The slope of dry granular materials surface is generally curved

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Abstract
A closer inspection shows that the granular heap slopes are not truly straight. Instead, more often they have convex shapes. Here, we show that the granular heap slopes, in general, are convex (curved downwards). This prediction is in accordance with the results of measurements on 10 types of granular heaps. The model answers why the surfaces of granular heaps, including desert sand dunes, are more often convex than straight. We also predict the existence of a critical grain size, where the repose angle is 90° at the critical size and is undefined if the grain size is less than the critical size. The critical grain size estimated by the present model is consistent with the experimental results of previous researchers. This model also advocates the simultaneous estimation of the coefficient of friction on the grain and type of packing in the heap.

Keywords Curved slopes · Granular heap · Sand dune · Angle of repose · Critical grain size · Janssen effect

1 Introduction

More than half a century ago, Johnson derived an intriguing equation to determine the maximum shift of a stack of blocks under gravity [1]. If there are \((n + 1)\) identical blocks of length unity, the maximum shift of the edge of the top block \((k = 1)\) to the edge of the bottom block \((j = n + 1)\) is

\[
\left(\frac{1}{2}\right) \sum_{j=1}^{n} \frac{1}{j} = (1/2) H_n, \quad \text{where } H_n \text{ is the harmonic number} \ [2].
\]

It can be shown that \(H_n = \ln(n + 1/2) + \gamma_E + O(n^{-2})\), where \(\gamma_E = 0.577215664 \ldots\) is the Euler-Mascheroni constant [2], which means that the maximum shift can exceed the block size if the number is very large. We demonstrate that the difference in the horizontal direction between the ends of the \(k\)-th block (numbering from the bottom) and the line joining the ends of the bottom and top blocks, \(\Delta y(k)\), satisfies the equation (see Supplementary 1):

\[
\frac{\Delta y(k)}{n} = \frac{H_n}{2n} \left[ \left( 1 - \frac{1}{H_n} \sum_{i=1}^{k} \frac{1}{n - i + 1} \right) - \left( 1 - \frac{k}{n} \right) \right] \quad (1)
\]

This arrangement is produced by an isolated block chain. If the chain consists of particles on the surface of a heap, the deviation might be greater because the gravity is countered by the blocks below it.

We also derived the equation for the blocks supported from below in block heap architecture (see Supplementary 2). We reveal that in a critical condition when the block stack is just about to collapse, the surface of the heap also curves. The slope of the top heap is close to 0° with respect to the horizontal direction, and the slope around the bottom is close to constant, depending on the coefficient of friction between blocks.

These properties are challenging to apply to granular heaps. Particles on the surface of the heap can be considered as chains, so we argue that at critical conditions, the surface of the heap is convex. Indeed, variations in the heap’s angle of repose have been investigated as a function of experimental parameters, and deviations in the shape of the tail of sandpiles were discovered [3]. By numerical investigation of very large granular heaps, Topić et al. [4] showed the change in angle of repose against the radius from the center, a finding that can only be explained by considering that the
slope of the heap is curved. However, reports claiming the granular heap slope is not straight are rare. To date, Hermann has derived the equation of heap height as a function of distance from the heap center [5]. However, on very wide heaps, curved slopes are found only around the peak, and the largest portion of the surface slope is straight. Furthermore, the slope formed around the peak only has one characteristic, namely, being convex. Hermann has also shown that a weak tail formed by grains that rolled off the heap base satisfies a logarithmic function.

In this paper, we demonstrate that the slope of the granular heap can have three distinct profiles: convex, concave, and straight. The equations were derived for granular material resting on a vertical wall of infinite width. This type of heap shape is demonstrated by sand dunes [6–17]. Sand dunes are gigantic sand heaps whose angle of repose ranges between 30° and 35° [18, 19], similar to that of dry sands [20]. We assume that the obtained equation is also valid for the conical heap, the heap shape in which experiments are conducted. To assess the obtained equations, we have carried out measurements on ten conical granular heaps. The results might open new understandings related to granular material properties as well as sand transport and geological records in several deserts in Africa, Asia, and America.

2 Method

2.1 Modeling

Figure 1 is the illustration of the arrangement of one chain of particles on the surface of the heap having a height of \( H \). For convenience, we select the \( x \) (horizontal) axis in the direction of the heap’s height and the \( y \) (vertical) axis to be parallel to the baseline. We define the angle of repose, \( \alpha \), as the angle at the upper edge made by a straight line from the baseline to the top. The angle made by this line to the horizontal axis, \( \bar{\alpha} \), satisfies \( \alpha - \bar{\alpha} = \pi/2 \) (\( \alpha \) is taken to be positive). We also assume that the grains tend to leave the heap in the horizontal direction (perpendicular to gravity) so that the direction of the frictional force (as opposed to the tendency to move) is perpendicular to gravity (inward).

We focus only on the arrangement of the particles in the top surface layer by assuming that the arrangement on this layer determines the granular surface profile. Similar considerations have been used by other authors [21, 22]. Jaeger et al. [23] revealed that when the slope angle exceeds the angle of repose, only the top layers of sand flow freely downhill. According to the image during the avalanche, when the slope angle exceeds the angle of repose, only a few layers of surface particles flow [24]. We focus on a single layer with a thickness (in a direction perpendicular to \( x \) and \( y \)) equal to the particle diameter so that the system is similar to a curved string [25].

Consider one element on the granular surface at a position between \( x \) to \( x + dx \) that contains only one particle. Suppose \( F(x) \) is the force in the direction of the slope (similar to the tension in a rope). The horizontal forces equilibrium, \( F(x + dx) \cos \theta(x + dx) + dN - F(x) \cos \theta(x) - dW = 0 \), gives:

\[
\frac{dN}{dx} = -\frac{1}{F(x)}(F(x) \cos \theta(x))dx + dW. \tag{2}
\]

The vertical forces equilibrium, \( F(x) \sin \theta(x) - F(x + dx) \sin \theta(x + dx) - df = 0 \), gives:

\[
\frac{df}{dx} = -\frac{1}{F(x)}(F(x) \sin \theta(x))dx. \tag{3}
\]

The frictional force \( df \) is contributed by sliding and rolling motion. The sliding friction (resistance) force is proportional to the normal contact force. The maximum rolling torque resistance is also proportional to the normal contact force [26, 27]. Therefore, in general, we can write \( df = \mu dN \), where \( \mu \) is the global friction coefficient. It is expected that the small particles have a shape greatly deviating from the spherical, leading to higher resistance to rolling compared to very large particles so that the rolling is nearly absent.
for small particles and the friction is dominated by sliding. Referring to the Coulomb equation [28], we made this decision with the assumption that the cohesive forces were low enough to be neglected [29]. From Eqs. (2) and (3) we get:

$$\left\lvert \frac{d}{dx} [F(x) \sin \theta(x)] \right\rvert = -\mu \frac{d}{dx} [F(x) \cos \theta(x)] \cos \theta + \mu dW.$$  

(4)

If the granular material is placed in a container that has a finite cross section, the pressure inside the material is not exactly the same as the hydrostatic pressure of the fluid. The friction between the grain and the wall reduces the pressure inside the granular material so that it becomes smaller than the hydrostatic pressure of the fluid. The pressure within the granular material eventually saturates to a certain value as the depth increases [30], known as the Janssen effect [31]. The pressure at depth $x$ satisfies the equation

$$P = \rho g \lambda \left(1 - \exp \left(-x/\lambda\right)\right),$$

where $\rho$ is the density of the granular material, and $\lambda$ is the characteristic length that satisfies $\lambda = 0.5 WD/(W + D) \kappa \mu_w$. ($W$ is the width of the container, $D$ is the depth of the container, $\kappa$ is the Janssen coefficient, and $\mu_w$ is the Coulomb friction coefficient). A comprehensive study of the Janssen effect, especially the dynamic behavior, has been reported by Windows-Yule et al. [30]. However, if the granular material is not placed in a finite container (open space such as a granular heap), we can take $W \to \infty$ and $D \to \infty$, which implies $\lambda \to \infty$. As a result, the pressure in the granular material is reduced to $P = \rho g x$, which is equal to the hydrostatic pressure of the fluid [33, 34]. The assumption of hydrostatic-like forces in granular systems has been widely reported [35].

The magnitude of force $F(x)$ acting along the surface arc can be approximated by a hydrostatic-like force, and all other forces have been accommodated in the frictional force with an inward direction perpendicular to gravity. The hydrostatic pressure of the material satisfies equation $P(x) = \rho g (H - x)$, where $\rho$ is the density of the granular material after taking into account the presence of pores, and $g$ is the gravity acceleration. The contact surface of contacting grain in a chain is assumed to be perpendicular to the slope line, so that the hydrostatic force at position $x$, $F(x) = \rho g \sigma_c A (H - x)$, where $\sigma_c$ is the effective cross section, has the same direction as the slope line. We approximate $\sigma_c = \pi R^2$, where $R$ is the particle radius.

We don’t yet know whether $\theta$ is monotonically increasing, monotonically descending, or oscillating when $x$ changes from 0 to $H$. However, from all observations that the slope of the granular material is almost constant (almost straight), the variation of $\theta$ to the value of $\hat{\theta}$ is not too large, which causes $\sin \theta$ and $\cos \theta$ to be almost constant. Furthermore, $-\pi/2 < \theta < 0$ so that $\cos \theta > 0$ and $\sin \theta < 0$, which results in $F \cos \theta$ decreasing and $F \sin \theta$ increasing as $x$ gets bigger. Thus, the following two inequalities are satisfied: $d/dx (F \sin \theta) > 0$ and $d/dx (F \cos \theta) < 0$. Since $dW/dx > 0$, Eq. (4) can be written as $d/dx (F \sin \theta) = \mu dW/dx + \mu dW/F \cos \theta$. By looking at Fig. 1, we can write $dW = (\sigma_c D) \rho g = (\sigma_c \rho g) D = \gamma ds$, where $\gamma = \sigma_c \rho g$ so that we get the following equation:

$$\frac{d}{dx} [F(\mu \cos \theta - \sin \theta)] = -\nu \frac{ds}{dx}.$$  

(5)

We can write the granular surface equation as $e(x) = \gamma(x) + z(x)$, where $\gamma(x) = -(H - x) \tan \hat{\theta}$ is the equation of the straight surface (mean), and $z(x)$ is the deviation of the straight surface, $\lvert x(x) \rvert \ll \lvert \gamma(x) \rvert$ and $\lvert dz/dx \rvert \ll \lvert dy/dx \rvert$. Furthermore, $\Delta s = \sqrt{\Delta x^2 + (\Delta y + \Delta z)^2}$, so we can easily prove that $ds = dx \approx 1/\cos \hat{\theta} + \sin \hat{\theta} dz/dx$. Equation (5) can then be written:

$$\frac{d}{dx} \left[ F(\mu \cos \theta - \sin \theta) + \mu \gamma \sin \hat{\theta} z \right] = -\nu \frac{\gamma}{\cos \hat{\theta}}.$$  

(6)

The general solution of Eq. (6) is:

$$F(\mu \cos \theta - \sin \theta) + \mu \gamma \sin \hat{\theta} z = \mu g (H - x)/ \cos \hat{\theta} + C,$$  

(7)

where $C$ is a constant.

Next, let us write the angle as follows: $\theta(x) = \hat{\theta} + \varphi(x)$, where $\lvert \varphi \rvert \ll 1$ and $\lvert d\varphi/dx \rvert \ll 1$. We use the trigonometric equation $\sin \theta = \sin \left(\hat{\theta} + \varphi\right)$, where $\cos \theta = \cos \left(\hat{\theta} + \varphi\right)$, which results in $\cos \theta = \cos \hat{\theta} \cos \varphi - \sin \hat{\theta} \sin \varphi$, and the approximation of $\varphi \approx \tan \varphi = dz/dx$. Using these approximations, $\sin \theta = \sin \hat{\theta} + \cos \hat{\theta} dz/dx$, $\cos \theta \approx \cos \hat{\theta} - \sin \hat{\theta} dz/dx$, and Eq. (7) becomes:

$$- (H - x) \frac{dz}{dx} + \frac{\mu \gamma \sin \hat{\theta}}{2 \left(\mu \sin \hat{\theta} + \cos \hat{\theta}\right)} z$$

$$+ \frac{\mu \cos \hat{\theta} - \sin \hat{\theta}}{\mu \sin \hat{\theta} + \cos \hat{\theta}} - \frac{\mu \gamma}{\omega \cos \hat{\theta} \left(\mu \sin \hat{\theta} + \cos \hat{\theta}\right)} (H - x)$$

$$= \frac{C}{\rho g A} \left(\mu \sin \theta + \cos \theta\right) = 0,$$  

(8)

For convenience, we define $t = H - x$ so that $dz/dt = -dx/dt$, and Eq. (8) simplifies to $tdz/dt + az + bt - c = 0$, where:

$$a = \frac{\mu \gamma \sin \hat{\theta}}{\rho g A} = -\frac{\mu \gamma \cos \alpha}{\rho g A (-\mu \cos \alpha + \sin \alpha)}.$$  

(9)
The general solution for $z(t)$ is $z(t) = -bt/(1 + a) - k_1t + c/a$, where $k_1$ is a constant. For the solution to be convergent in the range $0 \leq t \leq H$, $a$ must be negative. We also apply the boundary conditions $z(0) = z(H) = 0$ so that $c = 0$, which implies $C' = bH/(1 + a)$ and $k_1 = bH/(1 + a)$. Finally, the compact form of the solution is given by:

$$
\frac{z(x)}{H} = b\left(1 - \frac{x}{H}\right)^{-a} - (1 - \frac{x}{H}).
$$

The convergence condition, $a < 0$, requires $-\mu \cos \alpha + \sin \alpha > 0$ (Eq. (9)), which implies:

$$
\mu < \tan \alpha.
$$

Since $a < 0$, from Eq. (12) we get $z(0) = z(H) = 0$. The largest deviation appears at $x/H = 1 - (-\alpha)^{1/(1+a)}$, that is:

$$
\frac{z_{\text{max}}}{H} = b(-a)^{-a/(1+a)}.
$$

### 2.2 Experiment

One widely employed method to generate a granular heap is the fixed funnel method, where the granular material is poured through a funnel or orifice onto a rough surface [36]. We do a similar method to create the heap. The granular materials were gently poured on a flat rough surface through a fixed funnel. The vertical distance between the funnel and the top of the heap is maintained during the whole process. We created a heap with a height of about 10 cm. The time required to make one heap was around 3 minutes. We then took the image of the heap for analysis.

### 3 Results and discussion

#### 3.1 Comparison of measurement and modeling results

The slope deviations from the straight lines of 10 granular materials observed in the present work (sesame, white sugar, coriander, rice, sand, lentils, mung bean, barley, dal bean, and basmati) are shown in Fig. 2a, b (see Supplementary 3

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**Fig. 2** (a, b) Comparison of calculating results (curves) and measurement results (symbols that have the same color) of the deviation of 10 granular materials. In (a): (triangles) coriander, (squares) sesame, (circles) white sugar, (stars) rice, and (diamonds) beach sand. In (b): (triangles, height has been multiplied by 1/2) lentil, (circles) barley, (squares) dal bean, and (stars) basmati. c The image of the rice heap with the calculated curve (sum of the corresponding curve in (a) and the heights belong to the straight surface), and d is the image of sesame heap and its corresponding fitting curve.
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for more details). The shapes of these curves are very similar to the shape of the curve of the edge shift in a stack of blocks (Fig. S2 in Supplementary 1). We examined the images of heaps having heights of around 10 cm. The particle sizes of all materials conform to a lognormal distribution (see Supplementary 4). The average particle size of each is: sesame (\(L = 0.2761\) cm, \(W = 0.1737\) cm), white sugar (0.0181 cm), coriander (\(D = 0.3096\) cm), rice (\(L = 0.516\) cm, \(W = 0.226\) cm), beach sand (\(D = 0.0012\) cm), lentil (\(L = 0.475\) cm, \(W = 0.275\) cm), mung bean (\(L = 0.596\) cm, \(W = 0.418\) cm), barley (\(L = 0.492\) cm, \(W = 0.336\) cm), dal bean (\(L = 0.616\) cm, \(W = 0.326\) cm), and basmati (\(L = 0.845\) cm, \(W = 0.186\) cm), where \(L\) stands for length, \(W\) for width, and \(D\) for diameter. Symbols are the measurement results, and the fitting curves have been obtained using Eq. (12) after adding the values for the straight slope, \(a\) and \(b\) have been estimated based on the position of the peak deviation and the peak value, so they are not free parameters. The measured and calculated data are comparable. The coefficients of determination (\(R^2\)) are not free parameters. The measured and calculated data are close to unity, indicating the fitting equation is adequate.

Figure 2c, d are comparisons of curves obtained from Eq. (12) adding the values for the straight slope, \(y(x)/H = \tan(\pi/2 - a_0)(1 - x/H)\), to \(z(x)/H\); (c) for the rice and (d) for the sesame. The curves obtained are very consistent with the observed data.

### 3.2 Estimation of friction coefficient

If \(\phi\) is the porosity of the material, one has \(\rho = (1 - \phi)\rho_g\), where \(\rho_g\) is the density of the granular material [37]. Referring to Fig. 1, we have \(dW = y ds = \gamma (2R)\). But \(dW = (4/3)\pi R^2\rho_g\) so that \(\gamma = (2/3)\pi R^2\rho_g\) and \(\gamma/\rho g A = 2/3(1 - \phi)\). Based on Eqs. (10) and (14), one gets:

\[
\mu = \frac{2}{3(1 - \phi)} - \frac{2}{3(1 - \phi)} \left( \frac{a}{b} \right) \cos \alpha \sin \alpha - \sin \alpha \sin \alpha
\]

Equation (15) allows the estimation of the friction coefficient from the measurement of the angle of repose together with the fitting parameters \(a\) and \(b\). The square and diamond symbols in Fig. 3a were calculated using Eq. (15) and the parameters in Table 1. The square symbols were calculated using random closed packing (RCP, \(\phi = 0.36\)) and diamond symbols using random loose packing (RLP, \(\phi = 0.40\)). It appears that the measured data are in agreement with the simulation results [38, 39].

From Table 1 we find the \(b/a\) ratio varies from \(-0.5\) to \(-1.0\) after removing the data for lentils, which differ considerably. We draw the curve \(a_0 - \mu\) at of \(b/a = -0.5\) and \(b/a = -1.0\) (Fig. 3a), where the symbols represent data on 10 granular materials, and curves are the results of calculations with Eq. (15) for RCP (dashed) and RLP (solid). The measurement results are mostly above the RCP curve, concluding that the surfaces of the 10 granular materials were convex to also indicate that packings inside all granular materials are RCPs. Therefore, the present model is able to estimate simultaneously the coefficient of static friction in the dry granular materials and the type of packing.

As shown in Fig. 3a, when we move from the dashed curve (\(\phi = 0.36\)) to the solid curve (\(\phi = 0.40\)) that has the same color, at a certain angle of repose, the coefficient of friction increases. An increase in \(\phi\) means a decrease in the packing fraction, so we might state that with a decrease in packing, the friction coefficient increases. This prediction is in accordance with previous reports, whether experimental [40], modeling [41], or simulation [42–44], which showed that random loose packing decreases with friction coefficient increases.

| Granular materials | Measured repose angles, \(a_0\) (°) | \(a\) (fitting results) | \(b\) (fitting results) | Positions of maximum deviation \((x/H)\) | Maximum deviation \((z_{\text{max}})/H\) | \(R^2\) |
|--------------------|-----------------------------------|-------------------------|-------------------------|------------------------------------------|----------------------------------------|--------|
| White sugar        | 31.7                              | -1.188                  | 0.594                   | 0.60                                      | 0.20                                    | 0.883  |
| Sesame             | 29.2                              | -0.623                  | 0.503                   | 0.715                                     | 0.23                                    | 0.797  |
| Sand               | 33.9                              | -0.498                  | 0.240                   | 0.75                                      | 0.120                                   | 0.917  |
| Coriander          | 20.7                              | -0.910                  | 0.701                   | 0.65                                      | 0.27                                    | 0.884  |
| Rice               | 28.0                              | -0.662                  | 0.482                   | 0.705                                     | 0.215                                   | 0.972  |
| Basmati            | 25.5                              | -0.5005                 | 0.540                   | 0.75                                      | 0.27                                    | 0.944  |
| Barley             | 21.5                              | -0.9544                 | 0.4648                  | 0.64                                      | 0.175                                   | 0.935  |
| Dal bean           | 26.0                              | -0.723                  | 0.466                   | 0.69                                      | 0.20                                    | 0.823  |
| Mung bean          | 23.0                              | -0.714                  | 0.417                   | 0.69                                      | 0.18                                    | 0.796  |
| Lentil             | 16.0                              | -0.957                  | 1.516                   | 0.64                                      | 0.57                                    | 0.959  |
Equation (12) converges at $0 \leq x \leq H$ if $a < 0$, and based on Eq. (9), it requires $-\mu \cos \alpha + \sin \alpha > 0$ or $\mu < \tan \alpha$. It is easily shown that the divergence does not occur at $\alpha = -1$ by writing $1 + a = \Delta a$, where $a \to -1$ and $|\Delta a| \to 0$, $z_{\text{max}}/H = b(1 - \Delta a)^{1/\Delta a} \approx b(1 - \Delta a)^{1/\Delta a} = b \exp(-\Delta a) \approx b$.

Equation (12) states that the maximum deviation is determined by parameters $a$ and $b$ while these two parameters depend on the angle of repose and the coefficient of friction, which fulfills the condition $\mu < \tan \alpha$. For $\phi = 0.36$ (RCP), we get $2/(3(1 - \phi)) = 1.04$, and for $\phi = 0.4$ (RLP), we get $2/(3(1 - \phi)) = 1.11$. If we approximate $2/(3(1 - \phi)) \approx 1$ for these two kinds of packing, we can write:

$$\frac{z_{\text{max}}}{H} \approx \frac{1}{\tan \alpha} \left( \frac{\mu}{\tan \alpha - \mu} \right)^{\frac{\mu}{\tan \alpha - \mu}}. \tag{16}$$

Figure 3c, d is a plot of $z_{\text{max}}/H$ at various values of $\mu$ and $\tan \alpha$ (c: surface plot and (d: contour plot) in the area that satisfies $\tan \alpha > \mu$. The minimum value of $z_{\text{max}}/H$ occurs

- $0 \leq x/H < 1/2$ when $-\infty < a < -2$. Figure 3b is an illustration of the deviation peak positions at different $\alpha$. The ratio of maximum deviation to heap height has been normalized to unity.
around \( \mu \approx \tan \alpha \), where a straight slope \((b = 0)\) is formed (lower boundaries on two figures).

If we look at Eq. (16), it is as if \( \frac{z_{\text{max}}}{H} \to \infty \) if \( \mu \to (1/2)\tan \alpha \). This is not the case since Eq. (16) has been derived after approximating \( 2/(3(1 - \phi)) \approx 1 \), which is invalid for \( \mu = (1/2)\tan \alpha \). For \( \mu = (1/2)\tan \alpha \), based on Eqs. (3) and (4), we get \( a = -\gamma /\rho g A = -2/(3(1 - \phi)) \) and \( b = (2 + \tan^2 \alpha) / \tan \alpha - 2/(3(1 - \phi) \sin \alpha) \). Based on these two expressions, we get \( \frac{z_{\text{max}}}{H} \) is finite. For example, if \( \phi = 0.36 \) and \( \alpha = 40^\circ \), we obtain \( a = -1.04167 \) and \( b = 1.8226 \) so \( \frac{z_{\text{max}}}{H} = 0.657 \).

It can be seen from Fig. 3c that \( \frac{z_{\text{max}}}{H} \) is very large if \( \mu \to 0 \). In this condition, \( a \to 0^- \), \((-\alpha)^{\alpha/(1+\alpha)} \to 1 \), and \( b \to \cot \alpha \) so that \( \frac{z_{\text{max}}}{H} \to \cot \alpha \). Based on Fig. 3a, if \( \mu \to 0 \), \( a \to 0 \), which causes \( \frac{z_{\text{max}}}{H} \to \infty \). This means the peak height is \( H \to 0 \). So, the reason why \( \frac{z_{\text{max}}}{H} \) is very large when \( \mu \to 0 \) is not because of large \( z_{\text{max}} \) but because \( H \to 0 \).

From Eq. (15), if \( b = 0 \) (straight slope), the friction coefficient reads:

\[
\mu_f = \frac{1}{1 + \frac{1}{\cos \alpha} \left( \frac{2}{3(1 - \phi)} - 1 \right)} \tan \alpha .
\] (17)

By remembering \( a < 0 \), Eq. (15) gives \( \mu < \mu_f \) if \( b > 0 \) and gives \( \mu > \mu_f \) if \( b < 0 \). Equation (17) is challenged for further analysis. Since \( \mu_f \) must be less than \( \tan \alpha \), the second term in the denominator of Eq. (17) cannot be negative or \( \phi \geq 1/3 \), which results in a critical value \( \phi_c = 1/3 \). Only when \( \phi > \phi_c, \mu_f < \tan \alpha \) is satisfied to produce concave surfaces in which \( \mu_f < \mu < \tan \alpha \). However, if \( \phi < \phi_c \), only a convex surface \((b > 0)\) can be generated. Because the porosities for both RCP and RLP packings are greater than \( \phi_c \), both types of packing produce convex surfaces. Concave surfaces are produced when the porosity is less than the porosity of the RCP packing, and this is very rare. Figure 4a is the regions of the formation of convex and concave surfaces at various porosities.

In general, \( \alpha \leq 45^\circ \) so that from Eq. (17), \( \mu \) increases with \( \alpha \) in this angle range. If \( \alpha \) is slightly larger (smaller) than the critical angle of repose, the right side of Eq. (17) becomes larger (smaller) than \( \mu_f \) so that a convex (concave) surface is formed. These results are illustrated in Fig. 4b, c. The straight profile occurs if the angle of repose is exactly equal the critical angle of repose. Thus, concave, convex, or straight profiles can occur in the same granular material.

The OA line in Fig. 4b has a slope angle equal to the critical angle of repose, \( \alpha_0 \). The OT line makes the true angle of repose \((\alpha > \alpha_0)\), which connects the point on the baseline to the heap top. The OT curve represents the convex surface of the granular material. At the baseline, the angle formed is slightly greater than \( \alpha \) and clearly greater than \( \alpha_0 \) (the angle

**Fig. 4** (a) Locations where the surface is convex (green) and concave (yellow). The shape of the slope if the angle of repose is slightly different from the critical angle of repose. (b) The angle of repose is slightly larger than the critical angle of repose. (c) The angle of repose is slightly smaller than the critical angle of repose. See the explanation in the text.
of the line OB to the horizontal). The slope angle at this position can be assumed to be approximately equal to \(\alpha_0 + \delta\) as reported by Jaeger et al. [23].

The OA line in Fig. 4c has a slope angle equal to the critical angle of repose, \(\alpha_0\). The OT line makes the true angle of repose (\(\alpha < \alpha_0\)) that connects the point on the baseline to the heap top. The OT curve is the concave surface. At the baseline, the angle formed is slightly less than \(\alpha\) and clearly smaller than \(\alpha_0\). A slope angle smaller than the critical angle of repose is permitted because it is stable.

Figure 4 in [45] shows a slightly convex profile around the base of the silver sand heap. By simulation using 50,000 particles with various aspect ratios, Zhao et al. [46] produced a slightly convex surface of the granular heap. The maximum height of grain piles as a function of the distance from the vertical line that passes through the peak shows quite scattered data [47], but they can be fit with a convex curve. Figure 1 in [48] shows the convex surface of the heap of snow (snow is often treated as a granular material [49, 50]). Images of pellet heaps reported in [51] also show a convex profile.

### 3.3 Size dependence of some parameters

It is clear from Table 1 that the value of \(a\) is different for different grains, and we have already calculated that different grains have different sizes. Therefore, it is interesting to check whether the value of parameter \(a\) depends on grain size as well.

Figure 5 is the dependence of \(-a\) on the effective radius of the grain. For non-spherical grains, we calculate the effective radius by the equation \(\pi W^2L = (4\pi/3)R^3_{av}\), where grain is approximated by a cylinder. It appears that, in general, the larger the effective radius of the grain, the larger the \(-a\). We will describe this behavior qualitatively as follows.

For small \(k\), \(1/(n - i + 1) \approx 1/n\), so we can approximate Eq. (1) with:

\[
\frac{\Delta y(k)}{n} \approx \frac{H_n}{2n} \left( 1 - \frac{1}{k} - \frac{1}{n} \right) \approx \frac{H_n}{2n} \left[ 1 - \left( 1 - \frac{k}{n} \right) + \left( 1 - \frac{k}{n} \right) \right].
\]

This equation is very similar to Eq. (12) by taking \(x = k/n\) and analogizing the power of two equations, namely, \(-a \approx 1/H_n\). However, Eq. (18) has been derived for one isolated chain and uses a very small \(k\).

For a supported chain at any position (a chain on the surface of a heap), we hypothesize the general form:

\[
-a = \frac{\beta}{H_n},
\]

where \(\beta\) is a parameter that is possibly a function of the friction coefficient and other surface properties of the grain. Smaller grain size increases \(n\), which causes \(H_n\) to get bigger. Thus, we can guess that \(-a\) decreases with decrease in the particle size, which is qualitatively consistent with Fig. 5. Since cohesive strength generally increases with decreasing particle size, we expect that \(-a\) will decrease as cohesive strength increases. Referring to Fig. 3b, the increase in the cohesive strength causes the peak of slope deviation from the straight line to shift towards the heap’s base.

There are a number of interesting results that have been reported by previous researchers, such as the dependence of the repose angle on grain size. The smaller the grain size, the greater the repose angle. For grain size that approaches zero, the repose angle approaches 90°. One of the proposed equations regarding the dependence of the repose angle on grain size is \(a = q/R + \alpha_{\infty}\), where \(q\) is a constant [52]. This equation has been confirmed by measurement data for magnesium dioxide materials at sizes of 50–400 μm. Another proposed equation that was derived from “numerical experiment” is \(a = \text{atan}(\mu_{\text{eff, } \infty}(1 + q/R))\) [36]. More materials have been tested with this equation, giving a very good fit.

Using the equations discussed above, we derive the equation for the dependence of the repose angle qualitatively on size. For \(n > 1\), the harmonic number can be approximated by \(H_n \approx \ln(n) + \gamma_E = \ln(n \exp(\gamma_E))\), where \(n \approx \omega H/2R_{av}\), \(R_{av}\) is the average radius, and the unknown factor \(\omega\) has been introduced. This relationship states that the number of grain arrangements is proportional to the ratio of the heap height to the average grain diameter. Thus, Eq. (19) produces \(-1/a \approx \beta \ln \left( \frac{0.5\omega H \exp(\gamma_E)}{R_{av}} \right) - \beta \ln \left( R_{av} \right)\), which can be written in the form;
measurement of the repose angle of a number of granular heaps [20]. From Eq. (21) and substituting \( q/p = \alpha_\infty \), we can write:

\[
\frac{1}{\alpha - \alpha_\infty} = p\beta \ln \left( \frac{R_{\text{av}}}{R_c} \right) - pA. \quad (22)
\]

If \( R_{\text{av}} \rightarrow R_c \), we get \( \alpha \rightarrow \pi/2 \) [36] so that \( pA = p\beta \ln \left( \frac{R_c}{R_c} \right) \approx \frac{\pi}{2} - \alpha_\infty \), and Eq. (22) becomes:

\[
\frac{1}{\alpha - \alpha_\infty} = p\beta \ln \left( \frac{R_{\text{av}}}{R_c} \right) - p\beta \ln \left( \frac{R_c}{R_c} \right) + \frac{\pi}{2} - \alpha_\infty. \quad (23)
\]

To estimate the unknown parameters in Eq. (23), we draw the curve \( 1/(\alpha - \alpha_\infty) \) as a function \( \ln \left( \frac{R_{\text{av}}}{R_c} \right) \). We use the data for magnesium dioxide [52]. Figure 6b is data from (i) column 3 and (ii) column 4 in Table 1 [52]. From curve (i) we get \( p\beta = 5.94 \), and using \( p \approx 1.98 \), we obtain \( \beta \approx 3 \). From the results of curve fitting (i), we also get \( -p\beta \ln \left( \frac{R_c}{R_c} \right) + 1/\left( \pi/2 - \alpha_\infty \right) \approx 59.80 \). Since \( 1/\left( \pi/2 - \alpha_\infty \right) \approx 1 \), we get \( -3.28 \ln \left( \frac{R_c}{R_c} \right) \approx 58.8 \) to give \( R_c \approx 50 \mu m \) or \( D_c \approx 100 \mu m \). This result is quite close to that reported by Elekesa and Partelib that heaps formed by particles with diameters smaller than \( D_c \approx 50 \mu m \) display a strongly irregular shape because cohesiveness to the gravitational force ratio exceeds a critical value [36].

From curve (ii) we get \( p\beta = 3.28 \), and by using \( p \approx 1.98 \), one obtains \( \beta \approx 1.66 \). From curve fitting (ii) we get \( -p\beta \ln \left( \frac{R_c}{R_c} \right) + 1/\left( \pi/2 - \alpha_\infty \right) \approx 34.54 \). Since \( 1/\left( \pi/2 - \alpha_\infty \right) \approx 1 \), one gets \( -3.28 \ln \left( \frac{R_c}{R_c} \right) \approx 33.45 \) to give \( R_c \approx 36 \mu m \) or \( D_c \approx 72 \mu m \). These results are in agreement with those reported by [36]. The two curves in Fig. 6b are data from the same material, but the repose angle has been estimated by different equations. Thus, the critical size estimates obtained from the two curves must be the same. Here we prove that the estimation results are very close, namely, 100 \( \mu m \) estimated by curve (i) and 72 \( \mu m \) yang estimated by curve (i).

Furthermore, Eq. (23) can be written in the form

\[
1/(\alpha - \alpha_\infty) = p\beta \ln \left( \frac{R_{\text{av}}}{R_c} \right) \quad \text{or:}
\]

\[
\alpha = \frac{1/p}{\ln \left( \frac{R_{\text{av}}}{R_c} \right)} + \alpha_\infty. \quad (24)
\]

where

\[
K = \frac{1}{R_c} \exp \left( \frac{1}{p\beta \left( \pi/2 - \alpha_\infty \right)} \right). \quad (25)
\]

Equation (24) is similar to the equation proposed by Carstensen [52]; the difference is only in the use of the logarithm of size instead of the size itself.
4 Conclusion

We have strongly demonstrated, by modeling and experimentation, that the slopes of granular heaps and sand dune surfaces are generally convex. Such convex profiles, indeed, have implicitly appeared in several previous reports as well as from photographs of sand dunes, but none has explicitly claimed them as the true state. We also identified the existence of a critical grain size below which the repose angle could not be defined, and the estimated critical size value was in accordance with previous reports. The results reported here are expected to open new understandings related to granular material properties as well as sand transport and geological records in several deserts in Africa, Asia, and America. Another fundamental result is that this model allows the estimation of the coefficient of friction between particles in the granular heap in the simplest way.

Supplementary Information The online version contains supplementary material available at https://doi.org/10.1007/s10035-022-01229-3.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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