A Lower Bound on the Diameter of the Flip Graph

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Abstract. The flip graph is the graph whose nodes correspond to non-isomorphic combinatorial triangulations and whose edges connect pairs of triangulations that can be obtained one from the other by flipping a single edge. In this note we show that the diameter of the flip graph is at least $2n + \Theta(1)$, improving upon the previous $2n + \Theta(1)$ lower bound.

1 Introduction

A combinatorial triangulation is a maximal planar graph (a planar graph to which no edge can be added without destroying planarity) together with a clockwise ordering for the edges incident to each vertex. An intuitive way to define a combinatorial triangulation is as an equivalence class of planar drawings (say on the sphere) of a maximal planar graph, where two drawings are equivalent if a continuous morph exists from one drawing to the other that does not create crossings or overlaps between edges. We are interested in simple combinatorial triangulations, which have no self-loops or multiple edges. In the following, when we say triangulation we always mean simple combinatorial triangulation. Observe that, in a planar drawing equivalent to a triangulation, all the faces are delimited by cycles with three vertices (hence the name triangulation).

Consider a planar drawing $\Gamma$ on the sphere equivalent to a triangulation $G$ and consider an edge $(a, b)$ in $G$. If $(a, b)$ were removed from $\Gamma$, there would exist a unique region of the sphere delimited by a cycle with four edges; in fact the cycle delimiting such region would be $(a, a', b, b')$, for some vertices $a'$ and $b'$. The operation of flipping $(a, b)$ consists of removing $(a, b)$ from $G$ and inserting the edge $(a', b')$ inside the region delimited by the cycle $(a, a', b, b')$. The resulting triangulation $G'$ might not be simple though. In the following, we only refer to flips that maintain the triangulations simple.

The flip graph $G_n$ describes the possibility of transforming $n$-vertex triangulations using flips. The vertex set of $G_n$ is the set of distinct $n$-vertex triangulations; two triangulations $G$ and $H$ are connected by an edge in $G_n$ if there exists an edge $e$ of $G$ such that flipping $e$ in $G$ results in $H$.

Various properties of the flip graph have been studied. A particular attention has been devoted to the diameter of $G_n$, which is the length of the longest (among all pairs of vertices) shortest path; refer to the surveys \[3\] \[5\]. A first proof that the diameter of $G_n$ is finite goes back to almost a century ago \[11\]. A sequence of deep improvements \[4\] \[7\] \[9\] \[10\] have led to the current best upper bound of $5n + \Theta(1)$, which was proved this year by Cardinal et al. \[7\]. Significantly less results and techniques have been presented for the lower bound. We are only aware of a $2n + \Theta(1)$ lower bound on the diameter of $G_n$, which was proved by Komuro \[8\] by exploiting the existence of triangulations with “very different” vertex degrees. The main contribution of this note is the following theorem.

Theorem 1. For every $n \geq 3$, the diameter of the flip graph is at least $\frac{2n}{3} - 34$.

2 Proof of the Main Result

In this section we prove Theorem 1. Let $n \geq 3$. For a triangulation $G$, we denote by $V(G)$ and $E(G)$ its vertex and edge set, respectively.

Consider any $n$-vertex triangulation $G_1$. A path incident to $G_1$ in $G_n$ is a sequence of $n$-vertex triangulations such that the first triangulation in the sequence is $G_1$ and any two triangulations which are consecutive in the sequence can be obtained one from the other by flipping a single edge. Thus, a path incident to $G_1$ in $G_n$ corresponds to a valid sequence $\sigma = (u_1, v_1), \ldots, (u_k, v_k)$ of flips, where $u_1, \ldots, u_k, v_1, \ldots, v_k$ are vertices in $V(G_1)$ and $(u_i, v_i)$ is an edge of the triangulation obtained starting from $G_1$ by performing flips $(u_1, v_1), \ldots, (u_{i-1}, v_{i-1})$ in this order. For a valid sequence $\sigma$ of flips, denote
by $G_1^*$ the $n$-vertex triangulation obtained starting from $G_1$ by performing the flips in $\sigma$. Observe that $V(G_1) = V(G_1^*)$, given that a flip only modifies the edge set of a triangulation, and not its vertex set.

Now consider any two $n$-vertex triangulations $G_1$ and $G_2$ and consider a simple path in $G_n$, between them. This path corresponds to a valid sequence $\sigma$ of flips transforming $G_1$ into $G_2$. By the definition of $G_n$, the $n$-vertex triangulations $G_1^*$ and $G_2$ are isomorphic; that is, there exists a bijective mapping $\gamma : V(G_1^*) \to V(G_2)$ such that $(u, v) \in E(G_1^*)$ if and only if $(\gamma(u), \gamma(v)) \in E(G_2)$.

The key idea for the proof of Theorem 1 is to consider the bijective mapping $\gamma$ before the flips in $\sigma$ are applied to $G_1$ and to derive a lower bound on the number of flips in $\sigma$ based on properties of $\gamma$. In fact, the property we employ is the number of common edges of $G_1$ and $G_2$ according to $\gamma$.

More precisely, for a bijective mapping $\gamma : V(G_1) \to V(G_2)$ between the vertex sets of two triangulations $G_1$ and $G_2$, we define the number $c_\gamma$ of common edges with respect to $\gamma$ as the number of distinct edges $(u, v) \in E(G_1)$ such that $(\gamma(u), \gamma(v)) \in E(G_2)$. We have the following.

**Lemma 1.** For any two $n$-vertex triangulations $G_1$ and $G_2$, the number of flips needed to transform $G_1$ into $G_2$ is at least $3n - 6 - \max c_\gamma$, where the maximum is over all bijective mappings $\gamma : V(G_1) \to V(G_2)$.

**Proof.** The statement descends from the following two observations. First, two isomorphic $n$-vertex triangulations have $3n - 6$ common edges according to the bijective mapping $\gamma$ realizing the isomorphism. Second, for any two $n$-vertex triangulations $H$ and $L$ that have $\ell$ common edges with respect to any bijective mapping $\gamma$, flipping any edge in $H$ results in a combinatorial triangulation $H'$ such that $H'$ and $L$ have at most $\ell + 1$ common edges with respect to $\gamma$.

It remains to define two $n$-vertex triangulations $G_1$ and $G_2$ such that any bijective mapping $\gamma$ between their vertex sets has a small number $c_\gamma$ of common edges.

- Triangulation $G_1$ is defined as follows (see Fig. 1a). Let $H$ be any triangulation of maximum degree six with $\lceil \frac{n}{3} \rceil + 2$ vertices. Note that the number of faces of $H$ is $2(\lceil \frac{n}{3} \rceil + 2) - 4 = 2\lceil \frac{n}{3} \rceil$. If $n \equiv 2$ modulo 3, if $n \equiv 1$ modulo 3, or if $n \equiv 0$ modulo 3, then insert a vertex inside each face of $H$, insert a vertex inside each face of $H$ except for one face, or insert a vertex inside each face of $H$ except for two faces, respectively. When a vertex is inserted inside a face of $H$, it is connected to the three vertices of $H$ incident to the face. Denote by $G_1$ the resulting $n$-vertex triangulation. We say that the vertices of $G_1$ in $H$ are blue, while the other vertices of $G_1$ are red.

- Triangulation $G_2$ is defined as follows (see Fig. 1b). Starting from a path $P$ with $n - 2$ vertices, connect all the vertices of $P$ to two further vertices $a$ and $b$, and connect $a$ with $b$.

We have the following.

**Lemma 2.** For any bijective mapping $\gamma : V(G_1) \to V(G_2)$, we have $c_\gamma \leq 2\lceil \frac{n}{3} \rceil + 28$.

**Proof.** Consider any bijective mapping $\gamma : V(G_1) \to V(G_2)$. First, note that each vertex $v \in V(G_1)$ has degree at most twelve. Namely, $v$ has at most six blue neighbors; further, $v$ has at most six incident

![Fig. 1: Triangulations $G_1$ (a) and $G_2$ (b).](image-url)
faces in \( H \), hence it has at most six red neighbors. It follows that, whichever vertex in \( V(G_1) \) is mapped to \( a \) according to \( \gamma \), at most twelve of the \( n-1 \) edges incident to \( a \) are shared by \( G_1 \) and \( G_2 \) with respect to \( \gamma \). Analogously, at most twelve out of the \( n-1 \) edges incident to \( b \) are shared by \( G_1 \) and \( G_2 \) with respect to \( \gamma \). It remains to bound the number of edges of \( P \) that are shared by \( G_1 \) and \( G_2 \) with respect to \( \gamma \). This proof uses a pretty standard technique (see, e.g., \([5,7]\)). Since \( G_1 \) has no edge connecting two red vertices, the number of edges of \( P \) that are shared by \( G_1 \) and \( G_2 \) with respect to \( \gamma \) is at most the number of edges of \( P \) that have at least one of their end-vertices mapped to a blue vertex; since \( \lfloor \frac{n}{2} \rfloor + 2 \) vertices of \( G_1 \) are blue, there are at most \( 2 \lfloor \frac{n}{2} \rfloor + 4 \) such edges of \( P \). It follows that the number of edges shared by \( G_1 \) and \( G_2 \) with respect to \( \gamma \) is at most \( 2 \lfloor \frac{n}{2} \rfloor + 28 \). □

By Lemma 2 we have that \( G_1 \) and \( G_2 \) are two \( n \)-vertex triangulations such that, for any bijective mapping \( \gamma : V(G_1) \to V(G_2) \), we have \( c_\gamma \leq 2 \lfloor \frac{n}{2} \rfloor + 28 \). By Lemma 1 the number of flips needed to transform \( G_1 \) into \( G_2 \) is at least \( 3n - 6 - 2 \lfloor \frac{n}{2} \rfloor - 28 \geq 7n - 34 \). This concludes the proof of Theorem 1.

3 Conclusions

In this note we have presented a lower bound of \( \frac{2n}{n} + \Theta(1) \) on the diameter of the flip graph for \( n \)-vertex triangulations. One of the main ingredients for this lower bound is a lemma stating that there exist two \( n \)-vertex triangulations such that any bijective mapping \( \gamma \) between their vertex sets creates at most \( c_\gamma \leq \frac{2n}{n} + \Theta(1) \) common edges.

It is not clear to us whether the bound resulting from this approach can be improved further. That is, it is true that, for every two \( n \)-vertex triangulations, there exists a bijective mapping \( \gamma \) between their vertex sets creating \( c_\gamma \geq \frac{2n}{n} + \Theta(1) \) common edges? The only lower bound on the value of \( c_\gamma \) we are aware of comes as a corollary of the fact that every \( n \)-vertex triangulation has a matching of size at least \( \frac{n+1}{2} \) as proved in \([2]\), hence \( c_\gamma \geq \frac{n+1}{2} \).

It is an interesting fact that, for every \( n \)-vertex triangulation \( H \), a bijective mapping \( \gamma : V(H) \to V(G_2) \) exists creating \( c_\gamma = \frac{2n}{n} + \Theta(1) \) common edges, where \( G_2 \) is the graph from the proof of Theorem 1. In fact, every \( n \)-vertex triangulation \( H \) has a set of \( \frac{2n}{n} + \Theta(1) \) vertex-disjoint simple paths covering its vertex set \( V(H) \), as proved by Barnette \([1]\) (this bound is the smallest possible \([3]\)). Mapping these paths to sub-paths of the path \( P \) in \( G_2 \) provides the desired bijective mapping \( \gamma \).

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