BLACK HOLES OF A MINIMAL SIZE
IN STRING GRAVITY

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A lower limit for a neutral black hole size is obtained in the frames of the string gravity model with the second order curvature correction. It is shown that this effect remains when the third order curvature correction is also taken into account and argued that such restriction does exist in all perturbative orders of curvature expansions.

Keywords: black hole, string theory, higher order curvature corrections

There are a lot of unsolved (and even non-understood) problems in the modern theoretical physics now. One of the most intriguing of them is a question on the endpoint of a black hole evaporation\textsuperscript{1}. This problem is widely discussed now\textsuperscript{2} because a complete evaporation of black holes (without any remnants) can violate the Quantum Coherence. The indirect indications to a possible existence of the Quantum Coherence violation were found early\textsuperscript{3}. This puzzle is real only for black holes with initial masses less than $10^{15}$ grams\textsuperscript{4}. Hitherto such black holes could either completely evaporate or they could survive as some real objects. The subject of the study is to find a model describing their form of an existence. General Relativity does not suggest any variants of the solution of this puzzle. Perhaps, non-perturbative M-theory, presently developed, can clarify something but it is only at the beginning of its way. Therefore, in order to make a little step in this direction, it is possible (working in quasi-classical approach) to use the non-minimal gravity model which represents the effective low energy limit of some great unification theory.

\textit{In the perturbational approach} the string theory (as a part of M-theory) predicts the Einstein equations to be modified by higher order curvature corrections in the range where the curvature of space-time has the near-Planckian values\textsuperscript{5}. In this approach of the string theory one has to use the so-called effective low energy action

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with the higher order curvature corrections in order to extend the boundaries of applicability of the classical General Relativity. The addition of the next curvature term allows one to make the next step in this direction. But all the resulting conclusions must be treated as preliminary directions on conclusions in the near Planckian region, later they have to be checked by pure quantum calculations. At the present time the form of higher order curvature corrections in the string effective action is not completely investigated. We do not know the general structure of the expansion and, hence, the direct summing up is impossible. But as we deal with the expansion, the most important correction is the second order curvature one which is the product of the Gauss-Bonnet and dilatonic terms. As Gauss-Bonnet term is a total divergence in four dimensional space-time, its combination with a dilatonic field \( e^{-2\phi}L_2 \), see below) provides only the second order differential equations relatively metric. Thus, the generic form of the action for all kinds of strings (for simplicity, only boson part is taken into account) has the form

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ m^2_{Pl} \left( -R + 2\partial_\mu \phi \partial^\mu \phi \right) + \lambda_2 e^{-2\phi}L_2 \right],
\]

where

\[
L_2 = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2.
\]

Here \( \phi \) is the dilatonic field and \( \lambda_2 = \lambda_2(\alpha') \) is the string coupling constant. Its value depends upon the type of the string theory.

The investigations in the frames of the discussed model were performed formerly \[5, 6\]. One of its most important results is the determination of the restriction to the minimal black hole mass. This effect was found both in the numerical calculations and in the analytical ones and was independent from the metric parametrization. In Plank unit values it has the form

\[
r_{inf}^h = \sqrt{\lambda} 2^{-\frac{1}{4}} \sqrt{6},
\]

where \( r_h^\text{inf} \) is the lower limit value of the horizon radius (for more details see \[6\]). This restriction appears in the second order curvature gravity and is absent in the minimal Einstein-Schwarzschild gravity.

In order to understand whether the formula (2) represents the fundamental restriction resulting from the string theory or it is only the effect of the Gauss-Bonnet curvature correction, it is necessary to investigate the situation in the following perturbation orders. The general form of the action is

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ m^2_{Pl}R + 2\partial_\mu \phi \partial^\mu \phi \right.
+ \lambda_2 e^{-2\phi}L_2 + \lambda_3 e^{-4\phi}L_3 + \lambda_4 e^{-6\phi}L_4 + \ldots \right].
\]

Here \( L_2 \) denotes the second order curvature correction (Gauss-Bonnet term), \( L_3 \) denotes the third order curvature correction,

\[
L_3 = R^{\mu\nu}_\alpha R^{\alpha\beta}_\mu R^{\lambda\rho}_\nu + 2\Omega_3,
\]

where

\[
\Omega_3 = R^{\mu\nu\rho\sigma}_\alpha R^{\alpha\beta\rho\sigma}_\mu R^{\lambda\rho}_\nu R^{\mu\lambda}_\rho + \frac{3}{4} R R^2_{\mu\nu\alpha\beta}
\]
\[
+ 6R^\mu\nu\alpha\beta R_{\alpha\mu R_{\beta\nu}} + 4R^\mu\nu R_{\nu\alpha\beta} - 6R R^2_{\alpha\beta} + \frac{1}{4} R^3,
\]
and \(L_4\) denote the fourth order curvature correction. The rest members of formula (3) are
\[
\lambda_3 = c_3 \lambda_2^2, \lambda_4 = c_4 \lambda_2^3, \ldots
\]
The coefficients \(c_i\) depend upon the type of the string theory (complete set of their values can be found in [4]).

Similar to our previous work [6], we look for the static, spherically symmetric, asymptotically flat solutions providing a regular ("quasi-Schwarzschild") horizon. Therefore, the most convenient choice of metric (which is usually called as the "curvature gauge") is
\[
ds^2 = \Delta dt^2 - \frac{\sigma^2}{\Delta} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]
where \(\Delta = \Delta (r), \sigma = \sigma (r)\). The curvature gauge and the Einstein frame are used for a more convenient comparison with the previous results.

Field equations obtained from the action (3) have the form
\[
\left[ A_1 + \lambda_2 A_2 \right] \begin{bmatrix} \Delta'' \\ \sigma' \\ \phi'' \end{bmatrix} = B_1 + \sum_{i=2}^{\infty} \lambda_i B_i,
\]
where \((3 \times 3)\) matrix \(A_1 = A_1 (\Delta, \Delta', \sigma, \phi', r)\) and the column-vector \(B_1 = B_1 (\Delta, \Delta', \sigma, \phi', r)\) come from the variation of the Einstein part of the action (3). \((3 \times 3)\) matrix \(A_2 = A_2 (\Delta, \Delta', \sigma, \phi', r)\) and the column-vector \(B_2 = B_2 (\Delta, \Delta', \sigma, \phi, \phi', r)\) are the result of the variation of the Gauss-Bonnet part of the action (3). Column-vectors \(B_3, (B_4, \ldots), B_i = B_i (\Delta \ldots \Delta''', \sigma \ldots \sigma''', \phi \ldots \phi''', r)\) are the consequence of the variation of the third (fourth, \ldots) order curvature correction. Unfortunately, due to a huge size of \(B_i\) expressions, we have no opportunity to show them in this letter. Here it is necessary to note that the curvature corrections have the general form \(R \ast R \ast R \ast \ldots\). So, they do not produce the highest derivatives with increasing order in the field equations relatively \(L_3\). Therefore, the contribution of \(B_3 (B_4, \ldots)\) represents the singular perturbation for the \((A_1, A_2, B_1, B_2)\) part of the Eq. (3).

Let us consider the question about the influence of \(B_3\) part to the solution of the equations consisting from \((A_1, A_2, B_1, B_2)\) terms. The main mathematical reason providing the existence of the restriction to the minimal black hole size is the intersection of the solution with the singular surface defined by \(\det (A_1 + \lambda_2 A_2) = 0\) which contains a caustic. Since in the infinity the curvature corrections vanish more quickly than the Einstein part, since in the infinity the asymptotic solution is completely determined by the Arnowitt-Deser-Misner mass \(M\) and the dilatonic charge \(D\) (here we define \(\phi_\infty = 0\) for simplicity). In other words, the initial data for the numerical calculations for all functions \(\Delta, \sigma\) and \(\phi\) are
\[
\Delta = 1 - \frac{2M}{r} + O \left( \frac{1}{r} \right),
\]
\[
\sigma = 1 - \frac{1}{2} \frac{D^2}{r^2} + O \left( \frac{1}{r^2} \right),
\]
\[
\phi = \frac{D}{r} + O \left( \frac{1}{r} \right).
\]
The last asymptotics suppose that in a case of $\lambda_3 \to 0$, the solution of Eqs. (5) converges to the solution of Eqs. (6). This occurs when the solution of Eqs. (5) perturbed by the term $B_3$ with the highest derivatives is regular (boundary layers are absent). In other words, in the case $\lambda_3 \to 0$ this solution converges to the regular part of the main branch $[A_1, A_2, B_1, B_2]$. Consequently, one has to check the stability of the $[A_1, A_2, B_1, B_2]$ solution relatively to the regular perturbation of the singular contribution $B_3$.

Eqs. (5) with the contribution $B_3$ were solved numerically. The solution was obtained by the iteration method. At every step system (5) was solved by the Runge-Kutta method (see [6] for more details of our numerical integration strategy), the highest derivatives being taken from the previous step. Figure 1 shows the dependence of the metric function $\Delta$ against the radial coordinate $r$ at the different values of the event horizon $r_h$ (in the case when $\lambda_2 = \lambda_3 = 1$). Similar singularity case was found recently by K. Maeda et al. [9]. The curve (a) represents the case when $r_h$ is rather large and is equal to 20.0 Planck unit values (P.u.v.). The curve (b) shows $\Delta(r)$ with $r_h$ being rather close to $r_{h\text{min}}$. The curve (c) shows the case when $2M \ll r_{h\text{min}}$ and any horizon is absent. One should make a note that the value of $r_{h\text{min}}$ has the same order as in the Gauss-Bonnet case.

Here we would like to add that the asymptotic behavior of the discussed solution near the horizon is also “quasi-Schwarzschild” one and, as we checked numerically, the restriction to a minimal black hole size can be obtained from these asymptotic expansions.

From our analysis we can conclude that when the third curvature correction $L_3$ is taken into account, the restriction to the minimal black hole size does retain. Its numerical value slightly differs from the second order one, but the effect retains principally. In so far as the curvature corrections $L_i$, $i = 4, \ldots$ do not produce the highest derivatives with the increasing order relatively $L_3$ one, so far as a particularity existing in the third perturbation order exists in all perturbative orders. Thus, it is possible to get a conclusion that the discussed restriction to the minimal black hole mass is a fundamental restriction of the string theory because it takes place in all perturbative orders. Note that after taking into account the whole curvature expansions, one has a right to work with the masses of Plank order in the quasi-classical level.

**Conclusions**

We have analyzed the four dimensional black hole solutions appearing in the low energy limit in all kinds of the string theory. When the higher order curvature corrections are allowed for the analysis, the restriction to the minimal size of the black hole appears. For example, it is approximately equal to $0.4 m_{Pl}$ in the second order for the heterotic strings. Such restriction does exist in all perturbation orders of curvature expansions. Here it is necessary to point out that the same result was obtained during the quantization of black hole with selfgravitating dust shell [10].

And finally, speculating on this phenomenon (if this object is stable), the minimal (“quasi-Schwarzschild”) black hole being the endpoint of the Hawking evapora-
tion can represent the relic remnant of black holes formed during the initial stages of our Universe formation. This is the very interesting problem and it requires the future investigations.

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Figure 1: The dependence of the metric function $\Delta$ against the radial coordinate $r$ at the different values of the event horizon $r_h$ (in the case when $\lambda_2 = \lambda_3 = 1$). The curve (a) represents the case when $r_h$ is rather large and is equal to 20.0 Planck unit values (P.u.v.). The curve (b) shows $\Delta(r)$ with $r_h$ being rather close to $r_{h\text{min}}$. The curve (c) shows the case when $2M \ll r_{h\text{min}}$ and any horizon is absent.