Bounding the trace function of a hypergraph with applications

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Abstract

An upper bound on the trace function of a hypergraph \( H \) is derived and its applications are demonstrated. For instance, a new upper bound for the VC dimension of \( H \), or \( \text{vc}(H) \), follows as a consequence and can be used to compute \( \text{vc}(H) \) in polynomial time provided that \( H \) has bounded degeneracy. This was not previously known. Particularly, when \( H \) is a hypergraph arising from closed neighborhoods of a graph, this approach asymptotically improves the time complexity of the previous result for computing \( \text{vc}(H) \). Another consequence is a general lower bound on the distinguishing transversal number of \( H \) that gives rise to applications in domination theory of graphs. To effectively apply the methods developed here, one needs to have good estimations of degeneracy, and its variation or reduced degeneracy which is introduced here.

1 Introduction and Summary

Many important combinatorial problems in computer science, mathematics, and operations research arise from the set systems or hypergraphs. We recommend [9] and thesis [2] as references on hypergraphs. Formally, a hypergraph \( H = (V, E) \) has the vertex set \( V \) and the edge set \( E \), where each \( e \in E \) is a subset of \( V \). We do not allow multiple edges in our definition of a hypergraph, unless explicitly stated. When multiple edges exist, we slightly modify the concept. Let \( S \subseteq V \) and \( e \in E \). The trace of \( e \) on \( S \) is \( e \cap S \). The restriction of \( H \) to \( S \), denoted by \( H[S] \), is the hypergraph on vertex set \( S \) whose edges are set of all distinct traces of edges in \( E \) on \( S \). \( H[S] \) is also referred to as the induced subhypergraph of \( H \) on \( S \). A Pseudo induced subhypergraph on the vertex set \( S \) is obtained from \( H \) by removing the set \( V - S \) and the set of all edges of \( H \) that have non-empty intersection with \( V - S \). Note that any edge of such hypergraph is an edge \( e \) of \( H \) if \( e \subseteq S \). \( S \) is shattered in \( H \), if any \( X \subseteq S \) is a trace. Thus if \( S \) is shattered, then it has \( 2^{|S|} \) traces, that is, \( H[S] \) has \( 2^{|S|} \) edges. The VapnikChervonenkis (VC) dimension of a hypergraph \( H \), denoted by \( \text{vc}(H) \) is the cardinality of the largest subset of \( V \) which is shattered in \( H \). It was originally introduced for its applications in statistical learning theory [29] but has shown to be of crucial importance in combinatorics and discrete geometry [12]. Let \( S \subseteq V \), then, \( S \) is a transversal, or a hitting set, if \( e \cap S \neq \emptyset \), for all \( e \in E \). A set \( S \) is a distinguishing set if any two distinct edges of \( H \) have different traces on (intersections with) \( S \). Let \( dt(H) \) denote the size of a smallest distinguishing transversal set in \( H \). Note that if \( S \) is a smallest distinguishing transversal set, then it can not have an empty trace on it.

For any \( x \in V \), let degree of \( x \), denoted by \( d_H(x) \), denote the number of edges that contain \( x \). We denote by \( \delta(H) \), the smallest degree of any vertex in \( H \).

Any definition for a hypergraph, readily extends to a subhypergraph. A hypergraph \( I \) is a subhypergraph of \( H \) if it can be obtained by deleting some edges in \( H[S] \) for some \( S \subseteq V \). (Note that there are subhypergraphs of \( H \) that may not be induced.) Particularly, for any \( x \in S \), the degree of \( x \) in \( I \) is denoted by \( d_I(x) \). Furthermore \( \delta(I) \) denotes the minimum
degree of $I$. The degeneracy of $H$, denoted by $\delta(H)$, is the largest minimum degree of any subhypergraph of $H$. Observe that one can define $\hat{\delta}(H)$ as the largest minimum degree of any induced subhypergraph of $H$, since the addition of new edges to a hypergraph does not decrease the degrees of vertices. The pseudo degeneracy of $H$, denoted by $\delta^*(H)$, is the largest minimum degree any pseudo induced subhypergraph of $H$. Finally, the reduced degeneracy of $H$, denoted by $\tilde{\delta}(H)$ is the largest pseudo degeneracy of any induced subhypergraph of $H$.

**Observation 1.1.** For any induced subhypergraph $I$ of $H$, one has $\delta^*(I) \leq \hat{\delta}(I) \leq \tilde{\delta}(I)$, consequently, $\delta^*(H) \leq \hat{\delta}(H) \leq \tilde{\delta}(H)$.

The trace function of $H$ denoted by $T[H,k]$ is the largest number of traces of $H$ on a set $S$, $|S| = k$. Unless otherwise stated, we assume that $T[H,k]$ counts the number of non-empty traces only.

A powerful tool in studying hypergraph problems with a very broad range of applications is the Sauer–Shelah Lemma [23, 24]. The Lemma asserts for any hypergraph $H$ with $vc(H) = d$ and any $k \geq 0$, one has:

$$T[H,k] \leq \sum_{i=0}^{d} \binom{k}{i} = O(k^d)$$

(1)

The concept of a trace function is also studied as the Max Partial VC Dimension [13]. Particularly, it was shown in [13] that

$$T[H,k] \leq k(\Delta(H) + 1)/2 + 1$$

(2)

Our main result in this paper is Lemma 2.1 which is an upper bound on $T[H,k]$. A simple consequence of this upper bound is $T[H,k] \leq k\hat{\delta}(H)$. This upper bound is within a multiplicative factor of $\hat{\delta}(H)$ form the lower bound of $L(H,k) = \min\{|E|, k+1\}$ (when $H$ does not have multiple edges) that has also been recently constructed in [13] ; Thereby, $T(H,k)$ is proportional to $k$, provided that reduced degeneracy of $H$ is “small”, and hence in light of our upper bound for $T(H,k)$, the lower bound $L(H,k)$ (constructed in [13]), actually approximates $T(H,k)$ (for any $k$) to within a factor of $\hat{\delta}(H)$ which is an improvement of the factor $(\Delta(H) + 1)/2 + 1$ as authors stated in [13].

### 1.1 Connections to VC dimension

It is easy to verify that $vc(H) \leq \log(|E|)$ for any hypergraph $H$. It was previously known that when $H$ has an explicit representation by an $m \times n$ incident matrix, $vc(H)$ can be computed in $n^{O(\log(n))}$ [5]. Also, the decision version of the problem is LOGNP-complete [30] and remains in this complexity class for neighborhood hypergraphs of graphs [19]. A simple and immediate consequence of our work is that $vc(H) \leq \log(\hat{\delta}(H)) + 1$ (which was not known before) and hence $vc(H)$ can be computed in $n^{O(\log(\hat{\delta}(H)))}$. Consequently, $vc(H)$ can be computed in polynomial time for hypergraphs of bounded degeneracy which had not been known. Moreover, these results give rise to an algorithm for computing $vc(H)$ in $n^{O(\log^3(\Delta(G)))}$ time, when $H$ is the set of all closed neighborhoods of vertices of a graph $G$ with maximum degree $\Delta(G)$. This is an asymptotic improvement of the best known time complexity of $O(n^{2\Delta(G)})$ for solving the problem which was derived in [19].

### 1.2 Connections to domination theory

We recommend [6] as a reference on domination theory. For a graph $G = (V,E)$ and a vertex $x$, $N(x)$ denotes the open neighborhood of $x$, that is the set of all vertices adjacent to $x$, not
including \( x \). The closed neighborhood of \( x \) is \( N[x] = N(x) \cup \{x\} \). The closed (open) neighborhood hypergraph of an \( n \) vertex graph \( G \) is a hypergraph on the same vertices as \( G \) whose edges are all \( n \) closed (open) neighborhoods of \( G \). A subset of vertices \( S \) in \( G \) is a dominating set [3], if for every vertex \( x \) in \( G \), \( N[x] \cap S \neq \emptyset \). \( S \) is a total or open domination set [7] if, \( N(x) \cap S \neq \emptyset \). A subset of vertices \( S \) is locative in \( G \), if for every two distinct vertices \( x, y \in V - S \), one has \( N(x) \cap S \neq N(y) \cap S \). \( S \) is totally locative in \( G \), if for every two distinct vertices \( x, y \in V \), one has \( N(x) \cap S \neq N(y) \cap S \). A subset \( S \) of vertices in \( G \) is a locating dominating (locating total dominating) if it is a dominating (total dominating) set and it is also a locative set [25, 26]. \( S \) is an identifying code if it is a dominating set and for every two distinct vertices \( x, y \in V \), one has \( N[x] \cap S \neq N[y] \cap S \) [18]. \( S \) is an open locating domination, if \( S \) is a totally domination set and also totally locative in \( G \) [27].

Let \( \gamma^{LD}(G) \) and \( \gamma^{ID}(G) \) denote the sizes of a smallest Location domination and Identifying code sets in \( G \), respectively. Let \( \gamma^{OLD}(G) \) denote the size of a smallest open location domination in \( G \). Computing \( \gamma^{LD}(G) \), \( \gamma^{ID}(G) \) and \( \gamma^{OLD}(G) \) are known to be NP-hard problems and hence estimations of these parameters or their computational complexities have been an active area of research [11, 27, 28, 21, 20, 13, 14, 15, 16, 17, 3]. Recall that the distinguishing transversal number of \( H \), denote by \( dt(H) \), is the minimum size of any distinguishing transversal set [11]. A consequence of our upper bound for \( T(H, k) \), is that for any hypergraph \( H = (V, E) \) and any integer \( 0 \leq j < dt(H) \) one has \( dt(H) \geq \frac{|E| - T(H, j)}{\delta(H)} + j \); By properly applying this result to suitable neighborhood hypergraphs of a graph, one obtains some general lower bounds on \( \gamma^{LD}(G), \gamma^{ID}(G) \) and \( \gamma^{OLD}(G) \). For a specific application, one needs to determine the exact value or a good estimate for \( \delta(H) \) or \( \delta(H) \), and this can become a challenging task.

This paper is organized as follows. Section two contains our main lemma as well as the lower bound on distinguishing transversal number. Section three contains the applications to VC dimension. Section four contains the applications to domination theory by deriving general lower bounds for \( \gamma^{LD}(G), \gamma^{ID}(G) \) and \( \gamma^{OLD}(G) \). Additionally, we show in case of trees, our general approach gives rise to lower bounds that match some the best known results, or come close to them.

We finish this section by stating a folklore result for computing degeneracy and pseudo degeneracy of a hypergraph. The properties of the output of algorithm will help to establish some of our claims more easily.

**Theorem 1.1.** Let \( H = (V, E) \) be a hypergraph, then \( \hat{\delta}(H) \) can be computed in \( O(|V| + \sum_{e \in E} |e|) \) time.

**Proof.** For \( i = 1, 2, \ldots, n \), let \( x_i \) be a vertex of degree \( d_i = d_{H_i}(x_i) = \delta(H_i) \) in the induced subhypergraph \( H_i = H[V_i] \) on the vertex set \( V_i = V - \{x_1, x_2, \ldots, x_{i-1}\} \). Let \( d = \max\{d_i, i = 1, 2, \ldots, n\} \). We claim that \( \delta(H) = d \). Clearly, \( \hat{\delta}(H) \geq d \), and it suffices to show that \( \hat{\delta}(H) \leq d \). Now let \( I \) be any (induced) subhypergraph of \( H \), and let \( j \) be the smallest integer so that \( x_j \) is a vertex of \( I \). Then \( d_I(x_j) \leq d_j = \delta(H_j) \leq d \). Thus, \( \delta(I) \leq d \), and consequently, \( \hat{\delta}(H) \leq d \) as stated. Details of deriving time complexity that include representation of \( H \) as a bipartite graph and utilization of elementary data structures are omitted.

For a subhypergraph \( I = (U, F) \) of \( H \), and any \( x \in U \), let \( F_x \) denote the set of edges in \( F \) containing \( x \). The next result almost copies Theorem [11].

**Theorem 1.2.** Let \( H = (V, E) \), be a hypergraph, then, \( \delta^*(H) \) can be computed in \( O(|V| + \sum_{e \in E} |e|) \) time.

**Proof.** For \( i = 1, 2, \ldots, n \), let \( x_i \) be a vertex of degree \( d_i = d_{H_i}(x_i) = \delta(H_i) \) in the subhypergraph \( H_i \) on the vertex set \( V_i = V - \{x_1, x_2, \ldots, x_{i-1}\} \) and edge set \( E_i = E - \{E_{x_1}, E_{x_2}, \ldots, E_{x_{i-1}}\} \).
Let \( d = \max\{d_i, i = 1, 2, \ldots, n\} \). Clearly, \( \delta^*(H) \geq d \). Now let \( I \) be any pseudo induced subhypergraph of \( H \), and let \( j \) be the smallest integer so that \( x_j \) is a vertex of \( I \). Then, vertex set of \( I \) does not contain \( x_i, i = 1, 2, \ldots, j - 1 \); Consequently, the edge set of \( I \) is a subset of \( E_j \). Then \( d_I(x_j) \leq d_j = \delta(H_j) \leq d \) proving the claim. Details of deriving time complexity that include representation of \( H \) as a bipartite graph and utilization of elementary data structures are omitted. □

**Remark 1.1.** The sequences \( d_1, d_2, \ldots, d_n \) generated in Theorems \[1.1\] and \[1.2\] are called the degeneracy sequence, and pseudo degeneracy sequence, respectively.

### 2 Main lemma

For a subhypergraph \( I = (U, F) \) of \( H \), and any \( x \in U \), let \( F_x \) denote the set of edges in \( F \) containing \( x \).

**Lemma 2.1.** Let \( H = (V, E) \), let \( S \subseteq V, |S| = k \), and let \( I = H[S] = (S, F) \) be the restriction of \( H \) to \( S \). For \( i = 1, \ldots, k \), let \( x_i \) be a vertex in subhypergraph \( I_i \) on the vertex set \( S_i = S - \{x_1, x_2, \ldots, x_{i-1}\} \) and edge set \( F_i = F - \{F_{x_1}, F_{x_2}, \ldots, F_{x_{i-1}}\} \), and let \( k, j, l \geq 0 \) be integers with \( k = l + j \). Then,

\[ |F| = \sum_{i=1}^{k} |F_{x_i}| = \sum_{i=1}^{l} d_{I_i}(x_i) \]  
\[ = \sum_{i=1}^{l} d_{I_i}(x_i) + |F_{i+1}| \]  
\[ \leq \sum_{i=1}^{l} d_{I_i}(x_i) + T[H, j]. \]  

Consequently,

\[ T[H, k] \leq \delta^*(I)l + T[H, j] \]  
\[ \leq \delta^*(I)k \]  
\[ \leq \delta(H)k \]  

**Proof.** For \[3\] observe that \( F = \bigcup_{i=1}^{k} F_{x_i} \), that for \( i = 1, 2, \ldots, k \), \( F_{x_i} \)'s are disjoint and \( |F_{x_i}| = d_{I_i}(x_i) \). For \[4\] note that \( F_{i+1} = \bigcup_{i=l+1}^{k} F_{x_i} \). Next, note that the hypergraph \( I_{i+1} \) has the vertex set \( S_{i+1} = \{x_i, x_{i+1}, \ldots, x_k\} \), thus, \( |S_{i+1}| = k - l = j \). Consequently, \[5\] follows, since \( |F_{i+1}| \leq T[H, j] \). For \[6\], for \( i = 1, 2, \ldots, k \), let \( x_i \) be a vertex of minimum degree in \( I_i \), that is \( d_{I_i}(x_i) = \delta(I_i) \), note that \( \delta(I_i) \leq \delta^*(I) = \max\{\delta(I_i), i = 1, 2, \ldots, k\} \) (by Theorem \[1.2\] and use \[5\]). Now set \( j = 0 \) to obtain \[7\] and note that \( \delta^*(I) \leq \delta(H) \) to obtain \[8\]. □

**Remark 2.1.** Note that \( S_1 = S - \{x_0\} = S - \emptyset = S \), and similarly \( F_1 = F \), in the above Lemma.

**Theorem 2.1.** For any hypergraph \( H = (V, E) \), and any integer \( 0 \leq j \leq dt(H) \), one has

\[ dt(H) \geq \frac{|E| - T[H, j]}{\delta(H)} + j. \]

Consequently,

\[ dt(H) \geq \frac{|E| - 2^j + 1}{\delta(H)} + j. \]
Proof. Let $S, |S| = dt(H)$ be the smallest cardinality distinguishing transversal set; Thus $S$ must have exactly $|E|$ non empty distinct traces, that is, $T(H, d(H)) = |E|$. Now apply Lemma 2.1, we have $|E| \leq \delta^*(H[S]) (dt(H) - j) + T[H, j]$ which proves the main claim, since $\delta^*(H[S]) \leq \delta(H)$. To verify the second claim note that $T[H, j] \leq 2^j - 1$. \[ \Box \]

## 3 Applications to VC dimension

**Theorem 3.1.** Let $H = (V, E), |V| = n$, then, $vc(H) \leq \log(\delta(H)) + 1$. Consequently, for any $n$ vertex hypergraph $H$, $vc(H)$ can be computed in $n^{O(\log(\delta(H)))}$ time. Particularly, if $H$ is the closed neighborhood hypergraph of an $n$ vertex graph with maximum degree $\Delta$, then $vc(H)$ can be computed in $n^{2O(\log^2(\Delta))}$ time.

**Proof.** Let $S, |S| = d$ be the largest shattered set in $H$. We apply Lemma 2.1 with $j = d - 1$. Thus, $2^d - 1 = T(H, d) \leq \delta(H)(d - d + 1) + 2^{d - 1} - 1$ which gives $d \leq \log(\delta(H)) + 1$ as claimed. To compute $vc(H)$, one can represent $H$ as its incidence matrix form, requiring $O(nm)$ space, or in $O(n^2 \delta(H))$ space, where $m$ is the number of edges of $H$, by since by Lemma 2.1, with $k = n$ one has $m \leq n \delta(H)$. Now can one find $vc(H)$ by exhaustive enumeration. Note that, in doing so the largest shattered subset has size $O(\log(\delta))$; Hence in $n^{O(\log(\delta(H)))}$ time, one can compute $vc(H)$. To prove the claim when $H$ is the closed neighborhood hypergraph, note that $\delta(H) \leq \Delta(G) + 1$, and hence $vc(H) = O(\log(\Delta(G)))$. Since the largest shattered set must be contained in the closed neighborhood of one vertex of $G$, the enumeration algorithm takes $n\Delta(G)^{O(\log(\Delta(G)))}$ or in $n^{2O(\log^2(\Delta(G)))}$ time. \[ \Box \]

**Remark 3.1.** Note that the enumeration algorithm in Theorem 3.1 does not require knowing $\delta(H)$, although $\delta(H)$ can be computed in polynomial time. Also note the run time of $n^{2O(\log^2(\Delta(G)))}$ for computing VC dimension of neighborhood system of graphs compares favorable with the time complexity of $O(n^{2\Delta(G)})$ derived in [19]. \[ \Box \]

## 4 Applications to domination theory

**Theorem 4.1.** Let $G$ be an $n$ vertex graph with closed and open neighborhood hypergraphs $H$ and $H^o$, respectively, let $\delta^{**}(H) = \min(\delta(H), \delta(H^o))$. Then the following hold for any $0 \leq j \leq \gamma^{LD}$ in (i), $0 \leq j \leq \gamma^{OLD}$ in (ii) and $0 \leq j \leq \gamma^{LD}$ in (iii), where $H$ and $H^o$ do not have multiple edges in (ii) and (iii), respectively.

(i) \[ \gamma^{LD}(G) \geq \frac{n + \delta^{**}(H, j - T[H, j])}{\delta^{**}(H) + 1}. \]

(ii) \[ \gamma^{LD}(G) \geq \max\{ \frac{n - T[H, j]}{\delta(H)} + j, \frac{n + \delta^{**}(H, j - T[H, j])}{\delta^{**}(H) + 1}\}. \]

(iii) \[ \gamma^{OLD}(G) \geq \max\{ \frac{n - T[H, j]}{\delta(H^o)} + j, \frac{n + \delta^{**}(H, j - T[H, j])}{\delta^{**}(H) + 1}\}. \]

**Proof.** For (i), let $S$ be the smallest cardinality locative dominative set in $G$. Now, let $H^1 = (V, E^1)$, where $E^1 = \{N(x) | x \in V - S\}$ and $H^2 = (V, E^2)$ where $E^2 = \{N[x] | x \in V - S\}$. Note that for $i = 1, 2$, $T(H^i, |S|) = n - |S| \leq \delta(H^i)(|S| - j) + T[H^i, j]$ where last inequality is obtained by the application of Lemma 2.1. Furthermore, $H^1$ is a subhypergraph of $H^o$, and $H^2$ is a subhypergraph of $H$. Consequently, $\delta(H^1) \leq \delta(H^o)$ and $\delta(H^2) \leq \delta(H)$. It follows that $n - |S| \leq \delta^{**}(H)(|S| - j) + T[H, j]$. To finish the proof note that $LD(G) = |S|$, and do the algebra.
For (iii), note that $\gamma_{ID}(G) \geq \gamma_{LD}(G)$ and hence the lower bound in (i) is also a lower bound for $\gamma_{ID}(G)$. To complete the proof, observe that $S$ is an identifying code set in $G$, if and only if $S$ is a distinguishing transversal in $H$. Thus, $dt(H) = \gamma_{ID}(G)$. Now apply Theorem 2.1. Similarly for (iii) note that $\gamma_{OLD}(G) \geq \gamma_{LD}(G)$, and that, $S$ is an totally dominative and totally locative set in $G$, if and only if, $S$ is a distinguishing transversal set in $H'$ and thus, $dt(H') = \gamma_{OLD}(G)$. Now apply Theorem 2.1.

□

Remark 4.1. Let $G$ be an $n$ vertex graph of maximum degree $\Delta(G)$ with closed and open neighborhood hypergraphs $H$ and $H^o$, respectively. Then clearly $\hat{\delta}(H) \leq \Delta(G) + 1$ and $\hat{\delta}(H^o) \leq \Delta(G)$, since the largest sets in $H$ and $H^o$ are of cardinalities $\Delta(G) + 1$ and $\Delta(G)$, respectively. As we will see, one can get much stronger results in trees.

Remark 4.2. Let $L$ denote the set of leaves and in a tree $T$, and note that after removal all vertices in $L$ from $T$ we obtain another tree $T'$. Let $S$ denote the set of all leaves of the tree $T'$. Then each vertex in $S$ is a support vertex in $T$ and is called a canonical support vertex in $T$.

Theorem 4.2. Let $T$ be a $n \geq 2$ vertex tree with closed and open neighborhood hypergraphs $H$ and $H^o$, respectively, then the following hold.

(i) $\hat{\delta}(H) \leq 3$.
(ii) $\hat{\delta}(H^o) \leq 2$.
(iii) $\hat{\tilde{\delta}}(H^o) \leq 2$.
(iv) $\delta^s(H) \leq 2$.
(v) $\delta^s(H^o) \leq 2$.

Proof. For $n \leq 2$, the claims are valid. Now for $n \geq 3$ note that for (i) for any vertex $x$, $d_H(x)$ equals degree of $x$ in $T$ plus one, and hence $d_H(x) = \delta(H) - 2$, if $x$ is a leaf in $H$. Now apply Theorem 1.1, and let $d_1, d_2, \ldots, d_n$, or the sequence of numbers (or minimum degrees) generated by the algorithm associated with vertices $x_1, x_2, \ldots, x_n$, in the subhypergraph $H_1, H_2, \ldots, H_n$. Note that for any leaf of $x = x_i$ of $T$, we have $d_{H_i}(x_i) = d_i \leq 2$, where $1 \leq i \leq n$. Note further that by the previous remark any leaf in the new tree $T'$ is a canonical support vertex of $T$ and will of degree at most 3 in the hypergraph obtained after removing all leaves attached to it. Thus after removal of all leaves of $T$, we obtain a tree $T'$ whose leaves have degree at most three in the associated hypergraph. Now iterate on this process by removing all leaves of $T'$ to obtain a tree $T''$, and note that the degree of any leaf of $T''$ in the associated hypergraph is at most three. Consequently for $i = 1, 2, \ldots, n$ we have $d_i \leq 3$. For (ii), a similar argument is carried, but we need to observe that initially $d_{H^o}(x) = \delta(H^o) = 1$ and that after removal of leaves in $T$, any leaf of the resulting tree $T'$ has degree at most two in the corresponding hypergraph. (iii) follows from (ii). For (iv), we follow the arguments in (i), and note that degree of any leave $x$ of $T$ is initially two in $H$. Now apply Theorem 1.2 and note that after removing any leave $x$, the degree of all leaves with the same support vertex becomes one in the corresponding hypergraph, and after removing all leaves joined to a canonical support vertex $s$, the degree of $s$ becomes one in the resulting hypergraph.

Finally, (iv) follows from (iii).

□
Remark 4.3. The lower bound in part (i) of next result matches the best previously known lower bound of $\frac{n+1+2(L-S)}{3}$ in [27], os weaker (by a multiplicative factor of 3/2) in part (ii) than a recent result in [20], and in part (iii) is weaker only by an additive factor of 1 when $n$ is odd compared to the result in [27].

Corollary 4.1. Let $T$ be an $n \geq 4$ vertex tree, with $L$ leaves and $S$ support vertices. Then the following hold. For (ii) assume that every support vertex is adjacent to only one leaf.

(i) $\gamma^{LD}(T) \geq \frac{n+1+2(L-S)}{3}$.

(ii) $\gamma^{ID}(T) \geq \frac{n+3}{3}$.

(iii) $\gamma^{OLD}(T) \geq \frac{n+1}{2}$.

Proof. For (i) let $D$ be an LD set and let $s$ be a support vertex. We assume WLOG that $s \in D$, otherwise by placing $s$ and all but one leaf attached to $s$ in $D$, we obtain another LD set of the same size. Now follow Theorem 4.1 and Lemma 2.1 and note that a total of $L-S$ leaves have degree zero in hypergraph $H^1$ (defined in Theorem 4.1). Thus, we have $n-|D| \leq L^* + T[H^1, D-(L-S)] \leq T[H^1, D-(L+L^*-S)-1]+1 \leq \tilde{\delta}(H^1)(|D|-(L-S)-1)+1 \leq 2(|D|-(L-S)-1)+1$, where the last three inequalities are obtained by the application of Lemma 2.1, Theorem 4.2 and noting that $T[H^1, 1]=1$. Now (i) follows.

For (ii) use $j=2$, and $\delta^*(H) \leq 3$ form Theorem 4.2 and use Theorem 4.1. For (iii) use Theorem 4.1 with $j=1$ and $\delta(H^0) \leq 2$ form Theorem 4.2.

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