Characteristic form of equations of dynamics of media of complex structure

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Abstract. The article presents the construction of the characteristic form of the equations of dynamics of elastic media for which the stiffness matrix is asymmetric. For diagonalization of the asymmetric matrix, the procedure of unitary triangularization by Schur and additional transformations are used, which allow to consistently find all parameters of the stress-strain state of the medium. Additional assumptions, such as cosser’s pseudocontinuum, are not required. The division of deformations into symmetric and asymmetric parts is not carried out and they are assumed to be small. Then the following information is given about the matrix form of differential invariants, their properties and their usage in problems of dynamics of different media are discussed. An example of their application to the problem of statics is given.

Keywords: dynamic processes, spatial characteristics method, numerical modeling, dynamics and destruction of the environments of complex structure

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minors of the third order of the stiffness matrix. After that, the final form of the equations was obtained using simple algebraic transformations.

In the case of an asymmetric stiffness matrix, the matrix form of the equations of dynamics is constructed elementarily, but the procedure of diagonalizing the minors using orthonormal matrices is impossible. Therefore, in this case, it is necessary to use a different trick – unitary triangulation according to Shur. As a result, characteristic equations can still be constructed, while they possess some new properties.

2. THE CONSTRUCTION OF THE CHARACTERISTIC FORM OF EQUATIONS OF DYNAMICS FOR BODIES WITH AN ASYMMETRIC ELASTICITY MATRIX

We construct the characteristic equations using the matrix form for writing variables and Cartesian coordinates \( \{x_i\} \). The equations of motion have the form

\[
\partial_i p_{ij} = \rho \partial_j V_i, \quad i, j = 1, 2, 3, \tag{1}
\]

where \( \partial_i \equiv \partial/\partial x_i, \partial t = \partial/\partial t, p_{ij} \) are the stresses, \( V_i \) – particle velocities, \( \rho \) – body material density: summation is carried out here and thereafter on repeated Roman indices.

We introduce additional row matrices

\[
e_i = [\delta_i, \delta_{i1}, \delta_{i2}, \delta_{i3}],
\]

\[
q_{ij} = [\delta_i \delta_{j1}, \delta_i \delta_{j2}, \delta_i \delta_{j3}, \delta_i \delta_{j4}, \delta_i \delta_{j5}],
\]

(2)

where \( \delta_{ij} \) is the Kronecker symbol and also write the defined variables: and in the form of matrix rows:

\[
V = [V_1, V_2, V_3],
\]

\[
P = [p_{11}, p_{12}, p_{13}, p_{14}, p_{21}, p_{22}, p_{23}, p_{24}, p_{31}, p_{32}].
\]

(3)

Using matrices (2) and (3), equation (1) can be written as

\[
Q_i \partial_j P = \rho \partial_j V^T,
\]

somewhere \( Q_i = \epsilon_i q_{ij} \), there

\[
Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},
\]

\[
Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},
\]

\[
Q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]

We do not consider plasticity; therefore, the relationship between stresses and asymmetric deformations has the form

\[
p_{ij} = \epsilon_{ijkl} \gamma_{kl},
\]

(5)

where \( \gamma_{kl} = \partial u_k / \partial x_l \) is the strain tensor, \( u_k \) displacement, \( \epsilon_{ijkl} \) stiffness tensor, which in matrix form has the form of a positive definite asymmetric matrix

\[
C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} & c_{18} & c_{19} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & c_{27} & c_{28} & c_{29} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} & c_{37} & c_{38} & c_{39} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} & c_{47} & c_{48} & c_{49} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} & c_{57} & c_{58} & c_{59} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} & c_{67} & c_{68} & c_{69} \\ c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} & c_{77} & c_{78} & c_{79} \\ c_{81} & c_{82} & c_{83} & c_{84} & c_{85} & c_{86} & c_{87} & c_{88} & c_{89} \\ c_{91} & c_{92} & c_{93} & c_{94} & c_{95} & c_{96} & c_{97} & c_{98} & c_{99} \end{bmatrix}.
\]

Taking into account the expression of distortions through displacements, as well as using matrices (2), equations (5) can be written in matrix form

\[
P = CQ^T \partial_j u^T,
\]

(6)

or, differentiating by \( i \)

\[
\partial_i P^T = CQ^T \partial_j V^T.
\]

(7)

Equations (4) and (7) are a complete system of matrix equations for determining matrix rows \( P \) and \( V \).
Before performing further transformations, we note a number of properties of the constructed matrices
\[ Q_i Q_j = \delta_{ij} \text{diag}(1,1,1), \]
\[ Q_i^T Q = \text{diag}(1,1,1,1,1,1,1,1,1), \]
\[ Q_i C Q_j = \theta \Delta \theta^T, \]
where \( \theta \) is the orthonormal matrix of the third order, \( \Delta \) — the matrix is triangular; when \( i = j \) the matrix \( \Delta \) has positive eigenvalues and, if all elements of the central minor of the third order \( Q_i C Q_j \) are positive, the largest eigenvalue of the matrix \( \Delta \) is one [4].

Let the stress wave propagate along the axis \( x_\alpha \) of the coordinate system \( \{x_i\} \). We multiply (7) on the left by \( \theta^T \), introduce the notation:
\[ P_\alpha = \theta^T Q_i P_i, \]
\[ P_\beta = \theta^T Q_i P_i, \]
\[ P_\gamma = \theta^T Q_i P_i, \]
\[ \Psi_{\alpha\beta} = \theta^T Q_i C Q_j \theta, \]
\[ \Psi_{\alpha\gamma} = \theta^T Q_i C Q_j \theta, \]
with the help of which equations (9) take the form
\[ \partial_a P_a^T - \rho \partial_a V_a^T = \partial_\beta P_\beta + \partial_\gamma P_\gamma, \]
\[ \partial_a P_a^T - \Delta_a \partial_a V_a^T = \Psi_{\alpha\beta} \partial_\beta V_a^T + \Psi_{\alpha\gamma} \partial_\gamma V_a^T. \]

We write the triangular matrix \( \Delta \) in the form
\[ \Delta_a = \begin{bmatrix} \lambda_\alpha & 0 & 0 \\ \chi_{\alpha\beta} & \lambda_\beta & 0 \\ \chi_{\alpha\gamma} & \chi_{\beta\gamma} & \lambda_\gamma \end{bmatrix}, \]
and divide it into two parts \( \Delta_a = \Lambda_a + E_a \), where \( \Lambda_a \) — is the diagonal matrix
\[ \Lambda_a = \begin{bmatrix} \lambda_\alpha & 0 & 0 \\ 0 & \lambda_\beta & 0 \\ 0 & 0 & \lambda_\gamma \end{bmatrix}, \]
\[ E_a = \begin{bmatrix} \chi_{\alpha\beta} & 0 & 0 \\ \chi_{\alpha\gamma} & \chi_{\beta\gamma} & 0 \end{bmatrix}. \]

Next, we write the matrix \( \Lambda_a \) in the form \( \Lambda_a = \rho D_a^\circ \), where the matrix
\[ D_a = \pm \sqrt{\rho / \rho} = \text{diag}(\pm \lambda_\alpha, \pm \lambda_\beta, \pm \lambda_\gamma) \] — is the matrix of longitudinal and transverse velocities of stress waves propagating in both directions along the axis \( x_\alpha \). We multiply the first equation (10) on the left by \( |D_a| \) and add the resulting equations, highlighting the characteristic part on the left.
\[ (\partial_a + |D_a| \partial_a)(P_a^T - \rho |D_a| V_a^T) = \]
\[ = \partial_\beta (\Psi_{\alpha\beta} V_a^T + |D_a| P_\beta^T) + \]
\[ + \partial_\gamma (\Psi_{\alpha\gamma} V_a^T + |D_a| P_\gamma^T) + E_a \partial_a V_a^T. \]

Equations (13) are not characteristic due to the term taken in the frame and containing the derivative along the direction of wave motion. To exclude it, we return to the second equation (10), multiply both its parts on the left by \( \Delta_a^{-1} \) and select \( \partial_a V_a^T \). In this case, we obtain the equation
\[ \partial_a V_a^T = \Delta_a^{-1} (\partial_a P_a^T - \Psi_{\alpha\beta} \partial_\beta V_a^T - \Psi_{\alpha\gamma} \partial_\gamma V_a^T), \]
where:
\[ e_{\alpha\beta} = -\chi_{\alpha\beta} / \lambda_\alpha \lambda_\beta, \]
\[ e_{\alpha\gamma} = (\chi_{\alpha\beta} \chi_{\beta\gamma} - \lambda_\beta \lambda_\gamma) / \lambda_\alpha \lambda_\beta \lambda_\gamma, \]
\[ e_{\beta\gamma} = -\chi_{\beta\gamma} / \lambda_\beta \lambda_\gamma, \]
\[ \Delta_a^{-1} = \begin{bmatrix} \lambda_\alpha^{-1} & 0 & 0 \\ \chi_{\alpha\beta} & \lambda_\beta^{-1} & 0 \\ \chi_{\alpha\gamma} & \chi_{\beta\gamma} & \lambda_\gamma^{-1} \end{bmatrix}. \]

Equations (13) are not characteristic due to the term taken in the frame and containing the derivative along the direction of wave motion. To exclude it, we return to the second equation (10), multiply both its parts on the left by \( \Delta_a^{-1} \) and select \( \partial_a V_a^T \). In this case, we obtain the equation
\[ \partial_a V_a^T = \Delta_a^{-1} (\partial_a P_a^T - \Psi_{\alpha\beta} \partial_\beta V_a^T - \Psi_{\alpha\gamma} \partial_\gamma V_a^T), \]
where:
\[ e_{\alpha\beta} = -\chi_{\alpha\beta} / \lambda_\alpha \lambda_\beta, \]
\[ e_{\alpha\gamma} = (\chi_{\alpha\beta} \chi_{\beta\gamma} - \lambda_\beta \lambda_\gamma) / \lambda_\alpha \lambda_\beta \lambda_\gamma, \]
\[ e_{\beta\gamma} = -\chi_{\beta\gamma} / \lambda_\beta \lambda_\gamma, \]
\[ \Delta_a^{-1} = \begin{bmatrix} \lambda_\alpha^{-1} & 0 & 0 \\ \chi_{\alpha\beta} & \lambda_\beta^{-1} & 0 \\ \chi_{\alpha\gamma} & \chi_{\beta\gamma} & \lambda_\gamma^{-1} \end{bmatrix}. \]

We substitute (14) into (13) and carry out the corresponding grouping; as a result, we obtain the equation.
\[ (\partial_a + |D_a| \partial_a)(P_a^T - \rho |D_a| V_a^T) = \]
\[ = E_a (\Lambda_a^{-1} + E_a) \partial_a P_a^T + \]
\[ + \partial_\beta (|I_3 - E_a (\Lambda_a^{-1} + E_a^*) \Psi_{\alpha\beta} V_a^T + |D_a| P_\beta^T) + \]
\[ + \partial_\gamma (|I_3 - E_a (\Lambda_a^{-1} + E_a^*) \Psi_{\alpha\gamma} V_a^T + |D_a| P_\gamma^T), \]

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where:

\[ E_a (\Lambda_a^{-1} + E_a^*) = \begin{bmatrix} e_{\alpha \beta} \lambda_\beta & 0 & 0 \\ -\epsilon_{\alpha \gamma} \lambda_\gamma & e_{\beta \gamma} \lambda_\gamma & 0 \end{bmatrix}, \]

\[ I_3 = \text{diag}(1,1,1). \]

We write (15) in scalar form using the two-valued definition \( D_a \) and the following notation:

\[ [I_3 - E_a (\Lambda_a^{-1} + E_a^*)] \Psi_{\alpha \beta} V^T_{\alpha \beta} + D_a P_B = \]

\[ = \Pi_{\alpha \beta} = \begin{bmatrix} \Pi_{\alpha \beta} \Delta, \Pi_{\beta \beta}, \Pi_{\alpha \beta} \Delta \end{bmatrix}, \]

\[ [I_3 - E_a (\Lambda_a^{-1} + E_a^*)] \Psi_{\alpha \beta} V^T_{\alpha \beta} + D_a P_B = \]

\[ = \Pi_{\gamma \gamma} = \begin{bmatrix} \Pi_{\gamma \gamma} \Delta, \Pi_{\gamma \gamma} \Delta \end{bmatrix}, \]

with which (15) can be represented in the form of three pairs of scalar equations

\[
(\partial + D_a \partial \delta \phi \equiv \rho D_a \partial \phi),
\]

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\]

\[
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\]

\[ = \epsilon_{\alpha \beta} \partial \delta \phi \equiv \rho D_a \partial \phi, \]

\[ = \epsilon_{\alpha \gamma} \partial \delta \phi \equiv \rho D_a \partial \phi, \]

\[ = -\epsilon_{\alpha \gamma} \partial \delta \phi \equiv \rho D_a \partial \phi, \]

of which we successively find the voltage – particle velocity pairs: from the first pair of equations \( p_{aa} \) and \( V_{aa} \), from the second pair of equations, substituting the previously obtained \(- p_{aa} \) and \( V_{aa} \) into the right-hand sides, next pair \(- p_{aa} \) and \( V_{aa} \), finally, from the third pair, substituting into the right-hand sides all the previously obtained voltages and particle velocities, obtain remaining \( p_{a\gamma} \) and \( V_{a\gamma} \). After that, it is possible to reconstruct the row matrices \( P_a \) and \( V_a \) and use them further to determine the remaining stresses \( P_\beta \) and \( P_\gamma \) at fixed discontinuities. As you know, in this type of equations, the spatial derivative of the defined variables in the direction of wave motion (here it is \( \partial_a V^T_a \)) should be absent.

We return to equation (7), multiply both its parts on the left by \( \theta_a \equiv \Psi_{aa} \), express \( \partial_a V^T_a \) using (14) and using the notation (10) we obtain for \( P^T_\beta \)

\[
\partial_a (P^T_\beta - \theta_\beta \Psi_{aa} \theta_a \Lambda_a^{-1} P_a) =
\]

\[ = \theta_\alpha \Psi_{aa} C[I_3 - \theta_\alpha \theta_a \Lambda_a^{-1} \theta_a \Psi_{aa} C](\theta_\beta \theta_a \partial \beta V^T_a + \]

\[ + \theta_\gamma \theta_a \partial \gamma V^T_a). \]

Similarly, multiplying (7) from the left by \( \theta_a \equiv \Psi_{aa} \), we obtain

\[ \partial_a (P^T_\gamma - \theta_\gamma \Psi_{aa} \theta_a \Lambda_a^{-1} P_a) =
\]

\[ = \theta_\alpha \Psi_{aa} C[I_3 - \theta_\alpha \theta_a \Lambda_a^{-1} \theta_a \Psi_{aa} C](\theta_\beta \theta_a \partial \beta V^T_a + \]

\[ + \theta_\gamma \theta_a \partial \gamma V^T_a). \]

Having determined \( P^T_\beta, P^T_\gamma, P^T_\gamma \) and \( V^T_a \), the initial stresses and particle velocities are found by the formulas

\[ P^T = \Psi_{aa} \theta_a P_a + \theta_\beta \theta_a \theta_\beta P^T \]

\[ + \theta_\gamma \theta_a \theta_\gamma P^T, \]

\[ V^T = \theta_a V^T_a. \]

This completes the construction of the characteristic form of the equations of dynamics of media with an asymmetric stiffness matrix. Naturally, in practically important cases, this problem should be solved by numerical methods. However, it should be noted that the matrix approach used in this paper reduces the determination of the eigenvalues of matrices to third-order algebraic equations, which allows one to analytically perform the following operations:

1) find all the elements of the lower triangular matrices \( \Delta_a \) for all directions of wave propagation \{\alpha_i\} and, therefore, the matrices \( \Lambda_a \) and \( E_a \);

2) find the matrices \( \theta_a \) corresponding to them

3) find all the coupling coefficients \( \epsilon_{\alpha \beta}, \epsilon_{\alpha \gamma}, \epsilon_{\beta \gamma} \) of the inverse matrix \( \Lambda_a^{-1} \) and then the matrices \( E^*_a, E_a (\Lambda_a^{-1} + E_a) \), and all unchanged matrices in equations (16)-(18).

After these operations, the machine account for solving applied problems is sharply reduced.
3. SOME PROPERTIES OF CONSTRUCTED MATRICES AND MATRIX DIFFERENTIAL INVARIANTS

Analyzing the above construction of the characteristic equations, as well as the construction carried out in [1], it is easy to see that each uses one kind of auxiliary matrices. In [1] these are matrices $R_i = e^T r_{ij}$, where

$$r_{ij} = \left[ \delta_{i1}\delta_{j1}, \delta_{i2}\delta_{j1}, \delta_{i3}\delta_{j1}, \delta_{i1}\delta_{j2}, \delta_{i2}\delta_{j2}, \delta_{i3}\delta_{j2} \right]$$

In this paper, these are matrices $Q_i = e^T q_{ij}$. The structure of these matrices is simple – they are dyadic products of unit one-line three-component matrices $\varepsilon_j$, isolating the components of the vector, also written as a one-line matrix, and similarly written unit matrices $r_{ij}$ and $q_{ij}$, isolating the components of symmetric and asymmetric tensors, written in the same form.

An important feature of these matrices is that only one matrix is used in each of the considered processes of medium motion.

So the movement of a symmetrically elastic medium can be represented by the following cycle – Fig. 1, where the main role in topology is played by matrices $R_i$; in the equations of motion they lower the dimension (row matrices $\Sigma$), and in the defining equations increase the dimension (row matrices $V$).

Matrices $\varepsilon, R$ and $Q$ form the corresponding bases in three, six and nine-dimensional spaces, i.e.

\[
\begin{align*}
\text{Table 1} \\
\text{Matrix differential invariants} \\
\text{div} \vec{v} &= \varepsilon_i \partial_i V^T \\
\text{grad} \varphi &= \varepsilon_i \partial_i \varphi \\
\text{div} \hat{\sigma} &= R_1 \partial_i \Sigma^T \\
\text{def} \vec{v} &= \Pi (R_i^T \partial_i V^T) \\
\text{div} \dot{\varphi} &= R_i^T \partial_i \Sigma^T \\
\text{Grad} \vec{v} &= Q_i^T \partial_i \Sigma^T \\
\text{rot} \vec{v} &= S_i \partial_i V^T, \quad S_i = \varepsilon_{ijk} e^T_j e^T_k, \quad \varepsilon_{ijk} = -\text{Levi-Civita tensor} \\
\end{align*}
\]

where $I_3, I_6$ and $I_9$ are the unit matrices of the third, sixth and ninth order, which allows, using only their pairs, to find all the components of the velocity vector and stress tensors in the problems under consideration.

Table 1 presents the matrix differential invariants encountered in the problems of continuum dynamics and field theory and written in the Cartesian coordinate system $\{x_i\}$. Here is indicated: $\vec{v}$, $\hat{\sigma}$ and $\dot{\varphi}$ – a vector, symmetric and asymmetric tensors, $V$, $\Sigma$ and $P$ – they are also written in the form of single-line matrices; $\varphi$ – scalar, $\partial_i = \partial/\partial x_i$.

The matrix $S_i$ corresponds to the vector product of vectors and reflects the orthogonality of the movement and the action caused by it.

In an arbitrary orthogonal coordinate system $\{a^i\}$, the same differential invariants have the form presented in Table 2 where

\[
\begin{align*}
\text{Table 2} \\
\text{Matrix differential invariants} \\
\text{in an arbitrary orthogonal coordinate system} \\
\text{div} \vec{v} &= \left[ \partial_i + (\partial_i \ln (g_i / h_i)) \right] e^T_i V^T \\
\text{grad} \varphi &= \varepsilon_i \partial_i \varphi \\
\text{div} \hat{\sigma} &= \text{Div} \sigma = [\partial_i + (\partial_i \ln (g_i / h_i)) + e_{ijk} S_i (\partial_i \ln h_i)] V^T \\
\text{def} \vec{v} &= \Pi (R_i^T \partial_i V^T) \\
\text{div} \dot{\varphi} &= \text{Div} \dot{\varphi} = [\partial_i + (\partial_i \ln (g_i / h_i)) + e_{ijk} S_i (\partial_i \ln h_i)] P^T \\
\text{Grad} \vec{v} &= Q_i^T \partial_i \Sigma^T \\
\text{rot} \vec{v} &= [\partial_i + (\partial_i \ln h_i)] e_{ijk} e^T_j e^T_k V^T \\
\end{align*}
\]

Fig. 1. The cycle of motion of a symmetrically elastic medium.
\[ \partial_i = \partial / \partial x_i; \partial x_i = h_i \partial a_i; h_i = | \text{grad} a_i |; \]
\[ g_0 = h_1 h_2 h_3; i, j, k, \alpha, \beta = 1, 2, 3; i \neq j \neq k \neq i. \]

The expressions for matrix differential invariants presented in Tables 1 and 2 are convenient for matrix transformations: they have a block form, are written in a Cartesian (or accompanying Cartesian) coordinate system, and contain matrix rows of physical components of the defined variables that are affected by scalar - differential \(- \partial_i \) and algebraic \(- (\partial_i \ln(g_0 / h_i)) \) and \(- (\partial_i \ln h_i) \) operators. These equations contain four types of matrices, \( \epsilon_i, R_i, Q_i \) and \( S_i \), “stitching” individual scalar equations into groups and determining the structure of these groups.

Like the original symbolic ones, the constructed matrix relations are invariant under coordinate transformations, however, the use of these properties in matrix relations has its own peculiarities. Consider the transformation of matrix invariants when rotating the coordinate axes.

We introduce the Cartesian coordinate system \( \{ y_i \} \), with the axes rotated relative to the axes \( \{ x_i \} \) and the center coinciding with the center of the system \( \{ x_i \} \), and the transition matrix \( \Theta_3 = \epsilon_j^i \Theta_3 \epsilon_i \) from coordinates \( \{ x_i \} \) to coordinates \( \{ y_i \} \); obviously \( \theta_j = \partial y_j / \partial x_i \).

We transform the invariants written in tables 1 and 2 to variables \( \{ y_i \} \), for which we introduce the matrices \( M_j = \Theta_j \) and \( M_\rho = \Theta_\rho \) defined using the relations

\[ \Theta_3 \Theta_j S_j = S_j M_j, \Theta_3 \Theta_j R_j = R_j M_\rho; \Theta_3 \Theta_j Q_j = Q_j M_\rho. \]

Multiplying expressions (21) from the left, respectively, by \( S_j^T, R_j^T \) and \( Q_j^T \), and using equalities (20), we define these matrices in the form

\[ M_j = 0.5 S_j^T \Theta_j \mu S_\mu, M_\rho = \Pi R_j \Theta_3 \mu R_\mu, \]
\[ M_\rho = Q_j^T \Theta_3 \mu Q_\mu. \]

It is easy to show that both \( M_j = \Theta_j \) and \( M_\rho = \Theta_\rho \) are orthonormalized and are rotation matrices in three-dimensional and nine-dimensional space, the matrix \( M_6 \) provides rotation of the components of the symmetric rotation tensor in six-dimensional space, but is not orthonormal, \( |M_6| = 1 \).

Using the properties of the constructed matrices \( \epsilon_j, R_j, Q_j \) and \( S_j \), and carrying out a series of identical transformations, it is easy to construct formulas that allow one to recalculate the matrix differential invariants written in the concomitant (lower index \( x \)) coordinate system rotated relative to it (lower index \( y \))

\[ \text{div}_y V^T = \text{div}_x \Theta_j \epsilon_j V^T; \Theta_\text{grad}, T = \text{grad}_x \Theta_j T; \Theta_\text{rot}, V^T = \text{rot}_x \Theta_j \epsilon_j V^T, \Theta_\text{Div}, V^T = \text{Div}_x \Theta_j \epsilon_j V^T; \]
\[ \Theta_3 \text{Div}_x V^T = \text{Div}_x \Theta_j \epsilon_j V^T; \Theta_3 \text{def}_x V^T = \text{def}_x \Theta_j \epsilon_j V^T; \]
\[ \text{Grad}_x V^T = \Theta_j \text{Grad}_x \Theta_j V^T, \]

or, in general terms,

\[ \text{inv}_x \Psi^T = \Theta_j^T \text{inv}_x \Theta_j \Psi^T. \]

Comparing formulas (23)-(24), we can formulate the following statement:

To write the matrix form of any differential invariant (inv) from a column matrix \( \Psi^T \) in a Cartesian coordinate system \( \{ y_i \} \), rotated relative to the accompanying Cartesian system \( \{ x_i \} \) by a certain angle, specified using the rotation matrix \( \Theta_3 \), it is necessary:

1) multiply the transformed matrix invariant on the left by the transposed rotation matrix \( \Theta_j \), the order of which coincides with the number of equations of the group; for deformation should be used as \( \Pi^{-1} \text{def} \); \( \Theta_j = I \);
2) replace it \( \partial / \partial x_i \) with \( \partial / \partial y_i \);
3) in his argument, replace \( \Psi^T \) with the product \( \Theta_j \Psi^T \), where the matrix \( \Theta_j \) has an order that matches the number of elements in the column matrix \( \Psi^T \).
Consider an example where auxiliary matrices, matrix differential invariants, and their properties are useful in solving the problem.

Let in the defining equation the elastic body statics \( \hat{\sigma} = c \text{def} \, \hat{u} \), where \( c = c^{ijkl} \partial_{\alpha} \partial_{\beta} \partial_{\gamma} \partial_{\delta} \), and \( c^{ijkl} \) are the contravariant components of the tensor corresponding to the symmetric nondegenerate stiffness matrix. Let at the boundary with the normal \( x_\alpha \), all derivatives of the displacement vector \( \hat{u} \) except \( \partial / \partial x_\alpha \), and three components of the stress tensor \( \hat{\sigma} \) \( (\sigma_{\alpha\alpha}, \sigma_{\alpha\beta}, \sigma_{\alpha\gamma}) \) on this boundary are known.

It is necessary to construct the simplest equations for determining the remaining components of the stress tensor.

Decision. We write the original equation in matrix form by introducing a non-degenerate matrix \( \mathbf{C} = \Pi R_\alpha^T c^{ijkl} r_{kl} \Pi, \) which is the matrix expression of \( c \), both the row matrix of stresses \( -\Sigma \) and displacements \( -U \).

We multiply the resulting equation on the left by \( C^{-1} \), and we get the equation
\[
C^{-1} \Sigma^T = \Pi^{-1} \text{def}_x U^T. \tag{25}
\]

We draw attention to the fact that in the operator \( \text{def}_x \) the derivatives \( \partial / \partial x_\alpha \) of the row matrix \( U \) are multiplied by the matrix \( R_\alpha^T \), therefore, to exclude them from equation (25), it is enough to multiply the latter by the matrix orthogonal \( R_\alpha \).

We construct such a matrix in the form
\[
N_\alpha = I_6 \odot R_\alpha, \quad \text{where} \quad \odot \quad \text{is the operation of deleting rows corresponding to the matrix} \quad R_\alpha \quad \text{from the matrix} \quad I_6.
\]
The matrices \( R_\alpha \) and \( N_\alpha \) together form a six-dimensional basis (of two three-dimensional), obviously
\[
N_\alpha R_\alpha^T = 0, \quad R_\alpha R_\alpha^T = I_3, \tag{26}
\]
\[
N_\alpha N_\alpha^T = I_3, \quad R_\alpha^T R_\alpha + N_\alpha^T N_\alpha = I_6.
\]
We divide the components of the stress matrix row \( \Sigma \) into two groups: the stresses acting on the site with a normal \( x_\alpha \) – \( \Sigma_\alpha = R_\alpha \Sigma^T \) and all the others – \( \Sigma^T = N_\alpha \Sigma^T \), and, obviously, \( \Sigma^T = R_\alpha^T \Sigma_\alpha^T + N_\alpha^T \Sigma^T \). Let us
\[
A = C^{-1}, \quad A_{\alpha\alpha} = N_\alpha AN_\alpha^T, \quad A_{\alpha\beta} = N_\alpha AR_\alpha^T \tag{27}
\]
then write equation (16) in the form
\[
A_{\alpha\alpha} \Sigma_\alpha^T + A_{\alpha\beta} \Sigma^T = N_\alpha \Pi^{-1} \text{def}_x U^T.
\]

The matrix \( A_{\alpha\alpha} \) is the main submatrix of a non-degenerate matrix \( A \) and, therefore, is non-degenerate. Therefore, equation (27) can be divided on the left by \( A_{\alpha\alpha} \), after which it will take the form
\[
\Sigma_\alpha^T = A_{\alpha\alpha}^{-1} (N_\alpha \Pi^{-1} \text{def}_x U^T - A_{\alpha\beta} \Sigma^T). \tag{28}
\]

Equation (28) satisfies the given requirements, however, when constructing it, it is necessary to calculate the matrix \( A \), the inverse \( C \), the submatrices \( A_{\alpha\alpha} \) and \( A_{\alpha\beta} \) and the matrix \( A_{\alpha\alpha}^{-1} \). To simplify the calculations, we use the identity \( AC = I_6 \) and relations (26) that determine the decomposition of matrices \( A \) and \( C \) the basis of interest to us.

For submatrices \( A_{\alpha\alpha} \) and \( A_{\alpha\beta} \), we obtain the following relations:
\[
A_{\alpha\alpha} C_{\alpha\alpha} + A_{\alpha\beta} C_{\alpha\beta} = I_3, \quad A_{\alpha\alpha} C_{\alpha\beta} + A_{\alpha\beta} C_{\alpha\alpha} = 0 \tag{29}
\]
connecting submatrices of interest with submatrices
\[
C_{\alpha\alpha} = N_\alpha CN_\alpha^T, \quad C_{\alpha\beta} = N_\alpha CR_\alpha^T, \quad \text{matrices} \quad C.
\]

Note that relations (29) are not a standard block decomposition of matrices \( A \) and \( C \), since the submatrices under consideration do not form continuous blocks in the corresponding matrices, therefore (29) can be considered as a generalization of the block decomposition of nondegenerate matrices. For problems with real physical content, this generalization is very significant, since the bases for the decomposition are determined from the context of the problem.

Solving equations (29) we find \( A_{\alpha\alpha}^{-1} \) and \( A_{\alpha\alpha}^{-1} A_{\alpha\beta} \), and, substituting these expressions in equation (28), we finally obtain
\[
\Sigma_\alpha^T = (C_{\alpha\alpha} - C_{\alpha\beta} C_{\alpha\beta} C_{\alpha\alpha}) N_\alpha \Pi^{-1} \text{def}_x U^T +
+ C_{\alpha\beta} C_{\alpha\beta} \Sigma^T \tag{30}
\]
equation satisfying all the requirements.
4. CONCLUSION

In the work, the characteristic form of the equations of dynamics of deformable bodies with an asymmetric elastic matrix is constructed. As such bodies, there can be crystals with defects and a number of thin-walled and statically stressed building structures grown under nonequilibrium conditions.

It is shown that, in contrast to bodies with a symmetric matrix, in such bodies, longitudinal and transverse waves interact with each other inside the body. Since it is assumed that the tangential stresses $\sigma_{ij}$ and $\sigma_{ji}$ are not equal to each other, the obtained equations describe moment media, without additional assumptions, such as, for example, in the Cosserat pseudocontinuum theory [12].

During the construction, the simplest matrices $e$, $R$, $Q$, $S$ were used, which allow both to group the defined variables and to select the necessary components from the constructed groups. They are either an elementary matrix a row of the same dimension as the variable for which they are used, or they are a scalar, vector or dyad product of such row matrices. It turned out that these matrices are enough to construct the matrix form of all differential invariants in both Cartesian and any orthogonal curvilinear coordinate system.

It is convenient to use these matrices for various transformations of coordinate systems, in particular, together with the usual rotation matrix $\theta$, they allow one to construct rotation matrices for three, six, and nine-component vectors and write differential invariants in a rotated coordinate system relative to the original one. In three, six, and nine-dimensional spaces, the matrices $e$, $R$, $Q$, $S$ form unit bases.

The following example shows that such a group approach allows one to quickly and comfortably solve various problems of mathematical physics.

In conclusion, we note that the transformations carried out in [7] and in this work, and the example considered above, do not exhaust all the possibilities of using the mathematical apparatus described above.

REFERENCES

1. Magomedov KM. On the calculation of the desired surfaces in spatial methods of characteristics. DAN USSR, 1966, 171(6):1297-1300.
2. Magomedov KM, Colds AC. Grid-characteristic numerical methods. Moscow, Nauka Publ., 1988, 288 pp.
3. Petrov IB, Colds AC. On the regularization of discontinuous numerical solutions of hyperbolic equations. ZhVMiFM, 1984, 24(8):1172-1188.
4. Kukujanov VN. Numerical solution of non-one-dimensional problems of the propagation of stress waves in solids. Vol. 8, 67 p. Moscow, 1967, VC AN USSR.
5. Kukujanov VN, Kondaurov VI. Numerical solution of non-one-dimensional problems of the dynamics of a solid deformable body. In: Problems of the Dynamics of Elastoplastic Media. Moscow, Mir Publ., 1975, p. 40-84.
6. Bulychev GG, Kukujanov VN. Dynamic failure of a prestressed fiber composite caused by fiber breakage. Izvestia RAS, ser. MTT, 1993, 3:207-214.
7. Georgy G. Bulychev. Method of spatial characteristics in problems of a mechanics of deformable solid body. Radioelectronics. Nanosystems. Information Technologies (RENSIT), 2018, 10(1)77-90. DOI: 10.17725/rensit.2018.10.077.
8. Hirt J, Lot I. Theory of dislocations. Moscow, Atomizdat Publ., 1972, 598 p.
9. Shafranovsky II. Crystals of minerals. Curved, skeletal and dendritic forms. Moscow, Gosgeoltekhizdat Publ., 1961, 332 p.
10. Petrashen GI. Propagation of waves in anisotropic elastic bodies. Moscow, Nauka Publ., 1980, p. 225-234.
11. Britvin EI. Formation of a stiffness matrix of thin-walled rods taking into account the influence of shear deformation. Construction mechanics and calculation of structures, 2017, 1:23-28.
12. Novatsky V. Theory of elasticity. Moscow, Nauka Publ., 1975, p. 797-805.