A HOPF LEMMA AND REGULARITY FOR FRACTIONAL p-LAPLACIANS

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Abstract. In this paper, we study qualitative properties of the fractional p-Laplacian. Specifically, we establish a Hopf type lemma for positive weak super-solutions of the fractional p-Laplacian equation with Dirichlet condition. Moreover, an optimal condition is obtained to ensure \((-\Delta)^s_p u \in C^1(\mathbb{R}^n)\) for smooth functions \(u\).

1. Introduction and main results. The fractional p-Laplacian is defined by the singular integral

\[
(-\Delta)^s_p u(x) := C_{n,s,p} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} dy,
\]

where \(C_{n,s,p}\) is a positive constant depending only on \(n, s, p\), and \(s \in (0,1)\), and \(p > 1\). Denote

\[
L_{sp}(\mathbb{R}^n) := \left\{ u \in L^1_{loc}(\mathbb{R}^n) \left| \int_{\mathbb{R}^n} \frac{|u(x)|^{p-1}}{1 + |x|^{n+sp}} dx < \infty \right. \right\}.
\]

If \(u \in C^{1,1}_{loc} \cap L_{sp}(\mathbb{R}^n)\), then \((1)\) is well defined. Clearly, when \(p = 2\), \((1)\) becomes the fractional Laplacian which arises in many fields such as phase transitions, flame propagation, stratified materials and others (see \([1, 6, 7, 30]\)). In particular, the fractional Laplacian can be understood as the infinitesimal generator of a stable Lévy process (see \([32]\)). The fractional p-Laplacian also has many applications, for instance, it is used to study the non-local “Tug-of-War” game (see \([2, 3, 23]\)).
interest on these nonlocal operators continues to grow in recent years. We refer to [26] for the recent progress on these nonlocal operators.

Due to the non-locality of these kinds of operators, many traditional methods in studying the local differential operators no longer work. To overcome this difficulty, Cafarelli and Silvestre [8] introduced the extension method which turns nonlocal problems involving the fractional Laplacian \((p = 2)\) into local ones in higher dimensions, then the classical theories for local elliptic partial differential equations can be applied. We refer to [4, 17] and references therein for broad applications of this method.

Another useful method to study the fractional Laplacian is the integral equations method, which turns a given fractional Laplacian equation into its equivalent integral equation, and then various properties of the original equation can be obtained by investigating the integral equation, see [9, 15, 33] and references therein.

However, so far as we know, there has neither been any extension method nor the integral equations method that work for the fractional \(p\)-Laplacian equation when \(p \neq 2\) (Notice that the operator we consider here is different from the one introduced in [31]). The nonlinearity, the singularity \((1 < p < 2)\) and degeneracy \((p > 2)\) of the operator \((-\Delta)^s_p\) render many powerful methods to study the fractional Laplacian \((p = 2)\) no longer effective.

Recently, Chen et al. have developed a direct method of moving planes to investigate the nonlocal problems, which can be used to study not only the fractional Laplacian but also the fully nonlinear nonlocal operator

\[
F_\alpha(u(x)) := C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy,
\]

where \(\alpha > 0\), \(G : \mathbb{R} \to \mathbb{R}\) is a locally Lipschitz continuous function. The fractional \(p\)-Laplacian is a special case in which \(G(t) = |t|^{p-2}t\) and \(\alpha = sp\). This direct method has been successfully applied to obtain symmetry, monotonicity, nonexistence and other qualitative properties of solutions for various nonlocal problems, see e.g., [10, 11, 12, 13, 14].

In the present paper, we will continue to study qualitative properties for fractional \(p\)-Laplacian. We will establish a Hopf type lemma in general domains for super solutions to fractional \(p\)-Laplacian equations with a Dirichlet condition; and for any given smooth function \(u\), we will obtain an optimal condition for \((-\Delta)^s_p u\) to be continuously differentiable.

It is well-known that the Hopf lemma is a very powerful tool in the study of various differential equations. For example, it has been successfully used in the “second” step of the moving planes method.

In the case of fractional Laplacian \((p = 2)\), Fall and Jarohs [20, Proposition 3.3] proved a Hopf lemma for the entire antisymmetric supersolution of the problem

\[
(-\Delta)^s u(x) = c(x)u(x) \quad \text{in } \Omega.
\]  

Greco and Servadei [21] obtained a Hopf type lemma to (2) under the assumptions that \(c(x) \leq 0\) and \(\Omega \subset \mathbb{R}^n\) is a bounded domain. Chen and Li [25] established a Hopf lemma for anti-symmetric function on a half space through a rather delicate analysis. More recently, Jin and Li [24] extended the results of [25] to the fractional \(p\)-Laplacian with \(p > 3\) for positive anti-symmetric functions on the boundary of a half space. In this paper, we shall establish a Hopf type lemma for the positive weak supersolution of (3) on the boundary of more general domains.
Before stating our main results, we first introduce some definitions on fractional Sobolev spaces, and one can see [19, 22] for more details. For any domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, define

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \mid \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy < \infty \right\}$$

equipped with the norm

$$||u||_{W^{s,p}(\Omega)} := ||u||_{L^p(\Omega)} + \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy \right)^{\frac{1}{p}},$$

and define

$$W^{a,p}_0(\Omega) := \{ u \in W^{s,p}(\mathbb{R}^n) \mid u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \}.$$ 

If $\Omega \subset \mathbb{R}^n$ is bounded, set

$$\tilde{W}^{s,p}(\Omega) := \left\{ u \in L^p_{\text{loc}}(\mathbb{R}^n) \mid \exists U \supset \Omega \text{ s.t. } ||u||_{W^{s,p}(U)} + \int_{\mathbb{R}^n} \frac{|u(x)|^{p-1}}{|1 + |x||^{n+sp}} dx < \infty \right\},$$

and if $\Omega \subset \mathbb{R}^n$ is unbounded, set

$$\tilde{W}^{s,p}_{\text{loc}}(\Omega) := \{ u \in L^p_{\text{loc}}(\mathbb{R}^n) \mid u \in \tilde{W}^{s,p}(\Omega') \text{ for any } \Omega' \subset \subset \Omega \}.$$

Next, we present two definitions of solutions to fractional $p$–Laplacian equation with Dirichlet condition

$$\begin{cases}
(-\Delta)^s_p u(x) = f(x) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}$$

**Definition 1.1.** We say that $u \in C^{1,1}_{\text{loc}}(\Omega) \cap L^p(\mathbb{R}^n)$ is a classical supersolution (subsolution) of the Dirichlet problem (3) for a given continuous function $f$ in $\Omega$ if there hold

$$\begin{cases}
(-\Delta)^s_p u(x) \geq (\leq) f(x) & \text{in } \Omega, \\
u \geq (\leq) 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

where $(-\Delta)^s_p$ is defined by (1). Furthermore, if $u$ is both a supersolution and a subsolution of (3), then we say that it is a solution to (3).

**Definition 1.2.** Let $f \in W^{-s,p'}(\Omega)$, we say that $u \in \tilde{W}^{s,p}(\Omega)$ is a weak supersolution of (3), if there hold

$$(u + \epsilon)^- \in W^{a,p}_0(\Omega) \text{ for any } \epsilon > 0,$$

and

$$C_{n,s,p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{n+sp}} dxdy \geq \langle f, \phi \rangle$$

define similarly. Moreover, if $u$ is both a weak supersolution and a weak subsolution of (3), then we say it is a weak solution to (3).

Our first main result is the following

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^n$ be a domain with $C^{1,1}$ boundary. If $\Omega$ is bounded, we assume $u \in \tilde{W}^{s,p}(\Omega) \cap C(\Omega)$, and if it is unbounded, we assume $u \in \tilde{W}^{s,p}_{\text{loc}}(\Omega) \cap C(\Omega)$. Suppose that

$$\begin{cases}
(-\Delta)^s_p u \geq 0 & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega
\end{cases} (4)$$
in the weak sense. Then
\[ \liminf_{d(x) \to 0} \frac{u(x)}{d(x)^s} > 0, \]
where \(d(x) := \text{dist}(x, \Omega^c)\).

**Remark 1.4.** It is much easier in the case \(p = 2\), since one can construct a sub-solution (see, for example, [16]) to derive the Hopf lemma for the classical super solutions. The similar method does not work for \(p \neq 2\). So far, we do not know if this Hopf lemma holds for classical super solutions, because there are examples of \(u\) which is in \(C^{1,1}_{\text{loc}}(\Omega) \cap L_{sp}(\mathbb{R}^n)\), but not in \(\tilde{W}^{s,p}(\Omega) \cap C(\Omega)\). To this end, it seems technically easier to consider the weak super solutions.

The other main result of this work is on the regularity of \((-\Delta)^s u\). The regularity of solutions of the fractional \(p\)-Laplacian equations has attracted considerable attention in recent years, and it has been well understood for the fractional Laplacian equations \((p = 2)\). Specifically, the Schauder interior estimate of the solution is similar to that of the Poisson equation (associated with the regular Laplacian), which states roughly that if \(f \in C^\gamma(\Omega)\) and \(u \in C^{1,1}_{\text{loc}}(\Omega) \cap L_2(\mathbb{R}^n)\) is a solution of
\[
\begin{aligned}
(-\Delta)^s u(x) &= f(x) &\text{in } \Omega, \\
u &= 0 &\text{in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]
then the regularity of the solution \(u\) can be raised by the order of 2 in any proper subset of \(\Omega\), the same order as the operator \((-\Delta)^s\). By introducing the proper weighted Hölder norms as in the case of Poisson equations, one shall be able to control a weighted \(C^2\) norm of \(u\) in \(\Omega\) in terms of another weighted \(C^\gamma\) norm of \(f\) in \(\Omega\). However, when considering the regularity of the solution up to the boundary, the situation in the fractional order equation is quite different from that in the integer order equation (when \(s = 1\), the Poisson equation). In fact, Ros-Oton and Serra [27] proved that if \(u \in C^{1,1}_{\text{loc}}(\Omega) \cap L_{2s}(\mathbb{R}^n)\) is a solution of (5) with \(f \in L^\infty(\Omega)\), then \(u\) is \(C^s\) up to the boundary; and this is optimal in general. Later, Chen et al. [15] proved the similar results by a simpler method.

For the fractional \(p\)-Laplacian, the study of the regularity becomes quite complicated. So far as we know there are very few results. Di Castro and Kuusi [18] showed that if \(u \in \tilde{W}^{s,p}(\Omega)\) satisfies \((-\Delta)^s u = 0\) in \(\Omega\), then \(u\) is locally \(\gamma\)-Hölder continuous for small \(\gamma\). Brasco et al. [5] established a higher Hölder regularity for the fractional \(p\)-Laplacian equation in the superquadratic case \((p > 2)\). Indeed, the authors have verified that if \(u \in W^{s,p}(\Omega) \cap L_{sp}(\mathbb{R}^n)\) is a local weak solution of
\[
\begin{aligned}
(-\Delta)^s u &= f &\text{in } \Omega,
\end{aligned}
\]
where \(f \in L^q_{\text{loc}}(\Omega)\) with
\[
\begin{cases}
q > \frac{n}{sp} &\text{if } sp \leq n, \\
q \geq 1 &\text{if } sp > n,
\end{cases}
\]
then \(u \in C^{\delta}_{\text{loc}}(\Omega)\) for every \(0 < \delta < \Theta(n, s, p, q)\) with
\[
\Theta(n, s, p, q) = \min \left\{ \frac{1}{p - 1} \left( \frac{sp}{q} - \frac{n}{q} \right), 1 \right\}.
\]
Iannizzotto et al. [22] proved that the solutions of (6) with \(f \in L^\infty(\Omega)\) belong to \(C^{\alpha}(\Omega)\) for some \(\alpha \in (0, s]\).
Concerning the regularities of \((-\triangle)^s u\) for a given smooth function \(u\), there are more substantial technical difficulties than the local case.

For the fractional Laplacian \((-\triangle)^s\), Silvestre [29] has made a comprehensive investigation. More specifically, he has verified that if \(u \in L_{2s}(\mathbb{R}^n) \cap C^{2s+\epsilon} \) or \(C^{1,2s+\epsilon-1} \) if \(s > 1/2\) for some \(\epsilon > 0\) in an open set \(\Omega\), then \((-\triangle)^s u\) is a continuous function in \(\Omega\) for \(s \in (0, 1)\). Furthermore, if \(u \in C^{k,\alpha}\) and \(k + \alpha - 2s\) is not an integer, then \((-\triangle)^s u \in C^{l,\beta}\), where \(l\) is the integer part of \(k + \alpha - 2s\) and \(\beta = k + \alpha - 2s - l\).

While for the fractional \(p\)-Laplacian, the singularity \((0 < p < 2)\) make it more complex.

For example, even for the local operator \(\triangle\) and the sufficient smooth function \(u(x) = x^2 \) in \(\mathbb{R}\), \(-\triangle u(0) = \infty\) if \(1 < p < 2\), and \((-\triangle u)'(0) = \infty\) if \(1 < p < 3\) and \(p \neq 2\).

In this paper, we consider the differentiability of \((-\triangle)^s p u\) for \(p > 2\) and degeneracy \((p > 2)\) of operator \((-\triangle)^s p\) make it more complex.

The condition \(p > \frac{3}{2-s}\) is optimal as shown in the following

**Theorem 1.5.** Let \(p > 2\), \(u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{sp}(\mathbb{R}^n)\) and \(|\nabla u| \in L_{sp}(\mathbb{R}^n)\). If \(p > \frac{3}{2-s}\), then \((-\triangle)^s p u \in C^1(\mathbb{R}^n)\).

The condition \(p > \frac{3}{2-s}\) is optimal as shown in the following

**Theorem 1.6.** Let \(p > 2\), \(u(x) = \eta(x)x^2 \) in \(\mathbb{R}\), where \(\eta \in C_{0}^\infty(-2, 2)\) is even and satisfies \(0 \leq \eta(x) \leq 1\) in \(\mathbb{R}\), \(\eta(x) = 1\) in \((-1, 1)\) and \(|\eta'(x)| \leq 1\) in \(\mathbb{R}\). If \(p < \frac{3}{2-s}\), then

\[
\lim_{x \to 0^+} \left|\left((-\triangle)^s p u\right)'(x)\right| = \infty.
\]

And if \(p = \frac{3}{2-s}\), then

\[
\lim_{x \to 0^+} \left((-\triangle)^s p u \right)'(x) \neq \left((-\triangle)^s p u \right)'(0).
\]

The rest of the paper is organized as follows. Section 2 is devoted to establishing the Hopf type lemma for the positive solution of (4). In section 3, we first prove the differentiability of \((-\triangle)^s p u\) under the condition \(p > \frac{3}{2-s}\), then we show that this condition is optimal by giving a counterexample when \(p \leq \frac{3}{2-s}\). We state some results in [22] used in the present paper in the Appendix for the completeness of the paper.

2. Hopf type lemma. In this section, we prove the Hopf type lemma for the positive weak solution of (4) by constructing a suitable subsolution.

**Proof of Theorem 1.3.** For any given \(x_0 \in \partial \Omega\), it follows from the \(C^{1,1}\) property of the boundary of \(\Omega\) that there exist \(x_1 \in \Omega\) on the normal line to \(\partial \Omega\) at \(x_0\) and a positive constant \(\alpha\) such that \(B_\alpha(x_1) \subset \Omega\), \(\overline{B_\alpha(x_1)} \cap \partial \Omega = x_0\) and \(\text{dist}(x_1, \Omega^c) = |x_0 - x_1|\) (see [28] for the geometric property). Without loss of the generality, we suppose that \(x_0\) is the origin, \(\alpha = 1\) and \(x_1 = e_n\) with \(e_n = (0, \cdots, 1)\) the last vector of the canonical basis of \(\mathbb{R}^n\).

Let \(r \in \left(0, \frac{1}{3\sqrt{2}}\right)\) be a constant, \(O\) denote the origin and \(\eta \in C^2(\mathbb{R}^n)\) satisfy that

\[
\eta(X) = 1 \in B_{2r}(O), \quad \eta(X) = 0 \in B^c_{3r}(O), \quad \text{and} \quad \|
abla\eta\| \leq \frac{1}{r} \quad \text{in} \quad \mathbb{R}^n.
\]

Define \(\Psi : \mathbb{R}^n \to \mathbb{R}^n\) as

\[
\Psi(X) = X + \left(1 - X_n - \sqrt{(1 - X_n)^2 - |X'|^2)}^+\right) \eta(X)e_n,
\]

In fact, it suffices to show that for any $X \in B_{3r}(O)$. Since $r < \frac{1}{3\sqrt{3}}$, we have

$$1 - X_n \geq 2|X'| \quad \text{for any } X \in B_{3r}(O),$$

which implies

$$\Psi(X) = \begin{cases} X + (1 - X_n - \sqrt{(1 - X_n)^2 - |X'|^2}) \eta(X)e_n & \text{if } X \in B_{3r}(O), \\ X & \text{if } X \in B^c_{3r}(O). \end{cases}$$ (11)

Next we show the following two claims.

**Claim 1.** $\Psi$ is a $C^{1,1}$ diffeomorphism of $\mathbb{R}^n$. We firstly show that $\Psi$ is a bijection in $B_{3r}(O)$. Noting that if there exist $X = (X', X_n), Y = (Y', Y_n) \in \mathbb{R}^n$ such that $\Psi(X) = \Psi(Y)$, then $X' = Y'$. For any given $X' \in \mathbb{R}^{n-1}$ with $|X'| \leq 3r$, define $h : \{X_n \in \mathbb{R} | (X', X_n) \in B_{3r}(O)\} \to \mathbb{R}$ as

$$h(X_n) = X_n + (1 - X_n - \sqrt{(1 - X_n)^2 - |X'|^2}) \eta(X).$$ (12)

Now we show that $h$ is strictly monotone. Direct calculation implies that

$$h'(X_n) = 1 + \frac{|X'|^2}{1 - X_n + \sqrt{(1 - X_n)^2 - |X'|^2}} \frac{\partial \eta}{\partial X_n}(X)$$

$$+ \left( -1 + \frac{1 - X_n}{\sqrt{(1 - X_n)^2 - |X'|^2}} \right) \eta(X).$$ (13)

Noting that if $|X'| = 0$ then $h' = 1$. Furthermore, if $|X'| \neq 0$, the last term in (13) is positive and the second term can be rewritten as

$$\frac{|X'|}{1 - X_n + \sqrt{(1 - X_n)^2 - |X'|^2}} \cdot \frac{\partial \eta}{\partial X_n}(X)|X'| := J_1(X) \cdot J_2(X).$$

It then follows from $|X'| \leq 3r$ and (8) that $|J_2| \leq 3$. Moreover, thanks to (10), we can verify that $J_1 \leq \frac{1}{2 + \sqrt{3}}$. Consequently, there holds

$$h'(X_n) \geq 1 - \frac{3}{2 + \sqrt{3}} > 0,$$

that is, $h$ is strictly increasing in $\{X_n \in \mathbb{R} | (X', X_n) \in B_{3r}(O)\}$. This together with (11) shows that $\Psi \in C^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$ is a diffeomorphism of $\mathbb{R}^n$.

**Claim 2.** We have

$$B_1(e_n) \cap B_r(0) \subset \Psi(B_{3\sqrt{3}r}(O) \cap \mathbb{R}^n_+).$$ (14)

In fact, it suffices to show that for any $x \in B_1(e_n) \cap B_r(0)$, there is $X \in B_{3\sqrt{3}r}(O) \cap \mathbb{R}^n_+$ such that $\Psi(X) = x$. To this end, choose

$$X = \left( x', 1 - \sqrt{(1 - x_n)^2 + |x'|^2} \right).$$

Then

$$|X|^2 = |x'|^2 + 1 - 2\sqrt{(1 - x_n)^2 + |x'|^2} + (1 - x_n)^2 + |x'|^2$$

$$\leq 2|x'|^2 + 1 - 2(1 - x_n) + (1 - x_n)^2$$

$$\leq 2|x|^2$$

$$\leq 2r^2,$$
which shows that $X \in B_{\sqrt{2r}}(O) \cap \mathbb{R}^n_+$. It then follows from (11) that

$$
\Psi(X) = X + \left(1 - X_n - \sqrt{(1 - X_n)^2 - |X'|^2}\right) e_n
$$

$$
= \left(x', 1 - \sqrt{(1 - x_n)^2 + |x'|^2}\right) + \left(\sqrt{(1 - x_n)^2 + |x'|^2} - (1 - x_n)\right) e_n
$$

$$
= x.
$$

*Therefore the claim is verified.*

Now, we define $\rho : \mathbb{R}^n \to \mathbb{R}$ as

$$
\rho(x) = \text{dist}(x, B^1_r(e_n)) \quad \text{in} \ \mathbb{R}^n.
$$

Then there holds

$$
\rho(\Psi(X)) = (X_n)_+ \quad \text{for any} \ X \in B_{2r}(O).
$$

Indeed, it follows from (8) and (11) that

$$
\Psi(X) = \left(X', 1 - \sqrt{(1 - X_n)^2 - |X'|^2}\right) \quad \text{for any} \ X \in B_{2r}(O).
$$

By a direct calculation, we see that $\Psi(x) \in \partial B_{1 - x_n}(e_n)$ for any $X \in B_{2r}(O)$, which together with (15) leads to (16). Next we show by a similar calculation as in [22] that there exists a positive constant $C_1$ such that

$$
(-\Delta)^s \rho^s(x) \leq C_1 \quad \text{in} \ B_1(e_n) \cap B_r(0).
$$

Indeed, thanks to Lemma A.4, we only need to show that there exists $f \in L^\infty(B_1(e_n) \cap B_r(0))$ such that

$$
\lim_{\epsilon \to 0} \int_{\{|\Psi^{-1}(x) - \Psi^{-1}(y)| > \epsilon\}} \frac{G(\rho^s(x) - \rho^s(y))}{|x - y|^{n+ps}} dy = f \quad \text{in} \ L^1(B_1(e_n) \cap B_r(0)),
$$

where $G(t) = |t|^{p-2}t$ for any $t \in \mathbb{R}$. Making a change of variables $X = \Psi^{-1}(x)$, then for any $x \in B_1(e_n) \cap B_r(0)$, there exists $X \in B_{\sqrt{2r}}(O) \cap \mathbb{R}^n_+$ such that $\Psi(X) = x$ and

$$
\int_{\{|\Psi^{-1}(x) - \Psi^{-1}(y)| > \epsilon\}} \frac{G(\rho^s(x) - \rho^s(y))}{|x - y|^{n+ps}} dy
$$

$$
= \int_{B_{\epsilon}(X)} \frac{G(\rho^s(\Psi(X)) - \rho^s(\Psi(Y)))}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY
$$

$$
= \int_{B_{\epsilon}(X) \cap B_{2r}(O)} \frac{G((X_n)_+ - (Y_n)_+)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY
$$

$$
+ \int_{B_{2\epsilon}(O)} \frac{G(\rho^s(\Psi(X)) - \rho^s(\Psi(Y)))}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY
$$

$$
= \int_{B_{\epsilon}(X)} \frac{G((X_n)_+ - (Y_n)_+)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY
$$

$$
+ \int_{B_{2\epsilon}(O)} \frac{G(\rho^s(\Psi(X)) - \rho^s(\Psi(Y))) - G((X_n)_+ - (Y_n)_+)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY
$$

$$
:= J_1(X) + J_2(X),
$$

where the second equality follows from (14) and (16). Noting that $\Psi$ is a $C^{1,1}$ diffeomorphism of $\mathbb{R}^n$ and $\Psi = I$ in $B_{\sqrt{2r}}(O)$, Lemma A.2 then yields that there
exists $f_1 \in L^\infty(B_1(e_n) \cap B_r(0))$ such that
\[
\lim_{\epsilon \to 0} J_1(X) = \lim_{\epsilon \to 0} \int_{B_1^c(X)} \frac{G((X_n)_n^+ - (Y_n)_n^+)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY
\]
\[
= \lim_{\epsilon \to 0} \int_{\{|\Psi^{-1}(x) - \Psi^{-1}(y)| > \epsilon\}} \frac{G((\Psi^{-1}(x) \cdot e_n)_n^+ - (\Psi^{-1}(y) \cdot e_n)_n^+)}{|x - y|^{n+ps}} dY
\]
\[
= f_1(\Psi(X))
\]
(19)
in $L^1(\Psi^{-1}(B_1(e_n) \cap B_r(0)))$. Thanks to (14), there exists a positive constant $C$ such that for any $x \in B_1(e_n) \cap B_r(0)$ and $Y \in B_{2r}(O)$, we have
\[
|X - Y| \geq C(1 + |Y|).
\]
Therefore
\[
|J_2(X)| \leq \int_{B_{2r}(O)} \frac{|\rho^s(\Psi(X)) - \rho^s(\Psi(Y))|}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY
\]
\[
\leq C\Psi \int_{B_{2r}(O)} \frac{1}{|X - Y|^{n+s}} dY
\]
\[
\leq C\Psi \int_{B_{2r}(O)} \frac{1}{(1 + |Y|)^{n+s}} dY
\]
\[
\leq C\Psi,
\]
where the notation $C_\Psi$ above may denote different positive constants. This together with (18) and (19) shows that
\[
\lim_{\epsilon \to 0} \int_{B_1^c(X)} \frac{G(\rho^s(\Psi(X)) - \rho^s(\Psi(Y)))}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY = f_1(\Psi(X)) + J_2(X)
\]
in $L^1(\Psi^{-1}(B_1(e_n) \cap B_r(0)))$, with $f_1 \circ \Psi$ and $J_2$ in $L^\infty(\Psi^{-1}(B_1(e_n) \cap B_r(0)))$. It then follows that
\[
\lim_{\epsilon \to 0} \int_{\{|\Psi^{-1}(x) - \Psi^{-1}(y)| > \epsilon\}} \frac{G(\rho^s(x) - \rho^s(y))}{|x - y|^{n+ps}} dy = f_1(x) + J_2 \circ \Psi^{-1}(x)
\]
in $L^1(B_1(e_n) \cap B_r(0))$. Consequently (17) follows.

Now let $D \subset \subset B_1^c(e_n) \cap \Omega$ be a bounded smooth domain, and $\beta > 0$ be a positive constant to be determined below. Set
\[
\overline{u}(x) = \beta \rho^s(x) + \chi_D(x) u(x),
\]
(20)
where $\rho$ is defined by (15), and $\chi_D$ is the characteristic function of $D$, namely,
\[
\chi_D(x) = \begin{cases} 1, & x \in D, \\ 0, & x \notin D. \end{cases}
\]
It follows from $D \subset \subset B_1^c(e_n) \cap \Omega$ that there is a positive constant $C_D$ such that
\[
|x - y| \geq C_D \quad \text{for any } x \in B_1(e_n), \ y \in D.
\]
For any $x \in B_1(e_n) \cap B_r(0)$, direct calculation (we omit the index ‘$C_{n,s,p,\lim}$’ in the following calculation for convenience) shows that
\[
(-\Delta)_p \overline{u}(x) = \int_{B_r^c(x)} \frac{G(\overline{u}(x) - \overline{u}(y))}{|x - y|^{n+ps}} dy
\]
Then this together with (21) and (22) shows that

\[
\int_{B_{1}(x)} \frac{G(\beta \rho^s(x) - \beta \rho^s(y) - \chi D(y) u(y))}{|x - y|^{n + ps}} dy = \int_{B_{1}(x) \cap B_{1}(e_n)} \frac{G(\beta \rho^s(x) - \beta \rho^s(y))}{|x - y|^{n + ps}} dy + \int_{B_{1}(e_n)} \frac{G(\beta \rho^s(x) - \chi D(y) u(y))}{|x - y|^{n + ps}} dy
\]

\[
= \int_{B_{1}(x)} \frac{G(\beta \rho^s(x) - \beta \rho^s(y))}{|x - y|^{n + ps}} dy + \int_{B_{1}(e_n)} \frac{G(\beta \rho^s(x) - \chi D(y) u(y)) - G(\beta \rho^s(x) - \beta \rho^s(y))}{|x - y|^{n + ps}} dy
\]

\[
= \beta^{p-1}(-\Delta)^p \rho^s(x) + \int_{D} \frac{G(\beta \rho^s(x) - u(y)) - G(\beta \rho^s(x))}{|x - y|^{n + ps}} dy 
\]

\[
\leq \beta^{p-1} C_1 + \overline{C}_D A(x),
\]

where

\[
A(x) = \int_{D} G(\beta \rho^s(x) - u(y)) - G(\beta \rho^s(x)) dy,
\]

and the last inequality holds due to (17). Let

\[
M_0 = \min_{x \in D} u(x) > 0, \quad \beta \leq \frac{1}{2} M_0.
\]

Then it follows from the monotonicity of $G$ that

\[
A(x) \leq \int_{D} G(\beta \rho^s(x) - u(y)) dy
\]

\[
\leq \int_{D} G \left( \frac{1}{2} M_0 - M_0 \right) dy
\]

\[
= - \left( \frac{1}{2} \right)^{p-1} M_0^{p-1} |D|,
\]

this together with (21) and (22) shows that

\[
(-\Delta)^p u(x) \leq M_1 \beta^{p-1} - M_2 \quad \text{in } B_{1}(e_n) \cap B_r(0),
\]

where $M_1$ and $M_2$ are some positive constants. In view of (15) and (20), there holds

\[
u(x) \leq u(x) \quad \text{in } B_{1}(e_n).
\]

Let

\[
M_3 = \inf_{x \in B_{1}(e_n) \cap B_{1}(0)} u(x) > 0, \quad \beta < \frac{1}{2} \min \left( M_0, M_3, \left( \frac{M_2}{M_1} \right)^{\frac{1}{p-1}} \right).
\]

Then

\[
\begin{cases}
(-\Delta)^p u < 0 & \text{in } B_{1}(e_n) \cap B_r(0), \\
u(x) \leq u(x) & \text{in } (B_{1}(e_n) \cap B_r(0))^c.
\end{cases}
\]

The comparison principle (Lemma A.1) then yields that

\[
u(x) \geq u(x) \quad \text{in } \mathbb{R}^n.
\]

By the definition of $\rho$, we have $\rho(t e_n) = d(t e_n)$ for any $t \in (0, 1)$, and

\[
\frac{u(te_n)}{\rho^s(te_n)} = \frac{u(te_n)}{\rho^s(te_n)} = \beta \frac{u(te_n)}{u(te_n)} \geq \beta > 0.
\]
The proof is complete. \qed

3. **Regularity.** This section is devoted to the study of regularity of \((-\Delta)^s_p u\). We first prove the differentiability of \((-\Delta)^s_p u\) under the assumptions of Theorem 1.5, then we show that the condition \(p > \frac{3}{2-s}\) is optimal by giving a counterexample when \(p \leq \frac{3}{2-s}\).

**Proof of Theorem 1.5.** For any \(x \in \mathbb{R}^n\), by making a change of variables, we rewrite \((-\Delta)^s_p u(x)\) as

\[
(-\Delta)^s_p u(x) = \frac{1}{2} C_{n,s,p} \int_{\mathbb{R}^n} \frac{G(u(x) - u(x-y)) + G(u(x) - u(x+y))}{|y|^{n+sp}} dy,
\]

where \(G(t) = |t|^{p-2}t\) for any \(t \in \mathbb{R}\). Noting that

\[
\int_{\mathbb{R}^n} \frac{1}{|y|^{n+sp}} \left( \frac{\partial G(u(x) - u(x+y))}{\partial x_i} + \frac{\partial G(u(x) - u(x-y))}{\partial x_i} \right) dy
\]

\[
= \int_{|y| \leq 1} \frac{1}{|y|^{n+sp}} \left( \frac{\partial G(u(x) - u(x+y))}{\partial x_i} + \frac{\partial G(u(x) - u(x-y))}{\partial x_i} \right) dy
\]

\[
+ \int_{|y| > 1} \frac{1}{|y|^{n+sp}} \left( \frac{\partial G(u(x) - u(x+y))}{\partial x_i} + \frac{\partial G(u(x) - u(x-y))}{\partial x_i} \right) dy
\]

\[
:= I_1 + I_2.
\]

By a direct calculation, we obtain

\[
\frac{\partial G(u(x) - u(x-y))}{\partial x_i} = (p-1)|u(x) - u(x-y)|^{p-2} \left( \frac{\partial u(x)}{\partial x_i} - \frac{\partial (u(x)-y)}{\partial x_i} \right)
\]

\[
= (p-1)|u(x) - u(x-y)|^{p-2}(v(x) - v(x-y))
\]

and

\[
\frac{\partial G(u(x) - u(x+y))}{\partial x_i} = (p-1)|u(x) - u(x+y)|^{p-2}(v(x) - v(x+y)),
\]

where \(v(x) := \frac{\partial u}{\partial x_i}(x)\). Next we verify that \(|I_2| < \infty\). Since

\[
|I_2| \leq \int_{|y| > 1} \frac{1}{|y|^{n+sp}} \left( \left| \frac{\partial G(u(x) - u(x+y))}{\partial x_i} \right| + \left| \frac{\partial G(u(x) - u(x-y))}{\partial x_i} \right| \right) dy
\]

\[
\leq C_p \left( \int_{|y| > 1} \frac{|u(x)|^{p-2}|v(x)| + |u(x)|^{p-2}|v(x-y)|}{|y|^{n+sp}} dy
\]

\[
+ \int_{|y| > 1} \frac{|u(x-y)|^{p-2}|v(x)| + |u(x-y)|^{p-2}|v(x-y)|}{|y|^{n+sp}} dy
\]

\[
+ \int_{|y| > 1} \frac{|u(x)|^{p-2}|v(x)| + |u(x)|^{p-2}|v(x+y)|}{|y|^{n+sp}} dy
\]

\[
+ \int_{|y| > 1} \frac{|u(x+y)|^{p-2}|v(x)| + |u(x+y)|^{p-2}|v(x+y)|}{|y|^{n+sp}} dy
\]

it follows from \(u \in L_{sp}(\mathbb{R}^n), \nabla u \in L_{sp}(\mathbb{R}^n)\) and Hölder inequalities that \(|I_2| < \infty\).
For the term $I_1$, using the Taylor expansion formula, we have
\[
\frac{\partial G(u(x) - u(x + y))}{\partial x_i} = (p-1)|\nabla u(x) \cdot y + O(|y|^2)|^{p-2}(-\nabla v(x) \cdot y + O(|y|^2)) \\
= (p-1)|\nabla u(x) \cdot y + O(|y|^2)|^{p-2}(-\nabla v(x) \cdot y) \\
\quad + (p-1)|\nabla u(x) \cdot y + O(|y|^2)|^{p-2}O(|y|^2) \\
:= (p-1)J_1 + (p-1)J_2,
\]
and
\[
\frac{\partial G(u(x) - u(x - y))}{\partial x_i} = (p-1)|\nabla u(x) \cdot y + O(|y|^2)|^{p-2}(\nabla v(x) \cdot y) \\
\quad + (p-1)|\nabla u(x) \cdot y + O(|y|^2)|^{p-2}O(|y|^2) \\
:= (p-1)J_3 + (p-1)J_4,
\]
where the notation $O(|y|^2)$ denotes that there exist some positive constant $C$ such that $|O(|y|^2)| \leq C|y|^2$. Consequently, there is a positive constant $C$ such that
\[
|J_2| + |J_4| \leq C|y|^p.
\]

Next we consider two cases.

**Case 1.** $\nabla u(x) \cdot y = 0$. Then it follows from the definitions of $J_1$ and $J_3$ that there exist two positive constants $C_1$ and $C_3$ such that
\[
|J_1| < C_1|y|^{2p-3} \quad \text{and} \quad |J_3| < C_3|y|^{2p-3}.
\]

**Case 2.** $\nabla u(x) \cdot y \neq 0$. Then we rewrite $J_1$ and $J_3$ respectively as
\[
J_1 = |\nabla u(x) \cdot y + O(|y|^2)|^{p-2}(-\nabla v(x) \cdot y) \\
\quad = (|\nabla u(x) \cdot y + O(|y|^2)|^{p-2} - |\nabla u(x) \cdot y|^{p-2})(-\nabla v(x) \cdot y) \\
\quad \quad - |\nabla u(x) \cdot y|^{p-2}\nabla v(x) \cdot y
\]
and
\[
J_3 = (|\nabla u(x) \cdot y + O(|y|^2)|^{p-2} - |\nabla u(x) \cdot y|^{p-2})(\nabla v(x) \cdot y) + |\nabla u(x) \cdot y|^{p-2}\nabla v(x) \cdot y.
\]
It follows that
\[
|J_1 + J_3| \leq C(|\nabla u(x) \cdot y + O(|y|^2)|^{p-2} - |\nabla u(x) \cdot y|^{p-2})|\nabla v(x) \cdot y| \\
\quad \leq C|\nabla u(x) \cdot y|^{p-4}O(|y|^2)\nabla u(x) \cdot y + O(|y|^2)|\nabla v(x) \cdot y| \\
\quad \leq C|y|^p.
\]
To summarize, we conclude that there exists a positive constant $C$ independent of $y$ such that
\[
|J_1 + J_3| + |J_2| + |J_4| \leq C(|y|^{2p-3} + |y|^p) \quad \text{for any} \; y \in B_1(0).
\]
Notice that $p > \frac{3}{2-s}$, this further implies
\[
|J_1| \leq (p-1)\int_{|y| \leq 1} \frac{1}{|y|^{n+sp}}(|J_2| + |J_4| + |J_1 + J_3|) < \infty,
\]
that is,
\[
\int_{\mathbb{R}^n} \frac{1}{|y|^{n+sp}} \left| \frac{\partial G(u(x) - u(x + y))}{\partial x_i} + \frac{\partial G(u(x) - u(x - y))}{\partial x_i} \right| dy < \infty
\]
for any \( x \in \mathbb{R}^n \). By exchanging the order of integration and differentiation, we derive that \((-\Delta)^p_s u\) is differentiable in \( \mathbb{R}^n \), and then we conclude \((-\Delta)^p_s u \in C(\mathbb{R}^n)\) by exchanging the order of integration and limit. The proof is complete.

Theorem 1.5 implies that in the case \( p > 2 \), if one assumes in addition that \( p > \frac{2n}{n-2} \), then \((-\Delta)^p_s u \in C^1(\mathbb{R}^n)\) for any \( u \in C^0_{\text{loc}}(\mathbb{R}^n) \cap L^s_p(\mathbb{R}^n) \) and \(|\nabla u| \in L^s_p(\mathbb{R}^n)\). It seems from the proof of Theorem 1.5 that \( p > \frac{2n}{n-2} \) is a technical assumption. While, the counterexample in Theorem 1.6 shows that this condition is optimal to ensure \((-\Delta)^p_s u \in C(\mathbb{R}^n)\) for any \( u \in C^3_{\text{loc}}(\mathbb{R}^n) \cap L^s_p(\mathbb{R}^n) \) and \(|\nabla u| \in L^s_p(\mathbb{R}^n)\).

**Proof of Theorem 1.6.** By virtue of definition 1, we have

\[
(-\Delta)^p_s u(x) = \frac{1}{2} C_s p \int_{-\infty}^{+\infty} \frac{G(u(x) - u(x + y)) + G(u(x) - u(x - y))}{|y|^{1+sp}} dy.
\]

For the convenience of writing, we set

\[
F(x, y) := \frac{G(u(x) - u(x + y)) + G(u(x) - u(x - y))}{|y|^{1+sp}}.
\]

It follows from a straightforward calculation that

\[
\frac{\partial F(x, y)}{\partial x} = \frac{(p-1)}{|y|^{1+sp}} \left( |u(x) - u(x + y)|^{p-2} (u'(x) - u'(x + y))ight.
\]

\[
+ |u(x) - u(x - y)|^{p-2} (u'(x) - u'(x - y))
\]

\[
= \frac{(p-1)}{|y|^{1+sp}} \eta(x) x^2 - \eta(x+y)(x+y)^2|^{p-2}
\]

\[
\times \left[ \eta'(x)x^2 + 2\eta(x)x - \eta'(x+y)(x+y)^2 - 2\eta(x+y)(x+y) \right]
\]

\[
+ \frac{(p-1)}{|y|^{1+sp}} \eta(x) x^2 - \eta(x-y)(x-y)^2|^{p-2}
\]

\[
\times \left[ \eta'(x)x^2 + 2\eta(x)x - \eta'(x-y)(x-y)^2 - 2\eta(x-y)(x-y) \right].
\]

Let

\[
f(x, y) := \eta(x) x^2 - \eta(x+y)(x+y)^2|^{p-2}
\]

\[
\times \left[ \eta'(x)x^2 + 2\eta(x)x - \eta'(x+y)(x+y)^2 - 2\eta(x+y)(x+y) \right],
\]

then we can rewrite (23) as

\[
\frac{\partial F(x, y)}{\partial x} = \frac{(p-1)}{|y|^{1+sp}} [f(x, y) + f(x, -y)].
\]

Note that for any \( 0 < x < \frac{1}{8} \), we have

\[
\int_{-\infty}^{+\infty} \frac{\partial F(x, y)}{\partial x} dy
\]

\[
= (p-1) \int_{-\infty}^{+\infty} \frac{f(x, y) + f(x, -y)}{|y|^{1+sp}} dy
\]

\[
= 2(p-1) \int_{0}^{+\infty} \frac{f(x, y) + f(x, -y)}{|y|^{1+sp}} dy
\]
\[
2(p - 1) \left( \int_0^{\frac{1}{2}} \frac{f(x, y) + f(x, -y)}{|y|^{1+sp}} dy + \int_{\frac{1}{2}}^{2} \frac{f(x, y) + f(x, -y)}{|y|^{1+sp}} dy + \int_{\frac{1}{2}}^{\infty} \frac{f(x, y) + f(x, -y)}{|y|^{1+sp}} dy \right)
\]
\[
= 2(p - 1) (I_1 + I_2 + I_3).
\]
For \(I_3\), in view of \(y > \frac{3}{2}\) and \(0 < x < \frac{1}{8}\), there hold \(|x - y| > 2\) and \(|x + y| > 2\), which along with the properties of \(\eta\) implies that
\[
f(x, y) + f(x, -y) = 4x^{2p-3}.
\]
Hence, we have
\[
I_3 = \int_{\frac{1}{2}}^{\infty} \frac{4x^{2p-3}}{|y|^{1+sp}} dy = C_1 x^{2p-3},
\]
where \(C_1\) is a positive constant independent of \(x\).
For \(I_2\), thanks to \(\frac{1}{2} < y < \frac{5}{2}\) and \(0 < x < \frac{1}{8}\), there exists a positive constant \(C_2\) independent of \(x\) such that
\[
|I_2| < \infty.
\]
It remains to estimate the term \(I_1\). By virtue of \(0 < x < \frac{1}{8}\) and \(0 < y < \frac{1}{2}\), we see \(|x - y| < 1\) and \(|x + y| < 1\), which together with the properties of \(\eta\) yields
\[
f(x, y) + f(x, -y) = 2y|y^2 - 2xy|^{p-2} - 2y^2 + 2xy|^{p-2}.
\]
Therefore, for any \(x \in (0, \frac{1}{8})\), there hold
\[
I_1 = \int_0^{\frac{1}{2}} \frac{2y|y^2 - 2xy|^{p-2} - 2y^2 + 2xy|^{p-2}}{|y|^{1+sp}} dy
\]
\[
= \int_0^{\frac{1}{2}} \frac{2}{y^{2+sp-p}} (|y - 2x|^{p-2} - |y + 2x|^{p-2}) dy
\]
\[
= 2 \int_0^{\frac{1}{2}} \frac{x}{(xz)^{2+sp-p}} (|xz - 2x|^{p-2} - |xz + 2x|^{p-2}) dz
\]
\[
= \frac{2}{xz^{p-2p+3}} \int_0^{\frac{1}{2}} \frac{1}{z^{2+sp-p}} (|z - 2|^{p-2} - |z + 2|^{p-2}) dz
\]
\[
= \frac{2}{xz^{p-2p+3}} \int_0^{\frac{1}{2}} \frac{1}{z^{2+sp-p}} (2 - z)^{p-2} - (z + 2)^{p-2} dz
\]
\[
< \infty,
\]
which along with (24), (25) and (26) shows that for any fixed $x \in (0, \frac{1}{8})$, there holds

$$\left| \int_{-\infty}^{+\infty} \frac{\partial F(x, y)}{\partial x} dy \right| < \infty.$$ 

By exchanging the order of integration and differentiation, we derive that $((-\Delta)^s_p u)'$ is well-defined for any $x \in (0, \frac{1}{8})$, and

$$((-\Delta)^s_p u)'(x) = \frac{1}{2} C_{s,p} \int_{-\infty}^{+\infty} \frac{\partial F(x, y)}{\partial x} dy.$$

If $p < \frac{3}{2s-1}$, then (27) implies that

$$I_1 = \frac{2}{x^{sp-2p+3}} \int_0^2 \frac{(2 - z)^{p-2} - (z + 2)^{p-2}}{z^{2+sp-p}} dz
+ \frac{2}{x^{sp-2p+3}} \int_2^1 \frac{(z - 2)^{p-2} - (z + 2)^{p-2}}{z^{2+sp-p}} dz
\leq -C_x^{2p-sp-3},$$

which verifies that

$$\lim_{x \to 0^+} I_1 = -\infty.$$

Consequently, we conclude that

$$\lim_{x \to 0^+} ((-\Delta)^s_p u)'(x) = \frac{1}{2} C_{s,p} \int_{-\infty}^{+\infty} \frac{\partial F(x, y)}{\partial x} dy
= \lim_{x \to 0^+} C_{s,p} (p - 1)(I_1 + I_2 + I_3)
= -\infty,$$

that is, (7) holds.

In the case $p = \frac{3}{2s-1}$, we first prove that $((-\Delta)^s_p u)'(0) = 0$. Indeed,

$$((-\Delta)^s_p u)'_+(0) = \lim_{x \to 0^+} \frac{(-\Delta)^s_p u(x) - (-\Delta)^s_p u(0)}{x}
= \frac{1}{2} C_{s,p} \lim_{x \to 0^+} \frac{1}{x} \int_{-\infty}^{+\infty} F(x, y) - F(0, y) dy
= C_{s,p} \lim_{x \to 0^+} \frac{1}{x} \int_{-\infty}^{+\infty} F(x, y) - F(0, y) dy
= C_{s,p} \left( \lim_{x \to 0^+} \int_0^2 \frac{F(x, y) - F(0, y)}{x} dy
+ \lim_{x \to 0^+} \int_0^{1/2} \frac{F(x, y) - F(0, y)}{x} dy
+ \lim_{x \to 0^+} \int_{1/2}^{+\infty} \frac{F(x, y) - F(0, y)}{x} dy \right).$$
\[ := C_{s,p}(J_1 + J_2 + J_3) \]

For \( J_3 \), in view of \( y > \frac{5}{2} \) and \( 0 < x < \frac{1}{8} \), there hold \( |x - y| > 2 \) and \( |x + y| > 2 \), which along with the properties of \( \eta \) and \( sp = 2p - 3 \) implies that
\[
F(x, y) = \frac{2x^{2p-2}}{y^{2p-2}} \quad \text{and} \quad F(0, y) = 0.
\]

It then follows that
\[
J_3 = 2 \lim_{x \to 0^+} x^{2p-3} \int_{\frac{1}{2}}^{+\infty} \frac{1}{y^{2p-2}} dy = 0.
\]

For \( J_2 \), by exchanging the order of integration and limit, we have
\[
J_2 = \int_{\frac{1}{2}}^{\frac{5}{2}} \frac{\partial F}{\partial x}(0, y) dy
= (p - 1) \int_{\frac{1}{2}}^{\frac{5}{2}} \left( \frac{|\eta(y)y^2|^{p-2}(-y^2\eta'(y) - 2y\eta(y))}{y^{2p-2}} + \frac{|\eta(-y)y^2|^{p-2}(-y^2\eta'(-y) + 2y\eta(-y))}{y^{2p-2}} \right) dy.
\]

Since \( \eta(y) = \eta(-y) \) in \( \mathbb{R} \), there holds \( \eta'(y) = -\eta'(-y) \), which along with (28) implies \( J_2 = 0 \). As for \( J_1 \), we see
\[
J_1 = \lim_{x \to 0^+} \frac{1}{x} \int_{0}^{\frac{1}{2}} F(x, y) - F(0, y) dy
= \lim_{x \to 0^+} \frac{1}{x} \int_{0}^{\frac{1}{2}} \frac{\partial F}{\partial x}(0, y) x + O(x^2) dy
= \int_{0}^{\frac{1}{2}} \frac{\partial F}{\partial x}(0, y) dy
= (p - 1) \int_{0}^{\frac{1}{2}} \frac{|y^2|^{p-2}(-2y) + |y^2|^{p-2}(2y)}{y^{2p-2}} dy
= 0.
\]

To summarize, we conclude that
\[
(((-\Delta)^p u)^+)_{+}(0) = 0.
\]

Similarly, we can prove \((((-\Delta)^p u)^-)_{-}(0) = 0\). It then follows that
\[
(((-\Delta)^p u)^-)'(0) = 0.
\]

On the other hand, (25) implies that \( \lim_{x \to 0^+} I_3 = 0 \). Thanks to (26), by exchanging the order of integration and limit, we obtain \( \lim_{x \to 0^+} I_2 = 0 \). Similar calculations to (27) imply that for any \( x \in (0, \frac{1}{8}) \),
\[
I_1 = 2 \int_{0}^{\frac{1}{2}} \frac{|z - 2|^{p-2} - |z + 2|^{p-2}}{z^{p-1}} dz
= 2 \int_{0}^{2} \frac{(2 - z)^{p-2} - (z + 2)^{p-2} - (z + 2)^{p-2} + 2 \int_{2}^{\frac{1}{2}} (z - 2)^{p-2} - (z + 2)^{p-2}}{z^{p-1}} dz
\]
\[
< 2 \int_0^2 \frac{(2-z)^{p-2} - (z+2)^{p-2}}{z^{p-1}} dz \\
< 2 \int_1^2 \frac{(2-z)^{p-2} - (z+2)^{p-2}}{z^{p-1}} dz \\
\leq \frac{2(1-3^{p-2})}{2^{p-1}} < 0.
\]

To summarize, we derive that
\[
\lim_{x \to 0^+} ((-\triangle)_p^s u)'(x) = \frac{1}{2} C_{s,p} \lim_{x \to 0^+} \int_{-\infty}^{+\infty} \frac{\partial F(x,y)}{\partial x} dy = C_{s,p}(p-1) \lim_{x \to 0^+} (I_1 + I_2 + I_3) < C_{s,p}(p-1) \frac{2(1-3^{p-2})}{2^{p-1}} < 0.
\]

The proof is complete.

**Appendix.** In this Appendix, we list some results in [22] that were used in the proof of Theorem 1.3. The first one is the weak comparison principle.

**Lemma A.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain. Assume \( u, v \in \widetilde{W}^{s,p}(\Omega) \) satisfy, in the weak sense,
\[
\begin{aligned}
(-\triangle)^s_p u &\geq (-\triangle)^s_p v \quad \text{in } \Omega, \\
u &\geq v \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{aligned}
\]
Then
\[
u \geq v \quad \text{a.e. in } \Omega.
\]

Another key ingredient is the following “change of variables” lemma.

**Lemma A.2.** Let \( \Psi \) be a \( C^{1,1} \) diffeomorphism of \( \mathbb{R}^n \) such that \( \Psi = I \) in \( B_r^c(0) \), \( r > 0 \). Then the function \( v(x) = (\Psi^{-1}(x) \cdot e_n)^s_p \) belongs to \( \widetilde{W}^{s,p}_{\text{loc}}(\mathbb{R}^n) \) and is a weak solution of
\[
(-\triangle)^s_p v = f \quad \text{in } \Psi(\mathbb{R}^n_+),
\]
with
\[
\|f\|_{\infty} \leq C (\|D\Psi\|_{\infty}, ||D\Psi^{-1}||_{\infty}, r) \|D^2\Psi\|_{\infty},
\]
where \( C (\|D\Psi\|_{\infty}, ||D\Psi^{-1}||_{\infty}, r) \) is a positive constant. Moreover,
\[
\lim_{\epsilon \to 0} C_{n,s,p} \int_{\{|\Psi^{-1}(x) - \Psi^{-1}(y)| > \epsilon\}} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x-y|^{n+sp}} dy = f \quad (29)
\]
in \( L^1_{\text{loc}}(\Psi(\mathbb{R}^n_+)) \).

**Remark A.3.** The limit (29) follows from the “change of variables” lemma.

The following lemma implies that the point-wise solution is also a weak solution.

**Lemma A.4.** Let \( u \in \widetilde{W}^{s,p}_{\text{loc}}(\Omega) \) and \( D \) denote the diagonal of \( \mathbb{R}^n \times \mathbb{R}^n \). For any \( \epsilon > 0 \), assume \( A_\epsilon \subset \mathbb{R}^n \times \mathbb{R}^n \) is a neighborhood of \( D \) and satisfies
(i) \( (x,y) \in A_\epsilon \) for all \( (y,x) \in A_\epsilon \),
(ii) \( \sup_{x \in A} \text{dist}(x, D) \to 0 \) as \( \epsilon \to 0 \).
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For any \( x \in \mathbb{R}^n \), we set \( A_\epsilon(x) = \{ y \in \mathbb{R}^n \mid (x,y) \in A_\epsilon \} \) and
\[
ge_\epsilon(x) = C_{n,s,p} \int_{A_\epsilon^c(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{n+sp}} dy.
\]
If \( g_\epsilon \to f \) in \( L^1_{\text{loc}}(\Omega) \), then \( u \) is a weak solution of
\[
(-\triangle)_p^s u = f \text{ in } \Omega.
\]

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