Gluing theory of Riemann surfaces and Liouville conformal field theory

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Abstract: We study the gluing theory of Riemann surfaces using formal algebraic geometry, and give computable relations between the associated parameters for different gluing processes. As its application to the Liouville conformal field theory, we construct the sheaf of tempered conformal blocks on the moduli space of pointed Riemann surfaces which satisfies the factorization property and has a natural action of the Teichmüller groupoid.

1. Introduction

The Liouville conformal field theory is well studied since it is an important example of non-rational conformal field theories, and there are remarkable relations pointed by physicists with the quantum Teichmüller theory (cf. [V, T3, T5]) and the 4 dimensional gauge theory (cf. [AGT]). A basic tool in the study of the Liouville theory is to consider expansions of local conformal blocks by gluing parameters of Riemann surfaces. For example, the AGT correspondence conjectures the coincidence between these expansions and instanton partition functions. Furthermore, Teschner [T1, T2, T3, T4, T5] claims that by studying analytic continuations of these expansions, one may obtain spaces of global conformal blocks satisfying the factorization principle. The aim of this paper is to apply the gluing theory of Riemann surfaces to the study of Liouville conformal blocks, especially of Teschner’s consideration.

First, we study the gluing theory of Riemann surfaces using formal algebraic geometry, and give computable relations between the associated parameters for different gluing processes. By this result, one can study arithmetic geometry of Teichmüller groupoids which was introduced by Grothendieck [G], and studied by Moore-Seiberg [MS] and others [BK1, BK2, FG, G, HLS, NS]. Second, by studying analytic continuations of (local) gluing conformal blocks, we construct (generally infinite dimensional) Hilbert spaces consisting of “tempered” Liouville conformal blocks. More precisely, these Hilbert spaces give a sheaf of conformal blocks, namely a vector bundle with projectively flat connection on the moduli space of pointed Riemann surfaces which satisfies the factorization property and has a natural action of the Teichmüller groupoid. Therefore, we can give a mathematical foundation of considerations by Teschner [T3, T4, T5] on the “modular functor conjecture”, namely that there exists a global theory of Liouville conformal blocks which gives a modular functor in the sense of Segal [Se].

The organization of this paper is as follows. In Section 2, we recall results of [I1, I2] on computable relations between deformation parameters of degenerate (algebraic)
curves which are used in Section 3 to construct Teichmüller groupoids in the category of arithmetic geometry. In Section 4, by combining these results with results of Teschner [T1, T2, T3] and Hadasz-Jaskólski-Suchanek [HJS], we construct the sheaf of tempered Liouville conformal blocks.

2. Deformation of degenerate curves

2.1. Degenerate curve. We recall the well known correspondence between certain graphs and degenerate pointed curves, where a (pointed) curve is called degenerate if it is a stable (pointed) curve and the normalization of its irreducible components are all projective (pointed) lines. A graph $\Delta = (V, E, T)$ means a collection of 3 finite sets $V$ of vertices, $E$ of edges, $T$ of tails and 2 boundary maps

$$b : T \rightarrow V, \quad b : E \rightarrow (V \cup \{\text{unordered pairs of elements of } V\})$$

such that the geometric realization of $\Delta$ is connected. A graph $\Delta$ is called stable if its each vertex has degree $\geq 3$, i.e. has at least 3 branches. Then for a degenerate pointed curve, its dual graph $\Delta = (V, E, T)$ by the correspondence:

- $V$ $\longleftrightarrow$ \{irreducible components of the curve\},
- $E$ $\longleftrightarrow$ \{singular points on the curve\},
- $T$ $\longleftrightarrow$ \{marked points on the curve\}

such that an edge (resp. a tail) of $\Delta$ has a vertex as its boundary if the corresponding singular (resp. marked) point belongs to the corresponding component. Denote by $|X|$ the number of elements of a finite set $X$. Under fixing a bijection $\nu : T \sim \rightarrow \{1, ..., |T|\}$, which we call a numbering of $T$, a stable graph $\Delta = (V, E, T)$ becomes the dual graph of a degenerate $|T|$-pointed curve of genus rank $\mathbb{Z}H_1(\Delta, \mathbb{Z})$ and that each tail $h \in T$ corresponds to the $\nu(h)$th marked point. In particular, a stable graph without tail is the dual graph of a degenerate (non-pointed) curve by this correspondence. If $\Delta$ is trivalent, i.e. any vertex of $\Delta$ has just 3 branches, then a degenerate $|T|$-pointed curve with dual graph $\Delta$ is maximally degenerate.

2.2. Generalized Tate curve. Let $\Delta = (V, E)$ be a stable graph without tail, and under an orientation of $\Delta$, i.e., an orientation of each $e \in E$, denote by $v_h$ the terminal vertex of $h \in \pm E$ (resp. the boundary vertex $b(h)$ of $h \in T$). Take a subset $E$ of $\pm E = \{e, -e \mid e \in E\}$ whose complement $E_\infty$ satisfies the condition that

$$E_\infty \cap \{-h \mid h \in E_\infty\} = \emptyset,$$

and that $v_h \neq v_{h'}$ for any distinct $h, h' \in E_\infty$. We attach variables $\alpha_h$ for $h \in E$ and $q_e = q^{-e}$ for $e \in E$. Let $A_0$ be the $\mathbb{Z}$-algebra generated by $\alpha_h$ ($h \in E$), $1/(\alpha_e - \alpha_{-e})$ ($e, -e \in E$) and $1/(\alpha_h - \alpha_{h'})$ ($h, h' \in E$ with $h \neq h'$ and $v_h = v_{h'}$), and let

$$A = A_0[[q_e (e \in E)]], \quad B = A \left[ \prod_{e \in E} q_e^{-1} \right].$$
According to [I1, Section 2], we construct the universal Schottky group \( \Gamma \) associated with oriented \( \Delta \) and \( \mathcal{E} \) as follows. For \( h \in \pm E \), let \( \phi_h \) be the element of \( PGL_2(B) = GL_2(B)/B^\times \) given by

\[
\phi_h = \frac{1}{\alpha_h - \alpha_{-h}} \left( \begin{array}{cc} \alpha_h - \alpha_{-h}q_h & -\alpha_h\alpha_{-h}(1-q_h) \\ 1-q_h & -\alpha_{-h} + \alpha_h q_h \end{array} \right) \mod(B^\times),
\]

where \( \alpha_h \) (resp. \( \alpha_{-h} \)) means \( \infty \) if \( h \) (resp. \( -h \)) belongs to \( \mathcal{E}_\infty \). Then

\[
\frac{\phi_h(z) - \alpha_h}{z - \alpha_h} = q_h \frac{\phi_h(z) - \alpha_{-h}}{z - \alpha_{-h}} \quad (z \in \mathbb{P}^1),
\]

where \( PGL_2 \) acts on \( \mathbb{P}^1 \) by linear fractional transformation.

For any reduced path \( \rho = h(1) \cdot h(2) \cdots h(l) \) which is the product of oriented edges \( h(1), \ldots, h(l) \) such that \( h(i) \neq -h(i+1) \) and \( v_{h(i)} = v_{-h(i+1)} \), one can associate an element \( \rho^* \) of \( PGL_2(B) \) having reduced expression \( \phi_{h(l)} \phi_{h(l-1)} \cdots \phi_{h(1)} \). Fix a base point \( v_b \) of \( V \), and consider the fundamental group \( \pi_1(\Delta, v_b) \) which is a free group of rank \( g = \text{rank}_2 H_1(\Delta, \mathbb{Z}) \). Then the correspondence \( \rho \mapsto \rho^* \) gives an injective anti-homomorphism \( \pi_1(\Delta, v_b) \to PGL_2(B) \) whose image is denoted by \( \Gamma \). It is shown in [I1, Section 3] (and had been shown in [IN, Section 2] when \( \Delta \) is trivalent and has no loop) that for any stable graph \( \Delta = (V, E) \) without tail, there exists a stable curve \( C_\Delta \) of genus \( g \) over \( A \) which satisfies the following:

- The closed fiber \( C_\Delta \otimes_A A_0 \) of \( C_\Delta \) given by putting \( q_e = 0 \ (e \in E) \) is the degenerate curve over \( A_0 \) with dual graph \( \Delta \) which is obtained from \( P_v := \mathbb{P}^1_{A_0} \) \( (v \in V) \) by identifying \( \alpha_e \in P_v \) and \( \alpha_{-e} \in P_{v-e} \) \( (e \in E) \), where \( \alpha_h = \infty \) if \( h \in \mathcal{E}_\infty \).

- \( C_\Delta \) gives a universal deformation of \( C_\Delta \otimes_A A_0 \).

- \( C_\Delta \otimes_A B \) is smooth over \( B \) and is Mumford uniformized (cf. [M]) by \( \Gamma \).

- Let \( \alpha_h \ (h \in \mathcal{E}) \) be complex numbers such that \( \alpha_e \neq \alpha_{-e} \) and that \( \alpha_h \neq \alpha_{h'} \) if \( h \neq h' \) and \( v_h = v_{h'} \). Then for sufficiently small complex numbers \( q_e \neq 0 \ (e \in E) \), \( C_\Delta \) becomes a Riemann surface which is Schottky uniformized (cf. [S]) by \( \Gamma \).

We apply the above result to construct a uniformized deformation of a degenerate pointed curve which had been done by Ihara and Nakamura (cf. [IN, Section 2, Theorems 1 and 10]) when the degenerate pointed curve is maximally degenerate and consists of smooth pointed projective lines. Let \( \Delta = (V, E, T) \) be a stable graph with numbering \( \nu \) of \( T \). We define its extension \( \hat{\Delta} = (\hat{V}, \hat{E}) \) as a stable graph without tail by adding a vertex with a loop to the end distinct from \( v_h \) for each tail \( h \in T \). Then from the uniformized curve associated with \( \hat{\Delta} \), by substituting 0 for the deformation parameters which correspond to \( e \in \hat{E} - E \) and by replacing the singular projective lines which correspond to \( v \in \hat{V} - V \) with marked points, one has the required universal deformation.
2.3. Comparison of deformations. A rigidification of an oriented stable graph \( \Delta = (V,E,T) \) with numbering \( \nu \) of \( T \) means a collection \( \tau = (\tau_v)_{v \in V} \) of injective maps

\[
\tau_v : \{0, 1, \infty\} \to \{ h \in \pm E \cup T \mid v_h = v \}
\]

such that \( \tau_v(a) \neq -\tau_v(a) \) for any \( a \in \{0, 1, \infty\} \) and distinct elements \( v, v' \in V \) with \( \tau_v(a), \tau_v(a) \in \pm E \). One can see that any stable graph has a rigidification by the induction on the number of edges and tails. Let \( \Delta = (V,E,T) \) be a stable graph with numbering of \( T \) such that only one vertex, which we denote by \( v_0 \), has 4 branches and that the other vertices have 3 branches. Fix an orientation of \( \Delta \), and denote by \( h_1, h_2, h_3, h_4 \) the mutually different elements of \( \pm E \cup T \) with terminal vertex \( v_0 \). Then one can take a rigidification \( \tau = (\tau_v)_{v \in V} \) of \( \Delta \) such that

\[
\tau_{v_0}(0) = h_2, \quad \tau_{v_0}(1) = h_3, \quad \tau_{v_0}(\infty) = h_4,
\]

and hence \( x = x_{h_1} \) gives the coordinate on \( \mathbb{P}^1_{\mathbb{Z}} - \{0, 1, \infty\} \). Denote by \( C_{(\Delta, \tau)} \) the uniformized deformation given in 2.2 which is a stable \( |T| \)-pointed curve over

\[
A_{(\Delta, \tau)} = \mathbb{Z}[x, x^{-1}, (1 - x)^{-1}] \left[ \left[ y_v(e \in E) \right] \right].
\]

Let \( \Delta' = (V', E', T') \) (resp. \( \Delta'' = (V'', E'', T'') \)) be the trivalent graphs obtained from \( \Delta = (V,E,T) \) by replacing \( v_0 \) with an edge \( e'_0 \) (resp. \( e''_0 \)) having two boundary vertices one of which is a boundary of \( h_1, h_2 \) (resp. \( h_3, h_4 \)) and another is a boundary of \( h_3, h_4 \) (resp. \( h_2, h_4 \)). Then one can identify \( T', T'' \) with \( T \) naturally, and it is easy to see that according as \( x \to 0 \) (resp. \( x \to 1 \)), the degenerate \( |T| \)-pointed curve corresponding to \( x \) becomes the maximally degenerate \( |T| \)-pointed curve with dual graph \( \Delta' \) (resp. \( \Delta'' \)). Let \( \Delta' \) (resp. \( \Delta'' \)) without \( e'_0 \) (resp. \( e''_0 \)) have the orientation naturally induced from that of \( \Delta \), and let \( h_i' \) (resp. \( h_i'' \)) be the edge \( e'_0 \) (resp. \( e''_0 \)) with orientation. For \( i = 1, 2, 3, 4 \), we denote by \( h_i' \) (resp. \( h_i'' \)) the oriented edge in \( \Delta' \) (resp. \( \Delta'' \)) corresponding to \( h_i \) and identify the invariant part

\[
E^{\text{inv}} = E - \{|h_i| \mid 1 \leq i \leq 4\}
\]

of \( E \) as that of \( E' \) and \( E'' \). Then as seen above, for a rigidification \( \tau' \) (resp. \( \tau'' \)) of \( \Delta' \) (resp. \( \Delta'' \)), we have the uniformized deformation \( C_{(\Delta', \tau')} \) (resp. \( C_{(\Delta'', \tau'')} \)) which is a stable \( |T| \)-pointed curve over

\[
A_{(\Delta', \tau')} = \mathbb{Z}[s_{e'}(e' \in E')] \quad \text{(resp.} \quad A_{(\Delta'', \tau'')} = \mathbb{Z}[t_{e''}(e'' \in E'')] \text{)}.
\]

Then we will consider two isomorphisms of \( C_{(\Delta, \tau)} \) to \( C_{(\Delta', \tau')} \) and to \( C_{(\Delta'', \tau'')} \). Note that under the isomorphisms, the comparison between parameters of the base rings depends on the situation whether some \( h_i \) (\( 1 \leq i \leq 4 \)) are loops or not. In Theorem 2.1 below, we make the comparison in restricted cases for the saving of space since the other cases are seen to be treated similarly from the proof.

**Theorem 2.1.** (cf. [12, Theorem 1]) Put \( I = \{1 \leq i \leq 4 \mid h_i \in \pm E\} \), denote by \( y_i \) the deformation parameters associated with \( h_i \) for \( i \in I \), and denote by \( s_j \) (resp. \( t_j \)) the deformation parameters associated with \( h'_j \) (resp. \( h''_j \)) for \( j \in \{0\} \cup I \). Then we have
Remark 1. In (1) and (2) above, the constant terms of the ratios in $|\Delta|)$ can be shown in the same way. Over a certain open subset of $\Delta$, the assertion in general case follows, and we only prove (1) since (2)

Proof. We review the proof given in [I2] since it also gives the method of comparing deformation parameters of degenerate curves. We prove the theorem when $\Delta$ has no tail from which the assertion in general case follows, and we only prove (1) since (2) can be shown in the same way. Over a certain open subset of

$|\Delta|)$
with sufficiently small absolute values \(|x|\) and \(|y_e|\), \(C(\Delta, \tau)\) gives a deformation of the
degenerate curve with dual graph \(\Delta\). Hence there exists an isomorphism
\[
\mathbb{C}((x))[y_e(e \in E)] \cong \mathbb{C}((s_0))[s_{e'}(e' \neq e_0)]
\]
which induces an isomorphism \(C(\Delta, \tau) \cong C(\Delta', \tau')\) such that the degenerations of \(C(\Delta, \tau)\)
given by \(y_i \to 0\) (\(1 \leq i \leq 4\)) and \(y_e \to 0\) \((e \in E^\text{inv})\) correspond to those of \(C(\Delta', \tau')\) given by \(s_i \to 0\) and \(s_e \to 0\) respectively. Since these two curves are Mumford
uniformized, a result of Mumford [M, Corollary 4.11] implies that the uniformizing
groups \(\Gamma(\Delta, \tau)\) and \(\Gamma(\Delta', \tau')\) of \(C(\Delta, \tau)\) and \(C(\Delta', \tau')\) respectively are conjugate over the
quotient field of \(\mathbb{C}((x))[y_e(e \in E)] \cong \mathbb{C}((s_0))[s_{e'}(e' \neq e_0)]\). Denote by \(\iota : \Gamma(\Delta, \tau) \xrightarrow{\sim} \Gamma(\Delta', \tau')\) the isomorphism defined by this conjugation. Since eigenvalues are invariant
under conjugation and the cross ratio
\[
[a; b; c; d] = \frac{(a - c)(b - d)}{(a - d)(b - c)}
\]
of 4 points \(a, b, c, d\) is invariant under linear fractional transformation, one can see the
following:

(A) For any \(\gamma \in \Gamma(\Delta, \tau)\), the multiplier of \(\gamma\) is equal to that of \(\iota(\gamma)\) via the above
isomorphism.

(B) For any \(\gamma_i \in \Gamma(\Delta, \tau)\) \((1 \leq i \leq 4)\), the cross ratio \([a_1, a_2; a_3, a_4]\) of the attractive
fixed points \(a_i\) of \(\gamma_i\) is equal to that of \(\iota(\gamma_i)\) via the above isomorphism.

We consider the case that \(|h_i|\) \((1 \leq i \leq 4)\) are mutually different.
\[
A_1 = \mathbb{Z} \left[ \frac{y_1}{x}, \frac{y_2}{x}, y_3, y_4, y_e \ (e \in E^\text{inv}) \right],
\]
whose quotient field is denoted by \(\Omega_1\), and let \(I_1\) be the ideal of \(A_1\) generated by \(x, y_1/x, y_2/x, y_3, y_4\) and \(y_e \ (e \in E^\text{inv})\). Then from (A) and (B) as above and results in [I, §1], we will show that the isomorphism descends to \(A_1 \cong A(\Delta', \tau')\), where the variables
are related as in the statement of Theorem 1 (1). We take local coordinates \(\xi_h\) as
\[
\xi_h(z) = \begin{cases} 
  z - x & \text{if } h = h_1, \\
  z & \text{if } h = \tau_v(0) \text{ for some } v \in V, \\
  z - 1 & \text{if } h = \tau_v(1) \text{ for some } v \in V, \\
  1/z & \text{if } h = \tau_v(\infty) \text{ for some } v \in V.
\end{cases}
\]
For \(z \in \mathbb{P}^1(\Omega_1)\) with \(\xi_h(z) \in I_1\) and \(h' \in \pm E - \{h\}\) with \(v_{-h'} = v_h\), by the calculation
in the proof of [I, Lemma 1.2], one can see that \(\xi_{h'}(\phi_{h'}(z)) \in I_1\). Hence for \(\gamma \in \Gamma(\Delta, \tau)\)
with reduced expression \(\phi_{h_1(1)} \cdots \phi_{h_1(t)}\), if we take \(h \in \pm E - \{-h(t)\}\) with \(v_h = v_{-h(t)}\)
and \(z \in \mathbb{P}^1(\Omega_1)\) with \(\xi_h(z) \in I_1\), then by [I, Lemma 1.2], the attractive fixed point
\(a\) of \(\gamma\) is given by \(\lim_{a \to \infty} \gamma^n(z)\), and hence \(\xi_{h(t)}(a) \in I_1\). For each \(v \in V\), fix a path
\(\rho_v\) in \(\Delta\) from the base point \(v_h\) to \(v\). If \(\rho_i \in \pi_1(\Delta, v_0)\) has reduced expression \(\cdots h_i\)
(1 ≤ i ≤ 4), then the attractive fixed points \( a_i \) of \( \gamma_i = (\rho^*_{v_0})^{-1} \cdot \rho^t \cdot \rho^*_{v_0} \) satisfy that \([a_1, a_3; a_2, a_4] \in x \cdot (A_1)^\times\). Furthermore, by Proposition 1.4 and Theorem 1.5 of [I1], the attractive fixed points \( a_i^{(e)} \) of \( \iota(\gamma_i) \) satisfy that \([a_1^{(e)}; a_3^{(e)}; a_2^{(e)}] \in s_0 \cdot (A_{(\Delta', \tau')}^\times)\), and hence from (B), we have the comparison of \( y_i/x \) and \( s_i \) follows from applying (B) to \( \gamma_i = (\rho^*_{v_0})^{-1} \cdot \rho^t \cdot \rho^*_{v_0} \) (1 ≤ i ≤ 4), where \( \rho_i \in \pi_1(\Delta, v_0) \) has reduced expression:

\[
\begin{align*}
\rho_1 &= \cdots h_2, \\
\rho_2 &= \cdots h_3, \\
\rho_3 &= \cdots h_5 \cdot h_1, \\
\rho_4 &= \cdots h_6 \cdot h_1,
\end{align*}
\]

for distinct oriented edges \( h_5, h_6 \) with terminal vertex \( v_{-h_1} \). Similarly, we have the comparison of \( y_2/x \) (resp. \( y_3, y_4, y_e (e \in E^\text{inv} - \{\text{loops}\}) \)) and \( s_2 \) (resp. \( s_3, s_4, s_e \)) and further if \( e \in E^\text{inv} \) is a loop, then the comparison of \( y_e \) and \( s_e \) follows from applying (A) to \( \gamma = (\rho^*_{v_0})^{-1} \cdot \phi_e \cdot \rho^*_{v_0} \). Therefore, the above isomorphism descends to \( A_1 \cong A_{(\Delta', \tau')} \) such that under this isomorphism:

\[
\begin{align*}
x/s_0, & \frac{y_i}{x s_i} (i \in \{1, 2\}), \\
& \frac{y_i}{s_i} (i \in \{3, 4\}), \\
& \frac{y_e}{s_e} (e \in E^\text{inv})
\end{align*}
\]

belong to \((A_{(\Delta', \tau')})^\times\).

One can show the assertion in the case that \(|h_1| = |h_2| \) similarly. □

The following result was given substantially in [I2, 3.1], [I3, 1.2], and is explicitly shown here since this is crucial to prove results in Section 4.

**Theorem 2.2.** Let the notation be as above. Then there are elements \( u_i \) (1 ≤ i ≤ 3g − 4) of \( A_{(\Delta, \tau)} \) such that under \( x \to 0 \) (resp. 1), \( \{x, u_i\} \) (resp. \( \{1 - x, u_i\} \)) give deformation parameters of the closed fibers \( C_0 '(\Delta, \tau) \) (resp. \( C_0 ''(\Delta, \tau') \)), namely one has

\[
A_{(\Delta', \tau')} \cong \mathbb{Z}[\{x, u_i (1 ≤ i ≤ 3g - 4)\}], \quad A_{(\Delta'', \tau'')} \cong \mathbb{Z}[\{1 - x, u_i (1 ≤ i ≤ 3g - 4)\}].
\]

**Proof.** Assume that all branches starting from \( v_0 \) are not loops. We put \( y_i = y_{|v_i|} \) as in Theorem 2.1. Take

\[
u_i = \begin{cases} 
  y_1/x(1 - x) & \text{or } -y_1/x(1 - x) & (i = 1), \\
  y_2/x & \text{or } -y_2/x & (i = 2), \\
  y_3/(1 - x) & \text{or } -y_3/(1 - x) & (i = 3), \\
  y_4 & \text{or } -y_4 & (i = 4).
\end{cases}
\]

and \( u_i \) (i ≥ 5) by specifying one of \( y_e \) and \( -y_e \) for each \( e \in E^\text{inv} \). Then by Theorem 2.1, under \( x \to 0 \) (resp. 1), \( x \) (resp. \( 1 - x \)) and \( u_i \) (1 ≤ i ≤ 3g − 4) are deformation parameters of the maximally degenerate pointed curve \( C_0 ' \) (resp. \( C_0 '' \)). In the case that there are loops with boundary vertex \( v_0 \), one can also take required deformation parameters using Theorem 2.1. □
2.4. Ihara-Nakamura’s deformation. We consider deformations of maximally degenerate curves using “standard” local coordinates on $\mathbb{P}^1$ which is studied by Ihara and Nakamura [IN] with application to Galois theory on arithmetic fundamental groups of curves. Let $\Delta = (V, E, T)$ be a trivalent graph such that $\text{rank}_\mathbb{Z} H_1(\Delta, \mathbb{Z}) = g$, $|T| = n$, and $C$ denote the associated degenerate curve over $\mathbb{Z}$ which is a union of $P_v = \mathbb{P}^1$ ($v \in V$) with marked points $\alpha_t$ ($t \in T$), where $v_t = v$, by identifying $\alpha_h \in P_{v_h}$ and $\alpha_{-h} \in P_{v_{-h}}$ ($h \in \pm E$). We take a local coordinate $z_h$ on each $P_{v_h}$ such that

$$\{\text{marked points and singular points on } P_{v_h}\} = \{0, 1, \infty\},$$

and that $z_h(\alpha_h) = 0$. Then one can define the Ihara-Nakamura deformation $C_{\text{IN}}$ of $C_0$ over the ring $\mathbb{Z}[[q_e]]$ of integral formal power series of variables $q_e$ ($e \in E$) by the relation $z_h z_{-h} = q_{|h|}$ ($h \in \pm E$). As is shown in [IN, 2.4.2], for any $v_0 \in V$, for each $\gamma \in \pi_1(\Delta, v_0)$, one can associates an element $\gamma^* \in \text{PGL}_2(\mathbb{Z}[[q_e]])$ as follows. Let $v_0, h_0, v_1, h_1, \ldots, v_d = v_0$ ($d \geq 0$)

be the reduced path on $\Delta$ representing $\gamma$ such that $v_{-h_i} = v_i$ and $v_{h_i} = v_{i+1}$. Let $g_k$ ($1 \leq k \leq d - 1$) be the element of $\text{PGL}_2(\mathbb{Z})$ defined by $z_{h_k} = g_k (z_{h_{k-1}})$, and $h_k$ ($1 \leq k \leq d - 1$) be the transform $z \mapsto q_{|h_k|} z^{-1}$ which comes from the relation $z_{h_k} = q_{|h_k|} z_{-h_k}^{-1}$. Put $\gamma^* = g_d \circ h_{d-1} \circ \cdots \circ g_1 \circ h_0 \circ g_0$, where $g_0, g_d$ are defined by $z_{-h_0} = g_0(z)$, $z = g_d (z_{h_{d-1}})$. Then $\gamma \mapsto \gamma^*$ gives a representation $\pi_1(\Delta, v_0) \to \text{PGL}_2(\mathbb{Z}[[q_e]])$ whose image is a Schottky group over $\mathbb{Z}[[q_e]]$ as in 2.2. Then by the same way as in the proof of Theorem 2.1 especially using the assertions (A) and (B), one can compare the deformation parameters $q_e$ and those given in Theorem 2.1.

3. Teichmüller groupoid

3.1. Moduli space of curves. We review fundamental facts on the moduli space of pointed curves and its compactification [DM, KM, K]. Let $g$ and $n$ be non-negative integers such that $n$ and $2g - 2 + n$ are positive. Let $M_{g,n}$ (resp. $M_{g,n}^*$) denote the moduli stacks over $\mathbb{Z}$ of proper smooth curves of genus $g$ with $n$ marked points (resp. with $n$ marked points having non-zero tangent vectors). Then $M_{g,n}$ becomes a principal $(\mathbb{G}_m)^n$-bundle on $M_{g,n}$. Furthermore, let $\overline{M}_{g,n}$ denote the Deligne-Mumford-Knudsen compactification of $M_{g,n}$ which is defined as the moduli stack over $\mathbb{Z}$ of stable curves of genus $g$ with $n$ marked points, and $\overline{M}_{g,n}^*$ denote the $(\mathbb{A}^1)^n$-bundle on $\overline{M}_{g,n}$ containing $M_{g,n}^*$ naturally. For these moduli stacks $M_{g,n}$ and $\overline{M}_{g,n}$, $M_{g,n}^*$ and $\overline{M}_{g,n}^*$ denote the associated complex orbifolds. A point at infinity on $M_{g,n}$ (resp. $M_{g,n}^*$) is a point on $\overline{M}_{g,n}$ (resp. $\overline{M}_{g,n}^*$) which corresponds to a maximally degenerate $n$-pointed curve, and a tangential point at infinity is a point at infinity with tangential structure over $\mathbb{Z}$. 

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We describe the boundary of $\mathcal{M}_{g,n}$. Denote by $D_0$ the divisor of $\overline{\mathcal{M}}_{g,n}$ corresponding to singular stable marked curves which are desingularized to stable curves of genus $g-1$ with $n+2$ marked points. For an integer $i$ with $1 \leq i \leq [g/2]$, and $S$ be a subset of $P = \{1, \ldots, n\}$ such that $2i - 2 + |S|, 2(g - i) - 2 + n - |S|$ are positive. denote by $D_{i,S}$ the divisor of $\overline{\mathcal{M}}_{g,n}$ corresponding to singular stable marked curves which are desingularized to the sum of pairs of stable curves of genus $i$ with $|S|$ marked points and of genus $g-i$ with $n-|S|$ marked points. Then $\overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ consists of normal crossing divisors $D_0, D_{i,S}$, and hence $\overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ consists of the pullbacks of $D_0, D_{i,S}$ by the natural projection $\overline{\mathcal{M}}_{g,n} \to \mathcal{M}_{g,n}$ which we denote by the same notation.

3.2. Teichmüller groupoid. The Teichmüller groupoid for $\mathcal{M}_{g,\bar{n}}$ is defined as the fundamental groupoid for $\mathcal{M}^{an}_{g,\bar{n}}$ with tangential base points at infinity. Its fundamental paths called basic moves are half-Dehn twists, fusing moves and simple moves defined as follows.

Let $\Delta = (V, E, T)$ be a trivalent graph as above, and assume that $\Delta$ is trivalent. Then for any rigidification $\tau$ of $\Delta$, $\pm E \cup T = \bigcup_{v \in V} \text{Im}(\tau_v)$, and hence $A_{(\Delta, \tau)}$ is the formal power series ring over $\mathbb{Z}$ of $3g + n - 3$ variables $q_e$ ($e \in E$). First, the half-Dehn twist associated with $\epsilon$ is defined as the deformation of the pointed Riemann surface corresponding to $C_\Delta$ by $q_e \mapsto -q_e$. Second, a fusing move (or associative move, A-move) is defined to be different degeneration processes of a 4-hold Riemann sphere. A fusing move changes $(\Delta, e)$ to another trivalent graph $(\Delta', e')$ such that $\Delta, \Delta'$ become the same graph, which we denote by $\Delta''$, if $e, e'$ shrink to a point. We denote this move by $\varphi(e, e')$. As is done in [I2, Section 3] and [I3, Theorem 1], one can construct this move using Theorem 2.2. Finally, simple move (or S-move) is defined to be different degeneration processes of a 1-hold complex torus.

Then as the completeness theorem called in [MS], the following Theorem 2.1 is conjectured in [G] and shown in [BK1, BK2, FG, HLS, MS, NS] (especially in [NS, Sections 7 and 8] using the notion of quilt-decompositions of Riemann surfaces).

**Theorem 3.1.** (cf. [BK1, BK2, FG, HLS, MS, NS] and [NS, Sections 7 and 8]) The Teichmüller groupoid is generated by half-Dehn twists, fusing moves and simple moves with relations induced from $\mathcal{M}_{0,\bar{4}}, \mathcal{M}_{0,\bar{5}}, \mathcal{M}_{1,\bar{1}}$ and $\mathcal{M}_{1,\bar{2}}$.

4. Construction of Liouville conformal blocks

4.1. Conformal blocks. Fix a real number $c > 1$ called the central charge, and define the Virasoro algebra $\text{Vir}_c$ with generators $L_n$ ($n \in \mathbb{Z}$) satisfying the relations

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}.$$ 

Put $S = Q/2 + \sqrt{-1} \cdot \mathbb{R}^+$, and for each $\alpha \in S$, denote by $V_\alpha$ the irreducible highest weight representation of $\text{Vir}_c$ with generator $e_\alpha$ which is annihilated by $L_n$ ($n > 0$) and has the $L_0$-eigenvalue $\Delta_\alpha = \alpha(Q - \alpha)$, where $c = 1 + 6Q^2$. Then for any $v \in V_\alpha$,
$L_n(v) = 0$ if $n \gg 0$. There exists a unique inner product $\langle \cdot, \cdot \rangle_{V_\alpha}$ on $V_\alpha$ such that

$$\langle L_n(v), w \rangle_{V_\alpha} = \langle v, L_{-n}(w) \rangle_{V_\alpha} \quad (v, w \in V_\alpha), \quad \langle e_\alpha, e_\alpha \rangle = 1.$$  

Under $2g-2+n > 0$, let $C$ be a Riemann surface of genus $g$ with $n$ marked points $P_1, \ldots, P_n$ and local coordinates $t_i$ ($i = 1, \ldots, n$) vanishing at $P_i$. We associate highest weight representations $V_{\alpha_i}$ of $\text{Vir}_c$ to $P_i$ ($i = 1, \ldots, n$), and define the action of

$$\chi = \left( \sum_{k \in \mathbb{Z}} \chi_k^{(i)} t_i^{k+1} \partial_{t_i} \bigg) \in \bigoplus_{1 \leq i \leq n} \mathbb{C}((t_i)) \partial_{t_i}$$

on $\bigotimes_{i=1}^n V_{\alpha_i}$ as the following finite sum

$$\rho_{\chi}(v_1 \otimes \cdots \otimes v_n) = - \sum_{i=1}^n v_1 \otimes \cdots \otimes \left( \sum_{k \in \mathbb{Z}} \chi_k^{(i)} L_k(v_i) \right) \otimes \cdots \otimes v_n \quad (v_i \in V_{\alpha_i}).$$

Denote by $\mathcal{D}_C$ the Lie algebra of meromorphic differential operators on $C$ which may have poles only at $P_1, \ldots, P_n$. Then (invariant) conformal blocks associated with $(C; P_i, t_i)$ are linear maps $\mathcal{F}_C : \bigotimes_{i=1}^n V_{\alpha_i} \to \mathbb{C}$ satisfying the invariance property:

$$\mathcal{F}_C(\rho_{\chi}(v)) = 0 \quad \left( \chi \in \mathcal{D}_C, \ v \in \bigotimes_{i=1}^n V_{\alpha_i} \right),$$

where $\chi$ is regarded as an element of $\bigoplus_{i=1}^n \mathbb{C}((t_i)) \partial_{t_i}$ (cf. [FB, T3, T5]). If $(g, n) = (0, 3)$, then $\mathcal{F}_C$ is uniquely determined by the values $\mathcal{F}_C(\epsilon_{\alpha_1} \otimes \epsilon_{\alpha_2} \otimes \epsilon_{\alpha_3})$.

Let $C_i$ ($i = 1, 2$) be two Riemann surfaces with $n_i + 1$ marked points and associated local coordinates, and denote by $C_1 \sharp C_2$ the pointed Riemann surfaces obtained by gluing $C_i$ at their $(n_i + 1)$th points via the gluing parameter $q$. The gluing of conformal blocks $\mathcal{F}_{C_1}, \mathcal{F}_{C_2}$ by $\beta \in \mathbb{S}$ is defined as

$$\mathcal{F}_{C_1 \sharp C_2}^\beta (v_1 \otimes v_2) = \sum_{l, m} \mathcal{F}_{C_1} (v_1 \otimes v_l) \langle v_l^*, q^{L_0} v_m \rangle_{V_\beta} \mathcal{F}_{C_2} (v_m^* \otimes v_2),$$

where $\{v_l\}, \{v_l^*\}$ are dual bases of $V_\beta$ for $\langle \cdot, \cdot \rangle_{V_\beta}$. Then $\mathcal{F}_{C_1 \sharp C_2}^\beta (v_1 \otimes v_2)$ is the product of $q^{\Delta_\beta}$ and a formal power series of $q$ with constant term $\mathcal{F}_{C_1} (v_1 \otimes e_\beta) \mathcal{F}_{C_2} (e_\beta \otimes v_2)$.

For a Riemann surface $C$ with $n + 2$ marked points and associated local coordinates, the gluing $\mathcal{F}_{C_\sigma}^\beta$ of the conformal block $\mathcal{F}_C$ can be defined in a similar way, where $C^\sharp$ denotes the pointed Riemann surface obtained by gluing the $(n + 1)$th and $(n + 2)$th points on $C$. Let $\sigma$ be a pants decomposition of a Riemann surface of genus $g$ with $n$ marked points and local coordinates, and $\beta$ be an $\mathbb{S}$-valued function on the set $E(\sigma)$ of edges associated with $\sigma$. Then we define the \textit{gluing conformal block} $\mathcal{F}_{C_\sigma}^\beta$ as the gluing of conformal blocks on 3-pointed Riemann spheres, and it is represented as a formal power series of deformation parameters of the degenerate curve associated with $\sigma$. 


Let $C/S$ be a family of stable curves over $\mathbb{C}$ of genus $g$ with $n$ marked points $P_i$ and local coordinates $t_i$ ($1 \leq i \leq n$). Denote by $\sigma_i : S \to C$ the section corresponding to $P_i$. Then it is shown in [BK2, 7.4] that in the category of algebraic geometry, one can let $\mathcal{T}_S$ act on the sheaf of conformal blocks as follows. For a vector field $\theta$ on $S$, there exists a lift $\tilde{\theta}$ as a vector field on $C - \bigcup_{i=1}^{n} \sigma_i(S)$ since it is affine over $S$. Take the $i$th vertical component (for the local coordinate $t_i$) $\tilde{\theta}_{\text{vert}}$ of $\tilde{\theta}$; $\tilde{\theta}_{\text{holiz}}(t_i) = 0$, and define the action of $\tilde{\theta}$ on $\bigotimes_{i=1}^{n} V_{\alpha_i} \otimes \mathcal{O}_S$ as

$$\theta(fv) = \theta(f)v + f \sum_{i=1}^{n} \rho_{\tilde{\theta}_{\text{vert}}}(v) \left( f \in \mathcal{O}_S, \ v \in \bigotimes_{i=1}^{n} V_{\alpha_i} \right).$$

Then by the definition of conformal blocks, this action gives the action of $\mathcal{T}_S$ on the sheaf of conformal blocks on $C/S$. We denote $\nabla$ the corresponding connection. Then the following result is well known more or less, and can be checked using statements and the proof of [BK2, Sections 7.4 and 7.8].

**Proposition 4.1.**

1. The connection $\nabla$ is projectively flat.
2. The residue of $\nabla$ around the singular locus of $C/S$ is given by the action of $L_0$.
3. The gluing conformal block $F^\beta_{\sigma}$ gives a (formal) flat section of $\nabla$.

**Proof.** The assertions (1) and (2) are shown in [BK2, Proposition 7.4.8] and [BK2, Example 7.4.12 and Corollary 7.8.9] respectively. We prove (3). By [BK2, Propositions 7.8.6 and 7.8.7] and the proof, $F^\beta_{\sigma}$ is the image of a constant section by a $\mathcal{T}_S$-equivariant map to the sheaf of conformal blocks over $S = \text{Spec}[\mathbb{Q}[e \in E(\sigma)]]$, and hence is a flat section of $\nabla$. □

4.2. Tempered conformal blocks. We recall results of Teschner [T1, T2, T3] on analytic continuations of Liouville conformal blocks on 4-pointed Riemann spheres. We normalize $N(\alpha_1, \alpha_2, \alpha_3) = F_C(e_{\alpha_1} \otimes e_{\alpha_2} \otimes e_{\alpha_3})$ as in [TV, (8.3) and (12.22)], and $\sigma$, $\sigma'$ be pants decompositions of $\mathbb{P}_C^1 - \{0, 1, \infty, x\}$ which are connected by a fusing move $x \in (0, 1)$. Then it is shown in [T1, T2, T3] that for each $\beta \in \mathbb{S}$, the associated conformal block

$$F^\beta_{\sigma} : V_{\alpha_1} \otimes V_{\alpha_2} \otimes V_{\alpha_3} \otimes V_{\alpha_4} \to \mathbb{C}$$

can be analytically continued along $(0, 1)$ to a meromorphic form around $x = 1$ which is represented as

$$\int_{\mathbb{S}} d\mu(\beta') \Phi_{\beta, \beta'} F^\beta_{\sigma'}. $$
where \( S = Q/2 + \sqrt{-1} \cdot \mathbb{R}^+ \) for a kernel function \( \Phi_{\beta, \beta'} \) and a measure \( d\mu(\beta') \) explicitly given in \([T2, 5.2]\) and \([T3, 2.1]\). Therefore, the analytic continuation along \((0, 1)\) gives rise to a canonical isomorphism between the Hilbert spaces

\[
\int_{S} d\beta \ C\mathcal{F}_{\sigma}^{\beta} \cong \int_{S} d\beta' \ C\mathcal{F}_{\sigma}^{\beta'},
\]

obtained as direct integrals.

The analytic continuation of \( \mathcal{F}_{\sigma}^{\beta} \) along a simple move in \( \mathcal{M}_{1,1}^{an} \) is given by Hadasz-Jaskólski-Suchanek \([HJS]\), and that along the half-Dehn twist associated with an edge \( e \) is the multiplication by \( \exp(\pi \sqrt{-1} \Delta_{\beta(e)}) \).

Using the above results, we will define the space of tempered Liouville conformal blocks. Let \( C \) be a Riemann surface of genus \( g \) with \( n \) marked points \( P_i \) and local coordinates \( t_i \) \((1 \leq i \leq n)\). Take a tangential point \( p_{\infty} \) at infinity, and a path \( \pi \) in \( \mathcal{M}_{g,n}^{an} \) from \( p_{\infty} \) to the point \( p_C \) corresponding to \( (C; P_i, t_i) \). Denote by \( \sigma \) the pants decomposition corresponding to \( p_{\infty} \), and take an \( S \)-valued function \( \beta \) on the set \( E(\sigma) \) of edges associated with \( \sigma \). For each \( v \in \bigotimes_{i=1}^{n} V_{\alpha_i} \), one can see that

\[
\prod_{e \in E(\sigma)} q_e^{-\Delta_{\beta(e)}} \cdot \mathcal{F}_{C}^{\beta}(v)
\]

becomes a formal power series of \( q_e \) \((e \in E(\sigma))\), and denote its constant term by \( C_{\sigma}^{\beta}(v) \).

Let \( \mathcal{F}_{C}^{\beta} \) be a conformal block associated with \((C; P_i, t_i)\) defined by the condition

\[
\lim_{p \to p_{\infty}} \prod_{e \in E(\sigma)} t^{-\Delta_{\beta(e)}} \cdot \text{Trans}^p_{p_{\infty}} \left( \mathcal{F}_{C}^{\beta}(v) \right) = C_{\sigma}^{\beta}(v),
\]

where \( p \in \pi \) approaches to \( p_{\infty} \) as \( t \downarrow 0 \), and \( \text{Trans}^p_{p_{\infty}} \) denotes the parallel transport by \( \nabla \) along \( \pi \) from \( p \) to \( p_{\infty} \). Then its analytic continuation as the flat section of \( \nabla \) along \( \pi \) to \( p_{\infty} \) has the main part \( C_{\sigma}^{\beta}(v) \prod_{e \in E(\sigma)} t^{\Delta_{\beta(e)}} \), and hence is equal to \( \mathcal{F}_{\sigma}^{\beta}(v) \) as a formal power series \( \left( \text{multiplied by } \prod_{e \in E(\sigma)} q_e^{\Delta_{\beta(e)}} \right) \).

We define the space \( \mathcal{C}_{\text{temp}} \left( \bigotimes_{i=1}^{n} V_{\alpha_i}, C \right) \) of tempered conformal blocks associated with \((C; P_i, t_i)\) as the direct integral

\[
\int_{\mathcal{M}_{g,n}^{an}} d\beta \bigotimes_{e \in E(\sigma)} C\mathcal{F}_{C}^{\beta}
\]

which is isomorphic to the Hilbert space of square-integrable functions on

\[
\left\{ (\beta(e))_{e \in E(\sigma)} \mid \beta(e) \in S \right\} \cong (\mathbb{R}^+)^{3g-3+n}.
\]

**Theorem 4.2.**

1. The Hilbert space \( \mathcal{C}_{\text{temp}} \left( \bigotimes_{i=1}^{n} V_{\alpha_i}, C \right) \) is independent of \( \pi \) and \( p_{\infty} \).
(2) The Hilbert space \( \mathbb{C}B_{\text{temp}}^{a_1}(\otimes_{i=1}^{n_1} V_{a_i}, C) \) satisfies the factorization property in the following sense. For Riemann surfaces \( C_i \) \((i = 1, 2)\) with \( n_i + 1 \) marked points and local coordinates,

\[
\mathbb{C}B_{\text{temp}}^{a_1}(\otimes_{i=1}^{n_1} V_{a_i} \otimes \left( \otimes_{j=1}^{n_2} V_{a_j} \right), C_1 \sharp C_2)
\]

is canonically isomorphic to the direct integral

\[
\int_{S}^{\oplus} d\beta \ \mathbb{C}B_{\text{temp}}^{a_1}(\otimes_{i=1}^{n_1} V_{a_i} \otimes \mathcal{V}_{\beta}, C_1) \otimes \mathbb{C}B_{\text{temp}}^{a_2}(\mathcal{V}_{\beta} \otimes \left( \otimes_{j=1}^{n_2} V_{a_j} \right), C_2)
\]

Similarly, for a Riemann surface \( C \) with \( n + 2 \) marked points and local coordinates, one has a canonical isomorphism

\[
\mathbb{C}B_{\text{temp}}(\otimes_{i=1}^{n} V_{a_i}, C) \cong \int_{S}^{\oplus} d\beta \ \mathbb{C}B_{\text{temp}}^{a_1}(\otimes_{i=1}^{n} V_{a_i} \otimes \mathcal{V}_{\beta} \otimes \mathcal{V}_{\beta}, C)
\]

(3) By the connection \( \nabla \), \( \mathbb{C}B_{\text{temp}}^{a_1}(\otimes_{i=1}^{n} V_{a_i}, C) \) has a projective action of the Teichmüller groupoid for \( \mathcal{M}_{g, \vec{n}} \) such that the action of fusing moves and simple moves is given by the action in the case when \((g, n) = (0, 4)\) and \((1, 1)\) respectively.

Proof. First, we prove (1). Since \( \nabla \) is projectively flat, \( \mathbb{C}B_{\text{temp}}^{a_1}(\otimes_{i=1}^{n} V_{a_i}, C) \) is independent of the homotopy class of \( \pi \). Then by Theorem 3.1, to prove (1), it is enough to show that \( \mathbb{C}B_{\text{temp}}^{a_1} \) is independent of moving \( p_{\infty} \) by fusing moves and simple moves. Let \( \sigma \) and \( \sigma' \) be pants decompositions of Riemann surfaces of genus \( g \) with \( n \) marked points such that \( \sigma, \sigma' \) are connected by a fusing move \( \varphi \). Then a gluing conformal block \( F_{\sigma}^{\beta} \) is represented as the gluing \( F_{C_1 \sharp C_2}^{\beta} \) of \( F_{C_1}^{\beta} \) and \( F_{C_2}^{\beta} \), where \( C_1 \) denotes a 4-pointed Riemann sphere associated with \( \varphi \). By the above result of Teschner [T1, T2, T3], there exists a form \( F_{C_1}^{\beta} \) which is the parallel transport of \( F_{C_1}^{\beta} \) along \( \varphi \) becomes the gluing of \( F_{C_1}^{\beta} \) and \( F_{C_2}^{\beta} \) by the deformation parameters \( u_i \) given in Theorem 2.2. Therefore, \( \mathbb{C}B_{\text{temp}}^{a_1} \) is independent of moving \( p_{\infty} \) by fusing moves. By the result of Hadasz-Jaskólski-Suchanek [HJS], the space of tempered conformal blocks for 1-pointed curves of genus 1 is stable under a simple move. Therefore, in a similar way as above, one can show that \( \mathbb{C}B_{\text{temp}}^{a_1} \) is independent of moving \( p_{\infty} \) by simple moves.

Second, we prove (2) in the former case (and the latter case can be shown in a similar way). Take pants decompositions of the Riemann surfaces \( C_i \) \((i = 1, 2)\) which give a pants decomposition of \( C_1 \sharp C_2 \), and denote by \( p_{\infty} \) the associated tangential point \( p_{\infty} \) at infinity. Then one can obtain the required isomorphism from the description of the space of tempered conformal blocks associated with \( C_1 \sharp C_2 \) by \( p_{\infty} \).

The assertion (3) follows from the construction of the space of tempered conformal blocks and the proof of (1), (2). □
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