Abstract: In this paper, optimal actuator shape for nonlinear parabolic systems is discussed. The system under study is an abstract differential equation with a locally Lipschitz nonlinear part. A quadratic cost on the state and input of the system is considered. The existence of an optimal actuator shape has been established in the literature. This paper focuses on driving the optimality conditions for actuator shapes belonging to a Banach space. The application of the theory to the optimal actuator shape design for railway track model is considered.

Keywords: Infinite-dimensional systems, Optimal control theory, Parabolic systems, Semigroup and operator theory, Actuator shapes

1. INTRODUCTION

Actuator shape is an important design variable for feedback synthesis in control of distributed parameter systems. Optimizing actuator shape can improve performance of the controller and significantly reduces the cost of control. Numerical simulations in Kalise et al. (2018) show significant improvement in the cost and performance of the control.

Optimal shape of actuators has only been studied in few works. In (Privat et al. (2013)), the optimal shape and position of an actuator for the wave equation in one spatial dimension are discussed. An actuator is placed on a subset \( \omega \in [0, \pi] \) with a constant Lebesgue measure \( L\pi \) for some \( L \in (0, 1) \). The optimal actuator minimizes the norm of a Hilbert Uniqueness Method (HUM)-based control; such control steers the system from a given initial state to zero state in finite time. In (Privat et al. (2017)), optimal actuator shape and position for linear parabolic systems are discussed. This paper adopts the same approach as in (Privat et al. (2013)) but with initial conditions that have randomized Fourier coefficients. The cost is defined as the average of the norm of HUM-based controls. In Kalise et al. (2018), optimal actuator design for linear diffusion equations has been discussed. A quadratic cost function is considered, and shape and topological derivative of this function are derived. Optimal sensor design problems are in many ways similar to the optimal actuator design problems. In (Privat et al. (2015)), optimal sensor shape design has been studied where the observability is maximized over all admissible sensor shapes. Controllability-based approaches to actuator design were used in (Münch and Periago (2011); Münch (2009); Münch and Periago (2013)). Numerical techniques to optimize the actuator design concurrently with the controller are mostly limited to linear quadratic regulator problems and location of actuators, see for example Allaire et al. (2010); Kumar and Seinfeld (1978); Kubrusly and Malebranche (1985); Darivandi et al. (2013). An \( H_\infty \)-approach was used in Kasinathan and Morris (2013).

The previous studies have only discussed optimal actuator shape for linear systems. Optimal actuator design problems for nonlinear distributed parameter systems has also been studied. In (Edalatzadeh and Morris (2019c)), it is shown that under certain conditions on the nonlinearity and the cost function, an optimal input and actuator design exist, and optimality equations are derived. Results are applied to the nonlinear railway track model as well as to the semi-linear wave model in two spatial dimensions. The existence of an optimal shape in a Banach space for nonlinear systems has been discussed in (Edalatzadeh and Morris (2019a)). The optimality conditions in (Edalatzadeh and Morris (2019a)) are derived for admissible actuator shapes in Hilbert spaces. The actuator shape space in this paper is an arbitrary Banach space. Optimality conditions for actuator shapes over a subset of a Banach space are obtained for nonlinear parabolic systems. A quadratic cost function on the state and input is considered to be minimized. The theory can be applied to various models. Some applications are nonlinear diffusion equation, Kuramoto-Sivashinsky equation, and nonlinear beam models (Edalatzadeh and Alasty (2016)).
Edalatzadeh and Morris (2019b,c,a)). In this paper, the application of the theory to the optimal actuator shape design for railway track model is considered.

### 2. NOTATION AND DEFINITIONS

Let $X$ be a Hilbert space. The notation $X_1 \hookrightarrow X_2$ means that the space $X_1$ is densely and continuously embedded in $X_2$. Let $I$ be a set on the real line, and $m$ be a non-negative number. The Banach space $H^m(I; X)$ is the space of all strongly measurable functions $x : I \to X$ for which $\|x(t)\|_X$ is in $H^m(I, \mathbb{R})$. For simplicity of notation, when $I$ is an interval, the corresponding space will be indicated without the braces; for example $L^2([0, \tau]; X)$ will be indicated by $L^2(0, \tau; X)$.

The Banach space $W(0, \tau)$ is the set of all $x(\cdot) \in H^1(0, \tau; \mathbb{R}) \cap L^2(0, \tau; D(A))$ with norm (Bensoussan et al., 2015, Section II.2)

$$\|x\|_{W(0, \tau)} = \|\dot{x}\|_{L^2(0, \tau; \mathbb{R})} + \|Ax\|_{L^2(0, \tau; \mathbb{R})}.$$

When there is no ambiguity, the norm on $X$ will not be explicitly indicated.

For every $p \in [1, \infty)$ and $\alpha \in (0, 1)$, the interpolation space $D_A(\alpha, p)$ is defined as the set of all $x_0 \in X$ such that the function

$$t \mapsto v(t) = \left\|t^{1-\alpha}/p \cdot 2t^{\alpha} x_0\right\|$$

belongs to $L^p(0, 1)$ (Lunardi, 2012, Section 2.2.1). The norm on this space is

$$\|x_0\|_{D_A(\alpha, p)} = \|x_0\| + \|v\|_{L^p(0, 1)}.$$

### 3. OPTIMAL ACTUATOR DESIGN

Let $x(t)$ and $u(t)$ be the state and input taking values in Hilbert spaces $X$ and $U$, respectively. Also, let $r$ denote the actuator design parameter that takes value in a compact set $K_{ad}$ in a Banach space $\mathbb{K}$. Consider the following initial value problem (IVP):

$$\begin{cases}
\dot{x}(t) = A x(t) + F(x(t)) + B(r) u(t), & t > 0, \\
x(0) = x_0.
\end{cases}$$

(2)

The nonlinear operator $F(\cdot)$ maps a Hilbert space $V$ to $X$. It is assumed that $D_A(1/2, 2) \hookrightarrow V \hookrightarrow X$.

The linear operator $A$ is associated with a sesquilinear form $a : V \times V \to \mathbb{C}$ (see Lang, 2012, Chapter 4)). Let there be positive numbers $\alpha$ and $\beta$ such that

$$|a(x_1, x_2)| \leq \alpha \|x_1\|_V \|x_2\|_V, \quad \forall x_1, x_2 \in V,$$

$$\text{Re} \ a(x, x) \geq \beta \|x\|^2_X, \quad \forall x \in X.$$

The operator $A$ has an extension to $\bar{A} \in L(V, V')$ described by

$$\langle \bar{A}v, w \rangle_{V', V} = a(v, w), \quad \forall v, w \in V,$$

(3)

where $V'$ denotes the dual of $V$ with respect to pivot space $X$.

According to Edalatzadeh and Morris (2019a), there are positive numbers $\tau, R_1, R_2$ such that (2) admits a unique solution for any initial conditions $x_0 \in B_{V(R_2)}$ and inputs $u \in B_{L^2(0, \tau; \mathbb{R})}(R_1)$ where

$$B_{V(R_2)} = \{ x_0 \in V : \|x_0\|_V \leq R_2 \},$$

$$B_{L^2(0, \tau; \mathbb{R})}(R_1) = \{ u \in L^2(0, \tau; \mathbb{R}) : \|u\|_2 \leq R_1 \}.$$
and integration by parts, the following Gâteaux derivative

\[ r \]

Morris (2019c):"

\[ v \]

track

\[ \text{Letting} \]

for all \( u \)

Outline of Proof:

Then

Thus, replacing the operator \( \mathcal{P}(t) \) with \( \mathcal{F}_x(t) \) and noting that

proves the lemma.

Proposition 3. Let assumptions A1 and A2 hold. The solution map \( S(u(t), r; x_0) \) is Gâteaux differentiable with respect to each \( u(t) \) and \( r \) in \( U_{ad} \times K_{ad} \). Let \( x(t) = S(u(t), r; x_0) \). The state map \( S(u(t), r; x_0) \) at \( r \) in the direction \( \dot{r} \) is the mapping \( S'(u_0; r; x_0) : \mathbb{R} \to \mathbb{R} \), \( \dot{r} \mapsto z(t) \), where \( z(t) \) is the strict solution to

\[ \begin{aligned}
\dot{z}(t) &= (A + \mathcal{F}_x(t))z(t) + B'(r_0; \dot{r})u(t), \\
\dot{z}(0) &= 0.
\end{aligned} \]

Theorem 4. Suppose assumptions A1 and A2 hold. For any initial condition \( x_0 \in \mathcal{X} \), let the pair \((u^*, r^*)\) \( \in U_{ad} \times K_{ad} \) be a local minimizer of the optimization problem (P) with the optimal trajectory \( x^* = S(u^*; r^*; x_0) \) and let \( p^*(t) \) indicate the strict solution in \( W(0, \tau)^* \) of the final value problem

\[ p^*(t) = -(A^* + \mathcal{F}_x(t)^*)p^*(t) - Qx^*(t), \quad p^*(\tau) = 0. \]

Then \( u^*(t) = -R^{-1}B'(r^*; p^*(t))p^*(t) \) and

\[ \int_0^\tau \langle p^*(t), B'(r^*; \dot{r})u^*(t) \rangle dt = 0 \]

for all \( \dot{r} \in K_{ad} \).

Outline of Proof: Theorem 11 in (Edalatzadeh and Morris 2019a) ensures that \( u^*(t) = -R^{-1}B'(r^*; p^*(t))p^*(t) \). To obtain (15), the Gâteaux derivative of

\[ G(u, r) = J(S(u, r; x_0), u) \]

at \( r^0 \) in the direction \( \dot{r} \) is taken. After some manipulation and integration by parts, the following Gâteaux derivative is obtained

\[ G'(u_0; r^0; \dot{r}) = \int_0^\tau \langle p^*(t), B'(r^0; \dot{r})u^*(t) \rangle dt. \]

The optimality condition now follows by setting the Gâteaux derivative to zero.

4. RAILWAY TRACK MODEL

Letting \([0, \tau] \) indicate the time interval of interest, the following semilinear PDE governs the motion of a railway track \( w(x, t) \) with initial deflection \( w_0(x) \) and rate of deflection \( v_0(x) \) on \((x, t) \in [0, 1] \times [0, \tau] \) (Edalatzadeh and Morris 2019c):

\[ \begin{aligned}
\partial_t w + \partial_x x(\partial_x w + C_d \partial_x x w) + \mu \partial_x w + w + \alpha w^3 &= b(x, r)u(t), \\
w(x, 0) &= w_0(x), \quad \partial_x w(x, 0) = v_0(x), \\
w(0, t) &= w(1, t) = 0, \\
\partial_x w(0, t) + C_d \partial_x x w(0, t) &= 0, \\
\partial_x w(1, t) + C_d \partial_x x w(1, t) &= 0,
\end{aligned} \]

where the subscript \( \partial_x \) denotes the derivative with respect to \( x \); the derivative with respect to \( t \) is indicated similarly. The nonlinear part of the foundation elasticity corresponds to the coefficients \( \alpha \). The constant \( \mu \geq 0 \) is the viscous damping coefficient of the foundation, and \( C_d \geq 0 \) is the coefficient of Kelvin-Voigt damping in the beam. The track deflection is controlled by a single external force \( u(t) \). The shape influence function \( b(x, r) \) is a continuous function over \([0, 1] \) parametrized by the parameter \( r \) that describes its dependence on the actuator design. The function \( b(x, r) \) is differentiable with respect to \( r \).

Choose state \( x := (w, v) \) where \( v = \partial_t w \) and define the state space \( \mathcal{X} := H^2(0, 1) \times H_0^1(0, 1) \times L^2(0, 1) \) with norm

\[ \|w, v\|^2 = \int_0^1 (\partial_x v)^2 + w^2 + v^2 \, dx. \]

Define the closed self-adjoint positive operator \( A_{kw0} w := \partial_x x w \),

\[ D(A_0) := \{ w \in H^1(0, 1) \mid w(0) = w(1) = 0 \}, \]

and also define

\[ A_{kvw}, K(w, v) := (v, -A_{kw0}(w + C_d w)), \]

\[ \mathcal{X} \times \mathbb{R} \]

state operator \( A \) is defined as

\[ A := A_{kvw} + K, \quad \text{with} \quad D(A) = D(A_{kvw}). \]

Let \( U := \mathbb{R} \), the input operator \( B(r) : U \to \mathcal{X} \)

\[ B(r)u := (0, b(x, r)u) \]

The nonlinear operator \( F(t) : \mathcal{X} \to \mathcal{X} \) is defined as

\[ F(w, v) := 0, \quad (w, v, x) \in \mathcal{X} \]

With these definitions and by setting the state \( x(t) = w(t, t), v(t, t) \) and initial condition \( x_0 = (w_0(t), v_0(t)) \), the state space representation of the railway track model is

\[ \begin{aligned}
\dot{x}(t) &= Ax(t) + F(x(t)) + B(r)u(t), \quad t \in [0, \tau], \\
x(0) &= x_0.
\end{aligned} \]

Well-posedness and stability of this model has been established (Edalatzadeh and Morris 2019b).

The set of admissible inputs, \( U_{ad} \), is a convex and closed subset of \( L^2(0, \tau) \). The set of admissible actuator designs is denoted by \( K_{ad} \) and is compact in a Banach space \( \mathcal{K} \).

The cost function is

\[ J(u, r; x_0) := \frac{1}{2} \int_0^\tau \|x(t)\|^2 + \gamma \|u(t)\|^2 \, dt. \]

For a fixed initial condition \( x_0 \in \mathcal{X} \), consider the following optimization problem over the admissible input set \( U_{ad} \) and actuator design set \( K_{ad} \).
Let $p^o(t)$ indicate the strict solution of the final value problem

$$
p^o(t) = -\langle A^* + F^c(x^*(t))^* p^o(t) \frac{1}{2} x^o(t), x^o(t) \rangle = 0. \quad (27)
$$

Any solution $(u^o, r^o)$ to (P) in the interior of $U_{ad} \times K_{ad}$ satisfies

$$
\frac{\gamma}{2} u^o(t) + \mathcal{B}'(r^o) p^o(t) = 0, \quad (28a)
$$

$$
\int_0^\tau \langle p^o(t), \mathcal{B}'(r^o, \tilde{r}) u^o(t) \rangle \, dt = 0 \quad \forall \tilde{r} \in K_{ad}. \quad (28b)
$$

Write $p^o(t) = (f^o(x, t), g^o(x, t))$ and $b(x, r^o, \tilde{r})$ be the Gâteaux derivative of $b(x, r)$ at $r^o$ in the direction $\tilde{r}$. The optimality conditions can be written

$$
u^o(t) = \frac{2}{\gamma} \int_0^1 b(x, r^o) g^o(x, t) \, dx, \quad (29)
$$

$$
\int_0^\tau \int_0^1 g^o(x, t) b'(x, r^o, \tilde{r}) v^o(\tilde{r}) \, dx \, dt = 0, \quad \forall \tilde{r} \in K_{ad}. \quad (30)
$$

5. FUTURE DIRECTIONS

Future directions will focus on application of the theory to various PDE models. An example of a nonlinear parabolic PDE model is Kuramoto-Sivashinsky equation which models propagation of flames as well as dynamics of thin film fluids. Another example is nonlinear models of flexible beams with Kelvin-Voigt damping. Future research will develop of suitable numerical schemes for computation of optimal actuator shapes.

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