SOME GEOMETRIC PROPERTIES OF THE DOMAIN OF THE TRIANGLE $\mathcal{A}$ IN THE SEQUENCE SPACE $\ell(p)$

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ABSTRACT. The sequence space $\ell(\mathcal{A}, p)$ of non-absolute type is the domain of the triangle matrix $\mathcal{A}$ defined by the strictly increasing sequence $\lambda = (\lambda_n)$ of positive real numbers tending to infinity in the sequence space $\ell(p)$, where $\ell(p)$ denotes the space of all sequences $x = (x_k)$ such that $\sum_k |x_k|^p < \infty$ and were defined by Maddox in [Spaces of strongly summable sequences, Quart. J. Math. Oxford (2) 18 (1967), 345–355]. The main purpose of this paper is to investigate the geometric properties of the space $\ell(\mathcal{A}, p)$, like rotundity, Kadec-Klee property.

1. Introduction

By $\omega$, we denote the space of all sequences with complex elements which contains $\phi$, the set of all finitely non-zero sequences, that is,

$$\omega := \{ x = (x_k) : x_k \in \mathbb{C} \text{ for all } k \in \mathbb{N} \},$$

where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. By a sequence space, we understand a linear subspace of the space $\omega$. We write $\ell_{\infty}$, $c$, $c_0$ and $\ell_p$ for the classical sequence spaces of all bounded, convergent, null and absolutely $p$-summable sequences which are the Banach spaces with the norms $\|x\|_{\infty} = \sup_{k \in \mathbb{N}} |x_k|$ and $\|x\|_p = (\sum_k |x_k|^p)^{1/p};$ respectively, where $1 \leq p < \infty$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. Also by $bs$ and $cs$, we denote the spaces of all bounded and convergent series, respectively. $bv$ is the space consisting of all sequences $(x_k)$ such that $(x_k - x_{k+1})$ in $\ell_1$ and $bv_0$ is the intersection of the spaces $bv$ and $c_0$.

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A linear topological space $X$ over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ satisfying the following conditions for all $x, y \in X$:

(i) $g(\theta) = 0$.
(ii) $g(x) = g(-x)$.
(iii) Scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha$'s in $\mathbb{R}$ and all $x$'s in $X$, where $\theta$ is the zero vector in the linear space $X$.

Assume here and after that $(p_k)$ be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear space $\ell(p)$ was defined by Maddox [2] (see also Simons [3] and Nakano [4]) as follows:

$$\ell(p) := \left\{ x = (x_k) \in \omega : \sum_k |x_k|^p_k < \infty \right\}, \quad (0 < p_k \leq H < \infty)$$

which is complete paranormed space paranormed by

$$g(x) = \left( \sum_k |x_k|^p_k \right)^{1/M}.$$

We assume throughout that $p_k^{-1} + (p'_k)^{-1} = 1$ provided $p_k \leq H < \infty$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

The beta-dual $\lambda^\beta$ of a sequence space $\lambda$ is defined by

$$\lambda^\beta = \left\{ x = (x_k) \in \omega : xy = (x_k y_k) \in \text{cs} \quad \text{for all} \quad y = (y_k) \in \lambda \right\}.$$  

Let $\lambda, \mu$ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers $a_{nk}$, where $k, n \in \mathbb{N}$. Then, we say that $A$ defines a matrix transformation from $\lambda$ into $\mu$ and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the $A$-transform of $x$, is in $\mu$; where

$$(Ax)_n = \sum_k a_{nk}x_k \quad \text{ (1.1)}$$

provided the series on the right side of (1.1) converges for each $n \in \mathbb{N}$. By $(\lambda : \mu)$, we denote the class of all matrices $A$ such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if $Ax$ exists, i.e. $A_n \in \lambda^\beta$ for all $n \in \mathbb{N}$ and is in $\mu$ for all $x \in \lambda$, where $A_n$ denotes the sequence in the $n$-th row of $A$.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for triangles $A, B$ and any sequence $x$. Further, a triangle matrix $U$ uniquely has an inverse $U^{-1} = V$ which is also a triangle matrix. Then, $x = U(Vx) = V(Ux)$ holds for all $x \in \omega$.

The matrix domain $\lambda_A$ of an infinite matrix $A$ in a sequence space $\lambda$ is defined by

$$\lambda_A := \{ x = (x_k) \in \omega : Ax \in \lambda \}.$$
If A is triangle, then one can easily observe that the sequence spaces $\lambda_A$ and $\lambda$ are linearly isomorphic, i.e. $\lambda_A \cong \lambda$.

We consider the strictly increasing sequence $\lambda = (\lambda_k)_{k=0}^\infty$ of positive reals tending to $\infty$, that is

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots \quad \text{and} \quad \lim_{k \to \infty} \lambda_k = \infty.$$  

Via the sequence $\lambda = (\lambda_k)_{k \in \mathbb{N}}$, we define the triangle matrix $\tilde{A} = (\tilde{a}_{nk})$ by

$$\tilde{a}_{nk}(\lambda) = \begin{cases} \frac{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}{\lambda_n - \lambda_{n-1}}, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. It is easy to show that $\tilde{A}$ is a regular matrix and a straightforward calculation yields that the inverse $\tilde{A}^{-1} = (b_{nk}(\lambda))$ of the matrix $\tilde{A}$ is given by the following double band matrix as

$$b_{nk}(\lambda) = \begin{cases} (-1)^{n-k} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}}, & n - 1 \leq k \leq n, \\ 0, & 0 \leq k < n - 1 \text{ or } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. We study some geometric properties of the sequence space $\ell(\tilde{A}, p)$ of non-absolute type which is the domain of the triangle matrix $A$ in the sequence space $\ell(p)$, that is

$$\ell(\tilde{A}, p) := \left\{ (x_k) \in \omega : \sum_k \left| \sum_{j=0}^k \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right|^p < \infty \right\}$$

which is a complete linear metric space paranormed by the paranorm

$$g_1(x) = \left( \sum_k \left| \sum_{j=0}^k \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right|^p \right)^{1/M}$$

and has the AK property. In the special case $p_k = p$ for all $k \in \mathbb{N}$, the space $\ell(\tilde{A}, p)$ is reduced to the space $\ell_p(\tilde{A})$, i.e.,

$$\ell_p(\tilde{A}) := \left\{ (x_k) \in \omega : \sum_k \left| \sum_{j=0}^k \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right|^p < \infty \right\}, \quad (0 < p < \infty)$$

which is a BK-space with the norm

$$\|x\| = \left( \sum_k \left| \sum_{j=0}^k \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right|^p \right)^{1/p}, \quad \text{where} \quad 1 \leq p < \infty.$$
and is a complete \( p \)-normed space with the \( p \)-norm
\[
\|x\| = \left( \sum_k \left( \sum_{j=0}^k \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right)^p \right)^{1/p}, \text{ where } 0 < p < 1.
\]

One can see from Theorem 2.3 of Jarrah and Malkowsky [5] that the domain \( \mu_T \) of an infinite matrix \( T = (t_{nk}) \) in a sequence space \( \mu \) has a basis if and only if \( \mu \) has a basis, if \( T \) is a triangle. As an immediate consequence of this fact, we derive the following result:

**Corollary 1.** Let \( 0 < p_k \leq H < \infty \) and \( \alpha_k = (Ax)_k \) for all \( k \in \mathbb{N} \). Define the sequence \( b^{(k)} = \{ b_n^{(k)} \}_{n \in \mathbb{N}} \) of the elements of the space \( \ell(A,p) \) by
\[
b_n^{(k)} := \begin{cases} (-1)^{n-k} \frac{\lambda_k - \lambda_{k-1}}{x_0 - 2(x_{n-1} + x_{n-2})}, & n - 1 \leq k \leq n, \\ 0, & \text{otherwise} \end{cases}
\]
for every fixed \( k \in \mathbb{N} \). Then, the sequence \( \{ b^{(k)} \}_{k \in \mathbb{N}} \) given by (1.2) is a basis for the space \( \ell(A,p) \) and any \( x \in \ell(A,p) \) has a unique representation of the form \( x := \sum_k \alpha_k b^{(k)} \).

Since the algebraic and topological properties of the space \( r^q(p) \) were studied by Altay and Başar in [6], we essentially emphasize the geometric properties of the space \( \ell(A,p) \).

## 2. The Rotundity of the Space \( \ell(A,p) \)

In this section, we focus on the rotundity and some geometric properties of the space \( \ell(A,p) \). For details, the reader may refer to [7], [8] and [9]. The main purpose of this study is to characterize the rotundity and some other geometric properties of the space \( \ell(A,p) \), the domain of the triangle matrix \( A \) in the sequence space \( \ell(p) \).

**Definition 2.1.** Let \( S(X) \) be the unit sphere of a Banach space \( X \). Then a point \( x \in S(X) \) is called an extreme point if \( 2x = y + z \) implies \( y = z \) for every \( y, z \in S(X) \). A Banach space \( X \) is said to be rotund (strictly convex) if every point of \( S(X) \) is an extreme point.

**Definition 2.2.** A Banach space \( X \) is said to have the Kadec-Klee property (or property(H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

**Definition 2.3.** Let \( X \) be real vector space. A functional \( \sigma : X \to [0, \infty) \) is called a modular if
\[\begin{align*}
(i) \ & \sigma(x) = 0 \text{ if and only if } x = \theta; \\
(ii) \ & \sigma(\alpha x) = \sigma(x) \text{ for all scalars } \alpha \text{ with } |\alpha| = 1; \\
(iii) \ & \sigma(\alpha x + \beta y) \leq \sigma(x) + \sigma(y) \text{ for all } x, y \in X \text{ and } \alpha, \beta \geq 0 \text{ with } \alpha + \beta = 1.
\end{align*}\]
(iv) The modular $\sigma$ is called convex if $\sigma(\alpha x + \beta y) \leq \alpha \sigma(x) + \beta \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

A modular $\sigma$ on $X$ is called
(a) right continuous if $\sigma(\alpha x) \to \sigma(x)$, as $\alpha \to 1^+$ for all $x \in X_\sigma$.
(b) left continuous if $\sigma(\alpha x) \to \sigma(x)$, as $\alpha \to 1^-$ for all $x \in X_\sigma$.
(c) continuous if it is both right and left continuous, where

$$X_\sigma := \left\{ x \in X : \lim_{\alpha \to 0^+} \sigma(\alpha x) = 0 \right\}.$$ 

We define $\sigma_p$ on the real sequence space $\ell(\tilde{A}, p)$ by

$$\sigma_p(x) = \sum_k \left| \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=0}^k (\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j \right|^{p_k}.$$ 

If $p_k \geq 1$ for all $k \in \mathbb{N}$, by the convexity of the function $t \mapsto |t_k|^{p_k}$ for each $k \in \mathbb{N}$, $\sigma_p$ is a convex modular on $\ell(\tilde{A}, p)$.

**Proposition 1.** The modular $\sigma_p$ on $\ell(\tilde{A}, p)$ satisfies the following properties with $p_k \geq 1$ for all $k$, we have $M = H$:

(i) If $0 < \alpha \leq 1$, then $\alpha^M \sigma_p(x/\alpha) \leq \sigma_p(x)$ and $\sigma_p(\alpha x) \leq \alpha \sigma_p(x)$.
(ii) If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha)$.
(iii) If $\alpha \geq 1$, then $\sigma_p(x) \geq \alpha^M \sigma_p(x/\alpha)$.
(iv) The modular $\sigma_p$ is continuous on the space $\ell(\tilde{A}, p)$.

**Proof.** Consider the modular $\sigma_p$ on $\ell(\tilde{A}, p)$.
(i) Let $0 < \alpha \leq 1$, then $\alpha^M/\alpha^{p_k} \leq 1$. So, we have
\[
\alpha^M \sigma_p \left( \frac{x}{\alpha} \right) = \alpha^M \sum_k \left| \frac{1}{\alpha} \sum_{j=0}^{k} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} \frac{x_j}{\alpha^{p_k}} \right|^p \]
\[
= \alpha^M \sum_k \left| \frac{1}{\alpha\lambda_k - \lambda_{k-1}} \sum_{j=0}^{k} (\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j \right|^p \]
\[
= \sum_k \left| \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=0}^{k} (\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j \right|^p \]
\[
= \sigma_p(x),
\]
\[
\sigma_p(\alpha x) = \sum_k \left| \frac{\alpha}{\lambda_k - \lambda_{k-1}} \sum_{j=0}^{k} (\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j \right|^p \]
\[
= \sum_k \left| \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=0}^{k} (\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j \right|^p \]
\[
\leq \alpha \sum_k \left| \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=0}^{k} (\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j \right|^p \]
\[
= \alpha \sigma_p(x).
\]
(ii) Let $\alpha \geq 1$. Then, $\alpha^M/\alpha^{p_k} \geq 1$ for all $p_k \geq 1$. So, we have
\[
\sigma_p(x) \leq \frac{\alpha^M}{\alpha^{p_k}} \sigma_p(x) = \alpha^M \sigma_p \left( \frac{x}{\alpha} \right).
\]
(iii) Let $\alpha \geq 1$. Then, $\alpha/\alpha^{p_k} \leq 1$ for all $p_k \geq 1$. So, we have
\[
\sigma_p(x) \geq \frac{\alpha}{\alpha^{p_k}} \sigma_p(x) = \alpha \sigma_p \left( \frac{x}{\alpha} \right).
\]
(iv) One can immediately see by Part (ii) for $\alpha > 1$ that
\[
\sigma_p(x) \leq \alpha \sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha^M \sigma_p(x).
\]
Proposition 2. For any $x \in \ell(\tilde{A}, p)$, the following statements hold:

(i) If $\|x\| < 1$, then $\sigma_p(x) \leq \|x\|$. 
(ii) If $\|x\| > 1$, then $\sigma_p(x) \geq \|x\|$. 
(iii) $\|x\| = 1$ if and only if $\sigma_p(x) = 1$. 
(iv) $\|x\| < 1$ if and only if $\sigma_p(x) < 1$. 
(v) $\|x\| > 1$ if and only if $\sigma_p(x) > 1$.

Proof. Let $x \in \ell(\tilde{A}, p)$.

(i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$. By the definition of $\| \cdot \|$, there exists an $\alpha > 0$ such that $\|x\| + \varepsilon > \alpha$ and $\sigma_p(x) \leq 1$. From Parts (i) and (ii) of Proposition 1, we obtain

$$\sigma_p(x) \leq \sigma_p\left(\frac{\|x\| + \varepsilon}{\tilde{A}}\right) \leq (\|x\| + \varepsilon) \sigma_p\left(\frac{x}{\alpha}\right) \leq \|x\| + \varepsilon.$$

Since $\varepsilon$ is arbitrary, we have (i).

(ii) If we choose $\varepsilon > 0$ such that $0 < \varepsilon < 1 - (1/\|x\|)$, then $1 < (1-\varepsilon)\|x\| < \|x\|$. By the definition of $\| \cdot \|$ and Part (i) of Proposition 1, we have

$$1 < \sigma_p\left(\frac{x}{(1-\varepsilon)\|x\|}\right) \leq \frac{1}{(1-\varepsilon)\|x\|} \sigma_p(x).$$

So $(1-\varepsilon)\|x\| < \sigma_p(x)$ for all $\varepsilon \in (0, 1 - (1/\|x\|))$. This implies that $\|x\| < \sigma_p(x)$.

(iii) Since $\sigma_p$ is continuous, we directly have (iii).

(iv) This follows from Parts (i) and (iii).

(v) This follows from Parts (ii) and (iii).

\[ \square \]

Theorem 2.4. $\ell(\tilde{A}, p)$ is a Banach space with the Luxemburg norm.

Proof. Let $S_\alpha = \{\alpha > 0 : \sigma_p(x/\alpha) \leq 1\}$ and $\|x\| = \inf S_x$ for all $x \in \ell(\tilde{A}, p)$. Then, $S_x \subset (0, \infty)$. Therefore, $\|x\| \geq 0$ for all $x \in \ell(\tilde{A}, p)$.

For $x = \theta$, $\sigma_p(\theta) = 0$ for all $\alpha > 0$. Hence, $S_0 = (0, \infty)$ and $\|\theta\| = \inf S_0 = \inf(0, \infty) = 0$.

Let $x \neq \theta$ and $Y = \{kx : k \in \mathbb{C} \text{ and } x \in \ell(\tilde{A}, p)\}$ be a non-empty subset of $\ell(\tilde{A}, p)$. Since $Y \subseteq S[\ell(\tilde{A}, p)]$, there exists $k_1 \in \mathbb{C}$ such that $k_1x \notin S[\ell(\tilde{A}, p)]$. Obviously $k_1 \neq 0$. We assume that $0 < \alpha < 1/k_1$ and $\alpha \in S_x$. Then, $(x/\alpha) \in S[\ell(\tilde{A}, p)]$. Since $|k_1\alpha| < 1$, we get

$$k_1x = k_1\alpha\frac{x}{\alpha} \in S[\ell(\tilde{A}, p)]$$
Therefore, we obtain
\[ \sigma_p \left( \frac{kx}{\alpha} \right) \leq 1 \quad \text{and} \quad \frac{kx}{\alpha} \in S[\ell(\tilde{A}, p)]. \]

Therefore, we obtain
\[ \frac{|k|x}{\alpha} = \frac{|k|}{k} \times \frac{kx}{\alpha} \in S[\ell(\tilde{A}, p)] \quad \text{and} \quad \frac{\alpha}{|k|} \in S_x. \]

That is, \( \|x\| \leq \alpha/|k| \) and \( |k||x| \leq \alpha \) for all \( \alpha \in S_{kx} \). So, \( |k||x| \leq \|kx\| \). If we take \( 1/k \) and \( kx \) instead of \( k \) and \( x \), respectively, then we obtain that
\[ \left| \frac{1}{k} \right| \|kx\| \leq \left| \frac{1}{k}kx \right| = \|x\| \quad \text{and} \quad \|kx\| \leq |k||x|. \]

Hence, we see \( \|kx\| = |k||x| \) which also holds when \( k = 0 \).

To prove the triangle inequality, let \( x, y \in S[\ell(\tilde{A}, p)] \) and \( \varepsilon > 0 \) be given. Then, there exist \( \alpha \in S_x \) and \( \beta \in S_y \) such that \( \alpha < \|x\| + \varepsilon \) and \( \beta < \|y\| + \varepsilon \). Since \( S[\ell(\tilde{A}, p)] \) is convex,
\[ \frac{x}{\alpha} \in S[\ell(\tilde{A}, p)], \quad \frac{y}{\beta} \in S[\ell(\tilde{A}, p)], \quad \frac{x + y}{\alpha + \beta} = \frac{\alpha}{\alpha + \beta} \left( \frac{x}{\alpha} \right) + \frac{\beta}{\alpha + \beta} \left( \frac{y}{\beta} \right) \in S[\ell(\tilde{A}, p)]. \]

Therefore, \( \alpha + \beta \in S_{x+y} \). Then, we have \( \|x + y\| \leq \alpha + \beta < \|x\| + \|y\| + 2\varepsilon \). Since \( \varepsilon > 0 \) was arbitrary, we obtain \( \|x + y\| \leq \|x\| + \|y\| \). Hence, \( \|x\| = \inf \{\alpha > 0 : \sigma_p(x/\alpha) \leq 1\} \) is a norm on \( \ell(\tilde{A}, p) \).

Now, we show that every Cauchy sequence in \( \ell(\tilde{A}, p) \) is convergent with respect to the Luxemburg norm. Let \( \{x^{(n)}_k\} \) be a Cauchy sequence in \( \ell(\tilde{A}, p) \) and \( \varepsilon \in (0, 1) \).

Thus, there exists \( n_0 \) such that \( \|x^{(n)}_k - x^{(m)}_k\| < \varepsilon \) for all \( n, m \geq n_0 \). By Part (i) of Proposition 2, we have
\[ \sigma_p \left( x^{(n)}_k - x^{(m)}_k \right) \leq \left\| x^{(n)}_k - x^{(m)}_k \right\| < \varepsilon \]  
(2.3)

for all \( n, m \geq n_0 \). This implies that
\[ \sum_k \left\| \tilde{A} \left( x^{(n)}_k - x^{(m)}_k \right) \right\|_p < \varepsilon. \]  
(2.4)

Then, for each fixed \( k \) and for all \( n, m \geq n_0 \),
\[ \left\| \tilde{A} \left( x^{(n)}_k - x^{(m)}_k \right) \right\|_p = \left\| \tilde{Ax}^{(n)}_k - \tilde{Ax}^{(m)}_k \right\| < \varepsilon. \]

Hence, the sequence \( \{\tilde{Ax}^{(n)}_k\} \) is a Cauchy sequence in \( \mathbb{R} \). Since \( \mathbb{R} \) is complete, there is \( \tilde{Ax}_k \in \mathbb{R} \) such that \( \tilde{Ax}^{(m)}_k \to \tilde{Ax}_k \), as \( m \to \infty \). Therefore, as \( m \to \infty \)
by (2.4) we have
\[ \sum_k \left| \sum_{n \geq n_0} \left( x^{(n)} - x \right) \right|^{p_k} < \varepsilon \]
for all \( n \geq n_0 \).

Now, we have to show that \( (x_k) \) is an element of \( \ell(\tilde{A}, p) \). Since \( (\tilde{A}x_k) \) \( \to (\tilde{A}x)_k \) as \( m \to \infty \), we have
\[ \lim_{m \to \infty} \sigma_p \left( x^{(n)} - x^{(m)} \right) = \sigma_p \left( x^{(n)} - x \right). \] (2.5)
Then, we see by (2.3) that \( \sigma_p \left( x^{(n)} - x \right) \leq \| x^{(n)} - x \| < \varepsilon \) for all \( n \geq n_0 \). This implies that \( x^{(n)} \to x \), as \( n \to \infty \). So, we have \( x = x^{(n)} - (x^{(n)} - x) \in \ell(\tilde{A}, p) \).
Therefore, the sequence space \( \ell(\tilde{A}, p) \) is complete with respect to Luxemburg norm. This completes the proof. \( \square \)

**Theorem 2.5.** The space \( \ell(\tilde{A}, p) \) is rotund if and only if \( p_k > 1 \) for all \( k \in \mathbb{N} \).

**Proof.** Let \( \ell(\tilde{A}, p) \) be rotund and choose \( k \in \mathbb{N} \) such that \( p_k = 1 \). Consider the following sequences given by
\[ x = \left( 1, \frac{-\lambda_0}{\lambda_1 - 2\lambda_0}, 0, 0, \ldots \right) \quad \text{and} \quad y = \left( 0, \frac{\lambda_1 - \lambda_0}{\lambda_1 - 2\lambda_0}, -\frac{\lambda_1 - \lambda_0}{\lambda_2 - 2\lambda_1 + \lambda_0}, 0, 0, \ldots \right). \]
Then, obviously \( x \neq y \) and \( \sigma_p(x) = \sigma_p(y) = \sigma_p \left( \frac{x+y}{2} \right) = 1 \). By Part (iii) of Proposition 2, \( x, y, (x + y)/2 \in S[\ell(\tilde{A}, p)] \) which leads us to the contradiction that the sequence space \( \ell(\tilde{A}, p) \) is not rotund. Hence, \( p_k > 1 \) for all \( k \in \mathbb{N} \). Conversely, let \( x \in S[\ell(\tilde{A}, p)] \) and \( v, z \in S[\ell(\tilde{A}, p)] \) with \( x = (v + z)/2 \). By convexity of \( \sigma_p \) and Part (iii) of Proposition 2, we have
\[ 1 = \sigma_p(x) \leq \frac{\sigma_p(v) + \sigma_p(z)}{2} \leq \frac{1}{2} + \frac{1}{2} = 1 \]
which gives that \( \sigma_p(v) = \sigma_p(z) = 1 \) and
\[ \sigma_p(x) = \sigma_p \left( \frac{x + z}{2} \right) = \frac{\sigma_p(v) + \sigma_p(z)}{2}. \] (2.6)

Also, we obtain from (2.6) that
\[
\left| \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=0}^{k} \lambda_j \frac{(v_j + z_j)}{2} - 2\lambda_{j-1} \frac{(v_j + z_j)}{2} + \lambda_{j-2} \frac{(v_j + z_j)}{2} \right|^{p_k} \]
\[ = \frac{1}{2} \left| \sum_{j=0}^{k} \lambda_j - 2\lambda_{j-1} + \lambda_{j-2} \frac{(v_j)}{\lambda_k - \lambda_{k-1}} \right|^{p_k} + \frac{1}{2} \left| \sum_{j=0}^{k} \lambda_j - 2\lambda_{j-1} + \lambda_{j-2} \frac{(z_j)}{\lambda_k - \lambda_{k-1}} \right|^{p_k} \]
(2.7)
for all \( k \in \mathbb{N} \). Since the function \( t \mapsto \| v \|^p_t \) is strictly convex for all \( k \in \mathbb{N} \), it follows by (2.7) that \( v_k = z_k \) for all \( k \in \mathbb{N} \). Hence, \( v = z \). That is, the sequence space \( \ell(\tilde{A}, p) \) is rotund.

\[ \Box \]

**Theorem 2.6.** Let \( x \in \ell(\tilde{A}, p) \). Then, the following statements hold:

(i) \( 0 < \alpha < 1 \) and \( \| x \| > \alpha \) imply \( \sigma_p(x) > \alpha^M \).

(ii) \( \alpha \geq 1 \) and \( \| x \| < \alpha \) imply \( \sigma_p(x) < \alpha^M \).

**Proof.** Let \( x \in \ell(\tilde{A}, p) \).

(i) Suppose that \( \| x \| > \alpha \) with \( 0 < \alpha < 1 \). Then, \( \| x/\alpha \| > 1 \). By Part (ii) of Proposition 2, \( \| x/\alpha \| > 1 \) implies \( \sigma_p(x/\alpha) \geq \| x/\alpha \| > 1 \). That is, \( \sigma_p(x/\alpha) > 1 \). Since \( 0 < \alpha < 1 \), by Part (i) of Proposition 1, we get \( \alpha^M \sigma_p(x/\alpha) \leq \sigma_p(x) \). Thus, we have \( \alpha^M < \sigma_p(x) \).

(ii) Let \( \| x \| < \alpha \) with \( \alpha \geq 1 \). Then \( \| x/\alpha \| < 1 \). By Part (i) of Proposition 2, \( \| x/\alpha \| < 1 \) implies \( \sigma_p(x/\alpha) \leq \| x/\alpha \| < 1 \). That is, \( \sigma_p(x/\alpha) < 1 \). If \( \alpha = 1 \), then \( \sigma_p(x/\alpha) = \sigma_p(x) < 1 = \alpha^M \). If \( \alpha > 1 \), then by Part (ii) of Proposition 1, we have \( \sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha) \). This means that \( \sigma_p(x) < \alpha^M \).

\[ \Box \]

**Theorem 2.7.** Let \( (x_n) \) be a sequence in \( \ell(\tilde{A}, p) \). Then, the following statements hold:

(i) \( \| x_n \| \to 1 \), as \( n \to \infty \) implies \( \sigma_p(x_n) \to 1 \), as \( n \to \infty \).

(ii) \( \sigma_p(x_n) \to 0 \), as \( n \to \infty \) implies \( \| x_n \| \to 0 \), as \( n \to \infty \).

**Proof.** Let \( (x_n) \) be a sequence in \( \ell(\tilde{A}, p) \).

(i) \( \| x_n \| \to 1 \), as \( n \to \infty \) and \( \varepsilon \in (0,1) \). Then, there exists \( n_0 \in \mathbb{N} \) such that \( 1 - \varepsilon < \| x_n \| < \varepsilon + 1 \) for all \( n \geq n_0 \). By Parts (i) and (ii) of Theorem 2.6, \( 1 - \varepsilon < \| x_n \| \) implies \( \sigma_p(x_n) > (1 - \varepsilon)^M \) and \( \| x_n \| < \varepsilon + 1 \) implies \( \sigma_p(x_n) < (1 + \varepsilon)^M \) for all \( n \geq n_0 \). This means \( \varepsilon \in (0,1) \) and for all \( n \geq n_0 \) there exists \( n_0 \in \mathbb{N} \) such that \( (1 - \varepsilon)^M < \sigma_p(x_n) < (1 + \varepsilon)^M \) for all \( n \geq n_0 \). That is, \( \sigma_p(x_n) \to 1 \), as \( n \to \infty \).

(ii) We assume that \( \| x_n \| \to 0 \), as \( n \to \infty \) and \( \varepsilon \in (0,1) \). Then, there exists a subsequence \( (x_{n_k}) \) of \( (x_n) \) such that \( \| x_{n_k} \| > \varepsilon \) for all \( k \in \mathbb{N} \). By Part (i) of Theorem 2.6, \( 0 < \varepsilon < 1 \) and \( \| x_{n_k} \| > \varepsilon \) imply \( \sigma_p(x_{n_k}) > \varepsilon^M \). Thus, \( \sigma_p(x_{n_k}) \to 0 \), as \( n \to \infty \). Hence, we obtain that \( \sigma_p(x_n) \to 0 \), as \( n \to \infty \) implies \( \| x_n \| \to 0 \), as \( n \to \infty \).

\[ \Box \]

**Theorem 2.8.** Let \( x \in \ell(\tilde{A}, p) \) and \( (x^{(n)}) \subset \ell(\tilde{A}, p) \). If \( \sigma_p(x^{(n)}) \to \sigma_p(x) \), as \( n \to \infty \) and \( x_k^{(n)} \to x_k \), as \( n \to \infty \) for all \( k \in \mathbb{N} \), then \( x^{(n)} \to x \), as \( n \to \infty \).
Proof. Let $\varepsilon > 0$ be given. Since $\sigma_p(x) = \sum_k \left| (\tilde{A}x)_k \right|^p < \infty$, $x \in \ell(\tilde{A}, p)$ there exists $k_0 \in \mathbb{N}$ such that

$$
\sum_{k=k_0+1}^\infty \left| (\tilde{A}x)_k \right|^p < \frac{\varepsilon}{3(2^M+1)}.
$$

(2.8)

It follows from the equality

$$
\lim_{n \to \infty} \left[ \sigma_p(x^{(n)}) - \sum_{k=0}^{k_0} \left| (\tilde{A}x^{(n)})_k \right|^p \right] = \sigma_p(x) - \sum_{k=0}^{k_0} \left| (\tilde{A}x)_k \right|^p
$$

that there exists $n_0 \in \mathbb{N}$ and for all $k \in \mathbb{N}$

$$
\sigma_p(x^{(n)}) - \sum_{k=0}^{k_0} \left| (\tilde{A}x^{(n)})_k \right|^p < \sigma_p(x) - \sum_{k=0}^{k_0} \left| (\tilde{A}x)_k \right|^p + \frac{\varepsilon}{3(2^M)}
$$

(2.9)

and for all $k \in \mathbb{N}$

$$
\sum_{k=0}^{k_0} \left| (\tilde{A}(x^{(n)} - x))_k \right|^p < \frac{\varepsilon}{3}.
$$

(2.10)

Therefore, we obtain from (2.8), (2.9) and (2.10) that

$$
\sigma_p(x_n - x) = \sum_{k=0}^\infty \left| (\tilde{A}(x^{(n)} - x))_k \right|^p
$$

$$
< \sum_{k=0}^{k_0} \left| (\tilde{A}(x^{(n)} - x))_k \right|^p + \sum_{k=k_0+1}^\infty \left| (\tilde{A}(x^{(n)} - x))_k \right|^p
$$

$$
< \frac{\varepsilon}{3} + 2^M \left[ \sum_{k=k_0+1}^\infty \left| (\tilde{A}x^{(n)})_k \right|^p + \sum_{k=k_0+1}^\infty \left| (\tilde{A}x)_k \right|^p \right]
$$

$$
< \frac{\varepsilon}{3} + 2^M \left[ \sigma_p(x^{(n)}) - \sum_{k=0}^{k_0} \left| (\tilde{A}x^{(n)})_k \right|^p + \sum_{k=k_0+1}^\infty \left| (\tilde{A}x)_k \right|^p \right]
$$

$$
< \frac{\varepsilon}{3} + 2^M \left[ \sigma_p(x) - \sum_{k=0}^{k_0} \left| (\tilde{A}x)_k \right|^p + \frac{\varepsilon}{3(2^M)} + \sum_{k=k_0+1}^\infty \left| (\tilde{A}x)_k \right|^p \right]
$$

$$
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2^M \left[ 2 \sum_{k=k_0+1}^\infty \left| (\tilde{A}x)_k \right|^p \right]
$$

$$
< \frac{2\varepsilon}{3} + \frac{2^M+1}{3(2^M+1)} = \varepsilon.
$$
This means that \( \sigma_p (x^{(n)} - x) \to 0 \), as \( n \to \infty \). By Part (i) of Theorem 2.7, \( \sigma_p (x^{(n)} - x) \to 0 \), as \( n \to \infty \) implies \( \| x_n - x \| \to 0 \), as \( n \to \infty \). Hence, \( x_n \to x \), as \( n \to \infty \). \(\square\)

**Theorem 2.9.** The sequence space \( \ell(\tilde{A}, p) \) has the Kadec-Klee property.

**Proof.** Let \( x \in S \left[ \ell(\tilde{A}, p) \right] \) and \( x^{(n)} \subset \ell(\tilde{A}, p) \) such that \( \| x^{(n)} \| \to 1 \) and \( x^{(n)} \overset{w}{\to} x \) be given. By Part (ii) of Theorem 2.7, we have \( \sigma_p (x^{(n)}) \to 1 \) as \( n \to \infty \). Also \( x \in S \left[ \ell(\tilde{A}, p) \right] \) implies \( \| x \| = 1 \). By Part (iii) of Proposition 2, we obtain \( \sigma_p(x) = 1 \). Therefore, we have \( \sigma_p(x^{(n)}) \to \sigma_p(x) \), as \( n \to \infty \).

Since \( x^{(n)} \overset{w}{\to} x \), as \( n \to \infty \) and \( q_k : \ell(\tilde{A}, p) \to \mathbb{R} \) defined by \( q_k(x) = x_k \) is continuous, \( x_k^{(n)} \to x_k \), as \( n \to \infty \) for all \( k \in \mathbb{N} \). Therefore, \( x^{(n)} \to x \), as \( n \to \infty \).

Because of any weakly convergent sequence in \( \ell(\tilde{A}, p) \) is convergent, the sequence space \( \ell(\tilde{A}, p) \) has the Kadec-Klee property. \(\square\)

**Conclusion**

Let \( 0 < r < 1 \), \( q = (q_k) \) be a sequence of non-negative reals with \( q_0 > 0 \) and \( Q_n = \sum_{k=0}^{n} q_k \) for all \( n \in \mathbb{N} \), \( \bar{r} = (r_k) \) and \( \bar{s} = (s_k) \) be the convergent sequences. Suppose that the sequences \( u = (u_k) \) and \( v = (v_k) \) consist of non-zero entries; \( u, s \in \mathbb{R} \), and \( \lambda = (\lambda_n) \) be the strictly increasing sequence of positive real numbers tending to infinity with \( \lambda_{n+1} \geq 2\lambda_n \).

Let us define the Riesz matrix \( R^q = (r_{nk}) \) with respect to the sequence \( q = (q_k) \), the double band matrix \( F = (f_{nk}) \) defined by the sequence \( (f_n) \) of Fibonacci numbers, the matrix \( A^r = (a_{nk}^r) \), the generalized difference matrix \( B(u, s) = (b_{nk}(u, s)) \), the matrix \( A^u = (a_{nk}^u) \), the double sequential band matrix \( B(\bar{r}, \bar{s}) = (b_{nk}(\bar{r}, \bar{s})) \), the matrix \( \tilde{A} = \{a_{nk}(\lambda)\} \) and the Norlund matrix \( N^q = (a_{nk}^q) \) with respect to the sequence \( q = (q_k) \) by

\[
\begin{align*}
r_{nk}^q &:= \begin{cases} 
\frac{q_k}{\bar{r}_n}, & 0 \leq k \leq n, \\
0, & k > n,
\end{cases} \\
\frac{f_{nk}}{f_{n+1}} - f_{nk} \frac{f_{n+1}}{f_{n+1}} &:= \begin{cases} 
-\frac{f_{n+1}}{f_n}, & k = n-1, \\
f_{nk}, & k = n,
\end{cases} \\
0, & 0 \leq k < n-1 \text{ or } k > n,
\end{align*}
\]

\[
\begin{align*}
a_{nk}^r &:= \begin{cases} 
\frac{1+r^k}{n+1} u_k, & 0 \leq k \leq n, \\
0, & k > n,
\end{cases} \\
\frac{f_{nk}}{f_{n+1}} - f_{nk} \frac{f_{n+1}}{f_{n+1}} &:= \begin{cases} 
u, & k = n, \\
s, & k = n-1,
\end{cases} \\
0, & 0 \leq k < n-1 \text{ or } k > n,
\end{align*}
\]

\[
\begin{align*}
a_{nk}^u &:= \begin{cases} 
(-1)^{n-k} u_k, & n-1 \leq k \leq n, \\
0, & 0 \leq k < n-1 \text{ or } k > n,
\end{cases} \\
b_{nk}(\bar{r}, \bar{s}) &:= \begin{cases} 
r_k, & k = n, \\
s_k, & k = n-1,
\end{cases} \\
0, & 0 \leq k < n-1 \text{ or } k > n,
\end{align*}
\]
\[
a_{nk}(\lambda) := \begin{cases} 
\frac{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}{\lambda_n - \lambda_{n-1}} & ; \quad 0 \leq k \leq n, \\
0 & ; \quad k > n,
\end{cases}
\]
\[
a_{nq}^q = \begin{cases} 
\frac{q_{n-k}}{n} & ; \quad 0 \leq k \leq n, \\
0 & ; \quad k > n,
\end{cases}
\]
for all \( k, n \in \mathbb{N} \).

For concerning literature about the geometric properties of the domain of the infinite matrix \( A \) in the sequence space \( \ell(p) \), the following table may be useful:

| \( A \)     | the space \( \lambda \) | geometric properties of \( \lambda_A \) | refer to: |
|-------------|-----------------------------|----------------------------------------|-----------|
| \( A^r \)   | \( \ell(p) \)               | \( a^r(u, p) \)                        | [10]      |
| \( B(u, s) \)| \( \ell(p) \)                | \( \ell(p) \)                          | [11]      |
| \( A^u \)   | \( \ell(p) \)               | \( b^u(u, p) \)                        | [12]      |
| \( B(\bar{r}, \bar{s}) \)| \( \ell(p) \)                | \( \ell(\bar{B}, p) \)                | [13, 14]  |
| \( F \)     | \( \ell(p) \)               | \( \ell(F, p) \)                       | [15]      |
| \( N^q \)   | \( \ell(p) \)               | \( N^q(p) \)                           | [16]      |

Table 1: The domains of some triangle matrices in the spaces \( \ell(p) \).

In the special case \( q_k = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2} \) and \( Q_n = \lambda_n - \lambda_{n-1} , \) \( R^q \) is reduced to \( A \). So, the space \( \ell(A, p) \) can be seen as a special case of the space \( r^q(p) \), the domain of the Riesz mean \( R^q \) in the Maddox’ space \( \ell(p) \) introduced by Altay and Başar [6]. Since the geometric properties of the space \( r^q(p) \) was not investigated the main results of the present paper are not contained in Altay and Başar [6]. So, the main results of the present study can be seen as the complementary results for Altay and Başar [6].

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