Slow flows of an relativistic perfect fluid in a static gravitational field

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Relativistic hydrodynamics of an isentropic fluid in a gravitational field is considered as the particular example from the family of Lagrangian hydrodynamic-type systems which possess an infinite set of integrals of motion due to the symmetry of Lagrangian with respect to relabeling of fluid particle labels. Flows with fixed topology of the vorticity are investigated in quasi-static regime, when deviations of the space-time metric and the density of fluid from the corresponding equilibrium configuration are negligibly small. On the base of the variational principle for frozen-in vortex lines dynamics, the equation of motion for a thin relativistic vortex filament is derived in the local induction approximation.

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1. In the given work inviscid flows of a relativistic isentropic fluid are investigated in the general relativistic theory i.e. in significantly curved space-time parameterized by some coordinate system \((t, r)\), with the metric tensor \(g_{ik}(t, r)\).

For any hydrodynamic system it is possible to separate vortex and sound degrees of freedom, with the vortices being characterized as the "soft", while the sound as the "hard" degrees of freedom. If there is a static equilibrium state in the system with the density of fluid \(\rho = \rho_0(r)\) and with the velocity \(v = 0\), then the dynamical regime is possible with the hard degrees of freedom being excited weakly and with dynamics being entirely described by slow flows on the background of practically unchanged density \(\rho_0(r)\). A typical velocity in such flows should be much smaller then the sound speed (and also smaller then the speed of light, as in the case). The general theory of such slow vortical flows in spatially inhomogeneous systems has been developed recently by the present author in his work [3]. In particular, the Hamiltonian equations of motion have been obtained for an arbitrary given topology of the canonical vorticity field, and the general form of the Lagrangian for frozen-in vortex lines has been established.

The goal of this Brief Report is to apply the formalism developed to the model of relativistic hydrodynamics. As the equilibrium state, a spherically symmetric static distribution of a fluid will be considered which is the reason for space-time curvature. In the expression for the action of relativistic hydrodynamics [4] the only arbitrary dependence \(\varepsilon(n)\) appears which connects the density \(\varepsilon\) of the total energy of fluid measured in the locally comoving reference frame with the density \(n\) of number of conserved particles [4]. The scalar \(n\) is equal to the absolute value of the current 4-vector \(n^i = n(dx^i/ds)\) [3]. Therefore the static metric of the space-time, as well as the equilibrium distribution \(\rho_0(r)\) of the matter in space, after the equation of state \(\varepsilon(n)\) has been fixed, depend on single parameter – the total amount of fluid in the system.

As the result, the equation of motion of a slender vortex filament in relativistic fluid will be derived in the local induction approximation, and it will become clear how the the dynamics of frozen-in vorticity in curved space-time differs from the non-relativistic hydrodynamics, even in the case of small velocities.

2. Before starting the consideration of the concrete dynamical system, it is necessary to make some preliminary remarks. The ideal relativistic hydrodynamics is a member of the large family of the hydrodynamic-type Lagrangian models. This family is remarkable because of many interesting properties, first of all due to presence of infinite number of specific integrals of motion. Therefore it is reasonable in the beginning to consider the relativistic hydrodynamics from general positions of the canonical formalism [5, 6].

As known, the entire Lagrangian description of the flow of some continuous medium can be given by the 3D mapping \(\mathbf{r} = \mathbf{x}(t, \mathbf{a})\), which indicates the space coordinates of each medium point labeled by a label \(\mathbf{a} = (a_1, a_2, a_3)\), at an arbitrary moment in time \(t\). The labeling \(\mathbf{a}\) can be chosen in such a manner that the amount of matter in a small volume \(d^3\mathbf{a}\) in the label space is simply equal to this volume. Inasmuch as the dynamical system is supposed to be conservative, the equations of motion for the mapping \(\mathbf{x}(t, \mathbf{a})\) (as well as for the gravitational field, as in our case) follow from a variational principle

\[\delta S = \delta \int \mathcal{L}(\mathbf{x}(t, \mathbf{a}), g_{ik}(t, \mathbf{r})) dt = 0, \quad (1)\]

where the Lagrangian \(\mathcal{L}\) is a functional of \(\mathbf{x}(t, \mathbf{a}), g_{ik}(t, \mathbf{r})\) and their derivatives. The explicit expression for \(\mathcal{L}\) will be given later, but now let us note the very important circumstance related to the fluidity property of the medium under consideration. The fluidity is manifested in the fact that the Lagrangian actually contains the dependence on \(\mathbf{x}(t, \mathbf{a})\) only through two Eulerian characteristics of the flow, namely through the field of relative density \(\rho(t, \mathbf{r})\) and the velocity field \(\mathbf{v}(t, \mathbf{r})\), i.e. \(\mathcal{L} = \mathcal{L}(\rho, \mathbf{v}, g_{ik})\), with

\[\rho(t, \mathbf{r}) = \frac{1}{\det |\partial \mathbf{x}/\partial \mathbf{a}|}_{\mathbf{a} = \mathbf{x}^{-1}(t, \mathbf{r})}, \quad (2)\]
\[ \mathbf{v}(t, \mathbf{r}) = \dot{\mathbf{x}}(t, \mathbf{a})|_{\mathbf{a}=x^{-1}(t, \mathbf{r})} \] (3)

It follows from these definitions that the dynamics of the "density" \( \rho(t, \mathbf{r}) \) obeys the continuity equation in its standard form

\[ \rho_t + \nabla(\rho \mathbf{v}) = 0. \] (4)

The vanishing condition for variation of the action \( S = \int \mathcal{L}(\rho, \mathbf{v}, g_{ik}) \, dt \) when the mapping \( \mathbf{x}(t, \mathbf{a}) \) is varied by \( \delta \mathbf{x}(t, \mathbf{a}) \), with taking into account the obvious relations between the variations

\[ \delta \rho(\mathbf{r}) = -\nabla(\rho(\mathbf{r}) \cdot \delta \mathbf{x}(\mathbf{a}(\mathbf{r}))), \] (5)

\[ \delta \mathbf{v}(\mathbf{r}) = \delta \dot{\mathbf{x}}(\mathbf{a}(\mathbf{r})) - (\delta \mathbf{x}(\mathbf{a}(\mathbf{r})) \cdot \nabla)\mathbf{v}, \] (6)

can be expressed in Eulerian representation as follows (the generalized Euler equation [7]

\[ \left( \partial_t + \mathbf{v} \cdot \nabla \right) \left( \frac{1}{\rho} \frac{\delta \mathcal{L}}{\delta \rho} \right) = \nabla \left( \frac{\delta \mathcal{L}}{\delta \mathbf{v}} \right) - \frac{1}{\rho} \left( \frac{\delta \mathcal{L}}{\delta \mathbf{v}} \right) \nabla \mathbf{v}. \] (7)

Together with the conditions \( \delta S/\delta g_{ik} = 0 \), the equations [7] and [8] determine completely the evolution of hydrodynamic system. It is very important that in all such systems an infinite number of conservation laws exists [8]-[12]. The reason of their existence is that the Lagrangian \( \mathcal{L}(\mathbf{x}(t, \mathbf{a}), g_{ik}(t, \mathbf{r})) \) admits the infinite-parametric symmetry group – it assumes the same value on any two mappings \( x_1(t, \mathbf{a}) \) and \( x_2(t, \mathbf{a}) \), if they differ one from another only by some relabeling of the labels

\[ x_2(t, \mathbf{a}) = x_1(t, \mathbf{a}^*(\mathbf{a})), \quad \text{det}||\mathbf{a}^*/\mathbf{a}|| = 1. \] (8)

Obviously, such mappings create the same "density" and velocity fields. According to the Noether’s theorem [3], [2], every one-parametric sub-group of the relabeling group \( \mathbf{a}^*(\mathbf{a}) \) with unit Jacobian corresponds to an integral of motion. There are several classifications of these conservation laws. For instance, one can postulate that the circulation of the canonical momentum \( \mathbf{p}(t, \mathbf{r}) \)

\[ \mathbf{p}(t, \mathbf{r}) = \frac{\delta \mathcal{L}}{\delta \dot{\mathbf{x}}(\mathbf{a}(t, \mathbf{r}))} = \frac{1}{\rho} \left( \frac{\delta \mathcal{L}(\rho, \mathbf{v}, g_{ik})}{\delta \mathbf{v}} \right) \] (9)

along an arbitrary frozen-in closed contour \( \gamma(t) \) does not depend on time (the generalized theorem of Kelvin)

\[ \oint_{\gamma(t)} (\mathbf{p} \cdot d\mathbf{r}) = \text{const}. \]

We arrive at a different formulation when consider the solenoidal field of the vorticity \( \mathbf{\Omega}(t, \mathbf{r}) \)

\[ \mathbf{\Omega}(t, \mathbf{r}) = \text{curl} \mathbf{p}(t, \mathbf{r}). \] (10)

From the equation [8] the equation of frozenness for \( \mathbf{\Omega}(t, \mathbf{r}) \) follows

\[ \mathbf{\Omega}_t = \text{curl}[\mathbf{v} \times \mathbf{\Omega}]. \] (11)

The formal solution of this equation is

\[ \mathbf{\Omega}(t, \mathbf{r}) = \left( \frac{\mathbf{\Omega}_0(\mathbf{a}) \nabla_{\mathbf{a}}}{\text{det}||\partial \mathbf{x}(t, \mathbf{a})/\partial \mathbf{a}||}_{\mathbf{a}=x^{-1}(t, \mathbf{r})} = \right. \] (12)

with the solenoidal field \( \mathbf{\Omega}_0(\mathbf{a}) \) independent on time being exactly an integral of motion (the Cauchy invariant [13]). The formula [12] displays that the lines of the initial field \( \mathbf{\Omega}_0(\mathbf{a}) \) are transported by the flow, retaining all the topological characteristics. Such a property of vortex lines is known as their frozenness.

3. Let’s turn our attention to the relativistic hydrodynamics and consider the corresponding expression for the action \( S \). [4], [2]

\[ S = -\int \sqrt{-g} \left( \varepsilon(n) + \frac{1}{16\pi} R[g_{ik}] \right) \, dt. \] (13)

Here \( g = \text{det}||g_{ik}|| \) is the determinant of the metric tensor, \( R[g_{ik}] \) is the scalar curvature of space-time [2]. In this formula, however, the scalar \( n \) should be expressed through the dynamical variables \{\( \rho, \mathbf{v}, g_{ik} \}\}. It is easy to see, comparing the continuity equation [4] with the equation of fluid conservation [1]

\[ n_i^\prime = -\frac{\rho}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( \sqrt{-g} \frac{dx^i}{ds} \right) = 0, \] (14)

that \( \rho = \sqrt{-g}(dt/ds) \). From this we have the relation

\[ n = \frac{\rho}{\sqrt{-g}} \sqrt{g_{00} + g_{0\alpha}v^\alpha + g_{\alpha\beta}v^\alpha v^\beta}, \] (15)

which should be substituted into the expression [13] and after that the equations of motion can be obtained in accordance with equations [4], [2] and \( \delta S/\delta g_{ik} = 0 \). In such form, however, the action contains an arbitrariness related to the possibility of coordinate changing. Therefore four gauge conditions can be imposed on the metric tensor [2]. Even more, inasmuch as we are interested in a static central-symmetric gravitational field, the corresponding metric can be chosen as follows

\[ (ds^2)^{\text{stat}} = A(r) dt^2 - B(r) (dx^2 + dy^2 + dz^2), \] (16)

where \( r = \sqrt{x^2 + y^2 + z^2} \). The expression for the scalar curvature in this metric takes the form

\[ R^{\text{stat}}[A, B] = \frac{1}{2A^2B^2} \left( -B^2 A^2 + \right. \]

\[ + AB(A'B' + 2B(2A'/r + A'')) + \]

\[ + A^2(-3B^2 + 4B(2B'/r + B'')) \right). \] (17)
Therefore the static state is determined by an extremum of the functional
\[
\frac{S^{\text{stat}}}{4\pi} = -\int \sqrt{AB^3} \left[ \varepsilon \left( \frac{\rho_0}{\sqrt{B^3}} + \frac{R^{\text{stat}}[A,B]}{16\pi} \right) \right] r^2dr dt. 
\]

(18)

The variations on \( \delta A(r) \) and on \( \delta B(r) \) should be performed in the standard way when an extremum is being found, while the variation on displacements \( \delta x(a) \), with taking into account the relation (3) and subsequent integration, gives the relativistic hydrostatic equation for isentropic fluid (1)
\[
\sqrt{A}w \left( \frac{\rho_0}{\sqrt{B^3}} \right) = \lambda = \text{const},
\]

(19)

where \( w(n) = \varepsilon'(n) \) is the relativistic enthalpy per one conservative particle, while a constant of integration \( \lambda \) is related to the total amount of matter in the system.

With fixed static central-symmetric metric (10) the Lagrangian of relativistic hydrodynamics takes the form
\[
\mathcal{L}_r\{\rho, v\} = -\int \sqrt{AB^3} \varepsilon \left( \frac{\rho_0\sqrt{A} - Bv^2}{\sqrt{AB^3}} \right) dr.
\]

(20)

As far as we examine here only slow flows, we need only the first term from the expansion of \( \mathcal{L}_r \) on powers of \( v^2 \)
\[
\mathcal{L}_s = \int \omega \left( \frac{\rho_0}{\sqrt{B^3}} \right) \cdot \frac{\rho_0 B}{\sqrt{A}} \cdot \frac{v^2}{2} dr = \lambda \int \frac{\rho_0 B}{A} \cdot \frac{v^2}{2} dr.
\]

Just the functional \( \mathcal{L}_s \) determines the slow dynamics of frozen-in vorticity \( \Omega \)
\[
\Omega \approx \text{curl} \left( \frac{1}{\rho_0} \frac{\delta \mathcal{L}_s}{\delta v} \right) = \text{curl} \left( \frac{\lambda B}{A} v \right).
\]

4. In order to investigate the motion of frozen-in vortex lines in the given problem, it is necessary to define the Hamiltonian \( \mathcal{H}_s \)
\[
\mathcal{H}_s = \int \frac{\delta \mathcal{L}_s}{\delta v} \cdot v dr - \mathcal{L}_s = \frac{1}{\lambda} \int \frac{\rho_0 A}{B} \cdot \frac{p^2}{2} dr
\]

(21)

and then express it through the vorticity \( \Omega \) with help of the relation (14) and the condition of zero variation of density \( \rho_0 \)
\[
\nabla \left( \frac{\delta \mathcal{H}_s}{\delta p} \right) = \nabla \left( \frac{A\rho_0}{\lambda B} \right) p = 0.
\]

(22)

The equation of motion for \( \Omega \) in quasi-static regime can be written after that in the form (3)
\[
\Omega_t = \text{curl} \left[ \text{curl} \left( \frac{\delta \mathcal{H}_s}{\delta \Omega} \right) \times \frac{\Omega}{\rho_0(r)} \right].
\]

(23)

Inasmuch as the vortex lines are frozen into the fluid, it is reasonable to consider their shape as the new dynamical object (3) and parameterize the vorticity field as follows
\[
\Omega(r,t) = \int_{\mathcal{N}} d^2\nu \oint \delta(r - R(\nu, \xi, t))R_\xi d\xi.
\]

(24)

Here the label \( \nu = (\nu_1, \nu_2) \in \mathcal{N} \) belongs to the 2D manifold \( \mathcal{N} \) and singles out a vortex line, while the parameter \( \xi \) determines a point on the line. Obviously, the choice of the longitudinal parameter \( \xi \) is not unique (15). It is easy to understand the meaning of the above formula -- the frozen-in vorticity field is represented here as the continuous distribution of vortex lines.

It follows from the equations (23) and (24) that the equation of motion of vortex lines has the Hamiltonian form (see (3) for detailed derivation)
\[
\left[ R_\xi \times R_t \right] \rho_0(R) = \frac{\delta \mathcal{H}_s\{\Omega(R)\}}{\delta R}.
\]

(25)

Clearly, this equation does not depend on the choice of longitudinal parameterization. It is easy to check by direct calculation that the above equation corresponds to the variational principle \( \delta \int \mathcal{L}_N dt = 0 \) with the Lagrangian \( \mathcal{L}_N \) of the form (3)
\[
\mathcal{L}_N = \int_{\mathcal{N}} d^2\nu \oint \left( \left[ R_\xi(\nu, \xi) \times D(R(\nu, \xi)) \right] \cdot R_\xi(\nu, \xi) \right) d\xi
\]

\[ - \mathcal{H}_s\{\Omega(R)\}, \]

(26)

Here the vector function \( D(R) \) is related to the equilibrium density \( \rho_0 \) by the condition (3)
\[
\left( \nabla_R \cdot D(R) \right) = \rho_0(R).
\]

(27)

5. Unfortunately, practical use of the new variables \( R(\nu, \xi, t) \) is rather difficult in general case because of necessity to find the potential component of the canonical momentum field \( p \) from the system of equations (10) and (23) for substitution of \( p(\Omega) \) into the equation (24). However, in analysis of some situations this difficulty can be avoided. For instance, the vortex line representation allows in the most simple way to study the so called local induction approximation (LIA) in dynamics of thin vortex filaments. Let’s note that in usual Eulerian homogeneous hydrodynamics the LIA gives the integrable equation which is gauge equivalent to 1D nonlinear Schroedinger equation (14). Our purpose now is to derive the local induction equation for a slender vortex filament in a relativistic fluid. Let’s suppose that the vorticity is present in the system in form of a quasi-one-dimensional structure with the circulation \( \Gamma = \int_{\nu} d^2\nu \), with a typical width \( d \) and with a typical longitudinal scale \( L \gg d \). Then we may neglect the dependence of the vortex line shape on the label \( \nu \) in the main approximation and consider the vortex filament as the single curve \( R(\xi, t) \). In
such conditions, the Hamiltonian $\mathcal{H}_*$ is determined by the close neighborhood of this curve, so that with the logarithmic accuracy $\mathcal{H}_*$ is equal to the following expression

$$\mathcal{H}_* \approx \mathcal{H}_L = \Gamma \Lambda \int \frac{\rho_0(R)A(R)}{B(R)} |\mathbf{R}_\xi| d\xi,$$  

where the constant $\Lambda$ contains the large logarithm

$$\ln \left( \frac{L}{d} \right).$$

With help of the equations (25) and (28) we can obtain the equation of motion for a thin vortex filament in a relativistic fluid placed into a central-symmetric static gravitational field:

$$[\mathbf{R}_\xi \times \mathbf{R}_\xi] \left( \frac{\rho_0}{\Lambda} \right) = \nabla \left( \frac{\rho_0 A}{B} \right) |\mathbf{R}_\xi| \partial_\xi \left( \frac{\rho_0 A}{B} \right) \frac{\mathbf{R}_\xi}{|\mathbf{R}_\xi|}.$$

Being solved with respect to the time derivative and rewritten in terms of the unit tangent vector $\mathbf{t}$, the unit binormal vector $\mathbf{b}$ and the curvature $\kappa$ of the line, the above equation takes the form

$$\mathbf{R}_t \cdot \left( \frac{B(R)}{A(R)} \right) \cdot \frac{1}{\Lambda} = \nabla \ln \left( \frac{\rho_0(R)A(R)}{B(R)} \right) \times \mathbf{t} + \kappa \mathbf{b},$$

The condition of its applicability, besides $L \gg d$, is also the inequality

$$\frac{A}{\Lambda B} \cdot \frac{\Gamma}{d} \ll \sqrt{\frac{A}{B}},$$

which means the smallness of the maximal velocity in the flow in comparison with the speed of light.

Let us say finally few words about possible situation when the static distribution of the matter in space is highly inhomogeneous. Let’s suppose that the equation of state $\varepsilon(n)$ results in formation of a dense and massive core of relatively small size $r_\ast$ and extended easy shell of a mass being much smaller than the mass of the core $M_\ast$. Then, as far as we are interested in the dynamics of a thin vortex filament in the shell, one can believe that the functions $A(r)$ and $B(r)$ in the equation (30) at $r > r_\ast$ are the same as in the empty space around the mass $M_\ast$ [2]:

$$A(r) = \left( 1 - \frac{M_\ast}{2r} \right)^2 \left( 1 + \frac{M_\ast}{2r} \right)^{-2},$$  

$$B(r) = \left( 1 + \frac{M_\ast}{2r} \right)^4.$$  

The density $\rho_0(r)$ at $r > r_\ast$ in this case is determined by the hydrostatic equation [19] with the given $A$ and $B$.

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