Gravitomagnetism in Brane-Worlds

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In this paper we discuss a physical observable which is drastically different in a brane-world scenario. To date, the Randall-Sundrum model seems to be consistent with all experimental tests of general relativity. Specifically, we examine the so-called gravitomagnetic effect in the context of the Randall-Sundrum (RS) model. This treatment, of course, assumes the recovery of the Kerr metric in brane-worlds which we have found to the first order in the ratio of the brane separation to the radius of the AdS$_5$, ($\ell/r$). We first show that the second Randall-Sundrum model of one brane leaves the gravitomagnetic effect unchanged. Then, we consider the two-brane scenario of the original Randall-Sundrum proposal and show that the magnitude of the gravitomagnetic effect depends heavily on the ratio of ($\ell/r$). Such dependence is a result of the geometrodynamic spacetime and does not appear in static scenarios. We hope that we will be able to test this proposal experimentally with data from NASA’s Gravity Probe B (GP-B) and possibly disprove either the Randall-Sundrum two-brane scenario or standard general relativity.

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I. INTRODUCTION

Motivated by earlier works on the extra dimensional solution to the hierarchy problems of the fundamental interactions [1,2], Randall and Sundrum have proposed a five-dimensional model of the universe in which one can recover four-dimensional Einstein gravity at large enough distances [3]. This model consists of a four-dimensional hypersurface (3-brane) embedded in a five-dimensional Anti de Sitter Spacetime (AdS$_5$). The novelty of this theory is that there is no need for compactification of the extra dimension. Instead, an effective dimensional reduction occurs in the form of a massless graviton localized near the brane in a ‘bound state.’ In the low energy limit, this bound state dominates over the Kaluza-Klein (KK) modes and all the gauge fields, which are constrained to live on the brane and feels normal four-dimensional gravity. Thus, as long as the length scale of the AdS$_5$ is sufficiently small, standard Newtonian and general relativistic results can be recovered to high accuracy [4].

A generic, non-cosmological form of the metric in this non-factorizable spacetime has the form

$$ds^2_5 = e^{-2|y|/\ell} g_{\mu\nu} dx^\mu dx^\nu + dy^2,$$

where $\ell$ is the curvature radius of the AdS$_5$ and $g_{\mu\nu}$ must be asymptotically flat (for cosmological form and its generalization see, e.g., [5] and references therein). The extra dimension is denoted by $y$. In other words, the induced metric is any vacuum solution of the four dimensional Einstein’s equations.

One example of this sort is the recovery of the 4-dimensional Schwarzschild metric for $g_{\mu\nu}$ to first [6] and second order in ($\ell/r$) [7]. Here, the solution is a black string hidden behind a “cigar” horizon. The dependence of the metric elements on the extra dimension makes the Einstein equations extremely hard to solve exactly.

However, it is because of this recovery that people believe that the RS scenario is consistent with all tests of general relativity. We would like, however, to point out that general relativity itself is awaiting for yet another test to be observed: “gravitomagnetism effect”. This is a pure relativistic effect and there is no Newtonian counterpart. It is mainly due to the rotation of a massive body in the spacetime. While general relativity is being tested with the Gravity Probe B experiment, it is worthwhile to see whether the RS model is consistent with this yet to be tested effect of the general relativity. If the RS scenario is compatible with the result of Gravity Probe B then this model is really consistent with the all general relativistic effects.

In order to discuss topics related to rotation and frame-dragging, one has to demonstrate the recovery of the Kerr metric. This itself would be a very difficult problem due to its lack of symmetry compare with the Schwarzschild metric. While in the case of spherical symmetry, we have three unknown parameters to deal with, there are five unknown parameters in the axial symmetry case. All parameters are functions of two intrinsic spatial dimensions and the extra dimension.

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In this paper, we begin with a brief review of the standard gravitomagnetism and outline how it will be measured with Gravity Probe B. This is followed by a section describing the recovery of the Kerr metric in the slow-motion, weak-field limit. This has the form (in Boyer-Lindquist coordinates we need the Kerr Metric in the slow-motion, weak-field limit. This has the form (in Boyer-Lindquist coordinates where the Kerr metric in the slow-motion, weak-field limit) such as the earth. In this case, gravitomagnetic field (see [8] and references therein).

A magnetic field, so a matter current will give rise to a magnetic field, so a body of mass creates a magnetic field. Just as a current of charge produces an electromagnetic field in electromagnetism, so a body of charge produces an electromagnetic field. Here, we instead define the angular momentum, \( J \), which can be thought of as an angular dipole moment. Then one can naturally obtain the gravitomagnetic vector potential for a spinning mass

\[
\mathbf{h} \cong -2G_4 \frac{\mathbf{J} \times \hat{r}}{r^3}.
\]

If we orient our body such that the \( J \) is along the \( z \) axis, we have in spherical coordinates \( h_{0i} \cong -2G_4 J \sin^2 \theta/r \), which is just the \( g_{0i} \) component of the Kerr metric [9]. Not surprisingly, we can also define the gravitomagnetic field

\[
\mathbf{H}_{GR} \equiv \nabla \times \mathbf{h} \cong 2G_4 \left[ \frac{\mathbf{J} - 3(\mathbf{J} \cdot \hat{r})\hat{r}}{r^3} \right].
\]

With this formalism in place, one can consider the torque on a gyroscope with angular momentum \( \mathbf{S} \) due to \( \mathbf{H} \),

\[
\tau \equiv \frac{1}{2} \mathbf{S} \times \mathbf{H} = \frac{d\mathbf{S}}{dt} \equiv \hat{\Omega} \times \mathbf{S}.
\]

Therefore the gyroscope undergoes precession with angular velocity

\[
\hat{\Omega} = -\frac{1}{2} \mathbf{H} = -G_4 \left[ \frac{\mathbf{J} - 3(\mathbf{J} \cdot \hat{r})\hat{r}}{r^3} \right],
\]

which is very similar to the Larmor precession one would observe of a magnetic moment precessing around a constant Magnetic field in classical electromagnetism.

GP-B is a satellite experiment being designed and implemented by NASA in conjunction with Stanford University [10,11]. It is scheduled to be launched in the next year or two [12] and it is hoped that it will provide the first experimental test of the gravitomagnetic effect. The satellite will contain four gyroscopes made of fused quartz. These gyros will be electrically suspended and spinning in vacuum. They will be coated in niobium and cooled below the superconducting threshold with liquid helium. Superconducting Quantum Interference Devices (SQUID’s) will be used to measure the London moment which points along the spin axis of the gyros. The SQUID’s will measure changes in the direction of the axis as small as 0.1 milliarcseconds. This system will be implemented by NASA in conjunction with Stanford University [10,11]. It is scheduled to be launched in the next year or two [12] and it is hoped that it will provide the first experimental test of the gravitomagnetic effect. The satellite will contain four gyroscopes made of fused quartz. These gyros will be electrically suspended and spinning in vacuum. They will be coated in niobium and cooled below the superconducting threshold with liquid helium. Superconducting Quantum Interference Devices (SQUID’s) will be used to measure the London moment which points along the spin axis of the gyros. The SQUID’s will measure changes in the direction of the axis as small as 0.1 milliarcseconds. This system will be implemented to measure the gyroscopic precession caused by the Earth’s gravitomagnetism over one year to 1% or better. Standard General Relativity predicts that this will be 42 milliarcseconds [11].

\[
\nabla^2 h_{0i} \cong 16\pi G_4 \rho v_i.
\]

This is easily recognizable as Poisson’s equation. In considering a rotating charge in electrodynamics, one would find it useful to define the magnetic dipole moment. Here, we instead define the angular momentum, \( \mathbf{J} \), which can be thought of as an angular dipole moment. Then one can naturally obtain the gravitomagnetic vector potential for a spinning mass

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III. RECOVERY OF THE KERR METRIC IN BRANE-WORLDS

If the four-dimensional metric in [10] is the Minkowski metric, \( \eta_{\mu\nu} \), then the five-dimensional metric will satisfy the vacuum Einstein equations

\[
\Box_4 h_{\mu\nu} = -16\pi G_4 \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T \right),
\]

with \( \Box_4 \equiv \eta^{\rho\sigma} \frac{\partial^2}{\partial x^\rho \partial x^\sigma} \), and \( T \) the trace of the energy-momentum tensor, \( T^{\mu\nu} \). One can then consider the temporal, off-diagonal terms of the energy momentum tensor of a non-relativistic matter which take the form

\[
\nabla^2 h_{0i} \cong 16\pi G_4 \rho v_i.
\]
with the bulk cosmological constant $\Lambda = -(6/\ell^2)$. Due to the Randall-Sundrum fine tuning, these five-dimensional Einstein equations with a bulk cosmological constant reduce to the four-dimensional Einstein equations in vacuum without an effective cosmological constant [3]. Thus we can rewrite (9) as

$$R_{\mu\nu} - 4 \ell^{-2} g_{\mu\nu} = 0, \quad R_{5\mu} = 0, \quad R_{55} - 4 \ell^{-2} = 0. \quad (10)$$

We can now relax our requirement on $g_{\mu\nu}$ and only restrict it to being Ricci flat. Now any ordinary, four-dimensional solution to the vacuum Einstein equations can serve as our metric.

If we treat the Kerr metric in the slow motion and weak field limit as a perturbation to the Minkowski background then following [4] one can write the induced metric perturbation on the brane as the summation of two parts: the part due to the matter fields on the brane and the part due to the brane displacement,

$$h_{\mu\nu} = h_{\mu5}^{(m)} + 2 \ell^{-1} \gamma_{\mu\nu} \hat{\xi}^5, \quad (11)$$

where $\hat{\xi}^5(x^\mu) = -y$ is the brane displacement due to the matter source on the brane and $\gamma_{\mu\nu} = e^{-2|y|/\ell} \eta_{\mu\nu}$. The matter fields part $h_{\mu5}^{(m)}$ is obtained through following relation [3],

$$h_{\mu5}^{(m)} = -2 \kappa_5^2 \int d^4 x' G_5^{(R)}(x,x') \left( T_{\mu5} - \frac{1}{3} \gamma_{\mu5} T \right)(x'), \quad (12)$$

where $\kappa_5^2 = 3 \pi G_5 [4]$. $T_{\mu5}$ is the energy-momentum tensor of the brane matter fields and the contribution from the brane tension has been excluded, though its effect has been considered in order to derive (13). Here $T = T_{\mu5} = (6/\kappa_5^2) \Box_4 \hat{\xi}^5$ and $G_5^{(R)}(x,x')$ is the 5D retarded Green’s function which satisfies

$$[e^{+2|y|/\ell} \Box_4 + \partial_y^2 - 4 \ell^{-2} + 4 \ell^{-4} \delta(y)] G_5^{(R)}(x,x') = \delta_5(x - x'), \quad (13)$$

where the so-called RS gauge in which $h_{55} = h_{\mu5} = 0$, $h_{\mu\nu},_{\nu} = 0$ and $h^{\mu}_{\mu} = 0$ was used in arriving (13). Following [3], one can obtain the Green’s function from a complete set of eigenstates

$$G_5^{(R)} = -\int \frac{d^4 k}{(2\pi)^4} e^{ik_\mu(x^\mu - x'^\mu)} \left[ \frac{\ell^{-1} e^{-2(|y|+|y'|)/\ell}}{k^2 - (\omega + i\epsilon)^2} + \int_0^\infty dm \frac{u_m(y)u_m(y')}{m^2 + k^2 - (\omega + i\epsilon)^2} \right], \quad (14)$$

where the first term corresponds to zero mode contributions and the second term shows the contributions from the continuum KK modes $u_m(y) = \sqrt{m \ell} J_1(m \ell) Y_2(m \ell \ell y^\mu + y^\mu) J_2(m \ell \ell y^\mu + y^\mu)/\sqrt{J_1(m \ell)^2 + Y_1(m \ell)^2}$. Now in the case of the Kerr metric, the energy-momentum tensor can be written as $T_{\mu5} = \rho(r)v_{\mu}v_r$ and since we are dealing with the stationary, axial symmetry case by using Eq. (11) we find that

$$\bar{h}_{00} = \frac{2 G_4 M}{r} \left( 1 + \frac{2 \ell^2}{3 r^2} + \ldots \right),$$

$$\bar{h}_{0i} = -2 A_i \left( 1 + \frac{2 \ell^2}{3 r^2} + \ldots \right), \quad (15)$$

$$\bar{h}_{ij} = \frac{2 G_4 M}{r} \left( 1 + \frac{2 \ell^2}{3 r^2} + \ldots \right) \delta_{ij},$$

where $r = |x - x'|$, $G_4 = (3\pi G_5/8\ell)$ is the four-dimensional Newton’s constant, and $M = \int d^3 x \rho$ is the total mass. $A$ is the gravitomagnetic vector potential which has the form $A \sim G_4(J \times r)/r^3$. In deriving (14) we have assumed that $|x| > |x'|$ and both points are on the wall ($y = y' = 0$). Thus in the stationary case

$$G_5^{(R)}(x,0,x',0) = \int_{-\infty}^{+\infty} dt G_5^{(R)}(x,x') \approx -\frac{1}{4\pi r} \left[ 1 + \frac{1}{2} \frac{\ell^2}{r^2} + \ldots \right]. \quad (16)$$

Now, having all the components of the metric, it seems that up to the first order in $(\ell/r)$, we recovered the Kerr metric [4] in the slow motion and weak field limit on the brane. We would like, however, to point out that as the weak field limit of the RS model, in general, differs from the weak field limit of the normal four dimensional Schwarzschild solution, the ratio of gravitomagnetic to gravitoelectric metric coefficients is not the same as the one in the usual four dimensional Kerr solution. This means that the gravitomagnetic effect of the two models will be different.
Though we recovered the Kerr metric in the slow motion and weak field limit, one can obtain the general form of the Kerr metric by inserting the most general axially symmetric, (non-singular) stationary metric in four-dimensions into \( \Box \) to obtain \( \Box \).

\[
d s_5^2 = e^{-2|y|/\ell} \left[ -e^{2\nu} dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu_2}(dx_2)^2 + e^{2\mu_3}(dx_3)^2 \right] + dy^2,
\]

where \( \nu, \psi, \omega, \mu_2 \) and \( \mu_3 \) are functions of \( x_2, x_3 \) and \( y \). There is an unexploited gauge freedom between \( \mu_2 \) and \( \mu_3 \). This metric can then be substituted into \( \Box \) to yield a series of coupled partial differential equations in \( x_2, x_3 \) and \( y \). After rigorous calculations, one can show, for sufficiently small \( \ell \) that calculations that

\[
d s_5^2 \approx e^{-2|y|/\ell} \left[ -\rho^2 \frac{\Delta}{\Sigma^2} dt^2 + \frac{\Sigma^2}{\rho^2} \left( d\varphi + \frac{2G_4 J r}{\Sigma^2} dt \right)^2 \sin^2 \theta + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \right] + dy^2,
\]

yields

\[
e^{2d/\ell} \nabla^2 h_{0i} \cong 16\pi G_4 \rho v_i,
\]

where we have used the fact that the four-dimensional Newton’s gravitational constant satisfies the relation \( G_4 = 3\pi G_5 / 8\ell \tan h(d/\ell) \) for the same matter distributions on two branes.

Our correction to the gravitomagnetic potential is similarly manifested in gravitomagnetic field, \( \mathbf{H}_{GR} \):

\[
\mathbf{H}_{RS} = \mathbf{H}_{GR} e^{-2d/\ell}
\]

To solve the hierarchy problem, the warp factor must bridge a gap of sixteen orders of magnitude. This indicates that \( d/\ell \sim 36 \) and gravitomagnetism is also decreased by 32 orders of magnitude from the standard general relativistic prediction. This puts the gravitomagnetic field of the Earth far outside the reach of GP-B. Thus, this Randall-Sundrum model predicts that Gravity Probe B will see no gravitomagnetic field. If, however, a nonzero precession is measured, then the two-brane scenario must be amended. On the other hand, if GP-B will be able to measure the GR predictions then this will put an upper bound on \( (d/\ell) \) of 1.15 which is in sharp contrast to what you may find by observing the Brans-Dicke bound.

At the end we again would like to emphasize that GP-B is not merely another test of GR and could serve to test models with extra dimensions such as the RS model. Another point which we would like to bring to attention is that it is not sufficient for a theory of higher dimensions to just satisfy the Schwarzschild type of experimental test and it seems that the Kerr test provide deeper understanding of such models.
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