The convergence rate of equilibrium measures of $N$-player Games with Brownian common noise to its asymptotic Mean Field Game system is known as $O(N^{-1/9})$ with respect to 1-Wasserstein distance, obtained by the monograph [6, Cardaliaguet, Delarue, Lasry, Lions (2019)]. In this work, we study the convergence rate of the $N$-player LQG game with a Markov chain common noise towards its asymptotic Mean Field Game. The approach relies on an explicit coupling of the optimal trajectory of the $N$-player game driven by $N$ dimensional Brownian motion and Mean Field Game counterpart driven by one-dimensional Brownian motion. As a result, the convergence rate is $O(N^{-1/2})$ with respect to 2-Wasserstein distance.

Keywords. Convergence rate, Mean Field Games, Common noise

AMS subject classifications. 91A16, 93E20

1. Introduction

Mean Field Game (MFG) theory is intended to describe an asymptotic limit of complex $N$-player differential game invariant to reshuffling of the players’ indices, and has attracted resurgent attention from numerous researchers in probability after its pioneering works of [18, Lasry and Lions] and [17, Huang, Caines, and Malhame], and we refer comprehensive descriptions to the book [7, Carmona and Delarue] and the references therein.

A fundamental question in this regard is the convergence rate of $N$-player game to the desired MFG system. A well known result is about the convergence rate of value functions of the generic player, which can be shown $O(N^{-1})$, see [7, 5, 6] for instance. In contrast, the convergence rate of equilibrium measures is a rather challenging question due to the complication of the correlation structures among $N$ players. This question was answered until recently by the monograph [6, Cardaliaguet, Delarue, Lasry, Lions (2019)] in a general problem setting.

By [6, Page 14], the convergence rate under appropriate conditions is given by

$$E \left[ \sup_{t \in [0,T]} |\hat{X}_{1,t}^{(N)} - Z_t| \right] = O(N^{-1/(d+8)}),$$

where $\hat{X}_{1,t}^{(N)}$ is the generic player’s equilibrium path, while $Z_t$ is a distribution copy of the MFG equilibrium path in the probability space of the $N$-player game, and $d$ is the dimension of the state space. This in turn implies that the convergence rate of the equilibrium measures with respect to 1-Wasserstein distance $W_1$ is $1/(d + 8)$.

In this paper, we study the convergence rate of equilibrium measures of $N$-player differential game in the context of Linear-Quadratic (LQ) structure with a common noise to its limiting MFG system. Different from the works mentioned above, the common noise in this paper is a continuous-time Markov chain (CTMC) instead of Brownian motion. Our main contribution is that the convergence rate of equilibrium measures can become $1/2$ with respect to 2-Wasserstein distance $W_2$ in our framework, see Theorem 5 for its precise statement.

LQ control problems have been widely recognized in the stochastic control theory due to their broad applications. More importantly, LQ structure leads to solvability in a closed form, namely the Ricatti system, and this usually sheds light on many fundamental properties of the control theory. For this reason, LQ structure has also been studied in MFGs with or without common noises for its importance. The related literature include major and minor LQG Mean Field Games system ([15, 20, 14, 9]); social optimal in LQG Mean Field Games ([16, 8]); the LQG Mean Field Games with different model settings ([3, 12, 4, 13]); and LQG Graphon Mean Field Games ([11]). Recently, LQ Mean Field Games with a Brownian motion as the common noise have also been studied in ([1, 22]) with restrictions of the...
dependence of measure on its mean alone. Another similar setting to our paper is the recent work \cite{21}, which studies a Mean Field Control problem. Yet, none of the aforementioned papers, except \cite{6}, include a discussion of the convergence rate of the equilibrium measures. To obtain the desired convergence rate in this paper, the first building block is the characterization of the equilibrium measure of the limiting MFG by a finite-dimensional ODE system. Unfortunately, it is a controversial task even for the characterization of the first two moments of the equilibrium measure since an explicit example (see Example \ref{example1}) exhibits that the first moment is path-dependent to the common noise. For instance, if $Y$ is the common noise, the process $\mu_t$ of the first moment has to be in the form of $\mu_t = F(t,Y_{[0,t]})$ for some infinite dimensional function $F$, therefore there exists no finite-dimensional system directly available to characterize the equilibrium. The key step leading us to a desired finite-dimensional system is that, instead of searching for the infinite-dimensional function directly, we postulate $t \mapsto F(t,Y_{[0,t]})$ a Markovian structure governed by its coefficients given by finite-dimensional functions. As a result, the equilibrium measure is obtained as a Gaussian process conditional on common noise, whose first and second moment can be entirely determined by a Markov process whose coefficients are given by finite-dimensional ODE system. To the best of our knowledge, this approach is new even in the literature of MFG with Brownian common noise, see Theorem \ref{main_result}.

The next stage towards the convergence rate is to compare the limiting MFG system to a $N$-player game. In contrast to the characterization of the MFG system, it is relatively routine to solve the $N$-player game due to its LQ structure. Therefore, the convergence rate problem can be recasted to the following question about a coupling of two following processes: For two equilibrium processes $\hat{X}$ of MFG in $\Omega$ and $\hat{X}_1^{(N)}$ of $N$-player game in $\Omega^{(N)}$, find a random process $Z^N$ in $\Omega$ whose distribution is identical to $\hat{X}_1^{(N)}$ satisfying the estimate in the form of
\[
\mathbb{E} \left[ |\hat{X}_t - Z^N_t|^2 \right] = O(N^{-2}).
\]
Having said that, the main obstacle for this coupling is rooted in two aspects: (1) The sample space $\Omega^{(N)}$ for the equilibrium path $\hat{X}_1^{(N)}$ generated by $N$-dimensional Brownian motion $\{W_i^{(N)} : i = 1, \ldots, N\}$ is much richer than the sample space $\Omega$ for the path $\hat{X}$ of MFG with only two-dimensional Brownian motion. (2) The drift function of $\hat{X}$ is determined from a $\kappa$ dimensional ODE system (12), while the drift of $\hat{X}_1^{(N)}$ is provided by a huge $\kappa N^3$ dimensional ODE system (27) for some integer $\kappa$.

A striking phenomenon that leads to overcoming the above difficulty is an $N$-invariant algebraic structure of the seemingly intractable $\kappa N^3$ dimensional ODE system (27), which originated from an accidental discovery from a numerical example, see Table 1 for the illustrative pattern. Thanks to this $N$-invariant structure, the complex ODE system (27) can be reduced to the ODE system (31) whose dimension agrees with the ODE (12) of MFG system. Moreover, $\hat{X}_1^{(N)}$ can be represented as a stochastic flow driven by two Brownian motions $W_1^{(N)}$ and $W_{-1}^{(N)} := \frac{1}{\sqrt{N}} \sum_{i=2}^{N} W_i^{(N)}$, which enables us to embed the equilibrium process $\hat{X}_1^{(N)}$ to any probability space having only two Brownian motions. Eventually, the convergence rate essentially comes from the convergence rate of ODE system (31) to the limiting ODE system (12). Indeed, the pattern leading to the success of the above embedding procedure is precisely accounted for the dimension-free feature of the mean field effects at the equilibrium, which provides a new insight from the existing results in the literature.

The rest of this paper is outlined as follows: Section 2 presents a precise formulation of the problem and two main results. Section 3 is devoted to the derivation of our first result: the equilibrium of MFGs. In Section 4, we show in detail the convergence of the $N$-player game to MFGs, which yields our second main result. Section 5 demonstrates the convergence by some numerical examples. Section 6 is an appendix, that collects some related facts to support our main theme.

2. Problem setup and Main results

First, we collect common notations used in this paper in Subsection 2.1. Then, we set up problems on MFGs and the $N$-player game separately in Subsections 2.2 and 2.3. The main results are presented in Subsection 2.4 and some interpretations of our main results are added in Subsection 2.5.

2.1. Notations. Let $T > 0$ be a fixed terminal time and $(\Omega, \mathcal{F}_T, \mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}, \mathbb{P})$ be a completed filtered probability space satisfying the usual conditions, on which $W$ and $B$ are two independent standard Brownian motions, and $Y$ is a continuous time Markov chain (CTMC) independent
of $(W, B)$ taking values in a finite state space $\mathcal{Y} = \{1, 2, \ldots, \kappa\}$ with a generator
\[ Q = (q_{i,j})_{i,j \in \mathcal{Y}} \]
satisfying $q_{i,j} \geq 0$ for all $i \neq j \in \mathcal{Y}$ and $\sum_{i \neq j} q_{i,j} + q_{i,i} = 0$ for each $i \in \mathcal{Y}$. In the above, the Brownian motion $B$ does not play any role in MFG problem formulation until the convergence proof of the $N$-player game to MFGs.

By $L_p := L_p(\Omega, \mathcal{F}, \mathbb{P})$, we denote the space of random variables $X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite $p$-th moment with norm $\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$. We also denote by $L^p_T := L^p([0, T] \times \Omega)$ the space of all $\mathbb{F}$-progressively measurable random processes $\alpha = (\alpha_t)_{0 \leq t \leq T}$ satisfying
\[ \mathbb{E} \left[ \int_0^T |\alpha_t|^p dt \right] < \infty. \]

For any polish (complete separable metric) space $(P, \mathcal{B}(P), d)$, we use $\delta_x$ to denote the Dirac measure on the point $x \in P$. Then, the collection of all probabilities $m$ on $(P, \mathcal{B}(P), d)$ having finite $k$-th moment is denoted by $\mathcal{P}_k(P)$, i.e.
\[ [m]_k := \int x^k m(dx) < \infty, \quad \forall m \in \mathcal{P}_k(P). \]

The equilibrium of MFGs with the common noise yields the conditional distribution. For real valued random variables $X$ and $Z$ in $(\Omega, \mathcal{F}, \mathbb{P})$, we denote the distribution of $X$ conditional on $\sigma(Z)$ by $\mathcal{L}(X|Z)$, or equivalently
\[ \mathcal{L}(X|Z)(A) = \mathbb{E}[I_A(X)|Z], \quad \forall A \in \mathcal{F}. \]

Note that $\mathcal{L}(X|Z)(A) : \Omega \mapsto \mathbb{R}$ is a $\sigma(Z)$-measurable random variable, therefore, $\mathcal{L}(X|Z)$ is $\sigma(Z)$-measurable random probability distribution with $k$-th moment $[\mathcal{L}(X|Z)]_k = \mathbb{E}[X^k|Z]$, if it exists. We refer to more details on the conditional distribution in Volume II of [1]. Next proposition provides embedding approach to prove the convergence in distribution, which will be used later in the convergence of the $N$-player game to MFGs.

Proposition 1. Suppose $(\Omega^{(N)}, \mathcal{F}_T^{(N)}, \mathbb{P}^{(N)})$ is a complete probability space. Let $X^{(N)}$ and $X$ be random variables of $\Omega^{(N)} \mapsto P$ and $\Omega \mapsto P$, respectively. Then, $X^{(N)}$ is convergent in distribution to $X$, denoted by $X^{(N)} \Rightarrow X$, if there exists $Z^{(N)} : \Omega \mapsto P$ satisfying $\mathcal{L}(Z^{(N)}) = \mathcal{L}(X^{(N)})$, such that $Z^{(N)} \Rightarrow X$ holds almost surely, i.e.
\[ \lim_{N \to \infty} d(Z^{(N)}, X) = 0, \quad \text{almost surely in } \mathbb{P}, \]
where $d$ represents the metric assigned to the space $P$.

In this paper, we formulate the $N$-player game in the completed filtered probability space
\[ (\Omega^{(N)}, \mathcal{F}_T^{(N)}, \mathbb{P}^{(N)}) := (\mathcal{F}_t^{(N)} : 0 \leq t \leq T, \mathbb{P}^{(N)}), \]
and $Y^{(N)}$ is the continuous time Markov chain in $\Omega^{(N)}$ with the same generator given by (1) and $W^{(N)} = (W^{(N)}_t : i = 1, \ldots, N)$ is an $N$-dimensional standard Brownian motion. We assume $Y^{(N)}$ and $W^{(N)}$ are independent of each other.

For better clarity, we use the superscript $(N)$ for a random variable to emphasize the probability space $\Omega^{(N)}$ it belongs to. For example, Proposition 1 denotes a random variable in $\Omega^{(N)}$ by $X^{(N)}$, while its distribution copy in $\Omega$ by $Z^{(N)}$, but not by $Z^{(N)}$.

2.2. The equilibrium of MFGs. In this section, we define the equilibrium of MFGs associated with a generic player’s stochastic control problem in the probability setting $\Omega$, see Section 2.1.

Given a random measure flow $m : (0, T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, consider a generic player who wants to minimize her expected accumulated cost on $[0, T]$:
\[ J(y, x, \alpha) = \mathbb{E} \left[ \int_0^T \frac{1}{2} \alpha_s^2 + F(Y_s, X_s, m_s)ds + G(Y_T, X_T, m_T) \right] \bigg| Y_0 = y, X_0 = x \]
with some given cost functions $F, G : \mathcal{Y} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$ and underlying random processes $(Y, X) : [0, T] \times \Omega \mapsto \mathcal{Y} \times \mathbb{R}$. Among three processes $(Y, X, m)$, the generic player can control the process $X$ via $\alpha$ in the form of
\[ X_t = X_0 + \int_0^t \left( b_1(Y_s, s)X_s + b_2(Y_s, s)\alpha_s \right) ds + W_t, \quad \forall t \in [0, T] \]
where \( \tilde{b}_1(\cdot, \cdot) \) and \( \tilde{b}_2(\cdot, \cdot) \) are two deterministic functions. We assume that the initial state \( X_0 \) is independent of \( Y \). The process \( Y \) of (1) represents the common noise and \( m = (m_t)_{0 \leq t \leq T} \) is a given random density flow normalized up to total mass one.

The objective of the control problem for the generic player is to find its optimal control \( \hat{\alpha} \in \mathcal{A} := L^4_e \) to minimize the total cost, i.e.

\[
V[m](y, x) = J[m](y, x, \hat{\alpha}) \leq J[m](y, x, \alpha), \quad \forall \alpha \in \mathcal{A}.
\]  

(4)

Associated to the optimal control \( \hat{\alpha} \), we denote the optimal path by \( \hat{X} = (\hat{X}_t)_{0 \leq t \leq T} \). To introduce MFG Nash equilibrium, it is often convenient to highlight the dependence of the optimal path and optimal control of the generic player and its associated value on the underlying density flow \( m \), which are denoted by

\[
\hat{X}_t[m], \hat{\alpha}_t[m], \text{ and } V[m],
\]

respectively. Now, we present the definition of the equilibrium below, see also Volume II-P127 of [7] for a general setup with a common noise.

**Definition 2.** Given an initial distribution \( \mathcal{L}(X_0) = m_0 \in \mathcal{P}_2(\mathbb{R}) \), a random measure flow \( \hat{m} = \hat{m}(m_0) \) is said to be a MFG equilibrium measure if it satisfies fixed point condition

\[
\hat{m}_t = \mathcal{L}(\hat{X}_t[\hat{m}]|Y), \quad \forall 0 < t \leq T, \quad \text{almost surely in } \mathbb{P}.
\]  

(5)

The path \( \hat{X} \) and the control \( \hat{\alpha} \) associated to \( \hat{m} \) is called as the MFG equilibrium path and equilibrium control, respectively. The value function of the control problem associated to the equilibrium measure \( \hat{m} \) is called as MFG value function, denoted by

\[
U(m_0, y, x) = V[\hat{m}](y, x).
\]  

(6)

![MFGs diagram](image)

**Figure 1.** MFGs diagram.

The flowchart of MFGs diagram is given in Figure [1]. It is noted from the optimality condition (4) and the fixed point condition (5) that

\[
J[\hat{m}](y, x, \hat{\alpha}) \leq J[m](y, x, \alpha), \quad \forall \alpha
\]

holds for the equilibrium measure \( \hat{m} \) and its associated equilibrium control \( \hat{\alpha} \), while it is not

\[
J[\hat{m}](y, x, \hat{\alpha}) \leq J[m](y, x, \alpha), \quad \forall \alpha, m.
\]

Otherwise this problem turns into a McKean-Vlasov control problem discussed in [21].
2.3. Equilibrium of the N-player game. The discrete counterpart of MFGs is an N-player game, which is formulated below in the probability space $\Omega^{(N)}$, see Section 2.1 for more details on the probability setup.

Recall that, $W_{it}^{(N)}$ and $W_{jt}^{(N)}$ are independent Brownian motions for $j \neq i$ and the common noise $Y^{(N)}$ is the continuous time Markov chain in $\Omega^{(N)}$ with the generator given by [1]. Let the player $i$ follow the dynamic, for $i = 1, 2, \ldots, N$,

$$dX_{it}^{(N)} = (\bar{b}_1(Y_{it}^{(N)}, t)X_{it}^{(N)} + \bar{b}_2(Y_{it}^{(N)}, t)\alpha_{it}^{(N)}) dt + dW_{it}^{(N)}, \quad X_{i0}^{(N)} = x_i^{(N)}.$$ (7)

The cost function for player $i$ associated to the control $\alpha^{(N)} = (\alpha_i^{(N)}; i = 1, 2, \ldots, N)$ is

$$J_i^{(N)}(y, x^{(N)}, \alpha^{(N)}) = \mathbb{E} \left[ \int_0^T \left( \frac{1}{2} |\alpha_{it}^{(N)}|^2 + F(Y_{it}^{(N)}, X_{it}^{(N)}, \rho(X_{it}^{(N)})) \right) dt + G(Y_T^{(N)}, X_T^{(N)}, \rho(X_T^{(N)})) \right] \bigg| X_0^{(N)} = x^{(N)} , Y_0^{(N)} = y.$$ (8)

where $x^{(N)} = (x_1^{(N)}, x_2^{(N)}, \ldots, x_N^{(N)})$ is an $\mathbb{R}^N$-valued random vector in $\Omega^{(N)}$ to denote the initial state for N player, $\alpha_i^{(N)} \in \mathcal{A}^{(N)} := \mathcal{L}^4_{\rho^{(N)}}$, and

$$\rho(x^{(N)}) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^{(N)}}$$

is the empirical measure of a vector $x^{(N)}$ with Dirac measure $\delta$. We use the notation for the control $\alpha^{(N)} = (\alpha_1^{(N)}, \alpha_2^{(N)}, \ldots, \alpha_N^{(N)})$.

**Definition 3.**

(1) The value function of player $i$ for $i = 1, 2, \ldots, N$ of the Nash game is defined by $V^{(N)} = (V_i^{(N)}; i = 1, 2, \ldots, N)$ satisfying the equilibrium condition

$$V_i^{(N)}(y, x^{(N)}) = J_i^{(N)}(y, x^{(N)}, \alpha_i^{(N)}, \alpha_{-i}^{(N)}) \leq J_i^{(N)}(y, x^{(N)}, \alpha_i^{(N)}, \hat{\alpha}_{-i}^{(N)}), \quad \forall \alpha_i^{(N)} \in \mathcal{A}^{(N)}.$$ (9)

(2) The equilibrium path of the N-player game is the random path $\hat{X}_{it}^{(N)} = (\hat{X}_{1t}^{(N)}, \hat{X}_{2t}^{(N)}, \ldots, \hat{X}_{Nt}^{(N)})$ driven by [7] associated to the control $\hat{\alpha}_i^{(N)}$ satisfying the equilibrium condition of [9].

2.4. The main result with quadratic cost structures. We consider the following two functions $F, G : \mathcal{Y} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$ in the cost functional [2]:

$$F(y, x, m) = h(y) \int_{\mathbb{R}} (x - z)^2 m(dz),$$ (10)

and

$$G(y, x, m) = g(y) \int_{\mathbb{R}} (x - z)^2 m(dz),$$ (11)

for some $h, g : \mathcal{Y} \mapsto \mathbb{R}^+$. In this case, the $F$ and $G$ terms in [8] of the N-player game can be written by

$$F(Y_{it}^{(N)}, X_{it}^{(N)}, \rho(X_{it}^{(N)})) = \frac{h(Y_{it}^{(N)})}{N} \sum_{j=1}^N (X_{it}^{(N)} - X_{jt}^{(N)})^2,$$

and

$$G(Y_T^{(N)}, X_T^{(N)}, \rho(X_T^{(N)})) = \frac{g(Y_T^{(N)})}{N} \sum_{j=1}^N (X_{it}^{(N)} - X_{jt}^{(N)})^2,$$

respectively.

First, we note that $F$ and $G$ possess the quadratic structures in $x$. Secondly, the coefficients $h(y)$ and $g(y)$ provide the sensitivity to the mean field effects, which depend on the current CTMC state. For another remark, let us consider the scenario where the number of states is 2 and sensitivities are invariant, say

$$h(0) = h(1) = h, \quad g(0) = g(1) = 0.$$ Then the cost function and hence the entire problem is free from the common noise. Interestingly, as shown in the Appendix 6.1, there is no global solution for MFGs when $h < 0$, while there is global solution when $h > 0$. Therefore, we require positive values for all sensitivities for simplicity. It is
of course an interesting problem to investigate the explosion when some sensitivities take negative. Wrapping up the above discussions, we impose the following assumptions:

(A0) $\tilde{b}_1(y, \cdot), \tilde{b}_2(y, \cdot) : [0, T] \mapsto \mathbb{R}$ are continuous functions for all $y \in \mathcal{Y}$.

(A1) The cost functions are given by \[ \begin{align*}
L(y, x) &= a_y(y) x^2 - 2 a_y(y) x [m_0 y] + k_y(y) [m_0]^2 + b_y(y) [m_0]_2 + c_y, \\
\end{align*} \] with $h, g > 0$; The initial $X_0$ of MFGs satisfies $\mathbb{E}[X_0^2] < \infty$.

(A2) In addition to (A1), the initial $x^{(N)}(0) = (x_1^{(N)}, x_2^{(N)}, \ldots, x_N^{(N)})$ of the $N$-player game is a vector of i.i.d. random variables in $\Omega^{(N)}$ with the same distribution as the initial $\mathcal{L}(X_0)$ of MFG.

Our objective of this paper is to understand the Nash equilibrium of MFGs and its connection to the $N$-player game equilibrium:

(P1) With Assumption (A0), (A1), and (A2), obtain the convergence rate of $(\hat{X}_t^{(N)}, \hat{Y}^{(N)})$ from the $N$-player game of Definition 3 to $(\hat{X}_t, \hat{Y})$ from MFGs of Definition 2 in distribution.

To answer (P1), it is critical to have a solid understanding of the joint distribution $(\hat{X}_t, \hat{Y})$ for the underlying MFG, which yields another question:

(P2) With Assumption (A0) and (A1), characterize the MFG equilibrium path $\hat{X}$, as well as associated equilibrium measure $\hat{m}$ along the Definition 2.

For our first main result, we first answer (P2) via the following Riccati system for unknowns $(a_y, b_y, c_y, k_y : y \in \mathcal{Y})$:

\[ \begin{align*}
& a'_y + 2 b_1 y a_y - 2 b_2 y a_y^2 + \sum_{i=1}^{\kappa} q_y a_i + h_y = 0, \\
& b'_y + \left( 2 b_1 y - 4 b_2 y a_y \right) b_y + \sum_{i=1}^{\kappa} q_y a_i + h_y = 0, \\
& c'_y + a_y + b_y + \sum_{i=1}^{\kappa} q_y c_i = 0, \\
& k'_y - 2 b_2 y a_y^2 + 4 b_2 y b_y b_y + 2 b_1 y k_y + \sum_{i=1}^{\kappa} q_y k_i = 0, \\
& a_y(T) = b_y(T) = g_y, c_y(T) = k_y(T) = 0,
\end{align*} \]

where $h_y = h(y), g_y = g(y)$ for $y \in \mathcal{Y}$.

**Theorem 4** (MFG). Under (A0)-(A1), there exists a unique solution $(a_y, b_y, c_y, k_y : y \in \mathcal{Y})$ to the Riccati system \[ 12 \]. With these solutions, the MFG equilibrium path $\hat{X} = \hat{X}[\hat{m}]$ is given by

\[ \frac{dX_t}{dt} = \left( \tilde{b}_1(Y_t, t) X_t - 2 \tilde{b}_2(Y_t, t) a_{Y_t}(t) \left( \hat{X}_t - \hat{\mu}_t \right) \right) dt + dW_t, \quad X_0 = X_0, \]

with equilibrium control

\[ \hat{\alpha}_t = -2 \tilde{b}_2(Y_t, t) a_{Y_t}(t) \left( \hat{X}_t - \hat{\mu}_t \right), \]

where

\[ d\hat{\mu}_t = \tilde{b}_1(Y_t, t) \hat{\mu}_t dt, \quad \hat{\mu}_0 = \mathbb{E}[X_0]. \]

Moreover, the value function $U$ is

\[ U(m_0, y, x) = a_y(0)x^2 - 2a_y(0)x[m_0]_1 + k_y(0)[m_0]_2^2 + b_y(0)[m_0]_2 + c_y(0), \quad y \in \mathcal{Y}. \]

**Theorem 5** (Convergence rate). Under Assumption (A0)-(A1)-(A2), the joint law $(\hat{X}_t^{(N)}, Y_t^{(N)})$ of the $N$-player game converges in distribution to that of the MFG equilibrium $(\hat{X}_t, Y_t)$ for any $t \in (0, T]$ at the convergence rate

\[ \mathbb{W}_2(\mathcal{L}(\hat{X}_t^{(N)}, Y_t^{(N)}), \mathcal{L}(\hat{X}_t, Y_t)) = O \left( N^{-\frac{1}{2}} \right), \quad \text{as } N \rightarrow \infty. \]

2.5. Remarks on the main results. One can interpret main results in plain words: For $N$-player game with dynamic \[ 7 \] and cost structure \[ 8 \] for large $N$, the equilibrium control of the generic player can be effectively approximated by steering itself toward the population center $\hat{\mu}_t$ depending only on the function $\tilde{b}_1(\cdot, \cdot)$ and the entire past of the common noise, whose velocity is dependent on only the function $b_2(\cdot, \cdot)$ and the entire past of the common noise. The effectiveness can be quantified by the convergence rate of $1/2$ for the one-dimensional Mean Field Game under LQ structure and CTMC...
common noise. A natural question is whether the convergence rate can be generalized to more general settings?

This paper focuses on the one-dimensional problem to avoid unnecessary symbol complexity. Therefore, the main convergence rate 1/2 still holds for multidimensional problems using the same coupling procedure. For convenience to check, we summarized the computation involved in multidimensional problems in the Appendix 6.5.

The current coupling procedure can also be adapted with suitable modifications to the LQ Mean Field Game problem with Brownian common noise. In particular, the reduction of the $O(N^2)$-dimensional ODE can be conducted similarly and the convergence rate is still maintained as 1/2. However, the dependence of the mean and variance process on the common noise and subsequent calculations are significantly different from the current paper, and this problem will be discussed separately in our upcoming project.

Indeed, choosing the CTMC common noise instead of Brownian motion does not simplify the underlying problem, since it preserves the path-dependence feature of the equilibrium measure. On the contrary, the advantage of CTMC common noise is that the applications aim to model less frequently changing environment settings, such as government policies implemented by multiple different regimes. Due to its realistic applications, stochastic control theory perturbed by CTMC is extensively studied in the context of hybrid control problems, see books [19, 23] and the references therein.

We close this section with a remark on the uniqueness. The uniqueness of Mean Field Game can be achieved under Lasry-Lions monotonicity [18] or displacement monotonicity [10] and our setting in Section 2.2 does not satisfy either one. This paper does not answer the uniqueness of the current Mean Field Game setting and this is still an open question to us. The convergence of Theorem 5 only implies that the unique equilibrium path of $N$-player game converges one of the possibly many equilibrium paths of the limiting MFG, which is characterized by Theorem 4.

3. Riccati system for MFGs

This section is devoted to the proof of the first main result Theorem 4 on the MFG solution. First, we outline the scheme based on the Markovian structure of the equilibrium by reformulating the MFG problem in Subsection 3.1. Next, we solve the underlying control problem in Subsection 3.2 and provide the corresponding Riccati system. Finally, Subsection 3.3 proves Theorem 4 by checking the fixed point condition of MFG problem.

3.1. Overview. By Definition 3 to solve for the equilibrium measure, one shall search the infinite dimensional space of the random measure flows $m : (0, T] \times \Omega \rightarrow \mathcal{P}_2(\mathbb{R})$, until a measure flow satisfies the fixed point condition $m_t = \mathcal{L}(\hat{X}_t|Y), \forall t \in (0, T]$, see Figure 1, which requires to check the following infinitely many conditions:

$$[m_t]_k = \mathbb{E}[\hat{X}_t^k|Y], \ \forall k = 1, 2, \ldots,$$

if they exist.

The first observation is that the cost functions $F$ and $G$ in (10)-(11) are dependent on the measure $m$ only via the first two moments:

$$F(y, x, m) = h(y)(x^2 - 2x|m|_1 + |m|_2),$$

$$G(y, x, m) = g(y)(x^2 - 2x|m|_1 + |m|_2).$$

Therefore, the underlying stochastic control problem for MFGs can be entirely determined by the input given by $\mathbb{R}^2$ valued random process $\mu_t = [m_t]_1$ and $\nu_t = [m_t]_2$, which implies that the fixed point condition can be effectively reduced to check two conditions only:

$$\mu_t = \mathbb{E}[\hat{X}_t^1|Y], \ \nu_t = \mathbb{E}[\hat{X}_t^2|Y].$$

This observation effectively reduces our search from the space of random measure-valued processes $m : (0, T] \times \Omega \rightarrow \mathcal{P}_2(\mathbb{R})$ to the space of $\mathbb{R}^2$-valued random processes $(\mu, \nu) : (0, T] \times \Omega \rightarrow \mathbb{R}^2$.

Note that, if underlying MFGs have no common noise $Y$, then $(\mu, \nu)$ is a deterministic mapping $[0, T] \rightarrow \mathbb{R}^2$ and the above observation is enough to reduce the original infinite-dimensional MFGs into a finite-dimensional system. However, the following example shows that this is not the case for MFGs with a common noise and it becomes the main drawback to characterizing MFGs via a finite-dimensional system.
Example 1. To illustrate, we consider the following uncontrolled mean field dynamics: Let the mean field term
\[ \mu_t := \mathbb{E}[\hat{X}_t | Y], \]
where the underlying dynamic is given by
\[ d\hat{X}_t = -\mu_t Y_t dt + dW_t. \]

- \( \mu_t \) is path dependent on \( Y \), i.e.
  \[ \mu_t = \mu_0 \exp \left\{ - \int_0^t Y_s ds \right\}. \]
  This implies that no finite dimensional system is possible to characterize the process \( \mu_t \), since the \( (t, Y) \mapsto \mu_t \) is a function on an infinite dimensional domain.

- \( \mu_t \) is Markovian, i.e.
  \[ d\mu_t = -Y_t \mu_t dt. \]
  It might be possible to characterize \( \mu_t \) via a function \( (t, Y_t, \mu_t) \mapsto \frac{d\mu_t}{dt} \) on a finite dimensional domain.

To solidify the above idea, we need to postulate the Markovian structure for the first and second moments of the MFG equilibrium. More precisely, our search for the equilibrium will be confined to the space \( \mathcal{M} \) of measure flows whose first and second moment exhibits Markovian structure.

**Definition 6.** The space \( \mathcal{M} \) is the collection of all \( \mathcal{F}^Y \)-adapted measure flows \( m : [0, T] \times \Omega \to \mathcal{P}_2(\mathbb{R}) \), whose first moment \( [m]_1 := \mu_t \) and second moment \( [m]_2 := \nu_t \) satisfy
\[
\begin{align*}
\mu_t &= \mu_0 + \int_0^t (w_0(Y_s, s) \mu_s + w_1(Y_s, s)) ds, \quad \forall t \in [0, T] \\
\nu_t &= \nu_0 + \int_0^t (w_2(Y_s, s) \mu_s + w_3(Y_s, s) \nu_s + w_4(Y_s, s) \mu_s^2 + w_5(Y_s, s)) ds, \quad \forall t \in [0, T]
\end{align*}
\]
for some smooth deterministic functions \( (w_i : i = 0, 1, \ldots, 5) \).

The flowchart for our equilibrium is depicted in Figure 2. Subsection 3.2 covers the derivation of the Riccati system for the LQG system with a given population measure flow \( m \in \mathcal{M} \), which provides the key building block to MFGs. In Subsection 3.3, we check the fixed point condition and provide a finite-dimensional characterization of MFGs, which gives the first main result Theorem 4.

3.2. The generic player’s control with a given population measure. The advantage of the generic player’s control problem associated with \( m \in \mathcal{M} \) is that its optimal path can be characterized via the following classical stochastic control problem:

- **(P3)** Given smooth functions \( w = (w_i : i = 0, 1, \ldots, 5) \), find the optimal value \( \bar{V} = \bar{V}[w] \)
\[
\bar{V}(y, x, t, \bar{\mu}, \bar{v}) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[ \int_t^T \left( \frac{1}{2} \alpha_s^2 + F(Y_s, X_s, \mu_s, \nu_s) \right) ds + G(Y_T, X_T, \mu_T, \nu_T) \bigg| Y_t = y, X_t = x, \mu_t = \bar{\mu}, \nu_t = \bar{v} \right]
\]

![Figure 2. Equivalent MFGs diagram with \( \mu_0 = [m_0]_1 \) and \( \nu_0 = [m_0]_2 \).](image)
underlying $\mathbb{R}^4$-valued processes $(Y, X, \mu, \nu)$ defined through (1)–(3)–(15) with the finite dimensional cost functions: $F, G : \mathbb{R}^4 \mapsto \mathbb{R}$ given by

$$
F(y, x, \mu, \nu) = h(y)(x^2 - 2x\mu + \nu), \\
G(y, x, \mu, \nu) = g(y)(x^2 - 2x\mu + \nu),
$$
where $\mu, \nu$ are scalars, while $\mu, \nu$ are used as processes.

**Lemma 7.** Given $m \in M$ associated with $w = (w_i : i = 0, \ldots, 5)$, the player’s value $v \in C$ under assumption (A1) is

$$
U[m_0](y, x) = \bar{V}(y, x, 0, [m_0]_1, [m_0]_2),
$$
and the optimal control has a feedback form

$$
\hat{\alpha}(t) = \bar{\alpha}(Y_t, X_t, t, \mu_t, \nu_t)
$$

underlying the processes $(Y, X, \mu, \nu)$ defined through (1)–(3)–(15), whenever there exists a feedback optimal control $\bar{\alpha}$ for the problem (P3).

**Proof.** Due to the quadratic cost structure in (10)–(11), we have enough regularity to all concerned value functions and the details are omitted. \( \square \)

Next, we turn to the solution to the control problem (P3).

3.2.1. **HJB equation.** For the simplicity of notations, for each $i \in \{0, 1, 2, 3, 4, 5\}$ and $y \in \mathcal{Y}$, denote the function $(x, t, \bar{\mu}, \bar{\nu}) \mapsto \nu_i(y, x, t, \bar{\mu}, \bar{\nu})$ as $\nu_i$, and denote $t \mapsto w_i(y, t)$ as $w_{iy}$. We apply similar notations for other functions whenever they have a variable $y \in \mathcal{Y}$. Formally, under enough regularity conditions, the value function $\bar{V}$ defined in (P3) is the solution $v$ of the following coupled HJBs

$$
\begin{aligned}
\partial_t v_y + \tilde{b}_{1y}x\partial_x v_y - \frac{1}{2} \left( b_{2y} \partial_x v_y \right)^2 + \frac{1}{2} \partial_{xx} v_y + \partial_x v_y (w_{0y}\bar{\mu} + w_{1y}) \\
+ \partial_x v_y \left( w_{2y}\bar{\mu} + w_{3y}\bar{\nu} + w_{4y}\bar{\mu}^2 + w_{5y} \right) + \sum_{i=1}^n q_i v_y + \bar{F}_y = 0,
\end{aligned}
$$

(16)

Furthermore, the optimal control has to admit the feedback form of

$$
\hat{\alpha}(t) = -\tilde{b}_2(Y_t, t)\partial_x v(Y_t, X_t, t, \mu_t, \nu_t).
$$

(17)

Next, we identify what conditions are needed for equating the control problem (P3) and HJB equation. Denote

$$
\mathbb{S} = \left\{ v \in C^\infty : (1 + |x|^2)^{-1}(|v| + |\partial_x v|) + (1 + |x|)^{-1}(|\partial_{xx} v| + |\partial_x v|) + |\partial_{xx} v| < K, \forall (y, x, t, \mu, \nu), \text{for some } K \right\}.
$$

**Lemma 8.** (Verification theorem) Consider the control problem (P3) with some given smooth $w$. Suppose there exists a solution $v \in \mathbb{S}$ of (16). Then, $v_y(y, x, t, \bar{\mu}, \bar{\nu}) = \bar{V}(y, x, t, \bar{\mu}, \bar{\nu})$ holds, and an optimal control is provided by (17).

**Proof.** We first prove the verification theorem. Since $v \in \mathbb{S}$, for any admissible $\alpha \in L^4_2$, the process $X^\alpha$ is well defined and one can use Dynkin's formula given by Lemma 15 to write

$$
\mathbb{E}[v(Y_T, X_T, T, \mu_T, \nu_T)] = v(y, x, t, \bar{\mu}, \bar{\nu}) + \mathbb{E}\left[ \int_t^T \mathcal{G}^\alpha(s)v(Y_s, X_s, s, \mu_s, \nu_s)ds \right],
$$

where

$$
\mathcal{G}^\alpha f(y, x, s, \bar{\mu}, \bar{\nu}) = \left( \partial_t + \left( \tilde{b}_{1y}x + \tilde{b}_{2y}a \right) \partial_x + \frac{1}{2} \partial_{xx} + Q + (w_{0y}\bar{\mu} + w_{1y}) \partial_{\bar{\mu}} + (w_{2y}\bar{\mu} + w_{3y}\bar{\nu} + w_{4y}\bar{\mu}^2 + w_{5y}) \partial_{\bar{\nu}} \right) f(y, x, s, \bar{\mu}, \bar{\nu}).
$$

Note that HJB actually implies that

$$
\inf_{\alpha} \left\{ \mathcal{G}^\alpha v + \frac{1}{2} a^2 \right\} = -\bar{F},
$$

which again implies

$$
-\mathcal{G}^\alpha v \leq \frac{1}{2} a^2 + \bar{F}.
$$
Hence, we obtain that for all $\alpha \in L^2_T$,
\[
v(y, x, t, \mu, \nu) = \mathbb{E} \left[ \int_0^T -G^{a(s)}v(Y_s, X_s, s, \mu_s, \nu_s)ds \right] + \mathbb{E} \left[ v(Y_T, X_T, T, \mu_T, \nu_T) \right] 
\leq \mathbb{E} \left[ \int_0^T \left( \frac{1}{2} \alpha^2(s) + F(Y_s, X_s, \mu_s, \nu_s) \right) ds \right] + \mathbb{E} \left[ \overline{G}(Y_T, X_T, \mu_T, \nu_T) \right] = J(y, x, t, \alpha, \mu, \nu).
\]

In the above, if $\alpha$ is replaced by $\hat{\alpha}$ given by the feedback form (17), then since $\partial_tv$ is Lipschitz continuous in $x$, there exists corresponding optimal path $\bar{X} \in L^2_T$. Thus, $\hat{\alpha}$ is also in $L^2_T$. One can repeat all above steps by replacing $X$ and $\alpha$ by $\bar{X}$ and $\hat{\alpha}$, and $\leq$ sign by $=$ sign to conclude that $v$ is indeed the optimal value.

\[ \square \]

3.2.2. LQG solution. Note that, the costs $F$ and $\bar{G}$ of (P3) are quadratic functions in $(x, \bar{\mu}, \bar{\nu})$, while the drift function of the process $\nu$ of (15) is not linear in $(x, \bar{\mu}, \bar{\nu})$. Therefore, the control problem (P3) does not fall into the standard LQG control framework. Nevertheless, similar to the LQG solution, we guess the value function as a quadratic function in the form of
\[
v_y(x, t, \bar{\mu}, \bar{\nu}) = a_y(t)x^2 + d_y(t)x + e_y(t)\bar{\mu} + f_y(t)x\bar{\mu} + k_y(t)\bar{\mu}^2 + b_y(t)\bar{\nu} + c_y(t), \quad y \in \mathcal{Y}. \tag{18}\]

With the above setup, for $t \in [0, T]$, the optimal control is
\[
\hat{\alpha}_t = -\bar{b}_2(Y_t, t)\partial_xv(Y_t, \bar{X}_t, t, \mu_t, \nu_t) = -\bar{b}_2(Y_t, t) \left( 2a_Y(t)\bar{X}_t + d_Y(t) + f_Y(t)\mu_t \right), \tag{19}\]

and the optimal path $\bar{X}$ is
\[
d\bar{X}_t = \left( \hat{b}_1(Y_t, t)\bar{X}_t - \bar{b}_2(Y_t, t) \left( 2a_Y(t)\bar{X}_t + d_Y(t) + f_Y(t)\mu_t \right) \right) dt + dW_t. \tag{20}\]

Denote the following ODE systems for $y \in \mathcal{Y}$,
\[
\begin{cases}
a'_y + 2\bar{b}_{1y}a_y - 2\bar{b}_{2y}a_y^2 + \sum_{i=1}^{\kappa} q_{y,i} a_i + h_y = 0, \\
d'_y + \bar{b}_{1y}d_y - 2\bar{b}_{2y}a_y d_y + f_y w_{1y} + \sum_{i=1}^{\kappa} q_{y,i} d_i = 0, \\
c'_y - \bar{b}_{2y}f_y f_y + 2k_y w_{1y} + e_y w_{0y} + b_y w_{2y} + \sum_{i=1}^{\kappa} q_{y,i} c_i = 0, \\
f'_y + \bar{b}_{1y}f_y - 2\bar{b}_{2y}d_y f_y + f_y w_{0y} + \sum_{i=1}^{\kappa} q_{y,i} f_i - 2h_y = 0, \\
k'_y - \frac{1}{2} \bar{b}_{2y}f_y^2 + 2k_y w_{0y} + b_y w_{4y} + \sum_{i=1}^{\kappa} q_{y,i} k_i = 0, \\
b'_y + b_y w_{3y} + \sum_{i=1}^{\kappa} q_{y,i} b_i + h_y = 0, \\
c'_y + a_y - \frac{1}{2} \bar{b}_{2y}d_y^2 + e_y w_{1y} + b_y w_{5y} + \sum_{i=1}^{\kappa} q_{y,i} c_i = 0,
\end{cases}\tag{21}
\]

with terminal conditions
\[
a_y(T) = g_y, \quad b_y(T) = g_y, \quad c_y(T) = 0, \quad d_y(T) = 0, \quad e_y(T) = 0, \quad f_y(T) = -2g_y, \quad k_y(T) = 0. \tag{22}\]

**Lemma 9.** Suppose there exists a unique solution $(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y \in \mathcal{Y})$ to the ODE system (21)-(22) on $[0, T]$. Then the value function of (P3) is
\[
\overline{V}(y, x, t, \bar{\mu}, \bar{\nu}) = v_y(x, t, \bar{\mu}, \bar{\nu}) = a_y(t)x^2 + d_y(t)x + e_y(t)\bar{\mu} + f_y(t)x\bar{\mu} + k_y(t)\bar{\mu}^2 + b_y(t)\bar{\nu} + c_y(t),
\]

for $y \in \mathcal{Y}$ and the optimal control and optimal path are given by (19) and (20), respectively.
Proof. With the form of value function \( v_y \) given in (18) and the first and second moment of the conditional population density given in (15), we have

\[
\begin{align*}
\frac{\partial}{\partial t} v_y &= a_y'(t)x^2 + d_y'(t)x + e_y'(t)\bar{\mu} + f_y'(t)x\bar{\mu} + k_y'(t)\bar{\mu}^2 + b_y'(t)\bar{\nu} + c_y'(t), \\
\frac{\partial}{\partial x} v_y &= 2ax_y(t) + dy_y(t) + fy_y(t)\bar{\mu}, \\
\frac{\partial}{\partial xx} v_y &= 2a_y(t), \\
\frac{\partial}{\partial \bar{\mu}} v_y &= e_y(t) + f_y(t)x + 2k_y(t)\bar{\mu}, \\
\frac{\partial}{\partial \bar{\nu}} v_y &= b_y(t),
\end{align*}
\]

for \( y \in \mathcal{Y} \). Plugging them back to the coupled HJBs in (16), we get a system of ODEs in (21) by equating \( x, \bar{\mu}, \bar{\nu} \)-like terms in each equation.

Therefore, any solution \((a_y, b_y, c_y, d_y, e_y, f_y, k_y : y \in \mathcal{Y})\) of ODE system (21) leads to the solution of HJB (16) in the form of the quadratic function given by (23). Since the \((a_y, b_y, c_y, d_y, e_y, f_y, k_y : y \in \mathcal{Y})\) are differentiable functions on the closed set \([0, T]\), they are also bounded, and the function \( v \) meets regularity conditions required by Lemma 8 to conclude the desired result. \(\square\)

### 3.3. Fix point condition and the proof of Theorem 4

Going back to the ODE system (21), there are \(7\kappa\) equations, while we have total \(13\kappa\) deterministic functions of \([0, T] \times \mathbb{R}\) to be determined to characterize MFGs. Those are

\[
(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y \in \mathcal{Y}) \quad \text{and} \quad (w_{iy} : i = 0, 1, \ldots, 5, \ y \in \mathcal{Y}).
\]

In this below, we identify the missing \(6\kappa\) equations by checking the fixed point condition:

\[
\begin{align*}
\mu_s &= \mathbb{E}\left[\hat{X}_s \mid Y\right], \quad \nu_s &= \mathbb{E}\left[\hat{X}_s^2 \mid Y\right], \quad \forall s \in [0, T],
\end{align*}
\]

where \(\mu\) and \(\nu\) are two auxiliary processes \((\mu, \nu)[w]\) defined in (15), see Figure 2. This leads to a complete characterization of the equilibrium for the MFG posed by (P2).

Note that based on the dynamic of the optimal \(\hat{X}\) defined in (20), the fixed point condition (24) implies that the first moment \(\hat{\mu}_s := \mathbb{E}\left[\hat{X}_s \mid Y\right]\) and the second moment \(\hat{\nu}_s := \mathbb{E}\left[\hat{X}_s^2 \mid Y\right]\) of the optimal path conditioned on \(Y\) satisfy

\[
\begin{align*}
\hat{\mu}_s &= \bar{\mu} + \int_t^s \left( \left( \bar{b}_1(Y_r, r) - \bar{b}_2^2(Y_r, r)(2a_{Y_r}(r) + f_{Y_r}(r)) \right) \bar{\mu}_r - \bar{b}_2(Y_r, r)dy_r(r) \right) dr, \\
\hat{\nu}_s &= \bar{\nu} + \int_t^s \left( 1 + 2\bar{b}_1(Y_r, r)\bar{\nu}_r - \bar{b}_2^2(Y_r, r) \left( 4a_{Y_r}(r)\bar{\nu}_r + 2dy_r(r)\bar{\mu}_r + 2f_{Y_r}(r)\bar{\mu}_r^2 \right) \right) dr,
\end{align*}
\]

for \(s \geq t\). Note that under the optimal control in (19), comparing the terms in (15) and (25), we obtain another \(6\kappa\) equations:

\[
\begin{align*}
w_{0y} &= \bar{b}_1y - 2\bar{b}_2a_y - \bar{b}_2^2f_y, \quad w_{1y} = -\bar{b}_2y d_y, \quad w_{2y} = -2\bar{b}_2y d_y, \\
w_{3y} &= -4\bar{b}_2a_y + 2\bar{b}_1, \quad w_{4y} = -\bar{b}_2^2f_y, \quad w_{5y} = 1,
\end{align*}
\]

for \(y \in \mathcal{Y}\). Using further algebraic structures, one can reduce the ODE system of \(13\kappa\) equations composed by (21) and (26) into a system of \(4\kappa\) equations of the form (12) for the MFG characterization in Theorem 4.

**Proof of Theorem 4**. Since \(a_y (y \in \mathcal{Y})\) has the same expressions as (12), its existence, uniqueness and boundedness are shown in Lemma 22. Given \(a_y (y \in \mathcal{Y})\) and smooth bounded \(w\)’s,

\[
(b_y, d_y, e_y, f_y : y \in \mathcal{Y})
\]

is a coupled linear system, and their existence, uniqueness and boundedness is shown by Theorem 12.1 in [2]. Similarly, given \((b_y, d_y, f_y : y \in \mathcal{Y}), (k_y, c_y : y \in \mathcal{Y})\) is a linear system, and their existence and uniqueness is also guaranteed by Theorem 12.1 in [2].
The ODE system \((21)\) can be rewritten by

\[
\begin{align*}
    a_y' + 2b_{1y}a_y - 2b_{2y}^2a_y^2 + \sum_{i=1}^{\kappa} q_{y,i}a_i + h_y &= 0, \\
    d_y' + b_{1y}d_y - 2b_{2y}^2a_yd_y - b_{3y}^2f_yd_y + \sum_{i=1}^{\kappa} q_{y,i}d_i &= 0, \\
    c_y' - b_{2y}^2g_fd_y + b_{3y}^2k_yd_y + e_y \left( b_{1y} - 2b_{2y}^2a_y - \tilde{b}_{2y}^2f_y \right) - 2\tilde{b}_{2y}^2b_yd_y + \sum_{i=1}^{\kappa} q_{y,i}e_i &= 0, \\
    f_y' + b_{1y}f_y - 2b_{2y}^2a_yf_y + f_y \left( b_{1y} - 2b_{2y}^2a_y - \tilde{b}_{2y}^2f_y \right) + \sum_{i=1}^{\kappa} q_{y,i}f_i - 2h_y &= 0, \\
    k_y' = \frac{1}{2}b_{2y}^2f_y^2 + 2k_y \left( b_{1y} - 2b_{2y}^2a_y - \tilde{b}_{2y}^2f_y \right) - 2\tilde{b}_{2y}^2b_yf_y + \sum_{i=1}^{\kappa} q_{y,i}k_i &= 0, \\
    b_y' + b_y \left( -4b_{2y}^2a_y + 2b_{1y} \right) + \sum_{i=1}^{\kappa} q_{y,i}b_i + h_y &= 0, \\
    c_y' + a_y - \frac{1}{2}b_{2y}^2y^2 - 2\tilde{b}_{2y}^2d_y e_y + b_y + \sum_{i=1}^{\kappa} q_{y,i}c_i &= 0,
\end{align*}
\]

with the terminal conditions

\[a_y(T) = g_y, \quad b_y(T) = g_y, \quad c_y(T) = 0, \quad d_y(T) = 0, \quad e_y(T) = 0, \quad f_y(T) = -2g_y, \quad k_y(T) = 0.\]

Since \(a_y, b_y (y \in \mathcal{Y})\) have the same expressions as \((12)\), their existence, uniqueness and boundedness are shown in Lemma \(24\). Meanwhile, with the given \((a_y, b_y : y \in \mathcal{Y})\), we denote \(l_y = 2a_y + f_y\), and then

\[l_y' + 2b_{1y}l_y - b_{2y}^2l_y + \sum_{i=1}^{\kappa} q_{y,i}l_i = 0, \quad l_y(T) = 0.\]

By Lemma \(20\) and Lemma \(21\) in Appendix, there exists a unique solution for \(l_y (y \in \mathcal{Y})\), which is \(l_y = 0, y \in \mathcal{Y}\). This gives \(f_y = -2a_y\) and \(d_y' + b_{1y}d_y + \sum_{i=1}^{\kappa} q_{y,i}d_i = 0\), which implies \(d_y = 0, y \in \mathcal{Y}\) Then, the equation for \(c_y\) can be simplified as \(c_y' + b_{1y}c_y + \sum_{i=1}^{\kappa} q_{y,i}c_i = 0\), which indicates that \(c_y = 0, y \in \mathcal{Y}\). For \(k_y, c_y\), with the given of \((a_y, b_y : y \in \mathcal{Y})\), we have

\[k_y' + 2b_{1y}k_y - 2b_{2y}^2a_y^2 + 4b_{2y}^2a_yb_y + \sum_{i=1}^{\kappa} q_{y,i}k_i = 0, \quad k_y(T) = 0,
\]

\[c_y' + a_y + b_y + \sum_{i=1}^{\kappa} q_{y,i}c_i = 0, \quad c_y(T) = 0.
\]

The existence and uniqueness of the solution for \(k_y, c_y (y \in \mathcal{Y})\) are yielded by Theorem 12.1 in [2].

Note that in this case, since \(2a_y + f_y = 0\) and \(d_y = 0\) for \(y \in \mathcal{Y}\), from \(23\) we have

\[\hat{\mu}_s = \bar{\mu} + \int_t^s \hat{b}_1(Y_r, r)\hat{\mu}_r \, dr\]

for all \(s \in [t, T]\). Then

\[\hat{\nu}_s = \bar{\nu} + \int_t^s \left( 1 + 2\hat{b}_1(Y_r, r)\hat{\nu}_r - 4\tilde{b}_2^2(Y_r, r)a_{Y_r}(r)\hat{\nu}_r + 4\tilde{b}_2^2(Y_r, r)a_{Y_r}(r)\bar{\mu}_r^2 \right) \, dr.
\]

Plugging \(d_y = 0\) for \(y \in \mathcal{Y}\) back to \(19\), we obtain the optimal control by

\[\hat{\alpha}_s = -2\tilde{b}_2^2(Y_s, s)a_{Y_s}(s) \left( \tilde{X}_s - \hat{\mu}_s \right).\]

Since we have \(d_y = 0\) for \(y \in \mathcal{Y}\), the value function can be simplified from \(18\) to

\[v_y(x, t, \hat{\mu}, \hat{\nu}) = a_y(t)x^2 - 2a_y(t)x\hat{\mu} + k_y(t)\hat{\mu}^2 + b_y(t)\hat{\nu} + c_y(t).
\]

By the equivalence Lemma \(4\), it yields the value function \(U\) of Theorem \(4\). Moreover, since \(f_y = -2a_y\) and \(k_y \neq 0\), the ODE system \((21)\) together with \((26)\) can be reduced into \((12)\). From the Lemma \(22\), the existence and uniqueness of \((a_y, b_y, c_y, k_y : y \in \mathcal{Y})\) in \(12\) is guaranteed.

\[\square\]
4. THE N-PLAYER CONVERGENCE AND ITS CONVERGENCE TO MFGs

In this section, we show the convergence of the N-player game to MFGs. To simplify the presentation, we may omit the superscript \((N)\) for the processes in the probability space \(\Omega^{(N)}\), whenever there is no confusion. First, we solve the N-player game in Subsection 4.1 which provides a Riccati system consisting of \(O(N^3)\) equations. Subsection 4.2 reduces the corresponding Riccati system into an ODE system whose dimension is independent of \(N\). This becomes the key building block of the convergence rate obtained in Subsection 4.3. To obtain the convergence rate, Subsection 4.3 provides an explicit embedding of some processes in \(\Omega^{(N)}\) into the probability space \(\Omega\). Note that, \(\Omega^{(N)}\) is much richer than \(\Omega\) since \(\Omega^{(N)}\) contains \(N\) Brownian motions while \(\Omega\) has only two Brownian motions. Therefore, careful treatment has to be carried out to some processes of our interest, otherwise, such an embedding is in general implausible.

4.1. Characterization of the \(N\)-player game by Riccati system.

The \(N\)-player game is indeed an \(N\)-coupled stochastic LQG problem by its very own definition, see Subsection 2.3. Therefore, the solution can be derived via Riccati system with the existing LQG theory given below: For \(i = 1, 2, \ldots, N\), \(y \in \mathcal{Y}\),

\[
\begin{align*}
A_{iy} + 2\bar{b}_{iy}e_i^\top A_{iy} - 2\bar{b}_{2y}^2 A_{iy}^\top e_i A_{iy} &+ \sum_{j \neq i}^N \left(2\bar{b}_{iy} e_j^\top A_{iy} - 4\bar{b}_{2y}^2 A_{iy}^\top e_j A_{iy}\right) \\
&+ \sum_{j=1}^N q_{y,j} A_{ij} + \bar{h}_y \sum_{j \neq i}^N (e_i - e_j) (e_i - e_j)^\top = 0, \\
B_{iy} + \sum_{j \neq i}^N \left(\bar{b}_{iy} e_j^\top B_{iy} - 2\bar{b}_{2y}^2 A_{iy}^\top e_j B_{iy} - 2\bar{b}_{2y}^2 A_{iy}^\top e_j B_{iy}\right) &+ \bar{b}_{iy} e_i^\top B_{iy} - 2\bar{b}_{2y}^2 A_{iy}^\top e_i B_{iy} + \sum_{j=1}^N q_{y,j} B_{ij} = 0, \\
C_{iy} - \frac{1}{2} \bar{b}_{2y}^2 B_{iy} e_i^\top B_{iy} &- \sum_{j \neq i}^N \bar{b}_{2y}^2 B_{iy}^\top e_j B_{iy} + \sum_{j=1}^N \operatorname{tr}(A_{jj}) + \sum_{j=1}^N q_{y,j} C_{ij} = 0, \\
A_{iy}(T) &= \frac{g_y}{N} A_i, \quad B_{iy}(T) = 0 \cdot 1_N, \quad C_{iy}(T) = 0,
\end{align*}
\]

where the solutions consist of \(N \times N\) symmetric matrices \(A_{iy}\)'s, \(N\)-dimensional vectors \(B_{iy}\)'s, and \(C_{iy} \in \mathbb{R}\). In the above, \(1_N\) is the \(N\)-dimensional vector with all entries are 1, \(A_i\)'s are \(N \times N\) matrices with diagonal 1 except \((A_i)_{ii} = N - 1, (A_i)_{ij} = (A_i)_{ji} = -1\) for any \(j \neq i\) and the rest entries as 0, and \(e_i\)'s are the \(N\)-dimensional natural basis.

**Lemma 10.** Suppose \((A_{iy}, B_{iy}, C_{iy} : i = 1, 2, \ldots, N, \ y \in \mathcal{Y})\) is the solution of (27). Then, the value functions of \(N\)-player game defined by (9) are

\[
V_i(y, x^{(N)}) = (x^{(N)})^\top A_{iy}(0)x^{(N)} + (x^{(N)})^\top B_{iy}(0) + C_{iy}(0), \quad i = 1, 2, \ldots, N.
\]

Moreover, the path and the control under the equilibrium are

\[
d\hat{X}_it = \left(\bar{b}_1(Y_t, t, e_i)\hat{X}_t - \bar{b}_2^2(Y_t, t) \left(2(A_{iY_t})^\top_1 \hat{X}_t + (B_{iY_t})_i\right)\right) dt + dW_{it}, \quad i = 1, 2, \ldots, N \quad \text{(28)}
\]

and

\[
\dot{\alpha}_it = -\bar{b}_2(Y_t, t) \left(2(A_{iY})^\top_1 \hat{X}_t + (B_{iY})_i\right),
\]

where \((A)_i\) denotes the \(i\)-th column of matrix \(A\), \((B)_i\) denotes the \(i\)-th entry of vector \(B\) and \(\hat{X}_t = [\hat{X}_{1t} \hat{X}_{2t} \cdots \hat{X}_{Nt}]^\top\).
Proof. It is standard that, under the enough regularities, the players’ value function
\( V(y, x^{(N)}) = (V_1, V_2, \ldots, V_N)(y, x^{(N)}) \) can be lifted to the solution \( v_{iy}(x^{(N)}, t) \) of the following system of HJB equation, for \( i = 1, 2, \ldots, N \) and \( y \in \mathcal{Y} \),
\[
\begin{aligned}
\partial_t v_{iy} + \hat{b}_{1y} x_i \partial_i v_{iy} - \frac{1}{2} \left( \hat{b}_{2y} \partial_i v_{iy} \right)^2 + \sum_{j \neq i}^N \left( \hat{b}_{1y} x_j - \hat{b}_{2y} \partial_j v_{ij} \right) \partial_j v_{iy} \\
+ \frac{1}{2} \Delta v_{iy} + \sum_{j=1}^\kappa q_{iy, j} \partial_j v_{ij} + \frac{h}{N} \sum_{j \neq i}^N \left( (e_i - e_j)^\top x^{(N)} \right)^2 = 0, \\
v_{iy}(x^{(N)}, T) = g_y \sum_{j \neq i}^N \left( (e_i - e_j)^\top x^{(N)} \right)^2.
\end{aligned}
\]
Then, the value functions \( V \) of \( N \)-player game defined by (9) is \( V_i(y, x^{(N)}) = v_{iy}(x^{(N)}, 0) \) for all \( i = 1, 2, \ldots, N \). Moreover, the path and the control under the equilibrium are
\[
d\hat{X}_{it} = \left( \hat{b}_1(Y_t, t) \hat{X}_t - \hat{b}_2 x_i Y_t \right) dt + dW_{it}, \quad i = 1, 2, \ldots, N,
\]
and
\[
\hat{\alpha}_{it} = -\hat{b}_2(Y_t, t) \partial_i v_{iy}(\hat{X}_t, t).
\]
The proof is the application of Dynkin’s formula and the details are omitted here. Due to its LQG structure, the value function leads to a quadratic function of the form
\[
v_{iy}(x^{(N)}, t) = (x^{(N)})^\top A_{iy}(t) x^{(N)} + (x^{(N)})^\top B_{iy}(t) + C_{iy}(t).
\]
For each \( i = 1, 2, \ldots, N \), after plugging \( V_{iy} \) into (29), and matching the coefficient of variables, we get the desired results. \( \square \)

4.2. Reduced Riccati form for the equilibrium. So far, the \( N \)-player game and MFG have been characterized by Lemma 10 and Theorem 4 respectively. One of our main objectives is to investigate the convergence of the generic optimal path \( \hat{X}_{it}^{(N)} \) of \( N \)-player game generated (27)-(28) to the optimal path \( \hat{X}_t \) of MFG generated by (12)-(13).

Note that \( \hat{X}_t \) relies only on \( \kappa \) functions \((a_y : y \in \mathcal{Y})\) from the simple ODE system (12) while \( \rho(\hat{X}_t^{(N)}) \) depends on \( O(N^3) \) functions from \((A_{iy} : i = 1, 2, \ldots, N, y \in \mathcal{Y})\) solved from a huge Riccati system (27). Therefore, it is almost a hopeless task for a meaningful comparison between these two processes without gaining further insight on the complex structure of Riccati system (27).

To proceed, let us first observe some hidden patterns from a numerical result for the solution of Riccati (27). Table 1 shows \( A_{20} \) at \( t = 1 \) for \( N = 5 \) with same parameters as in figure 3 and figure 4 in section 5.1.

| Value 1 | Value 2 | Value 3 | Value 4 | Value 5 |
|--------|--------|--------|--------|--------|
| 0.1319 | -0.1924 | 0.0202 | 0.0202 | 0.0202 |
| -0.1924 | 0.7696 | -0.1924 | -0.1924 | -0.1924 |
| 0.0202 | -0.1924 | 0.1319 | 0.0202 | 0.0202 |
| 0.0202 | -0.1924 | 0.0202 | 0.1319 | 0.0202 |
| 0.0202 | -0.1924 | 0.0202 | 0.0202 | 0.1319 |

Table 1. \( A_{20}(1) \) for \( N = 5 \)

Interestingly enough, we observe that the entire 25 entries of Table 1 indeed consists of 4 distinct values. Moreover, similar computation with different values of \( N \) only yields a larger table depending on \( N \), but always consists of 4 values. Inspired by this accidental discovery from the above numerical example, we may want to believe and prove a pattern of the matrix \( A_{iy} \) in the following form:

\[
(A_{iy})_{pq} = \begin{cases} 
  a_{iy}(t), & \text{if } p = q = i, \\
  a_{2y}(t), & \text{if } p = q \neq i, \\
  a_{3y}(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\
  a_{4y}(t), & \text{otherwise},
\end{cases}
\]

for \( y \in \mathcal{Y} \). The next result justifies the above pattern: the \( N^2 \) entries of the matrix \( A_{iy} \) can be embedded to a \( 2\kappa \)-dimensional vector space no matter how big \( N \) is.
Lemma 11. There exists a unique solution \((\alpha_{1y}^N, \alpha_{2y}^N)\) from the ODE system \((31)\)
\[
\begin{align*}
\alpha_{1y}^t + 2b_{1y}a_{1y} - \frac{2(N + 1)}{N - 1}b_{2y}^2a_{1y}^2 + \sum_{j=1}^{\kappa} q_{y,j} a_{1j} + \frac{N - 1}{N} h_y &= 0, \\
\alpha_{2y}^t + 2b_{1y}a_{2y} + \frac{2}{(N - 1)^2}b_{2y}^2a_{2y}^2 - \frac{4N}{N - 1}b_{2y}a_{1y}a_{2y} + \sum_{j=1}^{\kappa} q_{y,j} a_{2j} + \frac{h_y}{N} &= 0, \\
\alpha_{1y}(T) &= \frac{N - 1}{N - g_y}, \quad a_{2y}(T) = g_y/N,
\end{align*}
\]
for \(y \in \mathcal{Y}\). Moreover, the path and the control of player \(i\) under the equilibrium are
\[
d\hat{X}^{(N)}_t = \left(\hat{b}_1(Y^{(N)}_t, t)\hat{X}^{(N)}_t - 2\hat{b}_2(Y^{(N)}_t, t)a_{1y}^{N_{1y}^{(N)}}(t) \left(\hat{X}^{(N)}_t - \frac{1}{N - 1} \sum_{j\neq i} X^{(N)}_j \right)\right) dt + dW^{(N)}_t, \tag{32}
\]
and
\[
\hat{a}^{(N)}_i(t) = -2\hat{b}_2(Y^{(N)}_t, t)a_{1y}^{N_{1y}^{(N)}}(t) \left(\hat{X}^{(N)}_t - \frac{1}{N - 1} \sum_{j\neq i} X^{(N)}_j \right)
\]
for \(i = 1, 2, \ldots, N\).

**Proof.** It is obvious to see that in the Riccati system \((27)\), \(B_{iy} = 0\) for all \(i = 1, 2, \ldots, N\) and \(y \in \mathcal{Y}\). Note that in this case, for \(i = 1, 2, \ldots, N\), the optimal control is given by
\[
\hat{a}^{(N)}_i(Y^{(N)}_t, t) = -2\hat{b}_2(Y^{(N)}_t, t)A_{1y}^{N_{1y}^{(N)}}(t)^\top \hat{X}^{(N)}_t.
\]
Plugging the pattern \((30)\) into the differential equation of \(A_{iy}\), we have
\[
\begin{align*}
\alpha_{1y}^t + 2b_{1y}a_{1y} - \frac{2(N + 1)}{N - 1}b_{2y}^2a_{1y}^2 + \sum_{j=1}^{\kappa} q_{y,j} a_{1j} + \frac{N - 1}{N} h_y &= 0, \\
\alpha_{2y}^t + 2b_{1y}a_{2y} - \frac{2}{(N - 1)^2}b_{2y}^2a_{2y}^2 - \frac{4b_{2y}}{(N - 1)^2}a_{1y}a_{2y} + \sum_{j=1}^{\kappa} q_{y,j} a_{2j} + \frac{h_y}{N} &= 0, \\
\alpha_{3y}^t + 2b_{1y}a_{3y} - \frac{2}{(N - 1)^2}b_{2y}^2a_{3y}^2 - \frac{4b_{2y}}{(N - 1)^2}a_{1y}a_{3y} + \sum_{j=1}^{\kappa} q_{y,j} a_{3j} + \frac{h_y}{N} &= 0, \\
\alpha_{3y}^t + 2b_{1y}a_{3y} - \frac{2}{(N - 1)^2}b_{2y}^2a_{3y}^2 - \frac{4b_{2y}}{(N - 1)^2}a_{1y}a_{3y} + \sum_{j=1}^{\kappa} q_{y,j} a_{3j} + \frac{h_y}{N} &= 0, \\
\alpha_{4y}^t + 2b_{1y}a_{4y} - \frac{2}{(N - 1)^2}b_{2y}^2a_{4y}^2 - \frac{4b_{2y}}{(N - 1)^2}a_{1y}a_{4y} + \sum_{j=1}^{\kappa} q_{y,j} a_{4j} &= 0,
\end{align*}
\]
which gives \(a_{1y} + (N - 2)a_{3y} = a_{2y} + (N - 2)a_{4y}\) since two expressions for \(a_{3y}\) should be identical. This implies that \((a_{1y} + (N - 2)a_{3y})' = (a_{2y} + (N - 2)a_{4y})'\) or
\[
\begin{align*}
-2b_{1y}a_{1y} + 2b_{2y}a_{2y}^2 + 4(N - 1)b_{2y}^2a_{3y}^2 - \frac{N - 1}{N} h_y - \sum_{j=1}^{\kappa} q_{y,j} a_{1j} &+ (N - 2) \left( -2b_{1y}a_{3y} + 2b_{2y}a_{1y}a_{3y} + 4b_{2y}^2 (a_{2y}a_{3y} + (N - 2)a_{3y}a_{4y}) - \sum_{j=1}^{\kappa} q_{y,j} a_{3j} + \frac{h_y}{N} \right) \\
&= -2b_{1y}a_{2y} + 2b_{2y}a_{2y}^2 + 4b_{2y} (a_{1y}a_{2y} + (N - 2)a_{3y}a_{4y}) - \sum_{j=1}^{\kappa} q_{y,j} a_{2j} - \frac{h_y}{N} \\
&+ (N - 2) \left( -2b_{1y}a_{4y} + 2b_{2y}a_{3y}^2 + 4b_{2y}^2 (a_{1y}a_{4y} + a_{2y}a_{3y} + (N - 3)a_{3y}a_{4y}) - \sum_{j=1}^{\kappa} q_{y,j} a_{4j} \right).
\end{align*}
\]
After combining terms and substituting \(a_{2y} + (N - 2)a_{4y}\) with \(a_{1y} + (N - 2)a_{3y}\), we get \(a_{2y} + (N - 2)a_{4y} - (N - 1)a_{3y} = 0\), which yields \(a_{3y} = a_{1y}\) or \(a_{3y} = -\frac{1}{N - 1}a_{1y}\). Note that \(a_{3y} \neq a_{1y}\) due to their different differential equations. Hence, we can conclude that \(a_{3y} = -\frac{1}{N - 1}a_{1y}\). In conclusion, for \(i = 1, 2, \ldots, N\), \(A_{iy}\) \((y \in Y)\) has the following expressions:

\[
(A_{iy})_{pq} = \begin{cases} 
    a_{1y}(t), & \text{if } p = q = i, \\
    \frac{1}{N - 1}a_{1y}(t), & \text{if } p = q \neq i, \\
    \frac{1}{(N - 1)(N - 2)}a_{1y}(t) - \frac{1}{N - 2}a_{2y}(t), & \text{otherwise}.
\end{cases}
\]

The existence and uniqueness of (27) is equivalent to the existence and uniqueness of (31). For \(a_{1y}\), the existence and uniqueness can be deduced from Lemma 20 and 21. Given \(a_{1y}'s\), \(a_{2y}'s\) are linear equations, thus their existence and uniqueness are guaranteed by Theorem 12.1 in [2]. Together with previous discussions, we conclude the results. □

4.3. Convergence. Based on the current progress, let us reiterate our goal (P1) for the convergence. Our objective is the convergence of the joint distribution \(L(\hat{X}^{(N)}_{it}, \hat{Y}^{(N)}_{it})\) of \(N\)-player game generated by (31)-(32) in the probability space \(\Omega^{(N)}\) to the distribution \(L(\hat{X}_t, \hat{Y}_t)\) of MFG generated by (12)-(13) in the probability space \(\Omega\). More precisely, we want to find a number \(\eta > 0\) satisfying

\[
\mathbb{W}_2\left(L(\hat{X}^{(N)}_{it}, \hat{Y}^{(N)}_{it}), L(\hat{X}_t, \hat{Y}_t)\right) = O(N^{-\eta}),
\]

where \(\mathbb{W}_2\) is the 2-Wasserstein metric. This procedure is given in the following two steps:

1. We will construct a process \(Z^N\) in the probability space \(\Omega\), who provides exact copy of the joint distribution in the sense of

\[
L(\hat{X}^{(N)}_{it}, \hat{Y}^{(N)}_{it}) = L(Z^N, Y).
\]

Note that, the (32) shows that \(\hat{X}^{(N)}_{it}\) correlates to \(N\) many Brownian motions \(\{W^{(N)}_i : i = 1, 2, \ldots, N\}\) from a much richer space \(\hat{\Omega}^{(N)}\) while \(\Omega\) is a much smaller space having only two Brownian motions \(W\) and \(B\). Therefore, such an embedding essentially requires to represent \(\hat{X}^{(N)}_{it}\) by two independent Brownian motions and is in general not possible. However, due to the symmetric structure of MFG (or the nature of the mean field effect), the embedding is possible and the details are provided in Lemma 12.

2. By Proposition 1, we can use distribution copy \((Z^N, Y)\) in \(\Omega\) to write

\[
\mathbb{W}_2^2\left(L(\hat{X}^{(N)}_{it}, \hat{Y}^{(N)}_{it}), L(\hat{X}_t, \hat{Y}_t)\right) \leq \mathbb{E}\left[|Z^N_t - \hat{X}_t|^2\right].
\]

To obtain the estimate of the above right hand side, we shall compare the (35) of \(Z^N\) and (13) of \(\hat{X}\) and it becomes essential to obtain the convergence rate of the ODE system (31) towards the ODE system (12). The details are provided in Lemma 13.

Lemma 12. Let \(\{X^i_0 : i \in \mathbb{N}\}\) be i.i.d. random variables in \(\Omega\) independent to \((W, B, Y)\) with \(X^0_0 = X_0\). Let \(Z^N\) be the solution of

\[
Z^N_t = X_0 + \int_0^t \hat{b}_1(Y_s, s)Z^N_s ds - \int_0^t 2\hat{b}_2^2(Y_s, s)\hat{a}^{N}_{1Y}(s)\left(Z^N_s - \hat{X}^N_s\right) ds + W_t, \quad (35)
\]

where

\[
d\hat{X}^N_t = \hat{b}_1(Y_t, t)\hat{X}^N_t dt + \frac{\sqrt{N - 1}}{N}dB_t + \frac{1}{N}dW_t, \quad \hat{X}^N_0 = \frac{1}{N} \sum_{i=1}^{N} X^i_0,
\]

and

\[
\hat{a}^{N}_{1y} = \frac{N}{N - 1} a^{N}_{1y},
\]

where \(a^{N}_{1y}\) is from the ODE system (31). Then, \((Z^N_t, Y_t)\) in \((\Omega, \mathcal{F}_T, \mathbb{P})\) has the same distribution as \((\hat{X}^{(N)}_{it}, \hat{Y}^{(N)}_{it})\) in \((\Omega^{(N)}, \mathcal{F}^{(N)}_T, \mathbb{P}^{(N)})\).
Proof. Continued from the Lemma [11] player $i$’s path in the $N$-player game follows

$$
\dot{X}^{(N)}_{it} = x^{(N)}_i + \int_0^t \tilde{b}_1(Y^{(N)}_s, s) \dot{X}^{(N)}_{is} ds - \int_0^t 2\tilde{b}_2(Y^{(N)}_s, s) a^{(N)}_{11'}(s) \left( \dot{X}^{(N)}_{is} - \frac{1}{N-1} \sum_{j \neq i}^{N} \dot{X}^{(N)}_{js} \right) ds + W^{(N)}_{it}.
$$

With the notation

$$
\dot{X}^{(N)}_s = \frac{1}{N} \sum_{i=1}^{N} \dot{X}^{(N)}_{is},
$$

one can rewrite the path by

$$
\dot{X}^{(N)}_{it} = x^{(N)}_i + \int_0^t \tilde{b}_1(Y^{(N)}_s, s) \dot{X}^{(N)}_{is} ds - \int_0^t 2\tilde{b}_2(Y^{(N)}_s, s) a^{(N)}_{11'}(s) \left( \dot{X}^{(N)}_{is} - \dot{X}^{(N)}_s \right) ds + W^{(N)}_{it}.
$$

By adding up the above equations (36) indexed by $i = 1$ to $N$, one can have

$$
\dot{X}^{(N)}_t = \bar{\dot{x}}^{(N)} + \int_0^t \tilde{b}_1(Y^{(N)}_s, s) \dot{X}^{(N)}_s ds + \frac{1}{N} \sum_{i=1}^{N} W^{(N)}_{it}
$$

(37)

where $W^{(N)}_{it} := \frac{1}{N-1} \sum_{j \neq i}^{N} W^{(N)}_{jt}$.

Next, we define solution maps of (36) and (37):

$$
\bar{G}_t(x, \phi, W_1, W_2) = E_t(\phi) \left( x + \int_0^t E_t(-\phi) d(W_1 + W_2) \right)
$$

(38)

and

$$
G_t(x, \phi_1, \phi_2, \phi_3, W) = x E_t(\phi_1 - \phi_2) + E_t(\phi_1 - \phi_2) \int_0^t E_t(-\phi_1 + \phi_2) (\phi_2(s) \phi_3(s) ds + dW_s),
$$

(39)

where

$$
E_t(\phi) = \exp \left\{ \int_0^t \phi(s) ds \right\}.
$$

Now, we can rewrite $\dot{X}^{(N)}_t$ of (37) and $\dot{X}^{(N)}_{it}$ of (36) as

$$
\dot{X}^{(N)}_t = \bar{G}_t \left( \frac{1}{N} \sum_{i=1}^{N} x^{(N)}_i, \tilde{b}_1(Y^{(N)}_s, \cdot), \sqrt{N-1} \left( \sqrt{N-1} W^{(N)}_1 \right), \frac{1}{N} W^{(N)}_1 \right),
$$

and

$$
\dot{X}^{(N)}_{it} = G_t \left( x^{(N)}_i, \tilde{b}_1(Y^{(N)}_s, \cdot), 2\tilde{b}_2(Y^{(N)}_s, \cdot) \tilde{a}_1^{(N)}(Y^{(N)}_s, \cdot), \dot{X}^{(N)}_s, W^{(N)}_1 \right).
$$

Meanwhile, $(Z^{N}, \tilde{X}^{N})$ of (35) can also be written in the form of

$$
\dot{X}^{N} = \bar{G}_t \left( \frac{1}{N} \sum_{i=1}^{N} X^{(N)}_i, \tilde{b}_1(Y_s, \cdot), \sqrt{N-1} B, \frac{1}{N} W \right),
$$

and

$$
Z^{N}_t = G_t \left( X_0, \tilde{b}_1(Y_s, \cdot), 2\tilde{b}_2(Y_s, \cdot) \tilde{a}_1^{(N)}(Y_s, \cdot), \tilde{X}^{N}(\cdot), W \right).
$$

(40)

Finally, the fact that the distribution of $(Z^{N}, Y)$ in the space $\Omega$ is identical distribution to $(\tilde{X}^{(N)}_1, Y^{(N)})$ in $\Omega^{(N)}$ comes from the followings:

- $\tilde{b}_1, \tilde{b}_2, \tilde{a}_1^{(N)}$ are deterministic functions.
- The random processes $(\sqrt{N-1} W^{(N)}_1, W^{(N)}_1, Y^{(N)})$ are independent mutually in $\Omega^{(N)}$, while the random elements $(B, W, Y)$ are also independent triples. Moreover, two random triples have identical joint distributions.
- Initial states are generated from identical joint distributions $\{x^{(N)}_i : i = 1, \ldots, N\}$ and $\{X_0^i : i = 1, \ldots, N\}$.

Therefore, $(Z^{N}, Y)$ and $(\tilde{X}^{(N)}_1, Y^{(N)})$ have the same distributions. This completes the proof. \qed
In view of (34), we shall estimate the second moment \( \mathbb{E} \left[ Z_t^N - \bar{X}_t \right]^2 \). First, we can rewrite \( \bar{X}_t \) using above representations via \( G_t \):
\[
\bar{X}_t = G_t \left( X_0, \tilde{b}_1(Y, \cdot), 2\tilde{b}_2(Y, \cdot) a(Y, \cdot), \bar{\mu}(\cdot), W \right),
\]
which leads to a better comparison with \( Z^N \) in the form of (40). To proceed, the following properties of \( G_t \) are useful for the estimate of the second moment, whose proof is relegated to the Appendix 6.3. Throughout the proof of the next lemma, we will use \( K \) in various places as a generic constant which varies from line to line.

**Lemma 13.** The convergence rate under the Wasserstein metric \( W_2(\cdot, \cdot) \) is
\[
W_2 \left( \mathcal{L}(\bar{X}_t^N), \mathcal{L}(\bar{X}_t) \right) = O \left( N^{-\frac{1}{2}} \right).
\]

**Proof.** In view of (34), we start with
\[
W_2^2 \left( \mathcal{L}(\bar{X}_t^N), \mathcal{L}(\bar{X}_t) \right) \leq \mathbb{E} \left[ Z_t^N - \bar{X}_t \right]^2
\]
\[
= \mathbb{E} \left[ G_t \left( X_0, \tilde{b}_1(Y, \cdot), 2\tilde{b}_2(Y, \cdot) a(Y, \cdot), \bar{X}_t, \bar{\mu}(\cdot), W \right) - G_t \left( X_0, \tilde{b}_1(Y, \cdot), 2\tilde{b}_2(Y, \cdot) a(Y, \cdot), \bar{\mu}(\cdot), W \right) \right]^2 \\
:= \mathbb{E} \left[ [I_1(t) - I_2(t)] \right]
\]
Applying the Lipschitz continuity of \((\phi_2, \phi_3) \rightarrow G_t(x, \phi_1, \phi_2, \phi_3, W)\) by Appendix 6.3 on the conditional expectation \( \mathbb{E} \left[ I_1(t) - I_2(t) \big| Y \right], \)
\[
\mathbb{E} \left[ Z_t^N - \bar{X}_t \right]^2 \leq K \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( 2\tilde{b}_2(Y_t, t) \tilde{a}_1^N(Y_t) - 2\tilde{b}_2(Y_t, t) a(Y_t) \right)^2 + \sup_{0 \leq t \leq T} \left( \bar{X}_t - \bar{\mu}(t) \right)^2 \right]
\]
\[
\leq K \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \tilde{b}_2(Y_t, t) \right|^2 \sup_{0 \leq t \leq T} \left| \tilde{a}_1^N(Y_t) - a(Y_t) \right|^2 + \sup_{0 \leq t \leq T} \left| \bar{X}_t - \bar{\mu}(t) \right|^2 \right]
\]
\[
\leq K \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \tilde{a}_1^N(Y_t) - a(Y_t) \right|^2 + \sup_{0 \leq t \leq T} \left| \bar{X}_t - \bar{\mu}(t) \right|^2 \right]
\]
From the dynamic of \( \bar{X}_t^N \) and \( \bar{\mu}(t) \),
\[
\left\{ d \left( \bar{X}_t^N - \bar{\mu}_0(t) \right) = \tilde{b}_1(Y_t, t) \left( \bar{X}_t^N - \bar{\mu}_0(t) \right) dt + \frac{\sqrt{N-1}}{N} dW_t, \right.
\]
\[
\bar{X}_0^N - \bar{\mu}_0 = \frac{1}{N} \sum_{i=1}^N X_0^i - \bar{\mu}_0,
\]
which can be written in terms of \( G_t \) of (38):
\[
\bar{X}_t^N - \bar{\mu}(t) = G_t \left( \frac{1}{N} \sum_{i=1}^N X_0^i - \bar{\mu}_0, \tilde{b}_1(Y, \cdot), \frac{\sqrt{N-1}}{N} B, \frac{1}{N} W \right). \]
Using the fact of \( \left| \tilde{b}_1 \right|_{\infty} < \infty \) and Ito’s isometry, this yields the following estimation:
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \bar{X}_t^N(t) - \bar{\mu}(t) \right|^2 \right] \leq K \left( \frac{1}{N} + \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N X_0^i - \bar{\mu}_0 \right]^2 \right).
\]
Note that, by the central limit theorem, we have
\[
N \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N X_0^i - \bar{\mu}_0 \right]^2 = \mathbb{E} \left[ \frac{\sum_{i=1}^N (X_0^i - \mu_0)^2}{N} \right] \rightarrow Var(X_0^i) < \infty, \quad N \rightarrow \infty,
\]
and we conclude that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \bar{X}_t^N(t) - \bar{\mu}(t) \right|^2 \right] = O(N^{-1}). \tag{41}
\]
Next we investigate the boundness of

\[
\sup_{0 \leq t \leq T} |\hat{a}_{11}^N(t) - a_Y(t)|^2.
\]

From (31) and \(\hat{a}_{1y}^N = \frac{N}{N - 1} \hat{a}_{11}^N\), we have

\[
\begin{align*}
\left\{ \begin{array}{l}
\left(\hat{a}_{1y}^N\right)' + 2b_{1y}\hat{a}_{1y}^N - \frac{2(N + 1)}{N} \hat{b}_{2y}^2 \left(\hat{a}_{1y}^N\right)^2 + \sum_{i=1}^\kappa q_{y,i} \hat{a}_{1y}^N + h_y = 0 \\
\hat{a}_{1y}^N(T) = g_y.
\end{array} \right.
\end{align*}
\]

Define \(u_y = a_y - \hat{a}_{1y}^N\), let \(\tau = T - t\) and denote \(u_y(\tau) := u_y(T - t)\), we have

\[
\begin{align*}
\left\{ \begin{array}{l}
u_y'(\tau) = 2b_{1y}(\tau)u_y(\tau) - 2\hat{b}_{2y}^2(\tau) \left(a_y(\tau) + \hat{a}_{1y}^N(\tau)\right) u_y(\tau) + \frac{2}{N} \hat{b}_{2y}^2(\tau) \left(\hat{a}_{1y}^N(\tau)\right)^2 + \sum_{i=1}^\kappa q_{y,i} u_i(\tau) \\
u_y(0) = 0,
\end{array} \right.
\end{align*}
\]

which gives that

\[
u_y(\tau) = \int_0^\tau \left(2b_{1y}(s)u_y(s) - 2\hat{b}_{2y}^2(s) \left(a_y(s) + \hat{a}_{1y}^N(s)\right) u_y(s) + \frac{2}{N} \hat{b}_{2y}^2(s) \left(\hat{a}_{1y}^N(s)\right)^2 + \sum_{i=1}^\kappa q_{y,i} u_i(s)\right) ds.
\]

Thus for \(\tau \in [0, T]\),

\[
|u_y(\tau)| \leq \int_0^\tau \left(2 \left|b_{1y}\right| \infty |u_y(s)| + 2 \left|\hat{b}_{2y}\right| \infty \left(|a_y| \infty + |\hat{a}_{1y}^N| \infty\right) |u_y(s)| + \frac{2}{N} \left|\hat{b}_{2y}\right| \infty ^2 \left|\hat{a}_{1y}^N\right| \infty ^2 + \sum_{i=1}^\kappa |q_{y,i}| |u_i(s)|\right) ds.
\]

Let \(\left(|b_{1y}| \infty , |\hat{b}_{2y}| \infty , |a_y| \infty , |\hat{a}_{1y}^N| \infty , \sup_{i \in Y} |q_{y,i}|\right) \leq K_1\), then

\[
|u_y(\tau)| \leq \frac{2}{N} K_1^4 T + \int_0^\tau \left((2K_1 + 4K_1^3) |u_y(s)| + K_1 \sum_{i=1}^\kappa |u_i(s)|\right) ds.
\]

By adding up the above equation indexed by \(y = 1\) to \(\kappa\), one can have

\[
\sum_{y=1}^\kappa |u_y(\tau)| \leq \frac{2\kappa K_1^4 T}{N} + \left(2K_1 + 4K_1^3 + \kappa K_1\right) \int_0^\tau \sum_{y=1}^\kappa |u_y(s)| ds.
\]

Let \(K_2 = 2\kappa K_1^4 T\) and \(K_3 = 2K_1 + 4K_1^3 + \kappa K_1\), by the Grönwall’s inequality,

\[
\sum_{y=1}^\kappa |u_y(\tau)| \leq \frac{K_2}{N} e^{K_3 \tau} \leq \frac{K_2}{N} e^{K_3 T}, \quad \forall \tau \in [0, T],
\]

which implies that

\[
\sum_{y=1}^\kappa |u_y(\tau)| \leq \frac{K}{N}, \quad \forall \tau \in [0, T].
\]

Thus, we have

\[
\sup_{0 \leq t \leq T} \left|\hat{a}_{11}^N(t) - a_Y(t)\right|^2 \leq \frac{K}{N^2}, \text{ almost surely}.
\]

Therefore, the convergence is obtained from (41) and (43):

\[
\mathbb{W}_2^2 \left(\mathcal{L}(Z^N_i), \mathcal{L}(\hat{X}_i)\right) \leq K \mathbb{E} \left[\sup_{0 \leq t \leq T} \left|\hat{a}_{11}^N(t) - a_Y(t)\right|^2 + \sup_{0 \leq t \leq T} \left|X^N(t) - \hat{\mu}(t)\right|^2\right] = O(N^{-1}).
\]

\(\square\)
5. Numerical results

5.1. Simulations of Riccati system, the value function and optimal control of the generic player. We have derived a $4\kappa$ dimensional Riccati ODE system \cite{12} to determine the parameter functions

$$(a_y, b_y, c_y, k_y : y \in \mathcal{Y})$$

needed for the characterization of the equilibrium and the value function. Meanwhile, we also show the solvability of the Riccati ODE system in Section \cite{3}.

As mentioned earlier, different from the MFG characterization with the common noise, the derived Riccati system is essentially finite-dimensional. In this subsection, we present a numerical experiment and show some numerical results for solving the Riccati system to demonstrate its computational advantages.

For the illustration purpose, assume the finite time horizon is given with $T = 5$ and that the coefficients of the dynamic equation are listed below:

$\mathcal{Y} = \{0, 1\},$

$$Q = \begin{bmatrix} -0.5 & 0.5 \\ 0.6 & -0.6 \end{bmatrix},$$

$\tilde{b}_1(\cdot, \cdot) = 0, \tilde{b}_2(\cdot, \cdot) = 1,$

$h_0 = 2, h_1 = 5, g_0 = 3, g_1 = 1,$

$\mu_0 = 0, \nu_0 = 2.$

Firstly, using the forward Euler’s method with the step size $\delta = 10^{-2}$, we can obtain trajectories of $(a_y, b_y, c_y : y \in \mathcal{Y})$, which is the solution of ODE system \cite{12}. Next, using the trajectories of the parameter functions and Markov chain $Y_t$, we can achieve the simulations for $\hat{\alpha}_t$ and $\hat{X}_t$. The Matlab code can be found at \url{https://github.com/JiaminJIAN/Regime_switching_MFG}.

![Figure 3](image1) ![Figure 4](image2)

**Figure 3.** Simulations for $a_y, V, \alpha$ and $\nu$.

**Figure 4.** Simulations for $b_y$ and $c_y$. 
As shown in figure 3, people tend to centralize since the conditional second moment of the population density $\nu_t$ is always decreasing.

5.2. **Convergence of the $N$-player game.** In section 4, we showed that the generic player’s path for the $N$-player game is convergent to the generic player’s path for MFGs. In this subsection, we demonstrate the convergence of the conditional first moment, conditional second moment, and the value functions of the $N$-player game to the corresponding terms of the generic player in the Mean Field Game setup by using some numerical examples.

The following figures show the value functions, $\mu^{(N)}$ and $\nu^{(N)}$ under $N \in \{10, 20, 50, 100\}$ with the same parameters’ settings as in figure 3 and figure 4 in section 5.1. We can clearly see the convergence to the solution of the generic player.

![Graph A](image1)

(A) $\mu_t$: conditional mean of the population density

![Graph B](image2)

(B) $\nu_t$: conditional 2nd moment of the population density

**Figure 5.** Simulations for $\mu_t$ and $\nu_t$.

![Graph C](image3)

**Figure 6.** Simulation of player 1’s optimal value function $V$.

6. **Appendix**

6.1. **Some explicit solutions on LQG-MFGs.** In this part, we only provide explicit solutions to some LQG-MFGs without the common noise. The methodology could be the utilization of the standard Stochastic Maximum Principle or Dynamic Programming approach, and all proofs will be omitted.

Suppose the position of a generic player $X_t$ follows

$$dX_t = \alpha_t dt + \sigma dW_t, \quad X_0 \sim \mathcal{N}(0, 1).$$

The goal of the generic player is to minimize the running cost

$$\inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[ \int_0^T \left( \frac{1}{2} \alpha_t^2 + h \int_{\mathbb{R}} (X_t - y)^2 m(t, dy) \right) dt \right],$$
subject to
\[ m_t = \mathcal{L}aw(X_t), \quad \forall t \in [0, T], \]
where \( h \in \mathbb{R} \) is a constant.

Denote
\[ V(x, t) = \inf_{\alpha} \mathbb{E} \left[ \int_t^T \left( \frac{1}{2} \alpha_x^2 + h \int_\mathbb{R} (X_s - y)^2 m(s, dy) \right) ds \right] X_t = x. \]

Note that the model can be characterized by Hamilton-Jacobian-Bellman equation coupled by Fokker-Planck-Kolmogorov equation:
\[
\begin{aligned}
\partial_t V + \frac{1}{2} \sigma^2 \partial_{xx} V - \frac{1}{2} (\partial_x V)^2 + F(x, m) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \\
\partial_t m - \frac{1}{2} \sigma^2 \partial_{xx} m - \partial_x (m \partial_x V) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \\
\end{aligned}
\]
where \( F(x, m) = h \int_\mathbb{R} (x - y)^2 m(t, dy). \)

The monotonicity condition on the source term \( F \) in the variable \( m \) plays crucial role for the uniqueness of the MFG system. A monotone function \( f : \mathbb{R} \mapsto \mathbb{R} \) is said to be increasing if it satisfies \( (f(x_1) - f(x_2))(x_1 - x_2) \geq 0 \), and decreasing if \( -f \) is increasing. This definition can be generalized to an infinite dimensional function \( F(x, m) \).

**Definition 14.** The real function \( F \) on \( \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \) is said to be monotone, if, for all \( m \in \mathcal{P}_2(\mathbb{R}) \), the mapping \( \mathbb{R} \ni x \mapsto F(x, m) \) is at most of quadratic growth, and for all \( m_1, m_2 \) it satisfies
\[ \int_\mathbb{R} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)(x) \geq 0. \]

\( F \) is said to be anti-monotone, if \((-F)\) is monotone.

According to [5], if \( F \) is monotone, then MFGs have at most one solution. Interestingly, the monotonicity of \( F \) is dependent on the sign of \( h \).

**Lemma 15.** \( F(x, m) = h \int_\mathbb{R} (x - y)^2 m(t, dy) \) is monotone if \( h < 0 \), and anti-monotone if \( h > 0 \).

A natural question is that, how the MFG system behaves differently to the monotonicity of \( F \)?

6.1.1. Case I: \( h > 0 \).

**Lemma 16.** For \( h > 0 \), there exists a solution (may not be unique) to the MFG system in the form of \( V(x, t) = f_1(t)x^2 + f_3(t) \) and \( m(t) \sim \mathcal{N}(0, \gamma(t)) \), where
\[
\begin{aligned}
f_1(t) &= \sqrt{\frac{h}{2}} \left( 1 - e^{-2\sqrt{2h}(T-t)} \right), \quad \gamma(t) = e^{-\int_0^t 4f_1(s)ds} \left( 1 + \int_0^t \sigma^2 e^{\int_0^s 4f_1(u)du} ds \right), \\
f_3(t) &= \int_t^T \left( \sigma^2 f_1(s) + h\gamma(s) \right) ds.
\end{aligned}
\]

6.1.2. Case II: \( h < 0 \).

**Lemma 17.** For \( h < 0 \), there exists a unique solution in \((t_0, T]\) to the MFG system in the form of \( V(x, t) = g_1(t)x^2 + g_3(t) \) and \( m(t) \sim \mathcal{N}(0, \lambda(t)) \), where
\[
\begin{aligned}
g_1(t) &= -\sqrt{-\frac{h}{2}} \tan \left( \sqrt{-2h}(T-t) \right), \quad \lambda(t) = e^{-\int_0^t 4g_1(s)ds} \left( 1 + \int_0^t \sigma^2 e^{\int_0^s 4g_1(u)du} du ds \right), \\
g_3(t) &= \int_t^T \left( \sigma^2 g_1(s) + h\lambda(s) \right) ds, \quad t_0 = \max \left( 0, T - \frac{1}{\sqrt{-2h}} \frac{\pi}{2} \right).
\end{aligned}
\]

6.1.3. **Remark.** When \( h > 0 \), the cost is anti-monotone, and there exists at least one global solution. When \( h < 0 \), the cost is monotone, and there exists at most one solution. Unfortunately, this solution lives in a short period. Lemma 17 coincides with the notes in Section 3.8 of [7] saying that due to the opposite time evolution of the system of HJB-FPK, the existence of the solution may exist for only a short period.
6.2. Dynkin’s formula for a regime-switching diffusion with a quadratic function. Since the running cost \([10]\) has a quadratic growth in the state variable, the value function \(V[\hat{m}](y, x, t)\) is expected to possess similar growth. Next, we present a version of Dynkin’s formula for the functions of quadratic growth, which is sufficient for our purpose. Throughout this subsection, we will use \(K\) in various places as a generic constant which varies from line to line. The notions of this subsection are independent of other parts of the paper.

**Lemma 18.** Let \(X\) be the \(\mathbb{R}^d\)-valued process satisfying
\[
X_t = X_0 + \int_0^t \left( \tilde{b}_1(Y_s, s)X_s + \tilde{b}_2(Y_s, s)\alpha_s \right) ds + \int_0^t \sigma(s)dW_s,
\]
where \(Y\) is CTMC with a generator
\[
Y \sim Q = (q_{ij})_{i,j=1,2,\ldots,K}.
\]
Suppose \(\sigma(\cdot), \tilde{b}_1(y, \cdot)\) and \(\tilde{b}_2(y, \cdot)\) are continuous functions on \([0, T]\) for every \(y \in \mathcal{Y} := \{1, 2, \ldots, \kappa\}\). If \(X_0 \in L^4, \alpha \in L^2_{\mathcal{F}}\) and \(f: \mathcal{Y} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}\) satisfies, for some large \(K\)
\[
\sup_{y \in \mathcal{Y}, t \in [0, T]} \left\{ |f(y, x, t)| + (1 + |x|)|\nabla f(y, x, t)| + (1 + |x|)^2|\Delta f(y, x, t)| + |\partial_t f(y, x, t)| \right\} \leq K(|x|^2 + 1),
\]
then the following identity holds for all \(t \in [0, T]\):
\[
\mathbb{E}[f(Y_t, X_t, t)] = \mathbb{E}[f(Y_0, X_0, 0)] + \mathbb{E}\left[\int_0^t (\partial_t + \mathcal{L}^{\alpha_x} + \mathcal{Q})f(Y_s, X_s, s)ds\right],
\]
where
\[
\mathcal{L}^{\alpha}f(y, x, s) = \left( \frac{1}{2} Tr \left( \sigma_s \sigma_s^T \Delta \right) + \left( \tilde{b}_{1y}x + \tilde{b}_{2y}a \right) \cdot \nabla_x \right) f(y, x, s)
\]
and
\[
\mathcal{Q}f(y, x, s) = \sum_{i=1}^n q_{yi} f(i, x, s).
\]

**Proof.** It’s enough to show that the local martingale defined by Itô’s formula
\[
M^f_t = f(Y_t, X_t, t) - f(Y_0, X_0, 0) - \int_0^t (\partial_t + \mathcal{L}^{\alpha_x} + \mathcal{Q})f(Y_s, X_s, s)ds
\]
is uniformly integrable, hence is a true martingale.

First, note that from the assumptions on \(X_0\) and \(\alpha\), we have
\[
\mathbb{E}\left[\|X_t\|^4\right] \leq K \mathbb{E}\left[\|X_0\|^4 + \int_0^t \|\tilde{b}_1(Y_s, s)X_s + \tilde{b}_2(Y_s, s)\alpha_s\|^4 ds + \int_0^t \|\sigma_sW_s\|^4 ds\right]
\]
\[
\quad \leq K \mathbb{E}\left[\|X_0\|^4 + \int_0^t \|X_s\|^4 ds + \int_0^t \|\alpha_s\|^4 ds + \int_0^t \|\sigma_sW_s\|^4 ds\right]
\]
\[
\quad \leq K + K \int_0^t \mathbb{E}\left[\|X_s\|^4\right] ds,
\]
where \(K\) is a generic constant which varies from line to line. Then, by the Grönwall’s inequality,
\[
\mathbb{E}\left[\|X_t\|^4\right] \leq Ke^{Kt} \leq K,
\]
which implies that \(\{X_t : 0 \leq t \leq T\}\) is \(L^4\) bounded uniformly in \(t\).

On the other hand, since \(x \mapsto f(y, x, t)\) is at most quadratic growth uniformly in \((y, t)\), we conclude that \(f(Y_t, X_t, t)\) is uniformly \(L^2\) bounded from the fact
\[
\sup_{t \in [0, T]} \mathbb{E}\left[\|f(Y_t, X_t, t)\|^2\right] \leq K \sup_{t \in [0, T]} \mathbb{E}\left[\|X_t\|^4\right] + K \leq K.
\]
The uniform \(L^2\)-boundedness of \(\int_0^t \partial_t f(Y_s, X_s, s)ds\) follows from our assumption on \(\partial_t f\). Similarly, since \(\mathcal{Q}f\) has a quadratic growth uniformly in \(y\) and \(t\), and
\[
\left\{ \int_0^t \mathcal{Q}f(Y_s, X_s, s)ds : 0 \leq t \leq T \right\}
\]
is $L^2$ bounded. At last, we have

$$
\mathbb{E}\left[\left(\int_0^t \mathcal{L}^{\alpha_s} f(Y_s, X_s, s) ds\right)^2\right]
\leq K \mathbb{E}\left[\int_0^t \left(\tilde{b}_1(Y_s, s) X_s + \tilde{b}_2(Y_s, s) \alpha_s\right) \cdot \nabla f + \frac{1}{2} \text{Tr} \left(\sigma_s \sigma_s^\top \Delta f\right)\right]^2 (Y_s, X_s, s) ds
\leq K \mathbb{E}\left[\int_0^t \|\tilde{b}_1(Y_s, s) X_s + \tilde{b}_2(Y_s, s) \alpha_s\|^2 \|
abla f\|^2 (Y_s, X_s, s) ds\right]
+ K \mathbb{E}\left[\int_0^t \frac{1}{4} \text{Tr} \left(\sigma_s \sigma_s^\top \Delta f\right)\right]^2 (Y_s, X_s, s) ds
\leq K \mathbb{E}\left[\int_0^t \|\alpha_s\|^4 ds\right] + K \mathbb{E}\left[\int_0^t \|X_s\|^4 ds\right] + K \mathbb{E}\left[\int_0^t \|\nabla f\|^4 (Y_s, X_s, s) ds\right]
+ K \mathbb{E}\left[\int_0^t \frac{1}{4} \|\nabla \Delta f\|^2 (Y_s, X_s, s) ds\right].
$$

Since $\nabla f$ is linear growth in $x$, the second term $\sup_{t \in [0,T]} \mathbb{E}\left[\int_0^t \|\nabla f\|^4 (Y_s, X_s, s) ds\right]$ is finite. Together with assumptions on $\Delta f$ and $\alpha$, we have uniform $L^2$-boundedness of $\int_0^t \mathcal{L}^{\alpha_s} f(Y_s, X_s, s) ds$.

As a result, each term of the right-hand side of (41) is uniform $L^2$-bounded in $t$, and thus $M^f_t$ belongs to $L^2_{\mathbb{F}}$ and this implies the uniform integrability. $\square$

6.3. Proof of the property of $G$.

**Lemma 19.** Define

$$
\mathcal{E}_t(\phi) = \exp \left\{ \int_0^t \phi_s ds \right\},
$$

and

$$
G_t(x, \phi_1, \phi_2, \phi_3, W) = x \mathcal{E}_t(\phi_1 - \phi_2) + \mathcal{E}_t(\phi_1 - \phi_2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2) (\phi_2(s) \phi_3(s) ds + dW_s),
$$

where $x$ is a given constant, $\phi_1, \phi_2, \phi_3$ are RCLL functions on $[0, T]$. Then

$$
\mathbb{E}\left[|G_t(x^1, \phi_1, \phi_2, \phi_3, W) - G_t(x^2, \phi_1, \phi_2, \phi_3, W)|^2\right]
\leq K \left|x^1 - x^2|^2 + \sup_{0 \leq s \leq T} |\phi_1^1(t) - \phi_2^2(t)|^2 + \sup_{0 \leq s \leq T} |\phi_1^3(t) - \phi_2^3(t)|^2\right).
$$

**Proof.** Firstly, it can be shown that $G(\cdot, \phi_1, \phi_2, \phi_3, W)$ is Lipschitz continuous with respect to $x$

$$
\mathbb{E}\left[|G_t(x^1, \phi_1, \phi_2, \phi_3, W) - G(x^2, \phi_1, \phi_2, \phi_3, W)|\right] \leq |x^1 \mathcal{E}_t(\phi_1 - \phi_2) - x^2 \mathcal{E}_t(\phi_1 - \phi_2)|
\leq \mathcal{E}_t(\phi_1 - \phi_2)|x^1 - x^2|
\leq K(|\phi_1|_{\infty}, |\phi_2|_{\infty}, T)|x^1 - x^2|.
$$

Next, we have

$$
\mathbb{E}\left[|G_t(x, \phi_1, \phi_2, \phi_3, W) - G(x, \phi_1, \phi_2, \phi_3, W)|^2\right]
= |\mathcal{E}_t(\phi_1 - \phi_2) \int_0^t \mathcal{E}_s(\phi_1 - \phi_2) (\phi_2(s) (\phi_3(s) - \phi_3(s)) ds|^2
\leq \mathcal{E}_t(2\phi_1 - 2\phi_2) \left( \int_0^t \mathcal{E}_s(\phi_1 - \phi_2) |(\phi_3(s) - \phi_3(s)) ds|^2\right)^2
\leq K(|\phi_1|_{\infty}, |\phi_2|_{\infty}, T) \left( \int_0^T |\phi_3(s) - \phi_3(s)| ds\right)^2
\leq K(|\phi_1|_{\infty}, |\phi_2|_{\infty}, T) \sup_{0 \leq t \leq T} |\phi_3^3(t) - \phi_3^3(t)|^2.
$$
Similarly, for $\phi_1^2(\cdot), \phi_2^2(\cdot) \in C([0, T])$,

\[
\mathbb{E} \left[ |G_t(x, \phi_1, \phi_1^2, \phi_2, \phi_3, W) - G(x, \phi_1, \phi_2^2, \phi_3, W)|^2 \right] \\
\leq K \left| x\mathcal{E}_t(\phi_1 - \phi_1^2) - x\mathcal{E}_t(\phi_1 - \phi_2^2) \right|^2 \\
+ K \left| \mathcal{E}_t(\phi_1 - \phi_1^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_1^2)\mathcal{E}_2(s)ds - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)\mathcal{E}_3(s)ds \right|^2 \\
+ K\mathbb{E} \left[ \left\| \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)dW_s - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)dW_s \right\|^2 \right] \\
:= K(J_1 + J_2 + J_3).
\]

Note that by the mean-value theorem and the continuity of $\phi_1, \phi_1^2$ and $\phi_2^2$ on $[0, T]$, we can get

\[
J_1 = \left| x\mathcal{E}_t(\phi_1 - \phi_1^2) - x\mathcal{E}_t(\phi_1 - \phi_2^2) \right|^2 \\
= x^2 \left( e^{t_0}(\phi_1(\cdot) - \phi_1^2(\cdot))ds - e^{t_0}(\phi_1(\cdot) - \phi_2^2(\cdot))ds \right)^2 \\
\leq K(x, |\phi_1^2|_\infty, |\phi_2^2|_\infty, T) e^{t_0} \left| \phi_1^2 - \phi_2^2 \right|^2 \\
\leq K(x, |\phi_1^2|_\infty, |\phi_2^2|_\infty, T) \left| \phi_1^2 - \phi_2^2 \right|^2,
\]

and

\[
J_3 = \mathbb{E} \left[ \left| \mathcal{E}_t(\phi_1 - \phi_1^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_1^2)dW_s - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)dW_s \right|^2 \right] \\
= \mathbb{E} \left[ \left| \mathcal{E}_t(\phi_1 - \phi_1^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_1^2)dW_s - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)dW_s \right|^2 \right] \\
+ \mathbb{E} \left[ \left| \mathcal{E}_t(\phi_1 - \phi_1^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)dW_s - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)dW_s \right|^2 \right] \\
\leq 2 \left( \mathbb{E} \left( \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_1^2)dW_s - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)dW_s \right)^2 \right) \\
+ 2 \left( \mathbb{E} \left( \mathcal{E}_t(\phi_1 - \phi_2^2) - \mathcal{E}_t(\phi_1 - \phi_2^2) \right)^2 \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)dW_s \right) \\
\leq K \left( |\phi_1^2|_\infty, |\phi_2^2|_\infty, |\phi_3^2|_\infty, T \right) \left| \phi_1^2 - \phi_2^2 \right|^2.
\]

Lastly, using the similar argument, we have

\[
J_2 = \left| \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_1^2)\mathcal{E}_2(s)ds - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)\mathcal{E}_3(s)ds \right|^2 \\
= \left| \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_1^2)\mathcal{E}_2(s)ds - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)\mathcal{E}_3(s)ds \right|^2 \\
+ \left| \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)\mathcal{E}_3(s)ds - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)\mathcal{E}_3(s)ds \right|^2 \\
\leq 2 \left( \left( \mathcal{E}_t(\phi_1 - \phi_2^2) - \mathcal{E}_t(\phi_1 - \phi_2^2) \right) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_1^2)\mathcal{E}_2(s)ds \right)^2 \\
+ 2 \left( \left( \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_1^2)\mathcal{E}_2(s)ds - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)\mathcal{E}_3(s)ds \right)^2 \right) \\
\leq K \left( |\phi_1^2|_\infty, |\phi_2^2|_\infty, |\phi_3^2|_\infty, T \right) \left| \phi_1^2 - \phi_2^2 \right|^2 \\
+ 2 \left( \mathbb{E} \left( \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_1^2)\mathcal{E}_2(s)ds - \mathcal{E}_t(\phi_1 - \phi_2^2) \int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)\mathcal{E}_3(s)ds \right)^2 \right) \\
\leq K \left( |\phi_1^2|_\infty, |\phi_2^2|_\infty, |\phi_3^2|_\infty, T \right) \left| \phi_1^2 - \phi_2^2 \right|^2.
\]
Sum up the above inequalities for $J_1, J_2$ and $J_3$, then
\[
E \left[ |G_t(x, \phi_1, \phi_2, \phi_3, W) - G(x, \phi_1, \phi_2, \phi_3, W)|^2 \right] \leq K(x, |\phi_1|_\infty, |\phi_2|_\infty, |\phi_3|_\infty, T) |\phi_1|_\infty^2 - |\phi_2|_\infty^2.
\]
Thus, we can obtain the desired result. \hfill \square

6.4. **Proof of the existence and uniqueness of the ODE system.** Consider the following ODE system
\[
\begin{aligned}
a' + C_1 b_1 a_y - C_2 b_2 a_y^2 + \sum_{i=1}^{\kappa} q_y a_i + h_y = 0, \\
a_y(T) = g_y,
\end{aligned}
\tag{45}
\]
for $y \in \mathcal{Y} = \{1, 2, \ldots, \kappa\}$, where $C_1, C_2, h_y, g_y$ are in $\mathbb{R}^+$. We need to show the existence and uniqueness of the solution to (45). Define $T_y^{(N)}$ as
\[
T_y^{(N)}[a](t) = \left[ g_y + \int_t^T \left( h_y + C_1 b_1(s) a_y(s) - C_2 b_2(s) a_y^2(s) + \sum_{i=1}^{\kappa} q_y a_i(s) \right) ds \right] \wedge N \forall 0,
\]
where $a = [a_1, a_2, \cdots, a_\kappa]^\top$. Let $D = \{ f \in C([0, T]) : 0 \leq \sup_{t \in [0, T]} f(t) \leq N \}$. Note that $T_y^{(N)}(y \in \mathcal{Y})$ maps $D^\kappa$ to $D^\kappa$.

**Lemma 20.** For fixed $N$, there exists a unique solution in $C([0, T])$ to
\[
a = T_y^{(N)}[a].
\tag{46}
\]

**Proof.** Denote the norm $\|f\|_k = \|e^{kt} \max_{y \in \mathcal{Y}} |f_y||f|_\infty^\kappa$, where $k$ needs to be determined later and $f$ is a $\kappa$ dimensional vector with entry of $f_y$, $y \in \mathcal{Y}$, which is equivalent to the infinite norm. Define the iteration rule $a_y^{(n+1)} = T_y^{(N)}[a_y^{(n)}]$ for $y \in \mathcal{Y}$. Note that
\[
\|e^{kt} (a_y^{(n+1)}(t) - a_y^{(n)}(t))\|_\infty \\
\leq \sup_{t \in [0, T]} e^{kt} \int_t^T \left( C_1 \left| b_1(t) \right| \infty \left| a_y^{(n)}(s) - a_y^{(n-1)}(s) \right| + C_2 \left| b_2(s) \right| \infty \left| a_y^{(n)}(s) - a_y^{(n-1)}(s) \right|^2 \right) \wedge N \forall 0,
\]
\[
\leq \sup_{t \in [0, T]} e^{kt} \int_t^T \left( C_1 \left| b_1(t) \right| \infty \left| a_y^{(n)}(s) - a_y^{(n-1)}(s) \right| + 2N C_2 \left| b_2(s) \right| \infty \left| a_y^{(n)}(s) - a_y^{(n-1)}(s) \right| \wedge N \right) \wedge N \forall 0,
\]
\[
\leq \frac{C_1 \left| b_1 \right| \infty + 2N C_2 \left| b_2 \right| \infty + \kappa \max_{y \in \mathcal{Y}} |q_y| \left| a_y^{(n)} - a_y^{(n-1)} \right|}{k} \wedge N \forall 0,
\]
Choose $k > C_1 \left| b_1 \right| \infty + 2N C_2 \left| b_2 \right| \infty + \kappa \max_{y \in \mathcal{Y}} |q_y|$, then
\[
\|a^{(n+1)} - a^{(n)}\|_k \leq \frac{C_1 \left| b_1 \right| \infty + 2N C_2 \left| b_2 \right| \infty + \kappa \max_{y \in \mathcal{Y}} |q_y|}{k} \|a^{(n)} - a^{(n-1)}\|_k,
\]
which gives us a contraction mapping from $D^\kappa$ to $D^\kappa$. Hence, by the Banach fixed point theorem, there exists a unique solution to (46). \hfill \square

Next, we want to show that for large enough $N$, the solution to (46) is also the solution to (45).
Lemma 21. For

\[ N \geq e^{KT} \left( \sum_{y=1}^{\kappa} g_y + T \sum_{y=1}^{\kappa} h_y \right), \]

where \( K := C_1 \max_{y \in Y} \| \hat{b}_{1y} \|_{\infty} + \max_{i \in Y} \sum_{y=1}^{\kappa} |q_{y,i}| \), the solution \( a^{(N)} \) to (46) satisfies the inequalities

\[ 0 \leq g_y + \int_{t}^{T} \left( h_y + C_1 \hat{b}_{1y}(s) a^{(N)}_y(s) - C_2 \hat{b}_{2y}(s) a^{2}_{y}(s) + \sum_{i=1}^{\kappa} q_{y,i} a^{(N)}_i(s) \right) ds \leq N \quad (47) \]

for all \( t \in [0, T] \), where \( y \in Y \).

Proof. For simplicity of notations, \( a_y \) is used instead of \( a^{(N)}_y \) for \( y \in Y \) if there is no confusion.

First, for \( y \in Y \), we prove the positiveness of \( a_y \) by contradiction. Suppose \( a_y \) (\( y \in Y \)) are not positive functions on \([0, T]\). Since \( a_1 \) is continuous and \( a_1(T) = g_1 > 0 \), there exists some \( \tau_1 \in [0, T] \) as the closest time to \( T \) such that \( a_1(\tau_1) = 0 \). Note that finding such a \( \tau_1 \) is possible. Let \( t_n \in [0, T] \) be a non-decreasing sequence such that \( a_1(t_n) = 0 \), there exists some \( \tau_1 \) such that \( t_n \to \tau_1 < T \) as \( n \to \infty \) since \( a_1 \) is continuous and \( a_1(\tau_1) = g_1 > 0 \). By the continuity of \( a_1 \), we have \( a_1(\tau_1) = 0 \), which gives the desirable point \( \tau_1 \). Then for all \( t \in (\tau_1, T) \), \( a_1(t) > 0 \) and it implies that \( a_1'(\tau_1) > 0 \). In this case, plugging \( t = \tau_1 \) to (45), we have

\[ a_1'(\tau_1) = -h_1 - \sum_{i \neq 1}^{\kappa} q_{1,i} a_1(\tau_1) > 0, \]

which implies there is some \( y \in Y \) and \( y \neq 1 \) such that \( a_y(\tau_1) < 0 \). Without loss of generality, we let \( a_2(\tau_1) < 0 \). Since \( a_2 \) is continuous on \([0, T]\) and \( a_2(T) = g_2 > 0 \), from the intermediate value theorem, there exists some \( \tau_2 \in (\tau_1, T) \) such that \( a_2(\tau_2) = 0 \) and \( a_2'(\tau_2) > 0 \). This indicates that \( a_2'(\tau_2) = -h_2 - \sum_{i \neq 2}^{\kappa} q_{2,i} a_2(\tau_2) > 0 \) by plugging \( t = \tau_2 \) back to (45), and it implies that there is some \( y \in Y \) and \( y \neq 1, 2 \) such that \( a_y(\tau_2) < 0 \) since we already know \( a_1(\tau_2) > 0 \). Without loss of generality, we can let \( a_3(\tau_2) < 0 \). By induction with the same argument, there is a \( \tau_k \in (\tau_{k-1}, T) \) such that \( a_k(\tau_k) = 0 \) and \( a_k'(\tau_k) > 0 \), which gives

\[ a_k'(\tau_k) + h_k + \sum_{i \neq k}^{\kappa} q_{k,i} a_k(\tau_k) = 0. \]

But it contradicts with the fact that

\[ a_k'(\tau_k) > 0, \quad h_k > 0, \quad q_{k,i} > 0, \quad a_k(\tau_k) > 0 \]

for \( i \in \{1, 2, \ldots, \kappa - 1\} \). Thus the positiveness of \( a_y \) on \([0, T]\) for all \( y \in Y \) is obtained.

Next, we prove the upper boundness for the integral in (47). Note that for all \( t \in [0, T] \) and \( y \in Y \), let \( \tau = T - t \), we have

\[ a_y'(\tau) = h_y + C_1 \hat{b}_{1y}(\tau) a_y(\tau) - C_2 \hat{b}_{2y}(\tau) a_y^2(\tau) + \sum_{i=1}^{\kappa} q_{y,i} a_i(\tau), \]

and thus

\[ \sum_{y=1}^{\kappa} a_y'(\tau) = \sum_{y=1}^{\kappa} h_y + C_1 \sum_{y=1}^{\kappa} \hat{b}_{1y}(\tau) a_y(\tau) - C_2 \sum_{y=1}^{\kappa} \hat{b}_{2y}(\tau) a_y^2(\tau) + \sum_{y=1}^{\kappa} \sum_{i=1}^{\kappa} q_{y,i} a_i(\tau) \]

\[ \leq \sum_{y=1}^{\kappa} h_y + C_1 \max_{y \in Y} \hat{b}_{1y} \| \hat{b}_{1y} \|_{\infty} \sum_{y=1}^{\kappa} a_y(\tau) + \sum_{y=1}^{\kappa} \sum_{i=1}^{\kappa} |q_{y,i}| a_i(\tau) \]

\[ \leq \sum_{y=1}^{\kappa} h_y + \sum_{i=1}^{\kappa} \left( C_1 \max_{y \in Y} \hat{b}_{1y} \| \hat{b}_{1y} \|_{\infty} + \sum_{y=1}^{\kappa} |q_{y,i}| \right) a_i(\tau) \]

\[ \leq \sum_{y=1}^{\kappa} h_y + K \sum_{i=1}^{\kappa} a_i(\tau), \]
where

\[ K := C_1 \max_{y \in \mathcal{Y}} \left| \tilde{b}_1 y \right|_{\infty} + \max_{i \in \mathcal{Y}} \sum_{y=1}^{\kappa} |q_{y,i}| \]

with \( \sum_{y=1}^{\kappa} a_y(T) = \sum_{y=1}^{\kappa} g_y \). By Grönwall’s inequality, for all \( \tau \in [0, T] \),

\[ \sum_{y=1}^{\kappa} a_y(\tau) \leq e^{KT} \left( \sum_{y=1}^{\kappa} g_y + T \sum_{y=1}^{\kappa} h_y \right). \]

Hence \( a_y(t) \leq e^{KT} \left( \sum_{y=1}^{\kappa} g_y + T \sum_{y=1}^{\kappa} h_y \right) \) for all \( t \in [0, T] \) and \( y \in \mathcal{Y} \). Hence, when

\[ e^{KT} \left( \sum_{y=1}^{\kappa} g_y + T \sum_{y=1}^{\kappa} h_y \right) \leq N, \]

(47) holds. □

**Lemma 22.** With the given of \( h_y, g_y \in \mathbb{R}^+ \), \( y \in \mathcal{Y} \), there exists a unique solution to the Riccati system (12).

**Proof.** The existence, uniqueness and boundedness of the solution to \( a_y (y \in \mathcal{Y}) \) are shown in Lemma [20] and Lemma [21]. Given \( (a_y : y \in \mathcal{Y}) \), the coefficient functions \( b_y (y \in \mathcal{Y}) \) form a linear ordinary differential equation system. Their existence and uniqueness are guaranteed by Theorem 12.1 in [2]. Similarly, with the given of \( (a_y, b_y : y \in \mathcal{Y}) \), the coefficient functions \( c_y, k_y (y \in \mathcal{Y}) \) also form a linear ordinary differential equation system. Applying the Theorem 12.1 in [2], we can obtain the existence and uniqueness of \( c_y, k_y (y \in \mathcal{Y}) \). □

### 6.5. Multidimensional Problem

In this subsection, we consider the multidimensional problem, which is a straightforward extension of the previous one-dimensional setup. The same type of Riccati system to characterize the equilibrium and the value function is obtained, and we have a similar result as the Theorem [4].

Suppose that \( X_t, W_t \) and \( \alpha_t \) take values in \( \mathbb{R}^d \), and all components of \( W_t \) are independent. Suppose that the dynamic of the generic player is given by

\[ X_t = X_0 + \int_0^t \left( \tilde{b}_1(Y_s, s)X_s + \tilde{b}_2(Y_s, s)\alpha_s \right) ds + W_t. \]

Consider the cost function

\[ J[m](y, x, t, \mu, \nu) = \mathbb{E} \left[ \int_t^T \left( \frac{1}{2} ||\alpha_s||^2 + h(Y_s) \int_{\mathbb{R}^d} ||X_s - z||^2 m(dz) \right) ds + g(Y_T) \int_{\mathbb{R}^d} ||X_T - z||^2 m(dz) \right] \]

\[ X_t = x, Y_t = y, \mu_t = \bar{\mu}, \nu_t = \bar{\nu} \]

\[ = \mathbb{E} \left[ \int_t^T \left( \frac{1}{2} ||\alpha_s||^2 + h(Y_s) \left( X_s^\top X_s - 2\mu_s^\top X_s + \nu_s \cdot 1_d \right) \right) ds + g(Y_T) \left( X_T^\top X_T - 2\mu_T^\top X_T + \nu_T \cdot 1_d \right) \right] \]

\[ X_t = x, Y_t = y, \mu_t = \bar{\mu}, \nu_t = \bar{\nu} \],
where \( m \) is the joint density function in \( \mathbb{R}^d \), and \( \mu, \nu \) take value in \( \mathbb{R}^d \). For \( y \in \mathcal{Y} \), define the Riccati system

\[
\begin{aligned}
a'_{y} + 2b_{1y}a_{y} - 2\tilde{b}_{2y}^2a_{y}^2 + \sum_{i=1}^{\kappa} q_{y,i}a_{i} + h_{y} &= 0, \\
b'_{y} + \left( 2\tilde{b}_{1y} - 4\tilde{b}_{2y}a_{y} \right) b_{y} + \sum_{i=1}^{\kappa} q_{y,i}b_{i} + h_{y} &= 0, \\
c'_{y} + da_{y} + db_{y} + \sum_{i=1}^{\kappa} q_{y,i}c_{i} &= 0, \\
k'_{y} - 2\tilde{b}_{2y}a_{y}^2 + 4\tilde{b}_{2y}^2a_{y}b_{y} + 2\tilde{b}_{1y}k_{y} + \sum_{i=1}^{\kappa} q_{y,i}k_{i} &= 0, \\
a_{y}(T) = b_{y}(T) = g_{y}, & \quad c_{y}(T) = k_{y}(T) = 0.
\end{aligned}
\]

(48)

**Theorem 23** (Verification theorem for MFGs). There exists a unique solution \((a_{y}, b_{y}, c_{y}, k_{y} : y \in \mathcal{Y})\) to the Riccati system \([18]\). With these solutions, for \( t \in [0, T] \), the MFG equilibrium path follows \( \hat{X} = \hat{X}[m] \) is given by

\[
d\hat{X}_t = \left( b_{1}(Y_t, t)\hat{X}_t - 2\tilde{b}_{2}(Y_t, t)a_{Y}(t)\left( \hat{X}_t - \hat{\mu}_t \right) \right) dt + dW_t, \quad \hat{X}_0 = X_0,
\]

with equilibrium control

\[
\hat{\alpha}_t = -2\tilde{b}_{2}(Y_t, t)a_{Y}(t)\left( \hat{X}_t - \hat{\mu}_t \right),
\]

where

\[
d\hat{\mu}_t = b_{1}(Y_t, t)\hat{\mu}_t dt, \quad \hat{\mu}_0 = \mathbb{E}[X_0].
\]

Moreover, the value function \( U \) is

\[
U(m_0, y, x) = a_y(0)x^T x - 2a_y(0)x^T |m_0|_1 + k_y(0)|m_0|_2 + b_y(0)|m_0|_2 \mathbb{1}_d + c_y(0)
\]

for \( y \in \mathcal{Y} \).

The proof is similar to the one-dimensional problem, and we don’t show the details here.

**Acknowledgments**

We would like to acknowledge valuable discussions and insightful examples provided by Prof Jianfeng Zhang of the University of Southern California.

**References**

[1] Saran Ahuja. *Mean Field Games with Common Noise*. Stanford University, 2015.

[2] Panos J Antsaklis and Anthony N Michel. *Linear Systems*. Springer Science & Business Media, 2006.

[3] Martino Bardi. Explicit solutions of some linear-quadratic mean field games. *Networks & Heterogeneous Media*, 7(2):243–249, IEEE, 2012.

[4] Martino Bardi and Fabio S Priuli. Lqg mean-field games with ergodic cost. In 52nd *IEEE Conference on Decision and Control*, pages 2493–2498. IEEE, 2013.

[5] Pierre Cardaliaguet. Notes on mean field games. Technical report, Technical report, 2010.

[6] Pierre Cardaliaguet, François Delarue, Jean-Michel Lasry, and Pierre-Louis Lions. *The Master Equation and the Convergence Problem in Mean Field Games*(AMS-201), volume 201. Princeton University Press, 2019.

[7] René Carmona, François Delarue, et al. *Probabilistic Theory of Mean Field Games with Applications I-II*. Springer, 2018.

[8] Xinwei Feng, Jianhui Huang, and Zhenghong Qiu. Mixed social optima and nash equilibrium in linear-quadratic-gaussian mean-field system. *arXiv preprint arXiv:1911.01886*, 2019.

[9] Dena Firooz, Sebastian Jaimungal, and Peter E Caines. Convex analysis for lgg systems with applications to major-minor lgg mean-field game systems. *Systems & Control Letters*, 142:104734, 2020.

[10] Wilfrid Gangbo, Alpár R Mészáros, Chenchen Mou, and Jianfeng Zhang. Mean field games master equations with non-separable hamiltonians and displacement monotonicity. *arXiv preprint arXiv:2101.12362*, 2021.

[11] Shuang Gao, Peter E Caines, and Minyi Huang. Lgg graphon mean field games. *arXiv preprint arXiv:2004.00679*, 2020.

[12] Jianhui Huang and Minyi Huang. Mean field lgg games with model uncertainty. In *52nd IEEE Conference on Decision and Control*, pages 3103–3108. IEEE, 2013.

[13] Jianhui Huang, Xun Li, and Tianxiao Wang. Mean-field linear-quadratic-gaussian (lgg) games for stochastic integral systems. *IEEE Transactions on Automatic Control*, 61(9):2670–2675, 2015.

[14] Jianhui Huang, Shujun Wang, and Zhen Wu. Mean field linear-quadratic-gaussian (lgg) games: major and minor players. *arXiv preprint arXiv:1403.3999*, 2014.
[15] Minyi Huang. Large-population LQG games involving a major player: the Nash certainty equivalence principle. *SIAM J. Control Optim.*, 48(5):3318–3353, 2009/10.

[16] Minyi Huang, Peter E Caines, and Roland P Malhamé. Social optima in mean field lqg control: centralized and decentralized strategies. *IEEE Transactions on Automatic Control*, 57(7):1736–1751, 2012.

[17] Minyi Huang, Roland P Malhamé, Peter E Caines, et al. Large population stochastic dynamic games: closed-loop mckean-vlasov systems and the nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.

[18] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Japanese journal of mathematics*, 2(1):229–260, 2007.

[19] Xuerong Mao and Chenggui Yuan. *Stochastic differential equations with Markovian switching*. Imperial college press, 2006.

[20] Son L Nguyen and Minyi Huang. Mean field lqg games with mass behavior responsive to a major player. In *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, pages 5792–5797. IEEE, 2012.

[21] Son L Nguyen, Dung T Nguyen, and George Yin. A stochastic maximum principle for switching diffusions using conditional mean-fields with applications to control problems. *ESAIM: Control, Optimisation and Calculus of Variations*, 26:69, 2020.

[22] Rinel Foguen Tchuendom. Uniqueness for linear-quadratic mean field games with common noise. *Dynamic Games and Applications*, 8(1):199–210, 2018.

[23] G. George Yin and Chao Zhu. *Hybrid switching diffusions*, volume 63 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2010. Properties and applications.