EXPERIMENTS WITH THE CENSUS

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ABSTRACT. In this paper we study the manifolds in the census of “small” 3-manifolds as available in SnapPy. We compare our results with the statistics of random 3-manifolds obtained using the Dunfield Thurston and Rivin models.

1. INTRODUCTION

In past work ([Riv14]) we have studied the statistics of random manifolds fibering over the circle, using a model (the “Rivin Model”) similar in spirit to that used by N. Dunfield and W.P.Thurston in [DT06]. The distributions of manifolds produced by both models is clearly skewed (the manifolds tend to be "long and skinny"). Many have argued that the "right" model is that of randomly gluing tetrahedra together, then throwing out those gluings that are not manifolds. Unfortunately, this is not at all probabilistically tractable, so we do the next best thing and look at all manifolds possessing small triangulations, thanks to the census of such manifolds built into SnapPy - [CDGW]. This paper started life as a Jupyter notebook.

2. PRELIMINARIES

First, we import the usual (and some unusual) libraries:

In [1]: from snappy import *
from multiprocessing import Pool
import pandas as pd
import numpy as np
import functools
from operator import mul
import xgboost as xg
import matplotlib.pyplot as plt
import seaborn as sns
from sklearn.model_selection import train_test_split
from sklearn.model_selection import GridSearchCV
from sklearn.cluster import KMeans
from sklearn.manifold import TSNE

Now we define some utility functions to deal with SnapPy’s goofy formats.

First, the length spectrum

In [2]: def mung_spec(specline):
    mult = specline['multiplicity']

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thelen = specline['length']
return [thelen]*mult

In [3]: def mung_all_spec(thespec):
    thelens = [mung_spec(i) for i in thespec]
    return sum(thelens, [])

In [4]: def mung_complexes(clist):
    cs = [[float(x.real()), float(x.imag())] for x in clist]
    return sum(cs, [])

Now, put all about the manifold in one line. The function returns None if the Dirichlet domain can not be constructed, so the manifold is not hyperbolic:

In [5]: def getline(m, cutoff=3.0, numcurves=10):
    try:
        namelist = [m.name()]
        thespec = m.length_spectrum(cutoff=cutoff)
        lenlist = mung_complexes(mung_all_spec(thespec))
        vollist = [float(m.volume())]
        homo = m.homology()
        qr = int(homo.betti_number())
        ed = [int(i) for i in homo.elementary_divisors() if i > 0]
        torsion = functools.reduce(mul, ed, 1)
        res = namelist + lenlist[:numcurves] + vollist + [qr, torsion]
    except:
        res = None
    return res

For some reason, it’s faster to first read in a list, then iterate over it:

In [6]: zoo = list(OrientableClosedCensus)

Now, read everything in:

In [7]: thelines = [getline(i) for i in zoo]
In [12]: thelines = [i for i in thelines if i is not None]
In [13]: linedf = pd.DataFrame.from_records(thelines, columns = ['name', 'a', 'A', 'b',
In [14]: len(zoo)
Out[14]: 11031
In [15]: len(thelines)
Out[15]: 10963

3. First Results

We see that out of the 11031 manifolds, 68 are not hyperbolic.

The lower case letters columns are of the shortest (5) geodesics, and the capital letters are of the twisting.

How about the homology, how is that distributed?

In [17]: linedf.betti.value_counts()
Out[17]: 0  10836
       1    126
       2     1
Name: betti, dtype: int64

We see that 10836 out of the 10963 hyperbolic manifolds (or close to 99%) are rational homology spheres. This is consistent with the Dunfield-Thurston model, and also with random fibered manifolds having $b_1 = 1$ with probability approaching 1.

There are a number of results (notably by Culler and Shalen with co-authors) on the influence of Betti numbers on volume. Let’s see what we see here:

In [18]: linedf.boxplot(column=['volume'], by='betti')

Out[18]: <matplotlib.axes._subplots.AxesSubplot at 0x1d270e45a90>

Of course, the number of samples is small in the higher Betti numbers groups, but it is interesting that the volume decreases as the Betti number increases.

What about torsion?

In [19]: linedf.torsion.hist(bins=30)

Out[19]: <matplotlib.axes._subplots.AxesSubplot at 0x1d2713358d0>
We notice that the highest values of torsion are quite large (given how small our manifolds are). Let’s see how torsion and volume are related.

In [20]: sns.jointplot(x='torsion', y='volume', data=linedf)

Out[20]: <seaborn.axisgrid.JointGrid at 0x1d271397e10>
We see that high torsion leads to large volume, though not the other way around. This is different from the Dunfield-Thurston and random fibered models, where both volume and log of torsion grow linearly with complexity. Speaking of log torsion, let’s take a look.

```python
In [21]: linedf['logtor'] = np.log(linedf.torsion)

In [22]: sns.jointplot(x='logtor', y='volume', data = linedf)
```

```
Out[22]: <seaborn.axisgrid.JointGrid at 0x1d270e45828>
```
We see that both volume and log torsion are potentially tending to a gaussian distribution. We also see the linear density cutoff in the center graph, showing that the random models do have something going for them. The graph also seems to indicate that $V(M) \geq a \log \text{tor}(M) + b$, for some $a > 0$.

4. Pairwise relationships

In [24]: sns.pairplot(linedf)

Out[24]: <seaborn.axisgrid.PairGrid at 0x1d2727ee908>
Above, on the diagonal we have the histograms of the various columns and the off-diagonal cells are the scatter plots of columns against one another. We see many interesting phenomena.

1. Notice that the imaginary parts of the complex lengths (the rotations) are NOT equidistributed - they are well-separated from 0 (and \( \pi \)) - they seem to become somewhat less so for longer geodesics.
2. The length of the systole (shortest geodesic) is exponentially distributed, while
3. The lengths of the \( k \)-th shortest geodesics become more and more normally distributed as \( k \) becomes large.
4. Both the real and the imaginary parts of the lengths seem to be uncorrelated (aside from the obvious relation of ordering on the real parts).
5. Log of torsion seems more-or-less normally distributed.
The scatter graphs of volume vs the other observables are also interesting. Let’s see if there is any volume between volume and the systole:

In [25]: sns.regplot(x='a', y='volume', data=linedf)

Out[25]: <matplotlib.axes._subplots.AxesSubplot at 0x1d27af28710>

While knowing that the systole is short gives us little information on the volume, knowing that it is long tells us that the volume is small, which is a bit counter-intuitive, at least to this author. Also, it is pretty that an inequality of the form $V(M) \leq as(M) + b$, where $s(M)$ is the systole length, and $a < 0$, holds.

What about other geodesics?

In [26]: sns.regplot(x='c', y='volume', data=linedf)

Out[26]: <matplotlib.axes._subplots.AxesSubplot at 0x1d27e1cff98>
The second longest curve has some slight (but *positive*) predictive power.

```
In [27]: sns.regplot(x='d', y='volume', data=linedf)
Out[27]: <matplotlib.axes._subplots.AxesSubplot at 0x1d27e1d1b70>
```
Same with the third...

5. THE SPACE OF 3-MANIFOLDS

The question is now, whether we can deduce any of the observables if we know the
others. For example, can we predict the volume from the length spectrum?

There is an existence proof, and a construction. For existence, let's see if there are
approximate linear relationships between our many fields (let's drop the homological
invariants for now):

In [28]: linedfmin = linedf[['a', 'A', 'b', 'B', 'c', 'C', 'd', 'D', 'e', 'E', 'volume']]
In [29]: u, s, v = np.linalg.svd(linedfmin)
In [30]: s
Out[30]: array([573.37142695, 257.38960156, 238.87874045, 218.20623087,
                    209.10291146, 203.66714934, 45.86448121, 24.94621714,
                    17.98065278, 13.50396573, 10.96093658])

We see that most of the energy is contained in the top six singular values, so the space
is approximately six dimensional (as a linear space).

Are the twist parameters redundant somehow?

In [51]: linedfmin2 = linedf[['a', 'b', 'c', 'd', 'e', 'volume']]
In [52]: u, s, v = np.linalg.svd(linedfmin2)
In [53]: s
Out[53]: array([572.95525389, 45.89914386, 24.99847935, 17.9864745 ,
                    13.50661835, 10.96897694])

Not at all! It looks like the space of volume and the five lengths, only one or two
dimensions are significant!

Let's try to go another way and see if knowing the length spectrum we can predict the
volume. For this we will use boosting - a very effective machine learning technique.

In [69]: featdf = linedfmin2[linedfmin2.columns[:-1]]
In [70]: targdf = linedfmin2['volume']
In [71]: X_train, X_test, y_train, y_test = train_test_split(featdf, targdf)
In [72]: clf1 = xg.XGBRegressor()
In [73]: parameters = {'objective':'[reg:linear]', 'learning_rate': [0.1, .3, 0.5], 'max_
                    In [74]: clf = GridSearchCV(clf1,parameters, cv = 2, n_jobs = 5, verbose=True)
In [75]: clf.fit(X_train,y_train)
Fitting 2 folds for each of 9 candidates, totalling 18 fits

[Parallel(n_jobs=5)]: Using backend LokyBackend with 5 concurrent workers.
[Parallel(n_jobs=5)]: Done 18 out of 18 | elapsed: 41.9s finished

Out[75]: GridSearchCV(cv=2, error_score='raise-deprecating',
                      estimator=XGBRegressor(base_score=0.5, booster='gbtree',
                      colsample_bylevel=1, colsample_bynode=1,
                      gamma=0, learning_rate=0.1, max_delta_step=0,
                      max_depth=3, min_child_weight=1, missing=None,
                      n_estimators=100, n_jobs=1, nthread=None,
                      objective='reg:linear', random_state=0,
reg_alpha=0, reg_lambda=1, scale_pos_weight=1, seed=None, silent=True, subsample=1),
fit_params=None, iid='warn', n_jobs=5, param_grid={'objective': [reg:linear], 'learning_rate': [0.1, 0.3, 0.5], 'max_depth': [5, 6, 7], 'min_child_weight': [9], 'silent': [1], 'subsample': [0.6], 'colsample_bytree': [0.8], 'n_estimators': [1000]},
pre_dispatch='2*n_jobs', refit=True, return_train_score='warn', scoring=None, verbose=True)

In [76]: preds = clf.predict(X_test)
In [101]: np.linalg.norm(preds-y_test)/np.sqrt(len(y_test))
Out[101]: 0.7417280255021659
In [ ]:
In [43]: y_test.describe()
Out[43]:
         count      2741.000000
        mean    5.061871
         std    0.805381
        min    1.529477
       25%    4.626565
       50%    5.228348
       75%    5.657743
        max    6.453448
Name: volume, dtype: float64

So we explain about 10% of the standard deviation, or 20% of the variance...

What if we use all the complex length info?

In [103]: featdf = linedfmin[linedfmin.columns[:-1]]
targdf = linedfmin['volume']
X_train, X_test, y_train, y_test = train_test_split(featdf, targdf)
clf1 = xg.XGBRegressor()
clf = GridSearchCV(clf1, parameters, cv = 2, n_jobs = 5, verbose=True)
clf.fit(X_train,y_train)
Fitting 2 folds for each of 9 candidates, totalling 18 fits

[Parallel(n_jobs=5)]: Using backend LokyBackend with 5 concurrent workers.
[Parallel(n_jobs=5)]: Done 18 out of 18 | elapsed: 1.2min finished

Out[103]: GridSearchCV(cv=2, error_score='raise-deprecating',
estimator=XGBRegressor(base_score=0.5, booster='gbtree',
colsample_bylevel=1, colsample_bytree=1, gamma=0, learning_rate=0.1, max_delta_step=0,
max_depth=3, min_child_weight=1, missing=None, n_estimators=100,
n_jobs=1, nthread=None, objective='reg:linear', random_state=0,
reg_alpha=0, reg_lambda=1, scale_pos_weight=1, seed=None,
silent=True, subsample=1),
fit_params=None, iid='warn', n_jobs=5,
param_grid={'objective': ['reg:linear'], 'learning_rate': [0.1, 0.3, 0.5],
pre_dispatch='2*n_jobs', refit=True, return_train_score='warn',
scoring=None, verbose=True)
In [104]: preds = clf.predict(X_test)
    np.linalg.norm(preds-y_test)/np.sqrt(len(y_test))
Out[104]: 0.5423190901708408

So we are doing pretty well (getting about 40% of the information).

In [63]: from sklearn.linear_model import LinearRegression
In [78]: reg = LinearRegression()
In [79]: reg.fit(X_train, y_train)
Out[79]: LinearRegression(copy_X=True, fit_intercept=True, n_jobs=None,
    normalize=False)
In [80]: rpred = reg.predict(X_test)
In [100]: np.linalg.norm(y_test-rpred)/np.sqrt(len(y_test))
Out[100]: 0.7917050890601595

We see that the first three lengths in the length spectrum give us pretty much all the information!

In [84]: newdf = linedf[ ['logtor', 'a', 'volume']]
In [85]: u, s, v = np.linalg.svd(newdf)
In [86]: s
Out[86]: array([ 659.45727388,  106.79900214,  15.42933463])
In [105]: reg2 = LinearRegression()
In [107]: reg2.fit(newdf[ ['logtor', 'volume']], newdf.a)
Out[107]: LinearRegression(copy_X=True, fit_intercept=True, n_jobs=None,
    normalize=False)
In [108]: reg2.coef_
Out[108]: array([-0.02357474, -0.01766166])

References

[CDGW] Marc Culler, Nathan M. Dunfield, Matthias Goerner, and Jeffrey R. Weeks. SnapPy, a computer program for studying the geometry and topology of 3-manifolds. Available at http://snappy. computop.org (DD/MM/YYYY).

[DT06] Nathan M Dunfield and William P Thurston. Finite covers of random 3-manifolds. Inventiones mathematicae, 166(3):457–521, 2006.

[Riv14] Igor Rivin. Statistics of random 3-manifolds occasionally fibering over the circle. arXiv preprint arXiv:1401.5736, 2014.

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