CHARACTER FORMULA FOR 
INFINITE DIMENSIONAL UNITARIZABLE MODULES 
OF THE GENERAL LINEAR SUPERALGEBRA

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Abstract. The Fock space of \(m+p\) bosonic and \(n+q\) fermionic quantum oscillators forms a unitarizable module of the general linear superalgebra \(gl_{m+p|n+q}\). Its tensor powers decompose into direct sums of infinite dimensional irreducible highest weight \(gl_{m+p|n+q}\)-modules. We obtain an explicit decomposition of any tensor power of this Fock space into irreducibles, and develop a character formula for the irreducible \(gl_{m+p|n+q}\)-modules arising in this way.

Key words: Lie superalgebra, unitarizable representations, Howe duality, character formula.

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1. Introduction

The Fock space of \(m+p\) bosonic and \(n+q\) fermionic quantum oscillators (see Subsection 3.3 for definition) with the standard inner product furnishes a unitarizable complex representation of the real form \(u(m, p|n, q)\) of the general linear superalgebra \(gl_{m+p|n+q}\). This representation decomposes into a direct sum of infinite dimensional irreducible representations which are of highest weight type with respect to an appropriate choice of a Borel subalgebra. Because of the unitarity, any tensor power of the representation is also semi-simple with all irreducible sub-representations being unitarizable highest weight representations. We shall characterize the irreducible sub-representations and determine their structure.

In recent years there have been considerable activities (see, e.g., 7 for references) in the physics community to study unitarizable highest weight representations of Lie superalgebras. This is motivated by applications of such representations in quantum field theory. For example, the symmetry algebra of the yet largely conjectural \(M\)-theory is closely related to \(osp_{1|32}(\mathbb{R})\) 23. An understanding of the unitarizable highest weight representations of this Lie superalgebra will help to solve mysteries of \(M\)-theory. It has also been recognized 9 that some real

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forms of simple basic classical Lie superalgebras provide the conformal superalgebras of higher dimensional space-time manifolds with extended supersymmetries. The unitarizable highest weight representations of these Lie superalgebras thus describe the spectra of possible elementary particles existing in such space-times.

The problem of determining the possible unitarizable irreducible highest weight representations of real forms of simple Lie superalgebras was investigated by a number of people (see [7] and references therein), with the most systematical study given in [16]. However, a classification analogous to the Enright-Howe-Wallach [11] classification of unitarizable positive energy irreducible representations for ordinary real simple Lie algebras has yet to be achieved (see Subsection 3.3).

A demanding but physically more important problem is to understand the structure of the unitarizable irreducible representations. Recall that a character formula for the unitarizable irreducible highest weight representations of real forms of simple Lie algebras [11] was given in [10] some fifteen years ago. In an earlier publication [6], two of the authors studied the irreducible representations arising from the decomposition of the tensor powers of the oscillator representations of the orthosymplectic superalgebras. By using results of [8, 10], a character formula for these irreducible representations was derived. In this paper we investigate the case of the general linear superalgebra.

It is known from [14] that \( u(d) \) and \( u(m, p|n, q) \) form a dual reductive pair on the \( d \)-th tensor power of the Fock space of \( m + p \) bosonic and \( n + q \) fermionic quantum oscillators. We explore the duality between the complexifications of these Lie (super)algebras to obtain in Theorem 3.3 an explicit decomposition of the tensor power of the Fock space into irreducible \( gl_d \times gl_{m+p|n+q} \)-modules. The Howe duality again as in [2, 6] forms the key ingredient and further enables us to compute the characters for the irreducible \( gl_{m+p|n+q} \)-representations of Theorem 3.3. This result is presented in Theorem 5.3. Another application of the Howe duality is the computation of the tensor product decomposition of any two such unitarizable modules, which is the content of Theorem 6.1.

Here is an outline of the paper. In Section 2 we discuss the \( (gl_d, gl_{m|n}) \)-dualities on \( S(C^d \otimes C^m|n) \) and its graded dual space. The material is largely known, but the highest weight vectors in the graded dual of \( S(C^d \otimes C^m|n) \) given in Lemma 2.2 have not been computed previously as far as we are aware of. In Section 3 we study the \( gl_{m+p|n+q} \)-representations furnished by tensor powers of the Fock space of \( m + p \) bosonic and \( n + q \) fermionic quantum oscillators. In Subsection 3.2 we show that such representations are unitarizable and their irreducible sub-representations are infinite dimensional highest weight representations, and in Subsection 3.4 we obtain the explicit decomposition of the \( d \)-th tensor power of the Fock space with respect to the semi-simple multiplicity free action of \( gl_d \times gl_{m+p|n+q} \). Section 4
gives the $gl_{m+p|n+q} \rightarrow gl_{p|q} \times gl_{m|n}$ branching rule for the infinite dimensional unitarizable irreducible $gl_{m+p|n+q}$-representations arising from the decomposition of tensor powers of the Fock space. In Section 5 we develop a character formula for these infinite dimensional irreducible $gl_{m+p|n+q}$-representations in terms of hook Schur functions. Finally in Section 6 we calculate the tensor product decomposition of two such irreducible $gl_{m+p|n+q}$-modules that appear in our decompositions of tensor powers.

2. TENSORIAL REPRESENTATIONS OF GENERAL LINEAR SUPERALGEBRA

This section presents some results on the $(gl_d, gl_{m|n})$-dualities on $S(C^d \otimes C^{m|n})$ and its graded dual vector space. The material contained here will be important for the remainder of the paper.

2.1. Preliminaries. We work on the field $C$ of complex numbers throughout the paper. Let $gl_d$ denote the Lie algebra of all complex $d \times d$ matrices. Let $\{e^1, \ldots, e^d\}$ be the standard basis for $C^d$. Denote by $e_{ij}$ the elementary matrix with 1 in the $i$-th row and $j$-th column and 0 elsewhere. Then $h_d = \sum_{i=1}^d C e_{ii}$ is a Cartan subalgebra, while $b_d = \sum_{1 \leq i \leq j \leq d} C e_{ij}$ is the standard Borel subalgebra containing $h_d$. The weight of $e^i$ is denoted by $\bar{e}_i$ for $1 \leq i \leq d$.

Let $C^{m|n} = C^{m|0} \oplus C^{0|n}$ denote the $m|n$-dimensional superspace. The super-space of complex linear transformations on $C^{m|n}$ has a natural structure of a Lie superalgebra [17], which we will denote by $gl_{m|n}$. Choose a basis $\{e_1, \ldots, e_m\}$ for the even subspace $C^{m|0}$ and a basis $\{f_1, \ldots, f_n\}$ for the odd subspace $C^{0|n}$, then $\{e_1, \ldots, e_m, f_1, \ldots, f_n\}$ is a homogeneous basis for $C^{m|n}$. We may regard $gl_{m|n}$ as consisting of $(m + n) \times (m + n)$ matrices relative to this basis. Denote by $E_{ij}$ the elementary matrix with 1 in the $i$-th row and $j$-th column and 0 elsewhere. Then $h_{m|n} = \sum_{i=1}^{m+n} C E_{ii}$ is a Cartan subalgebra, while $b_{m|n} = \sum_{1 \leq i \leq j \leq m+n} C E_{ij}$ is the standard Borel subalgebra containing $h_{m|n}$. We shall denote the weights of $e_i$ and $f_j$ by $\bar{e}_i$ and $\bar{f}_j$ respectively for $i = 1, \ldots, m$, and $j = 1, \ldots, n$.

By a partition $\lambda$ of length $k$ we mean a non-increasing finite sequence of non-negative integers $(\lambda_1, \ldots, \lambda_k)$. We will let $\lambda^t$ denote the transpose of the partition $\lambda$. For example, if $\lambda = (4, 3, 1, 0, 0)$, then the length of $\lambda$ is 5 and $\lambda^t = (3, 2, 2, 1)$. By a generalized partition of length $k$, we shall mean a non-increasing finite sequence of integers $(\lambda_1, \ldots, \lambda_k)$. In particular, every partition is a generalized partition of non-negative integers. Corresponding to each generalized partition $\lambda = (\lambda_1, \ldots, \lambda_d)$, we will define $\lambda^* := (-\lambda_d, \ldots, -\lambda_1)$. Then $\lambda^*$ is also a generalized partition.

We regard a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_d)$ of complex numbers as an element of the dual vector space $h^*_{d}$ of $h_d$ defined by $\lambda(e_{ii}) = \lambda_i$, for $i = 1, \ldots, d$. Denote by $V_d^\lambda$ the irreducible $gl_d$-module with highest weight $\lambda$ relative to the standard Borel subalgebra $b_d$. Similarly, we shall also regard a finite sequence of complex
numbers $\lambda = (\lambda_1, \cdots, \lambda_{m+n})$ as an element of the dual vector space $h^*_{m|n}$ of $h_{m|n}$ such that $\lambda(E_{jj}) = \lambda_j$, $1 \leq j \leq m + n$. We denote by $V^\lambda_{m|n}$ the irreducible $gl_{m|n}$-module with highest weight $\lambda$ relative to the standard Borel subalgebra $b_{m|n}$.

2.2. The $(gl_d, gl_{m|n})$-duality on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$. Recall that the natural action of the Lie superalgebra $gl_d \times gl_{m|n}$ on $\mathbb{C}^d \otimes \mathbb{C}^{m|n}$ induces an action on the symmetric tensor algebra $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$. This action is completely reducible and multiplicity free \cite{14, 22, 3, 4}. Indeed the pair $(gl_d, gl_{m|n})$ forms a dual reductive pair on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ in the sense of Howe \cite{14, 15}.

Theorem 2.1. \cite{3} Under the $gl_d \times gl_{m|n}$-action, $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ decomposes into

\begin{equation}
S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \cong \sum_{\lambda} V^\lambda_d \otimes \tilde{V}^\lambda_{m|n},
\end{equation}

where the sum in (2.1) is over all partitions $\lambda$ of length $d$ subject to the condition $\lambda_{m+1} \leq n$, and

\begin{equation}
\tilde{\lambda} = (\lambda_1, \cdots, \lambda_m; < \lambda'_1 - m >, \cdots, < \lambda'_n - m >).
\end{equation}

Here $\lambda'$ is the transpose partition of $\lambda$, and $< r >$ stands for $r$, if $r \in \mathbb{N}$, and 0 otherwise.

Remark 2.1. The condition $\lambda_{m+1} \leq n$ is considered to be automatically satisfied by every generalized partition $\lambda$ of length $d$ if $m \geq d$.

We shall need an explicit formula for the joint highest weight vectors of the irreducible $gl_d \times gl_{m|n}$-module $V^\lambda_d \otimes \tilde{V}^\lambda_{m|n}$ inside $S(\mathbb{C} \otimes \mathbb{C}^{m|n})$. (See also \cite{21} and \cite{20} for different descriptions of these vectors.) We set

\begin{equation}
x^i_l := e^i \otimes e_l, \quad \eta^i_k := e^i \otimes f_k,
\end{equation}

for $i = 1, \cdots, d$, $l = 1, \cdots, m$, and $k = 1, \cdots, n$. We will denote by $\mathbb{C}[x, \eta]$ the polynomial superalgebra generated by (2.3). By identifying $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ with $\mathbb{C}[x, \eta]$ the commuting pair $(gl_d, gl_{m|n})$ can be realized in terms of the following first order differential operators ($1 \leq i, i' \leq d$, $1 \leq s, s' \leq m$ and $1 \leq k, k' \leq n$):

\begin{equation}
\phi(e_{ii'}) := \sum_{j=1}^{m} x^j_i \frac{\partial}{\partial x^j_{i'}} + \sum_{j=1}^{n} \eta^j_{i'} \frac{\partial}{\partial \eta^j_i},
\end{equation}

\begin{equation}
\phi(E_{ss'}) := \sum_{j=1}^{d} x^j_s \frac{\partial}{\partial x^j_{s'}}, \quad \phi(E_{m+k,m+k'}) := \sum_{j=1}^{d} \eta^j_{m+k} \frac{\partial}{\partial \eta^j_{m+k'}},
\end{equation}

\begin{equation}
\phi(E_{s,m+k}) := \sum_{j=1}^{d} x^j_s \frac{\partial}{\partial \eta^j_k}, \quad \phi(E_{m+k,s}) := \sum_{j=1}^{d} \eta^j_{m+k} \frac{\partial}{\partial x^j_s}.
\end{equation}
Straightforward calculations show that $\phi(e_{ij}), 1 \leq i, j \leq d$, and $\phi(E_{ab}), 1 \leq a, b \leq m + n$, satisfy the same commutation relations as the elementary matrices $e_{ij}$ and $E_{ab}$ respectively. Thus (2.4) spans a copy of $gl_d$, and (2.5) a copy of $gl_{m|n}$.

For $1 \leq r \leq \min(d, m)$, we define

$$\Delta_r := \det \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_r^1 \\ x_1^2 & x_2^2 & \cdots & x_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^r & x_2^r & \cdots & x_r^r \end{pmatrix},$$

(2.6)

If $d > m$, we consider the following determinant of an $r \times r$ matrix for every $m < r \leq d$:

$$\Delta_{k,r} := \det \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_r^1 \\ x_1^2 & x_2^2 & \cdots & x_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^m & x_2^m & \cdots & x_r^m \\ \eta_1^1 & \eta_1^2 & \cdots & \eta_1^r \\ \eta_2^1 & \eta_2^2 & \cdots & \eta_2^r \\ \vdots & \vdots & \ddots & \vdots \\ \eta_k^1 & \eta_k^2 & \cdots & \eta_k^r \end{pmatrix}, \quad k = 1, \ldots, n.$$  

(2.7)

That is, the first $m$ rows are filled by the vectors $(x_j^1, \ldots, x_j^r)$, for $j = 1, \ldots, m$, in increasing order and the last $r - m$ rows are filled with the same vector $(\eta_k^1, \ldots, \eta_k^r)$. Here the determinant of an $r \times r$ matrix

$$A := \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_r^1 \\ a_1^2 & a_2^2 & \cdots & a_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^r & a_2^r & \cdots & a_r^r \end{pmatrix},$$

with matrix entries possibly involving Grassmann variables, is by definition the expression $\sum_{\sigma \in S_r} (-1)^{l(\sigma)} a_1^{\sigma(1)} a_2^{\sigma(2)} \cdots a_r^{\sigma(r)}$, where $l(\sigma)$ is the length of $\sigma$ in the symmetric group $S_r$.

Observe that both $\Delta_r$ and $\Delta_{k,r}$ (if defined) are weight vectors of $gl_d \times gl_{m|n}$. Their $gl_d$-weights are respectively

$$wt_d(\Delta_r) = (1, \ldots, 1, 0, \ldots, 0),$$

(2.8)

$$wt_d(\Delta_{k,r}) = (1, \ldots, 1, 0, \ldots, 0).$$
while the $gl_{m|n}$-weights are respectively
\[
wt_{m|n}(\lambda, r) = (1, \ldots, 1, r, 0, \ldots, 0),
\]
(2.9)
\[
wt_{m|n}(\Delta_k, r, m) = (1, \ldots, 1, 0, \ldots, 0, r - m, 0, \ldots, 0),
\]
Corresponding to each partition $\lambda$ of length $d$ satisfying the condition $\lambda_{m+1} \leq n$, we define
\[
\Delta_\lambda := \left\{ \frac{\Delta_{\lambda_1}' \Delta_{\lambda_2}' \cdots \Delta_{\lambda_{m+1}}'}{\prod_{k=1}^{\lambda_{m+1}} \Delta_{\lambda_k, \lambda_{k-1}} \prod_{j=1+\lambda_{m+1}}^{\lambda_{m+1}} \Delta_{\lambda_j}'}, \quad \text{if } \lambda_1' \leq m, \right.
\]
(2.10)
\[
\left. \frac{\Delta_{\lambda_1}' \Delta_{\lambda_2}' \cdots \Delta_{\lambda_{m+1}}'}{\prod_{k=1}^{\lambda_{m+1}} \Delta_{k, \lambda_{k-1}} \prod_{j=1+\lambda_{m+1}}^{\lambda_{m+1}} \Delta_{\lambda_j}',} \quad \text{if } \lambda_1' > m. \right)
\]

**Lemma 2.1.** The space of $gl_d \times gl_{m|n}$ highest weight vectors in the submodule $V^\lambda_d \otimes V^\hat{\lambda}_{m|n}$ of $\mathbb{C}[x, \eta]$ is $\mathbb{C}\Delta_\lambda$.

### 2.3. The $(gl_d, gl_{p|q})$-duality on $S(\mathbb{C}^d \otimes \mathbb{C}^{p|q})$.

Let us denote by $\mathbb{C}^{p|q}$ the dual of the natural $gl_{p|q}$-module $\mathbb{C}^{p|q}$, and by $\mathbb{C}^d$ the natural of the natural $gl_d$-module $\mathbb{C}^d$. Then the $gl_d \times gl_{p|q}$-action on $\mathbb{C}^d \otimes \mathbb{C}^{p|q}$ induces a $gl_d \times gl_{p|q}$-action on $S(\mathbb{C}^d \otimes \mathbb{C}^{p|q})$.

If $S^k(W)$ denotes the set of all homogeneous elements of degree $k$ in the supersymmetric tensor algebra of the superspace $W$, then $S^k(\mathbb{C}^d \otimes \mathbb{C}^{p|q}) \cong S^k(\mathbb{C}^d \otimes \mathbb{C}^{p|q})^*$, and thus $S(\mathbb{C}^d \otimes \mathbb{C}^{p|q}) \cong \sum_k S^k(\mathbb{C}^d \otimes \mathbb{C}^{p|q})^*$. Therefore it follows from the decomposition (2.10) that the $gl_d \times gl_{p|q}$-action on $S(\mathbb{C}^d \otimes \mathbb{C}^{p|q})$ is also semi-simple and multiplicity free. Furthermore, we have the following decomposition $S(\mathbb{C}^d \otimes \mathbb{C}^{p|q}) \cong \sum_\lambda (V^\lambda_d)^* \otimes (V^\hat{\lambda}_{p|q})^*$, where $\lambda$ is summed over all partitions of length $d$ subject to the condition $\lambda_{p+1} \leq q$. Clearly, $(V^\lambda_d)^* \cong V^\lambda_d$. Also, $(V^\hat{\lambda}_{p|q})^* \cong V^\bar{\lambda}_{p|q}$, where $\bar{\lambda}$ is the negative of the lowest weight of $V^\lambda_{p|q}$. We shall give an explicit formula for $\bar{\lambda}$ in equation (2.20).

Since the supertrace $\text{Str}$ is trivial on the derived algebra of $gl_{p|q}$, one may twist any action of $gl_{p|q}$ by any scalar multiple of the supertrace. This is to say that, if $X \in gl_{p|q}$ acts on a space, then we may define a new action of $X$ on this space by $X + \gamma \text{Str}(X)$ instead, where $\gamma \in \mathbb{C}$. This in particular allows us to twist the standard action of $gl_{p|q}$ on $S(\mathbb{C}^d \otimes \mathbb{C}^{p|q})$ by $-d \text{Str}$. Under this twisted action of $gl_{p|q}$ the space $S(\mathbb{C}^d \otimes \mathbb{C}^{p|q})$ decomposes into
\[
S(\mathbb{C}^d \otimes \mathbb{C}^{p|q}) \cong \sum_\lambda V^\lambda_d \otimes V^{-d+\hat{\lambda}}_{p|q},
\]
(2.11)
where $\lambda := (1, \ldots, 1, -1, \ldots, -1)$. Here the summation in $\lambda$ is over all generalized partitions of non-positive integers with length $d$ subject to $\lambda_{d-p} \geq -q$. Observe that $\lambda^*$ is a partition of length $d$ satisfying $(\lambda^*)_p \leq q$. 


Remark 2.2. For any generalized partition \( \lambda \) of length \( d \), the condition \( \lambda_{d-p} \geq -q \) is considered to be automatically satisfied if \( p \geq d \).

Remark 2.3. Hereafter we shall always refer to this twisted action of \( gl_{p|q} \) when considering \( S(\mathbb{C}^{d*} \otimes \mathbb{C}^{p|q*}) \) as a \( gl_d \times gl_{p|q} \)-module.

Let \( e^1, \ldots, e^d \) be the standard basis of \( \mathbb{C}^d \). Let \( e^{1*}, \ldots, e^{d*} \) be a basis for the contragredient \( gl_d \)-module \( \mathbb{C}^{d*} \). We require the two bases to be dual in the sense that \( e^i(e^j) = \delta_{ij} \) for all \( i, j \in \{1, \ldots, d\} \). Similarly, we let \( e_1, \ldots, e_p, f_1, \ldots, f_q \) denote the standard homogeneous basis for the natural \( gl_{p|q} \)-module \( \mathbb{C}^{p|q} \) and \( e_1^*, \ldots, e_p^*, f_1^*, \ldots, f_q^* \) denote the dual basis for the contragredient \( gl_{p|q} \)-module \( \mathbb{C}^{p|q*} \). For \( 1 \leq l \leq d, 1 \leq i \leq p \) and \( 1 \leq j \leq q \), we set

\[
y_i^l := e_i^* \otimes e_i^*, \quad \zeta_j^l := e_j^* \otimes f_j^*,
\]

which form a basis for \( \mathbb{C}^{d*} \otimes \mathbb{C}^{p|q*} \).

We will denote by \( \mathbb{C}[y, \zeta] \) the polynomial superalgebra generated by \( (2.12) \). Let \( e_{ij}, 1 \leq i, j \leq d \) and \( E_{ab}, 1 \leq a, b \leq p + q \) be the bases respectively for \( gl_d \) and \( gl_{p|q} \) consisting of elementary matrices. Then the action of the commuting pair \( (gl_d, gl_{p|q}) \) on \( \mathbb{C}[y, \zeta] \) can be realized in terms of first order differential operators as follows \( (1 \leq i, j \leq d, 1 \leq r, r' \leq p \) and \( 1 \leq s, s' \leq q) \):

\[
(2.13) \quad \bar{\phi}(e_{ij}) := - \sum_{k=1}^{p} y^j_k \frac{\partial}{\partial y^i_k} - \sum_{k=1}^{q} \zeta^j_k \frac{\partial}{\partial \zeta^i_k};
\]

\[
(2.14) \quad \bar{\phi}(E_{r'r'}) := - \sum_{l=1}^{d} \frac{\partial}{\partial y^r_l} y^l_{r'}, \quad \bar{\phi}(E_{s+p, s'+p}) := \sum_{l=1}^{d} \frac{\partial}{\partial \zeta^s_l} \zeta^l_{s'},
\]

\[
\bar{\phi}(E_{s+p,r}) := - \sum_{l=1}^{d} \frac{\partial}{\partial \zeta^s_l} y^l_r, \quad \bar{\phi}(E_{r,s+p}) := \sum_{l=1}^{d} \frac{\partial}{\partial y^r_l} \zeta^l_s.
\]

It is straightforward to show that the \( \bar{\phi}(e_{ij}) \) and \( \bar{\phi}(E_{ab}) \) satisfy the same commutation relations as the \( e_{ij} \) and \( E_{ab} \) respectively. Furthermore, the elements of \( (2.13) \) commute with those of \( (2.14) \).

For \( 1 \leq r \leq \min(d, p) \), we define the following determinant of an \( r \times r \) matrix:

\[
(2.15) \quad \Delta_r^* := \det \begin{pmatrix}
y^d_p & y^d_{p-1} & \cdots & y^d_{p-r+1} \\
y^{d-1}_p & y^{d-1}_{p-1} & \cdots & y^{d-1}_{p-r+1} \\
\vdots & \vdots & \ddots & \vdots \\
y^{d-r+1}_p & y^{d-r+1}_{p-1} & \cdots & y^{d-r+1}_{p-r+1}
\end{pmatrix}.
\]

For \( 1 \leq r \leq d \), we define

\[
(2.16) \quad \Delta_{k,r}^* := \zeta^d_k \zeta^{d-1}_k \cdots \zeta^{d-r+1}_k \quad k = 1, \ldots, q.
\]
It is clear that the $\Delta^*_\lambda$ and $\Delta^*_{k,r}$ are all $gl_d$ highest weight vectors with respect to the standard Borel subalgebra $b_d$. They are also weight vectors under the action of $gl_d \times gl_{p|q}$ with the $gl_d$-weights respectively given by

\begin{equation}
wt_d(\Delta^*_\lambda) = (0, \ldots, 0, -1, \ldots, -1), \quad wt_d(\Delta^*_{k,r}) = (0, \ldots, 0, -1, \ldots, -1),
\end{equation}

and the $gl_{p|q}$-weights respectively given by

\begin{equation}
wt_{p|q}(\Delta^*_\lambda) = -d1 + (0, \ldots, 0, -1, \ldots, -1, 0, \ldots, 0),
wt_{p|q}(\Delta^*_{k,r}) = -d1 + (0, \ldots, 0, -r, 0, \ldots, 0).
\end{equation}

Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a generalized partition of non-positive integers subject to the condition $\lambda_{d-p} \geq -q$. Then $\mu := \lambda^*$ is a partition satisfying the condition $\mu_{p+1} \leq q$. We let $\mu'$ denote the transpose partition of $\mu$. Define

\begin{equation}
\Delta^*_\lambda := \begin{cases} 
\prod_{k=1}^{\mu_1} \Delta^*_{q+1-k, \mu'_k} \prod_{l=q+1}^{\mu_1} \Delta^*_{\mu'_l} & \text{if } \mu_1 \leq q, \\
\prod_{k=1}^{\mu_1} \Delta^*_{q+1-k, \mu'_k} \prod_{l=q+1}^{\mu_1} \Delta^*_{\mu'_l} & \text{if } \mu_1 > q.
\end{cases}
\end{equation}

**Lemma 2.2.** Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a generalized partition of non-positive integers subject to the condition $\lambda_{d-p} \geq -q$. Then $\Delta^*_\lambda$ is a non-zero highest weight vector in $\mathbb{C}[y, \zeta]$ with respect to the joint actions of $\text{gl}_d$ and $\text{gl}_{p|q}$. The $\text{gl}_d$-weight of $\Delta^*_\lambda$ is $\lambda$, while the $\text{gl}_{p|q}$-weight of $\Delta^*_\lambda$ is $-d1 + \hat{\lambda}$ with

\begin{equation}
\hat{\lambda} := -(\langle \mu_p - q \rangle, \langle \mu_{p-1} - q \rangle, \ldots, \langle \mu_1 - q \rangle, \mu'_q, \mu'_{q-1}, \ldots, \mu'_1),
\end{equation}

where $\mu = \lambda^*$.

**Proof.** Note that $\phi(e_{ij})$, for all $1 \leq i, j \leq d$, act on $\mathbb{C}[y, \zeta]$ by derivations. Thus the product of any subset of the $\text{gl}_d$ highest weight vectors $\Delta^*_r$, $\Delta^*_{k,r}$, $r = 1, 2, \ldots, \min(d, p)$, $k = 1, 2, \ldots, q$, is also a $\text{gl}_d$ highest weight vector. Hence so is $\Delta^*_\lambda$.

Obviously $\Delta^*_\lambda$ is a highest weight vector with respect to the action of the subalgebra $\text{gl}_p \times \text{gl}_q$ of $\text{gl}_{p|q}$. Consider the action of $\phi(E_{p,p+1}) = \sum_{l=1}^d \frac{\partial}{\partial y_p} \zeta^l_l$ on $\Delta^*_\lambda$. When $-\lambda_d = \mu_1 \leq q$, $\Delta^*_\lambda$ does not involve any of the variables $y_p$, thus $\phi(E_{p,p+1}) \Delta^*_\lambda = 0$. By using the equation

$$
\Delta^*_{k,r} \sum_{l=1}^d c^l \frac{\partial}{\partial y_p} \Delta^*_s = 0, \quad r \geq s,
$$


we also easily show that $\Delta^*_\lambda$ is annihilated by $\phi(E_{p,p+1})$ when $\mu_1 > q$. This proves
that $\Delta^*_\lambda$ is indeed a $gl_d \times gl_{plq}$ highest weight vector. The rest of the lemma easily
follows from equations (2.17) and (2.18). □

To summarize this subsection, we combine Lemma 2.2 with equation (2.1) into
the following theorem.

**Theorem 2.2.** Under the $gl_d \times gl_{plq}$-action, $\mathbb{C}[y, \zeta]$ decompose into

$$\mathbb{C}[y, \zeta] \cong \sum_{\lambda} V^\lambda_d \otimes V^{-d1+\lambda}_{plq},$$

where $\lambda$ is summed over all generalized partitions of non-positive integers with
length $d$ subject to $\lambda_{d-p} \geq -q$. The space of highest weight vectors in $V^\lambda_d \otimes V^{-d1+\lambda}_{plq}$
is given by $\mathbb{C}\Delta^*_\lambda$.

3. **THE ($gl_d, gl_{l+2}$)-DUALITY ON $S(C^d \otimes C^n \otimes \mathfrak{C}^{q,q}*$)**

3.1. **The $gl_d \times gl_{l+2}$-action on $S(C^d \otimes C^n \otimes \mathfrak{C}^{q,q}*)$**. We described the
semi-simple multiplicity free actions of $gl_d \times gl_{l+2}$ on $S(C^d \otimes C^n)$ and $gl_d \times gl_{plq}$
on $S(C^d \otimes C^{l+2})$ in the last section. Through the obvious isomorphism between
$S(C^d \otimes C^n) \cong S(C^{l+2})$ and $S(C^{l+2} \otimes C^{q,q}*)$, these actions lead to a
$gl_d \times gl_{l+2}$-action on the latter, where $gl_d$ acts diagonally. It is not
immediately obvious, but nevertheless true ([14]), that $S(C^d \otimes C^n \otimes \mathfrak{C}^{q,q}*)$
also admits an action of the larger algebra $gl_d \times gl_{l+2}$.

For the purpose of describing this action, it is convenient to introduce a basis
for $gl_{l+2}$ different from that given in Subsection 2.1. Set $I = \{1, 2, \ldots, m + n + q\}$. Let $\{v_A | A \in I\}$ be a basis of $C^{l+2}$ such that $\{v_A | 1 \leq a \leq m + q\}$ and $\{v_{p+q} \leq c \leq m + n\}$ are respectively the standard bases for $C^{l+2}$ and
$C^{n}$ described in Subsection 2.1. Let $E^A_B, A, B \in I$, be the set of the elementary matrices satisfying $E^A_B v_C = \delta_{BC} v_A$. These matrices form a homogeneous basis of
$gl_{l+2}$ with the following commutation relations

$$[E^A_B, E^C_D] = \delta_{BC} E^A_D - (-1)^{\deg E^A_B \deg E^C_D} \delta_{AD} E^C_B,$$

where $\deg E^A_B$ is the $Z_2$-degree of $E^A_B$.

Let $B := \sum_{A \in B; A \in I} E^A_B$, and $h_{m+p+n+q} := \sum_{A \in I} C E^A_A$. Then $B$ forms a (non-
standard) Borel subalgebra of $gl_{l+2}$ with the Cartan subalgebra $h_{m+p+n+q}$. Note that $\sum_{A,B} E^A_B$ forms a subalgebra isomorphic to $gl_{plq}$, and $\sum_{A,B} E^A_B$ forms a subalgebra isomorphic to $gl_{l+2}$, and these two subalgebras mutually
commute. Together they form $gl_{plq} \times gl_{l+2}$, which is a regular subalgebra of
$gl_{l+2}$ in the sense that its standard Borel subalgebra $h_{plq} \times h_{l+2}$ is contained
in $B$, and the corresponding Cartan subalgebra $h_{plq} \times h_{l+2}$ is identified with
$h_{m+p+n+q}$. This identification leads to canonical embeddings of $h_{plq}$ and $h_{l+2}$ in
$\mathfrak{h}^*_m$ forms a subalgebra of $\text{End}(C_1)$ for $\mathfrak{h}_m$ and $\mathfrak{h}_n$. Choose a basis $\{E_A|A \in I\}$ for $\mathfrak{h}_m^*$ such that $\tilde{c}_A(E_{BB}) = \delta_{AB}$, for all $A, B \in I$. An element $\Lambda = \sum_{A \in I} \Lambda_A E_A$ of $\mathfrak{h}_m^*$ will also be written as $\Lambda = (\Lambda_1, \Lambda_2, \cdots, \Lambda_{p+q+m+n})$. Now any pair of elements $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{p+q}) \in \mathfrak{h}_p$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_{m+n}) \in \mathfrak{h}_m$ gives rise to an element $(\lambda; \mu) \in \mathfrak{h}_m^*$ defined by

$$\lambda; \mu := (\lambda_1, \lambda_2, \cdots, \lambda_{p+q}, \mu_1, \mu_2, \cdots, \mu_{m+n}).$$

We retain the notations $C[x, \eta]$ and $C[y, \zeta]$ from Section 2 and denote by $C[x, y, \eta, \zeta]$ the polynomial superalgebra $C[x, \eta] \otimes_C C[y, \zeta]$. Let $D[x, y, \eta, \zeta]$ denote the oscillator superalgebra generated by the variables $x_i^l, n_j^l, y_i^l, \zeta_i^l$, and their derivatives $\frac{\partial}{\partial x_i^l}, \frac{\partial}{\partial n_j^l}, \frac{\partial}{\partial y_i^l}, \frac{\partial}{\partial \zeta_i^l}$, where $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq r \leq p$, $1 \leq s \leq q$, and $1 \leq l \leq d$. Then $D[x, y, \eta, \zeta]$ naturally acts on $C[x, y, \eta, \zeta]$, thus forms a subalgebra of $\text{End}(C[x, y, \eta, \zeta])$.

The general linear group $GL(d)$ acts on $D[x, y, \eta, \zeta]$ by conjugations. The corresponding action of the Lie algebra $\mathfrak{gl}_d$ is realized in terms of the following first order differential operators $(1 \leq i, j \leq d)$:

$$\Phi(e_{ij}) = \sum_{k=1}^m x_k^i \frac{\partial}{\partial x_k^j} + \sum_{k=1}^n n_k^i \frac{\partial}{\partial n_k^j} - \sum_{k=1}^p y_k^j \frac{\partial}{\partial y_k^i} - \sum_{k=1}^q \zeta_k^i \frac{\partial}{\partial \zeta_k^j}.$$

Let $D[x, y, \eta, \zeta]^{GL(d)}$ denote the $GL(d)$-invariant subalgebra of $D[x, y, \eta, \zeta]$. The $GL(d)$-action is semi-simple. Thus from the first fundamental theorem of the invariant theory of the general linear group (see Chapter 4 of [12]) we deduce that $D[x, y, \eta, \zeta]^{GL(d)}$ is generated by the following operators:

$$\Phi(E^a_b) := \tilde{\phi}(E_{ab}), \quad 1 \leq a, b \leq p + q,$$

$$\Phi(E^{p+q+u}_{p+q+v}) := \phi(E_{uv}), \quad 1 \leq u, v \leq m + n,$$

$$\Phi(E^{p+q+i}_{p+q+i}) := \sum_{l=1}^d \frac{\partial}{\partial y_l^j} \frac{\partial}{\partial x_l^i}, \quad \Phi(E^{p+q+i}_{p+q+i}) := \sum_{l=1}^d \frac{\partial}{\partial y_l^i} \frac{\partial}{\partial x_l^j},$$

where $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq r \leq p$, and $1 \leq s \leq q$. It is an easy exercise to show that the space spanned by $\Phi(e_{ij})$, $1 \leq i, j \leq d$, and $\Phi(E^{A}_{B})$, $A, B \in I$ is a homomorphic image of $\mathfrak{gl}_d \times \mathfrak{gl}_{m+p+n}$ in $D[x, y, \eta, \zeta]$. As every Lie superalgebra map uniquely extends to a homomorphism of its universal enveloping algebra, we have an associative superalgebra homomorphism

$$\Phi: U(\mathfrak{gl}_d \times \mathfrak{gl}_{m+p+n}) \rightarrow D[x, y, \eta, \zeta].$$
Now by identifying $S(C^d \otimes C^{m|n} \oplus C^{d^*} \otimes C^{p|q^*})$ with the polynomial superalgebra $C[x, y, \eta, \zeta]$, we obtain an action of $gl_d \times gl_{m+p|n+q}$ on $S(C^d \otimes C^{m|n} + C^{d^*} \otimes C^{p|q^*})$. It can be extracted from [14] that this action is semi-simple and multiplicity free. We state this as a theorem for convenience of reference.

**Theorem 3.1.** [14] The pair $(gl_d, gl_{m+p|n+q})$ of Lie (super)algebras forms a dual reductive pair on $C[x, y, \eta, \zeta]$.

### 3.2. Unitarity.

We first recall some basic facts about $\ast$-superalgebras and their unitarizable representations. A $\ast$-superalgebra is an associative superalgebra $A$ together with an anti-linear anti-involution $\omega : A \to A$. Here we should emphasize that for any $a, b \in A$, we have $\omega(ab) = \omega(b)\omega(a)$, where no sign factors are involved. A $\ast$-superalgebra homomorphism $f : (A, \omega) \to (A', \omega')$ is a superalgebra homomorphism obeying $f \circ \omega = \omega' \circ f$. Let $(A, \omega)$ be a $\ast$-superalgebra, and let $V$ be a $\mathbb{Z}_2$-graded $A$-module. A Hermitian form $(\cdot | \cdot)$ on $V$ is said to be contravariant one can easily show that the form $(\cdot | \cdot)$ is positive definite. The polynomial superalgebra $\mathbb{C}[x, y, \eta, \zeta]$ with this inner product (after completion) is the Fock space of $d(m+p)$ bosonic and $d(n+q)$ fermionic quantum oscillators. When $d = 1$, we denote it by $\mathcal{F}_{m+p|n+q}$. Then it is clear that for arbitrary $d$ we have $\mathbb{C}[x, y, \eta, \zeta] \cong (\mathcal{F}_{m+p|n+q})^{\otimes d}$. What presented in this paragraph is standard material on Fock spaces, which is part of the basic ingredients of second quantization.

We now consider a $\ast$-structure of $U(gl_d \times gl_{m+p|n+q})$. We shall regard $gl_d \times gl_{m+p|n+q}$ as embedded in its universal enveloping algebra. Consider the anti-linear anti-involution $\sigma$ of $U(gl_d \times gl_{m+p|n+q})$ defined, for all $1 \leq a, b \leq p + q$, $p + q + 1 \leq r, s \leq p + q + m + n$, and $1 \leq i, j \leq d$, by

$$
E_b^a \mapsto (-1)^{|a|+|b|} E^b_a, \quad E_s^r \mapsto E_r^s, \\
E_s^a \mapsto -(-1)^{|a|} E^s_a, \quad E_r^b \mapsto -(-1)^{|b|} E_r^b, \\
e_{ij} \mapsto e_{ji},
$$
where \([a] = \begin{cases} 0, & 1 \leq a \leq p, \\ 1, & 1 \leq a - p \leq q. \end{cases}\) By direct calculations we can show that this anti-linear map respects the commutation relations (3.1), thus indeed defines an anti-linear anti-involution of \(U(gl_d \times gl_{m+p|n+q})\). Now \(\sigma\) gives rise to a \(*\)-structure for \(U(gl_d \times gl_{m+p|n+q})\).

**Theorem 3.2.** (1) The map \(\Phi\) is a \(*\)-superalgebra homomorphism from \((U(gl_d \times gl_{m+p|n+q}), \sigma)\) to the oscillator superalgebra \((\mathbb{D}[x, y, \eta, \zeta], \omega)\).

(2) \(\mathbb{C}[x, y, \eta, \zeta]\) is a unitarizable \((U(gl_d \times gl_{m+p|n+q}), \sigma)\)-module with respect to the Hermitian form \((\cdot, \cdot)\).

**Proof.** Using equations (3.3)-(3.7), we can show by direct calculations that for all \(X \in gl_d \times gl_{m+p|n+q}\), we have \(\Phi\sigma(X) = \omega\Phi(X)\). This proves part (1). Part (2) immediately follows from part (1).

We also have the following result.

**Lemma 3.1.** All the irreducible \(gl_d \times gl_{m+p|n+q}\)-submodules of \(\mathbb{C}[x, y, \eta, \zeta]\) are of highest weight type with respect to the Borel subalgebra \(b_d \times B\).

**Proof.** Let \(H\) be the harmonic subspace of \(\mathbb{C}[x, y, \eta, \zeta]\), i.e., the subspace consisting of such polynomials that are annihilated by all the elements \(\Phi(E_A^B), A \leq p + q, B > p + q, \) of (3.6). Then \(H\) forms a module of the subalgebra \(gl_d \times gl_{p|q} \times gl_{m|n}\) spanned by elements of (3.3), (3.4), and (3.5). It follows from the \((gl_d, gl_{m|n})\)-duality on \(\mathbb{C}[x, \eta]\) described in Theorem 2.1 and the \((gl_d, gl_{p|q})\)-duality on \(\mathbb{C}[y, \zeta]\) described in Theorem 2.2 that \(H\) decomposes into a direct sum of finite-dimensional irreducible \(gl_d \times gl_{p|q} \times gl_{m|n}\)-modules.

Let \(W\) be an irreducible \(gl_d \times gl_{m+p|n+q}\)-submodule of \(\mathbb{C}[x, y, \eta, \zeta]\). Let \(H_W = W \cap H\). Then \(H_W \neq 0\), as the lowest order polynomials of \(W\) are all contained in \(H_W\). Now \(H_W\) forms a module of the subalgebra \(gl_d \times gl_{p|q} \times gl_{m|n}\), which in fact is irreducible. To see this, we note that if \(H_W\) were reducible with respect to \(gl_d \times gl_{p|q} \times gl_{m|n}\), then due to complete reducibility \(H_W\) would contain more than one linearly independent \(gl_d \times gl_{p|q} \times gl_{m|n}\)-highest weight vectors. However, since they lie in \(H_W\), they would also be \(gl_d \times gl_{m+p|n+q}\)-highest weight vectors with respect to \(b_d \times B\), thus contradicting the irreducibility of \(W\). Thus \(W\) is generated by a \(gl_d \times gl_{m+p|n+q}\)-highest weight vectors with respect to \(b_d \times B\), as claimed.

**Remark 3.1.** From Theorem 3.1 and the proof of the lemma we can deduce that the action of \(gl_d \times gl_{p|q} \times gl_{m|n}\) on the harmonic subspace \(H\) of \(\mathbb{C}[x, y, \eta, \zeta]\) is semi-simple and multiplicity free.

Let \(g^\mathbb{R}\) be the real superspace spanned by \(\{X \in (gl_{m+p|n+q})_0 | \sigma(X) = -X\} \cup \sqrt{7}\{X \in (gl_{m+p|n+q})_1 | \sigma(X) = -X\}\). Then \(g^\mathbb{R}\) is a real form of \(gl_{m+p|n+q}\), that is, \(g^\mathbb{R}\) forms a real Lie superalgebra with the complexification being \(gl_{m+p|n+q}\) itself.
The usual notation for this real form is \( u(m, p|n, q) \). Note that the maximal even subalgebra of \( g^R \) is \( u(m, p) \times u(n, q) \). Thus every nontrivial unitarizable \( u(m, p|n, q) \)-module must be infinite dimensional.

3.3. Comments on unitarizable modules. At this point, we should relate to results in the literature. Note that the restrictions of \( \sigma \) to the subalgebras \( gl_{m|n} \) and \( gl_{p|q} \) act differently on the odd subspaces. They respectively give rise to two different real forms \( u_+(m|n) \) and \( u_-(p|q) \) of the subalgebras. Now \( g^R \) contains the subalgebra \( u_-(p|q) \times u_+(m|n) \), which one would like to regard as the ‘maximal compact subalgebra’. Unfortunately finite dimensional representations of \( u_+(m|n) \) and \( u_-(p|q) \) are not necessarily unitarizable. In fact it has long been known \([13]\) that the only finite dimensional unitarizable irreducible representations of \( u_+(m|n) \) (resp. \( u_-(p|q) \)) are the tensor products of the irreducible representations appearing in Theorem 2.1 (resp. Theorem 2.2) with some 1-dimensional representations (upon restricting modules of the general linear superalgebra to modules of its real form), which constitute a small class of the finite dimensional irreducible representations. Thus the situation is very different from the case of the compact real Lie algebra \( u(k) \).

The intersection of \( u(m, p|n, q) \) with \( sl_{m+p|n+p} \) gives rise to the real Lie superalgebra \( su(m, p|n, q) \). It was shown in \([16]\) that \( su(m, p|n, q) \) admits no unitarizable highest or lowest weight representations with respect to the standard Borel subalgebra if all the integers \( m, n, p \) and \( q \) are non-zero. Since \([16]\) was only concerned with irreducible unitarizable highest weight modules of simple basic classical Lie superalgebras with respect to their standard Borel subalgebras \([17]\), the irreducible \( gl_{m+p|n+q} \)-modules appearing in the decomposition of \( \mathbb{C}[x, y, \eta, \zeta] \) were ignored. In fact with respect to the standard Borel subalgebra of \( gl_{m+p|n+q} \), the unitarizable irreducible representations studied in this paper are neither highest weight nor lowest weight type unless some of the integers \( m, n, p \) and \( q \) are zero.

A final comment is that when both of the integers \( n \) and \( q \) are zero, the general linear superalgebra \( gl_{m+p|n+q} \) reduces to the ordinary Lie algebra \( gl_{m+p} \), and \( \mathbb{C}[x, y, \eta, \zeta] \) to the ordinary polynomial algebra \( \mathbb{C}[x, y] \) in the two sets of variables \( x \) and \( y \). Then the irreducible \( gl_{m+p} \)-modules appearing in \( \mathbb{C}[x, y] \) are the unitarizable irreducible \( u(m, p) \)-modules studied by Kashiwara and Vergne in \([18]\). It is known \([8, 18]\) that the unitarizable irreducible \( u(m, p) \)-module at every reduction point \([14]\) is a submodule in \( \mathbb{C}[x, y] \) for some \( d \). However, it is not known whether this is still true in the super case.

3.4. The \( (gl_d, gl_{m+p|n+q}) \)-duality on \( \mathbb{C}[x, y, \eta, \zeta] \). Each generalized partition \( \lambda = (\lambda_1, \cdots, \lambda_d) \) of length \( d \) can be uniquely expressed as \( \lambda = \lambda^+ + \lambda^- \), with

\[
\lambda^+ := (\max\{\lambda_1, 0\}, \cdots, \max\{\lambda_d, 0\}),
\]

\[
\lambda^- := (\min\{\lambda_1, 0\}, \cdots, \min\{\lambda_d, 0\}).
\]
Lemma 3.2. If the generalized partition λ = (λ_1, ..., λ_d) satisfies the conditions λ_{m+1} ≤ n and λ_{d-p} ≥ −q if and only if λ^+_{m+1} ≤ n and (λ^−)^* ≤ q. Corresponding to each such generalized partition λ, we define
\[
\square_\lambda := \Delta^*_\lambda \Delta^+_{\lambda^*}.
\]

Lemma 3.2. If the generalized partition λ satisfies the conditions λ_{m+1} ≤ n and λ_{d-p} ≥ −q, then \square_\lambda is a non-zero \textit{gl}_d \times \textit{gl}_{m+p|n+q} highest weight vector with respect to the Borel subalgebra \mathfrak{b}_d \times \mathfrak{b}. The \textit{gl}_d-weight of \square_\lambda is λ, and the \textit{gl}_{m+p|n+q}-weight is given by
\[
\Lambda(\lambda) := (-d \mathbf{1} + \widehat{\lambda}^−; \widehat{\lambda}^+),
\]
where the expression on the right hand side is as explained by (3.2).

Proof. By construction, \square_\lambda is a highest weight vector of the subalgebra \textit{gl}_d \times \textit{gl}_{p|q} \times \textit{gl}_{m|n} with respect to the standard Borel subalgebra \textit{b}_d \times \textit{b}_{p|q} \times \textit{b}_{m|n}. Therefore, we only need to show that \(\Phi(E_{p+q+p+q+1}) = \sum_{k=1}^{d} \frac{\partial^k}{\partial x_k^1} \frac{\partial}{\partial x_k^1}\) annihilates \square_\lambda in order to prove that \square_\lambda is a \textit{gl}_d \times \textit{gl}_{m+p|n+q} highest weight vector with respect to the Borel subalgebra \mathfrak{b}_d \times \mathfrak{b}. If \(\Phi(E_{p+q+p+q+1}) \square_\lambda \neq 0\), then there must exist at least one integer \(i \in \{1, 2, \cdots, d\}\) such that \(x_1^k \zeta^i_q\) appears in \square_\lambda. Let \(ht(\lambda^+\mathbf{1})\) denote the depth of \(\lambda^+\mathbf{1}\), and \(ht((\lambda^−)^*)\) denote the depth of \((\lambda^−)^*\). By examining its explicit form, we can see that \square_\lambda does not involve any of the variables \(x_1^k, d \geq k > ht(\lambda^+\mathbf{1})\), and \(\zeta^i_q, 1 \leq l < d + 1 - ht((\lambda^−)^*)\). Therefore in order for \(x_1^k \zeta^i_q\) to appear in \square_\lambda, the integer \(i\) must satisfy \(d + 1 - ht((\lambda^−)^*) \leq i \leq ht(\lambda^+\mathbf{1})\). But this is impossible since \(\textit{gl}_d\) requires \(ht(\lambda^+\mathbf{1}) + ht((\lambda^−)^*) \leq d\). Therefore, \(\Phi(E_{p+q+p+q+1}) \square_\lambda = 0\), and thus \square_\lambda is a \textit{gl}_d \times \textit{gl}_{m+p|n+q} highest weight vector with respect to the Borel subalgebra \textit{b}_d \times \mathfrak{b}.

The \textit{gl}_d-weight of \square_\lambda is obviously \(\lambda\). From Lemma 2.1 and Lemma 2.2, we easily see that the \textit{gl}_{m+p|n+q}-weight of \square_\lambda is indeed \((-d \mathbf{1} + \widehat{\lambda}^−; \widehat{\lambda}^+\)).

We shall denote by \(W^\Lambda_{m+p|n+q}\) the irreducible highest weight \textit{gl}_{m+p|n+q}-module with highest weight \(\Lambda\) relative to the non-standard Borel subalgebra \(\mathfrak{b}\).

Theorem 3.3. Under the \(\textit{gl}_d \times \textit{gl}_{m+p|n+q}\)-action \(\mathbb{C}[^\mathbf{x, y, \eta, \zeta}]\) decomposes into
\[
\mathbb{C}[^\mathbf{x, y, \eta, \zeta}] \cong \sum_{\lambda} V_{d}^\lambda \otimes W^\Lambda_{m+p|n+q},
\]
where \(\lambda\) is summed over all generalized partitions of length \(d\) subject to \(\lambda_{m+1} \leq n\) and \(\lambda_{d-p} \geq −q\). Furthermore, \(\mathbb{C} \square_\lambda\) is the space of highest weight vectors of the irreducible module \(V_{d}^\lambda \otimes W^\Lambda_{m+p|n+q}\).
Proof. Note that every irreducible \( gl_d \)-submodule of \( \mathbb{C}[x, y, \eta, \zeta] \) is finite dimensional. Thus it follows from Theorem 3.1 and Lemma 3.1 that the decomposition of \( \mathbb{C}[x, y, \eta, \zeta] \) under \( gl_d \times gl_{m+p|n+q} \) has to be of the form (3.11), with the sum in \( \lambda \) ranging over some subset of generalized partitions of length \( d \). (Here if \( \lambda \) happens to be a generalized partition not satisfying \( \lambda_{m+1} \leq n \) and \( \lambda_{d-p} \geq -q \), the expression \( \Lambda(\lambda) \) stands for the highest weight for \( gl_{m+p|n+q} \) corresponding to \( \lambda \) under this Howe duality.) In view of Lemma 3.2, we only need to show that every generalized partition \( \lambda \) belonging to this subset must satisfy the conditions \( \lambda_{m+1} \leq n \) and \( \lambda_{d-p} \geq -q \), in order to prove the theorem.

Let \( \mu \) be a generalized partition of length \( d \). Assume that either one or both of the conditions \( \mu_{m+1} \leq n \) and \( \mu_{d-p} \geq -q \) are violated. We choose a pair of positive integers \( p' \) and \( n' \) with \( p' \geq p \) and \( n' \geq n \) such that \( \mu_{m+1} \leq n' \) and \( \mu_{d-p'} \geq -q \). Such an \( n' \) is trivial to come by, and such a \( p' \) also exists since by Remark 2.2 the condition \( \mu_{d-p'} \geq -q \) is always satisfied if \( p' \geq d \). We let \( \mathbb{C}[x, y, \bar{\eta}, \zeta] \) denote the polynomial superalgebra generated by

\[
x^i_l := e^i \otimes e_s, \quad y^i_l := e^i \otimes e^*_r, \quad \bar{\eta}^i_j := e^i \otimes f_j, \quad \zeta^i_j := e^i \otimes f^*_s,
\]

for \( 1 \leq l \leq d, 1 \leq i \leq m, 1 \leq j \leq n', 1 \leq r \leq p' \) and \( 1 \leq s \leq q \). Then \( \mathbb{C}[x, y, \eta, \zeta] \) becomes a subspace of \( \mathbb{C}[x, y, \bar{\eta}, \zeta] \) upon identifying

\[
y^i_l := \bar{y}^i_{p'-p+r}, \quad \zeta^i_j := \bar{\zeta}^i_j, \quad x^i_l := \bar{x}^i_l, \quad \eta^i_j := \bar{\eta}^i_j,
\]

for \( 1 \leq l \leq d, 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq r \leq p \) and \( 1 \leq s \leq q \). We denote this inclusion by \( \iota : \mathbb{C}[x, y, \eta, \zeta] \rightarrow \mathbb{C}[x, y, \bar{\eta}, \zeta] \). There is also the surjection \( \pi : \mathbb{C}[x, y, \bar{\eta}, \zeta] \rightarrow \mathbb{C}[x, y, \eta, \zeta] \) defined by setting \( \bar{y}^i_l = 0 \), \( 1 \leq r \leq p' - p \), and \( \bar{\eta}^i_s = 0 \), \( n \leq s \leq n' \), for all \( l \), then making the identification (3.12). Obviously, \( \pi \iota \) is the identity map on \( \mathbb{C}[x, y, \eta, \zeta] \).

Now we turn to the \( gl_d \times gl_{m+p'|n'+q} \)-action on \( \mathbb{C}[x, y, \bar{\eta}, \zeta] \). Upon choosing the basis for \( \mathbb{C}^{p'|q} \oplus \mathbb{C}^{m|n'} \) that is the union of the standard bases of \( \mathbb{C}^{p'|q} \) and \( \mathbb{C}^{m|n'} \), the general linear superalgebra \( gl_{m+p'|n'+q} \) becomes the Lie superalgebra of \( (m + p' + n' + q) \times (m + p' + n' + q) \)-matrices. Consider the subalgebra \( \mathfrak{l} \) of \( gl_{m+p'|n'+q} \) consisting of matrices of the form

\[
\begin{pmatrix}
D & 0 & 0 \\
0 & X & 0 \\
0 & 0 & D'
\end{pmatrix},
\]

where \( X \in gl_{m+p|n+q} \), and \( D \) and \( D' \) are diagonal matrices of sizes \( (p' - p) \times (p' - p) \) and \( (n' - n) \times (n' - n) \), respectively.

Obviously \( \mathfrak{l} \) contains the \( gl_{m+p|n+q} \) subalgebra \( \left\{ \begin{pmatrix}
0 & 0 & 0 \\
0 & X & 0 \\
0 & 0 & 0
\end{pmatrix} \right| X \in gl_{m+p|n+q} \right\} \).

Let \( \mathfrak{p} = \mathfrak{n} + l \) be a parabolic subalgebra of \( gl_{m+p'|n'+q} \) with the Levi factor \( l \) and nilpotent radical \( \mathfrak{n} \). We assume that \( \mathfrak{p} \) contains all the upper triangular matrices. Then there exists a nilpotent subalgebra \( \mathfrak{n} \) consisting of strictly lower triangular matrices such that \( gl_{m+p'|n'+q} = \mathfrak{p} + \mathfrak{n} \). By examining equations (3.3)-(3.7) we
can see that \( t \) is a \( gl_d \times gl_{m+p|n+q} \)-module map. Let \( V \) be any \( gl_d \times gl_{m+p|n+q} \)
submodule of \( \mathbb{C}[x, y, \eta, \zeta] \). Then \( \iota (V) \) is in fact a \( gl_d \times p \)-module with \( n \) acting by zero. Thus \( W = \Phi (U(\mathfrak{g})) \iota (V) \) forms a \( gl_d \times gl_{m+p'|n'+q} \)-submodule of \( \mathbb{C}[x, y, \bar{\eta}, \bar{\zeta}] \).

Note that \( W \) is irreducible if \( V \) is irreducible with respect to \( gl_d \times gl_{m+p|n+q} \). Again by examining equations (3.3)–(3.7) we can see that \( \pi \) is a \( gl_d \times gl_{m+p|n+q} \)-module map from the restriction of \( \mathbb{C}[x, y, \eta, \zeta] \) to \( \mathbb{C}[x, y, \eta, \zeta] \), and satisfies \( \pi (W) = V \). This in particular implies that \( \pi t \) is the identity \( gl_d \times gl_{m+p|n+q} \)-module map on \( \mathbb{C}[x, y, \eta, \zeta] \).

Let \( v_\mu \in \mathbb{C}[x, y, \eta, \zeta] \) be any \( gl_d \times gl_{m+p|n+q} \) highest weight vector with the \( gl_d \)
weight \( \mu \) (that violates one or both of the conditions \( \mu_{m+1} \leq n \) and \( \mu_{d-p} \geq -q \)). Then by the above discussion, \( \iota (v_\mu) \) is a \( gl_d \times 1 \) highest weight vector in \( \mathbb{C}[x, \bar{y}, \bar{\eta}, \bar{\zeta}] \), which has the same \( gl_d \) weight, and is also annihilated by \( n \). Therefore, \( \iota (v_\mu) \) is a \( gl_d \times gl_{m+p'|n'+q} \) highest weight vector in \( \mathbb{C}[x, \bar{y}, \bar{\eta}, \bar{\zeta}] \). By Theorem 3.7 and Lemma 3.2 (with \( p \) replaced by \( p' \) and \( n \) by \( n' \)), there exists a unique non-zero \( \square_\mu \in \mathbb{C}[x, y, \bar{\eta}, \bar{\zeta}] \) such that \( \iota (v_\mu) = c \square_\mu \) for some complex number \( c \).

We claim that every monomial in the polynomial \( \square_\mu \) contains at least one of the variables \( \bar{y}_r^t, \bar{\eta}_s, \bar{\zeta}_t \), where \( r = 1, \ldots, p' - p, ~ s = n + 1, \ldots, n' \) and \( t = 1, \ldots, d \). This can be seen from the explicit form (3.9) of \( \square_\lambda \). Consider first the case with \( t := \mu_{m+1} > n \). Then \( (\mu^+)_1 > m \), and \( \Delta_{\mu^+} \) has the factor \( \Delta_{t,(\mu^+)_1} \). From (2.7) we see that \( \Delta_{t,(\mu^+)_1} \) is the determinant of a matrix with rows \( \bar{\eta}_1, \bar{\eta}_2^2, \ldots, \bar{\eta}_t^{(\mu^+)_1} \) with Grassmann number entries. Now consider the case with \( \mu_{d-p} < -q \). Let \( \gamma = (\mu^-)^* \), then \( \gamma_{p+1} > q \). Thus we must also have \( \gamma_1 > q \), and \( u := \gamma_{q+1} \geq p + 1 \). Now \( \Delta_{\mu^-} \) has the factor \( \Delta_u^* \). From (2.10) we see that \( \Delta_u^* \) is the determinant of a matrix with a column \( \bar{y}_p^{d+1-1}, \bar{y}_p^{d-1}, \ldots, \bar{y}_p^{d+1-u} \). Note that \( p' + 1 - u \leq p' - p \).

Now it is obvious that \( \pi (\square_\mu) = 0 \), which in turn implies \( v_\mu = 0 \). Therefore, the decomposition of \( \mathbb{C}[x, y, \eta, \zeta] \) can not contain \( V_\mu^d \otimes W_\lambda^{(\mu)} \) as an irreducible submodule if \( \mu \) violates any of the conditions \( \mu_{m+1} \leq n \) and \( \mu_{d-p} \geq -q \). □

By using Theorem 2.1 Theorem 2.2 and the decomposition \( \mathbb{C}[x, y, \eta, \zeta] = \mathbb{C}[x, \eta] \otimes_\mathbb{C} \mathbb{C}[y, \zeta] \), we have

\[
\mathbb{C}[x, y, \eta, \zeta] \cong \sum_{\lambda, \mu} V_\lambda^d \otimes V_\mu^d \otimes V_{\lambda|n}^\lambda \otimes V_{p|q}^{d_1+d_2},
\]

where the summation in \( \lambda \) is over all the partitions of length \( d \) satisfying \( \lambda_{m+1} \leq n \), and the summation in \( \mu \) is over all the generalized partitions of non-positive integers of length \( d \) satisfying \( \mu_{d-p} \geq -q \).

The decomposition of the tensor product of any two finite dimensional irreducible \( gl_d \)-modules is described by the Littlewood-Richardson theory. For any generalized partitions \( \lambda \) and \( \mu \) of length \( d \),

\[
V_\lambda^d \otimes V_\mu^d \cong \sum_{\nu} C_{\lambda \mu}^{\nu} V_\nu^d,
\]
where the non-negative integers $C^\nu_{\lambda \mu}$ are the so-called Littlewood-Richardson coefficients, which give the respective multiplicities of the irreducible $gl_d$-modules $V^\nu_d$ appearing in the tensor product.

Therefore, $\mathbb{C}[x, y, \eta, \zeta]$ decomposes into

$$\mathbb{C}[x, y, \eta, \zeta] \cong \sum_{\nu} V^\nu_d \otimes \sum_{\lambda, \mu} C^\nu_{\lambda \mu} V^{\tilde{\lambda}}_{m|n} \otimes V^{-d+\hat{\lambda}}_{p|q},$$

where the summation in $\nu$ is over all the generalized partitions of length $d$. Combining Theorem 3.3 and (3.14), the Littlewood-Richardson coefficients $C^\nu_{\lambda \mu}$ appearing in (3.14) may be non-zero only when $\nu$ satisfies the conditions $\nu_{m+1} \leq n$ and $\nu_{d-p} \geq -q$. Now if we put $p = d - m - 1$, $n = \lambda_{m+1}$ and $q = -\mu_{d-p}$, then the conditions become $\nu_{m+1} \leq \lambda_{m+1}$ and $\nu_{m+1} \geq \mu_{m+1}$. Letting $m$ run from 0 to $d - 1$, we have the following corollary.

**Corollary 3.1.** Assume that $\lambda$ is a partition of length $d$ and $\mu$ is a generalized partition of non-positive integers of length $d$. If $\nu$ is a generalized partition of length $d$ satisfying $\nu_m > \lambda_m$ or $\nu_m < \mu_m$ for some $m \in \{1, 2, \ldots, d\}$, then $C^\nu_{\lambda \mu} = 0$.

Corollary 3.1 translated to ordinary partitions implies the following.

**Corollary 3.2.** Let $\lambda$ and $\mu$ be two partitions of length $d$. If $\nu$ is a partition of length $d$ satisfies $\nu_m > \min\{\mu_m + \lambda_1, \lambda_m + \mu_1\}$ for some $m \in \{1, 2, \ldots, d\}$, then the Littlewood-Richardson coefficient $C^\nu_{\lambda \mu} = 0$.

**Proof.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$ be two partitions. Then $\mu - \mu_1 := (\mu_1 - 1, \mu_2 - 1, \ldots, \mu_d - 1)$ is a generalized partition. By Corollary 3.1, we have $C^\nu_{\lambda \mu} = C^\nu_{\lambda - \mu_1 \mu} = 0$ if $\nu_m - \mu_1 > \lambda_m$ for some $m \in \{1, 2, \ldots, d\}$. Therefore, the corollary follows from the symmetry property of the Littlewood-Richardson coefficients.

**Remark 3.2.** A alternative method to prove Theorem 3.3 is the following. One can first prove Corollary 3.2 using, for example, the celebrated combinatorial algorithm known as the Littlewood-Richardson rule (see e.g. [12]). Using Corollary 3.2 it can then be derived that in the tensor product decomposition of $V^\lambda_d \otimes V^\mu_d$, with $\lambda$ a partition of length $d$ satisfying $\lambda_{m+1} \leq n$, and $\mu$ a generalized partition of non-positive integers of length $d$ satisfying $\mu_{d-p} \geq -q$, only $gl_d$-modules associated to generalized partitions $\nu$ with $\nu_{m+1} \leq n$ and $\nu_{d-p} \geq -q$ can occur. Using this fact together with Lemma 3.2, it is then not difficult to prove Theorem 3.3.

4. **Branching rules of unitarizable irreducible $gl_{m+p|n+q}$-modules**

As an easy application of Theorem 3.3, we derive the $gl_{m+p|n+q} \rightarrow gl_{p|q} \times gl_{m|n}$ branching rule for the infinite dimensional unitarizable irreducible $gl_{m+p|n+q}$-representations arising from the decomposition of tensor powers of the Fock space.
of $m + p$ bosonic and $n + q$ fermionic quantum oscillators. Results of this section will be important for developing a character formula for these unitarizable irreducible $gl_{m+p|n+q}$-modules.

Let us denote by $\mathfrak{c}$ the subalgebra $gl_{p|q} \times gl_{m|n}$ of $gl_{m+p|n+q}$. Recall the decomposition of $\mathbb{C}[x, y, \eta, \zeta]$ as a $gl_d \times \mathfrak{c}$-module (3.14). Denote by $W_{m+p|n+q}^{\Lambda(\nu)}|_{\mathfrak{c}}$ the restriction of $W_{m+p|n+q}^{\Lambda(\nu)}$ to a $\mathfrak{c}$-module. Let us now consider Theorem 3.3 by restricting both sides of equation (3.11) to $gl_d \times \mathfrak{c}$-modules. Using (3.14) we obtain

$$\sum_\nu V_d^\nu \otimes W_{m+p|n+q}^{\Lambda(\nu)}|_{\mathfrak{c}} \cong \sum_\nu V_d^\nu \otimes \sum_{\lambda, \mu} C^\nu_{\lambda, \mu} V_{m|n}^\lambda \otimes V_{p|q}^{-d+1+\hat{\mu}},$$

where the summation in $\nu$ on the left hand side is over the generalized partitions of length $d$ satisfying the conditions $\nu_{n+1} \leq n$ and $\nu_{d-p} \geq -q$. The above equation immediately leads to the following $gl_{m+p|n+q} \rightarrow gl_{p|q} \times gl_{m|n}$ branching rule:

**Theorem 4.1.** Let $\nu$ be a generalized partition of length $d$ subject to the conditions $\nu_{n+1} \leq n$ and $\nu_{d-p} \geq -q$. We have

$$W_{m+p|n+q}^{\Lambda(\nu)}|_{gl_{p|q} \times gl_{m|n}} \cong \sum_{\lambda, \mu} C^\nu_{\lambda, \mu} V_{m|n}^\lambda \otimes V_{p|q}^{-d+1+\hat{\mu}},$$

where the summation in $\lambda$ is over all the partitions of length $d$ satisfying $\lambda_{n+1} \leq n$, and the summation in $\mu$ is over all the generalized partitions of non-positive integers with length $d$ satisfying $\mu_{d-p} \geq -q$.

Recall that $\Lambda(\nu)$ is defined by (3.10).

5. **Character formula for unitarizable irreducible $gl_{m+p|n+q}$-modules**

In this section, we shall develop a character formula for the infinite dimensional unitarizable irreducible $gl_{m+p|n+q}$-modules appearing in the decomposition of $\mathbb{C}[x, y, \eta, \zeta]$. Let us first present some background material on Schur functions and the so-called hook Schur functions of Berele-Regev [1]. A comprehensive reference on Schur functions is [19].

5.1. **Hook Schur function.** Let $x = \{x_1, x_2, \cdots, x_m\}$ be a set of $m$ variables. To a partition $\lambda$ of non-negative integers we may associate the Schur function $s_\lambda(x_1, x_2, \cdots, x_m)$. We will write $s_\lambda(x)$ for $s_\lambda(x_1, x_2, \cdots, x_m)$. For a partition $\mu \subset \lambda$ we let $s_{\lambda/\mu}(x)$ denote the corresponding skew Schur function. Denote by $\mu'$ the conjugate partition of a partition $\mu$. The hook Schur function [2] corresponding to a partition $\lambda$ is defined by

$$HS_\lambda(x; y) := \sum_{\mu \subset \lambda} s_\mu(x)s_{\lambda/\mu'}(y),$$
where as usual \( y = \{y_1, y_2, \ldots, y_n\} \).

Let \( \lambda \) be a partition and \( \mu \subseteq \lambda \). We fill the boxes in \( \mu \) with entries from the linearly ordered set \( \{x_1 < x_2 < \cdots < x_m\} \) so that the resulting tableau is semi-standard. Recall that this means that the rows are non-decreasing, while the columns are strictly increasing. Next we fill the skew partition \( \lambda/\mu \) with entries from the linearly ordered set \( \{y_1 < y_2 < \cdots < y_n\} \) so that it is conjugate semi-standard, which means that the rows are strictly increasing, while its columns are non-decreasing. We will refer to such a tableau as an \((m|n)\)-semi-standard tableau (cf. [1]). To each such tableau \( T \) we may associate a polynomial \((xy)^T\), which is obtained by taking the products of all the entries in \( T \). Then we have

\[
\tag{5.2}
HS_\lambda(x; y) = \sum_T (xy)^T,
\]

where the summation is over all \((m|n)\)-semi-standard tableaux of shape \( \lambda \).

We recall the following combinatorial identity involving hook Schur functions that is of crucial importance in the sequel. Note that \( HS_\lambda(x; y) \neq 0 \) iff \( \lambda_{m+1} \leq n \).

**Proposition 5.1.** [2] Let \( x = \{x_1, x_2, \ldots, x_m\} \), \( \eta = \{\eta_1, \eta_2, \ldots, \eta_n\} \) be two sets of variables and \( z = \{z_1, z_2, \ldots, z_d\} \) be \( d \) variables. Then

\[
\prod_{i,j,k} (1 - x_i z_k)^{-1}(1 + \eta_j z_k) = \sum_{\lambda} HS_\lambda(x; \eta)s_\lambda(z),
\]

where \( 1 \leq i \leq m \), \( 1 \leq j \leq n \), \( 1 \leq k \leq d \) and \( \lambda \) is summed over all partitions with length \( d \) subject to \( \lambda_{m+1} \leq n \).

Replacing \( x_i, \eta_j \) and \( z_k \) in Proposition 5.1 by \( y_i^{-1}, \zeta_j^{-1} \) and \( z_k^{-1} \), we obtain the following result.

**Proposition 5.2.** [2] Let \( y = \{y_1, y_2, \ldots, y_p\} \), \( \zeta = \{\zeta_1, \zeta_2, \ldots, \zeta_q\} \), and \( z = \{z_1, z_2, \ldots, z_d\} \). Set \( y^{-1} = \{y_1^{-1}, y_2^{-1}, \ldots, y_p^{-1}\} \), \( \zeta^{-1} = \{\zeta_1^{-1}, \zeta_2^{-1}, \ldots, \zeta_q^{-1}\} \), and \( z^{-1} = \{z_1^{-1}, z_2^{-1}, \ldots, z_d^{-1}\} \). Then

\[
\prod_{i,j,k} (1 - y_i^{-1} z_k^{-1})^{-1}(1 + \zeta_j^{-1} z_k^{-1}) = \sum_{\lambda} HS_{\lambda}(y^{-1}; \zeta^{-1})s_\lambda(z^{-1}),
\]

where \( 1 \leq k \leq d \), \( 1 \leq i \leq p \) and \( 1 \leq j \leq q \) and \( \lambda \) is summed over all partitions with length \( d \) subject to \( \lambda_{p+1} \leq q \).

We recall the following lemma which plays a crucial role in developing a character formula for unitarizable irreducible \( gl_{m+p+n+q}\)-modules by using Howe duality. Denote by \( ch(V_\lambda^d) \) the formal character of the irreducible \( gl_d \)-module \( V_\lambda^d \).

**Lemma 5.1.** [2] Let \( q \) be an indeterminate and suppose that \( \sum_\lambda \phi_\lambda(q)chV_\lambda^d = 0 \), where \( \phi_\lambda(q) \) are power series in \( q \) and \( \lambda \) above is summed over all generalized partitions of length \( d \). Then \( \phi_\lambda(q) = 0 \), for all \( \lambda \).
5.2. Character formula for finite dimensional modules. Recall from Subsection 2.1 that \( \tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_d \), are the weights of the natural \( gl_d \)-module \( \mathbb{C}^d \), and \( \epsilon_1, \cdots, \epsilon_m, \delta_1, \cdots, \delta_n \) are the weights of the natural \( gl_{m|n} \)-module \( \mathbb{C}^{m|n} \). Let \( e \) be a formal indeterminate. For \( k = 1, \cdots, d, \ i = 1, \cdots, m \) and \( j = 1, \cdots, n, \) we set

\[
\tilde{\epsilon}_k = e^{\tilde{\epsilon}_k}, \quad \tilde{\epsilon}_i = e^{\epsilon_i}, \quad \tilde{\epsilon}_j = e^{\delta_j},
\]

and let \( \mathbf{x} = \{ \tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m \}, \ \tilde{\eta} = \{ \tilde{\eta}_1, \tilde{\eta}_2, \cdots, \tilde{\eta}_n \} \) and \( \mathbf{z} = \{ \tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_d \} \).

Consider \( \mathbb{C}[\mathbf{x}, \eta] \) as a \( gl_d \times gl_{m|n} \)-module. Its formal character \( \text{ch}(\mathbb{C}[\mathbf{x}, \eta]) \) with respect to the Cartan subalgebra \( \sum_{i=1}^{m+n} \mathcal{E}_{ii} \oplus \sum_{k=1}^{d} \mathcal{C}_{kk} \) can be easily computed by using equations (2.4) and (2.5) to give

\[
\mu = (1 - \tilde{x}_i \tilde{z}_k)^{-1}(1 + \tilde{\eta}_j \tilde{z}_k),
\]

where \( 1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq k \leq d \). Thus by Proposition 5.1

\[
\text{ch}(\mathbb{C}[\mathbf{x}, \eta]) = \prod_{i,j,k} (1 - \tilde{x}_i \tilde{z}_k)^{-1}(1 + \tilde{\eta}_j \tilde{z}_k),
\]

where \( \lambda \) is summed over all partitions with length \( d \) subject to \( \lambda_{m+1} \leq n \).

Let us denote by \( \text{ch}(V_{m|n}^\lambda) \) the formal character of the irreducible \( gl_{m|n} \)-module. Theorem 2.1 leads to

\[
\text{ch}(\mathbb{C}[\mathbf{x}, \eta]) = \sum_{\lambda} \text{ch}(V_{m|n}^\lambda) \text{ch}(V_{m|n}^\lambda),
\]

where, we recall that, \( \text{ch}(V_{m|n}^\lambda) = s_\lambda(\mathbf{z}) \). By using Lemma 5.1 we obtain the following well-known result [1].

**Theorem 5.1.** For each partition \( \lambda \) of length \( d \) subject to the condition \( \lambda_{m+1} \leq n \),

\[
\text{ch}(V_{m|n}^\lambda) = \mathcal{H}_{\lambda}(\mathbf{x}; \tilde{\eta}),
\]

where \( \tilde{\lambda} \) is defined by (2.2).

Keep the notations of this subsection but replace \( m \) by \( p \) and \( n \) by \( q \). Let \( \mathbf{x}^{-1} = \{ \tilde{x}_1^{-1}, \tilde{x}_2^{-1}, \cdots, \tilde{x}_p^{-1} \}, \ \tilde{\eta}^{-1} = \{ \tilde{\eta}_1^{-1}, \tilde{\eta}_2^{-1}, \cdots, \tilde{\eta}_q^{-1} \} \) and \( \mathbf{z}^{-1} = \{ \tilde{z}_1^{-1}, \tilde{z}_2^{-1}, \cdots, \tilde{z}_d^{-1} \} \).

Using (2.13) and (2.14), we can easily compute the formal character of the \( gl_d \times gl_{p|q} \)-module \( \mathbb{C}[\mathbf{y}, \zeta] \) with respect to the Cartan subalgebra \( \sum_{i=1}^{p+q} \mathcal{E}_{ii} \oplus \sum_{k=1}^{d} \mathcal{C}_{kk} \). We have

\[
\mu = (1 - \tilde{x}_i \tilde{z}_k)^{-1}(1 + \tilde{\eta}_j \tilde{z}_k),
\]

where \( 1 \leq i \leq p, \ 1 \leq j \leq q, \) and \( 1 \leq k \leq d \). By Proposition 5.2

\[
\text{ch}(\mathbb{C}[\mathbf{y}, \zeta]) = (\tilde{x}_1 \cdots \tilde{x}_p)^{-d} (\tilde{\eta}_1 \cdots \tilde{\eta}_q)^d \sum_{\lambda} \mathcal{H}_{\lambda}(\mathbf{x}^{-1}; \tilde{\eta}^{-1}) s_\lambda(\mathbf{z}^{-1}),
\]
where $\lambda$ is summed over all partitions with length $d$ subject to $\lambda_{p+1} \leq q$.

Note that $\text{ch}(V_d^\lambda) = s_\lambda(\bar{z}^{-1})$. Thus the following theorem is a consequence of Theorem 5.2 by using Lemma 5.1 and equation (5.7).

**Theorem 5.2.** For each partition $\lambda$ of length $d$ subject to the condition $\lambda_{p+1} \leq q$,

$$\text{ch}V^d_{pl,q} = (\tilde{x}_1 \cdots \tilde{x}_p)^{-d} (\tilde{\eta}_1 \cdots \tilde{\eta}_q)^d HS_\lambda(\tilde{x}^{-1}; \tilde{\eta}^{-1}),$$

where $\tilde{\lambda}^*$ is as given in (2.18).

5.3. **Character formulas for unitarizable $\mathfrak{gl}_{m+p|n+q}$-modules.** We keep the notations $z_i$, $1 \leq i \leq d$, and $z$, $z^{-1}$ from the last subsection. Let $e$ be the formal indeterminate as before. For $1 \leq r \leq p$, $1 \leq s \leq q$, $1 \leq i \leq m$, and $1 \leq j \leq n$, we define

$$\bar{y}_r = e^{\tilde{e}_r}, \quad \bar{e}_i = e^{\tilde{e}_{p+i}},$$

$$\bar{e}_i = e^{\tilde{e}_{p+i}}, \quad \bar{\eta}_j = e^{\tilde{\eta}_{m+p+q}},$$

where we recall that, $\tilde{e}_A \in \mathfrak{h}^*_{\mathfrak{m}_p|\mathfrak{n}_q}$. $A \in \mathfrak{I}$, are defined by $\tilde{e}_A(E_B^\mathfrak{g}) = \delta_{AB}$, $A, B \in \mathfrak{I}$. Set $\bar{x} = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\}$, $\bar{\eta} = \{\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_n\}$, $\bar{y}^{-1} = \{\bar{y}_1^{-1}, \bar{y}_2^{-1}, \ldots, \bar{y}_p^{-1}\}$ and $\bar{\zeta}^{-1} = \{\bar{\zeta}_1^{-1}, \bar{\zeta}_2^{-1}, \ldots, \bar{\zeta}_q^{-1}\}$.

We wish to compute the formal characters $\text{ch}(W^\Lambda_{\mathfrak{m}_p\mathfrak{n}_q})$ with respect to the Cartan subalgebra $\mathfrak{h}_{\mathfrak{m}_p\mathfrak{n}_q} = \sum_{A=1}^{m+n+p+q} \mathbb{C}E_A^\mathfrak{g}$ for the unitarizable irreducible $\mathfrak{gl}_{m+p|n+q}$-modules $W^\Lambda_{\mathfrak{m}_p\mathfrak{n}_q}$ appearing in the decomposition of $\mathbb{C}[x, y, \eta, \zeta]$.

**Theorem 5.3.** For each generalized partition $\lambda$ of length $d$ subject to the conditions $\lambda_{m+1} \leq n$ and $\lambda_{d-p} \geq -q$,

$$\text{ch}(W^\Lambda_{\mathfrak{m}_p\mathfrak{n}_q}) = (\bar{y}_1 \bar{y}_2 \cdots \bar{y}_p)^{-d}(\bar{\zeta}_1 \bar{\zeta}_2 \cdots \bar{\zeta}_q)^d \sum_{\mu, \nu} C^\Lambda_{\mu^*, \nu} \text{HS}_\mu(x; \bar{\eta}) \text{HS}_\nu(y^{-1}; \bar{\zeta}^{-1}),$$

where $\mu$ and $\nu$ are summed over all partitions of length $d$ subject to the conditions $\mu_{m+1} \leq n$ and $\nu_{p+1} \leq q$ respectively. The $C^\Lambda_{\mu^*, \nu}$ are the Littlewood-Richardson coefficients.

**Proof.** Consider the restriction of $W^\Lambda_{\mathfrak{m}_p\mathfrak{n}_q}$ to a module of the subalgebra $\mathfrak{gl}_d \times \mathfrak{gl}_m \times \mathfrak{gl}_p \times \mathfrak{gl}_q$. Its formal character with respect to the Cartan subalgebra $\mathfrak{h}_d \times \mathfrak{h}_p \times \mathfrak{h}_m \times \mathfrak{h}_n$ coincides with $\text{ch}(W^\Lambda_{\mathfrak{m}_p\mathfrak{n}_q})$. Therefore by Theorem 4.1 we have

$$\text{ch}(W^\Lambda_{\mathfrak{m}_p\mathfrak{n}_q}) = \sum_{\mu, \nu} C^\Lambda_{\mu^*, \nu} \text{ch}(V^\mu_{\mathfrak{m}_m}) \text{ch}(V^\nu_{\mathfrak{p}_p}) \text{ch}(V^{-d+\tilde{\lambda}^*})$$

where $\mu$ and $\nu$ are summed over all partitions of length $d$ subject to the conditions $\mu_{m+1} \leq n$ and $\nu_{p+1} \leq q$ respectively. Using Theorem 5.1 and Theorem 5.2 in this equation we immediately arrive at the claimed result. \qed
6. Tensor Product Decomposition of Unitarizable Irreducible $gl_{m+p|n+q}$-Modules

As another application of Theorem 3.3, we shall compute the tensor product decomposition

\[
W_{m+p|n+q}^{\Lambda(\mu)} \otimes W_{m+p|n+q}^{\Lambda(\nu)} \cong \sum_{\lambda} a_{\mu \nu}^{\lambda} W_{m+p|n+q}^{\Lambda(\lambda)},
\]

where $\mu$ and $\nu$ are generalized partitions of length $l$ and $r$, respectively, satisfying in addition the conditions $\mu_{m+1} \leq n$, $\nu_{m+1} \leq n$, $\mu_{l-p} \geq -q$ and $\nu_{r-p} \geq -q$. It will follow easily from our discussion that the summation $\lambda$ in (6.1) is over all generalized partitions of length $l + r$ and satisfying $\lambda_{m+1} \leq n$ and $\lambda_{l+r-p} \geq -q$. We will compute the coefficients $a_{\mu \nu}^{\lambda}$ in terms of the usual Littlewood-Richardson coefficients (see e.g. [19]).

We have by Theorem 3.3 for $d = l$, $r$ and $l + r$ respectively

\[
S(\mathbb{C}^l \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{l*} \otimes \mathbb{C}^{p|q*}) \cong \sum_{\mu} V_{l}^{\mu} \otimes W_{m+p|n+q}^{\Lambda(\mu)},
\]

\[
S(\mathbb{C}^r \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{r*} \otimes \mathbb{C}^{p|q*}) \cong \sum_{\nu} V_{r}^{\nu} \otimes W_{m+p|n+q}^{\Lambda(\nu)},
\]

and

\[
S(\mathbb{C}^{l+r} \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{l+r*} \otimes \mathbb{C}^{p|q*}) \cong \sum_{\lambda} V_{l+r}^{\lambda} \otimes W_{m+p|n+q}^{\Lambda(\lambda)},
\]

where $\mu$, $\nu$ and $\lambda$ are generalized partitions satisfying the corresponding conditions described above. The isomorphism $S(\mathbb{C}^l \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{l*} \otimes \mathbb{C}^{p|q*}) \otimes S(\mathbb{C}^r \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{r*} \otimes \mathbb{C}^{p|q*}) \cong S(\mathbb{C}^{l+r} \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{l+r*} \otimes \mathbb{C}^{p|q*})$ gives rise to

\[
\sum_{\mu, \nu} V_{l}^{\mu} \otimes V_{r}^{\nu} \otimes W_{m+p|n+q}^{\Lambda(\mu)} \otimes W_{m+p|n+q}^{\Lambda(\nu)} \cong \sum_{\lambda} V_{l+r}^{\lambda} \otimes W_{m+p|n+q}^{\Lambda(\lambda)}.
\]

Now suppose that $V_{l+r}^{\lambda}$, when regarded as a $gl_l \times gl_r$-module via the obvious embedding of $gl_l \times gl_r$ into $gl_{l+r}$, decomposes as

\[
V_{l+r}^{\lambda} \cong \sum_{\mu, \nu} b_{\lambda}^{\mu \nu} V_{l}^{\mu} \otimes V_{r}^{\nu}.
\]

This together with (6.1) and (6.2) give

\[
a_{\mu \nu}^{\lambda} = b_{\lambda}^{\mu \nu}.
\]

The duality between the branching coefficients and tensor products of a general dual pair is well-known [15]. We recall that in (6.3) $\mu$, $\nu$ and $\lambda$ are generalized partitions subject to the appropriate constraints.

Now Theorem 2.1 with $n = 0$ combined with an analogous argument as the one given above imply that

\[
C_{\mu \nu}^{\lambda} = b_{\lambda}^{\mu \nu},
\]
where here $\mu$, $\nu$, $\lambda$ are partitions of appropriate lengths and the $C^\lambda_{\mu\nu}$’s are the usual Littlewood-Richardson coefficients.

Now for generalized partitions $\mu$, $\nu$ and $\lambda$ subject to appropriate constraints the decomposition $V^\lambda_{\ell+r} \cong \sum_{\mu,\nu} b^\lambda_{\mu\nu} V^\mu_{\ell} \otimes V^\nu_r$ implies that $V^\lambda_{\ell+r} \cong \sum_{\mu,\nu} b^\lambda_{\mu\nu} V^\mu_{\ell+1+r} \otimes V^\nu_{r+d1_{1+r}}$, where $1_k$ denotes the $k$-tuple $(1, 1, \ldots, 1)$ regarded as a partition. Hence $b^\lambda_{\mu\nu} = b^\lambda_{\mu+d1_{1+r},\nu+d1_{1+r}}$. Now if we choose a non-negative integer $d$ so that $\lambda + d1_{1+r}$ is a partition, then $b^\lambda_{\mu+d1_{1+r},\nu+d1_{1+r}} = C^\lambda_{\mu+d1_{1+r},\nu+d1_{1+r}}$ and hence by (6.3) and (6.4)

$$a^\lambda_{\mu\nu} = C^\lambda_{\mu+d1_{1+r},\nu+d1_{1+r}}.$$

From our discussion above we arrive at the following theorem.

**Theorem 6.1.** Let $\mu$ and $\nu$ be generalized partitions of length $l$ and $r$, respectively, satisfying in addition the conditions $m_{m+1} \leq n$, $n_{m+1} \leq n$, $\mu_{l-p} \geq -q$ and $\nu_{p-r} \geq -q$. Let $W^{\Lambda(\mu)}_{m+p|n+q}$ and $W^{\Lambda(\nu)}_{m+p|n+q}$ be the corresponding unitarizable $gl_{m+p|n+q}$-modules. We have the following decomposition of $W^{\Lambda(\mu)}_{m+p|n+q} \otimes W^{\Lambda(\nu)}_{m+p|n+q}$ into irreducible $gl_{m+p|n+q}$-modules:

$$W^{\Lambda(\mu)}_{m+p|n+q} \otimes W^{\Lambda(\nu)}_{m+p|n+q} \cong \sum_{(\lambda,d)} C^\lambda_{\mu+d1_{l+r},\nu+d1_{r}} W^{\Lambda(\lambda-d1_{l+r})}_{m+p|n+q},$$

where the summation above is over all pairs $(\lambda, d)$ subject to the following four conditions:

(i) $\lambda$ is a partition of length $l + r$ and $d$ a non-negative integer.
(ii) $(\lambda - d1_{l+r})_{m+1} \leq n$ and $(\lambda - d1_{l+r})_{l+r-p} \leq -q$.
(iii) $\mu + d1_{l}$ and $\nu + d1_{r}$ are partitions.
(iv) If $d > 0$, then $\lambda$ is a partition with $\lambda_{l+r} = 0$.

Here the coefficients $C^\lambda_{\mu+d1_{l+r},\nu+d1_{r}}$ are determined by the tensor product decomposition of $gl_k$-modules $V^\mu_{k+d1_{l}} \otimes V^\nu_{k+d1_{r}} \cong \sum_{\lambda} C^\lambda_{\mu+d1_{l+r},\nu+d1_{r}} V^\lambda_k$, where $k = l + r$.

**Remark 6.1.** In the above theorem the coefficients $C^\lambda_{\mu+d1_{l+r},\nu+d1_{r}}$ are the usual Littlewood-Richardson coefficients associated to partitions, and hence can be computed via the Littlewood-Richardson rule.

**Remark 6.2.** We finally note the remarkable similarity between the irreducible representations of the so-called super $\mathcal{W}_{1+\infty}$, that is the Lie superalgebra of differential operators on the circle with $N = 1$ extended symmetry, that appear in the decomposition of tensor powers of its natural representation on the infinite-dimensional Fock space [2], and the irreducible unitarizable representations of $gl_{m+p|n+q}$ of this paper. Indeed the characters and the tensor product decomposition are virtually identical modulo some modification necessitated by the infinite-dimensional nature of the super $\mathcal{W}_{1+\infty}$. This similarity can be explained as
by the existence a Howe duality between the super $W_{1+\infty}$ and $gl_d$ on the $d$-th tensor power of the Fock space generated by infinitely many fermionic and bosonic quantum oscillators [5].

References

[1] Berele, A.; Regev, A.: Hook Young diagrams with applications to combinatorics and representations of Lie superalgebras, Adv. Math. 64 (1987) 118–175.

[2] Cheng, S.-J.; Lam, N.: Infinite-dimensional Lie Superalgebras and Hook Schur functions, math.RT/0206034, Comm. Math. Phys., to appear.

[3] Cheng, S.-J.; Wang, W.: Howe Duality for Lie Superalgebras, Compositio Math. 128 (2001) 55–94.

[4] Cheng, S.-J.; Wang, W.: Remarks on the Schur-Howe-Sergeev Duality, Lett. Math. Phys. 52 (2000) 143–153.

[5] Cheng, S.-J.; Wang, W.: Lie subalgebras of differential operators on the super circle, math.QA/0103092, Publ. RIMS, to appear.

[6] Cheng, S.-J.; Zhang, R.B.: Howe duality and combinatorial character formula for orthosymplectic Lie superalgebras, math.RT/0206036, Adv. Math., to appear.

[7] Dobrev, V.K.: Positive energy unitary irreducible representations of $D=6$ conformal supersymmetry, J.Phys. A35 (2002) 7079-7100

[8] Davidson, M; Enright, E.; Stanke, R.: Differential Operators and Highest Weight Representations, Mem. Amer. Math. Soc. 94 (1991) no. 455.

[9] D’Auria, R.; Ferrara, S.; Lled, M. A.; Varadarajan, V. S.: Spinor algebras. J. Geom. Phys. 40 (2001), 101–129.

[10] Enright, T.: Analogues of Kostant’s $u$-cohomology formulas for unitary highest weight modules, J. Reine Angew. Math. 392 (1988) 27–36.

[11] Enright, T.; Howe, R.; Wallach, N.: A classification of unitary highest weight modules. In Representation theory of reductive groups (Park City, Utah, 1982), 97–143, Progr. Math., 40, Birkhäuser Boston, Boston, MA, 1983.

[12] Goodman, R.; Wallach, N.: Representations and Invariants of the Classical Groups, Cambridge University Press, Cambridge, 1998.

[13] Gould, M. D.; Zhang, R. B.: Classification of all star irreps of $gl(m|n)$. J. Math. Phys. 31 (1990), 2552–2559.

[14] Howe, R.: Remarks on classical invariant theory, Trans. Amer. Math. Soc. 313 (1989) 539–570.

[15] Howe, R.: Perspectives on Invariant Theory: Schur Duality, Multiplicity-free Actions and Beyond, The Schur Lectures, Israel Math. Conf. Proc. 8, Tel Aviv (1992) 1–182.

[16] Jakobsen, H. P.: The full set of unitarizable highest weight modules of basic classical Lie superalgebras. Mem. Amer. Math. Soc. 111 (1994), no. 532, vi+116 pp.

[17] Kac, V.: Lie superalgebras, Adv. Math. 16 (1977) 8–96.

[18] Kashiwara, M.; Vergne, M.: On the Segal-Shale-Weil representations and harmonic polynomials. Invent. Math. 44 (1978), no. 1, 1–47.

[19] Macdonald, I. G.: Symmetric functions and Hall polynomials, Oxford Math. Monogr., Clarendon Press, Oxford, 1995.

[20] Nazarov, M.: Capelli identities for Lie superalgebras, Ann. Scient. Éc Norm. Sup. 4e série 30 (1997) 847–872.
[21] Ol’shanskii, G.; Prati, M., *Extremal weights of finite-dimensional representations of the Lie superalgebra $\mathfrak{gl}_{n|m}$*, II Nuovo Cimento, 85 A (1985) 1–18.

[22] Sergeev, A.: *An analog of the classical invariant theory for Lie superalgebras, I, II*, Michigan Math. J. 49 (2001) 113–146; 147–168.

[23] Townsend, P. K.: *M-theory from its superalgebra*. In Strings, branes and dualities (Cargse, 1997), 141–177, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 520, Kluwer Acad. Publ., Dordrecht, 1999.

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