Twisted mass QCD and the $\Delta I = 1/2$ rule

C. Pena$^a$, S. Sint$^b$ and A. Vladikas$^a$†

$^a$I.N.F.N., Sezione di Roma 2, c/o Dipartimento di Fisica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica 1, I-00133 Rome, Italy

$^b$CERN, Theory Division, CH-1211 Geneva 23, Switzerland

We show that the application of twisted mass QCD (tmQCD) with four (Wilson) quark flavours to the computation of lattice weak matrix elements relevant to $\Delta I = 1/2$ transitions has important advantages: the renormalisation of $K \to \pi$ matrix elements does not require the subtraction of other dimension six operators, the divergence arising from the subtraction of lower dimensional operators is softened by one power of the lattice spacing and quenched simulations do not suffer from exceptional configurations at small pion mass. This last feature is also retained in the tmQCD computation of $K \to \pi\pi$ matrix elements, which, as far as renormalisation and power subtractions are concerned, has properties analogous to the standard Wilson case.

1. Introduction

At scales well below $M_W$, but above the charm quark mass, the effective weak Hamiltonian for $CP$-conserving, $\Delta S = 1$ decays can be written as:

$$H_{\text{eff}} = V_{ud} V_{us}^* \frac{G_F}{\sqrt{2}} \left[ C_+ (\mu/M_W) O_R^+(\mu) + C_- (\mu/M_W) O_R^-(\mu) \right],$$

(1)

where $O_R^{\pm}(\mu)$ are the dimension-6 four-fermion operators

$$O^{\pm} = (\bar{s} \gamma_\mu^L d)(\bar{u} \gamma_\mu^L u) \pm (\bar{s} \gamma_\mu^L u)(\bar{u} \gamma_\mu^L d) - [u \leftrightarrow c] = O^\pm_{VV,+AA} - O^\pm_{V,+AA}$$

(2)

renormalised at a scale $\mu$ and $\gamma_\mu^L = \frac{i}{2} \gamma_\mu(1 - \gamma_5)$ (the subscript $R$ indicates renormalised quantities). Parity ensures that $K^0 \to \pi^+\pi^-$ and $K^0 \to 0$ matrix elements will receive contributions only from $O^{\pm}_{V,+AA}$, while $K^+ \to \pi^+$ matrix elements will arise only from $O^{\pm}_{VV,+AA}$.

Under isospin transformations, the operator $O^-$ is purely $I = 1/2$, while $O^+$ exhibits both $I = 1/2$ and $I = 3/2$ parts. The $\Delta I = 1/2$ rule can be accounted for by an enhancement of the $K \to \pi\pi$ matrix element of $C_- O^-$ with respect to that of $C_+ O^+$. The contribution to this enhancement coming from the ratio of Wilson coefficients $C_-/C_+$ is small in typical renormalisation schemes (such as $\overline{\text{MS}}$); thus, the bulk of the enhancement must come from the ratio of the matrix elements of the operators themselves.

2. Wilson fermion renormalisation

A serious difficulty arises upon attempting to extract physical amplitudes directly from Euclidean correlation functions corresponding to matrix elements with more than one particle in the final state. Although it has recently been shown that the problem can be bypassed, there is still scope in adopting the long-standing alternative to a direct calculation. This consists in using chiral perturbation theory to obtain $K \to \pi\pi$ matrix elements from $K \to \pi$ ones, the latter not being plagued by the presence of final state interactions.

We are then faced with the problem of operator renormalisation and mixing. Besides a logarithmically divergent renormalisation constant, the operators $O^\pm$ also mix with two dimension-3
operators \((\bar{s} \gamma d)\) and \(\bar{s} d\) with coefficients \(c_1^\pm\) and \(c_3^\pm\) respectively) and two dimension-5 operators \((\bar{s} \sigma_{\mu \nu} \bar{G} \sigma_{\mu \nu} d + \bar{s} \sigma_{\mu \nu} G_{\mu \nu} d\) with coefficients \(d_1^\pm\) and \(d_3^\pm\) respectively. In the case of Ginsparg-Wilson fermions, chiral symmetry ensures that these subtractions are mild, since \(c_1^P, S \sim O(1)\) and \(d_3^P, S \sim O(a^2)\). Once chiral symmetry is lost with Wilson fermions, renormalisation patterns significantly worsen. Ignoring operators with contributions to matrix elements which vanish by the equations of motion we now have (on the basis of CPS symmetry)

\[
\begin{align*}
(O_{VA+AV}^\pm) & = Z_{VA+AV}^\pm (g_0, 2, a \mu) \left[ O_{VA+AV}^\pm + c_p^\pm (g_0^2, m, a) \bar{s} \gamma d + d_3^\pm (g_0^2, m, a) \bar{s} \sigma_{\mu \nu} F^{\mu \nu} d + \cdots \right], \\
(O_{VV+AA}^\pm) & = Z_{VV+AA}^\pm (g_0, 2, a \mu) \left[ O_{VV+AA}^\pm + \sum_{k=1}^4 Z_k^\pm (g_0^2) O_k^\pm + c_s^\pm (g_0^2, m, a) \bar{s} d + d_3^\pm (g_0^2, m, a) \bar{s} \sigma_{\mu \nu} F^{\mu \nu} d + \cdots \right],
\end{align*}
\]

where \(k = VV - AA, SS, PP, TT\) (in standard notation) and the ellipses indicate subtractions which, being \(O(a)\), are unimportant to the present discussion. Three problems immediately arise. First, the subtractions of the dimension-3 operators involve a linearly diverging coefficient in the case of \(O_{VA+AV}\) and, even worse, a quadratically diverging coefficient for \(O_{VV+AA}\):

\[
\begin{align*}
c_p^\pm & \sim \frac{1}{a} (m_c - m_u) (m_s - m_d), \\
c_s^\pm & \sim \frac{1}{a^2} (m_c - m_u).
\end{align*}
\]

A second, milder shortcoming is the mixing of the parity-even operator \(O_{VV+AA}\) with four other dimension-6 operators (just as in the more familiar case of \(B_K\)). Finally, Wilson fermions in the quenched approximation are plagued by exceptional configurations. In ref. \(3\) it has been shown that the tmQCD formulation of Wilson quarks does not suffer from the last two problems. Here \(1\) the coefficient \(d_3\) in eq. \(\ref{eq:tmf}\) is also \(O(a)\) and could have been omitted.

we will demonstrate that the implementation of tmQCD can also reduce the quadratic divergence of \(K \to \pi\) matrix elements to a linear one.

3. tmQCD with four quark flavours

For a recent review of lattice tmQCD see ref. \(3\) and references therein. Here we extend the formulation to four quark flavours. The action is given by

\[
\begin{align*}
\mathcal{L} & = \bar{\psi} (\gamma \nabla + m + i \mu \gamma_5) \psi + \bar{\psi} (\gamma \nabla + m + i \mu \gamma_5) \psi_h.
\end{align*}
\]

We distinguish a light quark doublet \(\psi_l^T = (u, d)\) and a heavy one \(\psi_h^T = (s, c)\). The \(2 \times 2\) quark mass matrices are \(m_{l,h} = \text{diag}(m_{u,s}, m_{d,c})\) and the twisted mass matrices are \(\mu_{l,h} = \text{diag}(\mu_{u,s}, \mu_{d,c})\). For simplicity we impose mass degeneracy in the light sector; i.e. \(m_u = m_d\) and \(\mu_u = -\mu_d\). Two twist angles are then defined through ratios of renormalised mass parameters:

\[
\tan \alpha = \frac{\mu_{R,u}}{m_{R,u}}, \quad \tan \beta = \frac{\mu_{R,s}}{m_{R,s}} = -\frac{\mu_{R,c}}{m_{R,c}}.
\]

The equivalence of tmQCD and standard QCD is formally established through the axial field rotations \(\psi_l \to \exp \left[ i \alpha \gamma_5 \frac{\gamma_3}{2} \right] \psi_l, \psi_h \to \exp \left[ i \beta \gamma_5 \frac{\gamma_3}{2} \right] \psi_h\) (and similarly for \(\bar{\psi}_l, \bar{\psi}_h\)).

For the \(K \to \pi\) weak matrix element a convenient choice is given by \(\alpha = \beta = \pi/2\). Operators are then related as follows

\[
\begin{align*}
[P_\pi]_{\text{tmQCD}} & = [P_\pi]_{\text{QCD}}, \\
[S_K]_{\text{tmQCD}} & = -i [P_K]_{\text{QCD}}, \\
[O_{VV+AA}]_{\text{tmQCD}} & = i [O_{VV+AA}]_{\text{QCD}},
\end{align*}
\]

where \(P_\pi \equiv \bar{\pi}_u \gamma_5 u, S_K \equiv \bar{u} s, P_K \equiv \bar{u} \gamma_5 s\). The following equation between tmQCD and standard QCD renormalised correlation functions (at unequal space-time arguments) is thus true up to discretisation effects:

\[
\begin{align*}
\langle P_\pi(x) O_{VV+AA}^\pm(0) S_K(y) \rangle_{\text{tmQCD}} & = \\
\langle P_\pi(x) O_{VV+AA}^\pm(0) P_K(y) \rangle_{\text{QCD}}.
\end{align*}
\]


In the asymptotic limit the r.h.s. yields \( \langle \pi | O_{VA+AV}^\pm | K \rangle_{\text{QCD}} \); this matrix element can thus also be computed in tmQCD (the l.h.s). We shall show below that in the latter formalism the renormalisation properties of the four-fermion operator are much more convenient.

For the \( K \to \pi \pi \) matrix element a convenient choice of twist angles is \( \alpha = -\beta = \pi/2 \). The equation between the corresponding renormalised correlation functions (at unequal space-time arguments) is then

\[
\langle P_\pi(x) P_\pi(y) O_{VA+AV}^\pm(0) S_K(z) \rangle_{\text{tmQCD}} = i \langle P_\pi(x) P_\pi(y) O_{VA+AV}^\pm(0) P_K(z) \rangle_{\text{QCD}},
\]

where now \( P_K = d\gamma_5 s, S_K = d\bar{s} \). In this case there are no advantages to be gained in the tmQCD computation as far as operator renormalisation properties are concerned. However, in the quenched approximation, the tmQCD computation is free of exceptional configurations. This is an important advantage close to the chiral limit.

4. tmQCD renormalisation

For the “convenient” choices of twist angles (i.e. \( \alpha = \pi/2, \beta = \pm \pi/2 \)) and using the discrete symmetries of the tmQCD action (cf. [3]) we find the following renormalisation pattern for the operators \( O_{VA+AV}^\pm \)

\[
(O_R)^\pm_{VA+AV} = Z_{VA+AV}[O_{VA+AV}^\pm + c_{P}^\pm \bar{s}\gamma_5 d + c_{S}^\pm \bar{s}d + \cdots],
\]

where the leading behaviour of the coefficients (in the chiral expansion) is

\[
c_{P}^\pm = \frac{1}{a} F_{P}^\pm(g_0^2) \left( \mu_c - \mu_u \right) \left( \mu_s - \mu_d \right),
\]

\[
c_{S}^\pm = F_{S}^\pm(g_0^2) \left( \mu_c - \mu_s \right) \left( \mu_u - \mu_d \right) + G_{S}^\pm(g_0^2) \left( \mu_c - \mu_u \right) \left( \mu_s - \mu_d^2 \right).
\]

Thus, we now have a linear divergence in \( c_{P}^\pm \) (while \( c_{S}^\pm \) contains two terms, both of \( O(1) \)). This situation compares favourably to the standard QCD case, characterized by a quadratic divergence (\( c_{S}^\pm \sim 1/a^2 \); cf. eq. (3)). Moreover, there are no dimension-6 operators to be subtracted in the tmQCD case (in standard QCD there are four such subtractions; cf. eq. (3)).

In order to determine the linearly divergent coefficient \( c_{P}^\pm \) we must resort to parity restoration in the continuum limit. Parity is broken by the lattice (Wilson) tmQCD but is recovered after renormalisation. In particular for the twist angle values of \( \alpha = \beta = \pi/2 \), parity transformations in the continuum limit assume the form

\[
u(x) \to i\gamma_0\gamma_5 u(\bar{x}) \quad d(x) \to -i\gamma_0\gamma_5 d(\bar{x}),
\]

\[
s(x) \to i\gamma_0\gamma_5 s(\bar{x}) \quad c(x) \to -i\gamma_0\gamma_5 c(\bar{x}),
\]

with \( \bar{x} = (x_0, -x) \) and similarly for the antiquarks. Thus \( O_{VA+AV}^\pm \) is a positive parity eigenstate whereas \( P_{dS} \equiv d\gamma_5 s \) is a negative parity eigenstate. This implies that for \( x \neq 0 \) we have

\[
\langle (O_R)^\pm_{VA+AV}(0) \rangle_{\text{tmQCD}} = 0.
\]

Expressing the above in terms of bare correlations yields

\[
\langle O_{VA+AV}^\pm P_{dS} \rangle + c_{P}^\pm \langle P_{dS} P_{dS} \rangle + c_{S}^\pm \langle S_{dS} P_{dS} \rangle = 0.
\]

Analogous parity arguments can be used to show that the last term of the above expression is a lattice artifact (i.e. \( c_{S}^\pm \langle S_{dS} P_{dS} \rangle = \mathcal{O}(1) \)) and may be dropped. Thus, eq. (15) can be solved to determine \( c_{P}^\pm \).

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