A brief review of string theory on group manifolds is given, and comparisons are then drawn between Minkowski space, SU(2), and SU(1,1) = AdS\(_3\). The proof of the no-ghost theorem is outlined, assuming a certain restriction on the representation content for bosonic and fermionic strings on SU(1,1). Some possible connections with the AdS/CFT correspondence are mentioned.

1 Introduction

The no-ghost theorem for strings in flat space is such a long-standing and fundamental result in the subject that there is some danger of taking it for granted. In the simplest version one considers traditional covariant quantization of the bosonic string in Minkowski space. The theorem states that the Fock space inner-product is positive-semi-definite on the subspace of physical states, defined as those that satisfy the mass-shell and Virasoro primary conditions:

\[
(L_0 - 1)|\psi\rangle = 0, \quad L_n|\psi\rangle = 0 \quad n > 0 \tag{1}
\]

respectively, where \(L_n\) obey the Virasoro algebra with \(c = 26\). Since string theory is a theory of gravity, however, we should regard Minkowski space as merely a particular choice of background. But it is far from clear how the no-ghost theorem and its proof can be generalized when one moves from flat space to more general background spacetimes.

The standard criteria for consistency of a target spacetime in perturbative string theory is that the world-sheet sigma-model which it defines should be quantum conformally invariant, as given by the vanishing of appropriate \(\beta\)-functions for the metric, dilaton, and other background fields. But it is not hard to see that this approach is sometimes insensitive to important aspects of the theory. By these criteria alone, a flat spacetime with 13 time-like and 13 space-like directions would seem to be a perfectly consistent background for the bosonic string, with \(c = 26\), and yet this model certainly contains ghosts—i.e. physical states (in the sense of (1)) which have negative norm.

It seems reasonable to restrict attention to backgrounds with a single time-like direction (though there have recently been more radical suggestions). The

\(^a\)Talk presented by JME
original proofs of the no-ghost theorem\textsuperscript{1–3} are easily applied to critical strings moving in target spaces of the form $\mathbb{R}^{d-1,1} \times \mathcal{M}$ where $2 \leq d \leq 26$ and $\mathcal{M}$ corresponds to a unitary CFT of appropriate central charge. If on the other hand we consider a background whose geometry involves a time-like direction in a non-trivial way, then issues such as unitarity and the absence of ghosts must be scrutinized very carefully.

To examine such questions it is natural to turn to the simplest string models which one can hope to solve exactly, namely those for which the target spaces are group manifolds.\textsuperscript{4} Requiring a single time-like direction leads unavoidably to the non-compact group $\text{SU}(1,1) = \text{SL}(2,\mathbb{R})$ or its covering group. This was exactly the route followed by Balog et al.\textsuperscript{5} and it led them to some unexpected and superficially discouraging conclusions. They found that although classical string propagation on $\text{SU}(1,1)$ is consistent and causal, there are nevertheless negative-norm physical states in the quantum theory.

There have been a number of suggestions in the intervening decade as to how one might sensibly interpret $\text{SU}(1,1)$ string theory in the face of these facts. One proposal\textsuperscript{6,7} is to restrict the allowed Kac-Moody representations in terms of their spin and the level, in imitation of the unitarity condition for compact Kac-Moody algebras. This was pursued by Hwang et al.\textsuperscript{8,9} and it is also the point of view adopted by us.\textsuperscript{10} An alternative approach\textsuperscript{11,12} involves an apparently different definition of the quantum theory in terms of free-field-like Fock spaces, with the introduction of additional singularities in the Kac-Moody currents. It is not clear how these approaches are related, if at all, and although this is an interesting question it is not one we are able to comment on here in any more detail.

The issues we have highlighted become all the more relevant in view of some celebrated developments that have taken place in the past year. It is by now a famous conjecture that there is a duality relating supergravity (as the low energy limit of string theory) on $n$-dimensional anti-de Sitter space, AdS\textsubscript{$n$}, and a conformal field theory that lives on its $(n-1)$-dimensional boundary.\textsuperscript{13} A specific example\textsuperscript{14,15} involves type IIB string theory on $\text{AdS}_3 \times S^3 \times \mathcal{M}^4$ (where $\mathcal{M}^4$ is either K3 or $T^4$) which is conjectured to be dual to a two-dimensional conformal field theory with target manifold a symmetric product of a number of copies of $\mathcal{M}^4$. This example is of special interest as both partners of the dual pair are simple enough to be analyzed explicitly and it should therefore be possible to subject the proposal to non-trivial tests. On the string theory side, for instance, we have $\text{AdS}_3 = \text{SL}(2,\mathbb{R}) = \text{SU}(1,1)$ and $S^3 = \text{SU}(2)$, and since these are both group manifolds we should be able to determine the spectrum of string states exactly. The AdS/CFT correspondence therefore provides an additional specific and powerful motivation for clarifying the status of string
theory on SU(1, 1).

Here we give a brief summary of strings on group manifolds with a view to comparing and contrasting the target space SU(1, 1) with Minkowski space and with SU(2). We then outline the arguments used to prove the no-ghost theorem, following our recent work for bosonic and fermionic SU(1, 1) strings. This builds on the earlier references cited above, but we hope that it also resolves some of the confusion that seems to have persisted in the literature for a number of years. Finally we comment on an intriguing prediction of the AdS/CFT conjecture and how this might be reflected in the traditional string picture. Since this work was presented, there have been a number of very interesting developments which attempt to understand string theory on AdS3 in more detail.

2 WZW models and strings

A string moving on a group manifold $G$ is described by a world-sheet WZW conformal field theory with the level $k$ proportional to the string tension. More precisely, the background is the bi-invariant metric on the group and an antisymmetric tensor field strength proportional to the parallelizing torsion. The space of string states is constructed in terms of the Kac-Moody (KM) algebra for $G$ with generators $J^a_m$ obeying

$$[J^a_m, J^b_n] = i f^{abc} J^c_{m+n} + k m \eta^{ab} \delta_{m+n}$$

where $\eta^{ab}$ is the Killing form (used to raise and lower indices) and $f_{abc}$ are the structure constants. This should be regarded as a non-abelian generalization of a set of harmonic oscillators.

The Sugawara expression for the Virasoro generators is

$$L_n = \frac{1}{2k + Q_{ad}} \sum_{\ell} \eta_{ab} : J^a_{\ell} J^b_{n-\ell} :$$

where the normal ordering procedure sends $J^a_n$ to the right and left for $n > 0$ and $n < 0$ respectively. It follows that

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m (m^2 - 1) \delta_{m+n}, \quad [L_n, J^a_m] = -m J^a_{n+m}$$

where the central charge is

$$c = \frac{k}{k + \frac{1}{2} Q_{ad}} \dim G.$$
At grade zero, the KM algebra contains a copy of the finite-dimensional Lie algebra $G$ with generators $J^a_0$. We write $Q = \eta_{ab} J^a_0 J^b_0$ for the quadratic Casimir of this subalgebra, and $Q_{\text{ad}}$ appearing above is its value in the adjoint representation.

To construct string states we choose a unitary base representation $\tau$ of the zero-grade subalgebra acting on states $|i\rangle$. We declare

$$J^a_0 |i\rangle = \tau^a_{ij} |j\rangle, \quad J^a_n |i\rangle \quad n > 0 \quad (6)$$

and the string Fock space is spanned at each grade $N$ by states

$$|\psi\rangle = J^{n_1}_1 \ldots J^{n_r}_r |i\rangle \quad \text{with} \quad N = \sum_k n_k. \quad (7)$$

Note that on such a state of grade $N$ we have

$$L_0 |\psi\rangle = \left( N + \frac{Q_{\tau}}{2k + Q_{\text{ad}}} \right) |\psi\rangle. \quad (8)$$

The superstring in the RNS formalism is constructed in a similar fashion from a super WZW model with target space $G$. In addition to the KM currents there are fermionic superpartners in the adjoint representation. These can be decoupled by a re-definition of the currents, however, resulting in a bosonic Kac-Moody algebra with shifted level $\hat{k} = k - \frac{1}{2} Q_{\text{ad}}$ which commutes with the fermions. It follows that the super-Virasoro algebra has (supersymmetric) central charge

$$\hat{c} = \left( 1 - \frac{Q_{\text{ad}}}{3k} \right) \dim G. \quad (9)$$

(As usual $c = 3\hat{c}/2$.)

So far we have said nothing about the nature of the group $G$. If $G$ is compact and semi-simple then $k$ must be quantized to ensure quantum consistency. With our normalization this means $k \in \frac{1}{2} \mathbb{Z}$. It is also well-known that in this case the KM representation constructed via (6) and (7) is unitary if and only if the highest weight of the base representation $\tau$ and the highest root of $G$ have an inner-product that is bounded by the level $k$. (In the supersymmetric case we replace $k \rightarrow \hat{k}$.)

2.1 $\quad G = \mathbb{R}^{d-1,1}$

This is the string in $d$-dimensional Minkowski space, and we may set $k = 1$ without loss of generality. The KM currents become the usual string oscillators $J^a_n \rightarrow \alpha^a_n$, and the zero-grade subalgebra is generated by the commuting
momentum operators $J^{a}_0 \rightarrow p^a$. Base representations are simply states of definite momentum $|p^a\rangle$. The Virasoro primary conditions determine the physical polarizations of states, while the mass-shell condition in conjunction with (8) implies that for physical states $(mass)^2 = -p^2 \sim N - 1$. There are finitely many physical states at each possible grade $N$.

2.2 $G = SU(2) = S^3$

Since the group is compact, $k$ must be a half-integer, and other relevant data are $\eta_{ab} = \text{diag}(+1,+1,+1)$, $f_{abc} = \varepsilon_{abc}$ and $Q_{ad} = 2$. The resulting central charges for the bosonic and supersymmetric theories are then

$$c = \frac{3k}{k+1}, \quad \hat{c} = 3 - \frac{2}{k}. \quad (10)$$

The base representations are the standard unitary representations of SU(2), with states $|j,m\rangle$ labelled by their eigenvalues:

$$Q |j,m\rangle = j(j+1)|j,m\rangle, \quad J^3_{0} |j,m\rangle = m|j,m\rangle.$$

The condition for a unitary representation of the KM algebra is

$$j \leq k \quad (11)$$

for the bosonic theory; in the supersymmetric theory we simply replace $k \rightarrow \hat{k} = k - 1$. The physical state conditions are not relevant to this model if it is taken in isolation because there is no time-like direction in the target space. It can of course appear as a factor in some larger target space, as we shall see below.

2.3 $G = SU(1,1) = \text{AdS}_3$

Since the target manifold is non-compact, $k$ is not quantized and can apparently be chosen at will. The basic data are $\eta_{ab} = \text{diag}(+1,+1,-1)$, $f_{abc} = \varepsilon_{abc}$ and $Q_{ad} = -2$. The values of the bosonic and fermionic central charges are then

$$c = \frac{3k}{k-1}, \quad \hat{c} = 3 + \frac{2}{k}. \quad (12)$$

As in the compact case, base representations of the zero-grade SU(1,1) algebra are written $|j,m\rangle$ corresponding to eigenvalues

$$Q |j,m\rangle = -j(j+1)|j,m\rangle, \quad J^3_{0} |j,m\rangle = m|j,m\rangle. \quad (13)$$
Since SU(1,1) is non-compact, however, its unitary representations are necessarily infinite-dimensional, excepting the trivial representation consisting of the single state with $j = m = 0$.

There are several types of non-trivial unitary representations of SU(1,1). The discrete representations are:

$$D_j^\pm = \{ |j, \mp m\rangle : m = j, j-1, j-2, \ldots \} \quad \text{with} \quad J_0^\mp |j, \mp j\rangle = 0 \quad (14)$$

and they exist only for the values $j = -1/2, -1, -3/2, \ldots$, which explains their name. In contrast there are also continuous representations, called principal, for which $j = -1/2 + i\kappa$ ($\kappa$ real), and exceptional, for which $-1/2 \leq j < 0$, with the values of $m$ quantized in either case. These continuous representations have no highest- or lowest-weight states. Notice also that for both kinds of continuous representation $j(j+1) < 0$.

These are the only unitary representations of the group SU(1,1) itself. If we pass to the covering group, however, then there are more general representations of type $D_j^\pm$ in which $j$ and $m$ need not be half-integral (although the allowed values of $m$ within any irreducible representation always differ by integers) and similarly there are additional continuous representations with the same ranges of values for $j$ but with more general values allowed for $m$. The group manifold of SU(1,1) is topologically $\mathbb{R}^2 \times S^1$ (with the compact direction being time-like) and this is responsible for the quantisation of $m$ in units of half integers. By contrast, the simply-connected covering group is topologically $\mathbb{R}^3$ and so there is no quantization of $m$ in this case. Our treatment of the no-ghost theorem applies equally well to either SU(1,1) or its covering space.

2.4 AdS$_3$ and the spin-level restriction

The minus sign in the metric indicates that the generator $J_3^n$ plays the role of a time-like oscillator, creating negative norm states. The presence of such states in a covariantly quantized string theory is hardly surprising. The problem for the SU(1,1) theory is that some of these negative norm states also satisfy the physical state conditions and so would seem to be part of the physical spectrum. These difficulties only arise for states built on base representations belonging to the discrete series $D_j^\pm$, however. For all other base representations the mass-shell condition implies that physical states occur only at grade zero, and hence their norms are positive.

To exclude the unwanted ghost states, it was suggested that a restriction should be imposed on the spin $j$ of the discrete series base representation in terms of the level $k$ by analogy with the compact case. For the bosonic
SU(1, 1) theory the condition is

\[ |j| \leq k \]  

while for the fermionic theory we again replace \( k \rightarrow \hat{k} = k + 1 \). It is easy to check by some explicit calculations that this has the desired effect on states at low-lying grades. Moreover, it can be shown to imply a completely ghost-free spectrum, as we shall sketch below.

The adoption of the above condition may look superficially natural, but after more careful consideration it becomes clear that something rather subtle is occurring and that the condition is working quite differently in the compact and non-compact cases. In the SU(2) model the restriction (11) guarantees unitary representations of the entire Kac-Moody algebra. In the non-compact case it is impossible to construct unitary representations of the Kac-Moody algebra by (4) and (5), whether one imposes (15) or not. Instead (15) forces physical states within these non-unitary representations to have positive norm.

The condition (15) also has unusual implications for the spectrum of the theory. It can easily be combined with the mass-shell condition (1) to give

\[ N < 1 + \frac{k}{2} \]  

(which holds for both the bosonic and fermionic cases). Thus for a fixed level \( k \) the physical states in the theory can only arise at a finite number of grades. There are infinitely many physical states at every allowed grade, however, since the unitary base representations of SU(1, 1) are infinite-dimensional. This should be contrasted with the string in flat space, which has finitely many physical states at each of infinitely many grades.

3 The no-ghost theorem

3.1 Skeleton of the proof

The approach of Goddard and Thorn can be conveniently divided into three steps, each of which must be established in order to show that a given model is ghost free. (We consider only bosonic strings for simplicity.)

Let \( \mathcal{H} \) be the string Fock space and \( \mathcal{F} \) the subspace of transverse states defined by

\[ L_n|f\rangle = K_n|f\rangle = 0 \quad n > 0 \]  

where \( K_n \) are components of some chosen spin-1 current. We first require:

(a) The set of states

\[ |\{\lambda, \mu\} f\rangle = L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m} K_{+1}^{\mu_1} \cdots K_{+m}^{\mu_m}|f\rangle, \quad |f\rangle \in \mathcal{F} \]  

(18)
is a basis for $\mathcal{H}$.

We then define a subspace $\mathcal{K}$ to be the span of those states with $\lambda_i = 0$ and a subspace $\mathcal{S}$ of spurious states with at least one $\lambda_i \neq 0$. Clearly these subspaces are complementary: $\mathcal{H} = \mathcal{K} + \mathcal{S}$. Notice that a spurious state is orthogonal to any physical state. It is also easy to show that if $c = 26$ the operators $L_n$ ($n > 0$) map each of these subspaces into themselves: $\mathcal{S} \rightarrow \mathcal{S}$ and $\mathcal{K} \rightarrow \mathcal{K}$.

This has the important consequence that if $|\psi\rangle = |k\rangle + |s\rangle$ is Virasoro primary, then $|k\rangle$ and $|s\rangle$ are separately Virasoro primary. We must then establish:

(b) If $|k\rangle \in \mathcal{K}$ is physical, then it is transverse, $|k\rangle \in \mathcal{F}$.

(c) The inner-product on the transverse space $\mathcal{F}$ is positive definite.

It is now easy to see how to prove the theorem. If $|\psi\rangle = |k\rangle + |s\rangle$ is physical, then $|k\rangle$ is transverse and $||\psi\rangle||^2 = |||k\rangle||^2 \geq 0$, which is the desired result.

In the original proof for the string in flat space, important simplifications were achieved by choosing $\mathcal{K}_n$ to correspond to a light-like direction in Minkowski space. The proof of (a) could then be carried out by ordering the states at each grade in an intelligent way, while steps (b) and (c) become essentially trivial. For $\text{SU}(1,1)$ it is tempting to choose $\mathcal{K}_n$ along a light-like direction in a similar fashion but unfortunately the same simplifications do not occur. It is possible that the proof could be completed along these lines, but there are serious technical obstacles.

An alternative approach is to choose $\mathcal{K}_n$ corresponding to a time-like direction. This means that steps (a) and (b) above become considerably more difficult to establish, even for the string in flat space. Fortunately, we can make use of the powerful general approach to CFT that has been developed in the meantime, and in particular the Kac determinant formula. This allows us to establish steps (a) and (b) either in flat space, or for the target space $\text{SU}(1,1)$ with the restriction (15).

Even for $\mathcal{K}_n$ time-like, the final step (c) is immediate in flat space, but it is highly non-trivial for the case of $\text{SU}(1,1)$ and it is here that the restriction (15) again enters crucially. In fact the necessary result had already been established in a rather different context by Dixon et al. They showed that precisely for the restricted set of KM representations given by (15) the coset $\text{SU}(1,1)/\text{U}(1)$ has a positive-definite inner-product, where the $\text{U}(1)$ corresponds to a time-like direction. This completes the proof.

### 3.2 Ghost-free models

There are now a number of models of the form $\text{SU}(1,1)_k \times \mathcal{M}$ which are guaranteed to be ghost-free by the method sketched above. A bosonic theory with target space $\text{SU}(1,1)_k \times \mathbb{R}^d$ is ghost-free for any $d$ provided one chooses
the rather bizarre-looking value $k = (26 - d)/(23 - d)$. A more natural way to achieve $c = 26$ in the bosonic case is to combine the compact and non-compact WZW models, taking $\text{SU}(1, 1)_k \times \text{SU}(2)_\ell \times M^{20}$ with $k = \ell + 2$. Finally, there is a very natural supersymmetric version of this, $\text{SU}(1, 1)_k \times \text{SU}(2)_k \times M^4$ which has $\hat{c} = 10$ from (12). Notice that the compact and non-compact super WZW models appear here with the same level. This is exactly the combination of WZW models which appears in the simplest example of the AdS/CFT correspondence mentioned in the introduction.

The arguments sketched above guarantee only that the string theory has no ghosts at the free level. To demonstrate consistency of the interacting theory it would be necessary to show that crossing-symmetric amplitudes can be defined whose fusion rules close among the ghost-free representations. For the case of SU(2) the fusion rules close amongst the unitary representations, but it is not clear why this should be so for the ghost-free representations of the SU(1, 1) theory. This is an interesting topic for future work.

4 Comments

Finally, we return to the equivalence between type IIB string theory on $\text{AdS}_3 \times S^3 \times M^4$ and a certain superconformal field theory in two dimensions. On the string side this involves a RR background of $Q_1$ D1-branes and $Q_5$ D5-branes, while the corresponding CFT has as its target space a symmetric product of $k$ copies of $M^4$ with $k \sim Q_1Q_5$. The correspondence should hold when both $Q_1$ and $Q_5$ are large. It was then pointed out that since there are only finitely many chiral primary states in the CFT, there must be a corresponding “stringy exclusion principle” which forbids certain states from appearing in the string spectrum.

Under an S-duality transformation, this RR background becomes a conventional NS/NS background of $Q_1$ fundamental strings and $Q_5$ NS5-branes. For $Q_1 = 1$ a perturbative string analysis of the type we have described here is valid, and the supersymmetric version of the restriction on representations turns out to reproduce the same constraint as the stringy exclusion principle. Some caution is required in making this comparison, however, since in the first instance the AdS/CFT correspondence is expected to hold only if $Q_1$ is large. Nevertheless, because of U-duality, the analysis for $Q_1 = 1$ will have implications for the case $Q_1$ large and therefore the domains of validity of the constraints may in fact overlap. It would be interesting to check this in more detail.
Acknowledgments

We thank Peter Goddard, Michael Green, Juan Maldacena, Hugh Osborn and
Arkady Tseytlin for discussions and comments.

JME is grateful to the organizers of the Trieste Conference on Super Five-
branes and Physics in 5+1 Dimensions for the invitation to speak and to the
ICTP for warm hospitality. The research of JME is supported by a PPARC
Advanced Fellowship. MRG is grateful to Jesus College and Fitzwilliam Col-
lege, Cambridge for financial support.

References

1. P. Goddard, C.B. Thorn, Phys. Lett. B 40, 235 (1972).
2. R.C. Brower, Phys. Rev. D 6, 1655 (1972).
3. C.M. Hull, hep-th/9807127.
   C.M. Hull, R.R. Khuri, hep-th/9808060.
4. D. Gepner, E. Witten, Nucl. Phys. B 278, 493 (1986).
5. J. Balog, L. O’Raifeartaigh, P. Forgács, A. Wipf, Nucl. Phys. B 325,
   225 (1989).
6. P.M.S. Petropoulos, Phys. Lett. B 236, 151 (1990).
7. N. Mohammedi, Int. Journ. Mod. Phys. A 5, 3201 (1990).
8. S. Hwang, Nucl. Phys. B 354, 100 (1991).
   M. Henningson, S. Hwang, Phys. Lett. B 258, 341 (1991).
   M. Henningson, S. Hwang, P. Roberts Phys. Lett. B 267, 350 (1991).
9. S. Hwang, Phys. Lett. B 276, 451 (1992).
10. J.M. Evans, M.R. Gaberdiel and M.J. Perry, hep-th/9806024.
11. I. Bars, Phys. Rev. D 53, 3308 (1996).
12. Y. Satoh, Nucl. Phys. B 513, 213 (1998).
13. L.J. Dixon, M.E. Peskin, J. Lykken, Nucl. Phys. B 325, 329 (1989).
14. J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998).
15. J. Maldacena, A. Strominger, hep-th/9804085.
16. M. Cvetic, A. Tseytlin, Phys. Lett. B 366, 95 (1996).
   A. Tseytlin, Mod. Phys. Lett. A 11, 689 (1996).
17. J. de Boer, hep-th/9806104.
18. A. Giveon, D. Kutasov, N. Seiberg, hep-th/9806194.
19. D. Kutasov, F. Larsen, R.G. Leigh, hep-th/9812027.
20. J. de Boer, H. Ooguri, H. Robins, J. Tannenhauser, hep-th/9812046.