Accelerated Expansion of the Universe in the Model with Nonuniform Pressure

Elena Kopteva1, Irina Bormotova1,2, Maria Churilova1, and Zdenek Stuchlik1

1 Institute of Physics and Research Centre of Theoretical Physics and Astrophysics, Faculty of Philosophy and Science in Opava, Silesian University in Opava, 746 01 Opava, Czech Republic; elena.koptieva@gmail.com
2 Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research 141980 Dubna, Russia

Accepted 2019 October 19; published 2019 December 13

Abstract

We present the particular case of the Stephani solution for shear-free perfect fluid with uniform energy density and nonuniform pressure. Such models appeared as possible alternative to the consideration of the exotic forms of matter like dark energy that would cause the acceleration of universe expansion. These models are characterized by the spatial curvature, depending on time. We analyze the properties of the cosmological model obtained on the basis of exact solution of the Stephani class, and adapt it to the recent observational data. The spatial geometry of the model is investigated. We show that despite possible singularities, the model can describe the current stage of the universe’s evolution.

Unified Astronomy Thesaurus concepts: Cosmology (343); Expanding universe (502); Cosmological models (337); Accelerating universe (12); Einstein field equations (450)

1. Introduction

Although the ΛCDM model, based on the Friedmann solution, is the most popular for explanation of the observed cosmological acceleration, it faces some fundamental problems, such as the “dark energy” problem and the “coincidence” problem (Weinberg 1989). Thus, different attempts to find a possible alternative arise, one of which is the consideration of the inhomogeneous cosmological models. The Stephani solution (Stephani 1967) has drawn attention of cosmologists so long as it allows us to build a model of the universe with accelerated expansion (Dabrowski & Hendry 1998; Stelmach & Szydlowski 2004; Stelmach & Jakacka 2001; Balcerzak et al. 2015; Ong et al. 2018). This is a nonstatic solution for the expanding perfect fluid with zero shear and rotation, which contains the known Friedmann solution as a particular case. The Stephani solution was discussed extensively in the literature (see e.g., Krasinski 1983; Sussman 1987, 1988a, 1988b, 2000; Dabrowski 1993; Korkina et al. 2016; Ong et al. 2018 and references therein). Originally, it has no symmetries, but the special case of spherical symmetry is of particular interest in cosmology. It is known that spatial sections of the Stephani spacetime in this case have the same geometry as if they were subspaces of the Friedmann ones. The spatial curvature in the Stephani cosmological models is arbitrary function of time. This very property allows us to obtain the appropriate behavior of the cosmological acceleration.

To our knowledge, only a few of the cosmological models based on the Stephani solution were studied concerning their correspondence to the observational data.

In the present work, we consider a rather general case of this solution restricted by the choice of the energy density in the same form as for the Friedmann dust. We analyze the properties of the resulting cosmological model and its applicability to the description of the current stage of the universe evolution.

The paper is organized as follows. In Section 2 we introduce the special case of the Stephani solution for our model and fit to the current values of the cosmological parameters. The geometry of the spatial part of the obtained solution is explored in Section 3. In Section 4 we investigate the dynamics of the universe evolution in our model, build the R-T-regions for the resulting spacetime and discuss singularities of the model. In Section 5 we consider the cosmological implications of our model. The conclusions are presented in Section 6.

2. Special Case of the Stephani Solution

It is known that for the perfect fluid described by a four-velocity vector field $u^a$ there exist four main kinematic characteristics (see, e.g., Stephani & Kramer 2003): the acceleration $a^a$, the volume expansion $\Theta$, the rotation $\omega_{\alpha\beta}$, and the shear $\sigma_{\alpha\beta}$. These parameters are defined as follows:

\begin{align*}
\Theta &= u^a_{;a}, \\
\omega_{\alpha\beta} &= u_{[\alpha;\beta]} + u_{[\alpha} u_{\beta]}, \\
\sigma_{\alpha\beta} &= u_{(\alpha;\beta)} + u_{(\alpha} u_{\beta)} - \Theta h_{\alpha\beta}/3, \\
h_{\alpha\beta} &= g_{\alpha\beta} + u_{\alpha} u_{\beta},
\end{align*}

where $h_{\alpha\beta}$ is the projection tensor, Greek indexes run from 0 to 3, and square/round brackets represent antisymmetrization/symmetrization by corresponded indexes. Throughout the paper, the dot means the partial derivative with respect to time, and the geometric units are used where $c = 1, \hbar = G = 1$.

The Stephani solution is the solution of the Einstein equations for the universe filled with perfect fluid with zero shear and rotation. It is usually written in comoving coordinates in which the four-velocity of fluid particles has the following components: $u^0 = 1/\sqrt{\rho_0}$, $u^i = 0$, $i = 1, 2, 3$.

In commonly used notations, the Stephani solution in the case of spherical symmetry has the form (Krasinski 1983)

\begin{equation}
ds^2 = D^2 dt^2 - \frac{R^2(t)}{V^2(t, \chi)}(d\chi^2 + \chi^2 d\sigma^2),
\end{equation}

where $d\sigma^2$ is the usual metric on the unit two-sphere, and

\begin{equation}
V = 1 + \frac{1}{4} c(t) \chi^2,
\end{equation}

\begin{equation}
C(t) = \frac{c(t)}{c_0}.
\end{equation}
\[ D = F(t) \left( \frac{V}{R} \right)^{\frac{1}{V - 1}}. \]  

The energy density is uniform and given by \[ \varepsilon(t) = 3C^2(t). \]  

The function \( k(t) \) is defined by the expression  
\[ k(t) = \left[ C^2(t) - \frac{1}{F^2(t)} \right] R^2(t), \]  

and corresponds to the curvature parameter, which in the Friedmann solution is constant normalizable to 0, \( \pm 1 \). Here, \( C(t), F(t), R(t) \) are arbitrary functions. Using the following relations,  
\[ r(t, \chi) = \frac{R(t)}{V(t, \chi)}, \]  
\[ \psi(t) = \frac{1}{F(t)}, \]  
\[ a(t) = R(t), \]  
\[ \zeta(t) = \frac{k(t)}{R(t^2)}. \]  

The metric (Equation (5)) may be rewritten in the form  
\[ ds^2 = \frac{r^2}{r^2\psi^2}dt^2 - r^2(d\chi^2 + \chi^2d\sigma^2), \]  

which we shall use in further consideration as more convenient for our purposes. Here,  
\[ r(t, \chi) = \frac{a(t)}{1 + \frac{1}{3}\zeta(t)a^2(t)\chi^2}, \]  
\[ \zeta(t) = \varepsilon(t) - \psi^2(t), \]  

where \( \zeta(t), \psi(t) \) and \( \varepsilon(t) \) are arbitrary functions, \( a(t) \) is the function related to the scale factor of the Friedmann solution. According to the Einstein equations, the pressure is defined by the expression  
\[ p(t, \chi) = -\varepsilon(t) - \frac{\dot{\varepsilon}(t)r(t, \chi)}{3\dot{R}(t, \chi)}. \]  

It is clear that the pressure is nonuniform in the Stephani solution. The function \( \zeta(t) \) is the spatial curvature. It is easy to verify that the scalar curvature \( R \) of the spatial sections \( t = \text{const} \) of the metric (Equation (14)) is \( R = 6\zeta(t) \) and the Kretschmann invariant in this three-dimensional case is \( K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 12\zeta^2(t) \). As far as \( t = \text{const} \), one obtains the subspace of everywhere constant curvature. It is from here that the identity for the spatial geometries of the Stephani and the Friedmann solutions follows.  

In this article, we sometimes omit the variable of the function, provided it will not cause the confusion. As can be deduced from the following consideration, Equation (16) is a generalization of the known Friedmann equation,  
\[ \frac{\dot{a}(t)^2}{a(t)^2} + \frac{k}{a(t)^2} = \frac{1}{3}\varepsilon(t), \]  

where \( k = 0, \pm 1 \), the factor 1/3 in (16) is left out because of the arbitrariness of \( \varepsilon(t) \). The function \( \psi(t) \) turns out to be connected with the Hubble parameter \( H \). In the case of inhomogeneous cosmological model, the definition of the Hubble parameter should be generalized, as it depends both on the time and spatial position. We shall use the generalization introduced by (Ellis 2009)  
\[ H = \frac{1}{l} \frac{dl}{dt} = \frac{1}{3}\Theta, \]  

where \( l \) is some “representative” length that corresponds to the scale factor \( a(t) \) in the Friedmann models, and \( \tau \) is the proper time given in the standard way by  
\[ d\tau = \sqrt{g_{00}}dt. \]  

Due to the definition of the comoving coordinates, we obtain for the metric in Equation (14) from the expression (Equation (1))  
\[ \Theta = \frac{1}{2\sqrt{g_{00}} \frac{\partial}{\partial t} g_{11}} = \frac{3}{r^2} \frac{\partial}{\partial r} \frac{\partial}{\partial \tilde{r}} = 3\psi. \]  

From Equations (19) and (21), it follows that  
\[ H = \psi. \]  

If \( \zeta = 0 \), then Equation (14) is a parabolic type of the Friedmann solution and it follows from Equation (16) that the function \( \psi^2 \) attains the sense of the critical energy density \( \varepsilon_{cr} = 3H^2 \), which is also in accordance with Equation (22).  

For the parabolic Friedmann solution, we have  
\[ r = a(t), \quad g_{00} = \frac{f^2}{r^2\psi^2} = 1 \quad \Rightarrow \quad g_{00} = \frac{\dot{a}^2}{a^2\psi^2} = 1, \]  

then for the function \( \psi(t) \) it follows that  
\[ \psi(t) = \frac{\dot{a}}{a}. \]  

Thus, for the appropriate transition to the Friedmann limit in the metric in Equation (14), one should choose \( \psi(t) \) in the form in Equation (24).  

2.1. The Model of the Universe with the Accelerated Expansion  

We now define our model of the universe with the accelerated expansion based on the mentioned particular case of the Stephani solution. We suppose the universe to be filled everywhere with the expanding shear-free perfect fluid with uniform energy density \( \varepsilon = \varepsilon(t) \) and nonuniform pressure, \( p = p(\chi, t) \).  

Let us start from the general Stephani metric in comoving coordinates written with conformally flat spatial part,  
\[ ds^2 = \frac{r^2\dot{a}^2}{r^2\dot{a}^2}dt^2 - r^2(d\chi^2 + d\sigma^2), \]  

where  
\[ r(t, \chi) = \frac{2a(t)e^3}{1 + \zeta(t)a^2(t)e^{2t}}. \]
As discussed above, the main equation that governs the evolution of the model reads
\[ \frac{\dot{a}^2}{a^2} + \zeta(t) = \varepsilon(t). \]  
(27)

From here and Equation (18), it evidently follows that the Stephani models with \( \zeta = \pm 1/a^2, 0 \) are the Friedmann models.

The appropriate choice of the spatial coordinate transformation brings the spatial part of the metric in Equation (25) to one of the following forms:
\[ d\ell^2 = \frac{a^2}{\left[ \cos^2 \frac{\chi}{2} + \zeta a^2 \sin^2 \frac{\chi}{2} \right]} (d\chi^2 + \sin^2 \chi d\sigma^2), \]
\[ e^x = \tan \frac{\chi}{2}, \]  
(28)
\[ d\ell^2 = \frac{a^2}{\left[ \cosh^2 \frac{\chi}{2} + \zeta a^2 \sinh^2 \frac{\chi}{2} \right]} (d\chi^2 + \sinh^2 \chi d\sigma^2), \]
\[ e^x = \tanh \frac{\chi}{2}, \]  
(29)
\[ d\ell^2 = \frac{a^2}{\left[ 1 + \zeta a^2 \chi^2 \right]} (d\chi^2 + \chi^2 d\sigma^2), \quad e^x = \frac{\chi}{2}. \]  
(30)

For our description, we shall choose the case in Equation (30), implying the spacetime metric
\[ ds^2 = \frac{\dot{r}^2 a^2}{r^2 a^2} dt^2 - r^2 (d\chi^2 + \chi^2 d\sigma^2), \]  
(31)
\[ r = \frac{a}{\sqrt{1 + \zeta a^2 \chi^2}}. \]  
(32)

The energy density is chosen the same as for the Friedmann dust:
\[ \varepsilon = \frac{a_0}{a^3}, \]  
(33)
\[ a_0 = \text{const} = a(t_0), \]  
where \( t_0 \) corresponds to the current moment of time (our time).

We take the spatial curvature in the form
\[ \zeta = -|\beta| \frac{a_0^k}{a^{k+2}}, \]  
(34)
where \( k = \text{const}, \beta = \text{const} < 0, \) that means that the spatial curvature is negative everywhere in the universe. Such an expression for the spatial curvature is induced by the form of the Friedmann Equation (18), which may contain the sum of energy densities of several noninteracting sources. In the Friedmann models, the energy density for all known components of matter (including those with negative pressure) is expressed in terms of scale factor raised to the correspondent power.

The models known in the literature are mostly the particular cases of Equation (34) with a fixed value of \( k \) ((Dabrowski 1995; Dabrowski & Hendry 1998) Model I: \( k = -1 \), Model II: \( k = 1 \), (Stel*ama* & Jakacka 2001; Stel*ama* & Sydzlowski 2004): \( k = -1 \), (Ong et al. 2018; Gregoris et al. 2019): \( k = -1 \)).

Slightly different discussion of the models with unfixed \( k \) is presented in Sussman (2000) and Hashemi et al. (2014) in the frame of investigation of the Stephani universes with physically meaningful equations of state of matter.

We carry out our consideration without fixing \( k \), but by figuring out the range of its values that correspond to the right behavior of the universe acceleration.

The pressure in the model according to Equations (17), (33), and (34) reads
\[ p = \frac{a_0}{a^3} \left( \frac{\chi^2}{2} |\beta| k \right). \]  
(35)

We now express some cosmological parameters in terms of \( \chi \) and \( a(t) \), which will somehow parameterize the time coordinate. Here, we restore the dimensions as far as we are going to put the numeric parameters of the model \( (a_0, \beta, k) \) in accordance with the observational data.

1. From the Hubble parameter in Equation (27), one has
\[ \frac{\dot{a}}{a} = \left[ \frac{a_0}{a^3} + |\beta| \frac{a_0^k}{a^{k+2}} \right]^\frac{1}{2}, \]  
(36)

hence,
\[ H = c \left[ \frac{a_0}{a^3} + |\beta| \frac{a_0^k}{a^{k+2}} \right]. \]  
(37)

2. The matter density parameter,
\[ \Omega_m = \frac{\varepsilon}{\varepsilon_{cr}} = \frac{a_0 c^2}{3 a^3 H^2}. \]  
(38)

3. The radius of the universe at present time \( r_0 \),
\[ r_0 = \int_{0}^{\chi_0} \sqrt{-g_{11}} d\chi = \int_{0}^{\chi_0} r(t_0, \chi) d\chi \]  
(39)
\[ r_0 = \int_{0}^{\chi_0} \frac{a(t_0)}{1 + \zeta(a_0) a^2(t_0) \chi^2} d\chi. \]  
(40)

Taking \( \zeta(t) \) from Equation (34), one obtains
\[ r_0 = \frac{4a_0}{|\beta|} \arctanh \left( \frac{\chi_0}{2 |\beta|} \right). \]  
(41)

Using this relation, it is possible to find the value of the coordinate \( \chi_0 \) corresponding to the current size of the universe: \( r(\chi_0, t_0) = r_0 \).

4. Deceleration parameter \( q \). We take the general definition of the deceleration parameter according to (Ellis 2009)
\[ q = -\frac{1}{l} \frac{d^2l}{dt^2} \frac{1}{H^2}. \]  
(42)

From Equations (19), (22), and (24), we have
\[ \frac{dl}{d\tau} = l \frac{\dot{a}}{a}. \]  
(43)
differentiating both parts of Equation (43) with respect to the time, we obtain
\[
\frac{d}{dt}\left(\frac{d}{dr}\right) = \frac{\dot{a}}{a} \frac{\dot{a}}{a} + \frac{1}{\sqrt{g_{00}}} \frac{d}{dt} \left(\frac{\dot{a}}{a}\right)
\]
\[
= H^2 + \frac{\dot{r}}{r a} \left(\frac{\dot{a}}{a} - H^2\right). \tag{44}
\]
Finally, for the deceleration parameter, there is
\[
q = -\left[1 + \frac{\dot{r}}{r a} \left(\frac{\dot{a}}{a} - 1\right)\right], \tag{45}
\]
or in the explicit form due to Equation (37):
\[
q = \frac{k|\beta|}{\left(\frac{\dot{a}}{a}\right)^2 - (k + 1)|\beta|} \left[1 + \frac{(\frac{\dot{a}}{a})^{-1} + k|\beta|}{2\left((\frac{\dot{a}}{a})^{-1} + |\beta|\right)}\right]. \tag{46}
\]

It is clear that the expression in Equation (45) in the Friedmann limit \((r = a)\) turns to the right form for the deceleration parameter in the Friedman models: \(q = -\frac{\dot{a}}{a^2}\).

### 2.2. Estimation of the Model Constants with Respect to the Observational Data

In this subsection, we introduce the comparison of some observable parameters from previous subsection with their values obtained within standard cosmological model.

The current values of cosmological parameters obtained within \(\Lambda\)CDM model (or FLRW model with nonzero curvature) may be found in Hinshaw et al. (2013). We shall assume the following numbers:
\[
H_0 = 2 \times 10^{-18} \text{ s}^{-1}, \quad \Omega_m = 0.3, \quad r_0 \approx 4, 4 \times 10^{26} \text{ m}. \tag{47}
\]

According to this data due to Equations (37) and (38), the constants of our model \((a_0, \beta, \chi_0)\) related to the current moment of time can be defined as follows:
\[
H_0 = \frac{c}{a_0} \sqrt{1 + |\beta|}, \tag{48}
\]
\[
\Omega_m = \frac{c^2}{3a_0^2 H_0^2}, \tag{49}
\]
\[
a_0 = 1.58 \times 10^{26} \text{ m}, \tag{50}
\]
\[
\beta = -0.111113, \tag{51}
\]
\[
\chi_0 = 2.59906. \tag{52}
\]

### 2.3. Singularities of the Model

It was also widely discussed (Krasinski 1983; Sussman 1988b; Dabrowski & Hendry 1998) that the Stephani models contain some special singularities that should be taken into account if one intends to build a cosmological model. In our case, the model contains three true singularities.

1. The initial singularity: \(a(t) = 0 \Rightarrow r = 0, \varepsilon \rightarrow \infty, p \rightarrow \infty\).

2. The singularity arising from \(g_{11}\):
\[
\chi = \frac{2(a/a_0)^{\frac{1}{2}}}{\sqrt{|\beta|}}. \tag{53}
\]

3. The singularity arising from the expression for pressure \(p\):
\[
\chi = \frac{2(a/a_0)^{\frac{1}{2}}}{\sqrt{(1 + k)|\beta|}}. \tag{54}
\]

In the case of \(k = -1\), the singularity points in Equation (54) belong to the spatial infinity independently on the value of the time coordinate. In the literature, this particular case is called the Stephani–Dabrowski model (Dabrowski 1993; Stelmach & Jakacka 2001; Stelmach & Szydlowski 2004). Here, if one chooses \(k < -1\), then the singular behavior of the pressure will disappear.

### 2.4. Mass Function and Horizons of the Model

Let us first briefly introduce the notion of R- and T-regions of the spherically symmetric spacetime (Novikov 2001).

The spherically symmetric metric written in general form
\[
d^2 = e^{\nu(t,x)} dt^2 - e^{\lambda(t,x)} dx^2 - r^2(t, x) d\sigma^2 \tag{55}
\]
can locally be brought to the view
\[
d^2 = A(\tilde{t}, \tilde{x}) d\tilde{t}^2 - B(\tilde{t}, \tilde{x}) d\tilde{x}^2 - \tilde{x}^2 d\sigma^2 \tag{56}
\]
by coordinate transformation preserving the spherical symmetry:
\[
\tilde{t} = \tilde{t}(t, x), \quad \tilde{x} = \tilde{x}(t, x). \tag{57}
\]
At the vicinity of a taken point two main situations are possible. The first one is the case when the world line \(\tilde{x} = \text{const}\), \(\theta = \text{const}, \varphi = \text{const}\) is time-like. In this case, \(\tilde{x}\) is the spatial coordinate, and the following inequality holds for the general metric (Equation (55))
\[
e^{\nu - \lambda} < \left(\frac{dx}{dt}\right)^2. \tag{58}
\]
Here, \(dx/dt\) is found from the equations \(\tilde{x}^2 = r^2(t, x) = \text{const}\), regarding the invariance of \(g_{22}\) and \(g_{33}\) under the transformation (Equation (57)). The points for which the inequality in Equation (58) is satisfied are called R-points. They form the R-region of the spacetime with usual properties of the world and observers.

The second case is when the world line \(\tilde{x} = \text{const}\), \(\theta = \text{const}, \varphi = \text{const}\) is space-like. In this case, \(\tilde{x}\) cannot be the spatial coordinate; thus, in the metric (Equation (56)), coordinates \(\tilde{t}\) and \(\tilde{x}\) “change” their roles (it is implied, that the functions \(A(\tilde{t}, \tilde{x})\) and \(B(\tilde{t}, \tilde{x})\) have the needed signs). In this case the following inequality holds for the general metric (Equation (55))
\[
e^{\nu - \lambda} > \left(\frac{dx}{dt}\right)^2. \tag{59}
\]
The points for which the inequality in Equation (59) is satisfied are called T-points. They form the T-region of essential instability where static observer is impossible.
The strict equality
\[ e^{\nu - \lambda} = \left( \frac{dx}{dt} \right)^2 \]  
(60)
defines the boundary between R- and T-regions of the spacetime, known as horizon.

Regarding the condition \( x^2 = r^2(t, x) = \text{const} \) we rewrite Equation (65) as follows
\[ e^{-\nu r^2} = e^{-\lambda r'^2}. \]  
(61)
This will be referred to as the horizon equation. The prime here means the partial derivative with respect to the spatial coordinate \( x \).

The coordinate condition in Equation (61) may also be expressed in terms of the so-called mass function (Korkina & Kopteva 2016), which for the metric (Equation (55)) reads
\[ m = r(1 + e^{-\nu r^2} - e^{-\lambda r'^2}). \]  
(62)
The horizon equation then transforms to
\[ m = r. \]  
(63)
For the metric (Equation (25)), regarding Equations (32) and (36), the mass function takes the form
\[ m = \frac{a_0 \chi^3}{\left(1 - \left(\begin{array}{l}
a_0 \\
\rho
\end{array}\right)\right)(\chi^2 + 1)|\beta|^3}. \]  
(64)
The horizon in Equation (63) then gives the following expressions for two branches of the horizon:
\[ \chi_{1,2} = \frac{2}{|\beta|} \sqrt{2 \left(\begin{array}{l}
a(t) \\
a_0
\end{array}\right) + |\beta| \pm 2 \left(\begin{array}{l}
a(t) \\
a_0
\end{array}\right)} \]  
(65)

\section{3. Geometry}

In this section, we investigate the spatial geometry of the obtained solution. To build the spatial sections of the spacetime with metric in Equation (31) we fix the time at present moment \( t = t_0 \) or \( \text{const} \) that yields \( a = a_0 = \text{const} \) in the formulae. To make it possible to visualize the three-dimensional hypersurface we also fix \( \theta = \pi/2 \). Applying these conditions to Equation (31) we obtain the intrinsic metric of the hypersurface of our interest in the following form:
\[ dl^2 = \frac{a_0^2}{\left(1 - \frac{|\beta|}{2}\right)^2}(d\chi^2 + \chi^2 d\phi^2). \]  
(66)
By use of a new coordinate \( \rho = \sqrt{|\beta|} \chi \), the metric in Equation (66) can be rewritten in more familiar way
\[ dl^2 = \frac{a_0^2}{|\beta|} \left(1 - \frac{\rho^2}{4}\right)^2(d\rho^2 + \rho^2 d\phi^2). \]  
(67)
This is a metric of the pseudo-sphere in terms of the stereographic projection coordinates (see, e.g., Dubrovin et al. 1992) accurate within the similarity transformation with constant factor \( a_0^2/|\beta| \). This stereographic projection maps the upper half of the pseudo-sphere represented by the hyperboloid of revolution onto the open disk \( \rho^2 = x^2 + y^2 < 1 \) on the plane \( z = 0 \) as shown at Figure 1. Such a disk equipped with normalized metric in Equation (67) (so that \( a_0^2/|\beta| = 1 \)) refers to the Poincare model of Lobachevsky geometry. It is seen that the spatial sections of the interval (Equation (31)) are the Lobachevsky spaces.

Figure 1 demonstrates the form of the spatial hypersurface of the universe within our model as an instantaneous snapshot at present moment of time, corresponding to the line \( T = 1 \) at Figures 2 and 3. To restore the three-dimensional picture from Figure 1, one should imagine that the circles of the sections \( z = \text{const} \) are in fact two-spheres.

\section{4. Universe Evolution in the Model}

To investigate the evolution of the universe in the obtained model, we consider an observer situated close to the symmetry center \( \chi = 0 \), who observes the dynamics of the infinite number of the concentric spheres marked by successive values of \( \chi \). The velocity of the expansion of some sphere \( \chi \) may be found as follows:
\[ v(t, \chi) = \frac{d}{dt} \int_0^\chi r(t, \chi) d\chi. \]  
(68)
Using the results obtained in Section 2, we now build the universe expansion velocity profile found from Equation (68) and the deceleration parameter given by Equation (46).
Further in our discussion we shall use the dimensionless function,

$$T \equiv \frac{a(t)}{a_0},$$

(69)

which will be treated as time parameter. The differentiation with respect to the time will be carried out taking into account that according to Equation (36):

$$\frac{dT}{dt} = \frac{1}{a_0} \sqrt{\frac{1}{T} + \frac{\beta}{T^k}}.

(70)

Figure 2 shows the universe expansion velocity profile in the model in terms of dimensionless units, where the time parameter $T$ is given by Equation (69). The concrete values of the index $k$ are chosen only for illustrative purposes, with decreasing of $k$ the picture qualitatively remains the same. The point of intersection of the lines $T = 1$ and $r = r_0$ defines the coordinate $\chi_0$, which indicates the sphere of radius $r_0$ corresponding to the edge of the universe. It is seen that at present time the boundary of the universe belongs to the region of nonstationarity, and expands with the velocity exceeding the speed of light as it is in standard Friedmann model. The central observer always belongs to the R-region of permitted observers. It is also seen that the universe does not reach the singularity given by Equation (53) up to its present age, and the singularity cannot be observed according to the causality principle.

Figure 3 shows the behavior of the deceleration parameter of the model. This profile is not affected by the singularity. For $k < -2$, there exists the line of zero deceleration parameter $q = 0$. Hence, one could expect that after some time the acceleration of the universe expansion changes into deceleration. However, even from the velocity profile (Figure 2), it is clear that there is no deceleration in the future. It may be verified by direct calculations using Equation (68) that the function $dv(t, \chi)/dt$ changes its sign only once, from negative to positive. Thus, we conclude that in our model, unlike the Friedmann models, there is no correlation between the signs of the deceleration parameter and the acceleration of the universe expansion.

5. Redshift–magnitude Relation in the Model

The advantage of the classical redshift–magnitude test is the sensitivity of the relation between the apparent magnitude and the redshift of the source to the cosmological model. In this section, we compare the observational results concerning the redshift–magnitude relation for SNe Ia with the theoretical predictions of our model. In this regard, we shall use the Hubble diagram of distance moduli and redshifts for $HST$-discovered SNe Ia in the gold and silver sets represented at (Riess et al. 2004).

The redshift–magnitude relation for the inhomogeneous Stephani model was derived first and studied in Dabrowski (1995) and Dabrowski & Hendry (1998) for two special cases near the observer position. Further, it was developed numerically by Stelmach & Jakacka (2001) for one of these cases, but higher redshifts.

For our model, we derive the redshift–magnitude relation in terms of distance modulus analytically without any supposition about values of the redshift.

The distance modulus within cosmological scales (Mpc) is defined by (see e.g., Ellis 2009)

$$\mu(z) = m(z) - M = 5 \log_{10}[d_L(z)/\text{Mpc}] + 25,

(71)$$

where $z$ is the redshift, $m(z)$ is the apparent bolometric magnitude of a standard candle whose absolute bolometric magnitude is $M$; $d_L(z)$ is the luminosity distance to the source,

$$d_L(z) = r_c(z)(1 + z)^2.

(72)$$

This relation holds rather general and does not depend on metric choice. Here $r_c(z)$ is comoving distance to the source by apparent size, which in our case is usual comoving radial
The general definition for the redshift in any cosmological model reads (Ellis 2009)

\[ 1 + z = \frac{(\kappa_\alpha u^\alpha)_{\text{emitter}}}{(\kappa_\alpha u^\alpha)_{\text{observer}}} \]

where \( u^\alpha = dx^\alpha/ds \) is usual four-velocity of the cosmological medium and \( \kappa_\alpha = dx^\alpha/d\lambda \) is a vector tangent to the correspondent null-geodesic with affine parameter \( \lambda \), i.e., the solution of the geodesic equations for the photon. Indexes “emitter” and “observer” mean that the quantity should be calculated at the correspondent position.

Applying these definitions to the interval (31) we shall act according to the following plan:

1. Solving the geodesic equations for the photon radial motion we obtain \( \kappa_\alpha \).
2. Taking into account the fact that in comoving system the only nonzero component of the four-velocity is \( u^0 = 1/\sqrt{g_{00}} \), we find the expression for the redshift in terms of the time and spatial coordinate.
3. Using previous results, we compose the distance modulus \( \mu \) as a function of \( T \) and \( \chi \), according to Equations (71)–(73).
4. Thus, we obtain the two-parametric area \( \mu - z \) with \( T \) and \( \chi \) being the parameters.

5. Finally, we plot the numerical solution of the Equation (76) as the line \( T(\chi) \) at the two-parametric diagram \( \mu(T, \chi) - z(T, \chi) \). As a result we obtain the theoretical prediction for the redshift–magnitude relation in our model and compare it with observational data from (Riess et al. 2004).

The geodesic equations for the interval in Equation (31) in case of the photon radial motion read

\[ \frac{d\kappa_1}{d\lambda} = \frac{1}{2g_{11}}(-2\kappa^1\kappa^0 \delta_{11} - (\kappa^1)^2 \delta_{11} - (\kappa_0)^2 \delta_{00}), \]

\[ \frac{d\kappa_0}{d\lambda} = \frac{1}{2g_{00}}(-2\kappa^1 \kappa^0 \delta_{00} + (\kappa^1)^2 \delta_{11} - (\kappa_0)^2 \delta_{00}), \]

\[ \kappa_0 = \frac{\sqrt{-\delta_{11}}}{\sqrt{g_{00}}} \kappa_1. \]

The metric coefficients in the interval in Equation (31) have the following explicit form:

\[ g_{00} = \frac{\left(1 - T^{-k}(1 + k)|\beta(\frac{\chi}{T})|^2\right)^2}{\left(1 - T^{-k}|\beta(\frac{\chi}{T})|^2\right)^2}, \]
Putting $\kappa^0$ from Equation (79) into Equation (77) and taking into account that $\kappa^0 = d\chi/d\lambda$, we obtain the differential equation with separable variables and integrate it with a result

$$\kappa^1 = 4a_0^2 T^k \left( \frac{1 - T^{-k} |\beta| \left( \frac{\chi}{2} \right)^2}{1 - T^{-k} (1 + k) |\beta| \left( \frac{\chi}{2} \right)^2} \right) \frac{1 + \frac{\chi}{2} \sqrt{|\beta| T^{-1} - \frac{1}{2}}}{1 - \frac{\chi}{2} \sqrt{|\beta| T^{-1} + \frac{1}{2}}},$$

and hence due to Equation (79),

$$\kappa_0 = g_{00} \kappa^0 = -4a_0^{1+k} T^{1+k} \left( \frac{1 + \frac{\chi}{2} \sqrt{|\beta| T^{-1} - \frac{1}{2}}}{1 - \frac{\chi}{2} \sqrt{|\beta| T^{-1} + \frac{1}{2}}} \right)^{2k+1} \sqrt{1 + T^{-1+k} |\beta|}.$$  \hspace{1cm} (83)

In our model, the observer occupies the position $\chi = 0$ and receives the signal at present time $T = 1$. Thus, we obtain for the redshift

$$1 + z = \frac{\kappa_0 \mu^0}{\kappa_0 \mu^0 |_{\chi=0,T=1}} = T^{1+k} \frac{1 - T^{-k} |\beta| \left( \frac{\chi}{2} \right)^2}{1 - T^{-k} (1 + k) |\beta| \left( \frac{\chi}{2} \right)^2} \frac{1 + \frac{\chi}{2} \sqrt{|\beta| T^{-1} - \frac{1}{2}}}{1 - \frac{\chi}{2} \sqrt{|\beta| T^{-1} + \frac{1}{2}}},$$

(84)

Now we have everything to plot the Hubble diagram for our model. The Figure 4 shows the redshift–magnitude dependence in terms of the distance modulus (Equation (71)) with overplotted data taken from (Riess et al. 2004). It is seen that even in this particular case without concretizing the form of the function $a(t)$ the Stephani model can in principle give an adequate interpretation of the observational data.

### 6. Conclusions

In this work, a particular case of the Stephani solution was investigated as possible model of the universe with accelerated expansion. The R-T-structure of the obtained spacetime was built, and it was shown that the central observer belongs to the R-region of permitted observers. In this model the boundary of the observable universe belongs to the T-region and expands with velocity exceeding the speed of light, as it is in standard Friedmann model. The correlation between the signs of the deceleration parameter and the acceleration of the universe expansion is absent in this model.

It is shown that the spatial sections of the universe are the Lobachevsky spaces. It turned out that the form of spatial section taken at present moment of time does not depend on the power of $a(t)$ in curvature function $\zeta(t)$.

It was established that the theoretical prediction for the redshift–magnitude relation in our model is in good accordance with SNe Ia observational data.

The obtained results serve as an evidence in favor of the possibility that our world in principle may be described by such model up to its recent stage without any harm from existing singularities. Another advantage of this approach is that it allows to stay within the general relativity with no need for modifications and introducing any exotic types of matter.
This paper is supported by the grant of the Plenipotentiary Representative of the Czech Republic in JINR under contract No. 208 from 02/04/2019. The authors acknowledge the Research Centre of Theoretical Physics and Astrophysics of the Faculty of Philosophy and Science, Silesian University in Opava for support. Z.S. acknowledges the Albert Einstein Centre for Gravitation and Astrophysics supported by the Czech Science Foundation grant No. 14-37086G. I.B. acknowledges the Silesian University in Opava grant SGS 12/2019. The authors cordially thank Maria Korkina for suggesting the problem and valuable discussion.

ORCID iDs
Elena Kopteva @ https://orcid.org/0000-0001-8364-0481

References
Balcerzak, A., Dabrowski, M., & Denkiewicz, T. 2015, PhRvD, 91, 083506
Celerier, M.-N. 2000, A&A, 353, 63
Dabrowski, M. 1993, JMP, 34, 1447
Dabrowski, M. 1995, ApJ, 447, 43
Dabrowski, M., & Hendry, M. 1998, ApJ, 498, 67
Dubrovin, B., Fomenko, A., & Novikov, S. 1992, Modern Geometry-Methods and Applications. Part I. The Geometry of Surfaces, Transformation Groups, and Fields (New York: Springer)
Ellis, G. F. R. 2009, GRG, 41, 581
Gregoritis, D., Ong, Y. C., & Wang, B. 2019, arXiv:1906.02879
Hashemi, S. S., Jalalzadeh, S., & Riazi, N. 2014, EPJC, 74, 2995
Hinshaw, G., Larson, D., Komatsu, E., et al. 2013, ApJS, 208, 19
Korkina, M. P., & Kopteva, E. M. 2016, arXiv:1604.08247
Korkina, M. P., Kopteva, E. M., & Egurnov, A. A. 2016, RvPhJ, 59, 328
Krasinski, A. 1983, GReGr, 15, 673
Misner, C. W., Thorne, K. S., & Wheeler, J. A. 1973, Gravitation (San Francisco, CA: Freeman)
Novikov, I. D. 2001, GReGr, 33, 2259
Ong, Y. C., Hashemi, S. S., An, R., & Wang, B. 2018, EPJC, 78, 405
Riess, A., Strolger, L.-G., Tonry, J., et al. 2004, ApJ, 607, 665
Stelmach, J., & Jakacka, I. 2001, COG, 18, 2643
Stelmach, J., & Szydłowski, M. 2004, arXiv:astro-ph/0403534v1
Stephani, H. 1967, CMaPh, 4, 137
Stephani, H., & Kramer, D. 2003, Exact Solutions of Einstein’s Field Equations (Cambridge: Cambridge Univ. Press)
Sussman, R. 1987, JMP, 28, 1118
Sussman, R. 1988a, JMP, 29, 945
Sussman, R. 1988b, JMP, 29, 1177
Sussman, R. 2000, GReGr, 32, 1527
Weinberg, S. 1989, RvMP, 61, 1