A model study on superfluidity of a unitary Fermi gas of atoms interacting with a finite-ranged potential

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Received 30 July 2021, revised 21 October 2021
Accepted for publication 29 October 2021
Published 14 March 2022

Abstract
We calculate Bardeen–Cooper–Schrieffer (BCS) state of a unitary Fermi gas of atoms interacting with the finite-ranged Jost-Kohn potential which has been recently shown to account for the resonant interactions (2019 J. Phys. B: At. Mol. Opt. Phys. 52 165004). Using exact scattering solution of the potential, we derive two-body $T$-matrix element which is employed to construct the BCS Hamiltonian in momentum space. We present results on the energy- and range-dependence of the pairing gap and superfluid density and the range-dependence of the chemical potential for a wide variation of the scattering length including the unitary regime. In the zero range limit our calculated gap at the Fermi energy is found to be nearly equal to that calculated in mean-field theory with contact potential. The mean gap averaged over the full width at half maximum of the gap function in the zero range and unitary limits is found to be $0.42E_F$ which is quite close to the recent result of the quantum Monte-Carlo simulation (2018 Phys. Rev. A 97 013601). The chemical potential in the zero range limit also agrees well with that for the contact potential.

Keywords: Fermi atoms, superfluidity, finite-ranged interaction, unitarity regime, Bardeen–Cooper–Schrieffer state

(Some figures may appear in colour only in the online journal)

1. Introduction

The experimental advancement in many-body quantum physics [1] with trapped ultracold atoms over the last two decades has opened new perspectives in both bosonic and fermionic superfluidity. In particular, the studies of superfluidity with a Fermi gas of atoms have implications in understanding the superconductivity of condensed matter systems, superfluidity of He-3 liquid and superconductivity or superfluidity of nuclear matter of astrophysical origin. It is expected that quantum simulations of various types of fermionic superfluidity using trapped atomic gases with control over atom–atom interactions, trap dimensionality and atomic density will provide new insight into electronic superconductivity and nuclear superfluidity. Superfluidity and superconductivity occur due to the spontaneous breaking of continuous symmetry resulting in the appearance of long wave-length Goldstone modes which are also known as Bogoliubov–Anderson modes. While Goldstone modes in charged superconductors become ill-defined due to Coulomb interactions, such modes in neutral superfluidity are expected to be well-defined and detectable. Bogoliubov–Anderson modes have been recently measured in a superfluid Fermi gas of atoms [2, 3]. Thus the charge neutrality of atomic superfluidity offers some advantages in exploring fundamental aspects of quantum many-body physics. On the other hand, the aspects of superconductivity that are intrinsically connected with charge or gauge fields are not available...
with the neutral systems. Nevertheless, in recent times, artificial gauge fields for neutral atomic gases are created by optical manipulations of atomic internal as well as center-of-mass degrees-of-freedom [4].

Bardeen–Cooper–Schrieffer (BCS) [5] type or standard superconductivity in electrons results from the phonon-mediated electron-electron attraction near the Fermi surface. This attractive interaction overcomes the Coulomb repulsion between the electrons at low energy, resulting in the formation of Cooper pairs which then constitute the many-body BCS state. Fermi superfluidity in trapped atomic gases arises when the atom–atom interaction is tuned towards attractive side by a magnetic Feshbach resonance [6]. In general, the ultracold atomic gases that are currently being experimentally studied are sufficiently dilute such that the range of interaction between the atoms is very small compared to the interatomic separation. Therefore, the range is generally neglected and the interaction is approximated as a contact potential expressed in terms of the $s$-wave scattering length $a_s$. A magnetic field is used to tune $a_s$ across the Feshbach resonance. A degenerate Fermi gas is transformed into BCS state by tuning $a_s$ towards small negative value for which the effective atom–atom interaction becomes attractive. Near the resonance where $a_s$ is large negative, the gas becomes strongly interacting leading to resonance superfluidity [7] that is characterized by the unitarity of the scattering $S$-matrix. On the positive side of $a_s$ across the resonance, the Cooper pairs may be transformed into bosonic diatomic molecules which can condense into a Bose–Einstein condensate. Such unitary regime and crossover physics may be available with colour superconductivity in quantum chromodynamics and neutral superfluidity of neutron stars.

Given the current state of the art in the experimental cold atom physics, an experimental attempt towards simulating neutron superfluidity appears to be a difficult task. However, the recent successful observations of BCS–BEC crossover open prospect for simulating nuclear physics [8–12]. The effective range of interaction plays an important role in many-body quantum dynamics when it becomes comparable to inter-particle separation [13–16]. In recent times, there has been considerable research interests to simulate dilute nuclear matter [17] such as proto-neutron stars [18, 19]. From a theoretical point of view, most of the works have been carried out by numerical experiments i.e., by Monte-Carlo simulations [20] or using density functional theory [21]. The effects of the finite range of interaction on universal equation of state (EOS) of the system have been theoretically studied using several model finite-ranged two-body interaction potentials [22–24]. The thermodynamic behaviour of a Fermi gas has been studied as a function of the parameters of a model finite-ranged potential of exponential type in the BCS–BEC crossover regime [23]. Two different class of finite-ranged potentials—one purely attractive such as the square well, the exponential and the Yukawa type potential, and the other having both attractive and repulsive character such as Van der Waals and dipolar potentials have been considered to explore how the depth and the spatial range of these potentials affect the pairing and molecule formation along the crossover [24]. Spin fluctuations of a strongly interacting Fermi gas across BCS–BEC crossover have been demonstrated by speckle imaging [25]. Momentum resolved photo-emission spectroscopy has been used to observe many-body pairing of a two dimensional trapped unitary Fermi gas above the transition temperature [26]. Controllable quasi-2D system across crossover has been observed with results that are beyond the mean-field level [27]. Quantized vortex ring is also observed in a unitary Fermi gas [28, 29].

The purpose of this paper is to explore the effects of the effective range of interaction on the superfluidity of a unitary Fermi gas. Towards this end, we resort to the finite-range Jost-Kohn (JK) model potential [30] which has been recently shown to account for the unitary regime. The use of this model interaction potential allows us to study the finite range effects and energy dependence of superfluid gap and density over a wide range of $a_s$ including the resonance limit. We use exact scattering solution of the JK potential in order to calculate momentum-dependent two-body T-matrix element for the BCS Hamiltonian. We present results on the energy-dependence of the superfluid pairing gap over the entire range of energy for different values of $a_s$ and range. The gap is found to exhibit strong energy dependence in the unitary regime. We also present results on the effects of the range and $a_s$ on the superfluid density distribution. Our results in the zero-range limit qualitatively agree well with those for the contact interaction potential. We find that a mean gap averaged over the energy within the full width at half maximum of the gap function in the zero range limit of our model interaction potential agrees quite well with the value calculated recently by quantum Monte-Carlo (QMC) simulation [14].

The paper is organised in the following way. In section 2 we discuss the standard BCS theoretical methods, introduce the JK model interaction potential and its exact scattering solution. In section 3 we present and discuss our results showing the effects of finite range and comparing them with those for contact interaction. We conclude and give an outlook of the work in section 4.

2. Theoretical methods

According to the BCS theory [5] of low temperature superconductivity, an attractive interaction between fermions in a quantum degenerate Fermi system leads to the formation of Cooper pairs [31] and the instability of the Fermi surface. At a critical temperature, the system then undergoes transition to superconducting phase. In solid-state electronic systems, a pair of electrons near the Fermi surface interacts by means of lattice-assisted or phonon-mediated attractive interaction, resulting in the formation of $s$-wave Cooper-pairs in spin-singlet state. The two electrons that form a Cooper-pair have equal and opposite lattice momenta. The Cooper-pair wave function is given by

$$\psi_{\downarrow}(\mathbf{r}_1, \mathbf{r}_2) = \sum_{k > k_F} g_k e^{i k (\mathbf{r}_1 - \mathbf{r}_2)},$$

where $\mathbf{r}_{1(2)}$ refers to the position vector of the electron $1(2)$ and $g_k$ is the wave function in momentum space. On substitution
of the above relation into Schrödinger equation, one obtains
\[ (E - 2\epsilon_k)g_k = \sum_{k' < k FR} g_{kk'}V_{kk'}, \tag{2} \]
where \( E \) is the eigen energy, \( \epsilon_k = h^2 k^2 / 2m \) (\( m \) is reduced mass) and \( V_{kk'} \) is the interaction in the momentum space. In the weakly interacting regime, \( V_{kk'} \) is related to the real-space interaction potential \( V_{int}(r) \) by
\[ V_{kk} = \frac{1}{(2\pi)^3} \int V_{int}(r)e^{-i(k-k')\cdot r} dr. \tag{3} \]
where \( r = r_1 - r_2 \) is the relative position vector of the two fermions. Approximating the two-particle interaction by the Fourier transform of \( V_{int}(r) \) into the momentum space as given above is akin to first order Born approximation of scattering \( T \)-matrix. In case of a conventional superconductor, \( V_{int} \) is the phonon-mediated potential and is often assumed to be a space-independent contact potential. As a consequence, the momentum integration in equation (2) diverges. This divergence is tackled by putting a cut-off as the upper limit in the \( V_{int} \) matrix. In case of a conventional superconductor, there exists a natural cut-off which is the Debye energy or Debye wave number.

The BCS theory is applicable to superfluidity of neutral Fermi liquids or gases, albeit with some modifications that are necessary due to the absence of any natural cut-off in energy. For neutral atomic Fermi gases, \( V_{int}(r) \) may be, in general, of contact or finite-range or even long-range type potential. If \( V_{int}(r) \) is approximated as a contact potential represented by a delta function, \( V_{kk'} \) becomes a constant that is proportional to the \( s \)-wave scattering length. Such approximation leads to logarithmic divergence in momentum integration of gap equation. This divergence is overcome by renormalising the interaction parameter or the coupling constant by subtracting the ultraviolet divergence part from the integrand \[8, 32\]. However, for a short-range or finite-range interaction, such divergence problems will not arise and the gap equation or superfluid behaviour will depend on the range of interactions apart from other system parameters.

Retaining only those interaction terms where two fermions interact with zero center-of-mass momentum, one can write down the reduced Hamiltonian
\[ \mathcal{H} = \sum_{k,\sigma} \epsilon_k b_{k\sigma}^\dagger b_{k\sigma} + \sum_{k,k'} V_{kk'}(\epsilon_k^\dagger c_{-k\downarrow}^\dagger c_{-k\uparrow} c_{-k'\downarrow}^\dagger c_{-k'\uparrow} ). \tag{4} \]
In Hartree self consistent or generalised mean-field approach, the BCS ground state is given by generalised spin coherent state \[34\]
\[ |\psi_G\rangle = \prod_{k=\epsilon_k < \epsilon_M} \left( u_k + v_k c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger \right) |\phi_0\rangle, \tag{5} \]
where \( |\psi_G\rangle \) is the probability of the pair-state \( (k, -k) \) being occupied and \( |u_k|^2 + |v_k|^2 = 1 \). Here \( |\phi_0\rangle \) is the vacuum state which is annihilated by \( c_k \). Introducing the number operator \( \hat{N} = \sum c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow} \), and a variational parameter \( \theta \) such that \( v_k = \cos \theta_k \) and \( u_k = \sin \theta_k \), one can obtain the BCS ground state by using the variational principle \[33\]
\[ \delta \langle \psi_G | \mathcal{H} - \mu \hat{N} | \psi_G \rangle = 0. \tag{6} \]
Here \( \mu \) is the chemical potential which conserves the total number of particles. \( \mathcal{H} \) of equation (4) can be represented in terms of pseudospin operators \[34\]. The value of \( \langle \mathcal{H} \rangle \) is calculated by implementing equations (4) and (5). Thus we have
\[ \langle \mathcal{H} - \mu \hat{N} \rangle = \sum_k 2 \xi_k (1 + \cos 2\theta_k) \]
\[ + \frac{1}{4} \sum_{k,k'} V_{kk'} \sin 2\theta_k \sin 2\theta_{k'}, \tag{7} \]
where \( \xi_k = (\epsilon_k - \mu) \). Taking the variation with respect to \( \theta_k \), we have
\[ \frac{\partial}{\partial \theta_k} (\mathcal{H} - \mu \hat{N}) = -2\xi_k \sin 2\theta_k \]
\[ + \sum_{k'} V_{kk'} \cos 2\theta_{k} \sin 2\theta_{k'} = 0 \tag{8} \]
which results in
\[ \tan 2\theta_k = \frac{1}{2\xi_k} \sum_{k'} V_{kk'} \sin 2\theta_{k'}. \tag{9} \]
Now, defining the superfluid gap \( \Delta_k = -\sum_{k'} V_{kk'} u_{k'} v_{k'}/2 \theta_{k'} \), the above equation can be expressed in the form of the well-known gap equation
\[ \Delta_k = \frac{1}{2} \sum_{k'} V_{kk'} \Delta_{k'}/(\Delta_k^2 + \xi_k^2) \tag{10} \]
with two probability amplitudes being \( u_k = \frac{1}{\sqrt{2}} \left( 1 + \xi_k/\sqrt{\Delta_k^2 + \xi_k^2} \right)^{\frac{1}{2}} \)
and \( v_k = \frac{1}{\sqrt{2}} \left( 1 - \xi_k/\sqrt{\Delta_k^2 + \xi_k^2} \right)^{\frac{1}{2}} \).

2.1. Finite-ranged model potential and its solution
To study the effects of the range of interaction, we use JK interaction potential
\[ V_{int}(r) = -\frac{4\hbar^2}{m r_0^2} \frac{\alpha \beta^2 e^{-2\alpha r}/\alpha}{(\alpha + e^{-2\beta r}/\beta)^2}, \tag{11} \]
where \( r = |r_1 - r_2|, \alpha = \sqrt{1 - 2r_0/\alpha}, \beta = 1 + \alpha, \alpha r_1 \) and \( r_2 \) are the coordinates of two particles and the \( s \)-wave scattering length \( \alpha r_1 \). In order to access the strongly interacting regime with a finite-ranged interaction, it is necessary to employ the exact scattering solution of the two-body problem as the first order Born approximation breaks down. In the limit of large \( \alpha r_1 \), the JK potential of equation (11) takes the form of Pschöl–Teller potential which admits an exact solution \[35, 36\] which is given by
\[ \psi_2(r) = \cosh^k(\kappa r) \sinh(\kappa r) F_1 \left( \frac{\lambda + ik}{2\kappa} + \frac{1}{2} \right), \]

\[ \left\{ \frac{\lambda}{2} - \frac{ik}{2\kappa} + \frac{1}{2} \right\} - \sinh^2(\kappa r) \right) \tag{12} \]

where \( \kappa = 2/r_0, \lambda = \frac{1}{2} \left( 1 - \sqrt{1 + 8/\alpha} \right) \) and \( F_1(a, b; c; z) \) is the hypergeometric function. This wave function has the asymptotic form

\[ \psi_2(r \to \infty) \propto C \cos(\kappa r + \phi_0) \tag{13} \]

with

\[ C = 2\Gamma(3/2) \left| \frac{\Gamma(-ik/\kappa) \exp(ik \log 2/\kappa) \Gamma(1 - \frac{1}{2} - i\frac{k}{2\kappa})}{\Gamma \left( \frac{\lambda + i\frac{k}{2\kappa}}{2} + i\frac{k}{2\kappa} \right) \Gamma \left( 1 - \frac{1}{2} - i\frac{k}{2\kappa} \right)} \right| \tag{14} \]

and

\[ \phi_0 = \arg \left[ \frac{\Gamma(-ik/\kappa) \exp(ik \log 2/\kappa) \Gamma(1 - \frac{1}{2} - i\frac{k}{2\kappa})}{\Gamma \left( \frac{\lambda + i\frac{k}{2\kappa}}{2} + i\frac{k}{2\kappa} \right) \Gamma \left( 1 - \frac{1}{2} - i\frac{k}{2\kappa} \right)} \right]. \tag{15} \]

Now, we normalize the wave function \( \psi_2^{\text{nc}}(r) \) to \( \psi_2(r)/C \) in a way so that the asymptotic form reduces to \( \psi_2^{\text{nc}}(r \to \infty) = \sin(\kappa r + \delta_0) \) where \( \delta_0 = \phi_0 + \pi/2 \) is the s-wave scattering phase shift. The effective interaction \( V_{\text{eff}} \) in momentum space can be obtained from the exact on-the-energy shell T-matrix element. Explicitly, we have

\[ V_q = 2\pi \int j_0(qr) V_{\text{in}}(r) \frac{\psi_2^{\text{nc}}(r)}{qr} r^2 \, dr, \tag{16} \]

where \( q = |k - k'| \) and \( j_0(qr) = \sin(qr)/qr \) is the spherical Bessel function of zeroth order. To extract \( V_{\text{eff}} \) from \( V_q \) we write \( q = \sqrt{k^2 + k'^2 - 2kk' \cos \theta} \). To get the exclusive dependence on \( k \) and \( k' \), we must integrate \( V_q \) over \( \theta \) which then yields

\[ V_{kk'} = \int_{-k-k'}^{k+k'} \frac{V_q dq}{kk'} \quad k \neq 0 \neq k'. \]

For \( k = 0, V_{kk'} = V_q \) and for \( k' = 0, V_{00} = V_q \). Also for \( k = 0 \neq k' \) we have to take \( V_{00} = V_q(q = 0) \). From the numerical point of view we first generate a large set of discrete values of \( q \) and corresponding \( V_q \). By this procedure we obtain a two-dimensional array of \( V_{kk'} \) as a function of \( k \) and \( k' \).

The JK potential is expressed in terms of two parameters, namely the s-wave scattering length and the effective range of a real two-body interaction potential. It is derived by inverse scattering method. So, our theoretical method is applicable to all the interacting Fermi systems that admit effective range expansion of the scattering phase shift. So unlike various types of model potentials used in the current literature [22–24], the use of JK potential offers some advantage in numerical computation in that there is no need for the calculation of scattering length and the effective range which enter into the BCS Hamiltonian as the input parameters. But the most significant feature of the JK potential is that it can capture the resonance effect or unitary regime of interaction for which it is necessary to use exact scattering solution of the potential. However, this potential cannot describe the effects of actual spatial range of a realistic potential.

2.2. \( k \)-dependent gap and superfluid density

Here we consider the superfluid gap as a function of momentum. By solving the gap equation in a self-consistent manner we calculate the \( k \)-dependent gap. The equation (10) is rewritten in the continuous form

\[ \Delta_k = -\frac{1}{2} \int \frac{V_{kk'} \Delta_{k'}}{\sqrt{\Delta_k^2 + (\epsilon_{k'} - \mu)^2}} \, d^3k'. \tag{17} \]

One point worth noticing here is that we need not take into account the renormalization of the interaction term unlike that in the standard gap equation with contact potential. However, in our case the interaction in \( k \)-space goes to zero as \( k \to \infty \). This allows us to obtain convergent solutions of \( \Delta_k \) by a numerical iteration process. The \( k \)-dependent superfluid fraction \( n_s \) or equivalently \( n(E) \) is given by \( n(E) = |\psi_k|^2 \) which is a dimensionless quantity with. At zero temperature, the total number density \( N = \frac{k^3}{6\pi^2} \) is related to \( n(E) \) by \( N = \int n(E) d^3k/(2\pi)^3 \). Thus, to calculate the chemical potential \( \mu \), we make use of the following equation

\[ \frac{k^3}{6\pi^2} = \frac{1}{2} \int \left( 1 - \frac{\epsilon_k - \mu}{\sqrt{\Delta_k^2 + (\epsilon_{k'} - \mu)^2}} \right) \, d^3k. \tag{18} \]

3. Results and discussions

The wave number- or energy-dependence of superfluid gap for different values of \( a_s \) and three values of \( r_0 \), namely, \( r_0 = 0.1k_F^{-1}, r_0 = 0.01k_F^{-1} \) and \( r_0 = 0.0001k_F^{-1} \) is shown in the subplots (a), (b) and (c) of figure 1, respectively. We numerically solve the coupled equations (17) and (18). We resort to an iterative procedure to obtain numerically the convergent values of \( \Delta_k \) and \( \mu \). We notice that \( \Delta(k) \equiv \Delta_k \) as a function of \( k \) has the peak value at \( k = 0 \) and monotonically decreases as \( k \) increases. Higher the value of \( a_s \), higher is \( \Delta(k) \). With decreasing \( r_0, \Delta(k) \) increases and saturates as \( r_0 \to 0 \). The figure 1 further shows that, as \( r_0 \) decreases below 0.01\( k_F^{-1} \), the results for \( \Delta(k) \) in the limit \( a_s \to -\infty \) become convergent. \( \Delta(0) \) for \( a_s \to -\infty \) is found to be close to the \( k \)-independent value calculated using the zero-range pseudopotential in the unitary limit [37].

The energy-dependence of superfluid density \( n(E) \) for different values of \( a_s \) and the three values of \( r_0 \) is illustrated in figure 2. For small values of \( a_s \) in the zero-range limit, \( n(E) \) resembles that of standard Fermi–Dirac distribution which then deviates as \( |a_s| \) increases or range becomes finite.

The superfluid gap \( \Delta_F = \Delta(k_F) \) at Fermi energy and chemical potential \( \mu \) in unit of are plotted as a function of dimensionless interaction strength \((k_Fa_s)^{-1}\) in figure 3 for different values of \( r_0 \). The variation of \( \Delta_F \) as a function of \((k_Fa_s)^{-1}\) in figure 3(a) shows qualitatively similar behaviour as in the contact interaction case [37], and in the zero range limit its
The superfluid gap \( \Delta(k) \equiv \Delta_0 \) (in unit of Fermi energy \( E_F \)) as a function of wave number \( k \) (in unit of Fermi wave number) is shown for different scattering lengths \( a_i \) and different values of the range \( r_0 \). The values of \( a_i \) in (c) are same as in (b).

![Figure 1](image1)

The variation of superfluid density with energy \( E \) (in unit of \( E_F \)) for the same set of parameters as in figure 1. The values of \( a_i \) in (c) are same as in (b).

![Figure 2](image2)

value is close to that of the contact case. Figure 3(b) shows that, as \( \mu \) increases \( \mu \) decreases. In the zero range limit of the JK potential, the variation of \( \mu \) with closely resembles to that for the contact interaction given in reference [37]. In figure 4 we show the comparison between the variations of gap for the zero-range limit of the JK potential and the contact potential as a function of \( r_0 \). Qualitatively they are similar. In the unitarity limit their quantitative values which are found to be and for as a function of \( \mu \).

In the unitarity limit, we show the comparison between the variation of gap for the contact case and JK potential, respectively, agree quite well. In the unitarity regime of the JK potential and the contact potential, we show the comparison between the variations of gap for the contact case and JK potential, respectively, agree quite well.

Figure 1 shows that the wave number dependence of \( \Delta \) for finite-ranged JK potential or even in the zero range limit of this potential is quite significant. So, the question arises whether it is appropriate to compare \( \Delta \) with the energy-independent gap for the contact interaction.

A reliable quantity for comparison may be a mean gap averaged over the full width at half maximum (FWHM) of the gap function. We define this energy-averaged mean gap by

\[
\bar{\Delta} \equiv \frac{1}{\xi} \int_0^\xi \Delta(E)\,dE,
\]

where \( \xi \) is the energy corresponding to the FWHM of the \( \Delta(k) \). \( \overline{\Delta} \) is computed by first identifying the energy corresponding half width of the \( \Delta(k) \) and then taking average over that half width. In figure 5 we have plotted \( \bar{\Delta} \) as a function of \( (k_{\text{F}}a_i)^{-1} \) for \( r_0 = 0.0001k_0 \). In the limit \( a_i \to -\infty, \bar{\Delta} = 0.42E_F \).

Most of the theoretical and experimental works on the superfluid gap of a unitary gas report the values of energy-independent gap for different system parameters. The recent experiment with superfluid \(^6\text{Li}\) using momentum and specially resolved radio frequency spectroscopy [2] reports \( \Delta = (0.39 \pm 0.03)E_F \) in the unitarity regime. There are numerous approaches to calculate the paring gap in the unitarity regime. Most of the calculations involve the quantum Monte Carlo (QMC) technique. The first QMC calculation was done with up to 40 particles with a modified Poschel–Teller interaction potential with \( k_{\text{F}}r_0 \approx 0.3 \) [38]. The estimated gap was \( \Delta \approx 0.54E_F \). The inclusion of polarization correction provides a better result \( \Delta \approx 0.49E_F \).

Difision Monte-Carlo calculations with improved trial function yield \( \Delta = 0.45E_F \) [39] for larger number of particles with a better extrapolation to \( k_{\text{F}}r_0 \to 0 \). A detailed comparison between the strong coupling theory and experimental results is made in reference [14]. The QMC results are in reasonable agreement with the experimental ones [2] for \( r_0 = 0 \). In our calculations, we numerically evaluate the gap for the entire range of energy. For the sake of comparison to other works, we choose the gap \( \Delta_F \), that is \( \Delta = \Delta_F \) at \( E = E_F \) and the energy-averaged gap \( \bar{\Delta} \) as defined in equation (19).

We find that \( \Delta_F = 0.64E_F \) in the limit \( k_{\text{F}}r_0 \to 0 \) as displayed in figure 3(a). Though this value is substantially higher than that of QMC prediction, but it is slightly lower than that \( \Delta = 0.69E_F \) predicted by mean-field BCS-Leggett theory [40]. However, our study shows that \( \bar{\Delta} \) as plotted in figure 5 agrees quite well with that predicted by the recent QMC calculations as well as with the experimental value. We have calculated the equation-of-state (EoS) which is given as \( E = \zeta E_{\text{BG}} \) with \( E_{\text{BG}} \) is the energy per particle of non-interacting fermion gas and \( E \) being the energy per particle.
Figure 3. Superfluid gap at \( k = k_F \) (a) and chemical potential \( \mu \) (b) in Fermi energy scale are plotted as a function of inverse scattering length (in unit of \( k_F \)) for different ranges of interaction.

Figure 4. Comparison between the gaps (\( \Delta_F \)) as a function of \((k_F a_s)^{-1}\) for JK and contact potentials. The values of \( \Delta_F \) for JK potential with \( r_0 = 0.0001 k_F^{-1} \) and that for contact potential the unitary limit are found to be 0.60\( E_F \) and 0.66\( E_F \), respectively.

Figure 5. \( \tilde{\Delta} \) as a function of \((k_F a_s)^{-1}\) for \( k_F r_0 = 0.0001 \).

for the interacting system. We find that the Bertsch parameter \( \zeta = 0.69 \) in the zero-range limit of the unitarity regime which has good agreement with the Monte Carlo simulation results [38].

4. Conclusions

In conclusion, we have demonstrated that the finite range of interaction leads to quite significant energy-dependence in the pairing gap and superfluid density both in the non-unitary and unitary regimes. The gap as a function of energy for a fixed scattering length and a fixed range is maximum at zero energy, and decreases monotonically as energy increases. For a small scattering length in the zero range limit, the energy-dependence of superfluid density resembles a step function like a standard Fermi distribution, for large scattering lengths and any value of range the density function deviates significantly from the step-like shape. The FWHM of density function is the Fermi energy. We have shown that our results on the gap and density functions approach convergent values as the range decreases below \( 10^{-2} k_F \). For the sake of comparison of our results with those for zero-range contact interaction, we have resorted in two different ways. First, we have considered the value of the gap at the Fermi energy in the zero range limit. This value is found be close to the corresponding value obtained by BCS mean-field method using contact potential. Next, we have considered the mean gap averaged over energy within FWHM of the gap in the unitary limit. This mean gap is found to be close to the results obtained by other workers using QMC simulation. From a theoretical point of view, we have made use of the exact two-body scattering solution of the JK potential to obtain the energy-dependent T-matrix element that is employed in our calculations. Unlike the case of contact interaction, the use of this exact T-matrix element allows us to calculate accurately the energy dependence of the gap without requiring any renormalisation of the interaction. The effective range effects described in this paper may be experimentally realisable with a relatively dense cloud of ultracold fermionic atoms near a Feshbach
resonance for which the effective range is finite, positive and likely to be tunable with an external field [30, 41]. In fact, the effective range near a narrow Feshbach resonance may become large and even negative [41–43]. In our study, we have not considered negative effective range which may also affect the pairing gap significantly [44]. However, in the known neutral s-wave superfluids such as neutron stars, negative effective range is unlikely to arise. Our work may serve as a precursor towards exploring quantum simulation of nuclear superfluidity using cold atoms.

Since the model potential we have used in our work, namely JK potential, is derived by inverse scattering method based on effective range expansion, the results presented here apply to all realistic interaction potentials that admit effective range expansion. There are two class of JK potentials—one for negative scattering length which effectively describes attractive interaction and the other for positive scattering length to account for repulsive interaction. In this paper we have studied only the BCS side of the BCS–BEC crossover including the unitary regime and hence used only former type of JK potentials. While the JK potential for negative scattering length is a two-parameter potential, that corresponding to the positive scattering length is a three-parameter potential—the additional parameter being the binding energy of the last bound state of the actual potential. Therefore, in general the two potentials do not smoothly match in the unitarity limit, that is $a_s \to \pm \infty$. Only if one assumes that the binding energy is given by $\hbar^2/(2m a_s^2)$ ($m$ is the reduced mass), then the two potentials reduce to the same form in the unitarity limit [35], and so under this assumption it may be possible to explore the smooth BCS–BEC crossover with JK potentials. This also suggests that in general the BCS–BEC crossover may not be continuous or smooth when the energy of the last bound state does not go to zero in the limit $a_s \to \infty$. We hope to address these issues in our future communication. One of the most salient features of these potentials is that in the unitarity limit they take the form of Posch–Teller potentials which admit exact analytical solutions in one dimension. Since the partial-wave Schödinger equation is an effective one dimensional equation in one side, one can use the first odd solution of the Posch–Teller equation as the exact finite-ranged s-wave solution in the unitary limit within the framework of effective range expansion. So, it is expected that the use of JK potentials will enable one to study the physics in the unitarity regime more accurately.

Acknowledgments

SM is thankful to Council of Scientific and Industrial Research (CSIR), Govt. of India, for a support.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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References

[1] Bloch I, Dalibard J and Zwerger W 2008 Rev. Mod. Phys. 80 885–964
[2] Hoinka S, Dyke P, Lingham M G, Kinnunen J J, Bruun G M and Vale C J 2017 Nat. Phys. 13 943–6
[3] Kuhn C C N, Hoinka S, Herrera I, Dyke P, Kinnunen J J, Bruun G M and Vale C J 2020 Phys. Rev. Lett. 124 150401
[4] Dalibard J, Gerbier F, Juzeliūnas G and Öhberg P 2011 Rev. Mod. Phys. 83 1523–43
[5] Bardeen J, Cooper L N and Schrieffer J R 1957 Phys. Rev. 108 1175–204
[6] Chin C, Grimm R, Julienne P and Tiesinga E 2010 Rev. Mod. Phys. 82 1225–86
[7] Kokkelmans S J J M F, Milstein J N, Chiofalo M L, Walser R and Holland M J 2002 Phys. Rev. A 65 033617
[8] Randieria M, Duan J-M and Shieh L-Y 1989 Phys. Rev. Lett. 62 981–4
[9] Micnas R, Ranninger J and Robaszkiewicz S 1990 Rev. Mod. Phys. 62 113–71
[10] Randieria M, Trivedi N, Moreo A and Scalettar R T 1992 Phys. Rev. Lett. 69 2001–4
[11] Drechsler M and Zwerger W 1992 Ann. Phys., Lpz. 504 15–23
[12] Casas M, Getino J M, de Llano M, Puente A, Quick R M, Rubio H and van der Walt D M 1994 Phys. Rev. B 50 15945–52
[13] Horikoshi M, Koashi M, Tajima H, Ohashi Y and Kuwata-Gonokami M 2017 Phys. Rev. X 7 041004
[14] van Wyk P, Tajima H, Inotani D, Ohnishi A and Ohashi Y 2018 Phys. Rev. A 97 013601
[15] Horikoshi M and Kuwata-Gonokami M 2019 Int. J. Mod. Phys. E 28 1930001
[16] Yin X Y, Hu H and Liu X J 2019 Phys. Rev. Lett. 123 073401
[17] Baker G A 1999 Phys. Rev. C 60 054311
[18] Heckel S, Schneider P P and Sedrakian A 2009 Phys. Rev. C 80 015805
[19] Baym G, Hatsuda T, Kojo T, Powell P D, Song Y and Takatsuka T 2018 Rep. Prog. Phys. 81 056902
[20] Navon N, Nascimbène S, Chevy F and Salomon C 2010 Science 328 729–32
[21] Forbes M M, Gandolfi S and Gezerlis A 2011 Phys. Rev. Lett. 106 235303
[22] Forbes M M, Gandolfi S and Gezerlis A 2012 Phys. Rev. A 86 053603
[23] Caballero-Benitez S F, Paredes R and Romero-Rochín V 2013 Phys. Lett. A 377 1756–9
[24] Neri E, Caballero-Benitez S F, Romero-Rochín V and Paredes R 2020 Phys. Scr. 95 034013
[25] Sanner C, Su E J, Keshet A, Huang W, Gillen J, Gomers R and Ketterle W 2011 Phys. Rev. Lett. 106 010402
[26] Feld M, Frohlich B, Vogt E, Koschorreck M and Khol M 2011 Nature 480 75–8
[27] Makhalov V, Martiyanov K and Turlapov A 2014 Phys. Rev. Lett. 112 045301
[28] Yefsah T, Sommer A T, Ku M J H, Cheuk L W, Ji W, Bakr W S and Zwierlein M W 2013 Nature 499 426–30
[29] Bulgac A, Forbes M M, Kelley M M, Roche K J and Wlazłowski G 2014 Phys. Rev. Lett. 112 025301
[30] Mal S, Adhikary K, Sardar D, Saha A K and Deb B 2019 J. Phys. B: At. Mol. Opt. Phys. 52 165004
[31] Cooper L N 1956 Phys. Rev. 104 1189–90
[32] Deb B, Mishra A, Mishra H and Panigrahi P K 2004 Phys. Rev. A 70 011604
[33] Montorsi A and Penna V 1997 Phys. Rev. B 55 8226–39
[34] de Gennes P G 1999 Superconductivity of Metals and Alloys (New York: Advanced Book Program, Perseus Books)
[35] Deb B 2016 Int. J. Mod. Phys. B 30 1650036
[36] Flügge S 1998 Practical Quantum Mechanics (Berlin: Springer)
[37] Deb B 2005 J. Phys. B: At. Mol. Opt. Phys. 39 529–51
[38] Carlson J, Chang S Y, Pandharipande V R and Schmidt K E 2003 Phys. Rev. Lett. 91 050401
[39] Carlson J and Reddy S 2008 Phys. Rev. Lett. 100 150403
[40] Leggett A J 1980 Modern Trends in the Theory of Condensed Matter ed A Pekalski and J A Przystawa (Berlin: Springer) pp 13–27
[41] Dyke P, Pollack S E and Hulet R G 2013 Phys. Rev. A 88 023625
[42] Blackley C L, Julienne P S and Hutson J M 2014 Phys. Rev. A 89 042701
[43] Hazlett E L, Zhang Y, Stites R W and O’Hara K M 2012 Phys. Rev. Lett. 108 045304
[44] Hu J, Wu F, He L, Liu X J and Hu H 2020 Phys. Rev. A 101 013615