On approximation measures of $q$-exponential function

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Abstract

We shall present effective approximations measures for certain infinite products related to $q$-exponential function. There are two main targets. First we shall prove an explicit irrationality measure result for the values of $q$-exponential function at rational points. Then, if we restrict the approximations to rational numbers of the shape $d^s/N$, we may replace Bundschuh’s irrationality exponent $7/3$ by $2 + \frac{1}{3 + 2\sqrt{3}} = 2.1547...$.

1 Introduction

Our work considers irrationality measures of the values $\tau = E_q(t)$ of the $q$-exponential function

$$E_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{\prod_{n=1}^{k} (1 - q^n)} = \prod_{k=0}^{\infty} \frac{1}{1 - q^k t}, \quad 0 < |t|, |q| < 1, \ t, q \in \mathbb{Q},$$

over the field of rational numbers.

By an irrationality measure of a real number $\tau$ we mean any function in $N$ bounding $|\tau - \frac{M}{N}|$, $M, N \in \mathbb{Z}$, from below for big enough $N$. Irrationality exponent of a real number $\tau$ means an exponent $\mu$ for which there exist positive constants $c$ and $N_0$ such that

$$|\tau - \frac{M}{N}| \geq \frac{c}{N^\mu}.$$
holds for all $M, N \in \mathbb{Z}, N \geq N_0$. Further, the asymptotic irrationality exponent $\mu_I(\tau)$ is then the infimum of all such exponents $\mu$.

We start our considerations by proving a fully explicit irrationality measure for the $q$-exponential function in the case $1/q \in \mathbb{Z} \setminus \{0, \pm 1\}$.

**Theorem 1.** Let $q = 1/d$, where $|d| \in \mathbb{Z}_{\geq 2}$, and $t = u/v \in \mathbb{Q}$, where $u \in \mathbb{Z}$, $v \in \mathbb{Z}_+$, $\gcd(u, v) = 1$ and $0 < |t| < 1$. Then

$$\left| E_q(t) - \frac{M}{N} \right| \geq C_1(2|N|)^{-(\frac{2}{d} + \varepsilon_1)}$$

holds for all $M, N \in \mathbb{Z}, |N| \geq 1$, with an explicit constant $C_1 = C_1(d, v)$ (given later in (25)) and

$$\varepsilon_1 = \varepsilon_1(v, d, N) = \frac{2\log v + 8\log |d| + 4\sqrt{(\log 4) \log |d|}}{\sqrt{\frac{3}{2}\log |d| \log(2|N|)}}.$$

In particular, this means that the (asymptotic) irrationality exponent of

$$\prod_{k=1}^{\infty} (1 - d^{-k})$$

is at most $7/3$, a record holding result proved already by Bundschuh [2]. However, Bundschuh’s result is not as explicit as ours. The work [3] generalizes Bundschuh’s result over arbitrary number fields and offers similar results for several other $q$-series related to the $q$-exponential function (1).

Our work is partly inspired by the work [3] where the authors are interested in $C$-nomials including the $q$-binomial coefficients and the Fibonomials defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{(1 - q)(1 - q^2) \cdots (1 - q^n)}{(1-q)(1-q^2) \cdots (1-q^k)(1-q^2) \cdots (1-q^{n-k})}$$

and

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_F = \frac{F_1 \cdots F_n}{F_1 \cdots F_k \cdot F_1 \cdots F_{n-k}}$$

respectively, where $F_k$ are the Fibonacci numbers, $F_0 = 0$, $F_1 = 1$, $F_{k+2} = F_{k+1} + F_k$ for $k = 0, 1, \ldots$.

Connected to the investigations of distances of $q$-binomial coefficients, the authors in [3] state the following result. Let $q = 1/d$, where $d \in \mathbb{Z}_{\geq 2}$, then

$$\left| \frac{1}{\prod_{n=1}^{\infty} (1 - d^{-n})} - \frac{d^s}{(d-1)^s} \right| > \frac{1}{((d-1)^s)^3}$$

holds for all but finitely many pairs $(l, s) \in \mathbb{Z}_{\geq 1}^2$. As noted in [3], already Bundschuh’s result, see [2], is better than (2). However, we can do more in this restricted case. Namely, we have
Theorem 2. Let \( q = 1/d \), where \(|d| \in \mathbb{Z}_{\geq 2} \), and \( t = u/v \in \mathbb{Q} \), where \( u \in \mathbb{Z}, v \in \mathbb{Z}_+ \), \( \gcd(u,v) = 1 \) and \( 0 < |t| < 1 \). Then there exists an effective positive constant \( C_2 = C_2(d,v) \) such that
\[
\left| E_q(t) - d^s \right| \geq C_2(2|N|)^{-(2+\frac{1}{3+2\sqrt{3}})\epsilon_2}
\]
holds for all \( s \in \mathbb{Z}_+, N \in \mathbb{Z}, |N| \geq 82836 \), with \( \epsilon_2 = \epsilon_2(d,v,N) \in \mathbb{R}_+ \) satisfying \( \epsilon_2(d,v,N) \to 0 \), when \( |N| \) increases.

Consequently, (3) implies that in this restricted case the (restricted) asymptotic irrationality exponent is at most \( 2 + \frac{1}{3+2\sqrt{3}} = 2.1547... \) which is smaller than Bundschuh’s result \( 7/3 \).

2 A lemma for irrationality exponents

Our target is to approximate the values \( E_q(t) \) of the \( q \)-exponential series by rational numbers of the shape \( M/N \) in the following cases:

(a) \( M, N \in \mathbb{Z}, N \neq 0 \),

(b) \( M = d^s, s \in \mathbb{Z}_+, N \in \mathbb{Z}, N \neq 0 \),

where \( q = 1/d \), \( |d| \in \mathbb{Z}_{\geq 2} \) and \( 0 < |t| < 1 \), \( t \in \mathbb{Q} \). The cases (a) and (b) correspond to Theorems 1 and 2 respectively.

Next we present a result which will be applied for deducing irrationality measures and upper bounds of irrationality exponents.

Lemma 1. Let \( \Phi \in \mathbb{R} \). Assume that we have a sequence of numerical linear forms
\[
q_n \Phi - p_n = r_n, \quad q_n, p_n \in \mathbb{Q}, \quad n \in \mathbb{N},
\]
satisfying the conditions
\[
q_n p_{n+1} - p_n q_{n+1} \neq 0,
\]
\[
|q_n|, |p_n| \leq Q(n) = e^{an^2+a_1 n+a_2},
\]
\[
|r_n| \leq R(n) = e^{-bn^2+b_1 n+b_2} < 1
\]

\( a, a_1, a_2, b, b_1, b_2 \) are positive constants.
for all \( n \geq n_0 \geq 1 \) with some constants \( a, a_1, a_2, b, b_1, b_2 \in \mathbb{R}_{\geq 0}, a, b > 0 \). Denote \( c_1 = (b_1 + \sqrt{b_1^2 + 4bb_2})/2b, c_2 = 2ac_1 + 4a + a_1, \) and \( c_3 = c_1(ac_1 + 4a + a_1) + 4a + 2a_1 + a_2. \) Then

\[
\left| \Phi - \frac{M}{N} \right| > \frac{1}{e^{c_3} (2|N|)^{1+a/b+c_2}/\sqrt{\log(2|N|)}}
\]

for all \((M, N) \in \mathbb{Z}^2 \cap \mathcal{E}, \) where \( \mathcal{E} \) is the set defined below.

Write \( D_n := N p_n - M q_n \) and define a set \( \mathcal{E}: \) Let \( M, N \in \mathbb{Z}. \) Then \((M, N) \in \mathcal{E}, \) if

\[
|N| \geq N_0 := e^{b_0^2 - b_1 n_0 - b_2}/2,
\]

and if there exists a largest \( \bar{n} \in \mathbb{Z}_+, \bar{n} \geq n_0, \) such that

\[
2|N| R(\bar{n}) \geq 1
\]

and

\[
D_{\bar{n}+1}, D_{\bar{n}+2} \in \mathbb{Z}. \tag{10}
\]

Of course, (10) is satisfied for all \( M, N \in \mathbb{Z}, N \neq 0, \) if \( p_n, q_n \in \mathbb{Z}. \) Thus (9) will also be valid because \( R(n) \to 0 \) as \( n \to \infty. \) This will be the case in Theorem 1. In Theorem 2 the condition (10) plays a crucial role for improving the the lower bound (8). However, the restricted approximations \( M/N \) satisfying the condition (10) may effect (9), see Lemma 7.

**Proof.** By denoting \( \Lambda := N \Phi - M \) and using (4) we get

\[
q_n \Lambda = N r_n + D_n. \tag{11}
\]

Suppose \((M, N) \in E. \) Then there exists a largest \( \bar{n} \in \mathbb{Z}_+, \bar{n} \geq n_0, \) such that (9) is true.

Further, by (5) and (10) we know that \( D_n \in \mathbb{Z} \setminus \{0\}, \) where \( n = \bar{n} + 1 \) or \( n = \bar{n} + 2. \) Thus by (11), (6) and (7) we get

\[
1 \leq |D_n| = |q_n \Lambda - N r_n| \leq |q_n| |\Lambda| + |N||r_n| \leq Q(n) |\Lambda| + |N| R(n),
\]

where \( |N| R(n) < \frac{1}{2} \) by (2). Hence

\[
1 < 2|\Lambda| Q(n) \leq 2|\Lambda| e^{a_0^2 + a_1 n + a_2}. \tag{12}
\]
From (7) and (9) we get

\[ b\bar{n}^2 - b_1\bar{n} - (b_2 + \log(2|N|)) \leq 0 \]

and further

\[ \bar{n} \leq \frac{b_1 + \sqrt{b_1^2 + 4bb_2 + 4b \log(2|N|)}}{2b} \leq c_1 + \frac{\sqrt{\log(2|N|)}}{\sqrt{b}}, \tag{13} \]

where we applied the inequality \( \sqrt{A + B} \leq \sqrt{A} + \sqrt{B}, \) \( A, B \geq 0. \) Then the estimate (13) implies

\[
an^2 + a_1 n + a_2 \leq a(\bar{n} + 2)^2 + a_1(\bar{n} + 2) + a_2 \leq \frac{a}{b} \log(2|N|) + \frac{c_2}{\sqrt{b}} \sqrt{\log(2|N|)} + c_3
\]

which with (12) proves (8). \( \square \)

The exponent \( 1 + \frac{a}{b} + \frac{c_2}{\sqrt{b \log(2|N|)}} \) in (8) gives an upper bound for the irrationality exponent of \( \Phi. \) In the following we call \( 1 + \frac{a}{b} \) the main term and \( \varepsilon(N) := \frac{c_2}{\sqrt{b \log(2|N|)}} \) will be called the error term.

### 3 Padé -approximations

First we give definitions of \( q \)-series factorials

\[(a)_0 = 1, \ (a)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n \in \mathbb{Z}_+,
\]

and the \( q \)-binomial coefficients

\[\left[\begin{array}{c} n \\ k \end{array}\right] = \left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{(q)_n}{(q)_k(q)_{n-k}}, \quad 0 \leq k \leq n.\]

Note that

\[\left[\begin{array}{c} n \\ k \end{array}\right]_q \in \mathbb{Z}[q], \quad \deg_q \left[\begin{array}{c} n \\ k \end{array}\right]_q = k(n - k),\]

see e.g \([1]\). Our starting point is the following Padé approximation formula of \( q \)-exponential function from \([4\text{, Article VI, Lemma1}].\)
Lemma 2 (4). Let

\[ B_n(t) = \sum_{k=0}^{n} \binom{n}{k} q^{\frac{k}{2}} (q^{n+1})_{n-k}(-t)^k, \]

\[ A_n(t) = \sum_{k=0}^{n} \binom{n}{k} q^{kn} (q^{n+1})_{n-k} t^k \]

and

\[ S_n(t) = (-1)^n t^{2n+1} q^{\frac{mn}{2}} \frac{(q)_n}{(q)_{2n+1}} \sum_{k=0}^{\infty} \frac{(q^{n+1})_k}{(q)_k (q^{2n+2})_k} t^k. \]

Then

\[ B_n(t) E_q(t) - A_n(t) = S_n(t). \] (14)

As usual when studying Diophantine properties of \( q \)-series we need to accelerate convergence of Padé approximations (13). This will be done by iterate use of the \( q \)-shift operator \( J \), \( JF(t) = F(qt) \), and application of the \( q \)-difference equation

\[ E_q(qt) = (1-t) E_q(t). \]

First we obtain

\[ (1-t)B_n(qt)E_q(t) - A_n(qt) = S_n(qt) \]

and repeating this process \( K \) times we get our new approximations

\[ (t)_K B_n(q^K t) E_q(t) - A_n(q^K t) = S_n(q^K t). \] (15)

The Padé approximations suggest that

\[ E_q(t) \sim \frac{A_n(q^K t)}{(t)_K B_n(q^K t)}. \]

Thus our starting point will be the following expression

\[ \Omega_{n,K}(q, t) = NA_n(q^K t) - M(t)_K B_n(q^K t). \] (16)

4 Denominators

For any rational number \( \eta \in \mathbb{Q} \) we call

\[ \text{den}(\eta) = \min\{d \in \mathbb{Z}_+ | d\eta \in \mathbb{Z}\} \]

the denominator of \( \eta \). Thus, e.g. we have \( v = \text{den}(t) \). In the following lemma we give estimates of the denominators of \( \Omega_{n,K}(q, t) \) in our two cases.
Lemma 3. (a) We have \( \text{den}(\Omega_{n,K}(q,t)) \leq v^{n+K}|d|^{\delta_{n,K}} \), where

\[
\delta_{n,K} = \begin{cases} 
\frac{K^2 - K}{2} + \frac{3n^2 + n}{2}, & \text{when } 0 \leq K \leq n; \\
\frac{K^2 - K}{2} + \frac{n^2}{2} + nK, & \text{when } K > n. 
\end{cases}
\]

(b) We have \( \text{den}(\Omega_{n,K}(q,t)) \leq v^{n+K}|d|^{\delta_{n,K}} \), where

\[
\delta_{n,K} = \begin{cases} 
\frac{K^2 - K}{2} + \frac{3n^2 + n}{2}, & \text{when } 0 \leq K \leq n; \\
n^2 + nK, & \text{when } K > n, 
\end{cases}
\]

if the assumption

\[ s \geq \frac{K(K-1)}{2} - \frac{n(n+1)}{2} \tag{17} \]

holds in the case \( K > n \).

Proof. By Lemma 2 we have

\[
A_n(q^K t) = \sum_{k=0}^{n} \left[ \binom{n}{k} q^{kn}(q^{n+1})_{n-k}(q^K t)^k, \quad B_n(q^K t) = \sum_{k=0}^{n} \left[ \binom{n}{k} q^{k(2)}(q^{n+1})_{n-k}(-q^K t)^k, \right.
\]

where \( \left[ \binom{n}{k} \right] \), \( (q^{n+1})_{n-k} \in \mathbb{Z}[q] \) and

\[
\deg_q \left[ \binom{n}{k} \right] = k(n-k), \quad \deg_q (q^{n+1})_{n-k} = \frac{3n^2}{2} - 2kn + \frac{k^2}{2} + \frac{n}{2} - \frac{k}{2}.
\]

Thus we see immediately that \( A_n(q^K t), (t)^K B_n(q^K t) \in \mathbb{Z}[q,t] \) and

\[
\deg_t A_n(q^K t) = n, \quad \deg_t (t)^K B_n(q^K t) = n + K. \tag{18}
\]

Next we estimate the degrees \( \deg_q A_n(q^K t) \) and \( \deg_q (t)^K B_n(q^K t) \). Now

\[
\deg_q A_n(q^K t) = \max_{0 \leq k \leq n} \{ \deg_q \left[ \binom{n}{k} \right] + kn + \deg_q (q^{n+1})_{n-k} + kK \}
\[
= \max_{0 \leq k \leq n} \{ \frac{3n^2}{2} + \frac{n}{2} - \frac{k^2}{2} - \frac{k}{2} + kK \}.
\]

If \( K \leq n \), we obtain the maximum, when \( k = K \). On the other hand, if \( K > n \), we obtain the maximum, when \( k = n \). Hence we have

\[
\deg_q A_n(q^K t) \leq \begin{cases} 
\frac{K^2 - K}{2} + \frac{3n^2 + n}{2}, & \text{when } 0 \leq K \leq n; \\
n^2 + nK, & \text{when } K > n.
\end{cases} \tag{19}
\]
Similarly we obtain that
\[
\deg_q(t)K \cdot B_n(q^K t) = \max_{0 \leq k \leq n} \left\{ \deg_q(t)_K + \deg_q \left[ n \right] + \binom{k}{2} + \deg_q(q^{n+1})_{n-k} + kK \right\}
\]
\[
= \max_{0 \leq k \leq n} \left\{ \frac{K(K - 1)}{2} + \frac{3n^2 + n}{2} + k(K - n - 1) \right\}.
\]

If \( K \leq n \) we obtain the maximum when \( k = 0 \) and if \( K > n \) we obtain the maximum at \( k = n \). Hence we have
\[
\deg_q(t)K \cdot B_n(q^K t) \leq \begin{cases} 
\frac{K^2 - K}{2} + \frac{3n^2 + n}{2}, & \text{when } 0 \leq K \leq n; \\
\frac{K^2 - K}{2} + \frac{n^2 - n}{2} + nK, & \text{when } K > n.
\end{cases}
\]
\[
(20)
\]

(a) We have \( N, M \in \mathbb{Z}, v = \text{den}(t) \) and \( q = 1/d \). Thus by \((16), (18) - (20)\) we obtain that \( \text{den}(\Omega_{n,K}(q, t)) \leq v^{n+K}|d|^{\delta_{n,K}} \), where
\[
\delta_{n,K} = \begin{cases} 
\frac{K^2 - K}{2} + \frac{3n^2 + n}{2}, & \text{when } 0 \leq K \leq n; \\
\frac{K^2 - K}{2} + \frac{n^2 - n}{2} + nK, & \text{when } K > n.
\end{cases}
\]

(b) Here \( N, M = d^s \in \mathbb{Z}, v = \text{den}(t) \) and \( q = 1/d \). Thus
\[
\text{den}(M(t)_K \cdot B_n(q^K t)) \leq v^{n+K}|d|^{\hat{\delta}_{n,K}},
\]
where
\[
\hat{\delta}_{n,K} \leq \begin{cases} 
\frac{K^2 - K}{2} + \frac{3n^2 + n}{2} - s, & \text{when } 0 \leq K \leq n; \\
\frac{K^2 - K}{2} + \frac{n^2 - n}{2} + nK - s, & \text{when } K > n.
\end{cases}
\]
\[
(21)
\]
In this case we assumed that \( s \geq \frac{K^2 - K}{2} - \frac{n^2 + n}{2} \), hence by \((16), (18), (19)\) and \((21)\) we get \( \text{den}(\Omega_{n,K}(q, t)) \leq v^{n+K}|d|^{\delta_{n,K}} \), where
\[
\delta_{n,K} \leq \begin{cases} 
\frac{K^2 - K}{2} + \frac{3n^2 + n}{2}, & \text{when } 0 \leq K \leq n; \\
n^2 + nK, & \text{when } K > n.
\end{cases}
\]
5 Numerical approximations and a non-vanishing result

By using the notations
\[ p_{n,K}(q, t) := \text{den}(\Omega_{n,K}(q, t))A_n(q^K t), \]
\[ q_{n,K}(q, t) := \text{den}(\Omega_{n,K}(q, t))(t)_K B_n(q^K t), \]
\[ r_{n,K}(q, t) := \text{den}(\Omega_{n,K}(q, t))S_n(q^K t) \]
we get numerical approximations
\[ r_{n,K}(q, t) = q_{n,K}(q, t)E_q(t) - p_{n,K}(q, t), \]  
where \( p_{n,K}(q, t) \) and \( q_{n,K}(q, t) \) are so called polynomial terms and \( r_{n,K}(q, t) \) is the remainder term. Here we note that the following quantity
\[ D_{n,K}(q, t) := \text{den}(\Omega_{n,K}(q, t)) \cdot \Omega_{n,K}(q, t) = p_{n,K}(q, t)N - q_{n,K}(q, t)M, \]
is an integer which plays a crucial role later.

Next we prove a non-vanishing result.

**Lemma 4.** For the Padé-approximation polynomials of Lemma 2 holds that
\[ \Delta_n(t) = \begin{vmatrix} B_n(t) & A_n(t) \\ B_{n+1}(t) & A_{n+1}(t) \end{vmatrix} = C_n t^{2n+1}, \]
where \( C_n = (q^{n+2})_{n+1}(-1)^n q^{\frac{3n^2 + n}{2}} \frac{(q)_n}{(q)_{2n+1}} \) is a non-zero if \( 0 < |q| < 1. \)

**Proof.** We have
\[ \Delta_n(t) = B_n(t)A_{n+1}(t) - B_{n+1}(t)A_n(t) \]
and by Lemma 2 we know that \( \deg_t A_n(t), \deg_t B_n(t) \leq n \). Hence we get that
\[ \deg_t \Delta_n(t) \leq 2n + 1. \]

On the other hand, it follows from Padé-approximation formula (14) that
\[ \begin{cases} B_n(t)E_q(t) - A_n(t) = t^{2n+1}\hat{S}_n(t); \\ B_{n+1}(t)E_q(t) - A_{n+1}(t) = t^{2n+3}\hat{S}_{n+1}(t), \end{cases} \]
where
\[ \hat{S}_n(t) = (-1)^n q^{\frac{3n^2 + n}{2}} \frac{(q)_n}{(q)_{2n+1}} \sum_{k=0}^{\infty} \frac{(q^{n+1})_k}{(q)_k (q^{2n+2})_k} t^k. \]

Thus
\[ \Delta_n(t) = \begin{vmatrix} B_n(t) & -t^{2n+1} \hat{S}_n(t) \\ B_{n+1}(t) & -t^{2n+3} \hat{S}_{n+1}(t) \end{vmatrix} = t^{2n+1} \left( B_{n+1}(t) \hat{S}_n(t) - t^2 B_n(t) \hat{S}_{n+1}(t) \right) \]
which implies that \( \text{ord}_t \Delta_n(t) \geq 2n + 1 \) and
\[ \Delta_n(t) = C_n t^{2n+1}, \]
where
\[ C_n = B_{n+1}(0) \hat{S}_n(0) = (q^{n+2})_{n+1} (-1)^n q^{\frac{3n^2 + n}{2}} \frac{(q)_n}{(q)_{2n+1}}. \]

\[ \square \]

**Corollary 1.** Suppose \( 0 < |t| < 1, \ 0 < |q| < 1. \) Then
\[ \Delta_{n,K}(q, t) = \begin{vmatrix} q_{n,K}(q, t) & p_{n,K}(q, t) \\ q_{n+1,K}(q, t) & p_{n+1,K}(q, t) \end{vmatrix} \neq 0 \]
for all \( n, K \in \mathbb{N}. \)

**Proof.** Due to [22]
\[ \Delta_{n,K}(q, t) = q_{n,K}(q, t)p_{n+1,K}(q, t) - p_{n,K}(q, t)q_{n+1,K}(q, t) \]
\[ = \text{den}(\Omega_{n,K}(q, t))\text{den}(\Omega_{n+1,K}(q, t))(t)_K \ (B_n(q^K)A_{n+1}(q^K) - A_n(q^K)B_{n+1}(q^K)). \]

Thus we have \( \Delta_{n,K}(q, t) = \text{den}(\Omega_{n,K}(q, t))\text{den}(\Omega_{n+1,K}(q, t))(t)_K \Delta_n(q^K t), \) which is non-zero for all \( n, K \in \mathbb{N} \) and \( 0 < |t| < 1, \ 0 < |q| < 1. \)
\[ \square \]

### 6 Estimates

In the following we give estimates for the upper bounds of \( |p_{n,K}(q, t)| \) and \( |q_{n,K}(q, t)|. \)

**Lemma 5.** Let \( n, K \in \mathbb{Z}_+, \) then
\[ \max \{ |q_{n,K}(q, t)|, |p_{n,K}(q, t)| \} \leq Q_1(n, K) = 8 \max \{ 1, (t)_K \} \text{den}(\Omega_{n,K}(q, t)). \]
Proof. According to [4, Article VI, p.8-9] we have
\[ 1 - (q + q^2) < (q)_k \leq 1 \quad \text{for all } 0 < q < 1, \ k \in \mathbb{N} \]
and
\[ 1 \leq (q)_k < 1 + |q| \quad \text{for all } \frac{1 - \sqrt{5}}{2} < q < 0, \ k \in \mathbb{N}. \]
Because \( q = 1/d, \ |d| \in \mathbb{Z}_{\geq 2}, \) we have \( \frac{1}{d} < (q)_k \leq 1 \) for positive \( q, \) and \( 1 \leq (q)_k < \frac{3}{2} \) for negative \( q. \) Hence by (22) and Lemma 2 we obtain that
\[
\max\{|q_{n,K}(q,t)|, |p_{n,K}(q,t)|\} = \text{den}(\Omega_{n,K}(q,t)) \max\{|A_n(q^Kt)|, |(t)_K B_n(q^Kt)|\}
\leq 4 \text{den}(\Omega_{n,K}(q,t)) \max\{1, (t)_K\} \sum_{k=0}^{\infty} (|q|^K|t|)^k
\leq 8 \max\{1, (t)_K\} \text{den}(\Omega_{n,K}(q,t)),
\]
where \((t)_K\) (with \( 0 < |t| < 1 \)) has the upper bound
\[
(t)_K < \prod_{k=0}^{\infty} \left(1 + \frac{1}{2^k}\right) \sim 4,768462.
\]

Above we have considered so called polynomial terms of our approximation. Next we will concentrate on the remainder term \( r_{n,K}(q,t). \)

Lemma 6. For our remainder term holds the estimate \( |r_{n,K}(q,t)| \leq R(n, K) \) with
\[
R(n, K) = 8|d|^{-\omega_{n,K}} \quad \text{where}
\]
\[
(a) \omega_{n,K} = \begin{cases} 2nK - \frac{K^2}{2} - (n + K)\frac{\log v}{\log |d|}, & \text{when } K \leq n; \\ n^2 + nK - \frac{K^2}{2} - (n + K)\frac{\log v}{\log |d|}, & \text{when } K > n, \end{cases}
\]
\[
(b) \omega_{n,K} = \begin{cases} 2nK - \frac{K^2}{2} - (n + K)\frac{\log v}{\log |d|}, & \text{when } K \leq n; \\ \frac{n^2}{2} + nK - (n + K)\frac{\log v}{\log |d|}, & \text{when } K > n, \end{cases}
\]
if the assumption \( s > \frac{K^2 - K}{2} - \frac{n^2 + n}{2} \) holds in the case \( K > n. \)

Proof. Due to Lemma 2 (15), (22) and (23) we have
\[
|r_{n,K}(q,t)| = \text{den}(\Omega_{n,K}(q,t))|S_n(q^Kt)|
\leq \text{den}(\Omega_{n,K}(q,t))(|q|^K|t|)^{2n+1}|q|^{\frac{n^2 + n}{2}} (q)^n \sum_{k=0}^{\infty} (|q|^{K} |t|)^{k}.
\]
Because $0 < |q| \leq \frac{1}{2}$, we have $\frac{1}{4} < (q)_k \leq 1$ for positive $q$, and $1 \leq (q)_k < \frac{3}{2}$ for negative $q$. Additionally $0 < |t| < 1$. From these facts it follows that

$$|r_{n,K}(q,t)| < 4 \text{den}(\Omega_{n,K}(q,t)) |q|^{\frac{n^2 + n}{2} + 2nK + K} \sum_{k=0}^{\infty} (|q|^K |t|)^k$$

$$\leq 8 \text{den}(\Omega_{n,K}(q,t)) |q|^{\frac{n^2 + n}{2} + 2nK + K},$$

where $\text{den}(\Omega_{n,K}(q,t)) = v^{n+K} |d|^n K$. Because $d = q^{-1}$, we get the upper bounds of the lemma directly from Lemma 3.

\[\square\]

### 7 Proof of Theorems 1 and 2

Due to (15) and (22) we have numerical linear forms

$$q_{n,K}E_q(t) - p_{n,K} = r_{n,K} \quad (n \in \mathbb{Z}_+),$$

where $q_{n,K}, p_{n,K} \in \mathbb{Z}$ and, by Corollary 1, (5) is satisfied. From now on we use the notation $\gamma = K/n$. Then Lemmas 3 and 5 imply that

$$\max\{|q_{n,K}(q,t)|, |p_{n,K}(q,t)|\} \leq Q_1(n) = e^{an^2 + a_1 n + a_2},$$

where

$$a_1 = a_1(\gamma) = \begin{cases} (\frac{1}{2} - \frac{\gamma}{2}) \log |d| + (\gamma + 1) \log v, & 0 < \gamma \leq 1; \\ (\gamma + 1) \log v, & \gamma > 1, \end{cases}$$

and

$$a_2 = \log(8 \prod_{k=0}^{\infty} (1 + \frac{1}{2^k})).$$

In the case (a)

$$\frac{a}{\log |d|} = a(\gamma) = \begin{cases} \frac{\gamma^2}{2} + \frac{3}{2}, & 0 < \gamma \leq 1; \\ \frac{\gamma^2}{2} + \gamma + \frac{1}{2}, & \gamma > 1, \end{cases}$$

and in the case (b) we have

$$\frac{a}{\log |d|} = a(\gamma) = \begin{cases} \frac{\gamma^2}{2} + \frac{3}{2}, & 0 < \gamma \leq 1; \\ 1 + \gamma, & \gamma > 1. \end{cases}$$
Lemma 6 implies that

\[ |r_{n,K}(q,t)| \leq R(n,K) = e^{-bn^2 + b_1n + b_2}, \]

where \( b_1 = b_1(\gamma) = (1 + \gamma) \log v \) and \( b_2 = \log 8 \). In the case (a)

\[ \frac{b}{\log |d|} = \frac{b(\gamma)}{\log |d|} = \begin{cases} 2\gamma - \frac{2}{\gamma}, & 0 < \gamma \leq 1; \\ 1 + \gamma - \frac{2}{\gamma}, & 1 < \gamma < (1 + \sqrt{3}), \end{cases} \]

and in the case (b)

\[ \frac{b}{\log |d|} = \frac{b(\gamma)}{\log |d|} = \begin{cases} 2\gamma - \frac{2}{\gamma}, & 0 < \gamma \leq 1; \\ \frac{1}{\gamma} + \gamma, & \gamma > 1. \end{cases} \]

Due to Lemma the main term of our irrationality exponent is \( \mu = 1 + a/b \). In the following we fix \( \gamma \) so that the quantity \( a(\gamma)/b(\gamma) \) will be least possible.

Proof of Theorem 1. In the case (a), we have

\[ \frac{a(\gamma)}{b(\gamma)} = \begin{cases} \frac{\gamma^2 + 3}{\gamma - \gamma^2}, & 0 < \gamma < 1; \\ \frac{\gamma^2 + 2\gamma + 1}{2\gamma - \gamma^2}, & 1 \leq \gamma < 1 + \sqrt{3}. \end{cases} \]

Obviously \( \frac{a(\gamma)}{b(\gamma)} \) is continuous, and it is decreasing when \( 0 < \gamma < 1 \) and increasing when \( 1 \leq \gamma < 1 + \sqrt{3} \). Thus

\[
\min_{0 < \gamma < 1 + \sqrt{3}} \frac{a(\gamma)}{b(\gamma)} = \frac{a(1)}{b(1)} = \frac{4}{3},
\]

and we get the lower bound \( \beta \) with the constants

\[
\begin{align*}
c_1 &= (b_1 + \sqrt{b_1^2 + 4bb_2})/2b, \quad bc_1^2 - b_1c_1 - b_2 = 0, \\
c_2 &= 2ac_1 + 4a + a_1, \\
c_3 &= c_1(ac_1 + 4a + a_1) + 4a + 2a_1 + a_2 = (\frac{ab_1}{b} + 4a + a_1)c_1 + \frac{ab_2}{b} + 4a + 2a_1 + a_2. \end{align*}
\]

By using the estimate

\[ c_1 \leq \frac{b_1}{b} + \sqrt{\frac{b_2}{b}} \]
we have
\[ C_1 = e^{-c_4}, \]
\[ c_3 \leq c_4 := \frac{14 \log v}{3 \log |d|} \left( \frac{4}{3} \log v + \sqrt{\log 4 \log |d|} \right) \]
\[ + 8 \sqrt{\log 4 \log |d|} + \log(2^\gamma v^{\frac{44}{3}} |d|) \prod_{k=0}^{\infty} (1 + \frac{1}{2^k}); \]  
(25)

\[ \varepsilon_1(v, d, N) = \frac{c_2}{\sqrt{b \log(2N)}} \leq \frac{\frac{22}{3} \log v + 8 \log |d| + 4 \sqrt{2 \log 2 \log |d|}}{\sqrt{\frac{3}{2} \log |d| \log(2N)}}. \]

This proves Theorem 1.

Proof of Theorem 2. In the restricted case (b), we have

\[ a(\gamma) = \begin{cases} \frac{\gamma^2 + 3}{4\gamma - \gamma^2}, & 0 < \gamma \leq 1; \\ \frac{1 + \gamma}{2 + \gamma}, & \gamma > 1, \end{cases} \]  
(26)

which is a continuous and decreasing function for all \( \gamma > 0. \)

Because the main term of irrational exponent is equal in the case (a) and (b), when \( 0 < \gamma \leq 1, \) we need to consider the restricted case (b) only when \( \gamma > 1. \) Remember our assumptions \( |d| \geq 2, \) \( 0 < |t| < 1. \) Thus

\[ \tau = E_q(t) \geq (-1, \frac{1}{2})^{-1} > \frac{1}{5}, \]

and by Lemma 6 we may write

\[ R(n, K) = |d|^{-(\gamma^2 + nK - (n+K) \log(\gamma) \log |d| - 3)}, \quad K > n, \]

if \( s \geq \frac{K^2 - K}{2} - \frac{n^2 + n}{2}. \) Our target is to approximate \( \tau = E_q(t) \) by numbers of the shape \( d^s/N. \) This implies that, when \( |N| \) grows then also \( s \) grows and thus the conditions (9) and (10) will have a complicated association. In the following lemma we will study the set \( E \) defined in Lemma 1. In Lemma 7 we are interested in enough close approximations, say \( |\tau - d^s/N| < 1/|N|, \) and thus we may use approximations with \( d^s N > 0. \)

**Lemma 7.** Assume that \( \gamma > 1 \) and

\[ \left| \tau - \frac{d^s}{N} \right| < \frac{1}{|N|}, \]  
(27)
Then \((d^*, N) \in \mathcal{E}, \) if \(|N| \geq N_2 \) (given below in \((34)\)) and
\[
\sqrt{\gamma^2 - 1} \left(2 + x + (4 + x)\gamma + \sqrt{(1 + \gamma)^2x^2 + (1 + 2\gamma)(6 + T)}\right) \leq (1 + 2\gamma)\sqrt{T},
\]
x := \frac{\log v}{\log |d|}, \quad T := \frac{2\log(|N|)}{\log |d|}.
\quad (28)

Proof. We want to choose a \(n\) satisfying the conditions
\[
R(n, K) < \frac{1}{2|N|}, \quad s \geq \frac{K^2 - n^2}{2} - \frac{K + n}{2}.
\quad (29)
\]
Note that the first condition in \((29)\) is equivalent to
\[
\frac{\log(2|N|)}{\log |d|} < (1/2 + \gamma)n^2 - (1 + \gamma)xn - 3, \quad x = \frac{\log v}{\log |d|}.
\]
Thereby we define
\[
\bar{n} := \left\lfloor \frac{(1 + \gamma)x + \sqrt{(1 + \gamma)^2x^2 + (1 + 2\gamma)(6 + T)}}{1 + 2\gamma} \right\rfloor.
\]
Consequently, \(\bar{n}\) is the largest \(n\) satisfying \((9)\). Instead of the second condition in \((29)\) we put
\[
\log(2|N|) \geq \frac{K^2 - n^2}{2} \log |d|,
\quad (30)
\]
which is equivalent to
\[
n \leq n_2 := \sqrt{\frac{T}{\gamma^2 - 1}}.
\]
The assumption \((27)\) is equivalent to
\[
|d|^s - 1 \leq |N| \leq |d|^s + 1.
\quad (31)
\]
Suppose \(|N| \geq 2/\tau\), then \((31)\) and \((30)\) imply
\[
s \geq \frac{\log(|N| \tau - 1)}{\log |d|} \geq \frac{\log(2|N|)}{\log |d|} + \frac{\log(\tau/4)}{\log |d|}
\geq \frac{K^2 - n^2}{2} - \frac{K + n}{2} + \frac{K + n}{2} + \frac{\log(\tau/4)}{\log |d|}
\geq \frac{K^2 - n^2}{2} - \frac{K + n}{2},
\]
if \(n \geq \frac{2}{1 + \gamma} \frac{\log(4/\tau)}{\log |d|}\). So, the assumption \((17)\) in Lemma 3, see also \((24)\), is satisfied and thus
\[
D_{n,K} \in \mathbb{Z}, \quad \text{if} \quad |N| \geq \frac{2}{\tau}, \quad n \geq \frac{2}{1 + \gamma} \frac{\log(4/\tau)}{\log |d|}.
\quad (32)
\]
Now we set
\[
\bar{n} + 2 \leq n_2.
\quad (33)\]
Hence \((d^s, N) \in \mathcal{E}\), if the assumption \((33)\) holds. Finally, note that the condition \((33)\) is equivalent to \((28)\) and the conditions \((30)\) and \((32)\) imply a lower bound

\[
\log(2|N|) \geq \log(2N_2) := \max\{\frac{2(\gamma - 1)}{(\gamma + 1) \log |d|} \left(\frac{4}{\tau}\right)^2\}. \tag{34}
\]

In Lemma \[ the condition \((28)\) gives an upper bound, say \(Y(|N|)\), for a feasible \(\gamma\) to be used in \((26)\). We will consider what happens when \(|N| \to \infty\). Then we may use any value from the interval \(1 < \gamma < 1 + \sqrt{3} = Y(\infty)\) and as a limit we get an optimal value

\[
\frac{a(\gamma)}{b(\gamma)} \to 1 + \frac{1}{3 + 2\sqrt{3}}
\]

for the main term. Hence

\[
2 + \frac{1}{3 + 2\sqrt{3}}
\]

is an asymptotic irrationality exponent of \(\tau\).

Let \(q = 1/d\), where \(|d| \in \mathbb{Z}_{\geq 2}\), and \(t = \frac{u}{v} \in \mathbb{Q}\), \(\gcd(u, v) = 1\), \(v \in \mathbb{Z}_+, 0 < |t| < 1\). By Lemmas \[ and \[ we have a lower bound

\[
\left|\tau - \frac{d^s}{N}\right| > \frac{1}{e^{c_3(\gamma)} (2|N|)^{1 + a(\gamma)/b(\gamma) + c_2(\gamma)/\sqrt{b(\gamma) \log(2|N|)}}}
\]

valid for any \(\gamma\) satisfying \((28)\). Therefore

\[
\left|E_q(t) - \frac{d^s}{N}\right| \geq C_2(2|N|)^{-(2 + \frac{1}{3 + 2\sqrt{3}} + \varepsilon_2)}
\]

holds for all \(s \in \mathbb{Z}_+, N \in \mathbb{Z}, |N| \geq N_2\) with some constant \(C_2 = C_2(d, v)\) and error term \(\varepsilon_2 = \varepsilon_2(d, v, N) \in \mathbb{R}_+\) satisfying \(\varepsilon_2(d, v, N) \to 0\), when \(|N|\) increases. By using the values \(\gamma = 1 + \sqrt{3}, \tau = 1/5\) and \(|d| = 2\) in \((34)\) we get a numerical value \(\log(2N_2) := 12.0177...\), thus we fix \(N_2 = 82836\).

\[\Box\]

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