A DESCENT CRITERION FOR EQUIVALENCES BETWEEN EQUIVARIANT DERIVED CATEGORIES

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ABSTRACT. We investigate equivalences between the categories of perfect complexes of the quotients of two smooth projective schemes by the action of a finite group. As a result we give a necessary and sufficient condition for an equivalence between the equivariant derived categories to descend to the categories of perfect complexes.

1. INTRODUCTION

Let $G$ be a finite group acting on a smooth projective variety $X$. We can consider the category of $G$-equivariant coherent sheaves on $X$ and the corresponding bounded derived category, which we will denote by $\mathcal{D}^G(X)$. When the action is free, the quotient $X/G$ is smooth and $\mathcal{D}^G(X)$ turns out to be equivalent to the bounded derived category of coherent sheaves of the quotient $\mathcal{D}(X/G)$. This is no longer true when the action is not free. However, $\mathcal{D}^G(X)$ is still equivalent to the derived category of the stack quotient $[X/G]$. Notice that $[X/G]$ is smooth as a stack, so we can think of $\mathcal{D}^G(X)$ as a replacement for the category $\mathcal{D}(X/G)$ when the action is not free. Ploog, in [Plo07], studied the autoequivalences of the aforementioned categories. This is a way to understand the relations between $\mathcal{D}(X/G)$ and $\mathcal{D}^G(X)$ for a non-free action, as a categorical analogue of the comparison between the quotient $X/G$ and the stack quotient $[X/G]$.

When $X/G$ is singular, another replacement for the category $\mathcal{D}(X/G)$ is provided by the subcategory $\text{Perf}(X/G)$. It is called the category of perfect complexes and consists of the objects in $\mathcal{D}(X/G)$ which are quasi-isomorphic to bounded complexes of locally free sheaves of finite type on $X/G$. We refer to [KPS18] for some more details about perfect complexes and equivariant derived categories. Our aim is to compare $\mathcal{D}^G(X)$ and $\text{Perf}(X/G)$ by studying when equivalences between equivariant derived categories can be related to equivalences of the categories of perfect complexes.

In order to state our main result, consider another finite group $H$ acting on smooth projective variety $Y$. Orlov’s representability theorem for smooth stacks was proven by Kawamata, see [Kaw04] Theorem 1.1. As an outcome we have that every fully faithful exact functor $F$ from $\mathcal{D}^G(X)$ to $\mathcal{D}^H(Y)$ is of Fourier-Mukai type, meaning that there exists an object $\mathcal{E}^\bullet$ in $\mathcal{D}^{G \times H}(X \times Y)$, such that $F$ is isomorphic to the functor $\Phi_{g^\bullet}$ given by $q_Y \circ (q_X(-) \otimes \mathcal{E}^\bullet)$, where $q_X$ and $q_Y$ are the projections on the two factors of $X \times Y$, and all the functors are properly
Theorem. An object $\Phi_\varnothing$ is called Fourier-Mukai functor with kernel $\mathcal{E}^\varnothing$.

Now we want to find a way to get an equivalence between $\text{Perf}(X/G)$ and $\text{Perf}(Y/H)$ starting from $\Phi_\varnothing$. The categorical tools we will use are the functors ind and res, standing for induction and restriction, which allows us to pass from the non equivariant setting to the equivariant one and vice versa. At the level of coherent sheaves, the restriction is just the forgetful functor from $\text{Coh}^G(X)$ to $\text{Coh}(X)$, while the induction is its left and right adjoint.

Consider now the projection $\pi : X \to X/G$; its derived pushforward (resp. pullback) can be composed with the derived res (resp. ind). We will denote the composition by $\Pi^G_\varnothing$ (resp. $\Pi^G_\ast$):

\[
\mathcal{G}^G(X) \xrightarrow{\text{Res}} \mathcal{G}(X) \xrightarrow{\text{Ind}} \mathcal{G}(X/G).
\]

This construction can be symmetrically performed also on $Y$ with the action of $H$. We put all of this together in the following diagram by consider also a Fourier-Mukai equivalence $\Phi_\varnothing$:

\[
\begin{array}{ccc}
\mathcal{G}^G(X) & \xrightarrow{\Phi_\varnothing^\ast} & \mathcal{G}^H(Y) \\
\Pi^G_\varnothing \downarrow & & \Pi^H_\ast \downarrow \\
\mathcal{G}(X/G) & \xrightarrow{\Phi_\varnothing} & \mathcal{G}(Y/H) \\
\text{Perf}(X/G) & \xrightarrow{\text{Res}} & \text{Perf}(Y/H)
\end{array}
\]

We call $\Omega$ the functor obtaining by the compositions $\Pi^H_\ast \circ \Phi_\varnothing^\ast \circ \Pi^G_\ast |_{\text{Perf}(X/G)}$. Such a functor is from $\text{Perf}(X/G)$ to $\mathcal{G}(Y/H)$. The aim of this paper is to find a condition, involving the kernel $\mathcal{E}^\varnothing$, for the equivalence $\Phi_\varnothing$ to induce an equivalence between $\text{Perf}(X/G)$ and $\text{Perf}(Y/H)$ through $\Omega$. The key is a property called descent: a $G$-equivariant complex of sheaves $\mathcal{F}^\ast$ descends to $\text{Perf}(X/G)$ if there exists a complex $\mathcal{Y}^\ast$ in $\text{Perf}(X/G)$ such that $\Pi^G_\ast(\mathcal{Y}^\ast)$ is isomorphic to $\mathcal{F}^\ast$. We make the role of this property explicit in Theorem 3.3.

**Theorem.** The functor $\Omega$ is an equivalence between $\text{Perf}(X/G)$ and $\text{Perf}(Y/H)$ if and only if the two following conditions are satisfied.

(i) The object $\Phi_\varnothing \circ \Pi^G_\ast (\mathcal{F}^\ast)$ descends to $\text{Perf}(Y/H)$, for all $\mathcal{F}^\ast \in \text{Perf}(X/G)$.

(ii) The object $(\Phi_\varnothing)^{-1} \circ \Pi^H_\ast(\mathcal{Y}^\ast)$ descends to $\text{Perf}(X/G)$, for all $\mathcal{Y}^\ast \in \text{Perf}(Y/H)$.

We point out that the problem is not trivial, even when we restrict to auto-equivalences. In order to clarify this, we give an explicit example at the end of Section 3. Such an example suggests that the descent criterion of Theorem 3.3 should depend only on the kernel $\mathcal{E}^\varnothing$ of the Fourier-Mukai equivalence we are considering. In order to see this more clearly, we will provide a criterion to characterise the descending sheaves: Theorem 4.8.

**Theorem.** An object $\mathcal{F}^\ast$ in $\mathcal{G}^G(X)$ descends to $\text{Perf}(X/G)$ if and only if the following condition holds.

(*) The stabiliser $G_x$ acts trivially on the $\mathcal{O}_X$-modules $H^j(\mathcal{F} \otimes k(x))$, for all $x \in X$, $j \in \mathbb{Z}$. 
By combining such a result with Theorem 3.3 we find a more explicit criterion: Theorem 5.1.

**Theorem.** The functor $\Omega$ is an equivalence between $\text{Perf}(X/G)$ and $\text{Perf}(Y/H)$ if and only if the two following conditions are satisfied

(i') The stabiliser $H_y$ acts trivially on
$$R\Gamma\left(\left[\iota_y^* \mathcal{q}_X^* \Pi^G_X \mathcal{A}^*\right]_{X \times \{y\}}\right)$$
for any point $y$ in $Y$ and for any complex $\mathcal{A}^*$ in $\text{Perf}(X/G)$.

(ii') The stabiliser $G_x$ acts trivially on
$$R\Gamma\left(\left[\iota_x^* \mathcal{q}_Y^* \Pi^H_Y \mathcal{B}^*\right]_{Y \times \{x\}}\right)$$
for any point $x$ in $X$ and for any complex $\mathcal{B}^*$ in $\text{Perf}(Y/H)$.

Here $\iota_x$ and $\iota_y$ are the natural embeddings of $\{x\} \times Y$ and $X \times \{y\}$ in $X \times Y$, respectively.

**Plan of the paper.** The paper is organised as follows: in Section 2 we give some basic notions on derived categories and equivariant sheaves. The functor $\Omega$ is central in this paper and it is defined in Section 3, where we also prove Theorem 3.3. At the end of Section 3 there are examples describing the possible behaviours of $\Omega$. The problem of finding when $\Omega$ is an equivalence is related to the descent data of sheaves: we give an explicit condition of this descent in 4. Finally, Section 5 we apply all the previous result to get the more explicit criterion Theorem 5.1.

**Conventions.** We will use $\mathbb{R}$ and $\mathbb{L}$ to denote the right or left derived functor, respectively. We will avoid writing them explicitly when it is clear from the context. Throughout the paper, we work over a base field $k$ which is algebraically closed of characteristic zero. Even if some concept can be stated also more generally, we will always assume the groups acting to be finite.

## 2. Preliminaries

**Definition 2.1.** A (left) action of $G$ on a variety $X$ is given by a group homomorphism $\phi$ from $G$ to $\text{Aut}(X)$.

We refer to the book [Isa08] for all the details on actions by finite groups; we will report here only the details that we need in this paper.

**Definition 2.2.** Let $U$ be a subset of $X$. The **orbit** of $U$ is the subset of $X$ defined as
$$G(U) := \{ g(x) \in X \text{ s.t. } x \in U, g \in G \}.$$ 

The **stabiliser** $G_x$ of the point $x \in X$ is the subset of $G$ defined as
$$G_x := \{ g \in G \text{ s.t. } g(x) = x \}.$$ 

The action of a group is called **free**, if $\phi$ is injective. Let us consider the following example:
Example 2.3. Consider the action of the group $\mathbb{Z}_2$ on $\mathbb{P}^2(x_0 : x_1 : x_2)$ given by
\[
g(x_0 : x_1 : x_2) := \begin{cases} 
(x_0 : x_1 : x_2) & \text{if } g = 0, \\
(x_0 : x_1 : -x_2) & \text{if } g = 1.
\end{cases}
\]

It is straightforward to check that the orbits of points are given by
\[
\mathbb{Z}_2(x_0 : x_1 : x_2) := \begin{cases}
(x_0 : x_1 : x_2), (x_0 : x_1 : -x_2) & \text{if } x_2 \neq 0, \\
(x_0 : x_1 : 0) & \text{if } x_2 = 0.
\end{cases}
\]

Moreover, the stabilisers of the point $(x_0 : x_1 : 0)$ and $(0 : 0 : 1)$ is the whole $\mathbb{Z}_2$. It is the identity group for every other point. Finally, the fixed locus for the action of $\mathbb{Z}_2$ is given by
\[
\{(z_0 : z_1 : z_2) \in \mathbb{P}^2 \text{ such that } z_2 = 0\} \cup \{[0 : 0 : 1]\}.
\]

Once we have an action of $G$ on $X$ as described before, we can certainly consider the topological quotient $X/G$, where the points are actually $G$-orbits. We want to understand when $X/G$ can be endowed with a scheme structure induced by $X$ via the projection map.

Definition 2.4. We define the geometric quotient of $X$ by $G$ to be the pair $(X/G, \pi)$, where $X/G$ is a projective variety and $\pi$ is a map $X \to X/G$, such that:

1. The set $\pi^{-1}(y)$ coincides with the orbit $G\{y\}$ for every $y \in X/G$;
2. The subset $\mathcal{U}$ is open in $X/G$ if and only if $\pi^{-1}(\mathcal{U})$ is open in $X$;
3. The structure sheaf $\mathcal{O}_{X/G}$ is isomorphic to $\pi_*\mathcal{O}_X^G$ i.e. for every open set $\mathcal{U} \subset X/G$ we have $\mathcal{O}_{X/G}(\mathcal{U}) = \mathcal{O}_X(\pi^{-1}(\mathcal{U}))^G$.

Here $\mathcal{O}_X^G$ denotes the $G$-linearised structure sheaf (cfr. Section 2.2).

The geometric quotient $(X/G, \pi)$ is a categorical quotient, it means that the following conditions are satisfied:

1. The map $\pi$ is $G$-invariant, i.e. for every $g \in G$ we have $\pi \circ \sigma_g = \pi$.
2. The map $\pi$ is universal, i.e. for every other couple $(Z, \pi')$, where $Z$ is a projective variety and $\pi': X \to Z$ is $G$-invariant, there exists a unique map $h: X/G \to Z$ such that the following diagram commutes:
\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X/G \\
\downarrow & & \downarrow h \\
Z & \xrightarrow{\exists h} & Z.
\end{array}
\]

When the group $G$ is finite, a geometric quotient always exists. In the notation above, it is given by the topological quotient $\pi: X \to X/G$. When $G$ is infinite, the situation is more complicated and a possible solution comes from the Geometric Invariant Theory. Starting from a smooth variety $X$, it is easy to prove that the quotient $X/G$ is smooth if and only if the action of $G$ is free.

Example 2.5. Let’s go back on Example 2.3 where $\mathbb{Z}_2$ is acting on $\mathbb{P}^2$; the action is not free. The quotient is isomorphic to the weighted projective plane $\mathbb{P}(1 : 1 : 2)$. This is isomorphic to the quadric cone defined in $\mathbb{P}^3$ by
\[
\{[w_0 : w_1 : w_2 : w_3] \in \mathbb{P}^3 \text{ such that } w_0 w_1 = w_2^2\}.
\]
This is a surface with a singular point in $[0 : 0 : 0 : 1]$; the isomorphism with the weighted projective plane is given explicitly by

$$w_0 = z_0^2, w_1 = z_0 z_1, w_2 = z_2^2, w_3 = z_2^2.$$ 

### 2.2. Derived category of $G$-equivariant sheaves.

We will closely follow [Plo05] and the related paper [Plo07], to which we refer to the reader for a very extensive introduction on the topic. As a consequence of Definition 2.1, every element $g \in G$ provides a map from $X$ to $X$, which we name by the very same element $g$. Hence, the pushforward $g_*$ and pullback $g^*$ of sheaves along the morphism $g$ make sense. We will consider the category of $G$-linearised coherent sheaves. It can be defined for any algebraic group, see [Plo05, Definition 3.1] and more in general [BL06]. When dealing with finite group, it is possible to consider this equivalent definition.

**Definition 2.6.** Let $F$ be a coherent sheaf of $X$. We call $G$-linearisation of $F$ a family of isomorphisms $\lambda : \{ \lambda_g : F \xrightarrow{\sim} g^*F \}_{g \in G}$ such that: $\lambda_{1d} = 1d_F$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$ for every $g, h \in G$.

The category $\text{Coh}^G(X)$ of $G$-equivariant coherent sheaves on $X$ consists of objects $(F, \lambda)$ where $F$ belongs to $\text{Coh}(X)$ and $\lambda$ is a $G$-linearisation of $F$. The morphisms are given by $f : (F, \lambda) \rightarrow (F', \lambda')$ such that the following diagram commutes for every $g \in G$:

$$\begin{array}{ccc}
F & \xrightarrow{\lambda_g} & g^*F \\
\downarrow{f} & & \downarrow{g^*f} \\
F' & \xrightarrow{\lambda'_g} & g^*F'.
\end{array}$$

Notice that the structure sheaf $\mathcal{O}_X$ admits a canonical $G$-linearization given by the identity morphisms $\mathcal{O}_X \rightarrow g^*\mathcal{O}_X$ for any $g \in G$. The category $\text{Coh}^G(X)$ is abelian, so we can consider the bounded derived category $\mathcal{D}_G(X) := \mathcal{D}(\text{Coh}^G(X))$ associated to it.

Consider now two groups $G$ and $H$ acting on two smooth projective varieties $X$ and $Y$. Assume we have a homomorphism $\phi : G \rightarrow H$.

**Definition 2.7.** A $\phi$-map between $X$ and $Y$ is a morphism $f : X \rightarrow Y$ such that the following diagram commutes for every $g \in G$:

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{\phi(g)} \\
X & \xrightarrow{f} & Y.
\end{array}$$

This allows us to get a well defined map from $X/G$ to $Y/H$. Given a $\phi$-map $f$, we can also define the equivariant pull back, $f_* : \text{Coh}^H(Y) \rightarrow \text{Coh}^G(X)$. When $\phi$ is surjective we also have the push forward $f_* : \text{Coh}^G(X) \rightarrow \text{Coh}^H(Y)$. Taking derived functors we get also the induced maps at the level of derived categories. There are two natural functors, called induction and restriction, which allows us to compare the ordinary bounded derived category $\mathcal{D}(X)$ and $\mathcal{D}^G(X)$. 


We can first define them at the level of coherent sheaves. The restriction is actually the forgetful functor
\[ \text{res} : \text{Coh}^G(X) \rightarrow \text{Coh}(X) \]
\[ (\mathcal{E}, \lambda) \mapsto \mathcal{E}. \]
The induction functor is defined in the following way:
\[ \text{ind} : \text{Coh}(X) \rightarrow \text{Coh}^G(X) \]
\[ \mathcal{E} \mapsto (\bigoplus_{g \in G} g^* \mathcal{E}, \rho_g), \]
where \( \rho_g \) is the linearisation which comes from the permutations of the summands.

We have an adjunction between them in both directions. By deriving on right or left, we have induced functors at level of derived categories.

**Remark 2.8.** Notice that the constructions of the functor ind and res can be generalized to the case in which we act with a subgroup of \( G \) to form an intermediate quotient. This could lead to an improvement of our main result taking into account varieties with quotient singularities. We leave this problem open.

**2.3. Fourier-Mukai transform.** Consider two algebraic spaces \( X, Y \). A functor \( F \) from \( D^b(X) \) to \( D^b(Y) \) is called of Fourier-Mukai type if there exists an object \( \mathcal{E} \) in \( D^b(X \times Y) \) such that \( F \) is isomorphic to the functor defined by
\[ \Phi_{\mathcal{E}^*} : D^b(X) \rightarrow D^b(Y) \]
\[ \mathcal{F}^* \mapsto \mathbb{R}q_X^*(\mathcal{F}^* \otimes \mathcal{E}^*), \]
where \( \mathbb{R} \) and \( \mathbb{L} \) stands for right and left derived functors, and \( q_X \) and \( q_Y \) denote the projections from \( X \times Y \) to \( X \) and \( Y \), respectively. Fourier-Mukai functors have a deep role in algebraic geometry, they are exact and well-behaved with respect to composition, and an active and rich field of study concerns finding conditions to describe which functors are actually of Fourier-Mukai type, see [CS12] for a survey on this topic.

The result of [Kaw04] goes in this direction. In the same terminology as before, we have that every exact, fully faithful functor
\[ \mathcal{F} : D^G(X) \rightarrow D^H(Y) \]
with a left adjoint is of Fourier-Mukai type, where the kernel belongs to the category \( D^{G \times H}(X \times Y) \). When dealing with projections and products it is very useful to consider the box product of a sheaf \( \mathcal{E}^* \) in \( D^G(X) \) with a sheaf \( \mathcal{F}^* \) in \( D^H(Y) \).

It is denoted by \( \mathcal{E}^* \boxtimes \mathcal{F}^* \) and is defined as the tensor product \( q_X^* \mathcal{E}^* \otimes q_Y^* \mathcal{F}^* \) in \( D(X \times Y) \). The same definition can be stated in the equivariant setting as well.

3. DESCENT DATA AND EQUIVALENCES OF PERFECT COMPLEXES

In all this section we will always work in the setting of two finite groups \( G \) and \( H \) acting on two smooth varieties \( X \) and \( Y \), respectively. We can follow [KS15] in order to obtain that the derived category \( D([X/G]) \) of the stack \( [X/G] \) is equivalent to the equivariant derived category \( D^G(X) \), the same holds with \( Y/H \). See also [Plo05, Remark 3.14]. It is then a consequence of [Kaw04, Theorem 1.1] that all the exact equivalences between \( D^G(X) \) and \( D^G(Y) \) are of
Fourier-Mukai type; in particular the kernel $\mathcal{E}^\bullet$ lives in $\mathcal{D}^{G \times H}(X \times Y)$. So we will fix a kernel $\mathcal{E}^\bullet$ in $\mathcal{D}^{G \times H}(X \times Y)$ such that the corresponding Fourier-Mukai functor $\Phi_{\mathcal{E}^\bullet}$ is an equivalence.

Thanks to the results of the previous section, we have the following commutative diagram:

We wrote explicitly only the maps we will use, but obviously all the construction can be done symmetrically for both $(X, G)$ and $(Y, H)$. We define the functor $\Pi^{G,\bullet}$ as the composition $\mathbb{L}(\text{ind} \circ \pi^*)$ and $\Pi^{H,\bullet}$ as the composition $\mathbb{R}(\pi_* \circ \text{res})$. The functor $\Omega$ is defined to be the composition $\Pi^{H,\bullet} \circ \Phi_{\mathcal{E}^\bullet} \circ \Pi^{G,\bullet}$, restricted to the subcategory of perfect complexes.

We want to find conditions to guarantee that the functor $\Omega$ is an equivalence, with image in the category $\text{Perf}(Y/H)$. In order to do that we need the following definitions:

**Definition 3.1.** A $G$-equivariant locally free sheaf of finite type $\mathcal{F}$ descends to $\text{Coh}(X/G)$ if there exists a locally free sheaf $\mathcal{V}$ in $\text{Coh}(X/G)$ such that $\pi^G_*(\mathcal{V})$ is isomorphic to $\mathcal{F}$.

By using the adjunction properties of the involved functors, Definition 3.1 turns out to be equivalent to requiring $\pi^G_* \circ \pi^*_G$ to be isomorphic to the identity functor of $\text{Coh}^G(X)$, where $\pi^*_G$ denotes $\text{ind}^* \circ \pi^*$ and $\pi^G_*$ denotes $\pi_* \circ \text{res}_*$. We are interested in a similar property, but in the case of the subcategory of perfect complexes. Due to the notation we use on the functors, this definition is really similar to the previous one.

**Definition 3.2.** Let $\mathcal{F}^\bullet$ be a $G$-equivariant complex of sheaves in $\mathcal{D}^{G}(X)$. We say that $\mathcal{F}^\bullet$ descends to $\text{Perf}(X/G)$ if there exists a complex $\mathcal{V}^\bullet$ in $\text{Perf}(X/G) \subset \mathcal{D}(X/G)$ such that $\pi^G_*(\mathcal{V}^\bullet)$ is isomorphic to $\mathcal{F}^\bullet$.

It follows from the definition that if $\mathcal{F}^\bullet$ descends to $\text{Perf}(X/G)$, then

$$\Pi^{G,\bullet} \circ \Pi^{G}_G(\mathcal{F}^\bullet) \simeq \mathcal{F}^\bullet,$$

namely the projection has a left inverse. If we work in a open neighborhood of a point it is just the identity map, and this implies that the descent property is local.

The theorem below gives us a necessary and sufficient condition to prove when $\Omega$ is an equivalence.

**Theorem 3.3.** The functor $\Omega$ is an equivalence between $\text{Perf}(X/G)$ and $\text{Perf}(Y/H)$ if and only if the two following conditions are satisfied:
(i) The object $\Phi_\mathcal{E} \circ \Pi^{G,*}(\mathcal{F}^*)$ descends to $\mathcal{D}(Y/H)$, for all $\mathcal{F}^* \in \mathcal{D}(X/G)$.

(ii) The object $(\Phi_\mathcal{E})^{-1} \circ \Pi^H,*(\mathcal{Y}^*)$ descends to $\mathcal{D}(X/G)$, for all $\mathcal{Y}^* \in \mathcal{D}(Y/H)$.

**Proof.** Let us first assume (i) and (ii) holds and prove that $\Omega$ is indeed an equivalence with image in $\mathcal{D}(Y/H)$. By construction, the functor $\Omega$ is defined from the category $\mathcal{D}(X/G)$ to $\mathcal{D}(Y/H)$. However, condition (i) guarantees us that every object $\Omega(\mathcal{F}^*)$ is isomorphic to an object in $\mathcal{D}(Y/H)$. It makes sense then to determine whether $\Omega$ is an equivalence or not.

In order to prove that $\Omega$ is fully faithful we recall that both $\Pi^{G,*}$ and $\Phi_\mathcal{E}$ are fully faithful. Moreover, the descent in Condition (i) is equivalent to ask that $\Pi^H,*$ restricted to the image of $\Phi_\mathcal{E} \circ \Pi^{G,*}$ is isomorphic to the identity $\text{id}_{\mathcal{D}(Y/H)}$, hence $\Pi^H,$ is fully faithful as well. Hence, the functor $\Omega$ is fully faithful because it is composition of three fully faithful functors. It remains to check that $\Omega$ is essentially surjective. Since $\Phi_\mathcal{E}$ is an equivalence, we can consider the inverse $(\Phi_\mathcal{E})^{-1}$. Asking that $\Omega$ is essentially surjective is the same as asking that the functor $\Pi^G_* \circ (\Phi_\mathcal{E})^{-1} \circ \Pi^H,*$. Condition (ii) guarantees exactly that.

Viceversa, assume that $\Omega$ is an equivalence between $\mathcal{D}(X/G)$ and $\mathcal{D}(Y/H)$. If we take an object $\mathcal{F}^* \in \mathcal{D}(X/G)$, then $\Omega(\mathcal{F}^*)$ belongs to $\mathcal{D}(Y/H)$ and it follows by definition that $\Phi_\mathcal{E} \circ \Pi^{G,*}(\mathcal{F}^*)$ descend to $\Omega(\mathcal{F}^*)$. It remains to prove that Conditions (ii) also holds: since $\Omega$ is an equivalence we can consider its inverse $\Omega^{-1}$. Notice that it is given by $\Pi^H_* \circ \Phi_\mathcal{E}^{-1} \circ \Pi^G,*$ restricted to $\mathcal{D}(Y/H)$. By applying the same reasoning as before: take an object $\mathcal{Y}^* \in \mathcal{D}(Y/H)$, then $\Omega^{-1}(\mathcal{Y}^*)$ belongs to $\mathcal{D}(X/G)$ and it follows by definition that $(\Phi_\mathcal{E})^{-1} \circ \Pi^H,*(\mathcal{Y}^*)$ descend to $\Omega^{-1}(\mathcal{Y}^*)$. 

Notice that actually Condition (i) is used in order to provide the fact that $\Omega$ has the right image and that is fully faithful. Condition (ii) is needed for the essential surjectivity of $\Omega$ and it can be interpreted, when it makes sense, as Condition (i) rephrased for the functor $\Omega^{-1} := \Pi^G_* \circ \Phi_\mathcal{E}^{-1} \circ \Pi^H,*$. By analogy with the definitions on sheaves and complexes, we will say that $\Omega$ descends if Condition (i) holds, and that $\Omega$ descends to an equivalence, if both conditions (i) and (ii) holds.

### 3.1. Positive and negative examples

Let us now focus on some examples, in order to understand better the behaviour of this descent criterion. For simplicity, we will now set $X = Y$ and $G = H$. We give an example which shows how to build an autoequivalence of $\mathcal{D}(X/G)$ starting from a Fourier-Mukai autoequivalence of $\mathcal{D}(X)$. In particular, we deal with the case of projective varieties with ample canonical or anticanonical bundle, since all the autoequivalences of $\mathcal{D}(X)$ are classified.

**Theorem 3.4.** [Kaw04 Theorem 7.2.] Let $X$ be a smooth normal projective variety with ample canonical sheaf or ample anticanonical sheaf with the action of a finite group $G$. Then, the group of isomorphism classes of exact autoequivalence $\mathcal{D}(X)$ is generated by shifts, tensor products with $G$-equivariant invertible sheaves and push forward along $G$-equivariant automorphisms of $X$.

It means that, to study weather an autoequivalence $\Phi_\mathcal{D}$ of $\mathcal{D}(X)$ induces and autoequivalence of $\mathcal{D}(X/G)$, it suffices to study what happens when $\Phi_\mathcal{D}$ is one of these generators. The case of $\Phi_\mathcal{D}$ being a shift is straigthforward. Explicitly, $\mathcal{D} = \Delta^G_* |\mathcal{O}_X[d]|$ for a certain integer $d$, where we are taking the push
forward of the map $\Delta : X \to X \times X$ composed with the functor $\text{ind}$. Then, it is immediate to see that the corresponding $\Omega$ descends to an autoequivalence of $\text{Perf}(X/G)$.

Now assume that $\Phi_G$ is a tensor product with an invertible $G$-equivariant sheaf $L$ on $X$, that is $\mathcal{P} = \Delta^*_G(L)$, in the same notation as before. In the context of vector bundles we have the following result:

**Theorem 3.5** (Thm 2.3, [DN89]). A $G$-equivariant vector bundle $\mathcal{P}$ on $X$ descends to $X/G$ if and only if the stabiliser $G_x$ acts trivially on the fibre $\mathcal{E}_x$ for every $x$ in $X$.

**Corollary 3.6.** Let $L$ be a $G$-line bundle on $X$. If $n$ is a multiple of the order of $G$, then $L^{\otimes n}$ descends to the quotient.

**Proof.** Consider any point $x$ in $X$. The stabiliser $G_x$ is finite and its action on the fibres is represented by a one-dimensional homomorphism whose values must be roots of unity. Hence after taking tensor powers for any multiple of the order of the group the action of $G_x$ on fibres of $L$ becomes trivial. The result follows by applying Theorem 3.5. \qed

As a consequence of the previous corollary the Fourier-Mukai transform associated to the kernel $\Delta_*(\mathcal{L})$ gives rise to a functor $\Omega := \Pi^G_{\ast} \circ \Phi_G \circ \Pi^{G,\ast}$:

$$\Omega : \text{Perf}(X/G) \to \text{Perf}(X/G)$$

$${\mathcal{A}^\ast} \mapsto \Pi^G_{\ast}(\mathcal{L} \otimes \Pi^{G,\ast}(\mathcal{A}^\ast)).$$

If $\mathcal{L}$ descends to the quotient, the stabiliser $G_x$ acts trivially on $\mathcal{L} \otimes k(x)$ for every $x$ in $X$, then $\Omega$ defines an autoequivalence of $\text{Perf}(X/G)$. If $\mathcal{L}$ does not descend, then, according to Corollary 3.5, it suffices to take the tensor product of $\mathcal{L}$ to the order of the group $G$ because the stabiliser $G_x$ acts trivially on every fibre of $\mathcal{L} \otimes [G]$.

Lastly, we deal with $G$-equivariant automorphisms, namely we assume $\mathcal{P}$ to be of the form $\mathcal{A}_f$, where $\Gamma_f$ denotes the graph of a $G$-automorphism $f$ of $X$. Then, $\Omega := \Pi^G_{\ast} \circ \Phi_G \circ \Pi^{G,\ast}$:

$$\Omega : \text{Perf}(X/G) \to \text{Perf}(X/G)$$

$${\mathcal{A}^\ast} \mapsto \Pi^G_{\ast}(f^\ast(\Pi^{G,\ast}(\mathcal{A}^\ast))).$$

Using the fact that the automorphisms of $X$ preserve the descending property, plus the fact that $\Pi^{G,\ast}(\mathcal{A}^\ast)$ descends, by Theorem 3.3 we have that $\Omega$ descends to an equivalence of $\text{Perf}(X/G)$.

Let us now consider another situation in which we will get that the functor $\Omega$ does not descend. Consider the group $\mathbb{Z}_2$ acting on $\mathbb{P}^2$ by multiplying the last coordinate by $-1$. This particular action has already been described in the preliminaries, see Examples 2.3 and 2.5. The $\mathbb{Z}_2$-equivariant derived category $\mathcal{D}_{\mathbb{Z}_2}[\mathbb{P}^2]$ has a nice description: it may be seen as the category of bounded complexes of free $\mathbb{Z}_2 \cdot k[x, y] \cdot$-modules of finite type with generators having bounded degree, up to homotopy equivalence. See [Ter03] and [Ter02] for more details. Moreover, we have an explicit way to check whether some objects are perfect or not. Indeed, it was shown in [Del04] that the sheaves $\mathcal{O}_{\mathbb{P}^2}[1,1,2](d)$ belongs to $\text{Perf}(\mathbb{P}^2)[1,1,2]$ if and only if $d$ is an even number.
Consider a Fourier–Mukai autoequivalence

$$\Phi_{\Delta^Z} : \mathcal{G}^Z(\mathbb{P}^2) \to \mathcal{G}^Z(\mathbb{P}^2),$$

obtained by taking the equivariant push forward of the diagonal embedding

$$\Delta : \mathbb{P}^2 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2,$$

applied to $\mathcal{O}(1)$. It can be explicitly computed that, for every $\mathcal{E}^* \in \mathcal{G}^Z(\mathbb{P}^2)$, we have $\Phi_{\Delta^Z}(\mathcal{E}^*) = \mathcal{E}^* \otimes \mathcal{O}(1)$.

The functor $\Omega$ sends the perfect object $\mathcal{O}_{\mathbb{P}^2}(1)$ to $\mathcal{O}_{\mathbb{P}^2}(1)$, which is not perfect. Hence, the hypothesis of Theorem 3.3 are not satisfied, and $\Omega$ does not descend.

By following the same lines, it is interesting to see what happens to the autoequivalence $\Phi_{\Delta^Z}^d$ for $d$ an even integer. In fact, if we take any $\mathcal{E}^*$ in $\mathcal{G}^Z(\mathbb{P}^2)$, we have

$$\Omega(\mathcal{E}^*) = \Pi_{\Delta^Z}^d(\mathcal{O}_{\mathbb{P}^2}(d) \otimes \pi^* \mathcal{E}^*) = \Pi_{\Delta^Z}^d(\pi^* \mathcal{O}_{\mathbb{P}^2}(1,2,d) \otimes \pi^* \mathcal{E}^*) = \mathcal{O}_{\mathbb{P}^2}(1,2,d) \otimes \mathcal{E}^*.$$

This is enough to prove that, in this case, $\Omega$ descends.

4. CRITERION FOR THE DESCENT TO PERFECT COMPLEXES

Consider the action of a finite group $G$ on a smooth projective variety $X$, the map $\pi$ being the projection to the quotient. The aim of this section is to provide a criterion (Theorem 4.4) for characterizing the complexes of $\mathcal{D}^G(X)$ descending to perfect complexes of $\mathcal{D}(X/G)$. In order to avoid misunderstanding, we will use $\mathcal{F}$ without decorations to identify a sheaf and $\mathcal{G}^*$ to identify a complex of sheaves. This applies in particular to the functors $\Pi^G$, defined as $\text{ind} \circ \pi^*$, and $\Pi^G_{\mathcal{E}}$, defined as $\pi_\mathcal{E} \circ \text{res}$.

We now want to exploit Theorem 3.3 in order to get a descent criterion for $\text{Perf}(X/G)$. We start with two lemmas which give us homological informations about the abelian category $\text{Coh}^G(X)$:

**Lemma 4.1.** [Ter02, Lemma 1.1.] The homological dimension of $\text{Coh}^G(X)$ is at most $\text{dim}(X)$. That is, for every pair of $G$-equivariant coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, $\text{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for every $i \geq \text{dim}(X)$.

**Lemma 4.2.** [Ter02, Lemma 1.2.] Every $G$-equivariant coherent sheaf admits a finite resolution of $G$-equivariant locally free sheaves of finite type.

Applying the two lemmas above we immediately get the following

**Corollary 4.3.** Any $\mathcal{F}^*$ in $\mathcal{D}^G(X)$ is quasi isomorphic to a $\mathcal{G}^*$ in $\mathcal{D}^G(X)$ such that, for every integer $i$, $\mathcal{G}^i$ is a $G$-equivariant locally free sheaf of finite type.

Now we are ready to give a descent criterion at the level of derived categories:

**Proposition 4.4.** The complex $\mathcal{F}^*$ in $\mathcal{D}^G(X)$ descends to $\text{Perf}(X/G)$ if and only if there exists a finite locally free $G$-resolution $\mathcal{E}^*$ of $\mathcal{F}^*$ such that, for every $x$ in $X$ and every integer $i$, the stabiliser $G_x$ acts trivially on the fibre $\mathcal{E}^i_x$.

**Proof.** Assume that $\mathcal{F}^*$ descends to $\text{Perf}(X/G)$, i.e. there exists a complex $\mathcal{B}^*$ in $\text{Perf}(X/G)$ such that $\Pi^G(\mathcal{B}^*)$ is quasi-isomorphic to $\mathcal{F}^*$. By adjunction, we get that $\Pi^G_{\mathcal{E}} \circ \Pi^G(\mathcal{B}^*)$ is isomorphic to the identity. Hence we have that $\mathcal{B}^* = \Pi^G(\mathcal{F}^*)$. Take a resolution of $\mathcal{B}^*$ by vector bundles on $X/G$ and denote it explicitly by

$$\mathcal{V}^* = \{0 \to \mathcal{V}^1 \to \ldots \to \mathcal{V}^n \to 0 \}.$$
It follows that we can apply the map $\Pi^{G,*}$ on the complex $\mathcal{B}^*$ by applying $\pi^{G,*}$ to each one of its components. We have the following quasi-isomorphisms:

$$\mathcal{F}^* \cong \Pi^{G,*}(\mathcal{B}^*) \cong \Pi^{G,*}(\mathcal{F}^*) \cong \{0 \to \pi^{G,*}(\mathcal{F}^i) \to \ldots \to \pi^{G,*}(\mathcal{F}^n) \to 0\}.$$ 

The property of being locally free sheaves is preserved by $\Pi^{G,*}$, hence $\Pi^{G,*}(\mathcal{F}^*)$ is a complex of vector bundles which is quasi-isomorphic to $\mathcal{F}^*$. Furthermore, for any point $x$ in $X$, the stabiliser $G_x$ acts trivially on the fibres of $\pi^{G,*}(\mathcal{F}^i)$ since $\pi^G_x(\mathcal{F}^i(x)) = \mathcal{F}^i$ is a vector bundle on $X/G$. This holds for every $i = 1, \ldots, n$, proving the claim.

Vice versa, assume that there exists a finite locally free $G$-resolution $\mathcal{E}^*$ of $\mathcal{F}^*$ such that, for every $x$ in $X$, the stabiliser $G_x$ acts trivially on the fibre $\mathcal{E}^i_x$ for every integer $i$. So, the functor $\Pi^G_x(\mathcal{E}^*)$ is exact and, thanks to Theorem 3.5, we have that every $\mathcal{E}^i_x$ descends to a vector bundle on the quotient. It follows that the image $\Pi^G_x(\mathcal{E}^*)$ is a complex of vector bundles on $X/G$, that is an element of $\text{Perf}(X/G)$.

**Lemma 4.5.** [Nev08, Lemma 2.14.] Let $Z$ be an affine scheme. Let $\mathcal{F}$ be a $G$-equivariant coherent sheaf on $Z$, such that the action of the stabiliser $G_z$ on the vector space $\mathcal{F} \otimes k(z)$ is trivial for every $z$ in $Z$. Then, there exists a $G$-equivariant locally free sheaf $\mathcal{E}$ on $Z$ with a surjective map $\mathcal{E} \to \mathcal{F}$ such that $\mathcal{E}$ descends to the quotient $Z/G$.

Recall that $\mathcal{E}$ descends to the quotient means that there exists a $\mathcal{F}^i \in \text{Coh}(Z/G)$ such that $\mathcal{E} \cong \pi_z^G(\mathcal{F}^i)$. The following result is an adaptation to our case of the replacement tool found in [Nev08, Proposition 4.1].

**Lemma 4.6.** Consider the following $G$-equivariant resolution of a complex $\mathcal{F}^*$ in $\mathcal{D}^G(X)$:

$$\mathcal{E}^* := \{0 \to \mathcal{E}^1 \xrightarrow{a_1} \mathcal{E}^2 \xrightarrow{a_2} \ldots \xrightarrow{a_{n-1}} \mathcal{E}^n \xrightarrow{a_n} 0\}.$$ 

If the action of the stabiliser $G_x$ is trivial on the $\mathcal{O}_X$-modules $H^n(\mathcal{E}^i \otimes k(x))$, for every point $x$ in $X$, then $\mathcal{E}^*$ is quasi isomorphic to a complex of $G$-equivariant locally free sheaves

$$\mathcal{F}^* := \{0 \to \mathcal{F}^m \to \ldots \to \mathcal{F}^0 \to \mathcal{F}^1 \to \ldots \to \mathcal{F}^n \to 0\}$$

such that $\mathcal{F}^n$ descends to the quotient.

**Proof.** Notice that Corollary 3.3 would directly imply the existence of $\mathcal{F}^*$. However, we still need to prove that $\mathcal{F}^n$ descends to the quotient. At this purpose, we will start from scratch and construct explicitly the positive part of the complex $\mathcal{F}^i$, $i \geq 0$, while the negative part will be constructed as a resolution of sheaves and glued to the previous one.

The descent property is local, hence we can assume that $X$ is affine. We have a surjection $\rho_n : \mathcal{E}^n \to \text{Coker}(a_{n-1})$; this induces another surjection for all $x \in X$, given by tensoring with $k(x)$:

$$\mathcal{E}^n \otimes k(x) \to \text{Coker}(a_{n-1}) \otimes k(x).$$

We know that $H^n(\mathcal{F}^* \otimes k(x)) = H^n(\mathcal{E}^* \otimes k(x)) = \text{Coker}(a_{n-1}) \otimes k(x)$ is $G_x$-invariant. We can apply Lemma 4.5 which implies the existence of a $G$-equivariant locally free sheaf $\mathcal{F}^n$ with a surjective morphism $\mathcal{F}^n \to \text{Coker}(a_{n-1})$ and a
morphism $f_n : \mathcal{V}^n \to \mathcal{E}^n$ such that $\rho_n \circ f_n = \gamma_n$. We found the last element of the complex $\mathcal{V}^\bullet$.

In order to define $\mathcal{V}^{n-1}$, consider the subsheaf $\mathcal{G}^n \subset \mathcal{E}^{n-1} \oplus \mathcal{V}^n$ made by those sections $(e, v)$ such that $\alpha_{n-1}(e) = f_n(v)$. By using the second projection in the direct sum, we get a morphism $\beta_{n-1} : \mathcal{G}^n \to \mathcal{V}^n$. Now we take a $G$-equivariant locally free sheaf $\mathcal{E}^{n-1}$ with a surjective map onto $\mathcal{G}^n$. By projecting on to the first summand, we obtain a map $f_{n-1} : \mathcal{V}^{n-1} \to \mathcal{E}^{n-1}$, and composing with $\beta_{n-1}$ we obtain a morphism $\beta_{n-1} : \mathcal{V}^{n-1} \to \mathcal{V}^n$.

The map $f_n$ induces a map from $\mathcal{V}^n / \text{Im}(\beta_{n-1})$ to $\text{Coker}(\alpha_{n-1})$ which is an isomorphism. It takes a standard diagram chasing argument to prove that $\text{Ker}(\gamma_n) = \text{Im}(\beta_{n-1})$: if a section $v$ of $\mathcal{V}^n$ is such that $\gamma_n(v) = 0$, then there exists a section $e$ of $\mathcal{E}^{n-1}$ such that $\alpha_{n-1}(e) = f_n(v)$, and thus $\beta_{n-1}(e, v) = v$; now it is sufficient to take a section $v'$ of $\mathcal{V}^{n-1}$ such that $\beta_{n-1}(v') = e$. If $v = \beta_{n-1}(v') = \beta_{n-1}(e, v)$ then $\gamma_n(v) = \rho_n(\alpha_{n-1}(e)) = 0$. A similar argument allows us to prove that the map $\gamma_{n-1} : \text{Ker}(\beta_{n-1}) \to \text{Ker}(\alpha_{n-1}) / \text{Im}(\alpha_{n-2})$ is surjective as well.

We now iterate the process: consider the subsheaf $\mathcal{G}^{n-1} \subset \mathcal{E}^{n-2} \oplus \mathcal{V}^{n-1}$ made by those sections $(e, v)$ such that $\alpha_{n-2}(e) = f_{n-1}(v)$. We have a natural morphism $\beta_{n-2} : \mathcal{G}^{n-1} \to \mathcal{V}^{n-1}$. Take a $G$-equivariant locally free sheaf $\mathcal{E}^{n-2}$ with a surjective map onto $\mathcal{G}^{n-1}$. Then, we obtain a map $f_{n-2} : \mathcal{V}^{n-2} \to \mathcal{E}^{n-2}$, and composing with $\beta_{n-2}$ we obtain a morphism $\beta_{n-2} : \mathcal{V}^{n-2} \to \mathcal{V}^{n-1}$. Finally, we get that $\gamma_{n-1}$ corresponds to $\text{Im}(\beta_{n-2})$. We have the following situation:

We can iterate this construction until we get a complex of $G$-equivariant vector bundles

$$0 \to \mathcal{V}^1 \xrightarrow{\beta_1} \mathcal{V}^2 \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_{n-2}} \mathcal{V}^{n-1} \xrightarrow{\beta_{n-1}} \mathcal{V}^n \to 0$$
such that $\gamma_j : \mathcal{V}^j \to \text{Coker}(\delta_{j-1})$ induces an isomorphism $H^1(\mathcal{V}^*) \simeq H^1(\mathcal{E}^*)$, for $2 \leq j \leq n$, and $\gamma_1 : \mathcal{V}^1 \to \mathcal{E}^1$ is surjective map. Moreover, by construction, $\mathcal{V}^n$ descends to the quotient. Thus, we can take the resolution of the subsheaf $i : \text{Ker}(\gamma_1) \hookrightarrow \mathcal{V}^1$:

$$
0 \longrightarrow \mathcal{V}^{-m} \longrightarrow \ldots \longrightarrow \mathcal{V}^{-1} \longrightarrow \mathcal{V}^0 \xrightarrow{\omega} \text{Ker}(\gamma^1) \longrightarrow 0.
$$

We obtain in this way the final form of the complex $\mathcal{V}^*$.

$$
0 \longrightarrow \mathcal{V}^{-m} \longrightarrow \ldots \longrightarrow \mathcal{V}^0 \xrightarrow{\omega_0} \mathcal{V}^1 \xrightarrow{\beta_1} \mathcal{V}^2 \longrightarrow \ldots \longrightarrow \mathcal{V}^n \longrightarrow 0.
$$

It is a complex of $G$-equivariant vector bundles quasi-isomorphic to $\mathcal{E}^*$ such that $\mathcal{V}^n$ descends to the quotient. These sheaves glue together because they are all quasi isomorphic to $\mathcal{F}^*$.

Since $\mathcal{V}^*$ is quasi-isomorphic to $\mathcal{E}^*$, we have that $H^i(\mathcal{V}^*) = 0$, for $i = -m, \ldots, 0$. Let us recall the following result about the action of a finite group acting on $C$-vector spaces.

**Lemma 4.7.** Let $H$ be a finite group acting on three $k$-vector spaces $V_1$, $V_2$ and $V_3$, sitting in an exact sequence

$$
V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3,
$$

where $\alpha$ and $\beta$ are equivariant maps. If $H$ acts trivially on $V_1$ and on $V_3$, then $H$ acts trivially on $V_2$ as well.

**Proof.** Let $x$ be in $V_2$ and let $h$ be in $H$. If $\beta(x) = 0$ then $x = \alpha(y)$ for a certain $y \in V_1$; so $hx = h\alpha(y) = \alpha(hy) = \alpha(y) = x$.

Suppose now that $\beta(x) \neq 0$. We have that $\beta(hx) = h\beta(x) = \beta(x)$, so $\beta(hx-x) = 0$. Hence, there exist $y \in V_1$ such that $x = hx - \alpha(y)$. Let make $h$ act again, since the action on $V_1$ is trivial we get $hx = h^2x - \alpha(y)$, which gives $x = h^2x - 2\alpha(y)$. Since $H$ is finite there exist an integer $n$ such that $h^n = \text{Id}$. By repeating the previous computation we get $x = h^n x - n\alpha(y)$, so $\alpha(y) = 0$. This concludes the proof.

Now we are ready to state and prove a criterion to characterize the complexes descending to perfect complexes.

**Theorem 4.8.** Let $G$ be a finite group acting on a smooth projective variety $X$. An object $\mathcal{F}^*$ in $\mathcal{D}^G(X)$ descends to $\text{Perf}(X/G)$ if and only if the following condition holds.

(\*) The stabiliser $G_x$ acts trivially on the $\mathcal{O}_X$-modules $H^j(\mathcal{F} \otimes k(x))$, for all $x \in X$, $j \in \mathbb{Z}$.

**Proof.** Start by assuming that $\mathcal{F}^*$ descends, and prove that condition (\*) holds.

Let $\mathcal{E}^*$ be a finite resolution of $G$-locally free sheaves of finite type of $\mathcal{F}^*$ obtained by Lemma 4.6.

We have the following isomorphism of $\mathcal{O}_X$-modules:

$$
H^1(\mathcal{F}^* \otimes k(x)) \xrightarrow{\sim} H^1(\mathcal{E}^* \otimes k(x)).
$$

Since $G_x$ acts trivially on $\mathcal{E}^* \otimes k(x)$, then it must act trivially also on $H^j(\mathcal{E}^* \otimes k(x))$, for all $j$, because the action of the group commutes with taking cohomology.
Viceversa, we assume that condition (•) holds. We will proceed by induction on the number of non-trivial cohomologies of $\mathcal{F}^*$, in order to prove that it descends. More precisely, the induction statement is the following: suppose that $H^j(\mathcal{F}^*) \neq 0$, for $j = 1, \ldots, n$, and the action of the stabiliser $G_x$ is trivial on $H^j(\mathcal{F}^* \otimes k(x))$ for every point $x \in X$, then $\mathcal{F}^*$ descends Perf$(X/G)$. If $n = 1$ the statement reduces to the case in which we assume that $\mathcal{F}^*$ is a coherent sheaf $\mathcal{F}$. We want to prove that $\mathcal{F}$ descends. The descent property is local, so we can suppose $X$ to be affine. If condition (•) holds, then in particular $G_x$ acts trivially on $\mathcal{F} \otimes k(x)$ for every $x$ in $X$:

$$H^j(\mathcal{F} \otimes k(x)) = H^0(\mathcal{F} \otimes k(x)) \cong \mathcal{F} \otimes k(x).$$

By Lemma 4.5 there exists a $G$-equivariant locally free sheaf $\mathcal{V}_1$ which descends to the quotient, and a surjective morphism $\phi : \mathcal{V}_1 \rightarrow \mathcal{F}$. Now, we denote by $\mathcal{K}_1$ be the kernel of morphism $\phi$ and we consider the following exact sequence:

$$0 \longrightarrow \mathcal{K}_1 \longrightarrow \mathcal{V}_1 \overset{\phi}{\longrightarrow} \mathcal{F} \longrightarrow 0.$$ 

We take the long exact cohomology sequence:

$$\cdots \rightarrow \text{Tor}_{j-1}(\mathcal{K}_1, k(x)) \rightarrow 0 \rightarrow \text{Tor}_{j-1}(\mathcal{F}, k(x)) \rightarrow$$
$$\rightarrow \text{Tor}_{j-2}(\mathcal{K}_1, k(x)) \rightarrow 0 \rightarrow \text{Tor}_{j-2}(\mathcal{F}, k(x)) \rightarrow$$
$$\cdots \rightarrow \text{Tor}_1(\mathcal{K}_1, k(x)) \rightarrow 0 \rightarrow \text{Tor}_1(\mathcal{F}, k(x)) \rightarrow$$
$$\rightarrow \mathcal{K}_1 \otimes k(x) \rightarrow \mathcal{V}_1 \otimes k(x) \rightarrow \mathcal{F} \otimes k(x) \rightarrow 0$$

Notice that the vector space $H^j(\mathcal{K}_1 \otimes k(x))$ is always in between two $G_x$-invariant vector spaces, so Lemma 4.7 implies that it is $G_x$-invariant as well. Now we iterate the process; apply again Lemma 4.5 to get a $G$-equivariant locally free sheaf $\mathcal{V}_2$ which descends to the quotient, and a surjective morphism $\phi' : \mathcal{V}_2 \rightarrow \mathcal{K}_1$. Let $\mathcal{K}_2$ be the kernel of $\phi'$. As before, we have the exact sequence:

$$0 \longrightarrow \mathcal{K}_2 \longrightarrow \mathcal{V}_2 \overset{\phi'}{\longrightarrow} \mathcal{K}_1 \longrightarrow 0.$$ 

From the long exact cohomology sequence associated to the sequence above, we deduce that $H^j(\mathcal{K}_2 \otimes k(x))$ is $G_x$-invariant. In particular there exists a $G$-equivariant locally free sheaf $\mathcal{V}_3$ which descends to the quotient, and a surjective morphism from $\mathcal{V}_3$ to $\mathcal{K}_2$. Since the homological dimension of Coh$(G)(X)$ is finite, the process must end in a finite number of iterations. We denote by $m$ this number. It means that, at some point we obtain a kernel $\mathcal{K}_m$ which is a locally free sheaf and such that the action of the stabiliser $G_x$ on the vector spaces $\mathcal{K}_m \otimes k(x)$ is trivial for every $x \in X$. Therefore, by Theorem 3.5 $\mathcal{K}_m$ descends to Perf$(X/G)$. In other words we obtain a finite $G$-equivariant locally free resolution of $\mathcal{F}$ of the form:

$$0 \longrightarrow \mathcal{V}_m \longrightarrow \cdots \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{F} \longrightarrow 0.$$ 

such that every $\mathcal{V}_i$ descends to the quotient for $i = 1, \ldots, m$.

Assume now that the statement is true for $n = 1$. Consider the following resolution of $\mathcal{F}^*$:

$$\mathcal{E}^* := [0 \rightarrow \mathcal{E}^1 \rightarrow \cdots \rightarrow \mathcal{E}^n \rightarrow 0].$$
By Lemma 4.6 there exists a complex $\mathcal{Y}^\bullet$ which is quasi-isomorphic to $\mathcal{E}^\bullet$ and $\mathcal{Y}^n$ descends to the quotient. So, we have the following distinguished triangle:

$$\mathcal{Y}^n[-n] \otimes k(x) \to \mathcal{Y}^\bullet \otimes k(x) \to \sigma_{\leq n-1} \mathcal{Y}^\bullet \otimes k(x) \to \mathcal{Y}^n[-n] \otimes k(x)[1],$$

where $\sigma_{\leq n-1} \mathcal{Y}^\bullet$ is the stupid truncation of the complex $\mathcal{Y}^\bullet$. Consider the long exact cohomology sequence:

$$0 \to H^{n-1}(\mathcal{Y}^\bullet \otimes k(x)) \to H^{n-1}(\sigma_{\leq n-1} \mathcal{Y}^\bullet \otimes k(x)) \to \mathcal{Y}^n[-n] \otimes k(x) \to H^n(\mathcal{Y}^\bullet \otimes k(x)) \to 0$$

By using Lemma 4.7 we have that

$$H^{n-1}(\sigma_{\leq n-1} \mathcal{Y}^\bullet \otimes k(x)) = H^{n-1}(\sigma_{\leq n-1} \mathcal{Y}^\bullet \otimes k(x))$$

is $G_x$-invariant. Furthermore,

$$H^j(\sigma_{\leq n-1} \mathcal{Y}^\bullet \otimes k(x)) = H^j(\mathcal{Y}^\bullet \otimes k(x)) \quad j = 1, \ldots, n-1.$$

It means that $\sigma_{\leq n-1} \mathcal{Y}^\bullet$ is a complex such that $H^j(\sigma_{\leq n-1} \mathcal{Y}^\bullet \otimes k(x))$ is $G_x$ invariant for all $j$. Moreover, $H^j(\sigma_{\leq n-1} \mathcal{Y}^\bullet) \neq 0$ for $j = 1, \ldots, n-1$. Hence we can apply the induction hypothesis to conclude that $\sigma_{\leq n-1} \mathcal{Y}^\bullet$ descends to $\text{Perf}(X/G)$. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{Y}^n[-n] & \to & \mathcal{Y}^\bullet \\
\downarrow & & \downarrow \\
\Pi^{G,*} \Pi^G_*(\mathcal{Y}^n[-n]) & \to & \Pi^{G,*} \Pi^G_*(\mathcal{Y}^\bullet) \\
\sigma_{\leq n-1} \mathcal{Y}^\bullet & \to & \Pi^{G,*} \Pi^G_*(\sigma_{\leq n-1} \mathcal{Y}^\bullet)
\end{array}$$

The first and third vertical map are quasi-isomorphism, by the five-Lemma this implies that the second vertical map is a quasi-isomorphism, it means that $\mathcal{Y}^\bullet$ descends to $\text{Perf}(X/G)$ and this concludes the proof.

5. Application of the criterion

In the previous section we found a criterion characterising the descending complexes. We can then make Theorem 4.5 by using Theorem 4.8 more explicit. Let $\Phi_{\mathcal{E}}$ be a Fourier-Mukai equivalence from $\mathcal{O}^G(X)$ to $\mathcal{O}^H(Y)$. Let us denote by $\mathcal{E}^\bullet$ the kernel of the inverse of $\Phi_{\mathcal{E}}$. Consider the following diagram

$$\begin{array}{ccc}
X & \xrightarrow{\eta_X} & X \times Y \\
\downarrow\pi_X & & \downarrow\pi_Y \\
X/G & \xrightarrow{\eta_{X/G}} & Y/H
\end{array}$$

(1)

Here the maps $\eta$ are the projections induced by the $\eta$’s. Recall that we will always take the equivariant version of the pushforward and pullback functors, as described in the introduction.

**Theorem 5.1.** The functor $\Omega$ is an equivalence between $\text{Perf}(X/G)$ and $\text{Perf}(Y/H)$ if and only if the two following conditions are satisfied

(i’) The stabiliser $H_y$ acts trivially on

$$\mathcal{R} \Gamma \left( (\iota^*_y \eta^*_X \Pi^G_{X,Y} \cdot \mathcal{E}^\bullet) \otimes \mathcal{E}^\bullet |_{X \times \{y\}} \right)$$

for any point $y$ in $Y$ and for any complex $\mathcal{E}^\bullet$ in $\text{Perf}(X/G)$.
(ii') The stabiliser $G_x$ acts trivially on
\[ \Gamma\left[ (\iota_x^* q_Y^* \Pi_Y^G \mathcal{B}^*) \otimes \mathcal{E}^* \right] \]
for any point $x$ in $X$ and for any complex $\mathcal{B}^*$ in $\text{Perf}(Y/H)$.

Here $\iota_x$ and $\iota_y$ are the natural embeddings of $\{x\} \times Y$ and $X \times \{y\}$ in $X \times Y$, respectively.

**Proof.** We want to apply Theorem 4.8 to the two conditions of Theorem 3.3. We will deal here only with Condition (i'), which comes from Condition (i), the second condition works in the same way. By Theorem 4.3 we have that $\Omega$ is an equivalence if and only if the stabiliser $H_y$ acts trivially on the $\mathcal{O}_Y$-module $H^j(\Phi_{\mathcal{E}^*} \otimes \Pi_Y^G \mathcal{B}^* \otimes k(y))$ for every integer $j$, for every $\mathcal{E}^* \in \text{Perf}(X/G)$ and $y \in Y$.

We now want to rephrase this condition in order to highlight the role of the kernel $\mathcal{E}^*$. We can write explicitly the first condition of Theorem 5.1 as
\[ H_y \text{ acts trivially on } H^j\left[ q_{Y,y}(\pi_Y^* (\Pi_Y^G \mathcal{E}^* ) \otimes \mathcal{E}^* ) \otimes k(y) \right], \]
again for every point $y$ of $Y$, every $\mathcal{E}^* \in \text{Perf}(X/G)$ and every integer $j$. Let us focus just on the expression inside the cohomology. The left square of (1) can be used for a base change, which leads to
\[ q_{Y,y}(\pi_Y^* (\pi_Y^* G \mathcal{E}^* ) \otimes \mathcal{E}^* ) \otimes k(y). \]

Consider the following diagram
\[ (2) \quad X \times \{y\} \xleftarrow{\iota} X \times Y \xrightarrow{q_Y} Y \]
\[ X/G \times Y/H \xrightarrow{\Pi_Y^G} Y/H, \]
where $y \in Y$ and $\iota$ is the natural embedding of $X \times \{y\}$ in $X \times Y$.

We note that the map $\iota$ can be used to bring $k(y)$ inside the equivariant push-forward $q_{Y,y}$:
\[ q_{Y,y}(\pi_Y^* (\pi_Y^* G \mathcal{E}^* ) \otimes \mathcal{E}^* ) \otimes \iota_x(\mathcal{E}^* \otimes k(y)) \].

The tensor product with $\iota_x(\mathcal{E}^* \otimes k(y))$ is defined fibrewise, and it is just the restriction to $X \times \{y\}$, so we get
\[ q_{Y,y}(\pi_Y^* (\pi_Y^* G \mathcal{E}^* ) |_{X \times \{y\}} \otimes \mathcal{E}^* |_{X \times \{y\}}). \]

Now we just focus on the first component of the tensor product. By Base Change from (2) we have
\[ (\pi_X \times \pi_Y)^* (G \mathcal{E}^* ) |_{X \times \{y\}} \cong (\pi_X \times \pi_Y)^* (\mathcal{E}^* ) \cong \iota^* (\Pi_Y^G \mathcal{E}^* ) \cong \iota_x^* q_Y^* (\Pi_Y^G \mathcal{E}^* ). \]

The map $q_{Y,y}$ is the push forward to the point $y$, so in the end we get a skyscraper sheaf supported in $y$ given by
\[ \Gamma\left[ \iota^* q_Y^* (\Pi_Y^G \mathcal{E}^* ) \otimes \mathcal{B}^* |_{X \times \{y\}} \right] \otimes \mathcal{O}_{Y,y}. \]

The action of $G_x$ fixes the point $y$, so we can just require $G_x$ to act trivially on the global sections. To prove the second condition we use the same procedure and we are done.
The conditions of the previous theorem can be checked just on a certain families of perfect complexes. In particular we get the following

**Corollary 5.2.** The conditions (i) and (ii) of Theorem 3.3 and the conditions (i') and (ii') of Theorem 5.1 can be checked on a spanning class.

**Proof.** Both these conditions are used in the theorem to prove the fully faithfulness of a certain functor. This property can be checked on a spanning class as proven in [Huy06, Proposition 1.49].

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