Phase Space Analysis of Interacting Dark Energy in $f(T)$ Cosmology

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Abstract: In this paper, we examine the interacting dark energy model in $f(T)$ cosmology. We assume dark energy as a perfect fluid and choose a specific cosmologically viable form $f(T) = \beta \sqrt{T}$. We show that there is one attractor solution to the dynamical equation of $f(T)$ Friedmann equations. Further we investigate the stability in phase space for a general $f(T)$ model with two interacting fluids. By studying the local stability near the critical points, we show that the critical points lie on the sheet $u^* = (c - 1)v^*$ in the phase space, spanned by coordinates $(u, v, \Omega, T)$. From this critical sheet, we conclude that the coupling between the dark energy and matter $c \in (-2, 0)$.

Keywords: Perfect fluid; dark energy; torsion; cosmology; stability.

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I. INTRODUCTION

Astrophysical observations indicate that nearly seventy percent of the cosmic energy density is hidden in some unknown ‘dark’ sector commonly called as ‘dark energy’ (DE) [1, 2]. The remaining contribution to the total energy density is contained in dark matter and meager baryonic matter [3]. Dark energy is described phenomenologically by an equation of state (EoS) $p_d = w \rho_d$, where $p_d$ and $\rho_d$ are the pressure and energy density of dark energy, while $w$ is an EoS parameter which effectively describes cosmic acceleration. To describe DE, one must have $p_d < 0$ and $w < 0$.

General relativity (GR) offers only one solution to this puzzle, namely ‘cosmological constant’ which suffers from fine tuning and coincidence problems. Since GR fails to explain the cosmic accelerated expansion, one needs to modify curvature or the matter part in the Einstein...
field equations. Some notable examples are $f(R)$ gravity, scalar-tensor gravity and Lovelock gravity and more recently $f(T)$ gravity, to name a few. Other models where matter action is modified include the bulk viscous stress and the anisotropic stress or some exotic fluid like Chaplygin gas (CG). One of the crucial tests to check the viability of extended theories of gravity is the potential detection of gravitational waves.

We assume a phenomenological form of interaction between matter and dark energy since these are the dominant components of the cosmic composition, following. The exact nature of this interaction is beyond the scope of the paper and unresolved till we obtain a consistent theory of quantum gravity. In these interacting dark energy-dark matter models, the dark energy decays into matter at a rate proportional to Hubble length. The interacting dark energy scenario can successfully resolve the coincidence problem then stable attractor solutions of the Friedmann-Robertson-Walker (FRW) equations can be obtained. Some observational support to these models come from the astrophysical observations. For motivation from field theory and particle physics of the interacting dark energy, the interested reader is referred to.

The present paper is devoted to the study of dynamics of interacting dark energy in $f(T)$ cosmology. This theory is based on torsion scalar $T$ rather than curvature scalar $R$. We assume the dark energy in the form of a perfect fluid interacting with the matter. We study this model by the local stability method and than extend it for dark energy satisfying a more general form of equation of state.

II. BASIC EQUATIONS

A. Basics of $f(T)$ gravity

General relativity is a gauge theory of the gravitational field. It is based on the equivalence principle. However it is not necessary to work with Riemannian manifolds. There are some extended theories such as Riemann-Cartan, in them the geometrical structure of the theory has non-vanishing object of non-metricity. In these extensions, there are more than one dynamical quantity (metric). For example this theory may be constructed from the metric, non-metricity and torsion. Ignoring from the non-metricity of the theory, we can leave the Riemannian manifold and go to Weitzenbock spacetime, with torsion and zero local Riemann tensor. One sample of such theories is called teleparallel gravity in which...
we are working in a non-Riemannian manifold. The dynamics of the metric is determined using the scalar torsion $T$. The fundamental quantities in teleparallel theory are the vierbein (tetrad) basis vectors $e^i_\mu$. This basis is an orthogonal, coordinate free basis, defined by the following equation

$$g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij}.$$  

This tetrad basis must be orthonormal and $\eta_{ij}$ is the Minkowski metric. It means that $e^i_\mu e^\mu_j = \delta^i_j$.

There is a simple extension of the teleparallel gravity, which is called $f(T)$ gravity. In this theories, $f$ is an arbitrary function of the torsion $T$. One suitable form of action for $f(T)$ gravity in Weitzenbock spacetime is

$$S = \frac{1}{2\kappa^2} \int d^4x (T + f(T) + L_m).$$

Here $e = \text{det}(e^i_\mu)$, $\kappa^2 = 8\pi G$. The dynamical quantity of the model is the scalar torsion $T$ and $L_m$ is the matter Lagrangian. The field equation can be derived from the action by varying the action with respect to $e^i_\mu$

$$e^{-1} \partial_\mu(eS^i_\mu)(1 + f_T) - e_\mu^i S^\mu_\rho \epsilon^{\rho\nu} f_T$$

$$+ S^\mu_\rho \partial_\nu f_T - \frac{1}{4} e^j_i (1 + f(T)) = 4\pi G e^i_\mu T^\nu_\mu,$$

where $T^\nu_\mu$ is the energy-momentum tensor for matter sector of the Lagrangian $L_m$, defined by

$$T^\mu_\nu = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g} L_m d^4x.$$  

Here $T$ is defined by

$$T = S^\mu_\rho T^\rho_\mu.$$  

where

$$T^\rho_\mu = e^p_i (\partial_\mu e^i_\nu - \partial_\nu e^i_\mu),$$

$$S^\mu_\rho = \frac{1}{2} (K^\mu_\rho + \delta^\mu_\rho T^\theta_\theta - \delta^\mu_\rho T^\theta_\theta),$$

and the contorsion tensor reads $K^\mu_\rho$ as

$$K^\mu_\rho = -\frac{1}{2} (T^\mu_\rho - T^\rho_\mu - T^\mu_\rho).$$
It is straightforward to show that this equation of motion reduces to Einstein gravity when \( f(T) = 0 \). Indeed, it is the equivalency between the teleparallel theory and Einstein gravity \[^{20}\]. The theory has been found to address the issue of cosmic acceleration in the early and late evolution of universe \[^{21}\] but this crucially depends on the choice of suitable \( f(T) \), for instance exponential form containing \( T \) cannot lead to phantom crossing \[^{22}\]. Reconstruction of \( f(T) \) models has been reported in \[^{23}\] while thermodynamics of \( f(T) \) cosmology including the generalized second law of thermodynamics has been recently investigated \[^{24}\].

B. Interacting dark energy in \( f(T) \) cosmology

We adopt the metric in the form of a flat Friedmann-Lemaitre-Robertson-Walker metric with metric \( ds^2 = dt^2 - a(t)^2(dx^i dx_i), i = 1, 2, 3 \). We start with the Friedmann equation for the \( f(T) \) model \[^{7}\]

\[
H^2 = \frac{1}{1 + 2f_T} \left( \frac{\kappa^2}{3} \rho - f \right),
\]

(2)

where \( \rho = \rho_m + \rho_d \), and \( \rho_m, \rho_d \) represent the energy densities of matter and dark energy.

The second FRW equation is

\[
\dot{H} = -\frac{\kappa^2}{2} \left( \frac{\rho + p}{1 + f_T + 2T f_{TT}} \right).
\]

(3)

For a spatially flat universe \((k = 0)\), the total energy conservation equation is

\[
\dot{\rho} + 3H(\rho + p) = 0,
\]

(4)

where \( H \) is the Hubble parameter, \( \rho \) is the total energy density and \( p \) is the total pressure of the background fluid. The so-called energy-balance equations corresponding to dark energy and dark matter are \[^{25}\]

\[
\dot{\rho}_d + 3H(\rho_d + p_d) = -Q,
\]

\[
\dot{\rho}_m + 3H \rho_m = Q,
\]

(5)

Here \( Q \) is the interaction term which corresponds to energy exchange between dark energy and dark matter. The function \( Q \) has dependencies on the energy densities of the dark matter and dark energy and the Hubble parameter, i.e. \( Q(H \rho_m), Q(H \rho_d) \) or \( Q(H \rho_m, H \rho_d) \). Since the nature of both dark energy and dark matter is unknown, it is not possible to derive \( \Gamma \) from first principles.
To give a reasonable $Q$, we may expand like $Q(H\rho_m, H\rho_d) \simeq \alpha_m H\rho_m + \alpha_d H\rho_d$. Since the coupling strength is also not known, we may adopt just one parameter for our convenience; hence we take $\alpha_m = \alpha_d = c$. We here choose the following coupling function $Q = 3Hc(\rho_d + \rho_m)$.

To perform the stability analysis of the cosmological model, it is always convenient to define dimensionless density parameters via

$$u \equiv \frac{\kappa^2 \rho_d}{3H^2}, \quad v \equiv \frac{\kappa^2 \rho_d}{3H^2}, \quad \Omega_m \equiv \frac{\kappa^2 \rho_m}{3H^2}. \quad (6)$$

Moreover the equation of state parameter of dark energy

$$w \equiv \frac{p_d}{\rho_d} = \frac{v}{u}. \quad (7)$$

Also the equation of state of all the fluids together is

$$w_{\text{tot}} \equiv \frac{p_d}{\rho_d + \rho_m} = \frac{v}{1 - \Omega_T}. \quad (8)$$

Using (6), we can rewrite (2) in dimensionless form

$$\Omega_m = 1 - u - \Omega_T, \quad (9)$$

where $\Omega_T \equiv f(T) - 2f_T$, is another dimensionless density parameter constructed for torsion scalar.

In $f(T)$ and teleparallel gravities, phase space analysis for different dark energy models has been reported in [28, 29].

### III. STABILITY ANALYSIS OF PERFECT FLUID

In this section, we treat dark energy as a perfect fluid. Physically it means that the fluid has no viscosity or heat transfer property. Its a simple fluid which cannot be self-gravitating and has a smooth distribution in space. A perfect fluid is represented by a phenomenological linear equation of state connecting energy density and pressure by

$$p_d = w\rho_d, \quad (10)$$

where $w$ is a ‘constant’ of proportionality but can be a function of time or e-folding parameter $x = \ln a$. For a general dark energy paradigm, the minimum condition to be satisfied is $w \leq -1/3$.

This opens a window to construct variety of theoretical models to explain cosmic acceleration.

Some of these well-known models are cosmological constant ($w = -1$), quintessence ($w < -1/3$) and phantom energy ($w < -1$).
The dynamical system representing the dynamics of perfect fluid’s density and pressure reads as

\[
\frac{du}{dx} = -3[u + v + c(1 - \Omega_T)] \\
\quad + 3u\left[\frac{1 - \Omega_T + v}{1 + 2T f_T + f_{TT}}\right], \\
\frac{dv}{dx} = -3w[u + v + c(1 - \Omega_T)] \\
\quad + 3v\left[\frac{1 - \Omega_T + v}{1 + 2T f_T + f_{TT}}\right].
\]

We notice that our dynamical system is under-determined: three unknown functions \((u, v, \Omega_T)\) and two coupled equations. Thus we must choose \(f(T)\) to solve the system. It is discussed in [30] that a power-law form of correction term \(f(T) \sim T^n\) \((n > 1)\) such as \(T^2\) can remove the finite-time future singularity. However, when \(n = 0\), the correction term behaves like a cosmological constant. The model with \(n = 1/2\) can be helpful in realizing power-law inflation, and also describes little-rip and pseudo-rip cosmology [30]. Due to these reasons, we choose

\[f(T) = \beta \sqrt{-T},\]

where \(\beta\) is a constant. Note that choosing \(\beta = 0\) (or \(f(T) = 0\)) leads to teleparallel gravity which is equivalent to General Relativity. Note that this choice \(f(T) = \beta \sqrt{-T}\), has correspondence with the cosmological constant EoS in \(f(T)\) gravity [31]. This \(f(T)\) model can be recovered via reconstruction scheme of holographic dark energy [35]. Also it can be inspired from a model for dark energy model form the Veneziano ghost [32]. Recently Capozziello et al [33] investigated the cosmography of \(f(T)\) cosmology by using data of BAO, Supernovae Ia and WMAP. Following their interesting results, we notice that if we choose \(\beta = \sqrt{3/2H_0(\Omega_{m0} - 1)}\), than one can estimate the parameters of this \(f(T)\) model as a function of Hubble parameter \(H_0\) and the cosmographic parameters and the value of matter density parameter.

Recently attractor solutions for the dynamical system with three fluids (dark matter, dark energy and radiation) interacting non-gravitationally have been investigated to resolve the coincidence problem using similar \(f(T)\) [29].

To perform the stability of the system comprising equations (11) and (12), we first calculate the critical points \((u_*, v_*)\) by equating

\[
\frac{du}{dx} = 0, \quad \frac{dv}{dx} = 0.
\]
We find two critical points A and B for system (11) and (12) given in Table-I. We check the stability of the dynamical system in the neighborhood of these critical points $u = u_* + \delta u$, $v = v_* + \delta v$. This is performed by linearizing the system of equations of motion for $u$ and $v$ like

$$\frac{d\delta u}{dx} = \frac{3}{2} v_* \delta u + 3 \left(-1 + \frac{1}{2} u_*\right) \delta v, \quad (13)$$

$$\frac{d\delta v}{dx} = -3w\delta u + 3 \left(1 - w + v_*\right) \delta v, \quad (14)$$

Next we construct a Jacobian matrix from the coefficients of $\delta u$ and $\delta v$ in the linearized (or perturbed) system and finding the eigenvalues from it. For the system (11) and (12), the Jacobian matrix for any critical point is

$$J = \begin{bmatrix} \frac{3}{2} v_* & -3 + \frac{3}{2} u_* \\ -3w & -3w + 3 + 3 v_* \end{bmatrix}. \quad (15)$$

We find the eigenvalues $\lambda_{1,2}$ of the Jacobian matrix for the two critical points given in Table-I. A critical point (also called equilibrium point) is said to be stable if the corresponding eigenvalues are negative $\lambda_{1,2} < 0$ for all the values of the model parameters. Such stable critical points are called attractor solutions of the dynamical system i.e. solution of the dynamical system for different initial conditions which converge to a stable critical point. A critical point is said to be unstable if both the eigenvalues are positive, and a saddle point if one of the eigenvalues is positive. We are interested only in stable critical points. Sometimes we get conditionally stable critical points i.e. the critical point which can be stable only under some conditions on the model parameters. In cosmology conditionally stable points are also of interest. In Table-I we write down the values of relevant cosmological parameters including the total equation of state and the deceleration parameter. Note that for $f(T) = \beta \sqrt{T}$, we have $\Omega_T = 0$, therefore $w_{\text{tot}} = v_*$. For A, $q < 0$ when $c < 1/3$.

We observe that A is an unstable critical point and B is conditionally stable. This means that if the interaction parameter $c < \frac{w+1}{w}$, this point is a stable critical point. Since our two
dimensional phase space embedded in a three dimensional space, it has no chaotic behavior, and we can investigate the dynamical behavior of the the system by using the usual dynamical systems approach. In figure 1, we plotted the dynamical phase space of the (11), (12) for different values of the parameters and for a range of the initial conditions. We adopted the initial value (here initial value means the present value) for energy density of dark energy \( u(0) = 0.7 \), but different values of pressure parameter \( v(0) \) for different forms of dark energy in order to have a good agreement with the observational data. From figure-1 it turns out the trajectory of phantom energy in phase space is more steep than quintessence and cosmological constant. From figure-2, we plot the deceleration parameter for different dark energy models in \( f(T) \) cosmology. For cosmological constant, the trajectory starts near \( x = -1 \) (or \( a = 0.36 \)) and terminates close to \( x = 4 \) (or \( a = 54.59 \)), meanwhile the deceleration parameter remains fixed between \( x = -1 \) and \( x = 4 \) since Hubble parameter is constant. For quintessence, the deceleration parameter begins increasing from \( x = -0.5 \) \( (a = 0.6) \) to \( x = 1.5 \) \( (a = 4.48) \) where \( q \) vanishes. Thus deceleration \( q > 0 \) is not possible with quintessence dark energy. For phantom energy, \( q \) starts decreasing from \( x = -1.3 \) but gets stable to value \( q = -0.7 \).

### IV. THE PHASE SPACE OF THE INTERACTING DARK ENERGY MODELS: GENERAL CASE

For a generic form of the \( f(T) \), the system of equations (11), (12) is not closed i.e. a third differential equation for \( \Omega_T \) is needed. The EoS of the fluid is in the form \( P_d = \Phi(\rho_d) \). The general

| Point | \((u_*, v_*)\) | \(\lambda_1\) | \(\lambda_2\) | Stability Condition |
|-------|----------------|---------------|---------------|-------------------|
| A     | \((2(1-c), -2)\) | \(3(-1 - \frac{w}{2} + \frac{1}{2} \sqrt{w^2 + 4wc})\) | \(3(-1 - \frac{w}{2} - \frac{1}{2} \sqrt{w^2 + 4wc})\) | Unstable point |
| B     | \((\frac{2\delta}{w}, 2\delta)\) | \(3(1 + \delta)\) | \(3(\delta - w)\) | Stable point if \( c < \frac{w+1}{w} \) |

TABLE II: Critical points and stability of the perfect fluid model ( \( \delta = \frac{1}{2}(w - \sqrt{w^2 + 4wc}), w^2 + 4wc \geq 0 \) ).
FIG. 1: Phase space for perfect fluid form of dark energy. Three forms of dark energy are shown in figure: phantom energy \( v(0) = -0.84 \) \( (w = -1.2, \text{ green}) \); cosmological constant \( v(0) = -0.7 \) \( (w = -1, \text{ blue}) \); quintessence \( v(0) = -0.231 \) \( (w = -0.33, \text{ red}) \).

FIG. 2: Behavior of deceleration parameter for perfect fluid form of dark energy. Three forms of dark energy are shown in figure: phantom energy \( v(0) = -0.84 \) \( (w = -1.2, \text{ green}) \); cosmological constant \( v(0) = -0.7 \) \( (w = -1, \text{ blue}) \); quintessence \( v(0) = -0.231 \) \( (w = -0.33, \text{ red}) \).

dynamical system then reads

\[
\frac{du}{dx} = -3[c(1 - \Omega_T) + u + v] \\
+ \frac{3u(1 - \Omega_T + v)}{1 + f_T + 2T f_{TT}}, \tag{16}
\]

\[
\frac{dv}{dx} = -3\Phi'[c(1 - \Omega_T) + u + v] \\
+ \frac{3v(1 - \Omega_T + v)}{1 + f_T + 2T f_{TT}}, \tag{17}
\]

\[
\frac{d\Omega_T}{dx} = -\frac{3T(1 - \Omega_T + v)(T f_T - f - 2T^2 f_{TT})}{1 + f_T + 2T f_{TT}}. \tag{18}
\]
Here $\Phi' = \frac{d\Phi}{d\rho_d}$. The system (16-18) is non-autonomous due to presence of $f$ and we need to add another equation to it. From the definition of the $T = -6H^2$ it is easy to show that the fourth equation is

$$\frac{dT}{dx} = -\frac{3T(1 - \Omega_T + v)}{1 + f_T + 2Tf_{TT}}. \quad (19)$$

Now the system (16-19) is closed. We discussed the different possible attractors of the system (16-19) in Table-III.

| Critical sheet | $(u^*, v^*, \Omega^*_T, T^*)$ | Stability Condition |
|----------------|--------------------------------|---------------------|
| C              | $(u^*, v^*, \Omega^*_T(0), 0)$ | Physically unacceptable |
| D              | $((c-1)v^*, v^*, T^*, \Omega^*_T)$ | Conditionally stable |

Case C is not physically viable since when $T^* = 0$ it implies $H = 0$. It means that the local geometry in the neighbourhood of this point is Minkowski flat. We are not interested to such cases since astrophysically $H \neq 0$. But for case D, we conclude that the critical points in the phase space lie on the two dimensional surface

$$u^* = (c - 1)v^*. \quad (20)$$

Since EoS is $w = \frac{p_d}{\rho_d}$, this equation tells us that the critical point can be written in the form $p_d^* = \frac{\rho_d^*}{c-1}$. It shows that $w^* = \frac{1}{c-1}$. Since $-1 < w^* < -\frac{1}{3}$, it shows that the interacting coupling must be in the range $c \in (-2, 0)$, which is also reported in an earlier work [36]. This is a new constraint on the coupling constant $c$ from the phase analysis approach. But from this analysis we can not obtain any new information about the values of the $T^*, \Omega^*_T$. We remember here that, this discussion is free from any specific form of the $f(T)$. We conclude that for interacting dark energy models in $f(T)$ gravity, there is only one physically acceptable critical point D.

V. CONCLUSION

In this paper, we investigated the stability and phase space description of a perfect fluid form of the dark energy interacting with matter. We wrote the general dynamical system equations for two fluids and found the critical points. We linearized the system of equations and showed that for perfect fluid case there is only one attractor solution. For any value of the EoS parameter $w$, we obtained a new bound on the coupling constant $c < \frac{w+1}{w}$.
It is a new constraint on $c$ in the context of the $f(T)$ gravity which shows that energy is being transferred from matter to dark energy. Thus we obtain a scenario where the universe becomes increasingly dark energy dominated. Further for a general $f(T)$ model, there is only one critical point, and this critical point lives on the surface with equation $u^* = (c - 1)v^*$ in the four dimensional phase space which is spanned by four coordinates $X = (u^*, v^*, \Omega^*_T, T^*)$.

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[1] A. G. Riess et al, (Supernova Search Team Collaboration), Astron. J. 116 (1998) 1009; S. Perlmutter et al, (Supernova Cosmology Project Collaboration), Astrophys. J. 517 (1999) 565; C. L. Bennett et al, Astrophys. J. Suppl. Ser. 148 (2003) 175; M. Tegmark et al, (Sloan Digital Sky Survey Collaboration) Phys. Rev. D 69 (2004) 103501.

[2] R. R. Caldwell and M. Kamionkowski, arXiv:0903.0866v1 [astro-ph.CO]; T. Padmanabhan, Phys. Rep. 380 (2003) 325; P. J. E. Peebles, B. Ratra, Rev. Mod. Phys. 75 (2003) 559; V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D 15 (2006) 2105; M. Sami, Lect. Notes Phys. 72 (2007) 219; ibid, arXiv:0904.3445 [hep-th]; E. J. Copeland et al, Int. J. Mod. Phys. D 15 (2006) 1753; T. Buchert, Gen. Rel. Grav. 40 (2008) 467.

[3] N. Bachall et al, Science 284, 1481 (1999).

[4] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, S. Zerbini, Phys. Rev. D 73 (2006) 084007. S. Capozziello, M. De Laurentis, Phys. Rept. 509, 167 (2011); M.R. Setare, M. Jamil, Gen. Relativ. Gravit. 43, 293 (2011); A. Sheykhi, K. Karami, M. Jamil, E. Kazemi, M. Haddad, Gen. Relativ. Gravit. 44, 623 (2012); M. Jamil, D. Momeni, Chin. Phys. Lett.28,099801 (2011); M. Jamil, I. Hussain, D. Momeni ,Eur. Phys. J. Plus, 126,80(2011); M. Jamil, S. Ali, D. Momeni, R. Myrzakulov,Eur. Phys. J. C , 72,1998 (2012) ; M. Jamil, D. Momeni, N. S. Serikbayev, R. Myrzakulov , Astrophys. Space Sci 339,37(2012) ; M. Jamil, D. Momeni, M. Raza, R. Myrzakulov, Eur. Phys. J. C 72,1999 (2012) ; M. Jamil, E.N. Saridakis, M.R. Setare, JCAP 1011, 032 (2010); K. Karami, M. Jamil, N. Sehraei, Phys. Scr. 82,045901 (2010) ; M R Setare, D Momeni, R Myrzakulov, Phys. Scr. 85 ,065007(2012); M. Jamil , M. A. Rashid, D. Momeni, O. Razina, K. Esmakhanova, J. Phys. Conf. Ser. 354 012008(2012); D Momeni, Y. Myrzakulov, P. Tsyba, K. Yesmakhanova, R. Myrzakulov, J. Phys. Conf. Ser.354 012011(2012)
[5] L. Jarv, P. Kuusk, M. Saal, Phys. Rev. D 78, 083530 (2008); T. Tamaki, Phys. Rev. D 77 (2008) 124020; H. Motavali et al, Phys. Lett. B 666 (2008) 10.

[6] C. Garraffo, G. Giribet, E. Gravanis, S. Willison, J. Math. Phys. 49 (2008) 042502; S.H. Mazharimousavi and M. Halilsoy, Phys. Lett. B 665 (2008) 125.

[7] R. Ferraro, F. Fiorini, Phys. Rev. D 75, 084031 (2007); R. Ferraro, F. Fiorini, Phys. Rev. D 78, 124019 (2008).

[8] I. Brevik and O. Gorbunova, Gen. Rel. Grav. 37 (2005) 2039; G. M. Kremer and F. P. Devecchi, Phys. Rev. D 67 (2003) 047301; I. Brevik, Grav. Cosmol. 14 (2008) 332.

[9] K. A. Malik and D. Wands, Phys. Rept. 475, 1 (2009)

[10] A. Yu. Kamenshchik, U. Moschella, V. Pasquier, Phys. Lett. B 511 (2001) 265.

[11] C. Corda, Int. J. Mod. Phys. D 18, 2275 (2009); C. Corda, Phys. Rev. D 83, 062002 (2011); ibid, Astropart. Phys. 34, 412 (2011)

[12] H. M. Sadjadi, M. Alimohammadi, Phys. Rev. D 74 (2006) 103007; T. Clifton, J.D. Barrow, Phys. Rev. D 73 (2006) 104022; G. M. Phys. Rev. D 68 (2003) 123507; M. R. Setare, Phys. Lett. B 648 (2007) 329; ibid, Int. J. Mod. Phys. D 18 (2009) 419.

[13] W. Zimdahl and D. Pavon, Gen. Rel. Grav. 36 (2004) 1483; M. Jamil, M.A. Rashid, Eur. Phys. J. C 60 (2009) 141; P. Wu, H. Yu, Class. Quant. Grav. 24, 4661 (2007); M. Jamil, Int. J. Theor. Phys. 49, 62 (2010).

[14] B. Wang et al, Nucl. Phys. B 778 (2007) 69.

[15] O. Bertolami et al, Phys. Lett. B 654 (2007) 165.

[16] A. de la Macorra, Phys. Rev. D 76 (2007) 027301.

[17] S.M. Carroll et al, Phys. Rev. D 68 (2003) 023509.

[18] L. Smalley, Phys. Lett. A 61 (1977) 436.

[19] C-Q. Geng, C.-C Lee, E. N. Saridakis, Y-P Wu, Phys. Lett. B 704 (2011) 384; C-Q. Geng, C.-C. Lee, E. N. Saridakis, JCAP 01 (2012) 002.

[20] K. Hayashi, T. Shirafuji, Phys. Rev. D 19, 3524 (1979); K. Hayashi, T. Shirafuji, Phys. Rev. D 24, 3312 (1981).

[21] D. Liu, P. Wu, H. Yu, arXiv:1203.2016 [gr-qc].

[22] P. Wu, H. Yu, Phys. Lett. B 693, 415 (2010).

[23] M. R. Setare, M. J. S. Houndjo, arXiv:1203.1315v1 [gr-qc]; M. H. Daouda, M. E. Rodrigues, M. J. S. Houndjo, Eur. Phys. J. C 72, 1893 (2012).

[24] K. Bamba, M. Jamil, D. Momeni, R. Myrzakulov, arXiv:1203.6113v1 [physics.gen-ph].

[25] M. U. Farooq, M. Jamil, U. Debnath, Astrophys. Space Sci. 334, 243 (2011); M. U. Farooq, M. A. Rashid, M. Jamil, Int. J. Theor. Phys. 49 (2010) 2278; ibid, Int. J. Theor. Phys. 49 (2010) 2334; M. Jamil, A. Sheykhi, M. U. Farooq, Int. J. Mod. Phys. D 19 (2010) 1831; M. Jamil, M.U. Farooq, Int. J. Theor. Phys. 49 (2010) 42.

[26] C. Feng, B. Wang, E. Abdalla, R-K Su, Phys. Lett. B 665, 111 (2008); ibid, Eur. Phys. J. C 58, 111.
[27] S. del Campo, R. Herrera, D. Pavon, JCAP 0901, 020 (2009).
[28] C. Xu, E. N. Saridakis, G. Leon, arXiv:1202.3781v1 [gr-qc].
[29] M. Jamil, D. Momeni, R. Myrzakulov, Eur. Phys. J. C 72, 1959 (2012).
[30] K. Bamba, R. Myrzakulov, S. Nojiri, S. D. Odintsov, Phys. Rev. D 85, 104036 (2012).
[31] R. Myrzakulov, Eur. Phys. J. C 71, 1752 (2011).
[32] K. Karami, A. Abdolmaleki, arXiv:1202.2278
[33] S. Capozziello, V.F. Cardone, H. Farajollahi, A. Ravanpak, Phys. Rev. D 84, 043527 (2011).
[34] A. Naruko, M. Sasaki, Prog. Theor. Phys. 121, 193 (2009); M. Bojowald, G. Calcagni, S. Tsujikawa, JCAP 11, 046 (2011); M. Susperregi, A. Mazumdar, Phys. Rev. D 58, 083512 (1998); A. R Liddle, A. Mazumdar, F. E Schunck, Phys. Rev. D 58, 061301 (1998).
[35] M. H. Daouda, M. E. Rodrigues, M. J. S. Houndjo, Eur. Phys. J. C 72, 1893 (2012).
[36] M. Jamil, M.A. Rashid, Eur. Phys. J. C 60, 141 (2009).