Modelling of the Effect of Piece-Wise Potential Volume Forces in the Photo-Elasticity Method

Elifkhan Agakhanov¹, Murad Agakhanov², Marat Khusseinov ³

¹ Dagestan State Engineering University, 367015, 70, Imam Shamil Avenue, Makhachkala, Russia
² Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, 129337, Russia
³ Dagestan State Engineering University, 367015, 70, Imam Shamil Avenue, Makhachkala, Russia

muradak@mail.ru

Abstract. The authors note a special feature complicating the direct modelling process in the photo-elasticity method (using the models made from widely-used epoxy materials) with volume forces even of the simplest kind (mass forces). This problem is more complicated than the problems with constant parameters \( \nu, E, \gamma, \alpha \) in the whole body. The authors give some additional relationships between the similarity factors, the use of which may create some restrictions and difficulties in the process of modelling non-uniform problems through traditional methods. The authors use the theoretical principles of mechanics of deformable rigid bodies and show that the solution of the problem in case, when the volume forces in different domains of the body have different potential values, may be reduced to the solutions of the problems with potential volume forces and to the solution of the problem with given stress values. The authors have developed both methodology and engineering questions of modelling piece-wise potential volume forces using the equivalence of effects. They note that the methods of solution of the problems with the loadings substituting the volume forces have been developed in full and give efficient results.

1. Introduction

The modelling methods are widely used for the solution of problems in mechanics of deformable rigid bodies [1-3]. The modelling of problems in mechanics of deformable rigid bodies (the problems in the elasticity theory, the thermo-elasticity theory, the wave dynamics and the creep theory) is carried out with the help of similarity criteria which are the basis for the model and determine the loading conditions; on the basis of these criteria, the transition from measured model values to corresponding values for the real structure is performed.

As the full-scale experimental studies are complicated, time-consuming and expensive, the experimental studies on small-scale models should be considered as the most promising ones. The most widely-used method here is the optical polarization method (photo-elasticity method); the advantages of the method are the following: its clearness, high precision and a wide range of the problems which can be solved by it.

The similarity factors for stresses, strains, displacement, geometric dimensions, volume forces and elasticity modulus in the process of modelling the creep theory problems and the elasticity theory
problems (in the elasticity theory problems the similarity factors become constant values, i.e. the similarity scales) are connected through the relationships

$$\frac{K_t K_F(t)}{K_o(t)} = 1, \quad \frac{K_t K_F(t)}{K_o(t) K_E} = 1, \quad \frac{K_t^2 K_F(t)}{K_o(t) K_E} = 1,$$

(1)

It follows from the relationships (1) that the stresses, strains and displacements caused by volume forces are reduced proportionally to the similarity scale. This special feature complicates the direct modeling in the photo-elasticity method when the models from widely-used epoxy materials are used (even for the simplest kind of volume forces - mass forces) [4, 5]. General methods of study of stresses and strains usually record them with an inadequate sensitivity. It follows from two last relationships (1) that the increase in the strains and displacements of the model may be achieved by the use of low-modulus optically sensitive materials. There is another possibility of measuring strains or some attendant parameters with a required accuracy in the process of modelling using small-scale models of the stress-and-strain state caused by mass forces (intentional gain in model material mass). In the photo-elasticity method, the intentional gain in model material mass is achieved by the use of methods based on the equivalence of effects (centrifuging, immersion analogy or a combined use of both methods).

In the process of modelling the problems, when the parameters vary in the body domains, it is necessary to consider some additional relationships between the similarity factors (scales):

$$K_{v_1} = K_{v_2} = ... = K_{v_n} = 1, \quad K_{E_1} = K_{E_2} = ... = K_{E_n}, \quad K_{F_1}(t) = K_{F_2}(t) = ... = K_{F_n}(t), \quad K_{\alpha_1} = K_{\alpha_2} = ... = K_{\alpha_n}, \quad K_{o\alpha r} = 1,$$

where \( n \) – number of domains with different parameter values. These conditions create some restrictions and difficulties for the use of traditional methods. They are connected with the fact that the production of optically active materials with different parameters is restricted. The use of some properties of polymers displayed in the process of polymerization (methods of polymerization, of straitened shrinkage and of fixation of temperature stresses) give certain possibilities for the solution of such problems. Besides that, they develop the methods intended for the determination of stresses in the problems where the body domains have different parameter values caused by the mechanical loads and the temperature in the models made from standard optically sensitive materials. The method of mechanical modelling of temperature stresses and the method of study of stresses in visco-elastic models are to the point, too. We will take into account the availability of the aforesaid methods allowing us to obtain an efficient solution for the body composed of the domains with different parameters using available standard optically sensitive material. So we will assume the constant values of the aforesaid parameters in different body domains when considering the volume forces action problem.

2. Problem statement

Consider a body of an arbitrary form occupying a space domain \( V \) restricted by a closed surface \( B = B_p + B_o \). Let the stress-and-strain state in the body be caused by the volume forces \( F(t) \) and the displacements \( f(t) \) of the points at the border part \( B_p \). In general case, the volume forces in different body domains \( V_1, V_2, ..., V_n \) may be distributed to different laws and may have different potentials.

Denote the number of these domains as \( n \), and the surfaces of their contacts as \( \Gamma \). Then \( +\Gamma \) and \( -\Gamma \) will stand for the contact surface for the domains located at the both sides of \( \Gamma \).

3. Theoretical principles

The system of equations for the solution of the problem under consideration consists of the following [6]:

\[ \text{(Equations go here)} \]
equilibrium equations

\[ \nabla^2 U_i^{(m)}(t) + \frac{\partial}{\partial t} \left[ \frac{e_i^{(m)}(t)}{3} + \frac{\Theta_i^{(m)}(t)}{3G(t)} \right] + \int \left[ \nabla^2 U_i^{(m)}(\tau) + \frac{\partial e_i^{(m)}(\tau)}{\partial \tau} \right] \frac{\partial}{\partial \tau} \Delta_i(t,\tau) d\tau + \frac{F_i^{(m)}(t)}{G(t)} = 0 \]

\[ \times \frac{\partial}{\partial \tau} \Delta_i(t,\tau) d\tau + \frac{F_i^{(m)}(t)}{G(t)} = 0 \text{ in } V_m, \tag{2} \]

conditions on the surface \( B_a \)

\[ \sum 2G(t) \left\{ \varepsilon_{ij}(t) + \left[ \frac{\Theta(t)}{6G(t)} - \frac{e(t)}{3} \right] \delta_{ij} + \int \left[ \varepsilon_{ij}(\tau) - \frac{e(\tau)}{3} \right] \delta_{ij} d\tau \right\} \frac{\partial}{\partial \tau} \Delta_i(t,\tau) n_j = 0, \tag{3} \]

conditions on the surface \( B_b \)

\[ U_i(t) = f_i(t), \tag{4} \]

conditions on the contact surface \( \Gamma \)

\[ U_i^{(\Gamma)}(t) = U_i^{(\Gamma)}(t), \tag{5} \]

\[ \sigma_{ij}^{(\Gamma)}(t) n_j = \sigma_{ij}^{(\Gamma)}(t) n_j + \bar{e}, \tag{6} \]

Here \((i, j = x, y, z), (m = 1, 2, ..., n), 1\)

\[ e(t) = \sum \varepsilon_{ij}(t), \tag{7} \]

\[ \theta(t) = \sum \sigma_{ij}(t) = K(t) \left[ e(t) + \int \varepsilon(\tau) \frac{\partial}{\partial \tau} \Delta_i(t,\tau) d\tau \right]. \tag{8} \]

The stress and strain values are connected through the relationships

\[ \sigma_{ij}(t) = 2G(t) \left\{ \varepsilon_{ij}(t) + \left[ \frac{\Theta(t)}{6G(t)} - \frac{e(t)}{3} \right] \delta_{ij} + \int \left[ \varepsilon_{ij}(\tau) - \frac{e(\tau)}{3} \right] \delta_{ij} d\tau \right\} \frac{\partial}{\partial \tau} \Delta_i(t,\tau) d\tau, \tag{9} \]

Divide the body \( V \) into the domains \( V_1, V_2, ..., V_n \) (here we don’t consider their interactions with each other) and introduce the auxiliary problems for every domain with the actions of given potential volume forces \( F_i^{(0)}(t), F_i^{(1)}(t), ..., F_i^{(m)}(t) \). Here some additional constraints should be introduced for the domains which are not in the equilibrium state. The effects of these additional constraints are taken into account through the results of the studies of particular problems. The solutions of these problems denote as \( \left[ \varepsilon_{ij}^{(m)}(t), U_i^{(m)}(t), \varepsilon_{ij}^{(m)}(t) \right] \).

The decision system of equations for additional problems includes the expressions similar to (2) – (4) and the condition
Now consider the problem for the body $V$ which differs from the initial problem only by the fact that the stresses $(\sigma_{ij}^{(1)}(t), ..., \sigma_{ij}^{(n)}(t))$ are “frozen” in the domains $V_1, V_2, ..., V_n$ instead of the corresponding volume forces $F_{ij}^{(1)}(t), F_{ij}^{(2)}(t), ..., F_{ij}^{(n)}(t)$.

We take the body $V$ assembled out of “frozen” domains for the initial non-strain state, the system of equations for the determination of stress-and-strain state arising after the “defreezing” $[\bar{\sigma}_{ij}^{(m)}(t), \bar{U}_{ij}^{(m)}(t), \bar{\epsilon}_{ij}^{(m)}(t)]$ has the form:

$$
\sum_j 2G(t) \left\{ \bar{\epsilon}_{ij}(t) + \left[ \frac{\bar{\theta}(t) - \bar{\epsilon}(t)}{6G(t)} \right] \delta_{ij} + \int_{\tau_i} \left[ \bar{\epsilon}_{ij}(\tau) - \frac{\bar{\epsilon}(\tau)}{3} \delta_{ij} \right] \times \frac{\partial}{\partial \tau} \Delta_i(t, \tau) d\tau \right\} n_j = 0 \text{ on } \Gamma, \quad (10)
$$

$$
\nabla^2 \bar{U}_{ij}^{(m)}(t) + \frac{\partial}{\partial t} \left[ \frac{\bar{\epsilon}(t) + \bar{\theta}(t)}{3} \right] + \int_{\tau_i} \left[ \nabla^2 \bar{U}_{ij}^{(m)}(\tau) + \frac{\partial \bar{\epsilon}(\tau)}{\partial t} \right] \times \frac{\partial}{\partial \tau} \Delta_i(t, \tau) d\tau + \frac{F_{ij}^{(m)}(t)}{G(t)} = 0 \text{ in } V_m, \quad (11)
$$

$$
\nabla^2 \bar{U}_{ij}^{(m)}(t) + \frac{\partial}{\partial t} \left[ \frac{\bar{\epsilon}(t) + \bar{\theta}(t)}{3} \right] + \int_{\tau_i} \left[ \nabla^2 \bar{U}_{ij}^{(m)}(\tau) + \frac{\partial \bar{\epsilon}(\tau)}{\partial t} \right] \times \frac{\partial}{\partial \tau} \Delta_i(t, \tau) d\tau + \frac{F_{ij}^{(m)}(t)}{G(t)} = 0 \text{ in } V_m, \quad (11)
$$

Here

$$
\bar{\epsilon}(t) = \sum_i \bar{\epsilon}_{ij}(t),
$$

$$
\bar{\theta}(t) = \sum_i \bar{\theta}_{ij}(t) = K(t) \left[ \bar{\epsilon}(t) + \int_{\tau_i} \bar{\epsilon}(\tau) \frac{\partial}{\partial \tau} \Delta_i(t, \tau) d\tau \right],
$$

The stress and strain values are connected through the relationships

$$
\bar{\sigma}_{ij}(t) = 2G(t) \left\{ \bar{\epsilon}_{ij}(t) + \left[ \frac{\bar{\theta}(t) - \bar{\epsilon}(t)}{6G(t)} \right] \delta_{ij} + \int_{\tau_i} \left[ \bar{\epsilon}_{ij}(\tau) - \frac{\bar{\epsilon}(\tau)}{3} \delta_{ij} \right] \times \frac{\partial}{\partial \tau} \Delta_i(t, \tau) d\tau \right\} - \bar{\sigma}_{ij}(t),
$$

Comparing the systems of equations (2) – (5) and (11) – (14), we have:

$$
U_i(t) = \bar{U}_i(t),
$$

$$
\epsilon_{ij}(t) = \bar{\epsilon}_{ij}(t),
$$

In case when the conditions (15) and (16) are satisfied, we obtain from the expressions (9) and (18):

$$
\sigma_{ij}(t) = \bar{\sigma}_{ij}(t) + \bar{\sigma}_{ij}(t),
$$
Thus, the solution of the problem in case when the volume forces in different body domains have
different potentials is reduced to the solution of the problems with potential volume forces and that of
the problem with given stress values.
If a direct creation of volume forces in the body domains is impossible, it is necessary to use the
equivalence of effects [7, 8]. It should be noted that the methods of solution of problems with the loads
substituting the volume forces are thoroughly developed and give efficient results.

4. Experimental implementation
A certain order of the procedure of experimental implementation of the method of modelling piece-
wise potential volume forces follows from the theoretical developments of this method:

1. We divide the given body with piece-wise potential volume forces into the domains and assume
that within each domain the volume forces are potential ones.
2. If in the process of release of domains from interactions with each other the equilibrium
conditions for any of them are upset, we may use additional constraints. But in this case we need the
tests of additional models for the evaluation of the effects of the aforesaid constraints. Therefore, we
use the force superposition principle and present the solution of the initial problem as the sum of
solutions of the problems in which the volume forces for the domains with additional constraints will
be equal to zero. Then the number of the models under tests will be equal to the number of the
domains in which the volume forces act in the initial problem.
3. We choose a way of creation of volume forces in the domains of the body. If a direct creation of
volume forces is impossible, we should use the equivalence of effects [9, 10]. Depending on the
problem to be solved we choose the conditions, in accordance with which we replace the action of the
given volume forces by the action of forced deformations and superficial forces.
4. We set the geometric similarity scale \( K = \frac{l_m}{l_r} \) (\( l_m \), \( l_r \) – linear dimensions for the real body and
the model, respectively). The scales of the models subject to different effects may be different.
5. We “freeze” the models of the body domains which are under the influence of volume forces.
Here the equilibrium of the models is guaranteed by the constraints similar to the constraints of the
real structure. If we use equivalent (substituting) effects, we choose a loading mode for the model
material blanks and “freeze” the forced deformations in them. The number of blanks and the “step” of
“frozen” forced deformations in them depend on the forms and dimensions of elements of the models chosen with respect to the distribution functions of forced deformation. Further we perform a mechanical treatment of “frozen” blanks in accordance with the distribution functions of forced deformations and glue together the models of domains. Then we anneal the domain models, assembled from preliminary “frozen” elements with the constraints similar to those in the real structure. Then we choose the way of application of surface loads and “freeze” the models of the body
domains, which are in an equilibrium state with the help of constraints similar to the constraints in the
real structure.
6. The standard photo-elasticity methods (strip method, compensation method) determine the
stresses \( \sigma_{ij}^{(F)} \) in the models of domains “frozen” under volume forces. When we use the equivalence
effects, we determine the stresses \( \sigma_{ij}^{(P)} \), \( \sigma_{ij}^{(S)} \) in the models of domains “frozen” under surface loads and
passed through the annealing cycle. The corresponding strain fields \( \varepsilon_{ij}^{(F)} \), \( \varepsilon_{ij}^{(P)} \) and \( \varepsilon_{ij}^{(S)} \) may be
determined by well-known experimental Moir’s method or through the relationships connecting the
strains and the stresses.
7. We produce the models of domains with additional artificial deformations \( \varepsilon_{ij}^{(F)} \) or \( \varepsilon_{ij}^{(P)} \) (when
we use equivalence effects) which do not cause any own stresses. Then we “freeze” these models
under volume forces or surface forces (in case when we use equivalence effects) through the chosen
earlier methods and remove these deformations which causes the stresses \( \sigma_{ij}^{(F)} \) or \( \sigma_{ij}^{(P)} \) in the
models of the body domains. Here the equilibrium of the models of the body domains is also achieved with the help of constraints similar to the constraints in the real structure.

8. If we use equivalence effects, we perform a mechanical treatment of “frozen” blanks with respect to the form and the dimensions of the model elements chosen in accordance with the strain distribution \( \varepsilon_{iy}^{(3)} \) and glue together the models of the body domains. In the models of the body domains, we “freeze” the stresses \( \sigma_{iy}^{(m)} \) which (depending on the problem to be solved) are determined through the corresponding expressions in accordance with the relationship

\[
\sigma_{iy}^{(F)} = \sigma_{iy}^{(p)} - \sigma_{iy}^{(3)} - \sigma_{iy}^{(m)},
\]

(22)

9. We glue together the models of the body domains containing the stresses \( \sigma_{iy}^{(F)} \) (item 9) which are in their natural non-strain state in accordance with the item 2. If we use equivalence effects, we glue together the models of the body domains assembled from preliminary “frozen” elements (item 10) which contain the stresses \( \sigma_{iy}^{(p)} \) (item 9), \( \sigma_{iy}^{(m)} \) (item 10) and are in their natural non-strain state in accordance with the item 2 and the expression (22).

10. We anneal the models in the presence of the constraints similar to the constraints in the real structure.

11. Using the standard photo-elasticity methods (strip method, compensation method), we determine the stresses in the models passed through the annealing procedure:

- In the models with given stresses – \( \sigma_{iy}^{(F)} \);
- with the use of equivalence effects
- In the models with given stresses – \( \sigma_{iy}^{(p)} \);
- In the models with earlier “frozen” strains \( \varepsilon_{iy}^{(3)} \);
- In the models with given stresses \( \sigma_{iy}^{(m)} \).

12. In accordance with the expression (21), we calculate the stresses in the models caused by acting volume forces in conformity with the item 2 \( \sigma_{iu}^{(F)} = \sigma_{iu}^{(p)} + \sigma_{iu}^{(F)} \) or caused by loadings substituting the volume forces \( \sigma_{iu}^{(p)} = \sigma_{iu}^{(p)} + \sigma_{iu}^{(F)} \), \( \sigma_{iu}^{(3)} = \sigma_{iu}^{(3)} + \sigma_{iu}^{(F)} \), \( \sigma_{iu}^{(m)} = \sigma_{iu}^{(m)} + \sigma_{iu}^{(F)} \).

13. With consideration of the superposition principle used in the item 2, we calculate the stresses in the model of the initial problem \( \sigma_{iu}^{(F)} \) or the stresses caused by substituting loadings \( \sigma_{iu}^{(p)} \), \( \sigma_{iu}^{(3)} \), \( \sigma_{iu}^{(m)} \).

14. We re-calculate the results of model studies for the real structure using the similarity formulas \( \left[ \sigma_{iu}^{(F)} \right] \) or \( \sigma_{iu}^{(3)} \), \( \sigma_{iu}^{(m)} \).

15. In accordance with the formula (22) and using the substituting loadings, we determine the stresses in the real structure caused by the volume forces \( \sigma_{iu}^{(F)} = \sigma_{iu}^{(p)} - \sigma_{iu}^{(3)} - \sigma_{iu}^{(m)} \). When the strain values \( \varepsilon_{iy}^{(F)} \) and \( \varepsilon_{iy}^{(p)} \) are small, the procedures corresponding to the items 6 and 9 become more simple:

1) the models of the body domains may be made without any additional deformations which excludes the necessity of preliminary determination of strains \( \varepsilon_{iy}^{(F)} \) and \( \varepsilon_{iy}^{(p)} \);

2) in the case of a two-dimensional (plane) problem, all the information concerning the stresses \( \sigma_{iy}^{(F)} \) and \( \sigma_{iy}^{(p)} \) may be obtained without producing any sections; in this case, as the models of the body domains with the stresses \( \sigma_{iy}^{(F)} \) and \( \sigma_{iy}^{(p)} \), we can use already “frozen” models of the body domains for the determination of stresses \( -\sigma_{iy}^{(F)} \) and \( -\sigma_{iy}^{(p)} \) under volume forces and superficial loading.
5. Conclusions
The solution of the problem in case when the volume forces in different body domains have different potentials is reduced to the solution of the problems with potential volume forces and that of the problem with given stress values.

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