1. Introduction

The influence of lattice discreteness on the properties of nonlinear systems having kink solutions was investigated by several authors [1]-[10]. These studies have pointed out a large variety of effects, including modification of kink velocity and its form and leading sometimes to the pinning of the kink on the lattice. But on the other side numerical investigations [9] have revealed the existence of narrow solitary waves propagating in certain discrete lattices without energy loss. There is also a close connection of these problems with the complete integrability of specific models in nonlinear continuum and discrete systems [11]-[13].

Since the pioneering paper of Krumhansl and Schrieffer [14] the role played by the elementary excitations of kink type in the thermodynamics of one-dimensional nonlinear systems was investigated by many authors ([15]-[20] and the references therein). As concerns the influence of lattice discreteness on these thermodynamic properties this has received a smaller attention [20]-[23]. The class of 1-D systems mostly discussed is described by the Hamiltonian (in the notations of CKBT [15])

\[ H = \sum_i l A \left( \frac{1}{2} \dot{\phi}_i^2 + \frac{\epsilon_0^2}{2l^2} (\phi_{i+1} - \phi_i)^2 + \omega_0^2 V(\phi_i) \right) \]  

(1)

where the nonlinearity enters only through the potential \( V(\phi) \), assumed to have at least two degenerate minima with nonvanishing curvature. It is the merit of Trullinger and Sasaki [21] to have shown that the first lattice corrections are taken into account if the potential \( V(\phi) \) is replaced by an effective potential

\[ V_{\text{eff}} = V(\phi) - \frac{l^2}{24d^2} \left( \frac{dV}{d\phi} \right)^2 + O \left( \frac{l^4}{d^4} \right), \]

(2)

where \( d = \frac{\omega_0}{\omega} \) is the mean width of the static kink. In the present paper we intend to complete their results and determine also the lattice corrections to the multi-kink contributions to the free energy.

The partition function and the free energy are usually calculated using the transfer integral operator (TIO) method. In the thermodynamic limit the lowest eigenvalues of TIO play the most important role. In the displacive/continuum limit, \( \frac{l}{d} \ll 1 \), it is possible to transform the Fredholm integral equation representing the TIO into a differential equation. This is easily done when the two-body interaction in (1) has an harmonic character \* [17],[19]. One obtains

\[ \exp(-\gamma V(\phi)) \exp \left( \frac{\gamma}{2m^*} D^2 \right) \Phi_n(\phi) = \exp(-\gamma \tilde{\epsilon}_n) \Phi(\phi). \]

(3)

Here \( \gamma = \beta l A \omega_0^2 \), \( m^* = (\beta A_0 \omega_0)^2 \), \( \Phi_n \) is the eigenvector of TIO corresponding to the eigenvalue \( \epsilon_n \) and

\[ \tilde{\epsilon}_n = \epsilon_n + \frac{1}{\gamma} \ln \frac{l}{d} \sqrt{\frac{2\pi}{\gamma}}. \]

(3a)

The left hand side of (3) contains the product \( e^A \cdot e^B \) with A and B two noncommuting operators, and in order to obtain the claimed differential equation we have to put it as \( e^C \), where the operator C can be written in terms of A and B using the Baker-Hausdorff formula [20],[21],[22]. This formula turns to be a mixed expansion in the small parameters \( \frac{l^2}{d^2} \) and \( \frac{\omega_0}{\omega} \) and consequently is very convenient for the proposed purposes. The first lattice correction is obtained if one retains only the terms proportional with \( \frac{l^2}{d^2} \). Contrary the opinion of Trullinger and Sasaki [21] the same result is found if one work with the symmetrised or non-symmetric form of TIO. It is possible to show through a set of transformations and

\* As was shown by Guyer and Miller [17] even for more complicated interactions between nearest neighbours it is possible to write a formal expression for the partition function and the free energy using a cumulant expansion.
neglecting systematically higher powers in $\frac{d^2}{d\phi^2}$, that the eigenvalues of TIO, with the first lattice corrections included, can be found from the equation \[ [21] **
\[
\frac{1}{2m^*} \frac{d^2\Psi_n}{d\phi^2} + (\hat{\epsilon}_n - V_{eff}(\phi))\Psi_n = 0. \tag{4}
\]
The result is very general and valid for any kind of potential function $V(\phi)$.

In the next section the leading terms as an asymptotic expansion of the lowest eigenvalues of (4) will be found in the low temperature limit, $m^* \gg 1$. The method used, briefly sketched in the Appendix, allow us to make a clear distinction between the contributions of phonons and of the various kink sectors. Explicit expressions for the lattice corrections to the kink and kink-kink contributions to the free energy will be given. The last section is devoted to a discussion of the results in the spirit of the CKBT [15] phenomenology of independent phonons and renormalised kinks. As it is known [15], [16], [19] this phenomenology is exact in the low temperature limit of the continuum models and the results of Trullinger and Sasaki [21] and those of the present paper strongly support the conjecture that it remains valid even when the first lattice corrections are taken into account.

2. Lattice Corrections to the Free Energy

Using (3a) the free energy per unit length becomes
\[
F = \frac{1}{\beta l} \ln \beta h \omega_0 \frac{d}{l} + A \omega_0^2 \hat{\epsilon}_0 \tag{5}
\]
where $\hat{\epsilon}_0$ is the lowest eigenvalue of eq. (4).

In the low temperature limit $\beta \gg 1, (m^* \gg 1)$ there are several ways to find approximate solutions of this equation, all included in so called class of the improved WKB methods [15]-[20],[23],[24] (and the references therein). We shall use the method developed by us in a series of papers [20], [23],[24] which has the advantage to make a clear distinction between the various contributions to the free energy: phonons, 1-kink, 2-kink and so on sectors. The method is based on well known results from the theory of asymptotic solutions of second order differential equations depending on a large parameter [25] an is presented in the Appendix. As we are interested in the leading term in $m^*$ the exact equation (4) is approximated by the comparison equation (A.9). With $V_{eff}$ replacing the potential $V$ in the definition (A.8) of the constant $a$ the first term in the asymptotic expansion of the lowest eigenvalue of the isolated potential well is given by
\[
E_0 = \frac{1}{2 \sqrt{m^*}} \left( 1 - \frac{l^2}{24d^2} \right) + O\left( \frac{1}{m^*} \right). \tag{6}
\]
Then the corresponding contribution to the free energy density (5) writes
\[
F_0 = \frac{1}{\beta l} \ln(\beta h \omega_0 \frac{d}{l}) + \frac{1}{2 \beta d} \left( 1 - \frac{l^2}{24d^2} \right) \tag{7}
\]
and is easily identified with the first terms in the series expansion in powers of $l^2/d^2$ of the exact free energy of an independent gas of phonons with the dispersion relation
\[
\omega^2(k) = \omega_0^2 \left( 1 + 4 \frac{d^2}{l^2} \sin^2 \frac{kl}{2} \right). \tag{8}
\]
** The discrepancy between the results of [21] and [20] comes from an inadequate treatment of a certain equation obtained in an intermediate step of the proof in our previous paper [20].
Higher order terms in \((m^*)^{-1/2}\), neglected in (6), will correspond to anharmonic interactions between the phonons [23].

The presence of the other degenerate minima determines a symmetrical splitting of each eigenvalue of the isolated potential well. In the case of the \(\phi^4\) model we obtain \(E_0 \to \tilde{\epsilon}_0 = E_0 \pm t_0\) where

\[
t_0 = \frac{c}{\sqrt{m^*}} \left(1 - \frac{l^2}{24d^2}\right) \nu
\]

with \(c = 1\). As is explained in the Appendix the value of \(\nu\) is determined from the boundary conditions (A.11) imposed on the eigenfunctions of the comparison equation. For the \(\phi^4\) model we get

\[
-1 = 2\sqrt{\pi e} \frac{e^{-ix\nu}}{\Gamma(-\nu)} \exp \left(2\lambda J + \nu(1 + ln2) - \frac{1}{2}(1 + 2\nu)ln(1 + 2\nu)\right).
\]

This expression is valid in leading order in \((m^*)^{-1/2}\), but in any order in \(\nu\), and is the starting point to find multi-kink contributions to the free energy. Here \(J\) is an integral defined by

\[
J = \int_{0}^{\mu_1} \sqrt{V_{eff} - \tilde{\epsilon}} \ d\phi
\]

with \(\mu_1\) the left turning point of the isolated potential well. As we are interested in the ground state the quantity \(\tilde{\epsilon}\) from (11) is connected to \(\nu\) by

\[
\tilde{\epsilon} = \frac{1}{2\sqrt{m^*}} \left(1 - \frac{l^2}{24d^2}\right) (1 + 2\nu c) = \tilde{\epsilon}_0 \left(1 - \frac{l^2}{24d^2}\right)
\]

where \(\tilde{\epsilon}_0\) is now independent on the lattice spacing \(l\). It is seen that \(J\) depends on \(l^2/d^2\) both through the integrand and the upper limit of integration. Using (12) and the expression of \(V_{eff}\) the turning points of the \(\phi^4\) model, calculated in the order \(O(l^2/d^2)\), are given by

\[
\mu_{1,2}^2 = 1 \mp \sqrt{8\tilde{\epsilon}_0} + \frac{l^2}{24d^2} \sqrt{8\tilde{\epsilon}_0} \left(\frac{1}{2} + \sqrt{8\tilde{\epsilon}_0}\right) + O\left(\frac{l^4}{d^4}\right).
\]

In the same order

\[
J = \frac{1}{2\sqrt{2}} \int_{0}^{\mu_1} \left(1 - \frac{l^2}{24d^2}\phi^2\right) \sqrt{(\mu_1^2 - \phi^2)(\mu_2^2 - \phi^2)} \ d\phi + O(l^4/d^4).
\]

Writing

\[
J = J_0 + \frac{l^2}{24d^2} J_1
\]

where now \(J_0\) and \(J_1\) doesn’t depend any more on \(l^2/d^2\), we get

\[
J_0 = \frac{1}{2\sqrt{2}} \int_{0}^{\mu_1} \sqrt{(\mu_1^2 - \phi^2)(\mu_2^2 - \phi^2)} \ d\phi
\]

\[
J_1 \simeq - \frac{1}{2\sqrt{2}} \int_{0}^{\mu_1} \phi^2 \sqrt{(\mu_1^2 - \phi^2)(\mu_2^2 - \phi^2)} \ d\phi + \\
+ \frac{8\tilde{\epsilon}_0}{4\sqrt{2}} \int_{0}^{\mu_1} \frac{1 - 2\phi^2}{\sqrt{(\mu_1^2 - \phi^2)(\mu_2^2 - \phi^2)}} \ d\phi.
\]

These integrals can be calculated in terms of complete elliptic integrals of modulus in the vicinity of unity. In the low temperature limit, keeping terms up to the order \(\tilde{\epsilon}_0\) we find

\[
2\sqrt{2}J_0 \simeq \frac{2}{3} - \tilde{\epsilon}_0(1 + ln\frac{8}{\tilde{\epsilon}_0})
\]
Introducing these results into (10) we get
\[
\sqrt{6} \beta E_k^{(0)} \pi e^{-\beta E_k^{(0)} (1 - \frac{l^2}{120d^2}) } e^{-\frac{\beta}{48d^2}} =
\]
\[
- e^{-i\pi \nu} \Gamma(-\nu) e^{-\nu ln(12\beta E_k^{(0)})} e^{\frac{\nu l^2}{24d^2}},
\]
(17) \(-\phi^4\)

where \(E_k^{(0)} = \frac{2}{3} A_0 \omega_0\) is the energy of the static kink.

In the sine-Gordon case (periodic potential) each eigenvalue of the isolated potential well is symmetrically splitted in an allowed band of width \(2\tilde{\epsilon}_0\) where \(\tilde{\epsilon}_0\) is given by (9) with \(c = 2\), and \(\nu\) results from boundary conditions imposed on the eigenfunction at the point \(\phi = \pi\). Finally we get
\[
-1 = 2\sqrt{\pi} e^{\frac{i\pi \nu}{\Gamma(-\nu)}} \Gamma\left(\frac{1}{2} + \nu\right) \Gamma\left(\frac{1}{2}\right) \exp \left(2\lambda J + 2\nu ln(1 + 2\lambda) - \frac{1 + 4\nu}{2} ln(1 + 4\nu)\right),
\]
(10a)

where the integral \(J\) is defined by
\[
J = \int^\pi_{\bar{\phi}} \sqrt{V_{eff}} - \epsilon d\phi.
\]
(11a)

The integral \(J\) is calculated in the same linear approximation in \(\frac{l^2}{2d^2}\) and the leading terms in \(\tilde{\epsilon}_0\) is given by
\[
J = 2\sqrt{2} \left(1 - \frac{l^2}{72d^2}\right) - \frac{\tilde{\epsilon}_0}{8} ln\frac{32}{\epsilon_0} - \frac{\tilde{\epsilon}_0}{8} (1 - \frac{l^2}{24d^2})
\]
(16a)

where \(\tilde{\epsilon}_0\) is defined in (12) being independent on lattice corrections. Introducing (16a) into (10a) and using the expression \(E_k^{(0)} = 8A_0 \omega_0\) for the energy of the static kink one obtains
\[
\sqrt{2} \beta E_k^{(0)} \pi e^{-\beta E_k^{(0)} (1 - \frac{l^2}{72d^2}) } e^{-\frac{l^2}{48d^2}} =
\]
\[
- e^{i\pi \nu} \Gamma\left(\frac{1}{2} + \nu\right) \Gamma\left(\frac{1}{2}\right) e^{2\nu \frac{l^2}{24d^2}},
\]
(17) \(-SG\)

In order to find various kink contributions we have to expand the right hand side of equations (17) in powers of \(\nu\) and to write also
\[
\nu = \nu_k + \nu_{kk} + ...
\]
(18)

The single kink term \(\nu_k\) comes from the expansion of
\[
\frac{1}{\Gamma(-\nu)} = -\nu + \gamma \nu^2
\]
\((\gamma = 0.577 - \text{Euler’s constant})\) and one obtains easily
\[
\nu_k = \sqrt{\frac{6\beta E_k^{(0)}}{\pi}} e^{-\beta E_k^{(0)} (1 - \frac{l^2}{20\pi d^2}) } \left(1 - \frac{5l^2}{48d^2}\right)
\]
(19) \(-\phi^4\)
\[
\nu_k = \sqrt{\frac{2\beta E_k^{(0)}}{\pi}} e^{-\beta E_k^{(0)} (1 - \frac{l^2}{2\pi d^2}) } \left(1 - \frac{l^2}{48d^2}\right)
\]
(19) \(-SG\)

This result is in complete agreement with that found by Trullinger and Sasaki [21].
In finding the kink-kink contribution $\nu_{kk}$ we have to take into account that the expansion (18) is a formal expansion in powers of $e^{-\beta E_k^{(0)}}$ and consequently $\nu_{kk}$ is of the same order as $\nu_k^2$. Then it is easily found that

$$\nu_{kk} = \nu_k^2 \left( \gamma + \ln(12\beta E_k^{(0)}) - \frac{5l^2}{24d^2} + i\pi \right)$$

(20) $- \phi^4$

One sees that $\nu_{kk}$ gets an imaginary part satisfying the very simple relation

$$\text{Im } \nu_{kk} = \pi \nu_k^2$$

(21)

This relation has been obtained by Zinn-Justin [26] in his analysis of the multi-instanton contributions in quantum mechanics and is tightly related to the non-Borel summability of the Rayleigh-Schrödinger perturbation expansion when the potential has degenerate minima.

For the sine-Gordon model one obtains in a similar way

$$\nu_{kk} = 2\nu_k^2 \left( \gamma + \ln 4\beta E_k^{(0)} - \frac{l^2}{24d^2} - \frac{i\pi}{2} \right)$$

(20) $- SG$

Both $(19 - \phi^4)$ and $(20 - SG)$ in the limit $\frac{l^2}{d^2} \to 0$ are in complete agreement with previous calculations of the kink-kink sector contributions [26]-[28].

3. Concluding Remarks

According to CKBT phenomenology the thermodynamic properties of the systems described by the Hamiltonians of the form (1) are influenced by the existence of the static kinks [15]-[19]. A complete agreement between this phenomenology and the exact results of TIO in the low temperature limit is found if one takes into account the scattering of phonons on the static kink, leading to a renormalisation of the kink energy.

The results of Trullinger and Sasaki and of the present paper are showing that lattice corrections are easily included into the thermodynamics of this systems. Although a complete proof of a similar CKBT phenomenology doesn’t exist at present, the existing results strongly support the idea that their phenomenology is still valid at low temperatures. This comes both from the phonon part, which, as mentioned above, reproduces exactly the first terms in the series expansion in $\frac{l^2}{d^2}$ of the free energy of a phonon lattice gas, and from the kink contribution. As is seen from (19) in the kink contribution appear a lattice corrected kink energy

$$E_k = E_k^{(0)} - \frac{l^2}{12d^2} E_k^{(1)}$$

(21)

where $E_k^{(0)}$ is the known unperturbed static kink energy and the correction $E_k^{(1)}$ for the SG and $\phi^4$ model are given by

$$E_k^{(1)} = -\frac{1}{6} E_k^{(0)}, \quad E_k^{(0)} = 8A\omega_0 c_0$$

(22) $- SG$

$$E_k^{(1)} = -\frac{1}{10} E_k^{(0)}, \quad E_k^{(0)} = \frac{2}{3} A0\omega_0 c_0$$

(22) $- \phi^4$

These corrected values have been obtained also by other authors [5] [6], and as will be shown below, result from a very simple perturbation theory. Indeed expanding the field variable $\phi(x \pm l)$ in a Taylor series the discrete Euler- Lagrange equation transforms into a fourth order differential equation

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{l^2}{12d^2} \frac{\partial^4 \phi}{\partial y^4} + \frac{\partial^2 \phi}{\partial y^2} = V_\phi$$

(23)
where $y = x/d$ and $V_\phi = \frac{dV}{d\phi}$. Looking for a perturbed kink solution

$$\phi_k(y) = \phi_k^{(0)}(y) + \frac{l^2}{12d^2} \phi_k^{(1)}(y), \quad (24)$$

where $\phi_k^{(0)}$ is the kink solution of the continuum limit, the linearized equation determining $\phi_k^{(1)}$ is

$$\frac{d^2 \phi_k^{(1)}}{dy^2} - V_{\phi\phi}(\phi_k^{(0)}) \phi_k^{(1)} = -\frac{d^4 \phi_k^{(0)}}{dy^4}. \quad (25)$$

The homogeneous part of (25) has appeared some years ago in the study of the kink dynamics in the presence of perturbations and has as solution the "translation mode" (Goldstone mode) [15], [16], [29]. Then eq. (25) can be solved by the method of variation of constants and the obtained results are in complete agreement with those already existing in literature [4]-[7],[10]. Using the same Taylor expansion for the field variable one can write down a corrected integral of energy. Introducing in it the new expansion of $\phi_k (24)$ the integration is easily performed and the result is just that presented in (22). Still unsolved remains the problem of calculation of renormalization of kinks due to the scattering of lattice phonons on the perturbed kinks, renormalization which determines the factor multiplying the exponentials in (19).

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Appendix

The lowest eigenvalues of (4) can be found, in the low temperature limit \( m^* >> 1 \), using asymptotic methods known from the theory of second order differential equation depending on a large parameter [25]. The equation under investigation is of the type

\[
- \frac{1}{2m^*} \frac{d^2 \Psi}{d\phi^2} + V(\phi) \Psi = \tilde{\epsilon} \Psi, \tag{A.1}
\]

where the potential \( V(\phi) \) has at least two degenerate minima and nonvanishing curvature at the minima.

The problem will be solved in two distinct steps. In the first, based on Langer’s transformation, one looks for an uniform valid expansion of the solution near the minimum. Due to the peculiarities mentioned above, this transformation is chosen in such a way to give the harmonic oscillator behaviour as a leading term. In the second step, the existence of the other minima of \( V(\phi) \) are taken into account using symmetry properties of the wave functions. Each eigenvalue of an isolated potential well is slightly splitted. These tunneling terms are directly related to the kink contribution to the free energy.

Let \( \mu_1 \) and \( \mu_2 \) be the two turning points of an isolated potential well. One passes from the variables \( \Psi \) and \( \phi \) to the new ones \( R \) and \( x \).

\[
\Psi = \chi^{-\frac{1}{4}} R \tag{A.2}
\]

\[
\zeta(x) = \int_{\mu_1}^{\phi} \sqrt{\tilde{\epsilon} - V(\phi)} \ d\phi \tag{A.3}
\]

\[
\chi = \frac{\tilde{\epsilon} - V}{(\zeta')^2}. \tag{A.4}
\]

Here \( \zeta' = \frac{d\zeta}{dx} \), and \( x \) is still an undefined function of \( \phi \). Then (A.1) transforms into

\[
\frac{d^2 R}{dx^2} + \lambda^2 R = \delta R \tag{A.5}
\]

where \( \delta = -\chi^{-\frac{1}{4}} \frac{d^2 (\chi^\frac{1}{4})}{d\phi^2} \) and \( \lambda^2 = 2m^* \). The function \( \chi \) is subjected to the requirement to be regular and not to vanish in the interval of interest. In order to obtain the harmonic oscillator behaviour as the leading term, one takes

\[
(\zeta')^2 = 4a^2 (1 - x^2). \tag{A.6}
\]

This is a function with two simple zeroes and \( x = -1 \) is associated with the turning point \( \phi = \mu_1 \) and \( x = +1 \) to \( \phi = \mu_2 \). Now it is possible to determine the relation between the new variable \( x \) and the old one \( \phi \). Integrating (A.6) we get

\[
a(\pi - \cos^{-1} x + x\sqrt{1 - x^2}) = \int_{\mu_1}^{\phi} \sqrt{\tilde{\epsilon} - V} \ d\phi \tag{A.7}
\]

where the constant \( a \) is given by

\[
a = \frac{1}{\pi} \int_{\mu_1}^{\mu_2} \sqrt{\tilde{\epsilon} - V} \ d\phi. \tag{A.8}
\]

It can be easily proved that for all the domain, even in the asymptotic region, \( \delta \sim \frac{1}{\lambda^4} \), and taking into account that \( \lambda^2 = 2m^* >> 1 \), the first order approximation can be found from the following comparison equation [25]:

\[
\frac{d^2 R}{dx^2} + 4a^2 \lambda^2 (1 - x^2) R = 0, \tag{A.9}
\]
whose solutions are the functions of the parabolic cylinder. For the $\phi^4$ model, the only convenient solution (decreasing exponentially when $x \to \infty$) is Whittaker’s function $D_\nu(y)$, where
\[ y = 2\sqrt{a\lambda} x \quad \text{and} \quad \nu = a\lambda - \frac{1}{2}. \quad (A.10) \]

The eigenvalues of the isolated potential well are obtained for $\nu = n$ an integer number. In leading order in $\lambda$ one finds
\[ \tilde{\epsilon}_n^{(0)} \approx \frac{1}{\sqrt{m^*}} \left( \frac{1}{2} + n \right) + O(\lambda^2). \quad (A.11) \]

Now, due to existence of the other degenerate minima of the potential $V(\phi)$ each eigenvalue of the isolated well is splitted into two very near levels. They can be found using the boundary conditions for the wave function and its derivative
\[ \frac{d\Psi}{d\phi}|_{\phi=0} = 0, \quad \Psi|_{\phi=0}. \quad (A.12) \]

The solution $\Psi(\phi)$ of (A.1) is related to the solution $R(x)$ of (A.9) by
\[ \Psi(\phi) = \left( \frac{dx}{d\phi} \right)^{-\frac{1}{2}} R(x), \]
and the explicit connection formula between the old variable $\phi$ and the new variable $x$ is (asymptotic region of $x$)
\[ x^2 \simeq \frac{1}{a} \int_0^{\mu_1} \sqrt{[\tilde{\epsilon} - V]} \, d\phi + \frac{1}{2} + \ln 2|x| + O(x^{-2}). \quad (A.13) \]

In applying the boundary conditions (A.11) we have to use the asymptotic expansion of $D_\nu(y)$ [30]
\[ D_\nu(y) \sim y^\nu e^{-\frac{y^2}{4}} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu} \frac{1}{y^{1+\nu}} e^{\frac{y^2}{4}}. \quad (A.14) \]

The value of $\nu$ is found from a matching between the dominant $e^{\frac{y^2}{4}}$ and the subdominant term $e^{-\frac{y^2}{4}}$. Then from (A.12,A.13,A.14) one obtains
\[ -1 = 2\sqrt{\pi e} e^{i\pi\nu} \frac{\Gamma(\frac{1}{2} + \nu)}{\Gamma(-\nu)} \exp \left( 2\lambda \int_0^{\mu_1} \sqrt{\tilde{\epsilon} - V} \, d\phi + \nu(1 + \ln 2) - \frac{1}{2}(1 + 2\nu) \ln(1 + 2\nu) \right). \quad (A.15) \]

The sine-Gordon case can be treated in a very similar way, the solution for the comparison equation (A.9) being, in that case, expressed more convenient in terms of Kummer’s functions having a definite parity. Each eigenvalue of an isolated potential well is splitted into a narrow allowed band. If the potential $V(\phi)$ has a $2\pi$-periodicity the index $\nu$ corresponding to the lower and upper boundary of the lowest band follows from similar conditions as (A.12) calculated in the point $\phi = \pi$. Finally a very similar relation with (A.15) is found, namely
\[ -1 = 2\sqrt{\pi e} e^{-i\pi\nu} \frac{\Gamma(\frac{1}{2} + \nu)}{\Gamma(-\nu)} \exp \left( 2\lambda \int_0^{\pi} \sqrt{\tilde{\epsilon} - V} \, d\phi + 2\nu(1 + \ln 2) - \frac{1}{2}(1 + 4\nu) \ln(1 + 4\nu) \right). \quad (A.15a) \]

Here by $\pm \tilde{\phi}$ we have denoted the turning points of the isolated potential well, centred around the minimum $\phi = 0$. Also the definition of $\nu$ is slightly modified and instead of (A.10) we have $2\nu = a\lambda - \frac{1}{2}$. To conclude we have to indicate the connection relation between the symmetric splitting of each eigenvalue of the isolated potential well, $E_0 \to \tilde{\epsilon}_0 = E_0 \pm t_0$ and the quantity $\nu$. We get
\[ t_0 = \frac{c}{\sqrt{m^*} \nu} \quad (A.16) \]
where $c = 1$ or 2 for the $\phi^4$ or the SG model respectively.
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