Powers of the Vandermonde determinant, Schur functions and recursive formulas

C Ballantine

Department of Mathematics and Computer Science, College of the Holy Cross, Worcester, MA 01610, USA

E-mail: cballant@holycross.edu

Received 3 May 2012, in final form 28 June 2012
Published 18 July 2012
Online at stacks.iop.org/JPhysA/45/315201

Abstract

The decomposition of an even power of the Vandermonde determinant in terms of the basis of Schur functions matches the decomposition of the Laughlin wavefunction as a linear combination of Slater wavefunctions and thus contributes to the understanding of the quantum Hall effect. We investigate several combinatorial properties of the coefficients in the decomposition. In particular, we give recursive formulas for the coefficient of the Schur function $s_\mu$ in the decomposition of an even power of the Vandermonde determinant in $n + 1$ variables in terms of the coefficient of the Schur function $s_\lambda$ in the decomposition of the same even power of the Vandermonde determinant in $n$ variables if the Young diagram of $\mu$ is obtained from the Young diagram of $\lambda$ by adding a tetris type shape to the top or to the left.

PACS numbers: 02.10.Ox, 02.10.Xm
Mathematics Subject Classification: 05E05, 15A15

1. Introduction

In the theory of symmetric functions Vandermonde determinants are best known for the part they play in the classical definition of Schur functions. Since each even power of the Vandermonde determinant is a symmetric function, it is natural to ask for its decomposition in terms of the basis for the ring of symmetric functions given by Schur functions [11]. This decomposition has been studied extensively (see [5, 6, 13], and the references therein) in connection with its usefulness in the understanding of the (fractional) quantum Hall effect. In particular, the coefficients in the decomposition correspond precisely to the coefficients in the decomposition of the Laughlin wavefunction [10] as a linear combination of (normalized) Slater determinantal wavefunctions. The calculation of the coefficients in the decomposition becomes computationally expensive as the size of the determinant increases. Several algorithms for the expansion of the square of the Vandermonde determinant in terms of Schur functions are available (see, for example, [14]). However, a combinatorial
interpretation for the coefficient of a given Schur function is still unknown. Recently, Boussicault et al [4] provided a purely numerical algorithm for computing the coefficient of a given Schur function in the decomposition without computing the other coefficients. The algorithm uses hyperdeterminants and their Laplace expansion. It was used by the authors to compute coefficients in the decomposition of even powers of the Vandermonde determinant of size up to 11. For determinants of large size, the algorithm becomes computationally too expensive for practical purposes. In [9] and [3], the authors study $q$-deformations of this and related problems. In this paper we present recursive combinatorial properties of some of the coefficients in the decomposition. Specifically, the coefficient of the Schur function $s_\mu$ in the decomposition of an even power of the Vandermonde determinant in $n+1$ variables is computed in terms of the coefficient of the Schur function $s_\lambda$ in the decomposition of the same even power of the Vandermonde determinant in $n$ variables if the Young diagram of $\mu$ is obtained from the Young diagram of $\lambda$ by adding a tetris type shape to the top or to the left.

In section 2 we introduce the notation and basic facts about partitions and Schur functions and their relation to the Vandermonde determinant. In section 3 we consider the reverse partition $\lambda^{bc}$ (as defined by [5]) of a given partition $\lambda$ and relate the coefficient of $s_{\lambda^{bc}}$ in the (correct) even power of the Vandermonde determinant to the coefficient of $s_\lambda$. In section 4 we exhibit two simple recursion rules followed in section 5 by two new and somewhat surprising recursive formulas. In section 5 we also present a third, conjectural, formula which has been verified for $n \leq 10$ using Maple and the list of coefficients provided in [15]. We prove two special cases of this formula. In section 6 we use the recursive formulas of sections 4 and 5 to prove several closed formulas and recursive observations given in [5], one of the pioneering articles in using the decomposition of the square of the Vandermonde determinant in terms of Schur functions to understand the quantum Hall effect. Our results improve considerably on the observations in [5].

An extended abstract containing the statement of the results presented here appeared in the proceedings of FPSAC11 [1].

2. Notation and basic facts
We first introduce some notation and basic facts about the Vandermonde determinant related to this problem. For details on partitions and Schur functions we refer the reader to [11, chapter 7].

Let $n$ be a non-negative integer. A partition of $n$ is a weakly decreasing sequence of non-negative integers, $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_{\ell})$, such that $|\lambda| := \sum \lambda_i = n$. We write $\lambda \vdash n$ to mean $\lambda$ is a partition of $n$ (or a partition of size $n$). The integers $\lambda_i$ are called the parts of $\lambda$. We identify a partition with its Young diagram, i.e. the array of left-justified squares (boxes) with $\lambda_1$ boxes in the first row, $\lambda_2$ boxes in the second row, and so on. The rows are arranged in matrix form from top to bottom. By the box in position $(i, j)$ we mean the box in the $i$th row and $j$th column of $\lambda$. The length of $\lambda$, $\ell(\lambda)$, is the number of rows in the Young diagram or the number of non-zero parts of $\lambda$. For example,

```
+---+---+---+
|   |   |   |
|   |   |   |
|   |   |   |
|   |   |   |
```

is the Young diagram for $\lambda = (6, 4, 2, 1, 1)$, with $\ell(\lambda) = 5$ and $|\lambda| = 14$.

We write $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ to mean that $\lambda$ has $m_i$ parts equal to $i$. 

Given a weak composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) of length \( n \), we write \( \lambda^\alpha \) for the monomial \( x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \). If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is a partition of length at most \( n \) and \( \delta_n = (n-1, n-2, \ldots, 1, 0) \), then the skew symmetric function \( a_{\delta+n} \) is defined as
\[
a_{\delta+n} = \det \left( x_i^{n-j} \right)_{i,j=1}^n.
\] (1)
If \( \lambda = \emptyset \),
\[
a_{\emptyset} = \det \left( x_i^{n-j} \right)_{i,j=1}^n = \prod_{1 \leq i < j \leq n} (x_i - x_j)
\] (2)
is the Vandermonde determinant. We have [11, theorem 7.15.1]
\[
a_{\delta+n} = s_{\delta_n}(x_1, \ldots, x_n),
\] (3)
where \( s_{\delta_n}(x_1, \ldots, x_n) \) is the Schur function of shape \( \lambda \) in variables \( x_1, \ldots, x_n \). Moreover, if we denote by \( [x^\lambda]_{\delta+n}f \) the coefficient of \( x^\lambda \) in \( a_{\delta+n}f \), then [11, corollary 7.15.2] for any homogeneous symmetric function \( f \) of degree \( n \), the coefficient of \( s_\lambda \) in the decomposition of \( f \) is given by \( \langle f, s_\lambda \rangle = [x^\lambda]_{\delta+n}f \). In particular, if \( f = a_{\emptyset}^k \), then
\[
\langle a_{\emptyset}^k, s_\lambda \rangle = [x^\lambda]_{\delta+n}a_{\emptyset}^{2k+1}.
\] (4)

In this work we investigate several combinatorial properties of the numbers (4).

The following proposition summarizes some easy to prove properties that are frequently used in this paper.

**Proposition 2.1.** We have

(i) The size of \( \delta_n \) is given by \( |\delta_n| = n(n-1)/2 \).

(ii) The skew symmetric function \( a_{\delta+n} \) is a homogeneous polynomial of degree \( n(n-1)/2 \).

(iii) If \( \langle a_{\emptyset}^2, s_\lambda \rangle \neq 0 \), then \( |\lambda| = kn(n-1), n-1 \leq \ell(\lambda) \leq n, k(n-1) \leq \lambda_1 \leq 2k(n-1) \) and \( \lambda_n \leq k(n-1) \).

(iv) Moreover, if \( \lambda_n = k(n-1) \) in (iii), then \( \lambda = ((k(n-1))^n) \).

By \( a_{\delta+n} \), we mean \( a_\delta \) with \( x_i \) replaced by \( x_{i+1} \) for each \( i = 1, 2, \ldots, n \). Thus, \( a_{\delta+n} = \prod_{1 \leq i < j \leq n+1} (x_i - x_j) \). By \( \bar{\alpha} \), where \( \alpha \) is the weak composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), we mean \( \bar{\alpha} = (x_1, x_2, \ldots, x_n) \) with \( x_i \) replaced by \( x_{i+1} \) for each \( i = 1, 2, \ldots, n \). Thus, \( \bar{\alpha} = (x_2, x_3, \ldots, x_{n+1}) \).

Given a weak composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) of \( n \) of length at most \( n \), we denote by \( c_{\alpha} \) the coefficient of \( x^\alpha \) in \( a_{\delta+n}^{2k+1} \). If \( \xi \) is a permutation of \( \{1, 2, \ldots, n\} \), and \( \xi(\alpha) \) is the weak composition \( (\alpha_{\xi(1)}, \alpha_{\xi(2)}, \ldots, \alpha_{\xi(n)}) \), one can easily see that
\[
c_{\alpha} = \text{sgn}(\xi)c_{\xi(\alpha)}. \tag{5}
\]

**3. The box-complement of a partition**

**Definition 3.1.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \) be a partition of \( kn(n-1) \) with \( \lambda_1 \leq 2k(n-1) \) and \( \ell(\lambda) \leq n \). The box-complement of \( \lambda \) is the partition of \( kn(n-1) \) given by
\[
\lambda^{bc} = (2k(n-1) - \lambda_n, 2k(n-1) - \lambda_{n-1}, \ldots, 2k(n-1) - \lambda_1).
\] (6)

Thus, \( \lambda^{bc} \) is obtained from \( \lambda \) as follows. Place the Young diagram of \( \lambda \) in the upper left corner of a box with \( n \) rows each of length \( 2k(n-1) \). Remove the Young diagram of \( \lambda \) and rotate the remaining shape by \( 180^\circ \), to obtain the Young diagram of \( \lambda^{bc} \).
Example. Let \( k = 1, n = 4 \) and \( \lambda = (5, 3, 2, 2) \). Then \( \lambda^{bc} = (4, 4, 3, 2) \). The Young diagram of \( \lambda \) is shown on the left of the \( 4 \times 6 \) box. The remaining squares of the box are marked with \( X \). They form the diagram of \( \lambda^{bc} \) rotated by \( 180^\circ \).

\[
\begin{array}{cccccc}
\times & \times & \times & X & X & X \\
X & X & X & X & X & X \\
X & X & X & X & X & X \\
X & X & X & X & X & X \\
\end{array}
\]

Lemma 3.2. (Box-complement lemma). With the notation above, we have

\[
\{a^{2k}_s, s_\lambda\} = \{a^{2k}_s, s_{\lambda^{bc}}\}. \tag{7}
\]

For a proof in the case \( k = 1 \), see [5, section 6]. The lemma can be proved for general \( k \) by elementary means, using induction on \( n \) (see [2]). In [5], Dunne also explains the physical meaning of the box-complement lemma.

4. Simple recursive formulas

The aim of this section is to establish some preliminary recursive formulas for \( \langle a^{2k}_{\lambda^{bc}}, s_\mu \rangle \) in terms of \( \langle a^{2k}_{\mu^{bc}}, s_\lambda \rangle \) when the diagram of \( \mu \) is obtained from the diagram of \( \lambda \) by adding a certain configuration of boxes, called a tetris type shape, to the top or to the left. For \( k = 1 \) these results have been mentioned in [14].

Theorem 4.1. If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is a partition of \( kn(n-1) \) with \( \ell(\lambda) \leq n \) and \( \mu \) is the partition of \( kn(n+1) \) given by \( \mu = (2kn, \lambda_1, \lambda_2, \ldots, \lambda_n) \), then

\[
\{a^{2k}_{\lambda^{bc}}, s_\mu\} = \{a^{2k}_{\mu^{bc}}, s_\lambda\}. \tag{8}
\]

Thus, adding the tetris type shape

\[
\begin{array}{cccccc}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\end{array}
\]

to the top of the diagram for \( \lambda \) does not change the coefficient. If \( k = 1 \), we denote this tetris type shape by \( T_0(n, 0) \).

Proof. The proof follows by induction from

\[
a^{2k+1}_{\mu^{bc}} = \prod_{i=2}^{n+1} (x_i - x_1)^{2k+1} a^{2k+1}_{\mu^{bc}}.
\]

Remark. If \( \lambda \) is just a weak composition of \( kn(n-1) \) with no more than one part equal to 0, it is still true that

\[
[a^{2k+1}_{\lambda^{bc}}, a^{2k+1}_{\lambda^{bc}}] = [a^{2k+1}_{\lambda^{bc}}, a^{2k+1}_{\lambda^{bc}}].
\]

Corollary 4.2. If \( \lambda = (2k(n-1), 2k(n-2), \ldots, 4k, 2k, 0) = 2k \delta_0 \), then \( \langle a^{2k}_{\lambda^{bc}}, s_\lambda\rangle = 1 \).

For the physical interpretation, when \( k = 1 \), the partition \( \lambda \) in the corollary corresponds to the most evenly distributed of the Slater states (every third single particle angular momentum is filled) [5].

Using theorem 4.1 and lemma 3.2, we obtain the following corollary.

Corollary 4.3. If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is a partition of \( kn(n-1) \) with \( \ell(\lambda) \leq n \) and \( \mu \) is the partition of \( kn(n+1) \) given by \( \mu = \lambda + (2k)\delta = (\lambda_1 + 2k, \lambda_2 + 2k, \ldots, \lambda_n + 2k, 0) \), then

\[
\{a^{2k}_{\lambda^{bc}}, s_\mu\} = \{a^{2k}_{\mu^{bc}}, s_\lambda\}. \tag{9}
\]
Thus adding the tetris type shape
to the left of the diagram of $\lambda$ does not change the coefficient. If $k = 1$, we denote this tetris type shape by $L(n)$.

Note. The result of theorem 4.1 for $k = 1$ is also noted in (23a) of [14] and in [5, section 6]. For $k = 1$, corollary 4.3 is (23b) of [14].

5. Recursive formulas in the case $k = 1$

For the remainder of this paper we set $k = 1$. In this section we prove two non-trivial recursive formulas involving tetris type shapes and present a third, conjectural, such formula.

5.1. First recursive formula

The following lemma and its corollary justify the assumption of the next theorem.

**Lemma 5.1.** Suppose $\lambda \vdash n(n-1)$ with $n-1 \leq \ell(\lambda) \leq n$ and $\langle a_{1,k}^2, s_\lambda \rangle \neq 0$. If $\lambda = \lambda_{n-1} = \ldots = \lambda_{n-i} = s$, then $i \leq s$, i.e. the maximum number of rows of size $s$ at the bottom of the diagram is $s + 1$.

**Proof.** The proof can be found in [1].

**Note.** We stated the lemma for $k = 1$ since only this case is needed in this paper. However, the lemma is true for general $k$. If $\lambda \vdash kn(n-1)$ and $\langle a_{1,k}^2, s_\lambda \rangle \neq 0$, the maximum number of rows of size $s$ at the bottom of the diagram for $\lambda$ is $\lfloor s/k \rfloor + 1$.

We reformulate lemma 5.1 in terms of the box-complement of the partition $\lambda$.

**Corollary 5.2.** Suppose $\lambda \vdash n(n-1)$ with $n-1 \leq \ell(\lambda) \leq n$ and $\langle a_{1,k}^2, s_\lambda \rangle \neq 0$. If $\lambda = \lambda_1 = \ldots = \lambda_i = 2n-m-1$, then $i \leq m$.

The first recursion formula of this section follows from the following theorem.

**Theorem 5.3.** Let $1 \leq m \leq n$. Let $\lambda \vdash n(n-1)$ with $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 2n-m-1$. Then,

$$\langle a_{1,k}^2, s_\lambda \rangle = \langle a_{1,k}^2, s_{(m-1)m} \rangle \cdot \langle a_{1,k}^2, s_{(m-1)m+1, m+1, \ldots, n} \rangle.$$  

(10)

**Proof.** If $\ell(\lambda) < n-1$ or $\ell(\lambda) > n$, both side of (10) are zero. Assume $n-1 \leq \ell(\lambda) \leq n$. We have

$$x^{\lambda + a_{1,k}} = (x_1^{3n-2m-1} \ldots x_m^{3n-2m-1}) \cdot (x_{m+1}^{\lambda_{m+1}+n-m-1} \ldots x_n^{\lambda_n}).$$

We write $a_{1,k}^3 = a_{1,k}^3 \cdot B_m \cdot C_m$, where

$$B_m = \prod_{1 \leq i < j \leq m} (x_i - x_j)^3$$

and $C_m = \frac{a_{1,k}^3}{a_{1,k}^3 B_m} = \prod_{m+1 \leq i < j \leq n} (x_i - x_j)^3$.
Note that $C_m$ is obtained from $a_{h}^{2}$ via the substitution $x_i \to x_{i+m}$, $x_j \to x_{j+m}$.

Since monomials in $B_m$ contain each $x_i$, $i = 1, \ldots, m$, with exponent at most $3n - 3m$, the monomials in $a_{h}^{3}$ contributing to $\{a_{h}^{2}, s_{j}\}$ are of the form

$$x_{1}^{\lambda_{1}} \cdot x_{2}^{\lambda_{2}} \cdots x_{m}^{\lambda_{m}} \cdot E = x_{1}^{\lambda_{1}} \cdot x_{2}^{\lambda_{2}} \cdots x_{m}^{\lambda_{m}} \cdot E,$$

where $E$ is a monomial in the variables $x_1, \ldots, x_m$. Since

$$\deg (a_{h}^{3}) = \deg (x_{1}^{\lambda_{1}} \cdot x_{2}^{\lambda_{2}} \cdots x_{m}^{\lambda_{m}} \cdot E) = \frac{3m(m - 1)}{2},$$

we have $E = 1$. Hence, the only monomial in $B_m$ contributing to $\{a_{h}^{2}, s_{j}\}$ is $x_{1}^{\lambda_{1}} \cdot x_{2}^{\lambda_{2}} \cdots x_{m}^{\lambda_{m}}$ (with coefficient 1).

Therefore, $\{a_{h}^{2}, s_{j}\} = \alpha_{m} \cdot \beta_{m}$, where $\alpha_{m} = \{a_{h}^{2}, s_{(m-1)m}\}$ and $\beta_{m}$ is the coefficient of $x_{m+1}^{\lambda_{m+1}} \cdots x_{n-1}^{\lambda_{n-1}} x_{n}^{\lambda_{n}}$ in $C_{m}$, i.e., $\beta_{m} = \{a_{h}^{2}, s_{(\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_{n})}\}$. □

**Remark.** More generally, if $\lambda \vdash kn(n - 1)$, it follows from [4, propositions 4.11 and 4.5] that

$$\{a_{h}^{2k}, s_{j}\} = \pm \{a_{h}^{2k}, s_{(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n})} - (j(2k(n-m))} \} \{a_{h}^{2k}, s_{(\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_{n})}\}. \quad (11)$$

Then, theorem 5.3 follows from (11) by setting $k = 1$ and $\lambda_{1} = \lambda_{2} = \ldots = \lambda_{m} = 2n - m - 1$. In this case, the sign on the right hand side is positive.

**Corollary 5.4.** Let $1 \leq m \leq n$. Let $\lambda \vdash (n-1)n$ with $\lambda_{1} = \lambda_{2} = \ldots = \lambda_{m} = 2n - m - 1$. Let $\mu \vdash (n-1)n$ with parts $\mu_{1} = \mu_{2} = \ldots = \mu_{m+1} = 2n - m$ and (if $m < n$) $\mu_{j} = \lambda_{j-1}$ for $j = m + 2, \ldots, n + 1$. Then,

$$\{a_{h}^{2n-m}, s_{\mu}\} = (-1)^{m} (2m + 1) \{a_{h}^{2}, s_{\lambda}\}. \quad (12)$$

Thus, adding the tetris type shape

```
\[
\begin{array}{cccccccc}
\hline
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\hline
\end{array}
\]
```

(13)

to the top of the diagram of $\lambda$ changes the coefficient by a multiple of $(-1)^{m}(2m + 1)$. We denote this tetris type shape by $T_{0}(n, m)$.

The partition $\lambda$ corresponds to the Slater state in which the angular momentum levels of the first $m$ particles are most closely bunched [5].

**Proof.** As in the proof of theorem 5.3, we have $\{a_{h}^{2}, s_{\mu}\} = \alpha_{m+1} \cdot \beta_{m}$, where $\alpha_{m+1} = \{a_{h}^{2}, s_{(m+1)m+1)}\}.$

By [11, exercise 7.37(b)],

$$\alpha_{m} = \{a_{h}^{2}, s_{(m-1)m}\} = (-1)^{m} (2m) \cdot 1 \cdot 3 \cdot \cdots \cdot (2m - 1). \quad (13)$$

Thus, from theorem 5.3, it follows that $\{a_{h}^{2}, s_{\mu}\} = (-1)^{m} (2m + 1) \{a_{h}^{2}, s_{\lambda}\}$. □

**Note.** Corollary 5.4 implies the case $k = 1$ of theorem 4.1.

Now [11, exercise 7.37(c)] follows easily by repeated use of corollary 5.4.

**Corollary 5.5.** If $\lambda = ((n + i - 1)^{n-i}, (i - 1)^{i})$, $1 \leq i \leq n$, then

$$\{a_{h}^{2}, s_{\lambda}\} = (-1)^{\frac{1}{2}(n-i)(n-2i)} [1 \cdot 3 \cdot \cdots \cdot (2i - 1)] \cdot [1 \cdot 3 \cdot \cdots \cdot (2(n-i) - 1)].$$
5.2. A conjectured recursion

We state conjecturally a similar combinatorial recursive property. The conjecture has been verified for \( n \leq 10 \) using Maple and the list of coefficients provided at [15].

**Conjecture 5.6.** Let \( 1 \leq m \leq n \). Let \( \lambda \vdash n(n - 1) \) with \( \lambda_1 = \lambda_2 = \ldots = \lambda_m = 2n - m - 2 \). Let \( \mu \vdash n(n + 1) \) with parts \( \mu_1 = 2n - m, \mu_2 = \ldots = \mu_{m+1} = 2n - m - 1 \) and (if \( m < n \)) \( \mu_j = \lambda_{j-1} \) for \( j = m + 2, \ldots, n + 1 \). Then,

\[
\{a_{\lambda_1}^2, s_m\} = (-1)^{m}(m + 1)\{a_{\mu_1}^2, s_m\}.
\] (14)

Thus adding the tetris type shape

![Tetris Type Shape](image)

to the top of the diagram of \( \lambda \) changes the coefficient by a multiple of \((-1)^m(m+1)\). We denote this tetris type shape by \( T_1(n, m) \).

We can attempt to prove conjecture 5.6 by mimicking the proof of theorem 5.3. We have

\[
x^{\lambda+b} = \left( x_1^{2n-m-3} x_2^{2n-m-4} x_3^{2n-m-5} \ldots x_m^{2n-2m-2} \right) \cdot \left( x_{m+1}^{\lambda_1+n} \ldots x_n^{\lambda_n+n} \right),
\]

\[
x^{\mu+b_{m+1}} = \left( x_1^{2n-m} x_2^{2n-m-2} x_3^{2n-m-3} \ldots x_m^{2n-1} \right) \cdot \left( x_{m+2}^{\mu_1} \ldots x_n^{\mu_n} \right).
\]

We write

\[
a_{\lambda_1}^3 = a_{\lambda_1}^3 \cdot B_m \cdot C_m
\]

with \( B_m \) and \( C_m \) as in the proof of theorem 5.3. Similarly, we write

\[
a_{\mu_1}^3 = a_{\mu_1}^3 \cdot \tilde{B}_m \cdot \tilde{C}_m
\]

where

\[
\tilde{B}_m = \prod_{1 \leq i < m+1, j \leq n+1} (x_i - x_j)^3 \quad \text{and} \quad \tilde{C}_m = \frac{a_{\mu_1}^3\tilde{B}_m}{a_{\lambda_1}^3 B_m} = \prod_{m+2 \leq i < j \leq n+1} (x_i - x_j)^3.
\]

By the argument in the proof of theorem 5.3, the monomials in \( a_{\lambda_1}^3 \) contributing to \( \{a_{\lambda_1}^2, s_m\} \) are of the form

\[
x_1^{2m-3} x_2^{2m-4} \ldots x_m^{2m-2} \cdot F,
\]

where \( F \) is a monomial of degree \( m \) in the variables \( x_1, x_2, \ldots, x_m \).

Similarly, monomials in \( a_{\mu_1}^3 \) contributing to \( \{a_{\mu_1}^2, s_m\} \), are of the form

\[
x_1^{2m} x_2^{2m-2} x_3^{2m-3} \ldots x_m^{m-1} \cdot G,
\]

where \( G \) is a monomial of degree \( m \) in \( x_1, x_2, \ldots, x_m, x_{m+1} \).

Let \( l = (l_1, l_2, \ldots, l_m) \) be a partition of \( m \) and set \( l^* = (l_1, l_2, \ldots, l_m, l_{m+1} = 0) \). Let \( \alpha \in S_m \) be a permutation of \( \{1, 2, \ldots, m\} \) and let \( \beta \in S_{m+1} \) be a permutation of \( \{1, 2, \ldots, m, m+1\} \). Denote by \( C(l, \alpha) \) the coefficient of

\[
x_1^{2m-3+l_{\alpha(1)}} x_2^{2m-4+l_{\alpha(2)}} \ldots x_m^{m-2+l_{\alpha(m)}} = x^l(m-\alpha)^{m+\alpha(l)+\delta_k}
\] (15)
in \(a^{2}_{h_{m}}\) and denote by \(\bar{C}(l, \beta)\) the coefficient of 
\[
x_{2m+\ell_{j(m)}, 2m-2+\ell_{j(2m)}, 2m-3+\ell_{j(2m)}, \ldots, x_{m+\ell_{j(m+1)}, m-1+\ell_{j(m+1)}}, x_{m+1, \ldots, x_{m}} = x_{(m-1)\ell_{j(m)} + \beta(l) + \delta_{m+1}}
\]
in \(a^{2}_{h_{m+1}}\). Denote by \(D(l)\) the coefficient of 
\[
x_{1, x_{2}, \ldots, x_{m}} \text{ in } B_{m}. C_{m} \text{ is symmetric in } x_{1}, x_{2}, \ldots, x_{m}. D(l) \text{ does not depend on the permutation } \alpha. \text{ The coefficient of }
\]
\[
x_{1, \ldots, x_{m}} \text{ on the right hand side has only one element, the coefficient of } 1 \text{ in } a^{2}_{h_{m}}, \text{ which is } 1. \text{ Therefore, the right hand side also equals } -2. \text{ We have the following proposition.}

**Proposition 5.7.** Let \(\lambda = (2n-3, \lambda_{2}, \ldots, \lambda_{n}) \vdash n(n-1)\) and \(\mu = (2n-1, 2n-2, \lambda_{2}, \ldots, \lambda_{n}) \vdash n(n+1)\). Then \(\langle a^{2}_{h_{m}}, s_{\mu} \rangle = -2\langle a^{2}_{h_{m}}, s_{\lambda} \rangle\).

Next, we prove the conjecture for \(m = n - 1\). This will be needed for the proof of the last recursive formula of this paper. We first introduce some definitions following [11, chapter 7]. Denote by \(f_{\lambda}\) the number of standard Young tableaux (SYT) of shape \(\lambda\). Given a Young diagram \(\lambda\) and a square \(u = (i, j)\) of \(\lambda\), define the content, \(c(u)\), of \(\lambda\) at \(u = (i, j)\) by \(c(u) = j - i\).

If \(\lambda\) is a partition of \(n(n-1)\) that can be written as \(\lambda = \eta + ((n-2)\eta)\), where \(\eta\) is a partition of \(n\), then, by [11, exercise 7.3.7.d], which is proved in [12, corollary 6.2], we have
\[
\langle a^{2}_{h_{m}}, s_{\eta} \rangle = (-1)^{\frac{1}{2}} f_{\lambda} \prod_{s \in \eta} (1 - 2c(s)).
\]

Consider the partitions \(\lambda = ((n-1)\eta) \vdash n(n-1)\) and \(\mu = (n+1, n^{2}-1, \ldots, n-1) \vdash n(n+1)\). We have
\[
\lambda = ((n-1)\eta) = (1^{\ell_{1}}) + ((n-2)\eta) \quad \text{and} \quad \mu = (2, 1^{n-1}) + ((n-1)\eta+1).
\]

Then, by (18) and the immediate fact that \(f_{1^{\eta'}} = 1\) and \(f_{((2, 1^{n-1})} = n\), it follows that \(\langle a^{2}_{h_{m}}, s_{\eta} \rangle\) equals
\[
(-1)^{\frac{1}{2}} f_{1^{\eta'}} \prod_{s \in (1^{\eta'})} (1 - 2c(s)) = (-1)^{\frac{1}{2}} 1 \cdot 3 \cdot 5 \cdots (2n-1)
\]
and \(\langle a^{2}_{h_{m+1}}, s_{\mu} \rangle\) equals
\[
(-1)^{\frac{1}{2}} f_{((2, 1^{n-1})} \prod_{s \in (2, 1^{n-1})} (1 - 2c(s)) = (-1)^{\frac{1}{2}} (n-1) \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1).
\]

Comparing (20) and (21), proves conjecture 5.6 in the case \(m = n - 1\).
Proposition 5.8. Let $\lambda = (n-1)^n \vdash n(n-1)$ and $\mu = (n+1, n^{n-1}, n-1) \vdash n(n+1)$. Then,
\[ \{a_{\lambda_{n+1}}^2, s_\mu\} = (-1)^{n-1}n(a_{\lambda_n}^2, s_\mu). \] (22)

5.3. Second recursive formula

Before considering the last recursive formula, we prove another helpful lemma.

First, some notation. Suppose $\nu$ is a partition of $n-1$ and $\mu$ is a partition of $n$ containing $\nu$. Then the shape $\mu$ is obtained by adding a square to the shape $\nu$. We denote by $c(\mu/\nu)$ the content of the square $\mu/\nu$ (i.e. the square added to the shape $\nu$ in order to obtain the shape $\mu$) in the shape $\mu$.

As noted in [7], $f_\nu = \sum_{\eta \subseteq \nu} f_\eta$, where $\nu \setminus 1$ is the set of partitions obtained from $\nu$ by removing a square. (This formula follows directly from the construction of SYT.) This can be rewritten as
\[ f_\nu = \sum_{\eta \subseteq \nu} f_\eta. \] (23)

Lemma 5.9. Let $\nu$ be a partition of $n-1$. Then, $nf_\nu = \sum_{\eta \subseteq \nu} \mu f_\mu (1 - 2c(\mu/\nu))$.

Proof. We prove the lemma by induction on $n$. If $n = 2$, the statement of the lemma is true by inspection. (Actually, if $n = 1$ the lemma is also true, assuming $f_\emptyset = 1$.)

Assume the statement is true for all partitions of $n-1$. Now let $\nu$ be a partition of $n$. We need to show that
\[ (n+1)f_\nu = \sum_{\mu \subseteq \nu} f_\mu (1 - 2c(\mu/\nu)). \] (24)

Consider first the left hand side of (24). Using (23), we have
\[ (n+1)f_\nu = f_\nu + nf_\nu = f_\nu + \sum_{\eta \subseteq \nu} nf_\eta, \] (25)

By the inductive hypothesis,
\[ (n+1)f_\nu = f_\nu + \sum_{\eta \subseteq \nu} \left(\sum_{\mu \subseteq \eta} f_\mu (1 - 2c(\mu/\eta))\right). \] (26)

Note that in (25) we remove a square from the shape $\nu$ whenever possible and in (26) we add a square to the obtained shape whenever possible. There are two possibilities:

(i) The added square is precisely the removed square. Then, $f_\mu = f_\nu$.

(ii) The added square is different from the removed square. In this case, the operations of removing and adding squares commute.

We separate these possibilities in the sum above. Thus,
\[ (n+1)f_\nu = f_\nu + f_\nu \sum_{\eta \subseteq \nu} (1 - 2(\nu_i - i)) + \sum_{\mu \subseteq \nu} \left(\sum_{\eta \subseteq \nu} f_\mu (1 - 2c(\mu/\eta))\right). \]
Using the commutativity of the operations of removal and addition of a square in case (ii) above, we have

\[(n + 1) f_\nu = f_\nu + f_\nu \sum_{v_i > v_{i+1}}^\ell(v_\nu) (1 - 2(v_i - i)) + \sum_{\mu \models n+1 \atop v \subseteq \mu} \left( \sum_{\substack{\eta \models n \atop v \subseteq \eta}} f_\eta (1 - 2c(\mu/v)) \right). \quad (27)\]

Now we consider the right hand side of (24).

Using (23), we have

\[\sum_{\mu \models n+1 \atop v \subseteq \mu} f_\mu (1 - 2c(\mu/v)) = \sum_{\mu \models n+1 \atop v \subseteq \mu} \left( 1 - 2c(\mu/v) \right) \sum_{\eta \models n \atop v \subseteq \eta} f_\eta. \]

Separating the possibilities (i) and (ii), the right hand side equals

\[f_\nu (1 - 2v_1) + f_\nu \sum_{v_i > v_{i+1}}^\ell(v_\nu) (1 - 2(v_i+1 - i)) + \sum_{\mu \models n+1 \atop v \subseteq \mu} \left( 1 - 2c(\mu/v) \right) \sum_{\eta \models n \atop v \subseteq \eta} f_\eta. \quad (28)\]

To show that (27) and (28) are equal, we need to show that

\[1 + \sum_{v_i > v_{i+1}}^\ell(v_\nu) (2i + 1 - 2v_i) = 1 - 2v_1 + \sum_{v_i > v_{i+1}}^\ell(v_\nu) (2i + 1 - 2v_i+1), \quad (29)\]

which is true since

\[\sum_{v_i > v_{i+1}}^\ell(v_\nu) (v_i - v_{i+1}) = v_1.\]

This concludes the proof of the lemma \(\square\)

Lemma 5.9 also follows from results in [8].

**Theorem 5.10.** Let \(\lambda \vdash n(n-1)\) with \(\ell(\lambda) = n - 1\) and \(\lambda_{n-1} \geq n - 1\) and let \(\mu \vdash n(n+1)\) be given by \(\mu = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{n-1} + 1, n, 1)\). Then,

\[\langle a_{\lambda_{n-1}}, s_\mu \rangle = (-1)^n 3n(a_{\lambda_n}^2, s_\lambda). \quad (30)\]

Thus adding the tetris type shape

\[
\begin{array}{c}
\vdots \\
\lambda_{n-1} - 1 \\
\lambda_n - 1 \end{array}
\]

to the left of the diagram of \(\lambda\) changes the coefficient by a multiple of \((-1)^n 3n\). We denote this tetris type shape by \(L_I(n)\).

**Proof.** Case I: \(\lambda_{n-1} \geq n\). Then \(\lambda = (n^{n-1})\) and \(\mu = ((n + 1)^{n-1}, n, 1)\). Using corollary 5.5 with \(i = 1\), we have

\[\langle a_{\lambda_n}^2, s_\lambda \rangle = (-1)^{\binom{n-1}{2}} 1 \cdot 3 \cdot 5 \cdots (2n - 3).\]


We have
\[ \mu^k = (2n - 1, n, (n - 1)^{n-1}) = (n, 1) + ((n - 1)^{n-1}). \]

By (18),
\[ \{a_{\mu_{k+1}}^2, s_{\mu_k}\} = (-1)^{\binom{\ell(v)}{2}} f_{(n,1)} \prod_{x \in (n,1)} (1 - 2c(x)). \]

Since \( f_{(n,1)} = n \), and by lemma 3.2, \( \{a_{\mu_{k+1}}^2, s_{\mu_k}\} = \{a_{\mu_{0}}^2, s_{\mu_0}\} \), we have
\[ \{a_{\mu_{0}}^2, s_{\mu_0}\} = (-1)^{\binom{\ell(v)}{2}} n(-1)^{n-1} 3 \cdot 3 \cdot 2 \cdot \ldots \cdot 2(n - 3) = (-1)^n n! \]

Case II: \( \lambda_{n-1} = n - 1 \). Thus, \( \lambda = ((n - 1)^{n-1}) + v \), where \( v = (v_1, v_2, \ldots, v_{n-1}) \not\sim n - 1 \).

The last part of \( v \) can only be 0 or 1. If \( v_{n-1} = 1 \), then we are in case I. Therefore, we assume \( v_{n-2} = 0 \).

Using corollary 4.3, we have \( \{a_{\mu_k}^2, s_k\} = \{a_{\mu_k}^2, s_{k/(2n-1)}\} \), where \( \lambda/((2n-1)^{n-1}) = (\lambda_1 - 2, \lambda_2 - 2, \ldots, \lambda_{n-1} - 2) \) since \( \lambda/(2n-1) \) is a partition of \( (n - 1)(n - 2) \), we can use (18) to obtain
\[ \{a_{\mu_k}^2, s_k\} = (-1)^{\binom{\ell(v)}{2}} f_{v} \prod_{x \in v} (1 - 2c(x)). \quad (31) \]

Now let us consider the partition \( \mu = (n + v_1, n + v_2, \ldots, n + v_{n-1}, n, 1) \). We have
\[ x_{x + x_{n+1}} = \prod_{i=1}^{n} (x_i - x_{n+1})^3. \]

We write \( a_{\mu_{k+1}}^2, s_{\mu_{k+1}} = a_{\mu_{k+1}}^2, s_{\mu_{k+1}} \prod_{j=1}^{n} (x_j - x_{n+1})^3. \) For each \( i = 2, \ldots, n, 1 \), the product \( \prod_{x \in (i,1)} (x_i - x_{n+1})^3 \) contributes \(-3x_i^2 x_{n+1}^3 x_{n+1}^3 \cdot \ldots \cdot x_n^3 \) to \( a_{\mu_k}^2 \). We have
\[ \eta^{(i)} = (n - 2)^{n} + \tilde{v}^{(i)}, \]

where \( \tilde{v}^{(i)} = (v_1, v_2, v_3, v_4, \ldots, v_{n-1}, v_1 + 1, v_2 + 1, \ldots, v_{n-1}). \)

To find \( c_i \), we use (18). We have
\[ c_i = (-1)^{\binom{\ell(v)}{2}} f_{\tilde{v}^{(i)}} \prod_{x \in \tilde{v}^{(i)}} (1 - 2c(x)). \quad (33) \]

Moreover,
\[ \prod_{x \in \tilde{v}^{(i)}} (1 - 2c(x)) = (1 - 2c(i, v_i + 1)) \prod_{x \in v} (1 - 2c(x)). \quad (34) \]

Thus, using (31) and (33), in order to prove the theorem, we need to show that
\[ nf_v = \sum_{\tilde{v}^{(i)} \geq v_{n-1}} f_{\tilde{v}^{(i)}} (1 - 2c(i, v_i + 1)). \quad (35) \]

Note that the terms for \( i = 1 \) and \( i = \ell(v) + 1 \) are always included in the sum.

This is precisely the statement of lemma 5.9. □
6. Applications

In [5, section 6], Dunne provides (without proof) closed formulas for several specific Slater states. They correspond to close formulas for \( \langle a_n^2, s_i \rangle \) for specific (very symmetric) partitions \( \lambda \). In this section, we use the recursive rules of the previous sections to prove some of these formulas. We also use our rules to explain recursive patterns observed by Dunne in the same section. We adapt the notation to match that of our previous sections and paraphrase Dunne’s physical explanations.

Dunne starts by mentioning that \( \langle a_n^2, s_i \rangle = 1 \) for the most uniformly distributed of the Slater states, i.e. the state corresponding to \( \lambda = (2(n - 1), 2(n - 2), \ldots, 4, 2, 0) \). This is the result of corollary 4.2. Next, he gives the coefficient for the situation in which the angular momentum levels are most closely bunched, i.e. \( \lambda = ((n - 1)^n) \). This is our formula (20):

\[
\langle a_n^2, s_i \rangle = (-1)^{\frac{i}{2}} 1 \cdot 3 \cdot 5 \cdots (2n - 1).
\]

Note that in each of these two cases \( \lambda = \lambda^\text{in} \).

The next case, \( \lambda = ((n)^n, 0) \), is not invariant under taking the box complement. Here one electron is in the 0 angular momentum state and the remaining \( n - 1 \) electrons are bunched together. The coefficient is \( \langle a_n^2, s_i \rangle = (-1)^{\frac{i}{2} - 1} 1 \cdot 3 \cdot 5 \cdots (2n - 3) \), which is the result of corollary 5.5 with \( i = 1 \).

The above cases have all been noted previously in the combinatorics literature (the first case in [14] and the last two as exercises in [11], for example). We mention them here for completion and to show how they fit in the framework of the recursive formulas. The interesting applications of our rules come in the next batch of Dunne’s closed formulas.

Starting with the maximally bunched state \( ((n - 1)^n) \) and successively moving the extreme inner and outer electrons in and out (respectively) by one step, the formulas given by Dunne correspond to:

\[
\langle a_n^2, s_{(n,(n-1)^n-2,n-2)} \rangle = (-1)^{\frac{i}{2}+1}(n-1) \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3) \tag{36}
\]

\[
\langle a_n^2, s_{(n+1,(n-1)^n-2,n-3)} \rangle = (-1)^{\frac{i}{2}+1}n(n-1) \cdot 1 \cdot 3 \cdot 5 \cdots (2n-5) \tag{37}
\]

\[
\vdots
\]

\[
\langle a_n^2, s_{(2(n-1),(n-1)^n-2,0)} \rangle = (-1)^{\frac{i}{2}+1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-5). \tag{38}
\]

To prove (36), notice that \( (n, (n-1)^n-2, n-2) \) is obtained from \( ((n-2)^{n-1}) \) by adding to its top a tetrise type shape \( T_1(n-1, n-2) \). By proposition 5.8 we have

\[
(n, (n-1)^n-2, n-2) = (-1)^{n-2}(n-1)\cdot((n-2)^{n-1})
\]

and thus, by (20), \( (n, (n-1)^n-2, n-2) = (-1)^{\frac{i}{2}}n(n-1) \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3) \), which is equivalent to (36).

To prove (38), we use theorem 4.1 to obtain \( \langle a_n^2, s_{(2(n-1),(n-1)^n-2,0)} \rangle = \langle a_n^2, s_{(n-1)^n-1} \rangle \).

Then, by corollary 5.5 with \( i = 1 \), we have

\[
\langle a_n^2, s_{(2(n-1),(n-1)^n-2,0)} \rangle = (-1)^{\frac{i}{2}-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-5),
\]

which is equivalent to (38).

We can also prove the formula that would naturally come before (38). Notice that \( (2n-3, (n-1)^{n-2}, 1) \) is obtained from \( (2n-4, (n-2)^{n-3}) \) by adding to the left a tetrise type shape \( L_1(n-1) \).

By theorem 5.10, \( \langle a_n^2, s_{(2(n-3),(n-1)^{n-2},1)} \rangle = (-1)^{n-1}3(n-1)\langle a_n^2, s_{(2(n-4),(n-2)^{n-3})} \rangle \). Since \( \langle a_n^2, s_{(2(n-4),(n-2)^{n-3})} \rangle = \langle a_n^2, s_{(n-2)^{n-3}} \rangle \) by theorem 4.1, we can use corollary 5.5 with \( i = 1 \) to obtain

\[
\langle a_n^2, s_{(2(n-3),(n-1)^{n-2},1)} \rangle = (-1)^{\frac{i}{2}}3(n-1) \cdot 1 \cdot 3 \cdot 5 \cdots (2n-7).
\]
Thus, the formula preceding (38) should be
\[
\left(a_h, s_{(2n-3, n-1) = 2 \cdot 1}ight) = (-1)^{\binom{n}{2}} 3(n - 1) \cdot 1 \cdot 3 \cdot 5 \cdots (2n - 7).
\]
The recursions established in this paper do not help prove (37) and the rest of the formulas alluded to above. On the other hand, the existence of these formulas is encouraging evidence that further recursions must exist (perhaps in the form of adding/removing ‘broken’ tetris type shapes).

Dunne’s next suggestion is to start with the maximally distributed state, corresponding to \( \lambda = (2(n-1), 2(n-2), \ldots, 4, 2, 0) \), and make local shifts of electrons between angular momentum levels. In terms of partitions and Young diagrams, this corresponds to removing the last box in the \( j \)th row of \( \lambda \) above and adding it to the end of the \((j+1)\)th row. He notes ‘the remarkable fact that such an operation always changes the coefficient by a factor of \(-3\)’.

He generalizes the observation to the situation when the last box in the \( j \)th row of \( \lambda \) is removed and added to the end of the \((j+k)\)th row. We prove this formula in the following proposition.

**Proposition 6.1.** Fix an integer \( j \) with \( 1 \leq j \leq n - 1 \) and let \( h \) be an integer such that \( j + 1 \leq h \leq n \). If \( v = (v_1, v_2, \ldots, v_h) \vdash n(n-1) \) is given by \( v_i = 2(n-i) \) if \( i \neq j, h \), and \( v_j = 2(n-j) - 1, v_h = 2(n-h) + 1 \), then
\[
\left(a_h, s_v\right) = (-1)^{h-j} \cdot 3 \cdot 2^{h-j-1}.
\]
(39)

**Proof.** Start with the Young diagram for \( v \), remove the top \( j \) rows, i.e. the tetris type shapes \( T_0(n-j, 0), T_0(n-j-1, 0), \ldots T_0(n-j, 1, 0) \). By theorem 4.1, we have
\[
\left(a_h, s_v\right) = \left(a_h, s_{(v_1, v_2, \ldots, v_h)}\right).
\]
Next, remove the \( h - j - 1 \) tetris type shapes \( T_j(n-j, 1), T_j(n-j-1, 1), \ldots, T_j(n-j, 1) \). By proposition 5.7, we have
\[
\left(a_h, s_{(v_1, v_2, \ldots, v_h)}\right) = \left(2\right)^{h-j-1}\left(a_h, s_{(v_1, v_2, \ldots, v_h)}\right).
\]
Notice that \( v_{h-1} - 1 = v_h \). Remove a tetris type shape \( T_0(n-h+1, 1) \). By corollary 5.4, we obtain
\[
\left(a_h, s_{(v_1, v_2, \ldots, v_h)}\right) = -3\left(a_h, s_{(v_1, v_2, \ldots, v_h)}\right).
\]
If \( h = n \), the partition \((v_h-1, v_{h+1}, \ldots, v_n)\) is the empty partition. Otherwise, it is \((2(n-h), 2(n-h-1), \ldots, 4, 2, 0) \). In either case \( \langle a_h, s_{(v_1, v_2, \ldots, v_h)}\rangle = 1 \). Combining these results completes the proof. \( \square \)

Finally, we use the recursions of this paper to explain some recursive properties observed by Dunne. If we write \( \#(n) \) for the number of Schur functions appearing in the decomposition of \( a_2 \), he notes that, with a consistent ordering of the coefficients (as in the tables at the end of [5]),

(i) ‘the first \( \#(n-1) \) coefficients for \( n \) particles coincide with all the coefficients for \( n-1 \) particles;
(ii) the next \( \#(n-2) \) coefficients of the \( n \) particle problem are given by \(-3\) times the \( \#(n-2) \) coefficients of the \( n-2 \) particle problem;
(iii) the next \( \#(n-3) \) coefficients of the \( n \) particle problem are given by \( 6\times \#(n-3) \) coefficients of the \( n-3 \) particle problem;
(iv) the next \( \#(n-4) \) coefficients of the \( n \) particle problem are given by \(-12\times \#(n-4) \) coefficients of the \( n-4 \) particle problem, etc.’

This can be explained as follows.
(i) Start with a partition \( \lambda \) corresponding to a Schur function appearing in the decomposition of \( a_{\delta_n,1}^2 \) and add to its left a tetris type \( L(n-1) \) to obtain a partition \( \mu \). Then, by corollary 4.3, \( \langle a_{\delta_n,1}^2, s_\mu \rangle = \langle a_{\delta_n,1}^2, s_\lambda \rangle \). (This correspondence matches Dunne's ordering in the tables at the end of his article.)

(ii) Start with a partition \( \lambda \) corresponding to a Schur function appearing in the decomposition of \( a_{\delta_n,1}^2 \) and add to its top a tetris type shape \( T_0(n-2, 0) \) (a row of length \( 2n-4 \)) to obtain a partition \( \mu \) whose Shur function appears in the decomposition of \( a_{\delta_n,1}^2 \) with coefficient \( \langle a_{\delta_n,1}^2, s_\mu \rangle \) (by theorem 4.1). Then, add to the top of \( \mu \) a tetris type shape \( T_0(n-1, 1) \) to obtain a partition \( \nu \) whose Shur function appears in the decomposition of \( a_{\delta_n,1}^2 \) with coefficient \( -3\langle a_{\delta_n,1}^2, s_\nu \rangle \) (corollary 5.4).

(iii) Start with a partition \( \lambda \) corresponding to a Schur function appearing in the decomposition of \( a_{\delta_n,1}^2 \) and follow the steps in (ii), i.e. add a tetris type shape \( T_0(n-3, 0) \) to the top of \( \lambda \) to obtain \( \mu \), and a tetris type shape \( T_0(n-2, 1) \) to the top of \( \mu \) to obtain \( \nu \). The Schur function for \( \nu \) appears in the decomposition of \( a_{\delta_n,1}^2 \). Now add to the top of \( \nu \) a tetris type shape \( T_1(n-1, 1) \) to obtain a partition \( \eta \). By proposition 5.7 and (ii), we have \( \langle a_{\delta_n,1}^2, s_\eta \rangle = -2\langle a_{\delta_n,1}^2, s_\mu \rangle \).

(iv) Start with a partition \( \lambda \) corresponding to a Schur function appearing in the decomposition of \( a_{\delta_n,1}^2 \) and follow the steps in (iii). Thus, \( \eta \), which is a partition for the \( n-1 \) particle problem, is obtained from \( \lambda \) by adding to its top, in order, \( T_0(n-4, 0) \), \( T_0(n-3, 1) \) and \( T_1(n-2, 1) \). Add to the top of \( \eta \) another tetris type \( T_1(n-1, 1) \) to obtain a partition \( \xi \). Then, by proposition 5.7 and (iii), we have \( \langle a_{\delta_n,1}^2, s_\xi \rangle = -12\langle a_{\delta_n,1}^2, s_\eta \rangle \).

7. Concluding remarks

The recursive formulas of this paper together with the box-complement lemma give 15 of the 16 coefficients in the \( n = 4 \) problem in terms of the coefficients for \( n = 3 \) and 48 of the 59 coefficients in the \( n = 5 \) problem in terms of the coefficients for \( n = 4 \). This is a considerable improvement to the recursive observation in [5] through which 23 of the 59 coefficients in the \( n = 5 \) problem are determined from the results for \( n = 2, 3, 4 \).

Maple calculations suggest that further recursive rules involving other tetris type shapes will likely require ‘broken’ shapes. As Dunne suggests [5] it is very likely that such rules exist.

Acknowledgment

The author is grateful to one of the referees for extended comments that improved the presentation of this paper.

References

[1] Ballantine C 2011 Powers of the Vandermonde determinant, Schur functions, and the dimension game 23rd Int. Conf. on Formal Power Series and Algebraic Combinatorics (June 13-17) Discrete Math. Theor. Comput. Sci. Proc., AO pp 87-98
[2] Ballantine C 2012 Powers of the Vandermonde determinant, Schur functions, and the recursive formulas arXiv:1201.4572v1 [math.CO]
[3] Boussicault A and Luque J-G 2008 Staircase Macdonald polynomials and the q-discriminant 20th Annual Int. Conf. on Formal Power Series and Algebraic Combinatorics (Nancy) Discrete Math. Theor. Comput. Sci. Proc., AJ pp 381-92
[4] Boussicault A, Luque J-G and Tollu C 2009 Hyperdeterminantal computation for the Laughlin wavefunction J. Phys. A: Math. Theor. 42 145301
[5] Dunne G 1993 Slater decomposition of Laughlin states *Int. J. Mod. Phys.* B **7** 4783–813
[6] Di Francesco P, Gaudin M, Itzykson C and Lesage F 1994 Laughlin’s wave functions, Coulomb gases and expansions of the discriminant *Int. J. Mod. Phys.* A **9** 4237–351
[7] Han G-N 2010 Hook lengths and shifted partitions *Ramanujan J.* **23** 127–35
[8] Konvalinka M 2011 The weighted hook-length formula II: complementary formulas *Eur. J. Comb.* **32** 580–97
[9] King R C, Toumazet F and Wybourne B G 2004 The square of the Vandermonde determinant and its q-generalization *J. Phys. A: Math. Gen.* **37** 735–67
[10] Laughlin R B 1983 Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations *Phys. Rev. Lett.* **50** 1395–8
[11] Stanley R P 1999 *Enumerative Combinatorics* (Cambridge Studies in Advanced Mathematics vol 62) (Cambridge: Cambridge University Press)
[12] Stembridge J R 1987 First layer formulas for characters of $\text{SL}(n, \mathbb{C})$ *Trans. Am. Math. Soc.* **299** 319–50
[13] Stone M 1990 Schur functions, chiral bosons, and the quantum-Hall-effect edge states *Phys. Rev.* B **42** 8399–404
[14] Scharff T, Thibon J-Y and Wybourne B G 1994 Powers of the Vandermonde determinant and the quantum Hall effect *J. Phys. A: Math. Gen.* **27** 4211–9
[15] http://www-igm.univ-mlv.fr/~luque/Vandermonde.html