Majorana fermion formulation of the two channel Kondo model

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We show that a Majorana fermion description of the two channel Kondo model can emerge quite naturally as a representation of the algebra associated with the spin currents in the two channels. Using this representation we derive an exact equivalent Hamiltonian for the two channel model expressed entirely in terms of Majorana fermions. The part of the Hamiltonian that is coupled to the impurity spin corresponds to the vector part of the $\sigma\tau$ model (compactified two channel model). Consequently all the thermodynamic properties associated with the impurity spin can be calculated from the $\sigma\tau$ model alone. The equivalent model can be used to confirm the interpretation of the many-body excitation spectrum of the low energy fixed point of the two-channel model as due to free Majorana fermions with appropriate boundary conditions.

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The two channel Kondo model is known to have a low energy non-Fermi liquid fixed point and has been put forward as a model to explain non-Fermi liquid behavior as observed in several quite different physical systems at low temperatures, such as certain heavy fermion alloys and two-level systems. A full description of the model and the various theoretical approaches that have been applied to elucidate its physics, plus its potential applications, can be found in a recent extensive and thorough review by Cox and Zawadowski. There are exact solutions for the ground state and thermodynamics of the model from Bethe ansatz calculations which have been known for some time, but there are continuing efforts to find a simple intuitive understanding of the nature of the excitations in the neighborhood of the low energy fixed point. Numerical renormalization group and conformal field theory calculations give predictions for the many-body excitations at the fixed point but they do not provide a simple explanation of these excitations in terms of more elementary ones, as is possible at the Fermi liquid fixed point of the single channel Kondo model.

The bosonization approach of Emery and Kivelson showed that close to a particular value of the coupling in the strong coupling regime (analogous to the Toulouse limit of the one channel model), the low temperature behavior has the same form as the weak coupling model at low temperatures with the same Wilson ratio. This suggested that the low energy fixed point could be described by some effective Hamiltonian of this form with renormalized parameters. At the solvable Emery-Kivelson point the effective Hamiltonian contains a combination of particle and hole creation operators which can be expressed most conveniently in terms of Majorana fermions. Recently, Maldacena and Ludwig using abelian bosonization have reformulated the conformal field theory in terms of Majorana fermions.

The Majorana fermion approach was developed further by Coleman, Ioffe and Tsvelik who introduced the ‘compactified’ two channel or $\sigma\tau$ model. This is a single channel model in which the impurity spin is coupled to the conduction electron spin and isospin, a combination which is most conveniently expressed in terms of three Majorana fermions. It was conjectured, with supporting arguments, that the $\sigma\tau$ model has the same low energy fixed point as the two channel Kondo model on which it was modeled. Subsequently Coleman and Schofield introduced a version of the Anderson model (the O(3) or compactified Anderson model) which can be mapped into the $\sigma\tau$ model in the local moment regime using the Schrieffer-Wolff transformation. Recent numerical renormalization group, conformal field theory, weak and renormalized perturbation expansion results for both the $\sigma\tau$ and O(3) Anderson models of Bulla, Hewson and Zhang confirm that the localized compactified models do give the same low temperature thermodynamic behavior as the two channel model.

In this paper we show that the Majorana fermion description can emerge quite naturally within the two channel model as a representation of the algebra of the total spin current of the two channels. We further show using this representation that an exact equivalent model for the two channel Kondo model can be obtained entirely in terms of Majorana fermions. The part of this model which includes the interaction with the impurity spin is the vector part of the $\sigma\tau$ model. This implies that the impurity spin contribution to the thermodynamic properties of the two channel Kondo model can be calculated from the $\sigma\tau$ model alone. We show that this Majorana fermion version of the model can be used to confirm the interpretation of the low energy fixed point of the two channel model as free Majorana fermion excitations subject to appropriate boundary conditions.

We start with the Hamiltonian in the form,

$$H = H_0 + H_1$$

$$H_0 = \frac{\psi_j^\dagger \sigma(x)}{2\pi} \sum_{\sigma=\pm 1} \int_{-\infty}^{+\infty} dx : \psi_j^\dagger \sigma(x) (i\partial_x) \psi_j \sigma(x) :$$

$$H_1 = \sum_{\alpha=x,y,z} J_\alpha S^\alpha_\tau J^\alpha_\sigma(0),$$

where we have retained the s-wave scattering only, linearized the fermion spectrum, and replaced the incoming and outgoing waves with two left-moving electron fields $\psi_{j,\sigma}(x)$; $J^\alpha_\sigma(x)$ are the conduction electron spin current operators.
\[ J^a_a(x) = \sum_{j,\sigma,\sigma'} \psi_{j,\sigma}^\dagger(x) s^a_{\sigma,\sigma'} \psi_{j,\sigma'}(x) : \]

where \( s^a \) being spin-1/2 matrices. We can also introduce charge and flavor currents

\[ J_c(x) = \sum_{j,\sigma} \psi_{j,\sigma}^\dagger(x) \psi_{j,\sigma}(x) : \]

\[ J_f^b(x) = \sum_{j,j',\sigma} : \psi_{j',\sigma}^\dagger(x) a^b_{\sigma} \psi_{j,\sigma}(x) : \]

where \( t^a_{j,j'} \) are generators of an SU(2) symmetry group. Following Affleck and Ludwig, the free part of the Hamiltonian can be rewritten as a sum of three commuting terms by the usual point-splitting procedure (Sugawara construction):

\[ H_0 = \frac{v_f}{2\pi} \int_{-\infty}^{+\infty} dx \left[ \frac{1}{4} : J_c(x) J_c(x) : + \frac{1}{4} : J_f(x) \cdot J_f(x) : + \frac{1}{4} : \bar{J}_f(x) \cdot \bar{J}_f(x) : \right], \]

while the interaction term is expressed in terms of the electron spin currents and the impurity spin only. The information about the number of channels is contained in the commutation relations obeyed by the spin currents

\[ [J^a_a(x), J^b_b(x')] = i e^{abc} J^c_c(x) \delta(x-x') + \frac{k_i}{4\pi} \delta_{a,b} \delta(x-x') \]

indicating that \( J^a_a(x) \) form a SU(2) level \( k = 2 \) Kac-Moody algebra. Meanwhile, the charge and flavor currents satisfy

\[ [J_c(x), J_c(x')] = 2k_i \delta'(x-x'), \]

\[ [J^b_b(x), J^b_b(x')] = i e^{abc} J^c_c(x) \delta(x-x') + \frac{k_i}{4\pi} \delta_{a,b} \delta'(x-x'). \]

They form a \( U(1) \) Kac-Moody and another SU(2) level-2 Kac-Moody algebra, separately.

It is now quite natural to introduce a Majorana representation of the spin current operators in the form,

\[ J^c_+ = -i : \chi_2(x) \chi_3(x) : ; \]

\[ J^c_- = -i : \chi_3(x) \chi_1(x) : ; \]

\[ J^c_0 = -i : \chi_1(x) \chi_2(x) : ; \]

where \( \chi_1(x), \chi_2(x), \) and \( \chi_3(x) \) are left-moving free Majorana fermion fields, and it can be shown to reproduce the SU(2) level-2 Kac-Moody commutation relations. It is important to note that this representation is only appropriate for the two channel model as it leads to a level-2 algebra. It would be inappropriate for the single channel Kondo model where the corresponding spin current generates a level-1 algebra.

In a similar way, we can also introduce Majorana representations for the flavor currents

\[ J^c_f(x) = -i : \chi_4^\prime(x) \chi_5^\prime(x) ; \]

\[ J^c_f^\prime(x) = -i : \chi_5^\prime(x) \chi_1^\prime(x) ; \]

\[ J^c_0 = -i : \chi_1^\prime(x) \chi_2^\prime(x) ; \]

which reproduces the commutation relations satisfied by the flavor currents, and

\[ J_c(x) = -2i : \chi_4^\prime(x) \chi_5^\prime(x) ; \]

can represent the charge current operator. Note that \( \chi_\alpha^\prime \) with \( \alpha = 1, 2, 3, 4, 5 \) are also left-moving free Majorana fermion fields. It is well-known that the dynamics of charge, flavor, and spin are completely determined by the commutation relations of the current operators. Though the spin currents of the two channel Kondo model can be represented in terms of three Majorana fermion fields \( \chi_\alpha(x) (\alpha = 1, 2, 3) \), we emphasize that they can not be given any simple physical interpretation in terms of the original conduction electrons \( \psi_{j,\sigma}(x) \).

At this point we have the current operator terms in the Hamiltonian as quartic in the Majorana fields. The Sugawara construction enables one to write kinetic energy terms, which are quadratic in field operators, as quartic terms. This is what was done earlier in writing the free part of the Hamiltonian in form of Eq.\((1)\), and it is convenient if one is pursuing a purely algebraic approach as used in the conformal field theory. However for our purposes it is more convenient now to perform an inverse Sugawara construction by the usual point-splitting procedure again, and rewrite the terms quartic in the Majorana fermions as kinetic energy terms which are quadratic:

\[ \begin{align*}
: J_c(x) J_c(x) : & := 4 \sum_{\alpha=1}^{5} : \chi_\alpha^\prime(i \partial_x) \chi_\alpha^\prime(x) ; \\
: \bar{J}_f(x) \cdot \bar{J}_f(x) : & := 2 \sum_{\alpha=1}^{3} : \chi_\alpha^\prime(i \partial_x) \chi_\alpha^\prime(x) ; \\
: \bar{J}_s(x) \cdot \bar{J}_s(x) : & := 2 \sum_{\alpha=1}^{3} : \chi_\alpha(i \partial_x) \chi_\alpha(x). 
\end{align*} \]

The model Hamiltonian is transformed and divided into the following two parts,

\[ H_c + H_f = \frac{v_f}{4\pi} \sum_{\alpha=1}^{5} \int_{-\infty}^{+\infty} dx : \chi_\alpha^\prime(x)(i \partial_x) \chi_\alpha^\prime(x) ; \]

\[ H_s = \frac{v_f}{4\pi} \sum_{\alpha=1}^{3} \int_{-\infty}^{+\infty} dx : \chi_\alpha(x)(i \partial_x) \chi_\alpha(x) : - \frac{iJ}{2} \bar{S} \cdot \chi(0) \times \chi(0) . \]

\( H_c + H_f \) describes the non-interacting charge and flavor degrees of freedom. It has a symmetry group.
$U(1) \otimes SU(2)_2 = SO(5)$ and is expressed by five free Majorana fermion fields $\chi^\alpha_\sigma(x)$ ($\alpha = 1, 2, 3, 4, 5$). $H_s$ is the main part of the model and describes the spin degrees of freedom with three left-moving Majorana fermion fields $\chi_\sigma (\alpha = 1, 2, 3)$ interacting with the impurity spin. It has the symmetry $SU(2)_2$ or $SO(3)$ so that the full Hamiltonian has the symmetry group $SO(5) \otimes SO(3) = SO(8)$, which is represented by eight different Majorana fermion fields.

In the two channel model Hamiltonian, $H_s$ given in Eq. (11) is the only part which includes an interaction with the impurity spin. This part of the Hamiltonian is exactly equivalent to the vector part of the $\sigma$-$\tau$ model. The $\sigma$-$\tau$ (compactified) model is defined by

$$H' = -t \sum_{n=0} \sum_{\sigma} [c_{\sigma}^\dagger(n+1)c_{\sigma}(n) + H.c.] + J [\bar{s}(0) + \bar{\tau}(0)] \cdot \vec{S}_d,$$

where $\bar{s}(0)$ and $\bar{\tau}(0)$ denote the conduction electron spin and isospin current operators at the impurity site. The spin currents are defined as usual, and form a $SU(2)$ level-1 Kac-Moody algebra. The conduction electron isospin currents are defined as follows

$$\tau^+(n) = (-1)^n c_1^\dagger(n)c_1(n), \quad \tau^-(n) = (-1)^n c_4^\dagger(n)c_4(n),$$

$$\tau^z(n) = \frac{1}{2} [c_1^\dagger(n)c_1(n) + c_4^\dagger(n)c_4(n) - 1],$$

which also forms a level-1 $SU(2)$ Kac-Moody algebra. Under the particle-hole transformation: $c_1(n) \rightarrow c_1(n)$ and $c_4(n) \rightarrow (-1)^n c_4(n)$, the conduction electron spin current operators change into the isospin current operators, and vice versa. It is known that Majorana fermions can be introduced as follows,

$$\left( \begin{array}{c} c_1(n) \\ c_4(n) \end{array} \right) = e^{i\pi n/2} \sqrt{2} \left( \begin{array}{c} \Psi_1(n) - i\Psi_2(n) \\ -\Psi_3(n) - i\Psi_0(n) \end{array} \right),$$

where a phase factor has been introduced to absorb the staggered phase factors of the conduction electron states, $\Psi_0(n)$ is referred to as the scalar component, and $\Psi_1(n)$, $\Psi_2(n)$, $\Psi_3(n)$ are referred to as the vector components. The model is thus divided into two parts

$$H' = H'_sc + H'_vec$$

$$H'_sc = \frac{v_f}{2\pi} \int_{-\infty}^{+\infty} dx : \chi_\sigma(x)(i\partial_x)\chi_\sigma(x) :$$

$$H'_vec = \frac{v_f}{2\pi} \sum_{\alpha=1}^3 \int_{-\infty}^{+\infty} dx : \chi_{\alpha}(x)(i\partial_x)\chi_{\alpha}(x) : -iJ\vec{S}_d : \Psi(0) \times \Psi(0):.$$

We can identify $H'_vec$ with $H_s$ up to an overall factor 2. This implies that the application of the $\sigma$-$\tau$ model is not restricted to the very low energy regime but can be used to calculate the impurity contribution to the thermodynamics of the two channel Kondo model over the full temperature range. This result is exact subject only to the requirement of linear dispersion for the conduction electrons.

We briefly show that the Majorana fermion form of the Hamiltonian can be used to confirm the analysis of the many-body excitations at the low energy fixed point given in earlier work. In our previous paper on the isotropic $\sigma$-$\tau$ and O(3) Anderson models we found that the excitations at the fixed point could be constructed from free Majorana fermion excitations from the vector and scalar channels. The boundary conditions of the vector part had to be changed relative to the scalar part so that if the scalar part has periodic boundary conditions the vector part has antiperiodic ones (sector A), and vice versa (sector B). This difference can be interpreted as due to strong coupling with the impurity spin in the vector channel, but we also found the same fixed point for the compactified Anderson model for all values of $U$, so it can be interpreted as the $U = 0$ fixed point of O(3) Anderson model. The combined many-body spectrum agreed both with the results of the numerical renormalization group and the conformal field theory calculations corresponding to both sectors A and B. We note that the boundary conditions we impose on the fields $\chi_\sigma(x)$ cannot be placed in direct correspondence with boundary conditions imposed on the physical conduction electron fields $\psi_{j,\sigma}(x)$. The resulting spectrum is given in Table I, which is in agreement with the numerical renormalization group and conformal field theory calculations of the two channel Kondo model. It is the same as that conjectured in our earlier work and in agreement with the work of Malencena and Ludwig and Ye.

Away from the low energy fixed point, using the Majorana fermion representation, it is straightforward to identify the leading boundary operator as we have shown in our previous paper for the $\sigma$-$\tau$ model. Since the interaction is restricted to the spin part of the Hamiltonian with an O(3) symmetry group, the boundary operator must be O(3) invariant. There is a unique O(3) invariant operator with a scaling dimension $\frac{1}{2}$ in the spin part: $\chi_1(x)\chi_2(x)\chi_3(x)$. Therefore, we are led to identify $\theta\chi_1(0)\chi_2(0)\chi_3(0)$ as the leading boundary operator.
of the two channel Kondo model \( \theta \), where \( \theta \) is an anti-commuting variable. As its scaling dimension is greater than one, this boundary operator is irrelevant. The impurity contributions to the thermodynamic properties of the two channel Kondo model can be obtained by the second order perturbation calculations of this leading irrelevant boundary operator.

To summarize, we have used a Majorana fermion representation of the spin currents to obtain an exact equivalent Majorana fermion description of the two channel Kondo model. The spin part corresponds to the vector part of the \( \sigma-\tau \) model. It provides a simple prescription for understanding the low-energy fixed point. This formulation makes possible further developments such as the calculation of spin-spin current dynamical correlations for the two channel model from the \( \sigma-\tau \) model alone.

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| TABLE I. Many-body excitation energies and their corresponding degeneracies (\(dg\)) of the two channel Kondo model corresponding to the single particle spectrum at the low-energy fixed point. The energy levels without primes correspond to sector A, and those with primes to sector B. |
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| \(E_{\text{ex}}/(\pi \eta \psi /l)\) | \(\sum \epsilon_k n_k\) | \(dg\) | total \(dg\) |
| 0 | 0 | 2 | 2 |
| 1/8 | \(\epsilon_{1/2}\)′ | 4 | 4 |
| 1/2 | \(\epsilon_{1/2}\)′ | 10 | 10 |
| 5/8 | \(\epsilon_{1/2}\)′ | 12 | 12 |
| 1 | \(\epsilon_{1/2}\)′ | 6 | |
| 2\(\epsilon_{1/2}\)′ | 20 | 26 |
| 9/8 | \(\epsilon_{1/2}\)′ | 20 | |
| 2\(\epsilon_{1/2}\)′ | 12 | 32 |
| 3/2 | \(\epsilon_{3/2}\)′ | 10 | |
| \(\epsilon_{1/2}\)′ | 30 | 60 |
| \(3\epsilon_{1/2}\)′ | 20 | |
| 13/8 | \(\epsilon_{3/2}\)′ | 12 | |
| \(\epsilon_{1/2}\)′ ′ | 60 | 76 |

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