TRAVELING WAVE SOLUTIONS OF A FREE BOUNDARY PROBLEM WITH LATENT HEAT EFFECT

CHUEH-HSIN CHANG*

Department of Applied Mathematics
Tunghai University
Tunghai University, Taichung, 40704, Taiwan

CHIUN-CHUAN CHEN AND CHIH-CHIANG HUANG

Department of Mathematics
National Taiwan University
National Taiwan University, Taipei, 10617, Taiwan

Abstract. We study a free boundary problem of two competing species with latent heat effect. We establish the existence and uniqueness of the traveling wave solution and derive upper and lower bounds for the wave speed. Especially our results show that the latent heat retards propagation of the waves.

1. Introduction. For the study of segregation and aggregation phenomena of biological individuals, Mimura, Yamada and Yotsutani [14, 15, 16] proposed a free boundary problem (the MYY system) for two competing species which are spatially segregated. They successfully established the existence, uniqueness and stability of the solution on a bounded domain. In a later paper [9], Hilhorst, Iida, Mimura and Ninomiya suggested to write the Stefan boundary condition of the MYY system in a form with latent heat and showed that the MYY system can be approximated by a 3-species competition-diffusion system. To understand the dynamical behavior of the MYY system, the authors in [1] studied the traveling wave solutions and obtained their existence. In this paper, we further investigate how latent heat affects the traveling waves of the MYY system. We show that the latent heat actually retards the propagation of the waves. Also we obtain new lower and upper bounds, which depend on the strength of the latent heat, for the wave speed.

We consider the MYY system in a one dimensional domain $[0, l]$. Let $P(x, t)$ and $Q(x, t)$ denote the densities of the two species in consideration, which occupies the regions $[0, s(t)]$ and $[s(t), l]$ respectively, where $s(t)$ denotes the moving boundary separating the two species. The MYY system in this case can be described as

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* Corresponding author: Chueh-Hsin Chang.
follows.

\[
\begin{align*}
P_t - d_1 P_{xx} &= P(a_1 - b_1 P), & t > 0, & 0 < x < s(t), \\
Q_t - d_2 Q_{xx} &= Q(a_2 - b_2 Q), & t > 0, & s(t) < x < l, \\
P &= Q = 0, & s'(t) &= -\alpha_1 P_x - \alpha_2 Q_x, & x &= s(t), \\
P(0,t) &= M_1, & Q(l,t) &= M_2, & t > 0, \\
s(0) &= s_0, & 0 < s_0 < l, \\
P(x,0) &= P_0(x) \geq 0, & 0 \leq x \leq s_0, \\
Q(x,0) &= Q_0(x) \geq 0, & s_0 < x < l,
\end{align*}
\]

where \(d_i, a_i, b_i, \alpha_i, M_i (i = 1, 2)\) are given positive constants. In [14, 15, 16], the existence, uniqueness and stability of the solution were studied. It was also proved that the \(\omega\)-limit set of (1) is nonempty and the free boundary \(s(t)\) converges to some constant as \(t\) goes to infinity. The multidimensional case of this model was studied by Hilhorst, Mimura and Schätzle in [10]. Recently Yang ([19] [20]) further studied this problem in unbounded domain and time-periodic case. Moreover, Du and Wu in [6] and other related work investigated the spreading behavior of two invasive species modeled by a Lotka-Volterra diffusive competition system with two free boundaries.

To study the spreading behavior of an invasive species more carefully, Du and Lin [4] considered a one-species Stefan boundary problem similar to the MYY system:

\[
\begin{align*}
U_t - d U_{xx} &= U(a - b U), & t > 0, & 0 < x < h(t), \\
U_x(t,0) &= 0, & U(t,h(t)) &= 0, & t > 0, \\
h'(t) &= -\mu U_x(t,h(t)), & t > 0, \\
h(0) &= h_0, & U(x,0) &= U_0(x), & 0 \leq x \leq h_0,
\end{align*}
\]

where \(h(t)\) is the moving boundary to be determined, \(h_0, \mu, d, a\) and \(b\) are given positive constants, and the initial condition \(U_0(x)\) satisfies

\[
U_0 \in C^2([0,h_0]), \quad U_0(0) = U_0(h_0) = 0, \quad \text{and} \quad U_0 > 0 \quad \text{in} \quad [0,h_0].
\]

After suitable scaling, one may assume \(d = a = b = 1\). Du and Lin [4] obtained a very interesting spreading-vanishing dichotomy: if \(h_0 \geq \frac{\pi}{2}\), then regardless of the initial population size, spreading always happens; if \(h_0 < \frac{\pi}{2}\), then whether spreading or vanishing occurs is determined by the initial population size \(U_0\) and the coefficient \(\mu\) in the Stefan boundary condition. Moreover, they showed that when spreading occurs, \(U\) approaches to a traveling wave determined by the equation

\[
\begin{align*}
u'' + cu' + u(1-u) &= 0, & z \in (-\infty, 0], \\
u(-\infty) &= 1, & u(0) &= 0, \\
u'(0) &= -\frac{c}{2},
\end{align*}
\]

and \(h(t)\) moves with the constant speed \(c\) asymptotically, i.e.

\[
h(t) = (c + o(1)) t + O(1) \quad \text{as} \quad t \to \infty.
\]

Du and Lin proved that for given \(\mu > 0\), there exists a unique \(c = c_\mu\) such that (3) has a unique traveling wave solution \(u_\mu(z)\).

Inspired by the result (4) of Du and Lin, it is natural to consider the traveling wave solutions corresponding to (1). After suitable scaling, we may assume \(a_1 = b_1 = b_2 = 1\) and \(a_2 = a\). Let \(s(t) = ct, P(t,x) = u(z)\), \(Q(t,x) = v(z)\) with
To Theorem 1.1. In [1], we proved the following theorem.

For the free boundary problem (1) with no reaction terms, the free boundary condition for one species only. See also the work by El-Hachem et al. [7].

By Kolmogorov, Petrovskii and Piskunov [11], \( c_{min,u} = 2 > 0 \), \( c_{min,v} = -2\sqrt{ad_v} < 0 \). In [1], we proved the following theorem.

Theorem 1.1. For \( \alpha_1 \geq 0, \alpha_2 \geq 0, a > 0 \) and \( d_v > 0 \) satisfying \( \alpha_1 + \alpha_2 > 0 \), there exists a unique \( c^* \) depending on \( \alpha_1, \alpha_2, a \) and \( d_v \) such that (5) has a solution for \( c = c^* \). Moreover, the solution is unique and we have the following estimates:

1. \( \min \left\{ 2, \frac{1}{2} \sqrt{\frac{d_v \alpha_1}{d_v + a \alpha_2}} \right\} > c^* > 0 \) if \( \alpha_1 > \alpha_2 \sqrt{a \over 2} \),

2. \( c^* = 0 \) if \( \alpha_1 = \alpha_2 \sqrt{a \over 2} \), and

3. \( \max \left\{ -2\sqrt{ad_v}, -a \over 2 \sqrt{\alpha_2 \over 1 + \alpha_1} \right\} < c^* < 0 \) if \( \alpha_1 < \alpha_2 \sqrt{a \over 2} \).

Later Yang and Lou [21] extended above theorem in a more general form and a three-species semi-wave problem. Du, Wang and Zhou [5] studied a two species problem with the free boundary condition for one species only. See also the work by El-Hachem et al. [7]. The free boundary problem (1) with no reaction terms was originally studied in the melting process of ice in contact with water (see [17]).

In (1), the differential equation for the free boundary \( s(t) \) is usually called the two phase Stefan condition. (6) can be derived as follows. Let \( P(x,t) \), \( Q(x,t) \) be the temperature of water and ice respectively. Consider the time variable from \( t_0 \) to \( t_0 + \Delta t \) for some \( t_0 \geq 0 \). The zero Dirichlet boundary condition \( P(s(t),t) = Q(s(t),t) = 0 \) at the free boundary \( s(t) \) for \( t \in (t_0, t_0 + \Delta t) \) means the phase changed from water to ice, while the temperature does not change. During the change of phases, there is heat transformed from water to ice. The amount of heat per unit mass been transformed from one phase to another is called the latent heat \( \lambda \). Suppose the line density of the ice and water are both \( \rho \). Then the total mass in the interval \( (s(t_0), s(t_0 + \Delta t)) \) is \( \rho [s(t_0 + \Delta t) - s(t_0)] \). Hence the amount of heat needed for the water to change to the ice is \( \lambda \rho [s(t_0 + \Delta t) - s(t_0)] \). By Fick’s law, the flux corresponding to \( P \) and \( Q \) cross the free boundary is proportional to the gradient of \( P \) and \( Q \). Therefore we have

\[
\lambda \rho [s(t_0 + \Delta t) - s(t_0)] = [-d_1 P(s(t_0), t) - d_2 Q(s(t), t)] \Delta t.
\]

Dividing both sides of the above equation by \( \Delta t \) and letting \( \Delta t \) tend to zero, we obtain

\[
\lambda \rho s'(t_0) = -d_1 P(s(t_0), t_0) - d_2 Q(s(t_0), t_0)
\]
and (6) becomes
\[ s'(t) = -\frac{d_1}{\lambda \rho} P(s(t), t) - \frac{d_2}{\lambda \rho} Q(s(t), t), \quad t > 0. \]

If the new boundary condition above is carried into (1), we can consider the following problem with latent heat
\[
\begin{aligned}
P_t - d_1 P_{xx} &= P(a_1 - b_1 P), & t > 0, 0 < x < s(t), \\
Q_t - d_2 Q_{xx} &= Q(a_2 - b_2 Q), & t > 0, s(t) < x < l, \\
P &= Q = 0, & \lambda s'(t) = -\alpha_1 P_x - \alpha_2 Q_x, & x = s(t), \\
P(0, t) = M_1, & Q(l, t) = M_2, & t > 0, \\
s(0) &= s_0, & 0 < s_0 < l, \\
P(x, 0) &= P_0(x) \geq 0, & 0 \leq x \leq s_0, \\
Q(x, 0) &= Q_0(x) \geq 0, & s_0 < x < l,
\end{aligned}
\]

where \( \alpha_i \) can be \( \frac{d_i}{\rho} > 0 \) (\( i = 1, 2 \)) or more general values.

It was shown that system (7) can be considered as a singular limit of competition-diffusion systems as the inter-specific competition rates tend to infinity. See Dancer et al. [3] and Hilhorst et al. [10] for \( \lambda = 0 \) and Hilhorst et al. [9] for \( \lambda > 0 \).

Similar to (5), we consider the traveling wave problem for (7):
\[
\begin{aligned}
u'' + cu' + u(1-u) = 0, & z \in (-\infty, 0], \\
dv'' + cv' + v(a-v) = 0, & z \in [0, \infty], \\
u(0) = 0, & v(0) = 0, u(-\infty) = 1, v(\infty) = a, \\
\lambda c &= -\alpha_1 u'(0) - \alpha_2 v'(0).
\end{aligned}
\]

Applying Theorem 1.1 with suitable scaling, we are able to prove the following theorem, which shows that the latent heat retards the spreading of the traveling wave.

**Theorem 1.2.** For \( \alpha_1 > 0, \alpha_2 > 0, a > 0, d_v > 0, \lambda \geq 0, \) there exists a unique \( c_\lambda \) depending on \( \lambda, \alpha_1, \alpha_2, a \) and \( d_v \) such that (8) has a solution \( (u_\lambda, v_\lambda) \) for \( c = c_\lambda \). Moreover, the solution is unique, and \( c_\lambda \) is a continuous function in \( \lambda \) with \( \lim_{\lambda \to \infty} c_\lambda = 0 \) and satisfies the following properties.

1. If \( \alpha_1 > \alpha_2 \sqrt{\frac{a^2}{d_v}} \), then \( c_\lambda \) is strictly decreasing in \( \lambda \) and
\[ 0 < c_\lambda < \min \left\{ 2, \frac{1}{2} \sqrt{\frac{d_v \alpha_1}{\lambda d_v + ac_2}} \right\}. \]

2. If \( \alpha_1 = \alpha_2 \sqrt{\frac{a^2}{d_v}} \), then \( c_\lambda = c_0 = 0 \).

3. If \( \alpha_1 < \alpha_2 \sqrt{\frac{a^2}{d_v}} \), then \( c_\lambda \) is strictly increasing in \( \lambda \) and
\[ \max \left\{ -2 \sqrt{d_v a - \frac{\alpha_2}{\lambda + \alpha_1}} \right\} < c_\lambda < 0. \]

**Remark 1.3.** The above theorem shows that even in the zero latent heat case (\( \lambda = 0 \)), (8) has a unique traveling wave solution with speed \( c_0 \). Moreover, the monotone dependence of \( c_\lambda \) with respect to \( \lambda \) implies that \( |c_\lambda| \leq |c_0| \) and the increase of the latent heat reduces the traveling wave speed.

Theorem 1.2 indicates that the sign of \( \alpha_1 - \alpha_2 \sqrt{a^2/d_v} \) is crucial in determining the sign of the traveling wave speed. Actually its sign coincides with the sign of \( c_\lambda \). Therefore it is an interesting question if one can get upper/lower bound estimates for \( c_\lambda \) which also depend on the quantity \( \alpha_1 - \alpha_2 \sqrt{a^2/d_v} \). Our result is as follows.
Theorem 1.4. For $\alpha_1 > 0, \alpha_2 > 0, a > 0, d_v > 0, \lambda > 0$, the speed $c_\lambda$ satisfies the following estimates

1. 
\[
\frac{\alpha_1 - \alpha_2 \sqrt{a^3/d_v}}{\sqrt{3}(\lambda + \alpha_1 + \alpha_2 a/d_v)} < c_\lambda < \frac{\alpha_1 - \alpha_2 \sqrt{a^3/d_v}}{\sqrt{3} \lambda} \quad \text{if } \alpha_1 - \alpha_2 \sqrt{a^3/d_v} > 0,
\]

2. 
\[
\frac{\alpha_1 - \alpha_2 \sqrt{a^3/d_v}}{\sqrt{3}(\lambda + \alpha_1 + \alpha_2 a/d_v)} > c_\lambda > \frac{\alpha_1 - \alpha_2 \sqrt{a^3/d_v}}{\sqrt{3} \lambda} \quad \text{if } \alpha_1 - \alpha_2 \sqrt{a^3/d_v} < 0,
\]

3. 
\[
c_\lambda = O(\lambda^{-1}) \quad \text{as } \lambda \to \infty.
\]

Theorem 1.4 can provide information about $c_\lambda$ when $\lambda$ is large. On the other hand, when $\lambda$ is small or equals 0, the estimate for $c_0$ becomes important. For this case we use a variational approach developed by Heize [8] and M. Lucia, C. B. Muratov, and M. Novaga [12, 13] to deal with the problem. The next theorem describes such a variational criterion for $c_0$.

Let 
\[
w_1 = 1, \quad w_2 = -\frac{\alpha_2}{\alpha_1 d_v}, \quad F(w) = \begin{cases} 
-\frac{1}{2} w^2 + \frac{1}{4} w^3 & \text{if } w > 0 \\
-\frac{1}{2} d_v a w^2 - \frac{d_v^2 a_1}{3 \alpha_2} w^3 & \text{if } w \leq 0,
\end{cases}
\]

and 
\[
H_i = \{\eta(z) + w_i : ||\eta|| < \infty\}, \quad B_i = \{\eta(z) + w_i \in H_i : \int_{-\infty}^{\infty} e^z \eta^2_z dz = 1\}
\]

for $i = 1, 2$, where the norm $||\eta||$ is defined by 
\[
||\eta|| = \int_{-\infty}^{\infty} e^z [\eta^2 + \eta^2_z] dz.
\]

We consider the following infima of energy type quantities for $i = 1, 2$:
\[
\nu_i := -\inf_{\phi \in B_i} \left\{ \int_{-\infty}^{\infty} e^z |F(\phi) - F(w_i)| dz \right\}.
\]

Theorem 1.5. Assume $\alpha_1 > 0$ and $\alpha_2 > 0$. Then

1. 
\[
\nu_2 > 0 \quad \text{and } c_0 = \sqrt{2 \nu_2} \quad \text{if } \alpha_1 - \alpha_2 \sqrt{a^3/d_v} > 0,
\]

2. 
\[
\nu_1 > 0 \quad \text{and } c_0 = -\sqrt{2 \nu_1} \quad \text{if } \alpha_1 - \alpha_2 \sqrt{a^3/d_v} < 0,
\]

3. 
\[
c_\lambda = c_0 = 0 \quad \text{if } \alpha_1 - \alpha_2 \sqrt{a^3/d_v} = 0,
\]

An easy upper bound for $|c_0|$ is given by $c(\min, v) < c_0 < c(\min, u)$, which follows from Theorem 1.2. For lower bounds of $|c_0|$, we apply Theorem 1.5 to obtain the result:

Corollary 1.6 For $\alpha_1 > 0$ and $\alpha_2 > 0$, the following estimates hold.
1. If \( \alpha_1 - \alpha_2 \sqrt{a_3/d_v} > 0 \), let

\[
K_1(s) = \frac{2s^2[(F(w_2) - F(w_1)) - \frac{e^{-s}}{1 - e^{-s}} - (F(0) - F(w_2))]}{(w_1 - w_2)^2}.
\]

Then \( c_0 \geq \sqrt{\max_{s \geq 0} K_1(s)} > 0 \).

2. If \( \alpha_1 - \alpha_2 \sqrt{a_3/d_v} < 0 \), let

\[
K_2(s) = \frac{2s^2[(F(w_1) - F(w_2)) - \frac{e^{-s}}{1 - e^{-s}} - (F(0) - F(w_1))]}{(w_1 - w_2)^2}.
\]

Then \( c_0 \leq -\sqrt{\max_{s \geq 0} K_2(s)} < 0 \).

In the corollary above, we have \( F(0) = 0 \), which is a critical value of the function \( F(w) \). We keep the term \( F(0) \) there to indicate that \( K_1 \) and \( K_2 \) are related to the variational structure of the system.

We will prove Theorem 1.2 and Theorem 1.4 in Section 2 and prove Theorem 1.5 and Corollary 1.6 in Section 3.

2. Proof of Theorem 1.2 and Theorem 1.4. We decompose (8) into the condition

\[
\lambda c = -\alpha_1 p'(0) - \alpha_2 v'(0)
\]

and the following two differential equations

\[
\begin{align*}
&u' = p, \quad p' = -cp - u(1 - u), \quad z \in (-\infty, 0], \\
&(u, p)(-\infty) = (1, 0), \quad u(0) = 0.
\end{align*}
\]

\[
\begin{align*}
&v' = q, \quad q' = -\frac{1}{d_v} (cq + v(a - v)), \quad z \in [0, \infty), \\
&(v, q)(+\infty) = (a, 0), \quad v(0) = 0.
\end{align*}
\]

To solve (10) and (11), one standard method is to consider \( p = u' \) and \( q = v' \) and take them as functions of \( u \) and \( v \) respectively. That is, \( p = p(u, c) \) and \( q = q(v, c) \), where the dependence on \( c \) is also indicated. Then (10) and (11) can be rewritten respectively as

\[
\begin{align*}
\frac{dp}{du} &= -c - \frac{u(1 - u)}{p}, \quad p(1, c) = 0, \quad p(0, c) \leq 0, \\
\frac{dq}{dv} &= -\frac{c}{d_v} - \frac{v(a - v)}{d_v q}, \quad q(a, c) = 0, \quad q(0, c) \geq 0
\end{align*}
\]

and (9) becomes

\[
\lambda c = -\alpha_1 p(0, c) - \alpha_2 q(0, c)
\]

2.1. Proof of Theorem 1.2. Let \( \hat{\alpha}_1 = \alpha_1/\lambda \) and \( \hat{\alpha}_2 = \alpha_2/\lambda \). Then (9) becomes

\[
c = -\hat{\alpha}_1 u'(0) - \hat{\alpha}_2 v'(0)
\]

and Theorem 1.1 can be applied with the coefficients of the Stefan boundary condition in (1) being replaced by \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \). The existence and uniqueness of the solution \((u_\lambda, v_\lambda, c_\lambda)\) of (8) with \( \lambda > 0 \) becomes a direct consequence of Theorem 1.1. Since

\[
\hat{\alpha}_1 - \hat{\alpha}_2 \sqrt{a_3/d_v} = \frac{1}{\lambda} \left( \alpha_1 - \alpha_2 \sqrt{a_3/d_v} \right),
\]
it is easy to verify that the conclusions (1), (2) and (3) of Theorem 1.1 with coefficients \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \) give the corresponding upper bound and lower bound for \( c_\lambda \) stated in Theorem 1.2 when \( \lambda > 0 \).

Now we consider the case \( \lambda = 0 \). Let

\[
g(c) = -\alpha_1 p(0,c) - \alpha_2 q(0,c).
\]

Solving (8) with \( \lambda = 0 \) is equivalent to finding some \( c \) such that \( g(c) = 0 \). We recall that \( c_{\min,u} = 2 \) and \( c_{\min,v} = -2\sqrt{ad_v} \), which are the minimum speeds of the KPP traveling waves of \( u \) and \( v \) respectively, and \( p(0,c_{\min,u}) = 0 \) and \( q(0,c_{\min,v}) = 0 \). Since the right hand side of the equation (12) for \( p \) is decreasing in \( c \) and the slope of the unstable manifold at \( (u,p) = (1,0) \) in the \( u-p \) plane is \( 2/(c + \sqrt{4 + c^2}) \), we have that \( p(u,c) < 0 \) for \( 0 < u < 1 \) and \( p(u,c) \) is increasing in \( c \) and strictly increasing when \( c < c_{\min,u} \) for \( 0 \leq u < 1 \). See Volpert, Volpert and Volpert [18] and Lemma 2.2 in [1]. Similarly, we have that \( q(u,c) > 0 \) for \( 0 < v < a \) and \( q(u,c) \) is increasing in \( c \) and strictly increasing when \( c > c_{\min,v} \) for \( 0 \leq v < a \). Hence \( g(c) = -\alpha_1 p(0,c) - \alpha_2 q(0,c) \) is strictly decreasing for \( c \in \mathbb{R} \) since \( \alpha_1 \) and \( \alpha_2 \) are positive. Since \( c_{\min,v} < c_{\min,u} \), we have

\[
g(c_{\min,u}) = -\alpha_2 q(0,c_{\min,u}) < -\alpha_2 q(0,c_{\min,v}) = 0
\]

and

\[
g(c_{\min,v}) = -\alpha_1 p(0,c_{\min,v}) > -\alpha_1 p(0,c_{\min,u}) = 0.
\]

From the decreasing property of \( g(c) \), we conclude that there is a unique \( c_0 \) such that \( g(c_0) = 0 \).

Figure 1. The intersection between \( y = \lambda c \) and \( y = g(c) \), and the zeros of \( g(c) = 0 \) under three cases: Figure (a): \( g(0) > 0 \), \( 0 < c_\lambda < c_0 < c_{\min,u} \). Figure (b): \( g(0) = 0 \), \( c_\lambda = c_0 = 0 \). Figure (c): \( g(0) < 0 \), \( c_{\min,v} < c_0 < c_\lambda < 0 \).

To show \( c_\lambda \) is monotone in \( \lambda \), we consider \( c_\lambda \) as the first component of the intersection point of the line \( y = \lambda c \) and the curve \( y = g(c) \) in the \((c,y)\) plane (see Figure 1). Since \( g(c) \) is strictly decreasing in \( c \), one can directly verify that \( c_\lambda \) is strictly monotone as \( \lambda \) increases. The proof is complete.
2.2. Proof of Theorem 1.4. First we calculate $g(0)$ as in [1]. Let $c = 0$ in (12). Then $p$ satisfies
\[
\frac{dp}{d\mu} = - \frac{u(1 - u)}{p}, \quad p(1, c) = 0
\]
and
\[
\int p dp = - \int u(1 - u) d\mu.
\]
After some computation we obtain
\[
p = \pm (1 - u) \sqrt{\frac{1 + 2u}{3}}.
\]
Since $p(u, 0) \leq 0$, we take negative sign in the above identity. Therefore
\[
p(u, 0) = -(1 - u) \sqrt{\frac{1 + 2u}{3}} < 0 \quad \text{for } 0 < u < 1
\]
and $p(0, 0) = -1/\sqrt{3}$. By similar computation for $q$, we obtain
\[
q(0, 0) = \sqrt{\frac{\alpha^3}{3d_v}}.
\]
Therefore
\[
g(0) = 1 \sqrt{\frac{\alpha_1 - \alpha_2}{\sqrt{3d_v}}}.
\]
Next we estimate the change rate of $g(c)$ as $c$ varies. Let $\sigma(u, c) = \frac{\partial p}{\partial c}(u, c)$. Then
\[
\sigma(u, c) \geq 0 \quad \text{since as mentioned in the above } p(u, c) \text{ is increasing in } c \text{ and } \sigma(1, c) = 0.
\]
For $0 \leq u < 1$ and $c$ being fixed, we have
\[
\frac{d\sigma}{d\mu} = -1 + \frac{u(1 - u)\sigma}{p^2},
\]
Therefore
\[
\sigma(1, c) - \sigma(u, c) = \int_u^1 \left( -1 + \frac{\bar{u}(1 - \bar{u})\sigma}{p^2} \right) \, d\bar{u} \geq \int_u^1 (-1) \, d\bar{u} = u - 1
\]
for $0 < u < 1$, which implies
\[
\sigma(0, c) = \lim_{u \to 0^+} \sigma(u, c) \leq \lim_{u \to 0^+} (1 - u) = 1
\]
Similarly if we let $\tau(u, c) = \frac{\partial q}{\partial c}(u, c)$, then $\tau(u, c) \geq 0$ and $\tau(a, c) = 0$. For $0 \leq v < a$ and $c$ being fixed, one has
\[
\frac{d\tau}{d\mu} = - \frac{1}{d_v} + \frac{v(a - v)\tau}{d_v q^2},
\]
Therefore
\[
\tau(a, c) - \tau(v, c) = \int_v^a \left( - \frac{1}{d_v} + \frac{\bar{v}(1 - \bar{v})\tau}{d_v q^2} \right) \, d\bar{v} \geq \int_v^a \left( \frac{1}{d_v} \right) \, d\bar{v} = \frac{v - a}{d_v}
\]
for $0 < v < a$, which implies
\[
\tau(0, c) = \lim_{v \to 0^+} \tau(v, c) \leq \lim_{v \to 0^+} \left( \frac{a - v}{d_v} \right) = \frac{a}{d_v}
\]
Putting the above estimates together, we have
\[
0 \geq g'(c) \geq -\alpha_1 \cdot 1 - \alpha_2 \cdot \frac{a}{d_v}.
\]
We consider the case \( g(0) = \frac{1}{\sqrt{3}} (\alpha_1 - \alpha_2 \sqrt{a^3/d_v}) > 0 \) first. In this case, by Theorem 1.2, we know \( 0 < c_0 < c_{u, \min} \) if \( \lambda > 0 \). Let
\[
G(c) = \lambda c - g(c).
\]
Then \( G(c_0) = 0 \) and
\[
g(0) = G(c_0) - G(0) = \int_0^{c_0} G'(c) \, dc \leq \int_0^{c_0} [\lambda + \alpha_1 + \alpha_2 a/d_v] \, dc,
\]
which implies
\[
g(0) \leq c_0 (\lambda + \alpha_1 + \alpha_2 a/d_v)
\]
and
\[
\frac{\alpha_1 - \alpha_2 \sqrt{a^3/d_v}}{\sqrt{3}(\lambda + \alpha_1 + \alpha_2 a/d_v)} \leq c_0.
\]
On the other hand, since \( g'(c) \leq 0 \) and \( G'(c) = \lambda - g'(0) \geq \lambda \), we have
\[
g(0) = G(c_0) - G(0) = \int_0^{c_0} G'(c) \, dc \geq \int_0^{c_0} \lambda \, dc = \lambda c_0,
\]
which implies
\[
\frac{\alpha_1 - \alpha_2 \sqrt{a^3/d_v}}{\sqrt{3}\lambda} \geq c_0.
\]
From the above estimates, we obtain statement (1) of Theorem 1.2. Via similar arguments, we can obtain statement (2) of Theorem 1.2. Statement (3) follows from statements (1) and (2). The proof of the theorem is complete.

3. Variational characterization of \( c_0 \). Proof of Theorem 1.5
Let \( u \) and \( v \) be the solution in (8) with \( c = c_0 \). Note that \( g(c_0) = 0 \). Let \( y = d_v z \) for \( z \geq 0 \), \( y = z \) for \( z \leq 0 \), and \( \bar{v}(z) = -\frac{\alpha_2}{\alpha_1 d_v} v(d_v z) \). Then \( \bar{v} \) satisfies
\[
\bar{v}_{zz} + c_0 \bar{v}_z + \bar{v}(d_v a + \frac{d_v^2 \alpha_1}{\alpha_2} \bar{v}) = 0, \quad z \in [0, \infty)
\]
and by \( g(c_0) = 0 \),
\[
\bar{v}_z(0) = -\frac{\alpha_2}{\alpha_1} v_y(0) = u_y(0) = u_z(0).
\]
Therefore if we let
\[
w(z) = \begin{cases} u(z) & \text{if } z < 0 \\ \bar{v}(z) & \text{if } z \geq 0, \end{cases}
\]
the regularity theory of differential equations implies that \( w \) is a smooth function and satisfies
\[
w_{zz} + c_0 w_z + f(w) = 0, \quad z \in (-\infty, \infty),
\]
where
\[
f(w) = \begin{cases} w(1-w) & \text{if } w > 0 \\ w(d_v a + \frac{d_v^2 \alpha_1}{\alpha_2} \bar{w}) & \text{if } w \leq 0. \end{cases}
\]
Let
\[
w_1 = 1 \quad \text{and} \quad w_2 = -\frac{\alpha_2}{\alpha_1 d_v}.
\]
The function \( w \) can be considered as a bistable traveling front solution of (22) connecting the steady states \( w_1 \) and \( w_2 \). Let

\[
F(w) = -\int_0^w f(s) \, ds = \begin{cases} 
- \frac{1}{2} w^2 + \frac{1}{3} w^3 & \text{if } w > 0 \\
- \frac{1}{2} d_0 aw^2 - \frac{d_0^2 \alpha_1}{3} w^3 & \text{if } w \leq 0.
\end{cases}
\]

Then by the theory of bistable traveling waves, \( c_\lambda > 0 \) if and only if

\[
F(w_2) - F(w_1) = \frac{1}{6} (1 - \frac{\alpha_2 a^3}{\alpha_1 d_0}) > 0,
\]

which is equivalent to the condition \( g(0) = \frac{1}{\sqrt{3}}(\alpha_1 - \alpha_2 \sqrt{a^3/d_0}) > 0 \) mentioned in Theorem 1.2.

Now we consider the case \( g(0) > 0 \). We further rescale the space variable and set \( \bar{z} = c_\lambda z \). Equation (22) becomes

\[
\epsilon^2_0 (w_{\bar{z}} + w_{\bar{z}z}) + f(w) = 0, \quad \bar{z} \in (-\infty, \infty).
\]

To simplify the notations, we still denote \( \bar{z} \) by \( z \). We consider the energy

\[
E_c(\phi) = \int_{-\infty}^{\infty} e^{\frac{\epsilon^2_0}{2} z^2} \phi^2 + F(\phi) - F(w_2) \, dz.
\]

and consider the minimization problem of it in the affine spaces

\[
H = \{ \eta(z) + w_2 : ||\eta|| < \infty \} \quad \text{and} \quad B = \{ \eta(z) + w_2 \in H : \int_{-\infty}^{\infty} e^{\frac{1}{2} z^2} \eta^2 \, dz = 1 \},
\]

where the norm \( ||\eta|| \) is defined by

\[
||\eta|| = \int_{-\infty}^{\infty} e^{\frac{1}{2} z^2} [\eta^2 + \eta^2] \, dz.
\]

Note that \( B \subset H \). By the definition of \( E_c \) and \( B \), it is easy to verify the following lemma.

**Lemma 3.1.** We have the properties:

1. \( E_c(\phi(z - \theta)) = e^{\theta} E_c(\phi(z)) \) for \( \theta \in \mathbb{R} \);
2. if \( \phi(z) \in H \) and \( \neq w_2 \), then there is \( \theta \in \mathbb{R} \) such that \( \phi(z - \theta) \in B \).

We consider the infimum of the energy on the constraint space \( B \)

\[
\mu_c = \inf_{\phi \in B} E_c(\phi)
\]

such that one traveling wave can be selected among all its translation profiles. Using the variational approach developed by Heize [8] and M. Lucia, C. B. Muratov, and M. Novaga [12, 13] to deal with the problem, one can obtain the solution \( w \) of (24) by solving the problem \( \inf E_c(\phi) = \mu_c \) if \( c \) is chosen suitably. See also Chen, Chen and Huang [2] for related details. Hence we have

\[
E_c(w) = \inf_{\phi \in B} E_c(\phi) = \mu_c
\]

for a suitable \( c \). One difficulty in solving the minimization problem comes from the situation that \( c \) is not determined in advance. In the following, we give a necessary condition for determining \( c \). By the theory of bistable traveling waves, up to a translation, (24) has a unique solution \( (w, c_0) \). It can be verified that \( w \in H \) and \( w(z - \theta) \in B \) for a suitable translation \( \theta \).

**Lemma 3.2.** Let \( (w, c_0) \) be the solution of (24) with \( c_0 > 0 \), then \( E_{c_0}(w) = 0 \).
Proof. Let \( \alpha, \beta \in \mathbb{R} \) and \( \alpha < \beta \). Multiplying (24) by \( e^z w_z \) and integrating by parts, we obtain

\[
0 = \int_{\alpha}^{\beta} e^z w_z \cdot \left[ c_0^2 (w_{zz} + w_z) - F'(w) \right] dz
\]

\[
= \int_{\alpha}^{\beta} e^z c_0^2 \left[ \frac{1}{2} (w_z^2)_{x} + w_z^2 \right] dz + \int_{\alpha}^{\beta} e^z \left[ - (F(w) - F(w_z)) \right] dz
\]

\[
= \left[ e^z \left( \frac{c_0^2}{2} w_z^2 - F(w) + F(w_z) \right) \right]_{z=\alpha}^{z=\beta}
\]

\[
+ \int_{\alpha}^{\beta} e^z \left[ \frac{c_0^2}{2} w_z^2 + F(w) - F(w_z) \right] dz.
\]

Since \( w \in H \), we can choose two sequences \( \alpha_k \) and \( \beta_k \) such that \( \alpha_k \to -\infty, \beta_k \to \infty \) and the boundary terms at \( z = \alpha_k \) and \( \beta_k \) converge to 0 as \( k \to \infty \). Then we obtain \( E_{c_0}(w) = 0 \).

By Lemma 3.2, equation (26), and Lemma 3.1, we conclude

\[
\mu_{c_0} = 0. \tag{27}
\]

Since

\[
\mu_c := \inf_{B} E_c,
\]

\[
= \frac{c^2}{2} + \inf_{\phi \in B} \left\{ \int_{-\infty}^{\infty} e^z [F(\phi) - F(w_\beta)] \, dz \right\},
\]

we know \( \mu_c \) is strictly increasing in \( c \) when \( c \geq 0 \) and \( c_0 \) is the only value satisfying (27). We thus obtain the variational characterization of \( c_0 \) in the case \( g(0) > 0 \).

For the case \( g(0) < 0 \). We can obtain the variational characterization of \( c_0 \) by similar arguments. The statement for the case \( g(0) = 0 \) follows from Theorem 1.2. The proof is complete.

**Proof of Corollary 1.6** First we consider the case \( g(0) > 0 \), which is equivalent to \( F(w_2) > F(w_1) \) as mentioned in the proof of Theorem 1.5. By Lemma 3.2, equation (26), and the uniqueness of \( c_0 \), we have

\[
E_{c_0}(w) = \inf_{\phi \in B} E_{c_0}(\phi) = \mu_{c_0} = 0. \tag{28}
\]

Since \( \mu_c \) is strictly increasing in \( c \) when \( c \geq 0 \), we conclude that \( \mu_{c} < 0 \) if and only if \( 0 < c < c_0 \) when considering a nonnegative \( c \). Moreover if for some \( \bar{c} \) there is \( \phi \in H \) such that \( E_{c_0}(\phi(z)) < 0 \), then we can find a \( \theta \) with \( \phi(z - \theta) \in B \) and obtain \( \mu_{\bar{c}} \leq E_{c_0}(\phi(z - \theta)) = e^{\bar{c} \theta} E_{c_0}(\phi(z)) < 0 \) by Lemma 3.1, which implies \( \bar{c} < c_0 \). That is, \( \bar{c} \) is a lower bound of \( c_0 \) if we can find a \( \phi \in H \) such that \( E_{c_0}(\phi(z)) < 0 \). Let

\[
K_1(s) = \frac{2s^2 [F(w_2) - F(w_1)] - e^{-s} - (F(0) - F(w_2))}{(w_1 - w_2)^2}.
\]

be defined in Corollary 1.6. Since both \( F(w_2) - F(w_1) \) and \( F(0) - F(w_2) \) are positive, we have \( K_1(0) = 0 \), \( K_1(s) \to -\infty \) as \( s \to \infty \), and \( \max_{s \geq 0} K_1(s) > 0 \). Now
for \( \bar{c} < \sqrt{\max_{s \geq 0} K_1(s)} \) and \( l > 0 \), we take the test function
\[
\phi(z) = \begin{cases} 
  w_2 & \text{if } z \geq 0 \\
  w_1 & \text{if } z \leq -l \\
  w_2 \frac{l + z}{l} + w_1 \frac{-z}{l} & \text{if } -l < z < 0 
\end{cases}
\]
Then we have \( \phi \in H \) and
\[
E_{\bar{c}}(\phi) = \int_{-\infty}^{\infty} e^{\frac{\bar{c}^2}{2} z^2 + F(\phi) - F(w_2)} \, dz
\]
\[
= \int_{-l}^{0} + \int_{0}^{\infty} + \int_{-\infty}^{-l},
\]
\[
\int_{-\infty}^{-l} = -(F(w_2) - F(w_1)) e^{-l},
\]
\[
\int_{0}^{l} \leq \left[ \frac{\bar{c}^2 (w_1 - w_2)^2}{2 l^2} + (F(0) - F(w_2)) \right] (1 - e^{-l})
\]
since \( F(\phi) - F(w_2) \leq F(0) - F(w_2) \) for \( w_2 \leq \phi \leq w_1 \), and
\[
\int_{0}^{\infty} = 0.
\]
Putting the estimates together, we have
\[
E_{\bar{c}}(\phi) \leq \frac{(w_1 - w_2)^2}{2 l^2} [\bar{c}^2 - K_1(l)].
\]
If we choose \( l \) such that \( K_1(l) = \max_{s \geq 0} K_1(s) \), then \( E_{\bar{c}}(\phi) < 0 \) since \( \bar{c}^2 < \max_{s \geq 0} K_1(s) \). Hence we conclude that \( c_0 \geq \bar{c} \) for any \( 0 \leq \bar{c} < \sqrt{\max_{s \geq 0} K_1(s)} \) and obtain part (1) of Corollary 1.6. The proof for the case \( g(0) < 0 \) is similar. We omit the details here.

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E-mail address: changjuexin@thu.edu.tw
E-mail address: chchchen@math.ntu.edu.tw
E-mail address: loveworldsteven@hotmail.com