Energy of the Coulomb gas on the sphere at low temperature

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Abstract

We consider the Coulomb gas of \(N\) particles on the sphere and show that the logarithmic energy of the configurations approaches the minimal energy up to an error of order \(\log N\), with exponentially high probability and on average, provided the temperature is \(O(1/N)\).

1 Introduction and statement of the result

Smale’s 7th problem. Let \(\|\cdot\|\) be the Euclidean norm on \(\mathbb{R}^3\) and

\[ S := \{ x \in \mathbb{R}^3 : \|x\| = 1 \} \]

the unit sphere. Consider the logarithmic energy of a configuration \(x_1, \ldots, x_N \in S\),

\[ \mathcal{H}_N(x_1, \ldots, x_N) := \sum_{i \neq j} \log \frac{1}{\|x_i - x_j\|} \]

The 7th problem from Smale [2000]’s list of mathematical problems for the next century asks to find for every \(N \geq 2\) a configuration \(x_1, \ldots, x_N \in S\) and \(c > 0\) independent on \(N\) such that

\[ \mathcal{H}_N(x_1, \ldots, x_N) - \min_{S^N} \mathcal{H}_N \leq c \log N. \tag{1.1} \]

More precisely, quoting Smale: “For a precise version one could ask for a real number algorithm in the sense of Blum, Cucker, Shub, and Smale [1996] which on input \(N\) produces as output distinct points \(x_1, \ldots, x_N\) on the 2-sphere satisfying (1.1) with halting time polynomial in \(N\)”. 

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Large $N$ expansion. One difficulty in this problem is that the large $N$ behavior of $\min_{S^N} \mathcal{H}_N$ is not even known up to precision $\log N$. Indeed, the actual knowledge is that

$$\min_{S^N} \mathcal{H}_N = V_{\log}(S) N^2 - \frac{1}{2} N \log N + C_{\log} N + o(N), \quad N \to \infty,$$

where the constant $V_{\log}(S) = \min_{\mu \in \mathcal{P}(S)} \iint \log \frac{1}{\|x - y\|} \mu(dx) \mu(dy) = \frac{1}{2} - \log 2$ is the minimal logarithmic energy over the space $\mathcal{P}(S)$ of probability measures on $S$.

The exact value of the constant $C_{\log}$ in (1.2) is still conjectural. A series of papers by Wagner [1989], Rakhmanov, Saff, and Zhou [1994], Dubickas [1996] and Brauchart [2008] gave upper and lower bounds for $C_{\log}$ as well as similar bounds for other choices of energies, but indeed the existence of $C_{\log}$ has only recently been obtained by Bétermin and Sandier [2018] where it is expressed in terms of the minimum of the renormalized energy introduced by Sandier and Serfaty [2012]. In the same paper it is also proved that

$$C_{\log} \leq 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)} = -0.0556053\ldots$$

This upper bound is conjectured to be an equality, see Brauchart, Hardin, and Saff [2012]. Bétermin and Sandier [2018] have also shown this conjecture is equivalent to the conjecture that the triangular lattice minimizes the renormalized energy. The tightest known lower bound $C_{\log} \geq -0.2232823526\ldots$, proved by Dubickas [1996], seems thus to be far from optimal.

One may look for configurations of $N$ points defined deterministically on $S$ which attain small values for $\mathcal{H}_N$, but it turns out to be very difficult to compute explicit asymptotics for any reasonable choice of points; Hardin, Michaels, and Saff [2016] made many numerical experiments analyzing different constructions, but none of them seems to reach the upper bound for $C_{\log}$ in (1.3).

Random configurations on the sphere. A possible strategy to attack the problem is to look for random configurations on $S$ whose logarithmic energy could satisfy (1.1) on average, or with high probability. If one naively picks $x_1, \ldots, x_N$ at random uniformly and independently on $S$, then an easy computation yields a formula for the mean energy,

$$\mathbb{E}_{\text{Unif}}[\mathcal{H}_N(x_1, \ldots, x_N)] = V_{\log}(S) N^2 + (\log 2 - 1/2)N.$$

Thus, unstructured configurations do not even reach precision $N$ on average. Armentano, Beltrán, and Shub [2011] suggested instead to take for $x_i$’s the zeros of Elliptic polynomials after a stereographic projection. These are random polynomials on $\mathbb{C}$ defined by

$$P_N(z) := \sum_{k=0}^{N} \sqrt{\binom{N}{k}} \xi_k z^k$$

where the $\xi_k$’s are i.i.d standard complex gaussian random variables. Up to multiplication by non-vanishing holomorphic functions, they are the only gaussian analytic functions
configuration $x$ when we denote by $P$ the laws of 2D electrostatics. The simplest description of the spherical ensemble, due to Butez and Zeitouni [2006], is as follows: choose two matrices $A, B$ whose entries are i.i.d standard complex gaussian random variables and compute the $N$ eigenvalues of $A^{-1}B$. Up to a stereographic projection, it turns out that these points are quite well distributed on $S$ on average. Indeed, Alishahi and Zamani [2015] obtained for these random configurations that, as $N \to \infty$,

$$E_{\text{DPP}}[\mathcal{H}_N(x_1, \ldots, x_N)] = V_{\log}(S) N^2 - \frac{1}{2} N \log N + (\log 2 - \gamma/2) N - \frac{1}{4} + O(1/N),$$

where $\gamma$ is the Euler constant; hence $\log 2 - \gamma/2 = 0.4045393\ldots$ Finally, a particular random construction with seemingly small energy values based on the distribution of charges along parallels in $S$ is currently being studied by Etayo and Beltrán.

All these bounds (analytical and numerical) are still far from the upper bound in (1.3).

**The Coulomb gas on the sphere.** Another natural random configuration associated with this problem is the Coulomb gas on $S$, which is the main character of this work; for references see e.g. [Forrester, 2010, Section 15.6]. More precisely, let $\sigma$ be the uniform measure on $S$ normalized so that $\sigma(S) = 1$, namely $\sigma := (4\pi)^{-1}\text{Vol}$. For any $N \geq 2$ and $\beta > 0$, consider the probability measure on $S^N$,

$$P_{N,\beta}(dx) := \frac{1}{Z_{N,\beta}} e^{-\beta \mathcal{H}_N(x)\sigma^\otimes N}(dx),$$

where we introduced the normalisation constant known as the partition function,

$$Z_{N,\beta} := \int_{S^N} e^{-\beta \mathcal{H}_N(x)\sigma^\otimes N}(dx).$$

We denote by $E_{\beta}$ the expectation with respect to $P_{N,\beta}$. Here $\mathcal{H}_N(x)$ means $\mathcal{H}_N(x_1, \ldots, x_N)$ when $x = (x_1, \ldots, x_N) \in S^N$. Physically, $\mathcal{H}_N(x)$ represents the electrostatic energy of a configuration $x_1, \ldots, x_N$ of $N$ identical charges placed on the sphere, following the classical laws of 2D electrostatics. $P_{N,\beta}$ is known in statistical physics as the canonical Gibbs measure associated with this energy and the random configurations it generates are referred to as the Coulomb gas at inverse temperature $\beta$. Typical configurations of the Coulomb gas will try to minimize $\mathcal{H}_N$ because of its density distribution proportional to $e^{-\beta \mathcal{H}_N}$. It is thus tempting to evaluate the energy $\mathcal{H}_N(x)$ for such random configurations so as to approximate the minimum of $\mathcal{H}_N$. In fact, when $\beta = 1$ the Coulomb gas benefits from an integrable structure: up to stereographic projection, this is the spherical ensemble mentioned above and studied by Alishahi and Zamani [2015]. But the larger $\beta$ is the more
likely it is for $\mathbb{P}_{N,\beta}$ to generate a configuration close to a minimizer, although the determinantal structure is lost when $\beta \neq 1$ making exact computations out of reach. The main achievement of this work is to show that the Coulomb gas on the sphere at temperature $O(1/N)$ provides almost minimizing configurations in the sense of Smale’s problem with high probability as well as on average.

**Theorem 1.1.** For any $N \geq 2$ and any $\beta \geq 1$, let $x_1, \ldots, x_N$ be the random configuration on $\mathbb{S}$ with joint distribution $\mathbb{P}_{N,\beta}$. For any constant $c > 0$ we have

$$\mathcal{H}_N(x_1, \ldots, x_N) - \min_{\mathbb{S}} \mathcal{H}_N \leq c \log N$$

with probability at least $1 - e^{-\kappa N}$, where

$$\kappa := c \frac{\beta}{N} \log N - \log \beta - 8 \log N.$$

Moreover, the mean energy satisfies

$$\mathbb{E}_\beta [\mathcal{H}_N(x_1, \ldots, x_N)] - \min_{\mathbb{S}} \mathcal{H}_N \leq \frac{N}{\beta} \left( \log \beta + 8 \log N \right).$$

Note that given $\beta$ and $N$, the constant $c$ has to be chosen so that $\kappa > 0$ since otherwise the first result becomes trivial. We reach the precision $\log N$ for any $N \geq 2$ when $\beta$ is at least of order $N$. For example, by taking $\beta = N$ and $c = 10$ in Theorem 1.1, we obtain the following estimates.

**Corollary 1.2.** For any $N \geq 2$, if the random configuration $x_1, \ldots, x_N$ on $\mathbb{S}$ has for distribution the Coulomb gas $\mathbb{P}_{N,N}$ at inverse temperature $\beta = N$, then

$$\mathcal{H}_N(x_1, \ldots, x_N) - \min_{\mathbb{S}} \mathcal{H}_N \leq 10 \log N$$

with probability at least $1 - e^{-N \log N}$. Moreover,

$$\mathbb{E}_{N} [\mathcal{H}_N(x_1, \ldots, x_N)] - \min_{\mathbb{S}} \mathcal{H}_N \leq 9 \log N.$$

Thus, if one accepts stochastic algorithms as solutions for the precise version of Smale’s 7th problem, it remains to show that one can sample a configuration from $\mathbb{P}_{N,N}$ in polynomial time, or at least approximate configurations which are close from those of $\mathbb{P}_{N,N}$, say, in total variation, with high probability. Of course, letting $\beta$ growing with $N$ faster than a linear rate leads to improved convergence results, and even allows $c$ to decay to zero as $N \to \infty$, but we expect that the larger $\beta$ is the harder it is to sample such configurations in practice.

The proof of the theorem relies on two facts. First, a general concentration inequality for the energy of arbitrary Gibbs measures provided in Section 2: we observe in a general setting that an explicit control on the probability that $\mathcal{H}_N(x) - \min_{\mathbb{S}} \mathcal{H}_N > \delta$, as well as an upper bound on the mean energy $\mathbb{E}_\beta [\mathcal{H}_N]$, can be made using solely a lower bound for $\log Z_{N,\beta} + \beta \min \mathcal{H}_N$. This lower bound can be easily derived when an upper bound on the second derivative of the energy is known. In Section 3, we work out such an upper bound in the case of the Coulomb gas on $\mathbb{S}$ and prove Theorem 1.1.
2 Concentration for the Gibbs measure’s energy

In this section, we consider the following general setting: Let $S$ be any non-empty measurable space equipped with a probability measure $\mu$ and a measurable map $H : S \to \mathbb{R} \cup \{+\infty\}$ such that $\inf_S H > -\infty$. Consider for any $\beta > 0$ the probability measure,

$$P_\beta(dx) := \frac{1}{Z_\beta} e^{-\beta H(x)} \mu(dx), \quad Z_\beta := \int e^{-\beta H(x)} \mu(dx),$$

(2.1)

and assume that $Z_\beta$ is finite and does not vanish so that $P_\beta$ is well defined. In particular $\inf_S H < +\infty$. In the following, $E_\beta$ stands for the expectation with respect to $P_\beta$.

We first observe that one can relate the deviations of the random variable $H(x)$ from $\inf_S H$, when $x$ has distribution $P_\beta$, to a lower bound on $\log Z_\beta + \beta \inf_S H$.

**Lemma 2.1.** Let $C_\beta$ be any constant satisfying

$$\log Z_\beta \geq -\beta \inf_S H - C_\beta. \quad (2.2)$$

Then, for any $\delta > 0$,

$$P_\beta(H(x) - \inf_S H > \delta) \leq e^{-\beta \delta + C_\beta}. \quad (2.3)$$

Note that since a rough upper bound yields $\log Z_\beta \leq -\beta \inf_S H$, the constant $C_\beta$ has to be non-negative, and $C_\beta = 0$ if and only if $H$ is $\mu$-a.s. constant on $S$.

**Proof.** Indeed, since $\mu$ is a probability measure,

$$P_\beta(H(x) - \inf_S H > \delta) \leq \frac{1}{Z_\beta} e^{-\beta (\delta + \inf_S H)} \leq e^{-\beta \delta + C_\beta},$$

where we used (2.2) for the second inequality. \hfill \Box

The identity

$$E(X) = \int_0^\infty P(X > t)dt,$$

which holds for any positive random variable $X$, yields together with (2.3) that

$$E_\beta[H(x)] - \inf_S H \leq \frac{e^{C_\beta}}{\beta}, \quad (2.4)$$

and in particular $H(x) \in L^1(P_\beta)$. Having in mind that $\beta$ may be taken large and that $C_\beta$ could grow with $\beta$, one can obtain a better bound than (2.4) as follows.

**Lemma 2.2.** Under the assumptions of Lemma 2.1,

$$E_\beta[H(x)] - \inf_S H \leq \frac{C_\beta}{\beta}. \quad (2.5)$$

**Proof.** Let $0 < \gamma < \beta$. Since $H(x) \in L^1(P_\beta)$ Jensen’s inequality yields,

$$\log Z_\gamma = \log \int e^{-(\gamma - \beta) H(x)} P_\beta(dx) + \log Z_\beta \geq -(\gamma - \beta) E_\beta[H(x)] + \log Z_\beta$$

...
and thus
\[ E_\beta[\mathcal{H}(x)] \leq \frac{\log Z_\gamma - \log Z_\beta}{\beta - \gamma}. \]
Together with the rough upper bound
\[ \log Z_\gamma \leq -\gamma \inf_S \mathcal{H} \]
and the definition of \( C_\beta \), see (2.2), we obtain
\[ E_\beta[\mathcal{H}(x)] \leq \inf_S \mathcal{H} + \frac{C_\beta}{\beta - \gamma}. \]
The lemma follows by letting \( \gamma \to 0 \).

As we shall see in the next section, one way to obtain such a constant \( C_\beta \) is to use an upper bound on the order two Taylor expansion of \( \mathcal{H} \) near a minimizer.

## 3 Proof of Theorem 1.1

Let \( d_S \) be the usual geodesic distance on the sphere \( S \),
\[ d_S(x, y) = \arccos\langle x, y \rangle_{\mathbb{R}^3}, \quad x, y \in S, \]
where \( \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \) stands for the usual inner product of \( \mathbb{R}^3 \). To prove Theorem 1.1 we will use the following estimate as a key ingredient.

**Proposition 3.1.** Let \( N \geq 2 \) and \( x^* \in S^N \) be a minimizer of \( \mathcal{H}_N \). If \( x \in S^N \) satisfies
\[ \max_{1 \leq i \leq N} d_S(x_i, x^*_i) \leq \arcsin \left( \frac{8}{\sqrt{5N^3/2}} \right) \]
for some \( 0 \leq s \leq \sqrt{5N}/2 \), then
\[ \mathcal{H}_N(x) \leq \min_{S^N} \mathcal{H}_N + s^2. \]
A similar estimate is provided in [Beltrán, 2013, Theorem 1.8]. Proposition 3.1 improves the range of validity of this result with an alternative proof.

We will also rely on the following result of Dragnev [2002] for the separation distance.

**Proposition 3.2.** Let \( N \geq 2 \) and \( x^* \in S^N \) be a minimizer of \( \mathcal{H}_N \). We have
\[ \min_{i \neq j} \|x^*_i - x^*_j\| \geq \frac{2}{\sqrt{N - 1}}. \]
We are now in position to provide a proof for our main theorem.

**Proof of Theorem 1.1.** For any \( 0 \leq r \leq \pi \) and \( x \in S \), the volume of a spherical cap \( B_S(x, r) := \{ y \in S : d_S(x, y) \leq r \} \) is explicit: recalling \( \sigma \) is normalized so that \( \sigma(S) = 1 \),
\[ \sigma(B_S(x, r)) = \sin^2 \left( \frac{r}{2} \right). \]
In particular, for any $t \in (0, 1)$,

$$\sigma(B_S(x, \arcsin(t))) = \sin^2 \left( \frac{1}{2} \arcsin(t) \right) = \frac{1 - \sqrt{1 - t^2}}{2} \geq \frac{t^2}{4}.$$ 

For any $N \geq 2$ and any minimizer $x^* \in S^N$ of $\mathcal{H}_N$ we set, for any $0 < s \leq \sqrt{5N/2}$,

$$\Omega := \left\{ x \in S^N : \max_{1 \leq i \leq N} d_S(x_i, x_i^*) \leq \arcsin \left( \frac{s}{\sqrt{5N/2}} \right) \right\}.$$ 

Thus,

$$\sigma^\otimes N(\Omega) \geq \left( \frac{s^2}{20N^3} \right)^N.$$ 

Next, assume $\beta \geq 1$ and we use Proposition 3.1 to obtain the lower bound

$$\log Z_{N, \beta} \geq \log \int_{\Omega} e^{-\beta, \mathcal{H}_N(x)} \sigma^\otimes N(dx) \geq -\beta \min_{S^N} \mathcal{H}_N - \beta s^2 + \log \sigma^\otimes N(\Omega) \geq -\beta \min_{S^N} \mathcal{H}_N - \beta s^2 + N \log \left( \frac{s^2}{20N^3} \right).$$

Since this holds for any $0 < s \leq \sqrt{5N/2}$ and $N \geq 2$, we obtain

$$\log Z_{N, \beta} + \beta \min_{S^N} \mathcal{H}_N \geq \max_{0 < s \leq \sqrt{5N/2}} \left( -\beta s^2 + N \log \left( \frac{s^2}{20N^3} \right) \right) \geq -N(1 + \log \beta + 2 \log N + \log 20) \geq -N\left( \log \beta + 8 \log N \right).$$

We used that the maximum is reached at $s = \sqrt{N/\beta}$ and $\sqrt{N/\beta} \leq \sqrt{5N/2}$ because $\beta \geq 1$. Thus, we have obtained the lower bound (2.2) with

$$C_\beta = N\left( \log \beta + 8 \log N \right),$$

and Theorem 1.1 follows from Lemma 2.1 by taking $\delta = c \log N$ and Lemma 2.2. 

We finally turn to the proof of Proposition 3.1.

**Proof of Proposition 3.1.** From now, let $N \geq 2$, let $x^* \in S^N$ be any minimizer of $\mathcal{H}_N$ and $0 < t \leq 1/(2N)$. We set for convenience, for any $1 \leq j \leq N$,

$$B_j := B_S(x^*_j, \arcsin(t)) \quad (3.1)$$

where we recall that $B_S(x, r)$ is the ball of radius $r$ centered at $x \in S$ associated with the geodesic distance $d_S$. Since $\|x - y\| \leq d_S(x, y)$ for any $x, y \in S$ and $\arcsin(t) \leq \sqrt{2t}$ for any $0 < t \leq 1/4$, Proposition 3.2 and the constraint on $t$ yields that the $B_j$’s are disjoint subsets of $S$ for any $N \geq 2$. 

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We equip the sphere \( S \subset \mathbb{R}^3 \) with its usual Riemannian structure inherited from \( \mathbb{R}^3 \), whose associated distance is \( d_S \), and let \( \Delta_S \) be the associated Laplace-Beltrami operator. It is well known that \( \log \| \cdot \|^{-1} \) satisfies the Poisson equation, namely
\[
\Delta_S \log \frac{1}{\| \cdot \|} = 2\pi(\sigma - \delta_0) \tag{3.2}
\]
in distribution, where we recall that \( \sigma \) is the uniform probability measure on \( S \), see e.g. [Forrester, 2010, Section 15.6.1]. It follows that, for any \( p \in S \), the map \( F_p(x) := \log \| x - p \|^{-1} \) satisfies \( \Delta_S F_p(x) = 1/2 \) when \( x \in S \setminus \{ p \} \), and in particular it is subharmonic there. Thus, for any \( (x_1, \ldots, x_N) \in B_1 \times \cdots \times B_N \) and any \( 1 \leq k \leq N \), the mapping
\[
y \mapsto \mathcal{H}_N(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_N) \tag{3.3}
\]
is subharmonic on \( B_k \) and satisfies the maximum principle. More precisely, by applying the classical Hopf’s maximum principle for uniformly elliptic operators, see e.g. [Jost, 2005, Theorem 24.1], in the specific case of the Laplace-Beltrami operator \( \Delta_S \) (in coordinates), we have that for any open set \( \Omega \) contained in a hemisphere of \( S \) and any \( F \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) which is subharmonic on \( \Omega \),
\[
\sup_{\Omega} F \leq \max_{\partial \Omega} F.
\]
By using \( N \) times this inequality for the mappings (3.3), we obtain
\[
\max_{B_1 \times \cdots \times B_N} \mathcal{H}_N \leq \max_{\partial B_1 \times B_2 \times \cdots \times B_N} \mathcal{H}_N \leq \cdots \leq \max_{\partial B_1 \times \partial B_2 \times \cdots \partial B_N} \mathcal{H}_N. \tag{3.4}
\]
Next, we observe that
\[
\partial B_1 \times \partial B_2 \times \cdots \partial B_N
\]

\[
= \left\{ x \in \mathbb{R}^N : d_S(x_i, x_i^*) = \arcsin(t) \text{ for all } 1 \leq i \leq N \right\}
\]

\[
= \left\{ \sqrt{1 - t^2} x^* + tv : v \in \mathbb{R}^N, \langle x_i^*, v \rangle_{\mathbb{R}^3} = 0 \text{ for all } 1 \leq i \leq N \right\}.
\tag{3.5}
\]
Indeed, a geodesic of \( S \) can always be parametrized as \( \theta(u) = \sqrt{1 - u^2} x + uv \) for \( x \in S \) and \( v \in \mathbb{R}^3 \) satisfying \( \|v\| = 1 \) and \( \langle x, v \rangle_{\mathbb{R}^3} = 0 \); if \( \Theta \subset S \) is the curve \( \{\theta(u)\}_{u \in [0,1]} \) then it starts at \( \theta(0) = x \) with initial speed \( \dot{\theta}(0) = v \) and has length
\[
\text{Length}(\Theta) = \int_0^t ||\dot{\theta}(u)|| \, du = \int_0^t \frac{du}{\sqrt{1 - u^2}} = \arcsin(t).
\]
In view of (3.1) and (3.4)–(3.5), it is thus enough to show that, for any \( v \in S^N \) satisfying \( \langle x_i^*, v_i \rangle_{\mathbb{R}^3} = 0 \) for every \( 1 \leq i \leq N \),
\[
\mathcal{H}_N(\sqrt{1 - t^2} x^* + tv) \leq \mathcal{H}_N(x^*) + 5t^2 N^3. \tag{3.6}
\]
Indeed the proposition follows by setting \( s := \sqrt{5N^{3/2}} \) which satisfies \( 0 < s \leq \sqrt{5N}/2 \).
To do so, let \( v \in S^N \) satisfying \( \langle x_i^*, v_i \rangle_{\mathbb{R}^3} = 0 \) for all \( 1 \leq i \leq N \) and set
\[
g(t) := \mathcal{H}_N(\sqrt{1 - t^2} x^* + tv).
\]

Since $g$ reaches a minimum at $t = 0$, there exists $\alpha \in (0, t)$ such that

$$\mathcal{H}_N(\sqrt{1 - t^2}x^* + tv) = \mathcal{H}_N(x^*) + \frac{t^2}{2}\tilde{g}(\alpha). \quad (3.7)$$

Next, we set for convenience

$$\gamma_{ij}(t) := \sqrt{1 - t^2}(x_i^* - x_j^*) + t(v_i - v_j)$$

so that we have

$$g = -\sum_{i \neq j} \log \|\gamma_{ij}\|, \quad \tilde{g} = -\sum_{i \neq j} \frac{\langle \gamma_{ij}, \tilde{\gamma}_{ij} \rangle}{\|\gamma_{ij}\|^2}$$

and moreover, using the Cauchy-Schwarz inequality,

$$\tilde{g} = \sum_{i \neq j} \left(2\frac{\langle \gamma_{ij}, \tilde{\gamma}_{ij} \rangle^2}{\|\gamma_{ij}\|^4} - \frac{\langle \gamma_{ij}, \gamma_{ij} \rangle + \|\gamma_{ij}\|^2}{\|\gamma_{ij}\|^2} \right) \leq \sum_{i \neq j} \frac{\|\gamma_{ij}\|^2 - \langle \gamma_{ij}, \tilde{\gamma}_{ij} \rangle}{\|\gamma_{ij}\|^2}. \quad (3.8)$$

By computing explicitly the derivatives of $\gamma_{ij}(t)$ we obtain

$$\|\gamma_{ij}(t)\|^2 = (1 - t^2)||x_i^* - x_j^*||^2 + 2t\sqrt{1 - t^2}\langle x_i^* - x_j^*, v_i - v_j \rangle + t^2\|v_i - v_j\|^2$$

$$\|\tilde{\gamma}_{ij}(t)\|^2 = \frac{t^2}{1 - t^2}\langle x_i^* - x_j^*, v_i - v_j \rangle - \frac{2t}{\sqrt{1 - t^2}}\langle x_i^* - x_j^*, v_i - v_j \rangle + \|v_i - v_j\|^2$$

$$\langle \gamma_{ij}(t), \tilde{\gamma}_{ij}(t) \rangle = -\frac{1}{1 - t^2}\|x_i^* - x_j^*\|^2 - \frac{t}{(1 - t^2)^{3/2}}\langle x_i^* - x_j^*, v_i - v_j \rangle$$

and this yields with (3.8),

$$\tilde{g}(t) \leq \sum_{i \neq j} \frac{1 + t^2}{1 - t^2}\|x_i^* - x_j^*\|^2 - \frac{t - 2t^3}{(1 - t^2)^{3/2}}\langle x_i^* - x_j^*, v_i - v_j \rangle + \|v_i - v_j\|^2.$$

For any $0 < \alpha \leq t$, we use the upper bounds $\|x_i^* - x_j^*\| \leq 2$, $\|v_i - v_j\| \leq 2$, $\|x_i^* - x_j^*, v_i - v_j\| = \|x_i^*, v_j\| + \|x_j^*, v_i\| \leq 2$ and $\alpha - 2\alpha^3 \leq \alpha(1 - \alpha^2)$, as well as the lower bound $\|x_i^* - x_j^*\| \geq 2\sqrt{N}$ provided by Proposition 3.2, to obtain

$$\tilde{g}(\alpha) \leq N(N - 1)\frac{1}{(1 - \alpha^2)^2} - 4t \left(\frac{8}{1 - t^2} + \frac{2t}{\sqrt{1 - t^2}}\right).$$

Finally, we use the inequality $1 - t^2 - Nt \geq (1 - t^2)/4$ for any $0 < t \leq 1/(2N)$ to obtain

$$\tilde{g}(\alpha) \leq N^3\left(\frac{8}{(1 - t^2)^2} + \frac{2t}{(1 - t^2)^{3/2}}\right) \leq 10N^3. \quad (3.9)$$

Together with (3.7) this yields (3.6), and the proof of Proposition 3.1 is therefore complete.

□
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