Solution of matching equations of IDA-PBC by Pfaffian differential equations

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ABSTRACT
Finding the general solution of partial differential equations (PDEs) is essential for controller design in newly developed methods. Interconnection and damping assignment passivity-based control (IDA-PBC) is one of such methods in which the solution to corresponding PDEs which are called matching equations is needed to apply it in practice. In this paper, these matching equations are transformed to corresponding Pfaffian differential equations. Furthermore, it is shown that upon satisfaction of the integrability condition, the solution to the corresponding third-order Pfaffian differential equation may be obtained quite easily. The method is applied to the PDEs of IDA-PBC in some benchmark systems such as Magnetic levitation system, Pendubot, and underactuated cable-driven robot to verify its applicability.

1. Introduction
Solving partial differential equations (PDEs) is one of the most challenging problems in mathematics. This issue is more crucial when the general solution is required while no boundary condition exists. One of the applications of such problems in control engineering is where the controller design in some methods is based on the solution of some PDEs. Interconnection and damping assignment passivity-based control (IDA-PBC) is one of the methods whose application is restricted to the prohibitive task of finding the general solution of PDEs (Gandarilla et al., 2020).

Port-controlled Hamiltonian (PCH) is a general method of modelling physical systems based on a Hamiltonian function together with interconnection and damping matrices (Batlle et al., 2009). One method to stabilise PCH systems is classical passivity-based control (PBC) where at priory the desired Hamiltonian as the desired storage function shall be assigned to the system and a suitable controller is designed to minimise the storage function (Haddad et al., 2018; Ortega & Garcia-Cansco, 2004). In order to rectify some of the technical issues of this method, the second class of PBC was proposed, in which, instead of fixing the closed-loop storage function, the desired structure of the closed-loop system is assigned. Interconnection and damping assignment (Ortega et al., 2002) and controlled Lagrangian (Bloch et al., 2000) are examples of such rectification. The desired Hamiltonian is found by the solution of a set of PDEs that are called matching equations (Franco, 2019). Since these PDEs do not have boundary conditions and the desired Hamiltonian should be minimum at the desired equilibrium point, the general solution of matching equations is required, which is a stumbling block in this method.

This problem has been the subject of attention of many researchers. In Acosta et al. (2005), a method for mechanical systems with one degree of underactuation has been developed. In this work, it is shown that upon satisfying some conditions, potential and kinetic energy PDEs may be solved easily. Viola et al. (2007) have striven to simplify the kinetic energy PDEs of underactuated mechanical systems by coordinate transformation. The matching equations of IDA-PBC are replaced by algebraic inequalities and algebraic equations in Acosta and Astolfi (2009) and Nunna et al. (2015), respectively. Borja et al. (2015), Donaire et al. (2015), Romero et al. (2016), Mehra et al. (2017), and Romero et al. (2018) are some other works that have focused on this issue by shaping the total energy of a class of underactuated systems without solving the matching equations. These papers are based on some restrictive assumptions on the system prototype, but instead, they are simply applicable. Generally, the mentioned works may be separated into two categories; some of them include an exceptional class of PCH systems, while the corresponding matching equations can be solved quite easily. On the contrary, other methods are applicable to a large class of systems while performing their solution in most cases is as hard as solving the original PDEs.

As indicated in Ortega et al. (2002), IDA-PBC generates all the stabilising controllers that are designed for a PCH system. This property coincides with the fact that IDA-PBC includes some degrees of freedom like desired interconnection matrix between the subsystems. Hence, one can derive different desired Hamiltonian together with desired interconnection and damping matrices for the closed-loop system such that all of them ensure the stability of desired equilibrium point while their properties are quite different. For example, the proposed controller for VTOL in Acosta et al. (2005) is smooth and global, whereas a local non-smooth one has been designed in Harandi and Taghirad (2020). Additionally, as indicated in Harandi et al. (2021), in order to get a suitable value for the upper bound
of control law, appropriate parameters for the closed-loop system are required. Therefore, the difficulty of solving a PDE on one side, and the requirement for a suitable solution with specific properties on another side, obligate the researchers to propose several methods to solve the matching equations.

In this paper, we utilise one of the powerful but less considered methods proposed in the literature (Sneddon, 2006), to derive the general solution of a PDE. It is shown that a first-order nonlinear PDE with \( n \) variables is equivalent to \( n \) Pfaffian differential equations. By this means, finding the solution of PDE is simplified to find the solution of its corresponding Pfaffian differential equations. Generally, solving this form of differential equations is not an easy task. However, for a third-order Pfaffian equation that satisfies a certain condition, the solution’s derivation is quite easy. Therefore, for a PDE with three variables, one should derive a Pfaffian differential equation such that the required condition is satisfied. By this means, the solution could be derived easily. Therefore, it can be considered a powerful complement of other works that essentially enhance the depth of available literature.

In what follows, details of this method are introduced, and it is applied to solve some benchmark systems. Notice that the basic mathematics of this work is borrowed from Sneddon (2006), and in this paper, we aim to show the applicability of this method in general. The advantages of the proposed method are as follows

- It is a general method that may be applied to the matching equations of a general system.
- The derived solution of matching equations is more general compared to that reported in the literature in several cases.
- It is possible to derive a solution with better properties rather than the reported one in the literature, e.g. enlarging the domain of attraction, considering the physical limitations, etc.
- Solving the PDEs that are not possible to be solved using the available methods such as an underactuated cable-driven robot introduced in Harandi et al. (2019).

### 2. IDA-PBC methodology

One of the methods for stabilisation of physical systems is IDA-PBC (Ortega & Garcia-Canseco, 2004). In the following, we briefly review this method. The curious reader is referred to Acosta et al. (2005), Ortega et al. (2002), and Ortega and Garcia-Canseco (2004) for more details. Note that for the ease of readability, Table 1 proposes the general notations used in this paper.

Consider a class of port-controlled Hamiltonian systems with a dynamic formulation of the following form

\[
\dot{x} = (J(x) - R(x)) \nabla_x H(x) + g(x)u, 
\]

where \( x \in \mathbb{R}^n \) denotes the states of the system, \( u \in \mathbb{R}^m \) denotes the input, \( J(x) = -I^T \in \mathbb{R}^{n \times n} \) and \( 0 \leq R(x) = R^T \in \mathbb{R}^{n \times n} \), are the interconnection and damping matrices respectively, \( H(x) \in \mathbb{R} \) denotes the total stored energy in the system and \( g(x) \in \mathbb{R}^{n \times m} \) is full rank input mapping matrix. The operator \( \nabla_x \) means \( \partial / \partial x \), i.e.

\[
\nabla_x H(x) = \begin{bmatrix} \frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n} \end{bmatrix}^T, 
\]

with \( x = [x_1, \ldots, x_n]^T \). The IDA-PBC method relies on matching the system (1) with the generalised Hamiltonian structure

\[
\dot{x} = (J_d(x) - R_d(x)) \nabla_x H_d(x),
\]

in which \( H_d(x) \) is continuously differentiable desired storage function which is (locally) minimum at the desired equilibrium point \( x^* \), while \( 0 \leq R_d(x) = R_d^T \in \mathbb{R}^{n \times n} \) and \( J_d(x) = -J_d(x) \in \mathbb{R}^{n \times n} \) represent desired interconnection and damping matrices, respectively. Assume that the matrices \( J_d, R_d \) together with \( H_d \) are chosen such that the following equation is satisfied

\[
g^{-1} (J - R) \nabla_x H(x) = g^{-1} (J_d - R_d) \nabla_x H_d(x),
\]

where matrix \( g^{-1}(x) \in \mathbb{R}^{n \times n} \) is left annihilator of \( g(x) \) such that \( g^{-1}g = 0 \). This equation results from matching the systems (1) and (3). The control law is derived as follows

\[
u(x) = (q^T g)^{-1} g^T ((J_d(x) - R_d(x)) \nabla_x H_d(x) - (J(x) - R(x)) \nabla_x H(x)).
\]

As a particular case, the IDA-PBC method can be utilised to regulate the position of an underactuated system (Acosta et al., 2005). Consider a (simple) mechanical system whose dynamic in port-controlled Hamiltonian representation is

\[
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & I_n \\ -I_n & 0_{n \times n} \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0_{n \times m} \\ G(q) \end{bmatrix} u
\]

where \( H(q,p) = \frac{1}{2} p^T M^{-1} q + V(q) \) is the total energy of the system as the summation of kinetic and potential energy, \( q, p \in \mathbb{R}^n \) are generalised position and momentum, \( M(q) > 0 \) denotes the inertia matrix and \( G(q) \in \mathbb{R}^{n \times m} \) is the full rank input mapping matrix. Suppose that the desired storage function is set to \( H_d = \frac{1}{2} p^T M_d^{-1} q + V_d(q) \) in which \( q^* \) is desired equilibrium point. Assume that the closed-loop equations are considered as follows

\[
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & M^{-1} M_d \\ -M_d M^{-1} & J_2 - G K_v G^T \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}
\]

### Table 1. Table of notations.

| Notation | Definition | Notation | Definition |
|----------|------------|----------|------------|
| \( x \)  | states of a system | \( H \)  | total stored energy |
| \( u \)  | control input | \( J \)  | interconnection matrix |
| \( R \)  | damping matrix | \( g(x) \) | input mapping matrix |
| \( R_d \) | desired damping matrix | \( J_d \) | desired interconnection matrix |
| \( H_d \) | desired storage function | \( x^* \) | desired equilibrium point |
| \( g^\perp \) | left annihilator of \( g \) | \( \nabla_x \) | gradient with respect to \( x \) |
| \( q \)  | generalized position | \( p \)  | generalized momentum |
| \( K \)  | kinetic energy | \( V \)  | potential energy |
| \( K_d \) | desired kinetic energy | \( V_d \) | desired potential energy |
| \( M \)  | inertia matrix | \( M_d \) | desired inertia matrix |
| \( \Phi \) | arbitrary function | \( J_2 \) | free skew-symmetric matrix |
in which $J_2(q, p) \in \mathbb{R}^{n \times n}$ is a skew-symmetric matrix. In order to match (6) with (8), the control law is derived as follows

$$u = (G^T G)^{-1} G^T (\nabla_q V - M_d M^{-1} \nabla_q V_d + \nabla_q K)$$

while $M_d$ and $V_d$ shall satisfy the following PDEs

$$G^\perp \{ \nabla_q (p^T M^{-1} p) - M_d M^{-1} \nabla_q (p^T M_d^{-1} p) + 2J_2 M_d^{-1} p \} = 0,$$

in which, $G^\perp$ is left annihilator of $G$. Note that (10a) is called the kinetic energy PDE (KE-PDE), and the potential energy PDE (PE-PDE) is presented in (10b).

The most challenging part of IDA-PBC design that may restrict its applications is solving the matching Equations (4) and (10). Invoking (Sneddon, 2006, Ch.2.3), in order to solve a first-order nonlinear PDE, it is possible to transform it into some Pfaffian differential equations, which are generally in the following form:

$$\sum_{i=1}^{n} f_i(x_1, \ldots, x_n) \, dx_i = 0.$$  \hfill (11)

It is shown that if $\phi_i(x_1, \ldots, x_n, z) = c_i$ with $i = 1, \ldots, n$, are independent solutions of the equations

$$\frac{dx_1}{p_1} = \frac{dx_2}{p_2} = \cdots = \frac{dx_n}{p_n} = \frac{dz}{R},$$

then for the arbitrary function $\Phi, \Phi(\phi_1, \ldots, \phi_n) = 0$, forms a general solution of the following partial differential equation

$$p_1 \frac{\partial z}{\partial x_1} + p_2 \frac{\partial z}{\partial x_2} + \cdots + p_n \frac{\partial z}{\partial x_n} = R,$$

in which, $p_i$s and $R$ are functions of independent variables $x_1, \ldots, x_n$ and also $z$. Note that since PDE (13) is generally nonlinear, its solution is implicit.

Solving the Pfaffian Equation (12) is still a cumbersome task while it is easier than that of the corresponding PDE. However, for a Pfaffian differential equation with $n = 3$, i.e.

$$P \, dx_1 + Q \, dx_2 + R \, dx_3 = 0,$$

it is shown that if the following condition holds

$$X^T \text{curl}(X) = 0,$$

with $X = [P, Q, R]^T$, then the problem turns to an exact differential equation which may be easily solved by direct integration. By this means, in order to solve (13) with $n = 3$, one may derive a Pfaffian differential equation such that condition (15) holds.

In the following, the method will be used to solve the matching equations of some benchmark systems to show its applicability. Thus, we concentrate on the matching equations and no simulation study is given. Note that more examples are given in Harandi and Taghirad (2020).

### 3. Solving matching equations of benchmark systems

Before presenting the examples, let us introduce the following useful corollary derived from the proposed method.

**Corollary 3.1:** Consider PDE (13) and assume that $p_i$s and $R$ are only functions of independent variables $x_i$s. Then, (a) The functions $z - \phi_i(x_1, \ldots, x_n) = c_i, i \in \{1, \ldots, n-1\}$ are the homogeneous solutions of this PDE if $\phi_i$s are solutions of the first $n-1$ Equation (12). (b) Non-homogeneous solution is derived by equalising the last term in (12) to other terms.

**Proof:** (a) Assume that $\phi_i(q_1, \ldots, q_n) = c_i$ are the solutions of first $n-1$ Equation (12). Since these equations are independent of $z$, therefore, $z - \phi_i(q_1, \ldots, q_n) = c_i$ are also the solutions of Pfaffian equations. Notice that they are homogeneous solutions of the PDE, because they satisfy the following equations which are related to the homogeneous part of PDE

$$\frac{dx_1}{P_1(x_1, \ldots, x_n)} = \cdots = \frac{dx_n}{P_n(x_1, \ldots, x_n)} = \frac{dz}{0},$$

(b) The proof of this part is clear. Non-homogeneous solution of PDE (13) corresponds to its special solution that depends on both the left and right-hand sides of (13). Hence, this is derived based on the last term of (12).

### 3.1 Magnetic levitation system

This system consists of a ferric ball hovered under a magnetic field created by an electromagnet. The schematic of the system is illustrated in Figure 1. Consider $\lambda$ as the flux generated by the magnet and $\theta$ as the distance of the centre of mass of the ball to its nominal position. It is shown in Ortega et al. (2001) that the system may be represented in PCH form as follows:

$$\dot{x} = \begin{bmatrix} -r & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u,$$

Figure 1. Schematic of magnetic levitation system ($y = \theta$).
in which, $x = [\lambda, \theta, m\theta]^T$ and $r$ represents coil resistance while the Hamiltonian function is given by Ortega et al. (2001)

$$H(x) = \frac{1}{2k} (1 - x_2^2)x_1^2 + \frac{1}{2m x_3^2} + mgx_2, \quad (18)$$

where $k$ is a constant. Let us consider stabilisation of the equilibrium point $x^* = [\sqrt{2kmg}, x_2^*, 0]^T$. It is shown in Ortega et al. (2001) that without modification of interconnection matrix, it is not possible to stabilise $x^*$. Hence, the following interconnection matrix is considered

$$\begin{bmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 1 \\ \alpha & -1 & 0 \end{bmatrix}.$$  \quad (19)

Furthermore, with $R_d = R$, the matching Equation (4) yields to

$$K_3(x) = 0,$$

$$\alpha K_1(x) - K_2(x) = -\frac{\alpha}{k} (1 - x_2)x_1, \quad (20)$$

where it is assumed that $H_d = H + H_a$ and

$$[K_1, K_2, K_3]^T := \frac{\partial H_a}{\partial x} = \left[ \frac{\partial H_a}{\partial x_1}, \frac{\partial H_a}{\partial x_2}, \frac{\partial H_a}{\partial x_3} \right]^T.$$  \quad (21)

The PDE represented by (20) shows that $H_a$ is independent of $x_3$. If $\alpha \neq 0$, using the proposed method, the PDE (20) is equivalent to the following Pfaffian equations

$$\frac{dx_1}{1} = \frac{dx_2}{-\beta - \frac{1}{k} (1 - x_2)x_1} = \frac{dH_a}{-\frac{1}{k} (1 - x_2)x_1} \quad (22)$$

with $\beta = 1/\alpha$. Assume that $\beta = -c_1x_1 - c_2x_2 - c_3$ with $c_i$s as arbitrary constants, and substitute it in (22), results in

$$\frac{dx_1}{1} = \frac{dx_2}{c_1x_1 + c_2x_2 + c_3} = \frac{dH_a}{-\frac{1}{k} (1 - x_2)x_1} \quad (23)$$

First, non-homogeneous solution is calculated by using Corollary 3.1. The strategy is to derive a Pfaffian equation satisfying (15). With some manipulation one may show that Equation (23) is equal to the following equations

$$\frac{dH_a}{\frac{x_1}{c_2k}} - \frac{dx_2}{c_2k} - \frac{x_2}{c_2k} dx_1 = \frac{dH_a}{\frac{x_1}{c_2k}} - \frac{dx_2}{c_2k} - \frac{x_2}{c_2k} dx_1 + \frac{c_3x_1}{c_2k} dx_1 - \frac{c_1x_1}{c_2k} dx_1 - \frac{c_3}{c_2k} dx_1 = 0.$$  \quad (24)

This equation satisfies condition (15) and is separable in the following form:

$$(dH_a) - \left( \frac{x_1}{c_2k} dx_2 + \frac{x_2}{c_2k} dx_1 \right) + \left( \frac{\partial H_a}{x_1}, \frac{\partial H_a}{x_2}, \frac{\partial H_a}{x_3} \right) = 0.$$  \quad (25)

Furthermore, by using Corollary 3.1, the homogeneous solution is derived from the following equation

$$(c_1x_1 + c_2x_2 + c_3) dx_1 - dx_2 = 0.$$  \quad (29)

This equation needs an integration factor $\mu$ which satisfies the following relation

$$\frac{\partial \mu}{\partial x_1} + (c_1x_1 + c_2x_2 + c_3) \frac{\partial \mu}{\partial x_2} = -c_2\mu.$$  \quad (30)

Hence, using the proposed method, it is equivalent to

$$\frac{dx_1}{1} = \frac{dx_2}{c_1x_1 + c_2x_2 + c_3} = \frac{d\mu}{-c_2\mu}.$$  \quad (31)

By considering the first and last terms, the solution may be given as $\mu = e^{-c_2x_1}$. Hence, homogeneous solution of (23) is derived from multiplying (29) to $\mu$ and then integration as follows

$$H_a = \phi \left( \frac{c_1}{c_2} x_1 + \frac{c_2}{c_2} x_2 + \frac{c_3}{c_2} e^{-c_2x_1} \right),$$  \quad (32)

in which, the function $\phi$ and the constants $c_i$s shall be determined such that $x^*$ becomes stable.

**Remark 3.1:** Ortega et al. (2001) and Nunna et al. (2015) state that $\theta$ shall remain in the interval of $(-1, \infty)$ while this limitation is released in our proposed solution. Note that using the method proposed in Tée et al. (2009) based on control barrier functions, one may define $c_i$s such that this constraint is satisfied. For example, set $c_1 = 0$ and $c_3 = -c_2$ results in $\alpha = -1/(c_2x_3 - c_2)$ that ensures $\theta \in (-1, \infty)$. Furthermore, for the solution given in Ortega et al. (2001), it is assumed that $\alpha$ is a constant. This limiting assumption is also released in the proposed solution given in this paper. Therefore, based on the proposed method, a solution with the suitable property was derived.
3.2 Micro electro-mechanical optical switch

Another benchmark example is the optical switching system with the following PCH model (Borja et al., 2015; Borovic et al., 2004):

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 \\
-1 & -b & 0 \\
0 & 0 & -c_1
\end{bmatrix} \nabla H(x) + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} u,
\]

(33)

whose energy function is given by Borja et al. (2015):

\[
H(x) = \frac{1}{2m} x_1^2 + \frac{1}{2} a_1 x_1^2 + \frac{1}{4} a_2 x_1^4 + \frac{x_1^2}{2c_1(x_1 + c_0)},
\]

(34)

where \( b, r > 0 \) are resistive constants, \( a_1, a_2 > 0 \) are spring terms, \( c_0, c_1 > 0 \) are capacitive elements and \( m \) denotes the mass of actuator. The physical constraint to consider is \( x_1 > 0 \), while the equilibrium points of the system are (Borja et al., 2015)

\[
x_1^* = 0, \quad x_3^* = (c_0 + x_1^*) \sqrt{2c_1(x_1^* + a_1 + a_2 x_1^* x_2^*)}.
\]

(35)

The aim of controller design in this example is to stabilise the system in \( x_1^* > 0 \). Hence, let us consider the following desired interconnection matrix

\[
J_d = \begin{bmatrix}
0 & 1 & 0 \\
-1 & -\alpha(x) & 0 \\
0 & \alpha(x) & 0
\end{bmatrix},
\]

(36)

where \( \alpha \) is a design parameter and \( R_d = R \). For simplicity and due to physical constraint, \( \alpha = \frac{\beta(x_1 + c_0)}{x_1} \).

By this means, the following Pfaffian differential equations should be solved

\[
\frac{dx_1}{-x_1} = \frac{dx_3}{\beta(x_1 + c_0)} = \frac{dH_a}{-\beta x_3 c_1}.
\]

(37)

In the sequel, it is shown that

\[
H_a = \phi(\beta x_1 + \beta c_0 \ln(x_1) + x_3, x_2)
\]

\[
- \frac{1}{2c_0 c_1} x_1^2 - \frac{\beta}{c_0 c_1} x_1 x_3 - \frac{\beta}{2c_0 c_1} x_1^2 - \frac{\beta}{c_1} x_1,
\]

(38)

is the solution of Pfaffian differential Equation (37). In order to derive non-homogeneous solution, the following equation is derived

\[
\frac{dx_3}{\beta(x_1 + c_0)} = \frac{dH_a}{\frac{\beta x_3 c_1}{c_0}} = \frac{\frac{x_3}{c_0 c_1} dx_3 + dH_a}{\frac{\beta x_1 c_0}{c_1}} = \frac{\frac{x_3}{c_0 c_1} dx_3 + dH_a + \frac{\beta x_3}{c_0 c_1} dx_1}{0}.
\]

(39)

Unfortunately, the last equation does not satisfy condition (15). To rectify this, let us add the term \( \frac{\beta x_1}{c_0 c_1} dx_3 \) to it. Finally, one may reach the following Pfaffian differential equation

\[
x_3 + \frac{\beta x_1}{c_0 c_1} dx_3 + dH_a + \frac{\beta x_3}{c_0 c_1} dx_1 + \frac{\beta x_1}{c_0 c_1} dx_1 + \frac{\beta}{c_1} dx_1 = 0,
\]

(40)

which has the following solution

\[
H_a = -\frac{1}{2c_0 c_1} x_3^2 - \frac{\beta}{c_0 c_1} x_1 x_3 - \frac{\beta}{2c_0 c_1} x_1^2 - \frac{\beta}{c_1} x_1.
\]

(41)

The homogeneous solution of (37) is derived easily as follows

\[
H_a = \phi(\beta x_1 + \beta c_0 \ln(x_1) + x_3, x_2).
\]

(42)

Thus, one can suitably define the constants and function \( \phi \) such that \( x^* \) becomes a stable equilibrium point while based on definition of \( \alpha \), the constraint \( x_1 > 0 \) is satisfied (Tee et al., 2009).

3.3 Third-order food-chain system

Consider the following model for third-order food-chain system based on Nunna et al. (2015) in PCH form (1) with the following values

\[
J = \begin{bmatrix}
x_1 x_2 & 0 & 0 \\
-x_1 x_2 & 0 & x_2 x_3 \\
0 & -x_2 x_3 & 0
\end{bmatrix}, \quad R = \begin{bmatrix}
x_1 & 0 & 0 \\
0 & x_2 & 0 \\
0 & 0 & x_3
\end{bmatrix}
\]

(43)

where \( x_i \) denotes the population of \( i \)-th species. In Ortega et al. (2000) it is shown that the PDE (4) is not solvable with \( J_d = J \) and \( R_d > 0 \) since the span of the first 2 rows of \( J_d - R_d \) is not involutive. The matching equation with the following matrices

\[
J_d = \begin{bmatrix}
0 & J_1 & J_2 \\
0 & -J_1 & 0 \\
J_2 & 0 & J_3
\end{bmatrix},
\]

\[
R_d = \begin{bmatrix}
R_1 & 0 & 0 \\
0 & R_2 & 0 \\
0 & 0 & R_3
\end{bmatrix}, \quad H_d = H + H_a,
\]

(44)

is in the following form

\[
-x_1 + x_1 x_2 = -R_1 (1 + K_1) + J_1 (1 + K_2) + J_2 (1 + K_3),
\]

\[
x_2 - x_1 x_2 + x_2 + x_3 = -J_1 (1 + K_1)
\]

\[
-R_2 (1 + K_2) + J_3 (1 + K_3),
\]

(45)

in which \( K_i \)s are defined in (21). If we set \( R_d = I \) as the simplest choice, it is inferred that with \( J_1 = 0, J_2 = f(x_1), J_3 = g(x_2) \), the PDEs are involutive. This means that homogeneous part of PDEs has a solution. The overall of PDEs has also a solution if non-homogeneous solution of a PDE satisfies the other PDE in the above equations. The corresponding Pfaffian differential equations are

\[
\frac{dx_1}{-1} = \frac{dx_2}{0} = \frac{dx_3}{f(x_1)} = \frac{dH_a}{-x_1 + x_1 x_2 + 1 - f(x_1)},
\]

\[
\frac{dx_1}{0} = \frac{dx_2}{-1} = \frac{dx_3}{g(x_2)} = \frac{dH_a}{-x_2 - x_1 x_2 + 2 x_2 x_3 + 1}.
\]

(46)

The solution of these equations using the explained methods is

\[
H_a = \phi_1 \left( x_2, \int f(x_1) \, dx_1 + x_3 \right)
\]
\[ H_a = \phi_2 (x_1, x_2 + \int \frac{1}{g(x_3)} \, dx_3) + \frac{1}{2} \alpha^2 + \frac{1}{2} x_1 x_2^2 - x_2 \\
\quad + \int \left( \alpha(x_2, x_3) + \beta(x_2, x_3) g(x_3) \right), \]

s. to \( \alpha + \beta g = -x_2 x_3, \) \( (47) \)

where the functions \( \alpha \) and \( \beta \) should be defined such that the last term is integrable. Now we should define \( f(x_1) \) and \( g(x_3) \) such that the non-homogeneous solution of a PDE lies in the homogeneous solution of other PDE. Hence, by defining \( f(x_1) = -1 \) and \( g(x_3) = 0, \) the solution of PDEs is

\[ H_a = \phi (x_1 - x_3) + \frac{1}{2} x_2^2 \] 

Hence, although the proposed method is based on a PDE, it might be utilised to solve a set of PDEs.

### 3.4 Pendubot

Here, the IDA-PBC method is applied to pendubot system in which it is not matched to the proposed papers based on total energy shaping without solving matching equations (Borja et al., 2015; D'Onaione et al., 2015; Mehra et al., 2017; Romero et al., 2018, 2016). The robot consists of two revolute joints in which merely the first one is actuated. The schematic of this system is shown in Figure 2. The dynamic model of the robot may be expressed in the form of (6) with the following matrices (Sandoval et al., 2008)

\[ M = \begin{bmatrix} c_1 + c_2 + 2c_3 \cos(q_2) & c_2 + c_3 \cos(q_2) \\ c_2 + c_3 \cos(q_2) & c_2 \end{bmatrix} \]

\[ G = [1, 0]^T, \quad V = -c_4 g \cos(q_1) - c_5 g \cos(q_1 + q_2), \]

where the constants \( c_i \) are given as follows

\[ c_1 = m_1 l_1^2 + m_2 l_1^2 + I_1, \quad c_2 = m_2 l_2^2 + I_2, \]
\[ c_3 = m_2 l_1 l_2, \quad c_4 = m_1 l_1 + m_2 l_1, \quad c_5 = m_2 l_2. \]

In Sandoval et al. (2008), it is shown that the corresponding KE-PDE given in (10a) is simplified to the following equation

\[ 2c_3 \sin(q_2) \left( \lambda_2^2 + \lambda_3 \lambda_4 \right) + \lambda_4 \frac{d}{dq_2} (\lambda_3 (c_2 + c_3 \cos(q_2)) + \lambda_4 c_2) = 0, \] \( (51) \)

in which

\[ M_d M^{-1} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}. \] \( (52) \)

Note that two other PDEs generated from KE-PDE (10a) may be solved by a suitable definition of the matrix \( J_2 \) in (8). The PE-PDE (10b) for this system is

\[ \lambda_3 V_{q_1} V_d + \lambda_4 V_{q_2} V_d = c_5 g \sin(q_1 + q_2). \] \( (53) \)

Since, PDE (51) has two unknown variables, for simplicity, assume that \( \lambda_4 = k \lambda_3, \) and reduce it to the following Pfaffian differential equations:

\[ \frac{d q_1}{0} = \frac{d q_2}{k \lambda_3 (c_2 + c_3 \cos(q_2) + k c_2)} = \frac{d \lambda_3}{- c_3 \lambda_3^2 \sin(q_2) (2 + k)}, \] \( (54) \)

Let us define \( k = -1 \) to simplify these equations. The non-homogeneous solution is derived from the following equation

\[ \frac{d \lambda_3}{\lambda_3} = \tan(q_2) \frac{d q_2}{c_5 g \cos(q_2) \sin(q_1 + q_2)}, \] \( (55) \)

which has the solution \( \lambda_3 = \frac{k}{c_5 g \cos(q_2) \sin(q_1 + q_2)} \) with \( k > 0 \) an arbitrary value since \( M_d \) should be positive. Note that the homogeneous solution is trivially found to be \( \phi(q_1). \) The corresponding Pfaffian equations to PDE (53) are given as follows

\[ \frac{d q_1}{\kappa} = \frac{d q_2}{c_5 g \cos(q_2) \sin(q_1 + q_2)}, \] \( (56) \)

The homogeneous solution is \( V_d = \phi(q_1 + q_2). \) In order to compute the non-homogeneous solution, we should derive an equation in the form of

\[ f_1(q_1, q_2) \frac{d q_1}{c_5 g \cos(q_2) \sin(q_1 + q_2)} + f_2(q_1, q_2) \frac{d q_2}{c_5 g \cos(q_2) \sin(q_1 + q_2)} = 0, \] \( (57) \)

in which,

\[ \kappa f_1 - \kappa f_2 + c_5 g \cos(q_2) \sin(q_1 + q_2) = 0, \] \( (58) \)

and the following constraint resulted from (15) shall be satisfied

\[ \frac{\partial f_2}{\partial q_1} = \frac{\partial f_1}{\partial q_2}. \] \( (59) \)

Combination of the two above equations yields the following equation

\[ \kappa \frac{\partial f_2}{\partial q_1} - \kappa \frac{\partial f_2}{\partial q_2} = -c_5 g \cos(q_1 + q_2). \] \( (60) \)
The solution to this equation is
\[ f_2 = \frac{c_5 g}{\kappa} \sin(q_1 + 2q_2). \] (61)

Therefore, the Pfaffian Equation (57) yields to
\[ (c_5 g \sin(q_1 + 2q_2) - c_5 g \cos(q_2) \sin(q_1 + 2q_2)) dq_1 + c_5 g \sin(q_1 + 2q_2) dq_2 - \kappa dV_d = 0. \] (62)

In order to derive its solution, rewrite it in the following form
\[ c_5 g \sin(q_2) \cos(q_1 + 2q_2) dq_1 + (c_5 g \sin(q_2) \cos(q_1 + q_2) + c_5 g \cos(q_2) \sin(q_1 + q_2)) dq_2 - \kappa dV_d = 0, \] (63)
whose solution may be found easily as
\[ V_d = \frac{c_5 g}{\kappa} \sin(q_1 + q_2) \sin(q_2). \] (64)

**Remark 3.2:** In Sandoval et al. (2008), the simplest solution of (51) is reported, in which \( \lambda_3 \) and \( \lambda_4 \) are set to be constant values. Here, a nontrivial solution with an enlarged domain of attraction is derived. In Sandoval et al. (2008), the simplest solution, where
\[ M_{d02} \propto -c_2 + c_3 \cos(q_2) \] (58)

is positive definite if \( q_2 \in (-\epsilon, \epsilon) \) with \( \epsilon = \arccos(c_3) \). This limitation is also released in the proposed solution, where \( M_{d02} \propto \kappa c_3 \) and \( M_{d12} \propto \kappa c_1 / \cos(q_2) + \kappa c_3 \) and the condition \( \det(M_d) > 0 \) confines \( q_2 \) inside a subset of the interval \((-\pi, \pi)\) which can be enhanced by enlarging arbitrary value \( M_{d11} \).

### 3.5 Underactuated spatial cable-driven robot

Cable-driven robots are a well-known type of parallel robot that the links are made by cables (Harandi et al., 2019; Korayem et al., 2010). Since cables can only apply tensile force, cable-driven robots are usually implemented with redundant actuators (Harandi et al., 2021). However, recently some researchers have focused on underactuated cable-driven robots due to the several advantages (Harandi et al., 2019; Ida et al., 2019). In this section, we concentrate on a 3-DOF cable-driven robot with two actuators. The schematic of this robot is shown in Figure 3. This system consists of a suspended mass which may have out-of-plane oscillation. Assume that the centre of coordinate is located on the first actuator, and the position of actuators is given by:
\[ A_1 = [0, 0, 0]^T, \quad A_2 = [b, 0, 0]^T \] (65)

Dynamic matrices of the robot may be easily found as
\[ M = ml_3, \quad V = mg \]

\[ G = \begin{bmatrix}
  \frac{x}{l_1} & \frac{x - b}{l_1} \\
  \frac{y}{l_1} & \frac{y}{l_1} \\
  \frac{z}{l_1} & \frac{z}{l_1}
\end{bmatrix} q = \begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} \]

where \( q \) denotes the position of end-effector, \( m \) denotes the payload mass, and
\[ \dot{r}_1^2 = x^2 + y^2 + z^2, \quad \dot{r}_2^2 = (x - b)^2 + y^2 + z^2. \] (67)

The equilibrium points of the system are \( q^* = [x^*, y^*, 0] \). Since these are natural equilibrium points of the robot, one may merely shape potential energy to stabilise the robot. However, in this work, we try to shape the total energy of the system for a broader representation and also reduction of convergence time. For this robot, KE-PDE introduced in (10a) yields to:
\[ G^\perp(-m^{-1}M_d \nabla q(p^T M_d^{-1} p) + 2J_2 M_d^{-1} p) = 0, \] (68)

with
\[ G^\perp = [0, -bz, by]. \] (69)

As explained in Acosta et al. (2005), the general solution of KE-PDE is obtained from the following equation
\[ \sum_{i=1}^{n} \gamma_i(q) \frac{dM_d^{-1}}{dq_i} = -[\mathcal{J}(q)A^T(q) + A(q)\mathcal{J}^T(q)] \]

where
\[ J_d = \begin{bmatrix}
  -p^T \alpha_1 & 0 & -p^T \alpha_2 & \cdots & -p^T \alpha_{n-1} \\
  -p^T \alpha_1 & 0 & 0 & \cdots & -p^T \alpha_{n-3} \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  -p^T \alpha_{n-1} & -p^T \alpha_{n-3} & \cdots & 0
\end{bmatrix}, \]

\[ \tilde{p} = M_d^{-1} p, \quad \mathcal{J} = [\alpha_1; \alpha_2; \cdots; \alpha_{n_0}] \in \mathbb{R}^{n_0 \times n_0}, \]

\[ A = -[W_1(G^\perp)^T, \ldots, W_m(G^\perp)^T] \in \mathbb{R}^{n_0 \times n}, \]

\[ \gamma = G^\perp M_d M^{-1}, \quad n_0 = n(n - 1)/2. \] (70)

In order to define \( W_i \), one may define the \((i, j)\)th element of matrices \( F_{ij}^{kl} \in \mathbb{R}^{n \times n} \) with \( k, l \in \{1, \ldots, n\} \) as follows
\[ F_{ij}^{kl} = \begin{cases}
  1 & \text{if } j > i, i = k \text{ and } j = l \\
  0 & \text{otherwise}
\end{cases} \] (71)

and set \( W^{kl} = F_{ij}^{kl} - (F_{ij}^{kl})^T \) while \( W_i \)s as
\[ W_1 = W_1^{12}, \quad W_2 = W_2^{13}, \ldots, \quad W_{n_0} = W^{(n - 1)n}. \] (72)
By this means one should solve the following PDE:
\[
(-z M_{d2} + y M_{d3}) \frac{\partial M_d}{\partial y} + (-z M_{d2} + y M_{d3}) \frac{\partial M_d}{\partial z} = m \begin{bmatrix}
2(-z \alpha_1 + y \alpha_2) & * & *

-2\alpha_2 + y(\alpha_2 + \alpha_3) & 2y \alpha_3 & *
yz + z(\alpha_3 - \alpha_1) & y \alpha_3 + z \alpha_3 & 2z \alpha_3
\end{bmatrix},
\]
(74)
where the *’s in the last matrix denote that it is symmetric. It is clear that \( M_{d1}, M_{d2}, M_{d3} \) may be found arbitrary and the remaining terms shall satisfy the following equations
\[
(-z M_{d2} + y M_{d3}) \frac{\partial M_{d2}}{\partial y} + (-z M_{d2} + y M_{d3}) \frac{\partial M_{d3}}{\partial z} = 2my \alpha_2, \\
(-z M_{d2} + y M_{d3}) \frac{\partial M_{d2}}{\partial y} + (-z M_{d2} + y M_{d3}) \frac{\partial M_{d3}}{\partial z} = my \alpha_3 + mz \alpha_2,
\]
(75)
\[
(-z M_{d2} + y M_{d3}) \frac{\partial M_{d2}}{\partial y} + (-z M_{d2} + y M_{d3}) \frac{\partial M_{d3}}{\partial z} = 2mz \alpha_3.
\]
(76)
This is a system of PDEs with two arbitrary functions. Hence, it is possible to convert it to a single PDE. However, there is no simple analytical solution for it. In the sequel, we will apply the proposed method to find the solution of (75). In order to convert (75) to Pfaffian equations, substitute first and third equations of (75) in the second equation. This yields to:
\[
\begin{align*}
\frac{dy}{P_1} = \frac{dz}{P_2} &= \frac{dM_{d3}}{R} \\
0 &= -z M_{d2} + y M_{d3}, \\
0 &= -z M_{d2} + y M_{d3}, \\
R &= -\left( \frac{y}{2} \frac{\partial M_{d3}}{\partial z} + \frac{z^2}{2y} \frac{\partial M_{d2}}{\partial y} + \frac{y^2}{2z} \frac{\partial M_{d3}}{\partial y} + \frac{y}{2} \frac{\partial M_{d2}}{\partial y} \right) - \left( z \frac{\partial M_{d2}}{\partial y} - \frac{z}{2} \frac{\partial M_{d3}}{\partial y} \right),
\end{align*}
\]
(77)
where \( M_{d2} \) and \( M_{d3} \) are set arbitrarily based on \( \alpha_2 \) and \( \alpha_3 \). Equation (76) is equivalent to the following equation
\[
\frac{z \, dy + y \, dz}{-z^2 M_{d2} + y^2 M_{d3}} = \frac{dM_{d3}}{R}
\]
(78)
Note that the left-hand side is independent of \( M_{d3} \), while \( R \) is the summation of two terms including a linear term and an independent term with respect to \( M_{d3} \). Pfaffian Equation (78) is easier to solve if \( R \) is independent of \( M_{d3} \). Notice that the last two terms in \( R \) are fractional and may lead to non-positive definite \( M_d \). Hence, one may consider \( M_{d2} \) and \( M_{d3} \) as
\[
M_{d2} = \frac{y^2}{2} + k_1, \quad M_{d3} = \frac{z^2}{2} + k_2,
\]
(79)
with \( k_1, k_2 > 0 \) to reduce the complexity. By substituting these values in (78), the solution is derived easily as \( M_{d3} = \frac{1}{2} yz \). Hence, the structure of \( M_d \) is in the following form
\[
M_d = \begin{bmatrix}
* & * & *

* & \frac{y^2}{2} + k_1 & \frac{1}{2} yz

* & \frac{1}{2} yz & z^2 + k_2
\end{bmatrix}
\]
(80)
where undefined elements may be determined arbitrarily. Notice that these elements do not appear in PE-PDE.

Potential energy PDE (10a) for this robot may be derived as:
\[
-bmgz = bm^{-1} (-z M_{d2} + y M_{d3}) \frac{\partial V_d}{\partial y} + bm^{-1} (-z M_{d2} + y M_{d3}) \frac{\partial V_d}{\partial z},
\]
(81)
Substitute (80) in this equation
\[
-m^2 gz = -k_1 z \frac{\partial V_d}{\partial y} + k_2 y \frac{\partial V_d}{\partial z}
\]
(82)
This is a simple PDE that can be solved easily by Corollary 3.1. The corresponding Pfaffian equations are
\[
\begin{align*}
\frac{dx}{0} &= \frac{dy}{-k_1 z} = \frac{dz}{k_2 y} = \frac{dV_d}{-m^2 gz}
\end{align*}
\]
(83)
It is clear that \( x = c_1 \) and \( k_2 y^2 + k_1 z^2 = c_2 \) are the solutions of the first two equalities. Thus, homogeneous solution of PDE is given as
\[
V_d = \phi (x, k_2 y^2 + k_1 z^2),
\]
(84)
and from second and forth terms, non-homogeneous solution is obtained as
\[
V_d = \frac{m^2 g}{k_1} (y - y^*),
\]
(85)

Remark 3.3: In this example, we have used the total energy shaping method for the spatial cable-driven robot. Note that to the best of authors’ knowledge, none of the reported papers on the topic of total energy shaping of an underactuated robot, e.g. Acosta et al. (2005), Borja et al. (2015), Donaire et al. (2015), Romero et al. (2016), and Viola et al. (2007) can be used to find a configuration-dependent solution.
3.6 Underactuated planar cable-driven robot

In this example, let us apply IDA-PBC method to a 3-DOF underactuated planar cable-driven robot proposed in Harandi et al. (2019). The schematic of this robot is illustrated in Figure 4. Dynamical matrices of the robot are in the form of (6) as given in Harandi et al. (2019):

$$G^\top = \begin{bmatrix} \frac{x - a \cos(\theta)}{l_1} & \frac{y - a \sin(\theta)}{l_1} & -a \cos(\theta) - a \sin(\theta) \\ \frac{x - b + a \cos(\theta)}{l_2} & \frac{y + a \sin(\theta)}{l_2} & a \cos(\theta) + a \sin(\theta) \\ \frac{2 \cos(\theta)}{l_1} & \frac{2 \sin(\theta)}{l_1} & 0 \end{bmatrix}$$

$$l_1^2 = (x - a \cos(\theta))^2 + (y - a \sin(\theta))^2, \quad l_2^2 = (x - b + a \cos(\theta))^2 + (y + a \sin(\theta))^2,$$

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & l \end{bmatrix}, \quad V = mgy, \quad q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$ (86)

For this robot, the manifold of equilibrium points may be derived as Harandi et al. (2019):

$$G^\top \nabla_q V = 0 \implies -2xy \cos(\theta) + by \cos(\theta) + ab \sin(\theta) \cos(\theta) + 2x^2 \sin(\theta) - 2bx \sin(\theta) = 0.$$ (87)

As indicated in Harandi et al. (2019), these points are natural equilibrium points of the system; thus, only potential energy shaping is required.

The PE-PDE (10b) for this system is in the following form:

$$\begin{align*}
-2xy \cos(\theta) + by \cos(\theta) + ab \sin(\theta) \cos(\theta) + 2x^2 \sin(\theta) - 2bx \sin(\theta) \\
= a(2 \cos(\theta) y^2 - 2 \sin(\theta) xy + by \sin(\theta) - ab \sin^2(\theta)) \frac{\partial V_d}{\partial x} \\
+ a(-2xy \cos(\theta) + by \cos(\theta)) \frac{\partial V_d}{\partial y} \\
+ ab \sin(\theta) \cos(\theta) + 2x^2 \sin(\theta) - 2bx \sin(\theta)) \frac{\partial V_d}{\partial x} \\
+ (2ax \sin(\theta) - 2ay \cos(\theta) + by - ab \sin(\theta)) \frac{\partial V_d}{\partial \theta}
\end{align*}$$ (88)

This is essentially a stumbling block to the procedure of controller design. However, it can be solved in a systematic way using the proposed method in section 2. Corresponding Pfaffian differential equations are

$$\begin{align*}
\frac{dx}{P_1} &= \frac{dy}{P_2} &= \frac{d\theta}{P_3} = \frac{dV_d}{mgP_2} \\
\text{with} \\
P_1 &= a(2 \cos(\theta) y^2 - 2 \sin(\theta) xy + by \sin(\theta) - ab \sin^2(\theta)) \\
P_2 &= a(-2xy \cos(\theta) + by \cos(\theta)) \\
&\quad + ab \sin(\theta) \cos(\theta) + 2x^2 \sin(\theta) - 2bx \sin(\theta)) \\
P_3 &= (2ax \sin(\theta) - 2ay \cos(\theta) + by - ab \sin(\theta)).
\end{align*}$$ (89)

To compute the homogeneous solution, let us derive a Pfaffian equation that satisfies condition (15). For this purpose, and considering (88), it is reasonable to derive a Pfaffian equation whose corresponding coefficients of $dx$ and $dy$ are merely function of $\theta$. Hence, let’s start with the following expression to omit $x^2$ and $y^2$ in its denominator

$$(4a \cos(\theta)) \ dx + (4a \sin(\theta)) \ dy + (-4a \sin(\theta)x + 4a \cos(\theta)y) \ d\theta,$$ (91)

which results in

$$\begin{align*}
\frac{(4 \cos(\theta)) \ dx + (4 \sin(\theta)) \ dy}{(4 \cos(\theta) y^2 + 4 \sin(\theta) x y)} &= \text{Equation (89)} \\
+ 4a \cos(\theta) x y^2 + 4a \cos(\theta) x y + 4b \sin(\theta) \cos(\theta)
\end{align*}$$ (92)

In this equation, $x^2$ was omitted. To omit $y^2$, first add $-2bdx$ to (91) and then add $2ab \sin(\theta) \ d\theta$ to it:

$$\begin{align*}
\frac{(4a \cos(\theta)) \ dx + (4a \sin(\theta)) \ dy + (-4a \sin(\theta)x + 4a \cos(\theta)y) \ d\theta}{0} \\
+ \frac{-2b \ dx + 2ab \sin(\theta) \ d\theta}{0} &= \text{Equation (89)}
\end{align*}$$ (93)

Thus, the nominator shall be zero, and by this means, one can easily verify that condition (15) holds. Hence, the solution of the Pfaffian equation

$$\begin{align*}
(4a \cos(\theta) - 2b) \ dx + (4a \sin(\theta)) \ dy + (-4a \sin(\theta)x + 4a \cos(\theta)y + 2ab \sin(\theta)) \ d\theta &= 0,
\end{align*}$$ (94)
is as follows

\[ V_d = \phi \left( (4a \cos(\theta) - 2b)x + 4a \sin(\theta)y - 2ab \cos(\theta) \right) . \]  

(95)

With a similar approach, we try to get a separable Pfaffian equation in the following form

\[ P(x_1) \, dx_1 + Q(x_2) \, dx_2 + R(x_3) \, dx_3 = 0, \]

(96)

which is trivially integrable and has the following solution

\[ \phi \left( \int P(x_1) \, dx_1 + \int Q(x_2) \, dx_2 + \int R(x_3) \, dx_3 \right) . \]

(97)

After some manipulations, the following equation is obtained

\[ x \, dx + y \, dy - \frac{b}{2} \, dx + \frac{ab}{2} \sin(\theta) \, d\theta = \text{Equation (89)} \]

(98)

The solution of (98) is

\[ V_d = \phi \left( x^2 + y^2 - bx - ab \cos(\theta) \right) . \]

(99)

Notice that since we shape the potential energy, the non-homogeneous solution is equal to the open-loop potential energy.

**Remark 3.4:** The first impression of PDE (88) is very inconvenient, and finding its solution is a prohibitive task, to the best of author’s knowledge not being reported in the literature, and it is not possible to solve it using any software. The potential of the proposed method to restate and reformulate this problem to some Pfaffian differential equation is the key point to solve this challenging problem.

## 4. Conclusions

In this paper, we derived a suitable solution for the PDEs arising in controller design methods such as in IDA-PBC. For this purpose, a first-order PDE was represented by some equivalent Pfaffian differential equations. It was shown that if the integrability condition holds for a Pfaffian differential equation with three variables, then the solution could be easily found. In order to illustrate how this method can be applied in practice; it was implemented to a number of different benchmark systems through which the IDA-PBC are designed. The computed solution of some systems had better properties compared to the proposed one in literature, while the PDE of some of the other systems could not be solved with the available methods. Although the investigated systems in this paper include magnetic levitation system, pendubot, and two underactuated ods. Although the investigated systems in this paper include magnetic levitation system, pendubot, and two underactuated.

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