What the survivors’ areas do at long times?

Boris Levitan and Eytan Domany

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Abstract

We investigate the long time behavior of the survivors’ area in the scaling state of two dimensional soap froth. We relate this problem to the recently studied temporal decay of the fraction of Potts spins that have never been flipped till time $t$. The results of our topological simulations are consistent with the value $\theta = 1$ for the scaling exponent of the survivors’ areas, in agreement with a recently obtained analytical result. We find, however, that the relaxation time needed to get into the scaling regime depends on the degree of randomness in the topological rearrangements and becomes very large in the deterministic limit.

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Coarsening of soap froth constrained between two closely spaced parallel plates (2d froth) has attracted the attention of physicists for nearly two decades [1,2]. The main reason for the current interest is the fact that such a froth is a simple non-equilibrium system that evolves to a universal scaling state. Many non-equilibrium systems of varying complexity share this fascinating characteristic of their long-time evolution. Therefore one feels that elucidating the properties of such a scaling state and the reasons for its widely observed occurrence may further our understanding of the dynamics of non-equilibrium systems. To gain such insights one must find ways to model the evolution of the froth. The chosen model has to be simple enough to allow treatment of very large systems (needed to get good statistics) without sacrificing any important aspect of the physics. It turns out that very different models predict static properties of the scaling state with acceptable accuracy, whereas the system’s dynamic properties are much more model-dependent [2]. The extent to which different models succeed in explaining known dynamic properties is of central importance in model selection. Once reliable models are identified, they can be used to predict new properties of the froth.

The evolution of 2d froth is governed by the strikingly simple microscopic von Neumann law [3], which relates the rate of area change of any bubble to the number of its sides \( l \):

\[
\frac{d a_l}{dt} = k(l - 6),
\]

(1)

where the constant \( k \) includes the physical properties of the soap films: the surface tension and the penetrability of the gas through the film. One of the immediate consequences of this equation is that bubbles with \( l > 6 \) grow, while those with \( l < 6 \) shrink and finally disappear. When a bubble disappears its neighbors undergo sudden topological rearrangements (so called T2 processes) and change their number of sides. Knowing the areas and the number of sides of the neighbors of a disappearing bubble does not determine uniquely the outcome of the T2 process; which of the possible ”channels” will be realized in each particular case is determined only by the explicit geometrical configuration of the froth just before the bubble’s disappearance [4]. The so called topological approach [2] does not keep track of
the explicit configuration of the froth (such as positions of all vertices); rather, evolution is described only in terms of the areas and the connectivity matrix. Within such an approach one needs to supplement Eq.(1) by a phenomenological rule that determines the outcome of the T2 processes in terms of these variables.

Eq.(1) implies that the total number of bubbles in the sample decreases, so their mean area $\bar{a}$ grows which gives rise to a coarsening process. Experiments show that the coarsening froth evolves to a scaling regime $^{10}$, where $\bar{a}(t) \sim t$ and the distribution of the areas and topological classes has the scaling form: $F_l(a,t) = (1/\bar{a}) f_l(a/\bar{a})$.

Past theoretical and experimental studies were devoted mainly to characterizing the instantaneous pictures of the scaling state; predicting the form of the function $f_n(a/\bar{a})$ (for modern review see refs. $^{12}$ and references therein) as well as the topological correlations in the froth $^{13,14}$. Dynamical properties of the scaling state, which are most important for model selection, were addressed only recently, by investigating theoretically and experimentally the behavior of survivors $^{8}$.

Survivors were defined as follows. Consider two photos of the evolving froth, $P_i$ and $P_f$, both taken in the scaling state, at two subsequent times $t_i$ and $t_f$, with $t_i < t_f$. Let the number of cells in these two pictures be $N(t_i)$ and $N(t_f)$ respectively ($N(t_i) > N(t_f)$). Using the information about all the intermediate states of the froth, we can identify on the earlier picture $P_i$ all those $N(t_f)$ cells that are present in the latter picture $P_f$. These $N(t_f)$ cells that survived till the moment $t_f$, constitute the sub-ensemble of ”survivors” at $t_i$. The statistical properties of this sub-ensemble differ from those of the ensemble of all the bubbles on the same photo and depend on $t = t_f - t_i$ (survival time), characterizing, therefore, the dynamics of the scaling state.

The time dependence of the survivors’ topological distribution has been investigated recently using topological simulations and mean field calculations $^{8}$ under the assumption of random T2 processes $^{8}$ and it was shown that the distribution approaches a fixed form in the long time limit. Subsequently, however, we discovered that the behavior of the survivors is much more sensitive to the details of the model than the properties of all the bubbles.
Simulating a more realistic model, one that performs T2 process in a deterministic way \[10\], we found that the distribution continues to evolve for much longer times than in the case of the random model. The length of these simulations did not suffice, however, to determine conclusively whether the topological distribution of the survivors approaches a fixed limit. On the other hand, mean field theory \[8\] definitely predicts the existence of a fixed form for the survivors’ topological distribution as well as the analytical results for the $q = \infty$-Potts model \[11\], which also are consistent with convergence of the area distribution of the survivors to a fixed form.

The long time behavior of the survivors’ area distribution is of interest also in the context of another interesting problem that has been studied intensively recently \[12–14\] concerning the dynamics of the Potts model. Starting from a random initial state, one considers the time dependence of $r(t)$, the fraction of spins that have never flipped from the beginning of the evolution till time $t$. It has been found that at long times

$$r(t) \sim t^{-\theta}$$

 Similar power laws have been observed in reaction diffusion models \[15\] and in the breath figures’ growth \[16\] (the fraction of the area that has not become wet till time $t$). The $q = \infty$-Potts dynamics present a good model for soap froth simulations \[17\]. The quantity that is analogous to $r(t)$ in the soap froth is the uncrossed area $A_{uncr}(t)$; that is, the total area that has not been crossed by any soap boundary till time $t$. This quantity was measured recently in experiment on soap froth \[18\] and the power law has been established.

One can determine a bound of $A_{uncr}(t)$ by overlaying the two pictures $P_i$ and $P_f$. Denote by $a_i^{(k)}$ the area of survivor bubble number $k$ in $P_i$, which evolved into bubble number $k$ in $P_f$. Denote the common area (on the overlaid pictures) of these two bubbles by $a_{com}^{(k)}(t)$; we then have $A_{uncr}(t) \leq \sum_k a_{com}^{(k)}(t)$. Since $a_{com}^{(k)}(t) \leq a_i^{(k)}(t)$, we have

$$A_{uncr}(t) \leq \sum_k a_i^{(k)}(t) = A_s(t),$$

where $A_s(t)$ is the total area of the survivors (to time $t$), as measured on $P_i$. Experimental observations, obtained by overlaying the raw (unpublished) figures such as those presented in
ref. [8] indicate that many survivors stay within their final area $a_f(t)$; hence one expects that $A_{uncr}(t) \approx const \cdot A_s(t)$. Experimentally, one could measure directly both $A_s(t)$ and $A_{uncr}(t)$ independently to provide a test of the extent to which this holds. Topological simulations do not follow the movement of the boundaries between the bubbles; hence $A_{uncr}(t)$ is not available. On the other hand $A_s(t)$ is easy to measure, and by studying the long time behavior of $A_s(t)$ one can at least give an upper bound for $A_{uncr}(t)$, that can be compared with $r(t)$ for the Potts model. This connection between the survivors’ area and the first flip probability serves as additional motivation to analyze the behavior of $A_s(t)$.

In this work we investigate the evolution of the survivors at extremely long times. In our simulations we use a recently introduced model [19]; a T2 processes is assigned either at random [8] or deterministically [14], with probabilities $\alpha$ and $1 - \alpha$ respectively (i.e. at $\alpha = 1$ we get the random model [8] while at $\alpha = 0$ - the purely deterministic model [14]). The explicit rule used for the deterministic case is described elsewhere [14]. Simulations of this model were in good agreement with results obtained either from detailed simulations [20,21] or experiments [1,8,4] on soap froth, while the random model does not agree with some of these results [2].

Extending our previous simulations [8,14], we initiate an array of $N = 1000 \ 000$ bubbles, whose topological distribution is close to that of the scaling state; it evolves till the number of bubbles is about 200. In order to reduce fluctuations we have averaged all our results over 12 runs.

It should be emphasized that in order to represent a system of $N$ bubbles in a Potts simulation one needs $mN$ spins, where the minimal number of spins needed to define a bubble, is at least $m \approx 10$. Running a Metropolis type simulation of such a system till a small fraction of the domains survive, one needs about $mN$ time steps of $mN$ operations, i.e. $O(m^2N^2)$ operations for a run. For a system as large as ours this would amount to impractically long computational time. Our topological approach uses bubbles as the elementary dynamical units. By working with a time-step of order $\bar{a}$, we can complete a run in $\sim N \log N$ operations. This enables us to run large systems for extremely long times.
Starting with \( N = 1000000 \) bubbles, we arrived in the scaling regime with about \( N_i = 640000 \) cells. At this point we number all the bubbles and we refer to this as our initial state. Running our simulations further, at time \( t \) we have \( N(t) \) bubbles, whose ancestors, identified in the initial state, constitute the \( N(t) \) survivors. We evaluated their average topological class \( \bar{n}_s(t) \) and \( \bar{a}_s(t) \), their average area (as measured in the initial state): \( \bar{a}_s(t) = A_s(t)/N(t) \). In the scaling state \( N(t) \sim 1/t \), so we have

\[
A_s(t) \propto \bar{a}_s(t)/t
\]  

When presenting our data, we defined the time as \( t = N_i/N \). Note, that in the scaling regime our time is proportional to the real physical time. The time dependences of \( \bar{a}_s(t) \), as obtained by simulations for different values of \( \alpha \), is presented on Fig. 1. For \( \alpha = 1 \) (pure random model) \( \bar{a}_s(t) \) clearly approaches a constant (that is equivalent to the value \( \theta = 1 \) for the exponent), while for both \( \alpha = 0 \) and \( \alpha = 0.5 \), \( \bar{a}_s(t) \) continues to increase till the end of the run, in a way consistent with approaching a constant asymptotic value.

In order to provide convincing support for this asymptotic behavior we plotted in Fig. 2 the quantity \( y(t) = -\log[1 - \bar{a}_s(t)/a_{\infty}] \) vs. \( \log t \). For each \( \alpha \) a constant \( a_{\infty} \) was tuned to obtain the best fit of \( y \) to a straight line. A linear dependence \( y = c_1 + \eta \log t \) is equivalent to an algebraic form \( \bar{a}_s(t)/a_{\infty} = 1 - ct^{-\eta} \), which means

\[
A_s = \frac{C}{t} \left( 1 - c_1 t^{-\eta} \right)
\]  

Our data, as presented in Fig. 2, indeed yields straight lines, with very good accuracy, for all three values of \( \alpha \). The value of the power \( \eta \) increases with \( \alpha \). In each case the linear fit breaks down at very long times, when the "signal" (i.e. the difference \( 1 - \bar{a}_s(t)/a_{\infty} \)) is of the same order as the "noise" due to statistical fluctuations.

A similar picture was found for the mean topological class of the survivors, \( \bar{n}_s(t) \) (see Fig. 3). In particularly, we note that for the deterministic case the topological part of the problem is still evolving, even for very long times. Analyzing this data in a way similar to that of Fig. 2 also indicates an asymptotic power-law approach of \( \bar{n}_s(t) \) to a constant value. Interestingly, \( \bar{n}_s(t) \) and \( \bar{a}_s(t) \) approach their asymptotic values with different powers.
On the basis of simulations of the $q = \infty$-Potts model at $T = 0$ in two dimensions, starting from random initial conditions [13], the value $\theta = 0.86$ was reported. This statement seems to be in conflict with our results as well as with the analytical result of Sire and Majumdar [11], that yields $\theta = 1$. However, looking carefully at Fig. 1 of ref. [13] one sees that the local slope varies with time, apparently approaching $\theta = 1$ at long times, in a way that resembles our Fig. 1.

In a recent experiment [18] scaling behavior of $A_{\text{uncr}}(t)$ has been observed for evolving two dimensional soap froth, with $\theta \approx 1.1$. That is, $A_{\text{uncr}}(t)$ was found to decay faster than our $A_s(t) \sim 1/t$. Since $A_s$ is only an upper bound on $A_{\text{uncr}}$, there is no contradiction between our result and the experiment. The latter does seem to disagree with the prediction of ref. [11], which can probably be attributed to insufficient length of the experiment. It would be interesting to see more extensive experimental measurements of survivors’ areas and topological class evolution in order to compare them directly with our results. Since the power $\eta$ depends on the details of topological rearrangements, it could be a new test of the validity of the model.

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FIG. 1. Time dependence of the mean survivors’ area, \( \bar{a}_s(t) \) as obtained by our topological simulations, for different values of the noise parameter \( \alpha \). Time is defined as \( t = N_i/N \), where \( N_i \) and \( N \) are the initial and the current number of bubbles in the system. The data have been averaged over 12 runs.
FIG. 2. The same data as in Fig. 1: the asymptotic value $a_\infty$ was tuned so that $y(t) = -\log[1 - \bar{a}_s(t)/a_\infty]$ vs. $\log t$ was approximated well by a straight line.
FIG. 3. Time dependence of the mean topological class of survivors, $\bar{n}_s(t)$, as obtained by our topological simulations for different values of the noise parameter $\alpha$. The data have been averaged over 12 runs. The behavior of $\bar{n}_s(t)$ resembles that of $\bar{a}_s(t)$ (as presented in Fig. 1).
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