ABSTRACT. The original Pak-Stanley labeling was defined by Pak and Stanley in [5] as a bijective map from the set of regions of an extended Shi arrangement to the set of parking functions. This map was later generalized to other arrangements associated with graphs and directed multi-graphs (see [1, 3, 4]). In these more general cases the map is no longer bijective. However, it was shown in [3] and [4] that it is surjective to the set of the $G$-parking functions, where $G$ is the multigraph associated with the arrangement.

This leads to a natural question: when is the generalized Pak-Stanley map bijective? In this paper we answer this question in the special case of centered hyperplane arrangements, i.e. the case when all the hyperplanes of the arrangement pass through a common point.

INTRODUCTION.

Let $V \subset \mathbb{R}^n$ be given by $x_1 + \ldots + x_n = 0$. Consider an arrangement $\mathcal{A}$ of affine hyperplanes in $V$, such that every hyperplane of $\mathcal{A}$ is of the form $H_{i,j}^a := \{x_i - x_j = a\}$ for some $i, j \in \{1, \ldots, n\}$ and $a > 0$. Let $G_{\mathcal{A}}$ be the associated directed multigraph, defined as follows. The set of vertices of $G_{\mathcal{A}}$ is $\{1, \ldots, n\}$, and the edge $i \rightarrow j$ has multiplicity

$$m_{ij} := \sharp\{a \in \mathbb{R}_{>0} | H_{i,j}^a \in \mathcal{A}\}.$$ 

Note that one gets $m_{ij} + m_{ji}$ hyperplanes parallel to $\{x_i = x_j\}$ in $\mathcal{A}$, $m_{ij}$ of them on one side of the origin, and $m_{ji}$ of them on the other. Note also that the multigraph $G_{\mathcal{A}}$ does not determine the combinatorial type of the arrangement $\mathcal{A}$, as one can shift the hyperplanes by changing the constants on the right hand sides of the equations without changing the graph.

Definition 1. We will call the arrangements of the type described above the multigraphical arrangements.

The generalized Pak-Stanley labeling of the regions (connected components of the complement) of a multigraphical arrangement was defined in [4]:

Definition 2. Let $R$ be a region of $\mathcal{A}$. Let $\mathcal{A}_R \subset \mathcal{A}$ be the subset consisting of the hyperplanes that separate $R$ from the origin. We define the label $\lambda_R$ to be the function $\lambda_R : \{1, \ldots, n\} \rightarrow \mathbb{Z}_{\geq 0}$ given by the following formula:

$$\lambda_R(i) := \sharp\{(a, j) | a \in \mathbb{R}_{>0}, j \in \{1, \ldots, n\}, \text{ and } H_{i,j}^a \in \mathcal{A}_R\}.$$ 

In other words, $\lambda_R(i)$ equals to the number of hyperplanes of the arrangement $\mathcal{A}$ of the form $H_{i,j}^a$ separating the region $R$ from the origin. (Note that here $i$ is fixed, but $j$ and $a$ might vary.)
We will use the notation $\langle \lambda(1), \ldots, \lambda(n) \rangle$ for a label $\lambda$. The region $R_0$ containing the origin is called the fundamental region. It is the only region labeled by $\langle 0, \ldots, 0 \rangle$. Note that the labeling can be defined inductively: as one crosses a hyperplane $H_{a_{i,j}} = \{ x_i - x_j = a > 0 \}$ in the direction away from the origin, the $i$th component of the label is increased by one, while the rest of the components remain unchanged.

**Definition 3.** Let $G$ be a directed multigraph on a vertex set $\{1, \ldots, n\}$. A function $\lambda : \{1, \ldots, n\} \rightarrow \mathbb{Z}_{\geq 0}$ is called a $G$-parking function if for any non-empty subset $I \subset \{1, \ldots, n\}$ there exists a vertex $i \in I$ such that the number of edges $(i \to j) \in E_G$ such that $j \notin I$ is greater than or equal to $\lambda(i)$.

The following results were proved in [3] and [4]:

**Theorem 4 ([3, 4]).** Let $R$ be any region of a multigraphical arrangement $A$. Then the corresponding label $\lambda_R$ is a $G_A$-parking function.

**Theorem 5 ([3, 4]).** Let $A$ be a multigraphical arrangement, and let $\lambda$ be any $G_A$-parking function. Then there exists a region $R$ of $A$, such that $\lambda_R = \lambda$.

Combining the above, we get that the generalized Pak-Stanley labeling is a surjective map from the set of regions of $A$ to the set of $G_A$-parking functions.

In [3] these results were proved in a more restricted context. In [4] they were generalized to multigraphical arrangements. In the classical case of extended Shi arrangements, one can show the bijectivity of the Pak-Stanley labeling by using the above results and then comparing the cardinalities of the two sets. The bijectivity results can be extended to other families of arrangements (see [2]). However, in general the generalized Pak-Stanley labelings often fail to be injective. The natural question is to characterize the directed multigraphs for which there exist arrangements with bijective labelings. In this paper we answer this question for the special case of central affine multigraphical arrangements.

1. CENTRAL AFFINE MULTIGRAPHICAL ARRANGEMENTS.

In the case of central multigraphical arrangements, the arrangement is fully determined by the corresponding multigraph (up to a global shift). We start by characterizing the multigraphs corresponding to central arrangements.

**Theorem 6.** Let $A$ be a central multigraphical arrangement, then the corresponding multigraph is simple and acyclic. Vice versa, if $G$ is a simple acyclic digraph, then there exists a central multigraphical arrangement $A$, such that $G_A = G$.

**Proof.** Let $A$ be a central multigraphical arrangement such that all hyperplanes intersect at the point $c = (c_1, c_2, \ldots, c_n)$. Since all hyperplanes $H_{a_{i,j}}$ intersect at $c$, then we can have at most one $H_{a_{i,j}}$ for each pair $i, j$. Moreover, if we have a hyperplane $H_{a_{i,j}}$ then we cannot have a hyperplane of the form $H_{b_{j,i}}$, because they would also be parallel. Thus the digraph $G_A$ is simple.

Assume that $G_A$ contains the cycle $i_0 \to i_1 \to \cdots \to i_k \to i_0$. It then follows that the hyperplanes corresponding to the edges in the cycle exhibit
\[ x_{i_0} - x_{i_1} = a_1 > 0 \]
\[ x_{i_1} - x_{i_2} = a_2 > 0 \]
\[ \vdots \]
\[ x_{i_{k-1}} - x_{i_k} = a_k > 0 \]
\[ x_{i_k} - x_{i_0} = a_{k+1} > 0 \]

Since each hyperplane passes through the point \( c \) all these equations are satisfied at \( x = c \).

After taking the sum of the above equations we see that \( 0 = \sum_{i=1}^{k+1} a_i \) which contradicts the assumption that the \( a_i > 0 \) for all \( i \). Thus \( G_A \) is acyclic.

Now, given an acyclic digraph \( G = (V, E) \), with \( V = \{1, \ldots, n\} \), and without loss of generality assume that the edges are oriented in an increasing way. We create the corresponding arrangement \( A \) by: for every edge \( (i \rightarrow j) \in E \) create the hyperplane \( H_{i,j}^{i-j} \). Consider the following ordered \( n \)-tuples:

- For \( n = 2k, c = (k - 1/2, k - 3/2, \ldots, 1/2, -1/2, \ldots, -(k - 3/2), -(k - 1/2)) \)
- For \( n = 2k + 1, c = (k, k - 1, \ldots, 1, 0, -1, \ldots, -(k - 1), -k) \)

From these \( n \)-tuples, we see that the point \( c \) lies in the intersection of all the hyperplanes since \( x_i - x_j = j - i \) for all \( 1 \leq i < j \leq n \). Therefore the graph \( G \) has a corresponding central multigraphical arrangement.

Let \( A \) be a central multigraphical arrangement, and let \( A' \) be the linear arrangement obtained from \( A \) by shifting all the hyperplanes so that they pass through the origin. Let \( G \) be the simple graph obtained from \( G_A \) by removing the orientations on the edges. Then it is well-known that the acyclic orientations of \( G \) are in one to one correspondence with the regions of \( A' \). The bijection is constructed as follows. Given a region \( R \) of \( A' \) and an edge \( i \rightarrow j \) of \( G \), we orient it \( i \rightarrow j \) if and only if \( x_i < x_j \) at every point of \( R \).

The regions of the original arrangement \( A \) are simply the regions of \( A' \) shifted by a vector. Therefore, they are also in bijection with the acyclic orientations of \( G \), or acyclic reorientations of \( G_A \).

**Theorem 7.** The fundamental region of \( A \) corresponds to the original orientation of \( G_A \), and crossing a hyperplane \( H_{i,j}^{a} \in A \) switches the orientation of the corresponding edge between \( i \) and \( j \).

**Proof.** Let \( R_0 \) be the fundamental region of the arrangement \( A \), and let \( A' \) be the corresponding linear arrangement. Let \( c = (c_1, \ldots, c_n) \) be in the intersection of all the hyperplanes of the arrangement \( A \). Then it follows that \(-c \) belongs to the corresponding region \( R' = R_0 - c \) of \( A' \). Therefore, if \( H_{i,j}^{a} \in A \) and the edge \( i \rightarrow j \) is the corresponding edge in \( G_A \), then at \( c \) we have \( c_i - c_j = a \), in particular we have that \( c_i > c_j \). It then follows that at \(-c \in R' \) that we have \( c_i < -c_j \). Thus, in the orientation corresponding to \( R' \) we also get the edge oriented as \( i \rightarrow j \).

Finally, crossing a hyperplane \( H_{i,j}^{a} \) corresponds to crossing the hyperplane \( x_i = x_j \) of the linear arrangement \( A' \), which switches the orientation of the corresponding edge.

**Lemma 8.** The Pak-Stanley labels for the arrangement \( A \) can be computed in terms of acyclic reorientations of the graph \( G_A \). More precisely, for a region \( R \) of \( A \) the label \( \lambda_R(i) \) equals
to the number of edges of $G_A$ leading from $i$, such that their orientations got switched in the reorientation corresponding to $R$. 

Proof. For an arrangement $A$ the Pak-Stanley label for a region $R$ is calculated by counting the number of hyperplanes of the form $H_{i,j}^a$ separating $R$ from the origin and increasing the value $\lambda_R(i)$ accordingly. However, Theorem 7 implies that as we cross a hyperplane $H_{i,j}^a$ we reorient the edge from $(i \rightarrow j)$ to $(j \rightarrow i)$, so it follows that $\lambda_R(i)$ is the number of edges of $G_A$ leading from $i$ that get reoriented in the graph corresponding to $R$. 

Now we are ready to prove our main theorem:

**Theorem 9.** Let $V = \{1, 2, \ldots, n\}$ and $G = (V, E)$ be an acyclic directed graph on $n$ vertices with edges oriented in the increasing way. Then the hyperplane arrangement corresponding to $G$ produces duplicate Pak-Stanley labelings if and only if there exists $1 \leq k < i < j \leq n$ such that $(k \rightarrow i), (k \rightarrow j) \in E$ and $(i \rightarrow j) \notin E$.

**Proof of Theorem 9** $(\Rightarrow)$ Assume that $G$ produces duplicate Pak-Stanley labelings and for the sake of contraction assume that no such $i, j, k$ exists. Since labelings correspond to acyclic reorientations of $G$, let $G' = (V, E')$ and $G'' = (V, E'')$ be such reorientations.

Since reorientations are in correspondence with labelings then there is an edge $k \rightarrow i$ of $G_A$ that is reoriented as $i \rightarrow k$ in $G'$ but not in $G''$. Moreover since the labels are equal, then there must also be another edge emanating from $k$, say edge $k \rightarrow j$, such that it is reoriented as $j \rightarrow k$ in $G''$ but not in $G'$. In other words, the duplicate labeling implies that we have edges $(i \rightarrow k), (k \rightarrow j) \in E'$ and $(k \rightarrow i), (j \rightarrow k) \in E''$. 

**Figure 1.** We consider the central arrangement corresponding to the digraph $G_A = (1 \rightarrow 2, 1 \rightarrow 3, 2 \rightarrow 3)$. The regions of the arrangement are labeled by the corresponding reorientations and the generalized Pak-Stanley labels. Note that the fundamental region is labeled by $G_A$ and $\langle 0, 0, 0 \rangle$, and as we cross the hyperplanes the orientations of the corresponding edges switch. Moreover, as we cross the hyperplane $H_{i,j}^a$ in a direction away from the origin, the $i$th entry of the Pak-Stanley label increases by 1.
Let $k$ be the largest integer such that this occurs. Since $k$ is the largest possible, it follows that all edges between vertices $p, q$ where $p, q > k$ are oriented in the same way in both reorientations. Without loss of generality we can assume that $i < j$. This gives arise to two cases depending on whether or not the edge from $i \to j$ is oriented as $i \to j$ or $j \to i$ in both $G'$ and $G''$. If we have the edge $i \to j$ then in $G'$ we have the cycle $k \to i \to j \to k$, a contradiction since $G$-parking functions arise from acyclic reorientations. Otherwise we have the edge $j \to i$, but as before we have the cycle $k \to j \to i \to k$ in $G''$ (see Figure 2).

\[ \tau = \langle 0, \ldots, 0, (j - k - 1)^{th}, 0, \ldots, 0, (j - i - 1)^{th}, 0, \ldots, 0 \rangle \]

Therefore $G$ produces duplicate labelings. \qed

**Example 10.** Consider the following graph $G = (V, E)$ where the vertex and edge sets are given by $V = \{1, 2, 3, 4\}$ and $E = \{(1 \to 2), (1 \to 3), (1 \to 4), (2 \to 3), (2 \to 4)\}$.

In this example we see that $(1 \to 3)$ and $(1 \to 4)$, but $(3 \to 4) \notin E$, so Theorem 9 implies that there should exist two reorientations $G'$ and $G''$ that produce the same Pak-Stanley labeling. Consider the following reorientations

These two reorientations of $G_A$ produce the label $\langle 2, 0, 0, 0 \rangle$. Similarly for $(2 \to 3), (2 \to 4) \in E$, but $(3 \to 4) \notin E$ there will be duplicates.
These two reorientations of $G$ produce the duplicate label $(0, 1, 0, 0)$. Actually, this graph produces four more duplicate labelings

$$\{(1, 1, 0, 0), (2, 1, 0, 0), (1, 2, 0, 0), (3, 1, 0, 0)\}.$$ 

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