**Leggett’s Modes in Magnetic Systems with Jahn-Teller distortion**

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Leggett’s mode is a collective excitation corresponding to the oscillation of the relative phase of the order parameters in a two band superconductor, with frequency proportional to interband coupling. We report on the existence of modes, similar to Leggett’s mode, in magnetic systems with Jahn-Teller distortion. The minimal Kugel-Khomskii model, which describes simultaneously both the spin and the orbital order, is studied. The dynamical degrees of freedom are spin-$s$ operators of localized spins and pseudospin-$\tau$ operators, which respond to the orbital degeneracy and satisfy the similar commutation relation with those of the spin operators. In the case of "G-type antiferro" spin and pseudospin order the system possesses two antiferromagnetic magnets with equal spin-wave velocities and two Leggett’s modes with equal gaps proportional to the square root of the spin-pseudospin interaction constant. In the case of "ferro" spin and pseudospin order the system possesses one ferromagnetic magnon and one Leggett’s mode with gap proportional to the spin-pseudospin interaction constant. We conclude that Leggett’s modes, in the spectrum of the magnetic systems with Jahn-Teller distortion, are generic feature of these systems.

**Introduction**-The spontaneous breaking of a continuous symmetry is accompanied with long range excitation known as Goldstone mode. In systems with two or more order parameters the Goldstone boson is supplemented by excitation which, in some sense, is orthogonal to it and has a mass proportional to the constant of interaction between different order parameters.

In the theory of superconductivity the phase of the order parameter is a massless excitation known as Anderson-Bogoliubov-Goldstone (ABG) mode \[1,2\]. In two band superconductor the (ABG) mode is a combination of the phases of the order parameters. It is complemented by a mode associated with the relative phases oscillation with frequency proportional to interband coupling \[3\]. The Leggett’s mode was observed in $MgB_2$ superconductor with Raman spectroscopy \[4\]. A novel peak in the one of the scattering channels is observed. The authors assign this feature to the Leggett’s mode. The measured mass is in accordance with theoretically predicted one \[5\]. In superconductors with three and more bands there are multiple Leggett’s modes classified by multiple interband couplings \[6\].

An analogous Leggett’s mode is theoretically predicted in superconductor with mixed-symmetry order parameter generated in an external magnetic field \[7\]. The oscillations of the relative phase (Leggett’s mode) are with frequency proportional to the magnetic field.

An important class of magnetic materials are compounds in which the state of magnetic ions is characterized by orbital as well as spin degeneracy. According to the Jahn-Teller theorem \[8\], an atom configuration in which orbital degeneracy is realized is unstable. The symmetry is lowered and the degeneracy is lifted, corresponding to ordering of the orbitals.

In a mathematical description of a two orbital system it is convenient to introduce pseudospin $\tau = 1/2$ associated with the two bands, in a way that the one of the band corresponds to the value $\tau^z = 1/2$, while the other one corresponds to the value $\tau^z = -1/2$. To model the electron-phonon coupling it is convenient to introduce three-component pseudospin operators $T_\alpha$ which satisfy the similar commutation relation with those of the spin operator, i.e., $[T_\alpha, T_\beta] = i\varepsilon_{\alpha\beta\gamma}T_\gamma$. Eliminating the phonons from the theory one obtains an effective theory with Hamiltonian which can be written in a form of Heisenberg (or Ising) Hamiltonian for pseudospins. There is an interaction between the spin and pseudospin of the ions. Collecting all terms including spin exchange ones we obtain the effective Kugel-Khomskii model \[9,10\].

The dynamical degrees of freedom are spin-$s$ operators of localized spins and pseudospin-$\tau$ operators, which respond to the orbital degeneracy and satisfy the similar commutation relation with those of the spin operators. In the case of "G-type antiferro" spin and pseudospin order the system possesses two antiferromagnetic magnons with equal spin-wave velocities and two Leggett’s modes with equal gaps proportional to the square root of the spin-pseudospin interaction constant. In the case of "ferro" spin and pseudospin order the system possesses one ferromagnetic magnon and one Leggett’s mode with gap proportional to the spin-pseudospin interaction constant.

**Kugel-Khomskii model**- The Hamiltonian of the minimal model is

$$
\hat{h} = J^s \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J^p \sum_{\langle ij \rangle} \mathbf{T}_i \cdot \mathbf{T}_j - J \sum_{\langle ij \rangle} (\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{T}_i \cdot \mathbf{T}_j),
$$

(1)

where $\mathbf{S}_i$ is spin-$s$ operator, $\mathbf{T}_i$ is pseudospin-$\tau$ operator, $J^s$ is spin exchange constant, $J^p$ is pseudospin exchange constant and $J$ is spin-pseudospin interaction constant. The sums are over all sites of a three-dimensional cubic lattice, and $\langle i, j \rangle$ denotes the sum over the nearest neighbors.
We consider a system with "antiferro" spin and pseudospin order. All constants in the Hamiltonian \( \text{Eq.}(1) \) are positive \((J^s > 0, J^p > 0, J > 0)\). To proceed we treat the spin-pseudospin interaction in the Hartree-Fock approximation. To this end one represents the term of interaction in the form

\[
(S_i \cdot S_j)(T_i \cdot T_j)_{HF} = -\langle S_i^a T_i^b \rangle \langle S_j^a T_j^b \rangle + \langle S_i^a T_i^b \rangle \langle S_j^{a} \rangle \langle S_j^b \rangle + \langle S_j^a T_j^b \rangle \langle S_i^{a} \rangle \langle S_i^b \rangle
\]

(2)

with \(\langle S_i^a T_i^b \rangle = \frac{\gamma_{a}^{b}}{\tau s v} \), where \(v\) is the Hartree-Fock parameter, to be determined self-consistently. The Hamiltonian \(\text{Eq.}(1)\) in the Hartree-Fock approximation reads

\[
h_{HF} = J^s \sum_{\langle ij \rangle} S_i \cdot S_j + J^p \sum_{\langle ij \rangle} T_i \cdot T_j - 2s v J \sum_{i} S_i \cdot T_i + J s^2 J^2 \tau^2 N
\]

(3)

where \(N\) is the number of the lattice’s sites. Equation \(\text{Eq.}(3)\) shows that the Hartree-Fock parameter renormalizes the spin-pseudospin interaction constant \(J_c = 2s v J\).

To study a theory with Hamiltonian \(\text{Eq.}(3)\), it is convenient to introduce Holstein-Primakoff representation for the spin \(S_j(a^+, a)\) and pseudospin \(T_j(b^+, b)\) operators

\[
S_j^+ = S_j^1 + iS_j^2 = \cos^2 \frac{\theta_j}{2} \sqrt{2s - a_j^+ a_j} a_j - \sin^2 \frac{\theta_j}{2} \sqrt{2s - a_j^+ a_j} a_j,
S_j^- = S_j^1 - iS_j^2 = \cos^2 \frac{\theta_j}{2} a_j^+ \sqrt{2s - a_j^+ a_j} - \sin^2 \frac{\theta_j}{2} \sqrt{2s - a_j^+ a_j} a_j,
S_j^3 = \cos \theta_j (s - a_j^+ a_j),
\]

(4)

where \(\theta_j = \mathbf{Q} \cdot \mathbf{r}_j\) and \(\mathbf{Q} = (\pi, \pi, \pi)\) is the antiferromagnetic wave vector. The representation for the pseudospin operators \(T_j(b^+, b)\) is obtained from \(\text{Eq.}(1)\), replacing Bose operators \((a^+, a)\) with Bose operators \((b^+, b)\) and spin-\(s\) with pseudospin-\(\tau\). In terms of the Bose fields and keeping only the quadratic terms, the effective Hamiltonian \(\text{Eq.}(3)\) adopts the form

\[
h_{HF} = N s^2 \tau^2 J(v - 1)^2 - N s^2 \tau^2 J + h_q
\]

(5)

\[
h_q = s J^s \sum_{\langle ij \rangle} (a_i^+ a_i a_j + a_j^+ a_j - a_i^+ a_j^+ - a_j^+ a_i^+) + \tau J^p \sum_{\langle ij \rangle} (b_i^+ b_i b_j + b_j^+ b_j - b_i^+ b_j^+ - b_j^+ b_i^+)
\]

\[
+ J_s \sum_{\langle ij \rangle} [\sqrt{s} \tau (a_i^+ b_i^+ + b_j^+ a_j^+) - \tau a_i^+ a_i - sb_i^+ b_i].
\]

To proceed one rewrites the Hamiltonian \(h_q\) in the momentum space representation:

\[
h_q = \sum_{k \in B} \left[ \varepsilon^a k^a k + \bar{\varepsilon} k^+ b_k b_k - \gamma (a^+_k b_k + b^+_k a_k) - \gamma_k^a (a^+_k a^+ - k - a_k a_k) - \gamma_b (b^+_k b^+_k + b_k b_k) \right],
\]

(6)

where the wave vector \(k\) runs over the first Brillouin zone and the dispersions are given by equalities

\[
\varepsilon^a = 6s J^s + \tau J_s, \quad \varepsilon^b = 6s J^p + s J_s, \quad \gamma = J_s \sqrt{s \tau},
\]

\[
\gamma_k^a = s J^s (\cos k_x + \cos k_y + \cos k_z), \quad \gamma_k^b = \tau J^p (\cos k_x + \cos k_y + \cos k_z).
\]

(7)

To diagonalize the Hamiltonian we introduce new Bose fields \(\alpha_k, \beta_k^+, \beta_k^-, \beta_k^0\) by means of the Bogoliubov transformation. The technique for diagonalization developed in \(\text{Eq.}(1)\) is used. The details are given in the supplementary materials \(\text{Eq.}(1)\). The transformed Hamiltonian has the form

\[
h_q = \sum_{k \in B} \left( E_k^a \alpha^+_k \alpha_k + E_k^b \beta^+_k \beta_k + E_k^0 \right),
\]

(8)

with dispersions

\[
E_k^a = \sqrt{\frac{1}{2} (A_k + B_k - \sqrt{(A_k - B_k)^2 + 4D_k})},
E_k^b = \sqrt{\frac{1}{2} (A_k + B_k + \sqrt{(A_k - B_k)^2 + 4D_k})},
\]

(9)

where

\[
A_k = (\varepsilon^a)^2 + \gamma^2 - 4(\gamma_k^a)^2,
B_k = (\varepsilon^b)^2 + \gamma^2 - 4(\gamma_k^b)^2,
D_k = \gamma^2 ((\varepsilon^a - \varepsilon^b)^2 - 4(\gamma_k^a - \gamma_k^b)^2).
\]

and \(E_k^0\) is the vacuum energy \(\text{Eq.}(1)\).

To determine self-consistently the Hartree-Fock parameter we calculate the free-energy of the system as a function of the parameter \(v\)

\[
F(v) = Js^2 \tau^2 (v - 1)^2 - Js^2 \tau^2 + \frac{1}{N} \sum_{k \in B} E_k^0 + \frac{T}{N} \sum_{k \in B} [\text{ln}(1 - \exp[-E_k^0/T]) + \text{ln}(1 - \exp[-E_k^0/T])].
\]

(11)

The physical value of the parameter is the value at which the free-energy has a minimum. The dimensionless free-energies \(F(v)/J\), as a function of the Hartree-Fock parameter \(v\) at zero temperature, are depicted in figure \(\text{(1)}\). For a spin \(s = 2\) and pseudospin \(\tau = 1/2\) system with parameters \(J^s/J = 1\) and \(J^p/J = 10\) (left scale-red line), one obtains \(v = 0.894\). For \(J^s/J = 0.1\) and \(J^p/J = 1\) (right scale-blue line) the Hartree-Fock parameter is \(v = 0.987\).

The equations \(\text{Eq.}(9)\) and \(\text{Eq.}(10)\) show that dispersions \(E_k^a\) and \(E_k^b\) depend on the wave vector \(k\) through the expression \(\varepsilon_k = \cos k_x + \cos k_y + \cos k_z\). It is convenient to draw these energies as functions of \(\varepsilon_k\). The figure \(\text{(2)}\) shows the functions \(E^a(\varepsilon_k)\) and \(E^b(\varepsilon_k)\) for a system with parameters \(s = 2, \tau = 1/2, J^s/J = 1, J^p/J = 10\) and \(v = 0.894\).
The gap is proportional to parameters $J^s/J = 1$ and $J^p/J = 10$ left scale (red line) and for $J^s/J = 0.1$ and $J^p/J = 1$ right scale (blue line). The Hartree-Fock parameter for the first one is $v = 0.894$, while for the second system it is $v = 0.987$.

The free energy of the system is

$$F(v) = J s^2 \tau^2 (v - 1)^2 - J s^2 \tau^2 J + h_q$$

$$h_q = s |J^s| \sum_{<ij>} (a_i^+ a_j + a_j^+ a_i) + \tau |J^p| \sum_{<ij>} (b_i^+ b_j + b_j^+ b_i) + J_r \sum_i \sqrt{\tau} (a_i^+ b_i + b_i^+ a_i) - \tau a_i^+ a_i - s b_i^+ b_i.$$ 

To illustrate the relationship between the geometry of the magnetic order and the nature of Leggett’s mode we consider a system with “ferro” spin and pseudospin order. The Hamiltonian of the system Eq. (10) has negative spin-exchange $J^s < 0$ and pseudospin-exchange $J^p < 0$ constants. We obtain, in the same way, the Hartree-Fock Hamiltonian Eq. (11) and use the Holstein-Primakoff representation Eq. (12) with ferromagnetic wave vector $Q = (0, 0, 0)$. In terms of the Bose fields and keeping only the quadratic terms, the effective Hamiltonian Eq. (13) adopts the form

$$h_{HF} = N s^2 \tau^2 J (v - 1)^2 - N s^2 \tau^2 J + h_q$$

$$h_q = s |J^s| \sum_{<ij>} (a_i^+ a_j + a_j^+ a_i) + \tau |J^p| \sum_{<ij>} (b_i^+ b_j + b_j^+ b_i) + J_r \sum_i \sqrt{\tau} (a_i^+ b_i + b_i^+ a_i) - \tau a_i^+ a_i - s b_i^+ b_i.$$ 

We rewrite the Hamiltonian $h_q$ in the momentum space representation:

$$h_q = \sum_{k \in B} \left[ \varepsilon_k^a a_k^+ a_k + \varepsilon_k^b b_k^+ b_k - \gamma (a_k^+ b_k + b_k^+ a_k) \right]$$

$$\varepsilon_k^a = 2 s |J^s| (3 - \cos k_x - \cos k_y - \cos k_z) + \tau J_r$$

$$\varepsilon_k^b = 2 \tau |J^p| (3 - \cos k_x - \cos k_y - \cos k_z) + s J_r$$

$$\gamma = \sqrt{s \tau} J_r$$

To diagonalize the Hamiltonian Eq. (14) we introduce new Bose fields $\alpha_k, \alpha_k^+, \beta_k, \beta_k^+$ by means of rotation. The transformed Hamiltonian has the form

$$h_q = \sum_{k \in B} \left( E_k^a \alpha_k^a \alpha_k + E_k^b \beta_k^b \beta_k \right)$$

with dispersions

$$E_k^a = \frac{1}{2} \left[ \varepsilon_k^a + \varepsilon_k^b - \sqrt{(\varepsilon_k^a - \varepsilon_k^b)^2 + 4 \gamma^2} \right]$$

$$E_k^b = \frac{1}{2} \left[ \varepsilon_k^a + \varepsilon_k^b + \sqrt{(\varepsilon_k^a - \varepsilon_k^b)^2 + 4 \gamma^2} \right]$$

The free energy of the system is

$$F(v) = J s^2 \tau^2 (v - 1)^2 - J s^2 \tau^2 + \frac{T}{N} \sum_{k \in B} \left[ \ln(1 - \exp[-E_k^a/T]) + \ln(1 - \exp[-E_k^b/T]) \right].$$

One obtains that at zero temperature the physical value of the Hartree-Fock parameter is $v = 1$ for all values of the parameters.

It follows from equations (14) and (16) that $E_0^a = 0$ and near the zero wave vector the $a$-boson has a ferromagnetic dispersion $E_k^a \propto \rho k^2$ with spin-stiffness.
$$\varrho = (s^2|J^p| + \tau^2|J^p|)/(s + \tau).$$ On the other hand, $\beta$-boson is gapped excitation with gap

$$\Delta_L = E_0^{\beta} = 2s\tau(s + \tau)J,$$

where $J$ is the spin-pseudospin interaction constant. This means that $\beta$-boson is the Leggett’s mode in the system.

Kugel-Khomskii model with Ising pseudospin anisotropy- The Hamiltonian of the system is

$$\hat{h} = h - \Delta J \sum_{(i,j)} T_i^z \cdot T_j^z$$

where $h$ is the Hamiltonian Eq.11, and $\Delta J > 0$ is the anisotropy parameter. While the $SU(2)$ pseudospin symmetry is broken, one can use the representations 4 for the spin $S_j(a^+, a)$ and pseudospin $T_j(b^+, b)$ operators, and following the same technique of calculation to obtain the spectrum of a system with negative (ferro) exchange constants $J^p < 0$ and $J^p < 0$ (see Eqs.13)

\[ \hat{E}_k^\alpha = \frac{1}{2} \left[ \varepsilon_k^\alpha + \varepsilon_k^\beta + \sqrt{(\varepsilon_k^\alpha - \varepsilon_k^\beta)^2 + 4\gamma^2} \right] \]

In Eqs.20 $\varepsilon_k^\alpha = \varepsilon_k^\beta + 6\tau \Delta J$ with $\varepsilon_k^\alpha$, $\varepsilon_k^\beta$ and $\gamma$ from Eqs.14. The energies $\hat{E}_k^\alpha$ and $\hat{E}_k^\beta$ have a minimum at $k = (0, 0, 0)$

\[ \hat{E}_0^\alpha = \frac{1}{2} \left[ \Delta_L^L + 6\tau \Delta J - \sqrt{(\Delta_L^L + 6\tau \Delta J)^2 - 24\tau^2 J^p \Delta J} \right] \]

\[ \hat{E}_0^\beta = \frac{1}{2} \left[ \Delta_J^L + 6\tau \Delta J + \sqrt{(\Delta_L^L + 6\tau \Delta J)^2 - 24\tau^2 J^p \Delta J} \right], \]

where $\Delta_L^L$ is the Leggett’s gap in an isotropic system Eq.12. Both dispersions have a gap but we can identify the Leggett’s mode as an excitation with the larger one $\hat{E}_0^\beta > \hat{E}_0^\alpha$. In the limit of small anisotropy the leading contribution of the anisotropy parameter $\Delta J$ to the dispersions is

\[ \hat{E}_0^\alpha \approx \frac{6\tau^2}{s + \tau} \Delta J \]

\[ \hat{E}_0^\beta \approx \Delta_L + \frac{6\tau s}{s + \tau} \Delta J, \]

Eqs.22 show that the gap of the $\alpha$ excitations is due to the pseudospin anisotropy, while the gap of the Leggett’s mode is a sum of the gap due to the anisotropy and Leggett’s gap.

Summary- In the present paper we have studied theoretically the existence of Leggett’s modes in magnetic systems with Jahn-Teller distortion. It is theoretically predicted that a system with “G-type-antiferro” spin and pseudospin order possesses two antiferromagnetic magnons with equal spin-wave velocities and two Leggett’s modes with equal gaps proportional to the square root of the spin-pseudospin interaction constant.

A prominent example of magnetic system with Jahn-Teller distortion is the LaMnO$_3$ compound with “perovskite” structure. The magnetic $Mn$ ion has an incomplete 3d-shell. The trivalent $Mn^{3+}$ ion has four electrons. It is surrounded by six oxygen $O^{2–}$ ions which form an octahedral structure. The crystal field of these ligands results in a particular splitting of the five d-orbitals into well separated in energy two groups: the $e_g$ and $t_{2g}$ states. The $t_{2g}$ sector forms a triplet, and the $e_g$ one forms a doublet. The triplet state is lower and three of the d-electrons occupy $t_{2g}$ bands, while the last one occupies $x^2 - y^2$ or $3z^2 - r^2$ band of $e_g$ doublet [14]. At ambient conditions LaMnO$_3$ is a paramagnetic insulator. Below $T_N = 140K$ the magnetic structure of the system is A-type antiferromagnetic [14]. The strong distortion of the $MnO_6$ octahedra is the signature of the cooperative Jahn-Teller effect and orbital ordering [15, 16]. At $T_JT = 750K$ LaMnO$_3$ undergoes a structural phase transition above which the orbital ordering disappears [17]. The C-type orbital structure in LaMnO$_3$ compound has been obtained experimentally by Y.Murakami et al. [18] and theoretically discussed in [19]. The C-type orbital ordering means that if $e_g$ electron on site “i” occupies $x^2 - y^2$ band the $e_g$ electron on nearest neighbor site in xy plane occupies $3z^2 - r^2$ one, while along the z-direction the same orbital state repeats.

The model under consideration, in the present paper, do not match perfectly the LaMnO$_3$ compound. But magnon fluctuations in A-type, C-type and G-type antiferromagnets are identical, two Goldstone bosons with linear dispersion (see Supplemental material B). This is while we expect that theoretically predicted Leggett’s modes are presented as well in the spectrum of the LaMnO$_3$ compounds.

There is an additional experimental evidence for this. Comparative Raman study of LaMnO$_3$ [20] and CaMnO$_3$ [21] shows that most intensive Raman line at 612 cm$^{-1}$ in the spectra of LaMnO$_3$ does not exist in the spectra of CaMnO$_3$. LaMnO$_3$ contains $Mn^{3+}$ ions with three $t_{2g}$ electrons and one $e_g$ electron which occupies $x^2 - y^2$ or $3z^2 - r^2$ band. This leads to Jahn-Teller effect in LaMnO$_3$. In CaMnO$_3$ the manganese is in $Mn^{4+}$ state with three $t_{2g}$ electrons and there is no Jahn-Teller effect. This pushes the authors to conclude that the most intensive Raman line at 612 cm$^{-1}$ is a consequence of the Jahn-Teller effect [22].

Under the pressure [23] the Raman peak at 612 cm$^{-1}$ shifts towards higher energy and loses intensity with increasing pressure. There is strong indication that the Jahn-Teller effect and the concomitant orbital order are completely suppressed above 18 GPa. The Raman signal from Jahn-Teller distorted octahedra is still observed at
The successful explanation of the extra Raman peak, in the two-band superconductor MgB$_2$, as a result of the Leggett’s mode in the compound, inspires to assign the extra peak in LaMnO$_3$ to the Leggett’s modes, theoretically predicted within the minimal Kugel-Khomskii model in the present paper.

There is a microscopical derivation of the effective Heisenberg model of A-type antiferromagnetism of LaMnO$_3$ compound [22], but there is not such results neither theoretically nor experimentally. The fluctuations. Too many Goldstone bosons are not acceptable neither theoretically nor experimentally. The pseudo-spin fluctuations. As a result we have uncoupled spin fluctuations and the Hamiltonian of the spin fluctuations and the Hamiltonian of the pseudo-spin fluctuations. As a result we have obtained from Eq.(8), the new dispersions $E^\alpha_k$ and $E^\beta_k$ with operators $\alpha_k$, $\beta_k$, $\alpha_k^\dagger$, $\beta_k^\dagger$, and imposes conditions the resulting Hamiltonian to be in a diagonal form Eq.(8).

To obtain the coefficients in the Bogoliubov transformation Eq.(24), the new dispersions $E^\alpha_k$, $E^\beta_k$ and the vacuum wave vector $q$. Following this work one derives the inverse transformation

$$\alpha_k = u_k^{11} a_k + u_k^{21} b_k - v_k^{11} a_k - v_k^{21} b_k$$
$$\beta_k = u_k^{12} a_k + u_k^{22} b_k - v_k^{12} a_k - v_k^{22} b_k$$

In Eqs.(25) we have used that Bogoliubov coefficients are real and even functions of the wave vector $q$.

Farther on we replace in the equations

$$[\alpha_k, h_q] = E^\alpha_k \alpha_k \quad [\beta_k, h_q] = E^\beta_k \beta_k,$$

obtained from Eq.(5). $\alpha$ and $\beta$ operators with $a$ and $b$ ones and use the equalities

$$[a_k, h_q] = \varepsilon^a a_k - 2\varepsilon_k^a a_k^\dagger - \gamma a_k$$
$$[b_k, h_q] = \varepsilon^b b_k - 2\varepsilon_k^b b_k^\dagger - \gamma a_k,$$

which follow from Eq.(26). Comparing the coefficients in the front of the $a$ and $b$ operators one obtains two systems of equations for the Bogoliubov coefficients:

$$E^\alpha_k - \varepsilon^a u_k^{11} + \gamma u_k^{21} + 2\gamma_k u_k^{11} = 0$$
$$E^\beta_k - \varepsilon^b u_k^{12} + \gamma u_k^{22} + 2\gamma_k u_k^{12} = 0$$

and

$$E^\alpha_k - \varepsilon^a u_k^{12} + \gamma u_k^{22} + 2\gamma_k u_k^{12} = 0$$
$$E^\beta_k - \varepsilon^b u_k^{12} + \gamma u_k^{22} + 2\gamma_k u_k^{12} = 0$$

and

$$E^\alpha_k = E^\beta_k = E_k^3 \quad [\alpha_k, h_q] = E^\alpha_k \alpha_k \quad [\beta_k, h_q] = E^\beta_k \beta_k,$$

obtained from Eq.(5). $\alpha$ and $\beta$ operators with $a$ and $b$ ones and use the equalities

$$[a_k, h_q] = \varepsilon^a a_k - 2\varepsilon_k^a a_k^\dagger - \gamma a_k$$
$$[b_k, h_q] = \varepsilon^b b_k - 2\varepsilon_k^b b_k^\dagger - \gamma a_k,$$

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and

$$E^\alpha_k - \varepsilon^a u_k^{12} + \gamma u_k^{22} + 2\gamma_k u_k^{12} = 0$$
$$E^\beta_k - \varepsilon^b u_k^{12} + \gamma u_k^{22} + 2\gamma_k u_k^{12} = 0$$

and

$$E^\alpha_k - \varepsilon^a u_k^{12} + \gamma u_k^{22} + 2\gamma_k u_k^{12} = 0$$
$$E^\beta_k - \varepsilon^b u_k^{12} + \gamma u_k^{22} + 2\gamma_k u_k^{12} = 0$$
We supplement the system of equations (28, 29) with two equations which are consequence of the Bose commutation relations \([\alpha_k, \alpha_k^+] = 1\) and \([\beta_k, \beta_k^+] = 1\):

\[(u_k^{11})^2 + (u_k^{21})^2 - (u_k^{11})^2 - (u_k^{21})^2 = 1\]  

(30)

\[(u_k^{12})^2 + (u_k^{22})^2 - (u_k^{12})^2 - (u_k^{22})^2 = 1\]  

(31)

Looking for the solution of the system of equations (28) one arrives at an equation for the dispersion \(E_k^0\), which is the same as the equation for the dispersion \(E_k^0\) obtained from the system (29)

\[E_k^4 - E_k^0[(e^a)^2 + (e^b)^2 - 2\gamma^2 + 4(\gamma_k^a)^2 + 4(\gamma_k^b)^2] + (e^a)e^b - 2\gamma e^a e^b - 4(e^a\gamma_k^a)^2 - 4(e^b\gamma_k^b)^2 + (\gamma^2 - 4\gamma_k^a\gamma_k^b) = 0\]  

(32)

The positive solutions of the equation (32) have the form

\[E_k^0 = \sqrt{\frac{1}{2}} \left( A_k + B_k \pm \sqrt{(A_k - B_k)^2 + 4D_k} \right),\]

(33)

with \(A_k, B_k\) and \(D_k\) given by equations (10). For definiteness one sets \(E_k^0 = E_k^a\) and \(E_k^0 = E_k^b\).

To present the Bogoliubov coefficients, which are the solutions of the systems of equations (28, 30) and (29, 31), we introduce the functions

\[M_k^1 = 2\gamma [\gamma_k^a(E_k^0 + e^a) - \gamma_k^b(E_k^0 + e^b)]\]

\[M_k^2 = 2\gamma^2 [\gamma_k^a(E_k^0 - e^a)(E_k^0 + e^a) - 8\gamma_k^3(\gamma_k^a)^2]\]

\[M_k^3 = \gamma(E_k^0 - e^a)(E_k^0 - e^b) - \gamma^3 + 4\gamma_k^a\gamma_k^b\]

\[M_k^4 = (E_k^0 - e^a)(E_k^0 - e^b) - \gamma^2(E_k^0 + e^a) + 4(\gamma_k^a)^2(E_k^0 - e^b)\]

(34)

and

\[R_k^1 = 2\gamma^2(\gamma_k^a - 2\gamma_k^a(E_k^0 - e^a)(E_k^0 + e^a) - 8\gamma_k^3(\gamma_k^a)^2)\]

\[R_k^2 = 2\gamma [\gamma_k^a(E_k^0 + e^b) - \gamma_k^b(E_k^0 - e^a)]\]

\[R_k^3 = (E_k^0 - e^a)(E_k^0 - e^b)(E_k^0 + e^a) - \gamma^2(E_k^0 + e^b) + 4(\gamma_k^b)^2(E_k^0 - e^a)\]

\[R_k^4 = \gamma(E_k^0 - e^a)(E_k^0 - e^b) - \gamma^3 + 4\gamma_k^a\gamma_k^b\]

(35)

Finally, one can represent the vacuum energy Eq.(8) in the form

\[E_k^0 = \frac{1}{2} \left[ E_k^0 + E_k^0 - e^a - e^b \right] + \gamma M_k^1 M_k^2 - \gamma M_k^3 M_k^4 + \frac{1}{4}(e^b - e^a)(M_k^1)^2 - (M_k^2)^2 - (M_k^3)^2 + (M_k^4)^2]\]

\[\frac{(M_k^1)^2 + (M_k^2)^2 - (M_k^3)^2 - (M_k^4)^2}{(M_k^1)^2 + (M_k^2)^2 - (M_k^3)^2 - (M_k^4)^2}\]

(36)

\[u_k^{11} = \frac{M_k^1}{\sqrt{(M_k^1)^2 + (M_k^2)^2 - (M_k^3)^2 - (M_k^4)^2}}\]

(37)

\[u_k^{12} = \frac{R_k^1}{\sqrt{(R_k^1)^2 + (R_k^2)^2 - (R_k^3)^2 - (R_k^4)^2}}\]

In terms of the above functions the expressions for the coefficients are simple:
Supplemental material B

For common discussion of the spin-wave excitations in A, C and G antiferromagnetic phases it is convenient to consider a theory with Hamiltonian

\[ h = \sum_{i\mu} J^\mu S_i \cdot S_{i+e_\mu}, \quad (39) \]

where \( e_\mu \) is the unit vector along \( \mu = x, y, z \) and \( J^\mu \) is the exchange constant which depends on the space directions \( (J^x, J^y, J^z) \). To obtain the ground state magnetic order we represent the spin operators as vectors \( S_i = s n_i \), where \( n_i \) is an unit vector in the form \( n_i = (\sin \theta_i, 0, \cos \theta_i) \). We consider a simplest dependence of the angle \( \theta_i \) on the lattice site \( \theta_i = r_i \cdot Q \), where \( Q = (Q_x, Q_y, Q_z) \). The ground state energy, obtained from the Hamiltonian Eq.(39) is

\[ h_{gr} = s^2 N \sum_{\mu} J^\mu \cos Q_\mu, \quad (40) \]

where \( N \) is the number of the lattice sites. The physical value of the wave vector \( Q \) is the value at which the ground state energy \( h_{gr} \) is minimal.

In the case when all three parameters are positive \((J^x > 0, J^y > 0, J^z > 0)\) the physical wave vector is \( Q = Q^G = (\pi, \pi, 0) \) and spin vectors, on nearest neighbor sites, are anti-aligned so that the net magnetization is zero. The state is said to be the G-type antiferromagnetic.

When \( J^x > 0, J^y > 0 \) and \( J^z < 0 \) the ground state energy is minimal at \( Q = Q^C = (\pi, \pi, 0) \). The spins are anti-aligned in \( x-y \) plane, and parallel along the \( z \) direction. This state is a C-type antiferromagnetic state.

Finally, when \( J^x < 0, J^y < 0 \) and \( J^z > 0 \) the physical wave vector is \( Q = Q^A = (0, 0, \pi) \). The spins are parallel in \( x-y \) plane, and antiparallel along the \( z \) direction. This phase is known as A-type antiferromagnetism.

In all three cases \( \sin \theta_i = 0 \). Therfore we can use the representation (41) for the spin operators. In terms of the base operators \( (a_i^+, a_i) \) the Hamiltonian (39) reads

\[ h = \sum_{i\mu} J^\mu \left[ \cos Q_\mu \left( s - a_i^+ a_i \right) \left( s - a_{i+e_\mu}^+ a_{i+e_\mu} \right) \right. \]

\[ + \frac{1}{2} \cos \frac{Q_\mu}{2} \left( f_i a_i a_{i+e_\mu}^+ f_{i+e_\mu} + a_i^+ f_i f_{i+e_\mu} a_{i+e_\mu} \right) \]

\[ - \frac{1}{2} \sin \frac{Q_\mu}{2} \left( f_i a_i a_{i+e_\mu} f_{i+e_\mu} + a_i^+ f_i f_{i+e_\mu} a_{i+e_\mu} \right) \left. \right] . \quad (41) \]

where \( f_i = \sqrt{2s} - a_i^+ a_i \).

In the spin-wave approximation \( f_i \approx \sqrt{2s} \) and we keep only quadratic terms of the Bose fields \( (a_i^+, a_i) \)

\[ h_{sw} = \sum_{i\mu} J^\mu \left[ -s \cos Q_\mu \left( a_i^+ a_i + a_{i+e_\mu}^+ a_{i+e_\mu} \right) \right. \]

\[ + s \cos \frac{Q_\mu}{2} \left( a_i a_{i+e_\mu}^+ + a_i^+ a_{i+e_\mu} \right) \]

\[ - s \sin \frac{Q_\mu}{2} \left( a_i a_{i+e_\mu} + a_i^+ a_{i+e_\mu}^+ \right) \left. \right] . \quad (42) \]

For G-type antiferromagnetic systems \( (Q = Q^G) \) the Hamiltonian is

\[ h_{sw}^G = s \sum_{i\mu} J^\mu \left[ a_i^+ a_i + a_{i+e_\mu}^+ a_{i+e_\mu} - a_i a_{i+e_\mu} - a_i^+ a_{i+e_\mu}^+ \right] . \quad (43) \]

For C-type antiferromagnetic systems \( (Q = Q^C) \) it is

\[ h_{sw}^C = \sum_{i} \left[ \sum_{\mu=x,y} s J^\mu \left( a_i^+ a_i + a_{i+e_\mu}^+ a_{i+e_\mu} \right) \right. \]

\[ + s J^z \left( a_i a_{i+e_\mu} + a_i^+ a_{i+e_\mu} - a_i a_{i+e_\mu} - a_i^+ a_{i+e_\mu} \right) \]

\[ - \sum_{\mu=x,y} s J^\mu \left( a_i a_{i+e_\mu} + a_i^+ a_{i+e_\mu} \right) \] . \quad (44) \]

Finally, the Hamiltonian of the A-type antiferromagnetic systems \( (Q = Q^A) \) is

\[ h_{sw}^A = \sum_{i} \left[ s J^z \left( a_i^+ a_i + a_{i+e_\mu}^+ a_{i+e_\mu} - a_i a_{i+e_\mu} - a_i^+ a_{i+e_\mu} \right) \right. \]

\[ + \sum_{\mu=x,y} s J^\mu \left( a_i a_{i+e_\mu} + a_i^+ a_{i+e_\mu} - a_i a_{i+e_\mu} - a_i^+ a_{i+e_\mu} \right) \left. \right] . \quad (45) \]

In momentum space representation the Hamiltonians Eqs. (43),(44) and (45) have the form

\[ h_{sw} = \sum_{k \in B} \left[ \varepsilon_k a_k^+ a_k - \gamma_k (a_k^+ a_{-k} + a_{-k} a_k) \right] , \quad (46) \]

where for G-type \((J^x > 0, J^y > 0, J^z > 0)\)

\[ \varepsilon_k^G = 2s \left( J^x + J^y + J^z \right) \]

\[ \gamma_k^G = s \left( J^x \cos k_x + J^y \cos k_y + J^z \cos k_z \right) \]

for C-type \((J^x > 0, J^y > 0, J^z < 0)\)

\[ \varepsilon_k^C = 2s \left( J^x + J^y \right) + 2s |J^z| \left( 1 - \cos k_z \right) \]

\[ \gamma_k^C = s \left( J^x \cos k_x + J^y \cos k_y \right) \]

and for A-type \((J^x < 0, J^y < 0, J^z > 0)\)

\[ \varepsilon_k^A = 2s \left( |J^x| - 1 - \cos k_x \right) + \left| J^y \right| \left( 1 - \cos k_y \right) + J^z \]

\[ \gamma_k^A = s J^z \cos k_z \]

Next, we diagonalize the Hamiltonian Eq.(46) by means of the Bogoliubov transformation. In terms of the Bogoliubov operators the Hamiltonian is

\[ h_{sw} = \sum_{k \in B} \left[ E_k \alpha_k^+ \alpha_k + E_k^0 \right] , \quad (50) \]

with energy of the Bogoliubov excitations

\[ E_k = \sqrt{\varepsilon_k^2 - 4\gamma_k^2} . \quad (51) \]

The energy of the G-type antiferromagnetic system \( E_k^G = \sqrt{(\varepsilon_k^G)^2 - 4(\gamma_k^G)^2} \) is zero at wave vectors \( k = \)}
(0, 0, 0) and \( k = Q^C \). Near these wave vectors the dispersion is linear

\[
E_{k \to 0}^C \simeq 2s\sqrt{J^x + J^y + J^z} \sqrt{\sum_{\mu} J^\mu k_{\mu}^2}
\]

(52)

\[
E_{k \to Q^C}^C \simeq 2s\sqrt{J^x + J^y + J^z} \sqrt{\sum_{\mu} J^\mu (k_{\mu} - Q_{\mu}^C)^2}.
\]

The energy \( E_k^C = (\varepsilon_k^C)^2 - 4(\gamma_k^C)^2 \) is zero at wave vectors \( k = (0, 0, 0) \) and \( k = Q^C \). Near these wave vectors the dispersion is linear

\[
E_{k \to 0}^C \simeq 2s\sqrt{J^x + J^y + J^z} \sqrt{\sum_{\mu} J^\mu k_{\mu}^2}
\]

(53)

\[
E_{k \to Q^C}^C \simeq 2s\sqrt{J^x + J^y + J^z} \sqrt{\sum_{\mu} J^\mu (k_{\mu} - Q_{\mu}^C)^2}.
\]

Finally, the energy \( E_k^A = \sqrt{(\varepsilon_k^A)^2 - 4(\gamma_k^A)^2} \) is zero at wave vectors \( k = (0, 0, 0) \) and \( k = Q^A \). Near these wave vectors the dispersion is linear

\[
E_{k \to 0}^A \simeq 2s\sqrt{J} \sqrt{\sum_{\mu} |J|^\mu k_{\mu}^2}
\]

(54)

\[
E_{k \to Q^A}^A \simeq 2s\sqrt{J} \sqrt{\sum_{\mu} |J|^\mu (k_{\mu} - Q_{\mu}^A)^2}.
\]

The dispersions (52), (53) and (54) allow to conclude that magnon fluctuations in A-type, C-type and G-type antiferromagnets are identical (two Goldstone bosons with linear dispersion). This is exactly what we claim in the Summary section of the present paper.

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