On sequential separability of functional spaces

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Abstract

In this paper, we give necessary and sufficient conditions for the space $B_1(X)$ of first Baire class functions on a Tychonoff space $X$, with pointwise topology, to be (strongly) sequentially separable. Also we claim that there are spaces $X$ such that $B_1(X)$ is not sequentially separable space, but $C_p(X)$ is sequentially separable (the Sorgenfrey line, the Niemytzki plane).

Keywords: Baire function, sequential separability, function spaces, strongly sequentially separable

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1. Introduction

In [2], [3] were given necessary and sufficient conditions for the space $C_p(X)$ of continuous real-valued functions on a space $X$, with pointwise topology, to be sequentially separable. Also in [3] was given necessary and sufficient condition for the space $C_p(X)$ to be strongly sequentially separable.

In this paper, we give necessary and sufficient conditions for the space $B_1(X)$ of first Baire class functions on a space $X$, with pointwise topology, to be sequentially separable and strongly sequentially separable.

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2. Main definitions and notation

Throughout this article all topological spaces are considered Tychonoff. As usually, we will be denoted by \( C_p(X) \) \((B_1(X))\) a set of all real-valued continuous functions \( C(X) \) (a set of all first Baire class functions \( B_1(X) \) i.e., pointwise limits of continuous functions) defined on \( X \) provided with the pointwise convergence topology. If \( X \) is a space and \( A \subseteq X \), then the sequential closure of \( A \), denoted by \([A]_{\text{seq}}\), is the set of all limits of sequences from \( A \). A set \( D \subseteq X \) is said to be sequentially dense if \( X = [D]_{\text{seq}} \). If \( D \) is a countable sequentially dense subset of \( X \) then \( X \) call sequentially separable space.

Call \( X \) strongly sequentially separable if \( X \) is separable and every countable dense subset of \( X \) is sequentially dense.

We recall that a subset of \( X \) that is the complete preimage of zero for a certain function from \( C(X) \) is called a zero-set. A subset \( O \subseteq X \) is called a cozero-set (or functionally open) of \( X \) if \( X \setminus O \) is a zero-set. If a set \( Z = \bigcup_i Z_i \) where \( Z_i \) is a zero-set of \( X \) for any \( i \in \omega \) then \( Z \) is called \( Z_\sigma \)-set of \( X \). Note that if a space \( X \) is a perfect normal space, then class of \( Z_\sigma \)-sets of \( X \) coincides with class of \( F_\sigma \)-sets of \( X \).

It is well known ([5]), that \( f \in B_1(X) \) if and only if \( f^{-1}(G) = Z_\sigma \)-set for any open set \( G \) of real line \( \mathbb{R} \).

Further we use the following theorems.

**Theorem 2.1.** ([2]). A space \( C_p(X) \) is sequentially separable if and only if there exist a condensation (one-to-one continuous map) \( f : X \mapsto Y \) from a space \( X \) on a separable metric space \( Y \), such that \( f(U) = F_\sigma \)-set of \( Y \) for any cozero-set \( U \) of \( X \).

**Theorem 2.2.** ([2]). A space \( B_1(X) \) is sequentially separable for any separable metric space \( X \).

Note that proof this theorem given more, namely there exist a countable subset \( S \subseteq C(X) \), such that \([S]_{\text{seq}} = B_1(X)\).

3. Sequentially separable

The main result of this paper is a next theorem.
Theorem 3.1. A space \( B_1(X) \) is sequentially separable if and only if there exist a bijection \( \varphi : X \leftrightarrow Y \) from a space \( X \) onto a separable metrizable space \( Y \), such that

1. \( \varphi^{-1}(U) = Z_\sigma \)-set of \( X \) for any open set \( U \) of \( Y \);
2. \( \varphi(T) = F_\sigma \)-set of \( Y \) for any zero-set \( T \) of \( X \).

Proof. (1) \( \Rightarrow \) (2). Let \( B_1(X) \) be a sequentially separable space, and \( S \) be a countable sequentially dense subset of \( B_1(X) \). Consider a topology \( \tau \) generated by the family \( \mathcal{P} = \{ f^{-1}(G) : G \) is an open set of \( \mathbb{R} \) and \( f \in S \} \). A space \( Y = (X, \tau) \) is a separable metrizable space because \( S \) is a countable dense subset of \( B_1(X) \). Note that a function \( f \in S \), considered as map from \( Y \) to \( \mathbb{R} \), is a continuous function. Let \( \varphi \) be the identity map from \( X \) on \( Y \).

We claim that \( \varphi^{-1}(U) = Z_\sigma \)-set of \( X \) for any open set \( U \) of \( Y \). Note that class of \( Z_\sigma \)-sets is closed under a countable unions and finite intersections of its elements. It follows that it is sufficient to prove for any \( P \in \mathcal{P} \). But \( \varphi^{-1}(P) = Z_\sigma \)-set for any \( P \in \mathcal{P} \) because \( f \in S \subset B_1(X) \).

Let \( T \) be a zero-set of \( X \) and \( h \) be a characteristic function of \( T \). Since \( T \) is a zero-set of \( X \), \( h \in B_1(X) \). There are \( \{ f_n \}_{n \in \omega} \subset S \) such that \( \{ f_n \}_{n \in \omega} \mapsto h \). Since \( S \subset C_\varphi(Y) \), \( h \in B_1(Y) \) and, hence, \( h^{-1}(\frac{1}{2}, \frac{3}{2}) = T \) is a \( Z_\sigma \)-set of \( Y \).

(2) \( \Rightarrow \) (1). Let \( \varphi \) be a bijection from \( X \) on \( Y \) satisfying the conditions of theorem. Then \( h = f \circ \varphi \in B_1(X) \) for any \( f \in C(Y) \) \( (h^{-1}(G) = \varphi^{-1}(f^{-1}(G)) \) — \( Z_\sigma \)-set of \( X \) for any open set \( G \) of \( \mathbb{R} \)). Moreover \( g = f \circ \varphi^{-1} \in B_1(Y) \) for any \( f \in B_1(X) \) because of \( \varphi(Z) \) is a \( Z_\sigma \)-set of \( Y \) for any \( Z \) \( Z_\sigma \)-set of \( X \). Define a map \( F : B_1(X) \mapsto B_1(Y) \) by \( F(f) = f \circ \varphi^{-1} \). Since \( \varphi \) is a bijection, \( C_\varphi(Y) \) embeds in \( F(B_1(X)) \) i.e., \( C(Y) \subset F(B_1(X)) \). By Theorem 2.2 each subspace \( D \) such that \( C(Y) \subset D \subset B_1(Y) \) is sequentially separable. Thus \( B_1(X) \) (homeomorphic to \( F(B_1(X)) \)) is sequentially separable.

Corollary 3.2. A space \( B_1(X) \) is sequentially separable for any regular space \( X \) with a countable network.

Proof. Let \( X \) be a regular space \( X \) with a countable network. Then there are a countable network \( \{ F_n : n \in \omega \} \) of \( X \) where \( F_n \) is closed subset of \( X \) for \( n \in \omega \). Consider a topology \( \tau \) on \( X \) generated by the family \( \mathcal{P} = \{ F_n \setminus F_n : n \in \omega \} \). Let \( Z = (X, \tau) \). The space \( Z \) is zero-dimensional space with countable base and, hence, it is metrizable space. Let \( h : Z \mapsto X \) be the identity map. Note that any element of \( \mathcal{P} \) is a \( F_\sigma \)-set of \( X \). Hence \( h(U) \)
is a $F_\sigma$-set of $X$ for any open set $U$ of $Z$. Since $h$ is a condensation, $h^{-1}(T)$ is a zero-set of $Z$ for any zero-set $T$ of $X$. Consider $\varphi = h^{-1} : X \mapsto Z$. Then $\varphi$ satisfies all the conditions of the Theorem 3.1. 

In [2] was proved that identity maps of Sorgenfrey line $S$ onto $\mathbb{R}$ and of Niemytzki Plane $N$ onto the closed upper half-plane $\mathbb{R}^2_+$ satisfies the condition of the Theorem 2.1. Hence $C_p(S)$ and $C_p(N)$ are sequentially separable spaces.

We claim that $B_1(S)$ and $B_1(N)$ are not sequentially separable spaces.

We recall some concepts and facts related to the space of first Baire class functions.

- A map $f : X \mapsto Y$ be called $Z_\sigma$-map, if $f^{-1}(Z)$ is a $Z_\sigma$-set of $X$ for any zero-set $Z$ of $Y$.

  Note that a continuous map is a $Z_\sigma$-map.

- A space $X$ be called an analytic space if there is a continuous map $\varphi$ from $\mathbb{P}$ (space of irrational numbers) onto $X$.

- A space $X$ be called an $K$-analytic space if there are Čech-complete Lindelöf space $Y$ and a continuous map $f$ from $Y$ onto $X$.

  An analytic space or a compact space are $K$-analytic space (5).

  If in definition of $K$-analytic space the continuous map is replace by $Z_\sigma$-map we get a wider class of spaces — class of $K_\sigma$-analytic spaces (7).

  Note that if $X$ is a $K_\sigma$-analytic space then $X^n$ is Lindelöf space for each $n \in \omega$ (6).

**Example 3.3.** There is a space $X$ such that $C_p(X)$ is sequentially separable space but $B_1(X)$ is not sequentially separable.

**Proof.** We claim that $B_1(S)$ is not sequentially separable space.

Consider $B_1(S)$. Let $p : S \mapsto \mathbb{R}$ be the identity map. Then $p(T)$ is a $F_\sigma$-set of $\mathbb{R}$ for any cozero-set $T$ of $S$ (2).

Suppose that $B_1(S)$ is sequentially separable space. By Theorem 3.1, there are separable metrizable space $Y$ and a bijection $\varphi : S \mapsto Y$ from $S$ onto $Y$ such that $\varphi(T)$ is a $F_\sigma$-set of $Y$ for any closed subset $T$ of $S$ and $\varphi^{-1}(U)$ is a $F_\sigma$-set of $S$ for any open subset $U$ of $Y$. Consider the map $\varphi \circ p^{-1} : \mathbb{R} \mapsto Y$. Since $\varphi \circ p^{-1}$ is a Borel function, the space $Y$ is an analytic separable metrizable space (3). Thus the map $\varphi^{-1} : Y \mapsto S$ is a $Z_\sigma$-bijection.
of analytic separable metrizable space $Y$. Then $S$ is a $K_{\sigma}$-analytic space and, hence, $S^2$ is a Lindelöf space, a contradiction. 

The same proof remains valid for the space $B_1(N)$.

4. Strongly sequentially separable

From [6], we note that $B_1(X)$ is separable if and only if $X$ has a coarser second countable topology.

In [3] be characterized those spaces $X$ so that $C_p(X)$ is strongly sequentially separable. The following definition and theorems are relevant. For a proof of Theorem 4.2 see [4], Theorem 4.3 see [3], and more information on the property $\gamma$ see [4].

Definition 4.1. A family $\alpha$ of subsets of $X$ is called an $\omega$-cover of $X$ if for every finite $F \subset X$ there is a $U \in \alpha$ such that $F \subset U$.

Theorem 4.2. ([4]). The following are equivalent:

1. $C_p(X)$ is Frechet-Urysohn;
2. $X$ has the property $\gamma$: for any open $\omega$-cover $\alpha$ of $X$ there is a sequence $\beta \subset \alpha$ such that $\lim \inf \beta = X$.

Theorem 4.3. ([3]). The space $C_p(X)$ is strongly sequentially separable if and only if $X$ has a coarser second countable topology, and every coarser second countable topology for $X$ has the property $\gamma$.

We characterize those spaces $X$ so that $B_1(X)$ is strongly sequentially separable.

Theorem 4.4. The function space $B_1(X)$ is strongly sequentially separable if and only if $X$ has a coarser second countable topology, and for any bijection $\varphi$ from a space $X$ onto a separable metrizable space $Y$, such that $\varphi^{-1}(U) = Z_\sigma$-set of $X$ for any open set $U$ of $Y$, the space $Y$ has the property $\gamma$.

Proof. ($\Rightarrow$). Assume that $B_1(X)$ is strongly sequentially separable. Let $\varphi$ be a bijection from a space $X$ onto a separable metrizable space $Y$, such that $\varphi^{-1}(U) = Z_\sigma$-set of $X$ for any open set $U$ of $Y$. Then for any $f \in C(Y)$, $h = f \circ \varphi \in B_1(X)$. Since $\varphi$ is a bijection, a map $F : C(Y) \to B_1(X)$ (define as $F(f) = f \circ \varphi$) is an embeds in $B_1(X)$ i.e., $F(C_p(Y)) \subset B_1(X)$. Note that
\( F(C_p(Y)) \) is dense separable subset of \( B_1(X) \) and, hence, \( C_p(Y) \) is strongly sequentially separable. By Theorem 4.3, the space \( Y \) has the property \( \gamma \).

\((\Leftarrow)\). Assume that \( X \) has a coarser second countable topology. Then \( B_1(X) \) is separable (Theorem 1 in [6]).

Let \( A \) be a countable dense subset of \( B_1(X) \). We wish to show that for any \( f \in B_1(X) \) there is some sequence \( \{f_i : i \in \omega\} \subset A \) such that \( f \) is the limit of the sequence.

Consider a topology \( \tau \) generated by the family
\[
P = \{g^{-1}(G) : G \text{ is an open set of } \mathbb{R} \text{ and } g \in A \cup \{f\}\}.\]
A space \( Y = (X, \tau) \) is a separable metrizable space because \( A \) is a countable dense subset of \( B_1(X) \). Note that a function \( g \in A \cup \{f\} \), considered as map from \( Y \) to \( \mathbb{R} \), is a continuous function. Let \( \varphi \) be the identity map from \( X \) on \( Y \).

We claim that \( \varphi^{-1}(U) \) — \( Z_\sigma \)-set of \( X \) for any open set \( U \) of \( Y \). Note that class of \( Z_\sigma \)-sets is closed under a countable unions and finite intersections of its elements. It follows that it is sufficient to prove for any \( P \in \mathcal{P} \). But \( \varphi^{-1}(P) \) — \( Z_\sigma \)-set for any \( P \in \mathcal{P} \) because \( g \in A \cup \{f\} \subset B_1(X) \). It follows that the space \( Y \) has the property \( \gamma \) and, hence, \( C_p(Y) \) is strongly sequentially separable. Since \( A \) is countable dense subset of \( B_1(X) \), the set \( A \) (where a map in \( A \) considered as map from \( Y \) to \( \mathbb{R} \)) is a countable dense subset of \( C_p(Y) \). Hence there is some sequence \( \{f_i : i \in \omega\} \subset A \) such that \( f \) is the limit of the sequence.

\( \Box \)

It is consistent and independent for arbitrary \( X \), that \( C_p(X) \) is strongly sequentially separable if and only if \( C_p(X) \) is Frechet-Urysohn.

Note that the theorem 4.2 implies the following proposition (Corollary 17 in [3]).

**Proposition 4.5.** (Cons (ZFC)) The following are equivalent:

1. \( C_p(X) \) is strongly sequentially separable;
2. \( X \) is countable;
3. \( C_p(X) \) is separable and Frechet-Urysohn.

For a space \( B_1(X) \) we have a following

**Corollary 4.6.** (Cons (ZFC)) The following are equivalent:

1. \( B_1(X) \) is strongly sequentially separable;
2. \( X \) is countable;
3. $B_1(X)$ is separable and Frechet-Urysohn.

Recall that the cardinal $p$ is the smallest cardinal so that there is a collection of $p$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo-intersection. Note that $\omega_1 \leq p \leq c$. (See [1] for more on small cardinals including $p$.)

**Example 4.7.** Assume that $\omega_1 < p$. There is uncountable space $X$ such that $B_1(X)$ is strongly sequentially separable but not Frechet-Urysohn.

**Proof.** Let $X$ be $\omega_1$ with the discrete topology. Clearly, that $X$ is separable submetrizable space and so $B_1(X)$ is a separable, dense subspace of $\mathbb{R}^X$. We know (from Theorem 11 in [3]) that $\mathbb{R}^X$ must be strongly sequentially separable space and so $B_1(X)$ is strongly sequentially separable. Note that $\mathbb{R}^{\omega_1}$ is not Frechet-Urysohn.

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