The pressure distribution and shear forces inside the proton

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The distributions of pressure and shear forces inside the proton are investigated using lattice Quantum Chromodynamics (LQCD) calculations of the energy momentum tensor, allowing the first model-independent determination of these fundamental aspects of proton structure. This is achieved by combining recent LQCD results for the gluon contributions to the energy momentum tensor with earlier calculations of the quark contributions. The utility of LQCD calculations in exploring, and supplementing, the assumptions in a recent extraction of the pressure distribution in the proton from deeply virtual Compton scattering is also discussed. Based on this study, the target kinematics for experiments aiming to determine the pressure and shear distributions with greater precision at Thomas Jefferson National Accelerator Facility and a future Electron Ion Collider are investigated.

Many of the most fundamental aspects of hadron structure are encoded in form factors that describe the hadron's interactions with the electromagnetic, weak, and gravitational forces. In the forward limit, the electromagnetic form factors reduce to the charge and magnetic moment of a hadron, weak form factors reduce to the axial charge and induced pseudoscalar coupling, while the gravitational form factors describe the hadron's mass, spin, and $D$-term. Unlike the mass, spin, and electromagnetic and weak properties of the proton, which are well-known, the quark $D$-term form factor, $D_q(t, \mu)$ (where $t$ is the squared momentum transfer and $\mu$ is a renormalisation scale), has only recently been extracted from experiment \cite{BEG}. The gluon term $D_g(t, \mu)$ has never been extracted. These functions, which parameterise the spatial components of the energy momentum tensor (EMT), describe the internal dynamics of the system through the pressure and shear distributions inside the proton \cite{BEGled}

While the quark and gluon contributions to the pressure distribution are not individually well-defined because they depend on the non-conserved components of the EMT and are scale- and scheme-dependent, the total pressure distribution (the sum of the quark and gluon contributions) is a measurable quantity. As such, it is of fundamental interest as one of the few remaining aspects of proton structure about which very little is known. Recently, the pressure distribution in the proton was extracted for the first time from deeply virtual Compton scattering (DVCS) experiments at the Thomas Jefferson National Accelerator Facility (JLab) \cite{BEGled} over a limited kinematic range. The result is remarkable; it indicates that the internal pressure in a proton is approximately $10^{35}$ Pascal, exceeding the estimated pressure in the interior of a neutron star. However, since DVCS is almost insensitive to gluons, this determination necessarily relies on several assumptions about the gluon contributions to the proton pressure that are important to investigate.

This letter presents the first determination of the QCD pressure and shear distributions inside the proton, including both the quark and gluon contributions to these quantities. The study is undertaken using lattice Quantum Chromodynamics (LQCD) with larger-than-physical values of the light quark masses. The results reveal that gluons play an important role, different from that of quarks, in the proton’s internal dynamics. In particular, the gluon contribution to the $D$-term form factor, which dictates the pressure and shear distributions, is distinguished in both magnitude and momentum-dependence from the quark contribution. At the scale $\mu = 2$ GeV in the $\overline{\text{MS}}$ scheme, gluons provide the dominant contributions to the proton shear distribution (for which the separation is well-defined). The utility of these LQCD results in augmenting the experimental extraction of the pressure in Ref. \cite{BEGled}, henceforth referred to as BEG, is also explored. While the calculations provide some support to the assumptions made in that pioneering work, they also indicate deficiencies that must be remedied before a completely model-independent determination of the pressure and shear distributions is possible from experiment. Based on the LQCD studies, the kinematics of future experiments at JLab, a future Electron Ion Collider (EIC), and other facilities that will be needed to achieve this are discussed.

The EMT and $D$-term form factors: The pressure and shear distributions in the proton are constructed from the $D$-term form factors, which are defined from the nucleon matrix elements of the traceless, symmetric energy-momentum tensor. Precisely, the matrix elements of the gluon component of the EMT,

$$\langle p', s'|G^{\mu\nu}_{\text{G}}[A_2, B_g, D_g]|p, s\rangle = u'(\gamma_\mu P_\nu + B_g \frac{i P_{\mu}(\sigma_\nu)}{2M_N}) u(p) \frac{\Delta_\mu}{2M_N} + D_g \frac{\Delta_{\mu\nu}}{4M_N} u(p)$$

depend on three generalised form factors (GFFs), $A_g(t, \mu)$, $B_g(t, \mu)$, and $D_g(t, \mu)$, that are functions of the momentum transfer $t = \Delta^2$ with $\Delta = p'_\mu - p_\mu$. In Eq. (1), $G^{\mu\nu}_{\text{G}}$ is the gluon field strength tensor, braces denote symmetrisation and trace-subtraction of the enclosed indices, $P_\mu = (p_\mu + p'_\mu)/2$, the spinors are expressed as $u = u_s(p)$ and $\bar{u}' = \bar{u}'_{s'}(p')$, and $M_N$ is the proton mass. An exactly analogous decomposition exists

\[ \langle p', s'|G^{\alpha\beta}_{\text{Q}}[A_2, B_q, D_q]|p, s\rangle = P'_{\alpha\beta}\Delta_{\nu\rho}\frac{\Delta_{\mu\nu}}{4M_N}\bar{u}'_{s'}(p') u(p) \]
for matrix elements of the quark EMT:
\[
\langle \p', s' | \bar{q} \gamma_{(\mu)1} \tilde{D}_\nu | q, s \rangle = \bar{u}' F_{\mu\nu} [A_{q} B_{q} D_{q}] u,
\]
where \( q \) is the quark field of flavour \( q \) and \( D_{\nu} \) is the gauge covariant derivative.

The individual EMT form factors depend on the renormalisation scheme and scale, \( \mu \). Since the isoscalar combinations of twist-two operators in Eqs. \([1] \) and \([2] \) mix under renormalisation, so too do the individual isoscalar quark and gluon form factors. This mixing takes the form
\[
\begin{pmatrix}
D_{u+d}(t, \mu) \\
D_{g}(t, \mu)
\end{pmatrix} = \begin{pmatrix}
Z_{qq}^{(\frac{t}{\mu^2})} & Z_{qg}^{(\frac{t}{\mu^2})} \\
Z_{gq}^{(\frac{t}{\mu^2})} & Z_{gg}^{(\frac{t}{\mu^2})}
\end{pmatrix} \begin{pmatrix}
D_{u+d}(t, \mu') \\
D_{g}(t, \mu')
\end{pmatrix},
\]
where the perturbative mixing coefficients are given in Ref. \([5] \). Because of conservation of the EMT, the isoscalar combination of the quark and gluon pieces, \( D(t) = D_{u+d}(t, \mu) + D_{g}(t, \mu) \), is scale-invariant.

In terms of the total \( D(t) \) form factor, the shear and pressure distributions in the proton can be expressed in the Breit frame as \([2] \) \([3] \)
\[
s(r) = -\frac{1}{2} \frac{d}{dr} \frac{d}{dr} D(r), \quad p(r) = \frac{1}{3} \frac{d}{dr} \frac{d}{dr} \frac{d}{dr} D(r),
\]
respectively, where
\[
\bar{D}(r) = \int \frac{d^3 \tilde{p}}{2E(2\pi)^3} e^{-i\tilde{p} \cdot \tilde{r}} D(-\tilde{p}^2).
\]

While the quark and gluon shear forces are individually well-defined (i.e., one can define scale-dependent partial contributions \( s_{\mu}(r, \mu) \), \( p(r) \) is defined only for the total system as it depends not only on the separate \( D_{q,g}(t, \mu) \) but also on additional GFFs related to the trace terms of the EMT that only cancel in the sum \([4] \).

**Lattice QCD quark and gluon D-term form factors:** The quark GFFs of the proton have been computed by a number of LQCD collaborations \([6] \) \([11] \) since the first studies in Refs. \([12] \) \([13] \) (see Ref. \([14] \) for a review). While there are as-yet no calculations directly at the physical quark masses, studies over masses corresponding to \( 0.21 \leq m_{\pi} \leq 1.0 \text{ GeV} \) show very mild mass-dependence relative to the other statistical and systematic uncertainties of the calculations. The \( t \)-dependence of the GFFs has been determined over the range \( 0 \leq |t| \leq 2 \text{ GeV}^2 \).

The calculations contain all contributions for the isovector combination \( D_{u-d}(t, \mu) \), while so-called disconnected contractions have been neglected in distinct determinations of the isoscalar quark GFFs, \( D_{u+d}(t, \mu) \), since these terms are both particularly numerically challenging to compute and are found to be small for many other quantities. An important observation from these determinations of the GFFs is that the isovector combination \( D_{u-d}(t, \mu) \sim 0 \) over the entire range of quark masses and momentum transfers that have been studied. An example of the isoscalar connected quark \( D \)-term form factor from Ref. \([8] \) is shown in Fig. \([1] \) at quark masses corresponding to \( m_{\pi} \sim 0.45 \text{ GeV} \).
dependence, since there is no a-priori reason that $D(t)$ need be monotonic, nor that prevents it from changing sign.

An alternative parametrisation of the $t$-dependence of GFFs is provided by a modified $z$-expansion:

$$D_{q/g}(t,\mu) = \frac{1}{(1 - t/\Lambda^2)^3} \sum_{k=0}^{k_{\text{max}}} a_k [z(t)]^k,$$

where $z(t) = [\sqrt{t_{\text{cut}} - t} - \sqrt{t_{\text{cut}} - t_0}] / [\sqrt{t_{\text{cut}} - t} + \sqrt{t_{\text{cut}} - t_0}]$. Since the conformal mapping guarantees analyticity around $z = 0$, the $z$-expansion provides a more reliable estimate of uncertainties in regions unconstrained by data. Modified $z$-expansion fits to the quark and gluon GFFs from LQCD, with the tripole mass $\Lambda$ fixed to that determined by a pure tripole fit to the GFF and with $k_{\text{max}} = 2$, $t_{\text{cut}} = 4m^2$, and $t_0 = t_{\text{cut}}(1 - \sqrt{1 + (2 \text{ GeV})^2/t_{\text{cut}}})$, are shown in Fig. 1.

In each case, the parametrisation is reasonably well constrained over a kinematic range that is sufficient for the GFFs to become indistinguishable from zero within uncertainties. Nevertheless, these fits are considerably less well constrained than the tripole fits.

The pressure distribution and shear forces in the proton: Figure 2 shows the pressure computed using the LQCD determinations of both quark and gluon $D$-term form factors for both the tripole parameterisation and modified $z$-expansion. Given the larger uncertainties in the latter fits to the $D$-term form factors, the $z$-expansion pressure is less well determined, although still resolved from zero by several standard deviations at the peak values. The differences provide an estimate of model dependence.

The shear distribution in the proton is not independent of the pressure distribution as they are related by EMT conservation. Nevertheless, since the quark and gluon contributions to shear are individually well-defined (although renormalisation-scale–dependent), it is instructive to calculate them through Eq. (6). In Fig. 3 the quark and gluon shear forces in the proton, determined from modified $z$-expansion fits to the $D$-term GFFs (Eq. [6]) are shown, along with a rendering of the tangential forces in the proton.

The shear and pressure distributions can be combined to define a mechanical radius of the proton $r_{\text{Mech}}^2 = \langle r^2 Z(r) d^3r / \int Z(r) d^3r \rangle$ where $Z(r) = \frac{2}{3}s(r) + p(r)$. Using the pressure and shear distributions determined from the LQCD results, this is found to be $r_{\text{Mech}}^2 = 0.51(2) \text{ fm}^2$ using the modified $z$-expansion to parameterise the $D$-term GFFs and $0.57(1) \text{ fm}^2$ using the tripole ansatz. This is smaller than the experimentally determined charge radius of the proton, but similar to the charge radius calculated from LQCD at heavier quark masses comparable to those used here.

Comparison to BEG $D$-term and future experimental goals: In Fig. 1 the BEG quark $D$-term form factor extracted from DVCS is compared with the LQCD determinations of the quark and gluon form factors. The BEG result has been shifted to the renormalisation scale $\mu = 2 \text{ GeV}$ in the $\overline{\text{MS}}$ scheme using the three-loop running $^{19}$.

The connected isoscalar quark GFF deter-

\[ \begin{align*}
D_{q/g}(t,\mu) &= \frac{1}{(1 - t/\Lambda^2)^3} \sum_{k=0}^{k_{\text{max}}} a_k [z(t)]^k, \\
D_{q/g}(t,\mu) &= \frac{1}{(1 - t/\Lambda^2)^3} \sum_{k=0}^{k_{\text{max}}} a_k [z(t)]^k,
\end{align*} \]

\[ \begin{align*}
D_{q/g}(t,\mu) &= \frac{1}{(1 - t/\Lambda^2)^3} \sum_{k=0}^{k_{\text{max}}} a_k [z(t)]^k, \\
D_{q/g}(t,\mu) &= \frac{1}{(1 - t/\Lambda^2)^3} \sum_{k=0}^{k_{\text{max}}} a_k [z(t)]^k.
\end{align*} \]
mined from LQCD is approximately $1.7 \times$ smaller in magnitude than the BEG GFF, albeit with significant uncertainties, and has a similar dependence on the momentum transfer $t$. The LQCD determination of the gluon $D$-term form factor is noticeably larger in magnitude than the BEG result. It also favours a more general functional form in $t$ than the tripole assumed in BEG, although it is not inconsistent with a tripole ansatz within uncertainties.

The BEG analysis assumes that $D_q(t, \mu) = D_g(t, \mu)$ as there is no information on the gluon $D$-term from experiment. This is in mild tension with the LQCD results, and, moreover, given the scale evolution, Eq. (3), can only possibly hold at one scale. Since DVCS accesses the charge-squared weighted combination of quark flavours, BEG also necessarily assumes that the isovector quark contributions to the $D_q(t, \mu)$ form factor vanish, i.e., $D_u(t, \mu) = D_d(t, \mu)$. The LQCD finding that $D_{u-d}(t, \mu) \sim 0$ provides compelling motivation for this assumption (large $N_c$ arguments [20] also support this). The left panel of Fig. 4 shows the pressure distribution of the proton computed from the BEG quark $D$-term GFF and the LQCD gluon GFF, both parametrised using a tripole form and assuming that the quark-mass dependence of the latter is negligible in comparison with the statistical uncertainties. This pressure distribution is consistent within uncertainties with the determination using only LQCD data. The pressure obtained under the assumptions of BEG (i.e., $D_q(t, \mu) = D_u+d(t, \mu)$) is also displayed. In comparison with the BEG assumption, the inclusion of the LQCD gluon contribution shifts the peaks of the pressure distribution outwards and extends the region over which the pressure is non-zero.

As discussed above, the tripole form assumed for $D_q(t, \mu)$ in BEG introduces significant model-dependence into the pressure extraction. With the limited kinematic range of the CLAS data this is particularly problematic; the LQCD calculations show that the quark and gluon $D$-term GFFs have significant support up to $|t| \sim 2$ GeV$^2$ (assuming weak quark-mass dependence), which is far beyond the range of the experimental data. Fig. 4 shows the result of a modified $z$-expansion fit to the BEG $D$-term form factor; outside the data range, the parametrisation is very poorly constrained. As shown in the right panel of Fig. 4, this more general fit leads to a pressure distribution that is consistent with zero everywhere, demonstrating that experimental data over a larger kinematic range is needed before a model-independent extraction of the pressure is possible.

In order to investigate the range of $t$ required for a model-independent pressure extraction from experiment, fake data for the quark $D$-term GFF are generated in intervals of $\Delta t = 0.1$ GeV$^2$ extending the experimental data along the tripole fit, assuming uncertainties of the same size as the average uncertainty in the BEG GFF determination. The consistency of the LQCD data with a tripole form gives confidence that such an extension is justified. These fake data are then used to constrain a modified $z$-expansion fit and calculate the corresponding pressure distribution. For a determination of the pressure distribution that is distinct from zero at 2 standard deviations at the maximum of the first peak, the range of the experimental data must be extended in this manner to at least $|t| \sim 1.0$ GeV$^2$. Future experiments, such as those using the CLAS12 detector at JLab and a future EIC, should seek to extend the kinematic reach to address this deficiency, even at the expense of precision in individual $t$ bins. With the EIC’s potential [21][22] to determine the gluon GPDs that are necessary in defining the pressure, similar kinematic coverage should be the goal of EIC experiments. Finally, the flavour separation necessary for a complete determination of the pressure distribution can be enabled by studies of deeply-virtual meson production and DVCS on deuterons [21][22].

**Summary:** The shear and pressure distributions of the proton are determined from LQCD calculations for the first time. The results indicate that gluons play an important role in the internal dynamics of the proton, distinct from that of quarks. In particular, the gluon contributions to the $D$-term form factor, from which the pressure and shear distributions are defined, dominate the quark terms at the scale $\mu = 2$ GeV in the $\overline{\text{MS}}$ scheme. These calculations are undertaken at heavier-than-physical quark masses corresponding to a pion mass roughly three times the physical value. LQCD calcul-
tions at the physical pion mass offer the prospect of a controlled, and model-independent, theoretical determination of the shear and pressure distributions of the proton. With improved LQCD algorithms and growing computational resources, this goal is eminently feasible and will set important benchmarks for measurements using the CLAS12 detector at JLab and at a future EIC.

This study provides support for some of the assumptions made in the recent first extraction of the pressure distribution of the proton from DVCS experiments at JLab. However, given the strong model-dependence involved in the relation of the $D$-term form factor to the shear and pressure distributions, it is found that a clean experimental determination of these quantities will require flavour-separated measurements of the quark $D$-term form factors over a kinematic range extending over at least $0 \leq |t| \lesssim 1 \text{ GeV}^2$, as well as constraints on the gluon $D$-term form factors for similar kinematics.

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