A TWO-PARAMETER FINITE FIELD ERDŐS-FALCONER DISTANCE PROBLEM

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Abstract. We study the following two-parameter variant of the Erdős-Falconer distance problem. Given $E, F \subset \mathbb{F}_q^{k+l}$, $l \geq k \geq 2$, the $k+l$-dimensional vector space over the finite field with $q$ elements, let $B_{k,l}(E, F)$ be given by

$$\{ (\|x' - y'\|, \|x'' - y''\|) : x = (x', x'') \in E, y = (y', y'') \in F; x', y' \in \mathbb{F}_q^k, x'', y'' \in \mathbb{F}_q^l \}.$$ 

We prove that if $|E||F| \geq Cq^{k+2l+1}$, then $B_{k,l}(E, F) = \mathbb{F}_q \times \mathbb{F}_q$. Furthermore this result is sharp if $k$ is odd. For the case of $l = k = 2$ and $q$ a prime with $q \equiv 3 \mod 4$ we get that for every positive $C$ there is $c$ such that

$$|B_{2,2}(E, F)| > cq^2.$$

1. Introduction

The Erdős-Falconer distance problem in $\mathbb{F}_q^d$ is to determine how large $E \subset \mathbb{F}_q^d$ needs to be to ensure that

$$\Delta(E) = \{ \|x - y\| : x, y \in E \},$$

with $\|x\| = x_1^2 + x_2^2 + \cdots + x_d^2$, is the whole field $\mathbb{F}_q$, or at least a positive proportion thereof. Here and throughout, $\mathbb{F}_q$ denotes the field with $q$ elements and $\mathbb{F}_q^d$ is the $d$-dimensional vector space over this field.

The distance problem in vector spaces over finite fields was introduced by Bourgain, Katz and Tao in [2]. In the form described above, it was introduced by the second listed author of this paper and Misha Rudnev ([5]), who proved that $\Delta(E) = \mathbb{F}_q$ if $|E| > 2q^{d+1}$. It was shown in [4] that this exponent is essentially sharp for general fields when $d$ is odd. When $d = 2$, it was proved in [3] that if if $E \subset \mathbb{F}_q^2$ with $|E| \geq cq^4$, then $|\Delta(E)| \geq C(c)q$. We do not know if improvements of the $\frac{d+1}{2}$ exponent are possible in even dimensions $\geq 4$. We also do not know if improvements of the $\frac{d+1}{2}$ exponent are possible in any even dimension if we wish to conclude that $\Delta(E) = \mathbb{F}_q$, not just a positive proportion.
In this paper we introduce a two-parameter variant of the Erdős-Falconer distance problem. Given $E, F \subseteq \mathbb{F}_q^{k+l}, l \geq k \geq 2$, the $k+l$-dimensional vector space over the finite field with $q$ elements, define $B_{k,l}(E, F)$ by
\[
\{ (\|x' - y'\|, \|x'' - y''\|) : x = (x', x'') \in E, y = (y', y'') \in F; x', y' \in \mathbb{F}_q^k, x'', y'' \in \mathbb{F}_q^l \}.
\]
This formulation introduces immediate interesting geometric complications. For example, let $k = l = 2$, let $E = \{(x, 0, 0) : \|x\| = 1\}$ and $F = \{(0, 0, y) : \|y\| = 1\}$.

Then $B_{2,2}(E, F) = \{(1, 1)\}$. However, we are going to see that if $|E||F|$ is sufficiently large, then $B_{k,l}(E, F) = \mathbb{F}_q \times \mathbb{F}_q$. Our first result is the following.

**Theorem 1.1.** Let $E, F \subseteq \mathbb{F}_q^{k+l}, l \geq k \geq 2$. There is a $C > 0$ such that
\[
\text{if } |E||F| > C q^{k+2l+1} \text{ then } B_{k,l}(E, F) = \mathbb{F}_q \times \mathbb{F}_q.
\]
If $k$ is odd, this result is best possible, up to the value of the constant $C$.

When $k$ is even, we can hope to improve the exponent a bit. We are able to accomplish this in the case $k = l = 2$. Our second result is the following.

**Theorem 1.2.** Let $q$ a prime with $q \equiv 3 \pmod{4}$. For every positive $C$ there is $c$ such that for $E, F \subseteq \mathbb{F}_q^{2+2}$
\[
\text{if } |E||F| > C q^{6+4} \text{ then } |B_{2,2}(E, F)| > cq^2.
\]
While this result probably is not sharp, we show the exponent cannot go below 6.

2. **Proof of Theorem 1.1**

We begin with a quick review of Fourier analytic preliminaries.

Let $\chi$ be the principal additive character on $\mathbb{F}_q$. Given $f : \mathbb{F}_q^d \rightarrow \mathbb{C}$, define
\[
\hat{f}(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(-x \cdot m) f(x).
\]
Observe that
\[
f(x) = \sum_{m \in \mathbb{F}_q^d} \chi(x \cdot m) \hat{f}(m),
\]
\[
\sum_{m \in \mathbb{F}_q^d} |\hat{f}(m)|^2 = q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2
\]
and
\[
\sum_{x \in \mathbb{F}_q^d} \chi(x \cdot m) = 0 \text{ if } m \neq \vec{0} \text{ and } q^d \text{ otherwise}.
\]
Lemma 2.1. Let $S_t^{d-1} = \{x \in \mathbb{F}_q^d : \|x\| = t\}$, where $\|x\| = x_1^2 + \cdots + x_d^2$. If $t \neq 0$ and $m \neq \vec{0}$, then
\[|\hat{S}_t^{d-1}(m)| \leq 2q^{-\frac{d+1}{2}}.\]

Lemma 2.2. With the notation above,
\[|S_t^{d-1}| = q^{d-1} + O(q^{d-2}).\]

For a proof of Lemma 2.1 and Lemma 2.2, see [5]. See also [7] and [6]. See [8] on a spectral graph theory viewpoint on similar phenomena.

We now move on to the proof of Theorem 1.1. Let $E(X), F(Y)$ denote the indicator functions of $E, F$, respectively, where $X = (x', x'')$ and $Y = (y', y'')$. Consider
\[\sum_{\|x'-y'\|=a; \|x''-y''\|=b} E(X)F(Y) = \sum_{X,Y} S_a^{k-1}(x' - y')S_b^{l-1}(x'' - y'')E(X)F(Y) = \sum_{X,Y,m',m''} \hat{S}_a^{k-1}(m')\hat{S}_b^{l-1}(m'')\chi((x' - y') \cdot m')\chi((x'' - y'') \cdot m'')E(X)F(Y) = \sum_{X,Y,m',m''} \hat{S}_a^{k-1}(m')\hat{S}_b^{l-1}(m'')\chi((X - Y) \cdot M)E(X)F(Y) = q^{2(k+l)} \sum_M \hat{S}_a^{k-1}(m')\hat{S}_b^{l-1}(m'')\hat{E}(M)\hat{F}(M). \tag{2.1}\]

We shall now break up the sum into three pieces. The first piece is the sum over $m' = m'' = \vec{0}$. The second piece is the sum over $m' \neq \vec{0}, m'' \neq \vec{0}$. The third sum is over $m' = \vec{0}, m'' \neq \vec{0}$ or $m'' = \vec{0}, m' \neq \vec{0}$.

2.1. The term $m' = \vec{0}, m'' = \vec{0}$. Plugging this condition into (2.1) we obtain
\[|E||F||S_a^{k-1}||S_b^{l-1}|q^{-k-l}. \tag{2.2}\]

2.2. The term $m' \neq \vec{0}, m'' \neq \vec{0}$. Using Cauchy-Schwarz we see that
\[\left( \sum_{m' \neq \vec{0} \neq m''} \hat{S}_a^{k-1}(m')\hat{S}_b^{l-1}(m'')\hat{E}(M)\hat{F}(M) \right)^2 \leq \sum_{m' \neq \vec{0} \neq m''} |\hat{S}_a^{k-1}(m')\hat{S}_b^{l-1}(m'')\hat{E}(M)|^2 \sum_{m' \neq \vec{0} \neq m''} |\hat{F}(N)|^2\]
Now for the first sum we see by using Lemma 2.1 and Plancherel that it is bounded by
\[\left( 2q^{-\frac{k+1}{2}} \right)^2 \left( 2q^{-\frac{l+1}{2}} \right)^2 \sum_M |\hat{E}(M)|^2 \leq 16q^{-(k+l+2)}q^{-k-l}|E|.\]
And again by Plancherel
\[ \sum_{N \neq 0} |\hat{F}(N)|^2 \leq q^{-k-l}|F| \]

Therefore
\[ q^{2(k+l)} \sum_{M \neq 0} \hat{S}_a^{k-1}(m') \hat{S}_b^{l-1}(m'') \hat{E}(M) \overline{F(M)} \leq 4q^{k+l-1} \sqrt{|E||F|} \]

2.3. The term \( m' \neq 0, m'' = 0 \). We obtain
\[ q^{2(k+l)} \cdot q^{-l} |\hat{S}_b^{l-1}| \sum_{m' \neq 0} \hat{S}_a^{k-1}(m') \hat{E}(m', 0) \overline{F(m', 0)} \]

Very similarly to the previous case we see
\[
\left( \sum_{m' \neq 0} \hat{S}_a^{k-1}(m') \hat{E}(m', 0) \overline{F(m', 0)} \right)^2 \leq \sum_{m' \neq 0} |\hat{S}_a^{k-1}(m') \hat{E}(m', 0)|^2 \sum_{n' \neq 0} |\hat{F}(n', 0)|^2 \\
\leq 4q^{-k-1} \sum_{m'} |\hat{E}(m', 0)|^2 \sum_{n'} |\hat{F}(n', 0)|^2
\]

And furthermore we have the following

**Lemma 2.3.** For \( E \subset \mathbb{F}_q^{k+l} \) we have
\[ \sum_{m' \in \mathbb{F}_q^k} \left| \hat{E}(m', 0) \right|^2 \leq q^{-k-l}|E| \]

**Proof.**
\[
\sum_{m' \in \mathbb{F}_q^k} \left| \hat{E}(m', 0) \right|^2 = \sum_{m' \in \mathbb{F}_q^k} q^{-2(k+l)} \sum_{x', y' \in \mathbb{F}_q^k} \chi((x' - y')m')E(x', x'')E(y', y'') \\
= q^{-k-2l} \sum_{x', x'' \in \mathbb{F}_q^k, x', y' \in \mathbb{F}_q^l} E(x', x'')E(x', y'') \\
\leq q^{-k-l}|E|.
\]
So now we can bound (2.3).

\[ q^{2(k+l)} \cdot q^{-l}S_b^{l-1} \cdot 2q^{\frac{k-1}{2}} q^{-k-l} \sqrt{|E||F|} = 2q^{\frac{k-1}{2}} S_b^{l-1} \sqrt{|E||F|} \]

Putting everything together we see that

\[ \sum_{\|x'-y'\|=a; \|x''-y''\|=b} E(X)F(Y) = |E||F| \frac{|S_{a}^{k-1}| |S_{b}^{l-1}|}{q^{k}} + \mathcal{D}, \]

where

\[ |\mathcal{D}| \leq 2q^{\frac{k-1}{2}} \sqrt{|E||F||S_b^{l-1}|} + 2q^{\frac{k-1}{2}} \sqrt{|E||F||S_a^{k-1}|} + 4q^{\frac{k+l}{2}-1} \sqrt{|E||F|}. \]

By a direct calculation (remembering that \( l \geq k \)) and using Lemma 2.2, the right hand side of (2.4) is positive if

\[ |E||F| > 16q^{k+2l+1}, \]

as desired.

Finally for the sharpness of this result in the case \( k \) odd, we need the following theorem from [4].

**Theorem 2.4.** There exists \( c > 0 \) and \( E \subset \mathbb{F}_q^d \), \( d \) odd, such that

\[ |E| \geq cq^{d+1} \text{ and } \Delta(E) \neq \mathbb{F}_q. \]

Let \( E_1 \subset \mathbb{F}_q^k \) be a set as in theorem above and \( E_2 = \mathbb{F}_q^l \). With \( E = E_1 \times E_2 \) we get \( |E| \geq cq^{\frac{2k+l+1}{2}} \) and \( B_{k,l}(E, E) = \Delta(E_1) \times \Delta(E_2) = \Delta(E_1) \times \mathbb{F}_q \neq \mathbb{F}_q \times \mathbb{F}_q \) since \( \Delta(E_1) \neq \mathbb{F}_q \). Hence our result is sharp if \( k \) is odd.

3. Proof of Theorem 1.2

For \( a, b \in \mathbb{F}_q \) let

\[ s(a, b) := |\{(x', x'', y', y'') \in E \times F : \|x' - y'\| = a, \|x'' - y''\| = b\}|. \]

We observe

\[ \left( \sum_{a,b \in \mathbb{F}_q} s(a, b) \right)^2 = |E|^2 |F|^2 \]

while at the same time Cauchy-Schwarz yields

\[ \left( \sum_{a,b \in \mathbb{F}_q} s(a, b) \right)^2 \leq B_{k,l}(E, F) \sum_{a,b \in \mathbb{F}_q} s(a, b)^2. \]
Hence,

\[(3.1) \quad \frac{|E|^2|F|^2}{\sum_{a,b \in \mathbb{F}_q} s(a,b)^2} \leq B_{k,l}(E,F)\]

so an upper bound on \(\sum_{a,b \in \mathbb{F}_q} s(a,b)^2\) will provide a lower bound for \(B_{k,l}(E,F)\). Now

\[s(a,b)^2 = \sum_{a,b \in \mathbb{F}_q} s(a,b)^2 \leq B_{k,l}(E,F)\]

so

\[(3.2) \quad \sum_{a,b \in \mathbb{F}_q} s(a,b)^2 = \sum_{a,b \in \mathbb{F}_q} s(a,b)^2 = \sum_{a,b \in \mathbb{F}_q} s(a,b)^2 \leq B_{k,l}(E,F)\]

We now proceed as in [1]. For \(\theta, \varphi \in SO_2(\mathbb{F}_q)\) we define \(r^E_{\theta,\varphi} : \mathbb{F}_q^2 \times \mathbb{F}_q^2 \to \mathbb{C}\) by the following property.

\[\sum_{u',u'' \in \mathbb{F}_q^2} r^E_{\theta,\varphi}(u',u'')f(u',u'') = \sum_{x',x'',z',z'' \in \mathbb{F}_q^2} f(x' - \theta z', x'' - \varphi z'')E(x',x'')E(z',z'')\]

for all \(f : \mathbb{F}_q^2 \times \mathbb{F}_q^2 \to \mathbb{C}\)

By setting

\[f(u',u'') = \begin{cases} 1, & \text{if } u' = u'' \\ 0, & \text{otherwise} \end{cases}\]

it is easily seen that \(r^E_{\theta,\varphi}\) is well defined and we get

\[r^E_{\theta,\varphi}(u',u'') = \{ (x',x'',z',z'') \in E \times E : x' - \theta z' = u', x'' - \varphi z'' = u'' \}\]

Therefore

\[(3.3) \quad \sum_{u',u'' \in \mathbb{F}_q^2} r^E_{\theta,\varphi}(u',u'')f^E_{\theta,\varphi}(u',u'') = \sum_{(x',x'',z',z',y',y'',w',w'') \in E^2 \times F^2} f(x' - \theta z' = y' - \theta w', x'' - \varphi z'' = y'' - \varphi w'')\]
With \( f(u',u'') = q^{-4}(-u'\cdot m' - u''\cdot m'') \) we can also calculate the Fourier-transform

\[
\tilde{r}_{\theta,\varphi}^E(m',m'') = \sum_{u',u'' \in \mathbb{F}_q^2} r_{\theta,\varphi}^E(u',u'')q^{-4}\chi(-u' \cdot m' - u'' \cdot m'')
\]

\[
= \sum_{x',x'',z',z'' \in \mathbb{F}_q^2} q^{-4}\chi(-(x' - \theta z') \cdot m' - (x'' - \varphi z'') \cdot m'')E(x',x'')E(z',z'')
\]

\[
= q^4 \hat{E}(m',m'')\hat{E}(\theta m',\varphi m'')
\]

Now our key observation is the following

**Lemma 3.1.** Let \( q \) a prime, \( q \equiv 3 \mod 4 \). Then for \( x,y \in \mathbb{F}_q^2 \setminus \{0\} \) we have \( \|x\| = \|y\| \) if and only if there is a unique \( \theta \in SO_2(\mathbb{F}_q) \) such that \( x = \theta y \)

This observation allows us to make the following connection

\[
\sum_{a,b \in \mathbb{F}_q} s(a,b)^2 \leq \sum_{u',u'' \in \mathbb{F}_q^2} \tilde{r}_{\theta,\varphi}^E(u',u'')r_{\theta,\varphi}^F(u',u'')
\]

by comparing (3.2) and (3.3) and seeing that

\[
\|x' - y\| = \|z' - w\| \implies \exists \theta \in SO_2(\mathbb{F}_q) : x' - \theta z' = y' - \theta w'.
\]

Now

\[
\sum_{U \in \mathbb{F}_q^2 \times \mathbb{F}_q^2} r_{\theta,\varphi}^E(U)r_{\theta,\varphi}^F(U) = \sum_{U \in \mathbb{F}_q^2 \times \mathbb{F}_q^2} \sum_{M \in \mathbb{F}_q^4} \chi(U)M\tilde{r}_{\theta,\varphi}^E(M) \sum_{N \in \mathbb{F}_q^4} \chi(U)N\tilde{r}_{\theta,\varphi}^F(N)
\]

\[
= \sum_{M \in \mathbb{F}_q^4} \tilde{r}_{\theta,\varphi}^E(M) \sum_{N \in \mathbb{F}_q^4} \tilde{r}_{\theta,\varphi}^F(N) \sum_{U \in \mathbb{F}_q^2 \times \mathbb{F}_q^2} \chi(U(N + M))
\]

\[
= q^4 \sum_{M \in \mathbb{F}_q^4} \tilde{r}_{\theta,\varphi}^E(M)\tilde{r}_{\theta,\varphi}^F(N)
\]

and it remains to find a bound for

\[
\sum_{\theta,\varphi \in SO_2(\mathbb{F}_q)} \sum_{u',u'' \in \mathbb{F}_q^2} r_{\theta,\varphi}^E(u',u'')r_{\theta,\varphi}^F(u',u'') = q^4 \sum_{\theta,\varphi \in SO_2(\mathbb{F}_q)} \sum_{m',m'' \in \mathbb{F}_q^2} r_{\theta,\varphi}^E(m',m'')r_{\theta,\varphi}^F(m',m'')
\]

\[
= q^{12} \sum_{m',m'' \in \mathbb{F}_q^2} \sum_{\theta,\varphi \in SO_2(\mathbb{F}_q)} \hat{E}(m',m'')\hat{E}(\theta m',\varphi m'')\hat{F}(m',m'')\hat{F}(\theta m',\varphi m'').
\]

Again we will need to split the sum into three terms
3.1. **The term** $m' = \hat{0}, m'' = \hat{0}$. Plugging into (3.4) we get

$$q^{12} \sum_{\theta, \varphi \in SO_2(F_q)} |\widehat{E}(\hat{0}, \hat{0})|^2 |\widehat{F}(\hat{0}, \hat{0})|^2 = q^{-4} |E|^2 |F|^2 |SO_2(F_q)|^2.$$ 

3.2. **The term** $m' \neq \hat{0}, m'' \neq \hat{0}.$

$$q^{12} \sum_{m', m'' \in F_q^2 \setminus \{\hat{0}\}} \widehat{E}(m', m'') \overline{\widehat{F}(m', m'')} \sum_{\theta, \varphi \in SO_2(F_q)} \widehat{E}(\theta m', \varphi m'') \overline{\widehat{F}(\theta m', \varphi m'')}$$

$$= q^{12} \sum_{a, b \in F_q \setminus \{0\}} \sum_{\|m'\| = a, \|m''\| = b} \widehat{E}(m', m'') \overline{\widehat{F}(m', m'')} \sum_{\theta, \varphi \in SO_2(F_q)} \widehat{E}(\theta m', \varphi m'') \overline{\widehat{F}(\theta m', \varphi m'')}$$

$$= q^{12} \sum_{a, b \in F_q \setminus \{0\}} \left| \sum_{\|m'\| = a, \|m''\| = b} \widehat{E}(m', m'') \overline{\widehat{F}(m', m'')} \right|^2$$

where we used Lemma 3.1 in the last step.

We continue with a trivial estimate on one of the inner factors

$$q^{12} \sum_{a, b \in F_q \setminus \{0\}} \left| \sum_{\|m'\| = a, \|m''\| = b} \widehat{E}(m', m'') \overline{\widehat{F}(m', m'')} \right|^2$$

$$\leq q^{12} \sum_{a, b \in F_q \setminus \{0\}} \left( \sum_{\|m'\| = a, \|m''\| = b} |\widehat{E}(m', m'')|^2 \right) \left( \sum_{\|n'\| = a, \|n''\| = b} |\widehat{F}(n', n'')|^2 \right)$$

$$\leq q^{12} \left( \sum_{m', m'' \in F_q^2 \setminus \{\hat{0}\}} |\widehat{E}(m', m'')|^2 \right) \left( \sum_{n', n'' \in F_q^2} |\widehat{F}(n', n'')|^2 \right)$$

$$= q^{12} \left( q^{-4} \sum_{u', u'' \in F_q^2} |E(u', u'')|^2 \right) \left( q^{-4} \sum_{u', u'' \in F_q^2} |E(u', u'')|^2 \right)$$

$$= q^{4} |E| |F|.$$
3.3. The term $m' \neq \tilde{0}, m'' = \tilde{0}$. As in the two previous cases we see
\[
q^{12} \sum_{m' \in \mathbb{F}_q^2} \sum_{\theta, \varphi \in SO_2(\mathbb{F}_q)} \hat{E}(m', \tilde{0}) \hat{E}(\theta m', \tilde{0}) \hat{F}(m', \tilde{0}) \hat{F}(\theta m', \tilde{0})
\]

(3.5) \[= q^{12} |SO_2(\mathbb{F}_q)| \sum_{m' \in \mathbb{F}_q^2 \setminus \{\tilde{0}\}} \hat{E}(m', \tilde{0}) \hat{F}(m', \tilde{0}) \sum_{\theta \in SO_2(\mathbb{F}_q)} \hat{E}(\theta m', \tilde{0}) \hat{F}(\theta m', \tilde{0})\]

We will deal with the inner sum first. Let $0 \neq a = ||m'||$.
\[
\sum_{\theta \in SO_2(\mathbb{F}_q)} \hat{E}(\theta m', \tilde{0}) \hat{F}(\theta m', \tilde{0}) \leq \sum_{||m||=a} \hat{E}(m, \tilde{0}) \hat{F}(m, \tilde{0})
\]

(3.6) \[\leq \sqrt{\sum_{||m||=a} |\hat{E}(m, \tilde{0})|^2 \sum_{||n||=a} |\hat{F}(n, \tilde{0})|^2} \]

Lemma 3.2. For $E \subset \mathbb{F}_q^2$, $0 \neq a \in \mathbb{F}_q$ we get
\[
\sum_{||m||=a} |\hat{E}(m, \tilde{0})|^2 \leq 3^{1/2} q^{-6} |E|^2
\]

Proof. With the notation introduced in Lemma 2.1 and $g : \mathbb{F}_q^2 \to \mathbb{C}$ where $g(m) = \overline{E(m, \tilde{0})} S_a(m)$. we can write this as
\[
\sum_{m \in \mathbb{F}_q^2} \hat{E}(m, \tilde{0}) S_a(m) g(m) = \sum_{m \in \mathbb{F}_q^2} q^{-4} \sum_{x', x'' \in \mathbb{F}_q^2} \chi(-x' \cdot m) E(x', x'') S_a(m) g(m)\]

\[= q^{-2} \sum_{x' \in \mathbb{F}_q^2} \left( \sum_{x'' \in \mathbb{F}_q^2} E(x', x'') \right) \hat{S_a} g(x')\]

Using Hölder’s Inequality with $q = \frac{4}{3}$, $r = 4$ we can bound this by
\[
(3.7) \leq q^{-2} \left( \sum_{x' \in \mathbb{F}_q^2} \left( \sum_{x'' \in \mathbb{F}_q^2} E(x', x'') \right)^{4/3} \right)^{3/4} \left( \sum_{x' \in \mathbb{F}_q^2} |\hat{S_a} g(x')|^4 \right)^{1/4}
\]

We will first find an estimate for the latter factor. By using the definition of the Fourier transform we get:
\[
(3.8) \sum_{x' \in \mathbb{F}_q^2} |\hat{S_a} g(x')|^4 = q^{-6} \sum_{u, v, u', v' \in S_a} g(u) g(v) g(u') g(v')
\]

Here we use the Fefferman trick. For fixed $u, v \in S_a$, $u \neq -v$ we want to find $u', v' \in S_a$ such that $u + v = u' + v'$. In other words we want to find $u' \in S_a$ such that
(u + v − u') ∈ S_a, so u' is in the intersection of the circles \( x \in \mathbb{F}_q^2 : ||x|| = a \) and \( x \in \mathbb{F}_q^2 : ||x − (u + v)|| = a \) which has at most two solutions as the circles are not identical \( u + v \neq 0 \). But we already know two solutions, namely \( u \) and \( v \). So either \( u' = u \) and \( v' = v \) or \( u' = v \) and \( v' = u \). If \( u = −v \) we get \( u' \in S_a \) and \( v' = −u' \). Therefore (and by noting that \( g(−u) = g(u) \)) we can write (3.8) as

\[
q^{-6} \left( \sum_{u,v \in S_a} 2g(u)g(v)g(u')g(v') + \sum_{u,u' \in S_a} g(u)g(−u)g(u')g(−u') \right)
= 3q^{-6} \sum_{u,v \in S_a} |g(u)|^2 |g(v)|^2
= 3q^{-6} \left( \sum_{u \in S_a} |g(u)|^2 \right)^2
= 3q^{-6} \left( \sum_{||u|| = a} |\hat{E}(u, 0)|^2 \right)^2
\]

The other factor of (3.7) can be dealt with as follows

\[
\left( \sum_{x' \in \mathbb{F}_q^2} \left( \sum_{x'' \in \mathbb{F}_q^2} E(x', x'') \right)^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \left( \sum_{x' \in \mathbb{F}_q^2} \left( \sum_{x'' \in \mathbb{F}_q^2} E(x', x'') \right) \left( \sum_{x'' \in \mathbb{F}_q^2} E(x', x'') \right) \right)^{\frac{1}{3}} \leq q^{\frac{1}{2}} |E|^{\frac{1}{2}}
\]

Therefore we have

\[
\sum_{||m|| = a} |\hat{E}(m, 0)|^2 \leq 3^{\frac{1}{2}} q^{-2} q^{\frac{1}{4}} |E|^{\frac{3}{4}} q^{-\frac{1}{2}} \left( \sum_{||m|| = a} |\hat{E}(m, 0)|^2 \right)^{\frac{1}{2}}
\]

so

\[
\sum_{||m|| = a} |\hat{E}(m, 0)|^2 \leq 3^{\frac{1}{2}} q^{-4} q |E|^{\frac{3}{2}} q^{-3} = 3^{\frac{1}{2}} q^{-6} |E|^{\frac{3}{2}}
\]

Continuing from (3.5) and using (3.6) and Lemma 3.2 we see
Finally we need to deal with
\[
\sum_{m' \in \mathbb{F}_q^2 \setminus \{0\}} \hat{E}(m', \vec{0}) \hat{F}(m', \vec{0}) \leq q^{12} |SO_2(\mathbb{F}_q)| \sum_{m' \in \mathbb{F}_q^2 \setminus \{0\}} \left| \hat{E}(m', \vec{0}) \hat{F}(m', \vec{0}) \right| \leq q^{12} |SO_2(\mathbb{F}_q)| \sum_{m' \in \mathbb{F}_q^2 \setminus \{0\}} \left| \hat{E}(m', \vec{0}) \hat{F}(m', \vec{0}) \right| \cdot 3^{3/2} q^{-6} |\hat{E}|^2 |\hat{F}|^2.
\]

Putting those results together we find that (3.5) is bounded by
\[
Cq^{12} q^{-4} |E||F|^{3/2} = Cq^3 |E|^{3/2} |F|^{3/2}.
\]

So we can bound the whole sum (3.4) by
\[
q^4 |E||F| + Cq^3 (|E||F|)^{3/4} + q^{-4} |E|^2 |F|^2 |SO_d(\mathbb{F}_q)|^2.
\]

Therefore we get from (3.1)
\[
\min \left\{ \frac{|E||F|}{3q^4}, \left( \frac{|E||F|}{3Cq^3} \right)^{3/4}, \frac{q^4}{3|SO_2(\mathbb{F}_q)|^2} \right\} \leq P(E).
\]

Hence it is enough that
\[
cq^2 \leq \left( \frac{|E||F|}{3Cq^3} \right)^{3/4} \iff cq^5 \leq \left( \frac{|E||F|}{3} \right)^{3/4} \iff c q^{20} \leq |E||F|
\]
since in this case also
\[
\frac{|E||F|}{3q^4} \geq \frac{c q^{20}}{q^4} \geq c q^2.
\]

**Remark 3.3 (Sharpness of results).** Let \( p \) a prime, with \( p \equiv 3 \mod 4 \). Consider \( E = \mathbb{F}_p^2 \times L \), where
\[
L = \{(a, 0) : a \in \{0, \ldots, p^{1-\varepsilon}\}\}.
\]

Then \( |E| = p^{3-\varepsilon} \) and \( |\Delta(L)| = 2p^{1-\varepsilon} \), so \( |B(E, E)| = o(p^2) \). Hence the \( 6 + \frac{2}{3} \) exponent in Theorem 1.2 is potentially not best possible, but we definitely cannot go below 6.
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