ANYONS AND DEFORMED LIE ALGEBRAS

M. Frau $^a$, A. Lerda $^b$ and S. Sciuto $^a$

$^a$ Dipartimento di Fisica Teorica, Università di Torino
and I.N.F.N., Sezione di Torino, Italy

$^b$ Dipartimento di Fisica Teorica, Università di Salerno
and I.N.F.N., Sezione di Napoli, Italy

Abstract

We discuss the connection between anyons (particles with fractional statistics) and deformed Lie algebras (quantum groups). After a brief review of the main properties of anyons, we present the details of the anyonic realization of all deformed classical Lie algebras in terms of anyonic oscillators. The deformation parameter of the quantum groups is directly related to the statistics parameter of the anyons. Such a realization is a direct generalization of the Schwinger construction in terms of fermions and is based on a sort of bosonization formula which yields the generators of the deformed algebra in terms of the undeformed ones. The entire procedure is well defined on two-dimensional lattices, but it can be consistently reduced also to one-dimensional chains.

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1 Introduction

In this contribution we discuss the connection between anyons and quantum groups that we originally discovered in [1] and later extended in [2] in collaboration with M.A. R.-Monteiro (see also [3]). Anyons and quantum groups seem apparently two very distinct subjects, but as we shall see, they share one common important property: both of them are deeply related to the braid group.

Anyons are particles with any statistics [4, 5] that interpolate between bosons and fermions of which they can be considered, in some sense, a deformation (for reviews see for instance [6, 7]). Anyons exist only in two dimensions because in this case the configuration space of collections of identical particles has some special topological properties allowing arbitrary statistics, which do not exist in three or more dimensions. Specifically the configuration space of identical particles is infinitely connected in two dimensions, but it is only doubly connected in three or more dimensions. In this case only two statistics are possible (the bosonic and fermionic ones) whereas in two dimensions there are infinitely many possibilities. Anyons are not objects of only pure mathematical interest; on the contrary they play an important role in certain systems of the real world. Since this has at least three space dimensions, it is clear that anyons cannot be real particles. However, there exist some (condensed matter) systems that can be regarded effectively as two-dimensional and in which the localized quasi-particle excitations obey the rules of the two-dimensional world. It is these quasi-particles that may be anyons and have fractional statistics. The most notable example of this behavior is provided by the systems that exhibit the fractional quantum Hall effect [8] in which the localized excitations carry fractional charge, fractional spin and fractional statistics, and are therefore anyons [9].

Since the first paper on the subject [4], it became clear that anyons are deeply connected to the braid group of which they are abelian representations, just like bosons and fermions are abelian representations of the permutation group. In fact, when one exchanges two identical anyons, it is not enough to compare their final configuration with the initial one, but instead it is necessary to specify also the way in which the two particles are exchanged, i.e. the way in which they braid around each other.

The braid group plays a crucial role also in the theory of quantum groups (or more properly
of quantum universal enveloping algebras) of which it is the centralizer [10, 11, 12, 13]. Quantum groups are deformations of ordinary Lie algebras which have recently found interesting applications in several areas of physics, like in the theory of exactly solvable models [14, 13], in conformal field theories [15] or in condensed matter theory [16]. It has been conjectured also that quantum groups might be the characteristic symmetry structures of anyon systems, even though explicit realizations of this fact are still missing. However, the fundamental role played by the braid group both in the theory of quantum groups and in the theory of anyons suggests that at least a direct relation between the two subjects should exist. It is well known that bosonic or fermionic oscillators, characterized by commutative or anti-commutative Heisenberg algebras, can be combined à la Schwinger [17] to construct non-abelian Lie algebras $G$ with the permutation group as centralizer. Similarly one can think of using anyonic oscillators with braid group properties and deformed commutation relations to build non-abelian algebras with the braid group as centralizer, i.e. to construct non-abelian quantum groups $U_q(G)$ from anyons. Here we elaborate on this idea and show that actually all deformed classical Lie algebras admit a simple anyonic realization. In particular, we will show [1, 2, 3] that using anyons of statistics $\nu$ it is possible to realize the quantum universal enveloping algebras $U_q(A_r), U_q(B_r), U_q(C_r)$ and $U_q(D_r)$ with $q = \exp(i\nu \pi)$.

A unified treatment for all these cases is provided by a sort of bosonization formula which expresses the generators of the deformed algebras in terms of the undeformed ones. Such a bosonization resembles the one typical of quantum field theories in $1 + 1$ dimensions [18] where bosons and fermions can be related to each other by means of the so-called disorder operators, and is similar also to the anyonization formula characteristic of quantum field theories in $2 + 1$ dimensions [5, 19]. However, we would like to stress that our bosonization formula is different from the standard relation between the generators of quantum and classical algebras. In fact, our expression is strictly two-dimensional and non-local since it involves anyonic operators defined on a two-dimensional lattice. Thus it cannot be extended to higher dimensions in a straightforward way. However we remark that anyons can consistently be defined also on one dimensional chains; in such a case they become local objects and their braiding properties are dictated by their natural ordering on the line. Consequently, our construction can be used equally well for one dimensional chains.

We organize this contribution as follows. In Section 2 we present a brief introduction to
anyons and discuss how they can be realized from fermions by means of a generalized Jordan-Wigner transformation. In Section 3 we review the basic properties of deformed Lie algebras that will be needed later. In Section 4 and 5 we present the details of our anyonic realization of deformed classical Lie algebras and finally in the last section we present some conclusive remarks.

2 Anyonic Statistics in Two Dimensions

In first quantization the notion of statistics is associated to the properties of the wave functions describing systems of identical particles under the exchange of any two of these. If the wave function is totally symmetric, it describes bosons; if instead it is totally antisymmetric, it describes fermions. However, in two space dimensions there exist more possibilities and the wave function for a system of identical particles is in general neither symmetric nor antisymmetric under permutations, but acquires a phase which generalizes the plus sign typical of bosons and the minus sign typical of fermions. In this general case one says that the wave function describes anyons, particles of arbitrary statistics. For example, a typical wave function for anyons of statistics \( \nu \) is

\[
\psi(z_1, z_1^*; \ldots; z_N, z_N^*) = \prod_{I<J} (z_I - z_J)^\nu f(z_1, z_1^*; \ldots; z_N, z_N^*) ,
\]

(2.1)

where \( z_I = x_I + i y_I \) is the position of the \( I \)-th particle in complex notation and \( f \) is a single-valued function symmetric under all permutations. When particle \( I \) is exchanged with particle \( J \), \( \psi \) acquires a phase \( e^{i\pi\nu} \) or \( e^{-i\pi\nu} \) depending on whether the exchange is done by rotating \( I \) around \( J \) clockwise or counterclockwise. Thus, if \( \nu \neq 0,1 \mod 2 \) it is of fundamental importance to specify not only the permutation but also the orientation of the exchange, i.e. the braiding. This is the essential feature that distinguishes anyons from bosons and fermions.

In second quantization the notion of statistics is usually associated to the algebra that the particle creation and annihilation operators satisfy. In fact, bosonic operators close canonical commutation relations, whilst fermionic operators close canonical anticommutation relations. Therefore, in two dimensions where statistics is arbitrary there should exist also anyonic creation and annihilation operators which close braiding relations among themselves. As we will see in
the following, these anyonic operators do indeed exist and can be simply constructed from fermionic (or bosonic) operators using a generalized Jordan-Wigner transformation.

Let us now briefly review \cite{7} the main features of anyonic statistics, both from the first and the second quantization standpoints. We start by considering a system of \( N \) indistinguishable hard-core particles moving in \( \mathbb{R}^d \). The configuration space for this system is

\[
M^d_N = \frac{\left( \mathbb{R}^d \right)^N - \Delta}{S_N}, \tag{2.2}
\]

where \( \Delta \) is the set of all points in \( \left( \mathbb{R}^d \right)^N \) with at least two equal coordinates and \( S_N \) is the permutation group of \( N \) objects. In (2.2) \( \Delta \) is removed from \( \left( \mathbb{R}^d \right)^N \) because of the hard-core condition which prevents any two particles from occupying the same position, and \( S_N \) is moded out because any two configurations which simply differ by a permutation must be identified since the particles are indistinguishable.

The topology of \( M^d_N \) is radically different depending on whether \( d > 2 \) or \( d = 2 \). For example, if \( d > 2 \) the fundamental group of \( M^d_N \) is the permutation group, \( i.e.

\[
\pi_1 \left( M^d_N \right) = S_N. \tag{2.3}
\]

On the contrary, if \( d = 2 \) we have

\[
\pi_1 \left( M^2_N \right) = B_N, \tag{2.4}
\]

where \( B_N \) is the braid group of \( N \) objects. It is precisely this topological property that allows the existence of anyonic statistics in \( d = 2 \).

To see this, let us consider the probability amplitude that our system evolves from a certain configuration \( q \in M^d_N \) at time \( t \) to the same configuration at a later time \( t' \). Let us denote this amplitude by \( K(q; t, t') \). In the path-integral formulation of quantum mechanics, \( K(q; t, t') \) is represented by a sum over all loops in \( M^d_N \) starting from \( q \) at time \( t \) and arriving at \( q \) at time \( t' \). This sum can be organized as a sum over homotopy classes \( \alpha \in \pi_1 \left( M^d_N \right) \) and a sum over all elements \( q_\alpha \) of each class. Within each homotopy class, all loops are weighted with the exponential of the action, but different classes in general can have different weights \footnote{For bosons all classes have the same weight, but for fermions each class is weighted with a plus or a minus sign depending on the parity of the corresponding permutation.}. In fact,
denoting by $\mathcal{L}$ the Lagrangian of the system, one has

$$K(q; t, t') = \sum_\alpha \chi(\alpha) \int \mathcal{D}q_\alpha \ e^{\frac{i}{\hbar} \int_{t}^{t'} \frac{d\tau}{\hbar} \mathcal{L}} \ (2.5)$$

where $\chi(\alpha)$ is a complex number representing the weight with which the class $\alpha$ contributes to the path-integral. This complex number cannot be totally arbitrary; indeed if we want to maintain the standard rules for combining probability amplitudes, it is necessary that

$$\chi(\alpha_1) \chi(\alpha_2) = \chi(\alpha_1 \alpha_2) \ (2.6)$$

for any $\alpha_1$ and $\alpha_2$. Thus, $\chi(\alpha)$ must be a one-dimensional representation of the fundamental group of the configuration space of the system, which for $d = 2$ is the braid group (see (2.4)).

Let us recall that the braid group of $N$ objects $B_N$ is an infinite group generated by $N - 1$ elements $\{\sigma_1, ..., \sigma_{N-1}\}$ which satisfy

$$\sigma_I \sigma_{I+1} \sigma_I = \sigma_{I+1} \sigma_I \sigma_{I+1} \ (2.7)$$

for $I = 1, ..., N - 2$, and

$$\sigma_I \sigma_J = \sigma_J \sigma_I \ (2.8)$$

for $|I - J| \geq 2$. The generator $\sigma_I$ simply represents the exchange of particle $I$ and particle $I + 1$ with a definite orientation (say counterclockwise). Any word constructed with the $\sigma_I$'s and their inverses, modulo the relations (2.7-2.8), is an element of $B_N$. Notice that $\sigma_I^2 \neq 1$. If we impose the further condition $\sigma_I^2 = 1$ for all $I$, then the braid group reduces to the permutation group $S_N$. The one-dimensional unitary representations of $B_N$ are simply given by

$$\chi(\sigma_I) = e^{-i \nu \pi} \quad \forall \ I \ , \ (2.9)$$

where $\nu$ is an arbitrary real parameter labelling the representation. Clearly we can always restrict $\nu$ in the interval $[0, 2)$. For an arbitrary braiding $\alpha \in B_N$, (2.9) is generalized as follows

$$\chi(\alpha) = e^{-i \nu \pi P_\alpha} \ , \ (2.10)$$

In principle one could also allow non-abelian representations of the fundamental group. Such possibility, which we do not discuss here, leads to parastatistics for the permutation group ($d > 2$) and to plectons for the braid group ($d = 2$).
where $P_\alpha$ is the difference between the number of counterclockwise exchanges and the number of clockwise exchanges that occur in $\alpha$. When we insert (2.10) into (2.5) we obtain

$$K(q; t, t') = \sum_{\alpha \in B_N} e^{-i \nu \pi P_\alpha} \int Dq_\alpha \ e^{i \hbar \int_t^{t'} d\tau \ L}$$

(2.11)

which represents the first-quantized propagator for anyons of statistics $\nu$. If $\nu = 0$, we describe bosons, since all homotopy classes of loops enter the path-integral with equal weight. If $\nu = 1$, we describe fermions since each homotopy class $\alpha$ is weighted with a plus sign or a minus sign depending on the parity of $P_\alpha$. If $\nu \neq 0, 1$, the weights of the homotopy classes are generic phases and thus we describe anyons of statistics $\nu$.

Let us now observe that (2.10) can be written also as follows

$$\chi(\alpha) = e^{-i \nu \sum_{i < j} \left[ \Theta_{ij}^{(i)}(t') - \Theta_{ij}^{(i)}(t) \right]} = e^{-i \nu \sum_{i < j} \int_t^{t'} d\tau \ \frac{d}{d\tau} \Theta_{ij}^{(i)}(\tau)},$$

(2.12)

where $\Theta_{ij}^{(i)}(t)$ is the winding angle of particle $I$ with respect to particle $J$ measured along the braiding $\alpha$ at time $t$. The function $\Theta_{ij}^{(i)}(t)$ is very complicated, but its explicit expression is not needed in the following. Inserting (2.12) into (2.5), we get

$$K(q; t, t') = \sum_{\alpha \in B_N} \int Dq_\alpha \ e^{i \hbar \int_t^{t'} d\tau \left[ -h \nu \sum_{i < j} \frac{d}{d\tau} \Theta_{ij}^{(i)}(\tau) \right]}$$

(2.13)

which has an interesting interpretation. In fact $K(q; t, t')$ is decomposed into subamplitudes which are weighted equally as if we were describing bosons with a Lagrangian

$$L_B = L - h \nu \sum_{i < j} \frac{d}{d\tau} \Theta_{ij}^{(i)}(\tau).$$

(2.14)

Furthermore, if we set

$$\nu = 1 + \nu'$$

(2.15)

we obtain

$$K(q; t, t') = \sum_{\alpha \in B_N} (-1)^{P_\alpha} \int Dq_\alpha \ e^{i \hbar \int_t^{t'} d\tau \left[ -h \nu' \sum_{i < j} \frac{d}{d\tau} \Theta_{ij}^{(i)}(\tau) \right]}$$

(2.16)

which too has an interesting interpretation. In fact, $K(q; t, t')$ is decomposed into subamplitudes which contribute to the path-integral with alternating signs as if we were describing fermions with a Lagrangian

$$L_F = L - h \nu' \sum_{i < j} \frac{d}{d\tau} \Theta_{ij}^{(i)}(\tau).$$

(2.17)
Eqs. (2.14) and (2.17) mean that we can trade anyonic statistics for some kind of “fictitious”
force and describe anyons as ordinary bosons or fermions with the modified Lagrangians $\mathcal{L}_B$ or $\mathcal{L}_F$. Notice that the new terms that appear in $\mathcal{L}_B$ or $\mathcal{L}_F$ are topological quantities (i.e. total derivatives) which do not modify the local dynamical properties of the particles like for example
their equations of motions, but which modify significantly their global dynamical properties like
their statistics. That is why these terms are called statistical interaction terms.

We now present an explicit realization of these statistical interactions by means of Chern-
Simons fields [20]. Let us consider a system of $N$ non-relativistic point particles moving on a
plane, with mass $m$ and charge $e$, whose coordinates $\vec{r}_I(t)$ and velocities $\vec{v}_I(t)$ serve as dynamical
variables ($I = 1, ..., N$). For definiteness we take these particles to be spinless fermions which
automatically satisfy the hard-core constraint because of the Pauli exclusion principle. Their
dynamics is governed by the action

$$S = \int dt \mathcal{L} ,$$

where

$$\mathcal{L} = \sum_{I=1}^{N} \frac{1}{2} m \vec{v}_I(t)^2 - V(\vec{r}_1(t), ..., \vec{r}_N(t)) , \quad (2.18)$$

and $V(\vec{r}_1(t), ..., \vec{r}_N(t))$ is the potential. In this system there exists a conserved current $j^\alpha(\vec{x}, t)$,
whose components are the matter density

$$j^0(\vec{x}, t) \equiv \rho(\vec{x}, t) = \sum_{I=1}^{N} \delta(\vec{x} - \vec{r}_I(t)) \quad (2.19)$$

and the current density

$$\vec{j}(\vec{x}, t) = \sum_{I=1}^{N} \vec{v}_I(t) \delta(\vec{x} - \vec{r}_I(t)) \quad (2.20)$$

which satisfy the continuity equation $\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0$.

Now let us suppose that $j^\alpha$ be coupled to an abelian gauge field $A_\alpha$ in the standard minimal
way, i.e.

$$S_{\text{int}} = -e \int d^3 x \, j^\alpha(x) A_\alpha(x) , \quad (2.21)$$

and that the dynamics of $A_\alpha$ be governed by the Chern-Simons action [20]

$$S_{\text{CS}} = \frac{\kappa}{2} \int d^3 x \, \epsilon^{\alpha\beta\gamma} A_\alpha(x) \partial_\beta A_\gamma(x) , \quad (2.22)$$

where $\kappa$ is a coupling constant. (Here and in the following, we set $c = \hbar = 1$, assume that the
space-time indices are contracted, raised and lowered with the Minkowski metric with signature
\[ +, -, -, \text{ and denote } (\vec{x}, t) \text{ simply by } x \text{ wherever not ambiguous.} \]

The total action for this interacting system is therefore
\[ S_{\text{tot}} = S + S_{\text{int}} + S_{\text{CS}} \tag{2.23} \]
and is invariant under standard abelian gauge transformations.

What we want to show now is that the Chern-Simons field effectively introduces a statistical interaction among the particles, and in particular that the Lagrangian corresponding to \( S_{\text{tot}} \) is of the type (2.17). To see this, let us first observe that varying \( S_{\text{tot}} \) with respect to \( A_0 \) yields a relation between the “magnetic” field \( B \equiv \partial_2 A_1 - \partial_1 A_2 \) and the particle density \( \rho \), namely
\[ B = -\frac{e}{\kappa} \rho \tag{2.24} \]
which means that a magnetic flux is attached to each particle. Fixing the Weyl gauge \( A_0 = 0 \) and removing any residual gauge invariance by imposing the subsidiary condition \( \partial_i A^i = 0 \), we can solve (2.24) and obtain \( A^i(x) \). The solution is
\[ A^i(x) = \sum_{I=1}^{N} A^i_I(\vec{x}, t), \tag{2.25} \]
where
\[ A^i_I(\vec{x}, t) \equiv A^i_I(\vec{r}_1(t), ..., \vec{r}_N(t)) \bigg|_{\vec{r}_I = \vec{x}} \tag{2.26} \]
with
\[ A^i_I(\vec{r}_1, ..., \vec{r}_N) = -\frac{e}{2\pi \kappa} \sum_{J \neq I} \epsilon^{ij} \frac{r_{j I} - r_{j J}}{|\vec{r}_I - \vec{r}_J|^2} \tag{2.27} \]
Despite the appearance, the meaning of this solution is quite simple. In fact, the Hamiltonian corresponding to the action \( S_{\text{tot}} \) is
\[ H' = \sum_{I=1}^{N} \frac{1}{2m} \left( \vec{p}_I - e \vec{A}_I(\vec{r}_1, ..., \vec{r}_N) \right)^2 + V(\vec{r}_1(t), ..., \vec{r}_N(t)). \tag{2.28} \]
Thus, the net effect of the Chern-Simons dynamics is to produce a non-local vector potential \( \vec{A}_I \) for each particle. This determines a magnetic field \( B \) which vanishes everywhere except at the particle locations, as we can see also from (2.24) upon using (2.19). Therefore the particles effectively carry both a charge \( e \) and a magnetic flux \( \phi = -e/\kappa \). The exotic statistics is then produced by the Aharonov-Bohm mechanism: when two particles are exchanged they pick up
a phase because the charge of one particle moves around the flux of the other and vice versa.

To make the connection with (2.17) more transparent, let us observe that

$$A_I^j(\vec{r}_1, ..., \vec{r}_N) = \frac{e}{2\pi \kappa} \frac{\partial}{\partial r_{1I}} \sum_{J \neq I} \Theta_{IJ} ,$$

(2.29)

where $\Theta_{IJ}$ is the winding angle of particle $I$ with respect to particle $J$ which is explicitly given by

$$\Theta_{IJ} = \tan^{-1} \left( \frac{r_{2I} - r_{2J}}{r_{1I} - r_{1J}} \right) .$$

(2.30)

Thus the Lagrangian corresponding to (2.28) is

$$\mathcal{L}' = \mathcal{L} + e \sum_{I=1}^N \vec{v}_I \cdot \vec{A}_I(\vec{r}_1, ..., \vec{r}_N)$$

$$= \mathcal{L} - \frac{e^2}{2\pi \kappa} \sum_{I<J} \left( v_I^i - v_J^i \right) \frac{\partial}{\partial r_I^i} \Theta_{IJ} ,$$

(2.31)

where we have used (2.29) and the property $\frac{\partial}{\partial r_J^i} \Theta_{IJ} = -\frac{\partial}{\partial r_I^i} \Theta_{IJ}$. If we now observe that

$$\left( v_I^i - v_J^i \right) \frac{\partial}{\partial r_I^i} \Theta_{IJ} = \frac{d}{dt} \Theta_{IJ} ,$$

we recognize that $\mathcal{L}'$ in (2.31) is of the same form of $\mathcal{L}_F$ in (2.17) with

$$\nu' = \frac{e^2}{2\pi \kappa}$$

(2.32)

Thus the system described by $S_{\text{tot}}$ describes fermion-based anyons with statistics $\nu'$ given by (2.32).

The same results can also be obtained using the second-quantized formalism. To see this, let us introduce a non-relativistic fermionic matter field $\psi(\vec{x}, t)$ of mass $m$ and charge $e$, which we minimally couple to an abelian gauge field $A_\alpha(\vec{x}, t)$ with a Chern-Simons kinetic term. Thus, we consider the following action

$$S = \int d^3x \left[ i \psi^\dagger D_0 \psi + \frac{1}{2m} \psi^\dagger \left( D_1^2 + D_2^2 \right) \psi + \kappa \frac{\epsilon^{\alpha\beta\gamma}}{2} A_\alpha \partial_\beta A_\gamma \right] ,$$

(2.33)

where $D_\alpha = \partial_\alpha + i e A_\alpha$ is the covariant derivative. Varying $S$ with respect to $A_\alpha$, we obtain

$$\epsilon^{\alpha\beta\gamma} \partial_\beta A_\gamma = \frac{e}{\kappa} j^\alpha$$

(2.34)

where the current $j^\alpha$ is explicitly given by

$$j^0 = \psi^\dagger \psi \equiv \rho , \quad j^i = \frac{i}{2m} \left( \psi^\dagger D^i \psi - (D^i \psi)^\dagger \psi \right) .$$

(2.35)
Notice that $\rho$ and $\vec{j}$ are respectively the second-quantized density and current operators that correspond to (2.19) and (2.20) and satisfy the continuity equation

$$\partial_0 \rho + \vec{\nabla} \cdot \vec{j} = 0 \ .$$

(2.36)

As is obvious from (2.34), the Chern-Simons field strength $(\partial_\beta A_\gamma - \partial_\gamma A_\beta)$ is completely determined by the particle currents. Now we show that the Chern-Simons potential itself is not an independent degree of freedom. The $\alpha = 0$ component of (2.34) is

$$\partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa} \rho \tag{2.37}$$

which is simply the second-quantized version of the quantum mechanical constraint (2.24). Imposing the condition $\partial_i A^i = 0$, we can solve (2.37) and formally obtain

$$A^i(x) = \epsilon^{ij} \frac{\partial}{\partial x^j} \left( \frac{e}{\kappa} \int d^2 y \ G(\vec{x} - \vec{y}) \rho(y) \right) , \tag{2.38}$$

where $G$ is the Green function for the Laplacian $\Delta = \vec{\nabla} \cdot \vec{\nabla}$, satisfying

$$\Delta G(\vec{x} - \vec{y}) = \delta(\vec{x} - \vec{y}) \ . \tag{2.39}$$

As is well known, the explicit solution to (2.39) is

$$G(\vec{x} - \vec{y}) = \frac{1}{2\pi} \ln |\vec{x} - \vec{y}| \ , \tag{2.40}$$

and thus $A^i$ can be written as follows

$$A^i(x) = \epsilon^{ij} \frac{\partial}{\partial x^j} \left( \frac{e}{2\pi \kappa} \int d^2 y \ \ln |\vec{x} - \vec{y}| \rho(y) \right) , \tag{2.41}$$

or equivalently

$$A^i(x) = \frac{e}{2\pi \kappa} \int d^2 y \ \frac{\partial}{\partial x_i} \Theta(\vec{x} - \vec{y}) \rho(y) \ , \tag{2.42}$$

where $\Theta(\vec{x} - \vec{y})$ is the angle under which $\vec{x}$ is seen from $\vec{y}$, namely

$$\Theta(\vec{x} - \vec{y}) = \tan^{-1} \left( \frac{x_2 - y_2}{x_1 - y_1} \right) \ . \tag{2.43}$$

In the following we will examine in detail the properties of this angle function, but for the moment it is enough to realize that it is a multi-valued function so that it is necessary to fix a cut and a reference axis in order to remove any ambiguity. Therefore, particular care must be used in moving the derivative $\partial/\partial x_i$ out of the integral in (2.42) and displaying $A^i$ as a
gradient. However, when the density $\rho$ is a sum of localized $\delta$-functions, as is appropriate for our collection of non-relativistic point particles, there are no problems in moving $\partial/\partial x_i$ outside the integral in (2.42) and formally write the vector potential as a pure gauge, namely

$$A^i(x) = \frac{e}{2\pi\kappa} \frac{\partial}{\partial x_i} \left( \int d^2y \ \Theta(\vec{x} - \vec{y}) \ \rho(\vec{y}) \right) . \quad (2.44)$$

Because of translational invariance of the angle function (2.43), it is harmless to shift the density $\rho(\vec{y})$ by a constant $\rho_0$ and write in general

$$A^i(x) = \frac{\partial}{\partial x_i} \Lambda(x) , \quad (2.45)$$

where

$$\Lambda(x) = \frac{e}{2\pi\kappa} \int d^2y \ \Theta(\vec{x} - \vec{y}) \ (\rho(\vec{y}) - \rho_0) . \quad (2.46)$$

Now we show that also $A_0$ is pure gauge. To this aim, let us first observe that the space-components of (2.34) are

$$\partial_i A_0 - \partial_0 A_i = -\frac{e}{\kappa} \epsilon_{ij} j^j . \quad (2.47)$$

Then, upon acting with $\partial/\partial x_i$ and recalling that $\partial_i A^i = 0$, we easily get, after an integration by parts,

$$A_0(x) = -\frac{e}{\kappa} \int d^2y \ \epsilon^{ij} \frac{\partial}{\partial x^j} G(\vec{x} - \vec{y}) \ j_i(y) , \quad (2.48)$$

or equivalently

$$A_0(x) = -\frac{e}{2\pi\kappa} \int d^2y \ \frac{\partial}{\partial x_i} \Theta(\vec{x} - \vec{y}) \ j_i(y) . \quad (2.49)$$

If we observe that $\frac{\partial}{\partial x^i} \Theta(\vec{x} - \vec{y}) = -\frac{\partial}{\partial y^i} \Theta(\vec{x} - \vec{y})$, and after a further integration by parts we use the continuity equation for the current, we finally get

$$A_0(x) = \frac{e}{2\pi\kappa} \int d^2y \ \Theta(\vec{x} - \vec{y}) \ \partial_0 (\rho(\vec{y}) - \rho_0) = \partial_0 \Lambda(x) . \quad (2.50)$$

Eqs. (2.45) and (2.50) can be combined into a single covariant expression $A_{\alpha}(x) = \partial_{\alpha} \Lambda(x)$ which exhibits the fact that the Chern-Simons potential in this model is pure gauge, albeit of a non-standard form because of the multivaluedness of the gauge function $\Lambda(x)$. This feature implies that $\partial_0 \Lambda(x)$ is not a trivial quantity: indeed, the vector potential (2.45) describes point-like “magnetic” fluxes localized on each particle, as one can see by computing the gauge invariant quantity

$$\oint_{C_i} d\vec{\ell} \cdot \vec{A} ,$$
where $C_I$ is a closed path encircling particle $I$.

Let us now quantize the action (2.33) by imposing equal-time anticommutation relations on the fermionic field $\psi$, namely

$$\left\{ \psi(\vec{x}, t), \psi^\dagger(\vec{y}, t) \right\} = \delta(\vec{x} - \vec{y}) , \quad \left\{ \psi(\vec{x}, t), \psi(\vec{y}, t) \right\} = \left\{ \psi^\dagger(\vec{x}, t), \psi^\dagger(\vec{y}, t) \right\} = 0 . \quad (2.51)$$

Since the Chern-Simons field is a functional of the density $\rho = \psi^\dagger \psi$, the commutation relation relations between $A_\alpha$ and $\psi$ are not trivial, but they can be easily worked out using (2.51), (2.45) and (2.50). Once this is done, it is possible to obtain the first-quantized Hamiltonian that is equivalent to the second-quantized action (2.33). As a result, one finds precisely the Hamiltonian (2.28) with $V = 0$, so that one can say that the system described by (2.33) is a system of fermion-based anyons of statistics $\nu'$ as in (2.32) (for simplicity hereinafter $\nu'$ will be denoted simply by $\nu$). We refer the reader to [7] for the explicit derivation of this result. Here instead we want to point out that there is another way of recognizing that our system (2.33) is actually a system of anyons. In fact, since the Chern-Simons field is formally pure gauge, it can be removed with a (singular) gauge transformation of parameter $\Lambda$, namely

$$A_\alpha \rightarrow A'_\alpha = A_\alpha - \partial_\alpha \Lambda = 0 . \quad (2.52)$$

Under this transformation the matter field $\psi$ transforms according to

$$\psi(x) \rightarrow \psi'(x) = e^{i e_\Lambda(x)} \psi(x) = e^{i \nu \int d^2 y \Theta(\vec{x} - \vec{y}) (\psi^\dagger(y)\psi(y) - \rho_0)} \psi(x) , \quad (2.53)$$

while the covariant derivatives $D_\alpha$ turn into ordinary derivatives $\partial_\alpha$. Thus the action (2.33) becomes simply

$$S' = \int d^3 x \left[ i \psi'^\dagger \partial_0 \psi' + \frac{1}{2m} \psi'^\dagger \Delta \psi' \right] . \quad (2.54)$$

This is a free action. However, it should be realized that the non-trivial effects of the Chern-Simons dynamics are actually hidden in the new matter fields $\psi'$ which do not satisfy any more canonical anticommutation relations like (2.51). In fact, as we can see with a few simple algebraic manipulations using (2.53) and (2.51), we have

$$\psi'(\vec{x}, t) \, \psi'(\vec{y}, t) = -e^{-i\nu \left[ \Theta(\vec{x} - \vec{y}) - \Theta(\vec{y} - \vec{x}) \right]} \psi'(\vec{y}, t) \, \psi'(\vec{x}, t) . \quad (2.55)$$
This is an example of a braiding relation, because the phase factor in the right hand side is different for different relative positions of $\vec{x}$ and $\vec{y}$. To make this statement more precise, let us recall that the multi-valued angle function $\Theta(\vec{x} - \vec{y})$ is unambiguously defined only after one puts a cut in the plane and fixes a reference axis. For example one can choose as a cut the negative $x$-axis and measure the angles starting from the positive $x$-axis. With this choice all angles are in the interval $(-\pi, \pi)$. Then it is not difficult to find that

$$
\Theta(\vec{x} - \vec{y}) - \Theta(\vec{y} - \vec{x}) = \begin{cases} 
\pi \text{sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2, \\
\pi \text{sgn}(x_1 - y_1) & \text{for } x_2 = y_2.
\end{cases}
$$

Using this result in (2.55), it becomes clear that $\psi'$ behaves as an anyonic field. Furthermore, since $\psi'$ depends also on the cut of the angle function, it is a non local operator. It is precisely for this non local character that the exchange relations (2.55) are unambiguous. In fact, if $x_2 > y_2$ (or if $x_2 = y_2$ and $x_1 > y_1$), there is only one way in which $\psi'(\vec{y}, t)$ can move around $\psi'(\vec{x}, t)$ without crossing its cut, namely counterclockwise (see Fig. 1a). In this case, $\Theta(\vec{x} - \vec{y}) - \Theta(\vec{y} - \vec{x}) = \pi$ and (2.55) becomes

$$
\psi'(\vec{x}, t) \; \psi'(\vec{y}, t) = -q^{-1} \; \psi'(\vec{y}, t) \; \psi'(\vec{x}, t),
$$

where

$$
q = e^{i\nu \pi}
$$

On the contrary, if $x_2 < y_2$ (or if $x_2 = y_2$ and $x_1 < y_1$) the field $\psi'(\vec{y}, t)$ must move around $\psi'(\vec{x}, t)$ clockwise in order not to cross its cut (see Fig. 1b). In this case $\Theta(\vec{x} - \vec{y}) - \Theta(\vec{y} - \vec{x}) = -\pi$, and (2.55) becomes

$$
\psi'(\vec{x}, t) \; \psi'(\vec{y}, t) = -q \; \psi'(\vec{y}, t) \; \psi'(\vec{x}, t).
$$

Thus, fixing the cut for the angle function automatically fixes also the orientation of the exchange of two fields $\psi'$ and determines unambiguously their braiding properties. In conclusion, we can say that the gauge transformation (2.53) transmutes fermions into anyons of statistics $\nu$ and so it can be considered as the two-dimensional generalization of the Jordan-Wigner transformation [21], which in one dimension transmutes fermions into bosons.

Before extending further our analysis, a few remarks are in order. First of all, the fact that the transformation from $\psi$ to $\psi'$ in (2.53) involves expressly the angle function $\Theta(\vec{x} - \vec{y})$ is a consequence of fixing the transverse gauge $\partial_i A^i = 0$ on the Chern-Simons field. Different
gauge fixings would lead to continuous deformations of the angle function (still denoted by \( \Theta(\vec{x} - \vec{y}) \)) which maintain the same multivalued properties of angles. However, in computing the exchange relations between two transformed fields (see for instance (2.55)) only one particular combination of these functions always appears, namely \( \Theta(\vec{x} - \vec{y}) - \Theta(\vec{y} - \vec{x}) \). Thus, since the particles created by \( \psi' \) in \( \vec{x} \) and \( \vec{y} \) are indistinguishable, one is led to choose only those gauges for which

\[
\Theta(\vec{x} - \vec{y}) - \Theta(\vec{y} - \vec{x}) = \pm \pi
\]

in such a way that the braiding relations are like (2.57) or (2.59), \textit{i.e.} with only constant phase factors involved. As we have seen, the transverse gauge is one of these gauges (perhaps the most obvious), but of course other choices are possible. We will exploit this freedom in the following sections.

We conclude this review of anyons by recalling that the Chern-Simons construction of fractional statistics and the Jordan-Wigner transformation transmuting fermions into anyons can be realized also if the two-dimensional space is discrete, \textit{i.e.} on a lattice \( \Omega \). For traditional reasons we denote the fermions on the lattice by \( c(\vec{x}) \) and \( c^\dagger(\vec{x}) \) and for simplicity we do not write the time dependence. They satisfy the following standard equal-time anticommutation relations

\[
\{ c(\vec{x}), c^\dagger(\vec{y}) \} = \delta_{\vec{x},\vec{y}} , \quad \{ c(\vec{x}), c(\vec{y}) \} = \{ c^\dagger(\vec{x}), c^\dagger(\vec{y}) \} = 0 ,
\]

(2.60)

where \( \delta_{\vec{x},\vec{y}} \) is the lattice \( \delta \)-function.

If we want to generalize the Jordan-Wigner transformation to the lattice, we must first define the angle function \( \Theta(\vec{x}, \vec{y}) \) for \( \vec{x} \) and \( \vec{y} \in \Omega \). The definition of the lattice angle function requires some care \cite{22,1}, but it is not difficult and can be regarded as a straightforward generalization of the continuum angle function considered so far. The only point that we want to mention here is that the cut that must be fixed to remove any ambiguity in the definition, has to be chosen on a suitably defined dual lattice \( \Omega^* \). (We refer the reader to \cite{1} for a thorough discussion of this issue.) Hence, also the reference point from which the angles are measured is always a point of \( \Omega^* \). For example we can choose as cuts the lines \( \gamma \)’s represented in Fig. 2 for a few points of \( \Omega \). With this choice, the angles are in the interval \( [-\pi, \pi) \) and the corresponding lattice function
\( \Theta(x, y) \) satisfies
\[
\Theta(x, y) - \Theta(y, x) = \begin{cases} 
\pi \text{ sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2, \\
\pi \text{ sgn}(x_1 - y_1) & \text{for } x_2 = y_2.
\end{cases}
\] (2.61)

This relation is very important because it allows to establish an ordering relation on the lattice. In fact, given two distinct points \( \vec{x} \) and \( \vec{y} \) ∈ \( \Omega \), we can posit
\[
\vec{x} > \vec{y} \iff \Theta(x, y) - \Theta(y, x) = \pi,
\]
\[
\vec{x} < \vec{y} \iff \Theta(x, y) - \Theta(y, x) = -\pi.
\] (2.62)

This definition is unambiguous and endows \( \Omega \) with an ordering relation enjoying all the correct properties. This ordering relation will play a crucial role in the following sections when we will discuss the connection between anyons and quantum groups.

After the lattice angle function is defined, we can define the so-called disorder operators \footnote{\cite{23}} according to
\[
K(x) = e^{i \nu \sum_{\vec{y} \neq \vec{x}} \Theta(x, y) (c^\dagger(y)c(y) - \rho_0)}
\] (2.63)

where \( \rho_0 \) is a constant to be fixed later, and prove that
\[
K(x) c(y) = e^{-i \nu \Theta(x, y)} c(y) K(x),
\]
\[
K(x) c^\dagger(y) = e^{i \nu \Theta(x, y)} c^\dagger(y) K(x),
\]
\[
K(x) K(y) = K(y) K(x)
\] (2.64)

for all \( x \) and \( y \) ∈ \( \Omega \). Then, using the disorder operators \( K(x) \), we define the lattice anyons as follows
\[
a(x) = K(x) c(x) = e^{i \nu \sum_{\vec{y} \neq \vec{x}} \Theta(x, y) (c^\dagger(y)c(y) - \rho_0)} c(x).
\] (2.65)

This is the generalized Jordan-Wigner transformation and is nothing but the lattice version of the gauge transformation (2.53). It is rather easy to show that the anyonic operators \( a \) satisfy the following braiding relations
\[
a(x) a(y) + q^{-1} a(y) a(x) = 0,
\]
\[
a(x) a^\dagger(y) + q a^\dagger(y) a(x) = 0,
\]
\[
a^\dagger(x) a^\dagger(y) + q^{-1} a^\dagger(y) a^\dagger(x) = 0,
\]
\[
a^\dagger(x) a(y) + q a(y) a^\dagger(x) = 0
\] (2.66)
for all $\vec{x} > \vec{y}$. We notice that the last two relations are simply the hermitian conjugate of the first two, since $q^* = q^{-1}$. For completeness we also point out that

$$a(\vec{x})^2 = a(\vec{x})^\dagger = 0 ,$$

and

$$a(\vec{x}) a(\vec{x})^\dagger a(\vec{x}) = 1 .$$

Thus, the anyonic operators $a$ obey the Pauli exclusion principle, and at the same point satisfy standard anticommutation relations without any phase factor, like their parent fermions.

We end our discussion by recalling that different choices of the cut and of the reference axis for the angle function lead to different kinds of disorder operators, and hence to different kinds of anyons. For example, instead of the cuts $\gamma$ considered so far, we can choose the cuts $\delta$ along the positive $x$-axis and measure the angles starting from the negative $x$-axis (see Fig. 3). Comparing Figs. 2 and 3, it is clear that $\delta$ and $\gamma$ are related to each other by a sort of parity transformation. We denote by $\tilde{\Theta}(\vec{x},\vec{y})$ the new angle function which is equal to $\Theta(\vec{y},\vec{x})$, and therefore satisfies

$$\tilde{\Theta}(\vec{x},\vec{y}) - \tilde{\Theta}(\vec{y},\vec{x}) = \begin{cases} -\pi & \text{for } \vec{x} > \vec{y} , \\ \pi & \text{for } \vec{x} < \vec{y} , \end{cases}$$

and by $\tilde{a}$ the anyonic operators that can be constructed thereupon, i.e.

$$\tilde{a}(\vec{x}) = e^{i\nu \sum_{\vec{y} \neq \vec{x}} \tilde{\Theta}(\vec{x},\vec{y}) (c(\vec{y}) c(\vec{y})^\dagger - \rho_0)} c(\vec{x}) .$$

The operators $\tilde{a}$ are again fermion-based anyons of statistics $\nu$, and satisfy the braiding relations (2.66) with $q$ replaced by $q^{-1} = q^*$. Indeed, $\tilde{a}$ is the parity transformed of $a$, and thus braids in the opposite way (see Fig. 4 in comparison with Fig. 1). We will use both anyons of type $a$ and anyons of type $\tilde{a}$ in the subsequent sections.

Finally, we observe that the anyonic operators defined in (2.65) and (2.70) have nothing to do with the so-called $q$-oscillators [24], despite some formal analogies. This is so for several reasons. First of all, the $q$-oscillators can be defined in any dimensions and are not related to the braid group, whereas anyons are strictly two-dimensional objects. Secondly, the $q$-oscillators are local operators and are characterized by $q$-commutation relations quite different from (2.68), whilst anyons are intrinsically non-local due to their braiding properties. This non-locality, essential to distinguish whether anyons are exchanged clockwise or anticlockwise, allows to define a natural
ordering among the particles, which in turn will be essential for the realization of the quantum
groups presented in Sections 4 and 5.

3 Deformed Lie Algebras

As we mentioned in the introduction, both anyons and deformed Lie algebras are deeply related
to the braid group, and so it is natural to conjecture the existence of a direct relation between
them. In this section and in the following ones we prove that this relation does indeed exist,
and in particular we show that representations of deformed Lie algebras can be constructed
using the non-local anyonic operators $a$ and $\tilde{a}$ previously defined. In order to illustrate this
fact, we first recall some fundamental properties of deformed Lie algebras and briefly discuss
their relation with the ordinary (i.e. undeformed) ones.

By construction, a deformed Lie algebra is a deformation of an ordinary Lie algebra to which
it reduces when the deformation parameter, usually denoted by $q$, goes to 1. Given an ordinary
Lie algebra $G$ of rank $r$, its $q$-deformation $U_q(G)$ is characterized by generalized commutation
relations which, in the Chevalley basis, are

\[
\begin{align*}
[H_i, H_j] &= 0, \quad (3.1a) \\
[H_i, E^\pm_j] &= \pm a_{ij} E^\pm_j, \quad (3.1b) \\
[E^+_i, E^-_j] &= \delta_{ij} [H_i]_{q_i}, \quad (3.1c) \\
\sum_{\ell=0}^{1-a_{ij}} (-1)^{\ell} \left[ \frac{1-a_{ij}}{\ell} \right]_{q_i} \left( E^+_i \right)^{1-a_{ij}-\ell} E^+_j \left( E^+_i \right)^{\ell} &= 0, \quad (3.1d)
\end{align*}
\]

where $H_i$ ($i = 1, ..., r$) are the generators of the Cartan subalgebra of $G$, $E^\pm_i$ are the step
operators corresponding to the simple root $\alpha_i$, and $a_{ij}$ are the elements the Cartan matrix, i.e.

\[
a_{ij} = \langle \alpha_i, \alpha_j \rangle = \frac{2 \langle \alpha_i, \alpha_j \rangle}{(\alpha_i, \alpha_i)}
\]

(see Tab. 1). In eqs. (3.1) we have used the standard notations

\[
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}},
\]
\[
\begin{align*}
\begin{bmatrix} m \\ n \end{bmatrix}_q &= \frac{[m]_q!}{[m-n]_q! \cdot [n]_q!},
\end{align*}
\]

(3.2)

Furthermore, \( q_i \equiv q^{(\alpha_i, \alpha_i)} \) so that

\[
q_i^a_j = q_j^{a_i}.
\]

To complete the definition of \( \mathcal{U}_q(G) \), we recall that the comultiplication \( \Delta \), the antipode \( S \) and the co-unit \( \epsilon \) are given by

\[
\begin{align*}
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \quad (3.3a) \\
\Delta(E_i^\pm) &= E_i^\pm \otimes q_i^{H_i/2} + q_i^{-H_i/2} \otimes E_i^\pm, \quad (3.3b) \\
S(1) &= 1, \quad S(H_i) = -H_i, \quad (3.3c) \\
S(E_i^\pm) &= -q_i^{H_i/2} E_i^\pm q_i^{-H_i/2}, \quad (3.3d) \\
\epsilon(1) &= 1, \quad \epsilon(H_i) = \epsilon(E_i^\pm) = 0. \quad (3.3e)
\end{align*}
\]

Notice that for complex \( q \) consistency of eqs. (3.3b) and (3.3d) implies that \( E_i^- \) is not the adjoint of \( E_i^+ \), but

\[
E_i^-|_q = \left( E_i^+|_q \right)^\dagger. \quad (3.4)
\]

It is easy to realize that when \( q \to 1 \), eqs. (3.1), (3.3) and (3.4) reduce to those appropriate for an ordinary Lie algebra, thus clearly exhibiting the fact that \( \mathcal{U}_q(G) \) is a deformation of \( G \).

When \( G \) is a classical Lie algebra (i.e. belonging to the \( A, B, C \) and \( D \) series), the connection between deformed and undeformed Lie algebras is even closer [2]. In fact there exists a set of non trivial representations of \( \mathcal{U}_q(G) \) that do not depend on \( q \) and are therefore common both to the deformed and undeformed enveloping algebras [3]. This happens when all the \( SU(2) \) subalgebras relative to the simple roots of \( G \) are in the spin-0 or spin-1/2 representation. We call \( R(0,1/2) \) the set of all representations with this property.

Let us denote by \( h_i \) and \( e_i^\pm \) the generators \( H_i \) and \( E_i^\pm \) in a representation belonging to \( R(0,1/2) \); then the following two properties hold

1. the eigenvalues of \( h_i \) (i.e. the Dynkin labels of any weight) are either 0 or \( \pm 1 \), and

\[\text{Actually this property holds also for } E_6 \text{ and } E_7, \text{ but not for the remaining exceptional algebras. The whole discussion of this section can thus be referred also to } \mathcal{U}_q(E_6) \text{ and } \mathcal{U}_q(E_7).\]
Property 1 implies that 
\[ [h_i]_{q_i} = h_i \]
for any value of \( q \). On the other hand, because of property 2, the deformed Serre relation (3.1d) is identically satisfied for all \( i \) and \( j \) such that \( a_{ij} = -2 \), and simply becomes 
\[ -(q_i + q_i^{-1}) e_i^± e_j^± e_i^± = 0 \]
for all \( i \) and \( j \) such that \( a_{ij} = -1 \). Finally, if \( a_{ij} = 0 \) eq. (3.1d) reduces to 
\[ [e_i^±, e_j^±] = 0 \]
for all values of \( q \).

These facts show that for the representations belonging to \( \mathbb{R}_{(0,1/2)} \) the deformed commutation relations (3.1) are actually independent of the deformation parameter \( q \) and therefore coincide with the undeformed ones.

The set \( \mathbb{R}_{(0,1/2)} \) is not empty. In fact, for any classical Lie algebra \( G \) the fundamental representations (see Fig. 5) certainly belong to \( \mathbb{R}_{(0,1/2)} \) (by fundamental representation we mean an irreducible representation such that any other representation can be constructed from it by taking tensor products, or, equivalently, by repeated use of comultiplication). Thus, the fundamental representation of \( G \) can be interpreted also as a representation of \( \mathbb{U}_q(G) \).

Let us now introduce an ordered set \( \Omega \) whose elements we denote by \( \vec{x} \). Later on we will identify \( \Omega \) with a two-dimensional lattice, but for the moment this interpretation is not needed. To each point \( \vec{x} \in \Omega \) we assign a fundamental representation of \( G \) and denote by \( h_i(\vec{x}) \) and \( e_i^±(\vec{x}) \) the corresponding generators. The previous discussion assures that these local generators satisfy the following generalized commutations relations

\[
[h_i(\vec{x}) , h_j(\vec{y})] = 0 , \tag{3.5a}
\]
\[
[h_i(\vec{x}) , e_j^±(\vec{y})] = \pm \delta(\vec{x},\vec{y}) a_{ij} e_j^±(\vec{x}) , \tag{3.5b}
\]
\[
[e_i^+(\vec{x}) , e_j^-(\vec{y})] = \delta(\vec{x},\vec{y}) \delta_{ij} [h_i(\vec{x})]_{q_i} , \tag{3.5c}
\]
\[
\sum_{\ell=0}^{1-a_{ij}} (-1)\ell \left[ \frac{1-a_{ij}}{\ell} \right]_{q_i} \left( e_i^±(\vec{x}) \right)^{1-a_{ij}-\ell} e_j^±(\vec{x}) \left( e_i^±(\vec{x}) \right)^{\ell} = 0 , \tag{3.5d}
\]
\[
[e_i^±(\vec{x}) , e_j^±(\vec{y})] = 0 \text{ for } \vec{x} \neq \vec{y} . \tag{3.5e}
\]
It should be clear that the relations (3.5) are just formally deformed and are actually independent of $q$ because $h_i(\vec{x})$ and $e_i^\pm(\vec{x})$ are in a representation belonging to $\mathcal{R}(0,1/2)$; however, for our later discussion, it is useful to write them as deformed commutation relations.

Now we can make an iterated use of the coproduct $\Delta$ of $U_q(G)$ as given in (3.3a) and (3.3b), and combine all the local representations to yield the global generators

$$H_i = \sum_{\vec{x} \in \Omega} H_i(\vec{x}) \quad , \quad E_i^\pm = \sum_{\vec{x} \in \Omega} E_i^\pm(\vec{x}) \ ,$$

(3.6)

where

$$H_i(\vec{x}) = h_i(\vec{x}) \ ,$$

(3.7a)

$$E_i^\pm(\vec{x}) = \prod_{\vec{y} < \vec{x}} q_i^{-h_i(\vec{y})/2} e_i^\pm(\vec{x}) \prod_{\vec{z} > \vec{x}} q_i^{h_i(\vec{z})/2} .$$

(3.7b)

The consistency between product and coproduct implies that the generators $H_i$ and $E_i^\pm$ in (3.6) are a “higher spin” representation of $U_q(G)$ and thus satisfy eqs. (3.1), as one can see also with a direct check Moreover, from (3.5b) and (3.7b) it is not difficult to prove that

$$E_i^\pm(\vec{x}) \, E_j^\pm(\vec{y}) = \begin{cases} q_i^{\pm a_{ij}} E_j^\mp(\vec{y}) \, E_i^\pm(\vec{x}) & \text{for } \vec{x} > \vec{y} , \\ q_i^{\mp a_{ij}} E_j^\pm(\vec{y}) \, E_i^\mp(\vec{x}) & \text{for } \vec{x} < \vec{y} . \end{cases}$$

(3.8)

If $\vec{x}$ and $\vec{y}$ are points of a two-dimensional lattice $\Omega$, then eqs. (3.8) can be interpreted as (generalized) braiding relations similar to those satisfied by the anyonic operators defined in Section 2. Moreover, as we have already observed, the set $\Omega$ has to be ordered and anyons can naturally provide an ordering relation. Therefore, it is not unconceivable to conjecture a close connection between anyons and quantum groups. In the next two sections we prove the existence of such a connection by constructing explicitly the generators $H_i(\vec{x})$ and $E_i^\pm(\vec{x})$ in terms of the anyonic operators $a(\vec{x})$ and $\tilde{a}(\vec{x})$.

4 Anyonic Construction of Deformed Classical Lie Algebras

It has been known for a long time that any classical Lie algebra $G$ can be constructed à la Schwinger in a rather straightforward way using fermionic (or bosonic) oscillators. More recently

4Notice that for $q \neq 1$ the coproduct is not cocommutative and for this reason we require that $\Omega$ is ordered.
we have found that replacing fermions with anyons of statistics $\nu$ in the Schwinger construction of $G$, one gets a realization of the deformed algebra $U_q(G)$ with $q = e^{i\pi \nu}$. In this section we present this result for the algebras of the $A$, $B$ and $D$ series, whereas we postpone to the next section the discussion of the algebras of the $C$ series, since these are slightly more complicated.

In the fermionic Schwinger construction of $G$ one makes use of many copies of a set of independent fermionic oscillators satisfying standard anticommutation relations. We label each copy by a vector $\vec{x}$ and assume that the set of all these vectors defines a two-dimensional lattice $\Omega$. For each point $\vec{x} \in \Omega$ we construct the local generators $h_i(\vec{x})$ and $e_i^\pm(\vec{x})$ of $G$, and then we sum these over the whole lattice to get the global ones, namely

$$H_i = \sum_{\vec{x} \in \Omega} h_i(\vec{x}) \quad , \quad E_i^\pm = \sum_{\vec{x} \in \Omega} e_i^\pm(\vec{x}) \quad . \quad (4.1)$$

Notice that these sums correspond simply to a repeated use of the comultiplication of $G$.

Let us now recall some details of this construction. For the algebra $A_r$ one introduces in each point $\vec{x}$ a set of $(r + 1)$ independent fermionic oscillators $c_i(\vec{x})$, and then defines the local generators

\begin{align*}
    h_i(\vec{x}) &= n_i(\vec{x}) - n_{i+1}(\vec{x}) \quad , \quad (4.2a) \\
    e_i^+(\vec{x}) &= c_i^\dagger(\vec{x}) c_{i+1}(\vec{x}) \quad , \quad (4.2b) \\
    e_i^-(\vec{x}) &= c_{i+1}^\dagger(\vec{x}) c_i(\vec{x}) \quad , \quad (4.2c)
\end{align*}

where $i = 1, \ldots, r$, and $n_i(\vec{x}) = c_i^\dagger(\vec{x})c_i(\vec{x})$ is the standard number operator.

For the algebras $B_r$ and $D_r$, one introduces instead $r$ independent fermionic oscillators for each $\vec{x}$. In particular for the algebra $B_r$, $h_i(\vec{x})$ and $e_i^+(\vec{x})$ for $i = 1, \ldots, r - 1$ are as in (4.2), while $h_r(\vec{x})$ and $e_r^+(\vec{x})$ are defined as

\begin{align*}
    h_r(\vec{x}) &= 2n_r(\vec{x}) - 1 \quad , \quad (4.3a) \\
    e_r^+(\vec{x}) &= c_r^\dagger(\vec{x}) S(\vec{x}) \quad , \quad (4.3b) \\
    e_r^-(\vec{x}) &= c_r(\vec{x}) S(\vec{x}) \quad , \quad (4.3c)
\end{align*}

where

$$S(\vec{x}) = \prod_{\vec{y} < \vec{x}} \prod_{i=1}^r (-1)^{n_i(\vec{y})}$$

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is a sign factor that must be introduced to make the generators commute at different points.

For the algebra $D_r$, $h_i(\vec{x})$ and $e_i^\pm(\vec{x})$ for $i = 1, \ldots, r - 1$ are again given by (4.2), whereas $h_r(\vec{x})$ and $e_r^\pm(\vec{x})$ are as follows

$$
\begin{align*}
  h_r(\vec{x}) &= n_{r-1}(\vec{x}) + n_r(\vec{x}) - 1, \\
  e_r^+(\vec{x}) &= c_r^\dagger(\vec{x}) c_{r-1}(\vec{x}), \\
  e_r^-(\vec{x}) &= c_{r-1}(\vec{x}) c_r(\vec{x}).
\end{align*}
$$

(4.4a) (4.4b) (4.4c)

It is a very easy task to check that the generators $h_i(\vec{x})$ and $e_i^\pm(\vec{x})$ defined in this way satisfy the commutation relations (3.5) with the appropriate Cartan matrices and $q = 1$. Moreover, one can easily realize that properties 1 and 2 discussed in Section 3 hold. In fact, due to the fermionic character of the oscillators $c(\vec{x})$, the eigenvalues of the Cartan generators $h_i(\vec{x})$ can only be either 0 or ±1, and all step operators $e_i^\pm(\vec{x})$ have a vanishing square. Thus, the representations of $A_r$, $B_r$, and $D_r$ constructed in this way belong to the set $\Re(0,1/2)$ and are fundamental ones. As mentioned before, all other representations of these algebras can be obtained by a repeated use of the coproduct, that is by summing the local generators over all points of $\Omega$ (see (3.1)).

To obtain the representations of the deformed Lie algebras, one could follow the standard procedure and make use of the comultiplication $\Delta$ defined in (3.3a) and (3.3b) and combine the local representations as in (3.7). However, it is possible to obtain a representation of the deformed Lie algebras also in an alternative way, namely by replacing fermions with anyons in the Schwinger construction. Because of eq. (3.4), both the oscillators $a(\vec{x})$ and $\tilde{a}(\vec{x})$ defined in (2.65) and (2.70) must be used. More precisely, let us define the anyonic generator densities $H_i^\pm(\vec{x})$ and $E_i^\pm(\vec{x})$ as

$$
\begin{align*}
  H_i(\vec{x}) &= N_i(\vec{x}) - N_{i+1}(\vec{x}), \\
  E_i^+(\vec{x}) &= a_i^\dagger(\vec{x}) a_{i+1}(\vec{x}), \\
  E_i^-(\vec{x}) &= \tilde{a}_{i+1}(\vec{x}) \tilde{a}_i(\vec{x}).
\end{align*}
$$

(4.5a) (4.5b) (4.5c)

where

$$
N_i(\vec{x}) = a_i^\dagger(\vec{x}) a_i(\vec{x}) = \tilde{a}_i^\dagger(\vec{x}) \tilde{a}_i(\vec{x}) = n_i(\vec{x})
$$

$i = 1, \ldots, r$ for $A_r$, and $i = 1, \ldots, r - 1$ for $B_r$ and $D_r$. The remaining generators $E_r^\pm(\vec{x})$ and
\( H_r(\vec{x}) \) for \( B_r \) and \( D_r \) are defined respectively as

\[
H_r(\vec{x}) = 2 \, N_r(\vec{x}) - 1 \quad ,
\]

\[
E^+_r(\vec{x}) = a^+_r(\vec{x}) \, S(\vec{x}) \quad ,
\]

\[
E^-_r(\vec{x}) = \tilde{a}_r(\vec{x}) \, S(\vec{x}) \quad ,
\]

and

\[
H_r(\vec{x}) = N_r(\vec{x}) + N_{r-1}(\vec{x}) - 1 \quad ,
\]

\[
E^+_r(\vec{x}) = a^+_r(\vec{x}) \, a^+_r(\vec{x}) \quad ,
\]

\[
E^-_r(\vec{x}) = \tilde{a}_{r-1}(\vec{x}) \, \tilde{a}_r(\vec{x}) \quad .
\]

Using the definitions (2.65) and (2.70) of \( a(\vec{x}) \) and \( \tilde{a}(\vec{x}) \) with \( \rho_0 = \frac{1}{2} \) and their braiding properties \[1\], it is possible to check that

\[
H_i = \sum_{\vec{x} \in \Omega} H_i(\vec{x}) \quad , \quad E^\pm_i = \sum_{\vec{x} \in \Omega} E_i^\pm(\vec{x})
\]

satisfy the generalized commutation relations (3.1) of the quantum algebras \( U_q(A_r) \), \( U_q(B_r) \) and \( U_q(D_r) \) with \( q = e^{i \pi \nu} \). A faster way of proving this is to recognize that, after substituting the definitions (2.65) and (2.70) of \( a(\vec{x}) \) and \( \tilde{a}(\vec{x}) \) into eqs. (4.5a), (4.6a) and (4.7a), the local anyonic generators read as follows

\[
H_i(\vec{x}) = h_i(\vec{x}) \quad ,
\]

\[
E^+_i(\vec{x}) = e^+_i(\vec{x}) \, \prod_{\vec{y} \neq \vec{x}} q^{-\frac{1}{2} \Theta(\vec{x},\vec{y})} \, h_i(\vec{y}) \quad ,
\]

\[
E^-_i(\vec{x}) = e^-_i(\vec{x}) \, \prod_{\vec{y} \neq \vec{x}} q^{\frac{1}{2} \tilde{\Theta}(\vec{x},\vec{y})} \, h_i(\vec{y}) \quad ,
\]

where \( h_i(\vec{x}) \) and \( e^\pm_i(\vec{x}) \) are the generators (4.2a), (4.3a) and (4.4a) in the fundamental representation satisfying both the undeformed and the deformed commutation relations.

Expression (1.9a) for \( H_i(\vec{x}) \) coincides with the one obtained with the coproduct. The same is true also for the operators \( E^\pm_i(\vec{x}) \) of eqs. (1.9b) and (1.9c) if a suitable gauge is chosen. In fact, as mentioned in Section 2, the functions \( \Theta(\vec{x},\vec{y}) \) and \( \tilde{\Theta}(\vec{x},\vec{y}) = \Theta(\vec{y},\vec{x}) \), can be continuously

\[5\]Notice that different kinds of anyons anticommute with each other, e.g. \( a_i(\vec{x}) \, a_j(\vec{y}) + a_j(\vec{y}) \, a_i(\vec{x}) = 0 \) for \( i \neq j \).
deformed by gauge transformations keeping the conditions (2.62) and (2.69). One can therefore choose
\[ \Theta(\vec{x}, \vec{y}) = \pm \frac{\pi}{2} \quad \text{for} \quad \vec{x} \succcurlyeq \vec{y}, \] (4.10a)
and
\[ \tilde{\Theta}(\vec{x}, \vec{y}) = \mp \frac{\pi}{2} \quad \text{for} \quad \vec{x} \succcurlyeq \vec{y}. \] (4.10b)

Then, in this gauge eqs. (4.9b) and (4.9c) coincide with eqs. (3.7), and thus it is clear that the global generators \( H_i \) and \( E_i^\pm \) obtained from the anyonic densities (4.9) satisfy the commutation relations of the deformed algebras \( U_q(A_r) \), \( U_q(B_r) \) and \( U_q(D_r) \).

Even if we have chosen a particular gauge, we want to stress that our result is gauge independent because the braiding relations among anyons and the generalized commutation relations among \( H_i \) and \( E_i^\pm \) depend only on the differences \( (\Theta(\vec{x}, \vec{y}) - \Theta(\vec{y}, \vec{x})) \) and \( (\tilde{\Theta}(\vec{x}, \vec{y}) - \tilde{\Theta}(\vec{y}, \vec{x})) \), and not on the special form of the functions \( \Theta(\vec{x}, \vec{y}) \) and \( \tilde{\Theta}(\vec{x}, \vec{y}) \).

## 5 Anyonic Construction of \( U_q(C_r) \)

The anyonic realization of \( U_q(C_r) \) deserves a special attention because the Schwinger construction of \( C_r \) is more natural in terms of bosonic oscillators and thus involves all representations. On the contrary, the previous discussion shows that our realization of a deformed Lie algebra requires to start from a representation of the undeformed algebra belonging to the set \( \mathcal{R}_{(0, 1/2)} \), which is directly provided by the fermionic Schwinger construction.

To realize the algebra \( C_r \) in terms of fermions, we embed it into the algebra \( A_{2r-1} \) [25], and introduce \( 2r \) fermionic oscillators \( c_\alpha(\vec{x}) \) (\( \alpha = 1, \ldots, 2r \)) for each point \( \vec{x} \in \Omega \). Then, the generators associated to the short roots \( \alpha_i \) of \( C_r \) are

\[
\begin{align*}
    h_i(\vec{x}) &= n_i(\vec{x}) - n_{i+1}(\vec{x}) + n_{2r-i}(\vec{x}) - n_{2r-i+1}(\vec{x}) , \\
    e_i^+(\vec{x}) &= c_i(\vec{x}) c_{i+1}(\vec{x}) + c_{2r-i}(\vec{x}) c_{2r-i+1}(\vec{x}) , \\
    e_i^-(\vec{x}) &= c_i(\vec{x}) c_{i+1}(\vec{x}) + c_{2r-i+1}(\vec{x}) c_{2r-i}(\vec{x})
\end{align*}
\] (5.1a) (5.1b) (5.1c)
for \( i = 1, \ldots, r - 1 \), while the generators corresponding to the long root \( \alpha_r \) of \( C_r \) are

\[
\begin{align*}
  h_r(\vec{x}) &= n_r(\vec{x}) - n_{r+1}(\vec{x}) , \\
  e^+_r(\vec{x}) &= c_r^\dagger(\vec{x}) c_{r+1}(\vec{x}) , \\
  e^-_r(\vec{x}) &= c^\dagger_{r+1}(\vec{x}) c_r(\vec{x}) .
\end{align*}
\]

It is easy to check that the operators \( h_i(\vec{x}) \), \( e^\pm_i(\vec{x}) \) defined in these equations satisfy the commutation relations (3.5) with the Cartan matrix appropriate for \( C_r \) (see Tab.1) and \( q = 1 \).

Actually, in order to select the fundamental representation we have to impose a further condition on the fermionic operators \( c_\alpha(\vec{x}) \), namely we must perform a sort of Gutzwiller projection to force the fermions to satisfy the extra condition

\[
c_\alpha(\vec{x}) c_\beta(\vec{x}) = c^\dagger_\alpha(\vec{x}) c^\dagger_\beta(\vec{x}) = 0
\]

for any \( \alpha, \beta = 1, \ldots, 2r \). In this way, the eigenvalues of \( h_i(\vec{x}) \) are only 0, \( \pm 1 \), and \((e^\pm_i(\vec{x}))^2 = 0\) for \( i = 1, \ldots, r \). Once this is done, the representation given by (5.1) and (5.2) belongs to the class \( \mathcal{R}(0,1/2) \).

We now observe that to obtain \( \mathcal{U}_q(C_r) \) we cannot simply replace in eqs. (5.1) the fermionic oscillators with anyonic ones defined as we defined them in (2.65) and (2.70). In fact, the operators \( E^\pm_i(\vec{x}) \) for \( i \neq r \) constructed in this way could not have the form of eqs. (4.9b) and (4.9c) because the disorder operators in \( a^\dagger_i a_{i+1} \) would yield a different structure from the one contained in \( a^\dagger_{2r-i} a_{2r-i+1} \). This difficulty is simply overcome if we require that the anyons \( a_i \) and \( a_{2r-i+1} \) arise from the fermions \( c_i \) and \( c_{2r-i+1} \) coupled both to the same Chern-Simons field with opposite charges. Therefore the disorder operators to be used in the Jordan-Wigner transformation (2.65) are

\[
K_i(\vec{x}) = K^\dagger_{2r-i+1}(\vec{x}) = \exp\left[ i \nu \sum_{\vec{y} \neq \vec{x}} \Theta(\vec{x}, \vec{y}) (n_i(\vec{y}) - n_{2r-i+1}(\vec{y})) \right].
\]

The same procedure has to be applied also to the oscillators \( \tilde{a}_i \) and \( \tilde{a}_{2r-i+1} \) by changing \( \Theta(\vec{x}, \vec{y}) \) with \( \tilde{\Theta}(\vec{x}, \vec{y}) \) in (5.3). The anyonic oscillators defined in this way have the same generalized commutation relations discussed in Section 2, and also non trivial braiding relations among themselves. For instance, we have

\[
a_i(\vec{x}) a_{2r-i+1}(\vec{y}) + q a_{2r-i+1}(\vec{y}) a_i(\vec{x}) = 0 \quad \text{for} \quad \vec{x} > \vec{y} .
\]
Now we can replace the fermions with the anyons defined in this way into eqs. (5.1) and (5.2), and get

\[ H_i(\vec{x}) = N_i(\vec{x}) - N_{i+1}(\vec{x}) + N_{2r-i}(\vec{x}) - N_{2r-i+1}(\vec{x}) \]

\[ E^+_i(\vec{x}) = a_i^\dagger(\vec{x}) a_{i+1}(\vec{x}) + a_{2r-i}^\dagger(\vec{x}) a_{2r-i+1}(\vec{x}) \]

\[ E^-_i(\vec{x}) = \tilde{a}_{i+1}(\vec{x}) \tilde{a}_i(\vec{x}) + \tilde{a}_{2r-i+1}(\vec{x}) \tilde{a}_{2r-i}(\vec{x}) \]

for \( i = 1, ..., r - 1 \); and

\[ H_r(\vec{x}) = N_r(\vec{x}) - N_{r+1}(\vec{x}) \]

\[ E^+_r(\vec{x}) = a_r^\dagger(\vec{x}) a_{r+1}(\vec{x}) \]

\[ E^-_r(\vec{x}) = \tilde{a}_{r+1}(\vec{x}) \tilde{a}_r(\vec{x}) \]

It is immediate to check that eqs. (4.9) are reproduced with \( q_r = q^2 = e^{2i\pi\nu} \) for the long root, and \( q_i = q \) for the short roots. Then, the discussion of Section 4 guarantees that the operators

\[ H_i = \sum_{\vec{x} \in \Omega} H_i(\vec{x}) \]

\[ E^\pm_i = \sum_{\vec{x} \in \Omega} E^\pm_i(\vec{x}) \]

for \( i = 1, ..., r \) satisfy the generalized commutation relations of the deformed algebra \( U_q(C_r) \).

6 Final Remarks

We have shown that it is possible to establish an explicit relation between anyons and deformed Lie algebras. Our treatment has been limited to the anyonic realization of \( U_q(G) \) for any classical Lie algebra \( G \). In fact, for our construction it is crucial the existence of non trivial representations of \( U_q(G) \) that do not depend on the deformation parameter \( q \) and are therefore common also to \( G \). These representations form a class that we called \( \mathcal{R}_{(0,1/2)} \). In the case of classical Lie algebras these representations coincide with the fundamental ones. This property is shared also by the exceptional algebras \( E_6 \) and \( E_7 \), and thus we believe that also \( U_q(E_6) \) and \( U_q(E_7) \) can be realized in terms of anyons, possibly by introducing a larger number of them.

The situation is instead quite different for the remaining exceptional cases \( U_q(E_8) \), \( U_q(F_4) \) and \( U_q(G_2) \), because the fundamental representations of \( E_8 \), \( F_4 \) and \( G_2 \) are not in the class
\( R_{(0,1/2)} \). Therefore these deformed algebras have no representations independent from \( q \) and this is in contrast with the possibility of building their anyonic realization in the way we have discussed.

A second remark is that our construction naturally makes sense only on a two-dimensional lattice. However, one can envisage an extension to the case in which anyons are defined on a continuum two-dimensional space instead of a lattice. In such a case, one should replace all discrete sums with suitably defined integrals both in the disorder operator and, more generally, in the definition of the comultiplication. At present, this is still an open problem.

On the contrary, it is very easy to “reduce” our procedure to one dimensional chains. On a chain the ordering is natural and the \( q \)-commutation relations like (2.66) can be postulated \emph{apriori}, defining one-dimensional “local anyons”. This amounts simply to replace everywhere the angles \( \Theta(\vec{x},\vec{y}) \) and \( \tilde{\Theta}(\vec{x},\vec{y}) \) with \( \pm \frac{\pi}{2} \) as specified in (4.10a-4.10b).

In such a case it is also possible to assign real values to the deformation parameter \( q \), as in one dimension it is no longer forced to be a pure phase. Our construction is valid also in that case. In fact for real \( q \) all our equations still hold, provided that the creation operators \( a_1^\dagger(\vec{x}) \) and \( a_1^\dagger(\vec{x}) \) are exchanged with each other, leaving unchanged the destruction operators \( a_i(\vec{x}) \) and \( \tilde{a}_i(\vec{x}) \). The case of real \( q \) can be interesting because it leads to unitary representations.

Finally, it is also interesting to notice that our construction has been extended in [26] to the two-parameter deformed Lie algebra \( S\ell_{q,s}(2) \) where the new parameter \( s \) is introduced by rotating the reference axes for the angles \( \Theta(\vec{x},\vec{y}) \) and \( \tilde{\Theta}(\vec{x},\vec{y}) \) in such a way that for generic \( s \) \( \tilde{\Theta}(\vec{x},\vec{y}) \neq \Theta(\vec{y},\vec{x}) \).
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Figure Captions

1. Exchanging trajectories for anyonic fields. In Fig. 1a, the anyon in \( \vec{y} \) must move counterclockwise around the anyon in \( \vec{x} \) in order not to cross its cut. On the contrary in Fig. 1b, the anyon in \( \vec{y} \) must move clockwise around the anyon in \( \vec{x} \) in order not to cross its cut. Therefore, the orientation of the exchange trajectories of two anyons depends on their relative positions.

2. Examples of the cuts \( \gamma \) for a few points on the lattice. The dashed lines represent the reference axes from which the angles \( \Theta \) are measured. Both the cuts and the reference axes are on a suitably defined dual lattice \([1]\).

3. Examples of the cuts \( \delta \) for a few points on the lattice. The dashed lines represent the reference axes from which the angles \( \tilde{\Theta} \) are measured.

4. Exchanging trajectories for anyons defined on the lattice with the angle \( \tilde{\Theta} \). They are oriented in the opposite way as compared with those in Fig. 1.

5. Highest weights of the fundamental representations in the Dynkin bases for the classical Lie algebras and their dimensions.
This figure "fig1-1.png" is available in "png" format from:

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