Comment on “Quasinormal modes in Schwarzschild-de Sitter spacetime: A simple derivation of the level spacing of the frequencies”

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Abstract

It is shown here that the extraction of quasinormal modes (QNM) within the first Born approximation of the scattering amplitude is mathematically not well founded. Indeed, the constraints on the existence of the scattering amplitude integral lead to inequalities for the imaginary parts of the QNM frequencies. For instance, in the Schwarzschild case, $0 \leq \omega_I < \kappa$ (where $\kappa$ is the surface gravity at the horizon) invalidates the poles deduced from the first Born approximation method, namely, $\omega_n = i n\kappa$.

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In [1], based on a scattering amplitude written in the first Born approximation, a simple derivation for the imaginary parts of the quasinormal mode (QNM) frequencies for Schwarzschild and Schwarzschild-de Sitter black holes is given. Similar results can also be found in [2]. For the sake of conciseness, we begin directly from Eqs (13) and (22) in [1] which give the scattering amplitudes in the first Born approximation. In [1], the authors claim that starting with the scattering amplitude \( S(\omega) \) given in the first Born approximation, the quasi-normal modes can be found by computing the poles of \( S(\omega) \).

I. SCHWARZSCHILD CASE

In the Born approximation and for the Schwarzschild case, the scattering amplitude is given as

\[
S(\omega) = \int_{2M}^{\infty} dr \ U(r) e^{2i\omega r} = \int_{2M}^{\infty} dr \ \left[ \frac{\ell(\ell + 1)}{r^2} + \frac{2M}{r^3} \right] \left( \frac{r}{2M} - 1 \right)^{4iM\omega} e^{2i\omega r} \tag{1}
\]

with

\[
r_* = r + 2M \ln \left( \frac{r}{2M} - 1 \right), \quad r > 2M \tag{2}
\]

and

\[
\omega = \omega_R + i\omega_I, \quad \omega_R > 0, \tag{3}
\]

where \( \omega_R \) and \( \omega_I \) denote the real and imaginary parts of a QNM frequency, respectively. In particular, for \( s = 2, 3 \) we have to compute integrals of the form

\[
I_s = \int_{2M}^{\infty} dr \ r^{-s} \left( \frac{r}{2M} - 1 \right)^{4iM\omega} e^{2i\omega r} \propto \int_0^\infty dx \ h(x) = \mathcal{I}_s, \quad h(x) = \frac{x^{4iM\omega}}{(x + 1)^s} e^{4iM\omega x}, \tag{4}
\]

where we made the substitution \( x = (r/2M) - 1 \). Taking into account that

\[
|h(x)| \approx \begin{cases} x^{-4M\omega_I} & \text{as } x \to 0^+, \\ x^{-s-4M\omega_I} e^{-4M\omega_I x} & \text{as } x \to \infty, \end{cases}, \quad s = 2, 3 \tag{5}
\]

the integral \( \mathcal{I}_s \) will exist if the imaginary part of the quasi-normal mode frequency is restricted to the following range

\[
0 \leq \omega_I < \frac{1}{4M}. \tag{6}
\]
For \( \omega_I \in [0, 1/(4M)) \) the above integral can be computed exactly in terms of Whittaker’s functions and Gamma functions as follows \cite{3}

\[
I_s = C_1 \Gamma \left( 1 + i \frac{\omega}{\kappa} \right) \Gamma \left( s - 1 - i \frac{\omega}{\kappa} \right) M_{s + \frac{1}{2}, \frac{1}{2} - \frac{i \omega}{\kappa}, \frac{i \omega}{\kappa}} + C_2 \frac{\Gamma \left( 1 - s + i \frac{\omega}{\kappa} \right) e^{-\frac{\pi i}{2}}}{1 + s - i \frac{\omega}{\kappa}} \left[ \frac{i s}{-s + i \frac{\omega}{\kappa}} M_{s + \frac{1}{2}, \frac{1}{2} + \frac{i \omega}{\kappa}, \frac{i \omega}{\kappa}} + C_3 M_{1 + \frac{1}{2}, \frac{1}{2} + \frac{i \omega}{\kappa}, \frac{i \omega}{\kappa}} \right],
\]

where

\[
C_1 = -\frac{i \kappa}{\omega \Gamma(s)} \left( \frac{i \omega}{\kappa} \right)^{\frac{s}{2} - \frac{i \omega}{\kappa}} ,
C_2 = \frac{\kappa}{\omega} \left( -\frac{i \omega}{\kappa} \right)^{\frac{s}{2} - \frac{i \omega}{\kappa}},
C_3 = \frac{\kappa(s + 1)}{\omega}.
\]

Here, \( \kappa = (4M)^{-1} \) is the surface gravity at the event horizon and the Whittaker’s function \( M_{\mu, \nu}(z) \) is defined as \cite{4}

\[
M_{\mu, \nu}(z) = z^{\nu+1/2} e^{-z/2} \sum_{n=0}^{\infty} \frac{(\nu - \mu + 1/2)_n}{n!(2\nu + 1)_n} z^n, \quad (a)_n = \prod_{s=0}^{n-1} (a + s), \quad a \in \mathbb{C}.
\]

Note that the Whittaker function \( M_{\mu, \nu}(z) \) is analytic everywhere except at the points \cite{5}

\[
2\nu = -1, -2, -3, \ldots,
\]

where it has simple poles. Considering the Euler-Weierstrass representation of the Gamma function

\[
\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)^{-1} e^{z/n}
\]

(with \( \gamma \) the Euler-Mascheroni constant) together with \cite{5} the poles of the amplitude \( S(\omega) \) are represented by the poles of the Gamma functions and of the Whittaker’s functions entering in \cite{7}. These poles have the general form,

\[
\omega_n = -i \kappa, \quad \omega_n = i \kappa,
\]

where the presence of a factor \( n \) instead of \( n + 1/2 \) is due to the Born approximation which holds for \( n \gg 1 \), i.e. for high energies. Therefore, all these poles have to be disregarded by virtue of the integrability condition \cite{6}. Hence, the conclusions drawn in \cite{1, 2, 6} concerning the large imaginary parts of the QNM frequencies should be taken with some caution. Note also that the QNMs come not only from the Gamma functions but also from the Whittaker functions.

An alternative quick way to arrive at the same conclusion is to observe that the integral \( \mathcal{I}_s \) will pick up a contribution only near the event horizon. In this case the problem reduces
to the computation of the integral
\[ \int_0^\infty dx \, x^{i\omega/\kappa} e^{4iM\omega x}. \] (12)

The above integral belongs to the class of integrals FI II 779 in [3]
\[ \int_0^\infty dx \, x^{\nu-1} e^{-\mu x} = \frac{\Gamma(\nu)}{\mu^\nu} \] (13)
under the integrability condition
\[ \text{Re}(\mu) > 0, \quad \text{Re}(\nu) > 0. \] (14)

It is not difficult to see that for the integral (12) the condition (14) requires that
\[ 0 < \omega_I < \kappa. \] (15)

Since the integral (12) exists if and only if \( \omega_I \in (0, \kappa) \) the Gamma function \( \Gamma(1 + i\omega/\kappa) \) will not exhibit any pole for the aforementioned range and the poles \( \omega_n = in\kappa \) will lay outside the range of validity of the method employed for the search of the poles of the scattering amplitude. This means that in the Born approximation the imaginary part of the quasi-normal modes has to be restricted to the interval (9).

Before proceeding further we note that in physics one often introduces a regularization method when an integral does not exist. The Fourier transform of the Coulomb potential (equivalent to the Born approximation of Coulomb scattering) is a typical example where one introduces a regulator of the form \( e^{-\mu r} \) in order to remedy the problem generated by the \( 1/r \) potential. In the present case, the integral (Eq. (13)) exists over a certain range of the parameter \( \nu \) and hence in principle no regularization is required. Now if one decides nevertheless to perform a regularization for the sake of finding poles (though there is no physical justification for that), then one could write:
\[ F = \lim_{a \to 0^+} \int_0^\infty t^{\nu-1} e^{-a/t} e^{-t} dt = \lim_{a \to 0^+} 2a^{\nu/2} K_{-\nu}(2\sqrt{a}), \] (16)
where using
\[ K_{\nu}(z) \propto \frac{1}{2} \left( \frac{\Gamma(\nu)}{2} \left( 1 + \frac{z^2}{4(1-\nu)} + \ldots \right) + \Gamma(-\nu) \left( \frac{z}{2} \right)^\nu \left( 1 + \frac{z^2}{4(1+\nu)} + \ldots \right) \right) \]
with \( z \to 0 \) and \( \nu \notin \mathbb{Z} \) (positive and negative integers), \( F = \Gamma(\nu) \). However, \( \nu \) cannot be a positive or negative integer and therefore this invalidates again the extraction of the poles from the Gamma function above. Thus, even if one tries a regularization one cannot determine the poles.
II. SCHWARZSCHILD-DE SITTER CASE

According to [1] the tortoise coordinate and the scattering amplitude for the Schwarzschild-de Sitter case will be given by

$$r^* = \frac{1}{2\kappa_-} \ln \left| \frac{r}{r_-} - 1 \right| - \frac{1}{2\kappa_+} \ln \left| 1 - \frac{r}{r_+} \right| - \frac{1}{2} \left( \frac{1}{\kappa_-} - \frac{1}{\kappa_+} \right) \ln \left| \frac{r}{r_- + r_+} + 1 \right|, \quad (17)$$

where \(r_-\) and \(r_+\) are the event and cosmological horizons, respectively, and \(\kappa_+\) and \(\kappa_-\) the corresponding surface gravity terms. The scattering amplitude is now given as

$$S(\omega) = \int_{r_-}^{r_+} dr \ U(r) \left( \frac{r}{r_-} - 1 \right)^{i\omega/\kappa_-} \left( 1 - \frac{r}{r_+} \right)^{-i\omega/\kappa_+} \left( \frac{r}{r_- + r_+} + 1 \right)^{i\omega(1/\kappa_+ - 1/\kappa_-)} \quad (18)$$

with

$$U(r) = \frac{\ell(\ell + 1)}{r^2} + \frac{2M}{r^3} - \frac{2}{3r_\Lambda^2}, \quad r_\Lambda = \frac{1}{\sqrt{\Lambda}} \quad (19)$$

By means of the transformation \(u = (r - r_-)/(r - r_+)\) mapping the event horizon to 0 and the cosmological horizon to 1 we can reduce the computation of \(S(\omega)\) to the computation of the following integral

$$I_s = \int_0^1 du F_s(u), \quad s = 0, 2, 3, \quad (20)$$

where

$$F_s(u) = \begin{cases} 
\frac{u_{\kappa_-}}{(1-u)^{\kappa_-}} (1-u^{-1})^{1/\kappa_-} & \text{for } s = 2, 3, \\
\frac{(1-yu)^{1/\kappa_-} (1-zu)^{1/\kappa_+}}{u^{1/\kappa_-} (1-u)^{1/\kappa_+}} & \text{for } s = 0 
\end{cases} \quad (21)$$

with

$$y = \frac{r_- - r_+}{r_-}, \quad z = \frac{r_- - r_+}{r_+ + 2r_-}. \quad (22)$$

Taking into account that

$$|F_s(u)| \approx \begin{cases} 
\frac{u_{\kappa_-}}{(1-u)^{\kappa_-}} & \text{as } u \to 0^+ \implies \omega_I < \kappa_-, \quad s = 0, 2, 3 \\
(1-u)^{\kappa_+} & \text{as } u \to 1^- \implies \omega_I > -\kappa_+, \quad s = 0, 2, 3
\end{cases} \quad (23)$$

it follows that the above integral exists if and only if the imaginary part of the QNM frequency satisfies the condition

$$-\kappa_+ < \omega_I < \kappa_- \quad (24)$$

Therefore, the poles of the scattering amplitude as computed in [1]

$$\omega_n = i n \kappa_-, \quad \omega_n = -i n \kappa_+ \quad n \gg 1 \quad (25)$$
should be disregarded by virtue of the integrability condition \((24)\).

In conclusion, we have shown that the limitation imposed by the requirement of the existence of the scattering amplitude integral impedes the deduction of the QNM frequencies using the Born approximation.

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