BISET TRANSFORMATIONS OF TAMBARA FUNCTORS

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Abstract. If we are given an \( H\)-\( G\)-biset \( U \) for finite groups \( G \) and \( H \), then any Mackey functor on \( G \) can be transformed by \( U \) into a Mackey functor on \( H \). In this article, we show that the biset transformation is also applicable to Tambara functors, and in fact forms a functor between the category of Tambara functors on \( G \) and \( H \). This biset transformation functor is compatible with some algebraic operations on Tambara functors, such as ideal quotients or fractions. In the latter part, we also construct the left adjoint of the biset transformation.

1. Introduction and Preliminaries

Let \( G \) and \( H \) be arbitrary finite groups. By definition, an \( H\)-\( G\)-biset \( U \) is a set \( U \) with a left \( H\)-action and a right \( G\)-action, which satisfy

\[(hu)g = h(ug)\]

for any \( h \in H, u \in U, g \in G \) \cite{2}. In this article, an \( H\)-\( G\)-biset is always assumed to be finite.

If we are given an \( H\)-\( G\)-biset \( U \), then there is a functor

\[ U \circ - : G\text{set} \rightarrow H\text{set} \]

which preserves finite direct sums and fiber products \cite{2}. In fact, for any \( X \in \text{Ob}(G\text{set}) \), the object \( U \circ X \in \text{Ob}(H\text{set}) \) is given by

\[ U \circ X = \{ (u, x) \in U \times X \mid uG \leq Gx \}/G, \]

where the equivalence relation \((/G)\) is defined by

- \((u, x)\) and \((u', x')\) are equivalent if there exists some \( g \in G \) satisfying \( u' = gu \) and \( x = gx' \).

We denote the equivalence class of \((u, x)\) by \([u, x]\). Then \( U \circ X \) is equipped with an \( H\)-action

\[ h[u, x] = [hu, x] \quad (\forall h \in H, \forall [u, x] \in U \circ X). \]

For any \( f \in G\text{set}(X, Y) \), the morphism \( U \circ f \in H\text{set}(U \circ X, U \circ Y) \) is defined by

\[ U \circ f([u, x]) = [u, f(x)] \quad (\forall [u, x] \in U \circ X). \]

The author wishes to thank Professor Serge Bouc for his comments and advices.
The author wishes to thank Professor Fumihito Oda for his suggestion.
Supported by JSPS Grant-in-Aid for Young Scientists (B) 22740005.
This functor $U \circ -$ enables us to transform a Mackey functor $M$ on $H$ into a Mackey functor $M \circ U = M(U \circ -)$ on $G$. In fact, this construction gives a functor (2)

$$-\circ U : \text{Mack}(H) \to \text{Mack}(G) ; \ M \mapsto M \circ U,$$

which, in this article, we would like to call the biset transformation along $U$. Here, $\text{Mack}(G)$ and $\text{Mack}(H)$ denote the category of Mackey functors on $G$ and $H$, respectively.

In this article, we show that the functor $U \circ G^-$ on $G\text{set} \to H\text{set}$ also preserves exponential diagrams. As a corollary we obtain a biset transformation for Tambara functors

$$-\circ U : \text{Tam}(H) \to \text{Tam}(G) ; \ T \mapsto T \circ U,$$

where $\text{Tam}(G)$ and $\text{Tam}(H)$ are the category of Tambara functors on $G$ and $H$.

This biset transformation is compatible with some algebraic operations on Tambara functors, such as ideal quotients or fractions. If we are given an ideal $\mathcal{I}$ of a Tambara functor $T$ on $H$ (3), then $\mathcal{I}$ is transformed into an ideal $\mathcal{I} \circ U$ of $T \circ U$, and there is a natural isomorphism of Tambara functors

$$(T/\mathcal{I}) \circ U \cong (T \circ U)/ (\mathcal{I} \circ U).$$

Or, if we are given a multiplicative semi-Mackey subfunctor $\mathcal{S}$ of a Tambara functor $T$ on $H$ (4), then $\mathcal{S}$ is transformed into a multiplicative semi-Mackey subfunctor $\mathcal{S} \circ U$ of $T \circ U$, and there is a natural isomorphism of Tambara functors

$$(\mathcal{S}^{-1}T) \circ U \cong (\mathcal{S} \circ U)^{-1}(T \circ U).$$

In the latter part, we construct a left adjoint functor

$$\Sigma_U : \text{Tam}(G) \to \text{Tam}(H)$$

of the biset transformation $-\circ U : \text{Tam}(H) \to \text{Tam}(G)$. As an immediate corollary of the adjoint property, $\Sigma_U$ becomes compatible with the Tambarization functor $\Omega[-]$ (Corollary 3.19).

For any finite group $G$, we denote the category of (resp. semi-)Mackey functors on $G$ by $\text{Mack}(G)$ (resp. $\text{SMack}(G)$). If $G$ acts on a set $X$ from the left (resp. right), we denote the stabilizer of $x \in X$ by $G_x$ (resp. $xG$). The category of finite $G$-sets is denoted by $G\text{set}$.

For any category $\mathcal{C}$, we denote the category of covariant functors from $\mathcal{C}$ to Set by $\text{Fun}(\mathcal{C}, \text{Set})$. If $\mathcal{C}$ admits finite products, let $\text{Add}(\mathcal{C}, \text{Set})$ denote the category of covariant functors $F : \mathcal{C} \to \text{Set}$ preserving finite products.
Definition 1.1. For each \( f \in G \text{set}(X, Y) \) and \( p \in G \text{set}(A, X) \), the canonical exponential diagram generated by \( f \) and \( p \) is the commutative diagram

\[
\begin{array}{c}
X \xrightarrow{p} A \leftarrow \text{exp} \quad X \times \Pi_f(A) \\
\downarrow f \quad \downarrow \pi \\
Y \leftarrow \Pi_f(A)
\end{array}
\]

where

\[
\Pi_f(A) = \left\{ (y, \sigma) \mid y \in Y, \sigma: f^{-1}(y) \to A \text{ is a map of sets, } p \circ \sigma \text{ is equal to the inclusion } f^{-1}(y) \hookrightarrow X \right\},
\]

\[
\pi(y, \sigma) = y, \quad e(x, (y, \sigma)) = \sigma(x),
\]
and \( f' \) is the pull-back of \( f \) by \( \pi \). On \( \Pi_f(A) \), \( G \) acts by

\[
g(y, \sigma) = (gy, g\sigma),
\]
where \( g\sigma \) is the map defined by \( g\sigma(x') = g\sigma(g^{-1}x') \) for any \( x' \in f^{-1}(gy) \). A diagram in \( G \text{set} \) isomorphic to one of the canonical exponential diagrams is called an exponential diagram.

Definition 1.2. (\([8]\)) A semi-Tambara functor \( T \) on \( G \) is a triplet \( T = (T^*, T_+, T_\cdot) \) of two covariant functors \( T^+: G \text{set} \to \text{Set} \) and one additive contravariant functor \( T^*: G \text{set} \to \text{Set} \) which satisfies the following.

1. \( T^\alpha = (T^*, T_+ \cdot T_\cdot) \) and \( T^\mu = (T^*, T_\cdot) \) are objects in \( \text{SMack}(G) \). \( T^\alpha \) is called the additive part of \( T \), and \( T^\mu \) is called the multiplicative part of \( T \).

2. (Distributive law) If we are given an exponential diagram

\[
\begin{array}{c}
X \xrightarrow{p} A \leftarrow \text{exp} \quad X \times \Pi_f(A) \\
\downarrow f \quad \downarrow \pi \\
Y \leftarrow \Pi_f(A)
\end{array}
\]

in \( G \text{set} \), then

\[
\begin{array}{c}
T(X) \xrightarrow{T_+(p)} T(A) \xrightarrow{T^*(\lambda)} T(Z) \\
\downarrow T_+(f) \quad \downarrow T^*(\lambda) \\
T(Y) \xleftarrow{T_+(q)} T(B)
\end{array}
\]

is commutative.

If \( T = (T^*, T_+, T_\cdot) \) is a semi-Tambara functor, then \( T(X) \) becomes a semi-ring for each \( X \in \text{Ob}(G \text{set}) \), whose additive (resp. multiplicative) monoid structure is induced from that on \( T^\alpha(X) \) (resp. \( T^\mu(X) \)). For each \( f \in G \text{set}(X, Y) \), those maps \( T^*(f), T_+(f), T_\cdot(f) \) are often abbreviated to \( f^*, f_+, f_\cdot \).
A morphism of semi-Tambara functors $\varphi: T \to S$ is a family of semi-ring homomorphisms

$$\varphi = \{ \varphi_X: T(X) \to S(X) \}_{X \in \text{Ob}(\text{Gset})},$$

natural with respect to all of the contravariant and the covariant parts. We denote the category of semi-Tambara functors by $\text{STam}(G)$.

If $T(X)$ is a ring for each $X \in \text{Ob}(\text{Gset})$, then a semi-Tambara functor $T$ is called a Tambara functor. The full subcategory of Tambara functors in $\text{STam}(G)$ is denoted by $\text{Tam}(G)$.

Remark 1.3. In [8], it was shown that the inclusion functor $\text{Tam}(G) \hookrightarrow \text{STam}(G)$ has a left adjoint $\gamma_G: \text{STam}(G) \to \text{Tam}(G)$.

Remark 1.4. Taking the multiplicative parts, we obtain functors $(-)\mu: \text{STam}(G) \to \text{SMack}(G)$, $(-)^\mu: \text{Tam}(G) \to \text{SMack}(G)$.

In [5], it was shown that $(-)\mu: \text{STam}(G) \to \text{SMack}(G)$ has a left adjoint $S: \text{SMack}(G) \to \text{STam}(G)$.

Composing with $\gamma_G$, we obtain a functor called Tambarization

$$\Omega_G[-] = \gamma_G \circ S: \text{SMack}(G) \to \text{Tam}(G),$$

which is left adjoint to $(-)^\mu: \text{Tam}(G) \to \text{SMack}(G)$.

2. Biset transformation

In this section, we consider transformation of a Tambara functor along a biset, and show how the functors in the previous section are related. Our first aim is to show the following.

**Proposition 2.1.** Let $G$, $H$ be finite groups, and let $U$ be an $H$-$G$-biset. Then

$$U \circ G - : \text{Gset} \to \text{Hset}$$

preserves exponential diagrams.

First, we remark the following.

**Remark 2.2.** Assume we are given an exponential diagram

$$
\begin{array}{ccc}
X & \xrightarrow{p} & A \\
\downarrow{f} & & \downarrow{\exp} \\
Y & \xleftarrow{\pi} & \text{exp} \Pi_f(A)
\end{array}
$$

in $\text{Gset}$. Since $U \circ G -$ preserves pullbacks, we obtain a pullback diagram

$$
\begin{array}{ccc}
U \circ G X & \xrightarrow{(U \circ G f) \circ (U \circ G \lambda)} & U \circ G Z \\
\downarrow{U \circ G f} & & \downarrow{U \circ G \rho} \\
U \circ G Y & \xleftarrow{U \circ G \pi} & U \circ G \Pi_f(A)
\end{array}
$$
in \( H_{set} \). If we take an exponential diagram associated to

\[
U \circ Y \xleftarrow{U \circ Y} U \circ X \xleftarrow{U \circ Y} U \circ A
\]
as

\[
\begin{array}{c}
\text{U} \\
\text{exp} \\
\text{U} \\
\end{array}
\begin{array}{c}
\text{U} \\
\text{P} \\
\text{U} \\
\end{array}
\begin{array}{c}
\leq \\
\leftarrow \\
\leq \\
\end{array}
\begin{array}{c}
\text{U} \\
\text{A} \\
\text{U} \\
\end{array}
\begin{array}{c}
\text{Z'} \\
\Pi_{U \circ Y} (U \circ A), \\
\end{array}
\]
then by the adjointness between

\(- \times_{U \circ Y} (U \circ X): H_{set}/U \circ Y \to H_{set}/U \circ X\)

and

\[\Pi_{U \circ Y}: H_{set}/U \circ Y \to H_{set}/U \circ X,\]
we obtain a natural bijection

\[H_{set}/U \circ Y (U \circ \Pi_f(A), \Pi_{U \circ Y} (U \circ A)) \cong H_{set}/U \circ X ((U \circ \Pi f(A)) \times_{U \circ Y} (U \circ X), U \circ A) \cong H_{set}/U \circ X (U \circ Z, U \circ A).\]

Thus there should exist a morphism

\[U \circ \Pi_f(A) \to \Pi_{U \circ Y} (U \circ A)\]
corresponding to \( U \circ \lambda: U \circ Z \to U \circ A. \)

With this view, we construct an \( H \)-map

\[\Phi: U \circ \Pi_f(A) \to \Pi_{U \circ Y} (U \circ A)\]
explicitly, and show it is bijective.

By definition, we have

\[U \circ \Pi_f(A) = \left\{ [u, (y, \sigma)] \mid u \in U, (y, \sigma) \in \Pi_f(A), u \circ G \leq G_{(y, \sigma)} \right\},\]

\[\Pi_{U \circ Y} (U \circ A) = \left\{ ([u, y], \tau) \mid [u, y] \in U \circ Y, \tau : (U \circ f)^{-1}([u, y]) \to U \circ A \text{ is a map, satisfying } (U \circ p) \circ \tau = \text{incl.} \right\}.\]

Remark 2.3. For any \([u, y] \in U \circ Y\), the following holds.

1. An element \([u_0, x_0] \in U \circ X\) belongs to \((U \circ f)^{-1}([u, y])\) if and only if there exists \(g_0 \in G\) satisfying

\[(2.1) \quad u = u_0 g_0 \quad \text{and} \quad g_0 y = f(x_0).\]

In particular, \(g_0^{-1} \cdot x_0 \in f^{-1}(y)\).
(2) Let \([u_0, x_0]\) be an element in \((U \circ f)^{-1}([u, y])\). If \(g_0\) satisfies (2.1) and \(g'_0\) similarly satisfies
\[
u = u_0g_0' \quad \text{and} \quad g'_0y = f(x_0),
\]
then we have \(g^{-1}_0 \cdot x_0 = g^{-1}_0 \cdot x_0\).

**Proof.** (1) We have
\[
[u_0, x_0] \in (U \circ f)^{-1}([u, y]) \quad \iff \quad [u_0, f(x_0)] = [u, y]
\]
\[
\exists g_0 \in G \text{ such that } u = u_0g_0, \quad g_0y = f(x_0).
\]

(2) Since \(u_0g_0 = u_0g'_0\) implies \(g'_0g_0^{-1} \in u_0G \leq G_{x_0}\), it follows \(g'_0g_0^{-1} \cdot x_0 = x_0\). \(\square\)

**Lemma 2.4.** For any \([u, (y, \sigma)] \in U \circ \Pi_f(A)\), define \(\Phi([u, (y, \sigma)])\) by
\[
\Phi([u, (y, \sigma)]) = ([u, y], \tau_{\sigma, u})
\]
where \(\tau_{\sigma, u} : (U \circ f)^{-1}([u, y]) \to U \circ A\) is a map defined by
\[
\tau_{\sigma, u}([u_0, x_0]) = [u, \sigma(g_0^{-1}x_0)] \quad (\forall [u_0, x_0] \in (U \circ f)^{-1}([u, y])),
\]
where \(g_0 \in G\) is an element satisfying (2.1). (It can be easily confirmed that \([u, \sigma(g_0^{-1}x_0)]\) belongs to \(U \circ A\), by using (2.1))

Then \(\Phi : U \circ \Pi_f(A) \to \Pi_{U \circ f}(U \circ A)\) becomes a well-defined \(H\)-map.

**Proof.** By Remark 2.3, this \(\sigma(g_0^{-1}x_0)\) is independent of the choice of \(g_0\). It suffices to show the following.

(1) \(\tau_{\sigma, u}\) is well-defined for each \([u, (y, \sigma)] \in U \circ \Pi_f(A)\).

(2) \(\Phi\) is well-defined.

(3) \(\Phi\) is an \(H\)-map.

(1) Suppose \([u'_0, x'_0] = [u_0, x_0]\) and take \(g_0, g'_0 \in G\) satisfying
\[
u = u_0g_0, \quad g_0y = f(x_0),
\]
\[
u = u'_0g'_0, \quad g'_0y = f(x'_0).
\]
Since \([u'_0, x'_0] = [u_0, x_0]\), there exists some \(g \in G\) satisfying
\[
u' = u_0g, \quad x'_0 = g^{-1}x_0.
\]
Then we obtain
\[
[u, \sigma(g'_0^{-1}x'_0)] = [u, \sigma(g_0^{-1}g^{-1}x_0)].
\]
Since \(u_0g_0 = u = u'_0g'_0 = u_0gg_0\), we have
\[
g_0g_0^{-1}g^{-1} \in u_0G \leq G_{x_0},
\]
which means \(g'_0g^{-1}x_0 = g^{-1}x_0\), and thus
\[
[u, \sigma(g'_0^{-1}x'_0)] = [u, \sigma(g_0^{-1}x_0)].
\]

(2) Suppose \([u, (y, \sigma)] = [u', (y', \sigma')]\). There exists \(g \in G\) satisfying
\[
u' = u_0g \quad \text{and} \quad (y', \sigma') = g^{-1} \cdot (y, \sigma),
\]
Thus it suffices to show
\[ \tau_{\sigma, u}(u, x) = [u, \sigma(g^{-1}x)] = [h\sigma(g^{-1}x)] \]
for any \( u, x \in U \) and \( \sigma \) satisfying (2.2).

We have the following.

**Remark 2.5.**

(1) When \( [u, x] \) runs through the elements in \( G \), then
\[ h^{-1}[u, x] = [h^{-1}u, x] \]
runs through the elements in \( G \).

(2) If \( g \in G \) satisfies (2.2), then we have
\[ u = h^{-1}u, g = f(x). \]

Thus by the definition of \( \tau_{\sigma, u} \), we have
\[ \tau_{\sigma, u}([h^{-1}u, x]) = [u, \sigma(g^{-1}x)]. \]

By (2.2) and Remark 2.5, we obtain
\[ h_{\tau_{\sigma, u}}[u, x] = h_{\tau_{\sigma, u}}([h^{-1}u, x]) \]
\[ = h[u, \sigma(g^{-1}x)] = \tau_{\sigma, h}[u, x]. \]

for any \( [u, x] \in G \). Namely, \( \tau_{\sigma, h} = h_{\tau_{\sigma, u}}. \)
Proof of Proposition 2.1: By Lemma 2.4, we obtain a well-defined $H$-map

$$\Phi: U \circ \Pi_f(A) \to \Pi_{U^\circ}f(U \circ A).$$

It suffices to construct the inverse $\Psi$ of $\Phi$. For any $([u, y], \tau) \in \Pi_{U^\circ}f(U \circ A)$, define $\Psi([u, y], \tau)$ by

$$\Psi([u, y], \tau) = [u, (y, \sigma_{\tau, u})],$$

where $\sigma_{\tau, u}: f^{-1}(y) \to A$ is a map satisfying

$$(2.4) \quad [u, \sigma_{\tau, u}(x^\dagger)] = \tau([u, x^\dagger]) \quad (\forall x^\dagger \in f^{-1}(y)).$$

Here, we have the following.

Remark 2.6: If $[u, a], [u', a'] \in U \circ A$ satisfies

$$[u, a] = [u', a'] \quad \text{and} \quad u = u',$$

then we have $a = a'$.

Thus $\sigma_{\tau, u}(x^\dagger)$ is well-defined by (2.4) for each $x^\dagger$. To show Proposition 2.1 it suffices to show the following.

1. $\Psi: \Pi_{U^\circ}f(U \circ A) \to U \circ \Pi_f(A)$ is a well-defined map.
2. $\Psi \circ \Phi = \text{id}$.
3. $\Phi \circ \Psi = \text{id}$.

(1) Suppose $([u, y], \tau) = ([u', y'], \tau')$. Then obviously we have $\tau' = \tau$. There exists some $g \in G$ satisfying

$$u = u' g, \quad g y = y'.$$

In particular we have $f^{-1}(y') = g \cdot f^{-1}(y)$. By definition of $\sigma_{\tau, u}$ and $\sigma_{\tau, u'}$, we have

$$[u, \sigma_{\tau, u}(x^\dagger)] = \tau([u, x^\dagger]),$$

$$[u', \sigma_{\tau, u'}(gx^\dagger)] = \tau([u', gx^\dagger])$$

for any $x^\dagger \in f^{-1}(y)$.

Thus it follows

$$[u, \sigma_{\tau, u}(x^\dagger)] = \tau([u, x^\dagger]) = \tau([u' g, x^\dagger])$$

$$= \tau([u', g x^\dagger]) = [u', \sigma_{\tau, u'}(gx^\dagger)]$$

$$= [ug^{-1}, \sigma_{\tau, u'}(gx^\dagger)] = [u, g^{-1}\sigma_{\tau, u'}(x^\dagger)].$$

By Remark 2.6 this means $\sigma_{\tau, u} = g^{-1}\sigma_{\tau, u'}$. Thus it follows

$$[u, (y, \sigma_{\tau, u})] = [u' g, (g^{-1} y', g^{-1}\sigma_{\tau, u'})]$$

$$= [u' g, g^{-1} y', \sigma_{\tau, u'}] = [u', (y', \sigma_{\tau, u'})],$$

and thus $\Psi$ is well-defined.

(2) Let $[u, (y, \sigma)] \in U \circ \Pi_f(A)$ be any element. We have

$$\Psi \circ \Phi([u, (y, \sigma)]) = \Psi([u, y], \tau_{\sigma, u}) = [u, (y, \sigma_{\tau, u, u})],$$

where $\tau_{\sigma, u}$ and $\sigma_{\tau, u, u'}$ are defined by

$$\tau_{\sigma, u}([u_0, x_0]) = [u, \sigma(g_0^{-1} x_0)] \quad (\forall [u_0, x_0] \in (U \circ f)^{-1}([u, y])),$$

$$[u, \sigma_{\tau, u, u'}(x^\dagger)] = \tau_{\sigma, u}([u, x^\dagger]) \quad (\forall x^\dagger \in f^{-1}(y)).$$
using \( g_0 \in G \) satisfying \( u = u_0 g_0 \) and \( g_0 y = f(x_0) \). In particular we have
\[
\tau_{\sigma,u}(\langle u, x^\dagger \rangle) = \langle u, \sigma(x^\dagger) \rangle \quad (\forall x^\dagger \in f^{-1}(y)),
\]
and thus
\[
\langle u, \sigma_{\tau,u}(x^\dagger) \rangle = \tau_{\sigma,u}(\langle u, x^\dagger \rangle) = \langle u, \sigma(x^\dagger) \rangle
\]
for any \( x^\dagger \in f^{-1}(y) \). By Remark 2.6 it follows \( \sigma_{\tau,u} = \sigma \), and thus \( \Psi \circ \Phi([u, (y, \sigma)]) = [u, (y, \sigma)] \).

(3) Let \( ([u, y], \tau) \in \Pi_{U \circ f}(U \circ A) \) be any element. We have
\[
\Phi \circ \Psi([u, y], \tau) = \Phi([u, (y, \sigma_{\tau,u})]) = ([u, y], \tau_{\sigma,u}(u)),
\]
where \( \sigma_{\tau,u} \) and \( \tau_{\sigma,u}(u) \) are defined by
\[
\tau_{\sigma,u}(\langle u, x^\dagger \rangle) = \tau(\langle u, x^\dagger \rangle) \quad (\forall x^\dagger \in f^{-1}(y)),
\]
\[
\tau_{\sigma,u}(\langle u_0, x_0 \rangle) = \langle u, \sigma_{\tau,u}(g_0^{-1} x_0) \rangle \quad (\forall \langle u_0, x_0 \rangle \in (U \circ f)^{-1}([u, y])),
\]
using \( g_0 \in G \) satisfying \( u = u_0 g_0 \) and \( g_0 y = f(x_0) \). It follows
\[
\tau_{\sigma,u}(\langle u_0, x_0 \rangle) = \tau(\langle u, g_0^{-1} x_0 \rangle) = \tau(\langle u_0, x_0 \rangle)
\]
for any \( \langle u_0, x_0 \rangle \in (U \circ f)^{-1}([u, y]) \), and thus \( \Phi \circ \Psi([u, y], \tau) = ([u, y], \tau) \). \( \square \)

Proposition 2.1 allows us to transform Tambara functors along a biset.

**Corollary 2.7.** Let \( U \) be an \( H \)-\( G \)-biset. For any \( T \in \text{Ob}(\text{Tam}(H)) \), if we define \( T \circ U \) by
\[
\begin{align*}
T \circ U(X) &= T(U \circ X) \quad (\forall X \in \text{Ob}(G \text{set})), \\
(T \circ U)^{*}(f) &= T^{*}(U \circ f) \\
(T \circ U)(f) &= T_{+}(U \circ f) \quad (\forall f \in G \text{set}(X, Y)), \\
(T \circ U)_{*}(f) &= T_{*}(U \circ f)
\end{align*}
\]
then \( T \circ U \) becomes an object in \( \text{Tam}(G) \).

If \( \varphi : T \to S \) is a morphism in \( \text{Tam}(H) \), then
\[
\varphi \circ U = \{ \varphi U_X \}_{X \in \text{Ob}(G \text{set})}
\]
forms a morphism \( \varphi \circ U : T \circ U \to S \circ U \) in \( \text{Tam}(G) \).

This correspondence gives a functor \( - \circ U : \text{Tam}(H) \to \text{Tam}(G) \). In the same way, we obtain a functor \( - \circ U : \text{STam}(H) \to \text{STam}(G) \).

**Remark 2.8.** Since \( U \circ - : G \text{set} \to H \text{set} \) preserves finite direct sums and pullbacks, this induces a functor
\[
- \circ U : \text{SMack}(H) \to \text{SMack}(G),
\]
defined in the same way. (For the case of Mackey functors, see [3].)
Clearly by the construction, these functors are compatible. Namely, we have the following commutative diagrams of functors.

\[
\begin{array}{c}
\text{Tam}(H) \xrightarrow{- \circ U} \text{Tam}(G) \\
\downarrow \quad \quad \quad \downarrow \\
\text{STam}(H) \xrightarrow{- \circ U} \text{STam}(G) \\
\downarrow \quad \quad \quad \downarrow \\
\text{SMack}(H) \xrightarrow{- \circ U} \text{SMack}(G)
\end{array}
\]

Corollary 2.9. In \cite{[6]}, an ideal \( \mathcal{I} \) of a Tambara functor \( T \) on \( H \) is defined to be a family of ideals \( \{ \mathcal{I}(X) \subseteq T(X) \}_{X \in \text{Ob}(H \setminus \text{set})} \), which satisfies the following for any \( f \in H \text{-set}(X, Y) \).

(i) \( f^*(\mathcal{I}(Y)) \subseteq \mathcal{I}(X) \),
(ii) \( f_*(\mathcal{I}(X)) \subseteq \mathcal{I}(Y) \),
(iii) \( f_*(\mathcal{I}(X)) \subseteq f_*(0) + \mathcal{I}(Y) \).

If \( \mathcal{I} \subseteq T \) is an ideal, then the objectwise ideal quotient \( T/\mathcal{I} = \{ T(X)/\mathcal{I}(X) \}_{X \in \text{Ob}(H \setminus \text{set})} \) carries a natural Tambara functor structure on \( H \) induced from that on \( T \).

Concerning Corollary 2.7, suppose we are given an \( H \)-\( G \)-biset \( U \). If we define \( \mathcal{I} \circ U \) by
\[
\mathcal{I} \circ U(X) = \mathcal{I}(U \circ G_X)
\]
for each \( X \in \text{Ob}(G \text{-set}) \), then \( \mathcal{I} \circ U \subseteq T \circ U \) becomes again an ideal, and we obtain a natural isomorphism of Tambara functors on \( G \)
\[
(T/\mathcal{I}) \circ U \cong (T \circ U)/(\mathcal{I} \circ U).
\]

Corollary 2.10. Let \( T \) be a Tambara functor on \( H \). In \cite{[7]}, it was shown that for any semi-Mackey subfunctor \( \mathcal{F} \subseteq T^\mu \), the objectwise fraction of rings
\[
\mathcal{F}^{-1}T = \{ \mathcal{F}(X)^{-1}T(X) \}_{X \in \text{Ob}(H \setminus \text{set})}
\]
carries a natural Tambara functor structure on \( H \) induced from that on \( T \).

Concerning Corollary 2.7, suppose we are given an \( H \)-\( G \)-biset \( U \). Then \( \mathcal{F} \circ U \subseteq (T \circ U)^\mu = T^\mu \circ U \) becomes again a semi-Mackey subfunctor, and we obtain a natural isomorphism of Tambara functors on \( G \)
\[
(\mathcal{F}^{-1}T) \circ U \cong (\mathcal{F} \circ U)^{-1}(T \circ U).
\]

3. Adjoint construction

In the rest, we construct a left adjoint of \( - \circ U : \text{Tam}(H) \to \text{Tam}(G) \). We use the following theorem shown in \cite{[8]}.

**Fact 3.1.** Let \( G \) be a finite group. There exists a category \( \mathcal{U}G \) with finite products satisfying the following properties.

1. \( \text{Ob}(\mathcal{U}G) = \text{Ob}(G \text{-set}) \).
2. There is a categorical equivalence \( \mu_G : \text{Add}(\mathcal{U}G, \text{Set}) \cong \text{STam}(G) \).
We recall the structure of $\mathcal{U}$ briefly. Details can be found in [8].

The set of morphisms $\mathcal{U}_G(X,Y)$ is defined as follows, for each $X,Y \in \text{Ob}(\mathcal{U}_G) = \text{Ob}(G \text{-set})$:

$$\mathcal{U}_G(X,Y) = \left\{ (X \xleftarrow{w} A \xrightarrow{v} B \xrightarrow{u} Y) \mid A, B \in \text{Ob}(G \text{-set}), u \in G \text{-set}(B,Y), v \in G \text{-set}(A,B), w \in G \text{-set}(A,X) \right\} / \text{equiv.}$$

where $(X \xleftarrow{w} A \xrightarrow{v} B \xrightarrow{u} Y)$ and $(X \xleftarrow{w'} A' \xrightarrow{v'} B' \xrightarrow{u'} Y)$ are equivalent if and only if there exists a pair of isomorphisms $a: A \to A'$ and $b: B \to B'$ such that $u = u' \circ b$, $b \circ v = v' \circ a$, $w = w' \circ a$.

Let $[X \xleftarrow{w} A \xrightarrow{v} B \xrightarrow{u} Y]$ denote the equivalence class of $(X \xleftarrow{w} A \xrightarrow{v} B \xrightarrow{u} Y)$. Composition law in $\mathcal{U}_G$ is defined by $[Y \xleftarrow{C} D \xrightarrow{Z} G] \circ [X \xleftarrow{A} B \xrightarrow{Y} G] = [X \xleftarrow{A''} D \xrightarrow{Z} G]$, with the morphisms appearing in the following diagram:

![Diagram](image)

For any $X,Y \in \text{Ob}(\mathcal{U}_G)$, we use the notation

- $T_u = [X \xleftarrow{id} X \xrightarrow{id} X \xrightarrow{u} Y]$ for any $u \in G \text{-set}(X,Y)$,
- $N_v = [X \xleftarrow{id} X \xrightarrow{id} Y \xrightarrow{v} Y]$ for any $v \in G \text{-set}(X,Y)$,
- $R_w = [X \xleftarrow{w} Y \xrightarrow{id} Y]$ for any $w \in G \text{-set}(Y,X)$.

**Remark 3.2.** For any pair of objects $X,Y \in \text{Ob}(\mathcal{U}_G)$, if we let $X \amalg Y$ be their disjoint union in $G \text{-set}$ and let $i_X \in G \text{-set}(X,X \amalg Y), i_Y \in G \text{-set}(Y,X \amalg Y)$ be the inclusions, then

$$X \xleftarrow{R_X} X \amalg Y \xrightarrow{R_Y} Y$$

gives the product of $X$ and $Y$ in $\mathcal{U}_G$.

**Remark 3.3.** For any $T \in \text{Ob}(\text{Add}(\mathcal{U}_G,\text{Set}))$, the corresponding semi-Tambara functor $T = \mu_G(T) \in \text{Ob}(STam(G))$ is given by

- $T(X) = T(X)$ for any $X \in \text{Ob}(G \text{-set})$.
- $T^+(f) = T(R_f)$, $T_0(f) = T(N_f)$, $T_+(f) = T(T_f)$, for any morphism $f$ in $G \text{-set}$.

As a corollary of Proposition 2.1, the following holds.
Corollary 3.4. Let $U$ be an $H$-$G$-set. Then $U \circ - : G\text{set} \to H\text{set}$ induces a functor $F_U : \mathcal{U}_G \to \mathcal{U}_H$ preserving finite products, given by

$$F_U(X) = U \circ X$$

for any $X \in \text{Ob}(G\text{set})$ and

$$F_U([X \xleftarrow{w} A \xrightarrow{v} B \xrightarrow{u} Y]) = [U \circ X \xleftarrow{U \circ w} U \circ A \xrightarrow{U \circ v} U \circ B \xrightarrow{U \circ u} U \circ Y]$$

for any morphism $[X \xleftarrow{w} A \xrightarrow{v} B \xrightarrow{u} Y] \in \mathcal{U}_G(X,Y)$.

Proof. Since $U \circ - : G\text{set} \to H\text{set}$ preserves finite coproducts, pullbacks and exponential diagrams, it immediately follows that $F_U$ preserves the compositions, and thus in fact becomes a functor. Moreover by Remark 3.2, $F_U$ preserves finite products. □

Remark 3.5. The biset transformation obtained in Corollary 2.7 is compatible with the composition by $F_U : \text{Add}([\mathcal{U}_H,\text{Set}]) \to \text{Add}([\mathcal{U}_G,\text{Set}])$.

Theorem 3.6. Let $U$ be an $H$-$G$-biset. Then the functor $- \circ U : \text{STam}(H) \to \text{STam}(G)$ admits a left adjoint, which we denote by $\Sigma_U$.

More generally, we have the following.

Proposition 3.7. Let $G,H$ be arbitrary finite groups, and let $F : \mathcal{U}_G \to \mathcal{U}_H$ be a functor preserving finite products. Then the induced functor

$$- \circ F : \text{Add}([\mathcal{U}_H,\text{Set}]) \to \text{Add}([\mathcal{U}_G,\text{Set}])$$

admits a left adjoint $L_F$.

In the rest, we show Proposition 3.7. The proof basically depends on the proof of Theorem 3.7.7 in [1].

Definition 3.8. Let $G,H,F$ be as in Proposition 3.7. For any $X \in \text{Ob}(\mathcal{U}_H)$, define a category $\mathcal{C}_X$ and a functor $A_X : \mathcal{C}_X \to \mathcal{U}_G$ as follows.

- An object $\mathfrak{t} = (k,(K_1,K_2,\ldots,K_k),\kappa)$ in $\mathcal{C}_X$ is a triplet of $k \in \mathbb{N}_{\geq 0}$, $K_i \leq G$ ($1 \leq i \leq k$), $\kappa \in \mathcal{U}_H(F(A_\mathfrak{t}),X)$, where $A_\mathfrak{t} = \coprod_{1 \leq i \leq k} G/K_i \in \text{Ob}(\mathcal{U}_G)$.

- A morphism in $\mathcal{C}_X$ from $\mathfrak{t}$ to $\mathfrak{t}' = (k',(K'_1,K'_2,\ldots,K'_k),\kappa')$ is a morphism $a \in \mathcal{U}_G(A_\mathfrak{t},A_{\mathfrak{t}'})$ satisfying $\kappa = \kappa' \circ F(a)$. 

$$
\begin{array}{ccc}
F(A_\mathfrak{t}) & \xrightarrow{F(a)} & F(A_{\mathfrak{t}'}) \\
\kappa & \circ & \kappa' \\
X & \searrow & \downarrow \\
\end{array}
$$
- For any \( k \in \text{Ob}(C) \), define \( A_X(k) \in \text{Ob}(U_G) \) by \( A_X(k) = A_k \).
- For any morphism \( a \in C_X(k, k') \), define \( A_X(a) \in U_G(A_X(k), A_X(k')) \) by \( A_X(a) = a : A_k \to A_{k'} \).

**Definition 3.9.** Let \( G, H, F \) be as in Proposition 3.7, and let \( T \) be any object in \( \text{Fun}(U_G, \text{Set}) \). Using the functor \( A_X : C \to U_G \) in Definition 3.8, we define \((L_F T)(X) \in \text{Ob}(\text{Set})\) by

\[
(L_F T)(X) = \text{colim} (T \circ A_X)
\]

for each \( X \in \text{Ob}(\text{H}) \).

For any morphism \( \nu \in \text{H}(X, Y) \), composition by \( \nu \) induces a functor

\[
v_\nu : C_X \to C_Y,
\]

\[
(k, (K_1, K_2, \ldots, K_k), \kappa) \mapsto (k, (K_1, K_2, \ldots, K_k), \nu \circ \kappa)
\]

compatibly with \( A_X \) and \( A_Y \).

This yields a natural map

\[
(L_F T)(\nu) : \text{colim} (T \circ A_X) \to \text{colim} (T \circ A_Y),
\]

and \( L_F T \) becomes a functor \( L_F : U_G \to \text{Set} \).

Moreover, if \( \varphi : T \to S \) is a morphism between \( T, S \in \text{Fun}(U_G, \text{Set}) \), this induces a natural transformation

\[
\varphi \circ A_X : T \circ A_X \Longrightarrow S \circ A_X
\]

and thus a map of sets

\[
(L_F T)(X) \to (L_F S)(X)
\]

for each \( X \). These form a natural transformation from \( L_F T \) to \( L_F S \), which we denote by \( L_F \varphi : L_F T \Longrightarrow L_F S \).

This gives a functor \( L_F : \text{Fun}(U_G, \text{Set}) \to \text{Fun}(U_H, \text{Set}) \).

Similarly as in Theorem 3.7.7 in [1], we have the following.

**Remark 3.10.** \( L_F \) is left adjoint to \(- \circ F : \text{Fun}(U_H, \text{Set}) \to \text{Fun}(U_G, \text{Set})\).

**Proof.** For any \( X \in \text{Ob}(\text{H}) \), we abbreviate \( T \circ A_X \) to \( T_X \). We denote the colimiting cone for \( T_X \) by

\[
\delta_X : T_X \Longrightarrow \Delta_{L_F T(X)},
\]

where \( \Delta_{L_F T(X)} : C_X \to \text{Set} \) is the constant functor valued in \( L_F T(X) \) ([1]).

We briefly state the construction of the bijection

\[
\text{Fun}(U_G, \text{Set})(T, S \circ F) \xrightarrow{\cong} \text{Fun}(U_H, \text{Set})(L_F T, S)
\]

\[
\theta \xleftarrow{\sim} (\forall T \in \text{Ob}(\text{Fun}(U_G, \text{Set})), \forall S \in \text{Ob}(\text{Fun}(U_H, \text{Set}))).\]
Suppose we are given \( \omega \in \text{Fun}(\mathscr{C}_H, \text{Set})(L_F T, S) \). For any \( A \in \text{Ob}(\mathscr{C}_G) \), take a decomposition into transitive \( G \)-sets
\[
(3.1) \quad \rho: \prod_{1 \leq i \leq k} G/K_i \xrightarrow{\cong} A,
\]
and put \( \mathfrak{t} = (k, (K_1, \ldots, K_k), F(\rho)) \). Then \( \mathfrak{t} \) is a terminal object in \( \mathscr{C}_{F(A)} \), and the composition
\[
\theta_{\omega, A} = (T(A) \xrightarrow{T(\rho^{-1})} T(A_\mathfrak{t}) = T_{F(A)}(\mathfrak{t}) \xrightarrow{\delta_{F(A), \mathfrak{t}}} L_F T(F(A)) \xrightarrow{\omega_{F(A)}} S \circ F(A))
\]
does not depend on the choice of the decomposition (3.1) for each \( A \). These form a natural transformation \( \theta_\omega: T \to S \circ F \).

Conversely, suppose we are given \( \theta \in \text{Fun}(\mathscr{C}_G, \text{Set})(T, S \circ F) \). For any \( X \in \text{Ob}(\mathscr{C}_H) \) and any morphism \( a \in \mathscr{C}_X(\mathfrak{t}, \mathfrak{t}') \) between
\[
\mathfrak{t} = (k, (K_1, K_2, \ldots, K_k), \kappa), \quad \mathfrak{t}' = (k', (K'_1, K'_2, \ldots, K'_{k'}), \kappa'),
\]
we have a commutative diagram in \( \text{Set} \)
\[
\begin{array}{ccc}
T_X(\mathfrak{t}) \xrightarrow{T_X(a)} T(A_\mathfrak{t}) & \xrightarrow{\theta_{A_\mathfrak{t}}} & S \circ F(A_\mathfrak{t}) \\
\downarrow & & \downarrow \\
T_X(\mathfrak{t'}) \xrightarrow{T_X(a)} T(A_{\mathfrak{t}'}) & \xrightarrow{\theta_{A_{\mathfrak{t}'}}} & S \circ F(A_{\mathfrak{t}'})
\end{array}
\]
This gives a cone \( T_X \implies \Delta_{S(X)} \), and thus there induced a map \( \omega_{\theta, X}: L_F T(X) \to S(X) \) for each \( X \in \text{Ob}(\mathscr{C}_H) \). These form a natural transformation \( \omega_\theta: L_F T \to S \). \( \square \)

If we can show that \( L_F T \) belongs to \( \text{Ob}(\text{Add}(\mathscr{C}_H, \text{Set})) \) whenever \( T \) belongs to \( \text{Ob}(\text{Add}(\mathscr{C}_G, \text{Set})) \), then we will obtain a functor
\[
L_F: \text{Add}(\mathscr{C}_G, \text{Set}) \to \text{Add}(\mathscr{C}_H, \text{Set}),
\]
which is left adjoint to \( \circ F: \text{Add}(\mathscr{C}_H, \text{Set}) \to \text{Add}(\mathscr{C}_G, \text{Set}) \). Thus Proposition 3.7 is reduced to the following.

**Claim 3.11.** \( L_F T \) belongs to \( \text{Ob}(\text{Add}(\mathscr{C}_H, \text{Set})) \), for any \( T \in \text{Ob}(\text{Add}(\mathscr{C}_G, \text{Set})) \).

For any pair of objects \( X, Y \in \text{Ob}(\mathscr{C}_H) \), define \( A_X * A_Y \) to be the composition of functors
\[
\mathscr{C}_X \times \mathscr{C}_Y \xrightarrow{A_X \times A_Y} \mathscr{C}_G \times \mathscr{C}_G \xrightarrow{\Pi} \mathscr{C}_G,
\]
\[
(A, B) \mapsto A \amalg B
\]
Since \( T \) is additive, \( T \circ (A_X * A_Y) \) becomes naturally isomorphic to
\[
\mathscr{C}_X \times \mathscr{C}_Y \xrightarrow{T_X \times T_Y} \text{Set} \times \text{Set} \xrightarrow{\Delta} \text{Set}.
\]
We abbreviate \( T \circ (A_X * A_Y) \) to \( T_X * T_Y \), and denote the colimiting cone for \( T_X * T_Y \) by
\[
\delta: T_X * T_Y \to \Delta_Z,
\]
where \( Z = \text{colim}(T_X * T_Y) \).
**Fact 3.12** (Lemma 3.7.6 in [1]). Let $\mathcal{C}_X \times \mathcal{C}_Y \xrightarrow{pr_X} \mathcal{C}_X$ be the projection, and let $\varphi_X : T_X \ast T_Y \Rightarrow T_X \circ pr_X$ be the natural transformation induced from the projection.

There uniquely exists a map of sets

$$\pi_X : Z \rightarrow L_F T(X)$$

which makes the following diagram of natural transformations commutative.

$$\begin{array}{ccc}
T_X \ast T_Y & \xrightarrow{\varphi_X} & T_X \circ pr_X \\
\delta & \circ & \delta_X \circ pr_X \\
\Delta Z & \xrightarrow{\pi_X} & (\Delta L_F T(X)) \circ pr_X
\end{array}$$

Similarly, we have a canonical map $\pi_Y : Z \rightarrow L_F T(Y)$. Then

$$(\pi_X, \pi_Y) : Z \rightarrow L_F T(X) \times L_F T(Y)$$

becomes an isomorphism.

**Definition 3.13.** Let $X, Y \in \text{Ob}(\mathcal{H})$ be any pair of objects. For any

$$s = (s, (S_1, S_2, \ldots, S_s), \sigma) \in \text{Ob}(\mathcal{C}_{X \sqcup Y}),$$

define $s_X \in \text{Ob}(\mathcal{C}_X)$ and $s_Y \in \text{Ob}(\mathcal{C}_Y)$ by

$$s_X = (R_{\iota_X})_t(s) \in \text{Ob}(\mathcal{C}_X),$$

$$s_Y = (R_{\iota_Y})_t(s) \in \text{Ob}(\mathcal{C}_Y),$$

where $\iota_X : X \rightarrow X \sqcup Y, \iota_Y : Y \rightarrow X \sqcup Y$ are the inclusions in $\mathcal{H}_{set}$.

**Definition 3.14.** Let $X, Y \in \text{Ob}(\mathcal{H})$ be arbitrary objects. For any

$$t = (k, (K_1, K_2, \ldots, K_k), \kappa) \in \text{Ob}(\mathcal{C}_X)$$

and

$$l = (\ell, (L_1, L_2, \ldots, L_\ell), \lambda) \in \text{Ob}(\mathcal{C}_Y),$$

define $t \sqcup l \in \text{Ob}(\mathcal{C}_{X \sqcup Y})$ by

$$t \sqcup l = (k + \ell, (K_1, \ldots, K_k, L_1, \ldots, L_\ell), \kappa \sqcup \lambda),$$

where $\kappa \sqcup \lambda$ is the abbreviation of

$$F(A_{\sqcup \sqcup}) \cong F(A_t) \sqcup F(A_l) \xrightarrow{\kappa \sqcup \lambda} X \sqcup Y.$$

**Lemma 3.15.** Let $(t, l) \in \text{Ob}(\mathcal{C}_X \times \mathcal{C}_Y)$ be any object. If we denote the inclusions in $\mathcal{C}_{set}$ by

$$\iota_t : A_t \hookrightarrow A_t \sqcup A_l = A_{\sqcup \sqcup},$$

$$\iota_l : A_l \hookrightarrow A_t \sqcup A_l = A_{\sqcup \sqcup},$$

then we obtain morphisms $R_{\iota_t} \in \mathcal{C}_X((t \sqcup l)_X, t)$ and $R_{\iota_l} \in \mathcal{C}_Y((t \sqcup l)_Y, l)$. 

Proof. Since we have a commutative diagram

\[
\begin{array}{c}
F(A_{\text{III}}) \cong F(A_t) \amalg F(A_t) \xrightarrow{\kappa \lambda} X \amalg Y \\
F(R_{F(A_t)}) \circ \xrightarrow{R_{F(A_t)}} \circ \xrightarrow{R_{X}} \xrightarrow{X}
\end{array}
\]

\((\iota_{F(A_t)}): F(A_t) \hookrightarrow F(A_t) \amalg F(A_t)\) is the inclusion in \(H\text{set}\),

the morphism in \(\mathcal{H}_G\)

\[
R_{\iota_t} : A_{\text{III}}|_X = A_{\text{III}} \to A_t
\]
gives a morphism \(R_{\iota_t} \in \mathcal{C}_X((\iota \amalg \iota)_X, \iota)\). Similarly for \(R_{\iota_t}^{\perp}\).

As a corollary of Lemma 3.15, we obtain commutative diagrams in \(\text{Set}\)

\[
\begin{array}{c}
\xymatrix{ 
\mathcal{T}_X((\iota \amalg \iota)_X) \ar[r]^{\mathcal{T}_X(R_{\iota_t})} & \mathcal{T}_X(\iota) \\
L_F \mathcal{T}(X) \ar[ru]_{\delta_{X,(\iota \amalg \iota)_X}} \ar[ru]_{\delta_{X,\iota}} & & \delta_{X,\iota} \ar[l]^{\delta_{X,(\iota \amalg \iota)_X}}
}\end{array}
\]

\[
(3.3)
\]

Claim 3.16. Let \(\tau : \mathcal{C}_X \times \mathcal{C}_Y \to \mathcal{C}_{\text{III}}\) be the functor defined as follows.

- For any \(\mathfrak{t} = (k, (K_1, K_2, \ldots, K_k), \kappa) \in \text{Ob}(\mathcal{C}_X)\) and \(\mathfrak{l} = (\ell, (L_1, L_2, \ldots, L_\ell), \lambda) \in \text{Ob}(\mathcal{C}_Y)\), define \(\tau(\mathfrak{t}, \mathfrak{l})\) by \(\tau(\mathfrak{t}, \mathfrak{l}) = \mathfrak{t} \amalg \mathfrak{l}\).
- For any \(a \in \mathcal{C}_X(\mathfrak{t}, \mathfrak{t}')\) and \(b \in \mathcal{C}_Y(\mathfrak{l}, \mathfrak{l}')\), define \(\tau(a, b)\) by

\[
\tau(a, b) = a \amalg b : \mathfrak{t} \amalg \mathfrak{l} \to \mathfrak{t}' \amalg \mathfrak{l}',
\]

where \(a \amalg b\) is

\[
A_{\text{III}} = A_t \amalg A_{\iota} \xrightarrow{\alpha \amalg \mu} A_{\iota} \amalg A_{\iota'} = A_{\text{III}}.
\]

Then \(\tau\) is a final functor in the sense of [4]. Namely, the comma category \((s \downarrow \tau)\) is non-empty and connected, for any \(s \in \text{Ob}(\mathcal{C}_{\text{III}})\).

If Claim 3.16 is shown, then Claim 3.11 follows. In fact if \(\tau\) is final, then by [4], the unique map

\[
h \in \text{Set}(Z, L_F \mathcal{T}(X \amalg Y))
\]

which makes the following diagram commutative for any \((\mathfrak{t}, \mathfrak{l}) \in \text{Ob}(\mathcal{C}_X \times \mathcal{C}_Y)\),

becomes an isomorphism.

\[
\begin{array}{c}
\xymatrix{ 
Z \ar[r]^h \ar[d]_{\delta_{(\mathfrak{t}, \mathfrak{l})}} & L_F \mathcal{T}(X \amalg Y) \\
(\mathcal{T}_X \times \mathcal{T}_Y)(\mathfrak{t}, \mathfrak{l}) = \mathcal{T}_X(\mathfrak{t}) \times \mathcal{T}_Y(\mathfrak{l}) = \mathcal{T}_{\text{III}}(\mathfrak{t} \amalg \mathfrak{l}) \ar@{=}[u]
}\end{array}
\]

\[
(3.4)
\]
From [3.3], [3.1] and the definition of $L_F T(R_{tX})$, we obtain a commutative diagram

For any $(t, l) \in \text{Ob}(\mathcal{C}_X \times \mathcal{C}_Y)$. Comparing with (3.2), we see that $\pi_X$ satisfies $\pi_X = L_F T(R_{tX}) \circ h$.

For $Y$, similarly $\pi_Y$ satisfies $L_F T(R_{tY}) \circ h = \pi_Y$. Thus we obtain

Since $h$ and $(\pi_X, \pi_Y)$ are isomorphisms, it follows that

\[(L_F T(R_{tX}), L_F T(R_{tY})): L_F T(X II Y) \to L_F T(X) \times L_F T(Y)\]

is an isomorphism for any $X, Y \in \text{Ob}(\mathcal{C}_H)$. This means $L_F T \in \text{Ob}(\text{Add}(\mathcal{C}_H, \text{Set}))$, and Claim 3.11 follows.

Thus it remains to show Claim 3.16.

Proof of Claim 3.16 Let $s = (s, (S_1, S_2, \ldots, S_k), \sigma) \in \text{Ob}(\mathcal{C}_{XII})$ be any object. Since the folding map $\nabla: A_{\oplus_1} \coprod A_{\oplus_2} = A_{\oplus_3} \coprod A_{\oplus_4} \to A_{\oplus_5}$ makes the diagram

in $\mathcal{C}_H$ commutative, this gives a morphism $R_\nabla: s \to s_X \coprod s_Y$ in $\mathcal{C}_{XII}$. Thus $(s \downarrow \tau)$ is non-empty.

Moreover, let $(t, l) \in \text{Ob}(\mathcal{C}_X \times \mathcal{C}_Y)$ be any object, where

\[t = (k, (K_1, K_2, \ldots, K_k), \kappa), \quad l = (l, (L_1, L_2, \ldots, L_l), \lambda),\]

and let $a \in \mathcal{C}_{XII}(a, t \coprod l)$ be any morphism. Denote the inclusions by

\[\iota_t: A_t \hookrightarrow A_t \coprod A_1, \quad \iota_l: A_l \hookrightarrow A_l \coprod A_1,\]
and put
\[ a_t = R_t \circ a \in \mathcal{U}_G(A_t, A_t), \]
\[ a_t = R_t \circ a \in \mathcal{U}_G(A_t, A_t). \]

Then, by the commutativity of the diagram
\[
\begin{array}{c}
F(A_{\mathcal{S}X}) = F(A_s) \\
\downarrow \\
F(a_t) \\
\end{array}
\begin{array}{c}
\sigma \circ \\
\downarrow \\
X \Join Y \\
\end{array}
\begin{array}{c}
\downarrow \\
F(\mathcal{A}_{\mathcal{S}X}) \\
\end{array}
\begin{array}{c}
\downarrow \\
F(a_t) \\
\end{array}
\begin{array}{c}
\kappa \circ \\
\downarrow \\
F(R_\mathcal{S}) \\
\end{array}
\begin{array}{c}
\downarrow \\
F(\mathcal{A}_t) \\
\end{array}
\begin{array}{c}
\downarrow \\
F(\mathcal{A}_t) \\
\end{array}
\begin{array}{c}
\downarrow \\
F(\mathcal{A}_t) \\
\end{array}
\begin{array}{c}
\downarrow \\
F(\mathcal{A}_t) \\
\end{array}
\begin{array}{c}
\downarrow \\
F(\mathcal{A}_t) \\
\end{array}
\]

in \( \mathcal{U}_H \), we obtain a morphism \( a_t \in \mathcal{C}_X(s, t) \). Similarly we obtain \( a_t \in \mathcal{C}_Y(s, l) \),
and thus a morphism \((a_t, a_t) : (s, s) \to (t, t) \) in \( \mathcal{C}_X \times \mathcal{C}_Y \).

Now there are three morphisms in \( \mathcal{C}_{\mathcal{S}X} \)
\[
\begin{array}{c}
R_\mathcal{S} : s \to s \Join s_Y = \tau(s, s_Y), \\
a : s \to t \Join l = \tau(t, l), \\
a_t \Join a_t = \tau(a_t, a_t) : \tau(s, s_Y) \to \tau(t, l),
\end{array}
\]
and the commutativity of the diagram in \( \mathcal{U}_G \)
\[
\begin{array}{c}
A_s \\
\downarrow \\
\mathcal{A}_{\mathcal{S}X} \Join \mathcal{A}_{s_Y} \\
\downarrow \\
\mathcal{A}_t \Join \mathcal{A}_l \\
\end{array}
\begin{array}{c}
R_\mathcal{S} \\
\downarrow \\
\tau(s, s_Y) \\
\downarrow \\
\tau(t, l) \\
\end{array}
\]
implies the compatibility of these morphisms.

Thus for any \((s \to \tau(t, l)) \in \text{Ob}((s \downarrow \tau))\), there exists a morphism from \((s \to \tau(s, s_Y))\) to \((s \to \tau(t, l))\) in \((s \downarrow \tau)\). In particular, \((s \downarrow \tau)\) is connected. \(\square\)

**Remark 3.17.** A similar argument proves that \(-\circ U : \text{SMack}(H) \to \text{SMack}(G)\) admits a left adjoint \(L_U : \text{SMack}(G) \to \text{SMack}(H)\). (For the case of Mackey functors, see also [3].)

**Corollary 3.18.** Let \(U\) be an \(H\)-\(G\)-biset. Then the functor \(-\circ U : \text{Tam}(H) \to \text{Tam}(G)\) admits a left adjoint.

**Proof.** This immediately follows from Theorem [3.6]. In fact \(\gamma_H \circ L_U\) gives the left adjoint. We also abbreviate this functor to \(\mathfrak{U}_U\). \(\square\)
Corollary 3.19. Let $U$ be an $H$-$G$-biset. Those functors $\mathcal{L}_U$ and $\mathcal{L}_U$ are compatible.

$$\text{Tam}(H) \xrightarrow{\mathcal{L}_U} \text{Tam}(G)$$

$\Omega_H[-] \circlearrowleft \Omega_G[-]$  

$\text{SMack}(H) \xrightarrow{\mathcal{L}_U} \text{SMack}(G)$.

Proof. This follows from the commutativity of (2.5), and the uniqueness of left adjoint functors.

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