Exact solitary wave solutions of the nonlinear Schrödinger equation with a source

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Abstract

We use a fractional transformation to connect the traveling wave solutions of the nonlinear Schrödinger equation (NLSE), phase-locked with a source, to the elliptic functions satisfying, $f'' \pm af \pm \lambda f^3 = 0$. The solutions are necessarily of the rational form, containing both trigonometric and hyperbolic types as special cases. Bright, and dark solitons, as also singular solitons, are obtained in suitable range of parameter values.
Much attention has been paid to the study of the externally driven NLSE, after the seminal work of Kaup and Newell [1]. This equation features prominently in the problem of optical pulse propagation in asymmetric, twin-core optical fibers [2, 3, 4], currently an area of active research. Of the several applications of an externally driven NLSE, perhaps the most important ones are to long Josephson junctions [5], charge density waves [6], and the plasmas driven by rf fields [7]. The phenomenon of autoresonance [8], indicating a continuous phase locking between the solutions of NLSE and the driving field has been found to be a key characteristic of this system. In the presence of damping, this dynamical system exhibits rich structure including bifurcation. This is evident from analyses, around a constant background, as well as numerical investigations [9, 10, 11]. Although the NLSE is a well-studied integrable system [12], no exact solutions have so far been found for the NLSE with a source, to the best of the authors’ knowledge. All the above inferences have been drawn through perturbations around solitons and numerical techniques.

In this Letter, we map exactly, the traveling wave solutions of the NLSE phase-locked with a source, to the elliptic functions, through a fractional transformation (FT). It was found that the solutions are necessarily of the rational type, with both the numerator and denominator containing terms quadratic in elliptic functions, in addition to having constant terms. It is well-known that the solitary wave solutions of the NLSE [13, 14] are cnoidal waves, which contain the localized soliton solutions in the limit, when the modulus parameter equals one [15]. Hence, the solutions found here, for the NLSE with a source, are nonperturbative in nature. We find both bright and dark solitons as also singular ones. Solitons and solitary pulses show distinct behavior. In the case, when the source and the solutions are not phase matched, perturbation around these solutions may provide a better starting point.

For nonlinear equations, a number of transformations are well-known in the literature, which map the solutions of a given equation to the other [16, 17]. The familiar example being the Miura transformation [18], which maps the solutions of the modified KdV to those of the KdV equation. To find static and propagating solutions, appropriate transformations have also been cleverly employed, to connect the nonlinear equations to the ones satisfied by the elliptic functions: \( f'' \pm af \pm \lambda f^3 = 0 \). Here and henceforth, prime denotes derivative with respect to the argument of the function. Solitons and solitary wave solutions of KdV, NLSE, and sine-Gordon etc., can be easily obtained in terms of the elliptic functions in this manner.

We consider the NLSE with an external traveling wave source:

\[
\frac{i}{\partial t} \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + g |\psi|^2 \psi + \mu \psi = ke^{i(\chi(\xi) - \omega t)} ,
\]

where \( g, \mu, \) and \( k \) are real and \( \xi = \alpha(x - vt) \). The traveling wave solution, phase-locked with the external source is taken to be \( \psi(x, t) = e^{i\chi(\xi) - \omega t)} \rho(\xi) \). Separating the real and imaginary parts of Eq. (1), and integrating the imaginary part one gets,

\[
\chi' = \frac{v}{2\alpha} + \frac{c}{\alpha \rho^2},
\]

where \( c \) is the integration constant. In order that, the external phase is independent of \( \psi \), we put \( c = 0 \) to obtain,

\[
\alpha^2 \rho'' + g \rho^3 - \epsilon \rho - k = 0,
\]

where \( \epsilon = \omega - \frac{v^2}{4} - \mu \). For illustrating our procedure, we assume that \( \epsilon > 0 \), and \( g > 0 \), the other cases can be studied analogously. After a suitable scaling of the field variable, the
The above equation can be cast into the convenient form,
\[ \rho'' + \rho^3 - \rho - k = 0. \]  \hfill (4)

We find, after a straightforward but lengthy algebra that, the following FT,
\[ \rho(\xi) = \frac{A + Bf^\delta(\xi, m)}{1 + Df^\delta(\xi, m)}, \]  \hfill (5)
for \( AB - D \neq 0 \), maps the solutions of Eq. (4) to the elliptic functions \( f(\xi) \), provided \( \delta = 2 \). \( f(\xi) \) satisfies \( f'' = f - f^3 \), with a conserved quantity \( E_0 = f'^2/2 + (1/4)f^4 - f^2/2 \).
The above equation has four bounded periodic solutions which are Jacobi elliptic functions: \( \text{cn}(\xi, m) \), \( \text{sd}(\xi, m) \), \( \text{dn}(\xi, m) \), and \( \text{nd}(\xi, m) \), where \( m \) is the modulus parameter. Apart from finite energy oscillatory solutions, the above has localized soliton solutions. It should be noted that for the attractive case \( (g < 0) \), the bounded solutions are \( \text{sn}(\xi, m) \), and \( \text{cd}(\xi, m) \).

Before elaborating on specific cases, a number of interesting features emerging from the above mapping is worth mentioning. First of all, the nontrivial solutions are necessarily of the rational type, i.e., \( D \neq 0 \), in the presence of the source. Secondly, \( E_0 = 0 \), and \( E_0 \neq 0 \) cases, show characteristically different behavior. For example, for \( E_0 \neq 0 \) case, one can have solutions with \( A = 0 \), and \( B \neq 0 \), of the type
\[ \rho(\xi) = \frac{(k/4E_0)f^2}{1 + (1/8E_0)f^2}. \]  \hfill (6)
However, \( A \neq 0 \), and \( B = 0 \) is not allowed. For \( E_0 = 0 \) case, one can have for \( B = 0 \), and \( A \neq 0 \), a singular solution of the following form,
\[ \rho(\xi) = \frac{2k}{1 - f^2}. \]  \hfill (7)
As we will see explicitly below, nonsingular solutions are also possible. Like the former case, here \( B \neq 0 \), and \( A = 0 \) is not allowed in the presence of the source.

For explicitness, we consider the unscaled equation containing all the parameters and illustrate below various type of solutions, taking \( f = \text{cn}(\xi, m) \). Other cases can be similarly worked out. The consistency conditions are given by,
\[ A\epsilon - 2\alpha^2(AD - B)(1 - m) + gA^3 - k = 0, \]  \hfill (8)
\[ 2\epsilon AD + \epsilon B + 6\alpha^2(AD - B)D(1 - m) - 4\alpha^2(AD - B)(2m - 1) + 3gA^2B - 3kD = 0, \]  \hfill (9)
\[ A\epsilon D^2 + 2\epsilon BD + 4\alpha^2(AD - B)D(2m - 1) + 6\alpha^2(AD - B)m + 3gAB^2 - 3kD^2 = 0, \]  \hfill (10)
\[ \epsilon BD^2 - 2\alpha^2(AD - B)Dm + gB^3 - kD^3 = 0. \]  \hfill (11)

The above equations clearly indicate that the solutions, for \( m = 1, m = 0 \) and other values of \( m \), have distinct properties. For example, when \( m = 1 \), \( A \) is obtained as the
solution of the cubic equation [Eq. (8)], containing the source strength \( k \). Similarly, for \( m = 0 \), either \( B \), or \( D \) appears as the solution of Eq. (11). As noted earlier, when \( D = 0 \), \( B \) also equals zero, indicating only a constant solution \( A \neq 0 \). One needs to be careful in choosing the real solutions of the above cubic equations, for suitable range of the parameter values. Although a wide class of solutions are allowed; for brevity, we only outline a few of the interesting solutions and their properties in a restricted range of equation parameters.

We start with the localized solitons, by taking \( m = 1 \) \((\text{cn}(\xi, 1) = \text{sech}(\xi))\), with the parameter values \( A = 1, B = \delta, \) and \( D = 1 - \delta \) we obtain

\[
\rho(\xi) = \frac{1 + \delta \text{sech}^2(\xi)}{1 + \Gamma \text{sech}^2(\xi)},
\]

(12)

here the amplitude, width, and velocity are related as, \( \delta = -(Q + \sqrt{Q^2 - 4PR})/2Q \), with \( Q = \epsilon - 2\alpha^2 - 3k \), \( P = 2\epsilon - 3k \), \( R = -k - 2\alpha^2 \), and \( \Gamma = 1 + \delta \). Here the width \( \alpha \) is the only independent parameter. As the above form of the solution indicates, both nonsingular and singular solitons are possible solutions depending on the values of \( \epsilon \), and the source strength \( k \). We have checked that periodic solutions are also allowed. Below we give a few more solutions, illustrating the differences between the localized soliton and the periodic solitary wave solutions.

Case(I): Trigonometric solution.- In the limit \( m = 0 \), unlike the unperturbed NLSE, in this case, one finds rational solutions of the trigonometric type. Apart from the general solutions, interestingly, for these, one can obtain special cases, where \( A = 0 \) and \( B \neq 0 \) is allowed. However, the vice-versa is forbidden. Following is an example of this type of nonsingular periodic solutions, for the repulsive case:

\[
\rho(\xi) = \left( -\frac{2k}{\epsilon} \right) \frac{\cos^2(\xi)}{1 - \frac{3}{2} \cos^2(\xi)}.
\]

(13)

Here, \( \epsilon \) has to be negative since, \( \alpha^2 = -\epsilon/4 \); and is given by \( \epsilon = (-27gk^2/2)^{1/3} \). This periodic solution is found to be stable, as evidenced from the numerical simulations seen in Fig.1.

Case(II): Hyperbolic solution.- Unlike the above periodic case, here one finds that, the solutions with \( B = 0 \), and \( A \neq 0 \) are allowed, the vice-versa not being true. Hence, these localized solitons behave differently from the soliton trains. There are both singular and nonsingular solutions. For \( B = 0 \), and \( m = 1 \), we found that \( \alpha^2 = \epsilon/4 \), and \( \epsilon = (-27gk^2/2)^{1/3} \). This yields, the singular hyperbolic solution,

\[
\rho(\xi) = \left( \frac{3k}{\epsilon} \right) \frac{1}{1 - \frac{3}{2} \text{sech}^2(\xi)}.
\]

(14)

The singularity here may correspond to extreme increase of the field amplitude due to self-focussing, as is known for the other nonlinear systems. We have also found nonsingular solutions of the above type.

Case(III): Pure cnoidal solutions.- In the general case, for \( 0 < m < 1 \), from Eq. (11), one obtains the width \( \alpha \), in terms of the other parameters. The other three equations then yield \( A, B, \) and \( D \); in the process one encounters a cubic equation whose real roots need only to be considered. This puts restriction on the solution parameters. Interestingly, the constraints imply an \( m \) independent condition on the parameters in the form

\[
5\epsilon AD^2 + \epsilon AD^3 + \epsilon BD^2 + 5\epsilon BD^2 + 3gA^2 BD + 3gAD^2 - 6kD^2 - 6kD^3 + 3gA^3 D^2 + 3gB^3 = 0.
\]
Below we list a few special cases. For $A = 0$, $D = 1$, and $m = 5/8$; we found that for $\alpha^2 = (2/7)\epsilon$, and $\epsilon = 7(-gk^2/18)^{1/3}$; this corresponds to attractive case. Explicitly, the solution is given by,

$$\rho(\xi) = \left(\frac{14k}{3\epsilon}\right) \frac{\text{cn}^2(\xi, m)}{1 + \text{cn}^2(\xi, m)}. \quad (15)$$

For $A = 0$, and $m = 1/2$; it is found that $\alpha^2 = \epsilon/2\sqrt{3}$ and $\epsilon = (-27gk^2)^{1/3}$, for which

$$\rho(\xi) = \left(\frac{2\sqrt{3}k}{\epsilon}\right) \frac{\text{cn}^2(\xi, m)}{1 + \frac{1}{\sqrt{3}}\text{cn}^2(\xi, m)}. \quad (16)$$

Since the localized solitons are usually robust, we have performed numerical simulations to check the stability of the solutions pertaining to Case(I), i.e., the trigonometric solution. It is worth pointing out that the numerical techniques based on the fast Fourier transform (FFT) are expensive as they require the FFT of the external source. Hence, we have used
the Crank-Nicholson finite difference method \cite{19} to solve the NLSE with a source, which is quite handy, and unconditionally stable. The initial conditions chosen from the exact solution are knitted on a lattice with a grid size $dx = 0.005$, and $dt = 5.0 \times 10^{-6}$. The evolution of this solution, as depicted in Fig.1 indicate that it is a stable solution.

In conclusion, we have used a fractional transformation to connect the solutions of the phase-locked NLSE with the elliptic functions, in an exact manner. The solutions are necessarily of the rational type that contain solitons, solitary waves, as also singular ones. Our procedure is applicable, both for the attractive and repulsive cases. Because of their exact nature, these will provide a better starting point for the treatment of general externally driven NLSE. Considering the utility of this equation in fiber optics and other branches of physics, these solutions may find practical applications.

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