Four-dimensional couplings among BF and massless Rarita-Schwinger theories: a BRST cohomological approach

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Abstract

The local and manifestly covariant Lagrangian interactions in four
spacetime dimensions that can be added to a free model that describes
a massless Rarita-Schwinger theory and an Abelian BF theory are
constructed by means of deforming the solution to the master equation
on behalf of specific cohomological techniques.

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1 Introduction

Topological field theories [1–2] are important in view of the fact that certain
interacting, non-Abelian versions are related to a Poisson structure algebra [3]
present in various versions of Poisson sigma models [4–10], which are known
to be useful at the study of two-dimensional gravity [11–20] (for a detailed
approach, see [21]). It is well known that pure three-dimensional gravity
is just a BF theory. Moreover, in higher dimensions general relativity and

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supergravity in Ashtekar formalism may also be formulated as topological BF theories with some extra constraints [22]–[25]. In view of these results, it is important to know the self-interactions in BF theories as well as the couplings between BF models and other theories. This problem has been considered in literature in relation with self-interactions in various classes of BF models [26]–[32] and couplings to other (matter or gauge) fields [33]–[36] by using the powerful BRST cohomological reformulation of the problem of constructing consistent interactions within the Lagrangian [37] or the Hamiltonian [38] setting. Other aspects concerning interacting, topological BF models can be found in [39] and [40].

The scope of this paper is to investigate the consistent interactions that can be added to a free, Abelian gauge theory consisting of a BF model and a massless Rarita-Schwinger field in \( D = 4 \). This matter is addressed by means of the deformation of the solution to the master equation from the BRST-antifield formalism [37]. Under the hypotheses of smooth, local, Lorentz covariant, and Poincaré invariant interactions, supplemented with the requirement on the preservation of the number of derivatives on each field with respect to the free theory, we obtain the most general form of the theory that describes the cross-couplings between a BF model and a massless spin-3/2 field. The resulting interacting model is accurately formulated in terms of a gauge theory with gauge transformations that close according to an open algebra (the commutators among the deformed gauge transformations only close on the stationary surface of deformed field equations), which are on-shell, second-order reducible. An interesting feature of the coupled theory is the appearance of certain similarities with the gauge symmetries from the gravitini sector of \( N = 1, D = 4 \) conformal SUGRA at the level of local \( Q \)-supersymmetry and \( U(1) \) gauge symmetry.

2 Free model: Lagrangian formulation and BRST symmetry

We start from a free four-dimensional theory whose Lagrangian action is written as the sum between the action for a massless Rarita-Schwinger field and the action for a topological BF theory involving one scalar field, two
one-forms and one two-form

\[ S_0[\psi_\mu, A^\mu, H^\mu, \varphi, B^{\mu\nu}] = \int d^4x \left( -\frac{i}{2} \bar{\psi}_\mu \gamma^{\mu\nu} \partial_\nu \psi_\rho + H_\mu \partial^\mu \varphi + \frac{1}{2} B^{\mu\nu} \partial_\mu A_\nu \right) \]

\[ \equiv \int d^4x \left( \mathcal{L}_0^{\text{RS}} + \mathcal{L}_0^{\text{BF}} \right). \]  

(1)

We work with a Minkowski-flat metric tensor of 'mostly minus' signature \( \sigma^{\mu\nu} = \sigma_{\mu\nu} = (+ - - -) \) and employ the Majorana representation of the Clifford algebra

\[ \gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = 2\sigma_{\mu\nu} 1. \]  

(2)

This means that all the \( \gamma \)-matrices are purely imaginary, with \( \gamma_0 \) Hermitian and \( \gamma_i \) anti-Hermitian. The charge conjugation matrix in this representation is given by

\[ C = -\gamma^0, \]  

(3)

while the Dirac and the charge conjugation operations are respectively defined by the expressions

\[ \bar{\psi} \equiv \psi^\dagger \gamma^0, \quad \psi^c \equiv (C\psi)^\top. \]  

(4)

In the above we denoted by \( \dagger \) and \( \top \) the operations of Hermitian conjugation and transposition, respectively. The Rarita-Schwinger field \( \psi_\mu \) is a Majorana vector spinor

\[ \bar{\psi}_\mu = \psi^{c\mu}. \]  

(5)

For definiteness, we take a basis in the vector space of 4 × 4 complex matrices of the form

\[ \{ 1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\rho}, \gamma_5 \}, \]  

(6)

where the generic notation \( \gamma_{\mu_1 \cdots \mu_k} \) means the (normalized) antisymmetrical product of \( k \) \( \gamma \)-matrices

\[ \gamma_{\mu_1 \cdots \mu_k} = \frac{1}{k!} \sum_{\sigma \in S_k} (-)^{\sigma} \gamma_{\mu_{\sigma(1)} \cdots \mu_{\sigma(k)}}. \]  

(7)

\( S_k \) and \( (-)^{\sigma} \) denote the set of permutations of \( \{1, \ldots, k\} \) and the signature of the permutation \( \sigma \), respectively. Finally, the matrix \( \gamma_5 \) is defined in the standard manner as \( \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \). Also, it is useful to recall the four-dimensional duality relations among the various matrices \( \gamma_{\mu_1 \cdots \mu_k} \), namely

\[ \gamma^{\mu_\nu\rho\lambda} = i\varepsilon^{\mu_\nu\rho\lambda} \gamma_5, \quad \gamma^\mu_{\nu} = -i\varepsilon^{\mu_\nu\rho\lambda} \gamma_\lambda \gamma_5, \]  

(8)
\begin{align*}
\gamma^{\mu \nu} &= -\frac{i}{2} \varepsilon^{\mu \nu \rho \lambda} \gamma_{\rho \lambda} \gamma_{5}, \quad \gamma^\mu = \frac{i}{6} \varepsilon^{\mu \nu \rho \lambda} \gamma_{\nu \rho \lambda} \gamma_{5}, \\
\text{where we used the convention } \varepsilon^{0123} = -\varepsilon^{0123} = 1 \text{ for the four-dimensional Levi-Civita symbol. In the chosen representation of algebra } (2), \text{ the } \gamma-\text{matrices exhibit the following symmetry/antisymmetry properties}
\end{align*}

\begin{align*}
(\gamma^0 \gamma^\mu)^\top &= \gamma^0 \gamma^\mu, \quad (\gamma^0 \gamma^{\mu \nu})^\top = \gamma^0 \gamma^{\mu \nu}, \\
(\gamma^0 \gamma^{\mu \nu \rho})^\top &= -\gamma^0 \gamma^{\mu \nu \rho}, \quad (\gamma^0 \gamma_5)^\top = -\gamma^0 \gamma_5.
\end{align*}

Spinor-like indices will be denoted everywhere by Latin capital letters, such that

\begin{align*}
\psi_\mu &\equiv (\psi_A^\mu)_{A=1,4} \equiv \begin{pmatrix}
\psi_1^\mu \\
\psi_2^\mu \\
\psi_3^\mu \\
\psi_4^\mu
\end{pmatrix}. \\
\text{Action } (1) \text{ is found invariant under the gauge transformations}
\end{align*}

\begin{align*}
\delta_\epsilon A^\mu &= \partial^\mu \epsilon, \quad \delta_\epsilon H^\mu = 2 \partial\rho \epsilon^{\mu \rho}, \quad \delta_\epsilon B^{\mu \nu} = -3 \partial\rho \epsilon^{\mu \nu \rho}, \\
\delta_\epsilon \varphi &= 0, \quad \delta_\epsilon \psi_\mu = \partial_\mu \chi,
\end{align*}

where the gauge parameters \( \epsilon, \epsilon^{\mu \nu}, \text{ and } \epsilon^{\mu \nu \rho} \) are bosonic, with \( \epsilon^{\mu \nu} \) and \( \epsilon^{\mu \nu \rho} \) completely antisymmetric. In addition, the gauge parameter \( \chi \) is a Majorana spinor

\begin{align*}
\chi &\equiv (\chi^A)_{A=1,4} \equiv \begin{pmatrix}
\chi^1 \\
\chi^2 \\
\chi^3 \\
\chi^4
\end{pmatrix}. \\
\text{From } (12) \text{ and } (13) \text{ we read the nonvanishing gauge generators of the fields, written in De Witt condensed notations, as}
\end{align*}

\begin{align*}
(Z^{\mu}_{(A)}) &= \partial^\mu, \quad (Z^{\mu}_{(H)})_{\alpha \beta} = -\partial_\alpha \delta^\mu_\beta, \\
(Z^{\mu \nu}_{(B)})_{\alpha \beta \gamma} &= -\frac{1}{2} \partial_\alpha \delta^\mu_\beta \delta^\nu_\gamma, \quad (Z^{A \mu}_{(\psi)})_B = \delta^A_\beta \partial^\mu,
\end{align*}

where we put an extra lower index \( ((A), (H), \text{etc.}) \) in order to indicate the field to which a certain gauge generator is associated with. Everywhere in this paper we use the convention that the symbol \( [\alpha \beta \ldots \gamma] \) signifies the operation of complete antisymmetry with respect to the (Lorentz) indices between
brackets, with the conventions that the minimum number of terms is always used and the result is never divided by the number of terms. The above gauge transformations are Abelian and off-shell, second-order reducible. More precisely, the gauge generators of the one-form $H^\mu$ are second-order reducible, with the first- and respectively second-order reducibility functions

\[(Z_1^{\alpha\beta})_{\mu'\nu'\rho'} = -\frac{1}{2} \partial_{[\mu'} \delta^\alpha_{\nu'] \delta^\beta_{\rho']}, \quad (Z_2^{\mu'\nu'\rho'})_{\alpha'\beta'\gamma'} = -\frac{1}{6} \partial_{[\alpha'} \delta^\mu_{\beta']} \delta^\nu_{\gamma'} \delta^\rho_{\delta']},\]  
\(16\)

while the gauge generators of the two-form $B^{\mu\nu}$ are first-order reducible, with the reducibility functions

\[(Z_1^{\alpha\beta\gamma})_{\mu'\nu'\rho'\lambda'} = -\frac{1}{6} \partial_{[\mu'} \delta^\alpha_{\nu'] \delta^\beta_{\rho'] \delta^\gamma_{\lambda']},\]  
\(17\)

such that the concrete form of the first- and second-order reducibility relations are expressed by

\[(Z_\mu^{\alpha\beta})(Z_1^{\alpha\beta})_{\mu'\nu'\rho'} = 0, \quad (Z_\mu^{\mu\nu})(Z_1^{\alpha\beta\gamma})(Z_1^{\alpha\beta\gamma})_{\mu'\nu'\rho'\lambda'} = 0,\]  
\(18\)

and

\[(Z_1^{\alpha\beta})_{\mu'\nu'\rho'}(Z_2^{\mu'\nu'\rho'})_{\alpha'\beta'\gamma'\delta'} = 0,\]  
\(19\)

respectively. We observe that the theory described by action (1) is a usual linear gauge theory (its field equations are linear in the fields and first-order in their spacetime derivatives), whose generating set of gauge transformations is second-order reducible, such that we can define in a consistent manner its Cauchy order, which is found equal to four.

In order to construct the BRST symmetry of this free theory, we introduce the field/ghost and antifield spectra

\[
\Phi^{\alpha_0} = (A^\mu, H^\mu, \varphi, B^{\mu\nu}, \psi_\mu), \quad \Phi^*_{\alpha_0} = (A^*_\mu, H^*_\mu, \varphi^*, B^{*\mu\nu}, \psi^{*\mu}), \quad (20)
\]

\[
\eta^{\alpha_1} = (\eta, C^{\mu\nu}, \eta^{\mu\nu\rho}, \xi), \quad \eta^*_{\alpha_1} = (\eta^*, C^{*\mu\nu}, \eta^{*\mu\nu\rho}, \xi^*), \quad (21)
\]

\[
\eta^{\alpha_2} = (C^{\mu\nu\rho}, \eta^{\mu\nu\rho\lambda}), \quad \eta^*_{\alpha_2} = (C^{*\mu\nu\rho}, \eta^{*\mu\nu\rho\lambda}), \quad (22)
\]

\[
\eta^{\alpha_3} = C^{\mu\nu\rho\lambda}, \quad \eta^*_{\alpha_3} = C^{*\mu\nu\rho\lambda}. \quad (23)
\]

The fermionic ghosts ($\eta, C^{\mu\nu}, \eta^{\mu\nu\rho}$) respectively correspond to the bosonic gauge parameters ($\epsilon, \epsilon^{\mu\nu}, \epsilon^{\mu\nu\rho}$), the bosonic ghosts for ghosts $\eta^{\alpha_2}$ are due to the first-order reducibility relations (18), while the fermionic ghosts for ghosts for ghosts $\eta^{\alpha_3}$ are required by the second-order reducibility relations (19).
addition, the ghost $\xi$, associated with the gauge parameter $\chi$, is a bosonic spinor of purely imaginary components. The star variables represent the antifields of the corresponding fields/ghosts. Their Grassmann parities are obtained via the usual rule

$$\varepsilon(\chi^*_{\Delta}) = (\varepsilon(\chi^\Delta) + 1) \text{ mod } 2,$$

where we employed the notations

$$\chi^\Delta = (\Phi^{\alpha_0}, \eta^{\alpha_1}, \eta^{\alpha_2}, \eta^{\alpha_3}), \quad \chi^*_\Delta = (\Phi^{\alpha_0*}, \eta^{\alpha_1*}, \eta^{\alpha_2*}, \eta^{\alpha_3*}). \quad (24)$$

Since both the gauge generators and the reducibility functions are field-independent, it follows that the BRST differential simply reduces to

$$s = \delta + \gamma,$$  \quad (25)

where $\delta$ is the Koszul-Tate differential and $\gamma$ means the exterior longitudinal derivative. The Koszul-Tate differential is graded in terms of the antighost number ($\text{agh}, \text{agh}(\delta) = -1, \text{agh}(\gamma) = 0$) and enforces a resolution of the algebra of smooth functions defined on the stationary surface of field equations for action $[1], C^\infty(\Sigma), \Sigma : \delta S_0/\delta \Phi^{\alpha_0} = 0$. The exterior longitudinal derivative is graded in terms of the pure ghost number ($\text{pgh}, \text{pgh}(\delta) = 1, \text{pgh}(\gamma) = 0$) and is correlated with the original gauge symmetry via its cohomology at pure ghost number zero computed in $C^\infty(\Sigma)$, which is isomorphic to the algebra of physical observables for the free theory. These two degrees of the generators (20)–(23) from the BRST complex are valued like

$$\text{pgh}(\Phi^{\alpha_0}) = 0,$$
$$\text{pgh}(\eta^{\alpha_1}) = 1,$$  \quad (26)
$$\text{pgh}(\eta^{\alpha_2}) = 2,$$
$$\text{pgh}(\eta^{\alpha_3}) = 3,$$  \quad (27)
$$\text{agh}(\Phi^{\alpha_0}) = \text{agh}(\eta^{\alpha_1}) = \text{agh}(\eta^{\alpha_2}) = \text{agh}(\eta^{\alpha_3}) = 0,$$  \quad (28)
$$\text{agh}(\Phi^{\alpha_0*}) = 1,$$
$$\text{agh}(\eta^{\alpha_1*}) = 2,$$  \quad (29)
$$\text{agh}(\eta^{\alpha_2*}) = 3,$$
$$\text{agh}(\eta^{\alpha_3*}) = 4,$$  \quad (30)

where the (right) actions of $\delta$ and $\gamma$ on them read as

$$\delta \Phi^{\alpha_0} = \delta \eta^{\alpha_1} = \delta \eta^{\alpha_2} = \delta \eta^{\alpha_3} = 0,$$  \quad (32)
$$\delta A^*_\mu = \partial^\nu B_{\nu \mu}, \quad \delta H^*_\mu = -\partial_\mu \varphi,$$  \quad (33)
\[ \delta \varphi^* = \partial^\mu H_\mu, \quad \delta B^*_{\mu \nu} = -\frac{1}{2} \partial_{[\mu} A_{\nu]}, \] (34)
\[ \delta \psi^* = -i \partial_\nu \bar{\psi}_\rho \gamma^{\mu \nu \rho}, \quad \delta \eta^* = -\partial^\mu A^*_\mu, \quad \delta C^*_{\mu \nu} = \partial_{[\mu} H^*_{\nu]}, \] (35)
\[ \delta \eta^*_{\mu \nu \rho} = \partial_{[\mu} \eta^*_{\nu \rho]}, \quad \delta \xi^* = \partial_{\mu} \psi^*_{\mu}, \quad \delta C^*_{\mu \nu \rho} = -\partial_{[\mu} C^*_{\nu \rho]} \] (36)
\[ \delta \Phi^*_{\alpha_0} = \gamma \eta^*_{\alpha_1} = \gamma \eta^*_{\alpha_2} = \gamma \eta^*_{\alpha_3} = 0, \] (38)
\[ \gamma A^\mu = \partial^\mu \eta, \quad \gamma H^\mu = 2 \partial_\rho C^*_{\mu \nu}, \quad \gamma B_{\mu \nu}^\mu = -3 \partial_\rho \eta^*_{\mu \nu \rho}, \] (39)
\[ \gamma \varphi = 0, \quad \gamma \psi_{\mu} = \partial_{\mu} \xi, \] (40)
\[ \gamma \eta = \gamma \xi = 0, \quad \gamma C_{\mu \nu}^\mu = -3 \partial_\rho C^*_{\mu \nu \rho}, \quad \gamma \eta^*_{\mu \nu \rho} = 4 \partial_\lambda \eta^*_{\mu \nu \rho \lambda}, \] (41)
\[ \gamma C_{\mu \nu \rho}^\mu = 4 \partial_\lambda C_{\mu \nu \rho}^\mu, \quad \gamma \eta^*_{\mu \nu \rho \lambda} = \gamma C_{\mu \nu \rho \lambda}^\mu = 0. \] (42)

The overall degree of the BRST complex is named ghost number (gh) and is defined like the difference between the pure ghost number and the antighost number, such that \( \text{gh}(s) = \text{gh}(\delta) = \text{gh}(\gamma) = 1 \). The BRST differential is known to have a canonical action in a structure named antibracket and denoted by the symbol \( (\cdot, \cdot) \) \( (s \cdot = \cdot, \bar{\mathcal{S}}) \), which is obtained by decreeing the fields/ghosts respectively conjugated to the corresponding antifields. The generator of the BRST symmetry is a bosonic functional of ghost number zero \( (\text{gh}(\bar{\mathcal{S}}) = 0, \varepsilon(\bar{\mathcal{S}}) = 0) \), which is solution to the classical master equation \( (\bar{\mathcal{S}}, \bar{\mathcal{S}}) = 0 \). In the case of the free theory under discussion, the solution to the master equation takes the form

\[ \bar{\mathcal{S}} = S_0 + \int d^4 x \left( A^*_{\mu} \partial^\mu \eta + 2 H^*_{\mu} \partial_\rho C^\mu_{\nu} - 3 B^*_{\mu \nu} \partial_\rho \eta^*_{\mu \nu \rho} + \psi^*_{\mu} \partial_\mu \xi - 3 C^*_{\mu \nu} \partial_\rho C^\mu_{\nu \rho} + 4 \eta^*_{\mu \nu \rho} \partial_\lambda \eta^*_{\mu \nu \rho \lambda} + 4 C^*_{\mu \nu \rho} \partial_\lambda C^\mu_{\nu \rho \lambda} \right). \] (43)

The solution to the master equation encodes all the information on the gauge structure of a given theory. We remark that in our case the solution \( (43) \) breaks into terms with antighost numbers ranging from zero to three. The piece with antighost number zero is nothing but the Lagrangian action \( (1) \), while the elements of antighost number one include the gauge generators \( (14) \)–\( (15) \). If the gauge algebra were non-Abelian, then there would appear at least terms linear in the antighost number two antifields and quadratic in the pure ghost number one ghosts. The absence of such terms in our case reflects that the gauge transformations are Abelian. The terms from \( (43) \) of higher antighost number give us information on the reducibility functions \( (16) \) and \( (17) \). If the reducibility relations held on-shell, then the solution of
the master equation would contain components linear in the ghosts for ghosts (ghosts of pure ghost number strictly greater than one) and quadratic in the various antifields. Such pieces are not present in (43), since the reducibility relations hold off-shell. Other possible components in the solution to the master equation offer information on the higher-order structure functions related to the tensor gauge structure of the theory. There are no such terms in (43), as a consequence of the fact that all higher-order structure functions vanish for this (free) model.

3 Deformation of the master equation — brief review

We begin with a “free” gauge theory, described by a Lagrangian action $S_0[\Phi^{\alpha_0}]$, invariant under some gauge transformations

$$\delta_\epsilon \Phi^{\alpha_0} = Z^{\alpha_0}_{\alpha_1} \epsilon^{\alpha_1}, \quad \frac{\delta S_0}{\delta \Phi^{\alpha_0}} Z^{\alpha_0}_{\alpha_1} = 0,$$

(44)

and consider the problem of constructing consistent interactions among the fields $\Phi^{\alpha_0}$ such that the couplings preserve the field spectrum and the original number of gauge symmetries. This matter is addressed by means of reformulating the problem of constructing consistent interactions as a deformation problem of the solution to the master equation corresponding to the “free” theory \[37], \[41]. Such a reformulation is possible due to the fact that the solution to the master equation contains all the information on the gauge structure of the theory. If a consistent interacting gauge theory can be constructed, then the solution $\bar{S}$ to the master equation associated with the “free” theory, $(\bar{S}, \bar{S}) = 0$, can be deformed into a solution $S$,

$$\bar{S} \to S = \bar{S} + \lambda S_1 + \lambda^2 S_2 + \cdots = \bar{S} + \lambda \int d^D x a + \lambda^2 \int d^D x b + \cdots,$$

(45)

of the master equation for the deformed theory

$$(S, S) = 0,$$

(46)

such that both the ghost and antifield spectra of the initial theory are preserved. The symbol $(,)$ denotes the antibracket. The equation (46) splits,
according to the various orders in the coupling constant (or deformation parameter) $\lambda$, into

\begin{align*}
(S, \bar{S}) &= 0, \quad (47) \\
2(S_1, \bar{S}) &= 0, \quad (48) \\
2(S_2, \bar{S}) + (S_1, S_1) &= 0, \quad (49) \\
(S_3, \bar{S}) + (S_1, S_2) &= 0, \quad (50) \\
&\vdots
\end{align*}

Equation (47) is fulfilled by hypothesis. The next equation requires that the first-order deformation of the solution to the master equation, $S_1$, is a co-cycle of the “free” BRST differential $s = (\cdot, \bar{S})$. However, only cohomologically nontrivial solutions to (48) should be taken into account, because the BRST-exact ones can be eliminated by a (possibly nonlinear) field redefinition. This means that $S_1$ pertains to the ghost number zero cohomological space of $s$, $H^0(s)$, which is generically nonempty due to its isomorphism to the space of physical observables of the “free” theory. It has been shown in [37, 41] (by means of the triviality of the antibracket map in the cohomology of the BRST differential) that there are no obstructions in finding solutions to the remaining equations ((49)–(50), etc.). However, the resulting interactions may be nonlocal, and there might even appear obstructions if one insists on their locality. As it will be seen below, this is not the case here since all the interactions in the case of the model under study turn out to be local.

4 Consistent interactions between a massless Rarita-Schwinger field and a topological BF theory

In this section we determine the consistent interactions that can be added to the free theory (1) that describes a massless Rarita-Schwinger field plus a topological BF model in four spacetime dimensions. This is done by solving the Lagrangian deformation equations ((48)–(50), etc.) via specific cohomological BRST techniques. The interacting theory and its gauge structure are deduced from the analysis of the deformed solution to the master equation.
that is consistent to all orders in the deformation parameter. For obvious reasons, we consider only smooth, local, Lorentz covariant, and Poincaré invariant deformations (i.e., we do not allow explicit dependence on the spacetime coordinates). In the meantime we require that the maximum number of derivatives allowed to enter the interaction vertices is equal to one, i.e. the maximum number of derivatives from the free Lagrangian. The smoothness of deformations refers to the fact that the deformed solution to the master equation, (45), is smooth in the coupling constant \( \lambda \) and reduces to the original solution, (43), in the free limit \( \lambda = 0 \).

### 4.1 Standard material: basic cohomologies

If we make the notation \( S_1 = \int d^4 x a \), with \( a \) a local function, then equation (48), which we have seen that controls the first-order deformation, takes the local form

\[
\text{sa} = \partial_\mu m^\mu, \quad \text{gh}(a) = 0, \quad \varepsilon(a) = 0, \quad (51)
\]

for some local \( m^\mu \). It shows that the nonintegrated density of the first-order deformation pertains to the local cohomology of \( s \) in ghost number zero, \( a \in H^0(s|d) \), where \( d \) denotes the exterior spacetime differential. The solution to (51) is unique up to \( s \)-exact pieces plus divergences

\[
a \rightarrow a + sb + \partial_\mu n^\mu, \quad \text{gh}(b) = -1, \quad \varepsilon(b) = 1. \quad (52)
\]

At the same time, if the general solution to (51) is found to be completely trivial, \( a = sb + \partial_\mu n^\mu \), then it can be made to vanish, \( a = 0 \).

In order to analyze equation (51) we develop \( a \) according to the antighost number

\[
a = \sum_{i=0}^{I} a_i, \quad \text{agh}(a_i) = i, \quad \text{gh}(a_i) = 0, \quad \varepsilon(a_i) = 0, \quad (53)
\]

and assume, without loss of generality, that the above decomposition stops at some finite value of \( I \). This can be shown, for instance, like in [43] (Section 3), under the sole assumption that the interacting Lagrangian at the first order in the coupling constant, \( a_0 \), has a finite, but otherwise arbitrary derivative order. Inserting decomposition (53) into equation (51) and projecting it on the various values of the antighost number, we obtain the tower of equations

\[
\gamma a_I = \partial_\mu \tilde{m}^\mu, \quad (54)
\]
\[
\begin{align*}
\delta a_I + \gamma a_{I-1} &= \partial_\mu \left( \frac{(I-1)^\mu}{m} \right), \\
\delta a_i + \gamma a_{i-1} &= \partial_\mu \left( \frac{(i-1)^\mu}{m} \right), & 1 \leq i \leq I - 1,
\end{align*}
\]
where \( \left( \frac{(i)^\mu}{m} \right) \) are some local currents with \( \text{agh} \left( \frac{(i)^\mu}{m} \right) = i \). Equation (55) can be replaced in strictly positive values of the antighost number by
\[
\gamma a_I = 0, \quad I > 0.
\]
Due to the second-order nilpotency of \( \gamma \) \( (\gamma^2 = 0) \), the solution to (57) is clearly unique up to \( \gamma \)-exact contributions
\[
a_I \rightarrow a_I + \gamma b_I, \quad \text{agh} (b_I) = I, \quad \text{pgh} (b_I) = I - 1, \quad \varepsilon (b_I) = 1. \tag{58}
\]
Meanwhile, if it turns out that \( a_I \) exclusively reduces to \( \gamma \)-exact terms, \( a_I = \gamma b_I \), then it can be made to vanish, \( a_I = 0 \). In other words, the nontriviality of the first-order deformation \( a \) is translated at its highest antighost number component into the requirement that \( a_I \in H^I (\gamma) \), where \( H^I (\gamma) \) denotes the cohomology of the exterior longitudinal derivative \( \gamma \) in pure ghost number equal to \( I \). So, in order to solve equation (51) (equivalent with (57) and (55)–(56)), we need to compute the cohomology of \( \gamma \), \( H (\gamma) \), and, as it will be made clear below, also the local homology of \( \delta \), \( H (\delta | d) \).

On behalf of definitions (38)–(42) it is simple to see that \( H (\gamma) \) is spanned by
\[
F_A = (\varphi, \partial_\mu A_\nu, \partial^\mu H_\mu, \partial_\mu B^{\mu\nu}, \partial_\mu \psi_\nu),
\]
the antifields \( \chi^*_\Delta \), all of their spacetime derivatives as well as by the undifferentiated ghosts
\[
\eta^\dagger = (\eta, \xi, \eta^{\mu\nu\rho\lambda}, C^{\mu\nu\rho\lambda}).
\]
(The derivatives of the ghosts \( \eta^\dagger \) are removed from \( H (\gamma) \) since they are \( \gamma \)-exact, in agreement with the first relation from (39), the last formula in (40), the fourth equation in (41), and the first definition from (42).) If we denote by \( e^M (\eta^\dagger) \) the elements with pure ghost number \( M \) of a basis in the space of the polynomials in the ghosts (60), then it follows that the general solution to equation (57) takes the form
\[
a_I = \alpha_I ([F_A], [\chi^*_\Delta]) e^I \left( \eta^\dagger \right),
\]
(61)
where \( \text{agh} (\alpha_I) = I \) and \( \text{pgh} (e^I) = I \). The notation \( f([q]) \) means that \( f \) depends on \( q \) and its spacetime derivatives up to a finite order. The objects \( \alpha_I \) (obviously nontrivial in \( H^0 (\gamma) \)) will be called “invariant polynomials”. The result that we can replace equation (54) with the less obvious one (57) is a nice consequence of the fact that the cohomology of the exterior spacetime differential is trivial in the space of invariant polynomials in strictly positive antighost numbers.

Inserting (61) in (55) we obtain that a necessary (but not sufficient) condition for the existence of (nontrivial) solutions \( a_{I-1} \) is that the invariant polynomials \( \alpha_I \) are (nontrivial) objects from the local cohomology of Koszul-Tate differential \( H(\delta|d) \) in antighost number \( I > 0 \) and in pure ghost number zero,

\[
\delta \alpha_I = \partial_\mu (I-1)^\mu_j, \quad \text{agh} \left( (I-1)^\mu_j \right) = I - 1, \quad \text{pgh} \left( (I-1)^\mu_j \right) = 0. \tag{62}
\]

We recall that the local cohomology \( H(\delta|d) \) is completely trivial in both strictly positive antighost and pure ghost numbers (for instance, see \([42] \), Theorem 5.4, and \([43] \) ), so from now on it is understood that by \( H(\delta|d) \) we mean the local cohomology of \( \delta \) at pure ghost number zero. Using the fact that the free BF model under study is a linear gauge theory of Cauchy order equal to four and the general result from \([42, 43] \), according to which the local cohomology of the Koszul-Tate differential is trivial in antighost numbers strictly greater than its Cauchy order, we can state that

\[
H_J(\delta|d) = 0 \quad \text{for all} \quad J > 4, \tag{63}
\]

where \( H_J(\delta|d) \) represents the local cohomology of the Koszul-Tate differential in antighost number \( J \). Moreover, if the invariant polynomial \( \alpha_J \), with \( \text{agh}(\alpha_J) = J \geq 4 \), is trivial in \( H_J(\delta|d) \), then it can be taken to be trivial also in \( H^\text{inv}_J(\delta|d) \):

\[
\left( \alpha_J = \delta b_{J+1} + \partial_\mu (j)^\mu_c, \text{agh} (\alpha_J) = J \geq 4 \right) \Rightarrow \alpha_J = \delta \beta_{J+1} + \partial_\mu (\gamma) J^\mu, \tag{64}
\]

with both \( \beta_{J+1} \) and \( (\gamma) J^\mu \) invariant polynomials. Here, \( H^\text{inv}_J(\delta|d) \) denotes the invariant characteristic cohomology in antighost number \( J \) (the local cohomology of the Koszul-Tate differential in the space of invariant polynomials). (An element of \( H^\text{inv}_J(\delta|d) \) is defined via an equation like (62), but with the
corresponding current an invariant polynomial.). This result together with (63) ensures that the entire invariant characteristic cohomology in antighost numbers strictly greater than four is trivial

\[ H_j^{\text{inv}} (\delta | d) = 0 \quad \text{for all} \quad J > 4. \quad (65) \]

The nontrivial representatives of \( H_J (\delta | d) \) and \( H_J^{\text{inv}} (\delta | d) \) for \( J \geq 2 \) depend neither on \( (\partial_{[\mu} A_{\nu]}, \partial^\mu H_\mu, \partial_{[\mu} B_{\nu\rho}, \partial_{[\mu} \psi_{\nu\rho]} ) \) nor on the spacetime derivatives of \( F_\lambda \) defined in (59), but only on the undifferentiated scalar field \( \phi \). With the help of relations (32)–(37), it can be shown that \( H_4^{\text{inv}} (\delta | d) \) is generated by the elements

\[
(P (W))^{\mu\nu\rho\lambda} = \frac{dW}{d\phi} C^{*\mu\nu\rho\lambda} + \frac{d^2W}{d\phi^2} (H^{*\mu\nu\rho}\lambda + C^{*\mu\nu\rho\lambda}) \\
+ \frac{d^3W}{d\phi^3} H^{*\mu\nu\rho} C^{*\mu\nu\rho\lambda} + \frac{d^4W}{d\phi^4} H^{*\mu} H^{*\nu} H^{*\rho} H^{*\lambda},
\]

where \( W = W (\phi) \) is an arbitrary, smooth function depending only on the undifferentiated scalar field \( \phi \). Indeed, direct computation yields

\[
\delta (P (W))^{\mu\nu\rho\lambda} = \partial^{[\mu} (P (W))^{\nu\rho\lambda]}, \quad \text{agh} ((P (W))^{\mu\nu\rho\lambda}) = 3,
\]

where we made the notation

\[
(P (W))^{\mu\nu} = \frac{dW}{d\phi} C^{*\mu\nu} + \frac{d^2W}{d\phi^2} H^{*\mu\nu}, \quad (P (W))^{\mu\nu} = \partial^{[\mu} (P (W))^{\nu]}, \quad \text{agh} ((P (W))^{\mu\nu}) = 2,
\]

where we employed the convention

\[
(P (W))^{\mu\nu} = \frac{dW}{d\phi} C^{*\mu\nu} + \frac{d^2W}{d\phi^2} H^{*\mu} H^{*\nu}.
\]

It is clear that \( (P (W))^{\mu\nu} \) is an invariant polynomial. By applying the operator \( \delta \) on it, we have that

\[
\delta (P (W))^{\mu\nu} = -\partial^{[\mu} (P (W))^{\nu]}, \quad \text{agh} ((P (W))^{\mu\nu}) = 1,
\]

where \( (P (W))^{\mu\nu} \) is also an invariant polynomial, from (69) it follows that \( (P (W))^{\mu\nu} \) belongs to \( H_3^{\text{inv}} (\delta | d) \). Moreover, further calculations produce

\[
\delta (P (W))^{\mu\nu} = \partial^{[\mu} (P (W))^{\nu]}, \quad \text{agh} ((P (W))^{\nu}) = 1,
\]

13
with

\[(P(W))^{\mu} = \frac{dW}{d\phi} H^{\mu}. \quad (72)\]

Due to the fact that \((P(W))^{\mu}\) is an invariant polynomial, we deduce that \((P(W))^{\mu\nu}\) pertains to \(H_{2}^{\text{inv}}(\delta|d)\). Using again the actions of \(\delta\) on the BRST generators, it can be proved that \(H_{3}^{\text{inv}}(\delta|d)\) is spanned, beside the elements \((P(W))^{\mu\nu\rho}\) given in (68), also by the undifferentiated antifields \(\eta^{\mu\nu\rho\lambda}\) (according to the first definition from (37)). Putting together the above results we can state that \(H_{2}^{\text{inv}}(\delta|d)\) is spanned by \((P(W))^{\mu\nu}\) listed in (70) and the undifferentiated antifields \(\eta^{\mu\nu\rho}\) and \(\xi^{\mu\nu\rho}\) (in agreement with the second definition in (35), the first formula from (36), and the second relation in (36)).

The above results are synthesized in the following array

| \(J\) | nontrivial representatives |
|------|--------------------------|
| \(J > 4\) | spanning \(H_{J}(\delta|d)\) and \(H_{J}^{\text{inv}}(\delta|d)\) |
| \(J = 4\) | \((P(W))^{\mu\nu\rho\lambda}\) |
| \(J = 3\) | \(\eta^{\mu\nu\rho\lambda}, (P(W))^{\mu\nu\rho}\) |
| \(J = 2\) | \(\eta^{\mu\nu\rho}, (P(W))^{\mu\nu}, \eta^{\mu\nu}, \xi^{\mu\nu}\) |

(73)

In contrast to the spaces \((H_{J}(\delta|d))_{J \geq 2}\) and \((H_{J}^{\text{inv}}(\delta|d))_{J \geq 2}\), which are finite-dimensional, the cohomology \(H_{1}(\delta|d)\) (known to be related to global symmetries and ordinary conservation laws) is infinite-dimensional since the theory is free. Fortunately, it will not be needed in the sequel.

The previous results on \(H(\delta|d)\) and \(H^{\text{inv}}(\delta|d)\) in strictly positive antighost numbers are important because they control the obstructions to removing the antifields from the first-order deformation. More precisely, we can successively eliminate all the pieces of antighost number strictly greater that four from the nonintegrated density of the first-order deformation by adding solely trivial terms, so we can take, without loss of nontrivial objects, the condition \(I \leq 4\) into (53). In addition, the last representative is of the form (61), where the invariant polynomial necessarily is a nontrivial object from \(H_{4}^{\text{inv}}(\delta|d)\).

### 4.2 First-order deformation

In the case \(I = 4\) the nonintegrated density of the first-order deformation (see (53)) becomes

\[a = a_{0} + a_{1} + a_{2} + a_{3} + a_{4}. \quad (74)\]
We can further decompose \( a \) in a natural manner as a sum between two kinds of deformations
\[
a = a^{BF} + a^{\text{int}} ,
\] (75)
where \( a^{BF} \) contains only fields/ghosts/antifields from the BF sector and \( a^{\text{int}} \) describes the cross-interactions between the two theories. Strictly speaking, we should have added to (75) a component \( a^{RS} \) that involves only the Rarita-Schwinger sector. As it will be seen in the end of this subsection, \( a^{RS} \) will automatically be included into \( a^{\text{int}} \). The piece \( a^{BF} \) is completely known and satisfies (separately) an equation of the type (51). It admits a decomposition similar to (74)
\[
a^{BF} = a_{0}^{BF} + a_{1}^{BF} + a_{2}^{BF} + a_{3}^{BF} + a_{4}^{BF} ,
\] (76)
where
\[
a_{4}^{BF} = (P (W))^{\mu \nu \rho \lambda} \eta C_{\mu \nu \rho \lambda} + \frac{1}{2} \epsilon_{\mu \nu \rho \lambda} (P (M))^{\mu \nu \rho \lambda} \eta_{\alpha \beta \gamma \delta} \eta_{\alpha \beta \gamma \delta} ,
\] (77)
\[
a_{3}^{BF} = (P (W))^{\mu \nu} (- \eta C_{\mu \nu} + 4 A^{\lambda} C_{\mu \nu \lambda})
+ 2 \left( 6 (P (W))^{\mu \nu} B^{\ast \rho \lambda} + 4 (P (W))^{\mu} \eta^{\nu \rho \lambda} + W \eta^{\mu \nu \rho \lambda} \right) C_{\mu \nu \rho \lambda}
- \epsilon_{\mu \nu \rho \lambda} (P (M))_{\alpha \beta \gamma} \eta_{\alpha \beta \gamma} \eta^{\mu \nu \rho \lambda} ,
\] (78)
\[
a_{2}^{BF} = (P (W))^{\mu \nu} (\eta C_{\mu \nu} - 3 A^{\rho} C_{\mu \rho \nu}) - 2 (3 (P (W))^{\mu} B^{\ast \nu \rho}
+ W \eta^{\ast \nu \rho} \right) C_{\mu \nu \rho} + \frac{9}{2} \epsilon_{\mu \nu \rho \lambda} (P (M))_{\mu} \eta_{\rho \alpha \beta} \eta^{\lambda}_{\alpha \beta}
+ \epsilon_{\mu \nu \rho \lambda} \left( 2 (P (M))_{\alpha} A^{\ast \alpha} - 2 M \eta^{\ast} + (P (M))_{\alpha \beta} B^{\ast \alpha} \right) \eta_{\mu \nu \rho \lambda} ,
\] (79)
\[
a_{1}^{BF} = (P (W))^{\mu} (- \eta H_{\mu} + 2 A^{\nu} C_{\mu \nu}) + W (2 B^{\ast \mu \nu} C_{\mu \nu} + \varphi^{\ast} \eta)
+ 2 \epsilon_{\nu \rho \sigma \lambda} \left( (P (M))_{\mu} B^{\mu \nu} - M A^{\ast \nu} \right) \eta^{\rho \sigma \lambda} ,
\] (80)
\[
a_{0}^{BF} = - W A^{\mu} H_{\mu} + \frac{1}{2} \epsilon^{\mu \nu \rho \lambda} M B_{\mu \nu} B_{\rho \lambda} .
\] (81)

In (77)–(81) the quantities denoted by \( (P (W))^{\mu_{1} \ldots \mu_{k}} \) and \( (P (M))^{\mu_{1} \ldots \mu_{k}} \) read as in (60), (68), (70), and (72) for \( k = 4, k = 3, k = 2, \) and \( k = 1 \) respectively, modulo the successive replacement of \( W (\varphi) \) with the real smooth functions \( W (\varphi) \) and \( M (\varphi) \), respectively.
Due to the fact that $a^{BF}$ and $a^{int}$ involve different types of fields and because $a^{BF}$ satisfies individually an equation of the type (51), it follows that $a^{int}$ is subject to the equation

$$sa^{int} = \partial^{\mu}m^{int}_\mu,$$

(82)

for some local current $m^{int}_\mu$. In the sequel we determine the general solution to (82) that complies with all the hypotheses mentioned in the beginning of the previous subsection.

In agreement with (73), the solution to the equation $sa^{int} = \partial^{\mu}m^{int}_\mu$ can be decomposed as

$$a^{int} = a^{int}_0 + a^{int}_1 + a^{int}_2 + a^{int}_3 + a^{int}_4,$$

(83)

where the components on the right-hand side of (83) are subject to the equations

$$\gamma a^{int}_4 = 0,$$

(84)

$$\delta a^{int}_k + \gamma a^{int}_{k-1} = \partial^{(k-1)\mu}m^{int}_\mu, \quad k = 1, 4.$$

(85)

The piece $a^{int}_4$ as solution to equation (84) has the general form expressed by (61) for $I = 4$, with $\alpha_4$ from $H^{inv}_4(\delta|d)$ and $e^4$ spanned by

$$\{\xi^A \xi^B \xi^C \xi^D, \xi^A \xi^B \eta, \xi^A \xi^B \eta^{\mu \nu \rho \lambda}, \eta^{\mu \nu \rho \lambda} \eta^{\alpha \beta \gamma \delta}\}.$$  

(86)

Taking into account the result that the general representative of $H^{inv}_4(\delta|d)$ is given by (66) and recalling that $a^{int}_4$ should mix the BF and the massless spin-$3/2$ sectors (in order to provide cross-couplings), it follows that the eligible representatives of $e^4$ from (80) allowed to enter $a^{int}_4$ are those elements containing at least one ghost of the type $\xi^A$. Recalling the symmetry properties (10) and (11) of the $\gamma$-matrices, we deduce the general solution of equation (84) under the form

$$a^{int}_4 = \frac{1}{4 \cdot 4!} \varepsilon_{\mu \nu \rho \lambda} \left[ (P(U_1))^{\mu \nu \rho \lambda}_\xi (\tilde{\xi} \gamma_\alpha \xi) \tilde{\xi} \gamma^\alpha \xi + \tilde{\xi} \gamma_\alpha \xi ( (P(U_2))^{\mu \nu \rho \lambda}_\xi \tilde{\xi} \gamma_\alpha \beta \xi + 2 (P(U_3))^{\mu \nu \rho \lambda}_\xi \tilde{\xi} \gamma_\alpha \beta \gamma_5 \xi) \right],$$

(87)

where $U_1$, $U_2$, and $U_3$ are smooth functions depending only on the undifferentiated scalar field $\varphi ((P(U_i))^{\mu \nu \rho \lambda}_\xi$, with $i = 1, 2, 3$, read as in (66), but
with the function $W$ replaced by $U_i$). Introducing (87) in equation (85) for $k = 4$ and employing definitions (32)–(42), we determine the component of antighost number three from (83) as

$$a_{int}^3 = \frac{1}{3!} \varepsilon_{\mu\nu\rho\lambda} \left[ (P(U_1))^{\nu\rho\lambda} \bar{\xi} \gamma^\alpha \psi^\mu \xi \gamma^\alpha \psi^\mu \xi \right]$$

$$+ (P(U_3))^{\nu\rho\lambda} \bar{\xi} \gamma^\alpha \gamma_5 \xi \right] + (P(U_3))^{\nu\rho\lambda} \bar{\xi} \gamma^\alpha \gamma_5 \psi^\mu \right] + a_{int}^{(88)}$$

where the objects $((P(U_i))^{\mu\nu\rho})_{i=1,2,3}$ are of the form (68), up to replacing the function $W$ with $U_i$, and $a_{int}^3$ is the general solution of the ‘homogeneous’ equation $\gamma \bar{a}_{int}^3 = \partial_\mu u^\mu_3$, which, according to the discussion from the previous subsection, can be replaced with

$$\gamma \bar{a}_{int}^3 = 0. \ (89)$$

This means that we can always take $\bar{a}_{int}^3$ as a nontrivial object of $H(\gamma)$. At this stage it is useful to decompose $a_{int}^3$ into

$$\bar{a}_{int}^3 = \hat{a}_{int}^3 + \tilde{a}_{int}^3. \ (90)$$

The first piece, $\hat{a}_{int}^3$, denotes the component of the solution to (89) required by the consistency of $a_{int}^3$ in antighost number two (ensures that (85) possesses solutions for $k = 3$ with respect to the terms from $a_{int}^3$ containing the functions $U_i$) and $\tilde{a}_{int}^3$ signifies the part of the solution to (89) that is independently consistent in antighost number two

$$\delta \hat{a}_{int}^3 = -\gamma \hat{c}_2 + \partial_\mu \tilde{m}_2^\mu. \ (91)$$

By means of definitions (32)–(42) and recalling the decomposition (90) one infers (by direct computation) that

$$\delta \hat{a}_{int}^3 = \delta \hat{a}_{int}^3 + \gamma \hat{c}_2 + \partial_\mu \tilde{j}_2^\mu + \chi_2, \ (92)$$

where we made the notations

$$c_2 = -\tilde{c}_2 + \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} \left[ (P(U_1))^{\rho\lambda} \bar{\xi} \gamma^\alpha \psi^\mu \xi \gamma^\alpha \psi^\mu \xi \right]$$

$$+ (P(U_2))^{\rho\lambda} \bar{\xi} \gamma^\alpha \psi^\mu \left( (P(U_2))^{\rho\lambda} \bar{\xi} \gamma^\alpha \psi^\mu \xi \right]$$

$$+ (P(U_3))^{\rho\lambda} \bar{\xi} \gamma^\alpha \psi^\mu \left( (P(U_3))^{\rho\lambda} \bar{\xi} \gamma^\alpha \psi^\mu \xi \right]$$
\( \frac{1}{2} (P(U_3))^\rho \left[ (\bar{\psi}^\mu \gamma_{\alpha \beta} \psi^\nu) \xi^{\gamma \alpha \beta} \gamma_5 \xi + (\bar{\xi} \gamma_{\alpha \beta} \xi) \bar{\psi}^\mu \gamma^{\alpha \beta} \gamma_5 \psi^\nu \right. \\
-4 (\bar{\xi} \gamma_{\alpha \beta} \psi^\mu) \xi^{\gamma \alpha \beta} \gamma_5 \psi^\nu \bigg] , \)  
\( \tag{93} \)

\( j_2^\mu = \bar{m}_2^\mu - \frac{1}{2} \varepsilon^{\mu \rho \lambda} \left\{ (P(U_1))^{\rho \rho} (\bar{\xi} \gamma_{\alpha \xi}) \xi^{\gamma \alpha} \psi_\lambda + (P(U_2))^{\rho \rho} (\bar{\xi} \gamma_{\alpha \xi}) \xi^{\gamma \alpha} \psi_\lambda \\
+ (P(U_3))^{\rho \rho} \left[ (\bar{\xi} \gamma_{\alpha \beta} \psi_\lambda) \xi^{\gamma \alpha \beta} \gamma_5 \xi + (\bar{\xi} \gamma_{\alpha \beta} \xi) \xi^{\gamma \alpha \beta} \gamma_5 \psi_\lambda \right] \right\} , \)  
\( \tag{94} \)

\( \chi_2 = \frac{1}{2} \varepsilon^{\mu \rho \lambda} \left\{ (P(U_1))_{\mu \nu} (\bar{\xi} \gamma_{\alpha \xi}) \xi^{\gamma \alpha} \partial_\rho \psi_\lambda + (P(U_2))_{\mu \nu} (\bar{\xi} \gamma_{\alpha \xi}) \xi^{\gamma \alpha} \partial_\rho \psi_\lambda \\
+ (P(U_3))_{\mu \nu} \left[ (\bar{\xi} \gamma^{\alpha \beta} \gamma_5 \xi) \xi^{\gamma \alpha \beta} \partial_\rho \psi_\lambda + (\bar{\xi} \gamma^{\alpha \beta} \xi) \xi^{\gamma \alpha \beta} \gamma_5 \partial_\rho \psi_\lambda \right] \right\} . \)  
\( \tag{95} \)

Comparing (92) with (85) for \( k = 3 \), it follows that the existence of \( a_2^{\text{int}} \) is ensured if and only if \( \chi_2 \) satisfies the equation
\( \chi_2 = -\delta \hat{a}_3^{\text{int}} + \gamma \hat{c}_2 + \partial_\mu \hat{j}_2^\mu , \)  
\( \tag{96} \)

where
\( \hat{c}_2 = -(a_2^{\text{int}} + c_2), \quad \hat{j}_2^\mu = (m^{\text{int}})^\mu - j_2^\mu . \)  
\( \tag{97} \)

We will show that (96) cannot hold unless \( \chi_2 = 0 \). In view of this, we assume equation (96) is valid. By taking its Euler-Lagrange (EL) derivatives with respect to \( C^*_{\mu \nu} \) we infer
\[ \frac{\delta \chi_2}{\delta C^*_{\mu \nu}} = -\delta \left( \delta \hat{a}_3^{\text{int}} \right) + \gamma \left( \frac{\delta \hat{c}_2}{\delta C^*_{\mu \nu}} \right) . \]  
\( \tag{98} \)

Direct computation based on (95) leads to
\[ \frac{\delta \chi_2}{\delta C^*_{\mu \nu}} = \frac{1}{4} \varepsilon^{\mu \rho \lambda} \left\{ \frac{dU_1}{d\varphi} \left( \bar{\xi} \gamma_{\alpha \xi} \right) \xi^{\gamma \alpha} \partial_\rho \psi_\lambda + \frac{dU_2}{d\varphi} \left( \bar{\xi} \gamma_{\alpha \xi} \right) \xi^{\gamma \alpha} \partial_\rho \psi_\lambda \\
+ \frac{dU_3}{d\varphi} \left[ (\bar{\xi} \gamma^{\alpha \beta} \gamma_5 \xi) \xi^{\gamma \alpha \beta} \partial_\rho \psi_\lambda + (\bar{\xi} \gamma^{\alpha \beta} \xi) \xi^{\gamma \alpha \beta} \gamma_5 \partial_\rho \psi_\lambda \right] \right\} . \]  
\( \tag{99} \)

It is easy to see that the right-hand side of (99) is a nontrivial object from \( H(\gamma) \), so relation (98) implies
\[ \frac{\delta \chi_2}{\delta C^*_{\mu \nu}} = -\delta \left( \delta \hat{a}_3^{\text{int}} \right) + \gamma \left( \frac{\delta \hat{c}_2}{\delta C^*_{\mu \nu}} \right) = 0 . \]  
\( \tag{100} \)
Due to (99), the former formula in (100) cannot take place. This is because $\hat{\alpha}_3^{\text{int}}$ comprises two spacetime derivatives, while $\frac{\delta \chi^2}{\delta C_{\mu \nu}}$ has only one. Indeed, $\hat{\alpha}_3^{\text{int}}$ is expressed by (61) for $I = 3$ and is also simultaneously linear in $C_{\mu \nu}$ and $\partial_\mu \psi_\nu$ (the linearity in $\partial_\mu \psi_\nu$ is imposed by the linearity of the right-hand side of (99)). Taking into account these observations, we get that $\hat{\alpha}_3^{\text{int}}$ is linear in both $\partial_\mu C_{\nu \rho}$ and $\partial_\mu \psi_\nu$, so it displays precisely two spacetime derivatives. As a consequence, the former equality from (100) does not hold, so neither do formulas (98) or (96). In conclusion, $\chi^2$ must vanish, which further implies that $(U_i (\varphi))_{i=1,2,3}$ must be constant, and thus $a_4^{\text{int}}$ itself vanishes.

Since the decomposition (83) cannot stop at antighost number four, we pass to the next possibility, namely that $a^{\text{int}}$ ends at antighost number three:

$$a^{\text{int}} = a_0^{\text{int}} + a_1^{\text{int}} + a_2^{\text{int}} + a_3^{\text{int}},$$

where the components on the right-hand side of (101) are subject to the equations

$$\gamma a_3^{\text{int}} = 0,$$

$$\delta a_k^{\text{int}} + \gamma a_{k-1}^{\text{int}} = \partial^\mu (k-1)_{\text{int}}^\mu \ , \quad k = 1,3.$$  

The piece $a_3^{\text{int}}$ as solution to equation (102) has the general form expressed by (61) for $I = 3$, with $\alpha_3$ from $H_3^{\text{inv}}(\delta|d)$ and $e^3$ spanned by

$$\{\xi^A \xi^B \xi^C, \xi^A \xi^B \eta, \xi^A \eta^\mu \rho \lambda, \eta^\mu \rho \lambda\}.$$  

Given the spinor-like behavior of some of the elements (104) and also the general expressions of the generators of $H_3^{\text{inv}}(\delta|d)$ (see (73) for $J = 3$), the general, real solution to equation (102) reads as

$$a_3^{\text{int}} = -\frac{i}{12} \varepsilon^{\mu \rho \lambda} (P(U_4))_{\mu \nu \rho \lambda} \bar{\xi} \gamma_5 \xi \eta,$$

where $U_4 = U_4 (\varphi)$ is a smooth function on the (undifferentiated) scalar field $\varphi$ and $(P(U_4))_{\mu \nu \rho \lambda}$ follows from (68) with $W$ replaced by $U_4$. Making use of the latter set of duality relations from (8), (105) can be written as

$$a_3^{\text{int}} = \frac{1}{12} (P(U_4))_{\mu \nu \rho \lambda} \bar{\xi} \gamma_5 \xi \eta.$$
Substituting (106) into (103) for \( k = 3 \) and recalling definitions (32)–(42), we identify the component of antighost number two from the first-order deformation as

\[
a_{2}^{\text{int}} = \frac{1}{4} (P (U_4))_{\mu\nu} \left( \bar{\xi} \gamma^{\mu\nu\rho} \gamma_5 \xi A_\rho - 2 \bar{\xi} \gamma^{\mu\nu\rho} \gamma_5 \psi_\rho \eta \right) + \bar{a}_{2}^{\text{int}},
\]

(107)

where the quantity \((P (U_4))_{\mu\nu}\) is of the type (70) and \(\bar{a}_{2}^{\text{int}}\) is, like in the above, the general solution to the ‘homogeneous’ equation \(\gamma \bar{a}_{2}^{\text{int}} = \partial_{\mu} u_2^\mu\), which, according to our discussion from the previous subsection, can be safely replaced with

\[
\gamma \bar{a}_{2}^{\text{int}} = 0.
\]

(108)

Just like before (see (90)), we decompose \(\bar{a}_{2}^{\text{int}}\) into

\[
\bar{a}_{2}^{\text{int}} = \hat{a}_{2}^{\text{int}} + \tilde{a}_{2}^{\text{int}},
\]

(109)

where \(\hat{a}_{2}^{\text{int}}\) is the solution to (108) necessary for the consistency of \(a_{2}^{\text{int}}\) in antighost number one (for the existence of solutions \(a_{1}^{\text{int}}\) to (103) for \( k = 2 \) with respect to the terms from \(a_{2}^{\text{int}}\) containing the function \(U_4\)) and \(\tilde{a}_{2}^{\text{int}}\) is the solution to (108) that is independently consistent in antighost number one

\[
\delta \tilde{a}_{2}^{\text{int}} = -\gamma \tilde{c}_1 + \partial_{\mu} \tilde{m}_1^\mu.
\]

(110)

With the help of definitions (32)–(42), we get

\[
\delta a_{2}^{\text{int}} = \delta \left[ \hat{a}_{2}^{\text{int}} - \frac{1}{2} (P (U_4))_{\mu} \bar{\xi} \gamma^{\mu\nu\rho} \gamma_5 \xi B^{*}_{\nu\rho} - \frac{1}{6} U_4 \bar{\xi} \gamma^{\mu\nu\rho} \gamma_5 \xi \eta^{*}_{\mu\nu\rho} 

- i (P (U_4))_{\mu} \psi^{*\mu} \gamma_5 \xi \eta + i U_4 \xi^{*} \gamma_5 \xi \eta \right] + \gamma c_1 + \partial_{\mu} j_1^\mu,
\]

(111)

where we made the notations

\[
c_1 = -\tilde{c}_1 + i U_4 \left( \psi^{*\mu} \gamma_5 \psi_\mu \eta - \psi^{*\mu} \gamma_5 \xi A_\mu - i \bar{\xi} \gamma^{\mu\nu\rho} \gamma_5 \psi_\rho \right) 

+ \frac{1}{2} (P (U_4))_{\mu} \left( 2 \bar{\xi} \gamma^{\mu\nu\rho} \gamma_5 \psi_\nu A_\rho + \bar{\psi}_\nu \gamma^{\mu\nu\rho} \gamma_5 \psi_\rho \eta \right),
\]

(112)

\[
j_1^\mu = \tilde{m}_1^\mu + \frac{1}{2} (P (U_4))_{\mu} \left( \bar{\xi} \gamma^{\mu\nu\rho} \gamma_5 \xi A_\rho - 2 \bar{\xi} \gamma^{\mu\nu\rho} \gamma_5 \psi_\rho \eta \right)
\]

20
\[
\frac{1}{2}U_4 \left( \tilde{\xi} \gamma^{\mu\rho} \gamma_5 \xi B_{\nu\rho} + 2i \psi^{*\mu} \gamma_5 \xi \eta \right),
\]

(113)

and \((P(U_4))_\mu\) reads as in (72) modulo the replacement \(W \rightarrow U_4\). Comparing (111) with (103) for \(k = 2\), we infer that

\[
\hat{a}_2^{\text{int}} = \frac{1}{2} (P(U_4))_\mu \tilde{\xi} \gamma^{\mu\rho} \gamma_5 \xi B_{\nu\rho}^{*} + \frac{1}{6} U_4 \tilde{\xi} \gamma^{\mu\rho} \gamma_5 \xi \eta_{\mu\rho}^{*} \\
+ i (P(U_4))_\mu \psi^{*\mu} \gamma_5 \xi \eta - iU_4 \xi^{*} \gamma_5 \xi \eta,
\]

(114)

\[
a_1^{\text{int}} = \tilde{c}_1 - iU_4 \left( \psi^{*\mu} \gamma_5 \psi_{\mu} \eta - \psi^{*\mu} \gamma_5 \xi A_{\mu} - i \tilde{\xi} \gamma^{\mu\rho} \gamma_5 \psi_{\mu} B_{\nu\rho}^{*} \right) \\
- \frac{1}{2} (P(U_4))_\mu \left( 2 \tilde{\xi} \gamma^{\mu\rho} \gamma_5 \psi_{\nu} A_{\rho} + \tilde{\psi}_{\nu} \gamma^{\mu\rho} \gamma_5 \psi_{\rho} \eta \right) + \bar{a}_1^{\text{int}},
\]

(115)

where \(\bar{a}_1^{\text{int}}\) is the general solution to the ‘homogeneous’ equation \(\gamma \bar{a}_1^{\text{int}} = \partial_{\mu} u^{\mu}_1\), which can again be replaced with

\[
\gamma \bar{a}_1^{\text{int}} = 0.
\]

(116)

Regarding the component \(\tilde{a}_2^{\text{int}}\) of the equation (118), it can be represented like in (61) for \(I = 2\), with \(\alpha_2\) from \(H^{\text{inv}}_2(\delta|d)\) and \(e^2\) spanned by

\[
\{ \xi A_1 \xi B, \xi A_1 \eta, \eta^{\mu\rho\lambda} \}.
\]

The most general representatives of \(H^{\text{inv}}_2(\delta|d)\) are listed in (73) for \(J = 2\), so \(\tilde{a}_2^{\text{int}}\) is generally expressed by

\[
\tilde{a}_2^{\text{int}} = \frac{1}{4} (P(U_5))_{\mu\nu} \tilde{\xi} \gamma^{\mu\nu} \gamma_5 \xi - \frac{1}{4} (P(U_6))_{\mu\nu} \tilde{\xi} \gamma^{\mu\nu} \xi,
\]

(117)

where \(U_5(\varphi)\) and \(U_6(\varphi)\) are some real, smooth (but otherwise arbitrary) functions depending only on \(\varphi\). Inserting (114) and (117) in (107), we arrive at

\[
a_2^{\text{int}} = \frac{1}{4} (P(U_4))_{\mu\nu} \left( \tilde{\xi} \gamma^{\mu\rho} \gamma_5 \xi A_{\rho} - 2\tilde{\xi} \gamma^{\mu\rho} \gamma_5 \psi_{\rho} \eta \right) \\
+ \frac{1}{2} (P(U_4))_{\mu\nu} \tilde{\xi} \gamma^{\mu\rho} \gamma_5 \xi B_{\nu\rho}^{*} + \frac{1}{6} U_4 \tilde{\xi} \gamma^{\mu\rho} \gamma_5 \xi \eta_{\mu\rho}^{*} \\
+ i (P(U_4))_{\mu\nu} \psi^{*\mu} \gamma_5 \xi \eta - iU_4 \xi^{*} \gamma_5 \xi \eta \\
+ \frac{1}{4} (P(U_5))_{\mu\nu} \tilde{\xi} \gamma^{\mu\nu} \gamma_5 \xi - \frac{1}{4} (P(U_6))_{\mu\nu} \tilde{\xi} \gamma^{\mu\nu} \xi.
\]

(118)
Applying $\delta$ on $\tilde{a}^\text{int}_2$ given in (117) and using definitions (32)–(42), we obtain the concrete expression of the object $\tilde{c}_1$ involved in (110) of the form

\[ \tilde{c}_1 = -i (P (U_5))_{\mu} \tilde{\xi} \gamma^{\mu\nu} \gamma_5 \psi_\nu + (P (U_6))_{\mu} \tilde{\xi} \gamma^{\mu\nu} \psi_\nu, \]  

(119)

which further inserted in (115) allows us to write the component of antighost number one from the first-order deformation as

\[ a^\text{int}_1 = - \frac{i}{2} (P (U_4))_{\mu} (2 \tilde{\xi} \gamma^{\mu\rho} \gamma_5 \psi_\rho + \psi_\nu \gamma^{\mu\rho} \gamma_5 \psi_\rho \eta) - i (P (U_5))_{\mu} \tilde{\xi} \gamma^{\mu\nu} \gamma_5 \psi_\nu + (P (U_6))_{\mu} \tilde{\xi} \gamma^{\mu\nu} \psi_\nu + \tilde{a}^\text{int}_1. \]  

(120)

We split again $\tilde{a}^\text{int}_1$ into

\[ \tilde{a}^\text{int}_1 = \hat{a}^\text{int}_1 + \tilde{a}^\text{int}_1, \]  

(121)

with $\hat{a}^\text{int}_1$ the solution to (116) ensuring the consistency of $a^\text{int}_1$ in antighost number zero (i.e., the existence of solutions $a^\text{int}_0$ to equation (103) for $k = 1$ for the pieces from $a^\text{int}_1$ containing the functions $U_m$, with $m = 4, 5, 6$) and $\tilde{a}^\text{int}_1$ the solution to (113) that is independently consistent at antighost number zero

\[ \delta \tilde{a}^\text{int}_1 = -\gamma \tilde{c}_0 + \partial_\mu \tilde{m}_0^\mu. \]  

(122)

By means of definitions (32)–(42), straightforward computation produces

\[ \delta a^\text{int}_1 = \delta \left[ \left( \hat{a}^\text{int}_1 + \frac{1}{2} U_5 \psi^{*\mu} \gamma_5 A_\mu \gamma_5 + \frac{1}{2} U_6 \psi^{*\mu} \gamma_5 \xi \right) + \gamma c_0 + \partial_\mu \tilde{j}_0^\mu \right], \]  

(123)

where

\[ c_0 = -\tilde{c}_0 + \frac{1}{2} U_4 \psi^{*\mu} \gamma_5 A_\mu \eta + \frac{1}{2} (i U_5 \psi^{*\mu} \gamma_5 \psi_\rho A_\rho - U_6 \psi^{*\mu} \gamma_5 \psi_\rho \gamma_5 \eta), \]  

(124)

\[ \tilde{j}_0^\mu = \hat{m}^\mu_0 - U_4 \left( \frac{1}{2} \tilde{\psi}_\nu \gamma^{\mu\rho} \gamma_5 \psi_\rho \eta + \tilde{\xi} \gamma^{\mu\rho} \gamma_5 \psi_\rho A_\rho \right) - i U_5 \tilde{\xi} \gamma^{\mu\nu} \gamma_5 \psi_\nu + U_6 \tilde{\xi} \gamma^{\mu\nu} \psi_\nu. \]  

(125)

Comparing (123) with (103) for $k = 1$, we deduce

\[ \hat{a}^\text{int}_1 = -\frac{1}{2} (U_5 \psi^{*\mu} \gamma_5 \xi + i U_6 \psi^{*\mu} \gamma_5 \xi), \]  

(126)
\[
a_0^\text{int} = \bar{c}_0 - \frac{1}{2} \left( U_4 \bar{\psi}_\mu \gamma^{\mu\rho} \gamma_5 \psi_\rho A_\mu + iU_5 \bar{\psi}_\mu \gamma^{\mu\nu} \gamma_5 \psi_\nu - U_6 \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \right) + \bar{a}_0^\text{int}, \tag{127}
\]
where \( \bar{a}_0^\text{int} \) is the general solution to the ‘homogeneous’ equation \( \gamma \bar{a}_0^\text{int} = \partial_\mu u_0^\mu \) (it cannot be replaced any longer with the simpler equation, corresponding to \( u_0^\mu = 0 \)). The component \( \bar{a}_1^\text{int} \), which is a solution to (116) that is independently consistent at antighost number zero, i.e. satisfies equation (122), can be taken to vanish
\[
\bar{a}_1^\text{int} = 0, \tag{128}
\]
since it produces only trivial deformations, as it will be shown in Appendix A.

Injecting (126) and (128) in (121), and the resulting expression in (120), we complete the component of antighost number one from the first-order deformation as
\[
a_1^\text{int} = -iU_4 \left( \psi^{*\mu} \gamma_5 \psi_\mu \eta - \psi^{*\mu} \gamma_5 \xi A_\mu - i\bar{\xi} \gamma^{\mu\rho} \gamma_5 \psi_\rho B_\nu^* \right)
- \frac{1}{2} \left( P(U_4) \right)_\mu \left( 2\bar{\xi} \gamma^{\mu\rho} \gamma_5 \psi_\rho \eta + \bar{\psi}_\nu \gamma^{\mu\rho} \gamma_5 \psi_\rho \eta \right)
- i \left( P(U_5) \right)_\mu \bar{\xi} \gamma^{\mu\nu} \gamma_5 \psi_\nu + \left( P(U_6) \right)_\mu \bar{\psi}_\nu \gamma^{\mu\nu} \psi_\nu
- \frac{1}{2} \left( U_5 \psi^{*\mu} \gamma_5 \psi_\xi + iU_6 \psi^{*\mu} \gamma_5 \xi \right). \tag{129}
\]
We mention that (128) also implies
\[
\bar{c}_0 = 0 \tag{130}
\]
in (124) and (127), such that the element of antighost number zero of the first-order deformation (the interacting Lagrangian at order one in the coupling constant) is
\[
a_0^\text{int} = -\frac{1}{2} \left( U_4 \bar{\psi}_\mu \gamma^{\mu\rho} \gamma_5 \psi_\rho A_\mu + iU_5 \bar{\psi}_\mu \gamma^{\mu\nu} \gamma_5 \psi_\nu - U_6 \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \right) + \bar{a}_0^\text{int}, \tag{131}
\]
where the piece \( \bar{a}_0^\text{int} \) is subject to the homogenous equation
\[
\gamma \bar{a}_0^\text{int} = \partial_\mu u_0^\mu. \tag{132}
\]
The general solution to (132) can be written as
\[
\bar{a}_0^\text{int} = \bar{a}_0^{\text{int}} + \bar{a}_0^{\text{ext}}, \tag{133}
\]
with \( \bar{a}_0^{\text{int}} \) and \( \bar{a}_0^{\text{int}} \) solutions to

\[
\begin{align*}
\gamma \bar{a}_0^{\text{int}} &= 0, \\
\gamma \bar{a}_0^{\text{int}} &= \partial_\mu u_0^\mu,
\end{align*}
\]

and \( u_0^\mu \) a nonvanishing current. We recall the main properties of \( \bar{a}_0^{\text{int}} \): it should mix the Rarita-Schwinger spinors with the BF fields (in order to provide cross-couplings) and contain at most one spacetime derivative (like the original Lagrangian). In agreement with our result (61) for \( I = 0 \), the component \( \bar{a}_0^{\text{int}} \) is of the type

\[
\bar{a}_0^{\text{int}} = \bar{a}_0^{\text{int}} ([F_A]),
\]

where \( F_A \) are given in (59). Due to the above mentioned main properties, it is easy to see that (136) contains at least two derivatives. Indeed, it is forced to contain gauge-invariant objects depending on the Rarita-Schwinger spinors, so it should be quadratic in \( \partial_\mu \psi_\nu \) (in order to render a bosonic quantity), which contradicts the derivative-order assumption, such that we must set

\[
\bar{a}_0^{\text{int}} = 0.
\]

In the meanwhile, as it will be shown in Appendix B the solution to equation (135) can be taken as trivial

\[
\bar{a}_0^{\text{int}} = 0,
\]

which, together with (137), leads to the conclusion that the general solution to the homogeneous equation in antighost number zero, (132), that complies with all the working hypotheses is also trivial

\[
\bar{a}_0^{\text{int}} = 0.
\]

In this manner we also completed the component of antighost number zero from the first-order deformation, which follows from (131) with \( \bar{a}_0^{\text{int}} \) as in (139):

\[
\bar{a}_0^{\text{int}} = \frac{1}{2} \left( U_4 \bar{\psi}_\mu \gamma^{\mu\nu} \gamma_5 \psi_\nu A_\rho + i U_5 \bar{\psi}_\mu \gamma^{\mu\nu} \gamma_5 \psi_\nu - U_6 \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \right).
\]

Putting together the BF piece (76) (whose various terms are listed in (77)–(81)) with the interacting one (101) (having the components (105), (118),
The massless Rarita-Schwinger field is given by the solution to the master equation for a four-dimensional BF model and we can state that the general expression of the first-order deformation of the solution to the master equation for a four-dimensional BF model and a massless Rarita-Schwinger field is given by:

\[
S_1 = \int d^4x \left[ A_\mu ( - W ( \varphi ) H^\mu ) + \frac{1}{2} M ( \varphi ) \varepsilon_{\alpha\beta\gamma\delta} B^{\alpha\beta} B^{\gamma\delta} 
- \frac{1}{2} ( U_1 ( \varphi ) \bar{\psi}_\mu \gamma^{\mu\rho} \gamma_5 \psi_\rho A_\rho + i U_2 ( \varphi ) \bar{\psi}_\mu \gamma^{\mu\nu} \gamma_5 \psi_\nu - U_3 ( \varphi ) \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu ) 
+ \frac{dW}{d\varphi} H^{*\mu} ( - \eta H_\mu + 2 A^\nu C_{\mu\nu} ) + W ( 2 B^{\mu\nu} C_{\mu\nu} + \varphi^* \eta ) 
+ 2 \varepsilon_{\nu\rho\sigma\lambda} \left( \frac{dM}{d\varphi} H^*_{\mu} B^{\mu\nu} - MA^\nu \right) \eta^{\rho\sigma\lambda} - i U_1 ( \psi^* \gamma_5 \psi_\mu \eta - \psi^* \gamma_5 \xi A_\mu ) 
- i \bar{\xi} \gamma^{\mu\rho} \gamma_5 \psi_\mu B_{\nu\rho} \right) 
- \frac{1}{2} \frac{dU_1}{d\varphi} H^*_{\mu} ( 2 \bar{\xi} \gamma^{\mu\rho} \gamma_5 \psi_\rho A_\rho + \bar{\psi}_\mu \gamma^{\mu\nu} \gamma_5 \psi_\nu ) 
+ i U_3 \bar{\psi}_\mu \gamma_5 \xi + ( P ( W ) )^{\mu\nu} ( \eta C_{\mu\nu} - 3 A^\rho C_{\mu\nu\rho} ) - 2 ( 3 ( P ( W ) )^{\mu} B^{\nu\rho} 
+ W \eta^{\mu\nu\rho} ) C_{\mu\nu\rho} + \frac{9}{2} \varepsilon^{\mu\nu\rho\lambda} ( P ( M ) )_{\mu\nu} \eta_{\rho\sigma\lambda} \eta^\alpha \beta 
+ \varepsilon_{\mu\nu\rho\lambda} \left( 2 ( P ( M ) )_{\alpha} A^{\alpha\alpha} - 2 M_{\eta}^* + ( P ( M ) )_{\alpha\beta} B^{\beta\alpha} \right) \eta_{b}^{\nu\rho\lambda} 
+ \frac{1}{4} ( P ( U_1 ) )^{\mu\nu} ( \bar{\xi} \gamma^{\mu\rho} \gamma_5 \xi A_\rho - 2 \bar{\xi} \gamma^{\mu\rho} \gamma_5 \psi_\rho \eta ) 
+ \frac{dU_1}{d\varphi} H^*_{\mu} \left( \frac{1}{2} \bar{\xi} \gamma^{\mu\rho} \gamma_5 \xi B_{\nu\rho} + i \psi^* \gamma_5 \xi \eta \right) + \frac{1}{6} U_1 \bar{\xi} \gamma^{\mu\rho} \gamma_5 \xi \eta^{\nu\rho} 
- i U_1 \bar{\xi} \gamma_5 \xi \eta + \frac{i}{4} ( P ( U_2 ) )^{\mu\nu} \bar{\xi} \gamma^{\mu\rho} \gamma_5 \xi - \frac{1}{4} ( P ( U_3 ) )_{\mu\nu} \bar{\xi} \gamma^{\mu\rho} \xi 
+ ( P ( W ) )^{\mu\nu} ( - \eta C_{\mu\nu} + 4 A^\lambda C_{\mu\nu\rho\lambda} ) - \varepsilon_{\mu\nu\rho\lambda} ( P ( M ) )_{\alpha\beta} \eta^{\beta\gamma} \eta^{\mu\nu\lambda} 
+ 2 ( 6 ( P ( W ) )^{\mu\nu} B^{\nu\rho} + ( P ( W ) )^{\mu} \eta^{\nu\rho\lambda} + W \eta^{\mu\nu\rho\lambda} ) C_{\mu\nu\rho}\lambda 
+ \frac{1}{12} ( P ( U_1 ) )_{\mu\nu\rho} \bar{\xi} \gamma^{\mu\rho} \gamma_5 \xi \eta + ( P ( W ) )^{\mu\nu\rho\lambda} \eta^{\mu\nu\rho\lambda} C_{\mu\nu\rho}\lambda 
+ \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} ( P ( M ) )^{\nu\rho\lambda} \eta_{\alpha\beta\gamma\delta} \eta^{\alpha\beta\gamma\delta} \right].
\]

(142)
It is important to notice that it complies with all the working hypotheses, including the derivative order assumption, and is parameterized by five smooth, real functions of the undifferentiated scalar field, namely \( W, M \), and \((U_i)_{i=1,2,3}\), which are otherwise arbitrary. The functional \( S_1 \) is by construction a \( s \)-cocycle of ghost number zero, such that \( \bar{S} + \lambda S_1 \) is solution to the master equation (46) to order one in \( \lambda \). We will see in the next section that the consistency of \( S_1 \) at order two in the coupling constant will restrict these five functions of \( \varphi \) to satisfy several equations.

### 4.3 Higher-order deformations

Next, we investigate the equations that control the higher-order deformations. The second-order deformation is governed by equation (49). Making use of (142), the second term in the left hand-side of (49) takes the concrete form

\[
\frac{1}{2} (S_1, S_1) = \int d^4x \left[ \varepsilon_{\mu\nu\rho\lambda} \sum_{a=0}^4 \left( T_{a\mu\nu\rho\lambda} \frac{d^a X}{d\varphi^a} + U_{a\mu\nu\rho\lambda} \frac{d^a Y}{d\varphi^a} \right) + \sum_{a=0}^2 \left( Q_{a\mu}^{(1)} \frac{d^a Z_1}{d\varphi^a} + Q_{a\mu}^{(2)} \frac{d^a Z_2}{d\varphi^a} \right) + \sum_{a=0}^3 Q_{a\mu}^{(3)} \frac{d^a Z_3}{d\varphi^a} + \sum_{a=0}^3 Q_{a\mu}^{(4)} \frac{d^a Z_4}{d\varphi^a} \right],
\]

(143)

where we used the notations

\[
T_{0\mu\nu\rho\lambda} = 4A^\mu C^{\nu\rho\lambda} + B^{\mu\nu} C^{\rho\lambda} + H^\mu \eta^{\nu\rho\lambda} - 2\eta^\mu C^{\nu\rho\lambda} - \varphi^\mu \eta^{\nu\rho\lambda},
\]

(144)

\[
T_{1\mu\nu\rho\lambda} = (H^\alpha A^\alpha C^{\nu\rho\lambda} + C^{\nu\rho\lambda} \eta^{\mu\lambda} \eta^{\nu\rho\lambda} - 2\eta^\mu C^{\nu\rho\lambda} - \varphi^\mu \eta^{\nu\rho\lambda})
\]

(145)
\[ T_3^{\mu\nu\rho\lambda} = H_\alpha H_\beta \left( (H_\gamma^* C^{\alpha\beta\gamma} + 3C^{\alpha\beta\gamma}_\delta) \eta^{\mu\nu\rho\lambda} - H_\gamma^* \eta^{\alpha\beta\gamma} C^{\mu\nu\rho\lambda} \right), \]  
(147)  
\[ T_4^{\mu\nu\rho\lambda} = H_\alpha H_\beta^* H_\gamma^* H_\delta^* C^{\alpha\beta\gamma\delta} \eta^{\mu\nu\rho\lambda}, \]  
(148)  
\[ U_0^{\mu\nu\rho\lambda} = \left( \frac{1}{2} \eta_{\alpha\beta\gamma\delta}^{\#} \eta^{\alpha\beta\gamma\delta} + \eta_{\alpha\beta\gamma}^{\#} \eta^{\alpha\beta\gamma} + B_\alpha^\ast B^\alpha \eta - 6A_\alpha^\ast A^\alpha + \right) \eta^{\mu\nu\rho\lambda} \]  
\[ + \left( A^\ast \eta + \frac{3}{2} B_\alpha^\ast \eta^{\alpha\beta\mu} - A_\alpha B^\alpha \eta \right) \eta^{\mu\nu\rho\lambda} + \frac{1}{2} B^{\mu\nu} B^{\rho\lambda} \eta, \]  
(149)  
\[ U_1^{\mu\nu\rho\lambda} = \left( \frac{1}{4} \eta C^{\mu\nu\rho\lambda} - A^\mu C^{\nu\rho\lambda} + 3B^\ast \eta C^{\nu\beta\rho\lambda} - 2\eta^{\mu\nu\rho\lambda} H^\lambda \right) \eta_{\alpha\beta\gamma\delta} \eta^{\alpha\beta\gamma\delta} \]  
\[ + \left( \left( \frac{1}{2} C_\alpha^\ast \eta + \frac{3}{2} C^\ast A_\gamma - 3B_\alpha^\ast H_\gamma^* \right) \eta^{\alpha\beta\gamma} + \left( \frac{1}{2} C_\alpha^\ast B^\alpha + A^\ast H^\alpha \right) \eta \right. \]  
\[ + H_\alpha^* A_\beta B^\beta \eta^{\mu\nu\rho\lambda} + \left( \frac{3}{2} \left( \frac{1}{2} C_\alpha^\ast \eta + H_\alpha^* A_\beta \right) \eta^{\alpha\beta\mu} - H_\alpha^* B^\alpha \eta \right) \eta^{\mu\nu\rho\lambda}, \]  
(150)  
\[ U_2^{\mu\nu\rho\lambda} = H_\alpha^\ast \left( \frac{3}{2} \left( C^\ast \eta + H_\beta^\ast A_\gamma \right) \eta^{\alpha\beta\gamma} + H_\beta^* B^\alpha \eta \right) \eta^{\mu\nu\rho\lambda} \]  
\[ + \frac{3}{4} H_\alpha^* H_\beta^* \eta^{\alpha\beta\rho} \eta^{\lambda} \eta^{\rho} \eta^{\mu\nu} \left( H^\ast \eta + 3H^{\mu\nu} B^\rho \eta \right) \]  
\[ + \frac{3}{4} \left( \frac{1}{2} C^{\mu\nu} \eta + H^{\mu\nu} A^\nu \right) \eta^{\rho} \eta \eta^{\alpha\beta\gamma} \]  
(151)  
\[ U_3^{\mu\nu\rho\lambda} = \frac{1}{2} H^\ast H^{\mu\nu} \left( 3C^{\rho\lambda} \eta + 2H^{\ast \rho} A^\lambda \right) \eta_{\alpha\beta\gamma\delta} \eta^{\alpha\beta\gamma\delta} \]  
\[ + \frac{1}{2} H_\alpha^* H_\beta^* H_\gamma^* \eta^{\alpha\beta\gamma} \eta^{\mu\nu\rho\lambda}, \]  
(152)  
\[ U_4^{\mu\nu\rho\lambda} = \frac{1}{2} \eta H^\ast H^{\mu\nu} H^{\ast \rho} H^{\ast \lambda} \eta_{\alpha\beta\gamma\delta} \eta^{\alpha\beta\gamma\delta}, \]  
(153)  
\[ Q_0^{(1)} = -\frac{1}{2} \left( i\bar{\psi}^\mu \gamma_\mu \gamma_5 \xi + i\bar{\psi}^\mu \gamma^{\mu\nu} \gamma_5 \psi^\nu \right) \eta + \frac{i}{2} \bar{\xi} \gamma^{\mu\nu} \gamma_5 \left( 2\psi^\mu A_\mu + \xi B^\mu \right), \]  
(154)  
\[ Q_1^{(1)} = \frac{i}{2} H^\ast \bar{\xi} \gamma^{\mu\nu} \gamma_5 \left( \xi A_\nu - 2\psi_\nu \right) + \frac{i}{4} \bar{C}^\ast \bar{\xi} \gamma^{\mu\nu} \gamma_5 \xi \eta, \]  
(155)  
\[ Q_2^{(1)} = \frac{i}{4} H^\ast \bar{H}^\ast \bar{\xi} \gamma^{\mu\nu} \gamma_5 \xi \eta, \]  
(156)
\[ Q_0^{(2)} = -\frac{i}{2} \left( \psi^\mu \gamma_\mu \xi + i \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \right) \eta - \frac{1}{2} \xi \gamma^{\mu\nu} \left( 2 \psi_\nu A_\mu + \xi B^*_\mu \right), \]  
\[ Q_1^{(2)} = -\frac{1}{2} H^*_\mu \xi \gamma^{\mu\nu} \left( \xi A_\nu - 2 \psi_\nu \eta \right) - \frac{1}{4} C^*_\mu \xi \gamma^{\mu\nu} \xi \eta; \]  
\[ Q_2^{(2)} = -\frac{1}{4} H^*_\mu H^*_\nu \xi \gamma^{\mu\nu} \xi \eta; \]  
\[ Q_0^{(3)} = \frac{3i}{2} \bar{\psi}_\mu \gamma^\mu \xi, \]  
\[ Q_1^{(3)} = -\frac{3i}{4} H^*_\mu \xi \gamma^\mu \xi, \]  
\[ Q_0^{(4)} = 2i \left( \xi^5 \tau_5 \xi + \psi_5 \gamma_5 \psi_5 \right) \varepsilon_\alpha \tau_\beta \tau_\delta \eta^{\alpha\beta\gamma\delta} + 2i A^*_\mu \xi \gamma^\mu \xi \]  
\[ - \left( 2i \psi^{*\mu} \gamma_5 \xi + \bar{\psi}_\mu \gamma^{\mu\nu} \gamma_5 \psi_\nu \right) \varepsilon_\rho \tau_\beta \tau_\delta \eta^{\beta\gamma\delta} \]  
\[ + \bar{\xi} \gamma^{\mu\nu} \gamma_5 \psi_\rho \varepsilon_{\mu \alpha \beta} B^{\alpha \beta}, \]  
\[ Q_1^{(4)} = \left[ -\frac{1}{6} C^*_\mu \bar{\xi} \gamma^{\mu\nu} \gamma_5 \xi + C^*_\mu \bar{\xi} \gamma^{\mu\nu} \gamma_5 \psi_\rho + H^*_\mu \left( \bar{\psi}_\nu \gamma^{\mu\nu} \gamma_5 \psi_\rho \right. \right. \]  
\[ - 2i \psi^{*\mu} \gamma_5 \xi \varepsilon_\alpha \tau_\beta \tau_\delta \eta^{\alpha\beta\gamma\delta} + \left( 2H^*_\mu \bar{\xi} \gamma^{\mu\nu} \gamma_5 \psi_\nu - \frac{1}{2} C^*_\mu \bar{\xi} \gamma^{\mu\nu} \gamma_5 \xi \right) \times \]  
\[ \times \varepsilon_{\rho \beta \gamma} \eta^{\beta\gamma} \right] - 2i H^*_\mu \xi \bar{\gamma}_\nu \xi B^{\mu\nu}, \]  
\[ Q_2^{(4)} = H^*_\mu H^*_\nu \left( \bar{\xi} \gamma^{\mu\nu} \gamma_5 \psi_\rho \varepsilon_{\alpha \beta \gamma} \eta^{\alpha\beta\gamma\delta} - \frac{1}{2} \bar{\xi} \gamma^{\mu\nu} \gamma_5 \xi \varepsilon_{\rho \beta \gamma} \eta^{\beta\gamma\delta} \right) \]  
\[ - \frac{1}{2} H^*_\mu C^*_\nu \bar{\xi} \gamma^{\mu\nu} \gamma_5 \xi \varepsilon_{\alpha \beta \gamma} \eta^{\alpha\beta\gamma\delta}, \]  
\[ Q_3^{(4)} = -\frac{1}{6} H^*_\mu H^*_\nu H^*_\rho \bar{\xi} \gamma^{\mu\nu} \gamma_5 \xi \varepsilon_{\alpha \beta \gamma} \eta^{\alpha\beta\gamma\delta}, \]  

Together with
\[ X (\varphi) = W (\varphi) M (\varphi), \quad Y (\varphi) = W (\varphi) \frac{dM (\varphi)}{d\varphi}, \]  
\[ Z_1 (\varphi) = W (\varphi) \frac{dU_2 (\varphi)}{d\varphi} + 2U_1 (\varphi) U_3 (\varphi), \]  
\[ Z_2 (\varphi) = W (\varphi) \frac{dU_3 (\varphi)}{d\varphi} - 2U_1 (\varphi) U_2 (\varphi), \]
\[
Z_3(\varphi) = (U_2(\varphi))^2 + (U_3(\varphi))^2, \quad Z_4(\varphi) = U_1(\varphi)M(\varphi). \tag{169}
\]

It is clear that none of the terms involving any of the functions \(X, Y, Z_1, Z_2, Z_3, Z_4\) or their derivatives with respect to the scalar field can be written as it is required by equation (49), namely, like the \(s\)-variation of some local functional, and therefore they must vanish. In other words, the consistency of the first-order deformation at order two in the coupling constant, namely the existence of a local \(S_2\) as solution to equation (49), restricts the five functions of the undifferentiated scalar field that parameterize \(S_1\) to satisfy the equations

\[
X(\varphi) = 0, \quad Y(\varphi) = 0, \quad Z_i(\varphi) = 0, \quad i = 1, 4. \tag{170}
\]

Due to (170), from (143) it is obvious that \((S_1, S_1) = 0\), and thus we can take

\[
S_2 = 0 \tag{171}
\]

as solution to (49). By relying on (171), it is easy to show that one can safely put

\[
S_k = 0, \quad k > 2. \tag{172}
\]

Collecting formulas (171) and (172), we can state that the complete deformed solution to the master equation for the model under study, which complies with all the working hypotheses and is consistent to all orders in the coupling constant, stops at order one in the coupling constant and reads as

\[
S = \bar{S} + gS_1, \tag{173}
\]

where \(\bar{S}\) is given in (13) and \(S_1\) is expressed by (142). The fully deformed solution to the master equations comprises five types of smooth functions that depend only on the undifferentiated scalar field: \(W, M,\) and \((U_i)_{i=1,2,3}\). They are no longer arbitrary, but satisfy equations (170), imposed by the consistency conditions.

There appear two complementary solutions to the above equations,

\[
U_2(\varphi) = U_3(\varphi) = M(\varphi) = 0, \tag{174}
\]

\[
U_1(\varphi) = U_2(\varphi) = U_3(\varphi) = W(\varphi) = 0, \tag{175}
\]

which will be analyzed separately below. They yield two types of deformed models, whose Lagrangian formulation will be discussed in the following.
5 Lagrangian formulation of the coupled model

The first case corresponds to solution (174) of equations (170), such that the deformed solution to the master equation is parameterized by the smooth, but otherwise arbitrary, functions $W(\varphi)$ and $U_1(\varphi)$. It is given by (173), with $S_1$ from (142) particularized to $U_2 = U_3 = M = 0$, and reads as

$$S^{(1)} = \int d^4x \left\{ H^\mu (\partial_\mu \varphi - \lambda W A_\mu) - \frac{i}{2} \left( \bar{\psi}_\mu \gamma^{\mu\nu\rho} (\partial_\nu + i \gamma_5 A_\nu U_1) \psi_\rho \right) \right.$$

$$+ \frac{1}{2} B^{\mu\nu} \partial_\mu A_\nu \right\} + A_\mu \partial^\mu \eta + \lambda W \varphi^* \eta + H^{*\mu} \left[ \lambda \frac{d W}{d \varphi} \left( -\eta H_\mu + 2 A^\nu C_{\mu\nu} \right) \right.$$  

$$+ 2 \partial^\nu C_{\mu\nu} - \frac{\lambda}{2} \frac{d U_1}{d \varphi} H^*_\mu \left( 2 \bar{\xi} \gamma^{\mu\nu\rho\gamma_5} \psi_\rho \eta + \bar{\psi}_\nu \gamma^{\mu\nu\rho\gamma_5} \psi_\rho \eta \right) \left. \right\} + B^*_{\mu\nu} \left( -3 \partial_\rho \gamma^{\mu\nu\rho} + 2 \lambda W C^{\mu\nu} + \lambda U_1 \bar{\xi} \gamma^{\mu\nu\rho\gamma_5} \psi_\rho \right) + \psi^* [ \partial_\mu \xi \\ - i \lambda U_1 (\gamma_5 \psi_\mu \eta - \gamma_5 \xi A_\mu)] + \lambda H^{*\mu} H^{*\nu} \left[ \frac{1}{4} \frac{d^2 U_1}{d \varphi^2} \bar{\xi} \gamma^{\mu\nu\rho\gamma_5} (\xi A^\rho - 2 \psi_\rho \eta) \right. \right.$$  

$$+ \frac{d^2 W}{d \varphi^2} \eta C_{\mu\nu} \right\} + \lambda \frac{d U_1}{d \varphi} H^*_\mu \left( \frac{1}{2} \bar{\xi} \gamma^{\mu\nu\rho\gamma_5} \xi B^*_{\nu\rho} + i \psi^{*\mu} \gamma_5 \xi \eta \right)$$  

$$- 3 C^{*\mu\nu} \left( \partial^\rho C_{\rho\mu\nu} + \lambda \frac{d W}{d \varphi} A^\rho C_{\rho\mu\nu} \right) \left[ + \eta^{*\mu\nu} \left( 4 \partial_\lambda \gamma^{\mu\nu\rho\lambda} - 2 \lambda W C^{\mu\nu\rho} \right) \right.$$  

$$+ \lambda C^{*\mu\nu} \left[ \frac{d W}{d \varphi} \eta C_{\mu\nu} + \frac{1}{4} \frac{d U_1}{d \varphi} \bar{\xi} \gamma^{\mu\nu\rho\gamma_5} \xi A^\rho - 2 \psi_\rho \eta \right] \right\} - \lambda U_1 (i \xi^* \gamma_5 \xi \eta$$  

$$- \frac{1}{6} \eta^{*\mu\nu\rho} \bar{\xi} \gamma^{\mu\nu\rho\gamma_5} \xi \right\} - 3 \lambda H^{*\mu} \left( \frac{d^2 W}{d \varphi^2} H^{*\nu} A^\rho + 2 \frac{d W}{d \varphi} B^{*\nu\rho} \right) C_{\mu\rho\nu}$$  

$$+ 4 C^{*\mu\nu\rho} \left( \partial^\lambda C_{\mu\rho\lambda} + \lambda \frac{d W}{d \varphi} A^{\lambda} \right)$$  

$$+ 2 \lambda W \eta^{*\mu\nu\rho\lambda} C_{\mu\rho\nu\lambda}$$  

$$- \lambda C^{*\mu\nu\rho} \left( \frac{d W}{d \varphi} \eta C_{\mu\rho\nu} + \frac{1}{12} \frac{d U_1}{d \varphi} \bar{\xi} \gamma^{\mu\nu\rho\gamma_5} \xi \eta \right)$$  

$$+ 4 \lambda \left[ \frac{3}{2} \frac{d^2 W}{d \varphi^2} H^{*\mu} C^{*\nu\rho} A^\lambda \right. + \frac{d W}{d \varphi} \left( 3 B^{*\mu\nu} C^{*\nu\rho\lambda} + 2 H^{*\mu} \eta^{*\nu\rho\lambda} \right) \left. \right\] C_{\mu\rho\nu\lambda}$$  

$$+ 4 \lambda H^{*\mu} H^{*\nu} \left( \frac{3}{2} \frac{d^2 W}{d \varphi^2} B^{*\nu\rho} + \frac{d^2 W}{d \varphi^3} H^{*\nu} A^{\lambda} \right) C_{\mu\rho\nu\lambda}$$  

$$- \lambda H^{*\mu} \left[ 3 C^{*\mu\rho} \left( \frac{d^2 W}{d \varphi^2} C_{\mu\rho\nu} + \frac{1}{12} \frac{d U_1}{d \varphi^2} \bar{\xi} \gamma^{\mu\nu\rho\gamma_5} \xi \right) \right] \right.$$  



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\[ + H^{*\nu} H^{*\rho} \left( \frac{d^3 W}{d\phi^3} C_{\mu\rho} - \frac{1}{12} \frac{d^4 U_1}{d\phi^4} \bar{\xi} \gamma_{\mu\nu\rho} \gamma_5 \xi \right) \eta \]
\[ + \lambda \left[ \frac{dW}{d\phi} C_{\mu\rho} - \frac{d^2 W}{d\phi^2} \left( H^{[\mu} C^{\nu\rho]} + C^{[\mu\nu} C^{\rho]} \right) \right. \]
\[ + \frac{d^3 W}{d\phi^3} H^{[\mu} H^{\nu]} C^{\rho\lambda]} + \left. \frac{d^4 W}{d\phi^4} H^{\mu\nu} H^{\rho\lambda} \right] \eta C_{\mu\nu\rho\lambda} \right\} . \] (176)

By virtue of the discussion from the end of Section 2 on the significance of terms with various antighost numbers from the solution to the master equation, at this stage we can extract all the information on the gauge structure of the coupled model. From the antifield-independent piece in (176) we read that the overall Lagrangian action of the interacting gauge theory has the expression
\[
\tilde{S}^{(I)}[A^\mu, H^\mu, \varphi, B^{\mu\nu}, \psi_\mu] = \int d^4x \left[ H_\mu \left( \partial^\mu \varphi - \lambda W (\varphi) A^\mu \right) \right.
\[ \left. + \frac{1}{2} B^{\mu\nu} \partial_{[\mu} A_{\nu]} \right] - \frac{i}{2} \left( \bar{\psi}_\mu \gamma^{\mu\rho} \left( \partial_\rho + \lambda i \gamma_5 U_1 \right) \psi_\rho \right) \right], \] (177)

while from the components linear in the antighost number one antifields we conclude that it is invariant under the gauge transformations
\[
\tilde{\delta}_\epsilon^{(I)} \varphi = \lambda W (\varphi) \epsilon, \] (178)
\[
\tilde{\delta}_\epsilon^{(I)} H^\mu = 2 \tilde{D}_\nu \epsilon^{\mu\nu} + \lambda \left[ \left( \frac{1}{2} \frac{dU_1}{d\varphi} \bar{\psi}_\nu \gamma^{\mu\rho} \gamma_5 \psi_\rho - \frac{dW}{d\varphi} H^\mu \right) \epsilon \right. \]
\[ \left. + \frac{dU_1}{d\varphi} A_\rho \bar{\psi}_\nu \gamma^{\mu\rho} \gamma_5 \chi \right], \] (179)
\[
\tilde{\delta}_\epsilon^{(I)} A^\mu = \partial^\mu \epsilon, \] (180)
\[
\tilde{\delta}_\epsilon^{(I)} B^{\mu\nu} = -3 \partial_\rho \epsilon^{\mu\rho} + 2 \lambda W (\varphi) \epsilon^{\mu\nu} - \lambda U_1 (\varphi) \bar{\psi}_\rho \gamma^{\mu\rho} \gamma_5 \chi, \] (181)
\[
\tilde{\delta}_\epsilon^{(I)} \psi_\mu = \partial_\mu \chi - i \lambda U_1 (\varphi) \left( \gamma_5 \psi_\mu \epsilon - \gamma_5 \chi A_\mu \right), \] (182)
where we employed the notation
\[
\tilde{D}_\nu = \partial_\nu + \lambda \frac{dW}{d\varphi} A_\nu. \] (183)

We observe that (176) contains two kinds of pieces quadratic in the ghosts of pure ghost number one: ones are linear in their antifields, and the others are quadratic in the antifields of the original fields, which indicates the open
behavior of the deformed gauge algebra. This is translated into the fact that the commutators among the deformed gauge generators only close on-shell, where on-shell means on the stationary surface of field equations for action \( (177) \). Also, we notice the presence of terms linear in the ghosts with pure ghost number two and three in \( (176) \), which shows that the gauge generators of the coupled model are also second-order reducible, but some of the reducibility functions are modified and, moreover, some of the reducibility relations only hold on-shell. The remaining elements in \( (176) \) give us information on the higher-order gauge structure of the interacting model. All these ingredients of the Lagrangian gauge structure of the deformed theory are listed in Appendix C.

It is interesting to mention certain similarities between the model under study and \( N = 1, D = 4 \) conformal SUGRA \[44\]. First, there are common fields in both theories, namely a gravitino \( \psi_\mu \) with undeformed gauge symmetries described by ordinary local \( Q \)-supersymmetry transformation \( \chi \) (see the latter formula from \( (13) \)) and a vector field \( A_\mu \) with initial \( U(1) \) gauge transformation \( \epsilon \) (see the first relation in \( (12) \)). It is suggestive to make the notations

\[
\chi \equiv \epsilon_Q, \quad \epsilon \equiv \epsilon_U, \quad D_\mu \equiv \partial_\mu + i\lambda U_1 (\varphi) \gamma_5 A_\mu, \quad (184)
\]

in terms of which the deformed gauge transformations of the Rarita-Schwinger spinors, \( (182) \), become

\[
\bar{\delta}_\epsilon^{(1)} \psi_\mu = D_\mu \epsilon_Q - i\lambda U_1 (\varphi) \gamma_5 \epsilon_U \psi_\mu. \quad (185)
\]

Second, if we make the choices

\[
U_1 (\varphi) = -\frac{3}{4}, \quad \lambda = 1, \quad (186)
\]

then the deformed gauge transformations of gravitini, \( (185) \), take the form

\[
\bar{\delta}_\epsilon^{(1)} \psi_\mu = D_\mu \epsilon_Q + \frac{3}{4} i\gamma_5 \epsilon_U \psi_\mu, \quad (187)
\]

in terms of the covariant derivative

\[
D_\mu \equiv \partial_\mu - \frac{3}{4} i\gamma_5 A_\mu. \quad (188)
\]

We observe that \( (187) \) are nothing but the standard \( N = 1, D = 4 \) conformal SUGRA gauge transformations of the spin-3/2 field in the absence
of local dilatational $D$-transformation, special conformal $S$-supersymmetry, local spacetime translations and local Lorentz rotations, i.e. solely in the presence of local $Q$-supersymmetry and $U(1)$-symmetry. Indeed, it coincides with formula (2.24) for $\psi_\mu$ given in [44] if one sets $\epsilon_D = \epsilon_S = \epsilon_\nu^I = \epsilon_\lambda^I = 0$. By contrast to conformal SUGRA, where the gauge transformation of the vector field $A_\mu$ gains $Q$-supersymmetric contributions, here it remains $U(1)$ also for the coupled model (see formula (180)). This is mainly because the BF model under discussion does not include a Maxwell Lagrangian with respect to $A_\mu$, but the term $(1/2)B^{\mu\nu}\partial_\mu A_\nu$ (see (177)). For this topological model it is precisely the two-form $B^{\mu\nu}$ whose gauge transformations (181) gain $Q$-supersymmetric contributions

$$\delta^{(I)}_\epsilon B^{\mu\nu} = \text{something} + \frac{3}{4}\bar{\psi}_\rho \gamma^{\mu\nu}\gamma_5 \epsilon_Q$$

that compensate the gauge variation of the last term from the right-hand side of (177), namely, $(i/2)\bar{\psi}_\mu \gamma^{\mu\nu}\gamma_5 \epsilon_Q$. In conclusion, we can state that (182) describes a certain generalization of gravitino gauge transformations from conformal SUGRA in the sense that puts the standard $Q$-supersymmetry and $U(1)$-symmetry parts in a ‘background’ potential $U_1(\varphi)$.

5.1 Case II. No couplings to the Rarita-Schwinger fields

The second solution to equations (170) is (175), so the deformed solution of the master equation is uniquely parameterized in this situation by the arbitrary function $M(\varphi)$ (of the undifferentiated scalar field). This case is less interesting from the point of view of interactions as the Rarita-Schwinger spinors are no longer coupled to the BF fields. From (143) and the higher-order deformation equations, (50), etc., it follows that we can safely take all the deformations (of order two or higher) to vanish

$$S_k = 0, \quad k > 1.$$  

Therefore, the overall deformed solution to the master equation that is consistent to all orders in the coupling constant is equal to the sum between the free solution, (143), and the first-order deformation, (142), where we set (175), $S^{(II)} = \tilde{S} + \lambda S_1$, being expressed by

$$S^{(II)} = \int d^4x \left[ H_\mu \partial^\mu \varphi + \frac{1}{2}B^{\mu\nu} \left( \partial_\mu A_\nu + \lambda \epsilon_\nu^I \lambda M B^{\rho\lambda} \right) + \right.$$
\[ -\frac{i}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_{\nu} \psi_\rho + A^\mu \left( \partial_\mu \eta - 2\lambda M \varepsilon_{\mu\alpha\beta\gamma} \eta^{\alpha\beta\gamma} \right) + 2 H_\mu^* \left( \partial_\nu C^{\mu\nu} + \lambda \frac{dM}{d\varphi} B^{\mu\alpha} \varepsilon_{\alpha\beta\gamma\delta} \eta^{\beta\gamma\delta} \right) - 3 B_\mu^* \partial_\rho \eta^{\mu\nu\rho} - 3 C_\mu^* \partial_\nu C^{\mu\nu} + 4 \eta_\mu^* \partial_\lambda \eta^{\mu\nu\rho\lambda} + \lambda \left[ \left( \frac{dM}{d\varphi} C^{\mu\nu} + \frac{d^2 M}{d\varphi^2} H^{\mu}_{\rho} H^{\lambda}_{\nu} \right) B^{\rho\lambda} + 2 \left( \frac{dM}{d\varphi} H^{*\mu}_{\rho} A^{\nu\mu} - M \eta^* \right) \right] \varepsilon_{\alpha\beta\gamma\delta} \eta^{\alpha\beta\gamma\delta} - \frac{9}{4} \lambda \left( \frac{dM}{d\varphi} C^{\mu\nu} + \frac{d^2 M}{d\varphi^2} H^{\mu}_{\rho} H^{\lambda}_{\nu} \right) \varepsilon_{\alpha\beta\gamma\delta} \eta^{\rho\alpha\beta\gamma\delta} + 4 C_\mu^* \partial_\lambda C^{\mu\nu\rho} \right] + \frac{\lambda}{2} \left[ \frac{dM}{d\varphi} C^{\mu\rho\lambda} + \frac{d^2 M}{d\varphi^2} H^{\mu}_{\nu} C^{\rho\lambda}_{\nu} + \frac{d^2 M}{d\varphi^2} C^{*\mu\rho\lambda}_{\nu} \right] \eta^{\mu\rho\lambda} \varepsilon_{\alpha\beta\gamma\delta} \eta^{\alpha\beta\gamma\delta} + \frac{d^3 M}{d\varphi^3} H^{\mu}_{\nu} H^{*}_{\rho} C^{\rho\lambda}_{\nu} + \frac{d^4 M}{d\varphi^4} H^{*\mu}_{\rho} H^{*\nu}_{\rho} H^{*\lambda}_{\lambda} \eta^{\mu\rho\lambda} \varepsilon_{\alpha\beta\gamma\delta} \eta^{\alpha\beta\gamma\delta} \right]. \] (191)

Its antighost number zero part emphasizes the Lagrangian action of the deformed theory

\[ \tilde{S}^{(II)}[A^\mu, H^\mu, \varphi, B^{\mu\nu}, y^i] = \int d^4x \left[ H_\mu \partial^\mu \varphi - \frac{i}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_{\nu} \psi_\rho \right. \\
+ \left. \frac{1}{2} B^{\mu\nu} \left( \partial_{\mu} A_{\nu} + \lambda \varepsilon_{\mu\rho\lambda} \lambda MB^{\rho\lambda} \right) \right], \] (192)

while the antighost number one components provide the gauge transformations of action (192)

\[ \bar{\delta}_\varepsilon^{(II)} A_\mu = \partial_\mu \varepsilon - 2\lambda M \left( \varphi \varepsilon_{\mu\alpha\beta\gamma} \varepsilon^{\alpha\beta\gamma} \equiv \tilde{Z}^{(A)}_{(A)} \varepsilon + \tilde{Z}^{(A)}_{(A)} \varepsilon^{\alpha\beta\gamma} \right), \] (193)

\[ \bar{\delta}_\varepsilon^{(II)} H^\mu = 2 \left( \partial_\mu \varepsilon^{\mu\nu} - \lambda \frac{dM}{d\varphi} B^{\mu\alpha} \varepsilon_{\alpha\beta\gamma\delta} \eta^{\beta\gamma\delta} \equiv \tilde{Z}^{(II)}_{(H)} \varepsilon_{\alpha\beta\gamma} \varepsilon^{\alpha\beta\gamma} + \tilde{Z}^{(II)}_{(H)} \varepsilon^{\alpha\beta\gamma} \right), \] (194)

\[ \bar{\delta}_\varepsilon^{(II)} \varphi = 0, \quad \bar{\delta}_\varepsilon^{(II)} B^{\mu\nu} = -3 \partial_\mu \varepsilon^{\mu\nu\rho} \equiv \tilde{Z}^{(II)}_{(B)} \varepsilon_{\alpha\beta\gamma} \varepsilon^{\alpha\beta\gamma}, \] (195)

\[ \bar{\delta}_\varepsilon^{(II)} \psi^A = \partial^\mu \chi^A \equiv \tilde{Z}^{(II)}_{(\psi)} \chi^A. \] (196)

We observe that in this case the Rarita-Schwinger fields remain uncoupled to the BF field sector. From (193)–(196), we notice that the nonvanishing
gauge generators are

\begin{align*}
(\tilde{Z}^{(II)}_{(A)\mu})(x, x') &= (Z_{(A)\mu})(x, x') = \partial_x^\mu \delta^4(x - x'), \quad (197) \\
(\tilde{Z}^{(II)}_{(A)\mu})_{\alpha\beta\gamma}(x, x') &= -2\lambda M (\varphi(x)) \varepsilon_{\mu\alpha\beta\gamma} \delta^4(x - x'), \quad (198) \\
(\tilde{Z}^{(II)}_{(H)\mu})_{\alpha\beta}(x, x') &= (Z^{\mu}_{(H)\alpha\beta})(x, x') = -\partial_{(\alpha}^\mu \delta_{\beta)}^4(x - x'), \quad (199) \\
(\tilde{Z}^{(II)}_{(H)\mu})_{\alpha\beta\gamma}(x, x') &= -2\lambda \frac{dM}{d\varphi} (x) B^{\mu\nu}(x) \varepsilon_{\nu\alpha\beta\gamma} \delta^4(x - x'), \quad (200) \\
(\tilde{Z}^{(II)}_{(B)\mu\nu})_{\alpha\beta\gamma}(x, x') &= (Z^{\mu\nu}_{(B)\alpha\beta\gamma})(x, x') = -\frac{1}{2} \partial_{\alpha}^\mu \delta_{\beta}^\nu \delta_{\gamma}^\delta \delta^4(x - x'), \quad (201) \\
(\tilde{Z}^{(II)}_{(A)\mu})_B &= (Z^{A\mu}_{(A)})_B = \delta^A_{\mu}\delta^4(x - x'). \quad (202)
\end{align*}

The deformed gauge algebra (corresponding to the generating set (193)–(196)) is open, as can be seen from the elements of antighost number two in (191) that are quadratic in the ghosts of pure ghost number one. The only non-Abelian commutators among the new gauge transformations are expressed by

\begin{align*}
(\tilde{Z}^{(II)}_{(B)\rho\lambda})_{\alpha\beta\gamma} \frac{\delta(\tilde{Z}^{(II)}_{(H)\mu})_{\alpha'\beta'\gamma'}}{\delta B^{\rho\lambda}} - (\tilde{Z}^{(II)}_{(H)\rho\lambda})_{\alpha'\beta'\gamma'} \frac{\delta(\tilde{Z}^{(II)}_{(B)\mu})_{\alpha\beta\gamma}}{\delta B^{\rho\lambda}} = \\
-\frac{\lambda}{4} \frac{dM}{d\varphi} (\tilde{Z}^{(II)}_{(H)\rho\lambda})_{\alpha\beta\gamma} \delta^\lambda_{[\alpha} \varepsilon_{\beta\gamma][\alpha'\beta'] \delta^\rho_{\gamma']}, \\
+\frac{d^2M}{d\varphi^2} \delta^{\mu}_{[\alpha} \varepsilon_{\beta\gamma][\alpha'\beta'] \delta^\nu_{\gamma']} \frac{\delta S^{(II)}}{\delta H^\nu}. \quad (203)
\end{align*}

Looking at the remaining terms of antighost number two from (191), we can state that, besides the original first-order reducibility relations (18), there appear some new ones

\begin{align*}
(Z^{(II)\mu}_{(H)})_{\rho\lambda} (\tilde{Z}^{(II)\rho\lambda}_{(H)})_{\alpha\beta\gamma} &= 2\lambda \varepsilon_{\alpha\beta\gamma} \frac{d^2M}{d\varphi^2} B^{\mu\nu} \frac{\delta S^{(II)}}{\delta H^\nu} + \frac{dM}{d\varphi} \sigma^{\mu\nu} \frac{\delta S^{(II)}}{\delta H^\nu}, \quad (204) \\
(Z^{(II)\mu}_{(A)})_{\rho\lambda} (\tilde{Z}^{(II)\rho\lambda}_{(A)})_{\alpha\beta\gamma} &= -2\lambda \varepsilon_{\alpha\beta\gamma} \frac{dM}{d\varphi} \sigma^{\mu\nu} \frac{\delta S^{(II)}}{\delta H^\nu}, \quad (205)
\end{align*}
which only close on-shell (i.e. on the stationary surface of field equations resulting from action (191)), where the accompanying first-order reducibility functions are of the form

\[
\left(\tilde{Z}^{(\text{II})}\right)_{\alpha\beta\gamma\delta}(x, x') = \lambda \frac{dM}{d\varphi}(x) B^{\rho\lambda}(x) \varepsilon_{\alpha\beta\gamma\delta}(x - x'),
\]

(206)

\[
\left(\tilde{Z}^{(\text{II})}_{1}\right)_{\alpha\beta\gamma\delta}(x, x') = -2\lambda M(\varphi(x)) \varepsilon_{\alpha\beta\gamma\delta}(x - x').
\]

(207)

The second-order reducibility is not modified (it continues to be expressed by (19)). The presence (in (191)) of elements with antighost number strictly greater than two that are proportional with the coupling constant \(\lambda\) signifies a higher-order gauge tensor structure of the deformed model, due to the open character of the gauge algebra, as well as to the field dependence of the deformed reducibility functions. The case (II) appears thus to be less important from the perspective of constructing effective couplings among the BF fields and the spin-vector, since no nontrivial interactions among them are allowed.

6 Conclusion

To conclude with, in this paper we have investigated the consistent interactions that can be introduced between a topological BF theory and a massless Rarita-Schwinger field. Starting with the BRST differential for the free theory, we give the consistent first-order deformation of the solution to the master equation, and obtain that it is parameterized by five kinds of functions depending only on the undifferentiated scalar fields. Next, we analyze the consistency of the first-order deformation, which imposes certain restrictions on these functions. Based on these restrictions, we show that we can take all the remaining deformations, of order two or higher, to vanish. As a consequence of our procedure, we are led to two classes of interacting gauge theories. Only one is interesting, being endowed with deformed gauge transformations, a non-Abelian gauge algebra that only closes on-shell, and on-shell, second-order reducibility relations. This coupled model emphasizes some contributions to the gauge transformations of the spin-3/2 field that generalize the local Q-supersymmetry and U(1) gauge symmetry contributions from \(N = 1, D = 4\) conformal SUGRA in the sense of multiplying them with an arbitrary function that depends only on the scalar field from the BF spectrum.
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A  Proof of formula (128)

The proof of formula (128) requires the derivative order assumption. We start from the general solution to (116), which is of the form (61) for $I = 1$, being thus written as

$$
\tilde{a}^{\text{int}}_1 = H^\mu N^\mu \xi + \varphi^N N^\xi + A^\mu_N \tilde{N}^\mu \xi + B^\mu \tilde{N}^\mu \xi + \psi^\mu M^\xi + \tilde{M}^\mu \tilde{\psi}^\mu, \quad (208)
$$

where $N^\mu$, $N$, $\tilde{N}^\mu$, $\tilde{N}^\nu$, and $M^\mu$ are some fermionic, gauge-invariant, spinor-like functions and $M^\mu$ is a matrix $4 \times 4$ with spinor indices and bosonic, gauge-invariant elements. In addition, $\tilde{N}^\mu$ is antisymmetric in its Lorentz indices. Because $\tilde{c}_0$ produced by (208) via (122) must contain at most one spacetime derivative, the objects $N^\mu$, $N$, $\tilde{N}^\mu$, $\tilde{N}^\nu$, $\tilde{M}^\mu$, and $M^\mu$ are also constrained to include at most one derivative. Given their properties, exposed above, the objects $N^\mu$, $N$, $\tilde{N}^\mu$, $\tilde{N}^\nu$, and $\tilde{M}^\mu$ can be represented like

$$
N^\mu = \partial_\alpha \tilde{\psi}_{\beta} N^{\mu}_{\alpha \beta}, \quad N = \partial_\alpha \tilde{\psi}_{\beta} N^{\alpha \beta}, \quad \tilde{N}^\mu = \partial_\alpha \bar{\psi}_{\beta} \tilde{N}^{\mu}_{\alpha \beta}, \quad (209)
$$

$$
\tilde{M}^\mu = \partial_\alpha \bar{\psi}_{\beta} M^{\mu}_{\alpha \beta}, \quad (210)
$$

in terms of some $4 \times 4$ matrices $N^{\mu}_{\alpha \beta}$, $N^{\alpha \beta}$, $\tilde{N}^{\mu}_{\alpha \beta}$, $\tilde{N}^{\mu}_{\alpha \beta}$, and $\tilde{M}^{\mu}_{\alpha \beta}$ with spinor-like indices, whose elements are smooth functions depending at most on the undifferentiated scalar field. Since $M^\mu$ is $4 \times 4$, bosonic matrix with, whose elements are gauge-invariant spinor functions with at most one derivative, it can be represented like

$$
M^\mu = M^{(0)}_\mu + M^{(1)}_{\mu \alpha} \partial_\alpha \varphi + M^{(2)}_\mu H^\alpha + M^{(3)}_{\mu \alpha \beta} \partial^{[\alpha} A^{\beta]} + M^{(4)}_{\mu \alpha} \partial_\beta B^{\alpha \beta}, \quad (211)
$$

where $M^{(0)}_\mu$, $M^{(1)}_{\mu \alpha}$, $M^{(2)}_\mu$, $M^{(3)}_{\mu \alpha \beta}$, and $M^{(4)}_{\mu \alpha}$ are $4 \times 4$ spinor matrices, whose elements may depend at most on the undifferentiated scalar field $\varphi$, with $M^{(3)}_{\mu \alpha \beta}$ also antisymmetric in $\alpha$ and $\beta$. Let us show initially that (211) can always be reduced to its first term via some trivial redefinitions (of the matrices from...
performed in $\tilde{a}_1^{\text{int}}$. In view of this, we introduce (209)–(211) in (208) and deduce

$$\tilde{a}_1^{\text{int}} = H^*_\mu \partial_{[\alpha \bar{\psi}_\beta]} \left( N^{\mu[\alpha \beta} \gamma^{\alpha \beta \rho} M_{\rho]}^{(1)} \right) \xi$$

$$+ \varphi^* \partial_{[\alpha \bar{\psi}_\beta]} \left( N^{[\alpha \beta} + \frac{i}{2} \gamma^{[\alpha \beta} M_{\rho]}^{(2)} \right) \xi$$

$$+ A^*_\mu \partial_{[\alpha \bar{\psi}_\beta]} \left( \bar{N}^{[\alpha \beta} - \frac{i}{2} \sigma^{[\alpha \beta \rho} \gamma^{\alpha \beta \rho} M_{\rho]}^{(3)} \right) \xi$$

$$+ B^*_\mu \partial_{[\alpha \bar{\psi}_\beta]} \left( \bar{N}^{[\alpha \beta} - \frac{i}{2} \sigma^{[\alpha \beta \rho} \gamma^{\alpha \beta \rho} M_{\rho]}^{(3)} \right) \xi$$

$$+ \psi^* M_{\mu}^{(0)} \xi + \partial_{[\alpha \bar{\psi}_\beta]} \bar{M}^{[\alpha \beta} \bar{\psi}_\mu \eta$$

$$+ s \left( - H^{*\alpha} \psi^* M_{\mu}^{(1)} \xi + \varphi^{*\psi^*} M_{\mu}^{(2)} \xi \right.$$

$$- 2 B^{*\alpha} \psi^* M_{\mu}^{(3)} \xi - A^{*\psi^*} M_{\mu}^{(4)} \xi \right) \ceq (212)$$

Formula (212) allows us to take the solution to (116) under the form

$$\tilde{a}_1^{\text{int}} = \omega_1 + \omega_2 \ceq (215)$$
where
\[
\omega_1 = -i\partial_\mu \bar{\psi}_\nu \gamma^{\mu\nu} M^{(0)}_\rho \xi,
\]
(216)
\[
\omega_2 = \partial_\alpha \bar{\psi}_{\bar{\beta}} N^{\alpha\beta} \xi \partial_\mu \phi - \partial_\alpha \bar{\psi}_{\bar{\beta}} N^{\alpha\beta} \xi \partial_\mu H^\mu + \partial_\alpha \bar{\psi}_{\bar{\beta}} N^{\alpha\beta} \xi \partial_\mu B_{\mu
u} + \partial_\alpha \bar{\psi}_{\bar{\beta}} \bar{N}^{\alpha\beta} \xi \partial_\mu A_\nu - \frac{i}{2} \partial_\alpha \bar{\psi}_{\bar{\beta}} \bar{M}^{\alpha\beta} \gamma_{\mu\nu\rho} (\partial^{[\nu} \psi^{\rho]}) \eta.
\]
(217)

Using decomposition (215), it follows that equation (122) becomes equivalent to two independent equations, one for each component:
\[
\omega_1 = \gamma d_0 + \partial_\mu v^\mu_0,
\]
(218)
\[
\omega_2 = \gamma d_1 + \partial_\mu v^\mu_1,
\]
(219)

where
\[
\tilde{c}_0 = - (d_0 + d_1), \quad \tilde{m}^\mu_0 = v^\mu_0 + v^\mu_1.
\]
(220)

The objects denoted by \(d_0\) or \(v^\mu_0\) are derivative-free, while \(d_1\) and \(v^\mu_1\) comprise a single spacetime derivative. We will approach equations (218) and (219) separately.

Related to (218), from (216) we find that
\[
\omega_1 = \gamma \left( -i\bar{\psi}_\mu \gamma^{\mu\rho} M^{(0)}_\rho \psi_\nu \right) + \partial_\mu \left\{ -i\bar{\psi}_\nu \left[ \gamma^{\mu\rho} M^{(0)}_\rho + \gamma^0 (\gamma^0 \gamma^{\mu\rho} M^{(0)}_\rho)^\top \right] \xi \right\} - \frac{i}{2} \bar{\psi}_\mu \left[ \gamma^{\mu\rho} \partial_\rho M^{(0)}_\mu + \gamma^0 \left[ \gamma^0 \gamma^{\mu\rho} \partial_\rho M^{(0)}_\mu \right] \right] \xi + i\partial_\mu \bar{\psi}_\nu \left[ \gamma^0 \left( \gamma^0 \gamma^{\mu\rho} M^{(0)}_\rho \right)^\top - \gamma^{\mu\rho} M^{(0)}_\rho \right] \xi - \omega_1,
\]
(221)

and hence equation (218) is fulfilled if and only if
\[
\gamma^0 \left( \gamma^0 \gamma^{\mu\rho} M^{(0)}_\rho \right)^\top - \gamma^{\mu\rho} M^{(0)}_\rho = 0,
\]
(222)
\[
\gamma^0 \left( \gamma^0 \gamma^{\mu\rho} M^{(0)}_\rho \right)^\top + \gamma^{\mu\rho} M^{(0)}_\rho = A,
\]
(223)

where \(A\) is a constant \(4 \times 4\) matrix with spinor-like indices. Relations (222) and (223) are equivalent with
\[
\gamma^{\mu\rho} M^{(0)}_\rho = \gamma^0 \left( \gamma^0 \gamma^{\mu\rho} M^{(0)}_\rho \right)^\top = \frac{1}{2} A,
\]
i.e. the matrix $M_p^{(0)}$ is purely constant, such that the corresponding term from $\tilde{a}_1^{\text{int}}$ cannot produce cross-couplings, as required. As a consequence, we can safely work with

$$M_p^{(0)} = 0 \quad (224)$$

in $\tilde{a}_1^{\text{int}}$, which further leads to

$$\omega_1 = d_0 = v_0^\mu = 0. \quad (225)$$

Regarding (219), we observe that $\omega_2$ is written as a sum of four different types of terms

$$\omega_2 = \omega_2^{(1)} + \omega_2^{(2)} + \omega_2^{(3)} + \omega_2^{(4)}, \quad (226)$$

where

$$\omega_2^{(1)} = \partial_{[\alpha} \bar{\psi}_{\beta]} N^{\mu |\alpha \beta} \xi \partial_\mu \varphi,$$

$$\omega_2^{(2)} = -\partial_{[\alpha} \bar{\psi}_{\beta]} N^{\alpha \beta} \xi \partial_\mu H^\mu,$$

$$\omega_2^{(3)} = \partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}^{\mu |\alpha \beta} \xi \partial^\nu B_{\mu \nu},$$

$$\omega_2^{(4)} = \partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}^{\mu \nu |\alpha \beta} \xi \partial_\mu A_{\nu} - \frac{i}{2} \partial_{[\alpha} \bar{\psi}_{\beta]} \hat{M}^{\mu |\alpha \beta} \gamma_{\mu \nu \rho} (\partial_\nu \psi^\rho) \eta, \quad (230)$$

so each type mixes the Rarita-Schwinger spinor with various fields/ghosts from the BF complex. Recalling decomposition (226), it is easy to see that $\omega_2$ is $\gamma$-exact modulo $d$ if and only if each of its components, $\omega_2^{(j)}$, $j = 1, 4$, is so:

$$\omega_2^{(j)} = \gamma d_1^{(j)} + \partial_\mu v_1^{(j)\mu}, \quad j = 1, 4, \quad (231)$$

where

$$d_1 = \sum_{j=1}^{4} d_1^{(j)}, \quad v_1^\mu = \sum_{j=1}^{4} v_1^{(j)\mu}. \quad (232)$$

We will analyze equations (231) separately as well. Firstly, we consider (231) for $j = 1$ and denote by $\hat{N}^{\mu |\alpha \beta}$ some $4 \times 4$ matrices with spinor-like indices, whose elements are nothing but some indefinite integrals of the corresponding elements of $N^{\mu |\alpha \beta}$. In terms of this new matrices, (227) becomes

$$\omega_2^{(1)} = \partial_{[\alpha} \bar{\psi}_{\beta]} \left( \partial_\mu \hat{N}^{\mu |\alpha \beta} \right) \xi$$

$$= \partial_\mu \left[ (\partial_{[\alpha} \bar{\psi}_{\beta]} \right) \hat{N}^{\mu |\alpha \beta} \xi] + \gamma \left[ - (\partial_{[\alpha} \bar{\psi}_{\beta]} \right) \hat{N}^{\mu |\alpha \beta} \psi_\mu \right]$$

$$- (\partial_\mu \partial_{[\alpha} \bar{\psi}_{\beta]} \hat{N}^{\mu |\alpha \beta} \xi], \quad (233)$$
such that $\omega_2^{(1)}$ is solution to (231) for $j = 1$ if and only if
\[(\partial_\mu \partial_{[\alpha} \bar{\psi}_{\beta]\}] ) \hat{N}^{\mu[\alpha} = 0, \tag{234}\]
or, in other words, matrices $\hat{N}^{\mu[\alpha\beta}$ are completely antisymmetric in their Lorentz indices. Since we work in $D = 4$, the general solution to (234) reads as
\[\hat{N}^{\mu[\alpha\beta} = \gamma^{\mu\alpha\beta} \left( i \hat{U}_7 + \hat{U}_8 \gamma_5 \right), \tag{235}\]
where $\hat{U}_7$ and $\hat{U}_8$ are some real, smooth functions of $\varphi$. Replacing (235) into (233), we get
\[\omega_2^{(1)} \equiv \partial_{[\alpha} \bar{\psi}_{\beta]\]} \gamma^{\mu\alpha\beta} \left( i \partial_\mu U_7 + \partial_\mu \hat{U}_8 \gamma_5 \right) \xi = \partial_\mu \left[ \left( \partial_{[\alpha} \bar{\psi}_{\beta]\]} \right) \gamma^{\mu\alpha\beta} \left( i \hat{U}_7 + \hat{U}_8 \gamma_5 \right) \xi \right] + \gamma \left[ - \left( \partial_{[\alpha} \bar{\psi}_{\beta]\]} \right) \gamma^{\mu\alpha\beta} \left( i \hat{U}_7 + \hat{U}_8 \gamma_5 \right) \psi_\mu \right], \tag{236}\]
which enables us to make the identifications
\[d_1^{(1)} = - \left( \partial_{[\alpha} \bar{\psi}_{\beta]\]} \right) \gamma^{\mu\alpha\beta} \left( i \hat{U}_7 + \hat{U}_8 \gamma_5 \right) \psi_\mu \tag{237}\]
\[v_1^{(1)\mu} = \left( \partial_{[\alpha} \bar{\psi}_{\beta]\]} \right) \gamma^{\mu\alpha\beta} \left( i \hat{U}_7 + \hat{U}_8 \gamma_5 \right) \xi. \tag{238}\]

Due to the relationship between $\hat{N}^{\mu[\alpha\beta}$ and $N^{\mu[\alpha\beta}$, we can write
\[N^{\mu[\alpha\beta} = \gamma^{\mu\alpha\beta} \left( i \frac{d\hat{U}_7}{d\varphi} + \frac{d\hat{U}_8}{d\varphi} \gamma_5 \right). \tag{239}\]

Next, we approach equation (231) for $j = 2$. By integrating by parts, we arrive at
\[\omega_2^{(2)} = \partial_\mu \left( -H^{\mu} \partial_{[\alpha} \bar{\psi}_{\beta]\]} N^{\alpha\beta} \xi \right) + \gamma \left( H^{\mu} \partial_{[\alpha} \bar{\psi}_{\beta]\]} N^{\alpha\beta} \psi_\mu \right) - 2 \partial_{[\alpha} \bar{\psi}_{\beta]\]} N^{\alpha\beta} \psi_\mu \partial_\nu C^{\mu\nu} + H^{\mu} \partial_\mu \left( \partial_{[\alpha} \bar{\psi}_{\beta]\]} N^{\alpha\beta} \right) \xi, \tag{240}\]
so (231) for $j = 2$ imposes the following restrictions (to be satisfied simultaneously):
\[\partial_{[\alpha} \bar{\psi}_{\beta]\]} N^{\alpha\beta} \psi_\mu = \partial_\mu l, \tag{241}\]
\[\partial_\mu \left( \partial_{[\alpha} \bar{\psi}_{\beta]\]} N^{\alpha\beta} \right) = 0. \tag{242}\]
The former equality, (241), cannot take place. (The Euler-Lagrange derivatives of the right-hand of (241) with respect to \(\psi_\rho\) vanishes obviously, while that of the left-hand side is nonvanishing.) Neither does equation (242) because in the opposite situation the quantity \(\partial_{[\alpha} \bar{\psi}_{\beta]} N^{\alpha\beta}\) should be constant.

The requirement that \(\omega_2^{(2)}\) is of the form (231) for \(j = 2\) implies thus the condition

\[ N^{\alpha\beta} = 0. \quad (243) \]

Integrating now by parts equation (231) for \(j = 3\), it results

\[
\omega_2^{(3)} = \partial^\nu \left[ (\partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} \xi B_{\mu\nu} \right] + \gamma \left[ - (\partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} \psi^\nu B_{\mu\nu} \right] \\
- 3 \left( \partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} \psi^\nu \partial^\eta \eta_{\mu\nu} - \partial^\nu \left( \partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} \right) \xi B_{\mu\nu}, \right.
\]

so \(\omega_2^{(3)}\) reads as in (231) for \(j = 3\) if and only if the next properties are verified simultaneously:

\[
(\partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} \psi^\nu - (\partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} \psi^\mu) = \partial^{[\mu} k^{\nu]}, \quad (245) \\
(\partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} = \partial^\mu k, \quad (246)
\]
with \(k^\mu\) and \(k\) local quantities and \(k\) having in addition a spinor-like behavior. Equations (245) and (246) are incompatible, so the requirement that \(\omega_2^{(3)}\) is of the form (231) for \(j = 3\) reveals the condition

\[ \bar{N}_\mu^{[\alpha\beta} = 0. \quad (247) \]

(The incompatibility between conditions (245) and (246) can be emphasized for instance by taking \(\bar{N}_\mu^{[\alpha\beta}\) to be a 4 x 4 matrix with spinor-like indices that is solution to (246). Since \(k\) is a derivative-free spinor, this solution inserted in the left-hand side of (246) gives

\[
(\partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} \psi^\nu - (\partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} \psi^\mu) = \partial^{[\mu} k^{\nu]} - (\partial^{[\mu} k^{\nu]} - k \left( \partial^{[\mu} \psi^{\nu]} \right) \\
\neq \partial^{[\mu} k^{\nu]}, \quad (248)
\]
which shows that (245) cannot be satisfied and proves thus our assertion.)

We are left now with equation (231) for \(j = 4\). Integrating it by parts, we obtain

\[
\omega_2^{(4)} = \partial_{[\alpha} \left( A_{\beta]} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} \xi - \eta \left( \partial_{[\alpha} \bar{\psi}_{\beta]} \bar{N}_\mu^{[\alpha\beta} \psi^\nu \right) \right]
\]
\[
\begin{align*}
&+ \gamma \left[ A_\mu \left( \partial_\alpha \bar{\psi}_\beta \right) \bar{N}^{\mu\nu|\alpha\beta} \psi_\nu \right] \\
&+ \partial_\mu \left[ \left( \partial_\alpha \bar{\psi}_\beta \right) N^{\mu\nu|\alpha\beta} \right] \left( \psi_\mu \eta - \xi A_\nu \right) \\
&+ \frac{1}{2} \left( \partial_\alpha \bar{\psi}_\beta \right) \left( \bar{N}^{\mu\nu|\alpha\beta} - i \sigma_{\rho\lambda} M^{\rho|\alpha\beta} \gamma^{\lambda\mu\nu} \right) \left( \partial_\mu \psi_\nu \right) \eta. \tag{249}
\end{align*}
\]

From (249) we notice that \( \omega^{(4)} \) can be written like in (231) for \( j = 4 \) if and only if the next formulas hold simultaneously:

\[
\begin{align*}
\left( \partial_\alpha \bar{\psi}_\beta \right) \bar{N}^{\mu\nu|\alpha\beta} &= \partial_\rho K^{\mu\nu\rho}, \tag{250} \\
\left( \gamma^0 \bar{N}^{\mu\nu|\alpha\beta} \right)^\top &= \gamma^0 \bar{N}^{\rho|\alpha\beta\mu\nu}. \tag{251}
\end{align*}
\]

In the above \( K^{\mu\nu\rho} \) has a spinor behavior, is derivative-free, and completely antisymmetric in its Lorentz indices, while \( \bar{N}^{\mu\nu|\alpha\beta} \) is defined via

\[
\bar{N}^{\mu\nu|\alpha\beta} = \bar{N}^{\mu\nu|\alpha\beta} - i \sigma_{\rho\lambda} \bar{M}^{\rho|\alpha\beta\gamma\lambda\mu\nu}. \tag{252}
\]

Equation (250) shows that \( \bar{N}^{\mu\nu|\alpha\beta} \) cannot depend on the scalar field \( \varphi \), being therefore some constant, \( 4 \times 4 \) matrices with spinor-like indices. Under these circumstances, (250) becomes equivalent to

\[
\partial_\rho \left( 2 \bar{\psi}_\lambda \bar{N}^{\mu\nu|\rho\lambda} - K^{\mu\nu\rho} \right) = 0 \tag{253}
\]

and, since the quantity under derivative is derivative-free, (253) produces

\[
2 \bar{\psi}_\lambda \bar{N}^{\mu\nu|\rho\lambda} = K^{\mu\nu\rho}. \tag{254}
\]

Taking into account the complete antisymmetry of \( K^{\mu\nu\rho} \), from the last formula it follows that \( \bar{N}^{\mu\nu|\rho\lambda} \) must be also antisymmetric with respect to its first three indices. On the other hand, \( \bar{N}^{\mu\nu|\rho\lambda} \) is known to be antisymmetric in its last two indices, so they must be fully antisymmetric with respect to their Lorentz indices. In \( D = 4 \) there is a single possibility for the solution of equation (250), namely

\[
\bar{N}^{\mu\nu|\rho\lambda} = \varepsilon^{\mu\nu\rho\lambda} \left( k_1 + i k_2 \gamma_5 \right), \tag{255}
\]

with \( k_1 \) and \( k_2 \) some arbitrary, real constants. In order to solve the last condition, (251), we decompose the matrices \( \bar{N}^{\mu\nu|\alpha\beta} \) according to the basis

\[
\bar{N}^{\mu\nu|\alpha\beta} = \tilde{\hat{n}}^{\mu\nu|\alpha\beta} + \tilde{\hat{n}}^{\mu\nu|\alpha\beta}_\rho \gamma^\rho + \tilde{\hat{n}}^{\mu\nu|\alpha\beta}_\rho \gamma^\rho \gamma^{\lambda\rho\lambda} + \tilde{\hat{n}}^{\mu\nu|\alpha\beta}_\rho \gamma^\rho \gamma^{\rho\sigma} + \tilde{\hat{n}}^{\mu\nu|\alpha\beta}_\gamma \gamma_5, \tag{256}
\]

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where \( \hat{n}^{\mu \nu | \alpha \beta} \), \( \hat{n}_{\rho \lambda} \), \( \hat{n}_{\rho \sigma} \), and \( \tilde{n}^{\mu \nu | \alpha \beta} \) are some Lorentz tensors, completely antisymmetric in their lower indices, displaying the same symmetry/antisymmetry properties like \( \hat{N}^{\mu \nu | \alpha \beta} \) with respect to their upper indices, and depending only on the undifferentiated scalar field. Since in \( D = 4 \) the only constant Lorentz tensors have an even number of indices, the decomposition (250) reduces to

\[
\hat{N}^{\mu \nu | \alpha \beta} = \hat{n}^{\mu \nu | \alpha \beta} + \hat{n}^{\mu \nu | \alpha \beta \rho \lambda} + \tilde{n}^{\mu \nu | \alpha \beta \gamma}. \tag{257}
\]

Asking that (257) satisfies (251) and using properties (10)–(11), it follows that \( \hat{n}^{\mu \nu | \alpha \beta} \), \( \hat{n}^{\mu \nu | \alpha \beta \rho \lambda} \), and \( \tilde{n}^{\mu \nu | \alpha \beta} \) consequently exhibit the symmetry/antisymmetry properties

\[
\hat{n}^{\mu \nu | \alpha \beta} = -\hat{n}^{\alpha \beta | \mu \nu}, \tag{258}
\]
\[
\hat{n}^{\mu \nu | \alpha \beta \rho \lambda} = -\hat{n}^{\alpha \beta \rho \lambda | \mu \nu}, \tag{259}
\]
\[
\tilde{n}^{\mu \nu | \alpha \beta} = -\tilde{n}^{\alpha \beta | \mu \nu}. \tag{260}
\]

Because all the constant Lorentz tensors can be constructed out of the flat metric and the Levi-Civita symbol and we work in \( D = 4 \), it is easy to see that there are no such tensors that fulfill the properties (258)–(260) and hence the solution to (251) is purely trivial

\[
\hat{N}^{\mu \nu | \alpha \beta} = 0. \tag{261}
\]

Replacing now (255) and (261) into (252), we conclude that

\[
i \sigma_{\rho \lambda} \tilde{M}^{\rho | \alpha \beta \gamma \lambda \mu \nu} = \varepsilon^{\mu \nu \rho \lambda} (k_1 + i k_2 \gamma_5). \tag{262}
\]

The last relations allow us to identify the concrete expression of the matrices \( \tilde{M}^{\rho | \alpha \beta} \), which are restricted to be constant. This can be done by decomposing these matrices along the basis (6)

\[
\tilde{M}^{\rho | \alpha \beta} = \tilde{m}^{\rho | \alpha \beta} + \tilde{m}^{\rho | \alpha \beta \gamma \sigma} + \tilde{m}^{\rho | \alpha \beta \gamma \sigma \varepsilon \gamma} + \tilde{m}^{\rho | \alpha \beta \gamma \sigma \varepsilon \gamma 5}, \tag{263}
\]

where \( \tilde{m}^{\rho | \alpha \beta} \), \( \tilde{m}^{\rho | \alpha \beta \gamma \sigma} \), \( \tilde{m}^{\rho | \alpha \beta \gamma \sigma \varepsilon \gamma} \), and \( \tilde{m}^{\rho | \alpha \beta \gamma \sigma \varepsilon \gamma 5} \) are some constant, nonderivative Lorentz tensors, antisymmetric in their lower indices and with the symmetry properties of \( \tilde{M}^{\rho | \alpha \beta} \) with respect to their upper indices. Invoking one more time the even number of Lorentz indices for any constant tensor in \( D = 4 \), only the second and the fourth terms from decomposition (263) will survive,

\[
\tilde{M}^{\rho | \alpha \beta} = \tilde{m}^{\rho | \alpha \beta \gamma \sigma} + \tilde{m}^{\rho | \alpha \beta \gamma \sigma \varepsilon \gamma}. \tag{264}
\]
Their general expressions are
\[ \bar{m}_{\sigma}^{\rho[\alpha\beta}] = \overline{k}_1 \varepsilon^{\rho[\alpha\beta}_\sigma + \bar{k}_2 \sigma^{\rho[\alpha \delta^3}_\sigma, \] \tag{265} \]
\[ \bar{m}_{\sigma\varepsilon\gamma}^{\rho[\alpha\beta]} = -\frac{1}{6} k_3 \varepsilon^{\rho[\alpha\beta}_{\sigma\varepsilon\gamma} + \frac{1}{6} k_4 \delta^{\rho\delta^3}[\epsilon] \delta^{\sigma\delta^3}_\gamma, \] \tag{266} \]
with \( \bar{k} \) some constants. From the Fierz identities
\[ \gamma_\alpha \gamma^{\mu\nu} = \delta^\mu_\alpha \gamma^{\nu}\rho + \gamma_\alpha^{\mu\nu}, \] \tag{267} \]
\[ \gamma_\alpha \beta \gamma^{\mu\nu} = -\delta^\mu_\alpha \delta^\nu_\beta \delta^\sigma_\gamma - \delta^\mu_{[\alpha} \delta^\nu_{\beta] \delta^\gamma_\sigma}, \] \tag{268} \]
the duality relations (8), and formulas (264)–(266), the left-hand side of equation (262) becomes
\[ i \sigma_\rho \lambda M^{\rho[\alpha\beta}\gamma^{\mu\nu} = 2 \varepsilon^{\mu\nu\alpha\beta} (-i \bar{k}_3 + \bar{k}_2 \gamma_5) - 2 \sigma^{\mu[\alpha \sigma^\beta]_\nu} (i \bar{k}_4 + \bar{k}_1 \gamma_5) + \sigma^{\mu[\alpha \gamma^\beta]_\nu} [i (\bar{k}_2 - \bar{k}_4) + (-\bar{k}_1 + \bar{k}_3) \gamma_5] - \sigma^{\nu[\alpha \gamma^\beta]_\mu} [i (\bar{k}_2 - \bar{k}_4) + (-\bar{k}_1 + \bar{k}_3) \gamma_5], \] \tag{269} \]
such that (262) projected on the elements of the basis (6) amounts to three independent equations
\[ -2i \varepsilon^{\mu\nu\alpha\beta} \bar{k}_3 - 2i \sigma^{\mu[\alpha \sigma^\beta]_\nu} \bar{k}_4 = \varepsilon^{\mu\alpha\beta} \bar{k}_1, \] \tag{270} \]
\[ 2 \varepsilon^{\mu\nu\alpha\beta} \bar{k}_2 - 2 \sigma^{\mu[\alpha \sigma^\beta]_\nu} \bar{k}_1 = i \varepsilon^{\mu\alpha\beta} \bar{k}_2, \] \tag{271} \]
\[ (\sigma^{\mu[\alpha \delta^3}_\rho \delta^\lambda_\nu - \sigma^{\mu[\alpha \delta^3}_\rho \delta^\lambda_\nu}) (\bar{k}_2 - \bar{k}_4) - (\sigma^{\mu[\alpha \epsilon^\beta]_\rho \delta^\lambda} - \sigma^{\mu[\alpha \epsilon^\beta]_\rho \delta^\lambda}) (-\bar{k}_1 + \bar{k}_3) = 0. \] \tag{272} \]
The previous relations are nothing but an algebraic system with the unknowns \( k_1, k_2, \bar{k}_1, \bar{k}_2, k_3, \) and \( \bar{k}_4 \):
\[ 2i \bar{k}_3 + k_1 = 0, \quad \bar{k}_4 = 0, \] \tag{273} \]
\[ 2 \bar{k}_2 - ik_2 = 0, \quad \bar{k}_1 = 0, \] \tag{274} \]
\[ \bar{k}_2 - \bar{k}_4 = 0, \quad -\bar{k}_1 + \bar{k}_3 = 0, \] \tag{275} \]
whose solution is purely trivial
\[ k_1 = k_2 = \bar{k}_1 = \bar{k}_2 = \bar{k}_3 = \bar{k}_4 = 0. \] \tag{276} \]
So far, we have shown that \( \omega_2^{(4)} \) satisfies equation (231) for \( j = 4 \) if and only if
\[ \bar{M}^{\rho[\alpha\beta} = \bar{N}^{\mu\nu|\rho\lambda} = 0. \] \tag{277} \]
We are now in a position to prove the assertion that $\tilde{a}_1^{\text{int}}$ can be taken to be trivial. Substituting results \((239), (243), (247),\) and \((277)\) into \((213)\), we identify the component $\tilde{a}_1^{\text{int}}$ of \((121)\), i.e. the solution to \((116)\) that fulfills \((122)\), as

$$\tilde{a}_1^{\text{int}} = H^*_\mu \left( \partial_{[\alpha} \bar{\psi}_{\beta]} \right) \gamma^{\mu \alpha \beta} \left( i \frac{d\hat{U}_7}{d\varphi} + \frac{d\hat{U}_8}{d\varphi} \gamma_5 \right) \xi, \quad (278)$$

From the actions of the differentials $\delta$, $\gamma$, and $s$, it is simple to see that \((278)\) reads as

$$\tilde{a}_1^{\text{int}} = \partial_{\mu} \left[ 2i\psi^\ast \mu \left( i\hat{U}_7 + \hat{U}_8 \gamma_5 \right) \xi \right] + s \left[ 2iH^*_\mu \psi^\ast \mu \left( i \frac{d\hat{U}_7}{d\varphi} + \frac{d\hat{U}_8}{d\varphi} \gamma_5 \right) \xi 
- 2i\xi^\ast \left( i\hat{U}_7 + \hat{U}_8 \gamma_5 \right) \xi \right] + \gamma \left[ -2i\psi^\ast \mu \left( i\hat{U}_7 + \hat{U}_8 \gamma_5 \right) \psi_\mu \right], \quad (279)$$

so, in agreement with \((52)\) and \((58)\), it will produce only trivial deformations, so it can indeed be made to vanish.

**B Proof of formula \((138)\)**

In the following we solve equation \((135)\) and hence proof formula \((138)\). It is advantageous to decompose $\tilde{a}_0^{\text{int}}$ with respect to the number of derivatives into

$$\tilde{a}_0^{\text{int}} = \frac{(0)}{\pi} + \frac{(1)}{\pi}, \quad (280)$$

where $\frac{(0)}{\pi}$ is derivative-free and $\frac{(1)}{\pi}$ comprises a single spacetime derivative of the fields. Again, $\frac{(0)}{\pi}$ and $\frac{(1)}{\pi}$ mix the Rarita-Schwinger field with the BF field sector. Due to \((280)\), equation \((135)\) is clearly equivalent to

$$\gamma^{(0)}_\pi = \partial_{\mu} u_0^{(0)} \mu, \quad (281)$$

$$\gamma^{(1)}_\pi = \partial_{\mu} u_0^{(1)} \mu. \quad (282)$$

Making use of definitions \((39)\) and \((40)\), we infer

$$\gamma^{(0)}_\pi = \frac{\partial R^{(0)}_\pi}{\partial \psi_\mu} \partial_{\mu} \xi + 2 \frac{\partial (0)}{\partial \mu} C^{\mu \nu} + \frac{\partial (0)}{\partial A_\mu} \partial_{\mu} \eta - 3 \frac{\partial (0)}{\partial B^{\mu \nu}} \partial_{\rho} \eta^{\mu \nu \rho}$$

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\[\begin{align*}
&= \partial_\mu \left( \frac{\partial R^{(0)}_\pi}{\partial \psi_\mu} \xi - 2 \frac{\partial R^{(0)}_\pi}{\partial H^\nu} C^{\mu\nu} + \frac{\partial^{(0)}_\pi}{\partial A_\mu} \eta - 3 \frac{\partial^{(0)}_\pi}{\partial B^{\nu\rho}} \eta^{\mu\nu}\right) \\
&\ - \left(\partial_\mu \frac{\partial R^{(0)}_\pi}{\partial \psi_\mu}\right) \xi + \left(\partial_\mu \frac{\partial^{(0)}_\pi}{\partial H^\nu}\right) C^{\mu\nu} - \left(\partial_\mu \frac{\partial^{(0)}_\pi}{\partial A_\mu}\right) \eta \\
&\ + \left(\partial_\mu \frac{\partial^{(0)}_\pi}{\partial B^{\nu\rho}}\right) \eta^{\mu\nu},
\end{align*}\]

so \( \pi \) satisfies (281) if

\[\begin{align*}
&\partial_\mu \frac{\partial R^{(0)}_\pi}{\partial \psi_\mu} = 0, \quad \partial_\mu \frac{\partial^{(0)}_\pi}{\partial H^\nu} = 0, \\
&\partial_\mu \frac{\partial^{(0)}_\pi}{\partial A_\mu} = 0, \quad \partial_\mu \frac{\partial^{(0)}_\pi}{\partial B^{\nu\rho}} = 0.
\end{align*}\]

The solutions to the latter equations from (284) and also to (285) may be trivial, but the former equation from (284) must necessarily provide a nontrivial solution in order to produce cross-interactions. But \( \pi \) is derivative-free, such that the solutions to (284) and (285) read as

\[\begin{align*}
&\frac{\partial R^{(0)}_\pi}{\partial \psi_\mu} = \Psi^\mu, \quad \frac{\partial^{(0)}_\pi}{\partial H^\mu} = h_\mu, \\
&\frac{\partial^{(0)}_\pi}{\partial A_\mu} = a^\mu, \quad \frac{\partial^{(0)}_\pi}{\partial B^{\mu\nu}} = b^{\mu\nu},
\end{align*}\]

where \( \Psi^\mu, h_\mu, a^\mu, \) and \( b^{\mu\nu} \) are all constants. In addition, \( \Psi^\mu \) is a spinor and \( b^{\mu\nu} \) is antisymmetric, \( b^{\mu\nu} = -b^{\nu\mu} \). There are no such constants and hence (284) and (285) possess only the trivial solution, which further implies that the solution to (281) is also trivial,

\[\pi = 0.\]

Regarding equation (282), from definitions (39) and (40) we can write

\[\gamma^{(1)}_\pi = \frac{\partial R^{(1)}_\pi}{\partial \psi_\mu} \partial_\mu \xi + \frac{\partial R^{(1)}_\pi}{\partial \lambda \psi_\mu} \partial_\lambda \partial_\mu \xi + 2 \frac{\partial^{(1)}_\pi}{\partial H^\mu} \partial_\nu C^{\mu\nu} + 2 \frac{\partial^{(1)}_\pi}{\partial (\partial_\lambda H^\mu)} \partial_\lambda \partial_\nu C^{\mu\nu}\]
\[ + \frac{\partial (1)}{\partial A_{\mu}} \partial_\mu \eta + \frac{\partial (1)}{\partial (\partial_{\lambda} A_{\mu})} \partial_{\lambda} \partial_\mu \eta - 3 \frac{\partial (1)}{\partial B_{\mu \nu}} \partial_\rho \eta^{\mu \nu \rho} - 3 \frac{\partial (1)}{\partial (\partial_{\lambda} B_{\mu \nu})} \partial_\lambda \partial_\rho \eta^{\mu \nu \rho} \]

\[ = \partial_\mu \left[ \left( \frac{\partial R (1)}{\partial \psi_{\mu}} - \partial_{\lambda} \frac{\partial R (1)}{\partial (\partial_\mu \psi_{\lambda})} \right) \xi + \frac{\partial R (1)}{\partial (\partial_\lambda \psi_{\mu})} \partial_\lambda \xi - 2 \frac{\partial (1)}{\partial (\partial_\lambda H^\nu)} \partial_\lambda C^{\mu \nu} \right. \]

\[ - 2 \frac{\partial (1)}{\partial H^\nu} C^{\mu \nu} + 2 \left( \partial_\rho \frac{\partial (1)}{\partial (\partial_\mu H^\lambda)} \right) C^{\rho \lambda} + \left( \frac{\partial (1)}{\partial A_{\mu}} - \partial_{\lambda} \frac{\partial (1)}{\partial (\partial_\mu A_{\lambda})} \right) \eta \]

\[ + \frac{\partial (1)}{\partial (\partial_{\lambda} A_{\mu})} \partial_{\lambda} \eta - 3 \left( \frac{\partial (1)}{\partial B^{\nu \rho}} \eta^{\mu \nu \rho} + \frac{\partial (1)}{\partial (\partial_\lambda B^{\nu \rho})} \partial_\lambda \eta^{\mu \nu \rho} \right) \]

\[ + 3 \left( \partial_\nu \frac{\partial (1)}{\partial (\partial_\mu B^{\rho \lambda})} \right) \eta^{\rho \nu \lambda} \right] - \left( \partial_\mu \frac{\delta R (1)}{\delta \psi_{\mu}} \right) \xi + \left( \partial_\mu \frac{\delta R (1)}{\delta H^\nu} \right) C^{\mu \nu} \]

\[ - \left( \partial_\mu \frac{\delta (1)}{\delta A_{\mu}} \right) \eta + \left( \partial_\mu A^{\mu \nu} \right) \eta^{\mu \nu \rho}, \quad (289) \]

so we conclude that \( (1) \pi \) fulfills (282) if the following equations take place:

\[ \partial_\mu \frac{\delta R (1)}{\delta \psi_{\mu}} = 0, \quad \partial_\mu \frac{\delta (1)}{\delta H^\nu} = 0, \quad (290) \]

\[ \partial_\mu \frac{\delta (1)}{\delta A_{\mu}} = 0, \quad \partial_\mu \frac{\delta (1)}{\delta B^{\nu \rho}} = 0. \quad (291) \]

Their solutions are given by

\[ \frac{\delta R (1)}{\delta \psi_{\mu}} = \partial_\lambda \Psi^{\mu \nu}, \quad \frac{\delta (1)}{\delta H^\mu} = \partial_\mu H, \]  

\[ \frac{\delta (1)}{\delta A_{\mu}} = \partial_\nu A^{\mu \nu}, \quad \frac{\delta (1)}{\delta B^{\mu \nu}} = \partial_{[\mu} \bar{B}_{\nu]} \]  

where the objects \( \Psi^{\mu \nu}, H, A^{\mu \nu}, \) and \( B_{\mu} \) are functions (with \( \Psi^{\mu \nu} \) fermionic and spinor-like and the remaining ones bosonic) depending only on the un-differentiated fields, with both \( \Psi^{\mu \nu} \) and \( A^{\mu \nu} \) antisymmetric

\[ \Psi^{\mu \nu} = -\Psi^{\nu \mu}, \quad A^{\mu \nu} = -A^{\nu \mu}. \quad (294) \]
Before solving equations (292) and (293), we will show that the dependence of $\pi^{(1)}$ on $H^\mu$, $A_\mu$, and $B^{\mu\nu}$ can be eliminated through some trivial deformations.

Let $N$ be a derivation in the algebra of the fields $\psi_\mu$, $H^\mu$, $A_\mu$, and $B^{\mu\nu}$ and of their derivatives that counts the powers of the fields and their derivatives

$$
N = \sum_{k \geq 0} \left( \frac{\partial^L}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_k} \psi_\mu)} + \frac{\partial \mu_1 \cdots \partial H^\mu}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_k} H^\mu)} + \frac{\partial (\partial_{\mu_1} \cdots \partial A_\mu)}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_k} A_\mu)} + \frac{\partial (\partial_{\mu_1} \cdots \partial B^{\mu\nu})}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_k} B^{\mu\nu})} \right). \tag{295}
$$

Then, it is easy to see that for every nonintegrated density $\phi$ we have

$$
N\phi = \frac{\delta R \phi}{\delta \psi_\mu} \psi_\mu + \frac{\delta \phi}{\delta H^\mu} H^\mu + \frac{\delta \phi}{\delta A_\mu} A_\mu + \frac{\delta \phi}{\delta B^{\mu\nu}} B^{\mu\nu} + \partial \mu t^\mu, \tag{296}
$$

where $\delta^{(R)} \phi / \delta \Phi^{\alpha_0}$ denotes the (right) variational derivative of $\phi$ with respect to the field $\Phi^{\alpha_0}$. If $\phi^{(l)}$ is a homogeneous polynomial of degree $l$ in the fields $\psi_\mu$, $H^\mu$, $A_\mu$, $B^{\mu\nu}$ and their derivatives, then $N\phi^{(l)} = l\phi^{(l)}$. Inserting (292) and (293) in (296) for $\phi = \pi^{(1)}$ and moving the derivatives such to act on the fields, we infer

$$
N^{(1)} \pi = \frac{1}{2} \Psi^{\mu\nu} \partial_{[\mu} \psi_{\nu]} - H \partial_{\mu} H^\mu + \frac{1}{2} A^{\mu\nu} \partial_{[\mu} A_{\nu]} + 2 B^{\mu\nu} \partial_{\mu} B_{\nu\mu} + \partial_{\mu} t^\mu. \tag{297}
$$

Decomposing $\pi^{(1)}$ as a sum of homogeneous polynomials of various degrees in the fields and their derivatives

$$
\pi^{(1)} = \sum_{k \geq 2} \pi^{(1)(k)}, \tag{298}
$$

such that $N^{(1)(k)} = k^{(1)(k)}$ (with $k \geq 2$ since $\pi^{(1)}$ contains at least two spinors), we deduce

$$
N^{(1)} \pi = \sum_{k \geq 2} k^{(1)(k)}. \tag{299}
$$

Comparing (299) with (297), we conclude that decomposition (298) induces a similar decomposition at the level of the functions $\Psi^{\mu\nu}$, $H$, $A^{\mu\nu}$, and $B_\mu$, i.e.

$$
\Psi^{\mu\nu} = \sum_{k \geq 2} \Psi^{\mu\nu}_{(k-1)}, \quad H = \sum_{k \geq 2} H_{(k-1)}, \tag{300}
$$

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\[
A^{\mu\nu} = \sum_{k \geq 2} A^{\mu\nu}_{(k-1)}, \quad B^{\mu} = \sum_{k \geq 2} B^{\mu}_{(k-1)}.
\] (301)

Substituting (300) and (301) in (297) and then comparing the result with (299), we find that

\[
(1)^{\frac{(k)}{\pi}} = \frac{1}{2k} \Psi^{\mu\nu}_{(k-1)} \partial_{[\mu} \psi_{\nu]} - \frac{1}{k} H_{(k-1)} \partial_{\mu} H^{\mu} + \frac{1}{2} A^{\mu\nu}_{(k-1)} \partial_{[\mu} A_{\nu]} + \frac{2}{k} B^{\mu}_{(k-1)} \partial^{\rho} B_{\mu\nu} + \partial_{\mu} \hat{t}^{\mu},
\] (302)

which further inserted in (298) provides \( (1)^{\frac{(k)}{\pi}} \) as

\[
(1)^{\frac{1}{\pi}} = \frac{1}{2} \hat{\Psi}^{\mu\nu} \partial_{[\mu} \psi_{\nu]} - \hat{H} \partial_{\mu} H^{\mu} + \frac{1}{2} \hat{A}^{\mu\nu} \partial_{[\mu} A_{\nu]} + 2 \hat{B}^{\mu} \partial^{\rho} B_{\mu\nu} + \partial_{\mu} \hat{t}^{\mu},
\] (303)

where we made the notations

\[
\hat{\Psi}^{\mu\nu} = \sum_{k \geq 21} \frac{1}{k} \Psi^{\mu\nu}_{(k-1)}, \quad \hat{H} = \sum_{k \geq 2} \frac{1}{k} H_{(k-1)},
\] (304)

\[
\hat{A}^{\mu\nu} = \sum_{k \geq 2} \frac{1}{k} A^{\mu\nu}_{(k-1)}, \quad \hat{B}^{\mu} = \sum_{k \geq 2} \frac{1}{k} B^{\mu}_{(k-1)}.
\] (305)

As a consequence, we succeeded in bringing the solution \( (1)^{\frac{1}{\pi}} \) of equation (282) to the form (303). On the one hand, it is direct to see (from the former definition in (33), definitions (34), and formula (303)) that all the terms from \( (1)^{\frac{1}{\pi}} \) but the first vanish on-shell modulo \( d \), and therefore they can be written in a \( \delta \)-exact modulo \( d \) form

\[
(1)^{\frac{1}{\pi}} = \frac{1}{2} \hat{\Psi}^{\mu\nu} \partial_{[\mu} \psi_{\nu]} + \delta \left( -\varphi^{\ast} \hat{H} - B^{\ast}_{\mu\nu} \hat{A}^{\mu\nu} - 2 A^{\ast}_{\mu} \hat{B}^{\mu} \right) + \partial_{\mu} \hat{t}^{\mu}.
\] (306)

On the other hand, equation (282) shows that \( (1)^{\frac{1}{\pi}} \) belongs to \( H^{0} (\gamma|d) \) in antighost number zero. Using the general result [42] according to which the elements of \( H (\gamma|d) \) independent of antifields are nontrivial elements of \( H (s|d) \) if and only if they do not vanish on-shell modulo \( d \), we can state that all the terms from (303) excepting the first one are trivial elements of \( H (s|d) \), so they can be eliminated by trivial redefinitions of the fields. In
conclusion, the entire dependence of \( \pi \) on \( H^\mu, A_\mu, \) and \( B^{\mu\nu} \) can be removed, which proves our statement from the end of the previous paragraph.

Now, we come back to completing \( \pi \) as solution to (282). By virtue of the above discussion, we can state that the general form of the nontrivial candidate to the solution of equation (282) can be chosen under the form

\[
\pi = \frac{1}{2} \hat{\Psi}^{\mu\nu} \partial_{[\mu} \psi_{\nu]} + \partial_\mu \bar{t}^\mu,
\]

where \( \hat{\Psi}^{\mu\nu} \) is antisymmetric, \( \hat{\Psi}^{\mu\nu} = -\hat{\Psi}^{\nu\mu} \), spinor-like, derivative-free (due to the derivative order assumption) and depends effectively on the undifferentiated scalar field \( \varphi \) (because otherwise \( \pi \) would not describe cross-couplings, but only self-interactions in the Rarita-Schwinger sector). It is convenient to represent \( \hat{\Psi}^{\mu\nu} \) like

\[
\hat{\Psi}^{\mu\nu} = \bar{\psi}_\rho \hat{\Psi}^{\mu\nu|\rho},
\]

where \( \hat{\Psi}^{\mu\nu|\rho} \) are \( 4 \times 4 \) matrices with spinor-like indices, whose elements are bosonic functions depending on the undifferentiated fields \( \psi_\lambda \) and \( \varphi \). According to the chosen basis (10), \( \hat{\Psi}^{\mu\nu|\rho} \) decomposes as

\[
\hat{\Psi}^{\mu\nu|\rho} = \tilde{\Psi}^{\mu\nu|\rho}_\alpha \gamma^\alpha + \tilde{\Psi}^{\mu\nu|\rho}_{\alpha\beta} \gamma^{\alpha\beta} + \tilde{\Psi}^{\mu\nu|\rho}_{\alpha\beta\gamma} \gamma^{\alpha\beta\gamma} + \bar{\Psi}^{\mu\nu|\rho}_\gamma \gamma_5,
\]

where \( \tilde{\Psi}^{\mu\nu|\rho}_\alpha \), \( \tilde{\Psi}^{\mu\nu|\rho}_{\alpha\beta} \), \( \tilde{\Psi}^{\mu\nu|\rho}_{\alpha\beta\gamma} \), and \( \bar{\Psi}^{\mu\nu|\rho}_\gamma \) are some bosonic, Lorentz tensors constructed out of \( \psi_\lambda \) and \( \varphi \), separately antisymmetric in their lower indices and in the upper pair \( \{\mu, \nu\} \) respectively. From (309) and properties (10)–(11), (307) can be expressed as

\[
\begin{align*}
\gamma (1) \pi = \frac{1}{2} & \left[ (\partial_{[\mu} \bar{\psi}_{\nu]}) \psi_\rho \tilde{\Psi}^{\mu\nu|\rho} - (\partial_{[\mu} \bar{\psi}_{\nu]}) \gamma^{\alpha} \psi_\rho \tilde{\Psi}^{\mu\nu|\rho}_\alpha - (\partial_{[\mu} \bar{\psi}_{\nu]}) \gamma^{\alpha\beta} \psi_\rho \tilde{\Psi}^{\mu\nu|\rho}_{\alpha\beta} \right. \\
& + (\partial_{[\mu} \bar{\psi}_{\nu]}) \gamma^{\alpha\beta\gamma} \psi_\rho \tilde{\Psi}^{\mu\nu|\rho}_{\alpha\beta\gamma} + (\partial_{[\mu} \bar{\psi}_{\nu]}) \gamma_5 \psi_\rho \tilde{\Psi}^{\mu\nu|\rho}_\gamma + \left. \partial_\mu \bar{t}^\mu \right].
\end{align*}
\]

By applying the differential \( \gamma \) on (310), we get

\[
\gamma (1) \pi = \frac{1}{2} \left[ \bar{\Psi}^{\mu\nu|\rho}_\lambda (\partial_{[\mu} \bar{\psi}_{\nu]}) \gamma^{\rho\alpha} - \bar{\Psi}^{\mu\nu|\rho}_\alpha (\partial_{[\mu} \bar{\psi}_{\nu]}) \gamma^{\rho\alpha} - \bar{\Psi}^{\mu\nu|\rho}_{\alpha\beta} (\partial_{[\mu} \bar{\psi}_{\nu]}) \gamma^{\rho\alpha\beta} \right. \\
+ \bar{\Psi}^{\mu\nu|\rho}_{\alpha\beta\gamma} (\partial_{[\mu} \bar{\psi}_{\nu]}) \gamma^{\rho\alpha\beta\gamma} + \bar{\Psi}^{\mu\nu|\rho}_\gamma (\partial_{[\mu} \bar{\psi}_{\nu]}) \gamma_5 + (\partial_{[\mu} \bar{\psi}_{\nu]}) \psi_\lambda \frac{\partial \bar{R} \bar{\Psi}^{\mu\nu|\rho}_\lambda}{\partial \psi_\rho} - (\partial_{[\mu} \bar{\psi}_{\nu]}) \gamma^{\rho\alpha} \psi_\lambda \frac{\partial \bar{R} \bar{\Psi}^{\mu\nu|\rho}_\alpha}{\partial \psi_\rho} - (\partial_{[\mu} \bar{\psi}_{\nu]}) \psi_\lambda \frac{\partial \bar{R} \bar{\Psi}^{\mu\nu|\rho}_{\alpha\beta}}{\partial \psi_\rho}.
\]
such that equation (282) is fulfilled if
\[ \partial_\rho \left[ \tilde{\Psi}^{\mu|\rho} \left( \partial_\mu \bar{\psi}_\nu \right) - \tilde{\Psi}^\alpha_{\alpha|\rho} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^\alpha - \tilde{\Psi}^{\mu|\rho}_{\alpha\beta} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^{\alpha\beta} \right. \]
\[ + \tilde{\Psi}^{\mu|\rho}_{\alpha\beta\gamma} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^{\alpha\beta\gamma} + \tilde{\Psi}^{\mu|\rho} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma_5 + \left( \partial_\mu \bar{\psi}_\nu \right) \psi_\lambda \frac{\partial R \tilde{\Psi}^{\mu|\lambda}}{\partial \psi_\rho} \]
\[ \left. - \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^\alpha \psi_\lambda \frac{\partial R \tilde{\Psi}^{\mu|\lambda}_\alpha}{\partial \psi_\rho} - \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^{\alpha\beta} \psi_\lambda \frac{\partial R \tilde{\Psi}^{\mu|\lambda}_{\alpha\beta}}{\partial \psi_\rho} \right) = 0. \] (312)

The above equation implies the existence of an antisymmetric spinor-tensor \( \Phi^{\alpha\lambda} \), \( \Phi^{\alpha\lambda} = -\Phi^{\lambda\alpha} \), depending on \( \varphi \) and \( \psi_\mu \), in terms of which
\[ \tilde{\Psi}^{\mu|\rho} \left( \partial_\mu \bar{\psi}_\nu \right) - \tilde{\Psi}^\alpha_{\alpha|\rho} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^\alpha - \tilde{\Psi}^{\mu|\rho}_{\alpha\beta} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^{\alpha\beta} \]
\[ + \tilde{\Psi}^{\mu|\rho}_{\alpha\beta\gamma} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^{\alpha\beta\gamma} + \tilde{\Psi}^{\mu|\rho} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma_5 + \left( \partial_\mu \bar{\psi}_\nu \right) \psi_\lambda \frac{\partial R \tilde{\Psi}^{\mu|\lambda}}{\partial \psi_\rho} \]
\[ - \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^\alpha \psi_\lambda \frac{\partial R \tilde{\Psi}^{\mu|\lambda}_\alpha}{\partial \psi_\rho} - \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^{\alpha\beta} \psi_\lambda \frac{\partial R \tilde{\Psi}^{\mu|\lambda}_{\alpha\beta}}{\partial \psi_\rho} \right) = \partial_\lambda \Phi^{\alpha\lambda}. \] (313)

In order to analyze equation (313), we compute its Euler-Lagrange derivative with respect to \( \varphi \) and deduce the necessary condition
\[ \delta \frac{\partial}{\partial \varphi} \left[ \tilde{\Psi}^{\mu|\rho} \left( \partial_\mu \bar{\psi}_\nu \right) - \tilde{\Psi}^\alpha_{\alpha|\rho} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^\alpha - \tilde{\Psi}^{\mu|\rho}_{\alpha\beta} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^{\alpha\beta} \right. \]
\[ + \tilde{\Psi}^{\mu|\rho}_{\alpha\beta\gamma} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^{\alpha\beta\gamma} + \tilde{\Psi}^{\mu|\rho} \left( \partial_\mu \bar{\psi}_\nu \right) \gamma_5 + \left( \partial_\mu \bar{\psi}_\nu \right) \psi_\lambda \frac{\partial R \tilde{\Psi}^{\mu|\lambda}}{\partial \psi_\rho} \]
\[ \left. - \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^\alpha \psi_\lambda \frac{\partial R \tilde{\Psi}^{\mu|\lambda}_\alpha}{\partial \psi_\rho} - \left( \partial_\mu \bar{\psi}_\nu \right) \gamma^{\alpha\beta} \psi_\lambda \frac{\partial R \tilde{\Psi}^{\mu|\lambda}_{\alpha\beta}}{\partial \psi_\rho} \right) = 0. \] (314)
Since $\hat{\Psi}^{\mu|\nu|\rho}$ are bosonic, they can be decomposed as sums of homogeneous polynomials of various, even degrees in the Rarita-Schwinger spinors, so it is enough to analyze equation (314) for a fixed value of this degree, say $2p$. Consequently, all the components from (309) (namely $\tilde{\Psi}^{\mu|\nu|\rho}$, $\tilde{\Psi}^{\mu|\nu|\alpha\beta}$, $\tilde{\Psi}^{\mu|\nu|\rho\alpha\beta\gamma}$, and $\bar{\Psi}^{\mu|\nu|\rho}$) will display the same degree with respect to $\psi^\rho$. Multiplying (314) at the right with $\psi^\rho$ and using (308), (309), and the homogeneity assumption, we finally find

$$ (2p + 1) \frac{\delta}{\delta \varphi} \left( \hat{\Psi}^{\mu\nu} \partial_{\mu} \psi_{\nu} \right) = 0, $$

which, according to (307), indicates that $\pi^{(1)}$ cannot depend nontrivially on the scalar field, and therefore it describes only self-interactions in the Rarita-Schwinger sector, so it can be made to vanish

$$ \pi^{(1)} = 0. $$

Based on results (288) and (316), from (280) we obtain precisely (138).

C  Gauge generators, commutators, and reducibility of the coupled model in case I

From the terms of antighost number one present in (176), we read the nonvanishing gauge generators of the coupled model (written in De Witt condensed notations) as

$$ (\tilde{Z}^{(1)}_{(A)}) = (Z^\mu_{(A)}) = \partial^\mu, \quad (\tilde{Z}^{(1)}_{(H)})_{\alpha\beta} = -D_{[\alpha} \delta^\mu_{\beta]}, \quad (317) $$

$$ (\tilde{Z}^{(1)}_{(H)}) = \lambda \left( \frac{1}{2} \frac{dU_1}{d\varphi} \bar{\psi}_\nu \gamma^{\mu\nu\rho} \gamma_5 \psi_\rho - \frac{dW}{d\varphi} H^\mu \right), \quad (318) $$

$$ (\tilde{Z}^{(1)}_{(H)})_A = \lambda \frac{dU_1}{d\varphi} A_\mu \left( \bar{\psi}_\nu \gamma^{\mu\nu\rho} \gamma_5 \psi_\rho \right)_A, \quad (\tilde{Z}^{(1)}_{(\varphi)}) = \lambda W, \quad (319) $$

$$ (\tilde{Z}^{(1)}_{(B)})_{\alpha\beta\gamma} = -\frac{1}{2} \partial_{[\alpha} \delta^\mu_{\beta]} \delta^\nu_{\gamma]}, \quad (\tilde{Z}^{(1)}_{(B)})_{\alpha\beta} = \lambda W \delta^\mu_{[\alpha} \delta^\nu_{\beta]}, \quad (320) $$

$$ (\tilde{Z}^{(1)}_{(B)})_A = -\lambda U_1 \left( \bar{\psi}_\rho \gamma^{\mu\nu\rho} \gamma_5 \right)_A, \quad (321) $$

$$ (\tilde{Z}^{(1)}_{(\psi)}) = -i \lambda U_1 \left( \gamma_5 \right)_A B \psi^{B\mu}, \quad (322) $$

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\[
(\tilde{Z}^{(1)\mu})_B = \left( \delta_B \partial^\mu + i\lambda U_1 (\gamma_5)^A B A^\mu \right),
\]
(323)

The nonvanishing commutators among the gauge transformations of the coupled model result from the pieces in \([176]\) that are quadratic in the ghosts of pure ghost number one and take the form

\[
(\tilde{Z}^{(1)}_{(\psi)}) \frac{\delta(\tilde{Z}^{(1)\mu})_{\alpha\beta}}{\delta \varphi} + (\tilde{Z}^{(1)}_{(A)}) \frac{\delta(\tilde{Z}^{(1)\mu})_{\alpha\beta}}{\delta A^\rho} - (\tilde{Z}^{(1)}_{(H)}) \frac{\delta(\tilde{Z}^{(1)\mu})_{\alpha\beta}}{\delta H^\rho} = \lambda \frac{dW}{d\varphi} (\tilde{Z}^{(1)\mu})_{\alpha\beta} + \lambda \frac{d^2 W}{d\varphi d^2 \delta_{\alpha\beta}} \delta H^\rho,
\]
(324)

\[
(\tilde{Z}^{(1)}_{(\psi)}) \frac{\delta(\tilde{Z}^{(1)\mu\nu})_{\alpha\beta}}{\delta \varphi} = \lambda \frac{dW}{d\varphi} (\tilde{Z}^{(1)\mu\nu})_{\alpha\beta},
\]
(325)

\[
(\tilde{Z}^{(1)}_{(\psi)}) \frac{\delta(\tilde{Z}^{(1)\mu})_{\alpha\beta}}{\delta \varphi} - (\tilde{Z}^{(1)}_{(A)}) \frac{\delta(\tilde{Z}^{(1)\mu})_{\alpha\beta}}{\delta A^\rho} + (\tilde{Z}^{(1)}_{(H)}) \frac{\delta(\tilde{Z}^{(1)\mu})_{\alpha\beta}}{\delta H^\rho}
+ \frac{\delta R(\tilde{Z}^{(1)\mu})_{\alpha\beta}}{\delta \psi^\rho} (\tilde{Z}^{(1)}_{(\psi)}) - \frac{\delta R(\tilde{Z}^{(1)\mu})_{\alpha\beta}}{\delta \psi^\rho} (\tilde{Z}^{(1)}_{(\psi)}),
\]
(326)

\[
= i\lambda U_1 (\tilde{Z}^{(1)\mu})_{(H)B} (\gamma_5)^B_A - \lambda \frac{dU_1}{d\varphi} (\tilde{Z}^{(1)\mu})_{(H)A} - \lambda \frac{d^2 U_1}{d\varphi^2} \frac{\delta S^{(1)}}{\delta H^\rho} (\tilde{Z}^{(1)\mu})_{(H)A}
+ \lambda \frac{d^2 U_1}{d\varphi^2} \frac{\delta S^{(1)}}{\delta H^\rho} (\tilde{Z}^{(1)\mu})_{(H)A},
\]
(327)

\[
\frac{\delta(\tilde{Z}^{(1)\mu})_{B}}{\delta \varphi} (\tilde{Z}^{(1)}_{(\psi)}) + \frac{\delta(\tilde{Z}^{(1)\mu})_{B}}{\delta A^\rho} (\tilde{Z}^{(1)}_{(A)}) - \frac{\delta R(\tilde{Z}^{(1)\mu})_{B}}{\delta \psi^\rho} (\tilde{Z}^{(1)}_{(\psi)}).
\]
(328)

\[
= i\lambda U_1 (\tilde{Z}^{(1)\mu})_{(H)A} (\gamma_5)^C_D - \lambda \frac{dU_1}{d\varphi} \frac{\delta S^{(1)}}{\delta H^\rho} (\tilde{Z}^{(1)\mu})_{(H)A},
\]
(329)
\[
\begin{align*}
\frac{\delta R(\tilde{Z}_{(B)}^{(1)\mu\nu})}{\delta \psi^C B} (\tilde{Z}_{(\psi)}^{(1)C})_A + \frac{\delta R(\tilde{Z}_{(B)}^{(1)\mu\nu})}{\delta \psi^C B} (\tilde{Z}_{(\psi)}^{(1)C})_B &= \lambda \frac{dU_1}{d\varphi} \tilde{S}(I) (\gamma^0 \gamma_{\mu \nu B} \gamma_5)_{AB} + \frac{\lambda}{3} U_1 (\tilde{Z}_{(B)}^{(1)\mu\nu})_{\alpha \beta \gamma} (\gamma^0 \gamma_{\alpha \beta \gamma} \gamma_5)_{AB}. 
\end{align*}
\]

The structure of the terms linear in the ghosts with pure ghost number two or three from \((176)\) shows that some of the reducibility functions are modified with respect to the free theory and, moreover, some of the reducibility relations only hold on-shell. From the analysis of these terms we infer the first-order reducibility functions

\[
\begin{align*}
(\tilde{Z}_{(1)}^{(1)\alpha \beta})_{\mu \nu \rho} &= -\frac{1}{2} D_{[\mu} \delta^{\alpha}_{\nu} \delta^{\beta}_{\rho]}, \\
(\tilde{Z}_{(1)}^{(1)\alpha \beta \gamma})_{\mu \nu \rho} &= -\frac{1}{3} \lambda W \delta^{\alpha}_{[\mu} \delta^{\beta}_{\nu} \delta^{\gamma}_{\rho]}, \\
(\tilde{Z}_{(1)}^{(1)\alpha \beta \gamma})_{\mu \nu \rho \lambda} &= -\frac{1}{6} \partial_{[\mu} \delta^{\alpha}_{\nu} \delta^{\beta}_{\rho} \delta^{\gamma}_{\lambda]},
\end{align*}
\]

and respectively the second-order ones

\[
\begin{align*}
(\tilde{Z}_{(2)}^{(1)\mu \nu \rho})_{\alpha \beta \gamma \delta} &= -\frac{1}{6} D_{[\alpha} \delta^{\mu}_{\beta} \delta^{\nu}_{\gamma} \delta^{\rho}_{\delta]}, \\
(\tilde{Z}_{(2)}^{(1)\mu \nu \rho \lambda})_{\alpha \beta \gamma \delta} &= \frac{1}{12} \lambda W \delta^{\mu}_{[\alpha} \delta^{\nu}_{\beta} \delta^{\rho}_{\gamma} \delta^{\lambda}_{\delta]}.
\end{align*}
\]

Finally, the associated first- and second-order reducibility relations are listed below:

\[
\begin{align*}
(\tilde{Z}_{(1)}^{(1)\mu})_{\alpha \beta} (\tilde{Z}_{(1)}^{(1)\alpha \beta})_{\nu \rho \lambda} &= -\lambda \frac{d^2 W}{d\varphi^2} A_{[\nu} \delta^{\mu}_{\rho} \delta^{\dot{S}}(I)_{\delta H \lambda]} - 2\lambda \frac{dW}{d\varphi} \delta^{\mu}_{[\nu} \delta^{\dot{S}}(I)_{\delta B \rho \lambda]}, \\
(\tilde{Z}_{(B)}^{(1)\mu \nu})_{\alpha \beta} (\tilde{Z}_{(B)}^{(1)\alpha \beta})_{\rho \lambda \sigma} + (\tilde{Z}_{(B)}^{(1)\mu \nu})_{\alpha \beta \gamma} (\tilde{Z}_{(1)}^{(1)\alpha \beta \gamma})_{\rho \lambda \sigma} &= \lambda \frac{dW}{d\varphi} \delta^{\mu}_{[\nu} \delta^{\dot{S}}(I)_{\delta H \sigma]}, \\
(\tilde{Z}_{(B)}^{(1)\mu \nu})_{\alpha \beta \gamma} (\tilde{Z}_{(1)}^{(1)\alpha \beta \gamma})_{\rho \lambda \sigma} &= 0, \\
(\tilde{Z}_{(1)}^{(1)\alpha \beta})_{\mu \nu \rho} (\tilde{Z}_{(1)}^{(1)\mu \nu \rho})_{\gamma \delta \epsilon} &= \lambda \frac{d^2 W}{d\varphi^2} A_{[\gamma} \delta^{\alpha}_{\delta} \delta^{\beta}_{\epsilon} \delta^{\dot{S}}(I)_{\delta H \epsilon]} - \lambda \frac{dW}{d\varphi} \delta^{\alpha}_{[\gamma} \delta^{\beta}_{\delta} \delta^{\dot{S}}(I)_{\delta B \epsilon]}, \\
(\tilde{Z}_{(1)}^{(1)\alpha \beta \gamma})_{\mu \nu \rho} (\tilde{Z}_{(1)}^{(1)\mu \nu \rho})_{\delta \epsilon \zeta} + (\tilde{Z}_{(1)}^{(1)\alpha \beta \gamma})_{\mu \nu \rho \lambda} (\tilde{Z}_{(1)}^{(1)\mu \nu \rho \lambda})_{\delta \epsilon \zeta} &= \lambda \frac{dW}{3 d\varphi} \delta^{\alpha}_{[\gamma} \delta^{\beta}_{\delta} \delta^{\beta}_{\epsilon} \delta^{\dot{S}}(I)_{\delta H \xi]}.
\end{align*}
\]
References

[1] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Topological field theory, *Phys. Rept.* **209** (1991) 129–340.

[2] J. M. F. Labastida and C. Losano, Lectures on topological QFT, in *Proceedings of La Plata-CERN-Santiago de Compostela Meeting on Trends in Theoretical Physics, La Plata, Argentina, April-May 1997*, eds. H. Falomir, R. E. Gamboa Saraví, F. A. Schaposnik (AIP, New York 1998), AIP Conference Proceedings vol. **419**, 54.

[3] P. Schaller and T. Strobl, Poisson structure induced (topological) field theories, *Mod. Phys. Lett.* **A9** (1994) 3129–3136 [arXiv:hep-th/9405110].

[4] N. Ikeda, Two-dimensional gravity and nonlinear gauge theory, *Annals Phys.* **235** (1994) 435–464 [arXiv:hep-th/9312059].

[5] A. Yu. Alekseev, P. Schaller and T. Strobl, Topological $G/G$ WZW model in the generalized momentum representation, *Phys. Rev.* **D52** (1995) 7146–7160 [arXiv:hep-th/9505012].

[6] T. Klösch and T. Strobl, Classical and quantum gravity in 1+1 dimensions: I. A unifying approach, *Class. Quantum Grav.* **13** (1996) 965–983 [arXiv:gr-qc/9508020]; Erratum-ibid. **14** (1997) 825.

[7] T. Klösch and T. Strobl, Classical and quantum gravity in 1+1 dimensions. II: The universal coverings, *Class. Quantum Grav.* **13** (1996) 2395–2421 [arXiv:gr-qc/9511081].

[8] T. Klösch and T. Strobl, Classical and quantum gravity in 1+1 dimensions: III. Solutions of arbitrary topology, *Class. Quantum Grav.* **14** (1997) 1689–1723 [arXiv:hep-th/9607226].

[9] A. S. Cattaneo and G. Felder, A path integral approach to the Kontsevich quantization formula, *Commun. Math. Phys.* **212** (2000) 591–611 [arXiv:math/9902090].

[10] A. S. Cattaneo and G. Felder, Poisson sigma models and deformation quantization, *Mod. Phys. Lett.* **A16** (2001) 179–189 [arXiv:hep-th/0102208].
[11] C. Teitelboim, Gravitation and hamiltonian structure in two spacetime dimensions, *Phys. Lett.* **B126** (1983) 41–45.

[12] R. Jackiw, Lower dimensional gravity, *Nucl. Phys.* **B252** (1985) 343–356.

[13] M. O. Katanaev and I. V. Volovich, String model with dynamical geometry and torsion, *Phys. Lett.* **B175** (1986) 413–416.

[14] J. Brown, *Lower Dimensional Gravity*, World Scientific, Singapore 1988.

[15] M. O. Katanaev and I. V. Volovich, Two-dimensional gravity with dynamical torsion and strings, *Annals Phys.* **197** (1990) 1–32.

[16] H.-J. Schmidt, Scale-invariant gravity in two dimensions, *J. Math. Phys.* **32** (1991) 1562.

[17] S. N. Solodukhin, Topological 2D Riemann-Cartan-Weyl gravity, *Class. Quantum Grav.* **10** (1993) 1011–1021.

[18] N. Ikeda and K. I. Izawa, General form of dilaton gravity and nonlinear gauge theory, *Prog. Theor. Phys.* **90** (1993) 237–246 [arXiv:hep-th/9304012].

[19] T. Strobl, Dirac quantization of gravity-Yang-Mills systems in 1+1 dimensions, *Phys. Rev.* **D50** (1994) 7346–7350 [arXiv:hep-th/9403121].

[20] D. Grumiller, W. Kummer and D. V. Vassilevich, Dilaton gravity in two dimensions, *Phys. Rept.* **369** (2002) 327–430 [arXiv:hep-th/0204253].

[21] T. Strobl, *Gravity in two space-time dimensions*, Habilitation thesis RWTH Aachen, May 1999, [arXiv:hep-th/0011240].

[22] K. Ezawa, Ashtekar’s Formulation for $N = 1,2$ Supergravities as “constrained” BF theories, *Prog. Theor. Phys.* **95** (1996) 863–882 [arXiv:hep-th/9511047].

[23] L. Freidel, K. Krasnov and R. Puzio, BF description of higher-dimensional gravity theories, *Adv. Theor. Math. Phys.* **3** (1999) 1289–1324 [arXiv:hep-th/9901069].
[24] L. Smolin, Holographic formulation of quantum general relativity, Phys. Rev. D61 (2000) 084007 [arXiv:hep-th/9808191].

[25] Y. Ling and L. Smolin, Holographic formulation of quantum supergravity, Phys. Rev. D63 (2001) 064010 [arXiv:hep-th/0009018].

[26] K.-I. Izawa, On nonlinear gauge theory from a deformation theory perspective, Prog. Theor. Phys. 103 (2000) 225–228 [arXiv:hep-th/9910133].

[27] C. Bizdadea, Note on two-dimensional nonlinear gauge theories, Mod. Phys. Lett. A15 (2000) 2047–2055 [arXiv:hep-th/0201059].

[28] N. Ikeda, A deformation of three dimensional BF theory, J.High Energy Phys. JHEP11(2000)009 [arXiv:hep-th/0001096].

[29] N. Ikeda, Deformation of BF theories, topological open membrane and a generalization of the star deformation, J. High Energy Phys. JHEP07(2001)037 [arXiv:hep-th/0105286].

[30] C. Bizdadea, E. M. Cioroianu and S. O. Saliu, Hamiltonian cohomological derivation of four-dimensional nonlinear gauge theories, Int. J. Mod. Phys. A17 (2002) 2191–2210 [arXiv:hep-th/0206186].

[31] C. Bizdadea, C. C. Ciobîrcă, E. M. Cioroianu, S. O. Saliu and S. C. Săraru, Hamiltonian BRST deformation of a class of n-dimensional BF-type theories, J. High Energy Phys. JHEP01(2003)049 [arXiv:hep-th/0302037].

[32] E. M. Cioroianu and S. C. Săraru, PT-symmetry breaking Hamiltonian interactions in BF models, Int. J. Mod. Phys. A21 (2006) 2573–2599 [arXiv:hep-th/0606164].

[33] C. Bizdadea, E. M. Cioroianu, S. O. Saliu and S. C. Săraru, Couplings of a collection of BF models to matter theories, Eur. Phys. J. C41 (2005) 401–420 [arXiv:hep-th/0508037].

[34] N. Ikeda, Chern-Simons gauge theory coupled with BF theory, Int. J. Mod. Phys. A18 (2003) 2689–2702 [arXiv:hep-th/0203043].
[35] E. M. Cioroianu and S. C. Săraru, Two-dimensional interactions between a BF-type theory and a collection of vector fields, *Int. J. Mod. Phys.* **A19** (2004) 4101–4125 [arXiv:hep-th/0501056].

[36] C. Bizdadea, E. M. Cioroianu, I. Negru, S. O. Saliu and S. C. Săraru, On the generalized Freedman-Townsend model, *J. High Energy Phys.* JHEP**10**(2006)004 [arXiv:0704.3407].

[37] G. Barnich and M. Henneaux, Consistent couplings between fields with a gauge freedom and deformations of the master equation, *Phys. Lett.* **B311** (1993) 123–129 [arXiv:hep-th/9304057].

[38] C. Bizdadea, Consistent interactions in the Hamiltonian BRST formalism, *Acta Phys. Polon.* **B32** (2001) 2843–2862 [arXiv:hep-th/0003199].

[39] N. Ikeda, Topological field theories and geometry of Batalin-Vilkovisky algebras, *J. High Energy Phys.* JHEP**10**(2002)076 [arXiv:hep-th/0209042].

[40] N. Ikeda and K.-I. Izawa, Dimensional reduction of nonlinear gauge theories, *J. High Energy Phys.* JHEP**09**(2004)030 [arXiv:hep-th/0407243].

[41] M. Henneaux, Consistent interactions between gauge fields: the cohomological approach, *Contemp. Math.* **219** (1998) 93 [arXiv:hep-th/9712226].

[42] G. Barnich, F. Brandt and M. Henneaux, Local BRST cohomology in the antifield formalism: I. General theorems, *Commun. Math. Phys.* **174** (1995) 57–91 [arXiv:hep-th/9405109].

[43] G. Barnich, F. Brandt and M. Henneaux, Local BRST cohomology in gauge theories, *Phys. Rept.* **338** (2000) 439–569 [arXiv:hep-th/0002245].

[44] E. S. Fradkin and A. A. Tseytlin, Conformal supergravity, *Phys. Rept.* **119** (1985) 233–362.