1 Introduction

Let $C$ be a smooth, connected, projective non-hyperelliptic curve of genus $g \geq 2$ over the complex numbers and let $\text{Pic}^d(C)$ be the connected component of its Picard variety parametrizing degree $d$ line bundles, for $d \in \mathbb{Z}$. The variety $\text{Pic}^{g-1}(C)$ carries a naturally defined divisor, the Riemann theta divisor $\Theta$, whose support consists of line bundles that have nonzero global sections. The Riemann singularity theorem describes the singular locus of the divisor $\Theta$ as

$$\text{Sing} \; \Theta = \{ \xi \in \text{Pic}^{g-1}(C) \mid h^0(C, \xi) > 1 \}.$$ 

The Jacobian variety of $C$, $JC := \text{Pic}^0(C)$, carries a canonically defined line bundle, which we denote by $\mathcal{O}(2\Theta_0)$. This line bundle is obtained by translating the bundle $\mathcal{O}(\Theta)$ by any theta characteristic. There is a canonical duality, called the Wirtinger duality, between the spaces of global sections ([Mu] p. 335):

$$w : |2\Theta|^* \cong |2\Theta_0|$$ (1.1)

In [vGvdG] the authors revisit and extend the work of Frobenius on the subseries $\mathbb{P} \Gamma_0 \subset |2\Theta_0|$ consisting of $2\theta$-divisors having multiplicity at least 4 at the origin and formulate a number of Schottky-type conjectures, some of which have been proved ([W] and [I1]). Moreover, J. Fay observed that $2\theta$-divisors of the Jacobian satisfy a remarkable equivalence (e.g. [W] Prop. 4.8, [Gu1] Cor. 1)

$$\text{mult}_0(D) \geq 4 \iff C - C \subset D \quad \forall D \in |2\Theta_0|$$ (1.2)

where the surface $C - C$ denotes the image of the difference map $\phi_1 : C \times C \to JC$, which sends a pair of points $(p, q)$ to the line bundle $\mathcal{O}(p - q)$. We denote by $\mathbb{P}(\mathbb{T}_0)$ the embedded tangent space at the origin to the Kummer variety $\text{Kum} := JC/\pm \subset |2\Theta|$. It is well-known that the space $\mathbb{T}_0$ of hyperplanes containing $\mathbb{T}_0$ corresponds under the Wirtinger duality ([L3]) to $\Gamma_0$. It follows that

$$\text{codim} \; \Gamma_0 = 1 + \left( \frac{g+1}{2} \right).$$ (1.3)

In this paper we are interested in some subseries of $\Gamma_0$, which have appeared in the work of several authors (e.g. [Gu1], [Gu2], [W]). We consider the subseries of $2\theta$-divisors which are singular along the surface $C - C$, i.e.

$$\Gamma^{(2)}_0 = \{ D \in \Gamma_0 \mid \text{mult}_{p-q}(D) \geq 2 \forall p, q \in C \}. $$ (1.4)

By ([L3]) the morphism given by the linear series $|2\Theta|$ maps $\text{Pic}^{g-1}(C)$ to $|2\Theta_0|$. Let $\langle \text{Sing} \Theta \rangle \subset H^0(JC, \mathcal{O}(2\Theta_0))$ denote the linear span of the image of $\text{Sing} \Theta$ under this morphism. We prove:
1.1. **Theorem.** For any non-hyperelliptic curve

(1) there exists a canonical isomorphism

\[ \Gamma_{00}/\Gamma_{00}^{(2)} \xrightarrow{\sim} \Lambda^3 H^0(K) \]

(2) we have an equality

\[ \langle \text{Sing} \Theta \rangle = \Gamma_{00}^{(2)}. \]

The method used in the proof of this theorem (section 4) has been developed in a recent paper by van Geemen and Izadi [vGI] and the key point are the incidence relations (section 2.4) between two families of stable rank 2 vector bundles with fixed trivial (resp. canonical) determinant. One of these families of bundles is related to the gradient of the $2\theta$ functions along the surface $C-C$, the other family is the Brill-Noether locus $W(3)$ of stable bundles having at least three independent sections. We also need (section 3) some relations between vectors of second order theta functions, which one derives from Fay’s trisecant formula and its generalizations [Gu1]. Finally (section 2.5) we describe the relationship between these bundles and the objects discussed in [vGI], which are related to the embedded tangent space at the origin to the moduli of stable rank 2 bundles.

A natural question to ask is whether the equivalence (1.2) admits generalizations. We denote by $C_2$ the second symmetric product of the curve. We introduce the fourfold $C_2 - C_2$, defined to be the image of the difference map $\phi_2 : C_2 \times C_2 \to JC$, which maps a 4-tuple $(p+q, r+s)$ to the line bundle $O(p+q-r-s)$. With this notation we define the subseries

\[ \Gamma_{11} = \{ D \in \Gamma_{00} \mid C_2 - C_2 \subset D \} \]  
\[ \Gamma_{000} = \{ D \in \Gamma_{00} \mid \text{mult}_0(D) \geq 6 \} \]

and we prove the following theorems (section 5 and 6). Let $I(2)$ (resp. $I(4)$) be the space of quadrics (resp. quartics) in canonical space $|K|^*$ containing the canonical curve.

1.2. **Theorem.** For any non-trigonal curve, there exists a canonical isomorphism

\[ \Gamma_{00}^{(2)}/\Gamma_{11} \xrightarrow{\sim} \text{Sym}^2 I(2). \]

1.3. **Theorem.** For any non-hyperelliptic curve, we have the following inclusions

\[ \Gamma_{11} \subset \Gamma_{000} \subset \Gamma_{00}^{(2)}. \]

The quotient of the first two spaces is isomorphic to the space of quadratic syzygies among quadrics in $I(2)$, i.e.

\[ \Gamma_{000}/\Gamma_{11} \cong \ker m : \text{Sym}^2 I(2) \to I(4) \]

where $m$ is the multiplication map.

The deepest statement in Theorem 1.2 is the surjectivity of the map in (1.7). The proof uses essentially two ideas: first, we can give an explicit basis of quadrics in $I(2)$ of rank less than or equal to 6 (Petri’s quadrics, section 5.1) and secondly, we can construct out of such a quadric a rank 2 vector bundle having at least four independent sections. This construction [BV] is recalled in section 5.2. Thus the Brill-Noether locus $W(4)$ appears in a natural way.
The proof of Theorem 1.3 is more in the spirit of Gunning’s previous work and uses only linear relations between vectors of second order theta functions. Except for a few cases (section 6.2) we are unable to deduce the dimension of \( \Gamma_{000} \) from \([L3]\).

In section 7 we give the version of Theorem 1.2 for trigonal curves.

It turns out that the vector bundle constructions used in the proofs of Theorems 1.1 and 1.2 can be seen as examples of a global construction (section 8).

Acknowledgements. The authors are grateful to J. Fay, W.M. Oxbury and S. Verra for various helpful comments. We especially thank B. van Geemen for many fruitful discussions and his constant interest in this work.

Notation.

- If \( X \) is a vector space or a vector bundle, by \( X^* \) we denote its dual.
- \( K \) is the canonical divisor of the curve \( C \).
- For a vector bundle \( E \) over \( C \), \( H^i(C, E) \) is often abbreviated by \( H^i(E) \) and \( h^i(E) = \dim H^i(C, E) \).
- \( C_n \) is the \( n \)-th symmetric product of the curve \( C \).
- \( W^r_d(C) \) is the subvariety of \( \text{Pic}^d(C) \) consisting of line bundles \( L \) such that \( h^0(L) > r \).
- The canonical curve \( C_{\text{can}} \) is the image of the embedding \( \varphi_K : C \rightarrow |K|^* \).
- The vector space \( I(n) \) is the space of degree \( n \) hypersurfaces in \( |K|^* \) containing \( C_{\text{can}} \).

2 Rank 2 vector bundles

In this section we construct two families of stable rank 2 bundles over \( C \) and describe some of their properties.

2.1 Preliminaries on extension spaces

Let \( SU_C(2, \mathcal{O}) \) (resp. \( SU_C(2, K) \)) be the moduli space of rank 2 vector bundles over \( C \) with fixed trivial (resp. canonical) determinant. It can be shown \([31]\) that the Kummer maps given by the linear system \( |2\Theta_0| \) over \( JC \) (resp. \( |2\Theta| \) over \( \text{Pic}^{g-1}(C) \)) can be factorized through the moduli space \( SU_C(2, \mathcal{O}) \) (resp. \( SU_C(2, K) \)). This gives the two following commutative diagrams

\[
\begin{array}{ccc}
JC & \xrightarrow{\text{Kum}} & |2\Theta_0|^* \\
\downarrow i & & \downarrow w \\
SU_C(2, \mathcal{O}) & \xrightarrow{D} & |2\Theta|
\end{array}
\quad \begin{array}{ccc}
\text{Pic}^{g-1}(C) & \xrightarrow{\text{Kum}} & |2\Theta|^* \\
\downarrow i & & \downarrow w \\
SU_C(2, K) & \xrightarrow{D} & |2\Theta_0|
\end{array}
\] (2.1)

The vertical morphisms \( i \) map \( JC \) (resp. \( \text{Pic}^{g-1} \)) to the semi-stable boundary of the moduli space \( SU_C(2, \mathcal{O}) \) (resp. \( SU_C(2, K) \)) by sending a line bundle \( \xi \) to the split bundle \( \xi \oplus \xi^{-1} \) (resp. \( \xi \oplus K\xi^{-1} \)). The rightmost morphism \( D \) associates to a semi-stable rank 2 bundle \( E \) with canonical determinant a divisor \( D(E) \), whose support is given by

\[
D(E) = \{ \xi \in JC \mid h^0(E \otimes \xi) > 0 \}
\] (2.2)
The definition of $D(E)$ for $E$ with trivial determinant is obtained by substituting $JC$ by $\text{Pic}^{g-1}(C)$. The composite map $w \circ \text{Kum} = D \circ i$ in the rightmost diagram (2.1) is the translation morphism

$$
\iota : \text{Pic}^{g-1}(C) \longrightarrow |2\Theta_0|
$$

$$
\xi \mapsto \Theta_\xi + \Theta_{K\xi^{-1}}
$$

where the divisor $\Theta_\xi \subset JC$ is obtained by translating the Riemann theta divisor $\Theta$ by $\xi$. Dually, we get $\iota : JC \longrightarrow |2\Theta|$ by substituting $\Theta$ by $\Theta_0$. Note that the symmetric theta divisor $\Theta_0$ depends on the choice of a theta characteristic.

Given a subspace $V \subset H^0(\text{Pic}^{g-1}(C), 2\Theta)$, we denote by $V^\perp \subset H^0(JC, 2\Theta_0)$ the image under the Wirtinger duality (1.1) of its annihilator.

### 2.1.1 degree 1

Given an $x \in \text{Pic}^1(C)$, let $\mathbb{P}(x) = |Kx^2|^* = \mathbb{P}H^1(C, x^{-2})$. This $g$-dimensional projective space parametrizes isomorphism classes of extensions

$$
0 \longrightarrow x^{-1} \longrightarrow E \longrightarrow x \longrightarrow 0
$$

and the classifying map $\psi : \mathbb{P}(x) \to SU_C(2, \mathcal{O}) \to |2\Theta|$ is linear and injective (lemme 3.6 [B2]). The following lemma (Prop. 1.2 [OPP]) describes the incidence relations between the extension spaces $\mathbb{P}(x)$.

#### 2.1. Lemma

Let $x, y \in \text{Pic}^1(C)$. If $x \otimes y = \mathcal{O}(p + q)$, the intersection $\mathbb{P}(x) \cap \mathbb{P}(y)$ is the secant line $\overline{pq}$ of the curve in either $\mathbb{P}(x)$ or $\mathbb{P}(y)$.

The linear system $|Kx^2|$ can be used to map the curve into $\mathbb{P}(x)$

$$
\varphi = \varphi_{Kx^2} : C \longrightarrow \mathbb{P}(x).
$$

A point in $\mathbb{P}(x)$ represents a stable bundle precisely away from $C_x := \varphi(C)$, while the image of a point $p \in C$ represents the equivalence class of the semi-stable bundle $x(-p) \oplus x^{-1}(p)$. The Abel-Jacobi map $t_x : C \to JC$ defined by $p \mapsto x(-p)$ fits in the following commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{t_x} & JC \\
\downarrow \varphi & & \downarrow i \\
\mathbb{P}(x) & \xrightarrow{\psi} & SU_C(2, \mathcal{O})
\end{array}
$$

### 2.1.2 degree 2

Given an $x \in \text{Pic}^2(C)$, we define similarly $\mathbb{P}(x) = |Kx^2|^* = \mathbb{P}H^1(C, x^{-2})$. This $(g+2)$-dimensional projective space parametrizes as in section 2.1.1 isomorphism classes of extensions (2.3) and the classifying map $\psi : \mathbb{P}(x) \to SU_C(2, \mathcal{O}) \to |2\Theta|$ is given by the complete system of quadrics [Ber] through the embedded curve $\varphi_{Kx^2} : C \hookrightarrow \mathbb{P}(x)$. Thus this map is defined away from the curve $C$. We will use the fact that a secant line $\overline{pq}$ to the curve $C \subset \mathbb{P}(x)$ is contracted by the linear system of quadrics to a point represented by the split bundle $x(-p - q) \oplus x^{-1}(p + q)$.
2.2 The bundles $E(p, q, r)$

We will associate to any three distinct points $p, q, r$ on the curve $C$ a unique stable rank 2 bundle $E(p, q, r)$, which we identify with its image in $|2\Theta|$. Consider the line bundle

$$x = \mathcal{O}(p + q - r) \in \text{Pic}^1(C)$$

and denote by $T_p(C_x)$ the embedded tangent line to the curve $C_x \subset \mathbb{P}(x)$ at the point $p \in C$. Note that the image $C_x \subset \mathbb{P}(x)$ is smooth, since $h^0(x^2) = 0$ (Lemma 5.2 [LN]). Recall that $\mathbb{P}(\mathcal{T}_0) \subset |2\Theta|$ denotes the linear span of the surface $C - C \subset JC$.

We say that three points $p, q, r$ are collinear if they are collinear as points on the canonical curve $C_{can}$.

2.2. Lemma. $T_p(C_x) \subset \mathbb{P}(\mathcal{T}_0) \iff p, q, r$ are collinear

Proof. Note that the image $t_x(p) = \mathcal{O}(q - r) \in C - C \subset JC$. We identify the projectivized tangent space to $JC$ at the point $\varphi(p)$ with the canonical space $|K|^*$. Under this identification the projectivized tangent space to $C - C$ at $q - r$ corresponds to the secant line $\overline{pq} \subset |K|^*$ and the projectivized tangent line to the curve $t_x(C) \subset JC$ at the point $q - r$ corresponds to $p \in |K|^*$. If $p \in \overline{pq}$, then $T_p(C_x) \subset T_{q-r}(C - C) \subset \mathbb{P}(\mathcal{T}_0)$.

Conversely, by [LN] Théorème A, we know that the projectivized tangent spaces to divisors in $\Gamma_0 = \mathcal{T}_0^+$ at the point $q - r$ cut out the secant line $\overline{pq} \subset |K|^*$. Therefore, $T_p(C_x) \subset \mathbb{P}(\mathcal{T}_0)$ implies that $p \in \overline{pq}$. \hfill $\Box$

We now introduce the line bundles

$$y = \mathcal{O}(p + r - q) \quad \quad z = \mathcal{O}(q + r - p)$$

2.3. Proposition. For any non-collinear points $p, q, r$, there exists a unique stable bundle $E := E(p, q, r) \in SU_C(2, \mathcal{O})$ containing the three line bundles $x^{-1}, y^{-1}, z^{-1}$. Moreover, $E \notin \mathbb{P}(\mathcal{T}_0)$.

Proof. By [LN] Prop. 1.1, the extension classes lying on the tangent line $T_p(C_x) \subset \mathbb{P}(x)$ correspond to stable rank 2 bundles containing $x(-2p) = y^{-1}$. By Riemann-Roch

$$h^0(Kx^2(-2p - 2q)) = h^0(K(-2r)) = g - 2$$

hence the two tangent lines $T_p(C_x)$ and $T_q(C_x)$ are contained in a projective plane and are distinct. Their intersection point $T_p(C_x) \cap T_q(C_x)$ determines a stable bundle, which we denote by $E(p, q, r)$ and which contains, by construction, $x^{-1}, y^{-1}$ and $z^{-1}$. We now observe that

$$x \otimes y = \mathcal{O}(2p)$$

hence, by Lemma 2.1, $T_p(C_x) = T_p(C_y)$. Similarly $T_q(C_x) = T_q(C_z)$ and $T_r(C_z) = T_r(C_y)$, from which it is clear that the three tangent lines intersect in a single point, hence the construction does not depend on the order of the three points. Finally Lemma 2.2 implies that $E \notin \mathbb{P}(\mathcal{T}_0)$. \hfill $\Box$

2.4. Proposition. If $a \in \text{Sing } \Theta$, then $h^0(E(p, q, r) \otimes a) > 0$. 

Proof. We write \( E = E(p, q, r) \) and take \( x \) as in (2.4). We tensor the exact sequence (2.3) with \( a \) and take global sections

\[
0 \longrightarrow H^0(x^{-1} \otimes a) \longrightarrow H^0(E \otimes a) \longrightarrow H^0(x \otimes a) \longrightarrow H^1(x^{-1} \otimes a)
\]

If \( h^0(x^{-1} \otimes a) > 0 \), we are done. So we can assume that \( h^0(x^{-1} \otimes a) = 0 \). By Serre duality, we have \( h^0(E \otimes a') = h^0(E \otimes a) \) with \( a' = K \otimes a^{-1} \), so we can also assume that \( h^0(x^{-1} \otimes a') = 0 \). But this is equivalent to \( h^0(x \otimes a) = h^1(x^{-1} \otimes a) = 1 \). Finally we see that \( h^0(E \otimes a) > 0 \) if and only if the coboundary map \( \delta \) is zero, i.e. the one-dimensional image of the multiplication map

\[
H^0(x \otimes a) \otimes H^0(x \otimes a') \longrightarrow H^0(Kx^2)
\]

is contained in the hyperplane of \( H^0(Kx^2) \) corresponding to the bundle \( E \) and which is, by definition, the linear span of the two subspaces \( H^0(Kx^2(-2p)) \) and \( H^0(Kx^2(-2q)) \). We observe now that \( H^0(a(-r)) \subset H^0(x \otimes a) \) and for dimensional reasons these two spaces must coincide, hence the unique global section of \( x \otimes a \) vanishes at \( p \) and \( q \). Since the same holds for \( x \otimes a' \), we are done. \( \square \)

2.3 The bundles \( E_W \)

In this section we recall some constructions from \( [vG1] \). Let \( Gr(3, H^0(K)) \) be the Grassmannian of 3-planes in \( H^0(K) \) and \( W(3) \) the locus of stable bundles \( E \) in \( SU_C(2, K) \) that are generated by global sections and for which \( h^0(E) = 3 \). We will define a rational map

\[
Gr(3, H^0(K)) \overset{\beta}{\longrightarrow} W(3).
\]

For a generic 3-plane \( W \subset H^0(K) \), it can be shown that the multiplication map

\[
W \otimes H^0(K) \rightarrow H^0(K^2)
\]

is surjective. \( (2.6) \)

Then the dual \( E_W^* \) of the bundle \( \beta(W) = E_W \) is defined by the exact sequence

\[
0 \longrightarrow E_W^* \longrightarrow W \otimes O_C \overset{ev}{\longrightarrow} K \longrightarrow 0
\]

(2.7)

where the last map is the evaluation map of global sections. Then condition (2.6) implies that \( E_W \in W(3) \).

Conversely, we can associate to a bundle \( E \in W(3) \) a 3-plane \( W_E \), i.e. we have an inverse map

\[
W(3) \overset{\alpha}{\longrightarrow} Gr(3, H^0(K)).
\]

To define \( \alpha(E) = W_E \), we consider the exact sequence

\[
0 \longrightarrow K^{-1} \overset{i}{\longrightarrow} H^0(E) \otimes O_C \overset{ev}{\longrightarrow} E \longrightarrow 0
\]

then the dualized exact sequence induces an injective linear map on global sections

\[
H^0(i^*) : H^0(E)^* \longrightarrow H^0(K)
\]

and we let \( W_E = \text{im} H^0(i^*) \).

Moreover, under the natural duality \( H^0(E)^* \cong \bigwedge^2 H^0(E) \), \( W_E \) coincides with the image of the exterior product map \( \bigwedge^2 H^0(E) \rightarrow H^0(K) \).

2.5. Remark. Since \( E_W \) is generated by global sections we can define a map

\[
C \rightarrow \mathbb{P} H^0(E_W) = \mathbb{P}^2
\]

\[
p \mapsto s_p := \ker (H^0(E_W) \overset{ev}{\longrightarrow} E_{W|p})
\]

which associates to a point \( p \) the unique section \( s_p \) of \( E_W \) vanishing at \( p \). This map coincides with the canonical map \( \varphi_K : C \leftrightarrow |K|^* \) followed by the projection with center \( \mathbb{P}W^\perp \subset |K|^* \).
2.4 Incidence relations

In the previous sections we have constructed two families of bundles i.e. \(E(p,q,r)\) (section 2.2) and \(E_W\) (section 2.3); each gives \(\theta\)-divisors under the \(D\)-maps in \(|2\Theta|\) and \(|2\Theta_0|\). These spaces are dual to each other (1.1) and it is therefore useful to determine their incidence relations. We will denote by \(H_W \in |2\Theta|^*\) the hyperplane in \(|2\Theta|\) corresponding under (1.1) to \(D(E_W) \in |2\Theta_0|\) and by \(p \wedge q \wedge r \in \Lambda^3H^0(K)^*\) (resp. \(\Lambda^3W \in \Lambda^3H^0(K)\)) the Plücker image of the 3-plane in \(H^0(K)^*\) spanned by the 3 points \(p,q,r \in C_{\text{can}}\) (resp. the 3-plane \(W\) in \(H^0(K)\)).

2.6. Proposition. We have the following equivalence

\[ E(p,q,r) \in H_W \iff (\Lambda^3W, p \wedge q \wedge r) = 0. \]

Proof. We write \(E = E(p,q,r)\) and take \(x\) as in (2.4). We note that the first condition \(E \in H_W\) is equivalent to \(h^0(E_W \otimes E) > 0\) [32]. Using remark 2.5, we see that the second condition is equivalent to the three sections \(s_p, s_q, s_r \in \mathbb{P}H^0(E_W) = \mathbb{P}^2\) being collinear.

We tensor the exact sequence (2.3) with \(E_W\) and take global sections

\[ 0 \longrightarrow H^0(E_W \otimes x^{-1}) \longrightarrow H^0(E_W \otimes E) \longrightarrow H^0(E_W \otimes x) \overset{\delta(\epsilon)}{\longrightarrow} H^1(E_W \otimes x^{-1}) \quad (2.8) \]

Let us first consider the case when \(h^0(E_W \otimes x^{-1}) > 0\). Then \(h^0(E_W \otimes E) > 0\), and, by definition of the bundle \(E_W\) (2.7), we have

\[ H^0(E_W \otimes x^{-1}) = \ker (W \otimes H^0(Kx^{-1})) \overset{ev}{\longrightarrow} H^0(K^2x^{-1}) \quad (2.9) \]

But \(H^0(Kx) = H^0(K(-p-q))\). Furthermore, if \(\dim W \cap H^0(K(-p-q)) = 1\), then the kernel of the map (2.9) is zero. Hence \(\dim W \cap H^0(K(-p-q)) \geq 2\), which implies that \(\dim W \cap H^0(K(-p-q-r)) \geq 1\) and we are done.

Now we can assume that \(h^0(E_W \otimes x^{-1}) = 0\), or equivalently \(h^0(E_W \otimes x) = 2\). Next we observe that the coboundary map \(\delta(\epsilon)\) (2.8), which depends on the extension class \(\epsilon \in \mathbb{P}(x)\) of the bundle \(E\), is skew-symmetric (here we identify \(H^1(E_W \otimes x^{-1}) = H^0(E_W \otimes x)^*\)). Therefore, if \(\{s, t\}\) is a basis of \(H^0(E_W \otimes x)\), we have

\[ h^0(E_W \otimes E) > 0 \iff \det (\delta(\epsilon)) = 0 \iff s \wedge t \in H^0(Kx^2) \text{ vanishes doubly at } p \text{ and } q. \quad (2.10) \]

Let \(\sigma\) be a section of the line bundle \(O(p+q)\). The 4-dimensional space \(H^0(E_W(p+q))\) contains \(H^0(E_W)\) and \(H^0(E_W \otimes x)\), which intersect in the section \(s_r \otimes \sigma\). We can choose \(s = s_r \otimes \sigma\) and \(t \notin H^0(E_W)\). Then \(t \wedge s_p\) vanishes at \(p\) and \(r\), hence \(t \wedge s_p \in H^0(K(q-r)) = H^0(K(-r))\), so \(t \wedge s_p\) also vanishes at \(q\). Similarly, \(t \wedge s_q\) vanishes at \(p\). The condition (2.10) says that \(t \wedge s_r\) vanishes at \(p\) and \(q\). Since \(t \notin H^0(E_W)\) we can assume e.g. that \(t(q) \neq 0\), but then \(s_p(q) \wedge s_r(q) = 0\), which implies that the three sections \(s_p, s_q, s_r\) cannot be linearly independent (otherwise they would generate \(E_W\) at \(q\)).

2.7. Remark. The incidence relations were first proved in [CG] for slightly different objects (see section 2.5). Working with the bundles \(E(p,q,r)\) instead of the projective spaces \(\mathbb{P}^4_{p,q,r}\) simplifies the proof somewhat.
2.5 Other descriptions

Consider the projection $P$ with center $\mathbb{P}(\mathbb{T}_0)$

$$P : |2\Theta| \longrightarrow \mathbb{P}(H^0(2\Theta)/\mathbb{T}_0) = |2\Theta|_{|\mathbb{T}_0|} \quad (2.11)$$

In section 2.2 we have associated to any triple of non-collinear points $p, q, r$ a point in $|2\Theta|_{|\mathbb{T}_0|}$, namely $P \circ D(E(p, q, r))$. In [vGI] the authors associate to the same data a 4-dimensional projective space $\mathbb{P}^4_{p,q,r} \subset |2\Theta|$. Their construction goes as follows:

We take $\zeta = \mathcal{O}(p + q) \in \text{Pic}^2(C)$ and consider in $\mathbb{P}(\zeta)$ (see 2.1.2) the 3-dimensional subspace $\langle 2p + 2q + r \rangle$ spanned by the tangent lines at $p$ and $q$ to $C \hookrightarrow \mathbb{P}(\zeta)$ and the point $r$. Then the restricted linear system of quadrics through $C$ determines a rational map

$$\psi : \langle 2p + 2q + r \rangle \longrightarrow |2\Theta|.$$ 

The image of $\psi$ is a cubic threefold in $\mathbb{P}^4_{p,q,r} \subset |2\Theta|$, singular at $D(\mathcal{O} \oplus \mathcal{O})$. Furthermore if $p, q, r$ are non-collinear, $\dim \mathbb{P}(\mathbb{T}_0) \cap \mathbb{P}^4_{p,q,r} = 3$, hence the projective space $\mathbb{P}^4_{p,q,r}$ is contracted by the projection $P$ to a point.

2.8. Lemma. For any non-collinear points $p, q, r$

$$P \circ D(E(p, q, r)) = P(\mathbb{P}^4_{p,q,r}) \in |2\Theta|_{|\mathbb{T}_0|}$$

Proof. We need to show that there exists an extension class $\epsilon \in \langle 2p + 2q + r \rangle \subset \mathbb{P}(\zeta)$ which corresponds to the bundle $E(p, q, r) \mod \mathbb{P}(\mathbb{T}_0)$. We recall that $E(p, q, r)$ has been characterized in Prop. 2.3. and we note ([LX] Prop. 1.1) that a bundle $E(\epsilon)$ fitting in the sequence

$$0 \longrightarrow \zeta^{-1} \longrightarrow E(\epsilon) \longrightarrow \zeta \longrightarrow 0$$

contains $\zeta(-2q-r) = z^{-1}$ if and only if $\epsilon \in \langle 2p + 2q + r \rangle$. Similarly $E(\epsilon)$ contains $\zeta(-2p-r) = y^{-1}$ if and only if $\epsilon \in \langle 2p + r \rangle$. Therefore we may take $\epsilon \in \langle 2q + r \rangle \cap \langle 2p + r \rangle \neq \emptyset$, so that $E(\epsilon)$ contains $y^{-1}$ and $z^{-1}$. Such extension classes $\epsilon$ are parametrized by $\mathbb{P}(y) \cap \mathbb{P}(z)$, which is the tangent line $T_r(C_z) = T_r(C_y)$ at the point $\mathcal{O}(p - q) \in \mathbb{P}(\mathbb{T}_0)$, hence $P(E(\epsilon)) = P(E(p, q, r))$ for any such $\epsilon$, and we are done.

\qed

3 Gunning’s results on second order theta functions

In this section we recall some classical theory of theta functions seen as holomorphic quasi-periodic functions on $\mathbb{C}^g$, as well as some results by Gunning on the gradient and Hessian of $2\theta$-functions along the surface $C - C$. We refer to Fay’s book [F] and to [Gu1] for a detailed exposition.

Let $C$ be the universal covering space of the curve $C$. We choose a base point $z_0 \in C$ and a canonical set of generators of $H_1(C, \mathbb{Z})$ and call the corresponding canonical basis of Abelian differentials $\omega_1, \ldots, \omega_g \in H^0(C, K)$; these can be thought of as holomorphic differential 1-forms on $C$ invariant under the group $\Gamma$ of covering transformations acting on $C$. We construct from these data the period matrix $\Omega$. The associated Abelian integrals

$$w_j(z) = \int_{z_0}^z \omega_j$$

8
are holomorphic functions on \( C \) and are the coordinate functions for a map \( w : C \rightarrow \mathbb{C}^g \). Let \( \Lambda \) be the lattice in \( \mathbb{C}^g \) defined by the period matrix \( \Omega \), then \( JC = \mathbb{C}^g/\Lambda \) and we have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{w} & \mathbb{C}^g \\
\downarrow \pi & & \downarrow \\
\mathcal{C}/\Gamma = C & \xrightarrow{t_x} & JC
\end{array}
\]
The horizontal map \( t_x \) is the Abel-Jacobi map (see 2.1.1) with \( x = O(p) \) and \( p = \pi(z_0) \). Both vertical arrows are quotient maps of the group actions of \( \Gamma \) (acting on \( \mathcal{C} \)) and \( \Lambda \) (acting on \( \mathbb{C}^g \)). Sections of the line bundle \( O(2\Theta_0) \) over \( JC \) correspond to the classical second-order theta functions. A basis of the space \( H^0(JC, O(2\Theta_0)) \) is given by the holomorphic functions on \( \mathbb{C}^g \)
\[
\theta_2 [\nu]_0 (w, \Omega) = \theta [\nu]_0 (2w, 2\Omega) \text{ for } \nu \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g
\]
where the right-hand side is obtained from the first-order theta function with characteristics \([\nu]_0\)
\[
\theta [\nu]_0 (w, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left[ \frac{1}{2} (n + \nu) \Omega (n + \nu) + (n + \nu) w \right]
\]
The functions (3.1) are the coordinate functions of a holomorphic map
\[
\theta_2 : \mathbb{C}^g \rightarrow \mathbb{C}^{2g}
\]
\[
w \mapsto (\ldots, \theta_2 [\nu]_0 (w, \Omega), \ldots)
\]
Using this basis, we identify (via the Wirtinger duality \((1.1)\)) \( \mathbb{P}(\mathbb{C}^{2g}) = |2\Theta| \), so that the map (3.2) coincides with the Kummer map \( JC \rightarrow |2\Theta| \).

Next we introduce the prime form (see [Gu]) formula (22)) \( q(z_1, z_2) \), which is a holomorphic function on \( \mathcal{C} \times \mathcal{C} \) with a simple zero along the subvariety \( z_1 = T z_2 \) for all covering transformations \( T \) and vanishing nowhere else. Moreover, \( q(z_1, z_2) = -q(z_2, z_1) \). Alternatively the function \( q \) is, up to a multiplicative constant, the pull-back to \( \mathcal{C} \times \mathcal{C} \) of the unique global section having as zero scheme the diagonal in \( \mathcal{C} \times \mathcal{C} \).

We use canonical coordinates on the universal covering \( \mathcal{C} \) (see section 6 [Gu]) and we denote by \( w_j \) the derivative of the holomorphic function \( w \) with respect to the canonical coordinates. To a point \( a \in \mathcal{C} \) we associate the differential operator
\[
D_a = \sum_{j=1}^{g} w_j'(a) \frac{\partial}{\partial w_j}.
\]
This operator corresponds, up to multiplication by a scalar, to the unique translation-invariant vector field over \( JC \), which has as tangent vector at the origin \( O \in JC \) the tangent direction at \( O \) to the curve \( C \), where \( C \) is embedded in \( JC \) by \( q \mapsto O(q - p) \) with \( p = \pi(a) \in C \).

Gunning (see [Gu] formulae (41), (42), (44)) introduces for \( a_1, a_2, a_3, a_4 \in \mathcal{C} \) the following vectors in \( H^0(2\Theta)/\mathbb{T}_0 = \mathbb{C}^{2g}/\mathbb{T}_0 \)
\[
\xi(a_1, a_2, a_3) = q(a_2, a_3)^{-2} PD_{a_1} \theta_2(w(a_2 - a_3))
\]
\[
\sigma(a_1; a_2, a_3; a_4) = \left[ \frac{\partial}{\partial a_1} \log \frac{q(a_1, a_2)}{q(a_1, a_3)} \right] \cdot \xi(a_2, a_3, a_4)
\]
\[ \tau(a_1, a_2; a_3, a_4) = q(a_3, a_4) - 2PD_{a_1}D_{a_2}\theta_2(w(a_3 - a_4)) \]  

Then \( \xi \) (resp. \( \tau \)) defines a holomorphic function on \( C^3 \) (resp. \( C^4 \)) with values in \( \mathbb{C}^{2\theta}/\mathbb{T}_0 \) and \( \sigma \) defines a meromorphic function on \( C^4 \) which has as singularities at most simple poles along the loci \( a_1 = Ta_2 \) and \( a_1 = Ta_3 \) for all covering transformations \( T \in \Gamma \). The functions \( \xi, \sigma, \tau \) have the following symmetry properties: \( \xi \) is skew-symmetric in \( a_1, a_2, a_3 \) (Cor. 4 [Gu1]), \( \sigma \) is symmetric in \( a_2, a_3 \) and \( \tau \) is symmetric in \( a_1, a_2 \) and in \( a_3, a_4 \). Given four points \( a_1, a_2, a_3, a_4 \in \mathcal{C} \) we will let \( \pi_{ijkl} = \sigma(a_i; a_j, a_k; a_l), \quad \tau_{ijkl} = \tau(a_i, a_j; a_k, a_l) \) and \( q_{ij} = q(a_i, a_j) \) where \( i, j, k, l \) are four indices such that \( \{i, j, k, l\} = \{1, 2, 3, 4\} \). Then the following statements hold:

3.1. Theorem (Theorem 3 [Gu1]). There exist vectors \( \xi_{jkl} \in \mathbb{C}^{2\theta}/\mathbb{T}_0 \) which are skew-symmetric in their indices \( j, k, l \) and such that

\[ \xi(a_1, a_2, a_3) = \sum_{j,k,l} \xi_{jkl} w'_j(a_1) w'_k(a_2) w'_l(a_3). \]

3.2. Corollary. The dimension of the linear span of the vectors \( \xi(a_1, a_2, a_3) \) for \( a_i \) varying in \( \mathcal{C} \) is at most \( \frac{5}{3} \).

3.3. Proposition (Cor. 5 [Gu1]). For any points \( a_1, a_2, a_3, a_4 \in \mathcal{C} \)

\[ \begin{align*}
(1) & \quad \frac{1}{2}P D_{a_1}D_{a_2}D_{a_3}D_{a_4}\theta_2(0) = \tau_{1234} + \tau_{1324} + \tau_{2314} - 2\sigma_{1234} - 2\sigma_{2134} - 2\sigma_{3124} \\
(2) & \quad \frac{1}{2}\tau_{1324} + \frac{1}{2}\tau_{1423} = \left( \frac{q_{1234}}{q_{1324}q_{2314}q_{1423}} \right)^2 P\theta_2(w(a_1 + a_2 - a_3 - a_4)) + \sigma_{1342} - \sigma_{3241} - \sigma_{4231}
\end{align*} \]

4 Proof of theorem 1.1

We consider the rational map

\[ \rho : C_3 \to |2\Theta|_{\mathbb{T}_0} \]

\[ (p, q, r) \mapsto P \circ D(E(p, q, r)) \]

By Lemma 2.2 the rational map \( \rho \) is defined away from the triples of collinear points. In particular, \( \rho \) is a morphism if \( C \) is not trigonal. Let \( \mathbb{P}(\mathbb{T}) \) be the inverse image under the projection \( P \) (2.11) of the linear span of the image of \( \rho \). Obviously \( \mathbb{T}_0 \subset \mathbb{T} \). Since the bundle \( E(p, q, r) \in T_p(C_x) \), the tangent line at the point \( O(q - r) \in \mathbb{P}(\mathbb{T}_0) \) to the curve \( C_x \), we see that a hyperplane in \( |2\Theta| \) containing \( \mathbb{P}(\mathbb{T}_0) \) and the point \( D(E(p, q, r)) \) determines via (1.1) a \( 2\theta \)-function \( f \) such that \( D_pf(q - r) = 0 \). Hence, varying the points \( p, q, r \), we obtain the equality \( T^\perp = \Gamma^{(2)}_{\mathbb{T}_0} \).

The rest of the argument coincides with the argument given in [vG]. We therefore just sketch their proof: the rational map \( \rho \) factorizes as follows

\[
\begin{array}{ccc}
C_3 & \xrightarrow{\rho} & \mathbb{P}(\mathbb{T}/\mathbb{T}_0) \\
\downarrow \pi & & \uparrow \\
Gr(3, H^0(K)^*) & \xrightarrow{Pl} & \mathbb{P}(\Lambda^3 H^0(K)^*)
\end{array}
\]

where \( \pi(p, q, r) = p \wedge q \wedge r \) and \( Pl \) is the Plücker embedding. By the incidence relations (Prop. 2.6) the support of the pull-back \( \rho^*H_W \) equals the set

\[ \{(p, q, r) \in C_3 \mid \langle \Lambda^3 W, p \wedge q \wedge r \rangle = 0 \}. \]
More precisely, we have an equality $\rho^*H_W = (Pl \circ \pi)^*(\Lambda^3W)^+$ as divisors on $C_3$, hence the factorization and the injectivity of $\gamma$. Since $\dim (\mathbb{T}/\mathbb{T}_0) \leq \binom{g}{3}$ by Cor. 3.2,

$$\gamma : \Lambda^3H^0(K)^* \xrightarrow{\sim} \mathbb{T}/\mathbb{T}_0$$

is an isomorphism. Then we take duals and since $\mathbb{T}^\perp = \Gamma^{(2)}_{00}$ and $\mathbb{T}_0^\perp = \Gamma_{00}$ we obtain Thm. 1.1 (1).

To prove Thm. 1.1 (2), we note that we have an inclusion $\langle \text{Sing}\Theta \rangle \subset \Gamma^{(2)}_{00}$, since by Riemann’s singularity theorem, for $\xi \in \text{Sing}\Theta$

$$\text{mult}_{p-q}(\Theta_\xi + \Theta_{K\xi^{-1}}) \geq 2H^0(\xi(p-q)) \geq 2.$$

Since both spaces have the same dimension (for a computation of $\dim \langle \text{Sing}\Theta \rangle$ see [ACGH]), we obtain equality.

## 5 Quadrics on canonical space

### 5.1 Petri’s quadrics

We will denote by $\tilde{Q}$ the polarized form of a quadric $Q$ on canonical space $|K|^*$, i.e. $\tilde{Q}$ is the symmetric bilinear form such that $\tilde{Q}(v, v) = Q(v), \forall v \in H^0(K)$.

**5.1. Lemma.** We consider $g-2$ points in general position $p_1, \ldots, p_{g-2}$. If a quadric $Q \in I(2)$ is such that

$$\tilde{Q}(p_i, p_j) = 0 \quad \forall i, j \in \{1, \ldots, g-2\}$$

then $Q$ is identically zero.

**Proof.** By the general position theorem ([ACGH], p.109), we know that a general hyperplane $H \subset |K|^*$ meets $C$ in $2g-2$ points any $g-1$ of which are linearly independent. We consider such an $H$ and $g-1$ independent points $q_1, \ldots, q_{g-1}$ in $H$ and suppose that the $g-2$ points $p_1, \ldots, p_{g-2}$ are taken among the $g-1$ residual points of $H \cap C$, i.e.

$$H \cap C = \{p_1, \ldots, p_{g-2}, q_1, \ldots, q_{g-1}, q_g\}.$$

It is clear that a general $(g-2)$-tuple $(p_1, \ldots, p_{g-2})$ can be realized in this way.

Now suppose that $\tilde{Q}(p_i, p_j) = 0, \forall i, j \in \{1, \ldots, g-2\}$, i.e. $Q$ contains the linear subspace $\Pi$ spanned by the $p_i$’s. Since $\Pi$ is a hyperplane in $H$, any line $\overline{q_iq_j}$, for $1 \leq i < j \leq g$ intersects $\Pi$. So $\overline{q_iq_j}$ is entirely contained in $Q$, since it meets $Q$ in at least three points. Hence we obtain that $\tilde{Q}(q_i, q_j) = 0, \forall i, j \in \{1, \ldots, g\}$, i.e. $Q$ contains the hyperplane $H$. But since $Q$ contains $C$, it cannot be the union of two hyperplanes, hence $Q$ is identically zero. \(\square\)

From now on we fix $g-2$ points $p_1, \ldots, p_{g-2}$, which are chosen in general position. By the preceding Lemma 5.1, the $\binom{g-2}{2}$ hyperplanes (for $1 \leq i < j \leq g-2$

$$\mathcal{H}_{ij} = \{Q \in I(2) \mid \tilde{Q}(p_i, p_j) = 0\}$$

are linearly independent in $I(2)^*$, hence they form a basis of $I(2)^*$. Let us denote by $\{Q_{ij}\}$ the corresponding dual basis. The quadrics $Q_{ij} \in I(2)$ are characterized by the properties

$$\tilde{Q}_{ij}(p_\alpha, p_\beta) = 0 \text{ if } \{i, j\} \neq \{\alpha, \beta\},$$

$$\tilde{Q}_{ij}(p_i, p_j) \neq 0.$$  \hspace{1cm} (5.1)
This basis of quadrics has been used by K. Petri ([P], see also [ACGH] p.123-135) in his work on the syzygies of the canonical curve. He defines them in a slightly different way:

Choose two additional points \( p_{g-1}, p_g \) in general position. For each \( i, 1 \leq i \leq g \), pick a generator \( \omega_i \) of the one-dimensional space

\[
H^0(K(- \sum_{j=1, j\neq i}^g p_j)) = \mathbb{C} \omega_i \tag{5.2}
\]

Up to a constant the \( \omega_i \)'s form a dual basis to the points \( p_i \in |K|^* \). Then there are constants ([ACGH] p.130) \( \lambda_{sij}, \mu_{sij}, b_{ij} \in \mathbb{C} \) such that, if we let

\[
\eta_{ij} = \sum_{s=1}^{g-2} \lambda_{sij} \omega_s \quad \nu_{ij} = \sum_{s=1}^{g-2} \mu_{sij} \omega_s \tag{5.3}
\]

the quadratic polynomials

\[
R_{ij} = \omega_i \omega_j - \eta_{ij} \omega_{g-1} - \nu_{ij} \omega_g - b_{ij} \omega_{g-1} \omega_g \tag{5.4}
\]

all vanish on \( C \). Moreover the \( R_{ij} \)'s form a basis of \( I(2) \) and, obviously, the rank of the quadric \( R_{ij} \) is less than or equal to 6.

5.2. Lemma. For each \( 1 \leq i < j \leq g - 2 \), the quadrics \( R_{ij} \) satisfy the conditions (5.1), hence \( R_{ij} = Q_{ij} \).

Proof. This follows immediately from the definition (5.2) of the \( \omega_i \)'s.

5.3. Proposition. If \( C \) is neither trigonal nor a smooth plane quintic and the points \( p_1, \ldots, p_{g-2} \) are in general position, then the quadrics \( Q_{ij} \) have the following properties

\[
(i) \quad \text{Sing} \ Q_{ij} \cap C = \emptyset \tag{5.5}
\]
\[
(ii) \quad \text{rk} \ Q_{ij} = 5 \text{ or } 6. \tag{5.6}
\]

Proof. We fix two indices \( i, j \). First we observe that the singular locus \( \text{Sing} \ Q_{ij} \) is the annihilator of the linear space

\[
\langle \omega_i, \omega_j, \omega_{g-1}, \omega_g, \eta_{ij}, \nu_{ij} \rangle \subset |K|. \tag{5.7}
\]

Hence \( C \cap \text{Sing} \ Q_{ij} \) is the base locus of this linear subsystem. In particular \( C \cap \text{Sing} \ Q_{ij} \) is contained in the base locus of \( \langle \omega_i, \omega_j, \omega_{g-1}, \omega_g \rangle \), which, by construction, consists of the \( g - 4 \) points (we delete the \( i \)-th and \( j \)-th point)

\[
p_1, \ldots, \hat{p}_i, \ldots, \hat{p}_j, \ldots, p_{g-2}. \tag{5.8}
\]

We will denote by \( D_{ij} \) the degree \( g - 4 \) divisor with support (5.8) and by \( \bar{D}_{ij} \) the linear span of \( D_{ij} \) in \( |K|^* \). Suppose now that there exists a \( (g - 2) \)-tuple \( p_1, \ldots, p_{g-2} \) such that (i) holds, then, since (i) is an open condition, (i) holds for a general \( (g - 2) \)-tuple of points. Therefore we will assume that (i) does not hold for all \( (g - 2) \)-tuples, i.e. \( \forall p_1, \ldots, p_{g-2} \) (in general position), there exists a \( p_\alpha \in \text{Sing} \ Q_{ij} \) for some \( \alpha \in \{1, \ldots, g - 2\}, \alpha \neq i, j \). But, since the quadric \( Q_{ij} \) does not depend on the order of the \( g - 4 \) points (5.8), \( p_\alpha \in \text{Sing} \ Q_{ij} \) implies that all \( g - 4 \) points (5.8) are in \( \text{Sing} \ Q_{ij} \). Hence

\[
\bar{D}_{ij} \subset \text{Sing} \ Q_{ij} \tag{5.9}
\]
and therefore $\text{rk} Q_{ij} \leq 4$. Since $\omega_i, \omega_j, \omega_{g-1}, \omega_g$ are linearly independent, we have $\text{rk} Q_{ij} = 4$. Hence the inclusion (5.9) is an equality (same dimension).

Consider now the two rulings of the rank 4 quadric $Q_{ij}$: they cut out on the curve two pencils of divisors

$$\mathbb{P}^1 \subset |L| \quad \mathbb{P}^1 \subset |M|$$

such that $L, M$ are line bundles satisfying

$$L \otimes M = K(\bar{D}_{ij}).$$

Therefore for general $D_{ij}$, we have constructed a pair $(L, M) \in W_1^d(C) \times W_1^d(C)$ satisfying (5.10) with $d + d' = \deg K(\bar{D}_{ij}) = g + 2$. By Mumford’s refinement of Martens’ Theorem (see [ACGH] p.192-3), if $C$ is neither trigonal, bi-elliptic, nor a smooth plane quintic, then $\dim W_1^d(C) \leq d - 4$ for $4 \leq d \leq g - 2$. Hence

$$\dim W_1^d(C) \times W_1^d(C) \leq (d - 4) + (d' - 4) = g - 6 < g - 4$$

which contradicts relation (5.10).

In order to prove (i) we need to show that the case $C$ bi-elliptic also leads to a contradiction. Let $\pi : C \rightarrow E$ be a degree 2 mapping onto an elliptic curve $E$. Then by [ACGH] p.269, exercise E1, the chords $\overline{pq}$ with $p + q = \pi^*e$ for some $e \in E$ all pass through a common point $a \notin C$. In particular $a$ lies on a chord through all points $p_\alpha \in \text{Sing} Q_{ij}$, hence $a \in Q_{ij}$. Since $C \subset |K|^*$ is non-degenerate, $a \in \text{Sing} Q_{ij}$ and for $p_1, \ldots, p_{g-2}$ in general position

$$a \notin \bar{D}_{ij}$$

hence $\text{rk} Q_{ij} \leq 3$, a contradiction.

It remains to show that (ii) holds. We observe that

$$\text{rk} Q_{ij} = 4 \iff \eta_{ij}, \nu_{ij} \in \langle \omega_i, \omega_j \rangle \iff \text{Sing} Q_{ij} = \bar{D}_{ij}$$

and we conclude as before.

5.4. Remark. If $C$ is trigonal or a smooth plane quintic, then for all $(g - 2)$-tuples $p_1, \ldots, p_{g-2}$ (in general position) we have

$$\text{Sing} Q_{ij} \cap C = D_{ij} \quad \text{and} \quad \text{rk} Q_{ij} = 4$$

We can give a more precise description of the quadrics $Q_{ij}$ in both cases: it will be enough to exhibit a set of quadrics satisfying the characterizing properties (5.1).

1. $C$ is trigonal. Let $|g_3^1|$ be the trigonal pencil and consider the complete linear series of degree $g - 1$

$$\xi = g_3^1 + D_{ij}$$

where $D_{ij}$ is as in the proof of Prop. 5.3. Then define $Q_\xi$ to be the cone with vertex $\bar{D}_{ij} = \mathbb{P}^{g-5}$ over the smooth quadric $|\xi|^* \times |K\xi^{-1}|^* = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 = |K(\bar{D}_{ij})|^*$

$$Q_\xi \quad \subset \quad |K|^*$$

$$\downarrow \quad \downarrow$$

$$\mathbb{P}^1 \times \mathbb{P}^1 \quad \xrightarrow{m} \quad \mathbb{P}^3$$

(5.11)
where $pr$ is the linear projection with center $D_{ij}$ and $m$ is the Segre map. From this description it is clear that the rank 4 quadric $Q_\xi$ satisfies (5.1).

2. $C$ is a smooth plane quintic $(g = 6)$. Let $|g^2_5|$ be the associated degree 5 linear series. We can write $D_{ij} = p_k + p_l$ for some indices $k,l$ and, as in the trigonal case, we consider the quadric $Q_\xi$ defined by the diagram (5.11) with

$$\xi = g^2_5(-p_k) \quad \text{and} \quad K_\xi^{-1} = g^2_5(-p_l)$$

Again we easily check that $Q_\xi$ satisfies (5.1).

### 5.2 Rank 6 quadrics and rank 2 vector bundles

In this section we recall a construction [BV] relating rank 6 quadrics and rank 2 vector bundles. We consider a rank 2 bundle $E$ and a subspace $V \subset H^0(E)$ which satisfy the conditions:

$$\text{det } E = K$$

$$\dim V = 4$$

$$V \text{ generates } E.$$ (5.12)

We can associate to such a bundle the following commutative diagram

$$\begin{array}{ccc}
C & \xrightarrow{\gamma} & Gr(2,V^*) \\
\downarrow \varphi_K & & \downarrow \\
|K|^* & \xrightarrow{\lambda^*} & \mathbb{P}(\wedge^2 V^*) = \mathbb{P}^5
\end{array}$$ (5.13)

where $\gamma$ is the morphism (defined since we have a surjection $O_C \otimes V \to E$)

$$\gamma : p \mapsto E_p^* \subset V^*$$ (5.14)

and $\lambda$ the map defined by taking the exterior product of global sections of $E$

$$\lambda : \wedge^2 V \to H^0(\wedge^2 E) = H^0(K).$$ (5.15)

The Grassmannian $Gr(2,V^*)$ embedded in $\mathbb{P}^5$ by the Plücker embedding is a smooth quadric. We define $Q_{(E,V)}$ to be the inverse image of this quadric:

$$Q_{(E,V)} = (\lambda^*)^{-1}(Gr(2,V^*)).$$ (5.16)

Then $\operatorname{rk} Q_{(E,V)} \leq 6$ and $Q_{(E,V)} \in |I(2)|$. If $h^0(E) = 4$, then $V = H^0(E)$ and we denote the quadric $Q_{(E,V)}$ simply by $Q_E$. We have the following lemmas:

#### 5.5. Lemma. For any pair $(E,V)$ satisfying conditions (5.12), let $\tilde{Q}_{(E,V)}$ be the polar form of the quadric $Q_{(E,V)}$. Then

$$\forall p,q \in C, p \neq q : \tilde{Q}_{(E,V)}(p,q) = 0 \iff V \cap H^0(E(-p-q)) \neq \{0\}.$$
5.6. Lemma. For any pair \((E, V)\) satisfying conditions (5.12), we have
\[
\text{rk } Q_{(E,V)} \leq 4 \iff E \text{ contains a line subbundle } L \text{ with } \dim H^0(L) \cap V = 2.
\]

\textit{Proof.} This is essentially Prop. 1.11 of \cite{BV}. Note that if \(E\) is generated by \(V\), then \(\dim H^0(L) \cap V \leq 2\).

\[
\begin{align*}
\text{5.7. Remark.} & \text{ We see that the definition (5.16) of the quadric } Q_{(E,V)} \text{ makes sense even if the bundle } E \text{ is not generated by global sections in } V \text{ at a finite number of points. We easily see that } E \text{ is not generated by } V \text{ at the point } p \text{ if and only if } p \in \mathbb{P} \ker^* \lambda^* \subset \text{Sing } Q_{(E,V)}. \text{ Moreover, if } \text{rk } Q_{(E,V)} \geq 5, \text{ then we have an equality } \mathbb{P} \ker^* \lambda^* = \text{Sing } Q_{(E,V)} \text{ (see [BV] (1.9)).}
\end{align*}
\]

The above described construction which associates to the pair \((E, V)\) the quadric \(Q_{(E,V)} \in I(2)\) admits an inverse construction, i.e. we can recover a bundle \(E\) from a general rank 6 quadric: consider a quadric \(Q \in I(2)\) satisfying
\[
\begin{align*}
gr & = \text{rk } Q = 5 \text{ or } 6 \\
\text{Sing } Q \cap C & = \emptyset
\end{align*}
\]

We project away from \text{Sing } Q
\[
\delta : Q \longrightarrow \mathbb{P}^{r-1}.
\]

If \(r = 6\), the image \(\delta(Q)\) is a smooth quadric in \(\mathbb{P}^5\) and can be realized as a Grassmannian \(Gr(2, 4)\). If \(r = 5\), \(\delta(Q)\) is a linear section of \(Gr(2, 4) \subset \mathbb{P}^5\). We consider the exact sequence over \(Gr(2, 4)\)
\[
0 \longrightarrow U \longrightarrow \mathcal{O}_{Gr}^4 \longrightarrow \check{U} \longrightarrow 0
\]

where \(U\) (resp. \(\check{U}\)) is the universal subbundle (resp. quotient bundle). Since \(\text{Sing } Q \cap C = \emptyset\), the restriction of \(\delta\) to the curve \(C\) is everywhere defined and we can consider the two pairs (let \(g = \delta|_C\))
\[
\begin{align*}
(g^*U^*, g^*H^0(U^*)) & \quad (g^*\check{U}, g^*H^0(\check{U}))
\end{align*}
\]

which satisfy conditions (5.12). The following proposition is proved in \cite{BV} (Prop. (1.18) and (1.19))

5.8. Proposition. The pairs (5.18) are the only pairs defining the quadric \(Q\). If \(\text{rk } Q = 5\) then they are isomorphic.

5.9. Lemma. Consider a bundle \(E\) with \(h^0(E) = 4\) and satisfying (5.12). If \(\text{rk } Q_E = 5\) or 6, then \(E\) is stable

\textit{Proof.} Suppose that there exists a destabilizing subbundle \(L \subset E\), with \(\deg L \geq g - 1\). By Lemma 5.6 we have \(h^0(L) \leq 1\) and by Riemann-Roch, \(h^0(KL^{-1}) \leq h^0(L)\). But then \(h^0(E) \leq h^0(L) + h^0(KL^{-1}) \leq 2\), a contradiction.
5.3 Proof of theorem 1.2

5.3.1 The map in \([1.7]\)

First, we prove the inclusion \(\Gamma_{11} \subset \Gamma_{00}^{(2)}\). Consider a point \(p - q \in C - C\) and a point \(r \in C\). The curve \(t_x(C) \subset JC\), with \(x = \mathcal{O}(r + p - q)\), is contained in the fourfold \(C_2 - C_2\). Therefore a hyperplane \(H\) in \([2\Theta]\) containing \(C_2 - C_2\) also contains all tangent lines \(T_{p - q}(C_x)\). Since these tangent lines (fix \(p, q\) and vary \(r\)) generate linearly the projectivized tangent space \(\mathbb{P}T_{p - q}JC\), we get \(\mathbb{P}(\mathbb{T}) \subset H\). We recall that \(\mathbb{T}\) is the linear span of the \(\mathbb{P}T_{p - q}JC\) when \(p, q\) vary (see section 4). Hence we get the inclusion.

Now we consider the difference map

\[ \gamma : C^4 \xrightarrow{pr} C_2 \times C_2 \xrightarrow{\phi_2} JC \]

where \(C^4\) is the 4-fold product of the curve and the first arrow \(pr\) is the quotient by the transpositions \((1, 2)\) and \((3, 4)\) acting on \(C^4\). We denote by \(\Delta_{i,j}\) the divisor in \(C^4\) consisting of 4-tuples having equal i-th and j-th entry. A straightforward computation shows that

\[ \gamma^*\mathcal{O}(2\Theta_0) = \bigotimes_{i=1}^4 \pi_i^*K(-2\Delta_{1,2} - 2\Delta_{3,4} + 2\Delta_{1,3} + 2\Delta_{1,4} + 2\Delta_{2,3} + 2\Delta_{2,4}) \]  

(5.19)

Note that the divisor \(\Delta_{1,3} + \Delta_{1,4} + \Delta_{2,3} + \Delta_{2,4} \subset C^4\) is invariant under the transpositions \((1, 2)\) and \((3, 4)\), hence comes from an irreducible divisor in \(C_2 \times C_2\), which we call \(\Delta\). We also observe that the line bundle \(\pi_1^*K \otimes \pi_2^*K(-2\Delta_{1,2})\) over \(C^2\) is invariant under the natural involution, hence comes from a line bundle \(\mathcal{M}\) over \(C_2\) and we have a canonical isomorphism (see e.g. \([BV]\))

\[ H^0(C_2, \mathcal{M}) \cong I(2). \]

(5.20)

With this notation we rewrite (5.19) as

\[ \phi_2^*(\mathcal{O}(2\Theta_0)) = \pi_1^*\mathcal{M} \otimes \pi_2^*\mathcal{M}(2\Delta). \]

(5.21)

Now we want to compute the pull-back of \(2\Theta\)-divisors vanishing doubly on \(C - C\). Let \(\mathcal{J}\) be the sheaf of ideals defining the surface \(C - C \subset JC\).

5.10. Lemma. If \(C\) is non-trigonal, the inverse image ideal sheaf \(\phi_2^{-1}\mathcal{J} \cdot \mathcal{O}_{C_2 \times C_2}(-\Delta)\), hence \(\phi_2^*\mathcal{J} = \mathcal{O}_{C_2 \times C_2}(-\Delta)\)

Proof. This follows from the observation that the inverse image of \(C - C\) under \(\phi_2\) is isomorphic to the divisor \(\Delta\). \(\square\)

5.11. Remark. If \(C\) is trigonal, the inverse image \(\phi_2^{-1}(C - C)\) contains, apart from the divisor \(\Delta\), a surface which is the image of the morphism

\[ C \times C \longrightarrow C_2 \times C_2 \]

\[ (p, q) \longmapsto (g_3^1(-p), g_3^1(-q)) \]

where \(g_3^1\) is the trigonal series (unique if \(g \geq 5\)) and \(g_3^1(-p)\) denotes the residual pair of points in the fibre containing \(p\). We deduce that \(\phi_2^{-1}\mathcal{J} \cdot \mathcal{O}_{C_2 \times C_2} \subset \mathcal{O}_{C_2 \times C_2}(-\Delta)\) and that there exists a natural map of \(\mathcal{O}_{C_2 \times C_2}\)-modules \(\phi_2^*\mathcal{J} \longrightarrow \mathcal{O}_{C_2 \times C_2}(-\Delta)\).
Combining (5.21) and Lemma 5.10, we obtain a linear map induced by $\phi_2$

$$\phi_2^* : H^0(JC, \mathcal{O}(2\Theta_0) \otimes \mathcal{J}^2) = \Gamma^{(2)}_{00} \to H^0(C_2 \times C_2, \pi_1^*\mathcal{M} \otimes \pi_2^*\mathcal{M})$$

(5.22)

This map is equivariant under the natural involutions of $JC$ and $C_2 \times C_2$. Since all second-order theta functions are even, the image of $\phi_2^*$ is contained in $\text{Sym}^2 H^0(C_2, \mathcal{M}) \cong \text{Sym}^2 I(2)$, by (5.20).

To summarize, we have shown that

$$\Gamma_{11} = \ker (\phi_2^* : \Gamma^{(2)}_{00} \to \text{Sym}^2 I(2)).$$

### 5.3.2 Surjectivity of $\phi_2^*$

The key point of the proof is the following proposition

**5.12. Proposition.** We have a commutative diagram

$$\begin{array}{ccc}
\mathcal{W}(4) & \xrightarrow{D} & \mathbb{P}\Gamma^{(2)}_{00} \\
\downarrow Q & & \downarrow \phi_2^* \\
|I(2)| & \xrightarrow{\text{Ver}} & \mathbb{P}\text{Sym}^2 I(2)
\end{array}$$

(5.23)

where the notation is as follows:

- $\mathcal{W}(4) = \{ [E] \in SU_C(2, K) \mid \dim H^0(E) = 4 \text{ and } E \text{ globally generated} \}$
- $Q$ is the map described in section 5.2
- $\text{Ver}$ is the Veronese map
- $D$ is the map (2.2)

**Proof.** Consider a bundle $E \in \mathcal{W}(4)$. By [L] Prop. V.2., we have an inequality

$$\text{mult}_{p-q}(D(E)) \geq h^0(E(p - q)) \geq 2$$

hence we see that $D(E) \in \mathbb{P}\Gamma^{(2)}_{00}$. To show commutativity, it is enough to check that the zero divisors of the two sections

$$Q_E \otimes Q_E \quad \phi_2^*(D(E))$$

(considered as sections of $\pi_1^*\mathcal{M} \otimes \pi_2^*\mathcal{M}$ over $C_2 \times C_2$ (2.2)) coincide as sets. Hence, by Lemma 5.5 and (2.2), it is enough to show the following equivalence: for any four distinct points $p, q, r, s \in C$

$$h^0(E(-p - q)) > 0 \lor h^0(E(-r - s)) > 0 \iff h^0(E(p + q - r - s)) > 0$$

The $\Rightarrow$ implication is obvious ($D(E)$ is symmetric). To prove the $\Leftarrow$ implication, we suppose that $h^0(E(-p - q)) = h^0(E(-r - s)) = 0$. Then, by Riemann-Roch and Serre duality, we have $h^0(E(p + q)) = 4$. Since $h^0(E) = 4$, we see that all global sections of $E(p + q)$ vanish at the points $p$ and $q$. Supposing that there exists a non-zero section $\varphi$ of $E(p + q - r - s)$, then $\varphi$ vanishes at $p, q$, contradicting $h^0(E(-r - s)) = 0$, hence $h^0(E(p + q - r - s)) = 0$. \hfill \Box

From now on, we assume that $C$ is non-trigonal. We consider $g$ points $p_1, \ldots, p_g$ in general position and their associated Petri quadrics $Q_{ij}$ for $1 \leq i, j \leq g - 2$ (section 5.1), which form a basis of $I(2)$. Then in order to prove surjectivity of $\phi_2^*$, it is enough, by Prop. 5.12, to construct a set of $h^2$ vector bundles (with $h = \binom{g - 2}{2} = \dim I(2)$) in $\mathcal{W}(4)$, which generate linearly $\text{Sym}^2 I(2)$. First we suppose that $C$ is not a smooth plane quintic. We proceed in 3 steps.

**Step 1**

By Prop. 5.3, the Petri quadrics $Q_{ij}$ satisfy conditions (5.17), so (Prop. 5.8) we can construct for each $i, j$ two pairs (see (5.18)) of bundles $(E_{ij}^{(1)}, V^{(1)})$ and $(E_{ij}^{(2)}, V^{(2)})$, which define the quadric $Q_{ij}$. 

17
5.13. Lemma. For general points \(p_1, \ldots, p_g\), the bundles \(E_{ij}^{(1)}\), \(E_{ij}^{(2)}\) are stable, distinct and 
\[h^0(E_{ij}^{(1)}) = h^0(E_{ij}^{(2)}) = 4.\]

Proof. We can give a different description of the bundles \(E_{ij}^{(1)}\) and \(E_{ij}^{(2)}\) using extension spaces. Let \(D = D_{ij} + p_i\) and consider extensions of the form 
\[0 \to \mathcal{O}(D) \to \mathcal{E} \to \mathcal{O}(K - D) \to 0 \quad (\varepsilon)\]
These extensions are classified by an extension class \(\varepsilon \in |2K - 2D|^* = \mathbb{P}^{g+2}\). Since \(h^0(D) = 1\) and 
\(h^0(K - D) = 3\), we see that \(h^0(\mathcal{E}_\varepsilon) = 4\) if and only if \(\varepsilon \in \ker m^* = (\coker m)^*\), where \(m\) is the multiplication map 
\[m : \text{Sym}^2 H^0(K - D) \to H^0(2K - 2D)\]
which is injective for general points \(p_i\). Note that \(\dim \coker m = g - 3\). Furthermore, consider a point \(p_\alpha \in D_{ij}\) and the multiplication map (which is injective)
\[m_\alpha : H^0(K - D + p_j + p_\alpha) \otimes H^0(K - D - p_j - p_\alpha) \to H^0(2K - 2D).\]
Then \(h^0(\mathcal{E}_\varepsilon(-p_j - p_\alpha)) > 0 \iff \varepsilon \in \ker m^\alpha\). We observe that the image \(\text{im} m_\alpha \subset H^0(2K - 2D)\)
under the canonical surjection \(H^0(2K - 2D) \to \coker m\) is a one-dimensional subspace, which
we denote by \(Z_\alpha\). Consider now a hyperplane \(H\) in \(\coker m\), which contains the linear span of the
\(Z_\alpha\) for \(\alpha\) such that \(p_\alpha \in D_{ij}\) (we will see a posteriori that such an \(H\) is unique, for dimensional
reasons), so that we obtain an extension class \(\varepsilon = \varepsilon(H) \in \mathbb{P}(\coker m)^* \subset |2K - 2D|^*\).
By construction, we have \(h^0(\mathcal{E}_\varepsilon) = 4\) and \(h^0(\mathcal{E}_\varepsilon(-p_\alpha - p_\beta)) > 0\) if \(\{\alpha, \beta\} \neq \{i, j\}\), hence, by (5.1) and 
lemma 5.5, we get \(Q_{\mathcal{E}_\varepsilon} = Q_{ij}\), and \(\mathcal{E}_\varepsilon = E_{ij}^{(1)}\).

The other bundle \(E_{ij}^{(2)}\) defining the quadric \(Q_{ij}\) is constructed in the same way using the divisor
\(D' = D_{ij} + p_j\) (instead of \(D\)). Then we have \(E_{ij}^{(1)} \neq E_{ij}^{(2)}\). Indeed, an isomorphism \(E_{ij}^{(1)} \to E_{ij}^{(2)}\)
would imply the existence of a nonzero section of \(\text{Hom}(\mathcal{O}(D), \mathcal{O}(K - D')) = \mathcal{O}(K - D - D')\), but
then the points \(p_1, \ldots, p_{g-2}\) are not in general position.

Finally, stability follows from Lemma 5.9 \(\square\)

We deduce from this lemma and Prop. 5.12 that \(Q_{ij} \otimes Q_{ij} \in \text{im} \phi_2^*\).

Step 2

Consider three distinct indices \(i, j, k\). Then all quadrics of the pencil
\[(\lambda Q_{ij} + \mu Q_{ik})_{\lambda, \mu \in \mathbb{C}}\]
have rank less than or equal to 6. This follows from expression (5.4) of Petri’s quadrics, namely
\[\lambda Q_{ij} + \mu Q_{ik} = \omega_i (\lambda \omega_j + \mu \omega_k) - \tilde{\eta} \omega_{g-1} - \bar{v} \omega_g - \bar{b} \omega_{g-1} \omega_g\]
with \(\tilde{\eta} = \lambda \eta_{ij} + \mu \eta_{ik}, \bar{v} = \lambda v_{ij} + \mu v_{ik}, \bar{b} = \lambda b_{ij} + \mu b_{ik}\). Now a general element of the pencil
(5.24) satisfies conditions (5.17), since these are open conditions and are satisfied by \(Q_{ij}\) and \(Q_{ik}\). Again by openness and Lemma 5.13, it follows that the two bundles associated with such a
general quadric have 4 sections and are stable. Let us pick such a bundle \(E_{ijk}\) defining the quadric
\(\lambda_0 Q_{ij} + \mu_0 Q_{ik}\), for \(\lambda_0, \mu_0 \neq 0\). Then we have in \(\text{Sym}^2 I(2)\)
\[\phi_2^* D(E_{ijk}) = \lambda_0^2 Q_{ij} \otimes Q_{ij} + \mu_0^2 Q_{ik} \otimes Q_{ik} + 2 \lambda_0 \mu_0 Q_{ij} \otimes Q_{ik}\]
hence, \(Q_{ij} \otimes Q_{ik} \in \text{im} \phi_2^*\).
Consider four distinct indices \(i, j, k, l\). Then all quadrics of the 2-dimensional family \(\mathcal{F}_{(ij)(kl)} = \mathbb{P}^1 \times \mathbb{P}^1\) (here \((\lambda, \lambda'), (\mu, \mu')\) are a set of homogeneous coordinates) are given by an expression:

\[
\mu \lambda Q_{ik} + \mu' \lambda' Q_{ij} + \mu \lambda' Q_{jk} + \mu' \lambda Q_{kl} =
\]

\[
(\mu \omega_i + \mu' \omega_j)(\lambda \omega_k + \lambda' \omega_l) - \bar{\eta} \omega_{g-1} - \bar{\nu} \omega_{g} - \bar{\omega} \omega_{g-1} \omega_g,
\]

where \(\bar{\eta}, \bar{\nu}, \bar{\omega}\) depend on \((\lambda, \lambda'), (\mu, \mu')\), see (5.4). The same holds for the two families obtained by permuting indices

\[
\mathcal{F}_{(ik)(ij)} : \mu \lambda Q_{ij} + \mu' \lambda' Q_{il} + \mu' \lambda Q_{jk} + \mu' \lambda Q_{kl}
\]

\[
\mathcal{F}_{(il)(kj)} : \mu \lambda Q_{ik} + \mu' \lambda' Q_{ij} + \mu' \lambda Q_{lk} + \mu' \lambda Q_{jl}
\]

As in step 2, we see that a general member of these 3 families satisfies (5.17) and we can pick 3 stable vector bundles \(E_{(ij)(kl)}, E_{(ik)(jl)}, E_{(il)(kj)}\) with 4 sections defining the 3 quadrics in these families with coordinates \((\lambda_0, \lambda'_0)(\mu_0, \mu'_0)\) for some \(\lambda_0, \lambda_0', \mu_0, \mu'_0 \neq 0\). Now we can write in Sym\(^2 I(2)\)

\[
\phi^* \mathcal{D}(E_{(ij)(kl)}) = 2 \mu_0 \mu'_0 \lambda_0 \lambda'_0 (Q_{ik} \otimes Q_{jl} + Q_{il} \otimes Q_{jk}) + \alpha
\]

\[
\phi^* \mathcal{D}(E_{(ik)(jl)}) = 2 \mu_0 \mu'_0 \lambda_0 \lambda'_0 (Q_{ij} \otimes Q_{kl} + Q_{il} \otimes Q_{jk}) + \beta
\]

\[
\phi^* \mathcal{D}(E_{(il)(kj)}) = 2 \mu_0 \mu'_0 \lambda_0 \lambda'_0 (Q_{ik} \otimes Q_{jl} + Q_{ij} \otimes Q_{kl}) + \gamma
\]

for some \(\alpha, \beta, \gamma \in \text{im} \phi^*\) (see step 1 and 2). But these linear equations immediately imply that the three symmetric tensors \(Q_{ij} \otimes Q_{kl}, Q_{ik} \otimes Q_{jl}, Q_{il} \otimes Q_{jk}\) \(\in \text{im} \phi^*\) and we are done.

To complete the proof we need to consider the case when \(C\) is a smooth plane quintic \((g = 6)\). We will show that the map \(Q\) in diagram (5.23) is dominant. By [AH] Prop. 3.2, the locus of rank 4 quadrics is a cubic hypersurface in \([I(2)] = \mathbb{P}^5\) and a general quadric has rank 6 (i.e., is smooth). Consider any smooth quadric \(Q \in [I(2)]\) and one of the associated pairs \((E, V)\) defining \(Q\) (5.18). It will be sufficient to show that \(h^0(E) = 4\), hence \(E \in \mathcal{W}(4)\). By [OPP] Thm. 8.1 (3), \(h^0(E) \geq 5\) if and only if \(E\) is an extension of the form

\[
0 \longrightarrow g^2_5 \longrightarrow E \longrightarrow g^2_5 \longrightarrow 0.
\]

From this we see that if \(h^0(E) = 5\), \(\dim V \cap H^0(g^2_5) \geq 2\), hence by Lemma 5.6 \(rk Q \leq 4\) and if \(h^0(E) = 6\), then \(E = g^2_5 \oplus g^2_5\) and we can also conclude that \(rk Q \leq 4\), contradicting \(Q\) smooth.

5.4 Another proof of a theorem by M. Green

As a consequence of Thms. 1.1 and 1.2 we get another proof of the following theorem (in the case of non-trigonal curves) due to M. Green ([G], see also [SV])

5.14. Theorem. For \(C\) non-trigonal, the projectivized tangent cones at singular points of \(\Theta\) span \(I(2)\).

Proof. For all \(\xi \in \text{Sing } \Theta\) with \(h^0(\xi) = 2\), the split bundle \(E = \xi \oplus K \xi^{-1} \in \mathcal{W}(4)\). Then the associated quadric \(Q_E\) has rank 4 and can be described as a cone over the smooth quadric \(\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}V^* = \mathbb{F}^3\) (as in diagram (5.11)) where \(V\) is the image of the multiplication map

\[
H^0(\xi) \otimes H^0(K \xi^{-1}) \longrightarrow H^0(K).
\]

Then \(Q_E\) is the projectivized tangent cone at \(\xi \in \text{Sing } \Theta\).

Suppose now that the image under \(Q\) of Sing \(\Theta\) in \([I(2)]\) is degenerate. By Prop. 5.12 and Thm. 1.1 (2), we see that the image of \(\phi^*\) is also degenerate, contradicting Thm. 1.2. \(\square\)
6 The space $\Gamma_{000}$

6.1 Proof of theorem 1.3

In this section we regard $2\theta$-functions as holomorphic functions on $\mathbb{C}^9$. For the proof of Theorem 1.3 we need the following lemma:

6.1. Lemma. The following statements are equivalent

(1) $f \in \Gamma_{000}$

(2) For all $a_1, a_2, a_3, a_4 \in \mathcal{C}$

\[(q_{12q34})^4 f(w(a_1 + a_2 - a_3 - a_4)) + (q_{14q23})^4 f(w(a_1 + a_4 - a_2 - a_3)) +
(q_{13q24})^4 f(w(a_1 + a_3 - a_2 - a_4)) = 0\]

Proof. We will derive identity (2) from Prop. 3.3. First we observe that the left-hand side $PD_a D_{a2} D_{a3} D_{a4} \theta_2(0)$ of Prop. 3.3 (1) is symmetric in the four variables. In particular it is symmetric in $a_2, a_4$, which leads to the equality

$$\tau_{1234} + \tau_{2314} = \tau_{1432} + \tau_{1342} + 2\sigma_{1234} + 2\sigma_{1243} + 2\sigma_{3124} - 2\sigma_{1432} - 2\sigma_{4132} - 2\sigma_{3142} \quad (6.1)$$

Combining (6.1) and prop.3.3 (1) we can write

$$\frac{1}{2} PD_a D_{a2} D_{a3} D_{a4} \theta_2(0) = \tau_{1324} + \tau_{1432} + \tau_{1342} + \text{some } \sigma'\text{s}$$

Using Prop. 3.3 (2) and the two relations obtained from it by interchanging $a_1$ with $a_3$ and $a_1$ with $a_4$, we can write

$$\frac{1}{2} PD_a D_{a2} D_{a3} D_{a4} \theta_2(0) = \left(\frac{q_{12q34}}{q_{13q14q23q24}}\right)^2 P\theta_2(w(a_1 + a_2 - a_3 - a_4)) +$$

$$\left(\frac{q_{32q14}}{q_{13q34q21q24}}\right)^2 P\theta_2(w(a_1 + a_4 - a_3 - a_2)) + \left(\frac{q_{24q31}}{q_{14q31q23q21}}\right)^2 P\theta_2(w(a_1 + a_3 - a_2 - a_4)) - X$$

where $X$ is the following sum of $\sigma'$s

$$(\sigma_{1243} + \sigma_{1324} + \sigma_{1432}) + (\sigma_{3142} + \sigma_{3241} + \sigma_{3214}) + (\sigma_{4132} + \sigma_{4213} + \sigma_{4231})$$

But the three terms within each pair of parentheses add up to zero (use the definition of $\sigma$ [3.4] and the fact that $\xi$ is skew-symmetric in its variables), hence $X = 0$ and we are done. $\square$

We are now in a position to prove Theorem 1.3. First we show the inclusion $\Gamma_{000} \subset \Gamma_{00}^{(2)}$. We fix $f \in \Gamma_{000}$ and three points $a_2, a_3, a_4 \in \mathcal{C}$. We consider $a_1$ as a canonical coordinate and derive two (resp. three) times with respect to $a_1$ and take the value at the point $a_1 = a_4$. This way, we obtain two equations among vectors in $\mathbb{C}^{29}/T_0$

$$D^2_{a4} \theta_2(w(a_2 - a_3)) + D_{a4} \theta_2(w(a_2 - a_3)) \left[4\partial \log \frac{q_{42}}{q_{43}}\right] = 0$$

$$D^2_{a4} \theta_2(w(a_2 - a_3)) \left[\partial \log q_{42} q_{43} + D_{a4} \theta_2(w(a_2 - a_3)) \left[\partial^2 \log \frac{q_{42}}{q_{43}} + 4\partial \log q_{42} q_{43} \cdot \partial \log \frac{q_{42}}{q_{43}}\right]\right] = 0$$
where $\partial$ means derivative with respect to the first variable of the prime form $q$. Hence we get, for all $a_2, a_3, a_4 \in \mathbb{C}$, a system of two linear equations involving the vectors $D_{a_4}^2 \theta_2(w(a_2 - a_3))$ and $D_{a_4} \theta_2(w(a_2 - a_3))$ whose determinant

$$\frac{\partial^2 \log q_{42}}{q_{43}}$$

is non-zero on an open subset of $\mathbb{C}^3$. Hence the two vectors $D_{a_4}^2 \theta_2(w(a_2 - a_3))$ and $D_{a_4} \theta_2(w(a_2 - a_3))$ are zero on an open subset of $\mathbb{C}^3$, so they are identically zero. This implies that $f \in \Gamma_{00}^{(2)}$.

The inclusion $\Gamma_{11} \subset \Gamma_{000}$ and the second statement of Theorem 1.3 can easily be deduced from the commutativity of the diagram

$$\langle \text{Sing}\Theta \rangle = \Gamma_{00}^{(2)} \xrightarrow{\phi_2^*} \text{Sym}^2 I(2)$$

$$\downarrow^\alpha \quad \checkmark$$

$$I(4)$$

(6.2)

where $\alpha$ is the map which associates to a $2\theta$-divisor its projectivized quartic tangent cone at the origin (i.e. the degree 4 term of the Taylor expansion at the origin), $\phi_2^*$ is as in (5.22) and the diagonal arrow is the multiplication map $m$. By definition we have

$$\ker \alpha = \Gamma_{000} \quad \ker \phi_2^* = \Gamma_{11}$$

To check the commutativity of (6.2), by Theorem 1.1 (2) it is enough to check that $\alpha(D(E)) = m(\phi_2^*(D(E)))$ for the bundle $E = \xi \oplus K\xi^{-1}$, with $\xi \in \text{Sing} \Theta$ and $h^0(\xi) = 2$, i.e. that $D(E) = \Theta_\xi + \Theta_{K\xi^{-1}}$. This follows from Prop. 5.12 and the description of $Q_E$ (see proof of Theorem 5.14).

6.2. Remark. We don’t know how to find a general formula for $\dim \Gamma_{000}$. In the examples that follow, we give $\dim \Gamma_{000}$ for $g \leq 7$.

6.2 Examples

6.2.1 Curves of genus less than 6

For any non-hyperelliptic curve of genus $g \leq 5$, we have

$$\dim \Gamma_{000} = 0.$$ 

This is an easy consequence of (1.3) and Thms. 1.1, 1.2 and 1.3 (For a trigonal genus 5 curve, we also use Prop. 7.2).

6.2.2 Curves of genus 6

Consider first a genus 6 curve which is not trigonal or a smooth plane quintic. In order to determine the quadratic syzygies in $I(2)$, i.e. $\ker m$, we consider the rational map induced by the linear system $|I(2)|$ on the canonical space $|K|^* = \mathbb{P}^5$

$$\pi : |K|^* \longrightarrow |I(2)|^* = \mathbb{P}^5.$$

Note that $\pi$ is defined away from the canonical curve. We need the following lemma (due to S. Verra).

6.3. Lemma. The rational map $\pi$ is finite of degree 2.
Proof. Let $H$ be a general hyperplane in $I(2)$ and choose a basis of quadrics $\{Q_0, Q_1, \ldots, Q_4\}$ of $H$. By a Bertini argument, the intersection in $|K|^*$ of three general quadrics $Q_i \in |I(2)|$ (for $i = 0, 1, 2$) is a smooth, irreducible surface $S$ (hence a $K3$-surface), which contains the canonical curve. Then the linear equivalence class of the two divisors determined by the quadrics $Q_3, Q_4$ on the surface $S$ is $2h - c$, where $h$ is the hyperplane section and $c$ the class of the canonical curve. We compute

$$(2h - c)^2 = 4h^2 - 4h \cdot c + c^2 = 2$$

since $h^2 = 8, h \cdot c = 2g - 2 = 10$ and $c^2 = 10$ (by the adjunction formula with $\omega_S = \mathcal{O}_S$). Therefore the fibre of $\pi$ over the point determined by $H$ consists of 2 points and we are done.

It follows from Lemma 6.3 that $\pi$ is onto, hence $\Gamma_{000} = \Gamma_{11}$. By Thm. 8.1 (1) [OPP] and Thm. 5.1 (1) [M2], there exists a unique stable bundle $E_{\max} \in SU_C(2, K)$ with maximal number of sections $h^0(E_{\max}) = 5$. Since $C_2 - C_2 \subset D(E_{\max})$, we have $D(E_{\max}) \in PG_{11}$. Moreover by (1.3) and Thms. 1.1 and 1.2, we have $\dim \Gamma_{11} = 1$, from which we deduce that

$$\mathbb{P}\Gamma_{000} = \mathbb{P}\Gamma_{11} = D(E_{\max}).$$

As a complement to the examples of Brill-Noether loci of $SU_C(2, K)$ provided in [OPP] we add a geometric description of the divisor $D(E_{\max}) \subset JC$. Let $L = g^1_4$ be a tetragonal series on $C$ and $\varphi_L$ be the associated surjective Abel-Jacobi map

$$\varphi_L : C \longrightarrow JC$$

$$(p_1, \ldots, p_6) \longmapsto K^{-1}L(p_1 + \ldots + p_6)$$

Let $\pi_L$ be the map induced by the base point free linear series $|KL^{-1}|$

$$\pi_L : C \longrightarrow |KL^{-1}|^* = \mathbb{P}^2.$$ 

If $C$ is bi-elliptic, the image is a smooth plane cubic. Otherwise (general case) $\pi_L$ maps $C$ birationally to a nodal plane sextic.

6.4. Proposition. Let $S \subset C_6$ be the divisor consisting of sextuples $(p_1, \ldots, p_6)$ such that the points $\pi_L(p_i)$ lie on a conic in $|KL^{-1}|^* = \mathbb{P}^2$. Then

$$\varphi_L(S) = D(E_{\max})$$

Proof. By [M2] Prop. 3.1 (see also [OPP] example 3.4), we can write $E_{\max}$ as an extension

$$0 \longrightarrow L \longrightarrow E_{\max} \longrightarrow KL^{-1} \longrightarrow 0 \quad (6.3)$$

By definition, we have $\lambda \in D(E_{\max}) \iff h^0(E_{\max} \otimes \lambda) > 0$. First, we see that if $h^0(L) > 0$ or $h^0(L^{-1}) > 0$, then the exact sequence (6.3) implies that $\lambda \in D(E_{\max})$ (note that $D(E_{\max}$ is symmetric). Therefore we can assume that $h^0(L) = h^0(L^{-1}) = 0$ or equivalently that $h^0(KL^{-1}) = 1$. Writing the long exact sequence associated to (6.3), we see that $H^0(E_{\max} \otimes \lambda) = \ker(\delta : H^0(KL^{-1}) \longrightarrow H^1(L))$. Therefore $h^0(E_{\max} \otimes \lambda) > 0$ if and only if the image of the multiplication map

$$H^0(KL^{-1}) \otimes H^0(KL^{-1}) \longrightarrow H^0(K^2L^{-2})$$
is contained in the hyperplane \( \text{Sym}^2 H^0(KL^{-1}) \subset H^0(K^2L^{-2}) \) which defines the extension class of \( E_{\text{max}} \) in \( |K^2L^{-2}|^* \) (see [OPP] example 3.4). Since \( h^0(KL^{-1}) = 1 \), there exists a unique sextuple \( (p_1, \ldots, p_6) \in C_6 \) such that \( \lambda = \varphi_L((p_i)) \). Then we deduce from the commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\varphi_{K^2L^{-2}}} & |K^2L^{-2}|^* = \mathbb{P}^6 \\
\downarrow{\pi_L} & & \downarrow{pr} \\
|KL^{-1}|^* = \mathbb{P}^2 & \xrightarrow{\text{Ver}} & \mathbb{P}\text{Sym}^2 H^0(KL^{-1})^* = \mathbb{P}^5
\end{array}
\]

that \( h^0(E_{\text{max}} \otimes \lambda) > 0 \) if and only if the 6 points \( \varphi_{K^2L^{-2}}(p_i) \) lie on a hyperplane in \( \mathbb{P}^6 \) which is the inverse image under the projection map \( pr \) of a hyperplane in \( \mathbb{P}^5 \). But this last condition says that the 6 points \( \pi_L(p_i) \) lie on a conic in \( |KL^{-1}|^* \). Finally, we notice that if \( h^0(L^\lambda_1) > 0 \) (\( \iff h^0(O(\sum p_i)) \geq 2 \)) then there exists a divisor \( D = \sum q_i \) in the linear system \( |\sum p_i| \) such that the \( \pi_L(q_i) \) lie on a conic (e.g. we can show that \( S \) is an ample divisor on \( C_6 \)).

Consider now the case of a smooth plane quintic. By Thm. 8.1 (3) [OPP] there exists a unique \( S \)-equivalence class \( \epsilon_{\text{max}} \) such that if \( h^0(E) \geq 5 \), then \( [E] = \epsilon_{\text{max}} \). In particular, \( \epsilon_{\text{max}} = [g_5^2 \oplus g_5^2] \). Moreover, using an explicit basis of quadrics of \( I(2) \), we can show that the map \( \pi \) is birational (for more details see [OPP] Thm. 5.5), hence \( \ker m = \{0\} \). As before, we deduce from Thms. 1.1 and 1.2 that

\[ \mathbb{P}\Gamma_{000} = \mathbb{P}\Gamma_{11} = D(\epsilon_{\text{max}}) = 2\Theta_{g_5^2}. \]

Consider now the case of a trigonal curve. By Thm. 8.1 (2) [OPP] there exists a projective line \( \mathbb{P}^1_{\text{ban}} \) of stable bundles \( E \) with \( h^0(E) = 5 \), hence \( \mathbb{P}^1_{\text{ban}} \subset \mathbb{P}\Gamma_{11} \). Using Prop. 7.2, we compute that \( \dim \Gamma_{11} = 2 \) and (using section 7) that \( \dim \ker m = 1 \). Hence we have

\[ \mathbb{P}^1_{\text{ban}} = \mathbb{P}\Gamma_{11} \quad \dim \Gamma_{000} = 3 \]

### 6.2.3 Curves of genus 7

For a non-tetragonal genus 7 curve, we have \( \dim \Gamma_{000}/\Gamma_{11} = 1 \) (by [M1] thm 4.2). So we obtain \( \dim \Gamma_{11} = 9 \) and \( \dim \Gamma_{000} = 10 \).

## 7 Trigonal curves

Let \( C \) be a trigonal curve with \( g \geq 5 \) and \( g_3^1 \) its unique trigonal series. By remark 5.11, we obtain as in (5.22) a linear map \( \varphi_\lambda^2 : \Gamma_{00}^{(2)} \rightarrow \text{Sym}^2 I(2) \). The aim of this section is to compute the rank of \( \varphi_\lambda^2 \) (cf. section 5.3.2).

First we need to quote some results about quadrics containing a rational normal scroll from [AH], which we state here for the case of the degree \( g - 2 \) surface \( S \subset |K|^* \) ruled by the pencil of trisecants to the canonical curve. For a trigonal curve, the space \( I(2) \) and the space of quadrics \( I_S(2) \) containing the surface \( S \) are isomorphic. Let \( V = H^0(C, K - g_3^1) \). We choose two sections \( s_0, s_1 \in H^0(g_3^1) \) and consider the isomorphism \( \beta_0 \) (resp. \( \beta_1 \)) induced by multiplication by the section \( s_0 \) (resp. \( s_1 \))

\[ \beta_0 : V \xrightarrow{\sim} V_0 \subset H^0(K) \quad \beta_1 : V \xrightarrow{\sim} V_1 \subset H^0(K) \]
where \( V_i = H^0(C, K - D_i) \) and \( D_i \) is the zero divisor of the section \( s_i \). We then define a linear map 

\[
\beta : \Lambda^2 V \to \text{Sym}^2 H^0(K)
\]

by setting

\[
\beta(v \wedge w) = \beta_0(v) \otimes \beta_1(w) - \beta_0(w) \otimes \beta_1(v)
\]

which is a quadric of rank less than or equal to 4 containing \( S \). One checks that \( \beta(v \wedge w) \) does not depend on the choice of the sections \( s_0, s_1 \). Then Prop. 2.14 [AH] says that \( \beta \) induces an isomorphism

\[
\beta : \Lambda^2 V \sim \to I_S(2)
\]

We also define a rational map \( \delta : W(4) \to \text{Gr}(2, V) \) as follows. Consider a semi-stable bundle \( E \in W(4) \) (see (5.23)). Then, by [M2], Prop. 3.1, \( h^0(E(-g_3^1)) \geq 1 \); it can be shown that \( h^0(E(-g_3^1)) = 1 \) for a general bundle \( E \in W(4) \), i.e. \( E \) can be uniquely written as an extension

\[
0 \to g_3^1 \to E \to K - g_3^1 \to 0
\]

and we define \( \delta(E) = \text{im} \left( H^0(E) \xrightarrow{H^0(\pi)} H^0(K - g_3^1) = V \right) \). Then we can prove the following

**7.1. Lemma.** The map \( Q : W(4) \to |I(2)| \) defined in section 5.2 factorizes as follows

\[
W(4) \xrightarrow{\delta} \text{Gr}(2, V) \xrightarrow{Pl} \mathbb{P}(\Lambda^2 V) \xrightarrow{\beta} |I_S(2)| = |I(2)|
\]

where \( Pl \) is the Plücker embedding of the Grassmannian.

**Proof.** Consider \( E \in W(4) \) with \( h^0(E(-g_3^1)) = 1 \) and identify \( H^0(g_3^1) \) with a 2-dimensional subspace of \( H^0(E) \). We choose a basis \( \{s_0, s_1\} \) of \( H^0(g_3^1) \). Let \( R = \beta \circ Pl \circ \delta(E) \in |I(2)| \) be the associated quadric. In order to show that \( R = Q_E \) it is enough to show that their associated polar forms, which we view as global sections of the line bundle \( M \) over \( C_2 \) (see (5.20)), coincide. Hence it is enough to show the implication

\[
\forall p, q \in C, \ p \neq q, \ h^0(g_3^1(-p - q)) = 0 \quad \hat{Q}_E(p, q) = 0 \implies \hat{R}(p, q) = 0
\]

But by Lemma 5.5, the assumption \( \hat{Q}_E(p, q) = 0 \) means that there exists a section \( a \in H^0(E) \) vanishing at \( p \) and \( q \). Since \( h^0(g_3^1(-p - q)) = 0 \), we have \( a \notin H^0(g_3^1) \) and we can find a section \( b \in H^0(E) \) such that \( \{s_0, s_1, a, b\} \) is a basis of \( H^0(E) \). Then \( H^0(\pi) \) induces a linear isomorphism \( C a \oplus C b \sim \delta(E) \subset V \). Let \( u, v \in \delta(E) \) be the images of \( a, b \) under this isomorphism. Then we see that

\[
\beta_0(u) = s_0 \wedge a \in H^0(K) \quad \beta_0(v) = s_0 \wedge b \in H^0(K).
\]

The same holds for \( \beta_1 \) and \( s_1 \). By (7.1) we have

\[
2\hat{R}(p, q) = (s_0 \wedge a)(p) \cdot (s_1 \wedge b)(q) + (s_0 \wedge a)(q) \cdot (s_1 \wedge b)(p) - (s_0 \wedge b)(p) \cdot (s_1 \wedge a)(q) - (s_0 \wedge b)(q) \cdot (s_1 \wedge a)(p)
\]

and this expression, which does not depend on the choice of the basis \( \{s_0, s_1, a, b\} \), is obviously zero if \( a \in H^0(E(-p - q)) \).  \( \square \)
We consider now the commutative diagram (5.23). Lemma 7.1 implies that the inverse image under the Veronese map of a hyperplane in Sym\(^2\)\(|I(2)|\) containing \(\text{im} \phi^*_2\) is a quadric in \(|I(2)|\) containing the Grassmannian \(Gr(2,V)\). Moreover any such quadric comes from an element in the annihilator \((\text{im} \phi^*_2)^\perp\). Hence we obtain an isomorphism

\[ I_{Gr(2,V)}(2) \cong (\text{im} \phi^*_2)^\perp. \] (7.4)

But the degree 2 part \(I_{Gr(2,V)}(2)\) of the ideal of the Grassmannian \(Gr(2,V)\) is isomorphic to the vector space \(\Lambda^4V\) generated by the Plücker equations (see e.g. [M2]). Hence we have shown

**7.2. Proposition.** The corank of \(\phi^*_2\) is \((g^2 - 4)\).

8 Concluding remarks

1. Some analytic versions of Thms. 1.1 and 1.2 were proved by Gunning in the case of a non-hyperelliptic curve of genus less than 6 ([Gu2] Thm. 6). Furthermore, the statement of Thm. 1.3 was proposed as plausible in [Gu1], p. 70.

2. It is natural to ask whether the three main theorems can be generalized to analogous subseries. For this purpose, we introduce the following subspaces of \(\Gamma_{00}\)

\[
\Gamma_{[n]:0} = \left\{ D \mid \text{mult}_0(D) \geq 2n \right\} \quad \text{for } n \geq 2
\]

\[
\Gamma_{dd} = \left\{ D \mid C_{d+1} - C_{d+1} \subset D \right\} \quad \text{for } d \geq 0
\]

\[
\Gamma_{(d+2)} = \left\{ D \mid \text{mult}_{C_{d+1}-C_{d+1}}(D) \geq 2 \right\} \quad \text{for } d \geq 0
\]

where \(C_{d+1} - C_{d+1}\) is the image of the difference map \((d \geq 0)\)

\[
\phi_{d+1} : C_{d+1} \times C_{d+1} \longrightarrow JC
\]

\[
(D, D') \mapsto \mathcal{O}(D - D')
\]

The following inclusions are obvious

\[
\Gamma_{(d+1)(d+1)} \subset \Gamma_{(d+2)} \subset \Gamma_{dd}
\] (8.1)

and one might expect that the following holds (see Thm. 1.3)

\[
\Gamma_{(d+1)(d+1)} \subset \Gamma_{[d+3]:0} \subset \Gamma_{dd}
\] (8.2)

\[
\Gamma_{[d+3]:0}/\Gamma_{(d+1)(d+1)} \cong \ker \text{Sym}^2I^{(d+1)}(d+2) \longrightarrow I(2d + 4)
\] (8.3)

where \(I^{(d+1)}(d+2)\) is the space of degree \(d+2\) polynomials vanishing at order \(d+1\) along \(C_{\text{can}}\).

Some previous work towards (8.2) has been done in [Gu3]. Statement (8.3) follows from (8.2). A generalization of Thm. 1.1 (2) would assert that the inclusion

\[
\langle W_{g-1}^{d+1}(C) \rangle \subset \Gamma_{dd}
\]

is an isomorphism.
3. An important ingredient of the proof of Thm. 1.1 (resp. Thm. 1.2) is the use of rank 2 vector bundles with 3 (resp. 4) sections. The constructions involved may be viewed as examples of a general construction: consider, for \( n \geq 3 \), the subvarieties

\[
W(n) = \{ [E] \in SU_C(2, K) \mid h^0(E) = n \text{ and } E \text{ is globally generated} \}.
\]

We associate to any \( E \in W(n) \) a commutative diagram (as in (5.13): replace \( V \) by \( H^0(E) \))

\[
\begin{array}{ccc}
C & \rightarrow & Gr(2, H^0(E)^*) \\
\downarrow \phi_K & & \downarrow \\
[K]^* & \rightarrow & \mathbb{P}(\Lambda^2 H^0(E)^*) = \mathbb{P}^{n-1}
\end{array}
\quad (8.4)
\]

The definitions of the morphisms \( \gamma \) and \( \lambda^* \) are as in (5.14) and (5.15).

If \( n \) is even, \( n = 2d + 4 \) for \( d \geq 0 \), the Plücker space \( \Lambda^2 H^0(E)^* \) carries canonically a symmetric multilinear form

\[
\tilde{\text{Pf}} (\omega_1, \ldots, \omega_{d+2}) = \omega_1 \wedge \ldots \wedge \omega_{d+2} \in \Lambda^{2d+4} H^0(E)^* \cong \mathbb{C}
\]

which defines a degree \( d + 2 \) polynomial \( \text{Pf} \in \text{Sym}^{d+2}(\Lambda^2 H^0(E)^*) \) vanishing to order \( d + 1 \) along the Grassmannian \( Gr(2, H^0(E)^*) \). Notice that \( \text{Pf} \) is the Pfaffian if we represent \( \omega \in \Lambda^2 H^0(E)^* \) as an \( n \times n \) skew-symmetric matrix. Therefore we can define for any \( E \in W(2d + 4) \) a polynomial \( Q_E = (\lambda^*)^{-1}(\text{Pf}) \in |I^{d+1}(d+2)| \). A straightforward generalization of Prop. 5.12 leads to

8.1. Proposition. For all \( d \geq 0 \), we have a commutative diagram

\[
\begin{array}{ccc}
W(2d + 4) & \rightarrow & \mathbb{P} \Gamma^{(2)}_{dd} \\
\downarrow Q & & \downarrow \phi_{d+2} \\
|I^{d+1}(d+2)| & \rightarrow & \mathbb{P} \text{Sym}^2 I^{d+1}(d+2)
\end{array}
\quad (8.5)
\]

In order to prove surjectivity of \( \phi_{d+2}^* \) (for \( d > 0 \)) one needs a better understanding of the vector space \( I^{d+1}(d+2) \). Can one construct naturally a basis of \( I^{d+1}(d+2) \) as in the \( d = 0 \) case (Petri’s quadrics)?

If \( n \) is odd and \( n > 3 \), the geometry one can extract from diagram (8.4) seems more complicated, due to the fact that there is no natural equation attached to the Grassmannian. If \( n = 3 \), the composite \( \lambda^* \circ \phi_K : C \rightarrow \mathbb{P}^2 = \mathbb{P}(\Lambda^2 H^0(E)^*) \) is the morphism described in remark 2.5.

4. Finally, we note that the bundles \( E_{W} \), which were introduced in section 2.3, can be used to work out a vector bundle-theoretical proof of Welters’ theorem [W], i.e. the base locus of \( \Gamma_{00} \) equals the surface \( C - C \) for \( g \geq 5 \).

References

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris: Geometry of algebraic curves, vol. 1, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985

[AH] E. Arbarello, J. Harris: Canonical curves and quadrics of rank 4, Comp. Math. 43 (1981) 145-179

[B1] A. Beauville: Fibrés de rang 2 sur une courbe, fibré déterminant et fonctions thêta, Bull. Soc. Math. France 116 (1988), 431-448

26
A. Beauville: Fibrés de rang 2 sur une courbe, fibré déterminant et fonctions thêta II, Bull. Soc. Math. France 119 (1991), 259-291

A. Bertram: Moduli of rank 2 vector bundles, theta divisor and the geometry of curves in projective space, J. Diff. Geom. 35 (1992), 429-469

S. Brivio, A. Verra: The theta divisor of $SU_C(2,2d)$ is very ample if $C$ is not hyperelliptic, Duke Math. J. 82 (1996), 503-552

J. Fay: Theta Functions on Riemann Surfaces, LNM 352, Springer-Verlag, Berlin, Heidelberg, New York, 1973

B. van Geemen, G. van der Geer: Kummer varieties and the moduli spaces of abelian varieties, Am. J. Math. 108 (1986), 615-642

B. van Geemen, E. Izadi: The tangent space to the moduli space of vector bundles on a curve and the singular locus of the theta divisor of the Jacobian, preprint 1997

M. Green: Quadrics of rank four in the ideal of the canonical curve, Invent. Math. 75 (1984), 85-104

R. C. Gunning: Some identities for Abelian integrals, Am. J. Math. 108 (1986), 39-74

R. C. Gunning: Riemann surfaces and their associated Wirtinger varieties, Bull. Am. Math. Soc. 11 (1984), 287-316

R. C. Gunning: Some identities for Abelian integrals II, unpublished preprint

R. Hartshorne: Algebraic Geometry, GTM 52, Springer-Verlag, New York, 1977

E. Izadi: The geometric structure of $A_4$, the structure of the Prym map, double solids and $\Gamma_0$-divisors, J. Reine Angew. Math. 462 (1995), 93-158

E. Izadi: Fonctions thêta du second ordre sur la jacobienne d’une courbe lisse, Math. Ann. 289 (1991), 189-202

H. Lange, M. S. Narasimhan: Maximal subbundles of rank two vector bundles on curves, Math. Ann. 266 (1983), 55-72

Y. Laszlo: Un théorème de Riemann pour les diviseurs thêta sur les espaces de modules de fibrés stables, Duke Math. J. 64 (1991), 333-347

S. Mukai: Curves and symmetric spaces I, Amer. J. Math. 117 (1995), 1627-1644

S. Mukai: Curves and Grassmannians, in Algebraic Geometry and Related Topics, eds. J-H. Yang, Y. Namikawa, K. Ueno, 1992

D. Mumford: Prym varieties I, Contributions to Analysis (L.V. Ahlfors, I. Kra, B. Maskit, and L. Niremberg, eds.), Academic Press, 1974, pp. 325-350

W.M. Oxbury, C. Pauly, E. Previato: Subvarieties of $SU_C(2)$ and $2\theta$-divisors in the Jacobian, Duke Algebraic Geometry preprint [alg-geom/9701010](alg-geom/9701010), to appear in Trans. A.M.S.
Christian Pauly  
Laboratoire J.-A. Dieudonné  
Université de Nice Sophia Antipolis  
Parc Valrose  
F-06108 Nice Cedex 02  
France  
E-mail: pauly@math.unice.fr

Emma Previato  
Department of Mathematics  
Boston University  
Boston, MA 02215-2411  
U.S.A.  
E-mail: ep@math.bu.edu