FORMAL SYMPLECTIC REALIZATIONS

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Abstract. We study the relationship between several constructions of symplectic realizations of a given Poisson manifold. We show that, for an arbitrary Poisson structure on $\mathbb{R}^n$, the realization given by tree-level part of Kontsevich star-product coincides with the one given by a formal version of the original construction of Weinstein, when suitably put in global Darboux form. We give an explicit formula for this realization in terms of rooted trees involving only computable Kontsevich’s weights, for which we set up a simple iterated integral, building on results of Kathotia.

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1. Introduction

A symplectic realization of a Poisson manifold $(M, \pi)$ is a Poisson map from a symplectic manifold $(S, \omega)$ to $(M, \pi)$, which is also a surjective submersion. Symplectic realizations are very natural objects in poisson geometry from the point of view of integration and quantization theory of Poisson manifolds.

Namely, following the standard “quantization dictionary” that associates a quantum algebra $\mathcal{A}_\pi$ of observables with a Poisson manifold $(M, \pi)$ (deformation quantization) and a Hilbert space $\mathcal{H}_\omega$ of states with a symplectic manifold $(S, \omega)$ (geometric quantization), a symplectic
realization should quantize to a representation of $A_\pi$ on $H_\omega$, which is the object encoding the symmetries at the quantum level (see [12]).

From the perspective of global integration of Poisson manifolds, symplectic realizations naturally arise as source maps of symplectic groupoids $(S, \omega) \rightrightarrows (M, \pi)$, which are the objects that integrate Poisson manifolds. Symplectic realizations contain a lot of information about integrability. For instance, Crainic and Fernandes showed in [3] that a Poisson manifold is globally integrable if and only if it admits a complete symplectic realization. However, in the global integration case, the symplectic realization do not carry all the information about the integrating symplectic groupoid (some extra data is needed to define a global inverse map for instance).

The situation is different for the local/formal integration of Poisson manifolds by local/formal symplectic groupoids. Recall that a local symplectic groupoid is, roughly, the structure obtained by restricting a global symplectic groupoid to a neighborhood of its unit space, while a formal symplectic groupoid is the structure obtained by taking $\infty$-jets of the symplectic groupoid structure maps (or their pullbacks as in [8]) at the unit space.

In the local/formal case, all the structure maps of the local/formal symplectic groupoid, including the inverse map, can be recovered from the source map alone ([11]), i.e., from a local/formal symplectic realization. Another simplification in this case is that the domain of the symplectic realization can always be taken to be the cotangent bundle of the Poisson manifold, or, more precisely, a local/formal neighborhood of its zero section.

Given a Poisson manifold $(M, \pi)$, the problem of constructing a local/formal symplectic realization can essentially be tackled in two ways, both of which start by considering a deformation $\epsilon \pi$ of the zero Poisson structure on $M$ by a parameter $\epsilon$. For the zero Poisson structure, the canonical bundle projection $q : T^*M \to M$ is a symplectic realization, where the symplectic form on the cotangent bundle is the canonical one $\omega_0$.

The first way is to try to deform $\omega_0$ into another symplectic form $\omega_\epsilon$ such that the bundle projection $q$ remains a symplectic realization from $(T^*M, \omega_\epsilon)$ (or from a neighborhood of its zero section) to $(M, \epsilon \pi)$. This approach was the original one of Weinstein in [11], who gave an integral formula for $\omega_\epsilon$ and proved the result in the case $M = \mathbb{R}^d$. The result for a general manifold has been proven recently by Crainic and Marcut in [4].

The second way is the approach of Karasev in [7] that keeps the canonical symplectic form $\omega_0$ fixed and deforms the bundle projection into a symplectic realization $q_\epsilon : (T^*M, \omega_0) \to (M, \epsilon \pi)$. The formal expansion of $q_\epsilon$ in the case $M = \mathbb{R}^d$ has been shown in [6] to coincide with the formal symplectic realization that one can extract (see [2]) from the tree-level part of the Kontsevich star-product given in [10].

In this paper, we show, in the case $M = \mathbb{R}^d$, the explicit relationship between these two constructions (Proposition [10] and Theorem [20]), and we give a very explicit formula for the formal symplectic realization in terms of topological rooted trees (Theorem [32] and Proposition [34]).

In Section [8], we review the construction of $\omega_\epsilon$ as considered in [4, 11], and we give a canonical way to obtain a Karasev-like realization $q_\epsilon$ of any Poisson manifold out of the deformed symplectic form $\omega_\epsilon$ by putting it in global Darboux form.
In Section 3 we review the formal symplectic realization
\[ s_K(p, x) = \frac{\partial S_{\pi}}{\partial p_2}(p, 0, x), \]
where \( S_{\pi} \) is the generating function (extracted from the Kontsevich star-product) of the formal symplectic groupoid constructed in [2] that integrates the Poisson manifold \((\mathbb{R}^d, \epsilon \pi)\).

We prove that it coincides with the Weinstein realization for \(\mathbb{R}^d\) when suitably put in Darboux form and that the coefficients admit a recursive definition (Theorem 20).

In Section 4 we show (Theorem 32) that \(s_K\) is given by the formula
\[ s^i_K(p, x) = x^i + \sum_{t \in [RT]} \epsilon_{|t|} \left( \frac{W_t}{|\text{sym}(t)|} \right) D^i_LV, \]
where \([RT]\) is the set of topological rooted trees, \(W_t\) is the Kontsevich weight of the Kontsevich graph associated to the topological rooted tree \(t\) as in Figure 4.5, and \(D^i_LV\) is the elementary differential (see Definition 24) associated with \(t\) and the vector field \(L_V = \pi^{ij}(x)p_i\partial_{x^j}\).

Building on Kathotia’s results in [9], we show in Proposition 34 that these weights are easily computable: \(W_t = I_t(0)\), where \(I_t\) is a recursive formula
\[ I_t(\theta) = \int_0^1 d\bar{\lambda} \int_0^{\bar{\lambda}} d\bar{\theta} I_{t_1}(\bar{\theta}) \cdots I_{t_m}(\bar{\theta}), \]
associated with the rooted tree \(t = [t_1, \ldots, t_m]\) made out of the subtrees \(t_1, \ldots, t_m\) by grafting their root to a new additional root, and where \(I_t(\theta) = \frac{1}{2} - \theta\) when \(t\) is the rooted tree with only one vertex.

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2. General results

In this section, we review the symplectic realization \(q : (T^* M, \omega_\pi) \to (\mathbb{R}^d, \epsilon \pi)\) considered originally by Weinstein in [11] and, more recently, by Crainic and Marcut in [4]. We explain how it can be constructed out of a Poisson spray, and we show how to put this realization in “global Darboux form” to obtain a Karasev-like realization \(q_\epsilon : (T^* M, \omega_\pi) \to (\mathbb{R}^d, \epsilon \pi)\). We end up by further analyzing the case \(M = \mathbb{R}^d\), for which we also consider a formal version of the constructions.

2.1. Symplectic realizations from Poisson sprays. Let \((M, \pi)\) be a Poisson manifold and \(q : T^* M \to M\) its cotangent bundle. Throughout the paper \(V \in \mathcal{X}(T^* M)\) will denote a Poisson spray for \(\pi\), namely, a vector field on \(T^* M\) satisfying:

1. \(T_\xi q(V|_\xi) = \pi^\xi(\xi)\) for all \(\xi \in T^* M\)
2. \(m_t^* V = tV\), for \(m_t : T^* M \to T^* M\) being the diffeomorphism obtained by fiberwise multiplication by \(t \in \mathbb{R}\)
Remark 1. Notice that $V_\varepsilon := m_\varepsilon V = \varepsilon V$ is thus a spray for the re-scaled Poisson structure $\varepsilon \pi$.

It follows from the definition that the flow $\varphi_s^\varepsilon$ of $V$ fixes the zero section and, thus, that there exists an open neighborhood $U_1 \subset T^*M$ of the zero section such that $\varphi_s^\varepsilon$ is defined for $s \in [0, 1]$.

Example 2. A linear connection $\nabla$ on $q : T^*M \to M$ defines a Poisson spray by setting, for $\xi \in T^*M$,

$$V_\xi^\nabla = \text{Hor}_\nabla(\pi(\xi))$$

For each $\varepsilon \in [0, 1]$, we choose an open neighborhood $U_\varepsilon \subset T^*M$ of the zero section such that $\varphi_s^\varepsilon$ is defined for $s \in [0, \varepsilon]$. We can take $U_1 \subset U_\varepsilon$ for all $\varepsilon \in [0, 1]$. Following [3], we consider the following differential forms on $U_\varepsilon$

$$\omega_{V,\varepsilon} := \frac{1}{\varepsilon} \int_0^\varepsilon ds \left( (\varphi_s^\varepsilon)^* \omega_0 \right) = d\theta_{V,\varepsilon},$$

$$\theta_{V,\varepsilon} := \frac{1}{\varepsilon} \int_0^\varepsilon ds \left( (\varphi_s^\varepsilon)^* \theta_0 \right),$$

where $\theta_0$ is the Liouville one form on $T^*M$ and $\omega_0 = d\theta_0$ the associated symplectic form.

As explained in [3], when evaluated at points of the zero section $0^{T^*M} \subset T^*M$,

$$\omega_{V,\varepsilon}|_{0^{T^*M}} = \omega_0 - \varepsilon \pi(P_{T^*M}(\cdot), (P_{T^*M}(\cdot))$$

is non degenerate for all $\varepsilon$, where $P_{T^*M} : T|_{0^{T^*M}}(T^*M) \to T^*M$ is the natural projection. It thus follows that we can choose possibly smaller neighborhoods $U(V)_\varepsilon \subset U_\varepsilon$ where this 2-form is non-degenerate. Then, $\omega_{V,\varepsilon}$ is a (exact) symplectic form on this $U(V)_\varepsilon$.

Theorem 3. The bundle projection $q : (U(V)_\varepsilon \subset T^*M, \omega_{V,\varepsilon}) \to (M, -\varepsilon \pi)$ is a symplectic realization.

This was proven in [4].

Remark 4. Consider the path construction of the (local) symplectic groupoid $(\mathcal{G}_\varepsilon, \Omega_\varepsilon) \rightrightarrows M$ integrating $(M, \varepsilon \pi)$ in terms of (small) cotangent paths modulo cotangent homotopies as in [1] and [3]. For each initial condition, close enough to the zero section in $T^*M$, the flow of $V_\varepsilon$ produces a particular cotangent path $[0, 1] \to T^*M$, defining in this way an exponential map $\exp_{V_\varepsilon} : U_1 \to \mathcal{G}_\varepsilon$. The symplectic form above arises as $\omega_{V,\varepsilon,1} = \exp_{V_\varepsilon}^\ast \Omega_\varepsilon = \omega_{V,\varepsilon}$. The content of the above theorem can then be understood as follows: the symplectomorphism $\exp_{V_\varepsilon}$ takes the source map of $\mathcal{G}_\varepsilon$, which is a symplectic realization (as for any symplectic groupoid), to the projection $q$.

Remark 5. (Change of spray). Let $(M, \pi)$ be a Poisson manifold and consider two Poisson sprays $V_1$ and $V_2$. Then, uniqueness of symplectic realizations (see [3]) together with Theorem 3 imply the existence of a symplectomorphism $F_\varepsilon : (U(V_1)_\varepsilon, \omega_{V_1,\varepsilon}) \to (U(V_2)_\varepsilon, \omega_{V_2,\varepsilon})$ such that $q \circ F_\varepsilon = q$.

1More generally, the results below also hold for any other vector field $V$ satisfying (1) and such that there is a neighborhood of the zero section on which the flow is defined on $[0, 1]$.

2Notice the sign convention $\omega_0 \simeq dp_i \wedge dx^i = -dx^i \wedge dp_i$. 


2.2. Symplectic realizations in Darboux form. Notice that the family of symplectic forms $\omega_{V,\epsilon}$ are exact for all $\epsilon$ and that $\omega_{V,\epsilon=0} = \omega_0$. One can thus try to apply a Moser-type argument to find a flow that transforms this family back into $\omega_0$.

**Definition 6.** We say that a symplectic realization $\alpha : (W, \Omega) \to (M, \pi)$ is in Darboux form if $W \subset T^*M$ is an open neighborhood of the zero section and $\Omega = \omega_0|_W$. Given a realization $r : (S, \Omega) \to (M, \pi)$, a global Darboux frame for it consists of a diffeomorphism $\Phi : W \subset T^*M \to S$ such that $\Phi^*\Omega = \omega_0$.

When a global Darboux frame for $(S, \Omega)$ exists, the composition of Poisson maps $\alpha = r \circ \Phi : (W, \omega_0) \to (M, \pi)$ yields an induced realization in Darboux form. In [5], a symplectic realization is called strict when the realization map admits a global Lagrangian section. By the Lagrangian embedding theorem, strict realizations always admit global Darboux frames.

For completeness, we now show that, for each $\epsilon$, there is a natural global Darboux frame for $\omega_{V, \epsilon}$ attached to the given data $(M, \pi, V)$.

**Proposition 7.** Let $(M, \pi)$ be a Poisson manifold and $V$ a Poisson spray. Consider the $(\epsilon$-dependent) vector field $X^V_\epsilon$ on $U(V)_1$ defined by

$$i_{X^V_\epsilon} \omega_{V, \epsilon} = -\frac{d}{d\epsilon} \theta_{V, \epsilon}$$

Then, there exists an open neighborhood $W \subset U(V)_1 \subset T^*M$ of the zero section such that the flow $\Phi_{V, \epsilon}$ of $X^V_\epsilon$ is defined on $W$ for $\epsilon \in [0,1]$ and $\Phi^*_{V, \epsilon} \omega_{V, \epsilon} = \omega_0$ for all $\epsilon \in [0,1]$.

**Proof.** Notice that the forms $\omega_{V, \epsilon}$ and $\theta_{V, \epsilon}$ are well defined on $U(V)_1$ for all $\epsilon \in [0,1]$ and that they define a smooth $\epsilon$-family of differential forms on $U(V)_1$. The vector field $X^V_\epsilon$ defined in eq. (2.2) satisfies $X^V_\epsilon|_{T^*M} = 0$ for all $\epsilon$ since, when evaluated at points of the zero section, $\theta_{V, \epsilon}|_{T^*M} = 0$.$\epsilon$. Then, there exists an open neighborhood $W \subset U(V)_1$ on which the flow $\Phi_{V, \epsilon}$ is defined for $\epsilon \in [0,1]$. Finally, a standard computation shows that

$$\frac{d}{d\epsilon} (\Phi^*_{V, \epsilon} \omega_{V, \epsilon}) = \Phi^*_{V, \epsilon} \left( L_{X^V_\epsilon} \omega_{V, \epsilon} \right) + \Phi^*_{V, \epsilon} \left( \frac{d}{d\epsilon} \omega_{V, \epsilon} \right)$$

$$= \Phi^*_{V, \epsilon} \left( d \left( i_{X^V_\epsilon} \omega_{V, \epsilon} \right) \right) + \Phi^*_{V, \epsilon} \left( d \left( \frac{d}{d\epsilon} \theta_{V, \epsilon} \right) \right)$$

$$= \Phi^*_{V, \epsilon} \left( d \left( i_{X^V_\epsilon} \omega_{V, \epsilon} + \frac{d}{d\epsilon} \theta_{V, \epsilon} \right) \right)$$

$$= 0.$$

The statement then follows from the initial condition $\omega_{V, \epsilon=0} = \omega_0$. \qed

As a consequence, there exists a natural family of induced symplectic realizations in Darboux form $q_\epsilon = q \circ \Phi_{V, \epsilon} : (W \subset T^*M, \omega_0) \to (M, \epsilon \pi)$ for $\epsilon \in [0,1]$ associated to the spray $V$. In the particular case $M = \mathbb{R}^d$ below, we shall see that there exists a simpler alternative construction leading to a different Darboux realization which was introduced earlier by Karasev ([7]).
2.3. Symplectic realizations for $\mathbb{R}^d$. In this paragraph, we analyse further the case $M = \mathbb{R}^n$ together with a “flat Poisson spray” $V$ defined from the flat connection $\nabla_{\text{flat}}$ on $T^*\mathbb{R}^d$ as in example 2. We shall also consider its “opposite” $\overline{V}$ (which will help us later on to take care of some sign issues), so that they are given by

$$(2.3) \quad V|_{\xi=(x,p)} = \pi^{ij}(x)p_i\partial_{x^j} \quad \text{and} \quad \overline{V}|_{\xi=(x,p)} = -\pi^{ij}(x)p_i\partial_{x^j}.$$ 

In this case, the symplectic realization realization $\psi: (U(V) \subset T^*\mathbb{R}^n, \omega_{V,\epsilon}) \to (\mathbb{R}^n, -\epsilon\pi)$ obtained from the Poisson spray as outlined in Section 2.1 coincides with the original construction by Weinstein in [11]. We shall thus refer to it as the Weinstein realization.

Remark 8. (Sign convention). Here, we will consider the “opposite” symplectic realization $\psi: (U(V) \subset T^*\mathbb{R}^n, \omega_{V,\epsilon}) \to (\mathbb{R}^n, \epsilon\pi)$ obtained from $V$ in the same way. Observe that $\omega_{V,\epsilon} = -\omega_{\overline{V},\epsilon}$.

In $\mathbb{R}^d$ there is a simple way of obtaining global Darboux frames for $\omega_{V,\epsilon}$ as follows. Since we are on flat space, we can define an $\epsilon$-family of maps

$$\psi_{\overline{V},\epsilon}: (x,p) \mapsto (\phi_\epsilon(x,p), p)$$

from $U(V) \subset T^*\mathbb{R}^d$ by its action on coordinates

$$(2.4) \quad \psi_{\overline{V},\epsilon}^*_x x^i = \phi^i_\epsilon(x,p) := \frac{1}{\epsilon} \int_0^\epsilon \left((\varphi_s^*)^*_x x^i\right)|_{(x,p)} ds$$

$$\psi_{\overline{V},\epsilon}^*_p p_j = p_j$$

It follows that

$$(2.5) \quad \omega_{\overline{V},\epsilon} = \psi_{\overline{V},\epsilon}^* \omega_0 = dp_i \wedge d\phi^i_\epsilon$$

since the flat spray $\overline{V}$ only acts on the $x$ coordinates.

Remark 9. Notice that when $\pi$ is an analytic bivector on $\mathbb{R}^d$ then $\psi_{\overline{V},\epsilon}$ is an analytic function of $\epsilon$.

On the other hand, since

$$(2.6) \quad D_x \phi_\epsilon(x,0) = \text{id}$$

and $\psi_{\overline{V},\epsilon}(x,0) = (x,0)$, the inverse $\psi_{\overline{V},\epsilon}^{-1}$ exists in a neighborhood $W \subset T^*\mathbb{R}^n$ of the zero section. It is then an immediate consequence of eq. (2.5) that $\psi_{\overline{V},\epsilon}^{-1}: W \to U(V)_\epsilon$ is a global Darboux frame for $q : (U(V)_\epsilon, \omega_{\overline{V},\epsilon}) \to (\mathbb{R}^d, \epsilon\pi)$. The induced Darboux-form realization is then given by the map

$$\alpha_{\overline{V},\epsilon} = q \circ \psi_{\overline{V},\epsilon}^{-1}: (W, \omega_0) \to (\mathbb{R}^d, \epsilon\pi)$$

It is not hard to see that the realization $\alpha_{\overline{V},\epsilon}$ coincides with the one given by Karasev in [7] (after rescaling the momenta using $m_\epsilon$) and we shall refer to it as the Karasev realization. We have then shown:

Proposition 10. For $M = \mathbb{R}^d$, the map $\psi_{\overline{V},\epsilon}^{-1}: W \to U(V)_\epsilon$ defined above is a global Darboux frame for the Weinstein realization defined by $\omega_{\overline{V},\epsilon}$. Moreover, the induced Darboux-form realization coincides with the Karasev realization $\alpha_{\overline{V},\epsilon}$.
The components $\alpha^i_\epsilon(x,p)$ of the Karasev realization $\alpha_{V,\epsilon}$ can be thus obtained by solving the following equation:

\begin{equation}
\phi_\epsilon(\alpha_\epsilon(x,p), p) = x \ \forall p.
\end{equation}

**Remark 11.** Notice that the Karasev realization $\alpha_{V,\epsilon}$ does not coincide with the realization $q_\epsilon$ coming from the general construction of Prop. 7 ($\psi_{V,\epsilon}$ is not a flow).

### 2.3.1. Formal expansions.

We now turn to the formal setup by expanding $\omega_{V,\epsilon}$ in formal power series in $\epsilon$.

**Remark 12.** (Conventions). We shall adopt the following conventions when working with formal power series in $C^\infty(T^*\mathbb{R}^n)[[\epsilon]]$. Let $w^{i_1}_\epsilon(x,p) = w^{i_1}_0(x,p) + \epsilon \tilde{w}^{i_1}_\epsilon(x,p)$ with $w^{i_1}_0(x,p) \in C^\infty(T^*\mathbb{R}^n)$ and $\tilde{w}^{i_1}_\epsilon(x,p) \in C^\infty(T^*\mathbb{R}^n)[[\epsilon]]$. For $f(x,p) \in C^\infty(T^*\mathbb{R}^n)$ we shall denote (multi-variable Taylor expansion)

\begin{equation}
f(w^{i_1}_\epsilon(x,p), p) = \sum_{n \geq 0} \frac{\epsilon^n}{n!} \tilde{w}^{i_1}_\epsilon(x,p) \cdots \tilde{w}^{i_n}_\epsilon(x,p) \left[ \frac{\partial^{(n)}}{\partial x^{i_1} \cdots \partial x^{i_n}} f \right] (w^{i_0}_0(x,p), p)
\end{equation}

First of all notice that the formal flow of $V$ can be written as

$$\phi^V_s(x,p) = (\exp(sV)(x), p),$$

where $\bar{x} = \exp(sV)(x)$ is the formal flow of $V$ (seen as a $p$-dependent vector field on $\mathbb{R}^d$) defined in components by the following formal power series

$$\bar{x}^i = \sum_{n \geq 0} \frac{s^n}{n!} L^i V(x).$$

**Lemma 13.** The map $\phi_\epsilon$ defined in eq. (2.4) is given by the formal expansion

\begin{equation}
\phi^i_\epsilon(x,p) = \sum_{n \geq 0} \frac{\epsilon^n}{(n+1)!} (L^i V)^n(x).
\end{equation}

This means that the formal version of the Karasev realization $\alpha_{V,\epsilon}$ is defined by the (unique) functions $\alpha^i_\epsilon(x,p) \in C^\infty(T^*\mathbb{R}^n)[[\epsilon]]$ solving eq. (2.7) with $\phi_\epsilon$ given by the formal expansion in eq. (2.9). We have thus shown

**Theorem 14.** The formal solution $\alpha_\epsilon(x,p)$ of Equation (2.7) is a formal symplectic realization $\alpha_{V,\epsilon} : (T^*\mathbb{R}^n, \omega_0) \to (\mathbb{R}^n, \epsilon \pi)$, i.e.,

$$\{\alpha^i_\epsilon, \alpha^j_\epsilon\}_{\omega_0}(x,p) = \epsilon \pi^{ij}(\alpha_\epsilon(x,p))$$

is satisfied at all orders in $\epsilon$. Moreover, the formal Darboux frame $\psi_{V,\epsilon}^{-1}(x,p) = (\alpha_\epsilon(x,p), p)$ takes the formal Weinstein realization $q : (T^*\mathbb{R}^d, \omega_{\mathbb{R}^d}) \to (\mathbb{R}^d, \epsilon \pi)$ to the formal Darboux realization given by $\alpha_\epsilon$.

### 3. The Kontsevich realization

In this section, we briefly review the construction of [2] that produces a formal symplectic realization out of the (tree-level part of) Kontsevich’s quantization formula, and we show that it coincides with the one obtained in Theorem 14.
3.1. The generating function and the realization. In [2], it was shown that the formal power series

\[ S_{\pi,2}(p_1, p_2, x) = (p_1 + p_2)x + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \sum_{\Gamma \in T_{n,2}} W_{\Gamma} \hat{B}_{\Gamma} \left( \frac{\pi}{2} \right) (p_1, p_2, x), \]

with \( x \in \mathbb{R}^d \) and \( p_1, p_2 \in (\mathbb{R}^d)^* \) is a formal generating function for the formal symplectic groupoid \( T^* \mathbb{R}^n \to \mathbb{R}^d \) (where the cotangent bundle is endowed with its canonical symplectic structure \( \omega_0 = \sum_i dp^i \wedge dx^i \)) integrating \((\mathbb{R}^n, \epsilon \pi)\). In the above formula,

- \( T_{n,2} \) is the set of Kontsevich trees of type \((n,2)\);
- \( W_{\Gamma} \) is a real number called the Kontsevich weight of \( \Gamma \);
- \( \hat{B}_{\Gamma}(\pi) \) is the symbol of the Kontsevich operator \( B_{\Gamma}(\pi) \).

We define these objects in the following section and refer the reader to [10, 6] for more details.

Remark 15. The formal generating function above depends on the Poisson structure through the symbols of the Kontsevich operators. In [2] and [6], the scaling of the Poisson structure is different: \( S_{\pi} \) is used instead of \( S_{\pi,2} \), and thus the corresponding formal symplectic groupoid integrates the Poisson structure \( 2\epsilon \pi \) instead of \( \epsilon \pi \). We also use Kathotia’s convention ([9]) for the Kontsevich weights; namely,

\[ W_{\Gamma} = n! W^K_{\Gamma}, \]

where \( W^K_{\Gamma} \) is the weight actually defined by Kontsevich in [10].

As shown in [2], the formal source and target maps can be extracted from the formal generating function as follows:

\[ s_K(p, x) = \frac{\partial S_{\pi,2}}{\partial p_2}(p, 0, x), \]

\[ t_K(p, x) = \frac{\partial S_{\pi,2}}{\partial p_1}(0, p, x). \]

Therefore, we obtain a formal symplectic realization from \((T^* \mathbb{R}^n, \omega_0)\) to \((\mathbb{R}^n, \epsilon \pi)\) of the form

\[ s_K(p, x) = x + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \sum_{\Gamma \in T_{n,2}} W_{\Gamma} \frac{\partial \hat{B}_{\Gamma}(\pi)}{\partial p_2}(0, 0, x), \]

As it will become clear after we define the Kontsevich operators in the next section, the factor \( \frac{1}{2\pi} \) in the formula above comes from the recalling of the Poisson structure: namely,

\[ B_{\Gamma}(\frac{1}{2\pi}) = \frac{1}{2n} B_{\Gamma}(\pi), \quad \text{for } \Gamma \in T_{n,2}. \]

3.2. Kontsevich’s graphs, operators and weights. A Kontsevich graph of type \((n, m)\) is a graph \((V, E)\) whose vertex set is partitioned in two sets of vertices \( V = V^a \sqcup V^g \), the aerial vertices \( V^a = \{1, \ldots, n\} \) and the terrestrial vertices \( V^g = \{1, \ldots, m\} \) such that

- all edges start from the set \( V^a \),
- loops are not allowed,
- there are exactly two edges going out of a given vertex \( k \in V^a \),
• the two edges going out of \( k \in V^a \) are ordered, the first one being denoted by \( e^1_k \) and the second one by \( e^2_k \).

An aerial edge is an edge whose end vertex is aerial, and a terrestrial edge is an edge whose end vertex is terrestrial. We denote by \( G_{n,m} \) the set of Kontsevich graphs of type \((n, m)\). Figure 3.1 illustrates a graphical way to represent Kontsevich graphs.

Given a Poisson structure \( \pi \) and a Kontsevich graph \( \Gamma \in G_{n,m} \), one can associate a \( m \)-differential operator \( B_\Gamma(\pi) \) on \( \mathbb{R}^n \) in the following way: For \( f_1, \ldots, f_m \in C^\infty(\mathbb{R}^n) \), we define

\[
B_\Gamma(\pi)(f_1, \ldots, f_m) := \sum_{I:E_\Gamma \to \{1, \ldots, d\}} \left[ \prod_{k \in V^a_\Gamma} \pi^{I(e^1_k)} \prod_{e \in E_\Gamma} \partial_{I(e)} \prod_{i \in V^g_\Gamma} \prod_{e \in E_\Gamma} \partial_{I(e)} f_i \right].
\]

The symbol \( \hat{B}_\Gamma \) of \( B_\Gamma \) is defined by

\[
B_\Gamma(e^{p_1}x, \ldots, e^{p_m}x) = \hat{B}_\Gamma(p_1, \ldots, p_m) e^{(p_1 + \cdots + p_m)x}.
\]

The following figure illustrates this.

![Figure 3.1](image_url)

**Figure 3.1.** A Kontsevich graph \( \Gamma \) of type \((n, 2)\). The terrestrial vertices are on an imaginary line (the dotted line) and the aerial vertices are placed above them. The first arrow stemming out of an aerial vertex has a solid black head, while the second one has a hollow white one.

**Example 16.** Consider the Kontsevich graph \( \Gamma \) as in Figure 3.1. The three aerial vertices 1, 2, and 3 correspond to three copies of the Poisson structure, say \( \pi^{ij}, \pi^{kl} \) and \( \pi^{mn} \). The two terrestrial vertices 1 and 2 correspond to two functions smooth functions on \( \mathbb{R}^d \), say \( f \) and \( g \). The arrows correspond to derivatives. Therefore, the Kontsevich operators associated with this graph is

\[
B_\Gamma(\pi)(f, g) = \pi^{ij} \partial_n \partial_i \pi^{kl} \partial_m \partial_k f \partial_m \partial_l g,
\]

where we use the Einstein summation convention of repeated indices. The corresponding symbol is obtained by replacing \( \partial_i f \) by \( p^1_i \) and \( \partial_j g \) by \( p^2_j \); namely,

\[
\hat{B}_\Gamma(\pi)(p_1, p_2, x) = \pi^{ij} \partial_n p^1_i \pi^{kl} \partial_m p^1_k \partial_l p^2_m \partial_l p^2_j.
\]

The order of the arrow is important because flipping, for example, the order of the first aerial vertex order would introduce a sign, since \( \pi^{ij} = -\pi^{ji} \).

\( T_{n,2} \) is a subset of \( G_{n,2} \) that we now define:
**Definition 17.** Let $\Gamma \in G_{m,n}$ be a Kontsevich graph. The **interior** of $\Gamma$ is the graph $\Gamma_i$ obtained from $\Gamma$ by removing all terrestrial vertices and terrestrial edges. A Kontsevich graph is a **Kontsevich tree** if its interior is a tree in the usual sense (i.e. it has no cycles). We denote by $T_{n,m}$ the set of Kontsevich’s trees of type $(n,m)$.

We give now a very rough presentation of the Kontsevich weights, which follows Kathotia’s conventions in [9], instead of the original ones of Kontsevich in [10]. Let $\mathcal{H}$ be the upper half plane and $\mathbb{R}_x$ be the $x$-coordinate line in $\mathcal{H}$. Given two points $p$ and $q$ in $\mathcal{H}$ with $p \notin \mathbb{R}_x$, we define the **angle function** $\phi(p, q) \in [0, 1)$, which depends on whether $q \in \mathbb{R}_x$ or not ($\phi(p, q) = \theta$ if $q \in \mathbb{R}_x$ and $\phi(p, q) = \lambda$ if $q \notin \mathbb{R}_x$), as depicted in Figure 3.2 (we refer the reader to [9, 10] for more details on the angles):

![Figure 3.2](image)

Observe that, for fixed $q$ and $q'$ as in Figure 3.2, the angles $\theta$ and $\lambda$ determine completely the position of the point $p$.

A **configuration** $z = (z_1, \cdots, z_n)$ with $z_i \in \mathcal{H}$ of a Kontsevich’s graph $\Gamma \in G_{n,2}$ is an identification $i \mapsto z_i$ of its aerial vertices $\{1, \ldots, n\}$ with the points of the $n$-tuple $z$ and of its terrestrial vertices 1 and 2 with respectively 0 and 1. We denote by $M_n$ the manifold of all the configurations of $\Gamma$; its compactification $\bar{M}_n$ is a manifold with corners (see [10] for details).

Given $\Gamma \in G_{n,2}$, we have $n$ pairs of angle-functions on $\bar{M}_n$, which we denote by $\phi_1^1, \phi_1^2, \ldots, \phi_n^1, \phi_n^2$. They are determined by the configuration of the aerial vertices of $\Gamma$: Namely, $\phi_k^i = \phi(z_k, z_{\gamma^i(k)})$ is the angle-function as above, where $\gamma^i(k)$ is the vertex target of $e_k^i$ (the $i^{th}$ edge of $k$). This allows us to define the angle-form on $\bar{M}_n$ associated with $\Gamma$:

$$\omega_\Gamma = \bigwedge_{k=1}^n d\phi_k^1 \wedge d\phi_k^2.$$  

Since $\omega_\Gamma$ is a $2n$-form and $\bar{M}_n$ is a compact real $2n$-manifold, we can integrate $\omega_\Gamma$, which give us the Kontsevich weight of $\Gamma$:

$$W_\Gamma = \int_{\bar{M}_n} \omega_\Gamma.$$
Remark 18. Following Kathotia, we do not include the original factor \( \frac{1}{n!} \) in the definition of the weights, and the factor \( \frac{1}{(2\pi)^n} \) is taken into account by the rescaling of the angles (we use \([0, 1]\) instead of \([0, 2\pi]\)).

3.3. Comparison between the two symplectic realizations. As mentioned earlier, the realization \( \alpha_\epsilon \) of Theorem 14 is a formal version of the one introduced by Karasev in [7] which, in turn, was shown in [6] to coincide with the realization \( s_\alpha \) coming from Kontsevich trees. Here we repeat the argument for completeness (c.f. Theorem 20) and also show that the coefficients in the \( \epsilon \)-expansion of \( \alpha_\epsilon \) admit a recursive definition (c.f. equation (3.9) below).

Recall that a formal symplectic realization from \( (T^*_\mathbb{R}^n, \omega_0) \) to \( (\mathbb{R}^n, \epsilon \pi) \) is a formal power series of the form

\[
s_\epsilon(p, x) = x + \epsilon s_{(1)}(p, x) + \epsilon^2 s_{(2)}(p, x) + \cdots,
\]

where the \( s_{(i)} \)'s are smooth functions on \( T^*_\mathbb{R}^n \), formally satisfying (i.e. at each order) the partial differential equation

\[
\{s^i, s^j\}_{\omega_0}(x, p) = \epsilon \pi^{ij}(s_\epsilon(x, p)).
\]

Lemma 19. All formal symplectic realizations from \( (T^*_\mathbb{R}^n, \omega_0) \) to \( (\mathbb{R}^n, \epsilon \pi) \) starting with the same first order term \( s_{(1)} \) and such that, for \( i \geq 1 \), we have

\[
\langle s_{(i)}(p, x), p \rangle = 0, \quad i \geq 1,
\]

\[
s_{(i)}(\lambda p, x) = \lambda^i s_{(i)}(p, x),
\]

coincide.

Proof. If we develop the right hand side of (3.5) in Taylor’s series and read off the \( n^{th} \) order in \( \epsilon \) for each \( n \geq 1 \), we obtain an infinite system of recursive equations of the form

\[
\frac{\partial s_{(n)}^i}{\partial p_i} - \frac{\partial s_{(n)}^j}{\partial p_j} = H_n(s_{(1)}, \ldots, s_{(n-1)}),
\]

where \( H_n \) is an expression involving only lower orders of \( s_{(k)} \). Now suppose we have another formal symplectic realization \( \bar{s} \) satisfying (3.7) and such that \( s_{(1)} = \bar{s}_{(1)} \). We show by induction on \( k \) that \( s_{(k)} = \bar{s}_{(k)} \) for all \( k \). By assumption, it is true for \( k = 1 \). Suppose that it is true all \( k < n \). Set \( g_{(i)} = s_{(i)} - \bar{s}_{(i)} \). Then (3.8) tells us that

\[
\frac{\partial g_{(n)}^i}{\partial p_i} - \frac{\partial g_{(n)}^j}{\partial p_j} = 0.
\]

This implies, by the Stokes Theorem, that there is a function \( f_n(p, x) \) such that \( \nabla_p f_n(p, x) = g_n \). Now, using (4.1) and (3.7), we have that

\[
\frac{d}{d\lambda} f_n(\lambda p, x) = \langle \nabla_p f_n(\lambda p, x), p \rangle = \langle s_n(\lambda p, x), p \rangle - \langle \bar{s}_n(\lambda p, x), p \rangle = 0,
\]

which implies that \( f_n \) does not depend on \( p \), and thus \( \nabla_p f_n = s_n - \bar{s}_n = 0 \). \( \square \)

Now, using definition Equation (2.8) and Formula (2.9) for \( \phi_\epsilon \), one gets that Equation (2.7) for \( \alpha_\epsilon \) is equivalent to:

\[
x^i = \sum_{n,m \geq 0} \sum_{r_1, \ldots, r_m \geq 1} \frac{\epsilon^{n+r_1+\cdots+r_m}}{(n+1)!m!} \alpha_{(r_1)}^{i_1}(x, p) \ldots \alpha_{(r_m)}^{i_m}(x, p) \left[ \frac{\partial^{(m)} (L^{\phi_\epsilon}_p x^i)}{\partial x^{i_1} \ldots \partial x^{i_m}} \right](x, p).
\]
Reading this equation at order \( N \) in \( \epsilon \), one gets

\[
\alpha_i^{(N)}(x, p) + \sum_{n=1}^{N-1} \sum_{m=1}^{N-n} \sum_{r_1 \geq 1} \cdots \sum_{r_m = N-n} \frac{1}{(n+1)! m!} \alpha_i^{(r_1)}(x, p) \cdots \alpha_i^{(r_m)}(x, p) \left[ \frac{\partial^{(m)}}{\partial x^{i_1} \cdots \partial x^{i_m}} \right] (x, p) + \sum_{n=1}^{N-1} \sum_{m=1}^{N-n} \sum_{r_1 \geq 1} \cdots \sum_{r_m = N-n} \frac{1}{(n+1)!} \left( L^N \right)(x^i)(x, p) = 0
\]

(3.9)

This is a recursive formula for the terms \( \alpha_i^{(N)} \) in \( \alpha_\epsilon \).

**Theorem 20.** Let \( \alpha_\epsilon : (T^* \mathbb{R}^n, \omega_0) \to (\mathbb{R}^n, \epsilon \pi) \) be the formal symplectic realization of Theorem 14. This realization coincides with the Kontsevich realization \( s_K \) and, moreover, the coefficients in the formal expansion of these coinciding realizations can be recursively computed using formula (3.9).

**Proof.** The last statement follows from the first one and from the computation above. It remains to prove the first statement, which is done by applying Lemma 19 as follows. As shown in [6], \( s_K \) satisfies the hypothesis of this Lemma. As for \( \alpha_\epsilon \), let us write

\[
\alpha_i^\epsilon(x, p) = x^i + \epsilon \sum_{r \geq 1} \epsilon^{r-1} \alpha_i^{(r)}(x, p).
\]

(3.10)

Using equation (3.9), for \( N = 1 \), we obtain \( \alpha_i^\epsilon(x, p) + \frac{1}{2} L^1 \pi p_i = 0 \). Thus, the first term of \( \alpha_\epsilon \) is \( \frac{1}{2} \pi^i p_i \), which is the same as for the Kontsevich realization. By inspection of the recursive formula above, we see that each term of \( \langle \alpha_i^{(N)}(x, p), p \rangle \) has a factor of the form

\[
\cdots \partial \cdots \partial L^i \pi p_i = \cdots \partial \cdots \partial \pi^i p_i p_i
\]

which is zero by the antisymmetry of the Poisson structure. This shows that (3.6) holds. An induction on \( N \) using the same equation immediately yields (3.7).

4. AN EXPLICIT FORMULA FOR THE REALIZATION

For a linear Poisson structure \( \pi^ij(x) = \pi^i_k x^k \) on \( \mathbb{R}^d \) (i.e. the one living on the dual \( \mathcal{G}^* \simeq \mathbb{R}^d \) of a Lie algebra \( \mathcal{G} \)), the Kontsevich trees that have an aerial vertex with more than one incoming edge have vanishing symbols: i.e. \( \hat{B}_T(\pi) = 0 \). Thus, the Kontsevich trees that have a non-vanishing contribution to the generating function (3.1) are of the form

![Figure 4.1](image-url)
Therefore, for linear Poisson structures, the only Kontsevich trees such that

\[ \frac{\partial \hat{B}_\Gamma(\pi)}{\partial p_2}(p,0,x) \neq 0 \]

are the ones of the form

\[ \text{Figure 4.2.} \]

Since, for two of these graphs \( \Gamma \) and \( \Gamma' \) with \( n \) arial vertices, we have that \( W_\Gamma B_\Gamma = W_\Gamma' B_\Gamma' \), and since there are \( n! 2^n \) of these graphs, we obtain the following explicit formula for the corresponding formal symplectic realization in the case of linear Poisson structures:

\[ s^K_i(p,x) = x^i + \sum_{n=1}^{\infty} \epsilon^n (-1)^n \frac{B_n}{n!} \text{ad}_p^n(x^i), \]

where \( \text{ad}_f(g) = \{ f, g \} \) for \( f, g \in C^\infty(\mathcal{G}^*) \) and \( p \in T^*_x \mathcal{G}^* \simeq \mathcal{G} \) is identified with a linear function on \( \mathcal{G}^* \), and where we used a well-known result of Kathotia in [9] about the Kontsevich weights of these types of Kontsevich’s graphs.

In this section, we generalize Formula (4.2) in the case of non-linear Poisson structures and we express the realization in terms of a sum over topological rooted trees. We show that the only Kontsevich trees satisfying (4.1) in the non-linear case are exactly the ones that Kathotia in [9] calls “W-computable.” Building on his results, we set up a recursive formula for the corresponding Kontsevich’s weights in terms of topological rooted trees.

Remark 21. It is interesting to notice that a direct computation of the formal Karasev realization \( \alpha_\epsilon \) using the recursion (3.9) in the linear Poisson case yields the r.h.s. of formula (4.2). Thus, in this case, the conclusion above about the Kontsevich weights can also be obtained as a corollary of the fact that \( s^K = \alpha_\epsilon \) (c.f. Theorem 20). On the other hand, directly solving the recursion for non-linear Poisson structures is more complicated and a general solution will be presented in the next subsections (c.f. Theorem 32).

4.1. Rooted trees and elementary differentials. A graph is the data \((V,E)\) of a finite set of vertices \( V = \{v_1, \ldots, v_n\} \) together with a set of edges \( E \), which is a subset of \( V \times V \). The number of vertices is called the degree of the graph and is denoted by \( |\Gamma|\). We think of \((v_1,v_2) \in E\) as an arrow that starts at the vertex \( v_1 \) and ends at \( v_2 \).

Two graphs are isomorphic if there is a bijection between their vertices that respects theirs edges. The set \( \Gamma \) of all isomorphic graphs to a given graph \( \Gamma \) is called a topological graph.

A symmetry of a graph is an automorphism of the graph (i.e. a relabeling of its vertices that leaves the graph unchanged). The group of symmetries of a given graph \( \Gamma \) will be denoted...
by $\text{sym}(\Gamma)$. Note that the number of symmetries of all graphs sharing the same underlying topological graph is equal; we define the symmetry coefficient $\sigma(\Gamma)$ of a topological graph $\Gamma$ to be the number of elements in $\text{sym}(\Gamma)$, where $\Gamma \in \overline{\Gamma}$.

A rooted tree is a graph that (1) contains no cycle, (2) has a distinguished vertex called the root, (3) whose set of edges is oriented toward the root. We will denote the set of rooted trees by $\text{RT}$ and the set of topological rooted trees by $[\text{RT}]$.

We represent graphically topological rooted trees as depicted in the following figure:

\begin{center}
\includegraphics[width=0.5\textwidth]{figure4.3.png}
\end{center}

Figure 4.3.

The set of topological rooted trees can be described recursively as follows: The single vertex graph $\bullet$ is in $[\text{RT}]$ and if $t_1, \ldots, t_n \in [\text{RT}]$ then so is $t = [t_1, \ldots, t_n], \bullet$, where the bracket is to be thought as grafting the roots of $t_1, \ldots, t_n$ to a new root, which is symbolized by the subscript $\bullet$ in the bracket $[\ldots, \ldots]$.

**Example 22.** For instance, the formal expressions

\begin{align*}
\bullet, & \quad [\bullet], \quad [[\bullet]], \quad [\bullet, \bullet], \\
& \quad [\bullet, [\bullet]], \quad [\bullet, [\bullet]], \quad [\bullet, [\bullet]], \quad [\bullet, [\bullet]], \quad [\bullet, [\bullet]],
\end{align*}

correspond, from left to right, to the topological trees depicted in Figure 4.3. Also observe that the total number of “$\bullet$” in these formal expressions corresponds to the degree of the rooted tree (i.e. the total number of vertices).

**Remark 23.** Since we are dealing with topological rooted trees, the ordering in $[t_1, \ldots, t_m]$ is not important (for instance, we do not distinguish between $[\bullet, [\bullet]], [\bullet, [\bullet]]$ and $[[\bullet]], [\bullet, [\bullet]]$).

**Definition 24.** Let $X = X^i \partial_i$ be a vector field on $\mathbb{R}^d$. We define the elementary differential of $X$ recursively as follows: For the single vertex tree, we define $D^u_\bullet X = X^u(x)$, and for $t = [t_1, \ldots, t_m] \in [\text{RT}]$, we define

\begin{equation}
(4.3) \quad D^u_t X = \partial_{i_1} \cdots \partial_{i_m} X^{u_i}(x) D^{i_1}_{t_1} X \cdots D^{i_m}_{t_m} X,
\end{equation}

where we used the Einstein summation convention. For a (non-topological) rooted tree $t \in \text{RT}$, we define $D^u_t X := D^u_{\overline{t}} X$, where $\overline{t}$ is the topological tree underlying $t$.

4.2. **Rooted tree expansion for the Kontsevich realization.** We now want to find for $s_K$ a formula generalizing Formula (4.2) as a sum over topological rooted trees. For this, we start by characterizing the Kontsevich trees whose contributions in the expression of $s_K$ is not zero.

**Definition 25.** We define $\tilde{T}_{n,2}$ to be the subset of $T_{n,2}$ such that

\begin{equation}
(4.4) \quad \frac{\partial \hat{B}_{i}(\pi)}{\partial p_2}(p, 0, x) \neq 0.
\end{equation}
Proposition 26. $\Gamma \in \tilde{T}_{n,2}$ if and only if the following two properties hold:

1. The interior $\Gamma_i$ of $\Gamma$ is a rooted tree
2. There is only one edge in $\Gamma$ landing on $\bar{2}$ (and it stems from the root)

Proof. If $\Gamma \in T_{n,2}$ satisfies (1) and (2), it is clear that it also satisfies (4.4): Namely, there will then be only one power of $p_2$ in $\hat{B}_T(\pi)$, since there is only one edge landing on $\bar{2}$. The converse is more subtle.

Suppose we have $\Gamma \in T_{n,2}$ such that (4.4) holds. Then (2) is obvious, otherwise there would be more that one power of $p_2$ occurring in $\hat{B}_T(\pi)$, and the differentiation by $\partial / \partial p^2$ would not kill them all. Setting $p_2 = 0$ would then yields zero.

The hard part is to prove (1), i.e., that the interior of $\Gamma$ is a rooted tree. In other words, we need to show that $\Gamma_i$ has only one root and that its edges are all oriented toward this root. Let us first find the root.

Since $\Gamma$ has only one edge landing on $\bar{2}$, there is a distinguished vertex; namely, the one this edge stems from: let us call this vertex $r$ and consider it as our candidate for the root.

Claim: The second edge of $r$ must land on $\bar{1}$.

To see this, suppose it is not the case, i.e. that the second edge of $r$ is aerial. The total number of aerial edges must be $n - 1$, since $\Gamma_i$ is a tree with $n$ vertices. The total number of edges stemming out from $V^a_{\Gamma_i} \{ r \}$ is $2(n - 1)$. Among those, $n - 2$ must be aerial. So $V^a_{\Gamma_i} \{ r \}$ must have $2(n - 1) - (n - 2) = n$ terrestrial edges landing on $\bar{1}$. This means that at least one vertex in $V^a_{\Gamma_i} \{ r \}$ has two arrows landing on $\bar{1}$, in which case $\hat{B}_T(\pi) = 0$, because of the antisymmetry of the Poisson structure. This contradicts the fact that $\Gamma \in \tilde{T}_{n,2}$.

Claim: All the aerial edges of $\Gamma$ toward the root.

Since the two edges of $r$ are terrestrial, $r$ has only incoming edges. Let us define $\text{star}(r)$ as being the set of aerial vertices (including $r$ itself) that are connected to $r$ by an oriented path pointing toward $r$. If $\text{star}(r) = V^a_{\Gamma_i}$, we are done, and $\Gamma_i$ is a rooted tree. Suppose it is not the case: This means that there is a vertex $k \in \text{star}(r)$ with an edge pointing out from $\text{star}(r)$ into a vertex $k_1 \in V^a_{\Gamma_i} \setminus \text{star}(r)$. Now the two edges of $k_1$ can not be both terrestrial (otherwise, by (2), they should both land on $\bar{1}$, which implies that $\hat{B}_T(\pi) = 0$). So, one of these edges should point into a vertex $k_2 \in V^a_{\Gamma_i} \setminus \text{star}(r)$ ($k_1$ can not point back into $\text{star}(r)$, otherwise we have a cycle). The situation is illustrated in the following figure:

**Figure 4.4.**
We repeat the same argument for $k_2$, and we see that we obtain a sequence
\[ k_1 \rightarrow k_2 \rightarrow k_3 \rightarrow \cdots \]
of vertices in $V_1^a \setminus \text{star}(r)$. This process must terminate because (1) $V_1^a$ is finite, (2) the vertices in the sequence must be different from each other and can not point back into \text{star}(r) (otherwise we create a cycle).

Consider the last vertex $k_m$ of this sequence. Then $k_m$ can not have any aerial edge (otherwise we introduce a cycle), and thus $k_m$ must have two terrestrial edges landing on $\bar{1}$. But then, this implies that $\hat{B}_{\Gamma}(\pi) = 0$ by antisymmetry of the Poisson structure. \hfill $\square$

**Definition 27.** Let $\Gamma, \Gamma' \in \tilde{T}_{n,2}$. We define the following equivalence relation: $\Gamma \simeq \Gamma'$ if their interior are isomorphic rooted trees (which we denote by $\Gamma_i \sim \Gamma'_i$).

**Proposition 28.** Let $\Gamma, \Gamma' \in \tilde{T}_{n,2}$ such that $\Gamma \simeq \Gamma'$, then $W_\Gamma B_\Gamma = W_{\Gamma'} B_{\Gamma'}$.

**Proof.** The fact that both $\Gamma$ and $\Gamma'$ have the same underlying topological rooted tree as interior means that they differ by (1) a relabeling of their aerial vertices, (2) by a flip of their edge order (since all of the edges land on $\bar{1}$ except for the second edge of the root). As explained in [10], the first operation leaves both the weights and the bidifferential operators unchanged. The second operation introduces the same sign $(-1)^{\epsilon}$ in both the weight and the operator; since we are taking products of the two, these signs cancel out. \hfill $\square$

**Definition 29.** Let $t \in RT$ be a rooted tree. We define $\Gamma_t \in \tilde{T}_{n,2}$ to be the Kontsevich tree whose interior is $t$ and such that all of its terrestrial edges land on $\bar{1}$ except for the root of $t$, which has its first edge landing on $\bar{1}$ and its second landing on $\bar{2}$. All the edges landing on $\bar{1}$ are considered to be the first ones.

![Figure 4.5](image_url) On the left, a rooted tree $t \in RT$ and, on the right, the associated Kontsevich tree $\Gamma_t \in \tilde{T}_{n,2}$. The labeling of the vertices has been omitted.

**Proposition 30.** If two rooted trees $t_1$ and $t_2$ are isomorphic, then $W_{\Gamma_t_1} = W_{\Gamma_t_2}$ and $B_{\Gamma_t_1} = B_{\Gamma_t_2}$.

**Proof.** Kontsevich trees having isomorphic tree as interior differ only by a relabeling of their aerial vertices, which is an operation that leaves both the corresponding weights and bidifferential operators unchanged. \hfill $\square$

The proposition above allows us to define Kontsevich weights for topological rooted tree (i.e classes of isomorphic rooted trees): For $\bar{t} \in [RT]$, we define
\[ W_{\bar{t}} := W_{\Gamma_{\bar{t}}}, \quad t \in \bar{t}. \]
Proposition 31. Let $\Gamma \in \tilde{T}_{n,2}$, then
\[ W_\Gamma \frac{\partial B_\Gamma (\pi)}{\partial p_i^2} (p, 0, x) = W_\Gamma D_{\Gamma_i}^t L_V, \]
where $L_V$ is the vector field associated with the flat Poisson spray $V$ and $\Gamma_i \in [RT]$ is the topological rooted tree associated with $\Gamma_i \in RT$.

Proof. We can prove this by induction using the graphical representation for the Kontsevich graphs. Graphically, $\frac{\partial B_\Gamma (\pi)}{\partial p_i^2} (p, 0, x)$ can be represented as a graph in the following way: Start with $\Gamma$, detach its single edge landing on $\bar{2}$ from its target, let it point to nothing as illustrated in Figure 4.6 below (we will say that this edge points “in the air”), and label it with $l$; this procedure implements the differentiation by $\frac{\partial}{\partial p_i^2}$. Then remove $\bar{2}$, which represents graphically setting $p_2$ to zero. Let us denote by $A$ the type of graphs obtained this way.

![Figure 4.6](image)

Figure 4.6. On the right, a Kontsevich tree in $\tilde{T}_{n,2}$ and, on the left, the corresponding graph in $A$ after “differentiation” by $\frac{\partial}{\partial p_i^2}$.

What we need to show now is that $D_{\bar{t}}^i L_V$ with $\bar{t} \in [RT]$ can be also described as a graph in $A$ (up to a relabeling of the aerial vertices and a number of flips of the edge order). More precisely, we need to show that, in terms of the graphical representation, $D_{\bar{t}}^i L_V$ is obtained from the Kontsevich tree $\Gamma_t$ with $t \in \bar{t}$ (up to a relabeling of the aerial vertices) by removing $2$ and letting the single edge landing on $2$ pointing in the air with label $l$.

We do this by induction on the number of aerial vertices.

Suppose that $|\bar{t}| = 1$. Then $\Gamma_{\bullet}$ is the Kontsevich tree with one aerial vertex with its first edge landing on $\bar{1}$ and its second edge landing on $\bar{2}$. Then, $\frac{\partial B_{\Gamma_{\bullet}} (\pi)}{\partial p_i^2} (p, 0, x)$ is now the graph with one aerial vertex, a first edge landing on $\bar{1}$ and the second edge pointing in the air with label $l$. The term represented by this graph is $\pi u l p v u \pi u L_V \cdot D_{\bar{t}_1}^1 L_V \cdot \cdots \cdot D_{\bar{t}_m}^m L_V$.

Now suppose that for $\bar{t} \in [RT]$ with $|\bar{t}| \leq n$, we have that the graph representing $D_{\bar{t}}^i L_V$ is in $A$. Take now $\bar{t}$ with $|\bar{t}| = n + 1$. Recursively, we have that $\bar{t} = \bar{t}_1, \ldots, \bar{t}_m$ with $|\bar{t}_k| < n + 1$ for $k = 1, \ldots, m$ and
\begin{equation}
D_{\bar{t}}^i L_V = \partial u_k \ldots \partial u_m \pi u m p u D_{\bar{t}_1}^1 L_V \cdots D_{\bar{t}_m}^m L_V.
\end{equation}

Since, by induction, we have that the $D_{\bar{t}_k}^i L_V$'s are represented (up to relabeling of the aerial vertices) by the graph $\Gamma_{t_k}$ with $t_k \in \bar{t}_k$ to which we have removed $2$ and let the corresponding terrestrial edge point in the air with label $u_k$. Now the operation (4.5) can be represented graphically as grafting all the edges pointing in the air in the graphical representation of the $D_{\bar{t}_k}^u L_V$'s to a new root vertex (corresponding to $\partial u_1 \ldots \partial u_m \pi u m p u$), which itself has its
first edge landing on 1 and its second edge landing on 2. The following figure illustrates the process:

Figure 4.7.

What we obtain is a representation of $D^t L_V$ in terms of a graph in $\mathcal{A}$; namely, the graph obtained from $\Gamma_t$ by removing 2 and letting the corresponding edge point in the air with the label $l$.

**Theorem 32.** We have the following expression for $s_K$ in terms of rooted trees:

$$s^i_K(p, x) = x + \sum_{t \in [RT]} e^{|t|} \left( \frac{W_t}{|\text{sym}(t)|} \right) D^i t L_V.$$  

**Proof.** This is a direct computation: From (3.4) and Proposition 26, we have

$$s_K(p, x) = x + \sum_{n=1}^{\infty} \frac{e^n}{n!} \sum_{\Gamma \in \tilde{T}_{n,2}} \frac{W_{\Gamma}}{2^n} \frac{\partial \hat{B}_{\Gamma}(\pi)}{\partial p_2} (p, 0, x),$$

where we used Proposition 30 and Proposition 31 for the second equality. We conclude the proof by observing that the number of Kontsevich tree in $\tilde{T}_{n,2}$ whose interior is in the class of $t$ is

$$\frac{2^n n!}{|\text{sym}(t)|}.$$  

Namely, two trees $\Gamma, \Gamma'$ having isomorphic interior differ by a relabeling of their aerial vertices and a flip of the edge order. There are $2^n$ such flips (where $n$ is the number of aerial vertices) and $n!/|\text{sym}(t)|$ different relabelings of the aerial vertices.

**Remark 33.** Theorem 32 gives a very explicit description of the formal symplectic groupoid described in [2], which integrates the Poisson manifold $(\mathbb{R}^d, \epsilon \pi)$. Recall that the graph of its product is given by $\text{Im } dS^*_\pi$, its inverse by $i(x, p) = (x, -p)$, and the unit map by $\epsilon(x) = (x, 0)$.  

Now we also have an explicit formula for the source map, given by Theorem 32, and for the target map

\[ t^i_K(p, x) = x + \sum_{t \in [RT]} e^{\|t\|} \left( (-1)^{|\|t\||} \frac{W_t}{|\text{sym}(t)|} \right) D^i_t L_V, \]

since \( t(x, p) = s \circ i(x, p) \) and \( D^i_t L_V \) is homogeneous in \( p \) of degree \( |t| \).

4.3. **Recursive formula for the weights.** Let \( t = [t_1, \ldots, t_m] \) be a topological rooted tree. We define recursively functions of \( \theta \in [0, 1) \) as follows

\[ I_t(\theta) = \hat{1}_{0} d\bar{\lambda} \int_0^{\lambda} d\bar{\theta} I_{t_1}(\bar{\theta}) \cdots I_{t_m}(\bar{\theta}), \]

with \( I_\bullet(\theta) = \frac{1}{2} - \theta \).

**Proposition 34.** Let \( t \in [RT] \) and \( W_t \) be the corresponding Kontsevich weight. Then

\[ W_t = I_t(0). \]

**Proof.** We proceed by induction on the tree degree, following the strategy outlined by K妥oria in [9]. For one vertex, we have \( I_\bullet(0) = \frac{1}{2} \), which is the Kontsevich weight of \( \Gamma_\bullet \). Suppose this is true for topological rooted trees with \( n \) vertices. Let \( t = [t_1, \ldots, t_m] \) such that \( |t_i| < n \) for \( i = 1, \ldots, m \). Consider the graph \( \Gamma_t(\theta) \) made out of \( \Gamma_t \) by letting the root second edge point in the air toward an additional fixed vertex \( z \) at distance 1 and angle \( \theta \) from \( \bar{1} \) as in the following figure:

![Figure 4.8](image)

Let \( \omega_{\Gamma_t(\theta)} \) be the angle-form defined by the angle subtended by the aerial vertices of \( \Gamma_t(\theta) \) minus the additional vertex \( z \). We define

\[ W_{\Gamma_t}(\theta) = \int_{M_{|t|}} \omega_{\Gamma_t(\theta)}, \]

where the integration is over the configuration space of the \( |t| \) aerial vertices of \( \Gamma_t \). Clearly, \( W_t = W_{\Gamma_t}(0) \). Denote by \( \bar{\theta} \) and \( \bar{\lambda} \) the angle subtended by the root \( r \) of \( t \) in \( \Gamma_t(\theta) \) at \( \bar{1} \) and \( z \) as in Figure 4.8. By Fubini, we have that

\[ W_{\Gamma_t}(\theta) = \int_0^1 d\bar{\lambda} \int_0^{\bar{\lambda}} d\bar{\theta} \int_{M_{|t|-1}} \omega_{\Gamma_t(\theta)-z}, \]
where $\omega_{\Gamma_\theta} - z$ is the angle-form sustained by the aerial vertices of $\Gamma_\theta(\theta)$ minus $z$. Because there is no aerial arrow between the aerial vertices of $t_i$ and $t_j$ if $i \neq j$, we have that

$$(\omega_{\Gamma_\theta(\theta)-z})|_{r=(\bar{\theta}, \bar{\lambda})} = \omega_{\Gamma_1(\bar{\theta})} \wedge \cdots \wedge \omega_{\Gamma_m(\bar{\theta})},$$

and by Fubini, we obtain

$$W_{\Gamma_1}(\theta) = \int_0^1 d\bar{\lambda} \int_0^{\bar{\lambda}} d\bar{\theta} \left( \int_{\Gamma_1} \omega_{\Gamma_1(\bar{\theta})} \right) \cdots \left( \int_{\Gamma_m} \omega_{\Gamma_m(\bar{\theta})} \right),$$

$$= \int_0^1 d\bar{\lambda} \int_0^{\bar{\lambda}} d\bar{\theta} W_{\Gamma_1}(\bar{\theta}) \cdots W_{\Gamma_m}(\bar{\theta}).$$

In [9], Kathotia computed that $W_{\Gamma_1}(\theta) = \frac{1}{2} - \theta$, which allows us to conclude that $I_\ell(\theta) = W_{\Gamma_1}(\theta)$ and $I_\ell(0) = W_{\Gamma_1}$. \qed

The iterated integrals for the weights are very easy to compute; for the first rooted trees, we have

$$I_\bullet(\theta) = -\theta + \frac{1}{2},$$

$$I_{\bullet\bullet}(\theta) = \frac{\theta^2}{2} - \frac{\theta}{4} + \frac{1}{12},$$

$$I_{\bullet\bullet\bullet}(\theta) = -\frac{\theta^3}{3} + \frac{\theta^2}{4} - \frac{\theta}{24},$$

$$I_{\bullet\bullet\bullet\bullet}(\theta) = -\frac{\theta^3}{6} + \frac{\theta^2}{4} - \frac{\theta}{12},$$

which give the first Kontsevich weights (with Kathotia normalization factors):

$$W_\bullet = \frac{1}{2}, \quad W_{\bullet\bullet} = \frac{1}{12}, \quad W_{\bullet\bullet\bullet} = \frac{1}{24}, \quad W_{\bullet\bullet\bullet\bullet} = 0,$$

which, in turn, give the first terms of the formal symplectic realization:

$$s^i_j(x, p) = x^i + \frac{\epsilon}{2} \pi^{ij} p^j + \frac{\epsilon^2}{12} \partial_u \pi^{ij} \pi^{ku} p^k p^u + \frac{\epsilon^3}{48} \partial_u \partial_w \pi^{ij} \pi^{ku} \pi^{lw} p^k p^l p^w + \mathcal{O}(\epsilon^4).$$

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