Generation and entanglement study of $N$-mode single photon perfect W-state

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We propose schemes to generate $N$-mode single photon perfect W-state and derive entanglement conditions to detect the entanglement of all generalized $N$-mode single photon perfect W-states. These states are suitable for perfect teleportation and superdense coding over other maximally entangled states belonging to W-class. Based on the evolution of single photon wave function in scalable integrated photonic lattices, we present schemes for the preparation of $N$-mode single photon perfect W-state at desired propagation distance. The integrated waveguide structures can precisely be fabricated and offer low photon propagation losses. We consider both planar and ring type waveguide structures for generation of the $N$-mode single photon perfect W-state. We derive set of generalized entanglement detection conditions using sum uncertainty relations of generalized $su(2)$ algebra operators. We show that any given genuinely entangled $N$-mode single photon state is a squeezed state of a specific $su(2)$ algebra operator and can be expressed as superposition of a pair of generalized $N$-mode single photon perfect W-states which are eigenstates of that specific $su(2)$ algebra operator. We further prove that the squeezed states of an $su(2)$ algebra operator satisfy the entanglement condition obtained using that operator and hence the usefulness of the proposed set of entanglement conditions to detect the entanglement of genuinely entangled single photon states. Finally, we propose an experimental scheme to verify the entanglement of generalized $N$-mode single photon perfect W-states using an integrated photonic circuit that consists of directional couplers and phase shifters and show that the same photonic circuit can also be used to generate generalized $N$-mode single photon perfect W-states. Our results show that optical waveguide structures are ideal platforms for generation and verification of multipartite entangled states which could find novel applications in quantum information science.

I. INTRODUCTION

Multipartite entangled states [1–3] involve complex entanglement structure [4] and has drawn wide interest recently. In the case of three qubits, there exist two inequivalent classes of maximally entangled states, known as the GHZ-class [5] and the W-class states [6]. For more than three qubits, we have other interesting classes of entangled states [7–9]. These multipartite entangled states are useful for various applications in quantum information processing [10–12]. One of such applications is quantum teleportation [13–16], where entangled state is shared as quantum channel between the sending and receiving ends. Teleportation with Bell states [17], GHZ states [18] and others multipartite states were shown to be possible. Among the multipartite entangled states, W-class states are known for their robustness against particle loss. A set of states belonging to W-class can be used for perfect teleportation [19, 20]. The generalization of such states for $N$-qubit system is given as,

$$|\tilde{W}\rangle_N = \frac{1}{\sqrt{2}} \left[ \sum_{j=1}^{N-1} \alpha_j |1\rangle_j |0...0\rangle_{1...j-1,j+1...N} \right] + \frac{1}{\sqrt{2}} |00...01\rangle_{1...N}$$

(1)

where $\alpha_j$’s are non-zero complex coefficients such that $|\alpha_j| < 1$ and $\sum_{j=1}^{N-1} |\alpha_j|^2 = 1$.

We denote such states as generalized $N$-qubit perfect W-states. When $\alpha_j = \frac{1}{\sqrt{N-1}}, \forall j$, the state $|\tilde{W}\rangle_N$...
reduces to

$$|W\rangle_N = \frac{1}{\sqrt{2(N-1)}} \left[ \sum_{j=1}^{N-1} |1\rangle_j |0\ldots0\rangle_{1,j-1,j+1,N} \right] + \frac{1}{\sqrt{2}} |00\ldots01\rangle_{1\ldots N},$$

(2)

which has been shown to be useful for perfect teleportation and superdense coding [20, 22]. We refer this state as $N$-qubit perfect W-state. Although the state holds important applications in quantum information processing, generation of such states with more number of qubits has been a challenging task with bulk optical elements. Earlier scheme on generation of such state considers bulk optical elements which is sensitive towards noise [22].

A promising system to realize generalized $N$-qubit perfect W-states is a single photon shared between $N$ spatial modes [23–28]. Single photon path entangled states has been reported to be useful for applications in quantum information processing [20, 29, 30]. Further such states are also studied in quantum random number generation [31, 33], quantum repeater [34], quantum state generation [35], optical Bloch oscillation [36], to name a few examples. Optical photonic waveguides can readily be used to generate such single photon entangled states. Integrated waveguide lattices are gaining interest for their diverse applications in physics. These are scalable, offer low loss and can be precisely fabricated by femtosecond laser direct writing technique [37–39]. They are compact and are realizable with current technologies, and hence can serve as an important tool for generation of $N$-qubit perfect W-state. Works related to generation of entanglement using optical waveguides can be found in Refs [26, 29, 30].

In this paper, we propose schemes to generate $N$-mode single photon perfect W-state using weakly coupled waveguide structures. In our previous work we have used 1D integrated waveguide structure to generate generalized 3-mode perfect W-states [10]. Here we consider both 1D structure (planar structure) and two-dimensional (2D) ring structure of single mode waveguides. In the 1D structure, the spacing between two successive waveguides is kept different to ensure different coupling strengths between waveguides. In 2D ring structure, $N$ waveguides are symmetrically arranged to form a ring and another waveguide is kept at the center of ring.

Two-qubit entanglement can be quantified using entanglement measures [41, 42]. Extensions of bipartite entanglement measures [43, 44] and monogamy relations [45–47] are used to study the multipartite entanglement. The entanglement of single photon systems can be detected using entanglement detection conditions [48–51]. These conditions involve violation of certain inequality relations by entangled states and can be verified experimentally. Entanglement witness operators are also used to detect the entanglement of two-mode and multi-mode single photon systems [27, 28, 52].

Perfect W-states, generated by injecting single photon through coupled waveguide structures, are not biseparable and retain bipartite entanglement between each pairs of modes. We derive a set of entanglement detection conditions based on sum uncertainty relation of $su(2)$ algebra operators to detect the entanglement of generalized $N$-mode single photon perfect W-states. Recently proposed entanglement condition [53] to detect the entanglement of multimode W-type entangled states belongs to this set of conditions. To detect the entanglement of any given genuinely entangled single photon states, an entanglement detection condition obtained from a suitable set of $su(2)$ algebra operators can be used. Finally, we propose an experimental scheme using directional couplers and phase shifters to verify the entanglement of generalized $N$-mode single photon perfect W-states which can also be used to generate generalized $N$-mode single photon perfect W-states. The directional couplers and the phase shifters can be integrated in the photonic chip [54].

Our work is organized as follows. In section II, we describe the generation of $N$-mode single photon perfect W-state using 1D-planar waveguide structure. In section III the generation of that state using 2D-ring structure is explicated. In section IV, we derive the entanglement conditions using $su(2)$ algebra operators and prove that the entanglement of genuinely entangled $N$-mode single photon states can be detected using the derived conditions. In section V the experimental verification of entanglement of $N$-mode single photon perfect W-state using an integrated photonic circuit is demonstrated. Finally we conclude in section VI with a number of directions for further investigation.

## II. Generation of $N$-Mode Single Photon Perfect W-State Using 1D Structure

For one dimensional array of $N$ identical waveguides, as shown in Figure 1, the Hamiltonian can be written as,

$$\hat{H} = \hbar \omega \sum_{j=1}^{N} \hat{a}_j^\dagger \hat{a}_j + \hbar \sum_{j=1}^{N-1} k_{j,j+1} (\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_j \hat{a}_{j+1}^\dagger)$$

(3)

where the first term represents the free propagation of light with $\omega$ proportional to refractive index of the material. $\hat{a}_j^\dagger (\hat{a}_j)$ is the bosonic creation(annihilation) operator. $k_{j,j+1}$ is the coupling strength between waveg-
guides $j$ and $j+1$ which depends on the separation $(d_{j,j+1})$ between them. The Heisenberg equations of motion are given by,
\[ i\frac{d\hat{a}^\dagger_j}{dz} = k_{1,2}\hat{a}^\dagger_2, \]
\[ i\frac{d\hat{a}^\dagger_{j-1}}{dz} = k_{j-1,j}\hat{a}^\dagger_j + k_{j,j+1}\hat{a}^\dagger_{j+1}, \quad (j = 2, \ldots, N - 1) \]
\[ i\frac{d\hat{a}^\dagger_N}{dz} = k_{N-1,N}\hat{a}^\dagger_{N-1}. \tag{4} \]

The solution to the above set of equations can take the form,
\[ \hat{A}^\dagger(z) = e^{-izM} \hat{A}^\dagger(0) \tag{5} \]
where $\hat{A}^\dagger$ is the column of creation operators, $M$ is the coupling matrix and $e^{-izM}$ is the evolution matrix. For given number of waveguides with appropriate coupling strengths between them, Eq. (5) can be used to find the value of $z$ for which the initial state evolves to $N$-mode single photon perfect $W$-state. In the following we consider the generation of 4-mode and 5-mode single photon perfect $W$-states.

![Figure 1. One dimensional array of $N$ identical waveguides. $k_{i,j}$ and $d_{i,j}$ are coupling strength and separation between waveguides $i$ and $j$ respectively.](image)

The 4-mode perfect $W$-state is given as,
\[ |W\rangle_4 = \frac{1}{\sqrt{6}}(|1000 \rangle + |0100 \rangle + |0010 \rangle + \sqrt{3}|0001 \rangle) \tag{6} \]

To generate 4-mode perfect $W$-state, we consider an array of 4 waveguides in which a single photon is injected in the third waveguide. The equations of motion can be written in the matrix form as follows.

\[ \frac{d}{dz} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \\ a_4^\dagger \end{pmatrix} = \begin{pmatrix} 0 & k_{1,2} & 0 & 0 \\ k_{1,2} & 0 & k_{2,3} & 0 \\ 0 & k_{2,3} & 0 & k_{3,4} \\ 0 & 0 & k_{3,4} & 0 \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \\ a_4^\dagger \end{pmatrix} \tag{7} \]

The initial state of the system is $|\psi(0)\rangle = \hat{a}_3^\dagger|0000\rangle$. After propagating a distance ‘$z$’ the state evolves to
\[ |\psi(z)\rangle = C_1|1000\rangle + C_2|0100\rangle + C_3|0010\rangle + C_4|0001\rangle. \]

To obtain the desired state, the parameters $z$ and $k_{i,j}$ are to be selected in such a way that $|C_1|^2 = \frac{1}{5}$, $|C_2|^2 = \frac{1}{9}$, $|C_3|^2 = \frac{1}{9}$ and $|C_4|^2 = \frac{1}{2}$. For the values of coupling constants and distance $z$ listed in TABLE I, the following generalized 4-mode single photon perfect $W$-state can be obtained by solving the above matrix equation.

\[ |\hat{W}\rangle_4 = \frac{1}{\sqrt{6}}(-|1000\rangle + i|0100\rangle + |0010\rangle + i\sqrt{3}|0001\rangle) \tag{8} \]

| Table I. The parameters required for generation of 4-mode and 5-mode single photon perfect $W$-states |
|---|---|---|---|---|
| No. of modes | $k_{1,2}$ (cm$^{-1}$) | $k_{2,3}$ (cm$^{-1}$) | $k_{3,4}$ (cm$^{-1}$) | $k_{4,5}$ (cm$^{-1}$) | $z$ (cm) |
| 4 | 1.2043 | 0.686372 | 0.781121 | - | 1.15042 |
| 5 | 1.0893 | 0.584456 | 0.988893 | 1.53062 | 1.23828 |

The 4-mode single photon perfect $W$-state can be obtained by adjusting the phase terms of individual ket vectors in Eq. (5). This phase adjustment can be performed by adding phase shifters $z_{i,j}$ to individual guides. In this particular case, phase shifters have to be added to provide a phase shift of $\gamma$ in the 1st waveguide and of $\gamma - i\gamma$ in the 2nd and 4th waveguides. Next we consider 5-mode perfect $W$-state which can be represented as.

\[ |W\rangle_5 = \frac{1}{2\sqrt{2}}(|10000\rangle + |01000\rangle + |00100\rangle + |00010\rangle + |00001\rangle) \tag{9} \]

This state can be generated using an array of 5 waveguides. The equations of motion can be written and solved as described above. Assuming that the photon is injected in the central waveguide, for the values of coupling parameters and distance $z$ given in TABLE I...
the following generalized 5-mode perfect state can be obtained.

\[ |\tilde{W}_5\rangle = \frac{1}{2\sqrt{2}}(-|10000\rangle + i|01000\rangle + |00100\rangle + i|00010\rangle - 2|00001\rangle) \]  

(10)

The 5-mode single photon perfect W-state can be generated by adding phase shifters to provide phase shift of ‘\(-1\)’ in the 1st and 5th waveguides and of ‘\(-i\)’ in the 2nd and 4th waveguides.

### III. GENERATION OF N-MODE SINGLE PHOTON PERFECT W-STATE USING 2D STRUCTURE

In the following, we present a scheme for generating single photon perfect W-state with large number of modes. We consider \(N+1\) identical waveguides, one at the center and \(N\) waveguides arranged symmetrically around it, as shown in Figure 2. The central waveguide is coupled with all surrounding waveguides with coupling strength \(\kappa\) and each surrounding waveguide is coupled with nearest neighbors with coupling strength \(C\). This type of two dimensional ring structure can be fabricated precisely using femtosecond laser direct writing technique [37, 38]. A photon is injected at the central waveguide.

The Hamiltonian of this system is

\[
\hat{H} = \hbar\omega \hat{a}^\dagger_{N+1} \hat{a}_{N+1} + \hbar\omega \sum_{j=1}^{N} \hat{a}^\dagger_{j} \hat{a}_{j} + i\kappa \sum_{j=1}^{N} (\hat{a}^\dagger_{N+1} \hat{a}_{j} + \hat{a}^\dagger_{j} \hat{a}_{N+1})
\]

with Heisenberg’s equations of motion

\[
\frac{i}{\hbar} \frac{d\hat{a}^\dagger_{N+1}}{dz} = \kappa \sum_{j=1}^{N} \hat{a}^\dagger_{j},
\]

\[
\frac{i}{\hbar} \frac{d\hat{a}^\dagger_{j}}{dz} = C(\hat{a}^\dagger_{j} + \hat{a}_{j}) + \kappa \hat{a}^\dagger_{N+1},
\]

\[
\frac{i}{\hbar} \frac{d\hat{a}^\dagger_{j+1}}{dz} = C(\hat{a}^\dagger_{j+1} + \hat{a}^\dagger_{j+1}) + \kappa \hat{a}^\dagger_{N+1}, \quad (j = 2, 3, ..., N - 1)
\]

and

\[
\frac{i}{\hbar} \frac{d\hat{a}^\dagger_{N}}{dz} = C(\hat{a}^\dagger_{N} + \hat{a}^\dagger_{N-1}) + \kappa \hat{a}^\dagger_{N+1}.
\]

The solution, \(\hat{a}^\dagger_{N+1}(z)\), can be written as

\[
\hat{a}^\dagger_{N+1}(z) = e^{(-iCz)} \left\{ \cos(\sqrt{C^2 + N\kappa^2}z) \right. \\
+ \frac{iC}{\sqrt{C^2 + N\kappa^2}} \sin(\sqrt{C^2 + N\kappa^2}z) \hat{a}^\dagger_{N+1}(0) \\
- \frac{i\kappa}{\sqrt{C^2 + N\kappa^2}} \sin(\sqrt{C^2 + N\kappa^2}z) \sum_{j=1}^{N} \hat{a}^\dagger_{j}(0) \right\}
\]

(13)

with \(N\kappa^2 = C^2\), \(\hat{a}^\dagger_{N+1}(z)\) is written as

\[
\hat{a}^\dagger_{N+1}(z) = e^{(-iCz)} \left\{ \cos(\sqrt{2Cz}) + \frac{i}{\sqrt{2}} \sin(\sqrt{2Cz}) \hat{a}^\dagger_{N+1}(0) \\
- \frac{i}{\sqrt{2N}} \sin(\sqrt{2Cz}) \sum_{j=1}^{N} \hat{a}^\dagger_{j}(0) \right\}
\]

(14)

When \(Cz = \frac{n\pi}{2\sqrt{2}}\), with \(n\) being odd integer, the cosine term becomes zero. For \(n = 1\), the input state \(\hat{a}^\dagger_{N+1}|000...0\rangle_{1...N} + \rangle\) evolves to

\[
|W\rangle_{N+1} = e^{i\pi/2\sqrt{2}} \left| \frac{i}{\sqrt{2N}} (10...0)_{1...N} |0\rangle_{N+1} + .... \right.
\]

Figure 2. \(N + 1\) waveguides arranged in 2D ring structure. The surrounding waveguides are labelled by integers from 1 to \(N\) and the central waveguide is labelled as \(N + 1\). \(r\) and \(a\) are radius of the ring and the nearest neighbor distance between surrounding waveguides respectively. The angular separation between two nearest surrounding waveguides is \(2\pi/N\).
+ |00...1\rangle_{1...N}|0\rangle_{N+1}) - \frac{i}{\sqrt{2}} |00...0\rangle_{1...N}|1\rangle_{N+1} \right) \tag{15}

In the 2D ring structure, the radius \( r \) of the ring and the nearest neighbor distance \( a \) between surrounding waveguides are related as \( a = 2rsin(\pi/N) \). Due to the condition, \( N\kappa^2 = C^2 \), the value of \( r \) and hence the value of \( a \) depend on the number of surrounding waveguides \( (N) \). This dependence can be found, by taking \( \kappa = ke^{-r/d_0} \) and \( C = ke^{-a/d_0} \) (\( k \) is characteristic coupling strength and \( d_0 \) describes the rate of exponential decay of coupling strength), as

\[
\frac{r}{d_0} = \frac{\text{log}_e(\sqrt{N})}{1 - 2\sin(\pi/N)} \quad \text{with} \quad N > 6 \tag{16}
\]

For example, when \( N = 7 \), we have, \( r \approx 7.35791 \times 6_0 \), \( a \approx 6.38496 \times d_0 \), \( C \approx 1.6867 \times 10^{-3} \) \( k \) and \( \kappa \approx 0.6375 \times 10^{-3} \). The probabilities as function of \( kz \) are shown in Figure 3a. The blue and green curves represent the probabilities of finding the photon at the central waveguide and surrounding waveguides respectively. The red curve represents the probability of finding the photon at a specific waveguide. The values of \( kz \) where the blue and green curves intersect, the 8-mode single photon perfect W-state can be generated by introducing appropriate phase shifts at individual waveguides.

It can be verified that the ratio \( a/d_0 \) decreases as \( N \) is increased. Hence, in this scheme, \( N \) cannot have very large values. In addition, for \( N > 12 \), the second nearest neighbor distance between surrounding waveguides becomes smaller than \( r \) which can be verified from the geometry of 2D ring structure (Figure 2) and hence the higher order coupling between surrounding waveguides are not ignorable.

Next we consider a special case where \( C = 0 \). From Eq. 13 it can be verified that the initial state \( \tilde{a}_{N+1}|000...0\rangle_{1...N+1} \) evolves to the following state.

\[
|W^\prime\rangle_{N+1} = D_1|00...0\rangle_{1...N}|1\rangle_{N+1} + D_2|00...1\rangle_{1...N}|0\rangle_{N+1} + ... + |10...0\rangle_{1...N}|0\rangle_{N+1} \tag{17}
\]

where

\[
D_1 = \cos(\sqrt{N}\kappa z) \tag{18}
\]

and

\[
D_2 = \frac{i\sin(\sqrt{N}\kappa z)}{\sqrt{N}} \tag{19}
\]

Unlike, in the previous case \( (C \neq 0) \), there is no relation between the radius \( (r) \) of the ring and the number of surrounding waveguides \( (N) \). However, the relation between \( a \) and \( r \) remains the same. Hence, in this case, we can have any number of surrounding waveguides by choosing appropriate value for \( r \). Taking \( N = 7 \) and \( r \approx 7.35791 \times 6_0 \), the probabilities are plotted against \( kz \) in Figure 3b. The blue, green and red curves represent the same probabilities as in the previous case. The blue and green curves intersect whenever

\[
0.4 \leq P = P_{\text{blue}} \leq 0.6 \tag{20}
\]

where \( n \) is an odd integer. In Figure 3b, the values of \( kz \) where the blue and red curves are intersecting, the probability of finding the photon is same for all

Figure 3. Probabilities as function of \( kz \) for 2D ring structure with 7 surrounding waveguides for (a) \( C \neq 0 \) and (b) \( C = 0 \). In both cases, blue (green) curve represents the probability of finding the photon at the central waveguide (any one of surrounding waveguides); red curve represents the probability of finding the photon in a specific surrounding waveguide which is same for all surrounding waveguides.
waveguides in the ring structure. Hence at those points, 8-mode single photon W-state can be generated.

Fabricating 2D ring structure with surrounding waveguides interacting with central waveguide but not among themselves will be useful for generating single photon perfect W-state for any higher mode. One possible way to achieve this structure, is by suppressing the evanescent coupling between adjacent waveguides on the ring by maintaining high refractive index contrast between waveguides and annular region between them. In the case of linear array of waveguides, the suppression of evanescent coupling is reported [56–58].

It can be noted that the interaction part of the Hamiltonian (Eq. 11) with \( C = 0 \) can be used to describe the intercavity hopping [59] of a single photon in a system consisting of \( N+1 \) cavities in which \( N \) cavities are coupled to a cavity through optical fibers. The fiber modes can be eliminated by adiabatic elimination process [60–64]. This system can be used to generate \( N+1 \) cavity mode perfect W-state.

IV. ENTANGLEMENT DETECTION CONDITION

Generalized \( N \)-mode single photon perfect W-states belonging to W-class are not biseparable and retain pairwise entanglement. Hence the entanglement detection condition proposed in [53] can be used to detect the entanglement of generalized \( N \)-mode perfect W-states. However, there are certain generalized \( N \)-mode perfect W-states which will not satisfy that condition. For example, the 5-mode generalized perfect W-state, \( |W\rangle_5 = \frac{1}{2\sqrt{2}} \{ |10000\rangle + |01000\rangle - |00100\rangle - |00010\rangle + 2|00001\rangle \} \), will not satisfy the following 5-mode entanglement condition [53].

\[
|\langle \hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4 |(\hat{\mathcal{N}}_{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4})^\dagger |\hat{\mathcal{N}}_{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4} \rangle |^2 > 4 \langle \hat{\mathcal{N}}_{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4} |(\hat{\mathcal{N}}_{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4})^\dagger |\hat{\mathcal{N}}_{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4} \rangle \tag{21}
\]

where

\[
\hat{\mathcal{N}}_{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4} = \frac{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4}{4}
\]

Hence, we derive a new set of entanglement conditions suitable to detect the entanglement of all generalized \( N \)-mode perfect W-states. Inspired by the form of generalized \( N \)-mode perfect W-states [Eq. 1], we consider the following operators.

\[
\hat{L}_1 = \sum_{j=1}^{N-1} [\alpha_j \hat{a}_j^\dagger \hat{a}_N + \alpha_j^\ast \hat{a}_j^\dagger \hat{a}_N]
\]

\[
\hat{L}_2 = \sum_{j=1}^{N-1} [\alpha_j \hat{a}_j^\dagger \hat{a}_N - i \alpha_j^\ast \hat{a}_j^\dagger \hat{a}_N]
\]

and

\[
\hat{\tilde{L}}_1 = \sum_{j,k=1}^{N-1} \left[ \alpha_j^\ast \alpha_k \hat{a}_j^\dagger \hat{a}_k - \hat{N}_{\hat{a}_N} \right] \tag{22}
\]

where \( \alpha_j \)’s are non-zero complex coefficients as described in section 1. For the given set of \( \alpha_j \)’s, these three operators satisfy \( su(2) \) algebra, \([\hat{L}_x, \hat{L}_y] = 2i\epsilon_{xyz} \hat{L}_z\).

The sum of variances of \( \hat{L}_1 \) and \( \hat{L}_2 \) can be written as

\[
(\Delta \hat{L}_1)^2 + (\Delta \hat{L}_2)^2 = 2 \sum_{j,k=1}^{N-1} \langle \alpha_j^\ast \alpha_k \hat{a}_j^\dagger \hat{a}_k \rangle + \langle \hat{N}_{\hat{a}_N} \rangle \tag{23}
\]

Following the arguments given in [53], it can be shown that the following inequality relation is satisfied by fully separable states.

\[
(\Delta \hat{L}_1)^2 + (\Delta \hat{L}_2)^2 \geq 2 \sum_{j,k=1}^{N-1} \langle \alpha_j^\ast \alpha_k \hat{a}_j^\dagger \hat{a}_k \rangle + \langle \hat{N}_{\hat{a}_N} \rangle \tag{24}
\]

Hence, violation of this inequality relation implies that the state is entangled and the entanglement condition can be written as

\[
\left| \sum_{j=1}^{N-1} \langle \alpha_j \hat{a}_j^\dagger \hat{a}_N \rangle \right|^2 > \sum_{j,k=1}^{N-1} \langle \alpha_j^\ast \alpha_k \hat{a}_j^\dagger \hat{a}_k \rangle \tag{25}
\]

The entanglement condition proposed in [53] is a special case \( (\alpha_j = 1/\sqrt{N-1}, \forall j) \) of this condition. Since the \( su(2) \) algebra operators (Eq. 22) conserve the total number of photons, in this paper, we consider the action of these operators only on single photon subspace.

Now any genuinely entangled \( N \)-mode single photon state of the form

\[
|\Psi\rangle = \sum_{j=1}^{N} C_j |1\rangle_j |0...0\rangle_{1-j-1,j+1...N} \tag{26}
\]

with \( C_j = |C_j| e^{i\varphi_j} \) and \( \sum_{j=1}^{N} |C_j|^2 = 1 \), can be rewritten as (global phase \( \varphi_N \) is excluded)

\[
|\Psi\rangle = \sqrt{\frac{1+\lambda}{2}} \sum_{j=1}^{N-1} \alpha_j |1\rangle_j |0...0\rangle_{1-j-1,j+1...N} + \sqrt{\frac{1-\lambda}{2}} |00...01\rangle_{1...N} \tag{27}
\]
where
\[ \lambda = 1 - 2|C_N|^2 \quad \text{and} \quad \alpha_j^* = \frac{C_j e^{-i\phi N}}{\sqrt{\sum_{j=1}^{N-1} |C_j|^2}}. \] (28)

Since \( |C_N| \) is neither 1 nor 0, the value of \( |\lambda| \) is less than one.

The state, \(|\Psi\rangle\), with the above form, can be shown to be \( \hat{L}_1 \)-squeezed state. That is, it can be shown to satisfy the following conditions [51, 65].

\[ (\Delta \hat{L}_1)(\Delta \hat{L}_2) = |\langle \hat{L}_3 \rangle| \quad \text{(30)} \]

\[ (\Delta \hat{L}_1)^2 = |\lambda \langle \hat{L}_3 \rangle| \quad \text{and} \]

\[ (\Delta \hat{L}_2)^2 = \left| \frac{\langle \hat{L}_3 \rangle}{\lambda} \right| \] (29)

Further, the state \(|\Psi\rangle\) can be written as superposition of two orthonormal generalized \( N \)-mode perfect W-states. That is,

\[ |\Psi\rangle = C_+|\tilde{W}\rangle_{N+} + C_-|\tilde{W}\rangle_{N-} \]

with
\[ C_{\pm} = \frac{\sqrt{1 - \lambda} \pm \sqrt{1 + \lambda}}{2} \]

and
\[ |\tilde{W}\rangle_{N\pm} = \frac{1}{\sqrt{2}} \left[ \sum_{j=1}^{N-1} \alpha_j^* |1\rangle_j |0...0\rangle_{1...j-1,j+1...N} \right] \]
\[ + \frac{1}{\sqrt{2}} |00...01\rangle_{1...N} \] (30)

It can be verified that the states \(|\tilde{W}\rangle_{N\pm}\) are eigenstates of \( \hat{L}_1 \) corresponding to the eigenvalues \( \pm 1 \).

For the state, \(|\Psi\rangle\), the separability condition (Eq. 24) becomes,
\[ \lambda^2 + 1 \geq 2. \] (31)

With \( \lambda \in (-1, 1) \), the state \(|\Psi\rangle\) violates the separability condition. Thus any genuinely entangled \( N \)-mode single photon state, in the form as given in Eq. 25 is a squeezed state of an operator belonging to a suitably chosen set of \( su(2) \) algebra operators; it can be expressed as superposition of a pair of orthonormal generalized \( N \)-mode perfect W-states and satisfy the entanglement condition (Eq. 25) obtained from the sum uncertainty relation of that \( su(2) \) algebra operators.

V. EXPERIMENTAL VERIFICATION OF ENTANGLEMENT OF GENERALIZED \( N \)-MODE SINGLE PHOTON PERFECT W-STATES

We now consider the experimental verification of violation of separability condition proposed in the last section. In order to do that the operators \( \hat{L}_1, \hat{L}_2 \) and the one on the right hand side of Eq. 24 have to be measured. An integrated photonic circuit to measure these operators is shown in Figure 4. The circuit consists of \( N - 1 \) directional couplers and \( N \) phase shifters. This circuit can be fabricated on a chip.

The action of directional coupler \( DC_j \) on the input modes \( \hat{b}_{j-1} \) and \( \hat{a}_{j+1} \) is shown below.

\[ \left( \begin{array}{c} \hat{b}_j \\ \hat{c}_j \end{array} \right) = \left( \begin{array}{cc} T_j & R_j \\ R_j & T_j \end{array} \right) \left( \begin{array}{c} \hat{b}_{j-1} \\ \hat{a}_{j+1} \end{array} \right) \] (32)

where, \( T_j \) and \( R_j \) satisfy the conditions, \( |T_j|^2 + |R_j|^2 = 1 \) and \( R_j^* T_j + T_j^* R_j = 0 \).

The input modes of directional coupler \( DC_1 \) are \( \hat{\alpha}_1 = (\hat{b}_0) \) and \( \hat{a}_2 \). The action of phase shifter \( PS_j \) on the mode \( \hat{a}_j \) is given by

\[ \hat{a}_{j} \rightarrow \hat{a}_{j} e^{-i\phi_j} \] (33)

From Eqs. 32 and 33 the mode \( \hat{b}_{N-2} \) can be written as follows.

\[ \hat{b}_{N-2} = \sum_{j=1}^{N-1} \alpha_j \hat{a}_j \] (34)

where

\[ \alpha_1 = \left[ \prod_{k=1}^{N-2} T_{N-1-k} \right] e^{-i\phi_1}, \]
\[ \alpha_j = \left[ \prod_{k=1}^{N-1-j} T_{N-1-k} \right] R_{j-1} e^{-i\phi_j}, \quad j = 2, ..., N - 2 \]
\[ \quad \text{and} \]
\[ \alpha_{N-1} = R_{N-2} e^{-i\phi_{N-1}}. \] (35)

It can be verified that \( \sum_{j=1}^{N-1} |\alpha_j|^2 = 1 \).
For directional coupler, DC\(_{N-1}\), the values of \(R_{N-1}\) and \(T_{N-1}\) are chosen as \(i/\sqrt{2}\) and \(1/\sqrt{2}\) respectively. The phase shift introduced by the phase shifter PS\(_N\) is denoted as \(\phi_N\). The output modes \(b_{N-1}\) and \(\hat{c}_{N-1}\) of DC\(_{N-1}\) can be written as
\[
\hat{b}_{N-1} = \frac{\hat{b}_{N-2} + i\hat{a}_Ne^{-i\phi_N}}{\sqrt{2}} \quad (36)
\]
\[
\hat{c}_{N-1} = \frac{i\hat{b}_{N-2} + \hat{a}_Ne^{-i\phi_N}}{\sqrt{2}} \quad (37)
\]

For \(\phi_N = \pi/2\) and \(\pi\), the photon number difference \(\hat{b}_{N-1}^*\hat{b}_{N-1} - \hat{c}_{N-1}^*\hat{c}_{N-1}\) gives the operators \(\hat{L}_1\) and \(\hat{L}_2\) respectively. Similarly, \(\hat{b}_{N-1}^*\hat{b}_{N-1} + \hat{c}_{N-1}^*\hat{c}_{N-1}\) yields the operator on the right hand side of Eq. (24) (excluding the factor 2) for any value of \(\phi_N\). Thus by measuring the sum and difference of photon numbers at photodetectors \(D_1\) and \(D_2\), the violation of separability condition can be verified.

When the input state is \(|\hat{W}\rangle_{N^+}\) (\(|\hat{W}\rangle_{N^-}\)) the photonic circuit does a unitary transformation such that the photon will be found in the mode \(\hat{b}_{N-1}\) (\(\hat{c}_{N-1}\)) at the output. This can be verified from Eq. (36) (Eq. (37)). Thus the photonic circuit can also be used to measure the fidelity \(|\langle\hat{W}^*\|\hat{W}\rangle|\) of the states \(|\hat{W}\rangle_{N^\pm}\). When the input state is one of the remaining \(N-2\) biseparable, degenerate eigenstates of \(\hat{L}_1\) corresponding to the eigenvalue 0, the photon will be found in one of the remaining \(N-2\) output modes which can be verified from the input-output relations of this circuit. The transformation of a general input state, while measuring the operator \(\hat{L}_1\), can be found by expressing the input state in the eigenbasis of \(\hat{L}_1\). In the same way, the measurement of other two operators involved in the separability condition can be explained.

The values of \(R_j\)’s, \(T_j\)’s (\(j = 1, 2, ..., N-2\)) and \(\phi_k\)’s (\(k = 1, 2, ..., N-1\)) can be chosen suitably to detect the entanglement of a specific generalized \(N\)-mode single photon perfect W-states. For example, in the case of \(N\)-mode single photon perfect W-state [Eq. (2)], the values of \(R_j\) and \(T_j\) (\(j = 1, 2, 3, ..., N-2\)) have to be chosen as \(i/\sqrt{j+1}\) and \(1/\sqrt{j(j+1)}\) respectively. The phase shift (\(\phi_j\)) introduced by the phase shifter PS\(_j\) for \(j = 2, 3, ..., N-1\) have to be \(\pi/2\) and 0 for \(j = 1\).

Finally, it can be verified from Eqs. (36) and (37) that the generalized \(N\)-mode single photon perfect W-states can be generated, using the photonic circuit given in Figure 4 by injecting a photon in either the mode \(\hat{b}_{N-1}\) or \(\hat{c}_{N-1}\).

VI. CONCLUSION

We have considered single photon realization of a set of states belonging to W-class denoted as generalized \(N\)-mode perfect W-states and useful for perfect teleportation and superdense coding. We have derived entanglement conditions which can be used to detect the entanglement of all generalized \(N\)-mode perfect W-states. These conditions are obtained from the sum uncertainty relation of \(su(2)\) algebra operators which are superposition of photon number conserving quadratic operators. By choosing suitable coefficients, a desired entanglement condition can be obtained to detect the entanglement of any given genuinely entangled \(N\)-mode single photon state which can be found to be the squeezed state of an \(su(2)\) algebra operator involved in the entan-
glement condition and further that state can be written as linear combination of a pair of orthonormal generalized $N$-mode perfect W-states which are the eigenstates of that $su(2)$ algebra operator.

We have thereafter considered schemes to generate $N$-mode single photon perfect W-state, a specific generalized $N$-mode single photon perfect W-state, and an experimental scheme to verify the entanglement of generalized $N$-mode single photon perfect W-states. Both state generation and entanglement detection schemes use waveguide structures. The state generation scheme involves injecting a photon into one of $N$ weakly coupled waveguides arranged in two different geometrical structures: $1D$ planar and $2D$ ring structures. These structures are stable, scalable and offer low photon propagation losses and hence are suitable for our purpose. Experimental implementation of entanglement condition is explained using photonic circuit consisting of directional couplers and phase shifters. The same circuit can also be used to generate generalized $N$-mode single photon perfect W-states.

Thus, in future, our schemes can be implemented using photonic integrated lattices, where the necessary optical elements can be realized on chip. The obtained perfect W-state will be useful for performing quantum communication tasks. The non-locality of the state can also be experimentally studied, as single photon entanglement is gaining much interest.

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