The Asymptotic Capacity of the Optical Fiber

Mansoor I. Yousefi

Abstract—It is shown that signal energy is the only available degree-of-freedom (DOF) for fiber-optic transmission as the input power tends to infinity. With \( n \) signal DOFs at the input, \( n - 1 \) DOFs are asymptotically lost to signal-noise interactions. The main observation is that, nonlinearity introduces a multiplicative noise in the channel, similar to fading in wireless channels. The channel is viewed in the spherical coordinate system, where signal vector \( \bar{X} \in \mathbb{C}^n \) is represented in terms of its norm \( |\bar{X}| \) and direction \( \bar{X} \). The multiplicative noise causes signal direction \( \bar{X} \) to vary randomly on the surface of the unit \((2n - 1)\)-sphere in \( \mathbb{C}^n \), in such a way that the effective area of the support of \( \bar{X} \) does not vanish as \( |\bar{X}| \to \infty \). On the other hand, the surface area of the sphere is finite, so that \( \bar{X} \) carries finite information. This observation is used to show several results. Firstly, let \( C(P) \) be the capacity of a discrete-time periodic model of the optical fiber with distributed noise and frequency-dependent loss, as a function of the average input power \( P \). It is shown that asymptotically as \( P \to \infty \), \( C = \frac{1}{2} \log \log P + c \), where \( n \) is the dimension of the input signal space and \( c \) is a bounded number. In particular, \( \lim_{P \to \infty} C(P) = \infty \) in finite-dimensional periodic models. Secondly, it is shown that capacity saturates to a constant in infinite-dimensional models where \( n = \infty \). An expression is provided for the constant \( c \), by showing that, as the input \( |\bar{X}| \to \infty \), the action of the discrete periodic stochastic nonlinear Schrödinger equation tends to multiplication by a random matrix (with fixed distribution, independent of input). Thus, perhaps counter-intuitively, noise simplifies the nonlinear channel at high powers to a linear multiple-input multiple-output fading channel. As \( P \to \infty \) signal-noise interactions gradually reduce the slope of the \( C(P) \), to a point where increasing the input power returns diminishing gains. Nonlinear frequency-division multiplexing can be applied to approach capacity in optical networks, where linear multiplexing achieves low rates at high powers.

I. INTRODUCTION

Several decades since the introduction of the optical fiber, channel capacity at high powers remains a vexing conundrum. Existing achievable rates saturate at high powers because of linear multiplexing and treating the resulting interference as noise in network environments [1]–[3]. Furthermore, it is difficult to estimate the capacity via numerical simulations, because channel has memory.

Multi-user communication problem for an ideal model of optical fiber can be reduced to single-user problem using the nonlinear frequency-division multiplexing (NFDM) [1], [3]. This addresses deterministic distortions, such as inter-channel and inter-symbol interference (signal-signal interactions). The problem is then reduced to finding the capacity of the point-to-point optical fiber set by noise.

There are two effects in fiber that impact Shannon capacity in point-to-point channels. (1) Phase noise. Nonlinearity transforms additive noise to phase noise in the channel. As the amplitude of the input signal tends to infinity, the phase of the output signal tends to a uniform random variable in the zero-dispersion channel [4, Section IV]. As a result, phase carries finite information in the non-dispersive fiber.

(2) Multiplicative noise. Dispersion converts phase noise to amplitude noise, introducing an effect which at high powers is similar to fading in wireless channels. Importantly, the conditional entropy grows strongly with input signal.

In this paper, we study the asymptotic capacity of a discrete-time periodic model of the optical fiber as the input power tends to infinity. The role of the nonlinearity in point-to-point discrete channels pertains to signal-noise interactions, captured by the conditional entropy.

The main result is the following theorem, describing capacity-cost function in models with constant and non-constant loss; see Definition [4].

**Theorem 1.** Consider the discrete-time periodic model of the NLS channel (2) described in Section III with non-zero dispersion. Capacity is asymptotically

\[
C(P) = \begin{cases} 
\frac{1}{n} \log \log P + c, & \text{non-constant loss,} \\
\frac{1}{2n} \log P + c, & \text{constant loss,}
\end{cases}
\]

where \( n \) is dimension of the input signal space, \( P \to \infty \) is the average input signal power and \( c \equiv c(n, P) < \infty \). In particular, \( \lim_{P \to \infty} C(P) = \infty \) in finite-dimensional models.

Intensity modulation and direct detection (photon counting) is nearly capacity-achieving in the limit \( P \to \infty \), where capacity is dominated by the first terms in \( C(P) \) expressions.

From the Theorem 1 and [4, Theorem 1], the asymptotic capacity of the dispersive fiber is much smaller than the asymptotic capacity of (the discrete-time model of) the zero-dispersion fiber, which is \( \frac{1}{2} \log P + c, c < \infty \). Dispersion reduces the capacity, by increasing the conditional entropy. With \( n \) DOFs at the input, \( n - 1 \) DOFs are asymptotically lost to signal-noise interactions, leaving signal energy as the only useful DOF for transmission.

There are a finite number of DOFs in all computer simulations and physical systems. However, as a mathematical problem, the following Corollary holds true.

**Corollary 1.** Capacity saturates to a constant \( c < \infty \) in infinite-dimensional models, including the continuous-time model.

The power level where signal-noise interactions begin to appreciably impact the slope of the \( C(P) \) is not determined in this paper. Numerical simulations indicate that the conditional entropy does not increase with input in the nonlinear Fourier domain, for a range of power larger than the optimal power in wavelength-division multiplexing [5, Fig. 9 (a)]. In this
regime, signal-noise interactions are weak and the capacity is dominated by the (large) number $c$ in the Theorem~I. A numerical estimation of the capacity of the point-to-point fiber at input powers higher than those in Fig.~5 should reveal the impact of the signal-dependent noise on the asymptotic capacity.

The contributions of the paper are presented as follows. The continuous-time model is discretized in Section~III. The main ingredient is a modification of the split-step Fourier method (SSFM) that shows noise influence more directly compared with the standard SSFM. A unit is defined in the modified SSFM (MSSFM) model that plays an important role throughout the paper. The MSSFM and units simplify the information-theoretic analysis.

Theorem~I and Corollary~I are proved in Section~IV. The proof of the Theorem~1, e.g., the asymptotic loss of DOFs, is based on a spherical coordinate system in the paper. Here, a vector $x \in \mathbb{C}^n$ is defined in the Theorem~1. A method (SSFM) that shows noise influence more directly compared with the standard SSFM. A

Theorem~I is proved again in Section~V by considering the limit $\mathcal{P} \to \infty$, which adds further intuition. Firstly, it is shown that, as the input $|X| \to \infty$, the action of the discrete periodic stochastic nonlinear Schrödinger (NLS) equation tends to multiplication by a random matrix (with fixed probability distribution function (PDF), independent of the input). As a result, perhaps counter-intuitively, as $|X| \to \infty$ noise simplifies the nonlinear channel to a linear multiple-input multiple-output (non-coherent) fading channel. Secondly, the asymptotic capacity is computed, without calculating the conditional PDF of the channel, entropies, or solving the capacity optimization problem. Because of the multiplicative noise, the asymptotic rate depends only on the knowledge that whether channel random operator has any deterministic component. The conditional PDF merely modifies the bounded number $c$ in the Theorem~I.

Note that we do not apply local analysis based on perturbation theories (valid in the low power regime). The proof of the Theorem~I, e.g., the asymptotic loss of DOFs, is based on a global analysis valid for any signal and noise; see Section~IV.

II. NOTATION AND PRELIMINARIES

The notation in this paper is motivated by $[3]$. Upper- and lower-case letters represent scalar random variables and their realizations, e.g., $X$ and $x$. The same rule is applied to vectors, which are distinguished using underline, e.g., $\bar{X}$ for a random vector and $\bar{x}$ for a deterministic vector. Deterministic matrices are shown by upper-case letter with a special font, e.g., $\mathbf{R} = (\mathbf{r}_{ij})$. Random matrices are denoted by upper-case letters with another special font, e.g., $\mathbf{M} = (\mathbf{M}_{ij})$. Important scalars are distinguished with calligraphic font, e.g., $\mathcal{P}$ for power and $C$ for capacity. The field of real and complex numbers is respectively $\mathbb{R}$ and $\mathbb{C}$.

A sequence of numbers $X_1, \cdots, X_n$ is sometimes abbreviated as $X^n$, $X^0 = \emptyset$. A zero-mean circularly-symmetric complex Gaussian random vector with covariance matrix $\mathbf{K}$ is indicated by $\mathcal{N}_\mathbb{C}(0, \mathbf{K})$. Uniform distribution on interval $[a, b]$ is designated as $\mathcal{U}(a, b)$.

Throughout the paper, the asymptotic equivalence $C(\mathcal{P}) \sim f(\mathcal{P})$, often abbreviated by saying “asymptotically,” means that $\lim_{\mathcal{P} \to \infty} C(\mathcal{P})/f(\mathcal{P}) = 1$. Letter $c \equiv c(n, \mathcal{P})$ is reserved to denote a real number bounded in $n$ and $\mathcal{P}$. A sequence of independent and identically distributed (i.i.d.) random variables $X_n$, drawn from the PDF $p_X(x)$ is presented as $X_n \sim \text{i.i.d.} p_X(x)$. The identity matrix with size $n$ is $I_n$.

The Euclidean norm of a vector $\bar{x} \in \mathbb{C}^n$ is

$$|\bar{x}| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}.$$ 

This gives rise to an induced norm $|M|$ for matrix $M$. We use the spherical coordinate system in the paper. Here, a vector $x \in \mathbb{C}^n$ is represented by its norm $|x|$ and direction $\hat{x} = x/|x|$ (with convention $\hat{x} = 0$ if $x = 0$). The direction can be described by $m = 2n - 1$ angles.

When direction is random, its entropy can be measured with respect to the spherical measure $\sigma^m(A), A \subseteq S^m$, where $S^m$ is the $m-$sphere

$$S^m = \{ \hat{x} \in \mathbb{R}^{m+1} : |\hat{x}| = 1 \}.$$ 

It is shown in the Appendix~A that the differential entropy can be measured with respect to the Lebesgue and spherical measures, denoted respectively by $h(\bar{X})$ and $h_\sigma(\bar{X})$, are related as

$$h(\bar{X}) = h(|\bar{X}|) + h_\sigma(\bar{X} | |\bar{X}|) + mE \log |\bar{X}|. \quad (1)$$

The entropy power of a random direction $\bar{X} \in \mathbb{C}^n$ is

$$V(\bar{X}) = \frac{1}{2\pi e} \exp \frac{2}{m} h_\sigma(\bar{X}).$$

It represents the effective area of the support of $\bar{X}$ on $S^m$.

III. THE MODIFIED SPLIT-STEP FOURIER METHOD

Signal propagation in optical fiber is described by the stochastic nonlinear Schrödinger (NLS) equation $[1]$ Eq.~2

$$\frac{\partial Q}{\partial z} = L_L(Q) + L_N(Q) + N(t, z), \quad (2)$$

where $Q(t, z)$ is the complex envelope of the signal as a function of time $t \in \mathbb{R}$ and space $z \in \mathbb{R}^+$ and $N(t, z)$ is zero-mean circularly-symmetric complex Gaussian noise with

$$E(\{N(t, z)N^*(t', z')\}) = \sigma^2 \delta(t-t') \delta(z-z'),$$

where $\delta_{WW}(x) \triangleq 2\mathcal{W} \sin \mathcal{W}(2Wx), \sin(x) \triangleq \sin(\pi x)/\pi x$, and $W$ is noise bandwidth. The operator $L_L$ represents linear effects

$$L_L(Q) = \sum_{k=0}^{\infty} j^{k+1} \frac{\beta_k}{k!} \frac{\partial^k Q}{\partial t^k} - \frac{1}{2} \alpha_r(t, z) * Q(t, z), \quad (3)$$

where $\beta_k$ are dispersion coefficients, $* \text{ is convolution and } \alpha_r$ is the residual fiber loss. The operator $L_N(Q) = j\gamma |Q|^2 Q$ represents Kerr nonlinearity, where $\gamma$ is the nonlinearity parameter. The average power of the transmit signal is

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |Q(t, 0)|^2 dt. \quad (4)$$
Definition 1 (Loss Models). The residual loss in \( \rho \) accounts for uncompensated loss and non-flat gain of the Raman amplification in distance and is generally frequency dependent. The constant loss model refers to the case where \( \alpha_r(t, z) \) is constant in the frequency \( f \), i.e., \( \alpha_r(f, z) = \mathcal{F}(\alpha_r(t, z)) \hat{=} \alpha_r(z) \), where \( \mathcal{F} \) is the Fourier transform with respect to \( t \). In realistic systems, however, loss varies over frequency, polarization or spatial models. This is the non-constant loss model. Channel filters act similar to a non-constant loss function.

We discretize (2) in space and time. Divide a fiber of length \( L \) into a cascade of \( n \) of pieces of discrete fiber segments of length \( \epsilon = L/n \) [4, Section III. A]. A small segment can be discretized in time and modeled in several ways. An appropriate approach is given by the split-step Fourier method (SSFM).

The standard SSFM splits the deterministic NLS equation into linear and nonlinear parts. In applying SSFM to the stochastic NLS equation, typically noise is added to the signal. We introduce a modified split-step Fourier method where, instead of noise addition, the nonlinear part of (2) is solved in the presence of noise analytically.

In the linear step, (2) is solved with \( \epsilon > 0 \) in the discrete-time model, linear step in a segment of length \( \epsilon \) consists of multiplying a vector \( X \in \mathbb{C}^n \) by the dispersion-loss matrix \( R = (r_{kl}) \). In the constant loss model, \( R = e^{-\frac{1}{2} \alpha_r U} \), where \( U \) is a unitary matrix. In the absence of loss, \( R \) is unitary. The values of \( r_{kl} \) depend on the dispersion coefficients, \( \epsilon \) and \( n \). In general, all entries of \( R \) are non-zero, although in a small segment, the off-diagonal elements can be very small.

Assumption 1. Matrix \( R \) is fully dispersive, i.e., \( r_{kl} \neq 0 \), for all \( k, l \).

In the nonlinear step, (2) is solved with \( L_N = 0 \) resulting in [7, Eq. 12], [4, Eq. 30]:

\[
Q(t, z) = (Q(t, 0) + W(t, z)) e^{i \Theta(t, z)},
\]

in which

\[
\Theta(t, z) = \int_0^z \left( Q(t, 0) + W(t, l) \right)^2 dl,
\]

where \( W(t, z) = \int_0^z N(t, l) dl \) is Wiener process. The modified nonlinear step in the MSSFM is obtained by discretizing (5).

Divide a segment 0 \( \leq z \leq \epsilon \) into \( L \) sub-segments of length \( \mu = \epsilon/L \). Define \( \Phi: \mathbb{C} \times \mathbb{C}^n \to [0, \infty) \) as

\[
\Phi(X, N) = \gamma \mu \left| X + N_1 \right|^2 + \gamma \mu \left| X + N_1 + N_2 \right|^2 + \cdots
\]

\[
+ \gamma \mu \left| X + N_1 + \cdots + N_L \right|^2,
\]

conditionally known

where \( N_i \sim i.i.d. \mathcal{N}(0, D/L) \), \( D = \sigma^2 W \epsilon/n \). The nonlinear step in a segment of length \( \epsilon \) maps vector \( X \in \mathbb{C}^n \) to vector \( Y \in \mathbb{C}^n \), according to

\[
Y_k = (X_k + N_{k1} + \cdots + N_{kL}) e^{i \Phi(X_k, N_k)},
\]

where \( N_k = (N_{k1}, \cdots, N_{kL})^T \), \( N_i \sim i.i.d. \mathcal{N}(0, D/L) \).

The nonlinear step is a deterministic phase change in the SSFM. In this form, nonlinearity is entropy-preserving and does not interact with noise immediately [6, Lemma 2–3] — unless several steps in the SSFM are considered, which complicates the analysis. In the MSSFM, noise is introduced in a distributed manner within each nonlinear step. This shows noise influence more directly.

Note that, conditioned on \( |Y_k| \), the last term in (6) is known. Other terms in (6) represent signal-noise interactions. They are conditionally unknown and are responsible for capacity limitation.

The MSSFM model for a fiber of length \( L \) consists of the cascade of linear and modified nonlinear steps (without noise addition between them).

Definition 2 (Unit). A unit in the MSSFM model is defined as the cascade of three segments of length \( \epsilon \): A modified nonlinear step \( X \to Y \), followed by a linear step \( U \to V \), followed by another modified nonlinear step \( V \to Y \); see Fig. 1. A unit of length \( 3 \epsilon \) is the smallest piece of fiber whose capacity behaves qualitatively similar to the capacity of the full model with length \( L \).

In the Appendix B it is shown that the input-output relation \( X \to Y \) in one unit is given by

\[
Y = M X + Z,
\]

where \( M = M(X, N_1, N_2) \) is a random matrix with entries

\[
M_{kl} = r_{kl} e^{i \Phi_k + j \Psi_l},
\]

in which

\[
\Psi_l = \Theta(X_l, N_l^1), \quad \Phi_k = \Theta(V_k, N_k^2).
\]

Here \( N_1 = (N_1^1, \cdots, N_1^L)^T \) and \( N_2 = (N_2^1, \cdots, N_2^L)^T \) are \( n \times L \) Gaussian ensembles with i.i.d. entries drawn from \( \mathcal{N}_C(0, D/L) \), independent of any other random variable. The additive noise \( Z = Z(X, N_1, N_2) \) is in general non-Gaussian but bounded in \( |X| \); see (55). Finally, \( Y \) is the output of the linear step in Fig. 1.

The input output relation \( X \to Y \) in a fiber of length \( L \) is obtained by composing \( m = m/2 \) blocks \( Y_k = R(M_k X_k + Z_k) \):

\[
Y_k = M_k X_k + Z(k),
\]

where \( k = 1, 2, \cdots, m \) is the transmission index, \{Z(k)\} is an i.i.d. stochastic process, and

\[
M_k = \prod_{k=1}^{m} R M_{k}, \quad Z(k) = R Z_m + \sum_{k=1}^{m-1} \left( \prod_{l=k+1}^{m} R M_{l} \right) R Z_{k}.
\]

The power constraint (4) is discretized to \( P = \frac{1}{n} \sum X \| X \|^2 \) in the discrete-time model.

Remark 1 (Bandwidth Assumption). Bandwidth, spectral broadening and spectral efficiency in the continuous-time model are discussed in Section IV-A.

Remark 2 (Nonlinearity). Note that \( M(X, N_1, N_2) \) is a nonlinear random operator. Particularly, it depends on input.
Remark 3 (Signal Dimension). Dimension of the input space is \( n \). To approximate the continuous-time model, \( n \to \infty \). However, we let \( n \) be arbitrary, e.g., \( n = 5 \). Dimension should not be confused with codeword length that tends to infinity.

IV. PROOF OF THE THEOREM

We first illustrate the main ideas of the proof via elementary examples.

Consider the additive white Gaussian noise (AWGN) channel \( Y = X + Z \), where \( X \in \mathbb{C} \) is input, \( Y \in \mathbb{C} \) is output and \( Z \sim \mathcal{N}_C(0, 1) \) is noise. Applying chain rule to the mutual information

\[
I(X; Y) = I(X; |Y|) + I(X; \angle Y | |Y|),
\]

where \( \angle \) denotes phase. The amplitude channel \( X \to |Y| \) is

\[
|Y| \approx |X| + Z_r,
\]

where \( Z_r \sim \mathcal{N}_C(0, \frac{1}{2}) \) and \( |X| \gg 1 \). It asymptotically contributes

\[
I(X; |Y|) \to \frac{1}{2} \log \mathcal{P} + c
\]
to the capacity.

Phase, on the other hand, is supported on the finite interval \([0, 2\pi)\). The only way that the contribution of the phase to the capacity could tend to infinity is that, phase noise tends to zero on the circle as \( |X| \to \infty \). Indeed,

\[
\angle Y = \angle X + \tan^{-1}\left( \frac{Z_i}{|X| + Z_r} \right)
\]

\[
\approx \angle X + \frac{Z_i}{|X|},
\]

where \( Z_r, Z_i \sim \text{i.i.d.} \mathcal{N}_C(0, \frac{1}{2}) \). The output entropy is clearly bounded, \( h(\angle Y | |Y|) \leq \log 2\pi \). However,

\[
h(\angle Y | X, |Y|) = h(Z_i) - \mathbb{E} \log |X|
\]

\[
\to -\frac{1}{2} \log \mathcal{P} + c \text{ as } \mathcal{P} \to \infty.
\]

Note that the differential entropy can be negative. The contribution of the phase to the mutual information is

\[
I(X; \angle Y | |Y|) \to \frac{1}{2} \log \mathcal{P} + c'.
\]

Condition (12) implies \( \mathrm{V}(|Y| | X, |Y|) \to 0 \), i.e., the effective phase noise on the unit circle asymptotically vanishes.

Now consider the fading channel \( Y = MX + Z \), where \( X \in \mathbb{C} \) is input, \( Y \in \mathbb{C} \) is output and \( M, Z \sim \text{i.i.d.} \mathcal{N}_C(0, 1) \).

To prepare for generalization to optical channel, we represent a complex scalar \( X \) as \( X = (\Re X, \Im X)^T \). Thus \( Y = MX + Z \), where

\[
M = \begin{pmatrix} M_r & -M_i \\ M_i & M_r \end{pmatrix}, \quad Z = \begin{pmatrix} Z_r \\ Z_i \end{pmatrix},
\]

in which \( M_{r,i}, Z_{r,i} \sim \text{i.i.d.} \mathcal{N}_C(0, \frac{1}{2}) \). As \( |X| \to \infty \), \( Y \approx MX, \ \Re \sim \mathcal{M}_X/|\Re X|, \text{ and randomness in } Y \text{ does not vanish with } |X| \).

Formally,

\[
\bar{h}_s(\bar{Y} | X, |Y|) = h_s(\bar{Y} | |Y|) = h_s(\hat{Y} | |M|^{-1}Y, |Y|) > -\infty,
\]

(13)

where (13) follows because \( a = |M|^{-1} \hat{Y} \) does not determine \( \hat{Y} \) for random \( M \). There are four random variables \( M_{r,i}, \bar{Y}_{1,2} \) for three equations \( M^{-1} \hat{Y} = a \) and \( |Y| = 1 \). As a result, \( I(X; \hat{Y} | |Y|) < \infty \), and \( |Y| \) is the only useful DOF at high powers, in the sense that its contribution \( I(X; \hat{Y} | |Y|) \) to the mutual information \( I(X; Y) \) tends to infinity with \( |X| \).

The zero-dispersion optical fiber channel is similar to the fading channel at high powers. The trivial condition

\[
h(\Theta(., z) | Q(., 0), |Q(., z)| > -\infty, \ \forall Q(., 0),
\]

is sufficient to prove that the capacity of (5) is asymptotically the capacity of the amplitude channel, namely \( \frac{1}{2} \log \mathcal{P} + c \).

The intuition from the AWGN, fading and zero-dispersion channels suggests to look at the dispersive optical channel in the spherical coordinate system. The mutual information can be decomposed using the chain rule

\[
I(Q(0); Q(z)) = I(|Q(0) ; Q(z)| + I(\hat{Q}(0); Q(z) | |Q(0)|)
\]

\[
= I(|Q(0) ; Q(z)| + I(\hat{Q}(0); Q(z) | |Q(z)|)
\]

\[
+ I(\hat{Q}(0); Q(z) | |Q(0)||Q(0)|,
\]

(14)

where we dropped time index in \( Q(t, z) \).

The first term in (14) is the rate of a single-input single-output channel which can be computed in the asymptotic limit as follows. Let \( X \) and \( Y \) represent discretizations of the input \( Q(0, .) \) and output \( Q(z, .) \). Consider first the lossless model. In this case, \( M \) is unitary and from (10), (11) and (55)

\[
|Y|^2 = |MX + Z|^2
\]

\[
= |X + MZ|^2
\]

\[
= |X + Z|^2,
\]

(15)

where \( M^* \) is the adjoint (nonlinear) operator and (15) follows because \( Z \) and \( M^*Z \) are identically distributed when \( Z \sim \mathcal{N}_C(0, mDI_n) \); see Appendix B. Thus \( |Y|^2/(mD) \) is a non-central chi-square random variable with \( 2n \) degrees-of-freedom and parameter \( |\bar{X}|^2/(mD) \). The non-central chi-square conditional PDF \( p(|Y|^2/(mD)) \) can be approximated at large \( |\bar{X}|^2 \) using the Gaussian PDF, giving the asymptotic rate

\[
I(|X|; |Y|) \to \frac{1}{2} \log \mathcal{P} + c.
\]

(16)

The bounded number \( c \) can be computed using the exact PDF.
Thus, the logarithm transforms the channel (17) with multiplicative noise in (8) can be ignored. Thus $|\bar{c}|$ loss simply influences the signal power, modifying constant term in (18) is a bounded real random variable because of the MSSFM using the data processing inequality.

We prove that the upper bounds in (19)–(20) do not scale with $\Gamma(n)$.

**Lemma 2.** In one unit of the MSSFM

$$\sup_{|x|} \frac{1}{n} I(\hat{X}; Y | |x|) < \infty,$$  

(21a)

$$\sup_{|y|} \frac{1}{n} I(\hat{X}; \hat{Y} | |y|) < \infty.$$  

(21b)

**Proof:** Consider first the lossless model, where $\bar{M}$ is a unitary operator. From Lemma 3 as $|x| \to \infty$, the additive noise in (8) can be ignored. Thus $Y = |Y| \hat{Y} \approx |X| \hat{Y}$. To prove (21a),

$$I(\hat{X}; Y | |x|) = I(\hat{X}; |X| \hat{Y} | |x|)$$  

$$= h_\sigma(\hat{Y} | |x|) = h_\sigma(\hat{Y} | \bar{X}, |x|).$$

Step (a) follows from the identity

$$I(\hat{X} ; ZY | Z) = I(\hat{X}; Y | Z), \quad Z \neq 0.$$  

(22)

We measure the entropy of $\hat{Y}$ with respect to the spherical probability measure $\sigma^m$, $m = 2n - 1$, on the surface of the unit sphere $S^m$. From the maximum entropy theorem (MET) for distributions with compact support,

$$\frac{1}{n} h_\sigma(\hat{Y} | |x|) \leq \frac{1}{n} \log A_n,$$

where $A_n = 2\pi^n / \Gamma(n)$ is the surface area of $S^m$, in which $\Gamma(n)$ is the gamma function.

We next show that the conditional entropy $h_\sigma(\hat{Y}; \hat{X}, |x|)$ does not tend to $-\infty$ with $|x|$. The volume of the spherical sector in Fig. 2 vanishes if and only if the corresponding area on the surface of the sphere vanishes. This can be formalized using identity (1). Let $W = U \hat{Y}$, where $U \sim U(0, 1)$ independent of $\hat{X}$ and $Y$. From (1)

$$h_\sigma(\hat{Y} | \hat{X}, |x|) = h(W | \hat{X}, |x|) - h(U) - \frac{m}{n} \log U.$$  

(23)

Applying chain rule to the differential entropy

$$h(W | .) = \sum_{k=1}^{n} h(W_k | W^{k-1}, .)$$

(24a)

$$= \sum_{k=1}^{n} h(\angle W_k, |W^{k-1}, .)$$

(24b)

$$+ \sum_{k=1}^{n} h(|W_k| |W^{k-1}, \angle W_k, .),$$

where entropy is conditioned on $|x|$ and $\bar{X}$.

For the phase entropies in (24a), note that, from (8)–(9), $\angle W_k = \angle y_k$ contains random variable $\Phi_k$ with finite entropy, which does not appear in $W^{k-1}$. Formally,

$$\angle W_k = \Phi_k + F(\Psi^n, \hat{x}),$$

for some function $F$, which can be determined from (8)–(9).

Thus

$$h(\angle W_k | W^{k-1}, .) = h(\Phi_k + F(\Psi^n, \hat{x}) | W^{k-1}, .)$$

(25)

Step (a) follows from the rule that conditioning reduces the entropy. Step (b) holds because $W^{k-1}$ is a function of...
\{\Phi^k, \Psi^n, U\}. Step (c) follows because \{\Psi^n, \cdot\} determines \(F(\Psi^n, \hat{x})\).

For the amplitude entropies in (24b), we explain the argument for \(n = 3\):

\[ W_k = U e^{j\Phi_k} (r_{k1}\hat{x}_1 e^{j\Psi_1} + r_{k2}\hat{x}_2 e^{j\Psi_2} + r_{k3}\hat{x}_3 e^{j\Psi_3}), \]

where \(1 < k < 3\). Noise addition in (8) implies \(Pr(\hat{X}_k = 0) = 0\), \forall k; we thus assume \(\hat{x}_k \neq 0\) for all \(k\). It is clear that \(h(W_1) > -\infty\).

There are 5 random variables \(U, \Phi_1, \Psi_{1,2,3}\) for two amplitude and phase relations in the \(W_1\) equation in (26). Given \(W_1\) and \(\angle W_2\), there are 6 random variables and three equations. One could, for instance, express \(\Psi_{1,2,3}\) in terms of \(U\) and \(\Phi_{1,2}\). This leaves free at least \(U \in |W_2|\), giving \(h(W_2) > h(U) + c > -\infty\).

The last equation for \(W_3\) adds one random variable \(\Phi_3\) and one equation for \(\angle W_3\). Together with the equation for \(|W_2|\), the number of free random variables, defined as the number of all random variables minus the number of equations, is 2; thus \(h(W_2) > h(U) + c > -\infty\).

In a similar way, in general, there are \(n + k + 1\) random variables in \(W^k\) and \(2k - 1\) equations in \((W^{k-1}, \angle W_k)\), resulting in \(n - k + 2 \geq 2\) free random variables. Thus

\[ h(W_k) > -\infty, \quad 1 \leq k \leq n. \]

Substituting (25) and (27) into (24a)–(24b), we obtain

\[ h(W\cdot) > -\infty. \]

Finally, from (23)

\[ h_p(\hat{Y}|\hat{X}, |x|) > -\infty. \]

The proof for lossy models, and (21b), is similar. Loss changes matrix \(R\), which has no influence on our approach to proving the boundedness of terms in (21a)–(21b).

The essence of the above proof is that, as \(|x| \to \infty\), the additive noise in (8) gets smaller relative to the signal, but phase noise (and thus randomness in \(M\)) does not decrease with \(|x|\). Furthermore, \(M\) has enough randomness, owing to the mixing effect of the dispersion, so that all \(2n - 1\) angles representing signal direction in the spherical coordinate system are random variables that do not vanish with \(|x|\).

Remark 4. For some special cases of the dispersion-loss matrix \(R\), it is possible to obtain deterministic components in \(Y\) as \(|x| \to \infty\). These are cases where mixing does not fully occur, e.g., \(R = I_n\). In the MSSSF, however, \(R\) is arbitrary, due to, e.g., step size \(\epsilon\).

A. Proof of the Corollary [4]

We fix the power constraint and let \(n \to \infty\) in the definition of the capacity. The logarithmic terms depending on \(P\) in the Theorem 1 approach zero, so that \(C < \infty\).

Consider now the continuous-time model [2]. We discretize the channel in the frequency domain, according to the approach in [8]. As the time duration \(T \to \infty\) in [8] Section II, we obtain a discrete-time model with infinite number of DOFs (Fourier modes) in any frequency interval at \(z = 0\). Therefore, \(C < \infty\) in the corresponding discrete-time periodic model.

It is shown in [4] Section VIII that, because of the spectral broadening, the capacity of the continuous-time model \(C_c\) can be strictly lower than the capacity of the discrete-time model \(C_d\). Since \(C_c \leq C_d\), and \(C_d < \infty\), we obtain \(C_c < \infty\).

We do not quantify constant \(c^1\) in the continuous-time model, which can be much lower than the constant \(c\) in the discrete-time model, due to spectral broadening (potentially, \(c^1(\infty, \infty) = 0\)). A crude estimate, based on the Carson bandwidth rule, is given in [4] Section VIII for the zero-dispersion channel.

To summarize, SE is bounded in input power in the continuous-time model with \(n = \infty\) (with or without filtering). The extent of the data rate loss due to the spectral broadening \((c^1\) versus \(c\)) remains an open problem.

V. RANDOM MATRIX MODEL AND THE ASYMPTOTIC CAPACITY

In this section it is shown that, as \(|X| \to \infty\), the action of the discrete-time periodic stochastic NLS equation tends to multiplication by a random matrix (with fixed PDF; independent of the input). Noise simplifies the NLS channel to a linear multiple-input multiple-output non-coherent fading channel. This section also proves Theorems 1 in an alternative intuitive way.

The approach is based on the following steps.

Step 1) In Section V-A the input signal space is partitioned into a bounded region \(\mathcal{R}^-\) and its complement \(\mathcal{R}^+\). It is shown that the overall rate is the interpolation of rates achievable using signals in \(\mathcal{R}^\pm\). Lemma 4 is proved, showing that the contribution of \(\mathcal{R}^-\) to the mutual information is bounded. Suitable regions \(\mathcal{R}^\pm\) are chosen for the subsequent step.

Step 2) In Section V-B it is shown that for all \(q(t, 0) \in \mathcal{R}^+\), the nonlinear operator \(L_N = j\gamma|Q|^2Q\) is multiplication by a uniform phase random variable, i.e.,

\[ L_N(Q) = j\Theta(t, z)Q, \quad \forall t, z, \]

where \(\Theta(t, z) \sim i.i.d. U(0, 2\pi)\). In other words, for input signals in \(\mathcal{R}^+\) the stochastic NLS equation is a simple linear channel with additive and multiplicative noise

\[ \frac{\partial Q}{\partial z} = L_N(Q) + j\Theta(t, z)Q + N(t, z). \]

Discretizing (29), we obtain that optical fiber is a fading channel when input is in \(\mathcal{R}^+\):

\[ Y = M X + Z, \quad \frac{1}{n} \mathbb{E} \|X\|^2 \leq P, \]

in which \(M\) is a random matrix of the form

\[ M = \prod_{k=1}^{m} RD_k, \quad D_k = \text{diag}(e^{j\Theta_{k1}}), \]
where \( \Theta_{kl} \sim \text{i.i.d. } U(0, 2\pi) \) and \( Z \) is noise

\[
Z = \sum_{k=1}^{m} \left( \prod_{i=1}^{k} \mathbb{R} \right) Z_k, \quad Z_k \sim \text{i.i.d. } \mathcal{N}_c(0, DI_n).
\]

In general, \( \mathbb{M} \) and \( Z \) are non-Gaussian. However, in the constant loss model, \( Z \sim \mathcal{N}_c(0, K) \) where \( K = (\sigma^2 W L_c / n) I_n, L_c = (1 - e^{-\alpha L}) / \alpha \). Note that \( \mathbb{M} \) and \( Z \) have fixed PDFs, independent of \( X \).

Summarizing, the channel law is

\[
p(y|x) = \begin{cases} \text{given by the NLS equation,} & x \in \mathbb{R}^-; \\ p(Mx + Z|x), & x \in \mathbb{R}^+. \end{cases}
\]

**Step 3)** In Section \( \text{V-C} \), the capacity of the multiplicative-noise channel \( (30) \) is studied. Lemma \( [7] \) and \( [8] \) are proved showing that, for any \( \mathbb{M} \) that does not have a deterministic component and is finite (see \( (33) \)), the asymptotic capacity is given by the Theorem \( [1] \). Importantly, the asymptotic rate is nearly independent of the PDF of \( \mathbb{M} \), which impacts only the bounded number \( c \). Finally, Lemma \( [8] \) is proved showing that the random matrix underlying the optical fiber at high powers meets the assumptions of the Lemma \( [7] \). An expression is provided for \( c \), which can be evaluated, depending on the PDF of \( \mathbb{M} \).

**A. Step 1): Rate Interpolation**

We begin by proving the following lemma, which is similar to the proof approach in \( [11] \), where the notion of satellite constellation is introduced.

**Lemma 3.** Let \( p(y|x) \), \( x, y \in \mathbb{R}^n \), be a conditional PDF. Define

\[
X = \begin{cases} X_1, & \text{with probability } \lambda, \\ X_2, & \text{with probability } 1 - \lambda, \end{cases}
\]

where \( X_1 \) and \( X_2 \) are random variables in \( \mathbb{R}^n \) and \( 0 < \lambda < 1 \). Then

\[
\lambda R_1 + (1 - \lambda) R_2 \leq R \leq \lambda R_1 + R_2 + H(\lambda),
\]

where \( R_1, R_2 \) and \( R \) are, respectively, mutual information of \( X_1, X_2, X \), and \( H(x) = -x \log x - (1 - x) \log(1 - x) \) is the binary entropy function, \( 0 \leq x < 1 \).

**Proof:** The PDF of the time sharing random variable \( X \) and its output \( Y \) are

\[
p_X(x) = \lambda p_{X_1}(x) + (1 - \lambda) p_{X_2}(x), \quad p_Y(y) = \lambda p_{Y_1}(y) + (1 - \lambda) p_{Y_2}(y),
\]

where

\[
p_{Y_1,Y_2}(y) = \int p(y|x)p_{X_1,X_2}(x) dx.
\]

By elementary algebra

\[
I(X;Y) = \lambda I(X_1;Y_1) + (1 - \lambda) I(X_2;Y_2) + \Delta I,
\]

where

\[
\Delta I = \lambda D(p_{Y_1}(y) || p(Y_1)) + (1 - \lambda) D(p_{Y_2}(y) || p(Y_2)).
\]

From \( (34) \),

\[
p_Y(y) \geq \max \{ \lambda p_{Y_1}(y), (1 - \lambda) p_{Y_2}(y) \},
\]

which gives \( \Delta I \leq H(\lambda) \). From \( \log x \leq x - 1, D(p(y_1,2)||p(y_1)) \geq 0 \), giving \( \Delta I \geq 0 \). Thus

\[
I(X;Y) \leq \lambda I(X_1;Y_1) + (1 - \lambda) I(X_2;Y_2) + H(\lambda), \quad I(X;Y) \geq \lambda I(X_1;Y_1) + (1 - \lambda) I(X_2;Y_2).
\]

**Corollary 2.** Define

\[
\bar{R} = \lim_{n \to \infty} \frac{1}{n} I(X;Y).
\]

With definitions in the Lemma \( [3] \) we have \( \bar{R} = \lambda \bar{R} + (1 - \lambda) \bar{R}_2 \).

For the rest of the paper, we choose \( R^- \) to be an \( n \)-hypercube in \( \mathbb{C}^n \)

\[
R^- = \{ x \in \mathbb{C}^n \mid |x_k| < \kappa, \quad 1 \leq k \leq n \},
\]

and \( \bar{R}^+ = \mathbb{C}^n \setminus R^- \). We drop the subscript \( \kappa \) when we do not need it. The following Lemma shows that, if \( \kappa < \infty \), the contribution of the signals in \( R^- \) to the mutual information in the NLS channel is bounded.

**Lemma 4.** Let \( X \in \mathbb{C}^n \) be a random variable supported on \( R^- \) and \( \kappa < \infty \). For the NLS channel \( (2) \)

\[
\frac{1}{n} I(X;Y) < \infty.
\]

**Proof:** From the MET, \( h(Y) \leq \log |R^-| < \infty \). Let \( X \to Z \to Y = Z + N \) be a Markov chain, where \( N \) is independent of \( Z \) and \( h(N) > -\infty \). Then \( h(Y|X) \geq h(Y|X,Z) = h(Y|Z) = h(N) > -\infty \). Applying this to the NLS channel with an independent noise addition in the last stage, we obtain \( I(X;Y) < \infty \).

Alternatively, from \( [8] \),

\[
\frac{1}{n} I(X;Y) \leq \log(1 + |R^-|^2 / nmD) < \infty.
\]

**B. Step 2): Channel Model in the High Power Regime**

We begin with the zero-dispersion channel. Let \( Q(t,0) = X = R_x \exp(j \Phi_x) \) and \( Q(t,z) = Y = R_y \exp(j \Phi_y) \) be, respectively, channel input and output in \( [5] \). For a fixed \( t \), \( X \) and \( Y \) are complex numbers.

**Lemma 5.** We have

\[
\lim_{|x| \to \infty} p(\phi_y|x) = \lim_{|z| \to \infty} p(\phi_y|x, r_y)
\]

\[
= \frac{1}{2\pi}.
\]

Thus, the law of the zero-dispersion channel tends to the law of the following channel

\[
Y = X e^{i \Theta} + Z,
\]
where \( \Theta \sim \mathcal{U}(0, 2\pi) \), \( Z \sim \mathcal{N}_C(0, D) \), and \((X, Z, \Theta)\) are independent. \(\square\)

**Proof:** The condition PDF is [4, Eq. 18]

\[
p(r_y, \phi_y | r_x, \phi_x) = \frac{1}{2\pi} p_0(r_y | r_x) + \frac{1}{2\pi} \sum_{m=1}^{\infty} \Re \left( p_m(r_y | r_x) e^{im(\phi_y - \phi_x - \gamma r_y^2)} \right).
\]

Here

\[
p_m(r_y | r_x) = 2r_x b_m \exp \left( -a_m(r_x^2 + r_y^2) \right) I_m(2b_mr_x r_y),
\]

where

\[
a_m = \frac{1}{D \zeta} x_m \coth(x_m), \quad b_m = \frac{1}{D \zeta} \frac{x_m}{\sinh(x_m)},
\]

in which \( x_m = \sqrt{m\gamma D \zeta} = t_m(1 + j), \quad t_m = \sqrt[2]{m\gamma D} \).

Note that \( p(r_y | r_x) = p_0(r_y | r_x) \).

The conditional PDF of the phase is

\[
p(\phi_y | r_x, \phi_x, r_y) = \frac{p(\phi_y, r_y | r_x, \phi_x)}{p(r_y | r_x)} = \frac{p(\phi_y, r_y | r_x, \phi_x)}{p(r_y | r_x)} = \frac{1}{2\pi} \sum_{m=1}^{\infty} \Re \left( D_m(r_x) e^{im(\phi_y - \phi_x - \gamma r_y^2)} \right) + \frac{1}{2\pi},
\]

where step (a) follows from \( p(r_y | r_x, \phi_x) = p(r_y | r_x) \) (see Fig. 6 (b)) and

\[
D_m(r_x) = \frac{p_m(r_y | r_x)}{p_0(r_y | r_x)} b_m I_m(2b_mr_x r_y) = \frac{b_0}{I_0(2b_mr_y r_y)} \times \exp \left\{ -b_0 (x_m \coth x_m - 1) (r_x^2 + r_y^2) \right\}.
\]

The following three inequalities can be verified:

\[
\left| \frac{b_m}{b_0} \right|^2 = \frac{4t^2}{\cosh 2t - \cos 2t} \leq 1, \quad t > 0.
\] \hfill (36a)

\[
\left| \frac{I_m(2r_x r_y b_m)}{I_0(2r_x r_y b_0)} \right| \leq \left| \frac{I_m(2r_x r_y b_0)}{I_0(2r_x r_y b_0)} \right| \leq 1.
\] \hfill (37a)

\[
F(t) = \Re \left( \frac{x_m \coth x_m - 1}{\cosh 2t - \cos 2t} \right) = \frac{\sinh(2t) + \sin(2t)}{\cosh(2t) - \cos(2t)} - 1 > 0,
\] \hfill (38a)

where \( t \geq t_m > 0 \).

Using (36a), (38a) in (35), we obtain \( |D_m(r_x)| \leq E_m(r_x) \), where

\[
E_m(r_x) = \exp \left\{ -\frac{1}{D \zeta} F(t_m)(r_x^2 + r_y^2) \right\}.
\] \hfill (39)

We have

\[
\lim_{r_x \to \infty} \left| \sum_{m=1}^{\infty} D_m(r_x) e^{im(\phi_y - \phi_x - \gamma r_y^2)} \right| \leq \lim_{r_x \to \infty} \sum_{m=1}^{\infty} |D_m(r_x)| \leq \sum_{m=1}^{\infty} E_m(r_x) \leq \sum_{m=1}^{\infty} \lim_{r_x \to \infty} E_m(r_x) = 0.
\]

Step (a) follows because \( E_m(r_x) \leq E_m(0) \) and \( \sum E_m(0) \) is convergent; thus, by the dominated convergence theorem, \( \sum E_m(r_x) \) is uniformly convergent. Step (b) follows from (39).

It follows that

\[
\lim_{r_x \to \infty} p(\phi_y | r_x, \phi_x, r_y) = \frac{1}{2\pi}.
\]

Furthermore

\[
\lim_{r_x \to \infty} p(\phi_y | r_x, \phi_x, r_y) = \lim_{r_x \to \infty} \int p(\phi_y | r_x, \phi_x, r_y') p(r_y' | r_x, \phi) dr_y' = \frac{1}{2\pi}.
\]

Lemma 5 generalizes to the vectorial zero-dispersion channel [5]. Since noise is independent and identically distributed in space and time, so are the corresponding uniform phases. This is true even if \( X_i \) in Fig. 1 are dependent, e.g., \( X = (x, \ldots, x) \).

We now consider the dispersive model. To generalize Lemma 5 to the full model, we use the following notion [6, Section 2.6].

**Definition 3** (Distributions that Escape to Infinity). A family of PDFs \( \{p_{X_\theta}(x)\}_{\theta}, 0 \leq \theta \leq \theta_0 \), is said to escape to infinity with \( \theta \) if \( \lim_{\theta \to \theta_0} \Pr(|X_\theta| < c) = 0 \) for any finite \( c \). \(\square\)

**Lemma 6.** Let \( X \in \mathcal{R}_k^+ \) and \( Y \in \mathcal{R}_k \), respectively, the channel input and output in the dispersive model. The PDF of \( Y_k \) escapes to infinity as \( \kappa \to \infty \) for all \( k \).

**Proof:** The proof is based on induction in the MSSFM units. We make precise the intuition that, as \( \kappa \to \infty \), the PDF of \( |Y_k| \) spreads out, so that an ever decreasing probability is assigned to any finite interval.

Consider vector \( \vec{V} \) in Fig. 1 at the end of the linear step in the first unit. Setting \( W = |\vec{V}_k| \), we have

\[
\Pr(W < c) = \int_0^c p_W(w) dw = \int_0^c \frac{1}{\pi} p_W(t) dt \leq \frac{c}{\pi} \|p_{T_\kappa}(t)\|_{\infty},
\]

where \( T_\kappa = \epsilon W, \quad p_{T_\kappa}(t) = \frac{1}{\pi} p_W(t) \) and \( \epsilon \doteq 1/\kappa \). Below, we prove that \( \|p_{T_\kappa}(t)\|_{\infty} < \infty \).
Fix $0 < \delta < 1$ and define the (non-empty) index set 

$$
\mathcal{I} = \{ i : |x_i| \geq \kappa^{1-\delta} \}.
$$

The scaled random variable $T_{\epsilon}$ is 

$$
T_{\epsilon} = \epsilon |V_k| = \sum_{i=1}^{n} e^{\psi_i(x_i, N_i^t)} + \epsilon \tilde{Z} = \sum_{i \in \mathcal{I}} \sum_{i \notin \mathcal{I}} e^{\psi_i(x_i, N_i^t)} + \epsilon \tilde{Z}.
$$

where $\tilde{Z}$ is an additive noise and $\tilde{x} = \epsilon \tilde{x}$.

As $\epsilon \to 0$, the second sum vanishes because, if $i \notin \mathcal{I}$, $|\tilde{x}_i| < \epsilon^\delta \to 0$. In the first sum, $|x_i| \to \infty$, thus $\psi_i(x_i, N_i^t) \to x_i U_i$, where $U_i \sim U(0, 2\pi)$. Therefore $T_{\epsilon} \to T_0$, in which 

$$
T_0 = \sum_{i \in \mathcal{I}} e^{\psi_i(x_i, N_i^t)}|\tilde{x}_i|,
$$

where $|\tilde{x}_i| > 0$. Since the PDF of $e^{\psi_i U_i}$ is in $L^\infty(T)$ on the circle $T$, so is the conditional PDF $p_{\tilde{U}_i|\tilde{x}}(t|\tilde{x})$, i.e., 

$$
\|p_{\tilde{U}_i|\tilde{x}}(t|\tilde{x})\|_\infty < \infty.
$$

Substituting (42) into (40) 

$$
\lim_{\epsilon \to 0} \Pr(W < c) = 0.
$$

In a similar way, (43) can be proved for $V$ at the output of the linear step in the second unit, by replacing $e^{\psi_i U_i}$ in (41) with $M_{kl}$, and noting that, as $\epsilon \to 0$, $\{M_{kl}\}_{i \in \mathcal{I}}$ tend to random variables independent of input, with a smooth PDF (without delta functions).

From the Lemma 3 as $\kappa \to \infty$ the probability distribution at the input of every zero-dispersion segment in the link escapes to infinity, turning the operation of the nonlinearity in that segment into multiplication by a uniform phase and independent noise addition. We thus obtain an input region $\mathcal{R}_x^+$ for which, if $x \in \mathcal{R}_x^+$, the channel is multiplication by a random matrix, as described in (30). The channel converts any small noise into worst-case noise in evolution.

C. Step 3): The Asymptotic Capacity

In this section, we obtain the asymptotic capacity of the channel (32).

Applying Lemma 3 to (32) 

$$
\hat{R}(\mathcal{P}) = \lambda \hat{R}_- (\mathcal{P}) + (1 - \lambda) \hat{R}_+ (\mathcal{P}),
$$

where $\hat{R}_+ (\mathcal{P}) = \frac{1}{n} I(\hat{X} ; Y)$, $X \in \mathcal{R}_x^+$ and $\lambda$ is a parameter to be optimized. To shorten the analysis, we ignore the term $c = H(\lambda)/n$ in (33), as it does not depend on $\mathcal{P}$.

We choose $\kappa$ sufficiently large, independent of the average input power $\mathcal{P}$. From the Lemma 3 $\sup_{\mathcal{P}} \hat{R}_- (\mathcal{P}) < \infty$. The following Lemma shows that $\hat{R}_+ (\mathcal{P})$ is given by the logarithmic terms in Theorem 4 with 

$$
c \geq \lambda \hat{R}_- + (1 - \lambda) \sup_{\tilde{X} \in \mathcal{X}} \frac{1}{n} I(\tilde{X} ; \hat{M} \tilde{X}),
$$

where $\hat{R}_-$ is the achievable rate at low powers. If $\hat{M}$ is Haar distributed, $c = \lambda \hat{R}_-$.

Define $h(\hat{M}) \triangleq h(M_{11}, \ldots, M_{nn})$.

Lemma 7. Assume that 

$$
(44)
$$

Then, the asymptotic capacity of (30) is given by the expressions stated in the Theorem 4.

Proof:

The capacity of the multiple-input multiple-output non-coherent memoryless fading channel (30) is studied in [6], [12]. Here, we present a short proof with a bit of approximation.

Using chain rule for the mutual information 

$$
I(X ; Y) = I(|X| ; Y | X) + I(|X| ; X | Y) + I(|X| ; Y | X).
$$

The first term in (45) gives the logarithmic terms in Theorem 4 as calculated in Section IV. We prove that the other terms are bounded in $|X|$. From the Lemma 3, the additive noise in (30) can be ignored when $X \in \mathcal{R}_x^+$, so that $X \approx M \bar{X}$. The second term in (45) is 

$$
I(\hat{X} ; Y | X) = I(\hat{X} ; M \bar{X} | X),
$$

where we used identity (22). Note that we can not assume that $|X|$ and $\bar{X}$ are independent.

For the output entropy 

$$
h(\hat{M} \bar{X} | |X|) \leq h(\hat{M} \bar{X})
$$

$$
\leq \sum_{k=1}^{n} h(\sum_{l=1}^{n} M_{kl} \hat{X}_l)
$$

$$
= \sum_{k=1}^{n} \log \left( \pi e \sum_{l=1}^{n} M_{kl} \hat{X}_l^2 \right)
$$

$$
= \sum_{k=1}^{n} \log \left( \sum_{l=1}^{n} E[M_{kl}^2] \right) + n \pi e
$$

$$
\leq n \log \left( \frac{1}{n} E[M_{F}^2] \right) + n \pi e
$$

$$
< \infty,
$$

where $|M|_F = \left( \sum_{k,l=1}^{n} |M_{kl}|^2 \right)^{\frac{1}{2}}$ is the Frobenius norm.

Step (a) is obtained using the inequality $h(W) = \sum_{k} h(W_k | W^{k-1}) \leq \sum_{k} h(W_k)$. Step (b) is due to the MET. Cauchy-Schwarz and Jensen’s inequalities are, respectively, applied in steps (c) and (d).

For the conditional entropy 

$$
h(\hat{M} \bar{X} | |X| , \bar{X}) = E_{\bar{X}} h(\hat{M} \bar{X} | |X| , \bar{X})
$$

$$
\geq \inf_{\tilde{X}} h(\hat{M} \tilde{X})
$$

$$
\geq -\infty.
$$

(46)

Step (a) holds because, from the Lemma 8 $h(\hat{M} \bar{X}) > -\infty$ for any $\tilde{X}$.
The third term in (45) can be upper bounded using the second term by setting $X = M^{-1}Y$. We prove it alternatively. Since $\bar{Y}$ is compactly supported, $h_{\sigma}(\bar{Y}|||Y||) \leq h_{\sigma}(\bar{Y}) < \infty$. The conditional entropy is

$$h_{\sigma}(\bar{Y}||X,|Y||) = h_{\sigma}(\bar{Y}||X,|X||M\bar{X})) = h_{\sigma}\left(\frac{M\bar{X}}{M\bar{X}}||X,|M\bar{X}||\right).$$

(47)

Applying identity (1) to $M\bar{X}$ and conditioning on $|X|

$$h_{\sigma}\left(\frac{M\bar{X}}{|M\bar{X}|}||X)|| = h(M\bar{X}||X)) - h((M\bar{X})||X||) - (2n-1)E[\log(|M\bar{X}|)||X||].$$

(48)

For the first term in (48)

$$h(M\bar{X}||X)| \geq h(M\bar{X}||X), \bar{X}, (49a)$$

$$> -\infty,$$ (49a)

where we used (46). Since $|M\bar{X}| \leq |M| \leq |M|_F$, from the MET

$$h((M\bar{X})||X)) \leq h(M\bar{X})$$

$$\leq \frac{1}{2}\log(2\pi e|M\bar{X}|^2)$$

$$\leq \frac{1}{2}\log(2\pi e|M|^2)$$

$$< \infty.$$ (50a)

Furthermore,

$$E[\log(|M\bar{X}|^2)||X||] \leq \log E[|M\bar{X}|^2||X|)$$

$$\leq \log E[|M|^2]$$

$$< \infty.$$(51a)

Substituting (49a)–(51a) into (48) and (47), we obtain

$$h_{\sigma}(\bar{Y}||X,|Y||) > -\infty.$$

The main ingredient in the proof of the Lemma 7 as well as Theorem 1 is the following lemma.

**Lemma 8.** Let $M$ be a random matrix and $x \in \mathbb{C}^n$ a non-zero deterministic vector. If $M$ satisfies the assumptions (44), then

$$h(M|x) > -\infty.$$

**Proof:**

Since $x \neq 0$, at least one element of $x$ is nonzero, say $x_1 \neq 0$. We switch the order of $M$ and $x$ in the product $Mx$ as follows. Let $M \in \mathbb{C}^{n^2}$ denote the vectorized version of $M$, where rows are concatenated as a column vector. Define $\bar{Y} \in \mathbb{C}^{n^2}$ as follows:

$$V_k = \begin{cases} M_{i,n}, & r = 0, \\ M_{i+r}, & r = 1, \\ M_{(i+1)r}, & r \geq 2, \end{cases}$$

$$Y_{i+r} = \sum_{l=1}^{n} M_{i+r,l}x_i, \quad r = 1, \quad (52)$$

where $k = in + r, 0 \leq i \leq n, 0 \leq r \leq n - 1$. Then $Y = Mx$ is transformed to (52), which in matrix notation is

$$V = AM,$$ (53)

in which $A_{n^2 \times n^2} = \text{diag}(X, \cdots, X)$, where the deterministic matrix $X_{n \times n}$ is

$$X = \text{diag}(x_1, x_2, \cdots, x_n),$$

in which 0 is the $(n - 1) \times 1$ all-zero matrix. From (53)

$$h(Y|x) = h(M|x) + \log \det A$$

$$= h(M) + n \log |x_1|$$

$$= h(M) + n \log |x_1|.$$ (52)

On the other hand, from (52)

$$h(Y|x) = h(Y, \{M_{ij}\}_{i \neq j}||x)$$

$$= h(Y||x) + h(\{M_{ij}\}_{i \neq j}||Y)$$

$$= h(M|x) + h(\{M_{ij}\}_{i \neq j}||Y).$$

Step (a) holds because conditions $r = 1$ and $r = 0, 1$ in (52) include, respectively, $Y$ and $\{M_{ij}\}_{i \neq j}$ and $r = 0, 1$. Combining the last two relations,

$$h(M|x) = h(M) + n \log |x_1| - h(\{M_{ij}\}_{i \neq j}||Y).$$

If $E[M_{ij}]^2 < \infty$, from the MET, the last term is bounded from below. Since $h(M) > -\infty$ and $x_1 \neq 0$, $h(M|x) > -\infty$.

**Lemma 9.** The random matrix $M$ (51), underlying optical fiber at high powers, satisfies the assumptions of the Lemma 7.

**Proof:**

Applying the triangle inequality to (31), $|M_{ij}| \leq (|R|^m)_{ij}$, where $|R|$ is the matrix with entries $|r_{ij}|$ and $m$ is the number of stages.

We check the entropy condition in (44). In what follows, let $\theta_j \sim \text{i.i.d.} \ U(0, 2\pi)$. For one linear and nonlinear steps $m = 1$:

$$M = \begin{pmatrix} e^{i\theta_1}r_{11} & e^{i\theta_2}r_{12} \\ e^{i\theta_1}r_{21} & e^{i\theta_2}r_{22} \end{pmatrix}.$$ In this case, there are four amplitude dependencies $|M_{ij}| = |r_{ij}|, 1 \leq i, j \leq 2$, and two phase dependencies:

$$\angle M_{11} = \angle M_{21} + k\pi, \quad \angle M_{12} = \angle M_{22} + k\pi, \quad k = 0, 1.$$ A dependency means that $M$ contains a deterministic component, i.e., $h(M) > -\infty$.

For $m = 2$:

$$M_{11} = e^{i(\theta_1 + \theta_3)}r_{11} + e^{i(\theta_1 + \theta_3)}r_{12}r_{21},$$

$$M_{12} = e^{i(\theta_2 + \theta_3)}r_{12} \left( r_{11} + e^{i(\theta_4 - \theta_3)}r_{22} \right),$$

$$M_{21} = e^{i(\theta_1 + \theta_3)}r_{21} \left( r_{11} + e^{i(\theta_4 - \theta_3)}r_{22} \right),$$

$$M_{22} = e^{i(\theta_2 + \theta_3)}r_{21}r_{12} + e^{i(\theta_2 + \theta_4)}r_{22}^2.$$ In this case too, there is a dependency $|r_{21}M_{12}| = |r_{12}M_{21}|.$
For $m = 3$:

\[
\begin{align*}
M_{11} &= e^{j(\theta_1 + \theta_3 + \theta_5)} r_{11}^2 r_{12}^2 + e^{j(\theta_1 + \theta_4 + \theta_5)} r_{11} r_{12} r_{21} + e^{j(\theta_1 + \theta_4 + \theta_3)} r_{11} r_{12} r_{22} + e^{j(\theta_1 + \theta_2 + \theta_3 + \theta_5)} r_{11} r_{12} r_{21} r_{22}, \\
M_{12} &= e^{j(\theta_2 + \theta_4 + \theta_5)} r_{11}^2 r_{12} + e^{j(\theta_2 + \theta_4 + \theta_3)} r_{11} r_{12}^2 + e^{j(\theta_2 + \theta_3 + \theta_5)} r_{11} r_{12} r_{21} + e^{j(\theta_2 + \theta_3 + \theta_4 + \theta_5)} r_{11} r_{12} r_{22}, \\
M_{21} &= e^{j(\theta_1 + \theta_4 + \theta_5)} r_{12}^2 r_{21} + e^{j(\theta_1 + \theta_2 + \theta_3 + \theta_5)} r_{12} r_{21} r_{22}, \\
M_{22} &= e^{j(\theta_2 + \theta_3 + \theta_5)} r_{11} r_{12} r_{21} + e^{j(\theta_2 + \theta_3 + \theta_4 + \theta_5)} r_{12} r_{21} r_{22}. 
\end{align*}
\]

Comparing the boxed terms, $|r_{21}M_{12}| \neq |r_{12}M_{21}|$. There are still 8 equations for 6 variables.

In general, the number of entries of $M$ is $n^2$. As $m > 2n$ steps are taken in distance, sufficient number of random variables $\theta_i$ are introduced in a matrix with fixed dimension. Since $n$ is finite and $m$ is free, we obtain an under-determined system of polynomial equations for $x_i = \exp(j\theta_i)$ whose solution space has positive dimension. Thus an entry of $M$ can not be determined from all other entries.

Remark 5. The rate interpolation Lemma \[3\] implies that, replacing $\mathbb{C}^n$ by $\mathbb{R}^+$ changes the asymptotic capacity by a finite number $c$. From the upper bound $C \leq \log(1 + \text{SNR})$ in \[8\] and Theorem 2.5 in \[6\], we think that the asymptotic capacity can be achieved by an input distribution that escapes to infinity. This implies that $\lambda = 0$, so that $c$ is indeed zero. We do not investigate this rigorously.

Remark 6 (Optimal Input Distribution). Multivariate Gaussian input distribution is a poor choice for channels with multiplicative noise. Indeed, it achieves a rate bounded in power in \[30\]. Log-normal input PDF for the signal norm achieves the asymptotic capacity of the non-constant loss model.

VI. REVIEW OF THE INFORMATION THEORY OF THE OPTICAL FIBER

An information-theoretic analysis of the full model of the optical fiber does not exist. Even in the special case of the zero-dispersion, spectral efficiency is unknown. In the full model, we do not know anything about the capacity in the high power regime, let alone the spectral efficiency. The state-of-the-art is still lower bounds that are good in the nearly-linear regime. This situation calls for basic research, in order to make progress on these open problems.

The present paper builds on earlier work. We acknowledge \[7\], Eq. 12 for the equation \[5\], \[4\], \[7\], \[13\] for the PDF of the zero-dispersion channel, \[4\] for the analysis of the zero-dispersion model, \[8\], \[14\] for noting that Shannon entropy is invariant under the flow of a broad class of deterministic partial differential equations and for highlighting the usefulness of the operator splitting (in numerical analysis) in the analysis of the NLS equation. Furthermore, we acknowledge \[11\] for helpful insight leading to the rate interpolation Lemma \[3\], \[6\], \[12\] for the study of the fading channels and Section II of \[8\] for unfolding the origin of the capacity limitations in fiber — particularly the finding that signal-signal interactions are not fundamental limitations in the deterministic model if communication takes place in the right basis (i.e., the nonlinear Fourier basis), which led us to the study of the remaining factor in this paper, namely the signal-noise interactions.

We do not intend to survey the literature in this paper. There is a good review in \[15\], Section I-A]. The achievable rates of 1- and multi-solitons is studied, respectively, in \[3\], \[16\]–\[18\] and \[19\]–\[21\]. There is also a myriad of lower bounds that hold good in the low power regime; see, e.g., \[22\]–\[27\].

The achievable rates of the nonlinear frequency-division multiplexing for multi-user communication are presented in \[5\] for the Hermitian channel. Fig. 3 compares the NFDM and WDM rates \[5\], Fig. 6]. The gap between the WDM and NFDM curve reflects signal-signal interactions. The gap between the NFDM and AWGN curve reflects signal-noise interactions. We conjecture that the NFDM rate is close to the capacity. At the power levels shown in Fig. 3, $C_{\text{wdm}}(P) = \log P + c$ and $C_{\text{nfdm}} = \log P + c', c < c'$. Although more gains are expected at $P > -2.4$ dB, the slope of the blue curve will gradually decrease, converging, in the limit $P \to \infty$, to the asymptotic form in Theorem 1.

It is interesting to compare the extent of the signal-noise interactions in the time domain \[9\] and in the nonlinear Fourier domain \[10\] Section IV. A].

VII. CONCLUSIONS

The asymptotic capacity of the discrete-time periodic model of the optical fiber is characterized as a function of the input power in Theorem 1. With $n$ signal DOFs at the input, $n - 1$ DOFs are asymptotically lost, leaving signal energy as the only available DOF for transmission. The appropriate input distribution is a log-normal PDF for the signal norm. Signal-noise interactions limit the operation of the optical communication systems to low-to-medium powers.
The nonlinear steps in Fig. 1 in matrix notation are

\[ X = \bar{D}^2 (V + N^2 \epsilon), \]

where \( \epsilon \in \mathbb{R}^k \) is the all-one column vector. Combining the linear and nonlinear steps, we obtain (8) with \( M = D_2 D_1 \) and

\[ Z = MN^1 \epsilon + D_2 N^2 \epsilon. \]  

Clearly \( N^1,2 \epsilon \sim N_0(0, D_x I_n) \). However \( MN^1 \epsilon \) and \( D_2 N^2 \epsilon \) are generally non-Gaussian due to the signal and noise terms in \( \Phi_k \) and \( \Psi_l \). But, 1) in the constant loss model, if 2) \( \forall k \bar{x}_k \rightarrow \infty \), then

\[ Z \sim N_0(0, K), \]

where \( K = D (1 + e^{-\alpha t} \epsilon) I_n. \)

In summary, \( N \) variables are Gaussian; \( Z \) variables are Gaussians in the asymptotic analysis of the constant loss model.

REFERENCES

[1] M. I. Yousefi and F. R. Kschischang, “Information transmission using the nonlinear Fourier transform, Part I: Mathematical tools,” IEEE Trans. Inf. Theory, vol. 60, no. 7, pp. 4312–4328, Jul. 2014, Also published at arXiv, Feb. 2012. [Online]. Available: http://arxiv.org/abs/1202.3653

[2] ———, “Information transmission using the nonlinear Fourier transform, Part II: Numerical methods,” IEEE Trans. Inf. Theory, vol. 60, no. 7, pp. 4329–4345, Jul. 2014, Also published at arXiv, Apr. 2012. [Online]. Available: http://arxiv.org/abs/1204.0830

[3] ———, “Information transmission using the nonlinear Fourier transform, Part III: Spectrum modulation,” IEEE Trans. Inf. Theory, vol. 60, no. 7, pp. 4346–4369, Jul. 2014, Also published at arXiv, Feb. 2013. [Online]. Available: http://arxiv.org/abs/1302.2875

[4] ———, “On the per-sample capacity of nondispersive optical fibers,” IEEE Trans. Inf. Theory, vol. 57, no. 11, pp. 7522–7541, Nov. 2011.

[5] M. I. Yousefi and X. Yangzhang, “Linear and nonlinear frequency-division multiplexing,” arXiv:1603.04389, pp. 1–14, Mar. 2016. [Online]. Available: http://arxiv.org/abs/1603.04389

[6] S. M. Moser, “Duality-based bounds on channel capacity,” Ph.D. dissertation, ETH Zurich, Switzerland, Jan. 2005.

[7] A. Mecozzi, “Limits to long-haul coherent transmission set by the Kerr nonlinearity and noise of the in-line amplifiers,” IEEE J. Lightw. Technol., vol. 12, no. 11, pp. 1993–2000, Nov. 1994.

[8] M. I. Yousefi, G. Kramer, and F. R. Kschischang, “Upper bound on the capacity of the nonlinear Schrödinger channel,” in IEEE 14th Canadian Workshop on Inf. Theory, St. John’s, Newfoundland, Canada, Jul. 2015, pp. 1–5.

[9] G. Kramer, “Nonlinear signal–noise interaction in optical links with nonlinear equalization,” IEEE J. Lightw. Technol., vol. 34, no. 6, pp. 1476–1483, Mar. 2016.

[10] I. Tavakkolnia and M. Safari, “Signalling over nonlinear fibre-optic channels by utilizing both solitonic and radiative spectra,” in European Conf. Networks and Commun., Paris, France, Jul. 2015, pp. 103–107.

[11] E. Agrell, “Conditions for a monotonic channel capacity,” IEEE Trans. Commun., vol. 63, no. 3, pp. 1–11, Sep. 2015.

[12] A. Lapidoth and S. Moser, “Capacity bounds via duality with applications to multiple-antenna systems on flat-fading channels,” IEEE Trans. Inf. Theory, vol. 49, no. 10, pp. 2426–2467, Oct. 2003.

[13] K. S. Turitsyn, S. A. Derevyanko, I. V. Yurkevich, and S. K. Turitsyn, “Information capacity of optical fiber channels with zero average dispersion,” Phys. Rev. Lett., vol. 91, no. 20, pp. 203901, Nov. 2003.

[14] G. Kramer, M. I. Yousefi, and F. Kschischang, “Upper bound on the capacity of a cascade of nonlinear and noisy channels,” in IEEE Info. Theory Workshop, Jerusalem, Israel, Apr. 2015, pp. 1–4.

[15] H. Ghoulian and G. Kramer, “Models and information rates for multiuser optical fiber channels with nonlinearity and dispersion,” arXiv:1503.03124, pp. 1–18, Mar. 2015. [Online]. Available: https://arxiv.org/abs/1503.03124

[16] E. Meron, M. Feder, and M. Shtaif, “On the achievable communication rates of generalized soliton transmission systems,” arXiv:1207.0297, pp. 1–13, Jul. 2012. [Online]. Available: https://arxiv.org/abs/1207.0297

[17] N. A. Shevchenko et al., “A lower bound on the per soliton capacity of the nonlinear optical fibre channel,” in IEEE Info. Theory Workshop, Jeju Island, South Korea, Oct. 2015, pp. 1–5.

[18] Q. Zhang and T. H. Chan, “Achievable rates of soliton communication systems,” in IEEE Int. Symp. Inf. Theory, Barcelona, Spain, Jul. 2016, pp. 605–609.

[19] P. Kazakopoulos and A. L. Moustakas, “Transmission of information via the non-linear Schrödinger equation: The random Gaussian input case,” arXiv:1210.7940, pp. 1–9, Oct. 2012. [Online]. Available: https://arxiv.org/abs/1210.7940

[20] P. Kazakopoulos and A. L. Moustakas, “On the soliton spectral efficiency in non-linear optical fibers,” in IEEE Int. Symp. Inf. Theory, Barcelona, Spain, Jun. 2016, pp. 610–614.

[21] H. Buelow, V. Aref, and W. Idler, “Transmission of waveforms determined by 7 eigenvalues with PSK-modulated spectral amplitudes,” in European Conf. Opt. Commun., Sep. 2016, pp. 1–3.

[22] A. Mecozzi and R.-J. Essiambre, “Nonlinear Shannon limit in pseudo-linear coherent systems,” IEEE J. Lightw. Technol., vol. 30, no. 12, pp. 2011–2024, Jun. 2012.

[23] M. Secondini, E. Forestieri, and G. Prati, “Achievable information rate in nonlinear WDM fiber-optics systems with arbitrary modulation formats and dispersion maps,” IEEE J. Lightw. Technol., vol. 31, no. 23, pp. 1–14, Dec. 2013.

[24] R. Dar, M. Shtaif, and M. Feder, “New bounds on the capacity of the nonlinear fiber-optic channel,” Opt. Lett., vol. 39, no. 2, pp. 398–401, 2014.

[25] I. S. Terekhov, A. V. Reznichenko, and S. K. Turitsyn, “Calculation of maximal information for nonlinear communication channel at large SNR,” Phys. Rev. E, vol. 94, no. 4, pp. 042203, Oct. 2016.

[26] M. Secondini and E. Forestieri, “The limits of the nonlinear Shannon limit,” in Opt. Fiber Commun. Conf. and Exposition, Anaheim, California, United States, Mar. 2016, pp. 1–3.

[27] S. A. Derevyanko, J. E. Prilepsky, and S. K. Turitsyn, “Capacity estimates for optical transmission based on the nonlinear Fourier transform,” Nature Commun., vol. 7, no. 12710, pp. 1–9, Sep. 2016.