Full characterization of graphs having certain normalized Laplacian eigenvalue of multiplicity $n - 3$

Fenglei Tian*, Yiju Wang

School of Management, Qufu Normal University, Rizhao, China.

Abstract: Let $G$ be a connected simple graph of order $n$. Let $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_{n-1}(G) > \rho_n(G) = 0$ be the eigenvalues of the normalized Laplacian matrix $\mathcal{L}(G)$ of $G$. Denote by $m(\rho_i)$ the multiplicity of the normalized Laplacian eigenvalue $\rho_i$. Let $\nu(G)$ be the independence number of $G$. In this paper, we give a full characterization of graphs with some normalized Laplacian eigenvalue of multiplicity $n - 3$, which answers a remaining problem in [S. Sun, K.C. Das, On the multiplicities of normalized Laplacian eigenvalues of graphs, Linear Algebra Appl. 609 (2021) 365-385], i.e., there is no graph with $m(\rho_1) = n - 3$ ($n \geq 6$) and $\nu(G) = 2$. Moreover, we confirm that all the graphs with $m(\rho_1) = n - 3$ are determined by their normalized Laplacian spectra.

Keywords: normalized Laplacian; normalized Laplacian eigenvalues; multiplicity of eigenvalues

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1 Introduction

Throughout, only connected and simple graphs are discussed. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $N_G(u)$ be the set of all the neighbors of the vertex $u$. Then $d_u = |N_G(u)|$ is called the degree of $u$. For a subset $S \subset V(G)$, $S$ is called a set of twin points if $N_G(u) = N_G(v)$ for any $u, v \in S$. By $u \sim v$, we mean that $u$ and $v$ are adjacent. The distance of two vertices $u, v$ is denoted by $d(u, v)$ and the diameter of a graph $G$ is written as $diam(G)$. A subset $S$ of $V(G)$ is called an independent set of $G$, if the vertices of $S$ induce an empty subgraph. The cardinality of the maximum independent set of $G$ is called the independence number, denoted by $\nu(G)$. The rank of a matrix $M$ is written as $r(M)$. Let $R_{vi}$ be the row of $M$ indexed by the vertex $v_i$. The multiplicity of an eigenvalue $\lambda$ of $M$ is denoted by $m(\lambda)$. Denote by $\mathcal{G}(n, n-3)$ the set of all $n$-vertex ($n \geq 5$) connected graphs with some normalized Laplacian eigenvalue of multiplicity $n - 3$. Let $A(G)$ and $L(G) = D(G) - A(G)$ be the adjacency matrix and the Laplacian matrix of graph $G$.

*Corresponding author. E-mail address: tflqfd@qfnu.edu.cn. Supported by ” the Natural Science Foundation of Shandong Province (No. ZR2019BA016) ”.
Theorem 1.1. Let \( \rho \) and \( \nu \) respectively. Then the normalized Laplacian matrix \( L(G) = [l_{uv}] \) of graph \( G \) is defined as
\[
L(G) = D^{-1/2}(G)L(G)D^{-1/2}(G) = I - D^{-1/2}(G)A(G)D^{-1/2}(G),
\]
where
\[
l_{uv} = \begin{cases} 
1, & \text{if } u = v; \\
-1/\sqrt{d_u d_v}, & \text{if } u \sim v; \\
0, & \text{otherwise.}
\end{cases}
\]
For brevity, the normalized Laplacian eigenvalues are written as \( L \)-eigenvalues. It is well known that the least \( L \)-eigenvalue of a connected graph is 0 with multiplicity 1 (see \cite{10}). Then let the \( L \)-eigenvalues of a graph \( G \) be
\[
\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_{n-1}(G) > \rho_n(G) = 0.
\]

The normalized Laplacian eigenvalues of graphs have been studied intensively (see for example \cite{1–8}), as it reveals not only some structural properties but also some relevant dynamical aspects (such as random walk) of graphs \cite{10}. Recently, the multiplicity of the normalized Laplacian eigenvalues attracts much attention. Van Dam and Omidi \cite{1} determined the graphs with some normalized Laplacian eigenvalue of multiplicity \( n-1 \) and \( n-2 \), respectively. Tian et al. \cite{11,12} characterized some families of graphs of \( G(n,n-3) \), but the graphs with \( \rho_{n-1}(G) \neq 1 \) and \( \nu(G) = \text{diam}(G) = 2 \) that contain induced \( P_4 \) are not considered, which is the last remaining case. Sun and Das \cite{9} presented the graphs of \( G(n,n-3) \) with \( m(\rho_{n-1}(G)) = n-3 \) and \( m(\rho_{n-2}(G)) = n-3 \) respectively, and gave the following problem.

Problem \cite{9}: Is it true that there exists no connected graph with \( m(\rho_1(G)) = n-3 \ (n \geq 6) \) and \( \nu(G) = 2 \) ?

To answer the above problem, it is urgent to complete the characterization of all the graphs in \( G(n,n-3) \). Note that the authors of \cite{11,12} have obtained the following results.

Theorem 1.1. \cite{11,12} Let \( G \in G(n,n-3) \) be a graph of order \( n \geq 5 \). Then

(i) \( \rho_{n-1}(G) = 1 \) if and only if \( G \) is a complete tripartite graph \( K_{a,b,c} \) or \( K_n - e \), where \( K_n - e \) is the graph obtained from the complete graph \( K_n \) by removing an edge.

(ii) \( \rho_{n-1}(G) \neq 1 \) and \( \nu(G) \neq 2 \) if and only if \( G \in \{G_1,G_2,G_3\} \) (see Fig. 1).

(iii) \( \rho_{n-1}(G) \neq 1 \), \( \nu(G) = 2 \) and \( \text{diam}(G) = 3 \) if and only if \( G = G_4 \) (see Fig. 1).

(iv) \( G \) is a cograph with \( \rho_{n-1}(G) \neq 1 \) and \( \nu(G) = 2 \) if and only if \( G = G_5 \) (see Fig. 1).

Hence, to characterize all graphs of \( G(n,n-3) \) and to address the above problem in \cite{9}, it suffices to consider the graphs that contain induced path \( P_4 \) with \( \rho_{n-1}(G) \neq 1 \) and \( \nu(G) = \text{diam}(G) = 2 \). Here, we obtain the following conclusion.

Theorem 1.2. Let \( G \in G(n,n-3) \) be a graph of order \( n \geq 5 \). Then \( G \) contains induced path \( P_4 \) with \( \rho_{n-1}(G) \neq 1 \) and \( \nu(G) = \text{diam}(G) = 2 \) if and only if \( G \) is the cycle \( C_5 \).
**Remark 1.3.** Combining Theorems 1.1 and 1.2, all graphs of $G(n, n - 3)$ are determined. As a result, the above problem in [4] is answered, that is, there is no connected graph with $m(\rho_1(G)) = n - 3$ $(n \geq 6)$ and $\nu(G) = 2$. Now, we can also confirm the uncertain result in [4] that if $G$ is the graph with $m(\rho_1(G)) = n - 3$ then $G$ is determined by its normalized Laplacian spectrum.

![Fig. 1: The graphs $G_i$ (1 ≤ i ≤ 5).](image)

The rest of the paper is arranged as follows. In Section 2, some lemmas are introduced. In Section 3, the proof of Theorem 1.2 is presented.

## 2 Preliminaries

For brevity, let $\Omega$ be the set of graphs of $G(n, n - 3)$ that contain induced path $P_4$ with $\rho_{n-1}(G) \neq 1$ and $\nu(G) = \text{diam}(G) = 2$. We always let $\theta$ be the $L$-eigenvalue of $G$ with multiplicity $n - 3$.

**Lemma 2.1.** [7] Let $G$ be a connected graph with order $n \geq 4$. If $G \not\cong K_{p,q}, K_a \vee (n-a)K_1$ $(p+q = n, a \geq 2)$, then $\rho_2(G) \geq \frac{n-1}{n-2}$.

**Lemma 2.2.** [2] Let $G \not\cong K_n$ be a connected graph with order $n \geq 2$. Then $\rho_{n-1}(G) \leq 1$.

**Lemma 2.3.** [5] Let $G$ be a graph with $n$ vertices. Let $K = \{v_1, \ldots, v_q\}$ be a clique in $G$ such that $N_G(v_i) - K = N_G(v_j) - K (1 \leq i, j \leq q)$, then $1 + \frac{1}{d_{v_i}}$ is an $L$-eigenvalue of $G$ with multiplicity at least $q - 1$.

**Lemma 2.4.** [11] Let $G \in \mathcal{G}(n, n - 3)$ with $\rho_{n-1}(G) \neq 1$, then $\theta \neq 1$.

**Lemma 2.5.** [12] Let $G \in \Omega$ with an induced path $P_4 = v_1v_2v_3v_4$. Then

$$(1 - \theta)^4d_{v_1}d_{v_2}d_{v_3}d_{v_4} - (d_{v_1}d_{v_2} + d_{v_3}d_{v_4} + d_{v_1}d_{v_4})(1 - \theta)^2 + 1 = 0. \quad (1)$$

Moreover, if there is a vertex $u_1$ (resp., $u_2$) such that $u_1v_2v_3v_4$ (resp., $v_1u_2v_3v_4$) is also an induced path, then $d_{v_1} = d_{u_1}$ (resp., $d_{v_2} = d_{u_2}$).

The following lemma is useful for us to complete the proof of Theorem 1.2.

**Lemma 2.6.** Let $G \in \Omega$ and $H_i$ (1 ≤ i ≤ 5) be the graphs as shown in Fig. 2. Then the following assertions hold.

(i) If $G$ contains $H_1$ as an induced subgraph, then $1 - \theta = -\frac{1}{d_{v_1}} = -\frac{1}{d_{v_5}}$. 


(ii) If $G$ contains $H_2$ as an induced subgraph, then $1 - \theta = -\frac{1}{d_{v2}} = -\frac{1}{d_{v3}}$.

(iii) If $G$ contains $H_3$ as an induced subgraph, then

$$(1 - \theta)^2d_{v1}d_{v2} + (1 - \theta)d_{v4} - 1 = 0.$$ 

(iv) If $G$ contains $H_4$ as an induced subgraph, then

$$
\begin{align*}
(1 - \theta) &= -\frac{d_{v3} + d_{v4}}{d_{v3}d_{v2}} = -\frac{d_{v3} + d_{v4}}{d_{v3}d_{v2}} \\
(1 - \theta)^2d_{v1}(d_{v3} + d_{v4}) + (1 - \theta)d_{v3} - 1 &= 0.
\end{align*}
$$

(v) If $G$ contains $H_5$ as an induced subgraph, then

$$(1 - \theta) = -\frac{d_{v3} + 2d_{v4}}{d_{v3}(d_{v2} + d_{v5})} = -\frac{d_{v3} + 2d_{v4}}{d_{v3}(d_{v2} + d_{v5})}.$$ 

(vi) If $G$ contains $H_6$ as an induced subgraph, then $1 - \theta = -\frac{1}{d_{v1}} = -\frac{1}{d_{v4}}$.

**Proof.** We first show the proof of assertion (i).

Since $m(\theta) = n - 3$, then $r(L(G) - \theta I) = 3$. Lemma 2.4 indicates that $\theta \neq 1$. Denote by $M_1$ the principal submatrix of $L(G) - \theta I$ indexed by the vertices of $H_1$, then

$$M_1 = \begin{pmatrix}
1 - \theta & -1 & 0 & 0 & -1 \\
-1 & 1 - \theta & \sqrt{d_{v1}d_{v2}} & 0 & 0 \\
0 & -1 & 1 - \theta & \sqrt{d_{v2}d_{v3}} & 0 \\
0 & 0 & -1 & 1 - \theta & 0 \\
-1 & -1 & 0 & 0 & 1 - \theta
\end{pmatrix}.$$ 

One can easily obtain that the first three rows of $M_1$ are linearly independent (considering the minor indexed by the first three rows and the middle three columns of $M_1$), which yields that the rows $R_{v1}, R_{v2}, R_{v3}$ of $L(G) - \theta I$ are linearly independent, and then $R_{v5}$ can be written as a linear combination of $R_{v1}, R_{v2}, R_{v3}$. Let

$$R_{v5} = aR_{v1} + bR_{v2} + cR_{v3}, \quad (2)$$
then
\[
\begin{pmatrix}
    a(1 - \theta) - \frac{b}{\sqrt{d_{e_1}d_{e_2}}} &=& \frac{-1}{\sqrt{d_{e_1}d_{e_5}}}, \\
    \frac{-a}{\sqrt{d_{e_1}d_{e_2}}} + b(1 - \theta) - \frac{c}{\sqrt{d_{e_2}d_{e_3}}} &=& \frac{-1}{\sqrt{d_{e_2}d_{e_5}}}, \\
    -\frac{b}{\sqrt{d_{e_2}d_{e_3}}} + c(1 - \theta) &=& 0, \\
    -\frac{c}{\sqrt{d_{e_3}d_{e_4}}} &=& 0
\end{pmatrix}
\tag{3}
\]

The fourth equation of (3) implies that \( c = 0 \), and further \( b = 0 \) from the third one. Recalling that \( d_{e_1} = d_{e_5} \) by Lemma 2.5, then we have \( a = 1 \) from the second of (3), and thus \( 1 - \theta = \frac{-1}{d_{e_1}} = \frac{-1}{d_{e_5}} \) by the first of (3).

For assertion (ii), let
\[
M_2 = \begin{pmatrix}
1 - \theta & -\frac{1}{\sqrt{d_{e_1}d_{e_2}}} & 0 & 0 & \frac{-1}{\sqrt{d_{e_1}d_{e_5}}} \\
-\frac{1}{\sqrt{d_{e_1}d_{e_2}}} & 1 - \theta & \frac{-1}{\sqrt{d_{e_2}d_{e_3}}} & 0 & \frac{-1}{\sqrt{d_{e_2}d_{e_5}}} \\
0 & \frac{-1}{\sqrt{d_{e_2}d_{e_3}}} & 1 - \theta & \frac{-1}{\sqrt{d_{e_3}d_{e_4}}} & \frac{-1}{\sqrt{d_{e_3}d_{e_5}}} \\
0 & 0 & \frac{-1}{\sqrt{d_{e_3}d_{e_4}}} & 1 - \theta & 0 \\
-\frac{1}{\sqrt{d_{e_1}d_{e_5}}} & \frac{-1}{\sqrt{d_{e_2}d_{e_5}}} & \frac{-1}{\sqrt{d_{e_3}d_{e_5}}} & 0 & 1 - \theta
\end{pmatrix}
\]

be the principal submatrix of \( L(G) - \theta I \) indexed by the vertices of \( H_2 \). Similar as above discussion, one can assume that the Eq. (2) still holds. Then
\[
\begin{pmatrix}
    a(1 - \theta) - \frac{b}{\sqrt{d_{e_1}d_{e_2}}} &=& \frac{-1}{\sqrt{d_{e_1}d_{e_5}}}, \\
    \frac{-a}{\sqrt{d_{e_1}d_{e_2}}} + b(1 - \theta) - \frac{c}{\sqrt{d_{e_2}d_{e_3}}} &=& \frac{-1}{\sqrt{d_{e_2}d_{e_5}}}, \\
    -\frac{b}{\sqrt{d_{e_2}d_{e_3}}} + c(1 - \theta) &=& \frac{-1}{\sqrt{d_{e_3}d_{e_4}}}, \\
    -\frac{c}{\sqrt{d_{e_3}d_{e_4}}} &=& 0.
\end{pmatrix}
\tag{4}
\]

By the fourth equation of (4), we see \( c = 0 \). Further, recalling that \( d_{e_2} = d_{e_5} \) for \( H_2 \) by Lemma 2.5 we get \( b = 1 \) from the third one. Then \( a = 0 \) from the first of (4), and thus \( 1 - \theta = -\frac{1}{d_{e_1}} = \frac{-1}{d_{e_5}} \) by the second one.

For assertion (iii), let
\[
M_3 = \begin{pmatrix}
1 - \theta & -\frac{1}{\sqrt{d_{e_1}d_{e_2}}} & 0 & 0 & \frac{-1}{\sqrt{d_{e_1}d_{e_5}}} \\
-\frac{1}{\sqrt{d_{e_1}d_{e_2}}} & 1 - \theta & \frac{-1}{\sqrt{d_{e_2}d_{e_3}}} & 0 & \frac{-1}{\sqrt{d_{e_2}d_{e_5}}} \\
0 & \frac{-1}{\sqrt{d_{e_2}d_{e_3}}} & 1 - \theta & \frac{-1}{\sqrt{d_{e_3}d_{e_4}}} & \frac{-1}{\sqrt{d_{e_3}d_{e_5}}} \\
0 & 0 & \frac{-1}{\sqrt{d_{e_3}d_{e_4}}} & 1 - \theta & 0 \\
-\frac{1}{\sqrt{d_{e_1}d_{e_5}}} & \frac{-1}{\sqrt{d_{e_2}d_{e_5}}} & \frac{-1}{\sqrt{d_{e_3}d_{e_5}}} & 0 & 1 - \theta
\end{pmatrix}
\]

be the principal submatrix of \( L(G) - \theta I \) indexed by the vertices of \( H_3 \). Clearly, the middle three rows of \( M_3 \) are linearly independent, which yields that the rows \( R_{e_2}, R_{e_3}, R_{e_4} \) of \( L(G) - \theta I \)
are linearly independent, and then we set

\[ R_{v_1} = aR_{v_2} + bR_{v_3} + cR_{v_4}. \]  

(5)

Applying (5) to the columns of \( M_3 \), we get

\[
\begin{align*}
-a \sqrt{d_{v_3}d_{v_5}} &= 1 - \theta \\
-b \sqrt{d_{v_2}d_{v_3}} &= \frac{-1}{\sqrt{d_{v_1}d_{v_2}}} \\
-c \sqrt{d_{v_3}d_{v_4}} &= \frac{-1}{\sqrt{d_{v_1}d_{v_3}}} \\
-a \sqrt{d_{v_2}d_{v_3}} + c(1 - \theta) &= 0 \\
-b \sqrt{d_{v_3}d_{v_4}} + c(1 - \theta) &= 0 \\
-c \sqrt{d_{v_4}d_{v_5}} &= \frac{-1}{\sqrt{d_{v_1}d_{v_5}}}.
\end{align*}
\]  

(6)

The first and the fourth equations of (6) tell us that \( a = -(1 - \theta) \sqrt{d_{v_1}d_{v_2}} \) and \( c = \sqrt{d_{v_4}d_{v_1}} \), and further \( b = (1 - \theta) d_{v_4} \sqrt{d_{v_2}d_{v_1}} \) from the third one. Taking the values of \( a, b, c \) into the second of (6), we derive that \( (1 - \theta)^2 d_{v_1}d_{v_2} + (1 - \theta)d_{v_4} - 1 = 0 \), as required.

For assertion (iv), let

\[
M_4 = \begin{pmatrix}
1 - \theta & -1 & 0 & 0 & -1 \\
\frac{-1}{\sqrt{d_{v_1}d_{v_2}}} & 1 - \theta & -1 & 0 & -1 \\
0 & \frac{-1}{\sqrt{d_{v_2}d_{v_3}}} & 1 - \theta & -1 & 0 \\
0 & 0 & \frac{-1}{\sqrt{d_{v_3}d_{v_4}}} & 1 - \theta & -1 \\
\frac{-1}{\sqrt{d_{v_1}d_{v_5}}} & \frac{-1}{\sqrt{d_{v_2}d_{v_5}}} & 0 & \frac{-1}{\sqrt{d_{v_3}d_{v_5}}} & 1 - \theta
\end{pmatrix}
\]

be the principal submatrix of \( \mathcal{L}(G) - \theta I \) indexed by the vertices of \( H_4 \). It is clear that the first three rows of \( M_4 \) are linearly independent, which indicates that the rows \( R_{v_1}, R_{v_2}, R_{v_3} \) of \( \mathcal{L}(G) - \theta I \) are linearly independent. Let

\[ R_{v_5} = aR_{v_1} + bR_{v_2} + cR_{v_3}. \]  

(7)

Applying (7) to the columns of \( M_4 \), we have

\[
\begin{align*}
-a \frac{1}{\sqrt{d_{v_1}d_{v_2}}} &= (1 - \theta) \\
-b \frac{1}{\sqrt{d_{v_2}d_{v_3}}} &= \frac{-1}{\sqrt{d_{v_1}d_{v_2}}} \\
-c \frac{1}{\sqrt{d_{v_3}d_{v_4}}} &= \frac{-1}{\sqrt{d_{v_1}d_{v_3}}} \\
-a \frac{1}{\sqrt{d_{v_2}d_{v_3}}} + c(1 - \theta) &= 0 \\
-b \frac{1}{\sqrt{d_{v_3}d_{v_4}}} + c(1 - \theta) &= 0 \\
-c \frac{1}{\sqrt{d_{v_4}d_{v_5}}} &= \frac{-1}{\sqrt{d_{v_1}d_{v_5}}}.
\end{align*}
\]  

(8)
Combining the last three equations of (8), it follows that
\[ a = -\sqrt{d_{v_1}d_{v_5}}(1 - \theta)(\frac{d_{v_2}}{d_{v_5}} + 1), \quad b = d_{v_3}(1 - \theta)\sqrt{\frac{d_{v_2}}{d_{v_5}}}, \quad c = \sqrt{\frac{d_{v_3}}{d_{v_5}}}. \]

Taking the values of \( a, b, c \) into the first and second equations of (8) respectively, one can easily derive that
\[
\begin{align*}
(1 - \theta)^2 d_{v_1}(d_{v_3} + d_{v_5}) + (1 - \theta)d_{v_3} - 1 &= 0 \\
(1 - \theta) &= -\frac{d_{v_3} + d_{v_5}}{d_{v_3}d_{v_5}}.
\end{align*}
\]

Moreover, by the symmetry between \( v_2 \) and \( v_5 \) (resp., \( v_3 \) and \( v_4 \)) in \( H_4 \), we can also get
\[
(1 - \theta) = -\frac{d_{v_3} + d_{v_5}}{d_{v_3}d_{v_5}}.
\]

At last, we prove assertion (v). Let the principal submatrix of \( \mathcal{L}(G) - \theta I \) indexed by the vertices of \( H_5 \) be \( M_5 \), then
\[
M_5 = \begin{pmatrix}
1 - \theta & -1 & \frac{d_{v_1}}{d_{v_2}} & 0 & 0 & \frac{1}{\sqrt{d_{v_1}d_{v_5}}} \\
-1 & 1 - \theta & \frac{1}{\sqrt{d_{v_2}d_{v_3}}} & 0 & \frac{1}{\sqrt{d_{v_2}d_{v_5}}} & -1 \\
0 & -1 & 1 - \theta & \frac{1}{\sqrt{d_{v_3}d_{v_4}}} & \frac{1}{\sqrt{d_{v_3}d_{v_5}}} & -1 \\
0 & 0 & -1 & 1 - \theta & \frac{1}{\sqrt{d_{v_4}d_{v_5}}} & -1 \\
-1 & 1 & \frac{1}{\sqrt{d_{v_5}d_{v_2}}} & \frac{1}{\sqrt{d_{v_5}d_{v_3}}} & \frac{1}{\sqrt{d_{v_5}d_{v_4}}} & 1 - \theta
\end{pmatrix}.
\]

Similar as the discussion in assertion (iv), the Eq. (7) can still hold. Then applying (7) to the columns of \( M_5 \), we obtain
\[
\begin{align*}
\begin{cases}
\begin{aligned}
a(1 - \theta) - \frac{b}{\sqrt{d_{v_1}d_{v_2}}} & = -\frac{1}{\sqrt{d_{v_1}d_{v_5}}} \\
-\frac{b}{\sqrt{d_{v_2}d_{v_3}}} + c(1 - \theta) & = -\frac{1}{\sqrt{d_{v_3}d_{v_5}}} \\
-\frac{c}{\sqrt{d_{v_3}d_{v_4}}} & = -\frac{1}{\sqrt{d_{v_3}d_{v_5}}} \\
-\frac{a}{\sqrt{d_{v_4}d_{v_5}}} - \frac{b}{\sqrt{d_{v_2}d_{v_5}}} - \frac{c}{\sqrt{d_{v_3}d_{v_5}}} & = 1 - \theta.
\end{aligned}
\end{cases}
\end{align*}
\]

It follows from the first three equations of (9) that
\[
\begin{align*}
a &= \frac{d_{v_3}}{d_{v_1}d_{v_5}}, \quad b = \sqrt{d_{v_2}d_{v_3}}((1 - \theta)\sqrt{\frac{d_{v_3}}{d_{v_5}}} + \frac{1}{\sqrt{d_{v_2}d_{v_5}}}), \quad c = \sqrt{\frac{d_{v_3}}{d_{v_5}}}.
\end{align*}
\]

Taking the values of \( a, b, c \) into the last of (9), one can derive
\[
(1 - \theta) = -\frac{d_{v_3} + 2d_{v_1}}{d_{v_1}(d_{v_3} + d_{v_5})}.
\]
Furthermore, the symmetry of $H_5$ implies that

$$(1 - \theta) = - \frac{d_{v_2} + 2d_{v_4}}{d_{v_4}(d_{v_2} + d_{v_4})},$$

as required.

For assertion (vi), one can refer to the process of proving Claim 1 of Lemma 3.2 in [12]. □

3 Proof of Theorem 1.2

Let $G \in \Omega$, i.e., $G$ is a graph of $G(n, n - 3)$ containing induced path $P_4$ with $\rho_{n-1}(G) \neq 1$ and $\nu(G) = diam(G) = 2$. Suppose $m(\theta) = n - 3$ in $G$. Now we prove Theorem 1.2.

Proof of Theorem 1.2 By direct calculation, the normalized Laplacian spectrum of the cycle $C_5$ is $\{0.691^2, 1.809^2, 0\}$, then it follows that $G \in \Omega$. Thus the sufficiency is clear.

In the following, we present the necessity part. Suppose that $G \in \Omega$ and $m(\theta) = n - 3$ in $G$, then $\theta \neq 1$ from Lemma 2.4. Denote by $P_4 = v_1v_2v_3v_4$ an induced path of $G$. Assume that $U \subseteq V(P_4)$ and

$$S_U = \{u \in V(G) \setminus V(P_4) : N_G(u) \cap V(P_4) = U\}.$$

It follows from $\nu(G) = 2$ that any vertex out of $V(P_4)$ must be adjacent to at least two of $V(P_4)$ and $S_{\{v_1,v_3\}} = S_{\{v_2,v_4\}} = S_{\{v_2,v_3\}} = \emptyset$. Further, since $diam(G) = 2$, then $d(v_1, v_4) = 2$, and thus there exists a vertex, say $v_5$, adjacent to $v_1$ and $v_4$. Note that $v_5$ maybe belong to $S_{\{v_1,v_4\}}$, $S_{\{v_1,v_2,v_4\}}$, $S_{\{v_1,v_3,v_4\}}$ or $S_{\{v_1,v_2,v_3,v_4\}}$. Accordingly, the remaining proof can be divided into the following cases.

Case 1. Suppose that $v_5 \in S_{\{v_1,v_4\}}$, i.e., $S_{\{v_1,v_4\}} \neq \emptyset$.

We will complete the discussion of this case by the following claims.

Claim 1.1 $|S_{\{v_1,v_4\}}| = 1$ and $S_{\{v_1,v_2\}} = S_{\{v_3,v_4\}} = \emptyset$.

Suppose that $|S_{\{v_1,v_4\}}| \geq 2$, then all the vertices of $S_{\{v_1,v_4\}}$ induce a clique (otherwise, $\nu(G) \geq 3$, a contradiction). Then $G$ contains an induced subgraph isomorphic to $X_1$ in Fig. 3. Similarly, one can obtain that if $S_{\{v_1,v_2\}} \neq \emptyset$ or $S_{\{v_3,v_4\}} \neq \emptyset$, $G$ also contains an induced subgraph isomorphic to $X_1$. Since $X_1$ contains $H_1$ as an induced subgraph, then by Lemma 2.6 (i)

$$1 - \theta = - \frac{1}{d_{v_1}}. \quad (10)$$

Moreover, $C_5$ (i.e., $H_3$) is an induced subgraph of $X_1$, then by Lemma 2.6 (iii)

$$(1 - \theta)^2d_{v_1}d_{v_2} + (1 - \theta)d_{v_4} - 1 = 0. \quad (11)$$

Combining (10) and (11), we get

$$d_{v_2} = d_{v_1} + d_{v_4}. \quad (12)$$
It is not hard to see that
\[
\begin{cases}
    d_{v_2} = |S_{\{v_1,v_2\}}| + |S_{\{v_1,v_2,v_3\}}| + |S_{\{v_1,v_2,v_4\}}| + |S_{\{v_1,v_2,v_3,v_4\}}| + 2 \\
    d_{v_1} = |S_{\{v_1,v_2\}}| + |S_{\{v_1,v_3\}}| + |S_{\{v_1,v_2,v_3\}}| + |S_{\{v_1,v_2,v_3,v_4\}}| + 1 \\
    d_{v_4} = |S_{\{v_3,v_4\}}| + |S_{\{v_4\}}| + |S_{\{v_2,v_3,v_4\}}| + |S_{\{v_2,v_3,v_4\}}| + 1
\end{cases}
\]

(13)

It follows from (12) and (13) that \(|S_{\{v_1,v_4\}}| = 0\), a contradiction.

Claim 1.2 \(S_{\{v_1,v_2,v_3\}} = S_{\{v_2,v_3,v_4\}} = \emptyset\).

It suffices to prove that \(S_{\{v_1,v_2,v_3\}} = \emptyset\). Suppose for a contradiction that \(S_{\{v_1,v_2,v_3\}} \neq \emptyset\) and \(v_6 \in S_{\{v_1,v_2,v_3\}}\). If \(v_5 \sim v_6\), then the vertices \(v_i \ (1 \leq i \leq 6)\) induce a subgraph isomorphic to \(X_1\). One can obtain a contradiction by similar discussion as above. If \(v_5 \sim v_6\), then \(X_2\) in Fig. 3 is an induced subgraph of \(G\). Deleting \(v_3\) with the incident edges from \(X_2\), we also get (10) by Lemma 2.6 (ii). Analogous discussion as Claim 1.1, the Eq. (12) still holds. As \(|S_{\{v_1,v_4\}}| = 1\) and \(S_{\{v_1,v_2\}} = S_{\{v_3,v_4\}} = \emptyset\) from Claim 1.1, then
\[
\begin{cases}
    d_{v_2} = |S_{\{v_1,v_2,v_3\}}| + |S_{\{v_2,v_3,v_4\}}| + |S_{\{v_2,v_3,v_4\}}| + |S_{\{v_1,v_2,v_3,v_4\}}| + 2 \\
    d_{v_1} = |S_{\{v_1,v_4\}}| + |S_{\{v_1,v_2,v_3\}}| + |S_{\{v_1,v_2,v_3,v_4\}}| + |S_{\{v_1,v_2,v_3,v_4\}}| + 1 \\
    d_{v_4} = |S_{\{v_1,v_4\}}| + |S_{\{v_2,v_3,v_4\}}| + |S_{\{v_2,v_3,v_4\}}| + |S_{\{v_2,v_3,v_4\}}| + 1
\end{cases}
\]

which, together with (12), yields that \(|S_{\{v_1,v_4\}}| = 0\), a contradiction.

Claim 1.3 \(S_{\{v_1,v_2,v_4\}} = S_{\{v_1,v_3,v_4\}} = \emptyset\).

We only need to show \(S_{\{v_1,v_2,v_4\}} = \emptyset\). Assume that \(S_{\{v_1,v_2,v_4\}} \neq \emptyset\) and \(v_6 \in S_{\{v_1,v_2,v_4\}}\), then \(v_5 \sim v_6\) (otherwise \(\nu(G) \geq 3\), a contradiction). Thus \(X_3\) in Fig. 3 is an induced subgraph of \(G\). Removing \(v_4\) with the incident edges from \(X_3\), we get (10) again by Lemma 2.6 (ii). By analogous discussion as above, one can easily obtain a contradiction.

Claim 1.4 \(S_{\{v_1,v_2,v_3,v_4\}} = \emptyset\).

Combining the first three claims, we see that if \(S_{\{v_1,v_2,v_3,v_4\}} \neq \emptyset\), then all the vertices of \(V(G) \setminus \{v_1, \cdots, v_5\}\) belong to \(S_{\{v_1,v_2,v_3,v_4\}}\). Then
\[
d_{v_1} = d_{v_2} = d_{v_3} = d_{v_4}.
\]

(14)

Since \(G\) now contains \(H_5\) (see Fig. 2) as an induced subgraph, then by Lemma 2.6 (v)
\[
(1 - \theta) = -\frac{d_{v_2} + 2d_{v_4}}{d_{v_3}(d_{v_2} + d_{v_4})},
\]
which is a rational number. Furthermore, since $G$ contains an induced $C_5$, then by (11) and (14) we derive that
\[(1 - \theta) = \frac{-1 \pm \sqrt{5}}{2d_{v_1}},\]
which is an irrational number, a contradiction.

The above four claims indicate that if $S_{\{v_1, v_4\}} \neq \emptyset$, then $|S_{\{v_1, v_4\}}| = 1$ and $|V(G)| = 5$, i.e., $G$ is the cycle $C_5$.

**Case 2.** Suppose that $v_5 \in S_{\{v_1, v_2, v_4\}}$, i.e., $S_{\{v_1, v_2, v_4\}} \neq \emptyset$ and $S_{\{v_1, v_4\}} = \emptyset$.

The following claims will help us complete the discussion of this case.

**Claim 2.1** $S_{\{v_1, v_2\}} = S_{\{v_3, v_4\}} = \emptyset$.

We first demonstrate $S_{\{v_1, v_2\}} = \emptyset$. If $S_{\{v_1, v_2\}} \neq \emptyset$, say $v_6 \in S_{\{v_1, v_2\}}$, then by Lemma 2.6 (i), the Eq. (10) holds. Since $G$ contains $H_4$ (see Fig. 2) as an induced subgraph, then by Lemma 2.6 (iv)
\[
\begin{align*}
(1 - \theta)^2d_{v_1}(d_{v_3} + d_{v_5}) + (1 - \theta)d_{v_3} - 1 &= 0 \\
(1 - \theta) &= -\frac{d_{v_3} + d_{v_5}}{d_{v_4}d_{v_5}}.
\end{align*}
\]
By (10) and the first equation of (15), we get $d_{v_1} = d_{v_5}$, which implies that $(1 - \theta) = -\frac{1}{d_{v_5}}$. Hence, by the second equation of (15), we have $d_{v_2} = 0$, a contradiction.

Next, we prove that $S_{\{v_3, v_4\}} = \emptyset$. Suppose that $S_{\{v_3, v_4\}} \neq \emptyset$ and $v_6 \in S_{\{v_3, v_4\}}$, then Lemma 2.6 (i) indicates that
\[(1 - \theta) = -\frac{1}{d_{v_4}}.
\]
It follows from (16) and the second equation of (15) that $d_{v_2} + d_{v_3} = d_{v_5}$. Note that any vertex out of $V(P_2)$ must be adjacent to $v_2$ or $v_4$ (thanks to $\nu(G) = 2$). Thus
\[d_{v_5} = d_{v_2} + d_{v_4} \geq 3 + 3 + n - 6 = n,
\]
a contradiction. Therefore, $S_{\{v_1, v_2\}} = S_{\{v_3, v_4\}} = \emptyset$, as required.

**Claim 2.2** $S_{\{v_1, v_2, v_3\}} = S_{\{v_2, v_3, v_4\}} = \emptyset$.

If $S_{\{v_1, v_2, v_3\}} \neq \emptyset$, then $G$ contains $H_2$ and $H_4$ as induced subgraphs. Thus, from Lemma 2.6 (ii) and (iv),
\[
\begin{align*}
(1 - \theta) &= -\frac{1}{d_{v_2}} \\
(1 - \theta) &= -\frac{d_{v_3} + d_{v_5}}{d_{v_3}d_{v_5}},
\end{align*}
\]
which yield that $d_{v_5} = 0$, a contradiction.

Similarly, if $S_{\{v_2, v_3, v_4\}} \neq \emptyset$, then from Lemma 2.6 (ii) and (iv),
\[
\begin{align*}
(1 - \theta) &= -\frac{1}{d_{v_3}} \\
(1 - \theta) &= -\frac{d_{v_2} + d_{v_5}}{d_{v_2}d_{v_5}},
\end{align*}
\]
which yield that $d_{v_2} + d_{v_5} = d_{v_2}$. Notice that any vertex distinct with $v_3$ and $v_5$ must be
adjacent to $v_3$ or $v_5$ (thanks to $\nu(G) = 2$). Therefore,

$$d_{v_2} = d_{v_3} + d_{v_5} \geq 3 + 3 + n - 6 = n,$$

a contradiction.

**Claim 2.3** $S\{v_1, v_3, v_4\} = \emptyset$.

Assume that $S\{v_1, v_3, v_4\} \neq \emptyset$ and $v_6 \in S\{v_1, v_3, v_4\}$. If $v_5 \sim v_6$, then $X_4$ (see Fig. 3) is an induced subgraph of $G$. By observation, $X_4$ contains an induced subgraph (by deleting $v_4$ with incident edges) isomorphic to $H_6$ (see Fig. 2), then from Lemma 2.6 (vi)

$$(1 - \theta) = -\frac{1}{d_{v_5}}, \quad (17)$$

Combining (17) and the second equation of (15), we obtain that $d_{v_2} = 0$, a contradiction.

If $v_5 \not\sim v_6$, then the principal submatrix, say $M_6$, of $L(G) - \theta I$ indexed by $\{v_1, \cdots, v_6\}$ can be written as the following block form

$$M_6 = \begin{pmatrix} M_4 & \alpha \\ \alpha^T & 1 - \theta \end{pmatrix},$$

where $M_4$ has been given in the proof of Lemma 2.6 (iv) and

$$\alpha = \left( \frac{-1}{\sqrt{d_{v_1}d_{v_6}}}, 0, \frac{-1}{\sqrt{d_{v_3}d_{v_6}}}, \frac{-1}{\sqrt{d_{v_4}d_{v_6}}}, 0 \right)^T,$$

a column vector. Obviously, the Eq. (7) still holds here, and by applying it to the columns of $M_6$, we get the equations of (8) and

$$\frac{-a}{\sqrt{d_{v_1}d_{v_6}}} - \frac{c}{\sqrt{d_{v_3}d_{v_6}}} = 0. \quad (18)$$

Then from (18) and the values of $a$ and $c$ obtained before

$$a = -\sqrt{d_{v_1}d_{v_5}}(1 - \theta)(\frac{d_{v_3}}{d_{v_5}} + 1), \quad c = \sqrt{\frac{d_{v_3}}{d_{v_5}}},$$

it follows that $\frac{d_{v_3}}{d_{v_5}} = \sqrt{d_{v_1}d_{v_5}}(1 - \theta)(\frac{d_{v_3}}{d_{v_5}} + 1)$, which yields $(1 - \theta) > 0$, contradicting with Lemma 2.6 (iv).

**Claim 2.4** $S\{v_1, v_2, v_3, v_4\} = \emptyset$.

Assume that there is a vertex, say $v_6$, in $S\{v_1, v_2, v_3, v_4\}$. If $v_5 \sim v_6$, then $X_5$ (see Fig. 3) is an induced subgraph of $G$. Deleting the vertex $v_3$ with the incident edges from $X_5$, the resultant graph is isomorphic to $H_6$ in Fig. 2. Then from Lemma 2.6 (vi), $(1 - \theta) = -\frac{1}{d_{v_2}}$, which together with $(1 - \theta) = -\frac{d_{v_3} + d_{v_5}}{d_{v_3}d_{v_2}}$ (thanks to Lemma 2.6 (iv)) indicates that $d_{v_5} = 0$, a contradiction.

If $v_5 \sim v_6$, then the principal submatrix, say $M_7$, of $L(G) - \theta I$ indexed by $\{v_1, \cdots, v_6\}$
can be written as

\[ M_7 = \begin{pmatrix} \frac{M_4}{\beta T} & \beta \\ \beta T & 1 - \theta \end{pmatrix}, \]

where \( M_4 \) is as above and

\[ \beta = \begin{pmatrix} -\frac{1}{\sqrt{d_{v_1}d_{v_6}}} & -\frac{1}{\sqrt{d_{v_2}d_{v_6}}} & -\frac{1}{\sqrt{d_{v_3}d_{v_6}}} & -\frac{1}{\sqrt{d_{v_4}d_{v_6}}} & -\frac{1}{\sqrt{d_{v_5}d_{v_6}}} \end{pmatrix}^T, \]

a column vector. Applying (7) to the last column of \( M_7 \), we get

\[ \frac{a}{\sqrt{d_{v_1}}} + \frac{b}{\sqrt{d_{v_2}}} + \frac{c}{\sqrt{d_{v_3}}} = \frac{1}{\sqrt{d_{v_5}}}, \] (19)

which together with the second equation of (8) yields that

\[ b\left(\frac{1}{\sqrt{d_{v_2}}} + \sqrt{d_{v_2}(1 - \theta)}\right) = 0. \]

Since \( b \neq 0 \) obtained before, then we have \((1 - \theta) = -\frac{1}{d_{v_2}}, \) and thus \( d_{v_5} = 0 \) (thanks to \((1 - \theta) = -\frac{d_{v_2} + d_{v_5}}{d_{v_2}d_{v_5}} \) in Lemma 2.6 (iv)), a contradiction.

In this case, combining Claims 2.1-2.4, we see that all the vertices out of \( V(P_4) \) belong to \( S_{\{v_1,v_2,v_4\}} \). Furthermore, it is obvious that \( S_{\{v_1,v_2,v_4\}} \) induces a clique of \( G \), as \( \nu(G) = 2 \). Hence, the structure of \( G \) is clear now, and \( d_{v_2} = n - 2, d_{v_3} = 2, d_{v_4} = n - 3 \) and \( d_{v_5} = n - 2 \). From Lemma 2.6 (iv),

\[ (1 - \theta) = -\frac{d_{v_2} + d_{v_4}}{d_{v_2}d_{v_4}} = \frac{-(2n-5)}{(n-2)(n-3)} = \frac{-n}{2(n-2)}, \]

which implies that \( n = 5 \), i.e., \( G = H_4 \). However, \( H_4 \notin \Omega \) by direct calculation. Therefore, \( S_{\{v_1,v_2,v_4\}} = \emptyset \), and by symmetry we get \( S_{\{v_1,v_3,v_4\}} = \emptyset \).

**Case 3.** Suppose that \( v_5 \in S_{\{v_1,v_2,v_3,v_4\}} \) (i.e., \( S_{\{v_1,v_2,v_3,v_4\}} \neq \emptyset \)) and \( S_{\{v_1,v_4\}} = S_{\{v_1,v_2,v_4\}} = S_{\{v_1,v_3,v_4\}} = \emptyset \).

If this is the case, then the vertices of \( V(G) \setminus \{V(P_4) \cup S_{\{v_1,v_2,v_3,v_4\}}\} \) belong to \( S_{\{v_1,v_2}\}}, \( S_{\{v_3,v_4\}} \), \( S_{\{v_1,v_3,v_2\}} \) or \( S_{\{v_2,v_3,v_4\}} \). Then we have the following claims.

**Claim 3.1** \( S_{\{v_1,v_2\}} = S_{\{v_3,v_4\}} = \emptyset \).

It suffices to show that \( S_{\{v_1,v_2\}} \neq \emptyset \). Suppose \( S_{\{v_1,v_2\}} \neq \emptyset \) and \( v_6 \in S_{\{v_1,v_2\}}, \) then

\[ (1 - \theta) = -\frac{1}{d_{v_1}} \] (20)

from Lemma 2.6 (i). Since

\[ (1 - \theta) = -\frac{d_{v_2} + 2d_{v_4}}{d_{v_4}(d_{v_2} + d_{v_5})} = -\frac{d_{v_2} + 2d_{v_4}}{d_{v_1}(d_{v_3} + d_{v_5})} \] (21)

12
from Lemma 2.6 (v), then by (20) and (21) we derive that
\[
\begin{align*}
d_{v_2}d_{v_1} + 2d_{v_1}d_{v_4} &= d_{v_2}d_{v_4} + d_{v_4}d_{v_5} \\
2d_{v_1} &= d_{v_5},
\end{align*}
\]
which implies that \(d_{v_1} = d_{v_4}\). As \(G\) contains an induced \(P_4\), then the equation (1) holds from Lemma 2.5. It follows from (1), (20) and \(d_{v_1} = d_{v_4}\) that
\[
d_{v_1}(d_{v_2} + d_{v_3}) = d_{v_2}d_{v_3}.
\]
Now we say that \(S\{v_2,v_3,v_4\} = \emptyset\), otherwise \((1 - \theta) = -\frac{1}{d_{v_2}}\) from Lemma 2.6 (ii), and then \(d_{v_1} = d_{v_3}\) from (20). Thus the equation (22) can be simplified as \(d_{v_3} = 0\), a contradiction. Analogously, one can derive that \(S\{v_1,v_2,v_3\} = \emptyset\). As a result, \(d_{v_2} = d_{v_1} + 1\). Recalling that \(d_{v_1} = d_{v_4}\), then \(|S\{v_1,v_2\}| = |S\{v_3,v_4\}|\), and thus \(d_{v_2} = d_{v_3}\). Considering (22) again, one can obtain that \(d_{v_1} = 1\), a contradiction.

Claim 3.2 \(S\{v_1,v_2,v_3\} = S\{v_2,v_3,v_4\} = \emptyset\).

It suffices to show that \(S\{v_1,v_2,v_3\} = \emptyset\). Suppose on the contrary that \(S\{v_1,v_2,v_3\} \neq \emptyset\), then \(d_{v_2} = d_{v_3}\) by observation. From Lemma 2.6 (ii), we have \(1 - \theta = -\frac{1}{d_{v_2}}\). Thus the equation (1) of Lemma 2.5 can be simplified as
\[
d_{v_1} + d_{v_4} = d_{v_2}.
\]
It is not hard to see that
\[
\begin{align*}
d_{v_1} &= |S\{v_1,v_2,v_3\}| + |S\{v_1,v_2,v_3,v_4\}| + 1 \\
d_{v_4} &= |S\{v_2,v_3,v_4\}| + |S\{v_1,v_2,v_3,v_4\}| + 1 \\
d_{v_2} &= |S\{v_1,v_2,v_3\}| + |S\{v_2,v_3,v_4\}| + |S\{v_1,v_2,v_3,v_4\}| + 2,
\end{align*}
\]
which together with (23) implies that \(|S\{v_1,v_2,v_3,v_4\}| = 0\), a contradiction. Therefore, the results of Claim 3.2 hold.

Now we are in a position to complete Case 3. Combining Claims 3.1 and 3.2, we see that all vertices out of \(V(P_4)\) belong to \(S\{v_1,v_2,v_3,v_4\}\). Then \(d_{v_2} = d_{v_1} + 1\). We claim that \(S\{v_1,v_2,v_3,v_4\}\) induces a clique of \(G\). Otherwise, there exist two vertices, say \(v_5\) and \(v_6\), of \(S\{v_1,v_2,v_3,v_4\}\), which are not adjacent. Then the subgraph induced by \(\{v_1,v_2,v_4,v_5,v_6\}\) is isomorphic to \(H_6\) in Fig. 2. Hence by Lemma 2.6 (vi), \(1 - \theta = -\frac{1}{d_{v_1}} = -\frac{1}{d_{v_2}}\), which indicates that \(d_{v_2} = d_{v_1}\), contradicting with \(d_{v_2} = d_{v_1} + 1\). As a result, if \(|S\{v_1,v_2,v_3,v_4\}| \geq 3\), then \(1 + \frac{1}{n-1}\) is an \(L\)-eigenvalue of \(G\) with multiplicity at least 2. Since \(G \not\cong K_n\), then \(\rho_{n-1}(G) \leq 1\) by Lemma 2.2. Clearly, \(G \not\cong K_{p,q}, K_a \vee (n-a)K_1\), then \(\rho_2(G) \geq \frac{n-1}{n-2}\) by Lemma 2.1. Noting that \(\rho_n(G) = 0\), we obtain \(G \not\in \Omega\). For the case of \(|S\{v_1,v_2,v_3,v_4\}| \leq 2\), one can get \(G \not\in \Omega\) by direct calculation.

The necessity part can be proved by Cases 1-3, and then the proof is completed. \(\Box\)

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