Abelian Closures of Infinite Binary Words

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Abstract

Two finite words \( u \) and \( v \) are called Abelian equivalent if each letter occurs equally many times in both \( u \) and \( v \). The abelian closure \( \mathcal{A}(x) \) of (the shift orbit closure of) an infinite word \( x \) is the set of infinite words \( y \) such that, for each factor \( u \) of \( y \), there exists a factor \( v \) of \( x \) which is abelian equivalent to \( u \). The notion of an abelian closure gives a characterization of Sturmian words: among binary uniformly recurrent words, Sturmian words are exactly those words for which \( \mathcal{A}(x) \) equals the shift orbit closure \( \Omega(x) \). In this paper we show that, contrary to larger alphabets, the abelian closure of a uniformly recurrent aperiodic binary word which is not Sturmian contains infinitely many minimal subshifts.

1. Introduction

The abelian equivalence relation has been an active topic of research in the recent decades. Two finite words \( u \) and \( v \) are called abelian equivalent if, for each letter \( a \) of the underlying alphabet \( \Sigma \), the words \( u \) and \( v \) contain equally many occurrences of \( a \). The notion has been studied in the relation of abelian complexity of infinite words \([2, 18, 27, 28]\), abelian repetitions and avoidance \([3, 4, 14, 23, 26]\), other topics \([9, 21, 22, 25]\); see also \([24]\) and references therein.

In this note we consider the so-called abelian closures of infinite binary words. This notion is a fairly recent one, and has thus far been considered only in the works \([10, 13, 24]\), where the terms “abelianization” and “abelian subshift” were used. The notion is motivated by a notion in discrete symbolic dynamics, namely, the shift orbit closure of a word. For an infinite word \( x \) we define the language \( \mathcal{L}(x) \) of \( x \) as the set of finite words occurring as factors in \( x \). The shift orbit closure of an infinite word \( x \) can be defined as the set \( \Omega(x) \) comprising those infinite words \( y \) for which \( \mathcal{L}(y) \subseteq \mathcal{L}(x) \). The shift orbit closure has a discrete symbolic dynamical definition as well: the set \( \Sigma^\mathbb{N} \) is a compact metric space under the product topology induced by the discrete topology on the finite alphabet \( \Sigma \). The set \( \Omega(x) \) then coincides with the closure of the orbit of \( x \) under the shift map \( \sigma \), which is defined by \( \sigma(a_0a_1a_2\cdots) = a_1a_2\cdots \). Now \( \Omega(x) \) is called a minimal subshift if it contains no proper shift orbit closures. The abelian closure of an infinite word can be seen as the “commutative” counterpart of its shift orbit closure. The abelian closure \( \mathcal{A}(x) \) of \( x \) is defined as the set of words \( y \) for which each factor is abelian equivalent to some factor of \( x \).

The abelian closures of infinite words can have diverse structures. Clearly, \( \Omega(x) \subseteq \mathcal{A}(x) \) for any word \( x \). For some words and families of words, for example, Sturmian words, the
equality holds: $\Omega(x) = A(x)$. Moreover, the property $\Omega(x) = A(x)$ characterizes Sturmian words among uniformly recurrent binary words [13]. On the other hand, it is easy to see that the abelian closure of the Thue–Morse word $\mathbf{TM}$, defined as the fixed point (starting with 0) of the morphism $0 \mapsto 01, 1 \mapsto 10$, is $\{0, 1\} \cdot \{01, 10\}^\mathbb{N}$ (see, e.g., [13] for a proof.) So, contrary to Sturmian words, the abelian closure of the Thue–Morse is huge compared to $\Omega(\mathbf{TM})$: essentially, it is a morphic image of the full binary shift. In general, the abelian closure of an infinite word might have a pretty complicated structure. T. Hejda, W. Steiner, and L.Q. Zamboni studied the abelian closure of the Tribonacci word $\mathbf{TR}$. They announced that $\Omega(\mathbf{TM})$ is a proper subset of $A(\mathbf{TM})$ but that $\Omega(\mathbf{TR})$ is the only minimal subshift contained in $A(\mathbf{TR})$ [10, 30].

In this paper we consider the abelian closures of binary words. Our main result states that for an aperiodic uniformly recurrent binary word, its abelian closure contains infinitely many minimal subshifts, unless it is Sturmian (Theorem 2.6). In many cases we are able to prove that the abelian closure actually contains uncountably many minimal subshifts. We remark that in the non-binary case, there exist words with finitely many (and more than one) minimal subshifts; for example, some balanced aperiodic words are like that (announced in [13], see also Example 2.5).

The paper is structured as follows. In Section 2 we give some background and state our main results. In Section 3 we give more technical preliminaries we use in the proofs. In particular, we discuss initial properties of abelian closures of binary words and give some background on Sturmian words. In Section 4 we prove Theorem 2.6 for the easy cases of words which do not have uniform letter frequencies or which have rational letter frequencies. These cases have been reported at the DLT 2018 conference [13], but we give full proofs here for the sake of completeness. We then prove the theorem for $C$-balanced words with irrational letter frequencies in Section 5. In Section 6 we develop some tools we use for the proof of the last and the hardest case of non-balanced words with irrational frequency, which we treat in Section 7. In Section 8 we give alternative proofs for some results. For example, we give a large family of words that have uncountably many minimal subshifts in their abelian closures. In Section 9 we conclude with some open problems.

2. Background and statement of main result

In this section we give some preliminaries on abelian closures and state our main results.

For a finite word $u \in \Sigma^*$, we let $|u|_a$ denote the number of occurrences of the letter $a \in \Sigma$ in $u$. A factor of a finite or an infinite word is any finite sequence of its consecutive letters. The Parikh vector $\Psi(u)$ of a finite word $u \in \Sigma^*$ is defined as $\Psi(u) = (|x|_a)_{a \in \Sigma}$. The words $u$ and $v$ are abelian equivalent, denoted by $u \sim v$, if their Parikh vectors coincide. We let $L(x)$ denote the language of factors of an infinite word $x$. We then call the set $L^{ab}(x) = \{\Psi(v) : v \in L(x)\}$ its abelian language, and an element of $L^{ab}(x)$ is referred to as an abelian factor of $x$. In symbols, the definition of the abelian closure of a word reads as follows.

**Definition 2.1.** The abelian closure of $x \in \Sigma^\mathbb{N}$ is defined as

$$A(x) = \{y \in \Sigma^\mathbb{N} : L^{ab}(y) \subseteq L^{ab}(x)\}.$$ 

In other words, for any factor $u$ of $y \in A(x)$ there is a factor $v$ of $x$ for which $u \sim v$. An infinite word $x$ is ultimately periodic if we may write $x = uv^\omega$, i.e., the prefix $u$ is followed by an infinite repetition of a non-empty word $v$. If $u$ is empty, then $x$ is called purely periodic.
The word \( x \) is called aperiodic if it is not ultimately periodic. An infinite word \( x \) is called recurrent if each factor of \( x \) occurs infinitely many times in \( x \). We say that a factor \( u \) occurs with bounded gaps if there exists \( N \in \mathbb{N} \) such that each factor of length \( N \) contains \( u \). An infinite word \( x \) is called uniformly recurrent if each factor occurs with bounded gaps.

The term “abelian subshift” used in [13] was motivated by the symbolic dynamical terminology, which we employ here as well. A subshift \( X \subseteq \Sigma^\mathbb{N} \), \( X \neq \emptyset \), is a closed set (with respect to the product topology of \( \Sigma^\mathbb{N} \)) satisfying \( \sigma(X) \subseteq X \),\(^1\) where \( \sigma \) is the shift operator defined in the introduction. For a subshift \( X \subseteq \Sigma^\mathbb{N} \) we let \( \mathcal{L}(x) = \bigcup_{y \in X} \mathcal{L}(y) \). A subshift \( X \subseteq \Sigma^\mathbb{N} \) is called minimal if \( X \) does not properly contain any subshifts. Observe that two minimal subshifts \( X \) and \( Y \) are either equal or disjoint. Let \( x \in \Sigma^N \). We let \( \Omega(x) \) denote the shift orbit closure of \( x \), which may be defined as the subshift \( \{ y \in \Sigma^\mathbb{N} : \mathcal{L}(y) \subseteq \mathcal{L}(x) \} \). Thus \( \mathcal{L}(\Omega(x)) = \mathcal{L}(x) \) for any word \( x \in \Sigma^\mathbb{N} \). It is known that \( \Omega(x) \) is minimal if and only if \( x \) is uniformly recurrent. For more on topic of subshifts we refer the reader to [15]. We remark that, for any \( x \in \Sigma^N \), the abelian closure \( \mathcal{A}(x) \) is readily seen to be a subshift.

Sturmian words can be defined in many equivalent ways; here we make of their characterization via balance.

**Definition 2.2.** An infinite word \( x \in \Sigma^\mathbb{N} \) is called \( C \)-balanced, where \( C \) is some positive integer, if for all \( v, v' \in \mathcal{L}(x) \) with \( |v| = |v'| \), we have \( |v_a - v'_a| \leq C \) for all \( a \in \Sigma \). If \( x \) is not \( C \)-balanced for any \( C \in \mathbb{N} \), then \( x \) is called non-balanced.

A 1-balanced word is simply called balanced. On the other hand, if \( x \) is not 1-balanced, then we call it unbalanced.\(^2\)

Periodic and aperiodic Sturmian words can then be defined as recurrent balanced binary words [20]. It follows that, for each periodic or aperiodic Sturmian word \( s \), its abelian language \( \mathcal{L}^{ab}(x) \) contains at most two elements of each length. We give more backgrounds on Sturmian words in Sections 3 and 6.3.

In [13] we showed that Sturmian words can be characterized in terms of abelian closures:

**Theorem 2.3 ([13]).** Let \( x \) be uniformly recurrent binary word. Then \( \mathcal{A}(x) \) contains exactly one minimal subshift if and only if \( x \) is periodic or aperiodic Sturmian.

In fact, for Sturmian words we have \( \mathcal{A}(x) = \Omega(x) \). We also investigated how the property containing exactly one minimal subshift extends to non-binary words, and we saw that there are many non-binary words with this property.

**Example 2.4.** Let \( s \) be a Sturmian word and let \( \varphi : 0 \mapsto 02, 1 \mapsto 12 \). Then \( \mathcal{A}(\varphi(s)) = \Omega(\varphi(s)) \) [13].

We also saw that in the non-binary case, there exist words with abelian closure containing more than one but finitely many minimal subshifts.

**Example 2.5.** Let \( f = abaababaa \cdots \) be the Fibonacci word over the alphabet \( \{a, b\} \) defined as the fixed point of the morphism \( \varphi : a \mapsto ab, b \mapsto a \). Consider the words \( u_1 \) and \( u_2 \) obtained from \( f \) by replacing the \( n \)th occurrence of \( a \) by the letter \( n \) (mod 3) (resp., \(-n\))

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\(^1\) Usually subshifts are defined as sets of bi-infinite words, in which case \( \sigma(X) = X \) is required in the definition.

\(^2\) Notice that non-balanced words are unbalanced, but unbalanced words are not necessarily non-balanced.
Proposition 5.1. The proof makes use of an operation similar to the squeezing operation. Further, removing Section 6.

It can be shown that the words \( u_1 \) and \( u_2 \) are balanced (see [12]). Moreover, for any factor \( x \) of \( u_2 \) with \( \Psi(x) = (|x|, |x|_1, |x|_2, |x|_b) \), \( \mathcal{L}_{ab}(u_2) \) contains the elements \((|x|, |x|_0, |x|_1, |x|_b)\) and \((|x|_1, |x|_2, |x|_0, |x|_b)\) (i.e., all the cyclic permutations of the first three elements): it can be straightforwardly shown that \((|\varphi^n(a)|_a \mod 3))_{n=0}^\infty = (1120210)\omega\). The claim follows from the observations that \( \varphi^{n+k+2}(a) = \varphi^{n+k+1}(a)\varphi^{n+k}(a) \) and \( \varphi^n(a) \) is a prefix of \( \varphi^{n+k}(a) \) for all \( n, k \geq 0 \). The facts established above imply that at least two of the first three components of \( \Psi(x) \) must be equal. The same observations apply to factors of \( u_1 \). To conclude, notice now that the 0s in \( u_1 \) and \( u_2 \) occur in the same positions. So the factor \( x' \) in \( u_1 \) occurring at the same position as \( x \in u_2 \) has \( \Psi(x') = (|x|_0, |x|_2, |x|_1, |x|_b) \). By a simple case analysis it can be seen that some cyclic permutation of the first three elements of \( \Psi(x') \) equals \( \Psi(x) \).

It can be shown that \( \mathcal{A}(u_1) \) contains exactly two minimal subshifts: it is the union of \( \Omega(u_1) \) and \( \Omega(u_2) \). Indeed, identifying the letters 0, 1, and 2 as \( a \) of any word \( y \) in \( \mathcal{A}(u_1) \) results in a Sturmian word that is in \( \mathcal{A}(f) \), and thus in \( \Omega(f) \) by Theorem 2.3. Further, removing all \( b \)s from \( y \) results in a word that is in \( \mathcal{A}((012)^\omega) = \Omega((012)^\omega) \cup \Omega((021)^\omega) \). From these observations, it is then straightforward to conclude that \( y \) must be in the shift orbit closure of either \( u_1 \) or \( u_2 \).

As the main result of this paper, we show that contrary to the non-binary case, aperiodic binary words can only contain either one minimal subshift (in the case of Sturmian words) or infinitely many minimal subshifts:

**Theorem 2.6.** Let \( x \) be a binary, aperiodic, uniformly recurrent word which is not Sturmian. Then \( \mathcal{A}(x) \) contains infinitely many minimal subshifts.

The proof consists of four parts treated in different ways: if \( x \) does not have uniform letter frequencies, the proof is almost immediate. If it has rational letter frequencies, then using standard words (certain factors of Sturmian words) we can show that its abelian closure contains uncountably many infinite subshifts (see Proposition 4.2). The proof for words with irrational frequencies is harder, and is split into the cases of \( C \)-balanced words and non-balanced words.

The proof for words which are \( C \)-balanced for some constant \( C \) is provided in Proposition 5.1. It is geometric in nature and is based on a so-called “squeezing operation” on infinite binary words. This operation does not extend the language of abelian factors of an infinite word, which allows to find infinitely many minimal subshifts in its abelian closure.

The hardest case turns out to be for non-balanced words with irrational letter frequencies (Proposition 7.1). The proof makes use of an operation similar to the squeezing operation in the \( C \)-balanced case. Due to non-balance, the analysis is heavily based on deep properties of Sturmian words and standard factorizations. We discuss these tools in Section 6.

3. Preliminaries and initial properties of abelian closures

We recall some notation and basic terminology from the literature of combinatorics on words. We refer the reader to [16, 17] for more on the subject. The set of finite words over an alphabet \( \Sigma \) is denoted by \( \Sigma^* \). The empty word is denoted by \( \varepsilon \). We let \(|w|\) denote the length of a word \( w \in \Sigma^* \). By convention, \(|\varepsilon| = 0\). The set of right infinite words is denoted by \( \Sigma^N \). We refer to infinite words in boldface font. Recall that the language \( \mathcal{L}(x) \) of an infinite word
x ∈ ΣN is the set of factors of x. The set of length n factors of x is denoted by \( L_n(x) \), and the set of factors of length at most n is denoted by \( L_{≤n}(x) \). We use the same notation for finite words as well.

In this paper we are mainly interested in binary words, and we mainly use the alphabet \{0, 1\}. For a finite binary word u, the weight of u refers to \(|u|_1\). A binary word is heavier than another if it has larger weight. Similarly it is called lighter, if its weight is smaller. Two binary words of equal length are abelian equivalent if and only if they have equal weight.

For \( x ∈ Σ^N \) and \( a ∈ Σ \), the limits

\[
\overline{\text{freq}}_x(a) := \lim_{n→∞} \frac{\max_{v ∈ L_n(x)} |v|_a}{n} \quad \text{and} \quad \underline{\text{freq}}_x(a) := \lim_{n→∞} \frac{\min_{v ∈ L_n(x)} |v|_a}{n}
\]

exist. Furthermore

\[
\overline{\text{freq}}_x(a) = \inf_{n ∈ N} \frac{\max_{v ∈ L_n(x)} |v|_a}{n} \quad \text{and} \quad \underline{\text{freq}}_x(a) = \sup_{n ∈ N} \frac{\min_{v ∈ L_n(x)} |v|_a}{n}.
\]

These facts follow from Fekete’s lemma, as \( \max_{v ∈ L_n(\alpha)} |v|_a \) (resp., \( \min_{v ∈ L_n(\alpha)} |v|_a \)) is subadditive (resp., superadditive) with respect to n. It thus follows that \( \max_{v ∈ L_n(x)} |v|_a ≥ \text{freq}_x(a) n \) and \( \min_{v ∈ L_n(x)} |v|_a ≤ \text{freq}_x(a) n \) for all \( n ∈ N \). These facts are used implicitly throughout the paper. If \( \text{freq}_x(a) = \overline{\text{freq}}_x(a) \), we denote the common limit by \( \text{freq}_x(a) \) and we say that x has uniform frequency of a.

A morphism \( f \) is a mapping \( Σ^* → Δ^* \), for alphabets Σ and Δ, such that \( f(uv) = f(u)f(v) \) for all words \( u, v ∈ Σ^* \). Notice that \( f \) is completely defined by the images of the letters of Σ. The morphic images of infinite words are defined in a natural way. A morphism is called erasing if \( f(\varepsilon) = \varepsilon \) for some letter a. Otherwise it is called non-erasing. For a morphism \( f : Σ → Δ^* \) and a subshift \( X ⊆ Σ^N \), we define \( φ(X) = \bigcup_{x ∈ X} Ω(φ(x)) \). When applying an erasing morphism to a subshift, we make sure that no element of X gets mapped to a finite word.

Sturmian words enjoy a plethora of different characterizations, and we shall use several of them in this note. Unless otherwise stated, the results presented below can be found from the excellent exposition [17, §2], to which we refer the reader for more on the topic.

The factor complexity function \( P_x : N → N \) is defined by \( P_x(n) = |L_n(x)| \) for each \( n ∈ N \). Similarly, we define the abelian complexity function \( P_x^{ab} : N → N \) of x as \( P_x^{ab}(n) = |L_n^{ab}(x)| \). The most commonly used definition of Sturmian words is given via the factor complexity function.

**Definition 3.1.** An infinite word x is Sturmian if \( P_x(n) = n + 1 \) for each \( n ∈ N \).

Notice that this definition implies that any Sturmian word is binary and is aperiodic by the famous Morse–Hedlund theorem (see Theorem 6.15 for a formulation). We shall also consider so-called periodic Sturmian words, which we define later on. To avoid confusion, we follow the convention that, when referring to Sturmian words, we mean the aperiodic Sturmian words.

It is known that any Sturmian word is uniformly recurrent. Furthermore, a Sturmian word s has irrational uniform letter frequencies. If \( \text{freq}_s(1) = \alpha \), then s is called a Sturmian word of slope \( \alpha \).

As we mentioned in the previous section, Sturmian words can be equivalently defined via balance, and this characterization of Sturmian words is crucial to our considerations:
Theorem 3.2 ([17, Thm. 2.1.5]). An infinite binary word $x$ is Sturmian if and only if it is balanced and aperiodic.

Next we consider the structure of factors of Sturmian words. We recall the so-called standard pairs from [17, Section 2.2]. Define two selfmaps $\Gamma$ and $\Delta$ on $\{0,1\}^* \times \{0,1\}^*$ by

$$\Gamma(u,v) = (u,uv), \quad \Delta(u,v) = (vu,v).$$

Definition 3.3. The set of standard pairs is the smallest set of pairs of binary words containing the pair $(0,1)$ and which is closed under $\Gamma$ and $\Delta$. A standard word is any component of a standard pair.

A word $w$ is called central, if $w01$ (or equivalently $w10$) is a standard word.

For example, the pairs $\Gamma^n(0,1) = (0,0^n1)$ and $\Delta^n(0,1) = (1^n0,1)$ are standard pairs for any $n \geq 0$. Here $1^n0$ and $0^n1$ are central words. These are the only standard pairs for which one of the components is a letter. Notice also that for a standard pair $(u,v)$, either $u$ is a letter or $u$ ends with $10$. Similarly either $v$ is a letter or $v$ ends with $01$. Recall that for a central word $w$ we have that $w01$ is a standard word that ends with $01$. It follows that $w01$ (resp., $w10$) can be expressed as the product $xy$ (resp., $yx$) for a standard pair $(x,y)$. In fact, such a standard pair is unique (see [17, Prop. 2.2.1]).

Definition 3.4. Let $(a_n)_{n \geq 1}$ be a sequence of integers with $a_1 \geq 0$ and $a_n > 0$ for $n > 1$. We define a sequence of words $S_{-1} = 1$, $S_0 = 0$, and $S_n = S_{n-2}a_n$ for $n \geq 1$. The sequence $(a_n)_{n \geq 1}$ is called a directive sequence and $(S_n)_{n \geq 1}$ is called a standard sequence.

It can be shown that each element $S_n$ of a standard sequence is a standard word. Conversely, every standard word occurs in some standard sequence. If $a_1 > 0$, then each of the words $S_n$, $n \geq 0$ starts with $0$. If $a_1 = 0$, then $S_1 = S_{-1} = 1$ and each of the words $S_n$, $n \geq 1$, starts with $1$. For $n \geq 1$ we have that $S_{2n+1}$ ends with $01$, while $S_{2n}$ ends with $10$.

A standard sequence $(S_n)_{n \geq 1}$ has the property that $\lim_{n \to \infty} S_n = s$ is a Sturmian word. Such a word is called a characteristic Sturmian word. It is the unique element of $\Omega(s)$ for which both $0s$ and $1s \in \Omega(s)$. For each directive sequence $(a_n)_{n \geq 1}$ there is a unique irrational number $\alpha$, such that the corresponding characteristic Sturmian word $s$ has $\freq_s(1) = \alpha$. Conversely, for any irrational $\alpha \in (0,1)$ there is a corresponding directive sequence which produces the characteristic Sturmian word having $\freq_s(1) = \alpha$.

Example 3.5. The Fibonacci word $f = 01001010\cdots$ is the characteristic Sturmian word defined by the directive sequence $(1)_{n=0}^\infty$. The directive sequence $(0,1,1,\ldots)$ gives the Fibonacci word by exchanging $0$ and $1$. The Fibonacci word is the characteristic Sturmian word of slope $1/\varphi^2$, where $\varphi$ is the golden ratio.

Periodic Sturmian words can be equivalently defined as follows:

Definition 3.6. A word is called periodic Sturmian if it is an element of $\Omega(S^\omega)$ for some standard word $S$.

To a periodic Sturmian word we may associate a directive sequence and a standard sequence. The difference is that the directive sequence is finite (with the final element $\omega$). The slope of a periodic Sturmian word is of course rational, and any rational number is a slope of some periodic Sturmian word (see [17, Prop. 2.2.15]). If two periodic Sturmian words
have the same slope, then they define the same shift orbit closure, similar to their aperiodic counterparts (this can be inferred from the fact that the standard words $xy$ and $yx$, for a standard pair $(x, y)$, define periodic Sturmian words that are shifts of each other). Hence, we have that the interval $[0, 1]$ coincides with the family of slopes of periodic and aperiodic Sturmian words.

Periodic and aperiodic Sturmian words are exactly the **recurrent** balanced binary words [20]. It follows that, for each periodic or aperiodic Sturmian word $s$, the abelian language $L_n^{ab}(s)$ consists of at most two elements. (For aperiodic Sturmian words it is always equal to 2, as it is easy to see that a word having $P_{x}^{ab}(n) = 1$ for some $n$ is purely periodic [5].)

There also exist non-recurrent balanced binary words. For us, it suffices to know that for any standard word $S$, the words $0S^ω$ and $1S^ω$ are balanced.

We now recall some preliminary observations on abelian closures of infinite words. The results appear in [13] unless otherwise stated.

**Lemma 3.7.** Assume $x \in \Sigma^\mathbb{N}$ has uniform frequency of a letter $a \in \Sigma$. Then any word $y \in A(x)$ has uniform frequency of $a$ and $\text{freq}_y(a) = \text{freq}_x(a)$.

We immediately have that if $x$ has an irrational uniform frequency of some letter $a$, then $A(x)$ contains only aperiodic words. We continue by observing how the abelian closures of periodic and ultimately periodic words can differ.

**Proposition 3.8.** For any purely periodic word $x$, the abelian closure $A(x)$ is finite.

The abelian closure of an ultimately, but not purely periodic word can be huge; in fact, it can contain uncountably many minimal subshifts. This was already observed in [10], and further examples were given in [13, Ex. 2].

We conclude this section by recalling two rather straightforward observations, which will be used throughout the paper. The first one is immediate by a "sliding window" argument and is well-known in the literature. The second one is a straightforward consequence of the first.

**Lemma 3.9 (Continuity of abelian complexity).** Let $u$ be an infinite binary word and $(s_1, t_1)$ and $(s_2, t_2)$ with $s_1 < s_2$ be two elements of $L_n^{ab}(u)$. Then each $(s, t)$ with $s + t = n$ and $s_1 < s < s_2$ is an element of $L_n^{ab}(u)$.

**Lemma 3.10 (Corridor Lemma).** Let $x$ be a binary word. Then $y \in A(x)$ if and only if, for all $n \in \mathbb{N},$

$$\min\{|v|_1 : v \in L_n(y)\} \geq \min\{|v|_1 : v \in L_n(x)\} \quad \text{and} \quad \max\{|v|_1 : v \in L_n(y)\} \leq \max\{|v|_1 : v \in L_n(x)\}.$$  

4. **Rational letter frequencies and no letter frequencies**

In this section, we prove easy parts of Theorem 2.6: the case when letter frequencies do not exist, and the case when they exist and are rational. As mentioned previously, the results were reported in [13]. We give the full proofs here for the sake of completeness, and we will further discuss further aspects of them in Section 8.

**Proposition 4.1.** Let $x$ be a binary word having no uniform letter frequencies. Then $A(x)$ contains uncountably many minimal subshifts.
Proof. Let \( \alpha = \text{freq}_x(1) > \text{freq}_x(1) = \alpha' \). Then, for any Sturmian word \( s \) of slope \( \beta \), where \( \alpha' \leq \beta \leq \alpha \), \( \Omega(s) \) is contained in \( \mathcal{A}(x) \) by the Corridor Lemma. There are uncountably many such \( s \).

We then turn to uniformly recurrent binary words having rational uniform letter frequencies. Our aim is to prove the following proposition:

**Proposition 4.2.** Let \( x \in \{0,1\}^\mathbb{N} \) be uniformly recurrent and aperiodic with rational uniform letter frequencies. Then \( \mathcal{A}(x) \) contains uncountably many minimal subshifts.

For the remainder of the section we fix the word \( x \) to be uniformly recurrent and aperiodic with \( \text{freq}(x)(1) = p/q \) and we assume \( \gcd(p,q) = 1 \). We begin with a few technical lemmas:

**Lemma 4.3.** For all \( n \in \mathbb{N} \) we have \( \max_{v \in \mathcal{L}_n(x)} |v|_1 > n^P_q \) and \( \min_{v \in \mathcal{L}_n(x)} |v|_1 < n^P_q \).

Proof. We show the claim for \( \max_{v \in \mathcal{L}_n(x)} |v|_1 \). The proof for \( \min_{v \in \mathcal{L}_n(x)} |v|_1 \) is symmetric. If \( np/q \) is not an integer, then the claim follows from the fact that \( \max_{v \in \mathcal{L}_n(x)} |v|_1 \geq np/q \). For the sake of contradiction, assume that \( \max_{v \in \mathcal{L}_n(x)} |v|_1 = n^P_q \) for some multiple \( n \) of \( q \). Write \( x = a_1a_2\ldots \). For any \( M \geq n \) we may write

\[
|x_{[1,M+n]}|_1 = \frac{1}{n} \sum_{i=1}^{M} |x_{[i,i+n]}|_1 + \frac{1}{n} \sum_{i=1}^{n-1} (n-i)(|a_i|_1 + |a_{M+n-i}|_1).
\]

Indeed, in the first sum each \( |a_i|_1 \) is counted \( n \) times for \( i = n, \ldots, M \). For \( i \in \{1, \ldots, n-1\} \) the values \( |a_i|_1 \) and \( |a_{M+n-i}|_1 \) are counted \( i \) times each. Hence, the first sum equals \( |x_{[n,M]}|_1 + \frac{1}{n} \sum_{i=1}^{n-1} i(|a_i|_1 + |a_{M+n-i}|_1) \). The second sum adds the missing contributions so that the total contribution of each letter is counted once after normalizing by \( \frac{1}{n} \). Observe that the second sum is bounded from above by \( n-1 \) (after dividing by \( \frac{1}{n} \)).

As \( x \) is uniformly recurrent, there exists \( N \in \mathbb{N} \) such that each factor of length \( N \) contains a factor of length \( n \) having at most \( n^P_q - 1 \) occurrences of 1. Recall that each factor of length \( n \) has weight at most \( np/q \) by assumption. Hence for all \( M \geq 1 \)

\[
\frac{1}{n} \sum_{i=1}^{MN} |x_{[i,i+n]}|_1 \leq M(N^P_q - \frac{1}{n})
\]

since at least \( M \) of the factors of length \( n \) of \( x_{[1, MN+n]} \) have at most \( n^P_q - 1 \) occurrences of the letter 1. But now

\[
\lim_{M \to \infty} \frac{1}{MN+n} |x_{[1, MN+n]}|_1 \leq \lim_{M \to \infty} \frac{1}{MN+n} \sum_{i=1}^{MN} \frac{1}{n} |x_{[i,i+n]}|_1 + \frac{n-1}{MN+n} \leq \lim_{M \to \infty} \frac{1}{MN+n} (MN^P_q - \frac{M}{n}) = \frac{p}{q} - \frac{1}{nN}.
\]

This is a contradiction. \( \square \)

As \( x \) is assumed to be aperiodic, an immediate consequence of the above is that for any multiple \( n \) of \( q \), the values \( n^P_q - 1 \) and \( n^P_q + 1 \) are the weights of some factors of length \( n \) of \( x \).

Let now \( \varphi : \{0,1\} \to \{0,1\}^* \) be defined by \( 0 \mapsto w01, 1 \mapsto w10 \), where \( w01 \) (or \( w10 \)) is the Standard word of slope \( \frac{p}{q} \) having \( |w01|_1 = p \) and \( |w01| = q \).
Lemma 4.4. For all \( n \in \mathbb{N} \) and \( v \in \mathcal{L}_n(\{0, 1\}^N) \) we have \( |v|_1 - np/q \leq 1 \). Furthermore, each integral value in the interval \([np/q - 1, np/q + 1]\) is the weight of some \( v \in \mathcal{L}_n(\{0, 1\}^N) \).

Proof. Let \( n \in \mathbb{N} \) and \( u \in \mathcal{L}_n(\{0, 1\}^N) \). Then there exist letters \( a, b, c, d \) with \( \{a, b\} = \{c, d\} = \{0, 1\} \) such that \( u = r\varphi(x)s \) for some \( x \in \{0, 1\}^\ast \), \( r \in \text{suff}(wab) \), and \( s \in \text{pref}(wcd) \) satisfying \( |r|, |s| < q \). Observe now that \( u \sim_{ab} v \) for some \( v \in r(wcd)^\ast s \) and that \( (wcd)\omega \) is periodic Sturmian.

If \( |r| \geq 2 \), \( r = \varepsilon \), or \( r = b = d \), then \( v \sim u' \) for some \( u' \in \mathcal{L}((wab)\omega) \). It follows that \( |u|_1 \in \{\lfloor n\frac{p}{q} \rfloor, \lceil n\frac{p}{q} \rceil\} \). Assume that \( r = b \neq d \). Now \( u \sim v = \text{pref}_n(b(wba)\omega) \), where \( b(wba)\omega \) is balanced. If \( n \) is not a multiple of \( q \), then \( |u|_1 = \lfloor n\frac{p}{q} \rfloor + |b|_1 \). If \( n \) is a multiple of \( q \), then \( |u|_1 = n\frac{p}{q} + |b|_1 - |a|_1 \).

We have shown that \( |u|_1 - np/q \leq 1 \) regardless of whether \( n \) is a multiple of \( q \) or not. Clearly each value is attained by some word in \( \mathcal{L}_n(\{0, 1\}^N) \). This concludes the proof. \( \square \)

The above lemmas allow us to conclude Proposition 4.2.

Proof of Proposition 4.2. Assume \( \text{freq}_x(1) = \frac{p}{q} \). Let \( \varphi(\{0, 1\}^N) = \mathcal{O} \) be as in the above lemma whence, for all \( v \in \mathcal{L}_n(\mathcal{O}) \), \( |v|_1 - np/q \leq 1 \). By Lemma 4.3, \( \max_{u \in \mathcal{L}_n(\mathcal{O})} |u|_1 > n\frac{p}{q} \) and \( \min_{u \in \mathcal{L}_n(\mathcal{O})} |u|_1 < n\frac{p}{q} \). By the Corridor Lemma we have that, for any word \( y \in \mathcal{O} \), we have \( \Omega(y) \in \mathcal{A}(x) \). Clearly \( \mathcal{O} \) contains uncountably many minimal subshifts. \( \square \)

5. Abelian closures of C-balanced words

In this section we prove the following statement:

**Proposition 5.1.** Let \( x \) be a uniformly recurrent binary word which is not Sturmian. Suppose in addition that \( x \) is \( C \)-balanced for some \( C > 1 \) and that \( \text{freq}_x(1) = \alpha \) is irrational. Then \( \mathcal{A}(x) \) contains infinitely many minimal subshifts.

We use the following notion of the graph \( g_w \) of an infinite word \( w \), which is a modification of a geometric approach from [1]. We focus on binary words, although the notion extends in an obvious way to nonbinary alphabets. Let \( w = a_1a_2 \cdots \) be an infinite word over a finite alphabet \( \Sigma \). We translate \( w \) to a graph visiting points of the infinite rectangular grid by interpreting letters of \( w \) as drawing instructions. In the binary case, we associate the letter 0 with a move by vector \( \vec{v}_0 = (1, 0) \), and the letter 1 with a move \( \vec{v}_1 = (1, 1) \). We start at the origin \((x_0, y_0) = (0, 0)\). At step \( n \), we are at a point \((x_{n-1}, y_{n-1})\) and we move by a vector corresponding to the letter \( a_n \), so that we come to a point \((x_n, y_n) = (x_{n-1}, y_{n-1}) + \vec{v}_{a_n}\), and the two points \((x_{n-1}, y_{n-1})\) and \((x_n, y_n)\) are connected with a line segment. So, we translate the word \( w \) to a path in \( \mathbb{Z}^2 \). We denote the corresponding graph by \( g_w \). So, for any word \( w \), its graph is a piecewise linear function with linear segments connecting integer points (see Figure 1). We remark that \( g_w(i) = |a_1 \cdots a_i|_1 \). Note also that instead of the vectors \((0, 1)\) and \((1, 1)\), one can use any other pair of noncollinear vectors \( \vec{v}_0 \) and \( \vec{v}_1 \). For a \( k \)-letter alphabet one can consider a similar graph in \( \mathbb{Z}^k \). Note that the graph can also be defined for finite words in a similar way, and we will sometimes use it.

To prove the proposition, we will need the following operation of (upper) \( C \)-squeezing.

**Definition 5.2.** Let \( x = a_1a_2 \cdots \) be a binary word with \( \text{freq}_x(1) = \alpha \), and let \( C \in \mathbb{R} \). We define an operation of \( C \)-squeezing of \( x \), \( s^C_{\alpha}(x) = a'_1a'_2 \cdots \), as follows. For each \( i \) such that \( g_x(i) > \alpha i + C \) and \( a_{i-1} = 1 \), \( a_i = 0 \), we define \( a'_{i-1} = 0 \), \( a'_i = 1 \). In this case we say that we have a switch at position \( i \). Otherwise we define \( a'_i = a_i \).
Informally, the $C$-squeezing operation works as follows. If there is a piece of graph above the line $y = \alpha x + C$, we make local changes in this piece getting this part of the graph closer to the stripe (see Figure 2). Clearly, we can symmetrically define an operation of lower $C$-squeezing, but for our proof it is enough to squeeze only from one side. We return to this operation in Subsection 8.2.

The following claim is immediate:

**Claim 1.** The operation of squeezing does not change the letter frequencies.

Note that, by Lemma 3.9 and Lemma 3.10, we have

**Claim 2.** Let $u$ be a binary word with $\text{freq}_u(1) = \alpha$. Then for each $m$ there exist $i$ and $j$ such that $|u_i \cdots u_{i+m-1}| = \lfloor \alpha m \rfloor$ and $|u_j \cdots u_{j+m-1}| = \lceil \alpha m \rceil$.

Let us now prove the main technical lemma relating $C$-squeezings and abelian closures.

**Lemma 5.3.** For a binary word $x$ we have $s^+_C(x) \in \mathcal{A}(x)$.  

---

Figure 1: Graph of the Thue–Morse word.

Figure 2: Upper squeezing.
Proposition 5.1: there exists a factor $x$.

If $a_{j-1} + a'_{j-1} = 0$, take $k = i + 1$.

Figure 3: Case 1 of the proof of Lemma 5.3: $a_{i-1}a_i = 10$, $a'_{i-1}a'_i = 01$.

Proof. Assume the converse. Let $x' = s_n^+(x)$. Then, due to Lemma 3.10 there exists a factor $a'_1 \cdots a'_{j-1}$ such that for each $k$ we have $|a'_1 \cdots a'_{j-1}|_1 > |a_k \cdots a_{k+j-1-1}|_1$ (the case of $< i$ is symmetric).

First we remark that the switches inside the factor (at positions $i + 1, \ldots, j - 1$) do not change the Parikh vector of the factor. So, to change the Parikh vector, we must have a switch at position $i$ or $j$ (or both). Secondly, note that we have a switch at position $\ell$ if and only if $g_\kappa(\ell) \neq g_\kappa(\ell)$.

1. Switch at $i$ and not in $j$.

In this case we must have $g_\kappa(i) > \alpha i + C$, $a'_1 = 1$, $a_i = 0$. Notice that we have $|a'_1 \cdots a'_{j-1}|_1 > [\alpha(j - i)]$ (recall that $x$ always contains a binary word with weight $[\lfloor \alpha \rfloor]$). This in turn means that $g_\kappa(j) > \alpha j + C$ (due to frequency). By the conditions of Case 1 we have that $g_\kappa'(j) = g_\kappa(j)$, which means that $a_{j-1}a_j \neq 10$. If $a_{j-1} = 0$, then by taking $k = i - 1$ we get an abelian equivalent factor in $x$ (see Figure 3a). If $a_{j-1} = 1$, then $a_j = 1$ and we can take $k = i + 1$ (see Figure 3b).

2. Switches at both $i$ and $j$.

In this case $g_\kappa(i) > \alpha i + C$ and $g_\kappa(j) > \alpha j + C$, then the Parikh vector does not change (we can take $k = i$).

3. Switch at $j$ and not in $i$. In this case $a_{j-1} = 1$ and $a'_{j-1} = 0$, so $|a'_1 \cdots a'_{j-1}|_1 < [a_k \cdots a_{k+j-1-1}]$, which contradicts our assumption of $a'_1 \cdots a'_{j-1}$ being heavier than the factors of $x$.

To prove Proposition 5.1, we prime the situation as follows. Notice that the property of a word being $C$-balanced for some $C$ is equivalent to the property that its graph lies between two lines\footnote{If, e.g., the graph goes above the line $y = ax + C$, say $g_\kappa(i) > \alpha i + C$, then the prefix of length $i$ has weight at least $[i\alpha] + C$. But $x$ also contains a factor with weight $[i\alpha]$ contradicting the $C$-balancedness of $x$.} $y = ax + C_1$ and $y = ax + C_2$ for some $C_1, C_2 \in \mathbb{R}$, $C_1 < C_2$. Here we choose $C_1$ and $C_2$ to be the largest and the smallest possible, i.e. $C_1 = \lim sup\{C : g_\kappa(x) \geq ax + C\}$ and $C_2 = \lim inf\{C : g_\kappa(x) \leq ax + C\}$. Notice that the line $ax + C''$ contains at most one integral point since $\alpha$ is irrational.

Definition 5.4. Let $x$ be an infinite binary word that is $C$-balanced. The width of a factor $v$ of $x$ is defined as $\sup_x(g_\kappa(x) - ax) - \inf_x(g_\kappa(x) - ax)$, where the maximum and the minimum is chosen from the positions $x$ in an occurrence of $v$ in $x$. 


Similarly we can define a width for an infinite word. In fact, a width of a factor depends only on the frequency of letters in \( w \) and not on the word \( w \) itself. Clearly, a width of a factor \( v \) of an infinite word \( w \) cannot be bigger than the width of the \( w \).

**Proof of Proposition 5.1.** The proof is based on the operation of upper \( C_2 - \varepsilon \)-squeezing; in fact, choosing different values of \( \varepsilon \), we can get different minimal subshifts from the abelian closure of the initial word.

The proof is split into several claims.

**Claim 3.** For each small enough \( \varepsilon > 0 \) there exist infinitely many points of the graph \( g_x \) in the stripe between \( y = \alpha x + C_2 - \varepsilon \) and \( y = \alpha x + C_2 \). Moreover, the gaps between these points are bounded.

**Proof.** (See Figure 4a.) Since \( C_2 = \lim \inf \{C : g_x(x) \leq \alpha x + C\} \), there exists by definition a point \( x \) such that \( g_x(x) > \alpha x + C_2 - \varepsilon / 2 \). Symmetrically, there exists a point \( x' \) such that \( g_x(x') < \alpha x + C_1 + \varepsilon / 2 \). Without loss of generality we may assume that \( x < x' \). Now since the word is uniformly recurrent, a factor abelian equivalent to \( a_x \ldots a_{x'-1} \) occurs with bounded gaps; let \( N(\varepsilon) \) denote an upper bound on the gaps. For each large enough occurrence of this factor its initial point is above the line \( y = \alpha x + C_2 - \varepsilon \), and its final point is below the line \( y = \alpha x + C_1 + \varepsilon \). Indeed, if the initial point was below the line \( y = \alpha x + C_2 - \varepsilon \) for arbitrarily large \( x \), then the final point would lie below \( \alpha x + C_1 - \varepsilon / 2 \) for arbitrarily large \( x \).

We will now use an operation of upper \( (C_2 - \varepsilon) \)-squeezing of \( x \). For simplicity, assume that \( \varepsilon < \alpha \) (this assumption is made in order to flip all the points on the stripe of width \( \varepsilon \)), and that \( C_2 - \varepsilon - C_1 > 1 \) (this corresponds to Sturmian width; we make this assumption to guarantee that the flipped points remain above the line \( \alpha x + C_1 \)). Due to Lemma 5.3, if \( x' = s^\varepsilon_{C_2-\varepsilon}(x) \), then \( x' \in A(x) \).

**Claim 4.** Any uniformly recurrent word from \( \Omega(x') \) is different from words in \( \Omega(x) \).

**Proof.** Indeed, there is no factor from the proof of Claim 3 in \( x' \) (their width is greater than \( C_2 - \varepsilon - C_1 \), so they do not fit the stripe of \( x' \)).

We now claim that varying \( \varepsilon \) in an appropriate way we obtain infinitely many minimal subshifts in \( A(x) \).

Given \( \varepsilon \), let \( N(\varepsilon) \) be the constant from Claim 3 (it is actually given by uniform recurrence), giving an upper bound for the points of the graph of \( x \) above the line \( y = \alpha x + C_2 - \varepsilon \).

Now choose \( \varepsilon_1 < \varepsilon \) to be the constant such that the points of the grid in the stripe between the lines \( y = \alpha x + C_2 - \varepsilon_1 \) and \( y = \alpha x + C_2 \) are at distance at least \( 2N(\varepsilon) \) (the value of \( \varepsilon_1 \) is given by the irrational value \( \alpha \)).

By the choice of \( N(\varepsilon) \) and \( \varepsilon_1 \), we have points of the graph \( g_x \) in the stripe between \( y = \alpha x + C_1 - \varepsilon \) and \( y = \alpha x + C_2 - \varepsilon_1 \) with gap at most \( 2N(\varepsilon) \).

We will now prove that for the word \( x'' = s^\varepsilon_{C_2-\varepsilon_1}(x) \) we have that each uniformly recurrent point in \( \Omega(x'') \) is different from any point from \( \Omega(x') \). Indeed, from what we just proved above, \( g_x \) (and hence \( g_{x''} \)) has points between \( y = \alpha x + C_2 - \varepsilon \) and \( y = \alpha x + C_2 - \varepsilon_1 \) with gap at most \( 2N(\varepsilon) \). Let \( \varepsilon \) be such that the integer points in the stripe between \( y = \alpha x + C_2 - \varepsilon \) and \( y = \alpha x + C_2 - \varepsilon + \varepsilon \) are with gap at least \( 4N(\varepsilon) \) (the value \( \varepsilon \), like \( \varepsilon_1 \), is defined by the irrational \( \alpha \)). So, the graph of \( x \) (and hence of \( x'' \)) has points between \( y = \alpha x + C_2 - \varepsilon + \varepsilon \) and \( y = \alpha x + C_2 - \varepsilon_1 \) with gap at most \( 4N(\varepsilon) \). (See Figure 4b for an illustration of the
situation.) Now, by an argument symmetric to Claim 3 applied for \( \tilde{\varepsilon} \), we get that there is an upper bound \( N(\tilde{\varepsilon}) \) for the gaps between the points of \( g_x \) between the lines \( y = \alpha x + C_1 \) and \( y = \alpha x + C_1 + \varepsilon \). Taking \( N = \max(4N(\varepsilon), N(\tilde{\varepsilon})) \) and considering points in the two stripes (between the lines \( y = \alpha x + C_1 \) and \( y = \alpha x + C_1 + \varepsilon \) and the lines \( y = \alpha x + C_2 - \varepsilon + \tilde{\varepsilon} \) and \( y = \alpha x + C_2 - \varepsilon_1 \)), we have factors of length at most \( \tilde{N} \) of width at least \( C_2 - C_1 - \varepsilon \) with gap at most \( N' \). Since there are only finitely many such factors, one of them occurs with bounded gap in \( x'' \), and hence in any word from \( \Omega(x'') \). On the other hand, these factors are too wide to fit into the stripe for \( x' \) (which is of width \( C_2 - C_1 - \varepsilon \)), so in fact \( \Omega(x') \) and \( \Omega(x'') \) do not intersect, and hence \( \Omega(x'') \) contains a new minimal subshift which is in \( A(x) \).

We continue this line of reasoning taking \( \varepsilon_1 \) instead of \( \varepsilon \) etc., each time getting a new minimal subshift in the abelian closure of \( w \). This concludes the proof.

### 6. Some structural results on binary abelian closures

The results presented in this section will be used as tools in proving the main result of the subsequent section (and the last and the hardest case of the main theorem), though they might have independent interest. We consider certain operations on binary words, and consider how they affect abelian closures. We first discuss morphic images of abelian closures. We then define an operation, which resembles an elementary cellular automaton on right-infinite words (see the precise definition in Subsection 6.2). And finally, we give a certain description of non-Sturmian binary words in terms of Standard pairs.

#### 6.1. Morphisms and binary abelian closures

Morphisms are an essential tool in the study of combinatorics on words. In this subsection we study the interaction between abelian closures and morphisms. As the main result of this subsection, we show that, given a Sturmian morphism \( f \) (see definition below), if \( z \in A(y) \) then \( f(z) \in A(f(y)) \). However, in general this property does not hold even for binary alphabets, as is illustrated by the following example.
Example 6.1. Take \( y = (0011)^\omega \) and observe that \( z = (01)^\omega \in A(y) \) (\( z \) is a periodic Sturmian of slope 1/2). Define \( f \) by \( f(0) = 100001 \) and \( f(1) = 010 \). Hence
\[
f(y) = (1000011000010100010)^\omega \text{ and } f(z) = (1000101010)^\omega.
\]
Observe now that the length 5 factors of \( f(y) \) all have at most two occurrences of 1. On the other hand, \( f(z) \) contains the factor 10101 which has three occurrences of 1. Hence \( f(z) \notin A(f(y)) \).

Let us now recall Sturmian morphisms and standard morphisms. For a concise treatment of these morphisms, see [17, §2.3]. We then consider the abelian closures of morphic images of binary words.

A morphism \( \varphi : \{0,1\}^* \to \{0,1\}^* \) is called Sturmian if, for each Sturmian word \( x \), the word \( \varphi(x) \) is Sturmian. We shall employ a striking result of Mignosi and Sécédol [19] which characterizes the set of Sturmian morphisms as the finitely generated monoid with generators
\[
D : \begin{cases} \ 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases} \quad E : \begin{cases} \ 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases} \quad G : \begin{cases} \ 0 \mapsto 10 \\ 1 \mapsto 0 \end{cases}.
\]

A morphism \( \varphi \) is called standard if, for any (unordered) standard pair \( \{u,v\} \), the pair \( \{\varphi(u), \varphi(v)\} \) is standard. Standard morphisms were studied by A. de Luca in [6]. In that article, the set of standard morphisms is characterized as the finitely generated monoid with generators \( \{D,E\} \) defined above.

We also recall the following result from [6, Thms. 8, 9, 12].

**Theorem 6.2.** A morphism \( \varphi \) is standard if and only if \( \varphi(0) = x \) and \( \varphi(1) = y \) for some \( x,y \) with \( \{x,y\} \) an unordered standard pair.

We are now ready to show the main result of this subsection.

**Proposition 6.3.** Let \( \varphi \) be a Sturmian morphism. Then \( z \in A(y) \) implies \( \varphi(z) \in A(\varphi(y)) \).

**Proof.** It suffices to show that the claim holds for the morphisms \( E, D \) and \( G \). To conclude from that, we may proceed by a simple induction on the length of the shortest representation of a Sturmian morphism as a composition of these generating morphisms.

The case of \( E \) is trivial. We prove the claim for \( D \), the case of \( G \) being symmetric. Let \( z \) be a factor of \( D(z) \). We show that there is an abelian equivalent factor in \( D(y) \) such that \( z' \sim z \). By the form of \( D \), \( z \) can be written as one of the following forms: \( D(w), D(w)0, 1D(w), \) and \( 1D(w)0 \), where \( w \) is some factor of \( z \). The claim is easily seen to hold in the case \( w = \varepsilon \), so we assume that \( w \neq \varepsilon \).

Assume first that \( z = D(w) \), with \( w \in L(z) \). Then \( y \) has a factor \( w' \sim w \), so \( D(w') = z' \sim z \). Assume second that \( z = D(w)0 \). There exists a factor \( w'x, x \in \{0,1\} \), of \( y \) with \( w' \sim w \). By the form of \( D \), \( D(w'x) \) begins with \( D(w')0 = z' \), with \( z' \sim z \).

Assume third that \( z = 1D(w) \). Then \( 0w \) is a factor of \( z \). Let us first show that there exists a factor \( w' \in y \) with \( w' \sim w \), and \( 0w' \) or \( w'0 \) occurs in \( y \). Indeed, since \( 0w \) occurs in \( z \), there is a corresponding abelian equivalent factor \( t \) in \( y \). Consider an occurrence of \( w'1 \). Assuming that \( t \) occurs before \( w'1 \), by a sliding window argument, there is a factor of the form \( 0t'1 \), where \( 0t' \sim t \) and \( t'1 \sim w'1 \). Thus \( 0t' \) is the factor we are looking for. The case that \( t \) occurs after \( w'1 \) is symmetric. Now if \( 0w' \) occurs in \( y \), we have \( 1D(w') = z' \) and we are...
done. If \( w'0 \) occurs in \( y \), then, by the form of \( D \), we may write \( D(w') = 0x D(w'') \) for some \( x \in \{ \varepsilon, 1 \} \) and \( w'' \in \Sigma^* \). Now \( x D(w'')0 \sim D(w') \), and thus \( x D(w'')01 = z' \sim 1 D(w) = z \).

Assume finally that \( z = 1D(w)0 \). Notice again that \( 0w \) occurs in \( z \). Proceeding as in the previous case, there exists a factor \( w' \sim w \) such that \( 0w' \) or \( w'0 \) occurs in \( y \). This time we may choose \( z' = D(w'0) \) or \( z' = D(0w') \).

\[6.2. \text{A selfmap on binary abelian closures}\]

In this section define a selfmap on binary abelian closures. This mapping has interesting dynamics, as we shall shortly see. We present the observations as having independent interest, but the main result of this section will be crucial in our subsequent constructions.

**Definition 6.4.** Let us define the operation \( T : \{0,1\}^N \rightarrow \{0,1\}^N \) by the following rule: \( T(x) \) is obtained from \( x \) by replacing each occurrence of 10 with 01. Let us further define \( F = \sigma \circ T \). Thus \( F \) operates on an infinite binary word by first flipping each occurrence of 10 to 01 \((T)\), and second removing the first letter \((\sigma)\).

Observe that the operation \( T \) is a simplified version of the \( C \)-squeezing operation used in Section 5.

**Remark 6.5.** The operation \( T \) can be defined for bi-infinite words (words indexed by the set of integers) as well. In this setting \( T \) is a *cellular automaton* known as the *Traffic cellular automaton*. It is *Rule 184* in the system of S. Wolfram [29].

Let us show that the mapping \( F \) is indeed a selfmap on a binary subshift.

**Lemma 6.6.** For any binary word \( x \in \{0,1\}^N \), we have \( F(x) \in A(x) \).

**Proof.** Assume the contrary, that \( F(x) \) has a factor which is either heavier or lighter than all factors of \( x \). Write \( x = a_0 a_1 a_2 \cdots \) and \( T(x) = b_0 b_1 b_2 \cdots \), where \( a_i, b_i \in \{0,1\} \). Assume first that the factor \( u = b_i \cdots b_j \) of \( T(x) \), with \( i \geq 1 \) (recall that \( F(x) \) is obtained by removing the first letter of \( T(x) \)), is heavier than all factors of \( x \) of the same length. We consider how \( u \) is generated from \( x \) under \( T \):

\[
x: \quad \cdots \quad a_{i-1} \quad a_i \quad a_{i+1} \cdots \quad a_{j-1} \quad a_j \quad a_{j+1} \cdots
\]

\[
T(x): \quad \cdots \quad * \quad b_i \quad b_{i+1} \cdots \quad b_{j-1} \quad b_j \quad * \quad \cdots
\]

Consider the factor \( v = a_i \cdots a_j \) of \( x \). Since \( u \) is heavier than \( v \), we must have \( a_i a_{j+1} \neq 10 \) by the definition of \( T \). Moreover, we see that \( |u| = 1 + |v| \). If \( a_j = 0 \), then \( |a_i \cdots a_{j-1}| = 1 + |v| \), so \( u \) is not heavier than this factor of \( x \). Now if \( a_j = 1 \), then necessarily \( a_{j+1} = 1 \) and thus \( |a_i \cdots a_{j+1}| = |v| \), and again, \( u \) is not heavier than this factor of \( x \). In either case, \( u \) has an equal weight and length corresponding factor in \( x \), a contradiction.

The case of \( b_i \cdots b_j \) being lighter than all other factors is symmetric. One simply notes that in this case necessarily \( a_j a_{j+1} = 10 \) and \( a_{i-1} a_i \neq 10 \).

The proof above is similar to the proof of Lemma 5.3, but the shift operation in the definition of \( F \) cannot be removed.
Remark 6.7. We remark that $T(x)$ is not necessarily an element of $A(x)$. For example, consider the Sturmian word $0f = 0010\ldots$, where $f$ is the Fibonacci word. Observe that $T(0f)$ begins with $0001$, hence $T(0f) \notin \Omega(0f) = A(0f)$.

Clearly $F(x)$ is uniformly recurrent if $x$ is. Further, if $x$ has uniform letter frequencies, then $F(x)$ has the same letter frequencies as $x$ by Lemma 3.7. Moreover, if $x$ is non-balanced, then so is $F(x)$. This can be shown by using similar arguments as in the above proof.

We now consider the operation of iterating $F$ on a word. For this, we need some terminology.

Definition 6.8. A word $y$ is a preimage of order $n$ of a word $x$, if $F^n(y) = x$. We say that $x$ has a preimage of order $n$, if such a $y$ exists.

Lemma 6.9. Let $x$ be a binary word containing the factor $11(01)^m00$ for some $n \geq 0$. Then $x$ has no preimage of order $n + 1$.

Proof. If $x$ does not have a preimage under $F$, then there is nothing to prove. Assume it has a preimage $y$ under $F$. We thus have $T(y) = x'$, where $x = \sigma(x')$. Consider the position in $y$ corresponding to where $11(01)^m00$ occurs in $x'$ (note that $x'$ also contains $11(01)^m00$):

$$
y: \quad \cdots * a b \cdots c d * \cdots \n$$

$$x': \quad \cdots 1 1 0 \cdots 1 0 0 \cdots$$

The only option is that $ab = 11$ and $cd = 00$: If $a = 0$, then either $a$ stays in the same position or is moved one step to the left depending on which letter precedes $a$ in $y$. This is impossible, since $x'$ has 1 in both positions. Furthermore if $b = 0$, then the letter $a = 1$ would be shifted by $T$ to the right by one position, which is also not possible, as $x'$ has 0 in that position. Similar arguments show that $cd = 00$.

Observe that the arguments showing $ab = 11$ and $cd = 00$ are independent of each other. Now if $n = 0$, the above observation poses a contradiction: we should have $1 = a = c = 0$. Thus $x$ has no preimage under $F$. For $n \geq 1$, we deduce from the above that $y$ contains the factor $11(01)^m00$ for some $m \leq n - 1$. By induction, $y$ has no preimage of order $m + 1$, so that $x$ has no preimage of order $m + 2 \leq n + 1$, as was to be shown.

The dynamics of the mapping $F$ will be of interest to us in our later considerations. The following proposition is the main result of this section.

Proposition 6.10. Let $x$ be a binary word with freq(1) < $1/2$. Then there exists an integer $n \geq 0$ such that all 1s are isolated in $F^n(x)$, that is, $11 \notin \mathcal{L}(F^n(x))$.

Proof. If all 1s are isolated in $x$ we may choose $n = 0$. Assume that 11 occurs in $x$. Due to our assumption freq(1) < $1/2$, there must exist a factor $v$ of maximal length for which freq$_v(1) > 1/2$ and, further, in which 11 occurs. We call such a factor of $x$ exceptional. Note that an exceptional factor has length at least 3, since 110 must occur in $x$ under the assumptions. Now any occurrence of an exceptional factor $v$ (occurring after the prefix of length 2) must be preceded and followed by 00 in $x$. Otherwise $x$ contains a factor of length $|v| + 2$ with frequency at least $|v| + 1 \over |v| + 2 > 1/2$ and which contains 11.

We partition the rest of the proof into a couple of claims.
Claim 5. For any exceptional factor $u$ of $F(x)$, there exists an exceptional factor $v$ of $x$ such that $|u| \leq |v|$.

Proof. Write $x = a_0a_1 \cdots$ and $T(x) = b_0b_1 \cdots$. Let $u = u_1u_2 \in \{0,1\}^*$. Actually, $u_2 \neq 0$: notice again that $u$ is followed by 00 in $F(x)$ so $u$ cannot end with 11, as otherwise $F(x)$ would contain the factor 1100. Lemma 6.9 would then imply that $F(x)$ does not have a preimage, which is absurd.

Let us depict an occurrence of $u = b_i \cdots b_j$ in $T(x)$, with $i \geq 1$.

$$\begin{array}{cccccccc}
\text{x:} & \cdots & a_{i-1} & a_i & \cdots & \ast & a & b & \cdots & a_j & a_{j+1} & \ast & \cdots \\
\text{T(x):} & \cdots & \ast & b_i & \cdots & 1 & 1 & \ast & \cdots & b_j & 0 & 0 & \cdots \\
\end{array}$$

Here we allow $F(x)$ to begin with $u$, so that $T(x)$ would begin with $0u$ or $1u$ (this does not matter in the following argument). Observe that this particular occurrence of $u$ depends only on the factor $a_{i-1} \cdots a_{j+1}$. We therefore have $|u| \leq |a_{i-1} \cdots a_{j+1}|$ by the form of $T$.

We now have that $ab = 11$ and $a_ja_{j+1} = 00$ by the same arguments as used in Lemma 6.9. It then follows that $|u| \leq |a_{i-1} \cdots a_{j-1}|$ as we had $a_ja_{j+1} = 00$. Now $a_{i-1} \cdots a_{j-1}$ contains an occurrence of 11 and the frequency of 1 is larger than 1/2. Hence $x$ contains an exceptional factor of length at least $|u|$, as was claimed.

As a consequence of the above claim, for any $m \geq 0$ and for any exceptional factor $u$ of $F^m(x)$, there exists an exceptional factor $v$ of $x$ such that $|u| \leq |v|$.

Claim 6. There exists an integer $m$ such that any exceptional factor of $F^m(x)$ is shorter than any exceptional factor in $x$.

Proof. Let $v$ be an exceptional factor of $x$ and take $m = \lceil |v|/2 \rceil$. Assume, for a contradiction, that $y = F^m(x)$ contains an exceptional factor $u$ of length $|v|$. Similar to $v$, all occurrences of $u$ in $y$ are followed by 00. We infer that $u00$ contains a factor of the form 11(01)\(k\)00, where $2(k + 1) \leq |u| = |v|$. By Lemma 6.9, $y$ has a preimage of the order at most $k \leq |v|/2 - 1 < \lceil |v|/2 \rceil = m$, which is a contradiction.

The claim above implies that there exists an integer $n \geq 1$ such that an exceptional factor in $F^{n-1}(x)$ has length at most 3. The only such factors are 011 and 110. This implies that each occurrence of 11 is always followed by 000 in $F^{n-1}(x)$. We conclude that, in the word $F^n(x)$, all 1s are isolated, which was to be proved.

We shall use the following immediate corollary in our later considerations.

Corollary 6.11. For a non-balanced binary word $x$ with $\text{freq}(1) < 1/2$, there exists a non-balanced word $x'$ in $A(x)$ such that $11 \notin \mathcal{L}(x')$.

6.3. The structure of binary words in terms of standard pairs

In this subsection we recall structural results related to standard words and central words from [17, § 2.2.1]. We then prove a couple of related technical lemmas about binary words that we use in the sequel for the proof of Theorem 2.6 in the case of non-balanced words.

The reversal $x^R$ of a finite word $x = a_0 \cdots a_n$ is $x^R = a_n \cdots a_0$. If $x^R = x$, then $x$ is called a palindrome. It is known that a word $w$ is central if and only if $w$ is a (possibly empty) power of a letter, or is a palindrome which can be written in the form $p10q = q01p$ for some
palindromes $p, q$. Moreover, this factorization is unique. Further, any palindrome prefix or suffix of a central word is central.

Let us recall an important structural result on central words. We say that the word $w = a_1a_2 \cdots a_n$, with $a_i \in \Sigma$, has period $k$ if $a_i = a_{i+k}$ for $i = 1, \ldots, n-k$. Notice that $k = n$ is allowed in this definition.

**Theorem 6.12** ([17, Thm 2.2.11]). A word is central if and only if it has two periods $k$ and $\ell$ such that $\gcd(k, \ell) = 1$ and $|w| = k + \ell - 2$. Moreover, if $w \notin 0^* \cup 1^*$ and $w = p10q$ with $p$ and $q$ palindromes, then $\{k, \ell\} = \{|p| + 2, |q| + 2\}$ and the pair $\{k, \ell\}$ is unique.

We remark the following straightforward consequence of this result.

**Lemma 6.13.** Let $w$ be a central word with $w \notin 0^* \cup 1^*$. Write $w01 = xy$ for some standard pair $(x, y)$. Then, writing $w = p10q$ for unique central words $p$ and $q$, we have $x = p10$ and $y = q01$. Furthermore $wx^R = xw$ and $wy^R = yw$.

**Proof.** Observe that $x \neq 0$ as $w$ would then be a power of a letter. Similarly $y \neq 1$. Hence $x = s10$, $y = t01$, and $w = st010$ for some central words $s$ and $t$. Since this factorization is unique, we have $s = p$ and $t = q$. Further, by the above theorem, $w$ has periods $|x|$ and $|y|$. For the last claim, we observe $xw = p10q01p = wx^R$, and $yw = q01p10q = wy^R$. □

We need the following two technical lemmas to argue about infinite words having distinct sets of factors if they are products of distinct standard pairs. These facts might be known by some experts in Sturmian words, but we were unable to find references for them, so we give proofs for the sake of completeness. In what follows, a factor $w$ of an infinite word $x$ is called **right special**, if $w0, w1 \in \mathcal{L}(x)$. Similarly, $w$ is **left special** if $0w, 1w \in \mathcal{L}(x)$. Finally, $w$ is called **bispecial**, if it is both right special and left special. Also, a set $X$ of binary words is called balanced if $u, v \in X$ with $|u| = |v|$ implies that $||u| - |v|| \leq 1$. It is known that for a balanced set $X$ that is factor closed (i.e., $X = \cup_{x \in \mathcal{L}(x)} \mathcal{L}(x)$), has at most $n + 1$ elements of length $n$, for each $n \in \mathbb{N}$ ([17, Prop. 2.1.2]). This fact will be used in several places of the following two lemmas.

**Lemma 6.14.** Let $x$ be an infinite, recurrent, aperiodic binary word which is not Sturmian. Then there exists a standard pair $(x, y)$ such that some shift of $x$ is a product of $x$ and $y$, and both $xx$ and $yy$ occur in the corresponding factorization. Moreover, the shortest unbalanced pair of factors in $x$ has length $|xy|$.

**Proof.** As $x$ is non-Sturmian and aperiodic, it follows that $x$ contains the factors $0w0$ and $1w1$, where $w$ is a palindrome. Furthermore, $|0w0|$ is the least length for which such an unbalanced pair exists (this fact is implicit in the proof of [17, Prop. 2.1.3]). Notice now that $\mathcal{L}_{|w|+1}(x)$ is balanced by the minimality of $|w|$. In particular, both $0w0$ and $1w1$ are balanced. It follows that $w$ is a right special factor of some Sturmian word $s$. Furthermore, since $w$ is a palindrome, it is even a central word (see [7] or [17, Prob. 2.2.7]).

If $w = 0^n$ for some $n \geq 0$, then $10^n1$ is the shortest block of 0s surrounded by 1s occurring in $x$. Since $x$ is recurrent, some shift $y$ of $x$ begins with $0^n1$. Note that $(0, 0^n1) = (x, y)$ is a standard pair. Now $y$ can be expressed as a product of the words $x$ and $y$ in a unique way. By assumption, both $0^{n+2}$ and $10^n1$ occur in $y$. The former implies that $0^{n+2}1 = xyx$ occurs in the factorization, and the latter implies that $0^n10^n1 = yy$ occurs in the factorization.

We are left with the case that $w \notin 0^* \cup 1^*$. Now we may write $w01 = xy$ for a (unique) standard pair $(x, y)$. Then $x = p10$ and $y = q01$ for some central words $p, q$ by the above
Figure 5: An illustration of the Rauzy graph of order \(|xy|\) of an infinite word obtained as a product of standard pair \((x, y)\). The word \(w\) is the only left (resp. right) special factor among factors of the same length.

The following lemma can be seen as a counterpart of the previous lemma. We need the following celebrated result of M. Morse and G. Hedlund which characterizes ultimately periodic words in terms of the factor complexity function.

Theorem 6.15 (Morse–Hedlund). An infinite word is ultimately periodic if and only if \(P_x(n) = P_x(n + 1)\) for some \(n \in \mathbb{N}\). In this case \(P_x\) is uniformly bounded.

Lemma 6.16. Assume that an infinite binary word \(x\) can be expressed as a product of the standard pair \((x, y)\). Then the set of factors of length less than \(|xy|\) is balanced.

\footnote{The Rauzy graph, or factor graph of order \(n\) has vertex set \(V = L_n(x)\), and there is a directed edge \((u, v)\) if there exist letters \(a, b\) such that \(au = bv \in L_{n+1}(x)\). See §1.3.4 and §2.2.3 of [17] for basic properties of general Rauzy graphs and Rauzy graphs of Sturmian words, respectively.}
Proof. The statement is true when \((x, y) \in \{(0, 0^n1), (1^n0, 1) : n \in \mathbb{N}\}\) by inspection. We may thus assume that \(x = p10\) and \(y = q01\) for some central words \(p\) and \(q\). Let \(w = p10q = q01p\). It follows that \(x\) begins with \(w\). Since \(x\) is aperiodic, the factorization into the standard pair \((x, y)\) contains both the factors \(xy\) and \(yx\). Assume that the former occurs first (the latter case is symmetric) so that the factorization begins with \(x^nxy = (p10)^nq01 = p01q(01p)^n01\) for some \(n \geq 0\). It is now evident that \(x\) begins with \(w\). Furthermore, \(w\) is always followed by \(x^R\) or by \(y^R\). We deduce that the Rauzy graph of order \(|w|\) of \(x\) is as in Figure 5. This implies that the number of factors of \(x\) of length \(|w| + 1\) equals \(|xy| = |w| + 2\). Since \(x\) is aperiodic, it follows by the Morse–Hedlund theorem Theorem 6.15 that \(P_x(n) = n + 1\) for each \(n \leq |w| + 1\).

We claim that the set \(X = \bigcup_{n \leq |w| + 1} L_n(x)\) is balanced. To see this, one can proceed as in [17, Thm. 2.1.5]: If \(X\) is not balanced, then by [17, Prop. 2.1.3], the set contains a palindrome \(w'\) such that \(0w'0, 1w'1 \in X\). Under our assumptions \(|w'| \leq n - 1\). Since \(P_x(k) = k + 1\), for each length \(k < |w| + 1\) there is a unique right special word \(u \in X\) of length \(n\). Observe that any suffix of \(u\) is also right special. Now, since \(w'\) is right special, it follows that either \(0w'\) or \(1w'\) is right special. Assuming that \(0w'\) is right special (so \(1w'\) is not), it follows that \(1w\) is always followed by \(1\). Letting \(v\) be a word such that \(1w'1v \in L_{2|w'|}(x)\). It can be shown that none of the factors of length \(|0w'|\) of \(1w'1v\) are right special (i.e., \(0w'\) does not occur in \(1w'1v\)). This, further, can be shown to imply that \(x\) is ultimately periodic. This contradiction concludes the proof.

7. Abelian closures of non-balanced words

To conclude the proof of Theorem 2.6, we consider the case of non-balanced words.

**Proposition 7.1.** Let \(x \in \{0, 1\}^\mathbb{N}\) be a uniformly recurrent, non-balanced word. Then \(A(x)\) contains infinitely many minimal subshifts.

We first make a straightforward observation related to irrational letter frequencies and morphisms.

**Recall that \(\Psi(u)\) is the Parikh vector of \(u\).**

**Definition 7.2.** A morphism \(f: \{0, 1\}^* \to \{0, 1\}^*\) is called degenerate if \(\Psi(f(0))\) and \(\Psi(f(1))\) are linearly dependent. Otherwise it is called non-degenerate.

Notice that any erasing morphism is degenerate. On the other hand, any Sturmian morphism \(\varphi\) is non-degenerate. Indeed, it can be shown, by induction on the length of a defining sequence of generators, that \(\gcd(|\varphi(0)|, |\varphi(1)|) = 1\). This suffices for non-degeneracy, as can be established with elementary properties of integers and the fact that \(|\varphi(0)|_a \geq 1\) for \(a = 0, 1\).

The following lemma is immediate.

**Lemma 7.3.** Let \(f\) be a degenerate morphism. Then, for all \(u\) for which \(f(u) \neq \varepsilon\), we have \(\text{freq}_{f(u)}(1) = C\) for some rational constant \(C\).

On the other hand, if \(f\) is non-degenerate (hence it is non-erasing), there is a one-to-one correspondence between frequencies of a word and its image. This can be seen as follows: The adjacency matrix \(M_f\) of \(f\) is defined as

\[
M_f = \begin{pmatrix}
|f(0)|_0 & |f(1)|_0 \\
|f(0)|_1 & |f(1)|_1
\end{pmatrix}.
\]
It is straightforward to check that $\Psi(f(u))^\top = M_f \Psi(u)^\top$ (where $\Psi(u) = (|u|, |u|_1)$). Furthermore, $M_f$ is invertible if and only if $f$ is non-degenerate. Hence, for any non-degenerate morphism $f$ and an image word $f(u)$, we can compute $\Psi(u)$ from $M_f^{-1} \Psi(f(u))^\top$. We may also compute \( \left( \frac{\text{freq}_{f(v_n)}(0)}{\text{freq}_{f(v_n)}(1)} \right) = \frac{|v_n|}{|f(v_n)|} M_f \left( \frac{\text{freq}_{v_n}(0)}{\text{freq}_{v_n}(1)} \right) \).

**Lemma 7.4.** Let $f : \{0,1\}^* \to \{0,1\}^*$ be a morphism, and let $y \in \{0,1\}^*$. Assume that $f(y) = z$ has irrational uniform letter frequencies. Then $y$ has irrational uniform letter frequencies. Furthermore, if $z$ is non-balanced, then so is $y$.

**Proof.** Let $\text{freq}_{f(y)}(1) = \alpha$ be irrational. Observe that, for a degenerate morphism $f$, $f(y)$ has rational uniform letter frequencies, as can be established by the above lemma. Hence $f$ is non-degenerate and, in particular, non-erasing.

Let $\text{freq}_{y}(1) = \beta$ and $\text{freq}_{y}(1) = \beta'$ for some numbers $\beta, \beta' \in [0,1]$. Let $(v_n)_n$ be a sequence of factors of increasing length of $y$ testifying the former limit frequency. Then we have $\lim_{n \to \infty} \text{freq}_{f(v_n)}(1) = \alpha$. On the other hand

\[
\left( \frac{\text{freq}_{f(v_n)}(0)}{\text{freq}_{f(v_n)}(1)} \right) = \frac{|v_n|}{|f(v_n)|} M_f \left( \frac{\text{freq}_{v_n}(0)}{\text{freq}_{v_n}(1)} \right).
\]

By a straightforward computation, we have $\lim_{n \to \infty} \frac{|v_n|}{|f(v_n)|} = \frac{1}{(|f(0)| + |f(1)| - |f(0)||\beta)} \in (0,1)$ under our assumption on $(v_n)_n$. Since the mapping $M_f$ is continuous, we deduce that

\[
\frac{1 - \alpha}{\alpha} = \frac{1}{|f(0)| + |f(1)| - |f(0)||\beta} M_f \left( \frac{1 - \beta}{\beta} \right).
\]

It is immediate that $\beta$ is irrational. Further, we get $M_f^{-1} \left( \frac{1 - \alpha}{\alpha} \right) = \frac{1}{|f(0)| + |f(1)| - |f(0)||\beta} \left( \frac{1 - \beta}{\beta} \right)$. Notice now that the same computations can be performed on the sequence of factors testifying the latter limit frequency $\beta'$, only $\beta$ is replaced with $\beta'$. As a consequence, we have $h(\beta) = h(\beta')$ for the linear fractional transformation $h(x) = \frac{x}{x + |f(1)| - |f(0)||\beta}$. One can check that $h$ is invertible (since $f$ is non-vanishing), so we conclude that $\beta = \beta'$. We have shown that $y$ has irrational uniform letter frequencies.

We then show that if $y$ is $C$-balanced for some $C$, then necessarily $z$ is $C'$-balanced for some $C'$. Let $u$ and $v$ be equal length factors of $z$. There exist factors $x, y$ of $y$ of minimal length for which $u$ is a factor of $f(x)$ and $v$ is a factor of $f(y)$. As the length of $f(x)$ is bounded below by $|u|$ and above by $|u| + D$ by some constant $D$ (take, e.g., $D = 2|f(01)|$, we have $0 \leq |f(x)|_1 - |u|_1 \leq D$. Similarly $0 \leq |f(y)|_1 - |v|_1 \leq D$. Thus, establishing a uniform bound on $||f(x)||_1 - |f(y)||_1$, (where $x$ and $y$ correspond to equal length factors of $z$) suffices to conclude the claim. Assume without loss of generality that $|x| \geq |y|$, and write $x = x'z$ with $|x'| = |y|$. Hence $\Psi(x) = \Psi(x') + \Psi(z)$. Observe now that

\[
D \geq |f(x)| - |f(y)| = |f(z)| + |f(x')| - |f(y)| = |f(z)| + (M_f(\Psi(x') - \Psi(y)), (1,1)),
\]

where $\langle \cdot, \cdot \rangle$ denotes the inner product. Recall we assume that $y$ is $C$-balanced. Hence the elements of $\Psi(x') - \Psi(y)$ have absolute value bounded by $C$. There are finitely many such integral points, and hence we have $(M_f(\Psi(x') - \Psi(y)), (1,1))$ is in some bounded interval $[-D', D']$ for some positive number $D'$ which depends on $f$ and $C$ alone. Therefore the right hand side is bounded below by $|z| - D'$ (since $f$ is non-erasing). We conclude that $|z| \leq D + D'$. 

21
Similarly we have
\[ |f(x)|_1 - |f(y)|_1 = |f(z)|_1 + (M_f(\psi(x') - \psi(y)), (0, 1)). \]
Again, the value \( (M_f(\psi(x') - \psi(y)), (0, 1)) \) is uniformly bounded due to the \( C \)-balancedness of \( y \). Here \( |f(z)|_1 \) is (crudely) bounded above by \( (D + D')|f(01)| \), so we conclude that \( |f(x)|_1 - |f(y)|_1 \leq (D + D')|f(01)| + D'' \), for a constant \( D'' \) depending solely on \( f \) and \( C \). This concludes the proof. \( \square \)

We have now developed sufficient tools to prove Proposition 7.1.

In what follows, \( x \) is a uniformly recurrent, non-balanced binary word having irrational letter frequencies. We may assume that \( \text{freq}_{x}(1) < 1/2 \) and, without loss of generality, all 1s are isolated in \( x \). Otherwise, by Proposition 6.10, there exists a word in \( \mathcal{A}(x) \) with this property, and we may argue about its abelian closure.

Our aim is to define, for each \( n \geq 0 \), a uniformly recurrent word \( x_n \) in \( \mathcal{A}(x) \). These words define pairwise distinct shift orbit closures in \( \mathcal{A}(x) \), which suffices for the claim.

For the construction, we actually define three sequences of words, \( (x_n)_{n \geq 0} \), \( (y_n)_{n \geq 1} \), and \( (z_n)_{n \geq 0} \), as well as two sequences \( (\psi_n)_{n \geq 0} \) and \( (\varphi_n)_{n \geq 1} \) of standard morphisms recursively. This will help to keep track of the properties we need for the conclusion. The entities satisfy the following properties for all \( n \geq 0 \).

1. \( y_{n+1} \) is non-balanced, uniformly recurrent, has \( \text{freq}_{y_1}(1) \) irrational and less than 1/2, and it contains both 00 and 11.
2. \( \psi_{n+1} = \psi_n \circ \varphi_{n+1} \) and \( \varphi_{n+1} \) is a non-trivial (meaning \( \varphi_{n+1}(01) \geq 3 \)) standard morphism.
3. \( z_n = \varphi_{n+1}(y_{n+1}) \).
4. \( z_n \) is non-balanced, uniformly recurrent, has \( \text{freq}_{z_1}(1) \) irrational and less than 1/2.
   Further, all 1s are isolated.
5. \( z_{n+1} \in \mathcal{A}(y_{n+1}) \).
6. \( \varphi_{n+1}(z_{n+1}) \in \mathcal{A}(z_n) \).
7. \( \psi_n(z_n) = x_n \).
8. \( x_n \in \mathcal{A}(x) \).

First we set \( x_0 = z_0 = x \) and \( \psi_0 = \text{id} \). The above list of properties concerning these entities hold immediately. The definitions of \( y_1 \) and \( \varphi_1 \) are evident from the construction that follows. The construction is depicted in Figure 6.

Assume then that \( x_n, z_n, \psi_n \) are defined and satisfy the above properties. We shall construct \( y_{n+1} \) and \( \varphi_{n+1} \) from these entities, so the knowledge of \( y_n \) and \( \varphi_n \) are not needed. Let us do this first. Since \( z_n \) is non-balanced and uniformly recurrent, by Lemma 6.14 there exists a standard pair \((x, y)\) such that a shift of \( z_n \) is a product of the words \( x, y \) and contains both \( xx \) and \( yy \) in the factorization. Let us denote this shift by \( z' \). Consider the morphism \( \varphi \) defined by \( 0 \mapsto x \) and \( 1 \mapsto y \). As \( z' \) is a product of \( x \) and \( y \), there exists a word \( y \) such that \( \varphi(y) = z' \). Observe now that \( y \) contains both 00 and 11. Because \( z' \) has irrational uniform letter frequencies and is non-balanced, \( y \) shares these properties by Lemma 7.4. We may assume that \( \text{freq}_{y_1}(1) < 1/2 \), otherwise we replace \( \varphi \) by \( \varphi \circ E \). Now set \( y_{n+1} = y \) and \( \varphi_{n+1} = \varphi \), which is a standard morphism by Theorem 6.2. It is also non-trivial, since \( y_{n+1} \) contains both 00 and 11 while \( z_n \) does not. We have thus established items 1, 2, and 3 in the above list. For the remainder of the construction, we omit the subscript from \( y_{n+1} \) for the sake of readability.
We then define $z_{n+1}$. By Corollary 6.11 there exists a non-balanced word in $\mathcal{A}(y)$ in which all 1s are isolated. We set $z_{n+1}$ to be such a word. Now $z_{n+1}$ is uniformly recurrent, has irrational uniform letter frequencies with $\text{freq}(1) < 1/2$ and is non-balanced, as $y$ has these properties. These observations establish item 4 and item 5. Further, since $z_{n+1} \in \mathcal{A}(y_{n+1})$, by Proposition 6.3 it follows that $\varphi_{n+1}(z_{n+1}) \in \mathcal{A}(\varphi_{n+1}(y)) = \mathcal{A}(z_n)$. This establishes item 6. We finally define $x_{n+1} = \psi_{n+1}(z_{n+1})$ in accordance with item 7.

Let us show that $x_{n+1}$ satisfies item 8. By Proposition 6.3, we have that $\varphi_{n+1}(z_{n+1}) \in \mathcal{A}(\varphi_{n+1}(y)) = \mathcal{A}(z_n)$. Applying Proposition 6.3 again, this time to $\varphi_{n+1}(x)$ and $z_n$ with $\psi_n$, we find

$$x_{n+1} = \psi_{n+1}(z_{n+1}) = \psi_n \circ \varphi_{n+1}(z_{n+1}) \in \mathcal{A}(\psi_n(z_n)) = \mathcal{A}(x_n) \subseteq \mathcal{A}(x),$$

as $x_n$ satisfies item 8 was assumed.

The following lemma combined with item 8 proves Proposition 7.1 immediately.

**Lemma 7.5.** For all $m \neq n$, we have $\Omega(x_n) \cap \Omega(x_m) = \emptyset$.

**Proof.** We show that the words have distinct factor sets. This suffices for the proof, since the words $x_n$ are uniformly recurrent (by items 7 and 4). Consider a fixed index $n \geq 0$. Combining items 7, 3, and 2, we have $x_n = \psi_n(z_n) = \psi_n(\varphi_{n+1}(y_{n+1})) = \psi_{n+1}(y_{n+1})$. By item 1, $y_{n+1}$ contains both 00 and 11. Thus $x_n$ is a product of the factors $\psi_{n+1}(0) = x_n$ and $\psi_{n+1}(1) = y_n$, and both $x_n x_n$ and $y_n y_n$ occur in $x_n$. By Theorem 6.2 $(x_n, y_n)$ (or $(y_n, x_n)$) is a standard pair. Further, by Lemma 6.14 the shortest unbalanced pair of factors has length $|x_n y_n|$, and by Lemma 6.16, the factors of length less than $|x_n y_n|$ form a balanced set.

To conclude the proof, it suffices to show that $|x_{n+1} y_{n+1}| > |x_n y_n|$ for all $n \geq 0$. By item 2, we find $\psi_{n+1} = \psi_n \circ \varphi_{n+1}$. We note that $|x_{n+1} y_{n+1}| > |x_n y_n|$, where $|x_{n+1} y_{n+1}| \geq 1$. Furthermore,
one of these values is at least 2 since \( \varphi_{n+1} \) is non-trivial. We thus have

\[
|x_{n+1}y_{n+1}| = |\psi_n(\varphi_{n+1}(01))| = |x_{n+1}y_{n+1}|0 \cdot |x_n| + |x_{n+1}y_{n+1}|1 \cdot |y_n| > |x_ny_n|.
\]

This concludes the proof. \qed

8. Some remarks on alternative approaches

In this section we discuss some alternative approaches to the results presented in the preceding sections. We first show that a large family of words with irrational letter frequencies contain uncountably many minimal subshifts in their abelian closures. We then show that trying to apply to Proposition 7.1 the approaches from the proofs of the other cases does not give the result in the full generality, although shows stronger results in particular cases, as well as demonstrates some new phenomena.

8.1. On abelian closures with uncountably many minimal subshifts

In this subsection we show that certain words \( x \) with irrational letter frequencies have uncountably many minimal subshifts in their abelian closures. We apply methods from the proof of Proposition 4.2. Notice however, that this does not give a stronger version of Propositions 5.1 or 7.1 in full generality.

**Proposition 8.1.** Let \( x \) be uniformly recurrent with \( \text{freq}_x(1) = \alpha \) with \( \alpha \) irrational. Assume further that, for each \( n \geq n_0 \) for some \( n_0 \in \mathbb{N} \), it contains factors of length \( n \) with one having weight \( \lceil n\alpha \rceil + 1 \) and another having weight \( \lfloor n\alpha \rfloor - 1 \). Then \( A(x) \) contains uncountably many minimal subshifts.

We may assume without loss of generality that \( \alpha < 1/2 \). We construct a family of words \( x \) as follows.

Let \( c \) be the characteristic Sturmian word of slope \( \alpha \). Let \( (a_n)_{n \geq 1} \) be the corresponding directive sequence, and \( (S_n)_{n \geq -1} \) the standard sequence. Recall that \( S_n = S_{n-1}S_{n-2} \) for each \( n \geq 1 \). We shall consider a modification of this sequence as follows.

Notice that \( c \in A(x) \) by the Corridor Lemma. We aim to "spread" the graph of \( c \) around the line \( y = \alpha x \) so that the obtained word is also in \( A(x) \). This is the part where we need extra room around the slope \( \alpha x \), which is granted by the assumptions. To this end, let \( k \geq 3 \) be such that \( |S_{k+1}| = |S_{k}a_{k+1}S_{k-1}| > n_0 \) (notice that \( |S_{k-1}| \geq 2 \) for \( k \geq 3 \)). Now \( c \) is a product of the words \( S_k \) and \( S_{k-1} \): let us write

\[
c = \prod_{i=0}^{\infty} S_k^{n_i} S_{k-1}.
\]

Here \( n_i \) is one of the two numbers \( a_{k+1}, a_{k+1} + 1 \), for each \( i \geq 0 \). Let \( \# \) denote the operation which flips the last two letters of a given word (of length at least two).

**Claim 7.** Let \( (b_i)_{i \geq 0} \) be a \( 0,1 \)-sequence. Then the word \( x' \) defined by

\[
x' = \prod_{i=0}^{\infty} S_k^{n_i} \#^{b_i}(S_{k-1})
\]

is in \( A(x) \).
Lemma 3.10. Assume for simplicity that $\theta$ and our choice of whether $1$ is only attained with $s$ and $1$ is only attained with $s'$, and express it as $u = spp$, where $P$ is a finite sub-product of (2) and $s$ (resp., $p$) is a proper suffix (resp., prefix) of the previous (resp., following) term. Consider the corresponding factor $s'p'$ from $s$. We have $|u|_1 - |v|_1 = |s|_1 - |s'|_1 + |p|_1 - |p'|_1$. Notice that $|s|_1 - |s'|_1 \leq 1$, and $1$ is only attained with $s = 0$ (and thus $s' = 1$). Similarly $|p|_1 - |p'|_1 \leq 1$ and $1$ is attained only when $p = S_k^{\alpha}g(S_{k-1})0^{-1}$. If both happen simultaneously, then $|u|_1 - |v|_1 = 0$. Consequently, $|u|_1 - |v|_1 \leq 1$. Hence $|n\alpha| - 1 \leq |u|_1 \leq |n\alpha| + 1$. By Lemma 3.10 and our assumptions on $x$, $x' \in A(x)$. We are going to prove that there are uncountably many $0$-1-sequences $(b_n)_{n=0}^\infty$ such that the corresponding words of the form (2) have distinct sets of factors. One of the ways to do this is using yet another characterization of Sturmian words via rotations.

We identify the interval $[0,1)$ with the unit circle $T$ (the point $1$ is identified with point $0$). For points $x, y \in T$, we let $I(x,y)$ denote the half-open interval on $T$ starting from $x$ and ending at $y$ in counter-clockwise direction (in most of the arguments it will not be important which endpoint is in the interval). Let $\alpha \in T$ be irrational and let $\rho \in T$. The map $R_{\alpha}: T \rightarrow T$, $x \mapsto \{x + \alpha\}$, where $\{x\} = x - [x]$ denotes the fractional part of $x \in \mathbb{R}$, defines a (counter-clockwise) rotation on $T$. Partition $T$ into two half-open intervals $I_0 = I(0,1 - \alpha)$ and $I_1 = I(1 - \alpha, 1)$ (so the endpoints $1 - \alpha$ and $0 = 1$ are in different partitions), and define the coding $\nu: T \rightarrow \{0,1\}$, $x \mapsto i$ if $x \in I_i$, $i = 0, 1$. The rotation word $s_{\alpha,\rho}$ of slope $\alpha$ and intercept $\rho$ is the word $a_0a_1\cdots \in \{0,1\}^\mathbb{N}$ defined by $a_n = \nu(R_{\alpha}^n(\rho))$ for all $n \in \mathbb{N}$.

Note that $00$ occurs in $s_{\alpha,\rho}$ if and only if $\alpha < 1/2$. Clearly, $s_{\alpha,\rho}$ is aperiodic as $\alpha$ is irrational. Each aperiodic rotation word is a Sturmian word and vice versa (regardless of the choice of whether $1 \in I_0$ or $0 \in I_0$). For each length $n$, one can partition the interval $[0,1)$ into $n + 1$ subintervals, each of which corresponds to a factor of the Sturmian word. More precisely, we can find when $v = b_1\cdots b_n$ occurs in $s$ at position $i$:

$$v = s_is_{i+1}\cdots s_{i+n-1} \Leftrightarrow R_{\alpha}^i(\rho) \in I_v,$$

where

$$I_v = I_{b_1} \cap R_{\alpha}^{-1}(I_{b_2}) \cap \cdots \cap R_{\alpha}^{-n+1}(I_{b_n}).$$

Example 8.2. In Figure 7a we have an example of a rotation system corresponding to a Sturmian word of slope $\alpha$. Here we assume that $3\alpha < 1 < 4\alpha$. The intervals defined by the points $\{-i\alpha\}$, $i = 0,\ldots,4$, define the factors of length 4 as follows: $I(0,\{-3\alpha\})$ corresponds to the factor $0^4$, $I(\{-3\alpha\},\{-2\alpha\})$ corresponds to $0^210$, $I(\{-2\alpha\},\{-\alpha\})$ corresponds to $0100$, $I(\{-\alpha\},\{-4\alpha\})$ corresponds to $10^3$, and $I(\{-4\alpha\},1)$ corresponds to $1001$.

Observe the special role played by $s_{\alpha,\alpha} = c$: both $01s$ and $10c \in \Omega(c)$ for any $\alpha \in (0,1)$. This follows from the fact that we may choose first that $0 \in I_0$ in which case $1 \in I_1$. Then $s_{\alpha,\alpha} = 01c$. The choice $1 \in I_0$ gives $s_{\alpha,\alpha} = 10c$.

Let us assume for simplicity that $S_{k-1}$ ends with $01$ for the remainder of this subsection. We claim that each occurrence of $01w$ in (1) (with $w01 = S_{k+1}$) starts from the second to last
Theorem 6.12

Lemma 6.13

1. \( w \) tells how consecutive occurrences of 01 in the factorization (1). Indeed, by Theorem 6.12 \( w \) has two periods: \(|S_k|\) and \(|S_k^{a_{k+1}-1}S_{k-1}|\) and further Lemma 6.13 tells how consecutive occurrences of \( w \) appear. So in (1) occurrences of \( w \) correspond to prefixes of each block \( S_k^{a_i}S_{k-1} \) (preceded by 0) and to factors starting from position \(|S_k|\) (also \(|S_k|\) if \( n_i = a_{k+1} + 1 \)) of the factor \( S_k^{a_i}S_{k-1} \). The latter occurrences of \( w \) are preceded by 10.

Claim 8. There are uncountably many 0-1-sequences \((b_n)_{n=0}^{\infty}\) so that the sequences of the form (2) have distinct sets of factors.

Proof. Take the Sturmian word \( c \) as defined above. Consider now the interval \( I_{01w} \) corresponding to the factor 01w in \( c \). We have \( I_{01w} = I(a, 1 - \alpha) \) for some \( a < 1 - \alpha \). Since both 0w0 and 1w0 occur in \( c \), it follows that \( I_{w0} \) contains the point \( \alpha \). Now \( I_1 \cap R_{a}^{-1}(I_{w0}) = I_{1w} \) and is of the form \( I(a + \alpha, 1) \) for some \( a < 1 - \alpha \). Hence \( I_{01w} = R_{a}^{-1}(I_{1w}) = I(a, 1 - \alpha) \). Observe now that each time the orbit of \( c \) hits the interval \( I_{01w} \), it synchronizes with the factorization (1) as describe in the above discussion. Let us modify the coding \( \nu \) to \( \nu' \) in such a way that allows to flip the last two letters of \( S_{k-1} \) to obtain a word of the form (2). Take a subinterval \( J \) of \( I_{01w} \) that does not have \( 1 - \alpha \) as an endpoint. Then \( J' = R(J) \) is a subinterval of \( 1w \) that does not have 1 as an endpoint. Define \( \nu' : T \to \{0, 1\} \) by \( \nu'(x) = 1 \) if \( x \in J \) or if \( x \in I_1 \setminus J' \). Similarly \( \nu'(x) = 0 \) if \( x \in J' \) or if \( x \in I_0 \setminus J \). So the coding \( \nu' \) partitions the torus into six subintervals: letting \( J = I(a, b) \), the intervals are in anti-clockwise order

- \( I(0, a) \) (\( \mapsto 0 \) under \( \nu' \)),
- \( I(a, b) = J \) (\( \mapsto 1 \)),
- \( I(b, 1 - \alpha) \) (\( \mapsto 0 \)),
- \( I(1 - \alpha, a + \alpha) \) (\( \mapsto 1 \)),
- \( I(a + \alpha, b + \alpha) = J \) (\( \mapsto 0 \)),
- \( I(b + \alpha, 1) \) (\( \mapsto 1 \)).

Code now the orbit of the point \( \alpha \) under \( \nu' \) to obtain an infinite word \( t \). This coding acts the same as the coding under \( \nu' \), except when the orbit hits a point in \( J \). The factor starting from this interval is 10w (as opposed to 01w in \( c \)). It is now immediate that \( t \) is of the form (2).

We claim that varying the length of \( J \) we get uncountably many minimal subshifts. We use the notion of factor frequency, generalizing letter frequencies. The uniform frequency of
a factor $z$ of $y$ is defined as the limit \( \operatorname{freq}_y(z) = \lim_{N \to \infty} \frac{|v_N|_z}{N} \) when it exists, uniformly over \( \{v_N\}_{n=0}^\infty \) being any sequence of factors of $y$ with $|v_N| = N$, where $|v|_z$ is the number of occurrences of $z$ as a factor in $v$. It can be seen that for any Sturmian word $y$ and for any finite word $z$, the frequency $\operatorname{freq}_y(z)$ exists and is equal to the length of the corresponding interval on the torus [17, §2.2.3]. Using precisely the same argument, one can see that the frequency of any factor of $t$ exists and equals the length of the corresponding interval/set on the corresponding torus. In particular, the frequency of the factor $1w1$ exists. An occurrence of $1w1$ corresponds exactly to an occurrence of $S_{k-1}:S_k^{qk+1}\mathcal{F}(S_{k-1})$ in (2) (choosing $J$ appropriately gives occurrences of this form). Indeed, $1w1$ does not occur in $c$ so it must overlap with at least one of the last two letters of an occurrence of $\mathcal{F}(S_{k-1})$. If it overlaps both letters, then it is a factor of $S_k^{qk+1}\mathcal{F}(S_{k-1})S_k^{qk+1}S_{k-1}(01)^{-1} = S_kS_k^{qk+1}S_{k-1}(01)^{-1}$ which is a factor of $c$ contradicting balancedness. So it overlaps only one of the last two letters, so we deduce that it occurs as a prefix of $1S_k^{qk+1}\mathcal{F}(S_{k-1})$. Since both $S_k$ and $\mathcal{F}(S_{k-1})$ end with 10 we deduce that the prefix 1 ends an occurrence of $S_{k-1}$ in the factorization (2) as claimed.

Now the frequency of $1w1$ in $t$ equals the size of the corresponding set on the torus, which can be varied continuously. Choosing $J$ appropriately gives words with different frequencies of $1w1$, which must have distinct sets of factors. The claim follows. 

8.2. Additional remarks on the structure of the abelian closures of non-balanced binary words

In this subsection we give a geometric proof of a weaker version of Proposition 7.1 and show that the abelian shift orbit closure of a uniformly recurrent word can contain non uniformly recurrent words of a quite complicated structure:

**Proposition 8.3.** Let $x$ be a binary uniformly recurrent word which is not Sturmian. Suppose in addition that $x$ is non-balanced and that the frequency $\alpha$ of 1 exists and it is irrational. Then $\mathcal{A}(x)$ contains infinitely many non uniformly recurrent words with distinct languages, such that none of their tails is uniformly recurrent.

We remark that this proposition does not guarantee infinitely many minimal subshifts in the abelian closure, since these words with distinct languages can have the same languages of uniformly recurrent points in their shift orbit closure.

To prove this proposition, we again make use of graphs of words, as well as squeezing operations. First we need the following lemma, which is a slight modification of item 3 of Theorem 3 from [1]:

**Lemma 8.4.** Let $w$ be a binary uniformly recurrent word with frequency of 1 equal to $\alpha$. Then there exists $u \in \Omega(w)$ such that $g_u$ intersects the line $y = \alpha x$ infinitely many times.

**Proof.** In the proof we use the notion of a return word. For $u \in \mathcal{L}(w)$, let $n_1 < n_2 < \ldots$ be all integers $n_i$ such that $u = w_{n_1} \ldots w_{n_i+1}$. Then the word $w_{n_1} \ldots w_{n_i+1}$ is a first return word (or briefly first return) of $u$ in $w$ [8, 11, 25]. We can also consider a second (third, etc.) return as a factor having exactly two occurrences of $u$, one of them being a prefix, and ending just before the next occurrence of $u$.

We now build a word $u$ as a limit of factors of $w$. Start with any factor $u_1$ of $w$, e.g. with a letter. Without loss of generality assume that $\operatorname{freq}_{u_1}(0) \geq \operatorname{freq}_{w}(0) = \rho_0$. Consider the factorization of $w$ into returns to $u_1$: $w = v_1^{(1)}v_2^{(1)} \ldots v_i^{(1)} \ldots$, so that $v_i^{(1)}$ is a return to $u_1$ for $i > 1$. We assume that all $v_i^{(1)}$ are longer than $u_1$, taking second returns (or third returns etc. if necessary). Then there exists $i_1 > 1$ satisfying $\operatorname{freq}_{v_1^{(1)}}(0) \geq \rho_0$. Suppose the converse holds,
i.e., for all $i > 1$ freq$^{(1)}_w(0) < \rho_0$. Due to uniform recurrence, the lengths of $v_i^{(1)}$ are uniformly bounded, and hence freq$^w_w(0) < \rho_0$, a contradiction. Take $u_2 = v_1^{(1)}$, so $u_1$ is a prefix of $u_2$. Now consider a factorization of $w$ into returns to $u_2$: $w = v_1^{(2)} v_2^{(2)} \cdots v_i^{(2)} \ldots$. Then there exists $i_2 > 1$ satisfying freq$^{(2)}_w(0) \leq \rho_0$; take $u_3 = v_i^{(2)}$. Continuing this line of reasoning to infinity, we build a word $u = \lim_{n \to \infty} u_i \in \Omega(w)$, such that freq$^{u_{2i}}(0) \geq \rho_0$, freq$^{u_{2i+1}}(0) \leq \rho_0$. So, the graph of $u$ intersects the line $y = \alpha x$ infinitely many times as was claimed.

Example 8.5. For $w = s_{0,\alpha}, g_w$ does not intersect $y = \alpha x$ infinitely many times. Taking for example $u = s_{\alpha,\alpha}$, we already have infinitely many intersections.

Let $u \in \Omega(w)$ be a word satisfying Lemma 8.4 and let $C \in \mathbb{R}, C > 1$. We define an operation of $C$-squeezing of $u$, $u' = s_C(u)$ as follows. For each $i$ such that $g_u(i) > \alpha i + C$ and $u_{i-1} = 1, u_i = 0$, we define $u'_{i-1} = 0, u'_i = 1$. Symmetrically, if $g_u(i) < \alpha i - C$ and $u_{i-1} = 0, u_i = 1$, we define $u'_{i-1} = 1, u'_i = 0$. In these cases we say that we have a switch at position $i$. Otherwise we define $u'_i = u_i$. Informally, this means that if there is a piece of graph outside the stripe between the lines $y = \alpha x - C$ and $y = \alpha x + C$, we make local changes in this piece getting this part of the graph closer to the stripe. Essentially, the operation is similar to upper squeezing, only we squeeze symmetrically from both sides and leave the stripe between the two lines $y = \alpha x \pm C$ unchanged. See Figure 8.

Clearly, similarly to upper squeezing (Claim 1), the operation of squeezing does not change frequency.

The following Lemma generalizes Lemma 5.3 for squeezing from both sides:

**Lemma 8.6.** Let $u' = s_C(u)$, where $u$ is as in Proposition 8.3. Then $u' \in A(u)$.

**Proof.** The proof is similar to the proof of Lemma 5.3, although here we have to consider more cases (for convenience of the reader, we repeat some arguments to have a complete proof here).
Assume the converse. Then, due to the Corridor Lemma there exists a factor $u'_i \cdots u'_{j-1}$ such that for each $k$ we have $|u'_i \cdots u'_{j-1}| > |u_k \cdots u_{k+j-i-1}|$—Assumption (*) (the case of $<$ is symmetric).

First we remark that the switches inside the factor (at positions $i+1, \ldots, j-1$) do not change the Parikh vector of the factor. So, to change Parikh vector, we must have a switch at position $i$ or/and $j$.

Secondly, note that we have a switch at position $\ell$ if and only if $g_u(\ell) \neq g_u'(\ell)$.

1. Switch at $i$ and not in $j$.
   
   If $g_u(i) < \alpha i - C$, then $u'_i = 0$, $u_i = 1$ and considering $k = i$, we get $|u_i \cdots u_{j-1}| > |u'_i \cdots u'_{j-1}|$, which is not possible by our Assumption (*).

   The case $g_u(i) > \alpha i + C$ is the same as Case 1 from the proof of Lemma 5.3: we then have $u'_i = 1$, $u_i = 0$. The only possibility is that $|u'_i \cdots u'_{j-1}| > [\alpha(j-i)]$ (since otherwise the factor $u'_i \cdots u'_{j-1}$ is in the set of abelian factors of $u$). This in turn means that $g_u'(j) > \alpha j + C$ (due to frequency). By the conditions of Case 1 we have that $g_u'(j) = g_u(j)$, which means that $u_j u_{j-1} \neq 10$. If $u_{j-1} = 0$, then we taking $k = i - 1$ we get an abelian equivalent factor in $u$ (see Figure 3a). If $u_j = 1$, then we can take $k = i + 1$ (see Figure 3b).

2. Switches at both $i$ and $j$.
   
   If $g_u(i) < \alpha i - C$ and $g_u(j) < \alpha j - C$ or $g_u(i) > \alpha i + C$ and $g_u(j) > \alpha j + C$, then the Parikh vector does not change (we can take $k = i$).

   If $g_u(i) < \alpha i - C$ and $g_u(j) > \alpha j + C$ or $g_u(i) > \alpha i + C$ and $g_u(j) < \alpha j - C$, then the new Parikh vector gets “closer” to the frequency, so it is evident that it belongs to the set of Parikh vectors of $u$.

3. switch at $j$ and not in $i$. The case is symmetric to Case 1.

\[ \square \]

Proof of Proposition 8.3. We may assume without loss of generality that all 1s are isolated in $x$ by Corollary 6.11. Further, we may assume that the graph of $x$ intersects the line $y = \alpha x$ infinitely often by Lemma 8.4. Consider the $C$-squeezing operation on $x$: since all 1s are isolated, the $C$-squeezing operation acts like the shift operation $\sigma$ on the parts that are outside the strip. Since the graph must contain arbitrarily long parts outside the stripe, each factor of $x$ is contained in the $s_C(x)$.

Consider a factor $w$ of $x$ that is not $2C$-balanced. It occurs within bounded gaps in $x$, and all iterations of $s_C$ on $x$. Notice though that the gaps could grow in length. Now iterating the $C$-squeezing operation, we get arbitrarily long prefixes that are $2C$-balanced. So we get longer and longer gaps. This means that the longest $2C$-balanced factors grow in length, when iterating $s_C$. Since each of them is contained in $A(x)$, the claim follows.

\[ \square \]

9. Conclusions and open problems

In this paper, we studied a notion of abelian closures of infinite binary words. An interesting open question is to characterize words for which $A(x) = \Omega(x)$. Among uniformly recurrent binary words, this property gives a characterization of Sturmian words, but the characterization does not extend to usual generalizations of Sturmian words over non-binary alphabets: neither for balanced words, nor for words of minimal complexity, nor for Arnoux-Rauzy words [10, 13, 24].
Open Problem 1. Find a characterization of the property $\Omega(x) = A(x)$ for nonbinary alphabets.

A modification of this question is to characterize words for which $A(x)$ contains exactly one minimal subshift.

Another question to study concerns abelian closures of binary words and Theorem 2.6. We showed that the abelian closures of uniformly recurrent binary words contain infinitely many minimal subshifts. We also showed that in fact there are uncountably many minimal subshifts unless the frequency exists and it is irrational. The proof is quite technical and consists of four parts relating to the cases of rational frequency, no letter frequencies, balanced and unbalanced words with irrational letter frequencies, and the proofs of all the parts rely on different methods. It would be interesting to try to find a proof treating all cases at once and giving a stronger result of uncountably many minimal subshifts in all the cases:

Open Problem 2. Find a shorter proof of Theorem 2.6. Does it hold if we substitute “infinitely many” by “uncountably many” in the case of irrational frequencies?

We remark that we were able to prove the problem for a wide class of such words (see Proposition 8.1). In fact, by Propositions 4.1, 4.2, and 8.1, if there is a uniformly recurrent binary word $x$ for which $A(x)$ contains infinitely, but only countably many minimal subshifts, then it must have irrational letter frequency $\alpha$ and, further, $\min_{v \in \mathcal{L}_n(x)} |v|_1 = \lfloor \alpha n \rfloor$ or $\max_{v \in \mathcal{L}_n(x)} |v|_1 = \lceil \alpha n \rceil$ for infinitely many $n$.

A quantitative version of the above question would be ”Does the abelian closure of a uniformly recurrent non-Sturmian aperiodic binary word have positive entropy?” See, e.g., [15] for a definition of entropy. The proof of Proposition 4.2 implies that the entropy is positive for words with rational letter frequencies. This can be translated to the case of no frequencies also.

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