Pullback attractors of nonautonomous discrete $p$-Laplacian complex Ginzburg–Landau equations with fast-varying delays

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Abstract

In this paper, we consider a class of nonautonomous discrete $p$-Laplacian complex Ginzburg–Landau equations with time-varying delays. We prove the existence and uniqueness of pullback attractor for these equations. The existing results of studying attractors for time-varying delay equations require that the derivative of the delay term should be less than $1$ (called slow-varying delay). By using differential inequality technique, our results remove the constraints on the delay derivative. So, we can deal with the equations with fast-varying delays (without any constraints on the delay derivative).

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1 Introduction

Due to numerous applications in physics, biology, and engineering such as pattern formation, propagation of nerve pulses, electric circuits, and so on, see, e.g., [2, 6, 7, 10, 12], lattice differential equations have become a large and growing interdisciplinary area of research. For an understanding of the dynamical behavior of dissipative infinite lattice systems, attractors are especially important because they retain most of the dynamical information. The existence of global attractors for lattice systems was initiated by Bates et al. [1], followed by extensions in [3, 8, 13, 16, 19, 24] and the references therein. Of those, the asymptotic behavior of an infinite-dimensional $p$-Laplacian lattice system was investigated in [8]. The dynamical behaviors for discrete complex Ginzburg–Landau equations were studied in [11, 27].

Since time-delays are frequently encountered in many practical systems, which may induce instability, oscillation, and poor performance of systems, delay lattice systems then arise naturally while these delays are taken into account. Recently, attractors of delay lattice systems have been considered in [4, 5, 9, 23, 26].
The existence and uniqueness of solutions were proved for the complex Ginzburg–Landau equation with \( p \)-Laplacian in [17, 18]. The dynamical behavior of \( p \)-Laplacian complex Ginzburg–Landau equations was considered in [25]. The existence and uniqueness of attractor for nonautonomous discrete \( p \)-Laplacian complex Ginzburg–Landau equations was considered in [25]. The existence and uniqueness of attractor for nonautonomous discrete \( p \)-Laplacian complex Ginzburg–Landau equations with fast-varying delays based on nonautonomous \( p \)-Laplacian complex Ginzburg–Landau equations with fast-varying delays are investigated in this paper. We prove the existence and uniqueness of pullback attractor for these equations. The existing results of studying attractors for time-varying delay equations require that the derivative of the delay term should be less than 1, see [14, 15, 21–23]. By using the differential inequality technique, our results remove the constrains on the delay derivative. So, we can deal with the equations with fast-varying delays.

Motivated by the discussions above, in this paper we study the dynamical behavior of nonautonomous discrete \( p \)-Laplacian complex Ginzburg–Landau equation with time-varying delays:

\[
\frac{du_n}{dt} = (\lambda + i\alpha)(|u_{n+1} - u_n|^{p-2}(u_{n+1} - u_n) - |u_n - u_{n-1}|^{p-2}(u_n - u_{n-1})) - (\kappa + i\beta)|u_n|^q u_n \\
+ f_n(u_n(t - \rho_0(t))) - (\gamma + i\delta)u_n + g_n(t), \quad n \in \mathbb{Z}, \tau \in \mathbb{R}, t > \tau, \tag{1.1}
\]

with the initial condition

\[
u_n(\tau + s) = \varphi_n(s), \quad s \in [-\rho, 0], \tag{1.2}
\]

where \( u_n \) is the unknown complex-valued function, \( \lambda, \alpha, \kappa, \beta, \gamma, \delta, \rho, p, q \) are real constants, where \( \lambda, \alpha, \beta, \gamma, \rho, q > 0 \) and \( p \geq q + 2 \), \( f_n \) is a nonlinear function satisfying certain conditions, \( \rho_0 \in C([0, \rho]) \) is an adequate given delay function, \( g(t) = (g_n(t))_{n \in \mathbb{Z}} \in L^2_{\text{loc}}(\mathbb{R}, L^2) \) \( (l^2 \text{ defined later}) \) is a given time-dependent sequence, and \( \varphi_n \in C([-\rho, 0], \mathbb{C}) \).

The plan of this paper is as follows. In the next section, we establish the existence of a continuous nonautonomous dynamical system in \( C([-\rho, 0], L^2) \) for nonautonomous equation (1.1) and (1.2). Section 3 contains all necessary uniform estimates of the solutions. We then prove the existence and uniqueness of a pullback attractor for the nonautonomous equations in Sect. 4.

\section{Nonautonomous dynamical systems associated with nonautonomous lattice systems}

In this section we show that there is a continuous nonautonomous dynamical system generated by the nonautonomous discrete \( p \)-Laplacian complex Ginzburg–Landau equations with time-varying delays.

Denote

\[
\mathbb{L} = \left\{ u = \{u_n\}_{n \in \mathbb{Z}} \in \mathbb{C}, \sum_{n \in \mathbb{Z}} |u_n|^2 < \infty \right\},
\]

and let \( l^2 \) be a Hilbert space with the inner product and norm given by

\[
(u, v) = \sum_{n \in \mathbb{Z}} u_n v_n, \quad \|u\|^2 = (u, u), \quad u, v \in l^2.
\]
We define the linear operators $A, B, B^*: l^2 \to l^2$ as follows:

$$(Au)_n = |u_n - u_{n-1}|^{p-2}(u_n - u_{n-1}) - |u_{n+1} - u_n|^{p-2}(u_{n+1} - u_n),$$

$$(Bu)_n = u_{n+1} - u_n, \quad (B^*u)_n = u_{n-1} - u_n, \quad n \in \mathbb{Z}, u \in l^2.$$

Then

$$(B^*u, v) = (u, Bv), \quad (Au, v) = (Bu)^p \otimes (Bu), \quad (Au, u) = \|Bu\|_p^p \leq \|Bu\|^p \leq 2^p \|u\|^p, \quad u, v \in l^2,$$

where $u \otimes v = (u_i v_i)_{i \in \mathbb{Z}}$ and $\|u\|_p = (\sum_{n \in \mathbb{Z}} |u_n|^p)^{\frac{1}{p}}$.

Denote by $u_t$, $t \in \mathbb{R}$, the function defined on $[-\rho, 0]$ according to the relation

$$u_t(s) = (u_n(s))_{n \in \mathbb{Z}} = (u_n(t + s))_{n \in \mathbb{Z}} = u(t + s), \quad s \in [-\rho, 0],$$

and let $C_\rho = C([-\rho, 0], l^2)$ with the maximum norm $\|\psi\|_{\rho} = \sup_{\rho \leq s \leq 0} \|\psi(s)\|$, $\psi \in C_\rho$.

Then problem (1.1)–(1.2) can be written as an equation in $l^2$: for $t \in \mathbb{R}$ and $t > \tau$,

$$\frac{du(t)}{dt} = -(\lambda + i\alpha)Au - (\kappa + i\beta)|u|^q u + f(u(t - \rho_0(t))) - (\gamma + i\delta)u + g(t)$$

(2.1)

and

$$u(t + s) = \psi(s), \quad s \in [-\rho, 0],$$

(2.2)

where $u = (u_n)_{n \in \mathbb{Z}}$, $|u|^q u = (|u_n|^q u_n)_{n \in \mathbb{Z}}$, $f(u(t - \rho_0(t))) = (f_n(u_n(t - \rho_0(t))))_{n \in \mathbb{Z}}$, $g(t) = (g_n(t))_{n \in \mathbb{Z}}$ and $\psi = (\psi_n)_{n \in \mathbb{Z}}$.

We make the following assumptions on $f_n$. For each $n \in \mathbb{Z}$, $f_n$ is a nonlinear function satisfying the following assumption:

$(H)$ $f_n(0) = 0$ and $f_n(s)$ is Lipschitz continuous with respect to $s$, that is, there is a positive constant $L$ such that, for all $s_1, s_2 \in \mathbb{C}$,

$$|f_n(s_1) - f_n(s_2)| \leq L|s_1 - s_2|.$$

In fact, by $(H)$ we find that

$$\|f(u) - f(v)\| \leq L\|u - v\|, \quad u, v \in l^2.$$

**Lemma 2.1** For any $p > 0$ and $a, b \in \mathbb{C}$, we have that there exists $c = c(p) > 0$ such that

$$|a|^p a - |b|^p b \leq c(|a|^p + |b|^p)|a - b|.$$

**Proof** Without loss of generality, we assume that $|a| \geq |b|$. By mean value theorem, we have

$$|a|^p - |b|^p = p(\theta |a| + (1 - \theta)|b|)^{p-1}||a| - |b||$$

$$\leq p(\theta |a| + (1 - \theta)|b|)^{p-1}|a - b|, \quad 0 < \theta < 1. \quad (2.3)$$
Then
\[ |a|^p - |b|^p | \leq \begin{cases} \frac{p|b|^p}{a} |a - b|, & 0 < p \leq 1, \\ \frac{p|a|^p}{b} |a - b|, & p > 1. \end{cases} \] (2.4)

By (2.3), (2.4), and Young’s inequality, we get
\[ |a|^p |a| - |b|^p |b| \leq |a|^p |a - b| + |(|a|^p - |b|^p) |b| \]
\[ \leq |a|^p |a - b| + |b|||a|^p - |b|^p| \]
\[ \leq |a|^p |a - b| + p|b|(|a|^{p-1} + |b|^{p-1}) |a - b| \]
\[ \leq c(|a|^p + |b|^p) |a - b|. \]

This completes the proof. □

Lemma 2.2 The operator \( A : l^2 \rightarrow l^2 \) is locally Lipschitz continuous.

Proof Based on Lemma 2.1 we have, for any \( u, v \in l^2 \),
\[
\|Au - Av\|^2 \leq 2 \sum_{n \in \mathbb{Z}} \left( |(Bu)_n|^{2p-2} (Bu)_n - |(Bv)_n|^{2p-2} (Bv)_n \right)^2 \\
+ 2 \sum_{n \in \mathbb{Z}} \left( |(Bu)_{n+1}|^{2p-4} (Bu)_{n+1} - |(Bv)_{n+1}|^{2p-4} (Bv)_{n+1} \right)^2 \\
\leq 2c^2 \sum_{n \in \mathbb{Z}} \left( |(Bu)_n| + |(Bv)_n| \right)^{2p-4} |(Bu)_n - (Bv)_n|^2 \\
+ 2c^2 \sum_{n \in \mathbb{Z}} \left( |(Bu)_{n+1}| + |(Bv)_{n+1}| \right)^{2p-4} |(Bu)_{n+1} - (Bv)_{n+1}|^2 \\
\leq 2p \varepsilon^2 \sum_{n \in \mathbb{Z}} \left( |u_n| + |v_n| \right)^{2p-4} |u_n - v_n|^2 \\
\leq 2p \varepsilon^2 \left( \|u\| + \|v\| \right)^{2p-4} \|u - v\|^2.
\]

This completes the proof. □

It follows from Lemma 2.2 that the right-hand side function in (2.1) is locally Lipschitz continuous from \( l^2 \) to \( l^2 \). Therefore, by the standard theory of functional differential equations, one can show that, for every \( \varphi \in C_{\rho} \), there exists \( T > 0 \) such that system (2.1)–(2.2) has a unique solution \( u_t(\cdot, \tau, \varphi) \in C([\tau, T], C_{\rho}) \). As showed below, under some conditions this local solution is actually defined for all \( t > \tau \). Furthermore, one may show that \( u_t(\cdot, \tau, \varphi) \) is continuous in \( \varphi \) with respect to the norm of \( C_{\rho} \).

In the sequence, we assume that
\[
\eta = \frac{2L^2}{\varepsilon^2} < 1. \quad (2.5)
\]
Lemma 2.3 Assume that \((H)\) and \((2.5)\) hold. Then, for every \(\tau \in \mathbb{R}, T > 0\) and \(\varphi \in C_\rho,\) there exists a positive constant \(c = c(\tau, T, \varphi)\) such that the solution \(u\) of \((2.1)-(2.2)\) satisfies

\[
\|u(t, \tau, \varphi)\|_\rho \leq c \quad \text{for } t \in [\tau, \tau + T).
\]  

\[(2.6)\]

Proof Taking the inner product of \((2.1)\) with \(u\) in \(L^2\) and keeping the real part, we find that for \(t > \tau,\)

\[
\frac{1}{2} \frac{d\|u\|^2}{dt} = -\lambda (|Bu|^{p-2} \otimes (Bu), Bu) - \kappa \|u\|^{q+2}_{q+2} + \text{Re}(f(u(t - \rho_0(t))), u)
\]

\[-\gamma \|u\|^2 + \text{Re}(g(t), u)\]

\[
\leq -\gamma \|u\|^2 + \text{Re}(f(u(t - \rho_0(t))), u) + \text{Re}(g(t), u). \tag{2.7}
\]

For the last two terms on the right-hand side of \((2.7),\) by \((H)\) we have

\[
\text{Re}(f(u(t - \rho_0(t))), u) + \text{Re}(g(t), u) \leq \frac{L^2}{\gamma} \|u(t - \rho_0(t))\|^2 + \frac{1}{\gamma} \|u(t)\|^2 + \frac{1}{\gamma} \|g(t)\|^2. \tag{2.8}
\]

Consequently, it follows from \((2.7)-(2.8)\) that

\[
\frac{d\|u(t)\|^2}{dt} \leq -\frac{\gamma}{2} \|u(t)\|^2 + \frac{L^2}{\gamma} \|u(t - \rho_0(t))\|^2 - \kappa \|u\|^{q+2} + \frac{1}{\gamma} \|g(t)\|^2, \quad t > \tau, \tag{2.9}
\]

which implies that

\[
\frac{d\|u(t)\|^2}{dt} \leq -\frac{\gamma}{2} \|u(t)\|^2 + \frac{L^2}{\gamma} \|u(t - \rho_0(t))\|^2 + \frac{1}{\gamma} \|g(t)\|^2, \quad t > \tau. \tag{2.10}
\]

It follows from \((2.10)\) and Gronwall’s inequality that, for \(t \geq \tau,\)

\[
\|u(t)\| \leq \|u(\tau)\| e^{-\frac{\gamma}{2}(t-\tau)} + \frac{L^2}{\gamma} \int_\tau^t e^{-\frac{\gamma}{2}(t-s)} \|u(s - \rho_0(s))\| ds
\]

\[
+ \frac{1}{\gamma} \int_\tau^t e^{-\frac{\gamma}{2}(t-s)} \|g(s)\| ds. \tag{2.11}
\]

From condition \((2.5),\) by using continuity, we obtain that there exist positive constants \(\lambda\) and \(N\) such that \(\|\varphi\|_\rho \leq N\) and

\[
\frac{\|\varphi\|_\rho}{N} + e^{\lambda_0} \frac{L^2}{(\frac{q}{2} - \lambda_0)\gamma} < 1 \tag{2.12}
\]

hold.

We will prove that, for \(t \geq \tau,\)

\[
\|u(t)\| \leq Ne^{-\lambda(t-\tau)} + (1 - \eta)^{-1} I(t), \tag{2.13}
\]

where \(I(t) = \max_{\tau \leq s \leq t} \frac{1}{\gamma} \int_\tau^s e^{-\frac{\gamma}{2}(t-s)} \|g(s)\| ds.\) To this end, we first prove for any \(d > 1,\)

\[
\|u(t)\| < dNe^{-\lambda(t-\tau)} + (1 - \eta)^{-1} I(t), \quad t \geq \tau. \tag{2.14}
\]
If inequality (2.14) is not true, from the fact that \( \| \varphi \|_\rho \leq N \) and \( \| u(t) \| \) are continuous, then there must be \( t^* > \tau \) such that

\[
\| u(t^*) \| \geq dN e^{-2(t^*-\tau)} + (1 - \eta)^{-1} I(t)
\]

and

\[
\| u(t) \| < dN e^{-2(t-\tau)} + (1 - \eta)^{-1} I(t), \quad \tau - \rho \leq t < t^*.
\]

Hence, it follows from (2.11), (2.12), (2.15), and (2.16) that

\[
\| u(t^*) \| \leq \| u(t) \| e^{-2(t^*-\tau)} + \frac{L^2}{\gamma} \int_{\tau}^{t^*} e^{-2(t^*-s)} \| u(s - \rho_0(s)) \| ds
\]

\[
+ \frac{1}{\gamma} \int_{\tau}^{t^*} e^{-2(t^*-s)} \| g(s) \| ds
\]

\[
< \| u(t) \| e^{-2(t^*-\tau)} + \frac{L^2}{\gamma} \int_{\tau}^{t^*} e^{-2(t^*-s)} (dN e^{\lambda \rho} e^{-2(t^*-\tau)} + (1 - \eta)^{-1} I(t^*)) ds
\]

\[
+ \frac{1}{\gamma} \int_{\tau}^{t^*} e^{-2(t^*-s)} \| g(s) \| ds
\]

\[
\leq \| u(t) \| e^{-2(t^*-\tau)} + \frac{L^2}{\gamma} \int_{\tau}^{t^*} e^{-2(t^*-s)} dN e^{\lambda \rho} e^{-2(t^*-\tau)} ds
\]

\[
+ \frac{L^2}{\gamma} (1 - \eta)^{-1} I(t^*) \int_{\tau}^{t^*} e^{-2(t^*-s)} ds + I(t^*)
\]

\[
\leq \left( \frac{\| u(t) \|}{N} + \frac{L^2}{\gamma} e^{\lambda \rho} \int_{\tau}^{t_{\ast}} e^{-2(t^*-\tau)} ds \right) dN e^{\lambda \rho} e^{-2(t^*-\tau)} + \eta (1 - \eta)^{-1} I(t^*) + I(t^*)
\]

\[
\leq \left( \frac{\| u(t) \|}{N} + e^{\lambda \rho} \frac{L^2}{\gamma (2 - \lambda)} \right) dN e^{\lambda \rho} e^{-2(t^*-\tau)} + (1 - \eta)^{-1} I(t^*)
\]

\[
\leq dN e^{-2(t^*-\tau)} + (1 - \eta)^{-1} I(t^*),
\]

which contradicts inequality (2.15). So inequality (2.14) holds for all \( t \geq \tau \). Letting \( d \to 1 \) in inequality (2.14), we have inequality (2.13). The proof is complete. \( \square \)

Lemma 2.3 implies that the solution \( u \) is defined in any interval of \( [\tau, \tau + T] \) for any \( T > 0 \). It means that this local solution is, in fact, a global one.

Given \( t \in \mathbb{R} \), define a translation \( \theta_t \) on \( \mathbb{R} \) by

\[
\theta_t(\tau) = \tau + t \quad \text{for all } \tau \in \mathbb{R}.
\]

Then \( \{\theta_t\}_{t \in \mathbb{R}} \) is a group acting on \( \mathbb{R} \). We now define a mapping \( \Phi : \mathbb{R}^* \times \mathbb{R} \times C_\rho \to C_\rho \) for problem (2.1)–(2.2). Given \( t \in \mathbb{R}^* \), \( \tau \in \mathbb{R} \), and \( u_\tau \in C_\rho \), let

\[
\Phi(t, \tau, u_\tau) = u_{t+\tau}(\cdot, \tau, u_\tau),
\]

where \( u_{t+\tau} \) is defined by (2.17).
where \( u_{t+s}(s, \tau, u_{\tau}) = u(t + \tau + s, \tau, u_{\tau}), s \in [-\rho, 0] \). By the uniqueness of solutions, we find that for every \( t, s \in \mathbb{R}^+, \tau \in \mathbb{R} \), and \( u_{\tau} \in C_\rho \),

\[
\Phi(t + s, \tau, u_{\tau}) = \Phi(t, s + \tau, \Phi(s, \tau, u_{\tau})).
\]

Then we see that \( \Phi \) is a continuous nonautonomous dynamical system on \( C_\rho \). In the following two sections, we investigate the existence of pullback attractor for \( \Phi \). To this end, we need to define an appropriate collection of families of subsets of \( C_\rho \).

Let \( B_\rho = \{ B_\rho(\tau) : \tau \in \mathbb{R} \} \) be a family of nonempty subsets of \( C_\rho \). Then \( B_\rho \) is called tempered (or subexponentially growing) if for every \( c > 0 \) the following holds:

\[
\lim_{t \to -\infty} e^{ct} \| B_\rho(\tau + t) \|_\rho = 0,
\]

where \( \| B_\rho \|_\rho = \sup_{x \in B_\rho} \| x \|_\rho \). In the sequel, we denote by \( D_\rho \) the collection of all families of tempered nonempty subsets of \( C_\rho \), i.e.,

\[
D_\rho = \{ B_\rho = \{ B_\rho(\tau) : \tau \in \mathbb{R} \} : B_\rho \text{ is tempered in } C_\rho \}.
\]

The following condition will be needed when deriving uniform estimates of solutions:

\[
\int_{-\infty}^\tau e^{\lambda s} \| g(s) \|_2^2 \, ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{2.20}
\]

### 3 Uniform estimates of solutions

In this section, we derive uniform estimates of solutions of problem (2.1)–(2.2) which are needed for proving the existence and uniqueness of a pullback attractor for problem (2.1)–(2.2). The estimates of solutions of problem (2.1)–(2.2) in \( C_\rho \) are provided below. The symbol \( c \) is a positive constant which may change its value from line to line.

**Lemma 3.1** Assume that (H), (2.5), and (2.20) hold. Then, for every \( \tau \in \mathbb{R} \) and \( D_\rho = \{ D_\rho(\tau) : \tau \in \mathbb{R} \} \in D_\rho \), there exists \( T = T_0(\tau; D_\rho) > \rho \) such that, for all \( t \geq T \), \( 0 \leq \xi \leq \rho \), and \( \varphi \in D_\rho(\tau - t) \), the solution \( u \) of (2.1)–(2.2) satisfies

\[
\frac{1}{\gamma} e^{\lambda \rho} \int_{-\infty}^0 e^{\lambda s} \| g(s + \tau) \|_2^2 \, ds.
\]

**Proof** From condition (2.5), by using continuity, we obtain that there exists a positive constant \( \lambda \) such that

\[
\lambda - \frac{\gamma}{2} \frac{\| u_{\tau} \|_\rho^2}{\| u_{\tau} \|_\rho^2} e^{\lambda \rho} < 0 \tag{3.2}
\]

holds.
Calculating the upper right-hand Dini derivative of (3.7), we obtain

\[
\begin{align*}
\frac{d\|u(\sigma, \tau - t, \varphi)\|^2}{dt} & \leq -\frac{\gamma}{2}\|u(\sigma, \tau - t, \varphi)\|^2 + \frac{L^2}{\gamma}\|u(\sigma - \rho(\sigma), \tau - t, \varphi)\|^2 \\
& \quad - \kappa\|u(\sigma, \tau - t, \varphi)\|\|g(\sigma)\|^2.
\end{align*}
\]  

(3.3)

For simplicity, we denote \(u(\sigma) = u(\sigma, \tau - t, \varphi)\). Then, let us define functions \(V(\sigma) = e^{\lambda \sigma}\|u(\sigma)\|\), \(\sigma \geq \tau - t - \rho\), and

\[
U(\sigma) \triangleq \begin{cases} 
  e^{\lambda (\sigma - \rho)}\|\varphi\|, & \sigma \in [\tau - t - \rho, \tau - t), \\
  e^{\lambda (\sigma - \tau)}\|\varphi\| - \kappa \int_{\tau - t}^{\sigma} e^{\lambda s} \|u(s)\|_{q} ds \\
  \quad + \frac{1}{\gamma} \int_{\tau - t}^{\sigma} e^{\lambda s} \|g(s)\| ds, & \sigma \geq \tau - t.
\end{cases}
\]

Now, we claim that

\[
V(\sigma) \leq U(\sigma), \quad \sigma \geq \tau - t.
\]  

(3.4)

If inequality (3.4) is not true from the fact that \(V(t)\) and \(U(t)\) are continuous, then there must be \(\sigma^* > \tau - t\) such that

\[
V(\sigma) < U(\sigma), \quad \sigma \in [\tau - t - \rho, \sigma^*),
\]  

(3.5)

\[
V(\sigma^*) = U(\sigma^*),
\]  

(3.6)

where

\[
\sigma^* \triangleq \inf \{ \sigma > \tau - t \mid V(\sigma) > U(\sigma) \},
\]

and there is a sufficiently small positive constant \(\Delta \sigma\) such that

\[
V(\sigma) > U(\sigma), \quad \sigma \in (\sigma^*, \sigma^* + \Delta \sigma).
\]  

(3.7)

Calculating the upper right-hand Dini derivative of \(V(t)\) at \(\sigma^*\) and considering (3.6) and (3.7), we obtain

\[
D^* V(\sigma^*) = \lim_{h \to 0^+} \frac{V(\sigma^* + h) - V(\sigma^*)}{h} \geq \lim_{h \to 0^+} \frac{U(\sigma^* + h) - U(\sigma^*)}{h}
\]

\[
= -\kappa e^{\lambda \sigma^*}\|u(\sigma^*)\|\|g(\sigma^*)\|^2 + \frac{1}{\gamma} e^{\lambda \sigma^*}\|g(\sigma^*)\|.
\]  

(3.8)

On the other hand, it follows from (3.3) that

\[
D^* V(\sigma^*) = \lambda e^{\lambda \sigma^*}\|u(\sigma^*)\| + e^{\lambda \sigma^*} D^*\|u(\sigma^*)\|
\]

\[
\leq \left( \lambda - \frac{\gamma}{2} \right) e^{\lambda \sigma^*}\|u(\sigma^*)\| + \frac{L^2}{\gamma} e^{\lambda \sigma^*}\|u(\sigma^* - \rho(\sigma^*))\|
\]

\[
- \kappa e^{\lambda \sigma^*}\|u(\sigma^*)\|\|g(\sigma^*)\|^2 + \frac{1}{\gamma} e^{\lambda \sigma^*}\|g(\sigma^*)\|.
\]  

(3.9)
Notice that $U(\sigma)$ is monotone nondecreasing on $[\tau - t - \rho, +\infty)$. This, together with (3.5) and (3.6), yields

$$V(\sigma^* - \rho_0(\sigma^*)) < U(\sigma^* - \rho_0(\sigma^*)) < U(\sigma^*) = V(\sigma^*),$$

which implies

$$\|u(\sigma^* - \rho_0(\sigma^*))\| \leq e^{\rho_0} \|u(\sigma^*)\|.$$  (3.11)

It follows from (3.2), (3.9), and (3.11) that

$$D^* V(\sigma^*) < \left(\lambda - \frac{\gamma}{2} + \frac{L^2}{\gamma} e^{\rho}t\right) e^{\rho t} \|u(\sigma^*)\| - \kappa e^{\rho t} \|u(\sigma^*)\| g(\sigma^*)$$

$$< -\kappa e^{\rho t} \|u(\sigma^*)\| g(\sigma^*),$$

which contradicts (3.8). Until now, (3.4) has been proven to be true. Thus, we get, for $t > \rho$ and $-\rho \leq \xi \leq 0$,

$$\|u(\tau + \xi, \tau - t, \varphi)\| + \kappa e^{-\kappa (t+\xi)} \int_{\tau - t}^{\tau + \xi} e^{\lambda \tau} \|u(s)\| g(s) ds$$

$$\leq \|\varphi\| e^{-\kappa (t+\xi)} + e^{-\kappa (t+\xi)} \frac{1}{\gamma} \int_{\tau - t}^{\tau + \xi} e^{\lambda \tau} \|g(s)\| ds$$

$$\leq \|\varphi\| e^{\lambda \rho} + e^{\lambda \rho} \frac{1}{\gamma} \int_{\tau - t}^{\tau + \xi} e^{\lambda \tau} \|g(s)\| ds.$$

Since $\varphi \in D(\tau - t)$ and $D \in D$, we find that for every $\tau \in \mathbb{R}$ and $D \in D$ there exists $T = T(\tau, D) > \rho$ such that, for all $t \geq T$ and $-\rho \leq \xi \leq 0$,

$$\|u(\tau + \xi, \tau - t, \varphi)\| + \kappa e^{-\kappa (t+\xi)} \int_{\tau - t}^{\tau + \xi} e^{\lambda \tau} \|u(s)\| g(s) ds$$

$$\leq 2 \frac{1}{\gamma} \int_{-\infty}^{0} e^{\lambda s} \|g(s + t)\| ds.$$

This completes the proof.

\[ \Box \]

**Lemma 3.2** Assume that (H), (2.5), and (2.20) hold. Then, for every $\tau \in \mathbb{R}$, $D = \{D(\tau) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D$, there exist $T = T(\tau, D) > \rho$ and $T = N(\tau) > 0$ such that, for all $t \geq T$ and $\varphi \in D(\tau - t)$, the solution $u$ of (2.1)–(2.2) satisfies

$$\sup_{-\rho \leq s \leq 0} \sum_{n \geq N} |u_n(\tau + s, \tau - t, \varphi)| \leq \varepsilon.$$  (3.12)

**Proof** Let $\vartheta$ be a smooth function defined on $\mathbb{R}^+$ such that $0 \leq \vartheta(s) \leq 1$ for all $s \in \mathbb{R}^+$, $\vartheta(0) = 0$ for $0 \leq s \leq 1$ and $\vartheta(s) = 1$ for $s \geq 2$. Note that $\vartheta'$ is bounded on $\mathbb{R}^+$, i.e., there exists a constant $c_0$ such that $|\vartheta'(s)| \leq c_0$ for $s \in \mathbb{R}^+$. 
Taking the inner product of (2.1) with $x = (\vartheta(\frac{|n|}{k})u_n)_{n\in\mathbb{Z}}$ in $l^2$ and keeping the real part, where $k$ is a fixed positive integer specified later, we get

$$\frac{1}{2} \frac{d}{dt} \sum_{n \in \mathbb{Z}} \vartheta \left( \frac{|n|}{k} \right) |u_n|^2 + \text{Re}(\lambda + i\alpha)(Au, x) \leq -\gamma \sum_{n \in \mathbb{Z}} \vartheta \left( \frac{|n|}{k} \right) |u_n|^2 + \text{Re}(f(u(t - \rho_0(t))), x) + \text{Re}(g(t), x).$$ (3.13)

We now estimate the terms in (3.13) as follows. First, we have

$$\text{Re}(\lambda + i\alpha)(Au, x) = \text{Re}(\lambda + i\alpha)(|Bu|^{p-2} \otimes Bu, Bx) = \text{Re}(\lambda + i\alpha) \sum_{n \in \mathbb{Z}} |u_{n+1} - u_n|^{p-2} |u_n - u_{n+1}| \left( \vartheta \left( \frac{|n+1|}{k} \right) - \vartheta \left( \frac{|n|}{k} \right) \right) |u_{n+1} - u_n|^{p-2}(u_{n+1} - u_n)\bar{u}_{n+1} + \lambda \sum_{n \in \mathbb{Z}} \vartheta \left( \frac{|n|}{k} \right) |u_{n+1} - u_n|^p \geq \text{Re}(\lambda + i\alpha) \sum_{n \in \mathbb{Z}} \left( \vartheta \left( \frac{|n+1|}{k} \right) - \vartheta \left( \frac{|n|}{k} \right) \right) |u_{n+1} - u_n|^{p-2}(u_{n+1} - u_n)\bar{u}_{n+1}. $$

By the property of the function $\vartheta$ and Young’s inequality, we have

$$\left| \text{Re}(\lambda + i\alpha) \sum_{n \in \mathbb{Z}} \left( \vartheta \left( \frac{|n+1|}{k} \right) - \vartheta \left( \frac{|n|}{k} \right) \right) |u_{n+1} - u_n|^{p-2}(u_{n+1} - u_n)\bar{u}_{n+1} \right| \leq \sqrt{\lambda^2 + \alpha^2} \sum_{n \in \mathbb{Z}} \vartheta' \left( \frac{|n|}{k} \right) |u_{n+1} - u_n|^{p-1}|u_{n+1}| \leq \frac{\sqrt{\lambda^2 + \alpha^2} c_0}{k} \sum_{n \in \mathbb{Z}} |u_{n+1} - u_n|^{p-1}|u_{n+1}| \leq \frac{c}{k} \|u\|_p^p,$$

which implies that

$$-\text{Re}(\lambda + i\alpha)(Au, x) \leq \frac{c}{k} \|u\|_p^p. \quad (3.14)$$

For the last two terms on the right-hand side of (3.13), we get from (H)

$$\text{Re}(f(u(t - \rho_0(t))), x) + \text{Re}(g(t), x) \leq \frac{L^2}{\gamma} \sum_{n \in \mathbb{Z}} \vartheta \left( \frac{|n|}{k} \right) |u_n(t - \rho_0(t))|^2 + \frac{\gamma}{2} \sum_{n \in \mathbb{Z}} \vartheta \left( \frac{|n|}{k} \right) |u_n|^2 + \frac{1}{\gamma} \sum_{n \in \mathbb{Z}} \vartheta \left( \frac{|n|}{k} \right) |g_n(t)|^2. \quad (3.15)$$
Then, by (3.13)–(3.15) we have
\[
\frac{d}{dt} \sum_{n \in \mathbb{Z}} \left( \frac{|n|}{k} \right) |u_n(t)|^2 \leq -\frac{\gamma}{2} \sum_{n \in \mathbb{Z}} \varphi \left( \frac{|n|}{k} \right) |u_n(t)|^2 + \frac{L^2}{\gamma} \sum_{n \in \mathbb{Z}} \varphi \left( \frac{|n|}{k} \right) |u_n(t - \rho_0(t))|^2
\]
\[+ \frac{c}{k} \|u(t)\|_p + \frac{1}{\gamma} \sum_{|n| \geq k} |g_n(t)|^2. \quad (3.16)
\]

By a similar argument as in Lemma 3.1, we get from (3.16) for any \( t > \rho \) and \(-\rho \leq \xi \leq 0\)
\[
\sum_{n \in \mathbb{Z}} \varphi \left( \frac{|n|}{k} \right) |u_n(t + \xi, \tau - t, \phi)|^2
\]
\[\leq \|\phi\|^2 e^{-\lambda(t+\xi)} + \frac{c}{k} e^{-\lambda(t+\xi)} \int_{t-t}^{t+\xi} e^{\lambda s} \|u(s, \tau - t, \phi)\|_p^p \, ds
\]
\[+ \frac{1}{\gamma} e^{-\lambda(t+\xi)} \int_{t-t}^{t+\xi} e^{\lambda s} \sum_{|n| \geq k} |g_n(s)|^2 \, ds. \quad (3.17)
\]

It follows from Lemma 3.1 and the relation \( p \leq \|t\| + 2, p \geq q + 2 \), that for any \( \tau \in \mathbb{R}, \phi \in \mathcal{D}_\rho \) and \( \varepsilon > 0 \), there exist \( T = T(\tau, \mathcal{D}_\rho) > \rho \) and \( K_1 = K_1(\tau, \mathcal{D}_\rho, \varepsilon) \) such that, for \( k > K_1, t > T \) and \(-\rho \leq \xi \leq 0\),
\[
\frac{c}{k} e^{-\lambda(t+\xi)} \int_{t-t}^{t+\xi} e^{\lambda s} \|u(s, \tau - t, \phi)\|_p^p \, ds \leq \varepsilon, \quad (3.18)
\]

which together with (3.17) implies
\[
\sum_{n \in \mathbb{Z}} \varphi \left( \frac{|n|}{k} \right) |u_n(t + \xi, \tau - t, \phi)|^2
\]
\[\leq \|\phi\|^2 e^{-\lambda(t+\xi)} + \frac{\varepsilon}{3} + \frac{1}{\gamma} e^{-\lambda(t+\xi)} \int_{t-t}^{t+\xi} e^{-\lambda s} \sum_{|n| \geq k} |g_n(s)|^2 \, ds. \quad (3.19)
\]

We have from \( \phi \in \mathcal{D}_\rho(\tau - t) \) that there exists \( T_1 = T_1(\tau, \mathcal{D}_\rho, \varepsilon) > 0 \) such that, for all \( t \geq T_1 \) and \(-\rho \leq \xi \leq 0\),
\[
\|\phi\|^2 e^{-\lambda(t+\xi)} \leq \|\phi\|^2 e^{\lambda\rho} e^{-\lambda t} \leq \frac{\varepsilon}{3}. \quad (3.20)
\]

We have from (2.20) that there is \( N_1 = N_1(\tau, \varepsilon) > 0 \) such that, for all \( k \geq N_1 \),
\[
\frac{1}{\gamma} e^{\lambda \rho} e^{-\lambda t} \int_{-\infty}^{0} e^{-\lambda \tau} \sum_{|n| \geq k} |g_n(s + \tau)|^2 \, d\tau \leq \frac{\varepsilon}{3}. \quad (3.21)
\]

Note that
\[
\sup_{-\rho \leq \xi \leq 0} \sum_{|n| \geq 2k} |u_n(t + \xi, \tau - t, \phi)|^2 \leq \sup_{-\rho \leq \xi \leq 0} \sum_{|n| \geq k} \varphi \left( \frac{|n|}{k} \right) |u_n(t + \xi, \tau - t, \phi)|^2,
\]
which along with (3.19)–(3.21) concludes the proof. \( \square \)
4 Existence of pullback attractors

In this section, we establish the existence of $\mathcal{D}_\rho$-pullback attractor for the nonautonomous dynamical system $\Phi$ associated with problem (2.1)–(2.2).

Lemma 4.1 Assume that (H), (2.5), and (2.20) hold. Then, for every $\tau \in \mathbb{R}$ and $\mathcal{D}_\rho = \{D_\rho(\tau) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}_\rho$, there exists $T = T(\tau, D_\rho) > \rho$ such that the solution $u$ of (2.1)–(2.2) satisfies that $u_1(\tau - t, \Theta)$ is equicontinuous in $\Omega$.

Proof Denote by $P_k u = (u_1, u_2, \ldots, u_k, 0, 0, \ldots)$ for $u \in \Omega$ and $k \in \mathbb{N}$. By Lemma 3.1, for $\tau > 0$, there exist $T = T(\tau, \epsilon) > \rho$ and large enough integer $N = N(\tau, \epsilon)$ such that, for all $t \geq T$,

$$\max_{\rho \geq 0} \left\| (I - P_N) u(\tau + s, \tau - t, \Theta) \right\| < \frac{\epsilon}{3}. \quad (4.1)$$

Let $u_1 = P_N u$. By Lemma 3.1, it follows from (2.1) and the equivalence of norm in a finite dimensional space that there exists $T = T(\tau) > \rho$ such that, for all $t \geq T$,

$$\int_{t - \rho}^{t} \left\| \frac{d}{dr} u_1(r, \tau - t, \Theta) \right\|^2 dr \leq c, \quad (4.2)$$

where $c = c(\tau)$ is a positive number. Without loss of generality, we assume that $s_1, s_2 \in [-\rho, 0]$ with $0 < s_1 - s_2 < 1$. Then, for any fixed $\tau \in \mathbb{R}$,

$$\left\| u_1(\tau + s_1, \tau - t, \Theta) - u_1(\tau + s_2, \tau - t, \Theta) \right\|$$

$$\leq \int_{s_2}^{s_1} \left\| \frac{du_1(r, \tau - t, \Theta)}{dr} \right\| dr$$

$$\leq \left( \int_{t - \rho}^{t} \left\| \frac{du_1(r, \tau - t, \Theta)}{dr} \right\|^2 dr \right)^{\frac{1}{2}} |s_2 - s_1|^{\frac{1}{2}} \leq c|s_2 - s_1|^{\frac{1}{2}}, \quad (4.3)$$

which implies that there exists a constant $\varrho = \varrho(\epsilon) > 0$ such that if $|s_1 - s_2| < \varrho$, then

$$\left\| u_1(\tau + s_2, \tau - t, \Theta) - u_1(\tau + s_1, \tau - t, \Theta) \right\| < \frac{\epsilon}{3},$$

which along with (4.1) implies that, for all $t \geq T$ and $|s_1 - s_2| < \varrho$,

$$\left\| u(\tau + s_2, \tau - t, \Theta) - u(\tau + s_1, \tau - t, \Theta) \right\|$$

$$\leq \left\| P_N u(\tau + s_2, \tau - t, \Theta) - P_N u(\tau + s_1, \tau - t, \Theta) \right\|$$

$$+ \left\| (I - P_N) u(\tau + s_2, \tau - t, \Theta) \right\|$$

$$+ \left\| (I - P_N) u(\tau + s_1, \tau - t, \Theta) \right\|$$

$$\leq \epsilon.$$ 

This completes the proof. \qed

As for the compactness in $\Omega$ in [19], one can easily verify the following compactness criteria in $C_\rho$ by means of uniform tail estimates.
Lemma 4.2 Let \( \{u^n\}_{n=1}^{\infty} = \{(u_i^n)_{i \in \mathbb{Z}}\}_{n=1}^{\infty} \subseteq C_{\rho} \). Then \( \{u^n\}_{n=1}^{\infty} \) is relative compact in \( C_{\rho} \) if and only if the following conditions are satisfied:

(i) \( \{u^n\}_{n=1}^{\infty} \) is bounded in \( C_{\rho} \);

(ii) \( \{u^n\}_{n=1}^{\infty} \) is equicontinuous;

(iii) \( \limsup_{k \to \infty} \limsup_{n \to \infty} \sup_{-\rho \leq s \leq 0} \sum_{|i| \geq k} |u_i^n|^2 = 0 \).

Theorem 4.1 Assume that \((H), (2.5), \) and \((2.20)\) hold. Then the nonautonomous dynamical system \( \Phi \) has a unique \( D_{\rho} \)-pullback attractor \( A_{\rho} = \{A_{\rho}(\tau) : \tau \in \mathbb{R}\} \in D_{\rho} \) in \( C_{\rho} \).

Proof For \( \tau \in \mathbb{R} \), denote by

\[ K(\tau) = \left\{ u \in C_{\rho} : \| u \|_{\rho}^2 \leq 2 \frac{1}{\gamma} \int_{-\infty}^{0} e^{\lambda s} \| g(s + \tau) \| ds \right\}. \]

First, we know from Lemma 3.1 that \( \Phi \) has a \( D_{\rho} \)-pullback absorbing set \( K(\tau) \). Second, since Lemmas 3.1, 3.2, and 4.1 coincide with all the conditions of Lemma 4.2, \( \Phi \) is \( D_{\rho} \)-pullback asymptotically compact in \( C_{\rho} \). Hence, the existence of a unique \( D_{\rho} \)-pullback attractor for the nonautonomous dynamical system \( \Phi \) follows from Proposition 2.7 in [20] immediately. \( \square \)

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References
1. Bates, P.W., Lu, K., Wang, B.: Attractors for lattice dynamical systems. Int. J. Bifurc. Chaos 11, 143–153 (2001)
2. Bell, J., Cosner, C.: Threshold behaviour and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons. Q. Appl. Math. 42, 1–14 (1984)
3. Beyn, W.J., Pilyugin, S.Y.: Attractors of reaction diffusion systems on infinite lattices. J. Dyn. Differ. Equ. 15, 485–515 (2003)
4. Caraballo, T., Morillas, F., Valero, J.: On differential equations with delay in Banach spaces and attractors for retarded lattice dynamical systems. Discrete Contin. Dyn. Syst. 34, 51–77 (2014)
5. Chen, T., Zhou, S., Zhao, C.: Attractors for discrete nonlinear Schrödinger equation with delay. Acta Math. Appl. Sin. Engl. Ser. 26, 633–642 (2010)
6. Chua, L.O., Roska, T.: The CNN paradigm. IEEE Trans. Circuits Syst. 40, 147–156 (1993)
7. Erneux, T., Nicolis, G.: Propagating waves in discrete bistable reaction diffusion systems. Physica D 67, 237–244 (1993)
8. Gu, A., Kloeden, PE.: Asymptotic behavior of a nonautonomous \( p \)-Laplacian lattice system. Int. J. Bifurc. Chaos Appl. Sci. Eng. 26, Article ID 1650174 (2016)
9. Han, X., Kloeden, P.E.: Non-autonomous lattice systems with switching effects and delayed recovery. J. Differ. Equ. 261, 2986–3009 (2016)
10. Kapval, R.: Discrete models for chemically reacting systems. J. Math. Chem. 6, 113–163 (1991)
11. Karachalios, N., Nistazakis, H., Yannacopoulos, A.: Asymptotic behavior of solutions of complex discrete evolution equations: the discrete Ginzburg–Landau equation. Discrete Contin. Dyn. Syst. 19, 711–736 (2007)
12. Keener, J.P.: Propagation and its failure in coupled systems of discrete excitable cells. SIAM J. Appl. Math. 47, 556–572 (1987)
13. Li, C., Hsu, C., Lin, J., Zhao, C.: Global attractors for the discrete Klein–Gordon–Schrodinger type equations. J. Differ. Equ. Appl. 20, 1404–1426 (2014)
14. Li, D., Shi, L.: Upper semicontinuity of random attractors of stochastic discrete complex Ginzburg–Landau equations with time-varying delays in the delay. J. Differ. Equ. Appl. 24, 872–897 (2018)
15. Li, D., Shi, L.: Upper semicontinuity of attractors of stochastic delay reaction–diffusion equations in the delay. J. Math. Phys. 59, Article ID 032703 (2018)
16. Li, X., Wei, K., Zhang, H.: Exponential attractors for lattice dynamical systems in weighted spaces. Acta Appl. Math. 114, 157–172 (2011)
17. Okazawa, N., Yokota, T.: Monotonicity method for the complex Ginzburg–Landau equation, including smoothing effect. Nonlinear Anal. 47, 79–88 (2001)
18. Okazawa, N., Yokota, T.: Global existence and smoothing effect for the complex Ginzburg–Landau equation with p-Laplacian. J. Differ. Equ. 182, 541–576 (2002)
19. Wang, B.: Dynamics of systems on infinite lattices. J. Differ. Equ. 221, 224–245 (2006)
20. Wang, B.: Pullback attractors for the non-autonomous FitzHugh-Nagumo system on unbounded domains. Nonlinear Anal. 70, 3799–3815 (2009)
21. Wang, J., Wang, Y.: Pullback attractors for reaction–diffusion delay equations on unbounded domains with non-autonomous deterministic and stochastic forcing terms. J. Math. Phys. 54, Article ID 082703 (2013)
22. Wang, Y.: On the upper semicontinuity of pullback attractors for multi-valued noncompact random dynamical systems. Discrete Contin. Dyn. Syst., Ser. B 21, 3669–3708 (2016)
23. Wang, Y., Bai, K.: Pullback attractors for a class of nonlinear lattices with delays. Discrete Contin. Dyn. Syst., Ser. B 20, 1213–1230 (2015)
24. Xiang, X., Zhou, S.: Attractors for second order nonautonomous lattice system with dispersive term. Topol. Methods Nonlinear Anal. 46, 893–914 (2015)
25. You, B., Hou, Y., Li, F., Jiang, J.: Pullback attractors for the non-autonomous quasi-linear complex Ginzburg–Landau equation with p-Laplacian. Discrete Contin. Dyn. Syst., Ser. B 19, 1801–1814 (2014)
26. Zhao, C., Zhou, S.: Attractors of retarded first order lattice systems. Nonlinearity 20, 1987–2006 (2007)
27. Zhao, C., Zhou, S.: Limit behavior of global attractors for the complex Ginzburg–Landau equation on infinite lattices. Appl. Math. Lett. 21, 628–635 (2008)

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