Stable solitons in a nearly $\mathcal{PT}$-symmetric ferromagnet with spin-transfer torque

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We consider the Landau-Lifshitz equation for the spin torque oscillator — a uniaxial ferromagnet in an external magnetic field with polarised spin current driven through it. In the absence of the Gilbert damping, the equation turns out to be $\mathcal{PT}$-symmetric. We interpret the $\mathcal{PT}$-symmetry as a balance between gain and loss — and identify the gaining and losing modes. In the vicinity of the bifurcation point of a uniform static state of magnetisation, the $\mathcal{PT}$-symmetric Landau-Lifshitz equation with a small dissipative perturbation reduces to a nonlinear Schrödinger equation with a quadratic nonlinearity. The analysis of the Schrödinger dynamics demonstrates that the spin torque oscillator supports stable magnetic solitons. The $\mathcal{PT}$ near-symmetry is crucial for the soliton stability: the addition of a finite dissipative term to the Landau-Lifshitz equation destabilises all solitons that we have found.

Keywords: spin torque oscillator; Landau-Lifshitz equation; parity-time symmetry; nonlinear Schroedinger equation; solitons; stability

I. INTRODUCTION

Conceived in the context of nonhermitian quantum mechanics [1], the idea of parity-time ($\mathcal{PT}$) symmetry has proved to be useful in the whole range of applied disciplines [2]. A $\mathcal{PT}$-symmetric structure is an open system where dissipative losses are exactly compensated by symmetrically arranged energy gain. In optics and photonics, systems with balanced gain and loss are expected to promote an efficient control of light, including all-optical low-threshold switching [3, 4] and unidirectional invisibility [4, 6]. There is a growing interest in the context of electronic circuitry [7], plasmonics [8], optomechanical systems [9], acoustics [10] and metamaterials [11].

This study is concerned with yet another area where the gain-loss balance gives rise to new structures and behaviours, namely, the magnetism and spintronics. In contrast to optics and nanophotonics, where the nonhermitian effects constitute a well-established field of study, the research into $\mathcal{PT}$-symmetric magnetic systems is still in its early stages, with only a handful of models set up over the last several years.

One of the systems proposed in the literature comprises two coupled ferromagnetic films, one with gain and the other one with loss [12]. (For an experimental implementation of this structure, see [13].) A related concept consists of a pair of parallel magnetic nanowires, with counter-propagating spin-polarized currents [14]. In either case the corresponding mathematical model is formed by two coupled Landau-Lifshitz equations, with $\mathcal{PT}$ symmetry being realised as a symmetry between the corresponding magnetisation vectors. A two-spin Landau-Lifshitz system gauge-equivalent to the $\mathcal{PT}$-symmetric nonlinear Schrödinger equation is also a member of this class of models [13].

An independent line of research concerned the dynamics of a single spin under the action of the spin-transfer torque. Projecting the magnetisation vector onto the complex plane stereographically and modelling the spin by an imaginary magnetic field [10], Galda and Vinokur have demonstrated the $\mathcal{PT}$-symmetry of the resulting nonhermitian Hamiltonian [17]. (For the generalisation to spin chains, see [18]; the nonreciprocal spin transfer is discussed in [19].) Unlike the two-component structures of Refs [12, 14], the $\mathcal{PT}$-symmetry of the spin torque oscillator of Galda and Vinokur is an intrinsic property of an individual spin. It results from the system’s invariance under the simultaneous time reversal and the imaginary magnetic field flip [13].

The structure we consider in this paper shares a number of similarities with the spin torque oscillator of Refs [12, 13]. (There is also a fair number of differences.) It consists of two ferromagnetic layers separated by a conducting film (Fig 1). The spin-polarised current flows from a layer with fixed magnetisation to a layer where the magnetisation vector is free to rotate [16, 20].

A one-dimensional uniaxial classical ferromagnet in the external magnetic field is described by the Landau-Lifshitz equation [16, 20] (also known as the Landau-Lifshitz-Gilbert-Slonczewski equation in the current context):

$$
\dot{M} = -M \times M'' - M \times H - \beta (M \cdot \dot{z}) M \times \dot{z} \\
- \gamma M \times M \times \dot{z} + \lambda M \times \dot{M}. 
$$

(1)
Here the overdot stands for the time derivative and the prime indicates the derivative with respect to $x$. In equation (1), the variables have been non-dimensionalised so that the magnetisation vector $\mathbf{M} = (M_x, M_y, M_z)$ lies on a unit sphere: $\mathbf{M}^2 = 1$. The magnetic field is taken to be constant and directed horizontally: $\mathbf{H} = (H_0, 0, 0)$. The anisotropy axis is $z$, with $\mathbf{z} = (0, 0, 1)$. The positive and negative constant $\beta$ corresponds to the easy-axis and easy-plane anisotropy, respectively. (Note that the authors of [17, 18] considered the ferromagnet anisotropic along the $x$ axis.) The fourth term in the right-hand side of (1) — the Slonczewski term — accounts for the spin transfer by the current that passes through an external ferromagnetic layer that has a fixed magnetisation in the direction $\mathbf{z}$. The last term is the Gilbert damping term. The damping coefficient $\lambda$ is positive; the field $H_0$ and the current amplitude $\gamma$ can also be chosen positive without loss of generality.

In this paper, we study the nonlinear dynamics of the localised solutions of the equation (1), both with small and finite-strength damping.

A class of soliton solutions of the Landau-Lifshitz equation (1) was obtained by Hoefer, Silva and Keller [21]. The solitons discovered by those authors are dissipative analogs of the Ivanov-Kosevich magnon droplets [22]. (For the experimental realisation, see [23].) Our setup has a different geometry from the one of Hoefer et al. One difference is that we consider the magnetic layer with a parallel anisotropy while the magnon droplets require a perpendicular one [21, 22]. An additional distinction is that our vector $\mathbf{H}$ is orthogonal to the direction of the fixed magnetisation — while the magnetic field in Ref [21] was not. Because of the different geometry, the Landau-Lifshitz equation of Ref [21] does not exhibit the $\mathcal{PT}$ invariance. The dissipative magnon droplets are sustained through the competition of torque and damping, the two actors represented by terms of different mathematical form, rather than by a symmetric balance of two similar but oppositely-directed effects.

Another class of localised structures in the spin torque oscillator is commonly referred to as the standing spin wave bullets. These have been theoretically predicted by Slavin and Tiberkevich [24] — outside the context of the Landau-Lifshitz equation. (For the experimental realisation, see [24].) The spin wave bullets are found in the magnetic layer with parallel anisotropy, when the magnetic field is directed parallel to the fixed layer’s magnetisation. The direction of the vector $\mathbf{H}$ is what makes our geometry different from the setup considered in Ref [24]. Like the magnon droplets, the spin wave bullets are sustained by an asymmetric balance of the spin torque and finite-strength damping.

The paper is organised as follows. We start with the demonstration of the gain-loss balance in the Landau-Lifshitz equation with the vanishing Gilbert damping. This $\mathcal{PT}$-symmetric system and systems that are close to it will prove to have special properties in this paper, where we consider equations both with small and finite $\lambda$.

In section II, we classify stability and bifurcation of four nonequivalent stationary states with uniform magnetisation. Three of those states are found to be admissible as stable backgrounds for localised structures. In the vicinity of the bifurcation points, the dynamics of the localised structures are governed by quadratic Schrödinger or Ginsburg-Landau equations, depending on whether the Gilbert damping is weak or finite-strength (section IV). Despite the absence of the dissipative terms, our quadratic Schrödinger equations are not conservative; however one of them obeys the $\mathcal{PT}$-symmetry. Both Ginsburg-Landau equations and their Schrödinger counterparts — $\mathcal{PT}$-symmetric or not — support two types of soliton solutions. We show that either of these types is only stable in the $\mathcal{PT}$-symmetric situation (sections V, VI). Section VII summarises results of this study.

II. GAIN-LOSS BALANCE IN THE ABSENCE OF GILBERT LOSSES

The equation (1) is nonconservative due to the presence of the spin torque and Gilbert’s dissipative term. In spin torque oscillators, solitons are expected to exist due to the energy supplied by torque being offset by finite-strength dissipation [21]. However when $\lambda = 0$, the spin hamiltonian modelling our structure is $\mathcal{PT}$-symmetric [17] and therefore some form of the gain-loss balance should occur in this case as well, despite the absence of the Gilbert damping. To uncover the gain-loss competition intrinsic to the spin torque, we define two complex fields, $u(x,t)$ and $v(x,t)$, related to the magnetisation vector $\mathbf{M}$ via the Hopf map:

$$M_x = v^* u + u^* v, \quad M_y = i (u^* v - v^* u), \quad M_z = |u|^2 - |v|^2. \quad (2)$$

When the magnetisation is spatially uniform, $\partial \mathbf{M} / \partial x = 0$, the equation (1) with $\lambda = 0$ can be reformulated as a
nonlinear Schrödinger dimer:

\[
i u_t + \frac{H_0}{2} v + \frac{\beta}{2} (|u|^2 - |v|^2) u = i \gamma |v|^2 u, \tag{3}
\]

\[
i v_t + \frac{H_0}{2} u - \frac{\beta}{2} (|u|^2 - |v|^2) v = -i \gamma |u|^2 v. \tag{4}
\]

According to equations (3)-(4), the external energy is fed into the \(u\)-mode and dissipated by its \(v\) counterpart. The magnetic field \(H_0\) couples \(u\) to \(v\), carrying out the energy exchange between the two modes.

The sustainability of the gain-loss balance in the system (3)-(4) is reflected by its invariance under the product of the \(\mathcal{P}\) and \(\mathcal{T}\) transformations. Here the inversion \(\mathcal{P}\) swaps the two modes around,

\[\mathcal{P} : \; u \rightarrow v, \quad v \rightarrow u, \tag{5}\]

while \(\mathcal{T}\) represents the reflection of time:

\[\mathcal{T} : \; t \rightarrow -t, \quad u \rightarrow u^*, \quad v \rightarrow v^*. \tag{6}\]

These transformations admit a simple formulation in terms of the components of magnetisation \(\mathcal{M}\):

\[\mathcal{P} : \; M_y \rightarrow -M_y, \quad M_z \rightarrow -M_z \tag{7}\]

and

\[\mathcal{T} : \; t \rightarrow -t, \quad M_y \rightarrow -M_y, \tag{8}\]

The involutions (5) and (8) remain relevant in the analysis of the equation (1) with the \(x\)-dependent magnetisation. Here one can either leave the parity operation in the form (7) or include the inversion of the \(x\) coordinate in this transformation:

\[\mathcal{P} : \; x \rightarrow -x, \quad M_y \rightarrow -M_y, \quad M_z \rightarrow -M_z. \tag{9}\]

Writing the vector equation (11) in the component form,

\[
\begin{align*}
\dot{M}_x &= M_z M''_y - M_y M''_z - \beta M_y M_z \\
&\quad - \gamma M_z M_z + \lambda (M_y \dot{M}_z - M_z \dot{M}_y), \\
\dot{M}_y &= M_x M''_z - M_z M''_x - H_0 M_z + \beta M_x M_z \\
&\quad - \gamma M_y M_z + \lambda (M_x \dot{M}_z - M_z \dot{M}_x), \\
\dot{M}_z &= M_y M''_z - M_z M''_z - H_0 M_y + \beta M_z M_z \\
&\quad + \gamma (M_x^2 + M_y^2) + \lambda (M_x M_y - M_y M_x),
\end{align*} \tag{10}
\]

one readily checks that in the conservative limit (\(\gamma = \lambda = 0\)), the Landau-Lifshitz equation is invariant under the \(\mathcal{P}\)- and \(\mathcal{T}\)-involutions individually. The equation with the spin torque term added (\(\gamma \neq 0\)) is invariant under the product \((\mathcal{P}\mathcal{T})\) transformation only. Accordingly, the equation with the \(\gamma\)-term is a \(\mathcal{P}\mathcal{T}\)-symmetric extension of the conservative Landau-Lifshitz equation. Finally, the addition of the Gilbert damping term (\(\lambda \neq 0\)) breaks the \(\mathcal{P}\mathcal{T}\)-symmetry.

### III. Uniform static states

The uniform static states are space- and time-independent solutions of equation (11) satisfying \(\mathbf{M}^2 = 1\). These are given by fixed points of the dynamical system

\[
\begin{align*}
\dot{M}_x &= -\beta M_y M_z - \gamma M_z M_z + \lambda (M_y \dot{M}_z - M_z \dot{M}_y), \\
\dot{M}_y &= \beta M_x - H_0 - \gamma M_y M_z + \lambda (M_x \dot{M}_z - M_z \dot{M}_x), \\
\dot{M}_z &= H_0 M_y + \gamma (M_x^2 + M_y^2) + \lambda (M_x M_y - M_y M_x)
\end{align*} \tag{11}
\]

on the surface of the unit sphere.

Once a fixed point \(\mathbf{M}^{(0)}\) has been determined, we let \(\mathbf{M} = \mathbf{M}^{(0)} + \delta \mathbf{M}\), linearise the system (11) in \(\delta \mathbf{M}\), and consider solutions of the form

\[
\delta \mathbf{M} = \mathbf{m} e^{i \mu t - i k x}, \tag{12}
\]

where \(\mathbf{m} = (m_x, m_y, m_z)^T\) is a real constant vector and \(k\) a real wavenumber that may take values from \(-\infty\) to \(\infty\).

We call the uniform static state unstable if at least one of the roots \(\mu\) of the associated characteristic equation has a positive real part in some interval of \(k\). Otherwise the state is deemed stable.

Since the equation (11) conserves the quantity \(\mathbf{M}^2\), the difference between \((\mathbf{M}^{(0)})^2\) and the square of the vector \(\mathbf{M}^{(0)} + \delta \mathbf{M}\) will be time-independent:

\[
\frac{\partial}{\partial t} \left( 2 \mathbf{M} \cdot \mathbf{M}^{(0)} \right) = 0.
\]

Substituting from (12) and assuming \(\mu \neq 0\), this gives

\[
\mathbf{m} \cdot \mathbf{M}^{(0)} = 0. \tag{13}
\]

Equation (13) implies (a) that the time-dependent perturbations of the uniform static states lie on the unit sphere; and (b) that the characteristic equation may not have more than two nonzero roots, \(\mu_1\) and \(\mu_2\). The third root (\(\mu_3\)) has to be zero.

Apart from classifying stability of the uniform static states, it is useful to know which of these solutions can serve as backgrounds to static magnetic solitons. To weed out a priori unsuitable cases, we set \(\mu = 0\) in the characteristic equation and consider \(k^2\) as a new unknown (rather than a parameter that varies from 0 to \(\infty\)). If all roots \((k^2)_n\) of the resulting equation are real positive, there can be no localised solutions asymptotic to the uniform static state \(\mathbf{M}^{(0)}\) as \(x \rightarrow \pm \infty\). On the other hand, if there is at least one negative or complex root, the solution \(\mathbf{M}^{(0)}\) remains a candidate for solitons’ background.

#### A. Equatorial fixed points on the unit sphere

One family of time-independent solutions of the system (11) describes a circle on the \((M_x, M_y)\)-plane:

\[
M_x^2 + \left( M_y + \frac{H_0}{2 \gamma} \right)^2 = \frac{H_0^2}{4 \gamma^2}, \quad M_z = 0.
\]
In the system with $\mathbf{M}^{(0)} > 0$ is stable irrespective of the choice of $\gamma$, $\lambda$ and $H_0$. On the other hand, when the anisotropy is easy-axis ($\beta > 0$), the eastern state is stable if

$$H_0 \geq \sqrt{\beta^2 + \gamma^2}$$

and unstable otherwise.

To check the suitability of the eastern uniform static state as a background for solitons, we set $\mu = 0$ in the expression (15a); this transforms it into a quadratic equation for $k^2$. When $\beta < \sqrt{H_0^2 - \gamma^2}$, both roots of this equation are negative: $k^2 = -\sqrt{H_0^2 - \gamma^2}$ and $k^2 = \beta - \sqrt{H_0^2 - \gamma^2}$. This implies that there is a pair of exponentials $\exp(-ik_{1,2}x)$ decaying to zero as $x \to -\infty$ and another pair decaying as $x \to +\infty$. Therefore the uniform static state (14) with $M_x^{(0)} > 0$ can serve as a background to solitons for any set of parameters $\beta, \gamma, H_0, \lambda$ in its stability domain.

Turning to the west-point solution ($M_x^{(0)} < 0$ in equation (14)), a simple analysis of the eigenvalues (15) indicates that there are wavenumbers $k$ such that $\text{Re} \mu > 0$ for any quadruplets of $\beta, \gamma, H_0$ and $\lambda$. Hence the western uniform static state is always unstable. We are not considering it any further.

### B. Latitudinal fixed points

Another one-parameter family of constant solutions of the equation (11) forms a vertical straight line in the $M_x, M_y, M_z$-space:

$$M_x = \frac{H_0\beta}{\beta^2 + \gamma^2}, \quad M_y = -\frac{H_0\gamma}{\beta^2 + \gamma^2}, \quad -\infty < M_z < \infty.$$  

The substitution of the above coordinates into $\mathbf{M}^2 = 1$ selects two fixed points on the unit sphere:

$$M_x^{(0)} = \frac{H_0\beta}{\beta^2 + \gamma^2}, \quad M_y^{(0)} = -\frac{H_0\gamma}{\beta^2 + \gamma^2}, \quad M_z^{(0)} = \pm \sqrt{1 - \frac{H_0^2}{\beta^2 + \gamma^2}}.$$  

Since these points lie above and below the equatorial ($M_x, M_y$)-plane, we will be calling them the latitudinal fixed points: the northern ($M_z^{(0)} > 0$) and the southern ($M_z^{(0)} < 0$) one. See Fig (2b).

In the anisotropic equation ($\beta \neq 0$) the northern and southern points are born as $H_0$ is decreased through $\sqrt{\beta^2 + \gamma^2}$. In this case, there is a parameter interval $\gamma < H_0 < \sqrt{\beta^2 + \gamma^2}$ where two pairs of fixed points, latitudinal and equatorial, coexist.

The bifurcation diagram for the isotropic equation is different. When $\beta = 0$, the latitudinal fixed points (17)
emerge as the eastern and western points \([14]\) converge and split out of the equatorial plane. In this case, there is just one pair of uniform static states for any \(H_0 \neq \gamma\): the equatorial pair for \(H_0 > \gamma\) and the latitudinal pair for \(H_0 < \gamma\).

The linearisation of equation \([11]\) about the uniform static state corresponding to the latitudinal fixed-point \([14]\) gives

\[
\mu_{1,2} = -\frac{\lambda Q + \gamma M_z^{(0)} + \sqrt{\frac{1}{4} \lambda^2 \beta^2 h^2 - P(P - \beta h)}}{1 + \lambda^2}, \quad (18)
\]

where

\[
P = k^2 + \beta - \gamma \lambda M_z^{(0)}, \quad Q = k^2 + \beta \left(1 - \frac{h}{2}\right)
\]

and

\[
h = \frac{H_0^2}{\beta^2 + \gamma^2}. \quad (19)
\]

A simple analysis demonstrates that when \(\beta \geq 0\), the north-point solution \((M_z^{(0)} > 0\) in \([17]\) is stable regardless of the values of \(\lambda \geq 0\), \(H_0\) and \(\gamma > 0\). As for the easy-plane anisotropy (\(\beta < 0\)), the northern uniform static state is stable only when the inequalities \(\lambda \leq \lambda_c\) and \(H_0 \leq H_c\) are satisfied simultaneously. Here

\[
\lambda_c = \frac{\gamma}{|\beta|} \sqrt{1 - \frac{1}{h/2}} \quad (20)
\]

and

\[
H_c = \frac{2(\beta^2 + \gamma^2)}{\beta^2} \left(\sqrt{\beta^2 + \gamma^2} - \gamma\right) \quad (21)
\]

Note that \(H_c\) is smaller than \(\sqrt{\gamma^2 + \beta^2}\); hence the region \(H_0 \leq H_c\) lies entirely within the northern point’s existence domain (defined by the inequality \(H_0 < \sqrt{\gamma^2 + \beta^2}\)).

Finally, we consider the eigenvalues pertaining to the southern uniform static state \((M_z^{(0)} < 0\) in \([17]\)\). Our conclusion here is that in the isotropic and easy-plane ferromagnet (\(\beta \leq 0\)), this solution is unstable regardless of the choice of other parameters. In the easy-axis situation (\(\beta > 0\)), the southern state is stable if \(\lambda \geq \lambda_c\) with \(\lambda_c\) as in \([20]\) — and unstable otherwise.

To determine the parameter region where the northern point solutions can serve as backgrounds for solitons, we set \(\mu = 0\) in \([18]\). In each of the two cases, the resulting quadratic equation for \(k^2\) has two positive roots only if \(\beta < 0\) is satisfied along with the inequality \(H_0 > H_c\), where \(H_c\) is as in \([21]\). This is the only no-go region for solitons. Outside this region, the quadratic equation has either two negative or two complex roots; the corresponding uniform static states can serve as solitons’ asymptotes.

The bottom line is that either of the two latitudinal uniform static states is suitable as a background for solitons in its entire stability domain.

### C. Summary of uniform static states

For convenience of the reader, the stability properties of the constant solutions corresponding to the four fixed points are summed up in Table I.

Before turning to the perturbations of these uniform static states, it is worth noting their symmetry properties. Each of the equatorial states is \(PT\)-symmetric in the sense that each of these two solutions is invariant under the product of the transformations \([11]\) and \([8]\). In contrast, neither of the two latitudinal states is invariant; the \(PT\) operator maps the northern solution to southern and the other way around. The different symmetry properties of the equatorial and longitudinal solutions will give rise to different invariances of equations for their small perturbations.

| Fixed-point solution: | \(\beta < 0\) | \(\beta = 0\) | \(\beta > 0\) |
|----------------------|---------------|---------------|---------------|
| eastern              | stable        | stable        | stable if \(H_0 \geq \sqrt{\beta^2 + \gamma^2}\) |
| western              | unstable      | unstable      | unstable      |
| northern             | stable if \(\lambda \leq \lambda_c\) | stable         | stable         |
| southern             | unstable      | unstable      | stable if \(\lambda \geq \lambda_c\)         |

**TABLE I. Stability of four constant solutions of equation \([11]\).**

### IV. SLOW DYNAMICS NEAR BIFURCATION POINTS

#### A. Perturbation of equatorial fixed point

Consider the eastern point of the pair of equatorial fixed points \([13]\):

\[
M^{(0)} = \left(\sqrt{1 - \frac{\gamma^2}{H_0^2}}, -\frac{\gamma}{H_0}, 0\right). \quad (22)
\]

We assume that the parameters \(\beta, \gamma, \lambda\) and \(H_0\) lie in the stability domain of the uniform static state \([22]\).

The plane orthogonal to the vector \(M^{(0)}\) is spanned by the vectors

\[
A = (0, 0, 1), \quad B = \left(\frac{\gamma}{H_0}, \sqrt{1 - \frac{\gamma^2}{H_0^2}}, 0\right).
\]

The unit vector \(M\) can be expanded over the orthonormal triplet \(\{A, B, M^{(0)}\}\):

\[
M = \eta A + \xi B + \chi M^{(0)}.
\]
Letting \( \mathbf{M}(x,t) \to \mathbf{M}^{(0)} \) as \( x \to \pm \infty \), the coefficient fields \( \eta, \xi \) and \( \chi \) have the following asymptotic behaviour:

\[
\eta \to 0, \quad \xi \to 0, \quad \chi \to 1 \quad \text{as} \quad |x| \to \infty.
\]

The complex field \( \Psi = \xi + i\eta \) satisfies

\[
i\dot{\Psi} = \chi \dot{\Psi}'' - \dot{\Psi} \chi'' + \lambda(\dot{\Psi} \chi - \chi \dot{\Psi}) - \sqrt{H_0^2 - \gamma^2} \Psi + \gamma(\chi - i\eta \xi - 1 + \eta^2) + i\beta \chi, \quad (23)
\]

where \( \chi = \sqrt{1 - |\Psi|^2} \) while the prime and overdot indicate the derivative with respect to \( x \) and \( t \), respectively. Note that when \( \lambda = 0 \), the equation (23) is \( P\overline{T} \)-symmetric, that is, invariant under a composite transformation consisting of three involutions: \( t \to -t, \quad x \to -x, \) and \( \Psi \to \Psi^* \).

Assume that \( H_0 \) is close to the bifurcation point of the uniform static state \( \mathbf{22} \) — that is, \( H_0 \) is slightly greater than \( \gamma \). In this case, \( \Psi \) will depend on a hierarchy of slow times \( T_n = e^{\nu t} \) and stretched spatial coordinates \( X_n = e^{\nu/2} x \), where \( n = 1, 3, 5, \ldots \) and the small parameter \( \epsilon \) is defined by

\[
\epsilon^2 = 1 - \frac{\gamma^2}{H_0^2}.
\]

In the limit \( \epsilon \to 0 \) the new coordinates become independent so we can write

\[
\frac{\partial}{\partial t} = \epsilon D_1 + \epsilon^3 D_3 + \ldots ;
\]

\[
\frac{\partial^2}{\partial x^2} = \epsilon \dot{D}_1 + 2\epsilon^2 \partial_1 \partial_3 + \epsilon^3 (\dot{D}_3^2 + 2\partial_1 \partial_3) + \ldots ,
\]

where \( D_n = \partial/\partial T_n \) and \( \partial_n = \partial/\partial X_n \). Assume, in addition, that the anisotropy constant \( \beta \) is of order \( \epsilon \) and let \( \beta = \epsilon \mathcal{B} \) with \( \mathcal{B} = O(1) \). Considering small \( \eta \) and \( \xi \), we expand

\[
\Psi = \epsilon \psi_1 + \epsilon^3 \psi_3 + \ldots .
\]

Substituting the above expansions in (23), we equate coefficients of like powers of \( \epsilon \). The order \( \epsilon^2 \) gives a Ginsburg-Landau type of equation with a quadratic nonlinearity:

\[
(i + \lambda)D_1 \psi - \epsilon \partial_1^2 \psi + \frac{\gamma}{2} \psi^2 = -\gamma \psi + \frac{\mathcal{B}}{2} (\psi - \psi^*). \quad (24)
\]

(Here \( \psi \) is just a short-hand notation for \( \psi_1 \).)

Note that in the derivation of (24) we took \( \lambda = O(1) \). If we, instead, let \( \lambda = O(\epsilon) \), the dissipative term would fall out of the equation (24) and we would end up with a nonlinear Schrödinger equation:

\[
iD_1 \psi - \partial_1^2 \psi + \frac{\gamma}{2} \psi^2 = -\gamma \psi + \frac{\mathcal{B}}{2} (\psi - \psi^*). \quad (25)
\]

The quadratic Schrödinger equation (24) does not have the \( \text{U}(1) \) phase invariance. However, the equation is \( P\overline{T} \)-symmetric, that is, invariant under the composite map \( t \to -t, \quad x \to -x, \quad \psi \to \psi^* \). As we will see in section IV, this discrete symmetry is enough to stabilise solitons.

### B. Perturbation of latitudinal fixed points

Choosing the background in the form of one of the two latitudinal fixed points

\[
\mathbf{M}^{(0)} = \left( \frac{\beta}{H_0} h, -\frac{\gamma}{H_0} h, \pm \sqrt{1 - h} \right), \quad (26)
\]

we let \( \mathbf{M}(x,t) \) approach the same point \( \mathbf{M}^{(0)} \) as \( x \to \pm \infty \). In (26), \( h \) is defined by the equation (19).

As in the previous subsection, we expand the magnetisation vector over an orthonormal basis \( \{ \mathbf{A}, \mathbf{B}, \mathbf{M}^{(0)} \} \):

\[
\mathbf{M} = \eta \mathbf{A} + \xi \mathbf{B} + \lambda \mathbf{M}^{(0)}, \quad (27)
\]

where, this time,

\[
\mathbf{A} = \left( \pm \frac{\beta}{H_0} \sqrt{h(1 - h)}, \pm \frac{\gamma}{H_0} \sqrt{h(1 - h)}, \sqrt{1 - h} \right)
\]

and

\[
\mathbf{B} = \left( \frac{\gamma}{H_0} \sqrt{1 - h}, \frac{\beta}{H_0} \sqrt{1 - h}, 0 \right).
\]

We assume that \( H_0 \) is close to the bifurcation point where the northern and southern fixed points are born (that is, \( H_0 \) is slightly smaller than \( \sqrt{\beta^2 + \gamma^2} \)) and define a small parameter \( \epsilon \):

\[
h = 1 - \epsilon^2.
\]

As in the analysis of the equatorial fixed points, we let \( \beta = \epsilon \mathcal{B} \), where \( \mathcal{B} = O(1) \). Assuming that the magnetisation \( \mathbf{M} \) is just a small perturbation of \( \mathbf{M}^{(0)} \), we expand the small coefficients in (27) in powers of \( \epsilon \):

\[
\eta = \epsilon \eta_1 + \epsilon^3 \eta_3 + \ldots , \quad \xi = \epsilon \xi_1 + \epsilon^3 \xi_3 + \ldots .
\]

The constraint \( \eta^2 + \xi^2 + \chi^2 = 1 \) implies then

\[
\chi = 1 - \frac{\epsilon^2 \eta_1^2 + \epsilon^2 \xi_1^2}{2} + \ldots .
\]

Substituting these expansions in the Landau-Lifshitz equation (11) and equating coefficients of like powers of \( \epsilon \), the order \( \epsilon^2 \) gives

\[
D_1 \xi_1 = \partial_1^2 \eta_1 - \lambda D_1 \eta_1 - \gamma (\eta_1 \xi_1 \pm \xi_1)
\]

and

\[
D_1 \eta_1 = \lambda D_1 \xi_1 - \partial_1^2 \xi_1 + \mathcal{B} \xi_1 \mp \gamma \eta_1 - \frac{\gamma}{2} (\eta_1^2 - \xi_1^2).
\]

The above two equations can be combined into a single equation for the complex function \( \psi = \xi_1 + i\eta_1 \):

\[
(i + \lambda)D_1 \psi - \partial_1^2 \psi + \frac{\gamma}{2} \psi^2 = \mp \epsilon i \gamma \psi - \frac{\mathcal{B}}{2} (\psi + \psi^*). \quad (28)
\]

The Ginsburg-Landau equation (28) resembles the equation (24) governing the dynamics near the equatorial uniform static state; however there is an important difference. Namely, even if we let \( \lambda = 0 \) in (28) [that is, even if we assume that the damping is \( O(\epsilon) \) or weaker in the Landau-Lifshitz-Gilbert equation (11)], the resulting nonlinear Schrödinger equation will not become \( P\overline{T} \)-symmetric. This fact will have important repercussions for the stability of solitons.
FIG. 3. Instability of the fundamental soliton in the presence of damping. This evolution was obtained by the direct numerical simulation of the equation (30) with \( b = 0 \) and \( \lambda = 0.1 \). The initial condition was in the form of the soliton (31) perturbed by a random perturbation within 5% of the soliton’s amplitude. The spatial interval of simulation was \((-58, 58)\); in the plot it has been cut down for visual clarity.

V. SOLITON EXCITATIONS OF EQUATORIAL STATE

Letting \( u(x, t) = -\frac{1}{3} \psi(X_1, T_1), \quad x = \frac{\sqrt{7}}{2} X_1, \quad t = \frac{\gamma}{4} T_1, \) (29)
the Ginsburg-Landau equation (24) is cast in the form

\[(i + \lambda)u_t - u_{xx} - 6u^2 = -4u + b(u - u^*), \tag{30}\]

where \( b = 2B/\gamma \). (We alert the reader that the scaled variables \( x \) and \( t \) do not coincide with the original \( x \) and \( t \) of the Landau-Lifshitz equation (1). We are just re-employing the old symbols in a new context here.)

In the present section we consider localised solutions of the equation (30) approaching 0 as \( |x| \to \infty \). Regardless of \( \lambda \), the zero solution is stable if \( b \leq 2 \) and unstable otherwise. This inequality agrees with the stability range (16) of the eastern uniform static state within the original Landau-Lifshitz equation. (Note that the term \( bu^* \) plays the role of the parametric driver in (30) \[26\]; the above stability criterion states that the zero solution cannot sustain drivers with amplitudes greater than \( b = 2 \).)

A. Fundamental soliton and its stability

Equation (30) has a stationary soliton solution:

\[ u_s = \text{sech}^2 x. \tag{31} \]

To distinguish it from localised modes with internal structure, we refer to this solution as the fundamental soliton — or simply sech mode. Letting

\[ u(x, t) = u_s(x) + \varepsilon [f(x) + ig(x)] e^{\mu t} \]

and linearising in small \( \varepsilon \), we obtain an eigenvalue problem

\[ \mu (g - \lambda f) = \mathcal{H} f, \tag{32a} \]
\[ -\mu (f - \lambda g) = (\mathcal{H} - 2b) g, \tag{32b} \]

with the operator

\[ \mathcal{H} = -d^2/dx^2 + 4 - 12 \text{sech}^2 x. \tag{33} \]

The vector eigenvalue problem (32) is reducible to a scalar eigenvalue problem of the form

\[ (\mathcal{H} - b + \mu \lambda)^2 g + (\mu^2 - b^2) g = 0. \]

The stability exponents \( \mu \) are roots of the quadratic equation

\[ (E - b + \mu \lambda)^2 + \mu^2 - b^2 = 0, \]

where \( E \) is an eigenvalue of the operator \( \mathcal{H} : \mathcal{H} y = Ey \). The two roots are

\[ \mu^{(\pm)} = \frac{\lambda(b - E) \pm \sqrt{\lambda^2 b^2 + E(2b - E)}}{1 + \lambda^2}. \tag{34} \]

The eigenvalues of the Pöschl-Teller operator (33) are \( E_0 = -5, E_1 = 0 \), and \( E_2 = 3 \), with the eigenfunctions \( y_0 = \text{sech}^2 x, y_1 = \text{sech}^2 x \tanh x \) and \( y_2 = \)
between two situations: damped ($\lambda > 0$) and undamped ($\lambda = 0$). Assume, first, that $\lambda > 0$ and let, in addition, $b \geq 0$. It is not difficult to check that the root $\mu_n^{(+)}$ will have a positive real part provided the corresponding eigenvalue $E_n$ satisfies $E_n < 2b$. On the other hand, the set of three eigenvalues of the operator (33) does include a negative eigenvalue ($E_0$) that satisfies $E_0 < 2b$ regardless of the particular value of $b \geq 0$. Therefore the soliton has an exponent $\mu_0^{(+)}$ with $\Re \mu_0^{(+)} > 0$ for any $b \geq 0$.

In the case where $\lambda > 0$ but $b < 0$, the root $\mu_n^{(+)}$ will have a positive real part provided $E_n$ satisfies $E_n < 0$. As in the previous case, this inequality is satisfied by the eigenvalue $E_0$ so that the soliton has an exponent with $\Re \mu_0^{(+)} > 0$ for any $b < 0$.

We conclude that the fundamental soliton of the equation (33) is unstable in the presence of damping — regardless of the sign and magnitude of the anisotropy coefficient $b$. Figure 3 illustrates the evolution of a weakly perturbed soliton in the Ginsburg-Landau equation with $\lambda \neq 0$.

Turning to the situation with $\lambda = 0$ we assume, first, that $b > 0$. The equations (33) will give a pair of opposite real roots $\mu_n^{(\pm)}$ if the corresponding eigenvalue satisfies $0 < E_n < 2b$ and a pair of pure imaginary roots otherwise. The only positive eigenvalue of the operator (33) is $E_2 = 3$; it satisfies the above inequality if $b > 3/2$.

In the situation where $\lambda = 0$ but $b < 0$, the pair of opposite exponents $\mu_n^{(\pm)}$ is real if $E_n$ falls in the interval $2b < E_n < 0$ and pure imaginary if $E_n$ lies outside this interval. The only negative eigenvalue is $E_0 = -5$; it falls in the interval in question if $b < -5/2$.

Finally, in the isotropic ferromagnet ($b = 0$) the stability exponents are all pure imaginary: $\mu_n^{(\pm)} = \pm i E_n$.

Combining the intervals where all exponents are pure imaginary gives us the stability region of the undamped fundamental soliton in terms of the anisotropy to spin-current ratio:

$$-\frac{5}{2} \leq b \leq \frac{3}{2}.$$  (35)

B. Twisted modes in isotropic ferromagnet

The Ginsburg-Landau equation (30) with $b = 0$ admits an additional pair of localised solutions:

$$u_T = 2 \text{sech}^2(2x) \pm 2i \text{sech}(2x) \tanh(2x).$$  (36)

The modulus of $u_T(x)$ is bell-shaped while its phase grows or decreases by $\pi$ as $x$ changes from $-\infty$ to $+\infty$. The solution looks like a pulse twisted by $180^\circ$ in the $(\Re u, \Im u)$-plane. In what follows, we refer to each of the equations (30) as a twisted, or simply sech-tanh, mode.

Linearising equation (30) about the twisted mode (36) and assuming that the small perturbation depends on time as $e^{\mu t}$, we arrive at an eigenvalue problem

$$\mathcal{L} f(X) = -\frac{\mu}{4}(\lambda + i)f(X)$$  (37)
for the Schrödinger operator with the Scarff-II complex potential:

\[
\mathcal{L} = -\frac{d^2}{dX^2} + 1 - 6\text{sech}^2 X \mp 6i \text{sech} X \tanh X. \quad (38)
\]

In \([37,38]\), \(X = 2x\).

The \(\mathcal{PT}\)-symmetric operator \([38]\) has an all-real spectrum including three discrete eigenvalues \([27]\). Let \(y_n\) be the eigenfunction associated with an eigenvalue \(E_n\):

\[
\mathcal{L} y_n = E_n y_n.
\]

The eigenvalue-eigenfunction pairs are then given by

\[
E_0 = -\frac{5}{4}, \quad y_0 = (\text{sech}^2 X \pm i \text{sech} X \tanh X)^{3/2};
\]

\[
E_1 = 0, \quad y_1 = \text{sech} X (\text{sech} X \pm i \tanh X)^2, \quad (39)
\]

and \(E_2 = 3/4\) with

\[
y_2 = (3 \pm 2i \sinh X)(\text{sech}^2 X \pm i \text{sech} X \tanh X)^{3/2}. \quad (40)
\]

Each of the eigenvalues \(E_n\) gives rise to a stability exponent

\[
\mu_n = 4i - \lambda E_n
\]

in equation \((37)\). When the dissipation coefficient \(\lambda > 0\), the exponent pertaining to the negative eigenvalue \(E_0\) has a positive real part. Accordingly, the twisted modes \([36]\) are unstable in the presence of damping. In contrast, when \(\lambda = 0\), all exponents \(\mu_n\) \((n = 0, 1, 2)\) are pure imaginary so the twisted modes are stable.

### C. Oscillatory modes

An interesting question is whether there are any other stable localised structures — in particular, in the situation where the equation \((30)\) has zero damping. Figure \([4]\) illustrates the evolution of a gaussian initial condition \(u(x, 0) = \exp(-x^2)\) that can be seen as a nonlinear perturbation of the soliton \((31)\). The gaussian evolves into an oscillatory localised structure (a kind of a breather) which remains close to the soliton \((31)\) — but does not approach it as \(t \to \infty\). This observation suggests that equation \((30)\) with \(\lambda = 0\) has a family of stable time-periodic spatially localised solutions, with the stationary soliton \((31)\) being just a particular member of the family.

It is fitting to note that the existence of breather families is common to nonlinear \(\mathcal{PT}\)-symmetric equations \([28]\). Breathers prevail among the products of decay of generic localised initial conditions \([28,29]\).

### D. Stable solitons in two dimensions

We close this section with a remark on the Landau-Lifshitz-Gilbert-Slonczewski equation in two dimensions:

\[
\frac{\partial \mathbf{M}}{\partial t} = -\mathbf{M} \times \nabla^2 \mathbf{M} - \mathbf{M} \times \mathbf{H} - \beta (\mathbf{M} \cdot \mathbf{\hat{z}}) \mathbf{M} \times \mathbf{\hat{z}}
\]

\[
-\gamma \mathbf{M} \times \mathbf{\hat{z}} + \lambda \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t}. \quad (41)
\]

Here \(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\). Assuming that \(H_0\) is only slightly above \(\gamma\) and that the anisotropy \(\beta\) and damping \(\lambda\) are small, we consider a perturbation of the east-point uni-
form state [22]. Following the asymptotic procedure outlined in section IV A, the equation (41) is reducible in this limit to a planar Schrödinger equation:

\[ iu_t = u_{xx} + u_{yy} + 6u^2 - 4u + b(u - u^*), \]  

where

\[ b = \frac{2H_0}{\gamma} \frac{\beta}{\sqrt{H_0^2 - \gamma^2}}. \]

Like its one-dimensional counterpart [23], equation (42) is \( P\bar{T}\)-symmetric. The \( P\bar{T}\)-operation can be chosen, for instance, in the form

\[ t \rightarrow -t, \quad x \rightarrow -x, \quad y \rightarrow -y, \quad u \rightarrow u^*. \]

The quadratic Schrödinger equation (42) has a static radially-symmetric soliton solution,

\[ u_s(x, y) = \mathcal{R}(r), \]  

where \( \mathcal{R}(r) \) is a nodeless (bell-shaped) solution of the boundary-value problem

\[ \mathcal{R}_{rr} + \frac{1}{r} \mathcal{R}_r - 4\mathcal{R} + 6\mathcal{R}^2 = 0, \]

\[ \mathcal{R}_r(0) = 0, \quad \mathcal{R}(r) \rightarrow 0 \text{ as } r \rightarrow \infty. \]

Postponing the detailed stability analysis of the soliton [24] to future publications, we restrict ourselves to the simplest case of isotropic ferromagnet, \( b = 0 \). A numerical simulation of equation (42) with the initial condition in the form of the noise-perturbed soliton (43) indicates that the soliton is stable against small perturbations. [See Fig 5(a).] On the other hand, generic localised initial conditions evolve into time-periodic breather-like states [Fig 5(b)]. This suggests that the quadratic Schrödinger equation (42) [and hence the planar Landau-Lifshitz equation (41)] supports a broad class of stable stationary and oscillatory localised structures.

VI. SOLITON EXCITATIONS OF LATITUDINAL STATE

The scaling transformation [25] takes the equations [26]

\[ (i + \lambda)u_t - u_{xx} - 6u^2 = \mp 4iu - b(u + u^*). \]  

As in section IV A, \( b = 2B/\gamma \) here.

In what follows, we confine ourselves to the analysis of the isotropic equations (\( b = 0 \)) as it is the only regime where we were able to obtain soliton solutions of (41). In the isotropic case, the \( u = 0 \) solution of the top-sign equation in (41) is stable and that of the bottom-sign equation unstable — regardless of whether \( \lambda \) is zero or not. (This agrees with the stability properties of the north and south fixed-point solutions of the Landau-Lifshitz equation; see section III B.) Hence we only keep the top-sign equation in what follows.

A. sech mode

Letting \( b = 0 \), the top-sign equation in (41) can be further transformed to

\[ (1 - i\lambda)w_t = w_{zz} - 4w + 6w^2, \]

where

\[ w(z, t) = -iu, \quad z = e^{i\pi/4}x. \]  

An obvious static solution of the equation [27] is \( w_s = \text{sech}^2z \); the corresponding solution of the original equation (41) is

\[ u_s(x) = i\text{sech}^2(\text{e}^{i\pi/4}x). \]  

The solution (47) decays to zero as \( x \rightarrow \pm \infty \) and does not have singularities on the real line. Similar to the solution [28] over the equatorial background, we term the solution (47) the sech soliton.

To classify the stability of the soliton (47), we linearise equation (45) about \( w_s = \text{sech}^2z \). Assuming that the small perturbation depends on time as \( e^{\mu t} \), we obtain

\[ \mu = -\frac{1 + i\lambda}{1 + \lambda^2}E, \]  

where \( E \) is an eigenvalue of the Pöschl-Teller operator

\[ \mathcal{H} = -\frac{d^2}{dz^2} + 4 - 12\text{sech}^2z. \]

The operator acts upon functions \( y(z) \) defined on the line \( z = e^{i\pi/4}\xi \ (\xi < \lambda < \infty) \) on the complex-\( z \)-plane and satisfying the boundary conditions \( y \rightarrow 0 \) as \( \xi \rightarrow \pm \infty \).

As discussed in the previous section, the equation \( \mathcal{H}y = Ey \) with \( E = -5 \) has a solution \( y_0 = \text{sech}^3z \). The function \( \text{sech}^3(\text{e}^{i\pi/4}\xi) \) is nonsingular for all \( -\infty < \xi < \infty \) and decays to zero as \( \xi \rightarrow \pm \infty \); hence \( E_0 = -5 \) is a discrete eigenvalue of the operator \( \mathcal{H} \). The corresponding exponent \( \mu \) in (48) has a positive real part regardless of \( \lambda \). This implies that the sech soliton (47) is unstable irrespective of whether \( \lambda \) is zero or not.

B. sech-tanh modes

Applying the transformation [29] to the solutions

\[ w_T = 2\text{sech}^2(2z) \pm 2i\text{sech}(2z) \tanh(2z) \]  

of the equation [45], we obtain a pair of localised solutions of the original equation (41):

\[ w_T = \mp 2\text{sech}(2\text{e}^{i\pi/4}x) \tanh(2\text{e}^{i\pi/4}x) + 2i\text{sech}^2(2\text{e}^{i\pi/4}x). \]  

By analogy with solutions [29] over the equatorial background, we are referring to (50) as the sech-tanh modes.

Linearising equation (45) about its stationary solutions [29] and assuming that the small perturbation depends
on time as $e^{\mu t}$, we obtain the following equation for the exponent $\mu$:

$$
\mu = -4 \frac{1 + i \lambda}{1 + \lambda^2} E.
$$

Here $E$ is an eigenvalue of the Scarff-II operator:

$$
L_y = E y, \quad (51a)
$$

$$
L = -\frac{d^2}{dz^2} + 1 - 6 \text{sech}^2 Z \mp 6i \text{sech}Z \tanh Z, \quad (51b)
$$

with $Z = 2z$. The eigenvalue problem (51) is posed on the line

$$
Z = e^{i\pi/4} \xi, \quad -\infty < \xi < \infty \quad (52)
$$
on the complex-$Z$ plane, with the boundary conditions $y \to 0$ as $\xi \to \pm \infty$.

Three solutions of the equation (51) are in (39)-10. Since $y_n(Z)$ ($n = 0, 1, 2$) are nonsingular and decay to zero as $Z$ tends to infinity in either direction along the line (52), these solutions are eigenfunctions of the operator $L$ — and the corresponding $E_n$ are eigenvalues. The exponent $\mu_0$ pertaining to the eigenvalue $E_0 = -5/4$ has a positive real part:

$$
\mu_0 = -4 \frac{1 + i \lambda}{1 + \lambda^2} E_0.
$$

Consequently, the \textit{sech-tanh} modes (50) are unstable — no matter whether $\lambda$ is zero or not.

\section*{C. Summary of one-dimensional solitons}

The stability properties of six localised modes supported by the quadratic Ginsburg-Landau equations (30) and (44) are summarised in Table II. The Table includes two \textit{sech} solitons (the fundamental soliton (51) and its latitudinal-background counterpart, equation (47)) and four \textit{sech-tanh} modes (the twisted modes (36) and their latitudinal analogs (50)).

| Nonlinear mode | over equatorial background | over latitudinal background (with $b = 0$) |
|----------------|-----------------------------|------------------------------------------|
| \textit{sech} soliton | stable if $\lambda = 0$ and $-\frac{3}{2} < b < \frac{3}{2}$ | unstable |
| \textit{sech-tanh} modes | exist if $b = 0$; stable if $\lambda = 0$ | unstable |

\begin{table}[h]
\centering
\begin{tabular}{|l|c|c|}
\hline
Nonlinear mode & over equatorial background & over latitudinal background (with $b = 0$) \\
\hline
\textit{sech} soliton & stable if $\lambda = 0$ and $-\frac{3}{2} < b < \frac{3}{2}$ & unstable \\
\textit{sech-tanh} modes & exist if $b = 0$; stable if $\lambda = 0$ & unstable \\
\hline
\end{tabular}
\caption{Stability of the stationary nonlinear modes in one dimension. The middle column classifies solutions of the equation (51) while the right-hand column corresponds to solutions of (44).}
\end{table}

\section*{VII. CONCLUDING REMARKS}

We have studied nonlinear structures associated with the spin torque oscillator — an open system described by the Landau-Lifshitz-Gilbert-Slonczewski equation. In the limit of zero damping ($\lambda = 0$), this nonconservative system is found to be $\mathcal{PT}$-symmetric. The nearly-$\mathcal{PT}$ symmetric equation corresponds to small nonzero $\lambda$. In this paper, we have considered both nearly-symmetric and nonsymmetric oscillators (small and moderate $\lambda$).

The spin torque oscillator has four stationary states of uniform magnetisation; they are described by four fixed points on the unit $\mathbf{M}$-sphere. Two of these states have their magnetisation vectors lying in the equatorial plane of the unit sphere while the other two correspond to fixed points in the northern and southern hemisphere, respectively. We have assumed that the external magnetic field $H_0$ has been tuned to values $\epsilon^2$-close to the bifurcation points of the “equatorial” and “latitudinal” uniform static states, and that the ferromagnet is only weakly anisotropic: $\beta = O(\epsilon)$. In that limit, small-amplitude localised perturbations of the uniform static states satisfy the Ginsburg-Landau equations — equations (30) and (44), respectively.

If the damping coefficient $\lambda$ is $O(\epsilon)$ or smaller, each of the two Ginsburg-Landau reductions becomes a quadratic nonlinear Schrödinger equation. Of the two Schrödinger equations, the one corresponding to perturbations of the “equatorial” uniform static state turns out to be $\mathcal{PT}$-symmetric. (Thus the asymptotic reduction of a nearly $\mathcal{PT}$-symmetric Landau-Lifshitz system is exactly $\mathcal{PT}$-symmetric.) This Schrödinger equation proves to be quite remarkable. Indeed, despite both our Ginsburg-Landau reductions supporting soliton solutions, it is only in the $\mathcal{PT}$-symmetric Schrödinger limit that the solitons are found to be stable.

The $\mathcal{PT}$-symmetric Schrödinger equation supports two types of stable solitons. The constant-phase solution (31) is stable in a band of $\beta = O(\epsilon)$ values, extending from the easy-axis to the easy-plane region. [The stability band is demarcated by the inequality (30).] On the other hand, a pair of stable solitons with the twisted phase, equations (30), are only supported by the nearly-isotropic ferromagnet: $\beta = O(\epsilon^2)$ or smaller. In addition to stable static solitons, the $\mathcal{PT}$-symmetric Schrödinger equation exhibits stable breathers.

In the two-dimensional geometry, the Landau-Lifshitz equation for the spin torque oscillator admits an asymptotic reduction to a planar quadratic Schrödinger equation, equation (42). Like its one-dimensional counterpart, the $\mathcal{PT}$-symmetric planar Schrödinger equation has stable static and oscillatory soliton solutions.

Finally, it is worth re-emphasising here that the $\mathcal{PT}$-symmetric Schrödinger equation is a reduction of the whole family of nearly-$\mathcal{PT}$ symmetric Landau-Lifshitz-Gilbert-Slonczewski equations with $\lambda = O(\epsilon)$ and not just of its special case with $\lambda = 0$. Therefore our conclusion on the existence of stable solitons is
applicable to the physically relevant class of spin torque oscillators with nonzero damping.

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