Linear magnetic response of external constant backgrounds to an applied Coulomb field. Magnetic-monopole solution.

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Abstract

Linear Maxwell equations, corresponding to the first-power deviations from a constant-field background, are studied within the infrared approximation. Linear magnetic responses, linearly induced currents and vector potentials to an imposed static finite-sized charge in an arbitrary combination of constant and homogeneous electric and magnetic fields are discussed. Two class of solutions are found, namely, magnetically neutral responses whose corresponding magnetic charges are zero, as a result of the fulfillment of the Bianchi identity at every point of the space and a magnetically charged one, in which the latter identity fails at the origin \( x = 0 \), resulting to a nontrivial magnetic charge thereby. The vector potential associated with the magnetically neutral responses are free of divergence while, for the latter case, are found to be Dirac strings, associated with an infinite number of points, in which the potential is singular and restricted to half-axes. All coefficients, proportional to the effective lagrangian and derivatives of it, are specialized to Quantum Electrodynamics, wherein the strong magnetic background regime is discussed in detail.

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1 Introduction

It is well understood that the vacuum filled with strong background field is, in Quantum Electrodynamics (QED), equivalent to a linear or nonlinear medium \cite{1, 2, 3}. The simplest (one-loop) Feynman diagrams responsible for description of such a media are shown in Fig. 1.

Figure 1: The second- and third-rank polarization tensors.

The bold lines there stand for the electron propagators in the external field. These are known exactly for the constant background, for the background plane electromagnetic field, also for special combinations of the latter two. In the first case the equivalent medium is space- and time-homogeneous, otherwise it is not, and then the energy and momentum exchange between the field and the background occurs. Using the exact solutions to the Dirac equation in external field makes these diagrams belonging to the so-called Furry picture. The first diagram corresponds to linearization of the field equations above the background. It represents the (second-rank) polarization tensor \( \Pi_{\mu\nu}(x, y) \), which contains in itself the linear polarizational properties of the equivalent medium, usually referred to as dielectric permeability and magnetic permittivity. It is responsible for the screening of charges and currents and transformations of their shapes due to the strong background, and for the photon propagation in the background, especially for polarization of the eigen-modes and (different) modifications of the mass shell in each mode (the birefringence making a goal for observations \cite{4}) by deviating the dispersion curves from the standard shape \( k_0^2 = k^2 \) known in the empty vacuum (the one with null background). The second diagram in Fig. 1, the third-rank polarization tensor \( \Pi_{\mu\nu\rho}(x, y, z) \), takes into account the quadratic deviation of the field from the background. When taken on the photon mass shell, it is responsible for the photon splitting and merging in an external field \cite{5, 6}. Beyond the mass shell, it also describes the response of the medium to small perturbations with the quadratic accuracy relative to these perturbations. Analogously, the fourth-rank polarization tensor includes the cubic response, photon-by-photon scattering (the first experimental detection of this fundamental process, which is the source of nonlinearity of QED, was recently reported in \cite{7}), photon splitting into two \cite{8}, and so on.

The polarization tensors may be defined as variational derivatives of the effective action \( \Gamma \)
known as [8], the generating functional of the one-particle-irreducible vertex function\footnote{Greek indices span the 4-dimensional Minkowski space-time, e. g., \( \mu = (0, i) \), \( i = 1, 2, 3 \), \( \eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1) \), and boldface letters denote three-dimensional Euclidean vectors (e. g., \( \mathbf{A}(x) = (A^i(x)) \), \( i = 1, 2, 3 \)). The Heaviside system of units, \( \hbar = c = 1 \), \( \alpha = e^2/4\pi \) is used throughout the paper. The four-rank and three-rank Levi-Civita tensors are normalized as \( \varepsilon_{0123} = +1 \) and \( \varepsilon_{123} = 1 \), respectively.}

\[
\Pi^{\alpha\rho}(x, y) = \frac{\delta^2 \Gamma}{\delta A_\alpha(x) \delta A_\rho(y)}, \quad \Pi^{\alpha\rho\beta}(x, y, z) = \frac{\delta^3 \Gamma}{\delta A_\alpha(x) \delta A_\rho(y) \delta A_\beta(z)} \bigg|_{A=\overline{A}},
\]

with respect to the potentials \( A_\alpha(x) \), taken at their background values \( A_\alpha(x) = \overline{A}_\alpha(x) \). This is equivalent to the Feynman diagram representation. Each differentiation adds an extra photon vertex to the diagram. The polarization tensors of all ranks participate in the nonlinear Maxwell equations

\[
\left[ \eta_{\rho\alpha} \Box - \partial_\rho \partial_\alpha \right] a_\rho(x) + \int d^4y \Pi^{\alpha\rho}(x, y) a_\rho(y) + \frac{1}{2} \int d^4y d^4z \Pi^{\alpha\rho\beta}(x, y, z) a_\rho(y) a_\beta(z) + O(a^3) = j_\alpha(x),
\]

(1)

where \( a_\alpha(x) \) is the potential over the background, \( a_\alpha(x) = A_\alpha(x) - \overline{A}_\alpha(x) \), i. e., the response to the applied source (perturbation) \( j_\rho(x) \). Bearing in mind that each vertex in the diagrams carries a small quantity, the electron charge \( e \), we may approach this equation perturbatively. Then, at the classical level, the solution to (1) is

\[
a^{(0)}_\alpha(x) = \int D^{\alpha\alpha'}_{(0)}(x - x') j_{\alpha'}(x') \, d^4x',
\]

(2)

where \( D^{\rho\rho}_{(0)}(x - x') \) is the free photon propagator. The first correction to it (within the linearity of the equation) is

\[
a^{(1)}_\alpha(x) = \int D^{\alpha\alpha'}_{(0)}(x - x') \Pi_{\alpha'\rho'}(x', y') D^{\rho\rho}_{(0)}(y' - y) j_\rho(y) \, d^4x' d^4y' d^4y.
\]

(3)

The correction of the second power of the perturbation \( j \) is

\[
a^{(2)}_\alpha(x) = \frac{1}{2} \int D^{\alpha\alpha'}_{(0)}(x - x') \Pi_{\alpha'\rho',\beta'}(x', y', z') D^{\rho\rho}_{(0)}(y' - y) j_\rho(y) D^{\beta\beta}_{(0)}(z' - z) j_\beta(z) \, d^4x' d^4y' d^4z' d^4y d^4z.
\]

(4)

The terms (3) and (4) are shown graphically in Fig. 2 where the wiggly line stands for the free photon propagator \( D^{\rho\rho}_{(0)}(x - x') \).

An essential simplification of the calculations is achieved if one confines oneself to the approximation of the local, effective action, where the functional \( \Gamma \) does not depend on the space-time
derivatives of the fields, for instance, where the Euler-Heisenberg expression for it is taken in one-loop [3] or two-loop [9, 10] approximation. This approximation is good as long as the fields slowly varying in time and space are dealt with. Contrary to [1], within this approximation the field equations are differential (not integral) ones, and they do not include higher derivatives, while (the Fourier transform of) the polarization tensor of $n$-th rank behaves as the $n$-th power of the momentum; in the language of optics this corresponds to disregard of the spatial and frequency dispersion. We thoroughly traced the derivation of the Maxwell equations within the local approximation in [11], [12], [13], [14]. In [12] the self-interaction of magnetic and electric dipole moments, which modifies their values calculated within any version of the strong-interaction theory, was considered using the 4-th rank polarization tensor with no background. In [15] the self-interaction of a point-like charge was studied with the same tools leading to the result that its field-energy (beyond the perturbation approach) is finite, although the field in the center of the charge is not. Therefore, the point charge, with its field being a solution of a nonlinear equation, becomes a soliton at rest or in motion [16], [17]. Interaction between long-wave electromagnetic waves were considered taking into account, effectively, the polarization tensors up to 6-th rank [18], [19]. In [13], [20] we showed that the quadratic response of the vacuum with the background of a constant magnetic field to an applied electric field of a point-like or extended central-symmetric charge, governed by the 3-rd rank polarization tensor and corresponding to term (4) is purely magnetic, i.e. we face here the magneto-electric effect. Moreover, the magnetic response far from the charge is the field of a magnetic dipole with its dipole moment quadratically dependent upon the electric charge. The photon splitting on the basis of the same diagram was studied in [6]. In [21] and now we take an arbitrary combination of constant electric $E$ and magnetic $B$ fields as a background (see [22] beyond the local approximation), and we consider linear response to an applied electric charge following the information contained in the 2-rd rank polarization tensor (term (4)). This response may be both electric and magnetic. The electric response was studied in [21], resulting in description of the induced charge density and modification of the Coulomb field far from the

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Figure 2: The linear and quadratic responses to applied current $j_\mu (x)$.
In the present paper we study two types of linear magnetic responses: magnetically neutral and magnetically charged. If the applied electric charge is point-like, there exists a response in the form of a spherically nonsymmetric magnetic monopole, whose magnetic charge is formed by the electric charge multiplied by a pseudoscalar combination of the background fields. Our publication [23] may be considered as giving preliminary results of this kind that relate to the special case of small background fields, parallel to one another, $E \parallel B$. The latter specialization results in the fact that there is only one Dirac string, whereas in the present case under consideration the number of strings is two, and the magnitude of the background field is not restricted.

The paper is organized as follows. In Sec. 2 after presenting the necessary Maxwell equations linearized near the background field and indicating the structure of the applied electric field, we obtain expressions for the electric current density induced in the "medium" inside and outside of the applied extended charge. In Subsecs. 2.1 and 2.2 we find the magnetic fields produced by this current source that carry zero and nonzero magnetic charges, respectively. In Sec. 3 the vector-potential is written for the spherically nonsymmetric magnetic-monopole solution found in Subsec. 2.2 which is singular along the two directions of the background fields. All the results reported above are written in terms of the derivatives of the local effective Lagrangian over the field invariants taken at the background. Hence these may be used with every model Lagrangian irrespective of its origin and of its connection to QED. On the contrary, in Sec. 4 we specialize the results to the one-loop Euler-Heisenberg Lagrangian of QED. In Conclusion the detailed analysis of the results is given.

## 2 Linearly induced currents and magnetic responses in constant backgrounds

Let there be a background electromagnetic field, with its field tensor $F_{\nu\mu}(x)$ equal to $\bar{F}_{\nu\mu}(x)$, produced by the background current $J_\mu$ via the (second) Maxwell equations

$$
\partial^\nu \bar{F}_{\nu\mu}(x) - \partial^\nu \left[ \frac{\delta L(x)}{\delta \bar{\mathbf{F}}(x)} \right]_{\bar{F}=\bar{F}} \bar{F}_{\nu\mu}(x) + \frac{\delta L(x)}{\delta \mathbf{G}(x)} \mathcal{J}_\mu(x) = J_\mu(x),
$$

(5)

whithin a nonlinear local electrodynamics with the Lagrangian

$$
L(x) = -\mathcal{F}(x) + \mathcal{L}(x),
$$

(6)

\(^2\text{A brief review of our previous works may be found in [14].}\)
where its nonlinear part $\mathcal{L}(x)$ is taken as a function $\mathcal{L}(x) = \mathcal{L}(\mathfrak{F}, \mathfrak{G})$ of two field invariants $\mathfrak{F} = (1/4) F^{\mu \nu} F_{\mu \nu} = (1/2) (B^2 - E^2)$, $\mathfrak{G} = (1/4) \tilde{F}^{\mu \nu} F_{\mu \nu} = -(E \cdot B)$, $\tilde{F}_{\mu \nu}(x) = (1/2) \varepsilon_{\mu \nu \alpha \beta} F^{\alpha \beta}(x)$, and may be thought of, for instance, as the effective Lagrangian of Quantum Electrodynamics in the local (infra-red) approximation, i.e., the one where the dependence on the space- and time-derivatives of the fields is neglected. (The special case with the Euler-Heisenberg Lagrangian accepted as the one-loop approximation of such local effective action will be treated for the purposes of the present article in Section 4 below.)

We shall be considering the constant background $F_{\nu \mu}(x) = \bar{F}_{\nu \mu} = \text{const.}$ here. This field does not require any current to be supported: it is seen that equation (5) is satisfied by $J_\mu(x) = 0$ in this case.

Let the constant background be perturbed by small introduced current $j_\mu(x)$. It causes the deviation $f_{\nu \mu}(x) = F_{\nu \mu}(x) - \bar{F}_{\nu \mu}$ of the field from the background. Expanding the Maxwell equations in powers of $f_{\nu \mu}(x)$ we obtain in the first order the linear equation (see Refs. [11, 12, 13, 14, 15, 20, 21, 23] for equations for higher orders, which are nonlinear as containing the second and higher powers of $f_{\nu \mu}(x)$):

\[
\frac{\partial^\nu}{\partial x^\nu} f_{\nu \mu}(x) = j^\text{lin}_\mu(x) + j_\mu(x),
\]

\[
j^\text{lin}_\mu(x) = \partial^\tau \left[ \mathcal{L}_\mathfrak{F} f_{\tau \mu}(x) + \frac{1}{2} \left( \mathcal{L}_\mathfrak{F} \bar{F}_{\alpha \beta} + \mathcal{L}_\mathfrak{G} \bar{F}_{\alpha \beta} \right) \bar{F}_{\tau \mu} f^{\alpha \beta}(x) \right],
\]

where the subscripts by $\mathcal{L}$ designate derivatives with respect to the indicated field invariants taken at their background value, for instance $\frac{\partial^2 \mathcal{L}}{\partial \mathfrak{F} \partial \mathfrak{G}} \bigg|_{\mathfrak{F} = \bar{F}} = \mathcal{L}_\mathfrak{F} \mathfrak{G}$. We have introduced here the notation for the linearly induced current $j^\text{lin}_\mu(x)$ (nonlinearly induced currents were dealt with in [11, 12, 14, 15, 20, 21, 13, 20, 23]. To avoid possible misunderstanding, we stress that nonlinearly induced currents are responsible for selfinteraction of the deviation fields $f_{\nu \mu}(x)$, whereas the nonlinearity of the theory given by the Lagrangian (6) shows itself in the framework of our present work as the interaction between the electromagnetic field $f_{\nu \mu}(x)$ and the electromagnetic background $\bar{F}_{\alpha \beta}$).

In our previous paper [23], where we considered a simpler case of special configuration of the background that consisted of mutually parallel electric and magnetic fields we were able to deal with the equation equivalent to (7) without appealing to the smallness of the coefficients of nonlinearity $\mathcal{L}_\mathfrak{F}, \mathcal{L}_\mathfrak{G}, \mathcal{L}_\mathfrak{F} \mathfrak{G}, \mathcal{L}_\mathfrak{G} \mathfrak{G}$ and $\mathcal{L}_\mathfrak{F} \mathfrak{G}$. That approach is good to the extent that the nonlinear model with the Lagrangian (6) is taken seriously. But as far as its origin is associated with perturbative effective Lagrangian of QED it would imply an excess of accuracy, which might be understood in terms of summation of certain specially selected Feynman graphs. Here we treat equation (7) perturbatively.
with respect to the above coefficients, whose connection with QED will be exploited in Section 4.

We represent

\[ f_{\nu\mu}(x) = f_{\nu\mu}^{(0)}(x) + f_{\nu\mu}^{(1)}(x) + \ldots, \]  

(8)

where \( f_{\nu\mu}^{(0)}(x) \) is a solution of the classical field equation

\[ \partial^{\nu} f_{\nu\mu}^{(0)}(x) = j_{\mu}(x), \]

while the linear response \( f_{\nu\mu}^{(1)}(x) \) is subject to the equation determined by the induced current taken on \( f_{\nu\mu}^{(0)}(x) \)

\[ \partial^{\nu} f_{\nu\mu}^{(1)}(x) = \partial^{\nu} \left[ \frac{1}{2} \left( \mathcal{L}_{\delta\delta} F_{\alpha\beta} + \mathcal{L}_{\delta\phi} F_{\alpha\beta} \right) \tilde{F}_{\nu\mu} f^{(0)\alpha\beta}(x) + \mathcal{L}_{\delta} f_{\nu\mu}^{(0)}(x) \right] + \partial^{\nu} \left[ \frac{1}{2} \left( \mathcal{L}_{\delta\phi} F_{\alpha\beta} + \mathcal{L}_{\phi\phi} F_{\alpha\beta} \right) \tilde{F}_{\nu\mu} f^{(0)\alpha\beta}(x) + \mathcal{L}_{\phi} \tilde{f}_{\nu\mu}^{(0)}(x) \right]. \]  

(9)

For the perturbation of the background we take the current corresponding to a static charge \( q \) homogeneously distributed over a sphere with the radius \( R \)

\[ j_{\mu}(x) = \delta_{\mu0} \rho^{(0)}(r), \quad r = |x|, \]

\[ \rho^{(0)}(r) = \frac{3q}{4\pi R^3} \theta(R - r), \quad R = \text{const.}, \]  

(10)

This charge density corresponds to a regularization of the pointlike static charge

\[ \rho^{(0)}(x) = q \delta^3(x), \quad \delta^3(x) = \delta(x^1) \delta(x^2) \delta(x^3), \]  

(11)

placed at origin \( x = 0 \). It is a source of the regularized Coulomb field \( f_{0i}^{(0)}(x) = E^{(0)i}(x) \) and null magnetic field \( B^{(0)j}(x) = - (1/2) \varepsilon_{ijk} f^{(0)jk}(x) = 0 \)

\[ \partial^{\nu} f_{\nu\mu}^{(0)}(x) = j_{\mu}(x), \quad E^{(0)}(x) = E^{(0)}_{\text{in}}(x) \theta(R - r) + E^{(0)}_{\text{out}}(x) \theta(r - R), \]

\[ E^{(0)}_{\text{in}}(x) = \frac{q x}{4\pi R^3}, \quad E^{(0)}_{\text{out}}(x) = \frac{q x}{4\pi r^3}, \quad B^{(0)}(x) = 0. \]  

(12)

Throughout the text, the indexes “in” and “out” classify electromagnetic quantities at points inside \( (r < R) \) and outside \( (r \geq R) \) of the spherical charge distribution, respectively.

In our previous work \([21]\), we studied the electric response \( E^{(1)k}(x) \neq 0 \),

\[ B^{(1)k}(x) = - (1/2) \varepsilon_{ijk} f^{(1)jk}(x) = 0 \] to equation \([9]\) giving a correction to the Coulomb law \([12]\);

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3Here and in what follows, \( \theta(z) \) denotes the Heaviside step function defined as \( \theta(z) = 1 \) if \( z \geq 0 \) and zero otherwise.
now we shall consider a magnetic solution. Thus, substituting the zero-order solutions $B^{(0)}(x) = 0$ and $E^{(0)}(x)$ [12] in Eq. (9), one finds that the first-order linear magnetic response $B^{(1)}(x)$ to the purely electric perturbation [10] is the solution of the differential equation

$$\nabla \times (B^{(1)}(x) - \mathcal{H}^{(0)}(x)) = 0,$$

wherein $\mathcal{H}^{(0)}(x)$ is the expression within the brackets in [7], taken in the zeroth order

$$\mathcal{H}^{(0)}(x) = -\mathcal{L}_e E^{(0)}(x) + \left[ \mathcal{L}_{\delta\delta} (E \cdot x) + \mathcal{L}_{\delta B} (B \cdot x) \right] B$$

+ $\left[ \mathcal{L}_{\delta B} (E \cdot x) + \mathcal{L}_{\delta B} (B \cdot x) \right] E$,

$$\mathcal{H}^{(0)}_{in}(x) = -\mathcal{L}_e E^{(0)}(x) - \left[ \mathcal{L}_{\delta\delta} (E \cdot x) + \mathcal{L}_{\delta B} (B \cdot x) \right] B$$

+ $\left[ \mathcal{L}_{\delta B} (E \cdot x) + \mathcal{L}_{\delta B} (B \cdot x) \right] E$, $r < R,

$$\mathcal{H}^{(0)}_{out}(x) = -\mathcal{L}_e E^{(0)}(x) - \left[ \mathcal{L}_{\delta\delta} (E \cdot x) + \mathcal{L}_{\delta B} (B \cdot x) \right] B$$

+ $\left[ \mathcal{L}_{\delta B} (E \cdot x) + \mathcal{L}_{\delta B} (B \cdot x) \right] E$, $r \geq R.$

Here the space- and time-independent electric and magnetic components of the background field are barred: $\overline{E}^i = F^i_{0i}$, $\overline{B}^j = - (1/2) \varepsilon_{ijk} F^{ijk}$.

Consider the linearly induced current density [7] to the same first-order approximation. In concord with (13) it is

$$\mathbf{j}^{lin(1)}(x) = \nabla \times \mathcal{H}^{(0)}(x) = \theta(R - r) \left[ \nabla \times \mathcal{H}^{(0)}_{in}(x) \right] + \theta(r - R) \left[ \nabla \times \mathcal{H}^{(0)}_{out}(x) \right].$$

Note that the quantity [14] is continuous at the border of the charge $r = R$, because $E^{(0)}(x)$ [12] is. For this reason the differentiation of the step functions has not contributed to the sum [17]. Hence there does not appear any current at the surface of the charge.

Thus we may define the inner $B^{(1)}_{in}(x)$ and outer $B^{(1)}_{out}(x)$ magnetic responses as solutions to the equations

$$\nabla \times B^{(1)}_{in}(x) = j^{lin(1)}_{in}(x), \quad \nabla \times B^{(1)}_{out}(x) = j^{lin(1)}_{out}(x),$$

It should be noted that the relativistic invariance is deprived in selecting a reference frame in which the static charge is at rest.
where the inner and outer parts of the first-order linearly induced current densities are

\[ j_{\text{in}}^{\text{lin}(1)}(x) = \nabla \times \mathcal{J}_{\text{in}}^{(0)}(x) = \frac{q}{4\pi r^3} (\mathcal{L}_{\delta\delta} + \mathcal{L}_{\varphi\varphi}) \left[ \mathbf{B} \times \mathbf{E} \right], \quad r < R, \quad (19) \]

and

\[ j_{\text{out}}^{\text{lin}(1)}(x) = \nabla \times \mathcal{J}_{\text{out}}^{(0)}(x) = \frac{q}{4\pi r^3} \left\{ \mathcal{L}_{\delta\delta} \left( [\mathbf{B} \times \mathbf{E}] + \frac{3}{r^2} (\mathbf{E} \cdot x) [x \times \mathbf{B}] \right) \right. \\
+ \left. \mathcal{L}_{\varphi\varphi} \left( [\mathbf{B} \times \mathbf{E}] - \frac{3}{r^2} (\mathbf{B} \cdot x) [x \times \mathbf{E}] \right) \right\}, \quad r \geq R. \quad (20) \]

respectively. The induced current \(^{(1)}\) is discontinuous at the edge of the sphere, the same as the charge density \(^{(0)}\) is.

For the special case of parallel external backgrounds \( \mathbf{B} \parallel \mathbf{E} \), the induced current density inside the charge disappears, \( j_{\text{in}}^{\text{lin}(1)} = 0 \), while the current \( j_{\text{out}}^{\text{lin}(1)}(x) \) circles the coordinate axis parallel to their common direction. Introducing the unit vector \( \mu \parallel \mathbf{B} \parallel \mathbf{E} \), \((\mu^2 = 1)\), the current \( j_{\text{out}}^{\text{lin}(1)}(x) \) acquires the form

\[ j_{\text{out}}^{\text{lin}(1)}(x) = \frac{3q}{4\pi r^5} \tilde{g} (\mu \cdot x) [x \times \mu], \quad (21) \]

where \( \tilde{g} \) is a combination of derivatives of the effective Lagrangian and field invariants,

\[ \tilde{g} = \tilde{\mathcal{G}} (\mathcal{L}_{\varphi\varphi} - \mathcal{L}_{\delta\delta}) + 2 \tilde{\mathcal{F}} \mathcal{L}_{\varphi\delta}, \quad \tilde{\mathcal{G}} = -\mathbf{B} \cdot \mathbf{E}, \quad \mathcal{F} = \frac{1}{2} \left( \mathbf{B}^2 - \mathbf{E}^2 \right). \quad (22) \]

The current flux \((21)\) flows in opposite directions in the upper and lower hemispheres (see Fig. 3), so that the total current through the part of a fixed meridional plane \( \varphi = \varphi_0, \quad 0 < \varphi_0 < 2\pi \), enclosed between any two coordinate spheres \( r_1 < r < r_2 \) is zero:

\[ \int j_{\text{out}}^{\text{lin}(1)}(x) \, ds = \frac{3q}{4\pi} \tilde{g} \int_{r_1}^{r_2} \frac{dr}{r^2} \int_0^\pi d\theta \cos \theta \sin \theta = 0. \]

Here \( \cos \theta = (\mu \cdot x) / r \). Once any two mutually non-orthogonal constant fields may be reduced to parallelity by an appropriate Lorentz transformation to a special inertial frame, the current \((20)\) differs from \((21)\) by a contribution due to the motion of the charge in that frame.
2.1 Magnetically neutral magnetic response

Besides the linearized Maxwell equations \( \text{(13)} \), the magnetic response \( \mathbf{B}^{(1)}(\mathbf{x}) \) should obey also the first Maxwell equation

\[
\nabla \cdot \mathbf{B}^{(1)}(\mathbf{x}) = 0 ,
\]  

which corresponds to one of the Bianchi identities in electrodynamics and which is a consequence of the formulation of the theory in terms of potentials. Equations \( \text{(13)} \) and \( \text{(23)} \) are satisfied by the magnetic response \( \mathbf{B}^{(1)}(\mathbf{x}) \)

\[
B^{(1)i}(\mathbf{x}) = \left( \delta^{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \mathbf{J}^{(0)j}(\mathbf{x}) ,
\]  

identified as the transverse component of \( \mathbf{J}^{(0)}(\mathbf{x}) \). The integral form of \( \text{(24)} \) reads

\[
B^{(1)i}(\mathbf{x}) = \mathbf{J}^{(0)i}(\mathbf{x}) + \frac{1}{4\pi} \partial_i \partial_j \int d\mathbf{y} \frac{\mathbf{J}^{(0)j}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} .
\]
After computing these integrals (see Eqs. (76) - (80) in Appendix A) the inner part $B^{(1)}_{\text{in}}(x)$ takes the form

$$B^{(1)}_{\text{in}}(x) = \frac{q}{4\pi} \left\{ \frac{3}{5R^3} \mathcal{L}_{\delta\delta} \left( \overline{E} \cdot x \right) + \frac{1}{5R^3} \left( \mathcal{L}_{\delta\delta} + 4\mathcal{L}_{\phi\phi} \right) \left( \overline{B} \cdot x \right) \right\} \overline{E}$$

$$+ \left[ -\frac{3}{5R^3} \mathcal{L}_{\delta\delta} \left( \overline{B} \cdot x \right) - \frac{1}{5R^3} \left( 4\mathcal{L}_{\delta\delta} + \mathcal{L}_{\phi\phi} \right) \left( \overline{E} \cdot x \right) \right] \overline{B} + \frac{\tilde{g}}{5R^3} x \right\} , \quad (26)$$

while the outer part $B^{(1)}_{\text{out}}(x)$ can be conveniently written as

$$B^{(1)}_{\text{out}}(x) = B^{(1)}_{\text{pl}}(x) + B^{(1)}_{\text{out}}(x; R) . \quad (27)$$

Here $B^{(1)}_{\text{pl}}(x)$ denotes an $R$-free part

$$B^{(1)}_{\text{pl}}(x) = \frac{q}{4\pi} \left\{ \frac{\tilde{g}}{2r^3} - \frac{\mathcal{L}_{\phi\phi} + \mathcal{L}_{\delta\delta}}{2r^3} \left[ (\overline{E} \cdot x) \overline{B} - (\overline{B} \cdot x) \overline{E} \right] \right\} + \frac{3}{2} \left[ (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\delta\delta}) (\overline{E} \cdot x) (\overline{B} \cdot x) - \mathcal{L}_{\delta\delta} \left( (\overline{B} \cdot x)^2 - (\overline{E} \cdot x)^2 \right) \right] \frac{x}{r^5} \right\} , \quad (28)$$

while $B^{(1)}_{\text{out}}(x; R)$, the $R$-dependent part, reads

$$B^{(1)}_{\text{out}}(x; R) = \frac{q}{8\pi} \left\{ \frac{3R^2}{5r^5} \right\} \left\{ 2\mathcal{L}_{\delta\delta} \left[ (\overline{E} \cdot x) \overline{E} - (\overline{B} \cdot x) \overline{B} \right] \right. \left. - (\mathcal{L}_{\delta\delta} - \mathcal{L}_{\phi\phi}) \left[ (\overline{B} \cdot x) \overline{E} + (\overline{E} \cdot x) \overline{B} \right] \right.$$  

$$+ \left[ -\frac{\tilde{g}}{r^2} + \frac{5}{r^2} \mathcal{L}_{\delta\delta} \left( \overline{E} \cdot x \right) (\overline{B} \cdot x) \right. \left. + \frac{5}{r^2} \mathcal{L}_{\delta\delta} \left( (\overline{B} \cdot x)^2 - (\overline{E} \cdot x)^2 \right) \right] \frac{x}{r^5} \right\} , \quad (29)$$

The division in $R$-dependent and $R$-free terms, expressed in Eq. (27), is aimed to emphasize that Eq. (29) corresponds to a pure homogeneous solution $\nabla \times B^{(1)}_{\text{out}}(x; R) = 0$. This is a consequence of the fact that the outer induced current density, given by Eq. (20), does not depend on $R$ or, in other words, there is no $R$-dependent source providing (29). Its real role is to provide continuity of the whole magnetic response $B^{(1)}(x) = B^{(1)}_{\text{in}}(x) \theta(R - r) + B^{(1)}_{\text{out}}(x) \theta(r - R)$ at the border of the Coulomb source (10). A similar feature has been reported by us in [21], wherein $R$-dependent terms in the electric response come automatically from the projection operator with the same interpretation. These $R$-dependent solutions are a consequence of the Coulomb source being an extended charge distribution rather than a pointlike one. In contrast, the $R$-independent part $B^{(1)}_{\text{pl}}(x)$ is the same as the first-order linear response of the pointlike Coulomb source (11), since it is
the only survivor in the limit \( r \gg R \) (or \( R \to 0 \)), bearing in mind that for any nonspecial direction, \( \mathbf{B}^{(1)}_{\text{pl}} (\mathbf{x}) \) decreases as \( r^{-2} \), while \( \mathbf{B}^{(1)}_{\text{out}} (\mathbf{x}; R) \) decreases as \( r^{-4} \). Therefore \( \mathbf{B}^{(1)}_{\text{pl}} (\mathbf{x}) \) is identified as the first-order linear response to the pointlike Coulomb source \((11)\). Moreover, according to Eq. \((20)\), \( \mathbf{B}^{(1)}_{\text{pl}} (\mathbf{x}) \) is provided by the outer induced current \( \mathbf{j}^{(1)}_{\text{out}} (\mathbf{x}) \)

\[
\nabla \times \mathbf{B}^{(1)}_{\text{pl}} (\mathbf{x}) = \nabla \times \mathbf{j}^{(1)}_{\text{out}} (\mathbf{x}) = \mathbf{j}^{(1)}_{\text{out}} (\mathbf{x}) .
\]

The first-order linear magnetic response calculated above does not carry any magnetic charge, in virtue of the triviality of the Gauss integral

\[
\oint_S \left( \mathbf{B}^{(1)}_{\text{in}} (\mathbf{x}) \cdot \mathbf{n} \right) dS = 0 , \tag{31}
\]

for an arbitrary closed surface \( S \) embracing the charge \( q \). This integral vanishes for each magnetic response \( \mathbf{B}^{(1)}_{\text{in}} (\mathbf{x}) \), \( \mathbf{B}^{(1)}_{\text{pl}} (\mathbf{x}) \) and \( \mathbf{B}^{(1)}_{\text{out}} (\mathbf{x}; R) \), independently. To begin with, taking \( S \) to be a sphere of radius \( R \), centered in the charge \( q \) (placed at the origin \( r = 0 \)), and choosing a reference frame in which \( \mathbf{B} \) is aligned along the \( z \)-axis and \( \mathbf{E} \) lies in the \( xz \)-plane (so that \( \mathbf{E} \cdot \mathbf{x} = E R \cos \gamma \), \( \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \varphi \) and \( \mathbf{E} \cdot \mathbf{B} = E B \cos \theta' \)), we find that only \( \mathbf{B}^{(1)}_{\text{in}} (\mathbf{x}) \) might contribute in Eq. \((31)\) and that

\[
\oint_S \left( \mathbf{B}^{(1)}_{\text{in}} (\mathbf{x}) \cdot \mathbf{n} \right) dS = \frac{3q}{20\pi} \left( -\frac{\mathcal{J} (R)}{R^2} + \frac{4\pi \tilde{g}}{3} \right) = 0 , \tag{32}
\]

where \( \mathcal{J} (R) \) denotes the integral over the surface \( S \)

\[
\mathcal{J} (R) = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \left\{ (\mathcal{L}_{\delta \delta} - \mathcal{L}_{\delta \phi}) (\mathbf{E} \cdot \mathbf{x}) (\mathbf{E} \cdot \mathbf{x}) \right. \\
\left. + \mathcal{L}_{\delta \phi} \left[ (\mathbf{E} \cdot \mathbf{x})^2 - (\mathbf{E} \cdot \mathbf{x})^2 \right] \right\} = \frac{4\pi R^2}{3} \tilde{g} . \tag{33}
\]

Therefore Eq. \((32)\) holds true. If one takes \( S \) to be an sphere of radius \( r > R \), then Eq. \((31)\) takes the form

\[
\oint_S \left( \mathbf{B}^{(1)}_{\text{out}} (\mathbf{x}) \cdot \mathbf{n} \right) dS = \oint_S \left( \mathbf{B}^{(1)}_{\text{pl}} (\mathbf{x}) \cdot \mathbf{n} \right) dS + \oint_S \left( \mathbf{B}^{(1)}_{\text{out}} (\mathbf{x}; R) \cdot \mathbf{n} \right) dS ,
\]

in which it can be seen that both integrals vanish identically

\[
\oint_S \left( \mathbf{B}^{(1)}_{\text{out}} (\mathbf{x}; R) \cdot \mathbf{n} \right) dS = -\frac{3R^2}{5r^2} \oint_S \left( \mathbf{B}^{(1)}_{\text{pl}} (\mathbf{x}) \cdot \mathbf{n} \right) dS \\
= -\frac{3R^2}{5r^2} \frac{q}{8\pi} \left( 4\pi \tilde{g} - \frac{3\mathcal{J} (r)}{r^2} \right) = 0 . \tag{34}
\]

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Here $J (r)$ is the same integral as (33), but with the $R$ replaced by $r$. One concludes that there is no magnetic charge attributed to the magnetic response $B^{(1)} (x)$: the magnetic lines of force incomming to and outgoing from the charge $q$, compensate each other, so that the corresponding magnetic flux be zero.

To visualize the structure of the magnetic lines of force, let us consider the particular case of parallel background fields, $\mathbf{E} = E \mu$, $\mathbf{B} = B \mu$ ($|\mu| = 1$), whose response acquires a simpler form

$$B_{\text{in}}^{(1)} (x) = \frac{q \tilde{g}}{4 \pi R^3} \frac{1}{5} \left[ x - 3 (\mu \cdot x) \mu \right],$$

(35)

and

$$B_{\text{out}}^{(1)} (x) = \frac{q \tilde{g}}{4 \pi r^3} \left\{ \left[ \left( 1 - \frac{3 R^2}{5 r^2} \right) - 3 \left( 1 - \frac{R^2}{r^2} \right) \left( \frac{\mu \cdot x}{r} \right)^2 \right] \frac{x}{2} - \frac{3 R^2}{5 r^2} (\mu \cdot x) \mu \right\}. $$

(36)

Moreover, in the limit $r/R \to \infty$, we are left with a single magnetic response, exclusively radial, although spherically nonsymmetric, corresponding to that of a pointlike Coulomb source $[11]$

$$B_{\text{pl}}^{(1)} (x) = \lim_{r/R \to \infty} B^{(1)} (x) = \frac{q \tilde{g}}{4 \pi} \left[ 1 - 3 \left( \frac{\mu \cdot x}{r} \right)^2 \right] \frac{x}{r^3}.$$  

(37)

The magnetic lines of force are straight lines, vanishing at the angles $\cos \theta = \zeta = \zeta_0 = 1/\sqrt{3}$. As no net magnetic charge exists for producing a nontrivial magnetic flux (34), there are inward magnetic lines (pointing to $q$) and outward magnetic lines (lines leaving $q$), in the same proportion (see Fig. 4).

### 2.2 Magnetically charged magnetic response

In contrast to the magnetically neutral magnetic fields (28) or (37) created in a constant two-field background by a small pointlike Coulomb source (11), in this Subsection we discuss a magnetic response to the same perturbation, which carries a nontrivial magnetic charge – at the cost of denial of Bianchi’s identity (23) at one specific point of the space, $r = 0$, as this is the case with the standard Dirac magnetic monopole. The Maxwell equation (13) admits many more solutions, apart from those that were considered in the previous Subsection. They may be generally presented as

$$\mathbf{B} (x) = J_{\text{out}}^{(0)} (x) + \nabla \Omega (x),$$

(38)
where $\Omega(\mathbf{x})$ is a (pseudo)scalar function. Those that are not of the special form (24) may carry a magnetic charge constructed from $\mathbf{f}_{\text{out}}^{(0)}(\mathbf{x})$, because the condition (23) is not generally fulfilled by $\mathbf{f}(\mathbf{x})$. We shall be interested in ones that correspond to a pointlike magnetic monopole, produced by the pointlike electric monopole (11), in other words in the situation where the total of the magnetic charge is concentrated in a point, the origin $r = 0$. When the background electric and magnetic fields are nonparallel we shall be looking for a neither center- nor axial-symmetric magnetic monopole solution of the nonlinear Maxwell equation in the form

$$
\mathbf{B}(\mathbf{x}) = f(\zeta, \xi) \frac{\mathbf{x}}{r^3} + \frac{1}{r^2} g_1(\zeta, \xi) \mathbf{\mu} + \frac{1}{r^2} g_2(\zeta, \xi) \mathbf{\nu},
$$

where

$$
\mathbf{\mu} = \frac{\mathbf{B}}{B}, \quad \mathbf{\nu} = \frac{\mathbf{E}}{E}, \quad \zeta = \frac{\mathbf{\mu} \cdot \mathbf{x}}{r}, \quad \xi = \frac{\mathbf{\nu} \cdot \mathbf{x}}{r},
$$

(39)
where \( f(\zeta, \xi), g_1(\zeta, \xi) \) and \( g_2(\zeta, \xi) \) are functions of the cosines of the angles between directions of \( \mathbf{E}, \mathbf{B} \) and the radius vector \( \mathbf{x} \). They are not arbitrary, but subjected to the equation

\[
\nabla \cdot \mathbf{B}(\mathbf{x}) = 0, \quad r > 0, \tag{40}
\]

supplying the trivial divergence to \( \mathbf{B}(\mathbf{x}) \) everywhere, except at the singular point \( r = 0 \).

Let us find the scalar function \( \Omega(r, \zeta, \xi) \) in (38) that would supply the form (39) to the solution. Firstly, from Eq. (40), one can see that \( g_1(\zeta, \xi) \) and \( g_2(\zeta, \xi) \) must be subjected to the first-order partial differential equation

\[
(1 - \zeta^2) \partial_\zeta g_1 + (1 - \xi^2) \partial_\xi g_2 + (\mathbf{\mu} \cdot \mathbf{\nu} - \zeta \xi) (\partial_\xi g_1 + \partial_\zeta g_2) = 2 (\zeta g_1 + \xi g_2), \tag{41}
\]

that is readily satisfied if \( g_1 \) and \( g_2 \) are linear in \( \zeta \) and \( \xi \), namely

\[
g_1 = g_1(\xi) = T \xi \quad \text{and} \quad g_2 = g_2(\zeta) = -T \zeta \quad \text{with} \quad T = \text{const}.
\]

In this case, one can use (16) for \( H_{\text{out}}^0(\mathbf{x}) \) and equate the components \( \mathbf{\mu} \) and \( \mathbf{\nu} \) in both sides of Eq. (38) to learn that \( \Omega(\mathbf{x}) \) is given by

\[
\Omega(\mathbf{x}) = \frac{q}{8\pi r^3} \left[ (\mathcal{L}_{\delta\delta} - \mathcal{L}_{\phi\phi})(\overline{\mathbf{B}} \cdot \mathbf{x})(\overline{\mathbf{E}} \cdot \mathbf{x}) + \mathcal{L}_{\delta\phi} \left( (\overline{\mathbf{B}} \cdot \mathbf{x})^2 - (\overline{\mathbf{E}} \cdot \mathbf{x})^2 \right) \right] + \Psi(r), \tag{42}
\]

where \( \Psi(r) \) is an arbitrary function of \( r \), and \( T \) is found to be \( T = -(q/8\pi) \overline{E \mathbf{B}} (\mathcal{L}_{\delta\delta} + \mathcal{L}_{\phi\phi}) \). By equating the radial components along \( \mathbf{x} \) in both sides of Eq. (38), the relation

\[
f(\zeta, \xi) = -\frac{q}{4\pi} \mathcal{L}_{\phi\phi} + r^2 \partial_r \Omega - r \zeta \partial_\zeta \Omega - r \xi \partial_\xi \Omega,
\]

is obtained. We set \( d\Psi(r)/dr = 0 \) and find that \( f(\zeta, \xi) \) has the form

\[
f(\zeta, \xi) = -\frac{q}{4\pi} \mathcal{L}_{\phi\phi} + \frac{3q}{8\pi r^2} \left[ (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\delta\delta})(\overline{\mathbf{B}} \cdot \mathbf{x})(\overline{\mathbf{E}} \cdot \mathbf{x}) - \mathcal{L}_{\delta\phi} \left( (\overline{\mathbf{B}} \cdot \mathbf{x})^2 - (\overline{\mathbf{E}} \cdot \mathbf{x})^2 \right) \right]. \tag{44}
\]

As a result, the magnetic response \( \mathbf{B}(\mathbf{x}) \) has the final form

\[
\mathbf{B}(\mathbf{x}) = \frac{q}{4\pi} \left\{ -\mathcal{L}_{\phi\phi} \frac{\mathbf{x}}{r^3} - \frac{(\mathcal{L}_{\phi\phi} + \mathcal{L}_{\delta\delta})}{2r^3} \left[ (\overline{\mathbf{E}} \cdot \mathbf{x}) \mathbf{B} - (\overline{\mathbf{B}} \cdot \mathbf{x}) \mathbf{E} \right] \right. \\
+ \left. \frac{3}{2} \left[ (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\delta\delta})(\overline{\mathbf{B}} \cdot \mathbf{x})(\overline{\mathbf{E}} \cdot \mathbf{x}) - \mathcal{L}_{\delta\phi} \left( (\overline{\mathbf{B}} \cdot \mathbf{x})^2 - (\overline{\mathbf{E}} \cdot \mathbf{x})^2 \right) \right] \frac{\mathbf{x}}{r^5} \right\}. \tag{45}
\]

The present response differs from Eq. (28) by the term \(-q \mathcal{L}_{\phi\phi}/4\pi \) in the place of \( \tilde{g}/2 \). Such
difference yields a nontrivial magnetic charge $q_B$, as it can be seen from the Gauss integral

$$q_B = \oint_S (\mathfrak{B}(x) \cdot \hat{n}) \, dS = -q \left( \mathfrak{L}_\mathfrak{G} + \frac{\tilde{g}}{2} \right), \quad (46)$$

concentrated at origin $r = 0$, being a pseudoscalar as it must be. Remind that $\tilde{g}$ is defined in (22). Following the Gauss theorem, one then states that $\nabla \cdot \mathfrak{B}(x) = q_B \delta^3(x)$. In the special case of a spatially odd theory, the magnetic charge may exist with no electric field in the background. Indeed, with $\mathbf{E} = \mathbf{G} = 0$, one finds from (22) that $\tilde{g} = \mathbf{B}^2 \mathfrak{L}_\mathfrak{G}$ and $q_B = -q \left( \mathfrak{L}_\mathfrak{G} + \mathbf{B}^2 / 2 \right) \mathfrak{L}_\mathfrak{G}$. This is zero unless the Lagrangian itself contains a parity-violating term linear in $\mathfrak{G}$. Otherwise this is not. It is worth noting that the solution (45) is rigorously defined up to a pure gradient term $(x/r) d\Psi (r) / dr$, where two possibilities take place: The first one is that it might be singular, corresponding to the Dirac monopole provided its magnetic charge had been introduced ad hoc. This possibility is beyond our interest here, as long as we are looking for the magnetic charge arranged from the background and the electric charge. A second possibility might be a linear function $\Psi (r) = cr + C$ with $c$ and $C$ constants, once it does not break the compatibility in Eq. (44). In this case, the coefficient $-q \mathfrak{L}_\mathfrak{G} / 4\pi$ in Eq. (44) would be modified by $-q \mathfrak{L}_\mathfrak{G} / 4\pi + c$ and the magnetic charge $q_B$ by $4\pi c - q \left( \mathfrak{L}_\mathfrak{G} + \tilde{g}/2 \right)$. Such consideration corresponds to a pure homogeneous solution of $\nabla \Omega (x) = 0$ and therefore not related to the electric charge $q$. For these reasons, $\Psi (r)$ is also disregarded from our consideration.

In the case of parallel constant backgrounds $\mathbf{E} = \mathbf{E}_\mu$, $\mathbf{B} = \mathbf{B}_\mu$ $(|\mu| = 1)$, the magnetic response (45) becomes purely radial

$$\mathfrak{B}(x) = -q \left( \mathfrak{L}_\mathfrak{G} + \frac{3\tilde{g}}{2} \right) \frac{x}{r^3}, \quad r > 0. \quad (47)$$

In the limit of small constant fields, when we must set $\mathbf{E} = \mathbf{B} = 0$ inside the coefficient functions, one has in a parity-even theory $\mathfrak{L}_\mathfrak{G} = \mathfrak{L}_\mathfrak{G} = 0$. In this case Eq. (47) coincides with the solution numbered as (12) in our previous paper [23] taken in the limit $b = 0$. Note that the approximation exploited in [23] is different from that in the present paper, as commented above, before Eq. (8).

It is notable that the monopole magnetic charge (46) contains the background fields as their relativistic-invariant combinations $\mathfrak{G}$ and $\mathfrak{G}$, and also the electric charge of the original electric monopole $q$, which is a relativistic invariant, too The angle between the fields enters only implicitly through the invariant $\mathfrak{G} = -\mathbf{E} \cdot \mathbf{B}$. This means that the expression of the magnetic charge calculated from (47) is in its appearance the same as (46). Clearly, nonparallel electric and magnetic fields of the background, provided that $\mathfrak{G} \neq 0$, can be set to parallelity by passing to the inertial

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5 To compute this integral one may conveniently select the same conventions as those used to evaluate the integrals (32) or (34), for instance.
frame, moving in the direction orthogonal to $\mathbf{E}$ and $\mathbf{B}$. In that frame, the charge is moving. Quite expectably, our calculations indicate that the value of the induced magnetic charge does not depend on the speed of the charge, i.e., the magnetic charge is a relativistic invariant, the same as the electric charge.

3 Vector potentials

The consideration is worth being extended to the level of vector potentials associated to the magnetically charged/neutral responses. The magnetic field in consideration has the generic form

$$
\mathbf{H} (\lambda | \mathbf{x}) = \frac{q}{4\pi} \left\{ \lambda \frac{\mathbf{x}}{r^3} - \frac{(\mathcal{L}_{\phi \phi} + \mathcal{L}_{\phi \delta})}{2r^3} \left[ (\mathbf{E} \cdot \mathbf{x}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{x}) \mathbf{E} \right] \\
+ \frac{3}{2} \left[ (\mathcal{L}_{\phi \phi} - \mathcal{L}_{\phi \delta}) (\mathbf{E} \cdot \mathbf{x}) (\mathbf{B} \cdot \mathbf{x}) - \mathcal{L}_{\phi \delta} \left( (\mathbf{B} \cdot \mathbf{x})^2 - (\mathbf{E} \cdot \mathbf{x})^2 \right) \right] \frac{\mathbf{x}}{r^5} \right\},
$$

which covers the magnetically neutral response, when $\lambda = \tilde{g}/2$, $\mathbf{H}(\tilde{g}/2|\mathbf{x}) = \mathbf{B}_{\text{pl}}(\mathbf{x})$ and magnetically charged responses, when $\lambda \neq \tilde{g}/2$. Among all magnetically charged responses, we are interested in the present paper only with those where no magnetic charge is present outside the origin. According to the previous Subsubsection, this magnetic monopole case corresponds to the value $\lambda = -\mathcal{L}_{\phi}$, $\mathbf{H}(-\mathcal{L}_{\phi}|\mathbf{x}) = \mathbf{B} (\mathbf{x})$. We are looking for the corresponding vector potential in the form

$$
\mathfrak{A} (\lambda | \mathbf{x}) = [\mathbf{\mu} \times \mathbf{x}] \frac{\mathcal{A} (\lambda | \zeta, \xi)}{r^2} + [\mathbf{\nu} \times \mathbf{x}] \frac{\mathcal{C} (\lambda | \zeta, \xi)}{r^2} + \frac{[\mathbf{\mu} \times \mathbf{\nu}]}{r} \mathcal{M},
$$

where $\mathcal{A} (\lambda | \zeta, \xi)$ and $\mathcal{C} (\lambda | \zeta, \xi)$ are angular functions and $\mathcal{M}$ is a constant. Although this form may be not most general, its use is sufficient for finding at least a certain class of the vector-potentials $\mathfrak{A}$, all the variety of other possible values for $\mathfrak{A}$ being gauge-equivalent to those found. For the left-hand side of (48) the representation of the field in terms of the vector potential should be exploited:

$$
\mathbf{H} (\lambda | \mathbf{x}) = \nabla \times \mathfrak{A} (\lambda | \mathbf{x}) = \frac{\mathbf{x}}{r^4} \left[ 2 \mathcal{A} (\mathbf{\mu} \cdot \mathbf{x}) + 2 \mathcal{C} (\mathbf{\nu} \cdot \mathbf{x}) \right] - \frac{\mathbf{x}}{r^3} \left( \mathbf{\mu} \cdot \mathbf{\nu} - \frac{(\mathbf{\mu} \cdot \mathbf{x})(\mathbf{\nu} \cdot \mathbf{x})}{r^2} \right) (\partial_\xi \mathcal{A} + \partial_\zeta \mathcal{C}) \\
- \frac{\mathbf{x}}{r^3} \left\{ \left[ 1 - \frac{(\mathbf{\mu} \cdot \mathbf{x})^2}{r^2} \right] \partial_\zeta \mathcal{A} + \left[ 1 - \frac{(\mathbf{\nu} \cdot \mathbf{x})^2}{r^2} \right] \partial_\xi \mathcal{C} \right\} + \frac{\mathcal{M}}{r^3} [\mathbf{\nu} (\mathbf{\mu} \cdot \mathbf{x}) - \mathbf{\mu} (\mathbf{\nu} \cdot \mathbf{x})].
$$

Note that only the $\mathcal{M}$-term from (49) contributes the $\mathbf{\mu}$- and $\mathbf{\nu}$- components to (50). The two equations obtained by projecting the equation obtained by equating (48) and (50) onto these
directions
\[- \frac{q}{4\pi} \frac{\left( \mathcal{L}_{\phi\phi} + \mathcal{L}_{\phi\delta} \right)}{2r^3} \left[ (\mathbf{E} \cdot \mathbf{x}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{x}) \mathbf{E} \right] = - \frac{\mathcal{M}}{r^2} [\mu (\nu \cdot x) - \nu (\mu \cdot x)] ,\]
are both satisfied with the unique choice
\[\mathcal{M} = \frac{q}{8\pi} (\mathcal{L}_{\phi\delta} + \mathcal{L}_{\phi\phi}) B E .\] (51)

This fact has an important consequence: after projecting (48), (50) onto the x-direction we are left with only one first-order partial differential equation for two functions \(A\) and \(C\)
\[2 (\zeta A + \xi C) - [(1 - \zeta^2) \partial_\zeta A + (1 - \xi^2) \partial_\xi C] - (\mu \cdot \nu - \zeta \xi) (\partial_\zeta A + \partial_\xi C)
= \frac{q}{4\pi} \lambda + \frac{3q}{8\pi} \left[ (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\phi\delta}) B E \zeta - \mathcal{L}_{\phi\delta} B^2 \zeta^2 + \mathcal{L}_{\phi\phi} E^2 \xi^2 \right] .\] (52)

Therefore, there is an arbitrary number of solutions to Eq. (52). Such arbitrariness corresponds to a certain part of the gauge freedom of vector potentials. It will be sufficient to seek for solutions of Eq. (52) in the following subclass of (49)
\[A (\lambda|\zeta, \xi) = \frac{q}{16\pi} (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\phi\delta}) B E \xi - \frac{q}{8\pi} \mathcal{L}_{\phi\delta} B^2 \zeta + Y \left( \zeta \right) ,\]
\[C (\lambda|\zeta, \xi) = \frac{q}{16\pi} (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\phi\delta}) B E \zeta + \frac{q}{8\pi} \mathcal{L}_{\phi\delta} E^2 \xi + X \left( \xi \right) ,\] (53)
where the functions \(Y (\zeta)\) and \(X (\xi)\) satisfy one and the same differential equation,
\[Z' (u) - \left( \frac{2u}{1 - u^2} \right) Z (u) = - \frac{q (\lambda - \tilde{g}/2)}{8\pi} \left( \frac{1}{1 - u^2} \right) ,\] (54)
in which \(Z = (Y, X)\) and \(u = (\zeta, \xi)\). The latter equation can be readily integrated,
\[Z (u) = \frac{z^2 - 1}{u^2 - 1} Z (z) + \frac{q}{8\pi} \left( \lambda - \frac{\tilde{g}}{2} \right) \left( \frac{u - z}{u^2 - 1} \right) ,\] (55)
yielding the final form of the vector potential (49),

\[ A(z, \tilde{z})(\lambda|x) = \frac{[B \times x]}{r^2} \left\{ \frac{q}{16\pi} \left( \mathcal{L}_{\phi \phi} - \mathcal{L}_{\delta \delta} \right) \left( \frac{E \cdot x}{r} \right) - \frac{q}{8\pi} \mathcal{L}_{\delta \phi} \left( \frac{B \cdot x}{r} \right) \right\} \\
+ \frac{z^2 - 1}{\zeta^2 - 1} Y(z) + \frac{q}{8\pi} \left( \lambda - \frac{\bar{g}}{2} \right) \left( \frac{\zeta - z}{\zeta^2 - 1} \right) \right\} \\
+ \frac{[E \times x]}{r^2} \left\{ \frac{q}{16\pi} \left( \mathcal{L}_{\phi \phi} - \mathcal{L}_{\delta \delta} \right) \left( \frac{B \cdot x}{r} \right) + \frac{q}{8\pi} \mathcal{L}_{\delta \phi} \left( \frac{E \cdot x}{r} \right) \right\} \\
+ \frac{\bar{z}^2 - 1}{\xi^2 - 1} X(\bar{z}) + \frac{q}{8\pi} \left( \lambda - \frac{\bar{g}}{2} \right) \left( \frac{\xi - \bar{z}}{\xi^2 - 1} \right) \right\} \\
+ \frac{q}{8\pi} \left( \mathcal{L}_{\delta \delta} + \mathcal{L}_{\phi \phi} \right) \frac{[B \times E]}{r^2}. \]  

(56)

In Eqs. (55) and (56), \( z \) and \( \tilde{z} \) are integration constants or boundary points, through which the boundary conditions \( Y(z) \), \( X(\tilde{z}) \) should be specified. The choice of the latter is a matter of gauge fixing. In the class (53), the following gauge condition is fulfilled by \( A(\lambda|x) \) (49),

\[ \nabla \cdot A^{(z, \tilde{z})}(\lambda|x) = \frac{q}{8\pi r^3} \left( \mathcal{L}_{\delta \delta} + \mathcal{L}_{\phi \phi} \right) x \cdot [E \times B]. \]  

(57)

The arbitrariness due to different choices of the boundary conditions leaves us within this gauge. There are two special choices \( z = \tilde{z} = \pm 1 \), which restrict the angular singularities in (56) at \( \zeta^2 = 1 \) and \( \xi^2 = 1 \), first belonging to the whole axes parallel to the background fields \( B \) and \( E \), to the half-axes with the cosine values \( \zeta = \pm 1 \), \( \xi = \pm 1 \). With these choices, the vector potential (56) acquires the form

\[ A^{(\pm, \pm)}(\lambda|x) = \frac{q}{16\pi r^2} \left[ B \times x \right] \left[ \left( \mathcal{L}_{\phi \phi} - \mathcal{L}_{\delta \delta} \right) \left( \frac{E \cdot x}{r} \right) \right] \\
- \frac{q}{16\pi r^2} \left[ B \times x \right] \left[ \left( \mathcal{L}_{\phi \phi} - \mathcal{L}_{\delta \delta} \right) \left( \frac{B \cdot x}{r} \right) + 2 \left( \lambda - \frac{\bar{g}}{2} \right) \left( \frac{\zeta + 1}{\zeta^2 - 1} \right) \right] \\
+ \frac{q}{16\pi r^2} \left[ E \times x \right] \left[ \left( \mathcal{L}_{\phi \phi} - \mathcal{L}_{\delta \delta} \right) \left( \frac{E \cdot x}{r} \right) \right] \\
- \frac{q}{16\pi r^2} \left[ E \times x \right] \left[ \left( \mathcal{L}_{\phi \phi} - \mathcal{L}_{\delta \delta} \right) \left( \frac{B \cdot x}{r} \right) + 2 \left( \lambda - \frac{\bar{g}}{2} \right) \left( \frac{\xi + 1}{\xi^2 - 1} \right) \right] \\
+ \frac{q}{8\pi} \left( \mathcal{L}_{\delta \delta} + \mathcal{L}_{\phi \phi} \right) \frac{[B \times E]}{r^2}. \]  

(58)

wherein the singularities rest at the positive or negative directions of the background. For example, selecting lower signs in (58), two singular lines start from the origin, where the point charge is located, and stretch along the positive direction of \( B \), \( (\zeta = 1) \), and \( E \), \( (\xi = 1) \).
neutral response (28), the singularities vanish identically, since \( \lambda = \tilde{g}/2 \). Otherwise they survive. These are Dirac strings, associated with an infinite number of points, in which the potential is singular and restricted to half-axes.

For parallel backgrounds, the cosines \( \zeta, \xi \) coincide and so do the functions \( X(\xi) \) and \( Y(\zeta) \). In this case, Eq. (58) reduces to

\[
\mathcal{A}^{(\pm)}(\lambda|x) = -\frac{q}{4\pi r^2} \left[ \frac{\mu \times \mathbf{x}}{r^2} \left( \frac{\tilde{g}}{2} \zeta + \left( \lambda - \frac{\tilde{g}}{2} \right) \left( \frac{\zeta \mp 1}{1 - \zeta^2} \right) \right) \right],
\]

in which the singularity now rests only at the positive or negative half-axis of the common direction of the background. Again, the Dirac string is absent for the magnetically neutral response. Note that in this case all the potentials, to which the ansatz (50) reduces in this case, obey the Coulomb gauge condition \( \nabla \cdot \mathcal{A}^{(\pm)}(\lambda|x) = 0 \).

In the Dirac case of spherically symmetric monopole, with total charge \( q_M \) and radial magnetic field \( q_M x/4\pi r^3 \), \( r > 0 \) the direction of the string is arbitrary [26], since it can be turned to any angle by a spatial rotation, under which the form of the field \( \mathfrak{B}(x) \) remains the same. Hence this rotation is nothing but a gauge transformation that leaves the field invariant. On the contrary, in our case the field (45) is not center- (even not axial-) symmetric, therefore it is affected by rotations. As a consequence, the change of directions of any of the two Dirac strings cannot be achieved by a gauge transformation.

4 Magnetic responses in QED

To visualize how the magnetic responses and related effects, valid for any local nonlinear theory, depend on the constant background, we apply the former results to a specific theory, whose non-linearity is provided by the local approximation of the effective Lagrangian of QED found within one-fermion-loop calculation by Euler and Heisenberg [3] (see e.g. [24])

\[
\mathcal{L} = \frac{m^4}{8\pi^2} \int_0^\infty dt \frac{e^{-t}}{t^3} \left\{ - (ta \cot ta)(tb \coth tb) + 1 - \frac{1}{3} (a^2 - b^2) t^2 \right\},
\]

where the integration contour is meant to circumvent the poles on the real axis of \( t \) supplied by \( \cot ta \) above the real axis. Here \( a \) and \( b \) are dimensionless combinations of the field invariants,

\[
a = \left( \frac{e}{m^2} \right) \sqrt{-\tilde{g}^2 + \tilde{F}^2 + \tilde{G}^2}, \\
b = \left( \frac{e}{m^2} \right) \sqrt{\tilde{F}^2 + \tilde{G}^2},
\]

(61)
and have the meaning of the electric and magnetic field in the Lorentz frame where these are parallel, normalized to the characteristic field value $e/m^2$, where $m$ and $e$ are the electron mass and charge, respectively. As it is well known, such a frame always exists when $\mathcal{G} \neq 0$, although the identification $a = eE/m^2$ and $b = eB/m^2$ is not strictly considered. For the sake of convenience we remove, “bars” above of quantities depending on the constant background, for instance, $\mathfrak{F}$ shall be replaced by $\mathfrak{f}$ from now on.

We are primarily interested in strong magnetic-dominated backgrounds, in which vacuum polarization effects overcome vacuum instability ones. In such backgrounds, the electric contribution is sufficiently small in comparison to the magnetic part

$$\frac{a}{b} \ll 1,$$

irrespective of whether $a$ and $b$ are small or not as compared to the unity, implying the magnetic dominance $B \gg E$ in any reference frame. Such condition is enough to probe vacuum nonlinear effects and should be applied in final expressions after all coefficients composing the magnetic responses (derivatives of the effective Lagrangian) have been calculated (with $a$ and $b$ arbitrary, i.e. not subjected to condition (62)). Thus, using general expressions for the derivatives of the effective Lagrangian, given by Eqs. (52) - (54) in Ref. [21], and also using the coefficient $\mathcal{L}_\mathcal{G}$,

$$\mathcal{L}_\mathcal{G} = -\frac{m^4}{16\pi^2} \frac{\kappa b^2}{\sqrt{\mathfrak{G}^2 + \mathfrak{G}^2}} \left( \frac{a}{b} \right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau^3} \left\{ \frac{b}{a} \right( \tau \coth \tau \right) Q \left( \frac{a\tau}{b} \right) \right\},$$

not included in that reference, the coefficient $\tilde{g}$ [22] and the magnetic charge $q_\mathcal{G}$ [46] take the form

$$\tilde{g} = \frac{m^4}{32\pi^2} \frac{b^2}{(\mathfrak{G}^2 + \mathfrak{G}^2)} \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau^3} \left\{ \mathcal{H} (\tau) Q \left( \frac{a\tau}{b} \right) \left( -4\mathfrak{G} + 2\mathfrak{G}_\kappa \left( \frac{a}{b} - \frac{b}{a} \right) \right) \right\}$$

\begin{align*}
&+ \left( \frac{\mathfrak{G}}{b} \left( \frac{1 - b^2}{a^2} \right) + 2\mathfrak{G}_\kappa \frac{a}{b} \right) \bar{Q} \left( \frac{a\tau}{b} \right) \\
&\quad + \left( \frac{a\tau}{b} \cot \frac{a\tau}{b} \right) \left[ 2 \left( \mathfrak{G}_\gamma + 2\mathfrak{G}_\kappa \frac{a}{b} \right) \mathcal{H}(\tau) \right] \\
&\quad + \left( \frac{\mathfrak{G}}{b} \left( \frac{1 - a^2}{b^2} \right) - 2\mathfrak{G}_\kappa \frac{a}{b} \right) \tilde{\mathcal{H}}(\tau) \right\},
\end{align*}

\begin{align}
\text{Note that all magnetic responses, given by Eqs. [26], [27], [45], the nonlinearly induced currents, given by Eqs. [19], [20] and the magnetic charge [46] vanish identically in the pure magnetic background $a/b = 0$, since $\mathfrak{G} = 0$.}
\end{align}
and

\[
\tilde{q}_\phi = -q \frac{m^4}{32\pi^2 (\tilde{\phi}^2 + \mathcal{G}^2)} b^2 \int_0^\infty \frac{d\tau}{\tau^3} e^{-\tau \gamma/b} \left\{ \mathcal{H}(\tau) Q \left( \frac{a\tau}{b} \right) \left( -2\mathcal{G} + \tilde{\mathcal{G}} \kappa \left( \frac{a}{b} - \frac{b}{a} \right) \right) + \left( \kappa \cot \frac{a\tau}{b} \right) \right\},
\]

where \( \mathcal{H}(\tau), Q(\tau), \tilde{\mathcal{H}}(\tau), \tilde{Q}(\tau) \) are the auxiliary functions

\[
\mathcal{H}(\tau) = \tau \coth \tau - \frac{\tau^2}{\sinh^2 \tau}, \quad Q(\tau) = \tau \cot \tau - \frac{\tau^2}{\sin^2 \tau},
\]

\[
\tilde{\mathcal{H}}(\tau) = \frac{2\tau^2}{\sinh^2 \tau} (\tau \coth \tau - 1), \quad \tilde{Q}(\tau) = \frac{2\tau^2}{\sin^2 \tau} (\tau \cot \tau - 1).
\]

\( \kappa = \text{sign}(\mathcal{G}) \) and \( \gamma_\pm = 1 \pm 2\tilde{\phi}/\sqrt{\tilde{\phi}^2 + \mathcal{G}^2} \) are constants.

Eqs. (64, 63) together with Eqs. (52) - (54) from Ref. [21] provide integral representations for all the necessary coefficients within the Euler-Heisenberg nonlinear electrodynamics to be substituted into Eqs. (26-29) and into Eq. (45) for specializing the magnetic fields of, respectively, the magnetically-neutral and magnetically charged linear responses.

Let us study the strong magnetic-dominated case, specified by the condition (62). In this case, one can expand trigonometric functions in power series of \( a/b \) (avoiding the poles at the real axis in Eqs. (63) - (65), thereby) to find that the coefficient \( \mathcal{L}_\phi \) is expressed as an odd-power series in \( a/b \),

\[
\mathcal{L}_\phi = \left( \frac{a}{b} \right)^3 \mathcal{L}_\phi^{(1)} + O \left( (a/b)^3 \right),
\]

\[
\mathcal{L}_\phi^{(1)} = -\kappa \left( \frac{a}{2\pi} \right) \int_0^\infty d\tau \frac{e^{-\tau \gamma/b}}{\tau} \left[ \left( \frac{1}{\tau} - \frac{2}{3\tau^2} \right) \coth \tau - \frac{1}{\sinh^2 \tau} \right],
\]

whose leading-order term is linear in \( a \) as expected, since \( \mathcal{L}_\phi \) must vanish in absence of the electric field (or, equivalently as \( a \to 0 \)). As for Eq. (64), one may conveniently rewrite \( \tilde{g} \) as

\[
\tilde{g} = -\mathcal{G} \mathcal{L}_- + 2\tilde{\phi} \mathcal{L}_{\phi \phi}, \quad \mathcal{L}_- = \mathcal{L}_{\phi \phi} - \mathcal{L}_{\phi \phi \phi}.
\]
We use the expansion for the coefficient $\mathcal{G}_-$

$$
\mathcal{G}_- = \left(\frac{a}{b}\right) (\mathcal{G}_-^{(1)}) + O ((a/b)^3),
$$

$$(\mathcal{G}_-^{(1)}) = 2\kappa \left(\frac{a}{2\pi}\right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left[\left(\frac{1}{\tau} - \frac{\tau}{3}\right) \coth \tau - \frac{\coth \tau}{\sinh^2 \tau}\right],$$

(69)

and for $\mathcal{G}_{\tilde{\phi}}$

$$
\mathcal{G}_{\tilde{\phi}} = \left(\frac{a}{b}\right) (\mathcal{G}_{\tilde{\phi}}^{(1)}) + O ((a/b)^3),
$$

$$(\mathcal{G}_{\tilde{\phi}}^{(1)}) = \kappa \left(\frac{a}{2\pi}\right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left[\left(\frac{3}{\tau} - \frac{2\tau}{3}\right) \coth \tau - \left(1 + \frac{2\tau^2}{3}\right) \frac{1}{\sinh^2 \tau} - 2\tau \frac{\coth \tau}{\sinh^2 \tau}\right].$$

(70)

to show that the leading-order contribution for $\tilde{g}$ is expressed as

$$
\tilde{g} = \left(\frac{a}{b}\right) \tilde{g}^{(1)} + O ((a/b)^3),
$$

$$
\tilde{g}^{(1)} = 2\kappa \left(\frac{a}{2\pi}\right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left[\left(\frac{2}{\tau} - \frac{\tau}{3}\right) \coth \tau - \left(1 + \frac{2\tau^2}{3}\right) \frac{1}{\sinh^2 \tau} - \frac{\coth \tau}{\sinh^2 \tau}\right],$$

(71)

and the magnetic charge (65) reads

$$
q_\mathcal{B} = \left(\frac{a}{b}\right) q_\mathcal{B}^{(1)} + O ((a/b)^3),
$$

$$
q_\mathcal{B}^{(1)} = -q \left(\frac{a}{2\pi}\right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left[\left(\frac{\tau}{3} + \frac{1}{\tau}\right) \coth \tau - \frac{2\tau^2}{3} \frac{1}{\sinh^2 \tau} - \frac{\coth \tau}{\sinh^2 \tau}\right].$$

(72)

To estimate the asymptotic behavior of the magnetic charge (72) and the coefficient $\tilde{g}$ (71) in the infinite magnetic field limit $b \to \infty$, it is convenient to express all integrals above in terms of the Hurwitz Zeta function $\zeta (z, a)$, DiGamma function $\psi (z) = \Gamma' (z) / \Gamma (z)$ and related functions [29]. Using Zeta-function regularization techniques (see e. g., [30, 31, 32]) each integral above can be expressed as follows:

$$
\tilde{g}^{(1)} = 2\kappa \left(\frac{a}{2\pi}\right) \left\{\frac{b}{3} + \frac{1}{6} - 12\zeta' (-1, \frac{1}{2b}) + \psi \left(\frac{1}{2b}\right) + \frac{1}{2b^2} \left[\psi \left(\frac{1}{2b}\right) + \frac{1}{2}\right] + \frac{1}{b} \left[2 \log 2b - \log 2\pi + \frac{1}{2} + \frac{1}{3} \psi^{(1)} \left(\frac{1}{2b}\right) + 2 \log \Gamma \left(\frac{1}{2b}\right)\right]\right\},
$$

$$
q_\mathcal{B}^{(1)} = -q \kappa \left(\frac{a}{2\pi}\right) \left\{-\frac{b}{3} + \frac{1}{6} + \frac{1}{3} \psi^{(1)} \left(\frac{1}{2b}\right) - 4\zeta' (-1, \frac{1}{2b})\right\} + \frac{1}{b} \left[\log 2b + \frac{1}{3} \psi^{(1)} \left(\frac{1}{2b}\right) + \frac{1}{2} + \frac{1}{2b^2} \left[\psi \left(\frac{1}{2b}\right) - \frac{1}{2}\right]\right].$$

(73)
where $\zeta'(z,a)$ is the derivative of the Hurwitz Zeta function with respect to $z$, $\psi^{(j)}(z)$ is the $j$-th derivative of the DiGamma function and $\gamma \approx 0.577$ is the Euler constant [29]. Using asymptotic expansions of special functions [88], [89], discussed in Appendix B, for the large-field limit, the coefficients above behave as

$$
\tilde{g}^{(1)} \sim 2\kappa \left( \frac{\alpha}{2\pi} \right) \left( -\frac{b}{3} + K_{\tilde{g}}^{(1)} \right), \quad b \to \infty, \\
q_{B}^{(1)} \sim -q\kappa \left( \frac{\alpha}{2\pi} \right) \left( \frac{b}{3} + K_{q_{B}}^{(1)} \right), \quad b \to \infty, \\
K_{\tilde{g}}^{(1)} = \frac{1}{6} - \gamma - 12\zeta'(-1), \quad K_{q_{B}}^{(1)} = -\frac{1}{6} - \frac{\gamma}{3} - 4\zeta'(-1).
$$

Therefore it is easily seen that the leading-order contribution to the coefficient $\tilde{g}$ and to the magnetic charge $q_{B}$ in the large-field limit

$$
\tilde{g} \sim 2\kappa \left( \frac{\alpha}{2\pi} \right) \left( -\frac{a}{3} + \frac{a}{b} K_{\tilde{g}}^{(1)} \right), \quad b \to \infty, \\
\tilde{q}_{B} \sim -q\kappa \left( \frac{\alpha}{2\pi} \right) \left( \frac{a}{3} + \frac{a}{b} K_{q_{B}}^{(1)} \right), \quad b \to \infty.
$$

is proportional to the electric part $a$. The pseudoscalar quality to the magnetic charge $\tilde{q}_{B}$ is imparted by the factor $\kappa = \text{sign}(\mathcal{G})$, because $\mathcal{G}$ changes its sign under spatial reflection.

## 5 Conclusion

Within a nonlinear local electrodynamics [6], we have considered two possible types of magnetic fields – magnetically neutral and magnetically charged – created by a static electric charge $q$ placed in a background of arbitrarily strong constant electric, $\mathbf{E}$, and magnetic, $\mathbf{B}$, fields by solving (the second pair of) the Maxwell equations [7] linearized near the background and treated in the approximation of small nonlinearity. All our formulas contain coefficients that are derivatives of the nonlinear part of the Lagrangian [6], where the background values of the fields are meant to be substituted after the derivatives have been calculated. These coefficients are related to dielectric permeability and magnetic permittivity of the equivalent ”medium” formed by the background fields in the vacuum [25].

Before considering the necessary magnetic fields we establish the character of their source, which comprises of the currents induced in the equivalent ”medium” by the static charge. The result for the current inside and outside of the charge is given by Eq. (19, 20). The flow of this current [21] for the special case of parallel background fields is shown in Fig. 3. There is no induced current inside the charge in this special case.
The magnetic response $B^{(1)i}(x)$ to an introduced small extended electric charge homogeneously distributed over a sphere of the radius $R$ carries no magnetic charge when it is subjected also to the "first" Maxwell equation (the Bianchi identity) $\nabla \cdot B^{(1)}(x) = 0$. The result for $B^{(1)i}(x)$ is given by Eq. (26) inside the charged sphere and by Eqs. (28, 29) outside it. The case of the pointlike charge is covered by the $R \to 0$ limit, Eq. (28). In the simplifying case of parallel background fields $\mathbf{B} \parallel \mathbf{E}$ the magnetic response is given by Eqs. (35, 36) inside and outside the extended charge, respectively, and by Eq. (37) for the point charge. The pattern of magnetic lines of force is presented in Fig.4.

Apart from the above magnetically neutral magnetic responses, other solutions of the second Maxwell equations exist that carry magnetic charge. We have described a magnetic response $\mathbf{B}(x)$ to a pointlike electric charge in the form of a magnetic monopole, once – in full analogy with the Dirac monopole – the first Maxwell equation is imposed everywhere but in the singular point. This point is just the origin of the coordinate system, where the point-like charge $q$ is located, $r = |x| = 0$. The resulting magnetic-monopole field $\mathbf{B}(x)$ is given by Eq. (45). Unlike the Dirac monopole, this one is neither center-, nor axial-symmetric, as it depends on two generally different angles $\theta$ and $\gamma$ between the radius-vector of the observation point $x$ and the directions of the two background fields $\mathbf{B}$ and $\mathbf{E}$, respectively. According to the result (45), in the structural formula for the magnetic-monopole field (39), the function $f(\zeta, \xi) = a + b(\zeta^2 - \xi^2) + \zeta \xi$ is an even quadratic polynomial of its two cosine arguments $\zeta = \cos \theta$, $\xi = \cos \gamma$, whereas the functions $g_1$ and $g_2$ are linear monomials of one cosine each: $g_1 = \xi$, $g_2 = s \zeta$, where $a, b, c, p, s$ stand for certain constants defined by (45) as depending only on the background fields and the angle $\theta'$ between them, and not on the coordinates. The magnetic charge $q_B$ carried by the solution $\mathbf{B}(x)$ (45), defined as the surface integral around the charge, (2.2), is different from zero, hence $\nabla \cdot \mathbf{B}(x) = q_B \delta^3(x)$.

Magnetic charge should be a pseudoscalar. This fact may explain the up-to-date failure of its observation, since the restrictions on the existing of a fundamental pseudoscalar particle, the axion, are very severe (see the most recent data in [33]). On the other hand, the nonlinear theory allows us, as a matter of fact, to combine electromagnetic fields in products. This may provide an understanding, why it so came out that the pseudoscalar $\mathbf{G} = -\mathbf{B} \cdot \mathbf{E}$ (multiplied by the Coulomb field of electric charge) has appeared to form the pseudoscalar magnetic charge (45). In the special case of parallel background fields the magnetic monopole field $\mathbf{B}(x)$ is given by Eq. (47). The magnetic charge is a function of only relativistic invariants of the background fields. This implies that the expression obtained for it (2.2) is valid also for the charge moving with a constant speed in the background fields, because changing to the co-moving frame, which reduces to the Lorentz transformation of these fields, does not affect the field invariants.

We also found the vector-potential $\mathbf{A}(x)$ for the magnetic monopole (45), such that $\mathbf{B}(x) = [\nabla \times \mathbf{A}(x)]$. The vector potential as a function of the two angle variables has two singularities.
stretching from the origin along the directions of the background fields, \((1 \pm \zeta)^{-1}\) and \((1 \pm \xi)^{-1}\), i.e. there are two Dirac strings, which merge to one if these directions coincide. Note, that according to Dirac [26] the only way to reconcile the finiteness of the magnetic charge \(\oint_S (\mathfrak{B} (\mathbf{x}) \cdot \hat{n}) \, dS \neq 0\) with the potential representation \(\mathfrak{B} (\mathbf{x}) = [\nabla \times \mathfrak{A}(\mathbf{x})]\) is to deny the Stokes theorem, from which it would directly follow (see e.g. [27]) that \(\oint_S ([\nabla \times \mathfrak{A}(\mathbf{x})] \cdot \hat{n}) \, dS = 0\) in contradiction with the previous inequality. The condition for fulfillment of the Stokes theorem [27] is the continuous differentiability of the vector field \(\mathfrak{A}(\mathbf{x})\) in every point of every part of the integration surface \(S\). Therefore we must assume that this condition is violated by that \(\mathfrak{A}(\mathbf{x})\) is singular at least in one point on every surface embracing the magnetic charge. If there is one singularity on every surface these singularities may be united into a string. For us it is important that the above consideration demands that there be at least one singularity, it does not insist that there be only one. Therefore, the presence of two Dirac strings is not excluded by the general analysis. If the Dirac monopole proper is concerned with the center-symmetric radial magnetic field configuration \(q_M x/4\pi r^3\), there is no physically meaningful direction in the space. For this reason the direction of the string is arbitrary and it may be changed by a gauge transformation. This is not the case in our context: there are two directions of the background fields, or one if they are parallel, which are physically specialized. For this reason the strings are directed along them, and this orientation cannot be changed by a gauge transformation. For the same reason the magnetic monopole of the present work cannot be used for claiming the electric charge quantization, contrary to the Dirac spherically symmetric monopole [28].

To adjust the present results with the realistic situation where the nonlinearity of the Maxwell equations is owing to nonlinearity stemming from the quantum interaction between electromagnetic fields inherent to QED, we give integral representations for all the nonlinearity coefficients, in terms of which our results for the fields and the magnetic charge are expressed, as they follow from appropriate differentiations with respect to the field invariants of the effective Lagrangian of QED in its local approximation taken as the Euler-Heisenberg (one-loop) effective Lagrangian. We consider the asymptotic regime when the magnetic background dominates over the electric one. We found that in that regime the above integrals are conveniently expressed in terms of the Hurwitz Zeta function. The resulting formula for the magnetic charge is linear in the background electric field.

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A Projection operator

In this Appendix we present further details concerning the derivation of the first-order linear magnetic responses $B^{(1)}(x)$ to the static charge in consideration, given by Eqs. (26) and (27). Referring to Eqs. (14) and (12), the integral in (25) can be expressed in terms of auxiliary integrals $I^j(x)$,

$$I^j(x) = w(r) x^j, \quad w(r) = \frac{1}{r^2} \int dy \frac{\Phi(y) (y \cdot x)}{|x - y|},$$

$$\Phi(y) = \frac{\theta (R - y)}{R^3} + \frac{\theta (y - R)}{y^3}, \quad r = |x|, \quad y = |y|,$$  \hfill (76)

in which the function $\Phi(r)$, stemming from the regularized Coulomb field $E^{(0)}(x)$ (12), provides magnetic responses over all points of the space. From the latter, the integral in the rhs. of Eq. (25) is expressed as

$$\int dy \frac{5j^{(0)j}(y)}{|x - y|} = -\frac{q}{4\pi} \mathcal{L}_{\mathcal{E}} I^j(x) - \frac{q}{4\pi} \mathcal{T}_B I^j(x),$$

$$\mathcal{T}_B^j = \left( \mathcal{L}_{\mathcal{E}} B^j - \mathcal{L}_{\mathcal{E}} E^j \right) E^k + \left( \mathcal{L}_{\mathcal{E}} B^j - \mathcal{L}_{\mathcal{E}} E^j \right) B^k.$$ \hfill (77)

Next, one can evaluate the scalar integral $w(r)$ by selecting a system of reference in which $x$ is aligned along the $z$-direction, such that $y \cdot x = yr \cos \theta$ and it is reduced to a simple radial integral

$$w(r) = \frac{2\pi}{r} \int_0^\infty dy y^3 \Phi(y) \int_{-1}^1 d(cos \theta) \frac{\cos \theta}{\sqrt{r^2 + y^2 - 2ry \cos \theta}}$$

$$= \frac{2\pi}{3r^3} \int_0^\infty dy y \Phi(y) \left\{ (r^2 - ry + y^2) (r + y) - (r^2 + ry + y^2) |r - y| \right\}.$$  

Computing the remaining integrals above, the auxiliary integrals (76) acquires the final form

$$I^j(x) = 2\pi \varrho(r) x^j, \quad \varrho(r) = \varrho_{\text{in}}(r) \theta (R - r) + \varrho_{\text{out}}(r) \theta (r - R),$$

$$\varrho_{\text{in}}(r) = \frac{1}{R} \left( 1 - \frac{r^2}{5R^2} \right), \quad \varrho_{\text{out}}(r) = \frac{1}{r} \left( 1 - \frac{R^2}{5r^2} \right).$$ \hfill (78)
From the definition of $\mathcal{S}^{jk}_B$ (77), one may use the following set of identities

$$
\begin{align*}
\mathcal{S}^{ik}_B x^k &= \left[ \mathcal{L}_{\delta\delta} (\mathbf{E} \cdot \mathbf{x}) + \mathcal{L}_{\delta\delta} (\mathbf{B} \cdot \mathbf{x}) \right] \mathbf{E}^i - \left[ \mathcal{L}_{\delta\delta} (\mathbf{E} \cdot \mathbf{x}) + \mathcal{L}_{\delta\delta} (\mathbf{B} \cdot \mathbf{x}) \right] \mathbf{B}^i, \\
\mathcal{S}^{ki}_B x^k &= \left[ \mathcal{L}_{\delta\delta} (\mathbf{B} \cdot \mathbf{x}) - \mathcal{L}_{\delta\delta} (\mathbf{E} \cdot \mathbf{x}) \right] \mathbf{E}^i + \left[ \mathcal{L}_{\delta\delta} (\mathbf{B} \cdot \mathbf{x}) - \mathcal{L}_{\delta\delta} (\mathbf{E} \cdot \mathbf{x}) \right] \mathbf{B}^i, \\
\mathcal{S}^{jk}_B x^i \delta^{ij} &= \mathcal{S}^{ji}_B \delta^{ij} = \tilde{g} x^i, \\
\mathcal{S}^{jk}_B (\delta^{ij} x^k + x^j \delta^{ik} + x^i \delta^{jk}) &= U_E \mathbf{E}^i + U_B \mathbf{B}^i + \tilde{g} x^i,
\end{align*}
$$

$$U_E = (\mathcal{L}_{\delta\delta} - \mathcal{L}_{\delta\delta}) (\mathbf{B} \cdot \mathbf{x}) - 2 \mathcal{L}_{\delta\delta} (\mathbf{E} \cdot \mathbf{x}),$$

$$U_B = (\mathcal{L}_{\delta\delta} - \mathcal{L}_{\delta\delta}) (\mathbf{E} \cdot \mathbf{x}) + 2 \mathcal{L}_{\delta\delta} (\mathbf{B} \cdot \mathbf{x}),$$

(79)

to learn that the action of partial derivatives on Eq. (77) takes the final form

$$
\frac{\partial_i \partial_j}{4\pi} \int \frac{dy \mathcal{F}^{(0)i}_j (y)}{|x - y|} = \left( \frac{\mathcal{L}_{\delta\delta}}{3} - \frac{q U_E \varrho' (r)}{8\pi} \frac{1}{r} \right) \mathbf{E}^i - \left( \frac{\mathcal{L}_{\delta\delta}}{3} + \frac{q U_B \varrho' (r)}{8\pi} \frac{1}{r} \right) \mathbf{B}^i
$$

$$- \frac{q}{8\pi} \left[ \frac{\tilde{g} \varrho' (r)}{r} + \mathcal{L}_{\delta\delta} \left( \frac{4 \varrho' (r)}{r} + \varrho'' (r) \right) + \left( \varrho'' (r) - \frac{\varrho' (r)}{r} \right) \frac{x^j \mathcal{S}^{jk}_B x^k}{r^2} \right] x^i.
$$

(80)

It should be noted that the inner $\varrho_{\text{in}} (r)$ and the outer $\varrho_{\text{out}} (r)$ components of $\varrho (r)$, defined in Eq. (78), are continuous at $r = R$ (as well as its first and second derivatives). For these reasons, coefficients proportional to Dirac delta functions, stemming from the differentiation of Heaviside step functions, vanishes everywhere, including at $r = R$. Accordingly, derivatives of $\varrho (r)$ can be treated as $\varrho' (r) = \varrho'_{\text{in}} (r) \theta (R - r) + \varrho'_{\text{out}} (r) \theta (r - R)$ and $\varrho'' (r) = \varrho''_{\text{in}} (r) \theta (R - r) + \varrho''_{\text{out}} (r) \theta (r - R)$.

**B Expansion coefficients**

In this Appendix, we present exact expressions for the expansion coefficients of the derivatives of the EH effective Lagrangian (60) in the strong magnetic-dominated case, discussed in Sec. 4 in terms of the Hurwitz Zeta and related functions. Moreover, we supplement some of our previous results concerning the electric response to the charge distribution (10) by the background in consideration, namely, the nonlinearly induced total charge inside the distribution $Q$ and the coefficient $\tilde{b}$, both given by Eqs. (32) and (33) in Ref. [21], respectively. Using formulae below, we write the leading and next-to-leading expansion coefficients of these quantities.

Starting with parity-even coefficients $\mathcal{S}_e = \{ \mathcal{L}_{\delta\delta}, \mathcal{L}_{\delta\delta}, \mathcal{L}_{\delta\delta} \}$, which admits general expansions of the form

$$
\mathcal{S}_e = \mathcal{S}_e^{(0)} + \left( \frac{a}{b} \right)^2 \mathcal{S}_e^{(2)} + O \left( \left( \frac{a}{b} \right)^4 \right),
$$

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the corresponding leading and next-to-leading contributions have the form:

\[ \begin{align*}
\mathcal{L}^{(0)}_\delta &= \frac{\alpha}{2\pi} \left\{ -\frac{1}{2b^2} - \frac{1}{3} + \frac{2}{3} \log 2 + \frac{2}{3} \log b + \frac{1}{b} \log \left(\frac{\pi}{b}\right) \\
&\quad - \frac{2}{b} \log \Gamma \left(\frac{1}{2b}\right) + 8\zeta' \left(-1, \frac{1}{2b}\right) \right\} \\
\mathcal{L}^{(2)}_\delta &= \frac{\alpha}{2\pi} \left\{ b + \frac{1}{2b} + \frac{1}{b} \log \left(\frac{b}{\pi}\right) + \frac{2}{b} \log \Gamma \left(\frac{1}{2b}\right) + \frac{2}{3} \psi \left(\frac{1}{2b}\right) \\
&\quad + \frac{1}{6b} \psi^{(1)} \left(\frac{1}{2b}\right) - 8\zeta' \left(-1, \frac{1}{2b}\right) \right\},
\end{align*} \]

(81)

and

\[ \begin{align*}
\left( \delta + \sqrt{\delta^2 + \Omega^2} \right) \mathcal{L}^{(0)}_{\delta \delta} &= \frac{\alpha}{2\pi} \left\{ \frac{2}{3} - \frac{1}{b^2} + \frac{1}{b} + \frac{1}{b} \log \frac{4b}{\pi} - \frac{2}{b} \log \Gamma \left(\frac{1}{2b}\right) + \frac{1}{b^2} \psi \left(\frac{1}{2b}\right) \right\} \\
\left( \delta + \sqrt{\delta^2 + \Omega^2} \right) \mathcal{L}^{(2)}_{\delta \delta} &= \frac{\alpha}{2\pi} \left\{ -\frac{2}{3} - b - \frac{8}{3} \psi \left(\frac{1}{2b}\right) + 32\zeta' \left(-1, \frac{1}{2b}\right) \\
&\quad + \frac{1}{b} \left[ -2 - \frac{7}{6} \psi^{(1)} \left(\frac{1}{2b}\right) - 6\log 2b + 2\log 2\pi - 4\log \Gamma \left(\frac{1}{2b}\right) \right] \\
&\quad + \frac{1}{b^2} \left[ \frac{1}{6} \zeta \left(3, \frac{1}{2b}\right) - 2\psi \left(\frac{1}{2b}\right) \right] \right\} \\
\left( \delta + \sqrt{\delta^2 + \Omega^2} \right) \mathcal{L}^{(0)}_{\delta \Omega} &= \frac{\alpha}{2\pi} \left\{ -\frac{2b}{3} - \frac{1}{3} - \frac{1}{2b^2} \\
&\quad + \frac{1}{b} \left[ \log \left(\frac{\pi}{b}\right) - 2\log \Gamma \left(\frac{1}{2b}\right) \right] + 8\zeta' \left(-1, \frac{1}{2b}\right) - \frac{2}{3} \psi \left(\frac{1}{2b}\right) \right\} \\
\left( \delta + \sqrt{\delta^2 + \Omega^2} \right) \mathcal{L}^{(2)}_{\delta \Omega} &= \alpha \left\{ -\frac{8b^3}{15} + \frac{5b}{3} + \frac{2}{3} - 40\zeta' \left(-1, \frac{1}{2b}\right) + \frac{2}{15} \zeta \left(3, \frac{1}{2b}\right) \\
&\quad + \frac{10}{3} \psi \left(\frac{1}{2b}\right) + \frac{1}{b^2} \left[ \frac{3}{2} + \psi \left(\frac{1}{2b}\right) \right] \\
&\quad + \frac{1}{b} \left[ 1 + 6\log 2b - 4\log 2\pi + 8\log \Gamma \left(\frac{1}{2b}\right) + \frac{5}{6} \psi^{(1)} \left(\frac{1}{2b}\right) \right] \right\}. \quad (82)
\end{align*} \]

As for the parity-odd coefficients, \( \mathcal{L}_{\delta} \) and \( \mathcal{L}_{\delta \Omega} \), the former admits the expansion

\[ \begin{align*}
\mathcal{L}_{\delta} &= \left( \frac{a}{b} \right) \mathcal{L}_{\delta}^{(1)} + O \left( \left( \frac{a}{b} \right)^3 \right), \\
\mathcal{L}_{\delta}^{(1)} &= -\kappa \left( \frac{\alpha}{2\pi} \right) \left\{ \frac{2b}{3} + \frac{1}{3} + \frac{2}{3} \psi \left(\frac{1}{2b}\right) - 8\zeta' \left(-1, \frac{1}{2b}\right) \\
&\quad + \frac{1}{b} \left[ \log \frac{b}{\pi} + 2\log \Gamma \left(\frac{1}{2b}\right) + \frac{1}{b^2} \right] \right\},
\end{align*} \]

(83)
whereas $\mathcal{L}_{\delta \phi}$ is expanded as

$$\mathcal{S} \mathcal{L}_{\delta \phi} = \left( \frac{a}{b} \right)^2 (\mathcal{S} \mathcal{L}_{\delta \phi})^{(2)} + O \left( \left( \frac{a}{b} \right)^4 \right),$$

$$(\mathcal{S} \mathcal{L}_{\delta \phi})^{(2)} = \frac{\alpha}{2\pi} \left\{ \frac{2}{3} + \frac{2b}{3} - 16\zeta' (-1, \frac{1}{2b}) + \frac{4}{3} \psi \left( \frac{1}{2b} \right) + \frac{1}{b^2} \psi \left( \frac{1}{2b} \right) + \frac{1}{b} \left[ 1 + \frac{1}{3} \psi^{(1)} \left( \frac{1}{2b} \right) + 3 \log 2b - \log 2\pi + 2 \log \Gamma \left( \frac{1}{2b} \right) \right]\right\}.$$  (84)

It should be noted that the zero-order contributions $\mathcal{L}_{\delta}^{(0)}$ and $\mathcal{L}_{\delta \phi}^{(0)}$ are trivial in virtue of the fact that $\mathcal{L}_{\phi}$ and $\mathcal{S} \mathcal{L}_{\delta \phi}$ must vanish identically if the electric field is zero.

With the help of these coefficients, the nonlinearly induced electric charge $Q$ and the coefficient $\tilde{b}$, corresponding to the electric response of the background to the charge (10),

$$Q = q \left( \mathcal{L}_{\delta} + \frac{\tilde{b}}{3} \right), \quad \tilde{b} = - \left( \mathcal{L}_{\delta \phi} E^2 + \mathcal{L}_{\delta \phi} B^2 - 2 \mathcal{S} \mathcal{L}_{\delta \phi} \right),$$  (85)

are expanded as follows:

$$\tilde{b} = \tilde{b}^{(0)} + \left( \frac{a}{b} \right)^2 \tilde{b}^{(2)} + O \left( \left( \frac{a}{b} \right)^4 \right),$$

$$\tilde{b}^{(0)} = \mathfrak{L}_{\delta}^{(0)} - \sqrt{\mathfrak{L}^2 + \mathfrak{S}^2 \mathcal{L}_{\delta}^{(0)}}, \quad \tilde{b}^{(2)} = \mathfrak{L}_{\delta}^{(2)} - \sqrt{\mathfrak{L}^2 + \mathfrak{S}^2 \mathcal{L}_{\delta}^{(2)}} + 2 \mathcal{S} \mathcal{L}_{\delta \phi}^{(2)};$$

$$\tilde{b}^{(0)} = \frac{\alpha}{2\pi} \int_0^\infty \frac{e^{-\tau/b}}{\tau} \left[ \left( \frac{1}{\tau} - \frac{2\tau}{3} \right) \coth \tau - \frac{1}{\sinh^2 \tau} \right]$$

$$= \frac{\alpha}{2\pi} \left\{ \frac{2b}{3} + \frac{1}{3} + \frac{2}{3} \psi \left( \frac{1}{2b} \right) - 8\zeta' (-1, \frac{1}{2b}) + \frac{1}{b} \left[ \log \left( \frac{b}{\pi} \right) + 2 \log \Gamma \left( \frac{1}{2b} \right) \right] + \frac{1}{2b^2} \right\}.$$  (86)

$$\tilde{b}^{(2)} = \frac{\alpha}{2\pi} \int_0^\infty \frac{e^{-\tau/b}}{\tau} \left[ \left( \frac{\tau}{3} - \frac{4\tau^3}{15} - \frac{1}{\tau} \right) \coth \tau + \left( 1 + \frac{\tau^2}{3} \right) \frac{1}{\sinh^2 \tau} \right]$$

$$= \frac{\alpha}{2\pi} \left\{ \frac{8b^4}{15} - \frac{b}{3} - \frac{2}{15} \zeta (3, \frac{1}{2b}) - \frac{2}{3} \psi \left( \frac{1}{2b} \right) + 8\zeta' (-1, \frac{1}{2b}) + \frac{1}{6} \psi^{(1)} \left( \frac{1}{2b} \right) + 2 \log \Gamma \left( \frac{1}{2b} \right) - \frac{1}{2b^2} \right\}.$$  (86)
where $\mathcal{L}_\pm = \mathcal{L}_{\delta\delta} \pm \mathcal{L}_{\phi\phi}$ and

$$Q = Q^{(0)} + \left(\frac{a}{b}\right)^2 Q^{(2)} + O\left(\left(\frac{a}{b}\right)^4\right),$$

$$Q^{(0)} = q\left(\frac{\alpha}{3\pi}\right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left\{ 1 - \left(\frac{1}{\tau} + \frac{\tau}{3}\right) \coth \tau + \frac{1}{\sinh^2 \tau} \right\},$$

$$Q^{(2)} = q\left(\frac{\alpha}{3\pi}\right) \left\{ \frac{b}{3} + \log b - \frac{1}{3} + \log 2 + \frac{1}{3}\psi\left(\frac{1}{2b}\right) + 8\zeta'\left(-1, \frac{1}{2b}\right) \right\} + \frac{1}{b} \left\{ \log \left(\frac{\pi}{b}\right) - 2\log \Gamma\left(\frac{1}{2b}\right) - \frac{1}{2b^2} \right\},$$

$$Q^{(0)} = q\left(\frac{\alpha}{3\pi}\right) \left\{ \frac{b}{3} + \log b - \frac{1}{3} + \log 2 + \frac{1}{3}\psi\left(\frac{1}{2b}\right) + 8\zeta'\left(-1, \frac{1}{2b}\right) \right\} + \frac{1}{b} \left\{ \log \left(\frac{\pi}{b}\right) - 2\log \Gamma\left(\frac{1}{2b}\right) - \frac{1}{2b^2} \right\}.$$ (87)

The representations above are useful to study the asymptotic regime for large $b$. For example, using the expansions

$$\log \Gamma\left(\frac{1}{2b}\right) = \log b - \frac{\gamma}{2b} + \log 2 + O\left(\left(\frac{1}{2b}\right)^2\right),$$

$$\psi\left(\frac{1}{2b}\right) = -2b - \frac{\pi^2}{12b} + O\left(\left(\frac{1}{2b}\right)^2\right),$$

$$\psi^{(1)}\left(\frac{1}{2b}\right) = 4b^2 + \frac{\pi^2}{6} + \psi^{(2)}\left(\frac{1}{2b}\right) + O\left(\left(\frac{1}{2b}\right)^2\right),$$

$$\zeta'(\frac{1}{2b}) = \zeta'(\frac{1}{2}) - \frac{1}{4b} \log 2\pi - \frac{1}{4b} \left(1 - \frac{1}{2b}\right) + \int_0^{1/2b} dx \log \Gamma\left(x\right)$$

$$= \zeta'(\frac{1}{2}) - \frac{1}{4b} \log 2\pi + \frac{1}{4b} + \frac{1}{2b} \log 2b + O\left(\left(\frac{1}{2b}\right)^2\right).$$ (88)
with \( \gamma \) representing the Euler constant, \( \gamma \simeq 0.577 \), the large \( b \) limit of Eqs. (81) - (84) read

\[
\lim_{b \to \infty} \mathcal{L}^{(0)}_{\delta \delta} \sim \frac{\alpha}{2\pi} \left( \frac{2}{3} \log b + \mathcal{K}^{(0)}_{\delta \delta} \right) , \quad \mathcal{K}^{(0)}_{\delta \delta} = -\frac{1}{3} + \frac{2}{3} \log 2 + 8\zeta'(-1) ,
\]
\[
\lim_{b \to \infty} \mathcal{L}^{(2)}_{\delta \delta} \sim \frac{\alpha}{2\pi} \left( -\frac{b}{3} + \mathcal{K}^{(2)}_{\delta \delta} \right) , \quad \mathcal{K}^{(2)}_{\delta \delta} = -\frac{2}{3} \gamma - 8\zeta'(-1) ,
\]
\[
\lim_{b \to \infty} \left( \mathcal{F} + \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \right) \mathcal{L}^{(0)}_{\delta \delta} \sim \frac{\alpha}{2\pi} \mathcal{K}^{(0)}_{\delta \delta} , \quad \mathcal{K}^{(0)}_{\delta \delta} = \frac{2}{3} ,
\]
\[
\lim_{b \to \infty} \left( \mathcal{F} + \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \right) \mathcal{L}^{(2)}_{\delta \delta} \sim \frac{\alpha}{2\pi} \left( \frac{2b}{3} + \mathcal{K}^{(2)}_{\delta \delta} \right) , \quad \mathcal{K}^{(2)}_{\delta \delta} = -\frac{2}{3} - \frac{8}{3} \gamma + 32\zeta'(-1) ,
\]
\[
\lim_{b \to \infty} \left( \mathcal{F} + \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \right) \mathcal{L}^{(0)}_{\delta \delta} \sim \frac{\alpha}{2\pi} \mathcal{K}^{(0)}_{\delta \delta} , \quad \mathcal{K}^{(0)}_{\delta \delta} = -\frac{1}{3} + \frac{2}{3} \gamma + 8\zeta'(-1) ,
\]
\[
\lim_{b \to \infty} \left( \mathcal{F} + \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \right) \mathcal{L}^{(2)}_{\delta \delta} \sim \frac{\alpha}{2\pi} \left( \frac{8b^3}{15} - \frac{5b}{3} + \mathcal{K}^{(2)}_{\delta \delta} \right) , \quad \mathcal{K}^{(2)}_{\delta \delta} = \frac{2}{3} - 40\zeta'(-1) - \frac{4}{15}\psi^{(2)}(1) - \frac{10}{3}\gamma
\]
\[
\lim_{b \to \infty} (\mathcal{F} \mathcal{L}^{(0)}_{\delta \delta} )^{(2)} \sim \frac{\alpha}{2\pi} \left( -\frac{2}{3} b + \mathcal{K}^{(2)}_{\delta \delta} \right) , \quad \mathcal{K}^{(2)}_{\delta \delta} = \frac{2}{3} - \frac{4}{3} \gamma - 16\zeta'(-1) .
\]

\( (89) \)

### References

[1] T. Eber, Rev. Mod Phys. 38, 626 (1966); Z. Bialynicka-Birula and I. Bialynicki-Birula, Phys. Rev. D 2, 2341 (1970).

[2] H. Euler and B. Kockel, Naturwiss. 23, 246 (1935).

[3] W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936); V. Wesskopf, Kong. Dans. Vid. Selsk. Math-fys. Medd. XIV, 6 (1936) [English translation in: Early Quantum Electrodynamics: A Source Book, A. I. Miller (University Press, Cambridge, 1994).]

[4] Xing Fan et al., arXiv:1705.00495 (2017).

[5] S.L. Adler, Ann. Phys. (N.Y.) 67, 599 (1971); R. J. Stoneham, J. Phys. A: Math. Gen. 12, 2187 (1979); V. O. Papanyan and V. I. Ritus, Sov. Phys. JETP 34, 1195 (1972), Sov. Phys. JETP 38, 879 (1974).

[6] H. Gies, F. Karbstein, N. Seegert, Phys. Rev. D 93, 085034 (2016).

[7] M. Aaboud et al. (ATLAS Collaboration), Evidence for light-by-light scattering in heavy-ion collisions with the ATLAS detector at the LHC, arXiv:1702.01625 Published in Nature Physics (2017).
[8] S. Weinberg, *The Quantum Theory of Fields* (University Press, Cambridge, 2001); V. I. Ritus, Zh. Eksp. Teor. Fiz. 69, 1517 (1975) [Sov. Phys. JETP 42, 774 (1976)].

[9] V.I. Ritus, in *Issues in Intense-Field Quantum Electrodynamics*, Proc. Lebed. Phys. Inst. 168, 5, ed. V. L. Ginzburg (Nauka, Moscow, 1986; Nova Science Publ., New York, 1987); Zh. Exp. Theor. Phys., 69, 540 (1975) [Sov. Phys.- JETP, 42, 775 (1975)]. S. R. Valluri, U. D. Jentschura, D.R. Lamm, arXiv:hep-ph/0308223 (2003)

[10] H. Gies, F. Karbstein, JHEP 03, 108 (2017); I. Huet, M.R. de Traubenberg, C. Schubert, Int. J. Mod. Phys.: Conference Series, 14, pp. 383-393 (2012); F. Karbstein, arXiv:1709.03819.

[11] D. M. Gitman and A. E. Shabad, Phys. Rev. D 86, 125028 (2012).

[12] C. V. Costa, D. M. Gitman, and A. E. Shabad, Phys. Rev. D 88, 085026 (2013).

[13] T. C. Adorno, D. M. Gitman, and A. E. Shabad, Eur. Phys. J. C 74, 2838 (2014).

[14] T. C. Adorno, D. M. Gitman, A. E. Shabad and A. Shishmarev, Izvestiya Vusov, FIZIKA, 59, 45 (2016) [Russ. Phys. Journ., 59, 1775 (2017)].

[15] C. V. Costa, D. M. Gitman, A. E. Shabad, Phys. Scr. 90, 074012 (2015).

[16] D. M. Gitman, A.E. Shabad, and A.A. Shishmarev, arXiv:1509.06401 [hep-th] (2015); Phys. Scr. 92, 054005 (2017).

[17] M.B. Ependiev, Teor. Mat. Fiz. 191, 417 (2017) [Theor. Math. Phys. 191, 836 (2017)].

[18] B. King, P. Böhl and H. Ruhl, Phys. Rev. D 90, 065018 (2014).

[19] A. Di Piazza, Phys. Rev. A 95, 032121 (2017).

[20] T. C. Adorno, D. M. Gitman, and A. E. Shabad, Phys. Rev. D 89, 047504 (2014).

[21] T. C. Adorno, D. M. Gitman, and A. E. Shabad, Phys. Rev. D 93, 125031 (2016).

[22] I. A. Batalin and A. E. Shabad, Zh. Eksp. Teor. Fiz. 60, 894 (1971) [Sov. Phys. JETP 33, 483 (1971)]; A. E. Shabad and V. V. Usov, Phys. Rev. D 81, 125008 (2010).

[23] T. C. Adorno, D. M. Gitman, and A. E. Shabad, Phys. Rev. D 92, 041702 (2015).

[24] V. B. Berestetsky, E. M. Lifshits, and L. P. Pitayevsky, *Quantum Electrodynamics* (Nauka, Moscow, 1989; Pergamon Press Oxford, New York, 1982).

[25] S. Villalba-Chávez and A. E. Shabad, Phys. Rev. D 86, 105040 (2012)
[26] P. A. M. Dirac, Proc. R. Soc. Lond. A 133, 60 (1931).

[27] I.E. Tamm, Foundations of the Electricity Theory (GITTL, Moscow, 1956).

[28] P. A. M. Dirac, Phys. Rev. 74, 817 (1948).

[29] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/ 2015-08-07 DLMF Update, Version 1.0.10.

[30] W. Dittrich and H. Gies, Probing the Quantum Vacuum; Perturbative Effective Action Approach in Quantum Electrodynamics and Its Application, Springer Tracts in Modern Physics Vol. 166 (Springer, Berlin, 2000).

[31] E. Elizalde, Ten Physical Applications of Spectral Zeta Functions, (Lecture Notes in Physics, Vol. 855, 2nd Ed., New York, 2012).

[32] K. Kirsten, Spectral Functions in Mathematics and Physics, (Chapman & Hall/CRC, Boca Raton, 2002).

[33] S. Villalba-Chávez, T. Podszus and C.Müller, arXiv:1612:0795 (2016).