1. Introduction

In 1952, Carlitz [1] introduced the definition of equidistribution modulo 1 in the formal power series case which reveals profitable; it uses Weyl’s criterion [1], the generalisation of van der Corput inequality by Dijksma [2], and the theorem of Koksma by Mathan [3].

Car in [4], inspired by equidistribution modulo 1 of the sequence \((n^a)_{n \in \mathbb{N}}\) where \(0 < \alpha < 1\), characterised equidistribution modulo 1 of the sequences \((L_n^{(1)})\) and \((P_n^{(1)})\), where \(L\) describes the sequence of irreducible polynomials in \(F_q[X]\) (resp. \(P\) describes the sequence of irreducible polynomials in \(F_q[X]\)) with an \(l\)-th root \((L_n^{(1)})\) (resp. \((P_n^{(1)})\)) in the field of formal power series.

In 2013, Mauduit and Car studied in [5] the \(Q\)-automaticity of the set of \(k\)-th power of polynomials in \(F_q[X]\). Moreover, they calculated the number of polynomials \(K \in F_q[X]\) with degree \(N\) such that the sum of digits of \(K^k\) in base \(Q\) is fixed. In the same subject, Madritsch and Thuswaldner in [6] called the maps \(f : F_q[X] \rightarrow G\), where \(G\) is the group of \(Q\)-additives satisfying \(f(AQ + B) = f(A) + f(B)\) for all polynomials \(A, B \in F_q[X]\) with \(\deg(B) < \deg(Q)\). They proved the equidistribution of the sequence \(h(W_i)\), where \(h \in F_q((X^{-1}))\) is a polynomial with coefficients in the field of formal power series and \((W_i)\) is an ordered sequence of polynomials in \(\mathcal{E}(f) = \{A \in L_\mathbb{N} : f(A) \equiv f \mod M\}\) if and only if one of the coefficients of \(h(Y) - h(0)\) is irrational.

In this article, we are interested in the subsequences \((L_n^{(1)})\) of \((L_n)\) and \((P_n^{(1)})\) of \((P_n)\) of polynomials in arithmetic progression having an \(l\)-th root. We will prove that the sequences \((L_n^{(1/l)})\) and \((P_n^{(1/l)})\) are equidistributed modulo 1.

2. Preliminary

Let \(F_q\) be a finite field of characteristic \(p\) with \(q\) elements. We consider \(F_q[X], F_q(X),\) and \(F_q((X^{-1}))\) as analogues of \(\mathbb{Z}, \mathbb{Q},\) and \(\mathbb{R}\), respectively.

An element \(f \in F_q((X^{-1}))\) is of the form \(f = \sum_{i=0}^{\infty} a_i X^{-i}\), with \(a_i \in F_q\) \(n_i \in \mathbb{Z}\), and \(a_{n_0} \neq 0\). We define \(v(f) = \deg(f) = -n_0\) and \(|f| = q^{\deg(f)}\). We note \(\lfloor f \rfloor\) the polynomial part of \(f\) and \(\{f\}\) its fractional part. Let \(\text{Res}(f) = a_i\) if \(f \neq 0\), and \(\text{sgn}(f) = a_{n_0}\). Let \(\psi : F_q \rightarrow \mathbb{C}\) be a nontrivial additive character. For all \(f \in F_q((X^{-1}))\), we suppose that \(E(f) = \psi(\text{Res}(f))\).

Let \(l\) be a positive integer >2 which is not divisible by the characteristic \(p\) of the field \(F_q\). We introduce \(\mathcal{L} = \{a_1, \ldots, a_r\}\) as the set of the \(r\)-th elements having an \(l\)-th root in \(F_q^\times\), and we have

\[
    r = \frac{q - 1}{(l, q - 1)} \tag{1}
\]

Then, for \(f, g \in F_q^\times((X^{-1}))\), \(g\) is called an \(l\)-th root of \(f\); we note \(f = g^f\) if and only if \(\nu(f) \equiv 0 \mod l\) and
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(see [1]; Weyl’s criterion)

Lemma 1. We denote by \( L \) the set of polynomials with an \( l \)-th root in \( F_q((X^{-1})) \):

\[ L = \{ A \in F_q[X] \setminus \{0\} : \deg(A) \equiv 0 \mod l \text{ and } \text{sgn}(A) \in \mathcal{L} \}, \]

and if \( \mathcal{I} \) is the set of irreducible polynomials over \( F_q[X] \), we define \( \mathcal{P} = \mathcal{I} \cap \mathcal{I} \).

If \( n = \sum_{i=1}^{\infty} n_i q^i \), where \( n_i \in \{0, \ldots, q - 1\} \) for all \( i \in \{0, \ldots, s\} \), is the representation in base \( q \) of the integer \( n \geq 1 \), then let

\[ H_n = \chi_{n_1} + \cdots + \chi_{n_s} X^s, \]

where \( \chi_{n_i} \) are given by the bijection \( n_i \mapsto \chi_{n_i} \) from \( \{0, \ldots, q - 1\} \) to \( F_q \). For \( n = 0 \) and 1, it is convenient to suppose that \( \chi_0 = 0 \) and \( \chi_1 = 1 \). We define the order in \( F_q \) by \( \chi_{n_i} < \chi_{n_i+1} \), for all \( n_i \in \{0, \ldots, q - 1\} \), and in \( \mathbb{Z} \) by

\[ m < n \iff \deg(H_n) < \deg(H_m). \]

Then, we order \( F_q \) by posing for all natural numbers \( n \):

\[ H_n < H_{n+1}. \]

This paper is devoted to the study of equidistribution modulo 1 of a certain sequence in the field of Laurent formal power series. In 1952, Carlitz introduced and characterised equidistribution modulo 1 in the field of Laurent formal power series and obtained the following result.

Lemma 1 (see [1]; Weyl’s criterion). For sequence \( \Theta = (\theta_n) \) with values in \( F_q((X^{-1})) \) is equidistributed modulo 1 if and only if

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} E(H\theta_n) = 0, \]

for all \( H \in F_q[X]^* \).

Finally, we enounce a result which concerns a class of irreducible polynomials given by Artin in [7], which will be very useful later.

Theorem 1 (see [7]). Let \( C, B \in F_q[X] \) be coprime polynomials. If \( \pi(m; C, B) \) denotes the number of monic irreducible polynomials with degree \( m \) which are congruent to \( B \) modulo \( C \), then

\[ \pi(m; C, B) = \frac{1}{\Phi(C)} \cdot \frac{q^m}{m} + O \left( \frac{q^m}{m} \right), \]

where \( \theta \) is a constant \(< 1 \). This theorem is analogous to the theorem of prime numbers in arithmetic progression.

3. Results

Let \( l \geq 2 \) be an integer nondivisible by the characteristic \( p \) of the field \( F_q \), we order the set of the \( l \)-th powers of \( \mathcal{L} \) under

the increasing order of \( F_q \), and we fix a polynomial \( C \) with degree \( c \). For all \( B \in F_q[X] \), we denote by \( L' \) the subset of \( L \) defined in (2),

\[ L' = L_{C,B} = \{ A \in L : A \equiv B \mod C \}, \]

and \( \mathcal{P}' \) the subset of \( \mathcal{P} = \mathcal{I} \cap \mathcal{I} \):

\[ \mathcal{P}' = \mathcal{P}_{C,B} = \{ A \in \mathcal{I} \cap \mathcal{I} : A \equiv B \mod C \text{ with } (B, C) = 1 \}. \]

We ordered the elements of \( L' \) and \( \mathcal{P}' \) with the order relation defined in (2); hence,

\[ L' = \{ L'_1, \ldots, L'_{s'} \} \] and \( \mathcal{P}' = \{ P'_1, \ldots, P'_{s'} \} \).

The aim of this paper is to prove the following theorems.

Theorem 2. Let \( (L'_n) \) be the sequence of polynomials of \( L' \) indexed under the increasing order of \( F_q[X] \). Then, for \( l \geq 2 \), the sequence \((L_{n^{(l)}})\) is equidistributed modulo 1.

Theorem 3. Let \( (P'_n) \) be the sequence of polynomials \( \mathcal{P}' \) indexed under the increasing order of \( F_q[X] \). Then, the sequence \((P_{n^{(l)}})\) is equidistributed modulo 1 for \( \ell > (1/(1 - \theta)) \), and \( \theta \) is a constant defined in Theorem 1. In particular, if \( X^q - X \) does not divide \( C \), then let \( l \geq 3 \).

4. Proofs of Theorems 2 and 3

4.1. Tools. A generalisation of Theorem 1 was proved in 1965 by Hayes introducing the arithmetic progression.

Lemma 2 (see [8]). Let \( C \in F_q[X] \) be a polynomial with degree \( c \). Then, for all polynomials \( B \), \( c \) monic polynomials with degree \( m \) which are congruent to \( B \) modulo \( C \) if \( m \geq c \).

Theorem 4 (see [8]). Let \( k \geq 1 \) be a positive integer, \( u = (u_1, \ldots, u_k) \) be a sequence of \( k \) elements in \( F_q \), and \( C_b \in F_q[X] \) be coprime polynomials. If \( (m; u, C, B) \) is the number of irreducible and monic polynomials \( P \) with degree \( m \) which are congruent to \( B \) modulo \( C \) such that deg \( (P - X^m - u_1 X^{m-1} - \cdots - u_k X^{m-k} \) \( < m-k) \)

then

\[ \pi(m; u, C, B) = \frac{q^{m-k}}{m \Phi(C)} + O \left( \frac{q^m}{m} \right), \]

where \( \theta \) is a constant \(< 1 \).\]

Remark 1. In particular, if \( X^q - X \) does not divide \( C \), then (11) is verified for \( \theta = (1/2) \).

The proofs of Theorems 2 and 3 are based on Corollary 1 whose proof needs the following lemmas.

Lemma 3 (see [9], Lemma II.1.1). Let \( k \in \mathbb{N} \), \( H \in F_q[X]^* \) with degree \( h \), and \( A \in \mathcal{L} \) with degree \( lk \). Then, for all \( Z \in F_q[X] \) such that deg \( (Z) = z < (l - 1)k - h - 1 \), we have...
\[ (1) \ (A + Z) \in \mathcal{I} \]
\[ (2) \ \text{Res}(HA^{(1/\theta)}) = \text{Res}(H(A + Z)^{1/\theta}) \]

**Lemma 4** (see [9], Lemma II.1.2). Let \( H \in \mathbb{F}_q[X]^* \) with degree \( h \) and \( k \in \mathbb{N} \) such that \((l - 1)k \geq h\). Then, for all \( a \in \mathcal{L} \), all \( \xi \in \mathbb{F}_q^* \), and all \( Y \in \mathbb{F}_q[X] \) with degree \( < k + h \), there exists unique \( \eta = \eta(a, Y) \in \mathbb{F}_q^* \) such that, for all \( Z \in \mathbb{F}_q[X] \) with degree \( < (l - 1)k - h - 1 \), we obtain
\[
\xi = \text{Res}\left(H(aX^{lk} + YX^{(l-1)k-h} + \eta X^{(l-1)k-h-1} + Z)^{(1/\theta)}\right).
\]
(12)

**Corollary 1.** Let \( H \in \mathbb{F}_q[X]^* \) with degree \( h \) and \( k \in \mathbb{N} \) such that \((l - 1)k \geq h\). For all \( a \in \mathcal{L} \), we have

(i) \[ \sum_{A \in \mathcal{A}} \psi(\text{Res}(HA^{(1/\theta)})) = 0, \quad \text{where } \mathcal{A} = A \in \mathcal{L}, \deg(A) = lk, \text{ and } \text{sgn}(A) = a \]

(ii) \[ \sum_{A \in \mathcal{A}} \psi(\text{Res}(HA^{(1/\theta)})) = O(q^{(lk)h+k+h}/lk), \quad \text{where } \mathcal{A} = A \in \mathcal{P}, \deg(A) = lk, \text{ and } \text{sgn}(A) = a, \quad \theta \text{ is a constant } < 1 \]

**Proof.** For \( H = \mathcal{A} \) or \( \mathcal{F} \), we note
\[
\sigma(\mathcal{A}; k, a) = \sum_{A \in \mathcal{A}} \psi(\text{Res}(HA^{(1/\theta)})) = \sum_{\xi \in \mathbb{F}_q} \psi(\xi)\pi(\xi),
\]
(13)
where \( \pi(\xi) \) is the number of polynomials \( A \in \mathcal{A} \) such that \( \text{Res}(HA^{(1/\theta)}) = \xi \), but with Lemma 4, for \( Y \in \mathbb{F}_q[X] \) with degree \( < k + h \), there exists \( \eta \in \mathbb{F}_q^* \) such that the polynomial
\[ K = K(a, Y, \xi) = aX^{lk} + YX^{(l-1)k-h} + \eta X^{(l-1)k-h}, \]
(14)
satisfying
\[ \text{Res}(HK^{(1/\theta)}) = \xi. \]
(15)

Let \( Z = A - K \); we denote by \( \pi(lk; Y, \xi) \) the number of polynomials \( A \in \mathcal{A} \) such that
\[ \deg(Z) < (l - 1)k - h - 1. \]
(16)

We obtain
\[
\sigma(\mathcal{A}; k, a) = \sum_{\xi \in \mathbb{F}_q} \psi(\xi) \sum_{\{Y \in \mathbb{F}_q[X] \mid \deg(Y) < k+h\}} \pi(lk; Y, \xi).
\]
(17)

(i) If \( \mathcal{A} = \mathcal{A} \), then by Lemma 2, we have \( \pi(lk; Y, \xi) = q^{(l-1)k-h-c-1} - 1 \). With the orthogonality criterion of \( \psi \), it results in
\[
\sigma(L'; k, a) = q^{k+h} \left( q^{(l-1)k-h-c-1} - 1 \right) \sum_{\xi \in \mathbb{F}_q} \psi(\xi) = 0.
\]
(18)

(ii) If \( \mathcal{A} = \mathcal{F} \), then by Theorem 4, we have
\[
\pi(lk; Y, \xi) = q^{(l-1)k-h-1} + O\left( \frac{q^{k+h}}{lk} \right), \quad \theta < 1.
\]
(19)

We deduce that
\[
\sigma(P'; k, a) = \frac{q^{k-1}}{lk\Phi(C)} \sum_{\xi \in \mathbb{F}_q} \psi(\xi) + \sum_{\xi \in \mathbb{F}_q} O\left( \frac{q^{k+h+c}}{lk} \right).
\]
(20)

Finally, with the orthogonality criterion, we obtain
\[
\sigma(P'; k, a) = O\left( \frac{q^{k+h+c}}{lk} \right), \quad \text{with } \theta < 1.
\]
(21)

4.2. Proof of Theorem 2. In \( \mathbb{F}_q[X] \), there are \( q^{m+c} \) monic polynomials which are congruent to \( B \) modulo \( C \) with degree \( m \), and let \( c = \deg(C) \). We denote by \( a_m \) (resp. \( b_m \)) the number of polynomials in \( \mathcal{L} \) with degree \( lm \) (resp. \( \leq lm \)). It is sufficient to verify that
\[
a_m = r q^{m-c},
\]
(22)
\[
b_m = a_1 + \ldots + a_m = \frac{r(q^{m+1-c} - q^{l-c})}{q - 1},
\]
where \( r \) is defined in (1). Let \( H \in \mathbb{F}_q[X]^* \) with degree \( h \), and \( N \) is an integer such that
\[ N > b_{\lfloor 1+(h+1)/(l-1) \rfloor}, \]
(23)
where \( \lfloor x \rfloor \) defines the least integer \( \geq x \). The sequence \( (b_m) \) is strictly increasing, and there exists a unique integer \( t \) such that
\[ b_{t-1} \leq N < b_t. \]
(24)

Moreover, there exists a unique integer \( s \in 0, \ldots, r - 1 \), such that
\[ b_{t-1} + s q^{l-1} \leq N < b_{t-1} + (s + 1) q^h. \]
(25)

Let
\[ W(N) = \sum_{n=1}^{N} E\left(H^{(1/\theta)}\right). \]
(26)

To prove Theorem 2, we have to show that
\[ \lim_{N \to \infty} \frac{1}{N} |W(N)| = 0. \]
(27)

Using relations (24) and (25), we rewrite the sum \( W(N) \) to obtain
\[ W(N) = W_1 + W_2 + W_3, \]
(28)
with
\[ W_1 = \sum_{n=1}^{b_{-1}} E\left( H_{n}^{(*)}\right). \]
\[ W_2 = \sum_{j=0}^{t-1} \sum_{n=0}^{b_{-1}+j+1} \sum_{l=0}^{p} E\left( H_{n}^{(*)}\right). \]
\[ W_3 = \sum_{n=b_{-1}+1}^{N} E\left( H_{n}^{(*)}\right). \]

We start by giving an estimation of the sum \( W_1 \) which concerns the polynomials of \( L^1 \) with degree \( \leq (t-1) \). We have

\[ W_1 = \sum_{k=1}^{b_{-1}} \sum_{L \leq \mathcal{L}(\mathcal{L})} E\left( H_{n}^{(*)}\right). \]

\[ = \sum_{k=1}^{b_{-1}} \sum_{L \leq \mathcal{L}(\mathcal{L})} E\left( H_{n}^{(*)}\right) + \sum_{k=1}^{t-1} \sum_{L \leq \mathcal{L}(\mathcal{L})} E\left( H_{n}^{(*)}\right). \]

We have to just major the first part of the sum by the number of polynomials with degree \( < ((l(h+c))/(l-1)) \), and we apply Corollary 1 on the second part to obtain

\[ |W_1| \leq q^{((l(h+c))/(l-1))}. \]

We apply the same Corollary 1 on \( W_2 \) that represents the sum of polynomials with degree \( lt \) and with \( a_1, \ldots, a_{l-1} \), then

\[ W_2 = \sum_{j=1}^{t-1} \sum_{L \leq \mathcal{L}(\mathcal{L})} E\left( H_{n}^{(*)}\right) = 0. \]

The polynomials in \( W_3 \) can be written in the form

\[ L_i = a_i X^l + H_{n_i}, \quad j \leq i \leq N, \]

where \( j = 1 + b_{-1} + sa_1 \) and the sequence \( (H_{n_i})_{j \leq i \leq N} \) is strictly increasing in \( F_q[X] \).

By the order relation on \( \mathcal{F}_q[X] \) (4), if

\[ n_N = c_0 + c_1 q + \cdots + c_m q^m \]

and is the presentation in base \( q \) of the integer \( n_N \), we have

\[ H_{n_N} = \chi_{c_0} + \chi_{c_1} X + \cdots + \chi_{c_m} X^m. \]

To estimate \( W_3 \), we will distinguish two cases: when the degree of \( H_{n_N} \) is up to the integer \( (l-1)t - h - c - 1 \) and when it is not:

1st case: \( m \leq (l-1)t - h - c - 1 \). Using (34) and the fact that the sequence \( (n_i)_{j \leq i \leq N} \) is strictly increasing, we obtain

\[ N - j \leq n_N - n_j \leq q^{m+1} - 1. \]

Thus,

\[ |W_3| \leq N - j + 1 \leq q^{((l-1)t - h - c - 1 - 1}). \]

2nd case: \( m > (l-1)t - h - c - 1 \). The polynomials \( H_{n_i} \) defined in (33) are of the form

\[ H_{n_i} = y_n + y_0 X + \cdots + y_m X^m \leq H_{n_i} = \chi_{c_0} + \chi_{c_1} X + \cdots + \chi_{c_m} X^m. \]

Let \( \mathcal{Y} \) be the set of polynomials of the form

\[ Y = y^{(l-1)t - h - c} + y_{(l-1)t - h - c+1} X + \cdots + y_m X^{m-(l-1)t + h + c} \]

such that, for every polynomial \( Z \) with degree \( < (l-1)t - h - c \), we have

\[ YX^{(l-1)t - h - c} + Z \leq H_{n_N} \]

If \( k \) is the greatest integer \( i \in j, \ldots, N \) for which \( L_i \) are written in the form

\[ L_i = a_i X^l + YX^{(l-1)t - h - c} + Z, \]

with \( Y \in \mathcal{Y} \) and \( Z \) being a polynomial with degree \( < (l-1)t - h - c \), then we rewrite the sum \( W_3 \):

\[ W_3 = W_4 + W_5, \]

with

\[ W_4 = \sum_{j=k+1}^{N} E\left( H_{n_j}^{(*)}\right) \]

\[ W_5 = \sum_{j=k+1}^{N} E\left( H_{n_j}^{(*)}\right). \]

We have

\[ W_3 = \sum_{\xi \in \mathcal{F}_q} \psi(\xi) \pi(\xi), \]

where \( \pi(\xi) \) is the number of couples \( (Y, Z) \) such that \( Y \in \mathcal{Y} \), \( Z \in \mathcal{F}_q[X] \) with degree \( < (l-1)t - h - c \), and \( \text{Res}(H(a_i X^l + YX^{(l-1)t - h - c} + Z)^{((l-1)t - h - c)}) = \xi \). Moreover, we have

\[ \pi(\xi) = \sum_{\xi \in \mathcal{Y}} \pi(\xi). \]

where \( \pi(L; Y, \xi) \) denotes the number of polynomials \( L \in L \) such that

\[ \text{deg}(L - K) < (l-1)k - h - c - 1, \]

with

\[ K = K(a,Y,\xi) = aX^{(l-1)k - h - c - 1} + \eta X^{(l-1)k - h - c - 1}, \]

which gives the same arguments presented in the proof of Corollary 1, and then we deduce that

\[ W_3 = 0. \]
In (34), let $v$ be the least index $>(I - 1)t - h - c$ such that $c_v \neq 0$; then, we have
\[ n_N = c_m q^m + \cdots + c_0. \tag{49} \]

Since all polynomial $L'$ of the form
\[ L' = a_1 X_1^m + y_m X_m^m + \cdots + y_{v-1} X_v^m + \cdots + y_v, \]
which coefficients satisfy the condition:
\[ y_m \leq X_m, y_{m-1} \leq X_m, \ldots, y_v \leq X_v, y' \leq X_{v'}, \tag{51} \]
is less then $L'_N$, we obtain
\[ n_k \geq c_m q^m + \cdots + c_{v-1} q^{v-1} + (q - 1) q^v + \cdots + (q - 1), \]
\[ n_k \geq (I - 1)t - h - c, \tag{52} \]
which leads to
\[ |W_5| \leq N - k \leq n_N - n_k \leq q^{(I - 1)t - h - c}. \tag{53} \]

Then, with (48) and (53), it results in
\[ |W_3| \leq q^{(I - 1)t - h - c}. \tag{54} \]

With (31), (32), and (54), we obtain
\[ \left| \frac{1}{N} \sum_{n=1}^{N} E(HL'_{(1)}) \right| \leq \frac{1}{N} \left( q^{(I(h+c))(l-1)} + q^{(I-1)(t-h-c)} \right), \tag{55} \]
and finally, with (24), we obtain
\[ |W| = O\left( \frac{q^{b+h+t+1}}{l(t)} \right). \tag{62} \]

\[ \left| W'_{\text{sgn}} \right| = \sum_{n=1}^{h-1} E\left( H_{P_{n}} \right) \leq q^{(I(h+c))(l-1)} + O\left( \frac{q^{(I-1)((h+1)h+1)}}{l(h+c)} \right), \tag{61} \]
and then
\[ W'_{2} = \sum_{j=0}^{b-1} \sum_{h=b+1}^{b+j} E\left( H_{P_{j}} \right) = \sum_{j=0}^{b-1} \sum_{k=0}^{b+j} E\left( H_{P_{j}} \right). \tag{63} \]
where $W'_{2}$ is the sum defined in (42) concerning the polynomials in $P'$. Finally, we treat the sum
\[ W'_{5} = \sum_{j=k+1}^{N} E\left( H_{P_{j}} \right). \tag{64} \]

With Theorem 4, for all polynomials
\[ Y = a_1 X_1^l + y_m X_m^m + \cdots + y_v X_v^v + \cdots + y_{(I-1)t-h-c} X_{(I-1)t-h-c}, \tag{56} \]
in which coefficients satisfy condition (51), there exists
\[ \pi(l; Y, C, B) = \frac{q^{(l-1)t-h-c}}{l\Phi(C)} + O\left(\frac{q^{l\theta}}{lt}\right), \quad \theta < 1. \]  

(66)

Irreducible polynomials \( P' \) are congruent to \( B \) modulo \( C \) such that \( \deg(P' - Y) < (l-1)t - h - c \). Such polynomials \( P' \) are in \( \Pi' \), and we have

\[ n_k \geq c_n q^m + \cdots + c_{q-1} q^{t-1} + \cdots + (q-1) q^{(l-1)t-h-c} \]

\[ n_\pi - 2q^{(l-1)t-h-c} + 1. \]

(67)

Finally, from (61), (62), and (69), we have

\[ \left| W'_n \right| \leq \left| W'_n + W''_n \right| \leq 2q^{(l-1)t-h-c} + O\left(\frac{q^{l\theta t+h}}{lt}\right). \]

(69)

Then,

\[ \left| W'_n \right| \leq N - k \leq 2q^{(l-1)t-h-c} - 1. \]

(68)

With (63 and 68), it results in

\[ \left| W'_n \right| = \left| W'_n + W''_n \right| \leq 2q^{(l-1)t-h-c} + O\left(\frac{q^{l\theta t+h}}{lt}\right). \]

(69)

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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