Dissipative Quantum Mechanics: The Generalization of the Canonical Quantization and von Neumann Equation.

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Summary. – Sedov variational principle, which is the generalization of the least action principle for the dissipative processes is used to generalize the canonical quantization and von Neumann equation for dissipative systems (particles and strings).

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1 Introduction.

In the last two decades great numbers of papers have dealt with the quantum description of the dissipative systems ([3] - [28]) – a subject pioneered by Cardirola [1].

The dissipative models in string theory are expected to have more broad range of application. It is caused by the following possibilities:

1) Noncritical strings are dissipative systems in the "coupling constant" phase space [27, 26]. In this case, dissipative forces are defined by non-vanishing beta-functions of corresponding coupling constant and by Zamolodchikov metric [27, 26].

2) Problems of quantum description of black holes on the two-dimensional string surface lead to the necessity of generalization of von Neumann equation for dissipative systems [3, 27, 23].

3) The motion of a string (particle) in affine-metric curved space is equivalent to the motion of the string (particle) subjected to dissipative forces on Riemannian manifold [28, 30]. So the consistent theory of the string in the affine-metric curved space is a quantum dissipative theory [28, 29, 23, 30].

But the quantum descriptions of the dissipative systems (particles, strings,...) have well known ambiguities [3, 44, 45, 4, 5, 11, 12, 13, 14, 15]. Let us describe briefly some approaches to the quantum description of dissipative systems. I hope you’ll forgive me the subjectivity in the description of the published works.

1.1. Quantum kinetics.

Vectorial Newtonian mechanics describe the motion of mechanical systems subjected to forces. The forces usually are divided into potential and dissipative forces. The Newtonian approach does not restrict the nature of the force [31]. Variational Lagrangian and Hamiltonian mechanics describe the systems subjected to the potential forces only [31]. Dissipative forces are beyond the sphere of the variational principles of least action [33]. For this reason the statistical mechanics cannot describe the irreversible processes. It is caused by the absence of Liapunov function in the phase space in Hamiltonian mechanics (Poincare-Misra theorem [38, 2]). To describe the dissipative and irreversible processes we must introduce an additional postulate in statistical mechanics (for example, the Bogolubov principle of weakening correlation [39] and the hypothesis of the relaxation time hierarchy [40]). Therefore these processes are considered within framework of the physical kinetics [41, 11].

It is known that the initial point of the quantum mechanics formalism is Hamiltonian mechanics [42]. Therefore the quantum theory describes physical objects in the potential force fields only. The irreversible and dissipative quantum dynamics outside the framework of quantum mechanics. The quantum description of the
dissipative and irreversible processes is the aim of quantum kinetics which is the quantum statistics with the additional physical postulates.

Quantum description of dissipative systems within framework of quantum kinetics is very popular and successful, but it is not valid in the fundamental theories such as string theory.

1.2. Generalization of von Neumann equation.

The other way to attack the problem of quantum dynamical description of the dissipative systems is the generalization of quantum statistics \([12, 13, 14, 15, 11, 7, 8, 10]\). An important property of the dissipative and irreversible processes is the increase of the entropy during these processes. Nevertheless, the quantum mechanical evolution equation for statistical ("density-matrix") operator, the quantum Liouville (called von Neumann) equation keeps the entropy invariant \([50]\).

The generalized von Neumann equations proposed until now, which should describe dissipative and irreversible processes are derived by the addition the superoperator which acts on the statistical operator and describes dissipative part of time evolution. The linear generalized von Neumann equations are connected with master (Pauli) equation of the quantum kinetics \([17]\) and with the quantum dynamical semigroup \([47]\). Note that the total time derivative of the dissipative system operator does not satisfy the Leibnitz rule, i.e. it is not the derivative operator, and called the dissipative operator \([48]\).

Messer and Baumgartner \([11]\), Beretta, Gytopoulos, Park, Hatsopoulos \([12, 13]\) and Korsch, Steffen, Hensel \([14, 15]\) proposed the nonlinear generalizations of von Neumann equation corresponding to the nonlinear Schroedinger equations introduced by Gisin \([16]\), Kostin \([17]\) to describe dissipative systems.

Note that proposed generalizations of the von Neumann equation are derived heuristically. For a given set of generalized equations different requirements for superoperator exist. Most of the requirements proposed until now, which should determine superoperator uniquely are not unique themselves and so one has to deal with the problems arising from these ambiguities. The superoperator form is not determine uniquely. Moreover, the generalizations of the von Neumann equation are not connected with classical Liouville equation for dissipative systems \([51, 53, 54, 52]\). Therefore the quantum description of the dissipative system dynamics used the generalizations of von Neumann equation proposed until now is ambiguous.

1.3. Problems of canonical quantization.

Let us mention some papers showed that canonical quantization of the dissipative systems is impossible.

1.3.1. Lemos \([3]\) proved that canonical quantization is impossible in the presence of dissipative forces. So dissipative systems are not compatible with canonical quantization commutation relations. Note that Lemos considered the total time
derivatives of the commutation relations for the coordinates and momentums, used the Jacobi identity and the dissipative equations of motion of Heisenberg operator.

It is easy to see that quantum equation of motion for dissipative system is not compatible with Heisenberg algebra. Let us consider the quantum Langevin equation

$$\dot{a} = (-\alpha - \beta)a + f(t)$$
$$\dot{\dagger}a = (-\alpha - \beta)\dagger a + \dagger f(t)$$

We have

$$[\dot{a}, \dagger a] + [a, \dot{\dagger}a] = -2\beta$$

On the other hand, the total time derivative of the Heisenberg algebra

$$[a, a] = [\dagger a, \dagger a] = 0, \ [a, \dagger a] = 1$$

and Leibnitz rule lead to the following

$$[\dot{a}, \dagger a] + [a, \dot{\dagger}a] = 0,$$

i.e. quantum dissipative equations of motion are not compatible with canonical commutation relations and Heisenberg algebra.

1.3.2. As is known that the equation of motion is the Euler-Lagrange equation based on local a Lagrangian function when the Helmholtz conditions are satisfied [43]. Havas [45] considered a general theory of multipliers which allows (by using the Helmholtz conditions) a Lagrangian formulation for a broad class of the equation of motion of the dissipative systems, which cannot fit into Lagrangian mechanics by usual approach.

Havas therefore noted that the quantization of systems described by Lagrangian of the above type is either impossible or ambiguous [44]. This is follow from the fact that in classical mechanics of the dissipative systems there are many different Lagrangians and Hamiltonians leading to the same equations of motion [46]. So we don’t know which of the possible Lagrangians is corrected and one to choose for quantization procedure.

1.3.3. Edvards [4] showed that, although classical Hamiltonians are necessary for canonical quantization, their existence is not sufficient for it. The quantization of the Hamiltonian which is not canonically related to the energy is ambiguous and therefore the results are conflicting with physical interpretations.

It is not sufficient for the Hamiltonian to generate the equation of motion, but Hamiltonian must also be necessarily related via canonical transformation to the total energy of the system. However, this condition can only be met by conservative systems, thus excluding dissipative systems from possible canonical quantization [4].
1.3.4. Hojman and Shepley [5] started with classical equation of motion and set very general quantization conditions (relation that the coordinate operators commute). The total time derivative of this commutation relation was considered. It was showed that commutator of the coordinate operator and the velocity operator form a symmetric tensor operator. They proved that classical analog of this tensor operator is a matrix which inverse matrix satisfies the Helmholtz conditions. Using the Jacobi identity for the coordinate and the velocity, Hojman and Shepley conclude the following: the general quantization condition implies that the equation of motion is equivalent to Euler-Lagrange equation of the Lagrangian. It is not sufficient for Lagrangian merely to generate the equation of motion, but it must necessarily give rise to Hamiltonian which is canonically related to the physical energy of the system [4].

1.4. Nonassociative Lie-admissible quantization.

The generalization of the canonical quantization of the dissipative systems was proposed by Santilli [18, 19, 20]. Santilli showed that the time evolution law of dissipative equation not only violates Lie algebra law but actually does not characterize an algebra. Therefore Santilli suggested, as a necessary condition to preserve the algebraic structure, that the quantum dynamics of the dissipative systems should be constructed within the framework of the nonassociative algebras. This is exactly the case of the noncanonical quantization at the level of the nonassociative Lie-admissible (or Lie-isotopic) enveloping algebra worked out by Santilli, as well as flexible case suggested by Okubo [56]. The Lie-admissible generalization of the ”density-matrix” operator and von Neumann equation was considered by Mignani [21].

The quantization of the dissipative systems was proposed by Santilli [18, 19, 20] as an operator image of the Hamiltonian-admissible and Birkhoffian generalization of the classical Hamiltonian mechanics. The generalized variations used by Santilli [21] to consider the dissipative processes in the field of the holonomic variational principles are connected with the generalized multipliers suggested by Havas [13] and therefore lead to an ambiguity in generalized variations.

1.6. Nonholonomic variational principle.

Sedov [33]- [37] suggested the variational principle which is the generalization of the least action principle for the dissipative and irreversible processes. The holonomic and nonholonomic functionals are used to include the dissipative processes in the field of the variational principle. The suggested variational principle was used by Sedov, Chipkin [60], Berdichevskii [57], Jelnorovich [58, 59] to construct the phenomenological models of the continuous media with irreversible processes.

On of the approaches used a path function with the properties of an action in order to describe quantum systems with friction was suggested by Battezzati in the
1.5. We can conclude the following:

1. Canonical quantization of the dissipative systems is impossible if all operators in quantum theory are associative.
2. In the consistent dissipative quantum theory equation of motion must be compatible with canonical commutation relations.
3. Hamiltonian must be canonically related to the physical energy of the dissipative system.
4. Total time derivative of the dissipative system operator does not satisfy the Leibnitz rule.
5. Generalization of the von Neumann equation must be connected with classical Liouville equation for dissipative systems.
6. Dissipative systems can be described within framework of the nonholonomic variational principle.

In this paper we consider some main points of the quantum description of the dissipative systems which take into account these conclusions.

1.6. Nonholonomic variational principle and quantum description of the dissipative systems.

Nonholonomic principle was suggested in [23, 24, 25] to generalize the classical mechanics in phase space. The classical mechanics of the dissipative systems in the phase space suggested in this paper can be used to consider the generalizations of canonical quantization for dissipative systems and von Neumann equation.

In order to solve the problems of the quantum description of dissipative systems we suggest to introduce an operator $W$ in addition to usual (associative) operators. The suggested operator algebra does not violate Heisenberg algebra because we extend the canonical commutation relations by introducing an operator of the nonholonomic quantity in addition to the usual (associative) operators of usual (holonomic) coordinate-momentum functions. That is the coordinate and momentum satisfy the canonical commutation relations. To satisfy the generalized commutation relations the operator $W$ of nonholonomic quantity must be nonassociative nonlieble (does not satisfy the Jacobi identity) and Lie-nonaddmisible operator [23, 29]. As the result of these properties the total time derivative of the multiplication and commutator of the operators does not satisfies the Leibnitz rule. This lead to compatibility of quantum equations of motion for dissipative systems and canonical commutation relations. The suggested generalization of the von Neumann equation is connected with classical Liouville equation for dissipative systems.

The dissipative quantum scheme suggested in [23, 29] and considered in this paper allows to formulate the approach to the quantum dissipative field theory and quantization of the phenomenological models of continuous media. As an example
of the dissipative quantum field theory the sigma-model approach to the quantum string theory was considered in the recent papers [28, 30]. Conformal anomaly of the energy momentum tensor trace for closed bosonic string on the affine-metric manifold and two-loop metric beta-function for two-dimensional nonlinear dissipative sigma-model was calculated [23, 28, 30].

2 Nonholonomic Variational Principle.

2.1. The equations of motions of the mechanical systems in n-dimensional configurational space are

\[ D_i T(q, u, t) - f_i = 0 \]  \hspace{1cm} (1)

where \( T \) is the kinetic energy, which can be written in the form

\[ T(q, u, t) = \frac{1}{2} a_{ij}(q, t) u^i u^j + a_i(q, t) u^i + a_0(q, t) \]  \hspace{1cm} (2)

\[ D_i \equiv \frac{d}{dt} \frac{\partial}{\partial u^i} - \frac{\partial}{\partial q^i} \]  \hspace{1cm} (3)

\( i, j = 1, \ldots, n; \ u^i \equiv dq^i/dt \) and \( f_i = f_i(q, u, t) \) is the sum of external forces. In general case, \( f_i \) is the sum of the potential \( f_i^p \) and the dissipative \( f_i^d \) forces. The potential force is the force for which a function \( V = V(q, u, t) \):

\[ D_i V = f_i^p \]  \hspace{1cm} (4)

exists. The dissipative force \( f_i^d \) is the force which cannot be written in the form (4). Then the Euler-Lagrange equations take the form

\[ D_i L - f_i^d = 0 \]  \hspace{1cm} (5)

where \( L = L(q, u, t) \equiv T - V \) is Lagrangian. In the dissipative case ( \( f^d \neq 0 \) ), the equation (5) cannot be followed from the least action principle [31]:

\[ \delta S(q) \equiv \delta \int_{t_1}^{t_2} dt \ L(q, u, t) = 0 \]  \hspace{1cm} (6)

2.2. Helmholtz conditions for the differential equations of motion

\[ F_k(t, q, u, ..., q^{(m)}) = 0 \]  \hspace{1cm} (7)

where \( q^{(m)} = d^m q/dt^m \), are equivalent [32] to a holonomic condition for the functional \( S(q) \), defined by the variational equation

\[ \delta S(q) = \int_{t_1}^{t_2} dt \ F_k(t, q, u, ..., q^{(m)}) \ \delta q^k \]  \hspace{1cm} (8)
This functional is a holonomic functional if and only if
\[ \oint_\Gamma (\delta \delta' S - \delta' \delta S) = 0 \quad (9) \]

The holonomic condition is given by [32]
\[ \delta \delta' S - \delta' \delta S = 0 \quad (10) \]

It is easy to see that this condition lead to Helmholtz conditions
\[ \frac{\partial F_k}{\partial q^l} - \frac{\partial F_l}{\partial q^k} - \sum_{s=1}^m [-1]^s \frac{d^s}{dt^s} \frac{\partial F_l}{\partial q^{k(s)}} = 0 \quad (11) \]
\[ \frac{\partial F_k}{\partial q^{l(i)}} - \sum_{s=1}^m [-1]^s \frac{d^s}{dt^s} \frac{\partial F_l}{\partial q^{k(s)}} = 0 \quad (12) \]

If the conditions (10) or (11,12) are satisfied then a local Lagrangian function exists and the integral of this function is the holonomic functional called action
\[ S(q) = \int_{t_1}^{t_2} L(t, q, u, ..., q^{(m)}) \, dt \quad (13) \]

In this case the equations (6) can be derived from the least action principle (3). Note that, if the Lagrangian exists we can define the metric in the \((n + 1)\)-dimensional configurational space [31]. So the motion on the metric manifold is equivalent to the motion of the non dissipative system [31, 28].

2.3. The basic variational principle for dissipative processes is the nonholonomic principle suggested by Sedov [33]-[37]. It is a generalization of the least action principle. The Sedov variational principle has the form:
\[ \delta S(q) + \delta \tilde{W}(q) = 0 \quad (14) \]

where \( S(q) \) is the holonomic functional called action and \( \tilde{W}(q) \) is the nonholonomic functional. The nonholonomic functional is defined by the nonholonomic equation. Let a variation of the nonholonomic functional be linear in the variations \( \delta q^i \) and \( \delta u^i \) that is
\[ \delta \tilde{W} = \delta \int dt \, w(q, u) = \int dt \, (f_i(q, u) \delta q^i + g_i(q, u) \delta u^i) \quad (15) \]

where \( f_i \) and \( g_i \) are the vector functions in the configurational space. Let us consider the Hamiltonian approach to the variational classical mechanics with dissipative forces.
3 Dissipative Mechanics in Phase Space.

3.1. One direct corollary of the nonholonomic variational principle is the following dissipative equation of motion

\[
\frac{d}{dt}\left(\frac{\partial L(q,u)}{\partial u^i} + g_i(q,u)\right) = \frac{\partial L(q,u)}{\partial q^i} + f_i(q,u) \quad \frac{dq^i}{dt} = u^i 
\] (16)

Let us define a canonically conjugate momentum by the equations

\[
p_i \equiv \frac{\partial L(q,u)}{\partial u^i} + g_i(q,u) \quad \text{(17)}
\]

and represent this relation in the form \(u^i = v^i(q,p)\). The Hamiltonian is given by

\[
h(q,p) = p_i v^i(q,p) - L(q,v(q,p)) \quad \text{(18)}
\]

If we consider the variation of the Hamiltonian, we obtain the dissipative Hamiltonian equations of motion

\[
\frac{dq^i}{dt} = \delta(h - w) \delta p_i \quad ; \quad \frac{dp_i}{dt} = -\delta(h - w) \delta q^i 
\] (19)

where \(\delta w(q,p) = \delta w(q,v(q,p)) = w^q_i \delta q^i + w^p_i \delta p_i \) (20)

3.2. Let the coordinates \(z^k, (k = 1, \ldots, 2n)\), where \(z^i = q^i, z^{n+i} = p_i (i = 1, \ldots, n)\) and \(w, t\) of the \((2n+2)\)-dimensional extended phase space be connected by the equations

\[
dw - a_k(z,t) \, dz^k = 0 \quad \text{(21)}
\]

where \(a_k (k = 1, \ldots, 2n)\) are the vector functions in phase space. Let us call the dependence \(w\) on the coordinate \(q\) and momentum \(p\) the holonomic-nonholonomic function and denote \(w = w(z) \in \Phi\). If the vector functions satisfy the relation

\[
\frac{\partial a_k(z)}{\partial z^l} = \frac{\partial a^l(z)}{\partial z^k} \quad \text{(22)}
\]

where \(k,l = 1, \ldots, 2n\), the coordinate \(w\) is the holonomic function \((w \in F)\). By definition, if these vector functions don’t satisfy the relation (22) the object \(w(z)\) we call the nonholonomic function \((w \in \tilde{F})\). Let us define the generalized Poisson brackets for \(\forall a, b \in \Phi\) in the form:

\[
[f, g] \equiv \frac{\delta f}{\delta q^i} \frac{\delta g}{\delta p_i} - \frac{\delta f}{\delta p_i} \frac{\delta g}{\delta q^i} \quad \text{(23)}
\]
The basic properties of the generalized Poisson brackets:

1) *Skew-symmetry*: \( \forall f, g \in \Phi \quad [f, g] = -[g, f] \in F \); 

2) *Jacobi identity*: \( \forall f, g, s \in F \quad J[f, g, s] = 0; \)

3) *Nonlinearity*: \( \forall f, g, s, \in \Phi : f \lor g \lor s \in \tilde{F} \quad J[f, g, s] \neq 0; \)

4) *Leibnitz rule*: \( \forall f, g \in \Phi \quad \partial_t [f, g] = [\partial_t f, g] + [f, \partial_t g]; \)

5) *Distributive rule*: \( \forall f, g, s \in \Phi \quad [\alpha f + \beta g, s] = \alpha [f, s] + \beta [g, s] \)

where

\[ J[f, g, s] \equiv [f, [g, s]] + [g, [s, f]] + [s, [f, g]] \]

\( \alpha \) and \( \beta \) are the real numbers. It is easy to verify that this properties of the generalized Poisson brackets for the holonomic functions coincide with properties of the usual Poisson brackets \([31]\). Let us consider now the characteristic properties of the physical quantities:

1) \([p_i, p_j] = [q^i, q^j] = 0 \quad \text{and} \quad [q^i, p_j] = \delta^i_j\)

2) \([w, p_i] = w^i_p \quad \text{and} \quad [w, q^i] = -w^i_p \quad i \neq j, \quad [w, w] = 0\)

3) \([w, p_i] \neq [w, [p_j, p_i]] \quad \text{or} \quad J[p_i, w, p_j] = \omega_{ij} \neq 0 \quad i \neq j\)

4) \([w, q^i] \neq [w, [q^j, q^i]] \quad \text{or} \quad J[q^i, w, q^j] = \omega^{ij} \neq 0 \quad i \neq j\)

5) \([w, q^i] \neq [w, [p_j, q^i]] \quad \text{or} \quad J[q^i, w, p_j] = \omega^i_j \neq 0\)

where

\[ \omega^i_j = \frac{\partial w^q_j}{\partial p_i} - \frac{\partial w^q_i}{\partial p_j} = \frac{\delta^2 w}{\delta p_i \delta q^j} - \frac{\delta^2 w}{\delta q^i \delta p_j} \quad (24) \]

This object \( \omega^{kl} \) \((k, l = 1, ..., 2n)\) characterizes deviation from the condition of integrability \((22)\) for the equation \((21)\) and by the Stokes theorem

\[ \oint_{\partial M} \delta w = \int_M \omega^{kl} dz^k \wedge dz^l \neq 0 \quad (25) \]
Note that $w$ is the nonholonomic object if one of $\omega^{kl}$ is not trivial. Therefore some of the properties 3-5 can be not satisfied but one of it must be carry out if we consider the dissipative processes.

If we take into account generalized Poisson brackets the equation of motion in phase space for dissipative systems (19) takes the form

$$\frac{dq^i}{dt} = [q^i, h - w] \quad \frac{dp_i}{dt} = [p_i, h - w]$$

(26)

The total time derivative of the physical quantity $A = A(q, p, t) \in F$ is given by

$$\frac{dA(q, p, t)}{dt} = \frac{\partial A(q, p, t)}{\partial t} + [A, h - w]$$

(27)

The equation of motion (26) can be derived from the equation (27) as a particular case. Note that any term which added to the Hamiltonian $h$ and nonholonomic object $w$ does not change the equations of motions (26), (27). This ambiguity in the definition of the Hamiltonian is easy to avoid by the requirement that Hamiltonian must be canonically related to the physical energy of the system [4]

$$[w, q^i] = 0$$

(28)

It is easy to see that total time derivative of the generalized Poisson brackets does not satisfies the Leibnitz rule

$$\frac{d}{dt}[f, g] = \left[\frac{d}{dt}f, g\right] + [f, \frac{d}{dt}g] + J[f, w, g]$$

(29)

3.3. Let us consider the solution of the equation (26) in the form

$$q^i = q^i(q_0, p_0, t) \quad p_i = p_i(q_0, p_0, t)$$

(30)

We assume that the points of the volume

$$V_0 = \int \delta q_0 \delta p_0$$

in the phase space are initial points at the moment $t = t_0$. Then the equations (26) transform the volume $V_0$ to the volume

$$V = \int \delta q \delta p = \int I \delta q_0 \delta p_0,$$

where

$$I = \frac{\partial (q, p)}{\partial (q_0, p_0)} = \frac{\partial q^i}{\partial q^k} \frac{\partial p_i}{\partial p_k} - \frac{\partial p_i}{\partial q_k} \frac{\partial q^i}{\partial p_k} = [q^i, p_i]_0.$$
The following equation is easily verified
\[ \frac{dV}{dt} = \int \delta q \delta p \, \omega(t, q, p) \] (31)
where
\[ \omega(t, q, p) = \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} J[q_i, w, p_i] \]

The fundamental hypothesis of the statistical mechanics \[49, 2\] is that the state at the moment \( t \) is defined by the distribution function \( \rho(q, p, t) \), called density, which satisfies the normalization condition
\[ \int dq dp \, \rho(q, p, t) = 1 \] (32)
The average of the physical quantity \( A(q, p, t) \) is defined \[2\] by
\[ \langle A(t) \rangle_{\rho(t)} = \int dq dp \, \rho(q, p, t) \, A(q, p, t) \] (33)

Using for equation (32) formula (31), we obtain the dissipative analog of the Liouville equation \[51, 52, 2\]:
\[ \frac{d}{dt} \rho(q, p, t) = -\omega(t, q, p) \rho(q, p, t) \] (34)
or
\[ i \frac{\partial}{\partial t} \rho(q, p, t) = \hat{L} \rho(q, p, t) \] (35)
where
\[ \hat{L} = i( \frac{\delta(h - w)}{\delta q^k} \frac{\partial}{\partial p_k} - \frac{\delta(h - w)}{\delta p_k} \frac{\partial}{\partial q_k} - \omega(t, q, p) ) \] (36)
called Liouville operator \[2\]. In addition to the Poincare-Misra theorem \[4\] can be obtained the statement: “There exists the Liapunov function of the coordinate and momentum in the dissipative Hamiltonian mechanics”. Let us define the function \( \eta(q, p, t) \equiv -\ln \rho(q, p, t) \) and assume \( \omega > 0 \). The equation (34) shows that
\[ d/dt \eta(q, p, t) = \omega(t, q, p) \]
and the function \( \eta \) satisfies the relations \( d\eta/dt > 0 \). It is convenient to introduce the entropy of the distribution defined as follows
\[ s \equiv < \eta > = -\int \delta q \delta p \, \rho(q, p, t) \ln \rho(q, p, t) \] (37)

The relation \( ds/dt > 0 \) is easily verified. In the general case, any function \( f(q, p, t) \) which is the composite function \( f(q, p, t) = g(\rho(q, p, t)) \) and satisfies the relation \( \omega (\partial g(\rho))/\partial \rho < 0 \) (\( \forall t \)) is the Liapunov function, that is \( (df)/(dt) > 0 \). A crucial point is that the condition \( \omega > 0 \) or \( \omega (\partial g(\rho))/\partial \rho < 0 \) is not necessary \[53\].
4 Quantum Dissipative Mechanics.

4.1. In order to solve the problems of the quantum description of dissipative systems we suggest to introduce an operator $W$ in addition to usual (associative) operators. Let us use the usual rule of definition of the quantum physical quantities which have the classical analogues [12]: If we consider the operators $A, B, C$ of the physical quantities $a, b, c$ which satisfy the classical Poisson brackets $[a, b] = c$, then the operators must satisfy the relation: $[A, B] \equiv (AB) - (BA) = i\hbar C$. If we take into account the characteristic properties the physical quantities operators are defined by the following relations:

1) $[Q^i, Q^j] = [P_i, P_j] = 0 \quad [Q^i, P_j] = i\hbar \delta^i_j$

2) $[W, P_i] = i\hbar W^i_p \quad [W, Q^i] = -i\hbar W^i_p \quad [W, W] = 0$

3) $[[W, P_i], P_j] \neq [[W, P_j], P_i] \quad i \neq j \quad or \quad J[P_i, W, P_j] = \Omega_{ij} \neq 0$

4) $[[W, Q^i], Q^j] \neq [[W, Q^j], Q^i] \quad i \neq j \quad or \quad J[Q^i, W, Q^j] = \Omega^{ij} \neq 0$

5) $[Q^i, [W, P_j]] \neq [P_j, [W, Q^i]] \quad or \quad J[Q^i, W, P_j] = \Omega^i_j \neq 0$

where

$$J[A, B, C] = -1/(\hbar^2) \ ( [A[BC]] + [B[CA]] + [C[AB]] )$$

and $Q^i = Q; \ P^i = P; \ W^i = W$. Let us require that the canonical quantum commutation rules be a part of this rule. To satisfy the commutation relations and canonical commutation rules for the operator of the holonomic function the operators of the nonholonomic quantities must be nonassociative. It is sufficient to require that the operator $W$ satisfies the following conditions:

1) left and right associativity:

$$(Z^k, Z^l, W) = (W, Z^k, Z^l) = 0$$

2) left-right nonassociativity:

$$(Z^k, W, Z^l) \neq 0 \quad if \quad k \neq l$$

where $k; l = 1, ..., 2n; \ Z^i = Q^i$ and $Z^{n+i} = P_i; \ i = 1, ..., n$;

$$(A, B, C) \equiv (A(BC)) - ((AB)C)$$
4.2. The state in the quantum dissipative mechanics can be represented by the "matrix-density" (statistical) operator $\rho(t)$ which satisfy the condition $\rho^\dagger(t) = \rho(t)$. The time variations of the operator of physical quantity $A(t) \equiv A(Q,P,t)$ and of the operator of state $\rho(t)$ are written in the form

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{i}{\hbar} [H - W, A] \quad (38)$$

$$\frac{d\rho}{dt} = -\frac{1}{2} [\rho, \Omega]_+ \quad (39)$$

$$\frac{\partial}{\partial t} \rho = -\frac{i}{\hbar} [\rho, H] + \frac{i}{\hbar} [W, \rho] - \frac{1}{2} [\rho, \Omega]_+ \quad (40)$$

where anticommutator $[ , ]_+$ is the consequence of the hermiticity for the density operator $\rho$ and for the operator $\Omega$, which is defined by

$$\Omega = \sum_{i=1}^n \Omega_i = \sum_{i=1}^n J[Q_i, W, P_i]$$

The solution of the first equation may be written in the form

$$A(t) = S(t,t_0) A(t_0) S^\dagger(t,t_0) \quad \text{where} \quad S(t,t_0) = T \exp \left( \frac{i}{\hbar} \int_{t_0}^t d\tau \ (H - W)(\tau) \right) \quad (41)$$

T-exponent is defined as usual, but we must take into account the following flow chart

$$\exp A = 1 + A + \frac{1}{2} (AA) + \frac{1}{6} ((AA)A) + \frac{1}{24} (((AA)A)A) + ... .$$

The solution of the equation (40) is given by

$$\rho(t) = U(t,t_0) \rho(t_0) U^\dagger(t,t_0) \quad \text{where} \quad U(t,t_0) = T \exp \left( \frac{i}{2} \int_{t_0}^t d\tau \ \Omega(\tau) \right) \quad (42)$$

In this way the time evolution of the physical quantity operator is unitary and the evolution of the state operator is nonunitary. It is easy to verify that the pure state at the moment $t = t_0$ ( $\rho^2(t_0) = \rho(t_0)$ ) is not a pure state at the next time moment $t \neq t_0$. We can define the entropy operator $\eta$ of the state $\rho(t)$: $\eta(t) = -\ln \rho(t)$. The entropy operator satisfies the equation

$$\frac{d}{dt} \eta(t) = \Omega$$
It is easy to see that the commutator with nonassociative operator $W$ and the total time derivative of both the quantum Poisson brackets and of the multiplication of the two operators do not satisfy the Leibnitz rule

$$[AB, W] = A[B, W] + [A, W] B + (A, W, B)$$  \hspace{1cm} (43)

$$\frac{d}{dt}[A, B] = \left[ \frac{d}{dt} A, B \right] + [A, \frac{d}{dt} B] + J[A, W, B]$$  \hspace{1cm} (44)

$$\frac{d}{dt}(AB) = (\frac{d}{dt} A) B + (A, \frac{d}{dt} B) + (A, W, B)$$  \hspace{1cm} (45)

where $A$ and $B$ are the associative operators (operators of the holonomic functions). This lead to compatibility of quantum equations of motion for dissipative systems and canonical commutation relations.

4.3. Let us define the canonical (unitary) transformation $[42]$ of an operator $A_H(t) = A(t)$ in the form $A_S(t, t_0) = S^\dagger(t, t_0) A_H(t) S(t, t_0)$. The operator $A_S(t, t_0)$ satisfies the condition $A_S(t_0, t_0) = A_H(t_0)$. In this case the equations (38), (40) take the form

$$\frac{d}{dt} A_S(t, t_0) = S^\dagger(t, t_0) \frac{\partial A_H(t)}{\partial t} S(t, t_0)$$  \hspace{1cm} (46)

$$\frac{d}{dt} \rho_S(t, t_0) = \frac{i}{\hbar} [\rho_S, (H - W)] + \frac{1}{2} [\Omega_S, \rho_S]$$  \hspace{1cm} (47)

This is the dissipative analogue of the Schroedinger equations, the operators $A_H(t)$ and $A_S(t)$ called Heisenberg and Schroedinger representations accordingly. The solution of the equation (47) is given by

$$\rho_S(t, t_0) = U_S^\dagger(t, t') \rho_S(t', t_0) U_S(t, t')$$  \hspace{1cm} (48)

where

$$U_S^\dagger(t, t') = T \exp\left(-\frac{i}{\hbar} \int_{t'}^t d\tau (H - W - \frac{i\hbar}{2} \Omega) S(\tau, t_0) \right)$$  \hspace{1cm} (49)

4.4. Let us consider some important features of the basis vectors $[42]$. Account is to be taken of the time dependence of the state operator

$$\rho_H(t) = \sum_a \rho_a [\psi_a, t > H < \psi_a, t]$$

and of the wave vectors in the Heisenberg representation $[\psi, t > H$. That is $[q, t_1 > H \neq [q, t_2 > H$ contrary to usual quantum mechanics. Let us define the basis vectors $\{[q, t > \}$ $[42]$ at the fixed time point $t = t_f$ :

1) $Q_H(t) [q, t > H = [q, t > H q_f$  \hspace{1cm} 2) $< q, t] H [q', t > H = \delta(q - q')$
3) \[ \int dq \left[ q, t > H < q, t \right] = 1 \]
4) \[ Q_H(t) = \int dq \left[ q, t > H q_f < q, t \right] \]
5) \[ [\psi, t > H] = \int dq \left[ q, t_f > H \psi(q, t, t_f) \right] \]

where \( \Psi_H(q, t, t_f) = \langle q, t_f | H | \psi, t \rangle \). It is easy to prove the following statements:

1. The basis vector unitary transformed is a basis vector;
2. There exists a unitary transformation for any two basis vectors defined at the non equal time points. Thus, Schrödinger representation of the basis vector \([q, t_0 > H S(t - t_0)] q, t > H \) might be considered as the unitary transformation of the basis vector

\[ [q, t_0 > H S[t - t_0)] q, t > H = [q, t, t_0 > S] \]

therefore the trace of the operator can be defined only in fixed time point. Note that the operator of the state \( \rho(t) \) satisfies the usual condition

\[ S \rho(t) \rho(t) = 1 \quad (\forall t = t_{fixed}) \]

where we take into account the fixed-time definition of the basic vectors. The average of the physical quantity \( A(t) = A(p, q, t) \) is defined by

\[ \langle A(t) \rangle = S \rho(t) \rho(t) \quad (\forall t = t_{fixed}) \]

and the time derivative of the average quantity can be defined only by following

\[ \frac{d}{dt} < A(t) > t \equiv \frac{d}{dt} S \rho(t) \rho(t) \quad (\forall t = t_{fixed}) \]

i.e. as the average of the time derivative of the operator.

**4.5.** Let us consider now Green’s functions and its Feynman representation. If we take into account the equation (47) we can write the dissipative Schrödinger equation for wave vector in the form

\[ \dot{\psi}(t) = (H - W - \frac{i \hbar}{2} \Omega_S(t, t_0)) \psi(t, t_0) \]

The simple example of this equation for the harmonic oscillator with friction is considered in **4.7.**. Account is to be taken of the time dependence of state in Heisenberg representation. Therefore we make the distinctions between following Green’s functions

\[ \Psi_S(q, t) = \int dq' G_S(q, q', t - t') \Psi_S(q', t') \]

\[ \Psi_H(q, t) = \int dq' G_H(q, q', t - t') \Psi_H(q', t') \]
where

$$\Psi_S(q, t) \equiv <q, t|_S|\psi, t >_S \equiv <q, t|_H|\psi, t >_H \quad (55)$$

$$\Psi_H(q, t) \equiv <q, t|_H|\psi, t >_S \equiv <q, t|_S|\psi, t >_H \quad (56)$$

$$G_S(q, q', t - t') \equiv <q, t|_S U^\dagger_S(t, t') [q', t'] >_S \theta(t - t') \equiv <q, t|_H U^\dagger_H(t, t') [q', t'] >_H \theta(t - t') \quad (57)$$

$$G_H(q, q', t - t') \equiv <q, t|_S U^\dagger_H(t, t') [q', t'] >_S \theta(t - t') \quad (58)$$

and  $$[q, t]_H \equiv [q, t]_{t,\text{fixed}} >_H$$. The Green’s function satisfies the time-dependent equation

$$i\frac{d}{dt} G_S(q, q', t) = (\mathcal{H} - W - \frac{i\hbar}{2} \Omega)_S G_S(q, q', t) \quad \text{and} \quad G_S(q, q', 0) = \delta(q - q') \quad (59)$$

If we use the method considered in [63] and the conditions

$$<p, t_f|\mathcal{H} - W - \frac{i\hbar}{2} \Omega|q, t_f >_H = (h - w - \frac{i\hbar}{2} \omega)(q_f, p_f) <p, t_f|_S|q, t_f >_H \quad (60)$$

$$<q^{n+1}, t_n|_S U^\dagger_S(t_{n+1} - t_n) [q^n, t_n] >_S \simeq \quad (61)$$

then the Feynman representation of the Green’s functions is given by

$$G_S(q, q', t - t') = \int Dq \; Dp \; \exp \left( \frac{i}{\hbar} \int^{t} d\tau \left( p \frac{dq}{d\tau} - h(q, p, \tau) + w(q, p, \tau) + \frac{i\hbar}{2} \omega \right) \right) \quad (63)$$

In the same way we can formulate the path integration and generating functional in the quantum field theory which was considered in the papers [31, 28, 30]. As one of methods to solve the quantum dissipative equations we suggested the normal geodesic coordinate and covariant background field method [67, 64, 66] which is generalization of the Taylor’s series expansion around the classical solution and was considered for Riemannian [64, 66], affine [67, 63] and affine-metric [62, 28, 30, 29] manifolds.

4.6. Let us consider the harmonic oscillator with friction and prove that the configurational space of this oscillator is curved by the quantum fluctuations. It is known that the harmonic oscillator configurational space metric is defined by the kinetic energy [31]

$$d^2 s = 2T(dt)^2 = \delta_{ij}dq^i dq^j \quad (64)$$
where \( T = \sum_i^n (1/2)(dq^i/dt)^2 \) is the kinetic energy of the harmonic oscillator. The existence of the potential forces with the potential

\[
U(q) = \sum_{i=1}^n (\omega_i^2/2)(q^i)^2; \quad \omega_i^2 = \sqrt{k_i/m_i}
\]

lead to the deformation of the configurational space metric in the form

\[
d^2s = (E - U(q))\delta_{ij}dq^idq^j
\]

for conservative systems only \cite{31}, where \( E \) is the total energy. The dissipative forces don’t allow us to derive the mechanical trajectory by Jacobi variational principle \cite{31}. That is the configuration space of the harmonic oscillator with friction is flat.

The equations of motion for this oscillator in n-dimensional configuration space are

\[
du^i/dt + \omega^2_i q^i = f_i(q,u)
\]

where \( i, j = 1, ..., n; u^i \equiv dq^i/dt \) and \( f_i(q,u) \) is the dissipative force. Let us consider the following friction force

\[
f_i(q,u) = c_{ij}(q)u^j + D_{ikl}(q)u^ku^l
\]

As is well known, the equation (65) with friction (66) cannot be derived from the least action principle. The nonholonomic functional for this harmonic oscillator with friction is defined by the nonholonomic equation linear in the variation

\[
\delta \tilde{W} = \delta \int dt \, w(q,u) = \int dt \, f_k(q,u)\delta q^k
\]

We derive the background field expansion of the Hamiltonian \( h(q,p) \), nonholonomic functional \( w, \) and omega \( \omega(q,p) \) around the classical solution \( q_0^i(t) \) of this oscillator as a power series in \( \xi^i(t) : q^i(t) = q_0^i(t) + \xi^i(t) \). The functional integral (63) over momentum \( p^i \) is Gaussian integral and we derive the path integral form of Green’s function

\[
G_S(q,q',t-t') = N \int D\xi^i \exp \frac{i}{\hbar} \int_{t'}^t d\tau \left( T(q_0) + Z_1(q_0,\xi,\tau) + Z_2(q_0,\xi,\tau) \right)
\]

where \( T(q_0) = \frac{1}{2} \delta_{ij}v^i_0v^j_0; \) \( v^i_0 = dq^i_0/d\tau; \)

\[
Z_1(q_0,\xi) = \frac{1}{2} \delta_{ij} \frac{d\xi^i}{d\tau} \frac{d\xi^j}{d\tau} - 2D_{ikj}(q_0)v^k_0\xi^i \frac{d\xi^j}{d\tau} - D_{imn}(q_0)v^n_0v^m_0\xi^i \xi^j - \frac{\omega^2}{2} (\xi^i)^2
\]

and \( a_{n,m}(q) \equiv \partial a_n(q)/\partial q_m \). \( Z_2(q_0,\xi) \) is the sum of the series terms in \( \xi \) which don’t contribute to the one-loop finite renormalization of the metric. In the calculation of the vacuum diagrams \( q_0^i(t) \) is regarded as an external field and \( \xi^i(t) \) is the
quantum field. The vacuum contributions to the Green’s function which have the form \((1/2)T_{ij}(q_0)v_0^iv_0^j\) lead to the metric redefinition

\[d^2s = 2T(q_0)(dt)^2 = (\delta_{ij} + T_{ij})dq_0^idq_0^j\]

The one-loop vacuum contribution for the harmonic oscillator with friction (66) has the form

\[T_{ij} = \sum_{k=1}^{n} \left( \sum_{l=1}^{n} \frac{2}{\omega_k + \omega_l} D_{kl} (D_{kl} - D_{lk}) - \frac{1}{\omega_k} D_{kk} \right)\] (70)

It is easy to see that the background configurational space of the quantum harmonic oscillator with friction (66) is not flat. If the friction is quadratic in the velocity with coefficients which depend on the coordinates or these coefficients are non completely symmetric tensor the configurational space is curved by the quantum fluctuations. The full expression of the two-loop metric redefinition terms is complicated. Let us note the simple condition which allow the configurational space to be flat, no matter one-loop or two-loop vacuum diagrams are considered, is given by

\[D_{ikl;j} = 0 \quad \text{and} \quad D_{ikl} = D_{(ikl)}\] (71)

4.7. In the conclusion, let us consider the dissipative analogue of Schrodinger equation for one dimensional harmonic oscillator with friction. Hamiltonian and nonholonomic functional are following

\[\hbar = \frac{p^2}{2m} + \frac{m\omega^2q^2}{2} ; \quad \delta w = \gamma mp \delta q\]

We use the background field method and expand the Hamiltonian and nonholonomic functional in Taylor series of \(Q = q - q_0\), where \(q_0\) is the solution of the classical equation of motion in the coordinate space. Let us choose \(q_0 = 0\). In this case the generalized Schrodinger equation takes the form

\[i\hbar \frac{d}{dt} \Psi(t) = \left[ -\frac{\hbar}{2m} \frac{\partial^2}{\partial Q^2} + i\hbar \gamma Q \frac{\partial}{\partial Q} + \frac{m\omega^2Q^2}{2} - \frac{i}{2} \gamma \right] \Psi(t)\] (72)

The stationary state \(\Psi(\xi, t) = u(\xi)exp - i\hbar Et\) is defined by the equation

\[u''(\xi) - a\xi u'(\xi) + (\varepsilon - \xi^2)u(\xi) = 0\]

where

\[a = \frac{2\gamma}{\omega} ; \quad \xi = \sqrt{\frac{m\omega}{\hbar}Q} ; \quad \varepsilon = \frac{2}{\hbar \omega}(E - \frac{i}{2}\gamma)\]
Let us consider the function $u(\xi)$ in the form

$$u(\xi) = \left(\sum_{k=0}^{n} A_k \xi^k\right) \exp \left(-\frac{1}{2}s\xi^2\right)$$

where $s$ is the solution of the equation $s^2 + as - 1 = 0$ and $n < \infty$.

As a result we obtain the following eigenvalues

$$E_n = \hbar \sqrt{\omega^2 - \gamma^2(n + \frac{1}{2})} - i\gamma$$

when $0 < \gamma^2/\omega^2 < \frac{1}{2}$ and the continuous spectrum, when $\gamma^2/\omega^2 > \frac{1}{2}$. Note that the life time for the state is $T = \hbar/2\gamma < \infty$. We can rewrite the result in the form

$$\Delta E_n(\omega) = (\hbar \sqrt{\omega^2 - \gamma^2} \text{ when } \omega^2 > 2\gamma^2) \cap (0 \text{ when } \omega^2 < 2\gamma^2)$$

where $\Delta E_n \equiv E_n - E_{n-1}$. Note that the jump in the point $\omega_0 = \sqrt{2\gamma}$ is the purely quantum dissipative effect.

* * *

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