\textbf{SO(5)$_q$ and Contraction}

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\textbf{Abstract}

Representations of $SO(5)_q$ are constructed explicitly on the Chevalley basis for all $q$, generic and root of unity. Matrix elements of the generators are obtained for all representations depending on three variable indices, the maximal number being 4. A prescription for contraction is given such that a complete Hopf algebra is immediately obtained for the non-semisimple contracted case. For $q$ a root of unity the periodic representations for $SO(5)_q$ and the contracted algebra are obtained directly in the "fractional part" formalism which unifies the treatments for the generic and root of unity cases. The $q$-deformed quadratic Casimir operator is explicitly evaluated for the representations presented.
In our notations and conventions with the Cartan matrix

\[ \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \]  

(1)

and \([x] \equiv (q^x - q^{-x})/(q - q^{-1}), [x]_2 \equiv (q^{2x} - q^{-2x})/(q^2 - q^{-2})\) the \(q\)-deformed \(SO(5)\) algebra is (with mutually commuting \(q^{\pm h_1}, q^{\pm h_2}\))

\[
q^{h_1} e_1 = q e_1 q^{h_1}, q^{h_1} f_1 = q^{-1} f_1 q^{h_1}, \\
q^{2h_2} e_1 = q^{-1} e_1 q^{2h_2}, q^{2h_2} f_1 = q f_1 q^{2h_2}, \\
q^{h_1} e_2 = q^{-1} e_2 q^{h_1}, q^{h_1} f_2 = q f_2 q^{h_1}, \\
q^{h_2} e_2 = q e_2 q^{h_2}, q^{h_2} f_2 = q^{-1} f_2 q^{h_2}, \\
[e_1, f_1] = [2h_1], [e_1, f_2] = 0, \\
[e_2, f_2] = [2h_2], [e_2, f_1] = 0,
\]

(2)

We define \(K\) and \(M\) through

\[ q^M = q^{h_1}, q^{K-M} = q^{h_2} \]  

(3)

The classical solutions are characterized by two invariant parameters \(n_1, n_2(n_1 \geq n_2)\) both being integers or half-integers. Four variable indices span the most general irreducible spaces. It can be shown that for the limiting \(n_2\) values (for all \(n_1\))

\[
(i) \quad n_2 = 0, \quad (ii) \quad n_2 = \frac{1}{2} \quad and \quad (iii) \quad n_2 = n_1
\]

(4)

all Chevalley basis representations can be labelled with **three** variable indices. We will construct them in a form directly valid for **all** \(q\), the irreducible spaces being spanned by the states \(|jmk>\).

The ansatz is

\[
q^M |jmk> = q^m |jmk> \\
q^K |jmk> = q^k |jmk> \\
e_1 |jmk> = ([j - m][j + m + 1])^{1/2} |jm + 1k> \\
e_2 |jmk> = ([j - m + 1][j - m + 2])^{1/2} a(j, k) |j + 1m - 1k + 1> \\
\quad + ([j + m][j + m - 1])^{1/2} b(j, k) |j - 1m - 1k + 1> \\
\quad + ([j + m][j - m + 1])^{1/2} c(j, k) |jm - 1k + 1>
\]

(5)
and

\[ < j'm'k'| f_i | jmk >= < jmk| e_i| j'm'k'> \] (6)

For the general case one must have four-index states, say, \( |jmk\ell > \) and the ansatz (for \( e_2, f_2 \)) has to be generalized for the variation of \( \ell \). Here the reduced elements can be obtained from

\[ [e_2, f_2] = [2h_2]_2 = [K - M]_2 \] (7)

and

\[ e_2^2 e_1 - (q^2 + q^{-2}) e_2 e_1 e_2 + e_1 e_2^2 = 0 \] (8)

All the other relations in (2) are then automatically satisfied. For the three cases (4) one has respectively the following solutions

(i) \( n_2 = 0, n_1 = n \)

\[ a(j, k) = (q + q^{-1})^{-1}\left( \frac{[n - j - k][n + j + k + 3]}{[2j + 1][2j + 3]} \right)^{1/2} \]

\[ b(j, k) = (q + q^{-1})^{-1}\left( \frac{[n + j - k + 1][n - j + k + 2]}{[2j - 1][2j + 1]} \right)^{1/2} \]

\[ c(j, k) = 0 \] (9)

For generic \( q \), the domains of parameters are

\[ j = 0, 1, 2, \cdots, n \]

\[ k = n - j, n - j - 2, \cdots, -(n - j - 2), -(n - j) \]

\[ m = j, j - 1, \cdots, -(j - 1), -j. \]

The parameters for the root of unity \( q \)'s will be discussed separately.

(ii) \( n_2 = \frac{1}{2}, n_1 = n - \frac{1}{2} \) Defining \( \delta \equiv \frac{1}{2}(1 - (-1)^{n-j-k}) \)

\[ a(j, k) = (q + q^{-1})^{-2}[j + 1]_2^{-1}\left( [n - j - k - \delta][n + j + k + 3 - \delta] \right)^{1/2} \]

\[ b(j, k) = (q + q^{-1})^{-2}[j]_2^{-1}\left( [n - j + k + 1 + \delta][n + j - k + \delta] \right)^{1/2} \]

\[ c(j, k) = (q + q^{-1})^{-2}[j + 1]_2^{-1}[j]_2^{-1}\]

\[ ( [n + (2\delta - 1)j - k + \delta][n + (2\delta - 1)j + k + 1 + \delta] )^{1/2} \] (10)

with

\[ j = n - \frac{1}{2}, n - \frac{3}{2}, \cdots, \frac{1}{2} \]

\[ k = n - j, n - j - 1, \cdots, -(n - j) \]

\[ m = j, j - 1, \cdots, -(j - 1), -j. \]
(iii) $n_2 = n_1 = n$

\[ a(j, k) = (q + q^{-1})^{-1} \left( \frac{[n-j][n+j+2][j+k+1][j+k+2]}{[2j+3][2j+1][j+1]^2} \right)^{1/2} \]

\[ b(j, k) = (q + q^{-1})^{-1} \left( \frac{[n-j+1][n+j+1][j-k][j-k-1]}{[2j+1][2j-1][j]^2} \right)^{1/2} \tag{11} \]

\[ c(j, k) = (q + q^{-1})^{-1} [n+1]_2 \left( \frac{[j-k][j+k+1]}{[j+1][j]^2} \right)^{1/2} \]

with

\[
\begin{align*}
  j &= n, n-1, \ldots, 0(1/2) \\
  k &= j, j-1, \ldots, -(j-1), -j \\
  m &= j, j-1, \ldots, -(j-1), -j
\end{align*}
\]

Defining

\[
\begin{align*}
  e_3^{(\pm)} &= (q^{\pm 1} e_1 e_2 - q^{\mp 1} e_2 e_1) \\
  f_3^{\pm} &= (q^{\pm 1} f_2 f_1 - q^{\mp 1} f_1 f_2) \\
  e_4 &= (q^{-1} e_1 e_3^{(+)} - q e_3^{(+)} e_1) = (q e_1 e_3^{(-)} - q^{-1} e_3^{(-)} e_1) \\
  f_4 &= (q^{-1} f_3^{(+)} f_1 - q f_1 f_3^{(+)} = (q f_3^{(-)} f_1 - q^{-1} f_1 f_3^{(-)}))
\end{align*}
\tag{12}
\]

the $q$-deformed second order Casimir operator can be written as

\[ A = (q + q^{-1})^{-1} \left\{ (f_1 e_1 + [M][M + 1] \frac{(q^{2K+3}) + q^{-(2K+3)}}{(q + q^{-1})} + [K + 1][K + 2]) \right\} \]

\[ + f_2 e_2 + (q + q^{-1})^{-2} \left\{ q f_3^{(+)} e_3^{(+)} q^{2M} + q^{-1} f_3^{(-)} e_3^{(-)} q^{-2M} + f_4 e_4 \right\} - 1 \tag{13} \]

$2A$ can be shown to be the $q$-deformation of the classical Casimir

\[ J_2 = \sum_{i<j} J_{ij}^2 \tag{14} \]

(the $J_{ij}$’s being the standard Cartan-Weyl generators of $SO(5)$) with the eigenvalue, for a irrep. $(n_1, n_2)$,

\[ n_1(n_1 + 3) + n_2(n_2 + 1). \tag{15} \]

For the cases (9), (10) and (11) respectively one obtains
\[(i) \quad A |jmk > = \frac{[n][n + 3]}{(q + q^{-1})^2} |jmk > \] (16)

\[(ii) \quad A |jmk > = \frac{1}{(q + q^{-1})} \left\{ \left( \frac{q^2 + q^{-2}}{q + q^{-1}} \right) [n_1][n_1 + 3] + (q^2 + q^{-2} - 1) \left[ \frac{1}{2} \right] \right\} |jmk > \] (17)

\[(iii) \quad A |jmk > = [n_2][n + 2] |jmk > \] (18)

\[q = e^{i 2\pi/N} \quad (N = 3, 4, \cdots)\]

For \(q\) a root of unity the matrix elements remain formally the same as before. Only the domains of the parameters involved are changed as follows.

Instead of being ordered integers or half integers as before they now have "fractional parts" (in fact real proper fractions and also possibly imaginary parts) with certain restrictions.

For brevily, I only indicate that for odd \(N\) and periodic representations one should have \((\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\) each being \(\pm 1\))

\[\{j, (j + \epsilon_1 m), (j + \epsilon_2 k), (n_1 + \epsilon_3 j), (n_1 + \epsilon_4 (j + \epsilon_2 k))\} \notin Z/2 \] (19)

Periodic, partially periodic and highest weight representations can similarly be obtained for both odd and even \(N\) with suitable modifications of (19).

For periodic (or cyclic) representations one can define the constans \(\gamma_i\) as

\[|jmk > = \gamma_1^{-1} |j + N, m, k > = \gamma_2^{-1} |j, m + N, k > = \gamma_3^{-1} |j, m, k + N > \] (20)

For a general 4-index representation one also has the supplementary periodicity condition

\[|jmk\ell > = \gamma_4^{-1} |j, m, k, \ell + N > \] (21)

Then one has 10 invariant parameters, namely (with f.p. \(\approx\) fractional parts)

\[n_1, n_2; f.p.'sof j, m, kand\ell; \gamma_1, \gamma_2, \gamma_3, \gamma_4 \] (22)

The dimension of these representations, is \(N^4\).

For the restricted classes presented above (without the index \(\ell\)) one obtains 8 parameters for the periodic case. The dimension is now \(N^3\).
It can be shown that all the representations obtained above, for generic $q$ and root of unity, are irreducible ones.

**Contraction**

From the set of generators

$$(q^{\pm h_1}, q^{\pm h_2}, e_1, f_1, e_2, f_2)$$  \hspace{1cm} (23)

we go over to the "contracted" set

$$(q^{\pm h_1}, q^{\pm h_2}, e_1, f_1, \hat{e}_2, \hat{f}_2)$$  \hspace{1cm} (24)

such that instead of

$$[e_2, f_2] = [2h_2]_2 = [K - M]_2,$$

now

$$[\hat{e}_2, \hat{f}_2] = 0$$  \hspace{1cm} (25)

All other relations in the set (2) remain unchanged.

In each of the foregoing representations, let (with $\mu$ an arbitrary constant, real for generic $q$)

$$(\hat{a}, \hat{b}, \hat{c}) = \mu \left( \lim_{n_1 \to \infty} q^{n_1} (a, b, c) \right)$$ for $q > 1$ and $q < 1$ respectively  \hspace{1cm} (26)

Then the contracted representations are given by only the following modification of (5) and (6)

$$\hat{e}_2 |jk > = ([j - m + 1][j - m + 2])^{1/2} \hat{a}(j, k) |j + 1, m - 1, k + 1 > + ([j + m][j + m - 1])^{1/2} \hat{b}(j, k) |j - 1, m - 1, k + 1 >$$

$$+ ([j + m][j - m + 1])^{1/2} \hat{c}(j, k) |j, m - 1, k + 1 >$$  \hspace{1cm} (27)

Thus corresponding to (11) one has

$$\hat{a}(j, k) = \mu \frac{[j + k + 1][j + k + 2]}{[j + 1][2j + 3][2j + 1]}^{1/2}$$

$$\hat{b}(j, k) = \mu \frac{[j - k][j - k - 2]}{[2j + 1][2j - 1]}^{1/2}$$  \hspace{1cm} (28)

$$\hat{c}(j, k) = \mu \frac{[j - k][j + k + 1]}{[j + 1][2j + 1]}^{1/2}$$
As compared to (13) the Casimir is now
\[
\hat{A} = \hat{f}_2 \hat{e}_2 + (q + q^{-1})^{-2} \{ \hat{f}_3^{(+)} \hat{e}_3^{(+)} q^{2M+1} + \hat{f}_3^{(-)} \hat{e}_3^{(-)} q^{-2M-1} + \hat{f}_4 \hat{e}_4 \} \tag{29}
\]
where, analogously to (12),
\[
\hat{e}_3^{(\pm)} = (q^{\pm 1} \hat{e}_1 \hat{e}_2 - q^{\mp 1} \hat{e}_2 \hat{e}_1)
\]
and so on.

Instead of (18) one now has (indicating the new parameter $\mu$ in the states)
\[
\hat{A}|jmk \rangle_{(\mu)} = (q + q^{-1})^2 \mu^2 |jmk \rangle_{(\mu)} \tag{31}
\]
Contractions of (9) and (10) are also easy to write down. They exhibit specific degeneracies.

It can be shown that the Drinfeld-Jimbo Hopf algebra (coproducts, counits and antipodes) of $SO(5)_q$ can be carried over intact for the contracted, non-semisimple case. Thus, for example
\[
\Delta \hat{e}_2 = \hat{e}_2 \otimes q^{K-M} + q^{-K+M} \otimes \hat{e}_2 \tag{31}
\]
and so on.

For $q$ a root of unity, exactly as for the uncontracted $SO(5)_q$ case, one can obtain periodic and other types of representations by suitably introducing fractional parts.

For generic $q$ the contracted representations are, of course, all infinite dimensional. For $q$ a root of unity one can obtain finite dimensional representations. Thus, for example, (27) can give $N^3$ dimensional periodic representations.

At the classical level ($q = 1$) the contracted algebra has the following structure. By writing the standard Cartan-Weyl generators ($J_{ij}$) in terms of the Chevalley generators the effect of our contraction on the $J$’s can be easily evaluated. It can be shown that one now has (denoting each $J_{ij}$ undergoing contraction by $\hat{J}_{ij}$)

1. an $SO(3)$ algebra
   
   \[
   (J_{12}, J_{23}, J_{13}), \tag{32}
   \]

2. two "translation triplets" (all mutually commuting)
   
   \[
   (\hat{J}_{14}, \hat{J}_{24}, \hat{J}_{34}) \text{and}(\hat{J}_{15}, \hat{J}_{25}, \hat{J}_{35}) \tag{33}
   \]

3. with $J_{45}$ linking these triplets as (with $i = 1, 2, 3$)
   
   \[
   [J_{45}, \hat{J}_{i4}] = -i\hat{J}_{i5}, \ [J_{45}, \hat{J}_{i5}] = i\hat{J}_{i4} \tag{34}
   \]
For $q \neq 1$ the translation triplets become non-commutative. This deformation can again be evaluated without difficulty by expressing the $J$'s in terms of (24) and (30) with its analogues for $\hat{e}_3^{(\pm)}, \hat{f}_3^{(\pm)}, \hat{e}_4$ and $\hat{f}_4$, providing an invertible set of relations. Due to lack of space I will not write down the equations explicitly. It can provide an interesting exercise for the interested reader.

A much more detailed discussion of the preceding results can be found in [1]. The necessity for constructing representations of $SO(n)$ on the Chevalley basis, with all Cartan generators diagonalized is emphasized and explained there. This only opens the way for a successful $q$-deformation as exhibited here for the simplest non-trivial case of $SO(5)_q$. The contraction scheme which assures the existence of a complete Hopf algebra was introduced before for $SU(n)_q$. Here, a beginning is made for $SO(n)_q$. Though the explicit representations here are given for restricted classes, the passage to a root of unity $q$ and the contraction scheme are both quite general and works for any $q$-deformed algebra. All these points are more fully discussed in [1].

References

1. A. Chakrabarti, $SO(5)_q$ and contraction : Chevalley Basis Representations of $q$ Generic and Root of Unity (preprint 1993 ; to be published in Jour. Math. Phys.)

Contractions valid for all $q$ and preserving the full Hopf algebra was introduced in.

2. A. Chakrabarti, Jour. Math. Phys. 32 (1991) 1227.

A different type of contraction with $q \to 1$ can be found, for example, in

3. E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, Jour. Math. Phys. 32 (1991) 1159.

4. J. Lukiersky, A. Nowicki and H. Ruegg, Phys. Lett. B293 (1992) 344.

The doubling of generators (such as $e_3^{(\pm)}, f_3^{(\pm)}$ in (12)) was recognized as an "antipode-extended system" in

5. J. Lukiersky, A. Nowicki and H. Ruegg, Phys. Lett. B271 (1991) 321.

The extended system appears, for $SU(N)_q$ already in [2].

The "fractional part" formalism unifying representations for generic and root of unity $q$ was introduced in

6. D. Arnaudon and A. Chakrabarti, Comm. Math. Phys. 139 (1991) 461.

This is slightly modified here and made quite generally applicable.

Different approaches to $SO(5)_q$ can be found in

7. D. Arnaudon and A. Chakrabarti, Phys. Lett. B262 (1991) 68.

8. W.A. Schnizer, Representations of quantum groups, RIMS-961.

More relevant sources can be found in [1].