Accurate solution of near-colliding Prony systems via decimation and homotopy continuation

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Abstract

We consider polynomial systems of Prony type, appearing in many areas of mathematics. Their robust numerical solution is considered to be difficult, especially in “near-colliding” situations. We transform the nonlinear part of the Prony system into a Hankel-type polynomial system. Combining this representation with a recently discovered “decimation” technique, we present an algorithm which applies homotopy continuation to an appropriately chosen Hankel-type system as above. In this way, we are able to solve for the nonlinear variables of the original system with high accuracy when the data is perturbed.

1. Introduction

1.1. The Prony problem

Consider the following approximate algebraic problem.

Problem 1. Given \((\tilde{m}_0, \ldots, \tilde{m}_{N-1}) \in \mathbb{C}^N\), find \(s \in \mathbb{N}\), a multiplicity vector \(D = (d_1, \ldots, d_s) \in \mathbb{N}^s\) with \(d := \sum_{j=1}^s d_j\) and \(2d \leq N\), and complex numbers \(\{z_j, \{a_{\ell,j}\}_{\ell=0}^{d_j-1}\}_{j=1}^s\) with \(a_{d_j-1,j} \neq 0\) such that for some perturbation vector \((\epsilon_k) \in \mathbb{C}^N\) with \(|\epsilon_k| < \varepsilon\) we have

\[
\tilde{m}_k = \sum_{j=1}^s z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell + \epsilon_k, \quad k = 0, \ldots, N - 1. \tag{1}
\]
This so-called “confluent Prony problem” appears in signal processing, frequency estimation, exponential fitting, Padé approximation, sparse polynomial interpolation, spectral edge detection, inverse moment problems and recently in theory of super-resolution (see \cite{8,10,12,13,17,27,29} and references therein).

In this paper we assume that \( |z_j| = 1 \) for all \( j = 1, \ldots, s \), which is often the case in several of these applications.

Let us introduce some notation. The number of unknown parameters is denoted by \( R := d + s \). We also extensively use the notion of node separation, defined as follows.

**Definition 1.** Let \( x \) denote a point in the “data space”

\[
\mathbb{C}^R \ni x = (a_{0,1}, \ldots, a_{d-1,1}, z_1, \ldots, a_{0,s}, \ldots, a_{d-1,s}, z_s)^T
\]

such that \( |z_j| = 1 \) for \( j = 1, \ldots, s \). For \( i \neq j \), let \( \delta_{ij} := |\arg z_i - \arg z_j| \) with the convention that \( \delta_{ij} \leq \pi \). Furthermore, we denote

\[
\delta = \delta(x) := \min_{i \neq j} \delta_{ij}.
\]

The high degree of symmetry in the system of equations (1) allows to separate the problem into a linear and a nonlinear part. The basic observation (due to Baron de Prony \cite{30}) is that the sequence of exact measurements \( \{m_k\} \) satisfies a linear recurrence relation

\[
\sum_{\ell=0}^{d} m_{k+\ell}c_\ell = 0, \quad k \in \mathbb{N},
\]

where \( \{c_\ell\} \) are the coefficients of the “Prony polynomial” defined by

\[
P(x) := \prod_{j=1}^{s} (x - z_j)^{d_j} \equiv \sum_{\ell=0}^{d} c_\ell x^\ell.
\]

Thus, the system (1) can be solved for \( N = 2d \) by the following steps.

1. Using (4), recover the coefficients \( \{c_\ell\} \) of \( P(x) \) from a non-trivial vector in the nullspace of the Hankel matrix

\[
H_d := \begin{pmatrix}
\tilde{m}_0 & \cdots & \tilde{m}_{d-1} & \tilde{m}_d \\
\vdots & \ddots & \vdots & \vdots \\
\tilde{m}_{d-1} & \tilde{m}_d & \cdots & \tilde{m}_{2d-1}
\end{pmatrix}.
\]

2. Recover the nodes \( \{z_j\} \) by finding the roots of \( \tilde{P}(x) \) with appropriate multiplicities.

3. Given the nodes \( \{z_j\} \), recover the coefficients \( \{a_{\ell,j}\} \) by solving a linear Vandermonde inverse problem.
In many applications more than 2d measurements are available, and often the error has some decay/statistical distribution. Several additional algorithms have been proposed for solving (1) in the literature, such as MUSIC/ESPRIT, Approximate Prony Method, matrix pencils, Total Least Squares, Variable Projections (VARPRO) or $\ell_1$ minimization ([5, 9, 11, 12, 26, 24, 29] and references therein). While the majority of these algorithms perform well on simple and well-separated nodes (i.e. with $D = (1,1,\ldots,1)$ and $\delta \gg 0$), they are poorly adapted to handle either multiple/clustered nodes (the root extraction/eigenvalue computation becoming ill-conditioned), large values of $N$ (the quadratic cost function is highly non-convex w.r.t to the unknowns $z_j, a_{\ell,j}$) or non-Gaussian noise.

In this paper we are mainly concerned with accurate recovery of the nonlinear variables \{z\}_j^{s=1}. While undoubtedly important for applications, step 3 above is a relatively straightforward and well-researched procedure.

1.2. Limits of accuracy and “super-resolution”

Despite the somewhat disappointing performance of numerical algorithms for Prony systems in the high-multiplicity/colliding scenarios mentioned above, our recent studies [3, 5, 6] suggest that these problems are only partially due to the inherent sensitivity of the problem (i.e. problem conditioning). Generally speaking, introduction of confluent (high-order) nodes into the model leads, in some cases, to improved estimation of the parameters – as indicated by the reduced condition number of the problem. In particular, we argue that while for $N\delta \gg 1$ and $D = (1,1,\ldots,1)$ the existing methods might be optimal, there is a gap between theory and practice in the “super resolution case” $N\delta \ll 1$ and high multiplicity.

Another interesting phenomenon, discovered in [3] and further elaborated in [6], is the so-called “decimation”. This phenomenon essentially means that if $N\delta \ll 1$, then after taking the “decimated” sequences \{m_0, m_p, m_{2p}, \ldots, m_{(R-1)p}\}, where $p \in \mathbb{N}$ is not too large, as measurements, and solving the resulting square system, we get accuracy improvement of the order of $p^{-d_j}$ for the node $z_j$ compared to the error in the case $p = 1$. Numerical studies carried out in [6] indicate that in this case, the resulting accuracy is very close to the “best possible” one - as quantified by the “super-resolution condition number”. See Subsection 2.1 below for further details.

1.3. Our contribution

In the most general setting of the problem, the multiplicity structure $D$ is unknown a-priori and must be recovered from the data. In this paper we assume that it is known and fixed (but see discussion in Section 5), and concentrate on the accurate solution of the system (1) in the “super-resolution” regime $N\delta \ll 1$.

We propose a novel symbolic-numeric technique, “decimated homotopy”, for this task. The approach is an extension of the method used in [3] for the case $s = 1$ (i.e.only one node), and its main ingredients are:
1. decimating the measurements;
2. constructing a square polynomial system for the unknowns \( \{ z_j \} \) (in \([3]\) this was a single polynomial equation);
3. solving the resulting \textit{well-conditioned} system with high accuracy.

Step 2 above is a purely symbolic computation based on \([1]\) and \([5]\), while for step 3 we chose the homotopy continuation method for polynomial systems, due to the fact that it will provably find the solution.

1.4. Related work
Decimation can be considered as a certain type of regularization for Prony systems. As we discuss in \([6]\), it is related to other similar ideas in numerical analysis \([31]\) and signal processing \([22, 23, 25]\). In symbolic-numeric literature connected with sparse numerical polynomial interpolation (i.e. in the noisy setting), the possible ill-conditioning of the Hankel matrices \( H_d \) can be overcome either by random sampling of the nodes \( \{ z_j \} \) \([17, 19, 21]\) or by the recently introduced affine sub-sequence approach \([20]\) for outlier detection (see also \([13]\)), which is in many ways similar to decimation.

It would be interesting to establish more precise connections of our method to these works.

1.5. Organization of the paper
In Section 2 we discuss in detail the relevant prior work, in particular accuracy bounds on (decimated) Prony systems \([5, 6]\) and the algebraic reconstruction method for the case \( s = 1 \) from \([3, 4]\). The decimated homotopy algorithm is subsequently developed in Section 3. Results of numerical experiments are described in Section 4 while several future research directions are outlined in Section 5.

2. Accuracy of solving Prony systems, decimation and algebraic reconstruction

Following \([6]\), in Subsection 2.1 we present numerical stability bounds, including in the decimated scenario, for the system \([1]\). In Subsection 2.2 we discuss the “algebraic” reconstruction algorithm for the system \([1]\) with \( s = 1 \), used in \([3, 4]\), and highlight some of its key properties, in particular the effect of decimation on its accuracy.

2.1. Stability bounds and decimation

\textbf{Definition 2.} For any \( N \geq R \), let the “forward mapping” \( \mathcal{P}_N : \mathbb{C}^R \rightarrow \mathbb{C}^N \) be given by the measurements, i.e. for any \( \mathbf{x} \in \mathbb{C}^R \) (see \([2]\)) we have

\[ \mathcal{P}_N(\mathbf{x}) := (m_0, \ldots, m_{N-1})^T, \]

where \( m_k \) are given by \([1]\).
A standard measure of sensitivity \cite{14,32} for well-conditioned polynomial systems is the following.

**Definition 3.** Let $x \in \mathbb{C}^R$ be a point in the data space. Assume that $J_N(x) := dP_N(x)$, the Jacobian matrix of the mapping $P_N$ at the point $x$, has full rank. For $\alpha = 1, 2, \ldots, R$, the *component-wise condition number* of parameter $\alpha$ at the data point $x \in \mathbb{C}^R$ is the quantity

$$CN_{\alpha,N}(x) := \sum_{i=1}^{N} |J_N^+(x)_{\alpha,i}|,$$  \hspace{1cm} (7)

where $J_N^+$ is the Moore-Penrose pseudo-inverse of $J_N$.

In \cite{6} we show that for $N\delta \gg 1$, the Prony system (1) is well-conditioned as follows.

**Theorem 1** (Theorem 2.4 in \cite{6}). Let $x \in \mathbb{C}^R$ be a data point, such that $\delta = \delta(x) > 0$ and $a_{d_j-1,j} \neq 0$ for $j = 1, \ldots, s$. Then

1. The Jacobian matrix $J_N(x) = dP_N(x) \in \mathbb{C}^{N \times R}$ has full rank.
2. There exist constants $K, C^{(1)}$, not depending on $N$ and $\delta$, such that for $N > K \cdot \delta^{-1}$:

$$CN_{z_j,N}(x) \leq C^{(1)} \cdot \frac{1}{|a_{d_j-1,j}|} \cdot \frac{1}{N^{d_j}}.$$ \hspace{1cm} (8)

On the other hand, as numerical experiments in \cite{6} show, when $N\delta \to 0$ then the growth of $CN_{z_j,N}$ is much more rapid than $N^{d_j}$, and in fact $CN_{z_j,N} \to \infty$. As we now argue, this “phase transition” near $N\delta \sim O(1)$ can be partially quantified by considering a sequence of decimated square systems.

Fixing $N = R$, we have the following upper bound, which is tight.

**Theorem 2** (Theorem 2.3 in \cite{6}). Assume the conditions of Theorem 7, and furthermore that $N = R$. Then there exists a constant $C^{(2)}$, not depending on $x$ (and in particular on $\delta$), such that:

$$CN_{z_j,R}(x) \leq C^{(2)} \cdot \left(\frac{1}{\delta}\right)^{R-d_j} \cdot \frac{1}{|a_{d_j-1,j}|}.$$ \hspace{1cm} (9)

A natural question is whether increasing $N$ can essentially improve the bound (9) above. One possible answer is given by what we call “decimation”, as follows.

**Definition 4.** Let $p \in \mathbb{N}$ be a positive integer. The *decimated Prony system with parameter $p$* is given by

$$n_k := m_{pk} = \sum_{j=1}^{s} \sum_{l=0}^{d_j-1} (a_{k,j}p^l) k^\ell, \quad k = 0, 1, \ldots, R - 1.$$ \hspace{1cm} (10)

\footnote{It is easy to show (see e.g. \cite{3}, Appendix A) that the upper bound on $CN_{z_j,N}$ is asymptotically tight.}
Definition 5. The decimated forward map $P^{(p)} : \mathbb{C}^R \rightarrow \mathbb{C}^R$ is given by
$$P^{(p)}(x) := (n_0, \ldots, n_{R-1}),$$
where $x \in \mathbb{C}^R$ is as in (2) and $n_k$ are given by (10).

Definition 6. The decimated condition numbers $CN^{(p)}_N(\alpha)$ are defined as
$$CN^{(p)}_N(\alpha)(x) := \sum_{i=1}^{R} \left| \left( J^{(p)}(x) \right)^{-1} \right|_{\alpha,i},$$
where $J^{(p)}(x)$ is the Jacobian of the decimated map $P^{(p)}$ (the definition applies at every point $x$ where the Jacobian is non-degenerate).

The usefulness of decimation becomes clear given the following result.

Theorem 3 (Corollary 3.2 in [6]). Assume the conditions of Theorem 7. Assume further that $N\delta^* < \pi R$ where $\delta^* := \max_{i \neq j} \delta_{ij}$ (i.e. all nodes form a cluster). Then the condition numbers of the decimated system with parameter $p^* := \left\lfloor \frac{N}{R} \right\rfloor$ satisfy
$$CN^{(p^*)}_N(\alpha)(x) \leq C^{(3)} \cdot \left( \frac{1}{\delta} \right)^{R-d_j} \frac{1}{|a_{t_{j-1},j}|} \cdot \frac{1}{N^R}.$$

The intuition behind this result is that decimation with parameter $p$ is in fact equivalent to applying the Prony mapping $P_R$ to a rescaled data point $y := R_p(x)$, where
$$R_p \left( (a_{0,1}, \ldots, a_{d_1-1,1}, z_1, \ldots, a_{0,s}, \ldots, a_{d_s-1,s}, z_s)^T \right) :=
(b_{0,1}, \ldots, b_{d_1-1,1}, w_1, \ldots, b_{0,s}, \ldots, b_{d_s-1,s}, w_s)^T =
(a_{0,1} \cdot p^{0}, \ldots, a_{d_1-1,1} \cdot p^{d_1-1}, z_1^p, \ldots, a_{0,s} \cdot p^0, \ldots, a_{d_s-1,s} \cdot p^{d_s-1}, z_s^p)^T.$$

Since for small $\delta$ we have that $\min_{i \neq j} |z_i^p - z_j^p| \approx \delta p$, (12) follows from the above and (9).

Experimental evidence suggests that decimation is nearly optimal in the “super-resolution” region, i.e.
$$CN^{(p^*)}(\alpha) \approx CN_{z_j}(x), \quad N\delta^* < \pi R.$$

We believe that it is an important question to provide a good quantification of (14).

From practical perspective, this suggests a “nearly-optimal” approach to numerically solving the system (10) when all nodes are clustered - namely, to pick up the
$R$ evenly spaced measurements $\{m_0, m_p, \ldots, m_{(R-1)p}\}$ and solve the resulting square system.

An important caveat of the decimation approach is that it introduces aliasing for the nodes - indeed, the system (10) has $w_j = z_j^p$ as the solution instead of $z_j$, and therefore after solving (10), the algorithm must select the correct value for the $p^{th}$ root ($\tilde{w}_j^p$). Thus, either the algorithm should start with an approximation of the correct value (and thus decimation will be used as a fine-tuning technique), or it should choose one among the $p$ candidates - for instance, by calculating the discrepancy with the other measurements, which were not originally utilized in the decimated calculation.

2.2. Algebraic reconstruction

Although many solution methods for the system (11) exist, as we mentioned they are not well-suited for dealing with multiple roots/eigenvalues. While averaging might work well in practice, it is difficult to analyze rigorously, and in particular to prove the resulting method’s rate of convergence.

In [3, 4] we developed a method based on accurate solution of Prony system for resolving the Gibbs phenomenon, i.e. for accurate recovery of a piecewise-smooth function from its first $N$ Fourier coefficients. This problem arises in spectral methods for numerical solutions of nonlinear PDEs with shock discontinuities, and was first investigated by K.Eckhoff in the 90’s [15]. The key problem was to develop a method which, given the left-hand side $\{m_k\}_{k=0}^{N-1}$ of (11) with error decaying as $|\Delta m_k| \sim k^{-1}$, would recover the nodes $\{z_j\}$ with accuracy not worse than $|\Delta z_j| \sim N^{-d-j-1}$.

Our solution was based on two main ideas:

1. Due to the specifics of the problem, it was sufficient to provide a solution method as described above in the case of a single node, i.e. $s = 1$.
2. The resulting system was solved by decimation, elimination of the linear variables $\{a_{t,j}\}$, and polynomial root finding.

The elimination step is a direct application of the recurrence relation (11) for the coefficients of the Prony polynomial (5), as follows. The (unperturbed) system (11) for $s = 1$ reads

$$m_k = z^k \sum_{\ell=0}^{d-1} a_{k\ell} k^\ell. \quad (15)$$

The corresponding decimated system (16) with parameter $p$ is

$$n_k = m_{pk} = (z^p)^k \sum_{\ell=0}^{d-1} (a_{p\ell}) k^\ell, \quad k = 1, 2, \ldots, d+1.$$

Denote $\rho := z^p$. Then clearly the sequence $\{n_k\}$ satisfies $\sum_{\ell=0}^{d} n_k c_{\ell} = 0$, where the Prony polynomial is just $(x - \rho)^d = \sum_{\ell=0}^{d} c_{\ell} x^\ell$. That is, $c_{\ell} = (-1)^\ell (\frac{d}{\ell}) \rho^{d-\ell}$
Algorithm 1 Algebraic reconstruction of a single node

1. Set decimation parameter to $p^* := \left\lfloor \frac{N}{d+1} \right\rfloor$.

2. Construct the polynomial $\tilde{q}_{p^*}(u)$ from the given perturbed measurements $\{\tilde{m}_{k}\}_{k=0}^{N-1}$:
   \[
   \tilde{q}_{p^*}(u) := \sum_{\ell=0}^{d} (-1)^\ell \binom{d}{\ell} \tilde{m}_{p^*(\ell+1)} u^{d-\ell}.
   \]

3. Set $\tilde{\rho}$ to be the root of $\tilde{q}_{p^*}$ closest to the unit circle in $\mathbb{C}$.

4. Choose the solution $z^*$ to (15) among the $p^*$ possible values of $(\rho^*)^1_{p^*}$ according to available a-priori approximation.

and we obtain that $\rho$ is one of the roots of the unperturbed polynomial
\[
q_p(u) := \sum_{\ell=0}^{d} (-1)^\ell \binom{d}{\ell} n_{\ell+1} u^{d-\ell}.
\] (16)

The algebraic reconstruction method for $s = 1$ ([7, Algorithm 2]) is summarized on this page.

The key result of [7] is that as $N \to \infty$ (and therefore $p^* \to \infty$ as well), and assuming perturbation of size $\varepsilon$ for the coefficients $\{\tilde{m}_{k}\}$, all the $d$ roots of $\tilde{q}_{p^*}$ remain simple and well-separated, while the corresponding perturbation of the root $\rho^*$ is bounded by
\[
|\tilde{\rho} - \rho| \lesssim N^{-(d-1)} \varepsilon \implies |\tilde{z} - z| \lesssim N^{-d} \varepsilon.
\]

Thus, the method is optimal - recall the condition estimate (5).

Remark 1. Decimation acts as a kind of regularization for the otherwise ill-conditioned multiple root. To see why, consider the case $d = 2$. Then we have
\[
m_k = z^k (a_0 + ka_1).
\]

The Prony polynomial is $P(x) = (x - z)^2$, and thus for each $k \in \mathbb{N}$ the point $z$ is a root of
\[
q^\#_k(u) := m_k u^2 - 2um_{k+1} + m_{k+2}.
\]

As $k \to \infty$, the above polynomial “approaches in the limit”
\[
\frac{q^\#_k(u)}{k} \to a_1 z^k (u - z)^2.
\]

Thus, a “non-decimated” analogue of Algorithm 1 (such as [8, 15]) would be recovering an “almost double” root $z^*$, and it is well-known that the accuracy of reconstruction in this case is only of the order $\sqrt{\varepsilon}$ when the data is perturbed.
by $\varepsilon$. On the other hand, $q_p(u) = m_p u^2 - 2u m_2 p + m_3 p$, and as $p \to \infty$ it is easy to see that

$$
\frac{q_p(u)}{p} \to a_1 \rho \left( u^2 - 4\rho u + 3\rho^2 \right) = a_1 \rho (u - \rho) (u - 3\rho),
$$
i.e. the limiting roots are well-conditioned.

### 3. Decimated homotopy algorithm

In this section we develop the decimated homotopy algorithm, which is a generalization of Algorithm 1 to the case $s > 1$. We assume that the multiplicity $D$ is known, and the noise level is small enough so that accurate recovery of the nodes by solving the decimated system (10) according to Theorem 3 is possible.

Recall that the feasible solutions are restricted to the complex torus $T^s := \{ z \in \mathbb{C}^s : |(z)_i| = 1, i = 1, \ldots, s \}$.

We also assume that an initial approximation to the desired solution $z$ is available with accuracy $\sim O(N^{-1})$, i.e. the algorithm is provided with $z_{\text{init}} \in T^s$ and $\eta > 0$ such that

$$
|z_{\text{init}} - z| \leq \eta. 
$$

#### 3.1. Construction of the system

Consider the decimated system (10) with fixed parameter $p$. Denote $w_j := z_{p_j}^p$. The decimated measurements $\{n_k\}_{k=0}^{R-1}$ satisfy for each $k = 0, \ldots, s - 1$

$$
\sum_{i=0}^{d} n_{k+i} c_i = 0,
$$
where $c_i$ are the coefficients of the Prony polynomial

$$
P(x) = \prod_{j=1}^{s} (x - w_j)^{d_j} = \sum_{\ell=0}^{d} c_{\ell} x^{\ell}.
$$

Let $\sigma_i(x_1, \ldots, x_d)$ denote the elementary symmetric polynomial of order $i$ in $d$ variables. Then we have

$$
c_{\ell} = (-1)^{d-\ell} \sigma_{d-\ell} \left( w_{1, \ldots, 1}, w_{1, \ldots, 1, w_{s, \ldots, s}} \right) := \tau_{\ell} (w_1, \ldots, w_s). 
$$

Thus the point $w = (w_1, \ldots, w_s) \in T^s$ is a zero of the $s \times s$ polynomial system

$$
\left\{ f_k^{(p)}(u) := \sum_{i=0}^{d} n_{k+i} \sigma_i(u) = 0 \right\}_{k=0, \ldots, s-1}.
$$
This Hankel-type system is therefore our proposed generalization to the polynomial equation (16).

**Example 1.** \( s = 2, \ d_j = 2. \) The system (19) reads

\[
\begin{bmatrix}
  f_0(u) \\
  f_1(u)
\end{bmatrix} = \begin{bmatrix}
  n_0 u_1^3 u_2^2 + n_1 (2u_1^2 u_2 - 2u_2 u_1^2) + n_2 (u_1^4 + 4u_1 u_2 + u_2^4) + n_3 (2u_1 - 2u_2) + n_4 \\
  n_1 u_1^2 u_2^3 + n_2 (2u_1^2 u_2 - 2u_2 u_1^2) + n_3 (u_1^4 + 4u_1 u_2 + u_2^4) + n_4 (2u_1 - 2u_2) + n_5
\end{bmatrix}
\]

3.2. Finding the solution

Generalizing the root finding step of Algorithm 1 we propose to use the homotopy continuation method in order to find all the solutions of the (perturbed) system (19).

While a-priori it is not clear whether the variety defined by (19) has positive-dimensional components, we assume that there exists a neighborhood of \( w \) without such components.

We now consider the question of how to recover the correct solution of the original problem (11) from among all the isolated solutions

\[ S = \{u_1, \ldots, u_S\} \]

of (19). Two essential issues immediately arise. Based on the case of one node, we provide heuristics for both issues which work well in practice. Rigorous proof of their validity is left for future research (see below).

1. **Spurious solutions.** In general all the solutions of the perturbed system (19) will be infeasible, i.e. \( \tilde{u}_i \notin \mathbb{T}^s \). Since the noise is small, we propose to select the solution which has smallest distance to \( \mathbb{T}^s \):

\[
\begin{align*}
  u^* &\leftarrow \arg\min_{u_k \in S} \max_{i=1,\ldots,s} |1 - \| (u_k)_i \|| \\
  (u^*)_i &\leftarrow \frac{(u^*)_i}{\| (u^*)_i \|}.
\end{align*}
\]

(20)

2. **Aliasing.** Given a solution \( u^* \) to the \( p \)-decimated system (19), there are in general \( p^s \) possible corresponding solutions to (11), namely

\[ Z_p (u^*) := \{(z_1^*, \ldots, z_s^*) \in \mathbb{T}^s : (z_i^*)^p = (u^*)_i \}. \]

Two possible heuristics are proposed:

(a) select the solution satisfying the a-priori approximation (17), i.e.

\[
  z^* \in \{ z \in Z_p (u^*) : |z - z_{init}| \leq \eta \}.
\]

(21)

(b) select the solution giving the smallest residual for the non-decimated equations (19), i.e.

\[
  z^* := \arg\min_{z \in Z_p (u^*)} \sum_{k=0}^{N-d-1} \left| f_k^{(1)}(z) \right|.
\]

(22)

For large \( p \) this heuristic is time-prohibitive.

The decimated homotopy algorithm is summarized on the next page.
Algorithm 2. Decimated homotopy algorithm

Given: \((\tilde{m}_0, \ldots, \tilde{m}_{N-1}) \in \mathbb{C}^N\), multiplicity \(D\), initial approximation \(z_{\text{init}}\), cutoff \(\eta\).

1. Set decimation parameter \(p^* = \lfloor \frac{N}{R} \rfloor\).
2. Construct the system
   \[
   \mathcal{H}_{p^*} : \left\{ f_k^{(p^*)}(u) := \sum_{i=0}^{d} \tilde{m}_{p^*(k+i)}(u) \tau_i(u) = 0 \right\}_{k=0,\ldots,s-1}.
   \]
3. Solve \(\mathcal{H}_{p^*}\) by homotopy continuation method. Let \(S\) be the collection of its isolated solutions.
4. Select \(u^* \in S\) according to the heuristic (20).
5. Select \(z^* \in \mathcal{Z}_{p^*}(u^*)\) according to either (21) or (22).

3.3. Initial analysis

While the results of [3, 4] provide rigorous proof that Algorithm 1 achieves the best possible accuracy in the case \(s = 1\), we do not yet have an analogous result for Algorithm 2. As a first step towards this goal, we show that the solution \(w\) of (19) is well-conditioned. The proof is presented in Appendix A.

Theorem 4. Let the conditions of Theorem 3 be satisfied. At \(u = w\), the Jacobian matrix of the system (19) is non-degenerate.

Remark. The above result holds for the solution \(w\), but not necessarily for the other solutions of (19).

In order to show that Algorithm 2 is effective, we hope that it is possible to analyze the limiting system (19) (as \(p \to \infty\)) and show that all its solutions remain well-separated. We leave this question for future research.

4. Numerical experiments

4.1. Setup

We chose the model (11) with two closely spaced nodes, varying multiplicity and random linear coefficients \(\{a_{\ell,j}\}\). Choosing the overall number of measurements to be relatively high (1000-4000), we varied the decimation parameter \(p\) and compared the reconstruction error for Algorithm 2 (we implemented only the heuristic (21) for the de-aliasing step, since (22) is time prohibitive) and the generalized ESPRIT algorithm [1, 2, 33] (see also [5]), one of the best performing subspace methods for estimating parameters of the Prony systems (11) with white Gaussian noise \(\epsilon_k\). The noise level in our experiments was relatively small.

In addition to the reconstruction error, for each run we also computed both the full and decimated condition numbers \(CN_{z_j, N}\) and \(CN_{z_j}^{(p)}\) from their respective definitions (7) and (11).
Additional implementation details:

1. PHCPACK $[13]$ Release 2.3.96 was used as the homotopy continuation solver. It was called via its MATLAB interface PHCLab $[18]$.
2. We used the value $\eta \approx N^{-1}$ for the heuristic $[21]$.
3. The node selection in generalized ESPRIT was done via $k$-means clustering on the output of the eigenvalue step.

4.2. Results

The results of experiments are presented in Figure 1 on page 13. They can be summarized as follows:

1. The accuracy of Decimated Homotopy (DH) surpasses ESPRIT by several significant digits in the “super-resolution” region $N\delta \ll 1$.
2. DH achieves desired accuracy in larger number of cases.

Some additional remarks:

1. The number of solutions of the system $[19]$ was equal to $s!d^s$ ($s=$ number of nodes, $d=d_j=$ degree).
2. Running times are better for DH when $p$ is large, because the selection step of Algorithm 2 is $O(N)$, while the cost of full SVD for ESPRIT is $O(N^2R)$.
3. Condition number estimates are somewhat pessimistic, nevertheless indicating the order of error decay in a relatively accurate fashion. The periodic pattern is well-predicted by the theory, see $[6]$.

5. Conclusions and future work

In this paper we presented a novel algorithm, Decimated Homotopy, for numerical solution of systems of Prony type $[1]$ with nodes on the unit circle, $|z_j|=1$ which are closely spaced. Numerical experiments show that the algorithm works well in practice, providing reconstruction accuracy several orders of magnitude better than the standard ESPRIT algorithm. It would be desirable to provide rigorous proof the for convergence of the algorithm, similar to what was done for the special case $s=1$ in $[3, 4]$.

Another important question of interest is robust detection of these near-singular situations, i.e. correct identification of the collision pattern $D$. While the integer $d$ can be estimated via numerical rank computation of the Hankel matrix $H_d$ $[9]$ (see e.g. $[11]$ and also a randomized approach $[21]$), the determination of the individual components of $D$ is a more delicate task, which requires an accurate estimation of the distance from the data point to the nearest “pejorative” manifold of larger multiplicity, and comparing it with the a-priori bound $\varepsilon$ on the error. We hope that the present (and future) symbolic-numeric techniques such as $[14, 28]$, combined with description of singularities of the Prony mapping $\mathcal{P}_N$ $[8]$, will eventually provide a satisfactory answer to this question.
Figure 1: Decimated Homotopy (DH) vs. ESPRIT. Also plotted are condition numbers, both full and decimated, as well as the threshold $\eta$ used in the experiments.
References

[1] R. Badeau, B. David, and G. Richard. High-resolution spectral analysis of mixtures of complex exponentials modulated by polynomials. *IEEE Transactions on Signal Processing*, 54(4):1341–1350, 2006.

[2] R. Badeau, B. David, and G. Richard. Performance of ESPRIT for estimating mixtures of complex exponentials modulated by polynomials. *IEEE Transactions on Signal Processing*, 56(2):492–504, 2008.

[3] D. Batenkov. Complete Algebraic Reconstruction of Piecewise-Smooth Functions from Fourier Data. *To appear in Mathematics of Computation*.

[4] D. Batenkov and Y. Yomdin. Algebraic Fourier reconstruction of piecewise smooth functions. *Mathematics of Computation*, 81:277–318, 2012.

[5] D. Batenkov and Y. Yomdin. On the accuracy of solving confluent Prony systems. *SIAM J. Appl. Math.*, 73(1):134–154, 2013.

[6] Dmitry Batenkov. Numerical stability bounds for algebraic systems of Prony type and their accurate solution by decimation. *arXiv preprint arXiv:1409.3137*, 2014.

[7] Dmitry Batenkov. Prony Systems via Decimation and Homotopy Continuation. In *Proceedings of the 2014 Symposium on Symbolic-Numeric Computation*, SNC ’14, pages 59–60, New York, NY, USA, 2014. ACM.

[8] Dmitry Batenkov and Yosef Yomdin. Geometry and Singularities of the Prony mapping. *Journal of Singularities*, 10:1–25, 2014.

[9] B. Beckermann, G. H. Golub, and G. Labahn. On the numerical condition of a generalized Hankel eigenvalue problem. *Numerische Mathematik*, 106(1):41–68, March 2007.

[10] Michael Ben-Or and Prasoon Tiwari. A deterministic algorithm for sparse multivariate polynomial interpolation. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, pages 301–309. ACM, 1988.

[11] James A. Cadzow. Total Least Squares, Matrix Enhancement, and Signal Processing. *Digital Signal Processing*, 4(1):21–39, January 1994.

[12] E. Candes and C. Fernandez-Granda. Towards a mathematical theory of super-resolution. *Communications on Pure and Applied Mathematics*, 67(6):906–956, June 2014.

[13] Matthew T. Comer, Erich L. Kaltofen, and Clément Pernet. Sparse polynomial interpolation and Berlekamp/Massey algorithms that correct outlier errors in input values. In *Proceedings of the 37th International Symposium on Symbolic and Algebraic Computation*, pages 138–145. ACM, 2012.
[14] Barry Dayton, Tien-Yien Li, and Zhonggang Zeng. Multiple zeros of non-linear systems. *Mathematics of Computation*, 80(276):2143–2168, 2011.

[15] K.S. Eckhoff. Accurate reconstructions of functions of finite regularity from truncated Fourier series expansions. *Mathematics of Computation*, 64(210):671–690, 1995.

[16] S. Elaydi. *An Introduction to Difference Equations*. Springer, 2005.

[17] Mark Giesbrecht, George Labahn, and Wen-shin Lee. Symbolic-numeric sparse interpolation of multivariate polynomials. *Journal of Symbolic Computation*, 44(8):943–959, August 2009.

[18] Yun Guan and Jan Verschelde. PHClab: a MATLAB/Octave interface to PHCpack. In *Software for Algebraic Geometry*, pages 15–32. Springer, 2008.

[19] Erich Kaltofen and Wen-shin Lee. Early termination in sparse interpolation algorithms. *Journal of Symbolic Computation*, 36(3–4):365–400, September 2003.

[20] Erich Kaltofen and Clément Pernet. Sparse Polynomial Interpolation Codes and their decoding beyond half the minimal distance. In *ISSAC 2014 Proc. 39th Internat. Symp. Symbolic Algebraic Comput.*, pages 280–287, 2014.

[21] Erich L. Kaltofen, Wen-shin Lee, and Zhengfeng Yang. Fast estimates of Hankel matrix condition numbers and numeric sparse interpolation. In *Proceedings of the 2011 International Workshop on Symbolic-Numeric Computation*, pages 130–136. ACM, 2012.

[22] S.H. Kia, H. Henao, and G.-A. Capolino. A high-resolution frequency estimation method for three-phase induction machine fault detection. *IEEE Transactions on Industrial Electronics*, 54(4):2305–2314, August 2007.

[23] Yong-Hwa Kim, Young-Woo Youn, Don-Ha Hwang, Jong-Ho Sun, and Dong-Sik Kang. High-resolution parameter estimation method to identify broken rotor bar faults in induction motors. *IEEE Transactions on Industrial Electronics*, 60(9):4103–4117, September 2013.

[24] Wen-shin Lee. From Quotient-difference to Generalized Eigenvalues and Sparse Polynomial Interpolation. In *Proceedings of the 2007 International Workshop on Symbolic-numeric Computation*, SNC ’07, pages 11–116, New York, NY, USA, 2007. ACM.

[25] I. Maravic and M. Vetterli. Sampling and reconstruction of signals with finite rate of innovation in the presence of noise. *IEEE Transactions on Signal Processing*, 53(8 Part 1):2788–2805, 2005.

[26] Dianne P. O’Leary and Bert W. Rust. Variable projection for nonlinear least squares problems. *Computational Optimization and Applications*, 54(3):579–593, 2013.
Appendix A. Proof of Theorem 4

Let $x \in \mathbb{C}^R$ be a data point \((2)\) satisfying the conditions of Theorem 3 and $p^*$ be the corresponding decimation parameter. The corresponding decimated measurements $n_k$ are defined as in \((10)\):

$$n_k = \sum_{j=1}^{s} w_j^k \sum_{\ell=0}^{d_j-1} b_{\ell,j} k^\ell,$$

where $w_j = z_j^{p^*}$ and $b_{\ell,j} = a_{\ell,j} p^{\ell}$. Let $D \in \mathbb{C}^{s \times s}$ be the Jacobian matrix of the system \((19)\) at the point $u = w := (w_1, \ldots, w_s)$:

$$D := \left( \frac{\partial f_k}{\partial u_j} \bigg|_w \right)_{j=1,\ldots,s}^{k=0,\ldots,s-1}.$$

Theorem 4 claims that $D$ is invertible.

Lemma 1. For $j = 1, \ldots, s$ and arbitrary $k \in \mathbb{N}$ we have

$$\frac{\partial f_k}{\partial u_j} \bigg|_w = -d_j w_j^{k+d_j-1} b_{d_j-1,j} \prod_{i \neq j} (w_j - w_i)^{d_i}.$$
Proof. Considering the coefficients \( n_k \) as functions of \( w \), we have the identity

\[
f_k \{ n_k (w) \}, w = \sum_{i=0}^{d} n_{k+i} (w) \tau_i (w) \equiv 0.
\]

Thus, for each \( w_j \) the total derivative \( \frac{df_k}{dw_j} \{ n_k \}, u \) vanishes on \( u(w) = w \). By the chain rule

\[
\frac{df_k}{dw_j} (\{ n_k \}, u) = \frac{\partial f_k}{\partial n_{k+i}} \frac{\partial n_{k+i}}{\partial w_j} + \sum_{\ell=1}^{s} \frac{\partial f_k}{\partial u_\ell} \frac{\partial u_\ell}{\partial w_j} = \sum_{i=0}^{d} \tau_i (w) (k + i) w_j^{k+i-1} \sum_{\ell=0}^{d_j-1} b_{j,j} (k + i)^\ell + \frac{\partial f_k}{\partial u_j} (w) = 0.
\]

Let \( r_j (k) \) denote the following polynomial in \( k \) of degree \( d_j \):

\[
r_j (k) := \sum_{\ell=0}^{d_j-1} b_{j,j} k^{\ell+1}.
\]

Then we have

\[
\frac{\partial f_k}{\partial u_j} (w) = -w_j^{k-1} \sum_{i=0}^{d} w_j^i \tau_i (w) r_j (k + i). \tag{A.1}
\]

We now employ standard tools from finite difference calculus \[16\]. Consider the right-hand side of (A.1) as a discrete sequence depending on a running index \( k \). Let \( E = E_k \) denote the discrete shift operator in \( k \), i.e. for any discrete sequence \( g(k) \) we have

\[
E g(k) = (E g)(k) = g(k + 1).
\]

Let us further denote by \( \Delta := E - I \) the discrete differentiation operator (I is the identity operator). Now consider the difference operator

\[
\mathcal{E}_j := \prod_{i=1}^{s} (w_j E - w_i I)^{d_i}. \tag{A.2}
\]

Recall the definition of \( \tau_i \) from \[15\]. Opening parenthesis, we obtain that for any \( g(k) \)

\[
\mathcal{E}_j g(k) = \sum_{i=0}^{d} w_j^i \tau_i (w) g(k + i).
\]

Therefore

\[
\frac{\partial f_k}{\partial u_j} (w) = -w_j^{k-1} \mathcal{E}_j r_j (k).
\]
Since the linear factors in \((A.2)\) commute, we proceed as follows:

\[-w_j^{k-1} E_j r_j (k) = -w_j^{k-1} \prod_{i \neq j} (w_j - w_i) d_i (w_j \Delta) d_j r_j (k). \tag{A.3}\]

It is an easy fact (e.g. \([16]\)) that for any polynomial \(p (k)\) of degree \(n\) and leading coefficient \(a_0\), we have that

\[\Delta^n p (k) = a_0 n!.\]

Since \(r_j (k)\) has degree \(d_j\), we obtain that

\[\Delta^{d_j} r_j (k) = d_j! b_{d_j - 1,j}. \tag{A.4}\]

Furthermore, applying the operator \(w_j E - w_i I\) to a constant sequence \(c(k)=c\) gives

\[(w_j E - w_i I) c = (w_j - w_i) c. \tag{A.5}\]

Plugging \((A.4)\) and \((A.5)\) into \((A.3)\) we get:

\[\frac{\partial f_k}{\partial u_j} (\mathbf{w}) = -w_j^{k+d_j - 1} d_j! b_{d_j - 1,j} \prod_{i \neq j} (w_j - w_i)^{d_i},\]

completing the proof of Lemma \([11]\) \(\Box\)

**Example 2.** For \(s = 3, d_j = 2\) we have:

\[D = \begin{bmatrix}
-2b_{21} w_1 (w_1 - w_2)^2 (w_1 - w_3)^2 & -2b_{21} w_2 (w_1 - w_2)^2 (w_1 - w_3)^2 & -2b_{21} w_3 (w_1 - w_2)^2 (w_1 - w_3)^2 \\
-2b_{22} w_1 (w_1 - w_2)^2 (w_1 - w_3)^2 & -2b_{22} w_2 (w_1 - w_2)^2 (w_1 - w_3)^2 & -2b_{22} w_3 (w_1 - w_2)^2 (w_1 - w_3)^2 \\
-2b_{23} w_1 (w_1 - w_2)^2 (w_1 - w_3)^2 & -2b_{23} w_2 (w_1 - w_2)^2 (w_1 - w_3)^2 & -2b_{23} w_3 (w_1 - w_2)^2 (w_1 - w_3)^2 \\
\end{bmatrix}\]

Let \(V (\mathbf{w})\) denote the \(s \times s\) Vandermonde matrix on the nodes \(\{w_1, \ldots, w_s\}\). For example, if \(s = 3\) we have:

\[V (\mathbf{w}) = \begin{bmatrix}
1 & 1 & 1 \\
w_1 & w_2 & w_3 \\
1 & w_2 & w_3 \\
\end{bmatrix}.\]

Let \(y = R_{p^*} (\mathbf{x})\) where \(R_{p^*}\) is the scaling mapping \([13]\). Denote by \(B (y)\) the following \(s \times s\) diagonal matrix:

\[B (y) := \text{diag}_{j=1,\ldots,s} \left\{-d_j! b_{d_j - 1,j} \prod_{i \neq j} (w_j - w_i)^{d_i}\right\}.\]

By Lemma \([11]\) we have the factorization

\[D (y) = V (\mathbf{w}) B (y).\]

According to our assumptions, both \(V (\mathbf{w})\) and \(B (y)\) are non-singular (in particular, since \(a_{d_j - 1,j} \neq 0\) and \(\min_{i \neq j} |w_i - w_j| > 0\)). This completes the proof of Theorem \([4]\)