On the Density Estimation by the super-parametric Method

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ABSTRACT

In this paper, the approach is to study an estimator of distribution free and to design source program which might be useful. This distribution free estimator, super-parametric density estimator, and its related algorithm were suggested (Tsai et al. 2007). Though we will focus on the implementation, the computer programming, of the algorithm and strategies of choosing window functions, the consistency of the of the estimator is studied and the window functions such as B-spline, Bezier spline and piecewise Bezier spline are studied as well. Since the algorithm is designed for solving the optimization of likelihood function, there is a set of nonlinear equations with a large number of variables. The numerical results show that algorithm is very powerful and effective in the sense of mathematics, that is, the iteration procedures converge and the rate of convergence is very fast. Though it is not main purpose to study the consisten- cy of the estimator, the approach in this paper to attain the consistency is straightforward and comprehensive. From the numerical examples, the reader can find how to use this new theory and new methods of density estimation. The fortran source programs are appended in this paper.

1. INTRODUCTION

Though it might not be a nice approach to learn statistics from the application, the authors started to learn statistics from computer science. There are good surveys of density estimation in the textbook (Duda and Hart 1973). The authors studied nonparametric density estimation, Parzen-window, from this textbook and penalized likelihood method from papers (Good and Gaskins 1971,1980). We think that parametric approach is well established method. Therefore, we try to combine the theories and techniques of both nonparametric and parametric approaches. There are two problems must be solved. The first one is the consistency of the estimator and the second one is the nonlinear optimization. Though it is a hard work to encode and to debug a source program, it is worthwhile to try a new theory and a new method. Originally, we designed a fortran source program to test the algorithm of nonlinear mathematical programming and to test the function of splines. Due to the powerful theorem, Bernstein polynomial and Stone-Weierstrass theorem, the results are so good that are beyond our imagination, especially, the continuous case.

2. PARZEN WINDOW AND SUPER-PARAMETRIC ESTIMATOR

In order to solve the second problem, we model the problem by intuition. Let \( \delta \) be Dirac delta function. Let \( f \) be the density function. Clearly, it is that

\[
f(x) = \int \delta(x - t)f(t)dt.
\]

Here, \( f \) will be estimated by observations \( x_1, x_2, x_3, \ldots, x_m \). The integration
is replaced by summation. Let $\tilde{f}$ be the estimator of $f$. Let $\tilde{f}(x) = \sum_{i=1}^{n} c_i \varphi_i(x)$ where $\varphi_i$ are window functions, $0 \leq \varphi_i(x)$, $\int \varphi_i(x) dx = p_i$ and $p_i < \infty$. Let $l = \prod_{j=1}^{m} \tilde{f}(x_j)$ be the likelihood function. Now the problem is to maximize $l$ subjected to the constraints $\sum_{i=1}^{n} p_i c_i = 1$ and $0 \leq c_i$, $i = 1, 2, ..., n$. Mathematically, since $c_i$ are going to be determined, if we redefine $\tilde{f}(x) = \sum_{i=1}^{n} (c_i/p_i) \varphi_i(x)$, then the constraints become $\sum_{i=1}^{n} c_i = 1$ and $0 \leq c_i$, $i = 1, 2, ..., n$. Naturally, this estimator is called super-parametric estimator. We have noticed that most nonparametric documents give the note: If the density function is the (linear) combinations of window functions, then Dirac delta functions shall be obtained when maximum likelihood estimator is applied and hence undesirable roughness will be introduced. We think, by choosing window function carefully, the roughness can be avoided. Before we discuss how to choose the window function, we will quote some results of nonlinear optimizations designed (Tsai et al. 2007).

3. THE ITERATION PROCEDURES

Let $\tilde{f}(x) = \sum_{i=1}^{n} u_i v_i \varphi_i(x)$, where $\varphi_i$ are the window functions. Let $\tilde{l} = \prod_{j=1}^{m} \tilde{f}(x_j)$ be the likelihood function. Now the problem is to maximize $\tilde{l}$ subjected to the constraints

$$\sum_{i=1}^{n} u_i u_i = r. \quad (1)$$
$$\sum_{i=1}^{n} v_i v_i = r. \quad (2)$$

Let $l$ and $\tilde{l}$ be the likelihood functions defined above. Let $\mathcal{A}$ be the set of all $l$. Let $\tilde{\mathcal{A}}$ be the set of all $\tilde{l}$. It is obvious that $\mathcal{A} \subseteq \tilde{\mathcal{A}}$. Therefore, the maximum of $\mathcal{A}$ is less than or equal to that of $\tilde{\mathcal{A}}$. It was shown that the extreme points of $\tilde{l}$ should be located at the points such that $u_i = v_i$, $i = 1, 2, ..., n$. Therefore, the problem to maximize $l$ subjected to its constraint is equivalent to that of maximizing $\tilde{l}$ subjected to the constraints $\{1\}$ and $\{2\}$. Instead of solving the problem directly, the iteration procedures are constructed.

The procedures are:

**Step (i).** Initialize the procedure by setting $k = 1$ and $v_i^k = \sqrt{\frac{1}{n}}$, $i = 1, 2, ..., n$.

**Step (ii).** maximize $\tilde{l}$ subjected to the constraint $\{2\}$. Then values of $v_i^k$, $i = 1, 2, ..., n$, are obtained.
Step (iii). Check the condition \( \sum_{i=1}^{n} u_i^k v_i^k + \varepsilon \geq r \) is satisfied or not, where \( \varepsilon \) is a small positive number for controlling the termination of the procedures. If the condition is satisfied, then stop the iteration procedures and the density estimator, \( \hat{f}(x) = \sum_{i=1}^{n} u_i^k v_i^k \varphi_i(x) \), is obtained. Otherwise, increase the value of \( k \) by one, set \( v_i^k = \theta \sqrt{u_i^{k-1} v_i^{k-1}} \), where \( \theta \) is a chosen constant for satisfying constraint \( B \). Then go to Step (ii) and proceed the procedures.

It is not so easy to complete step (ii), because nonlinear optimization is very complicated usually. Let \( \psi_i(x) = v_i^k \varphi_i(x) \). The simple notation, \( \hat{f}(x) = \sum_{i=1}^{n} u_i \psi_i(x) \) shall be used and the superscript of the symbols \( u_i^k \) and \( v_i^k \) shall be dropped hereafter. Let \( b_{ij} = \psi_i(x_j) \). Let \( D_{ij} = (r/m)(b_i \cdot b_j) \). Let \( u \) and \( b_j \) be \( n \) components vectors, where \( u = [u_1, u_2, \ldots, u_n]^t \) and \( b_j = [b_{1j}, b_{2j}, \ldots, b_{nj}]^t \). Then \( u = (r/m) \sum_{j=1}^{m} b_j \), where \( \sum_{j=1}^{m} \alpha_j b_j = 1 \), \( k = 1, 2, \ldots, m \). And hence step (ii) is executed completely if the values of all \( \alpha_k \) are found. The procedures has been designed and studied by (Tsai et al. 2007), especially, the most complicated step, step (ii), is studied completely. Some of them are listed in the appendix A. Since it has been studied, we will not discuss the details of the procedures in this paper. In order to avoid introducing the roughness, the splines are chosen as the window functions.

4. THE CONSISTENCY OF THE ESTIMATOR

In order to make the approach more comprehensive, we prefer to use the same mathematical notation as elementary calculus. Though the upper case letter \( X_i \) and the lower case letter \( x_i \) are associated with different meanings, we try to use lower case letter as much as possible. If we are going to study the estimation density function which is distribution free, we may assume that the density function, \( f \), be a measurable function defined on its domain. Let \( g_{ni}(x) = 1 \) for some interval \( (a_{ni}, b_{ni}) \) and \( g_{ni}(x) = 0 \) otherwise. Then there is a sequence \( f_n \) such that \( \lim_{n \to \infty} f_n(x) = f(x) \) almost everywhere, where \( f_n(x) = \sum_{i=1}^{m} \theta_{ni}^n g_{ni}(x) \). If we impose some conditions on \( f \), then it might be possible that \( \lim_{n \to \infty} f_n(x) = f(x) \) uniformly. For any \( \varepsilon > 0 \), there is \( f_n \) such that \( \left| f(x) - \hat{f}(x) \right| \leq \left| f(x) - f_n(x) \right| + \left| f_n(x) - \hat{f}(x) \right| \). Here \( \hat{f} \) is an estimator of \( f \). Though \( f \) and \( f_n \) are unknown, we can estimate \( f \) if all \( g_{ni} \) are known. Let \( \hat{f}(x) = \sum_{i=1}^{m} \theta_i^n g_{ni}(x) \). If \( \theta_i \), \( i = 1, 2, \ldots, m \), are parameters which are going to be determined, then this is a simple parametric problem. Roughly speaking, the nonparametric estimator can be transferred to a parametric estimator and hence we call it super-parametric estimator. There are many contributors (Wald 1949) who have proved the consistency of the maximum likelihood estimators. The
only problem that we need to solve is a nonlinear optimization problem. If the density function, \( f \), is continuous on closed interval \([a, b]\), then Bernstein polynomials play an important role in density estimation. Here, we assume that it is well known that there are strong connections among the Bezier spine, Bernstein polynomial and Stone-Weierstrass theorem. Let \( f \) be the density function which is defined and continuous on \([0, 1]\). Let \( \hat{f}(x) = \sum_{i=1}^{n} \theta_i \varphi_i(x) \) be super-parametric estimator. Let \( B_n(x) = \sum_{i=0}^{n} f(x_i) C_n^i x^i (1-x)^{n-i} \) be Bernstein polynomial which is associated with the density function \( f \), where \( C_n^i = \frac{n!}{i!(n-i)!} \) and \( x_i = i/n \). Let \( \varphi_i(x) = N_i C_n^i x^i (1-x)^{n-i} \), here \( N_i \) is a constant to make \( \int \varphi_i(x) dx = 1 \). Let \( \theta^0_i = f(x_i)/N_i \). From the property of Bernstein polynomial, we have

\[
|f(x) - \hat{f}(x)| \leq |f(x) - \sum_{i=0}^{n} \theta^0_i \varphi_i(x)| + \sum_{i=0}^{n} |\theta^0_i - \theta_i| |\varphi_i(x)| \quad (3)
\]

Since the first term in right hand side of the inequality can be handled, it seems that the problem of density estimation becomes a problem of curve fitting. Of course, it is not so simple actually.

In order to make our approach more comprehensive, we use the advantages of mathematical notations. Let

\[
S_n = \{ g; g(x) = \sum_{i=0}^{n} \theta_i \varphi_i(x), \sum_{i=0}^{n} \theta_i = 1, \theta_i \geq 0, x \in D \}.
\]

Here \( \varphi_i \) is a normalized nonnegative function, for example, the window function obtained from Bernstein polynomials and \( D \) is the domain of functions. Let \( f \) be the density function which is going to be estimated by a set of samples. If \( f \in S_n \), we are so lucky, then the consistency of super-parametric estimator had been proved (Wald 1949). Generally speaking, it is impossible to infer the uncountable information of a density function by a finite set of samples without sufficient assumptions and strong intuition. It should be allowed to approximate the density function by all means. Let \( X_1, X_2, \ldots, X_m \) be independent identically distributed from a distribution \( F \) of which the density is \( f \). Let \( G \) be another distribution of which the density is \( g \) and \( f \neq g \). Let \( A_{\text{sup}} = \{ x; f(x)g(x) \neq 0 \} \). For simplicity, we assume that the integral is taken on \( A_{\text{sup}} \). It is obvious that

\[
\int \log\left(\frac{g}{f}\right) dF < \log\left(\int \frac{g}{f} dF\right) = 0. \quad (4)
\]

since the second derivative of \(-\log\) is positive and hence \(-\log\) is convex. From the law of large number, if the number \( m \) is large enough, then we have

\[
\frac{1}{m} \sum_{i=1}^{m} \log g(X_i) < \frac{1}{m} \sum_{i=1}^{m} \log f(X_i). \quad (5)
\]
From (5), intuitively, it is a reasonable approach to approximate \( f \) by \( g_0 \) if 
\[ l_0 = \prod_{j=1}^{m} g_0(x_j) \text{ and } l_0 \text{ is the maximum of } L, \]
where \( L = \{ l; l = \prod_{i=1}^{n} g(x_i), g \in S_n \} \).
The existences of \( g_0 \) and \( l_0 \) are doubtless since the set of \( x_i \) are known and hence \( l \) is a continuous function defined on compact set \( \Omega \), 
\[ \Omega = \{ (\theta_1, \theta_2, ..., \theta_n); \theta_i \geq 0, \sum_{i=1}^{n} \theta_i = 1 \}. \]
We will not discuss the details here. The further study is discussed in appendix B.

5. B-SPLINE AND BEZIER SPLINE

Though normal distribution is a good candidate for window function (Duda and Hart 1973), we try to use the splines as window function in this paper. Without the Taylor’s series, most useful function might not be useful. Without the power series, there would be no special functions which are used in classic physics and quantum mechanics. In the practical problem, the power series shall be replaced by polynomial. Spline is a synonym of polynomial. There were many splines which were developed in last century (Newman and Sproull 1979, Quarteroni Sacco and Saleri 2000). B-spline is one of the most useful splines. In computer graphic, we will call them blending functions instead of window functions. According to the degree of blending polynomial functions, the blending function is denoted by the symbol \( N_{i,k} \) and defined as follows:

\[
N_{i,1}(x) = 1 \text{ if } t_i \leq x < t_{i+1},
\]
\[
N_{i,1}(x) = 0 \text{ otherwise. We define them recursively,}
\]
\[
N_{i,k}(x) = \frac{(x-t_i)N_{i,k-1}(x)}{t_{i+k-1} - t_i} + \frac{(t_{i+k} - x)N_{i+1,k-1}(x)}{t_{i+k} - t_{i+1}}.
\]

Here, we have the convention, \( 0/0 = 0 \), and the set of knot values \( t_i \),
\[ i = 0, 1, 2, ..., n + k, \] are defined
\( t_i = x_0 \) if \( i < k \).
\( t_i = x_{i-k+1} \) if \( k \leq i \leq n \).
\( t_i = x_{n-k+2} \) if \( i > n \).
The estimator shall be defined the linear combinations of the blending functions, that is \( \hat{f}(x) = \sum_{i=0}^{n} c_i N_{i,k}(x) \), where \( k \) will be chosen for controlling the order of continuity. Indeed, we can find that they are the simple window functions used in Parzen approach when \( k = 1 \), though they are not centrally located. In order to use the B-spine, the observations should be reordered such that \( x_0 \leq x_1, ..., \leq x_m \). In B-spline method, we put \( m = n \).

Bezier spines are much simpler than B-spines because there are no extra knot points in Bezier spline. We have listed them already and they are
\[
B_{i,n}(x) = \frac{n!}{i!(n-i)!} x^i (1-x)^{n-i}, i = 0, 1, ..., n, \quad 0 \leq x \leq 1.
\]
Since \( x \) could be defined on the specified interval, the scaling must be done for individual problem at hand. It should be emphasized that the algorithm (sup-
porting document 2007) requires that each observation data \( x_j \) there must be some \( \varphi_i \) such that \( \varphi_i(x_j) > 0 \). If the Gauss distributions, long tail distributions, are chosen to be the window functions, then the requirement is satisfied automatically. Since we are in favor of splines, the requirement makes the computer programming more complicated, especially, in B-spline. In many applications, B-spline has more advantages than Bezier spline. Therefore, we started encoding the program for choosing as the window functions the B-spline. In most documents, B-splines are defined on a parameter or parameters, more precisely, B-spline curves and surfaces are define by one and two parameters respectively. All the information of the B-spline that we get is the B-spline of low order with uniform spaced knot points. In this paper, the knot points are the observations. Of course, the set of observation shall be sorted and assigned to be the knot points. Unlike the Parzen-window functions, the sizes and the shapes of the window functions are different from one to another. And this makes some difficulties, for example, the area of some window functions might be zero or very small numbers and hence it is impossible to normalize the window functions in numerical computation. These difficulties can be handled by symbolic computation of integral by using some computer software such as Mathematica, Maple etc. From the figures, Figure 1 and Figure 2, we can find some effects of non-equalized window functions, especially at two end points. Therefore, the interval on which the windows are defined is extended in both end points.

6. THE PIECEWISE BEZIER SPLINE

Bezier spline with order 10, more or less, is good enough to estimate any continuous unimodal density function since it has 11 parameters to control the curve. If it is necessary, then the order can be increased to 30. Classification plays an important role in many application fields, classification by the features of different species. Samples of pattern classification are not obtained from a single distribution. If the density is well-defined, then it might the mixture of different densities, so to speak. Naturally, the piecewise Bezier spline is a nice candidate for estimator. Moreover, the consistency we have discussed is the case of continuous density function. If the case we study is not continuous, then we should modify it to fit the real case. In order to implement the piecewise Bezier spline, the domain of distribution shall be partitioned into subdomains if it is necessary. After the domain having been partitioned, the estimator is the sum of Bezier splines which are defined on the subdomains. Since the data, the samples, is the set of finite elements, the problem is to design a simple computer program to partition domain into subdomains. The algorithm is based on searching for tails in the middle. In order to collaborate with fortran source program, we use the array notation instead of subscript index. The procedures are:

(P1) Sort the samples to obtain the ordered samples, say, \( x(i), i = 0, 1, 2, ..., m \).
(P2) Compute the (random) intervals \( t(i), t(i) = x(i + 1) - x(i) \).
(P3) Find the maximum of \( t(i) \), say \( t(i_0) \).
(P4) Test the condition for partitioning the domain into two subdomains. Here, we assume that the sample size \( m = 180 \). The condition is \( m/6 < i_0 < 5m/6 \) and \( 30 < i_0 < (m - 30) \). If the condition is satis-
fied, then the interval \([x(i_0), x(i_0 + 1)]\) is removed and the domain is partitioned into subdomains, \([x(0), x(i_0)]\) and \([x(i_0 + 1), x(m)]\) and hence the set of \(t(i)\), not the samples, is divided into two subsets, set \(t(i_0) = \text{indicator}\). In order to make the program workable, the location of must be stored in an array, say \(lo(ik)\), in the fortran program. And the value of \(t(i_0)\) is set to be indicator, say, \(t(i_0) = -1\). Otherwise, that is, \(i_0 \geq (5m)/6\) or \(i_0 \leq m/6\), the value of \(t(i_0)\) is set to be zero. The setting must be done to avoid being reselected. (P5) Test the condition for terminate the algorithm. The condition is all the length between two adjacent indicators is less than, say, \(m/6\). If the condition is satisfied, then terminate the algorithm. Otherwise, go to procedure (P3). After the algorithm being executed, the domain may be one piece interval or partitioned into several disconnected subintervals.

7. NUMERICAL EXAMPLES

In this paper, there are three numerical examples: Example 1 is an unimodal distribution and the probability density function is \(\exp(-x)\). Example 2 is bimodal and the probability density function is defined on \([0, 4]\), \(f(x) = 2/3\) when \(1 \leq x \leq 2\); \(f(x) = 1/3\) when \(3 \leq x \leq 4\); otherwise \(f(x) = 0\). Example 3 is a trimodal distribution and the probability density function is defined on \([0, 4]\), \(f(x) = 1\) when \(0 \leq x \leq 1/2\); \(f(x) = 1/2\) when \(1 \leq x \leq 3/2\); \(f(x) = 1/2\) when \(3 \leq x \leq 7/2\); otherwise \(f(x) = 0\). Before starting to design the fortran source program for testing the formulation of this paper, we should study the character of B-spline and Bezier spline. Figure 1 to Figure 6 are the numerical results of this paper. There are three methods, B-spline with order of continuity 12, Bezier spline and piecewise Bezier spline. It is meaningless to use large number of windows and hence the number of windows is reduced to 11 when the sample size is 30.

Though the consistency that we have studied is the distribution with continuous density function, the numerical examples we apply are not confined in the continuous density function. Therefore, some figures are not good enough. But they are acceptable.

8. DISCUSSION AND CONCLUSION

Comparing to the existent results (Dong and Wetes 200, Good and Gaskins 1971, Parzen 1962 and Rosenblatt 1956), the super-parametric approach is a method with potential. In the continuous case, it seems that the concavity of estimator, Bezier spline with order 10, is almost the same as that of the density function. So far, we think it might be a coincidence since the concavity of a function concerns with the second derivative of the function and the likelihood function has nothing concerning the derivative of any function. Roughly speaking, the order of Bezier spline should be less than 10 otherwise it will violate the spirits the piecewise polynomials. Due the high degree of global polynomial, the adjoin properties, oscillations will be introduced. This is a drawback of global polynomial. If global polynomial works well, then it is not necessary
to design new splines such as cubic spline, B-spline etc., since we have had Lagrange polynomial. In order to apply Stone-Weierstrass theorem, the piecewise Bezier spline shall be adopted. Though Bezier spline can be jointed by pieces in computer aid design, the joint of Bezier spline in density estimator should be carefully treated because the window function in both ends have only one side tails. This will introduce the biases of samples implicitly. Therefore, we hope that some new flexible spline should be designed and studied.

In this paper, we focus on the splines instead of general super-parametric approach. Though there might be some better window functions, it seems that Bezier spline is a nice candidate for window functions of the super-parametric estimators. The programs were designed for testing. Therefore, they are designed by bottom up and hence they are not readable. After having been tested, some documents were inserted in the programs and they become readable. Therefore, we decide to attach the source programs in this paper. It is very easy or trivial to generalize this approach to multivariate distributions. In multivariate distributions, the coordinates of samples shall be transformed to principle direction axis. This can be done by diagonalizing the covariance matrix. After the transformation, the new coordinates of samples are projected into the axis accordingly and hence spline of higher dimension can be constructed by taking the product of one dimension spline on each axis (Newman and Sproull 1979).

If all the shape of window functions are the same and each one window covers only one sample, then we get the same result as Parzen’s approach and hence the consistency has been proved. From the figures, we find that B-spline estimator is not so good as the others. It must be confessed that we do not use B-spline appropriately since we just assign the samples to knot points. If we should choose the knot points carefully the results might be better. Due to some reason, the roughness of estimator, we do not try to improve choosing knot points. Though the results obtained from B-spline is not good enough it might be useful if we follow the approach of Bayes. Like Bayes learning, we may consider uniform distribution as priori density, the different procedures which we use are processes of learning and the results obtained from the piecewise Bezier as posteriori density.

APPENDIX A

In order to solve the nonlinear equations \[ \sum_{j=1}^{m} D_{kj} \alpha_k \alpha_j = 1, \quad k = 1, 2, ..., m, \] the iteration procedures are constructed. First, initialize the procedure by setting \[ \alpha_k = \sqrt{\frac{1}{(\overline{D} m)}} \] where \( \overline{D} \) is the maximum of \( D_{ij} \). Then start the iteration procedures:

**Step (a).** Compute \( E_i = \left| \sum_{j=1}^{m} D_{ij} \alpha_i \alpha_j - 1 \right|, \quad i = 1, 2, ..., m \), and \( E = \sum_{i=1}^{m} E_i \).

**Step (b).** Test the condition whether \( E \leq \delta \) is satisfied or not, where \( \delta \) is a small positive number for terminating the procedures. If \( E \leq \delta \), then the
desired results are obtained. Compute \( u_k \) by the identity
\( u_k = (r/m) \sum_{j=1}^{m} \alpha_j b_{kj}, \)
k = 1, 2, ..., n. And stop the iteration. Otherwise,

**Step (c),** Find the largest element of the set of all \( E_k \). Suppose that the
largest element is \( E_k \) for some \( k \). Eliminate \( E_k \) by updating the value of \( \alpha_k \) by
\( \alpha'_k, \alpha'_k = (-s + \sqrt{s^2 + 4D_{kk}})/2D_{kk} . \) And go to Step (a).

**APPENDIX B**

Let \( S_n = \{ g : g(x) = \sum_{i=0}^{n} \theta_i \varphi_i(x), \sum_{i=0}^{n} \theta_i = 1, \theta_i \geq 0, x \in D \} \). Let \( \varphi_i \) be
bounded, that is, \( |\varphi_i(x)| < M \). Though it is impossible to get the information of \( f \) completely, we may assume that there is a function \( g, g \in S_n \) such
that \( |f(x) - g(x)| < \varepsilon \). From (3) we have,

\[
\int |f(x) - \hat{f}(x)|dF \leq \int |f(x) - g(x)|dF + \int |g(x) - \hat{f}(x)|dF
\]

\[
\int |f(x) - \hat{f}(x)|dF \leq \int |f(x) - g(x)|dF + \int |g(x) - \hat{f}(x)|(dF + dG - dG)
\]

\[
\int |f(x) - \hat{f}(x)|dF \leq \varepsilon \int dF + \int |g(x) - \hat{f}(x)|dG + \int |g(x) - \hat{f}(x)||f(x) - g(x)|dx
\]

\[
\int |f(x) - \hat{f}(x)|dF \leq \varepsilon \int dF + \int |g(x) - \hat{f}(x)|dG + \varepsilon \int |g(x) - \hat{f}(x)|dx
\]

\[
\int |f(x) - \hat{f}(x)|dF \leq \varepsilon \int dF + \int |g(x) - \hat{f}(x)|dG + \varepsilon \int (|g(x)| + |\hat{f}(x)|)dx
\]

\[
\int |f(x) - \hat{f}(x)|dF \leq \varepsilon \int dF + \int |g(x) - \hat{f}(x)|dG + \varepsilon (\int d\hat{F} + \int dG)
\]

\[
\int |f(x) - \hat{f}(x)|dF \leq 3\varepsilon + \int \sum_{i=0}^{n} |\theta_i^0 - \theta_i| |\varphi_i(x)|dG
\]

\[
\int |f(x) - \hat{f}(x)|dF \leq 3\varepsilon + M \sum_{i=0}^{n} |\theta_i^0 - \theta_i|dG.
\]

From (6), we are able to use the samples to estimate \( \theta_i^0 \) though these samples are obtained from the distribution \( F \) instead of \( G \). Here, we assume that \( \varepsilon \) is very small. There is still a problem since the existence of \( g, |f(x) - g(x)| < \varepsilon, \)
is not unique. We think that it is not a real problem because the problem
should be transferred to a practical problem, the global maximum of likelihood function. The problem has been studied partially (Tsai et al. 2007).

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Figure 1: In this figure, B-spline estimator is adopted to estimate the unimodal distribution. The sample size is 180.
Figure 2: In this figure, the Bezier spline estimator is adopted to estimate the unimodal distribution. The sample size is 180.
Figure 3: In this figure, the B-spline estimator is adopted to estimate the bimodal distribution. The sample size is 180.
Figure 4: In this figure, the Bezier-spline estimator is adopted to estimate the bimodal distribution. The sample size is 180.
Figure 5: In this figure, the B-spline estimator is adopted to estimate the trimodal distribution. The sample size is 180.
Figure 6: In this figure, the piecewise Bezier spline estimator is adopted to estimate the trimodal distribution. The sample size is 180.