Bifurcations for Hamiltonian systems

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Abstract Our assumptions and results include the following special versions: For a
topological space A and \( H \in C(\Lambda \times S^1 \times \mathbb{R}^{2n}) \), \( S^1 = \mathbb{R}/\mathbb{Z} \), suppose that each \( H(\lambda, t, \cdot) \)
is \( C^2 \) and all its partial derivatives depend continuously on \( (\lambda, t, z) \) and that for each \( \lambda \in \Lambda \) there exists a periodic solution \( v_{\lambda} : A \rightarrow \mathbb{R}^{2n} \) of \( \dot{v}(t) = J\nabla_z H(\lambda, t, v(t)) \) such that \( \Lambda \times A \ni (\lambda, t) \rightarrow v_\lambda(t) \in \mathbb{R}^{2n} \) and \( \Lambda \times A \ni (\lambda, t) \rightarrow \bar{v}_\lambda(t) \in \mathbb{R}^{2n} \) are also continuous. Let \( i(\gamma_\lambda) \) be the Maslov-type index of the fundamental matrix solution \( \gamma_\lambda \) of \( \dot{Z}(t) = J\nabla_z^2 H(\lambda, t, v_\lambda(t))Z(t) \), which is equal to the Conley-Zehnder index 
\[ \mu_{CZ}(\gamma_\lambda) \text{ if } \nu(\gamma_\lambda) := \dim \ker(\gamma_\lambda(1) - I_{2n}) \] is zero. Then

(i) Suppose that \( \Lambda = (\lambda_k) \subset \Lambda \) converging to \( \mu \) and \( \lambda \)
periodic solutions of \( \dot{v}(t) = J\nabla_z H(\lambda_k, t, v(t)), \bar{v}_k \neq v_{\lambda_k}, k = 1, 2, \ldots, \) such that \( \bar{v}_k \rightarrow v_\mu \) in \( W^{1,2}(S^1; \mathbb{R}^{2n}) \), then \( \nu(\gamma_\lambda) > 0 \).

(ii) If \( \Lambda \) is path-connected and there exist two distinct points \( \lambda^+, \lambda^- \in \Lambda \) such that \( [i(\gamma_{\lambda^+}), i(\gamma_{\lambda^-}) + \nu(\gamma_{\lambda^+})] \) or \( \nu(\gamma_{\lambda^-}) = 0 \), then there exists \( \mu \in \Lambda \) for which the conditions in (i) hold and \( \bar{v}_k \rightarrow v_\mu \) in \( C^1(S^1; \mathbb{R}^{2n}) \). If \( \Lambda \) is first countable, and for some \( \mu \in \Lambda \) there exist \( \Lambda^+ \neq \Lambda^- \) in any deleted neighborhood of \( \mu \) satisfying just conditions then the conditions in (i) hold and \( \bar{v}_k \rightarrow v_\mu \) in \( C^1(S^1; \mathbb{R}^{2n}) \).

(iii) Suppose that \( \Lambda = (-\epsilon, \epsilon) \), \( \mu = 0 \) and \( \gamma_\lambda \) is as above. If \( \nu(\gamma_\lambda) = 0 \) for each \( \lambda \neq 0 \), and \( \nu(\gamma_0) \neq 0 \), and \( i(\gamma_\lambda) \) takes, respectively, values \( i(\gamma_0) \) and \( i(\gamma_0) + \nu(\gamma_0) \) as \( \lambda \rightarrow (-\epsilon, \epsilon) \) varies in two deleted half neighborhoods of 0, then

either \( \dot{v}(t) = J\nabla_z H(\mu, t, v(t)) \) has a sequence of solutions, \( v^\mu_k \neq v_\mu, k = 1, 2, \ldots, \) such that \( v^\mu_k \rightarrow v_\mu \) in \( C^1(S^1; \mathbb{R}^{2n}) \);

or for every \( \lambda \in \Lambda \setminus \{\mu\} \) near \( \mu \) there is a solution \( \bar{v}_\lambda \neq v_\lambda \) of \( \dot{v}(t) = J\nabla_z H(\lambda, t, v(t)) \) such that \( \bar{v}_\lambda \rightarrow v_\mu \) in \( C^1(S^1; \mathbb{R}^{2n}) \) as \( \lambda \rightarrow \mu \);

or for a given neighborhood \( W \) of \( v_\mu \) in \( C^1(S^1; \mathbb{R}^{2n}) \) there is an
one-sided neighborhood \( \Lambda^0 \) of \( \mu \) such that for any \( \lambda \in \Lambda^0 \setminus \{\mu\}, \dot{v}(t) = J\nabla_z H(\lambda, t, v(t)) \) has at least two distinct solutions \( v^1_\lambda \neq v_\lambda \) and \( v^2_\lambda \neq v_\lambda \) in \( W \), which can also be required to have different Hamiltonian actions provided that \( \nu(\gamma_0) > 1 \) and \( \dot{v}(t) = J\nabla_z H(\lambda, t, v(t)) \) has only finitely many solutions in \( W \).

(iv) Under the assumptions of (iii), if \( H \) is independent of \( t \) and that each \( v_\lambda \) is constant (therefore \( \gamma_\lambda(t) = \exp(tJ\nabla_z^2 H(\lambda, v_\lambda)) \) and \( \ker(\nabla_z^2 H(0, v_\lambda)) = \{0\} \), then either \( \dot{v}(t) = J\nabla_z H(0, v(t)) \) has a sequence of \( S^1 \)-distinct solutions, \( v^\lambda_k \notin \mathbb{R} \cdot v_\lambda \), \( k = 1, 2, \ldots, \) which converges to \( v_0 \) in \( C^1(S^1; \mathbb{R}^{2n}) \), or there exist left and right neighborhoods \( \Lambda^- \) and \( \Lambda^+ \) of 0 in \( \Lambda \) and integers \( n^+, n^- \geq 0 \), such that \( n^+ \geq n^- \geq 0, 4 \)-periodic solutions \( \dot{v}(t) = J\nabla_z H(\lambda, v(t)) \) has at least \( n^- \) \( S^1 \)-distinct solutions, \( v^\lambda_i \notin S^1 \cdot v_\lambda, i = 1, \ldots, n^- \) (resp. \( n^+ \)) which converge to \( v_0 \) in \( C^1(S^1; \mathbb{R}^{2n}) \).

(v) If \( H \) in (i)-(iii) is independent of \( t \) and all \( v_\lambda \) are equal to a nonconstant \( v_\mu \), corresponding bifurcation results about \( S^1 \)-distinct solutions with (i)-(iii) are given.

(vi) If \( H \) in (i)-(iii) also satisfies \( H(\lambda, -t, (-p, q)) = H(\lambda, t, (p, q)) \) for all \( (\lambda, t) \in \Lambda \times \mathbb{R} \) and \( (p, q) \in \mathbb{R}^{2n} \), corresponding results about brake orbits of \( \dot{v}(t) = J\nabla_z H(\lambda, t, v(t)) \) with (i)-(iii) are given.

In addition, similar results are also proved for bifurcations for Hamiltonian paths
connecting affine Lagrangian subspaces.
Contents

1 Introduction and main results 3

2 Preliminaries 32

3 Proofs of Theorems 1.4, 1.7 and Corollary 1.9, 1.10 42

4 Proofs of Theorems 1.14, 1.18, 1.19, 1.20, 1.21 and Corollary 1.16 53

4.1 Proof of Theorem 1.14 and Corollary 1.16 54

4.2 Proofs of Theorems 1.18, 1.19, 1.20, 1.21 57

5 Proofs of Theorems 1.23, 1.24, 1.26 and Corollaries 1.25, 1.28, 1.29 66

6 Proofs of Theorems 1.33, 1.34 and 1.37 69

6.1 Proofs of Theorems 1.33, 1.34 69

6.2 Proof of Theorem 1.37 74

A Maslov-type index and Morse index 75

B Proof of Proposition 1.3 84

C Generalizations and corrections for related results in [48, 50, 52] 85

1 Introduction and main results

This work continues our program on variational bifurcations beginning at [50, 51]. The current manuscript studies bifurcations of the following Hamiltonian boundary value problem

\[
\dot{u}(t) = J \nabla_z H(\lambda, t, u(t)) \forall t \in [0, \tau] \quad \text{and} \quad (u(0), u(\tau)) \in N \tag{1.1}
\]

with respect to a continuous family \( \{u_\lambda | \lambda \in \Lambda\} \) of solutions of this problem parameterized by a topological space \( \Lambda \), where \( N \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n} \) is a submanifold, \( J \) is given by (1.2) and \( H : \Lambda \times [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R} \) is as in Assumption 1.1. That is, we answer the following questions:

- Under what conditions there exists a point \( \mu \in \Lambda \) such that every neighborhood of \((\mu, u_\mu)\) in \( \Lambda \times C^1([0, \tau];\mathbb{R}^{2n}) \) contains a point \((\lambda, v_\lambda) \notin \{(\lambda, u_\lambda) | \lambda \in \Lambda\}\) satisfying (1.1)?

- What are the necessary (resp. sufficient) condition for a point \((\mu, u_\mu)\) to satisfy the above properties?

- How is the solutions of (1.1) distributed near the point \((\mu, u_\mu)\) as above?

There is a vast amount of literature concerning bifurcations for periodic solutions of Hamiltonian systems, i.e., the case that \( N \) is the diagonal in \( \mathbb{R}^{2n} \times \mathbb{R}^{2n} \) and \( H : \Lambda \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R} \) is \( \tau \)-periodic in the \( \mathbb{R} \)-variable, see, e.g., [5, 7, 22, 24, 26, 30, 59, 61, 62, 63, 68, 69] and references therein. Most of them considered Hamiltonians affinely depending on a real parameter or invariant for the action of a Lie group on \( \mathbb{R}^{2n} \), and used either Liapunov-Schmidt reductions (5, 7, 61, 62, 69) or spectral flow methods (22, 30, 62, 63, 64) or the Poincaré map (59, 26).

In this work, using the abstract bifurcation theory recently developed by author in [50, 52] and Appendix C we can prove many new bifurcation results for solutions of four types of non-autonomous Hamiltonian boundary value problems nonlinearly depending on parameters. Our
methods are also used to derive some bifurcation results starting at a nonequilibrium (i.e., non-constant) periodic solution of autonomous Hamiltonian systems. Though some necessary or sufficient conditions for bifurcations of these Hamiltonian problems may also be derived from previous abstract theories, the most interesting and important alternative results in this paper can only be proved with our generalizations in [50, 52] and Appendix C for the famous Rabinowitz’s alternative bifurcation theorem [60]. Before precisely stating these we begin with the following.

Notation and conventions. Let $S_\tau := \mathbb{R}/\tau\mathbb{Z}$ for $\tau > 0$. All vectors in $\mathbb{R}^m$ will be understood as column vectors. The transpose of a matrix $M \in \mathbb{R}^{m \times m}$ is denoted by $M^T$. $\text{Ker}(M) = \{x \in \mathbb{R}^m \mid Mx = 0\}$. We denote $(\cdot, \cdot)_{\mathbb{R}^m}$ by the standard Euclidean inner product in $\mathbb{R}^m$ and by $|\cdot|$ the corresponding norm. Let $\mathcal{L}_s(\mathbb{R}^m)$ be the set of all real symmetric matrices of order $m$, and let $\text{Sp}(2n, \mathbb{R})$ be the symplectic group of real symplectic matrices of order $2n$, i.e., $\text{Sp}(2n, \mathbb{R}) = \{M \in \text{GL}(2n, \mathbb{R}) \mid M^TJM = J_n\}$, where

$$J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad (1.2)$$

with the identity matrix $I_n$ of order $n$, which gives the standard complex structure on $\mathbb{R}^{2n}$,

$$(q_1, \cdots, q_n, p_1, \cdots, p_n)^T \mapsto J(q_1, \cdots, q_n, p_1, \cdots, p_n)^T = (-p_1, \cdots, -p_n, q_1, \cdots, q_n)^T.$$

For a map $f$ from $X$ to $Y$, $Df(x)$ (resp. $df(x)$ or $f'(x)$) denotes the Gâteaux (resp. Fréchet) derivative of $f$ at $x \in X$, which is an element in $\mathcal{L}(X,Y)$. Of course, we also use $f'(x)$ to denote $Df(x)$ without occurring of confusions. When $Y = \mathbb{R}$, $f'(x) \in \mathcal{L}(X, \mathbb{R}) = X^*$, and if $X = H$ we call the Riesz representation of $f'(x)$ in $H$ gradient of $f$ at $x$, denoted by $\nabla f(x)$. The Fréchet (or Gâteaux) derivative of $\nabla f$ at $x \in H$ is denoted by $f''(x)$ or $\nabla^2 f(x)$, which is an element in $\mathcal{L}_s(H)$. (Precisely, $f''(x) = (f'(x))' \in \mathcal{L}(H; \mathcal{L}(H; \mathbb{R}))$ is a symmetric bilinear form on $H$, and is identified with $D(\nabla f)(x)$ after $\mathcal{L}(H, \mathbb{R}) = H^*$ is identified with $H$ via the Riesz representation theorem.) For $\varepsilon > 0$ and a point $x$ in a Banach space $X$ we write $B_X(x, \varepsilon) = \{y \in X \mid \|x-y\| < \varepsilon\}$.

We now turn to a detailed description of our main results in four subsections. Section 1.1 lists bifurcation results for solutions of two types of non-autonomous Hamiltonian boundary value problems. Results in the second type immediately follows from that of the first type. Section 1.2 is to concern with bifurcations for generalized periodic solutions of autonomous Hamiltonian systems. In Section 1.3 we state bifurcations for brake orbits of periodic Hamiltonian systems. Section 1.4 gives bifurcations for Hamiltonian paths connecting affine Lagrangian subspaces.

1.1. Bifurcations for solutions of two types of non-autonomous Hamiltonian systems. We begin with the following two closely related assumptions.

**Assumption 1.1.** For a real $\tau > 0$, $M \in \text{Sp}(2n, \mathbb{R})$ and a topological space $\Lambda$, let $H : \Lambda \times [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R}$ be a continuous function such that each $H(\lambda, t, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}$, $(\lambda, t) \in \Lambda \times [0, \tau]$, is $C^2$, and all possible partial derivatives of $H$ depend continuously on $(\lambda, t, z) \in \Lambda \times [0, \tau] \times \mathbb{R}^{2n}$. Denote by $\nabla_{\lambda} H(\lambda, t, \cdot)$ the euclidian gradient of $H(\lambda, t, \cdot)$ with respect to the $\mathbb{R}^{2n}$-variable, and by $\nabla_{\lambda}^2 H(\lambda, t, \cdot) = D_z(\nabla_{\lambda} H(\lambda, t, \cdot)) \in \mathcal{L}_s(\mathbb{R}^{2n})$. For each $\lambda \in \Lambda$ let $u(\lambda)$ be a solution of the Hamiltonian boundary value problem

$$\dot{u}(t) = J\nabla_{\lambda} H(\lambda, t, u(t)) \quad \forall t \in [0, \tau] \quad \text{and} \quad u(\tau) = Mu(0), \quad (1.3)$$

and $\Lambda \times [0, \tau] \ni (\lambda, t) \mapsto u_\lambda(t) \in \mathbb{R}^{2n}$ is also continuous.
Under this assumption it follows from \[1.3\] that \(\Lambda \times [0, \tau] \ni (\lambda, t) \mapsto \dot{u}_\lambda(t) \in \mathbb{R}^{2n}\) is also continuous. For any given \(\mu \in \Lambda\) using nets we can prove \(\|u_\lambda - u_\mu\|_{C^1} \to 0\) as \(\lambda \to \mu\). Note that Assumption \[1.3\] cannot guarantee that each \(u_\lambda : [0, \tau] \to \mathbb{R}^{2n}\) is \(C^2\) since we have not assumed that each \(H(\lambda, \cdot, \cdot) : [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R}, \lambda \in \Lambda\), is \(C^1\), and all possible partial derivatives of order one of \(H\) depend continuously on \((\lambda, t, \tau) \in \Lambda \times [0, \tau] \times \mathbb{R}^{2n}\).

**Assumption 1.2.** For a real \(\tau > 0\), a matrix \(M \in \text{Sp}(2n, \mathbb{R})\) and a topological space \(\Lambda\), let \(H : \Lambda \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}\) be a continuous function such that each \(H(\lambda, t, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}, (\lambda, t) \in \Lambda \times \mathbb{R}\), is \(C^2\), and all possible partial derivatives of \(H\) depend continuously on \((\lambda, t, \tau) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n}\), and that

\[
H(\lambda, t + \tau, Mz) = H(\lambda, t, z) \quad \forall (\lambda, t, z) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n}.
\]

(1.4)

For each \(\lambda \in \Lambda\) let \(v_\lambda : \mathbb{R} \to \mathbb{R}^{2n}\) satisfy the following generalized periodic Hamiltonian system

\[
\dot{v}(t) = J \nabla_z H(\lambda, t, v(t)) \quad \text{and} \quad (t + \tau) = M v(t) \quad \forall t \in \mathbb{R},
\]

(1.5)

and \(\Lambda \times \mathbb{R} \ni (\lambda, t) \mapsto v_\lambda(t) \in \mathbb{R}^{2n}\) is also continuous.

(In some literatures solutions of \[1.5\] were called \((M, \tau)\)-periodic or affine \(\tau\)-periodic.) Clearly, a solution \(v\) of \[1.3\] gives one of \[1.5\], \(v_{\|0,\tau\|}\). Conversely, since \[1.3\] implies

\[
\nabla_z H(\lambda, t + \tau, Mz) = (M^{-1})^T \nabla_z H(\lambda, t, z) \quad \forall (\lambda, t, z) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n},
\]

(1.6)

every solution \(u\) of \[1.3\] gives rise to a solution \(u^M\) of \[1.5\] defined by

\[
u^M(t) := M^k u(t - k\tau) \quad \text{for} \quad k\tau < t \leq (k + 1)\tau, \quad k \in \mathbb{Z}.
\]

(1.7)

Moreover, this correspondence between solutions of \[1.3\] and \[1.5\] is one-to-one.

The linearized problems of \[1.3\] and \[1.5\] along \(u_\lambda\) and \(v_\lambda\) are, respectively,

\[
\begin{align*}
\hat{u}(t) & = J \nabla_z^2 H(\lambda, t, u_\lambda(t)) u(t) \quad \forall t \in [0, \tau] \quad \text{and} \quad u(0) = 0, \\
\hat{v}(t) & = J \nabla_z^2 H(\lambda, t, v_\lambda(t)) v(t) \quad \text{and} \quad v(0) = 0.
\end{align*}
\]

(1.8)

(1.9)

Clearly, Assumption \[1.1\] and Assumption \[1.2\] imply that maps

\[
\begin{align*}
\Lambda \times [0, \tau] \ni (\lambda, t) & \mapsto \nabla_z^2 H(\lambda, t, u_\lambda(t)) \in \mathcal{L}_2(\mathbb{R}^{2n}) \\
\Lambda \times \mathbb{R} \ni (\lambda, t) & \mapsto \nabla_z^2 H(\lambda, t, v_\lambda(t)) \in \mathcal{L}_2(\mathbb{R}^{2n})
\end{align*}
\]

are continuous, respectively.

Under Assumption \[1.2\] let \((\lambda_k) \subset \Lambda\) converge to \(\mu \in \Lambda\), and let \(\bar{v}_k : \mathbb{R} \to \mathbb{R}^{2n}\) be a solution of \[1.5\] with \(\lambda = \lambda_k \in \Lambda\) for each \(k = 1, 2, \ldots\). In this paper we say that \(\bar{v}_k \to v_\mu\) in \(C^0_{\text{loc}}(\mathbb{R}, \mathbb{R}^{2n})\) if \(\bar{v}_k|_{[a, b]} \to v_\mu|_{[a, b]}\) in \(C^1([a, b]; \mathbb{R}^{2n})\) (resp. \(C^0([a, b]; \mathbb{R}^{2n})\)) for any bounded interval \([a, b) \subset \mathbb{R}\). Clearly, \(\bar{v}_k \to v_\mu\) in \(C^1_{\text{loc}}\) (resp. \(C^0_{\text{loc}}\)) if and only if \(\bar{v}_k|_{[0, \tau]} \to v_\mu|_{[0, \tau]}\) in \(C^1([0, \tau]; \mathbb{R}^{2n})\) (resp. \(C^0([0, \tau]; \mathbb{R}^{2n})\)).

Because of these and the following Proposition \[1.3\](i), in the sense of Definition \[1.2\] \((\mu, v_\mu)\) is said to be a bifurcation point along sequences of \[1.5\] with respect to the branch \(\{(\lambda, v_\lambda) | \lambda \in \Lambda\}\) if there exists a sequence \((\lambda_k) \subset \Lambda\) converging to \(\mu \in \Lambda\) and solutions \(\bar{v}_k \neq v_\lambda_k\) of \[1.5\] with \(\lambda = \lambda_k \in \Lambda\), \(k = 1, 2, \ldots\), such that \(\bar{v}_k|_{[0, \tau]} \to v_\mu|_{[0, \tau]}\) in \(C^0([0, \tau]; \mathbb{R}^{2n})\) or \(\Lambda \times C^1([0, \tau]; \mathbb{R}^{2n})\).

**Proposition 1.3.** (i) Under Assumption \[1.3\] \((\mu, v_\mu)\) is a bifurcation point along sequences of the problem \[1.5\] in \(\Lambda \times C^0([0, \tau]; \mathbb{R}^{2n})\) with respect to \(\{(\lambda, u_\lambda) | \lambda \in \Lambda\}\) if and only if it is that of the problem \[1.5\] in \(\Lambda \times C^1([0, \tau]; \mathbb{R}^{2n})\) with respect to \(\{(\lambda, u_\lambda) | \lambda \in \Lambda\}\).
Let $\mathbb{E} := E^1_M$ be the Hilbert space as in Appendix E, which is the fractional Sobolev space $H^{1/2}(S_\tau; \mathbb{R}^{2n})$ if $M = I_{2n}$. Under Assumption 1.2, if $H$ also satisfies
\begin{equation}
|\nabla_z H(\lambda, t, z)| \leq c_1 + c_2 |z|^r \quad \forall (\lambda, t, z) \in \Lambda \times [0, \tau] \times \mathbb{R}^{2n},
\end{equation}
where $c_1, c_2 \geq 0$ and $r \in (1, 2]$ (resp. $r > 1$) are constants if $M \neq I_{2n}$ (resp. $M = I_{2n}$), then $(\mu, u_\mu)$ is a bifurcation point along sequences of the problem (1.3) in $\Lambda \times E$ with respect to $\{ (\lambda, u_\lambda) \mid \lambda \in \Lambda \}$ if and only if it is that of the problem (1.3) in $\Lambda \times C^1([0, \tau]; \mathbb{R}^{2n})$ with respect to $\{ (\lambda, u_\lambda) \mid \lambda \in \Lambda \}$.

This result will be proved in Appendix 3. It shows under Assumption 1.2 and the condition (1.10) that statements of bifurcations along sequences of the problem (1.3) with $M = I_{2n}$ are intrinsic, i.e., are independent of choices of spaces $H^{1/2}(S_\tau; \mathbb{R}^{2n}), W^{1,2}(S_\tau; \mathbb{R}^{2n})$ and $C^i(S_\tau; \mathbb{R}^{2n})$, $i = 0, 1, 2$.

Let $\gamma : [0, \tau] \to \text{Sp}(2n, \mathbb{R})$ be the fundamental matrix solution of
\begin{equation}
\dot{Z}(t) = J \nabla^2_z H(\lambda, t, u_\lambda(t))Z(t).
\end{equation}

The Maslov type index of $\gamma_\lambda$ relative to $M$ is a pair of integers $(i_{r,M}(\gamma_\lambda), \nu_{r,M}(\gamma_\lambda))$ (cf. Appendix A for details), where $\nu_{r,M}(\gamma_\lambda) = \dim \text{Ker}(\gamma_\lambda(\tau) - M)$ is the dimension of solution space of (1.8). When $M = I_{2n}$, $(i_{r,M}(\gamma_\lambda), \nu_{r,M}(\gamma_\lambda))$ is just the Maslov type index $(i_r(\lambda_\lambda), \nu_r(\lambda_\lambda))$ defined in [13], in particular, $i_r(\lambda_\lambda)$ is the Conley-Zehnder index $i_{CZ}(\lambda_\lambda)$ of $\gamma_\lambda$ if $\nu_r(\lambda_\lambda) = 0$ (17, 34, 66). Consider the Banach subspaces of $C^i([0, \tau]; \mathbb{R}^{2n})$,
\begin{equation}
C^i_M([0, \tau]; \mathbb{R}^{2n}) = \{ u \in C^i([0, \tau]; \mathbb{R}^{2n}) \mid u(\tau) = M u(0) \}, \quad i \in \mathbb{N} \cup \{ 0 \}.
\end{equation}

For bifurcations of the boundary value problem (1.3) we have:

**Theorem 1.4.** Under Assumption 1.3, let $\gamma_\lambda$ be as above.

(I) ( Necessary condition): $\nu_{r,M}(\gamma_\mu) \neq 0$ if $(\mu, u_\mu)$ be a bifurcation point along sequences of (1.3) in $\Lambda \times C^0_M([0, \tau]; \mathbb{R}^{2n})$ with respect to the branch $\{ (\lambda, u_\lambda) \mid \lambda \in \Lambda \}$, i.e., there exists a sequence $(\lambda_k) \subset \Lambda$ converging to $\mu$ and solutions $u^k \neq u_{\lambda_k}$ of (1.3) with $\lambda = \lambda_k$ such that $u^k \to u_\mu$ in $C^0([0, \tau], \mathbb{R}^n)$.

(II) ( Sufficient condition): Let $\Lambda$ be first countable. Suppose for some $\mu \in \Lambda$ that there exist two sequences in $\Lambda$ converging to $\mu$, $(\lambda_\mu^+, \lambda_\mu^-)$, such that for each $k \in \mathbb{N}$,
\begin{equation}
[i_{r,M}(\gamma_\lambda^-), i_{r,M}(\gamma_\lambda^+) + \nu_{r,M}(\gamma_\lambda^-)] \cap [i_{r,M}(\gamma_\lambda^+), i_{r,M}(\gamma_\lambda^+) + \nu_{r,M}(\gamma_\lambda^+)] = \emptyset
\end{equation}
and either $\nu_{r,M}(\gamma_\lambda^+) = 0$ or $\nu_{r,M}(\gamma_\lambda^-) = 0$. Let $\hat{\Lambda} := \{ \mu, \lambda_\mu^+, \lambda_\mu^- \mid k \in \mathbb{N} \}$. Then $(\mu, u_\mu)$ is a bifurcation point of (1.3) in $\hat{\Lambda} \times C^1_M([0, \tau]; \mathbb{R}^{2n})$ with respect to the branch $\{ (\lambda, u_\lambda) \mid \lambda \in \hat{\Lambda} \}$. In particular, $(\mu, u_\mu)$ is a bifurcation point along sequences of (1.3) in $\Lambda \times C^1_M([0, \tau]; \mathbb{R}^{2n})$ with respect to the branch $\{ (\lambda, u_\lambda) \mid \lambda \in \Lambda \}$.

(III) ( Existence for bifurcations): Let $\Lambda$ be path-connected. If there exist two points $\lambda^+, \lambda^- \in \Lambda$ such that $[i_{r,M}(\gamma_\lambda^-), i_{r,M}(\gamma_\lambda^+) + \nu_{r,M}(\gamma_\lambda^-)] \cap [i_{r,M}(\gamma_\lambda^+), i_{r,M}(\gamma_\lambda^+) + \nu_{r,M}(\gamma_\lambda^+)] = \emptyset$ and either $\nu_{r,M}(\gamma_\lambda^+) = 0$ or $\nu_{r,M}(\gamma_\lambda^-) = 0$, then for any path $\alpha : [0, 1] \to \Lambda$ connecting $\lambda^+$ to $\lambda^-$ there exists a sequence $(t_k) \subset [0, 1]$ converging to some $t$ and solutions $u^k \neq u_\alpha(t_k)$ of (1.3) with $\lambda = \alpha(t_k), k = 1, 2, \cdots$ such that $\| u^k - u_\alpha(t_k) \|_{C^1} \to 0$ (and so $\| u^k - u_\alpha(t_k) \|_{C^1} \to 0$). Moreover, $\alpha(t)$ is not equal to $\lambda^+$ (resp. $\lambda^-$) if $\nu_{r,M}(\gamma_{\lambda^+}) = 0$ (resp. $\nu_{r,M}(\gamma_{\lambda^-}) = 0$). In particular, $(\alpha(t), 0)$ is a bifurcation point along sequences of (1.3) in $\Lambda \times C^1_M([0, \tau]; \mathbb{R}^{2n})$. 


Remark 1.6. Under Assumption 1.2, let $B$ be the fundamental matrix solution of

$$\dot{Z}(t) = J\nabla^2_z H(\lambda, t, v_\lambda(t))Z(t).$$

(I) (Necessary condition): $\nu_{\tau, M}(\gamma_\mu) \neq 0$ if $(\mu, v_\mu)$ is a bifurcation point along sequences of \ref{72} with respect to the branch $\{((\lambda, v_\lambda) | \lambda \in \Lambda\}$.

(II) (Sufficient condition): Let $\Lambda$ be first countable. Suppose for some $\mu \in \Lambda$ that there exist two sequences in $\Lambda$ converging to $\mu$, $(\lambda^{-}_k)$ and $(\lambda^{+}_k)$, such that for each $k \in \mathbb{N}$,

$$[i_{\tau, M}(\gamma^{-}_k), i_{\tau, M}(\gamma^{+}_k) + \nu_{\tau, M}(\gamma^{+}_k)] \cap [i_{\tau, M}(\gamma^{-}_k), i_{\tau, M}(\gamma^{+}_k) + \nu_{\tau, M}(\gamma^{+}_k)] = \emptyset,$$

and either $\nu_{\tau, M}(\gamma^{-}_k) = 0$ or $\nu_{\tau, M}(\gamma^{+}_k) = 0$. Let $\hat{\Lambda} := \{\mu, \lambda^{+}_k, \lambda^{-}_k | k \in \mathbb{N}\}$. Then $(\mu, v_\mu)$ is a bifurcation point of \ref{72} with respect to the branch $\{((\lambda, v_\lambda) | \lambda \in \hat{\Lambda}\}$ (and so $\{((\lambda, v_\lambda) | \lambda \in \Lambda\}$).

(III) (Existence for bifurcations): Let $\Lambda$ be path-connected. If there exist two points $\lambda^+, \lambda^- \in \Lambda$ such that $[i_{\tau, M}(\gamma^{-}_k), i_{\tau, M}(\gamma^{+}_k) + \nu_{\tau, M}(\gamma^{+}_k)] \cap [i_{\tau, M}(\gamma^{-}_k), i_{\tau, M}(\gamma^{+}_k) + \nu_{\tau, M}(\gamma^{+}_k)] = \emptyset$ and either $\nu_{\tau, M}(\gamma^{+}_k) = 0$ or $\nu_{\tau, M}(\gamma^{-}_k) = 0$, then for any path $\alpha : [0, 1] \rightarrow \Lambda$ connecting $\lambda^+$ to $\lambda^-$ there exists a sequence $(t_k) \subset [0, 1]$ converging to some $\bar{t}$ and solutions $v^k \neq v_{\alpha(t_k)}$ of \ref{72} with $\lambda = \alpha(t_k)$, $k = 1, 2, \ldots$, such that $(v^k)$ converges to $v_{\alpha(\bar{t})}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology as $k \to \infty$. Moreover, $\alpha(\bar{t})$ is not equal to $\lambda^+$ (resp. $\lambda^-$) if $\nu_{\tau, M}(\gamma^{+}_k) = 0$ (resp. $\nu_{\tau, M}(\gamma^{-}_k) = 0$).

Remark 1.6. (i) Let us compare Theorem 1.5 with previous related results. If $M = I_{2n}$, $\tau = 2\pi$ and $\Lambda$ is an open subset in a real finite dimensional Banach space, the conclusion of Theorem 1.5 can also be derived from [3] Proposition 26.1 provided that $\nabla_z H$ and mappings $A \times [0, 2\pi] \ni (\lambda, t) \mapsto v_\lambda(t) \in \mathbb{R}^{2n}$ and $A \times [0, 2\pi] \ni (\lambda, t) \mapsto \dot{v}_\lambda(t) \in \mathbb{R}^{2n}$ are also $C^1$. In fact, in the present conditions we can apply [3] Proposition 26.1 to $X = E = \mathbb{R}^{2n}$, $k = 1$ and $f(\lambda, t, \zeta) = \nabla_z H(\lambda, t, \zeta + v_\lambda(t)) - \dot{v}_\lambda(t)$ to conclude $\nu_{2\pi, M}(\gamma_\mu) \neq 0$ because the assumption guarantees that $(\mu, 0)$ is also a bifurcation point of $\dot{x} = f(\lambda, t, x)$ in the sense of [4] page 369).

Let $H \in C^2([a, b] \times \mathbb{R} \times \mathbb{R}^{2n})$ be $2\pi$-periodic in the second variable and satisfy $H(\lambda, t, 0) \equiv 0$. (Therefore it satisfies Assumption 1.2 with $\Lambda = [a, b]$ and $M = I_{2n}$.) Suppose that

(H1) There exist constants $r > 1$ and $c_i > 0$, $i = 1, 2$, such that for all $(\lambda, t, z) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n}$,

$$|\nabla_z H(\lambda, t, z)| \leq c_1 + c_2 |z|^r \quad \text{and} \quad |\nabla^2_z H(\lambda, t, z)| \leq c_1 + c_2 |z|^r.$$

(H2) There is a path $\lambda \mapsto A_\lambda$ of time-independent symmetric $2n \times 2n$ real matrices such that

$$H(\lambda, t, z) = (A_\lambda z, z)_{\mathbb{R}^{2n}}/2 + R(\lambda, t, z),$$

where $\nabla_z R(\lambda, t, z) = o(|z|)$ as $|z| \to 0$. (Without using $H(\lambda, t, 0) \equiv 0$, this implies that $\nabla_z H(\lambda, t, 0) = 0$ and $\nabla^2_z H(\lambda, t, 0) = A_\lambda$ for all $(\lambda, t)$. In particular, $\lambda \mapsto A_\lambda$ is continuous because $H$ is $C^2$.)
Let $\gamma_\lambda$ be the fundamental matrix solution of $\dot{Z}(t) = JA_\lambda Z(t)$. It was claimed in [22] page 22 that $\nu_{2\pi}(\gamma_{\mu}) = \dim \ker(\gamma_{\mu}(2\pi) - I_{2n}) \neq 0$ is a necessary condition for $(\mu, 0)$ to be a bifurcation point of (1.3) with respect to the trivial branch in $[a, b] \times H^{1/2}(S^\ast_{2\pi}; \mathbb{R}^{2n})$ [or equivalently in $[a, b] \times C^1(S^\ast_{2\pi}; \mathbb{R}^{2n})$ by Proposition 3.3(ii)]. Clearly, this result is contained in Theorem 1.1.

(ii) Theorem 1.1(III) with $M = I_{2n}$ and $\tau = 2\pi$ generalizes [22, Theorem 1.1] and [22, Theorem 2.2]. Let $H \in C^\infty([a, b] \times \mathbb{R} \times \mathbb{R}^{2n})$ be $2\pi$-periodic in the second variable and satisfy the above growth condition (H1). Assume that $\dot{v}(t) = J\nabla_z H(\lambda, t, v(t))$ possesses a known family $\{v_\lambda\}$ of $2\pi$-periodic solutions smoothly parametrized by $\lambda$ in $[a, b]$. Denote by $\gamma_\lambda$ the fundamental matrix solution of $\dot{Z}(t) = J\nabla_z H(\lambda, t, v_\lambda(t))Z(t)$. For each $\lambda \in [a, b]$, we define $L_\lambda \in \mathcal{L}_H(\mathbb{H}^{1/2}(S^\ast_{2\pi}; \mathbb{R}^{2n}))$ by

$$(L_\lambda v, w)_{\mathbb{H}^{1/2}} = \int_0^{2\pi} (J\dot{v}(t) + \nabla_z^2 H(\lambda, t, v_\lambda(t))v(t), w(t))_{\mathbb{R}^{2n}} dt$$

as in the final line of [22] page 24, and $A_\lambda : H^1(S^\ast_{2\pi}; \mathbb{R}^{2n}) \to L^2(S^\ast_{2\pi}; \mathbb{R}^{2n})$ by

$$(A_\lambda v)(t) = J\dot{v}(t) + \nabla_z^2 H(\lambda, t, v_\lambda(t))v(t).$$

Let $s_\mathcal{F}(A)$ and $s_\mathcal{F}(L)$ be the spectral flows of the families $\{A_\lambda | a \leq \lambda \leq b\}$ and $\{L_\lambda | a \leq \lambda \leq b\}$, respectively. Consider a symplectic path $\Gamma_{2\pi} : [a, b] \to \mathrm{Sp}(2n, \mathbb{R})$ given by $\Gamma_{2\pi}(\lambda) = \gamma_\lambda(2\pi)$. Its Maslov-type index $i(\Gamma_{2\pi}, \{a, b\})$ defined by (A.2) (or [22, (2.7)]) then (A.3) implies

$$\mu_{\mathcal{CZ}}(\Gamma_{2\pi}) = \mu_{\mathcal{CZ}}(\gamma_b) - \mu_{\mathcal{CZ}}(\gamma_a).$$

By [66] (see also [11, Theorem 11.1]) and [22, Proposition 2.1] there holds

$$s_\mathcal{F}(A) = \mu_{\mathcal{CZ}}(\Gamma_{2\pi}) \quad \text{and} \quad s_\mathcal{F}(L) = \mu_{\mathcal{CZ}}(\Gamma_{2\pi})$$

and therefore $s_\mathcal{F}(L) = s_\mathcal{F}(A) = \mu_{\mathcal{CZ}}(\Gamma_{2\pi})$ (see also [69, Theorem 4.1]). (A.7) may lead to

Claim. If $\nu_{2\pi}(\gamma_a) = \nu_{2\pi}(\gamma_b) = 0$, then $\mu_{\mathcal{CZ}}(\gamma_*) = i_{\mathcal{CZ}}(\gamma_*) = i_{2\pi}(\gamma_*)$, $* = a, b$, and so

$$\mu_{\mathcal{CZ}}(\Gamma_{2\pi}) = i_{2\pi}(\gamma_b) - i_{2\pi}(\gamma_a),$$

$$i_{2\pi}(\gamma_b) \neq i_{2\pi}(\gamma_a) \iff \mu_{\mathcal{CZ}}(\Gamma_{2\pi}) \neq 0 \iff s_\mathcal{F}(L) \neq 0.$$ (1.15)

Having this [22, Theorem 1.1] and [22, Theorem 2.2] may be restated as:

**Theorem** (22, Theorem 1.1). Let $H \in C^2([a, b] \times \mathbb{R} \times \mathbb{R}^{2n})$ be $2\pi$-periodic in the second variable and satisfy (H1)-(H2) in (i) above and

$$(H3) \quad \text{The matrices } JA_a \text{ and } JA_b \text{ have no eigenvalues that are integral multiples of } \sqrt{-1},$$

and therefore specially $\nu_{2\pi}(\gamma_a) = \nu_{2\pi}(\gamma_b) = 0$.

If $i_{2\pi}(\gamma_b) \neq i_{2\pi}(\gamma_a)$, then there exists $\mu \in (a, b)$ such that $(\mu, 0)$ is a bifurcation point of (1.3) with $(\Lambda, M, \tau) = ([a, b], I_{2n}, 2\pi)$ in $[a, b] \times H^{1/2}(S^\ast_{2\pi}; \mathbb{R}^{2n})$ with respect to the trivial branch.
Theorem (22 Theorem 2.2). Let \( H \in C^\infty([a, b] \times \mathbb{R} \times \mathbb{R}^{2n}) \) be \( 2\pi \)-periodic in the second variable and satisfy (H1) in (i), and let \( \{ v_\lambda | \lambda \in [a, b] \} \) be a family of \( 2\pi \)-periodic solutions of \( \dot{v}(t) = J \nabla_H \lambda, t, v(t) \) smoothly parameterized by \( \lambda \) in \([a, b]\). If \( v_{2\pi}(\gamma_0) = v_{2\pi}(\gamma_0) = 0 \) and \( i_{2\pi}(\gamma_0) \neq i_{2\pi}(\gamma_0) \), then there exists \( \mu \in (a, b) \) such that \((\mu, v_\mu)\) is a bifurcation point of \((1.2)\) with \((\Lambda, M, \tau) = ([a, b], I_{2\pi}, 2\pi)\) in \([a, b] \times H^{1/2}(S_{2\pi}; \mathbb{R}^{2n})\) with respect to the branch \( \{(\lambda, v_\lambda) | \lambda \in \Lambda\} \).

When \( v_\lambda \equiv 0 \) in this result, the condition "\( H \in C^\infty([a, b] \times \mathbb{R} \times \mathbb{R}^{2n})\)" was weakened as "\( H \in C^0([a, b] \times \mathbb{R} \times \mathbb{R}^{2n})\), each \( H(\lambda, \cdot, \cdot) \in [a, b], \) is \( C^2 \) and all possible partial derivatives of \( H \) depend continuously on the parameter \( \lambda \)" in lines 1-4 of [69, page 743].

Clearly, the assumptions on \( H \) in the above two theorems are stronger than those in Assumption 1.2 with \((\Lambda, M, \tau) = ([a, b], I_{2\pi}, 2\pi)\).

To the best of the author’s knowledge, the following alternative bifurcation result (and so its consequence, Theorem 1.8) is completely new, and no similar results appear before.

**Theorem 1.7** (Alternative bifurcations of Rabinowitz’s type and of Fadell-Rabinowitz’s type). Let Assumption 1.7 with \( \Lambda \) being a real interval be satisfied, and let \( \gamma_\lambda \) be as in Theorem 1.4 for each \( \lambda \in \Lambda \). Suppose for some interior point \( \mu \) of \( \Lambda \) that \( \dim \ker(\gamma_\mu(\tau) - M) \neq 0 \) and \( \dim \ker(\gamma_\lambda(\tau) - M) = 0 \) for each \( \lambda \in \Lambda \setminus \{ \mu \} \) near \( \mu \), and that \( i_{\tau, M}(\gamma_\lambda) \) takes, respectively, values \( i_{\tau, M}(\gamma_\mu) \) and \( i_{\tau, M}(\gamma_\mu) + \nu_{\tau, M}(\gamma_\mu) \) as \( \lambda \in \Lambda \) varies in two deleted half neighborhoods of \( \mu \).

Then one of the following assertions holds:

(i) The problem \((1.3)\) with \( \lambda = \mu \) has a sequence of solutions, \( u_\mu^k \neq u_\mu, k = 1, 2, \ldots, \) which converges to \( u_\mu \in C^1_M([0, \tau]; \mathbb{R}^{2n}) \).

(ii) For every \( \lambda \in \Lambda \setminus \{ \mu \} \) near \( \mu \) there is a solution \( \tilde{u}_\lambda \neq u_\lambda \) of \((1.3)\) with parameter value \( \lambda \), such that \( \tilde{u}_\lambda - u_\lambda \) converges to zero in \( C^1_M([0, \tau]; \mathbb{R}^{2n}) \) as \( \lambda \to \mu \).

(iii) For a given neighborhood \( W \) of \( u_\mu \) in \( C^1_M([0, \tau]; \mathbb{R}^{2n}) \) there is an one-sided neighborhood \( \Lambda^0 \) of \( \mu \) such that for any \( \lambda \in \Lambda^0 \setminus \{ \mu \} \), \((1.3)\) with parameter value \( \lambda \) has at least two distinct solutions \( u_\lambda^1 \neq u_\lambda \) and \( u_\lambda^2 \neq u_\lambda \) in \( W \), which can also be required to satisfy

\[
\int_0^\tau \left[ \frac{1}{2}(J\dot{u}_\lambda^1(t), u_\lambda^1(t))_{\mathbb{R}^{2n}} + H(\lambda, t, u_\lambda^1(t)) \right] dt \neq \int_0^\tau \left[ \frac{1}{2}(J\dot{u}_\lambda^2(t), u_\lambda^2(t))_{\mathbb{R}^{2n}} + H(\lambda, t, u_\lambda^2(t)) \right] dt
\]

provided that \( \nu_{\tau, M}(\gamma_\mu) > 1 \) and \((1.3)\) with parameter value \( \lambda \) has only finitely many solutions in \( W \).

Moreover, if \( u_\lambda = 0 \forall \lambda \), and all \( H(\lambda, t, \cdot) \) are even, then either (i) holds or the following occurs:

(iv) There exist left and right neighborhoods \( \Lambda^- \) and \( \Lambda^+ \) of \( \mu \) in \( \Lambda \) and integers \( n^+, n^- \geq 0 \), such that \( n^+ + n^- \geq \nu_{\tau, M}(\gamma_\mu) \), and for \( \lambda \in \Lambda^- \setminus \{ \mu \} \) (resp. \( \lambda \in \Lambda^+ \setminus \{ \mu \} \)), \((1.3)\) with parameter value \( \lambda \) has at least \( n^- \) (resp. \( n^+ \)) distinct pairs of nontrivial solutions, \( \{ u_{\lambda, i}^-, -u_{\lambda, i}^- \}, i = 1, \ldots, n^- \) (resp. \( n^+ \)), which converge to zero in \( C^1_M([0, \tau]; \mathbb{R}^{2n}) \) as \( \lambda \to \mu \).

As above this directly leads to the following bifurcation result of the system \((1.5)\).

**Theorem 1.8** (Alternative bifurcations of Rabinowitz’s type and of Fadell-Rabinowitz’s type). Let Assumption 1.7 with \( \Lambda \) being a real interval be satisfied, and let \( \gamma_\lambda \) be as in Theorem 1.4 for each \( \lambda \in \Lambda \). Suppose for some interior point \( \mu \) of \( \Lambda \) that \( \dim \ker(\gamma_\mu(\tau) - M) \neq 0 \) and \( \dim \ker(\gamma_\lambda(\tau) - M) = 0 \) for each \( \lambda \in \Lambda \setminus \{ \mu \} \) near \( \mu \), and that \( i_{\tau, M}(\gamma_\lambda) \) takes, respectively, values \( i_{\tau, M}(\gamma_\mu) \) and \( i_{\tau, M}(\gamma_\mu) + \nu_{\tau, M}(\gamma_\mu) \) as \( \lambda \in \Lambda \) varies in two deleted half neighborhoods of \( \mu \).

Then one of the following assertions holds:
Corollary 1.9. The problem (1.3) with $\lambda = \mu$ has a sequence of solutions, $v^k_\mu \neq v_\mu$, $k = 1, 2, \cdots$, such that $v^k_{\mu|[0, \tau]} \to v_{\mu|[0, \tau]}$ in $C^1([0, \tau]; \mathbb{R}^{2n})$.

(ii) For every $\lambda \in \Lambda \setminus \{\mu\}$ near $\mu$ there is a solution $\tilde{v}_\lambda \neq v_\lambda$ of (1.3) with parameter value $\lambda$ such that $v_\lambda|_{[0, \tau]} - v_{\lambda|[0, \tau]} \to 0$ in $C^1([0, \tau]; \mathbb{R}^{2n})$ as $\lambda \to \mu$.

(iii) For a given neighborhood $W$ of $v_\mu|_{[0, \tau]}$ in $C^1([0, \tau]; \mathbb{R}^{2n})$ there is an one-sided neighborhood $\Lambda^0$ of $\mu$ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, (1.3) with parameter value $\lambda$ has at least two distinct solutions $v^1_\lambda \neq v_\lambda$ and $v^2_\lambda \neq v_\lambda$ such that $v^1_\lambda|_{[0, \tau]}$ and $v^2_\lambda|_{[0, \tau]}$ are in $W$, which can also be required to satisfy

$$\int_0^\tau \left[ \frac{1}{2}(J\dot{v}^1_\lambda(t), v^1_\lambda(t))_{\mathbb{R}^{2n}} + H(\lambda, t, v^1_\lambda(t)) \right] dt \neq \int_0^\tau \left[ \frac{1}{2}(J\dot{v}^2_\lambda(t), v^2_\lambda(t))_{\mathbb{R}^{2n}} + H(\lambda, t, v^2_\lambda(t)) \right] dt$$

provided that $\nu_\tau,M(\gamma_{\mu}) > 1$ and (1.3) with parameter value $\lambda$ has only finitely many solutions whose restrictions to $[0, \tau]$ belong to $W$.

Moreover, if $v_\lambda = 0 \forall \lambda$, and all $H(\lambda, t, \cdot)$ are even, then either (i) holds or the following occurs:

(iv) There exist left and right neighborhoods $\Lambda^-$ and $\Lambda^+$ of $\mu$ in $\Lambda$ and integers $n^+, n^- \geq 0$, such that $n^+ + n^- \geq \nu_\tau,M(\gamma_{\mu})$, and for $\lambda \in \Lambda^0 \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^+ \setminus \{\mu\}$), (1.3) with parameter value $\lambda$ has at least $n^-$ (resp. $n^+$) distinct pairs of nontrivial solutions, $\{v^i_{\lambda}, v^j_{\lambda}\}$, $i = 1, \cdots, n^-$ (resp. $n^+$), such that their restrictions to $[0, \tau]$ converge to zero in $C^1([0, \tau]; \mathbb{R}^{2n})$ as $\lambda \to \mu$.

The first part is an alternative bifurcation result of Rabinowitz’s type, and the second part is of alternative bifurcations of Fadell-Rabinowitz’s type.

Theorem 1.17 has the following:

Corollary 1.9. Let $M \in \text{Sp}(2n, \mathbb{R})$, and let $H_0, \dot{H} : [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R}$ be $C^1$ functions such that all $H_0(t, \cdot), \dot{H}(t, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}$ are $C^2$ and all their partial derivatives depend continuously on $(t, z) \in [0, \tau] \times \mathbb{R}^{2n}$. Let $\bar{u} : [0, \tau] \to \mathbb{R}^{2n}$ satisfy

$$\dot{u}(t) = J\nabla_2 H_0(t, u(t)) \forall t \in [0, \tau] \quad \text{and} \quad u(\tau) = Mu(0).$$

Suppose that $\nabla_2 \dot{H}(\bar{t}, u(\bar{t})) = 0$ for all $t \in [0, \tau]$ and that

either $\nabla_2^2 \dot{H}(t, \bar{u}(t)) > 0 \forall t$ or $\nabla_2^2 \dot{H}(t, \bar{u}(t)) < 0 \forall t$.

Take $H(\lambda, t, z) = H_0(t, z) + \lambda \dot{H}(t, z)$ and $\Lambda = \mathbb{R}$ in (1.3). Then for the fundamental matrix solution $\gamma_\lambda$ of $\dot{Z}(t) = J\nabla^2_2 H(\lambda, t, \bar{u}(t))Z(t)$,

$$\Sigma := \{ \lambda \in \mathbb{R} | \nu_\tau,M(\gamma_\lambda) > 0 \}$$

is a discrete set in $\mathbb{R}$, and $(\mu, \bar{u}) \in \mathbb{R} \times W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ is a bifurcation point for (1.3) with this $H$ if and only if $\nu_\tau,M(\gamma_\mu) > 0$. Moreover, for each $\mu \in \Sigma$ and a small enough $\rho > 0$ it holds that

$$i_{\tau,M}(\gamma_\lambda) = \begin{cases} i_{\tau,M}(\gamma_\mu) & \forall \lambda \in [\mu - \rho, \mu], \\ i_{\tau,M}(\gamma_\mu) + \nu_\tau,M(\gamma_\mu) & \forall \lambda \in (\mu, \mu + \rho) \end{cases} \quad (1.16)$$

if $\nabla_2^2 \dot{H}(t, \bar{u}(t)) > 0 \forall t \in [0, \tau]$, and

$$i_{\tau,M}(\gamma_\lambda) = \begin{cases} i_{\tau,M}(\gamma_\mu) + \nu_\tau,M(\gamma_\mu) & \forall \lambda \in [\mu - \rho, \mu], \\ i_{\tau,M}(\gamma_\mu) & \forall \lambda \in (\mu, \mu + \rho) \end{cases} \quad (1.17)$$

if $\nabla_2^2 \dot{H}(t, \bar{u}(t)) < 0 \forall t \in [0, \tau]$. Consequently, for each $\mu \in \Sigma$, the conclusions in the first part of Theorem 1.17 holds, and the second one of Theorem 1.17 is also true provided that $\bar{u} = 0$ and all $H_0(t, \cdot), \dot{H}(t, \cdot)$ are even.
Suppose that $\hat{u}$ is constant and $H_0 = 0$, Proposition [1.5] gives stronger results than (1.16) and (1.17).

As a consequence of Theorem [1.7] we can also obtain a corresponding result to [51, Theorem 5.6].

**Corollary 1.10.** For a real $\tau > 0$ let $H : [0, \tau] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a $C^1$ function such that each $H(t, \cdot) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $t \in [0, \tau]$, is $C^2$ and all partial derivatives of $H(t, \cdot)$ depend continuously on $(t, z) \in [0, \tau] \times \mathbb{R}^{2n}$. Let $M \in \text{Sp}(2n, \mathbb{R})$, $\hat{u} \in \text{Ker}(M - I_{2n})$ satisfy $\nabla_z H(t, \hat{u}) = 0$ for all $t \in [0, \tau]$, and let $B(t) = \nabla^2_z H(t, \hat{u})$, $\xi_{2n}(t) = I_{2n}$ for $t \in [0, \tau]$, and $\Upsilon_B$ be the fundamental matrix solution of $\dot{Z}(t) = JB(t)Z(t)$ on $[0, \tau]$.

(I) Suppose that $B(t) > 0$ for all $t \in [0, \tau]$ and $i_{\tau, M}(\Upsilon_B) \neq i_{\tau, M}(\xi_{2n}) + \dim \text{Ker}(I_{2n} - M)$. Then there exists at least one and at most finitely many numbers in $(0, \tau)$, $\tau_1 < \cdots < \tau_l$, such that

$$\nu_{\tau_k, M}(\Upsilon_B) := \dim \text{Ker}(\Upsilon_B(\tau_k) - M) \neq 0, \quad k = 1, \ldots, l,$$

and that for each $\tau_k$ one of the following assertions holds:

(1-1) The boundary value problem

$$\dot{u}(t) = J\nabla_z H(\frac{\tau}{\tau}, u(t)) \forall t \in [0, \tau_k] \quad \text{and} \quad u(\tau_k) = Mu(0)$$

has a sequence of solutions converging to $\hat{u}$ in $C^1_\tau([0, \tau_k]; \mathbb{R}^{2n})$, $u_m \neq \hat{u}$, $m = 1, 2, \ldots$.

(1-2) For every $s \neq \tau_k$ near $\tau_k$, the boundary value problem

$$\dot{u}(t) = J\nabla_z H(\frac{\tau}{\tau}, u(t)) \forall t \in [0, s] \quad \text{and} \quad u(s) = Mu(0) \quad (1.18)$$

has a solution $u_s \neq \hat{u}$, which converges to $\hat{u}$ in $C^1_\tau([0, s]; \mathbb{R}^{2n})$ as $s \rightarrow \tau_k$.

(1-3) There is an one-sided neighborhood $\Delta^0$ of $\tau_k$ in $[0, \tau]$ such that for any $s \in \Delta^0 \setminus \{\tau_k\}$, (1.18) has at least two distinct solutions $u^1_s \neq \hat{u}$ and $u^2_s \neq \hat{u}$ near $\hat{u}$, which converges to $\hat{u}$ in $C^1_\tau([0, s]; \mathbb{R}^{2n})$ as $s \rightarrow \tau_k$; besides, $u^1_s$ and $u^2_s$ can also be required to satisfy

$$\int_0^s \left[ \frac{1}{2} (J\dot{u}^1(t), u^1(t))_{\mathbb{R}^{2n}} + H\left(\frac{\tau}{\tau}, u^1(t)\right) \right] dt \neq \int_0^s \left[ \frac{1}{2} (J\dot{u}^2(t), u^2(t))_{\mathbb{R}^{2n}} + H\left(\frac{\tau}{\tau}, u^2(t)\right) \right] dt$$

provided that $\nu_{\tau_k, M}(\Upsilon_B) = 1$ and (1.18) has only finitely many solutions near $\hat{u}$.

Furthermore, if $\hat{u} = 0$ and all $H(t, \cdot)$ are even, then either (1-1) holds or the following occurs:

(1-4) There exist left and right neighborhoods $\Delta^-$ and $\Delta^+$ of $\tau_k$ in $[0, \tau]$ and integers $n^+, n^- \geq 0$, such that $n^+ + n^- \geq \nu_{\tau_k, M}(\Upsilon_B)$, and for $s \in \Delta^- \setminus \{\tau_k\}$ (resp. $s \in \Delta^+ \setminus \{\tau_k\}$), (1.18) has at least $n^-$ (resp. $n^+$) distinct pairs of nontrivial solutions, $\{u^i_k, -u^i_k\}, i = 1, \ldots, n^-$ (resp. $n^+$), which converge to zero in $C^1_\tau([0, s]; \mathbb{R}^{2n})$ as $s \rightarrow \tau_k$.

(II) Suppose that $B(t) < 0$ for all $t \in [0, \tau]$ and that $i_{\tau, M}(\Upsilon_B) \neq i_{\tau, M}(\xi_{2n}) + \dim \text{Ker}(I_{2n} - M)$, where $\Upsilon_B(t) = \Upsilon_B(\tau - t)\Upsilon_B(\tau)^{-1}$ for all $t \in [0, \tau]$. Then there exists at least one and at most finitely many numbers in $(0, \tau)$, $\delta_1 < \cdots < \delta_t$, such that

$$\nu_{\delta_k, M}(\Upsilon_B) := \dim \text{Ker}(\Upsilon_B(\delta_k) - M) \neq 0, \quad k = 1, \ldots, l,$$

and that for each $\delta_k$ one of the following assertions holds:
Remark 1.11. (1) By [19, Remark 4.3], the conditions in Corollary 1.10 that (1.18) with $s = m$, $m = 1, 2, \ldots$, which converges to $\bar{u}$ in $C^1_M([\tau - \delta_k^2/\tau; \tau]; \mathbb{R}^{2n})$.

(II-1) The boundary value problem
\[
\dot{w}(t) = \frac{s}{\delta_k} J \nabla_z H(t, w(t)) \quad \forall t \in [\tau - \delta_k^2/\tau, \tau] \quad \text{and} \quad M w(\tau) = w(\tau - \delta_k^2/\tau)
\]
has a sequence of solutions, $w_m \neq \bar{u}$, $m = 1, 2, \ldots$, which converges to $\bar{u}$ in $C^1_M([\tau - \delta_k^2/\tau, \tau]; \mathbb{R}^{2n})$.

(II-2) For every $\rho \neq \delta_k$ near $\delta_k$, the boundary value problem
\[
\dot{w}(t) = \frac{s}{\rho} J \nabla_z H(t, w(t)) \quad \forall t \in [\tau - \rho^2/\tau, \tau] \quad \text{and} \quad M w(\tau) = w(\tau - \rho^2/\tau) \quad (1.19)
\]
has a solution $w_\rho \neq \bar{u}$, which converges to $\bar{u}$ in $C^1_M([\tau - \rho^2/\tau, \tau]; \mathbb{R}^{2n})$ as $\rho \to \delta_k$.

(II-3) There is an one-sided neighborhood $\Delta^0_j$ of $\delta_k$ in $[0, \tau]$ such that for any $\rho \in \Delta^0_j \setminus \{\tau_k\}$, (1.19) has at least two distinct solutions $w^1_\rho \neq \bar{u}$ and $w^2_\rho \neq \bar{u}$ near $\bar{u}$, which converges to $\bar{u}$ in $C^1_M([\tau - \rho^2/\tau, \tau]; \mathbb{R}^{2n})$ as $\rho \to \tau_k$; besides, $w^1_\rho$ and $w^2_\rho$ can also be required to satisfy
\[
\int_{\tau - \rho^2/\tau}^{\tau} \left[ \frac{1}{2} (J \dot{w}_\rho^1(t), w^1_\rho(t))_{\mathbb{R}^{2n}} + \frac{s}{\rho} H(t, w^1_\rho(t)) \right] dt \\
\neq \int_{\tau - \rho^2/\tau}^{\tau} \left[ \frac{1}{2} (J \dot{w}_\rho^2(t), w^2_\rho(t))_{\mathbb{R}^{2n}} + \frac{s}{\rho} H(t, w^2_\rho(t)) \right] dt
\]
provided that $\nu_{\delta_k, M}(\Upsilon_{\bar{B}}) > 1$ and (1.19) has only finitely many solutions near $\bar{u}$.

Furthermore, if $\bar{u} = 0$ and all $H(t, \cdot)$ are even, then either (II-1) holds or the following occurs:

(II-4) There exist left and right neighborhoods $\Delta^-$ and $\Delta^+$ of $\delta_k$ in $[0, \tau]$ and integers $n^+, n^- \geq 0$, such that $n^+ + n^- \geq \nu_{\delta_k, M}(\Upsilon_{\bar{B}})$, and for $\rho \in \Delta^+ \setminus \{\delta_k\}$ (resp. $\rho \in \Delta^- \setminus \{\delta_k\}$), (1.19) has at least $n^+$ (resp. $n^-$) distinct pairs of nontrivial solutions, $\{w^1_\rho, w^2_\rho\}$, $i = 1, \ldots, n^-$ (resp. $n^+$), which converge to zero in $C^1_M([\tau - \rho^2/\tau, \tau]; \mathbb{R}^{2n})$ as $\rho \to \delta_k$.

Clearly, (I) of Corollary 1.10 always implies that there exists a sequence $(s_m) \subset (0, \tau)$ such that (1.15) with $s = s_m$ has a solution $u_m \neq \bar{u}$ for each $m$, and that $\|u_m - \bar{u}\|_{C^1([0, s_m]; \mathbb{R}^{2n})} \to 0$ as $m \to \infty$. By (II) of Corollary 1.10 we have also a similar conclusion.

Remark 1.11. (1) By [19, Remark 4.3], the conditions in Corollary 1.10
\[ "i_{\tau, M}(\Upsilon_B) \neq i_{\tau, M}(\xi_{2n}) + \dim \ker(I_{2n} - M)" \quad \text{and} \quad "i_{\tau, M}(\Upsilon_B) \neq i_{\tau, M}(\xi_{2n}) + \dim \ker(I_{2n} - M)"
\]
respectively become
\[ "i_{\tau}(\Upsilon_B) \neq n" \quad \text{and} \quad "i_{\tau}(\Upsilon_B) \neq n"
\] if $M = I_{2n}$, and
\[ "i_{\tau, M}(\Upsilon_B) \neq \kappa" \quad \text{and} \quad "i_{\tau, M}(\Upsilon_B) \neq \kappa"
\] if $M = \text{diag}\{-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa\}$. 

Guangcun Lu
(2) If $H$ in Corollary 1.10 is independent of time, then $B(t) \equiv \nabla_z^2 H(\bar{u})$, $\Upsilon_B(t) = \exp(tJ\nabla_z^2 H(\bar{u}))$ and $\Upsilon_B(t) = \Upsilon_B(\tau - t)\Upsilon_B^{-1} = \exp(-tJ\nabla_z^2 H(\bar{u}))$ for all $t \in [0, \tau]$.

Corollary 1.9 implies

**Corollary 1.12.** For $M \in \text{Sp}(2n, \mathbb{R})$, let $C^1$ functions $H_0, \dot{H} : \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ satisfy:

(a) $H_0(t + \tau, Mx) = H_0(t, x)$ and $\dot{H}(t + \tau, Mx) = \dot{H}(t, x)$ for all $(t, x) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n}$.

(b) all $H_0(t, \cdot), \dot{H}(t, \cdot) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ are $C^2$ and all their partial derivatives depend continuously on $(t, z) \in \mathbb{R} \times \mathbb{R}^{2n}$.

Let $\bar{v} : \mathbb{R} \rightarrow \mathbb{R}^{2n}$ satisfy

(c) $\dot{v}(t) = J\nabla_z H_0(t, v(t))$ and $v(t + \tau) = Mv(t) \forall t$.

(d) $\nabla_z \dot{H}(t, \bar{v}(t)) = 0$ for all $t \in \mathbb{R}$, and either $\nabla_z^2 \dot{H}(t, \bar{v}(t)) > 0 \forall t$ or $\nabla_z^2 \dot{H}(t, \bar{v}(t)) < 0 \forall t$.

Take $H(\lambda, t, z) = H_0(t, z) + \lambda \dot{H}(t, z)$ in (1.4). Then

(A) $\Sigma := \{ \lambda \in \mathbb{R} | \nu_{\tau, M}(\gamma_{\lambda}) > 0 \}$ is a discrete set in $\mathbb{R}$.

(B) $(\mu, \bar{v})$ with $\mu \in \mathbb{R}$ is a bifurcation point for (1.3) if and only if $\nu_{\tau, M}(\gamma_{\mu}) > 0$.

(C) For each $\mu \in \Sigma$ and a small enough $\rho > 0$, (1.10) and (1.17) hold, and therefore the conclusions in the first part of Theorem 1.8 holds, and the second one of Theorem 1.8 is also true provided that $\bar{v} = 0$ and all $H_0(t, \cdot), \dot{H}(t, \cdot)$ are even.

### 1.2. Bifurcations for generalized periodic solutions of autonomous Hamiltonian systems

**Assumption 1.13.** Let $M \in \text{Sp}(2n, \mathbb{R})$ be an orthogonal symplectic matrix and $\Lambda$ be a topological space. Moreover, let $H : \Lambda \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a continuous function such that each $H_\lambda : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $\lambda \in \Lambda$, is $M$-invariant and $C^2$, and all its partial derivatives depend continuously on the parameter $\lambda \in \Lambda$. For some fixed real $\tau > 0$ and each $\lambda \in \Lambda$ let $v_\lambda : \mathbb{R} \rightarrow \mathbb{R}^{2n}$ satisfy the following generalized periodic Hamiltonian system

$$
\dot{v}(t) = J\nabla_z H(\lambda, v(t)) \quad \text{and} \quad v(t + \tau) = Mv(t) \forall t \in \mathbb{R},
$$

and $\Lambda \times \mathbb{R} \ni (\lambda, t) \mapsto v_\lambda(t) \in \mathbb{R}^{2n}$ is also continuous.

Under this assumption, each $v_\lambda$ is $C^2$, and $\Lambda \times \mathbb{R} \ni (\lambda, t) \mapsto \mathring{v}_\lambda(t) \in \mathbb{R}^{2n}$ and $\Lambda \times \mathbb{R} \ni (\lambda, t) \mapsto \ddot{v}_\lambda(t) \in \mathbb{R}^{2n}$ are also continuous. Moreover, each element in $\mathbb{R} : v_\lambda := \{ v_{\lambda}(\theta + \cdot) | \theta \in \mathbb{R} \}$ (R-orbit) also satisfies (1.20). It follows that

$$
\ddot{v}_\lambda(t) = J\nabla_z^2 H(\lambda, v_\lambda)\dot{v}_\lambda(t) \quad \text{and} \quad \dot{v}_\lambda(t + \tau) = M\dot{v}_\lambda(t) \forall t.
$$

Thus if $v_\mu$ is nonconstant (i.e., $\mathring{v}_\mu \neq 0$) for some $\mu \in \Lambda$, then $(\mu, v_\mu)$ is a bifurcation point of (1.20) in the sense above (1.10). In order to give an exact description for bifurcation pictures of solutions of (1.20) near $\mathbb{R} : v_\lambda$ some concepts are needed. Two solutions $v_1$ and $v_2$ of (1.20) with parameter value $\lambda$ is said to be $\mathbb{R}$-distinct if they belong to different $\mathbb{R}$-orbits. We call $(\mu, v_\mu)$ a bifurcation $\mathbb{R}$-orbit along sequences of (1.20) with respect to the branch $\{ (\lambda, \mathbb{R} : v_\lambda) | \lambda \in \Lambda \}$ if there exists a sequence $(\lambda_k) \subset \Lambda$ converging to $\mu$, and a solution $\mathring{v}_k$ of (1.20) with parameter value $\lambda_k$ for each $k$, such that: (i) $\bar{v}_k \notin \mathbb{R} : v_{\lambda_k} \forall k$, (ii) all $\bar{v}_k$ are $\mathbb{R}$-distinct, (iii) $\bar{v}_k \rightarrow v_\mu$ in $C^1_{\text{loc}}$.

When each $v_\lambda$ is constant, we have the following Theorem 1.12 and Corollaries 1.15 1.16
In particular, then (ii) may be replaced by the following alternatives:

\[ R \to H \]

Corollary 1.15. The following problem

Theorem 1.14 holds for (1.20) with 

\[ \lambda \to \nu \]

By the proof of Corollary 1.9 we may derive from Theorem 1.14:

(iii) For every \( \lambda \in \Lambda \setminus \{ \mu \} \) near \( \mu \) there is a solution \( \bar{v}_\lambda \notin \mathbb{R} \cdot v_\lambda \) of (1.20) with parameter value \( \lambda \), such that \( \bar{v}_\lambda - v_\lambda \) converges to zero on any compact interval \( I \subset \mathbb{R} \) in \( C^1 \)-topology as \( \lambda \to \mu \).

(iv) For a given \( \varepsilon > 0 \) there is an one-sided neighborhood \( \Lambda^0 \) of \( \mu \) in \( \Lambda \) such that for any \( \lambda \in \Lambda^0 \setminus \{ \mu \} \), (1.20) with parameter value \( \lambda \) has either infinitely many \( \mathbb{R} \)-distinct solutions \( \bar{v}_\lambda^k \notin \mathbb{R} \cdot v_\lambda \) such that \( \| \bar{v}_\lambda^k \|_{[0, \tau]} - v_\lambda \|_{C^1} < \varepsilon \), \( k = 1, 2, \ldots \), or at least two \( \mathbb{R} \)-distinct solutions \( \bar{v}_\lambda^1 \notin \mathbb{R} \cdot v_\lambda \) and \( \bar{v}_\lambda^2 \notin \mathbb{R} \cdot v_\lambda \) such that \( \| \bar{v}_\lambda^i \|_{[0, \tau]} - v_\lambda \|_{C^1} < \varepsilon \), \( i = 1, 2 \), and that

\[ \int_0^{\tau} \left[ \frac{1}{2} (J\dot{\bar{v}}_\lambda^1(t), \bar{v}_\lambda^1(t))_{\mathbb{R}^{2n}} + H(\lambda, \bar{v}_\lambda^1(t)) \right] dt \neq \int_0^{\tau} \left[ \frac{1}{2} (J\dot{\bar{v}}_\lambda^2(t), \bar{v}_\lambda^2(t))_{\mathbb{R}^{2n}} + H(\lambda, \bar{v}_\lambda^2(t)) \right] dt. \]

The first part is an alternative bifurcation result of Fadell-Rabinowitz’s type, and the second part is that of Rabinowitz’s type.

By the proof of Corollary 1.9 we may derive from Theorem 1.14:

Corollary 1.15. Let \( M \in \text{Sp}(2n, \mathbb{R}) \) be as in Theorem 1.14, \( \bar{v} \in \text{Ker}(M - I_{2n}) \), and \( H_0, \tilde{H} : \mathbb{R}^{2n} \to \mathbb{R} \) be \( M \)-invariant \( C^2 \)-functions. Suppose that \( dH_0(\bar{v}) = 0 \) and that either \( \tilde{H}''(\bar{v}) > 0 \) or \( \tilde{H}''(\bar{v}) < 0 \). Then for \( \gamma_\lambda(t) = \exp(tJH_0''(\bar{v}) + \lambda tH''(\bar{v})) \) and any \( \tau > 0 \), \( \Sigma_\tau := \{ \lambda \in \mathbb{R} | |\nu_\tau(M(\gamma_\lambda)) > 0 \} \) is a discrete set in \( \mathbb{R} \); and for each \( \mu \in \Sigma_\tau \) such that (1.23) with \( H(\mu, x) = H_0(x) + \mu \tilde{H}(x) \) and \( v_\mu = \bar{v} \) has no nonzero constant solutions, the conclusions of Theorem 1.14 holds for (1.20) with \( H(\lambda, x) = H_0(x) + \lambda \tilde{H}(x) \) and \( v_\lambda = \bar{v} \forall \lambda \).
In particular, taking $\tau = 1$, $H_0 = 0$ and $H = \dot{H}$ we obtain the following existence result of $M$-rotating periodic orbits of $\dot{v} = J\nabla H(v)$ near a $M$-equilibrium $\bar{v}$.

**Corollary 1.16.** Let $M \in \text{Sp}(2n, \mathbb{R})$ and $\bar{v} \in \text{Ker}(M - I_{2n})$ be as in Corollary 1.15, and let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a $M$-invariant $C^2$-function satisfying $dH(\bar{v}) = 0$. Suppose that either $H''(\bar{v}) > 0$ or $H''(\bar{v}) < 0$. Then

(A) $\Gamma(H, \bar{v}, M) := \{ \lambda \in \mathbb{R} \setminus \{0\} \mid \nu_{1,M}(\gamma_{\lambda}) = \dim \text{Ker}(\exp(\lambda JH''(\bar{v})) - M) > 0\}$ is a discrete set.

(B) If $H''(\bar{v}) > 0$ and $\Gamma(H, \bar{v}, M) \cap (0, \infty) \neq \emptyset$ (resp. $H''(\bar{v}) < 0$ and $\Gamma(H, \bar{v}, M) \cap (-\infty, 0) \neq \emptyset$), then for $\mu \in \Gamma(H, \bar{v}, M) \cap (0, \infty)$ (resp. $\Gamma(H, \bar{v}, M) \cap (-\infty, 0)$), one of the following alternatives occurs:

(B.i) The problem

$$\dot{v}(t) = J\nabla H(v(t)) \quad \text{and} \quad v(t + \mu) = Mv(t), \quad \forall t \in \mathbb{R} \quad (1.23)$$

has a sequence of $\mathbb{R}$-distinct solutions, $v_k, k = 1, 2, \cdots$, such that each $v_k$ is $\mathbb{R}$-distinct from $\bar{v}$, and that $(v_k)$ converges to $\bar{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology.

(B.ii) There exist left and right neighborhoods $\Lambda^-$ and $\Lambda^+$ of $\mu$ in $\mathbb{R} \setminus \{0\}$ and integers $n^+, n^- \geq 0$, such that $n^+ + n^- \geq \frac{1}{2} \dim \text{Ker}(\exp(\mu H''(\bar{v})) - M)$, and for $\lambda \in \Lambda^\pm \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^- \setminus \{\mu\}$) the problem

$$\dot{v}(t) = J\nabla H(v(t)) \quad \text{and} \quad v(t + \lambda) = Mv(t), \quad \forall t \in \mathbb{R} \quad (1.24)$$

has at least $n^-$ (resp. $n^+$) $\mathbb{R}$-distinct solutions solutions, $v^\lambda, i = 1, \ldots, n^-$ (resp. $n^+$), such that each of them is $\mathbb{R}$-distinct from $\bar{v}$ and converges to $\bar{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology as $\lambda \to \mu$.

Moreover, if $\dim \text{Ker}(\exp(\mu H''(\bar{v}))-M) \geq 3$, then either (B.i) holds or one of the following alternatives occurs:

(B.iii) For every $\lambda \in \mathbb{R} \setminus \{\mu\}$ near $\mu$, the problem $(1.24)$ has a solution $v_\lambda$, which is $\mathbb{R}$-distinct from $\bar{v}$ and converges to $\bar{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology as $\lambda \to \mu$.

(B.iv) For a given $\varepsilon > 0$ there is an one-sided neighborhood $\Lambda^0$ of $\mu$ in $\mathbb{R} \setminus \{0\}$ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, the problem $(1.24)$ with parameter value $\lambda$ has either infinitely many $\mathbb{R}$-distinct solutions $v^\lambda_k$ such that each of them is $\mathbb{R}$-distinct from $\bar{v}$ and $\|v^\lambda_k\|_{[0,|\lambda|]} - \|\bar{v}\|_{C^1} < \varepsilon$, $k = 1, 2, \cdots$, or at least two $\mathbb{R}$-distinct solutions $v^\lambda_1$ and $v^\lambda_2$ such that: a) each of both is $\mathbb{R}$-distinct from $\bar{v}$, b) $\|v^\lambda_1\|_{[0,|\lambda|]} - \|\bar{v}\|_{C^1} < \varepsilon$, $i = 1, 2, c)$

$$\int_0^T \left[ \frac{1}{2} (J\dot{v}^\lambda_1(t), \dot{v}^\lambda_1(t))_{\mathbb{R}^{2n}} + H(\lambda, \dot{v}^\lambda_1(t)) \right] dt \neq \int_0^T \left[ \frac{1}{2} (J\dot{v}^\lambda_2(t), \dot{v}^\lambda_2(t))_{\mathbb{R}^{2n}} + H(\lambda, \dot{v}^\lambda_2(t)) \right] dt.$$

**Remark 1.17.** Suppose $H''(\bar{v}) > 0$. By Williamson theorem (cf. [29], page 41) there exists a symplectic matrix $S \in \text{Sp}(2n, \mathbb{R})$ such that $S^T H''(\bar{v}) S = D_{H''(\bar{v})} = \text{diag}(q_1, \cdots, q_n, \tilde{q}_1, \cdots, \tilde{q}_n)$, where $0 < q_j \leq \tilde{q}_j$ for $j \leq k$, are the symplectic eigenvalues of $H''(\bar{v})$, namely $(\pm \sqrt{-1} \Omega_j)$ for $1 \leq j \leq n$ are all eigenvalues of $JH''(\bar{v})$. Therefore we can reduce the study of the Hamiltonian system $\dot{v} = J\nabla H(v)$ with $M$-rotating periods near the $M$-equilibrium $\bar{v}$ to that of the Hamiltonian system $\dot{v} = J\nabla G(v)$ with $S^{-1} MS$-rotating periods near $\bar{w} = S^{-1} \bar{v}$, where $G(z) = H(Sz)$ and so $G(\bar{w}) = S^T H'(S\bar{w}) = 0$ and $G''(\bar{w}) = S^T H''(\bar{w}) S = D_{H''(\bar{v})}$. Since $S$ is symplectic,
\( JG''(\bar{w}) = S^{-1}JH''(\bar{v})S \) and \( \exp(\lambda JH''(\bar{v})) = S \exp(\lambda JD_{H''(\bar{v})}S^{-1} \) for any \( \lambda \in \mathbb{R} \). The former implies that both \( G''(\bar{w}) \) and \( H''(\bar{v}) \) have the same symplectic eigenvalues. The latter leads to
\[
\Gamma(H, \bar{v}, M) \cap (0, \infty) = \{ \lambda > 0 \mid \dim \ker(\exp(\lambda JH''(\bar{v}))) - M > 0 \} = \Gamma(G, \bar{w}, S^{-1}MS) \cap (0, \infty).
\]

In particular, from this and (1.38) we derive
\[
\Gamma(H, \bar{v}, I_{2n}) \cap (0, \infty) = \bigcup_{j=1}^{n} \left( \frac{2\pi}{\theta_j} \mathbb{N} \right).
\]

because \( \dim \ker(\exp(\lambda JH''(\bar{v})) - I_{2n}) \) is equal to
\[
\dim \ker \left( \begin{pmatrix}
\text{diag}(\cos(\theta_1 \lambda), \ldots, \cos(\theta_n \lambda)) & \text{diag}(\sin(\theta_1 \lambda), \ldots, \sin(\theta_n \lambda)) \\
\text{diag}(\sin(\theta_1 \lambda), \ldots, \sin(\theta_n \lambda)) & \text{diag}(\cos(\theta_1 \lambda), \ldots, \cos(\theta_n \lambda))
\end{pmatrix} - I_{2n}\right) = \sum_{j=1}^{n} \dim \ker \left( \begin{pmatrix}
\cos(\theta_j \lambda) & -\sin(\theta_j \lambda) \\
\sin(\theta_j \lambda) & \cos(\theta_j \lambda)
\end{pmatrix} - I_{2} \right). \tag{1.25}
\]

This is exactly the set of periods of all period solutions of \( \dot{v} = JH''(\bar{v})v \).

Let us compare previous results with Corollary 1.16. Very recently, J. Xing, X. Yang and Y. Li [70, Corollary 1.2] gave a Lyapunov type theorem on \( M \)-rotating periodic orbits for an orthogonal \( M \in \text{Sp}(2n, \mathbb{R}) \), which can be stated as follows in our notations:

Let \( \bar{v} \in \ker(M - I_{2n}) \), and let \( H : \mathbb{R}^{2n} \to \mathbb{R} \) be a \( M \)-invariant \( C^2 \)-function satisfying \( H' (\bar{v}) = 0 \) and \( H'' (\bar{v}) > 0 \). Suppose that the symplectic eigenvalues of \( H'' (\bar{v}) \) as above, \( \theta_1, \ldots, \theta_n \), are nonresonant, i.e., \( \theta_i/\theta_j \notin \mathbb{Q} \) for \( 1 \leq i, j \leq n, i \neq j \). Then for each sufficiently small \( \varepsilon > 0 \), the energy surface \( H(v) = H(\bar{v}) + \varepsilon^2 \) contains at least \( n \) \( M \)-rotating periodic orbits of \( \dot{v} = JH''(\bar{v})v \) with periods close to ones of the linear system \( \dot{v} = JH''(\bar{v})v \).

Note that the set of \( M \)-rotating periods of the linear system \( \dot{v} = JH''(\bar{v})v \) is exactly the discrete set \( \Gamma(H, \bar{v}, M) \) in Corollary 1.16(A). This theorem and Corollary 1.16 belong to two categories of fixed energy and fixed period problems about \( M \)-rotating periodic orbits, respectively.

Previous other related results require \( M = I_{2n} \). To compare with them we may assume \( \bar{v} = 0 \) and \( H''(0) = \text{diag}(\theta_1, \ldots, \theta_n, \theta_1, \ldots, \theta_n) \) by the above arguments. Then for \( T > 0 \), \( \dot{v} = JH''(0)v \) has a nontrivial \( T \)-periodic solution if and only if \( T \) belongs to \( \bigcup_{j=1}^{n} \left( \frac{2\pi}{\theta_j} \mathbb{N} \right) \). For \( 1 \leq i \leq n \) let
\[
\mathbb{R}^{2}_i = \{(q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n} \mid q_j = p_j = 0 \ \forall j \neq i \}.
\]

It is an invariant subspace of \( JH''(0) \). For a given \( T \in \bigcup_{j=1}^{n} \left( \frac{2\pi}{\theta_j} \mathbb{N} \right) \), \( E = \bigoplus_{T \theta_j \in 2\pi \mathbb{N} \mathbb{R}^{2}_j} \) and \( F = \bigoplus_{T \theta_j \notin 2\pi \mathbb{N} \mathbb{R}^{2}_j} \) are invariant subspaces of \( JH''(0) \), and \( \mathbb{R}^{2n} = E \oplus F \). It is easy to see:

- \( \dim E = 2n \{ j \in \{1, \ldots, n\} \mid T \theta_j \in 2\pi \mathbb{N} \} = \dim \ker(\exp(TJH''(0)) - I_{2n}) \) by (1.25).

- Each solution of \( \dot{v} = JH''(0)v \) in \( E \) is \( T \)-periodic, and no solutions of \( \dot{v} = JH''(0)v \) with initial data in \( F \setminus \{0\} \) is \( T \)-periodic.

Since \( H''(0) > 0 \), \( \dot{v} = JH''(0)v \) has no constant solutions in \( E \setminus \{0\} \) and the quadratic form \( E \ni z \mapsto (H''(0)z, z) \) has the signature \( \dim E \). From [21, Theorem 8.4] (or [23, Theorem]) we can only derive:
either (B.i) with \((\bar{v}, \mu, M) = (0, T, I_{2n})\) holds, or (B.ii) with \((\bar{v}, \mu, M) = (0, T, I_{2n})\) is true. But when \(2\pi j \in \{1, \cdots, n\}\) \([T\bar{g}_j \in 2\pi N]\) \(\geq 3\), Corollary 1.16 can also give: either (B.i) with \((\bar{v}, \mu, M) = (0, T, I_{2n})\) holds, or (B.iii) with \((\bar{v}, \mu, M) = (0, T, I_{2n})\) or (B.iv) with \((\bar{v}, \mu, M) = (0, T, I_{2n})\) is true. Moreover, 68 Corollaries 4.2.4.3 can only lead to: for each \(1 \leq j \leq n\) there exists a sequence \((v_k)\) of nonconstant periodic solutions of \(\dot{v} = J\nabla H(v)\) with (not necessarily minimal) period \(T_k\) such that \(\|v_k\|_{L^\infty} \to 0\) and \(T_k \to 2\pi/\bar{g}_j\).

Assume \(\{\theta_1, \cdots, \theta_n\} = \{\beta_1, \cdots, \beta_p\}\) with \(\beta_1 < \cdots < \beta_p\). For \(\beta \in \{\beta_1, \cdots, \beta_p\}\) let \(H_\beta\) be the sum of the generalized eigenspaces of \(JH''(0)\) in \(\mathbb{C}^{2n}\) associated to the eigenvalues of the form \(\pm ik\beta\), \(k \in \mathbb{N}\), i.e., \(H_\beta = \sum_{\lambda \in \Gamma_\beta} \cup_{n=1}^\infty \text{Ker}(\lambda I_{2n} - JH''(0))\), where \(\Gamma_\beta = \{ik\beta\mid k \in \mathbb{Z} \setminus \{0\}\} \cap \{-i\beta_1, \cdots, -i\beta_p, i\beta_1, \cdots, i\beta_p\}\). Put

\[\Xi_\beta = \{\xi, \eta \in \mathbb{R}^{2n} \mid \xi + i\eta \in H_\beta\}.

They are invariant subspaces of \(JH''(0)\) in \(\mathbb{C}^{2n}\) and \(\mathbb{R}^{2n}\), respectively. \(\dim \Xi_\beta\) is even and \(H''(0)|_{\Xi_\beta} > 0\) implies: a) each solution of \(\dot{v} = JH''(0)v\) with initial value in \(\Xi_\beta\) has (not necessary minimal) period \(2\pi/\beta\) and no solutions of \(\dot{v} = JH''(0)v\) with initial data in \(\mathbb{R}^{2n} \setminus \Xi_\beta\) has the period \(2\pi/\beta\); b) \(\dot{v} = JH''(0)v\) has \(\dim \Xi_\beta/2\) linearly independent periodic solutions with period \(2\pi/\beta\) as claimed below (H2) on the page 128 of [5]. The latter sentence shows \(\dim \Xi_\beta = \dim \text{Ker}(\exp(TJH''(0)) - I_{2n})\) with period \(T = 2\pi/\beta\). Therefore using [5] Theorem 9.5 we can only obtain:

For each \(\beta \in \{\beta_1, \cdots, \beta_p\}\) and \(T = 2\pi/\beta\), either (B.i) with \((\bar{v}, \mu, M) = (0, T, I_{2n})\) holds, or (B.ii) with \((\bar{v}, \mu, M) = (0, T, I_{2n})\) is true.

[5] Theorem 9.5] gave no conclusions for \(T \in \bigcup_{j=1}^n \left(\frac{2\pi}{\bar{g}_j} \mathbb{N}\right) \setminus \{2\pi/\beta_k \mid k = 1, \cdots, p\}\).

In summary, Corollary 1.16 with \(M = I_{2n}\) cannot be included by the previous results.

Most of the general works on bifurcations of Hamiltonian systems have very little on the case where all \(v_\lambda\) in Assumption 1.13 are equal to a nonequilibrium (i.e., nonconstant) solution \(\bar{v}\). Using the iso-energetic Poincaré mapping some results were obtained, see [59, 26]. Here are our three bifurcation results of \(M\)-rotating periodic solutions from \(\bar{v}\).

**Theorem 1.18 (Necessary condition).** Let \(\Lambda, M\) and \(H\) be as in Assumption 1.13 and let \(\bar{v} : \mathbb{R} \to \mathbb{R}^{2n}\) be a nonconstant solution of (1.20) for each \(\lambda \in \Lambda\). Suppose that for some \(\mu \in \Lambda\) there exists a sequence \((\lambda_k) \subset \Lambda\) converging to \(\mu\) such that (1.20) with \(\lambda = \lambda_k\) has solutions \(v_k\), \(k = 1, 2, \cdots\), which are \(\mathbb{R}\)-distinct each other and satisfies \(v_k|_{[0, \tau]} \to \bar{v}|_{[0, \tau]}\) in \(C^0([0, \tau]; \mathbb{R}^{2n})\). Then the following problem

\[\dot{v}(t) = J\nabla^2_H(\mu, \bar{v}(t))v(t) \quad \text{and} \quad v(t + \tau) = Mv(t) \forall t \quad (1.26)\]

has at least two linearly independent nontrivial solutions, i.e., \(\dim \text{Ker}(\gamma_\mu(\tau) - M) \geq 2\), where \(\gamma_\lambda(t)\) is the fundamental matrix solution of \(\dot{v}(t) = J\nabla^2_H(\lambda, \bar{v}(t))v(t)\).

Under the assumptions of Theorem 1.18 since \(\dot{v}(0) \neq 0\) and \(\gamma_\lambda(\tau)\dot{v}(0) = \dot{v}(\tau) = M\dot{v}(0)\) we get \(\dim \text{Ker}(\gamma_\lambda(\tau) - M) \geq 1\). (Though 1 and −1 as eigenvalues of a symplectic matrix \(S \in \text{Sp}(2n; \mathbb{R})\) always have even algebraic multiplicity it is possible that the geometric multiplicity of 1 (or −1) is equal to one.) When \(\dim \text{Ker}(\gamma_\lambda(\tau) - M) = 1\) we call \(\bar{v}\) a transversally nondegenerate \(M\)-rotating \(\tau\)-periodic solution of (1.20). Therefore the conclusion of Theorem 1.18 says \(\bar{v}\) to be a transversally degenerate \(M\)-rotating \(\tau\)-periodic solution of \(\dot{v}(t) = J\nabla^2_H(\mu, \bar{v}(t))\).

Clearly, Theorem 1.18 is a strengthen version of Theorem 1.15 (a). Corresponding with (b) and (c) of Theorem 1.3 and Corollary 1.12 respectively, we have also:
Theorem 1.19 (Sufficient condition). Let $\Lambda, M, H, \bar{v}$ and $\gamma_\lambda$ be as in Theorem 1.18 and let $\Lambda$ be first countable. Suppose:

(a) The orbit $O := \mathbb{R} \cdot \bar{v}$ is an embedded circle (i.e., $\mathbb{R}_\bar{v}$ is an infinite cyclic subgroup of $\mathbb{R}$ with generator $p > 0$).

(b) For some $\mu \in \Lambda$, $\dim \text{Ker}(\gamma_\mu(\tau) - M) \geq 2$ and there exist two sequences in $\Lambda$ converging to $\mu$, $(\lambda_k^-)$ and $(\lambda_k^+)$, such that for each $k \in \mathbb{N}$,

$$[i_{\tau,M}(\gamma_{\lambda_k^-}), i_{\tau,M}(\gamma_{\lambda_k^+}) + \nu_{\tau,M}(\gamma_{\lambda_k^+}) - 1] \cap [i_{\tau,M}(\gamma_{\lambda_k^-}), i_{\tau,M}(\gamma_{\lambda_k^+}) + \nu_{\tau,M}(\gamma_{\lambda_k^+}) - 1] = \emptyset,$$

and either $\nu_{\tau,M}(\gamma_{\lambda_k^+}) = 1$ or $\nu_{\tau,M}(\gamma_{\lambda_k^-}) = 1$.

(c) For any solution $v$ of (1.20) with $\lambda = \mu$, if there exists a sequence $(s_k)$ of reals such that $s_k \cdot v$ converges to $\bar{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^0$-topology (and so $C^1$-topology by Proposition 1.3), then $v$ is periodic. (Clearly, this holds if $M^l = I_{2n}$ for some $l \in \mathbb{N}$.)

Then there exists a sequence $(\lambda_k) \subset \hat{\Lambda} := \{\mu, \lambda_k^+, \lambda_k^- | k \in \mathbb{N}\}$ converging to $\mu$ and solutions $v_k$ of (1.20) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$, such that any two of these $v_k$ are $\mathbb{R}$-distinct and that $(v_k)$ converges to $\bar{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology as $k \to \infty$.

Theorem 1.20 (Existence for bifurcations). Let $\Lambda, M, H, \bar{v}$ and $\gamma_\lambda$ be as in Theorem 1.18. Suppose that $\Lambda$ is path-connected, the conditions (a) and (c) in Theorem 1.19 and the following are satisfied:

(d) There exist two points $\lambda^+, \lambda^- \in \Lambda$ such that

$$[i_{\tau,M}(\gamma_{\lambda^+}), i_{\tau,M}(\gamma_{\lambda^+}) + \nu_{\tau,M}(\gamma_{\lambda^+}) - 1] \cap [i_{\tau,M}(\gamma_{\lambda^-}), i_{\tau,M}(\gamma_{\lambda^-}) + \nu_{\tau,M}(\gamma_{\lambda^-}) - 1] = \emptyset,$$

and either $\nu_{\tau,M}(\gamma_{\lambda^+}) = 1$ or $\nu_{\tau,M}(\gamma_{\lambda^-}) = 1$.

Then for any path $\alpha : [0, 1] \to \Lambda$ connecting $\lambda^+$ to $\lambda^-$ there exists a sequence $(t_k) \subset [0, 1]$ converging to some $t$ and solutions $v_k \neq \bar{v}$ of (1.20) with $\lambda = \alpha(t_k)$, $k = 1, 2, \cdots$, such that any two of these $v_k$ are $\mathbb{R}$-distinct and that $(v_k)$ converges to $\bar{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology as $k \to \infty$. Moreover, $\alpha(t)$ is not equal to $\lambda^+$ (resp. $\lambda^-$) if $\nu_{\tau,M}(\gamma_{\lambda^+}) = 1$ (resp. $\nu_{\tau,M}(\gamma_{\lambda^-}) = 1$).

Theorem 1.21 (Alternative bifurcations of Rabinowitz’s type and of Fadell-Rabinowitz’s type). Let $\tau > 0$ and $\Lambda$ be a real interval, and let $H : \Lambda \times \mathbb{R}^{2n} \to \mathbb{R}$ be a continuous function such that each $H_\lambda : \mathbb{R}^{2n} \to \mathbb{R}$, $\lambda \in \Lambda$, is $C^2$ and all its partial derivatives depend continuously on the parameter $\lambda \in \Lambda$. Let $\bar{v} : \mathbb{R} \to \mathbb{R}^{2n}$ be a nonconstant $\tau$-periodic solution of $\ddot{v}(t) = J\nabla \mathcal{H}(\lambda, v(t))$ for each $\lambda \in \Lambda$, (therefore have the minimal period $\tau/p$ with $p \in \mathbb{N}$), and let $\gamma_\lambda(t)$ be the fundamental matrix solution of $\dot{v}(t) = JH_\lambda(\bar{v}(t))v(t)$. Suppose for some interior point $\mu$ of $\Lambda$ that $\nu_{\tau}(\gamma_\mu) := \dim \text{Ker}(\gamma_\mu(\tau) - I_{2n}) > 1$, $\nu_{\tau}(\gamma_\lambda) := \dim \text{Ker}(\gamma_\lambda(\tau) - I_{2n}) = 1$ for each $\lambda \in \Lambda \setminus \{\mu\}$ near $\mu$, and that $i_{\tau}(\gamma_\mu)$ takes values $i_{\tau}(\gamma_\mu)$ and $i_{\tau}(\gamma_\mu) + \nu_{\tau}(\gamma_\mu) - 1$ as $\lambda \in \Lambda$ varies in two deleted half neighborhoods of $\mu$. Then

(1) The conclusions of Theorem 1.19 (after $\hat{\Lambda}$ is replaced by $\Lambda$) hold.

(2) If $\nu_{\tau}(\gamma_\mu) > 1$, one of the following alternatives occurs:

(II.1) Equation (1.20) with $\lambda = \mu$ has a sequence of $\mathbb{R}$-distinct solutions, $v_k$, $k = 1, 2, \cdots$, such that $(v_k)$ converges to $\bar{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology as $k \to \infty$. 

18

Guangcun Lu
(II.2) For every $\lambda \in \Lambda \setminus \{\mu\}$ near $\mu$ there is a solution $v_{\lambda}$ of (1.27) with parameter value $\lambda$, which is $\mathbb{R}$-distinct with $\tilde{v}$ and converges to $\tilde{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology as $\lambda \to \mu$.

(II.3) There is an one-sided neighborhood $\Lambda^0$ of $\mu$ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, (1.27) with parameter value $\lambda$ has either infinitely many $\mathbb{R}$-distinct solutions $\tilde{v}_k \notin \mathbb{R} \cdot \tilde{v}$, $k = 1, 2, \ldots$, or at least two $\mathbb{R}$-distinct solutions, $\tilde{v}_1 \notin \mathbb{R} \cdot \tilde{v}$ and $\tilde{v}_2 \notin \mathbb{R} \cdot \tilde{v}$, such that

\[
\int_0^T \left[ \frac{1}{2} (J\dot{v}_1^\lambda(t), v_1(t))_{\mathbb{R}^{2n}} + H(\lambda, v_1(t)) \right] dt \neq \int_0^T \left[ \frac{1}{2} (J\dot{v}_2^\lambda(t), v_2(t))_{\mathbb{R}^{2n}} + H(\lambda, v_2(t)) \right] dt;
\]

and these $\tilde{v}_k^\lambda$, $\tilde{v}_1^\lambda$ and $\tilde{v}_2^\lambda$ converge to $\tilde{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology as $\lambda \to \mu$.

(III) If $\nu_{r/p}(\gamma_\mu) = 1$, (which implies $p \geq 2$ because $\nu_{r}(\gamma_\mu) > 1$), and $p$ is equal to 2 (resp. a prime greater than 2), then one of (II.1) and the following (III.1) [resp. (III.2)] occurs:

(III.1) There exist left and right neighborhoods $\Lambda^-$ and $\Lambda^+$ of $\mu$ in $\Lambda$ and integers $n^+, n^- \geq 0$, such that $n^+ + n^- \geq \nu_r(\gamma_\mu)$, and for $\lambda \in \Lambda^- \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^+ \setminus \{\mu\}$), (1.27) with parameter value $\lambda$ has at least $n^-$ (resp. $n^+$) $\mathbb{R}$-distinct solutions, $v_i^\lambda \notin \mathbb{R} \cdot \tilde{v}$, $i = 1, \ldots, n^-$ (resp. $n^+$), which converge to $\tilde{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology as $\lambda \to \mu$.

(III.2) Replace $\nu_r(\gamma_\mu)$ with $\nu_r(\gamma_\mu)/2$ in (III.1).

1.3. Bifurcations for brake orbits of periodic Hamiltonian systems

Assumption 1.22. For a real $\tau > 0$ and a topological space $\Lambda$, let $H : \Lambda \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ be a continuous function such that each $H(\lambda, t, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}$, $(\lambda, t) \in \Lambda \times \mathbb{R}$, is $C^2$, and all its partial derivatives of $H$ depend continuously on $(\lambda, t, z) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n}$, and that

\[
H(\lambda, t + \tau, z) = H(\lambda, t, z) \quad \forall (\lambda, t, z) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n},
\]

\[
H(\lambda, -t, Nz) = H(\lambda, t, z) \quad \forall (\lambda, t, z) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n},
\]

where $N = \left( \begin{array}{cc} -I_n & 0 \\ 0 & I_n \end{array} \right)$. For each $\lambda \in \Lambda$ let $v_\lambda : \mathbb{R} \to \mathbb{R}^{2n}$ satisfy the following periodic Hamiltonian system

\[
\dot{v}(t) = J\nabla_z H(\lambda, t, v(t)), \quad v(t + \tau) = v(t) \quad \text{and} \quad v(-t) = Nv(t) \quad \forall t \in \mathbb{R},
\]

and $\Lambda \times [0, \tau] \ni (\lambda, t) \mapsto v_\lambda(t) \in \mathbb{R}^{2n}$ is continuous.

As pointed out below Assumption 1.11, it follows from Assumption 1.22 that $\Lambda \times [0, \tau] \ni (\lambda, t) \mapsto v_\lambda(t) \in \mathbb{R}^{2n}$ is continuous. When $H$ is independent of $t$, solutions of (1.29) were called brake orbits.

We call $(\mu, v_\mu)$ with some $\mu \in \Lambda$ a bifurcation point of (1.29) if each neighborhood of it in $\Lambda \times W^{1,2}(S_\tau; \mathbb{R}^{2n})$ (or equivalently $\Lambda \times C^1(S_\tau; \mathbb{R}^{2n})$) contains elements $(\lambda, v)$, where $v \neq v_\lambda$ is a solution of (1.29) for the parameter value $\lambda$. Similarly, we can define a bifurcation point along sequence of (1.29) in the sense of Definition 1.2.

Under Assumption 1.22 the linearized problem of (1.29) along $v_\lambda$ is

\[
\dot{v}(t) = J\nabla_z^2 H(\lambda, t, v_\lambda(t))v(t), \quad v(t + \tau) = v(t) \quad \text{and} \quad v(-t) = Nv(t)
\]
for all $t \in \mathbb{R}$. Note that (1.22) implies

$$N^T \nabla_z H(\lambda, -t, Nz) = \nabla_z H(\lambda, t, z)$$

and $N^T \nabla^2 z H(\lambda, -t, Nz) N = \nabla^2 z H(\lambda, t, z)$. Hence each $B_1(t) := \nabla^2 z H(\lambda, t, v_1(t))$ satisfies Assumption A.6 Let $\gamma_1 : \mathbb{R} \rightarrow \text{Sp}(2n, \mathbb{R})$ be the fundamental matrix solution of $\dot{Z}(t) = JB_1(t)Z(t)$. It has the Maslov-type index $(\mu_1, \tau(\gamma_1), \nu_1, \tau(\gamma_1))$ defined by (A.13) and (A.14). For a solution of (1.29), since $NJ = -JN$ and $N^T = N$, it is not hard to see that $Nu$ does not necessarily satisfy (1.29) [even if $H$ is independent of $t$].

Corresponding to Theorems 1.21, 1.22 we have:

**Theorem 1.23.** Let Assumption 1.24 be satisfied.

(I) **(Necessary condition):** If $(\mu, \nu)$ is a bifurcation point along sequences of (1.29), that is, there exists a sequence $(\lambda_k) \subset \Lambda$ converging to $\mu$ and solutions $\nu^k \neq \nu_k$ of (1.29) with $\lambda = \lambda_k$ such that $\nu^k \rightarrow \nu_k$ on any compact interval $I \subset \mathbb{R}$ in $C^0$-topology as $k \rightarrow \infty$, then $\nu_1(\mu_k) \neq 0$.

(II) **(Sufficient condition):** Let $\Lambda$ be first countable. Suppose for some $\mu \in \Lambda$ that there exist two sequences in $\Lambda$ converging to $\mu$, $(\lambda^-_k) \subset (\lambda^+_k)$, such that for each $k \in \mathbb{N}$,

$$[\mu_1, \tau(\gamma^+_k), \nu_1, \tau(\gamma^+_k)] \cap [\mu_1, \tau(\gamma^-_k), \nu_1, \tau(\gamma^-_k)] = \emptyset$$

and either $\nu_1(\gamma^+_k) = 0$ or $\nu_1(\gamma^-_k) = 0$. Let $\hat{\Lambda} := \{\mu, \lambda^-_k, \lambda^+_k | k \in \mathbb{N}\}$. Then $(\mu, \nu)$ is a bifurcation point of (1.29) with respect to the branch $\{(\lambda, \nu_k) | \lambda \in \hat{\Lambda}\}$ and so $\{(\lambda, \nu_k) | \lambda \in \Lambda\}$.

(III) **(Existence for bifurcations):** Let $\Lambda$ be path-connected. Suppose that there exist two points $\lambda^+, \lambda^- \in \Lambda$ such that $[\mu_1, \tau(\gamma^+_k), \mu_1, \tau(\gamma^+_k)] \cap [\mu_1, \tau(\gamma^-_k), \mu_1, \tau(\gamma^-_k)] = \emptyset$ and either $\nu_1(\gamma^+_k) = 0$ or $\nu_1(\gamma^-_k) = 0$. Then for any path $\alpha : [0, 1] \rightarrow \Lambda$ connecting $\lambda^+ \rightarrow \lambda^-$ there exists a sequence $(t_k) \subset [0, 1]$ converging to some $t$ and solutions $\nu^k \neq \nu_{t_k}$ of (1.29) of (1.29) with $\lambda = \alpha(t_k), k = 1, 2, \cdots$, such that $(\nu^k)$ converges to $\nu_{\alpha(t)}$ on any compact interval $I \subset \mathbb{R}$ in $C^1$-topology as $k \rightarrow \infty$. Moreover, $\alpha(t)$ is not equal to $\lambda^+$ (resp. $\lambda^-$) if $\nu_1(\gamma^+_k) = 0$ (resp. $\nu_1(\gamma^-_k) = 0$).

**Theorem 1.24 (Alternative bifurcations of Rabinowitz’s type and of Fadell-Rabinowitz's type).** Let Assumption 1.24 with $\Lambda$ being a real interval be satisfied. Suppose for some interior point $\mu$ of $\Lambda$ that $\nu_1, \tau(\gamma_\mu) \neq 0$, $\nu_1, \tau(\gamma_\mu) = 0$ for each $\lambda \in \Lambda \setminus \{\mu\}$ near $\mu$, and $\mu_1, \tau(\gamma_\mu)$ takes, respectively, values $\mu_1, \tau(\gamma_\mu)$ and $\mu_1, \tau(\gamma_\mu) + \nu_1, \tau(\gamma_\mu)$ as $\lambda \in \Lambda$ varies in two deleted half neighborhoods of $\mu$.

Then one of the following three claims occurs:

(i) **Equation (1.29)** with $\lambda = \mu$ has a sequence of solutions, $\tilde{v}_k \neq \nu_\mu, k = 1, 2, \cdots$, which converges to $\nu_\mu$ in $C^1(S_r; \mathbb{R}^{2n})$. 

(ii) **For every $\lambda \in \Lambda \setminus \{\mu\}$ near $\mu$ there is a solution $\tilde{v}_\lambda \neq \nu_\lambda$ of (1.29) with parameter value $\lambda$, such that $\tilde{v}_\lambda - \nu_\lambda$ converges to zero in $C^1(S_r; \mathbb{R}^{2n})$ as $\lambda \rightarrow \mu$.**

(iii) **For a given neighborhood $W$ of $\nu_\mu$ in $C^1(S_r; \mathbb{R}^{2n})$ there is an one-sided neighborhood $\Lambda^0$ of $\mu$ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, (1.29) with parameter value $\lambda$ has at least two distinct solutions $\tilde{v}_\lambda^1 \neq \nu_\lambda$ and $\tilde{v}_\lambda^2 \neq \nu_\lambda$ in $W$, which can also be required to satisfy

$$\int_0^t \left[\frac{1}{2}(J\tilde{v}_\lambda^1(t), \tilde{v}_\lambda^1(t))_{\mathbb{R}^{2n}} + H(\lambda, t, \tilde{v}_\lambda^1(t))\right] dt \neq \int_0^t \left[\frac{1}{2}(J\tilde{v}_\lambda^2(t), \tilde{v}_\lambda^2(t))_{\mathbb{R}^{2n}} + H(\lambda, t, \tilde{v}_\lambda^2(t))\right] dt$$
provided that \( \nu_{1,\tau}(\gamma_\mu) > 1 \) and (1.29) with parameter value \( \lambda \) has only finitely many solutions in \( W \).

Moreover, if \( \nu_\lambda = 0 \forall \lambda \), and all \( H(\lambda, t, \cdot) \) are even, then either (i) holds or the following occurs:

(iv) There exist left and right neighborhoods \( \Lambda^- \) and \( \Lambda^+ \) of \( \mu \) in \( \Lambda \) and integers \( n^+, n^- \geq 0 \), such that \( n^+ + n^- \geq \nu_{1,\tau}(\gamma_\mu) \), and for \( \lambda \in \Lambda^- \setminus \{ \mu \} \) (resp. \( \lambda \in \Lambda^+ \setminus \{ \mu \} \)), (1.29) with parameter value \( \lambda \) has at least \( n^- \) (resp. \( n^+ \)) distinct pairs of nontrivial solutions, \( \{ v^{i}_{\lambda}, -v^{i}_{\lambda} \}, i = 1, \ldots, n^- \) (resp. \( n^+ \)), which converge to zero in \( C^1(\mathbb{S}_r; \mathbb{R}^{2n}) \) as \( \lambda \to \mu \).

Corresponding to Corollary (1.9) or (1.12) we have also

**Corollary 1.25.** Let \( C^1 \) functions \( H_0, \dot{H} : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R} \) satisfy the following conditions:

(A) For all \( (t, z) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n} \) it holds that

\[
H_0(-t, Nz) = H_0(t, z) = H_0(t + \tau, z) \quad \text{and} \quad \dot{H}(-t, Nz) = \dot{H}(t, z) = \dot{H}(t + \tau, z).
\]

(B) All \( H_0(t, \cdot), \dot{H}(t, \cdot) : \mathbb{R}^{2n} \to \mathbb{R} \) are \( C^2 \) and all their partial derivatives depend continuously on \( (t, z) \in \mathbb{R} \times \mathbb{R}^{2n} \).

Suppose that \( \tilde{v} : \mathbb{R} \to \mathbb{R}^{2n} \) satisfy

\[
\dot{v}(t) = J \nabla_z H_0(t, v(t)), \quad v(t + \tau) = v(t) \quad \text{and} \quad v(-t) = N v(t) \forall t,
\]

and that \( \nabla_z \dot{H}(t, \tilde{v}(t)) = 0 \) for all \( t \in \mathbb{R} \) and

either \( \nabla_z^2 \dot{H}(t, \tilde{v}(t)) > 0 \forall t \) or \( \nabla_z^2 \dot{H}(t, \tilde{v}(t)) < 0 \forall t \).

Take \( H(\lambda, t, z) = H_0(t, z) + \lambda \dot{H}(t, z) \) in (1.29) and let \( \gamma_\lambda : \mathbb{R} \to \text{Sp}(2n, \mathbb{R}) \) be the fundamental matrix solution of \( \dot{Z}(t) = JB_\lambda(t)Z(t) \) with \( B_\lambda(t) := \nabla_z^2 \dot{H}(\lambda, t, \tilde{v}(t)) \), the following holds:

(i) \( \Sigma_1 := \{ \lambda \in \mathbb{R} | \nu_{1,\tau}(\gamma_\lambda) > 0 \} \) is a discrete set in \( \mathbb{R} \).

(ii) \( (\mu, \tilde{v}) \) with \( \mu \in \mathbb{R} \) is a bifurcation point for (1.29) if and only if \( \nu_{1,\tau}(\gamma_\mu) > 0 \).

(iii) For each \( \mu \in \Sigma_1 \) and a small enough \( \rho > 0 \) it holds that

\[
\mu_{1,\tau}(\gamma_\lambda) = \begin{cases} 
\mu_{1,\tau}(\gamma_\mu) & \forall \lambda \in [\mu - \rho, \mu), \\
\mu_{1,\tau}(\gamma_\mu) + \nu_{1,\tau}(\gamma_\mu) & \forall \lambda \in (\mu, \mu + \rho]
\end{cases}
\]

if \( \nabla_z^2 \dot{H}(t, \tilde{v}(t)) > 0 \) for all \( t \), and

\[
\mu_{1,\tau}(\gamma_\lambda) = \begin{cases} 
\mu_{1,\tau}(\gamma_\mu) + \nu_{1,\tau}(\gamma_\mu) & \forall \lambda \in [\mu - \rho, \mu), \\
\mu_{1,\tau}(\gamma_\mu) & \forall \lambda \in (\mu, \mu + \rho]
\end{cases}
\]

if \( \nabla_z^2 \dot{H}(t, \tilde{v}(t)) < 0 \) for all \( t \).

(iv) One of (i)-(iii) in Theorem (1.24) with this \( H \) and \( v_\lambda \equiv \tilde{v} \) occurs.

(v) If \( \tilde{v} = 0 \) and all \( H_0(t, \cdot), \dot{H}(t, \cdot) \) are even, then for this \( H \) either (i) or (iv) in Theorem (1.24) and \( v_\lambda \equiv 0 \) holds.
Theorem 1.26 (Alternative bifurcations of Rabinowitz’s type and of Fadell-Rabinowitz’s type). Let Assumption 1.22 with $\Lambda$ being a real interval be satisfied. Suppose that $H$ is independent of time $t$. For some $\tau > 0$ let $\bar{v} : \mathbb{R} \to \mathbb{R}^{2n}$ be an even solution of
\[
\dot{v}(t) = J \nabla_{z} H(\lambda, v(t)), \quad v(t + \tau) = v(t) \quad \text{and} \quad v(-t) = Nv(t) \quad \forall t \in \mathbb{R},
\]
for each $\lambda \in \Lambda$, and let $\gamma_{\lambda}(t)$ be the fundamental matrix solution of $\dot{v}(t) = J \nabla_{z} H(\lambda, \bar{v}(t))v(t)$.
Suppose that for some interior point $\mu$ of $\Lambda$ the following holds.

(a) $\nu_{1, \tau}(\gamma_{\mu}) \neq 0$, $\nu_{1, \tau}(\gamma_{\lambda}) = 0$ for each $\lambda \in \Lambda \setminus \{\mu\}$ near $\mu$, and $\mu_{1, \tau}(\gamma_{\lambda})$ takes, respectively, values $\mu_{1, \tau}(\gamma_{\mu})$ and $\mu_{1, \tau}(\gamma_{\mu}) + \nu_{1, \tau}(\gamma_{\mu})$ as $\lambda \in \Lambda$ varies in two deleted half-neighborhoods of $\mu$.

(b) The following problem
\[
\dot{v}(t) = J \nabla_{z}^{2} H(\mu, \bar{v}(t))v(t), \quad v(t + \tau) = v(t) \quad \text{and} \quad v(-t) = Nv(t) \quad \forall t
\]
has no nonzero even solutions.

Then one of the following alternatives occurs:

(i) Equation 1.31 with $\lambda = \mu$ has a sequence of solution pairs, $\{v_{k}, Nv_{k}\}$, $k = 1, 2, \ldots$, such that $v_{k}$ and $Nv_{k}$ converge to $\bar{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^{1}$-topology.

(ii) There exist left and right neighborhoods $\Lambda^{-}$ and $\Lambda^{+}$ of $\mu$ in $\Lambda$ and integers $n^{+}, n^{-} \geq 0$, such that $n^{+} + n^{-} \geq \nu_{1, \tau}(\gamma_{\mu})$, and for $\lambda \in \Lambda^{-} \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^{+} \setminus \{\mu\}$), 1.31 with parameter value $\lambda$ has at least $n^{-}$ (resp. $n^{+}$) distinct solution pairs $\{v_{k}^{i}, Nv_{k}^{i}\}$, $i = 1, \ldots, n^{-}$ (resp. $n^{+}$), such that all $v_{k}^{i}$ and $Nv_{k}^{i}$ converge to $\bar{v}$ on any compact interval $I \subset \mathbb{R}$ in $C^{1}$-topology as $\lambda \to \mu$.

Furthermore, if $\bar{v} = 0$, each $H(\lambda, \cdot)$ is even and the assumption (b) is removed, then one of the following alternatives occurs:

(iii) Equation 1.31 with $\lambda = \mu$ has a sequence of solution quadruples, $\{v_{k}, -v_{k}, Nv_{k}, -Nv_{k}\}$, $k = 1, 2, \ldots$, such that $v_{k}$ converges to $0$ on any compact interval $I \subset \mathbb{R}$ in $C^{1}$-topology.

(iv) There exist left and right neighborhoods $\Lambda^{-}$ and $\Lambda^{+}$ of $\mu$ in $\Lambda$ and integers $n^{+}, n^{-} \geq 0$, such that $n^{+} + n^{-} \geq \nu_{1, \tau}(\gamma_{\mu})$, and for $\lambda \in \Lambda^{-} \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^{+} \setminus \{\mu\}$), 1.31 with parameter value $\lambda$ has at least $n^{-}$ (resp. $n^{+}$) distinct solution quadruples $\{v_{k}^{i}, -v_{k}^{i}, Nv_{k}^{i}, -Nv_{k}^{i}\}$, $i = 1, \ldots, n^{-}$ (resp. $n^{+}$), such that $v_{k}^{i}$ converges to $0$ on any compact interval $I \subset \mathbb{R}$ in $C^{1}$-topology as $\lambda \to \mu$.

Remark 1.27. For the first part in Theorem 1.26 if we do not assume the condition (b), using [32] Theorem 3.6] we can deduce that one of the above (i) and the following two claims holds:

(ii’) For every $\lambda \in \Lambda \setminus \{\mu\}$ near $\mu$ there is a solution $v_{\lambda} \neq \bar{v}$ of 1.31 with parameter value $\lambda$, which converges to $\bar{v}$ in $C^{1}(S_{\tau}; \mathbb{R}^{2n})$ as $\lambda \to \mu$.

(iii’) For a given neighborhood $\mathcal{W}$ of $\bar{v}$ in $C^{1}(S_{\tau}; \mathbb{R}^{2n})$ there is an one-sided neighborhood $\Lambda^{0}$ of $\mu$ such that for any $\lambda \in \Lambda^{0} \setminus \{\mu\}$, 1.31 with parameter value $\lambda$ has at least two distinct solutions $v_{k}^{1} \neq \bar{v}$ and $v_{k}^{2} \neq \bar{v}$ in $\mathcal{W}$, which can also be required to satisfy
\[
\int_{0}^{\tau} \left[ \frac{1}{2} (J \dot{v}_{1}^{1}(t), v_{1}^{1}(t))_{\mathbb{R}^{2n}} + H(\lambda, v_{1}^{1}(t)) \right] dt \neq \int_{0}^{\tau} \left[ \frac{1}{2} (J \dot{v}_{2}^{1}(t), v_{2}^{1}(t))_{\mathbb{R}^{2n}} + H(\lambda, v_{2}^{1}(t)) \right] dt
\]
provided that $\nu_{1, \tau}(\gamma_{\mu}) > 1$ and 1.31 with parameter value $\lambda$ has only finitely many solutions in $\mathcal{W}$.
Clearly, these have already been implied in the first part of Theorem 1.24. Of course, for \( v_\lambda \) in (ii), \( Nv_\lambda \) is also a solution of (1.31) with parameter value \( \lambda \). But we cannot affirm \( Nv_\lambda \neq v_\lambda \).

Similarly, if \( \nu_1, \tau(\gamma_\mu) = 1 \) it is uncertain that \( \nu_1^1 \neq N\nu_1^2 \) in (iii'). If \( \nu_1, \tau(\gamma_\mu) > 1 \), (iii') implies that (1.31) with parameter value \( \lambda \) has at least two distinct solution pairs \( \{v_\lambda^1, N\nu_1^2\} \) and \( \{\nu_1^2, N\nu_1^2\} \) in \( \mathcal{W} \).

**Corollary 1.28.** Let \( H_0, \hat{H} : \mathbb{R}^{2n} \rightarrow \mathbb{R} \) be \( N \)-invariant \( C^2 \)-functions. Suppose that \( dH(0) = d\hat{H}(0) = 0 \) and that either \( \hat{H}''(0) > 0 \) or \( H''(0) < 0 \). Then for \( \gamma_\lambda(t) = \exp(tIJH''(0)) \) and any \( \tau > 0 \), \( \Sigma_\tau := \{ \lambda \in \mathbb{R} \mid \nu_1, \tau(\gamma_\lambda) > 0 \} \) is a discrete set in \( \mathbb{R} \). Moreover, for each \( \mu \in \Sigma_\tau \) and a small enough \( \rho > 0 \) it holds that

\[
\mu_1, \tau(\gamma_\lambda) = \begin{cases} 
\mu_1, \tau(\gamma_\mu) + \nu_1, \tau(\gamma_\mu) & \forall \lambda \in [\mu - \rho, \mu), \\
\mu_1, \tau(\gamma_\mu) & \forall \lambda \in (\mu, \mu + \rho]
\end{cases}
\]

(1.33)

if \( \hat{H}''(0) > 0 \) \( \forall t \in [0, \tau] \), and

\[
\mu_1, \tau(\gamma_\lambda) = \begin{cases} 
\mu_1, \tau(\gamma_\mu) + \nu_1, \tau(\gamma_\mu) & \forall \lambda \in [\mu - \rho, \mu), \\
\nu_1, \tau(\gamma_\mu) & \forall \lambda \in (\mu, \mu + \rho]
\end{cases}
\]

(1.34)

if \( \hat{H}''(0) < 0 \) \( \forall t \in [0, \tau] \). Consequently, for each \( \mu \in \Sigma_\tau \), if (1.32) with \( H(\mu, x) = H_0(x) + \mu \hat{H}(x) \) and \( \bar{v} = 0 \) has no nonzero even solution, then the first conclusion of Theorem 1.26 holds for (1.31) with \( H(\lambda, x) = H_0(x) + \lambda \hat{H}(x) \). Suppose that both \( H_0 \) and \( \hat{H} \) are also even. Then the second conclusion of Theorem 1.26 holds for (1.31) with \( H(\lambda, x) = H_0(x) + \lambda \hat{H}(x) \).

Corresponding to Corollary 1.16 we have

**Corollary 1.29.** Let \( H : \mathbb{R}^{2n} \rightarrow \mathbb{R} \) be a \( N \)-invariant \( C^2 \)-function satisfying \( dH(0) = 0 \). Suppose that either \( H''(0) > 0 \) or \( H''(0) < 0 \). Then \( \Delta(\hat{H}) := \{ \lambda \in \mathbb{R} \setminus \{0\} \mid \nu_1, 1(\gamma_\mu^\hat{H}) > 0 \} \) is a discrete set, where \( \gamma^\hat{H}_\lambda(t) = \exp(t\lambda J\hat{H}(0)) \).

If \( H''(0) > 0 \) and \( \mu \in \Delta(\hat{H}) \cap (0, \infty) \) is such that the following problem

\[
\dot{v}(t) = \mu JH''(0)v(t), \quad v(t + 1) = v(t) \quad \text{and} \quad v(-t) = Nv(t) \quad \forall t
\]

(1.35)

has no nonzero even solutions, then one of the following alternatives occurs:

(i) The problem

\[
\dot{v}(t) = J\nabla H(v(t)), \quad v(t + \mu) = v(t) \quad \text{and} \quad v(-t) = Nv(t) \quad \forall t
\]

(1.36)

has a sequence of solution pairs, \( \{v_k, Nv_k\}, k = 1, 2, \ldots \), such that \( v_k \) and \( Nv_k \) converge to 0 on any compact interval \( I \subset \mathbb{R} \) in \( C^1 \)-topology.

(ii) There exist left and right neighborhoods \( \Lambda^- \) and \( \Lambda^+ \) of \( \mu \) in \( \Lambda \) and integers \( n^+, n^- \geq 0 \), such that \( n^+ + n^- \geq \nu_1, 1(\gamma_\mu^\hat{H}) \), and for \( \lambda \in \Lambda^- \setminus \{\mu\} \) (resp. \( \lambda \in \Lambda^+ \setminus \{\mu\} \), the problem

\[
\dot{v}(t) = J\nabla H(v(t)), \quad v(t + \lambda) = v(t) \quad \text{and} \quad v(-t) = Nv(t) \quad \forall t
\]

(1.37)

has at least \( n^- \) (resp. \( n^+ \)) distinct solution pairs \( \{v_\lambda^i, Nv_\lambda^i\}, i = 1, \ldots , n^- \) (resp. \( n^+ \)), such that all \( v_\lambda^i \) and \( Nv_\lambda^i \) converge to 0 on any compact interval \( I \subset \mathbb{R} \) in \( C^1 \)-topology as \( \lambda \to \mu \).

If \( H''(0) < 0 \) and \( \mu \in \Delta(\hat{H}) \cap (-\infty, 0) \) is such that (1.35) has no nonzero even solutions, then one of the above (i) and (ii) with this \( \mu \) holds.

Suppose further that \( H \) is also even. Then \( \{v_k, Nv_k\} \) in (i) and \( \{v_\lambda^i, Nv_\lambda^i\} \) in (ii) are replaced by \( \{v_k, Nv_k, -v_k, -Nv_k\} \) and \( \{v_\lambda^1, Nv_\lambda^i, -v_\lambda^1, -Nv_\lambda^i\} \), respectively.
As in Remark 1.17 we can show that this result cannot be included in [3, Theorem 9.12].

**Example 1.30.** Let $0 < \varrho_1 \leq \cdots \leq \varrho_n$ and $H : \mathbb{R}^{2n} \to \mathbb{R}$ be given by

$$H(x_1, \cdots, x_n, y_1, \cdots, y_n) = \sum_{j=1}^{n} \frac{\varrho_j}{2}(x_j^2 + y_j^2).$$

Then by [10, Example 3.2] $\gamma^H_\lambda(t) := \exp(\lambda t JH'(0))$ is equal to

$$
\begin{pmatrix}
\text{diag}(\cos(\varrho_1 \lambda t), \ldots, \cos(\varrho_n \lambda t)) & -\text{diag}(\sin(\varrho_1 \lambda t), \ldots, \sin(\varrho_n \lambda t)) \\
\text{diag}(\sin(\varrho_1 \lambda t), \ldots, \sin(\varrho_n \lambda t)) & \text{diag}(\cos(\varrho_1 \lambda t), \ldots, \cos(\varrho_n \lambda t))
\end{pmatrix}
$$

and

$$\mu_{1,\tau}(\gamma^H_\lambda) = \mu_{2,\tau}(\gamma^H_\lambda) = n - k + \sum_{i=1}^{n} \left[ \frac{\lambda \varrho_i \tau}{\pi} \right],$$

where $k = \sharp\{i \in \{1, \cdots, n\} \mid \lambda \varrho_i \tau = 0 \bmod \pi \}$. Moreover, by (A.14) it is easily proved that

$$\nu_{1,\tau}(\gamma^H_\lambda) = \nu_{2,\tau}(\gamma^H_\lambda) = \dim \ker(\text{diag}(\sin(\varrho_1 \lambda \tau/2), \ldots, \sin(\varrho_n \lambda \tau/2)))$$

Taking $\tau = 1$ we obtain $\Delta(H) \cap (0, \infty) = \bigcup_{i=1}^{n} \{ \lambda > 0 \mid \lambda \varrho_i = 0 \bmod 2\pi \}$. For $\mu \in \Delta(H) \cap (0, \infty)$, all nontrivial solutions of the problem

$$\dot{v}(t) = \mu JH''(0)v(t) = \mu J\text{diag}(\varrho_1, \cdots, \varrho_n, \varrho_1, \cdots, \varrho_n)v(t) \quad \text{and} \quad v(t + 1) = v(t)$$

have forms $v(t) = \gamma^H_\mu(t)c$, where components $c_1$ and $c_{n+i}$ in $c = (c_1, \cdots, c_{2n})^T \in \mathbb{R}^{2n}$ are equal to zeros if $\mu \varrho_i \notin 2\pi \mathbb{Z}$. Since $NJ = -JN$ and $NH''(0) = H''(0)N$ we deduce that $Nv(t) = N\gamma^H_\mu(t)c = \gamma^H_\mu(-t)Nc$. This implies that $Nv(t) = v(-t) \forall t$ if and only $Nc = c$, i.e., $c_1 = \cdots = c_n = 0$. Thus if the above $v(t)$ satisfies $Nv(t) = v(-t) \forall t$, then

$$v(t) = \begin{pmatrix}
-\varrho_{n+1} \sin(\lambda \varrho_1 t), \ldots, -\varrho_{2n} \sin(\lambda \varrho_n t), \varrho_{n+1} \cos(\lambda \varrho_1 t), \cdots, \varrho_{2n} \cos(\lambda \varrho_n t)
\end{pmatrix}^T,$$

where $c_{n+i} = 0$ if $\mu \varrho_i \notin 2\pi \mathbb{Z}$. Clearly, if it is also even then $v \equiv 0$. Hence the problem (1.35) with this $H$ has no nonzero even solutions. Suppose that $H$ is also $N$-invariant. By Corollary 1.25 for each $\mu \in \Delta(H) \cap (0, \infty)$ one of the statements (i) and (ii) in Corollary 1.29 holds after $\nu_{1,1}(\gamma^H_\mu)$ is replaced by $\nu_{1,1}(\gamma^H_\mu)$.

**Question 1.31.** For a positive definite matrix $A$ of order $2n$, Williamson theorem gives rise to a symplectic matrix $S \in \text{Sp}(2n, \mathbb{R})$ such that $S^TAS = \text{diag}(\varrho_1, \cdots, \varrho_n, \varrho_1, \cdots, \varrho_n)$, where $0 < \varrho_j \leq \varrho_k$ for $j \leq k$. Suppose $AN = NA$. Can the above symplectic matrix $S$ be chosen to satisfy $NS = SN$?

If this question is solved in an affirmative way, the conclusions of Corollary 1.29 in the case $H''(0) > 0$ are the same as those of Example 1.30.

In [55], we shall study bifurcations for Hamiltonian paths connecting affine Lagrangian subspaces with saddle point reduction methods.

1.4. **Bifurcations for Hamiltonian trajectories connecting affine Lagrangian subspaces.** Recall that an affine Lagrangian subspace of $\mathbb{R}^{2n}$ is a subset of $\mathbb{R}^{2n}$ of the form $w + L$, where $w \in \mathbb{R}^{2n}$ is a fixed point and $L$ is a Lagrangian subspace of $\mathbb{R}^{2n}$. 
Assumption 1.32. For a real \( \tau > 0 \) and a topological space \( \Lambda \), let \( H : \Lambda \times [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R} \) be a continuous function such that each \( H(\lambda, t, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}, (\lambda, t) \in \Lambda \times [0, \tau] \), is \( C^2 \) and all possible partial derivatives of \( H \) depend continuously on \( (\lambda, t, z) \in \Lambda \times [0, \tau] \times \mathbb{R}^{2n} \). Let \( L \) and \( L' \) be two Lagrangian subspaces of \( \mathbb{R}^{2n} \), and let \( \Lambda \ni \lambda \mapsto w_\lambda \in \mathbb{R}^{2n} \) and \( \Lambda \ni \lambda \mapsto w'_\lambda \in \mathbb{R}^{2n} \) be two continuous maps. For each \( \lambda \in \Lambda \) let \( \lambda : [0, \tau] \to \mathbb{R}^{2n} \) be a differentiable path satisfying the Hamiltonian boundary value problem

\[
\begin{align*}
\dot{u}(t) &= J \nabla L(\lambda, t, u(t)) \quad \forall t \in [0, \tau], \\
u(0) &\in w_\lambda + L, \quad u(t) \in w'_\lambda + L'.
\end{align*}
\]

(1.40)

Suppose also that the map \( \Lambda \times [0, \tau] \ni (\lambda, t) \mapsto u_\lambda(t) \in \mathbb{R}^{2n} \) is continuous.

As noted below Assumption 1.32, the above assumption implies that \( (\lambda, u_\lambda) \) is continuous. But \( u_\lambda \) cannot be assured to be \( C^2 \).

Under Assumptions 1.32 we say \((\mu, u_\mu)\) to be a bifurcation point along sequences of the problem of (1.40) with respect to the trivial branch \( \{ (\lambda, u_\lambda) | \lambda \in \Lambda \} \) if there exists a sequence \( \{ (\lambda_k) \} \subset \Lambda \) converging to \( \mu \) and solutions \( v_k \neq u_{\lambda_k} \) of (1.40) with \( \lambda = \lambda_k \) for each \( k = 1, 2, \ldots \), such that \( v_k \to u_\mu \) in \( C^1([0, \tau], \mathbb{R}^{2n}) \).

Let \( \gamma_\lambda : [0, \tau] \to \text{Sp}(2n) \) be the fundamental matrix solution of \( \dot{u}(t) = J \nabla L(\lambda, t, u(t))u(t) \).

For any symplectic path \( \gamma : [0, \tau] \to \text{Sp}(2n) \) starting at the identity, Liu-Wang-Lin introduced a \((L, L')\)-index

\[
(i_L^L(\gamma), i_L^{L'}(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\}
\]

(1.41)

where \( i_L^L(\gamma) = \dim(\gamma(\tau) L \cap L') \). When \( L' = L \), \((i_L^L(\gamma), i_L^{L'}(\gamma))\) is equal to \( L\)-index \((i_L(\gamma), i_L(\gamma))\) of \( \gamma \) introduced by Liu.

Let \( L_0 = \emptyset \times \mathbb{R}^n \subset \mathbb{R}^{2n} \) and let \( O \in \text{Sp}(2n, \mathbb{R}) \) be an orthogonal symplectic matrix such that \( OL_0 = L \). According to Liu

\[
i_L(\gamma) = i_{L_0} \left( \{ O^{-1} \gamma(t) O \}_{0 \leq t \leq \tau} \right).
\]

(1.42)

If this \( \gamma \) is also piecewise smooth, the equality above and (5.99)] show that the right side is equal to

\[
\mu_{\text{CLM}}(f) - n = \mu_{\text{CLM}}(\tilde{f}) - n,
\]

where \( f(t) = (L_0, O^{-1} \gamma(t) O L_0) \) and \( \tilde{f}(t) = (L, \gamma(\tau) L) \) for \( 0 \leq t < \tau \), where \( \mu_{\text{CLM}}(f) \) is the Cappell-Lee-Miller index of \( f \) characterized by properties I-VI of [11 pp. 127-128].

Theorem 1.33. Under Assumption 1.32 let \( \gamma_\lambda \) be as above.

(I) (Necessary condition): If \((\mu, u_\mu)\) is a bifurcation point along sequences of the problem (1.40) with respect to the trivial branch \( \{ (\lambda, u_\lambda) | \lambda \in \Lambda \} \), i.e., there exists a sequence \( \{ \lambda_k \} \subset \Lambda \) converging to \( \mu \) and solutions \( u_k \neq u_{\lambda_k} \) of (1.40) with \( \lambda = \lambda_k \) such that \( u_k \to u_\mu \) in \( C^0([0, \tau], \mathbb{R}^{2n}) \), then \( \nu_L^L(\gamma_\mu) \neq 0 \).

(II) (Sufficient condition): Let \( \Lambda \) be first countable. Suppose for some \( \mu \in \Lambda \) that there exist two sequences in \( \Lambda \) converging to \( \mu \), \( (\lambda^+_k) \) and \( (\lambda^-_k) \), such that for each \( k \in \mathbb{N} \),

\[
[i_L^L(\gamma_{\lambda^-_k}), i_L^L(\gamma_{\lambda^-_k}) + \nu_L^L(\gamma_{\lambda^-_k})] \cap [i_L^L(\gamma_{\lambda^+_k}), i_L^L(\gamma_{\lambda^+_k}) + \nu_L^L(\gamma_{\lambda^+_k})] = \emptyset
\]

and either \( \nu_L^L(\gamma_{\lambda^-_k}) = 0 \) or \( \nu_L^L(\gamma_{\lambda^+_k}) = 0 \). Let \( \hat{\Lambda} := \{ \mu, \lambda^+_k, \lambda^-_k | k \in \mathbb{N} \} \). Then \((\mu, u_\mu)\) is a bifurcation point of (1.40) with respect to the branch \( \{ (\lambda, u_\lambda) | \lambda \in \hat{\Lambda} \} \) (and so \( \{ (\lambda, u_\lambda) | \lambda \in \Lambda \} \)).
(III) (Existence for bifurcations): Let $\Lambda$ be path-connected. Suppose that there exist two points $\lambda^+, \lambda^- \in \Lambda$ such that $[\nu_L^{(\lambda^+)}(\gamma_{\lambda^+}), \nu_L^{(\lambda^-)}(\gamma_{\lambda^-})] \cap [\nu_L^{(\lambda^-)}(\gamma_{\lambda^-}), \nu_L^{(\lambda^+)}(\gamma_{\lambda^+})] = \emptyset$, and either $\nu_L^{(\lambda^+)}(\gamma_{\lambda^+}) = 0$ or $\nu_L^{(\lambda^-)}(\gamma_{\lambda^-}) = 0$. Then for any path $\alpha : [0, 1] \to \Lambda$ connecting $\lambda^+$ to $\lambda^-$ there exists a sequence $(t_k) \subset [0, 1]$ converging to some $t$ and solutions $u_k \neq u_{\alpha(t_k)}$ of $(1.40)$ with $\alpha = \alpha(t_k)$, $k = 1, 2, \ldots$, such that $(u_k)$ converges to $u_{\alpha(t)}$ in $C^1$-topology as $k \to \infty$. Moreover, $\alpha(t)$ is not equal to $\lambda^+$ (resp. $\lambda^-$) if $\nu_L^{(\lambda^+)}(\gamma_{\lambda^+}) = 0$ (resp. $\nu_L^{(\lambda^-)}(\gamma_{\lambda^-}) = 0$).

Theorem 1.34 (Alternative bifurcations of Rabinowitz’s type and of Fadell-Rabinowitz’s type). Let Assumption $[1.32]$ with $\Lambda$ being a real interval be satisfied, and let $\gamma_{\lambda}$ be as in Theorem $[1.32]$ for each $\lambda \in \Lambda$. Suppose for some $\mu \in \text{Int}(\Lambda)$ that $\nu_L^{(\lambda)}(\gamma_{\mu}) \neq 0$ and $\nu_L^{(\lambda)}(\gamma_{\mu}) = 0$ for each $\lambda \in \Lambda \setminus \{\mu\}$ near $\mu$, and that $\nu_L^{(\lambda)}(\gamma_{\mu})$ take, respectively, values $\nu_L^{(\lambda)}(\gamma_{\mu})$ and $\nu_L^{(\lambda)}(\gamma_{\mu}) + \nu_L^{(\lambda)}(\gamma_{\mu})$ as $\lambda$ varies in two deleted half neighborhoods of $\mu$. Then one of the following alternatives occurs:

(i) The problem $(1.40)$ with $\lambda = \mu$ has a sequence of solutions, $v_k \neq u_{\lambda}$, $k = 1, 2, \ldots$, which converges to $u_{\lambda}$ in $C^1([0, \tau], \mathbb{R}^{2n})$.

(ii) For every $\mu \in \Lambda \setminus \{\mu\}$ near $\mu$ there is a solution $v_{\lambda} \neq u_{\lambda}$ of the problem $(1.40)$ with parameter value $\lambda$, such that $v_{\lambda} - u_{\lambda}$ converges to zero in $C^1([0, \tau], \mathbb{R}^{2n})$ as $\lambda \to \mu$.

(iii) For a given neighborhood $W$ of $u_{\mu}$ in $C^1([0, \tau]; \mathbb{R}^{2n})$ there is an one-sided neighborhood $\Lambda^0$ of $\mu$ such that for any $\lambda \in \Lambda^0 \setminus \{\mu\}$, the problem $(1.40)$ with parameter value $\lambda$ has at least two distinct solutions, $v_{\lambda} \neq u_{\lambda}$ and $v_{\lambda} \neq u_{\lambda}$ in $W$, which can also be required to satisfy

$$
\int_0^\tau \left[ \frac{1}{2} (J\dot{v}_{\lambda}(t), v_{\lambda}(t))_{\mathbb{R}^{2n}} + H(\lambda, t, v_{\lambda}(t)) \right] dt
\neq \int_0^\tau \left[ \frac{1}{2} (J\dot{v}_{\lambda}(t), v_{\lambda}(t))_{\mathbb{R}^{2n}} + H(\lambda, t, v_{\lambda}(t)) \right] dt;
$$

provided that $\nu_L^{(\lambda)}(\gamma_{\mu}) > 1$ and $(1.40)$ with parameter value $\lambda$ has only finitely many solutions in $W$.

Moreover, if $u_{\lambda} = 0 \forall \lambda$, and all $H(\lambda, t, \cdot)$ are even, then either (i) holds or the following occurs:

(iv) There exist left and right neighborhoods $\Lambda^-$ and $\Lambda^+$ of $\mu$ in $\Lambda$ and integers $n^+, n^- \geq 0$, such that $n^+ + n^- \geq \nu_L^{(\lambda)}(\gamma_{\mu})$, and for $\lambda \in \Lambda^+ \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^- \setminus \{\mu\}$), $(1.40)$ with parameter value $\lambda$ has at least $n^-$ (resp. $n^+$) distinct pairs of nontrivial solutions, $(u_{\lambda}, -u_{\lambda})$, $i = 1, \ldots, n^-$ (resp. $n^+$), which converge to zero in $C^1([0, \tau]; \mathbb{R}^{2n})$ as $\lambda \to \mu$.

By [38] Lemma 1.6] Theorems [1.23, 1.24 can also be derived from the above two results.

Example 1.35 (Bifurcation of the Sturm-Liouville problem). Let $\Lambda$ be a topological space, let $P \in C(\Lambda \times [0, \tau], \mathbb{R}^{n \times n})$ and $V \in C(\Lambda \times [0, \tau] \times \mathbb{R}^n)$ satisfy the following conditions:

(i) Each $P(\lambda, t), (\lambda, t) \in \Lambda \times [0, \tau]$, is positive definite.

(ii) Each $V(\lambda, t, \cdot) : \mathbb{R}^n \to \mathbb{R}, (\lambda, t) \in \Lambda \times [0, \tau]$, is $C^2$ and all possible partial derivatives of $V$ depend continuously on $(\lambda, t, q) \in \Lambda \times [0, \tau] \times \mathbb{R}^n$.

Given two reals $0 \leq \alpha, \beta \leq \pi$, and $\lambda \in \Lambda$, a differentiable path $q : [0, \tau] \to \mathbb{R}^n$ satisfies the following Sturm-Liouville problem

$$
\frac{d}{dt} (P(\lambda, t)\dot{q}) - \nabla q V(\lambda, t, q) = 0,
\left\{ \begin{array}{l}
(\cos \alpha) q(0) - (\sin \alpha) P(\lambda, 0)\dot{q}(0) = 0,
(\cos \beta) q(\tau) - (\sin \beta) P(\lambda, 1)\dot{q}(\tau) = 0
\end{array} \right\}
$$

(1.43)
if and only if it is a critical point of the functional
\[
\Phi_\lambda(q) = \int_0^\tau \left[ \frac{1}{2} (P(\lambda, t) \dot{q}(t), \dot{q}(t)) - V(\lambda, t, q(t)) \right] dt
\]
on the Hilbert subspace \( W^{1,2}_{\alpha,\beta}(0, \tau, \mathbb{R}^n) \) consisting of \( q \in W^{1,2}(0, \tau, \mathbb{R}^n) \) satisfying the boundary conditions in (1.43). (In this case we may deduce that \( t \mapsto P(\lambda, t) \dot{q}(t) \) (and thus \( q \)) is \( C^1 \). But we cannot obtain that \( q \) is \( C^2 \) if \( t \mapsto P(\lambda, t) \dot{q}(t) \) is not \( C^1 \).) These functionals \( \Phi_\lambda \) are \( C^2 \) and have finite Morse indexes and nullities at all critical points of them. For each \( \lambda \in \Lambda \), let \( q_\lambda : [0, \tau] \to \mathbb{R}^n \) be a differentiable path satisfying (1.43). (From the conditions (i)-(ii) and (1.43) we derive that \( t \mapsto P(\lambda, t) \dot{q}(t) \) is \( C^1 \). This can only lead to the \( C^1 \)-smoothness of \( q \) since we have not assumed that \( t \mapsto P(\lambda, t) \) is differentiable.) Denote by \( m^- (\Phi_\lambda, q_\lambda) \) and \( m^0 (\Phi_\lambda, q_\lambda) \) the Morse index and nullity of \( \Phi_\lambda \) at \( q_\lambda \). For some \( \mu \in \Lambda \) we call \( (\mu, q_\mu) \) to be a bifurcation point along sequences of the problem (1.43) with respect to the trivial branch \( \{(\lambda, q_\lambda) \mid \lambda \in \Lambda \} \) if there exists a sequence \( (\lambda_k) \subset \Lambda \) converging to \( \mu \) and solutions \( y_k \neq q_{\lambda_k} \) of (1.40) with \( \lambda = \lambda_k \) for \( k = 1, 2, \ldots \), such that \( y_k \to q_\mu \) in \( C^1([0, \tau], \mathbb{R}^n) \).

Suppose also that \( (\lambda, t) \mapsto q_\lambda(t) \) is continuous. From Theorems 1.33, 1.34 we can obtain:

(I) (Necessary condition): If \( (\mu, q_\mu) \) is a bifurcation point along sequences of the problem (1.43) with respect to the trivial branch \( \{(\lambda, q_\lambda) \mid \lambda \in \Lambda \} \), then \( m^0 (\Phi_\mu, q_\mu) > 0 \).

(II) (Sufficient condition): Let \( \Lambda \) be first countable. Suppose for some \( \mu \in \Lambda \) that there exist two sequences in \( \Lambda \) converging to \( \mu \), \( (\lambda^-_k) \) and \( (\lambda^+_k) \), such that for each \( k \in \mathbb{N} \),
\[
[m^- (\Phi_{\lambda^-_k}, q_{\lambda^-_k}), m^- (\Phi_{\lambda^+_k}, q_{\lambda^+_k}) + m^0 (\Phi_{\lambda^-_k}, q_{\lambda^-_k})] \cap [m^- (\Phi_{\lambda^+_k}, q_{\lambda^+_k}), m^- (\Phi_{\lambda^-_k}, q_{\lambda^-_k}) + m^0 (\Phi_{\lambda^+_k}, q_{\lambda^+_k})] = \emptyset
\]
and either \( m^0 (\Phi_{\lambda^-_k}, q_{\lambda^-_k}) = 0 \) or \( m^0 (\Phi_{\lambda^+_k}, q_{\lambda^+_k}) = 0 \). Let \( \hat{\Lambda} := \{ \mu, \lambda^+_k, \lambda^-_k \mid k \in \mathbb{N} \} \). Then \( (\mu, q_\mu) \) is a bifurcation point of (1.43) with respect to the branch \( \{(\lambda, q_\lambda) \mid \lambda \in \hat{\Lambda} \} \) (and so \( \{(\lambda, q_\lambda) \mid \lambda \in \Lambda \} \)).

(III) (Existence for bifurcations): Let \( \Lambda \) be path-connected. Suppose that there exist two points \( \lambda^+, \lambda^- \) in \( \Lambda \) such that
\[
[m^- (\Phi_{\lambda^+}, q_{\lambda^+}), m^- (\Phi_{\lambda^-}, q_{\lambda^-}) + m^0 (\Phi_{\lambda^+}, q_{\lambda^+})] \cap [m^- (\Phi_{\lambda^-}, q_{\lambda^-}), m^- (\Phi_{\lambda^-}, q_{\lambda^-}) + m^0 (\Phi_{\lambda^-}, q_{\lambda^-})] = \emptyset
\]
and either \( m^0 (\Phi_{\lambda^+}, q_{\lambda^+}) = 0 \) or \( m^0 (\Phi_{\lambda^-}, q_{\lambda^-}) = 0 \). Then for any path \( \alpha : [0, 1] \to \Lambda \) connecting \( \lambda^+ \) to \( \lambda^- \) there exists a sequence \( (\lambda_k) \subset [0, 1] \) converging to some \( \tilde{t} \) and solutions \( q^k \neq q_{\alpha(t_k)} \) of (1.43) with \( \lambda = \alpha(t_k) \), \( k = 1, 2, \ldots \), such that \( (q^k) \) converges to \( q_{\alpha(t)} \) in \( C^1 \)-topology as \( k \to \infty \). Moreover, \( \alpha(t) \) is not equal to \( \lambda^+ \) (resp. \( \lambda^- \)) if \( m^0 (\Phi_{\lambda^+}, q_{\lambda^+}) = 0 \) (resp. \( m^0 (\Phi_{\lambda^-}, q_{\lambda^-}) = 0 \)).

(IV) (Alternative bifurcations of Rabinowitz’s type and of Fadell-Rabinowitz’s type): If \( \Lambda \) is a real interval, \( \mu \in \text{Int}(\Lambda) \) is such that \( m^0 (\Phi_{\mu}, q_{\mu}) > 0 \) and \( m^0 (\Phi_{\mu}, q_{\mu}) = 0 \) for each \( \lambda \in \Lambda \setminus \{ \mu \} \) near \( \mu \), and that \( m^- (\Phi_{\lambda}, q_{\lambda}) \) take, respectively, values \( m^- (\Phi_{\mu}, q_{\mu}) \) and \( m^- (\Phi_{\mu}, q_{\mu}) + m^0 (\Phi_{\mu}, q_{\mu}) \) as \( \lambda \) in \( \Lambda \) varies in two deleted half neighborhoods of \( \mu \). Then one of the following alternatives occurs:

(IV.1) The problem (1.43) with \( \lambda = \mu \) has a sequence of solutions, \( y_k \neq q_\mu, k = 1, 2, \ldots \), which converges to \( q_\mu \) in \( C^1([0, \tau], \mathbb{R}^n) \).

(IV.2) For every \( \lambda \in \Lambda \setminus \{ \mu \} \) near \( \mu \) there is a solution \( y_\lambda \neq q_\lambda \) of the problem (1.43) with parameter value \( \lambda \), such that \( y_\lambda - q_\lambda \) converges to zero in \( C^1([0, \tau], \mathbb{R}^n) \) as \( \lambda \to \mu \).
(IV.3) For a given neighborhood $W$ of $q_\mu$ in $C^1([0, \tau]; \mathbb{R}^n)$ there is an one-sided neighborhood $\Lambda_0$ of $\mu$ such that for any $\lambda \in \Lambda_0 \setminus \{\mu\}$, the problem \[1.43\] with parameter value $\lambda$ has at least two distinct solutions, $y^1_\lambda \neq q_\lambda$ and $y^2_\lambda \neq q_\lambda$ in $W$, which can also be required to satisfy $\Phi_\lambda(y^1_\lambda) \neq \Phi_\lambda(y^2_\lambda)$ provided that $m^0(\Phi_\mu, q_\mu) > 1$ and \[1.43\] with parameter value $\lambda$ has only finitely many solutions in $W$.

Moreover, if $q_\lambda = 0 \forall \lambda$, and all $V(\lambda, t, \cdot)$ are even, then either (III.1) holds or the following occurs:

(IV.4) There exist left and right neighborhoods $\Lambda^-$ and $\Lambda^+$ of $\mu$ in $\Lambda$ and integers $n^+, n^- \geq 0$, such that $n^+ + n^- \geq m^0(\Phi_\mu, q_\mu)$, and for $\lambda \in \Lambda^- \setminus \{\mu\}$ (resp. $\lambda \in \Lambda^+ \setminus \{\mu\}$), \[1.43\] with parameter value $\lambda$ has at least $n^-$ (resp. $n^+$) distinct pairs of nontrivial solutions, $\{u_i^\lambda, -u_i^\lambda\}$, $i = 1, \cdots, n^-$ (resp. $n^+$), which converge to zero in $C^1([0, \tau]; \mathbb{R}^n)$ as $\lambda \to \mu$.

Indeed, let $H(\lambda, t, p, q) = 1/2 \left( P(\lambda, t)^{-1} p, p \right)_{\mathbb{R}^n} - V(\lambda, t, q)$. Then the conditions (i)-(ii) implies that $H$ satisfies Assumption \[1.32\]. Clearly, $q(t)$ satisfies \[1.43\] if and only if $u(t) := (p(t)^T, q(t)^T)^T$ with $p(t) = P(\lambda, t) q(t)$ satisfies

\[
\begin{align*}
\dot{u}(t) &= J\nabla_z H(\lambda, t, u(t)) \quad \forall t \in [0, \tau], \\
u(0) &= L_\alpha, \quad u(\tau) = L_\beta,
\end{align*}
\]

where $L_\alpha = \{(p^T, q^T)^T \in \mathbb{R}^{2n} \mid (\cos \xi) q - (\sin \xi) p = 0, p, q \in \mathbb{R}^n\}$ for $\xi \in \mathbb{R}$. In particular, $u_\lambda(t) := (p_\lambda(t)^T, q_\lambda(t)^T)^T$ with $p_\lambda(t) = P(\lambda, t) q_\lambda(t)$ satisfies \[1.44\], and $(\mu, q_\mu)$ is a bifurcation point of the problem \[1.43\] with respect to the trivial branch $\{(\lambda, q_\lambda) \mid \lambda \in \Lambda\}$ if and only if $(\mu, u_\mu)$ is that of \[1.44\] with respect to the trivial branch $\{(\lambda, u_\lambda) \mid \lambda \in \Lambda\}$. Let $\gamma_\lambda : [0, \tau] \to \text{Sp}(2n)$ be the fundamental matrix solution of $\dot{u}(t) = J\nabla_z^2 H(\lambda, t, u_\lambda(t)) u(t)$. It was stated in [38, page 66, line 2] that $i_{\lambda, \alpha}^{-}(\gamma_\lambda)$ and $u_{\lambda, \alpha}^{-}(\gamma_\lambda)$ are equal to $m^-(\Phi_\lambda, q_\lambda)$ and $m^0(\Phi_\lambda, q_\lambda)$, respectively. In the above conclusions in (I)-(III) immediately follow from Theorems \[1.35\] \[1.34\].

Example 1.36 (Bifurcation of the Bolza Problem). Split $z \in \mathbb{R}^{2n}$ into $(p^T, q^T)^T$, with $p$ and $q \in \mathbb{R}^n$. Fix two points $q_0, q_1 \in \mathbb{R}^n$. In Assumption \[1.32\] choose $w = (0, q_0^T)^T$, $w' = (0, q_1^T)^T$ and $L = L' = \{(p^T, q^T)^T \mid p \in \mathbb{R}^n, q = 0\}$. \[1.40\] becomes the following Bolza bifurcation problem:

\[
\begin{align*}
\dot{p} &= -H'_q(\lambda, t, p, q) \\
\dot{q} &= H'_p(\lambda, t, p, q) \\
u(0) &= q_0 \\
q(\tau) &= q_1.
\end{align*}
\]

Our final result is an analogy of Corollary \[1.49\]. Let $H : \mathbb{R}^{2n} \to \mathbb{R} = C^2$, and let $u : [0, \tau] \to \mathbb{R}^{2n}$ satisfy $\dot{u}(t) = J\nabla_z H(u(t))$. For two Lagrangian subspaces $L$ and $L'$ of $\mathbb{R}^{2n}$, and $0 < s \leq \tau$, we say $u(s)$ to be $(L, L')$-conjugate to $u(0)$ along $u$ if $\nu^L(\gamma) > 0$ for the fundamental matrix solution $\gamma : [0, s] \to \text{Sp}(2n)$ of $\dot{v}(t) = J\nabla_z^2 H(u(t)) v(t)$ on $[0, s]$. The number $\nu^L(\gamma)$ is called the multiplicity of $u(s)$ as a $(L, L')$-conjugate point to $u(0)$ along $u$. In particular, when $L' = L = u(s)$ is said to be $L$-conjugate to $u(0)$ along $u$ if $\nu_1(\gamma) > 0$ for the above $\gamma$, and the number $\nu_1(\gamma)$ is called the multiplicity of $u(s)$. We call $\mu \in (0, \tau]$ a bifurcation instant for $(H, u, L)$ if there exists a sequence $(\tau_k) \subset (0, \tau]$ converging to $\mu$ such that for each $k \in \mathbb{N}$ the boundary value problem

\[
\begin{align*}
\dot{v}(t) &= J\nabla_z H(u(t)) \quad \forall t \in [0, \tau_k], \\
v(0) &= u(0) + L, \\
v(\tau_k) &= u(\tau_k) + L
\end{align*}
\]

has a solution $v_k \neq u|_{[0, \tau_k]}$ such that $\|v_k - u\|_{C^1([0, \tau_k], \mathbb{R}^{2n})} \to 0$ as $k \to \infty$. 

Theorem 1.37. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be $C^2$, let $u : [0, \tau] \to \mathbb{R}^{2n}$ satisfy $\dot{u}(t) = J\nabla v H(u(t))$, let $\gamma : [0, \tau] \to \text{Sp}(2n)$ be the fundamental matrix solution of $\dot{v}(t) = J\nabla v H(u(t)) v(t)$ on $[0, \tau]$, and let $L$ be a Lagrangian subspace of $\mathbb{R}^{2n}$. Then:

(A) If $\mu \in (0, \tau]$ a bifurcation instant for $(H, u, L)$, then $u(\mu)$ is $L$-conjugate to $u(0)$ along $u$.

(B) Suppose that $\nabla^2 v H(u(t))$ is positive definite for all $t \in [0, \tau]$. Then

$$\Gamma(\mu) := \{ \lambda \in (0, \tau) \mid \dim(\gamma(\lambda)L) \cap L > 0 \}$$

contains at most finitely many points, that is, there exist at most infinitely many $L$-conjugate points to $u(0)$ along $u$; and each $\mu \in \Gamma(H, u, L)$ is a bifurcation instant for $(H, u, L)$, precisely one of the following alternatives occurs:

(B.i) The problem

$$\begin{align*}
\dot{v}(t) &= J\nabla v H(u(t)) \forall t \in [0, \mu], \\
v(0) &\in u(0) + L, \quad v(\mu) \in u(\mu) + L
\end{align*}$$

(1.46)

has a sequence of distinct solutions, $v_k \neq u|_{[0, \mu]}$, $k = 1, 2, \ldots$, such that $v_k \to u|_{[0, \mu]}$ in $C^1([0, \mu], \mathbb{R}^{2n})$ as $k \to \infty$.

(B.ii) For every $\lambda \in (0, \tau) \setminus \{ \mu \}$ near $\mu$, the problem

$$\begin{align*}
\dot{v}(t) &= J\nabla v H(u(t)) \forall t \in [0, \lambda], \\
v(0) &\in u(0) + L, \quad v(\lambda) \in u(\lambda) + L
\end{align*}$$

(1.47)

has a solutions $v_\lambda \neq u|_{[0, \mu]}$ such that $\|v_\lambda - u\|_{C^1([0, \lambda], \mathbb{R}^{2n})} \to 0$ as $\lambda \to \mu$.

(B.iii) There is an one-sided neighborhood $\Lambda^0$ of $\mu$ in $[0, \tau]$ such that for any $\lambda \in \Lambda^0 \setminus \{ \mu \}$ the problem

$$\begin{align*}
\dot{v}(t) &= J\nabla v H(u(t)) \forall t \in [0, \lambda], \\
v(0) &\in u(0) + L, \quad v(\lambda) \in u(\lambda) + L
\end{align*}$$

(1.46)

has at least two distinct solutions, $v^i_\lambda \neq u|_{[0, \lambda]}$, $i = 1, 2$, to satisfy the condition that $\|v^i_\lambda - u|_{[0, \lambda]}\|_{C^1([0, \lambda], \mathbb{R}^{2n})} \to 0$ as $\lambda \to \mu$, $i = 1, 2$. Moreover, if $\dim(\gamma(\lambda)L) \cap L > 1$ and the problem (1.46) has only finitely many distinct solutions near $u|_{[0, \lambda]}$ in $C^1([0, \lambda], \mathbb{R}^{2n})$ then the above two distinct solutions $v^i_\lambda$ can also be chosen to satisfy

$$\int_0^\lambda \left[ \frac{1}{2}(J\dot{v}_1^i(t), v_1^i(t))_{\mathbb{R}^{2n}} + H(v_1^i(t)) \right] \, dt \neq \int_0^\lambda \left[ \frac{1}{2}(J\dot{v}_2^i(t), v_2^i(t))_{\mathbb{R}^{2n}} + H(v_2^i(t)) \right] \, dt.$$

Finally, if $L$ is equal to $L_0 := \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ (resp. $L_1 := \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}$), the condition “$\nabla^2 v H(u(t))$ is positive definite for all $t \in [0, \tau]$” in (B) can be substituted with the condition “$B_{22}(t)$ (resp. $B_{11}(t)$) is positive definite for all $t \in [0, \tau]$”, where $\nabla^2 v H(u(t)) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}$, $B_{ij}(t) \in \mathbb{R}^{n \times n}$, $i, j = 1, 2$.

Remark 1.38. Let $L$ and $L'$ be two Lagrangian subspaces of $\mathbb{R}^{2n}$, and let $G : \mathbb{R}^{2n} \to \mathbb{R}$ be a $C^2$ function. Motivated by the notion of an orbit cylinder ([28 Definition 1.5]) a Hamiltonian path $v : [0, \tau] \to \mathbb{R}^{2n}$ of $X_G$ connecting $L$ and $L'$ is said to admit a path trapezoid if there exist $\varepsilon > 0$ together with a family $\mathcal{O} = (v_\lambda)_{\lambda \in ((-\varepsilon, \varepsilon) \cap \Lambda^0)}$ of Hamiltonian paths $v_\lambda : [0, T(\lambda))] \to \mathbb{R}^{2n}$ of $X_G = J\nabla G$ connecting $L$ and $L'$ with $v_0 = v$ and $T(0) = \tau$ such that $(-\varepsilon, \varepsilon) \ni \lambda \mapsto T(\lambda) \in \mathbb{R}$ is continuous and $H(v_\lambda) = H(v) + \lambda$ for all $\lambda \in (-\varepsilon, \varepsilon)$. Define

$$H : (-\varepsilon, \varepsilon) \times \mathbb{R}^{2n} \to \mathbb{R}, \quad (\lambda, z) \mapsto \frac{T(\lambda)}{\tau}G(z)$$

and $u_\lambda : [0, \tau] \to \mathbb{R}^{2n}$ by $u_\lambda(t) = v_\lambda(T(\lambda)/\tau)$ for $\lambda \in (-\varepsilon, \varepsilon)$. Then $H, u_\lambda$ satisfy Assumption 1.32 with $\Lambda = (-\varepsilon, \varepsilon)$ and $w_\lambda = 0 = u'_\lambda$ for all $\lambda$. Applying Theorems 1.33, 1.34 to them some interesting results may be obtained.
1.5. Research methods. Our proofs for the above results in Sections 1.1-1.4 are based on the abstract bifurcation theory recently developed by the author in [50, 52] through Morse theory methods. These theorems cannot be used for the associated functionals as in [1, 50] because all $\Phi_\lambda$ are strongly indefinite, that is, each critical point of them has infinite Morse index and co-index. One cannot directly study bifurcations of $\nabla \Phi_\lambda = 0$ by changes of the Morse indexes of the Hessians $\nabla^2 \Phi_\lambda$ at given solutions. There are some ways to overcome this difficulty. Using spectral flow methods as in [22, 30, 62, 63, 68, 69] we can only obtain necessary conditions and sufficient criteria, and cannot get alternative results of Rabinowitz type. There exists a powerful way to arrive at the latter aim, the saddle point reduction by Amann and Zehnder [2, 3]. This method is to reduce the original problem to a corresponding one in the finite dimension space via a Lyapunov-Schmidt reduction. For example, references [5, 6, 57] studied bifurcations of periodic solutions of Hamiltonian systems via this method. A detailed process will be demonstrated in the proofs of Theorems 1.33, 1.34 and 1.37 in Section 6.

Though the same methods can be used to prove the results in Sections 1.1-1.3, because of the lower smoothness for variational functionals in our abstract bifurcation theorems in [50, 52] it is possible to prove them via another way—the dual variational principle by Clarke and Ekeland. In order to show our ideas let us consider the case of Theorem 1.4. Under Assumption 1.1 each $u_\lambda$ is $C^2$ and continuously depends on $\lambda$. As done above the proof of Theorem 1.4(1) in Section 6 the problem is reduced to the case:

$$u_\lambda \equiv 0 \quad \forall \lambda \quad \text{and} \quad H \quad \text{satisfies (3.3)} \quad \text{and so (3.4). (1.48)}$$

Consider the Hilbert subspace

$$W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) := \{ u \in W^{1,2}([0, \tau]; \mathbb{R}^{2n}) | u(\tau) = Mu(0) \} \quad (1.49)$$

of $W^{1,2}([0, \tau]; \mathbb{R}^{2n})$, and $C^2$ functionals $\Phi_\lambda : W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \to \mathbb{R}$ defined by

$$\Phi_\lambda(v) = \int_0^\tau \left[ \frac{1}{2}(J\dot{v}(t), v(t))_{\mathbb{R}^{2n}} + H(\lambda, t, v(t)) \right] dt \quad (1.50)$$

under the assumption (1.48). The critical set of $\Phi_\lambda$ is exactly the solution set of (1.3). Let us choose $\kappa \in \mathbb{R}$ such that each $H_\kappa(\lambda, t, z) := H(\lambda, t, z) - \frac{\kappa}{2} |z|^2$ is strictly convex in $z$ and so that we may get the dual action of $\Phi_\lambda$, $\Psi_\kappa(\lambda, \cdot) : W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \to \mathbb{R}$ defined by

$$\Psi_\kappa(\lambda, v) = \int_0^\tau \left[ \frac{1}{2}(J\dot{v}(t) + \kappa v(t), v(t))_{\mathbb{R}^{2n}} + H_\kappa^*(\lambda, t; -J\dot{v}(t) - \kappa v(t)) \right] dt.$$ 

The latter functional is $C^1$, twice Gâteaux-differentiable, and has the same critical set as that of $\Phi_\lambda$ (Corollary 2.13). However, $\Psi_\kappa(\lambda, \cdot)$ is not $C^2$ in general. It is this reason that the Morse theory method for this dual functional has not been used well before. Our abstract bifurcation theorems in [50, 52], which were developed on the base of the author’s Morse theory for non-twice continuously differentiable functionals on Hilbert spaces (17, 48, 49), do not require that potential operators are $C^2$, and therefore make this way to become possible. With a Banach space isomorphism $\Lambda_{M,\tau,\kappa I_{2n}} : W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \to L^2([0, \tau]; \mathbb{R}^{2n})$ in (3.10) we consider the functional

$$\psi_\kappa(\lambda, \cdot) = \Psi_\kappa(\lambda, \cdot) \circ (-\Lambda_{M,\tau,\kappa I_{2n}})^{-1} : L^2([0, \tau]; \mathbb{R}^{2n}) \to \mathbb{R}$$

given by (5.11). Then the bifurcation problem of $\nabla\Phi_\lambda = 0$ near $(\mu, 0) \in \Lambda \times W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ is reduced to the bifurcation one of $\nabla\psi_\kappa(\lambda, \cdot) = 0$ near $(\mu, 0) \in \Lambda \times L^2([0, \tau]; \mathbb{R}^{2n})$. At each
critical point \( w \) of \( \psi_c(\lambda, \cdot) \), \( \nabla^2 \psi_c(\lambda, w) \) has finite Morse index and nullity, \( m^-(\psi_c(\lambda, \cdot), w) \) and \( m^0(\psi_c(\lambda, \cdot), w) \). By (2.32) and (2.33) it holds that
\[
\begin{align*}
m^-(\psi_c(\lambda, \cdot), 0) &= i_{r,M}(\gamma_\lambda) - i_{r,M}(\Upsilon_{rI,2n}) - \nu_{r,M}(\Upsilon_{rI,2n}),
m^0(\psi_c(\lambda, \cdot), 0) &= \nu_{r,M}(\gamma_\lambda) = \dim \ker(\gamma_\lambda(\tau) - M),
\end{align*}
\]
where \( \Upsilon_{rI,2n}(t) = \exp(t \kappa J) \), \( \gamma_\lambda : [0, \tau] \to \Sp(2n, \mathbb{R}) \) is the fundamental matrix solution of \( \dot{Z}(t) = J \nabla_c^2 H(\lambda, t, 0)Z(t) \), and \((i_{r,M}(\gamma_\lambda), \nu_{r,M}(\gamma_\lambda)) \) (resp. \((i_{r,M}(\Upsilon_{rI,2n}), \nu_{r,M}(\Upsilon_{rI,2n}))\)) is the \( M \)-Maslov-type index of \( \gamma_\lambda \) (resp. \( \Upsilon_{rI,2n} \)) defined in [19]. Using these we can prove that \( \mathcal{F}_\lambda(\cdot) = \psi_c(\lambda, \cdot) \), \( H = L^2([0, \tau]; \mathbb{R}^{2n}) \) and \( X = C_0^1([0, \tau]; \mathbb{R}^{2n}) \) satisfy conditions of [50] Theorem 3.1 (Theorem C.6) under the assumptions of Theorem 1.1.1. We can also prove that \( \mathcal{L}_\lambda(\cdot) = \psi_c(\lambda, \cdot) \), \( H = L^2([0, \tau]; \mathbb{R}^{2n}) \) and \( X = C_0^1([0, \tau]; \mathbb{R}^{2n}) \) satisfy conditions of [50] Theorem 3.6 (resp. [52] Theorem 3.6) under the assumptions of Theorem 1.1.1 (resp. Theorem 1.1.7).

By restricting to smaller variational spaces we may use the above methods to complete proofs for Theorems 1.14, 1.18, 1.19, 1.21 and Theorems 1.23, 1.24, 1.26. But the proofs of the latter three theorems are much more complex.

1.6. Further researches. There are a variety of natural continuations to this work:

(i) Some bifurcations for affine-periodic solutions of Lagrangian systems directly follow from results in this paper via the Legendre transform. But such methods seem not to be effective for getting analogues of Theorems 1.18, 1.19, 1.21. We can prove them with theories developed in [50, 51, 52], see [54].

(ii) In order to study the existence of periodic solutions to certain class of scalar delay differential equations, Kaplan and Yorke [33] introduced a new technique, that is, the original problem was reduced to finding periodic solutions to an associated (generalized) Hamiltonian system. Our theories and methods in this paper can be used to derive some bifurcation results for delay (Hamiltonian) equations, [53], and for homoclinic orbits of Hamiltonian systems and solutions of other nonlinear (e.g. Dirac and wave) equations [54].

(iii) As done by Ciriza [14], we may generalize results in this paper to Hamiltonian systems on general symplectic manifolds with a parametrized version of Darboux’s theorem (cf. [66, §9]). We can also combine our methods with that of [34]. They would appear elsewhere.

1.8. Organization of the article. In Section 2 we collect some preliminaries, including a review about properties of a perturbation Hamiltonian operator (Section 2.1) and statements of Clarke-Ekeland dual variational principles and their versions used in this paper (Section 2.2). Section 3 proves Theorems 1.1, 1.7 and Corollaries 1.9, 1.10. Section 4 is devoted to the proofs of Theorems 1.13, 1.18, 1.19, 1.21 and Corollary 1.16. Section 5 proves Theorems 1.23, 1.24, 1.26 and their Corollaries 1.25, 1.29. In Section 6 we complete the proofs of Theorems 1.33, 1.34 and 1.37. For completeness, we add three appendixes. In Appendix A we first briefly review various related Maslov-type indexes for symplectic paths and their relations to Morse indexes, and then give some relations (Theorems A.4, A.8) between Morse indexes and the Maslov-type indexes defined by Long, Zhang and Zhu [60]. In Appendix B we present the proof of Proposition 1.3. In Appendix C we first give a remark to clarify how the compactness assumption of the parameter topology space \( \Lambda \) [65, Theorems A.1, A.2] was precisely used. Then we prove generalizations and refinements of [62, Theorems 3.2, 3.5] in great detail.
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2 Preliminaries

2.1. Sobolev spaces and a perturbation Hamiltonian operator. We begin with the following elementary fact of linear functional analysis.

**Proposition 2.1.** Let $H$ be a Hilbert space with norm $\| \cdot \|$ and let $A : D(A) \subset H \to H$ be an unbounded linear operator that is densely defined and closed. Suppose that $A$ is self-adjoint and that there exist a complete norm $| \cdot |$ on $D(A)$ for which the inclusion $\iota : (D(A), | \cdot |) \to H$ is compact and $A : (D(A), | \cdot |) \to H$ is continuous. Then $\dim \ker(A) < \infty$, $R(A)$ is closed in $H$ and there exists an orthogonal decomposition $H = \ker(A) \oplus R(A)$. Moreover

$$\dot{A} : (D(A) \cap R(A), | \cdot |) \to (R(A), \| \cdot \|), \ x \mapsto Ax$$

is invertible and $\iota \circ \dot{A}^{-1}$ as an operator on the Hilbert subspace $R(A)$ of $H$ is compact and self-adjoint.

The first two claims follow from [10, Exercise 6.9] directly. Since $A$ is self-adjoint, by [10, Theorem 2.19] we have $(\ker(A))^\perp = R(A^*) = R(A)$. Other conclusions are obvious.

Let $L^2([0, \tau]; \mathbb{R}^{2n}) = (L^2([0, \tau]; \mathbb{R})^n)^2$ and $W^{1,2}([0, \tau]; \mathbb{R}^{2n}) = (W^{1,2}([0, \tau]; \mathbb{R}))^n$ be the Hilbert spaces equipped with $L^2$-inner product and $W^{1,2}$-inner product

$$(u, v)_{L^2} = \int_0^\tau (u(t), v(t))_{\mathbb{R}^{2n}} dt, \quad (u, v)_{W^{1,2}} = \int_0^\tau [(u(t), v(t))_{\mathbb{R}^{2n}} + (\dot{u}, \dot{v})_{\mathbb{R}^{2n}}] dt,$$

respectively. The corresponding norms are denoted by $\| \cdot \|_2$ and $\| \cdot \|_{1,2}$, respectively. (As usual each $u \in L^2([0, \tau]; \mathbb{R}^{2n})$ will be identified with any fixed representative of it; in particular, we do not distinguish $u \in W^{1,2}([0, \tau]; \mathbb{R}^{2n})$ with its unique continuous representation.)

For $A \in L^\infty([0, \tau]; L_2(\mathbb{R}^{2n}))$ and a symplectic matrix $M \in \mathrm{Sp}(2n, \mathbb{R})$ let $\Lambda_{M, \tau, A}$ be the operator on $L^2([0, \tau]; \mathbb{R}^{2n})$ defined by

$$(\Lambda_{M, \tau, A}u)(t) := J \frac{d}{dt}u(t) + A(t)u(t)$$

with domain

$$\text{dom}(\Lambda_{M, \tau, A}) = W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}).$$

Denote by $\Upsilon_A$ the fundamental solution of the linear Hamiltonian system

$$\dot{x} = JA(t)x \quad \forall t \in [0, \tau].$$

Clearly, $\ker(\Lambda_{M, \tau, A}) = \{ \Upsilon_A(\cdot)\xi | \xi \in \mathcal{E}_{M, \tau, A} \}$, where

$$\mathcal{E}_{M, \tau, A} = \ker(\Upsilon_A(\tau) - M) \quad \text{and} \quad \mathcal{E}_{M, \tau, A}^\perp = \{ u \in \mathbb{R}^{2n} | (u, v)_{\mathbb{R}^{2n}} = 0 \ \forall v \in \mathcal{E}_{M, \tau, A} \}.$$

Note that $\Upsilon_{\kappa J}(t) = e^{\kappa t J}$ for a real constant $\kappa$. 

It was proved in Lemma 4 of [20, page 102] that the operator $\Lambda_{M,\tau,0}$ is closed and self-adjoint. But $\Lambda_{M,\tau,A}$ is the sum of $\Lambda_{M,\tau,0}$ and the following continuous linear self-adjoint operator

$$L^2([0, \tau]; \mathbb{R}^{2n}) \to L^2([0, \tau]; \mathbb{R}^{2n}), \ u \mapsto A(t)u.$$ 

An elementary functional analysis fact (cf. [10, Exercise 2.20]) shows that $\Lambda_{M,\tau,0}$ is closed and self-adjoint. Hence Proposition 2.1 leads to the following result, which was directly proved by Dong [10, Proposition 2.1].

**Proposition 2.2.** $\Lambda_{M,\tau,A}$ has the closed range $R(\Lambda_{M,\tau,A})$ in $L^2([0, \tau]; \mathbb{R}^{2n})$, the kernel $\text{Ker}(\Lambda_{M,\tau,A})$ is of dimension at most $2n$, and there exists an orthogonal decomposition

$$L^2([0, \tau]; \mathbb{R}^{2n}) = \text{Ker}(\Lambda_{M,\tau,A}) \oplus R(\Lambda_{M,\tau,A}).$$

Moreover, the restriction $\tilde{\Lambda}_{M,\tau,A}$ of $\Lambda_{M,\tau,A}$ to $R(\Lambda_{M,\tau,A}) \cap \text{dom}(\Lambda_{M,\tau,A})$ is invertible and the inverse $(\tilde{\Lambda}_{M,\tau,A})^{-1} : R(\Lambda_{M,\tau,A}) \to R(\Lambda_{M,\tau,A})$ is compact and self-adjoint if $R(\Lambda_{M,\tau,A})$ is endowed with the $L^2$-norm.

For $v(t) = \Upsilon_A(t)\xi$ with $\xi \in \mathbb{R}^{2n}$, and $u \in L^2([0, \tau]; \mathbb{R}^{2n})$,

$$(u, v)_2 = -\int_0^\tau (J\Upsilon_A(t)^{-1}Ju(t), \xi)_{\mathbb{R}^{2n}}dt = -\left(\int_0^\tau J\Upsilon_A(t)^{-1}Ju(t)dt, \xi\right)_{\mathbb{R}^{2n}}$$

implies

$$\tilde{L}^2_{M,A}([0, \tau]; \mathbb{R}^{2n}) := R(\Lambda_{M,\tau,A}) = \left\{ u \in L^2([0, \tau]; \mathbb{R}^{2n}) \mid \int_0^\tau J\Upsilon_A(t)^{-1}Ju(t)dt \in \mathcal{E}_{M,\tau,A}^\perp \right\}.$$

Moreover, for $w \in W^{1,2}([0, \tau]; \mathbb{R}^{2n})$ and $v(t) = \Upsilon_A(t)\xi$ with $\xi \in \mathbb{R}^{2n}$, it holds that

$$(w, v)_{1,2} = \int_0^\tau (w(t), \Upsilon_A(t)\xi)_{\mathbb{R}^{2n}}dt + \int_0^\tau (\dot{w}(t), JA(t)\Upsilon_A(t)\xi)_{\mathbb{R}^{2n}}dt$$

$$= \int_0^\tau (-A(t)J\dot{w}(t) + w(t), \Upsilon_A(t)\xi)_{\mathbb{R}^{2n}}dt$$

$$= \int_0^\tau (J\Upsilon_A(t)^{-1}J^{-1}[-A(t)J\dot{w}(t) + w(t)], \xi)_{\mathbb{R}^{2n}}dt$$

$$= \int_0^\tau (-J\Upsilon_A(t)^{-1}J[-A(t)J\dot{w}(t) + w(t)], \xi)_{\mathbb{R}^{2n}}dt$$

$$= \int_0^\tau (J\Upsilon_A(t)^{-1}J[A(t)J\dot{w}(t) - w(t)], \xi)_{\mathbb{R}^{2n}}dt.$$

It follows that the orthogonal complementarity of $\text{Ker}(\Lambda_{M,\tau,A})$ in $W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$,

$$\tilde{W}^{1,2}_{M,A}([0, \tau]; \mathbb{R}^{2n}) = \left\{ w \in W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \mid \int_0^\tau J\Upsilon_A(t)^{-1}J[A(t)J\dot{w}(t) - w(t)]dt \in \mathcal{E}_{M,\tau,A}^\perp \right\}.$$

That is, we have an orthogonal decomposition of Hilbert spaces

$$W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) = \text{Ker}(\Lambda_{M,\tau,A}) \oplus \tilde{W}^{1,2}_{M,A}([0, \tau]; \mathbb{R}^{2n}).$$

(2.5) Clearly, $\Lambda_{M,\tau,A}$ restricts to a Banach space isomorphism

$$\tilde{\Lambda}_{M,\tau,A} : \tilde{W}^{1,2}_{M,A}([0, \tau]; \mathbb{R}^{2n}) \to \tilde{L}^2_{M,A}([0, \tau]; \mathbb{R}^{2n}).$$
It was explicitly written out in [19]. Indeed, by the orthogonal splitting
\[ \mathbb{R}^{2n} = (I_{2n} - \mathcal{Y}_A(\tau)^{-1}M)(\mathbb{R}^{2n}) \oplus \text{JKer}(\mathcal{Y}_A(\tau) - M) \]
(19. (3.8)), \( I_{2n} - \mathcal{Y}_A(\tau)^{-1}M \) restricts to an invertible operator from \( (I_{2n} - \mathcal{Y}_A(\tau)^{-1}M)(\mathbb{R}^{2n}) \) to itself. Therefore we can define
\[ \mathfrak{J} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \; x \mapsto \mathfrak{J}x = (I_{2n} - \mathcal{Y}_A(\tau)^{-1}M)x_1 + x_2, \]
where \( x = x_1 + x_2 \) with \( x_1 \in (I_{2n} - \mathcal{Y}_A(\tau)^{-1}M)(\mathbb{R}^{2n}) \) and \( x_2 \in \text{JKer}(\mathcal{Y}_A(\tau) - M) \). Then \( \mathfrak{J} \) is invertible. By [19, (3.6)-(3.7)] we have for \( u \in L^2_{M,A}([0, \tau]; \mathbb{R}^{2n}) \),
\[ [(\Lambda_{M,\tau})^{-1}u](t) = \mathcal{Y}_A(t)\mathfrak{J}^{-1} \int^t_0 \mathcal{Y}_A(t)^{-1}Ju(t)dt - \mathcal{Y}_A(t) \int^t_0 \mathcal{Y}_A(s)^{-1}Ju(s)ds. \]  
Clearly, if \( E_{M,\tau,A} = \{0\} \), i.e., \( \det(\mathcal{Y}_A(\tau) - M) \neq 0 \), then \( \Lambda_{M,\tau,A} \) is a Banach space isomorphism from \( W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \) onto \( L^2([0, \tau]; \mathbb{R}^{2n}) \).

2.2. Clarke-Ekeland adjoint-action principle. Let \( X \) be a Banach space. For a function \( G : X \to \mathbb{R} \cup \{+\infty\} \), not identically \( +\infty \), not necessarily convex, its Legendre-Fenchel conjugate is the function \( G^* : X^* \to \mathbb{R} \cup \{+\infty\} \) defined by
\[ G^*(u^*) = \sup\{\langle u^*, u \rangle - G(u) \mid u \in X\} \]  
with effective domain \( D(G^*) = \{u^* \in X^* \mid G^*(u^*) < +\infty\} \). It is convex and semi-continuous.

From now on, we assume that \( X \) is a reflexive Banach space, i.e., James map \( J_X : X \to X^{**} \) is an Banach space isomorphism. Let \( A : D(A) \subset X \to X^* \) be a closed and densely defined unbounded linear self-adjoint operator, and let \( F : X \to \mathbb{R} \cup +\infty \) be a convex lower semi-continuous (l.s.c.) function, not identically \( +\infty \). Define a functional \( \Phi : X \to \mathbb{R} \cup \{+\infty\} \) and its dual functional \( \Psi : D(A) \to \mathbb{R} \cup \{+\infty\} \) by
\[ \Phi(v) = \frac{1}{2} \langle Av, v \rangle_{X^*,X} + F(v) \quad \text{and} \quad \Psi(x) = \frac{1}{2} \langle Ax, x \rangle + [F^* \circ (-A)](x), \]
where \( F^* \circ (-A) : X \to \mathbb{R} \cup \{+\infty\} \) is defined by
\[ (F^* \circ (-A))(x) = \begin{cases} F^*(-Ax) \text{ if } x \in D(A), \\ +\infty \text{ otherwise.} \end{cases} \]
When \( x \in D(A) \) satisfies the condition that \( 0 \in Ax + \partial F(x) \) (resp. \( 0 \in Ax + \partial[F^* \circ (-A)](x) \)), it is called a critical point of \( \Phi \) (resp. \( \Psi \)), where \( \partial F(x) \) is the subgradient of \( F \) at \( x \in X \) defined by
\[ \partial F(x) = \{u^* \in X^* \mid F(u) \geq F(x) + \langle u^*, u - x \rangle \forall u \in X\}. \]

The Clarke duality was introduced by Clarke [15] and developed by Clarke-Ekeland [16]. The following general abstract result was given by Ekeland [20].

**Theorem 2.3** ([20, page 100, Theorem 2]). Let the above assumptions be satisfied. If \( \bar{u} \in D(A) \) is a critical point of \( \Phi \), then all \( u \in \text{Ker}(A) + \bar{u} \) are critical points of \( \Psi \) and \( \Psi(u) = -\Phi(\bar{u}) \). Conversely, if \( \bar{v} \in D(A) \) is a critical point of \( \Psi \), and
\[ 0 \in \text{Int}(D(F^*) - R(A)) \]
then there is some \( \bar{w} \in \text{Ker}(A) \) such that \( \bar{u} := \bar{v} - \bar{w} \) is a critical point of \( \Phi \) with \( \Phi(\bar{u}) = -\Psi(\bar{v}) \).
Let us state three convenient corollaries of this theorem.
By [20] page 92, Proposition 9] and Theorem [23] we may deduce:

**Corollary 2.4.** Under the assumptions of Theorem 2.3, suppose that $F : X \to \mathbb{R} \cup \{+\infty\}$ (resp. $F^* : X \to \mathbb{R} \cup \{+\infty\}$) is Gateaux-differentiable at each point in $D(A)$ (resp. $R(A)$), and that $D(A)$ itself is a Banach space with norm $\| \cdot \|$ such that

$$i : (D(A), | \cdot |) \to X \quad \text{and} \quad A : (D(A), | \cdot |) \to X^*$$

are continuous, which implies

(i) $F : (D(A), | \cdot |) \to \mathbb{R}$ and so $\Phi : (D(A), | \cdot |) \to \mathbb{R}$ is Gateaux-differentiable,

(ii) $F^* \circ (-A) : (D(A), | \cdot |) \to \mathbb{R}$ (and so $\Psi : (D(A), | \cdot |) \to \mathbb{R}$) is Gateaux-differentiable.

Then for any critical point $\bar{u}$ of $\Phi$ on $(D(A), | \cdot |)$, each $u \in \text{Ker}(A) + \bar{u}$ is a critical point of $\Psi$ on $(D(A), | \cdot |)$ and $\Psi(u) = -\Phi(\bar{u})$. Conversely, if $0 \in \text{Int}(D(F^*) - R(A))$, then for any critical point $\bar{v}$ of $\Psi$ on $(D(A), | \cdot |)$, there is some $\bar{w} \in \text{Ker}(A)$ such that $\bar{u} := \bar{v} - \bar{w}$ is a critical point of $\Phi$ on $(D(A), | \cdot |)$ with $\Phi(\bar{u}) = -\Psi(\bar{v})$.

Suppose that $F \in C^1(X)$ is strictly convex, i.e., $\langle v - w, F'(v) - F'(w) \rangle > 0$ if $v \neq w$. Then $F'$ is injective, $F^*$ is finite on the range of $F'$, and $\partial F^*(F'(v)) = \{v\}$ for any $v \in X$. If, in addition, $F'$ is strongly monotone and coercive in the sense that

$$\langle v - w, F'(v) - F'(w) \rangle \geq \alpha(\|v - w\|)\|v - w\| \quad \forall v, w \in X,$$

where $\alpha : [0, \infty) \to [0, \infty)$ is a non-decreasing function vanishing only at 0 and such that $\alpha(r) \to \infty$ as $r \to \infty$, then $F^* \in C^1(X^*)$ and $F'$ is a homeomorphism of $X$ onto $X^*$.

Similarly, [20] page 92, Proposition 9] and Theorem [23] may give rise to:

**Corollary 2.5.** Let $A : D(A) \subset X \to X^*$ be as in Theorem 2.3, and let $F \in C^1(X)$ be strictly convex and $F'$ be strongly monotone. Suppose that $D(A)$ itself is a Banach space with norm $| \cdot |$ and

$$i : (D(A), | \cdot |) \to X \quad \text{and} \quad A : (D(A), | \cdot |) \to X^*$$

are continuous, which implies that $F^* \circ (-A) : (D(A), | \cdot |) \to \mathbb{R}$ (and so $\Psi : (D(A), | \cdot |) \to \mathbb{R}$) is $C^1$. Then for a critical point $\bar{u}$ of $\Phi$ on $(D(A), | \cdot |)$, each $u \in \text{Ker}(A) + \bar{u}$ is a critical point of $\Psi$ on $(D(A), | \cdot |)$ and $\Psi(u) = -\Phi(\bar{u})$. Conversely, if $\bar{v}$ is a critical point of $\Psi$ on $(D(A), | \cdot |)$, then there is some $\bar{w} \in \text{Ker}(A)$ such that $\bar{u} := \bar{v} - \bar{w}$ is a critical point of $\Phi$ on $(D(A), | \cdot |)$ with $\Phi(\bar{u}) = -\Psi(\bar{v})$.

Using [12] Lemma 2.1] we can also derive from Theorem 2.3:

**Corollary 2.6.** Under the assumptions of Theorem 2.3, suppose that $D(A)$ itself is a Banach space with norm $| \cdot |$ such that the inclusion $i : (D(A), | \cdot |) \to X$ is continuous. Assume also that both $F$ and $F^*$ are convex and locally Lipschitz continuous functions on $X$. Then for a critical point $\bar{u} \in D(A)$ of $\Phi$ on $(D(A), | \cdot |)$, all $u \in \text{Ker}(A) + \bar{u}$ are critical points of $\Psi$ on $(D(A), | \cdot |)$ and satisfy $\Psi(u) = -\Phi(\bar{u})$. Conversely, if $\bar{v} \in D(A)$ is a critical point of $\Psi$ on $(D(A), | \cdot |)$, and [2.3] holds, then there is some $\bar{w} \in \text{Ker}(A)$ such that $\bar{u} := \bar{v} - \bar{w}$ is a critical point of $\Phi$ on $(D(A), | \cdot |)$ with $\Phi(\bar{u}) = -\Psi(\bar{v})$.

By [10] Exercise 2.20] it is easy to deduce the following result, which can be used to enlarge applications of the above theorems and corollaries.
Proposition 2.7. Let $X$ be a reflexive Banach space, $T \in \mathcal{L}(X, X^*)$ and let $A : D(A) \subset X \to X^*$ be an unbounded linear operator that is densely defined and closed. Suppose that $T$ and $A$ are self-adjoint. Then the operator $B : D(B) \subset X \to X^*$ defined by $D(B) = D(A)$ and $B = A + T$ is also closed, and self-adjoint.

Let $\Omega$ be some borelian subset of $\mathbb{R}^n$ having finite Lebesgue measure. Assume that a Borel function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies for a.e. $\omega \in \Omega$:

- $f(\omega, \cdot)$ is bounded from below, \hspace{1cm} (2.9)
- $f(\omega, \cdot)$ is lower semi-continuous (l.s.c.), \hspace{1cm} (2.10)
- $f(\omega, \cdot)$ is convex. \hspace{1cm} (2.11)

Write $f^*(\omega; \xi) = (f(\omega, \cdot))^*(\xi)$. As noted in the first two lines on page 96] we have:

Proposition 2.8 ([20 Chapter II, §3, Theorem 2, Corollary 3]). Let $\Omega \subset \mathbb{R}^n$ be as above, and let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Borel function, $1 \leq \alpha \leq +\infty$ and $\beta = \alpha/(\alpha - 1)$. Define the functional $F : L^\alpha(\Omega; \mathbb{R}^N) \to \mathbb{R}$ by

$$F(u) = \int_\Omega f(x, u(x))dx, \hspace{1cm} u \in L^\alpha(\Omega; \mathbb{R}^N).$$

(1) If $\alpha \in (1, +\infty)$ and $f$ satisfies (2.9)-(2.11), then $F$ is l.s.c.; if $\alpha = 1$ and $f$ satisfies (2.10)-(2.11) then $F$ is l.s.c for $\sigma(L^\infty, L^1)$ provided that there exists $\bar{u} \in L^\infty(\Omega; \mathbb{R}^N)$ such that

$$\int_\Omega |f(x, \bar{u}(x))|dx < +\infty. \hspace{1cm} (2.12)$$

(2) If $f$ satisfies (2.11) then $F$ is convex.

(3) Assume that $f$ satisfies (2.10)-(2.11), and (2.13) for some $\bar{u} \in L^\infty(\Omega; \mathbb{R}^N)$, and that $f$ also satisfies either (2.9) or

$$\int_\Omega |f^*(x; \bar{u}(x))|dx < +\infty \hspace{1cm} (2.13)$$

for some $\bar{v} \in L^\beta(\Omega; \mathbb{R}^N)$. Then

$$F^*(u) = \int_\Omega f^*(x; u(x))dx, \hspace{1cm} \forall u \in L^\beta(\Omega; \mathbb{R}^N).$$

(4) Assume that $f$ satisfies (2.10)-(2.11), and (2.12) and (2.13) for some $\bar{u}, \bar{v} \in L^\infty(\Omega; \mathbb{R}^N)$. Then

$$\partial F(u) = \{x^* \in L^\beta(\Omega; \mathbb{R}^N) \mid x^*(\omega) = \partial f(\omega, u(\omega)) \text{ a.e.}\}.$$ 

Proposition 2.9. Let $H : [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R}^2$ be measurable in $t$ for each $z \in \mathbb{R}^{2n}$ and strictly convex and continuously differentiable in $z$ for almost every $t \in [0, \tau]$. Denote by $H^*(t; \cdot)$ the Fenchel conjugate of $H(t, \cdot)$ as defined by (2.7). (It is also strictly convex and continuously differentiable.) Assume that there exists $\alpha > 0$, $\delta > 0$, and $\beta, \gamma \in L^2([0, \tau]; \mathbb{R}^+)$, such that for a.e. $t \in [0, \tau]$ and every $z \in \mathbb{R}^{2n}$, one has

$$\delta|z|^2 - \beta(t) \leq H(t, z) \leq \alpha|z|^2 + \gamma(t). \hspace{1cm} (2.14)$$
Then the dual of the functional

\[ \mathcal{H} : L^2([0, \tau]; \mathbb{R}^{2n}) \to \mathbb{R}, \quad u \mapsto \int_0^\tau H(t, u(t))dt \]

is given by

\[ \mathcal{H}^* : L^2([0, \tau]; \mathbb{R}^{2n}) \to \mathbb{R} \cup \{+\infty\}, \quad u \mapsto \int_0^\tau H^*(t; u(t))dt. \]

Moreover, both \( \mathcal{H} \) and \( \mathcal{H}^* \) are convex, bounded on any bounded subsets, and of class \( C^1 \), and

\[ \nabla \mathcal{H}(u) = \nabla z H(\cdot, u(\cdot)) \quad \text{and} \quad \nabla \mathcal{H}^*(u) = \nabla z H^*(\cdot; u(\cdot)). \tag{2.15} \]

**Proof.** By \([2.14]\) and \([57] \ Proposition 2.4\), \( H^*(t; z) \) for a.e. \( t \in [0, \tau] \) is strictly convex and continuously differentiable in \( z \in \mathbb{R}^{2n} \), and also satisfies

\[ \alpha^{-1}|z|^2 - \gamma(t) \leq H^*(t; z) \leq \delta^{-1}|z|^2 + \beta(t). \tag{2.16} \]

Note that \([2.14]\) and \([2.16]\) imply

\[ |H(t, z)| \leq \beta(t) + \gamma(t) + \alpha|z|^2 \quad \text{and} \quad |H^*(t; z)| \leq \beta(t) + \gamma(t) + \delta^{-1}|z|^2, \]

respectively. Thus

\[ \int_0^\tau H(t, 0) \leq \int_0^\tau (\beta(t) + \gamma(t))dt < +\infty \quad \text{and} \quad \int_0^\tau H^*(t; 0) \leq \int_0^\tau (\beta(t) + \gamma(t))dt < +\infty. \]

For any \( u \in L^2([0, \tau]; \mathbb{R}^{2n}) \) we derive from (3)-(4) in Proposition \(2.8\) that

\[ \mathcal{H}^*(u) = \int_0^\tau H^*(t; u(t))dt, \]

\[ \partial \mathcal{H}(u) = \{ x^* \in L^2([0, \tau]; \mathbb{R}^{2n}) | x^*(t) \in \partial_z H(t, u(t)) \ \text{a.e.} \} = \{ \nabla z H(\cdot, u(\cdot)) \}, \]

\[ \partial \mathcal{H}^*(u) = \{ x^* \in L^2([0, \tau]; \mathbb{R}^{2n}) | x^*(t) \in \partial_z H^*(t; u(t)) \ \text{a.e.} \} = \{ \nabla z H^*(\cdot; u(\cdot)) \}. \]

By these and \([20] \ page 92, Proposition 9\) we obtain that both \( \mathcal{H} \) and \( \mathcal{H}^* \) are Gateaux-differentiable at each \( u \in L^2([0, \tau]; \mathbb{R}^{2n}) \) and their gradients are given by \([2.15]\).

It follows from \([2.14], \ [2.16]\) and a theorem of Krasnosel’skii (cf. \([20] \ page 98, Corollary 5\)) that the operators \( \mathbf{H}, \mathbf{H}_s : L^2([0, \tau]; \mathbb{R}^{2n}) \to L^1([0, \tau]; \mathbb{R}) \) defined by

\[ \mathbf{H}(u)(t) = H(t, u(t)) \quad \text{and} \quad \mathbf{H}_s(u)(t) = H(t, u(t)) \]

are continuous and maps bounded sets into bounded ones. Hence the functionals \( \mathcal{H} \) and \( \mathcal{H}^* \) are convex, continuous and bounded on any bounded subsets.

Finally, as in the proof of \([57] \ Theorem 2.3\), \([2.14]\) and \([2.16]\) may lead to

\[ |\nabla z H(t, z)| \leq 1 + \alpha|z| + \alpha(\beta(t) + \gamma(t)), \tag{2.17} \]

\[ |\nabla z H^*(t; z)| \leq 1 + \delta^{-1}|z| + \delta^{-1}(\beta(t) + \gamma(t)), \tag{2.18} \]

respectively. As above we derive that both \( \nabla \mathcal{H} \) and \( \nabla \mathcal{H}^* \) are continuous. Hence \( \mathcal{H} \) and \( \mathcal{H}^* \) are of class \( C^1 \). \( \square \)
Proposition 2.10. Under the assumptions of Proposition 2.9, suppose also that \( H \) is twice continuously differentiable in \( z \) for almost every \( t \in [0, \tau] \), and there exist constants \( C_2 > C_1 > 0 \) such that for a.e. \( t \in [0, \tau] \),
\[
C_1|\eta|^2 \leq (\nabla_2^2 H(t, \xi)\eta, \eta)_{\mathbb{R}^{2n}} \leq C_2|\eta|^2, \quad \forall \xi, \eta \in \mathbb{R}^{2n}.
\]
(2.19)
Then both \( \nabla H \) and \( \nabla H^* \) are Gateaux differentiable, strongly monotone and coercive, and therefore homeomorphisms from \( L^2([0, \tau]; \mathbb{R}^{2n}) \) to itself. In particular, \( H \) and \( H^* \) are twice Gateaux differentiable, and
\[
\mathcal{H}''(x)[u, v] = (D(\nabla H)(x)[u], v)_{\mathbb{R}^{2n}} = \int_0^\tau (\nabla_2^2 H(t, x(t))u(t), v(t))_{\mathbb{R}^{2n}} dt,
\]
\[
(\mathcal{H}^*)''(x)[u, v] = (D(\nabla H^*)(x)[u], v)_{\mathbb{R}^{2n}} = \int_0^\tau (\nabla_2^2 H^*(t; x(t))u(t), v(t))_{\mathbb{R}^{2n}} dt
\]
for all \( x, u, v \in L^2([0, \tau]; \mathbb{R}^{2n}) \). Clearly, both \( D(\nabla H)(x) \) and \( D(\nabla H^*)(x) \) are positive definite.

Proof. For \( s \in (-1, 1) \setminus \{0\} \), using the mean value theorem we get
\[
\left( \frac{1}{s} (\nabla H(x + su) - \nabla H(x)) - \nabla_2^2 H(\cdot, x(\cdot))u(\cdot), v \right)_2 = \int_0^\tau \left( \frac{1}{s} (\nabla H(t, x(t) + su(t)) - \nabla H(t, x(t))) - \nabla_2^2 H(t, x(t))u(t), v(t) \right)_{\mathbb{R}^{2n}} dt
\]
\[
= \int_0^\tau (\nabla_2^2 H(t, x(t) + \theta(s, t) su(t))u(t) - \nabla_2^2 H(t, x(t))u(t), v(t))_{\mathbb{R}^{2n}} dt
\]
for some function \( (s, t) \mapsto \theta(s, t) \in (0, 1) \). This leads to
\[
\left\| \frac{1}{s} (\nabla H(x + su) - \nabla H(x)) - \nabla_2^2 H(\cdot, x(\cdot))u(\cdot) \right\|_2 \leq \left( \int_0^\tau \left| \nabla_2^2 H(t, x(t) + \theta(s, t) su(t))u(t) - \nabla_2^2 H(t, x(t))u(t) \right|_{\mathbb{R}^{2n}}^2 dt \right)^{1/2}.
\]
(2.20)
Since \( \left| \nabla_2^2 H(t, x(t) + \theta(s, t) su(t))u(t) - \nabla_2^2 H(t, x(t))u(t) \right|^2_{\mathbb{R}^{2n}} \leq 4C_2|u(t)|_{\mathbb{R}^{2n}}^2 \), by the Lebesgue dominated convergence theorem the integration in (2.20) converges to zero as \( s \to 0 \). Hence
\[
D(\nabla H)(x)[u] = \nabla_2^2 H(\cdot, x(\cdot))u(\cdot).
\]
Moreover, (2.19) implies that for a.e. \( t \in [0, \tau] \), \( \nabla_2 H(t, \cdot) \) is strongly monotone and coercive, and therefore \( \nabla_2 H(t, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) is a homeomorphism. By [20] page 92, Proposition 10, for \( \xi \in \mathbb{R}^{2n} \) and \( \xi \) holds that \( \xi = \nabla_2 H^*(t; \xi) \) and \( I_2 = \nabla_2^2 H^*(t; \xi) \nabla_2^2 H(t, \xi) \). These and (2.19) imply
\[
C_2^{-1}|\eta|^2 \leq (\nabla_2^2 H^*(t; \xi)\eta, \eta)_{\mathbb{R}^{2n}} \leq C_1^{-1}|\eta|^2, \quad \forall \xi, \eta \in \mathbb{R}^{2n}.
\]
(2.21)
As above, we can derive from this that \( \nabla H^* \) is Gateaux differentiable and
\[
D(\nabla H^*)(x)[u] = \nabla_2^2 H^*(\cdot, x(\cdot))u(\cdot).
\]
Other claims follow from (2.19) and (2.21).
The following is a generalization of \cite{57} Theorem 2.3.

**Theorem 2.11.** Let $H : [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R}$ be as in Proposition 2.10, $M \in \text{Sp}(2n, \mathbb{R})$, and $K \in L^\infty([0, \tau]; \mathcal{L}_s(\mathbb{R}^{2n}))$. Then functionals

\[
\Phi_{H,K}(v) = \int_0^\tau \left[ \frac{1}{2}(J\dot{v}(t) + K(t)v(t), v(t))_{\mathbb{R}^{2n}} + H(t, v(t)) \right] dt, \quad (2.22)
\]

\[
\Psi_{H,K}(v) = \int_0^\tau \left[ \frac{1}{2}(J\dot{v}(t) + K(t)v(t), v(t))_{\mathbb{R}^{2n}} + H^*(t; -J\dot{v}(t) - K(t)v(t)) \right] dt \quad (2.23)
\]

are $C^1$ and twice Gâteaux differentiable on $W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$, and the following holds:

(i) If $\bar{u} \in W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ is a critical point of $\Phi_{H,K}$, i.e., a solution of

\[
J\dot{u}(t) = -K(t)u(t) - \nabla_z H(t, u(t)) \quad \forall t \in [0, \tau] \quad \text{and} \quad u(\tau) = Mu(0), \quad (2.24)
\]

then each $u \in \bar{u} + \text{Ker}(\Lambda_{M,\tau,K})$ is also a critical point of $\Psi_{H,K}$ on $W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ and $\Psi_{H,K}(u) = -\Phi_{H,K}(\bar{u})$. Conversely, if $\bar{v} \in W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ is a critical point of $\Psi_{H,K}$, then there exists a unique $\xi \in \mathcal{E}_{M,\tau,K}$ such that $\bar{v} := \bar{v} - v_\xi$, where $v_\xi(t) = \Upsilon_K(t)\xi$, is a critical point of $\Phi_{H,K}$ on $W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ and satisfies $\Phi_{H,K}(\bar{u}) = -\Psi_{H,K}(\bar{v})$; this unique $\xi$ is equal to $(\Upsilon_K(t))^{-1}(\bar{v}(t) - \nabla_z H^*(t; -J\dot{\bar{v}}(t) - K(t)\bar{v}(t)))$ and therefore $\bar{u}(t) = \nabla_z H^*(t; -J\dot{\bar{v}}(t) - K(t)\bar{v}(t))$.

(ii) If $\det(\Upsilon_K(t) - M) \neq 0$, $\Phi_{H,K}$ and $\Psi_{H,K}$ have same critical point sets on $W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ and $\Phi_{H,K}(u) = -\Psi_{H,K}(u)$ for each critical point $u$ of them; in fact, $v \mapsto \Gamma(v)$, where $\Gamma(v)(t) = \nabla_z H^*(t; -J\dot{v}(t) - K(t)v(t))$, is a map from $\text{Crit}(\Phi_{H,K}) = \text{Crit}(\Psi_{H,K})$ to itself.

(iii) The restrictions of $\Phi_{H,K}$ and $\Psi_{H,K}$ to $\bar{W}^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$, denoted by $\bar{\Phi}_{H,K}$ and $\bar{\Psi}_{H,K}$, have same critical point sets and $\Phi_{H,K}(u) = -\Psi_{H,K}(u)$ for each critical point $u$ of them.

(iv) Any critical point of $\bar{\Psi}_{H,K}$ is also one of $\Psi_{H,K}$ on $W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$.

**Proof.** By Propositions 2.9, 2.10 the first two claims can be derived. Taking $X = L^2([0, \tau]; \mathbb{R}^{2n})$, $A = \Lambda_{M,\tau,K}$, $D(A) = \bar{W}^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ and

\[
\Phi = \Phi_{H,K}(v) = \frac{1}{2} (\Lambda_{M,\tau,K}v, v)_{L^2} + \mathcal{H}(v), \quad (2.25)
\]

\[
\Psi = \Psi_{H,K}(v) = \frac{1}{2} (\Lambda_{M,\tau,K}v, v)_{L^2} + \mathcal{H}^*(-\Lambda_{M,\tau,K}v), \quad (2.26)
\]

the desired conclusions in (i), (ii) and (iii) follow from Proposition 2.2 and Corollary 2.4 immediately. For the second claim in (i) can be proved as follows. Using Proposition 2.2 and Corollary 2.4 we can get some $\xi \in \mathcal{E}_{M,\tau,K}$ such that $\bar{u} := \bar{v} - v_\xi$, where $v_\xi(t) = \Upsilon_K(t)\xi$, is a critical point of $\Phi_{H,K}$ on $\bar{W}^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ and satisfies $\Phi_{H,K}(\bar{u}) = -\Psi_{H,K}(\bar{v})$. Since this $\bar{u}$ satisfies $2.24$ and $\Lambda_{M,\tau,K}\bar{u} - \Lambda_{M,\tau,K}\bar{v} = 0$, we derive that for each $t \in [0, \tau],$

\[
-J\dot{\bar{u}}(t) - K(t)\bar{u}(t) = \nabla_z H(t, \bar{u}(t))
\]

and so $\bar{v}(t) - v_\xi(t) = \bar{u}(t) = \nabla_z H^*(t; -J\dot{\bar{v}}(t) - K(t)\bar{v}(t))$ by strict convexity of $H(t, \cdot)$.

(\textbf{Note:} Since $\mathcal{H}$ and $\mathcal{H}^*$ are convex and of class $C^1$ on $L^2([0, \tau]; \mathbb{R}^{2n})$, by \cite{20} page 92, Proposition 9) we may also use Corollary 2.6 instead of Corollary 2.4.)
As to (iv), let \( v \) be a critical point of the restriction of \( \Psi_{H,K} \) to \( \tilde{W}^{1,2}_{M,K}([0, \tau]; \mathbb{R}^{2n}) \). For any \( h \in W^{1,2}_M([0, \tau], \mathbb{R}^{2n}) \), since

\[
\int_0^T \left[ \frac{1}{2} \langle J\dot{v}(t) + K(t)v(t), h(t) \rangle_{\mathbb{R}^{2n}} \right] dt = \int_0^T \left[ \frac{1}{2} (\dot{J}h(t) + K(t)h(t), v(t))_{\mathbb{R}^{2n}} \right] dt,
\]

we deduce that

\[
d\Psi_{H,K}(v)[h] = \int_0^T \left[ \frac{1}{2} (\dot{J}v(t) + K(t)v(t), h(t))_{\mathbb{R}^{2n}} + \frac{1}{2} (\dot{K}h(t) - K(t)h(t), v(t))_{\mathbb{R}^{2n}} \right] dt
\]

\[
+ \int_0^T \left[ (v(t), J\dot{h}(t) + K(t)h(t))_{\mathbb{R}^{2n}} \right] dt
\]

\[
+ \int_0^T \left[ (v(t) - \nabla_zH^*(t; -\dot{J}v(t) - K(t)v(t)), \Lambda_{M,\tau,K})_{\mathbb{R}^{2n}} \right] dt.
\]

By \((2.25)\), we can decompose \( h \) into \( h_0 + h_1 \), where \( h_0 \in \text{Ker}(\Lambda_{M,\tau,K}) \) and \( h_1 \in \tilde{W}^{1,2}_{M,K}([0, \tau]; \mathbb{R}^{2n}) \). Hence \((2.27)\) leads to

\[
d\Psi_{H,K}(v)[h] = \int_0^T \left[ (v(t) - \nabla_zH^*(t; -\dot{J}v(t) - K(t)v(t)), \Lambda_{M,\tau,K} h_1)_{\mathbb{R}^{2n}} \right] dt
\]

\[
= d\Psi_{H,K}(v)[h_1] = 0.
\]

That is, \( v \) is a critical point of \( \Psi_{H,K} \).

**Remark 2.12.** As pointed out below \((2.24)\), \( \Lambda_{M,\tau,K} \) restricts to a Banach space isomorphism \( \tilde{\Lambda}_{M,\tau,K} \) from \( \tilde{W}^{1,2}_{M,K}([0, \tau]; \mathbb{R}^{2n}) \) onto \( \tilde{L}^2_{M,K}([0, \tau]; \mathbb{R}^{2n}) \). Thus under the assumptions of Theorem \(2.11\) \( v \) is a critical point of \( \tilde{\Psi}_{H,K} \) on \( \tilde{W}^{1,2}_{M,K}([0, \tau]; \mathbb{R}^{2n}) \) if and only if \( u := -\tilde{\Lambda}_{M,\tau,K} v \) is that of the functional \( \tilde{\psi}_{H,K} : \tilde{L}^2_{M,K}([0, \tau]; \mathbb{R}^{2n}) \to \mathbb{R} \) given by

\[
\tilde{\psi}_{H,K}(u) := \tilde{\Psi}_{H,K} \circ (-\tilde{\Lambda}_{M,\tau,K})^{-1}(u) = \int_0^T \left[ \frac{1}{2} (w(t), ((\Lambda_{M,\tau,K})^{-1}u)(t))_{\mathbb{R}^{2n}} + H^*(t; u(t)) \right] dt.
\]

In particular, if \( \det(\Upsilon_K(\tau) - M) \neq 0 \), then \( v \) is a critical point of \( \Psi_{H,K} \) on \( W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \) if and only if \( w := -\Lambda_{M,\tau,K} v \) is one of the \( C^1 \) functional

\[
\psi_{H,K}(u) := \int_0^T \left[ \frac{1}{2} (u(t), ((\Lambda_{M,\tau,K})^{-1}u)(t))_{\mathbb{R}^{2n}} + H^*(t; u(t)) \right] dt \tag{2.28}
\]

on \( L^2([0, \tau]; \mathbb{R}^{2n}) \). (Note: the functional \( \Psi_{H,K} \) and so \( \tilde{\psi}_{H,K} \) is mistakenly considered to be of class \( C^2 \) in many references. In general, they cannot be \( C^2 \), see [67] Chap.5, Sec.5.1, Theorem 1.)

For a critical point \( w \) of \( \tilde{\psi}_{H,K} \), \( v := (-\tilde{\Lambda}_{M,\tau,K})^{-1} w \) is a critical point of \( \tilde{\Psi}_{H,K} \) (and so that of \( \Psi_{H,K} \) by Theorem \(2.11\)(iv)). It follows from Theorem \(2.11\)(i) that there exists a unique \( \xi \in \mathcal{E}_{M,T,K} \) such that \( u := v - v_\xi \), where \( v_\xi(t) = \Upsilon_K(t) \xi \), is a critical point of \( \Phi_{H,K} \) on \( W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \) and satisfies \( \Phi_{H,K}(u) = -\Theta_{H,K}(v) \). Then for any \( t \in [0, \tau] \) we have

\[
-\nabla_zH(t; u(t)) = J\dot{u}(t) + K(t)u(t) = J\dot{v}(t) + K(t)v(t) = -w(t)
\]
and hence $\nabla_{\xi}^2 H^*(t;w(t))\nabla_{\eta}^2 H(t,u(t)) = I_{2n}$. A direct computation shows

$$(\tilde{\psi}_{H,K})''(w)[\xi,\eta] = \int_0^T \left[ ((\bar{A}_{M,\tau,K})^{-1})^{-1}(\xi)(t,\eta(t))\right] dt$$

for $\xi,\eta \in \mathbb{E}_{2,M,K}([0,\tau];\mathbb{R}^{2n})$. By the arguments below Theorem 2.13, $(\tilde{\psi}_{H,K})''(w)$ is a Legendre form in $\mathbb{E}_{2,M,K}([0,\tau];\mathbb{R}^{2n})$, and hence has finite Morse index and the nullity, denoted by $m^-(\psi_{H,K},w)$ and $m^0(\psi_{H,K},w)$, respectively. By (A.28) and (A.26) we have

$$m^-(\psi_{H,K},w) = i_{\tau,M}(\Upsilon_{K,\nu,H(K)\nu,H(K)}) - i_{\nu,K}(\Upsilon_{K}), \quad \text{(2.29)}$$

$$m^0(\psi_{H,K},w) = \nu_{\nu,K}(\Upsilon_{K,\nu,H(K)\nu,H(K)}) = \dim \ker(\Upsilon_{K,\nu,H(K)\nu,H(K)}(\tau) - M), \quad \text{(2.30)}$$

where $i_{\tau,M}$ is defined by (A.16) and $\Upsilon_{K,\nu,H(K)\nu,H(K)}$ is the fundamental matrix solution of

$$\dot{Z}(t) = (J(K(t) + \nabla_{\xi}^2 H(t,u(t)))Z(t) \quad \forall t \in [0,\tau].$$

A direct proof of (2.30) can also be given by following [19, Proposition 3.14]. When $\det(\Upsilon_{K}(\tau) - M) \neq 0$, $\psi_{H,K}$ in (2.28) and (2.30) is replaced by $\psi_{H,K}$ in (2.28), and $u = v = (-\bar{A}_{M,\tau,K})^{-1}w$.

**Corollary 2.13.** For $M \in \text{Sp}(2n,\mathbb{R})$, let $H : [0,\tau] \times \mathbb{R}^{2n} \to \mathbb{R}, (t,u) \to H(t,u)$ be measurable in $t$ for each $u \in \mathbb{R}^{2n}$ and twice continuously differentiable in $u$ for almost every $t \in [0,\tau]$. Assume that there exists $\ell > 0, \alpha > 0, \delta > 0$, and $\beta, \gamma \in L^2(0,T;\mathbb{R}^+)$ such that for almost every $t \in [0,\tau],

$$|\langle \nabla_{\eta}^2 H(t,\xi,\eta)\rangle_{\mathbb{R}^{2n}}| \leq \ell |\xi|^2 \quad \forall \xi,\eta \in \mathbb{R}^{2n}, \quad \delta |\xi|^2 - \beta(t) \leq H(t,\xi) \leq \alpha |\xi|^2 + \gamma(t) \quad \forall \xi \in \mathbb{R}^{2n}.$$

Then for any real $\kappa < -\ell$, and $H_\kappa(t,\xi) := H(t,\xi) - \frac{\kappa}{2} |\xi|^2$, functionals

$$\Phi(v) = \int_0^T \left[ \frac{1}{2}(J\dot{v}(t),v(t))_{\mathbb{R}^{2n}} + H(v(t),t) \right] dt = \Phi_{H_{\kappa,1_{2n}}}(v),$$

$$\Psi_\kappa(v) := \Phi_{H_{\kappa,1_{2n}}}(v) = \int_0^T \left[ \frac{1}{2}(J\dot{v}(t) + \kappa v(t),v(t))_{\mathbb{R}^{2n}} + (H_\kappa)\ast(t) - J\dot{v}(t) - \kappa v(t) \right] dt$$

are $C^1$ and twice Gâteaux differentiable on $W^{1,2}_{M}([0,\tau];\mathbb{R}^{2n})$, and the following holds:

(i) If $\bar{u} \in W^{1,2}_{M}([0,\tau];\mathbb{R}^{2n})$ is a critical point of $\Phi$, then each $u \in \bar{u} + \ker(\Lambda_{M,\tau,K})$ is also a critical point of $\Phi_{\kappa}$ on $W^{1,2}_{M}([0,\tau];\mathbb{R}^{2n})$ and $\Psi_\kappa(u) = -\Phi(\bar{u})$. Conversely, if $\bar{v} \in W^{1,2}_{M}([0,\tau];\mathbb{R}^{2n})$ is a critical point of $\Psi_\kappa$, then there exists a unique $\xi \in \mathcal{E}_{M,\tau,K}$ such that $\bar{u} := \bar{v} - \xi$, where $v(t) = \Upsilon_{K,1_{2n}}(t)\xi$, is a critical point of $\Phi$ on $W^{1,2}_{M}([0,\tau];\mathbb{R}^{2n})$ and satisfies $

\Phi(\bar{u}) = -\Phi(\bar{v});$ this unique $\xi$ is equal to $(\Upsilon_{K,1_{2n}}(t))^{-1}(\bar{v}(t) - \nabla_x H_{\kappa})\ast(t) - \nabla_x (H_{\kappa})\ast(t)$ and therefore $\bar{u}(t) = \nabla_x (H_{\kappa})\ast(t) - J\dot{v}(t) - K(t)v(t)).$

(ii) If $\det(\Upsilon_{K,1_{2n}}(\tau) - M) \neq 0$, $\Phi$ and $\Psi_\kappa$ have the same critical point sets on $W^{1,2}_{M}([0,\tau];\mathbb{R}^{2n})$ and $\Phi(u) = -\Psi_\kappa(u)$ for each critical point $u$ of them.

(iii) The restrictions of $\Phi$ and $\Psi_\kappa$ to $W^{1,2}_{M,\tau,K}(0,\tau];\mathbb{R}^{2n})$, denoted by $\tilde{\Phi}$ and $\tilde{\Psi}_\kappa$, have the same critical point sets and $\tilde{\Phi}(u) = -\Psi_\kappa(u)$ for each critical point $u$ of them.
(iv) Any critical point of $\tilde{\Psi}_\kappa$ is also one of $\Psi_\kappa$ on $W^{1,2}_M([0,\tau];\mathbb{R}^{2n})$.

Moreover, we can take a real $\kappa < -\ell$ such that $\det(\Upsilon_{\kappa I_{2n}}(\tau) - M) \neq 0$, and therefore $\bar{v}$ is a critical point of $\Psi_\kappa$ on $W^{1,2}_M([0,\tau];\mathbb{R}^{2n})$ if and only if $\bar{w} := -\Lambda_{M,\tau,\kappa I_{2n}} \bar{v}$ is that of $C^1$ and twice Gâteaux differentiable functional

$$\psi_\kappa(w) := \int_0^\tau \left[ \frac{1}{2}(w(t),((\Lambda_{M,\tau,\kappa I_{2n}})^{-1}w)(t))_{\mathbb{R}^{2n}} + (H_\kappa)^*(t; w(t)) \right] dt \quad (2.31)$$

on $L^2([0,\tau];\mathbb{R}^{2n})$; in this case there holds

$$m^-(\psi_\kappa, \bar{w}) = i_{\tau, M}(\Upsilon_{z^2 H(\cdot, \bar{v}(\cdot))}) - i_{\tau, M}(\Upsilon_{\kappa I_{2n}}) - \nu_{\tau, M}(\Upsilon_{\kappa I_{2n}}), \quad (2.32)$$

$$m^0(\psi_\kappa, \bar{w}) = \nu_{\tau, M}(\Upsilon_{z^2 H(\cdot, \bar{v}(\cdot))}) = \dim \ker(\Upsilon_{z^2 H(\cdot, \bar{v}(\cdot))}(\tau) - M), \quad (2.33)$$

where $\Upsilon_{\kappa I_{2n}}(t) = \exp(tk\kappa J)$, and $\Upsilon_{z^2 H(\cdot, \bar{v}(\cdot))}$ is the fundamental matrix solution of

$$\dot{Z}(t) = J(H(t))''(\bar{v}(t))Z(t) = J\nabla^2_H(t, \bar{v}(t))Z(t).$$

**Proof.** For any $-\kappa > \ell$, by the assumptions it is easily seen that $H_\kappa$ satisfies (2.19) and (2.14). Therefore applying Theorem 2.11 to $K(t) = \kappa I_{2n}$, $H = H_\kappa$ and therefore $\psi_{H,K} = \psi_\kappa$ we obtain the conclusions above “Moreover”.

Since $\Upsilon_{\kappa I_{2n}}(t) = \exp(tk\kappa J)$, by Proposition A.5 we can choose a real $\kappa < -\ell$ such that $\det(\Upsilon_{\kappa I_{2n}}(\tau) - M) \neq 0$. Other claims follow from Remark 2.12 and the fact that $\kappa I_{2n} + \nabla^2_H(\cdot, \bar{v}(\cdot)) = \nabla^2_H(\cdot, \bar{v}(\cdot))$. 

\[\square\]

3 Proofs of Theorems 1.14, 1.17 and Corollary 1.9, 1.10

The following remark about parameter space $\Lambda$ is effective for this section and next sections.

**Remark 3.1.** The parameter space $\Lambda$ may be, respectively, replaced by its subsets

- $\{\mu, \lambda_k | k \in \mathbb{N}\}$ in Theorem 1.4 (I), Theorem 1.18, Theorem 1.23 (I), Theorem 1.33 (I),
- $\{\mu, \lambda^+_k, \lambda^-_k | k \in \mathbb{N}\}$ in Theorem 1.4 (II), Theorem 1.19, Theorem 1.23 (II), Theorem 1.33 (II),
- $\alpha([0,1])$ in Theorem 1.4 (III), Theorem 1.20, Theorem 1.23 (III), Theorem 1.33 (III),
- $[\mu - \epsilon, \mu + \epsilon], \epsilon > 0$, in Theorem 1.7, Theorems 1.14, Theorem 1.21, Theorems 1.24, 1.26, Theorem 1.33

These four sets $\{\mu, \lambda_k | k \in \mathbb{N}\}$, $\{\mu, \lambda^+_k, \lambda^-_k | k \in \mathbb{N}\}$, $\alpha([0,1])$ and $[\mu - \epsilon, \mu + \epsilon]$ are compact and sequential compact subsets of $\Lambda$.

Our proofs will be completed with theorems in [50, 52] and Theorems C.4, C.5, C.6. To this goal let us define $\bar{H} : \Lambda \times [0, \tau] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ by

$$\bar{H}(\lambda, t, z) = H(\lambda, t, z + u_\lambda(t)) - (z, \nabla z H(\lambda, t, u_\lambda(t)))_{\mathbb{R}^{2n}}. \quad (3.1)$$

Since each $\Lambda \times [0, \tau] \ni (\lambda, t) \mapsto u_\lambda(t) \in \mathbb{R}^{2n}$ is continuous, it is clear that $\bar{H}$ satisfies Assumption 1.2 and $\bar{u} \equiv 0 \in \mathbb{R}^{2n}$ satisfies

$$\dot{u}(t) = J\nabla z \bar{H}(\lambda, t, u(t)) \quad \forall t \in [0, \tau] \quad \text{and} \quad u(\tau) = Mu(0) \quad (3.2)$$

for each $\lambda \in \Lambda$. Note that a function $u : [0, \tau] \rightarrow \mathbb{R}^{2n}$ satisfies (3.2) with the parameter value $\lambda$ if and only if the function $w(t) := u(t) - u_\lambda(t)$ is a solution of (3.2). Hence we have
Claim 3.2. For $X = W^{1,2}_M([0,\tau];\mathbb{R}^{2n})$ or $C^1_M([0,\tau];\mathbb{R}^{2n})$, the bifurcation problem of (1.3) in $\Lambda \times X$ with respect to the branch $\{ (\lambda, u_\lambda) | \lambda \in \Lambda \}$ is equivalent to that of (2.2) in $\Lambda \times X$ with respect to the trivial branch $\{ (\lambda, 0) | \lambda \in \Lambda \}$.

Since $\nabla_z \tilde{H}(\lambda, t, 0) = \nabla_z H(\lambda, t, u_\lambda(t))$ for all $(\lambda, t) \in \Lambda \times [0,\tau]$, $\gamma_\lambda : [0,\tau] \to \text{Sp}(2n, \mathbb{R})$ is also the fundamental matrix solution of

$$
\dot{Z}(t) = J \nabla_z \tilde{H}(\lambda, t, 0) Z(t).
$$

Therefore in what follows we only need to prove Theorems 1.4, 1.7 in the case where $u_\lambda \equiv 0$ for all $\lambda \in \Lambda$. (This means $\nabla_z H(\lambda, t, 0) = 0$ for all $(\lambda, t)$.)

Modify $H$ outside an open neighborhood of $0 \in \mathbb{R}^{2n}$. By Remark 3.1 we always assume that $\Lambda$ is compact and sequential compact in what follows. By Assumption 1.1

$$
\Lambda \times [0,\tau] \ni (\lambda, t) \mapsto \nabla^2_z H(\lambda, t, 0) \in \mathcal{L}_s(\mathbb{R}^{2n})
$$

is continuous. Therefore we can get a constant $C > 0$ and a compact neighborhood $U$ of 0 in $\mathbb{R}^{2n}$ such that

$$-C I_{2n} \leq \nabla^2_z H(\lambda, t, z) \leq C I_{2n}, \quad \forall (\lambda, t, z) \in \Lambda \times [0,\tau] \times U.
$$

Choose an open neighborhood of $U_0$ of 0 in $\mathbb{R}^{2n}$ such that $C I(U_0) \subset U$, and a smooth cut-off function $\chi : \mathbb{R}^{2n} \to [0,1]$ such that $\chi|_{U_0} = 1$ and $\chi|_{\mathbb{R}^{2n}\setminus U} \equiv 0$. Define

$$
\tilde{H} : \Lambda \times [0,\tau] \times \mathbb{R}^{2n} \to \mathbb{R}
$$

by $\tilde{H}(\lambda, t, z) = \chi(z) H(\lambda, t, z)$. Then it is easily computed that for any $\xi, \eta \in \mathbb{R}^{2n}$,

$$
(\nabla^2_z \tilde{H})(\lambda, t, z) \xi, \eta_{\mathbb{R}^{2n}} = (H_{\lambda,t})'(z)[\xi][\chi'(z)\eta] + \chi(z)(\nabla^2_z H(\lambda, t, z) \xi, \eta)_{\mathbb{R}^{2n}} + \chi'(z)[\xi](H_{\lambda,t})'(z)[\eta] + H(\lambda, t, z)(\chi''(z) \xi, \eta)_{\mathbb{R}^{2n}}.
$$

Since $H$ and $\nabla_z H$ are continuous by Assumption 1.1 it follows from the compactness of $\Lambda$ that there exists a constant $C' > 0$ such that

$$-C' I_{2n} \leq \nabla^2_z \tilde{H}(\lambda, t, z) \leq C' I_{2n}, \quad \forall (\lambda, t, z) \in \Lambda \times [0,\tau] \times \mathbb{R}^{2n}.
$$

Note for some small $\epsilon > 0$ that $\|u\|_{1,2} \leq \epsilon$ implies $u([0,\tau]) \subset U_0$, and that we are only concerned with solutions of (1.3) near 0 in $W^{1,2}_M([0,\tau];\mathbb{R}^{2n})$. Replacing $H$ by $\tilde{H}$ we can assume that $H$ satisfies

$$-C I_{2n} \leq \nabla^2_z H(\lambda, t, z) \leq C I_{2n}, \quad \forall (\lambda, t, z) \in \Lambda \times [0,\tau] \times \mathbb{R}^{2n}.
$$

(3.3)

For a given $(\lambda, t, z) \in \Lambda \times [0,\tau] \times \mathbb{R}^{2n}$, using Taylor expansion we have $\theta \in (0,1)$ such that

$$
H(\lambda, t, z) = H(\lambda, t, 0) + (\nabla_z H(\lambda, t, 0), z)_{\mathbb{R}^{2n}} + \frac{1}{2}(\nabla^2_z H(\lambda, t, \theta z) z, z)_{\mathbb{R}^{2n}}.
$$

Since $(\lambda, t) \to H(\lambda, t, 0)$ is continuous, and hence bounded, from the above expression it follows that there exist constants $c_1' > 0, c_2' > 0$ such that

$$-c_1'|z|^2 - c_2' \leq H(\lambda, t, z) \leq c_1'|z|^2 + c_2', \quad \forall (\lambda, t, z) \in \Lambda \times [0,\tau] \times \mathbb{R}^{2n}.
$$

(3.4)
Proof of Theorem 1.4(I). By the above arguments we can also assume that \( H \) satisfies (3.3) and (3.4). Then we may choose

\[
\text{Proof of Theorem 1.4(I).}
\]

(i) \( \det(e^{\tau \mathbf{M}} - I_{2n}) \neq 0; \)

(ii) each \( H_\kappa(\lambda, t, z) := H(\lambda, t, z) - \frac{\kappa}{2} |z|^2 \) satisfies

\[
c_1 I_{2n} \leq \nabla_z^2 H_\kappa(\lambda, t, z) \leq c_2 I_{2n}, \quad \forall (\lambda, t, z); \tag{3.5}
\]

(iii) \( c_1 |z|^2 - c_3 \leq H_\kappa(\lambda, t, z) \leq c_2 |z|^2 + c_3 \) for all \( (\lambda, t, z) \).

Let \( (H_\kappa)^*(\lambda, t; z) = (H_\kappa(\lambda, t, \cdot))^*(z) \). These and Assumption 1.1 imply:

(iv) For \( \xi \in \mathbb{R}^{2n} \) and \( \xi^* = \nabla_z H_\kappa(\lambda, t, \xi) \) it holds that

\[
\xi = \nabla_z (H_\kappa)^*(\lambda, t; \xi^*) \quad \text{and} \quad I_{2n} = \nabla_z^2 (H_\kappa)^*(\lambda, t; \xi^*) \nabla_z^2 H_\kappa(\lambda, t, \xi). \tag{3.6}
\]

(v) Because of (3.5), it holds for all \( (\lambda, t, z) \) that

\[
\frac{1}{c_2} I_{2n} \leq \nabla_z^2 (H_\kappa)^*(\lambda, t; z) \leq \frac{1}{c_1} I_{2n}. \tag{3.7}
\]

(vi) \( (H_\kappa)^*: \Lambda \times [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R} \) also satisfies Assumption 1.1 that is, it is a continuous function such that each \( (H_\kappa)^*(\lambda, t, \cdot): \mathbb{R}^{2n} \to \mathbb{R}, \quad (\lambda, t) \in \Lambda \times [0, \tau] \), is \( C^2 \) and all possible partial derivatives of it depend continuously on \( (\lambda, t, z) \in \Lambda \times [0, \tau] \times \mathbb{R}^{2n} \). (These may follow from (3.6) and the implicit function theorem.)

(vii) \( \frac{1}{c_2} |z|^2 - c_3 \leq (H_\kappa)^*(\lambda, t; z) \leq \frac{1}{c_1} |z|^2 + c_3 \) for all \( (\lambda, t, z) \).

By these and Corollary 2.43 the functionals \( \Phi(\lambda, \cdot): W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \to \mathbb{R} \) defined by

\[
\Phi(\lambda, v) := \int_0^\tau \left[ \frac{1}{2} (J_\kappa(t), v(t))_{\mathbb{R}^{2n}} + H(\lambda, t, v(t)) \right] dt \tag{3.8}
\]

and \( \Psi_\kappa(\lambda, \cdot): W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \to \mathbb{R} \) defined by

\[
\Psi_\kappa(\lambda, v) = \int_0^\tau \left[ \frac{1}{2} (J_\kappa(t) + \kappa v(t), v(t))_{\mathbb{R}^{2n}} + (H_\kappa)^*(\lambda, t; -J_\kappa(t) - \kappa v(t)) \right] dt, \tag{3.9}
\]

are \( C^1 \) and twice Gâteaux-differentiable, and have same critical point sets, which exactly correspond to solutions of (1.3) with the parameter value \( \lambda \). Moreover, \( \Phi(\lambda, u) = -\Psi_\kappa(\lambda, u) \) for any critical point \( u \) of them. As noted in Remark 2.12 (i) also implies that

\[
\Lambda_{M, \tau, \kappa, I_{2n}}: W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \to L^2([0, \tau]; \mathbb{R}^{2n}) \tag{3.10}
\]

is a Banach space isomorphism, and thus the functional

\[
\psi_\kappa(\lambda, \cdot) \overset{\text{def}}{=} \Psi_\kappa(\lambda, \cdot) \circ (-\Lambda_{M, \tau, \kappa, I_{2n}})^{-1}: L^2([0, \tau]; \mathbb{R}^{2n}) \to \mathbb{R}
\]

given by

\[
\psi_\kappa(\lambda, u) = \int_0^\tau \left[ \frac{1}{2} (u(t), ((\Lambda_{M, \tau, \kappa, I_{2n}})^{-1} u(t))_{\mathbb{R}^{2n}} + (H_\kappa)^*(\lambda, t; u(t)) \right] dt \tag{3.11}
\]
has the same analytical properties as those of \( \Psi_\kappa(\lambda, \cdot) \). In particular, \( \psi_\kappa(\lambda, \cdot) \) is \( C^1 \) and twice Gâteaux-differentiable.

Let us prove that [50, Theorem 3.1] (Theorem C.6) is applicable to

\[
\mathcal{F}_\lambda(\cdot) = \psi_\kappa(\lambda, \cdot), \quad H = L^2([0, \tau]; \mathbb{R}^{2n}), \quad X = L^2([0, \tau]; \mathbb{R}^{2n})
\]

(though any Banach space \( X \) which is dense in \( H = L^2([0, \tau]; \mathbb{R}^{2n}) \)). To this end we define

\[
A_{M, \tau, \kappa} : L^2([0, \tau]; \mathbb{R}^{2n}) \to L^2([0, \tau]; \mathbb{R}^{2n}), \quad u \mapsto \iota \circ (A_{M, \tau, \kappa}^{-1}u),
\]

where \( \iota : W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \to L^2([0, \tau]; \mathbb{R}^{2n}) \) is the inclusion. By (2.6)

\[
[(A_{M, \tau, \kappa}^{-1}u)(t) = \mathcal{Y}_{\kappa I}(t)Z^{-1} \int_0^T \mathcal{Y}_{\kappa I}(t)^{-1} \mathcal{J}u(t)dt - \mathcal{Y}_{\kappa I}(t) \int_0^t \mathcal{Y}_{\kappa I}(s)^{-1} \mathcal{J}u(s)ds.
\]

Then \( A_{M, \tau, \kappa} \) is a compact self-adjoint operator. It is easily proved that \( \psi_\kappa(\lambda, \cdot) \) has the gradient

\[
\nabla_v \psi_\kappa(\lambda, v) = A_{M, \tau, \kappa}v + \nabla_z(H_\kappa)^*(\lambda, \cdot; v(\cdot))
\]

and that \( L^2([0, \tau]; \mathbb{R}^{2n}) \ni v \mapsto \nabla_v \psi_\kappa(\lambda, v) \in L^2([0, \tau]; \mathbb{R}^{2n}) \) has a Gâteaux derivative

\[
B_\lambda(v) := D_v \nabla_v \psi_\kappa(\lambda, v) = A_{M, \tau, \kappa} + \nabla^2_v(H_\kappa)^*(\lambda, \cdot; v(\cdot)) \in \mathcal{L}_s(L^2([0, \tau]; \mathbb{R}^{2n})).
\]

Denote by

\[
P_\lambda(v) = \nabla^2_v(H_\kappa)^*(\lambda, \cdot, v(\cdot)) \quad \text{and} \quad Q_\lambda(v) = A_{M, \tau, \kappa}.
\]

Both are in \( \mathcal{L}_s(L^2([0, \tau]; \mathbb{R}^{2n})) \). Clearly, \( Q_\lambda \) satisfies (iii)-(iv) of [50, Theorem 3.1]. By (3.7), \( P_\lambda(v) \in \mathcal{L}_s(L^2([0, \tau]; \mathbb{R}^{2n})) \) is uniformly positive definite with respect to \( (\lambda, v) \), i.e., it satisfies (ii) of [50, Theorem 3.1].

We also need to prove that \( P_\lambda \) satisfies (i) of [50, Theorem 3.1], that is, for any \( h \in L^2([0, \tau]; \mathbb{R}^{2n}) \) there holds

\[
\|P_\lambda(v_k)h - P_\mu(0)h\|_2^2 = \int_0^T |\nabla^2_v(H_\kappa)^*(\lambda_k, t; v_k(t))h(t) - \nabla^2_v(H_\kappa)^*(\mu, t; 0)h(t)|^2 dt \to 0
\]

provided that \( (v_k) \subset L^2([0, \tau]; \mathbb{R}^{2n}) \) and \( (\lambda_k) \subset \Lambda \) converge to \( 0 \in L^2([0, \tau]; \mathbb{R}^{2n}) \) and \( \mu \in \Lambda \), respectively. Arguing indirectly, we can assume after passing subsequences if necessary that

\[
v_k \to 0 \text{ a.e.}, \quad \text{and } \|P_\lambda(v_k)h - P_\mu(0)h\|_2 \geq \varepsilon_0 \text{ for some } \varepsilon_0 > 0 \text{ and all } k.
\]

By the above (iv) the map \( (\lambda, t, z) \mapsto \nabla^2_v(H_\kappa)^*(\lambda, t; z) \) is continuous, and thus

\[
\nabla^2_v(H_\kappa)^*(\lambda_k, t; v_k(t))h(t) \to \nabla^2_v(H_\kappa)^*(\mu, t; 0)h(t) \text{ almost everywhere.}
\]

Observe that [37] implies

\[
|\nabla^2_v(H_\kappa)^*(\lambda_k, t; v_k(t))h(t) - \nabla^2_v(H_\kappa)^*(\mu, t; 0)h(t)|^2 \\
\leq 4(|\nabla^2_v(H_\kappa)^*(\lambda_k, t; v_k(t))h(t)|^2 + |\nabla^2_v(H_\kappa)^*(\mu, t; 0)h(t)|^2) \leq \frac{8}{c_1^2} |h(t)|^2.
\]

Using Lebesgue dominated convergence theorem we deduce

\[
\lim_{k \to \infty} \int_0^T |\nabla^2_v(H_\kappa)^*(\lambda_k, t; v_k(t))h(t) - \nabla^2_v(H_\kappa)^*(\mu, t; 0)h(t)|^2 dt = 0.
\]
This contradicts (3.11). The desired claim is proved.

Since $(\mu, 0) \in \Lambda \times W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ is a bifurcation point along sequences of the problem (1.3) if and only if it is that of solutions of $\nabla_u \Psi(\lambda, u) = 0$ (or equivalently $\nabla_u \Psi(\lambda, u) = 0$) in $\Lambda \times W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$, we deduce that $(\mu, 0) \in \Lambda \times W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ is a bifurcation point along sequences of the problem (1.3) if and only if $(\mu, 0) \in \Lambda \times L^2([0, \tau]; \mathbb{R}^{2n})$ is a bifurcation point along sequences for solutions of $\nabla_w \psi(\lambda, w) = 0$ in $\Lambda \times L^2([0, \tau]; \mathbb{R}^{2n})$.

It follows from the assumption of Theorem 1.4(II) and [50, Theorem 3.1] (Theorem C.3) that $0 \in L^2([0, \tau]; \mathbb{R}^{2n})$ is a degenerate critical point of $\psi(\mu, \cdot)$ and therefore

$$\dim \text{Ker}(\nabla_{\psi} H(\mu, 0))(\tau) = m^0(\psi(\mu, \cdot), 0) > 0$$

by (2.33). Note that $\gamma_\mu = \nabla_{\psi} H(\mu, 0)$ by the definition of $\gamma_\mu$. Hence

$$\nu_{\tau, M}(\gamma_\mu) = \dim \text{Ker}(\gamma_\mu(\tau) - M) = \dim \text{Ker}(\nabla_{\psi} H(\mu, 0))(\tau) = m^0(\psi(\mu, \cdot), 0) > 0.$$

**Note:** [50, Theorem 3.2] (Theorem C.6) can be also used because its conditions may be satisfied by the following proof of Theorem 1.4(II).

**Proof of Theorem 1.4(II).** The arguments before the final paragraph in the proof of Theorem 1.4(I) are still valid. Clearly, the isomorphism $\Lambda_{M, \tau, \kappa, I_{2n}}$ in (3.10) gives rise to a Banach space isomorphism from $C^0_M([0, \tau]; \mathbb{R}^{2n})$ onto $C^0_M([0, \tau]; \mathbb{R}^{2n})$, denoted by $\Lambda_{M, \tau, \kappa, I_{2n}}$, for the sake of clearness. If $w \in C^0_M([0, \tau]; \mathbb{R}^{2n})$ then $u := A_{M, \tau, \kappa, w}$ belongs to $u \in W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$ and satisfies $J_\dot{u} + \kappa u = \Lambda_{M, \tau, \kappa, I_{2n}}u = w$. It follows that $J_\dot{u} = w - u \in C^0_M([0, \tau]; \mathbb{R}^{2n})$ and hence $u \in C^0_M([0, \tau]; \mathbb{R}^{2n})$.

Let us prove in five steps that the family

$$\{\Lambda(\cdot) = \psi(\lambda, \cdot) \mid \lambda \in \Lambda\}$$

satisfies the conditions of Theorem C.4 with $\lambda^* = \mu$, $H = L^2([0, \tau]; \mathbb{R}^{2n})$ and $X = C^0_M([0, \tau]; \mathbb{R}^{2n})$ except for the condition that $\text{Ker}(B_{\lambda^*}(0)) \neq \{0\}$.

**Step 1 (Prove \(\psi_{\kappa}\) to be continuous).** By a contradiction we assume that $\psi_{\kappa}$ is not continuous at some point $(\bar{\lambda}, \bar{u}) \in \Lambda \times L^2([0, \tau]; \mathbb{R}^{2n})$. Since $\Lambda \times L^2([0, \tau]; \mathbb{R}^{2n})$ is first countable, there exists $\varepsilon > 0$ and a sequence $\{(\lambda_k, u_k)\}_{k=1}^\infty$ in $\Lambda \times L^2([0, \tau]; \mathbb{R}^{2n})$ converging to $(\bar{\lambda}, \bar{u})$ such that

$$|\psi(\lambda_k, u_k) - \psi(\bar{\lambda}, \bar{u})| \geq \varepsilon, \quad \forall k = 1, 2, \ldots.$$

By (vii) in the proof of Theorem 1.4(II) we have

$$|H^{\kappa}_{\lambda}(\lambda, t; z)| \leq \frac{1}{c_1}|z|^2 + c_3 \quad \forall (\lambda, t, z).$$

Since $H^\kappa_{\lambda} : \Lambda \times [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R}$ is continuous, it follows from Lebesgue dominated convergence theorem that

$$\lim_{\lambda \to \lambda^*} \int_0^T H^\kappa_{\lambda}(\lambda, t; \bar{u}(t))dt = \int_0^T H^\kappa_{\lambda^*}(\bar{\lambda}, t; \bar{u}(t))dt.$$  

Moreover, (3.18) and [51, Proposition C.1] imply that maps

$$L^2([0, \tau]; \mathbb{R}^{2n}) \to L^1([0, \tau]), \quad u \mapsto H^\kappa_{\lambda}(\lambda, :: u(\cdot))$$

are uniformly continuous at $\bar{u}$ with respect to $\lambda \in \Lambda^* := \{\bar{\lambda}, \lambda_k \mid k \in \mathbb{N}\}$. Then we have a neighborhood $\Lambda_0$ of $\bar{\lambda}$ in $\Lambda$ and a natural number $N$ such that

$$\left|\int_0^T H^\kappa_{\lambda}(\lambda, t; \bar{u}(t))dt - \int_0^T H^\kappa_{\lambda^*}(\bar{\lambda}, t; \bar{u}(t))dt\right| < \frac{\varepsilon}{4}, \quad \forall \lambda \in \Lambda_0,$$
This contradicts (3.18). Hence \(|\mathcal{L}_\lambda| \lambda \in \Lambda\) is a continuous family.

**Step 2.** By (3.6)-(3.7) and (3.14) it is easily seen that \(\nabla \psi_\kappa(\lambda, v) \in X\) for each \(v \in X\). Thus

\[
\mathcal{A} : \Lambda \times X \rightarrow X, \quad (\lambda, v) \mapsto \nabla \psi_\kappa(\lambda, v) = A_{M, \tau, \kappa} v + \nabla_z H^*_\kappa(\lambda, \cdot; v(\cdot))
\]

is well-defined. By a contradiction we assume that \(\mathcal{A}\) is not continuous at some point \((\bar{\lambda}, \bar{u}) \in \Lambda \times X\). Then there exists \(\varepsilon > 0\) and a sequence \(\{(\lambda_k, u_k)\}_{k=1}^\infty\) in \(\Lambda \times X\) converging to \((\bar{\lambda}, \bar{u})\) such that

\[
\|\mathcal{A}(\lambda_k, u_k) - \mathcal{A}(\bar{\lambda}, \bar{u})\|_{C^0} \geq \varepsilon, \quad \forall k = 1, 2, \ldots.
\]

(Here we have also used the first countability of \(\Lambda\) and \(X\).) By (3.10) and (3.12) we deduce that \(\|A_{M, \tau, \kappa} u_k - A_{M, \tau, \kappa} \bar{u}\|_{C^0} \rightarrow 0\) as \(k \rightarrow \infty\). Therefore there exists a natural number \(N\) such that

\[
\|A_{M, \tau, \kappa} u_k - A_{M, \tau, \kappa} \bar{u}\|_{C^0} < \varepsilon/2 \quad \text{for all } k \geq N.
\]

It follows from this and (3.19) that

\[
\|\nabla_z H^*_\kappa(\lambda_k, \cdot; u_k(\cdot)) - \nabla_z H^*_\kappa(\bar{\lambda}, \cdot; \bar{u}(\cdot))\|_{C^0} \geq \frac{\varepsilon}{2}, \quad \forall k \geq N.
\]

These mean that there exists a sequence \(\{t_k\}_{k=N}^\infty\) in \([0, \tau]\) such that

\[
|\nabla_z H^*_\kappa(\lambda_k, t_k; u_k(t_k)) - \nabla_z H^*_\kappa(\bar{\lambda}, t; \bar{u}(t))|_{\mathbb{R}^{2n}} \geq \frac{\varepsilon}{2}, \quad \forall k \geq N.
\]

Passing to a subsequence we can assume \(t_k \rightarrow \bar{t} \in [0, \tau]\). Since \(\nabla_z H^*_\kappa(\lambda, t; z)\) is continuous and

\[
|u_k(t_k) - \bar{u}(\bar{t})|_{\mathbb{R}^{2n}} \leq |u_k(t_k) - \bar{u}(t_k)|_{\mathbb{R}^{2n}} + |\bar{u}(t_k) - \bar{u}(\bar{t})|_{\mathbb{R}^{2n}} \leq \|u_k - \bar{u}\|_{C^0} + |\bar{u}(t_k) - \bar{u}(\bar{t})|_{\mathbb{R}^{2n}} \rightarrow 0,
\]

it follows from (3.21) that \(0 = |\nabla_z H^*_\kappa(\bar{\lambda}, \bar{t}; \bar{u}(\bar{t})) - \nabla_z H^*_\kappa(\bar{\lambda}, \bar{t}; \bar{u}(\bar{t}))|_{\mathbb{R}^{2n}} \geq \frac{\varepsilon}{2}\). This contradiction shows that \(\mathcal{A}\) is continuous.

**Step 3.** We claim that \(A_\lambda\) is \(C^1\). In fact, as in the proof of Proposition 2.10 we obtain that \(A_\lambda(\cdot) := \mathcal{A}(\lambda, \cdot)\) has the Gâteaux derivative at \(w \in X\),

\[
DA_\lambda(w) : X \rightarrow \mathcal{L}(X), \quad \xi \mapsto \nabla_z^2(H_\kappa)^*(\lambda, \cdot; w(\cdot))\xi(\cdot).
\]

For any given \(w \in X\), since \((t, z) \mapsto \nabla_z^2(H_\kappa)^*(\lambda, t; z)\) is uniformly continuous on a compact neighborhood of \([0, \tau] \times w([0, \tau])\) in \([0, \tau] \times \mathbb{R}^{2n}\), using the inequality

\[
\|DA_\lambda(u) - DA_\lambda(w)\|_{\mathcal{L}(X)} \leq \max_{0 \leq t \leq \tau} \|\nabla_z^2(H_\kappa)^*(\lambda, t; u(t)) - \nabla_z^2(H_\kappa)^*(\lambda, t; w(t))\|_{\mathbb{R}^{2n \times 2n}}
\]

for \(u \in X\) we can deduce that \(\|DA_\lambda(u) - DA_\lambda(w)\|_{\mathcal{L}(X)} \rightarrow 0\) as \(\|u - w\|_{C^0} \rightarrow 0\) and therefore that \(A_\lambda\) is \(C^1\).

**Step 4.** For \(B_\lambda(v)\) given by (3.15) we claim

\[
\{u \in H \mid B_\lambda(0)u = su, \ s \leq 0\} \subset X \quad \text{and} \quad \{u \in H \mid B_\lambda(0)u \in X\} \subset X.
\]
That is, both (D1) in [50, Hypothesis 1.1] and (C) in [50, Hypothesis 1.3] hold with $B_\lambda(0)$.

In fact, suppose that $u \in H$ and $q \leq 0$ satisfy $B_\lambda(0)u = gu$. Then

$$(A_{M,r,\kappa}u)(t) + \nabla^2_{x_\kappa}(H_\kappa)* (\lambda, t; 0)u(t) = gu(t).$$

By (3.10), $\nabla^2_{x_\kappa}(H_\kappa)* (\lambda, t; 0) - qI_{2n}$ and so $\nabla^2_{x_\kappa}(H_\kappa)* (\lambda, t; 0) - qI_{2n}$ is invertible. Since

$$A_{M,r,\kappa}u = t \circ (A_{M,r,\kappa}I_{2n})^{-1}u = (A_{M,r,\kappa}I_{2n})^{-1}u \in W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}),$$

we deduce that $u(t) = -[\nabla^2_{x_\kappa}(H_\kappa)* (\lambda, t; 0) - qI_{2n}]^{-1}(A_{M,r,\kappa}I_{2n}u)(t)$ is continuous, i.e., $u \in X$.

(Actually, this is unnecessary in the condition (D) in [50, Hypothesis 1.1].)

In order to prove the second inclusion in (3.22) let $u \in H$ be such that $v := B_\lambda(0)u \in X = C^0_M([0, \tau]; \mathbb{R}^{2n})$. Then

$$(A_{M,r,\kappa}u)(t) + \nabla^2_{x_\kappa}(H_\kappa)* (\lambda, t; 0)u(t) = v(t).$$

By (3.10), $A_{M,r,\kappa}u \in W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \subset X$. This implies that $u \in X$ since $t \mapsto \nabla^2_{x_\kappa}(H_\kappa)* (\lambda, t; 0)$ is continuous and $\nabla^2_{x_\kappa}(H_\kappa)* (\lambda, t; 0)$ is invertible.

**Step 5.** In the proof of Theorem 1.4(I) we have proved that $P_\lambda(v) = \nabla^2_{x_\kappa}(H_\kappa)* (\lambda, \cdot; v(\cdot))$ and $Q_\lambda(v) = A_{M,r,\kappa}$ satisfy the conditions (i), (ii), (iii) and (iv) in Theorem C.4 with $\lambda = \mu$. Combing these with the above Steps 3, 4 we see that Hypothesis C.3 and so the condition (v) in Theorem C.4 with $\lambda' = \mu$ is satisfied. Because the condition $\text{Ker}(B_\lambda(0)) \neq \{0\}$ is not used in the proofs above we have actually showed that Hypothesis C.3 is satisfied with any $\lambda \in \Lambda$.

As in the final paragraph in the proof of Theorem 1.4 it follows from (2.32) and (2.33) that the Morse index and nullity of $L_{\lambda}$ at $0 \in H$ are

$$m_{\lambda}^- = m^-(\psi_\kappa(\lambda, \cdot), 0) = i_{r,M}(\gamma_\lambda) - i_{r,M}(\Upsilon_{\kappa I_{2n}}) - \nu_{r,M}(\Upsilon_{\kappa I_{2n}}),$$

$$m_{\lambda}^0 = m^0(\psi_\kappa(\lambda, \cdot), 0) = \nu_{r,M}(\gamma_\lambda) = \text{dim Ker}(\gamma_\lambda(\tau) - M),$$

respectively, where $\Upsilon_{\kappa I_{2n}}(t) = \exp(t\kappa J)$. Let $\Xi = -i_{r,M}(\Upsilon_{\kappa I_{2n}}) - \nu_{r,M}(\Upsilon_{\kappa I_{2n}})$. Under the assumptions in Theorem 1.4(II), for each $k \in \mathbb{N}$ we derive from (5.23)-(5.24)

$$[m_{\lambda_k}^-, m_{\lambda_k}^- + m_{\lambda_k}^0] \cap [m_{\lambda_k}^-, m_{\lambda_k}^- + m_{\lambda_k}^0] = [i_{r,M}(\gamma_{\lambda_k}) + \Xi, i_{r,M}(\gamma_{\lambda_k}) + \nu_{r,M}(\gamma_{\lambda_k}) + \Xi] \cap [i_{r,M}(\gamma_{\lambda_k}) + \Xi, i_{r,M}(\gamma_{\lambda_k}) + \nu_{r,M}(\gamma_{\lambda_k}) + \Xi]$$

$$= \Xi + [i_{r,M}(\gamma_{\lambda_k}^-), i_{r,M}(\gamma_{\lambda_k}^-) + \nu_{r,M}(\gamma_{\lambda_k}^-)] \cap [i_{r,M}(\gamma_{\lambda_k}^-), i_{r,M}(\gamma_{\lambda_k}^-) + \nu_{r,M}(\gamma_{\lambda_k}^-)] = \emptyset$$

and either $m_{\lambda_k}^0 = \nu_{r,M}(\gamma_{\lambda_k}^-) = 0$ or $m_{\lambda_k}^0 = \nu_{r,M}(\gamma_{\lambda_k}^-) = 0$. Hence from the conclusion (B) of Theorem C.4 we immediately conclude that there exists a sequence $\{(\lambda_k, w_k)\}_{k \geq 1}$ in $\Lambda \times L^2([0, \tau]; \mathbb{R}^{2n})$ converging to $(\mu, 0)$ such that each $w_k \neq 0$ and satisfies $\nabla w \psi_\kappa(\lambda_k, w_k) = 0$, $k = 1, 2, \cdots$. Then

$$u_k := -(A_{M,r,\kappa I_{2n}})^{-1}w_k + u_{\lambda_k} \in W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}), \ k = 1, 2, \cdots,$$

satisfy (1.3) and $0 < \|u_k - u_{\lambda_k}\|_{1,2} = \|(A_{M,r,\kappa I_{2n}})^{-1}w_k\|_{1,2} \to 0$ as $k \to \infty$. By Proposition 1.3(i) [or its proof] we obtain that $u_k \in C^0_M([0, \tau]; \mathbb{R}^{2n})$ and $0 < \|u_k - u_{\lambda_k}\|_{C^1} \to 0$. The required conclusions are proved.

**Proof of Theorem 1.4(III).** Recall that we have assumed $\Lambda = \alpha([0, 1])$. From the proof of Theorem 1.4(II) it is easily seen that the conditions of Theorem C.5 are satisfied. Then there exists a sequence $\{(t_k, w_k)\}_{k \geq 1} \subset [0, 1] \times L^2([0, \tau]; \mathbb{R}^{2n})$ such that

$$\text{Proof is completed.}$$
• $t_k \to t$ and $0 < \|w_k\|_2 \to 0$,
• each $w_k$ satisfies $\nabla_w \psi_{\kappa}(\alpha(t_k), w_k) = 0$, $k = 1, 2, \ldots$,
• $\alpha(t)$ is not equal to $\lambda^+$ (resp. $\lambda^-$) if $m^0_{\lambda^+} = 0$ (resp. $m^0_{\lambda^-} = 0$), where $m^0_{\lambda}$ is given by \([\text{8.1}]\).

As above we define $u^k := -(\Lambda_{M, \tau, cI_2n})^{-1}w_k + u_\alpha(t_k) \in W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})$, $k = 1, 2, \ldots$, and using Proposition \([\text{8.3}]\) it is easily checked that they satisfy the required conclusions.

\[ \tag{8.3} \]

\textbf{Remark 3.3.} Even if $(\Lambda, M, \tau) = (I, I_{2n}, 2\pi)$, where $I$ is an interval containing $\mu$ as an interior point, we cannot still guarantee that $\psi_{\kappa}$ is $C^1$. Therefore the conclusion cannot be derived from \([8]\) Theorem 4.6 (a result due to Mawhin and Willem \([57]\) Theorem 8.9).

\textbf{Proof of Theorem 1.7} Follow the notations in the proofs of Theorems \([1.4]\). But now the parameter space $\Lambda$ is a real interval. Comparing the conditions in Theorem C.4 with those of \([8, \text{Theorem 4.6}]\) (a result due to Mawhin and Willem \([57, \text{Theorem 8.9}]\)). But now the proof of Theorem 1.4(III).

\[ \tag{8.6} \]

We claim that the latter can be also satisfied. In fact, by the assumptions of Theorem 1.7 and \([52, \text{Remark 3.9}]\) (or \([50, \text{Remark 5.14}]\)) to obtain the desired claims.

Since the Banach space isomorphism $-\Lambda_{M, \tau, cI_2n} : C^1_M([0, \tau]; \mathbb{R}^{2n}) \to C^0_M([0, \tau]; \mathbb{R}^{2n})$ maps the set of solutions of $\nabla_u \psi_{\kappa}(\lambda, w) = 0$ in $C^1_M([0, \tau]; \mathbb{R}^{2n})$ onto that of $\nabla_u \psi_{\kappa}(\lambda, w) = 0$ in $C^0_M([0, \tau]; \mathbb{R}^{2n})$, and for each critical point $w$ of $\psi_{\kappa}(\lambda, \cdot)$ it holds that

$$\psi_{\kappa}(\lambda, w) = \Psi_{\kappa}(\lambda, -\Lambda_{M, \tau, cI_2n}^{-1}w) = -\Phi(\lambda, -\Lambda_{M, \tau, cI_2n}^{-1}w),$$

the conclusions in the first part of Theorem \([1.7]\) follow from these and Proposition \([8.3]\) as in the proof of Theorem \([1.4]\) III).

For the part of "Moreover", note that the Banach space isomorphism $\Lambda_{M, \tau, cI_2n}$ is equivariant with respect to the natural actions of $\mathbb{Z}_2 = \{id, -id\}$ on spaces

$W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}), \quad L^2([0, \tau]; \mathbb{R}^{2n}), \quad C^1_M([0, \tau]; \mathbb{R}^{2n}) \quad \text{and} \quad C^0_M([0, \tau]; \mathbb{R}^{2n})$

and that functionals $\Psi_{\kappa}(\lambda, \cdot)$ and $\psi_{\kappa}(\lambda, \cdot)$ in \([\text{8.3}]\) and \([\text{8.11}]\) are invariant under the $\mathbb{Z}_2$-action, i.e., even. Comparing the conditions in Theorem \([\text{C.7}]\) \([52, \text{Theorem 3.6}]\) or \([50, \text{Theorem 4.6}]\) with those of Theorem \([C.8] \([52, \text{Theorem 3.7}]\) or \([50, \text{Theorem 5.12}]\) we immediately use the latter and \([52]\) Remark 3.9 (or \([50]\) Remark 5.14) to obtain the desired claims.
Proof of Corollary 1.9. If \((\mu, \bar{u}) \in \mathbb{R} \times W^{1,2}_M([0, \tau]; \mathbb{R}^{2n})\) is a bifurcation point for (1.3) with \(H(\lambda, t, z) = H_0(t, z) + \lambda \dot{H}(t, z)\) and \(\Lambda = \mathbb{R}\), then \(\nu_{\tau, M}(\gamma_{\mu}(\tau)) > 0\) by Theorem 1.4(I).

Suppose now that \(\nu_{\tau, M}(\gamma_{\mu}(\tau)) > 0\). As done above the proof of Theorem 1.4(I), (changing a compact neighborhood \(U\) of \(0\) in \(\mathbb{R}^{2n}\) as a compact neighborhood \(U\) of \(\bar{u}([0, \tau]) \subset \mathbb{R}^{2n}\)), by modifying values of \(H_0\) and \(\dot{H}\) outside an open neighborhood of \([0, \tau] \times \bar{u}([0, \tau]) \subset [0, \tau] \times \mathbb{R}^{2n}\) with a cutting-off technique we may assume that \(H\) satisfies

\[-CI_{2n} \leq \nabla_2^2 \dot{H}(t, z) \leq CI_{2n}, \quad \forall (t, z) \in [0, \tau] \times \mathbb{R}^{2n},
\]

for some constants \(C > 0, c_1 > 0, c_2 > 0\). Therefore there exist constants \(\kappa < 0\) and \(c_i > 0, i = 1, 2, 3\) such that (i)-(iii) in the proof of Theorem 1.4(I) are satisfied with

\[H_n(\lambda, t, z) := H_0(t, z) + \lambda \dot{H}(t, z) - \frac{\kappa}{2} |z|^2\]

for all \((\lambda, t, z) \in [\mu - 1, \mu + 1] \times [0, \tau] \times \mathbb{R}^{2n}\). In particular, (3.25) implies that there exist at most finitely many points \(\lambda, t, z\) such that (i)-(iii) in the proof of Theorem 1.4(I) are satisfied with

\[c_1I_{2n} \leq \nabla_2^2 \dot{H}(t, z) \leq c_2I_{2n}\] (3.25)

for all \((\lambda, t, z) \in [\mu - 1, \mu + 1] \times [0, \tau] \times \mathbb{R}^{2n}\). Let \(\bar{w} = -A_{\mu, \tau, \kappa}I_{2n} \bar{u}\) and \(\psi_{\nu, \lambda}(t) := \psi(\lambda, \bar{u})\) be given by (3.11). Then for all \(\xi, \eta \in L^2([0, \tau]; \mathbb{R}^{2n})\),

\[
\psi_{\nu, \lambda}(\bar{w})[\xi, \eta] = \int_0^\tau \left[ (A_{\mu, \tau, \kappa} \xi)(t), \eta(t) \right] \mathbb{R}^{2n} \] + \left[ (\nabla_2^2 H_0(t, \bar{u}(t)) + \lambda \nabla_2^2 \dot{H}(t, \bar{u}(t)) - \kappa I_{2n} \right]^{-1}[\xi(t), \eta(t)] \mathbb{R}^{2n} \right) dt
= 2q_{\mu, \nu, \lambda} A(\xi, \eta),
\]

where \(B_{\mu}(t) = \nabla_2^2 H_0(t, \bar{u}(t)) + \lambda \nabla_2^2 \dot{H}(t, \bar{u}(t))\), \(A(t) = \kappa I_{2n}\), and \(q_{\mu, \nu, \lambda}\) is as in (A.21). By (3.26) for all \((\lambda, t) \in [\mu - 1, \mu + 1] \times [0, \tau]\) we have

\[c_1I_{2n} \leq B_{\mu}(t) - A(t) \leq c_2I_{2n}\] (3.26)

Since \(\gamma_{\lambda} = \Upsilon_{B_{\lambda}}\), it follows from (A.28) and (A.20) that

\[
m_\mu(\psi_{\nu, \lambda}, \bar{w}) = j_{\tau, M}(B_{\lambda} | A) \quad \text{and} \quad m_\mu(\psi_{\nu, \lambda}, \bar{w}) = \nu_{\tau, M}(B_{\lambda} | A) = \nu_{\tau, M}(\gamma_{\lambda}).
\]

Firstly, let us assume \(\nabla_2^2 \dot{H}(t, \bar{u}(t)) > 0\) for all \(t \in [0, \tau]\). Then by (3.26) we get

\[B_{\lambda_2} > B_{\lambda_1} \geq A + c_1I_{2n} \quad \text{for any} \quad \mu - 1 \leq \lambda_1 < \lambda_2 \leq \mu + 1.
\]

Because of these we derive from (A.29) that

\[j_{\tau, M}(B_{\lambda_2} | A) \geq j_{\tau, M}(B_{\lambda_1} | A) + \nu_{\tau, M}(B_{\lambda_1} | A)\] (3.28)

for any \(\mu - 1 \leq \lambda_1 < \lambda_2 \leq \mu + 1\). By (A.28), both \(j_{\tau, M}(B_{\lambda} | A)\) and \(\nu_{\tau, M}(B_{\lambda} | A)\) are nonnegative integers. Hence (3.28) implies that there exist at most finitely many points \(\lambda \in [\mu - 1, \mu + 1]\) where \(\nu_{\tau, M}(B_{\lambda} | A) = \nu_{\tau, M}(B_{\lambda}) = \nu_{\tau, M}(\Upsilon_{B_{\lambda}}) = \nu_{\tau, M}(\gamma_{\lambda}) \neq 0\). It follows that \(\{\lambda \in \mathbb{R} : \nu_{\tau, M}(\gamma_{\lambda}) > 0\}\) is a discrete set in \(\mathbb{R}\). The first claim is proved in the present case.
In order to prove (1.16) let us take \( \rho \in (0,1) \) so small that \( \nu_{\tau,M}(B_\lambda|A) = \nu_{\tau,M}(B_\lambda) = 0 \) for \( \lambda \in [\mu - \rho, \mu + \rho] \setminus \{\mu\} \). Since \( \nabla^2_2 \tilde{H}(t, \tilde{u}(t)) > 0 \) \( \forall t \in [0,\tau] \), by (A.30) we get

\[
j_{\tau,M}(B_{\mu_1}) \leq j_{\tau,M}(B_{\mu_2}) \leq j_{\tau,M}(B_{\mu}) \quad < j_{\tau,M}(B_{\mu}) + \nu_{\tau,M}(B_{\mu}) \leq j_{\tau,M}(B_{\lambda_1}) \leq j_{\tau,M}(B_{\lambda_2}) \quad \text{for any} \quad \mu - \rho \leq \mu_1 < \mu_2 < \mu < \lambda_1 < \lambda_2 < \mu + \rho.
\]

Moreover, by (A.28) we can derive from Proposition 2.3.3 that

\[
B \mapsto j_{\tau,M}(B|A) \quad \text{and} \quad B \mapsto j_{\tau,M}(B|A) + \nu_{\tau,M}(B|A)
\]

are lower-semi continuous and upper-semi continuous, respectively. These and (A.27) imply that

\[
B \mapsto i_{\tau,M}(B) \quad \text{and} \quad B \mapsto i_{\tau,M}(B) + \nu_{\tau,M}(B)
\]

are lower-semi continuous and upper-semi continuous, respectively. Using (3.29) again we deduce

\[
\begin{align*}
j_{\tau,M}(B_{\mu_1}) &= j_{\tau,M}(B_{\mu}) \quad \forall \mu_1 \in [\mu - \rho, \mu], \\
j_{\tau,M}(B_{\lambda_1}) &= j_{\tau,M}(B_{\lambda_1}) + \nu_{\tau,M}(B_{\lambda_1}) = j_{\tau,M}(B_{\mu}) + \nu_{\tau,M}(B_{\mu}) \quad \forall \lambda_1 \in [\mu, \mu + \rho].
\end{align*}
\]

By Theorem (A.2) these lead to (1.16).

Similarly, if \( \nabla^2_2 \tilde{H}(t, \tilde{u}(t)) < 0 \) \( \forall t \in [0,\tau] \), we may obtain reversed inequalities to those in (3.28) and (3.29), and therefore the discreteness of \( \Sigma \). Then as above we have

\[
\begin{align*}
j_{\tau,M}(B_{\mu_1}) &= j_{\tau,M}(B_{\mu}) \quad \forall \mu_1 \in [\mu, \mu + \rho], \\
j_{\tau,M}(B_{\lambda_1}) &= j_{\tau,M}(B_{\lambda_1}) + \nu_{\tau,M}(B_{\lambda_1}) = j_{\tau,M}(B_{\mu}) + \nu_{\tau,M}(B_{\mu}) \quad \forall \lambda_1 \in [\mu - \rho, \mu].
\end{align*}
\]

and so (1.17).

Finally, other conclusions may follow from (1.16)-(1.17) and Theorem 1.7.

**Proof of Corollary 1.10. Step 1 (Prove (I)).** Define \( F : (0,1] \times [0,\tau] \times \mathbb{R}^{2n} \to \mathbb{R} \) by

\[
F(\lambda, t, \tau) = \lambda H(\lambda, t, \tau).
\]

It satisfies Assumption 1.6 and \( \nabla_2 F(\lambda, t, \tau, \tilde{u}) = 0 \) for all \( (\lambda, t) \). Let

\[
B_\lambda(t) = \nabla^2_2 F(\lambda, t, \tilde{u}) = \lambda \nabla^2_2 H(\lambda, t, \tilde{u}) = \lambda B_\lambda(t),
\]

and let \( \gamma_\lambda(t) \) be the fundamental matrix solution of \( \dot{Z} = JB_\lambda(t)Z \) on \( [0,\tau] \). It is easily checked that \( \gamma_\lambda(t) = \Upsilon_B(\lambda t) \) for all \( \lambda \in (0,1] \) and \( t \in [0,\tau] \). Since \( B(t) > 0 \) for all \( t \in [0,\tau] \), by (A.33) we have

\[
i_{\tau,M}(\Upsilon_B) - i_{\tau,M}(\xi_{2n}) - \dim \text{Ker}(I_{2n} - M) = \sum_{0 < s \leq \tau} \nu_{s,M}(\Upsilon_B|_{[0,s]}) = \sum_{0 < \lambda < 1} \nu_{\lambda,M}(\Upsilon_B|_{[0,\lambda \tau]}) = \sum_{0 < \lambda < 1} \nu_{\tau,M}(\gamma_\lambda).
\]

It follows this and the assumption \( i_{\tau,M}(\Upsilon_B) \neq i_{\tau,M}(\xi_{2n}) + \dim \text{Ker}(I_{2n} - M) \) that \( \{0 < \lambda < 1 | \nu_{\tau,M}(\gamma_\lambda) > 0\} \) is nonempty and only consists of finitely many numbers \( \lambda_1 < \cdots < \lambda_l \). Let \( \tau_k = \lambda_k \tau \), \( k = 1, \cdots, l \). Then

\[
\nu_{\tau_k,M}(\Upsilon_B|_{[0,\tau_k]}) := \dim \text{Ker}(\Upsilon_B(\tau_k) - M) = \dim \text{Ker}(\gamma_{\lambda_k}(\tau) - M) = \nu_{\tau,M}(\gamma_{\lambda_k}) \neq 0
\]
for \( k = 1, \cdots, l \). Moreover, for each \( \lambda \in (0,1] \) we may derive from (A.33) that

\[
\begin{align*}
\iota_{\tau,M}(\gamma_\lambda) - i_{\tau,M}(\xi_{2n}) - \dim \ker(J_{2n} - M) &= \sum_{0 < t < \tau} \nu_{t,M}(\gamma_\lambda|_{[0,\bar{t}]}) \\
&= \sum_{0 < t < \lambda \tau} \dim \ker(\Upsilon_B(t) - M).
\end{align*}
\]

Hence for each \( \lambda_k, k = 1, \cdots, l \), we obtain

\[
\begin{align*}
i_{\tau,M}(\gamma_\lambda) &= \iota_{\tau,M}(\gamma_{\lambda_k}) \quad \text{if } \lambda_{k-1} < \lambda \leq \lambda_k, \\
i_{\tau,M}(\gamma_\lambda) &= \iota_{\tau,M}(\gamma_{\lambda_k}) + \nu_{\tau,M}(\gamma_{\lambda_k}) \quad \text{if } \lambda_k < \lambda \leq \lambda_{k+1}
\end{align*}
\]

(3.30)

where \( \lambda_0 = 0 \) and \( \lambda_{l+1} = 1 \). Note that \( \hat{u} : [0, \tau] \to \mathbb{R}^{2n} \) satisfies the boundary value problem

\[
\dot{u}(t) = J\nabla \hat{z}F(\lambda, t, u(t)) \quad \forall t \in [0, \tau] \quad \text{and} \quad u(\tau) = M u(0)
\]

(3.31)

if and only if \( v : [0, \lambda \tau] \to \mathbb{R}^{2n}, \ t \mapsto u(t/\lambda) \) satisfies

\[
\dot{v}(t) = J\nabla \hat{z} \hat{H}(\lambda t, v(t)) \quad \forall t \in [0, \lambda \tau] \quad \text{and} \quad v(\lambda \tau) = M v(0),
\]

(3.32)

and in this situation there holds

\[
\int_0^{\lambda \tau} \left[ \frac{1}{2} (\dot{J} \dot{u}(t), u(t))_{\mathbb{R}^{2n}} + H(\lambda t, u(t)) \right] dt = \int_0^{\tau} \left[ \frac{1}{2} (\dot{J} \dot{u}(t), u(t))_{\mathbb{R}^{2n}} + F(\lambda, t, u(t)) \right] dt.
\]

Because of (3.30), applying Theorem 1.7 to \( \hat{F} \) leads to the desired results.

**Step 2 (Prove (II)).** Define \( \hat{H} : [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R} \) by \( \hat{H}(t, z) = -\hat{H}(\tau - t, z) \). It satisfies Assumption 1.1 and \( \nabla \hat{z} \hat{H}(t, \bar{u}) = 0 \) for all \( (\lambda, t) \). Note that \( \hat{B}(t) = \nabla^2 \hat{H}(t, \bar{u}) = -\nabla^2 \hat{H}(\tau - t, \bar{u}) = -\hat{B}(\tau - t) \) is positive definite for each \( t \), and that \( \hat{\Upsilon}_B(t) = \hat{\Upsilon}_B(\tau - t)\hat{\Upsilon}_B(\tau)^{-1} \) for all \( t \in [0, \tau] \).

Applying the conclusions in (I) to \( \hat{H} \) we get at least one and at most finitely many numbers in \( (0, \tau), \delta_1 < \cdots < \delta_l \), such that

\[
\nu_{\delta_k, M}(\hat{\Upsilon}_B) := \dim \ker(\hat{\Upsilon}_B(\delta_k) - M) \neq 0, \quad k = 1, \cdots, l,
\]

and that (I) holds if \( H \) and \( \tau_k \) are replaced by \( \hat{H} \) and \( \delta_k, k = 1, \cdots, l \). Moreover, for \( 0 < \rho < \tau \), \( u : [0, \rho] \to \mathbb{R}^{2n} \) satisfies the following boundary value problem

\[
\dot{u}(t) = J\nabla \hat{z} \hat{H}(\frac{\rho}{\tau} t, u(t)) \forall t \in [0, \rho] \quad \text{and} \quad u(\rho) = M u(0)
\]

if and only if \( w : [\tau - \rho^2 / \tau, \tau] \to \mathbb{R}^{2n} \) given by \( w(t) = u(\tau - \frac{\rho}{\tau} t) \) satisfies

\[
\dot{w}(t) = \frac{\rho}{\tau} J\nabla \hat{z} H(t, w(t)) \forall t \in [\tau - \rho^2 / \tau, \tau] \quad \text{and} \quad M w(\tau) = w(\tau - \rho^2 / \tau);
\]

and in this situation there holds

\[
\int_0^{\rho} \left[ \frac{1}{2} (\dot{J} \dot{u}(t), u(t))_{\mathbb{R}^{2n}} + \hat{H}(\frac{\rho}{\tau} t, u(t)) \right] dt = \int_{\tau - \rho^2 / \tau}^{\tau} \left[ \frac{1}{2} (\dot{J} \dot{w}(t), w(t))_{\mathbb{R}^{2n}} + \frac{\tau}{\rho} H(t, w(t)) \right] dt.
\]

Using these and the conclusions in (I) it is easy for us to derive (II).  \( \square \)
4 Proofs of Theorems [1.14, 1.18, 1.19, 1.20, 1.21] and Corollary [1.16]

All proofs are completed in two subsections. Before giving these we begin with the following.

Variational spaces. Consider the Hilbert spaces
\begin{align*}
\mathcal{L}_{r,M} &= \{ v \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^{2n}) \mid v(t + \tau) = Mv(t) \text{ for a.e. } t \in \mathbb{R} \}, \\
\mathcal{W}_{r,M} &= \{ v \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^{2n}) \mid v(t + \tau) = Mv(t) \text{ for all } t \in \mathbb{R} \}
\end{align*}
equipped with inner products (2.1) and (2.2) respectively. There exists a natural (continuous) \( \mathbb{R} \)-action on \( \mathcal{L}_{r,M} \) and \( \mathcal{W}_{r,M} \):
\[ (\theta, v) \mapsto \theta \cdot v \] (4.1)
given by \((\theta \cdot v)(t) := v(\theta + t)\) for any \( t \in \mathbb{R} \).

Let \( S \) be one of the spaces \( C^1_4([0, \tau]; \mathbb{R}^{2n}) \), \( W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \) and \( L^2([0, \tau]; \mathbb{R}^{2n}) \). For each \( u \in S \) we define \( u^M : \mathbb{R} \rightarrow \mathbb{R}^{2n} \) by (4.1). Clearly, if \( u \in C^1_4([0, \tau]; \mathbb{R}^{2n}) \), then \( u^M \in C^1(\mathbb{R}, \mathbb{R}^{2n}) \) and satisfies \( u^M(t + \tau) = M u^M(t) \forall t \in \mathbb{R} \). When \( u \in L^2([0, \tau]; \mathbb{R}^{2n}) \) we have \( u^M(t + \tau) = M u^M(t) \forall t \in \mathbb{R} \setminus \{\tau \mathbb{Z}\} \), and so \( u^M(t + \tau) = M u^M(t) \) for a.e. \( t \in \mathbb{R} \). Thus there exists a natural (continuous) \( \mathbb{R} \)-action on \( S \):
\[ \mathbb{R} \times S \rightarrow S, (\theta, u) \mapsto (\theta \ast u)(\cdot) := (\theta \cdot u^M)(\cdot) \] (4.2)
Clearly, the actions in (4.1) and (4.2) are via Banach isometry isomorphisms if \( M \) is also an orthogonal matrix, and become \( S^1 \)-actions, where \( S^1 = \mathbb{R}/(l \mathbb{Z}) \), if \( M^l = I_{2n} \) for some integer \( l \geq 1 \). Moreover Hilbert space isomorphisms
\[ W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \rightarrow \mathcal{W}_{r,M} \text{ and } L^2([0, \tau]; \mathbb{R}^{2n}) \rightarrow \mathcal{L}_{r,M} \]
given by \( u \mapsto u^M \) are equivariant with respect to the actions in (4.1) and (4.2). Because of these, variational spaces will be chosen as \( W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \) and \( L^2([0, \tau]; \mathbb{R}^{2n}) \) in the following proofs though natural choices should be \( \mathcal{L}_{r,M} \) and \( \mathcal{W}_{r,M} \).

Let \( \mathbb{R}_u \) denote the isotropy group at a point \( u \in S \) of the action in (4.2). It is a subgroup of \( (\mathbb{R}, +) \). If \( \mathbb{R}_u \) contains a sequence \((\theta_k) \subset \mathbb{R} \setminus \{0\}\) converging to 0, then \( \mathbb{R}_u \) must be dense in \( \mathbb{R} \) and so is equal to \( \mathbb{R} \) because the continuity of the map \( \theta \mapsto \theta \cdot u \) implies that \( \mathbb{R}_u \) is closed. Therefore \( \mathbb{R}_u \) must be equal to either \( \mathbb{R} \), or \( \{0\} \) or \( \mathbb{R} \mathbb{Z} \) for some real \( \rho > 0 \). It follows that \( u \) is constant in the first case, injective in the second case, and has a period \( \rho = k\tau \) for some \( k \in \mathbb{N} \) in the final case.

**Proposition 4.1.** Let \( H : \mathbb{R}^{2n} \rightarrow \mathbb{R} \) be \( C^{1,1} \) and \( M \)-invariant. Suppose that \( u : [0, \tau] \rightarrow \mathbb{R}^{2n} \) satisfies \( \dot{u} = J \nabla H(u) \) and \( u(\tau) = M u(0) \). Then \( u \) is \( C^2 \) and
\begin{enumerate}
\item if \( \mathbb{R}_u = \mathbb{R} \), \( u \) is equal to a constant vector \( u_0 \) in Ker(\( M - I_{2n} \));
\item if \( \mathbb{R}_u = \{0\} \), \( \theta \mapsto \theta \ast u \) is an one-to-one \( C^1 \) immersion from \( \mathbb{R} \) to \( C^1_4([0, \tau]; \mathbb{R}^{2n}) \);
\item if \( \mathbb{R}_u = \mathbb{R} \mathbb{Z} \) for some \( \rho > 0 \), i.e., \( \rho \) is a minimal period of \( u \), the \( \mathbb{R} \)-orbit \( \{\theta \ast u \mid \theta \in \mathbb{R}\} \) is a \( C^1 \) embedded \( S^1 = \mathbb{R}/(\mathbb{R} \mathbb{Z}) \) in \( C^1_4([0, \tau]; \mathbb{R}^{2n}) \).
\end{enumerate}
Moreover, if \( M = I_{2n} \) then \( \tau \mathbb{Z} \subset \mathbb{R}_u \) and therefore the case (ii) does not occur.

(ii) comes from the existence and uniqueness theorem for solutions of \( \dot{u} = J \nabla H(u) \) with initial values. (iii) uses the elementary topological fact that a continuous one-to-one map from a compact space to a Hausdorff space is a homeomorphism.
4.1 Proof of Theorem 1.14 and Corollary 1.16

Proof of Theorem 1.14. Define \(\hat{H} : \Lambda \times \mathbb{R}^{2n} \rightarrow \mathbb{R}\) by
\[
\hat{H}(\lambda, z) = H(\lambda, z + v_\lambda(t)).
\]

Clearly, it satisfies Assumption 1.13 with \(v_\lambda \equiv 0\). Therefore in what follows we always write \(\hat{H}\) as \(H\) and assume \(v_\lambda = 0\) for all \(\lambda \in \Lambda\).

When \(\bar{v}\) is viewed as a constant value map from \([0, \tau]\) to \(\mathbb{R}^{2n}\) we write it as \(\bar{u}\), i.e., \(\bar{u} = \bar{v}|_{[0, \tau]}\).

Then \(\bar{M}\bar{u} = \bar{u}\) and \(\bar{u}\) is a fixed point for the \(\mathbb{R}\)-action in (4.2).

As in the first paragraph of Section 3 we can modify \(H\) as \(\hat{H}(\lambda, t, z) = \chi(z)H(\lambda, t, z)\) so that it is also \(M\)-invariant. (Indeed, \(\bar{U}_0 := \sum_{j=0}^l M^j U_0\) is \(M\)-invariant and \(Cl(\bar{U}_0)\) may be contained in \(U\) by shrinking \(U_0\). We only need to replace \(\chi\) by \(\hat{\chi}\), where \(\hat{\chi}(z) := \chi(\sum_{j=0}^l M^j z)\).)

Follow the notations in proofs of Theorems 1.4, 1.7. As pointed out at the beginning of the proof of Theorem 1.7 we have actually checked in the proof of Theorem 1.4(II) that \(\mathcal{L}_\lambda(\cdot) = \psi_\lambda(\lambda, \bar{w} + \cdot)\) with
\[
\bar{w} = -\Lambda_{M, \tau, I_{2n}} \bar{u} = -J\bar{u} - \kappa \bar{u} = -\kappa \bar{u}
\]
and \(\hat{H}(\lambda, \bar{w})\), \(H = L^2([0, \tau]; \mathbb{R}^{2n})\) and \(X = C^0_{\mathcal{M}}([0, \tau]; \mathbb{R}^{2n})\) satisfy conditions in Theorem C.7 ([52, Theorem 3.6] or [50, Theorem 4.6]) except for the condition (f). Note that \(\mathcal{L}_\lambda(\cdot)\) are also invariant for the \(S^1\)-action with \(S^1 = \mathbb{R}/(ITZ)\) given by (4.2).

Step 1 (Prove the alternative of (i) or (ii)). Let us prove that Theorem C.8 ([52, Theorem 3.7] or [50, Theorem 5.12]) can be used. Because of the assumption (b), it suffices to prove that

\[
\text{the fixed point set of the induced } S^1\text{-action on } H^0_\mu := \ker(B_\mu(\bar{w})) \text{ is } \{0\},
\]

where \(B_\mu(\bar{w})\) is given by (3.15). Let \(\xi \in H^0_\mu\) be a fixed point for the above \(S^1\)-action. Then it is constant and satisfies \(M\xi = \xi\). Moreover, \(B_\mu(\bar{w})\xi = \Lambda_{M, \tau, \kappa} \xi + \nabla_\bar{z}(H_\mu)^*(\mu; \bar{w})\xi = 0\), that is,
\[
\xi = -\Lambda_{M, \tau, I_{2n}} (\nabla_\bar{z}(H_\mu)^*(\mu; \bar{w})\xi) = -\kappa \nabla_\bar{z}(H_\mu)^*(\mu; \bar{w})\xi = -\kappa [\nabla_\bar{z}(H_\mu, \bar{w}) - \kappa I_{2n}^{-1}]\xi,
\]
where the second equality comes from (2.3), and the third equality is because
\[
\nabla_\bar{z}(H_\mu)^*(\mu; \bar{w})\nabla_\bar{z}(H_\mu, \bar{u}) = I_{2n}
\]
by the equality \(\bar{w} = -J\bar{u} - \kappa \bar{u} = \nabla_\bar{z}H(\mu, \bar{u}) - \kappa \bar{u} = \nabla_\bar{z}H_\mu(\mu, \bar{u})\) and [20] page 92, Proposition 10]. Then the fixed point set of the induced \(S^1\)-action on \(H^0_\mu\) is equal to
\[
\{\nabla_\bar{z}(H_\mu, \bar{u})\xi = 0 \mid \xi \in \mathbb{R}^{2n}\}.
\]
By the assumption (a), i.e., \(\ker(M - I_{2n}) \cap \ker(\nabla_\bar{z}(H_\mu, \bar{u})) = \{0\}\), we deduce \(\xi = 0\). (4.3) is proved.

Now by Theorem C.8 ([52, Theorem 3.7] or [50, Theorem 5.12]) one of the following alternatives occurs:

(I) \((\mu, \bar{w})\) is not an isolated solution in \(\{\mu\} \times L^2([0, \tau]; \mathbb{R}^{2n})\) of \(\nabla_\bar{w}\psi_\lambda(\mu, w) = 0\).

(II) There exist left and right neighborhoods \(\Lambda^-\) and \(\Lambda^+\) of \(\mu\) in \(\Lambda\) and integers \(n^+, n^- \geq 0\), such that \(n^+ + n^- \geq \frac{1}{2} \dim H^0_\mu\), and that for \(\lambda \in \Lambda^- \setminus \{\mu\}\) (resp. \(\lambda \in \Lambda^+ \setminus \{\mu\}\)) the functional \(\psi_\lambda(\lambda, \cdot)\) has at least \(n^-\) (resp. \(n^+\)) distinct critical \(S^1\)-orbits disjoint with \(\bar{w}\), which converge to \(\bar{w}\) in \(C^0_{\mathcal{M}}([0, \tau]; \mathbb{R}^{2n})\) as \(\lambda \to \mu\).
In the case of (I), we have a sequence \((w_j) \subset L^2([0, \tau]; \mathbb{R}^{2n}) \setminus \{\bar{w}\}\) such that \(||w_j - \bar{w}||_2 \to 0\) as \(j \to \infty\) and that \(\nabla_w \psi_\kappa(\mu, w_j) = 0\) for each \(j \in \mathbb{N}\). Because \(\bar{w}\) is a fixed point for the \(S^1\)-action, the \(S^1\)-orbits are compact and different \(S^1\)-orbits are not intersecting, by passing to a subsequence we can assume that any two of \(w_j, j = 0, 1, \cdots\), do not belong the same \(S^1\)-orbit. The same claim also holds for \(\bar{u}\) and \(u_j := -\Lambda_{\mathbb{M}, \tau, \kappa, I_{2n}}^{-1} w_j \neq \bar{u}, j = 1, 2, \cdots\). Moreover, since \(\Lambda_{\mathbb{M}, \tau, \kappa, I_{2n}} : W^1_{\mathbb{M}}([0, \tau]; \mathbb{R}^{2n}) \to L^2([0, \tau]; \mathbb{R}^{2n})\) is a Banach space isomorphism, \(u_j \to \bar{u}\) in \(W^1_{\mathbb{M}}([0, \tau]; \mathbb{R}^{2n})\), and hence \(||u_j - \bar{u}||_{C^1} \to 0\) by Proposition 1.21 [or its proof]. Since \(H(\mu, \cdot)\) is \(\mathbb{M}\)-invariant, each \(u_j\) can be extended into a solution \(\bar{v}_j := (u_j)^{\mathbb{M}}\) of (1.20) with \(\lambda = \mu\) via (1.7), which is not equal to \(\bar{v}\). Clearly, any two of \(\bar{v}_j, j = 0, 1, \cdots, \) are \(\mathbb{R}\)-distinct, and \((\bar{v}_j)\) converges to \(\bar{v}\) on any compact interval \(I \subset \mathbb{C}\) in \(\mathbb{C}\)-topology.

In the case of (II), note firstly that \(\dim H^0_\mu\) is equal to the number of maximal linearly independent solutions of

\[
\dot{u}(t) = J\nabla^2_\mu H(\mu, u)u(t) \quad \forall t \in [0, \tau] \quad \text{and} \quad u(\tau) = M u(0). \tag{4.4}
\]

Since \(M^T \nabla^2_\mu H(\mu, M z) = \nabla^2_\mu H(\mu, z)\), the extension via (1.7) establishes an isomorphism between the solution space of (4.4) and that of

\[
\dot{v}(t) = J\nabla^2_\mu H(\mu, v)v(t) \quad \text{and} \quad v(t + \tau) = M v(t), \quad \forall t \in \mathbb{R}.
\]

Hence \(\dim H^0_\mu\) is equal to the number of maximal linearly independent solutions of (1.22) with \(\lambda = \mu\). For \(\lambda \in \Lambda^- \setminus \{\mu\}\) (resp. \(\lambda \in \Lambda^+ \setminus \{\mu\}\) let \(S^1 \cdot u^\lambda_i, i = 1, \cdots, n^-\) (resp. \(n^+\)) be distinct critical \(S^1\)-orbits of \(\psi_\kappa(\lambda, \cdot)\) disjoint with \(\bar{w}\), which converge to \(\bar{w}\) in \(C^0_\mathbb{M}([0, \tau]; \mathbb{R}^{2n})\) as \(\lambda \to \mu\).

Since \(\Lambda^\mu_{\mathbb{M}, \tau, \kappa, I_{2n}} : C^1_\mathbb{M}([0, \tau]; \mathbb{R}^{2n}) \to C^0_\mathbb{M}([0, \tau]; \mathbb{R}^{2n})\) is a Banach space isomorphism,

\[
u^\lambda_i := (\Lambda^\mu_{\mathbb{M}, \tau, \kappa, I_{2n}})^{-1}(w^\lambda_i) \notin S^1 \cdot \bar{u} = \{\bar{u}\}, \quad i = 1, \cdots, n^- \quad \text{(resp.} \quad n^+)\),

and \(S^1 \cdot u^\lambda_i \to \bar{u}\) in \(C^1_\mathbb{M}([0, \tau]; \mathbb{R}^{2n})\) as \(\lambda \to \mu\). It follows that \(v^\lambda_i := (u^\lambda_i)^{\mathbb{M}} + \nu^\lambda_i, i = 1, \cdots, n^-\) (resp. \(n^+)\), are \(\mathbb{R}\)-distinct solutions of (1.20) under the initial assumptions on \(H\) in Assumption 1.13 and also \(\mathbb{R}\)-distinct from \(v^\lambda_i\) because each \(v^\lambda_i\) is a fixed point for the \(S^1\)-action given by (1.1).

**Step 2** (Prove the part after “Moreover”). Since any orbit \(O\) of the induced \(S^1\)-action on \(H^0_\mu\) is either a point or an embedded circle, our assumption \(\dim H^0_\mu = \nu_{\mathbb{M}, \gamma_\mu} \geq 3\) shows that Theorem C.9 (or [52] Theorem 3.10) is available to the functionals \(L_\lambda(\cdot) = \psi_\kappa(\lambda, \bar{w} + \cdot)\). Then we obtain either (I) or one of the following alternatives occurs:

**(III)** For every \(\lambda \in \Lambda \setminus \{\mu\}\) near \(\mu \in \Lambda\) there is a \(S^1\)-orbit \(S^1 \ast \bar{w}_\lambda \neq S^1 \ast \bar{w} = \{\bar{w}\}\) near \(\bar{w} \in C^0_\mathbb{M}([0, \tau]; \mathbb{R}^{2n})\) such that \(\nabla_w \psi_\kappa(\lambda, \bar{w}_\lambda) = 0\) and that \(S^1 \ast \bar{w}_\lambda \to \bar{w}\) in \(C^0_\mathbb{M}([0, \tau]; \mathbb{R}^{2n})\) as \(\lambda \to \mu\).

**IV** For any small \(S^1\)-invariant neighborhood \(\mathcal{N}\) of \(\bar{w}\) in \(C^0_\mathbb{M}([0, \tau]; \mathbb{R}^{2n})\) there is an one-sided deleted neighborhood \(\Lambda^\mu\) of \(\mu\) in \(\Lambda\) such that for any \(\lambda \in \Lambda^\mu\), \(\nabla_w \psi_\kappa(\lambda, \cdot) = 0\) has either infinitely many \(S^1\)-orbits of solutions in \(\mathcal{N}\), \(S^1 \ast \bar{w}_\lambda^j, j = 1, 2, \cdots\), or at least two \(S^1\)-orbits of solutions in \(\mathcal{N}\) with different energy, \(S^1 \ast \bar{w}_1^\lambda \neq \{\bar{w}\}\) and \(S^1 \ast \bar{w}_2^\lambda \neq \{\bar{w}\}\). Moreover, these orbits converge to \(\bar{w}\) in \(C^0_\mathbb{M}([0, \tau]; \mathbb{R}^{2n})\) as \(\lambda \to \mu\).

In the case of (III), let \(\bar{u}_\lambda := -\Lambda_{\mathbb{M}, \tau, \kappa, I_{2n}}^{-1}(\bar{w}_\lambda), \) which is not in \(S^1 \ast \bar{u} = \{\bar{u}\} \). Then \(\bar{v}_\lambda := (\bar{u}_\lambda)^{\mathbb{M}} + \nu^\lambda_i\) a solution of (1.20) under the initial assumptions on \(H\) in Assumption 1.13 and also \(\mathbb{R}\)-distinct from \(v^\lambda_i\).
In the case of (IV), let \( \hat{u}_i^\lambda := -\Lambda_{M,T,I_{2n}}^{-1}(\hat{w}_\lambda), i = 1, 2, \) and \( \tilde{u}_j^\lambda := -\Lambda_{M,T,I_{2n}}^{-1}(\tilde{w}_\lambda), j = 1, 2, \cdots \). Then \( \hat{v}_i^\lambda := (\hat{u}_i^\lambda)^M + v_\lambda, i = 1, 2, \) and \( \tilde{v}_j^\lambda := (\tilde{u}_j^\lambda)^M + v_\lambda, j = 1, 2, \cdots \), are the desired solutions if \( N \) is small enough. In fact, since \( \Lambda_{M,T,I_{2n}}^\epsilon : C_M^1([0, \tau]; \mathbb{R}^{2n}) \to C_M^0([0, \tau]; \mathbb{R}^{2n}) \) is a Banach space isomorphism and \( \tilde{w} = 0 \), we deduce

\[
\|\hat{v}_i^\lambda\|_{[0,\tau]} - v_\lambda\|_{[0,\tau]}\|_{C^1} \leq \|\tilde{v}_j^\lambda\|_{[0,\tau]}\|_{C^1} \leq \|\tilde{v}_j^\lambda\|_{[0,\tau]}\|_{C^0} = \|\tilde{v}_j^\lambda - \tilde{w}\|_{C^0}, \quad i = 1, 2, \\
\|\hat{v}_i^\lambda\|_{[0,\tau]} - v_\lambda\|_{[0,\tau]}\|_{C^1} \leq \|\tilde{v}_j^\lambda\|_{[0,\tau]}\|_{C^1} \leq \|\tilde{v}_j^\lambda\|_{[0,\tau]}\|_{C^0} = \|\tilde{v}_j^\lambda - \tilde{w}\|_{C^0}, \quad \forall \lambda \in \mathbb{N}.
\]

Hence we only need to take the above neighborhood \( N \) as the open ball of radius \( \|\Lambda_{M,T,I_{2n}}^{-1}\|\|\tau\|/\epsilon \) and center at \( \tilde{w} = 0 \) in \( C_M^0([0, \tau]; \mathbb{R}^{2n}) \).

**Proof of Corollary 1.16.** By Corollary 1.14 with \( H_0 = 0 \) and \( \hat{H} = H, \gamma_\lambda(t) = \exp(\lambda t J H''(v_0)) \) and therefore

\[\Sigma_1 = \{ \lambda \in \mathbb{R} | v_0, M) \cup \{0\}\]

is a discrete set in \( \mathbb{R} \), which gives rise to the claim (A). In addition, for each \( \mu \in \Gamma(H, \tilde{v}, M) \) the problem (1.22) with \( H(\mu, \cdot) = \mu \hat{H}(\cdot) \) and \( v_\mu = \tilde{v} \) has no nonzero constant solutions because \( \text{Ker}(H''(\tilde{v})) = \{0\} \).

Suppose \( H''(\tilde{v}) > 0 \) and \( \Gamma(H, \tilde{v}, M) \cap (0, \infty) \neq \emptyset \) (resp. \( H''(\tilde{v}) < 0 \) and \( \Gamma(H, \tilde{v}, M) \cap (-\infty, 0) \neq \emptyset \)). For \( \mu \in \Gamma(H, \tilde{v}, M) \cap (0, \infty) \) (resp. \( \Gamma(H, \tilde{v}, M) \cap (-\infty, 0) \)), it follows from (A.39) (resp. (A.36)) that \( i_{1,M}(\gamma_\lambda) = i_{1,M}(\gamma_\mu) \) for \( \lambda \leq \mu \) close to \( \mu \) and that \( i_{1,M}(\gamma_\lambda) = i_{1,M}(\gamma_\mu) + \nu_{1,M}(\gamma_\mu) \) for \( \lambda > \mu \) close to \( \mu \). Hence for each \( \mu \in \Gamma(H, \tilde{v}, M) \) Corollary 1.15 implies that one of the following alternatives occurs:

1. The problem

\[\hat{v}(t) = \mu J \nabla H(v(t)) \quad \text{and} \quad v(t + 1) = M v(t), \forall t \in \mathbb{R} \tag{4.5}\]

has a sequence of \( \mathbb{R} \)-distinct solutions, \( w_k, k = 1, 2, \cdots \), which are \( \mathbb{R} \)-distinct from \( \tilde{v} \) and converges to \( \tilde{v} \) on any compact interval \( I \subset \mathbb{R} \) in \( C^1 \)-topology.

2. There exist left and right neighborhoods \( \Lambda^- \) and \( \Lambda^+ \) of \( \mu \) in \( \mathbb{R} \setminus \{ \mu \} \) and integers \( n^+, n^- \geq 0 \), such that \( n^+ + n^- \geq \nu_{1,M}(\gamma_\mu)/2 \), and for \( \lambda \in \Lambda^- \setminus \{ \mu \} \) (resp. \( \lambda \in \Lambda^+ \setminus \{ \mu \} \)), the problem

\[\hat{v}(t) = \lambda J \nabla H(v(t)) \quad \text{and} \quad v(t + 1) = M v(t), \forall t \in \mathbb{R} \tag{4.6}\]

has at least \( n^- \) (resp. \( n^+ \)) \( \mathbb{R} \)-distinct solutions, \( \tilde{w}_i^\lambda, i = 1, \cdots, n^- \) (resp. \( n^+ \)), which are \( \mathbb{R} \)-distinct from \( v_0 \) and converge to \( \tilde{v} \) on any compact interval \( I \subset \mathbb{R} \) in \( C^1 \)-topology as \( \lambda \to \mu \).

Moreover, if \( \dim \text{Ker}(\exp(\mu H''(\tilde{v}) - M)) \geq 3 \), by the conclusions after “Moreover” in Theorem 1.13 with \( H(\lambda, x) = \lambda \hat{H}(x) \) and \( v_\lambda \equiv \tilde{v} \forall \lambda \), we obtain that either (1) holds or one of the following alternatives occurs:

3. For every \( \lambda \in \mathbb{R} \setminus \{ \mu \} \) near \( \mu \), (1.6) has a solution \( w_\lambda \), which is \( \mathbb{R} \)-distinct from \( \tilde{v} \) and converges to \( \tilde{v} \) on any compact interval \( I \subset \mathbb{R} \) in \( C^1 \)-topology as \( \lambda \to \mu \).

4. For a given \( \epsilon > 0 \) there is an one-sided neighborhood \( \Lambda_0 \) of \( \mu \) in \( \mathbb{R} \setminus \{ \mu \} \) such that for any \( \lambda \in \Lambda_0 \setminus \{ \mu \} \), (1.6) with parameter value \( \lambda \) has either infinitely many \( \mathbb{R} \)-distinct solutions \( \tilde{w}_k^\lambda \) such that each of them is \( \mathbb{R} \)-distinct from \( \tilde{v} \) and \( \|\tilde{w}_k^\lambda\|_{[0,1]} - \tilde{v}\|_{C^1} < \epsilon, k = 1, 2, \cdots \), or at least two \( \mathbb{R} \)-distinct solutions \( \tilde{w}_1^\lambda \) and \( \tilde{w}_2^\lambda \) such that:

a) each of them is \( \mathbb{R} \)-distinct from \( \tilde{v} \),
b) \( \|\dot{w}_i^k\|_{[0,1]} - v \|_{C^1} < \epsilon, \ i = 1, 2, \)

c) 
\[
\int_0^1 \left[ \frac{1}{2}(J\dot{w}_i^k(t), \dot{w}_i^k(t))_{\mathbb{R}^{2n}} + \lambda H(\dot{w}_i^k(t)) \right] dt \neq \int_0^1 \left[ \frac{1}{2}(J\dot{w}_2^k(t), \dot{w}_2^k(t))_{\mathbb{R}^{2n}} + \lambda H(\dot{w}_2^k(t)) \right] dt.
\]

Define \( v_k : \mathbb{R} \to \mathbb{R}^{2n}, \ t \mapsto w_k(t/\mu), \ k = 1, 2, \ldots, \) and \( v^i : \mathbb{R} \to \mathbb{R}^{2n}, \ t \mapsto w_i(t/\lambda), \ i = 1, \ldots, n^+ \) (resp. \( n^- \)). By (1) and (2) they satisfy (B.i) and (B.ii), respectively. Similarly, (B.iii) and (B.iv) may follow from (3) and (4), respectively.

4.2 Proofs of Theorems 1.18, 1.19, 1.20, 1.21

Since \( \mathbb{R} \) is a non-compact Lie group and the \( \mathbb{R} \)-action in (4.2), (B.iv) may follow from (3) and (4), respectively.

In what follows we assume that \( \bar{v} : \mathbb{R} \to \mathbb{R}^{2n} \) is a nonconstant solution of (1.20) for each \( \lambda \in \Lambda \). Then \( \bar{v} \) is \( C^3 \). As in Section 4.3 we write \( \bar{u} := \bar{v} |_{[0,\tau]} \). It is nonconstant. By Proposition 4.1 and the above discussion we see:

- either the map \( \theta \mapsto \theta * \bar{u} \) is an one-to-one \( C^2 \) immersion from \( \mathbb{R} \) to \( W_{M,\tau}^{1,2}([0,\tau]; \mathbb{R}^{2n}), \) (thus the restriction of it to any compact submanifold of \( \mathbb{R} \) is a \( C^2 \)-embedding by an elementary result in general topology,) and \( \{ \theta * \bar{u} \mid \theta \in \mathbb{R} \} \) is a \( C^2 \)-immersed submanifold in \( W_{M,\tau}^{1,2}([0,\tau]; \mathbb{R}^{2n}) \) without self-intersection points;

- or \( \{ \theta * \bar{u} \mid \theta \in \mathbb{R} \} \) is a \( C^2 \)-embedded circle.

Let \( \bar{w} := -\Lambda_M,\tau,\kappa,\ell_2 (\bar{u}) = -\dot{J} - \bar{u} \). It belongs to \( C^1_2([0,\tau]; \mathbb{R}^{2n}) \). Since \( \Lambda_M,\tau,\kappa,\ell_2 \) is a Banach space isomorphism from \( W_{M,\tau}^{1,2}([0,\tau]; \mathbb{R}^{2n}) \) to \( L^2([0,\tau]; \mathbb{R}^{2n}), \) and equivariant with respect to the action in (1.2), i.e., \( \Lambda_M,\tau,\kappa,\ell_2 (\theta * u) = \theta * (\Lambda_M,\tau,\kappa,\ell_2 (u)) \) for any \( (\theta, u) \in \mathbb{R} \times W_{M,\tau}^{1,2}([0,\tau]; \mathbb{R}^{2n}), \) we have:

**Claim 4.2.** \( \mathcal{O} := \{ \theta * \bar{w} \mid \theta \in \mathbb{R} \} \) is either a \( C^2 \)-immersed submanifold in \( L^2([0,\tau]; \mathbb{R}^{2n}) \) without self-intersection points or a \( C^2 \)-embedded circle in \( L^2([0,\tau]; \mathbb{R}^{2n}) \). In particular, by (4.7)

\[
0 \neq \frac{d}{d\theta} (\theta \ast \bar{w}) = (\bar{w})^{\Lambda M} (\theta + \cdot) |_{[0,\tau]} \quad \forall \theta \in \mathbb{R} \quad \text{and so} \quad \dot{\bar{w}} = (\bar{w})^{\Lambda M} |_{[0,\tau]} \neq 0.
\]

Let \( H_{\perp} \) denote the orthogonal complement of \( \bar{w} \) in \( H := L^2([0,\tau]; \mathbb{R}^{2n}) \) and

\[
B^H_{\perp} (\bar{w}, \varepsilon) := \{ \bar{w} + u \mid u \in H_{\perp}, \| u \|_2 < \varepsilon \}, \ \forall \varepsilon > 0.
\]

Clearly, \( B^H_{\perp} (\bar{w}, \varepsilon) \) is a \( C^\infty \) submanifold of \( L^2([0,\tau]; \mathbb{R}^{2n}) \) containing \( \bar{w}, \) and \( T_{\bar{w}} B^H_{\perp} (\bar{w}, \varepsilon) = H_{\perp} \) and \( L^2([0,\tau]; \mathbb{R}^{2n}) = T_{\bar{w}} B^H_{\perp} (\bar{w}, \varepsilon) \oplus (\bar{w}) \).

Note that the isotropy groups \( \mathbb{R}_{\bar{w}} \) and \( \mathbb{R}_{\bar{w}} \) of the \( \mathbb{R} \)-action defined by (1.2) satisfy \( \mathbb{R}_{\bar{w}} \subset \mathbb{R}_{\bar{w}} \).

For \( \theta \in \mathbb{R}_{\bar{w}} \) and \( u \in H_{\perp}, \| \theta \ast u \|_2 = \| u \|_2 \) and \( (\bar{w}, \theta \ast u) = (\theta \ast \bar{w}, \theta \ast u) = (\bar{w}, * u) = 0 \) imply that \( B^H_{\perp} (\bar{w}, \varepsilon) \) is \( \mathbb{R}_{\bar{w}} \)-invariant. Since \( \Lambda_{M,\tau,\kappa,\ell_2} (\theta \ast \bar{u}) = \theta \ast \bar{w} \) for any \( \theta \in \mathbb{R}, \mathbb{R}_{\bar{w}} \) is equal to the isotropy group \( \mathbb{R}_{\bar{u}} \) of the \( \mathbb{R} \)-action at \( \bar{u} \in W_{M,\tau}^{1,2}([0,\tau]; \mathbb{R}^{2n}). \)
Proposition 4.3. For any $0 < \epsilon \leq \varepsilon$, $\mathbb{R} \ast B^+_H(\bar{w}, \epsilon)$ is a neighborhood of the orbit $\mathcal{O} = \mathbb{R} \ast \bar{w}$.

Because the $\mathbb{R}$-action is not $C^1$ we cannot directly deduce that the orbit $\mathbb{R} \ast \bar{w}$ has a transversal intersection with $B^+_H(\bar{w}, \epsilon)$ for every $w \in L^2([0, \tau]; \mathbb{R}^{2n})$ sufficiently close to $\bar{w}$. But Proposition 4.3 can be proved with the following:

Proposition 4.4 ([3 Proposition 3.5]). Let $\mathcal{M}$ be a finite-dimensional manifold, $N$ a (possibly infinite dimensional) Banach manifold, $Q \subset N$ a Banach submanifold, and $A$ a topological space. Assume that $\chi : A \times \mathcal{M} \to N$ is a continuous function such that there exists $a_0 \in A$ and $m_0 \in \mathcal{M}$ with:

(a) $\chi(a_0, m_0) \in Q$;
(b) $\chi(a, \cdot) : \mathcal{M} \to N$ is of class $C^1$;
(c) $\partial_2 \chi(a_0, m_0)(T_{m_0} \mathcal{M}) + T_{\chi(a_0, m_0)}Q = T_{\chi(a_0, m_0)}N$.

Then, for a $a \in A$ near $a_0$, $\chi(a, \mathcal{M}) \cap Q \neq \emptyset$.

Proof of Proposition 4.3. Applying Proposition 4.4 to $A = N = L^2([0, \tau]; \mathbb{R}^{2n})$, $\mathcal{M} = \mathbb{R}$, $Q = B^+_H(\bar{w}, \epsilon)$, $a_0 = \bar{w}$, $m_0 = 0$ and $\chi : A \times \mathcal{M} \to N : (w, \theta) \mapsto \theta * w$, we get that $(\mathbb{R} \ast \bar{w}) \cap B^+_H(\bar{w}, \epsilon) \neq \emptyset$ if $w \in L^2([0, \tau]; \mathbb{R}^{2n})$ is close to $\bar{w}$.

As in the first paragraph of Section 3 we can require that all modified $H(\lambda, \cdot)$ are also $M$-invariant. (See the proof of Theorem 1.14 in Section 4.1.) Following the notations in Section 3 we have a family of $C^1$ and twice Gâteaux-differentiable functionals on $H = L^2([0, \tau]; \mathbb{R}^{2n})$,

$$
\psi_\kappa(\lambda, u) = \int_0^1 \left[ \frac{1}{2}(u(t), (A_{M, \tau, \kappa} u)(t))_{\mathbb{R}^{2n}} + H^*_\kappa(\lambda; u(t)) \right] dt, \quad \lambda \in \Lambda. \tag{4.8}
$$

Moreover, each $\psi_\kappa(\lambda, \cdot) := \psi_\kappa(\lambda, \cdot)$ is also invariant for the $\mathbb{R}$-action given by (4.2), and

$$
\nabla(\psi_\kappa(\lambda, v)) = A_{M, \tau, \kappa}v + \nabla_z H^*_\kappa(\lambda; v(\cdot)), \quad (4.9)
$$

$$
B_\lambda(v) := D_\nu \nabla(\psi_\kappa(\lambda, v)) = A_{M, \tau, \kappa} + \nabla_z^2 H^*_\kappa(\lambda; v(\cdot)) \in \mathcal{L}_a(L^2([0, \tau]; \mathbb{R}^{2n})), \quad (4.10)
$$

$$
B_\lambda(\bar{w}) = 0. \tag{4.11}
$$

Fortunately, we can directly prove the following result, which naturally holds if the $\mathbb{R}$-action in (4.2) is $C^1$.

Proposition 4.5. There exists $\varepsilon > 0$ such that for any $\lambda \in \Lambda$, if $u \in B^{-1}_H(0, \varepsilon)$ is a critical point of the restriction of $\mathcal{L}_\lambda$ to $B^{-1}_H(0, \varepsilon)$ then it is a critical point of $\mathcal{L}_\lambda(\cdot) = \psi_\kappa(\lambda, \bar{w} + \cdot)$. Moreover, $u$ is $C^1$ and $\|u\|_{C^1} \to 0$ as $\|u\|_2 \to 0$.

Proof. We only need to prove sufficiency. Let $u \in H^{-1}$ be a critical point of the restriction of $\mathcal{L}_\lambda$ to $H^{-1}$. Then $\bar{w} + u$ is a critical point of the restriction of $(\psi_\kappa)_\lambda$ to $B^{-1}_H(\bar{w}, \varepsilon)$, i.e.,

$$
d(\psi_\kappa)_\lambda(\bar{w} + u)[\xi] = 0 \quad \forall \xi \in T_{\bar{w} + u}B^{-1}_H(\bar{w}, \varepsilon) = H^{-1}. \tag{4.12}
$$

Step 1 (Prove that $u$ is $C^1$). Since

$$
d(\psi_\kappa)_\lambda(\bar{w} + u)[\xi] = (A_{M, \tau, \kappa}(\bar{w} + u) + \nabla_z H^*_\kappa(\lambda; \bar{w} + u), \xi)_2 = 0
$$

for all $\xi \in H^{-1}$, we have $\mu(\lambda, u) \in \mathbb{R}$ such that

$$
\nabla(\psi_\kappa)_\lambda(\bar{w} + u) = A_{M, \tau, \kappa}(\bar{w} + u) + \nabla_z H^*_\kappa(\lambda; \bar{w} + u) = \mu(\lambda, u) \bar{w}. \tag{4.13}
$$
Note that \( \dot{w} \) is \( C^1 \) and that \( A_{M,\tau,\kappa}(\dot{w} + u) \in W^{1,2}_M([0, \tau]; \mathbb{R}^{2n}) \) by (3.12). The map

\[
\Gamma : [0, \tau] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad (t, z) \mapsto \nabla_z H^*_\kappa(\lambda; z) - \mu(\lambda, u)\dot{w}(t) + (A_{M,\tau,\kappa}(\dot{w} + u))(t)
\]

is continuous, and \( C^1 \) in \( z \). But \( D_1\Gamma(t, z) = \nabla_z^2 H^*_\kappa(\lambda; z) \) is invertible. Therefore applying the implicit function theorem to \( \Gamma \) we derive from (4.13) that \( \dot{w} + u \) and therefore \( u \) is \( C^0 \). By (2.6), \( A_{M,\tau,\kappa}(\dot{w} + u) \in C^0([0, \tau]; \mathbb{R}^{2n}) \) and hence

\[
t \mapsto \mu(\lambda, u)\dot{w}(t) - (A_{M,\tau,\kappa}(\dot{w} + u))(t)
\]

is \( C^1 \). Then \( \Gamma \) is \( C^1 \). Using the implicit function theorem to \( \Gamma \) again we obtain that \( u \) is \( C^1 \).

**Step 2** *(For any \( \varepsilon > 0 \), prove that there exists \( \delta > 0 \) such that \( \|u\|_2 < \delta \) implies \( |\mu(\lambda, u)| < \varepsilon \) for all \( \lambda \in \Lambda \)).* Since \( \nabla(\psi_\kappa)\lambda(\dot{w}) = 0 \forall \lambda \in \Lambda \), by (1.13) we have

\[
\mu(\lambda, u)(\dot{w}, v)_2 = (\nabla(\psi_\kappa)\lambda(\dot{w} + u), v)_2 - (\nabla(\psi_\kappa)\lambda(\dot{w}), v)_2
\]

\[
= (A_{M,\tau,\kappa}(u), v)_2 + (\nabla_z H^*_\kappa(\lambda; \dot{w} + u), v)_2 - (\nabla_z H^*_\kappa(\lambda; \dot{w}), v)_2
\]

\[
= \int_0^\tau \int_0^1 (\nabla_z^2 H^*_\kappa(\lambda; t; \dot{w}(t) + su(t))u(t), v(t))_{\mathbb{R}^{2n}}dt ds
\]

\[
+ (A_{M,\tau,\kappa}(u), v)_2, \quad \forall v \in H.
\]

It follows from this and (5.7) that

\[
|\mu(\lambda, u)| \cdot \|\dot{w}\|_2 \leq \|A_{M,\tau,\kappa}(u)\|_2 + \frac{1}{c_1}\|u\|_2.
\]

Then the claim follows because \( \|\dot{w}\|_2 > 0 \) and \( A_{M,\tau,\kappa} \in \mathcal{L}_2(L^2([0, \tau]; \mathbb{R}^{2n})) \) by (3.12).

**Step 3** *(For any \( \varepsilon > 0 \), prove there exists \( \varepsilon > 0 \) such that \( \|u\|_2 < \varepsilon \) implies \( \|u\|_{C^1} < \varepsilon \)).* Since \( \nabla(\psi_\kappa)\lambda(\dot{w}) = 0 \forall \lambda \in \Lambda \), by (1.13) we have

\[
A_{M,\tau,\kappa}(u) + \nabla_z H^*_\kappa(\lambda; \dot{w} + u) - \nabla_z H^*_\kappa(\lambda; \dot{w}) = \mu(\lambda, u)\dot{w}.
\]

(4.14)

Note that this also holds in the sense of pointwise because \( \dot{w} \) is \( C^2 \), \( u \) is \( C^1 \) and

\[
A_{M,\tau,\kappa}(u)(t) = e^{\varepsilon t}\varepsilon^{-1} - 1 + \int_0^\tau e^{-\varepsilon t}Ju(t)dt - e^{\varepsilon t}\int_0^t e^{-\varepsilon s}Ju(s)ds
\]

(4.15)

is \( C^2 \) by (3.12)–(3.13). It follows that there exists a constant \( C_\kappa > 0 \) such that

\[
|A_{M,\tau,\kappa}(u)(t)|_{\mathbb{R}^{2n}} \leq C_\kappa\|u\|_2, \quad \forall t \in [0, \tau].
\]

(4.16)

Clearly, (3.7), the mean value theorem of integrals and (4.14) may lead to

\[
\frac{1}{c_2}\|u(t)\|_{\mathbb{R}^{2n}}^2 \leq \int_0^1 (\nabla_z^2 H^*_\kappa(\lambda; t; \dot{w}(t) + su(t))u(t), u(t))_{\mathbb{R}^{2n}}ds
\]

\[
= (\nabla_z H^*_\kappa(\lambda; \dot{w}(t) + u(t)) - \nabla_z H^*_\kappa(\lambda; \dot{w}(t)), u(t))_{\mathbb{R}^{2n}}
\]

\[
= (\mu(\lambda, u)\dot{w}(t) - A_{M,\tau,\kappa}(u)(t), u(t))_{\mathbb{R}^{2n}}
\]

\[
\leq |\mu(\lambda, u)|_{\mathbb{R}^{2n}} \cdot |\dot{w}(t)|_{\mathbb{R}^{2n}} \cdot |u(t)|_{\mathbb{R}^{2n}} + |A_{M,\tau,\kappa}(u)(t)|_{\mathbb{R}^{2n}} \cdot |u(t)|_{\mathbb{R}^{2n}}
\]

and so

\[
\frac{1}{c_2}|u(t)|_{\mathbb{R}^{2n}} \leq |\mu(\lambda, u)|_{\mathbb{R}^{2n}} \cdot |\dot{w}(t)|_{\mathbb{R}^{2n}} + C_\kappa\|u\|_2, \quad \forall t \in [0, \tau]
\]
by (4.16). Therefore, since \( \dot{w} \) is \( C^1 \), for any \( \epsilon' > 0 \), from this and the result in Step 2 we derive that there exists \( \varepsilon > 0 \) such that
\[
\|u\|_2 \leq \varepsilon \quad \Rightarrow \quad \|u\|_{C^0} < \epsilon'.
\] (4.17)

Taking the derivatives on two sides of equation in (1.14) with respect to \( t \) we obtain
\[
\frac{d}{dt} A_{M,\tau,\kappa}(u)(t) + \nabla^2_u H^*_\kappa(\lambda; \bar{w}(t) + u(t))\dot{u}(t) + \nabla^2_u H^*_\kappa(\lambda; \bar{w}(t))\dot{\bar{w}}(t) - \nabla^2_u H^*_\kappa(\lambda; \bar{w}(t))\dot{w}(t) = \mu(\lambda, u)\ddot{w}(t).
\]

It follows from (3.7) and this that
\[
\frac{1}{c_2}\|\dot{u}(t)\|_{\mathbb{R}^{2n}} \leq \|\nabla^2_u H^*_\kappa(\lambda; \bar{w}(t) + u(t))\dot{u}(t)\|_{\mathbb{R}^{2n}} \leq \|\nabla^2_u H^*_\kappa(\lambda; \bar{w}(t) + u(t)) - \nabla^2_u H^*_\kappa(\lambda; \bar{w}(t))\|_{\mathbb{R}^{2n}} + \|\mu(\lambda, u)\|_{\mathbb{R}^{2n}}\|\ddot{w}(t)\|_{\mathbb{R}^{2n}} + \|\mu(\lambda, u)\|_{\mathbb{R}^{2n}}\|\ddot{w}(t)\|_{\mathbb{R}^{2n}} (4.18)
\]

When \( \|u\|_{C^0} \to 0 \), (4.15) and Step 2 lead to
\[
\max_t \left\| \frac{d}{dt} A_{M,\tau,\kappa}(u)(t) \right\|_{\mathbb{R}^{2n}} \to 0 \quad \text{and} \quad \|\mu(\lambda, u)\|_{\mathbb{R}^{2n}}\|\ddot{w}(t)\|_{\mathbb{R}^{2n}} \to 0,
\]
respectively. Moreover, for a given compact neighborhood \( U \) of \( \bar{w}([0, \tau]) \) in \( \mathbb{R}^{2n} \), since \( \nabla^2_u H^*_\kappa(\lambda; z) \) is uniformly continuous in \( \Lambda \times U \), we deduce that
\[
\max_t \left\| \nabla^2_u H^*_\kappa(\lambda; \bar{w}(t) + u(t)) - \nabla^2_u H^*_\kappa(\lambda; \bar{w}(t)) \right\|_{\mathbb{R}^{2n} \to \mathbb{R}^{2n}} \to 0
\]
uniformly with respect to \( \lambda \in \Lambda \) as \( \|u\|_{C^0} \to 0 \). From these and (4.15) we deduce that \( \|\dot{u}\|_{C^0} \to 0 \) as \( \|u\|_{C^0} \to 0 \). The desired claim follows from this and (4.17).

**Step 4** (Prove \( d(\psi_\kappa)_\lambda(\bar{w} + u) = 0 \)). Note that \( \|\dot{w}\|_{C^0} > 0 \). By Step 3 we get \( \varepsilon > 0 \) such that \( \|\dot{u}\|_2 \leq \varepsilon \) implies \( \|u\|_{C^1} < \|\dot{w}\|_{C^0} \) and \( \|u\|_{C^1} < \|\dot{w}\|_2/\sqrt{\varepsilon} \). Then \( w := \bar{w} + u \) satisfy
\[
\|\dot{w}\|_{C^0} \geq \|\dot{w}\|_{C^0} - \|\dot{u}\|_{C^0} > 0 \quad \text{and} \quad \langle \dot{w}, \dot{w} \rangle = \langle \dot{w}, \dot{u} \rangle + \langle \dot{w}, \dot{u} \rangle \geq \|\dot{w}\|_2^2 - \|\dot{w}\|_2\|\dot{u}\|_2 \geq \|\dot{w}\|_2^2 - \sqrt{\varepsilon} \|\dot{w}\|_2\|\dot{u}\|_{C^1} > 0.
\]
The former shows that \( w \) is nonconstant, and therefore that the map \( \theta \mapsto \theta \ast w \) is a \( C^1 \) immersion from \( \mathbb{R} \) to \( H = L^2([0, \tau]; \mathbb{R}^{2n}) \). Then for \( \nu > 0 \) small enough, \( \Delta := [-\nu, \nu \ast w] \) is a \( C^1 \) embedded submanifold and \( T_w\Delta = \mathbb{R}\dot{w} \). Because \( \langle \dot{w}, \dot{w} \rangle > 0 \), the orthogonal decomposition \( H = (\mathbb{R}\dot{w}) \oplus T_wB^H_{\mathbb{R}}(\bar{w}, \varepsilon) \) implies a direct sum decomposition of Banach spaces
\[
H = T_w\Delta \oplus T_wB_{H,\perp}(\bar{w}, \varepsilon). \quad (4.19)
\]
Since \( (\psi_\kappa)_\lambda \) is invariant for the \( \mathbb{R} \)-action given by (2.2), \( d(\psi_\kappa)_\lambda(w)[\xi] = 0 \forall \xi \in T_w\Delta \). From this, (1.12) and (4.19) it follows that \( d(\psi_\kappa)_\lambda(w) = 0 \). \( \square \)
Consider the following Banach space and functional
\[ X^+ := \{ u \in C^0_\mathcal{M}([0,\tau]; \mathbb{R}^{2n}) \mid (\dot{w}, u)_2 = 0 \}, \]
\[ \mathcal{L}^+ : H^+ \to \mathbb{R}, \ u \mapsto \mathcal{L}(u) = \psi_{\kappa}(\lambda, \dot{w} + u). \]  
(4.20)

They are invariant under the isotropy group \( \mathbb{R} \cdot \bar{w} \), and \( d\mathcal{L}^+(0) = 0 \) \( \forall \lambda \). Moreover, by shrinking \( \varepsilon > 0 \) if necessary, Proposition 4.7 shows that for \( u \in B_{H^+}(0, \varepsilon) \),
\[ d\mathcal{L}^+(u) = 0 \iff d\mathcal{L}(u) = d(\psi_{\kappa})(\dot{w} + u) = 0. \]
(4.22)

Denote by \( \Pi : H \to H^+ \) the orthogonal projection. Then \( \Pi(u) = u - \frac{(u, \dot{w})_2}{||\dot{w}||_2^2} \dot{w} \) for \( u \in H \), and
\[ \nabla \mathcal{L}^+(u) = \nabla(\psi_{\kappa})(\dot{w} + u) - \frac{(\nabla(\psi_{\kappa})(\dot{w} + u), \dot{w})_2}{||\dot{w}||_2^2} \dot{w} \ \forall u \in H^+, \]
(4.23)

where \( \nabla(\psi_{\kappa})(\dot{w} + u) = A_{M,\tau,\kappa}(\dot{w} + u) + \nabla_z H^*_{\kappa}(\lambda, \dot{w}(\cdot) + v(\cdot)) \) by (3.14). Since \( \dot{w} \) is \( C^1 \), it easily follows from (3.14) that \( \nabla \mathcal{L}^+(u) \in X^+ \) for any \( u \in X^+ \), and
\[ \mathcal{A}_\lambda : X^+ \to X^+, \ u \mapsto \nabla \mathcal{L}^+(u) \]
(4.24)
is \( C^1 \). For any \( (\lambda, x), (\lambda_0, x_0) \in \Lambda \times X^+ \), noting that both \( \| \cdot \|_X \) and \( \| \cdot \|_{X^+} \) are \( \| \cdot \|_{C^0} \), and
\[ \mathcal{A}_\lambda(x) - \mathcal{A}_{\lambda_0}(x_0) = \nabla(\psi_{\kappa})(\dot{w} + x) - \nabla(\psi_{\kappa})(\dot{w} + x_0) + \frac{(\nabla(\psi_{\kappa})(\dot{w} + x) - \nabla(\psi_{\kappa})(\dot{w} + x_0), \dot{w})_2}{||\dot{w}||_2^2} \dot{w}\]
we deduce
\[ \| \mathcal{A}_\lambda(x) - \mathcal{A}_{\lambda_0}(x_0) \|_{X^+} \leq \| \nabla(\psi_{\kappa})(\dot{w} + x) - \nabla(\psi_{\kappa})(\dot{w} + x_0) \|_X + \sqrt{\tau} \| \nabla(\psi_{\kappa})(\dot{w} + x) - \nabla(\psi_{\kappa})(\dot{w} + x_0) \| \|x||_{C^0}. \]

By Step 2 of the proof of Theorem 1.2(II), \( \Lambda \times X \ni (\lambda, v) \mapsto \nabla(\psi_{\kappa})(\dot{w} + x) = \nabla v \psi_{\kappa}(\lambda, v) \in X \) is continuous. This and (4.25) lead to

Claim 4.6. \( \Lambda \times X^+ \ni (\lambda, x) \mapsto \mathcal{A}_\lambda(x) \in X^+ \) is continuous.

From (3.14) and (3.15) we may deduce that \( \nabla \mathcal{L}^+ \) has a Gâteaux derivative at \( u \in H^+ \), \( \mathcal{B}_\lambda(u) \in \mathcal{L}_H(H^+) \), given by
\[ \mathcal{B}_\lambda(u)v = B\lambda(u + \bar{w})v - \frac{(B\lambda(u + \bar{w})v, \dot{w})_2}{||\dot{w}||_2^2} \dot{w} \ \forall v \in H^+, \]
(4.25)
where \( B\lambda(u + \bar{w}) = A_{M,\tau,\kappa} + \nabla_z^2 H^*_{\kappa}(\lambda; u(\cdot) + \bar{w}(\cdot)) \) by (3.15). Since \( P\lambda(v) = \nabla^2_z H^*_{\kappa}(\lambda; v(\cdot)) \) (resp. \( Q\lambda(v) = A_{M,\tau,\kappa} \)) is a positive definite (resp. compact self-adjoint) operator on \( H \),
\[ \mathcal{B}_\lambda(u) := \Pi \circ P\lambda(u + \bar{w})|_{H^+} \quad (\text{resp. } \Omega\lambda(u) := \Pi \circ Q\lambda(u + \bar{w})|_{H^+}) \]
(4.26)
is a positive definite (resp. compact self-adjoint) operator on \( H^+ \). Moreover, it is clear that \( \mathcal{B}_\lambda(u) = \mathcal{B}_\lambda(u) + \Omega\lambda(u) \).

Proposition 4.7. \( (H^+, X^+, \mathcal{L}^+, \mathcal{A}_\lambda, \mathcal{B}_\lambda) \) satisfies (3.1) \( \Lambda \) Hypothesis 1.1) (3.12) \( \Lambda \) Hypothesis 3.1) and (3.13) \( \Lambda \) Hypothesis 1.3) (3.14) \( \Lambda \) Hypothesis 3.2).
Proof. It remains to prove that \((H^\perp, X^\perp, \mathcal{L}_\Lambda, \mathfrak{A}_\Lambda, \mathfrak{B}_\Lambda)\) satisfies (D1) in \([50\text{ Hypothesis 1.1}] \) (\([52\text{ Hypothesis 3.1}]\)) and (C) in \([50\text{ Hypothesis 1.3}] \) (\([52\text{ Hypothesis 3.2}]\)), that is,

\[
\{ u \in H^\perp \mid \mathfrak{B}_\Lambda(0) u = su, s \leq 0 \} \subset X^\perp \quad \text{and} \quad \{ u \in H^\perp \mid \mathfrak{B}_\Lambda(0) u \in X^\perp \} \subset X^\perp.
\]

(4.27)

In fact, if \( u \in H^\perp \) and \( q \leq 0 \) satisfy \( \mathfrak{B}_\Lambda(0) u = q u \), then \( B_\Lambda(\bar{w}) u = q u \) by \([4.23]\) and \([4.11]\). The first inclusion in \([4.22]\) leads to \( u \in X \) and so \( u \in X^\perp \). Similarly, if \( u \in H^\perp \) is such that \( v := \mathfrak{B}_\Lambda(0) u \in X^\perp \), \([4.25]\) and \([4.11]\) lead to \( B_\Lambda(\bar{w}) u = v \), and so the second claim by the second inclusion in \([4.22]\).

Note that \([4.11]\) implies \( \mathbb{R} \bar{w} \subset \ker(B_\Lambda(\bar{w})) \). By \([4.23]\) and \([4.24]\) we get

\[
m_0(\mathcal{L}_\Lambda^\perp, 0) = m_0(\psi_\Lambda(\bar{w}), 1) = \dim \ker(\gamma(\tau) - M) - 1. \quad (4.29)
\]

Proof of Theorem 1.19. In the proof of Theorem 1.14 we proved that

\[
\{ \mathcal{F}_\Lambda(\cdot) := (\psi_\Lambda(\bar{w} + \cdot) \mid \lambda \in \Lambda \}
\]

satisfies the conditions of Theorem \([4.6] \) \([50\text{ Theorem 3.1}]\) with \( H = X = L^2([0, \tau] ; \mathbb{R}^{2n}) \). Thus if \( \lambda_k \to \mu \) and \((x_k) \subset H^\perp\) converges 0 it follows from \([4.26] \) that

\[
\| \psi_{\Lambda_k}(x_k) - \psi_{\mu}(0) \|_{L(H^\perp)} \leq \| Q_{\Lambda_k}(\bar{w}) - Q_{\mu}(\bar{w}) \|_{L(H^\perp)} \to 0.
\]

Similarly, we have \( \| \Omega_{\Lambda_k}(0) - \Omega_{\mu}(0) \|_{L(H^\perp)} \leq \| Q_{\Lambda}(x + \bar{w}) - Q_{\Lambda}(\bar{w}) \|_{L(H)} \to 0 \).

These imply that the conditions (i)-(ii) in \([50\text{ Theorem 3.1}]\) (cf. Theorem \([4.6]\) \([50\text{ Theorem 3.1}]\)) are satisfied with \( \mathfrak{B}_\Lambda \) and \( \mathfrak{A}_\Lambda \) near 0 in \( H^\perp \). Hence Theorem \([4.6] \) \([50\text{ Theorem 3.1}]\) is applicable to \( \mathcal{F}_\Lambda := \mathcal{L}_\Lambda \) near 0 in \( H^\perp \).

Let \((\lambda_k, v_k)\) be as in Theorem 1.19. Then \( \lambda_k \to \mu \), \( \| v_k \|_{L^2([0, \tau])} \to 0 \) and each \( w_k := -\Lambda_{\mu, \tau, \nu, t_2}(v_k) \) is a critical point of \( \psi_\Lambda \) on \( L^2([0, \tau] ; \mathbb{R}^{2n}) \). By Proposition 1.13 \( \| v_k \|_{L^2([0, \tau])} \to 0 \). So there exists \( \epsilon_\mu > 0 \) such that \( \epsilon_\mu \in (-\infty, 0) \) and hence \( \| w_k \|_{L^2([0, \tau])} \to 0 \). Since each \( \| B_{\mu} \| \to 0 \) for each \( \epsilon_\mu \), every critical point of \( \psi_\Lambda \) on \( L^2([0, \tau] ; \mathbb{R}^{2n}) \). Since all \( v_k \) are differentiable \( \varepsilon \), any two of all \( w_k \) are differentiable \( \varepsilon \). These show that \( (\mu, 0) \) is a bifurcation point of \( \nabla \mathcal{L}_\Lambda(u) = 0 \) in \( \Lambda \times H^\perp \). By Theorem \([4.6] \) \([50\text{ Theorem 3.1}]\) we get \( m_0(\mathcal{L}_\mu^\perp, 0) > 0 \). From \([4.29]\) it follows that \( \dim \ker(\gamma(\tau) - M) > 1 \). □

Proof of Theorem 1.19. Follow the notations above. In the proof of Theorem 1.4(II) we proved that \( \{ \mathcal{L}_\Lambda(\cdot) := \psi_\Lambda(\bar{w} + \cdot) \mid \lambda \in \Lambda \} \) satisfies the conditions of Theorem \([4.4] \) on \( H = L^2([0, \tau] ; \mathbb{R}^{2n}) \) and \( X = C^0_M([0, \tau] ; \mathbb{R}^{2n}) \). By Claim 4.3 Proposition 4.7 and the proof of Theorem 1.18 the conditions of Theorem \([4.3] \) are also satisfied for \( \{ (H^\perp, X^\perp, \mathcal{L}_\Lambda, \mathfrak{A}_\Lambda, \mathfrak{B}_\Lambda) \mid \lambda \in \Lambda \} \).

Note that using \([4.25\text{, }4.26]\) we can translate the assumptions (a)-(c) of Theorem 1.19 as follows:
(a') The orbit $O := \mathbb{R} \ast \bar{w}$ is an embedded circle (i.e., $\mathbb{R} \bar{w}$ is an infinite cyclic subgroup of $\mathbb{R}$ with generator $p > 0$).

(b') $m^0 (L^\perp_{\mu_k^+}, 0) \geq 1$, for each $k \in \mathbb{N}$, $\lambda_k^- \neq \lambda_k^+$,

$$[m^-(L^\perp_{\lambda^-_k}, 0), m^-(L^\perp_{\lambda^+_k}, 0) + m^0(L^\perp_{\lambda^+_k}, 0)] \cap [m^-(L^\perp_{\lambda^-_k}, 0), m^-(L^\perp_{\lambda^+_k}, 0) + m^0(L^\perp_{\lambda^+_k}, 0)] = \emptyset,$$

and either $m^0(L^\perp_{\lambda^-_k}, 0) = 0$ or $m^0(L^\perp_{\lambda^+_k}, 0) = 0$.

(c') For any critical point $w$ of $\psi_n(\mu, \cdot)$ which sits in $X$, if there exists a sequence $(s_k)$ of reals such that $s_k \cdot w$ converges to $\bar{w}$ in $X$, then $w$ is periodic, and so $\mathbb{R} \ast w$ is closed. (Clearly, this holds if $M^l = id_{\mathbb{R}^m}$ for some $l \in \mathbb{N}$.)

By (b') we may use Theorem C.4 to get $m^0 (L^\perp_{\mu^+_k}, 0) > 0$ and a sequence $\{(\lambda_k, u_k)\}_{k \geq 1}$ in $\hat{A} \times H^\perp$ converging to $(\mu, 0)$, where $\hat{A} := \{\mu, \lambda_k^+, \lambda_k^- \mid k \in \mathbb{N}\}$, such that each $u_k$ is a nonzero solution of $\nabla L^\perp_{\lambda_k^+}(w) = 0$, $k = 1, 2, \ldots$. Clearly, we can assume $u_k \in B_{\tilde{H}}(0, \varepsilon) \forall k \in \mathbb{N}$. By Proposition 4.5

$$w_k := \bar{w} + u_k \in B_{\tilde{H}}(\bar{w}, \varepsilon) \subset \mathbb{R}^1, \text{satisfies } d(\psi_n)_{\lambda_k}(w_k) = 0 \text{ and } \|w_k - \bar{w}\|_{C^1} \to 0. \quad (4.30)$$

From now on we use (a'), i.e., the orbit $O = \mathbb{R} \ast \bar{w}$ is the $C^1$ embedded circle. Since it is transversely intersecting with $B_{\tilde{H}}(\bar{w}, \varepsilon)$ at $\bar{w}$. By shrinking $\varepsilon > 0$ we may assume

$$\bigcap B_{\tilde{H}}(\bar{w}, \varepsilon) = \{\bar{w}\}.$$

It follows from this and (4.30) that there exists $k_0 > 0$ such that for each $k > k_0$, $\mathbb{R} \ast w_k$ is an immersed $C^1$ submanifold which transversely intersects with $B_{\tilde{H}}(\bar{w}, \varepsilon)$ at $w_k$. Hence $\mathbb{R} \ast w_k \neq O$ for any $k > k_0$. (Otherwise, $O$ and $B_{\tilde{H}}(\bar{w}, \varepsilon)$ have at least two distinct intersecting points $w_k$ and $\bar{w}$ in $B_{\tilde{H}}(\bar{w}, \varepsilon)$.)

Finally, we conclude that $\{\mathbb{R} \ast w_k \mid k \in \mathbb{N}\}$ is an infinite set. (Thus $(w_k)$ has a subsequence which only consists of $\mathbb{R}$-distinct elements, also denoted by $(w_k)$ for brevity. For each $k = 1, 2, \ldots$, let $v_k$ be defined by $-(\Lambda_{\mu, \kappa, \lambda_{k2}})^{-1} u_k$ via (1.7). Then $\{(\lambda_k, v_k)\}_{k \geq 1}$ is the required one. The proof is completed.) Otherwise, passing to a subsequence we may assume that all $w_k$ belong to an $\mathbb{R}$-orbit, i.e., $w_k = s_k \ast \bar{w}$ for some $s_k \in \mathbb{R}$, where $\bar{w} \in X$ satisfies $d(\psi_n)_{\lambda_k}(\bar{w}) = d(\psi_n)_{\lambda_k}(w_k) = 0$ for all $k \in \mathbb{N}$. Since $\Lambda \times X \ni (\lambda, v) \mapsto \nabla \psi_n(\lambda, v) \in X$ is continuous, and $\lambda_k \to \mu$, we obtain $d(\psi_n)_{\mu}(\bar{w}) = 0$. Clearly, $\bar{w}$ sits in the intersection of $O$ and the closure of $\mathbb{R} \ast \bar{w}$. By the assumption (c'), $\mathbb{R} \ast \bar{w}$ is closed and so equal to $O$, which contradicts the claim that $O \neq \mathbb{R} \ast w_k = \mathbb{R} \ast \bar{w}$ for any $k > k_0$ in the last paragraph. \[ \square \]

**Proof of Theorem 1.20.** The first two paragraphs in the proof of Theorem 1.19 are still valid unless we replace $\Lambda$ and $\Lambda$ by $\alpha([0, 1])$, and (b') by

(d') $[m^-(L^\perp_{\lambda^-_k}, 0), m^-(L^\perp_{\lambda^+_k}, 0) + m^0(L^\perp_{\lambda^+_k}, 0)] \cap [m^-(L^\perp_{\lambda^-_k}, 0), m^-(L^\perp_{\lambda^+_k}, 0) + m^0(L^\perp_{\lambda^+_k}, 0)] = \emptyset,$

and either $m^0(L^\perp_{\lambda^-_k}, 0) = 0$ or $m^0(L^\perp_{\lambda^+_k}, 0) = 0$.

By Theorem C.5 there exists a sequence $\{(t_k, u_k)\}_{k \geq 1}$ in $[0, 1] \times H^\perp$ converging to $(\bar{t}, 0)$, such that each $u_k$ is a nonzero solution of $\nabla L^\perp_{\lambda_k}(u) = 0$, $k = 1, 2, \ldots$. Moreover, $\alpha(\bar{t}) \neq \lambda^+$ (resp. $\mu \neq \lambda^-$) if $m^0(L^\perp_{\lambda^+_k}, 0) = 0$ (resp. $m^0(L^\perp_{\lambda^-_k}, 0) = 0$). As in the proof of Theorem 1.19 we may assume $u_k \in B_{\tilde{H}}(0, \varepsilon) \forall k \in \mathbb{N}$ and have (4.30). Letting $w_k := \bar{w} + u_k$ for each $k \in \mathbb{N}$ and repeating other arguments in the proof of Theorem 1.19 the required results can be obtained. \[ \square \]
Proof of Theorem 1.21. Part (I) follows from Theorem 1.19 directly. In order to prove Parts (II) and (III), following the first paragraph of the proof of Theorem 1.19 and comparing conditions in Theorem $\text{[C.3]}$, it is easily seen that $(\mathcal{L}_\lambda^\perp, H^\perp, X^\perp)$ with $\lambda \in \Lambda$ satisfy the conditions in Theorem $\text{[C.4]}$ except for the condition (f) with $\lambda^* = \mu$. As in the proof of Theorem 1.7 we may use (1.28)-(1.29) and the assumptions about Morse indexes in Theorem 1.21 to check that the condition (f) can be satisfied, that is, $m^0(\mathcal{L}_\mu^\perp, 0) = \nu_\tau(\gamma_\mu) - 1 > 0$, and $m^-(\mathcal{L}_\lambda^\perp, 0)$ takes values $m^-(\mathcal{L}_\mu^\perp, 0) = i_\tau(\gamma_\mu)$ and $m^-(\mathcal{L}_\mu^\perp, 0) + m_\tau(\mathcal{L}_\mu^\perp, 0) = i_\tau(\gamma_\mu) + \nu_\tau(\gamma_\mu) - 1$ as $\lambda \in \mathbb{R}$ varies in two deleted half neighborhoods of $\mu$.

Since $M = I_{2n}$, the $\mathbb{R}$-action on $H$ is actually a $S^1 = \mathbb{R}/(\tau \mathbb{Z})$ action,

$$\left[ \theta \right] \circ x = (\theta + k \tau) \ast x, \quad \forall x \in H, \quad \left[ \theta \right] = \theta + \tau \mathbb{Z} \in S^1, \quad k \in \mathbb{Z}.$$ 

This is a continuous action on $H$ (resp. $X$) via Hilbert (resp. Banach) isometry isomorphisms.

Claim 4.8. The isotropy group of the above $S^1$-action at $\bar{w}$ is

$$S^1_{\bar{w}} = \left\{ \left[ \frac{lt}{p} \right] \bigg| l = 0, \ldots, p - 1 \right\} \cong \mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z},$$

and $[\theta] \circ \bar{w} = \bar{w}$ for any $[\theta] \in S^1_{\bar{w}}$.

Proof. Recall that $\bar{u} = \bar{v}|_{[0, \tau]}$ and $\bar{w} = -\Lambda_{I_{2n}, \tau, \kappa I_{2n}} \bar{u}$. Suppose $[\theta] \circ \bar{w} = \bar{w}$, i.e., $(\theta + k \tau) \ast \bar{w} = \bar{w}$. Note that $\Lambda_{I_{2n}, \tau, \kappa I_{2n}}$ is the equivariant isomorphism. Hence $(\theta + k \tau) \ast \bar{u} = \bar{u}$. That is,

$$(\bar{u})^{I_{2n}}(\theta + k \tau + t) = \bar{u}(t), \quad \forall 0 \leq t \leq \tau.$$ 

Since $(\bar{u})^{I_{2n}} = \bar{v}$ and $\bar{v}$ has the minimal period $\tau/p$, this means that $\bar{v}(\theta + t) = \bar{v}(t)$ $\forall 0 \leq t \leq \tau$ and so $\bar{v}(\theta + t) = \bar{v}(t)$ $\forall t \in \mathbb{R}$. It follows that

$$\left[ \theta \right] \in \bigcup_{l=0}^{p-1} \left( \mathbb{Z} \frac{l \tau}{p} + \frac{l \tau}{p} \mathbb{Z} \right).$$ 

Moreover, it is easily checked that such a $\theta$ satisfies $[\theta] \circ \bar{w} = \bar{w}$. The first claim follows.

Since $\bar{u}$ is $C^2$, and $\bar{w} = -\Lambda_{M, \tau, \kappa I_{2n}}(\bar{u}) = -J \bar{u} - \kappa \bar{u}$, we deduce $\dot{\bar{w}} = -\Lambda_{M, \tau, \kappa I_{2n}}(\dot{\bar{u}})$. Note that $\tau/p$ is also a period of $\bar{v}$ and $\dot{\bar{u}} = \hat{v}|_{[0, \tau]}$. It must hold that $[\theta] \circ \bar{w} = \bar{w}$ for any $[\theta] \in S^1_{\bar{w}}$.

Recall that $H^\perp$ is the orthogonal complement of $\bar{w}$ in $W = L^2([0, \tau]; \mathbb{R}^{2n})$ and $X^\perp = X \cap H^\perp$. The second conclusion in Claim 1.8 spaces $H^\perp, X^\perp$ and each functional $\mathcal{L}_\lambda^\perp$ are $S^1_{\bar{w}}$-invariant.

Since $\left\{ [\theta] \circ \bar{u} \mid [\theta] \in S^1 \right\} = \left\{ \theta \ast \bar{u} \mid \theta \in \mathbb{R} \right\}$ is a $C^2$-embedded circle, by Claim 1.2 the orbit $O := \{ [\theta] \circ \bar{w} \mid [\theta] \in S^1 \}$ is a $C^2$-embedded circle in $L^2([0, \tau]; \mathbb{R}^{2n})$ and $T_{\bar{w}}O = \mathbb{R}\bar{w}$. Let $\pi : NO \rightarrow O$ be the normal bundle of $O$ in $H$. It is a $C^1$ Hilbert vector bundle over $O$ and its fibre $NO_{\bar{w}}$ at $\bar{w}$ is equal to $H^\perp$. Let $NO(\varepsilon) := \{ (w, x) \in NO \mid ||x||_2 < \varepsilon \}$ and define

$$\exp : NO(\varepsilon) \rightarrow H, \quad (w, x) \mapsto w + x.$$ 

Clearly, $\exp$ is equivariant, i.e., $[\theta] \circ (\exp(w, x)) = \exp([\theta] \circ w, [\theta] \circ x)$ for all $[\theta] \in S^1$. By shrinking $\varepsilon > 0$, $\exp$ gives rise to a $C^1$ diffeomorphism $F$ from $NO(\varepsilon)$ to an open neighborhood $N(O, \varepsilon)$ of $O$ in $H$. Let $x, y \in NO(\varepsilon)_{\bar{w}} = B_H^\perp(\bar{w}, \varepsilon)$ such that

$$[\theta] \circ (\bar{w} + x) = \bar{w} + y \quad \text{for some } [\theta] \in S^1 = \mathbb{R}/(\tau \mathbb{Z}).$$

Then $[\theta] \circ \bar{u} + [\theta] \circ x = \bar{w} + y$, i.e., $\exp([\theta] \circ \bar{w}, [\theta] \circ x) = \exp(\bar{w}, y)$. It follows that $[\theta] \circ \bar{w} = \bar{w}$ and $[\theta] \circ x = y$. The former shows $[\theta] \in S^1_{\bar{w}}$, and the latter implies that $x$ and $y$ sit in the same $S^1_{\bar{w}}$-orbit. Therefore we have proved:
Claim 4.9. If \( x, y \in NO(\varepsilon) = B_\lambda^1(\bar{w}, \varepsilon) \) belong to distinct \( S^1_\bar{w} \)-orbits, then \( S^1 \circ (\bar{w} + x) \neq S^1 \circ (\bar{w} + y) \).

Claim 4.10. The fixed point set of \( S^1_\bar{w} \)-action on \( \text{Ker}(\mathcal{B}_\lambda(0)) \),
\[
\{x \in \text{Ker}(\mathcal{B}_\lambda(0)) | [\theta] \circ x = x \ \forall [\theta] \in S^1_\bar{w}\},
\]
is a linear subspace of dimension \( \nu_{\tau/p}(\gamma_\mu) - 1 \).

Proof. Since \( B_\lambda(\bar{w})\bar{w} = 0 \), by [4.10], we have
\[
\text{Ker}(\mathcal{B}_\lambda(0)) = \{x \in H^\perp | B_\lambda(\bar{w})x = 0\}.
\]

Note that \([\tau/p]\) is a generator of \( S^1_\bar{w} \). \( x \in \text{Ker}(\mathcal{B}_\lambda(0)) \) is a fixed point of \( S^1_\bar{w} \)-action if and only if \([\tau/p] \circ x = x\), i.e., \((\tau/p) \ast x = x\). By [4.11] and [4.12],
\[
\begin{align*}
(L_{2n, \tau, \kappa I_2n})^{-1}\bar{w} + \nabla(H^*_\lambda(\bar{w}(\cdot))) &= A_{I_{2n, \tau, \kappa I_{2n}}}\bar{w} + \nabla(H^*_\lambda(\bar{w}(\cdot))) = 0, \\
(L_{2n, \tau, \kappa I_2n})^{-1}x + (H^*_\lambda)'(\bar{w}(\cdot))x &= A_{M, \tau, \kappa I_{2n}}x + (H^*_\lambda)'(\bar{w}(\cdot))x = 0.
\end{align*}
\]

Let \( \bar{u} = -(L_{2n, \tau, \kappa I_{2n}})^{-1}\bar{w} \). Then \( \bar{u} = \nabla(H^*_\lambda(\bar{w}(\cdot))) \) and so \( \bar{w} = \nabla(H_\lambda)(\bar{u}(\cdot)) \) and
\[
I_{2n} = (H^*_\lambda)'(\bar{w}(\cdot))(H^*_\lambda)'(\bar{u}(\cdot)).
\]

It follows from these that \( y := (L_{2n, \tau, \kappa I_{2n}})^{-1}x \) satisfies
\[
y(t) = JH^*_\lambda(\bar{u}(t))y(t), \quad 0 \leq t \leq \tau.
\]

Moreover, \((\tau/p) \ast y = y\) implies \( y^{I_{2n}} \) has a period \( \tau/p \). Hence the desired claim follows. \( \square \)

Proof of (II). Since \( m^0(B^+, \emptyset) = \nu_{\tau}(\gamma_\mu) - 1 > 1 \), Theorem [C.7] ([52, Theorem 3.6]) implies that one of the following alternatives occurs:

(i) \((\mu, 0)\) is not an isolated solution in \( \{\mu\} \times H^\perp \) of the equation \( \Phi_\lambda(x) = 0 \).

(ii) For every \( \lambda \in \Lambda \) near \( \mu \) there is a nontrivial solution \( x_\lambda \) of \( \Phi_\lambda(x) = 0 \) in \( X^\perp \), which converges to \( 0 \) in \( X^\perp \) as \( \lambda \to \mu \).

(iii) For any neighborhood \( \mathcal{N} \) of \( 0 \) in \( X^\perp \), which may be assumed to be \( S^1_\bar{w} \)-invariant, there is an one-sided neighborhood \( \Lambda^0 \) of \( \mu \) in \( \Lambda \) such that for any \( \lambda \in \Lambda^0 \setminus \{\mu\} \), \( \Phi_\lambda(x) = 0 \) has at least two nontrivial solutions \( x_1^\lambda \) and \( x_2^\lambda \) in \( \mathcal{N} \) with different energy if \( \Phi_\lambda(x) = 0 \) has only finitely many nontrivial solutions in \( \mathcal{N} \).

By (i) we have a sequence \( (x_k) \subset H^\perp \setminus \{0\} \) converging to \( 0 \) in \( H^\perp \) such that \( \nabla L^*_\mu(x_k) = 0 \) for each \( k \). Since \( S^1_\bar{w} \) is a finite group, by passing to a subsequence (if necessary) we can assume that all \( S^1_\bar{w} \circ x_k \) are distinct each other. By Proposition [13.1] and Claim [13.3] all \( S^1 \circ (\bar{w} + x_k) = \mathbb{R} \ast (\bar{w} + x_k) \) are distinct critical orbits of \( \psi_k(\mu, \cdot) \) and \( \bar{w} + x_k \to \bar{w} \) in \( H \). For each \( k = 1, 2, \ldots \), set
\[
u_k := -(L_{I_{2n, \tau, \kappa I_{2n}}})^{-1}(\bar{w} + x_k) \quad \text{and} \quad \nu_k := (\nu_k)^{I_{2n}}.
\]
Then \( \|\nu_k - \bar{v}\|_{[0, \tau]} \to 0 \), and therefore \( \|\nu_k - \bar{v}\|_{C^1} \to 0 \) by Proposition [1.3]. Hence \( (\nu_k) \) satisfy (II.1).

For every \( \lambda \in \Lambda \setminus \{0\} \), let \( x_\lambda \) be as in (ii). Set \( u_\lambda := -(L_{I_{2n, \tau, \kappa I_{2n}}})^{-1}(\bar{w} + x_\lambda) \) and \( v_\lambda := (u_\lambda)^{I_{2n}} \). As above these \( v_\lambda \) satisfy (II.2).
By (iii'), for any \( \lambda \in \Lambda^0 \setminus \{ \mu \} \), \( A_\lambda(x) = 0 \) has

- either infinitely many distinct nontrivial solutions in \( N \), and therefore infinitely many distinct nontrivial \( S^1_\omega \)-orbit of solutions in \( N \) (because of the finiteness of \( S^1_\omega \)), \( S^1_\omega \circ \bar{x}_\lambda^k, k = 1, 2, \cdots \),

- or at least two nontrivial \( S^1_\omega \)-orbit of solutions in \( N \), \( S^1_\omega \circ x_\lambda^1 \) and \( S^1_\omega \circ x_\lambda^2 \), such that \( \psi_\lambda(\lambda, x_\lambda^1) \neq \psi_\lambda(\lambda, x_\lambda^2) \).

As above we define \( \bar{u}_\lambda^k := - (\Lambda_{I_{2n}, r, n I_{2n}})^{-1} (\bar{w} + \bar{x}_\lambda^k) \) and \( \bar{v}_\lambda^k := (\bar{u}_\lambda^k)^I_{2n}, k = 1, 2, \cdots \), and \( u_\lambda^i := - (\Lambda_{I_{2n}, r, n I_{2n}})^{-1} (\bar{w} + x_\lambda^i) \) and \( v_\lambda^i := (u_\lambda^i)^I_{2n}, i = 1, 2 \). Then \( \bar{v}_\lambda^k \) and \( v_\lambda^i \) satisfy (II.3).

**Proof of (III).** Since \( \nu_{r/\mu}(\gamma_{\mu}) = 1 \), by Claim [4.10] the fixed point set of \( S^1_\lambda \)-action on \( \text{Ker}(A_\lambda(0)) \) is equal to \( \{ 0 \} \). Using Theorem C.3 ([52, Theorem 3.7]) and [52, Remark 3.9] we obtain:

- If \( p = 2 \), one of \((i')\) and the following \((iv')\) occurs:

\( (iv') \) There exist left and right neighborhoods \( \Lambda^- \) and \( \Lambda^+ \) of \( \mu \) in \( \Lambda \) and integers \( n^+, n^- \geq 0 \), such that \( n^+ + n^- \geq \nu_{r}(\gamma_{\mu}) \), and for \( \lambda \in \Lambda^- \setminus \{ \mu \} \) (resp. \( \lambda \in \Lambda^+ \setminus \{ \mu \} \)), \( A_\lambda(x) = 0 \) has at least \( n^- \) (resp. \( n^+ \)) nontrivial \( S^1_\omega \)-orbit of solutions of in \( X^\perp \), \( S^1_\omega \circ x_\lambda, i = 1, \cdots , n^- \) (resp. \( n^+ \)), which converge to 0 in \( X^\perp \) as \( \lambda \to \mu \).

- If \( p > 2 \) is a prime, one of \((i')\) and the following \((v')\) occurs:

\( (v') \) The conclusions obtained by replacing \( \nu_{r}(\gamma_{\mu}) \) with \( \nu_{r}(\gamma_{\mu})/2 \) in \((iv')\).

As above, the desired conclusions follow from these and Claim [4.9]. \[ \square \]

# 5 Proofs of Theorems 1.23, 1.24, 1.26 and Corollaries 1.25, 1.28, 1.29

Let \( L_r \) be the Hilbert subspace of \( L^2(S_r; \mathbb{R}^{2n}) \) as in \((A.37)\). Below \((A.37)\) we have showed that the operator \( \Lambda_{I_{2n}, r, 0} \) on \( L^2(S_r; \mathbb{R}^{2n}) \) defined by \( (2.3) \) with domain \( W^{1,2}(S_r; \mathbb{R}^{2n}) \) restricts to a closed linear and self-adjoint operator \( \Lambda_{I_{2n}, r, 0} \) on \( L_r \) with domain \( \mathcal{W}_r \), and

\[
\sigma(\Lambda_{I_{2n}, r, 0}) = \frac{2\pi}{r} \mathbb{Z}
\]

consists of only eigenvalues, and each eigenvalues has multiplicity \( n \). Then for each real \( \kappa \notin \frac{2\pi}{r} \mathbb{Z} \),

\[
\Lambda_{I_{2n}, r, n I_{2n}} : \mathcal{W}_r \to L_r, \quad v \mapsto \Lambda_{I_{2n}, r, 0} v + \kappa v = J \frac{d}{dt} v + \kappa v
\]

(5.1)
is a Banach space isomorphism.

Solutions of \((1.24)\) are critical points of the functional

\[
\Phi_\lambda(v) = \int_0^r \left[ \frac{1}{2} (J \dot{v}(t), v(t))_{\mathbb{R}^{2n}} + H(\lambda, t, v(t)) \right] dt
\]
on the Hilbert space \( \mathcal{W}_r \).

Note that the function \( \bar{H} : \Lambda \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R} \) by

\[
\bar{H}(\lambda, t, z) = H(\lambda, t, z + u_\lambda(t)) - (z, \nabla_z H(\lambda, t, u_\lambda(t)))_{\mathbb{R}^{2n}}
\]
satisfies Assumption \((1.22)\) and for each \( \lambda \in \Lambda, \bar{v} \equiv 0 \in \mathbb{R}^{2n} \) satisfies

\[
\dot{v}(t) = J \nabla_z \bar{H}(\lambda, t, v(t)), \quad v(t + \tau) = v(t) \quad \text{and} \quad v(-t) = N v(t) \forall t \in \mathbb{R} \tag{5.2}
\]
Proof of Theorem 1.23(I). As in the proof of Theorem 1.4(I) in Section 3.4, replacing H by \( \bar{H} \) in what follows we can assume that H satisfies inequalities in (3.3) and (3.4) for all \((\lambda, t, z) \in \Lambda \times \mathbb{R} \times \mathbb{R}^{2n}\), and only need to prove Theorems 1.23-1.24 in the case where \( \nu_\lambda \equiv 0 \) for all \( \lambda \in \Lambda \).

Then we may choose \( \kappa \notin \frac{1}{2i \pi} \mathbb{Z} \), \( c_i > 0 \), \( i = 1, 2, 3 \) such that each

\[
H_\kappa(\lambda, t, z) := H(\lambda, t, z) - \frac{\kappa}{2} |z|^2
\]

satisfies (ii)-(iii) in the proof of Theorem 1.4(I). Consider the dual action \( \Psi_\kappa(\lambda, \cdot) : W_\tau \to \mathbb{R} \) defined by (3.3). We obtain a family of \( C^1 \) and twice Gâteaux-differentiable functionals

\[
\psi_\kappa(\lambda, \cdot) = \Psi_\kappa(\lambda, \cdot) \circ (-\tilde{A}_{I_{2n}, \tau, \kappa I_{2n}})^{-1} : L_\tau \to \mathbb{R}
\]

given by

\[
\psi_\kappa(\lambda, v) = \int_0^\tau \left[ \frac{1}{2} \left( v(t), ((\tilde{A}_{I_{2n}, \tau, \kappa I_{2n}})^{-1} v)(t) \right)_{\mathbb{R}^{2n}} + H_\kappa^\ast(\lambda, t, v(t)) \right] dt, \quad (5.3)
\]
such that \( v \in W_\tau \) is a critical point of \( \Psi_\kappa(\lambda, \cdot) \) if and only if \( w := -\tilde{A}_{I_{2n}, \tau, \kappa I_{2n}} v \) is a critical point of \( \psi_\kappa(\lambda, \cdot) \). In particular, \((\mu, 0) \in \Lambda \times W_\tau \) is a bifurcation point of the problem (1.29) if and only if \( (\mu, 0) \in \Lambda \times L_\tau \) is a bifurcation point of \( \nabla \nu \psi_\kappa(\lambda, v) = 0 \).

Let \( \psi_{\kappa, \lambda}(v) = \psi_\kappa(\lambda, v) \). As in Remark 2.12, we can compute

\[
(\psi_{\kappa, \lambda})''(0)[\xi, \eta] = \int_0^\tau \left[ \left( (\tilde{A}_{I_{2n}, \tau, \kappa I_{2n}})^{-1} \xi)(t), \eta(t) \right)_{\mathbb{R}^{2n}} + (\nabla_\nu^2 H_\kappa(\lambda, t, 0) \xi(t), \eta(t))_{\mathbb{R}^{2n}} \right] dt = Q_{B_\lambda, \kappa I_{2n}}(\xi, \eta), \quad \forall \xi, \eta \in L_\tau,
\]

where \( Q_{B_\lambda, \kappa I_{2n}} \) is given by (A.39) with \( B_\lambda(t) = \nabla_\nu^2 H(\lambda, t, 0) \). By Theorem A.7, we obtain

\[
m^-_{(\psi_{\kappa, \lambda}, 0)} = \mu_{1, \tau}(\gamma_\lambda) - n[\kappa \tau/(2\pi)], \quad (5.4)
\]
\[
m^0_{(\psi_{\kappa, \lambda}, 0)} = \nu_{1, \tau}(\gamma_\lambda). \quad (5.5)
\]

Let \( \iota : W_\tau \to L_\tau \) be the inclusion. Then

\[
\tilde{A}_{I_{2n}, \tau, \kappa} : L_\tau \to L_\tau, \quad v \mapsto \iota \circ (\tilde{A}_{I_{2n}, \tau, \kappa I_{2n}})^{-1}v,
\]

is a compact self-adjoint operator, and \( \psi_\kappa(\lambda, \cdot) \) has the gradient

\[
\nabla_\nu \psi_\kappa(\lambda, v) = \tilde{A}_{I_{2n}, \tau, \kappa} v + \nabla_\nu H_\kappa^\ast(\lambda, \cdot; v(\cdot)). \quad (5.7)
\]

Moreover, \( L_\tau \ni v \mapsto \nabla_\nu \psi_\kappa(\lambda, v) \in L_\tau \) has a Gâteaux derivative

\[
B_\lambda(v) := D_v \nabla_\nu \psi_\kappa(\lambda, v) = \tilde{A}_{I_{2n}, \tau, \kappa} + \nabla_\nu^2 H_\kappa^\ast(\lambda, \cdot; v(\cdot)) \in \mathcal{L}_s(L_\tau).
\]

Let \( P_\lambda(v) = \nabla_\nu^2 H_\kappa^\ast(\lambda, \cdot; v(\cdot)) \) and \( Q_\lambda(v) = \tilde{A}_{I_{2n}, \tau, \kappa} \).

**Proof of Theorem 1.23(I).** As in the proof of Theorem 1.4 we can show that Theorem C.6 ([60, Theorem 3.1]) is applicable to \( H = X = L_\tau \) and \( \mathcal{F}_\lambda(\cdot) = \psi_\kappa(\lambda, \cdot) \). □

Consider Banach subspaces of \( C^k(S_\tau; \mathbb{R}^{2n}) \), \( C^k \tau = C^k(S_\tau; \mathbb{R}^{2n}) \cap L_\tau \), \( k = 0, 1, \ldots \).
Proof of Theorem 1.23(II). Note that \( \tilde{\Lambda}_{I_{2n},\tau,K} \) gives rise to a Banach space isomorphism from \( C^1_{\tau} \) to \( C^0_{\tau} \), denoted by \( \Lambda^C_{I_{2n},\tau,K} \). As in the proof of Theorem 1.4(II), using (5.4) and (5.5) we can prove that \( \mathcal{L}_\lambda(\cdot) = \psi_\kappa(\cdot,\cdot) \) with \( \lambda \in \Lambda \) satisfies the conditions of Theorem 5.12 with \( \lambda^* = \mu \) and \( H = L_\tau \) and \( X = C^0_{\tau} \).

Proof of Theorem 1.23(III). Following the notations above, using (5.4) and (5.5) we can prove as in the proof of Theorem 1.4(III) that \( \mathcal{L}_\lambda(\cdot) = \psi_\kappa(\cdot,\cdot) \) with \( \lambda \in \Lambda \) satisfies the conditions of Theorem 5.5 with \( \lambda^* = \mu \) and \( H = L_\tau \) and \( X = C^0_{\tau} \).

Proof of Theorem 1.24. As in the proof of Theorem 1.7 the first part follows from Theorem 5.14 ([50, Theorem 3.7]) and [52, Remark 3.9]. As in the proof of Theorem 1.4(II), using (5.4) and (5.5) we can prove that \( \mathcal{L}_\lambda(\cdot) = \psi_\kappa(\cdot,\cdot) \) with \( \lambda \in \Lambda \) satisfies the conditions of Theorem 5.12 with \( \lambda^* = \mu \) and \( H = L_\tau \) and \( X = C^0_{\tau} \).

Proof of Theorem 1.25. As in the proof of Corollary 1.9 the desired conclusions can be completed using Theorems A.7, A.8 and the above arguments.

Proof of Theorem 1.26. Step 1. The Lie group \( G = Z_2 = \{0,1\} \) orthogonally acts on \( L_\tau \) and \( W_\tau \) by \([0] \cdot u = u \) and \([1] \cdot u = N \cdot u \), where \((N \cdot u)(t) = N(u(t)) \). Clearly, this action induces an isometric action on each space \( C^k \). The set of fixed points of this action, \( \text{Fix}_G \), is equal to \((u \in L_\tau | u(-t) = u(t), \text{a.e. } t \in \mathbb{R}) \). Clearly, each functional \( \psi_\kappa(\cdot,\cdot) \) defined by (5.3) is invariant under this \( G \)-action. As in the proof of Theorem 1.11 we can obtain that the functional \( \mathcal{L}_\lambda(\cdot) = \psi_\kappa(\cdot,\cdot) + \lambda \tilde{\lambda}_{I_{2n},\tau,K} \tilde{\psi} \) satisfies conditions in Theorem 5.7([52, Theorem 3.6]) or [50, Theorem 4.6]) with \( H = L_\tau \) and \( X = C^0_{\tau} \).

Since \( \tilde{\psi} \) is even, \( \tilde{\psi}(t) = \tilde{\psi}(-t) = N \tilde{\psi}(t) \) for all \( t \), that is, \( \tilde{\psi} \in \text{Fix}_G \). Then \( \tilde{\psi} \) belongs to \( \text{Fix}_G \) and so \( \mathcal{L}_\lambda \) is \( G \)-invariant. The assumption (b) implies that the fixed point set of the induced \( G \)-action on \( H^0_\mu \) is \( \{0\} \). Combing the assumption (a) we can use Theorem 5.8([52, Theorem 3.7]) or [50, Theorem 5.12]) to derive that one of the claims (i) and (ii) in Theorem 1.26 holds.

Step 2. The Lie group \( G = Z_2 \times Z_2 = \{((0,0)),([0],[1]),([1],[0]),([1],[1])\} \) orthogonally acts on \( L_\tau \) and \( W_\tau \) by
\[
([0],[0]) \cdot u = u, \quad ([0],[1]) \cdot u = N \cdot u, \quad ([1],[0]) \cdot u = -u, \quad ([1],[1]) \cdot u = -(N \cdot u),
\]
and the fixed point set of the induced \( G \)-action on \( H^0_\mu \) is \( \{0\} \). Clearly, each functional \( \mathcal{L}_\lambda(\cdot) \) is invariant for this action, using Theorem 5.8([52, Theorem 3.7]) or [50, Theorem 5.12]) and [52, Remark 3.9] ([50, Remark 5.14]) we may directly obtain either (iii) or (iv).

Proof of Corollary 1.28. Suppose \( \lambda_1, \tau(\gamma) > 0 \). As in the proof of Corollary 1.9 by modifying values of \( H_0 \) and \( H \) outside a neighborhood \( U \in \mathbb{R}^{2n} \) we can choose \( \kappa \in \mathbb{R} \setminus \frac{2n}{3} \mathbb{Z} \) and \( c_i > 0 \), \( i = 1, 2, 3 \) such that \( H_\kappa(\lambda, z) := H_0(z) + \lambda H(z) - \frac{\kappa}{2} |z|^2 \) satisfies
\[
c_1 I_{2n} \leq (H_0)'(z) + \lambda H''(z) - \kappa I_{2n} \leq c_2 I_{2n}
\]
and \( c_1 |z|^2 - c_3 \leq H_\kappa(\lambda, z) \leq c_2 |z|^2 + c_3 \)
for all \( (\lambda, t, z) \in [\mu - 1, \mu + 1] \times [0, \tau] \times \mathbb{R}^{2n} \). Let \( \psi_{\kappa,\lambda}(u) := \psi_\kappa(\lambda, u) \) be given by (5.3). Then
\[
\psi'_{\kappa,\lambda}(0) = \psi''_{\kappa,\lambda}(0) = Q_{B_\lambda,\kappa I_{2n}}(\xi, \eta) \quad \text{for all } \xi, \eta \in L_\tau,
\]
where \( Q_{B_\lambda,\kappa I_{2n}} \) is given by (5.39) with \( B_\lambda(t) = (H_0)'(0) + \lambda H''(0) \). Therefore
\[
m^-(\psi_{\kappa,\lambda}, 0) = m^- (Q_{B_\lambda,\kappa I_{2n}}) \quad \text{and} \quad m^0(\psi_{\kappa,\lambda}, 0) = m^0(Q_{B_\lambda,\kappa I_{2n}}).
\]
If \( \dot{H}''(0) > 0 \) (resp. \( \ddot{H}''(0) < 0 \)), by Theorem \( \text{A.8} \) we deduce that

\[
\begin{align*}
\dot{m}^-(Q_{B_{\lambda_2},\kappa I_{2n}}) &\ge m^-(Q_{B_{\lambda_1},\kappa I_{2n}}) + m^0(Q_{B_{\lambda_1},\kappa I_{2n}}) \\
(\text{resp. } \ddot{m}^-(Q_{B_{\lambda_1},\kappa I_{2n}}) &\ge \dot{m}^-(Q_{B_{\lambda_2},\kappa I_{2n}}) + m^0(Q_{B_{\lambda_2},\kappa I_{2n}}))
\end{align*}
\]

for any \( \mu - 1 \le \lambda_1 < \lambda_2 \le \mu + 1 \). These imply that \( \{ \lambda \in [\mu - 1, \mu + 1] | m^0(Q_{B_{\lambda},\kappa I_{2n}}) > 0 \} \) is a finite set. It follows from this and Theorem \( \text{A.7} \) that \( \{ \lambda \in \mathbb{R} | \nu_{1,\tau}(\gamma_\lambda) > 0 \} \) is a discrete set in \( \mathbb{R} \). The first claim is proved.

As in the proof of Corollary \( \text{1.29} \) we can use Theorem \( \text{A.8} \) to derive that

\[
\begin{align*}
\dot{m}^-(Q_{B_{\lambda},\kappa I_{2n}}) &= \dot{m}^-(Q_{B_{\mu},\kappa I_{2n}}) \forall \lambda \in [\mu - \rho, \mu], \\
\dot{m}^-(Q_{B_{\lambda},\kappa I_{2n}}) &= \dot{m}^-(Q_{B_{\mu},\kappa I_{2n}}) + m^0(Q_{B_{\mu},\kappa I_{2n}}) \forall \lambda \in (\mu, \mu + \rho]
\end{align*}
\]

if \( \dot{H}''(0) > 0 \) and \( \rho > 0 \) is small enough, and

\[
\begin{align*}
\ddot{m}^-(Q_{B_{\lambda},\kappa I_{2n}}) &= \ddot{m}^-(Q_{B_{\mu},\kappa I_{2n}}) \forall \lambda \in [\mu - \rho, \mu], \\
\ddot{m}^-(Q_{B_{\lambda},\kappa I_{2n}}) &= \dot{m}^-(Q_{B_{\mu},\kappa I_{2n}}) + m^0(Q_{B_{\mu},\kappa I_{2n}}) \forall \lambda \in (\mu, \mu + \rho]
\end{align*}
\]

if \( \ddot{H}''(0) < 0 \) and \( \rho > 0 \) is small enough. These and Theorem \( \text{A.7} \) lead to \( \text{(1.33)} \) and \( \text{(1.34)} \). Other conclusions follow from Theorem \( \text{1.26} \).

**Proof of Corollary 1.29.** The first claim follows from Corollary 1.28. Suppose \( H''(0) > 0 \) and \( \mu \in \Delta(H) \cap (0, \infty) \). Let \( \gamma_\lambda(t) = \exp(\lambda t H''(0)) \). It follows from \( \text{(1.33)} \) that \( \mu_{1,1}(\gamma_\lambda) = \mu_{1,1}(\gamma_\mu) \) for \( \lambda \le \mu \) close to \( \mu \) and that \( \mu_{1,1}(\gamma_\lambda) = \mu_{1,1}(\gamma_\mu) + \nu_{1,1}(\gamma_\mu) \) for \( \lambda > \mu \) close to \( \mu \). By the third conclusion of Corollary 1.28 with \( H_0 = 0 \) and \( \dot{H} = \ddot{H} \) we obtain the desired claims. The final result may follow from the fourth conclusion of Corollary 1.28 with \( H_0 = 0 \) and \( \dot{H} = \ddot{H} \).

## 6 Proofs of Theorems 1.33, 1.34 and 1.37

### 6.1 Proofs of Theorems 1.33, 1.34

As in \( \text{6.1} \) we define \( \tilde{H}(\lambda, t, z) = H(\lambda, t, z + u_\lambda(t)) \) for \( (\lambda, t, z) \in \Lambda \times [0, \tau] \times \mathbb{R}^{2n} \). Then \( \tilde{H} \) still satisfies Assumption 1.32 and it holds that

\[
\begin{align*}
\nabla \tilde{H}(\lambda, t, z) &= \nabla H(\lambda, t, u_\lambda(t)) - \nabla_z H(\lambda, t, u_\lambda(t)), \\
\nabla^2 \tilde{H}(\lambda, t, z) &= \nabla^2 H(\lambda, t, z + u_\lambda(t)).
\end{align*}
\]

(6.1)

It is easy to see that \( \tilde{H} : [0, \tau] \rightarrow \mathbb{R}^{2n} \) satisfies \( \text{(1.40)} \) if and only if \( \dot{v} := \dot{w} - \dot{u}_\lambda \) satisfies the Hamiltonian boundary value problem

\[
\dot{v}(t) = J \nabla \tilde{H}(\lambda, t, v(t)) \forall t \in [0, \tau],
\]

\[
v(0) \in L, \quad v(\tau) \in L'.
\]

(6.2)

In particular, \( \tilde{v} \equiv 0 \in \mathbb{R}^{2n} \) satisfies \( \text{(6.2)} \) for each \( \lambda \in \Lambda \). For the vertical Lagrangian subspace \( L_0 := \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{2n} \), we may choose an orthogonal symplectic matrix \( O \in \text{Sp}(2n) \) such that \( O L_0 = L \). Define \( L'' = O^{-1} L' \) and

\[
K : \Lambda \times [0, \tau] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}, \quad (\lambda, t, z) \mapsto H(\lambda, t, O z).
\]

Then \( K \) also satisfies Assumption 1.32 and

\[
\begin{align*}
\nabla K(\lambda, t, z) &= O^{-1} \nabla \tilde{H}(\lambda, t, O z), \\
\nabla^2 K(\lambda, t, z) &= O^{-1} \nabla^2 \tilde{H}(\lambda, t, O z) O.
\end{align*}
\]

(6.3)
It follows that \( v: [0, \tau] \to \mathbb{R}^{2n} \) satisfies (6.2) if and only if \( w(t) := O^{-1}v(t) \) satisfies the Hamiltonian boundary value problem
\[
\begin{align*}
\dot{u}(t) &= J\nabla_z K(\lambda, t, u(t)) \quad \forall t \in [0, 1], \\
u_0(t) &= L_0, \quad \nu(\tau) \in L''.
\end{align*}
\] (6.4)

In particular, \( w \equiv 0 \in \mathbb{R}^{2n} \) satisfies (6.1) for each \( \lambda \in \Lambda \).

Let \( \alpha : [0, \tau] \to \text{Sp}(2n) \) be the fundamental matrix solution of \( \dot{u}(t) = J\nabla_z^2 K(\lambda, t, 0)u(t) \). Since (6.1) and (6.3) imply \( \nabla_z^2 K(\lambda, t, 0) = O^{-1}\nabla_z^2 H(\lambda, t, 0)O = O^{-1}\nabla_z^2 H(\lambda, t, u_\lambda(t))O \), we have
\[
\alpha(t) = O^{-1}\gamma_\lambda(t)O
\]
and hence
\[
i_L^g(\gamma_\lambda) = i_{L_0}^g(\alpha) \quad \text{and} \quad \nu_L^g(\gamma_\lambda) = \nu_{L_0}^g(\alpha) = \dim(\gamma(\tau)L \cap L')
\] (6.5)
by [38] Definition 2.4]. Note that \( v: [0, \tau] \to \mathbb{R}^{2n} \) satisfies (6.1) if and only if \( u(t) := Ov(t) + u_\lambda(t) \) satisfies (6.10). Therefore from now on we can assume:

in Assumption 1.32 \( L = L_0, \ u_\lambda = 0, \ w_\lambda = 0 = w_\lambda' \) for all \( \lambda \in \Lambda \). (6.6)

Under assumptions in (6.6), we only concern solutions of (1.40) near 0 in \( \mathbb{R}^{2n} \). Because of Remark 6.1, \( \Lambda \) may be assumed to be compact and sequential compact. Therefore by modifying \( H \) outside a large ball in what follows we may assume that
\[
\|\nabla_z^2 H(\lambda, t, x)\|_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} \leq C(H) \quad \forall (\lambda, t, x)
\] (6.7)
and so
\[
\|\nabla_z H(\lambda, t, x)\|_{\mathbb{R}^{2n}} \leq C(H)\|x\|_{\mathbb{R}^{2n}} + C(H)' \quad \forall (\lambda, t, x),
\] (6.8)
where \( C(H) \) and \( C(H)' \) are positive constants. Because of these, using [31] Proposition C.1 we derive that the maps
\[
L^2([0, \tau]; \mathbb{R}^{2n}) \to L^2([0, \tau]; \mathbb{R}^{2n}), \ u \mapsto \nabla_z H(\lambda, \cdot; u(\cdot))
\] (6.9)
have an uniform bound on any bounded subset and are uniform continuous at any \( \bar{u} \in L^2([0, \tau]; \mathbb{R}^{2n}) \) with respect to \( \lambda \in \Lambda \). Let \( u_{\lambda_2} \in L^2([0, 1], \mathbb{R}^{2n}), \ i = 1, 2, \) satisfy \( \|u_{\lambda_2} - u_{\lambda_1}\|_2 \to 0 \) as \( \lambda_2 \to \lambda_1 \). Then
\[
\left( \int_0^\tau \|\nabla_z H(\lambda_2, t, u_{\lambda_2}(t)) - \nabla_z H(\lambda_1, t, u_{\lambda_1}(t))\|^2 dt \right)^{1/2} \leq \left( \int_0^\tau \|\nabla_z H(\lambda_2, t, u_{\lambda_2}(t)) - \nabla_z H(\lambda_2, t, u_{\lambda_1}(t))\|^2 dt \right)^{1/2}
\]
\[+ \left( \int_0^\tau \|\nabla_z H(\lambda_1, t, u_{\lambda_1}(t))\|^2 dt \right)^{1/2}.
\]
Clearly, as \( \lambda_2 \to \lambda_1 \) the first term of the right side converges to zero because of the uniform continuity of the maps in (6.9) at any \( u_{\lambda_1} \in L^2([0, \tau]; \mathbb{R}^{2n}) \) with respect to \( \lambda \in \Lambda \). From (6.8) and Lebesgue dominated convergence theorem it follows that the second term of the right side converges to zero as \( \lambda_2 \to \lambda_1 \). Hence we obtain the first claim of the following.

**Lemma 6.1.** The map
\[
\Lambda \times L^2([0, \tau]; \mathbb{R}^{2n}) \to L^2([0, \tau]; \mathbb{R}^{2n}), \ (\lambda, u) \mapsto \nabla_z H(\lambda, \cdot; u(\cdot))
\] (6.10)
is continuous. Therefore if \( u_{\lambda_i} \in C^1([0, \tau], \mathbb{R}^{2n}), \ i = 1, 2, \) satisfy \( \dot{u}_{\lambda_i} = J\nabla_z H(\lambda_i, t, u_{\lambda_i}(t)) \), and \( \|u_{\lambda_2} - u_{\lambda_1}\|_2 \to 0 \) as \( \lambda_2 \to \lambda_1 \), then \( u_{\lambda_2} \to u_{\lambda_1} \) in \( C^1([0, \tau], \mathbb{R}^{2n}) \) as \( \lambda_2 \to \lambda_1 \).
Proof. By the first claim, \( \|\dot{u}_{\lambda_2} - \dot{u}_{\lambda_1}\|_2 \to 0 \) as \( \lambda_2 \to \lambda_1 \). Then \( \|u_{\lambda_2} - u_{\lambda_1}\|_{C^0} \to 0 \) and so \( \|u_{\lambda_2} - u_{\lambda_1}\|_{C^0} \to 0 \) as \( \lambda_2 \to \lambda_1 \). By Assumption 1.32 (6.1) and (6.3) imply that

\[
\Lambda \times [0, \tau] \times \mathbb{R}^{2n} \ni (\lambda, t, z) \mapsto \nabla_z H(\lambda_2, t, z) \in \mathbb{R}^{2n}
\]
is continuous. It follows from \( \dot{u}_{\lambda_1} = J \nabla_z H(\lambda_1, t, u_{\lambda_1}(t)) \) that \( \|\dot{u}_{\lambda_2} - \dot{u}_{\lambda_1}\|_{C^0} \to 0 \) as \( \lambda_2 \to \lambda_1 \). \( \square \)

Following [38] §4 the problem is reduced to one in finitely dimensional spaces. For each \( s \geq 0 \) consider a Hilbert space

\[
H^s_{L_0} = \left\{ x \in L^2([0, \tau], \mathbb{R}^{2n}) \mid x = \sum_{m \in \mathbb{Z}} -J \exp \left( \frac{m J t a}{\tau} \right) a_m, \ a_m \in \mathbb{R}^n \oplus \{0\}, \right\}
\]
with the following inner product

\[
\langle x, y \rangle_{H^s_{L_0}} = \tau \langle x_0, y_0 \rangle_{\mathbb{R}^{2n}} + \sum_{k \neq 0} |k|^{2s} x_k y_k, \quad x, y \in H^s_{L_0}. \quad (6.11)
\]

Denote the associated norm by \( \|\cdot\|_{H^s_{L_0}} \). Then \( H^0_{L_0} \) and \( H^1_{L_0} \) are Hilbert subspaces of \( L^2([0, \tau], \mathbb{R}^{2n}) \) and \( W^{1,2}([0, \tau], \mathbb{R}^{2n}) \), respectively. Hence we also write \( \langle x, y \rangle_{H^s_{L_0}} \) as \( \langle x, y \rangle_2 \) (resp. \( \langle x, y \rangle_{1,2} \)) for \( s = 0 \) (resp. \( s = 1 \)), and \( \|\cdot\|_{H^s_{L_0}} \) as \( \|\cdot\|_2 \) (resp. \( \|\cdot\|_{1,2} \)) for \( s = 0 \) (resp. \( s = 1 \)) below.

Take an orthogonal symplectic matrix \( P \in Sp(2n) \) such that \( PL_0 = L' \). We have a matrix \( M \) such that \( P = e^M \). Consider Hilbert subspaces of \( L^2([0, \tau], \mathbb{R}^{2n}) \) and \( W^{1,2}([0, \tau], \mathbb{R}^{2n}) \)

\[
\begin{align*}
\mathcal{W}_0 &= \left\{ x \in L^2([0, \tau], \mathbb{R}^{2n}) \mid [0, \tau] \ni t \mapsto \exp \left( -\frac{t}{\tau} M \right) x(t) \text{ belongs to } H^0_{L_0} \right\}, \\
\mathcal{W}_1 &= \left\{ x \in L^2([0, \tau], \mathbb{R}^{2n}) \mid [0, \tau] \ni t \mapsto \exp \left( -\frac{t}{\tau} M \right) x(t) \text{ belongs to } H^1_{L_0} \right\}
\end{align*}
\]
and an unbounded self-adjoint operator \( \mathcal{A} \in \mathcal{W}_0 \) with domain \( D(\mathcal{A}) = \mathcal{W}_1 \), which is given by

\[
\langle \mathcal{A} x, y \rangle_2 = \int_0^\tau (-\frac{\mathcal{J} \dot{x}(t) + JM x(t), y(t))_{\mathbb{R}^{2n}} \rangle dt.
\]
The range of \( \mathcal{A} \) is closed, the resolution of \( \mathcal{A} \) is compact and the spectrum of \( \mathcal{A} \) only consists of eigenvalues, more precisely \( \sigma(\mathcal{A}) = \frac{\pi}{2} \mathbb{Z} \). Note that \( \ker(\mathcal{A}) = \{[0, \tau] \ni t \mapsto e^{tM}a \mid a \in \mathbb{R}^n \oplus \{0\}\} \).

For each \( \lambda \in \Lambda \), define functionals \( \mathcal{H}_\lambda : \mathcal{W}_0 \to \mathbb{R} \) and \( f_\lambda : (\mathcal{W}_1, \|\cdot\|_2) \to \mathbb{R} \) by

\[
\begin{align*}
\mathcal{H}_\lambda(x) &= \int_0^\tau H(\lambda, t, x(t)) dt + \frac{1}{2} \int_0^\tau (JM x(t), x(t))_{\mathbb{R}^{2n}} dt, \\
f_\lambda(x) &= \frac{1}{2} \langle \mathcal{A} x, x \rangle_2 - \mathcal{H}_\lambda(x).
\end{align*}
\]
(6.12) and (6.13) imply that both functionals are of class \( C^1 \), and that the \( L^2 \)-gradient \( \nabla L^2 \mathcal{H}_\lambda \) of \( \mathcal{H}_\lambda \) defined by

\[
\langle \nabla L^2 \mathcal{H}_\lambda(x), y \rangle_2 = \int_0^\tau \langle \nabla H(\lambda, t, x(t)), y(t) \rangle_{\mathbb{R}^{2n}} dt + \int_0^\tau (JM x(t), y(t))_{\mathbb{R}^{2n}} dt \quad \forall x, y \in \mathcal{W}_0
\]
is Gâteaux differentiable. It is easily checked that the Gâteaux derivative \( D \nabla L^2 \mathcal{H}_\lambda(x) \in L_s(\mathcal{W}_0) \) at \( x \in \mathcal{W}_0 \) is given by

\[
(D \nabla L^2 \mathcal{H}_\lambda(x)y, z)_2 = \int_0^\tau \langle \nabla^2 H(\lambda, t, x(t)y(t)), z(t) \rangle_{\mathbb{R}^{2n}} dt + \int_0^\tau (JM y(t), z(t))_{\mathbb{R}^{2n}} dt
\]
(6.14)
for all \( y, z \in W_0 \). From (6.7) and (6.14) we derive that \( \|D\nabla L^2H_\lambda(x)\|_{L_2(W_0)} \leq C(H) + \|M\|_{\mathbb{R}^{2n \times 2n}} \). Moreover the \( L^2 \)-gradient of \( f_\lambda \) is given by

\[
\nabla \nabla f_\lambda(x) = Ax - \nabla z \left( H(\cdot, \cdot, x(\cdot)) - JMx \right), \quad \forall x \in W_1,
\]

and \( x \in (W_1, \| \cdot \|_2) \) is a critical point of \( f_\lambda \) if and only if it satisfies (1.40) with the assumptions in (6.6).

Let \( \Pi_0 : W_0 \to \text{Ker}(A) \) be the orthogonal projection. Define \( \tilde{A} = A + \Pi_0 \), and denote by \( E_s \) the spectral resolution of \( \tilde{A} \). Choose \( \beta \in \mathbb{R} \setminus \sigma(\tilde{A}) \) such that \( \beta > 2(C(H) + \|M\|_{\mathbb{R}^{2n \times 2n}} + 1) \), and define

\[
P = \int_{-\beta}^{\beta} dE_s, \quad P^+ = \int_{\beta}^{+\infty} dE_s, \quad P^- = \int_{-\infty}^{-\beta} dE_s,
\]

\[
W_0^+ := P^+(W_0), \quad W_0^- := P^-(W_0), \quad Z := P(W_0).
\]

Then \( W_0 = W_0^+ \oplus W_0^- \oplus Z \), and \( \dim Z < \infty \). Bounded self-adjoint linear operators on \( W_0 \),

\[
S^+ = \int_{\beta}^{\infty} s^{-1/2} dE_s, \quad S^- = \int_{-\infty}^{-\beta} (-s)^{-1/2} dE_s, \quad R = \int_{\alpha}^{\beta} |s|^{-1/2} dE_s,
\]

have ranges \( W_0^+ \), \( W_0^- \), and \( Z \), respectively, are pairwise commuting, and \( S^+|_{W_0^+} \), \( S^-|_{W_0^-} \) and \( R|_Z \) are injective.

According to Amann [2], Amann-Zehnder [3], Chang [13], Long [42] and Liu-Wang-Lin [38, Theorems 4.1, 4.2] we may use Lemma 6.1 to obtain:

**Theorem 6.2.** There exist \( (x(\cdot, \cdot), y(\cdot, \cdot)) \in C(\Lambda \times Z, W_0^+ \times W_0^-) \) satisfying the following:

(i) The continuous map \( u(\cdot, \cdot) : \Lambda \times Z \to W_0 \) defined by

\[
u(\lambda, z) = S^+ x(\lambda, R^{-1} z) + S^- y(\lambda, R^{-1} z) + z
\]

takes values in \( W_1 \), each \( u(\lambda, \cdot) : Z \to W_0 \) is a \( C^1 \) embedding, and \( d_z u(\lambda, z) \) is continuous in \( \lambda, z \).

(ii) For each \( \lambda \in \Lambda \), the functional \( a_\lambda : Z \to \mathbb{R} \) defined by

\[
a_\lambda(z) = f_\lambda(u(\lambda, z)) = \frac{1}{2} (\tilde{A}u(\lambda, z) - H_\lambda(u(\lambda, z))
\]

is \( C^2 \), and

\[
a_\lambda'(z) = \tilde{A}z - PH_\lambda'(u(\lambda, z)) = \tilde{A}u(\lambda, z) - H_\lambda'(u(\lambda, z)),
\]

\[
a_\lambda''(z) = \tilde{A}|z - PdH_\lambda'(u(\lambda, z))d_z u(\lambda, z) = [\tilde{A} - dH_\lambda'(u(\lambda, z))]d_z u(\lambda, z).
\]

(\text{It follows that})

\[
\Lambda \times Z \ni (\lambda, z) \mapsto a_\lambda'(z) \in Z \quad \text{and} \quad \Lambda \times Z \ni (\lambda, z) \mapsto a_\lambda''(z) \in \mathcal{L}(Z)
\]

are continuous. Note that (i) implies \( \Lambda \times Z \ni (\lambda, z) \mapsto a_\lambda(z) \in \mathbb{R} \) to be continuous.)

(iii) For each \( \lambda \in \Lambda \), the map \( Z \ni z \mapsto u(\lambda, z) \in W_0 \) gives an one-one correspondence between the critical points of \( a_\lambda \) and those of \( f_\lambda \). (The condition (6.6) implies that \( u(\lambda, 0) = 0 \) and \( 0 \in Z \) is a critical point of \( a_\lambda \) for any \( \lambda \in \Lambda \).)
(iv) If \( z \in Z \) is a critical point of \( a_\lambda \) and \( \beta_{\lambda,z} : [0, \tau] \to \text{Sp}(2n) \) is the fundamental matrix solution of \( \dot{\psi}(t) = J\nabla^2_\lambda H(\lambda, t, u(\lambda, z)(t))v(t) \), then

\[
m^0(a_\lambda, z) = \nu^L_{L_0}(\beta_{\lambda,z}) \quad \text{and} \quad m^-(a_\lambda, z) = i^L_{L_0}(\beta_{\lambda,z}).
\]

In particular, \( m^0(a_\lambda, 0) = \nu^L_{L_0}(\gamma_\lambda) = \dim(\gamma_\lambda(\tau)L_0 \cap L') \) and \( m^-(a_\lambda, 0) = i^L_{L_0}(\gamma_\lambda) \).

**Proof of Theorem 1.33.** (I) Under the assumptions in (6.10) let \((\mu, 0)\) be a bifurcation point along sequences of the problem \((6.10)\) with respect to the trivial branch \(\{(\lambda, 0) \mid \lambda \in \Lambda\}\). That is, there exists a sequence \((\lambda_k) \subset \Lambda\) converging to \(\mu\) and solutions \(v_k \neq 0\) of \((6.10)\) with \(\lambda = \lambda_k\) such that \(v_k \to 0\) in \(C^1([0, \tau], \mathbb{R}^{2n})\) by Proposition 1.3. Then \(v_k\) is a critical point of \(f_{\lambda_k}\) in \((W_1, \| \cdot \|_2)\) and \(\|v_k\|_2 \to 0\) as \(k \to \infty\). Therefore for each \(k \in \mathbb{N}\) by Theorem 6.2 (iii) we have an unique \(z_k \in Z\) such that

\[
v_k = u(\lambda_k, z_k) = S^+x(\lambda_k, R^{-1}z_k) + S^-y(\lambda_k, R^{-1}z_k) + z_k.
\]

This implies that \(\|z_k\|_2 \leq \|v_k\|_2 \to 0\). Clearly, \(z_k \neq 0\) by (i) and (iii) in Theorem 6.2. These show that \((\mu, 0) \in \Lambda \times Z\) is a bifurcation point along sequences of \(a'_\lambda(z) = 0\) with respect to the trivial branch \(\{(\lambda, 0) \mid \lambda \in \Lambda\} \subset \Lambda \times Z\). Suppose \(\nu^L_{L_0}(\gamma_\mu) = \dim(\gamma_\mu(\tau)L_0 \cap L') > 0\). Theorem 6.2 (iv) gives rise to \(m^0(a_\mu, 0) = 0\), that is, \(a''_\mu(0) : Z \to Z\) is invertible. By Theorem 6.2 (iii) we may use the implicit function theorem for \(\Lambda \times Z \ni (\lambda, z) \mapsto a'_\lambda(z) \in Z\) near \((\mu, 0)\) to yield \(z_k = 0\) for \(k\) large enough. A contradiction is obtained.

(II). As above we can derive from the assumptions and Theorem 6.2 (iv) that for each \(k \in \mathbb{N}\),

\[
[m^-((a_\lambda^-)^k, 0), m^-((a_\lambda^+)^k, 0) + m^0((a_\lambda^+)^k, 0)] \cap [m^-((a_\lambda^+)^k, 0), m^-((a_\lambda^+)^k, 0) + m^0((a_\lambda^+)^k, 0)]
\]

and either \(m^0((a_\lambda^-)^k, 0) = \nu^L_{L_0}(\gamma_\lambda^+) = 0\) or \(m^0((a_\lambda^+)^k, 0) = \nu^L_{L_0}(\gamma_\lambda^-) = 0\). By Theorem 6.2 there exists a sequence \(\{(\lambda_k, z_k)\}_{k \geq 1} \in \Lambda \times Z \setminus \{(\mu, 0)\}\) converging to \((\mu, 0)\) such that \(z_k \neq 0\) and \(a'_\lambda(z_k) = 0\) for each \(k \in \mathbb{N}\). Let \(v_k = u(\lambda_k, z_k) = S^+x(\lambda_k, R^{-1}z_k) + S^-y(\lambda_k, R^{-1}z_k) + z_k\) for each \(k \in \mathbb{N}\). Then by Theorem 6.2 we obtain that \(v_k \neq 0, v_k \to 0\) in \((W_1, \| \cdot \|_2)\) and \(f'_{\lambda_k}(v_k) = 0\) for each \(k \in \mathbb{N}\). The expected result is proved.

(III). As above we may obtain

\[
[m^-((a_\lambda^-)^k, 0), m^-((a_\lambda^-)^k, 0) + m^0((a_\lambda^-)^k, 0)] \cap [m^-((a_\lambda^+)^k, 0), m^-((a_\lambda^+)^k, 0) + m^0((a_\lambda^+)^k, 0)] = \emptyset
\]

and either \(m^0((a_\lambda^+)^k, 0) = \nu^L_{L_0}(\gamma_\lambda^+) = 0\) or \(m^0((a_\lambda^-)^k, 0) = \nu^L_{L_0}(\gamma_\lambda^-) = 0\). Then by Theorem 6.3 we obtain a \(\mu \in \Lambda\) such that \((\mu, 0)\) is a bifurcation point along sequences of \(a_\lambda(z) = 0\) in \(\Lambda \times Z\). Moreover, if \(m^0((a_\lambda^+)^k, 0) = 0\) (resp. \(m^0((a_\lambda^-)^k, 0) = 0\), then \(\mu \neq \lambda^+\) (resp. \(\mu \neq \lambda^-\)). Finally, the desired result follows from these as above.

**Proof of Theorem 1.34.** Under the assumptions of Theorem 1.34 with additional conditions in (6.6), following the notations in the proof of Theorem 1.33 we obtain that \(m^0(a_\lambda, 0) \neq 0\) and \(m^0(a_\lambda, 0) = 0\) for each \(\lambda \in \Lambda \setminus \{\mu\}\) near \(\mu\), and that \(m^-(a_\lambda, 0)\) take, respectively, values \(m^-((a_\lambda, 0))\) and \(m^-(a_\lambda, 0) + m^0(a_\lambda, 0)\) as \(\lambda \in \Lambda\) varies in two deleted half neighborhoods of \(\mu\). Therefore [52] Theorem 3.6 (or [52] Theorem 4.10) is applicable to \(\{a_\lambda \mid \lambda \in \Lambda\}\). The corresponding conclusions are easily translated into those of Theorem 1.34 with additional conditions in (6.6).
6.2 Proof of Theorem 1.37

Define $G : [0, \tau] \times [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ by

$$G(\lambda, t, z) = \lambda H (z + (1 - t)u(0) + tu(\lambda)) + (z + (1 - t)u(0) + tu(\lambda), J(\mu(\lambda) - u(0))) \mathbb{R}^{2n}.$$ 

Then $\nabla_z G(\lambda, t, z) = \lambda \nabla H (z + (1 - t)u(0) + tu(\lambda)) + J(\mu(\lambda) - u(0))$ and $\nabla^2 G(\lambda, t, z) = \lambda \nabla^2 H (z + (1 - t)u(0) + tu(\lambda))$. For $\lambda \in (0, \tau]$ it is easily proved that $z : [0, \lambda] \to \mathbb{R}^{2n}$ satisfies

$$\dot{v}(t) = J \nabla_z H (v(t)) \forall t \in [0, \lambda],$$

$v(0) \in u(0) + L, \ \ v(\lambda) \in u(\lambda) + L$ \quad (6.20)

if and only if $w : [0, 1] \to \mathbb{R}^{2n}$ defined by $w(t) := z(\lambda t) - (1 - t)u(0) - tu(\lambda)$ satisfies

$$\dot{w}(t) = J \nabla_z G(\lambda, t, w(t)) \forall t \in [0, 1],$$

$w(0) \in L, \ \ w(1) \in L$. \quad (6.21)

In particular, since $u|_{[0, \lambda]}$ satisfies (6.20), the path

$$w_\lambda : [0, 1] \to \mathbb{R}^{2n}, \ t \mapsto u(\lambda t) - (1 - t)u(0) - tu(\lambda)$$

satisfies (6.21). Let $B(t) = \nabla^2 H (u(t))$ for $t \in [0, \tau]$, and let $\gamma : [0, \tau] \to \text{Sp}(2n)$ be the fundamental matrix solution of $\dot{v}(t) = JB(t)v(t)$ on $[0, \tau]$. Then it is easily checked that

$$B_\lambda(t) := \nabla^2 G(\lambda, t, w_\lambda(t)) = \lambda \nabla^2 H (u(\lambda t)) = \lambda B(\lambda t) \ \ \forall t \in [0, 1],$$

and $[0, 1] \ni t \mapsto \gamma_\lambda(t) := \gamma(\lambda t)$ is the fundamental matrix solution of $\dot{v}(t) = JB_\lambda(t)v(t)$ on $[0, 1]$.

We shall obtain conclusions of Theorem 1.37 by applying Theorem 1.33(I) and Theorem 1.34 to (6.21). Note that $(i_L, \nu_L) = (i_L, \nu_L)$.

**Step 1 (Prove (A)).** Suppose that $\mu \in (0, \tau]$ is a bifurcation instant for $(H, u, L)$. By the above definition there exists a sequence $(\tau_k) \subset (0, \tau]$ converging to $\mu$ such that for each $k \in \mathbb{N}$ the boundary value problem (1.45) has a solution $v_k \neq u|_{[0, \tau_k]}$ such that $\|v_k - u\|_{C^1([0, \tau_k], \mathbb{R}^{2n})} \to 0$ as $k \to \infty$. Then

$$w_k : [0, 1] \to \mathbb{R}^{2n}, \ t \mapsto v_k(\tau_k t) - (1 - t)u(0) - tu(\tau_k)$$

is not equal to $w_{\tau_k}$, and satisfies (6.21) and

$$\|w_k - w_{\tau_k}\|_{C^1([0, 1], \mathbb{R}^{2n})} \leq (1 + \tau)\|v_k - u\|_{C^1([0, \tau_k], \mathbb{R}^{2n})} \to 0$$

as $k \to \infty$. Since both $u$ and $\dot{u}$ are uniformly continuous on $[0, \tau]$, it is not hard to prove that $\|w_{\tau_k} - w_{\mu}\|_{C^1([0, 1], \mathbb{R}^{2n})} \to 0$ as $k \to \infty$. Hence $\|w_k - w_{\mu}\|_{C^1([0, 1], \mathbb{R}^{2n})} \to 0$. These show that $(\mu, w_{\mu})$ is a bifurcation point of (6.21). By Theorem 1.33(I) we deduce

$$\nu_L(\gamma|_{[0, \mu]}) = \dim(\gamma(\mu)I) \cap L = \nu_L(\gamma|_{[0, \mu]}) \geq 0$$

and therefore Theorem 1.37(A).

**Step 2 (Prove (B)).** If all $B_{22}(t)$ (resp. all $B_{11}(t)$) are positive definite, it was claimed in 37 Lemma 5.1 that

$$i_{L_0}(\gamma) = \sum_{0 < \lambda < \tau} \nu_{L_0}(\gamma_\lambda) = \sum_{0 < \lambda < \tau} \dim((\gamma(\lambda)L_0) \cap L_0)$$

(resp. $i_{L_1}(\gamma) = \sum_{0 < \lambda < \tau} \nu_{L_1}(\gamma_\lambda) = \sum_{0 < \lambda < \tau} \nu_{L_0} \dim((\gamma(\lambda)L_1) \cap L_1)$). (6.25)
Theorem 1.34 about (6.21) are equivalent to (B.1), (B.ii) and (B.iii), respectively. We can apply Theorem 1.34 to (6.21). Then the corresponding conclusions (i), (ii) and (iii) of Note that all $O^T B(t) O$ are also positive definite. By (6.21) we obtain
\begin{equation}
\dim((\Upsilon(\gamma)L_0) \cap L_0) = \dim((\gamma(L)L) \cap L) \tag{6.26}
\end{equation}
because $\dim((\Upsilon(\gamma)L_0) \cap L_0) = \dim((O^{-1}\gamma(L)OL_0) \cap L_0) = \dim((\gamma(L)L) \cap L)$. This implies the first claim in Theorem 1.37(B).

Let $\mu \in (0, \tau)$ be such that $\dim(\gamma(\mu)L) \cap L > 0$. Then there exists a small $\varepsilon > 0$ such that $\dim(\gamma(\lambda)L) \cap L = 0$ for all $\lambda \in [\mu - \varepsilon, \mu + \varepsilon] \setminus \{\mu\}$. Note that (6.21) implies
\begin{equation}
\dim((\gamma(\lambda)L) \cap L) = \dim((\gamma(\theta)L) \cap L)
\end{equation}
and so
\begin{equation}
i_L(\gamma) = \begin{cases}
i_L(\gamma) \\ i_L(\gamma) + \nu(\gamma)
\end{cases} \forall \lambda \in (\mu, \mu + \varepsilon).
\end{equation}
We can apply Theorem 1.34 to (6.21). Then the corresponding conclusions (i), (ii) and (iii) of Theorem 1.34 about (6.21) are equivalent to (B.1), (B.ii) and (B.iii), respectively.

With (6.24) and (6.25), the final conclusions can be derived as above. \hfill \Box

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A Maslov-type index and Morse index

Let $\text{Sp}(2n, \mathbb{R})^0 = \{M \in \text{Sp}(2n) \mid D(M) = 0\}$, where $D(M) = (-1)^{n-1} \det(M - tI_{2n})$. The complementary set $\text{Sp}(2n, \mathbb{R})^* := \text{Sp}(2n, \mathbb{R}) \setminus \text{Sp}(2n, \mathbb{R})^0$ has exactly two path-connected components $\text{Sp}(2n, \mathbb{R})^\pm = \{M \in \text{Sp}(2n, \mathbb{R}) \mid D(M) < 0\}$, which contain, respectively,
\begin{equation}
M_n^+ = \text{blockdiag}(2I_n, -\frac{1}{2}I_n) \quad \text{and} \quad M_n^- = \text{blockdiag}(-2, 2I_{n-1}, -\frac{1}{2}I_{n-1}).
\end{equation}
Every loop in these two components is contractible in $\text{Sp}(2n)$ ([17, Lemma 1.7] and [60, Lemma 3.2]).

Let $\mathcal{P}_\tau(2n) = \{\gamma \in C([0, \tau], \text{Sp}(2n, \mathbb{R})) \mid \gamma(0) = I_{2n}\}$ and let $\mathcal{P}_\tau^+(2n)$ consist of $\gamma \in \mathcal{P}_\tau^+(2n)$ such that $\gamma(\tau) \in \text{Sp}(2n, \mathbb{R})^*$. If $\gamma_1, \gamma_2 \in \mathcal{P}_\tau(2n)$ satisfy $\gamma_1(\tau) = \gamma_2(0)$, define $\gamma_1 * \gamma_2: [0, \tau] \to \text{Sp}(2n, \mathbb{R})$ by $\gamma_2 * \gamma_1(t) = \gamma_1(2t)$ for $t \in [0, \tau/2]$, and $\gamma_2 * \gamma_1(t) = \gamma_2(2t - \tau)$ for $t \in [\tau/2, \tau]$.

Recall that each $S \in \text{Sp}(2n, \mathbb{R})$ has a unique decomposition $P U$, where $P = \sqrt{S^T S}$ and $U = \begin{pmatrix} U_1 & -U_2 \\ U_2 & U_1 \end{pmatrix}$ such that $u(S) := U_1 + \sqrt{-1}U_2$ is a unitary matrix, i.e., $u(S) \in U(n, \mathbb{C})$. Therefore each path $\gamma \in C([a, b], \text{Sp}(2n, \mathbb{R}))$ corresponds to a unique continuous path $u_\gamma : [a, b] \to U(n, \mathbb{C})$ given by $u_\gamma(t) := u(\gamma(t))$. By the lifting criterion in the cover space theory there exists a continuous real function $\Delta_\gamma$ on $[a, b]$ such that $\det u_\gamma(t) = \exp(\sqrt{-1} \Delta_\gamma(t))$ for $t \in [a, b]$, and $\Delta(\gamma) := \Delta_\gamma(b) - \Delta_\gamma(a)$ is uniquely determined by $\gamma$.
to every pair of Lagrangian paths $\Lambda$. Recall that the Cappell-Lee-Miller index $\mu$ assigns an integer $\in \gamma$ to $P^*_\tau(2n)$, the so-called Conley-Zehnder index of $\gamma$, where $\beta$ is a path in $Sp(2n)^*$. An alternative exposition was presented in [66].

There exist two ways to extend the Conley-Zehnder index to paths in $P_\tau(2n) \setminus P^*_\tau(2n)$. Long [143] defined the Maslov-type index of $\gamma \in P_\tau(2n)$ to be a pair of integers $(i_\tau(\gamma), \nu_\tau(\gamma))$,

$$i_\tau(\gamma) = \inf \{ i_{CLM}(\beta) \mid \beta \in P^*_\tau(2n) \text{ is sufficiently } C^0 \text{ close to } \gamma \in P_\tau(2n) \}$$

and $\nu_\tau(\gamma) := \dim \ker(\gamma(\tau) - I_{2n})$. It was proved in [42] that $i_\tau$ has the following homotopy invariance: Two paths $\gamma_0$ and $\gamma_1$ in $P_\tau(2n)$ have the same Maslov-type indexes $i_\tau(\gamma_0) = i_\tau(\gamma_1)$ if there is a map $\delta \in C([0,1] \times [0,\tau], Sp(2n,\mathbb{R}))$ such that $\delta(0,\cdot) = \gamma_0$, $\delta(1,\cdot) = \gamma_1$, $\delta(s,0) = I_{2n}$ and $\nu_\tau(\delta(s,\cdot))$ is constant for $0 \leq s \leq 1$.

For $a < b$ and any path $\gamma \in C([a,b], Sp(2n,\mathbb{R}))$, choose $\beta \in P_1(2n)$ with $\beta(1) = \gamma(a)$, and define $\phi \in P_1(2n)$ by $\phi(t) = \beta(2t)$ for $0 \leq t \leq 1/2$, and $\phi(t) = \gamma(a + (2t - 1)(b-a))$ for $1/2 \leq t \leq 1$. Long showed in [42] that the difference $i_1(\phi) - i_1(\beta)$ only depends on $\gamma$, and called

$$i(\gamma,[a,b]) := i_1(\phi) - i_1(\beta)$$ (A.1)

the Maslov-type index of $\gamma$. Clearly, $i(\gamma,[0,1]) = i_1(\gamma)$ for any $\gamma \in P_1(2n)$.

In order to introduce another extension way of $i_{CLM}$ by Robbin-Salamon [64]. Let $(F,\Omega)$ be the symplectic space $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n},(-\omega_0) \oplus \omega_0)$ and let $L(F,\Omega)$ denote the manifold of Lagrangian subspaces of $(F,\Omega)$. For $M \in Sp(2n,\mathbb{R})$, both

$$W := \{(x^T,x^T)^T \in \mathbb{R}^{4n} \mid x \in \mathbb{R}^{2n}\} \quad \text{and} \quad Gr(M) := \{(x^T,(Mx)^T)^T \in \mathbb{R}^{4n} \mid x \in \mathbb{R}^{2n}\}$$

belong to $L(F,\Omega)$. The Robbin-Salamon index $\mu^{RS}$ defined in [64] assigns a half integer $\mu^{RS}(\Lambda,\Lambda')$ to every pair of Lagrangian paths $\Lambda,\Lambda' : [a,b] \to L(F,\Omega)$. The Conley-Zehnder index of $\gamma \in C([a,b],Sp(2n,\mathbb{R}))$ was defined in [65] Remark 5.35 by

$$\mu_{CLM}(\gamma) = \mu^{RS}(Gr(\gamma),W).$$ (A.2)

By [64] Remark 5.4 it holds that

$$\mu_{CLM}(\gamma) = i_{CLM}(\gamma), \forall \gamma \in P^*_\tau(2n).$$ (A.3)

There exists a precise relation between $i(\gamma,[a,b])$ and $\mu_{CLM}(\gamma)$ for each $\gamma \in C([a,b],Sp(2n,\mathbb{R}))$. Recall that the Cappell-Lee-Miller index $\mu_{CLM}$ characterized by properties I-VI of [11] pp. 127-128] assigns an integer $\mu_{CLM}(\Lambda,\Lambda')$ to every pair of Lagrangian paths $\Lambda,\Lambda' : [a,b] \to L(F,\Omega)$. It was proved in [13] Corollary 2.1 that

$$i_\tau(\gamma) = \mu_{CLM}(W,Gr(\gamma),[0,\tau]) - n, \forall \gamma \in P_\tau(2n).$$ (A.4)

Then for any $\gamma \in C([a,b],Sp(2n,\mathbb{R}))$, (A.1), (A.3) and the path additivity of $\mu_{CLM}$ lead to

$$i(\gamma,[a,b]) = \mu_{CLM}(W,\text{Gr}(\gamma],[a,b])$$ (A.5)

and therefore

$$i(\gamma,[a,b]) = \mu_{CLM}(\gamma) - \frac{1}{2}(\dim \ker(I_{2n} - \gamma(b)) - \dim \ker(I_{2n} - \gamma(a)))$$ (A.6)

by (A.2) and [43] Theorem 3.1. In particular,

$$i_\tau(\gamma) = \mu_{CLM}(\gamma) - \frac{1}{2} \dim \ker(I_{2n} - \gamma(\tau)), \forall \gamma \in P_\tau(2n).$$ (A.7)
For \( \phi \) and \( \beta \) in \((A.1)\), since \( \mu_{CZ} \) is additive with respect to the catenation of paths we have also

\[
\mu_{CZ}(\gamma) = \mu_{CZ}(\phi) - \mu_{CZ}(\beta). \tag{A.8}
\]

Both indexes \( i \) and \( \mu_{CZ} \) have also vanishing, product and the following:

\textbf{(Homotopy)} If a continuous path \( \ell : [0, 1] \to C([a, b], \text{Sp}(2n, \mathbb{R})) \) is such that

\[
s \mapsto \dim \ker(I_{2n} - \ell(s)(a)) \quad \text{and} \quad s \mapsto \dim \ker(I_{2n} - \ell(s)(b)) \tag{A.9}
\]

are constant functions on \([0, 1] \), then

\[
i(\ell(s), [a, b]) = i(\ell(0), [a, b]) \quad \forall s \in [0, 1], \tag{A.10}
\]

\[
\mu_{CZ}(\ell(s)) = \mu_{CZ}(\ell(0)) \quad \forall s \in [0, 1]. \tag{A.11}
\]

\textbf{(Naturality)} For any \( \phi, \gamma \in C([a, b], \text{Sp}(2n, \mathbb{R})) \) it holds that

\[
i(\phi \gamma \phi^{-1}, [a, b]) = i(\gamma, [a, b]) \quad \text{and} \quad \mu_{CZ}(\phi \gamma \phi^{-1}) = \mu_{CZ}(\gamma). \tag{A.12}
\]

In fact, we have continuous maps \([0, 1] \times [0, 1] \to \text{Sp}(2n, \mathbb{R}), (s, t) \mapsto \beta_s(t)\) given by

\[
\beta_s(t) = \beta_0 \left( \frac{t}{1 - s} \right) \quad \text{for } 0 \leq t \leq 1 - s, \quad \beta_s(t) = \ell(t - 1 + s)(a) \quad \text{for } 1 - s \leq t \leq 1,
\]

and \([0, 1] \times [0, 1] \to \text{Sp}(2n, \mathbb{R}), (s, t) \mapsto \phi_s(t)\) defined by

\[
\phi_s(t) = \beta_s(2t) \quad \text{for } 0 \leq t \leq 1/2, \quad \phi_s(t) = \ell(s)(a + (2t - 1)(b - a)) \quad \text{for } 1/2 \leq t \leq 1.
\]

Then \( \beta_0(0) = \phi_s(0) = I_{2n} \) for all \( s \in [0, 1] \), and \( \nu_1(\beta_s) = \dim \ker(I_{2n} - \ell(s)(a)) \) and \( \nu_1(\phi_s) = \dim \ker(I_{2n} - \ell(s)(b)) \) are constant functions on \([0, 1] \). By \((A.1), i(\beta_s, [0, 1]) = i(\beta_0)\) for all \( s \in [0, 1] \). The homotopy invariance of \( i_1 \) leads to \((A.10)\), and therefore \((A.11)\) by \((A.6)\) and \((A.9)\).

In order to prove \((A.12)\), we consider a continuous path \( \ell : [0, 1] \to C([a, b], \text{Sp}(2n, \mathbb{R})) \) given by \( \ell(s)(t) = \phi(st)\gamma(t)(\phi(st))^{-1} \). Then \((A.11)\) implies \( \mu_{CZ}(\phi \gamma \phi^{-1}) = \mu_{CZ}(\phi(0)\gamma(\phi(0))^{-1}) \). Note that

\[
\text{Gr}(\phi(0)\gamma(\phi(0))^{-1})(t) = \{ (x^T, (\phi(0)\gamma(t)(\phi(0))^{-1}x)^T \in \mathbb{R}^{4n} | x \in \mathbb{R}^{2n} \} = 2\text{Gr}(\gamma)(t),
\]

where \( \Xi : (F, \Omega) \to (F, \Omega) \) is the symplectic isomorphism given by \( \Xi(x \oplus y) = (\phi(0)x) \oplus (\phi(0)y) \). Since \( \Xi W = W \), by \((A.2)\) and the naturality property of \( \mu_{RS} \) \((\mathbb{R} \mathbb{S} \S 5)\) we obtain

\[
\mu_{CZ}(\phi(0)\gamma(\phi(0))^{-1}) = \mu_{RS}(2\text{Gr}(\gamma), W) = \mu_{RS}(\text{Gr}(\gamma), W) = \mu_{CZ}(\gamma).
\]

The first equality in \((A.12)\) may follow from this and \((A.6)\).

Let \( U_1 = \{ 0 \} \times \mathbb{R}^n \) and \( U_2 = \mathbb{R}^n \times \{ 0 \} \). For \( \gamma \in P_\tau(2n) \) with \( \gamma(\tau)/2 = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \), where \( A, B, C, D \in \mathbb{R}^{n \times n} \), Long, Zhang and Zhu \((A.10)\) defined

\[
\mu_{k, \tau}(\gamma) = \mu_{RS}^{CLM}(U_k, \gamma U_k, [0, \tau/2]), \quad k = 1, 2, \tag{A.13}
\]

\[
\nu_{1, \tau}(\gamma) = \dim \ker(B) \quad \text{and} \quad \nu_{2, \tau}(\gamma) = \dim \ker(C). \tag{A.14}
\]
Dong \cite{19} and Liu \cite{36} extended the Maslov-type index \((i_\tau(\gamma), \nu_\tau(\gamma))\) of \(\gamma \in \mathcal{P}_\tau(2n)\) to the case relative to a given symplectic matrix \(M \in \text{Sp}(2n, \mathbb{R})\) via different methods, denoted by

\[
(i_{\tau, M}(\gamma), \nu_{\tau, M}(\gamma)) \quad \text{and} \quad (i_{\tau}^M(\gamma), \nu_{\tau}^M(\gamma))
\]

respectively. (The former was written as \((i_M(\gamma), \nu_M(\gamma))\) in \cite{19}). They are more suitable and convenient for dual variational methods and saddle point reduction ones, respectively. If \(M = I_{2n}\), by \cite{19} and \cite{36} Definition 2.5 and Definition 2.6 there holds

\[
(i_{\tau, M}(\gamma), \nu_{\tau, M}(\gamma)) = \left(\begin{array}{c}i_{\tau}(\gamma) \\ (i_{\tau}^M(\gamma), \nu_{\tau}^M(\gamma))\end{array}\right) = (i_{\tau}(\gamma), \nu_{\tau}(\gamma)). \tag{A.15}
\]

Let \([a]\) denote the largest integer no more than \(a \in \mathbb{R}\), and let \(\xi\) be any element in \(\mathcal{P}_\tau(2n)\) satisfying \(\xi(\tau) = M^{-1}\), Dong defined

\[
\begin{align*}
\nu_{\tau, M}(\gamma) &= \dim \ker(\gamma(\tau) - M) \quad \text{and} \\
i_{\tau, M}(\gamma) &= [i_{\tau}((\gamma M^{-1}) \ast \xi) - \Delta(\xi)/\pi] \tag{A.16}
\end{align*}
\]

\((\cite{19} Definitions 2.1, 2.2)], and Liu \cite{40} Definition 2.7 and Remark 2.8) defined

\[
\begin{align*}
\nu_{\tau}^M(\gamma) &= \dim \ker(\gamma(\tau) - M) \quad \text{and} \\
i_{\tau}^M(\gamma) &= i_{\tau}((M^{-1} \gamma) \ast \xi) - i_{\tau}(\xi) \tag{A.17}
\end{align*}
\]

if \(M \neq I_{2n}\), and \((i_{\tau}^M(\gamma), \nu_{\tau}^M(\gamma)) = (i_{\tau}(\gamma), \nu_{\tau}(\gamma))\) if \(M = I_{2n}\). (Note that \(i_{\tau}((M^{-1} \gamma) \ast \xi)\) may be replaced by \(i_{\tau}((\gamma M^{-1}) \ast \xi))\), see the proof of \(A.19\) below. It was shown in \cite{40} Remark 2.8 that when \(M = I_{2n}\), the right side of the second equality in \(A.17\) is \(i_{\tau}(\gamma) + n\). A direct observation leads to a precise relation between \(i_{M, \tau}\) and \(i_{\tau}^M\) as follows.

**Proposition A.1.** For any \((M, \gamma) \in (\text{Sp}(2n, \mathbb{R}) \setminus \{I_{2n}\}) \times \mathcal{P}_\tau(2n)\) and any \(\xi \in \mathcal{P}_\tau(2n)\) satisfying \(\xi(\tau) = M^{-1}\) it holds that

\[
i_{\tau, M}(\gamma) = [i_{\tau}^M(\gamma) + i_{\tau}(\gamma) - \Delta(\xi)/\pi] = i_{\tau}^M(\gamma) + [i_{\tau}(\gamma) - \Delta(\xi)/\pi]. \tag{A.18}
\]

\(([i_{\tau}(\gamma) - \Delta(\xi)/\pi] \text{ only depends on } M)\)

**Proof.** Clearly, \(M^{-1} \gamma M \in \mathcal{P}_\tau(2n)\) and \((M^{-1} \gamma M)(\tau) = M^{-1}\) for any \(\xi \in \mathcal{P}_\tau(2n)\) satisfying \(\xi(\tau) = M^{-1}\). By \((A.17)\) and \cite{42} Corollary 6.5

\[
i_{\tau}^M(\gamma) = i_{\tau}((M^{-1} \gamma) \ast (M^{-1} \gamma M)) - i_{\tau}(M^{-1} \gamma M) = i_{\tau}((\gamma M^{-1}) \ast \xi) - i_{\tau}(\xi). \tag{A.19}
\]

Substituting this in the second equality in \((A.16)\) we obtain \((A.18)\). \(\square\)

\(i_{\tau, M}\) and \(i_{\tau}^M\) may be different. For the symplectic path

\[
\xi_{2n, \tau} : [0, \tau] \to \text{Sp}(2n, \mathbb{R}), \quad t \mapsto I_{2n}. \tag{A.20}
\]

A direct computation yields \(i_{\tau, J_n}(\xi_{2n, \tau}) = 0\) and \(i_{\tau, J_{-n}}(\xi_{2n, \tau}) = [n/2]\).

For any two \(m \times m\) real symmetric matrices \(A_1\) and \(A_2\), we write \(A_1 \leq A_2\) (resp. \(A_1 < A_2\)) if \(A_2 - A_1\) is positive semi-definite (resp. positive definite). For any \(B_1, B_2 \in L^\infty([0, \tau]; \mathcal{L}_+(\mathbb{R}^{2n}))\) we define \(B_1 \leq B_2\) (resp. \(B_1 < B_2\)) if \(B_1(t) \leq B_2(t)\) for a.e. \(t \in [0, \tau]\) (resp. if \(B_1 \leq B_2\) and \(B_1(t) < B_2(t)\) on a subset of \([0, \tau]\) with positive measure).
For $B, B_1, B_2 \in L^\infty([0, \tau]; \mathcal{L}_s(\mathbb{R}^{2n}))$ with $B_1 < B_2$, it was defined in [19] Definitions 3.1, 3.2 that
\[
\begin{aligned}
\nu_{\tau,M}(B) := \nu_M(\Upsilon_B) \quad \text{and} \\
I_{\tau,M}(B_1, B_2) := \sum_{s \in \{0, 1\}} \nu_{\tau,M}( (1 - s)B_1 + sB_2).
\end{aligned}
\tag{A.21}
\]
where $\Upsilon_B : [0, \tau] \to \text{Sp}(2n, \mathbb{R})$ is the fundamental matrix solution of the linear system $\dot{Z}(t) = JB(t)Z(t)$ with $\Upsilon_B(0) = I_{2n}$.

For any $B_1, B_2 \in L^\infty([0, \tau]; \mathcal{L}_s(\mathbb{R}^{2n}))$, and $s \in \mathbb{R}$ such that $sI_{2n} > B_1$ and $sI_{2n} > B_2$,
\[
I_{\tau,M}(B_1, B_2) := I_{\tau,M}(B_1, sI_{2n}) - I_{\tau,M}(B_2, sI_{2n})
\tag{A.22}
\]
is was called the relative Morse index between $B_1$ and $B_2$ ([19] Definition 3.5]). The $M$-index of $B \in L^\infty([0, \tau]; \mathcal{L}_s(\mathbb{R}^{2n}))$ was defined in [19] Definition 3.8 by
\[
j_{\tau,M}(B) := i_{\tau,M}(\xi_{2n}) + I_{\tau,M}(0, B),
\tag{A.23}
\]
where $\xi_{2n}$ is as in (A.20). Then $j_{\tau,M}(0) = i_{\tau,M}(\xi_{2n})$ ([19] Remark 3.9)).

**Theorem A.2** ([19] Theorem 4.1). $j_{\tau,M}(B) = i_{\tau,M}(\Upsilon_B)$ for any $B \in L^\infty([0, \tau]; \mathcal{L}_s(\mathbb{R}^{2n}))$.

For $B \in L^\infty([0, \tau]; \mathcal{L}_s(\mathbb{R}^{2n}))$, let $A \in L^\infty([0, \tau]; \mathcal{L}_s(\mathbb{R}^{2n}))$ satisfy $B - A \geq \varepsilon I_{2n}$ for $\varepsilon > 0$ small enough, and let $C(t) = (B(t) - A(t))^{-1}$. Consider the quadratic form
\[
q_{M,B,A}(u, v) = \frac{1}{2} \int_0^\tau \left( (\hat{\Lambda}_{M,\tau,A})^{-1}u(t), v(t) \right)_{\mathbb{R}^{2n}} + (C(t)u(t), v(t))_{\mathbb{R}^{2n}} dt
\tag{A.24}
\]
on the Hilbert space $\tilde{L}_{M,A}^2([0, \tau]; \mathbb{R}^{2n}) = R(\Lambda_{M,\tau,A})$ (cf. § 2.1). By [27] Th.7.1, $\tilde{L}_{M,A}^2([0, \tau]; \mathbb{R}^{2n})$ is expressible in a unique manner as the direct sum of three linear subspaces
\[
E_M^-(B|A), \quad E_M^0(B|A), \quad E_M^+(B|A)
\]
having the following properties:

(a) $E_M^-(B|A)$, $E_M^0(B|A)$, $E_M^+(B|A)$ are mutually $L^2$-orthogonal and $q_{M,B,A}$-orthogonal;

(b) $q_{M,B,A}$ is negative on $E_M^-(B|A)$, zero on $E_M^0(B|A)$, and positive on $E_M^+(B|A)$.

Moreover, since $2q_{M,B,A}(u, v) = ((\hat{\Lambda}_{M,\tau,A})^{-1}u, v)_{L^2} + (\hat{C}u, v)_{L^2}$ for a compact self-adjoint operator $\hat{C} : \tilde{L}_{M,A}^2([0, \tau]; \mathbb{R}^{2n}) \to \tilde{L}_{M,A}^2([0, \tau]; \mathbb{R}^{2n})$ defined by
\[
(\hat{C}u, v)_{L^2} = \int_0^\tau (C(t)u(t), v(t))_{\mathbb{R}^{2n}} dt,
\]

namely $q_{M,B,A}$ is a Legendre form by [27] Theorem 11.6], the subspaces $E_M^-(B|A)$ and $E_M^0(B|A)$ have finite dimensions (cf. [27] Theorem 11.3) or Corollary on page 311 of [28]. Dong [19] also proved these directly and called
\[
j_{\tau,M}(B|A) := \dim E_M^-(B|A) \quad \text{and} \quad \nu_{\tau,M}(B|A) := \dim E_M^0(B|A)
\tag{A.25}
\]
$M$-index and $M$-nullity of $B$ with respect to $A$, respectively. Let $m^-(q_{M,B|A})$ and $m^0(q_{M,B|A})$ denote the Morse index and the nullity of $q_{M,B|A}$ on $\tilde{L}_{M,A}^2([0, \tau]; \mathbb{R}^{2n})$, respectively. Then
\[
m^0(q_{M,B|A}) = \nu_{\tau,M}(B|A) = \nu_{\tau,M}(\Upsilon_B),
\tag{A.26}
\]
where the second equality comes from [19 Proposition 3.14]. By [19 Remark 3.24] and Theorem A.2 we have also
\[
\begin{align*}
   j_{\tau,M}(B|A) &= j_{\tau,M}(B) - j_{\tau,M}(A) - \nu_{\tau,M}(A) \\
   &= i_{\tau,M}(\mathcal{Y}_B) - i_{\tau,M}(\mathcal{Y}_A) - \nu_{\tau,M}(\mathcal{Y}_A).
\end{align*}
\] (A.27)

It follows from these, and [27 Theorem 9.1] or [19 Lemma 3.16] that
\[
m^-(q_{M,B|A}) = j_{\tau,M}(B|A) = i_{\tau,M}(\mathcal{Y}_B) - i_{\tau,M}(\mathcal{Y}_A) - \nu_{\tau,M}(\mathcal{Y}_A).
\] (A.28)

For \(B_1, B_2 \in L^\infty([0,\tau];\mathcal{L}_\mu(\mathbb{R}^{2n}))\) with \(B_2 > B_1 > A + \varepsilon I_{2n}\) for a small enough \(\varepsilon > 0\), it was proved in [19 Lemma 3.18, Proposition 3.25] that
\[
\begin{align*}
   j_{\tau,M}(B_2|A) &\geq j_{\tau,M}(B_1|A) + \nu_{\tau,M}(B_1|A), \\
   j_{\tau,M}(B_2) &\geq j_{\tau,M}(B_1) + \nu_{\tau,M}(B_1).
\end{align*}
\] (A.29) (A.30)

The former is also a special case of [27 Theorem 9.4].

Remark A.3. When \(M = I_{2n}\) and \(A = \ell I_{2n}\) with \(\ell \in \mathbb{R} \setminus 2\pi\mathbb{Z}\), based on the arguments in [25] and the saddle point reduction (cf. [19 §6.1, Theorem 1]) Liu [33 Theorem 2.1] proved the following precise versions of (A.27) and (A.26) (in terms of the above notations)
\[
\begin{align*}
   j_{1,I_{2n}}(B|\ell I_{2n}) &= i_1(\mathcal{Y}_B) + n + 2n \left[ \frac{\ell}{2\pi} \right], \\
   \nu_{1,I_{2n}}(B|\ell I_{2n}) &= \nu_1(\mathcal{Y}_B).
\end{align*}
\] (A.31) (A.32)

For \(B \in L^\infty([0,\tau];\mathcal{L}_\mu(\mathbb{R}^{2n}))\) and \(0 \leq t \leq \tau\), let
\[
\nu_{t,M}(\mathcal{Y}_B|_{[0,t]}) = \dim \ker(\mathcal{Y}_B(t) - M)
\]
and so \(\nu_{0,M}(\mathcal{Y}_B|_{[0,0]}) = \dim \ker(I_{2n} - M)\). As a generalization of [20 Chapter I, §4, Theorem 6] and [13 Proposition 15.3], it was proved in [19 Lemma 4.2] that
\[
i_{\tau,M}(\mathcal{Y}_B) = i_{\tau,M}(\xi_{2n}) + \sum_{0 \leq t \leq \tau} \nu_{t,M}(\mathcal{Y}_B|_{[0,t]})
\] (A.33)
if \(B\) is positive definite, and
\[
i_{\tau,M}(\mathcal{Y}_B) = i_{\tau,M}(\xi_{2n}) - \sum_{0 < t \leq \tau} \nu_{t,M}(\mathcal{Y}_B|_{[0,t]})
\] (A.34)
if \(B\) is negative definite. Here \(\xi_{2n}\) is as in (A.20).

Remark A.4. The sum \(\sum_{0 < t \leq \tau} \nu_{t,M}(\mathcal{Y}_B|_{[0,t]})\) in (A.34) was written as \(\sum_{0 < t \leq \tau} \nu_{t,M}(\mathcal{Y}_B|_{[0,t]})\) in [19 (4.2)]. Actually, it was only asserted in [19] that [19 (4.2)] can be obtained in a similar way to the proof of [A.34]. Since Proposition A.1 implies that \(i_{\tau,M}(\gamma) - i_{\tau,M}(\xi_{2n}) = i^M_{\tau}(\gamma) - i^M_{\tau}(\xi_{2n})\), and
\[
i^M_{\tau}(\xi_{2n}) - i^M_{\tau}(\mathcal{Y}_B) = \sum_{0 < t \leq \tau} \nu_{t,M}(\mathcal{Y}_B|_{[0,t]})
\]
by [11 (4.8)], (A.34) follows.
From \((A.33)\) and \((A.34)\) we derive:

**Proposition A.5.** Let \(A \in \mathcal{L}_s(\mathbb{R}^{2n})\) be either positive definite or negative definite, and \(\Upsilon_{\lambda A}(t) = \exp(\lambda tJA)\) for \(\lambda \in \mathbb{R} \setminus \{0\}\). (Clearly, \(\Upsilon_{\lambda A}(t) = \Upsilon_A(\lambda t)\).) Then

\[
\Gamma(A) := \{ t \in \mathbb{R} \setminus \{0\} \mid \dim \ker(\exp(tJA) - M) > 0 \}
\]

is a discrete set in \(\mathbb{R}\), and the following holds. \((\lambda_0 = 0\) and so \(\Upsilon_{\lambda_0 A}(t) = I_{2n}\) for all \(t \in \mathbb{R}\).

(i) Let \(A > 0\). If \(\Gamma(A) \cap (0, \infty) = \{ \lambda_1 < \cdots < \lambda_k < \cdots < \lambda_n \}\) with \(\kappa \in \mathbb{N} \cup \{\infty\}\), then

\[
i_{1,1}(\Upsilon_{\lambda A}) = \begin{cases} 
i_{1,1}(\Upsilon_{\lambda_{k+1} A}) + \nu_{1,1}(\Upsilon_{\lambda_{k+1} A}) & \lambda_{k-1} < \lambda \leq \lambda_k \text{ with } k \geq 1, \\
i_{1,1}(\Upsilon_{\lambda_{k+1} A}) + \nu_{1,1}(\Upsilon_{\lambda_{k+1} A}) & \lambda_k < \lambda \leq \lambda_{k+1} \text{ with } k \geq 0, \\
i_{1,1}(\Upsilon_{\lambda_k A}) & \kappa \in \mathbb{N} \text{ and } \lambda > \lambda_k;
\end{cases}
\]

and if \(\Gamma(A) \cap (-\infty, 0) = \{ \lambda_{-k} < \cdots < \lambda_{-k-1} < \cdots < \lambda_{-1} \}\) with \(\kappa \in \mathbb{N} \cup \{\infty\}\), then

\[
i_{1,1}(\Upsilon_{\lambda A}) = \begin{cases} 
i_{1,1}(\Upsilon_{\lambda_{k+1} A}) + \nu_{1,1}(\Upsilon_{\lambda_{k+1} A}) & \lambda_{-k} < \lambda < \lambda_{-k+1} \text{ with } k \geq 1, \\
i_{1,1}(\Upsilon_{\lambda_{k+1} A}) & \lambda_{-k-1} < \lambda < \lambda_{-k} \text{ with } k \geq 1, \\
i_{1,1}(\Upsilon_{\lambda_{k} A}) & \kappa \in \mathbb{N} \text{ and } \lambda \geq \lambda_{-k}.
\end{cases}
\]

(ii) Let \(A < 0\). If \(\Gamma(A) \cap (0, \infty) = \{ \lambda_1 < \cdots < \lambda_k < \cdots < \lambda_n \}\) with \(\kappa \in \mathbb{N} \cup \{\infty\}\), then

\[
i_{1,1}(\Upsilon_{\lambda A}) = \begin{cases} 
i_{1,1}(\Upsilon_{\lambda_{k+1} A}) & \lambda_{-k} < \lambda < \lambda_{-k+1} \text{ with } k \geq 1, \\
i_{1,1}(\Upsilon_{\lambda_{k+1} A}) + \nu_{1,1}(\Upsilon_{\lambda_{k+1} A}) & \lambda_k < \lambda < \lambda_{k+1} \text{ with } k \geq 0, \\
i_{1,1}(\Upsilon_{\lambda_k A}) & \kappa \in \mathbb{N} \text{ and } \lambda \geq \lambda_{-k};
\end{cases}
\]

and if \(\Gamma(A) \cap (-\infty, 0) = \{ \lambda_{-k} < \cdots < \lambda_{-k-1} < \cdots < \lambda_{-1} \}\) with \(\kappa \in \mathbb{N} \cup \{\infty\}\), then

\[
i_{1,1}(\Upsilon_{\lambda A}) = \begin{cases} 
i_{1,1}(\Upsilon_{\lambda_{k+1} A}) + \nu_{1,1}(\Upsilon_{\lambda_{k+1} A}) & \lambda_{-k} < \lambda < \lambda_{-k+1} \text{ with } k \geq 1, \\
i_{1,1}(\Upsilon_{\lambda_{k+1} A}) + \nu_{1,1}(\Upsilon_{\lambda_{k+1} A}) & \lambda_{-k-1} < \lambda < \lambda_{-k} \text{ and } k \geq 1, \\
i_{1,1}(\Upsilon_{\lambda_{k} A}) & \kappa \in \mathbb{N} \text{ and } \lambda < \lambda_{-k}.
\end{cases}
\]

**Proof.** By \((A.33)\) and \((A.34)\) we derive

\[
i_{1,1}(\Upsilon_{\lambda A}) - i_{1,1}(\Upsilon_{\lambda_0 A}) = \begin{cases} \sum_{0 \leq t < \lambda} \dim \ker(\exp(tJA) - M) & \text{if } A > 0, \\
\sum_{0 \leq t \leq \lambda} \dim \ker(\exp(tJA) - M) & \text{if } A < 0
\end{cases}
\]

for \(\lambda > 0\), and

\[
i_{1,1}(\Upsilon_{\lambda A}) - i_{1,1}(\Upsilon_{\lambda_0 A}) = \begin{cases} \sum_{\lambda \leq t \leq 0} \dim \ker(\exp(tJA) - M) & \text{if } A < 0, \\
\sum_{\lambda \leq t < 0} \dim \ker(\exp(tJA) - M) & \text{if } A > 0
\end{cases}
\]

for \(\lambda < 0\). These imply that \(\sum_{-N \leq t < N} \dim \ker(\exp(tJA) - M) < \infty\) for any \(N > 0\). The first claim is proved.

(i) \([A > 0]\). Let \(\Gamma(A) \cap (0, \infty) = \{ \lambda_1 < \cdots < \lambda_k < \cdots < \lambda_n \}\) with \(\kappa \in \mathbb{N} \cup \{\infty\}\). Recall that \(\nu_{t,1}(\Upsilon_{\lambda A}) = \dim \ker(\exp(tJA) - M)\). By \((A.33)\) we have

\[
i_{1,1}(\Upsilon_{\lambda A}) = i_{1,1}(\Upsilon_{\lambda_0 A}) + \sum_{0 \leq t < \lambda} \dim \ker(\exp(tJA) - M) = i_{1,1}(\Upsilon_{\lambda_k A}).
\]
for $\lambda_{k-1} < \lambda \leq \lambda_k$ with $k \geq 1$, and

$$i_{1,M}(\Upsilon_{\lambda A}) = i_{1,M}(\Upsilon_{\lambda_0 A}) + \sum_{0 \leq t < \lambda} \dim\ker(\exp(tJ A) - M)$$

$$= i_{1,M}(\Upsilon_{\lambda_0 A}) + \sum_{0 \leq t < \lambda_k} \dim\ker(\exp(tJ A) - M) + \dim\ker(\exp(\lambda_k J A) - M)$$

$$= i_{1,M}(\Upsilon_{\lambda_k A}) + \mu_{1,M}(\Upsilon_{\lambda_k A})$$

for $\lambda_k < \lambda \leq \lambda_{k+1}$ with $k \geq 0$. If $\kappa \in \mathbb{N}$, then the last two lines hold true for $k = \kappa$ and any $\lambda > \lambda_k$.

Next, let $\Gamma(A) \cap (-\infty, 0) = \{\lambda_{-k} \cdots < \lambda_{-1} \}$ with $\kappa \in \mathbb{N} \cup \{\infty\}$. By (A.34),

$$i_{1,M}(\Upsilon_{\lambda A}) = i_{1,M}(\Upsilon_{\lambda_0 A}) - \sum_{\lambda \leq t < 0} \dim\ker(\exp(tJ A) - M)$$

$$= i_{1,M}(\Upsilon_{\lambda_0 A}) - \sum_{\lambda_{-1} \leq t < 0} \dim\ker(\exp(tJ A) - M) + \mu_{1,M}(\Upsilon_{\lambda_{-1} A})$$

$$= i_{1,M}(\Upsilon_{\lambda_{-1} A}) + \mu_{1,M}(\Upsilon_{\lambda_{-1} A})$$

for $\lambda_{-k} < \lambda \leq \lambda_{-k+1}$ with $k \geq 1$, and

$$i_{1,M}(\Upsilon_{\lambda A}) = \sum_{\lambda \leq t < 0} \dim\ker(\exp(tJ A) - M)$$

$$= i_{1,M}(\Upsilon_{\lambda_0 A}) - \sum_{\lambda_{-k} \leq t < 0} \dim\ker(\exp(tJ A) - M) \mu_{1,M}(\Upsilon_{\lambda_{-k} A})$$

for $\lambda_{-k-1} < \lambda \leq \lambda_{-k}$ with $k \geq 1$. Clearly, when $\kappa \in \mathbb{N}$, the last two lines also hold true for $k = \kappa$ and any $\lambda \leq \lambda_{-\kappa}$.

(ii) [A < 0]. Assume that $\Gamma(A) \cap (0, \infty) = \{\lambda_1 < \cdots < \lambda_k < \cdots < \lambda_{\kappa}\}$ with $\kappa \in \mathbb{N} \cup \{\infty\}$. By (A.34) we have

$$i_{1,M}(\Upsilon_{\lambda A}) = i_{1,M}(\Upsilon_{\lambda_0 A}) - \sum_{0 < t \leq \lambda} \dim\ker(\exp(tJ A) - M)$$

$$= i_{1,M}(\Upsilon_{\lambda_0 A}) - \sum_{0 < t \leq \lambda_k} \dim\ker(\exp(tJ A) - M) + \mu_{1,M}(\Upsilon_{\lambda_k A})$$

$$= i_{1,M}(\Upsilon_{\lambda_k A}) + \mu_{1,M}(\Upsilon_{\lambda_k A})$$

for $\lambda_{k-1} < \lambda \leq \lambda_k$ with $k \geq 1$, and

$$i_{1,M}(\Upsilon_{\lambda A}) = i_{1,M}(\Upsilon_{\lambda_0 A}) - \sum_{0 < t \leq \lambda} \dim\ker(\exp(tJ A) - M)$$

$$= i_{1,M}(\Upsilon_{\lambda_0 A}) - \sum_{0 < t \leq \lambda_k} \dim\ker(\exp(tJ A) - M) \mu_{1,M}(\Upsilon_{\lambda_k A})$$

for $\lambda_k \leq \lambda \leq \lambda_{k+1}$ with $k \geq 1$. If $\kappa \in \mathbb{N}$, the last two lines also hold true for $k = \kappa$ and any $\lambda \geq \lambda_{\kappa}$.

When $\Gamma(A) \cap (-\infty, 0) = \{\lambda_{-k} \cdots < \lambda_{-1} \}$ with $\kappa \in \mathbb{N} \cup \{\infty\}$, by (A.33) we obtain

$$i_{1,M}(\Upsilon_{\lambda A}) = i_{1,M}(\Upsilon_{\lambda_0 A}) + \sum_{\lambda < t \leq 0} \dim\ker(\exp(tJ A) - M) = i_{1,M}(\Upsilon_{\lambda_{-1} A})$$
for \( \lambda_{-k} \leq \lambda < \lambda_{-k+1} \) with \( k \geq 1 \), and

\[
    i_{1,M}(Y_{\lambda A}) = i_{1,M}(Y_{\lambda_0 A}) + \sum_{\lambda < t \leq 0} \dim \ker(\exp(tJA) - M)
\]

\[
    = i_{1,M}(Y_{\lambda_{-k} A}) + \nu_{1,M}(Y_{\lambda_{-k} A})
\]

for \( \lambda_{k-1} \leq \lambda < \lambda_{-k} \) with \( k \geq 1 \). Clearly, when \( \kappa \in \mathbb{N} \), the last two lines also hold true for \( k = \kappa \) and any \( \lambda < \lambda_{-\kappa} \).

Let \( N \) be as in Assumption [A.22] and let \( \mu_{k, \tau} \) and \( \nu_{k, \tau} \), \( k = 1, 2 \), be as in [A.13] and [A.14].

**Assumption A.6.** (B1) Let \( B \in C(\mathbb{R}, \mathbb{R}^{2n \times 2n}) \) be a path of symmetric matrix which is \( \tau \)periodic in time \( t \), i.e., \( B(t + \tau) = B(t) \) for any \( t \in \mathbb{R} \).

(B2) \( B(-t)N = B(t) \), that is, if \( B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} \), then \( B_{11}, B_{22} : \mathbb{R} \to \mathbb{R}^{n \times n} \) are even at \( t = 0 \) and \( \tau/2 \), and \( B_{12}, B_{21} : \mathbb{R} \to \mathbb{R}^{n \times n} \) are odd at \( t = 0 \) and \( \tau/2 \).

Note that the operator \( \Lambda_{I_{2n}, \tau, 0} \) defined by [2.3] has the spectrum \( \frac{2\pi}{\tau} \mathbb{Z} \). Under the assumption (B1) we can choose \( \kappa \notin \frac{2\pi}{\tau} \mathbb{Z} \) such that \( B(t) - \kappa I_{2n} \geq \varepsilon I_{2n} \) for all \( t \) and small enough \( \varepsilon > 0 \).

Consider the Hilbert subspaces of \( L^2(S; \mathbb{R}^{2n}) \) and \( W^{1,2}(S; \mathbb{R}^{2n}) \),

\[
    L_\tau = \{ z \in L^2(S; \mathbb{R}^{2n}) : z(-t) = Nz(t) \text{ a.e. } t \in \mathbb{R} \}, \quad (A.37)
\]

\[
    W_\tau = \{ z \in W^{1,2}(S; \mathbb{R}^{2n}) : z(-t) = Nz(t) \text{ a.e. } t \in \mathbb{R} \}, \quad (A.38)
\]

Then \( W_\tau = W^{1,2}(S; \mathbb{R}^{2n}) \cap L_\tau \). Clearly, the operator \( \Lambda_{I_{2n}, \tau, 0} \) maps \( W_\tau \) into \( L_\tau \). Thus it restricts to a closed linear and self-adjoint operator on \( L_\tau \) with domain \( W_\tau \), denoted by \( \tilde{\Lambda}_{I_{2n}, \tau, 0} \) for clearness. Note that the spectrum of \( \tilde{\Lambda}_{I_{2n}, \tau, 0} \) is also \( \frac{2\pi}{\tau} \mathbb{Z} \). Choose \( \kappa \notin \frac{2\pi}{\tau} \mathbb{Z} \) as above. Then the operator

\[
    \tilde{\Lambda}_{I_{2n}, \tau, 0} : W_\tau \subset L_\tau \to L_\tau, \quad u \mapsto \tilde{\Lambda}_{I_{2n}, \tau, 0} - \kappa u
\]

is invertible and the inverse \( (\tilde{\Lambda}_{I_{2n}, \tau, 0})^{-1} : L_\tau \to L_\tau \) is compact and self-adjoint. It follows under the assumptions (B1) and (B2) that the quadratic form on \( L_\tau \),

\[
    Q_{B, \kappa I_{2n}}(u, v) = \frac{1}{2} \int_0^\tau \left( \langle (\tilde{\Lambda}_{I_{2n}, \tau, 0} - \kappa I_{2n})^{-1} u(t), v(t) \rangle_{\mathbb{R}^{2n}} + \langle C(t)u(t), v(t) \rangle_{\mathbb{R}^{2n}} \right) dt \quad (A.39)
\]

with \( C(t) = (B(t) - \kappa I_{2n}, I_{2n})^{-1} \), is a Legendre form, and hence has finite Morse index \( m^{-}(Q_{B, \kappa I_{2n}}) \) and nullity \( m^{0}(Q_{B, \kappa I_{2n}}) \). Using [45, Theorem 5.1] and suitably modifying the proof of [35, Theorem 2.1] we can get the corresponding results with [A.31] - [A.32].

**Theorem A.7.**

\[
    m^{-}(Q_{B, \kappa I_{2n}}) = \mu_{1, \tau}(Y_B) - n \left[ \frac{\kappa \tau}{2\pi} \right], \quad (A.40)
\]

\[
    m^{0}(Q_{B, \kappa I_{2n}}) = \nu_{1, \tau}(Y_B), \quad (A.41)
\]

If \( u \in L_\tau \) belongs to the zero space of \( Q_{B, \kappa I_{2n}} \), i.e., \( Q_{B, \kappa I_{2n}}(u, v) = 0 \forall v \in L_\tau \), it easily follows that \( w := (\tilde{\Lambda}_{I_{2n}, \tau, 0})^{-1} u \in W_\tau \) satisfies \( Jw'(t) + B(t)w(t) = 0 \) for a.e. \( t \in \mathbb{R} \). Thus \( u \neq 0 \) implies that \( w(t) \neq 0 \) for a.e. \( t \in \mathbb{R} \). Repeating the proof of [19, Lemma 3.18] we may obtain:
Theorem A.8. Let $B_1, B_2 \in C(\mathbb{R}, \mathbb{R}^{2n \times 2n})$ satisfy Assumption A.6. Assume $B_2 > B_1$. Then for each $\kappa \notin \frac{2\pi}{\tau} \mathbb{Z}$ such that $B_i(t) - \kappa I_{2n} \geq \varepsilon I_{2n}$ for all $t$ and small enough $\varepsilon > 0$, $i = 1, 2$, it holds that

$$m^-(Q_{B_2, \kappa I_{2n}}) \geq m^-(Q_{B_1, \kappa I_{2n}}) + m^0(Q_{B_1, \kappa I_{2n}})$$

or equivalently $\mu_{1, \tau}(Y_{B_2}) \geq \mu_{1, \tau}(Y_{B_1}) + \nu_{1, \tau}(Y_{B_1})$.

Using the last two theorems it is direct to give corresponding results with [35, §4] and [19, §5] for the system (1.29) under suitable conditions on $H$. They will be provided elsewhere.

B Proof of Proposition 1.3

Since the operator $\Lambda_{M, \tau} \equiv \Lambda_{M, \tau, 0}$ defined by (2.2) is a self-adjoint unbounded linear operator in $L^2([0, \tau], \mathbb{R}^{2n})$ with domain $\text{dom}(\Lambda_{M, \tau}) = W_M^{1,2}([0, \tau]; \mathbb{R}^{2n})$, by [31, Remarks 2.2.3], all eigenvalues of $\Lambda_{M, \tau}$ have form

$$\cdots \leq \lambda_{-k} \leq \cdots \leq \lambda_{-1} < 0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots$$

and there exists a unit orthogonal basis of $L^2([0, \tau], \mathbb{R}^{2n})$, \{\(e_j | \pm j \in \mathbb{N}\) $\cup \{e_0\}_{i=1}^q\},$ such that \(|e_j(t)| = |e_0(t)| \equiv 1 \forall t, \text{Ker}(\Lambda_{M, \tau}) = \text{Span}(\{e_0\}_{i=1}^q)\) and that each $e_j$ is an eigenvector corresponding to $\lambda_j$, $j = \pm 1, \pm 2, \cdots$.

For $s \geq 0$ we say that $x \in L^2([0, \tau], \mathbb{R}^{2n})$ belongs to $E_M^s$ if and only if $x = \sum_{k \in \mathbb{Z}} x_k e_k$ satisfies

$$\sum_{k \in \mathbb{Z}} |\lambda_k|^{2s} x_k^2 < \infty.$$  

It is easy to prove that $E_M^s$ is a Hilbert space with respect to the inner product

$$\langle x, y \rangle_{E_M^s} = \langle x_0, y_0 \rangle_{\mathbb{R}^{2n}} + \sum_{k \neq 0} |\lambda_k|^{2s} x_k y_k, \quad x, y \in E_M^s.$$  

Denote the associated norm by $\| \cdot \|_{E_M^s}$. Note that $E_M^0 = L^2([0, \tau], \mathbb{R}^{2n})$ and $\| \cdot \|_{E_M^0} = \| \cdot \|_{L^2}$. For $t > s \geq 0$, the inclusion $E_M^s \rightarrow E_M^t$ is compact ([31, Proposition 2.5]).

Proof of Proposition 1.3 Step 1 (Prove (i)). Let $(\lambda_k, \bar{u}_k)$ be a sequence of solutions of the problem (1.3) converging to $(\mu, u_\mu)$ in $\Lambda \times C^0([0, \tau]; \mathbb{R}^{2n})$. It suffices to prove that

$$|\bar{u}_k(t) - u_\mu(t)| = |JzH(\lambda_k, t, \bar{u}_k(t)) - JzH(\mu, t, u_\mu(t))| = |\nabla_zH(\lambda_k, t, \bar{u}_k(t)) - \nabla_zH(\mu, t, u_\mu(t))|$$

uniformly converges to zero on $[0, \tau]$. Otherwise, there exists $\varepsilon > 0$ and subsequences $(k_i)$ and $(t_i) \subset [0, 1]$ such that

$$|\nabla_zH(\lambda_{k_i}, t_i, \bar{u}_{k_i}(t_i)) - \nabla_zH(\mu, t_i, u_\mu(t_i))| \geq \varepsilon, \quad \forall i = 1, 2, \cdots. \quad (B.1)$$

Passing to a subsequence (if necessary) we can assume $t_i \rightarrow t_0 \in [0, 1].$ Note that

$$|\bar{u}_{k_i}(t_i) - u_\mu(t_0)| \leq |\bar{u}_{k_i}(t_i) - u_\mu(t_i)| + |u_\mu(t_i) - u_\mu(t_0)|$$

$$\leq \|\bar{u}_{k_i} - u_\mu\|_{C^0} + |u_\mu(t_i) - u_\mu(t_0)| \rightarrow 0.$$  

Since $\nabla_zH(\lambda, t, z)$ is continuous in $(\lambda, t, z) \in \Lambda \times [0, \tau] \times \mathbb{R}^{2n}$ by Assumption A.11, letting $i \rightarrow \infty$ in (B.1) we obtain $0 = |\nabla_zH(\mu, t_0, u_\mu(t_0)) - \nabla_zH(\mu, t_0, u_\mu(t_0))| \geq \varepsilon$, which is a contradiction.
Step 2 (Prove (ii)). Let \((\lambda_k, \bar{u}_k)\) be a sequence of solutions of the problem (1.3) converging to \((\mu, u_\mu)\) in \(\Lambda \times \mathcal{E}\). By [31, Proposition 2.10], all \(\bar{u}_k\) and \(u_\mu\) are \(C^1\).

Firstly, we assume \(M \neq I_{2n}\). By [31, Proposition 2.5], we have a compact inclusion \(\mathcal{E} \to L^2([0, \tau]; \mathbb{R}^{2n})\) and therefore
\[
\|\bar{u}_k - u_\mu\|_{L^r} \to 0 \quad \text{(B.2)}
\]
because \(1 < r \leq 2\). Note that
\[
\|\dot{u}_k - \dot{u}_\mu\|_{L^1} = \int_0^\tau |\nabla_z H(\lambda_k, t, \bar{u}_k(t)) - \nabla_z H(\mu, t, u_\mu(t))| dt \\
\leq \int_0^\tau |\nabla_z H(\lambda_k, t, u_\mu(t)) - \nabla_z H(\mu, t, u_\mu(t))| dt \\
+ \int_0^\tau |\nabla_z H(\lambda_k, t, \bar{u}_k(t)) - \nabla_z H(\lambda_k, t, u_\mu(t))| dt. \quad \text{(B.3)}
\]

Since \(\nabla_z H\) is continuous, by (1.10) it follows from Lebesgue dominated convergence theorem that
\[
\lim_{k \to \infty} \int_0^\tau |\nabla_z H(\lambda_k, t, u_\mu(t)) - \nabla_z H(\mu, t, u_\mu(t))| dt = 0. \quad \text{(B.4)}
\]

Note that [51, Proposition C.1] is still true for \(p \geq 1\) and that \(\Lambda_0 := \{\mu, \lambda_k | k \in \mathbb{N}\}\) is sequential compact. From [1,10] we deduce that maps
\[
L^r([0, \tau]; \mathbb{R}^{2n}) \to L^1([0, \tau]; \mathbb{R}^{2n}), \, u \mapsto \nabla_z H(\lambda, \cdot; u(\cdot))
\]
are uniform continuous at \(u_\mu\) with respect to \(\lambda \in \Lambda_0\). This result and (B.2) imply
\[
\lim_{k \to \infty} \int_0^\tau |\nabla_z H(\lambda_k, t, \bar{u}_k(t)) - \nabla_z H(\lambda_k, t, u_\mu(t))| dt = 0.
\]

It follows from this, (B.4) and (B.3) that \(\|\dot{u}_k - \dot{u}_\mu\|_{L^1} \to 0\) and so \(\|\bar{u}_k - u_\mu\|_{L^1} \to 0\) by (B.2). Then
\[
\|\bar{u}_k - u_\mu\|_{C^0} \leq \max\{\tau^{-1}, 1\} \|\dot{u}_k - \dot{u}_\mu\|_{L^1} \to 0. \quad \text{(B.5)}
\]

By the proof in Step 1, \(\bar{u}_k \to u_\mu\) in \(C^1([0, \tau]; \mathbb{R}^{2n})\).

Next, we consider the case \(M = I_{2n}\). Then there exists a stronger result than [31, Proposition 2.5]: \(\mathcal{E} = H^{1/2}(S_\tau; \mathbb{R}^{2n})\) is compactly embedded in \(L^s(S_\tau; \mathbb{R}^{2n})\) for each \(s \in [1, \infty)\) (cf. [61, Proposition 6.6] or Theorem 3 in [29, Appendix A.3]). Therefore (B.2) is still true for the present \(r\). Repeating the above arguments we may obtain the desired conclusions.

Question B.1. When \(M \neq I_{2n}\), can \(\mathcal{E}\) be embedded in \(L^s(S_\tau; \mathbb{R}^{2n})\) for each \(s \in (2, \infty)\)? If this is affirmative, then the condition “1 < \(r\) ≤ 2” may be replaced by “1 < \(r\) < \(\infty\)”.

C Generalizations and corrections for related results in [48, 50, 52]

We begin with the following remark about [48, Theorems A.1,A.2].

Remark C.1. In [48] Theorems A.1,A.2 we assumed that the topology space \(\Lambda\) is compact. Actually, “compact” can be replaced by “first countable and sequential compact”. Here we say the topology space \(\Lambda\) to be sequential compact if every sequence \((\lambda_n)\) of points in it has a subsequence converging to some \(\lambda \in \Lambda\). Recall that compact topology spaces satisfying the first axiom of countability must be sequential compact. In what follows let us point out how these different conditions of compactness are used in the proofs of [48] Theorems A.1,A.2].
(i) For the proof of “φ is continuous” in [48, line 20, page 2980] using nets we can complete it without any requirements. If Λ is first countable we can easily prove this claim with sequences. (See (v) below.)

(ii) For the proof of “Step 1” in [48, page 2980], from the proof of Claim A.3 in the proof of [48, Theorem A.2] we see that the sequential compactness of Λ was actually used. When Λ is only compact we may work with nets instead of sequences to complete the desired proof.

(iii) Our proof of “Step 2” in [48, pages 2980-2981] actually used the assumption that Λ is first countable and sequential compact. If Λ is only compact the expected proof can be completed with nets.

(iv) The proof of “(A.2)” in [48, page 2981] actually used the sequential compactness of Λ. For compact Λ a complete proof can be given with nets.

(v) As in (i), for the proof of “φ is continuous” in Step 7 in [48, page 2983] we need to use nets. It suffices to use sequences for first countable Λ.

These five remarks are effective for the proof of [48, Theorem A.2].

In summary, the topology space Λ in [48, Theorems A.1, A.2] should be either compact or first countable and sequential compact. Because of these we have also:

(vi) The topology space Λ in [50, Theorem A.3] and [52, Theorem 3.3] should be assumed to be first countable and sequential compact because first countable compact spaces are sequential compact and our proof for (A.11) in the proof of [50, Theorem A.3] actually used the fact that Λ has a countable neighborhood basis at λ∗.

Let Λ be a topological space, and let X, Y be Banach spaces. For continuous maps F : Λ × X → Y and Λ ⊃ λ → xλ ∈ X satisfying F(λ, xλ) = 0 for all λ ∈ Λ, recall that (λ*, xλ*) ∈ Λ × X is called a bifurcation point of F(λ, x) = 0 in Λ × X with respect to the trivial branch \{ (λ, xλ) | λ ∈ A \} if every neighborhood U of (λ*, xλ*) contains a point (λ, yλ) ≠ (λ, xλ) satisfying F(λ, yλ) = 0. If Λ is first countable, this definition is equivalent to the following:

**Definition C.2.** Under the assumptions above, (λ*, xλ*) ∈ Λ × X is said to be a bifurcation point along sequences of F(λ, x) = 0 in Λ × X with respect to the trivial branch \{ (λ, xλ) | λ ∈ A \} if there exists a sequence \{ (λn, yn) \}n≥1 of solutions of F(λ, x) = 0 in Λ × X converging to (λ*, xλ*) in Λ × X such that yn ≠ xλn for all n ∈ N.

Clearly, a bifurcation point along sequences must be a bifurcation point.

**Hypothesis C.3** ([50, Hypothesis 1.3]). Let H be a Hilbert space with inner product (·, ·)H and the induced norm ∥ · ∥, and let X be a Banach space with norm ∥ · ∥X, such that X ⊂ H is dense in H and ∥x∥ ≤ ∥x∥X ∀ x ∈ X. For an open neighborhood U of 0 in H, U ∩ X is also an open neighborhood of 0 in X, denoted by UX. Let \( L : U \to \mathbb{R} \) be a continuous functional satisfying the following conditions:

(F1) \( L \) is continuously directional differentiable and \( DL(0) = 0 \).

(F2) There exists a continuous and continuously directional differentiable map \( A : U^X \to X \), which is also strictly Fréchet differentiable at 0, such that \( DL(x)[u] = (A(x), u)_H \) for all \( x \in U \cap X \) and \( u \in X \).
(F3) There exists a map \( B : U \cap X \to \mathcal{L}_s(H) \) such that \( (DA(x)[u], v)_H = (B(x)u, v)_H \) for all \( x \in U \cap X \) and \( u, v \in X \). (So \( B(x) \) induces an element in \( \mathcal{L}(X) \), denoted by \( B(x)|_X \), and \( B(x)|_X = DA(x) \in \mathcal{L}(X), \forall x \in U \cap X. \))

(C) \( \{ u \in H \mid B(0)(u) \in X \} \subset X \), in particular \( \text{Ker}(B(0)) \subset X \).

(D) \( B \) satisfies the same conditions as in [50, Hypothesis 1.1], that is,

\[ (D1) \{ u \in H \mid B(0)u = \mu u, \mu \leq 0 \} \subset X, \]

and \( B \) has a decomposition \( B = P + Q \), where for each \( x \in U \cap X \), \( P(x) \in \mathcal{L}_s(H) \) is positive definitive and \( Q(x) \in \mathcal{L}_s(H) \) is compact, such that the maps \( P \) and \( Q \) also satisfy the following properties:

\[ (D2) \text{For any sequence } (x_k) \subset U \cap X \text{ with } \|x_k\| \to 0, \text{ it holds that } \|P(x_k)u - P(0)u\| \to 0 \text{ for any } u \in H. \]

\[ (D3) \text{The map } Q : U \cap X \to \mathcal{L}_s(H) \text{ is continuous at } 0 \text{ with respect to the topology on } H. \]

\[ (D4) \text{For any sequence } (x_k) \subset U \cap X \text{ with } \|x_k\| \to 0, \text{ there exist constants } C_0 > 0 \text{ and } k_0 \in \mathbb{N} \text{ such that } (P(x_k)u, u)_H \geq C_0 \|u\|^2 \text{ for all } u \in H \text{ and for all } k \geq k_0. \]

We say an isolated critical point \( p \) of a \( C^1 \)-functional \( f \) on a Banach manifold \( \mathcal{M} \) to be homological visible if there exists a nonzero critical group \( C_m(f,p;K) \) for some Abel group \( K \).

In view of Remark C.1 we may revise and refine [52, Theorem 3.5] as follows.

**Theorem C.4.** Let \( H, X \) and \( U \) be as in Hypothesis C.3 and let \( \Lambda \) be the first countable topology space. Let \( \mathcal{L}_\Lambda \in C^1(U, \mathbb{R}), \lambda \in \Lambda, \) be a continuous family of functionals satisfying \( \mathcal{L}_\lambda(0) = 0 \) for all \( \lambda \in \Lambda \). For each \( \lambda \in \Lambda \), assume that there exist maps \( A_\lambda \in C^1(U^X, X) \) and \( B_\lambda : U \cap X \to \mathcal{L}_s(H) \) such that

\[ a) \lambda \times U^X \ni (\lambda, x) \to A_\lambda(x) \in X \text{ is continuous}; \]

\[ b) \text{for all } x \in U \cap X \text{ and } u, v \in X, \]

\[ D\mathcal{L}_\lambda(x)[u] = (A_\lambda(x), u)_H \text{ and } (DA_\lambda(x)[u], v)_H = (B_\lambda(x)u, v)_H; \quad (C.1) \]

\[ c) B_\lambda \text{ has a decomposition } B_\lambda = P_\lambda + Q_\lambda, \text{ where for each } x \in U \cap X, P_\lambda(x) \in \mathcal{L}_s(H) \text{ is positive definitive and } Q_\lambda(x) \in \mathcal{L}_s(H) \text{ is compact.} \]

Suppose also for some \( \lambda^* \in \Lambda \) that \( P_\lambda \) and \( Q_\lambda \) satisfy the following conditions:

\[ (i) \text{For each } h \in H, \text{ it holds that } \|P_\lambda(x)h - P_{\lambda^*}(0)h\| \to 0 \text{ as } x \in U \cap X \text{ approaches to } 0 \text{ in } H \text{ and } \lambda \in \Lambda \text{ converges to } \lambda^*. \]

\[ (ii) \text{For some small } \delta > 0, \text{ there exists } c_0 > 0 \text{ such that} \]

\[ (P_\lambda(x)u, u) \geq c_0 \|u\|^2 \quad \forall u \in H, \forall x \in B_H(0, \delta) \cap X, \forall \lambda \in \Lambda. \]

\[ (iii) Q_\lambda : U \cap X \to \mathcal{L}_s(H) \text{ is uniformly continuous at } 0 \text{ with respect to } \lambda \in \Lambda. \]

\[ (iv) \text{If } \lambda \in \Lambda \text{ converges to } \lambda^* \text{ then } \|Q_\lambda(0) - Q_{\lambda^*}(0)\| \to 0. \]

\[ (v) \text{Each tube } (\mathcal{L}_\lambda, H, X, U, A_\lambda, B_\lambda = P_\lambda + Q_\lambda), \lambda \in \Lambda, \text{ satisfies Hypothesis } C.3. \]
Then there holds:

(A) If \((\lambda^*, 0)\) is not a bifurcation point of \(\nabla \mathcal{L}_\lambda(x) = 0\) in \(\Lambda \times U\), (which implies that 0 is an isolated critical point of \(\mathcal{L}_\lambda\) for each \(\lambda\) near \(\lambda^*\)), \(\Lambda\) is sequential compact, and \(\text{Ker}(B_{\lambda^*}(0)) \neq \{0\}\), then \(C_*(\mathcal{L}_\lambda; 0; \mathbf{K}) \cong C_*(\mathcal{L}_\lambda^*; 0; \mathbf{K})\) for any \(\lambda\) in a small neighborhood of \(\lambda^* \in \Lambda\) and for any Abel group \(\mathbf{K}\). (Note: From the proof it is easily seen that (u) need only to be satisfied for \(\lambda^*\).) In addition, the condition “\(C_*(\mathcal{L}_\lambda; 0; \mathbf{K}) \cong C_*(\mathcal{L}_\lambda^*; 0; \mathbf{K})\)” may be changed into “\(C_*(\mathcal{L}|_{U^X}; 0; \mathbf{K}) \cong C_*(\mathcal{L}_\lambda^*|_{U^X}; 0; \mathbf{K})\)” provided for each \(\lambda\) near \(\lambda^*\) that \(\mathcal{L}|_{U^X} \in C^2(U^X, \mathbb{R})\) and \(B_{\lambda}\) is continuous as a map from \(U^X\) to \(\mathcal{L}_s(H)\).

(B) Suppose that there exist two sequences in \(\Lambda\) converging to \(\lambda^*\), \((\lambda^n_k^-)\) and \((\lambda^n_k^+)\), such that for each \(k \in \mathbb{N}\), \([\mu_{\lambda^n_k^+}, \mu_{\lambda^n_k^-}] \cap [\mu_{\lambda^n_k^+}, \mu_{\lambda^n_k^-}] = \emptyset\) and one of the following conditions is satisfied:

(B.1) There exists \(\lambda \in \{\lambda^n_k^+, \lambda^n_k^-\}\) such that 0 is an either nonisolated or homological visible critical point of \(\mathcal{L}_\lambda\).

(B.2) \(\mathcal{L}|_{U^X} \in C^2(U^X, \mathbb{R})\) and \(B_{\lambda}\) is continuous as a map from \(U^X\) to \(\mathcal{L}_s(H)\) for each \(\lambda \in \{\lambda^n_k^+, \lambda^n_k^-\}\), and there exists \(\lambda \in \{\lambda^n_k^+, \lambda^n_k^-\}\) such that 0 is an either nonisolated or homological visible critical point of \(\mathcal{L}|_{U^X}\).

(B.3) Either \(\nu_{\lambda^n_k^+} = 0\) or \(\nu_{\lambda^n_k^-} = 0\).

(Here \(\mu_{\lambda} = \dim H_{\lambda}^-\) and \(\nu_{\lambda} = \dim H_{\lambda}^0\) are dimensions of the negative definite and zero spaces \(H_{\lambda}^-\) and \(H_{\lambda}^0\) of \(B_{\lambda}(0)\), respectively.) Then \(\nu_{\lambda^*} > 0\) and there exists a sequence \(\{(\lambda_k, x_k)\}_{k \geq 1}\) in \(\Lambda \times U\) converging to \((\lambda^*, 0)\), where \(\lambda := \{\lambda^*_n, \lambda^+_n, \lambda^-_n\} \in \mathbb{N}\), such that each \(x_k\) is a nonzero solution of \(\nabla \mathcal{L}_\lambda(x) = 0\), \(k = 1, 2, \ldots\). In particular, \((\lambda^*, 0)\) is a bifurcation point of \(\nabla \mathcal{L}_\lambda(x) = 0\) in \(\Lambda \times U\) (and so in \(\Lambda \times U\)).

That \((\lambda^*, 0)\) is not a bifurcation point of \(\nabla \mathcal{L}_\lambda(x) = 0\) in \(\Lambda \times U\) also implies that \((\lambda^*, 0)\) is not a bifurcation point of \(D(\mathcal{L}_\lambda|_{U^X})(x) = 0\) in \(\Lambda \times U^X\). The latter claim shows that 0 is also an isolated critical point of \(\mathcal{L}|_{U^X}\) for each \(\lambda\) near \(\lambda^*\) and thus \(C_*(\mathcal{L}|_{U^X}; 0; \mathbf{K})\) is well-defined. The condition “\(C_*(\mathcal{L}|_{U^X}; 0; \mathbf{K}) \cong C_*(\mathcal{L}_\lambda^*|_{U^X}; 0; \mathbf{K})\)” is easily checked in applications.

**Proof.** Step 1[Prove (A)]. Since \(\text{Ker}(B_{\lambda^*}(0)) \neq \{0\}\) and \(\Lambda\) is first countable and sequential compact, by Remark [C.1(vi)] we may use [52] Theorem 3.3] or [52] Theorem A.3 to find a neighborhood \(\Lambda_0\) of \(\lambda^*\) in \(\Lambda\), \(\epsilon > 0\), a (unique) \(C^1\) map \(\psi : \Lambda_0 \times B_{H_{\lambda^*}^0}(0, \epsilon) \rightarrow X_{\lambda^*}^0\), which is \(C^1\) in the second variable and satisfies \(\psi(\lambda, 0) = 0\) \(\forall \lambda \in \Lambda_0\) and

\[
P_{\lambda^*}^\pm A_{\lambda}(z + \psi(\lambda, z)) = 0 \quad \forall (\lambda, z) \in \Lambda_0 \times B_{H_{\lambda^*}^0}(0, \epsilon),
\]

and a homeomorphism \(\Phi_{\lambda}\) from \(B_{H_{\lambda^*}^0}(0, \epsilon) \oplus B_{H_{\lambda^*}}(0, \epsilon) \oplus B_{H_{\lambda^*}^0}(0, \epsilon)\) onto an open neighborhood of 0 in \(H\) satisfying \(\Phi_{\lambda}(0) = 0\) for each \(\lambda \in \Lambda_0\), such that for each \(\lambda \in \Lambda_0\),

\[
\mathcal{L}_{\lambda} \circ \Phi_{\lambda}(z, u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2 + \mathcal{L}_{\lambda}(z + \psi(\lambda, z)) \quad \forall (z, u^+ + u^-) \in B_{H_{\lambda^*}^0}(0, \epsilon) \times \left(B_{H_{\lambda^*}^0}(0, \epsilon) + B_{H_{\lambda^*}^0}(0, \epsilon)\right).
\]

Moreover, the functional

\[
\mathcal{L}_{\lambda}^0 : B_{H_{\lambda^*}^0}(0, \epsilon) \rightarrow \mathbb{R}, \quad z \mapsto \mathcal{L}_{\lambda}(z + \psi(\lambda, z))
\]
is of class $C^2$, and

$$d\mathcal{L}_c(z)[\zeta] = (A_\lambda(z + \psi(\lambda, z)), \zeta)_{H^*}, \quad \forall (z, \zeta) \in B_{H_{\lambda^*}}(0, \epsilon) \times H_{\lambda^*}. \quad \text{(C.5)}$$

By (C.2) and (C.5), for each $\lambda \in \Lambda_0$, the map $z \mapsto z + \psi(\lambda, z))$ induces an one-to-one correspondence between the critical points of $\mathcal{L}_c$ near $0 \in H_{\lambda^*}$, and zeros of $A_\lambda$ near $0 \in X$.

Since $(\lambda^*, 0) \in \Lambda \times U$ is not a bifurcation point of $\nabla \mathcal{L}_c(x) = 0 \in \Lambda \times U$, by shrinking $\Lambda_0$ towards $\lambda^*$ (in $\Lambda$) we can find $\delta > 0$ such that $\nabla \mathcal{L}_c(x) = 0$ has only trivial solutions in $\Lambda_0 \times B_H(0, \delta) \subset \Lambda_0 \times U$ (and therefore $A_\lambda(x) = 0$ has only trivial solutions in $\Lambda_0 \times B_X(0, \delta) \subset \Lambda_0 \times B_H(0, \delta)$). It follows from this and the sentence below (C.5) that after shrinking $\epsilon > 0$ (if necessary),

for each $\lambda \in \Lambda_0$ the functional $\mathcal{L}_c$ has a unique critical point $0$ in $B_{H_{\lambda^*}}(0, \epsilon)$. \quad \text{(C.6)}

Since $\psi$ is continuous, by the assumption c), (C.4) and (C.5), maps

$$\Lambda_0 \times B_{H_{\lambda^*}}(0, \epsilon) \ni (\lambda, z) \mapsto \mathcal{L}_c(z) \in \mathbb{R} \quad \text{and}$$

$$\Lambda_0 \times B_{H_{\lambda^*}}(0, \epsilon) \ni (\lambda, z) \mapsto d\mathcal{L}_c(z) \in H_{\lambda^*} = X_{\lambda^*},$$

are continuous. We conclude that

$$\lim_{\lambda \to \lambda^*} \sup_{z \in B_{H_{\lambda^*}}(0, \epsilon/2)} |\mathcal{L}_c(z)| = 0, \quad \text{(C.7)}$$

$$\lim_{\lambda \to \lambda^*} \sup_{z \in B_{H_{\lambda^*}}(0, \epsilon/2)} \|d\mathcal{L}_c(z)\| = 0. \quad \text{(C.8)}$$

Clearly, (C.8) implies (C.7) because $\mathcal{L}_c(0) - \mathcal{L}_c(0) = \mathcal{L}_c(0) - \mathcal{L}_c(0) \to 0$ as $\lambda \to \lambda^*$. Note that $\Lambda$ has a countable neighborhood basis at $\lambda^*$. By contradiction (C.8) may follow from the sequential compactness of $B_{H_{\lambda^*}}(0, \epsilon/2)$. For any two different points in $B_{H_{\lambda^*}}(0, \epsilon/2)$, $z_1, z_2$, it follows from the mean value theorem that

$$\left| \frac{\langle \mathcal{L}_c(z_1) - \mathcal{L}_c(z_2) \rangle}{|z_1 - z_2|} \right| \leq \int_0^1 \|d\mathcal{L}_c(tz_1 + (1 - t)z_2)\| dt.$$

This and (C.8) lead to

$$\|\mathcal{L}_c - \mathcal{L}_c\|_{\infty} := \sup \left\{ \left| \frac{\langle \mathcal{L}_c(z_1) - \mathcal{L}_c(z_2) \rangle}{|z_1 - z_2|} \right| : z_1, z_2 \in B_{H_{\lambda^*}}(0, \epsilon/2), \ z_1 \neq z_2 \right\} \to 0$$

as $\lambda \to \lambda^*$. By this and (C.6)-(C.7) we derive from [18] Theorem 5.1] that critical groups

$$C_\ast(\mathcal{L}_c; 0; \mathbf{K}) = C_s(\mathcal{L}_c; 0; \mathbf{K}) \quad \forall \lambda \in \Lambda_0 \quad \text{(C.9)}$$

for any Abel group $\mathbf{K}$ by shrinking $\Lambda_0$ towards $\lambda^*$ (if necessary). Moreover $0 \in U$ is a unique critical point of $\mathcal{L}_c$ in $B_H(0, \delta) \subset U$. Using (C.6) and [50] Corollary A.5 we obtain

$$C_q(\mathcal{L}_c; 0; \mathbf{K}) = C_{q - \mu_{\lambda^*}}(\mathcal{L}_c; 0; \mathbf{K}), \quad \forall (\lambda, q) \in \Lambda_0 \times (\mathbb{N} \cup \{0\}).$$

This and (C.9) lead to $C_s(\mathcal{L}_c; 0; \mathbf{K}) = C_s(\mathcal{L}_c; 0; \mathbf{K})$ for all $\lambda \in \Lambda_0$.

Under the assumptions in part “In addition” of (A), if Ker($B_0(0)$) $\neq \{0\}$ (resp. Ker($B_0(0)$) = $\{0\}$) by Jiang [32] there exists a splitting lemma (resp. Morse lemma) for $\mathcal{L}_c|_{U^X}$ near $0 \in U^X$. (The case of Ker($B_0(0)$) = $\{0\}$ can be easily obtained from the proof therein.) By [32]
Corollary 2.8] we get $C_*(\mathcal{L}_\lambda|_{U_0};K) \cong C_*(\mathcal{L}_\lambda,0;K)$ for each $\lambda$ near $\lambda^*$. The required conclusion follows from the last one.

**Step 2 [Prove (B)].** Firstly, we prove $\nu_{\lambda^*} > 0$, i.e., $\text{Ker}(B_{\lambda^*}(0)) \neq \{0\}$. Otherwise, in the proof of [52 Theorem A.3], replacing $\Lambda$ by the first countable and sequential compact space $\hat{\Lambda}$ we need not use the implicit function theorem and obtain that the lines 3–4 on the page 1294 of [50] become

$$
(D\mathcal{L}_\lambda(u^+ + u^-) - D\mathcal{L}_\lambda(u^+ + u^-))[u^2 - u^-] \leq -a_1\|u^2 - u^-\|^2,
$$

$$
D\mathcal{L}_\lambda(u^+ + u^-)[u^+ - u^-] \geq a_2(\|u^+\|^2 + \|u^-\|^2)
$$

(C.10)

for all $(\lambda, u^+, u^-) \in \hat{\Lambda}_0 \times B_{H^1_{\lambda^*}}(0,\varepsilon) \times B_{H^1_{\lambda^*}}(0,\varepsilon)$, where $a_1$ and $a_2$ are positive constants and $\hat{\Lambda}_0$ is a neighborhood of $\lambda^*$ in $\hat{\Lambda}$. In particular, the second inequality implies that $D\mathcal{L}_\lambda(u^+)\|u^+\| \geq a_2\|u^+\|^2$. Actually, since $H^1_{\lambda^*} \subset X$ and $H^1_{\lambda^*} \cap X$ is dense in $H^1_{\lambda^*}$, from the second inequality we easily derive that $(B_{\lambda}(0)u^+, u^+) \geq a_2\|u^+\|^2$ and $(B_{\lambda}(0)u^-, u^-) \leq -a_2\|u^-\|^2$ for all $(\lambda, u^+, u^-) = \hat{\Lambda}_0 \times H^1_{\lambda^*} \times H^1_{\lambda^*})$. By Remark [C.1] and [48 Theorem A.1] there exists $\epsilon > 0$, an open neighborhood $U$ of $\hat{\Lambda}_0 \times \{0\}$ in $\hat{\Lambda}_0 \times H$ and a homeomorphism

$$
\phi : \hat{\Lambda}_0 \times (B_{H^1_{\lambda^*}}(0,\varepsilon) \times B_{H^1_{\lambda^*}}(0,\varepsilon)) \to U
$$

with $\phi(\lambda,0) = (\lambda,0)$ such that for all $(\lambda, u^+, u^-) \in \hat{\Lambda}_0 \times B_{H^1_{\lambda^*}}(0,\varepsilon) \times B_{H^1_{\lambda^*}}(0,\varepsilon),$

$$
\mathcal{L}_\lambda(\phi(\lambda, u^+ + u^-)) = \|u^+\|^2 - \|u^-\|^2.
$$

(C.11)

Note by [C.10] that 0 is a unique critical point of $\mathcal{L}_\lambda$ in $B_{H^1_{\lambda^*}}(0,\varepsilon) \oplus B_{H^1_{\lambda^*}}(0,\varepsilon)$ for each $\lambda \in \hat{\Lambda}_0$, and that each $\phi(\lambda, \cdot)$ is a homeomorphism from $B_{H^1_{\lambda^*}}(0,\varepsilon) \oplus B_{H^1_{\lambda^*}}(0,\varepsilon)$ onto an open neighborhood of 0 in $H$. It follows from these and [C.11] that for any Abel group $K$,

$$
C_q(\mathcal{L}_\lambda,0;K) = \delta_{\mu_{\lambda^*}}^q K \forall (\lambda,q) \in \hat{\Lambda}_0 \times \mathbb{Z},
$$

(C.12)

where $\delta_0^p = 1$ if $p = q$, and $\delta_0^p = 0$ if $p \neq q$.

Choose a large $k$ so that $\lambda_{\lambda}^+ and $k_{\lambda}^-$ belong to $\hat{\Lambda}_0$. Then 0 is an isolated critical point of $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$ in $U$ for each large $k$. Because of the assumption (v) in Theorem [C.4] it follows from [48 (2.7)] and the shifting theorem [48 Corollary 2.6] that for any Abel group $K$ and $* = +, -$,

$$
C_q(\mathcal{L}_{\lambda^+}^k,0;K) = \delta_0^q K \forall q \in \mathbb{N} \cup \{0\} \text{ if } \nu_{\lambda^+} = 0,
$$

(C.13)

$$
C_q(\mathcal{L}_{\lambda^-}^k,0;K) = 0 \forall q \notin [\mu_{\lambda^+},\mu_{\lambda^+} + \nu_{\lambda^+}] \text{ if } \nu_{\lambda^+} > 0,
$$

(C.14)

respectively. (See [13 Corollary 5.1] for the latter). From (C.12), (C.13) and (C.14) we deduce

$$
\mu_{\lambda^+} \in [\mu_{\lambda^+},\mu_{\lambda^+} + \nu_{\lambda^+}] \cap [\mu_{\lambda^+},\mu_{\lambda^+} + \nu_{\lambda^+}],
$$

which contradicts the first condition in (B). Hence $\nu_{\lambda^+} > 0$.

By contradiction, suppose that $(\lambda^*,0) \in \hat{\Lambda} \times U$ is not a bifurcation point of $\nabla\mathcal{L}_\lambda(x) = 0$ in $\hat{\Lambda} \times U$. We can shrink $\Lambda_0$ towards $\lambda^*$ in $\hat{\Lambda}$ and $\varepsilon > 0$ so that for each $\lambda \in \hat{\Lambda}_0$,

1. the functional $\mathcal{L}_\lambda$ has an isolated critical point 0 in $U$,
2. the functional $\mathcal{L}_\lambda^0$ has a unique critical point 0 in $B_{H^1_{\lambda^*}}(0,\varepsilon)$. 

Since \( \hat{\Lambda} \) is first countable and sequential compact, the first paragraph in Step 1 is still valid after \( \Lambda \) and \( \Lambda_0 \) are replaced by \( \hat{\Lambda} \) and \( \hat{\Lambda}_0 \), respectively. Repeating the arguments in the second paragraph in Step 1, for any Abel group \( K \) we get \( C_s(\mathcal{L}_\Lambda, 0; K) = C_s(\mathcal{L}_\Lambda, 0; K) \) for all \( \lambda \in \Lambda_0 \), and hence

\[
C_q(\mathcal{L}_{\Lambda_k^+}, 0; K) = C_q(\mathcal{L}_{\Lambda_k^-}, 0; K), \quad \forall q \in \mathbb{N} \cup \{0\}
\]

if \( k \) is so large that \( \lambda_k^+, \lambda_k^- \in \hat{\Lambda}_0 \). For such a large \( k \), we can obtain a contradiction under any of the assumptions \((B.1)\), \((B.2)\) and \((B.3)\) as follows.

**Case (B.1):** Because of \((1)\), 0 is a homological visible critical point of \( \mathcal{L}_{\Lambda_k^+} \) and \( \mathcal{L}_{\Lambda_k^-} \). Combing with \((C.15)\) we derive \( C_m(\mathcal{L}_{\Lambda_k^+}, 0; K) = C_m(\mathcal{L}_{\Lambda_k^-}, 0; K) \neq 0 \) for some Abel group \( \hat{K} \) and some \( m \in \mathbb{N} \cup \{0\} \). As above this and \((C.13)-(C.14)\) yield a contradiction to the first condition in \((B)\).

**Case (B.2):** Since \((1)\) implies that 0 \( \in U^X \) is an isolated critical point \( \mathcal{L}_{\Lambda_k^+} \mid U^X \) for any \( \lambda \in \{\lambda_k^+, \lambda_k^-\} \), 0 is a homological visible critical point of either \( \mathcal{L}_{\Lambda_k^+} \mid U^X \) or \( \mathcal{L}_{\Lambda_k^-} \mid U^X \). It follows from our assumptions that Then either \( C_m(\mathcal{L}_{\Lambda_k^+} \mid U^X, 0; K) \neq 0 \) for some Abel group \( K \) and some \( m \in \mathbb{N} \cup \{0\} \), or \( C_n(\mathcal{L}_{\Lambda_k^-} \mid U^X, 0; K') \neq 0 \) for some Abel group \( K' \) and some \( n \in \mathbb{N} \cup \{0\} \). As in the final paragraph in Step 1, for any Abel group \( G \) we have also \( C_s(\mathcal{L}_\Lambda \mid U^X, 0; G) \equiv C_s(\mathcal{L}_\Lambda, 0; G) \) for \( \lambda = \lambda_k^+, \lambda_k^- \). These and \((C.15)\) lead to either

\[
C_m(\mathcal{L}_{\Lambda_k^+}, 0; K) = C_m(\mathcal{L}_{\Lambda_k^-}, 0; K) \neq 0 \quad \text{or} \quad C_n(\mathcal{L}_{\Lambda_k^+}, 0; K') = C_n(\mathcal{L}_{\Lambda_k^-}, 0; K') \neq 0.
\]

As above, these and \((C.13)-(C.14)\) we derive a contradiction to the first condition in \((B)\).

**Case (B.3):** This case can also be divided into three cases.

*Case 1: \( \nu_{\lambda_k^+} = 0 \) and \( \nu_{\lambda_k^-} = 0 \). Because of \((1)\), we may use \((C.13)\) and \((C.15)\) to derive \( \mu_{\lambda_k^+} = \mu_{\lambda_k^-} = \mu_{\lambda_k'^-} \), which contradicts the first condition in \((B)\).*

*Case 2: \( \nu_{\lambda_k^+} = 0 \) and \( \nu_{\lambda_k^-} > 0 \). Because \((1)\), \((C.13)\) with \( * = + \), \((C.15)\) and \((C.14)\) with \( * = - \) show that \( \mu_{\lambda_k'^-} \in [\mu_{\lambda_k^+}, \mu_{\lambda_k^+} + \nu_{\lambda_k^-}] \). This contradicts the first condition in \((B)\).*

*Case 3: \( \nu_{\lambda_k^+} > 0 \) and \( \nu_{\lambda_k^-} = 0 \). This case may be proved as in Case 2.*

The following bifurcation theorem may be viewed as a global version of Theorem \((C.4)\).

**Theorem C.5.** Under the assumptions a)-c) and \((v)\) of Theorem \((C.4)\) without requirement of the first countability for \( \Lambda \), suppose that the topology space \( \Lambda \) is path connected and the following conditions are satisfied.

\[ \text{d) } P_\lambda \text{ and } Q_\lambda \text{ satisfy the conditions } (i)-(iv) \text{ of Theorem } (C.4) \text{ for any } \lambda^* \in \Lambda; \]

\[ \text{e) There exist two points } \lambda^+, \lambda^- \in \Lambda \text{ such that one of the following conditions is satisfied:} \]

\[ \text{(e.1) Either } 0 \text{ is not an isolated critical point of } \mathcal{L}_{\lambda^+}, \text{ or } 0 \text{ is not an isolated critical point of } \mathcal{L}_{\lambda^-}, \text{ or } 0 \text{ is an isolated critical point of } \mathcal{L}_{\lambda^+} \text{ and } \mathcal{L}_{\lambda^-} \text{ and } C_m(\mathcal{L}_{\lambda^+}, 0; K) \text{ and } C_m(\mathcal{L}_{\lambda^-}, 0; K) \text{ are not isomorphic for some Abel group } K \text{ and some } m \in \mathbb{Z}. \text{ Moreover, } \quad C_m(\mathcal{L}_{\lambda^+}, 0; K) \text{ and } C_m(\mathcal{L}_{\lambda^-}, 0; K) \text{ in the third case may be changed into } \quad C_s(\mathcal{L}_{\lambda^+} \mid U^X, 0; K) \text{ and } C_s(\mathcal{L}_{\lambda^-} \mid U^X, 0; K) \text{ provided also that } \mathcal{L}_{\lambda\mid U^X} \in C^2(U^X, \mathbb{R}) \text{ and } B_{\lambda} \text{ is continuous as a map from } U^X \text{ to } \mathcal{L}_s(H) \text{ for each } \lambda \in \{\lambda^+, \lambda^-\} \text{.} \]

\[ \text{(e.2) } [\mu_{\lambda^+}, \mu_{\lambda^+} + \nu_{\lambda^+}] \cap [\mu_{\lambda^-}, \mu_{\lambda^-} + \nu_{\lambda^-}] = \emptyset, \text{ and there exists } \lambda \in \{\lambda^+, \lambda^-\} \text{ such that } 0 \text{ is an either nonisolated or homological visible critical point of } \mathcal{L}_{\lambda}. \]
(e.3) \[ [\mu_\lambda^+, \mu_\lambda^- + \nu_\lambda^+] \cap [\mu_\lambda^-, \mu_\lambda^- + \nu_\lambda^-] = \emptyset, \quad L_{\lambda}^{\mid U_X} \in C^2(U^X, \mathbb{R}) \text{ and } B_\lambda \text{ is continuous as a map from } U^X \text{ to } L_{\lambda}(H) \text{ for each } \lambda \in \{\lambda^+, \lambda^-, \lambda^+ \}, \text{ and there exists } \lambda \in \{\lambda^+, \lambda^- \} \text{ such that } 0 \text{ is an either nonisolated or homological visible critical point of } L_{\lambda}^{\mid U_X}.

(e.4) \[ [\mu_\lambda^+, \mu_\lambda^- + \nu_\lambda^-] \cap [\mu_\lambda^-, \mu_\lambda^- + \nu_\lambda^+] = \emptyset, \text{ and either } \nu_\lambda^+ = 0 \text{ or } \nu_\lambda^- = 0.

Then for any path \( \alpha : [0, 1] \to \Lambda \) connecting \( \lambda^+ \) to \( \lambda^- \) there exists \( \bar{t} \in [0, 1] \) such that \( (\bar{t}, 0) \in [0, 1] \times U \) is a bifurcation point of \( \nabla L_{\alpha(t)}(x) = 0 \) in \( [0, 1] \times U \), and therefore that \( (\alpha(\bar{t}), 0) \in \Lambda \times U \) is a bifurcation point along sequences of \( \nabla L_\lambda(x) = 0 \) in \( \Lambda \times U \).

Moreover, if as a map from \( U \cap X \) to \( H \), \( A_{\lambda^+} \) (resp. \( A_{\lambda^-} \)) has a Gâteaux derivative \( B_{\lambda^+}(x) \) (resp. \( B_{\lambda^-}(x) \)) at every point \( x \in U \cap X \) and \( \nu_{\lambda^+} = 0 \) (resp. \( \nu_{\lambda^-} = 0 \)), then \( \alpha(\bar{t}) \neq \lambda^+ \) (resp. \( \alpha(\bar{t}) \neq \lambda^- \)).

In e.1), even if 0 is an isolated critical point of \( L_{\lambda^+} \) and \( L_{\lambda^-} \) it is possible that either \( \lambda^+, 0 \) or \( \lambda^- \) is a bifurcation point. The second part in e.1) and e.4) are, sometimes, more conveniently used in applications, see [54].

**Proof.** Clearly, it suffices to prove the case that \( \Lambda = [0, 1] \) and \( \lambda^+ = 0, \lambda^- = 1 \). By contradiction, suppose that \( [0, 1] \times U \) contains no bifurcations point of \( \nabla L_\lambda(x) = 0 \) in \( [0, 1] \times U \). Then there exists a small \( \varepsilon > 0 \) such that

for each \( \lambda \in [0, 1] \), \( \nabla L_\lambda(x) = 0 \) has a unique solution in \( B_H(0, \varepsilon) \subset U \). \( \tag{C.16} \)

(In particular, this implies that 0 is an isolated critical point of \( L_\lambda \) for each \( \lambda \in [0, 1] \).) We conclude that for any Abel group \( K \) and any \( \lambda, \lambda' \in [0, 1] \),

\[ C_q(L_\lambda, 0; K) = C_q(L_{\lambda'}, 0; K), \quad \forall q \in \mathbb{N} \cup \{0\}. \quad \tag{C.17} \]

In fact, for any given point \( \lambda_0 \in [0, 1] \), if \( \text{Ker}(B_{\lambda_0}(0)) \neq \{0\} \), since \( [0, 1] \) is first countable and sequential compact, by Step 1 in the proof of Theorem [2] there exists a connected open neighborhood \( \mathcal{N}(\lambda_0) \) of \( \lambda_0 \) in \( [0, 1] \) such that \( C_s(L_\lambda, 0; K) = C_s(L_{\lambda_0}, 0; K) \) for any Abel group \( K \) and for all \( \lambda \in \mathcal{N}(\lambda_0) \).

If \( \text{Ker}(B_{\lambda_0}(0)) = \{0\} \), replacing \( \hat{\Lambda} \) by \( [0, 1] \) in the proof of (C.12) we obtain a connected open neighborhood \( \mathcal{N}(\lambda_0) \) of \( \lambda_0 \) in \( [0, 1] \) such that \( C_s(L_\lambda, 0; K) = \delta_{\mu_{\lambda_0}} K \) for any Abel group \( K \) and all \( (q, \lambda) \in \mathbb{Z} \times \mathcal{N}(\lambda_0) \).

In summary, for any Abel group \( K \), \( [0, 1] \ni \lambda \mapsto C_s(L_\lambda, 0; K) \) is locally constant, and therefore (C.17) holds. In particular, this yields for any Abel group \( K \)

\[ C_q(L_{\lambda^+}, 0; K) = C_q(L_{\lambda^-}, 0; K), \quad \forall q \in \mathbb{N} \cup \{0\}, \quad \tag{C.18} \]

Let us derive contradictions in each case.

**Step 1** (Case e.1). By (C.16), the third case in e.1) must occur. It contradicts (C.18). For the part of “Moreover”, since \( L_{\lambda}^{\mid U_X} \in C^2(U^X, \mathbb{R}) \) and \( B_\lambda \) is continuous as a map from \( U^X \) to \( L_{\lambda}(H) \) for each \( \lambda \in \{\lambda^+, \lambda^-\} \), as in the final paragraph in Step 1 of the proof of Theorem [2] we may derive from [32] Corollary 2.8 that \( C_s(L_{\lambda}^{\mid U_X}, 0; K) \cong C_s(L_\lambda, 0; K) \) for \( \lambda = \lambda^+, \lambda^- \). This and (C.18) lead to \( C_s(L_{\lambda^+}^{\mid U_X}, 0; K) = C_s(L_{\lambda^-}^{\mid U_X}, 0; K) \), which contradicts the new assumption.

**Step 2** (Case e.2). By (C.16), 0 is an isolated critical point of \( L_{\lambda^+} \) and \( L_{\lambda^-} \) and 0 is a homological visible critical point of either \( L_{\lambda^+} \) or \( L_{\lambda^-} \). As in arguments below (C.12), for any Abel group \( K \) we have also for \( \ast = +, - \),

\[ C_q(L_{\lambda^\ast}, 0; K) = \delta_{\mu_{\lambda^\ast}} K, \quad \forall q \in \mathbb{N} \cup \{0\} \text{ if } \nu_{\lambda^\ast} = 0, \quad \tag{C.19} \]
Suppose that 0 is a homological critical point of $L_{\lambda^*}$. (The proof of another case is similar.) Then $C_m(L_{\lambda^*}, 0; K) \neq 0$ for some Abel group $K$ and some $m \in \mathbb{Z}$, and therefore $m \in [\mu_{\lambda^*}, \mu_{\lambda^*} + \nu_{\lambda^*}]$ by (C.19) and (C.20). A contradiction is yielded.

Step 3[Case (e.3)]. By (C.17), 0 is also an isolated critical point of $L_{\lambda^*|U}$ for each $\lambda \in \{\lambda^+, \lambda^-, \lambda^\ast\}$. Then the third case in (e.3) must occur, and so either $C_m(L_{\lambda^*|U}, 0; K) \neq 0$ or $C_n(L_{\lambda}, 0; K^\prime) \neq 0$ for some Abel group $K'$ and some $n \in \mathbb{N} \cup \{0\}$. Since for any Abel group $G$ we may use [32, Corollary 2.8] to obtain $C_n(L_{\lambda}, 0; G) \cong C_n(L_{\lambda}, 0; G)$ for $\lambda = \lambda^+, \lambda^-$, as in the proof of the case (B.2) in Step 2 of the proof of Theorem C.6 using these and (C.16) we obtain either $C_m(L_{\lambda^*}, 0; K) = C_m(L_{\lambda^-}, 0; K) \neq 0$ or $C_n(L_{\lambda^*}, 0; K^\prime) = C_n(L_{\lambda^-}, 0; K^\prime) \neq 0$. These and (C.19) – (C.20) lead to a contradiction to the first condition in (e.3).

Step 4[Case (e.4)]. Because of (C.16), (C.18) and (C.19) – (C.20) can be used.

- If $\nu_{\lambda^*} = 0$ and $\nu_{\lambda^-} = 0$, then (C.18) and (C.19) lead to $\mu_{\lambda^*} = \mu_{\lambda^-}$, and hence a contradiction.
- If $\nu_{\lambda^*} = 0$ and $\nu_{\lambda^-} > 0$, then (C.18) and (C.19) with $\ast = +$ imply $C_{\mu_{\lambda^*}}(L_{\lambda^-}|U, 0; K) = K$ and therefore $\mu_{\lambda^*} \in [\mu_{\lambda^-}, \mu_{\lambda^*} + \nu_{\lambda^-}]$ by (C.20) with $\ast = -$, which is a contradiction.
- If $\nu_{\lambda^*} > 0$ and $\nu_{\lambda^-} = 0$, as in the second case we obtain $\mu_{\lambda^-} \in [\mu_{\lambda^*}, \mu_{\lambda^*} + \nu_{\lambda^*}]$ and so a contradiction as above.

Step 5[Prove the final claim]. Let $(\lambda^*, 0) \in [0, 1] \times U$ be a bifurcation point of $\nabla L_{\lambda}(x) = 0$ in $[0, 1] \times U$, which is also a bifurcation point along sequences because $[0, 1]$ is first countable. By contradiction suppose that $\lambda^* = \lambda^+$ (resp. $\lambda^* = \lambda^-$) in the case of $\nu_{\lambda^*} = 0$ (resp. $\nu_{\lambda^*} = 0$). The additional assumption on $A_{\lambda^+}$ (resp. $A_{\lambda^-}$) implies that $(U, X, L_{\lambda^+}, A_{\lambda^+}, B_{\lambda^+})$ [resp. $(U, X, L_{\lambda^-}, A_{\lambda^-}, B_{\lambda^-})$] also satisfies [32, Hypothesis 1.1]. Then by [32, Theorem 3.1] (see Theorem C.6) we obtain $\nu_{\lambda^*} = \nu_{\lambda^*} > 0$ (resp. $\nu_{\lambda^*} = \nu_{\lambda^*} > 0$), a contradiction. 

Because of remarks above, in most of results in [50, 51, 52] we either require the topology space $\Lambda$ to be first countable or change “bifurcation point” into “bifurcation point along sequences”. For example, [50, Theorems 3.1,3.2] should be revised as follows.

**Theorem C.6.** In [50, Theorems 3.1,3.2], “bifurcation point” should be changed into “bifurcation point along sequences”. (Note that 0 is not required to be an isolated critical point of $F_{\lambda^*}$ from the proofs therein.)

**Theorem C.7.** The conclusions in [52, Theorems 3.6] should be revised as follows:

Then $(\lambda^*, 0) \in \Lambda \times U$ is a bifurcation point for the equation

$$DL_{\lambda^*}(u) = 0, \quad (\lambda, u) \in \Lambda \times U.$$ 

More precisely, one of the following alternatives occurs:

(i) $(\lambda^*, 0)$ is not an isolated solution of the equation $DL_{\lambda^*}(u) = 0$ in $\{\lambda^*\} \times U$.

(ii) For every $\lambda \in \Lambda$ near $\lambda^*$ there is a nontrivial solution $u_{\lambda}$ of $A_{\lambda}(u) = 0$ in $U^X$, which converges to 0 in $X$ as $\lambda \to \lambda^*$.

(iii) For any given neighborhood $W$ of 0 in $X$ there is an one-sided neighborhood $\Lambda^*$ of $\lambda^*$ such that for any $\lambda \in \Lambda^* \setminus \{\lambda^*\}$, $A_{\lambda}(u) = 0$ has at least two nontrivial solutions in $W$, which can also be required to correspond to distinct critical values provided that $\nu_{\lambda^*} > 1$ and $A_{\lambda}(u) = 0$ has only finitely many nontrivial solutions in $W$. 


In the conclusions of [50, Theorems 4.6], (i) is changed into the (i) above, and the sentence “\((\lambda^*, 0) \in \Lambda \times U^X\) is a bifurcation point for the equation (4.15)” should be deleted.

**Proof.** By the assumption f) in [52, Theorems 3.6] we obtain the first claim by the conclusion (B) in Theorem C.4. Suppose that (i) is not true. Then 0 \(\in U\) is an isolated critical point of \(L_{\lambda^*}\) in \(U\). (Of course, this implies that 0 \(\in U^X\) is an isolated zero of \(A_{\lambda^*}(u) = 0\) in \(U^X\).) By the arguments in the first paragraph in Step 2 in the proof of Theorem C.4, 0 \(\in U\) is also an isolated critical point of \(L_\lambda\) for each \(\lambda \in [\lambda^* - \delta, \lambda^* + \delta]\). The other arguments in the proofs of [50 Theorems 4.6] and [52 Theorems 3.6] are valid.

**Theorem C.8.** The conclusion (i) in [52, Theorems 3.7] and [50, Theorems 5.12] should be revised as follows:

(i) \((\lambda^*, 0)\) is not an isolated solution of the equation \(D L_{\lambda^*}(u) = 0\) in \([\lambda^*] \times U\).

The sentences “\((\lambda^*, 0) \in \Lambda \times U^X\) is a bifurcation point of (3.23)” in [52, Theorems 3.10] and “\((\lambda^*, 0) \in \Lambda \times U^X\) is a bifurcation point of (4.15)” in [50, Theorems 5.12] are changed into “\((\lambda^*, 0) \in \Lambda \times U\) is a bifurcation point of the equation \(D L_\lambda(u) = 0\) in \(\Lambda \times U\)."

**Theorem C.9.** In [52 Theorems 3.10], the words “\((\lambda^*, 0) \in \Lambda \times U^X\) is a bifurcation point for the equation (3.23); in particular” should be deleted, and the conclusion (i) should be revised as follows:

(i) \((\lambda^*, 0)\) is not an isolated solution of the equation \(D L_{\lambda^*}(u) = 0\) in \([\lambda^*] \times U\).

**Remark C.10.** Using theorems in [49, Appendix B] we can also prove corresponding results with the theorems above as supplements to [49, §6]. They and related applications will be given elsewhere.

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