Quantum Origin of (Newtonian) Mass and Galilean Relativity Symmetry

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Abstract

The Galilei group has been taken as the fundamental symmetry for ‘nonrelativistic’ physics, quantum or classical. Our fully group theoretical formulation approach to the quantum theory asks for some adjustments. We present a sketch of the full picture here, emphasizing aspects that are different from the more familiar picture. The analysis involves a more careful treatment of the relation between the exact mathematics and its physical application in the dynamical theories, and a more serious full implementation of the mathematical logic than what is usually available in the physics literature. The article summarizes our earlier presented formulation while focusing on the part beyond, with an adjusted, or corrected, identification of the basic representations having the (Newtonian) mass as a Casimir invariant and the notion of center of mass as dictated by the symmetry, that is particularly also to be seen as the Heisenberg-Weyl symmetry inside it. Another result is the necessary exclusion of the time translational symmetry. The time translational symmetry in the Galilei group plays no role in the formulation of the dynamical theory and does not correspond to the physical time in any nontrivial setting.

Keywords : Particle Mass, Casimir Invariants, (Quantum) Relativity Symmetry, Composite Systems as Symmetry Representations.

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I. INTRODUCTION

Symmetry consideration is one of the most important theoretical guiding principles in modern physics. A group theoretical formulation of fundamental theories can go a very long way in ‘explaining’ the fundamental structures. Group theory, especially in the commutator (Lie) algebra of the quantum operators which serves as generators of unitary symmetry (group) transformations of the fundamental (relativity) symmetry, has been common in the literature of quantum physics. Yet, even most physicists today may have limited appreciation of what is the strict mathematical logic and what may be only contingent on the application of that in a particular physical theory. For example, in the physics literature, the commutator algebra and the way the corresponding operators have been introduced are often taken as the defining structure of the symmetry Lie algebra. That actually is nothing more than one consistent representation of the symmetry. In fact, it is only such a representation when the Hilbert space the operators are supposed to act on is indeed the representation space. The word ‘representation’ has been used in a few different ways, which is also a source of confusion. In this article, we use the word only in the straight symmetry theoretical sense as in the literature of mathematics.

We have results that indeed go against some of the very first things physicists learned in the early days of their education. To appreciate them, readers certainly need to exercise patience and care to consider our logic critically but otherwise objectively and seriously with an open mind.

Relativity symmetries have been taken as the most important kind of fundamental symmetries in physics. The Galilean symmetry for ‘nonrelativistic’ physics has been firstly taken as the symmetry for the Newtonian space-time which is a representation/coset space of the Galilei group $G(3)$. The phase space for a Newtonian particle can also be identified with a $G(3)$ coset space [1]. The Hilbert space for the states for a quantum particle is known as the representation space of the Heisenberg-Weyl symmetry $H(3)$, as usually appreciated through the commutator algebra of the position and momentum operators. $G(3)$ symmetry is still taken to be the relativity symmetry, though it can only be realized so-called projectively. In fact, what is called a projective representation of a Lie group can, and in our opinion should only, be seen as a unitary representation of its $U(1)$ central extension [2]. The $U(1)$ central extension, $\tilde{G}(3)$, as an abstract symmetry contains the $H(3)$ as a subgroup. Taking that se-
riously gives a better picture of the symmetry theoretical formulation of quantum mechanics [3]. This article gives a slightly modified picture of that, with two mathematically minor adjustments having, however, important implications. One is to take out the time translation symmetry from $G(3)$ or $\tilde{G}(3)$. The other is to have the central charge generator in $H(3)$ represented as the Newtonian mass operator. The key new features are: Newtonian particle mass is a Casimir invariant of the fundamental/relativity symmetry for the quantum theory only to be inherited by classical mechanics; which then dictates the exact notion of center of mass; the Newtonian time in the dynamical theory does not come from the ‘nonrelativistic’ relativity symmetry.\footnote{Our study here is restricted to the basic picture of particle dynamics, classical and quantum. Otherwise, for example in relation to geometric description of gravity, somewhat extended symmetry framework may be of interest [4–6].}

The phase space for a quantum particle, is, like all classical phase space, a symplectic manifold [7]. The Poisson bracket in terms of the observables is essentially given by the commutator [8, 9]. Note that symplectic manifold and its Poisson bracket are generally very closely connected to the symmetry of Lie group/algebra and its Lie bracket [10]. With the symplectic geometry, one has a picture of one-parameter groups of Hamiltonian flows as symmetry transformations on the phase space, each for a generic Hamiltonian function [11]. The Hamiltonian functions which match to (Hermitian) elements of the observable algebra [12] have the transformations as unitary transformations on the Hilbert space. The physical Hamiltonian operator, the energy observable, is to be identified among them. (Newtonian) time is its flow/evolution parameter. In that sense, we can say that the fundamental symmetry as symmetry for the phase space essentially dictates the dynamics of Newtonian time, though it does not identify which particular one-parameter Hamiltonian flow/evolution is the time flow/evolution. In fact, it need not even have the physical Hamiltonian operator as representing any element in the Lie algebra. The $H$ generator in $G(3)$ cannot give the correct time anyway.

To avoid confusion and facilitate the appreciation of some of the subtle issues presented in our discussions below, let us present the structure and logic of the mathematics and then the physically formulated theory as an application in general, emphasizing what we see as the full picture or the full power of the symmetry formulation.

One can start with an abstract (real) Lie algebra. It is a (real) vector space, defined by
a set of abstract generators as basis vectors, on which a Lie product/bracket is defined by giving the Lie bracket between any pair of generators as a linear combination of all. The corresponding Lie group can be obtained as its formal exponentiation. Another structure of importance to physics is its universal enveloping algebra, which is essentially formal complex polynomials of the generators. One may want to extend that further to functions or even distributions of the generators which can also be seen as elements of the complex group \((C^*)\)-algebra and its extension. Subtle details of the part we need not concerned much here. We are interested in a representation of all of that, and particularly the irreducible ones. Hence, we use the term symmetry theoretical formulation instead of group theoretical formulation and use basically the group symbols to denote the whole mathematical package. Most of the explicit analysis focuses on the Lie algebras though. A physical system bearing the symmetry would correspond to such a representation for which one has a representation (complex vector) space with operators representing the generators. The operator product matches exactly to the formal product and the commutator to the Lie bracket. In the case of quantum mechanics, the representation space is the Hilbert space of states, and the operator algebra is the algebra of observables. A few elements in the universal enveloping algebra are of special importance in the representation theory. These are the Casimir elements. They form a set of independent elements which commute with all the generators hence all other elements. An irreducible eigenspace of the set of Casimir operators, representing the Casimir elements, gives an irreducible representation of the symmetry. The corresponding eigenvalues of Casimir operators, the Casimir invariants, hence completely characterize the representation as an elementary physical system bearing the symmetry. We expect the Casimir invariants to be important physical properties of the system. Actually, the formulation of the physical theory should be considered unsatisfactory if the system has a fundamental characteristic that does not correspond to a Casimir invariant. With the exception of having the Newtonian particle mass as a Casimir invariant, our previous formulation a particle of zero spin [3] fully illustrates the power of the (relativity) symmetry theoretical formulation. Though Ref.[3] still acknowledges the symmetry as \(\tilde{G}(3)\), the formulation used really only the \(H_\text{r}(3)\) part without the generator \(H\) for time translation. We present below the key symmetry issues about the theory, focusing on the key notions mentioned above but skipping detailed formulation aspects which can be easily adapted from Ref.[3]. A serious consideration of a composite system, of two particles, is an important part.
Note that most of the presentations of symmetry theoretical formulation for the quantum theory are based on the Heisenberg-Weyl symmetry $H(3)$ not in a manner to be seen as a part of the $\tilde{G}(3)$ symmetry. In particular, the position operators are taken as directly representing generators of the abstract Lie algebra, which is really problematic. Such a presentation is, as we argue below, incorrect. It also hides the related role of the mass from the symmetry picture and goes against the notion of the center of mass for a composite system. Some discussions on how the Galilean symmetry is realized in the quantum theory, for example Ref.[13], get the part of the results fine.

In general, the Galilean symmetry picture is well-presented in many textbooks. But the key issues from the formulation perspective are left not clarified. We intend here to do exactly that, in the next section. Note that an important paper from Lévy-Leblond [14] is an exception. It addresses directly a symmetry theoretical formulation based on the Galilean symmetry, and hence serves as a great background reference for our presentation of the formulation. However, there is no explicit discussion in the paper of the issues concerning the position operators from the formulation. Situation in Refs.[15, 16] is similarly, though they give a nice emphasis on taking the language of the central extension and have some interesting discussions on Newtonian mass as the central charge in relation to the equivalence principle and other things. In particular, they still fell short of naming the mass as a Casimir invariant. Some aspects of formulation, mostly for the part of spin, are presented in the appendix, at least to make the presentation of the symmetry theoretical formulation more complete.

Another difference between our presentation and that of Ref.[14] and others in the literature, is about the time translational symmetry part of the $\tilde{G}(3)$ symmetry. Our presentation of the formulation, as given in the next section, used only the part of the so-called $H_R(3)$ of the $\tilde{G}(3)$ without the time translation. We address how Newtonian time shows up in the dynamical theory from the Hamiltonian or symplectic geometry perspective in section III. The considerations for dropping the time translational symmetry, hence sticking to use only the $H_R(3)$ symmetry in the formulation are discussed. Let us emphasize here that we are not proposing that one should drop time translational symmetry from consideration in physics. The notion of time and the symmetry as in the admissible arbitrary choice of the origin of time in a dynamical, at least so long as we are not looking at cosmology, is no doubt of fundamental importance. We plead only a case that the symmetry should not be
included in the fundamental symmetry from which our dynamical theory of ‘nonrelativistic’ (quantum) mechanics is to be obtained as a representation theory of. We also express some further opinion on how time is to be seen from physics in general. That leads to the last concluding section.

II. SYMMETRY THEORETICAL FORMULATION OF QUANTUM MECHANICS AND THE NEWTONIAN MASS

The Heisenberg-Weyl symmetry $H(3)$ for quantum mechanics is usually presented in terms of the Lie algebra as

$$[X_i, P_j] = i\hbar \delta_{ij} I,$$  \hspace{1cm} (1)

which is a direct abstraction of the Heisenberg commutation relation between the position and momentum operators $[\hat{X}_i, \hat{P}_j] = i\hbar \delta_{ij} \hat{I}$. The generator $I$ is a central charge, having vanishing Lie brackets with all other generators and hence all elements of the seven generator Lie algebra. In any irreducible unitary representation, it has to be represented by a real multiple of the identity operator $\hat{I}$. That real number is then the eigenvalue of the operator representation of $I$. In fact, any nonzero value gives a unitary representation, the direct integral of all is the regular representation [17]. Note that the Stone-von Neumann theorem really only addresses the case of an operator picture of the Heisenberg commutation relation, instead of starting from the above Lie algebra for which the abstract generator $I$ cannot be assumed to be (represented by) the identity. The one-parameter set of irreducible representations, other than the effectively commutative limit of its vanishing value, is well appreciated in the mathematics literature [17]. While such an extra nontrivial parameter can be absorbed, it is puzzling if that is really just a redundancy in the relation between the abstract mathematics and physics or if it has other implications.

Explicitly, one can take for each irreducible representation characterized by a $\zeta \neq 0$

$$\sqrt{\zeta} \hat{X}_i (= \sqrt{\zeta} \hat{x}_i) = \hat{X}_{\zeta} \leftarrow X_i,$$

$$\sqrt{\zeta} \hat{P}_i (= -i\hbar \sqrt{\zeta} \frac{\partial}{\partial x^i}) = \hat{P}_{\zeta} \leftarrow P_i,$$

$$\zeta \hat{I} = \hat{I}_{\zeta} \leftarrow I,$$  \hspace{1cm} (2)

which clearly satisfies $[\sqrt{\zeta} \hat{X}_i, \sqrt{\zeta} \hat{P}_j] = i\hbar \delta_{ij} \zeta \hat{I}$ as the homomorphic to the abstract $[X_i, P_j] = \ldots$
\(i\hbar \delta_{ij} I\) relation as required for a representation. That is essentially what is given in the mathematic literature [17] and adopted in our earlier work [3]. We want to emphasize that having the operator representation of the generators on a definite representation (Hilbert) space, here say of the Schrödinger wavefunctions, is having a representation for the full symmetry set-up. The latter includes all elements of the Lie algebra, Lie group, the universal enveloping algebra, . . . etc. We have the full observable algebra with each observable as, say, a function of the basic observables of position and momentum. Together with a natural symplectic structure on the Hilbert space and its projective space, the full dynamical theory can be obtained [3]. The subtly about the different admissible values of \(\zeta\) corresponding to different, mathematically inequivalent, irreducible representations has otherwise apparently not been noted in the literature of physics. An obvious reason is that the presence of \(\zeta\) has no practical implication so long as a single irreducible representation is concerned. However, the matter should be look into more carefully. First of all, the central charge \(I\) is effectively a Casimir elements and its representing operator a Casimir operator. Or we called them so in the sense that they share the same key properties in relation to the representation theory. Namely, \(I\) has a vanishing Lie bracket with all generators, hence representing operator commutes with all operators in the theory, giving the eigenvalue \(\zeta\) the role of a Casimir invariant characterizing an irreducible representation. That operator is \(\hat{I}_\zeta\), that is simply \(\zeta\) times the identity operator on that particular Hilbert space. One should wonder why the formulation has such a Casimir invariant irrelevant to physics, in stark contrast to the usual role of Casimir invariants, like the mass and spin of the Poincaré symmetry. Another important question is how the \(\hat{X}_i\) and \(\hat{P}_i\) relate to the picture of Galilean symmetry. In the latter case, one may appreciate the identification of \(\hat{P}_i\) as directly representing the generators of (spatial) translational symmetry and \(\hat{K}_i = m\hat{X}_i\) (with \(m\) as the Newtonian mass) as representing the generators of the Galilean boosts. That actually is a question of how to reconcile a symmetry theoretical formulations of the quantum theory with \(H(3)\) versus \(\tilde{G}(3)\) as the fundamental symmetry. Lastly, but more importantly, the apparently harmless \(\zeta\) actually causes serious problems when the formulation is pushed onto composite systems. For example, one has to confront the question of the composite of two irreducible representations with different \(\zeta\) values.

We have stated that the abstract symmetry of \(H(3)\) is part of the abstract \(\tilde{G}(3)\), like the Lie algebra of the former is a subalgebra of the latter. At the operator level, it looks like
there is no problem between having $[\hat{X}_i, \hat{P}_j] = i\hbar \delta_{ij} \hat{I}$ and $[\hat{K}_i, \hat{P}_j] = i\hbar \delta_{ij} \hat{M}$ with $\hat{K}_i = m\hat{X}_i$ and $\hat{M} = m\hat{I}$. The subtly though is that the simple operator relations, which are physically correct, do not translate well into parallel results at the abstract Lie algebra level. The usual Galilean picture, with the Lie algebra for the $H(3)$ seen as

$$m\hat{X}_i(=mx_i) = \hat{K}_i \rightarrow K_i,$$

$$\hat{P}_i(=-i\hbar \frac{\partial}{\partial x^i}) \rightarrow P_i,$$

$$m\hat{I} = \hat{M} \rightarrow M,$$

$$[K_i, P_j] = i\hbar \delta_{ij} M$$

is the correct one. The usual $H(3)$ picture of $[X_i, P_j] = i\hbar \delta_{ij} I$ is actually incorrect. That gives $m$, the eigenvalue of $\hat{M}$ characterizing an irreducible representation for the Casimir element $M$, as the Casimir invariant. It is, of course, the Newtonian mass. Note that a relation such as $\hat{X}_i = \frac{1}{m} \hat{K}_i$ does not translate into a fixed relation in the Lie algebra. $m$ is not a scalar in the Lie algebra. The operator relation is really $\hat{X}_i = \hat{M}^{-1} \hat{K}_i$ as acting on the fixed Hilbert space of the representation. The $m$ value is different for a different representation, and the inverse $M^{-1}$ is not defined in the Lie algebra or even the universal enveloping algebra. One cannot write $X_i = M^{-1} \hat{K}_i$, or $X_i = \frac{1}{m} K_i$. So, while $\hat{K}_i, \hat{P}_i$, and $\hat{M}$ are operators directly representing generators of the Lie algebra, $\hat{X}_i$ do not directly represent elements of the abstract structure. In fact, that is exactly what it should be, as to be illustrated in an analysis of a composite system, for example, of two particles.

We have for a particle of mass $m_a(>0)$ the operators $\hat{M}_a = m_a, \hat{K}_{ia}, \hat{P}_{ia}$, and $\hat{X}_{ia} = \frac{1}{m_a} \hat{K}_{ia}$ acting on a Hilbert space $\mathcal{H}_a$, and similarly for a particle of mass $m_b(>0)$ the operators $\hat{M}_b = m_b, \hat{K}_{ib}, \hat{P}_{ib}$, and $\hat{X}_{ib} = \frac{1}{m_b} \hat{K}_{ib}$ acting on a Hilbert space $\mathcal{H}_b$. The basic point about the symmetry theoretical formulation of a composite system is to identify that as the product representation of the irreducible representations describing its contributing elementary parts. That is actually a straightforward exercise along the line of angular momentum addition widely available in quantum mechanics textbooks. For the system of two particles, the formulation dictates that $\hat{G} = \hat{G}_a \otimes \hat{I} + \hat{I} \otimes \hat{G}_b$, for any generator or, in fact, any element of the Lie algebra $G$. Here, $\hat{G}_a$ and $\hat{G}_b$ are the operators representing $G$ for particle $a$ and $b$, respectively; and $\hat{G}$ acts on the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$, the direct product of those for the individual components. That is a straight consequence of the mathematical logic, which has very important consequences. For example, we have

$$\hat{M} = \hat{M}_a \otimes \hat{I} + \hat{I} \otimes \hat{M}_b,$$  \hspace{1cm} (4)
as well as

$$\hat{P}_i = \hat{P}_{ia} \otimes \hat{I} + \hat{I} \otimes \hat{P}_{ib}. \quad (5)$$

The naive expressions give mass and momentum as additive quantities. In particular, we have a (total) mass \( m = m_a + m_b \) for the composite system. The three position operators \( \hat{X}_i \) however cannot satisfy the kind of simple sum relation to those of the individual particles. The operators \( \hat{X}_{ia} \otimes \hat{I} + \hat{I} \otimes \hat{X}_{ib} \) do not bear any physical meaning as position observables. And they do not have the right Heisenberg commutation relation with the \( \hat{P}_i \) above. That is actually an indicator that \( \hat{X}_i \) do not represent generators directly. Otherwise, there is no escape from that notion of additive position for a composite system. The simple fact has apparently escaped attention in earlier discussions. As from the formulation with \( K_i \) as generators instead, we have

$$\hat{X}_i \equiv \frac{1}{m} \hat{K}_i = \frac{1}{m} (K_{ia} \otimes \hat{I} + \hat{I} \otimes K_{ib}) = \frac{1}{m} (m_a \hat{X}_{ia} \otimes \hat{I} + \hat{I} \otimes m_b \hat{X}_{ib}) \quad (6)$$

Hence, we have the notion of the center of mass position operators dictated by the representation theory.

Many physicists may, understandably, be uncomfortable about taking \( K_i \) and \( M \) in the place of the usually used \( X_i \) and \( I \) as generators for the Heisenberg-Weyl symmetry. Most simply do not separate the abstract notion of the Lie algebra from its operator representation in a physical theory of a particular system, and may not have paid attention to the exact symmetry treatment of composite systems. Our elaboration above gives the mathematical logic clearly. Tracing that logic without the bias from what has become too familiar, one will be able to agree with the author that what we present here is the better way of seeing the symmetry structure behind the physics. In fact, it is the only way to avoid the unphysical notion of additive position observables.

The next symmetry to note is the rotational symmetry \( SO(3) \), given as

$$[J_{ij}, J_{hk}] = i\hbar (\delta_{jk} J_{ih} + \delta_{ih} J_{jk} - \delta_{ik} J_{jh} - \delta_{jh} J_{ik}) \quad (7)$$

\( i, j, \cdot \) goes from 1 to 3 with \( J_{ij} = -J_{ji} \). On the Hilbert space for quantum mechanics of a spinless particle, the momentum operators give translations in \( x^i \), and the position operators give translations in \( p^i \) \([3]\). That, together with the intuitive notion of the operators as the position and momentum coordinate observables suggest each set as components of
a three-vector. That is encoded into the commutation relations between them and the familiar picture of the orbital angular momentum operators \( \hat{J}_{ij} = \hat{L}_{ij} \equiv \hat{X}_i \hat{P}_j - \hat{P}_i \hat{X}_j = \frac{1}{m}(\hat{K}_i \hat{P}_j - \hat{P}_i \hat{K}_j) \). At the abstract Lie algebra level, we have the combined symmetry of \( \mathcal{H}_r(3) \) as a semidirect product (of Lie groups), with the full Lie algebra as

\[
\begin{align*}
[J_{ij}, J_{hk}] &= i\hbar(\delta_{jk} J_{ih} + \delta_{ih} J_{jk} - \delta_{ik} J_{jh} - \delta_{jh} J_{ik}) \\
[J_{ij}, K_k] &= i\hbar(\delta_{ik} K_j - \delta_{jk} K_i) , \\
[J_{ij}, P_k] &= i\hbar(\delta_{ik} P_j - \delta_{jk} P_i) , \\
[K_i, P_j] &= i\hbar \delta_{ij} M .
\end{align*}
\]

The ten generator Lie algebra, giving all the nontrivial Lie brackets, is a major part of \( \tilde{G}(3) \). Only one generator, the ‘Hamiltonian’ \( H \) is missing. We denote the symmetry by \( \mathcal{H}_r(3) \) and have presented a detailed picture of the full formulation for quantum theory for a spinless particle, together with its classical approximation based on a symmetry contraction in Ref.[3]. Issues related to the generator \( H \) is the focus of the next section. The \( \mathcal{H}_r(3) \) symmetry has irreducible representations beyond those of \( \mathcal{H}(3) \). They are the ones with nontrivial spin more or less as discussed in Ref.[14]. We sketch the part of the story under the current framework in the appendix for completeness.

A couple of extra comments are in order. The important role of mass \( m \) in the quantization of Newtonian mechanics has been appreciated, in the language of projective representation of the Galilean symmetry [2, 14]. Using the latter in the place of the more proper and direct one of the \( U(1) \) central extension (namely having the central charge \( M \) as a generator of the Lie algebra), however, does not help in revealing all the important features discussed above. Another question is if the representation of nonpositive mass is meaningful. The question of \( m \) with the ‘wrong’ sign has been well answered in Ref.[14]. For technical reasons, it is better for us to leave the more rigorous statement to the appendix. What is important is the sign of the Casimir invariant does not matter. The representations with the same \( |m| \) are essentially the same hence one can assume the mass to be always positive. Naively, there is no difficulty taking things to the \( m \to 0 \) limit [14]. However, having zero central charge really means the collapse of the Heisenberg commutation relation formally. The full implication of the formulation of such a theory may have to be taken more carefully. In ‘nonrelativistic’ physics, there seems to be no room for massless particles anyway.
III. \( H_r(3) \) VERSUS \( \tilde{G}(3) \) AND THE DYNAMICAL NATURE OF TIME

The notion of relativity symmetry as a fundamental symmetry of a dynamical theory was mostly brought into physics explicitly from Einstein’s introduction of Special Relativity. As so, it was first addressed as the symmetry of the background spacetime, or in relation to reference frame transformations of spacetime admissible in the dynamical theory. The theory itself is to be given otherwise. For the ‘nonrelativistic’ case at hand, Newton’s Laws dictate the dynamical behavior of particles. A particle as an ideal object occupying a definite point in the Newtonian model of the physical space at an instant of (the Newtonian) time \( t \) is introduced. It has only a single assumed characteristic, namely its mass \( m \). Of course, the notions of space and time, both modeled on Euclidean geometry, and mass as ‘quantity of matter’ are abstractions from our intuitions. The particular picture of spatial position, time, and mass from the mathematical models, or even the necessity of such notions, as well as the relativity symmetry obtained, are ‘correct’ only up to the success of the whole theory and within the logic of the particular formulation. We have seen, for example, substantial modifications of the notion of space and time, and in a way also the notion of mass, in Einstein’s theory compared to Newton’s. We have discussed how quantum mechanics may be better looked at as giving a different, quantum, model of the physical space [3, 9, 18]. Here in this article, we present what a symmetry theoretical formulation of the theory says about mass and time. We have discussed above how the Newtonian mass \( m \) is really given from the, \( H_r(3) \) or \( H(3) \), symmetry as a Casimir invariant for an ‘elementary’ object as an irreducible representation – the particle. That Casimir invariant is not there in the classical symmetry. In this section, we look at the Newtonian time, and the picture obtained is actually shared by the corresponding classical theory in itself.

The quantum Hilbert space as the representation space for the particle is its phase space. And it has a symplectic structure, fixed by the inner product [8, 9]. The Schrödinger equation is essentially a set of Hamilton’s equations of motion. The Heisenberg equation of motion should be similarly interpreted with \( \frac{1}{i\hbar} [\cdot, \cdot] \) as the exact Poisson bracket, the relation of which to the fundamental Lie algebra is obvious.

The basic observables, not including the spin part, are the (noncommutative) phase space coordinates \( \hat{X}_i \) and \( \hat{P}_i \). All other observables can be expressed as like functions of them. There is no time among them. The full theory is successfully constructed using only the
\( H_R(3) \), even when nonzero spin is included. Yet, it does offer a notion of the Newtonian time \( t \) as a real parameter of unitary transformations, actually Hamiltonian transformations. The physical Hamiltonian \( \hat{H}_{\text{phys}} \), or the energy observable, is the generating Hamiltonian function. Explicitly, we have

\[
\frac{d}{dt} |\phi(t)\rangle = \frac{1}{i\hbar} \hat{H}_{\text{phys}} |\phi(t)\rangle,
\]

for any state \( |\phi\rangle \), or

\[
\frac{d}{dt} \hat{A} = \frac{1}{i\hbar} [\hat{A}, \hat{H}_{\text{phys}}],
\]

for any dynamical variable as observable \( \hat{A} \). Not only that the dynamical equations give a definite notion of the time \( t \), this dynamical time as the evolution parameter from the \( \hat{H}_{\text{phys}} \) obviously depends on the interactions as encoded in the potential part. The correct Newtonian time should always correspond to the evolution parameter of any dynamical system with whatever admissible, or practical observed, interaction in the theory. There cannot be a single fixed expression for \( \hat{H}_{\text{phys}} \) in terms of \( \hat{X}_i \) and \( \hat{P}_i \), and certainly no definite element or generator of a fundamental symmetry as an extension of the \( H_R(3) \) that may correspond to the representation of. In fact, we see no other way to talk about time so long as that dynamical theory is concerned.

The essence of symplectic geometry is that it admits generic Hamiltonian flows or transformations of which the dynamical time evolution is just one example. In the case of the quantum theory, we have any Hermitian operator \( \hat{H}_s \) as an element of the observable algebra gives mathematically a one-parameter group of unitary transformations on the phase space. Under the Schrödinger picture it satisfies \( \frac{d}{ds} \hat{H}_s = \frac{1}{i\hbar} \hat{H}_s \). On the observable algebra under the Heisenberg picture it satisfies \( \frac{d}{ds} \hat{A} = \frac{1}{i\hbar} [\hat{A}, \hat{H}_s] \). What may be the physical meaning of the parameter \( s \) for a particular \( \hat{H}_s \) is a different question. \( \hat{L}_{ij} \) is, for example, \( H_{\theta_{ij}} \) where \( \theta_{ij} \) is the angle of the rotation generated. \( \hat{H}_{\text{phys}} \) is exactly \( \hat{H}_t \). Most of such Hamiltonian transformations are not a basic part of the fundamental symmetry. That is to say, \( \hat{H}_s \) may not represent an element of the Lie algebra itself. The formulation does not give the Newtonian time any special mathematical position many physicists may prefer to have. But that is the exact mathematical logic. Of course, the analogous formulation of the ‘relativistic’ theory would put Minkowski time on a special footing together with the Minkowski spatial positions. The ‘nonrelativistic’ theory as then an approximation to that would give the Newtonian time as the limit.
The usual picture of Galilean symmetry $\tilde{G}(3)$, however, has a notion of time translational symmetry with the generator $H$ which has as the only nontrivial Lie bracket the rest of the, $H_R(3)$, generators given by

$$[K_i, H] = i\hbar P_i .$$

(11)

The full symmetry has an extra Casimir element, $2MH - P_i P^i$, as well illustrated in Ref.[14]. With $m$ fixed in a representation, the extra Casimir invariant is conveniently taken as the real number $V$ satisfying

$$\hat{H} = \frac{1}{2m} \hat{P}_i \hat{P}^i + V .$$

(12)

Each irreducible representation for $\tilde{G}(3)$ is hence characterized by the triple \{m, s, V\}. Obviously, an irreducible representation as the quantum theory of a particle as described above serves as one for the full $\tilde{G}(3)$. Now, $\hat{H}$ is obviously not the correct physical Hamiltonian as $\hat{H}_{\text{phys}} \equiv \hat{H}_t = \frac{1}{2m} \hat{P}_i \hat{P}^i + V(\hat{X}_i) .$

(13)

It admits no potential, i.e. no nontrivial dynamics. From the perspective of generic Hamiltonian transformations in the phase space as described above, we have $\hat{H} \equiv \hat{H}_t$ for some ‘time’ parameter $\tilde{t}$, with, for example, $\frac{d}{dt} = \frac{i}{\hbar} \hat{H}_t \equiv \frac{i}{\hbar} \hat{H}$ acting on a state as in the Schrödinger equation. In the practical case when the potential vanished, i.e. for a free particle, and only in that special and dynamically not so interesting case, $\hat{H}$ agrees with $\hat{H}_t$ and $\tilde{t}$ agrees with $t$. So from the point of view of the dynamical theory, the ‘time translation’ generated by $\hat{H}$ is only a translation or shift in the value of the $\tilde{t}$ parameter characterizing points (states) on the curves of constant $\hat{H}$ value of the unitary transformation $e^{\frac{i}{\hbar} \hat{H}}$. That is actually irrelevant to physics so long as any nontrivial dynamics is concerned. The parameter $t$ gives the correct Newtonian time, while the parameter $\tilde{t}$ does not. Moreover, while we can kind of practically implement the ‘translation’ in the time $t$, say in resetting the clock, there is no way to implement a ‘translation’ in $\tilde{t}$, unless we can find a way to ‘tell’ $\tilde{t}$ which has to be based on a system without any interaction what-so-ever.

The observable $\hat{H}$ is not the energy observable as $\hat{H}_{\text{phys}}$, it is essentially only the kinetic energy, if one insists on having the theory as one of the $\tilde{G}(3)$ symmetry. But the quantity $\mathcal{V}$, which is supposed to be a basic characteristic of a particle on the same footing as the mass and the spin, has never been given any true physical meaning. It is called internal
energy in Ref.[14], which is clearly inaccessible to physical phenomena. It is odd to have a Casimir invariant like that, that we have to talk about particles with different values of internal energy that has otherwise no relevance to physics.

We believe there is no notion of time in the theory of ‘nonrelativistic’ quantum mechanics that goes beyond what that evolution parameter $t$ of the physical Hamiltonian as $\hat{H}_t$ can offer. Physical time is exactly the time in a dynamical evolution, and there is no way to contemplate time without dynamics. Each of us human beings is a dynamic system. Without any dynamics, without physical changes, including what happened in our brain, there is no way to appreciate a notion of time. We do not see any necessary notion of ‘time translational symmetry’ in ‘nonrelativistic’ physics beyond what is given by that dynamical Hamiltonian flow either, unless one insists that the theory or the fundamental symmetry has to be able to describe the empty space-time in itself, as a representation[1]. But for what purpose. Actually, even our practical physical notion of space, so long as a theory of particle dynamics is concerned, can only be the totality of all possible position of a particle. That gives the Newtonian space from the classical theory. For the quantum theory, it works only all possible position is to be taken as all possible eigenstates of the $\hat{X}_i$, otherwise, a better picture is offered by the new new notion of quantum model of the physical space [9]. There naive notion of Newtonian space-time in itself has no physical content and is not experimentally accessible. Moreover, we know our ‘nonrelativistic’ theory is only an approximation to the ‘relativistic’ theory. The latter certainly gives a notion of time as part of the Minkowski spacetime, if that can be a comfort for those who do not want to do without a notion of Newtonian space-time as a basic part of the symmetry theoretical formulation of the dynamical ‘nonrelativistic’ (quantum) dynamics. That simply do not work except for trivial dynamics. The right symmetry should be taken as $H_R(3)$ only, not $\tilde{G}(3)$.

IV. FURTHER CONCLUDING REMARKS

Applications of full symmetry theoretical formulation, naturally coupled with the matching picture of symplectic or Hamiltonian dynamics, of all aspects of the theory for ‘nonrelativistic’ quantum mechanics have been presented in Ref.[3]. The formulation includes a symmetry contraction that retrieves all aspects of the classical theory as an approximation.
The (quantum) relativity symmetry used is $H_R(3)$ as presented above. We present a sketch of the formulation here with mathematically minor, but physically very important, modifications here. The key is to have the naive picture of the abstract Heisenberg-Weyl symmetry $H(3)$ as having generators $\{X_i, P_i, I\}$ and operator representation as given in Eq.(2) to be replaced by generators $\{K_i, P_i, M\}$ and operator representation as given in Eq.(3) exactly in line with the Galilean symmetry picture. The modification is necessary to avoid having position observables $\hat{X}_i$ being additive, as like the momentum and mass, in a composite system. The improved $H_R(3)$ symmetry theoretical formulation has as Casimir invariants the Newtonian mass and spin, the former characterizing the different admissible irreducible representations of the Heisenberg-Weyl symmetry. The implication is very interesting. It is intriguing that mass as a fundamental notion in Newton’s theory of particle dynamics is to be seen as a remnant of the quantum theory. In fact, it is dictated by the formulation of the latter. The notion of center of mass is also dictated, in relation.

The $H_R(3)$ symmetry is short of the usually identified Galilean symmetry of $G(3)$ or here more properly $\tilde{G}(3)$ by missing the generator $H$ of the ‘time translation’ symmetry. Not only that the formulation makes no use of that, but we also elaborate on how the dynamical theory as retrieved from the symplectic geometry of the representation (Hilbert) space gives the notion of the correct Newtonian time. It is the parameter of the Hamiltonian transformation/evolution parameter. The operator representing the generator $H$ give essentially only the kinetic energy and cannot gives the correct time translation in the Hamiltonian picture so long as there is nontrivial dynamics. From the perspective of the formulation presented, there is no need at all to have a notion of Newtonian time or the time translation symmetry as a part of the fundamental/relativity symmetry. Of course, a picture of spacetime and additive energy-momentum can be obtained from the Lorentz covariant ‘relativistic’ symmetry theoretical formulation with ‘nonrelativistic’ theory as an approximation to that.

In coming to the current perspectives, it is important to take the mathematics of the symmetry and its representation theory more seriously than what is done in most of the physics literature. Starting with an abstract picture of the symmetry, one has to reconsider the relation between its basic ingredients and the physical quantities based on which we first actually identified the symmetry from.

From the symmetry perspective, there is no reason at all to expect the need for anything beyond a single irreducible representation to be necessary for the description of an elementary
system taken, as one that cannot be seen as composed of different parts. Composite systems are, in general, described by reducible representations, as the sum of irreducible parts, obtained as the product representation of those for the parts. The Casimir invariants are the fundamental characteristics of an irreducible representation and the (part of the) system it describes. For an elementary system, they are universal for all states. For a composite system, the story is more complicated. Generally, more than one set of Casimir invariants, each for an irreducible component, mathematically characterize the different subspaces of the full phase space of the composite system which are invariant under all observables constructed from the symmetry. If one starts with a state within one such invariant subspace, the description of dynamical features of the state certainly need not go beyond that subspace.

However, in a quantum theory, there is no trivial answer to the question of whether one can prepare an initial state that is a linear combination of states in different invariant subspaces. An explicit example, commonly discussed in the ‘relativistic’ setting, is the question of if we can prepare a state as a nontrivial linear combination of states of different spins. The question is particularly interesting in relation to the notion of quantum frames of reference which has been catching popularity lately (interested readers are referred to Ref.[19] and references therein). In particular, a carefully detailed analysis has been presented in Ref.[20] addressing symmetry issues with a practical consideration of the relative nature of observed quantities, as well as the validity of superselection rules. The key concluding statement of “observable quantities are invariant under symmetry and that, in quantum mechanical laboratory experiments, the measured statistics pertain not to some absolute quantity, but rather to an observable, relative quantity, corresponding to the system and apparatus combined, along with the appropriate high localisation limit on the side of the apparatus” is to be taken as the background to understand our discussion of the observables as apparent ‘absolute quantities’. Here, the “apparatus” can be replaced by, or embodies, the physical object as the frame of reference to give physical definition to the observables. The “high localisation limit” essentially corresponds to objects which can be well described as classical. For a composite of two particles with one being essentially classical, of a large mass, that conclusion says that the \( R-Q \) variables as relative observables to the center of mass describe well the physics of the light, quantum, particle. That offers essentially the same picture as given by the absolute quantity description picture of the single-particle system we presented. Though that is what one would expect, the solid confirmation of that
is as important as it is interesting. To truly look at a system of quantum particles, however, quantum particles observed from a physical frame of reference, the quantum nature of which cannot be neglected, more studies, probably along the lines of Refs.[19] and [21], would be needed.

Finally, we want to emphasize that empty space-time is of little interest to physics, the theories of which are to describe physical phenomena. Dynamics are geometrically described in the phase space. Hence, it should be the symmetry of the latter, rather than that of a simple notion of space-time, that is of fundamental importance. If relativity symmetry is about the symmetry of reference frame transformations, it is likewise reference frames for the description of dynamics rather than only that of the space-time that should be relevant. Especially when considering the important violations of the naive Newtonian notion of momentum being the product of mass and velocity, for example in the case of electrodynamics, direct consideration of reference frames (transformations) for the phase space has to be taken seriously. The only physical notion of the space is the totality of all possible positions of a (free) particle, and as such can be retrieved as part of the phase space. Our symmetry theoretical formulation of the quantum theory gives at the classical limit, the Newtonian single particle phase space as a representation with the Newtonian space as an irreducible component agreeing with the coset space picture. In the quantum setting, the Hilbert space is an irreducible representation, hence cannot be seen as a product of independent configuration/position space and momentum space. No irreducible representation gives exactly the Newtonian space at the classical limit anyway. That is exactly like the fact that the Minkowski spacetime as an irreducible representation of the ‘relativistic’ symmetry can be definitely split into the Newtonian space and the Newtonian time at the ‘nonrelativistic’ limit. That further justifies calling the $H_k(3)$ (quantum) relativity symmetries and the single particle quantum phase spaces the quantum model of the physical space [3]. That can actually be used to give a picture of quantum mechanics as particle dynamics on the quantum/noncommutative geometric model of space with the position and momentum observables as coordinates [9, 18].

The corresponding picture for the ‘relativistic’ case, with other important implications, will be addressed in a separate publication [22].
Appendix: On Representations of Nonzero Spin

We focus first on irreducible representations of the $H_{\alpha}(3)$ symmetry with nonzero spin. These are representations that go beyond those of $H(3)$. They are more or less as presented in Ref.[14] under the $\tilde{G}(3)$ symmetry perspective, there are subtle but interesting differences between our presentation here and that of the latter reference though. We have the spin label $s$ as effectively a Casimir invariant. the actual Casimir element is $\frac{1}{2}T_{ij}T^{ij}$, with $T_{ij} \equiv MJ_{ij} - (K_iP_j - P_iK_j)$, which actually ‘commute’ with all generators (Einstein summation convention assumed, and generators written with upper $i, j$ indices are the same as the ones with lower indices). Note that $T_{ij}$, like all quadratic Casimir elements, do not exist as elements of the Lie algebra. They have to be taken as elements of the universal enveloping algebra. Of course at the representation level, all the operator sum/product combinations are well defined. The Casimir element $\frac{1}{2}T_{ij}T^{ij}$ can be seen as sitting in for the $\frac{1}{2}J_{ij}J^{ij}$ of $SO(3)$ which gives our familiar angular momentum as a Casimir invariant.

Introducing intrinsic angular momentum operators $\hat{S}_{ij} \equiv \frac{1}{m}\hat{T}_{ij}$, we have the familiar $\hat{J}_{ij} = \hat{L}_{ij} + \hat{S}_{ij}$ for the orbital angular momentum operator $\hat{L}_{ij} = \hat{X}_i\hat{P}_j - \hat{P}_i\hat{X}_j$. As $T_{ij}$ commute with all other generators besides the $J_{ij}$, we can think about a Lie algebra as the direct sum of an $SO(3)$ from $T_{ij}$ with the invariant subalgebra from the generator set $\{K_i, P_i, M, H\}$, the $H(3)$. Irreducible representations of the symmetry are then each a simple direct product of irreducible representations of the two parts. Each such irreducible representation serves as an irreducible representation of our $H_{\alpha}(3)$, and that exhausts the list. The $SO(3)$, or its double cover, gives the familiar $(2s + 1)$-dimensional representation space of the spin components for the intrinsic angular momentum $s(s + 1)\hbar^2$ as the effective Casimir invariant. It is the eigenvalue for the operator $\frac{1}{2}\hat{S}_{ij}\hat{S}^{ij}$, for integral or half-integral values of $s \geq 0$. Each component is a replica of the same representation of zero spin as an irreducible representation of the $H(3)$ part. Nonzero spin in ‘nonrelativistic’ physics is not commonly addressed. But spin is really about rotational symmetry, whether it is $SO(3)$ or spacetime rotations of $SO(1, 3)$, and can be addressed independently [14, 23] or as approximations to the ‘relativistic’ one. Under the essentially $\tilde{G}(3)$ framework, an irreducible representation has the labeling characteristics $\{m, s, \mathbb{V}\}$, where $s$ is the spin label and can be mapped, through an anti-unitary transformation, to one of $\{-m, \bar{s}, -\mathbb{V}\}$ with $\bar{s}$ denoting the $SO(3)$ representation conjugates to that of $s$. Hence, the two representations are essentially
the same. The basic picture already presented in Ref.[14], of course, maintains with only \( H_k(3) \), only having \( V \) dropped from consideration.

Let us further consider a composite system as a product of two irreducible representations of \( \{ m_a, s_a \} \) and \( \{ m_b, s_b \} \). As noted, \( m_a \) and \( m_b \) can be taken as positive. The first question is what are the admissible values of the Casimir invariants? A closely related question is if the representation divides into a number of irreducible components. Note that we have, in a simplified form of the tensor product notation, \( \hat{P}_i = \hat{P}_{ia} + \hat{P}_{ib} \) and \( \hat{X}_i = \frac{1}{m}(m_a \hat{X}_{ia} + m_b \hat{X}_{ib}) \), and of course also \( \hat{J}_{ij} = \hat{J}_{ija} + \hat{J}_{ijb} \). To go on with the analysis of the composite system, it is convenient to introduce the complementary observables given by the operators \( \hat{R}_i \equiv \hat{X}_{ia} - \hat{X}_{ib} \) and \( \hat{Q}_i \equiv \frac{1}{m}(m_b \hat{P}_{ia} - m_a \hat{P}_{ib}) \). The operators all commute with \( \hat{X}_i \) and \( \hat{P}_i \) while \([\hat{R}_i, \hat{Q}_j] = i\hbar\delta_{ij}\). For example, with the Hilbert space tensor product as spanned by simultaneous momentum eigenstates \( |p_{ia}, p_{ib}\rangle \), a more convenient basis to work with would simply be given by \( |p_i, q_i\rangle \), simultaneous eigenstates of \( \hat{P}_i \) and \( \hat{Q}_i \). \( \hat{X}_i \) and \( \hat{P}_i \) are the basic dynamic observables for the center of mass degrees of freedom, while \( \hat{R}_i \) and \( \hat{Q}_i \) those for the degrees of freedom of the relative motion between the particles which is internal to the composite system. One can go as far as seeing the two sets as sets of canonical noncommutative coordinates for the phase space of the system \([9, 18]\).

With a bit of calculation, one obtains for the product representation

\[
\hat{J}_{ij} = (\hat{X}_i \hat{P}_j - \hat{P}_i \hat{X}_j) + (\hat{R}_i \hat{Q}_j - \hat{Q}_i \hat{R}_j) + \hat{S}_{ija} + \hat{S}_{ijb}, \tag{A.1}
\]

giving

\[
\hat{S}_{ij} = \hat{R}_i \hat{Q}_j - \hat{Q}_i \hat{R}_j + \hat{S}_{ija} + \hat{S}_{ijb}. \tag{A.2}
\]

The interpretation is that the relative motion between the two particles gives rise to an ‘orbital’ angular momentum which is, however, intrinsic to the system as a composite. The latter behaves exactly like part of the spin of the composite taken as a particle. The ‘spin’ of the full \( \hat{S}_{ij} \), i.e. \( \frac{1}{2} \hat{S}_{ij} \hat{S}^{ij} \), is effectively a Casimir invariant. This ‘spin’ \( s \) for the composite has a list of admissible values coming from the standard angular momentum addition picture, as the sum of the three parts. The different \( s \) values correspond to different irreducible representations as components of the product representation which is reducible. For example, with the \( |p_i, q_i\rangle \) basis, each set of vectors with fixed \( q^2 = q_i q^i \) span a subspace invariant under the rotations generated by \( \hat{S}_{ij} \). It is important to emphasize that neither \( \frac{1}{2} \hat{J}_{ij} \hat{J}^{ij} \) nor
\( \frac{1}{2} \hat{L}_{ij} \hat{L}^{ij}, \hat{L}_{ij} \equiv \hat{X}_i \hat{P}_j - \hat{P}_i \hat{X}_j, \) serves as Casimir invariant, only \( \frac{1}{2} \hat{S}_{ij} \hat{S}^{ij}, \) which for a composite system of spin-zero particles has \( \hat{S}_{ij} \) as ‘orbital’ angular momentum \( \hat{L}^s_{ij} \equiv \hat{R}_i \hat{Q}_j - \hat{Q}_i \hat{R}_j, \) does. The \(|p_i, q_i\rangle\) basis, even restricted to fixed \( q^2 = q_i q_i \), is however generally not the best basis for the description of a particular two-particle system. The irreducible component of the product representation has, for the degree of freedom for the relative motion, a fixed angular momentum. The latter contains both the ‘orbital’ part and the part of the spins of the particles. As we appreciate from the physics, a closed composite system has the total angular momentum, as the \( \hat{S}_{ij} \) here, being conserved, but the interaction between the parts allows the exchange of angular momentum between them, including the spin parts with the ‘orbital’ part. Apart from the \(|p_i\rangle\) for the center of mass degree of freedom, the degree of freedom for the relative motion is to be described in a spin space of finite dimension \( 2\ell_s + 1 \), with \( \ell_s(\ell_s + 1)h^2 \) being the eigenvalue of the Casimir invariant. All that is independent of the actual interaction dynamics between the particles.

In relation to the interaction paradigm, the physical Hamiltonian or the energy observable for the composite system, neglecting spin-dependent interactions, is expected to have the form

\[
\hat{H}_{\text{phys}} = \frac{1}{2m} \hat{P}_i \hat{P}^i + \frac{1}{2\mu} \hat{Q}_i \hat{Q}^i + V(\hat{R}_i \hat{R}^i). \quad (A.3)
\]

Having \( \frac{1}{2} \hat{S}_{ij} \hat{S}^{ij} \) as a Casimir invariant enforces the potential to be a function of \( \hat{R}_i \hat{R}^i \). Intuitively, one can see that the interaction potential has to be independent of \( \hat{X}_i \) and \( \hat{P}_i \). In that sense, when one writes down a nontrivial dynamical description of a particle, it is really the dynamics of \( \hat{R}_i \hat{Q}_i \) that one is writing. The Hamiltonian is invariant under the rotations generated by \( \hat{J}_{ij} = \hat{L}_{ij} \) and \( \hat{S}_{ij} = \hat{L}^s_{ij} \). The practical Hilbert space is the representation space of an irreducible component of the product representation fixed the initial value of \( \ell_s \), which is spanned by \( |m_{\ell_s}\rangle, \) \( m_{\ell_s}h \) being the eigenvalue of \( \hat{L}^s_{12} \).

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