). \)

The formulas for spin-tomographic symbols of creation and annihilation operators in case of \( N \) fermions can be given in the following form

\[
\Omega_{a_j}(m_1, \bar{n}_1, m_2, \bar{n}_2, \cdots m_N, \bar{n}_N) = \Omega_+^{(N-j+1)}(m_{N-j+1}, \bar{n}_{N-j+1}) \prod_{k=N-j+2}^{N} \Omega_2^k(m_k, \bar{n}_k) \quad (55)
\]

The formulas for factors \( \Omega_+^s(m_s, \bar{n}_s) \) where \( s = N - j + 1 \) and for \( f_2^k(m_k, \bar{n}_k) \) are given by (47) and (50). The symbol of creation Fermi operator \( a_j^+ \) is given by (55) with the replacement \( \Omega_+^{(N-j+1)} \longrightarrow \Omega_+^{(N-j+1)}. \) We give tomogram of \( |\text{vac}\rangle \) for \( N = 2 \)

\[
\omega_{\text{vac}}(m_1, \bar{n}_1, m_2, \bar{n}_2) = \begin{cases} 
\cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2}, m_1 = +\frac{1}{2}, m_2 = +\frac{1}{2} \\
\cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2}, m_1 = +\frac{1}{2}, m_2 = -\frac{1}{2} \\
\sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2}, m_1 = -\frac{1}{2}, m_2 = +\frac{1}{2} \\
\sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2}, m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2} \end{cases} \quad (56)
\]

The star-product kernel of the tomographic symbols for \( N \) fermions is the product of kernels for each of the fermions which is given by formula (26). It follows from the fact that the dequantizers and quantizers for several fermions are given as the tensor product of quantizers and dequantizers for each of fermions.

6 Conclusion

To conclude we point out the main results of our work. The known anticommutation relations for fermion field operators were presented in explicit matrix realization. Using the matrix realization we mapped the fermion field operators onto spin tomographic symbols. The density matrix of fermion vacuum state of \( N \) fermions was
used to construct the tomographic probability representation of this state and the
tomogram of the fermion vacuum is calculated in explicit form. The creation and
annihilation fermion operators and their multiplication are given in the tomographic
framework by using explicit tomographic star-product kernel.

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