2D Black Holes, Conformal Vacua and CFTs on the Cylinder

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Abstract: We investigate two-dimensional dilaton gravity models with a power-law dilaton potential, whose black hole solutions contain, among others, the dimensional reduction of the Schwarzschild black hole, the Anti-de Sitter black hole and Rindler spacetime. We show that the ground state of these models satisfies simple transformation laws under the $SL(2,R)$ conformal group. We use these transformation laws to explain the scaling behavior of the thermodynamical parameters of the black hole and the nonextensivity of the thermodynamical system. The black hole thermodynamical behavior, in particular its entropy, is reproduced by a mapping into a two-dimensional conformal field theory on the cylinder.
1. Introduction

Recent developments have pointed out the relevance of nonextensivity in understanding the microscopic origin of the entropy in gravitational systems [1, 2, 3]. In particular the Cardy-Verlinde formula [4] gives an expression of the entropy in terms of the energy and of the (sub-extensive) Casimir energy of the system, which is valid for all dimensions. Originally proposed in a cosmological context, the Cardy-Verlinde formula has been readily extended to other gravitational systems, in particular black holes [3]. It has been successfully used for Anti-de Sitter (AdS)/ Schwarzschild black holes, CFTs with AdS duals, charged black holes and asymptotically flat black holes in any dimension.

For two-dimensional (2D) gravitational systems (more in general systems that admit 2D CFTs as duals) one can make use, directly, of the Cardy formula [4] that gives the entropy of a CFT in terms of the central charge $c$ and the eigenvalue of the Virasoro operator $l_0$. However, this is possible only for 2D systems for which one can explicitly show (e.g using the AdS/CFT correspondence) that they are in correspondence with a 2D CFT [4, 5]. Even in this most favorable case the use of the Cardy formula for the computation of the entropy of the gravitational system is far from trivial. $c$ and $l_0$ have to be expressed in terms of the gravitational parameters, an operation that sometimes turns out to be very hard [7]. The problem becomes much more intricate when a duality of the gravitational model with a CFT cannot be shown explicitly to hold (e.g. asymptotically flat 2D black holes).

In this paper we consider two-dimensional dilaton gravity models with a power-law dilaton potential. This class of models is particularly interesting because it shows up in several contexts: dimensional reduction to $D = 2$ of the D-dimensional
Schwarzschild solution (see for instance \cite{8}), effective theories for dilatonic 0-branes \cite{1, 7}, AdS$_2$ black hole \cite{10, 13}, Callan-Giddings-Harvey-Strominger black hole (CGHS) (or its Weyl rescaled version, the Rindler spacetime) \cite{11, 12}. We show that the ground state of the model, although in general not invariant, satisfies simple transformation laws under the action of the $SL(2, R)$ conformal group (Sect. 2). These transformation properties are used to explain the scaling behavior of the thermodynamical parameters and the violation of the Euler identity, which codify the non-extensivity of the thermodynamical system (Sect. 3). The black hole is put in correspondence with a 2D CFT on the cylinder. The black hole mass is identified with the Casimir energy of the cylinder vacuum and the Hawking temperature is given in terms of the length of the compact dimension of the cylinder. The entropy of the 2D black hole is reproduced using the Cardy formula (Sect. 4). As particular case we consider the Rindler spacetime. We point out that our method can be used to give mass and entropy to the Rindler spacetime in terms of the Casimir energy and entropy of a 2D CFT (Sect. 5).

2. Conformal properties of the ground state

A particular interesting class of 2D dilaton gravity models is represented by the action

$$A = \frac{1}{2} \int \sqrt{-g} \, d^2 x \left( \Phi R + \lambda^2 V(\Phi) \right),$$

with a power-law dilaton potential

$$V(\Phi) = (a + 1)\Phi^a, \quad a > -1.$$  \hspace{1cm} (2.1)

The action (2.1) emerges as 2D effective action both for D-dimensional Einstein gravity and dilatonic 0-branes in the string theory context. D-dimensional, spherically symmetric, gravity reduces, after dimensional reduction of the (D-2)-dimensional sphere of radius $r = \Phi^{-(a)}$ to the model (2.1) with $a = 1/(2 - D)$ \cite{5}. In particular for $a = -1/2$ our model describes the four-dimensional Schwarzschild solution of general relativity. In the dual-frame, dilatonic 0-branes solutions of D-dimensional supergravity admit a (near-horizon) effective description in terms of a two-dimensional dilaton gravity model with a power-law potential \cite{4, 7}. The models under consideration have also an intrinsic interest as 2D theories of gravity. For $a = 1$ Eq. (2.1) becomes the Jackiw-Teitelboim model, which plays a crucial role in the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence in two dimensions \cite{4}. For $a = 0$ we get the Weyl-rescaled version of the CGHS model \cite{11, 12}, which admits the Rindler spacetime as non-trivial dynamical solution \cite{12}.

In the Schwarzschild gauge, the general black hole solution of the model (2.1) is

$$ds^2 = -\left( \Phi^{a+1} - \frac{2M}{\lambda} \right) dt^2 + \left( \Phi^{a+1} - \frac{2M}{\lambda} \right)^{-1} dr^2,$$

$$\Phi = \lambda r,$$  \hspace{1cm} (2.3)
where \( M \) is the Arnowitt-Deser-Misner mass of the solution. For \( a \neq 0, 1 \) the solution has a curvature singularity shielded by an event horizon at \( \Phi = \Phi_h = \left( \frac{2M}{\lambda} \right)^{1/(a+1)} \). For \( a = 1 \) the solution describes 2D AdS black holes [10], whereas for \( a = 0 \) Rindler spacetime endowed with a nontrivial dilaton [12].

Let us now investigate the conformal symmetries of the \( M = 0 \) black hole vacuum solution. For generic values of the parameter \( a \) the \( M = 0 \) solution is not maximally symmetric, it admits only one Killing vector generating time-translations. Only for \( a = 0, 1 \) when the spacetime becomes Minkowskian, respectively, AdS\(_2\), the isometry group is maximal and, in the case of AdS\(_2\), is the conformal group \( SO(1, 2) \sim SL(2, R) \). Nonetheless, we expect simple transformation laws of the vacuum solution under the action of the conformal group \( SL(2, R) \), owing to its power-law dependence on the coordinate \( r \).

We will first consider \( a \neq 0 \), the case \( a = 0 \) will be considered separately. In the conformal gauge and using light-cone coordinates the \( M = 0 \) solution in Eq. (2.3) takes the form,

\[
ds^2 = 2g_{+-} dx^+ dx^- = \left[ \frac{|a| \lambda}{2} (x^+ - x^-) \right]^{-(1 + \frac{1}{a})} dx^+ dx^- \]

\[
\Phi = \left[ \frac{|a| \lambda}{2} (x^+ - x^-) \right]^{-1/a}.
\]

Let us now consider the action of the conformal group \( SL(2, R) \), realized as the fractional transformation,

\[
x'^\pm = \frac{\alpha x^\pm + \beta}{\gamma x^\pm + \delta}, \quad \alpha \delta - \beta \gamma = 1.
\]

(2.5)

on the solution (2.4). We find

\[
g_{+-}(x) = g'_{+-}(x) \left( \frac{dx'^+}{dx^+} \right)^h \left( \frac{dx'^-}{dx^-} \right)^h
\]

\[
\Phi(x) = \Phi'(x) \left( \frac{dx'^+}{dx^+} \right)^l \left( \frac{dx'^-}{dx^-} \right)^l.
\]

(2.6)

where \( h = \frac{1}{2} (1 - \frac{1}{a}) \), \( l = -\frac{1}{2a} \) are “effective” conformal dimensions of the fields. Notice that we are not using the usual definition of conformal dimension of a field, which is given by Eqs. (2.6) with the field on the right hand side of the equation evaluated on \( x' \) not on \( x \). The two definitions agree only for constant fields (e.g for Minkowski space). \( h \) and \( l \) give information about the nontrivial \( x \)-dependence of the conformal fields and characterize their scaling behavior. Our definition is a natural extension to
conformal fields of the geometric notion of isometry (form invariance). For instance, AdS$_2$ (Eq. (2.3) with $a = 1$) has $SL(2, R)$ as isometry group. The usual definition of conformal primary field keeps track only of the tensor character of the metric and assigns to it conformal dimension equal to 1. Conversely, our definition (2.6) assigns to AdS$_2$ zero conformal dimension, consistently with its $SL(2, R)$-isometry.

For scale transformations $x'_{\pm} = \mu x_{\pm}$ ($\gamma = \beta = 0, \alpha = 1/\delta = \sqrt{\mu}$ in Eq. (2.5)) Eqs. (2.6) give

$$g_{+-}(x) = \mu^{2h} g_{+-}'(x) \quad \Phi(x) = \mu^{2l} \Phi'(x). \quad (2.7)$$

One can also consider infinitesimal conformal transformations. The conformal Killing vectors corresponding to translations, dilatations and special conformal transformations are, respectively, $\chi_T^\pm = \alpha, \chi_D^\pm = \mu x^\pm, \chi_S^\pm = -\omega (x^\pm)^2$, whereas Eq. (2.4) becomes $\delta g_{+-} = \Omega g_{+-}, \delta \Phi = \Omega \Phi$, with $\Omega_T = 0, \Omega_D = 2k\mu, \Omega_S = -2k\omega (x^+ + x^-), k$ being the above defined conformal dimension of the field. Only for $a = 1, a = \infty$ (corresponding to the Reissner-Nordstrom case [7]) and $h = 0$, the ground state $M = 0$ solution admits full $SL(2, R)$ as isometry group. For generic values of $a$, the metric admits only the Killing vector $\chi_T^\pm$ generating time-translations, but it satisfies simple scaling transformations under dilatations and special conformal transformations. In the next sections we will relate this conformal properties of the vacuum black hole solutions with the scaling behavior of the thermodynamical parameters associated with the black hole. Notice that also the dilaton $\Phi$ transforms as a primary field under $SL(2, R)$. It is (form)-invariant only for $a \to \infty$, whereas it transforms as a field of conformal dimension $l$ otherwise.

Eqs. (2.4) hold only for $a \neq 0$. For $a = 0$, introducing appropriate coordinates, the black hole vacuum has the form of Minkowski space with a nonconstant dilaton

$$ds^2 = -dy^+ dy^- , \Phi = -(\lambda^2/4)y^+ y^-.$$  

Obviously, the metric part of this solution behaves under $SL(2, R)$ as in Eq. (2.4) with $h = 1$. The dilaton does not follow the simple law (2.4) for all the $SL(2, R)$ transformations. However under dilatations transforms according to Eq. (2.7) with $l = 1$.

In the following we will need the form of the dilatations in the the coordinate system $(r, t)$ of Eqs. (2.3). We have,

$$r \to \nu r, \quad t \to \nu^{-a} t, \quad ds^2 \to \nu^{1-a} ds^2, \quad \Phi \to \nu \Phi, \quad (2.8)$$

with $\nu = \mu^{-1/a}$. This formula holds for every value of $a > -1$.

The previously described conformal properties of the $M = 0$ solution do not apply to the $M \neq 0$ black hole solutions. The general solution is (time)-translationally invariant but does not satisfy simple scaling laws under the action of full $SL(2, R)$.

3. Scaling behavior of the thermodynamical parameters

The physical meaning of the scale symmetry of the black hole vacuum we have found
in the previous section can be easily understood: the system has no physical scale \(^1\) and we can change the size of the system without changing the physical parameters. The mass \(M\), the entropy \(S\) and the temperature \(T\) of the state are zero and remain zero if we perform the scale transformations (2.8). This is not true anymore when we consider \(M \neq 0\) black hole solutions. Now the system has a physical scale (the black hole mass, or if you prefer the black hole temperature) and the solution does not satisfy the simple scale symmetry of Eq. (2.8) and (2.7). We expect that changing the size of the system will affect \(M, S, T\). It is easy to see that the scale symmetry (2.8) of the black hole vacuum determines the scaling behavior of the thermodynamical parameters. The key point is that \(M, S, T\) are completely determined by the dilaton potential and by the value \(\Phi_h\) of the dilaton at the black hole horizon. We have

\[
M = \frac{\lambda}{2} \Phi_h^{a+1}, \quad S = 2\pi\Phi_h, \quad T = \frac{\lambda}{4\pi}(a + 1)\Phi_h^a.
\]

(3.1)

Taking into account that the dilaton depends linearly on the coordinate \(r\) (see Eqs. (2.3)), one finds the following scale transformations

\[
r \rightarrow \nu r, \quad M \rightarrow \nu^{a+1}M, \quad S \rightarrow \nu S, \quad T \rightarrow \nu^aT.
\]

(3.2)

Assuming that \(M\) transforms as in Eq. (3.2), one finds that the scale symmetry (2.8) is also realized for \(M \neq 0\).

The scaling behavior (3.2) has a simple thermodynamical interpretation if one considers that for our 2D gravitational system \(r_h\) is the volume \(V\). The scaling law for the entropy given by Eq. (3.2) can be written as \(S(\nu V) = \nu S(V)\), meaning that the entropy of the black hole is an extensive quantity. For the energy (the black hole mass) we have

\[
M(\nu V) = \nu^{a+1}M(V).
\]

(3.3)

Notice that we can trade \(V\) for \(S\) and write \(M = M(S) = \frac{\lambda}{2} \left(\frac{S}{2\pi}\right)^{a+1}\), which satisfies \(M(\nu S) = \nu^{a+1}M(S)\). Hence, for generic values of \(a\) the energy is non-extensive. It becomes extensive only for \(a = 0\), corresponding to the CGHS black hole. In this latter case, as it is evident from Eq. (3.1) the temperature is mass-independent and given by \(T = \lambda/4\pi\). The black hole geometry can be described by a Rindler spacetime endowed with a nontrivial dilaton and the temperature can be expressed in terms of the acceleration of Rindler observers in Minkowski space.

The non-extensivity of the thermodynamical system defined by the black hole (2.3) can be also understood as a violation of the Euler identity. From Eqs. (3.1) it follows

\[
M = \frac{1}{a+1}TS.
\]

(3.4)

\(^1\)In the the action (2.1) appears the constant \(\lambda\), which has the dimension of an inverse length. However, \(\lambda\) does not represent a dynamical physical scale. It is related with the size of the curvature of the 2D spacetime, but it is not affected by the gravitational dynamics.
The Euler identity is satisfied only for the Rindler case, \( a = 0 \), whereas it is violated for generic values of \( a \). It is important to notice that the non-extensivity of the thermodynamical system, expressed either as the scaling law (3.3) or as the violation of the Euler identity (3.4) can be parametrized in terms of the conformal dimension of the black hole vacuum, \( h \) (or \( l \)).

The relations (3.2) have been derived using Eqs. (3.1), which express \( M, T, S \) as a function of the volume of the system. There is however an alternative description in which one considers the black hole temperature \( T \) as the inverse periodicity at the horizon of the Euclidean time. In this description one continues the black solution (2.3) into Euclidean space, to eliminate the conical singularity at the black hole horizon one is forced to put the theory on the cylinder. In this way a physical scale is generated (the Hawking temperature \( T \)) and its scaling law can be determined using the transformation of the time \( t \) in Eq. (2.8) under scale transformations.

Analytically continuing in Euclidean space the \( M \neq 0 \) black hole solution (2.3) and introducing an appropriate \( R \) coordinate one finds near the black hole horizon

\[
ds^2 = \frac{4\pi^2}{\Theta^2}R^2 dt^2 + dR^2,
\]

where \( \Theta = 1/T \). In order to eliminate the conical singularity the Euclidean time has to be chosen periodic with period \( \Theta \). The Euclidean description of the black is naturally related with the CFT on the cylinder we are going to discuss in the next section.

### 4. Black holes and CFT on the cylinder

The possibility of describing 2D black holes by means of a CFT has been widely investigated in recent years [5, 13, 14, 15, 16]. Presently, it is not completely clear if it is always possible to mimic the gravitational dynamics of the 2D black hole through a CFT. However, in some cases and/or for generic black holes in particular regimes, CFTs have been shown to give a good description. This is in particular true for black holes in AdS space, where the AdS/CFT correspondence [3, 13, 14] should do the job and for the near-horizon regime of general black holes, for which general arguments lead to an associate CFT [15].

In the 2D dilaton gravity context, CFTs have emerged as an effective description of the gravitational system in many cases: black holes on AdS\(_2\) (the model of Eq. (2.3) with \( a = 1 \)) [3], 0-branes (the model of Eq. (2.3) with \( a \geq 0 \)) [7], Liouville models [10] and there is some evidence that this could also be generally true.

In this paper we will assume that our 2D black hole (2.3) can be described by some sort of 2D CFT. We will show that, if this is true, then the thermodynamical features of the black hole described in the previous section, have a natural interpretation in terms of general (model independent) features of the CFT.
The description of the black hole given in the previous section can be summarized as follows. We have a ground state with simple conformal behavior and with no physical scale inscribed, when a black hole of mass \( M \) is formed a physical scale is generated. In the Euclidean space, this physical scale generation is described in geometric terms by considering a periodic time, with period \( \Theta \). It is not difficult to recognize the analogy with the relationship between 2D CFT on the Euclidean plane and CFT on the cylinder, the mapping between the infinite plane (with holomorphic coordinate \( z \)) and the cylinder (with coordinate \( w \) and length of the circle \( L \)) being given by

\[
z = e^{\frac{2\pi}{L}w}.
\]

(4.1)

It is well-known that if we map the CFT on the plane, with associated, conformal invariant, vacuum \(|0^{pl}\rangle\) and central charge \(c\), into the CFT on the cylinder a Casimir energy \(E_c\) for the cylinder vacuum, \(|0^{cyl}\rangle\), is generated due to finite-size effect (see for instance Ref. [17]).

Following this analogy we are led to a correspondence between 2D black holes on the left hand side and 2D CFT on the right hand side,

black hole ground state \(\iff|0^{pl}\rangle\)

\(M \neq 0\) black hole state \(\iff|0^{cyl}\rangle\)

\(M \iff E_c\)

\(\Theta \iff L\)

This analogy has been already successfully used for calculating the entropy of the AdS\(_2\) black hole in terms of degeneracy of states of a 2D CFT [14]. The relation between the \(L_0\) operator of the Virasoro algebra on the plane and on the cylinder is given by (see for instance Ref. [17]),

\[
L_0^{pl} = L_0^{cyl} + \frac{c}{24}.
\]

(4.2)

Because we are assuming a conformal invariant vacuum we have \(L_0^{pl}|0^{pl}\rangle = 0\) and \(L_0^{cyl}|0^{cyl}\rangle = 0\). Applying Eq. (4.2) on the cylinder vacuum \(|0^{cyl}\rangle\) we get

\[
L_0^{pl}|0^{cyl}\rangle = \frac{c}{24}|0^{cyl}\rangle.
\]

(4.3)

This means that \(|0^{cyl}\rangle\) is an eigenstate of \(L_0^{pl}\) with eigenvalue \(l_0^{pl} = \frac{c}{24}\).

The correspondence between 2D dilaton gravity and 2D CFT we are considering implies that the \(|0^{cyl}\rangle\) vacuum has to be considered as an excitation of the \(|0^{pl}\rangle\) vacuum. The energy of \(|0^{cyl}\rangle\) measured with respect to the \(|0^{pl}\rangle\) vacuum is the Casimir energy, which can be read directly from Eq. (4.3), \(E_c = \frac{\pi}{12L}c\). Identifying
the Casimir energy with the black hole mass, $E_c = M$ and using Eq. (4.3) we can express both the central charge and $l_0^{pl}$ in terms of the black hole mass,

$$c = \frac{12L}{\pi}M, \quad l_0^{pl} = \frac{L}{2\pi}M. \quad (4.4)$$

The Cardy formula \cite{4} enables us to calculate the entropy associated with the $|0^{pl}\rangle >$ vacuum considered as an excitation of the $|0^{pl}\rangle >$ vacuum. Because $|0^{pl}\rangle >$ has zero conformal weight the Cardy formula reads

$$S = 2\pi \sqrt{\frac{c}{6}l_0^{pl}}. \quad (4.5)$$

Feeding Eq. (4.5) with Eq. (4.4), one finds

$$S = 2LM, \quad (4.6)$$

which reproduces exactly the black hole thermodynamical relation (3.4) after identifying $L$ with $\Theta$ in the following way

$$L = \left(\frac{a + 1}{2}\right)\Theta. \quad (4.7)$$

This equation implies that the temperature $T$ of the black hole cannot be identified directly with the inverse of $L$. A factor, depending on the conformal dimension of the black hole ground state, relates them. This factor is equal to one only for the AdS$_2$ black hole, as expected because in this case the black hole ground state is truly invariant under SL(2, $R$). To end this section let us stress the fact that our derivation of the thermodynamical entropy of black holes holds also for asymptotically flat black holes. The most striking example is the Schwarzschild black hole, which is the particular case $a = -1/2$ of our general model.

5. Rindler spacetime and CFT

The analogy between 2D black holes and 2D CFT on the cylinder is particularly instructive in two particular cases: $a = 1$ corresponding to the AdS$_2$ black hole and $a = 0$ corresponding to the Rindler spacetime (CGHS black hole). In both cases the black hole geometry is equivalent, modulo space-time diffeomorfisms, to the ground state $M = 0$ solution. The spacetime can be still interpreted as a black hole owing to the presence of a nontrivial dilaton \cite{10, 12}. The AdS$_2$ case has been discussed at length in previous papers and we will not discuss it here any further.

For $a = 0$ the $M = 0$ ground state solution can be described, using appropriate coordinates, by Minkowski space $ds^2 = -dy^+dy^-$. The $M \neq 0$ black hole solution has a Rindler form and a $M$-dependent dilaton \cite{12}

$$ds^2 = -\exp\left(\frac{\lambda}{2}(x^+ - x^-)\right)dx^+dx^-, \quad \Phi = \frac{2M}{\lambda} + \exp\left(\frac{\lambda}{2}(x^+ - x^-)\right). \quad (5.1)$$
The coordinates \( x^\pm \) give a Rindler coordinatization of Minkowski space,

\[
y^\pm = \pm \frac{2}{\lambda} \exp \left( \pm \frac{\lambda}{2} x^\pm \right).
\]

(5.2)

It is not difficult to recognize in the previous equation the Minkowskian version of
the mapping between CFT on the plane and CFT on the cylinder given in Eq. (4.1).
In the Rindler case the black hole/CFT correspondence presented in the previous
section is more stringent then in the general case. The transformation that maps
vacua in the CFT is the same as the map between the ground state and the black
hole. It is clear that also for a generic one can show that the mapping (4.1) maps
the Euclidean black hole metric (3.5) into the Euclidean plane (which is conformally
related to the black hole ground state). This is a simple consequence of the fact
that Eq. (3.5) is a near-horizon expansion. Near the horizon the generic black hole
solution (2.3) has always a Rindler form. But, this is true only locally, near the
horizon, whereas in the pure Rindler case the mapping holds globally, for the whole
space.

The Minkowski vacuum \( |0^M> \) of the gravitational description corresponds to
the \( |0^{pl}> \) vacuum in the CFT description, whereas the Rindler vacuum \( |0^R> \)
corresponds to \( |0^{cyl}> \). The well-known fact that the Rindler observer will see \( |0^M> \) as
filled with thermal radiation with temperature \( T_R = \lambda/4\pi \) can be interpreted from
the CFT point of view as a finite-size effect, with the length \( L \) related to the Rindler
temperature by \( L = 1/2T_R \). Moreover our correspondence can be used to assign in
a natural way mass and entropy to the Rindler spacetime. The Rindler mass and
entropy can be expressed in terms of the central charge of the CFT. The mass is
simply given by the Casimir energy of \( |0^{cyl}> \), \( M_R = E_c \), whereas the entropy is
\( S_R = M_R/T_R = (4\pi/\lambda)E_c \).

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