On the spectrum $bo \wedge tmf$

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Abstract

M. Mahowald, in his work on bo-resolutions, constructed a bo-module splitting of the spectrum $bo \wedge bo$ into a wedge of summands related to integral Brown-Gitler spectra. In this paper, a similar splitting of $bo \wedge tmf$ is constructed. This splitting is then used to understand the $bo_*$-algebra structure of $bo_*tmf$ and allows for a description of $bo^*tmf$.

1 Introduction

All cohomology groups are assumed to have coefficients in $\mathbb{F}_2$ and all spectra completed at the prime 2 unless stated otherwise. Let $\mathcal{A}$ denote the Steenrod algebra, and $\mathcal{A}(n)$ the subalgebra generated by $\{Sq^1, Sq^2, \cdots, Sq^{2^n}\}$. Consider the Hopf algebra quotient $\mathcal{A} // \mathcal{A}(n) = \mathcal{A} \otimes_{\mathcal{A}(n)} \mathbb{F}_2$. Here the right action of $\mathcal{A}(n)$ on $\mathcal{A}$ is induced by the inclusion and the left action on $\mathbb{F}_2$ by the augmentation. Algebraically, one can consider the subsequent surjections

$$\mathcal{A} \rightarrow \mathcal{A} // \mathcal{A}(0) \rightarrow \mathcal{A} // \mathcal{A}(1) \rightarrow \mathcal{A} // \mathcal{A}(2) \rightarrow \mathcal{A} // \mathcal{A}(3) \rightarrow \cdots$$

and ask whether each algebra can be realized as the cohomology of some spectrum. The case $n \geq 3$ requires the existence of a non-trivial map $S^{2n+1-1} \rightarrow S^0$ which cannot occur due to Hopf invariant one. For $n < 3$, however, it is now well-known that each algebra can indeed be realized by the cohomology of some spectrum:

$$H^*HF_2 \rightarrow H^*HZ \rightarrow H^*bo \rightarrow H^*tmf$$
There are maps realizing the above homomorphisms of cohomology groups

\[ \text{tmf} \to \text{bo} \to \text{H}_{\mathbb{Z}} \to \text{H}_{F_2} \]

In particular, the spectrum tmf is at the top of a “tower” whose “lower floors” have been well studied in the literature, culminating with Mahowald’s [6] understanding of the spectrum bo ∧ bo and Carlsson’s [1] description of the cohomology operations [bo, bo]. More difficult questions arise: What is the structure of tmf∧tmf? What are the stable cohomology operations of tmf?

We would like to understand the spectrum bo ∧ tmf for a variety of reasons. First, it might serve as a nice intermediate step towards understanding the spectrum tmf ∧ tmf. Furthermore, determining its structure comes with an added bonus of understanding operations [tmf, bo] which may provide some insight into understanding the cohomology operations of tmf. Second, the splitting of bo ∧ tmf has been instrumental to the author in demonstrating the splitting of the Tate spectrum of tmf into a wedge of suspensions of bo.

Let \( B_1(j) \) denote the \( j \)th integral Brown-Gitler spectrum, whose homology will be described as a submodule of \( H_* \text{HZ} \). Such spectra have been studied extensively in the literature (see [2], [3], [9], for example). In particular, Mahowald [6] demonstrated the splitting of bo-module spectra bo ∧ bo ≃ \( \bigvee_{j \geq 0} \Sigma^4 j \text{bo} \wedge B_1(j) \). Let \( \Omega = \bigvee_{0 \leq j \leq i} \Sigma^{8i+4j} B_1(j) \). The main theorem of this paper is the following

**Theorem 1.1.** There is a homotopy equivalence of bo-module spectra

\[ \text{bo} \wedge \Omega \to \text{bo} \wedge \text{tmf} \]  

The splitting is analogous to that of bo ∧ bo of Mahowald and even \( \text{MO}(8) \wedge \text{bo} \) of Davis [3]. Its proof, therefore, contains ideas and results from both. Section 2 deals with demonstrating an isomorphism on the level of homotopy groups, which first requires an understanding of the left \( \mathcal{A}(1) \)-module structure of \( H^* \text{tmf} \). In Section 3, we construct a map of bo-module spectra realizing the isomorphism of homotopy groups. Section 4 uses this splitting along with pairings of integral Brown-Gitler spectra to explicitly determine the bo*-algebra structure of bo∗tmf and also identifies the cohomology bo∗tmf.
2 The algebraic splitting

The $E_2$-term of the Adams spectral sequence converging to the homotopy groups of $bo \wedge \text{tmf}$ is given by

$$\text{Ext}_{A}^{s,t}(H^*(bo \wedge \text{tmf}), \mathbb{F}_2) \Rightarrow \pi_{t-s}(bo \wedge \text{tmf}).$$  \hfill \text{(2)}

The Ext-group appearing in the above spectral sequence can be simplified via a change-of-rings isomorphism:

$$\text{Ext}_{A(1)}^{s,t}(H^*\text{tmf}, \mathbb{F}_2) \Rightarrow \pi_{t-s}(bo \wedge \text{tmf}).$$  \hfill \text{(3)}

Therefore, it suffices to understand the left $A(1)$-module structure of $H^*\text{tmf}$. Computations and definitions simplify upon dualizing. Indeed, the dual Steenrod algebra, $A_*$, is the graded polynomial ring $\mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$ with $|\xi_i| = 2^i - 1$. An equivalent problem after dualizing is determining the right $A(1)$-module structure of the subring $H_*\text{tmf} \subset A_*$. The homology of tmf as a right $A$-module is given by Rezk \cite{8}

$$H_*\text{tmf} \cong \mathbb{F}_2[\zeta_1^4, \zeta_2^4, \zeta_3, \zeta_4, \ldots].$$  \hfill \text{(4)}

The generators $\zeta_i = \chi \xi_i$, where $\chi : A_* \to A_*$ is the canonical antiautomorphism. Define a new weight on elements of $A_*$ by $\omega(\xi) = 2^i - 1$ for $i \geq 1$. For $a, b \in A_*$ define the weight on their product by $\omega(ab) = \omega(a) + \omega(b)$. Let $N_{k}^{\text{tmf}}$ denote the $\mathbb{F}_2$-vector space inside $H_*\text{tmf}$ generated by all monomials of weight $k$ with $N_0^{\text{tmf}} = \mathbb{F}_2$ generated by the identity.

**Lemma 2.1.** As right $A(2)$-modules,

$$H_*\text{tmf} \cong \bigoplus_{i \geq 0} N_{8^i}^{\text{tmf}}$$

*Proof.* Certainly, the two modules are isomorphic as $\mathbb{F}_2$-vector spaces. To see there is an isomorphism of right $A(2)$-modules, note that the right action of the total square $Sq = \sum_{i \geq 0} Sq^i$ on the generators of $H_*\text{tmf}$ is given by:

$$\begin{align*}
\zeta_1^8 \cdot S q &= \zeta_1^8 + 1; \\
\zeta_2^4 \cdot S q &= \zeta_2^4 + \zeta_1^8 + 1; \\
\zeta_3^2 \cdot S q &= \zeta_3^2 + \zeta_2^4 + \zeta_1^8 + 1; \\
\zeta_n \cdot S q &= \sum_{i=0}^{n} \zeta_{8^{n-i}}^{2^i}
\end{align*}$$
for \( n > 3 \). Since \( \omega(1) = 0 \), modulo the identity the total square preserves the weight of the generators of \( H_* \text{tmf} \). Note that \( \zeta_1^{2^{n-1}} Sq^{2^{n-1}} = 1 \), hence the total square over \( \mathcal{A}(2) \) cannot contain a 1 in the expansion for dimensional reasons.

Consider the homomorphism \( V : A_* \rightarrow A_* \) defined on generators by

\[
V(\zeta_i) = \begin{cases} 
1, & i = 0, 1; \\
\zeta_{i-1}, & i \geq 2.
\end{cases}
\]

Restricting \( V \) to the subring \( H_* \text{tmf} \subset A_* \) clearly provides a surjection \( V_{\text{tmf}} : H_* \text{tmf} \rightarrow H_* \text{bo} \). Let \( M_{\text{bo}}(4i) \) denote the image of \( N_{\text{tmf}}^{8i} \) under the homomorphism \( V_{\text{tmf}} \). It is generated by all monomials with \( \omega(\zeta^I) \leq 4i \). The following proposition is clear.

**Proposition 2.2.** As right \( A(2) \)-modules

\[
N_{8i}^{\text{tmf}} \cong \sum^{4i} M_{\text{bo}}(4i).
\]

**Proof.** Due to the weight requirements, \( V_{\text{tmf}} \) is injective when restricted to \( N_{8i}^{\text{tmf}} \). Indeed, the exponent of \( \zeta_1 \) in each monomial is uniquely determined by the other exponents.

Additionally, if we denote by \( N_{k}^{\text{bo}} \) the \( \mathbb{F}_2 \)-vector space inside \( H_* \text{bo} \) generated by all elements of weight \( k \) with \( N_{0}^{\text{bo}} = \mathbb{F}_2 \) generated by the identity, we have a similar lemma:

**Lemma 2.3.** As right \( A(1) \)-modules,

\[
M_{\text{bo}}(4i) \cong \bigoplus_{j=0}^{n} N_{4j}^{\text{bo}}.
\]

Further restricting \( V \) to the subring \( H_* \text{bo} \) provides a surjection \( V_{\text{bo}} : H_* \text{bo} \rightarrow H_* \text{HZ} \). Let \( M_{\text{HZ}}(2j) \) denote the image of \( N_{4j}^{\text{bo}} \) under \( V \). This submodule is generated by all monomials with \( \omega(\zeta^I) \leq 2j \). As in Proposition 2.2 we have the identification

**Proposition 2.4.** As right \( A(1) \)-modules,

\[
N_{4j}^{\text{bo}} \cong \sum^{4j} M_{\text{HZ}}(2j).
\]

(6)
Goerss, Jones and Mahowald [5] identify the submodule $M_{HZ}(2j) \subset H_*HZ$ as the homology of the $j$th integral Brown-Gitler spectrum:

**Theorem 2.5** (Goerss, Jones, Mahowald [5]). For $j \geq 0$, there is a spectrum $B_1(j)$ and a map

$$B_1(j) \xrightarrow{g_\ast} \mathbb{H}Z$$

such that

(i) $g_\ast$ sends $H_*(B_1(j))$ isomorphically onto the span of monomials of weight $\leq 2j$;

(ii) there are pairings

$$B_1(m) \wedge B_1(n) \rightarrow B_1(m + n)$$

whose homology homomorphism is compatible with the multiplication in $H_*HZ$.

**Remark 2.1.** The submodules $M_{bo}(4i)$ are the so-called $bo$-Brown-Gitler modules. There is a family of spectra with similar properties, having these modules as their homology. Proposition 2.2 demonstrates that as an $\mathcal{A}(2)$-module, $H_*tmf$ is a direct sum of these modules. On the level of spectra, however, $tmf \wedge tmf$ does not split as a wedge of $bo$-Brown-Gitler spectra.

Combining the results of Lemmas 2.1 and 2.3 with Theorem 2.5, $H_*tmf$ as a right $\mathcal{A}(1)$-module can be written in terms of homology of integral Brown-Gitler spectra:

**Theorem 2.6.** As right $\mathcal{A}(1)$-modules,

$$H_*tmf \cong \bigoplus_{0 \leq j \leq i} \sum_{0 \leq j \leq i} \Sigma^{8i+4j} H_*B_1(j).$$

The $E_2$-term of the Adams spectral sequence (7) then becomes isomorphic to

$$\bigoplus_{0 \leq j \leq i} \Sigma_{0 \leq j \leq i} \sum_{0 \leq j \leq i} \Sigma^{8i+4j} \text{Ext}^{s,t}_{\mathcal{A}(1)} (H^*B_1(j), \mathbb{F}_2) \Rightarrow \pi_{t-s}(bo \wedge tmf).$$

This is precisely the Adams $E_2$-term converging to the homotopy of $bo \wedge \Omega$. The chart can be obtained by applying the following theorem of Davis [4] which links $bo \wedge B_1(n)$ to Adams covers of $bo$ or $bsp$, depending on the parity of $n$. 5
Theorem 2.7 (Davis [4]). If $\overline{n} = (n_1, \ldots, n_s)$, let $|\overline{n}| = \sum_{i=1}^s n_i$ and $\alpha(\overline{n}) = \sum_{i=1}^s \alpha(n_i)$, and $B_1(\overline{n}) = \bigwedge_{i=1}^s B_1(n_i)$. Then there are homotopy equivalences

$$bo \wedge B_1(\overline{n}) \simeq K \vee \begin{cases} bo^{2|\overline{n}| - \alpha(\overline{n})}, & \text{if } |\overline{n}| \text{ is even;} \\ bsp^{2|\overline{n}| - 1 - \alpha(\overline{n})}, & \text{if } |\overline{n}| \text{ is odd;} \end{cases}$$

where $K$ is a wedge of suspensions of $HF_2$.

Figure 1: $Ext_{A^*}^{s,t} (H^*(bo \wedge tmf), F_2) \Rightarrow \pi_{t-s}(bo \wedge tmf)$

The charts for $bo$ and $bsp$ are well known. Using the above theorem along with the algebraic splitting of $H^*tmf$, we see that Adams covers of $bo$ begin in stems congruent to 0 mod 8 while Adams covers of $bsp$ begin in stems congruent to 4 mod 8. The first 32 stems of the chart for $bo \wedge tmf$ is displayed in Figure 1 modulo possible elements of order 2 in Adams filtration $s = 0$ corresponding to free $A(1)$’s inside $H^*tmf$. The symbol $\bigodot$ appears in Figure 1 to reduce clutter. It is used to mark the beginning of another $\mathbb{Z}$-tower. In general, all $\mathbb{Z}$-towers are found in stems congruent to 0 mod 4 while those supporting multiplication by $\eta$ occur in stems congruent to 4 mod 8.

Theorem 2.8. There is an isomorphism of homotopy groups

$$\pi_*(bo \wedge tmf) \cong \pi_*(bo \wedge \Omega)$$

Proof. The $E_2$-terms of their respective Adams spectral sequences have been shown to be isomorphic. Both spectral sequences collapse. Indeed, the classes
charted in Figure 1 cannot support differentials for dimensional and naturality reasons. Each element of order two in Adams filtration \( s = 0 \) correspond to copies of \( \mathcal{A}(1) \) inside \( H^* \text{tmf} \). These summands split off, obviating the existance of differentials.

\[ \square \]

3  **The topological splitting**

Theorem 1.1 concerns a bo-module splitting of the spectrum \( bo \wedge \text{tmf} \). The following observation will aid us in studying bo-module maps.

**Lemma 3.1.** Let \( X \) and \( Y \) be spectra. Then

\[ [bo \wedge X, bo \wedge Y]_{bo} = [X, bo \wedge Y] \]

**Proof.** Let \( u_{bo} : S^0 \to bo \) and \( m_{bo} : bo \wedge bo \to bo \) denote the unit and the product map of \( bo \), respectively. Given \( f : bo \wedge X \to bo \wedge Y \) and \( g : X \to bo \wedge Y \), the equivalence is given by the composites

\[ f \mapsto f \circ (u \wedge 1) \]

\[ g \mapsto (m_{bo} \wedge 1) \circ (1 \wedge g) \]

\[ \square \]

The spectra \((bo, m_{bo}, u_{bo})\) and \((\text{tmf}, m_{\text{tmf}}, u_{\text{tmf}})\) are both unital \( \mathcal{E}_\infty \)-ring spectra [7]. This induces a unital \( \mathcal{E}_\infty \)-ring structure \((bo \wedge \text{tmf}, m, u)\). This structure will play an important role in the proof of the main theorem. We begin by defining an increasing filtration of \( \Omega \) via:

\[ \Omega^n = \bigvee_{j=0}^{n} \bigvee_{i=j}^{\infty} \Sigma^{8i+4j} B_1(j) \]  

(8)

Notationally, it will be convenient to let \( B(j) = \Sigma^{12j} B_1(j) \), so that the filtration (8) can be rewritten as

\[ \Omega^n = \bigvee_{j=0}^{n} \bigvee_{i \geq j}^{\infty} \Sigma^{8i} B(j). \]  

(9)

The proof of Theorem 1.1 will proceed inductively on \( n \). We will assume the existance of a bo-module map \( \varrho_{2i-1} : bo \wedge \Omega^{2i-1} \to bo \wedge \text{tmf} \) which is a stable
A-isomorphism through a certain dimension. The inductive step will be then to construct a bo-module map \( \varphi_{2^i-1} : \text{bo} \wedge \Omega^{2^i-1} \to \text{bo} \wedge \text{tmf} \) which is a stable \( A \)-isomorphism through higher dimensions. To do this, we will employ the pairings given in Theorem 2.5(ii). Define the map

\[
g_{m,n} : \Sigma^{8n} B(m) \to \text{bo} \wedge \text{tmf}
\]  

(10)
to be the restriction of \( \varphi_{2^i-1} \) to the summand \( \Sigma^{8n} B(m) \). Denote by \( g_m = g_{m,0} \).

**Lemma 3.2.** Let \( m = 2^i \) and \( 0 \leq n < m \). Suppose there are bo-module maps \( f_m : \text{bo} \wedge B_1(m) \to \text{bo} \wedge \text{tmf} \) and \( f_n : \text{bo} \wedge B_1(n) \to \text{bo} \wedge \text{tmf} \) inducing injections on homology. Then there is a bo-module map

\[
f_{m+n} : \text{bo} \wedge B_1(m + n) \to \text{bo} \wedge \text{tmf}
\]

inducing an injection on homology.

**Proof.** For all \( 0 \leq n < m \), Theorem 2.7 supplies equivalences of bo-module spectra

\[
\text{bo} \wedge B_1(m) \wedge B_1(n) \simeq (\text{bo} \wedge B_1(m + n)) \vee K
\]

(11)
where \( K \) is a wedge of suspensions of \( HF_2 \). There are no maps \([HF_2, \text{bo} \wedge \text{tmf}]\) so that the composite \( m \circ (f_m \wedge f_n) \) lifts as a bo-module map to the first summand

\[
f_{m+n} : \text{bo} \wedge B_1(m + n) \to \text{bo} \wedge \text{tmf}
\]
hence is also an injection in homology. \( \square \)

**Corollary 3.3.** Suppose there are bo-module maps \( \varphi_{2^i-1} : \text{bo} \wedge \Omega^{2^i-1} \to \text{bo} \wedge \text{tmf} \) and \( g_{2^i} : \text{bo} \wedge B(2^i) \to \text{bo} \wedge \text{tmf} \) inducing injections on homology. Then there is a bo-module map

\[
\varphi_{2^{i+1}-1} : \text{bo} \wedge \Omega^{2^{i+1}-1} \to \text{bo} \wedge \text{tmf}
\]

inducing an injection on homology groups.

**Proof.** For \( 0 \leq m \leq 2^i - 1 \) and \( n \geq 0 \), there are bo-module maps \( g_{2^i+m,n} \) inducing an injection in homology. These maps are obtained by applying Lemma 3.2 to \( g_{2^i} \) and the restriction of \( \varphi_{2^i-1} \) to the summand

\[
g_{m,n} : \text{bo} \wedge \Sigma^{8n} B(m) \to \text{bo} \wedge \text{tmf}
\]
The map \( \varphi_{2^{i+1}-1} \) is the wedge of these maps. \( \square \)
The following observation will simplify our calculations inside the Adams spectral sequence.

Lemma 3.4. Let $X$ and $Y$ be spectra. Suppose $\mathcal{F} : \mathcal{b}o \wedge X \to \mathcal{b}o \wedge Y$ is given by the composite $(m_{\mathcal{b}o} \wedge Y) \circ (\mathcal{b}o \wedge f)$ for some map $f : X \to \mathcal{b}o \wedge Y$. Then $\mathcal{F}_*(r \cdot x) = r \cdot \mathcal{F}_*(x)$ if $r \in \mathcal{b}o_*$ and $x \in \mathcal{b}o_* X$.

Proof. By construction, the composite $\mathcal{F}$ is a $\mathcal{b}o$-module map. 

In particular, the $\mathcal{b}o$-module map $\varrho_{2^i - 1}$ constructed in Lemma 3.2 induces a map in homotopy groups in Adams filtration $s = 0$. The above lemma allows us to apply the $\mathcal{b}o_*$-module structure to extend the morphism into positive Adams filtrations. To complete the inductive step it suffices to construct a map $g_{2^i} : B(2^i) \to \mathcal{b}o \wedge \text{tmf}$ inducing an injection on homology. Indeed, we can then apply Corollary 3.3 to extend $\varrho_{2^i - 1}$ to a $\mathcal{b}o$-module map $\varrho_{2^i + 1} : \mathcal{b}o \wedge \Omega^{2^i + 1} \to \mathcal{b}o \wedge \text{tmf}$.

To construct $g_{2^i}$, we will use $g_{2^i - 1}$ supplied by the inductive hypothesis. Once again we will attempt to use the pairing of integral Brown-Gitler spectra:

$$B_1(2^i - 1) \wedge B_1(2^i) \to B_1(2^i)$$

(12)

to construct a map $\mathcal{b}o \wedge B_1(2^i) \to \mathcal{b}o \wedge \text{tmf}$. Unfortunately, Lemma 3.2 will not apply. Indeed, the above pairings (12) are not surjective in homology since the element corresponding to $\zeta_{i+3}$ inside $H_* B_1(2^i)$ is indecomposable.

To handle this case, we turn to a lemma of Mahowald [6] made precise by Davis [3]:

Lemma 3.5 (Davis [3]). If $n$ is a power of 2, let $F_n = \Sigma^{8n-5} M_2 \wedge B_1(1)$. There is a map $F_n \to \mathcal{b}o \wedge B_1(n) \wedge B_1(n)$ such that the cofibre of the composite

$$\delta : \mathcal{b}o \wedge F_n \xrightarrow{1 \wedge \alpha} \mathcal{b}o \wedge \mathcal{b}o \wedge B_1(n) \wedge B_1(n) \xrightarrow{m_{\mathcal{b}o} \wedge 1 \wedge 1} \mathcal{b}o \wedge B_1(n) \wedge B_1(n)$$

is equivalent modulo suspensions of $H F_2$ to $\mathcal{b}o \wedge B_1(2n)$.

Define $m_{i-1} : \mathcal{b}o \wedge B(2^i - 1) \wedge B(2^i - 1) \to \mathcal{b}o \wedge \text{tmf}$ to be the $\mathcal{b}o$-module map induced by the composite $m \circ (g_{2^i - 1} \wedge g_{2^i - 1})$. With Lemma 3.5 in mind, consider the diagram:

$$\begin{array}{c}
\mathcal{b}o \wedge \Sigma^{2^i + 5} M_2 \wedge B_1(1) \\
\xrightarrow{\delta} \mathcal{b}o \wedge B(2^i - 1) \wedge B(2^i - 1) \\
\xrightarrow{m_{i-1}} \mathcal{b}o \wedge \text{tmf}
\end{array}$$
It suffices to show the composite \( m_{i-1} \delta \) is nullhomotopic, since then \( m_{i-1} \) would then extend to the desired map \( g_{2i} \). The following theorem is essentially due to Davis [3, Prop. 2.8], however modified to our context.

**Theorem 3.6** (Davis, [3]). Suppose \( g_{2i-1} : bo \wedge B(2^{i-1}) \to bo \wedge \text{tmf} \) induces an injection on homology. Then

\[
\pi_{2i+4-4} \left( bo \wedge B(2^{i-1}) \wedge B(2^{i-1}) \right) \cong \mathbb{Z}_2
\]

(13)

with generator \( \alpha_{2i+4-4} \) whose image under \( (m_{i-1})_* \) is divisible by 2.

**Proof.** Since \( bo \wedge \text{tmf} \) has the structure of an \( E_\infty \)-ring spectrum, the map \( m_{i-1} \) factors through the quadratic construction on \( B(2^{i-1}) \), i.e., there is a map \( j \) making the following diagram commute:

\[
\begin{array}{ccc}
bo \wedge D_2(B(2^{i-1})) & \to & bo \wedge \text{tmf} \\
\downarrow & & \downarrow \\
bo \wedge B(2^{i-1}) \wedge B(2^{i-1}) & \xrightarrow{m_{i-1}} & bo \wedge \text{tmf}
\end{array}
\]

where

\[ D_2(B(2^{i-1})) = S^1 \ast_{\Sigma_2} (B(2^{i-1}) \wedge B(2^{i-1})). \]

Here the \( \Sigma_2 \)-action on \( S^1 \) is the antipode and the action on the smash product interchanges factors. Using this factorization, it suffices to show that the induced map \( j_* \) in homotopy sends the generator in dimension \( 2i+4-4 \) to twice an element of the homotopy of the quadratic construction. This is proved by Davis [3].

**Proof that Theorem 3.6 implies Theorem 1.1.** Let \( [x_i] \in \pi_i(bo \wedge \text{tmf}) \) for \( i = 0, 8, 12 \) denote the classes in bidegree \((i, 0)\) in the \( E_2 \)-term displayed in Figure [1]. The class \([x_{12}]\) does not support action by \( \eta \) so that \( x_{12} \) extends to a map \( B(1) \to bo \wedge \text{tmf} \). Upon smashing with \( bo \), we get maps

\[
\begin{align*}
g_0 : bo \wedge B(0) & \to bo \wedge \text{tmf} \\
g_{0,1} : \Sigma^8 bo \wedge B(0) & \to bo \wedge \text{tmf} \\
g_1 : bo \wedge B(1) & \to bo \wedge \text{tmf}
\end{align*}
\]

inducing injections in homology. In particular, Lemma [3.2] extends these to a \( bo \)-module map \( g_1 : bo \wedge \Omega^1 \to bo \wedge \text{tmf} \) which is also an injection on homology.
Lemma 3.4 extends this morphism to positive Adams filtrations. Figure 1 demonstrates that modulo possible order 2 elements on the zero line, this map accounts for all homotopy classes through the 23-stem. Hence, it is a stable $\mathcal{A}$-equivalence in this range.

For the purpose of induction, assume the existence of a $\text{bo}$-module map $\varrho_{2^i-1} : \text{bo} \wedge \Omega^{2^i-1} \to \text{bo} \wedge \text{tmf}$ inducing a stable $\mathcal{A}$-equivalence through the $(12(2^i)-1)$-stem. In particular, there is a map $g_{2^i-1} : \text{bo} \wedge B(2^i-1) \to \text{bo} \wedge \text{tmf}$ of $\text{bo}$-module spectra inducing an injection on homology groups. Define $m_{i-1}$ and $\delta$ as above. We will show $m_{i-1}\delta \simeq \ast$.

![Diagram](image)

Figure 2: $H^*(M_{2^i} \wedge B_1(1))$

Figure 2 shows the cell diagram for $H^*(M_{2^i} \wedge B_1(1))$. Since there are no elements of positive Adams filtration in stems congruent to $\{5, 6, 7\}$ mod 8 in the Adams spectral sequence converging to $\pi_*(\text{bo} \wedge \text{tmf})$, the composite $m_{i-1}\delta$ restricts to a map $\Sigma^{2^i+4-5}M_{2^i} \to \text{bo} \wedge \text{tmf}$. Consider the composite

$$S^{2^i+4-5} \xrightarrow{a_0} \Sigma^{2^i+4-5}M_{2^i} \wedge B_1(1) \xrightarrow{m_{i-1}\delta} \text{bo} \wedge \text{tmf}$$

restricting $m_{i-1}\delta$ to the bottom cell of $\Sigma^{2^i+4-5}M_{2^i}$. There are no elements of positive Adams filtration in stems congruent to 3 mod 8 so this restriction extends to the top cell

$$S^{2^i+4-4} \xrightarrow{a_1} \Sigma^{2^i+4-5}M_{2^i} \wedge B_1(1) \xrightarrow{m_{i-1}\delta} \text{bo} \wedge \text{tmf}.$$

Theorem 3.6 indicates that the class $(m_{i-1})_!(\delta a_1)$ is divisible by 2. Hence, this map is nullhomotopic. Applying Corollary 3.3 gives the result. \qed

### 4 The $\textbf{bo}$-homology of tmf

Both $\text{bo}$ and tmf have the structure of $E_{\infty}$-ring spectra, so that the smash product $\text{bo} \wedge \text{tmf}$ also inherits such a structure. The splitting of $\text{bo} \wedge \text{tmf}$ into
pieces involving integral Brown-Gitler spectra gives a nice description of its structure as a ring spectrum. Indeed, the pairing of the $B_1(j)$ is compatible with multiplication inside $H_\ast \mathbb{H} \mathbb{Z}$ of which $H_\ast \mathfrak{tmf}$ is a subring. In particular, the pairings of the integral Brown-Gitler spectra induce the ring structure of $b_0 \wedge \mathfrak{tmf}$. The induced structure on homotopy groups is given by the following theorem:

**Theorem 4.1.** There is an isomorphism of graded $b_0$-algebras

$$
\pi_\ast (b_0 \wedge \mathfrak{tmf}) \cong \frac{b_0 \{\sigma, b_i, \mu_i \mid i \geq 0\}}{(\mu b_i^2 - 8b_{i+1}, \mu b_i - 4\mu_i, \eta b_i)} \oplus F
$$

where $|\sigma| = 8$, $|b_i| = 2^{i+4} - 4$, $|\mu_i| = 2^{i+4}$ and $F$ is a direct sum of $\mathbb{F}_2$ in varying dimensions.

**Proof.** Theorem [2.7](#) gives homotopy equivalences

$$
b_0 \wedge B(n) \wedge B(2^i) \to K \vee (b_0 \wedge B(n + 2^i))
$$

for all $n < 2^i$. In particular, the induced pairings

$$
\pi_\ast (b_0 \wedge B(n)) \otimes \pi_\ast (b_0 \wedge B(2^i)) \to \pi_\ast (b_0 \wedge B(n + 2^i))
$$

provide an isomorphism for all $n < 2^i$, modulo possible order 2 elements in Adams filtration zero corresponding to free $\mathcal{A}(1)$ inside $H^\ast \mathfrak{tmf}$. Therefore, the homotopy classes inside $b_0$, $\Sigma^8 b_0$ and $b_0 \wedge B(2^i)$ for $i \geq 0$ generate the homotopy of $\pi_\ast (b_0 \wedge \mathfrak{tmf})$. Hence, it suffices to examine the pairings

$$
b_0 \wedge B(2^i) \wedge B(2^i) \to b_0 \wedge B(2^{i+1}).
$$

Figure [3](#) depicts the $E_2$-term of the Adams spectral sequence converging to $b_0 \mathcal{B}(1)$ along with its generators as a $b_0$-module. With these generators, we can determine the decomposibles inside $b_0 \mathcal{B}(2^i)$. Indeed, Lemma [3.5](#) provides us with a fiber sequence

$$
b_0 \wedge B(2^i) \wedge B(2^i) \to b_0 \wedge B(2^{i+1}) \to b_0 \wedge \Sigma^{2^{i+5}} M_{2^i} \wedge B_1(1)
$$

(15)

inducing a long exact sequence of Ext-groups. Figure [4](#) shows how to use (15) to form the $E_2$-page of $b_0 \wedge B(2^{i+1})$. The arrows represent subsequent differentials and the dotted lines non-trivial extensions. The classes in black are those contributed by $b_0 \wedge B(2^i) \wedge B(2^i)$, i.e., the decomposable classes
hit by multiplication by elements in the summand \( bo \wedge B(2^i) \). Those in red (or grey) are contributed by \( bo \wedge \Sigma^{2i+5-4} M_{2t} \wedge B_1(1) \). Denote by \( b_{i+1} \) the class found in bidegree \( (2i+5-4, 0) \) and \( \mu_{i+1} \) the class in \( (2i+4, 1) \). These two elements are thus indecomposable in the ring \( \pi_*(bo \wedge \text{tmf}) \).

Note that the class in bidegree \( (2i+5-8, 0) \) corresponds to the element \( b_i^2 \). In particular, \( \mu b_i^2 = 8b_{i+1} \). Also note that \( \mu b_{i+1} = 4\mu_{i+1} \) and \( \eta b_{i+1} = 0 \).

Remark 4.1. The splitting of \( bo \wedge \text{tmf} \) can also be used to give a description of the \( bo \)-cohomology of \( \text{tmf} \). Indeed, Lemma 3.1 gives that \( [\text{tmf}, bo] = [bo \wedge \text{tmf}, bo]_{bo} \). Since Theorem 1.1 provides a splitting as \( bo \)-module spectra,
one has the following chain of equivalences of $bo^*$-comodules:

\[
bo^*\text{tmf} = [\text{tmf}, bo] = [bo \wedge \text{tmf}, bo]_{bo}
\]

\[
= \left[ \bigvee_{m,n \geq 0} \Sigma^{8n} bo \wedge B(m), bo \right]_{bo}
\]

\[
= \left[ \bigvee_{m,n \geq 0} \Sigma^{8n} B(m), bo \right]
\]

\[
= \bigoplus_{m,n \geq 0} \Sigma^{-8n} bo^* B(m)
\]

A complete description of the summands $bo^* B(m)$ is given by Carlsson [1]. The comultiplication on $bo^* \text{tmf}$ is once again induced by the pairings of integral Brown-Gitler spectra. It would be interesting to determine the explicit generators and relations as a $bo^*$-coalgebra.

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