1. INTRODUCTION

Since its theoretical prediction by Bose and Einstein [1, 2] in the 1920s until its laboratory observation with magneto-optical traps [3–6] from 1995 onwards, Bose–Einstein condensation (BEC) of dilute atomic gases has stimulated enormous efforts of related work. Among the issues addressed one finds, e.g., rigorous mathematical questions related to BEC [7], diverse theoretical and heuristic aspects [8, 9], and is now even viewed as a viable tool for precision tests in gravitational physics [10–20].

The study of its associated thermodynamic properties is naturally also a pertinent aspect of BECs [21–25]. Indeed, the condensation temperature $T_c$, i.e., the critical temperature below which a macroscopic quantum state of matter appears, has been the subject of considerable discussion, see [8, 26] and references therein. In particular, the influence of interparticle interactions on $T_c$ turns out to be a deep nontrivial matter, see, e.g., [27–29].

Interboson interactions produce a shift $\Delta T_c = (T_c - T_c^0)/T_c^0$ in the condensation temperature $T_c$ with respect to that of the ideal noninteracting case $T_c^0$ in the thermodynamic limit. For instance, the contributions to $\Delta T_c/T_c^0$ due to interactions in a uniform dilute gas originate in the fact that the associated many-body system is affected by long-range critical fluctuations rather than from purely mean-field (MF) considerations [26, 30, 31]. However, it is generally accepted that $\Delta T_c/T_c^0$ for this system behaves like $c_1 \Delta + (c_2 \ln \delta + c_2') \delta^2$, with the dimensionless variable $\delta \equiv \rho^{1/3} a$, where $\rho$ is the corresponding boson number density, $a$ the $S$-wave two-body scattering length [30] related to the pair interaction, and the $c_i$'s are dimensionless con-

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**Weakly Interacting Bose–Einstein Condensates in Temperature-Dependent Generic Traps**

*Dedicated to the loving memory father Elías Castellanos de Luna*

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We study the shift $\Delta T_c$ in the condensation temperature of an atomic Bose–Einstein condensate trapped in a temperature-dependent three-dimensional generic potential. With no assumptions other than the mean-field approach and the semiclassical approximation, it is shown that the inclusion of a $T$-dependent trap improves upon the pure semiclassical result giving better agreement between the predicted $\Delta T_c$ value and its experimental value. However, despite this improvement, the effect of a $T$-dependent trap is not sufficient to fully reduce the discrepancy between theoretical prediction and data.

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stages. A good fit [27] gives $c_1 = 1.32$, $c'_1 = 19.75$, and $c''_2 = 75.7$.

It is noteworthy that these ideas can be extended to more general traps [32–34] in which the relative shift $\Delta T_c/T_c^0$ on the condensation temperature explicitly exhibits a sensitive trap-dependence. This extension to generic traps allows summarizing the corrections on $\Delta T_c/T_c^0$ as function of a simple index parameter describing the trap shape.

On the other hand, when interactions are considered for the more common harmonic traps one finds a shift in $T_c$ up to second order in the $S$-wave scattering length $a$ within the MF approach given by [28, 29]

$$\frac{\Delta T_c}{T_c^0} = b_1(a/\lambda_{c1}^0) + b_2(a/\lambda_{c2}^0)^2,$$ (1)

where

$$k_B T_c^0 = \hbar \omega [N/\zeta(3)]^{1/3}$$ (2)

(with $\zeta(3) = 1.202$) is the condensation temperature associated with the ideal system ($a = 0$) in the thermodynamic limit [22], while $b_1 = -3.426$ [35] while $b_2 = 11.7$ [29], together with $\lambda_{c1}^0 = (2\pi \hbar^2/mk T_c^0)^{1/2}$ the thermal wavelength. Furthermore, these results seem to contrast with the results reported, e.g., in [36, 37] since, as mentioned in [28], the well-known logarithmic corrections to (1) are not discernible within the error bars.

Note that from (1) $\Delta T_c$ is negative for repulsive interactions, i.e., $a > 0$ since $b_1$ is negative. The result (1) is in excellent agreement with laboratory measurements of $\Delta T_c/T_c^0$ [29, 38–40] to first order in $(a/\lambda_{c1}^0)$ but differs somewhat with data to second order $(a/\lambda_{c1}^0)^2$. In [28], high precision measurements of the condensation temperature of the bosonic atom $^{39}$K vapor in the range of parameters $N = (2–8) \times 10^5$, $\omega = 75–85$ Hz, $10^{-3} < a/T_c^0 < 6 \times 10^{-2}$ and $T_c = 180–330$ nK have detected second-order effects in $\Delta T_c/T_c^0$. The measured $\Delta T_c/T_c^0$ is well fitted by a quadratic polynomial (1) with best-fit parameters $b_1^{\text{exp}} = -3.5 \pm 0.3$ and $b_2^{\text{exp}} = 46 \pm 5$ so that the value $b_2 = 11.7$ [29] is strongly excluded by data. This discrepancy between (1) and data may be due to beyond-MF effects (see [29]). Beyond-MF effects are expected to be important near criticality, where the physics is often nonperturbative. It would therefore seem reasonable that a beyond-MF treatment might give a correct estimation of $b_2$. However, this is not certain since beyond-MF effects have been calculated in the case of uniform condensates [37, 41] but are still poorly understood for trapped BECs [36, 42–45]. It thus seems that it is currently not possible to ascertain whether the discrepancy between $b_2$ and $b_2^{\text{exp}}$ can be explained in the MF context or arises from beyond-MF effects.

Nevertheless, the effect of interactions on the condensation temperature $T_c$ of a Bose–Einstein condensate trapped in a harmonic potential was recently discussed [35]. In the latter paper it was shown that, within the MF Hartree–Fock (HF) and semiclassical approximations, interactions among the particles produce a shift $\Delta T_c/T_c^0 = b_1(a/\lambda_{c1}^0) + b_2(a/\lambda_{c2}^0)^2 + \psi[a/\lambda_{c1}^0]$ with $\lambda_{c1}^0 = (2\pi \hbar^2/mk T_c^0)^{1/2}$ the thermal wavelength, and $\psi[a/\lambda_{c1}^0]$ a non-analytic function such that $\psi[0] = \psi'[0] = \psi''[0] = 0$ but $|\psi''[0]| = \infty$. Therefore, with only the usual assumptions of the HF and semiclassical approximations, interaction effects are perturbative to second order in $a/\lambda_{c1}^0$ and the expected nonperturbativity of physical quantities at the critical temperature emerges only at third order. Indeed, in [35] an analytical estimation for $b_2 = 18.8$ was obtained which improves the previous numerical fit-parameter value of $b_2 = 11.7$ obtained in [29]. Even so, the value for $b_2$ obtained in [35] still differs substantially from the empirical value $b_2^{\text{exp}} = 46 \pm 5$ [28].

We mention that the temperature shift $\Delta T_c/T_c^0$ induced by interparticle interactions obtained in [35] seems to contradict, for instance, the result reported in [36] where the interaction induced temperature shift is estimated as

$$\frac{\Delta T_c}{T_c^0} = b_1(a/\lambda_{c1}^0) + [b_2^* + b_2^* \ln(a/\lambda_{c1}^0)](a/\lambda_{c2}^0)^2$$ (3)

with $b_1 = -3.426$, $b_2^* = -45.86$, and $b_2^* = -155.0$ [37] (see also [27] for a discussion). This result has been obtained using lattice simulations and a technique based on a scalar field analogy, but is questionable (see discussion in [35]) besides being in striking contradiction to the data. It is thus clear that these results differ substantially from the estimations obtained in [35] and the results obtained here (see below), but also conflict with the results obtained in [29] as well as experiment [28].

Also, it was recently proposed [46] that accounting for a nonlinear quadratic Zeeman effect gives a value of $b_2$ which depends on the properties of the atomic species of the condensate, which for a $^{39}$K condensate gives a value $b_2 = 42.3$ in much better agreement with measurements obtained in [28]. However, this result is based on a physical mechanism completely different from the one considered here. Furthermore, to con-
firm whether that the quadratic Zeeman effect actually plays such an important role in the physics of atomic condensates, one should repeat the measurements performed in [28] for different atomic species and compare the results with the predictions obtained in [46]. However, to our knowledge, [28] is the only reported measurement of the nonlinear coefficient $b_2$.

We therefore propose that before addressing beyond-MF effects these facts suggest that MF effects might still be well-understood and deserve further analysis.

In fact, in a recent paper [47] the use of an effective temperature-dependent trapping potential was suggested in order to calculate the condensation temperature of noninteracting systems; see also [48] for a wide-ranging justification of $T$-dependent Hamiltonians. Hence, it might be useful to explore this idea in the context of the effects on the condensation temperature caused by interparticle interactions.

These considerations drove us into the novel terrain of $T$-dependent Hamiltonians, and more specifically to $T$-dependent trapping potentials. We note that this is not the first time that such a terrain has been reached, e.g., we find the successful use of $T$-dependent dynamics in: (a) superconductivity in the work of Bogoliubov, Zubarev, and Tserkovnikov, as mentioned by Blatt [49]; (b) an explanation [50] of the empirical law in superconductors $H_c(T) = H_c(0)[1 - (T/T_c)^2]$, where $H_c(T)$ is the critical magnetic field at $T$; (c) finite-$T$ behavior [24, 25, 51–54] of a class of relativistic field theories (RFTs) to address the question of restoration of a symmetry which at $T = 0$ is broken either dynamically or spontaneously; (d) the Wick–Cutkosky model [55] in a RFT; (e) QCD to explain [57, 58] the masses of different quarkonium families and their deconfinement temperatures; and most recently, as mentioned above, (f) in a comparative study [47] of the experimental features of the Bose–Einstein condensates in several species of bosonic atomic gases.

We thus examine the possibility of such $T$-dependent generic potentials in order to analyze (and hopefully even improve upon) the value $b_2 \simeq 18.8$ obtained in [35] within the HF MF theory, and to explore its discrepancy with the empirical value $b_2^{\text{exp}} \simeq 46 \pm 5$. Our conclusion is that, even though the inclusion of a temperature dependence in the trapping potential might improve the predicted value of $b_2$, this is not sufficient to obtain full agreement with data. We stress that we consider $T$-dependent effective potentials from a phenomenological point of view. In other words, the inclusion of such external potentials is of theoretical and/or mathematical interest, in order to analyze, for instance, the shift in the condensation temperature caused by interactions. For all this, we now entertain $T$-dependent generic traps $V(r, T)$.

2. MEAN FIELD HARTREE–FOCK APPROXIMATION

Following [35] we define the following semiclassical energy spectrum in the MF HF approximation (see, e.g., [8, 22])

$$E(p, r, g) = \epsilon(p, r) + 2gn(r, g),$$

where $\epsilon(p, r) \equiv p^2/2m + V(r)$ with $V(r)$ the external potential, $n(r, g)$ the spatial density of bosons, and $g \equiv 4\pi\hbar^2a/m$ the parameter describing the interaction.

Moreover, the semiclassical condition allows approximating summations over energy states by integrals, namely, $\sum_k \rightarrow \int d^3rd^3p/(2\pi\hbar)^3$. Therefore, the number of particles $N$ in three-dimensional space obeys the normalization condition [8, 22]

$$N = N_0 + \int d^3rd^3p/(2\pi\hbar)^3 \left\{ \exp \left[ \frac{E(p, r, g) - \mu}{k_BT} \right] - 1 \right\}^{-1},$$

where $N_0$ is the number of particles in the ground state, $\mu$ is the corresponding chemical potential, and $k_B$ is the Boltzmann constant.

At the condensation temperature $T_c$, we assume within MF theory that the chemical potential $\mu$ is given by [35]

$$\mu_c(g) = 2gn(r = 0, g).$$

Further assuming just above $T_c$ that in the ground state $N_0$ is negligible it follows that

$$N \pi \hbar^3/2 = \int d\Omega \Lambda[\theta],$$

where

$$\Lambda[\theta] = \left\{ \exp \left[ \frac{E(p, r, g) - \mu_c(g)}{k_BT_c(g)} \right] - 1 \right\}^{-1},$$

or

$$\theta = \frac{\epsilon(p, r) + 2\bar{n}(r, g)}{k_BT_c(g)}, \quad \bar{n}(r, g) = n(r, g) - n(0, g).$$

From (7) we are able to extract, in principle, $T_c$ as a function of the parameter $g$ describing interactions. Note that the scattering length $a$ can be positive or negative, its sign and magnitude depending crucially on the details of the atom–atom potential [8]. However, a negative scattering length could lead to instabilities within the system [22], and finite-size effects could be important in this situation due to the number of particles $N$ not being large enough [8]. Here, we restrict ourselves, as usual, to positive values of the interaction parameter $g$ in order to compare our results with the reported [28] experimental data.
On the other hand, if $\Delta T_c$ is analytic in $g$ one can express the relative shift in $T_c$ for small values of $g$ as follows

$$\frac{\Delta T_c}{T_c} = \frac{\sum_{h=1}^{\infty} g^h \frac{\partial^h T_c}{T_c} (g) h!}{T_c (g) \mid_{g=0}}.$$  (9)

Note that $T_c (g = 0) = T_c^0$ is by definition the $T_c$ temperature for the noninteracting system, given by (2). Additionally, the expansion coefficients can be expressed as

$$\frac{\partial^h T_c (g)}{T_c (g) \mid_{g=0}} = \frac{I_h}{(k_B T_c \Lambda_r^3)^h},$$  (10)

where the numerical factors $I_h$ depend on the external potential under consideration and can be calculated explicitly.

This enables one to reexpress (9) as a power series in the dimensionless interaction-dependent variable $a/\lambda_r$

$$\frac{\Delta T_c}{T_c} = \sum_{h=1}^{\infty} \frac{2^h I_h (a/\lambda_r)^h}{h!} = \sum_{h=1}^{\infty} b_h (a/\lambda_r)^h$$  (11)

which defines the coefficients $b_h$. For an isotropic harmonic potential $V(r) \sim r^2$ the first two factors $I_1$ and $I_2$ are given respectively by [35]

$$I_1 = 2 \int \frac{d\Sigma \lambda' [u^2 + v^2] Q(v^2)}{\int d\Sigma (u^2 + v^2) \lambda' [u^2 + v^2]},$$  (12)

$$I_2 = 4 \int \frac{d\Sigma \lambda' [u^2 + v^2] S(v^2) + \lambda'' [u^2 + v^2]}{\int d\Sigma (u^2 + v^2) \lambda' [u^2 + v^2]},$$  (13)

where $\lambda (0) = \{\exp (0) - 1\}^{-1}$, $d\Sigma \equiv dudv u^2 v^2$, $Q(\alpha) \equiv g_{3/2} (\exp (-\alpha) - g_{3/2}) [1]$, and $g_{\alpha} [z] = \sum_{k=1}^{\infty} z^k / k^\alpha$ is the so-called Bose–Einstein function [59]. Thus, $S(\alpha) \equiv \frac{3}{2} I_1 Q(\alpha) + \{\alpha I_1 - 2 Q(\alpha)\} g_{1/2} (\exp (-\alpha))$ with $\alpha \equiv \nu (r) / 2 g_{\overline{\eta}} (r, g) / k_B T_c (g)$, see [35] for details.

Note that the assumptions used above lead to $b_1 = -3.426$ in agreement with the experimental $b_1 = -3.5 \pm 0.3$ obtained in [28]. In addition, one gets $b_2 = 18.8$, which improves upon the estimation of $b_2 = 11.7$ in [29]. However, this value still remains much smaller than the experimental estimation $b_2^{\text{exp}} = 46 \pm 5$ reported in [28].

3. T-DEPENDENT GENERIC POTENTIALS AND $T_c$

Here we consider the following $T$-dependent generic potentials

$$V(r, T) = \frac{m \omega_r^2}{2} \left[ 1 + d \left( \frac{m \omega_r^2}{2 k_B T} \right)^{\beta / 2} \right].$$  (14)

$$V(r, T) = \frac{m \omega_r^2}{2} \left( \frac{m \omega_r^2}{2 k_B T} \right)^{\delta / 2}$$  (15)

for $T = T_c$ and with $d, \beta$, and $\delta$ dimensionless parameters.

3.1. T-Dependent Generic Potential with Free Parameters $d$ and $\beta$

Here we use the potential (14) and find $b_1 (d, \beta)$ from (10) for $h = 1$ as a function of $d$ and $\beta$, which reads

$$\frac{\partial^1 T_c (g)}{T_c (g) \mid_{g=0}} = \frac{I_1 (d, \beta)}{k_B T_c \lambda_r^3},$$  (16)

where

$$I_1 = 2 \int \frac{d\Sigma \lambda' [u^2 + v^2 (1 + d \nu^\beta)] Q(v^2 (1 + d \nu^\beta))}{\int d\Sigma [u^2 + v^2 (1 + \nu^\beta)] \lambda' [u^2 + v^2 (1 + d \nu^\beta)]}$$  (17)

This integral can be evaluated numerically for $b_1$, which gives

$$b_1 (d, \beta) = 2 I_1 (d, \beta).$$  (18)

Therefore, one can find a range of values of $d$ and $\beta$ which are in agreement with the empirical value $b_1 = -3.5 \pm 0.3$ found in [28]. On the other hand, we may calculate $b_2 (d, \beta)$ from the parameters under consideration from

$$I_2 (d, \beta) = 4 \int d\Sigma \left[ \lambda' [u^2 + v^2 (1 + d \nu^\beta)] \right]$$

$$\times \left\{ Q[v^2 (1 + d \nu^\beta)] - \frac{1}{2} [u^2 + v^2 (1 + d \nu^\beta)] I_1 (d, \beta) \right\}$$  (19)

$$\times \left\{ \int d\Sigma [u^2 + v^2 (1 + d \nu^\beta)] \lambda' [u^2 + v^2 (1 + d \nu^\beta)] \right\}^{-1},$$

where

$$I_1 (d, \beta) = 2 \int d\Sigma \frac{d\Sigma \lambda' [u^2 + v^2 (1 + d \nu^\beta)]}{\int d\Sigma [u^2 + v^2 (1 + \nu^\beta)] \lambda' [u^2 + v^2 (1 + d \nu^\beta)]}$$  (20)

$$\times \left\{ Q[v^2 (1 + d \nu^\beta)] - \frac{1}{2} [u^2 + v^2 (1 + d \nu^\beta)] I_1 (d, \beta) \right\}$$  (21)

$$\times \left\{ \int d\Sigma [u^2 + v^2 (1 + d \nu^\beta)] \lambda' [u^2 + v^2 (1 + d \nu^\beta)] \right\}^{-1}. $$  (22)


We remark that the case \( \beta = -1 \) corresponds to the potential suggested in [47]. Table 1 shows the results obtained for \( b_1(\beta, \, d) \) and \( b_2(\beta, \, d) \) from different values of the parameter \( \beta \), where \( b_1(\beta, \, d) \) and \( b_2(\beta, \, d) \) from different values of the parameter \( \beta \), we found that, for \( \beta = 0.5 \), \( b_1 = -3.7862 \) which is in agreement with the experimental value \( b_1^{\text{exp}} = -3.5 \pm 0.3 \) obtained in [28], and consequently we select \( b_2 = 25.986 \):

\[
b_2(\delta) = 2I_2(\delta). \tag{25}
\]

Thus, one can find a range of values of \( \delta \), which are in agreement with the empirical value \( b_1 = -3.5 \pm 0.3 \). Table 2 shows the results obtained for \( b_1(\delta) \) and \( b_2(\delta) \) from different values of the parameter \( \delta \), we found that, for \( \delta = 0.5 \), \( b_1 = -3.7862 \) which is in agreement with the experimental value \( b_1^{\text{exp}} = -3.5 \pm 0.3 \) obtained in [28], and consequently we select \( b_2 = 25.986 \):

\[
b_2(\delta) = 2I_2(\delta). \tag{25}
\]

A similar procedure leads to

\[
I_2(\delta) = 4 \int d\Sigma \left[ \Lambda' \left[ u^2 + \nu^2 + \delta \right] S[\nu^2 + \delta] ight.
\]

\[
+ \Lambda'' \left[ u^2 + \nu^2 + \delta \right] \left\{ Q[\nu^2 + \delta] - \frac{1}{2} (u^2 + \nu^2 + \delta) I_1(\delta) \right\} 
\]

\[
\left. \times \left\{ \int d\Sigma \left[ u^2 + \nu^2 + \delta \right] \Lambda' \left[ u^2 + \nu^2 + \delta \right] \right\}^{-1} \right], \tag{26}
\]

where

\[
S[\alpha] = \frac{3}{2} I_1(d, \beta) Q[\alpha]
\]

\[
+ \{ \alpha I_1(d, \beta) - 2Q[\alpha] \} g_{1/2} \exp(-\alpha). \tag{20}
\]

From this one obtains

\[
b_2(d, \beta) = 2I_2(d, \beta). \tag{21}
\]

We remark that the case \( \beta = -1 \) corresponds to the potential suggested in [47]. Table 1 shows the results obtained for \( b_1(\beta, \, d) \) and \( b_2(\beta, \, d) \) from different values of parameters \( d \) and \( \beta \). We found that for \( \beta = 1 \) and \( d = 1 \), \( b_1 = -3.764 \) which is in agreement with the experimental value \( b_1^{\text{exp}} = -3.5 \pm 0.3 \) obtained in [28]. We also obtain \( b_2 = 25.27 \), which improves upon the result \( b_2 = 18.8 \) obtained in [35]. However, our estimation for the parameter \( b_2 \) still remains smaller than the experimental estimation \( b_2^{\text{exp}} = 46 \pm 5 \) reported in [28].

### 3.2. Temperature-Dependent Generic Potential with Free Parameter \( \delta \)

On the other hand, for the potential (15) Eq. (10) is only a function of \( \delta \) since

\[
\left. \frac{\partial^2 T_\beta(g)}{T_\beta(g)} \right|_{g=0} = \frac{I_1(\delta)}{k_B T_\beta t_{1,2}}, \tag{22}
\]

where now

\[
I_1 = 2 \int d\Sigma \Lambda' \left[ u^2 + \nu^2 + \delta \right] Q[\nu^2 + \delta] 
\]

\[
\int d\Sigma (u^2 + \nu^2 + \delta) \Lambda' \left[ u^2 + \nu^2 + \delta \right]. \tag{23}
\]

This integral must also be evaluated numerically in order to obtain the value of \( b_1(\delta) \):

\[
b_1(\delta) = 2I_1(\delta). \tag{24}
\]

Thus, one can find a range of values of \( \delta \), which are in agreement with the empirical value \( b_1 = -3.5 \pm 0.3 \). Table 2 shows the results obtained for \( b_1(\delta) \) and \( b_2(\delta) \) from different values of the parameter \( \delta \), we found that, for \( \delta = 0.5 \), \( b_1 = -3.7862 \) which is in agreement with the experimental value \( b_1^{\text{exp}} = -3.5 \pm 0.3 \) obtained in [28], and consequently we select \( b_2 = 25.986 \):

\[
b_2(\delta) = 2I_2(\delta). \tag{25}
\]

### 4. CONCLUSIONS

We have explored the shift in the condensation temperature up to second order in the S-wave scatter-
ing length, for a Bose–Einstein condensate trapped in a temperature-dependent generic potential, with no further assumptions than the semiclassical and Hartree–Fock approximations. Thus, we have recovered the usual value for the parameter \( b_1 \), and consequently, were able to improve the numerical value associated with the second parameter \( b_2 \) up to 25.271 for the corresponding potential (14), and 25.986 for the second potential (15) compared to the value obtained in [35] under typical laboratory conditions. However, the corresponding values for \( b_3 \) obtained here remain smaller than the experimental value reported in [28]. Such disagreement might be related to effects beyond the HF MF framework. Finally, we stress here that the use of temperature-dependent traps open up a very interesting line of research for other relevant properties associated with Bose–Einstein condensates.

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