Lee-Yang zeroes in the one flavour massive lattice Schwinger model

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Abstract

We study the partition function of the model formulated with Wilson fermions with only one species, both analytically and numerically. At strong coupling we construct the solution for lattice size up to $8 \times 8$, a polynomial in the hopping parameter up to $O(\kappa^{128})$. At $\beta > 0$ we evaluate the expectation value of the fermion determinant for complex values of $\kappa$. From the Lee-Yang zeroes we find support for the existence of a line of phase transitions from $(\beta = 0, \kappa \simeq 0.38)$ up to $(\beta = \infty, \kappa = 1/4)$. 
1 Introduction

QED$_2$, the theory of electrons and photons in 2D, for massless electrons is analytically solvable [1] and has been studied extensively [2]. In the original version the system has $n_f = 1$ fermions. The fermions are confined and the model is equivalent to a system of non-interacting bosons. The theory is superrenormalizable and there is only finite renormalization of the charge. Chiral symmetry is broken due to an anomaly in the axial current thus there is no Goldstone boson, but a massive pseudoscalar of Gaussian nature. One has charge shielding, i.e. there are no long-range forces between static charges. Allowing for $n_f > 1$ fermions, one does have additional massless states [3], a situation prototyped in the $O(n_f)$ non-linear $\sigma$-model.

Many of these properties are intriguingly similar to those of QCD in 4D, which in the non-perturbative domain is studied mainly in the lattice regularization by computer simulations. The Schwinger model formulated on a lattice is therefore a challenging 2D model for lattice QCD$_4$. In this formulation it naturally encompasses the massless and the massive situation. In the continuum the massive Schwinger model cannot be solved explicitly although there are results perturbative in $m/e$ [4, 5, 6]. From these we expect a situation similar to the massless case with further bound states and quark trapping, again very much like in QCD$_4$.

Lattice formulations of fermions are plagued with the doubling problem. There are more than one pole in the Brillouin zone of the momentum space propagator. In the staggered (Kogut-Susskind) formulation one distributes different components of the spinor on different sites of a hypercube and thereby effectively reduces the $2^D$ multiplicity by a factor $2^{D/2}$. In the Wilson formulation the doubler modes are given masses of $O(1/a)$ and one expects decoupling in the continuum limit $a \to 0$. Furthermore most lattice calculations rely on the hybrid Monte Carlo method to implement the fermions. In this process, however, another doubling of fermion modes is introduced in order to deal with the positive definite square of the determinant instead of the determinant itself. The only known way to avoid this doubling is to include the determinant in the observables and not use it as a probability weight. Thus, in the standard lattice formulation one deals with an $n_f > 1$ Schwinger model. Recently some exact results became available for the continuum massless $n_f > 1$ Schwinger model [3], thus the situation in respect to the flavour problem of lattice versions has somewhat improved.
The naive and the staggered formulation has a simple phase diagram in the \((\beta, m)\)-plane, where \(\beta = 1/e^2\) is the gauge coupling and \(m\) denotes the bare fermion mass parameter. The continuum limit is recovered approaching the point \((\infty, 0)\) along trajectories with \(m \approx 1/\sqrt{\beta}\), where the proportionality constant characterizes the physical mass of the bound state boson. In that limit the known condensate value \(\frac{1}{\bar{\epsilon}} \langle \bar{\psi} \psi \rangle\) should be obtained. Most of the lattice calculations worked in that formulation and reproduced the expected continuum results of the massless Schwinger model successfully (see e.g. \cite{7, 6, 8}, for recent work on the non-compact formulation cf. \cite{9}).

Little is known for the Wilson formulation, though. Here the two bare coupling parameters are the gauge coupling and the hopping parameter \(\kappa\) related to the bare fermion mass. Chiral symmetry is explicitly violated for finite \(\beta\) and most likely restored in spontaneously broken form in the continuum limit. This deprives us of a suitable order parameter. Also, for the \(n_f = 1\) model, there is no massless Goldstone boson. A continuum limit should be obtained in the approach to \((\beta = \infty, \kappa = 1/4)\) along trajectories following fixed values of physical (dimensionless) quantities. In QCD\(_4\) there is a line \(\kappa_c(\beta)\) where the pion mass vanishes, corresponding to the situation of a vanishing bare fermion mass (due to the Wilson term the bare quark mass is renormalized additively). In the Schwinger model we may expect a similar behaviour and in this case the massless continuum Schwinger model is obtained following such a curve \(\kappa = \kappa_c(\beta)\).

Despite good knowledge of the properties of the theory in the continuum, it is still a challenge to clarify the \(n_f = 1\) model phase diagram on the lattice in the Wilson formulation. One cannot use \(\langle \bar{\psi} \psi \rangle\) as an order parameter nor can we determine the mass of the boson state in a direct simulation. In this paper we therefore try to contribute to this issue in an analytic (at \(\beta = 0\) and \(\infty\)) study and a computer calculation (at \(\beta = 1\) and 5) of the partition function itself. With help of the equivalence of the strong coupling model to the 7-vertex model \cite{10, 11} we construct the hopping expansion up to \(O(\kappa^{128})\). At non-zero \(\beta\) we directly determine the fermion determinant in the numeric integration over the gauge field background. The scaling of the Lee-Yang zeroes indicates that, somewhat contrary to an earlier finding \cite{11}, there is indeed a phase transition line for all positive values of the gauge coupling \(\beta\). A subset of the numeric results has been presented elsewhere \cite{12}.
2 Action, partition function and its zeroes

The action for the massive lattice Schwinger model in the Wilson representation is given by

\[ S(\kappa, \beta) = S_F(\kappa) + \beta S_G, \quad (1) \]

\[ S_F(\kappa) = \sum_{x \in \Lambda} \left( \kappa \sum_{\mu} (\bar{\psi}(x + \hat{\mu})(1 + \gamma_\mu)U_\mu^\dagger(x)\psi(x) + \bar{\psi}(x)(1 - \gamma_\mu)U_\mu(x)\psi(x + \hat{\mu}) - \bar{\psi}(x)\psi(x) \right) \]

\[ \equiv \sum_{x, x' \in \Lambda} \bar{\psi}(x)[M_\Lambda(\kappa, U)]_{x, x'}\psi(x'). \quad (2) \]

For the gauge fields \( S_G \) is the standard Wilson action with the lattice gauge coupling \( \beta \) and \( M_\Lambda(\kappa, U) \) denotes the Dirac operator on the lattice \( \Lambda \). Eq.s (1) and (2) are formal expressions in the partition function of the model, which, after Grassmann ‘integration’, is given by

\[ Z_\Lambda(\kappa, \beta) = \int d\mu(U) \det M_\Lambda(\kappa, U)e^{-\beta S_G(U)} \]

\[ \equiv Z_{G,\Lambda}(\beta) \langle \det M_\Lambda(\kappa, U) \rangle_G. \quad (3) \]

\( Z_{G,\Lambda}(\beta) \) is the partition function of the purely bosonic system and \( \langle \mathcal{O}(U) \rangle_G \) the expectation value of some operator for that system. Without loss of generality we normalize \( Z_{G,\Lambda}(0) = 1 \). The partition function \( Z_\Lambda(\kappa) \) is a polynomial of degree \( 2|\Lambda| \) in \( \kappa \) and thus an entire function. All coefficients of that polynomial are positive. We know that \( \det M_\Lambda \) is strictly positive for \( \kappa(\beta) < \kappa_c(\beta = \infty) = 1/4 \); above this value of \( \kappa \) we checked explicitly that individual configurations do produce negative values although the partition function remains positive for finite lattices.

The zeroes of that polynomial, called Lee-Yang zeroes \[13\], have a non-vanishing imaginary part for all finite \( |\Lambda| \). In the thermodynamic limit these zeroes pinch the real axis of the complex \( \kappa \) plane and define thereby the point of non-analyticity at \( \kappa_c \). How the zeroes approach the real axis is governed by finite size scaling, at a critical point related to the critical exponents of the system \[14\].

From (4) we see that the partition function of the full model is proportional to the purely bosonic expectation value of the determinant of the
lattice Dirac operator. Since \( Z_{G_A}(\beta) > 0 \) we find that \( Z_A(\kappa, \beta) \) is zero only where \( \langle \det M_A(\kappa, U) \rangle_G = 0 \). Whereas mass gap calculations cannot be performed for one flavour, this operator can be calculated and this calculation can be extended to any number of flavours.

For given \( \beta \) we have to determine the expectation value for the determinant for complex \( \kappa \in \mathbb{C} \). Only at \( \beta = 0 \) and \( \beta = \infty \) this may be done analytically. Elsewhere one may rely on the following numerical methods.

**analytic continuation:** One obtains values of \( \langle \det M_A \rangle \) for various real \( \kappa \). Performing a polynomial fit and using this fit one analytically continues to complex \( \kappa \). In a test for the controllable situation of free fermions we find unstable results; even the closest zeroes could not be identified reliably. A better alternative is to obtain the coefficients of the polynomial expressed in terms of moments by direct numerical simulation. This approach has been successfully used in a recent study in 4D [15]. The crucial point is the convergence of the coefficients with \( n \). This approach is closest related to the quite successful method of analytic extrapolation and determination of zeroes via histograms [16].

**direct evaluation:** One integrates over the gauge fields with the usual Monte Carlo simulation and produces gauge configurations with the standard bosonic measure \( e^{-\beta S_G(U)} d\mu(U) \). For each configuration one determines \( \det M_A \) for a sufficiently dense set of complex \( \kappa \) in the expected region of the closest zero. The gauge field average eventually produces the required results. This method is applicable for small lattices only, since one has to determine the determinant for each gauge field configuration explicitly. On the other hand it does not involve any extrapolation and works for arbitrary number of flavours.

### 3 Analytic results

#### 3.1 Strong coupling limit (\( \beta = 0 \))

For small lattice size (e.g. \( 2 \times 2 \)) one may analytically solve

\[
\det M_A(\kappa, U) = \sum_{n=0}^{2|A|} c_n(U)\kappa^n
\]  

(5)
and integrate over the gauge fields explicitly to obtain

\[ Z_\Lambda(\kappa) = \langle \det M_\Lambda(\kappa, U) \rangle_G \]  

as a polynomial in \( \kappa \). Although done with symbolic programs this becomes prohibitive for larger lattices.

Another method is based on a map of the massive lattice Schwinger model (at \( \beta = 0 \)) on the eight-vertex model \([10]\). Integrating over the gauge fields first one finds a theory of non-intersecting loops represented by a vertex model,

\[ Z_{8V,\Lambda}(M) = \sum_{\{n_i\}} \prod_{i=1}^{8} \alpha_i^{n_i}, \]  

where \( \alpha_i \) denotes the coupling for vertex type \( i \) and \( n_i \) the multiplicity. The Schwinger model corresponds to \( \alpha_1 = M^2, \alpha_2 = 0, \alpha_3 = a_4 = 1, \alpha_{i>4} = \frac{1}{2} \) and \( M = 1/(2\kappa) \). In particular the vertextype 2 corresponding to the intersection of loops is forbidden. We then produce all possible configurations summing up the corresponding weights.

A configuration may be represented by a legal set of vertices or, equivalently, by a collection of non-intersecting closed loops (out of connected links). For the determination of the series we simultaneously use the link- and the vertex representation. For lattices of size \( L \times N \) (periodic b.c.) the number of a priori possible link configurations is \( 2^{2LN} \).

The transfer matrix approach is by far the most economic approach. For a lattice of size \( L \times N \) we consider the transfer matrix for a column of \( L \) vertices, \( T(a, b) \) where \( a, b \) denote the link configurations of the left and righthand set of \( L \) links each, e.g. \( a \equiv (a_1, a_2, \ldots, a_L) \);\( a \) and \( b \) may assume \( 2^L \) values each. The internal variables have been integrated and periodic b.c. conditions in the vertical direction have been implemented. Each entry in \( T \) is given by a polynomial term \( M^{2n_1} \) if it has \( n_1 \) vertices of type 1.

One has the symmetries \( T(a, b) = T(b, a) = T(\bar{a}, \bar{b}) = T(\bar{b}, \bar{a}) \), with the notation \( \bar{a} \equiv (a_L, \ldots, a_2, a_1) \), reversal of the direction. Furthermore only entries with even number of occupied external links (corresponding to values of \( a \) and \( b \)) and allowed vertex configurations are non-vanishing. This effectively reduces the number of relevant entries by a factor of roughly 8. In the computer implementation only relevant entries of \( T \) are kept. For \( L = 8 \), \( T \) has only 8384 independent non-zero entries, each a polynomial in \( M \). From
one constructs
\[ T^2(a, c) = \sum_b T(a, b) T(b, c) \quad T^4 = T^2 T^2, \]
and so on. Then
\[ Z_{SV}(M; L \times N) = \text{Tr}\{ T^N \} = \sum_a T^N(a, a). \]

Alternatively, we also considered blocks of 4 × 4, which may be described by functions, \( P(a, b, c, d) \), with four arguments denoting the link configurations of the four edges. Again one has various rotational and reflection symmetries reducing the number of independent non-zero entries of \( P \), which are constructed explicitly at the begin. From \( P(a, b, c, d) \) one may construct the series for e.g. 4 × 8 and 8 × 8 lattices by suitable summations, like
\[ Z_{SV}(M; 8 \times 8) = \sum_{abcdrstu} P(a, b, c, d) P(a, u, c, t) P(r, b, s, d) P(r, u, s, t). \]

This method is less efficient than the one-column transfer matrix approach, which we finally used to construct the series.

In the 7-vertex model the partition function is given as a series in the variable \( M \),
\[ Z_{SV}(M; L \times N) = \sum_n d_n M^n. \]

In the 2D Schwinger model one uses the hopping parameter \( \kappa \) and in table I we give the coefficients of the series in this variable,
\[ Z_{\Lambda}(\kappa) = (2\kappa)^{LN} Z_{SV}(\frac{1}{2\kappa}; L \times N) = \sum_n c_n \kappa^n, \]
for various lattice sizes. The coefficients \( c_n \) are integers and are related to the coefficient of the series in \( M \) through \( c_n = 2^n d_{LN-n} \). Fig. 1a give the positions of the zeroes in the first quadrant of \( \mathbb{C} \) for lattice size 8 × 8.

### 3.2 Free fermions (\( \beta = \infty \))

For \( U(1) \) gauge systems with torus geometry the choice between periodic or antiperiodic boundary conditions (b.c.) for the lattice Dirac operator \( M_{\Lambda} \) is irrelevant for all beta. Assume a field configuration in fixed gauge, then antiperiodic b.c. amount only to multiplying the gauge fields at the boundary
with a factor of \(-1 \in U(1)\). This new configuration is within the sum over all gauge field configurations with the same weight as the original one. Thus it is a symmetry of the gauge field integral.

At \(\beta = \infty\), if we fix the gauge, the choice of b.c. may be parametrized by two \(U(1)\) group elements \((A, B)\), e.g. for antiperiodic b.c. \(A = B = -1\). Thus, as is done usually in analytic calculations, we may discuss the system of free fermions with antiperiodic b.c. as one particular configuration of the \(\beta = \infty\) limit. In this case the fermionic action can be diagonalized and the determinant evaluated by Fourier transformation. Within the limitations of available workstations this may be done with symbolic computer programs up to lattice sizes \(32 \times 32\) and larger. However, the results for the complex positions of zeroes are quite different for periodic and antiperiodic b.c. although the scaling behaviour agrees. Due to the dependence on the b.c. we cannot compare the zeroes directly with our results at finite \(\beta\).

Unlike \(\beta < \infty\) in the free case the zeroes occur degenerate with multiplicity 4 and 8; fig. 1b shows the zeroes (in the first quadrant of \(\mathbb{C}\)) at \(\beta = \infty\) for lattice size \(8 \times 8\). For both, at \(\beta = 0\) and \(\beta = \infty\), we find a distribution of zeroes, that does not follow a simple geometric shape. The general shape of the distribution at \(\beta = \infty\) becomes clearer at larger lattices, as shown in fig. 1c for lattice size \(32 \times 32\).

4 Numerical results and discussion

For \(0 < \beta < \infty\) we use numeric techniques to obtain information on the closest zero on lattice of size \(2 \times 2\), \(4 \times 4\), and \(8 \times 8\), following the second method discussed at the end of sec.2. We simulated background gauge field configurations for \(\beta = 5\), \(\beta = 1\), and also for \(\beta = 0\), in order to check the reliability of the approach. For each configuration we evaluated the determinant on a grid of \(20 \times 20\) complex \(\kappa\) values in the presumed region of the closest zeroes. The final expectation values on this grid are then analyzed with help of interpolation. For lattice \(2 \times 2\) size we summed over 10000 gauge field configurations, for \(4 \times 4\) and \(8 \times 8\) over 5000.

The analytic results at \(\beta = 0\) are in excellent agreement with the numeric results for lattice sizes \(2 \times 2\) and \(4 \times 4\). At the largest lattice size, however, the exact position is off from the numerically determined position by 2 standard deviations (the jackknife statistical error). Actually, \(\beta = 0\) is the worst
case in the gauge field integration, since the configurations are completely random. We have to conclude that for the largest lattice size the number of gauge configurations considered is too small. (A test at \( \beta = 0 \) with a statistics of 32000 did not sufficiently improve the situation.) For this reason for lattice size \( 8 \times 8 \) we doubled the statistical errors obtained at \( \beta = 0 \) to define the errorbars for the other \( \beta \)-values and consider these values with some caution.

Fig. 2 summarizes our results for the size- and \( \beta \)-dependence of the zeroes closest to the real \( \kappa \)-axis. The imaginary parts show the tendency to vanish for \( L \to \infty \) indicating the existence of a phase transition for all \( \beta \) along a curve \( \kappa_c(\beta) \). For free fermions, where \( \kappa - \kappa_c \simeq 1/\xi \), one finds \( O(1/L) \) dependence. For \( \beta < \infty \), for the lattices considered, the behaviour indicates an even faster approach towards zero, like the \( O(1/L^2) \) expected at first order transitions (where the susceptibility \( \simeq L^2 \)). However, the errorbars and smallness of the lattices do not justify stronger statements. Also, considering the non-uniform behaviour of the real parts shows that we are not yet in the asymptotic regime.

As mentioned, we know of no suitable order parameter to identify existence and type of the possible phase transition line. Earlier investigations of the behaviour of the strongly coupled massive lattice Schwinger model based on the 7-vertex model \([11]\) rather suggested that there is no second critical point at \( \beta = 0 \) for \( \kappa < \infty \). To clarify this inconsistency we repeated that 7-vertex calculation with significantly increased statistics (up to a factor of ten) and denser grid in the coupling. It turned out that with this improvement one does indeed find indications of scaling in the susceptibility for lattices larger than \( |\Lambda| = 32 \times 32 \). There is still a controversy concerning the peak position and the boundary conditions of the 7-vertex model to be settled.

In summary, the \( n_f = 1 \) massive (and massless) Schwinger model on the lattice and with Wilson fermions is still a formidable task. Direct simulation of the model and determination of the bosonic mass seem to be impossible with present day resources. From the equivalence to the 8-vertex model we know that there is an isolated critical point at \( \beta = 0 \) and \( \kappa = \infty \) \([11, 11]\); this is most likely not the endpoint of the singular line discussed above. The expansion in \( \kappa \) around \( (\beta = 0, \kappa = 0) \) is convergent only for \( |\kappa| < \bar{\kappa} \leq 1/2 \)\([11]\). The responsible singularity could lie at any complex \( \kappa \) with \( |\kappa| = \bar{\kappa} \) and therefore \( \bar{\kappa} \) does not necessarily define a critical point. The model is believed to have a line of phase transitions at \( \kappa_c(\beta) \) beginning at...
\[ \kappa_c(\infty) = 1/(2d) \] which is a point with a second order transition and running to some \( \kappa_c(0) < \infty \). Except for \( \beta \to \infty \), the free model, there is no proof that this statement is correct.

Our findings from the direct evaluation of the partition function and the finite size dependence of the Lee-Yang zeroes support the scenario of a phase transition line from \( \beta = 0, \kappa \simeq 0.38(2) \) up to \( \beta = \infty, \kappa = 1/4 \). The phase diagram may look similar to the \( n_f > 1 \) situation, also for QCD\(_4\).

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Table 1: Series for $Z_\Lambda(\kappa; L \times N) = \sum_n c_n \kappa^n$.

| n  | $2 \times 2$ | $4 \times 4$ | $8 \times 8$ |
|----|--------------|--------------|--------------|
| 0  | 1            | 1            | 1            |
| 4  | 64           |              |              |
| 8  | 768          | 2304         | 1024         |
| 12 | 32768        | 32768        |              |
| 16 | 2617344      | 2547712      |              |
| 20 | 51904512     | 121634816    |              |
| 24 | 1068498944   | 621490952    |              |
| 28 | 9663676416   | 29126024192  |              |
| 32 | 37597741056  | 16264921612288 |
| 36 | 733659871051776 |          |              |
| 40 | 32983549241982976 |      |              |
| 44 | 1393668629299462144 |      |              |
| 48 | 57045024334275411968 |      |              |
| 52 | 2178933957104869834752 |      |              |
| 56 | 7803575625608176027520 |      |              |
| 60 | 2591089287867411030081536 |  16264921612288 |
| 64 | 7997351201755634532028416 |  32983549241982976 |
| 68 | 2269245313085843057330356224 |  1393668629299462144 |
| 72 | 58685404081509064264407580672 |  57045024334275411968 |
| 76 | 13780381268036305177120105984 |  2178933957104869834752 |
| 80 | 29110687425223635390841502040064 |  7803575625608176027520 |
| 84 | 5479242205342828396220612917760 |  2591089287867411030081536 |
| 88 | 9113040973297576534851687523811328 |  7997351201755634532028416 |
| 92 | 13239313640453274219119019631837184 |  2269245313085843057330356224 |
| 96 | 1658720256337887142482109949475291136 |  58685404081509064264407580672 |
| 100| 1766071053316812784288053250400854016 |  13780381268036305177120105984 |
| 104| 1570588244477979862076814317113632096256 |  29110687425223635390841502040064 |
| 108| 114050057826078629293495730676818771768 |  5479242205342828396220612917760 |
| 112| 6579687246247908061970155944177873977344 |  9113040973297576534851687523811328 |
| 116| 28956384933789738373830226402627572203520 |  13239313640453274219119019631837184 |
| 120| 92150412611536628441916337226044921610240 |  1658720256337887142482109949475291136 |
| 124| 187529160722546587292381268726888701886464 |  1766071053316812784288053250400854016 |
| 128| 223167089080216837357579582404487997816832 |  1570588244477979862076814317113632096256 |
Figures

Fig 1: Distribution of the complex zeroes for (a) $\beta = 0$ on an $8 \times 8$ lattice and at $\beta = \infty$ on (b) $8 \times 8$ and (c) $32 \times 32$ lattices.

Fig 2: (a) The zeroes closest to the real $\kappa$-axis at $\beta = 0$ (squares), $\beta = 1$ (crosses) and $\beta = 5$ (triangles); subsequent lattice sizes ($2 \times 2, 4 \times 4, 8 \times 8$) are connected by lines to guide the eye. (b) Scaling behaviour of the imaginary parts of the closest Lee-Yang zeroes (notation as in (a)) vs. $1/L$; we also show results for $\beta = \infty$ and antiperiodic b.c. as discussed in the text.
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