NP-Completeness of Hamiltonian Cycle Problem on Rooted Directed Path Graphs

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Abstract

The Hamiltonian cycle problem is to decide whether a given graph has a Hamiltonian cycle. Bertossi and Bonuccelli (1986, Information Processing Letters, 23, 195-200) proved that the Hamiltonian Cycle Problem is NP-Complete even for undirected path graphs and left the Hamiltonian cycle problem open for directed path graphs. Narasimhan (1989, Information Processing Letters, 32, 167-170) proved that the Hamiltonian Cycle Problem is NP-Complete even for directed path graphs and left the Hamiltonian cycle problem open for rooted directed path graphs. In this paper we resolve this open problem by proving that the Hamiltonian Cycle Problem is also NP-Complete for rooted directed path graphs.

Keywords: Intersection graph, undirected path graph, directed path graph, rooted directed path graph, NP-Completeness, Hamiltonian cycle

1 Introduction

Let $G = (V, E)$ be a graph. Let $N(x)$ denote the set of all neighbors of $x$ in $G$. Let $d(x) = |N(x)|$ denote the degree of $x$. A subset $S$ of $V$ is called a clique of $G$ if $G[S]$, the subgraph of $G$ induced on $S$, is a complete subgraph of $G$. $S$ is called a maximal clique if $S$ is a clique but no proper super set of $S$ in $G$ is a clique in $G$. By the maximum degree of a graph $G$, we mean the maximum of the degrees of the vertices in $G$. A cycle $C$ of $G$ is called a Hamiltonian cycle of $G$ if $C$ contains all the vertices of $G$. The problem of deciding whether a given graph $G$ has a Hamiltonian cycle is known as Hamiltonian cycle problem. This problem in general graph is well-known to be NP-Complete [6]. It is known to be NP-Complete even when the inputs are restricted to several classes of interesting special classes of graphs such as planar cubic 3-connected graphs [4], bipartite graphs [1], edge graphs (line graphs) [3], and chordal graphs [5]. Bertossi and Bonuccelli [2] proved that the Hamiltonian Cycle Problem is NP-Complete even for undirected path graphs. The Hamiltonian cycle problem for directed path graphs and circular arc graphs were left open by Bertossi

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and Bonuccelli. Narasimhan later proved that the Hamiltonian Cycle Problem is NP-Complete even for directed path graphs. However, the Hamiltonian cycle problem for circular graphs was solved by Shih et al. in $O(n^2 \log n)$ time. The Hamiltonian cycle problem on rooted directed path graph was left open by Narasimhan. The Hamiltonian cycle problem on rooted directed path graph is also mentioned to be open in (13, page 311).

In this paper we resolve this open problem. In fact, we prove that the Hamiltonian cycle problem is also NP-Complete for rooted directed path graphs. However, it is worth mentioning that the Hamiltonian cycle problem can be solved in polynomial time for 2-sep rooted directed path graphs, a proper subclass of rooted directed path graphs.

The rest of the paper is organized as follows. In Section 2, we introduce the rooted directed path graphs and present some results on this class of graphs. In Section 3, we prove that the Hamiltonian cycle problem is NP-Complete for rooted directed path graphs. We use the techniques similar to those used in [2, 10]. The reduction is carried out from the Hamiltonian Cycle Problem on bipartite graph with maximum degree 3, which was proved to be NP-Complete by Itai et al. [8].

2 Rooted directed path graphs

Two paths, say $P_1$ and $P_2$, in a tree $T$ are said to intersect if $V(P_1) \cap V(P_2) \neq \emptyset$.

Let $\mathcal{F}$ be a finite family of non-empty sets. An undirected graph $G$ is an intersection graph for $\mathcal{F}$ if there is a one-to-one correspondence between the vertices of $G$ and the sets in $\mathcal{F}$ such that two vertices in $G$ are adjacent if and only if their corresponding sets have non-empty intersection. If $\mathcal{F}$ is a family of paths in an undirected tree $T$, then $G$ is called an undirected path graph. If $\mathcal{F}$ is a family of directed paths in a directed tree $T$, i.e., a tree in which each edge is oriented, then $G$ is called a directed path graph. Note that a directed tree may have more than one vertex of in-degree zero. A rooted directed tree is a directed tree having exactly one vertex of in-degree zero. If $\mathcal{F}$ is a family of directed paths in a rooted directed tree $T$, then $G$ is called a rooted directed path graph. Undirected path graphs, directed path graphs, and rooted directed path graphs are also known as undirected vertex graphs or UV graphs, directed vertex graphs or DV graphs, and rooted directed vertex graphs or RDV graphs, respectively (see [3]).

Note that in the above definition of rooted directed path graph, the rooted directed tree $T$ is arbitrary. However, Gavril characterized the rooted directed path graphs $G$ in terms of a tree $T$ such that $V(T)$ is the set of all maximal cliques of $G$.

Theorem 2.1 (Clique Tree Theorem, [2, 3]). Let $G = (V, A)$ be a graph and let $\mathcal{K}$ be the set of all maximal cliques of $G$. For each vertex $v \in V$, let $\mathcal{K}_v$ be the set of cliques of $\mathcal{K}$ containing the vertex $v$. Then $G$ is a rooted directed path graph if and only if there exists a rooted directed tree $T$ with the vertex set $\mathcal{K}$, such that for every $v \in V$, $T(\mathcal{K}_v)$, the subtree of $T$ induced on $\mathcal{K}_v$, is a directed path in $T$.

The tree $T$ in the above theorem is called the RDP clique tree for $G$.

3 NP-Completeness

Consider the following problem.
Problem II:
Instance: A bipartite graph $B = (M, N, E)$ having maximum degree 3.
Question: Does $B$ contain a Hamiltonian cycle?

It has been shown in [8] that the problem II is NP-Complete.

Lemma 3.1 (Itai et. al [8]). The problem II is NP-Complete.

We will show that the following problem $Π_1$ is NP-complete.

Problem $Π_1$:
Instance: A rooted directed path graph $G = (V, A)$.
Question: Does $G$ contain a Hamiltonian cycle?

The transformation of an instance of problem II to an instance of the problem $Π_1$ is described below.

Let $B(M, N, E)$, a bipartite graph having $n$ vertices with maximum degree 3, be an instance of the problem II. Without loss of generality, we assume that $n = 2r, r \geq 2$, the vertex sets $M$ and $N$ both have $r$ vertices each and that $B$ has no vertex with degree one, since otherwise $B$ has no Hamiltonian cycle. Let $M = \{m_1, m_2, \ldots, m_r\}$ and $N = \{n_1, n_2, \ldots, n_r\}$. We show how to construct an instance of the problem $Π_1$ by showing how to construct a directed path graph $G(V, A)$ such that $B$ has a Hamiltonian cycle if and only if $G$ has a Hamiltonian cycle. We describe $G$ by describing all its maximal cliques. Note that describing all the maximal cliques of a graph fully defines the graph itself.

Construction 3.2. Corresponding to each vertex $m_i \in M$, $1 \leq i \leq r$, construct the clique $K_i = \{X_i\} \cup \{A_{ij} : m_in_j \in E, 1 \leq s \leq i, 1 \leq j \leq r\}$. Corresponding to each vertex $n_j \in N$ with $d(n_j) = 3$, $1 \leq j \leq r$, construct two cliques $K_j' = \{Y_j\} \cup \{A_{ij} : m_in_j \in E, 1 \leq i \leq r\}$ and $K_j'' = \{Z_j\} \cup \{A_{ij} : m_in_j \in E, 1 \leq i \leq r\}$. Corresponding to each vertex $n_j \in N$ with $d(n_j) = 2$, $1 \leq j \leq r$, construct the clique $K_j' = \{Y_j\} \cup \{A_{ij} : m_in_j \in E, 1 \leq i \leq r\}$.

Note that the cliques mentioned above are the only maximal cliques in $G$. Hence it is clear that, $V(G) = \{X_1, X_2, \ldots, X_r\} \cup \{Y_1, Y_2, \ldots, Y_r\} \cup \{Z_j : d(n_j) = 3\} \cup \{A_{ij} : m_in_j \in E, 1 \leq i \leq r, 1 \leq j \leq r\}$

Figure ?? illustrates the construction of the maximal cliques of $G$ from a given bipartite graph $B$.

We now prove that the resulting graph $G$ is a rooted directed path graphs.

Lemma 3.3. The graph $G$ constructed by Construction 3.2 is a rooted directed path graph.

Proof. Let $X$ be the set of all maximal cliques of $G$. Hence, $X = \{K_1, K_2, \ldots, K_r\} \cup \{K'_1, K'_2, \ldots, K'_r\} \cup \{K''_j : d(n_j) = 3, 1 \leq j \leq r\}$. Let $T = (X, A)$ be the directed graph such that $A = \{K_iK_{i+1} : 1 \leq i \leq r - 1\} \cup \{K_iK'_j : 1 \leq j \leq r\} \cup \{K''_jK'_j : d(n_j) = 3, 1 \leq j \leq r\}$. Clearly, $T$ is a rooted directed tree with root $K_1$. Figure ?? contains a bipartite graph $G$, the set of maximal cliques of $G$ constructed by Construction 3.2 and an RDP clique tree $T$ of $G$ constructed as above.

Let $v$ be a vertex of $G$. If $v$ is either $X_i$, $Y_j$ or $Z_j$, then $T(X_v)$ consists of the only one vertex and hence is a directed path of length zero. If $v = A_{ij}$ and $d(n_j) = 3$, then $T(X_v)$ consists of the directed path $\langle K_i, K_{i+1}, \ldots, K_r, K'_j, K''_j \rangle$. If $v = A_{ij}$ and $d(n_j) = 2$, then $T(X_v)$ consists of the directed path $\langle K_i, K_{i+1}, \ldots, K_r, K''_j \rangle$. Hence for each vertex $v \in V(G)$, $T(X_v)$ is a directed path in $T$. Hence by the Theorem 2.1 $G$ is a rooted directed path graph. □
\[K_1 = \{X_1, A_{11}, A_{13}\}\]
\[K_2 = \{X_2, A_{11}, A_{13}, A_{31}, A_{23}\}\]
\[K_3 = \{X_3, A_{11}, A_{13}, A_{31}, A_{23}, A_{32}, A_{33}\}\]
\[K'_1 = \{Y_1, A_{11}, A_{21}\}\]
\[K'_2 = \{Y_2, A_{22}, A_{32}\}\]
\[K'_3 = \{Y_3, A_{13}, A_{23}, A_{33}\}\]
\[K''_1 = \{Z_1, A_{11}, A_{21}\}\]
\[K''_2 = \{Z_2, A_{13}, A_{23}, A_{33}\}\]

\[B \text{ Set of all maximal cliques of } G\]
\[T \text{ Construction 3.2, and an RDP clique tree } T \text{ of the graph } G\]

**Figure 1:** The bipartite graph \(B\), the set of all maximal cliques of \(G\) constructed using Construction 3.2, and an RDP clique tree \(T\) of the graph \(G\).

**Lemma 3.4.** The bipartite graph \(B\) contains a Hamiltonian cycle if and only if \(G\) contains a Hamiltonian cycle.

**Proof.**  **Necessity:**

If \(B\) has a Hamiltonian cycle \(C_B\), we obtain a Hamiltonian cycle \(C_G\) for \(G\) as follows.

If \(m_i, n_j, m_k\) are three consecutive vertices in \(C_B\), we obtain \(C_G\) by substituting the sequence \(X_i, A_{ij}, Y_j, A_{kj}, Z_j, A_{kji}, X_k\) if \(d(n_j) = 3\) (in this case, \(m_k\) is the third vertex adjacent to \(n_j\), or with \(X_i, A_{ij}, Y_j, A_{kji}, X_k\) if \(d(n_j) = 2\). This results in a Hamiltonian cycle for \(G\) since all the vertices are covered.

**Sufficiency:***

Let \(C_G\) be a Hamiltonian cycle for \(G\).

**Claim:** A sequence of the form \(X_s, A_{ij}, Y_j, A_{kji}, Z_j, A_{kji}, X_t\), where \(s \geq i, t \geq k\) if \(d(n_j) = 3\), or of the form \(X_s, A_{ij}, Y_j, A_{kji}, X_t\), where \(s \geq i, t \geq k\) if \(d(n_j) = 2\), must appear in \(C_G\).

**Proof of Claim:** If \(d(n_j) = 3\), then \(N(Y_j) = N(Z_j) = \{A_{ij} : m_in_j \in E\} = \{A_{ij}, A_{kji}, A_{kji}\}\). So, the sequence \(A_{ij}, Y_j, A_{kji}, Z_j, A_{kji}\) must appear in \(C_G\). If \(d(n_j) = 2\), then \(A_{ij}, Y_j, A_{kji}\) must appear in \(C_G\) as \(A_{ij}\) and \(A_{kji}\) are the only two neighbors of \(Y_j\) in \(G\). We call such a sequence a \(j\)-block. Now each \(A_{ij}\) such that \(m_in_j \in E\) is contained in exactly one \(j\)-block. There are exactly \(r\) distinct \(X_s\)'s each appearing in exactly one clique and there are exactly \(r\) \(j\)-blocks. So, each \(j\)-block must appear immediately after an \(X_s\) and must appear immediately before an \(X_i\) in \(C_G\). If a \(j\)-block, which starts with \(A_{ij}\) and ends with \(A_{kji}\), appears immediately after \(X_s\) and appears immediately before \(X_k\) in \(C_G\), then \(s \geq i\) and \(t \geq k\). So, the claim is proved.

Now we obtain \(C_B\) from \(C_G\) as follows. If a \(j\)-block appears immediately after \(X_i\) and appears immediately before \(X_k\) in \(C_G\), we obtain \(C_B\) by substituting the sequence \(X_i, j\)-block, \(X_k\) with the sequence \(m_i, n_j, m_k\). It is easy to see that the resulting \(C_B\) is a Hamiltonian cycle for \(B\).
Next we show that the problem \( \Pi_1 \) is NP-Complete.

**Theorem 3.5.** *The problem \( \Pi_1 \) is NP-Complete.*

**Proof.** Clearly, the problem \( \Pi_1 \) is in NP. To show that \( \Pi_1 \) is NP-hard, we use a transformation from the problem \( \Pi \).

Consider an instance of \( \Pi \), i.e., a bipartite graph \( B = (M, N, E) \) with maximum degree 3. Construct the graph \( G = (V, A) \) from \( B \) using **Construction 3.2**. It can be easily verified that the construction of \( G \) from \( B \) can be done in polynomial time. By Lemma 3.3, \( G \) is a rooted directed path graph. Again by Lemma 3.4, there exists a Hamiltonian cycle in \( G \) if and only if there exists a Hamiltonian cycle in \( B \). So, \( B \) is an \( 'yes' \) instance of \( \Pi \) if and only if \( G \) is an \( 'yes' \) instance of \( \Pi_1 \). Since, by Lemma 3.1, \( \Pi \) is NP-Complete, \( \Pi_1 \) is also NP-Complete.

4 Conclusion

In this paper, we proved that the problem of deciding whether a given rooted directed path graph \( G \) contains a Hamiltonian cycle is NP-complete. This was an open problem and was mentioned in [10] and in ([13], page 311).

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