Holographic models with anisotropic scaling

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Abstract. We consider gravity duals to d+1 dimensional quantum critical points with anisotropic scaling. The primary motivation comes from strongly correlated electron systems in condensed matter theory but the main focus of the present paper is on the gravity models in their own right. Physics at finite temperature and fixed charge density is described in terms of charged black branes. Some exact solutions are known and can be used to obtain a maximally extended spacetime geometry, which has a null curvature singularity inside a single non-degenerate horizon, but generic black brane solutions in the model can only be obtained numerically. Charged matter gives rise to black branes with hair that are dual to the superconducting phase of a holographic superconductor. Our numerical results indicate that holographic superconductors with anisotropic scaling have vanishing zero temperature entropy when the back reaction of the hair on the brane geometry is taken into account.

1. Introduction
We consider d + 2 dimensional gravity models that are dual to quantum critical points with anisotropic scaling in d + 1 dimensions,

\[ t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x}, \]

with \( z \geq 1 \) and \( \vec{x} = (x_1, \ldots, x_d) \). The d + 2 dimensional bulk action consists of three parts,

\[ S = S_{\text{Einstein-Maxwell}} + S_{\text{Lifshitz}} + S_{\text{matter}}. \]

The first term is the standard action of Einstein-Maxwell gravity with a negative cosmological constant,

\[ S_{\text{Einstein-Maxwell}} = \int d^{d+2}x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right). \]

This is followed by a term involving a massive vector field,

\[ S_{\text{Lifshitz}} = - \int d^{d+2}x \sqrt{-g} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{c^2}{2} A_{\mu} A^{\mu} \right), \]

whose sole purpose is to provide backgrounds with anisotropic scaling. This is a modified version of the holographic model of [1], which was formulated in four-dimensional spacetime and obtained...
anisotropic scaling by including a pair of coupled two- and three-form field strengths. Upon integrating out the three-form field strength, the remaining two-form becomes a field strength of a massive vector and in this form the model is easily extended to general dimensions. Finite temperature corresponds to having a black hole in the higher-dimensional spacetime and with the Maxwell gauge field, added in, the black hole can carry electric charge, which is dual to a finite charge density in the lower-dimensional theory. Finally, we consider matter in the form of a scalar field, which is charged under the Maxwell gauge field but does not couple directly to the auxiliary massive vector field,

\[ S_{\text{matter}} = -\frac{1}{2} \int d^{d+2}x \sqrt{-g} \left( \left( \partial^\mu \phi^* + iqA^\mu \phi^* \right) \left( \partial_\mu \phi - iqA_\mu \phi \right) + m^2 \phi^* \phi \right) . \]  

At low temperature there is an instability for charged black holes in the model to grow scalar hair, which corresponds to the superconducting phase of holographic superconductors with \( z > 1 \) asymmetric scaling, as described in Section 3 below.

The equations of motion obtained from the action (2) consist of the scalar field equation

\[ (\nabla^\mu - iqA^\mu) (\nabla_\mu - iqA_\mu) \phi - m^2 \phi = 0 , \]  

along with the Einstein equations, the Maxwell equations, and field equations for the auxiliary massive vector,

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = T^\text{Maxwell}_{\mu\nu} + T^\text{Lifshitz}_{\mu\nu} + T^\text{matter}_{\mu\nu} , \]  
\[ \nabla_\nu F^{\nu\mu} = j^\mu_{\text{matter}} , \]  
\[ \nabla_\nu F^{\nu\mu} = c^2 A^\mu . \]

The asymmetric scaling symmetry (1), sometimes referred to as Lifshitz scaling, is realized in a \( d+2 \) dimensional spacetime,

\[ ds^2 = L^2 \left( -r^2 dt^2 + r^2 d\vec{x}^2 + \frac{dr^2}{r^2} \right) , \]

whose metric is invariant under the transformation

\[ t \rightarrow \lambda^z t , \quad \vec{x} \rightarrow \lambda \vec{x} , \quad r \rightarrow \frac{r}{\lambda} . \]

Length dimensions are carried by the characteristic length \( L \) while the coordinates \((t,r,\vec{x})\) are dimensionless. The scale invariant Lifshitz geometry (10) is a solution of the equations of motion when \( L \) is related to the cosmological constant \( \Lambda \) via

\[ \Lambda = -\frac{z^2 + (d-1)z + d^2}{2L^2} , \]

and the mass of the auxiliary vector field is fine-tuned to \( c = \sqrt{z d / L} \). In this solution the Maxwell field vanishes, \( A_\mu = 0 \), but the massive vector field has the background value

\[ A_t = \sqrt{\frac{2(z-1)}{z}} L r^z , \quad A_{x_i} = A_r = 0 . \]

\(^1\) See also [2] for early work on gravitational backgrounds with anisotropic scaling.
2. Charged black branes

In order to study finite temperature effects in the dual strongly coupled field theory we look for static black brane solutions of the equations of motion (6) - (9) which are asymptotic to the Lifshitz fixed point solution given by (10) and (13). From now on we set $L = 1$ and consider a metric of the form

$$ds^2 = -r^2 f(r)^2 dt^2 + r^2 dx^2 + \frac{g(r)^2}{r^2} dr^2.$$  \hfill (14)

An asymptotically Lifshitz black brane with a non-degenerate horizon has a simple zero of both $f(r)$ and $g(r)$ at the horizon, which we take to be at $r = r_0$, and $f(r), g(r) \to 1$ as $r \to \infty$. It is straightforward to generalize this ansatz to include black holes with a spherical horizon or topological black holes with a hyperbolic horizon but it is the flat horizon case (14) that is of direct interest for the gravitational dual description of strongly coupled $d + 1$ dimensional field theories. In a static electrically charged black brane background the Maxwell gauge field and the massive vector can be taken to be of the form

$$A_\mu = r f(r) \{ \alpha(r), 0, \ldots, 0 \}; \quad A_\mu = \sqrt{\frac{2(z-1)}{z}} r f(r) \{ a(r), 0, \ldots, 0 \}.$$  \hfill (15)

with $\alpha(r) \to 0$ and $a(r) \to 1$ as $r \to \infty$.

2.1. Field equations for static configurations

For static configurations the equations of motion can be expressed as a first order system of ordinary differential equations. Introducing a scale invariant radial variable $u = \log(r/r_0)$ and writing $\dot{\equiv} \frac{d}{du}$, the scalar field equation becomes

$$\dot{\phi} = \chi,$$  \hfill (16)

$$\dot{\chi} + \left( \frac{\dot{f}}{f} - \frac{\dot{g}}{g} \right) \chi = -(d+z) \chi + \left( m^2 - q^2 \alpha^2 \right) g^2 \phi,$$  \hfill (17)

while the Maxwell equations and the equations of motion for the massive vector reduce to,

$$\dot{\alpha} + \frac{\dot{f}}{f} \alpha = -z \alpha + g \beta,$$  \hfill (18)

$$\dot{\beta} = -d \beta + q^2 g \phi^2 \alpha,$$  \hfill (19)

$$\dot{a} + \frac{\dot{f}}{f} a = -z a + z g b,$$  \hfill (20)

$$\dot{b} = -d b + d g a.$$  \hfill (21)

The functions $\chi, \beta,$ and $b$ are defined via (16), (18), and (20), respectively. The Einstein equations can also be written in first order form,

$$\frac{\dot{g}}{g} + \frac{\dot{f}}{f} = (z-1) \left( g^2 a^2 - 1 \right) + \frac{1}{2d} \left( \chi^2 + q^2 g^2 \alpha^2 \phi^2 \right),$$  \hfill (22)

$$2d \frac{\dot{f}}{f} = d(1-d-2z) + \frac{\chi^2}{2} + g^2 \left[ (z-1) \left( da^2 - z b^2 \right) - \frac{\beta^2}{2} + \frac{1}{2} \left( q^2 \alpha^2 - m^2 \right) \phi^2 + z^2 + (d-1) z + d^2 \right].$$  \hfill (23)

The field equations (16) - (23) are manifestly invariant under the scaling (11) and the Lifshitz fixed point solution is given by $f = g = a = b = 1$ and $\alpha = \beta = \phi = \chi = 0$. 


In the absence of charged matter, the Maxwell equation (19) integrates to \( \beta(u) = \tilde{\rho} e^{-du} \). The integration constant \( \tilde{\rho} = \rho/r_0^d \) is proportional to the electric charge per unit \( d \)-volume of the black brane, which in turn corresponds to the charge density in the dual field theoretic system. For given \( d \) and \( z \geq 1 \), the remaining field equations then have a one parameter family of black brane solutions, labelled by \( \tilde{\rho} \). A neutral black brane without scalar hair has \( \tilde{\rho} = 0 \) while the extremal limit is given by \( \tilde{\rho} \to \pm \sqrt{2(z^2 + (d-1)z + d^2)} \). A non-vanishing charged scalar field changes this picture as discussed below.

Equation (18) can easily be solved in the Lifshitz geometry (10) without matter, giving

\[
\alpha(u) = \begin{cases} 
\tilde{\mu} e^{-zu} + \frac{1}{(z-d)\tilde{\rho}} e^{-du} & \text{if } z \neq d, \\
\tilde{\mu} e^{-du} + \tilde{\rho} u e^{-du} & \text{if } z = d,
\end{cases}
\]

where the integration constant \( \tilde{\mu} = \mu/r_0^z \) corresponds to having non-vanishing chemical potential in the dual system. In general, we are not working with the Lifshitz background but with solutions that are only asymptotically Lifshitz as \( u \to \infty \). However, as long as the value of \( z \) isn’t too high, the asymptotic behavior of the gauge potential carries over from the Lifshitz background to the more general case, and one can read off the charge density and chemical potential in the dual field theory from the leading two terms in the expansion of \( \alpha \) at large \( u \).

Calculations in this paper refer to fixed \( \rho \), corresponding to a fixed density of charge carriers in the dual system, but one can also work at fixed chemical potential.

2.2. Numerical solutions

Black brane solutions at generic \( z > 1 \) can be obtained using numerical techniques similar to those of [5]. The field equations are integrated numerically starting near the black hole, with suitable initial conditions and proceeding out towards the asymptotic region. For a regular, non-degenerate horizon we require the functions that appear in the metric ansatz (14) to behave as

\[
f(u) = \sqrt{u}(f_0 + f_1 u + \ldots), \quad g(u) = \frac{1}{\sqrt{u}}(g_0 + g_1 u + \ldots),
\]

near \( u = 0 \). The appropriate near-horizon behavior of the remaining field variables can be worked out and then inserted into the equations of motion to generate initial value data for the numerical integration. For any given values of \( d \) and \( z \), we obtain a three parameter family of initial values, where for instance \( \phi(0), \beta(0), \) and \( b(0) \) can be taken as the independent parameters. The condition that the metric and massive vector approach the Lifshitz fixed point solution (10) and (13) sufficiently rapidly as \( u \to \infty \) restricts the solutions further [5, 6, 7]. As a result, \( b(0) \) is fixed for given \( \phi(0) \) and \( \beta(0) \) and one has a two-parameter family of solutions. This means in particular that, in the absence of scalar hair, there is a unique (up to overall scale) asymptotically Lifshitz charged black brane solution for given \( d, z \) and \( \tilde{\rho} \).

The Hawking temperature is determined by the near-horizon behavior of the black brane metric,

\[
T_H = r_0^2 \frac{f_0}{4\pi g_0}.
\]

The full numerical solution of the field equations is required, however, to relate the coefficients \( f_0 \) and \( g_0 \) to the charge density \( \rho \) and other physical variables of the dual field theory.

\[ ^2 \text{At } z \geq 3d \text{ non-linear effects give rise to additional terms in (24) with a falloff in between that of the charge density and chemical potential terms.} \]
2.3. Conserved charge under radial evolution

A conserved charge under radial evolution was found in [6] for electrically neutral black branes with \( d = 2 \) and arbitrary dynamical critical exponent \( z \). Such a conserved charge is useful for matching solutions across the bulk geometry from the near-horizon region to the asymptotic large \( u \) region and also provides a check on numerical solutions. The charge found in [6] generalizes to charged black branes with scalar hair in general spatial dimensions,

\[
D_0 = r_0^{z+d} e^{(z+d)u} \left[ \frac{1}{2g} \left( \chi^2 - 2d(d+1) \right) - 2d(z-1)ab - d\alpha\beta \right. \\
+ g \left. \left[ (z-1)(d a^2 - z b^2) + z^2 + (d-1)z + d^2 - \frac{1}{2}(\beta^2 + m^2\phi^2 - q^2\alpha^2\phi^2) \right] \right].
\]  

(27)

It is straightforward to check that \( \frac{d}{du}D_0 = 0 \) when the field equations (16) - (23) are satisfied. The conserved charge is related to thermodynamic state variables of the dual system in a simple way. Inserting a perturbative near-horizon expansion of the fields, one finds

\[
D_0 = 2r_0^{d+z} f_0 \rho_0 = 32\pi ST,
\]  

(28)

where \( T \) is the temperature (26) and \( S = r_0^d/4 \) is the Bekenstein-Hawking entropy density, which are identified with the temperature and entropy of the dual field theory.

2.4. Exact solution

As always, it is useful to have explicit analytic solutions to work with. Although Lifshitz black branes at \( z > 1 \) can in general only be obtained numerically, it turns out that for each value of \( d \) an isolated \( z = 2d \) exact solution can be found [4, 8],

\[
b = 1, \quad f^2 = \frac{1}{g^2} = a^2 = 1 - e^{-2du}, \quad f\alpha = \pm \sqrt{2}e^{-2du} \left( 1 - e^{-du} \right),
\]  

(29)

with \( \phi = 0 \) and \( \bar{\rho} = \pm \sqrt{2d} \). It is straightforward to continue the exact black brane metric inside the horizon and obtain the globally extended geometry [4]. Define a tortoise coordinate \( u_* \) by

\[
u_* = \frac{1}{2d r_0^{2d}} \log \left( 1 - e^{-2du} \right),
\]  

(30)

and then transform the \((t, u)\) variables to a pair of null coordinates

\[
V = \exp \left[ dr_0^{2d}(u_* + t) \right], \quad U = -\exp \left[ dr_0^{2d}(u_* - t) \right].
\]  

(31)

In the new coordinate system the metric is given by

\[
ds^2 = \frac{-dU\,dV}{d^2(1+UV)^2} + \frac{r_0^2\,dx^2}{(1+UV)^{V/d}},
\]  

(32)

and is manifestly non-singular at the horizon, which is located at \( UV = 0 \). There is a null curvature singularity at \( UV \to \infty \), which corresponds to \( r \to 0 \) in the original coordinate system. The asymptotic region \( r \to \infty \) corresponds to \( UV \to -1 \). By a further transformation

\begin{itemize}
  \item The corresponding exact solutions for \( d = 2, z = 4 \) black holes with a spherical horizon and topological black holes with a hyperbolic horizon were also found in [4]. In the limit of vanishing electric charge these black hole solutions reduce to the previously discovered \( z = 4 \) black hole solution of [6].
\end{itemize}
to new null variables $P, Q$, defined through $V = \tan \frac{2\pi P}{2}$, $U = \tan \frac{2\pi Q}{2}$, the global geometry can be represented by a simple diagram shown in Figure 1. Each point in the diagram represents an entire $d$-volume parametrized by $\vec{x}$. The global diagram in Figure 1 differs from standard Carter-Penrose conformal diagrams in that the boundary at $r \to \infty$ is not conformally flat. This is a consequence of the scaling asymmetry between $t$ and $\vec{x}$ and is readily apparent in the globally extended metric (32).

When $z = 1$ the auxiliary massive vector field $A_{\mu}$ can be consistently set to zero and the field equations then reduce to those of Einstein-Maxwell gravity with a negative cosmological constant. In this case, there is a well known exact solution, the AdS-Reissner-Nordström black brane, which has a timelike curvature singularity inside an inner and an outer horizon. The interior geometry of the exact $z = 2d$ black brane is markedly different with a null curvature singularity at $r = 0$ and no smooth inner horizon.

3. Holographic superconductors with asymmetric scaling
We now consider charged black branes with hair. Static spherically symmetric solutions of the scalar field equation (6) have the asymptotic form $\phi(u) \to c_-(e^{-\Delta_- u} + \ldots) + c_+(e^{-\Delta_+ u} + \ldots)$, with

$$\Delta_{\pm} = \frac{d + z}{2} \pm \sqrt{\left(\frac{d + z}{2}\right)^2 + m^2},\quad (33)$$

while the asymptotic behavior of the electromagnetic field field strength is

$$\beta(u) \approx \frac{\rho}{r_0^d} e^{-du} + \ldots.\quad (34)$$

Working at fixed charge density in the dual field theory, we read the radial location of the horizon off from the asymptotic behavior of $\beta(u)$ and then the temperature can be obtained from the numerical solution for $f(u)$ and $g(u)$ using (26).

In the following we set the scalar mass squared to $m^2 = \frac{1}{4} - \left(\frac{d + z}{2}\right)^2$, which is inside the range where there is a choice of two boundary theories [9]. This choice leads to convenient values, $\Delta_{\pm} = \frac{d + z}{2} \pm \frac{1}{2}$, for the dimensions of the operators $\mathcal{O}_{\pm}$ that are dual to the scalar field in each of the two boundary theories. Non-linear descendants of the leading scalar field modes are suppressed by $O(e^{-2u})$ at $u \to \infty$ and this choice of mass squared ensures that the first descendant of $\psi_- \equiv e^{-2u}$ falls off faster than $\psi_+$.

In order to study holographic superconductivity, we first select some value for $d$, $z$ and the electric charge $q$ carried by the scalar field and then generate numerical black brane solutions for
a range of initial values $\beta(0)$ and $\phi(0)$. We then investigate the asymptotic large $u$ behavior of the scalar hair in the numerical solutions. A superconducting condensate corresponds to either

$$c_+ = 0, \quad \langle O_- \rangle = c_- \neq 0,$$

or

$$c_- = 0, \quad \langle O_+ \rangle = c_+ \neq 0,$$

depending on which of the two boundary theories is being considered [10, 11]. We look for a curve in the $\beta(0)$ vs. $\psi(0)$ plane of initial values at the horizon, for which the corresponding black brane solution has vanishing $c_+ (c_-)$, and tabulate the value of $c_- (c_+)$ along this curve.

The temperature is found from the same numerical solutions via (26) and (34). Figure 2 shows a plot of $c_- \text{ vs. } T$ obtained by this procedure for $d = 2, z = 2, \text{ and } q = 1$. The results are expressed in terms of dimensionless ratios that are insensitive to the overall scale (set by the charge density $\rho$, which is held fixed at some finite value throughout).

These results demonstrate that a superconducting condensate can form in systems with anisotropic scaling but we have not touched on a number of interesting topics including the electric conductivity and magnetic properties of these holographic superconductors.

4. Zero temperature entropy vs. scalar hair

In the absence of charged matter, the charged black branes in our model have an extremal limit given by $\tilde{\rho} \to \pm \sqrt{2(d^2 + (d-1)z + d^2)}$. It then follows from $\tilde{\rho} = \rho / r_0^d$ that the radial location of the horizon $r_0$ has a finite value in the extremal limit for fixed charge density $\rho$. This in turn means that the entropy density $S = r_0^d / 4$ remains finite in the zero temperature limit indicating a macroscopic groundstate degeneracy.

This conclusion is radically altered when the system is coupled to a charged scalar field. In this case the zero temperature limit is approached in the condensed phase and the black holes, that are dual to extreme low temperature states, have scalar hair. It is straightforward to keep track of the black brane entropy as the temperature is lowered below the critical value for forming the superconducting condensate. The result for the same $d = 2, z = 2$ holographic superconductor as was considered in the previous section is shown in Figure 3. Both the entropy density and the temperature are normalized to their values at the onset of condensation, $S_c$ and $T_c$ respectively.

Although the numerical calculations break down before absolute zero is reached, the numerical data strongly suggest that $S$ vanishes in the $T \to 0$ limit for a generic dynamical critical exponent $z$, which is consistent with a non-degenerate ground state in the dual field theory. The corresponding result for conformal systems with $z = 1$ was established in [12].
5. Summary
We have presented an overview of the construction of charged black brane solutions in gravity models that realize the anisotropic scaling symmetry that is characteristic of many interesting quantum critical points. The motivation for the study of these models comes from condensed matter theory, in particular from two- and three-dimensional systems involving strongly correlated electrons. The relevance of gravitational models to real world condensed matter systems remains highly speculative, but the gravitational approach continues to produce effects that are intriguingly similar to what is seen in experiments. A recent example from our own work [13] involves non-Fermi-liquid behavior in the specific heat of anisotropic black branes of the type considered in the present paper, which turns out to be qualitatively similar to the measured specific in certain heavy fermion metals near a quantum phase transition [14, 15]. While much of the work on gravitational modeling of strongly coupled field theories to date involves asymptotically AdS spacetime and an underlying conformal symmetry, it was crucial to the success of this particular application to have a non-trivial dynamical critical exponent $z > 1$. This provides impetus for further study of gravity models with anisotropic scaling.

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