Probabilistic hesitant triangular fuzzy power aggregation operator and its application

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Abstract. Based on the hesitant triangular fuzzy set, the probabilistic hesitant fuzzy set and the power aggregation operator, the probabilistic hesitant triangular fuzzy multi-attribute decision problem with the correlation between the aggregated data is studied. Firstly, the probabilistic hesitant triangular fuzzy set (PHTFS) is defined, and the algorithm, scoring function and distance measure of the probabilistic hesitant triangular fuzzy element (PHTFE) are proposed. Then, according to the PHTFS and power aggregation operator, the power aggregation operator of PHTFS is proposed. Finally, a multi-attribute decision making model is constructed based on the probabilistic hesitant triangular fuzzy power aggregation operator, and the validity of the model is verified by a case.

1. Introduction

At present, the fuzzy set [1] was successfully applied to the field of multi-attribute group decision making (MAGDM), theoretical results of decomposition have appeared. Many scholars have extended fuzzy sets, for example, interval fuzzy set (IFS) [2], intuitionistic fuzzy set (IFS) [3], triangular fuzzy set (TFS) [4]. In MAGDM, decision makers will hesitate between multiple evaluation values, and using only a single evaluation value is not enough to accurately express their true judgment results, so Torra et al. [5] proposed the hesitant fuzzy set (HFS). In recent years, scholars have carried out extensive research and application on hesitant fuzzy sets [6-8]. In particular, based on TFS and HFS, hesitant triangular fuzzy set (HTFS) was defined by Zhao et al. [9]. This set can be used to describe the fuzzy problem that there are several possible values in the degree to which things belong to an attribute.

In MAGDM, in order to reflect importance of some membership degrees, Zhang et al. [10] defined probabilistic hesitant fuzzy set (PHFS) based on HFS. Not only all the membership degrees of each evaluation alternative are provided, but also the different probabilities of each membership degree are provided. At present, many scholars have been interested in PHFS [11-16].

In the MAGDM problem, the interrelationship between information directly affects the rationality of decision results. Yager [17] measured the relationship between the elements to be aggregated by establishing the support function, and proposed a power mean (PA) aggregation operator based on this. In recent years, many scholars have studied PA aggregation operators. Xu [18] proposed intuitionistic fuzzy PA operators. By combining PA operators with Pythagorean fuzzy set (PFS), Wei et al. [19] pointed out Pythagorean fuzzy PA operators. Then Zhang [20] defined some hesitant fuzzy PA operators, and studied the property of them.
Although the HTFS allows the membership degree of an element to belong to a certain set to be multiple different values, the probability that each membership degree occurs is considered to be the same. For instance, when evaluating the performance of a new car, the evaluation results given by the experts are expressed as $\{[0.2,0.2,0.3], [0.4,0.5,0.6]\}$ in the form of hesitant triangular fuzzy element (HTFE), but the expert believes that the probability of occurrence of $[0.2,0.2,0.3]$ is greater than that of $[0.4,0.5,0.6]$, obviously the HTFS cannot express this preference of experts. In order to be able to consider the probability of each membership degree, based on HTFS and PHFS, this paper first proposes the probabilistic hesitant triangular fuzzy set (PHTFS) and gives relevant algorithms, score function values, distance measure and comparison rules. Considering the fact that there is a relationship between data in real decision making problems, this paper combines PA operator with PHTFS, and proposes the probability hesitant triangular fuzzy power average (PHTFPA) operator, probability hesitant triangular fuzzy power geometric (PHTFPG) operator, probability hesitant triangular fuzzy power weighted average (PHTFPWA) operator, and probability hesitant triangle fuzzy power weighted geometric (PHTFPWG) operator. Finally, a probabilistic hesitant triangular fuzzy MAGDM model applicable to the interrelationship between aggregated data is established, and verified by specific example.

2. Preliminaries

Definition 1[4] If $\tilde{a} = [a_L, a_M, a_U]$ and $0 \leq a_L \leq a_M \leq a_U$, then $\tilde{a}$ is called a triangular fuzzy number (TFN), and its membership degree function can be expressed as

$$
\mu_{\tilde{a}}(x) = \begin{cases} 
0, & x < a_L \\
\frac{x-a_L}{a_M-a_L}, & a_L \leq x < a_M \\
\frac{x-a_U}{a_M-a_U}, & a_M \leq x < a_U \\
0, & x \geq a_U 
\end{cases}
$$

(1)

Where $a_L$ and $a_U$ are the supported lower and upper bounds, respectively, and $a_M$ is the median values of $a$.

Definition 2[21] Let $\tilde{a} = [a_L, a_M, a_U]$ and $\tilde{\alpha} = [\alpha_L, \alpha_M, \alpha_U]$ be two TFNs on the domain $X$, the degree of possibility of $\tilde{a} \geq \tilde{b}$ is defined as

$$
p(\tilde{a} \geq \tilde{b}) = \varepsilon \max(1 - \max(\frac{b_M-a_L}{a_M-a_L}, 0), 0) + (1 - \varepsilon) \max(1 - \max(\frac{b_U-a_M}{a_U-a_M} + b_U - b_M, 0), 0)
$$

(2)

The parameter $\varepsilon$ represents the risk attitude of the decision maker. If the attitude is risk appetite, then $\varepsilon > 0.5$; if the attitude is risk neutral, then $\varepsilon = 0.5$; if the attitude is risk averse, then $\varepsilon < 0.5$. Based on definition 2, two triangular fuzzy numbers can be compared.

Definition 3[17] Let $\alpha_i (i = 1, 2, \cdots, n)$ be a set of non-negative real numbers, and $k = 1, 2, \cdots, n$, a) The power average (PA) operator is

$$
PA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \frac{1}{n} \sum_{i=1}^{n} (1+T(\alpha_i)) \alpha_i
$$

(3)

b) The power geometric (PG) operator is

$$
PG(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\prod_{i=1}^{n} (\alpha_i)\right)^{\frac{1}{\sum_{i=1}^{n} (1+T(\alpha_i))}}
$$

(4)
\[ T(\alpha_i) = \sum_{j=1}^{k} \text{Sup}(\alpha_i, \alpha_j) \]

Where \( \text{Sup}(\alpha_i, \alpha_j) \) represents the support degree of data \( \alpha_j \) for \( \alpha_i \), with:
1. \( \text{Sup}(\alpha_i, \alpha_j) \leq 1 \);
2. \( \text{Sup}(\alpha_i, \alpha_j) = \text{Sup}(\alpha_j, \alpha_i) \);
3. If \( |\alpha_i - \alpha_j| \leq |\alpha_i - \alpha_x| \), then \( \text{Sup}(\alpha_i, \alpha_x) \geq \text{Sup}(\alpha_i, \alpha_j) \).

### 3. Probabilistic Hesitant Triangular Fuzzy Sets (PHTFSs) and Related Theories

Considering that the hesitant triangular fuzzy set cannot effectively consider the different occurrence probabilities of each triangular fuzzy number, this paper proposes the PHTFS based on the references [9] and [10], and gives related algorithm, scoring function, and distance measure.

**Definition 4** The probabilistic hesitant triangle fuzzy set (PHTFS) on the domain \( X \) can be defined as

\[ H = \{ x \in \{1, \ldots, n\} \} \]

Where \( h_i(p) = \{ [y_{i1}, y_{i2}, y_{i3}] | p_x | i = 1, \ldots, l \} \) is called the probabilistic hesitant triangular fuzzy element (PHTFE), the \( [y_{i1}, y_{i2}, y_{i3}] \) represents the membership degree of the element \( x \) belonging to the set \( H \), \( p_x (p_x \geq 0) \) is the probability that \( [y_{i1}, y_{i2}, y_{i3}] \) occurs, and \( \sum_{i=1}^{l} p_x \leq 1 \), \( l \)
represents the number of \( [y_{i1}, y_{i2}, y_{i3}] \) in \( h(x) \).

For the convenience of representation, in this paper, \( h_i(p) = h(p) = \{ [y_{i1}, y_{i2}, y_{i3}] | p_x | i = 1, \ldots, l \} \), and the element \( [y_{i1}, y_{i2}, y_{i3}] \) is arranged from small to large according to the triangular fuzzy number \( [y_{i1}, y_{i2}, y_{i3}] \).

For the example of automobile safety evaluation proposed in the introduction, the expert's evaluation results are expressed as \{0.1,0.2,0.3\} \( [0.6, \{0.4,0.5,0.6\}] \) in the form of PHTFEs, where 0.6 and 0.4 represent the probabilities of \{0.1,0.2,0.3\} and \{0.4,0.5,0.6\}, respectively. Compared with the hesitant triangular fuzzy set, the PHTFS not only fully considers different membership degrees but also gives the probability of each membership degree, which effectively considers the preferences of decision makers.

If \( \sum_{i=1}^{l} p_x = 1 \), the evaluation information is complete. If \( \sum_{i=1}^{l} p_x < 1 \), the decision maker cannot give the complete evaluation information. In order to facilitate calculation and normalize the probability information, this paper proposes a standard PHTFE. The basic definition is as follows.

**Definition 5** Let \( \bar{h}(p) = \{ [y_{i1}, y_{i2}, y_{i3}] | p_x | i = 1, \ldots, l \} \) be a PHTFE, the standard PHTFE of \( h(p) \) is

\[ h(\bar{p}) = \{ [y_{i1}, y_{i2}, y_{i3}] | p_x | i = 1, \ldots, l \} \]

Where \( \bar{p}_x = p_x / \sum_{i=1}^{l} p_x \).

**Definition 6** Let \( h_1(p), h_2(p) \) and \( h(p) \) be three PHTFEs.

a) \( h_1(p) \otimes h_2(p) = \bigcup \{ [y_{i1}, y_{i2}, y_{i3}] | p_x | i = 1, \ldots, l \} \]

b) \( h_1(p) \otimes h_2(p) = \bigcup \{ [y_{i1}, y_{i2}, y_{i3}] | p_x | i = 1, \ldots, l \} \]
\[ c) \alpha h(p) = \bigcup_{\lambda \in \{1, 2, \ldots, \lambda\}} \left\{ \left[ 1 - \left(1 - \gamma^L_{\lambda} \right)^{\alpha}, 1 - \left(1 - \gamma^M_{\lambda} \right)^{\alpha}, 1 - \left(1 - \gamma^U_{\lambda} \right)^{\alpha} \right] \right\} | p^{\lambda} \]  
\[ d) (h(p))^{\alpha} = \bigcup_{\lambda \in \{1, 2, \ldots, \lambda\}} \left\{ \left[ \gamma^L_{\lambda}, \gamma^M_{\lambda}, \gamma^U_{\lambda} \right] \right\} | p^{\lambda} \]  
Where \( \alpha > 0 \).

**Definition 7** Let \( h(p) = \left\{ \left[ \gamma^L_{\lambda}, \gamma^M_{\lambda}, \gamma^U_{\lambda} \right] \right\} | p^{\lambda} | \lambda = 1, 2, \ldots, \lambda \) be a PHTFE, then the score function is
\[ s(h(p)) = \left[ \sum_{\lambda=1}^{\lambda} p^{\lambda} \gamma^L_{\lambda}, \sum_{\lambda=1}^{\lambda} p^{\lambda} \gamma^M_{\lambda}, \sum_{\lambda=1}^{\lambda} p^{\lambda} \gamma^U_{\lambda} \right] \]  
(5)

For any two PHTFEs \( h_1(p) \) and \( h_2(p) \), combined with the Definition 2, if \( s(h_1(p)) = s(h_2(p)) \), then \( h_1(p) = h_2(p) \), if \( s(h_1(p)) > s(h_2(p)) \), then \( h_1(p) > h_2(p) \).

**Definition 8** Let \( h_1(p) = \left\{ \left[ \gamma^L_{\lambda}, \gamma^M_{\lambda}, \gamma^U_{\lambda} \right] \right\} | p^{\lambda} | \lambda = 1, 2, \ldots, \lambda \) and \( h_2(p) = \left\{ \left[ \gamma^L_{\lambda}, \gamma^M_{\lambda}, \gamma^U_{\lambda} \right] \right\} | p^{\lambda} | \lambda = 1, 2, \ldots, \lambda \) be two PHTFEs, the Hamming distance between them is
\[ d(h_1(p), h_2(p)) = \frac{1}{\lambda} \sum_{\lambda=1}^{\lambda} \left| \gamma^L_{\lambda} - \gamma^L_{\lambda} \right| + \left| \gamma^M_{\lambda} - \gamma^M_{\lambda} \right| + \left| \gamma^U_{\lambda} - \gamma^U_{\lambda} \right| \]  
(6)

**Property 1** Let \( d(h_1(p), h_2(p)) \) be the Hamming distance of two PHTFEs \( h_1(p) \) and \( h_2(p) \), then

a) \( 0 \leq d(h_1(p), h_2(p)) \leq 1 \)

b) \( d(h_1(p), h_2(p)) = d(h_2(p), h_1(p)) \)

c) \( d(h_1(p), h_2(p)) = 0 \) if and only if \( h_1(p) = h_2(p) \)

4. Probabilistic Hesitant Triangular Fuzzy Power Aggregation Operators and Related Theories

Considering the mutual relationship between the aggregated data, this paper combines the PHTFS with the traditional power aggregation operator. First, the probabilistic hesitant triangular fuzzy power average (PHTFPA) operator and the probabilistic hesitant triangular fuzzy power geometric (PHTFPG) operator are proposed. Then, their related properties are studied. Finally, considering the weight of each triangular fuzzy element may be different, the probabilistic hesitant triangular fuzzy power weighted average (PHTFPWA) operator and the probabilistic hesitant triangular fuzzy power weighted geometric (PHTFPWG) operator are proposed.

**Definition 9** Let \( h_1(p), h_2(p), \ldots, h_n(p) \) be \( n \) PHTFEs,

a) The PHTFPA operator is
\[ \text{PHTFPA}(h_1(p), h_2(p), \ldots, h_n(p)) = \sum_{i=1}^{n} \frac{1 + T(h_i(p))}{\sum_{i=1}^{n} (1 + T(h_i(p)))} h_i(p) \]  
(7)

b) The PHTFPG operator is
\[ \text{PHTFPG}(h_1(p), h_2(p), \ldots, h_n(p)) = \sum_{i=1}^{n} \frac{1 + T(h_i(p))}{\sum_{i=1}^{n} (1 + T(h_i(p)))} (h_i(p)) \]
Where $T(h_i(p)) = \sum_{j=1, j\neq i}^{n}\text{Sup}(h_i(p), h_j(p))$, $\text{Sup}(h_i(p), h_j(p))$ represents the support degree of $h_j(p)$ for $h_i(p)$, with:

1. $0 \leq \text{Sup}(h_i(p), h_j(p)) \leq 1$;
2. $\text{Sup}(h_i(p), h_j(p)) = \text{Sup}(h_j(p), h_i(p))$;
3. If $d(h_i(p), h_j(p)) \geq d(h_i(p), h_k(p))$, then $\text{Sup}(h_i(p), h_j(p)) \leq \text{Sup}(h_i(p), h_k(p))$, where $d$ represents the distance measure of PHTFEs.

**Theorem 1** The aggregated value of PHTFPWA operator and PHTFPWG operator is still PHTFE.

**Theorem 2** (boundedness) Let $h_1(p), h_2(p), \ldots, h_n(p)$ be $n$ PHTFEs, then

$$h^-(p) \leq \text{PHTFPWA}(h_1(p), h_2(p), \ldots, h_n(p)) \leq h^+(p)$$

$$h^+(p) \leq \text{PHTFPWG}(h_1(p), h_2(p), \ldots, h_n(p)) \leq h^+(p)$$

Where $h^-(p) = \{\min_{i=1}^{n} \min_{j=1}^{l} [y_{i1}^d, y_{i2}^d, y_{i3}^d] | 1 \}$ and $h^+(p) = \{\max_{i=1}^{n} \max_{j=1}^{l} [y_{i1}^d, y_{i2}^d, y_{i3}^d] | 1 \}$.

**Theorem 3** (commutativity) Let $h_1(p), h_2(p), \ldots, h_n(p)$ be $n$ PHTFEs, $h'_1(p), h'_2(p), \ldots, h'_n(p)$ is any permutation of them, then

$$\text{PHTFPWA}(h_1(p), h_2(p), \ldots, h_n(p)) = \text{PHTFPWA}(h'_1(p), h'_2(p), \ldots, h'_n(p))$$

$$\text{PHTFPWG}(h_1(p), h_2(p), \ldots, h_n(p)) = \text{PHTFPWG}(h'_1(p), h'_2(p), \ldots, h'_n(p))$$

**Theorem 4** (monotonicity) Let $h_1(p), h_2(p), \ldots, h_n(p)$ and $g_1(p), g_2(p), \ldots, g_n(p)$ be two sets of PHTFEs, if $h_i(p) \leq g_i(p) (i = 1, 2, \ldots, n)$, then

$$\text{PHTFPWA}(h_1(p), h_2(p), \ldots, h_n(p)) \leq \text{PHTFPWA}(g_1(p), g_2(p), \ldots, g_n(p))$$

$$\text{PHTFPWG}(h_1(p), h_2(p), \ldots, h_n(p)) \leq \text{PHTFPWG}(g_1(p), g_2(p), \ldots, g_n(p))$$

Considering that $n$ PHTFEs are $h_1(p), h_2(p), \ldots, h_n(p)$, they may be assigned different weights. Therefore, this paper proposes the PHTFPWA operator and the PHTFPWG operator that consider the weight of the data.

**Definition 10** Let $h_1(p), h_2(p), \ldots, h_n(p)$ be $n$ PHTFEs, $w = (w_1, w_2, \ldots, w_n)^T$ be the related weight vectors, with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

a) The PHTFPWA operator is

$$\text{PHTFPWA}(h_1(p), h_2(p), \ldots, h_n(p)) = \bigoplus_{i=1}^{n} w_i (1 + T(h_i(p)))h_i(p)$$

$$= \bigcup_{[y_{i1}^d, y_{i2}^d, y_{i3}^d] \in \text{H}(h_i(p))} \left\{ \left[ 1 - \prod_{i=1}^{n} (1 - y_{il}^d) \right] \left[ 1 - \prod_{i=1}^{n} (1 - y_{il}^d) \right] \left[ 1 - \prod_{i=1}^{n} (1 - y_{il}^d) \right] \right\}$$

b) The PHTFPWG operator is
5. MAGDM Model Based on Probabilistic hesitant Triangular Fuzzy Power Aggregation Operator

We constructs a probabilistic hesitant triangular fuzzy MAGDM model based on the probabilistic hesitant triangular fuzzy power aggregation operator.

For any probabilistic hesitant triangular fuzzy MAGDM problem, suppose that \( X = \{x_i | i = 1, 2, \ldots, m\} \) is a given set of alternatives, \( C = \{c_j | j = 1, 2, \ldots, n\} \) is the set of attributes, the set of the weights of all attributes is \( w = (w_1, w_2, \ldots, w_n)^T \). The decision steps are as follows:

a) First get the expert’s decision results and get the decision matrix \( M = (h_{ij}(p))_{m \times n} \).

b) If the weights of all attributes are unknown, then calculate the comprehensive attribute information of each alternative \( x_i (i = 1, 2, \ldots, n) \) by PHTFPA operator or PHTFPG operator. The formulas are as follows:

\[
PHTFPWG(h_1(p), h_2(p), \ldots, h_n(p)) = \frac{n}{\sum_{j=1}^{n}w_j(1+T(h_j(p)))} \sum_{j=1}^{n}w_j(1+T(h_j(p)))
\]

\[
= \bigcup \left\{ \left[ \prod_{j=1}^{n}(y_{j}^{w_j}) \sum_{j=1}^{n}w_j(1+T(h_j(p))) , \prod_{j=1}^{n}(y_{j}^{w_j}) \sum_{j=1}^{n}w_j(1+T(h_j(p))) , \prod_{j=1}^{n}(y_{j}^{w_j}) \sum_{j=1}^{n}w_j(1+T(h_j(p))) \right] \bigg| \sum_{j=1}^{n}w_j(1+T(h_j(p))) \right\}
\]

\[
(16)
\]

\[
PHTFPWG(h_1(p), h_2(p), \ldots, h_n(p)) = \frac{1}{\sum_{j=1}^{n}(1+T(h_j(p)))} \sum_{j=1}^{n}(1+T(h_j(p)))
\]

\[
(17)
\]

\[
= \bigcup \left\{ \left[ \prod_{j=1}^{n}(y_{j}^{w_j}) \sum_{j=1}^{n}(1+T(h_j(p))) , \prod_{j=1}^{n}(y_{j}^{w_j}) \sum_{j=1}^{n}(1+T(h_j(p))) , \prod_{j=1}^{n}(y_{j}^{w_j}) \sum_{j=1}^{n}(1+T(h_j(p))) \right] \bigg| \sum_{j=1}^{n}(1+T(h_j(p))) \right\}
\]

\[
(18)
\]

\[
PHTFPWG(h_1(p), h_2(p), \ldots, h_n(p)) = \frac{1}{\sum_{j=1}^{n}(1+T(h_j(p)))} \sum_{j=1}^{n}(1+T(h_j(p)))
\]

\[
(19)
\]

\[
= \bigcup \left\{ \left[ \prod_{j=1}^{n}(y_{j}^{w_j}) \sum_{j=1}^{n}(1+T(h_j(p))) , \prod_{j=1}^{n}(y_{j}^{w_j}) \sum_{j=1}^{n}(1+T(h_j(p))) , \prod_{j=1}^{n}(y_{j}^{w_j}) \sum_{j=1}^{n}(1+T(h_j(p))) \right] \bigg| \sum_{j=1}^{n}(1+T(h_j(p))) \right\}
\]

\[
(20)
\]

c) If the weights of all attributes are known, then calculate the comprehensive attribute information of each alternative \( x_i (i = 1, 2, \ldots, n) \) by PHTFPWA operator or PHTFPWG operator. The formulas are as follows:

d) By the formula (5), the score function value \( s(h_i(p)) \) of \( h_i(p)(i = 1, 2, \ldots, m) \) be calculated, the alternatives are sorted by Definition 2 and \( s(h_i(p)) \), and the most appropriate scheme is selected.

6. Illustrative Example

6.1. Problem Description

In the field of product design, reasonable module division not only better meets the diverse product needs of customers, but also facilitates the timely testing and maintenance of products. Taking a
switchgear design as an example, according to the mapping relationship between functions, structures, and user needs, module partitioning of switchgear components can obtain 4 module division schemes \( X = \{ x_1, x_2, x_3, x_4 \} \). Enterprises generally evaluate the division of each module from four attributes: 
- \( c_1 \): Purchase Cost, 
- \( c_2 \): Design Cost, 
- \( c_3 \): Production Cost, 
- \( c_4 \): Repair Cost, and select the best scheme [22]. The attribute weighted vectors is \( w = (0.3, 0.4, 0.2, 0.1)^T \). The evaluation results are expressed in the form of PHTFSs, and they are shown in Table 1.

**Table 1. Probabilistic hesitant triangular fuzzy decision matrix \( M \)**

|   | \( c_1 \) | \( c_2 \) | \( c_3 \) | \( c_4 \) |
|---|---|---|---|---|
| \( x_1 \) | \([0.2,0.3,0.4]|0.5, [0.3,0.4,0.5]|0.5 \) | \([0.2,0.3,0.4]|0.5, [0.4,0.5,0.6]|0.5 \) | \([0.2,0.3,0.4]|0.5, [0.4,0.5,0.6]|0.5 \) | \([0.2,0.4,0.6]|0.4, [0.5,0.6,0.7]|0.6 \) |
| \( x_2 \) | \([0.3,0.4,0.5]|0.4, [0.6,0.7,0.8]|0.6 \) | \([0.2,0.3,0.5]|0.7, [0.4,0.5,0.7]|0.3 \) | \([0.5,0.6,0.8]|0.6, [0.6,0.7,0.9]|0.4 \) | \([0.6,0.7,0.8]|0.8, [0.7,0.8,0.9]|0.6 \) |
| \( x_3 \) | \([0.2,0.3,0.4]|0.5, [0.5,0.6,0.7]|0.5 \) | \([0.3,0.4,0.6]|0.8, [0.5,0.6,0.7]|0.2 \) | \([0.4,0.5,0.6]|0.5, [0.7,0.8,0.9]|0.5 \) | \([0.6,0.7,0.8]|0.8, [0.7,0.8,0.9]|0.6 \) |
| \( x_4 \) | \([0.2,0.3,0.4]|0.5, [0.4,0.5,0.6]|0.5 \) | \([0.3,0.4,0.6]|0.8, [0.5,0.6,0.7]|0.2 \) | \([0.4,0.5,0.6]|0.5, [0.7,0.8,0.9]|0.5 \) | \([0.6,0.7,0.8]|0.8, [0.7,0.8,0.9]|0.6 \) |

6.2. Evaluation Plan Ranking

For the convenience of calculation, we suppose \( \varepsilon = 0.5 \). Because the weights of all attributes in this case are known, the PHTFPWA operator and PHTFPWG operator are used for decision analysis.

- **Method 1**: The decision information of each alternative \( x_i (i = 1, 2, 3, 4) \) is aggregated by PHTFPWA operator, then get comprehensive attribute information \( h_i (p) \) for each alternative. According to formula (5), the score function value can be calculated, the results are 
  \[ s(h_1 (p)) = [0.3020, 0.4094, 0.5213], \quad s(h_2 (p)) = [0.4101, 0.5235, 0.9789], \quad s(h_3 (p)) = [0.6435, 0.7543, 0.9789], \quad s(h_4 (p)) = [0.3480, 0.4861, 0.6433]. \]  
According to formula (2), the result of the sorting is \( x_3 \succ x_2 \succ x_4 \succ x_1 \).

- **Method 2**: The decision information of \( x_i (i = 1, 2, 3, 4) \) is aggregated by PHTFPWA operator, then get comprehensive attribute information \( h_i (p) \) for each alternative. According to formula (5), the score function value can be calculated, the results are 
  \[ s(h_1 (p)) = [0.2706, 0.3785, 0.4842], \quad s(h_2 (p)) = [0.3552, 0.4749, 0.8758], \quad s(h_3 (p)) = [0.6108, 0.7124, 0.8758], \quad s(h_4 (p)) = [0.3160, 0.4531, 0.6103]. \]  
According to formula (2), the result of the sorting is \( x_3 \succ x_2 \succ x_4 \succ x_1 \).

According to the calculation results of the PHTFPWA operator and the PHTFPWG operator, it can be finally determined that the scheme \( x_3 \) is the best.

6.3. Comparative Analysis

For expressing the advantages of the PHTFPWA operator and the PHTFPWG operator, this paper selects the hesitant triangular fuzzy weighted average (HTFWA) operator and the hesitant fuzzy triangular fuzzy weighted geometric (HTFWG) operator mentioned in reference [9] to make a decision analysis on the above example. Because the HTFE meets that the probability of each TFN is consistent, the following hesitant triangular fuzzy decision matrix \( D \) is obtained. Let the weighted vector of all attributes is \( w = (0.3, 0.4, 0.2, 0.1)^T \).
### Table 2. Hesitant triangular fuzzy decision matrix $D$

|   | $c_1$                      | $c_2$                      | $c_3$                      | $c_4$                      |
|---|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $x_1$ | $[0.2,0.3,0.4],[0.3,0.4,0.5]$ | $[0.2,0.3,0.4],[0.4,0.5,0.6]$ | $[0.2,0.3,0.4],[0.2,0.4,0.6]$ | $[0.3,0.4,0.5],[0.7,0.8,0.9]$ |
| $x_2$ | $[0.3,0.4,0.5],[0.6,0.7,0.8]$ | $[0.2,0.3,0.5],[0.4,0.5,0.7]$ | $[0.6,0.7,0.8],[0.6,0.7,1.0]$ | $[0.2,0.4,0.5],[0.3,0.5,0.6]$ |
| $x_3$ | $[0.7,0.8,0.9],[0.8,0.9,1.0]$ | $[0.5,0.6,0.8],[0.6,0.7,0.9]$ | $[0.6,0.7,0.9],[0.8,0.9,1.0]$ | $[0.4,0.5,0.6],[0.7,0.8,0.9]$ |
| $x_4$ | $[0.2,0.3,0.4],[0.5,0.6,0.7]$ | $[0.3,0.4,0.6],[0.5,0.6,0.7]$ | $[0.2,0.4,0.6],[0.3,0.5,0.7]$ | $[0.1,0.3,0.5],[0.6,0.8,0.9]$ |

The calculation results using HTFWA operator and HTFWG operator are shown in the following table.

### Table 3. Comparison of the results of the four calculation methods

| Method   | Ranking Results | Best Scheme |
|----------|-----------------|-------------|
| HTFWA    | $x_3 > x_2 > x_4 > x_1$ | $x_3$ |
| HTFWG    | $x_1 > x_2 > x_3 > x_4$ | $x_1$ |
| PHTFPWA  | $x_3 > x_1 > x_4 > x_2$ | $x_1$ |
| PHTFPWG  | $x_1 > x_3 > x_2 > x_4$ | $x_3$ |

By comparing the ranking results, it can be seen that the ranking results obtained by using the operators constructed in this paper are the same as the ranking results obtained by using the two operators in reference [9]. However, the PHTFPWA operator and the PHTFPWG operator mentioned have the following advantages: (1) The PHTFPWA operator and the PHTFPWG operator take into account the probability of occurrence of the evaluation result, which is more reasonable than HTFWA operator and HTFWG operator; (2) The PHTFPWA operator and PHTFPWG operator consider the relationship between the data, while the HTFWA operator and the HTFWG operator consider the data to be independent of each other, so the decision results obtained by the model in this paper are closer to the actual situation.

### 7. Conclusion

PHTFS is an extension of the HTFS, which is more suitable for expressing evaluation information of decision makers. This paper firstly defined the basic concepts of the PHTFS, then proposed PHTFPA operator, PHTFPG operator, PHTFPWA operator, and PHTFPWG operator. Finally this paper pointed out a very effectively method in order to solve the probabilistic hesitant triangular fuzzy MAGDM problem. In the future study, the cross-entropy measures of PHTFS will be discussed.

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