High-dimensional Filtering using Nested Sequential Monte Carlo

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Abstract

Sequential Monte Carlo (SMC) methods comprise one of the most successful approaches to approximate Bayesian filtering. However, SMC without good proposal distributions struggle in high dimensions. We propose nested sequential Monte Carlo (NSMC), a methodology that generalises the SMC framework by requiring only approximate, properly weighted, samples from the SMC proposal distribution, while still resulting in a correct SMC algorithm. This way we can exactly approximate the locally optimal proposal, and extend the class of models for which we can perform efficient inference using SMC. We show improved accuracy over other state-of-the-art methods on several spatio-temporal state space models.

Keywords: particle filtering, spatio-temporal models, state space models, approximate Bayesian inference, backward simulation

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1 Introduction

Inference in complex and high-dimensional statistical models is a very challenging problem that is ubiquitous in applications such as climate informatics [Monteleoni et al., 2013], bioinformatics [Cohen, 2004] and machine learning [Wainwright and Jordan, 2008], to mention a few.

We are interested in sequential Bayesian inference in settings where we have a sequence of posterior distributions that we need to compute. To be specific, we are focusing on settings where the model (or state variable) is high-dimensional, but where there are local dependencies. One example of the type of models we consider are the so-called spatio-temporal models [Wikle, 2015, Cressie and Wikle, 2011, Rue and Held, 2005].

Sequential Monte Carlo (SMC) methods comprise one of the most successful methodologies for sequential Bayesian inference. However, SMC struggles in high dimensions and these methods are rarely used for dimensions, say, higher than ten [Rebeschini and van Handel, 2015]. The purpose of the NSMC methodology is to push this limit well beyond the single digits.

The basic strategy is to mimic the behavior of a so-called fully adapted (or locally optimal) SMC algorithm. Full adaptation can drastically improve the efficiency of SMC in high dimensions [Snyder et al., 2015]. Unfortunately, it can rarely be implemented in practice since the fully adapted proposal distributions are typically intractable. NSMC addresses this difficulty by requiring only approximate, properly weighted, samples from the proposal distribution. This enables us to use a second layer of SMC to simulate approximately from the proposal. The proper weighting condition ensures the validity of NSMC, thus providing a generalisation of the family of SMC methods. This paper extends preliminary work [Naesseth et al., 2015a] with the ability to handle more expressive models, more informative
central limit theorems and convergence proofs, as well as new experiments.

Related work

There has been much recent interest in using Monte Carlo methods as nested procedures of other Monte Carlo algorithms. The SMC$^2$ and IS$^2$ algorithms by Chopin et al. [2013] and Tran et al. [2013], respectively, are algorithms for learning static parameters as well as latent variable(s). In these methods one SMC/IS method for the parameters is coupled with another for the latent variables. Chen et al. [2011] and Johansen et al. [2012] on the other hand addresses the state inference problem by splitting $x_t$ into two components and run coupled SMC samplers for these. These methods solve different problems and the “internal” SMC samplers are constructed differently, for approximate marginalization instead of simulation.

By viewing the state inference problem as a sequential problem in the components of $x_t$ we can make use of the method for general graphical models by Naesseth et al. [2014b]. This method is combined with the island particle filter Vergé et al. [2015], and studied more closely by Beskos et al. [2014] under the name space-time particle filter (ST-PF). The ST-PF does not generate an approximation of the fully adapted SMC. Another key distinction is that in ST-PF each particle in the “outer” SMC sampler corresponds to a complete particle system, whereas for NSMC it will correspond to different hypotheses about the latent state $x_t$ as in standard SMC. This leads to lower communication costs and better memory efficiency in e.g. distributed implementations. We have also found that NSMC typically outperforms ST-PF, even when run on a single machine with matched computing times.

The method proposed by Jaoua et al. [2013] can be viewed as a special case of NSMC when the nested procedure to generate samples is given by IS with the proposal being the
transition probability. Independent resampling PF (IR-PF) introduced in Lamberti et al. [2016] generates samples in the same way as NSMC with IS, instead of SMC, as the nested procedure. However, IR-PF uses a different weighting that requires both the outer and the inner number of particles to tend to infinity for consistency. Furthermore, we provide results in the supplementary material that show NSMC significantly outperforming IR-PF on an example studied in Lamberti et al. [2016].

There are other SMC-related methods that have been introduced to tackle high-dimensional problems, see e.g. the so-called block PF studied by Rebeschini and van Handel [2015], the location particle smoother by Briggs et al. [2013], and various methods reviewed in Djuric and Bugallo [2013]. These methods are, however, all inconsistent because they are based on approximations that result in systematic errors.

The concept of proper weighting (or random weights) is not new and has been used in the so-called random weights particle filter [Fearnhead et al., 2010]. They require exact samples from a proposal $q_t$ but use a nested Monte Carlo method to unbiasedly estimate the importance weights $w_t$. In Martino et al. [2016] the authors study proper weighting as a means to perform partial resampling, i.e. only resample a subset of the particles at each time. The authors introduce the concept of “unnormalized” proper weighting, which is essentially the same as proper weighting that was introduced and used to motivate NSMC in Naesseth et al. [2015a]. Furthermore, Stern [2015] uses proper weighting and NSMC to solve an inference problem within statistical historical linguistics.

Another approach to solve the sequential inference problem is the sequential Markov chain Monte Carlo class of methods [Yang and Dunson, 2013]. It was shown by Septier and Peters [2016] that the optimal sequential MCMC algorithm actually is equivalent to the fully adapted SMC.
2 Sequential probabilistic models

In statistics, data science and machine learning, probabilistic modeling and Bayesian inference are essential tools to finding underlying patterns and unobserved quantities of interest. To illustrate the nested SMC sampler we will make use of two general classes of sequential probabilistic models, the so-called Markov random field (MRF) and the state space model (SSM). Sequential probabilistic models are in general built up of a sequence of (probabilistic) models that share common random variables and structure. These models will serve to illustrate the usefulness and wide applicability of the method we propose. We are interested in the type of sequential models where the latent variables are fairly high-dimensional. In subsequent sections we will also show explicitly how we can make use of structure between the (latent) random variables to design an efficient SMC sampler that lets us scale to much higher dimensions than possible with standard SMC methods, usually by up to 1–2 orders of magnitude. Note also that the NSMC is by no means restricted to the classes of models we illustrate in this section, rather it can in principle be applied to any sequence of distributions we would like to approximate. We will refer to this sequence of distributions of interest as the target distributions.

2.1 Markov random fields

The Markov random field is a type of undirected probabilistic graphical model [Jordan, 2004]. The MRF is typically not represented as a sequence of distributions (or models), but it has previously been shown [Hamze and de Freitas, 2005; Everitt, 2012; Naesseth et al., 2014a; 2015a; 2015c; Lindsten et al., 2016] that it can be very useful to artificially introduce a sequence to simplify the inference problem. Furthermore, it is also possible to postulate the model as an MRF that increases with “time”, useful in e.g. climate science [Fu et al.].
In the exposition below we will first for simplicity assume that we have an MRF that is of fixed dimension, i.e. the latent variable $x = (x_1, \ldots, x_{n_x})$ is a finite-dimensional multivariate random variable. The conditional independencies of an MRF are described by the structure of the graph $G = \{V, E\}$, where $V = \{1, \ldots, n_x\}$ is the vertex set and $E = \{(i, j) : (i, j) \in V \times V, \exists \text{ edge between vertex } i \text{ and } j\}$ is the edge set. Given $G$ we can define a joint probability density function for $x$ that incorporates this structure as

$$\pi(x) = \frac{1}{Z} \prod_{i \in V} \phi(x_i, y_i) \prod_{(i,j) \in E} \psi(x_i, x_j),$$

where $y = (y_1, \ldots, y_{n_x})$ is the observed variable and $\phi, \psi$ are called observation and interaction potentials, respectively. The normalization constant that ensures that $\pi(\cdot)$ integrates to one is given by

$$Z := \int \prod_{i \in V} \phi(x_i, y_i) \prod_{(i,j) \in E} \psi(x_i, x_j) dx.$$

Note that (1) is usually referred to as a pairwise MRF in the literature due to $\pi(\cdot)$ factorising into potentials that only depend on pairs of components of the random variable $x$. For clarity we restrict ourselves to this type, however the method we propose in this paper can be applied to more general types of graphs, see e.g. Naesseth et al. [2014b] for ideas on how to extend SMC inference to non-pairwise MRFs.

Now, the sequential MRF is obtained if we consider a random variable $x_{1:t} = (x_1, \ldots, x_t)$, for some $t = 1, \ldots, T$, that factorises according to

$$\pi_t(x_{1:t}) = \frac{1}{Z_t} \gamma_t(x_{1:t}) := \frac{1}{Z_t} \gamma_t(x_{1:t-1}) \prod_{i \in V} \phi(x_{t,i}, y_{t,i}) \rho(x_{t-1,i}, x_{t,i}) \prod_{(i,j) \in E} \psi(x_{t,i}, x_{t,j}),$$

where $G = \{V, E\}$ again encodes the structure of the graphical model and $\rho(\cdot)$ is a new type of interaction potential that links $x_{t-1}$ to $x_t$. Furthermore, the normalisation constant
is given by $Z_t := \int \gamma_t(x_{1:t})dx_{1:t}$. We illustrate a typical example of a sequential MRF in Figure 1. It can amongst other things be used to model spatio-temporal phenomena, it was e.g. used by Naesseth et al. [2015a] to detect drought based on annual average precipitation rates collected from various sites in North America and Africa over the last century. We

![Diagram of sequential MRF](image)

Figure 1: Illustration of a sequential MRF where $G$ is given by $2 \times 3$ grid with nearest neighbour interaction.

would like to remark on one peculiarity that arises when the sequential MRF is used to model a spatio-temporal process. Consider $\pi_t(\cdot)$ without measurements as a prior on a spatio-temporal model, i.e. the observation potentials $\phi$ in (2) do not depend on $y_t$. In this case we get that the marginals for $t < T$ change depending on the value of $T$, i.e. in general $\pi_t(x_{1:t}) \neq \pi_T(x_{1:T}) = \int \pi_T(x_{1:T})dx_{t+1:T}$. Typically we would expect that a priori what happens for a dynamical process at time $t$ should not be affected by the length of time-series we consider. The next class of models we consider can introduce dependencies in both time and space without giving rise to this counter-intuitive result.
2.2 Spatio-temporal state space models

Before we move on to define the spatio-temporal state space model (ST-SSM), we will briefly review SSMs, a comprehensive and important model type commonly used for studying dynamical systems. For a more detailed account, and with pointers to the wide range of applications, we refer the readers to e.g. Cappé et al. [2005], Douc et al. [2014], Shumway and Stoffer [2010].

In state space models the sequential structure typically enters as a known, or postulated, dynamics on the unobserved latent state $x_t$ that is then partially observed through the measurements $y_t$. A common definition for SSMs is through its functional form

$$
\begin{align}
  x_t &= a(x_{t-1}, v_t), & v_t &\sim p_v(\cdot), \\
  y_t &= c(x_t, e_t), & e_t &\sim p_e(\cdot),
\end{align}
$$

where $v_t$ and $e_t$, often called process and measurement noise, respectively, are random variables with some given distributions $p_v(\cdot), p_e(\cdot)$. Furthermore, we have that the initial state $x_1$ is a random variable with some initial distribution $\mu(\cdot)$. For simplicity we will assume that both $a(x_{t-1}, \cdot) : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ and $c(x_t, \cdot) : \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_y}$ are bijective and continuously differentiable. Then by the transformation theorem we can equivalently express (3) through the corresponding probability density functions (PDF)

$$
\begin{align}
  x_t|x_{t-1} &\sim f(x_t|x_{t-1}), \\
  y_t|x_t &\sim g(y_t|x_t),
\end{align}
$$

and we define the sequential probabilistic model (or target distribution) as follows

$$
\pi_t(x_{1:t}) := \frac{\gamma_t(x_{1:t})}{Z_t} = \frac{1}{Z_t} \mu(x_1)g(y_1|x_1) \prod_{s=2}^{t} f(x_s|x_{s-1})g(y_s|x_s).
$$

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We will assume that $g(y_t|x_t)$ is available and can be evaluated pointwise. This condition is often satisfied in practical applications.

A typical assumption when using the SSM to model spatio-temporal systems is to introduce the spatial dependency only between time steps $t - 1$ and $t$, see e.g. the paper by Wikle and Hooten [2010]. This can be achieved by defining a model $a(\cdot)$ such that the product of the induced distributions $f(x_t|x_{t-1})g(y_t|x_t)$, conditionally on $x_{t-1}$, completely factorize over the components of $x_t$, see also [Rebeschini and van Handel, 2015] where SMC applied to such a model is studied. Here we will study the case where we introduce spatial dependencies within each time step through the disturbance term $v_t$. We define the ST-SSM as a combination of the functional and PDF representation of an SSM where the distribution for $v_t$ is given by an MRF as in

\[
\begin{pmatrix}
x_{t,1} \\
\vdots \\
x_{t,n_x}
\end{pmatrix} = \begin{pmatrix} a_1(x_{t-1},v_{t,1}) \\
\vdots \\
a_{n_x}(x_{t-1},v_{t,n_x})
\end{pmatrix}, \quad v_t \sim \frac{1}{Z_v} \prod_{i \in V} \phi(v_{t,i}) \prod_{(i,j) \in E} \psi(v_{t,j},v_{t,i}), \quad (6a)
\]

\[
y_t|x_t \sim g(y_t|x_t). \quad (6b)
\]

We make no assumptions on local dependencies between $x_t$ and $x_{t-1}$, however, to keep it simple we will assume that the graph $G = \{V, E\}$ describing the distribution for $v_t$ does not depend on time $t$. Furthermore, we will in this paper mainly consider models where dependencies between components in $v_t$ are “few”, e.g. the MRF is sparse with few elements in $E$, and where components of $y_t$ in $g$ only depends on subsets of $x_t$. To illustrate the dependency structure in an ST-SSM we propose a combination of the traditional undirected graph for the MRF and the directed acyclic graph for the SSM, see Figure [2]. This allows us to model more complex dynamical processes than Naesseth et al. [2015a] who assumed...
Figure 2: Illustration of a spatio-temporal state space model with $n_x = 4$, one conditionally independent observation per component in $x_t$, and the MRF for $v_t$ is given by a chain. Grey circles illustrate the observations $y_t$.

that $f(x_t|x_{t-1})g(y_t|x_t)$ factorized with only local dependencies between components of $x_t$. Furthermore, we can clearly see that the peculiarity discussed in Section 2.1 is not present in this model; the marginal of the prior does not change with $T$ as expected.

3 Nested Sequential Monte Carlo Methods

Inference in sequential probabilistic models essentially boils down to computing the target distribution $\pi_t(x_{1:t})$ for $t = 1, 2, \ldots$; typically an intractable problem with no analytical or numerically efficient solution. This means that we have to resort to approximations. In this paper we focus on one particular successful solution to the problem, the so called sequential Monte Carlo family of algorithms first introduced in the papers by Gordon et al. [1993], Stewart and McCarty [1992], Kitagawa [1996].

The basic idea with SMC is to move a set of weighted samples (particles) $\{(x_{1:t-1}^i, w_{t-1}^i)\}_{i=1}^N$ approximating $\pi_{t-1}$, to a new set of particles $\{(x_{1:t}^i, w_t^i)\}_{i=1}^N$ which approximates $\pi_t$. These
samples define an empirical approximation of the target distribution

$$
\pi_t^N(dx_{1:t}) := \sum_{i=1}^N \frac{w_i}{\sum_\ell w_\ell} \delta_{x_{1:t}}(dx_{1:t}),
$$

(7)

where $\delta_x(dx)$ is a Dirac measure at $x$. In the next section we will detail an especially efficient way of moving the particles, known as fully adapted SMC [Pitt and Shephard 1999], ensuring that all normalized weights are equal to $\frac{1}{N}$.

3.1 Fully Adapted Sequential Monte Carlo

The procedure to move the particles and their weights from time $t - 1$ to $t$ in any SMC sampler is typically done in a three-stage approach. The first, resampling, stochastically chooses $N$ particles at time $t - 1$ that seem promising, discarding low-weighted ones. The second stage, propagation, generates new samples for time $t$ conditioned on the resampled particles. The final stage, weighting, corrects for the discrepancy between the target distribution and the proposal, i.e. the instrumental distribution used in the propagation step.

Fully adapted SMC [Pitt and Shephard 1999] makes specific choices on the resampling weights, $\nu_{t-1}$, and the proposal, $q_t(x_t|x_{1:t-1})$, such that all the importance weights $w_t$ are equal. By introducing ancestor indices $a_{t-1}^i \in \{1, \ldots, N\}$, we can describe the resampling step by simulating $N$ times i.i.d. from

$$
P(a_{t-1}^i = j) = \frac{\nu_{t-1}^j}{\sum_{\ell=1}^M \nu_{t-1}^\ell}, \quad \nu_{t-1}^j := \int \frac{\gamma_t((x_{1:t-1}^j, x_t))}{\gamma_{t-1}(x_{1:t-1}^j)} dx_t.
$$

(8)
Propagation then follows by simulating $x^i_t$ conditional on $x^{a_i}_{1:t-1}$, for $i = 1, \ldots, N$, as follows

$$x^i_t | x^{a_i}_{1:t-1} \sim q_t(x_t | x^{a_i}_{1:t-1}) := \pi_t(x_t | x^{a_i}_{1:t-1}) = \frac{\pi_t((x^{a_i}_{1:t-1}, x_t))}{\pi_t(x^{a_i}_{1:t-1})},$$

(9)

This proposal is sometimes referred to as the (locally) optimal proposal because it minimizes incremental variances in the importance weights $w^i_t$. Weighting is easy since all weights are equal, i.e. the unnormalized weights are all set to $w^i_t = 1$. The fully adapted SMC sampler in fact corresponds to a locally optimal choice of both resampling weights and proposal with an incremental variance in the importance weights $w^i_t$ that is zero.

Note that in most cases it is impossible to implement this algorithm exactly, since we can not calculate $\nu_{t-1}$ and/or simulate from $q_t$. Nested SMC solves this by requiring only approximate resampling weights and approximate samples from $q_t$, in the sense that is formalized in Section 3.3. However, we will start by detailing some specific cases when we can efficiently implement exact fully adapted SMC. These cases are of interest in themselves, however, here we will use them to build intuition for how the approximations in NSMC are constructed.

### 3.2 Forward Filtering–Backward Simulation

The problems we need to solve are those of computing $\nu_{t-1}$ and simulating from $q_t$ efficiently, i.e. in such a way that the computational complexity is controlled. There are at least two important special cases when we can use fully adapted SMC. The first is if the state space $X$ is discrete and finite, i.e. $x_t \in \{1, \ldots, S\}^{\otimes n_x}, \forall t$. Even though exact algorithms are known in this case [Cappé et al., 2005] the computational complexity typically scales quadratically with the cardinality of $x_t$, thus SMC methods can still be of interest [Fearnhead and Clifford].
The second case is if \( \frac{\gamma_t(x_{1:t})}{\gamma_{t-1}(x_{1:t-1})} \) is an unnormalized Gaussian distribution, e.g. in the ST-SSM this would correspond to

\[
x_t = a(x_{t-1}) + v_t, \quad v_t \sim \text{Gaussian MRF},
\]

\[
y_t | x_t \sim \mathcal{N}(y_t; Ct, R),
\]

for some matrix \( C \), covariance matrix \( R \), and an MRF in the components of \( v_t \) where all pair-wise potentials are Gaussian.

Now, even though in principle the fully adapted SMC is available these special cases, the computational complexity can be prohibitive. In fact in general it is of the order of \( \mathcal{O}(S^{n_x}) \) and \( \mathcal{O}(n_x^2) \) for the finite state space and Gaussian case, respectively. However, when there are local dependencies it is possible to make use of an underlying chain (or tree) structure, as proposed by Naesseth et al. [2014a] for the finite state space case, to make efficient implementations with only \( \mathcal{O}(S^2 n_x) \) and \( \mathcal{O}(n_x) \) complexity, respectively. This approach makes use of forward filtering–backward simulation (sampling), from Carter and Kohn [1994], Frühwirth-Schnatter [1994], on the components of \( x_t \) to compute \( \nu_{t-1} \) and sample \( q_t \) exactly. Let us as an example consider the above ST-SSM with \( C = I \) and \( R = I \) and the Gaussian MRF given by

\[
p_v(v_t) = \frac{1}{Z_v} \exp \left\{ -\frac{\tau}{2} \sum_{d=1}^{n_x} v_{t,d}^2 - \frac{\lambda}{2} \sum_{d=2}^{n_x} (v_{t,d} - v_{t,d-1})^2 \right\},
\]

for some positive constants \( \tau \) and \( \lambda \). Then straightforward computations gives the proposal and resampling weights

\[
q_t(x_t|x_{1:t-1}) = \frac{1}{\nu_{t-1}} \frac{\gamma_t(x_{1:t})}{\gamma_{t-1}(x_{1:t-1})} = \frac{1}{\nu_{t-1}} f(x_t|x_{t-1}) g(y_t|x_t),
\]

\[
\nu_{t-1} = \int f(x_t|x_{t-1}) g(y_t|x_t) dx_t.
\]
However, an equivalent way to simulate from this distribution and calculate $\nu_{t-1}$ is given below

$$x_t = a(x_{t-1}) + v_t', \quad v_t' \sim \frac{1}{\nu_{t-1}} g(y_t|a(x_{t-1}) + v_t)p_v(v_t),$$

$$\nu_{t-1} = \int g(y_t|a(x_{t-1}) + v_t)p_v(v_t)dv_t.$$ 

Due to the structure in $p_v(\cdot)$ and $g(y_t|x_t)$ we can see that the distribution to sample from corresponds to a Gaussian MRF with a chain-structure in the $v_{t,d}$'s (cf. Figure 2)

$$p(v_{t,1:n_x}|y_t, x_{t-1}) = \frac{g(y_t|a(x_{t-1}) + v_t)p_v(v_t)}{\prod_{d=1}^{n_x} p(y_{t,d}|y_{t,1:d-1}, x_{t-1})} \propto \frac{1}{Z_v} \exp \left\{ -\frac{1}{2} \sum_{d=1}^{n_x} [(y_{t,d} - a_d(x_{t-1}) - v_{t,d})^2 + \tau v_{t,d}^2] - \frac{\lambda}{2} \sum_{d=2}^{n_x} (v_{t,d} - v_{t,d-1})^2 \right\}. \quad (10)$$

Because of this structure we can efficiently compute the normalization constant of (10) by means of “forward” filtering, keeping track of the incremental contributions to $\nu_{t-1}$, $p(y_{t,d}|y_{t,1:d-1}, x_{t-1}), \ d = 1, \ldots, n_x$. Sampling the distribution is then done by an explicit “backward” pass, simulating $v_{t,d}' \sim p(v_{t,d}|v_{t,d+1:n_x}, y_{t,d:n_x}), \ d = n_x, n_x-1, \ldots, 1$. We provide an illustration of the process in Figure 3. See also Naesseth et al. [2014a] for an example of how this is done in practice for a discrete state space.

The main idea behind nested SMC is to emulate this behavior for arbitrary sequential probabilistic models. Because computing $\nu_{t-1}$ and simulating from $q_t$ exactly is intractable in general we propose to run an SMC-based forward filtering–backward simulation [Godsill et al., 2004, Lindsten and Schön, 2013] method on the components of $x_t$ (or $v_t$) to approximate $\nu_{t-1}$ and draws from $q_t$. 

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\[ v_{t,1} x_{t,1} - v_{t,1} v_{t,1} x_{t,2} - v_{t,2} v_{t,2} x_{t,3} - v_{t,3} x_{t,4} \cdot \cdot \cdot v_{t,1} x_{t,1} - v_{t,1} v_{t,1} x_{t,2} - v_{t,2} v_{t,2} x_{t,3} - v_{t,3} x_{t,4} \]

Figure 3: Illustration of forward filtering–backward sampling on \( v_t \) as explained by (10).

Note that after the last step we simply set \( x_t = a(x_{t-1}) + v_t' \).

### 3.3 Nested Sequential Monte Carlo

One way to think of the nested SMC family of methods is as an *exact approximation* \[\text{Andrieu et al. 2010}\] of an SMC algorithm with resampling weights \( \nu_{t-1} \) and proposal \( q_t \) given as in the fully adapted SMC. Instead of exactly evaluating each \( \nu_{t-1}^i \), we run a nested (or internal) SMC sampler with \( M \) particles, for each \( i \), on the components \( x_{t,1:d} \) (or \( v_{t,1:d} \)) with the final target (for \( d = n_x \)) equal to \( q_t(x_t|x_{1:t-1}) \) to mimic the exact forward filtering procedure. The normalization constant estimates from these internal filters gives us *unbiased* approximations of \( \nu_{t-1}^i \) that we use to perform the resampling step. The resampling step not only selects the ancestors \( x_{1:t-1}^i \), but we also resample the complete internal state, denoted by \( u_{t-1} \), of the nested SMC samplers which will be used for the propagation step. Lastly we simulate \( x_t^i \) by running a backward simulation procedure \[\text{Godsill et al. 2004, Lindsten and Schöns 2013}\] using the resampled internal SMC sampler’s \( u_{t-1}^i \) to mimic the exact backward sampling described above. More formally, one step from iteration \( t - 1 \) to \( t \) of the NSMC method proceeds as follows.

Given an unweighted particle set \( \{x_{1:t-1}^i\}_{i=1}^N (w_{t-1} \equiv 1) \), approximating \( \pi_{t-1} \), we generate the internal states by simulating \( u_{t-1}^i \sim \eta_{t-1}^M(u_{t-1}|x_{1:t-1}^i) \) (cf. forward filtering). Here \( \eta_{t-1}^M \) denotes the joint distribution of all random variables generated by the internal SMC.
sampler. Then we extract an estimate of the resampling weights $\nu_{t-1} = \tau_t(u_{t-1})$, where $\tau$ is a function such that

$$
\int \tau_t(u_{t-1})\eta_{t-1}^M(u_{t-1}|x_{1:t-1})du_{t-1} = \frac{\int \gamma_t((x_{1:t-1}, x_t))dx_t}{\gamma_{t-1}(x_{1:t-1})} = \nu_{t-1}, \quad \tau(u_{t-1}) \geq 0 \text{ a.s.} \quad (11)
$$

This is the normalization constant estimate at the final step of the internal SMC samplers, where the target is equal to $\gamma_t((x_{1:t-1}, x_t))$, and then (11) is satisfied by known properties of SMC [Del Moral, 2004, Proposition 7.4.1]. We now proceed to resample the internal SMC samplers, i.e. generating ancestor variables $a_t^i$ such that

$$
P(a_t^i = j) = \left\{ \frac{\tau_t(u_{t-1}^j)}{\sum \tau_t(u_{t-1}^\ell)} \right\}_{j=1}^N, \quad (12)
$$

which concludes the resampling step.

Next, for propagation we generate samples $x_t^i \sim \kappa_t^M(x_t|u_{t-1}^a_t)$ (cf. backward sampling), where $\kappa_t^M$ is a distribution satisfying the following condition

$$
\int \tau_t(u_{t-1})\kappa_t^M(x_t|u_{t-1})\eta_{t-1}^M(u_{t-1}|x_{1:t-1})du_{t-1} = \frac{\gamma_t(x_{1:t})}{\gamma_{t-1}(x_{1:t-1})}. \quad (13)
$$

The distribution $\kappa_t^M$ can be realized by running backward simulation, however, a simple straightforward alternative that also satisfies (13) can be to sample from the corresponding empirical distribution induced by the internal SMC sampler. We discuss the choice of $\eta_{t-1}^M, \kappa_t^M$ and $\tau_t$ further in the next section.

Finally, we set $x^i_{1:t} = (a_{t-1}^i, x_t^i)$ and have thus obtained a new set of unweighted particles approximating $\pi_t$, i.e.

$$
\pi_t^N(dx_{1:t}) := \frac{1}{N} \sum_{i=1}^N \delta_{x_{1:t}^i}(dx_{1:t}). \quad (14)
$$

The two conditions on $\eta_{t-1}^M, \tau_t, \kappa_t^M$, i.e. (11) and (13), can in fact be replaced by the single condition that $(x^i_t, \tau_t(u_{t-1}^i))$ are properly weighted for $\frac{\gamma_t(x_{1:t})}{\gamma_{t-1}(x_{1:t-1})}$. 

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Algorithm 1 Nested Sequential Monte Carlo (all for $i = 1, \ldots, N$)

Require: $\eta_{t-1}^M, \kappa_t^M, \tau_t$ that generate samples properly weighted for $\frac{\gamma_t(x_{1:t})}{\gamma_{t-1}(x_{1:t-1})}$

1: for $t = 1$ to $T$ do
2:  Simulate $u_{t-1}^i \sim \eta_{t-1}^M(u_{t-1} | x_{1:t-1}^i)$
3:  Draw $a_t^i$ with probability $\mathbb{P}(a_t^i = j) = \frac{\tau_t(u_{t-1}^j)}{\sum_j \tau_t(u_{t-1}^j)}$
4:  Simulate $x_{1:t}^i \sim \kappa_t^M(x_{1:t} | u_{t-1}^{a_t^i})$
5:  Set $x_{1:t}^i = (x_{1:t-1}^i, x_{1:t}^i)$
6: end for

Definition 1. We say that the (random) pair $(x_t, \tau_t(u_{t-1}))$ are properly weighted for the (unnormalized) distribution $\frac{\gamma_t(x_{1:t})}{\gamma_{t-1}(x_{1:t-1})}$ if $\tau_t(u_{t-1}) \geq 0$ a.s. and for all measurable functions $h$

$$\mathbb{E}[h(x_t)\tau_t(u_{t-1})] = C \int \frac{\gamma_t(x_{1:t})}{\gamma_{t-1}(x_{1:t-1})} dx_t \int h(x_t)\pi_t(x_t|x_{1:t-1}) dx_t,$$ (15)

for some positive constant $C > 0$ that is independent of the $x$’s and $u$’s.

We provide a summary of the proposed method in Algorithm 1. Although we here focus on approximating the fully adapted SMC sampler, the extension to arbitrary resampling weights and proposal is straightforward, see the supplementary material. Next we will illustrate how we can make use of nested or internal SMC samplers to construct $\eta_{t-1}^M, \tau_t, \kappa_t^M$ that generate properly weighted samples.

### 3.4 Constructing $\eta_{t-1}^M$, $\tau_t$ and $\kappa_t^M$

To construct $\eta_{t-1}^M$ we propose to run an SMC sampler targeting the components of $x_t$ (or $v_t$) one-by-one. This is done by choosing some sequence of (unnormalized) targets $p_d(x_{1:d})$ and proposals $r_d(x_d|x_{1:d-1})$ such that $p_{x_d}(. \mid \cdot) \propto \frac{\gamma_t(x_{1:d})}{\gamma_{t-1}(x_{1:t-1})}$. For notational convenience we
supress the dependence on time $t$ in this section. We provide a summary in Algorithm 2 in this case $u_{t-1} := \{ x^{1:M}_{d} \}_{d=1}^{n_x} \bigcup \{ a^{1:N}_{d} \}_{d=2}^{n_x}$.

**Algorithm 2** Sequential Monte Carlo (all for $i = 1, \ldots, M$)

**Require:** Unnormalized Monte Carlo $p_d(x_{1:d})$, proposals $r_d(x_d|x_{1:d-1})$, and $M$

1. $x_1^i \sim r_1(x_1)$
2. Set $w_1^i = \frac{p_1(x_1^i)}{r_1(x_1^i)}$
3. for $d = 2$ to $n_x$ do
   4. Draw $a_d^i$ with probability $P(a_d^i = j) = \frac{w_d^{j-1}}{\sum_{\ell} w_d^{\ell-1}}$
   5. Simulate $x_d^i \sim r_d(x_d|x_{1:d-1}^{a_d^i})$
   6. Set $x_{1:d}^i = (x_{1:d-1}^{a_d^i}, x_d^i)$
   7. Set $w_d^i = \frac{p_d(x_{1:d}^i)}{p_{d-1}(x_{1:d-1}^{a_d^i}) r_d(x_d^i|x_{1:d-1}^{a_d^i})}$
4. end for

A first simple alternative to construct $\kappa^M_t$ can be to simply simulate directly from the empirical measure defined by the approximation in Algorithm 2. Although this will be properly weighted it can introduce significant correlation between the samples. Instead we propose to make use of backward simulation [Godsill et al., 2004, Lindsten and Schön, 2013] to construct a more efficient $\kappa^M_t$, see Algorithm 3.

Now, putting all this together we define the complete procedure in Definition 2.

**Definition 2** (SMC and BS). Let $\eta^M_{t-1}$, $\tau_t$, and $\kappa^M_t$ be defined as follows for some sequence $p_d(\cdot)$ such that $p_{n_x}(\cdot) \propto \frac{\gamma(x_{1:t})}{\gamma(x_{1:t-1})}$:

1. Simulate $u_{t-1} \sim \eta^M_{t-1}(u_{t-1} | x_{1:t-1})$ by running Algorithm 3
2. Set $\tau_t(u_{t-1}) = \prod_{d=1}^{n_x} \frac{1}{M} \sum_{i=1}^{M} w_d^i$
3. Simulate $x_t \sim \kappa^M_t(x_t | u_{t-1})$ by running Algorithm 3
Algorithm 3 Backward Simulation

**Require:** \( \{(x^i_1, u^i_d)\}_{i=1}^M, \; d = 1, \ldots, n_x \) approximating \( p_d(x_{1:d}) \)

1: Draw \( b \) with probability \( \mathbb{P}(b_{n_x} = j) = \frac{w^n_j}{\sum_{\ell} w^n_{\ell}} \)

2: Set \( x_{n_x} = x^b_{n_x} \)

3: for \( d = n_x - 1 \) to 1 do

4: Draw \( b \) with probability \( \mathbb{P}(b = j) \propto w^j_d p_{n_x}(x^j_1, x^j_{d+1:n_x}) \)

5: Set \( x_{d:n_x} = (x^b_d, x_{d+1:n_x}) \)

6: end for

**Proposition 1** (Proper weighting). The procedure in Definition 2 generates \( (x_t, \tau_t(u_{t-1})) \) that are properly weighted for \( \gamma_t(x_{1:t}) / \gamma_{t-1}(x_{1:t-1}) \).

**Proof.** The result follows from Theorem 2 in [Naesseth et al., 2015a].

**Remark 1.** Note that we can in fact replace Step 1 of Definition 2 (SMC and BS) with running the NSMC algorithm itself, i.e. Algorithm 4, and letting the \( w_d := 1 \) in Step 3. This will also yield properly weighted samples as discussed in [Naesseth et al., 2015a]. We will in the experiments show how this can be used to design efficient algorithms by nesting several layers of SMC samplers.

Compare with the example in Section 3.2 and Figure 3 where we used forward filtering–backward sampling by considering the components of \( v_t, 1:d \) as our target. Instead of exact forward filtering we can use Algorithm 2, and instead of exact backward sampling we can use Algorithm 3 to generate properly weighted samples.
3.5 Theoretical Justification

In this section we will provide a central limit theorem that further motivates NSMC, and show how the asymptotic variance depends on the internal approximation of the exact fully adapted SMC. Furthermore, we provide a result that shows how this asymptotic variance converges to that of the corresponding asymptotic variance of the exact fully adapted SMC method as \( M \to \infty \).

**Theorem 1** (Central Limit Theorem). Under certain (standard) regularity conditions on the function \( \varphi : X_t \mapsto \mathbb{R} \), specified in the supplementary material, we have the following central limit theorem

\[
\sqrt{N} \left( \frac{1}{N} \sum_{i=1}^{N} \varphi(x_{1:t}^i) - \pi_t(\varphi) \right) \xrightarrow{d} N \left( 0, \Sigma^M_t(\varphi) \right),
\]

where \( \{x_{1:t}^i\}_{i=1}^{N} \) are generated by Algorithm 4 and the asymptotic variance is given by

\[
\Sigma^M_t(\varphi) = \sum_{s=0}^{t} \sigma_{s,t}^M(\varphi),
\]

for \( \sigma_{s,t}^M(\varphi) \)'s defined by

\[
\sigma_{t,t}^M(\varphi) = \pi_t \left( (\varphi - \pi_t(\varphi))^2 \right),
\]

\[
\sigma_{s,t}^M(\varphi) = \int \Psi_{s,t}^M(x_{1:s}; \varphi) \pi_s(x_{1:s}) dx_{1:s}, \quad \text{for } 0 < s < t,
\]

\[
\sigma_{0,t}^M(\varphi) = \int \frac{\tau_1(u_0)^2}{Z_1^2} \left( \int (\varphi(x_{1:t}) - \pi_t(\varphi)) \frac{\pi_t(x_{1:t})}{\pi_1(x_1)} \kappa_{1}^M(x_{1}|u_0) dx_{1:t} \right)^2 \eta_0^M(u_0) du_0.
\]

with

\[
\Psi_{s,t}^M(x_{1:s}; \varphi) :=
\]

\[
\mathbb{E}_{\eta_0^M(u_s|x_{1:s})} \left[ \frac{Z_s^2}{Z_{s+1}^2} \tau_{s+1}(u_s)^2 \left( \int (\varphi(x_{1:t}) - \pi_t(\varphi)) \frac{\pi_t(x_{1:t})}{\pi_{s+1}(x_{1:s+1})} \kappa_{s+1}^M(x_{s+1}|u_s) dx_{s+1:t} \right)^2 \right] \quad (16)
\]
Proof. See the supplementary material.

This theorem shows that, even for a fixed and finite value of $M$, the NSMC method obtains the standard $\sqrt{N}$ convergence rate. We can see how the asymptotic variance depends on how well we approximate $q_t$ and its normalization constant with $\kappa_t$ and $\tau_t$. Furthermore, this lets us study convergence of the variance in $M$ and also analytic expressions for a high-dimensional state space model.

To show the convergence to fully adapted SMC as the approximation improves with increasing $M$ we make some further assumptions detailed below.

**Assumption 1** (Uniform integrability). The sequence (in $M$) of random variables $\{\Psi^M_{s,t}(x_{1:s};\varphi)\}$ is uniformly integrable.

**Remark 2.** Note that a sufficient condition for Assumption 1 to hold is that for some $\delta > 0$ and for all $s, M \geq 1$ the following holds

$$\int \Psi^M_{s,t}(x_{1:s};\varphi)^{1+\delta} \pi_s(x_{1:s})dx_{1:s} < \infty.$$ 

**Assumption 2** (Strong mixing). For all $s, t$, there exists

$$\lambda^-_{s+1,t} \cdot \pi_t(x_{s+2:t}|x_{1:s+1}) \leq \frac{\pi_t(x_{1:t})}{\pi_{s+1}(x_{1:s+1})} \leq \lambda^+_{s+1,t} \cdot \pi_t(x_{s+2:t}|x_{1:s+1}),$$

where $0 < \lambda^-_{s+1,t}, \lambda^+_{s+1,t} < \infty$.

**Remark 3.** In the supplementary material we detail a weaker assumption for which Proposition 2 still holds.

**Proposition 2.** Under the assumptions of Theorem 1, Assumption 1 and 2 the following
limit holds:

\[
\lim_{M \to \infty} \Sigma^M_t(\varphi) = \\
= \pi_t ((\varphi - \pi_t(\varphi))^2) + \sum_{s=1}^{t-1} \int \frac{\pi_t(x_{1:s})^2}{\pi_s(x_{1:s})} \left( \int \varphi(x_{1:t}) \pi_t(x_{s+1:t} | x_{1:s}) dx_{s+1:t} - \pi_t(\varphi) \right)^2 dx_{1:s}.
\]

Proof. See the supplementary material.

Remark 4. The attained asymptotic variance is exactly the one derived for the fully adapted SMC asymptotic variance by [Johansen and Doucet 2008].

3.6 Choosing N vs M

The computational complexity for the two-level NSMC is proportional to \(O(NM)\), and it is interesting to study the trade-off between the number of particles in the outer procedure \((N)\) and the inner \((M)\). To this end we consider a fairly simple model and test function that leads to analytical expressions for the asymptotic variance in the CLT above. We propose to study a high-dimensional SSM, given in Definition 3, i.e. obtained by making \(n_x\) independent copies of an SSM. For this model we can obtain analytical solutions given by Proposition 3.

Definition 3. Define the independent state space model as follows

\[
\pi_t(x_{1:t}) \propto \prod_{d=1}^{n_x} \mu(x_{1:d}) \prod_{s=1}^{t} g(y_{s,d} | x_{s,d}) \prod_{s=2}^{t} f(x_{s,d} | x_{s-1,d}).
\]

For simplicity we also assume that \(y_{s,d} = y_{s,e}, \forall d, e\) and that \(\mathbb{E}_{\pi_t}[x_t] = 0\).

Proposition 3 \((N vs M)\). For the model in Definition 3 and \(\varphi(x_{1:t}) = \sum_{d=1}^{n_x} x_{1:d}\), we have that the asymptotic variance of fully adapted SMC is given by

\[
\Sigma^F_t(\varphi) = n_x A_t + \sum_{s=1}^{t-1} n_x B_s^{n_x-1} A_s + n_x(n_x - 1)B_s^{n_x-2} C_s^2,
\]
and using \( r(x_{s,d}|x_{s-1,d}) \) as proposal in the NSMC method in Definition 3 we get that the asymptotic variance of NSMC is

\[
\Sigma^M_t(\varphi) = n_x A_t + \sum_{s=0}^{t-1} \left[ n_x B_s^{n_x-1} \left( A_s + M^{-1} \left( \hat{A}_s - A_s \right) \right) \left( 1 - \frac{1}{M} \right)^{n_x-1} \left( 1 + \frac{\hat{B}_s}{B_s(M-1)} \right)^{n_x-1} + n_x(n_x - 1) B_s^{n_x-2} \left( C_s + M^{-1} \left( \hat{C}_s - C_s \right) \right)^2 \left( 1 - \frac{1}{M} \right)^{n_x-2} \left( 1 + \frac{\hat{B}_s}{B_s(M-1)} \right)^{n_x-2} \right],
\]

for the (finite) positive constants \( A_t, A_s, \hat{A}_s, B_s, \hat{B}_s, C_s, \) and \( \hat{C}_s \) defined in the supplementary material.

**Proof.** See the supplementary material.

**Remark 5.** As expected the asymptotic variance of the NSMC will (like fully adapted SMC) in general scale exponentially bad with the dimension \( n_x \) of the state. However, to control the additional approximation introduced by not evaluating \( \nu_{t-1} \) and sampling \( q_t \) exactly, we only need to scale \( M \propto n_x \), even as \( n_x \to \infty \). We expect that intuition and rule-of-thumbs from running standard SMC also apply to the internal approximation targeting \( \frac{\gamma_t(x_{1:t})}{\gamma_{t-1}(x_{1:t-1})} \).

## 4 Numerical Results

### 4.1 Gaussian Model

We start by considering a Gaussian spatio-temporal state space model where the exact solution is available via the Kalman filter [Kalman 1960], and we can implement exact fully adapted SMC as explained in Section 3.2. The model is given by

\[
x_t = 0.5x_{t-1} + v_t, \quad v_t \sim \frac{1}{Z_v} \exp \left( -\frac{\tau}{2} \sum_{d=1}^{n_x} v_{t,d}^2 - \frac{\lambda}{2} \sum_{d=2}^{n_x} (v_{t,d} - v_{t,d-1})^2 \right) \quad (17a)
\]

\[
y_t|x_t \sim N(x_t, \sigma_y^2 I). \quad (17b)
\]
The results for $N = 100, T = 10, \tau = \lambda = 1$ and $\sigma_y^2 = 0.25^2$, i.e. with fairly high signal to noise ratio, is given in Figure 4. We compare NSMC with (and without) backward simulation to the bootstrap particle filter (BPF) that uses the transition probability as proposal. We give all methods equivalent computational budget as the number of internal particles $M$ grow, i.e. BPF gets $N_{\text{BPF}} = 100 \cdot M$ particles. Furthermore, for illustrative purposes we include fully adapted SMC (FAPF), the method that NSMC approximates, for a fixed number of particles $N_{\text{FAPF}} = 100$. The experiments are run ten times independently and we show the median squared error (MSE) as well as 25%/75% quantiles, for estimates of the log-likelihood, $E[x_{T,1}]$ and $E[x_{T,n_x}]$ with $n_x \in \{10, 100\}$. The expectations are with respect to the posterior distribution.

We can see that NSMC is significantly better than BPF and that it converges quickly towards the fully adapted SMC. Backward simulation also clearly helps with estimates of $E[x_{T,d}]$ for $d = 1$, alleviating the correlation between generated samples. It is worthwhile to point out that for small $M$ the NSMC seems to improve much more quickly than the standard asymptotic rate $M^{-1}$. For the likelihood estimate the rate almost exceeds $M^{-4}$. We provide results for different settings of $\sigma_y^2$ in the supplementary material. In general we see less striking improvement of NSMC over BPF when the signal to noise ratio is low, i.e. $\sigma_y^2$ is high compared to $\tau^{-1}$, which is to be expected [Snyder et al., 2015].

### 4.2 Soil Carbon Cycles

We move on to study the performance of NSMC and compare it to ST-PF on a spatio-temporal model inspired by the soil carbon cycle model of Murray [2016] and Clifford et al. [2014]. The simplified model that we use to profile the two state-of-the-art methods is...
Figure 4: Median SE and 25%/75% quantiles of Monte Carlo estimates of $\log p(y_{1:T}), \mathbb{E}[x_{T,1}], \mathbb{E}[x_{T,n_x}]$ for BPF, FAPF and two variants of NSMC. $N = 100$ for FAPF and NSMC and BPF has equivalent computational budget $N = 100 \cdot M$. Left column $n_x = 10$, right column $n_x = 100$ and $T = 10, \sigma_y^2 = 0.25^2$ in all experiments.
defined by

\[ x_t = 0.5(x_{t-1} + e^{\xi_t})e^{v_t}, \quad v_t \sim \frac{1}{Z_v} \exp \left( -\frac{\tau}{2} \sum_{i \in V} v_{t,i}^2 - \frac{\lambda}{2} \sum_{(i,j) \in E} (v_{t,i} - v_{t,j})^2 \right), \]  

(18a)

\[ y_t | x_t \sim \text{TruncatedNormal} (x_t, \sigma^2 I, 0, \infty), \]  

(18b)

where \( \xi_t \) is a known input signal and \((V, E)\) is a square lattice, \( \sqrt{n_x} \times \sqrt{n_x} \), with nearest neighbour interaction, i.e. \((i, j) \in E\) if \(i\) and \(j\) are neighbors on the lattice. The latent variables \( x_t \) are positive and it is not possible to implement the exact fully adapted SMC method. We set \( \sigma = 0.2, \tau = 2, \) and \( \lambda = 1.0 \) and run NSMC and ST-PF with matched computational complexity. Figure 5 displays the median, over the \( n_x \) dimensions, mean squared error for each time-point \( t \) estimated by running the algorithms 20 times independently. Ground truth is estimated using 20 independent runs of the method of Naesseth et al. [2014b] with 64,000 samples. We can see that the different NSMC versions either

perform as well, or better than ST-PF. This is without taking into account that NSMC simplifies distribution of the computation and is more memory efficient, only \( N \) rather than

Figure 5: Results for \( T = 2, n_x = 64 (8 \times 8) \).
Finally, we consider an example with a non-Gaussian ST-SSM, borrowed from Beskos et al. [2014] where the full details of the model are given. The transition probability $f(x_t | x_{t-1})$ is a spatially localised Gaussian mixture and the measurement probability $g(y_t | x_t)$ is Student’s t-distributed. The model dimension is $n_x = 1024$. Beskos et al. [2014] report improvements for ST-PF over both the BPF and the block PF by Rebeschini and van Handel [2015]. Following Beskos et al. [2014] we use $N = M = 100$ for both ST-PF and NSMC and the BPF is given $N = 10000$. In Figure 6 we report the effective sample size (ESS, higher is better), estimated according to Carpenter et al. [1999]. The ESS for the BPF is close to 0, for ST-PF around 1–2, and for NSMC slightly higher at

\[ \text{ESS} \]

Figure 6: Median ESS with 15 – 85% percentiles (shaded region) for the non-Gaussian SSM.

\[ \text{Figure 6} \]

\[ \text{The results in this section have been previously published by the authors in Naesseth et al. [2015a].} \]
7–8. However, we note that all methods perform quite poorly on this model, and to obtain satisfactory results it would be necessary to use more particles.

A Supplementary Material

A.1 General Nested Sequential Monte Carlo

Assume that we are interested in approximating an arbitrary auxiliary SMC sampler with proposal \( q_t(x_{1:t-1} | x_{1:t-1}) = \frac{r_t(x_t | x_{1:t-1})}{\int r_t(x_t | x_{1:t-1}) \, dx_t} \) and adjustment multipliers \( \nu_{t-1}(x_{1:t-1}) \). The fully adapted SMC that we focus on in this paper is then attained as a special case when \( q_t(x_{1:t-1} | x_{1:t-1}) \propto \gamma_t(x_{1:t}) \gamma_{t-1}(x_{1:t-1}) \) and \( \nu_{t-1}(x_{1:t-1}) = \int \frac{\gamma_t(x_{1:t})}{\gamma_{t-1}(x_{1:t-1})} \, dx_t \).

We can just as easily use a nested Monte Carlo method that produces properly weighted samples with respect to an arbitrary proposal \( q_t(x_{1:t-1}) \) and multipliers \( \nu_{t-1}(x_{1:t-1}) \), see Algorithm 4.

**Algorithm 4** Nested Sequential Monte Carlo (all for \( i = 1, \ldots, N \))

**Require:** \( \eta_{t-1}^M, \kappa_t^M, \tau_t \) that generate samples properly weighted for \( q_t(x_{1:t-1}) \)

1: for \( t = 1 \) to \( T \) do
2: Simulate \( u_{t-1}^i \sim \eta_{t-1}^M(u_{t-1} | x_{1:t-1}^i) \)
3: Draw \( a_t^i \) with probability \( P(a_t^i = j) = \frac{\nu_{t-1}(x_{t}^i | a_t^i, u_{t-1}^i) w_{t-1}^i}{\sum_j \nu_{t-1}(x_{t}^i | a_t^i, u_{t-1}^i) w_{t-1}^i} \)
4: Simulate \( x_t^i \sim \kappa_t^M(x_t | a_t^i, u_{t-1}^i) \)
5: Set \( x_{1:t}^i = (x_{1:t-1}^i, x_t^i) \)
6: Set \( w_{t-1}^i = \frac{\gamma_t(x_{1:t}^i)}{\gamma_{t-1}(x_{1:t-1}^i) \nu_{t-1}(x_{t}^i | a_t^i, u_{t-1}^i) r_t(x_{t}^i | x_{1:t-1}^i)} \)
7: end for

**Remark 6.** Note that if the adjustment multipliers \( \nu_{t-1} \) do not depend on \( u_{t-1} \), simulating from \( \eta_{t-1} \) can be done after resampling (simulating \( a_t \)). This ensures that the new samples
are conditionally independent, thus decreasing correlation between samples.

Generating Properly Weighted Samples using IS

There are many ways of generating properly weighted samples with respect to a distribution, one example is using sequential Monte Carlo with or without backward simulation as explained in the main manuscript. However, perhaps one of the most straightforward and simple approaches is to use standard importance sampling. This means we would define $\eta_{t-1}^M, \kappa_t^M, \tau_t$ as follows:

$\eta_{t-1}^M(u_{t-1}|x_{1:t-1})$: Set $u_{t-1} = \{\tilde{x}_i^t\}_{i=1}^M$, where $\tilde{x}_i^t \sim p_t(x_t|x_{1:t-1})$ for some proposal $p_t$.

$\kappa_t^M(x_t|u_{t-1})$: Set $x_t = \tilde{x}_B^t$, where $B$ is simulated with probability $\mathbb{P}(B = j) = \tilde{w}_t^j / \sum_{i=1}^M \tilde{w}_t^i$ with $w_t^j = \gamma_t(x_j^t|x_{1:t-1}) / p_t(\tilde{x}_j^t|x_{1:t-1})$.

$\tau_t(u_{t-1})$: Set $\tau_t(u_{t-1}) = \frac{1}{M} \sum_{i=1}^M \tilde{w}_t^i$.

It is straightforward to show that the above procedure generates properly weighted samples for $q_t$ as long as $p_t > 0$ whenever $q_t$ is. Now, if we want to use the above to approximate fully adapted SMC we simply let $r_t = \gamma_t / \gamma_{t-1}$ and $\nu_{t-1} = \tau_t$.

A.2 Theoretical Results

Proof of Theorem 1

We reproduce the central limit theorem of Naesseth et al. [2015a] here for clarity, see the Appendix of the extended version Naesseth et al. [2015b] for details.
To explicitly state the general theorem we need some notation defined below:

\[
\begin{align*}
\Gamma_t(x_{1:t}, u_{0:t}) &= \frac{\tau_t(u_{t-1}) \eta_t^M(u_t| x_{1:t}) \kappa_t^M(x_t| u_{t-1})}{r_t(x_t| x_{1:t-1})} \gamma_t(x_{1:t}) \Gamma_{t-1}(x_{1:t-1}, u_{0:t-1}), \\
\Pi_t(x_{1:t}, u_{0:t}) &= \Gamma_t(x_{1:t}, u_{0:t}) Z_t^{-1}, \\
w_t(x_{1:t}, u_{0:t}) &\propto \frac{\gamma_t(x_{1:t})}{\gamma_{t-1}(x_{1:t-1})} \frac{\tau_t(u_{t-1})}{\nu_{t-1}(x_{1:t-1}, u_{t-1})} r_t(x_t| x_{1:t-1}), \\
\Gamma'_t(x_{1:t}, u_{0:t}) &= \nu_t(x_{1:t}, u_t) \Gamma_t(x_{1:t}, u_{0:t}), \\
\Pi'_t(x_{1:t}, u_{0:t}) &\propto \Gamma'_t(x_{1:t}, u_{0:t}), \\
Q_t^M(x_t, u_t| x_{1:t-1}, u_{t-1}) &= \eta_t^M(u_t| x_{1:t}) \kappa_t^M(x_t| u_{t-1}), \\
w'_t(x_{1:t}, u_{0:t}) &= \frac{\Pi'_t(x_{1:t}, u_{0:t})}{Q_t^M(x_t, u_t| x_{1:t-1}, u_{t-1})} \nu_t(x_{1:t}, u_t) w_t(x_{1:t}, u_{0:t}), \\
\omega_t(x_{1:t}, u_{0:t}) &= \frac{\Pi_t(x_{1:t}, u_{0:t})}{Q_t^M(x_t, u_t| x_{1:t-1}, u_{t-1})} \nu_t(x_{1:t}, u_t) w_t(x_{1:t}, u_{0:t}).
\end{align*}
\]

Domain of \(\Pi_t(x_{1:t}, u_{0:t})\) is denoted by \(\Theta_t = X_t \times U_t\). For a function \(h : X_t \mapsto \mathbb{R}\), we define the extension of \(h\) to \(\Theta_t\) by \(h^e(x_{1:t}, u_{0:t}) := h(x_{1:t})\). Let \(\Phi_t\) be defined recursively to be the set of measurable functions \(h : \Theta_t \mapsto \mathbb{R}\) such that there exists a \(\delta > 0\) with \(\mathbb{E}_{Q_t^M \Pi_{t-1}^M} [\|w'_t h\|^{2+\delta}] < \infty\), and such that \((x_{1:t-1}, u_{0:t-1}) \mapsto \mathbb{E}_{Q_t^M} [w'_t h]\) is in \(\Phi_{t-1}\). We are now ready to state the more general central limit theorem of [Naesseth et al. 2015a].

**Theorem 2 (Central Limit Theorem).** Assume that \(\varphi : X_t \mapsto \mathbb{R}\) is a function such that \(\mathbb{E}_{Q_t^M \Pi_{t-1}^M} [\|w'_t \varphi e\|^{2+\delta}] < \infty\) for some \(\delta > 0\), and that \((x_{1:t-1}, u_{0:t-1}) \mapsto \mathbb{E}_{Q_t^M} [\omega_t \varphi e]\) is in \(\Phi_{t-1}\). Then we have the following central limit theorem

\[
\sqrt{N} \left( \sum_{i=1}^{N} \frac{w_t^i}{\sum_{t=1}^{N} w_t^i} \varphi(x_{1:t}) - \pi_t(\varphi) \right) \overset{d}{\rightarrow} \mathcal{N} \left( 0, \Sigma_t^M(\varphi) \right),
\]

for \(i = 1, \ldots, N\).
where \( \{(w_i^t, x^t_i)\}_{i=1}^N \) are generated by Algorithm 5 in Naesseth et al. [2015b] and the asymptotic variance is given by

\[
\Sigma^M_t(\phi) = \tilde{V}^M_t(\phi^e - \mathbb{E}\Pi_t[\phi^e]),
\]

where \( \tilde{V}^M_t \) is defined by the following set of recursions for measurable functions \( h : \Theta_t \mapsto \mathbb{R} \)

\[
\begin{align*}
\tilde{V}^M_t(h) &= \tilde{V}^M_{t-1}(\mathbb{E}_{Q^M_{t-1}}[h]) + \mathbb{E}_{\Pi^t_{t-1}} \left[ \text{Var}_{Q^M_{t-1}}(h) \right], \quad t > 0, \\
V^M_t(h) &= \tilde{V}^M_t(h - \mathbb{E}\Pi^t_t[h]), \quad t \geq 0, \\
\dot{V}^M_t(h) &= V^M_t(h) + \text{Var}_{\Pi^t_t}(h), \quad t \geq 0.
\end{align*}
\]

initialized by \( \tilde{V}^M_0(h) = \text{Var}_{\eta^0_0}(h) \) for \( h : \Theta_0 \mapsto \mathbb{R} \).

**Approximating the Fully Adapted SMC**

When we are approximating the fully adapted SMC, i.e. when we have \( q^t(x_t|x_{1:t-1}) \propto \frac{\gamma^t(x_{1:t})}{\gamma^t_{t-1}(x_{1:t-1})} \) and \( \nu_t(x_{1:t}, u_t) = \tau_{t+1}(u_t) \), we can make significant simplifications of the expressions in the general central limit theorem above. Specifically we get that

\[
\begin{align*}
\Pi^t_t(x_{1:t}, u_{0:t}) &= \frac{\tau_{t+1}(u_t)}{Z_{t+1}} \Gamma_t(x_{1:t}, u_{0:t}), \\
w^t_t(x_{1:t}, u_{0:t}) &= \frac{Z_t}{Z_{t+1}} \tau_{t+1}(u_t), \\
\omega_t(x_{1:t}, u_{0:t}) &= 1.
\end{align*}
\]

**Lemma 1.** The asymptotic variance \( \Sigma^M_t(\phi) \) in Theorem 2 when approximating the fully adapted SMC is given by

\[
\Sigma^M_t(\phi) = \text{Var}_{\eta^0_t}(h_0) + \sum_{s=1}^t \text{Var}_{\Pi^t_{t-1}, Q^M_{t-1}}(h_s),
\]
for \( h_s \) defined by

\[
h_t = \varphi^e - \mathbb{E}_{\Pi_t}[\varphi^e],
\]

\[
h_s = \frac{Z_s}{Z_{s+1}} \tau_{s+1}(u_s) \left( \mathbb{E}_{Q^{M}_{s+1}}[h_{s+1}] - \mathbb{E}_{\Pi'_s} \left[ \mathbb{E}_{Q^{M}_{s+1}}[h_{s+1}] \right] \right), \quad 1 \leq s \leq t - 1,
\]

\[
h_0 = \frac{1}{Z_1} \tau_1(u_0) \left( \mathbb{E}_{Q^M_1}[h_1] - \mathbb{E}_{\Pi'_0} \left[ \mathbb{E}_{Q^M_1}[h_1] \right] \right),
\]

where \( \Pi'_0(u_0) = \frac{\tau_1(u_0)}{Z_1} h^M_0(u_0) \).

**Proof.** For a function \( h_t : \Theta_t \mapsto \mathbb{R} \) we have by Theorem 2 that

\[
\tilde{V}^M_t(h_t) = \tilde{V}^M_{t-1} \left( \mathbb{E}_{Q^M_t}[h_t] \right) + \mathbb{E}_{\Pi'_{t-1}} \left[ \text{Var}_{Q^M_t}(h_t) \right]
\]

\[
= \tilde{V}^M_{t-1} \left( \mathbb{E}_{Q^M_t}[h_t] \right) + \text{Var}_{\Pi'_{t-1}} \left( \mathbb{E}_{Q^M_t}[h_t] \right) + \mathbb{E}_{\Pi'_{t-1}} \left[ \text{Var}_{Q^M_t}(h_t) \right]
\]

\[
= \tilde{V}^M_{t-1} \left( \mathbb{E}_{Q^M_t}[h_t] - \mathbb{E}_{\Pi'_{t-1}} \left[ \mathbb{E}_{Q^M_t}[h_t] \right] \right) + \text{Var}_{\Pi'_{t-1}} \left( \mathbb{E}_{Q^M_t}[h_t] \right) + \mathbb{E}_{\Pi'_{t-1}} \left[ \text{Var}_{Q^M_t}(h_t) \right]
\]

Recursion with \( h_{t-1} \left( \frac{Z_{t-1}}{Z_t} \tau_t(u_{t-1}) \left( \mathbb{E}_{Q^M_t}[h_t] - \mathbb{E}_{\Pi'_{t-1}} \left[ \mathbb{E}_{Q^M_t}[h_t] \right] \right) \right) \) gives the result.

**Lemma 2.**

\[
h_t = \varphi - \pi_t(\varphi), \quad (20)
\]

\[
h_s = \frac{Z_s}{Z_{s+1}} \tau_{s+1}(u_s) \int (\varphi(x_{1:s}) - \pi_t(\varphi)) \frac{\pi_t(x_{1:s})}{\pi_{s+1}(x_{1:s+1})} \kappa_{s+1}^M(x_{s+1}|u_s) dx_{s+1:t}, \quad 1 \leq s \leq t - 1,
\]

\[
h_0 = \frac{1}{Z_1} \tau_1(u_0) \int (\varphi(x_{1:s}) - \pi_t(\varphi)) \frac{\pi_t(x_{1:s})}{\pi_1(x_1)} \kappa_1^M(x_1|u_0) dx_{1:t}, \quad (21)
\]

**Proof.** The first, \( h_t \), follows straightforwardly by the definition of \( \varphi^e \) and \( \Pi_t \). The remaining will be proved by induction. Assume that for \( s \leq t - 1 \) \((21)\) holds. We will now show that
this in fact holds for both $h_{t-1}$ and $h_{s-1}$; thus the result follows. Start by considering $h_{t-1}$ using the definition in Lemma 1.

$$h_{t-1} = \frac{Z_{t-1}}{Z_t} \tau_t(u_{t-1}) \left( \mathbb{E}_{Q^M}[h_t] - \mathbb{E}_{n_{t-1}^s} \left[ \mathbb{E}_{Q^M}[h_t] \right] \right) = \frac{Z_{t-1}}{Z_t} \tau_t(u_{t-1}) \left( \mathbb{E}_{Q^M}[\varphi - \pi_t(\varphi)] - 0 \right)$$

$$= \frac{Z_{t-1}}{Z_t} \tau_t(u_{t-1}) \left( \int \varphi(x_{1:t}) \kappa_t^M(x_t|u_{t-1}) dx_t - \pi_t(\varphi) \right).$$

Now, for $h_{s-1}$ let us start by studying $\mathbb{E}_{Q^M}[h_s]$ and $\mathbb{E}_{n_{s-1}^s} \left[ \mathbb{E}_{Q^M}[h_s] \right]$

$$\mathbb{E}_{Q^M}[h_s] = \mathbb{E}_{Q^M} \left[ \frac{Z_s}{Z_{s+1}} \tau_{s+1}(u_s) \int (\varphi(x_{1:t}) - \pi_t(\varphi)) \frac{\pi_t(x_{1:t})}{\pi_{s+1}(x_{1:s+1})} \kappa_{s+1}^M(x_{s+1}|u_s) dx_{s+1:t} \right]$$

$$= \ldots = \int (\varphi(x_{1:t}) - \pi_t(\varphi)) \frac{\pi_t(x_{1:t})}{\pi_s(x_{1:s})} \kappa_s^M(x_s|u_{s-1}) dx_{s:t},$$

$$\mathbb{E}_{n_{s-1}^s} \left[ \mathbb{E}_{Q^M}[h_s] \right] = \ldots = 0.$$

This gives us that

$$h_{s-1} = \frac{Z_{s-1}}{Z_s} \tau_s(u_{s-1}) \left( \mathbb{E}_{Q^M}[h_s] - \mathbb{E}_{n_{s-1}^s} \left[ \mathbb{E}_{Q^M}[h_s] \right] \right)$$

$$= \frac{Z_{s-1}}{Z_s} \tau_s(u_{s-1}) \int (\varphi(x_{1:t}) - \pi_t(\varphi)) \frac{\pi_t(x_{1:t})}{\pi_s(x_{1:s})} \kappa_s^M(x_s|u_{s-1}) dx_{s:t}.$$ 

The results follows by noting that the procedure is the same for $h_0$ taking into account edge effects, i.e. $Z_0 = 1$. \hfill \Box

**Lemma 3.**

$$\text{Var}_{n_{t-1}^s} \mathbb{E}_{Q^M}[h_t] = \pi_t \left( (\varphi - \pi_t(\varphi))^2 \right),$$

$$\text{Var}_{n_{s-1}^s} \mathbb{E}_{Q^M}[h_s] = \int \left[ \frac{Z_s^2 \tau_{s+1}(u_s)^2}{Z_{s+1}^2} \left( \int (\varphi(x_{1:t}) - \pi_t(\varphi)) \frac{\pi_t(x_{1:t})}{\pi_{s+1}(x_{1:s+1})} \kappa_{s+1}^M(x_{s+1}|u_s) dx_{s+1:t} \right)^2 \eta_s^M(u_s|x_{s-1}) \pi_s(x_{1:s}) \right] du_s dx_{1:s}, \quad 1 \leq s \leq t - 1,$$

$$\text{Var}_{n_0^t} \left[ \mathbb{E}_{Q^M}[h_0] \right] = \int \frac{\tau_1(u_0)^2}{Z_1^2} \left( \int (\varphi(x_{1:t}) - \pi_t(\varphi)) \frac{\pi_t(x_{1:t})}{\pi_1(x_{1})} \kappa_1^M(x_{1}|u_0) dx_{1:t} \right)^2 \eta_0^M(u_0) du_0.$$
Proof. We get the first equality

\[ \text{Var}_{\Pi_{t-1}, Q^M_t}(h_t) = \mathbb{E}_{\Pi_{t-1}, Q^M_t} \left[ (\varphi - \pi_t(\varphi))^2 \right] - \left( \mathbb{E}_{\Pi_{t-1}, Q^M_t} [\varphi - \pi_t(\varphi)] \right)^2 = \pi_t \left( (\varphi - \pi_t(\varphi))^2 \right), \]

due to Lemma [2] and because \( \Pi \) follows.

\[
\text{Var}_{\Pi_{t-1}, Q^M_s}(h_s) = \mathbb{E}_{\Pi_{t-1}, Q^M_s} \left[ h_s^2 \right] - \left( \mathbb{E}_{\Pi_{t-1}, Q^M_s} [h_s] \right)^2 = \mathbb{E}_{\Pi_{t-1}, Q^M_s} \left[ h_s^2 \right]
\]

\[
= \int h_s(x_{1:s}, u_s)^2 \Pi_s(x_{1:s}, u_{0:s}) du_{0:s} dx_{1:s} = \int h_s(x_{1:s}, u_s)^2 \eta^M_s(u_s|x_{1:s}) \pi_s(x_{1:s}) du_s dx_{1:s}
\]

\[
= \int \left[ \frac{Z^2_s \tau_{s+1}(u_s)^2}{Z^2_{s+1}} \right] \left( \int (\varphi(x_{1:t}) - \pi_t(\varphi)) \frac{\pi_t(x_{1:t})}{\pi_{s+1}(x_{1:s+1})} \kappa^M_{s+1}(x_{s+1}|u_s) dx_{s+1:t} \right)^2 \eta^M_s(u_{s+1}|x_{1:s-1}) \pi_s(x_{1:s}) du_s dx_{1:s}, \quad 1 \leq s \leq t - 1,
\]

where the second equality follows by noting that \( \mathbb{E}_{\Pi_{t-1}, Q^M_s} [h_s] = 0 \). Analogously to Lemma [2] the expression for \( s = 0 \) follows by taking into account the edge effects.

Finally, with Lemmas [1] [2] and [3] together the result, i.e. Theorem 1 of the main manuscript, follows.

Proof of Proposition [2] in the Main Manuscript

Assumption 3 (Approximation property). The approximation of \( q^M_s(x_s|x_{1:s-1}) \propto \frac{\pi_s(x_{1:s})}{\pi_{s-1}(x_{1:s-1})} \) and \( \nu_{s-1}(x_{1:s-1}) = \int \frac{\pi_s(x_{1:s})}{\pi_{s-1}(x_{1:s-1})} dx_s \) based on \( \eta^M_s \), \( \kappa^M_{s+1} \) and \( \tau_s \) satisfies

\[ \Psi^M_{s,t}(x_{1:s}; \varphi) \xrightarrow{d} \frac{\pi_t(x_{1:s})}{\pi_s(x_{1:s})} \left( \int \varphi(x_{1:t}) \pi_t(x_{s+1:t}|x_{1:s}) dx_{s+1:t} - \pi_t(\varphi) \right)^2, \quad \text{as } M \to \infty. \quad (23) \]

Furthermore, assume that \( \sigma^M_{0,t}(\varphi) \xrightarrow{d} 0 \) as \( M \to \infty \).

Lemma 4. The strong mixing assumption of the main manuscript,

\[ \lambda^+_{s+1,t}, \pi_t(x_{s+2:t}|x_{1:s+1}) \leq \frac{\pi_t(x_{1:t})}{\pi_{s+1}(x_{1:s+1})} \leq \lambda^-_{s+1,t}, \pi_t(x_{s+2:t}|x_{1:s+1}), \]

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where $0 < \lambda_{s+1,t}, \lambda^+_{s+1,t} < \infty$, implies that

$$\Psi^M_{s,t}(x_{1:s}; \varphi) \xrightarrow{d} \frac{\pi_t(x_{1:s})^2}{\pi_s(x_{1:s})^2} \left( \int \varphi(x_{1:t})\pi_t(x_{s+1:t}|x_{1:s})dx_{s+1:t} - \pi_t(\varphi) \right)^2, \text{ as } M \to \infty. \quad (24)$$

**Proof.** Under the strong mixing assumption and given that we use a SMC method to generate properly weighted samples the result follows from standard SMC results [Del Moral, 2004].

**Theorem 3** (Vitali Convergence Theorem). If $\{\Psi^M_{s,t}(x_{1:s}; \varphi)\}$ is uniformly integrable and if $\Psi^M_{s,t}(x_{1:s}; \varphi) \xrightarrow{d} \Psi_{s,t}(x_{1:s}; \varphi)$, then

$$\lim_{M \to \infty} \int \Psi^M_{s,t}(x_{1:s}; \varphi)\pi_s(x_{1:s})dx_{1:s} = \int \Psi_{s,t}(x_{1:s}; \varphi)\pi_s(x_{1:s})dx_{1:s}.$$ 

**Proof.** See [Folland, 1999, Chapter 6].

Under assumptions of uniform integrability and strong mixing (or Assumption 3), the result now follows by using the Vitali convergence theorem and noting that

$$\int \Psi_{s,t}(x_{1:s}; \varphi)\pi_s(x_{1:s})dx_{1:s} = \int \frac{\pi_t(x_{1:s})^2}{\pi_s(x_{1:s})} \left( \int \varphi(x_{1:t})\pi_t(x_{s+1:t}|x_{1:s})dx_{s+1:t} - \pi_t(\varphi) \right)^2 dx_{1:s}.$$
Choosing $N$ and $M$

The constants in Proposition 3 in the main manuscript are defined as follows

\[
A_t = \int x_{t,d}^2 \pi_t(x_{1:t,d}) dx_{1:t,d}, \\
A_s = \int \frac{\pi_t(x_{1:s,d})^2}{\pi_s(x_{1:s,d})} \left( \int x_{t,d} \pi_t(x_{t,d}|x_{s,d}) dx_{t,d} \right)^2 dx_{1:s,d}, \\
\tilde{A}_s = \int \frac{\pi_t(x_{1:s+1,d})^2}{\pi_s(x_{1:s,d})} \left( \int x_{t,d} \pi_t(x_{t,d}|x_{s+1,d}) dx_{t,d} \right)^2 dx_{1:s+1,d}, \\
B_s = \int \frac{\pi_t(x_{1:s,d})^2}{\pi_s(x_{1:s,d})} dx_{1:s,d}, \\
\tilde{B}_s = \int \frac{\pi_t(x_{1:s+1,d})^2}{\pi_s(x_{1:s,d})} dx_{1:s+1,d}, \\
C_s = \int \frac{\pi_t(x_{1:s,d})^2}{\pi_s(x_{1:s,d})} \int x_{t,d} \pi_t(x_{t,d}|x_{s,d}) dx_{t,d} dx_{1:s,d}, \\
\tilde{C}_s = \int \frac{\pi_t(x_{1:s+1,d})^2}{\pi_s(x_{1:s,d})} \int x_{t,d} \pi_t(x_{t,d}|x_{s+1,d}) dx_{t,d} dx_{1:s+1,d},
\]

with $A_0 = 0, B_0 = 1, C_0 = 0$,

\[
\tilde{A}_0 = \int \frac{\pi_t(x_{1,d})^2}{r(x_{1,d})} \left( \int x_{t,d} \pi_t(x_{t,d}|x_{1,d}) dx_{t,d} \right)^2 dx_{1,d}, \\
\tilde{B}_0 = \int \frac{\pi_t(x_{1,d})^2}{r(x_{1,d})} dx_{1,d}, \\
\tilde{C}_0 = \int \frac{\pi_t(x_{1,d})^2}{r(x_{1,d})} \int x_{t,d} \pi_t(x_{t,d}|x_{1,d}) dx_{t,d} dx_{1,d},
\]

and

\[
\tilde{A}_{t-1} = \int \frac{\pi_t(x_{1:t-1,d})^2}{\pi_s(x_{1:t-1,d}) r(x_{t,d}|x_{t-1,d})} x_{t,d} dx_{1:t,d}, \\
\tilde{B}_{t-1} = \int \frac{\pi_t(x_{1:t-1,d})^2}{\pi_s(x_{1:t-1,d}) r(x_{t,d}|x_{t-1,d})} dx_{1:t,d}, \\
\tilde{C}_{t-1} = \int \frac{\pi_t(x_{1:t-1,d})^2}{\pi_s(x_{1:t-1,d}) r(x_{t,d}|x_{t-1,d})} x_{t,d} dx_{1:t,d}.
\]
Proof of Proposition 3

For fully adapted SMC we have from the result in Johansen and Doucet [2008] (see also our convergence result in the previous section) and for the model defined in the main manuscript

$$\pi_t((\varphi - \pi_t(\varphi))^2) = n_x A_t$$

$$\frac{\pi_t(x_{1:s})^2}{\pi_s(x_{1:s})} \left( \sum_{d=1}^{n_x} \int x_{t,d} \pi_t(x_{s+1:t}|x_{1:s}) dx_{s+1:t} \right)^2 dx_{1:s} =$$

$$= \sum_{e=1}^{n_x} \sum_{f=1}^{n_x} \int \left[ \frac{\pi_t(x_{1:s})^2}{\pi_s(x_{1:s})} \int x_{t,e} \pi_t(x_{s+1:t}|x_{1:s}) dx_{s+1:t} \cdot \int x_{t,f} \pi_t(x_{s+1:t}|x_{1:s}) dx_{s+1:t} \right] dx_{1:s}$$

$$= \sum_{e=1}^{n_x} \sum_{f=1}^{n_x} \int \left[ \frac{\pi_t(x_{1:s})^2}{\pi_s(x_{1:s})} \int x_{t,e} \pi_t(x_{s+1:e}|x_{s,e}) dx_{t,e} \cdot \int x_{t,f} \pi_t(x_{t,f}|x_{s,f}) dx_{t,f} \right] dx_{1:s}$$

$$= n_x B_s^{n_x-1} A_s + n_x (n_x - 1) B_s^{n_x-2} C_s^2,$$

with constants as defined above.

For nested SMC we have $r(x_s|x_{s-1}) = \prod_{d=1}^{n_x} r(x_{s,d}|x_{s-1,d})$ and due to the independence between dimensions we will have no dependence on internal ancestor variables in $\eta_s, \kappa_{s+1}, \tau_{s+1}$, i.e.

$$\eta_s(u_s|x_{1:s}) = \prod_{d=1}^{n_x} \prod_{j=1}^{M} r(x_{s+1,d}|x_{s,d}),$$

$$\kappa_{s+1}(x_{s+1}|u_s) = \prod_{d=1}^{n_x} \sum_{j=1}^{M} w^j_d \delta_{x_{s+1,d}} (dx_{s+1,d}),$$

$$\tau_{s+1}(u_s) = \prod_{d=1}^{n_x} \frac{1}{M} \sum_{j=1}^{M} w^j_d,$$

$$w^j_d = \frac{f(x_{s+1,d}|x_{s,d}) g(y_{s+1,d}|x_{s+1,d})}{r(x_{s+1,d}|x_{s,d})}.$$
For the variance contribution of the final step we obtain\(\pi_t((\varphi - \pi_t(\varphi))^2) = n_x \sigma_x^2\), the same result as fully adapted SMC. The remaining can be calculated as follows

\[
\int \left[ \frac{Z_{s+1}^2(u_s)^2}{Z_{s+1}^2} \left( \int \varphi(x_{1:t}) \frac{\pi_t(x_{1:t})}{\pi_{s+1}(x_{1:s+1})} \eta_{s+1}^M(x_{s+1}\mid u_s)dx_{s+1:t} \right)^2 \eta_{s+1}^M(u_s\mid x_{1:s-1})\pi_s(x_{1:s}) \right] du_sdx_{1:s}
\]

\[
= \frac{1}{p(y_{s+1}\mid y_s)^2} \int \left[ \tau_{s+1}(u_s)^2 \frac{1}{M^{2n_x} \tau_{s+1}(u_s)^2} \eta_{s+1}^M(u_s\mid x_{1:s-1})\pi_s(x_{1:s}) \right.
\]

\[
\left. \cdot \left( \sum_{e=1}^{n_x} \sum_{j=1}^{M} \sum_{e' = 1}^{n_x} \int \tilde{h}_e(x_{1:s}, u_s) \tilde{h}_e'(x_{1:s}, u_s) \prod_{d \neq e} \int \frac{\pi_t(x_{1:s,d}, x_{s+1,d})}{\pi_{s+1}(x_{1:s,d}, x_{s+1,d})} \right)^2 \right] du_sdx_{1:s}
\]

\[
= \frac{1}{p(y_{s+1}\mid y_s)^2 M^{2n_x}} \sum_{e=1}^{n_x} \sum_{e' = 1}^{n_x} \int \tilde{h}_e(x_{1:s}, u_s) \tilde{h}_e'(x_{1:s}, u_s) \prod_{d \neq e} \int \frac{\pi_t(x_{1:s,d}, x_{s+1,d})}{\pi_{s+1}(x_{1:s,d}, x_{s+1,d})} \left( \sum_{j=1}^{M} \sum_{d \neq e} \int \frac{\pi_t(x_{1:s,d}, x_{s+1,d})}{\pi_{s+1}(x_{1:s,d}, x_{s+1,d})} \right)^2 \right] du_sdx_{1:s},
\]

(25)

for \(\tilde{h}_e\) defined by

\[
\tilde{h}_e(x_{1:s}, u_s) = \sum_{j=1}^{M} \int \frac{x_{t,e} \pi_t(x_{1:s,e}, x_{s+1,e}, x_{t,e})}{\pi_{s+1}(x_{1:s,e}, x_{s+1,e})} dx_{t,e} \prod_{d \neq e} \int \frac{\pi_t(x_{1:s,d}, x_{s+1,d})}{\pi_{s+1}(x_{1:s,d}, x_{s+1,d})} .
\]

Now, note that

\[
\tilde{h}_e(x_{1:s}, u_s)^2 = \sum_{i_{1:n_x}, j_{1:n_x}} \left[ \prod_{d=1}^{n_x} \int \frac{x_{t,e} \pi_t(x_{1:s,e}, x_{s+1,e}, x_{t,e})}{\pi_{s+1}(x_{1:s,e}, x_{s+1,e})} dx_{t,e} \prod_{d \neq e} \int \frac{\pi_t(x_{1:s,d}, x_{s+1,d})}{\pi_{s+1}(x_{1:s,d}, x_{s+1,d})} \right],
\]

\[
\tilde{h}_e(x_{1:s}, u_s)\tilde{h}_e'(x_{1:s}, u_s) = \sum_{i_{1:n_x}, j_{1:n_x}} \left[ \prod_{d=1}^{n_x} \int \frac{x_{t,e} \pi_t(x_{1:s,e}, x_{s+1,e}, x_{t,e})}{\pi_{s+1}(x_{1:s,e}, x_{s+1,e})} dx_{t,e} \prod_{d \neq e} \int \frac{\pi_t(x_{1:s,d}, x_{s+1,d})}{\pi_{s+1}(x_{1:s,d}, x_{s+1,d})} \right],
\]

with \(e \neq e'\) and all \(i_d, j_d \in \{1, \ldots, M\}\).
We will in the sequel also make use of the following observation

$$\frac{w^i_d}{\pi_{s+1}(x_{1:s,d}, x_{s+1,d}^i)} = \frac{p(y_{s+1,d}|y_{1:s,d})}{r(x_{s+1,d}^i|x_{s,d})\pi_s(x_{1:s,d})}. \quad (26)$$

Now, we consider the case in (25) when \( e = e' \):

$$\frac{1}{p(y_{s+1}|y_{1:s})^2M^{2n_x}} \int \hat{h}_e(x_{1:s}, u_s)^{2n_x} \prod_{d=1}^{n_x} \pi_s(x_{1:s,d}) \prod_{j=1}^M r(x_{s+1,d}^j|x_{s,d}) \pi_t(x_{1:s,d}, x_{s+1,d}^j) \, du_s \, dx_{1:s}$$

$$= \frac{1}{M^{2n_x}} \sum_{i_1,n_x,j_1,n_x} \left[ \prod_{d \neq e} \int \prod_{j=1}^M r(x_{s+1,d}^j|x_{s,d}) \pi_t(x_{1:s,d}, x_{s+1,d}^j) \pi_t(x_{1:s,d}, x_{s+1,d}^j) \, du_s \, dx_{1:s,d} \right]$$

$$\cdot \int \left[ \prod_{j=1}^M r(x_{s+1,e}^j|x_{s,e}) \pi_t(x_{1:s,e})^2 \right]$$

$$\cdot \int x_{t,e} \pi_t(x_{s+1,e}, x_{t,e}|x_{s,e}) \, dx_{t,e} \int x_{t,e} \pi_t(x_{s+1,e}, x_{t,e}|x_{s,e}) \, dx_{t,e} \right] \, du_s \, dx_{1:s,e} \right]$$

$$= B_s^{n_x-1} \left( A_s + M^{-1} \left( \tilde{A}_s - A_s \right) \right) \left( 1 - \frac{1}{M} \right)^{n_x-1} \left( 1 + \frac{\tilde{B}_s}{B_s(M - 1)} \right)^{n_x-1},$$

where in the first equality we have used (26) and independency over dimensions. The second equality follows by straightforward (but tedious) calculations using combinatorial identities and noting that by definition of the model the constants do not depend on the dimension \( d \).
Let us now consider the case in (25) when \( e \neq e' \):

\[
\frac{1}{p(y_{s+1}|y_1:s)^2 M^2 n_x} \int \tilde{h}_e(x_{1:s}, u_s) \tilde{h}_{e'}(x_{1:s}, u_s) \prod_{d=1}^{n_x} \left[ \pi_s(x_{1:s,d}) \prod_{j=1}^{M} r(x_{s+1,d}^j| x_{s,d}) \right] du_s dx_{1:s} \\
= \frac{1}{M^2 n_x} \sum_{i_1, n_x, j_1, n_x} \left[ \prod_{d \neq e, e'} \int \prod_{j=1}^{M} r(x_{s+1,d}^j | x_{s,d}) \pi_t(x_{1:s,d}) \prod_{j=1}^{M} r(x_{s+1,d}^j | x_{s,d}) \pi_t(x_{1:s,d}) du_s dx_{1:s,d} \right. \\
\left. \cdot \int \prod_{d \neq e, e'} \prod_{j=1}^{M} r(x_{s+1,d}^j | x_{s,d}) \pi_t(x_{1:s,d}) dx_{1:s,e} \right] \\
= \left( C_s + M^{-1} (\tilde{C}_s - C_s) \right)^2 \left( 1 - \frac{1}{M} \right)^{n_x-2} \left( 1 + \frac{\tilde{B}_s}{B_s(M-1)} \right)^{n_x-2},
\]

where again we have made use of independency over dimensions \( d \) and (26). The last equality follows again by straightforward manipulations and we can see that product \( \prod_{d \neq e, e'} \) is more or less equal to the one above, hence we obtain \( B_{s}^{n_x-2} \) instead of \( B_{s}^{n_x-1} \).

Putting all this together we get that

\[
\Sigma_t^M (\varphi) = n_x A_t + \sum_{s=0}^{t-1} \left[ n_x B_{s}^{n_x-1} \left( A_s + M^{-1} (\tilde{A}_s - A_s) \right) \left( 1 - \frac{1}{M} \right)^{n_x-1} \left( 1 + \frac{\tilde{B}_s}{B_s(M-1)} \right)^{n_x-1} \\
+ n_x (n_x - 1) B_{s}^{n_x-2} \left( C_s + M^{-1} (\tilde{C}_s - C_s) \right)^2 \left( 1 - \frac{1}{M} \right)^{n_x-2} \left( 1 + \frac{\tilde{B}_s}{B_s(M-1)} \right)^{n_x-2} \right],
\]

equality follows by noting that \( \sum_{e, e'} = \sum_e \sum_{e'=e} + \sum_e \sum_{e'=e} \) and that the constants do not depend on \( e/e' \).
A.3 Comparison with Independent Resampling Particle Filter

We compare several variants of NSMC to Independent Resampling Particle Filter (IR-PF) on the same setup studied in [Lamberti et al. 2016, High dimensional problems], for more information on the model and setup we refer to that paper. Figure 7 illustrates the results for \( N = M \in \{10, 100\} \) and as we can see NSMC outperforms IR-PF significantly in root mean square error (RMSE). NSMC-IS and NSMC-PF both approximate the optimal proposal SMC and as such generate conditionally independent samples (see supplementary methods section above for how to use IS as a nested procedure). NSMC-FAPF, clearly the best of all of them, on the other hand, approximates the fully adapted SMC and generates conditionally dependent samples.

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Figure 7: RMSE of the IR-PF and three types of NSMC methods, approximation of optimal proposal SMC using IS (orange) and PF (green), approximation of fully adapted SMC using PF with BS (red).
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