Abstract: It has been known for some time that a 3D incompressible Euler flow that has initially a barely smooth velocity field nonetheless has Lagrangian fluid particle trajectories that are analytic in time for at least a finite time (Ph. Serfati C. R. Acad. Sci. Série I 320, 175–180 (1995); A. Shnirelman arXiv:1205.5837 (2012)). Here an elementary derivation is given, based on Cauchy’s form of the Euler equations in Lagrangian coordinates. This form implies simple recurrence relations among the time-Taylor coefficients of the Lagrangian map, used here to derive bounds for the $C^{1,\gamma}$ Hölder norms of the coefficients and infer temporal analyticity of Lagrangian trajectories when the initial velocity is $C^{1,\gamma}$.

1. Introduction

The issue of finite time blow-up of solutions to the 3D Euler equation for incompressible fluid is still an open question. The available results indicate that (a hypothetical) occurrence of singularity is intimately related to the loss of spatial regularity (in the Eulerian coordinates) of the solution. It is known, however, that an initial regularity of the flow, that is quite modest — marginally better than $C^1$, which classical solutions to the Euler equations ought to possess — will preserve this regularity for at least a finite time $t_c$ (see, e.g., [1]). It may be surprising to find that, in such a rough velocity field, the time dependence of the position of any fluid particle, i.e. the Lagrangian time dependence, is actually analytic up to $t_c$. By contrast, in Eulerian coordinates, lack of spatial smoothness also translates into lack of temporal smoothness, because of the sweeping of fine-scale structure by large-scale velocity fields. For instance, the turbulent solutions à la Kolmogorov–Onsager have a spatial regularity of the Hölder type with an exponent close to $1/3$ in both the spatial and the temporal Eulerian domain.
The unexpected possibility that Lagrangian trajectories could be analytic in time was first considered by Serfati [15] (part 2), [16,17] (see also [9]) who suggested to attack this problem as an abstract ordinary differential equation (ODE) in a complex Banach space with an analytic right hand side. Prior to this, Chemin [4] proved that the Lagrangian trajectories are $C^\infty$ smooth. More recently, Shnirelman [18], using the geometric interpretation of the Euler equation, obtained another analytic abstract ODE formulation; he then constructed a time-analytic Lagrangian solution to the ODE applying Picard’s theorem. The fact, that the Lagrangian structure can be nicer than the Eulerian one, was known to Ebin and Marsden [7] who observed (on p. 104) that in Lagrangian coordinates the Euler equations can be written in such a way that “no derivative loss occurs”. An early embodiment of this is actually found already in Cauchy’s little-known Lagrangian formulation of the Euler equation [3].

This formulation will be our starting point. It has the advantage of allowing to set up not only an elementary proof of the analyticity, but also a procedure for calculation of the various temporal Taylor coefficients. In Section 2 we derive simple recursion relations for the time-Taylor coefficients of the Lagrangian displacement. Then, in Section 3, we obtain bounds for the $C^{1,\gamma}$ of these coefficients. The key to obtaining these bound is that the Lagrangian gradient of $n$-th Taylor coefficient can be recursively written in terms of nonlocal operators applied to products of two and three gradients of lower-order such Taylor coefficients. These nonlocal operators are essentially inverse Laplacians composed with two suitable space derivatives and thus all the Taylor coefficients stay within the same Hölder space and simple estimates are obtained for their Hölder norms. Concluding remarks are presented in Section 4.

2. Basic equations ...

Our starting point is not the usual Eulerian formulation of the Euler equations for incompressible fluid, but a little-known Lagrangian formulation due to Cauchy [3], specifically his eq. (15), together with the condition of unit Jacobian of the Lagrangian map that expresses incompressibility. We use $q$ and $x$ for the Lagrangian and Eulerian positions of fluid particles and $\xi \equiv x - q$ for the displacement. In modern notation the dynamical equations of Cauchy [3] take the form

$$\sum_{k=1}^{3} \nabla^k \dot{x}_k \times \nabla^L x_k = \omega_0,$$

$$\det (\nabla^L x) = 1,$$

where $\omega_0 \equiv \nabla^L \times v_0$ is the initial vorticity, $\nabla^k$ denotes the gradient in the Lagrangian variables and $\nabla^L x$ the Jacobian matrix with entries $\nabla^L x_{ij}$. The Cauchy equations can be obtained as the (Lagrangian) curl of the Weber equations [19] (see also [5,2,14]). These equations, obviously, express the time invariance of the l.h.s. and are therefore sometimes referred to as “Cauchy invariants of an ideal fluid”. For simplicity, in what follows we assume spatial $2\pi$-periodicity (flow in the 3D torus $T^3 = [0, 2\pi]^3$), but this can be relaxed.
Following [8], we study the formal Taylor expansion of the particle displacement $\xi(q, t)$ in a power series in time,

$$\xi(q, t) = \sum_{s=1}^{\infty} \xi^{(s)}(q)t^s.$$  \hspace{1cm} (1)

We shall introduce a power series in $t$ with real non-negative coefficients bounding certain norms of $\xi^{(s)}(q)$, and prove, under the appropriate assumptions, that this series converges for a certain real $t = t_c$. We shall conclude that, for complex $t$, series (1) has a finite radius of convergence, not smaller than $t_c$.

In terms of the displacement, Cauchy’s equations become:

$$\nabla^L \times \xi + \sum_{k=1}^3 \nabla^L \xi_k \times \nabla^L \xi_k = \omega_0,$$  \hspace{1cm} (2)

$$\nabla^L \cdot \xi + \sum_{1 \leq i < j \leq 3} \left((\nabla^L_i \xi_i)\nabla^L_j \xi_j - (\nabla^L_i \xi_j)\nabla^L_j \xi_i\right) + \det(\nabla^L \xi) = 0.$$  \hspace{1cm} (3)

Following [8], we now derive recurrence relations for the time-Taylor coefficients of the displacement at $t = 0$. Substituting (1) into (2) and (3), we find

$$s \nabla^L \times \xi^{(s)} = \omega_0 \delta_1^s - \sum_{k=1,2,3} \sum_{0 < n < s} n \nabla^L \xi^{(n)}_k \times \nabla^L \xi^{(s-n)}_k,$$  \hspace{1cm} (4)

$$\nabla^L \cdot \xi^{(s)} = \sum_{1 \leq i < j \leq 3} \sum_{0 < n < s} \left((\nabla^L_j \xi^{(n)}_i)\nabla^L_i \xi^{(s-n)}_j - (\nabla^L_i \xi^{(n)}_j)\nabla^L_j \xi^{(s-n)}_i\right)$$

$$- \sum_{i,j,k} \varepsilon_{ijk} (\nabla^L_i \xi^{(l)}_1) (\nabla^L_j \xi^{(m)}_2) (\nabla^L_k \xi^{(n)}_3),$$  \hspace{1cm} (5)

where $\varepsilon_{ijk}$ is the unit antisymmetric tensor. It is immediately seen from the two equations for $s = 1$ that $\xi^{(1)} = \nu_0$.

To handle the equations (4)–(5) for the Lagrangian curl and divergence of the Taylor coefficients, we apply the Hodge decomposition,

$$\xi^{(s)}(q) = \nabla^L \times a^{(s)} + \nabla^L b^{(s)},$$

where the potentials satisfy the gauge conditions: $\nabla^L \cdot a^{(s)} = 0$ and vanishing of averages over the periodicity box. This yields a pair of Poisson equations for
the vector and scalar potentials \(a^{(s)}\) and \(b^{(s)}\):
\[
 s \nabla^2 a^{(s)} = -\omega_0 \delta_t^s + \sum_{0 < n < s} n \nabla^L \left( (\nabla^L \times a^{(n)})_k + \nabla^L b^{(n)} \right) \\
\times \nabla^L \left( (\nabla^L \times a^{(s-n)})_k + \nabla^L b^{(s-n)} \right),
\]
(6)
\[
 \nabla^2 b^{(s)} = \sum_{1 \leq i < j \leq 3} \sum_{0 < n < s} \left( \nabla^L \left( (\nabla^L \times a^{(n)})_i + \nabla^L b^{(n)} \right) \nabla^L \left( (\nabla^L \times a^{(s-n)})_j + \nabla^L b^{(s-n)} \right) \right) \\
\times \nabla^L \left( (\nabla^L \times a^{(n)})_j + \nabla^L b^{(n)} \right) \nabla^L \left( (\nabla^L \times a^{(s-n)})_i + \nabla^L b^{(s-n)} \right) \\
- \sum_{j_1, j_2, j_3} \varepsilon_{j_1 j_2 j_3} \prod_{l=1}^3 \nabla^L_j \left( (\nabla^L \times a^{(n_l)})_l + \nabla^L b^{(n_l)} \right),
\]
(7)

(here \(\nabla^2 \equiv (\nabla^L)^2\) denotes the Laplacian in the Lagrangian variables and \((\nabla^L \times a^{(n)})_k\) the \(k\)-th component of the vector \(\nabla^L \times a^{(n)}\)).

3. ... and their consequence

In this section we establish the finite-time analyticity in time of the Lagrangian map for solutions to the three-dimensional Euler equation. The assumption is that the initial velocity field \(v_0\) is in \(C^{1,\gamma}\). In other words, the initial vorticity \(\omega_0 \equiv \nabla \times v_0\) is in \(C^{0,\gamma}\) for some \(0 < \gamma < 1\), i.e., satisfies the Hölder condition \(|\omega_0(q + \delta q) - \omega_0(q)| \leq C|\delta q|^{\gamma}\). Demonstration of temporal analyticity amounts to proving the following statement:

**Theorem.** Consider a space-periodic three-dimensional flow of incompressible fluid governed by the Euler equation. Suppose the initial vorticity \(\omega_0(q)\) is in \(C^{0,\gamma}(\mathbb{T}^3)\). Then, at sufficiently small times, the displacement of fluid particles, \(\xi(q, t)\), is given by power series \(1\), whose coefficients can be recursively calculated by solving equations \(4\) and \(5\). The radius of convergence of \(1\), \(\tau\), is bounded from below by a strictly positive quantity, which is inversely proportional to \(|\omega_0|_{0,\gamma}\). (More precisely, \(\tau \geq Q_c|\omega_0|^{-1}_{0,\gamma}/2\), where \(Q_c\) is given by \(12\) below.)

**Proof.** From recurrence relations \(6\) and \(7\) we derive bounds for suitable Hölder norms of the potentials. Using our assumption that \(\omega_0\) is in \(C^{0,\gamma}\), it is easy to establish by induction that each potential \(a^{(s)}\) and \(b^{(s)}\) is \(C^{2,\gamma}\)-regular. For this, we recall that \(C^{m,\gamma}(\mathbb{T}^3)\) is an algebra (i.e. the sum and the product of two functions from \(C^{m,\gamma}(\mathbb{T}^3)\) belong to this space; in particular, \(|fg|_{0,\gamma} \leq |f|_{0,\gamma}|g|_{0,\gamma}\) and apply standard inequalities for Hölder norms of functions whose integrals over the periodicity box vanish:

\[
|\phi|_{2,\gamma} \leq \tilde{\Theta} \nabla^2 \phi|_{0,\gamma} \quad \forall \phi \in C^{2,\gamma}(\mathbb{T}^3), \quad (8)
\]
\[
|\nabla^L_i \nabla^L_j \phi|_{0,\gamma} \leq \Theta |\nabla^2 \phi|_{0,\gamma} \quad \forall i, j \text{ and } \forall \phi \in C^{2,\gamma}(\mathbb{T}^3) \quad (9)
\]
(see, e.g., [11,12,13] or [14]). Here $\hat{\Theta}$ and $\Theta$ are some positive constants depending on $\gamma > 0$ and bounded by $C/\gamma$, $\nabla^2_1 \nabla^2_2$ denotes the second partial derivative $\partial^2 / \partial q_1 \partial q_2$, and $| \cdot |_{\alpha, \gamma}$ designates the Hölder norm in $C^{\alpha, \gamma}(\mathbb{T}^3)$ for a scalar function $\phi$ and $|\phi(q)|_{0, \gamma} = \max k \cdot |\phi_k(q)|_{0, \gamma}$ for a vector-valued function $\phi$.

Now, we consider the generating functions

$$A(t) \equiv \sum_{s=1}^{\infty} \left| \nabla^2 a^{(s)} \right|_{0, \gamma} t^s, \quad B(t) \equiv \sum_{s=1}^{\infty} \left| \nabla^2 b^{(s)} \right|_{0, \gamma} t^s,$$

thus reintroducing time into the problem. Note that, by construction, for $t > 0$, $A(t)$ and $B(t)$ are monotonically increasing functions of time. When no confusion is possible, we shall write just $A$ and $B$ for $A(t)$ and $B(t)$. In order to show that the series $A$ and $B$ converge, we proceed as follows. We derive from relations (6) and (7) two inequalities for the quantities $A$ and $B$. The first of these inequalities involves time derivatives $\dot{A}$ and $\dot{B}$, but it can be integrated in time. The inequality resulting from this integration and the second inequality are of the form of upper bounds for $A$ and $B$. These bounds are polynomials of $2A + B$ that involve, apart from the contribution from the initial vorticity $\omega_0(q)$, only quadratic and cubic terms. Since $a^{(s)}$ and $b^{(s)}$ vanish at $t = 0$, the inequalities yield an upper bound for $2A + B$ at least at small times, when $2A + B = O(t)$ and higher-order terms are insignificant.

Relation (6) and inequalities (7) imply

$$\dot{A} = \sum_{s=1}^{\infty} -\omega_0 \delta^s + \sum_{s=1}^{\infty} \sum_{0 < n < s} n \nabla^L \left( (\nabla^L \times a^{(n)})_k + \nabla^L b^{(n)} \right)\right|_{0, \gamma} \times \nabla^L \left( (\nabla^L \times a^{(s-n)})_k + \nabla^L b^{(s-n)} \right)_{0, \gamma},$$

$$\leq |\omega_0|_{0, \gamma} + 6 \sum_{s=1}^{\infty} \sum_{0 < n < s} n \left( \max_{i,j} |\nabla^L_i \nabla^L_j a^{(n)}|_{0, \gamma} + \max_{i,j} |\nabla^L_i \nabla^L_j b^{(n)}|_{0, \gamma} \right),$$

$$\dot{B} = \sum_{s=1}^{\infty} \sum_{0 < n < s} 2 \left| \nabla^2 a^{(n)} \right|_{0, \gamma} + \left| \nabla^2 b^{(n)} \right|_{0, \gamma},$$

$$\leq |\omega_0|_{0, \gamma} + 6 \Theta^2 \sum_{s=1}^{\infty} \sum_{0 < n < s} n \left( 2 |\nabla^2 a^{(n)}|_{0, \gamma} + |\nabla^2 b^{(n)}|_{0, \gamma} \right),$$

Integration of this inequality in time yields

$$A \leq |\omega_0|_{0, \gamma} t + 3 \Theta^2 (2A + B)^2,$$

Relations (7) and inequalities (10) imply

$$B \leq 6 \Theta^2 (2A + B)^2 + 6 \Theta^3 (2A + B)^3.$$
From the last two inequalities we deduce
\[ p(\zeta) \equiv 6\Theta^3\zeta^3 + 12\Theta^2\zeta^2 - \zeta + Q \geq 0, \tag{10} \]
where we have denoted \( \zeta \equiv 2A + B \) and \( Q \equiv 2|\omega_0|_{\gamma} t \). The discriminant of the polynomial \( p(\zeta) \),
\[ \Delta \equiv 972\Theta^6 \left( -Q^2 - Q \left( \frac{64}{9} + \frac{4}{3\Theta} \right) + \frac{4}{27\Theta^2} + \frac{2}{81\Theta^3} \right), \]
is positive at small times, when \( Q \) is sufficiently small, whereby \( p(\zeta) \) has three real roots \( \zeta_i \). The polynomial has then two local extrema at points of different signs, hence it has roots of both signs. Since by Viète’s theorem the product of the roots is negative, two roots are positive and one is negative. Inequality (10) implies
\[ 2A + B \leq \zeta_2(Q), \tag{11} \]
where \( \zeta_2 \) is the intermediate root (i.e., the smaller of the two positive roots).

We determine now the largest time \( t_c \), for which bound (11) holds. Differentiating the equation for the roots of polynomial (10) in \( Q \), we find
\[ \frac{\partial \zeta_i}{\partial Q} = -\left( \frac{\partial p}{\partial \zeta} \bigg|_{\zeta = \zeta_i} \right)^{-1}. \]
Consequently, on increasing \( Q \), the root \( \zeta_2 \) monotonically increases and \( \zeta_3 \) monotonically decreases till the two roots collide (i.e. \( \zeta_2 = \zeta_3 \)) at a critical value \( Q = Q_c \). Then \( t_c = Q_c/(2|\omega_0|_{\gamma}) \). For \( Q = Q_c \) the discriminant \( \Delta \) vanishes, which implies
\[ Q_c = \sqrt{\left( \frac{2}{3\Theta} + \frac{32}{9} \right)^2 + \frac{4}{27\Theta^2} + \frac{2}{81\Theta^3} - \frac{2}{3\Theta} - \frac{32}{9}}. \tag{12} \]

For small \( \gamma \) and hence large \( \Theta \sim 1/\gamma \), we have \( Q_c = \Theta^{-2}/48 + O(\Theta^{-3}) \sim \gamma^2 \).

So far, inequality (11) for \( t < t_c = Q_c/(2|\omega_0|_{\gamma}) \) is just an a priori bound. However, one can easily check that its derivation can be repeated, without any changes, for partial sums of the series \( A \) and \( B \) having an arbitrary but equal number of terms. The bound (11) being uniform in the number of terms left in the partial sums, convergence of the series follows from standard arguments, as well as the validity of the bound (11) for the infinite series \( 2A + B \). This remark concludes the proof.

Clearly, if the bound (11) holds true up to the time \( t = t_c \), then the series defining \( A \) and \( B \) are convergent in the complex time disk \( |t| < t_c \). Using the inequalities for the coefficients of the series for \( 2A + B \), it is easy to show that, in this disk, the series (11) is not just a formal – but a genuine – solution to (2)–(3). By virtue of the proven [1] uniqueness of the solution in the said Hölder class, any Lagrangian solution takes the form of the series (11). This guarantees that, within this disk, the solution \( \xi \) and its gradient \( \nabla^H \xi \) are analytic functions in time.

Finally, we observe that the solution constructed to this point is based on a Taylor expansion about \( t = 0 \). However, we can restart from any time \( \tilde{t} > 0 \), at which vorticity has the \( C^{0.5} \) regularity for some \( \tilde{\gamma} > 0 \) (which can be smaller than
the Hölder exponent $\gamma$ for the initial data). Thus, temporal analyticity persists on the interval $[\tilde{t}, \tilde{t} + \tilde{t}_*]$ for a certain positive $\tilde{t}_*$. This process can be continued until the vorticity ceases to have any $C^{0.5}$ regularity at some time $t_*$, which may be finite or infinite. Hence trajectories of fluid particles remain real analytic functions of the time as long as the vorticity preserves its Hölder continuity. This completes our proof of analyticity in time of fluid particle trajectories as long as the velocity field is $C^{1,\gamma}$-continuous for some $\gamma$ such that $0 < \gamma < 1$.

4. Concluding remarks

Examination of the structure of the proof of time analyticity given in Section 3 reveals that the essential ingredients are (i) the fact that $|fg|_{0,\gamma} \leq |f|_{0,\gamma}|g|_{0,\gamma}$ (algebra property) and (ii) the boundedness in the Hölder norm of the operator $\nabla_i \nabla_j \nabla^{-2}$, where $\nabla^{-2}$ is the inverse Lagrangian Laplacian acting on functions of zero spatial mean. There are obviously many other function spaces for which similar properties hold. This will be discussed elsewhere.

Of course, none of the results derived here from Cauchy’s Lagrangian formulation of the 3D incompressible Euler equation shed light on the issue of whether or not there is finite-time blow up. It is interesting to observe that, in Cauchy’s formulation, all the nonlinear terms vanish if the solutions depend only on one spatial coordinate, because all the gradients appearing in the equations are then parallel. Our bounds on the various nonlinear terms are rough as they do not take into account the cancellation occurring in the vector products, for instance, the possible strong depletion of nonlinearity which can be associated to the development of almost one-dimensional solutions. At present such issues can be investigated only through very precise numerical simulations. By performing the time-stepping in the Lagrangian domain one can then fully exploit the analyticity result.

Here we have established results for the Lagrangian structure of strong solutions, as long as they exist. Recently there has been significant progress on weak solutions where the velocity has much less spatial smoothness, e.g. Hölder continuity $C^{0,\gamma}$ with $\gamma$ close to the Kolmogorov–Onsager value $1/3$ [6]. It would be interesting to investigate the Lagrangian structure of such weak solutions.

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