Green functions of mass diffusion waves in porous media

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Keywords: mass wave, diffusion, green function, porous media

Abstract
A formulism of frequency-domain mass diffusion-waves in porous media is derived by means of Fourier transform. In analogy to conventional thermal-wave fields, internally consistent Green functions in Cartesian coordinates for linear mass diffusion-waves is also presented for infinite, semi-infinite and finite-size domains in three-dimensional spaces. The Green functions are utilized to analyze the response of a particular type of mass diffusion physical system to any arbitrary tracer source distributions. This method allows the introduction of frequency-dependent physically intrinsic properties of porous media. The Green functions presented in this letter may significantly advance understanding in linear mass diffusion-wave physics in porous media, and can be applied to retrieve spatial-temporal diffusive-wave fields from ambient mass fluctuations in geological reservoirs.

1. Introduction

Green functions are used to describe the response of a particular type of physical system to a point source in spatial-temporal domains [1]. Therefore, the physical field response to any arbitrary source distribution can be found by a convolution integral of the distribution with the Green function over the source volume [2]. Green functions in diffusive-wave physics have been proposed by Mandelis [3]. Green function techniques have been widely applied in thermal-wave physics [4–7]. Mandelis [8] has presented method and Green functions of diffusion-wave fields. Diffusion waves arise from the classical parabolic diffusion equation with an oscillatory force function in homogeneous media [9]. Diffusion-wave methods have been developed in the study of heat transfer [10], diffusive neutron waves [11], diffusive viscosity waves [12], pressure diffusion in porous media [13–16] and mass transport [17]. Diffusion waves are heavily damped with relatively slow velocity and short wavelength. The penetration depth and complex wavenumber describe diffusion-wave behavior. Diffusion waves in porous media obey an accumulation-depletion law [14].

Recently diffusion-waves of mass transport have been receiving much attention in geochemistry, and found applications in volcanic eruption linked by radioactive element transfer [18] and organic compound migration [19] in soil. Linear mass diffusion-wave mathematical formalisms are based on harmonic wave solutions, or Laplace transform methods [20]. These theories have introduced theoretical treatments of mass diffusion-wave fields. Green functions can be used to model the response of a system to a prescribed excitation with knowing the internal properties of the system. Temporal-spatial Green functions for mass diffusion-waves have been presented by Paterson [21]. Mass field response to any arbitrary mass source distribution can be represented as a convolution integral of the distribution with a Green function over source coordinates. An important question arises over the existence of the spectrum of chemical tracer concentrations observed in a geological reservoir, which encompasses the entire spectral bandwidth. To analyze the conventional mass diffusion-wave behavior, the wideband spectral concentration measurements must be reduced to a single spectral component. The Green functions of frequency domain must be used to calculate mass-diffusion wave fields.

In this letter, we present a formulism of frequency-spatial mass diffusion in porous media, and present Cartesian-coordinate mass diffusion-wave Green functions for infinite, semi-infinite, and finite-size domains in three dimensional spaces. The linear mass diffusion-wave fields are mathematically analyzed with Green
functions, which can be used for precisely predicting the radioactive element transfer (dispersion-decay) in the large-scale nuclear geological reservoirs. This method allows the introduction of frequency-dependent material properties of geological systems. By virtue of the uniqueness, rapidity of convergence, and closed-form representations of Green functions, mass diffusion will be precisely predicted for all physically acceptable boundary conditions. It is hoped that methodologies of Green functions will form a mathematically rigorous and useful reference for enhancement of shale gas, oil recovery, and prediction of underground radioactive element leakage. Furthermore, Green functions can be used to inversely determine the mass diffusivity, mass source and decay constant of radioactive elements in the Earth.

2. Mathematical methods and green functions

2.1. Green function of mass diffusion wave

Based on the well-known diffusion theorem [9], a mass diffusion (dispersion-decay) wave [22] generated by a source function \( q(r, t) \) in porous media is given by

\[
D_{\text{eff}} \nabla^2 \vartheta(r, t) - \lambda \vartheta(r, t) - \frac{\partial}{\partial t} \vartheta(r, t) = -q(r, t)
\]  

(1)

where \( D_{\text{eff}} \) is the mass effective diffusivity in a porous medium and the function of porosity, and \( \lambda \) presents a decay constant of a radioactive tracer. \( \vartheta(r, t) \) denotes mass concentration. The mass diffusion equation (1) is essentially a modified heat diffusion equation [9] with the lambda term, \( \lambda \vartheta(r, t) \), corresponding to the depletion. The underlining physical phenomenon of the depletion term includes the geological decay of radioactive tracers, and adsorption of the mobile tracer or solute to the surface of porous media. Using Fourier transform [2], the mass diffusion-wave obeys, in the frequency domain, the following equation

\[
D_{\text{eff}} \nabla^2 \Phi(r, \omega) - (\lambda + j\omega)\Phi(r, \omega) = -Q(r, \omega)
\]  

(2)

It notes that the derivation of equation (2) from equation (1) requires that the mass concentration is bounded.

The solution to the homogeneous (dispersion-decay) diffusion wave equation (2) is expressed as

\[
\Phi(r, \omega) = \int_{-\infty}^{+\infty} \vartheta(\mathbf{r}, t)e^{-j\omega t}dt
\]  

(3)

\( Q(r, \omega) \) is a spectrum of the source distribution \( q(r, t) \). \( \Phi(r, \omega) \) presents the mass concentration at location \( \mathbf{r} \) due to the random forcing \( Q(r, \omega) \). The Green function (see appendix) for the linear mass (dispersion-decay) diffusion waves in three-dimensions is given by equation (4):

\[
g(r, t | r_0, t_0) = \frac{1}{8\pi D_{\text{eff}}(t - t_0)^2} \times e^{\left[\frac{\lambda(t-t_0)+\left(r-r_0\right)^2}{4D_{\text{eff}}(t-t_0)}\right]} \times H(t - t_0)
\]  

(4)

The Green function (4) is the solution to the mass diffusion equation (1) when the source is a delta function. Where \( r(r_0) \) is the coordinate of the observation (source) point with respect to the origin, \( t(t_0) \) presents the observation (source appearance) time, and \( H \) denotes the Heaviside function \( H(t - t_0) = \begin{cases} 1, & t \geq t_0 \\ 0, & t < t_0 \end{cases} \). The Green function of equation (4) satisfies

\[
D_{\text{eff}} \nabla^2 g(r, t | r_0, t_0) - \lambda g(r, t | r_0, t_0) - \frac{\partial}{\partial t}g(r, t | r_0, t_0) = -\delta(r - r_0)\delta(t - t_0)
\]  

(5)

here the Green function for parabolic equations in three variables \( g(r, t | r_0, t_0) \) describes the response of the system at the point \( r \) to a point source located at \( r_0 \). The point source is given by \( \delta(r - r_0)\delta(t - t_0) \), the product of Dirac delta functions. \( \nabla^2 \) is the Laplace operator. Taking the temporal Fourier transform of equation (4)

\[
G(\mathbf{r} | r_0, \omega; t_0) = \int_{-\infty}^{+\infty} g(r, t | r_0, t_0)e^{-j\omega t}dt
\]  

(6)

and using method presented by Mandelis [8, 26], we find

\[
D_{\text{eff}} \nabla^2 G(\mathbf{r} | r_0, \omega; t_0) - (\lambda + j\omega)G(\mathbf{r} | r_0, \omega; t_0) = -\delta(r - r_0)e^{-j\omega t_0}
\]  

(7)

Use has been made of the notation [8]

\[
G(\mathbf{r} | r_0, \omega)e^{-j\omega t_0} = G(\mathbf{r} | r_0, \omega; t_0)
\]  

(8)
The Green function can be written by
\[ D_{\text{eff}} \nabla^2 G(\mathbf{r}|\mathbf{r}_\text{o}, \omega) - (\lambda + j\omega) G(\mathbf{r}|\mathbf{r}_\text{o}, \omega) = -\delta(\mathbf{r} - \mathbf{r}_\text{o}) \] (9)

By interchanging the coordinates \( \mathbf{r}_\text{o} \leftrightarrow \mathbf{r} \) in equations (2) and (9), multiplying equation (2) by \( G(\mathbf{r}_\text{o}|\mathbf{r}, \omega) \) and equation (9) by \( \Phi(\mathbf{r}_\text{o}, \omega) \), then subtracting the resulting expressions, we have
\[ D_{\text{eff}} [G(\mathbf{r}_\text{o}|\mathbf{r}, \omega) \nabla^2 \Phi(\mathbf{r}_\text{o}, \omega) - \Phi(\mathbf{r}_\text{o}, \omega) \nabla^2 G(\mathbf{r}_\text{o}|\mathbf{r}, \omega)] = -Q(\mathbf{r}_\text{o}, \omega) G(\mathbf{r}_\text{o}|\mathbf{r}, \omega) \]
\[ + \delta(\mathbf{r}_\text{o} - \mathbf{r}) \Phi(\mathbf{r}_\text{o}, \omega) \]

Applying a reciprocity property \([1]\) of Green functions \( G(\mathbf{r} - \mathbf{r}_\text{o}, \omega) = G(\mathbf{r}_\text{o} - \mathbf{r}, \omega) \) and the shifting property of Delta function \([8]\), carrying out an integration by parts in the region of the source volume, and using Green’s theorem \([2, 8]\), it gives
\[ \Phi(\mathbf{r}, \omega) = \iiint_{V_\text{o}} Q(\mathbf{r}_\text{o}, \omega) G(\mathbf{r}|\mathbf{r}_\text{o}, \omega) dV_\text{o} + D_{\text{eff}} \iiint_{V_\text{o}} [G(\mathbf{r}|\mathbf{r}_\text{o}, \omega) \nabla \Phi(\mathbf{r}_\text{o}, \omega) \]
\[ - \Phi(\mathbf{r}_\text{o}, \omega) \nabla G(\mathbf{r}|\mathbf{r}_\text{o}, \omega)] \cdot dS_\text{o} \]
(10)

The time modulation factor \( e^{j\omega t} \) is implicitly complied in equation (9). Here \( \iiint \cdots dV_\text{o} \) denotes an integration over a volume \( V_\text{o} \). The integral \( \iiint \cdots dS_\text{o} \) is over the surface \( S_\text{o} \), which bounds and encloses the volume \( V_\text{o} \). \( \mathbf{r}_\text{o} \) presents a coordinate point on \( S_\text{o} \), \( dS_\text{o} = \mathbf{n}_\text{o} dS_\text{o} \), where \( \mathbf{n}_\text{o} \) is the outward normal to the surface \( S_\text{o} \). \( G_s(\mathbf{r}|\mathbf{r}_\text{o}, \omega) \) is a wideband Fourier spectrum of Green’s function \( g(\mathbf{r}, t \mid \mathbf{r}_0, t_0) \). The equations of (2), (9) and (10) are the fundamental formula for the dispersion-decay diffusion-wave field in porous media. It points out that \( G \) the final expression of mass (dispersion-decay) diffusion fields must be multiplied by \( e^{j\omega t} \).

**2.2. Three dimensional infinite Green functions**
Appendix describes the Green function of mass diffusion-waves (dispersion-decay) for infinite three-dimensional spaces
\[ G(\mathbf{r}|\mathbf{r}_\text{o}, \omega) = \frac{e^{\frac{-\sqrt{(x-x_\text{o})^2 + (y-y_\text{o})^2 + (z-z_\text{o})^2}}{D_{\text{eff}}}}}{4\pi D_{\text{eff}} |\mathbf{r} - \mathbf{r}_\text{o}|} \] (11)

Here \( |\mathbf{r} - \mathbf{r}_\text{o}| = \sqrt{(x-x_\text{o})^2 + (y-y_\text{o})^2 + (z-z_\text{o})^2} \) is a module of the distance vector \( (\mathbf{r} - \mathbf{r}_\text{o}) \) from the point source \((x_\text{o}, y_\text{o}, z_\text{o})\) to the response point \((x, y, z)\) in the Cartesian coordinate, where \( \sqrt{\frac{\lambda + j\omega}{D_{\text{eff}}}} \) is the modified mass diffusion-wave field wavenumber.

**2.3. Three dimensional semi-infinite green functions**
If a linear dispersion-decay diffusion field is divided by a surface at the observation coordinate \( z = 0 \), a semi-infinite geometry is considered. The Green function can be obtained directly from equation (11) by the method of images presented by Mandelis [26]. In mass diffusion wave fields, the complex wavenumber is
\[ \sigma(\omega) = \sqrt{\frac{\lambda + j\omega}{D_{\text{eff}}}} \] (12)

If the Green functions satisfy homogeneous Dirichlet conditions at \( z_\text{o} = 0 \), \( G(\mathbf{r}|\mathbf{r}_\text{o}, \omega) \big|_{z_\text{o}=0} = 0 \), applying the method of images \([1, 26]\), it easily yields
\[ G(\mathbf{r}|\mathbf{r}_\text{o}, \omega) = \frac{1}{4\pi D_{\text{eff}}} \left( e^{\frac{-\sqrt{(x-x_\text{o})^2 + (y-y_\text{o})^2 + (z-z_\text{o})^2}}{|\mathbf{r} - \mathbf{r}_\text{o}|}} - e^{\frac{-\sqrt{(x-x_\text{o})^2 + (y-y_\text{o})^2 + (z-z_\text{o})^2}}{|\mathbf{r} - \mathbf{r}_\text{o}'|}} \right) \] (13)

Here \( |\mathbf{r} - \mathbf{r}_\text{o}| \) and \( |\mathbf{r} - \mathbf{r}_\text{o}'| \) are thus given by
\[ |\mathbf{r} - \mathbf{r}_\text{o}| = \sqrt{(x-x_\text{o})^2 + (y-y_\text{o})^2 + (z-z_\text{o})^2} \]
\[ |\mathbf{r} - \mathbf{r}_\text{o}'| = \sqrt{(x-x_\text{o})^2 + (y-y_\text{o})^2 + (z+z_\text{o})^2} \]

For this choice of \( z_\text{o} = 0 \), the above-mentioned expressions reduce to
\[ |\mathbf{r} - \mathbf{r}_\text{o}| = |\mathbf{r} - \mathbf{r}_\text{o}'| = \sqrt{(x-x_\text{o})^2 + (y-y_\text{o})^2 + (z)^2} \]

If the Green function satisfies Neumann conditions at \( z_\text{o} = 0 \), \( \nabla z(\mathbf{r}|\mathbf{r}_\text{o}, \omega) \big|_{z=0} = 0 \). The impulsive image source argument requires that the mass-wave fluxes cancel out at the interface. The Green function is given by
2.4. Three dimensional green functions for finite geometry

Morse and Feshbach [1] has presented method of images, where image sources must be located at source coordinates as shown in figure 2 of reference [26]. Green functions satisfy either Dirichlet or Neumann boundary conditions at \( z_0 = 0 \), \( L \). For this case, Green function is known for an infinite geometry and it is required for a geometry with finite boundaries. Using the method similar to derivation of equations (13) and (14), the Green function is given by

\[
G(r \mid r_0, \omega) = \frac{1}{4\pi D_{\text{eff}}} \left( e^{-\frac{\sqrt{D_{\text{eff}}}}{r-r_0}} + e^{-\frac{\sqrt{D_{\text{eff}}}}{r-r'_0}} \right)
\]  

(14)

Here \( \pm \) corresponds to Neumann or Dirichlet boundary conditions. With the array of image sources found in Figure 3.7 of reference [8], the following expressions are given by

\[
|r-r_{\text{on}}| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + [z-(2nL+z_0)]^2}
\]

\[
|r-r'_{\text{on}}| = \sqrt{(x-x_0)^2 + (y-y_0')^2 + [z-(2nL+z_0)]^2}
\]

The exact behavior of mass diffusion waves depends on the boundary conditions. The influence of different boundaries is clearly shown from equation (15).

3. Application of green functions

We consider the semi-infinite region with mass concentration specified by \( \Phi(r_0, t) = \Phi_r \) over a interface plane \( z_0 \). The Green function satisfies a homogeneous Dirichlet condition on the source plane \( z_0 \) and obeys equation (13). Without bulk sources considered in half-space \( (x, y, z_0) \), and using \( G(r \mid r_0, \omega)|_{z_0=0} = 0 \), the mass-wave field at a given position directly follows from equation (10):

\[
\Phi(r, \omega) = -D_{\text{eff}} \iiint \Phi(r'_{0}, \omega) \nabla_0 G(r \mid r'_{0}, \omega) \cdot dS_0
\]

where the boundary surface integral over \( dS_0 \) is \( dS_0 = n_0 dx_0 dy_0 \). \( n_0 \) presents the inward normal vector to the surface \( dS_0 \). By means of

\[
\nabla_0 = n_0 \frac{\partial}{\partial z_0} = -n_0 \frac{\partial}{\partial z_0}
\]

(17)

denoting the outward normal vector by \( n_{\text{on}} \), equation (16) can be expressed as

\[
\Phi(r, \omega) = D_{\text{eff}} \iiint \Phi(r'_{0}, \omega) \frac{\partial}{\partial z_0} G(r \mid r'_{0}, \omega)|_{z_0=0} dx_0 dy_0
\]

(18)

From equation (13), it follows that

\[
\frac{\partial}{\partial z_0} G(r \mid r_0, \omega)|_{z_0=0} = \frac{z}{2\pi D_{\text{eff}}} \left( e^{-\frac{\sqrt{D_{\text{eff}}}}{r-r_0}} + e^{-\frac{\sqrt{D_{\text{eff}}}}{r-r'_0}} \right)
\]

\[
r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}
\]

(19)

Combining equations (18) and (19), and multiplying by \( e^{i\omega t} \) yields

\[
\Phi(r, \omega) = \frac{z}{2\pi} \Phi_r e^{i\omega t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( e^{-\frac{\sqrt{D_{\text{eff}}}}{r-r_0}} + \frac{1}{r^3} \right) e^{-\frac{\sqrt{D_{\text{eff}}}}{r-r'_0}} dx_0 dy_0
\]

(20)
With the change of variables \( x - x_0 = \eta \) and \( y - y_0 = \xi \), equation (20) becomes

\[
\Phi(r, \omega) = \frac{z}{2\pi} \Phi_0 e^{i\omega t_0} \int_{-\infty}^{+\infty} \left[ \frac{\lambda + j\omega}{\sqrt{D_{\text{eff}}}} \left( \frac{1}{(\eta^2 + \xi^2 + z^2)^{3/2}} \right) + \left( \frac{1}{(\eta^2 + \xi^2 + z^2)^{3/2}} \right) \right] e^{-(\lambda + j\omega) \sqrt{D_{\text{eff}}}} d\eta d\xi
\]

\[
\times e^{\frac{\lambda + j\omega}{\sqrt{D_{\text{eff}}}} \sqrt{\eta^2 + \xi^2 + z^2}} d\eta d\xi
\]

(21)

By changing Cartesian coordinates to polar coordinate systems [19] as

\[
\rho^2 = \eta^2 + \xi^2 \text{ and } d\eta d\xi = \rho d\rho d\varphi
\]

denoting the radial coordinate by \( \rho \), and the angular coordinate by \( \varphi \), this gives

\[
\Phi(r, \omega) = \Phi_0 e^{i\omega t_0} \int_{0}^{\infty} \rho^2 \left( \frac{\lambda + j\omega}{\sqrt{D_{\text{eff}}}} \left( \frac{1}{(\rho^2 + z^2)^{3/2}} \right) + \left( \frac{1}{(\rho^2 + z^2)^{3/2}} \right) \right) e^{-(\lambda + j\omega) \sqrt{D_{\text{eff}}}} (\rho^2 + z^2) d\rho
\]

(22)

The mass diffusion-wave field represented by equation (22) is in a compact form, which can be separated out two components including the real and imaginary parts. The real \( k_r \) and imaginary \( k_{im} \) square root of \( \sqrt{\frac{\lambda + j\omega}{\sqrt{D_{\text{eff}}}}} \) is

\[
k_r + jk_{im} = \sqrt{\frac{\lambda + \sqrt{\lambda^2 + \omega^2}}{2D_{\text{eff}}} + j \frac{-\lambda + \sqrt{\lambda^2 + \omega^2}}{2D_{\text{eff}}}}
\]

(23)

With respect to Euler’s formula [23], it results in

\[
\Phi(r, \omega) = \int_{0}^{\infty} \Phi_0 \rho (\text{REAL} + j \cdot \text{IMAG}) e^{-k_{im} \sqrt{\rho^2 + z^2}} d\rho
\]

(24)

with \text{REAL} \text{ and } \text{IMAG} \text{ defined in the form}

\[
\text{REAL} = \left( \frac{k_r}{\rho^2 + z^2} + \frac{1}{(\rho^2 + z^2)^{3/2}} \right) \cos (\omega t_0 - k_{im} \sqrt{\rho^2 + z^2}) + \frac{k_{im}}{\rho^2 + z^2} \sin (\omega t_0 - k_{im} \sqrt{\rho^2 + z^2})
\]

\[
\text{IMAG} = \frac{k_{im}}{\rho^2 + z^2} \cos (\omega t_0 - k_{im} \sqrt{\rho^2 + z^2}) - \frac{k_r}{(\rho^2 + z^2)^{3/2}} \left( \frac{1}{(\rho^2 + z^2)^{3/2}} \right) \sin (\omega t_0 - k_{im} \sqrt{\rho^2 + z^2})
\]

The physical mass-wave field is the real part of equation (24). The dispersion-decay of radioactive elements (rubidium) is simulated in a nuclear geological reservoir [18].

Figure 1 depicts the depth-dependent amplitude of the mass-wave field (22) with time-modulation angular frequency as a parameter. It shows that the slope of the spatial decay remarkably increases with increasing \( \omega \). The mass diffusion length, \( \sqrt{\frac{2D_{\text{eff}}}{\omega}} \), known as the penetration depth, describes the root mean square depth [24] of the diffusion-wave penetration into the domain \( \rho > 0 \). As a frequency increases, a mass penetration depth decreases, and the diffusion-wave attenuation accelerates. The imaginary part of the wavenumber is in a compact form, which can be separated out two components including the real and imaginary parts. The real \( k_r \) and imaginary \( k_{im} \) square root of \( \sqrt{\frac{\lambda + j\omega}{\sqrt{D_{\text{eff}}}}} \) is

Such phenomena can be explained on the physical grounds. As mass diffusivity increases, the penetration depth accordingly increases, a fact that corresponds to an increase in the degree of mass transfer. The diffusivity is a transport property of the porous media, which is linearly proportional to the porosity [22]. It indicates that effects of porosity on behaviors of mass diffusion-wave fields become similar to that of the corresponding diffusivity.

Figure 2 illustrates similar depth dependence at a constant modulation frequency of \( 10^{-10} \text{ Hz} \), with domain mass diffusivity as a parameter. The effect of increasing mass diffusivity on the slope of the decay curves is seen to be similar to that of decreasing frequency, as expected from the structure of equation (22). Such phenomena can be explained on the physical grounds. As mass diffusivity increases, the penetration depth accordingly increases, a fact that corresponds to an increase in the degree of mass transfer. The diffusivity is a transport property of the porous media, which is linearly proportional to the porosity [22]. It indicates that effects of porosity on behaviors of mass diffusion-wave fields become similar to that of the corresponding diffusivity.

Figure 3 shows the effects of various values of the parameter \( \lambda \) on the amplitude of the mass diffusion-wave depth profile. It is seen that with increasing \( \lambda \) the slope of the spatial delay increases. The differences in amplitude are sensitive to the value of \( \lambda \). As \( \lambda \) decreases, the degree of mass decay decreases, and the mass diffusion-wave attenuates slowly, leading to mass accumulation within a porous medium. As shown in figures 1–3, amplitude profiles clearly exhibit the periodical behaviors. The wave-front is well behaved and captured. The periodically
forced function with internal damping has remarkable influences on mass diffusion-wave motions. The mass diffusion-wave fields obey the field-gradient-driven accumulation-depletion rules [9]. The application example is directed to geophysical phenomena. However the chemical reactions in geological reservoirs are nonlinear. Future works could be needed to extend the application of Green functions to nonlinear problems [28].

4. Conclusions

We propose a formulism of frequency-domain mass diffusion-waves in porous media, and derive internally consistent Cartesian-coordinate mass-wave Green functions for infinite, semi-infinite, and finite-size domains in three-dimensional spaces. It notes that the three-dimensional semi-infinite Green function is of practical importance, because it describes the exact behavior of diffusive mass excited by an arbitrary source in geological reservoirs. The integral expressions for propagating mass diffusion-wave fields are presented in homogeneous systems and in practically geological geometries. A specific application is explicitly implemented in terms of Dirichlet boundary conditions in a case of semi-infinite region with mass impulse function prescribed over the interface plane. The periodically forced function with internal damping has remarkable influences on mass diffusion-wave motions. The mass diffusion-wave fields obey the field-gradient-driven accumulation-depletion rules. Green functions obtained for mass diffusion-waves can be used for all physically acceptable boundary conditions.
conditions under fixed geometries. The exact behavior of mass diffusion waves depends on the boundary conditions. It is hoped that Green functions presented in this article will form a mathematically rigorous and useful reference for enhancement of shale gas or oil recovery, and migration of radioactivity in nuclear geology. Furthermore, Green functions can be used to determine the mass diffusivity and fluid properties of geological reservoirs.

Acknowledgments

The support of the National Natural Science Foundation of China under Grant No. 41340027 is acknowledged. The anonymous reviewers have given comments and suggestions, which improved this letter.

Appendix

The Green function formulism of the linear mass (dispersion-decay) wave field is given by

$$D_{\text{eff}} \nabla^2 G(r - r_0, \omega) - (\lambda + j\omega) G(r - r_0, \omega) = -\delta(r - r_0)$$  \hspace{1cm} (A1)

Using Fourier transform in spatial-frequency

$$G(r - r_0, \omega) = \frac{1}{(2\pi)^3} \iiint e^{iR} G(s, \omega) d^3s,$$

where $r = r_0$ and applying the Dirac delta function [27] as the Fourier integral $\delta(R) = \frac{1}{(2\pi)^3} \iiint e^{iR} d^3s$, and equating integrands, we obtain

$$G(s, \omega) = \frac{1}{s^2D_{\text{eff}} + \lambda + j\omega}$$  \hspace{1cm} (A2)

$$G(r - r_0, \omega) = \frac{1}{(2\pi)^3} \iiint d^3s \frac{e^{iR}}{s^2D_{\text{eff}} + \lambda + j\omega}$$  \hspace{1cm} (A3)

By integrating over the spherical shell of radius $s$, with $d^3s = s^2 \sin \theta ds d\theta d\varphi$, it yields

$$G(R, \omega) = \frac{1}{(2\pi)^3D_{\text{eff}}} \int_0^\infty s^2 ds \int_0^\pi d\varphi \int_0^\pi e^{iR \cos \theta} \frac{s^2 + \sigma^2}{s^2 + \sigma^2} \sin \theta d\theta = \frac{1}{j\pi R(2\pi)^3D_{\text{eff}}} \int_0^\infty \frac{s^2}{s^2 + \sigma^2} e^{i\sigma Rs} ds$$

$$\times \left( \frac{e^{iR \sigma} - e^{-iR \sigma}}{R} \right)$$

$$= \frac{1}{jR(2\pi)^2D_{\text{eff}}} \int_0^\infty \frac{e^{i\sigma Rs} ds}{s^2 + \sigma^2}$$  \hspace{1cm} (A4)

Now let

$$f(z) \equiv \oint \frac{e^{i\sigma 2z} dz}{z^2 + \sigma^2}$$  \hspace{1cm} (A5)

It is observed that the integrand of $f(z)$ has simple poles over the complex contour at
\[ z = \pm j \sqrt{\frac{\lambda + j\omega}{D_{\omega}}} = \pm j\sigma. \] Using the Cauchy Residue Theorem, it gives
\[ f(z) = \lim_{z \to \infty} \int_{C} e^{zR_{k}dz} + \int_{-\infty}^{\infty} e^{i\sigma z} = 2\pi j \left( \frac{e^{R_{k}}}{2z} \right) \bigg|_{z=-j\sigma} = \pi je^{-\sigma R} = \pi je^{-\sqrt{\frac{\lambda + j\omega}{D_{\omega}}}} \] (A6)

Inserting equation (A6) into equation (A4), it finally leads to the Green function in the frequency domain:
\[ G(R, \omega) = \frac{-\sqrt{\frac{\lambda + j\omega}{D_{\omega}}}}{4\pi RD_{\text{eff}}} \] (A7)

Applying the Fourier inversion transform [8] of equation (A7), it gives
\[ g(R, t) = \frac{1}{8[\pi D_{\text{eff}}(t-t_{0})]} \times e^{-\sqrt{\frac{\lambda_{t-t_{0}} + (r-r_{0})^{2}}{4D_{\text{eff}}(t-t_{0})}}} \times H(t-t_{0}) \] (A8)

Subject to the time discontinuity at \( t = t_{0} \) and consistent with the Causality property of time-domain Green function, using the Heaviside function \( H(t-t_{0}) \), we obtain the temporal-spatial Green function of the infinite space domain
\[ g(r, t | r_{0}, t_{0}) = \frac{1}{8[\pi D_{\text{eff}}(t-t_{0})]} \times e^{-\sqrt{\frac{\lambda_{t-t_{0}} + (r-r_{0})^{2}}{4D_{\text{eff}}(t-t_{0})}}} \times H(t-t_{0}) \] (A9)

With the similar methodologies, the one-dimensional Green function is given by
\[ g(x-x_{0}, t-t_{0}) = \frac{1}{2\sqrt{\pi D_{\text{eff}}(t-t_{0})}} \times e^{-\frac{(x-x_{0})^{2}}{4D_{\text{eff}}(t-t_{0})}} \times H(t-t_{0}) \] (A10)

The method of images [1] is used to derive the temporal-spatial Green function for semi-infinite domains
\[ g(r, t | r_{0}, r_{0}', t_{0}) = \frac{H(t-t_{0})}{8[\pi D_{\text{eff}}(t-t_{0})]} \times e^{-\sqrt{\frac{\lambda_{t-t_{0}} + (r-r_{0})^{2}}{4D_{\text{eff}}(t-t_{0})}}} \times e^{-\sqrt{\frac{\lambda_{t-t_{0}} + (r-r_{0}')^{2}}{4D_{\text{eff}}(t-t_{0})}}} \] (A11)

By means of the method similar to derivation of equation (A11), we find the Green function for finite boundaries
\[ g(r, t | r_{1n}, r_{1n}', t_{0}) = \frac{H(t-t_{0})}{8[\pi D_{\text{eff}}(t-t_{0})]} \times e^{-\sqrt{\frac{\lambda_{t-t_{0}} + (r-r_{1n})^{2}}{4D_{\text{eff}}(t-t_{0})}}} \times e^{-\sqrt{\frac{\lambda_{t-t_{0}} + (r-r_{1n}')^{2}}{4D_{\text{eff}}(t-t_{0})}}} \] (A12)

In equations (A11), (A12), the symbols, \( \pm \), present Neumann and Dirichlet boundary conditions, respectively. The terms, \( (r-r_{0})^{2}, (r-r_{0}')^{2}, (r-r_{1n})^{2} \) and \( (r-r_{1n}')^{2} \) are defined in sections 2.3, 2.4.

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