Research of the stability of some hereditary dynamic systems

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Abstract. In the training course of the theory of differential equations, there exists a section on the investigation of the stability of systems of differential equations. If the system of differential equations consists of differential equations of integer order, then the stability theory of Lyapunov is usually used to research the stability of their rest points. However, in the case when the system of differential equations consists of differential equations of non-integer order, then it is necessary to use other methods of investigating the stability of such systems. Therefore, this article is devoted to the method of investigating the rest points of systems of differential equations of fractional order. In this paper we will investigate the stability of the rest points of the hereditary dynamical systems by the example of some fractal oscillators. Moreover, we will consider two types of hereditary dynamical systems: commensurable and incommensurate, for which the corresponding stability theorems for rest points are valid. Next, examples of applying these stability theorems to a fractal linear oscillator, the Duffing fractal oscillator, will be considered. The results of the research of the stability of the rest points of the hereditary dynamical systems were confirmed by constructing phase trajectories for the fractal oscillators under consideration. This article can be useful in the study of a fairly new section in the theory of differential equations-fractional calculus.

1. Introduction

The study of dynamical systems that possess fractal properties or power memory has an important theoretical and practical significance. The presence of memory in a dynamic system indicates the dependence of its current state on the finite number of its previous states. This leads to nonlocal properties of dynamical systems, for example, in mechanics, the effect of aftereffect is known in the description of viscoelastic media [1, 2], in material science, fatigue of materials characterized by the gradual accumulation of defects under the action of stresses, which leads to the destruction of the material [3], in the economy — the effects of dynamic memory in economic theory [4] and even in medicine [5].

Nonlocal (fractal) properties also can have oscillatory systems - fractal oscillators [6]. These oscillators can have new oscillatory regimes and contain previously known ones that are typical for classical oscillators, which is very important in studying oscillatory processes in various applied problems [7].
One of the important directions in the research of fractal oscillators in the framework of qualitative analysis is the study of the stability of their rest points [8-10], which provides information on the vibrational regimes considered by the hereditary dynamic system [11-13]. Therefore, the present paper is devoted to the study of the stability of the quiescent points of hereditary oscillatory systems using the example of fractal oscillators.

2. Statement of the problem and methods of research

It is necessary to research the stability of the rest point of the hereditary dynamical system, which can be written down in the form of the following Cauchy problem:

\[ \partial_{\alpha}^\beta x (\eta) + \lambda(x(t), t) \partial_{\alpha}^{\beta}x(\eta) = f(x(t), t), \quad x(0) = x_0, \dot{x}(0) = y_0, \tag{1} \]

derivatives of fractional orders of Gerasimov-Caputo [1], [3]:

\[ \partial_{\alpha}^\beta x (\eta) = \frac{1}{\Gamma(2-\beta)} \int_0^t \frac{\dot{x}(\tau) d\tau}{(t-\eta)^{2-\beta}}, \partial_{\alpha}^{\beta}x(\eta) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{x}(\tau) d\tau}{(t-\eta)^{1-\gamma}}, 1 < \beta < 2, 0 < \gamma < 1, \]

where \( x(t) \in C^2 [0, T] \) – bias function, nonlinear function, \( \lambda(x(t), t) \) – nonlinear friction in the oscillatory system, \( f(x(t), t) \) – a nonlinear function that contains an external action on the oscillating system and its restoring force, \( \dot{x}(t) = dx/dt, \ddot{x}(t) = d^2x(t)/dt^2 \) – derivatives of the first and second orders, \( T \) – simulation time, \( x_0 \) and \( y_0 \) – given constants that determine the initial conditions of the Cauchy problem (1).

**Remark 1.** The questions of the justification for the existence and uniqueness of the solution of the Cauchy problem (1) were considered in [14], taking into account that the functions \( \lambda(x(t), t) \) and \( f(x(t), t) \) have the necessary smoothness properties.

**Definition 1.** A dynamical system characterizing the class of oscillators with power memory that represents the Cauchy problem (1) will be called the class of hereditarity (fractal) oscillators.

**Remark 2.** The model equation (ref eq1) can be conveniently rewritten as the following dynamic system:

\[
\begin{align*}
\partial_{\alpha}^{\beta} x_1 (\tau) &= x_2 (t), \quad \alpha_1 = \gamma, \\
\partial_{\alpha}^{\beta} x_2 (t) &= f(x_1 (t), t) - \lambda(x_1 (t), t) x_2 (t), \quad \alpha_2 = \beta - \gamma,
\end{align*}
\]

and the initial conditions are rewritten as:

\[ x_1 (0) = x_0, x_2 (0) = y_0. \tag{3} \]

**Remark 3.** Note that in dynamical system (2) fractional orders \( \alpha_1 \) and \( \alpha_2 \) have the following properties: \( 0 < \alpha_1 < 1, \) and \( 0 < \alpha_2 < 2, \) where, if \( \{ \beta \} \leq \{ \gamma \}, \) then \( 0 < \alpha_2 \leq 1, \) and for \( \{ \beta \} > \{ \gamma \}, \) then \( 1 < \alpha_2 < 2, \) where \( \{ \} \) is the fractional part of the number. These properties must be taken into account when constructing finite-difference schemes for finding the numerical solution of the Cauchy problem (2), (3).

We recall that the object of research in the theory of stability of dynamical systems are the points of their equilibrium or rest.

**Definition 2.** The equilibrium points \( E^* (x^*_1, x^*_2) \) of the dynamical system (2) are solutions of a system of algebraic equations of the form:

\[
\begin{align*}
x^*_1 (t) &= 0, \\
f(x^*_1 (t), t) - \lambda(x^*_1 (t), t) x^*_2 (t) &= 0.
\end{align*}
\]

In [6], the concepts of commensurable and incommensurate systems were introduced.
Definition 3. A dynamical system (2) is said to be commensurable if \( \alpha = \alpha_1 = \alpha_2 \) and incommensurable if \( \alpha_1 \neq \alpha_2 \).

Theorem 1. The equilibrium points \( E^* (x_1^*, x_2^*) \) of the commensurable dynamical system (ref eq2) are said to be asymptotically stable if all eigenvalues \( \xi_i \) of the Jacobi matrix:

\[
J = \begin{bmatrix}
\frac{\partial f(x_1^*, t)}{\partial x_1^*} & 0 \\
-\frac{\partial \lambda(x_1^*, t)}{\partial x_1^*} & 1
\end{bmatrix}
\]

satisfy the following conditions:

\[
|\arg (\xi_i)| > \frac{\alpha \pi}{2}, i = 1, ..., n,
\]

where \( n \) is the order of the characteristic equation (5).

Similarly, for an incommensurable dynamical system (2), the following theorem [6].

Theorem 2. The equilibrium points \( E^* (x_1^*, x_2^*) \) of an incommensurable dynamical system (2) are called asymptotically stable, where \( \alpha_i = \mu_i / m \), \( m \) is an integer if the eigenvalues \( \xi_i \) calculated by them satisfy the characteristic equation:

\[
det (\text{diag} ([\xi_i^{m \alpha_1}, \xi_i^{m \alpha_2}]) - J) = 0,
\]

and the following conditions:

\[
|\arg (\xi_i)| > \frac{\varphi \pi}{2}, i = 1, ..., n, \varphi = 1/m,
\]

where \( n \) is the order of the characteristic equation (7).

The proofs of Theorem 1 and Theorem 2 can be found in [9].

3. Results of research
Consider the application of Theorem 1 and Theorem 2 to research the rest points of specific types of fractal oscillators and construct corresponding phase trajectories for them.

Remark 4. Note that for the construction of phase trajectories in the examples in question, we solve numerically the Cauchy problem (1) using an explicit non-local finite difference scheme whose properties were considered in [15] and [16].

In [8], with the help of Theorem 1 and Theorem 2, the rest points for the Van Der Pol fractal oscillator was investigated. Let us consider examples of other fractal oscillators.

Example 1. Fractal linear oscillator. Consider the case when in the system (2): \( f(x_1, t) = -\omega^\beta x_1(t) + \delta \cos (\mu t) \) and \( \lambda_1 (x_1, t) = \lambda \), which corresponds to the simplest linear fractal oscillator considered in [17]:

\[
\begin{align*}
\partial_{\alpha_1} x_1 (\tau) &= x_2 (t), \quad \alpha_1 = \gamma, \\
\partial_{\alpha_2} x_2 (t) &= -\omega^\beta x_1 (t) + \delta \cos (\mu t) - \lambda x_2 (t), \quad \alpha_2 = \beta - \gamma, \\
x_1 (0) &= x_0, \quad x_2 (0) = y_0,
\end{align*}
\]

The fractal oscillator (9) is a generalization of a harmonic oscillator with friction and external harmonic action. The rest points of the system (9) can be easily determined from the solution of the following algebraic system:

\[
\begin{align*}
x_1^* (t) &= 0, \\
-\omega^\beta x_1^* (t) + \delta \cos (\mu t) &= 0,
\end{align*}
\]
and the Jacobian of the system (9) has the form:

\[
J = \begin{bmatrix}
0 & 1 \\
-\omega^2 & -\lambda
\end{bmatrix},
\]

(11)

**Remark 5.** Note that the Jacobian (11) doesn’t depend on the rest points \(E^*(x_1^*, x_2^*)\), which can be obtained from the solution of the system (10). Therefore, the classification of the rest points for the original system (9) will be determined by the eigenvalues — the roots of the characteristic equation.

Consider the simplest case of a commensurable system (9), where \(\alpha_1 = \alpha_2 = 1\), which corresponds to a classical harmonic oscillator with friction and external periodic action. Then the characteristic equation (5) takes the form:

\[
\xi^2 + \lambda \xi + \omega^2 = 0,
\]

(12)

whence \(\xi_{1,2} = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\omega^2}}{2}\). There can be three cases: 1) \(\lambda \geq 2\omega\); 2) \(\lambda < 2\omega\), 3) \(\lambda = 0\).

As we know, from the stability theory of systems of differential equations from the solution of the characteristic equation (12) in the first case, we get a stable node, since the roots are real and negative, in the second case we get a stable focus, since the roots are complex conjugate with the negative real part, and the third case yields purely imaginary roots to which the singular point of the system corresponds to the center.

We show this fact with the help of Theorem 1. For the first case we put: \(\lambda = 1.5\), \(\omega = 0.5\), then the solution of equation (2) has the following roots: \(\xi_1 = -0.1909830056\) and \(\xi_2 = -1.309016994\) for which the condition (6) is met. For the second case, we put: \(\lambda = 0.15\), \(\omega = 0.5\), then the roots obtained by the roots \(\xi_{1,2} = -0.0750 \pm 0.494329983I\), will also satisfy the condition (6). In the third case, the roots have the form \(\xi_{1,2} = \pm 0.50I\), then we get the equality \(|\arg (\xi_{1,2})| = \pi/2\), which corresponds to the rest point of the center. In Fig. 1 shows the phase trajectories for these three cases.

**Figure 1.** Phase trajectories for a commensurable system (9) with the parameters \(\alpha_1 = \alpha_2 = 1\), \(\delta = 0\), \(\omega = 0.5\), \(\beta = 2\), constructed according to two initial conditions: \(x_1(0) = 1, x_2(0) = 0; x_1(0) = 0.4, x_2(0) = 0\) for the following three cases: a) \(\lambda \geq 2\omega, t \in [0, 200]\); b) \(\lambda < 2\omega, t \in [0.200]; \) c) \(\lambda = 0, t \in [0.20]\)
Consider the case of an incommensurate system (9) for this purpose, we choose the values of the parameters: \( \alpha_2 = 0.8 = 8/10, \alpha_1 = 0.6 = 6/10 \). Taking into account the Jacobian (11), we’ll compose the characteristic equation by the formula (7):

\[
\xi^{14} + 0.15\xi^8 + 0.3789291416 = 0,
\]

whose roots will have the following form:

\[
\xi_i = \begin{bmatrix}
2.901454238, 0.2401384155, 2.481336561, 0.6602560929, 2.012781014, 1.128811639, \\
1.570796327, 1.570796327, 2.012781014, 1.128811639, 2.481336561, 0.6602560929, \\
2.901454238, 0.2401384155
\end{bmatrix}
\]

Note that all these roots satisfy the condition (8) of Theorem 2, that is, \(|\arg(\xi_i)| > \pi/20\).

Therefore, we come to the conclusion that all rest points of the system (9) are stable for this case and correspond to a stable limit cycle (Fig. 2).

**Figure 2.** A stable limit cycle for the Duffing oscillator with a memory with parameters \( \alpha_1 = 0.8, \alpha_2 = 0.6 (\beta = 1.4, \gamma = 0.8), \lambda = 0.15, \delta = 0.3, \omega = 0.5, \mu = 1, t \in [0.100] \) with initial conditions: curve 1 - \( x_1(0) = 1, x_2(0) = 0 \); curve 2 - \( x_1(0) = 0.4, x_2(0) = 0 \).

**Example 2.** Duffing Fractal Oscillator. Suppose that the following equations hold in the system (2):

\[
f(x_1, t) = x_1(t) - x_1^3(t) + \delta \cos(\mu t) \quad \text{and} \quad \lambda_1(x_1, t) = \lambda,
\]

which corresponds to the Duffing fractal oscillator [18]:

\[
\begin{align*}
\partial_{\alpha_1}^0 x_1(\tau) &= x_2(t), \alpha_1 = \gamma, \\
\partial_{\alpha_2}^0 x_2(t) &= x_1(t) - x_1^3(t) + \delta \cos(\mu t) - \lambda x_2(t), \alpha_2 = \beta - \gamma, \\
x_1(0) &= x_0, x_2(0) = y_0,
\end{align*}
\]

The equilibrium points of the system (9) are determined from the system of algebraic equations:

\[
\begin{align*}
x_1^2(t) &= 0, \\
x_1(t) - x_1^3(t) + \delta \cos(\mu t) &= 0,
\end{align*}
\]

and the Jacobian for the system (6) has the form:

\[
J = \begin{bmatrix}
0 & 1 \\
1 - 3x_1^2 & -\lambda
\end{bmatrix},
\]

Put \( \lambda = 0.15, \delta = 0.3, \mu = 1, t = 1 \). Then the system (9) has three points of rest:

\[E_1(1.07288371, 0), E_2 = (-0.90615851, 0), E_3 = (-0.16672520, 0).\]
We consider the case of a commensurable system, for example, when in (9) \( \alpha_1 = \alpha_2 = 1 \). In this case the system (9) becomes a classical oscillator Duffing.

Then for the first quiescent point \( E_1 \) we have the following proper values \( \xi_{1,2} = -0.750 \pm 1.564485021i \), for the second point \( E_2 \) the eigenvalues: \( \xi_{1,2} = 0.750 \pm 1.207371004i \), and for the third point \( E_3 \) the eigenvalues have the form:

\[
\xi_1 = 1.035329694, \xi_2 = -0.8853296945.
\]

Note that the condition (6) of Theorem 1 satisfies only the eigenvalue \( \xi_2 \) for the point \( E_3 \), which is a saddle. This implies that the system (9) is unstable. The phase trajectory for this example is shown in Fig. 3.

![Figure 3](image)

**Figure 3.** The classical chaotic attractor of Duffing with parameters \( \alpha_1 = \alpha_2 = 1, \lambda = 0.15, \delta = 0.3, \mu = 1, t \in [0, 200], x_1 (0) = 0.21, x_2 (0) = 0.13 \)

In Fig. 3. It’s seen that the classical oscillator has a chaotic regime. It is interesting to note that if we choose other parameters, for example, as in [6]: \( \alpha_1 = \alpha_2 = 0.95, \lambda = 0.5, \delta = 1.3, \mu = 1, t \in [0, 200], x_1 (0) = 0.21, x_2 (0) = 0.13 \), then all eigenvalues of \( \xi_i \) satisfy the condition (6) and we get a stable limit cycle (Figure 4).

![Figure 4](image)

**Figure 4.** A stable limit cycle for the Duffing oscillator with memory with parameters \( \alpha_1 = \alpha_2 = 0.95 (\beta = 1.9, \gamma = 0.95), \lambda = 0.5, \delta = 1.3, \mu = 1, t \in [0, 200] \) with the initial conditions: curve 1 - \( x_1 (0) = 0.21, x_2 (0) = 0.13 \); curve 2 - \( x_1 (0) = -1, x_2 (0) = 2 \)
From Fig. 4. It can be seen that the phase trajectories go to the same limit cycle under different initial conditions - points inside and outside it.

Consider the case of an incommensurate system (9). Let the parameters $\alpha_1$ and $\alpha_2$ have the values: $\alpha_1 = 0.8 = 8/10$, $\alpha_2 = 0.6 = 6/10$. Then $\varphi = 1/10$, and the remaining parameters from the previous case. The system (9), as we have established, has three quiescent points: $E_1 (1.07288371, 0)$, $E_2 = (0.90615851, 0)$, $E_3 = (-0.16672520, 0)$. Let us find the eigenvalues $\xi_i$ according to the characteristic equation (7) from the condition of Theorem 2.

For the first quiescent point $E_1$, the characteristic equation has the form:

$$\xi^{14} + 0.15\xi^8 + 2.45323837966222 = 0,$$

for $E_2$ the characteristic equation is $\xi^{14} + 0.15\xi^8 - 0.916608122093736 = 0$, for $E_3$ the characteristic equation $\xi^{14} + 0.15\xi^8 + 1.46336974243152 = 0$, from which all points satisfy condition (8), i.e. executed $|\arg(\xi_i)| > \frac{\pi}{20}$, $i = 1, \ldots, 14$. As in the previous example, this situation corresponds to a stable limit cycle (Figure 5).

![Figure 5](image)

**Figure 5.** A stable limit cycle for the Duffing oscillator with a memory with parameters $\alpha_1 = 0.8, \alpha_2 = 0.6 (\beta = 1.4, \gamma = 0.8), \lambda = 0.15, \delta = 0.3, \mu = 1, t \in [0.200]$ with the initial conditions: curve 1 - $x_1 (0) = 0.7, x_2 (0) = 0.4$; curve 2 - $x_1 (0) = 1, x_2 (0) = 0.1$

4. Conclusion
The article described a technique for researching the stability of quiescent points of hereditary dynamical systems using the example of fractal oscillators. The results of the researches were compared with the phase trajectories of the fractal oscillators and gave good agreement. This work can be used as a methodical in the preparation of training courses on the theory of differential equations or qualitative analysis of differential equations with derivatives of fractional orders.

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