A NOTE ON THE KUZNETSOV COMPONENT OF THE VERONESE DOUBLE CONE

MARIN PETKOVIĆ AND FRANCO ROTA

Abstract. This note describes moduli spaces of complexes in the derived category of a Veronese double cone \( Y \). Focusing on objects with the same class \( \kappa_1 \) as ideal sheaves of lines, we describe the moduli space of Gieseker stable sheaves and show that it has two components. Then, we study the moduli space of stable complexes in the Kuznetsov component of \( Y \) of the same class, which also has two components. One parametrizes ideal sheaves of lines and it appears in both moduli spaces. The other components are not directly related by a wall-crossing: we show this by describing an intermediate moduli space of complexes as a space of stable pairs in the sense of Pandharipande and Thomas.

1. Introduction

The notion of a stability condition on a triangulated category was introduced by Bridgeland in [Bri07]. Constructing stability conditions is already a very challenging problem, even when the triangulated category is \( D^b(X) \), the derived category of coherent sheaves on a smooth projective variety. Stability conditions are classified completely only in the case of curves [Oka06, BMW15, Bri07, Mac07]. A connected component is described in [Bri08] for K3 surfaces, and a general construction on surfaces is given in [AB13], alternative constructions appear in [TX17, Tod13]. In dimension three, in a series of remarkable papers, stability conditions have been constructed on Fano threefolds [Mac14b, BMT14, Li19b, Piy17, BMSZ17], abelian threefolds [BMS16, MP15, MP16], quintic threefolds [Li19a].

On the other hand, having stability conditions allows one to consider moduli spaces of stable complexes and to study their properties, like non-emptiness, irreducibility, or projectivity. Moduli spaces of stable complexes can sometimes be used to study more classical spaces associated to a variety: for example, Bridgeland stability detects the birational geometry of Hilbert schemes of points on \( \mathbb{P}^2 \) [ABCH13] and other surfaces [BC13], or of moduli spaces of sheaves on a K3 surface [BM14]. Other applications of stability conditions to classical surface questions are surveyed in [Bay18]. We also mention [GLHS18], where the authors use stability techniques to describe the components of the moduli space of quartic curves in \( \mathbb{P}^3 \).

In the case that a triangulated category \( \mathcal{D} \) has an exceptional collection, Bayer, Lahoz, Macrì, and Stellari [BLMS17] give a general criterion to induce a stability condition on the right orthogonal of the collection, starting from a weak stability on \( \mathcal{D} \). They apply their result to cubic fourfolds and Fano threefolds of Picard rank 1.

The derived category of a cubic fourfold \( X \) is expected to play an important role in the question of its rationality. More precisely, \( D^b(X) \) admits an exceptional collection with an orthogonal complement \( \text{Ku}(X) \) [Kuz10, AT14], and it is conjectured that \( X \) is rational if and only if \( \text{Ku}(X) \) is the derived category of a K3 surface [Kuz10, Conjecture 1.1]. This motivates the study of moduli spaces of complexes in \( \text{Ku}(X) \) and their relation to the geometry of \( X \). In this direction, the authors of [BLMS17] show that the Fano variety of lines of a very general \( X \), denoted \( F(X) \), can be realized as a moduli space of stable complexes in \( \text{Ku}(X) \). They use this to obtain a reconstruction result: since \( F(X) \) determines \( X \), it follows that \( \text{Ku}(X) \) also determines \( X \) (similar theorems are called categorical Torelli theorems). If moreover \( X \) does not contain a plane, [LLSvS17] construct a hyperkähler eightfold starting from twisted cubics on \( X \), which also arises as a moduli space of complexes of \( \text{Ku}(X) \) as shown in [LLMS18]. The paper [LPZ18] recovers these results without the generality assumptions.

Categorical Torelli theorems also hold for Enriques surfaces [LNSZ19, LSZ21], where they are proven without using stability conditions.
A smooth Fano threefold $Y$ of Picard rank 1 also admits a similar triangulated subcategory $\text{Ku}(Y)$ [Kuz09] (called the Kuznetsov component of $Y$). If $Y$ has index 2, i.e. $K_Y \sim -2H$ (we denote by $H$ the ample generator of $\text{Pic}(Y)$), then it belongs to one of five families, indexed by their degree $d := H^3 \in \{1, \ldots, 5\}$ [Isk77]. We recall the classification:

- $Y_5 = \text{Gr}(2, 5) \cap \mathbb{P}^6 \subset \mathbb{P}^9$, a linear section of the Grassmannian $\text{Gr}(2, 5)$;
- $Y_4 = Q_1 \cap Q_2 \subset \mathbb{P}^6$, the intersection of two quadric hypersurfaces;
- $Y_3 \subset \mathbb{P}^4$, a cubic hypersurface;
- $Y_2 \rightarrow \mathbb{P}^3$, a double cover ramified over a quartic surface;
- $Y_1$, a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$ (also called a Veronese double cone, see Section 3).

In all cases, $\text{Ku}(Y)$ is right orthogonal to a sequence of line bundles. More explicitly, there is a decomposition

$$D(Y) = \langle \text{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle,$$

where

$$\text{Ku}(Y) := \{ E \in D(Y) \mid \text{Hom}_{D(Y)}(\mathcal{O}_Y, E[i]) = \text{Hom}_{D(Y)}(\mathcal{O}_Y(1), E[i]) = 0 \text{ for all } i \in \mathbb{Z} \}.$$

The numerical Grothendieck group $N(\text{Ku}(Y)) \subset N(D^b(Y))$ is a rank 2 lattice generated by the classes

$$\kappa_1 = 1 - \frac{H^2}{d} \quad \text{and} \quad \kappa_2 = H - \frac{H^2}{2} - \frac{(6 - d)H^3}{6d},$$

(we will sometimes abuse notation and use $H$ to indicate the class $[H]$ in the numerical Grothendieck group and the Chern character $\text{ch}(H)$ in the cohomology ring).

The papers [APR19] and [PY20] study moduli spaces of objects in $\text{Ku}(Y)$, and as an application obtain categorical Torelli theorems. Ideal sheaves of lines on $Y$ have class $\kappa_1$, and Pertusi–Yang [PY20] show that for $d \geq 2$ the moduli space of stable objects of class $\kappa_1$ is isomorphic to the Hilbert scheme of lines, which is an irreducible surface, smooth for $d \geq 3$, and generically smooth for $d = 2$. They use this to give an alternative proof of the Torelli theorem for cubic threefolds, which appeared in [BMMS12]. The paper [APR19] describes $\mathcal{M}_\sigma(\kappa_2)$ for all degrees and general $Y$, and obtains a categorical Torelli theorem for the degree 2 case, strengthening a result of [BT16].

Categorical Torelli theorems are now known to hold for degrees 2 or higher: $Y_5$ is unique up to isomorphism [Isk77], while the case $d = 4$ is treated in [Rei72, BO95]. The question of whether $\text{Ku}(Y)$ determines $Y$ is still open in degree 1: the proof techniques for degree 2,3 use that the stability condition on $\text{Ku}(Y)$ has a two-dimensional heart, as does the categorical Torelli theorem for cubic fourfolds in [BLMS17]. However, this is not the case for $d = 1$ (see Remark 4.5), and a different technique is needed.

The recent [Qin21] considers moduli spaces of complexes of class $2\kappa_1$ to study instanton bundles on $Y$.

**Summary of the results.** In this work we study moduli spaces of objects of class $\kappa_1$ when $Y$ is general of degree 1.

**Hilbert scheme of lines.** In the degree 1 case, the first multiple of $H$ to be very ample is $3H$: this gives rise to an extra component in moduli, in analogy with the case of twisted cubics in projective space [PS85]. Hilbert scheme $\text{Hilb}(Y, t + 1)$ coincides with a moduli space of stable sheaves (Prop. 3.5) and it has two irreducible components $\mathcal{M}_1$ and $\mathcal{M}_2$, parametrizing smooth lines and genus 1 curves union a point, respectively (Theorem 3.7).

**Moduli of complexes in $\text{Ku}(Y)$.** Two irreducible components also appear in moduli spaces of complexes in $\text{Ku}(Y)$. Combined with the rotation autoequivalence introduced in [Kuz15, Sec. 3.3], a construction of [PY20] induces isomorphisms of moduli spaces of $\sigma$-stable complexes in $\text{Ku}(Y)$ of different classes. When applied to the Veronese double cone, this isomorphism identifies $\mathcal{M}_\sigma(\kappa_1)$ with $\mathcal{M}_\sigma(\kappa_2)$, which has been studied in [APR19] and has a component $\mathcal{M}_3$ isomorphic to $Y$ itself. This is proven in Theorem 4.6, which also embeds the component $\mathcal{M}_1$ in the moduli space $\mathcal{M}_\sigma(\kappa_1) \simeq \mathcal{M}_1 \cup \mathcal{M}_3$. 
Moduli of tilt-stable complexes and Pandharipande-Thomas stable pairs. The work [PY20] relates $M_\sigma(\kappa_1)$ to the space of Gieseker-stable sheaves by deforming (weak) stability conditions and studying wall-crossing. The authors show that the moduli spaces $M^0_{\alpha,\beta}(\kappa_1)$ and $M_{\alpha,\beta}(\kappa_1)$ (associated to the interpolating stability conditions $\sigma_{\alpha,\beta}$ and $\sigma^0_{\alpha,\beta}$, see Sec. 2) are all isomorphic to $M_\sigma(\kappa_1)$. This happens in [APR19] (for class $\kappa_2$) and in [BLMS17] (in the context of cubic fourfolds) as well.

While the same strategy is viable here, the additional components complicate the picture and we no longer have the isomorphisms above. Theorem 5.1 classifies objects that are stable for $\sigma_{\alpha,\beta}$ and $\sigma^0_{\alpha,\beta}$, and Theorem 6.1 identifies $M^0_{\alpha,\beta}$ with a moduli space of stable pairs as studied by Pandharipande and Thomas [PT09].

As a consequence of Theorem 5.1, we give an additional interpretation of $\sigma^0_{\alpha,\beta}$-semistable complexes as quotients of $O_Y$ of class $\kappa_1$, in a perverse (repeatedly tilted) heart on $Y$ (Prop. 6.3).

Summary. There is a more precise description of $M^0_{\alpha,\beta}(\kappa_1)$ in Theorem 6.1. $M^0_{\alpha,\beta}(\kappa_1)$ has three irreducible components, it contains

$$\text{Hilb}(Y, t + 1) \simeq M_G(\kappa_1) \simeq M_{\alpha,\beta}(\kappa_1)$$

(with irreducible components $M_1$ and $M_2$), and the third component $M_3$ is a blow-up of $M_3 \simeq Y \subset M_{\sigma}(\kappa_1)$. This can be recollected in the diagram:

$$M_1 \cup \tilde{M}_3 \leftarrow M^0_{\alpha,\beta}(\kappa_1) \leftarrow M_1 \cup M_2 \simeq \text{Hilb}(Y, t + 1)$$

$$M_1 \cup M_3 \simeq M_\sigma(\kappa_1)$$

Structure of the paper. After introducing preliminary notions in Section 2, we study the Hilbert scheme of lines on a Veronese double cone in Section 3. Section 4 is dedicated to the description of the moduli space $M_\sigma(\kappa_1)$, and Section 5 contains the classification of semistable objects for the interpolating weak stability conditions, and Section 6 contains the description of the moduli space of stable pairs.

Acknowledgements. We are grateful to Laura Pertusi, Song Yang, Aaron Bertram and Arend Bayer for the fruitful discussions on these topics.

2. Preliminaries

2.1. Stability conditions. Here we give a short review of Bridgeland stability condition, with the main purpose of fixing the notation for what follows. We direct the interested reader to the seminal work of Bridgeland [Bri07] and to the survey [MS17] and references therein for a thorough description.

Definition 2.1. Let $\mathcal{A}$ be an abelian category. A (weak) stability function is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that

$$\Im Z(E) > 0 \text{ or } \Im Z(E) = 0 \text{ and } \Re Z(E) < (\leq) 0$$

for any $0 \neq E \in \mathcal{A}$. To a (weak) stability function $Z$, we associate a slope function

$$\mu(E) = \begin{cases} \frac{-\Re Z(E)}{\Im Z(E)} & \text{if } \Im Z(E) \neq 0 \\ +\infty & \text{otherwise} \end{cases}$$

We say that $E \in \mathcal{A}$ is stable if for all quotients $E \twoheadrightarrow F$ in $\mathcal{A}$ we have

$$\mu(E) < \mu(F).$$

Similarly, $E$ is said to be semistable if only the non-strict inequality $\mu(E) \leq \mu(F)$ holds.

Definition 2.2. Let $\mathcal{T}$ be a triangulated category and $v : K(\mathcal{T}) \to \Lambda$ a surjection to a finite rank lattice. A (weak) stability condition on a triangulated category $\mathcal{T}$ (with respect to $v : K(\mathcal{T}) \to \Lambda$) is a pair $\kappa = (\mathcal{A}, Z)$ consisting of

- a heart of a bounded $t$-structure $\mathcal{A}$
- a (weak) stability function $K(\mathcal{A}) \to \Lambda \overset{Z}{\to} \mathbb{C}$

satisfying the following properties:

(i) (Harder-Narasimhan filtration) Any $E \in \mathcal{A}$ has a filtration in $\mathcal{A}$ with semistable quotients with decreasing slopes.
(ii) (Support property) There exists a quadratic form $Q$ on $\Lambda \otimes \mathbb{R}$ which is negative definite on $\ker Z$ and for all semistable $E \in \mathcal{A}$ we have $Q(E) \geq 0$.

We say an object $E \in \mathcal{T}$ is $\sigma$-(semi)stable if $E[k] \in \mathcal{A}$ for some $k \in \mathbb{Z}$ and $E[k]$ is semistable with respect to $Z$.

**Definition 2.3.** Let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition on $\mathcal{T}$. For $\beta \in \mathbb{R}$, we define subcategories $\mathcal{A}_{\mu, \leq \beta}$ and $\mathcal{A}_{\mu, > \beta}$ consisting of objects $E$ such that slopes of all Harder-Narasimhan factors of $E$ are $\leq 0$ and $> 0$ respectively. The tilt of $\mathcal{A}$ is then defined as the extension closure of $\mathcal{A}_{\mu, \leq \beta}$ and $\mathcal{A}_{\mu, > \beta}$ and denoted

$$\mathcal{A}_{\beta}^\sigma = \left[ \mathcal{A}_{\mu, \leq \beta}[1], \mathcal{A}_{\mu, > \beta} \right].$$

That is, objects $E \in \mathcal{A}_{\beta}^\sigma$ are complexes with

$$\mathcal{H}_\mathcal{A}^{-1}(E) \in \mathcal{A}_{\mu, \leq \beta},$$

$$\mathcal{H}_\mathcal{A}^0(E) \in \mathcal{A}_{\mu, > \beta},$$

$$\mathcal{H}_\mathcal{A}^i(E) = 0, \text{ for } i \neq -1, 0.$$

For a smooth projective variety $Y$ with a hyperplane class $H$, let the lattice $\Lambda = \mathbb{Z}^{\beta, Y}$ be the image of the map $v = (H^3 \text{ch}_0, H^2 \text{ch}_1, H \text{ch}_2) : K(Y) \to \mathbb{Q}^3$. In this paper, we will be working with the following weak stability conditions on $D^b(Y)$:

**2.1.1. Slope stability.** $\sigma_{\mu} = (\text{Coh}(Y), -H^2 \text{ch}_1 + i H^3 \text{ch}_0)$ is a weak stability condition with respect to the rank 2 lattice defined as the image of $v = (H^3 \text{ch}_0, H^2 \text{ch}_1) : K(Y) \to \mathbb{Z}^2$.

This stability is also called Mumford stability, or slope stability. We will denote the corresponding slope function with $\mu_{\mu}$.

**2.1.2. Tilt-stability.** $\sigma_{\alpha, \beta} = (\text{Coh}^\beta(Y), Z_{\alpha, \beta})$, for $\alpha > 0$ and $\beta \in \mathbb{R}$, where

$$\text{Coh}^\beta(Y) = \left[ \text{Coh}(Y)_{\mu, \leq \beta}[1], \text{Coh}(Y)_{\mu, > \beta} \right]$$

and

$$Z_{\alpha, \beta}(E) = -H \text{ch}^\beta_3 E + \frac{\alpha^2}{2} H^3 \text{ch}_0 E + i \left( H^2 \text{ch}_1 E - \beta H^3 \text{ch}_0 E \right).$$

Here, $\text{ch}^\beta(-) := e^{-\beta \mathcal{H}} \cdot \text{ch}(-)$ is the twisted Chern character. This is a weak stability condition with respect to the lattice $\Lambda$ ([BMT14, BMS16]), and is usually called tilt-stability. The corresponding slope function will be denoted with $\mu_{\alpha, \beta}$. The quadratic form satisfying the support property is [BMT14, Cor. 7.3.2]:

$$Q(E) = (H^2 \text{ch}^\beta_1(E)) - 2(H \text{ch}^\beta_2(E))(H^3 \text{ch}_0(E)).$$

For a class $w \in \Lambda$, the half-plane $\{ (\alpha, \beta) \mid \alpha > 0, \beta \in \mathbb{R} \}$ admits a wall-and-chamber decomposition:

**Definition 2.4.** A numerical wall with respect to $w \in \Lambda$ is the solution set in $\{ (\alpha, \beta) \mid \alpha > 0, \beta \in \mathbb{R} \}$ of an equation $\mu_{\alpha, \beta}(w) = \mu_{\alpha, \beta}(u)$ for some $u \in \Lambda$.

A subset of a numerical wall for $w$ is an actual wall if there exists a short exact sequence of semistable complexes in $\text{Coh}^\beta(Y)$, $0 \to F \to E \to G \to 0$, with $v(E) = w$ and $v(F)$ defining the numerical wall.

Walls of tilt-stability satisfy Bertram’s Nested Wall Theorem (first proven for surfaces in [Mac14a]). In particular:

**Theorem 2.5 ([Sch20, Theor. 3.3]).** Fix $w \in \Lambda$.

- numerical walls are nested semicircles centered on the $\beta$-axis, except for possibly one, which is a half-line with constant $\beta$;
- if two numerical walls intersect, then they coincide;
- if a point of a numerical wall is an actual wall, then the whole numerical wall is an actual wall.

We then define chambers as connected components of complements of actual walls. If $\alpha$, $\beta$ belong to the same chamber, then an object $E$ of class $w$ is $\sigma_{\alpha, \beta}$-semistable if and only if it is $\sigma_{\alpha', \beta'}$-semistable.
2.1.3. Rotation of tilt-stability. \( \sigma^0_{\alpha,\beta} = (\text{Coh}^0_{\alpha,\beta}(Y), Z^0_{\alpha,\beta}) \), for \( \alpha > 0 \) and \( \beta \in \mathbb{R} \), where

\[
\text{Coh}^0_{\alpha,\beta}(Y) = \left[ \text{Coh}^\beta(Y)_{\mu_{\alpha,\beta} \leq 0}[1], \text{Coh}^\beta(Y)_{\mu_{\alpha,\beta} > 0} \right]
\]

and

\[
Z^0_{\alpha,\beta}(E) = -iZ_{\alpha,\beta}(E)
\]

This is also a weak stability condition with respect to \( \Lambda \) ([BLMS17, Prop 2.15]). The corresponding slope function will be denoted with \( \mu^0_{\alpha,\beta} \).

Like for tilt-stability, one can define walls and chambers for \( \sigma^0_{\alpha,\beta} \) by replacing \( \mu_{\alpha,\beta} \) with \( \mu^0_{\alpha,\beta} \) and \( \text{Coh}^\beta(Y) \) with \( \text{Coh}^0_{\alpha,\beta}(Y) \) in Definition 2.4.

2.2. Kuznetsov component. Let \( Y \) be a smooth Fano threefold of index 2, and Picard rank 1. The derived category of \( Y \) admits a semi-orthogonal decomposition

\[
D(Y) = \langle \text{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle
\]

where the admissible subcategory \( \text{Ku}(Y) \) is called the Kuznetsov component [Kuz08]. The numerical Grothendieck group \( \mathcal{N}(\text{Ku}(Y)) \subset \mathcal{N}(D^b(Y)) \) has rank 2 and is generated by the classes

\[
\kappa_1 = [I] = 1 - \frac{H^2}{d} \quad \& \quad \kappa_2 = H - \frac{H^2}{2} - \frac{(6 - d)H^3}{6d}.
\]

In this basis, the Euler form writes

\[
\begin{pmatrix}
-1 & -1 \\
1 & -d & -d
\end{pmatrix}.
\]

It is negative definite, and if \( d = 1 \) the only \(-1\) classes are \( \pm \kappa_1, \pm \kappa_2 \) and \( \pm (\kappa_1 - \kappa_2) \).

Recall that for \( E \in D^b(Y) \) exceptional, the left mutation \( L_E(-) \) across \( E \) is the functor sending \( G \in D^b(Y) \) to the cone of the evaluation map \( \text{ev} \):

\[
\mathbb{R} \text{Hom}(E, G) \otimes E \xrightarrow{\text{ev}} G \to L_E(G).
\]

The inclusion \( \text{Ku}(Y) \subset D^b(Y) \) has an adjoint projection functor \( \pi := \mathbb{L}_{\mathcal{O}_Y} \circ \mathbb{L}_{\mathcal{O}_Y(1)} \).

The category \( \text{Ku}(Y) \) admits an autoequivalence called the rotation functor

\[
\mathbb{R}(-) := \mathbb{L}_{\mathcal{O}_Y}(- \otimes \mathcal{O}_Y(1)),
\]

and a Serre functor. In fact, the two are related:

**Lemma 2.6.** The Serre functor on \( \text{Ku}(Y) \) satisfies

\[
S^{-1}_{\text{Ku}(Y)} \simeq \mathbb{R}^2[-3].
\]

**Proof.** By [Kuz14, Lemma 2.7], we have that \( S^{-1}_{\text{Ku}(Y)} \simeq \pi \circ S^{-1}_Y \). It is then straightforward to check that

\[
\pi S^{-1}_Y(E) = \pi(E(2))[-3] = \mathbb{L}_{\mathcal{O}}(\mathbb{L}_{\mathcal{O}(1)}(E(2)))[-3] \simeq \mathbb{R}^2(E)[-3]. \qed
\]

One of the results of [BLMS17] is that \( \text{Ku}(Y) \) supports stability conditions. Define the set

\[
V = \left\{ (\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R} \mid 0 < \alpha < \min\{-\beta, \beta + 1\}, -1 < \beta < 0 \right\}
\]

then we have

**Theorem 2.7** ([BLMS17, Theor. 6.8]). For any \((\alpha, \beta) \in V\), the weak stability condition \( \sigma^0_{\alpha,\beta} \) from Section 2.1.3. induces a Bridgeland stability condition \( \sigma(\alpha, \beta) \) on \( \text{Ku}(Y) \), with heart given by

\[
\mathcal{A} := \text{Coh}^0_{\alpha,\beta}(Y) \cap \text{Ku}(Y)
\]

and central charge \( Z^0_{\alpha,\beta}(\mathcal{A}) \). We will denote the slope function of \( \sigma(\alpha, \beta) \) with \( \mu(\alpha, \beta) \).

The set of stability conditions on \( \text{Ku}(Y) \) is denoted \( \text{Stab(\text{Ku}(Y))} \), it is a complex manifold and it admits the following group actions:
The universal cover \( \tilde{\GL}_2^+ (\mathbb{R}) \) acts on the right: an element of \( \tilde{\GL}_2^+ (\mathbb{R}) \) is a pair \( \tilde{g} = (g, M) \) where \( g : \mathbb{R} \to \mathbb{R} \) is increasing and such that \( g(\phi + 1) = g(\phi) + 1 \), and \( M \in \GL_2 (\mathbb{R}) \). Given a stability condition \( \sigma (Z, \mathcal{P}) \in \text{Stab}(\text{Ku}(Y)) \), we define \( \sigma \cdot \tilde{g} = (Z', \mathcal{P}') \) to be the stability condition with \( Z' = M^{-1} \circ Z \) and \( \mathcal{P}'(\phi) = \mathcal{P}(g(\phi)) \). Stability is preserved under this action: an object \( E \in \text{Ku}(Y) \) is \( \sigma \)-stable if and only if it is \( \sigma \cdot \tilde{g} \)-stable for all \( \tilde{g} \in \tilde{\GL}_2^+ (\mathbb{R}) \).

An autoequivalence \( \Phi \) of \( T \) acts on the left: for \( \sigma \) as above we set
\[
\Phi \cdot \sigma := (Z(\Phi^{-1}_*(-)), \Phi(\mathcal{P})),
\]
where \( \Phi_* \) is the automorphism of \( K(\text{Ku}(Y)) \) induced by \( \Phi \).

Pick \( 0 < \alpha < \frac{1}{2} \), and let \( K \) denote the \( \tilde{\GL}_2^+ (\mathbb{R}) \)-orbit of the stability condition \( \sigma (\alpha, -\frac{1}{2}) \) in \( \text{Stab}(\text{Ku}) \). Then we have:

**Proposition 2.8** ([PY20, Prop. 3.6]). For all \( (\alpha, \beta) \in V, \sigma (\alpha, \beta) \in K \).

Another result of [PY20] is the following:

**Proposition 2.9** ([PY20, Prop. 5.7]). If \( Y \) is a Fano threefold of Picard rank 1 and index 2, then there exists \( \tilde{g} \in \tilde{\GL}_2^+ (\mathbb{R}) \) such that
\[
R \cdot \sigma (\alpha, -\frac{1}{2}) = \sigma (\alpha, -\frac{1}{2}) \cdot \tilde{g}.
\]

For \( \sigma \in K \) and \( \kappa \in N(\text{Ku}(Y)) \), we write \( M_\sigma (\kappa) \) the moduli space of \( \sigma \)-stable objects of class \( \kappa \) in \( \text{Ku}(Y) \). As an immediate consequence of Prop. 2.9 we have:

**Corollary 2.10.** For all \( n \in \mathbb{Z} \), there is an isomorphism
\[
M_\sigma (\kappa) \simeq M_\sigma (\mathbb{R}^n \kappa).
\]

### 3. Lines on a Veronese double cone

#### 3.1. Veronese double cones.

We fix some notation and recall some general results on Veronese double cones, following [Isk77] and [HK15]. Let \( Y \) be a hypersurface cut out by a sextic equation in the weighted projective space \( \mathbb{P} := \mathbb{P}(1, 1, 1, 2, 3) \). Let \( x_0, \ldots, x_4 \) be coordinates of \( \mathbb{P} \), with \( x_3 \) and \( x_4 \) those of weight 2,3 respectively. By completing a square, we can write the equation for \( Y \) as \( x_2^2 = f_6(x_0, \ldots, x_3) \) where \( f_6 \) is a degree 6 polynomial. The linear series \( H := \mathcal{O}_Y(1) \) has three sections and a unique base point \( y_0 \), hence it induces a rational map \( \phi_H : Y \to \mathbb{P}(H^0(\mathcal{O}_Y(1))) \simeq \mathbb{P}^2 \). On the other hand, \( 2H \sim -K_Y \) is base point free, and induces a morphism \( \phi_{2H} : Y \to \mathbb{P}(H^0(\mathcal{O}_Y(2))) \simeq \mathbb{P}^6 \), whose image \( K \simeq \mathbb{P}(1, 1, 1, 2) \) is the cone over a Veronese surface with vertex \( k := \phi(y_0) \).

More precisely, for \( V := H^0(\mathcal{O}_Y(1)) \) we have
\[
H^0(\mathcal{O}_Y(2)) = \text{Sym}^2 V \oplus \langle x_3 \rangle,
\]
and the map
\[
i : V \oplus \langle x_3 \rangle \to \text{Sym}^2 V \oplus \langle x_3 \rangle
\]

\[
(v, r) \mapsto (v^2, r)
\]

embeds \( \mathbb{P}(V \oplus 0) \) as a Veronese surface and \( K \) as the cone over \( \mathbb{P}(V) \) and vertex \( k = \mathbb{P}(0 \oplus \langle x_3 \rangle) \).

The morphism \( \phi_{2H} \) is smooth of degree 2 outside \( k \) and the divisor \( W := \{f_6 = 0\} \in |\mathcal{O}_K(3)| \). For this reason, \( Y \) is often referred to as to a Veronese double cone. We will denote by \( \iota \) the involution on \( Y \) corresponding to the double cover \( \phi_{2H} \).

There is a commutative diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\phi_{2H}} & K \\
\phi_H \downarrow & \Downarrow \eta & \downarrow \\
\mathbb{P}^2 & \simeq & \mathbb{P}^2
\end{array}
\]
where $\eta$ is the projection from $k$. Consider the blowup $\sigma_K : \tilde{K} \to K$ of the vertex $k$ with exceptional divisor $E$. Then, the blow-up $\tilde{Y} = Y \times_K \tilde{K}$ resolves the indeterminacy of diagram (4):

$$
\begin{array}{ccc}
Y & \xrightarrow{\phi} & K \\
\sigma_Y & \downarrow & \sigma_K \\
\tilde{Y} & \xrightarrow{\tilde{\phi}} & \tilde{K}
\end{array}
$$

where $\tilde{\phi} : \tilde{Y} \to \tilde{K}$ is a degree 2 cover ramified over the divisor $E \cup \sigma_K^{-1}(W)$.

The map $\eta$ restricted to $W$ is a 3-to-1 cover of $\mathbb{P}^2$, and it ramifies at a curve $C_0$. Throughout this section, we assume that $Y$ is smooth and that $C_0$ is irreducible and general in moduli (this is the generality assumption used in [Tih82], whose results we will use).

3.2. Stable sheaves of class $\kappa_1$ on $Y$. Let $M_G(v)$ denote the moduli space of stable sheaves of class $v$ on $Y$. Objects in $M_G(v)$ are related to subschemes of $Y$ with Hilbert polynomial $t + 1$, we start by studying those.

**Definition 3.1.** A line in $Y$ is a smooth subscheme of pure dimension 1 with Hilbert polynomial $t + 1$.

In particular, for every line $L$ we have $H.L = 1$. We say that a curve $C \subset Y$ has degree $k$ if $H.C = k$: thus, lines are rational curves of degree 1 in $Y$.

A similar definition holds for lines and conics in $K$: let $j : K \to \mathbb{P}^6$ the embedding induced by the map $i$ of Eq. (3). We use the notation $K^\circ := K \setminus \{k\}$.

**Definition 3.2 ([HK15, Def. 3.1]).** A curve $C$ in $K$ is a line (resp. a conic) if the closure of its image $j(C \cap K^\circ)$ is a line (resp. a conic) in $\mathbb{P}^6$.

Lines and conics in $K$ are described in [HK15, Sec. 3]. Lines in $K$ are the closure of fibers of the projection $\eta : K^\circ \to \mathbb{P}^2$, conics of $K$ are smooth (in which case they do not contain the vertex $k$), or the union of two lines (possibly doubled). For the rest of the section, we use the shorthand $\phi := \phi_{2H}$.

**Lemma 3.3.** Let $C$ be a degree 1 curve in $Y$. Then the image $c := \phi(C)$ is a conic in $K$, which intersects $W$ in three (possibly coinciding) points with multiplicity 2. There are two possibilities:

- $p_a(C) = 0$: $c$ is a smooth conic in $K$. In this case, $C$ is a line, and $\phi^{-1}(C) = C \cup C'$ where $C'$ is also a line.
- $p_a(C) = 1$: $c$ is a doubled line. Then $C$ is a smooth curve of genus 1, or a singular rational curve.

**Proof.** Since $\phi$ is induced by the linear series $|2H|$, $c = \phi(C)$ must be a conic on $K$. Note first of all that if $c$ is reducible then so is $C$, but this is impossible since $C.H = 1$. Hence, $c$ is either smooth or a doubled line.

If $c$ is smooth and it intersects $W$ with odd multiplicity at a point, then $\phi^{-1}(c)$ must be irreducible of degree $> 1$. This is not the case as $C \subseteq \phi^{-1}(c)$. So $c$ is tritangent to $W$, and $\phi^{-1}(c) = C \cup C'$ is the union of two lines.

If $c = 2l$ is a doubled line with $l$ a line in the ruling of $K$, then the restriction of $\phi : \phi^{-1}(l) \to l$ is a covering map branched over the four points $(l \cap W) \cup k$. Since $k \notin W$, $\phi^{-1}(l)$ must be irreducible. If the points in $(l \cap W)$ are all distinct, then $C = \phi^{-1}(l)$ is a smooth elliptic curve. If two points of $l \cap W$ coincide, then $C$ has a double point. If all three coincide, $C$ has a cusp. \qed

We can now classify Gieseker-semistable sheaves of class $\kappa_1$:

**Proposition 3.4.** Semistable sheaves of class $\kappa_1$ on $Y$ are exactly ideal sheaves of subschemes $Z$ with Hilbert polynomial $\chi(\mathcal{O}_Z(t)) = t + 1$. There are three possibilities for $Z$:

(i) $Z$ is a line in $Y$;

(ii) $Z$ is a non-reduced scheme supported on a curve of degree 1 and genus 1 with an embedded point;

(iii) $Z$ is the union of a curve of degree 1 and genus 1 and a point which does not belong to the curve.

**Proof.** Ideal sheaves are torsion free of rank 1, and therefore stable. So, it suffices to show that a Gieseker-semistable sheaf $E$ is an ideal sheaf. This is a standard argument: since $Y$ is smooth, $E \to E' \otimes E'$ is injective and $E'$ is reflexive, so that $E' \simeq \mathcal{O}_Y(-D)$ for some divisor $D$. Therefore $E \otimes \mathcal{O}_Y(-D)$ is the ideal sheaf
of a subscheme supported in codimension 2. Then, \( E \simeq I_Z \otimes O_Y(D) \), and since \([E] = \kappa_1\) we must have \( D = 0 \) and \( \chi(O_Z(t)) = t + 1 \) (the Hilbert polynomial is that of \( O_L \) for \( L \) a smooth rational curve in \( Y \)).

The three possibilities for \( Z \) follow from the fact that \( H.Z_{\text{red}} = 1 \) is the degree of the Hilbert polynomial, so \( Z_{\text{red}} \) contains one of the curves described in Lemma 3.3. Then, the only possible cases are those listed, note moreover that all three can occur \([Tih82]\).

We will refer to the three possibilities listed in Proposition 3.4 as to subschemes of type (i), (ii), and (iii). Observe moreover that Proposition 3.4 implies the following:

**Proposition 3.5.** The moduli space \( M_G(\kappa_1) \) is isomorphic to the Hilbert scheme of lines \( \text{Hilb}(Y, t + 1) \).

**Proof.** We argue as in the proof of \([PT09, \text{Theorem 2.7}]\). Let \( I \) be a flat family of semistable sheaves of class \( \kappa_1 \) over a base \( B \), normalized so that it has trivial determinant along \( B \). The sheaf \( I \) has rank 1, and it is pure so it injects into its double dual

\[
0 \to I \to I^{\vee\vee}.
\]

Flatness of \( I \) implies that \( I^{\vee\vee} \) is locally free, and \( I^{\vee\vee} \) has trivial determinant since \( I \) does. Therefore, \( I^{\vee\vee} \simeq O_{Y \times B} \), and there is a short exact sequence

\[
0 \to I \to O_{Y \times B} \to Q \to 0,
\]

where \( Q \) is a flat family of quotients of \( O_{Y \times B} \). Conversely, any such family of quotients gives rise to a family of ideal sheaves as those listed in Prop. 3.4. This identifies the functors represented by \( M_G(\kappa_1) \) and \( \text{Hilb}(Y, t + 1) \).

**Remark 3.6.** As mentioned in the introduction, \( 3H \) is the smallest very ample multiple of \( H \). The embedding \( Y \to \mathbb{P}(H^0(O_Y(3H))) \) maps the Hilbert scheme \( \text{Hilb}(Y, t + 1) \) to that of twisted cubics, which has two irreducible components whose intersection parametrizes non-reduced subschemes \([CK11, \text{Sec. 3}]\).

We describe the Gieseker moduli space \( M_G(\kappa_1) \). We prove the theorem here, even if in the proof we apply Proposition 3.11, which is postponed to after some more technical computations:

**Theorem 3.7.** The moduli space \( M_G(\kappa_1) \) has two irreducible components \( M_1 \) and \( M_2 \).

\( M_1 \) is a smooth surface compactifying the locus of ideals of smooth lines of \( Y \). \( M_2 \) has dimension 5, and its general object is a subscheme of type (iii). It is smooth outside the intersection with \( M_1 \).

Points in \( M_1 \cap M_2 \simeq C_0 \) parametrize singular rational curves with an embedded point at the singularity.

**Proof.** The component \( M_1 \) parametrizing ideal sheaves of lines is described in \([Tih82, \text{Theorem 4}]\): \( M_1 \) is a smooth surface intersecting the rest of \( M_G(v) \) on the locus parametrizing singular curves with a nilpotent embedded at the singularity. This locus is isomorphic to the curve \( C_0 \).

There is a 5 dimensional family of schemes of type (iii) (two parameters determine the one dimensional component, and three determine the point). Denote by \( M_2 \) the component of \( M_G(v) \) containing this family. By Prop. 3.5, the tangent space of \( Z = C \cup p \) of type (iii) is

\[
T_Z M_2 \simeq \text{Hom}(I_Z, O_Z) \simeq \text{Hom}(I_C, O_C) \oplus \text{Hom}(I_p, O_p).
\]

The spaces in the right hand side parametrize deformations of \( C \) and \( p \), respectively, so \( \dim T_Z M_2 = 5 \). This shows that \( \dim M_2 = 5 \) and that \( M_2 \) is smooth at type (iii) points. Moreover, Proposition 3.11 shows that \( M_2 \) is smooth at points of type (ii) for which the nilpotent is supported on smooth points.

Finally, there are no other components in \( M_G(v) \), because we exhausted the possibilities in Proposition 3.4.

**Remark 3.8.** The component \( M_1 \) is sometimes denoted \( F(Y) \) and called the Fano surface of lines of \( Y \) (e.g. in \([Tih82]\)).

**Lemma 3.9.** Let \( C \) be a curve in \( Y \) of degree 1 and arithmetic genus \( p_a = 1 \). Then

\[
\begin{align*}
\text{Ext}^0(I_C, I_C) &= C \\
\text{Ext}^1(I_C, I_C) &= C^2 \\
\text{Ext}^4(I_C, I_C) &= 0 \text{ otherwise.}
\end{align*}
\]
Proof. The curve $C$ is cut out by the pull-back of two linear forms from $\mathbb{P}^2$ via $\eta: K^3 \to \mathbb{P}^2$, denote them $l, m$. In fact, the Koszul complex in $l$ and $m$ is exact on $Y$:

$$(5) \quad 0 \to \mathcal{O}_Y(-2) \xrightarrow{(m, -l)} \mathcal{O}_Y(-1)^{\oplus 2} \to IC \to 0.$$  

Applying the functor $\text{Hom}(-, IC)$ gives the map:

$$(6) \quad H^0(IC(1))^{\oplus 2} \simeq \text{Hom}(\mathcal{O}_Y(-1), IC)^{\oplus 2} \xrightarrow{(m, -l)} \text{Hom}(\mathcal{O}_Y(-2), IC) \simeq H^0(IC(2)).$$  

It is straightforward to check that the map (6) has rank 3, and the conclusion follows. \hfill \Box

Lemma 3.10. Let $Z$ be a subscheme of type (ii) with the embedded point in the smooth locus of $Z_{\text{red}}$. Then

$$\text{Ext}^i(Z, \mathcal{O}_p) = \begin{cases} \mathbb{C}^3 & \text{if } i = 0, 1 \\ \mathbb{C} & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, applying $\text{Hom}(-, \mathcal{O}_p)$ to the sequence

$$(7) \quad I_Z \to I_{Z_{\text{red}}} \to \mathcal{O}_p,$$

we get a non-zero homomorphism $\alpha: \text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) \to \text{Ext}^1(I_{Z_{\text{red}}}, \mathcal{O}_p)$. \hfill \Box

Proof. The groups $\text{Hom}^*(\mathcal{O}_p, \mathcal{O}_p)$ are the exterior algebra on the tangent space at $p$, so they have dimensions $1, 3, 3, 1$ for $* = 0, 1, 2, 3$. Applying the functor $\text{Hom}(-, \mathcal{O}_p)$ to the resolution (5) as in Lemma 3.9, we see that $\text{hom}^*(I_{Z_{\text{red}}}, \mathcal{O}_p) = 2, 1, 0, 0$ for $* = 0, 1, 2, 3$.

Apply $\text{Hom}(-, \mathcal{O}_p)$ to the sequence (7) and consider the corresponding long exact sequence: this shows immediately that

$$\text{ext}^2(I_Z, \mathcal{O}_p) = 1 \quad \text{ext}^3(I_Z, \mathcal{O}_p) = 0.$$  

On the other hand, we may consider a set of local coordinates around $p$ given as $\{l, m, s\}$, where $l, m$ define $Z_{\text{red}}$. Then, $l, m^2$, and $ms$ generate $I_Z$ locally around $p$. Resolving $I_Z$ using these generators we see that $\text{hom}^1(I_Z, \mathcal{O}_p) = 3$, arguing as above.

Finally, observe that $\chi(I_Z, \mathcal{O}_p) = \chi(I_Z, \mathcal{O}_q) = \chi(\mathcal{O}_Y, \mathcal{O}_q) = 1$ where $q \in Y \setminus Z_{\text{red}}$ (since this quantity only depends on the numerical class of $\mathcal{O}_p$), which implies that $\text{ext}^1(I_Z, \mathcal{O}_p) = 3$.

The map $\alpha$ appears in the long exact sequence, and a simple dimension count shows that it does not vanish. \hfill \Box

Proposition 3.11. If $Z$ is a subscheme of type (ii) with the embedded point in the smooth locus of $Z_{\text{red}}$, then

$$\text{ext}^1(I_Z, I_Z) = 5.$$  

Proof. We may write $I_Z \simeq [I_{Z_{\text{red}}} \to \mathcal{O}_p]$ where $p$ is the embedded point. Then, $R \text{Hom}(I_Z, I_Z)$ may be computed with the spectral sequence

$$(8) \quad E_1^{p, q} = H^q(K^{\bullet, p}) \Rightarrow H^{p+q}(K^{\bullet}).$$

The first page is

| $p = -1$ | $p = 0$ | $p = 1$ |
| --- | --- | --- |
| $\text{Ext}^1(\mathcal{O}_p, I_{Z_{\text{red}}})$ | $\text{Ext}^1(I_{Z_{\text{red}}}, I_{Z_{\text{red}}}) \oplus \text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p)$ | $\text{Ext}^1(I_{Z_{\text{red}}}, \mathcal{O}_p)$ |
| $\text{Hom}(\mathcal{O}_p, I_{Z_{\text{red}}})$ | $\text{Hom}(I_{Z_{\text{red}}}, I_{Z_{\text{red}}}) \oplus \text{Hom}(\mathcal{O}_p, \mathcal{O}_p)$ | $\text{Hom}(I_{Z_{\text{red}}}, \mathcal{O}_p)$ |

with arrows pointing to the right and zeros in all other columns. We claim that the dimensions of the vector spaces above are given by
Indeed, the third column (and hence, by Serre duality, the first one) is computed in the proof of Lemma 3.10.

The contributions from $\text{Hom}^\bullet(I_{Z_{red}}, I_{Z_{red}})$ in the central column follow from Lemma 3.9, while the dimensions of $\text{Hom}^\bullet(\mathcal{O}_p, \mathcal{O}_p)$ follow because $p$ is a smooth point of $Y$, as in the proof of Lemma 3.10.

Our next claim is that the maps in the middle rows are non-zero, and that the map in the bottom row has one-dimensional image. Granting the claim, the second page of the spectral sequence reads

\[
\begin{array}{c|ccc}
* & 1 & 0 & 0 \\
1 & 3 & 0 & 0 \\
0 & 2 + 3 & 1 & 0 \\
0 & 1 + 1 & 2 & 0 \\
\end{array}
\]

and hence $\text{ext}^1(I_Z, I_Z) = 5$.

The map on the second row from the top is $\text{Ext}^2(\mathcal{O}_p, I_{Z_{red}}) \to \text{Ext}^2(\mathcal{O}_p, \mathcal{O}_p)$. It is Serre dual to the homomorphism $\alpha$ (see Lemma 3.10), which is also the restriction to the second summand of the map on the third row:

\[
\text{Ext}^1(I_{Z_{red}}, I_{Z_{red}}) \oplus \text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) \to \text{Ext}^1(I_{Z_{red}}, \mathcal{O}_p).
\]

It follows from Lemma 3.10 that these two maps do not vanish. Finally, observe that the map

\[
\text{Hom}(I_{Z_{red}}, I_{Z_{red}}) \oplus \text{Hom}(\mathcal{O}_p, \mathcal{O}_p) \to \text{Hom}(I_{Z_{red}}, \mathcal{O}_p)
\]

has one-dimensional image (the span of the natural map $I_{Z_{red}} \to \mathcal{O}_p$ of (7)).

\[\square\]

4. Moduli spaces of objects of $\text{Ku}(Y)$

For the rest of this note, $Y$ will denote a general Veronese double cone (we will follow the notation of Section 3.1). When a result holds for all Fano threefolds of Picard rank 1 and index 2, we will make it explicit. In this section, we construct three families of objects of $\text{Ku}(Y)$ and show that they are related by a rotation. More precisely, we show that the set \( \{ \pm \kappa_1, \pm \kappa_2, \pm (\kappa_1 - \kappa_2) \} \) is an orbit of the action of $\mathbb{R}_+$ on $N(\text{Ku}(Y))$.

As a result, Corollary 2.10 yields an isomorphism of the corresponding moduli spaces.

We start by defining the three families of objects:

(A) For any Fano threefold $Y$ of Picard rank one, index 2, and degree $d$, we can consider projections of skyscraper sheaves to $\text{Ku}(Y)$: for $p \in Y$, the projection $\pi(C_p)$ of $C_p$ is the complex $M_p[1]$, defined as the cone

\[ O^d \to I_p(1) \to M_p. \]

We have $[M_p] = \kappa_2 - d\kappa_1$.

(B) A second family of objects are the complexes $E_p$ studied in [APR19]. They have class $\kappa_2$, and are defined by the distinguished triangle

\[ O_Y(-1)[1] \to E_p \to I_p \]

for any point $p \in Y$.

(C) Assume now that $Y$ has degree 1. Then, we can construct another class of objects as follows. For a point $p \in Y \setminus \{y_0\}$, let $x := \phi_H(p) \in \mathbb{P}^2$ and let $C := C_x$ be the corresponding genus 1 curve (notation as in Sec. 3). Then, $H^0(O_C(p)) = \mathbb{C}$, and we consider the cone of the triangle

\[ O_Y \to O_C(p) \to F_p. \]

Similarly, define complexes associated with $y_0$: for all $x \in \mathbb{P}^2$, $y_0 \in C_x$ and $H^0(O_{C_x}(y_0)) = \mathbb{C}$ as above, so we write

\[ O_Y \to O_{C_x}(y_0) \to G_x \]

for the corresponding cones.

Remark 4.1. - The numerical class of $F_p$ and $G_x$ is $-\kappa_1$. In fact, $O_C(p)$ (and $O_{C_x}(y_0)$) has the same Hilbert polynomial as $O_l$ for any line $l \subset Y$, so $[F_p] = [G_x] = -[I_l] = -\kappa_1$;
– the objects $F_p$ belong to $\text{Ku}(Y)$: the vanishing $\text{Hom}(O_Y(1), F_p) = 0$ follows from (10) and the observation that the sheaves $O_Y(-1)$ and $O_C(p - y_0)$ have no cohomologies. Similarly, the vanishing of $\text{Hom}(O_Y, F_p)$ follows from the isomorphism $R \text{Hom}(O_Y, O_Y) \simeq R \text{Hom}(O_Y, O_C(p))$; 

– on the other hand, the objects $G_x \notin \text{Ku}(Y)$. Note, in fact, that for any curve $C_x$ we have 
\[ O_{C_x} \otimes O_Y(1) \simeq O_{C_x}(y_0), \]

since if $D \in |O(1)|$ does not contain $C_x$ then $[D \cap C_x] = H^3$, and the only point of cohomology class $H^3$ is $y_0$. Then, by (11) we have
\[
\text{Hom}(O_Y(1), G_x) \simeq \text{Hom}(O_Y(1), O_{C_x}(y_0)) \simeq \text{Hom}(O_Y, O_{C_x}(y_0 - y_0)) = \mathbb{C}.
\]

The three classes of objects (A), (B), and (C) are related by rotations:

**Lemma 4.2.** We have $R(E_p) = M_p$ for every $p \in Y$. This holds for $Y$ of any degree.

**Proof.** Twist the defining sequence of $E_p$:
\[
O[1] \to E_p(1) \to I_p(1)
\]
and mutating across $O$ shows $R(E_p) \simeq L_{O_Y}(I_p(1))$. Then, observe that (9) computes $L_{O_Y}(I_p(1))$. \hfill □

Recall that $\iota : Y \to Y$ is the involution corresponding to the double cover $\phi_2 : Y \to K$. Then we have:

**Lemma 4.3.** For $p \neq y_0$, we have $R(M_p) = F_{\iota(p)}$.

**Proof.** By its definition, the cohomologies of $M_p(1)$ are those of the complex $[O^2(1) \xrightarrow{\phi_p} I_p(2)]$. The kernel of the evaluation map is $O_Y$, and the cokernel is the cokernel of the inclusion $I_C(2) \to I_p(2)$, which is $O_C(2y_0 - p)$, where $C := C_{\phi_2y_0}(p)$. This shows that $R(M_p) = L_{O_Y}(O_C(1) \cdot y_0 - p))$. One then checks that the divisor $2y_0 - p$ on $C$ is linearly equivalent to $\iota(p)$, by considering the Weierstrass equation for $C$ in $\mathbb{P}(1, 1, 1, 2, 3)$ and observing that taking inverses coincides with applying $\iota$. Therefore, $R(M_p) = F_{\iota(p)}$. \hfill □

From Lemmas 4.2 and 4.3 we get:

**Corollary 4.4.** The matrix associated to $R_x$ in the basis $\kappa_1, \kappa_2$ is \[ \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \]. In particular, $R_x$ acts transitively on the set \{ $\pm \kappa_1, \pm \kappa_2, \pm (\kappa_1 - \kappa_2)$\} of classes in $N(\text{Ku}(Y))$ with square $-1$.

**Proof.** By Lemma 4.2, we have $R_x(\kappa_2) = \kappa_2 - \kappa_1$, and by Lemma 4.3 we have $R_x(\kappa_2 - \kappa_1) = -\kappa_1$, the rest is straightforward. \hfill □

**Remark 4.5** (Homological dimension). The heart $A(\alpha, \beta)$ has homological dimension 2 if $d = 2, 3$ [PY20]. This is false in the case $d = 1$. In fact, by Lemmas 4.2 and 4.3 above we have $E_{\iota(p)} \simeq R^{-2}(F_p)$ for $p \neq y_0$ in $Y$. Then, by Serre duality and Lemma 2.6, 
\[
\text{Ext}^3(F_p, E_{\iota(p)}) \simeq \text{Hom}(F_p, R^{-2}(F_p)[3]) \simeq \text{Hom}(F_p, F_p)^* \neq 0.
\]

We now recollect the results of this section in the following theorem (we use the same notation $M_1$ for the copy of $F(Y)$ embedded as an irreducible component in $M_3(\kappa_1)$ (Theorem 3.7) and in $M_x(-\kappa_1)$):

**Theorem 4.6.** Let $Y$ be a general smooth Veronese double cone. The moduli spaces $M_p(\kappa_1)$, $M_p(-\kappa_2)$ and $M_p(\kappa_1 - \kappa_2)$ are isomorphic. They have two irreducible components $M_1$ and $M_3$ isomorphic respectively to the Fano surface of lines $F(Y)$ and to $Y$ itself, intersecting along $C_0$. The generic point of the component $Y$ parameterizes, respectively, objects of form $F_p$, $E_{\iota(p)}$ and $M_p$.

**Proof.** Corollaries 2.10 and 4.4 yield the isomorphism of moduli spaces. The description of the irreducible components is [APR19, Theor. 1.5]. The statement on the general objects follows again from Lemmas 4.2 and 4.3. \hfill □

We conclude the section describing the objects $F_{y_0}$, $M_{y_0}$, and $F_{y_0} := R^2(F_{y_0})$: these correspond to the point $y_0$ in the component $Y$ of the three moduli spaces of Theorem 4.6, and they are of a different nature from the others.
Proposition 4.7 (Rotations at $y_0$). We have $R(E_{y_0}) = M_{y_0}$, a complex with cohomologies

$$H^{-1}(M_{y_0}) \simeq \text{coker}(\mathcal{O}_Y(-2) \to \mathcal{O}_Y(-1)^{\oplus 3})$$

$$H^0(M_{y_0}) = \mathcal{O}_Y.$$

The complex $F_{y_0}$ has three cohomologies, and it fits in a triangle

$$0 \to H^{-1}(M_{y_0}) \to \mathcal{O}_Y^{\oplus 3} \xrightarrow{ev} \mathcal{O}_Y(1) \to H^0(M_{y_0}) \to \mathcal{O}_Y \to 0,$$

where the evaluation map $ev$ is surjective, and coincides with the last map of a Koszul complex on three linear forms. Therefore, $H^{-1}(M_{y_0}) \simeq \text{coker}(\mathcal{O}_Y(-2) \to \mathcal{O}_Y(-1)^{\oplus 3})$ and $H^0(M_{y_0}) = \mathcal{O}$.

To compute $F_{y_0} = R(M_{y_0})$, compute the cohomology sheaves of $M_p(1)$ by twisting (13), and write the cohomology sequence of the triangle

$$\mathcal{O}_Y^{\oplus 3} \oplus \mathcal{O}_Y[1] \to M_{y_0}(1) \to F_{y_0}.$$

It reads

$$0 \to \mathcal{O}_Y(-1) \to \mathcal{O}_Y^3 \to \text{coker}(\mathcal{O}_Y(-1) \to \mathcal{O}_Y^{\oplus 3}) \to H^{-1}(F_{y_0}) \to \mathcal{O}_Y^3 \to \mathcal{O}_Y(1) \to \mathcal{C}_y \to 0,$$

whence the claim. \hfill $\square$

5. Set-theoretic considerations

5.1. Stable complexes of class $\kappa_1$. In this section, we classify objects of class $\kappa_1$ that are semistable with respect to $\sigma_{\alpha,\beta}$ and $\sigma_{\alpha,\beta}$. Here, $\sigma$ denotes one of the stability conditions of Theorem 2.7.

Our classification shows that following the strategy of [APR19] and [PY20] to describe $M_\sigma(\kappa_1)$ is more difficult in this setting. In those works, moduli spaces of $\sigma$-stable objects are related via wall-crossing to moduli spaces of complexes which are stable with respect to $\sigma_{\alpha,\beta}$ and $\sigma_{0,\beta}$. More precisely, for $v = \kappa_2$, or $d > 1$ and $v = \kappa_1$, the three notions of stability coincide, and we have

$$M_\sigma(v) \simeq M_{\sigma_{\alpha,\beta}}(v) \simeq M_{\sigma_{\alpha,\beta}}(v)$$

(this is also the case for cubic fourfolds, [BLMS17]). If $d = 1$ and $v = \kappa_1$, there are objects in $D^b(Y)$ that are $\sigma$-semistable but not $\sigma_{\alpha,\beta}$-semistable, and conversely. We will show:

**Theorem 5.1.** Let $E$ be a complex in $D^b(Y)$ of class $-\kappa_1$, fix $\beta = -\frac{1}{2}$ and $\alpha \ll 1$. Then:

- $E$ is $\sigma_{\alpha,\beta}$-semistable if and only if it is a Gieseker stable sheaf in $M_G(\kappa_1)$ (classified in Prop. 3.4);
- $E$ is $\sigma_{0,\beta}$-semistable if and only if $E$ is isomorphic to:
  (i) $F_p$ for $p \neq y_0$,
  (ii) $G_x$, for $x \in \mathbb{P}^2$, or
  (iii) $I_Z[1]$, where $I_Z$ is a Gieseker-semistable sheaf in $M_G(\kappa_1)$.

We start the proof with some lemmas computing $\sigma_{\alpha,\beta}$-walls in the $(\alpha, \beta)$-plane for $-\kappa_1$. Observe that, by definition of $Z^0_{\alpha,\beta}$ (see Eq. (1)), the same equations define numerical walls for both weak stability conditions $\sigma_{\alpha,\beta}$ and $\sigma_{0,\beta}$.

**Lemma 5.2.** For $\beta = 0$, objects $F_p$ and $G_x$ are strictly semistable of infinite slope in $\text{Coh}^\beta(Z)$. In other words, the half-line $\beta = 0$ is a vertical wall for $-\kappa_1$ in the $(\alpha, \beta)$-plane.

**Proof.** The complex $F_p$ fits into the exact triangle

$$\mathcal{O}_C(p) \to F_p \to \mathcal{O}_Y[1].$$
Both $\mathcal{O}_C(p)$ and $\mathcal{O}_Y[1]$ are semistable of infinite slope in $\text{Coh}^2(Y)$: it is straightforward to compute that $\mathfrak{S}Z_{\alpha,\beta}(-)$ vanishes on both $\mathcal{O}_Y$ and $\mathcal{O}_C(p)$ since
\[
\chi_{\leq 2}^{\beta=0}(\mathcal{O}_Y) = (1, 0, 0) \quad \text{and} \quad \chi_{\leq 2}^{\beta=0}(\mathcal{O}_C(p)) = (0, 0, H^2).
\]

**Lemma 5.3.** There are no actual walls for $-\kappa_1$ in the strip $-1 < \beta < 0$.

**Proof.** By Lemma 5.2, the line $\beta = 0$ is a vertical wall.

Next, we show that no actual walls intersect the line $\beta = -1$. Suppose otherwise that for some $\alpha > 0$ there is an actual wall, realized by a sequence of $\sigma_{\alpha,-1}$-semistable complexes
\[
0 \to E \to F \to G \to 0
\]
in $\text{Coh}^{-1}(Y)$. Observe that for any $\alpha > 0$ and any $F$ of class $-\kappa_1$, $\mathfrak{S}Z_{\alpha,-1}(F) = 1$ is the smallest positive value of
\[
\mathfrak{S}Z_{\alpha,-1}(-) = H^2 \chi_1^{-1}(-).
\]
Then, either $\mathfrak{S}Z_{\alpha,-1}(E) = 1$, and therefore $Z_{\alpha,-1}(G) = 0$, or $\mathfrak{S}Z_{\alpha,-1}(G) = 1$ and $Z_{\alpha,-1}(E) = 0$.

Assume the former: since $G$ is $\sigma_{\alpha,-1}$-semistable, the support property implies $\chi_{\leq 2} G = 0$, which means that (14) is not an actual wall. The same argument works in the latter case swapping the roles of $E, G$.

By Theorem 2.5, walls are nested semicircles in the $(\alpha, \beta)$-plane. Therefore it suffices to find a semicircular wall outside the strip $-1 < \beta < 0$. A standard computation (sketched below for the ease of reading) shows that the class $[\mathcal{O}_Y(-1)]$ defines a numerical wall on the semicircle with radius $\frac{1}{2}$ and center $(0, -\frac{3}{2})$

We have:
\[
\chi_{\leq 2}^{\beta=0}(\mathcal{O}_Y(-1)) = \left(1, -H - \beta H, \frac{H^2}{2} + \beta H^2 + \frac{\beta^2}{2}H^2\right),
\]
\[
Z_{\alpha,\beta}(\mathcal{O}_Y(-1)) = \left(\frac{\alpha^2 - \beta^2}{2} - \frac{1}{2} - \beta\right) - i (1 + \beta),
\]
\[
Z_{\alpha,\beta}(F) = \left(\frac{\alpha^2 - \beta^2}{2} + 1\right) - i\beta.
\]

Then, the condition that $\mu_{\alpha,\beta}(\mathcal{O}_Y(-1)) = \mu_{\alpha,\beta}(F)$ simplifies to
\[
\alpha^2 + \left(\beta + \frac{3}{2}\right)^2 = \frac{1}{4},
\]
the desired semicircle.

**Lemma 5.4.** Objects $F_p$ and $G_x$ are $\sigma_{\alpha,\beta}^0$-semistable for $\alpha > 0$, $-1 < \beta < 0$.

**Proof.** Lemma 5.2 implies that $F_p$ is semistable in $\text{Coh}_{0,0}(Y)$ of slope 0 (arguing as in the proof of [BLMS17, Prop.2.15]). For $-1 << \beta < 0$, we have $\mathcal{O}_Y[1], \mathcal{O}_C(p) \in \text{Coh}_{0,0}(Y)$ and therefore $F_p \in \text{Coh}_{0,0}(Y)$ (although $F_p \notin \text{Coh}^2(Y)$ since $\mathcal{O}_Y[1] \notin \text{Coh}^2(Y)$). Since walls for $\sigma_{\alpha,\beta}^0$-stability coincide with those for tilt-stability, and $\mu_{\alpha,\beta}(\mathcal{O}_Y[1]) > 0 = \mu_{\alpha,\beta}(\mathcal{O}_C(p))$, $F_p$ is semistable left of the vertical wall $\beta = 0$ and outside of the largest semicircular wall of Lemma 5.3. Then, $F_p$ is $\sigma_{\alpha,\beta}^0$-semistable, for all $-1 < \beta < 0$.

**Proposition 5.5.** Let $\beta = -\frac{1}{2}$ and $\alpha << 1$. Then, the objects (i), (ii), and (iii) in Theorem 5.1 are $\sigma_{\alpha,\beta}^0$-semistable.

**Proof.** The same argument as [PY20, Prop. 4.1] applies to the $I_2[1]$ and implies that they are $\sigma_{\alpha,\beta}^0$-semistable. The other objects are $\sigma_{\alpha,\beta}^0$-semistable by Lemma 5.4.

Next, we show that the objects listed in Theorem 5.1 are the only $\sigma_{\alpha,\beta}^0$-semistable objects.

**Proposition 5.6.** Let $F$ be $\sigma_{\alpha,\beta}^0$-semistable object of class $-\kappa_1$. Then $F$ is one of the objects (i), (ii), and (iii) in Theorem 5.1.

**Proof.** Follows from lemmas 5.7 and 5.8 below.
Lemma 5.7. For $F$ as in Prop. 5.6, there is a triangle
$$F'[1] \to F \to T$$
where $F' \in \text{Coh}^\beta(Y)$ is $\sigma_{\alpha,\beta}$-semistable, and $T$ is either 0 or $\mathbb{C}_p$, for some $p \in Y$.

Proof. Since $F$ is in $\text{Coh}^0_{\alpha,\beta}(Y)$, there is a triangle
$$F'[1] \to F \to T$$
with $F' \in \text{Coh}^\beta(Y)_{\mu_{\alpha,\beta} \leq 0}, T \in \text{Coh}^\beta(Y)_{\mu_{\alpha,\beta} > 0}$. Since $F$ is semistable with respect to $\mu_{\alpha,\beta}^0 Z_{\alpha,\beta}(T)$ has to be 0, so $T$ is supported on points, that is, $T$ has finite length $m$. Also, if $F'$ is not semistable, then neither is $F$. Next we prove that $m \leq 1$. It suffices to show that $\text{ch}_3(F') \leq 1$, since $\text{ch}(F') = (1, 0, -H^2, mH^3)$. By [Li19b], [BMS16, Conjecture 4.1] holds for $F'$, for all $(\alpha, \beta)$ where it is semistable. In particular, since $F'$ is semistable along the line $\beta = -\frac{1}{2}$, the inequality holds for $\alpha = 0$ and $\beta = -\frac{1}{2}$, which gives
$$4 \cdot \frac{49}{64} - 6 \cdot \frac{1}{2} \text{ch}_3(F') \geq 0$$
which simplifies to $\text{ch}_3 F' \leq 3/2$. This proves $m \leq 1$ (in fact the inequality for $\beta = -1$ gives the exact bound $\text{ch}_3 F' \leq 1$). \qed

We now classify all possibilities for $F'$ and $T$ as in Lemma 5.7.

Lemma 5.8. In the setting of Lemma 5.7, $F'$ is the ideal sheaf of a one-dimensional subscheme of $Y$. More precisely, there are two possibilities:
- if $T = 0$ in , then $F' = I_Z$, for $Z \subset Y$ a subscheme as in Prop. 3.4;
- if $T \neq 0$, then $F' = I_{C}$, for $C \subset Y$ a genus 1 curve of degree 1 (see Lemma 3.3). In this case, $F$ is $F_p$ if $T = \mathbb{C}_p$, and $F$ is one of the $G_x$ if $T = \mathbb{C}_{y_0}$.

Proof. Since $\text{ch}_c(F) = \text{ch}_c(F')$, Lemma 5.3 shows that there are no walls for $F'$ in the $-1 < \beta < 0$ strip. Hence $F'$ is $\sigma_{\alpha,\beta}$-semistable for $\alpha \gg 0$. It follows from [BMS16, Lemma 2.7] that $F'$ is a Gieseker-semistable sheaf.

If $T = 0$, then $[F'] = \kappa_1$ is one of the ideal sheaves $I_Z$ classified in Prop. 3.4.

Otherwise, $F'$ is an ideal sheaf of a subscheme supported on a curve of degree 1. This is either a line or a genus one curve. It cannot be a line: otherwise, we would have $H^3 = \text{ch}_3 F' \leq 0$. Hence $F'$ is the ideal sheaf of a genus 1 curve $C$. The only complex with cohomologies $I_C[1]$ and $\mathbb{C}_p$ is $F_p$. Similarly, the $G_x$ are all the complexes with cohomologies $I_C$ and $\mathbb{C}_{y_0}$. \qed

Proof of Theorem 5.1. The statement about $\sigma_{\alpha,\beta}$-semistable objects is proven with the same argument as [PY20, Prop. 4.1]: the authors show that $\sigma_{\alpha,\beta}$-stability coincides with Gieseker stability for $\alpha \gg 1$, and there are no walls for objects of class $-\kappa_1$ on the line $\beta = -\frac{1}{2}$.

On the other hand, Propositions 5.5 and 5.6 show that $\sigma_{\alpha,\beta}$-semistable objects are precisely those listed in the statement. \qed

Remark 5.9. A simple consequence of Lemma 5.4 is that every $F_p$ is $\sigma(\alpha, \beta)$-stable for all $(\alpha, \beta) \in V$ (defined by Eq. (2)). In fact, $F_p$ is $\alpha_{\alpha,\beta}$-semistable, for all $0 < \alpha, -1 < \beta < 0$. Since this strip intersects $V$, $F_p$ is also $\sigma(\alpha, \beta)$-semistable, for some $(\alpha, \beta) \in V$, and hence for all of them by Prop. 2.8. Having primitive numerical class, $F_p$ must be $\sigma(\alpha, \beta)$-stable. This proves that the $F_p$ are $\sigma$-stable, giving an alternative argument than that of Theorem 4.6.

Remark 5.10. Note that the object $F_{y_0} \in \text{Coh}^0_{\alpha,\beta}(Y)$ is not $\sigma^0_{\alpha,\beta}$-semistable. It is destabilized by the triangle (12). However, $F_{y_0}$ is $\sigma(\alpha, \beta)$-stable, since it is the rotation of the stable object $E_{y_0}$ (see Theorem 4.6). There is no wall in the $(\alpha, \beta)$ plane which would make $F_{y_0}$ stable. Nevertheless, the objects $G_x$ defined in Sec. 4 are $\sigma^0_{\alpha,\beta}$-semistable, and they can be obtained from (12) as all the possible extensions in the other direction: in fact, the objects $G$ fitting in a triangle
$$[\mathcal{O}_{Y}^3 \to \mathcal{O}_Y(1)] \to G \to \mathcal{O}_Y(-1)][2]$$
are all and only the $G_x$. Indeed, the complex $[\mathcal{O}_{Y}^3 \to \mathcal{O}_Y(1)]$ fits in the Koszul complex
$$\mathcal{O}_Y(-2) \to \mathcal{O}_Y(-1)^{\oplus 3} \to \mathcal{O}_{Y}^{\oplus 3} \to \mathcal{O}_Y(1) \to \mathbb{C}_{y_0}.$$
Then the cohomology sequence of (15) gives immediately

\begin{equation}
0 \to \mathcal{O}_Y(-1) \xrightarrow{\delta} K \to H^{-1}(G),
\end{equation}

where $K = \ker(b) = \text{coker}(a)$ and $H^{-2}(G) = 0$ because $c \neq 0$. Considering the sequence $\mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 3} \to K$, one sees that $c$ must lift to an inclusion $\mathcal{O}(-1) \to \mathcal{O}(-1)^{\oplus 3}$, and hence $H^{-1}(G) \simeq \text{coker}(c) = \text{coker}(\mathcal{O}_Y(-2) \to \mathcal{O}_Y(-1)^{\oplus 2}) = I_{C_x}$ for some $x \in \mathbb{P}^2$. In other words, $G$ has cohomologies

$$I_{C_x}[1] \to G \to \mathbb{C}_{y_0}$$

and hence $G \simeq G_x$ for some $x$. Conversely, all $G_x$ fit in a triangle (15).

6. Stable pairs and moduli of $\sigma_{\alpha,\beta}^0$-semistable complexes

In this section we show that there is a fine moduli space for $\sigma_{\alpha,\beta}^0$-semistable complexes. We recall that a stable pair on $Y$ is a pair $(P, s)$ where:

- $P$ is a pure sheaf supported on a curve of $Y$;
- $s$ is a map

$$\mathcal{O}_Y \xrightarrow{\delta} P$$

with zero-dimensional cokernel (see [PT09]).

We say that $\mathcal{O}_Y \xrightarrow{\delta} P$ has class $\kappa_1$ if $v(P) = \kappa_1 - v(\mathcal{O}_Y)$.

A family of stable pairs over a quasi-projective base scheme $B$ as $(P, s)$ where $P \in \text{Coh}(Y \times B)$ is flat over $B$ and

$$\mathcal{O}_{Y \times B} \xrightarrow{s} P,$$

with the property that the restriction $(P_b, s_b)$ is a stable pair on $Y \times \{b\}$ for all closed $b \in B$.

There is a fine moduli space $P(\kappa_1)$ representing the functor

$$\mathcal{P}(\kappa_1) : (\text{Sch}/\mathbb{C})^{\text{op}} \to \text{Sets}$$

whose value on a scheme $B$ is the set of families of stable pairs over $B$ of class $\kappa_1$, and which maps morphisms to pull-backs of families (the space $P(\kappa_1)$ is constructed using GIT techniques and it is projective [LP93]).

Pandharipande and Thomas show that two stable pairs $\mathcal{O}_Y \xrightarrow{\delta} P$ and $\mathcal{O}_Y \xrightarrow{\delta'} P'$ are isomorphic if and only if they are quasi-isomorphic as complexes in $D^b(Y)$ [PT09, Prop. 1.21]. As a consequence, they identify $\mathcal{P}(\kappa_1)$ with the moduli functor whose value in $B$ is the quasi-isomorphism class of $B$-perfect complexes on $Y \times B$ that restrict to stable pairs of class $\kappa_1$ on closed points of $B$ [PT09, §2].

On the other hand, consider the weak stability condition $\sigma_{\alpha,\beta}^0$ of Theorem 5.1. We can define a moduli functor

\begin{equation}
\mathcal{M}_{\alpha,\beta}^0(\kappa_1) : (\text{Sch}/\mathbb{C})^{\text{op}} \to \text{Gpds}
\end{equation}

whose value on a scheme $B$ is the groupoid of all $B$-perfect complexes $I \in D(Y \times B)$ such that for all closed $b \in B$, $I_b \in D(Y \times \{b\})$ is $\sigma_{\alpha,\beta}^0$-semistable of class $\kappa_1$ (as above, the value of $\mathcal{M}_{\alpha,\beta}^0(\kappa_1)$ on morphisms is pull-back).

Observe that Theorem 5.1 classifies exactly all stable pairs of class $\kappa_1$. In fact, one can argue as in [PT09, Lemma 1.6] and show that a stable pair of class $\kappa_1$, viewed as a complex $I := [\mathcal{O}_Y \xrightarrow{\delta} F] \in D^b(Y)$, satisfies $h^0(I) \simeq I_C$ where $C$ is a degree 1 curve, $\text{length}(h^1(I)) \leq 1$, and all other cohomologies vanish. Such complexes are precisely (shifts of) those in Theorem 5.1.

In other words, $\mathcal{P}(\kappa_1)$ (interpreted as a moduli functor of complexes) is identified with $\mathcal{M}_{\alpha,\beta}^0(\kappa_1)$, and therefore $P(\kappa_1)$ is a fine moduli space for $\mathcal{M}_{\alpha,\beta}^0(\kappa_1)$.

Moreover (recall the descriptions of $M_G(\kappa_1)$ and $M_\sigma(-\kappa_1)$ in in Theorem 3.7 and Theorem 4.6) we have:

**Theorem 6.1.** The projective scheme $P(\kappa_1)$ is a fine moduli space of $\sigma_{\alpha,\beta}^0$-semistable objects, for $\sigma_{\alpha,\beta}^0$ as in Theor. 5.1. $P(\kappa_1)$ contains $M_G(\kappa_1)$, and has a third irreducible component $\tilde{M}_3$, which is the blow-up of $M_3 \simeq Y$ at $y_0$. 
Proof. It follows from Theorem 5.1 that the universal family of $M_G(\kappa_1)$ induces an inclusion $M_G(\kappa_1) \to P(\kappa_1)$. The third irreducible component of $P(\kappa_1)$ parametrizes complexes of the form $F_p$ for $y_0 \neq p \in Y$, and $G_x$ for $x \in \mathbb{P}^2$.

Observe first of all that $\tilde{M}_3$ is smooth outside the intersection with the other components, in fact, we have $\text{ext}^1(F_p, F_p) = 3$ (by Theorem 4.6) and $\text{ext}^1(G_x, G_x) = 3$ by Lemma 6.2 below. Then, the locus $D'$ parametrizing the objects $G_x$ is a Cartier divisor in $\tilde{M}_3$. Set $D = D' \times Y$ and write $i_D: D \to \tilde{M}_3 \times Y$ for the inclusion.

Let $\mathcal{I} \in D^b(\tilde{M}_3 \times Y)$ be the universal family of $P(\kappa_1)$ restricted to $\tilde{M}_3$. We will use a modification of $\mathcal{I}$ to construct a family of objects of $\text{Ku}(Y)$ supported on $\tilde{M}_3$. Consider the triangle

$$\mathcal{I}(-D) \to \mathcal{I} \xrightarrow{i_D} \mathcal{I}|_D$$

and the relative version of (15) over the projection $p_D: D \to Y$:

$$p_D^*[-1][2] \to \mathcal{O}_Y(1) \to \mathcal{I}|_D \xrightarrow{\mathcal{I}} p_D^*\mathcal{O}_Y(-1)[2]$$

(we denote $A := [\mathcal{O}_Y^3 \to \mathcal{O}_Y(1)]$ in what follows). We abuse notation and we use the same letter $u$ for the map $i_D^*\mathcal{I}|_D \xrightarrow{u} i_D^*p_D^*\mathcal{O}_Y(-1)[2]$ obtained by pushing forward. The octahedral axiom applied to $u \circ r$ yields a triangle

$$\mathcal{I}(-D) \to \mathcal{I}' \to i_D^*p_D^*A,$$

where $\mathcal{I}'$ is the cone of $u \circ r$. By tensoring $i_D^*p_D^*A$ with the sequence $\mathcal{O}_{\tilde{M}_3 \times Y}(-D) \to \mathcal{O}_{\tilde{M}_3 \times Y} \to i_D^*\mathcal{O}_D$ we obtain a triangle on $D$:

$$p_D^*A(-D) \xrightarrow{0} p_D^*A \to \mathcal{L}_D i_D^*p_D^*A,$$

where $\mathcal{L}_D i_D^*p_D^*A$ is the derived restriction of $i_D^*p_D^*A$ to $D$.

On the other hand, restriction of (18) to a fiber $[G_x] \times Y$ of $D$, gives a triangle

$$G_x \to (\mathcal{L}_D^*T')_{[G_x]} \to (\mathcal{L}_D^*i_D^*p_D^*A)_{[G_x]}.$$

The cohomologies of $(\mathcal{L}_D^*i_D^*p_D^*A)_{[G_x]}$ can be computed from (19), using that the cohomologies of $A$ (and those of $G_x$) are computed in Remark 5.10. Then, taking cohomologies of (20), we see

$$\begin{array}{c|c|c}
G_x & (\mathcal{L}_D^*T')_{[G_x]} & (\mathcal{L}_D^*i_D^*p_D^*A)_{[G_x]} \\
0 & H^{-2}(\mathcal{L}_D^*T')_{[G_x]} & K \\
I_{C_x} & H^{-1}(\mathcal{L}_D^*T')_{[G_x]} & M \\
C_{y_0} & H^0(\mathcal{L}_D^*T')_{[G_x]} & C_{y_0}
\end{array}$$

where $K = h^{-1}(A)$ and $M$ is an extension of $C_{y_0}$ by $K$. By construction, the connecting map $K \to I_{C_x}$ coincides with that in (16). So $H^{-2}(\mathcal{L}_D^*T')_{[G_x]} \simeq \mathcal{O}_Y(-1)$ and $f = 0$. Similarly, the map $M \to C_{y_0}$ is surjective, so that $H^{-1}(\mathcal{L}_D^*T')_{[G_x]} \simeq K$ and $g = 0$, which implies $H^0(\mathcal{L}_D^*T')_{[G_x]} \simeq C_{y_0}$.

Therefore there is a triangle

$$\mathcal{O}_Y(-1)[2] \to (\mathcal{L}_D^*T')_{[G_x]} \to A,$$

which shows that $(\mathcal{L}_D^*T')_{[G_x]} \simeq F_0$ by Lemma 4.7 (and because $\text{ext}^1(A, \mathcal{O}_Y(-1)[2]) = 1$).

Then, $\mathcal{I}' \in D(\tilde{M}_3 \times Y)$ defines a flat family of $\sigma$-stable objects of $\text{Ku}(Y)$ of class $\kappa_1$, and therefore a morphism $\tilde{M}_3 \to M_3$ which maps $D'$ to $y_0$. \hfill $\square$

Lemma 6.2. For $x \in \mathbb{P}^2$, we have $\text{ext}^1(G_x, G_x) = 3$.

Proof. We compute this applying the spectral sequence (8) to the complex $G_x \simeq [\mathcal{O}_Y \to \mathcal{O}_{C_x}(y_0)]$. The first page has spaces of dimension

$$\begin{array}{c|c|c}
0 & 0 & 0 \\
1 & * & 0 \\
0 & 0 & 2 \\
0 & 1 & 1
\end{array}$$
Since the map in the bottom row
\[ \text{Hom}(\mathcal{O}_Y, \mathcal{O}_Y) \oplus \text{Hom}(\mathcal{O}_{C_x}(y_0), \mathcal{O}_{C_x}(y_0)) \to \text{Hom}(\mathcal{O}_Y, \mathcal{O}_{C_x}(y_0)) \]
is non-zero, \( \text{ext}^1(G_x, G_x) \leq 3 \). But every \( G_x \) fits in a three dimensional component (there are two dimensions for deforming \( C_x \) and one to move \( y_0 \)), so \( \text{ext}^1(G_x, G_x) \geq 3 \) and equality holds. \( \square \)

6.1. \( P(\kappa_1) \) as a generalized Quot scheme. In Section 3 we showed that \( M_G(\kappa_1) \) is isomorphic to the Hilbert scheme of lines on \( Y \). Here, we give a similar interpretation for the moduli space \( P(\kappa_1) \) of \( \sigma_{\alpha, \beta}^0 \)-semistable objects as quotients of \( \mathcal{O}_Y \) in an appropriate heart of \( D^b(Y) \).

Consider the sheaves of the form \( \mathcal{O}_C(p) \), where \( p \in Y \) (possibly \( p = y_0 \)) and \( C = C_x \) for some \( x \in \mathbb{P}^2 \). By Riemann-Roch we also have
\[ \chi(\mathcal{O}_C(p)(t)) = 1 + t. \]
However, the \( \mathcal{O}_C(p) \) are not sheaf quotients of \( \mathcal{O}_Y \) and do not represent points of the Hilbert scheme of lines of \( Y \). In this section, we consider a different space of quotients, and show that the distinguished triangles
\[ \begin{align*}
I_Z &\to \mathcal{O}_Y \to \mathcal{O}_Z \\
F_p[-1] &\to \mathcal{O}_Y \to \mathcal{O}_C(p) \\
G_x[-1] &\to \mathcal{O}_Y \to \mathcal{O}_{C_x}(y_0)
\end{align*} \] (21)
are all short exact sequences in an appropriate abelian category (the notation here is the same as that of Theorem 5.1).

More precisely, define \( B^\theta_{\alpha, \beta} \) as follows: pick \( (\alpha, \beta) \in \mathbb{V} \) so that the chain of inequalities
\[ \frac{\beta}{-\left(\frac{\alpha^2 - 2}{2}\right)} = \mu^0_{\alpha, \beta}(F_p) > 0 = \mu^0_{\alpha, \beta}(\mathcal{O}_C(p)) > \mu^0_{\alpha, \beta}(\mathcal{O}_Y) = \frac{2\beta}{-(\alpha^2 - \beta^2)} \]
is satisfied. By the wall computation of Lemma 5.3, we may pick \( 0 < \epsilon \ll 1 \) so that, if \( F \) is any unstable object of class \(-\kappa_1\), then any destabilizing quotient \( G \) satisfies \( \mu^0_{\alpha, \beta}(G) \leq \theta \) and \( \mu^0_{\alpha, \beta}(G) > \theta \).

Then, consider the torsion pair in \( \text{Coh}^0_{\alpha, \beta}(Y) \) consisting of the categories \( \text{Coh}^0_{\alpha, \beta}(Y) \leq \theta \) and \( \text{Coh}^0_{\alpha, \beta}(Y) > \theta \) generated by \( \sigma_{\alpha, \beta}^0 \)-semistable objects of slope \( \leq \theta \) and \( > \theta \) respectively. Denote by \( B^\theta_{\alpha, \beta} \) the (shift of the) corresponding tilt:
\[ B^\theta_{\alpha, \beta} := \left[ \text{Coh}^0_{\alpha, \beta}(Y)_{\mu^0_{\alpha, \beta} \leq \theta}, \text{Coh}^0_{\alpha, \beta}(Y)_{\mu^0_{\alpha, \beta} > \theta}\right][1]. \]

Since \( I_Z[1] \), \( F_p \), and \( G_x \) are \( \sigma_{\alpha, \beta}^0 \)-semistable of phase \( > \theta \), their shifts by \(-1\) belong to \( B^\theta_{\alpha, \beta} \). Similarly, by the choice of \( \theta \) we have that \( \mathcal{O}_Z, \mathcal{O}_C(p), \mathcal{O}_{C_x}(y_0), \) and \( \mathcal{O}_Y \) belong to \( B^\theta_{\alpha, \beta} \) as well. Then, the triangles in (21) are short exact sequences in \( B^\theta_{\alpha, \beta} \). The converse is true:

**Proposition 6.3.** Quotients of \( \mathcal{O}_Y \) of class \( \kappa_1 \) in \( B^\theta_{\alpha, \beta} \) are precisely the objects \( \mathcal{O}_Z, \mathcal{O}_C(p), \) and \( \mathcal{O}_{C_x}(y_0) \) listed in (21).

**Proof.** First, we claim that if
\[ F \to \mathcal{O}_Y \to Q \] (22)
is a short exact sequence in \( B^\theta_{\alpha, \beta} \) with \( \chi(Q(t)) = t + 1 \), then \( F \) is \( \sigma_{\alpha, \beta}^0 \)-semistable of class \( \kappa_1 \). Indeed, the statement about the numerical class is immediate. As for semistability: a destabilizing quotient \( G \) of \( F \) must have \( \mu^0_{\alpha, \beta}(F) > \mu^0_{\alpha, \beta}(G) > \theta \), otherwise \( G \notin B^\theta_{\alpha, \beta} \). But this contradicts our choice of \( \theta \).

So, \( F \) must be (a shift of) the objects classified in Theorem 5.1, and the sequence (22) must be one of those listed in (21). \( \square \)

**Remark 6.4.** The arguments above identify the moduli functor \( \mathcal{M}^0_{\alpha, \beta}(\kappa_1) \) with the generalized Quot functor defined in [BLM+19, Sec. 11] and [Rot19].
References

[AB13] Daniele Arcara and Aaron Bertram, Bridgeland-stable moduli spaces for $K$-trivial surfaces, J. Eur. Math. Soc. (JEMS) 15 (2013), no. 1, 1–38, With an appendix by Max Lieblich.

[ABCH13] Daniele Arcara, Aaron Bertram, Izzet Coskun, and Jack Huizenga, The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability, Adv. Math. 235 (2013), 580–626. MR 3010070

[APR19] Matteo Altavilla, Marin Petkovic, and Franco Rota, Moduli spaces on the Kuznetsov component of Fano threefolds of index 2, arXiv e-prints (2019), arXiv:1908.10986.

[AT14] Nicolas Addington and Richard Thomas, Hodge theory and derived categories of cubic fourfolds, Duke Math. J. 163 (2014), no. 10, 1885–1927. MR 3292044

[Bay18] Arend Bayer, Wall-crossing implies Brill-Noether: applications of stability conditions on surfaces, Algebraic geometry: Salt Lake City 2015, Proc. Sympos. Pure Math., vol. 97, Amer. Math. Soc., Providence, RI, 2018, pp. 3–27. MR 3821144

[BCL13] Aaron Bertram and Izzet Coskun, The birational geometry of the Hilbert scheme of points on surfaces, Birational geometry, rational curves, and arithmetic, Simons Symp., Springer, Cham, 2013, pp. 15–55. MR 314922

[BLM+19] Arend Bayer, Marti Lahoz, Emanuele Macri, Howard Nuer, Alexander Perry, and Paolo Stellari, Stability conditions in families, J. Algebraic Geom., 23 (2014), no. 1, 117–163. MR 3121850

[BM14] Arend Bayer and Emanuele Macri, MMP for moduli of sheaves on $K3$s via wall-crossing: nef and movable cones, Lagrangian fibrations, Invent. Math. 198 (2014), no. 3, 505–590. MR 3279352

[BMMS12] Marcello Bernardara, Emanuele Macrì, Sukhendu Mehrotra, and Paolo Stellari, A categorical invariant for cubic threefolds, Adv. Math. 229 (2012), no. 2, 770–803. MR 2855078

[BMS16] Arend Bayer, Emanuele Macri, and Paolo Stellari, The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds, Invent. Math. 206 (2016), no. 3, 869–933. MR 3573975

[BMSZ17] Marcello Bernardara, Emanuele Macri, Benjamin Schmidt, and Xiaolei Zhao, Bridgeland stability conditions on Fano threefolds, Épijournal Géom. Algébrique 1 (2017), Art. 2, 24. MR 3743105

[BMT14] Arend Bayer, Emanuele Macri, and Yukinobu Toda, Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities, J. Algebraic Geom., 23 (2014), no. 1, 117–163. MR 3121850

[BW15] A. Bertram, S. Marcus, and J. Wang, The stability manifolds of $p1$ and local $p1$, Hodge Theory and Classical Algebraic Geometry 647 (2015), 1.

[BO95] A. Bondal and D. Orlov, Semiorthogonal decomposition for algebraic varieties, arXiv preprint alg-geom/9506012 (1995).

[Bri07] T. Bridgeland, Stability conditions on triangulated categories, Ann. of Math. (2) 166 (2007), no. 2, 317–345.

[Bri08] ________, Stability conditions on $K3$ surfaces, Duke Math. J. 141 (2008), no. 2, 241–291.

[BTL16] Marcello Bernardara and Gonzalo Tabuada, From semi-orthogonal decompositions to polarized intermediate Jacobians via Jacobians of noncommutative motives, Mosc. Math. J. 16 (2016), no. 2, 205–235. MR 3480702

[CK11] Kiryong Chung and Young-Hoon Kiem, Hilbert scheme of rational cubic curves via stable maps, Amer. J. Math. 133 (2011), no. 3, 797–834. MR 2808332

[GLHS18] Patricio Gallardo, César Lozano Huerta, and Benjamin Schmidt, Families of elliptic curves in $\mathbb{P}3$ and Bridgeland stability, Michigan Math. J. 67 (2018), no. 4, 787–813. MR 3877437

[HK15] Jun-Muk Hwang and Hosung Kim, Varieties of minimal rational tangents on Veronese double cones, Geom. Dedicata 179 (2015), no. 2, 176–192. MR 3350155

[Isk77] V. A. Iskovskih, Fano threefolds. I, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 3, 516–562, 717. MR 463151

[Kuz08] Alexander G. Kuznetsov, Derived categories of Fano threefolds, https://arxiv.org/pdf/0809.0225.pdf, September 2008.

[Kuz09] ________, Derived categories of Fano threefolds, Tr. Mat. Inst. Steklova 264 (2009), no. Mnogomernaya Algebraicheskaya Geometriya, 116–128. MR 2590842

[Kuz10] ________, Derived categories of cubic fourfolds, Cohomological and geometric approaches to rationality problems, Progr. Math., vol. 282, Birkhäuser Boston, Boston, MA, 2010, pp. 219–243. MR 2605171

[Kuz14] ________, Semiorthogonal decompositions in algebraic geometry, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 635–660. MR 3229044

[Kuz15] ________, Calabi–Yau and fractional Calabi–Yau categories, https://arxiv.org/pdf/1509.07657.pdf, September 2015.

[Li19a] Chunyi Li, On stability conditions for the quintic threefold, Invent. Math. 218 (2019), no. 1, 301–340. MR 3994590

[Li19b] ________, Stability conditions on Fano threefolds of Picard number 1, J. Eur. Math. Soc. (JEMS) 21 (2019), no. 3, 709–726. MR 3908763

[LLMS18] Marti Lahoz, Manfred Lehn, Emanuele Macri, and Paolo Stellari, Generalized twisted cubics on a cubic fourfold as a moduli space of stable objects, J. Math. Pures Appl. (9) 114 (2018), 85–117. MR 3801751

[LSvS17] Christian Lehn, Manfred Lehn, Christoph Sorger, and Duco van Straten, Twisted cubics on cubic fourfolds, J. Reine Angew. Math. 731 (2017), 87–128. MR 3709061

[LNS19] Chunyi Li, Howard Nuer, Paolo Stellari, and Xiaolei Zhao, A refined Derived Torelli Theorem for Enriques surfaces, arXiv e-prints (2019), arXiv:1912.04332.

[LP93] Joseph Le Potier, Systèmes cohérents et structures de niveau, Astérisque (1993), no. 214, 143. MR 1244404

[LPZ18] Chunyi Li, Laura Pertusi, and Xiaolei Zhao, Twisted cubics on cubic fourfolds and stability conditions, arXiv e-prints (2018), arXiv:1802.01134.
A NOTE ON THE KUZNETSOV COMPONENT OF THE VERONESE DOUBLE CONE

[LSZ21] Chunyi Li, Paolo Stellari, and Xiaolei Zhao, A Refined Derived Torelli Theorem for Enriques surfaces, II: the non-generic case, arXiv e-prints (2021), arXiv:2104.13610.

[Mac07] E. Macrì, Stability conditions on curves, Math. Res. Lett. 14 (2007), no. 4, 657–672.

[Mac14a] Antony Maciocia, Computing the walls associated to Bridgeland stability conditions on projective surfaces, Asian J. Math. 18 (2014), no. 2, 263–279. MR 3217637

[Mac14b] Emanuele Macrì, A generalized Bogomolov-Gieseker inequality for the three-dimensional projective space, Algebra Number Theory 8 (2014), no. 1, 173–190. MR 3207582

[MP15] Antony Maciocia and Dulip Piyaratne, Fourier-Mukai transforms and Bridgeland stability conditions on abelian threefolds, Algebr. Geom. 2 (2015), no. 3, 270–297. MR 3370123

[MP16] ______, Fourier-Mukai transforms and Bridgeland stability conditions on abelian threefolds II, Internat. J. Math. 27 (2016), no. 1, 1650007, 27. MR 3454685

[MS17] Emanuele Macrì and Benjamin Schmidt, Lectures on Bridgeland stability, Moduli of curves, Lect. Notes Unione Mat. Ital., vol. 21, Springer, Cham, 2017, pp. 139–211. MR 3729077

[Oka06] S. Okada, Stability manifold of $\mathbb{P}^1$, J. Algebraic Geom. 15 (2006), no. 3, 487–505.

[Piy17] Dulip Piyaratne, Stability conditions. Bogomolov-Gieseker type inequalities and Fano 3-folds, arXiv e-prints (2017), arXiv:1705.04011.

[PS85] Ragni Piene and Michael Schlessinger, On the Hilbert scheme compactification of the space of twisted cubics, Amer. J. Math. 107 (1985), no. 4, 761–774. MR 796901

[PT09] R. Pandharipande and R. P. Thomas, Curve counting via stable pairs in the derived category, Invent. Math. 178 (2009), no. 2, 407–447. MR 2545686

[PY20] Laura Pertusi and Song Yang, Some remarks on Fano threefolds of index two and stability conditions, arXiv e-prints (2020), arXiv:2004.02798.

[Qin21] Xuqiang Qin, Bridgeland stability of minimal instanton bundles on fano threefolds, 2021.

[Reid72] M. A. Reid, The complete intersection of two or more quadrics, Ph.D. thesis, University of Cambridge Cambridge, 1972.

[Rot19] Franco Rota, Moduli spaces of sheaves: generalized Quot schemes and Bridgeland stability conditions, Ph.D. thesis, 2019.

[Sch20] Benjamin Schmidt, Bridgeland stability on threefolds: some wall crossings, J. Algebraic Geom. 29 (2020), no. 2, 247–283. MR 4069650

[Tih82] A S Tihomirov, The Fano surface of the Veronese double cone, Mathematics of the USSR-Izvestiya 19 (1982), no. 2, 377–443.

[Tod13] Yukinobu Toda, Stability conditions and extremal contractions, Math. Ann. 357 (2013), no. 2, 631–685. MR 3096520

[TX17] R. Tramel and B. Xia, Bridgeland stability conditions on surfaces with curves of negative self-intersection, ArXiv e-prints (2017).

MP: Department of Mathematics, University of Utah, Salt Lake City, UT 84102, USA
Email address: petkovic@math.utah.edu

FR: Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
Email address: rota@math.rutgers.edu