Abstract

I give an introduction to the QCD-based theory of the heavy flavor hadrons and their weak decays. Trying to remain at the next-to-elementary level and skip technicalities, I concentrate on the qualitative description of the most important applications and physical meaning of the theoretical statements. The numerical results of the dedicated theoretical analyses of extracting $|V_{cb}|$ are given and the possibilities to determine $|V_{ub}|$ in future are discussed. At the same time I describe in simple language subtle peculiarities distinguishing actual QCD of heavy quarks from naive quantum mechanical treatment often applied to heavy flavor hadrons. These subtleties are often mistreated. Particular attention is paid to the concept of the heavy quark mass and its evaluation, to the kinetic operator and the question of the $1/m_Q$ corrections to inclusive widths of heavy flavor hadrons. I argue that the properly defined $b$ quark mass is known with a good accuracy from the $\bar{b}b$ threshold cross section.
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1 Introduction

The Standard Model (SM) has six quarks whose existence by now is reliably established in experiment. They are naturally allocated to three generations

\[
\begin{pmatrix}
  u \\
d
\end{pmatrix}
\begin{pmatrix}
  c \\
s
\end{pmatrix}
\begin{pmatrix}
  t \\
b
\end{pmatrix}
\]

Masses of \( u \) and \( d \) quarks are only a few MeV, much smaller than \( \Lambda_{\text{QCD}} \), and in most applications of hadron physics \( u \) and \( d \) can be considered as massless. The strange quark mass is about 150 MeV, that is, literally only a little smaller than \( \Lambda_{\text{QCD}} \). Nevertheless, there is ample evidence that treating \( s \)-quark as light is justified, and corrections to the (light) \( SU(3) \) symmetry are reasonably small. Thus, \( m_s \) lies essentially below the actual typical hadronic QCD scale \( \mu_{\text{hadr}} \sim 500 \) to 700 MeV.

Those quarks \( Q \) for which \( m_Q \gg \mu_{\text{hadr}} \sim (2-3)\Lambda_{\text{QCD}} \) are heavy quarks. The sixth \( t \) quark is the heaviest, \( m_t \approx 170 \) GeV. However, it is too heavy. Its width due to the ‘semiweak’ decay \( t \to b + W^+ \) is \( \Gamma_t \approx 1 \) GeV. It decays too fast for the \( t \)-hadrons, the bound states with light quarks to be formed.

The best candidate for application of the Heavy Quark Expansion (HQE) are beauty hadrons. It is heavy enough to confidently use the expansion in \( 1/m_b \), yet it appears not too heavy for the leading corrections to the heavy quark limit to be negligible.

The charmed quark \( c \) can be called heavy only with some reservations. While in some cases it gives a reasonable approximation, \( m_c \) often appears manifestly too low for a quantitative treatment of charmed physics in the \( 1/m_Q \) expansion. There is no universal answer here, and considerable care must be exercised in such a treatment in every particular case.

The hadrons we consider are composed from one heavy quark \( Q \), and a light cloud: a light antiquark \( \bar{q} \), or diquark \( qq \), together with the gluon medium and light quark-antiquark pairs. The role of the gluon medium is to keep all ‘valence’ constituents together, in a colorless bound state which will be generically denoted by \( H_Q \). Therefore, we have the following simplified picture of a heavy-flavor hadron. The heavy quark has a small size \( \sim 1/m_Q \), and is surrounded by a static Coulomb-like color field \( A_0 \) at small distances. Non-Abelian self-interaction slightly modifies the potential, but the non-linearity is driven by the coupling \( \alpha_s(r^{-1})/\pi \) and is not significant. At larger distances the self-interaction strengthens, at \( R \gtrsim \Lambda_{\text{QCD}}^{-1} \) it is completely nonperturbative: the soft modes of the light fields are strongly coupled and strongly fluctuate.

What happens when such a hadronic state is disturbed? In weak decays the standard situation is that the external (to QCD) forces like \( W \)-bosons interact with

\footnote{There exists a commonly accepted abbreviation HQET which sounds very similar. It denotes the “Heavy Quark Effective Theory”, a rather specific approach extensively used for a number of exclusive transitions. In the past every theoretical consideration related to heavy quarks was often not quite consistently called HQET. The difference will be illustrated later. The term HQE was introduced, apparently, in [27] to distinguish a more general direct QCD-based approach.}
the heavy quark, say, instantaneously replace the $b$ quark by $c$ quark generally changing its velocity, make it to move. Such an event excites first the typical modes of the heavy quark, hard gluons with $\vec{k}_{\text{typ}} \sim m_Q$. Since

$$\alpha_s(k_{\text{typ}}) \sim \alpha_s(m_Q) \ll 1$$

one can apply perturbation theory to describe what happens there.

An actual decay process, nevertheless, eventually runs into the strong-interaction nonperturbative domain of $\vec{k} \sim \omega \sim \Lambda_{\text{QCD}}$, where $\omega$ denotes the characteristic frequencies. The final hadronization dynamics shaping the hadrons we observe in experiment, is a result of soft nonperturbative physics which is responsible for confinement. It is important that the complicated final state dynamics involve $\omega \ll m_Q$. This means that to deal with nonperturbative effects one can safely use a nonrelativistic expansion for the heavy quark.

The main subject of the HQE is nonperturbative physics. The reason is twofold: first, the perturbative corrections are straightforward. The actual computations are often cumbersome and, beyond the first-order effects, typically require sophisticated state-of-the-art technique. Nevertheless, they are conceptually simple.

The second reason is that, anyway the perturbative corrections are calculated in the full QCD rather than in the effective low-energy theory, since the part (often dominant) comes just from the gluon momenta $k \sim m_Q$ where the nonrelativistic approximation is not applicable. Still, it is worth noting that the interplay of the perturbative and nonperturbative effects is quite nontrivial and involves theoretical subtleties which are not always fully respected.

The treatment of the nonperturbative effects is a nontrivial problem, and different methods of QCD are used here. The basic tool for all of them in heavy quarks is the Wilson operator product expansion (OPE) \cite{1}. Stating briefly,

$$\text{HQE} = \text{OPE} + \text{nonrelativistic expansion}.$$  

Unless an analytic solution of QCD is found, these two ingredients are indispensable for the heavy quark theory. What do they mean and help us with?

### 1.1 Nonrelativistic Expansion

The main simplification of a nonrelativistic treatment is that the number of heavy quarks $n_Q$ and antiquarks $\bar{n}_Q$ are separately conserved. Propagation of the nonrelativistic quark with usual relativistic Green function for the heavy quark contains also the process of the $Q\bar{Q}$ pair creation if the time ordering of the vertices along the line is reversed somewhere. In the nonrelativistic kinematics, however, such configurations yield a power-suppressed contribution, since the virtuality of the intermediate state (the energy denominator, in the language of noncovariant perturbation theory of quantum mechanics) is of the order of $2m_Q$. In order to observe such processes as real ones, it would be necessary to supply energy at least as large as $\omega \approx 2m_Q$. 

3
Heavy quarks can appear also in closed loops, for example, in the processes of the virtual gluon conversion. For heavy quarks such effects are also suppressed, \( \sim \vec{k}^2/m_Q^2 \) if the gluon momentum \( k \) is much smaller than \( m_Q \).

As a result, the field-theoretic, or second-quantized description of the heavy quark becomes redundant, and it is sufficient to resort to its usual quantum-mechanical (QM) treatment. For example, the wavefunction of a heavy flavor hadron takes the form

\[
\Psi_\alpha [\vec{x}_Q, \{x_{\text{light}}\}],
\]

where \( \vec{x}_Q \) is the heavy quark coordinate, \( x_{\text{light}} \) generically represents an infinite number of light degrees of freedom; index \( \alpha \) describes the heavy quark spin.

A relativistic \( S = \frac{1}{2} \) particle has four components, i.e. \( \alpha = 1...4 \). The nonrelativistic spinor \( \Psi(x) \) has only two of them, \( \alpha = 1, 2 \) describing two spin states. The remaining \( \alpha = \{3, 4\} \) components describe antiparticles which decouple in the nonrelativistic theory:

\[
Q(x) = \begin{pmatrix}
\Psi(x) \\
\chi(x)
\end{pmatrix}
\]

\[
\Psi(x) \sim O(1), \quad \chi(x) \sim \frac{\vec{p}}{m_Q} \to 0
\]

The nonrelativistic Hamiltonian of the spin-\( \frac{1}{2} \) particle is the well known Pauli Hamiltonian:

\[
\mathcal{H}_{\text{Pauli}} = -g_s A_0 + \frac{(i\vec{\partial} - g_s \vec{A})^2}{2m} + \frac{g_s \vec{\sigma} \vec{B}}{2m}.
\]

The last operator was written with a coefficient appropriate for an ‘elementary’ pointlike particle. Presence of the interaction generally renormalizes its strength, the chromomagnetic moment of the heavy quark. In the heavy quark limit \( m_Q \to \infty \) only the first term survives while the last two terms describing space propagation and interaction of spin with the chromomagnetic field \( \vec{B} \), disappear. An infinitely heavy quark is static and interacts only with the color Coulomb potential. In turn, the heavy quark is a source of a static color field independent of the heavy quark spin. The actual dynamics of the heavy quark spin reveal itself in full only when the quark interacts with quanta having large momentum \( |\vec{q}| \gtrsim m_Q \).

It is easy to illustrate the \( b \) quark spin-independence of the strong forces in the following way. One can imagine the QCD world where, instead of the actual \( b \) quark there exists a scalar spinless \( \tilde{b} \) quark with the same mass. The usual spin-0 \( B \) meson, \( b\bar{q} \) would become a spin-\( \frac{1}{2} \) particle \( \tilde{B} \sim b\tilde{q} \). The \( \Lambda_b \) baryon \( bud \) having spin-\( \frac{1}{2} \) would, in turn become a scalar spinless \( \tilde{\Lambda}_b \) state. Nevertheless, at \( m_b \to \infty \) the properties of \( B \) and \( \tilde{B} \) or \( \Lambda_b \) and \( \tilde{\Lambda}_b \) would become identical. The two degenerate spin states of \( \tilde{B} \) are counterparts of \( B \) and \( B^* \) of the actual QCD.

\[\text{Speaking of the Coulomb interaction in QCD we mean the interaction with the time-like component } A_0 \text{ of the color gauge potential. It differs from the simple } 1/r \text{ electrodynamic potential. It is also customary to absorb the coupling } g_s \text{ into the definition of the gauge potential, and I often omit it from the expressions.}\]
An immediate question comes to one’s mind at this point: what about the relation between spin and statistics? In such a gedanken operation we replace spin-\(\frac{1}{2}\) hadrons by \(S = 0\) ones, and vice versa, i.e. interchange fermions and bosons. It is clear, however, that for the states or processes with a single heavy quark the statistics symmetry properties do not play a role.

Such an independence of the strong dynamics of the heavy quark spin is called the heavy quark Spin Symmetry.

The rest-mass \(m_Q\) of the heavy quark also enters in a trivial way: the Hamiltonian merely contains \((n_Q + \bar{n}_Q)m_Q\). Since both are fixed, it is just an overall additive constant. For a moving quark this constant is \(E = \sqrt{m_Q^2 + \vec{p}^2} = m_Q\sqrt{1 + \vec{v}^2}\). Therefore, the actual dynamics is not affected by the concrete value of the mass \(m_Q\). This is a Heavy Flavor Symmetry which states, for example, the equal properties of charmed and beauty hadrons to the extent they both can be considered heavy enough.

The heavy flavor symmetry leads also to certain scaling behavior of the transition amplitudes with the heavy flavor hadrons: the amplitudes depend on their velocities rather than on the absolute values of momenta:

\[
A\left(\frac{P_Q^{\text{in}}}{m_Q}, \frac{p_i^{\text{in}}}{m_Q}; \frac{P_Q'}{m_Q'}, \frac{p_i'}{m_Q'}\right) = A\left(\frac{P_Q^{\text{out}}}{m_Q}, \frac{p_i^{\text{out}}}{m_Q}; \frac{P_Q'}{m_Q'}, \frac{p_i'}{m_Q'}\right).
\] (4)

Here \(P\) denote the momenta of the heavy flavor hadrons and \(p\) refer to other participating particles. It is important to keep in mind that such a scaling behavior is valid only with respect to the soft part of the interaction, and no hard gluons with \(\vec{k} \sim m_Q\) are involved.

The one-particle (or QM) description of the heavy quark degrees of freedom is the key simplification of the nonrelativistic expansion. The light cloud, however, still requires a full-fledged field-theoretic treatment. Even considering the static limit \(m_Q \to \infty\) where only interaction with the Coulomb field \(A_0\) remains, one should realize that it is ‘very quantum’ and strongly fluctuate, in contrast to simple potential QM models where the corresponding potential \(V(x)\) is merely a \(c\)-number function of coordinates.

For this reason even the quantum mechanics of heavy quarks is highly nontrivial. Exploiting the symmetry properties of the heavy quark interactions does not require understanding of these complicated strong-interaction dynamics. This was the main field of applications at an early stage of theoretical development of heavy quark physics. The recent progress is mainly related to a better treatment of basic properties of this complicated strongly-interacting system \(\square\) via application of dynamic QCD methods based on the short-distance expansion.

\[3\]

In early papers on heavy quarks the system of soft components of quark and gluon fields constituting the light cloud was often negligently referred to as “brown muck”, which seems unfair. Already in the pre-heavy-quark era most of the impressive theoretical results leading, for example, to formulation of QCD as a theory of strong interactions, were related to ingenious treatment of this soft cloud in the world of light hadrons.
1.2 Operator Product Expansion

The basic theoretical tool of the heavy quark theory is the Wilson operator product expansion [1]. It is not so easy to give a precise definition of what is OPE in a few understandable words. In particular, I found that many authors often associate with it somewhat different ideas, or even limit it to concrete technicalities.

The idea of the OPE was formulated by K. Wilson in the late 60’s, originally in the context of the statistical problems which are closely related to the field theories in Euclidean space. This idea, in general, is a separation of effects originating at large and small distances. The application to real physical processes in Minkowski space is often less transparent and technically more complicated, however it is always based on the same underlying concept.

The original QCD Lagrangian

\[ \mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} + \sum_q \bar{q}(i \not{D} - m_q)q + \sum_Q \bar{Q}(i \not{D} - m_Q)Q = \mathcal{L}_{\text{light}} + \sum_Q \bar{Q}(i \not{D} - m_Q)Q \]  

is formulated at very short distances, or, which is the same, at a very high normalization point \( \mu = M_0 \), where \( M_0 \) is the mass of an ultraviolet regulator. In other words, the normalization point is assumed to be much higher than all mass scales in the theory, in particular, \( \mu \gg m_Q \). An effective theory for describing the low-energy properties is obtained by evolving the Lagrangian from the high scale \( M_0 \) down to a lower normalization point \( \mu \). This means that we integrate out, step by step, all high-frequency modes in the theory thus calculating the Lagrangian \( \mathcal{L}(\mu) \). The latter is a full-fledged Lagrangian with respect to the soft modes with characteristic frequencies less than \( \mu \). The hard (high-frequency) modes determine the coefficient functions in \( \mathcal{L}(\mu) \), while the contribution of the soft modes is hidden in the matrix elements of (an infinite set of) operators appearing in \( \mathcal{L}(\mu) \). The value of this approach, outlined by Wilson long ago [1], has become widely recognized and exploited in countless applications. The peculiarity of the heavy quark theory lies in the fact that the \( \text{in} \) and \( \text{out} \) states contain heavy quarks. Although we integrate out the field fluctuations with the frequencies down to \( \mu \ll m_Q \), the heavy quark fields themselves are not integrated out since we consider the sector with non-zero heavy-flavor charge. The effective Lagrangian \( \mathcal{L}(\mu) \) acts in this sector. Since the heavy quarks are neither produced nor annihilated, any sector with the given \( Q \)-number is treated separately from all others.

The simplest and most familiar example of integrating out high-frequency modes is the low-energy four-fermion weak decay Lagrangian. The weak decays are mediated by virtual exchanges of \( W \) bosons and, in principle, depend on the external momenta via the momentum flowing through the \( W \) boson line:

\[ A = \frac{g_2^2}{8} V_{cb} V_{ub}^* \bar{u}_c(p_c)\gamma_\mu(1-\gamma_5)u_b(p_b) \bar{u}_d(p_d)\gamma_\nu(1-\gamma_5)u_u(p_u) \frac{\delta_{\mu\nu} - (p_b - p_c)_\mu(p_b - p_c)_\nu}{(p_b - p_c)^2 - M_W^2}. \]  

(6)
The nontrivial denominator signifies propagation of a particle, the $W$ boson. Eliminating explicit $W$ boson from the theory we are left with amplitudes generated by the local effective Lagrangian:

$$L_w = -\frac{G_F}{\sqrt{2}} V_{cb} V^*_{ub} \int d^4x \bar{c}(x) \gamma_\mu (1-\gamma_5) b(x) \bar{d}(x) \gamma_\mu (1-\gamma_5) u(x) + \mathcal{O}\left(\frac{p^2}{M_W^2}\right), \quad (7)$$

provided all momenta are much smaller than $M_W$.

This example is particularly simple in a few aspects: the $W$ boson does not couple to gluons, and it has a large energy $M_W$ even if it carries a small momentum. The OPE works perfectly in a more general situation as well: for example, one can integrate out hard gluons with large momenta (virtualities) even though the gluon field is massless, and gluons couple to each other. In spite of these complications, the effect of hard gluons is given by series of local operators generating simple effective vertices, which are more and more suppressed when their dimension increases. To state it differently, soft particles (fluctuations) can "look inside" a hard process only for the price of power suppression.

This basic idea applies to any field theory. It brings in additional advantages for QCD related to the fact that QCD is a renormalizable theory. Another convenience is that the emerging effective operators are gauge-invariant. This reduces their number and leads to many useful relations among generated amplitudes. Combining the OPE with nonrelativistic expansion is particularly constraining. For example, if the heavy quark spin is switched off, there is no independent axial current, etc.

Another feature is that in QCD the short-distance physics is governed by the small running coupling $\alpha_s(q^2)$ and, therefore, the coefficients in front of these operators, affected by the QCD dynamics, can be calculated in perturbation theory.

Heavy quark physics has an intrinsic large scale $m_Q$. Nevertheless, the majority of phenomenologically interesting processes are 'soft', that is, essentially dependent on the low-scale dynamics as well. There are a few exceptions which are the subject of special attention. An important progress in the heavy quark theory was reached identifying those processes that are genuinely 'hard' in this respect.

2 Basics of the Heavy Quark Theory

2.1 Effective Hamiltonian

What are virtual particles which must be integrated out to obtain an effective theory suitable for heavy quarks? In the case of ordinary weak decays the answer was quite obvious, at least in the tree approximation: one had heavy virtual $W$ which could virtually appear only for a short interval of time $\sim 1/M_W$ allowed by the uncertainty principle. For a heavy quark $Q$ its mass $m_Q$ is a large parameter, but the quark itself should not be eliminated completely since it is present in the initial state.

The strategy is described in the text-books. The nonrelativistic fermion field $Q(x)$ has four degrees of freedom. Two of them, $\Psi(x)$ in Eq. (2) are nearly on-shell
and two \( \chi(x) \) are highly virtual describing excitation of antiquarks. One needs to integrate out first the antiparticle fields \( \chi(x) \). For simplicity, we consider it in the rest frame.

Integrating out the antiparticle fields can be done straightforwardly since the QCD Lagrangian is bilinear in the quark fields. In this way one would obtain the (tree-level version of the) so-called Lagrangian of HQET \( \mathcal{L}_{\text{HQET}} \). It is a correct nonrelativistic Lagrangian up to the first order in \( 1/m_Q \).

This does not complete the program, however: one yet has a full ‘particle’ field \( \Psi(x) \) which includes all frequencies from 0 to \( \infty \). Without taking care of this problem we would be in a position of a theorist in the end of the 20’s when the nonrelativistic QM of a charged particle interacting with the electromagnetic field had been written, but its usefulness was severely limited by the presence of ultraviolet (UV) divergences when the corrections due to quantum fluctuations were considered. Such a problem does not show up in the potential models where modes with \( k \gg \Lambda_{\text{QCD}} \) are not excited. However, in actual QCD all radiative corrections would diverge due to large momentum gluons.

Therefore, besides the antiparticle fields one needs to integrate out hard gluons and the high frequency components of the heavy quark field \( \Psi(x) \) itself, those for which \( |\vec{k}|, \omega > \mu \). The scale \( \mu \) is the normalization point of the effective theory. Since the configurations we integrate out depend on \( \mu \), the remaining effective low-energy theory cannot but have to be \( \mu \)-dependent.

In practice, we want to have \( \mu \ll m_b \) and, actually, as low as possible. On the other hand, \( \mu \) must still belong to the perturbative domain. In practice this means that the best choice routinely adopted for applications is \( \mu \sim \text{a few} \times \Lambda_{\text{QCD}}, \) from 0.7 to 1 GeV. All coefficients in the effective Lagrangian obtained in this way are functions of \( \mu \).

The nonrelativistic heavy quark expansion can be performed in somewhat different ways. The best known examples are NRQCD (Nonrelativistic QCD) \[2\] and HQET \[3, 4\] which are mainly used for describing exclusive heavy flavor transitions or static characteristics. At first glance, HQET differs from NRQCD only in the choice of the zero-order approximation for \( m_Q \to \infty \), by discarding even the standard QM kinetic term \( \vec{p}^2/2m_Q \) together with the Pauli term \( \vec{\sigma} \vec{H}/2m_Q \). However, in higher orders in \( 1/m_Q \) the expansion has not been applied quite consistently until recently. The correct way was advocated by Körner et al. \[5\] and followed the text-book methods elaborated in early days of QM. It thus coincided with the approach of NRQCD (the practical applications of the latter usually did not address higher orders in \( 1/m_Q \) where the difference emerges). Unfortunately for HQET, for a few years these justified suggestions were ignored and, with a delay, admitted only recently \[6, 7\].

Another peculiarity of HQET as an effective theory is that, at the technical level, it attempts to “integrate out” perturbative corrections down to \( \mu = 0 \) (except for some special cases). While such a procedure is not justified and, rather, looks illegitimate, it is often possible to perform such an integration in the concrete orders
in perturbation theory. These inconsistencies seemed to be curable technicalities. However, in a number of publications \cite{3, 8, 9, 10, 6, 11} they were canonized and concrete computational techniques having limited applicability were promoted to the status of indispensable elements of HQET itself. As a result, there is no yet a clear understanding of what exactly is HQET as an actual quantum field theory (QFT), and this ambiguity is reflected in the existing definitions of some fundamental HQET parameters. Inaccurate treatment of such theoretical subtleties sometimes led to paradoxical claims. For example, it was stated \cite{9} that HQET contains a source of the nonperturbative $1/m_Q^2$ corrections to inclusive decay widths, contrary to the theorem established in QCD itself.

To summarize, in treating heavy quarks we separate all strong interaction effects into ‘hard’ and ‘soft’ introducing a normalization scale $\mu$. To calculate the effect of the short-distance (perturbative) physics we use the original QCD Lagrangian Eq. (5). The soft physics is treated by the nonrelativistic Lagrangian where the heavy quarks are represented by the corresponding nonrelativistic fields $\phi_Q$:

\begin{equation}
\mathcal{L}_{\text{eff}} = -\frac{1}{4}G_{\mu\nu}^2 + \sum_q \bar{q}(i \not{D} - m_q)q + \sum_Q \left\{ -m_Q \varphi_Q^+ \varphi_Q + \varphi_Q^+ i D_0 \varphi_Q - \frac{1}{2m_Q} \varphi_Q^+ \left( \sigma i \bar{D} \right)^2 \varphi_Q - \frac{1}{8m_Q^2} \varphi_Q^+ \left[ -(\bar{D}\bar{E}) + \bar{\sigma} \cdot \{ \bar{E} \times \bar{\pi} - \bar{\pi} \times \bar{E} \} \right] \varphi_Q \right\} \tag{8}
\end{equation}

where

\begin{equation}
\bar{\pi} \equiv i \bar{D} = \bar{p} - \bar{A}, \quad \left( \sigma i \bar{D} \right)^2 = (\sigma \bar{\pi})^2 = \bar{\pi}^2 + \sigma \bar{B}; \tag{9}
\end{equation}

for simplicity I wrote only the tree-level $\mu$-independent coefficients. By the standard rules one constructs from the Lagrangian (8) the corresponding Hamiltonian. The heavy quark part takes the form

\begin{equation}
\mathcal{H}_Q = -A_0 + \frac{1}{2m_Q}(\bar{\pi}^2 + \sigma \bar{B}) + \frac{1}{8m_Q^2} \left[ -(\bar{D}\bar{E}) + \bar{\sigma} \cdot \{ \bar{E} \times \bar{\pi} - \bar{\pi} \times \bar{E} \} \right] + \mathcal{O}(1/m_Q^3). \tag{10}
\end{equation}

Since the external interactions (electromagnetic, weak \textit{etc.}) are given in terms of the full QCD fields $Q(x)$, one needs also the relation between $Q(x)$ and $\varphi_Q(x)$:

\begin{equation}
\varphi = \left( 1 + \frac{(\sigma \bar{\pi})^2}{8m_Q^2} + \ldots \right) \frac{1 + \gamma_0}{2} Q, \quad \frac{1 - \gamma_0}{2} Q = \frac{\not{\pi}}{2m_Q} Q. \tag{11}
\end{equation}

Although obtaining the nonrelativistic Lagrangian is a standard text-book procedure, let me briefly recall it. One starts with

\begin{equation}
\mathcal{L}_{\text{heavy}}^0 = \bar{Q}(x)(i \not{D} - m_Q)Q(x) \tag{12}
\end{equation}

\footnote{The leading term $A_0$ is often omitted here. It is then implied that the time evolution operator is $\pi_0 = i \frac{\not{D}}{m_Q} + A_0$ rather than $i \frac{\not{D}}{m_Q}$ in the usual Schrödinger equation. This can be consistently carried out through the analysis.}
and factors out of the \( Q(x) \) field the "mechanical" time-dependent factor associated with the rest energy \( m_Q \):

\[
Q(x) = e^{-imQt} \tilde{Q}(x) .
\]

(13)

In an arbitrary frame moving with four-velocity \( v_\mu \) it takes the following form:

\[
Q(x) = e^{-imQv_\mu x_\mu} \tilde{Q}(x) .
\]

(14)

Then

\[
iD_\mu Q(x) = e^{-imQ(vx)} (m_Q v_\mu + \pi_\mu) \tilde{Q}(x) , \quad \pi_\mu = \dot{P}_\mu - m_Q v_\mu .
\]

The Dirac equation \((i \not{D} - m_Q) Q = 0\) takes the form (now the tilde on \( Q \) is omitted)

\[
\frac{1 - \gamma_0}{2} Q = \frac{i}{2m_Q} Q , \quad \pi_0 Q = -\frac{\pi^2 + i\frac{1}{2}\sigma G}{2m_Q} Q .
\]

(15)

\[
\frac{i}{2} \sigma G = \frac{i}{2} g_\mu \sigma_{\mu\nu} G_{\mu\nu} , \quad i G_{\mu\nu} = [\pi_\mu, \pi_\nu] = [\dot{P}_\mu, \dot{P}_\nu]
\]

which allows one to exclude the small low components \( \frac{1 - \gamma_0}{2} Q(x) \) expressing them via the 'large' upper components \( \frac{1 + \gamma_0}{2} Q(x) \).

A subtlety emerges on this route that must be treated properly: at order \( \frac{1}{m_Q^2} \) the time derivative \( \partial_0 \) appears with a nontrivial coefficient depending, for example, on the gluon field. This can be eliminated, and the time derivative returned to its canonical form performing the Foldy-Wouthuysen transformation

\[
\varphi(x) = \left( 1 + \frac{(\not{\sigma} \not{\pi})^2}{8m_Q^2} + ... \right) \frac{1 + \gamma_0}{2} Q(x) .
\]

(16)

Why does one need this Foldy-Wouthuysen transformation? It is worth discussing it since HQET uses the different fields

\[
h(x) = \frac{1 + \gamma_0}{2} \tilde{Q}(x)
\]

(17)

to all orders in \( \frac{1}{m_Q} \) to represent the effective low-energy heavy quark field. As a result, for example, the higher-order terms in the HQET effective Lagrangian are different from Eq. (8) starting already \( \frac{1}{m_Q^2} \) (in this order they consist of the so-called Darwin and convection current, or \( LS \) interaction). Moreover, \( h(x) \) is not mass-independent but contains explicit \( \frac{1}{m_Q^2} , \frac{1}{m_Q^3} \) ... terms depending on the external fields.

In principle, it is not a mistake \textit{per se}. If one were able to calculate all HQET Green functions computing the functional integral with such a HQET Lagrangian exactly, it wouldn’t have mattered: any choice of dynamic variables is legitimate as soon as the proper reduction technique is applied to obtain the \( S \)-matrix elements from the Green functions. In reality, such a possibility is as Utopian for HQET as for
full QCD. Therefore, in practice one can only rely on various expansion techniques, and here using the correct nonrelativistic fields incorporating the Foldy-Wouthuysen transformation and the Lagrangian becomes mandatory, as was realized 70 years ago. Let me briefly illustrate it.

The standard approach in the heavy quark expansion uses the well-elaborated methods of QM operating with the QM states, in particular, in constructing the perturbation theory in $1/m_Q$. In QM the terms in the expansion are expressed \textit{via} various $T$-products, and this holds in the standard form only if the canonical coordinates are used. Otherwise, somewhat different objects replace usual $T$-products. Moreover, the effective Hamiltonian describing the evolution of the $h$ fields becomes, in a sense, non-Hermitian in higher orders in $1/m_Q$.

On the other hand, one wants to employ the QFT methods which, in particular, are used in the OPE. They rely on the equations of motion for quantum field operators. If a system is described by the Lagrangian $\mathcal{L}$, one has the classical equations of motion

$$\partial_{\mu} \frac{\delta \mathcal{L}}{\delta \partial_{\mu} \varphi(x)} - \frac{\delta \mathcal{L}}{\delta \varphi(x)} = 0.$$ \hspace{1cm} (18)

However, we always use the very same equations of motion for the quantum Heisenberg operators $\varphi(x)$. The validity of these operator equations is a nontrivial fact (recall that for any operator $A$ the time derivative by definition is $i[H, A]$), and is a consequence of an analogue (generalization) of the Erenfest theorem in QM. This is true, however, \textit{only} if the term with the time derivative has a canonical (coordinate-independent in QM, of field-independent in the field theory) form. For the HQET quantum fields the naive equations of motion do not hold. This general fact was “empirically” observed in [12] (see also [13]) in attempts to reproduce in HQET the $1/m_Q^2$ corrections to the inclusive decay widths calculated earlier directly in QCD. They required modifications of the existed strategy of obtaining higher-order $1/m_Q$ corrections. This was interpreted as a ‘subtlety in applying equations of motion’ [12, 13]. Surprisingly, the well known origin of this subtlety has not been realized. Not all related flaws have been eliminated, however. The expansion of the exclusive heavy flavor transition at order $1/m_Q^2$ [14] suffering from the same inconsistencies has never been revisited even though it is used nowadays by Neubert to claim the most accurate evaluation of the zero recoil $B \to D^*$ formfactor. This will be discussed in more detail in Sect. 3.5.

Finally, when no rigorous QCD evaluation of the nonperturbative effects can be made, one often resorts to simplified QM constituent quark models (\textit{e.g.}, ISGW [15]) where all wavefunction overlaps, expectation values and other characteristics can be computed. In applying these computations to the QCD system one certainly must use the proper relation between the QCD fields and the variables employed in the

\hspace{1cm} \footnote{In fact, the classical equations of motion are often modified due to anomalies. This is a different aspect related to the UV divergences in the theory. What is discussed here has nothing to do with divergences and applies even to the systems with the finite number of degrees of freedom.}
model. As was first noted, apparently, by A. Le Yaouanc \[16\], this was not always done properly in HQET \[14\].

Therefore, it is certain that the Foldy-Wouthuysen transformation is not an obsolete, mole-beaten technique used only before the advantages of the modern formulations of the quantum fields theories were known or appreciated in full. Using the standard nonrelativistic expansion also often gives a simple derivation, or at least transparent interpretation of many results. Missing it may easily lead to direct mistakes.

2.2 Applications to Spectroscopy of Heavy Flavor Hadrons

To illustrate the consequences of the heavy quark Hamiltonian, let us consider the masses of hadrons containing a single heavy quark. This is not a dynamic question and does not require more than just symmetry properties of $\mathcal{H}_Q$. It is described in much detail in a number of old reviews \[17\], and I can be brief.

The mass of a hadron $H_Q$ is given by the expectation value of the Hamiltonian:

$$M_{H_Q} = \frac{1}{2M_{H_Q}} \langle H_Q | \mathcal{H}_{\text{tot}} | H_Q \rangle, \quad (19)$$

and we have

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{light}} + \mathcal{H}_Q + m_Q, \quad (20)$$

where we can expand

$$\mathcal{H}_Q = \mathcal{H}_0 + \frac{1}{m_Q} \mathcal{H}_1 + \frac{1}{m_Q^2} \mathcal{H}_2 + \ldots \quad (21)$$

with

$$\mathcal{H}_0 = -\varphi_R^\dagger A_0 \varphi_Q \quad \overset{\text{QM}}{\Rightarrow} \quad -A_0(0) \{x_{\text{light}}\},$$

$$\frac{1}{m_Q} \mathcal{H}_1 = \frac{1}{2m_Q} (\bar{\pi}^2 + \bar{\sigma} \bar{B}),$$

$$\frac{1}{m_Q^2} \mathcal{H}_2 = \frac{1}{8m_Q^2} \left[ -(\bar{D} \bar{E}) + \bar{\sigma} \cdot \{\bar{E} \times \bar{\pi} - \bar{\pi} \times \bar{E}\} \right], \quad (22)$$

*etc.* (the first terms in $\mathcal{H}_2$ is called Darwin and the second is the convection current, or $LS$ term). Therefore,

$$M_{H_Q} = m_Q + \bar{\Lambda} + \frac{1}{2m_Q} \frac{\langle H_Q | \bar{\pi}^2 + \bar{\sigma} \bar{B} | H_Q \rangle}{2M_{H_Q}} + \ldots = m_Q + \bar{\Lambda} + \frac{(\mu_\pi^2 - \mu_G^2)_{\mathcal{H}_Q}}{2m_Q} + \ldots \quad (23)$$

Here we introduced the notations $\mu_\pi^2, \mu_G^2$ for the expectation values of two $D = 5$ heavy quark operators which will often appear in our discussion:

$$\mu_\pi^2 = \langle H_Q | \bar{\pi}^2 | H_Q \rangle_{\text{QM}} \equiv \frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q} \bar{\pi}^2 Q(0) | H_Q \rangle_{\text{QFT}}$$

12
\[ \mu_G^2 = \langle H_Q | \vec{\sigma} \vec{B} | H_Q \rangle_{\text{QM}} \equiv \frac{1}{2M_{H_Q}} \langle H_Q | -Q \frac{i}{2} \sigma_{\mu\nu} G^{\mu\nu} Q(0) | H_Q \rangle_{\text{QFT}} . \]  

(24)

The physical meaning of \( \mu_G^2 \) is quite evident: the heavy quark inside \( H_Q \) experiences a zitterbewegung due to its coupling to light cloud. Its average spatial momentum squared is \( \mu_G^2 \). The second expectation value measures the amount of the chromomagnetic field produced by the light cloud at the position of the heavy quark. In principle, the actual heavy hadron states \( H_Q \) depend on \( m_Q \) via the \( 1/m_Q \)-suppressed terms of the Hamiltonian. Therefore, the above expectation values also have such terms. Often it is convenient to consider the asymptotic values at \( m_Q \to \infty \), and to use, correspondingly, the eigenstates of the \( m_Q \to \infty \) Hamiltonian \( \mathcal{H}_0 + \mathcal{H}_{\text{light}} \).

The value of \( \overline{\Lambda} = \lim_{m_Q \to \infty} \left( M_{H_Q} - m_Q \right) \) has the scale of \( \Lambda_{\text{QCD}} \) and depends on the state of light degrees of freedom. These states generally carry spin \( j \). In the limit \( m_Q \to \infty \) the heavy quark spin decouples since the spin-dependent parts are present only starting \( \mathcal{H}_1 \). Thus, the heavy flavor hadrons can be classified not only by their total spin \( J \) but by the spin of light degrees of freedom \( j \). It would be just the overall spin of a hadron in the hypothetical world with the spinless heavy quarks discussed in Sect. 1.1. Unless \( j = 0 \), there are two values of the total spin \( J = j \pm \frac{1}{2} \). The corresponding states form a ‘hyperfine’ multiplets and are degenerate up to \( 1/m_Q \) corrections. They are, for example

\[
\begin{align*}
    j = 1/2 & \quad \begin{cases} 
        D, & J = 0 \\
        D^*, & J = 1 \\
        B, & j = 0, J = 1/2 \\
        B^*, & j = 0, J = 1/2
    \end{cases} \\
    \Lambda_c, \Lambda_b
\end{align*}
\]

For \( \Lambda_Q \)-baryons all spin is carried by the heavy quark (up to \( 1/m_Q \) corrections). The observed spectroscopy of these states clearly supports this picture:

\[
\begin{align*}
    M_{\Lambda_b} - M_B & \simeq 350 \text{ MeV} & M_{B_s} - M_{B^-} & \simeq 90 \text{ MeV} \\
    M_{\Lambda_c} - M_D & \simeq 420 \text{ MeV} & M_{D_s} - M_{D^0} & \simeq 104 \text{ MeV}
\end{align*}
\]

(25)

These relations are easily improved including \( 1/m_Q \) terms, Eq. (23). The operator \( \bar{Q} \pi^2 Q \) is spin-independent and its expectation value is the same for all members of a hyperfine multiplet. It does not split masses inside the multiplet.

The chromomagnetic operator \( Q \frac{i}{2} \sigma GQ = -2 S_Q \vec{B}(0) \) depends on the heavy quark spin \( S_Q \) and thus lifts degeneracy leading to the hyperfine splitting among the members of a multiplet. Restricted to a particular hyperfine multiplet, the chromomagnetic field \( \vec{B}(0) \) as an operator is proportional to the operator of spin of light degrees of freedom:

\[ \vec{B} = c \cdot \vec{j} . \]

Therefore,

\[ \langle \vec{\sigma} \vec{B} \rangle = 2c \langle S_Q \vec{j} \rangle = c \left( J(J + 1) - j(j + 1) - \frac{3}{4} \right) . \]  

(26)
As a matter of fact, it is easy to see that

$$\sum_{H_Q} \langle H_Q | \bar{Q} \sigma \bar{B} Q | H_Q \rangle \equiv \text{Tr} \bar{Q} \sigma \bar{B} Q = 0$$

(27)

always holds if the summation is performed over a hyperfine multiplet. Therefore, for example,

$$\mu_G^2(B) + 3 \mu_G^2(B^*) = 0 .$$

(28)

In the $\Lambda_b$ family the expectation value of $\bar{b} i \sigma G b$ vanishes.

So far most of the practical applications refer to $B$ mesons; as a result, usually $\mu_\pi^2$ proper denotes the expectation value of the kinetic operator just in $B$ or $B^*$. Likewise, $\mu_G^2 \equiv \mu_G^2(B) = -3 \mu_G^2(B^*)$. Experimentally,

$$M_{B^*} - M_B \simeq \left( \frac{1}{3} + 1 \right) \frac{\mu_G^2}{2 m_b} = 0.4 \frac{\mu_G^2}{2 m_b} \simeq 46 \text{ MeV} .$$

(29)

Neglecting the difference between $M_B + M_{B^*}$ and $2 m_b$ (which is formally an effect of higher order in $1/m_b$), one can write

$$\mu_G^2 \simeq \frac{4}{3} \left( M_{B^*}^2 - M_B^2 \right) \simeq 0.36 \text{ GeV}^2 .$$

(30)

In charmed mesons $M_{D^*} - M_D \simeq 140 \text{ MeV}$ which agrees with the fact that this hyperfine splitting is proportional to $1/m_c$. The hadron mass averaged over a hyperfine multiplet, e.g. $\overline{M} = \frac{3 M_B + M_{B^*}}{4}$ is affected at order $1/m_Q$ by only the kinetic energy term $\mu_\pi^2/2 m_b$.

In many applications one needs to know the difference between $m_b$ and $m_c$. This difference is well constrained in the heavy quark expansion. For example,

$$m_b - m_c = \overline{M}_B - \overline{M}_D + \mu_\pi^2 \left( \frac{1}{2 m_c} - \frac{1}{2 m_b} \right) + \frac{\rho_D^3 - \bar{\rho}^3}{4} \left( \frac{1}{m_c^2} - \frac{1}{m_b^2} \right) + \mathcal{O} \left( \frac{1}{m_b^2} \right) .$$

(31)

Here $\mu_\pi^2$ is the asymptotic expectation value of the kinetic operator, $\rho_D^3$ is the expectation value of the Darwin term given by the local four-fermion operator and $\bar{\rho}^3 \equiv \rho_\pi^3 + \rho_S^3$ is the sum of two positive non-local correlators [18]. As will be discussed in the subsequent lectures, all quantities in Eq. (31) but the meson masses depend on the normalization point which can be arbitrary, except that it must be much lower than $m_{c,b}$. In this way we arrive at

$$m_b - m_c \simeq 3.50 \text{ GeV} + 40 \text{ MeV} \frac{\mu_\pi^2 - 0.5 \text{ GeV}^2}{0.1 \text{ GeV}^2} + \Delta M_2 , \quad |\Delta M_2| \lesssim 0.015 \text{ GeV} .$$

(32)

We will discuss the values of the hadronic parameters later. Now, let me mention that the $m_b - m_c$ estimate at $\mu_\pi^2 = 0.5 \text{ GeV}^2$ appears to be in a good agreement with the separate determinations of $m_b$ and $m_c$ from the sum rules in charmonia and $\Upsilon$.
The main uncertainty in $m_b - m_c$ is due to that in the value of $\mu_\pi^2$. The Darwin term can be reasonably estimated relying on factorization [19]; it is of order 0.1 GeV$^3$. The non-local positive matrix elements $\rho_3^{\pi\pi}$ and $\rho_3^S$ are expected, generally speaking, to be of the same order. Altogether, assuming $|\rho_3^D - \bar{\rho}_3^D| \lesssim 0.1$ GeV$^3$, one arrives at the uncertainty in $m_b - m_c$ due to the higher-order terms $\Delta M_2$ quoted above.

2.3 Heavy Quark Symmetry for Formfactors

The amplitudes of the semileptonic weak $b \to c$ decays are described by the corresponding transition formfactors. Typical semileptonic $b \to c$ decays as they look like in Feynman diagrams are shown in Figs. 1. The hadronic part of the weak decay Hamiltonian mediating such decays is

$$\mathcal{H}_{\text{weak}} = \int d^3x \, e^{i\vec{q}\cdot\vec{x}} \bar{c}\gamma_\mu(1 - \gamma_5)b(x).$$

(33)

It says that $b$ with a momentum $\vec{p}$ is instantaneously replaced by the $c$ quark with the momentum $\vec{p} - \vec{q}$. The resulting state hadronizes into eigenstates of the Hamiltonian corresponding to $m_Q = m_c$, that is, must be projected onto such states.

The space-time picture of the decay is simple for heavy quarks. At $t < 0$ the initial $b$ hadron is at rest and constitutes a coherent state of light degrees of freedom in the static field of the heavy quark. At $t = 0$ the $b$ quark emits the lepton pair with the momentum $\vec{q}$ and transforms into a $c$ quark. The $c$ quark gets the recoil momentum $-\vec{q}$ and starts moving with the velocity $\vec{v} = -\vec{q}/m_c$. Such a state is not anymore an eigenstate of the Hamiltonian, and afterwards undergoes nontrivial evolution. The light cloud can get a coherent boost along the direction of $-\vec{q}$ and form again the same ground-state, or excited hadron. Alternatively, it can crack apart and produce a few-body final hadronic state.

The heavy quark symmetry per se cannot help calculating the amplitudes to create such final states. However, it tells one that the hadronization process does not depend on the heavy quark spin, or on the concrete value of the mass $m_Q$ but rather on the velocity of the final state heavy hadron. This velocity cannot change in the process of hadronization if only soft gluons are exchanged between the heavy quark and the light cloud. This independence holds, of course, only when $m_Q$ is very large (and the final state quark does not move too fast); there are various $1/m_Q$ corrections at finite masses.
Let us consider, for example, the ground-state transition \( B \to D \). Its amplitude depends on the velocity \( \vec{v} \) of \( D \), \( f(\vec{v}^2) \). The very same function would describe also decays \( B \to D^* \), or the elastic amplitude of scattering of a photon on the \( b \) quark \( B \to B, B \to B^* \). Moreover, in the proper normalization \( f(0) = 1 \) holds. This fact follows from the conservation of the \( b \)-quark vector current, for the amplitude at zero momentum transfer measures the total ‘\( b \)-quark charge’ of the hadron \( n_b - n_{\bar{b}} \). Its origin is simply understood: if \( \vec{v} = 0 \), the final state is not really disturbed by movement of the static source. A nontrivial rearrangement of the light cloud for \( B \to D \) is associated only with the mass-dependent terms in \( \mathcal{H}_Q \) vanishing when \( m_Q \to \infty \).

As far as I know, this physical picture and the fact of normalization of the \( B \to D \) and \( B \to D^* \) transition amplitudes near zero recoil \( q = 0 \) were first discussed in QCD by M. Shifman and M. Voloshin in 1986 [20].

Let us consider the vector \( \vec{b}\gamma/\mu b \) current in \( B \) meson. It is described by the single formfactor \( f_+(q^2) \):

\[
\langle B(p')|\vec{b}\gamma/\mu b(0)|B(p)\rangle = f_+(q^2) (p + p')_\mu , \quad q_\mu = p'_\mu - p_\mu \quad (34)
\]

(\( f_{+0} \)) (the second structure \( (p + p')_\mu \) is forbidden by \( T \) invariance or current conservation). The value \( f_{+0} \) measures the total ‘beauty charge’ of the hadron and is not renormalized by the strong interaction, \( f_{+0} = 1 \) [21].

Passing to the velocities, we use instead of \( q^2 \) the scalar product \( \nu_\mu \nu'_\mu \):

\[
(vv') = (pp')/M_B^2 = 1 - (q^2)/2M_B^2 \geq 1 , \quad (35)
\]

and

\[
f_+(q^2) = \xi(vv') , \quad \xi(1) = 1 . \quad (36)
\]

\( \xi(vv') \) is the Isgur-Wise function. The heavy quark symmetry then states that

\[
\langle D(v')|\vec{c}\gamma/\mu b(0)|B(v)\rangle = \left( \frac{M_B + M_D}{2\sqrt{M_B M_D}} (p + p')_\mu - \frac{M_B - M_D}{2\sqrt{M_B M_D}} (p - p')_\mu \right) \xi(vv') . \quad (37)
\]

\( D^* \) differs from \( D \) only by the alignment of the \( c \) quark spin. Taking this into account yields

\[
\langle D^*(v', \epsilon)|\vec{c}\gamma/\mu b(0)|B(v)\rangle = i\epsilon_{\mu\alpha\beta}\epsilon^{\ast\epsilon}_{\nu\alpha'}v_\beta \xi(vv') \sqrt{M_B M_D^*} ,
\]

\[
\langle D^*(v', \epsilon)|\vec{c}\gamma/\mu b(0)|B(v)\rangle = \left\{ \epsilon^{\ast\epsilon}_{\mu}(vv' + 1) - v'_{\mu}(\epsilon v) \right\} \xi(vv') \sqrt{M_B M_D^*} . \quad (38)
\]

It is important that these relations are valid in the limit \( m_{b,c} \to \infty \) and if no short-distance radiative corrections were present. The corrections to the symmetry limit are minimal at \( q = 0 \) (\( v = v' \), the so-called zero recoil point); numerically the value of the axial formfactor \( F_{B \to D^*}(0) \approx 0.9 \) [23]. The correction at arbitrary \( \vec{v} \sim 1 \) are generally quite significant.

\footnote{For simplicity, I use the convention that \( B \) consists of \( b \) and \( \bar{q} \). That is, \( B^- \) is a \( B \) meson while \( B^+ \) is \( B \).}
2.4 Feynman Rules at $m_Q \to \infty$

The Feynman rules for heavy quarks are usual propagators and vertices for nonrelativistic particles where $1/m \to 0$. If the heavy quark momentum is $p_\mu = (m_Q + \omega, \vec{p})$ then

$$G(p) = \lim_{m \to \infty} \delta_{\alpha\beta} \frac{1}{\omega - i\epsilon - \frac{\vec{p}^2}{2m}} \delta_{ij} = \frac{\delta_{\alpha\beta}}{\omega - i\epsilon} \delta_{ij},$$

$$\Gamma_\mu = g_s \frac{\lambda^a}{2} \delta_{\mu0}. \quad (39)$$

Here $a, \alpha, \beta$ are color indeces and spinor indeces $i, j$ take values 1 or 2. (It is often advantageous to keep the nonrelativistic term $\vec{p}^2/2m_Q$ in the propagator.) These rules follow immediately from the static Lagrangian

$$L_Q = \varphi_Q^\dagger D_0 \varphi_Q, \quad D_0 = \partial_0 - ig_s A_0^a \frac{\lambda^a}{2}. \quad (40)$$

The same nonrelativistic system can be considered in the arbitrary moving frame, where one can write

$$L_v = \varphi_v^\dagger (vD) \varphi_v. \quad (41)$$

Instead of the nonrelativistic spinor $\varphi_Q$ one then considers the “bispinor” $\varphi_v(x) = \frac{1+v}{2} Q(x)$; the propagator is written as

$$\frac{1 + \gamma^\dagger}{2} \frac{m_Q + \gamma^\dagger - \frac{k}{2}}{m_Q - (p - k)^2 - i\epsilon} \frac{1 + \gamma^\dagger}{2} \frac{1}{vk - i\epsilon} \rightarrow \frac{1 + \gamma^\dagger}{2} \frac{1}{vk - i\epsilon} \quad (41)$$

$$\Gamma_\mu = g_s \frac{\lambda^a}{2} v_\mu, \quad (42)$$

where $k = p - m_Q v$. Of course, such a generalization can be useful only when the initial and final state hadrons have different velocities, $\vec{v} \neq \vec{v}'$. In that case the external (‘weak’) source carries a large momentum $\vec{q} \sim m_Q \delta \vec{v}$.

It must be noted, however, that HQET is not quite consistently formulated as a quantum field theory for processes where $\vec{v} \neq \vec{v}'$ when the quantum radiative corrections are really incorporated. Any change in the velocity of a static source leads to actual radiation of real hadrons with momenta $\vec{k}$ all the way up to $m_Q$:

$$\frac{d\omega}{d\omega} \sim \frac{\alpha_s(\omega)}{\omega} (\vec{v}' - \vec{v})^2; \quad (42)$$

here $\omega$ denotes the radiated energy. The non-Abelian QCD dipole radiation was considered in [22]. On the other hand, effective theories are called upon to eliminate (‘integrate out’) all high-momentum subprocesses. In order to do this one would have to ‘integrate out’ not only highly virtual, but also real processes, which clearly is not a completely legitimate procedure.

This is a real difficulty and not a pure mathematical subtlety. HQET as an effective static theory was originally introduced [3] to study the nonperturbative
effects in the heavy flavor hadrons on the lattice. Suppose we would try indeed to extract physical results for the amplitudes with a nonzero velocity transfer on a fine enough lattice using the HQET Lagrangian (40). Since this Lagrangian describes infinitely heavy quarks, we would have to get zero for any exclusive amplitude: all they contain a more or less universal suppression factor

\[ S \sim e^{-\frac{4\alpha_s}{9\pi}(\Delta \vec{v})^2\ln \frac{m_Q}{\mu}} \]  

and thus must vanish at \( m_Q \to \infty \) (\( \mu \) is a hadronic mass scale serving as an infrared cutoff). This factor \( S \) is the square of the well known Sudakov formfactor (more exactly, of its nonrelativistic analogue).

Any actual simulation would yield, of course, a final result since lattices contain an intrinsic UV cutoff, the inverse lattice spacing \( 1/a \). The result, therefore, would essentially depend on this cutoff for any observable process we would try to compute. It is not too surprising from a general point of view. HQET considers not only the sectors with different numbers of heavy quarks as completely separate. Even the sectors with a single heavy quark but with different velocities are different quantum systems for HQET, with formally different dynamic degrees of freedom, Hilbert space of states, etc. On the other hand, the transition amplitudes at \( \vec{v} \neq \vec{v}' \) require, for example, calculating the ‘inner’ product of states in these theories. Such a procedure is not quite legitimate or well-defined. Hence, it does not come as a surprise that the results sometimes look unphysical or prove to be fallacious. A closer look at some formal derivations [23, 24] reveals that the subtleties at \( \vec{v} \neq \vec{v}' \) were indeed neglected there.

This means that HQET per se is not a well-formulated theory for different velocities beyond the first order in \( \vec{v} \) when the gluon bremsstrahlung appears. The introduction of a “hard” factorization scale \( \mu \) necessary for any effective theory, becomes here a nontrivial and not always harmless procedure. A thoughtful reader notices at this point that the problem roots to the essentially Minkowskian nature of the processes with actual heavy quarks. On the other hand, in QCD the Sudakov-type bremsstrahlung is to a large extent universal, the fact analogous to the fundamental factorization of hard and soft processes in usual OPE formulated in Euclidean domain. Therefore, an alternative way of consistently formulating the OPE expansion for these problems seems in principle possible, even when nonperturbative effects are addressed.

3 Heavy Quark Expansion in Dynamics

The heavy quark expansion resulted in many important insights into physics of heavy flavor hadrons. Already the simple qualitative considerations or symmetry relations noted in the 80’s influenced both the theoretical understanding and the experimental strategy in studying \( B \) particles. Nevertheless, a true measure of our
control over QCD dynamics is, eventually, how accurately we can extract fundamental parameters like $|V_{cb}|$, $|V_{ub}|$, $m_b$, $m_c$ ... from experimental properties of hadrons, or predict necessary amplitudes. At this point we need to address more dynamic aspects of QCD in heavy flavors.

A major progress over the last years was made in this direction. It is impossible to review, even briefly, all relevant topics here. Therefore, I am forced to limit myself to a brief discussion of the existing situation with, probably, the most intriguing subjects – extracting $|V_{cb}|$, $|V_{ub}|$ and $m_b$. An approximate – though already nontrivial – determination of some of these parameters was made already a decade ago. The accuracy and a confidence in our estimates now are at a new level. This required a better understanding of various subtleties of the heavy quark theory as a quantum field theory derived from QCD, which are absent in simplified hadronic models. Moreover, explaining itself the origin of controversy in the literature often requires going into such ‘technicalities’. As a result, a consistent review of the situation requires at this point a more rigorous presentation of the OPE for heavy quarks, discussing the precise definitions and the status of determination of the relevant hadronic parameters, existing uncertainties, etc. – and only then application to heavy flavor phenomenology.

In these lectures I am trying to avoid technicalities in favor of outlining a more transparent physical picture. To get a feeling of what we need to know and how accurately, let me start directly with the question of the precise determination of $|V_{cb}|$ from the total semileptonic $B$ width, the most reliable up to date method both theoretically and experimentally. This will naturally lead me to the problem of the heavy quark masses, kinetic energy etc. in the field theory approach. These questions will constitute a significant part of my lectures. Then we will be able to return to discussing practical extraction of $V_{cb}$. Such a strategy will lead to a certain fragmentarity of the presentation, and I apologize for related inconvenience.

### 3.1 Inclusive Widths

Inclusive widths of the heavy flavor hadrons are examples of the genuine short-distance processes. The decays proceed at the space-time intervals $\sim 1/m_b$ (more precisely, inverse energy release), and the widths are affected by the soft strong dynamics to the minimal extent. The utility of the OPE is demonstrated by the QCD theorem [25, 26]: there are no nonperturbative corrections to the inclusive widths of heavy flavor hadrons. It applies to all types of decays: semileptonic, nonleptonic, radiatively-induced ($b \rightarrow s + \gamma$) etc. – they only must be truly inclusive.

In practical application of this analysis there are some limitations, the so-called onset of quark-hadron duality must be passed. This question will be briefly addressed later. The best situation in this respect exists for semileptonic decays, and we will discuss them now in detail. It should be noted that the methods of HQET are not applicable to the inclusive decays. These processes have an intrinsic large ‘dynamic’ scale of energy release $\sim m_b$, and radically differ in this respect from
the exclusive heavy-flavor formfactors. The OPE analysis is done directly in QCD utilizing this large kinematic parameter. Only at the final stage when the nonperturbative effects are expressed via the expectation values of the local heavy quark operators, one employs the nonrelativistic expansion for the $b$ fields $[25]$.

It was realized long ago that for heavy enough quarks the inclusive widths must look similar to the analogous weak decay widths of an almost non-interacting muon:

$$\Gamma(\mu \to e\nu\bar{\nu}) = \frac{G_F^2 m_\mu^5}{192\pi^3} \left( z_0 \left( \frac{m_e^2}{m_\mu^2} \right) - \frac{\alpha_s}{2\pi} a_1 \left( \frac{m_e^2}{m_\mu^2} \right) + \ldots \right), \quad (44)$$

where $z_0(x)$ is the phase space factor; the term with $a_1(x)$ accountings for the electromagnetic interaction was calculated already in 1956 $[27]$. Similarly,

$$\Gamma_{sl}(B) = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 z_0 \left( \frac{m_c^2}{m_b^2} \right) \cdot \kappa \quad (45)$$

(likewise for the $b \to u$ decays), where all QCD effects, both perturbative and nonperturbative, are lumped together in the factor $\kappa$. It was not clear, though, what mass must be used, the quark mass $m_b$ or the mass of the decaying hadron $M_B$. The numerical difference is yet quite significant even for $B$ meson: $M_B^5/m_b^5 \approx 1.5$ to 2, in spite of being formally a $1/m_b$ effect:

$$\frac{M_B - m_b}{m_b} \sim \frac{\Lambda_{QCD}}{m_b}, \quad \frac{M_{\Lambda_b} - M_B}{M_B} \sim \frac{\Lambda_{QCD}}{m_b}. \quad (46)$$

The OPE analysis in QCD unambiguously states that one must use the quark masses here, so that the mass factors are the same in all types of beauty hadrons. In contrast to hadron masses, the inclusive widths of heavy flavor hadrons do not have relative $1/m_Q$ splitting. This result was established by Bigi, Shifman, Vainshtein and myself in 1992.

Is this property expected $a$ priori? Yes and no. In the simple constituent models where the binding energy is neglected, $M_B = m_b + m_{sp}$. It is obvious that in such systems the widths are determined by $m_b^5$: the mass of the spectator – as long as it remains a spectator – does not enter.

This consideration is, of course, of little relevance for QCD. The mass of the spectator quarks in $B^-$, $B^0$ or $\Lambda_b$ is only a few MeV and we are not interested in such negligible effects. The actual question is about a few hundred MeV cloud whose energy, at least partially, is the interaction energy $E_{pot}$. In general, this $E_{pot}$ does directly affect the decay width. In the nonrelativistic models

$$M_B \simeq m_b + m_{sp} + E_{pot} + E_{kin} + \mathcal{O}\left( \frac{\Lambda_{QCD}^2}{m_Q} \right), \quad (47)$$

and $E_{pot} \sim E_{kin} \ll m_{sp} \sim \Lambda_{QCD}$, so this effect is less important than mere the rest energy of the spectator quark, and is routinely discarded. On the contrary, in QCD one can rather neglect $m_{sp}$.
At the same time, at least a major part of the mass dependence in Eq. (45) comes from the phase space available for the final states (in particular, for leptons), which naively knows nothing about the quark mass and rather senses the overall hadron mass. It is this consideration that may seem to be relevant for the QCD picture of hadrons. It often underlies implicit questioning of the validity of the QCD analysis resurrecting, in one form or another, the $1/m_Q$ corrections to the widths $[28, 29, 30, 31]$. And, nevertheless, the dependence on the binding energy disappears. How it happens in the language of the summation of the contributions of separate exclusive final states, was illustrated in $[32]$ using the so-called heavy quark transition sum rules $[18]$. Here, instead, I’ll try to explain the mechanism of the cancellation on a transparent toy example.

Let us consider the semileptonic decay $b \rightarrow c$ in the kinematics where leptons carry away a major portion of the energy release, and the final $c$ quark is moving slowly (this is the so-called small-velocity (SV) kinematics first suggested as a theoretical tool by Shifman and Voloshin in $[20]$). The initial $b$ quark undergoes a binding color Coulomb potential $V(0)$ which is of order $\Lambda_{\text{QCD}}$, and this, in principle, affects the decay probability. However, the final $c$ quark is produced inside the very same potential, and thus the energy of all final states is shifted by the same amount. The equality of the potential binding force in the initial and final state is guaranteed in this case by equality of the color ‘charges’ of the initial $b$ and final $c$ quarks which, in turn, is a consequence of the conservation of the color flow. Thus, it is easy to see that whatever strong is the binding energy by the Coulomb interaction, in such a case it is canceled between the initial and final state interaction due to the ‘color charge’ conservation.

The situation may seem different when one goes beyond the SV kinematics and the final quark is moving fast. First, the interaction of the final relativistic quark is not given by mere the Coulomb potential but depends on all components of the gauge potential. Second, the quark promptly leaves the vicinity of the point where it was produced, and the potential in other space locations is quite different. Moreover, if the interquark potential grows at large distances, whatever large $m_b$ is chosen, the final quark can never escape and would remain confined with the initial light cloud. In contrast, for the interaction vanishing at large distances the quark momentarily becomes ‘free’. This may naively seem to contradict OPE since no assumptions about the large-distance behavior of the interaction was involved in proving the cancellation of the binding energy effects.

Nevertheless, the cancellation is complete in this case as well. The gauge nature of the QCD interaction and the conservation of the color current ensure that the integrated effect of the interaction with the potential $A_\mu$ is the same for slow and relativistic particle. A closer look at the second problem also shows that no actual contradiction is present. Since the act of the decay happens for a small time interval $\sim 1/E_{\text{rel}}$, its probability is sensitive only to the potential at the distances $\sim 1/E_{\text{rel}}$ from the origin. Details of the interaction at larger distances merely determine the spectrum of the final states and how exactly the overall probability is allocated.
over the different final states, but not the total decay probability. For example, if
the interaction vanishes beyond a distance \( \sim \Lambda_{\text{QCD}}^{-1} \), the final states belong to the
continuous spectrum being represented by plane waves with some distortion near
the origin. If the potential is confining, there are no real quasifree decays but the
transitions to the bound states in such a potential. The sum of these probabilities
reproduces the decay width into the quasifree quark states.

This picture explains the OPE result for QCD. The latter, of course, is more
general and does not rely on an assumption about QM potential, etc. The key point
is that the decay is almost instantaneous, its time is of the order of \( 1/m_b \). It is
for this reason the total rate is determined by the potential at \( |\vec{x}| \sim 1/m_b \), and
the width is expanded in local operators \( bO_ib \) where \( O_i \) contain the QCD fields at
\( \vec{x} = 0 \) and their derivatives. In essence, the perturbative corrections represent the
expansion in the singularity of the interaction at \( x = 0 \) (recall, \( V(x) \propto \alpha_s/|\vec{x}| \)),
and the power nonperturbative corrections are related to the terms of the Taylor
expansion of the interaction near \( x = 0 \). This is a quite general property and does
not depend on the underlying nature of the interaction.

The conservation of the color flow and the gauge nature of the QCD interaction
of quarks leads to an additional property: the leading term in the Taylor expansion,
the ‘potential’ \( V(0) \) does not appear as a result of the exact cancellation between the
effects of the initial and final state interactions. This is a meaning of the theorem
about the absence of \( 1/m_Q \) corrections in the inclusive widths.

This is an example of a general QM statement known as the KLN theorem. In the
simple words, it states that all “sufficiently inclusive” transition probabilities must
be completely infrared (IR) insensitive\(^7\). However, to be “sufficiently inclusive”, the
process, in general, requires a summation over the degenerate initial states as well,
which is not realized in any experiment. The degree to which the KLN theorem is
valid for actual “insufficiently inclusive” transitions is \textit{a priori} unknown. The OPE
– when it applies – allows to quantify this sensitivity.

It is worth emphasizing that the absence of \( 1/m_Q \) corrections in the inclusive
widths is not an automatic consequence of the KLN theorem or any other general
QM fact. In dynamic realizations of strong interactions other than QCD – for
example, due to exchange of scalar particles – such corrections would be present
and depend non-universally on various kinematic details. That is why, for example,
the summation of the exclusive semileptonic transition probabilities in the ISGW
model \(^{36}\) did not reproduce the partonic width, especially in the \( b \to u \) decays.

After the general illustrations given above it is easier to outline the main technical
steps in calculating the widths: one only needs to believe that a particular inclusive

\(^7\)It was suggested in \(^33\) that the effects of duality violation are related to the singularities of
the interaction at finite distances \( x \) from the origin, in the complex plane. I do not discuss this
aspect here.

\(^8\)I deliberately do not explain here the precise meaning of both the assumption and of the
statement, what is IR insensitive. It can be found in the dedicated papers \(^34\), \(^35\).
width is indeed given by the local operators. Then we can write

$$\Gamma_H = \frac{1}{2M_{H_b}} \langle H_b| c_1 \bar{b}b + c_G \bar{b}Gb + \sum c_i \bar{b}O_i b + ...|H_b \rangle$$  \hspace{1cm} (48)$$

where $c$ are short-distance coefficient functions and $\bar{b}O_i b$ denote local operators of $D = 6$ and higher. Note that there is no independent operator of $D = 4$: the width must be a Lorentz scalar, and the only possible operator $\bar{b} \gamma_\alpha iD_\alpha b$ reduces to $m_b \bar{b}b$ by the QCD equations of motion. This is the counterpart of the cancellation of the Coulomb potential $A_0$ is the previous discussion: gauge invariance states that $A_0$ must always enter together with $\partial_0$, and this combination applied to the $b$ quark field is free of any dynamics to order $\Lambda_{QCD}$ by virtue of equations of motion (that is, including the interaction of the initial $b$ quark).

The expectation values of the operators in Eq. (48) scale like $\Lambda_{QCD}^{d-3}$ (more exactly, it can be a power of the normalization scale $\mu$, i.e. $\mu^{d-3}$). Therefore, the relative contribution to the width of higher-dimension operators scale like $(\Lambda_{QCD}, \mu)^{d-3}/m_b^{d-3}$.

Eq. (48) is not the whole story, however, since the leading operator $\bar{b}b$ describing the free quark decay is also affected by the nonperturbative dynamics. Its expectation value, however, can be expanded in $1/m_b$:

$$\langle H_b| \bar{b}b|H_b \rangle = \langle H_b| \bar{b}\gamma_0 b|H_b \rangle + \frac{\langle H_b| \bar{b} \left( \frac{\pi^2}{2} + \frac{i}{2} \sigma G \right) b|H_b \rangle}{2m_b^2} + \mathcal{O}(1/m_b^4) =$$

$$= 2M_{H_b} \left( 1 - \frac{\mu^2_{\pi} - \mu^2_G}{2m_b^2} + ... \right).$$  \hspace{1cm} (49)$$

The expectation value of $\bar{b}\gamma_0 b$ is exactly unity, which completes the proof of absence of $1/m_Q$ corrections. Eq. (49) was also derived by Bigi, Shifman, Vainshtein and myself and first given in [25]. It is worth noting that the term $-\frac{\mu^2_G}{2m_b^2}$ also has a transparent physical meaning [39]: it is nothing but a Lorentz dilation of the decay due to nonrelativistic ‘shivering’ (so-called Fermi motion) inside the hadron through the binding effects.

Including the leading corrections, the semileptonic width has the following form [25, 26, 40, 39]:

$$\Gamma_{sl} = \frac{G_F m_b^5 |V_{cb}|^2}{192\pi^3} \left\{ z_0 \left( 1 - \frac{\mu^2_{\pi} - \mu^2_G}{2m_b^2} \right) - 2 \left( 1 - \frac{m_c^2}{m_b^2} \right) \frac{\mu^2_G}{m_b^2} - \frac{2\alpha_s}{3\pi} a_1 + ... \right\}$$  \hspace{1cm} (50)$$

where ellipses stand for higher order perturbative and/or power corrections, $z_0$ and $a_1$ depend on $m_c^2/m_b^2$. The value of $\mu^2_{\pi}$ is not yet measured in experiment.

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9 One possibility to show it formally is described in detail, e.g. in [33]. Historically, an important role here was played by paper [37].

10 Recently this relation was attributed by Neubert [38] to the later papers, in particular, to [14]. However, there was no such a relation there.
exists an inequality $\mu_\pi^2 > \mu_\pi^2$ which essentially constraints the interval of its possible values. The status of $\mu_\pi^2$ will be addressed later. Irrespectively of the exact value of $\mu_\pi^2$, the $1/m_b^2$ corrections to $\Gamma_{sl}$ are rather small, about $-5\%$ and lead to the increase in the extracted value of $|V_{cb}|$ by $2.5\%$. The impact of the higher order power corrections is negligible.

A rather detailed most accurate numerical analysis of the semileptonic widths is given in [41], Sect. 8.1. Let me briefly summarize it. The leading-order $O(\alpha_s)$ perturbative corrections are known from the QED calculations in mu on decay [27]. The $1/m_b^2$ nonperturbative corrections were calculated in the above mentioned papers and decrease the width by about $5\%$. The $1/m_b^3$ corrections were evaluated in [42, 43] and are at the level of $1\%$. They are negligible for extracting $|V_{cb}|$. The part of the higher-order perturbative corrections associated only with running of $\alpha_s$ in the first-order loop diagrams, the so-called BLM corrections [44], were calculated to order $\alpha_s^2$ [45] and to all orders [46]. Their impact appeared to be small. The remaining, nontrivial non-BLM (‘genuine’) two-loop $\alpha_s^2$ corrections were expected to be moderate. They were recently evaluated in a series of papers by Czarnecki and Melnikov [47] and were confirmed to be small. These were highly nontrivial sophisticated technical analyses often considered unfeasible even a few years ago. A resummation of certain enhanced non-BLM higher-order corrections was performed in [32]. At present, no significant theoretical uncertainty remains in the perturbative corrections to the semileptonic width.

Evaluating the theoretical prediction we get

$$|V_{cb}| = 0.0419 \left( \frac{\text{BR}(B \to X_c \ell \nu)}{0.105} \right)^{1/2} \left( \frac{1.55 \text{ ps}}{\tau_B} \right)^{1/2} \cdot \left( 1 - 0.012 \frac{(\mu_\pi^2 - 0.5 \text{ GeV}^2) + 0.012}{0.1 \text{ GeV}^2} \right) \times$$

$$\left( 1 - 0.01 \frac{\delta m_b(\mu)}{50 \text{ MeV}} \right) \left( 1 + 0.006 \frac{\alpha_s^{\text{MS}}(1 \text{ GeV}) - 0.336}{0.02} \right) \left( 1 + 0.007 \frac{\hat{\rho}^3}{0.1 \text{ GeV}^3} \right). \quad (51)$$

Here $m_b(\mu)$ is a certain short-distance mass of the $b$ quark which is the subject of a special discussion in the next section, $\hat{\rho}$ is a positive non-local correlator mentioned before (its expected scale is $0.1 \text{ GeV}^3$). The main theoretical uncertainty at the moment resides in the value of $\mu_\pi^2$. We arrive thus at the model-independent evaluation

$$|V_{cb}| = 0.0419 \left( \frac{\text{BR}(B \to X_c \ell \nu)}{0.105} \right)^{1/2} \left( \frac{1.55 \text{ ps}}{\tau_B} \right)^{1/2} \times$$

$$\left( 1 - 0.012 \frac{(\mu_\pi^2 - 0.5 \text{ GeV}^2)}{0.1 \text{ GeV}^2} \right) \cdot \left( 1 \pm 0.015_{\text{pert}} \pm 0.01_{m_b} \pm 0.012 \right), \quad (52)$$

where the last error reflects $m_Q^{-3}$ and higher power corrections as well as possible deviations from duality. The quoted uncertainty associated with $m_b$ can be viewed as rather conservative; this assertion will be explained in the sections to follow.
Similar to the treatment of $\Gamma(B \to X_c \ell \nu)$, it is straightforward to relate the value of $|V_{ub}|$ to the total semileptonic width $\Gamma(B \to X_u \ell \nu)$ \[48\]:

$$|V_{ub}| = 0.00465 \left( \frac{\text{BR}(B^0 \to X_u \ell \nu)}{0.002} \right)^{\frac{1}{2}} \left( \frac{1.55 \text{ps}}{\tau_B} \right)^{\frac{1}{2}} \cdot (1 \pm 0.025_{\text{pert}} \pm 0.03_{\text{mb}}) . \quad (53)$$

The dependence on $\mu^2_{\pi}$ is practically absent here. The theoretical uncertainty in $\Gamma(B \to X_u \ell \nu)$ is relatively insignificant. The major problem is experimental, how to measure $\text{BR}_{sl}(b \to u)$. For a long time it seemed unfeasible. Nevertheless, a possibility of extracting it from experiment with a reasonable confidence is being discussed since recently, and may eventually yield an accurate determination of $|V_{ub}|$ on this route.

I’ll return to the question of $\Gamma_{sl}$ and $|V_{cb}|$ later in the summary. Before proceeding to a more theoretical issues, it seems necessary to note that rather controversial opinions existed in the literature regarding the reliability of calculation of $\Gamma_{sl}$ and, in particular, the impact of higher-order perturbative corrections. Often they were claimed to be too large or even uncontrollable \[49, 45, 50, 51\]. In reality, the apparent instability was associated with not quite physical formulation of the problem and the confusion rooted to the problem of the definition of the heavy quark mass. As a matter of fact, after all refinements and thorough analysis of the corrections, we are basically back to square one. The expression for $|V_{cb}|$ given in Ref. \[21\] has changed by less than one percent (for the same input values of the experimental parameters). What became clearer in the last two years is the acceptable range of the underlying key QCD parameters entering the heavy quark expansion, and the size of theoretical corrections. I cannot help mentioning in this respect that, meanwhile, the value of $\tau_B$ (at least used by theoreticians as an input) drifted from $1.32 \pm 0.09 \text{ps} \quad [52]$ to $1.57 \text{ps}$.

### 3.2 Problem of $m_Q$ in QCD

The heavy quark mass is certainly one of the most important starting parameters of the heavy quark theory. The practical necessity to know it well is illustrated by the fact that the decay widths of the beauty hadrons are proportional to a high power of the mass:

$$\Gamma(B, \Lambda_b) \sim m_b^n , \quad n = 5 \gg 1 . \quad (54)$$

All uncertainties in $m_b$ are magnified by the factor $n$. On the other hand, one can turn vices into virtues developing $1/n$ expansion and performing resummation of the large-$5$ terms. This was accomplished in \[32\]. Let me mention only one, the perturbative aspect.

In general, the perturbative series for, say, $\Gamma_{sl}$ have $n$-enhanced terms:

$$\Gamma_{sl}^{\text{pert}} \propto 1 + n d_1 \frac{\alpha_s}{\pi} + n^2 d_2 \left( \frac{\alpha_s}{\pi} \right)^2 + n^3 d_3 \left( \frac{\alpha_s}{\pi} \right)^3 + \ldots \quad (55)$$
where \( d_1, d_2 \ldots \sim 1 \). Even at \( n = 5 \) and a moderate value of \( \frac{c_n}{s_n} = 0.1 \) the higher-order terms are rather significant to be neglected already if a 5% accuracy in \( V_{cb} \) is targeted. Fortunately, this series can be readily summed up \([32]\), and the remaining corrections are small. However, once again it requires an accurate understanding of what are \( m_b \) and \( m_c \).

Meanwhile, quite controversial estimates could be found in the literature for the numerical values of the charm and, in particular, bottom quark masses. As a result, the majority of the HQET applications, for example, were focused on the quantities which were mainly independent on \( m_b \). HQET, claiming to have finally clarified the issue of what is the heavy quark mass, actually admitted that its determination with a sensible accuracy was impossible in practice.

In appeared that these ‘practical’ difficulties had a deep theoretical reason. HQET was based on the concept of the “pole” mass of the heavy quark \( m_Q^{\text{pole}} \). However, in QCD-like theories the “pole” mass is not a physical notion. Moreover, remaining a purely perturbative construction, it cannot theoretically be defined with the necessary accuracy \([34]\), the irreducible uncertainty in defining its value is of the order of typical hadronic scale:

\[
\delta m_Q^{\text{pole}} = \text{few units} \times \Lambda_{\text{QCD}}.
\]

No wonder all attempts to extract this ill-defined value yielded controversial results.

### 3.2.1 What is \( m_Q \)?

In quantum field theory the object we begin our work with is the Lagrangian formulated at some high scale \( M_0 \). The mass \( m_0 \) is a parameter in this Lagrangian; it enters on the same footing as, say, the bare coupling constant \( \alpha_s^{(0)} \) with the only difference being that it carries dimension. As with any other coupling, \( m_0 \) enters in all observable quantities in a certain combination with the ultraviolet cutoff \( M_0 \), which is universal for a renormalizable theory.

The mass parameter \( m_0 \) by itself is not observable, like \( \alpha_s^{(0)} \). For calculating observable quantities at the scale \( \mu \ll M_0 \) it is usually convenient to relate \( m_0 \) to some mass parameter relevant to the scale \( \mu \). Integrating out momentum scales above \( \mu \) converts \( \alpha_s^{(0)} \) into \( \alpha_s(\mu) \) – and likewise \( m_0 \) into \( m(\mu) \). Such \( m(\mu) \) is not something absolute since depends on \( \mu \). It is either used on the same footing as \( \alpha_s(\mu) \) or, in the final expressions, is eliminated in favor of some suitable observable mass. For example, in quantum electrodynamics (QED) at low energies (i.e. \( E \ll m_e \)) there is an obvious “best” candidate: the actual mass of an isolated electron, \( m_e \).

In the perturbative calculations it is determined as the position of the pole in the electron Green function (more exactly, the beginning of the cut). The advantages are evident: \( m_e \) is gauge-invariant and experimentally measurable.

The analogous parameter for heavy quarks in QCD is referred to as the pole quark mass, the position of the pole of the quark Green function. Unlike \( m_e \), it is gauge invariant. Unlike QED, however, the quarks do not exist as isolated objects (there
are no states with the quark quantum numbers in the observable spectrum, and the quark Green function beyond a given order has neither a pole nor a cut. Hence, \( m^{\text{pole}} \) cannot be directly measured; \( m^{\text{pole}} \) exists only as a theoretical construction.

In principle, there is nothing wrong with using \( m^{\text{pole}} \) in perturbation theory where it naturally appears in the Feynman graphs for the quark Green functions, scattering amplitudes and so on. It may or may not be convenient, depending on concrete goals.

The pole mass in QCD is perturbatively infrared stable, order by order, like in QED. It is well-defined to every given order in perturbation theory. One cannot define it to all orders, however; the sum of the series does not converge to a definite number. In a sense, the pole mass is not infrared-stable nonperturbatively. Intuitively this is clear: since the quarks are confined in the full theory, the best one can do is to define the would-be pole position with an intrinsic uncertainty of order \( \sim \Lambda_{\text{QCD}} \).

Based on experience in QED or ordinary QM, non-existence of \( m^{\text{pole}}_Q \) may seem counter-intuitive. Even rigorous inequalities like \( \Lambda > 232 \text{ MeV} \) were claimed [53] for the ‘fundamental HQET parameter’, the difference \( \Lambda = M_B - m^{\text{pole}}_b \) (the flaw was pointed out in [54]). The confusion did not dissipate completely and still surfaces in the literature. In the review article on the quark masses in PDG-96 by A. Manohar it was stated that there existed a third, the “HQET” mass \( m^{\text{HQET}}_Q \) which differed from \( m^{\text{pole}}_Q \) starting \( \alpha^2_s \) in QCD with massless (light) flavors [55]. That unsubstantiated claim is erroneous; the mass used by HQET coincided with \( m^{\text{pole}}_Q \) to any concrete order of perturbation theory. Moreover, it is not clear what normalization point could have had the coupling \( \alpha_s \) in that difference: the renormalizability would leave as an option only the mass of the light quark... I hope that such statements will be corrected in future issues of PDG. The “HQET” mass is as ill-defined as the “pole” mass.

Employing the perturbation theory, we start with the free quark propagator

\[
G(p) = \frac{1}{m_Q(\mu) - p^2}
\]

\( m_Q(\mu) \) is the parameter entering the Lagrangian) which has a pole at \( p^2 = m_Q^2(\mu) \) and, therefore, describes a particle with mass \( m_Q(\mu) \). Accounting for the gluon exchanges to the first order in \( \alpha_s \) adds the diagram Fig. 2 and the pole moves to \( p^2 \sim \left( m_Q(\mu) + \frac{4\alpha_s}{3\pi} \mu \right)^2 \) describing now a particle with the mass \( m_Q^{(1)} \) \( \simeq m_Q(\mu) + \frac{4\alpha_s}{3\pi} \mu \), etc. In any order of the perturbation theory we see a quark pole and the corresponding particle with certain mass, differing from the mass in the Lagrangian. This mass is just the “pole” mass. Clearly, it depends on the order of perturbation theory one considers, and on a concrete version of the employed expansion:

\[
m_Q^{(k)} = m_Q(\mu) \sum_{n=0}^{k} C_n \left( \frac{\mu}{m} \right) \left( \frac{\alpha_s(\mu)}{\pi} \right)^n, \quad C_0 = 1 . \quad (57)
\]

It is tempting to define the ‘actual’ pole mass as a sum of the series (57), and this used to be a standard assumption in HQET. It appears, however, that the sum
Figure 2: Perturbative diagrams leading to renormalization of the heavy quark mass. The contribution of the gluon momenta below $m_Q$ expresses the classical Coulomb self-energy of the colored particle. The number of bubble insertions into the gluon propagator can be arbitrary generating corrections in all orders in $\alpha_s$. The factorial growth of the coefficients produces the IR renormalon uncertainty in $m_Q^{\text{pole}}$ of order $\Lambda_{\text{QCD}}$.

does not converge to a reasonable number and cannot be defined with the necessary accuracy in a motivated way, but suffers from an irreducible uncertainty of order $\Lambda_{\text{QCD}}$.

Before explaining the origin of this perturbative uncertainty, let me note that such a definition of the quark mass is nothing but equating it with the lowest eigenvalue of the (perturbative) Hamiltonian in the sector with the number of heavy quarks 1. The eigenvalues of the Hamiltonian, however, are not a short-distance quantity, and their calculation in the perturbative expansion is not adequate. This is in the nice correspondence with the observed fact that no isolated heavy quark exists in the spectrum of hadrons which contains instead $B, B^{**}, \Lambda_b, B_s, \ldots$ with masses differing from each other by amount $\mathcal{O}(\Lambda_{\text{QCD}}^{1/2})$.

The physical reason behind the perturbative instability of the long-distance regime is the growing of the interaction strength $\alpha_s$. One can illustrate this instability in the following transparent way. Consider the energy stored in the chromoelectric field in a sphere of radius $R \gg 1/m_Q$ around a static color source,

$$
\delta E_{\text{Coul}}(R) \propto \int_{1/m_Q \leq |x| < R} d^3 x \mathbf{E}_{\text{Coul}}^2 \propto \text{const} - \frac{\alpha_s(R)}{\pi} \frac{1}{R} .
$$

This energy is what one adds to the bare mass of a heavy particle to determine what will be the mass including the QCD interactions. The definition of the pole mass amounts to setting $R \to \infty$; i.e., in evaluating the pole mass one undertakes to integrate the energy density associated with the color source over all space assuming that it has the Coulomb form. In real life the color interaction becomes strong at $R_0 \sim 1/\Lambda_{\text{QCD}}$; at such distances the chromoelectric field has nothing to do with the Coulomb tail. Thus, one cannot include the region beyond $R_0$ in a meaningful way. Its contribution which is of order $\Lambda_{\text{QCD}}$, thus, has to be considered as an irreducible uncertainty which is power-suppressed relative to $m_Q$,

$$
\frac{\delta_{\text{IR}} m_Q^{\text{pole}}}{m_Q} = \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{m_Q}\right) .
$$

Exactly this behavior is traced formally in the perturbation theory. In the non-relativistic regime when the internal momentum $|\vec{k}| \ll m_Q$ the expression for the
diagram Fig. 2 is simple,
\[ \delta m_Q \sim -\frac{4}{3} \int \frac{d^4 k}{(2\pi)^4 i k_0} \frac{4\pi \alpha_s}{k^2} = \frac{4}{3} \int \frac{d^3 \vec{k}}{4\pi^2 \vec{k}^2} \alpha_s. \]  

(60)

The running of the coupling is generated by dressing the gluon propagator by virtual pairs and leads to
\[ \delta m_Q \simeq \frac{4}{3} \int \frac{d^3 \vec{k}}{4\pi^2 \vec{k}^2} \alpha_s(\vec{k}^2). \]  

(61)

Since
\[ \alpha_s(k^2) = \alpha_s(\mu^2) \left\{ 1 + \frac{\alpha_s(\mu^2)}{4\pi} b \ln \frac{k^2}{\mu^2} \right\}^{-1}, \quad b = \frac{11}{3} N_c - \frac{2}{3} n_f, \]  

(62)

we can expand \( \alpha_s(k^2) \) in a power series in \( \alpha_s(\mu^2) \) and easily find the \( (n+1) \)-th order contribution to \( \delta m_Q \),
\[ \frac{\delta m_Q^{(n+1)}}{m_Q} \sim \frac{4}{3} \frac{\alpha_s(\mu)}{\pi} n! \left( \frac{b \alpha_s(\mu)}{2\pi} \right)^n. \]  

(63)

We observe that the coefficients grow factorially and contribute with the same sign. Therefore, one cannot define the sum of these contributions even using the trick with the Borel transformation. The best one can do is to truncate the series judiciously. An optimal truncation leaves us with an irreducible uncertainty \( \sim O(\Lambda_{\text{QCD}}) \) [56, 57].

The above perturbative corrections are example of the so-called infrared renormalons [58].

This uncertainty can be quantified. A formal Borel resummation of such non-summable series leads to the result which literally has an imaginary part which can be taken as a measure of the uncertainty. The imaginary part for the series Eq. (63) is
\[ \text{Im} m_Q^{\text{pole}} = \frac{8\pi}{3b} e^{5/6} \Lambda_{\text{QCD}}^{\text{MS}}. \]  

(64)

It became conventional to assign to the irreducible uncertainty the formal imaginary part divided by \( \pi \). Even with this minimal choice
\[ \delta m_Q^{\text{pole}} \sim \frac{1}{\pi} |\text{Im} m_Q^{\text{pole}}| = \frac{8}{27} e^{5/6} \Lambda_{\text{QCD}}^{\text{MS}} \simeq 0.7 \Lambda_{\text{QCD}}^{\text{MS}}. \]  

(65)

Thus, the perturbative expansion per se anticipates the onset of the nonperturbative regime (the impossibility of locating the would-be quark pole to accuracy better than \( \Lambda_{\text{QCD}} \)). Certainly, the concrete numerical value of the uncertainty in \( m_{\text{pole}} \) obtained through renormalons is not trustworthy. The renormalons do not represent the dominant component of the infrared dynamics. However, they are a clear indicator of the presence of the power-suppressed nonperturbative effects, or infrared instability of \( m_{\text{pole}} \); the very fact that there is a correction \( O(\Lambda_{\text{QCD}}/m_Q) \) is beyond any doubt.
It is worth noting that the pole mass was the first example where a quantity which is perturbatively infrared-stable was shown not to be stable nonperturbatively at the level $\Lambda_{QCD}$. The observation of Refs. [56, 57] gave impetus to dedicated analyses of other perturbatively infrared-stable observables in numerous hard processes without OPE, in particular, in jet physics. Such nonperturbative infrared contributions, linear in $\Lambda_{QCD}/Q$ were indeed found shortly after in thrust and many other jet characteristics (for a review and a representative list of references see e.g. [59]).

To demonstrate that the problem of divergence of the $\alpha_s$ series for $m_{pole}^b$ is far from being academic, let us examine how the ‘perturbative’ contribution to the $b$ quark mass looks numerically:

$$m_{pole}^b = m_b(1 \text{ GeV}) + \delta m_{pert}(\text{below } 1 \text{ GeV}) \simeq 4.55 \text{ GeV} + 0.25 \text{ GeV} + 0.22 \text{ GeV} + 0.38 \text{ GeV} + 1 \text{ GeV} + 3.3 \text{ GeV} + ..., \quad (66)$$

where $m_b(1 \text{ GeV})$ is the running mass at $\mu = 1 \text{ GeV}$, and $\delta m_{pert}$ is the perturbative series taking account of the loop momenta from 1 GeV down to zero. It is quite obvious that the corrections start to blow up already in rather low orders! Their size is consistent with the estimate Eq. (65). Expressing observable infrared-stable quantities (e.g. the inclusive semileptonic width $B \rightarrow X_u \ell \nu$) in terms of the pole mass will necessarily entail large coefficients in the $\alpha_s$ corrections compensating for the explosion of the coefficients in $m_{pole}^b$. I will return to this point later in Sect. 3.5.

Summarizing, the pole mass per se does not appear in OPE for infrared-stable quantities. Such expansions operate with the short-distance (running) mass. Any attempt to express the OPE-based results in terms of the pole mass creates a problem making the Wilson coefficients ill-defined theoretically and poorly convergent numerically.

Since it is impossible to relate $m_Q(\mu)$ and $m_{pole}^Q$ to the necessary accuracy, it is clear that either $m_{pole}^Q$ or $m_Q(\mu)$ must be irrelevant for the $1/m_Q$ expansion, for example, for calculating the widths. Which one? This question was formulated and answered in early 1994 in Ref. [56]. In agreement with the qualitative discussion given above, the answer is: $m_{pole}^Q$ is irrelevant and must be replaced everywhere by $m_Q(\mu)$. In the OPE, the IR part of the pole mass is not related to any local operator $\bar{Q}O_i Q$, and does not enter any observable calculable in the short-distance expansion. Since this IR piece does not enter observables, it cannot, in turn, be determined experimentally. The numerical instability of various attempts to pinpoint the value of $m_{pole}^Q$ is a result of using the perturbative expansion for the effects originating from the nonperturbative domain.

The above facts were later illustrated in detail in Ref. [60] in a concrete model for the higher-order corrections to the semileptonic widths obtained via the so-called bubble resummation of the one-loop perturbative diagrams. The corresponding IR renormalon uncertainty disappeared from the perturbative corrections when the width was expressed in terms of the short-distance heavy quark mass.$^1$

\footnote{A similar cancellation was claimed to be independently obtained afterwards also in [10]. How-}
It is important to note that the irrelevance of the pole mass goes beyond the problems with the IR renormalon contributions illustrated above. Even if there existed some way to define reasonably the sum of the pure perturbation series for $m_Q^{\text{pole}}$, or asymptotic states with single-quark quantum numbers with finite energy existed in a strong-interaction theory like QCD, this mass still would have been inadequate for constructing the effective field theory, to the extent the difference with $m_Q(\mu)$ cannot be neglected numerically.

A properly constructed perturbative treatment suitable for the $1/m_Q$ expansion incorporates only gluons with $|\vec{k}| \gtrsim \mu$ which are not ‘resolved’ and are included into the heavy quark field wavefunction corresponding to the heavy quark field $Q(\mu)(x)$ normalized at the scale $\mu$. The normalization point $\mu$ can be changed: descending from $\mu$ down to $\mu_1 < \mu$ one has to integrate out newly-resolved gluons with $\mu_1 < |\vec{k}| < \mu$. For example, the Coulomb field associated with such gluons increases the mass of the quark by the amount

$$\delta m_Q = \int_{\mu_1 < |\vec{k}| < \mu} \frac{d^3 \vec{k}}{4\pi^2} \frac{4 \alpha_s(\vec{k}^2)}{3\vec{k}^2}.$$ (67)

The pole mass clearly appears when $\mu_1 \to 0$. From the OPE point of view, it is an attempt to construct an effective theory with the normalization scale $\mu = 0$ formulated, nevertheless, still in terms of quarks and gluons. Speaking theoretically, one can imagine a limit of small $\mu$ which would correspond to integrating out all modes down to $\mu = 0$ in evaluation of the effective Lagrangian. It would be nothing but constructing the $S$-matrix of the theory from which one can directly read off all conceivable amplitudes. Clearly, it could have been formulated in terms of physical mesons and baryons but not of quarks and gluons! Such inconsistencies stemmed from postulating in HQET technicalities like using the dimensional regularization as an indispensable tool to formulate the theory at the quantum level.

In the Wilson OPE one uses $m_Q(\mu)$ with $\Lambda_{\text{QCD}} \ll \mu \ll m_b$. I’ll illustrate later that just such a mass can be accurately measured in experiment. It is $\mu$-dependent:

$$\frac{d m_Q(\mu)}{d \mu} = -\frac{16}{9} \frac{\alpha_s(\mu)}{\pi} - \frac{4}{3} \frac{\alpha_s(\mu)}{\pi} \frac{\mu}{m_Q} + \mathcal{O}\left(\frac{\alpha_s^2}{m_Q^2}\right);$$ (68)

the higher order perturbative corrections were computed recently [22]. There are different schemes for defining $m_Q(\mu)$ (similar to renormalization schemes for $\alpha_s$), and the coefficients above are generally different there. As long as a concrete scheme is adopted, there is no ambiguity in the numerical value of $m_Q(\mu)$. Instead of the HQET parameter $\Lambda$ in QCD one has $\Lambda(\mu) = \lim_{m_Q \to \infty} M_{H_Q} - m_Q(\mu)$. The value of $\Lambda(\mu)$ is of the hadronic mass scale if $\mu$ does not scale with $m_Q$.

There exists a popular choice of a short-distance mass, the so-called $\overline{\text{MS}}$ mass $\overline{m}(\mu)$. The $\overline{\text{MS}}$ mass is not a parameter in the effective Lagrangian; rather it is a
certain *ad hoc* combination of the parameters which is particularly convenient in the perturbative calculations using dimensional regularization. Its relation to the perturbative pole mass is known to two loops [61]:

\[
m_Q^{\text{pole}} = \bar{m}_Q(\bar{m}_Q) \left\{ 1 + \frac{4\alpha_s(\bar{m}_Q)}{3\pi} + (1.56\, b - 3.73) \left( \frac{\alpha_s}{\pi} \right)^2 + \ldots \right\}
\] (69)

At \( \mu \gg m_Q \) the \( \overline{\text{MS}} \) mass coincides, roughly speaking, with the running Lagrangian mass seen at the scale \( \sim \mu \). However, it becomes rather meaningless at \( \mu \ll m_Q \):

\[
\bar{m}_Q(\mu) \simeq \bar{m}_Q(\bar{m}_Q) \left\{ 1 + \frac{2\alpha_s}{\pi} \ln \frac{m_Q}{\mu} \right\}.
\] (70)

It logarithmically diverges when \( \mu/m_Q \to 0 \). For this reason \( \bar{m}(\mu) \) is not appropriate in the heavy quark theory where the possibility of evolving down to a low normalization point, \( \mu \ll m_Q \), is crucial. Otherwise, for example, \( M_{H_Q} - \bar{m}_Q \propto m_Q \) and does not stay constant in the heavy quark limit.

The reason for this IR divergence is that the \( \overline{\text{MS}} \) scheme technically attempts to determine the (perturbative) running of all quantities by their divergences calculated when the space-time dimension approaches \( D = 4 \). The divergence can emerge only at \( k \to 0 \) or \( k \to \infty \). For the mass, the IR divergences are absent, and the dimensional regularization is sensitive only to the UV divergence at \( k \gg m_Q \). Studying only \( 1/(D-4) \) singularities, it is in principle unable to capture the change of the regime at \( \mu \ll m_Q \) and assumes the same running in this domain. The actual running below \( m_Q \) is much slower. Such a mistake was made, for example, in [8] where the logarithmic running of the mass was stated for arbitrary scale.

The properly defined short-distance masses always exhibits an explicit linear \( \mu \)-dependence similar to Eq. (68) at \( \mu \ll m_Q \). The perturbative pole mass, order by order, would correspond to the limit \( \mu \to 0 \). However such a limit does not exist.

### 3.2.2 Which mass is to be used?

Since \( m_Q^{\text{pole}} \) does not exist as a well-defined mass parameter, a different, short-distance mass must be used. The normalization point \( \mu \) can be arbitrary as long as \( \mu \gg \Lambda_{\text{QCD}} \). It does not mean, however, that all masses are equally practical, since the perturbative series are necessarily truncated after a few first terms. Using an inappropriate scale makes numerical approximations bad. In particular, relying on \( \bar{m}_Q(m_Q) \) in treating the low-scale observables can be awkward. The following toy example illustrates this point.

Suppose, we would like to exploit QCD-like methods in solving the textbook problem of the positronium mass, calculating the standard Coulomb binding energy. To use the result written in terms of the \( \overline{\text{MS}} \) mass one would, first, need to evaluate \( \bar{m}_e(m_e) \). This immediately leads to technical problems: \( \bar{m}_e(m_e) \) is known only to the order \( \alpha^2 m_e \); therefore, the “theoretical uncertainty” in \( \bar{m}_e(m_e) \) would generate
the error bars $\sim \alpha^3 m_e$ in the binding energy, i.e. at least 0.01 eV. Moreover, without cumbersome two-loop calculations one would not know the binding energy even at a few eV level – although, obviously, it is known to a much better accuracy (up to $\alpha^4 m_e$ without a dedicated field-theory analysis), and the result

$$M_P = 2m_e(0) - \frac{\alpha^2 m_e}{4} + \mathcal{O}(\alpha^3 m_e)$$

(71)
is obtained without actual loop calculations!

Thus, working with the $\overline{\text{MS}}$ mass we would have to deal here with the “handmade” disaster. The reason is obvious. The relevant momentum scale in this problem is the inverse Bohr radius, $\mu_B = r_B^{-1} \sim \alpha m_e$. Whatever contributions emerge at much shorter distances, say, at $\mu^{-1} \sim m_e^{-1}$, their sole role is to renormalize the low-scale parameters $\alpha$ and $m_e$. If the binding energy is expressed in terms of these parameters, further corrections are suppressed by powers of $\mu_B/\mu$. Exploiting the proper low-energy parameters $\alpha(0)$, $m_e(0)$ one automatically resums ‘tedious’ perturbative corrections. In essence, this is the basic idea of the effective Lagrangian approach, which, unfortunately, is often forgotten.

In the perturbative determination of $m_b(\mu)$, for example, the importance of higher-order corrections becomes governed by $\left(\frac{\alpha_s}{\pi}\right)^k \mu$ rather than $\left(\frac{\alpha_s}{\pi}\right)^k m_b$. Since $\mu \ll m_b$, this is a significant improvement.

Needless to say, it is the high-scale masses that appear directly in the processes at high energies. In particular, the inclusive width $Z \to b\bar{b}$ is sensitive to $m_b(M_Z)$; using $\overline{\text{MS}}$ mass normalized at $\mu \sim M_Z$ is appropriate here. On the contrary, the inclusive semileptonic decays $b \to c\ell\nu$ are rather low-energy in this respect [32], and, to some extent, that is true even for $b \to u$.

The construction of the running low-scale heavy quark mass suitable for the OPE to any order in perturbation theory can be done in a straightforward way. However, the Wilsonian approach implies introduction a cutoff on the momenta of the gluon fields. Since gluon carries color, its momentum is not a gauge-invariant quantity, and such a mass typically appears not gauge-invariant. More accurately, obtaining the same $m_Q(\mu)$ requires somewhat different cutoff rules in different gauges.

Even though this is not a real problem for the theory, it is often viewed as a disadvantage. To get rid of this spurious problem, a manifestly gauge-invariant definition of the running mass was suggested in [32, 65] which is formulated only in terms of observables. Moreover, it is convenient for the OPE in the $1/m_Q$ expansion since more or less directly enters many relevant processes, e.g. heavy flavor transitions or the threshold heavy flavor production. Its relation to the $\overline{\text{MS}}$ mass is known with enough accuracy [22]. To avoid ambiguities I always use this definition unless other convention is indicated explicitly.

It is important to note the following fact. In the relativistic theory for a particle with mass $m$ one always has $p^2 = m^2$, that is, $E = \sqrt{m^2 + p^2}$. In the nonrelativistic
expansion, therefore,
\[ E = m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3} + \ldots \]  
(72)

all coefficients are fixed in terms of powers of \( m \) by Lorentz invariance. As will be mentioned in Sect. 3.3.4, in applications to heavy quarks it is typically advantageous to use such a Wilsonian cutoff which preserves usual QM properties for the price of violating Lorentz invariance (the reasons are discussed in [67]). Then the mass parameters in Eq. (72) generally become different:

\[ H_Q = m_0 \varphi_Q^+ \varphi_Q - \varphi_Q^+ A_0 \varphi_Q + \frac{1}{2m_2} \left( (i \vec{D})^2 + c_G \vec{\sigma} \vec{B} \right) - \ldots \]  
(73)

even for a “quasifree” quark in the effective theory. Of course, the differences between \( m_i \) appear only due to perturbative corrections:

\[ m_i(\mu) - m_k(\mu) = \mathcal{O}(\alpha_s \mu) \]

Moreover, the differences can be calculated perturbatively and are completely free of any IR effects existing below the scale \( \mu \). For example, in our scheme \( m_0(\mu) \approx m_2(\mu) + \frac{4}{9} \frac{\alpha_s(\mu)}{\pi} \). We use the mass \( m_2(\mu) \), the mass that enters the kinetic energy operator \( \vec{p}^2/2m \); it is the most important mass for heavy quark transitions.

This calculable difference between different “masses” for the same quark must be properly accounted for in the OPE analysis. Probably, the most obvious place where it is important is the \( \bar{Q}Q \) threshold physics discussed in Sect. 3.2.3. While in the short-distance expansion the free quark threshold starts at \( 2m_0(\mu) \), the propagation of heavy quarks or the bound state dynamics are actually determined by \( m_2(\mu) \). The shift in the position of the threshold which serves as a reference point for energy in the nonrelativistic system must be properly taken into account.

Concluding this section, I would like to make a side remark concerning the t-quark mass [41]. The peculiarity of the t quark is that it has a significant width \( \Gamma_t \approx 1 \text{ GeV} \) due to its weak decay. The perturbative position of the pole in the propagator is, thus, shifted into the complex plane by \(-\frac{i}{2} \Gamma_t \). The finite decay width of the t quark introduces a physical infrared cutoff for the infrared QCD effects [62]. In particular, the observable decay characteristics do not have ambiguity associated with the uncertainty in the pole mass discussed above. The uncertainty cancels in any physical quantity that can be measured. That is not the case, however, in the position of the pole of the t-quark propagator in the complex plane (more exactly, its real part). The quark Green function is not observable there, and one would encounter the very same infrared problem and the same infrared renormalon. The latter does not depend on the absolute value of the quark mass (and whether it is real or have an imaginary part). Thus, in the case of top, one would observe an intrinsic – but artificial – infrared renormalon uncertainty of several hundred MeV in attempts to relate the peak in the physical decay distributions to the position of the propagator singularity in the complex plane.
While this fact will pose no practical problem any time soon, it will do so in the future, in particular when analyzing top production at linear e\(^+\)e\(^-\) colliders. In my opinion, there is no point in expressing the fundamental experimental findings in terms of unphysical parameters which, additionally, cannot be defined even theoretically. The observables can be conveniently expressed in terms of a mass \(m_t\) defined at the scale \(\mu \approx \Gamma_t\) that is free from the renormalon ambiguities. It can be defined and measured without intrinsic limitations. It is worth trying to come to an agreement in advance what kind of top quark mass should be listed in the PDG tables, and thus avoid the problems which accompanied attempts to pinpoint the masses of \(c\) and \(b\) quarks.

### 3.2.3 The numerical values of \(m_c\) and \(m_b\)

The mass of the \(c\) quark at the scale \(\sim m_c \sim 1\,\text{GeV}\) can be obtained from the charmonium sum rules [63], \(m_c(m_c) \approx 1.25\,\text{GeV}\). The result to some extent is affected by the value of the gluon condensate. To be safe, we conservatively ascribe a rather large uncertainty,

\[
m_c(m_c) = 1.25 \pm 0.1\,\text{GeV}.
\]

There are reasons to believe that the precision charmonium sum rules actually determine the charmed quark mass to a better accuracy.

An accurate measurement of \(m_b\) is possible in the \(\bar{b}b\) production in e\(^+\)e\(^-\) annihilation. Since we want to know \(m_b\) with comparable or better absolute precision, both the data and calculations, at first glance, must have increased accuracy. The data are available, however, only below and near the threshold. Certain integrals (moments) of the cross section over this domain are particularly sensitive to the low-scale mass \(m_b(\mu)\) with \(\mu\) in the interval 1 to 2\, GeV. A brief description of the method can be found in the review [41]; the details are given in the original papers [63, 64].

The value of \(m_b\) extracted from the fit, taken literally, has a very small uncertainty, and the actual error bars lie in the accuracy of theoretical formulae, in particular, in the perturbative corrections. Nevertheless, the leading Coulomb corrections have been resummed, and the expected dominant BLM-type corrections accounted for. We argued in [41] that the uncertainty in \(m_b(\mu)\) with \(\mu \approx 1\,\text{GeV}\) must not be worse than 30 MeV, and suggested that a \(\pm 50\,\text{MeV}\) assessment yields a safe interval:

\[
m_b(1\,\text{GeV}) = 4.64\,\text{GeV} \pm 0.05\,\text{GeV}.
\]  

The definition corresponds to the concrete renormalization scheme [53, 32, 11] which we consistently use. In other words, the “one-loop” pole mass is

\[
m_b^{(1)} = 4.64\,\text{GeV} + \frac{16}{9} \frac{\alpha_s}{\pi} \cdot 1\,\text{GeV} \approx 4.83\,\text{GeV}
\]  

\[75\]
at $\alpha_s = 0.336$. The result is, of course, sensitive to the choice of $\mu$. The corresponding value of $\Lambda(1 \text{GeV}) \approx 0.6 \text{GeV}$.

The heavy quark masses can be measured, in principle, by studying the distributions in the semileptonic $B$ decays $[33, 66]$. Such analyses were undertaken recently $[67, 68]$. Unfortunately, the data are not good enough yet to yield a competitive determination. On the theoretical side, there are potential problems with higher-order corrections due to not too large energy release $m_b - m_c \approx 3.5 \text{GeV}$ and/or relatively small mass of the $c$ quark. In particular, the effect of higher-order power corrections can be noticeable. The result of analysis reported in Refs. $[67, 68]$ is compatible with the value (74) within the uncertainties. Unfortunately, the results are quoted for the pole mass obtained in a certain approximation and, thus, only the errors of the fit itself are sensible. The theoretical uncertainties were not even addressed consistently.

I think that direct determinations of $m_b(m_b)$ will long suffer from the uncertainties at least 100 to 200 MeV. This is a minimal scale of the second-order corrections in the high-energy measurements. In order to extract $m_b$ one has to consider short-distance observables which essentially depend on $m_b$. Exact calculation of the $\mathcal{O}(\alpha_s^2)$ corrections with massive particles is extremely difficult. Simultaneously, the absolute accuracy in the measurements has to be very good for a competitive determination of $m_b$.

A much larger uncertainty in the $b$ quark mass is often cited in the literature, and it would not be justified to avoid discussing this controversy. For example, Neubert inflates it up to $\pm 300 \text{MeV} [50, 51]$. It is not clear where such an uncertainty comes from; I suspect that it merely refers to the variation of numerical values one would get using different schemes to define theoretically what is the heavy quark mass. It makes no more sense than, say, calling the difference in the values of $\overline{\text{MS}}$ and MS strong couplings at the same scale the uncertainty in $\alpha_s$.

Some scepticism to the determination of $m_b$ rose after Ref. $[69]$ obtained – using the analysis similar to the Voloshin’s $[64]$ – a significantly lower value of $m_b$. The new analysis was not quite correct, however. Its deficiencies were discussed in detail recently in $[70]$. Among other things, it was illustrated that including the missed resonant contributions in the used sum rules reproduces the Voloshin’s numbers. Nevertheless, the general sceptical attitude seems to prevail up to now, and I feel appropriate to present some complimentary arguments why there hardly is a room for large uncertainties here. The controversial estimates once again seem to be related to the attempts to determine nonexisting pole mass of the $b$ quark.

In determination of $m_b$ from $e^+e^- \rightarrow b\bar{b}$ the experimental input is moments $I_n$ of $R_b(s)$ which is related to the cross section. The dispersion relations equate them with the derivatives of the correlator of the $b$ quark vector currents $\Pi_b$ at $q^2 = 0$:

\[
\frac{2\pi^2}{n!} \Pi_b^{(n)}(0) = I_n = \int \frac{ds R_b(s)}{s^{n+1}} \sim
\]
\[ M^{-2(n+1)}_T(1S) \int ds \, R_b(s) \exp \left\{ -(n+1) \left( \frac{s}{M^2_T(1S)} - 1 \right) \right\} \simeq \] (76)

\[ 2M^{-2n-1}_T(1S) \int dE \, R_b \left( (M_T(1S) + E)^2 \right) e^{-E\Delta} , \quad \Delta \simeq m_b/n . \]

The moments \( I_n \) at large \( n \gtrsim 10 \) are very sensitive to the value of \( m_b \):

\[ I_n \propto e^{-2n \frac{m_b}{m_b} \overline{M_T(1S)}} . \] (77)

Moreover, the leading nonperturbative effects are described by the vacuum gluon condensate \( \langle G^2_{\mu\nu} \rangle \), so these corrections are very small unless \( n \) is taken too large to push the relevant momentum scale \( \sim m_b/\sqrt{n} \) down to a hadronic scale. However, one has to pay the price of resumming the enhanced perturbative Coulomb corrections for such a convenience. In the potential interaction the expansion parameter in the nonrelativistic problem is \( \alpha_s/|\vec{v}| \) rather than \( \alpha_s \) itself, and typical \( |\vec{v}| \sim 1/\sqrt{n} \).

The Coulomb corrections also contain numerically large factors. For example, the resummation of the first-order Coulomb exchanges yields the factor ranging from 10 to 50 for \( n = 10 \) to 20 (roughly speaking, it is \( e^{\frac{3}{2}\sqrt{\frac{\alpha_s}{\pi}}\mu} \) in the relevant domain \( \alpha_s/\sqrt{n} \sim 1 \)). Naively, without a precise resummation in the theoretical expressions for the moments one cannot get anything reasonable for the \( b \) quark mass. Even worse, since the effect of resummation is so large, one may naturally harbor doubts whether the higher-order effects left over are really harmless. In my opinion, all such fears are exaggerated.

In reality, the majority of the analyses treated the sum rules for the moments not quite consistently from the basic theoretical perspective, regarding the role of the infrared effects. The OPE states that the (leading) effect of the IR QCD dynamics in the moments \( I_n \) is given by the gluon condensate \( \langle G^2 \rangle \) only if the fixed \( m_b \) is a short-distance mass \( m_b(\mu) \). The Coulomb potential interaction, on the other hand, is resummed in the standard QM way which is written in terms of the pole masses. In this case the naively used OPE statement is not applicable, and the impact of the IR domain on \( I_n \) is much larger for large \( n \). Keeping \( m_b^{\text{pole}} \) fixed in the theoretical expressions rather adds a huge factor \( \sim e^{2n\delta m_b^{\text{pole}}/m_b} \) to \( I_n \), where \( \delta m_b^{\text{pole}} \) is the shift in the pole mass due to switching on the \( \alpha_s \)-corrections in the IR domain of momenta below a certain scale \( \mu \). For essentially one-loop calculations, for instance, \( \delta m_b^{\text{pole}} \sim \frac{\alpha_s}{\pi} \mu \). Any modification of the used potential at large distances (small exchanged momenta) would lead actually to a drastic change in \( I_n \left( m_b^{\text{pole}}(\mu) \right) \). This would be just a reflection of the fact that the moments in terms of the short-distance mass remain the same, and the difference between the pole mass and \( m_b(\mu) \) changes.

If instead one keeps \( m_b(\mu) \) fixed, the dependence of \( I_n \) on the IR part of gluon exchanges disappears. This can be checked explicitly. It is important to remember, however, that this is a remarkable property of QCD. Exchange of massless scalar particles, for example, can perfectly imitate the Coulomb interaction of nonrelativistic particles, but the relative sign of the Coulomb interaction and the corrections
to the self-energy would be the opposite for them. This remark suggests that one should be cautious applying conclusions drawn from *ad hoc* nonrelativistic models to the QCD analysis of the $\Upsilon$ system.

*The quantum mechanical interpretation*

Let us illustrate in the simple language the OPE statements referred to above. The largest and most dangerous in practice IR contribution to the pole mass is linear in the momentum scale, Eq. (60). The expression for the mass shift, actually, is nothing but self-interaction $\frac{1}{2}V_{\text{IR}}(0)$ where $V_{\text{IR}}$ is the heavy quark potential mediated by the gauge interactions with momenta below certain $\mu \ll m_b$ [54, 56]. Yet the mass of the $\bar{b}b$ system includes also the same Coulomb interaction between quark and antiquark. Since for the colorless $\bar{b}b$ state the sum of color “charges” is zero, these effects cancel each other for the quanta with wavelength less than the interquark spacing $r$. Therefore, for the Fourier transform of the potential $V(\vec{q})$ in terms of which

$$V(0) = \int \frac{d^3\vec{q}}{(2\pi)^3} V(\vec{q}) ,$$

only the components with $|\vec{q}| > 1/r$ contribute. The softer exchanges are suppressed by powers of the multipole factor $q^2 r^2$.

This, of course, automatically emerges in all calculations. Let us single out the effect of gluon exchanges with $|\vec{q}| < \mu$:

$$V_{\text{IR}}(r) = - \int_{|\vec{q}|<\mu} \frac{d^3\vec{q}}{(2\pi)^3} V(\vec{q}) e^{-i\vec{q}\cdot\vec{r}} = -V_0 + \frac{1}{6} r^2 \mu^2 V_2 - ... ,$$

$$V_0 = \int_{|\vec{q}|<\mu} \frac{d^3\vec{q}}{(2\pi)^3} V(\vec{q}) , \quad V_2 = \int_{|\vec{q}|<\mu} \frac{d^3\vec{q}}{(2\pi)^3} \frac{q^2}{\mu^2} V(\vec{q}) , \quad ...$$

(the minus sign reflects the fact that the second particle is an antiquark, i.e. negative $C$ parity of the vector current). If quarks reside at distances much smaller than $1/\mu$, the soft potential is just a constant. Its sole role is only to shift the energy of all $\bar{b}b$ states by a constant amount $-V_0$. It does not affect wavefunctions and, therefore, does not modify the coupling of the virtual photon in $e^+e^-$ annihilation to these states.

However, a constant potential $A_0$ cannot change the energy of the neutral system, whether the field is classical or quantum. It means that just the opposite shift is present in the sum of masses of $b$ and $\bar{b}$ renormalized by the same gauge interaction. Indeed, explicit expression (50) shows that $\delta m_b$ amounts to a half of $V_0$, so that the change in the sum of masses cancels the constant term in the potential for closely spaced $\bar{b}b$. Once again the IR effect distinguishing the pole mass from $m_b(\mu)$ disappears from the short-distance quantity!

For the potential mediated by exchanges of a hypothetical scalar $\phi(x)$, the “scalar charges” of $b$ and $\bar{b}$ would have the same sign, so their sum is nonzero. The quark self-energy would not cancel anymore the shift in the potential but rather double
it. In the OPE language this contribution to the moments \( I_n \) (and, eventually, to \( m_Q \)) would correspond to the expectation value of \( \phi^2 \). In QCD the operator \( A_0^2 \) is forbidden by gauge invariance.

The above reasoning, while illustrating the cancellation of the IR effects present in the pole mass, is too naive in many aspects. If these were the only grounds for the QCD analysis, it would have been hardly relevant at all, at least for \( b \) particles. The actual space separation between quarks even in the lowest \( \Upsilon(1S) \) state is not much smaller than \( \Lambda_{\text{QCD}}^{-1} \) which is still strongly affected by the nonperturbative dynamics. For higher states the situation is even less favorable.

Additionally, the relation between the pole mass and the constant term in the soft potential does not hold in QCD even in perturbation theory in high orders, where the definition of the potential itself becomes nontrivial (the subtle large-time effects in the heavy quark potential not governed by the scale of the momentum transfer were discussed long ago in [71]). As a matter of fact, the heavy quark potential is not a truly short-distance quantity [22], either in coordinate or momentum space at \( r \ll \Lambda_{\text{QCD}}^{-1} \) or \( |\vec{q}| \gg \Lambda_{\text{QCD}} \).

Nevertheless, the OPE ensures that the IR dynamics affect determination of the short-distance mass \( m_b(\mu) \) even less than can be concluded from the naive potential description. The reason is that the existing strategy [63] suggested already for charmonium (where perturbative potential treatment is out of question) studies genuinely short-distance inclusive quantities like moments \( I_n \) and not energies or wavefunctions of individual states. This leads, actually, to a limited applicability of the simple potential picture for soft power corrections and to disappearance of unwanted complications mentioned above. For example, there are no effects \( \sim \Lambda_{\text{QCD}}^3 \) (or \( \mu^3 \), in general, for \( \mu \ll m_b/n \)) in the moments \( I_n \) which naively can be anticipated from the curvature \( V_2 \) of the potential in Eq. (79).

The key point is that the heavy quarks are pairly produced from a point with the limited energy \( \Delta \sim m_b/n \); the pair thus exists for typical time \( \frac{1}{\Delta} \). The quark velocity is small, \( v \sim \sqrt{\Delta/m_b} \). Even though the time separation between \( \bar{b}b \) creation and annihilation can be large, \( \gtrsim \Lambda_{\text{QCD}} \), the quarks still cannot move apart to a significant distance, and typical \( r \sim vt \sim 1/\sqrt{\Delta m_b} \sim \sqrt{n}/m_b \) is small compared to \( \Lambda_{\text{QCD}} \). Their interaction with gluon fields is then only dipole, \textit{i.e.} weak, and the pair propagates through an almost transparent medium. This applies to both nonperturbative field configurations and to perturbative gluons with \( |\vec{q}| < 1/r \). Therefore, whatever complicated is the structure of the bound states in an interquark potential or behavior of the wavefunctions, the heavy quark propagation between the points of creation and annihilation will be perturbative.

It is also clear that the applicability of the potential picture for calculating the corrections to the moments in the fiducial domain is limited. The literal potential description is adequate when particles reside long time at a certain distance. This

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12In this analysis “energy” \( \Delta \), time \( t \), velocity \textit{etc.} are Euclidean or, actually, complex. This does not change the qualitative reasoning described here.
is not the case: since the quarks are produced pointlike, they remain close to each other. Already the dipole interaction grows with distance, therefore the static interaction properly accounts for a relatively small ‘middle’ part of the lifetime of the pair. The period of expansion from the point of creation to the typical distances is not static. As a result, the naive potential estimates do not always hold literally.

The actual short-distance OPE expansion is applicable when $\Delta \sim 1/t \gg \Lambda_{\text{QCD}}$. It predicts a small leading IR effect $\sim \langle G_{a\beta}^2 \rangle / (\Delta^3 m_b)$. There is no a potential-generated contribution $\sim \langle r^2 \rangle t \mu^2 V_2 \sim \mu^2 V_2 / (\Delta^2 m_b)$ which would dominate at $t \ll \Lambda_{\text{QCD}}^{-1}$. This can be interpreted as the fact that the soft IR part of the potential even at small distances (large spacelike momentum transfer) depends on the time interval and is suppressed for the potential “switched on” for only a too short period of time. Similarly, the above mentioned long-distance effects in the large-$\vec{q}$ potential $V(\vec{q})$ associated with the large time scale nonlocalities in $V$ do not emerge here.

At $\Delta \ll \Lambda_{\text{QCD}}$ the literal OPE does not apply. Nevertheless one still can use the usual multipole expansion in space coordinates. The effect, in general, is not expressed in terms of local in time expectation values but depends on the time correlators. A simple estimate for the IR effect

$$\langle r^2 \rangle t \langle \int_0^t d\tau \vec{E}(0,0) \vec{E}(\tau,0) \rangle \sim \frac{\langle \vec{E}^2 \rangle}{\Delta^3 m_b}$$

qualitatively agrees with the OPE at $\Delta \sim \Lambda_{\text{QCD}}$. This correction is still small.

It is important that, although the literal OPE is justified at $m_b/n \gg \Lambda_{\text{QCD}}$, the smallness of the IR contributions and applicability of the perturbative treatment of moments $I_n$ requires only $p_b \sim \sqrt{\Delta m_b} \sim m_b/\sqrt{n} \gg \Lambda_{\text{QCD}}$. The energy resolution $\Delta = m_b/n$ can be much better than $\Lambda_{\text{QCD}}$ for large $m_b$, but the duality between the short-distance expressions for the moments and their actual saturation by hadronic states will still hold. The dedicated discussion of the quarkonium production near threshold and OPE can be found in the old paper [72].

After this digression in the meaning of the OPE for the moments of the spectral function let us return to extraction of $m_b$. Does the above understanding help in practice? We know that the moments are sensitive to a short-distance mass $m_b(\mu)$, and it means that the QCD corrections to it must be moderate. The effective scale $\mu$ depends on $n$ and for the fiducial range 8 – 20 used in [64] is 1 to 2 GeV. Then let us, for a moment, leave out the resummation machinery of Voloshin at all and attempt to determine $m_b(1 \text{GeV})$ from simply the tree-level theoretical expressions, in the same range 8 to 20. The result lies in the interval from 4.58 to 4.64 GeV. Even without accounting for any QCD corrections one gets the correct result with a 50 MeV accuracy!

If the “bare” result for $m_b$ is $m_b \approx 4.6$ GeV, one should more consistently evaluate the effect of the Coulomb resummation in $I_n$ comparing them with the (bare) moments evaluated with this short-distance mass. If we do this, the apparent impact of resummation changes dramatically: the resummation increases moments by
a factor ranging only from 3 to 5. In fact, for a more natural, higher renormalization point \( \mu \simeq 1.5 \text{ GeV} \) the impact becomes even smaller and changes a little with variation of \( n \). For \( n \gg 1 \) it is this \( n \)-dependence that pinpoints the value of \( m_b \).

Of course, before incorporating the perturbative effects one cannot accurately check the normalization point dependence of \( m_b(\mu) \) and answer the question to which exactly \( \mu \) such a tree-level determination of \( m_b \) refers. This requires account for the perturbative corrections in the theoretical predictions for the moments. It is clear that this scale lies in the interval 1 to 2 GeV (depending on \( n \)), and is neither \( m_b \) nor what would lead to \( m_b^\text{pole} \).

We see that already the tree-level computation yields the correct value of the short-distance mass \( m_b(1 \text{ GeV}) \) with a 50 MeV accuracy. The precision is further improved by the straightforward Coulomb resummation. The existing analysis is additionally improved by incorporating the next-to-leading \( \alpha_s^{n+1}/|\vec{v}|^n \) corrections and the presumably dominant BLM \( \alpha_s^2 \) effects. Therefore, there is no room in sight for larger uncertainty in determination of \( m_b(\mu) \). This explains why the most labor-consuming analysis of the next-to-leading effects in \( [64] \) which was necessary for getting a sensible determination of \( m_b \), led to a modest change in the numerical value of \( m_b \) about only \( \sim 15 \text{ MeV} \), as compared to the ten-year old analysis of Ref. \( [73] \).

One of the problems inflating the apparent uncertainties is that most of the analyses of the \( \Upsilon \) family attempt to determine \( m_b^\text{pole} \). The uncertainty here, indeed, cannot be decreased below \( \sim 200 \text{ MeV} \). Using the \( \overline{\text{MS}} \) mass, on the other hand, makes mandatory accounting for \( \alpha_s^2 \) effects exactly, which seems unfeasible.

The way which would manifest smallness of the perturbative corrections to extraction of \( m_b(\mu) \) at any stage is to follow the idea of the Wilson OPE more literally. Namely, one first gets rid of the high gluon momenta \( k \gtrsim m_b \) passing to the effective low-energy nonrelativistic theory of \( b \) and \( \bar{b} \). For example, apart from evolving \( m_b \) to a lower scale, a certain finite renormalization of the \( \gamma^* \bar{b}b \) coupling is generated. This is typically accomplished in the existing analyses to a certain extent. However we should also separate out the momenta \( |\vec{k}| < \mu \) from the low-energy theory, in particular from the potential. Their effect on the moments \( I_n \) is small and, if necessary, can be restored by adding the OPE contribution in the form of \( \langle G^2 \rangle^\text{pert}_\mu \) (for small \( n \)) or the perturbative part of the correlators in the dipole expansion. Fitting the moments one would get directly \( m_b(\mu) \).

Since only a relatively small domain of gluon momenta, between 1 to 1.5 GeV and about 4 GeV would remain, the effect of the exchanges to be computed will be very moderate. For example, the potential will be peeled off from both hard and long-range parts, the spectrum of states will be modified mildly, etc.

As a matter of fact, a similar procedure was already done by Voloshin in early 1995. Instead of using the literal Coulomb potential with small non-Abelian corrections, the low-momentum part was removed,

\[
V_\mu(\vec{r}) = \int_{|\vec{q}| > \mu} \frac{d^3\vec{q}}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} V(\vec{q})
\]  

(80)
which yielded a certain $\mu$-dependent potential fast decreasing and oscillating at large distances. Since the problem with the Landau pole was absent at $\mu > \Lambda_{\text{QCD}}$, it was possible to use the all-order BLM-resummed potential instead of only simple BLM scale fixing. The quark Green functions were calculated in this potential, and the theoretical expression for the moments explicitly depended on $\mu$. Fitted $m_b(\mu)$ depended on $\mu$, and the normalization point dependence of mass agreed well with the perturbative $\mu$-dependence similar to Eq. (38).

This analysis just led to the result Eq. (74) (with the uncertainty about 20 MeV); this investigation, however, has not been officially published. (It was referred to already in paper [48], Ref. 23.) In the subsequent papers I always used and quoted this evaluation of $m_b(1\text{ GeV})$; it did not actually differ from the value given in [64] within the stated error bars. I hope that an accurate analysis of this type will eventually appear.

Completing the discussion of determination of $m_b$ I would like to comment briefly on another outcome of the analysis of the near-threshold $e^+e^- \rightarrow b\bar{b}$ cross section. The overall normalization of the moments $I_n$ is also sensitive to the value of $\alpha_s$ at the scale 1 to 2 GeV, which suggests a way to accurately determine it. It was done in [64] and resulted in a value on a smaller side corresponding to $\alpha_s(M_Z) \simeq 0.110$, with a very small quoted error. It is possible, however, that the actual uncertainty in the short-distance $\alpha_s$ is larger here. Since $\Upsilon$'s determine $\alpha_s$ at a relatively low scale, the effect of higher-order terms can be rather significant. In other words, one determines accurately here a certain effective coupling. Although it can be, in principle, related to the standard $\overline{\text{MS}}$ coupling, this relation is not known with enough precision. Moreover, since the $\overline{\text{MS}}$ coupling is quite unphysical being selected to satisfy only the requirements of convenience in technical computations, the higher-order terms in the relation between the two couplings are expected to be large. The perturbative relations can be poorly convergent – if, in particular, $\Lambda_{\text{QCD}}$ happens to be on the higher side of the existing estimates. For this reason, in my opinion it may be premature yet to assign a too small error in the determination of the $\overline{\text{MS}}$ coupling $\alpha_{\overline{\text{MS}}}^s$ evolved to higher energies.

At the same time, it is important that the associated uncertainty in $\alpha_s$ does not affect $m_b(\mu)$ with $\mu \approx 1.5\text{ GeV}$. Other masses like $m_b^{\overline{\text{MS}}}(m_b)$ or various approximations for $m_b^{\text{pole}}$ are bound to exhibit sensitivity to it since involve a significant $\alpha_s$-dependent evolution from that intermediate value of $\mu$.

### 3.3 $\mu^2_\pi$ and $\mu^2_G$

The heavy quark masses $m_c, m_b$, being the key parameters in the HQE, are to a large extent ‘external’ to the properties of the effective low-energy theory itself. There are two nonrelativistic heavy quark operators in the Hamiltonian; their expectation values Eq. (24) in the heavy meson $B$ play a key role in many applications. In contrast to $m_Q$, they are determined by the QCD dynamics itself. We cannot yet calculate theoretically their values from first principles of QCD since it would require
more or less exact solution of QCD in the strong coupling regime. Instead, we can try to measure them extracting from known properties of hadrons.

The value of \( \mu^2_G \) is known: since

\[
\frac{1}{2 m_Q} Q_i^\dagger \sigma_{\mu\nu} G_{\mu\nu} Q
\]

describes the interaction of the heavy quark spin with the light cloud and causes the hyperfine splitting between \( B \) and \( B^* \),

\[
\mu^2_G \simeq \frac{3}{4} 2 m_b (M_{B^*} - M_B) \simeq \frac{3}{4} (M_{B^*}^2 - M_B^2) \approx 0.36 \text{GeV}^2 .
\] (81)

In actual QCD \( \mu^2_G \) logarithmically depends on the normalization point; usual one-loop diagrams yield

\[
\mu^2_G(\mu') \simeq \left( \frac{\alpha_s(\mu')}{\alpha_s(\mu)} \right)^{\frac{3}{8}} \mu^2_G(\mu) .
\] (82)

In the mass relation given above the operator is normalized at the scale \( \mu \simeq m_b \):

\[
\frac{1}{m_Q} \mathcal{H}^{\text{spin}}_1 = -C_G(\mu) \left( \frac{Q_i^\dagger \sigma G Q}{2 m_Q} \right)_{\mu} , \quad C_G \simeq \left( \frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^{\frac{3}{8}} .
\] (83)

Evolving perturbatively to the normalization scale \( \mu \sim 1 \text{GeV} \) slightly enhances the value of \( \mu^2_G(\mu) \), but this effect is numerically insignificant, and we usually neglect it.

The kinetic expectation value \( \mu^2_\pi \) has not been measured yet. It enters various distributions in semileptonic decays or processes of the type \( b \to s + \gamma \). However, it remains rather uncertain at the moment. First attempts to extract it from the semileptonic distributions were reported, however so far the outcome suffers from large, partially artificial uncertainties and is inconclusive. The inconsistency in treating the quark masses propagates to the kinetic operator; since it is a subleading \( 1/m_Q \) effect, making up for this is already more tricky than for \( m_b \) itself.

Historically, the first attempt to determine the average of the kinetic operator from the QCD sum rules was Ref. \cite{74} where a negative (!) value \( \approx -1 \text{GeV}^2 \) was obtained. The treatment of the sum rules was not legitimate, however (the result was eventually retracted by the author.) Shortly after, a more thoughtful application of the QCD sum rules yielded \cite{75} \( \mu^2_\pi \approx 0.6 \text{GeV}^2 \). The prediction was later refined by the authors \cite{76}

\[
\mu^2_\pi = (0.5 \pm 0.15) \text{GeV}^2 .
\] (84)

Meanwhile a model-independent lower bound was established \cite{13, 7, 21, 18}

\[
\mu^2_\pi > \mu^2_G \approx 0.4 \text{GeV}^2
\] (85)

which constrained possible values of \( \mu^2_\pi \). It is worth emphasizing that this inequality takes place for \( \mu^2_g \) normalized at any point \( \mu \), provided \( \mu^2_G \) is normalized at the same
point. For large \( \mu \) it becomes uninformative; so, it is in our best interests to use it at \( \mu = \text{several units} \times \Lambda_{\text{QCD}} \lesssim 1 \text{ GeV} \).

Recently, a new QCD sum rules evaluation of \( \mu_\pi^2 \) was undertaken in \[78\] and was claimed to yield the result \( 0.1 \pm 0.05 \text{ GeV}^2 \); irrespective of the central value, the quoted error bars are clearly unrealistic.

The expectation value of the kinetic operator was estimated in the quark models, with also a controversial spectrum of predictions: the nonrelativistic ISGW model was stated to predict for \( \mu_\pi^2 \) the value \( 0.27 \text{ GeV}^2 \); the relativistic quark model \[79\] gave about \( 0.6 \text{ GeV}^2 \) or even slightly larger; the estimates of Ref. \[80\] yield a close value \( 0.5 \text{ GeV}^2 \).

The origin of the difference between the QCD sum rule evaluations \[75, 76\] and \[78\] was explained in detail in \[41\]. While Ball et al. considered the sum rules for \( \mu_\pi^2 \) directly, the recent approach by Neubert calculated some other transition amplitude which, however reduces to the expectation value of the kinetic operator by equations of motion. In this respect, these two methods \textit{a priori} are equally justified and must have yielded the same result provided the calculations were exact. However, the techniques of the QCD sum rules is based on certain approximations, and they are quite different for these quantities. In particular, it refers to the contribution of the excited states which is a kind of a theoretical background, and has to be suppressed as much as possible.

Not much is known about these contributions in QCD. The simple model considerations suggest \[1\] that the bias can be more significant for the sum rules of \[78\] than in the traditional approach by Ball et al. The general QCD sum rule experience also states that the sum rules operate in better environment when there is a continuum contribution imitating the excited states than when it is absent (as in the correlators considered in \[78\]). The actual scale of associated uncertainty in \( \mu_\pi^2 \) in each approach remains questionable. It is obvious that the actual uncertainty in the analysis of \[78\] is grossly underestimated and in reality must be increased by a large factor. The error bars quoted in \[76\] may also require some revision.

Before mentioning the problem of the quark models, let me briefly discuss the inequality \( \mu_\pi^2 > \mu_G^2 \). Its physical meaning will be explained in Sect. 3.4.1. The most transparent way to illustrate it is based on the sum rules for \( \mu_\pi^2 \) and \( \mu_G^2 \) \[81, 41\]:

\[
\frac{\mu_\pi^2}{3} = \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 + 2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 , \tag{86}
\]

\[
\frac{\mu_G^2}{3} = -2 \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 + 2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 , \tag{87}
\]

where \( \tau_{1/2}^{(n)} \) and \( \tau_{3/2}^{(m)} \) are the standard notations for the transition amplitudes of the ground state \( B \) meson to the excited ‘\( P \)-wave’ states with spin of the light cloud \( j = \frac{1}{2} \) and \( \frac{3}{2} \), respectively (for explicit definition see, e.g., \[41\]). The excitation energies \( \epsilon_k \) are defined as \( \epsilon_k = M_{B_k} - M_B \). The products \( \epsilon_k \tau^{(k)} \) are merely the matrix
elements of the momentum operators $\vec{\pi} = \vec{\pi}_{\text{B}}(i\vec{D})b(0)$ between the ground state and the corresponding excited states $\{18, 41\}$. These sum rules belong to a whole family; I will need later another, so-called optical sum rule $[82]$ for the difference between the meson and the quark masses:

$$\Xi(\mu) = 2 \left( \sum_n \epsilon_n |\tau_{1/2}^{(n)}|^2 + 2 \sum_m \epsilon_m |\tau_{3/2}^{(m)}|^2 \right).$$  \hspace{1cm} (88)

The two first sum rules immediately lead to

$$\mu_{\pi}^2 - \mu_G^2 = 9 \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 > 0.$$  \hspace{1cm} (89)

In QCD both $\mu_{\pi}^2$ and $\mu_G^2$ depend on normalization point $\mu$. The inequality holds for arbitrary $\mu$ as long as renormalization of both operators is introduced in the same way.

The values of $\tau$'s, in particular, $\tau_{1/2}$ are not well known at present. The QCD sum rules $[83]$ as well as the ISGW model $[15]$ predict for the lowest states $\tau_{1/2}^{(1)} \approx 0.25$ to 0.3 while the corresponding $\epsilon_1$ is expected to be about 400 MeV. Assuming $\mu_G^2 \approx 0.4$ GeV$^2$ we then have $\mu_{\pi}^2 \gtrsim 0.5$ GeV$^2$.

### 3.3.1 Kinetic operator in the nonrelativistic models

Let me briefly describe the situation with the nonrelativistic quark models. The calculation of the average square of the spacelike momentum in the ISGW model yield a value close to 0.27 GeV$^2$, notably smaller than the model-independent bound $[14]$ Eq. (89). The reason is that in the nonrelativistic systems the chromomagnetic field $\vec{B}$ is much smaller than $\vec{p}$ and the spin-orbital interaction is negligible. As a result, one additionally has the spectator spin degeneracy of the light cloud states, and $\epsilon_{1/2} = \epsilon_{3/2}$, $\tau_{1/2} = \tau_{3/2}$ for all excitations. The sum rule (87) would yield $\mu_G^2 \approx 0$ in agreement with vanishing $\vec{B}$ in the hadronic scale.

The nonrelativistic constituent quark models typically yield reasonable description of actual mesons despite their admittedly simplified nature. Why does it fail here? The actual reason lies in an implicit assumption of what is the counterpart of the kinetic operator in these models. In QCD $\mu_{\pi}^2$ is the expectation value of $\vec{\pi}$, and $\vec{\pi} = i\vec{D} - \vec{A}$, with the vector gauge potential $\vec{A} \sim \Lambda_{\text{QCD}}$. In the nonrelativistic models $\vec{A} \ll A_0 \sim \Lambda_{\text{QCD}}$ and is neglected, so that the identification of the covariant momentum with the ordinary momentum operator

$$\vec{\pi} \rightarrow i\vec{D} = \vec{p}, \quad \mu_{\pi}^2 \rightarrow \langle \vec{p}^2 \rangle$$  \hspace{1cm} (90)

In the nonrelativistic oscillator model the transition amplitudes vanish for all but the first excitations.

In the field theory where the perturbative radiative corrections are included, certain alternative definitions of quantum operators $O_{\pi}, O_G$ differing in subtraction of the perturbative effects, may modify the inequality. It cannot be violated, however, if the radiative effects are absent or neglected, as in the quark models.
is assumed as a self-evident thing. However, it is by far not as harmless assumption as it might seem. Since the momentum operators commute, one has

$$[\pi_j, \pi_k] = 0, \quad (91)$$

whereas in QCD

$$[\pi_j, \pi_k] = iG_{jk} = -\frac{2}{3}i\epsilon_{jkl} j_l \cdot 0.4 \text{ GeV}^2. \quad (92)$$

The fact itself of noncommutativity of the momentum operators leads to the stated lower bound on $\mu_\pi^2$, and this noncommutativity is an exact statement of QCD. The nonrelativistic quark models like the ISGW model, identifying $\pi_i$ with $p_i$, violate this commutation relation. This is as dangerous as modifying the basic uncertainty principle of QM $[x_j, p_k] = i\hbar \delta_{jk}$.

This, in fact, is the main problem of quark models: the chromomagnetic field is experimentally known to be of primary importance for the properties of the light cloud in $B$ mesons, but it does not naturally fit the nonrelativistic description of valence quarks.

Can the utility of the models like ISGW proved to be reasonable for the wealth of other hadron properties, be rescued? In my opinion, the sum rules provide such a possibility even without actual rebuilding the model like changing its Hamiltonian or explicitly introducing the gauge potential. Indeed, we do not expect the model to yield grossly distorted values of the transition probabilities $\tau^2$ or the excitation energies $\epsilon_k$. Accounting for the relativistic spin-orbital interaction is rather expected to reasonably split the values of $\epsilon^{(k)}_{1/2} \tau^{(k)}_{1/2}$ and/or underestimated $\epsilon^{(m)}_{3/2} \tau^{(m)}_{3/2}$. Let us denote

$$\sum_m \epsilon_m^2 |\tau^{(m)}_{3/2}|^2 = (1 + \alpha) \sum_n \epsilon_n^2 |\tau^{(n)}_{1/2}|^2, \quad \alpha > 0, \quad (93)$$

where $\alpha$ is a positive constant, presumably of order 1. Next, we assume that the ISGW model approximately correctly evaluated the average sum of the products $|\epsilon^{(k)} \tau^{(k)}|^2$. By virtue of the sum rule (86) it amounts to

$$(3 + 2\alpha) \sum_n \epsilon_n^2 |\tau^{(n)}_{1/2}|^2 \approx \frac{\langle p^2 \rangle_{\text{ISGW}}}{3} \approx 0.09 \text{ GeV}^2. \quad (94)$$

Eq. (89) then yields

$$\mu_\pi^2 \simeq \mu_G^2 + \frac{\langle p^2 \rangle_{\text{ISGW}}}{1 + \frac{3}{4} \alpha}. \quad (95)$$

Even varying the value of $\alpha$ from 0.5 to 3 we get $\mu_\pi^2$ ranging from 0.5 GeV$^2$ to 0.6 GeV$^2$ (I put $\mu_G^2 = 0.4 \text{ GeV}^2$).

Of course, the assumptions made above are not very rigorous. Nevertheless, I believe that such a way to cure the deficiency of the nonrelativistic constituent quark
models is reasonable. It is fair to say that the value of $\mu_\pi^2$ in the QCD-compatible model motivated by ISGW is $(0.5 \pm 0.1) \text{ GeV}^2$, unless one is ready to accept that the ISGW model provides a strongly distorted description of $B$ mesons.

It is important to keep in mind that in actual QCD the expectation values of the kinetic operator also depend on the normalization point, the effect which is not present in the simple quark models. The prediction obtained in such models refer to a relatively low normalization point $\mu \approx 0.7 \text{ to } 1 \text{ GeV}$. In HQET an analogue of $\mu_\pi^2$ is called $-\lambda_1$. If the light cloud in the hadrons were a simple QM system without actual gluonic degrees of freedom, then $-\lambda_1$ would be equal to $\mu_\pi^2$. The short-distance (perturbative) effects distinguish them making $\mu_\pi^2$ scale-dependent. In contrast to $\mu_\pi^2$, $-\lambda_1$ is postulated to be a fundamental scale-independent quantity, from which “all perturbative contributions” are subtracted. In this respect, it is a direct analogue of the pole mass of the heavy quark, and is ill-defined as well. It makes no sense to state the exact value of $-\lambda_1$ since it can be chosen arbitrarily within a few units times $\Lambda_{\text{QCD}}^2$. On the contrary, $\mu_\pi^2(\mu)$ is well defined at all $\mu$. At $\mu \gg \Lambda_{\text{QCD}}$ its $\mu$-dependence is given by perturbation theory

$$
\frac{d\mu_\pi^2(\mu)}{d\mu^2} = \frac{4}{3} \frac{\alpha_s}{\pi} \left( e^{-5/3+\ln^2 \mu} \right) - \left( \frac{\pi^2}{2} - \frac{13}{4} \right) \left( \frac{\alpha_s}{\pi} \right)^2 + O(\alpha_s^3). \quad (96)
$$

As a matter of fact, a constructive definition of $-\lambda_1$ has never been given. What is used in the literature is an implicit assumption that one must subtract from $\mu_\pi^2(\mu)$ its formal perturbative part, that is, integrate Eq. (96) down to $\mu = 0$ order by order in $\alpha_s$. It is well known that such a procedure cannot be performed correctly, the resulting series is divergent while its sum is meaningless and the value of $-\lambda_1$ strongly depends on the way one viciously defines the sum or truncates the series. To illustrate it numerically, for $\mu = 1 \text{ GeV}$ the series would look like

$$
-\lambda_1 = \mu_\pi^2(\mu) - \frac{4}{3} \frac{\alpha_s}{\pi} \ln \mu - \frac{13}{4} \left( \frac{\alpha_s}{\pi} \right)^2 + \cdots = \mu_\pi^2(1 \text{ GeV}) - 0.12 \text{ GeV}^2 - 0.36 \text{ GeV}^2 - \cdots; \quad (97)
$$

the higher-order terms grow and constitute already in the low orders shifts larger than the quantity in question. Certainly, the number assigned to $-\lambda_1$ can be arbitrary compared to $\mu_G^2$: no rigorous bound can be derived for the quantity which is not possible to define.

### 3.3.2 Hard QCD and normalization point dependence

The sum rules (86)–(87) express $\mu_\pi^2$ and $\mu_G^2$ as the sum of observable quantities, products of the hadron mass differences and transition probabilities. The observable quantities are scale-independent. How then $\mu_\pi^2$ and $\mu_G^2$ happen to be $\mu$-dependent?

The answer is that in actual quantum field theory like QCD the sums over excited states are generally UV divergent when $\epsilon_k \gg \Lambda_{\text{QCD}}$, and not saturated by a few lowest states with contributions fading out fast at large $\epsilon_k$ as in ordinary QM. The contributions of states with $\epsilon_k \gg \Lambda_{\text{QCD}}$ are dual to what one calculates in
perturbation theory using its basic objects, quarks and gluons. The latter, of course, yield the continuous spectrum and the corresponding transition probabilities do not depend on the initial bound state. They can be evaluated perturbatively using isolated quasifree heavy quarks as the initial state. The final states are heavy quarks and a certain number of gluons and light quarks. The difference between the actual hadronic and quark-gluon transitions resides at low excitation energies.

In order to make the sum rules meaningful, we must cut off the sums at some energy \( \mu \) which then makes the expectation value \( \mu \)-dependent. The simplest way is merely to extend the sum only up to \( \epsilon_n < \mu \); this is the convention I consistently use. For analytic computations it is often convenient to use the exponential factor \( e^{-\epsilon_n/\mu} \), which is essentially the Borel transform of the related correlation functions. The concrete values of \( \mu_n^2 \) and \( \mu_G^2 \) depend, of course, on the adopted scheme. Since at large \( \mu \) the cutoff factors differ only in the perturbative domain, the difference between various renormalization schemes can be calculated perturbatively.

The high-energy tail of the transitions to order \( \alpha_s \) is given by the quark diagrams in Figs. 3 with

\[
\sum_k \ldots \to \int \frac{d^3 \vec{k}}{2\omega}
\]

where \((\omega, \vec{k})\) is the momentum of the real gluon. The amplitudes are just a constant proportional to \( g_s \), and performing the simple calculations we arrive at the first-order term in the evolution of \( \mu_n^2(\mu) \), Eq. (98). Purely perturbatively, the continuum analogues of \( \tau_{1/2} \) and \( \tau_{3/2} \) are equal and a similar ‘additive’ renormalization of \( \mu_G^2 \) is absent.

The perturbatively obtained evolution equations (98), (22) allow one to determine the asymptotic values of \( \tau_{1/2} \) and \( \tau_{3/2} \) at \( \epsilon \gg \Lambda_{\text{QCD}} \):

\[
\sum_{\mu < \epsilon_n < \mu + \Delta} \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 + 2 \sum_{\mu < \epsilon_m < \mu + \Delta} \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 \simeq \frac{8}{9} \frac{\alpha_s(\mu)}{\pi} \mu \Delta, \quad (98)
\]

\[
\sum_{\mu < \epsilon_m < \mu + \Delta} \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 - \sum_{\mu < \epsilon_n < \mu + \Delta} \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 \simeq -\frac{3}{2\pi} \frac{\Delta}{\mu} \left\{ \sum_{\epsilon_m < \mu} \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 - \sum_{\epsilon_n < \mu} \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 \right\}, \quad (99)
\]
The minimal value of $\Delta$, a duality interval depends on the considered energy. When $\mu$ is in the resonant zone, only a little above the first prominent resonances, $\Delta$ is of order $\Lambda_{\text{QCD}}$. At large values of $\mu$ where the spectrum is smooth, $\Delta$ can be taken smaller.

Such asymptotic equations, even incorporating higher-order corrections, are valid only for large enough $\mu$, above the onset of duality where the perturbative expressions are applicable. Although formally $\mu_\alpha^2(\mu)$ and $\mu_G^2(\mu)$ are defined at arbitrary $\mu$, below a certain scale their running is completely different from the perturbative renormalization. In particular, they both vanish when $\mu \to 0$ while staying finite for any value of $\mu$. It is in a sharp contrast with the HQET definition of, say, $\mu_F^2(\mu)$ (it is called $3\lambda_2$) which would exhibit a singularity at some value of $\mu$ where the $\overline{\text{MS}}$ coupling approaches the Landau singularity. The actual running of $\mu_G^2(\mu)$ has nothing to do with the formal perturbative expression $-\gamma_G \mu_\alpha^2(\mu)$ at small $\mu$. The same general remark refers also to the definition of the Isgur-Wise function.

### 3.3.3 Where is the caveat?

Although the inequality $\mu_\alpha^2(\mu) > \mu_G^2(\mu)$ holds for any $\mu$, it may, or may not be practically informative. The value of accounting for the so-called “condensate” nonperturbative corrections when the perturbative effects are present is a dynamic question of QCD, and it cannot be answered a priori by only mathematical manipulations. The dynamic challenge emerges for the sum rules as the question at which scale duality sets in, that is, what is the minimal value of $\mu$ where the perturbatively obtained expression for the amplitudes (in particular, $\mu$-dependence) start to hold for the actual hadrons to a reasonable accuracy. For example, at large enough $\mu$ the inequality $\mu_\alpha^2(\mu) > \mu_F^2(\mu)$ is trivial: the kinetic expectation value is then dominated by the large perturbative piece $\frac{4\alpha_s(\mu)}{3\pi} \mu^2$ while $\mu_G^2(\mu) \sim \Lambda_{\text{QCD}}^2$. At which minimal $\mu$ the value of $\mu_G^2(\mu)$ is already a quantity about 0.4 GeV$^2$? We can be reasonably confident that at $\mu$ as large as $m_b \sim 4$ GeV it holds. Is it still as significant at the scale $\mu \lesssim 1$ GeV, or its value is mainly saturated at $\epsilon_k \gg m_c$, we do not know for sure. This is the basic question we must learn to determine the applicability of the heavy quark expansion.

It is tempting to optimistically interpret the well-satisfies experimentally relation

$$M_D^2 - M_D^* \simeq M_B^2 - M_B^*$$

suggested by the $1/m_{c,b}$ expansion as an evidence for a safe early onset of duality. It should be noted, however, that the same difference persists also for $K^* - K$ and even $\rho - \pi$ splitting, which certainly cannot be attributed to a heavy quark symmetry and must have an independent dynamic origin. The question, therefore, remains open.

If, alternatively, the saturation is delayed, one cannot count on any quantitative application of the heavy quark symmetry to charmed hadrons, including the symmetry relations for the formfactors. I consider the question of how the sum rules
for $\mu_G^2$ (and the similar sum rules for $M_B - m_b$, $M_{\Lambda_b} - m_b$) are saturated one of the most topical issues in the HQE. The first step toward obtaining the answer from experiment has been recently done in [84]. The uncertainties are still large, however. Only the leading $1/m_Q$ terms in the amplitude were considered; all they have $1/m_c$ corrections expected to be quite sizeable. The outcome is yet inconclusive – stretching the uncertainties in one direction it is possible to reasonably saturate $\mu_G^2$ even at as low scale as 600 MeV. The space and time limitations do not allow me to dwell on this issue here in any detail. It is very interesting to understand also on which hadronic states the significantly larger $\bar{\Lambda}$ for the $\Lambda_b$ baryon is saturated. Unfortunately, the necessary experimental information is practically absent here.

3.3.4 Theoretical passions around the kinetic operator

The question of the field-theoretic definition of the kinetic operator used to be a subject of controversial opinions in the literature; even now some incorrect statements remain. It makes sense to briefly address these problems here.

A constructive definition of an effective theory and the composite operators in it, or defining renormalized operators in full QCD requires eliminating the high-energy modes of the quark and gluon fields. It can be done in somewhat different ways. For instance, even fixing the normalization scale to be $\mu$ one still can cut off modes higher than, say, $2\mu$ or $\mu/2$. Of course, the difference can be more essential. This is illustrated in the most transparent way on the example of the perturbative corrections.

Let us consider the kinetic operator $\bar{Q}(i\vec{D})^2Q(0)$ and keep in mind that $m_Q \to \infty$. The operator is perfectly defined at the semiclassical level when the gauge field is a fixed external ($c$-number) function of coordinates. However, the quantum fluctuations of $A_\mu$ generate divergent corrections: interacting with the gauge quanta the heavy quark acquires a recoil which diverges in the ultraviolet when integrated over the frequencies of the quanta. For example, two diagrams in Fig. 4 describe the order-$\alpha_s$ corrections to the operator whose expectation value generally becomes nonvanishing even without any external field:

$$\left(\bar{Q}(i\vec{D})^2Q(0)\right)_{\text{pert}} = \frac{4}{3} g_s^2 (I_a + I_b).$$

The corresponding Feynman integrals are

$$I_a = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{k^2}{k_0^2}, \quad I_b = -\int \frac{d^4k}{(2\pi)^4} \frac{3}{k^2} k_0^2. \quad (100)$$

The integrals are quadratically divergent in the ultraviolet and somehow must be cut off.

The UV regularization can be done in the different ways. If one wants to preserve the QM description of the effective theory, it must be local in time and no cut over the timelike component of $k_0$ of the gluon momentum is allowed (a more detailed
Figure 4: Feynman diagrams contributing to the one-loop renormalization of the kinetic operator $\bar{Q}(i\not{D})^2 Q$. Dashed line denotes gluon, and dark box represents the operator.

discussion can be found in [65]). We can thus insert $\vartheta(\mu^2 - \vec{k}^2)$ in the integrand, that is, cut off over the spacelike components of the gluon momentum. It would preserve such natural properties of the effective theory as positivity of transition probabilities, etc. – even for momenta up to $\mu$. This is the regularization suggested in [58], and the explicit expressions I used in these lectures always refer to it. With such a regularization $3I_a = -I_b$, and Eq. (96) is reproduced.

In calculating the one-loop perturbative diagrams it is often more convenient to impose a cutoff as a function of (Euclidean) $k^2$ – that is, first to perform the Wick rotation $k_0 = ik_4$, and then add the factor $\vartheta(\mu^2 - k^2)$ or any other cutoff function of $\vec{k}^2 + k_4^2$. For example, this is the way the cutoff is introduced in the Pauli-Villars scheme which regulates the integrals subtracting the similar diagrams with fictitious gluons having large masses. The direct computation yields that if, instead of integrating over $k_0$ one first averages over the directions of Euclidean momentum $k$, then $I_a = -I_b$, and the perturbative contribution vanishes.

This fact was noted in [60] where the authors employed essentially the Pauli-Villars–type regularization and observed vanishing of the first-order renormalization of the kinetic operator. It stimulated speculations that this fact could be not accidental but have a deep reason which would ensure vanishing of the renormalization to all orders. These speculations were intensively lobbied for some time (see, e.g. [85]). In particular, they served a ground for the hope that there existed a “proper” regularization scheme in which there were no perturbative corrections to the kinetic operator at all, and which thus would provide a possibility to define unambiguously an “absolute” value for $-\lambda_1$, as it is desired in HQET. In reality, there were no reason for such a miraculous expectation. Eventually, even Neubert had to admit [86] that $-\lambda_1$ is an unphysical and indefinite parameter.

The proponents of such a possibility blamed the physical regularization described above for violating Lorentz invariance – the cutoff over $\vec{k}$ allegedly is not Lorentz-invariant in contrast to the cutoff in $k^2$. The nonvanishing renormalization was attributed to this ‘defect’. Such a claim is erroneous in this context, for $\vec{k}^2$ is as Lorentz invariant as $k^2$ since there is an intrinsic Lorentz vector $v_\mu$ in the problem, and $\vec{k}^2 = (vk)^2 - k^2$. The idea to invent a “Lorentz-invariant” (that is, $v$-independent)

\[15\] Actually, I showed in [65] that this fact holds to all orders in the Abelian theory without light flavors, and up to $\alpha^3$ in QED with light flavors.
regularization is misleading since the kinetic operator itself $\hat{Q} \{ (ivD)^2 - (iD)^2 \} Q(0)$ is explicitly $\nu$-dependent. To state it differently, already the HQET heavy quark field $h_{\nu}(x)$ itself is not Lorentz-invariant by construction Eq. (14), and it obeys the $\nu$-dependent equation of motion $i(vD) h_{\nu}(x) = 0$.

The question of renormalization in higher orders could have been answered calculating the two-loop radiative corrections to the kinetic operator. However, it would require first to define a “proper Lorentz-invariant” cutoff procedure. As explained above, there is no natural way to define it in general, and it could have been done only in the technical way at the diagrammatic level. The naive Pauli-Villars regularization is not possible since the gluon mass violates non-Abelian current conservation; the result becomes gauge-dependent starting two loops where the difference with the Abelian case first energies. Moreover, the momentum itself of a color particle (gluon) is a gauge-dependent quantity. Therefore, the requirement of a “Lorentz-invariance” of the cutoff explicitly conflicts with gauge invariance. Physically, the problem is quite evident: it is impossible in the non-Abelian theory with self-interacting gluons to separate the virtual gluons from the static Coulomb field of the heavy quark, and the latter is Lorentz-noninvariant.

Thus, there is no alternative definition of the kinetic operator in QCD beyond the first loop which would be free of “Lorentz-noninvariance” of the physical scheme of Ref. 18.

An analysis of the two-loop renormalization of the kinetic operator was carried out recently by Neubert 86 in a scheme which was claimed to be free of these problems, in the contradiction with the no-go arguments given above. A closer look reveals, however, that the analysis is wrong in the basic aspects.

According to the expectations, the two-loop diagrams with the non-Abelian gluon interaction did not vanish for any symmetry reason and led to the divergent Feynman integrals which had to be regularized. To assign them a definite meaning the author used a dispersion integral with the cutoff over the energy of intermediate states. It was failed to realize that it was exactly the same regularization procedure that is consistently used in the physical definition criticized by the author, and thus should have been considered not invariant to the same extent. The claims of its “Lorentz-invariance” were wrong.

More to the point, contrary to the assertions of the paper the results are gauge-dependent. It was forgotten that the integrated discontinuities are gauge-invariant only if the external color particles are on-shell and only if all diagrams were incorporated. Both these requirements were not respected in the “regularization” of Ref. 86. In particular, the “vanishing” diagrams discarded for symmetry reasons – if regulated in the same way – would yield nonzero contributions.

As a result, the procedure used in Ref. 86 cannot even be considered as a consistent regularization scheme. Leaving aside the gauge-dependence, one cannot use different schemes for the order-$\alpha_s$ and order-$\alpha_s^2$ corrections or apply different cutoff rules for different groups of diagrams in the single order. Moreover, it must be realized that the regularization of the part of non-Abelian two-loop diagrams was
performed in the same dispersion method as suggested in [18], which would lead to
non-trivial renormalization of the kinetic operator already to order $\alpha_s$. The whole
analysis then makes no sense and is irrelevant even if were made consistently.

Summarizing the “puzzle” of the perturbative renormalization of the kinetic op-
erator, the question of whether there exists any motivated ‘invariant’ non-Abelian
regularization beyond the first loop, and if yes then whether the operator is renor-
malized in this scheme, remains open. The claims of answering it have no grounds.
Personally, though, I do not think it has any practical relevance, and am not sure
in the theoretical relevance either.

A completely defined regularization scheme was suggested and discussed in [18,
59, 22]; another possibility is using lattice versions of QCD with heavy quarks.
The latter is not “Lorentz-invariant” in the discussed sense either. Moreover, it
manifestly violates the rotational invariance. The cutoff over the gluon momenta
in the physical scheme we adopt does not look in high orders as simple as in the
first loop. Moreover, it necessarily has a nontrivial form for particular graphs to
be gauge-invariant. The renormalized operator itself is manifestly gauge-invariant
since can be written in terms of the observable transition probabilities. Finally, the
technical feasibility of such a renormalization scheme was demonstrated in [57, 22]
where the complete two-loop renormalization of the kinetic operator was computed.

3.3.5 Regularization and renormalization

Continuing a more theoretical part of my lectures, I feel necessary to discuss the dif-
fERENCE BETWEEN REGULARIZATION AND RENORMALIZATION. The relevance of this question
is not confined by the heavy quark theory and is more general. The confusion among
these two notions is particularly apparent in heavy quarks, and often is the reason
behind contradicting statements, especially regarding the role of the dimensional
regularization.

In a few words, the regularization is used as a technical tool to make all interme-
diate results of the calculations finite when a finite final answer is split, in the process
of calculations, into simpler pieces which separately are divergent. Therefore, any
consistent regularization is possible and nothing depends on it. Renormalization is
the actual eliminating of actual divergences constructing an effective theory with
a real ultraviolet cutoff. All effective couplings and operators do depend on the
renormalization procedure. I’ll try to explain it here in more detail.

QED, QCD and the SM in general are renormalizable theories (strictly speaking,
they are perturbatively renormalizable). It means that, fixing a number of observ-
ables corresponding to the bare parameters of the Lagrangian we can, in principle,
determine other observables. For example, in QED we fix $m_e, \alpha_{em}(0)$ defined via the
static Coulomb potential at microscopic distances, etc. All interesting cross sections
and probabilities can be expressed in their terms. In renormalizable theories all such
relations are eventually finite.

Technically, however, in perturbation theory we usually proceed indirectly. Namely,
the observables are calculated via the (bare) parameters of the Lagrangian, and then the quantum corrections associated with loop effects are typically UV divergent. This reflects the fact that the complete definition of the theory requires introducing an ultraviolet cutoff, and the bare couplings depend on this cutoff in a divergent way when $\Lambda_{\text{UV}} \to \infty$.

At the same time, the relations between the physical input parameters characterizing the theory and the bare constants are divergent as well. Renormalizability means that the divergencies here are exactly the same and cancel when the two steps are performed together. The presence of the ultraviolet cutoff in any order of perturbation theory affects the final result only via inverse powers of $\Lambda_{\text{UV}}$, so that the dependence on it disappears when $\Lambda_{\text{UV}} \to \infty$.

Subtracting or dividing infinities is not a correct mathematical procedure per se and, if not done accurately, may lead to an arbitrary result. The standard way is to regularize the involved Feynman integrals to make them finite – though dependent on the UV regulator – at any stage. The regularization involves certain modifications of the theory – for example, introducing extra heavy particles in the Pauli-Villars scheme. As long as it is ensured that the final physical relations do not depend on them, such modifications are legitimate. The regularized expressions can be safely manipulated since they are well-defined numbers. The final relations appear to be finite and regularization-independent, all infinities must cancel. The regularization can be removed at the final stage.

The regularization is thus a purely technical element inherent to the standard way to treat the perturbative integrals. It is related to splitting the finite, convergent expressions into separately divergent pieces which are merely easier to calculate. All regularization schemes are equally acceptable since the result does not depend on them. It must be stressed, however, that we speak here about the observables. Similar regularization procedure is often applied also to infrared problems when an observable is split into parts which separately can be IR divergent.

One of the most technically convenient regularization schemes is dimensional regularization (DR) $^{[88]}$. Its idea is to calculate Feynman diagrams in a theory formulated in arbitrary dimension $D$ where the integrals are well-defined. It has an additional advantage of preserving gauge-invariance at any value of the regularization parameter $\epsilon = 4 - D$.

Quite different problems must be solved when one needs to construct an effective theory integrating out the UV degrees of freedom above a certain scale $\mu$. This procedure really modifies the theory at the scale $\mu$ and above. For example, the spectrum of a lattice gauge theory, while describing actual hadrons at small energies, has little in common with the perturbative QCD at energies $E \gtrsim 1/a$. All composite operators in the effective theory are well-defined and do not suffer from divergences. They are renormalized. Even the literal perturbative expansion does not require (UV) regularization in the effective theory at any step. The renormalization depends, on the contrary, on the way the cutoff is introduced. The bare couplings, e.g. $\alpha_s^{(0)}$ are concrete finite numbers in the effective theory, depending on
Since all objects of the effective theory are finite, it can be used also to regularize the perturbative calculations. Although the result explicitly depends on $\mu$, for the renormalizable bare Lagrangians all the dependence resides in the terms inversely proportional to powers of $\mu$ and disappears at $\mu \to \infty$. Moreover, it is often convenient to use $\alpha_s(\mu)$, $m_q(\mu)$ etc., in particular in QCD as an intermediate parameters.

However, can one really use the standard regularization techniques in the opposite direction, namely, to construct an effective theory and define renormalized operators? The answer depends on the scheme and, in general, is “No”. The details of regularization are crucial here, contrary to regularization of purely perturbative calculations. In particular, I’ll demonstrate that the most convenient DR is, as a rule, unsuitable for constructing renormalized operators.

First of all, the main application of the effective theories (say, for heavy quarks) is nonperturbative effects. While the DR perfectly handles regularization of Feynman integrals generated by the perturbative expansion of the formal QCD functional integrals in arbitrary (complex) dimension, it is so far impossible to formulate QCD as a complete quantum field theory including nonperturbative effects in non-integer dimensions. Even if it were done, the physical point $D = 4$ would have been a point of essential singularity and had extremely complicated analytic structure in $\epsilon$ reflecting various phases of the theory. It would not be possible to define extrapolation to $D = 4$ subtracting only a few simple rational functions of $D$.

Thus, it is clear that the DR as a purely perturbative construction helps nothing in defining the nonperturbative operators in an effective theory – a task which, say, the lattice regularization does successfully. One can try to limit consideration to only the perturbative expansion. There the DR proved to be very useful in many aspects like applications of renormalization group. However, even perturbatively the DR typically fails to perform the renormalization of operators in the way necessary for the Wilson OPE. The only exception are operators which are not subject to (powerlike) mixing with lower-dimension operators and have only logarithmic anomalous dimensions. An example of such operator is the chromomagnetic operator $O_G$. Their perturbative normalization is given by the (Euclidean) integrals of the type

$$ J = \int \frac{d^4k}{(2\pi)^4} g_s^2 \frac{1}{k^4} = \int \frac{k^2 dk^2}{16\pi^2} \frac{g_s^2}{k^4}. $$

The denominators are in fact somewhat different in the infrared since they include external particle momenta $p_{\text{ext}}$, and the integrals can be IR convergent. However, they are logarithmically divergent at $k^2 \to \infty$. The DR, roughly speaking, multiplies the integrand by $(k^2/\mu^2)^{D/2}$:

$$ J_D(\mu) = \int \frac{d^Dk}{(2\pi)^D} \mu^{D-4} g_s^2 \frac{1}{k^4} = \int \frac{k^D dk^2}{(4\pi)^{D/2} \Gamma(D/2)} \mu^{D-4} g_s^2 \frac{1}{k^4}. $$

55
At $\text{Re} \, D < 4$ the integral becomes convergent but contains the pole $1/(D - 4)$. The DR in the perturbative calculations suggests merely to subtract from $J_D(\mu)$ this pole. The minimal subtraction scheme of the DR, for example, subtracts the combination $1/(D - 4) + \frac{\gamma}{2} - \frac{1}{2} \ln 4\pi$ containing the Euler-Mascheroni constant:

$$J^R_D(\mu) = J_D(\mu) - \frac{g_s^2}{16\pi^2} \left( \frac{2}{D - 4} + \gamma - \ln 4\pi \right).$$

(103)

In the complete perturbative calculations this UV pole must finally cancel in observables, and ad hoc removing it when passing to $J^R_D(\mu)$ does not change the final answer at $D \to 4$. In the renormalization procedure, on the contrary, we want to define such integrals themselves. The quantity $\bar{J}^R(\mu) = \lim_{D \to 4} J^R_D(\mu)$ is finite and is the renormalized integral in the MS scheme.

Does $\bar{J}^R(\mu)$ has a more transparent meaning? Yes, it is easy to see that it approximately coincides with the original integral Eq. (101) if the integration is cut off in the UV at $k^2 \sim \mu^2$:

$$J^R(\mu) = \int^{k^2 < \mu^2} \frac{d^4k}{(2\pi)^4} \frac{1}{g_s^2 k^4} \quad \text{with} \quad \bar{\mu}^2 = e \cdot \mu^2 \gg \mu^2_{\text{ext}}$$

(104)

(the relation between $\mu$ and $\bar{\mu}$ is not universal and depends also on the order of perturbation theory). Thus, for logarithmic integrals the DR effectively works as an UV cutoff in the effective theory, at least in the low orders.

Unfortunately, in all other cases the DR fails as a renormalization procedure. It can be seen on the example of Feynman integrals encountered in renormalization of the kinetic operator or mass of the nonrelativistic particle, the QCD operator $G^2_{\mu\nu}$ etc. which are power-divergent. Already in one loop simple power integrals of the types

$$J^{(2)} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{g_s^2 k^2}, \quad \int \frac{d^4k}{16\pi^2} \frac{g_s^2}{k^2},$$

etc. appear. They are not regularized by an infinitesimal shift of the dimension. The expressions can be formally defined only at complex $D$ far away $D = 4$, at $\text{Re} \, D = 2$, $\text{Re} \, D = 0$ and so on (depending on the dimension of the operator), where they actually vanish. The only conceivable way to define such integrals as analytic functions of $D$ is to have all them zero. This can be anticipated a priori: these integrals must have a positive dimension of mass which can be provided only by $\mu$ accompanying the coupling $g_s$ at $D \neq 4$. However, $D = 4$ is not a singular point for such integrals.

A direct way to see this fact is to analyze how these ‘renormalized’ integrals would enter calculations of observables. They describe the IR part of the complete Feynman integrals in the domain $k \ll m_Q$. Since the latter are safely IR convergent, regularization merely does not change the IR part and, therefore, includes it completely. No room remains for the additional contribution of such operators in perturbation theory.
As a result, except for the special case of the logarithmic renormalization, the DR merely considers the perturbative integrals for the operators zero irrespective of any details, and this corresponds to setting the Wilsonian cutoff \( \mu = 0 \). In reality in higher loops, for example in the kinetic operator, the resulting integrals by themselves are not sensibly renormalized at any \( D \), even at \( \text{Re}D = 2 \).

To summarize this digression into the theoretical subtleties, one should clearly distinguish regularization and renormalization which mean quite different things. Renormalization introduces nonvanishing corrections even in the finite quantities or theories, where regularization is not necessary. While the DR is a consistent and one of the most technically convenient regularization schemes, it is unsuitable for renormalization of composite operators in the effective theories.

### 3.4 \( |V_{cb}| \) from the Zero Recoil Rate of \( B \rightarrow D^* \ell \nu \)

The concept of the heavy quark symmetry was very important for the evolution of studies of heavy flavor hadrons. The fact of the fixed normalization of the \( B \rightarrow D \) and \( B \rightarrow D^* \) formfactors at zero recoil in the limit \( m_{b,c} \rightarrow \infty \) was of special significance in applications since suggested the method for determinations of \( |V_{cb}| \) from the exclusive \( B \rightarrow D^* \) semileptonic transition near zero recoil. To this end one measures the differential rate, extrapolates to the point of zero recoil and gets the quantity \( |V_{cb}F_{D^*}(0)| \), where \( F_{D^*} \) is the axial \( B \rightarrow l\nu D^* \) formfactor. Since the charm quark is only marginally heavy, it is very important to estimate the corrections, in particular, nonperturbative. The exclusive transition amplitudes are not genuinely short-distance, such transitions proceed in time intervals \( \sim 1/\Lambda_{\text{QCD}} \). Nevertheless, it turns out that the large-distance effects appear in these kinematics only suppressed by \( 1/m_{c,b}^2 \):

\[
F_{D^*}(0) = 1 + O(\frac{\alpha_s}{\pi}) + \delta_{1/m^2}^A + \delta_{1/m^3}^A + \ldots
\]

(106)

The absence of \( 1/m_Q \) corrections was first noted in Ref. [20]. As far as I understand, the reasoning used by Shifman and Voloshin was very simple: let us consider, for example, the vector \( B \rightarrow D \) transition at zero recoil:

\[
\langle D|\bar{c}\gamma_0 b|B\rangle = (M_B + M_D)\left\{1 + \frac{a}{m_c} - \frac{a}{m_b} + \ldots\right\}
\]

(107)

(short-distance effects are neglected). The relative magnitude of \( 1/m_b \) and \( 1/m_c \) terms is fixed since at \( m_b = m_c \) all corrections must vanish identically. \( T \)-invariance, however, says that the coefficients for \( 1/m_c \) and \( 1/m_b \) terms must be equal since \( B \) differs from \( D \) by only the value of the heavy quark mass. Thus, both terms must vanish. This observation was later studied in more detail in [90] and is often called Luke’s theorem. It essentially improves the credibility of this method in spite of certain experimental difficulties.

\[\text{16}\] The first preprint version of the paper [89] discussing the heavy quark symmetry in QCD was given to me by M. Shifman in July 1986.
The task of precision determination of $|V_{cb}|$ from the exclusive transition requires a detailed dynamic analysis of various preasymptotic corrections in Eq. (106). The perturbative part, albeit technically complicated, is at least conceptually transparent. The theory of power corrections is more challenging.

The need in evaluation of the $1/m^2_Q$ corrections in Eq. (106) for practical purposes was realized quite early [14]. In these days the theory of the power corrections in heavy quarks was immature, our knowledge was scarce, so that it was hard to decide even the sign of $\delta_{1/m^2}^A$. The existed opinion stated that the deviations from the symmetry limit must be very small, and suggestions that they could be as large as 10% were categorically refuted [14]. The situation as it existed by 1994, before the OPE-based heavy quark expansion was applied to the problem in [21, 18] was summarized in the review lectures [8]:

$$\eta_A = 0.986 \pm 0.006 \quad \delta_{1/m^2}^A = (-2 \pm 1)\% \quad (108)$$

($\eta_A$ is a purely perturbative factor introduced for this transition in HQET), yielding $F_{D^*} \simeq 0.97$ with a very small error. This estimate was promoted to “one of the most important and, certainly, most precise predictions of HQET”. Nowadays we believe that the actual corrections to the symmetry limit are larger, and the central theoretical value lies rather closer to 0.9 [21, 18]. After heated debates for a few years the estimates we suggested in April 1994 seem to be accepted in the literature.

Regarding the perturbative calculations per se, it was later pointed out [91] that the claimed improvement [92] of the original one-loop estimate was incorrect, and the proper central value is rather $\eta_A \approx 0.965$; such a value was confirmed by complete two-loop $O(\alpha_s^2)$ computation [93].

The existing estimates of the power nonperturbative corrections in $F_{D^*}$ are based on the sum rules for heavy flavor transitions [21, 18]. The validity of the sum rules had been questioned from various perspectives, but now is commonly accepted. The idea of the application of the sum rules to $B \to D^*$ was to consider the “inclusive” transition probability into all possible final states and not limiting them to only $D^*$ we are really interested in, Fig. 1a. Studying the zero-recoil transition we only fix the spacelike momentum $\vec{q}$ carried away by the lepton pair, $\vec{q} = 0$. Since such a sum is an “inclusive” probability, it is a short-distance quantity and is directly expandable in the OPE in powers of $1/m_{c,b}$; the relation is established applying the usual tools – analyticity and unitarity. The idea and technology of the heavy quark sum rules were recently reviewed in [41] and discussed in detail in the original paper [18]. Here I mainly dwell on the results.

Schematically, the zero-recoil sum rule for the axial $\bar{c}\gamma b$ current responsible for

\begin{footnote}
Neubert, for example, claimed that the sum rules were wrong [49, 10] while simultaneously using them. It does not take long to see why those arguments are invalid. Nevertheless, so far he has not retracted these claims.
\end{footnote}
the $B \to D^*$ transition has the form [21, 18]

$$|F_{D^*}|^2 + \sum_{0<\epsilon_i<\mu} |F_i|^2 = \xi_A(\mu) - \Delta_{1/m^2}^A - \Delta_{1/m^3}^A + \mathcal{O}\left(\frac{1}{m^2_{Q}}\right),$$

(109)

where

$$\Delta_{1/m^2}^A = \frac{\mu_g^2(\mu)}{3m^2_c} + \frac{\mu_{\pi}^2(\mu) - \mu_{G}^2(\mu)}{4} \left(\frac{1}{m^2_c} + \frac{1}{m^2_b} + \frac{2}{3m_c m_b}\right).$$

(110)

Here $F_i$ are the axial current transition formfactors to excited charm states $i$ with the mass $M_i = M_{D^*} + \epsilon_i$, and $\xi_A$ is a short-distance renormalization factor, $\xi_A(\mu) \equiv \eta_A^2(\mu)$. Contributions from excitations with $\epsilon$ higher than $\mu$ are dual to perturbative contributions and get lumped into the coefficient $\xi_A(\mu)$ of the unit operator, the first term in the right-hand side of Eq. (109).

The role of $\mu$ is thus two-fold: in the left-hand side it acts as an ultraviolet cutoff in the effective low-energy theory, and by the same token determines the normalization point for the local operators; simultaneously, it defines the infrared cutoff in the Wilson coefficients.

The $1/m^3_{Q}$ corrections to the sum rule $\Delta_{1/m^3}^A$ are also known [11]. The short-distance renormalization factor $\xi_A(\mu)$ is calculated perturbatively. To the first order it was computed in [18], all-order BLM resummation performed in [53]. The most technically involved part of genuine $\mathcal{O}(\alpha_s^2)$ corrections to $\xi_A(\mu)$ was computed in [93], and the complete result given in [87]. The overall short-distance normalization appears to be very small (this is the result of certain numerical cancellations), $\eta_A(\mu) = \sqrt{\xi_A(\mu)} \approx 0.99$ for $\mu \simeq 0.6$ GeV with the uncertainty at a percent level. The Wilson coefficient for the kinetic operator was also computed to the next-to-leading order in [87]; the correction to it is small.

The sum rule Eq. (109) leads to the upper bound:

$$|F_{D^*}|^2 \simeq \xi_A(\mu) - \Delta_{1/m^2}^A - \sum_{0<\epsilon_i<\mu} |F_i|^2,$$

$$-\delta_{1/m^2}^A > \frac{1}{2} \Delta_{1/m^2}^A \geq \frac{M_{B^*}^2 - M_B^2}{8m_c^2} \approx 0.035.$$  

(111)

The last relation is a model-independent lower bound for the $1/m^2$ corrections to $F_{D^*}$ at zero recoil [21].

I pause here to make the following remark. The sum rule Eq. (109) is the most obvious way to see the absence of $1/m^2$ IR corrections at zero recoil. The sum of the transition probabilities has only the calculated $1/m_{c,b}^2$ corrections; no $1/m_Q$ corrections can appear merely because there is no nontrivial local operator with $D = 4$. The excited transition amplitudes are due to the mass-dependent terms in the Hamiltonian, so they start with $1/m_Q$; the probabilities are then only $1/m_Q^2$. A slightly different (fixed $q^2$, that is, a SV rather than zero-recoil) version of this argument was implied by Voloshin and Shifman in mid-80’s when noting the absence
of $1/m_Q$ corrections. In is curious that the fact itself of calculability of this inclusive probability was not discussed in this context, however. A formal QCD derivation not appealing directly to an effective heavy quark Hamiltonian or heavy quark symmetry was given in [18].

Such a way based on the sum rules is advantageous since clarifies the subtlety appearing in the presence of QCD-inherent radiative corrections: the masses used here must be short-distance, not pole masses. This differs from the HQET approach. The perturbative $\mu$-independent factor $\eta_A$ as it is defined in HQET has an infrared $O(\Lambda_{QCD}/m_c)$ part via the dependence on masses. Likewise the difference $F^*_D - \eta_A$ (or analogous ratio) does have $O(\alpha_s/m_c)$ long-distance contribution merely since $\eta_A$ has it. The Luke’s theorem is not valid as it is literally understood in HQET.

One can consider the similar sum rules for $\vec{q} \neq 0$, or for other weak currents. For example, for the transitions driven by the hypothetical zero-recoil pseudoscalar current $J_5 = \int d^3x (\bar{c} i\gamma_5 b)(x)$ one obtains the sum rule

$$\sum_{\epsilon_i < \epsilon_0} |\tilde{F}_i|^2 = (1/2m_c - 1/2m_b)^2 \left( \mu^2_\pi(\mu) - \mu^2_G(\mu) \right) > 0 . \quad (112)$$

This is the field-theoretic analogue of the difference of the sum rules (86) and (87) and of the inequality discussed earlier.

Returning to $F^*_D$, if we want to obtain an actual estimate of the nonperturbative corrections rather than a bound, we need to know something about the contribution of the excited states in the sum rule Eq. (109). Unfortunately, no model-independent answer to this question exists at present. The best we can do today is to assume that the sum over the excited states is a fraction $\chi$ of the local term given by $\mu^2_\pi$ and $\mu^2_G$,

$$\sum_{\epsilon_i < \epsilon_0} |F_i|^2 = \chi \Delta A_{1/m^2} , \quad (113)$$

where on general grounds $\chi \sim 1$. The contribution of the continuum $D\pi$ state can be calculated [21], however theoretically it is expected to constitute only a small fraction of the sum over resonant states. Trying to be optimistic, we rather arbitrarily limit $\chi$ by unity on the upper side; the larger is $\chi$, the smaller is $F^*_D$. In this way we arrive at

$$F^*_D \simeq \eta_A(\mu) - (1 + \chi) \left[ \frac{\mu^2_G}{6m^2_c} + \frac{\mu^2_\pi - \mu^2_G}{8} \left( \frac{1}{m^2_c} + \frac{1}{m^2_b} + \frac{2}{3m_c m_b} \right) \right] - \delta A_{1/m^3} . \quad (114)$$

Assembling all pieces together we get for $\chi = 0.5 \pm 0.5$

$$F^*_D \simeq 0.91 - 0.013 \frac{\mu^2_\pi - 0.5\text{GeV}^2}{0.1\text{GeV}^2} \pm 0.02 \delta_{\text{excit}} \pm 0.01 \delta_{\text{pert}} \pm 0.025 \delta_{1/m^3} . \quad (115)$$

Estimates of the uncertainties in the $O(1/m^3_Q)$ corrections and the contributions from the higher excitations are not very firm and are rather on an optimistic side; they can be larger. With $1/m^2_Q$ corrections amounting to $\sim 8\%$ noticeably smaller
$1/m^3$ effects can be only a result of accidental cancellations. Their more elaborated estimates were done in \[65\]. Altogether we get

$$F_{D^*} \simeq 0.91 \pm 0.06 ,$$

(116)

where the optimistic uncertainty $\pm 0.1$ GeV$^2$ is ascribed to $\mu^2$.

Eq. (115) basically coincides with the estimates given in the original paper [21]. The QCD-based analysis definitely favors a significantly larger deviation of $F_{D^*}(0)$ from unity than those en vogue three or four years ago. Trying to understand the reason behind the difference, I carefully studied the earlier analyses claiming smaller effects. They all ascend to the paper [14]; its subsequent usage was somewhat too liberal, I would say. The analysis of [14] was not actually a kind of a reasonably accurate evaluation but rather a general discussion of the scale of possible effects not pretending on high precision. The effects of the relativistic spin-orbital interaction in light cloud were practically ignored since they did not fit the oversimplified quark model used for the estimates. I think that this was even more essential than the fact itself that the $1/m_Q$ expansion was performed incorrectly at order $1/m^2_Q$ in [14]. It was only in the successive reviews by Neubert where a tentative evaluation of that paper was gradually promoted to the status of the accurate prediction, without new relevant input.

3.4.1 Quantum-mechanical interpretation

In QM the inequality $\mu^2 > \mu^2_G$ can be illustrated in different ways. Formally, it expresses the positivity of the Pauli Hamiltonian [77]:

$$\frac{1}{2m}(\mathbf{\sigma} \mathbf{D})^2 = \frac{1}{2m}\left((i\mathbf{D})^2 - \frac{i}{2}\sigma_G\right).$$

More physically, it shows the Landau precession of a colored, i.e. “charged” particle in the (chromo)magnetic field. Let me recall that the particle Hamiltonian in the magnetic field looks formally like for free particle, $\mathbf{P}^2/2m$, with the only difference that the covariant momenta $\mathbf{P}$ do not commute in the presence of the magnetic field. This ‘uncertainty relation’ leads to the precession. Hence, one has $\langle p^2 \rangle \geq |\mathbf{B}|$. Literally speaking, in the $B^*$ meson the quantum-mechanical expectation value of the chromomagnetic field is suppressed, $\langle B_z \rangle = -\mu^2_G/3$. It completely vanishes in the $B$ meson. However, the essentially non-classical nature of the ‘commutator’ $\mathbf{B}$ (e.g. $\langle \mathbf{B}^2 \rangle \geq 3\langle B \rangle^2$), in turn, enhances the bound which then takes the same form as in the external classical field.

The basic sum rule Eq. (109) also has a transparent interpretation revealed in [18]. From the gluon point of view the semileptonic decay of the $b$ quark is an instantaneous replacement of $b$ by $c$ quark. In ordinary QM the overall probability of the produced state to hadronize to some final state is exactly unity, which is the first, leading term in the r.h.s. of (109). Why then are there any nonperturbative corrections in the sum rule? The answer is that the ‘normalization’ of the weak
current $\bar{c}\gamma_\mu\gamma_5b$ is not exactly unity and depends, in particular, on the external gluon field. This appears as presence of local higher-dimension operators in the current. Indeed, expressing the QCD current in terms of the nonrelativistic fields used in QM one has

$$\bar{c}\gamma_k\gamma_5b \leftrightarrow \varphi_c^+ \left\{ \sigma_k - \left( \frac{1}{8m_c^2} (\vec{\sigma}i\vec{D})^2 \sigma_k + \frac{1}{8m_b^2} \sigma_k (\vec{\sigma}i\vec{D})^2 - \frac{1}{4m_c m_b} (\vec{\sigma}i\vec{D}) \sigma_k (\vec{\sigma}i\vec{D}) \right) + O\left( \frac{1}{m^3} \right) \right\} \varphi_b . \quad (117)$$

The weak current $\bar{c}\gamma_5\gamma_5b$, according to Eq. (117) converts the initial wavefunction $\Psi_b$ into $\tilde{\Psi}$:

$$\Psi_B \xrightarrow{\bar{c}\gamma_5\gamma_5b} \tilde{\Psi} = \sigma_k \Psi_B - \left( \frac{1}{8m_c^2} (\vec{\sigma}i\vec{D})^2 \sigma_k + \frac{1}{8m_b^2} \sigma_k (\vec{\sigma}i\vec{D})^2 - \frac{1}{4m_c m_b} (\vec{\sigma}i\vec{D}) \sigma_k (\vec{\sigma}i\vec{D}) + ... \right) \Psi_B . \quad (118)$$

Then it is easy to calculate the normalization of $\tilde{\Psi}$:

$$\| \tilde{\Psi} \|^2 = \| \Psi_B \|^2 - \frac{\mu_G^2}{3m_c^2} - \frac{\mu_\pi^2}{4} + \frac{2}{3m_c m_b} \right) - ... \right\} \Psi_B . \quad (119)$$

The additional terms are just the nonperturbative correction in the right-hand side of the sum rule.

The similar relation holds for the zero-recoil vector current $\bar{c}\gamma_0b$:

$$\bar{c}\gamma_0b \leftrightarrow \varphi_c^+ \left\{ 1 - \frac{1}{8} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} - \frac{2}{m_c m_b} \right) (\vec{\sigma}i\vec{D})^2 + O\left( \frac{1}{m^3} \right) \right\} \varphi_b \quad (120)$$

and, therefore, the normalization of the resulting wavefunction in these transitions is

$$\left\| \int d^3x \bar{c}\gamma_0b(x) \Psi_B \right\|^2 = \| \Psi_B \|^2 - \frac{\mu_\pi^2 - \mu_G^2}{4} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} - \frac{2}{m_c m_b} \right) - O\left( \frac{1}{m_Q^2} \right). \quad (121)$$

The first two $1/m_Q^2$ terms in the bracket in Eq. (117) or terms $1/m_c^2$, $1/m_b^2$ in Eq. (120) are the result of the Foldy-Wouthuysen transformation Eq. (16). Correspondingly, they were missed in the standard HQET analysis used, for example, in [14]. The dominant effect $1/m_c^2$ was thus basically lost. It would be correctly reproduced in the approach of the Mainz group [5]; however, the issue was not elaborated in enough detail for practical applications. The above inconsistency was first noted by A. Le Yaouanc in 1994 [16]. It should be noted that the Neubert’s current numbers for $\delta_{1/m^2}^A$ and $F_{D^*}$ ascending to [43] and quoted in later reviews, and which
are routinely cited by experimentalists, were derived in [49] using the same incorrect expansion. The mistake has not been corrected yet. It was the origin of the claim of insensitivity of the analysis to the value of $\mu^2$ which clearly contradicts Eq. (115).

The QM interpretation of the sum rules explains why the perturbative factor $\sqrt{\xi_A(\mu)}$ is the short-distance renormalization translating the full QCD axial current into the current in the effective theory. Often the $\mu$-independent $\eta_A$ is called a short-distance factor. It is not quite correct, $\eta_A$ does not depend on the choice of the separation scale $\mu$ determining which effects are considered as short-distance and which are still included into the low-scale physics. It would be possible only if the perturbative corrections at arbitrary scale vanish, which is not the case. In reality $\eta_A$ is a mixture of short-distance and long-distance effects. Since, nevertheless, it is defined perturbatively, it suffers from the IR renormalon uncertainties similar to those discussed for the pole mass. They are too significant in $\eta_A$.

3.5 Summary on $|V_{cb}|$

I would like to give here a brief summary of the two methods of extracting $|V_{cb}|$. The most precise at the moment is the value obtained from $\Gamma_d(B)$: Eqs. (51) and Eq. (52) with the central experimental input values shown there lead to

$$|V_{cb}| \simeq 0.0419 \cdot \left(1 - 0.012 \frac{(\mu^2 - 0.5 \text{GeV}^2)}{0.1 \text{GeV}^2}\right).$$

The overall relative theoretical uncertainty in this result is $\delta_{th} \lesssim 4\%$. With future refinements implementing already elaborated strategy, we can expect reducing it down to 2%. It is important that I am speaking here about the defensive theoretical accuracy, the error bars that are assumed to cover the whole interval where the exact number can be. I think that the theoretical predictions – to the extent they are not affected by independent experimental input subject to additional statistical uncertainties – should have such a status. The way when the previous predictions are praised for being “only 2.5σ away” from a posteriori updated estimates, seems to be inappropriate evaluation of the theoretical analysis.\footnote{The idea that the theoretical predictions can be wrong – outside the stated intervals – in almost a half of cases sounds somewhat humiliating.}

To allow for such a confidence, one should not combine different theoretical uncertainties in quadrature as it is accustomed with statistical fluctuations, and I rather added them linearly.

The $B \to D^*\ell\nu$ zero-recoil rate also provides a good accuracy. The exact experimental status of the measurements extrapolated to $q^2 = 0$ is not completely clear to me at the moment. I heard different opinions, and leave the final word for the experimental experts. Using the reported average value and the estimate $F_{D^*} \simeq 0.9$ a somewhat lower value of $|V_{cb}| \simeq 0.038$ seems to emerge. It is interesting that the central theoretical value, according to Eq. (115) exhibits the dependence on $\mu^2$.\footnote{The idea that the theoretical predictions can be wrong – outside the stated intervals – in almost a half of cases sounds somewhat humiliating.}
similar in magnitude but opposite in sign to Eq. (122):

\[ |V_{cb}| \simeq 0.038 \cdot \left( 1 + 0.014 \frac{(\mu^2_{\pi} - 0.5 \text{ GeV}^2)}{0.1 \text{ GeV}^2} \right). \] (123)

The theoretical uncertainty \( \delta_{\text{th}} \) here constitutes probably \( \delta_{\text{th}} \approx 6\% \), however this estimate relies on additional mild theoretical assumptions. It is not clear how at the moment it can be decreased. Speaking of feelings, personally I think that \( F_{D^*} \) larger than 0.95 or below 0.85 would be unnatural. Even the end points of this interval look extreme. However, the existing analyses do not allow to forbid them confidently, and the feeling itself may be merely a result of a historically formed common belief rather than based on scientific facts.

It is remarkable that the values of \( |V_{cb}| \) that emerged from exploiting two theoretically complementary approaches are very close. The progress was not for free: it became possible only due to essential refinements of the theoretical tools in the last several years, which prompted us, in particular, that the zero-recoil \( B \to D^* \) formfactor \( F_{D^*} \) is probably close to 0.9, significantly lower than previous expectations. The decrease in \( F_{D^*} \) and more accurate experimental data which became available shortly after, reduced the gap between the exclusive and inclusive determinations of \( |V_{cb}| \).

A few years ago such an accuracy and agreement would be considered as an important success. Nowadays we are more demanding to the heavy quark theory. The difference in the two values of \( |V_{cb}| \) from a certain perspective may look as a hint for the discrepancy: the central value of \( |V_{cb}| \) from \( B \to D^* \) decay seems to be somewhat lower than that from \( \Gamma_{sl}(B) \). Since both theoretical values depend to a certain extent on the precise magnitude of \( \mu^2_{\pi} \), it is tempting to think that the actual value of \( \mu^2_{\pi} \) is somewhat larger than the “canonical” 0.5 GeV\(^2\). Increasing it by about 0.2 GeV\(^2\) makes the two results much closer. This would not really contradict known facts. I hasten to add, though, that the existing experimental error bars are such that speculations about adjusting \( \mu^2_{\pi} \) are premature. Moreover, theoretical uncertainties in the exclusive formfactor also preclude us from the above adjustment of \( \mu^2_{\pi} \) (\( F_{D^*} \) can well be, say, 0.87 even at the canonical value \( \mu^2_{\pi} = 0.5 \text{ GeV}^2 \)). The ways to sharpen the theoretical predictions of \( \Gamma_{sl} \) are more or less clear. Future accurate measurements will, hopefully, allow one to directly measure – through comparison with \( \Gamma_{sl}(B) \) – the exclusive formfactor with accuracy better than that achieved by today’s theory. Thus, we will get a new source of information on intricacies of the strong dynamics in a so far rather poorly known regime.

Changing the parameters like \( \mu^2_{\pi} \) per se within reasonable intervals is not tightly limited at the moment. However, it may well imply other related changes – the size of \( 1/m_c \) corrections for formfactors at \( \vec{v} \neq 0 \), delaying onset of duality with smaller values of \( \mu^2_{\pi}(0.8 \text{ GeV}) \), etc. Not all these links are properly taken into account in the analyses. The comprehensive study is still to be done. One element is already clear. In most of the existing theoretical analyses it is essentially assumed
that the approximate duality between the actual hadronic amplitudes and the quark-gluon ones sets in already at the excitation energies $\sim 0.7$ to $1$ GeV. While there are no experimental indications so far that this is not the case (at least, in the semileptonic physics), the proof is not known either. If that is not true, and duality starts only above $1$ GeV, most probably one would have to abandon the idea of accurate determination of $|V_{cb}|$ from the exclusive $B \to D^{(*)}$ transitions. The only option still open will be the inclusive semileptonic decays where the energy release is large, $\sim 3.5$ GeV. Of course, in such a pessimistic scenario the theoretical precision in $|V_{cb}|$ will hardly be better than $5\%$.

3.5.1 Comments on the literature

Since the question of the accurate determination of $V_{cb}$ is very topical, there is a wide spectrum of opinions here. Speaking to the experimental colleagues, it would not be fair to avoid mentioning alternative suggestions. Over the last few years with the developing of the genuinely QCD-based methods, general convergence of central values in the estimates is clearly observed. The controversy mainly shifted to the complicated issue of estimating theoretical uncertainties.

Those who are familiar with ongoing debates can notice that my assignments for the theoretical uncertainties in $\Gamma_{sl}$ are on the lower side of the spectrum. For the $B \to D^{(*)} \ell \nu$ zero recoil rate they are closer to common estimates (with the exception of Neubert who pretends to a significantly more accurate prediction). The smaller uncertainties in $\Gamma_{sl}$ are anticipated a priori. First, $\Gamma_{sl}$ is a genuinely short-distance quantity while $F_{D^*}$ is not, starting $1/m_c^2$ effects. Second, the normalization of $F_{D^*}$ rests on the heavy quark symmetry and thus suffers from downgrading the expansion parameter: it is $1/m_c$ rather than $1/m_b$. The task of the QCD-based HQE was to make full use of these advantages.

Let me first dwell on $B \to D^{(*)} \ell \nu$. In general, the statements of a good accuracy here have been merely transferred from the old reviews [8, 94] claiming three times smaller corrections than today and, correspondingly, small error bars. While the central value quoted by the author went, in a few iterations, down to the current prediction for $F_{D^*}$, a similar due revision did not affect the suggested error bars. A closer look shows that they are not better motivated than before.

The central value quoted by Neubert now does not differ at a discussible level from the original estimates of the heavy flavor sum rules [21]. The analysis is based anyway on the very same relations and assumptions. It is stated, in addition, that the paper [12] where the sum rule analysis was reproduced, has been complimented by some symmetry relations; this was called later the “hybrid” approach by the author. In fact, the sum rules manifestly respect the heavy quark symmetries, and the latter cannot provide any further constraint. Careful studying of the analysis of [12] may seem complicated due to presence of numerous parameters and notations as well as contours on plots called upon to make numerical conclusions looking convincingly. It can be done nevertheless, and shows that the analysis of the $1/m^2$ terms merely
was not correct. The flaws were briefly discussed in [11] and were conceptually related to missing the contributions from the Foldy-Wouthuysen transformation.

The basic points which led Neubert to the small uncertainties were the following:

- discarding $1/m^3$ and higher-order corrections altogether
- ad hoc fixing the value $\mu_π^2 = 0.4$ GeV$^2$ in the analysis
- the claim of insensitivity of the result to $\mu_π^2$ related to missing the term $(\mu_π^2 - \mu_\pi^2)/m_\pi^2$.

Curing these flaws returns one to the estimates given in Sect. 3.4.

Another flaw of the estimates by Neubert is improper account for the radiative corrections. The formfactor $F_{D^*}$ was judiciously split into two parts. The first was “purely perturbative” factor $\eta_A$, and all the rest was just called the nonperturbative corrections:

$$\delta_{1/m^2}^A + \delta_{1/m^3}^A + \ldots \equiv F_{D^*} - \eta_A;$$

$\eta_A$ was defined as a sum of the formal perturbative series calculated without any infrared cutoff. Such a quantity cannot, however, be consistently defined even if our knowledge of the perturbative coefficients were practically unlimited. The associated numerical uncertainties are quite significant. Even adopting an optimistic convention for the uncertainty with the ad hoc factor $1/\pi$ in front of the imaginary part of the Borel-resummed series (like in Eq. (65)), one gets for the $1/m_Q^2$ and $1/m_Q^3$ irreducible uncertainties

$$\delta_{1/m^2}^{IR} \approx \frac{2}{27} \left( \frac{\Lambda_{QCD}^{MS}}{250 \text{ MeV}} \right)^2 \left( \frac{2}{m_\pi^2 + 1} \right)^2 \approx 0.022 \left( \frac{\Lambda_{QCD}^{MS}}{250 \text{ MeV}} \right)^2,$$

$$\delta_{1/m^3}^{IR} \approx \frac{5}{72} \left( \frac{\Lambda_{QCD}^{MS}}{250 \text{ MeV}} \right)^3 \left( \frac{11}{m_\pi^2} + \frac{5}{m_\pi^2 m_\tau} + \frac{5}{m_\pi^2} + \frac{11}{m_\tau^3} \right) \approx 0.014 \left( \frac{\Lambda_{QCD}^{MS}}{250 \text{ MeV}} \right)^3,$$

respectively (higher-order IR renormalons are smaller). Thus, the uncertainties in both “perturbative” and “nonperturbative” corrections defined in such a way somehow appeared smaller than their irreducible intrinsic uncertainty.

To avoid confusion I must note that Eq. (125) was obtained in [11]. This, however, contradicted the claimed high accuracy of calculating $\eta_A$ [12]. Hence, already in the next papers the numerical estimate for this uncertainty was skillfully reduced by a significant factor. It was achieved by simple two-step manipulations: the value of $\delta_{1/m^2}^{IR}$ was first expressed via the square of the relative IR uncertainty in the mass, Eq. (65). Then a rather low numerical uncertainty in the mass was used. In this way $\delta_{1/m^2}^{IR}$ seemed to emerge only 0.006, to the author’s satisfaction [13, 95]. Using the direct formulae, however, one gets about 4 times larger uncertainty. For this reason

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19 The insensitivity was illustrated by varying $\mu_\pi^2$ in a very narrow interval from 0.36 GeV$^2$ to 0.5 GeV$^2$. This allowed to bury the impact of changing $\mu_\pi^2$ under uncertainties in other model parameters.
the two expressions for $\delta_{1/m^2}$ in $\eta_A$ and $\delta_{IR}$ in $m_{Q}^{pole}$ were never quoted together, and
the uncertainty $\delta_{IR}^{1/m^2}$ was always numerically evaluated by Neubert in this “indirect way” via more or less arbitrarily defined $\delta m_{Q}^{pole}$ rather than using the known explicit expression.

The practical inconsistency of the approach adopted in [49] is well known. Physically it leads to double counting of the large-distance domain – once as a part of the perturbative effects, then in the nonperturbative corrections. Numerically it is quite significant in the case of $F_{D^*}$. For example, Ref. [96] pointed out that the bounds used by Neubert for $F_{D^*} - \eta_A$ are too strongly offset by the perturbative corrections. In fact, the theoretical inconsistency of such an approach were even noted by the author himself in [49], where the results were stated conditionally valid provided the corrections would not alter them significantly (according to [96], they just do). This qualification has never appeared later in the numerous author’s papers quoting [49].

In any case, it is easy to see that the bounds of the type used in [49] are justified only if the perturbative factor $\xi_{1/2}(\mu)$ is used instead of $\eta_A$, see Eqs. (109, 111). This short-distance factor is well calculated now [87], and is significantly larger than the value used by Neubert. It is not clear whether the author plans to revise those inconsistent estimates, and what will be the central value of the updated numbers. Taking everything at face value, the central value for $F_{D^*}$, according to Neubert, must have been close to 0.94.

Now, let us turn to $\Gamma_{sl}$. In spite of the accurate experimental data and a well-developed theoretical description, determination of $V_{cb}$ in this way often unjustifiably discarded being ascribed inflated uncertainties. As I mentioned before, the origin was the misconception of the pole mass flourishing in HQET a few years ago. In particular, it was tacitly assumed that the quark pole mass has an unambiguous value. Then it was observed that:

(a) It is difficult to accurately extract $m_{b}^{pole}$ from experiment. In any particular calculation one can identify effects left out, which can change its value by $\sim 200$ MeV. This uncertainty leads to a theoretical error $\delta_1 \simeq 10\%$ in $\Gamma_{sl}(B)$.
(b) When routinely calculating $\Gamma_{sl}(B)$ in terms of the pole masses, there are significant higher order corrections $\delta_{II} \simeq 10\%$.

Thus the conclusion was made: $\Gamma_{sl}(B)$ cannot be calculated with accuracy better than $\delta_1 + \delta_{II} \sim 20\%$, and, correspondingly, at best $\delta |V_{cb}|/|V_{ub}| \sim 10\%$ [49].

Both observations (a) and (b) above are correct, beyond any doubt. If one more step is taken, however, the conclusion is invalidated – one should take into account the fact that the origin of these two uncertainties is actually the same, and therefore they practically cancel each other.

Since the nature of the heavy quark in the QCD-based HQE was clarified in [76], the numerical relevance of these theoretical facts was illustrated in a number of publications [48, 49, 77, 98, 92]. Nowadays only the above two problems allegedly impair the precise calculation of $\Gamma_{sl}$: the danger of large perturbative corrections and the uncertainty in $m_{b}$. The anatomy of large corrections and their cancellation
was described recently, for example, in the review [41]. Here I would like to give a complimentary illustration.

The HQE in QCD expresses the width in terms of the short-distance mass $m_b(\mu)$ with $\mu \simeq 1\,\text{GeV}$. Such a mass is well defined and has a certain value. We do not know it precisely, of course. The HQE assumes, nevertheless, that it can be extracted from experiment with theoretically unlimited accuracy. Let us adopt here the determination from the moments of $e^+e^- \to b\bar{b}$ cross section discussed in Sect. 3.2.3. Thus, we calculate the semileptonic width directly via $V_{cb}$ and $\sigma_{e^+e^-}^{b\bar{b}}(s)$.

Reference to the quark mass is made only for computational purposes. Let us also neglect the small nonperturbative corrections at this point and fix the value of $m_b - m_c$. Both $\Gamma_{sl}$ as a function of $m_b(\mu)$ and the mass $m_b(\mu)$ itself as expressed via $\sigma_{e^+e^-}^{b\bar{b}}(s)$ undergo perturbative corrections. How important is their overall impact?

To see this, we can first neglect the perturbative corrections altogether. It is important that it must be done at both stages of extracting $V_{cb}$ simultaneously: it is not legitimate to keep $O(\alpha_s)$ correction in one part and discard it in another part of a single relation. If we neglect them consistently, we would get a certain value which we can denote $|V_{cb}^{(0)}|$. It is somewhat smaller than quoted in Eq. (51). Already here the two competing effects are apparent: we would use a smaller $m_b \simeq 4.60\,\text{GeV}$ but, at the same time, discard negative perturbative corrections to the width.

Now we include $O(\alpha_s)$ corrections. Again, we must use the same (not running in the first order) value of $\alpha_s$ at both stages. We get a new value, $|V_{cb}^{(1)}|$. It appears that

$$|V_{cb}^{(1)}|/|V_{cb}^{(0)}| \simeq 1.06.$$  \hspace{1cm} (127)

The effect of the leading perturbative corrections changes $|V_{cb}|$ at a 5% level only! This numerical result is easy to interpret also in terms of the usual calculations expressed via the pole masses: the first-order radiative effects shift $m_b^{\text{pole}}$ from 4.60 GeV up to 4.82 GeV, but bring in the perturbative suppression factor about 0.8.

One can incorporate the BLM improvement, that is, account for running of $\alpha_s$, again in strictly the same way in $\Gamma_{sl}$ and $\sigma_{e^+e^-}^{b\bar{b}}(s)$. The impact of, e.g., the second-order corrections changes $|V_{cb}|$ by less than a percent [48]. And this magnitude of the $O(\alpha_s^2)$ effects is quite expected if the $O(\alpha_s)$ correction was only a few percent. The all-order BLM resummation does not change this conclusion.

The complete $O(\alpha_s^2)$ analysis of extracting $m_b(\mu)$ from $\sigma_{e^+e^-}^{b\bar{b}}(s)$ has not been done yet, and we cannot literally extend this comparison to the non-BLM effects at order $\alpha_s^2$. Nevertheless, it is difficult to imagine how a significant effect exceeding a percent level can emerge from it.

The usually cited ‘naive’ evaluation of the magnitude of the perturbative corrections in the width differs from the above in one but quite important aspect. It forgets that the pole mass $m_b$ routinely used there has to be perturbatively calculated anew in every order, with the numerical effect similar to Eq. (66). Instead it is treated as a fixed value. It is reasonable to ask – which exactly? The answer is not clear, since only the “correction factor” is quoted. And the corrections look significant indeed.
It only remains unclear to a reader how with these huge higher-order correction the final number quoted for $|V_{cb}|$ differs from what had been obtained without any refinements [21] by less than 1%, and even neglecting the QCD corrections altogether yields a 5% accuracy. The reason is explained above.

3.5.2 Analyticity and unitarity constraints on the formfactor

Extracting $|V_{cb}|$ from the exclusive decays implies extrapolating data to zero recoil. Near zero recoil statistics in the decays $B \to D^* \ell \nu$ is very limited, and the result for the differential decay rate at this point is sensitive to the way one extrapolates the experimental data to $\vec{q} = 0$. Most simply this is done through linear extrapolation. Noticeable curvature of the formfactor would change the experimental value for $|V_{cb} F_{D^*}(0)|$. Under the circumstances it is natural to try to get independent theoretical information on the $q^2$-behavior of the amplitude.

Some time ago it was emphasized [99] that additional constraints on the $q^2$-behavior follow from analytic properties of the $B \to D^*$ formfactor considered as a function of the momentum transfer $q^2$, combined with certain unitarity bounds. In particular, in the dispersion integral

$$F(q^2) = \frac{1}{\pi} \int ds \frac{\text{Im} F(s)}{s - q^2}$$

(128)

the contribution of the physical $s$-channel domain $q^2 > (M_B + M_{D^*})^2$ is bound since $|F|^2$ describes the exclusive production of $BD$ and cannot exceed the total $b\bar{c}$ cross section. The integral (128) receives important contribution from the domain below the open $b\bar{c}$ threshold, where $F(q^2)$ has narrow pole-like singularities corresponding to a few lowest $b\bar{c}$ bound states. Introducing a set of unknown residues and making plausible estimates of the positions of these bound states, on the one hand, and adding a small non-resonant subthreshold contribution, on the other hand, Refs. [100] suggested an ansatz for the formfactor in the whole decay domain. Since the residues are unknown and the momentum transfer in the actual decays $B \to D^{(*)} \ell \nu$ varies in a rather narrow range, the advantages of this parameterization over the standard polynomial fit are not clear at the moment.

A much more stringent relation between the slope and the curvature of the formfactor (i.e., $F'(q^2)$ and $F''(q^2)$ at zero recoil) was claimed in [101]. Unfortunately, this relation was hastily incorporated in some experimental analyses. It was shown in [98, 11] that the analysis of Ref. [101] was erroneous, and correcting the flaws kills the predictivity of the relations discussed there. This conclusion was later confirmed in [102].

A different few-parameter ansatz for the shape of the effective $B \to D^*$ formfactor was also suggested in [102]. Although it seems to be one of reasonable possible ansatze, its model-independence was exaggerated. The authors relied too heavily on the approximate heavy quark symmetry in the threshold region. The indirect form in which these constraints were imposed hid this fact. Additionally, the shape
is sensitive to the exact structure of all the subthreshold $b\bar{c}$ states, which were taken from potential models whose reliability for the higher excited states is suspicious. A detailed consideration reveals that one should allow for larger corrections to the parameterization suggested in [102].

3.5.3 On the $1/m_Q$ corrections to the widths and duality violation

The question of the asymptotic expansion of the inclusive widths at the nonperturbative level was not too obvious, as illustrated in Sect. 3.1. Even though the HQE answered it, this question is still occasionally raised in the literature. The suggestions of possible $1/m_Q$ nonperturbative effects in different form still flash every now and then [9, 31, 30, 28, 29]. The experimental evidences that $\text{BR}_{s\ell}(B)$ and the lifetime $\tau_{\Lambda_b}$ seem to be lower than suggested by theory additionally provoke such speculations.

There are a few reasons behind such a doubt. One of them lies purely in technical aspects. The traditional approach of HQET to the semileptonic widths [104] is to calculate only the differential semileptonic width $\Gamma_{s\ell}(q^2)$ where the lepton invariant mass $q^2$ is fixed. The total width is finally obtained integrating $\Gamma_{s\ell}(q^2)$:

$$
\Gamma_{s\ell} = \int_0^{(M_B-M_D)^2} dq^2 \Gamma_{s\ell}(q^2) \quad (129)
$$

(the case of $b \rightarrow u$ is simply obtained by setting $M_D = m_c = 0$). At not too large $q^2$ this differential width indeed can be easily calculated including the leading nonperturbative corrections. For example, up to $1/m^2$ effects it is given by

$$
\Gamma_{s\ell}(q^2) = \frac{G_F^2|V_{cb}|^2}{96\pi^3} \left\{ \frac{(m_b^2-m_c^2)^2 + q^2(m_b^2 + m_c^2) - 2q^4}{m_b^5} \left[ (m_b^2 - m_c^2 - q^2)^2 - 4m_c^2q^2 \right]^{1/2} \right. \\
+ m_b^3 \mathcal{O} \left( \frac{\Lambda_{QCD}}{(m_b - m_c - \sqrt{q^2})} \right) \right\} \quad (130)
$$

However, at $q^2$ approaching $(m_b - m_c)^2$ the $1/m_Q$ expansion breaks down and the differential width cannot be evaluated. In this soft domain no literal $1/m_Q$ expansion is available. For example, with both $b$ and $c$ quarks heavy enough, at maximal $q^2$ the actual hadronic width differs from the quark width by the universal factor

$$
\frac{\Gamma_{\text{had}}(q^2)}{\Gamma_{\text{quark}}(q^2)} = 1 + \frac{3\mathcal{A}(m_b - m_c)}{2m_bm_c} \quad (131)
$$

which explicitly includes $1/m_c$ correction. Although decreasing $q^2$ somewhat diminishes the ratio, even after the $D^{**}$ threshold the deviation is still of a similar order, i.e. scales like $1/m_Q$.

At the same time, the overall contribution to the semileptonic width from this domain of large $q^2$ is a priori at the relative level of $\Lambda_{QCD}/(m_b - m_c)$, so it has to be
calculated to address even $1/m_Q$ effects. Moreover, the upper limit of integration over $q^2$ is clearly dictated by the hadron rather than quark masses; this is particularly manifest for $b \to u$ where $M_D \to m_q \simeq 0$. Naively one could have expected the width to depend, at least partially, on the hadron rather than quark masses.

This problem of the HQET approach underlies the reason why, for example, the paper [104] allowed for the possibility to have $1/m_Q$ nonperturbative effects even in the semileptonic widths. (For the nonleptonic widths such an approach is not applicable at all.) In this language the QCD statement of vanishing such nonperturbative effects is quite nontrivial: while the width $\Gamma_{sl}(q^2)$ cannot be literally calculated in the above domain, the integral over $q^2$

$$\int_{q^2_{\text{min}}}^{(M_B-M_D)^2} dq^2 \Gamma_{sl}(q^2)$$

(132)

can be, provided $m_b - m_c - \sqrt{q^2_{\text{min}}} \gg \Lambda_{\text{QCD}}$. This fact automatically emerges in the approach suggested in [25, 26, 39]. To get it in the HQET-type approach required usage of the sum rules for the heavy quark transitions [18], and was demonstrated recently in [32].

It is worth emphasizing that the QCD-specific sum rules guarantee that the leading nonperturbative effect affecting the phase space in the semileptonic decay cancels against the corrections to the hadronic formfactors, the fact which is difficult to ensure in ad hoc quark models. This cancellation was apparently missed in [30]. As a matter of fact, the cancellation persists even in the perturbative effects up to momenta $\sim m_b/5$.

The paper [9] claimed that there is an independent $d = 4$ operator in HQET capable to affect the semileptonic widths producing $1/m_b$ nonperturbative corrections. Such a statement is unacceptable. I think that the confusion was merely a matter of ill-formulated expansion procedure.

A similar deficiency affects in my opinion the analysis [103]. The calculation of the width included a set of the nonperturbative parameters among which there was $\Lambda$ governing $1/m_b$ corrections explicitly introduced in the analyses. The inconsistency was again rooted to relying on the hadron masses. The QCD states, on the contrary, that for the $B$ meson width $\Gamma_{sl}(B)$ the $B$ meson mass $M_B$ is no more relevant than the mass of $\Lambda_b$, or $M_{B_s}$, or any their combination!

I cannot avoid mentioning that lobbying a possibility of $1/m_Q$ corrections got recently new impetus. Resurrecting the previous misconceptions Grinstein et al. [31] claimed establishing the OPE-violating $1/m_Q$ terms in the inclusive widths on the example of the ’t Hooft model, the large-$N_c$ two-dimensional QCD. I cannot dwell on details here, but can assure the reader that these results are wrong.

To summarize, there must be no reason so far for admitting a possibility of $1/m_Q$ corrections in the inclusive widths, even though it may not be obvious in certain approaches.

A more subtle problem is violations of local duality which can affect the widths, e.g. semileptonic, at a certain level. Reliable information on this aspect of QCD
is scarce. The only model specifically designed to address the issue is presented in Ref. [33]. It predicts negligible deviations from duality in the semileptonic decays of $B$ mesons. Irrespective of the model, they must be small. In this respect I would disagree with Bj who tried to argue in favor of naturalness of significant effects [105]. The arguments examined in more detail rather suggest a small scale for the effects. Of course, it cannot be interpreted as a rigorous proof.

The violation of duality must fast decrease with energy. It is worth to recall that in a similar four-fermion hadronic decay width of $\tau$ lepton one only has $E = m_\tau \simeq 1.78$ GeV. Nevertheless, it is often believed that the duality holds there at the level of 1 or 2%. (More realistically, the scale of the violation is around 5 to 7%.)

In $b \to c \ell \nu$ the energy release is 3.5 GeV. Moreover, there must be an additional suppression which can be revealed by varying $m_c$ as a free parameter.

In $b \to c \ell \nu$ the energy release is the smallest when $m_c$ increases and approaches $m_b$. In this limit (it is just the SV limit) there is an exact duality at least up to $1/m_b^2$ terms, the fact emphasized first by Shifman and Voloshin in the mid 80's in one of the first papers [20] established the base of the HQE. The effect can be evaluated similar to $F_{D^{(*)}}$ and is below a percent level.

In another extreme limit $m_c \to 0$ the heavy quark symmetry does not apply, however the energy release approaches 5 GeV. At such a scale it also must be negligible. It is hardly possible that the duality violation can have a strong peak at an intermediate $m_c$. Comparing to $\tau$ leptons, we thus have an essentially larger energy parameter and an important suppression of the duality-violating effects by the heavy quark symmetry. It is hardly reasonable in this situation to allow a much stronger effect compared to ‘unprotected’ hadronic $\tau$ decays.

It is interesting to discuss another argument by Bj who notes that in $e^+e^-$ annihilation one still has a significant resonance structure up to $E = \sqrt{s} \approx 2.5$ GeV. In $B$ decays the energy is safely above this domain, but in general one has to be careful relying on this experience. For the kinematics of hadronic transitions are quite different. In $e^+e^-$ we have a point-like production of the initial quarks while in $B$ decay the final state quarks start evolving at large separation $\sim \Lambda_{QCD}$ after the decay. This ‘multiperipheral’ nature in the heavy quark decays determines the difference in the nonperturbative effects which start with $1/m_Q^2 \text{ vs. } 1/E^4$ in the annihilation.

This peculiarity is reflected in the duality-violating effects as well, in particular in $\tau$ decays. Considering only the vector channel decay rate one experimentally observes larger, sizeable violations of duality. They are strongly washed out as soon as the axial channel is added when more low-energy resonances contribute. In this respect $\Gamma_{sl}(B)$ has a clear advantage: a variety of final states with different angular momenta is much more rich here even without producing extra quark pairs.

The scale of onset of duality is typically governed by the masses of the first prominent resonances. In $\tau$ decays they are $M_\rho$ and $m_\pi$, $M_{a_1}$ for $V$ and $A$, respec-

\[20\]Since the effects of duality violation must oscillate as a function of $m_\tau$ [33], at a certain mass they nevertheless may vanish.
tively. Such scale is lower in the semileptonic decays. For example, the $P$ wave excitations in $b \to c$ have the mass gap $\lesssim 500 \text{MeV}$. It is directly related to the richer quantum number structure of the decay. Even radiatively excited states are expected below $M_D + 1 \text{GeV}$.

As a result, a closer look at the potential problems with duality rather suggests that it should work better in $\Gamma_{\text{sl}}(B)$ than in $\Gamma_{\text{had}}(\tau)$. I think that a more dedicated analysis is to be done to reveal reasons – if any – why the former may be more vulnerable. So far all arguments suggest that $\Gamma_{\text{sl}}(B)$ is safer.

3.6 Lepton Spectra and Decay Distributions.

Model-insensitive Determination of $|V_{ub}|$

The inclusive differential distributions in semileptonic (or similar radiatively-induced) decays are treated in the HQE similarly to the inclusive widths \[40, 39, 106\]. An important fact pointed out in \[39\] is that the QCD-based OPE automatically leads to an analogue of nonrelativistic motion of the heavy quark inside the hadron. This “Fermi motion” was phenomenologically introduced long ago, first in \[107\] and then elaborated further to the status of a well-formulated model \[108\]. A similar phenomenological approach was later used in a number of papers \[109\]. A detailed description of the Fermi motion in the framework of the $1/m_Q$ expansion was later given in \[19, 110\] and in \[111\].

The “Fermi motion” in QCD has certain peculiarities which are absent in the phenomenological models \[13, 112\]. The distribution over the ‘primordial’ Fermi momentum $F(\vec{p})$ is replaced by a certain distribution function $F(x)$, where $x \leq 1$ measures the momentum of the $b$ quark in the units of $M_B - m_b$. First distinction is that $F(x)$ is one-dimensional; one can define only the distribution over a certain projection of the momentum.

Second, $F(x)$ depends essentially on the final state quark mass (more exactly, on its velocity). While it is the same for $b \to u \ell \nu$ and $b \to s + \gamma$ where the final quark is ultrarelativistic, it is rather different for $c$ quark in $b \to c \ell \nu$ where it is closer to a nonrelativistic particle.

Finally, $F(x)$ is actually normalization-point dependent.

Although $F(x)$ depends on the final state quark mass, there are still certain relations between the moments of $F(x)$ at arbitrary mass. The moments are expressed via the expectation values of the local heavy quark operators; these relations are examples of the generic sum rules for the decays of heavy flavors \[18\]. In particular, the ‘center of gravity’ of the distribution over the primordial momentum points at the heavy quark mass $m_b$. The second moment of $F(x)$, its dispersion, is proportional to the kinetic expectation value $\mu_\pi^2$. The third moment is proportional to the Darwin term, \textit{etc.} Let me mention one fact \[112, 110\] which is often missed: the $AC^2M^2$ model \[108\] does correspond to a certain QCD distribution function $F(x)$, the so-called ‘Roman’ function which however is not Gaussian. Moreover, the value of $\mu_\pi^2$ in this model is not given by $p_F$ itself but is a more complicated function of
$p_F$ and $m_{sp}$, the two parameters of the AC$^2$M$^2$ model.

It appears that the effects of Fermi motion are very important for a possible determination of $V_{ub}$. We know certainly that $|V_{ub}| \neq 0$ holds since (i) the decays $B \to \pi \ell \nu, \rho \ell \nu$ have been identified and (ii) $B \to X \ell \nu$ has been observed with lepton energies $E_\ell$ that are accessible only if $X$ does not contain a charm hadron: $E_\ell \geq (M_B^2 - M_{D}^2)/2M_B = 2.31$ GeV. Translating these findings into reliable numbers for $|V_{ub}|$ is a much more difficult task theoretically: On the one hand the exclusive decays $B \to \pi \ell \nu, \rho \ell \nu$ depend on bound-state effects in an essential way. On the other hand in analyzing the endpoint spectrum for charged leptons in the inclusive decays $B \to X \ell \nu$ one encounters different sorts of systematic problems. Only a fairly small fraction of the charmless semileptonic decays $B \to X_u \ell \nu$, namely around 10% or so, produce a charged lepton with an energy beyond that possible for $B \to X_c \ell \nu$. The data suggest that $|V_{ub}|/|V_{cb}| \sim 0.1$, however the actual estimates vary within a factor of two. In addition, the $b \to c$ rate is so much bigger than that for $b \to u$ that leakage from it due to measurement errors becomes a serious background problem; furthermore the endpoint region is particularly sensitive to the nonperturbative dynamics.

Since the inclusive semileptonic width $\Gamma_{sl}(b \to u \ell \nu)$ is reliably calculated theoretically, Eq. (53), the best way to determine $|V_{ub}|$ would be to accurately determine $BR_{sl}(b \to X_u \ell \nu)$ where $X_u$ denotes the charmless hadronic states. It is not yet completely clear how reliably this fraction can be measured in experiment. Of course, the problem is not statistics per se.

The most direct way to disentangle $b \to u \ell \nu$ from $b \to c \ell \nu$ without tagging the secondary charm decay would be to study the invariant mass of hadrons in the final state $M_X$:

$$
\frac{d}{dM_X} \Gamma(B \to X \ell \nu), \quad M_X^2 = \left( \sum_i P_{i\text{hadr}}^{(i)} \right)^2.
$$

For the free quark decay one has $M_X^2 \approx 0$ in $b \to u$ and $M_X^2 = m_c^2$ for the $b \to c$ transitions. In actual decays the mass can take values exceeding $m_c$ and $M_D$, respectively. The increase in mass can originate both in the perturbative processes when a hard gluon is emitted in the decay, or through soft bound-state or hadronization processes. The leading soft effects emerge due to same physics which gives rise to the Fermi motion. They are quite significant. For example, the average invariant mass square of hadrons in $b \to u$ gets the nonperturbative correction $\sim \Lambda_{\text{QCD}} \cdot m_b$ [113]:

$$
\langle M_X^2 \rangle_{\text{nonpert}} = \frac{7}{10} m_b (M_B - m_b) + O \left( \Lambda_{\text{QCD}}^2 \right) \simeq 1.5 \text{ GeV}^2
$$

(the suggestion of [114] about the absence of these relative $1/m_Q$ corrections to $M_X^2$ was erroneous). The perturbative corrections lead to $\langle M_X^2 \rangle_{\text{pert}} \sim \frac{4\pi}{3} m_b^2$, however in the $B$ decays this increase is still smaller than the nonperturbative corrections.

The above estimates already show that the broadening of the $M_X$-distribution is quite essential; whether its impact is insignificant or dramatic for discriminating
the two semileptonic decay channels, was not clear a priori and required numerical analysis. It was done in [114]. The charm quark mass happens to be just near the borderline separating the domains of ‘heavy’ quark $m_c \gg \sqrt{\Lambda_{\text{QCD}} m_b}$ where the effect of broadening is small, and of ‘light’ quark $m_c \ll \sqrt{\Lambda_{\text{QCD}} m_b}$ when the kinematic difference between the inclusive $b \to u$ and $b \to c$ decays is buried under the hadronization effects.

To quantify the effect of the strong interactions let us introduce, following Refs. [114, 115] the fraction of the $b \to u$ events with $M_X$ below a certain cut-off mass $M_{\text{max}}$:

$$\Phi(M_{\text{max}}) = \frac{1}{\Gamma(b \to u)} \int_0^{M_{\text{max}}} dM_X \frac{d\Gamma}{dM_X}.$$  (135)

Clearly, $\Phi(0) = 0$ and $\Phi(M_B) = 1$ independently of strong interactions. The main question to theory is whether it can calculate accurately enough $\Phi(M_{\text{max}})$ with $M_{\text{max}} \lesssim 1.5$ GeV.

The dedicated analysis was carried out in [114, 115], and the conclusion appeared to be quite optimistic – the strong interaction effects in $b \to u \ell \nu$ are not expected to populate the domain above $M_X = 1.5$ GeV too significantly. The theoretical predictions for the spectrum can be found in the above papers. Later a similar analysis was undertaken in [116].

As anticipated, the mass spectrum is broad and extends even beyond $M_D$ – yet only a small fraction does so, namely $\sim 10\%$. Due to measurement errors there will be a tail from $b \to c$ transitions below $M_D$. To avoid this leakage one can concentrate on recoil masses below a certain value $M_{\text{max}}$. The actual choice of $M_{\text{max}}$ is driven by competing considerations: the lower $M_{\text{max}}$, the less leakage from $b \to c$ will occur – yet the smaller the relevant statistics. Even more important, the accuracy of the theoretical predictions deteriorates at lower cutoff mass. The predictions and their major uncertainties are illustrated in Figs. 5 taken from [115].

This analysis showed that it seems important to reach a value of $M_x \simeq 1.5$ GeV in the reliable measurement of the fraction $\Phi(M_{\text{max}})$. If the necessary discrimination of $b \to c$ decays can be achieved at such masses, a rather precise model-insensitive determination of $|V_{ub}|$ and/or $|V_{ub}|/|V_{cb}|$ will be possible. Those who are interested in further details can find them in Ref. [115].

I have to note that the conclusions of paper [116] were less definite, and its predictions suffered from larger uncertainties. The treatment of both the perturbative and nonperturbative corrections was somewhat different; I consider some elements of the analysis not quite consistent.

Sharpening the theoretical predictions for $\Phi(M_{\text{max}})$ would require, in particular, a more refined treatment of the perturbative corrections. In the $b \to u \ell \nu$ or $b \to s + \gamma$ distributions they are not very simple since include strong bremsstrahlung corrections that must be dealt with carefully. This will not be a stumbling block, however. A certain approximation, the “APS” (Advanced Perturbative Spectrum) was elaborated in [110] which is feasible and, at the same time, must yield more than
enough precision in accounting for the perturbative corrections to the distributions even in these transitions. If future studies demonstrate the possibility of accurate measurements of the hadronic invariant mass in the semileptonic decays and the anticipated accuracy becomes clear, such refined calculations will be done.

Concluding this discussion, I want to remind that it is very desirable to study such distributions separately for $B^\pm$ and $B^0$ mesons. The same refers also to studying the end-point region of the lepton spectra. In this way it will be possible to directly study and detect the so-called weak annihilation, the nonperturbative effect sensitive to the flavor of the spectator quark [113]. Weak annihilation is a relatively small $1/m_b^3$ effect in the total widths. However, in the semileptonic transitions it mainly originates from the end-point domain, $E_\ell \approx M_B/2$ and small $M_X$. Thus, in these studies its relative effect is enhanced. It would have an independent theoretical interest as well providing valuable information for the HQE about the expectation values of the four-fermion operators, in particular, their nonfactorizable pieces [113].

4 Challenges in the HQE

Concluding my lectures, I cannot avoid mentioning places where the HQE seems to have problems when compared to experiment. It is probably premature to speak of a direct contradiction; nevertheless, today’s question marks carry the seeds of tomorrow’s advances. Basically there are two problems where our theoretical understanding is lagging behind. Both are related to nonleptonic decays.

4.1 Semileptonic Branching Ratio of the $B$ Mesons and $n_c$

The theoretical attitude to this problem changes with time. Twenty years ago the parton model gave a prediction $\text{BR}_{\text{sl}}(B) \simeq 13–15\%$, which was accurate enough
according to the existed standards. The variation reflected mostly the choice of quark masses. While the rates of $c \to c\ell\nu$ and $b \to c\bar{u}d$ were affected by the choice of $m_c$ in a more or less same way, the rate of the $b \to c\bar{c}s$ channel decreased much faster when larger masses (closer to the ‘constituent’ rather than short-distance ‘current’ ones) were used. Therefore, $\text{BR}_{\text{sl}}(B)$ increases for larger masses. It should be remembered that the difference of $m_b$ and $m_c$ is fixed, so the choices of $m_c$ and $m_b$ are always correlated. Even though the heavy quark symmetry had not been formulated, the latter fact was clearly realized at least in the early 80’s [117].

The experimental situation was not very definite and indicated a rather large $\text{BR}_{\text{sl}}(B)$ which reasonably fitted the ‘larger’ mass option [118]. Since then it became more or less standard to use the larger quark masses, and the parton model prediction for $\text{BR}_{\text{sl}}(B)$ was accepted to be 13–14%, even though a smaller value could be obtained as well. Since the impact of the nonperturbative corrections was completely unknown and presumably larger, it was not seriously discussed even when the better data became available.

The situation changed when in 1992 Bigi et al. showed that there are no $1/m_Q$ nonperturbative corrections to the inclusive widths, both semileptonic and nonleptonic. Moreover, even the smaller $1/m_b^2$ nonperturbative effects were readily calculated [25, 26, 119]. They appeared to be suppressed, in particular, as a result of certain cancellations. The overall effect $\Delta_{\text{nonpert}}\text{BR}_{\text{sl}}(B)$ was found to be about $-0.5\%$; the nonperturbative effects could not be blamed for a discrepancy any more [120].

It prompted a more careful analysis of the perturbative corrections to the widths. In particular, the $O(\alpha_s)$ corrections to the nonleptonic $b \to c\bar{u}d$ width were calculated accounting for the non-zero $m_c$ [121]; this additionally enhanced the nonleptonic width. Later the account for the charm mass was also completed for $b \to c\bar{c}s$ [122, 123, 124]. Altogether, these $O(\alpha_s)$ corrections further decreased $\text{BR}_{\text{sl}}(B)$ down to 11–12%. Although this shift naively seems very significant and may raise concerns about the convergence of the perturbative corrections, it actually is not dramatic if one starts with more appropriate short-distance masses, the choice forgotten for historical rather than rational reasons. It is important to keep in mind that the relative increase in the nonleptonic width was particularly significant in the $b \to c\bar{c}s$ channel.

The issue of the semileptonic branching ratio must be considered in conjunction with the charm yield $n_c$, the number of charm states emerging from $B$ decays. To measure $n_c$ one assigns charm multiplicity one to $D$, $D_s$, $\Lambda_c$ and $\Xi_c$ and two to charmonia. Zero is assigned to the charmless hadronic final state. It is obvious that

$$n_c \simeq 1 + \text{BR}(B \to c\bar{c}s \bar{q}) - \text{BR}(B \to \text{no charm}). \quad (136)$$

Such a joint analysis was motivated already in [25]: the energy release in $b \to c\bar{c}s$ is rather small, and this can lead to significant duality-violating and higher-order effects. The stability of the perturbative expansion also downgrades. Measuring $n_c$ allows one to effectively exclude this channel from the theoretical calculations.
The data on \( n_c \) were not quite consistent for some time, but finally seem to converge to — unfortunately — a smaller, more difficult for theory value. If we take experiment and theory at face value, the allowed intervals do not overlap. The semileptonic fraction seems too low, about 10.6\%, which is possible to accommodate only at \( n_c \gtrsim 1.25 \). Such a large number is not compatible with the CLEO measurements. It is not clear how it must be interpreted. At this School I heard suggestions from the experimental colleagues whose opinion I highly respect, that one may not take too closely the discrepancy within two, or even within three quoted error bars. Taking such an optimistic (for a theorist) attitude, one could have been relaxed. I cannot, however make such a decision myself and have to discuss numbers as they are officially quoted.

In my opinion, it is difficult to accommodate theoretically the value of \( \text{BR}_{sl}(B) \) below 11.2\% if \( n_c \) does not exceed 1.20. Those who say that it is easy to obtain it in the SM, probably stretch uncertainties too liberally. Of course, even if eventually experiment arrives at the current central values, we will not have to revise the SM. One possibility would be that the second-order perturbative corrections in the non-leptonic modes are grossly underestimated and too significant. Another — probably, more realistic — is the violation of local duality in the nonleptonic widths; I’ll return to it a bit later. \textit{A priori}, one could expect significant, \( \sim 15\% \) nonperturbative effects in the widths. The actual problem lies in the fact that the leading corrections are calculated and happen to be small.\footnote{The coefficient for the leading nonperturbative correction is calculated only in the so-called leading logarithmic approximation. There are reasons to think that the additional terms in the complete \( \mathcal{O}(\alpha_s) \) coefficient can be as large.}

\section*{4.2 Lifetimes of Heavy Flavor Hadrons}

As stated before, differences between meson and baryon decay widths arise already in order \( 1/m_Q^2 \). The perturbative corrections to the lifetime ratios are completely absent. The lifetimes of the various mesons get differentiated effectively first in order \( 1/m_Q^3 \). A detailed review can be found in [125, 126].

Because the charm quark mass is not much larger than typical hadronic scales one can expect to make only semi-quantitative predictions on the \textit{charm} lifetimes, in particular for the charm baryons. The agreement of the predictions with the data is good. I would even say it is too good keeping in mind that the \( 1/m_c \) expansion can hardly be justified.

As far as the \textit{beauty} lifetimes are concerned the \( 1/m_b \) expansion is to be applicable. Table [4] contains the world averages of published data together with the predictions. The latter were actually made before data (or data of comparable sensitivity) became available.

Data and predictions on the meson lifetimes are completely and non-trivially consistent. Yet even so, a comment is in order for proper orientation. The numerical prediction is based on the assumption of factorization at a typical hadronic
Table 1: QCD Predictions for Beauty Lifetimes

| Observable | QCD Expectations (1/m_b expansion) | Ref. | Data from |
|------------|-----------------------------------|------|-----------|
| \( \tau(B^-)/\tau(B_d) \) | \( 1 + 0.05(f_B/200 \text{ MeV})^2 \) | 37   | 1.04 ± 0.04 |
| \( \tau(B_s)/\tau(B_d) \) | \( 1 \pm \mathcal{O}(0.01) \) | 125  | 0.97 ± 0.05 |
| \( \tau(\Lambda_b)/\tau(B_d) \) | \( \gtrsim 0.9 \) | 125  | 0.77 ± 0.05 |

scale which is commonly taken as the one where \( \alpha_s(\mu_{\text{hadr}}) \simeq 1 \). While there is no justification for factorization at \( \mu \sim m_b \), there exists ample circumstantial evidence in favor of approximate factorization at a typical hadronic scale – from the QCD sum rule calculations, to lattice evaluations, to \( 1/N_c \) arguments. More to the point, the validity of factorization can be probed in semileptonic decays of \( B \) mesons in an independent way, as was pointed out in [113].

The possible effect of the nonfactorizable contribution has been discussed in detail in [113, 127]. They include also the nonvalence gluon mechanism discussed long ago in [128] in the simplified language of the quark model. Significant nonfactorizable contributions would in general lead to large effects of weak annihilation in \( D_s \) mesons where experimentally such effects are quite small. Of course, we cannot reliably treat \( \Gamma_D \) in the \( 1/m_Q \) expansion, and \( \Gamma_{D_s} - \Gamma_{D^+} \) is sensitive to a particular combination of the expectation values of a few four-fermion operators. Nevertheless, it is a serious indication that the effects due to nonfactorizable expectation values must be suppressed. Thus, the estimates of the preasymptotic corrections to the widths based on factorization, most probably, are valid within a 50% accuracy. The actual uncertainty in \( f_B^2 \) is not much smaller; it includes the subleading \( 1/m_b \) effects which formally belong to neglected \( 1/m_b^4 \) corrections to \( \Gamma_B \).

The recent paper [129] attempted to estimate the nonfactorizable expectation values in \( B \) mesons in QCD sum rules. Although the calculations were carried out in a too simplified manner and cannot be really considered as predictions, they showed that there hardly exists room for large nonfactorizable contributions, which could yield drastic effects conjectures in [130]. In particular, I must note that the possible interval 0.8 to 1.4 for \( \tau_{B^-}/\tau_{B^0} \) claimed by Neubert [131] is unacceptable. While one cannot guarantee \textit{a priori} that the onset of the \( 1/m_Q \) expansion is safely below \( m_b \) and the latter can be applied to actual \( b \) hadrons, it is certain that such large deviations \textit{cannot} be obtained in the framework of the self-consistent \( 1/m_Q \) expansion of the width itself. The quoted interval emerged merely as a result of reckless manipulation with \textit{ad hoc} introduced hadronic parameters. Their values saturating such corrections would violate certain bounds [132] which are analogous to the unitarity constraints. In any case, these flavor-dependent \( 1/m_Q^3 \) effects under discussion originate only in particular quark decay channels. It is clear that the preasymptotic effects must constitute only a small fraction of the widths in order to be amenable to the \( 1/m_Q \) expansion.
The agreement of the data on $B$ meson lifetimes with experiment is obscured by the apparent conflict for $\tau_{\Lambda_b}/\tau_{B_d}$. To predict the $1/m_Q^3$ corrections to $\tau_{\Lambda_b}$ in the $1/m_b$ expansion one needs to evaluate the baryonic expectation values of two operators,

$$\langle \Lambda_b | \bar{b} b \bar{u} \gamma_0 u | \Lambda_b \rangle, \quad \langle \Lambda_b | \bar{b} \lambda^a b \bar{u} \gamma_0 \lambda^a u | \Lambda_b \rangle.$$  \hfill (137)

They do not have the usual factorizable contribution, and their values are rather uncertain. Nevertheless, it was shown that their contributions cannot be too large \[132\] and the maximal effect in $\Gamma_{\Lambda_b}$ does not exceed $10-12\%$. To achieve the larger corrections one would have to go beyond a usual description of baryons when light quarks are “soft”. This agrees with the fact that the constituent quark model estimates typically yield about 3 to 5% enhancement \[133\]. A similar conclusion has been reached by the authors of Ref. \[134\] who analyzed the relevant baryonic matrix elements through QCD sum rules.

We see that there are indications of disagreement between theory and experiment, at the level of at least two quoted experimental error bars. The opinion is sometimes expressed that before a disagreement reaches three standard deviations, it is premature to start worrying. I am not sure that it fully applies. On the one hand, the literal theoretical expectations for $\tau_{\Lambda_b}/\tau_{B_d}$ concentrate closer to 0.95, and the quoted limit 0.9 is not a central theoretical value. On the other hand, these theoretical predictions were known already for a number of years \[125\], so I hope that the experiments have analyzed obvious possible biases in the data and corrected those which could generate the enhanced gap. Neither argument is convincing by itself, but together they give me a feeling that the disagreement does exist.

If it will be firmly established in the future experiments, I think that in the framework of the SM the most probable explanation is the violation of duality. As a matter of fact, its significance in nonleptonic widths is theoretically expected \textit{a priori}. Indeed, the expansion parameter for the widths is not $m_b$ directly but rather the energy release which is noticeably smaller in $b \to c$. Moreover, the preasymptotic corrections depend on the concrete form of the weak interaction involved. For the four-fermion interaction they are enhanced: the large-5 arguments \[32\] mentioned earlier suggest that the actual scale parameter is smaller, $\sim E_{\text{rel}}/5$. In the semileptonic decays this does not deteriorate the expansion since it is automatically protected by the heavy quark symmetry when $m_c$ increases. Even if we keep the invariant mass of leptons $\sqrt{q^2}$ fixed, the corresponding quark width remains correct at least up to $1/m_Q$ terms for arbitrary, for example, maximal $q^2$ when the energy release is tiny, see Eq. (131).

For the nonleptonic decays the heavy quark symmetry does not generally apply, and at insufficient energy release one expects significant violations of duality. A few years ago, therefore, we would not be too much concerned with present discrepancies. Now, on the other hand, we have evaluated the leading terms in the $1/m$ expansion of the width and did not find indications for significant effects. Since violation of duality is conceptually related to the asymptotic nature of the OPE \[133, 33\], we could have expected that duality works at a few percent level as well.
### 4.3 $1/m_Q$ Expansion and Duality Violation

The problem of duality violation attracts more and more attention of those who study the heavy quark theory; a recent extensive discussion was given in [33]. The expansion in $1/m_Q$ is asymptotic. There are basically two questions one can ask here: what is the onset of duality, i.e. when does the expansion start to work? The most straightforward approach was undertaken in Ref. [136], and no apparent indication toward an increased energy scale was found. Another question, how is the equality of the QCD parton-based predictions with the actual decay rates achieved, was barely addressed. Though a relevant example of such a problem is easy to give.

The OPE ensures that no terms $\sim 1/m_Q$ can be in the widths in QCD and the corrections start with $1/m_Q^2$. However, the OPE per se cannot forbid a scenario where, for instance,

\[ \frac{\delta \Gamma_{HQ}}{\Gamma_{HQ}} \sim C \frac{\sin (m_Q \rho)}{m_Q \rho}, \quad \rho \sim \Lambda_{QCD}^{-1}. \]  

(138)

In the actual strong interaction, $m_b$ and $m_c$ are fixed and not free parameters, so, from the practical viewpoint these types of corrections are not too different – but the difference is profound in the theoretical description! It reflects specifics of the OPE in Minkowski space, and such effects can hardly be addressed, for example, in the lattice simulations. Their control requires a deeper understanding of the underlying QCD dynamics beyond the knowledge of first few nonperturbative condensates.

In fact, the literal corrections of the type of Eq. (138) are hardly possible; the power of $1/m_Q$ in realistic scenarios is larger, and these duality corrections must be eventually exponentially suppressed though, probably, starting at a higher scale [33]. But a theory of such effects is still in its embryonic stage and needs an additional experimental input as well.

Nevertheless, we know that the duality-violating terms in the HQE are not completely arbitrary, and they must obey certain constraints. For example, they cannot be monotonous and positive or negative but oscillate, etc. All these exact properties are often missed in the theoretical discussions.

It is worth adding that the folklore about violations of duality is often rather superficial. Even the used terminology is not always consistent. It was written recently, for example, in [103] that the assumptions behind the $1/m_Q$ expansion for the widths, in particular, for the nonleptonic widths, include a certain factorization of the decay amplitudes. In fact, such a factorization is not required at all, like it is not required in the asymptotic expansion of $R_{e^+e^-}(s)$. The only assumption is the usual factorization of contributions of large-distance and short-distance physics, the OPE. Moreover, in principle it is enough to assume it only in Euclidean domain. A stronger assumption of such a factorization for actual decay processes in Minkowski kinematics at fixed energy would lead to additional constraints on violation of local duality.

It has been quite popular to state that the quark-hadron duality sets in only when the number of hadrons in the final state becomes large [137]. This is a wrong
criterium. The semileptonic decays in the SV kinematics are a nice example. In the limit

\[ m_b, m_c \gg \Lambda_{\text{QCD}}, \quad m_b - m_c \gg \Lambda_{\text{QCD}}, \quad \frac{m_b - m_c}{m_b + m_c} \sim v \ll 1 \]  

the exact quark-hadron duality perfectly holds with the theorem-like rigorousness while the total decay rate is saturated by only two final states, \( D^* \) and \( D \) (even only one state can be retained if special weak currents are considered). Simultaneously, the yield of the excited states can be arbitrary small: it is driven by two parameters \((\Lambda_{\text{QCD}}/m_c)^2 \ll 1\) and \((m_b - m_c)/(m_b + m_c)^2 \ll 1\). This fact was actually noted in [20] in the mid 80’s and was among the first steps in the development of the heavy quark symmetry and dynamic \(1/m_Q\) expansion. It is really surprising why this classic result used to be ignored even in the relatively recent papers.

On the other hand, the duality violation has peculiarity in decays of heavy quarks. Apart from the usual assumption of local duality, the practical applications of the OPE expansion relies here on the property of \(\text{global}\) duality, which can also be violated to a certain extent. An explanation of what is global duality would lead me too far astray; it can be found in Ref. [18], Sect. IV.A where this notion was introduced. Let me only note that since then this term has often been abused in the literature. For example, it is tautological to discuss the violations of global duality in \(R_{e^+e^-}\) or in the hadronic \(\tau\) decay width (more accurately, it would be merely verifying the dispersion relation). Everything there is local duality and the global duality is a very simple statement, a direct consequence of analyticity and unitarity.

I come to the end of my lectures without a formal concluding section. Theory of heavy quarks has become a well-developed field of QCD. Not only it benefits from the power of theoretical methods elaborated in QCD over 25 years of evolution; the application of the heavy quark expansion provides feedback for the theory of QCD itself. A few slices of the heavy quark theory I discussed here demonstrate that it is an actively evolving field where many interesting questions still are to be understood. Without doubts, the data from the new generation of experiments will provide the new impetus; we all are looking forward to new fascinating surprises.

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References

[1] K. Wilson, Phys. Rev. **179** (1969) 1499.

[2] W.E. Caswell and G.P. Lepage, Phys. Lett. **B167** (1986) 437; G.P. Lepage, L. Magnea, C. Nakleh, U. Magnea, and K. Hornbostel, Phys. Rev. **D46** (1992) 4052.

[3] E. Eichten and B. Hill, Phys. Lett. **B234** (1990) 511.

[4] H. Georgi, Phys. Lett. **B240** (1990) 447.

[5] S. Balk, J.G. Körner and D. Pirjol, Nucl. Phys. **B428** (1994) 499.

[6] A. Manohar, Phys. Rev. **D56** (1997) 230.

[7] B. Grinstein and I. Rothstein, Phys. Rev. **D57** (1998) 78.

[8] M. Neubert, in *The Building Blocks of Creation - From Microfermius to Megaparsecs*, 1993 TASI Proceedings, Eds. S. Raby and T. Walker (World Scientific, Singapore, 1994) p. 125 (hep-ph/9404296).

[9] M. Luke, A. Manohar and M. Savage, Phys. Rev. **D51** (1995) 4924.

[10] M. Neubert and C.T. Sachrajda, Nucl. Phys. **B438** (1995) 235.

[11] M. Neubert, *Lectures at the 34th Intern. School of Subnuclear Physics: Effectives Theories and Fundamental Interactions*, Erice, Italy, 3-12 July 1996 (hep-ph/9610266).

[12] A. Falk, M. Luke and M. Savage, Phys. Rev. **D49** (1994) 3367.

[13] A. Manohar and M. Wise, Phys. Rev. **D49** (1994) 1310.

[14] A. Falk and M. Neubert, Phys. Rev. **D47** (1993) 2965.

[15] N. Isgur and M. Wise, Phys. Rev. **D43** (1991) 819.

[16] A. Le Yaouanc, *B Decays, an Introductory Survey*, talk at the XXIXth Rencontre de Moriond, “QCD and High Energy Hadronic Interactions”, March 1994, Méribel France; A. Le Yaouanc et al., hep-ph/9406341, also hep-ph/9406334.

[17] N. Isgur and M.B. Wise, in *B Decays*, Ed. B. Stone, 2nd edition (World Scientific, Singapore, 1994), p. 231; H. Georgi, in *Perspectives in the Standard Model*, Proceedings of the Theoretical Advanced Study Institute, Boulder, Colorado, 1991, Ed. R.K. Ellis, C.T. Hill, and J. D. Lykken (World Scientific, Singapore, 1992);
T. Mannel, *Chinese Journal of Physics*, **31** (1993) 1; also *Heavy Quark Mass Expansions in QCD*, in Proc. of the Workshop *QCD – 20 Years Later*, Eds. P. M. Zerwas and H.A. Kastrup (World Scientific, Singapore, 1993), vol. 2, p. 634;

B. Grinstein, *Ann. Rev. Nucl. Part. Sci.* **42** (1992) 101;

M. Neubert, *Phys. Reports* **245** (1994) 259;

M. Voloshin, *Surv. High En. Physics*, **8** (1995) 27;

B. Grinstein, *An Introduction to Heavy Mesons*, in Proc. 6th Mexican School of Particles and Fields, Eds. J. C. D’Olivo, M. Moreno, and M. A. Perez, (World Scientific, Singapore, 1995) [hep-ph/9508227]

[18] I.I. Bigi, M. Shifman, N.G. Uraltsev and A. Vainshtein, *Phys. Rev.* **D52** (1995) 196.

[19] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, *Int. Journ. Mod. Phys.* **A9** (1994) 2467.

[20] M. Voloshin and M. Shifman, *Yad. Fiz.* **47** (1988) 801 [Sov. J. Nucl. Phys. **47** (1988) 511].

[21] M. Shifman, N.G. Uraltsev and A. Vainshtein, *Phys. Rev.* **D51** (1995) 2217.

[22] A. Czarnecki, K. Melnikov and N. Ural'tsev, *Phys. Rev. Lett.* **80** (1998) 3189.

[23] A.F. Falk, M. Neubert and M.E. Luke, *Nucl. Phys.* **B388** (1992) 363.

[24] T. Mannel, W. Roberts and Z. Ryzak, *Nucl. Phys.* **B368** (1992) 204.

[25] I. Bigi, N. Uraltsev and A. Vainshtein, *Phys. Lett.* **B293** (1992) 430.

[26] B. Blok and M. Shifman, *Nucl. Phys.* **B399** (1993) 441 and 459.

[27] S. Berman, *Phys. Rev.* **112** (1958) 267;

T. Kinoshita and A. Sirlin, *Phys. Rev.* **113** (1959) 1652.

[28] G. Altarelli, G. Martinelli, S. Petrarca and F. Rapuano *Phys. Lett.* **B382** (1996) 409.

[29] P. Colangelo, C.A. Dominguez and G. Nardulli, *Phys. Lett.* **B409** (1997) 417;

P. Colangelo, in *Proc. of the High Energy Physics Euroconference QCD’97*, Montpellier, France, 3-9 July 1997.

[30] C. Jin, *Phys. Rev.* **D56** (1997) 2928.

[31] B. Grinstein and R. Lebed, *Phys. Rev.* **D57** (1998) 1366.

[32] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, *Phys. Rev.* **D56** (1997) 4017.
[33] B. Chibisov, R. Dikeman, M. Shifman and N. Uraltsev, *Int. J. Mod. Phys.* **A12** (1997) 2075.

[34] T. Kinoshita, *J. Math. Phys.* **3** (1962) 650; T.D. Lee and M. Nauenberg, *Phys. Rev.* **133** (1964) B1549.

[35] R. Akhoury, L. Stodolski and V.I. Zakharov, UM-TH-96-06. Unfortunately, this preprint has not officially appeared yet.

[36] N. Isgur, D. Scora, B. Grinstein and M. Wise, *Phys. Rev.* **D39** (1989) 799.

[37] I. Bigi and N. Uraltsev, *Phys. Lett.* **B280** (1992) 271.

[38] M. Neubert, *Nucl. Phys. Proc. Suppl.* **59** (1997) 101.

[39] I. Bigi, M. Shifman, N. Uraltsev, A. Vainshtein, *Phys. Rev. Lett.* **71** (1993) 496.

[40] I. Bigi, B. Blok, M. Shifman, N. Uraltsev and A. Vainshtein, *The Fermilab Meeting*, Proc. of the 1992 DPF meeting of APS, C.H. Albright et al. (World Scientific, Singapore 1993), vol. 1, p. 610.

[41] I. Bigi, M. Shifman and N.G. Uraltsev, *Ann. Rev. Nucl. Part. Sci.*, **47** (1997) 591.

[42] B. Blok, R. Dikeman and M. Shifman, *Phys. Rev.* **D51** (1995) 6167.

[43] M. Gremm and A. Kapustin, *Phys. Rev.* **D55** (1997) 6924.

[44] S.J. Brodsky, G.P. Lepage and P.B. Mackenzie, *Phys. Rev.* **D28** (1983) 228; G.P. Lepage and P.B. Mackenzie, *Phys. Rev.* **D48** (1993) 2250.

[45] M. Luke, M. Savage and M. Wise, *Phys. Lett.* **B345** (1995) 301.

[46] P. Ball, M. Beneke and V. Braun, *Phys. Rev.* **D52** (1995) 3929.

[47] A. Czarnecki and K. Melnikov, *Phys. Rev. Lett.* **78** (1997) 3630; preprint TTP98-14, [hep-ph/9804213](https://arxiv.org/abs/hep-ph/9804213).

[48] N. Uraltsev, *Int. J. Mod. Phys.* **A11** (1996) 515.

[49] M. Neubert, *Phys. Lett.* **B338** (1994) 84.

[50] M. Neubert, [hep-ph/9505238](https://arxiv.org/abs/hep-ph/9505238) (Proc. of the 30th Rencontres de Moriond, Les Arcs, France, March 1995).

[51] M. Neubert, in *Proc. XVII International Symposium on Lepton–Photon Interactions*, 10-15 Aug. 1995, Beijing, Ed. Zhi-Peng Zheng and He-Sheng Chen (World Scientific, Singapore, 1996), p. 298 [hep-ph/9511409](https://arxiv.org/abs/hep-ph/9511409).
[52] M. Luke and M. Savage, *Phys. Lett.* **B321** (1994) 88.

[53] Z. Guralnik and A. Manohar, *Phys. Lett.* **B302** (1993) 103.

[54] I.I. Bigi, N.G. Uraltsev, *Phys. Lett.* **B321** (1994) 412.

[55] A. Manohar, PDG-96: *Phys. Rev.* **D54** (1996) p.303.

[56] I. Bigi, M. Shifman, N. Uraltsev, A. Vainshtein, *Phys. Rev.* **D50** (1994) 2234.

[57] M. Beneke, V. Braun, *Nucl. Phys.* **B426** (1994) 301.

[58] G. ’t Hooft, in *The Whys Of Subnuclear Physics*, Erice 1977, ed. A. Zichichi (Plenum, New York, 1977); B. Lautrup, *Phys. Lett.* **69B** (1977) 109; G. Parisi, *Phys. Lett.* **76B** (1978) 65; *Nucl. Phys.* **B150** (1979) 163; A. Mueller, *Nucl. Phys.* **B250** (1985) 327.

For a recent review see A. H. Mueller, in *Proc. Int. Conf. “QCD – 20 Years Later”*, Aachen 1992, eds. P. Zerwas and H. Kastrup, (World Scientific, Singapore, 1993), vol. 1, p. 162.

[59] V. Braun, NORDITA-96/65-P, [hep-ph/9610212](http://arxiv.org/abs/hep-ph/9610212).

[60] M. Beneke, V.M. Braun and V.I. Zakharov, *Phys. Rev. Lett.* **73** (1994) 3058.

[61] N. Gray, D.J. Broadhurst, W. Grafe and K. Schilcher, *Z. Phys.* **C 48** (1990) 673.

[62] V.A. Khoze, in *Future Physics and Accelerators*, Proc. First Arctic Workshop on Future Physics and Accelerators, Saariselka, Finland, August 1994, Eds. M. Chaichian, K. Huitu, and R. Orava (World Scientific, Singapore, 1995), p. 458 [hep-ph/9412239](http://arxiv.org/abs/hep-ph/9412239).

[63] M. Shifman, A. Vainshtein, M. Voloshin and V. Zakharov *Phys. Lett.* **B77** (1978) 80.

[64] M.B. Voloshin, *Int. J. Mod. Phys.* **A10** (1995) 2865.

[65] N.G. Uraltsev, *Nucl. Phys.* **B491** (1997) 303.

[66] M. Voloshin, *Phys. Rev.* **D51** (1995) 4934.

[67] V. Chernyak, *Phys. Lett.* **B387** (1996) 173.

[68] M. Gremm, A. Kapustin, Z. Ligeti, and M. Wise *Phys. Rev. Lett.* **77** (1996) 20.

[69] M. Jamin and A. Pich, *Nucl. Phys.* **B507** (1997) 334.

[70] J.H. Kühn, A.A. Penin and A.A. Pivovarov, TTP98-01, [hep-ph/9801356](http://arxiv.org/abs/hep-ph/9801356).
[71] T. Appelquist, M. Dine and I. Muzinich, Phys. Rev. D17 (1978) 2074.
[72] M.B. Voloshin, Nucl. Phys. B54 (1979) 365.
[73] M. Voloshin and Yu. Zaitsev, Usp. Fiz. Nauk 152 (1987) 361 [Sov. Phys. Uspekhi 30 (1987) 553].
[74] M. Neubert, Phys. Rev. D46 (1992) 1076.
[75] P. Ball and V. Braun, Phys. Rev. D49 (1994) 2472.
[76] E. Bagan, P. Ball, V. Braun and P. Gosdzinsky, Phys. Lett. B342 (1995) 362.
[77] M. Voloshin, Surv. High En. Phys. 8 (1995) 27.
[78] M. Neubert, Phys. Lett. B389 (1996) 727.
[79] F. De Fazio, Mod. Phys. Lett. A11 (1996) 2693.
[80] D.S. Hwang, C.S. Kim and W. Namgung, Phys. Lett. B406 (1997) 117.
[81] I. Bigi, A. Grozin, M. Shifman, N. Uraltsev, A. Vainshtein, Phys. Lett. B339 (1994) 160.
[82] M. Voloshin, Phys. Rev. D46 (1992) 3062.
[83] B. Blok and M. Shifman, Phys. Rev. D47 (1993) 2949.
[84] A. Leibovich, Z. Ligeti, I. Stewart and M. Wise, Phys. Rev. Lett. 78 (1997) 3995, and hep-ph/9705467.
[85] G. Martinelli, M. Neubert and C.T. Sachrajda, Nucl. Phys. B461 (1996) 238.
[86] M. Neubert, Phys. Lett. B393 (1997) 110.
[87] A. Czarnecki, K. Melnikov and N. Uraltsev, Phys. Rev. D57 (1998) 1769.
[88] G. ’t Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189; C.G. Bollini and J.J. Giambiagi, Phys. Lett. B40 (1972) 566; Nuovo Cim. B12 (1972) 20.
[89] M.A. Shifman, in Proc. of the Intern. Symposium on Production and Decay of Heavy Hadrons, Heidelberg (1986).
[90] M. Luke, Phys. Lett. B252 (1990) 447.
[91] N.G. Uraltsev, Mod. Phys. Lett. A10 (1995) 1803.
[92] M. Neubert, Phys. Rev. D46 (1992) 2212.
[93] A. Czarnecki, Phys. Rev. Lett. 76 (1996) 4124; A. Czarnecki and K. Melnikov, Nucl. Phys.. B505 (1997) 65.

[94] M. Neubert, Phys. Rep. 245 (1994) 259.

[95] M. Neubert, Phys. Lett. B341 (1995) 367.

[96] A. Kapustin, Z. Ligeti, M.B. Wise and B. Grinstein, Phys. Lett. B375 (1996) 327.

[97] N.G. Uraltsev, Nucl. Instrum. & Methods A384 (1996) 17 (Proceedings of “Beauty ‘96”).

[98] N. Uraltsev, Acta Phys. Polon. B28 (1997) 755 (Proceedings of the Third International Symposium on Radiative Corrections in Cracow, Poland, 1-5 August 1996); hep-ph/9612349.

[99] E. de Rafael and J. Taron, Phys. Lett. B282 (1992) 215.

[100] C. Boyd, B. Grinstein, R. Lebed, Phys. Lett. B353 (1995) 306; Nucl. Phys. B461 (1996) 461.

[101] I. Caprini and M. Neubert, Phys. Lett. B380 (1996) 376.

[102] C. G. Boyd, B. Grinstein and R. Lebed, Phys. Rev. D56 (1997) 6895.

[103] A.F. Falk, M. Luke and M.J. Savage, Phys. Rev. D53 (1996) 6316.

[104] J. Chay, H. Georgi and B. Grinstein, Phys. Lett. B247 (1990) 399.

[105] J.D. Bjorken, SLAC-PUB-7521, hep-ph/9706524.

[106] B. Blok, L. Koyrakh, M. Shifman and A. Vainshtein, Phys. Rev. D49 (1994) 3356.

[107] A. Ali, E. Pietarinen Nucl. Phys. B154 (1979) 519.

[108] G. Altarelli et al., Nucl. Phys. B208 (1982) 365.

[109] A. Bareiss and E.A. Paschos, Nucl. Phys. B327 (1989) 353; C.H. Jin, W.F. Palmer and E.A. Paschos, Phys. Lett. B329 (1994) 364.

[110] R.D. Dikeman, M. Shifman and N.G. Uraltsev, Int. Journ. Mod. Phys., A11 (1996) 571.

[111] R. Jaffe and L. Randall, Nucl. Phys. B412 (1994) 79 A. Falk, E. Jenkins, A. Manohar and M. Wise, Phys. Rev. D49 (1994) 4553; M. Neubert, Phys. Rev. D49 (1994) 4623.
[112] I. Bigi, M. Shifman, N.G. Uraltsev and A. Vainshtein, *Phys. Lett.* B328 (1994) 431.

[113] I. Bigi and N. Uraltsev, *Nucl. Phys.* B423 (1994) 33.

[114] R.D. Dikeman, N.G. Uraltsev, *Nucl. Phys.* B509 (1998) 378.

[115] I. Bigi, R.D. Dikeman, N. Uraltsev, TPI-MINN-97/21-T, hep-ph/9706520; *Eur. Phys. J. C* (former *Zeit. f. Phys. C*) February 23, 1998 (electronic version).

[116] A.F. Falk, Z. Ligeti and M.B. Wise, *Phys. Lett.* B406 (1997) 225.

[117] R. Rueckl, in *Proc. Intern. School of Physics “Enrico Fermi” Course XCII, 1984*, Ed. N. Cabibbo (North-Holland, Amsterdam, 1987), p. 43; in *Flavor Mixing in Weak Interactions*, Ed. Ling-Lie Chau (Plenum Press, New York, 1984), p. 681; preprint MPI-PAE/PTh 36/89.

[118] G. Altarelli, S. Petrarca, *Phys. Lett.* B261 (1991) 303.

[119] L. Koyrakh, *Phys. Rev.* D49 (1994) 3379.

[120] I.Bigi, B.Blok, M.Shifman and A Vainshtein, *Phys. Lett.* B323 (1994) 408.

[121] E. Bagan, P. Ball, V. Braun and P. Gosdzinsky, *Phys. Lett.* B342 (1995) 362 [(E) B374 (1996) 363]; *Nucl. Phys.* B432 (1994) 3; see also earlier papers Q. Ho-Kim and X.-Y. Pham, *Phys. Lett.* B122 (1983) 297; *Annals Phys.* 155 (1984) 202.

[122] E. Bagan, P. Ball, B. Fiol and P. Gosdzinsky, *Phys. Lett.* B351 (1995) 546.

[123] M. B. Voloshin, *Phys. Rev.* D51 (1995) 3948.

[124] M. Lu et al., *Phys. Rev.* D55 (1997) 2827.

[125] I. Bigi, B. Blok, M. Shifman, N. Uraltsev and A. Vainshtein, in *B Decays*, Ed. S. Stone, 2nd edition (World Scientific, Singapore 1994), p. 132.

[126] G. Bellini, I.I. Bigi and P. Dornan, *Phys. Rep.* 289 (1997) 1.

[127] I. Bigi and N. Uraltsev, *Z. Phys.* C62 (1994) 623.

[128] H. Fritzsch and P. Minkowski, *Phys.Lett.* 90B (1980) 455.

[129] M. Baek, J. Lee, C. Liu and H. Song, SNUTP 97-064, hep-ph/9709386.

[130] M. Neubert and C.T. Sachrajda, *Nucl. Phys.* B483 (1997) 339.

[131] M. Neubert, Talk at 2d *Int. Conf. on B Physics and CP Violation*, Honolulu, Hawaii, 24-27 March 1997 hep-ph/9707217.
[132] N. Uraltsev, *Phys. Lett.* **B376** (1996) 303.

[133] See, e.g. J. Rosner, *Phys. Lett.* **B379** (1996) 267, and references therein.

[134] P. Colangelo and F. De Fazio *Phys. Lett.* **B387** (1996) 37.

[135] M. Shifman, *Lectures on Heavy Quarks in Quantum Chromodynamics*, in *QCD and Beyond*, Proc. Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 95), Ed. D.E. Soper (World Scientific, Singapore, 1996), p. 409 [hep-ph/9510377].

[136] B. Blok and T. Mannel, *Mod. Phys. Lett.* **A11** (1996) 1263.

[137] See, e.g., A. Falk, *Theory of Rare B Decays*, (Talk at the Int. Symposium on Vector Boson Interactions, UCLA, February 1-3, 1995), [hep-ph/9503483].