MULTIPLY CONJUGATE SYSTEMS CONTAINING DEFORMATIONS OF QUADRICS

ION I. DINCA

Abstract. We provide a generalization of Bianchi’s triply conjugate systems containing a family of deformations of 2-dimensional quadrics together with its Bäcklund transformation to higher dimensions.

Contents

1. Introduction 1
2. Confocal quadrics in canonical form 2
3. Multiply conjugate systems containing deformations of quadrics 5
3.1. (Isotropic) quadrics without center 6
3.2. Quadrics with center 9
4. The Bäcklund transformation 9
4.1. (Isotropic) quadrics without center 9
4.2. Quadrics with center 10
References 10

1. Introduction

At the end of the XIXth century Weingarten produced the first examples of triply orthogonal systems (Lamé families) containing a family of constant Gauß curvature (CGC) $K = -1$ surfaces. Bianchi extended his work by allowing the (negative) CGC to vary within its family of surfaces and by developing the Bäcklund (B) transformation for such triply orthogonal systems. With the development of the theory of deformations of quadrics at the beginning of the XXth century Bianchi introduced triply conjugate systems containing a family of deformations of quadrics together with its B transformation as the natural metric-projective generalization of Weingarten’s triply orthogonal systems (see Darboux [9] for triply conjugate systems, Green [10] for projective differential geometry of triple systems of surfaces and Bianchi ([5], Vol 4,(143)&(146)) for triply conjugate systems containing deformations of quadrics).

According to Bianchi such triply conjugate systems must induce on the deformations of quadrics the conjugate system common to the quadric and its deformation, thus generalizing lines of curvature on CGC surfaces (surfaces of triply orthogonal systems cut each other along lines of curvature).

Note that Peterson’s 1-dimensional family of deformations of quadrics (see [11]) a-priori does not provide a triply conjugate system, although we have the conjugate system common to the quadric and its deformation property for each deformation.

In what concerns deformations of higher dimensional quadrics the first results are those of Cartan [7] concerning the deformation problem of space forms in space forms; in particular he introduced the minimal co-dimension $(n-1)$ for the deformation problem of $n$-dimensional space forms in space forms and phrases. Bäcklund transformation, Bianchi Permutability Theorem, common conjugate systems, (confocal) quadrics, (discrete) deformations in $\mathbb{C}^{n-1}$ of quadrics in $\mathbb{C}^{n+1}$, multiply conjugate systems.

Supported by the University of Bucharest.
forms, the exteriorly orthogonal forms tool (naturally appearing from the Gauß equations), their canonical form (given by lines of curvature on the deformation) and the \(n(n-1)\) functions of one variable dimensionality of the space of such deformations. These have been extended on one hand (upon a suggestion from S. S. Chern and using a result due to Moore on the Chebyshev coordinates on deformations of \(\mathbb{H}^n(\mathbb{R})\) in \(\mathbb{R}^{2n-1}\); these are lines of curvature and thus in bijective correspondence with such deformations) to the B transformation of \(\mathbb{H}^n\) in \(\mathbb{R}^{2n-1}\) in Tenenblat-Terng [12] (Terng also developed the Bianchi Permutability Theorem (BPT) for this B transformation in [13]) and on the other hand to deformations in \(\mathbb{R}^{2n-1}\) of quadrics in \(\mathbb{R}^{n+1}\) or in \(\mathbb{R}^n \times (i\mathbb{R})\) with positive definite linear element in Berger, Bryant and Griffiths [1] (again we have the minimal co-dimension \((n-1)\) and the \(n(n-1)\) functions of one variable dimensionality of the space of such deformations).

Note also that multiply orthogonal systems are present in the current literature of integrable systems (see for example Terng-Uhlenbeck [14] and its references).

Thus the natural question of completing deformations of higher dimensional quadrics and their B transformation to multiply conjugate systems containing deformations of higher dimensional quadrics and their B transformation arises. Just as the theory of deformation of 2-dimensional quadrics admits discretization via the iteration of the B transformation (moving Möbius configurations in Bianchi’s denomination; see Bobenko-Pinkall [6] for discrete deformations of the 2-dimensional pseudo-sphere), a similar approach should give discrete multiply conjugate systems containing deformations of higher dimensional quadrics (note that Weingarten discovered his triply orthogonal systems of CGC \(-1\) surfaces by the iteration of an infinitesimal transformation of CGC \(-1\) surfaces, so he essentially used the discrete version to find the differential version).

According to the principles laid down by Bianchi, if we restrict \((n-1)\) parameters in the multiply conjugate systems to constants so as to obtain a deformation of a quadric, then the remaining \(n\) parameters should provide the conjugate system common to the quadric and its deformation (a-priori the quadric may vary with the \((n-1)\) constants, but we restrict our discussion only to deformations of a fixed quadric).

Thus to obtain multiply conjugate systems containing deformations of quadrics one must extend the differential system for deformations of \(n\)-dimensional quadrics in \(\mathbb{C}^{2n-1}\) via isothermal-conjugate system on the considered quadric and the conjugate system common to the quadric and its deformation according to Bianchi’s principles and Cartan’s exterior differential calculus.

For more details on the (classical) theory of deformations of (higher dimensional) quadrics we refer the reader to one of our previous notes concerning Bianchi’s Bäcklund transformation for higher dimensional quadrics.

All computations are local and assumed to be valid on their open domain of validity without further details; all functions have the assumed order of differentiability and are assumed to be invertible, non-zero, etc when required (for all practical purposes we can assume all functions to be analytic).

2. Confocal quadrics in canonical form

Consider the complexified Euclidean space 
\[
(\mathbb{C}^m, <.,.>), <x,y> := x^T y, \quad |x|^2 := x^T x, \quad x,y \in \mathbb{C}^m
\]
with standard basis \(\{e_j\}_{j=1,\ldots,m}, \quad e^T_k e_k = \delta_{jk}\).

Isotropic (null) vectors are those vectors \(v\) of length 0 \((|v|^2 = 0)\); since most vectors are not isotropic we shall call a vector simply vector and we shall only emphasize isotropic when the vector is assumed to be isotropic. The same denomination will apply in other settings: for example we call quadric a non-degenerate quadric (a quadric projectively equivalent to the complex unit sphere).

A quadric \(x \in \mathbb{C}^{n+1}\) is given by the quadratic equation \(Q(x) := x^T A x + B^T x + C = 0\), \(A = A^T \in \mathbb{M}_{n+1}(\mathbb{C}), B \in \mathbb{C}^{n+1}, C \in \mathbb{C}, \quad A B^T C \neq 0\). 

A metric classification of all (totally real) quadrics in \( \mathbb{C}^{n+1} \) requires the notion of symmetric Jordan (SJ) canonical form of a symmetric complex matrix. The symmetric Jordan blocks are:

\[
J_j := 0 = 0_{1,1} \in M_1(\mathbb{C}), \quad J_2 := f_1 f_2^T \in M_2(\mathbb{C}), \quad J_3 := f_1 e_3^T + e_3 f_1^T \in M_3(\mathbb{C}), \quad J_4 := f_1 f_2^T + f_2 f_1^T + f_2 f_3^T + f_3 f_2^T \in M_4(\mathbb{C}), \quad J_5 := f_1 f_2^T + f_2 e_1^T + e_1 f_2^T + f_2 f_3^T + f_3 e_1^T + e_1 f_3^T \in M_6(\mathbb{C}), \quad J_6 := f_1 f_2^T + f_2 f_3^T + f_3 f_1^T + f_3 f_4^T + f_4 f_3^T + f_4 f_5^T + f_5 f_4^T + f_5 f_6^T \in M_6(\mathbb{C}), \text{ etc, where } f_j := \frac{e_{j+1}+e_{j+2}}{\sqrt{2}} \text{ are the standard isotropic vectors (at least the blocks } J_2, J_3 \text{ were known to the classical geometers). Any symmetric complex matrix can be brought via conjugation with a complex rotation to the symmetric Jordan canonical form, that is a matrix block decomposition with blocks of the form } a_j I_p + J_p; \text{ totally real quadrics are obtained for eigenvalues } a_j \text{ of the quadric part } A \text{ defining the quadric being real or coming in complex conjugate pairs } a_j, \bar{a}_j, \text{ with subjacent symmetric Jordan blocks of same dimension } p. \text{ Just as the usual Jordan block } \sum_{j=1}^p e_j e_j^T \text{ is nilpotent with } e_{p+1} \text{ cyclic vector of order } p, \text{ } J_p \text{ is nilpotent with } \bar{f}_1 \text{ cyclic vector of order } p, \text{ so we can take square roots of SJ matrices without isotropic kernels}

\[
(\sqrt{a} I_p + J_p) := \sqrt{a} \sum_{j=0}^{p-1} (\frac{a}{|a|})^j J_p, \quad a \in \mathbb{C}^*, \quad \sqrt{a} := \sqrt{r e^{i\theta}} \text{ for } a = re^{2i\theta}, \quad 0 < r, -\pi \leq 2\theta < \pi,
\]

two matrices with same SJ decomposition type (that is } J_p \text{ is replaced with a polynomial in } J_p \text{ commute, etc.}

The confocal family \( \{x_z\}_{z \in \mathbb{C}} \) of a quadric \( x_0 \subset \mathbb{C}^{n+1} \) in canonical form (depending on as few constants as possible) is given in the projective space \( \mathbb{CP}^{n+1} \) by the equation \( Q_z(x_z) := \begin{bmatrix} x_z^T \\ 1 \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} \begin{bmatrix} x_z \\ 1 \end{bmatrix} = 0, \) where

- \( A = A^T \in \text{GL}_{n+1}(\mathbb{C}) \text{ SJ, } B = 0 \in \mathbb{C}^{n+1}, \quad C = -1 \text{ for quadrics with center } (\mathbb{QC}), \)
- \( A = A^T \in \text{M}_{n+1}(\mathbb{C}) \text{ SJ, } \ker(A) = \mathbb{C} e_{n+1}, \quad B = -e_{n+1}, \quad C = 0 \text{ for quadrics without center } (\text{IQWC}) \) and
- \( A = A^T \in \text{M}_{n+1}(\mathbb{C}) \text{ SJ, } \ker(A) = \mathbb{C} f_1, \quad B = -\bar{f}_1, \quad C = 0 \text{ for isotropic quadrics without center } (\text{IQWC}) \).

From the definition one can see that the family of quadrics confocal to \( x_0 \) is the adjugate of the pencil generated by the adjugate of \( x_0 \) and Cayley’s absolute \( C(\infty) \subset \mathbb{CP}^n \) in the hyperplane at infinity; since Cayley’s absolute encodes the Euclidean structure of \( \mathbb{C}^{n+1} \) (it is the set invariant under rigid motions and homotheties of \( \mathbb{C}^{n+1} := \mathbb{CP}^{n+1} \) \( \subset \mathbb{CP}^n \)) the mixed metric-projective character of the confocal family becomes clear.

For \( QC \text{ spec}(A) \) is unambiguous (does not change under rigid motions) but for (I)QWC it may change with \((p+1)-\)roots of unity for the block of \( \ker(A) \) in \( A \) being \( J_p \) even under rigid motions \( (R,t) \in \text{O}_{n+1}(\mathbb{C}) \times \mathbb{C}^{n+1} \) which preserve the canonical form, so it is unambiguous up to \((p+1)-\)roots of unity.

We have the diagonal \( Q(W)C \) respectively for \( A = \sum_{j=1}^{n+1} a_j^{-1} e_j e_j^T, \quad A = \sum_{j=1}^{n} a_j^{-1} e_j e_j^T \); the diagonal IQWC come in different flavors, according to the block of \( f_1 : \quad A = J_p + \sum_{j=p+1}^{n+1} a_j^{-1} e_j e_j^T \); in particular if \( A = J_n+1 \), then \( \text{spec}(A) = \{0\} \) is unambiguous. General quadrics are those for which all eigenvalues have geometric multiplicity 1; equivalently each eigenvalue has an only corresponding SJ block; in this case the quadric also admits elliptic coordinates.

There are continuous groups of symmetries which preserve the SJ canonical form for more than one SJ block corresponding to an eigenvalue, so from a metric point of view a metric classification according to the elliptic coordinates and continuous symmetries may be a better one.

With \( R_z := I_{n+1} - z A, \quad z \in \mathbb{C} \setminus \text{spec}(A)^{-1} \) the family of quadrics \( \{x_z\}_{z} \) confocal to \( x_0 \) is given by \( Q_z(x_z) = x_z^T \begin{bmatrix} \bar{A} R_z^{-1} x_z + 2(2R_z^{-1}B)^T x_z + C + zB^T R_z^{-1} B = 0. \end{bmatrix} \) For \( z \in \text{spec}(A)^{-1} \) we obtain singular confocal quadrics; those with \( z^{-1} \) having geometric multiplicity 1 admit a singular set which is an \((n-1)\)-dimensional quadric projectively equivalent to \( C(\infty) \), so they will play an important rôle in the discussion of homographies \( H \in \text{PGL}_{n+1}(\mathbb{C}) \) taking a confocal family into another one, since \( H^{-1}(C(\infty)), \quad C(\infty) \) respectively \( C(\infty), \quad H(C(\infty)) \) will suffice to determine each confocal family.

The Ivory affinity is an affine correspondence between confocal quadrics and having good metric properties (it may be the reason why Bianchi calls it affinity in more than one language): it is given by \( x_z = \sqrt{R_z} x_0 + C(z), \quad C(z) := -\frac{1}{2} \int_{\mathbb{C}} (\sqrt{R_w})^{-1} dw) B. \) Note that \( C(z) = 0 \) for QC,
isotropic normals elliptic coordinates (given by the roots becomes between confocal quadrics involving the Ivory affinity reveal a natural parametrization of IQWC $C:= (|z|^1_{+} - 1)$. For general quadrics the polynomial equation $Q_2(x) = 0$ has degree $n + 1$ in $z$ and it has multiple roots iff $0 = \partial_x Q_2(x) = (\hat{N}_z)^2$; thus outside the locus of isotropic normals elliptic coordinates (given by the roots $z_1, \ldots, z_{n+1}$ of the said equation) give a parametrization of $C^{n+1}$ suited to confocal quadrics.

We have now some classical metric properties of the Ivory affinity: with $x_0, x_0^1 \in x_0$, $V_0^1 := x_1^1 - x_0^1$, etc the Ivory Theorem (preservation of length of segments between confocal quadrics) becomes $|V_0^1|^2 = |x_0 + x_0^1 - C(z)|^2 - 2(x_0^1)^1 (I_{n+1} + \sqrt{R} x_1^1)$ and it has multiple roots iff $0 = \partial_x Q_2(x) = (\hat{N}_z)^2$; hence defined satisfies $z = (\hat{N}_z)^2$. Note that we can take for QWC $\hat{A} = \odot$, to see this we make

$$
L^T (I_{n+1} - 2^1 e_n) = 0 \text{ and } L^T \hat{A} = I_{n+1} \text{ with the above property is found.}$
$$

For IQWC such a parametrization fails because of the isotropic ker($A$); however the computations between confocal quadrics involving the Ivory affinity reveal a natural parametrization of IQWC which is again an affine transformation of $Z$.

Consider a canonical IQWC $x_0^1 (A x_0 - 2^1 f_1) = 0$, ker($A$) = $C f_j$, $A = J^p + \ldots$. SJ. We are looking for a linear map $L \in \text{GL}_{n+1}(\mathbb{C})$ such that $x_0 = L Z$, equivalently $L^T A L = e^{2 \theta} I_{n+1}, I_{n+1} := I_{n+1} - e_n e_{n+1}^T$. Applying $L$ with $L(e^{-\theta} I_{n+1} + e^{2 \theta} e_n e_{n+1}^T)$ we can make $a = 0$. Thus $L e_{n+1} = f_1, L^T (A + f_1 f_1^T) L = I_{n+1},$ so $L^{-1} = (A + f_1 f_1^T)^{-1}, R^T R = I_{n+1}$ with $R e_{n+1} = (A + f_1 f_1^T)^{-1} f_1$. (note that $\text{Re}_{n+1}$ has, as required, length 1). Once $R \in O_{n+1}(\mathbb{C})$ with the above property is found, $L$ thus defined satisfies $L^T f_1 = e_{n+1}$ and thus $L^T A L = I_{n+1}$. L with the above properties is unique up to rotations fixing $e_{n+1}$ in its domain and a canonical choice of $R$ reveals itself from a SJ canonical form when doing computations on confocal quadrics. We have $L L^T = (A + f_1 f_1^T)^{-1}, I_{n+1} L^{-1} \sqrt{R} L = I_{n+1} L^{-1} \sqrt{R} L I_{n+1} = L^{-1} \sqrt{R} L e_{n+1} f_1 \sqrt{R} L = L^T \hat{A} \sqrt{R} L = I_{n+1} \sqrt{I_{n+1} - z L^T A L}: = I_{n+1} \sqrt{R} L, A' := L^T A L, \text{ ker}(A') = C e_{n+1} \ominus C L^{-1} (A + f_1 f_1^T)$. Note that we can take for $C Q W C$ $C := (\hat{A} + e_n e_{n+1}^T)^{-1}, A' := A, \text{ ker}(A') = C e_{n+1}$, so IQWC can be regarded as metrically degenerated QWC. Note that $e_{n+1}^T \sqrt{R} L = (-I_{n+1} L^{-1} C(z) + e_{n+1})$; this can be confirmed analytically by differentiating with respect to $z$ and using $(L L^T)^{-1} = AL - B_{L, 1}$ and will imply the symmetry of the TC, but since we have already proved the symmetry of the TC, we can use this to imply the previous. Thus $L^{-1} x_z = L^{-1} \sqrt{R} L Z + L^{-1} C(z) = I_{n+1} \sqrt{R} Z + e_n e_{n+1} (I_{n+1} L^{-1} C(z) + e_{n+1})^T Z + L^{-1} C(z), (x_z^2 - z_0^2)^1 N_o = (L^{-1} x_z^2 - z_0^2) (I_{n+1} Z_0 e_{n+1}) = Z_0^T I_{n+1} \sqrt{R} Z_1 + (Z_0 + Z_1)^T (I_{n+1} L^{-1} C(z) + e_{n+1}) L^{-1} C(z).$ Note that for IQWC $|I_{n+1} L^{-1} C(z)|^2 = 2 e_{n+1} L^{-1} C(z), L^{-1} C(z)$ lies itself on $Z$ (also in this case since $L \sqrt{F} f_1 = \delta_k p_{-1}$ we have $e_{n+1} L^{-1} C(z) = \hat{f}^T C(z) = (\frac{1}{p_{-1}})^2 - \frac{1}{p_{-1}}^2 d_{k_{-1}}$, so $e_{n+1}^T C(z)$ picks up the highest power of in $L^{-1} C(z)$). To see this we need $0 = |L^{-1} C(z) - \hat{f}^T C(z)e_{n+1}|^2 - 2 \hat{f}^T C(z) = C(z)^T (L L^T)^{-1} C(z) - \hat{f}^T C(z) = C(z)^T AC(z) - 2 \hat{f}^T C(z)$. Thus using $AC(z) = (I_{n+1} - \sqrt{R}) \hat{f}^T$ and $(I_{n+1} + \sqrt{R}) C(z) = \hat{f}_1$ it is satisfied.
Note also that as needed later we have $(L^TL)^{-1} = A' - I_{1,n}L^{-1}Be^T + [B \sqrt{e_{n+1}e_{n+1}}], |N_0| = \|(L^TL)^{-1}(I_{1,n}Z - e_{n+1})\|^2 = Z^TA'Z + 2Z^TI_{1,n}L^{-1}B + |B|^2,
(I_{n+1} + \sqrt{R_2})I_{1,n}L^{-1}C(z) = I_{1,n}L^{-1}(I_{n+1} + \sqrt{R_2})C(z) = -zI_{1,n}L^{-1}B.

3. Multiply conjugate systems containing deformations of quadrics

We assume all subspaces in discussion to be non-isotropic (the Euclidean product induced on them by the one on $\mathbb{C}^m$ is non-degenerate; this assures the existence of orthonormal normal frames and thus the discussion of sub-manifolds via the Gauß-Weingarten (GW) and Gauß-Codazzi-Mainardi(-Peterson)-Ricci (G-CMP-R) equations).

For deformations $x \subset \mathbb{C}^{2n-1}$ of $x_0 \subset \mathbb{C}^{n+1}$ (that is $|dx|^2 = |dx_0|^2$) with common conjugate system $(u^1, ..., u^n)$ and non-degenerate joined second fundamental forms (that is $[d^2x^T_{\alpha}N_0 \quad d^2x^T_N]$ is a symmetric quadratic $\mathbb{C}^n$-valued form which contains only $(du)^2$ terms for $N_0$ unit normal field of $x_0$ and $N = [N_1 ..., N_{n-1}]$ orthonormal normal frame of $x$ and the dimension $n$ cannot be lowered in an open dense set) the linear element must satisfy the condition

$$\Gamma^j_{jk} = 0, \ j, k, l \ \text{distinct}$$

and such deformations $x \subset \mathbb{C}^{2n-1}$ are in bijective correspondence with solutions $\{a_j\}_{j=1,...,n} \subset \mathbb{C}^*$ of the differential system

$$\left(\log a_j\right)_k = \Gamma^j_{jk}, \ j \neq k, \ \sum_{j=1}^n \frac{(a_j^0)^2}{a_j^0} + 1 = 0,$$

where $N_k^T d^2x_0 = \sum_{j=1}^n h^j(du)^0$ is the second fundamental form of $x_0$ (we shall use Latin indices $j, k, l, ...$ including to differentiate respectively with $u^j, u^k, u^l, ...$ when clear from the context; also we shall preserve the classical notation $d^2$ for the symmetric (tensorial) second derivative and we shall use $d/du$ for the exterior (antisymmetric) derivative; thus $d \wedge d = 0$).

Once a solution of this system is known, one finds the second fundamental form of $x$ (complete the row $[\partial^2_{x^j} \quad ... \quad \partial^2_{x^k}]$ as the first row in an orthogonal $R \subset O_n(\mathbb{C})$, delete it, multiply the column $j$ of the $M_{n-1,n}(\mathbb{C})$ obtained matrix respectively with $a_j$ and take the $k$-th row to obtain the second fundamental form of $x$ in the $N_k$ direction) and then one finds $x$ by the integration of a Ricatti equation and quadratures (the Gauß-Bonnet(-Peterson) Theorem).

By the argument of Cartan’s reduction of exteriorly orthogonal forms to the canonical form such coordinates $\{u^r\}_j$ exist for real deformations $\mathbb{R}^{2n-1}$ of imaginary quadrics $\mathbb{R}^n \times i\mathbb{R}$ of negative curvature (for such cases also all computations in the deformation problem will be real; see Berger, Bryant, Griffiths [II]). However, since we have completely integrable differential systems (systems in involution) for the deformation problem for quadrics, the dimensionality of the space of deformations of quadrics remains the same (namely solution depending on $n(n-1)$ functions of one variable) in the complex setting also (the Cartan characters remain the same).

To obtain multiply conjugate systems we need to extend $x$ with the independent variables $u^{n+1}, ..., u^{2n-1}$.

An $m$-dimensional region

$$x = x(u^1, ..., u^n) \subset \mathbb{C}^m, \ du^1 \wedge ... \wedge du^n \neq 0$$

gives a multiply conjugate system iff

$$x_{jk} = (\log a_j)_k x_j + (\log a_k)_j x_k, \ a_j \subset \mathbb{C}^*, \ j, k = 1, ..., m, \ j \neq k$$

with the compatibility condition

$$\left(\frac{a_j}{a_k}\right)_{kl} = (\log a_k)_i(\frac{a_j}{a_k})_k + (\log a_k)_k(\frac{a_j}{a_k})_l, \ j, k, l = 1, ..., m \ \text{distinct}.$$
For the specific computations of deformations of quadrics we shall use the convention $\mathbb{C}^n \subset \mathbb{C}^{n+1}$ with 0 on the $(n+1)$th component; thus for example we can multiply $(n+1, n+1)$-matrices with $n$-column vectors and similarly one can extend $(n, n)$ matrices to $(n+1, n+1)$ matrices with zeroes on the last column and row. The converse is also valid: an $(n+1, n+1)$ matrix with zeroes on the last column and row (or multiplied on the left with an $n$-row vector and on the right with an $n$-column vector) will be considered as an $(n, n)$-matrix.

3.1. (Isotropic) quadrics without center.

With $V := \sum_{k=1}^n v^k e_k = [v^1 \ldots v^n]^T$ consider the complex equilateral paraboloid $Z = Z(v^1, \ldots, v^n) = V + |V|^2 e_{n+1}$. We have the (I)QWC $x_0 := LZ$, $L \in \text{GL}_{n+1}(\mathbb{C})$ (recall $L := (\sqrt{A + e_{n+1}e_{n+1}^T})^{-1}$, ker$(A) = \mathbb{C} e_{n+1}$, $A$ SJ, $B = -e_{n+1}$ for QWC and $L e_{n+1} = f_1$, $L^T(A + f_1 f_1^T)L = I_{n+1}$, $A' := L^T A L$ SJ for ker$(A) = \mathbb{C} f_1$, $A$ SJ, $B = -f_1$ in the case of IQWC) with linear element, unit normal, second fundamental form and Christoffel symbols $|dx_0|^2 = dV^T L dV + (V^T dV)^2 |L e_{n+1}|^2 + 2(L e_{n+1})^T L dV (V^T dV)$, $N_0 = (|LT|^{-1} + B| + B)^2 = V^T A' V + 2V^T L^{-1} B + |B|^2$. Therefore $\mathcal{C}$.

Because $(v^1, \ldots, v^n)$ are isothermal-conjugate and $(u^1, \ldots, u^n)$ are conjugate on $x_0$, the Jacobian $\frac{\partial(v^1, \ldots, v^n)}{\partial(u^1, \ldots, u^n)}$ has orthogonal columns, so with $\lambda_j := |\frac{\partial v^j}{\partial u}| \neq 0$, $\Lambda := [\lambda_1 \ldots \lambda_n]^T$, $\delta' := \text{diag}[\delta u_1 \ldots \delta u^n]$, $d'f := \sum_{j=1}^n f_j d\delta u_j$ we have $dV = R\delta' \Lambda$, $R \in \text{O}_n(\mathbb{C})$. Multiplying the formula for the change of Christoffel symbols $\frac{\partial v^j}{\partial u_j} = \frac{\partial^2 v^e}{\partial u^e \partial u^j} + \frac{\partial v^e}{\partial u^j} \Gamma^e = \frac{\partial^2 v^e}{\partial u^e \partial u^j} + \lambda_j^2 \delta_{jk} + \frac{\partial \log(\sqrt{\Lambda})}{\partial u^k}$ on the left with $\frac{\partial}{\partial u}$ and summing after $e$ we obtain $\Gamma_{jk}^p = \lambda_p^{-2} (\sum_{e} \frac{\partial v^e}{\partial u^j} \frac{\partial \log(\sqrt{\Lambda})}{\partial u^k})_{jk} = \delta_{pk}(\log \lambda)_{jk} + \delta_{jk} \frac{\partial \log(\sqrt{\Lambda})_{k}}{\partial u^j}$ so $\Gamma^j = (\log(\Lambda)_{jk}, \Gamma^k_{jk} = \frac{\lambda^2}{\lambda_j}(\log(\sqrt{\Lambda}))_{jk}$.

Because $\frac{\partial}{\partial u}$, since $(\log \lambda_j) k = \Gamma^j_{jk} = (\log(\alpha_j _k), j \neq k$ we get $\lambda_j = e_j(\alpha_j)$, after a change of the $u$ variable into itself we can make $\lambda_j = a_j$, $j = 1, \ldots, n$, so from (3.1) $|\Lambda|^2 = -H$ and $\Lambda d' \Lambda = -d' V^T (A'V + L^{-1} B) = -\Lambda^T \delta'^T R (A'V + L^{-1} B)$. Imposing the compatibility condition $R^T d' \Lambda = R \delta' \Lambda$ we get $R^T d'r \delta' \Lambda - d' \delta' \Lambda = 0$, or $(\lambda_j) k = e_j R^T R e_k \lambda_j$, $j \neq k$, $R^T R e_k \lambda_j = e_j$ for $k$, $l$ distinct. Now by the standard Cartan trick $-e_k R^T R e_k = -e_l R^T R e_l = e_j R^T R e_j \lambda_j = e_j R^T R e_j$ and $\omega^l := \sum_{j=1}^n (e_j \Gamma^j_{k} R^T R e_k \delta'^T R \delta')_{k} = -\omega^T$ we have $d' \Lambda = \omega' \Lambda - \delta'^T (A'V + L^{-1} B)$ and $d' \Lambda = 0$. Becomes the differential system in involution (that is no further conditions appear if one imposes $d' \Lambda$ conditions and one uses the equations of the system itself) as the compatibility condition for the completely integrable linear differential system

$$d'V = R \delta' \Lambda, \quad d' \Lambda = \omega' \Lambda - \delta'^T (A'V + L^{-1} B), \quad A' \Lambda^T = -(V^T A' V + 2V^T L^{-1} B + |B|^2).$$

(3.5)

Note that for QWC once we know a solution $V \in \text{O}_n(\mathbb{C})$ of (3.4), a solution $V$, $\Lambda$ of (3.5) and certain linearly independent solutions $V_j$, $\Lambda_j$, $j = 1, \ldots, 2n-1$ of the (homogeneous) differential part of (3.5), then we can find up to rigid motions the space realization of the deformation $x \subset \mathbb{C}^{2n-1}$ of $x_0$ up to quadric systems and without the use of the Gauß-Bonnet(-Peterson) Theorem (this is again due to Bianchi for $n = 2$).

Let $V_j$, $\Lambda_j$, $j = 1, \ldots, 2n-1$ be certain linearly independent solutions of the (homogeneous) differential part of (3.5) and $V$, $\Lambda$ solution of (3.5) such that $x \subset \mathbb{C}^{2n-1}$ given up to quadric system systems.
by
\begin{equation}
(3.6)
d'x := [V_1 \ldots V_{2n-1}]^T dV
\end{equation}
is a deformation of \(x_0\) with non-degenerate joined second fundamental forms. Note \(d' \land d'' = 0\); since \(|d'x|^2 = |d'x_0|^2\), \(d'x_0 = (L + e_{n+1} V^T)dV\) we get \(\sum_{j=1}^{2n-1} V_j V_j^T - VV^T = I_{n+1} L^T L\); applying \(d'\) we get \(\sum_{j=1}^{2n-1} V_j^T - V^T = 0\) (use \(\delta' = M \delta\), \(M, N \subset M_n(\mathbb{C}) \iff M = N = \text{diag}\)); applying \(d''\) again we get \(\sum_{j=1}^{2n-1} \Lambda_j \Lambda_j^T - \Lambda \Lambda^T = I_n\).

With \(V := [V_1 \ldots V_{2n-1} \ iV]\), \(L := [A_1 \ldots A_{2n-1} \ iA]\) these can be written as
\begin{equation}
(3.7)
[V \ L] [V^T \ L^T] = \begin{bmatrix} I_{1, n} L^T L & 0 \\ 0 & I_n \end{bmatrix}.
\end{equation}

We also have the prime integral property \([V^T \ L^T] \begin{bmatrix} A' & 0 \\ 0 & I_n \end{bmatrix} [V \ L] = \begin{bmatrix} C \omega & icT \\ icT & 1 \end{bmatrix}, \ C = C^T \in M_{2n-1}(\mathbb{C}), c \in \mathbb{C}^{2n-1}\); multiplying it on the left with \([V \ L]\) and using (3.7) we get \(C = I_{2n-1}, c = 0\); thus \([V^T \ L^T]\) is determined modulo a multiplication on the left with a rotation \(\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \in O_{2n} (\mathbb{C})\) by
\begin{equation}
(3.8)
[V^T \ L^T] \begin{bmatrix} A' & 0 \\ 0 & I_n \end{bmatrix} [V \ L] = I_{2n}.
\end{equation}

Thus (3.6) is equivalent to (3.7) and to \([V^T L^{-1} \ L^T] \subset O_{2n}(\mathbb{C})\).

For the non-degenerate joined second fundamental forms property we need to prove that there is no vector field \(N \subset \mathbb{C}^{2n-1}\) along \(x\) such that \(N^T d'x = 0, N^T d''x = |d'V|^2 = \Lambda' \delta^2 \Lambda\), that is the linear system \([V \ L] [\begin{bmatrix} N \\ 0 \end{bmatrix}] = 0\) is inconsistent; using (3.8) this becomes \([N] = 0 = \Lambda^T \Lambda\), which is indeed inconsistent because \(\Lambda^2 \neq 0\).

Note that since for IQWC \(\ker A' = \mathbb{C}^{n+1} \oplus \mathbb{C} L f_1, \ \text{coker} A' = \mathbb{C}^{n+1} \oplus \mathbb{C} L^{-1} B\) for such quadrics we cannot derive from the equivalent of (3.8) for IQWC the full information about the equivalent of (3.7) for IQWC. This is to be expected, since a fundamental set of solutions of the homogeneous differential part of (3.6) contains \(2n\) linearly independent solutions; the obvious solution \([f_1 L 0]^T\) is isotropic and perpendicular on all the others with respect to (wrt) the bilinear form given by the left hand side of (3.8); by the Gramm-Schmidt orthogonalization process one can complete this solution with \(2n-1\) other solutions \(V_j, \ \Lambda_j, j = 1, \ldots, 2n-1\) orthonormal wrt the same bilinear form. This isotropic solution is added in the \(2n-1\) other with yet undetermined constant coefficients \(V_j L^{-1} B\).

The part of the equivalent of (3.8) for IQWC that cannot be inferred from the equivalent of (3.8) for IQWC is \([V \ L] = - \sum_{j=1}^{2n-1} [V_j \ A_j] V_j L^{-1} B + \begin{bmatrix} L^T f_1 \\ 0 \end{bmatrix} V^T L^{-1} B\), but this requires \(V^T L^{-1} B\) to be constant, so \(V, \ \Lambda\) has to be solution of the homogeneous differential part of (3.6), a contradiction, so \(V_j, \ \Lambda_j, j = 1, \ldots, 2n-1\) cannot be solutions of the homogeneous differential part of (3.6).

We now extend the independent variables \(u^1, \ldots, u^n\) with independent variables \(u^{n+1}, \ldots, u^{2n-1}\), \(du^1 \land \ldots \land du^{2n-1} \neq 0\), \(d = d' + d'', d' = \sum_{j=1}^{n} f_j du^j, d'' = \sum_{j=n+1}^{2n-1} f_j du^{j+n}\); thus \(d \land d = 0\) becomes \(d' \land d' = 0, d' \land d'' + d'' \land d' = 0, d'' \land d'' = 0\).

With \(\delta'' := \text{diag}(du^{n+1} \ldots du^{2n-1} 0)\) we now have \(d'' \Lambda = M \delta'' V, M \subset M_n(\mathbb{C}), M_{en} = 0\) (this choice of \(M\) is due to Bianchi for \(n = 2\) and its reason will appear immediately).

Imposing the compatibility condition \(d'' \land d'' \Lambda + d'' \land d'' = 0\) we obtain \([d'' \land (M \delta'') - \omega' \land M \delta'') V + (d'' \land \omega' - M \delta'' \land R \delta') \Lambda + \delta' R^T \land (A' d'' V - d'' R R^T (A' V + L^{-1} B)) = 0\) from where we obtain by applying \(\delta' \land: \delta' \land \delta'' \land (M \delta'') - \omega' \land M \delta'' = 0\), \(\delta' \land (d'' \land \omega' - M \delta'' \land R \delta') = 0\), so \(d'' \land (M \delta'') = \omega' \land M \delta'' + d'' R^T \land N \delta'', N \subset M_n(\mathbb{C}), N_{en} = 0\), \(d'' \land \omega' = M \delta'' \land R \delta' + d'' R^T \land \delta'' M \Lambda \land \ldots\). Since \(\omega' = -\omega''\) we have \(d'' \land \omega' = M \delta'' \land R \delta' + d'' R^T \land \delta'' M \Lambda\) (in particular we get \(M\) from \(R\) and its derivatives; this is the reason for the choice of \(M\)), so \(A' d'' V = (d'' R R^T A' - N \delta'') V - \delta'' M \Lambda + d'' R R^T L^{-1} B\). Imposing the compatibility condition \(d'' \land A' d'' V + d'' \land A' d'' V = 0\) and collecting the coefficient of \(R \delta' \Lambda\) we get
\[d''RR^TA' - N\delta'' = A'd''RR^T + \delta''N^T\] (in particular we get \(N\) from \(R\) and its derivatives). Note that for IQWC ker \(A' = \mathbb{C}e_{n+1} + \mathbb{C}L_1 f_1\), coker \(A' = \mathbb{C}e_{n+1} + \mathbb{C}L_1 B\) for such quadrics we cannot derive the full information about \(d''V\) from \(d''V\) only: from the prime integral property \(d''\Lambda = -d''V^T(A'V + L^{-1}B)\) and using \(V^T(d''RR^TA' - N\delta'')V = 0\) we have \((L^{-1}B)^T(d''V + d''RR^TV) = 0\). Now using \(A'(d''V - d''RR^TV) = [A' - e_{n+1}(L^{-1}B)](d''V - d''RR^TV)\), \(A' = : L^T\) we can finally extend the completely integrable linear differential system \((3.9)\) to the completely integrable linear differential system

\[
d' = V = R\delta\Lambda + (d''RR^T + A\delta''N^T)\Lambda - A\delta''M^T\Lambda + A\delta''RR^T L^{-1}B,\tag{3.9}
\]

with the extended compatibility (and algebraic) conditions

\[
d' \wedge \omega' = \omega' \wedge \omega' - \delta' R^T A'R \wedge \delta', \; \omega' \wedge \delta' = \delta' \wedge R^T d'R, \; d'' \wedge \omega' = M \delta'' \wedge R\delta' + \delta'' R^T \wedge d'' \Lambda \wedge (A''N^T) R, \; d'' \wedge (N\delta'') = R\delta' M \delta'' + (d''RR^T + A\delta''N^T), \; d'' \wedge (N\delta'') = R\delta' M \delta'' + (d''RR^T + A\delta''N^T) + d''RR^T \wedge N\delta'', \; A'd''RR^T + \delta'' N^T = d''RR^T A' - N\delta'', \; M\delta'' A \wedge d'' \wedge R^T L^{-1}B = N\delta''^T A \wedge d'' \wedge R^T L^{-1}B = 0, \; R \in \mathbb{O}_n(\mathbb{C}).\tag{3.10}
\]

If we impose \(d''\wedge d''\wedge d''\) on \((3.11)\), use \(d' \wedge d'' = 0\), \(d' \wedge d'' \wedge d'' = 0\) and the equations of the system itself, then we get the algebraic conditions

\[
M\delta'' A \wedge \delta'' N^T \wedge R\delta' + \delta'' R^T \wedge N\delta'' \wedge A\delta'' M^T = 0 (\Leftrightarrow M\delta'' A \wedge \delta'' N^T R = \text{diag}), \]  
\[
M\delta'' A^k L^{-1}B = N\delta''^k A^k L^{-1}B = N\delta'' A \wedge d'' RR^T A^k L^{-1}B = 0, \quad k \geq 0 \tag{3.11}
\]

the last relations are relevant for IQWC and state that \(M\delta''\), \(N\delta''\), \(M\delta'' A \wedge d'' RR^T\), \(N\delta'' A \wedge d'' RR^T\) are not supported on the coordinates corresponding to the SJ block of \(L^T f_1\) in \(A')\).

Further imposing \(d''\wedge d''\wedge d''\) on \((3.11)\), using \(d' \wedge d'' = 0\), \(d' \wedge d'' \wedge d'' = 0\), \(d'' \wedge d'' = 0\), the equations of \((3.11)\) itself and \((3.10)\) we finally get involution (that is no further conditions appear); note that for diagonal QWC (which form an open dense set in the set of (I)QWC) \((3.11)\) is vacuous, so we have \((3.10)\) already in involution.

While the differential system \((3.9)\) together with its compatibility conditions \((3.10)\&(3.11)\) is interesting on its own, these compatibility conditions must further be extended with new conditions in order to describe multiply conjugate systems containing deformations of quadrics; in order to find these conditions we need to consider the space realization of solutions.

To extend \(x\) with the independent variables \(u_{n+1}, ..., u_{2n-1}\) \((3.2)\, (3.3)\ and \((3.6)\) will provide the needed information to obtain multiply conjugate systems containing deformations of QWC. For our problem we have \(m = 2n - 1\), \(d'x = [V_1 \ldots V_{2n-1}]^T d'V\) with \(R\) solution of \((3.10)\&(3.11)\), \(V, \Lambda, V_j, j = 1 \ldots 2n - 1\) solutions of the (homogeneous) differential part of \((3.9)\) satisfying \((3.5)\) and \(a_{ij} = \lambda_{ij}, j = 1 \ldots n; \) from \(x_{n+i} = (\log \lambda_{ij})x_{n+i}\), \(j = 1 \ldots n, i = 1 \ldots n - 1\) we get \(x_{n+i} = (V_1 \ldots V_{2n-1}^T N - [A_1 \ldots A_{2n-1}^T M] e_{i+1})\), \(d'\log(a_{n+i}) = c_i^T \Lambda^T d'V; \) from \((a_{n+i})^T j = (\log \lambda_{ij})k(a_{n+i})j + (\log \lambda_{jk})j(a_{n+i})k, k = 1 \ldots n, \; j \neq k, \; l = 1 \ldots n - 1\) we get \(\frac{1}{\lambda_{ij}} = c_i^T (AV + f), \; d'f = 0, \; d'x = [V_1 \ldots V_{2n-1}^T N\delta'' - [A_1 \ldots A_{2n-1}^T M\delta''](AV + f)\) and \((3.3)\) for \(f = 0\), \(0 = d'' \wedge d'' \wedge d'' = [V_1 \ldots V_{2n-1}^T ]^T R^T / M\delta'' A'f\) (so \(f = 0\)) and \(d'' \wedge d'' = [V_1 \ldots V_{2n-1}^T] \wedge N\delta'' - [A_1 \ldots A_{2n-1}^T] \wedge M\delta'' A) = \wedge \delta'' N^T V, \) so we need

\[
M\delta'' A \wedge A\delta'' M^T = N\delta'' A \wedge A\delta'' M^T = N\delta'' A \wedge A\delta'' N^T = 0. \tag{3.12}
\]
Imposing $d' \wedge$ and $d'' \wedge$ conditions on (3.12), and using (3.10) & (3.11) and (3.12) themselves we don’t get any new conditions; note also that (3.12) are vacuous for diagonal QWC. Now we need

$$x_{n+l} n+p = (\log \alpha_{n+l})_{n+p} x_{n+l} + (\log \alpha_{n+p})_{n+l} x_{n+p}, \quad l, p = 1, ..., n-1, \quad l \neq p,$$

$$\lambda_1 x_{n+l} = (\log \lambda_{n+l}) x_1 + (\log \alpha_{n+l}) (\lambda_1) x_{n+l}, \quad j = 1, ..., n, \quad j \neq k, \quad l = 1, ..., n-1,$$

$$\lambda_2 x_{n+p} = (\log \alpha_{n+p}) (\lambda_2) x_{n+p}, \quad j = 1, ..., n, \quad l = 1, ..., n-1, \quad l \neq p,$$

$$\lambda_3 x_{n+p} = (\log \lambda_{n+p}) (\alpha_{n+l}) x_{n+p}, \quad j = 1, ..., n, \quad l = 1, ..., n-1, \quad l \neq p,$$

$$\lambda_4 x_{n+q} = (\log \alpha_{n+q})_{n+p} x_{n+p} + (\log \alpha_{n+q})_{n+q} x_{n+q}, \quad l, p = 1, ..., n-1, \quad l \neq p,$$

these are satisfied for diagonal QWC if $d'' g_t = (g_t)_{n+l} du^{n+l}$.

3.2. Quadrics with center.

4. The Bäcklund transformation

4.1. (Isotropic) quadrics without center.

For the B transformation the space realization

$$(4.1) \quad x^1 = x^0 + \left[ x^0 \right]_\delta \cdots \left[ x^0 \right]_\delta (\sqrt{R_1 V_0} - V_0 + I_1, n L^{-1} C(z)) \subset C^{2n-1}$$

of the leaf $x^1$ relative to the seed $x^0 \subset C^{2n-1}$, $|d' x^0|^2 = |d' x^0|^2$, the algebraic transformation

$$V_1 = \sqrt{R_1 V_0} - \sqrt{R_1 A_0} + I_1, n L^{-1} C(z), \quad A_1 = R_1^2 (\sqrt{A V_0} + \sqrt{R_1 A_0} + \sqrt{I_1 n L^{-1} B}),$$

remain valid, but the differential system subjacent to the B transformation (Ricatti equation)

$$(4.2) \quad (0, \sqrt{z}) \leftrightarrow (1, -\sqrt{z})$$

of solutions of (3.13) and the algebraic formula of the BPT

$$(4.3) \quad R_3 R_0^T = (D_2 - D_1 R_3 R_1^T) (D_2 R_2 R_1^T - D_1)^{-1}, \quad D_j := \sqrt{R_j z_j} / \sqrt{R_j z_j}, \quad j = 1, 2$$

in $R_1$ must be extended and then the BPT algebraic formula must satisfy this extension (for the third Möbius configuration the algebraic computations only suffice).

Applying $d''$ to (4.2) we get the full Ricatti equation

$$(4.5) \quad d'' R_1 = \frac{1}{d'' R_0 R_1^T D_2 + N_0 d'' \delta} D_1 + D'' R_0 \delta', \quad d'' d'' R_1 = D'' (d'' R_0 R_1^T D_2 + N_0 d'' \delta) D_1 + D'' R_0 \delta'$$

in $R_1$ and further

$$M_1 d'' = R_1^2 (D_1 R_0 M_0 \delta'' - N_0 d'' \delta) D_1^{-1}, \quad M_1 d'' = (A'R_1 M_0 \delta'' - R_1 M_0 \delta'' A') + D'' (N_0 \delta'' \delta) D_1^{-1}$$

(4.6)

(this we need $A' e_n \subset C e_n$).

Imposing compatibility conditions $d'' \wedge$ and $d'' \wedge$ on (4.5) and using the equation itself we get $d'' \wedge$ and $d'' = R_1 \left( \right.$

$$d'' R_1 = R_1 \left( \cdots \right)$$

and using the equation itself we get $d'' \wedge$

$$d'' R_1 = R_1 \left( \right.$$
is a solution of a linear differential equation and remains 0 if initially it was 0. The fact that $R_1$ is itself a solution of (3.10) follows from the symmetry $(0, \sqrt{2}) \leftrightarrow (1, -\sqrt{2})$ and the fact that $d \wedge dR_0 = 0$ (basically the converse of the proven result).

Therefore we only need to prove that $R_3$ given by (3.3) satisfies (3.5) for $(R_0, z)$ replaced by $(R_1, z)$, $(R_2, z)$; by symmetry it is enough to prove only one relation. Since $dR_1 = - N \omega_0 - R_0 \omega R_0^2 D_1 R_1 + R_1 R_0 \rho + D_1^2 (d'' R_0 R_0^2 D_2^2 + N \rho \rho^2) D_1^2 R_1 + D_1^{-1} \rho \rho^2 \rho M_0^2 - R_0 M_0 \rho \rho^2 \rho D_1 R_1$, $dR_2 = - N \omega_0 - R_2 \omega R_2 R_2^2 D_2 R_2 + R_2 R_2 \rho + D_2^{-1} (d'' R_0 R_0^2 D_2^2 + N \rho \rho^2) D_2^{-1} R_2 + D_2^{-1} \rho \rho^2 \rho M_0^2 - R_2 M_0 \rho \rho^2 \rho D_2 R_2$, we get $d (R_2 R_2^T) = - N \omega_0 - R_0 \omega R_0^2 D_1 R_1 + R_1 R_0 \rho + D_1^2 (d'' R_0 R_0^2 D_2^2 + N \rho \rho^2) D_1 R_1 + D_1^{-1} \rho \rho^2 \rho M_0^2 - R_0 M_0 \rho \rho^2 \rho D_1 R_1$.

Thus if we prove the similar relation $d (R_3 R_3^T) = - (R_3 R_3^T) R_0 \rho \rho^2 \rho D_1 (d'' R_0 R_0^2 D_2^2 + N \rho \rho^2) D_1 - R_3 M_0 \rho \rho^2 \rho D_1 R_1$, then since $dR_0 = - N \omega_0 + R_0 \rho \rho^2 \rho D_1 R_1 - R_1 R_0 \rho + D_1^2 (d'' R_0 R_0^2 D_2^2 + N \rho \rho^2) D_1 R_1 - R_0 M_0 \rho \rho^2 \rho D_1 R_1$ we obtain what we want: $dR_3 = - N \omega_0 - R_0 \rho \rho^2 \rho D_1 R_1 - R_1 R_0 \rho + D_1^2 (d'' R_0 R_0^2 D_2^2 + N \rho \rho^2) D_1 R_1 - R_0 M_0 \rho \rho^2 \rho D_1 R_1$.

Thus if we prove the terms $\Omega := d'' R_0 R_0^2$ we need $D_0^2 D_0^{-1} D_1 \Omega D_1 D_2 R_2 R_2^T - R_3 R_3^T \rho \rho^2 \rho D_2^2 + N \rho \rho^2 \rho D_2^2$ which follows directly from (3.3) and the terms containing $\rho \rho^2 \rho D_2^2$ become $R_0 \rho \rho^2 \rho D_2^2 + D_2 R_2 R_2^T - D_2$.

Rekplacing $d'' R_1 R_1^T$, $M_1 \rho \rho^2 \rho$, $N \rho \rho^2 \rho$ from (3.4) and (4.4) the remaining terms split into the two containing $d'' R_0 R_0^2 \rho \rho^2 \rho$ and $\rho \rho^2 \rho D_2^2$ and turn out to be as they should.

For the terms containing $\Omega := d'' R_0 R_0^2$ we need $D_0^2 D_0^{-1} D_1 \Omega D_1 D_2 R_2 R_2^T - R_3 R_3^T \rho \rho^2 \rho D_2^2 + N \rho \rho^2 \rho D_2^2$ which follows directly from (3.3) and the terms containing $\rho \rho^2 \rho D_2^2$ become $R_0 \rho \rho^2 \rho D_2^2 + D_2 R_2 R_2^T - D_2$.

4.2. Quadrics with center.

REFERENCES

[1] Berger, E., Bryant, R. L., Griffiths, P. A. The Gauß equations and rigidity of isometric embeddings, Duke Math. J., 50 (1983), 803-862.
[2] L. Bianchi Sur la déformation des quadriques, Comptes rendus de l’Académie, 142, (1906), 562-564 and 143, (1906) 633-635.
[3] L. Bianchi Lezioni Di Geometria Differenziale, Teoria delle Transformazioni delle Superfici applicabili sulle quadrici, Vol 3, Enrico Spoerri Librio-Editore, Pisa (1909).
[4] L. Bianchi Lezioni Di Geometria Differenziale, Vol 1-4, Nicola Zanichelli Editore, Bologna (1922-27).
[5] L. Bianchi Opere, Vol 1-11, a cura dell’Unione Matematica Italiana e col contributo del Consiglio Nazionale Delle Richerche, Edizioni Cremonese (1952-59).
[6] A. I. Bobenko, U. Pinkall Discrete surfaces with constant negative Gaussian curvature and the Hirota equation, J. Diff. Geom., 43 (1996), no. 3, 527-611.
[7] E. Cartan Sur les variétés de courbure constante d’un espace euclidien ou non-euclidien, Bull. Soc. Math. France, 47 (1919), 125-160 and 48 (1920), 132-208.
[8] G. Darboux Leçons Sur La Théorie Génerale Des Surfaces, Vol 1-4, Gauthier-Villars, Paris (1894-1917).
[9] G. Darboux Leçons Sur Les Systèmes Orthogonaux Et Les Coordonnées Curvilignes, Gauthier-Villars, Paris (1910).
[10] G. M. Green Projective Differential Geometry of Triple Systems of Surfaces, Press of The New Era Printing Company, Lancaster, PA (1913).
[11] K. -M. Peterson Sur la déformation des surfaces du second ordre, Annales de la faculté des sciences de Toulouse 2nd série, tome 7, 1 (1905), 69-107.
[12] K. Tenenblat and C.-L. Terng Backlund’s theorem for n-dimensional sub-manifolds of R^{2n-1}, Ann. of Math., 111 (1980), 477-490.
[13] C.-L. Terng A higher dimension generalization of the Sine-Gordon equation and its soliton theory, Ann. of Math., 111 (1980), 491-510.
[14] C.-L. Terng, K. Uhlenbeck Poisson Actions and Scattering Theory for Integrable Systems, arXiv:dg-ga/9707004 v1 7 Jul 1997

Faculty of Mathematics and Informatics, University of Bucharest, 14 Academiei Str., 010014, Bucharest, Romania
E-mail address: dinca@gta.math.unibuc.ro