Abstract
This paper presents a compact, recursive, non-linear, filter, derived from the Gauss-Newton (GNF), which is an algorithm that is based on weighted least squares and the Newton method of local linearisation. The recursive form (RGNF), which is then adapted to the Levenberg-Marquardt method is applicable to linear/nonlinear of process state models, coupled with the linear/nonlinear observation schemes. Simulation studies have demonstrated the robustness of the RGNF, and a large reduction in the amount of computational memory required, identified in the past as a major limitation on the use of the GNF.

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Keywords: Gauss-Newton, filter, tracking, recursive

1. Introduction

The minimum variance algorithm has been used to estimate parameters from batches of observations, accumulated over a defined period of time. The most popular version of the minimum variance methods is the weighted least squares, which are at the heart of adaptive filtering [1][2]. The recursive least squares (RLS) methods are efficient versions of the least squares approach, and are applicable to estimation of future states from scalar input data streams. However, recent studies [3] have seen the development of state space recursive least squares (SSRLS) methods that show robustness in the estimation of linear state space models. For the estimation of non-linear state space models, a non-recursive filter called the Gauss-Newton filter (GNF) was developed and has been successfully used in many applications [4][5]. The GNF algorithm is a combination of the Newton method of local linearization and the least squares-like version of the minimum variance method [6]. It is used to estimate process states that are governed by non-linear, autonomous, differential equations, coupled with linear or non-linear observation schemes. The GNF algorithm, although robust, requires significant processing power, i.e. the amount of memory required. To improve the computational efficiency of the GNF, studies of the use of Field Programmable Gate Arrays (FPGA) and other
co-processor technology have been made [6, 7]. Memory requirements were identified in these studies as being the major stumbling block in implementations on both on FPGA (low power and parallelism) and coprocessor (ease of use) technology. This paper obtains a recursive form of the GNF with zero memory. We then adapt the recursive filter to the Levenberg-Maquardt method, renown for its robustness [8, 9, 10, 11, 12], widely used in non linear curve fitting problems and neural networks algorithms. The contribution of this paper is the derivation of a compact recursive form of the GNF that is applicable to four major scenarios:

Case 1: linear process dynamic and linear observation scheme.

Case 2: linear process dynamic and non-linear observation scheme.

Case 3: non-linear process dynamic and linear observation scheme.

Case 4: non-linear process dynamic and non linear observation scheme.

The paper begins with an exposition of a state space model based on non-linear differential equations. This is followed, in Section 3, by the derivation of a recursive GNF. In Section 4 we describe the adaptation of the recursive equations of the filter to the Levenberg-Maquardt method. A complete filter algorithm is presented. In Section 5 the state space situations to which we can apply this new recursive form are demonstrated, with a look at stability. We then demonstrate the power of the new recursive GNF in an application to range and bearing only tracking of a manoeuvring target (Section 6), before concluding with a summary of results achieved.

2. State space model based on non-linear differential equations

Consider the following autonomous, non-linear differential equation (DE) governing the process state:

\[ DX(t) = F(X(t)) \] (1)

in which \( F \) is a non linear vector function of the state vector \( X \) describing a process, such as the position of a target in space. We assume the observation scheme of the process is a non-linear function of the process state with expression:

\[ Y(t) = G(X(t)) + v(t) \] (2)

where \( G \) is a non-linear function of \( X \) and \( v(t) \) is a random Gaussian vector. The goal is to estimate the process state from the given non-linear state models. For linear differential equations (DEs), the state transition matrix could be easily obtained. This, however, is not the case with non-linear DEs. Nevertheless, there is a procedure, based on local linearization, that enables us to get around this obstacle, which we will now present.
2.1. The method of local linearisation

The solution of the DE gives rise to infinitely many trajectories that are dependent on the initial condition. However there will be one trajectory whose state vector the filter will attempt to identify from the observations. We assume that there is a known nominal trajectory with state vector \( \bar{X}(t) \) that has the following properties:

- \( \bar{X}(t) \) satisfies the same DE as \( X(t) \)
- \( \bar{X}(t) \) is close to \( X(t) \)

The above-mentioned properties result in the following expression:

\[
X(t) = \bar{X}(t) + \delta X(t)
\]

where \( \delta X(t) \) is a vector of time-dependent functions that are small in relation to the corresponding elements of either \( \bar{X}(t) \) or \( X(t) \). The vector \( \delta X(t) \) is called the *perturbation vector* and is governed by the following DE (the derivation is shown in Appendix A):

\[
D(\delta X(t)) = A(\bar{X}(t))\delta X(t)
\]

where \( A(\bar{X}(t)) \) is called a sensitivity matrix defined as follows:

\[
A(\bar{X}(t)) = \left. \frac{\partial F(X(t))}{\partial X(t)} \right|_{\bar{X}(t)}
\]

Equation is thus a linear DE, with a time varying coefficient and has a the following transition equation:

\[
\delta X(t + \zeta) = \Phi(t_n + \zeta, t_n, \bar{X})\delta X(t)
\]

in which \( \Phi(t_n + \zeta, t_n, \bar{X}) \) is the transition matrix from time \( t_n \) to \( t_n + \zeta \) (increment \( \zeta \)). The transition matrix is governed by the following DE:

\[
\frac{\partial}{\partial \zeta} \Phi(t_n+\zeta, t_n, \bar{X}) = A(\bar{X}(t_n + \zeta))\Phi(t_n+\zeta, t_n, \bar{X})
\]

\[
\Phi(t_n, t_n, \bar{X}) = I
\]

The transition matrix is a function of \( \bar{X}(t) \) and can be evaluated by numerical integration and in order to fill the values of \( A(\bar{X}(t_n + \zeta)) \), \( \bar{X}(t) \) has to be integrated numerically. We will soon present a recursive algorithm that will avoid the computation of the transition matrix. We have shown in this section that we can estimate the true state of process by estimating the perturbation vector, which is governed by a linear differential equation. The next task is to obtain a linear perturbation observation from the non-linear observation scheme.
2.2. The observation perturbation vector

In this section we will adopt the notation $X_n$ and $Y_n$ for $X(t_n)$ and $Y(t_n)$ respectively. We define a simulated noise free observation vector $\tilde{Y}_n$ as follows:

$$\tilde{Y}_n = G(\tilde{X}_n)$$  \hspace{1cm} (9)

Subtracting $\tilde{Y}_n$ from the actual observation $Y_n$ gives the observation perturbation vector:

$$\delta Y_n = Y_n - \tilde{Y}_n$$  \hspace{1cm} (10)

In appendix A we show that the observation perturbation vector is related to the state perturbation vector as follows:

$$\delta Y_n = M(\tilde{X}_n)\delta X_n + v_n$$  \hspace{1cm} (11)

where $M(\tilde{X}_n)$ is the Jacobean matrix of $G$, evaluated at $\tilde{X}_n$. The matrix is also called the observation sensitivity matrix and is defined as follows:

$$M(\tilde{X}_n) = \frac{\partial F(X_n)}{\partial X_n} \bigg|_{\tilde{X}_n}$$  \hspace{1cm} (12)

We now examine the sequence of observations.

2.3. Sequence of observation

We assume that $L+1$ observation are obtained with time stamps $t_n, t_{n-1}, ..., t_{n-L}$. Theses observations are assembled as follows:

$$\begin{bmatrix}
\delta Y_n \\
\delta Y_{n-1} \\
. \\
. \\
. \\
. \\
\delta Y_{n-L}
\end{bmatrix}
= \begin{bmatrix}
M(\tilde{X}_n)\delta X_n \\
M(\tilde{X}_{n-1})\delta X_{n-1} \\
. \\
. \\
. \\
M(\tilde{X}_{n-L})\delta X_{n-L}
\end{bmatrix}
\begin{bmatrix}
\delta X_n \\
\delta X_{n-1} \\
. \\
. \\
. \\
\delta X_{n-L}
\end{bmatrix}
+ \begin{bmatrix}
v_n \\
v_{n-1} \\
. \\
. \\
. \\
v_{n-L}
\end{bmatrix}$$  \hspace{1cm} (13)

Using the relationship:

$$\delta X_m = \Phi(t_m, t_n, \tilde{X})\delta X_n$$  \hspace{1cm} (14)

then, substituting Equation\textsuperscript{13} the observation sensitivity equation can be written as:

$$\delta Y_n = T_n \delta X_n + V_n$$  \hspace{1cm} (15)
in which $T_n$, the total observation matrix is defined as follows:

$$
T_n = \begin{bmatrix}
M(\hat{X}_n) \\
M(\hat{X}_{n-1})\Phi(t_{n-1}, t_n; \hat{X}) \\
\vdots \\
M(\hat{X}_{n-L})\Phi(t_{n-L}, t_n; \hat{X})
\end{bmatrix}
$$

(16)

The vectors $\delta Y_n$ and $V_n$ are large. The solution of the equation can be obtained from the minimum variance estimation as follows:

$$
\delta \hat{X}_n = (T^*_n R^{-1}_n T_n)^{-1} T^*_n R^{-1}_n \delta Y_n
$$

(17)

The estimate $\delta \hat{X}_n$ has a covariance matrix:

$$
S_n = (T^*_n R^{-1}_n T_n)^{-1}
$$

(18)

where $R^{-1}_n$ is a block diagonal weight matrix, also called the least squares weight matrix, but, in fact, if we define $R_n$ as the covariance matrix of the error vector $v_n$. Then $R^{-1}_n$ is expressed as:

$$
R^{-1}_n = \begin{bmatrix}
R^{-1}_n & 0 & \ldots & 0 \\
0 & R^{-1}_{n-1} & & \\
\vdots & \ddots & \ddots & \\
0 & \ldots & 0 & R^{-1}_{n-L}
\end{bmatrix}
$$

(19)

In this section we arrived at a form of filter that uses the minimum variance estimation initiated by Gauss and the local linearisation technique championed by Newton, to estimate the state of the process from the non linear observation scheme. This filter is called Gauss-Newton filter (GNF) and is described in detail in Morrison’s work [4, 13]. The GNF has been successfully implemented in some practical applications: [5] showing strong stability. The memory nature of the filter has made it unattractive to researchers in the past, and even now, challenging [7]. However recent developments have presented recursive form of the linear least-squares for state space model [3]. We derive a recursive form of GNF using a similar approach to M. B. Malik [3]. However, before we derive a recursive form of the GNF filter, we rewrite the expression of $T_n$ using the backward differentiation:

$$
\Phi(t_{n-L}, t_n; \hat{X}) = A(\hat{X}_{n-L})^{-1}\Phi(t_{n-L+1}, t_n; \hat{X})
$$

(20)
The expression is thus:

\[
\delta Y_n^T = \begin{bmatrix}
M_0 \\
M_1 A_1 \\
M_2 A_2 \\
\vdots \\
M_L A_L
\end{bmatrix}
\]  

where

\[
A_L = \prod_{i=1}^{L} A(\bar{X}_{n-i})^{-1}
\]

and

\[
M_L = M(\bar{X}_{n-L})
\]

with

\[
A_0 = I
\]

We now move to derive the \textit{Recursive Gauss-Newton Filter} in the next section.

3. The Recursive Gauss-Newton filter

To obtain the recursive form, we use an approach similar to M. B. Malik in [3]. Suppose that the observations start arriving at \( n = 0 \) and that all initial values of the filter are available. In order to maintain the filter adaptiveness, a weight matrix function using a fading parameter \( \lambda < 1 \) is adopted, and is defined as follows:

\[
R^{-1}_n = \begin{bmatrix}
R^{-1} & 0 & \ldots & 0 \\
0 & \lambda R^{-1} & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \lambda^n R^{-1}
\end{bmatrix}
\]

The following, further definitions are adopted:

\[
W_n = T_n^T R^{-1}_n T_n
\]
\[ \xi_n = T_n^T R_n^{-1} \delta Y \]  
\[ \text{Resulting in:} \]
\[ \delta \hat{X}_n = W_n^{-1} \xi_n \]  

In the next section, the recursive update of the perturbation vector is demonstrated.

### 3.1. The recursive update of \( W_n \)

Using equation (21) and the definitions in equations (26) and (25) we have:

\[ W_n = \sum_{j=1}^{L} \lambda^j R^{-1} \prod_{i=1}^{j} A(\bar{X}_{n-i})^{-T} M(\bar{X}_{n-j})^T \]
\[ \times M(\bar{X}_{n-j}) \prod_{i=0}^{j} A(\bar{X}_{n-i})^{-1} \]
\[ + M(\bar{X}_{n-j})^T R^{-1} M(\bar{X}_{n}) \]  

and

\[ W_{n-1} = \sum_{j=1}^{L-1} \lambda^j R^{-1} \prod_{i=1}^{j} A(\bar{X}_{n-i-1})^{-T} M(\bar{X}_{n-j})^T \]
\[ \times M(\bar{X}_{n-j}) \prod_{i=0}^{j} A(\bar{X}_{n-i-1})^{-1} \]
\[ + M(\bar{X}_{n-j})^T R^{-1} M(\bar{X}_{n-1}) \]  

Comparing equations (29) and (30) the following recursive equation is obtained:

\[ W_n = \lambda A(\bar{X}_{n-1})^{-T} W_{n-1} A(\bar{X}_{n-1})^{-1} + M(\bar{X}_{n})^T R^{-1} M(\bar{X}_{n}) \]  

which is the discrete, quadratic, Lyapunov, difference equation.

### 3.2. The recursive form of \( \xi_n \)

Using equations (21) (25) (27) \( \xi_n \) can be expressed as:

\[ \xi_n = \sum_{j=0}^{L} \lambda^j R^{-1} \prod_{i=1}^{j} A(\bar{X}_{n-i})^{-T} M(\bar{X}_{n-j})^T \delta Y_{n-j} \]
\[ + M(\bar{X}_{n})^T R^{-1} \delta Y_n \]  

Comparing equations (32) and (33) the following recursive equation is obtained:

\[ \xi_n = \lambda A(\tilde{X}_{n-1})^T \xi_{n-1} + M(\tilde{X}_{n-1})^T R^{-1} \delta Y_n \]  

(34)

4. Adaptation to Levenberg and Maquardt

In order to guarantee local convergence of the recursive filter and also to avoid the singularity of \( W_n \), we replace it by \( W_n + \mu I \) as suggested by Levenberg and Maquardt. The presence of the damping factor \( \mu \) will have two effects:

- for large value of \( \mu \) the algorithm behaves as a steepest descent which is ideal when the current solution is far from the local minimum. The convergence will be slow but however guaranteed. We therefore have

\[ \delta \tilde{X}_n = \frac{1}{\mu} \xi_n \]  

(35)

- for \( \mu \) very small the algorithm will behave as gauss newton with faster convergence. The current step will be

\[ \delta \tilde{X}_n = W_n^{-1} \xi_n \]  

(36)

4.1. The Gain Ratio

The \( \mu \) can be updated by the so called gain ratio. We consider the following cost function which is

\[ E(\delta X_n) = (\delta Y_n - T_n \delta X_n)^T R^{-1} (\delta Y_n - T_n \delta X_n) \]  

(37)

The denominator of gain ratio is:

\[ E(0) - E(\delta X_n) = \delta X_n^T (\xi_n + \mu \delta X_n) \]  

(38)

We define:

\[ F(\delta X_n) = (Y_n - G(\tilde{X}_n + \delta X_n))^T R^{-1} (Y_n - G(\tilde{X}_n + \delta X_n)) \]  

(39)

The gain ratio is therefore:
\[ \varrho = \frac{F(0) - F(\delta X_n)}{E(0) - E(\delta X_n)} \]  

(40)

A large value of \( \varrho \) indicates that \( E(\delta X_n) \) is a good approximation of \( \tilde{Y} \), and \( \mu \) can be decreased so that the next Levenberg-Marquardt step is closer to the Gauss-Newton step. If \( \varrho \) is small or negative then \( E(\delta X_n) \) is a poor approximation, then \( \mu \) should be increased to move closer to the steepest descent direction. The complete filter algorithm adapted from [9] is presented in Algorithm [11].

5. State Space Models

We will present four possible models to which the recursive GNF can be applied:

- **Model 1**, with linear process dynamic and linear observation scheme. In this model the recursive formulation is similar to the derived forms except the estimation is made directly for \( X_n \) and that the observed perturbation vector \( \delta Y_n \) is replaced by the actual observation vector \( Y_n \). The sensitivity matrices in this case become the measurement and transition matrices of the process. In this case the LM algorithm is not required.

- **Model 2**, with linear process dynamic and non-linear observation scheme. The recursive model of the filter remains the same except the state sensitivity matrix becomes the transition matrix of the process. The state perturbation is estimated to obtain the estimate of the process state.

- **Model 3**, with non-linear process dynamic and linear observation scheme. The measurement sensitivity matrix has become the measurement matrix.

- **Model 4**, with a non-linear process dynamic and non-linear observation scheme. The derived recursive form without any further modification is applicable to this case.

5.1. Stability of the Recursive GNF

The matrix \( W_n \) is the inverse of of the covariance matrix of the filter and is therefore positive definite. As a consequence the solution of the derived discrete Lyapunov equation in (31) is unique with the sensitivity matrix being stable. The eigenvalues of the inverse of the sensitivity matrix are within an open unit circle and therefore the stability of the system is ensured by having \( \lambda < 1 \).

6. Simulation: Range and Bearing tracking

In these simulation studies, we consider an example of a vehicle executing various manoeuvres. During turn manoeuvres of unknown constant turn rate, the aircraft dynamic model is:
Algorithm 1 L-M algorithm for tracking system

\( k := 0; \nu := 2; \bar{X}_n := X_{n/\nu}; \)
\( \delta Y_n := Y_n - G(\bar{X}_n); \)
\( W_{\text{temp}} = M(\bar{X}_n)^T R^{-1} M(\bar{X}_n); \)
\( W_n = W_{n/\nu} + W_{\text{temp}}; \)
\( \xi_{\text{temp}} = M(\bar{X}_n)^T R^{-1} \delta Y_n; \)
\( \xi_n = \xi_{n/\nu} + \xi_{\text{temp}}; \)
\( \text{stop} := \text{false}; \)
\( \mu := \tau \cdot \max(\text{diag}(W_{n/\nu})); \)

While (not stop) and \( (k \leq k_{\text{max}}) \)
\( k := k + 1; \)
repeat;
solve \( (W_n + \mu I) \delta \bar{X}_n = \xi_n; \)
if \( (\|\delta \bar{X}_n\| \leq \varepsilon \|\bar{X}_n\|) \)
\( \text{stop} := \text{true}; \)
else
\( X_{\text{new}} := \bar{X}_n + \delta \bar{X}_n; \)
\( F(\delta X) = Y_n - G(X_{\text{new}}); F(0) = \delta Y_n^T R^{-1} \delta Y_n; \)
\( E(0) - E(\delta X_n) = \delta X_n^T (\xi_n + \mu \delta X_n); \)
\( \varphi = \frac{F(0) - F(\delta X_n)}{E(0) - E(\delta X_n)}; \)
if \( \varphi > 0 \)
\( \bar{X}_n = X_{\text{new}}; \)
\( \delta Y_n := Y_n - G(\bar{X}_n); \)
\( W_{\text{temp}} = M(\bar{X}_n)^T R^{-1} M(\bar{X}_n); \)
\( W_n = W_{n/\nu} + W_{\text{temp}}; \)
\( \xi_{\text{temp}} = M(\bar{X}_n)^T R^{-1} \delta Y_n; \)
\( \xi_n = \xi_{n/\nu} + \xi_{\text{temp}}; \)
\( \mu = \mu \cdot \max(1/3, 1 - (2\varphi + 1)^3); \nu := 2; \)
else
\( \mu := \nu \cdot \mu; \)
\( \nu := 2 \cdot \nu; \text{ssm} \)
endif
endif
until(\( \varphi > 0 \)) or (\( \text{stop} \));
endwhile

\( X_{n/\nu} = X_{\text{new}}; \)
\( X_{n/(\nu+1)} = \Phi(s)X_{n/\nu}; \)
\( W_{n/(\nu+1)} = \Lambda A(X_{n/\nu})^{-T} W_n A(X_{n/\nu})^{-1}; \)
\( \xi_{n/(\nu+1)} = \Lambda A(X_{n/\nu})^{-T} \xi_n; \)
The vehicle starts at true initial state \( X_0 = [10\text{m}, 25\text{ms}^{-1}, 400\text{m}, 0\text{ms}^{-1}, -3\text{ms}^{-1}] \) and moves at nearly constant velocity for 100s. Then it executes a turn manoeuvre from time index \( n = 101 \) to \( n = 150 \). After the manoeuvre, the vehicle’s velocity remains nearly constant from \( n = 151 \) to \( n = 250 \). At \( n = 251 \) it starts a new turn manoeuvre at rate \( \Omega = 3\text{ms}^{-1} \) until \( n = 400 \). Finally from \( n = 400 \) to \( n = 500 \) it moves at nearly constant velocity. Figure [1] describes the complete trajectory of the vehicle.
The filter uses a single model of a constant velocity to track the entire manoeuvre:

\[
A(X_n) = \begin{bmatrix}
1 & T & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & T & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]  

Equation (45)

The initial value \( W_{-1/0} = 10^{-2}I \), where \( I \) is an identity matrix. The filter parameters are the following \( k_{\text{max}} = 200 \), \( \varepsilon = 1 \times 10^{-24} \), \( \tau = 1 \times 10^{-3} \), \( \lambda = 0.4 \). The filter initial state is generated randomly and then ensuring that it has the same sign as the true state. This procedure guarantees the local convergence of the first estimate. The experiment was repeated for 250 Monte Carlo runs and the root means squared error (RMSE) is used as a performance metric. The position RMSE is computed using the following expression:

\[
\text{RMS}E = \sqrt{\frac{1}{N} \sum_{i=1}^{N} ((x_n^i - \hat{x}_n)^2 + (y_n^i - \hat{y}_n)^2)}
\]

Equation (46)

where \((x_n^i, y_n^i)\) and \((\hat{x}_n, \hat{y}_n)\) true and estimated position coordinates respectively. The velocity root mean square error (RMSE) is computed similarly. Figures [2] and [3] show the RMSE of the position and velocity respectively. The position RMSE is not affected by different manoeuvres while the velocity RMSE shows variation from different manoeuvre states. The average values of the damping factor after complete cycles of iteration is presented in Figure [3]. The damping factor increases rapidly at the transition between manoeuvres. The average number of iterations \( k \) at convergence from Figure [4] shows similar variations.

7. Conclusions

The GNF with memory combines the minimum variance estimation and the Newton method of local linearisation to estimate the process true state. The recursive form for the Gauss-Newton filter has been derived in one compact form that can be applied to all the four state and observation linearity and non-linearity scenarios:

Case 1: linear process dynamic and linear observation scheme.

Case 2: linear process dynamic and non-linear observation scheme.

Case 3: with non-linear process dynamic and linear observation scheme.

Case 4: non-linear process dynamic and non-linear observation scheme.

The Hessian matrix of the filter which is computed recursively is augmented by a damping factor as suggested earlier by Levenberg-Maquardt for non linear curve fitting problems. The new filter is therefore a combination of Newtons steepest descent and the Gauss-newton, ensuring its robustness. The presence of a forgetting factor in the filter equations renders it capable of tracking manoeuvring targets with a single filter dynamic model.
Appendix A.

Appendix A.1. The differential equation governing $\delta X(t)$

Starting from:

$$\delta X(t) = X(t) - \bar{X}(t)$$  \hspace{1cm} (A.1)

The differentiation rule is applied:

$$D\delta X(t) = F(\bar{X}(t) + \delta X(t)) - F(\bar{X}(t))$$  \hspace{1cm} (A.2)

Let $F$ be defined as follows:

$$F = \begin{pmatrix} f_1 \\ . \\ . \\ f_n \end{pmatrix}$$  \hspace{1cm} (A.3)

Equation becomes:

$$D\delta X(t) = \begin{bmatrix} f_1(\bar{X}(t) + \delta X(t)) \\ . \\ . \\ f_n(\bar{X}(t) + \delta X(t)) \end{bmatrix} - \begin{bmatrix} f_1(\bar{X}(t)) \\ . \\ . \\ f_n(\bar{X}(t)) \end{bmatrix}$$  \hspace{1cm} (A.4)

The Taylor first order approximation is applied:

$$D\delta X(t) = \begin{bmatrix} f_1(\bar{X}(t)) \\ . \\ . \\ f_n(\bar{X}(t)) \end{bmatrix} + \begin{bmatrix} \nabla f_1(\bar{X}(t))^T \\ . \\ . \\ \nabla f_n(\bar{X}(t))^T \end{bmatrix} \delta X(t)$$

$$- \begin{bmatrix} f_1(\bar{X}(t)) \\ . \\ . \\ f_n(\bar{X}(t)) \end{bmatrix}$$  \hspace{1cm} (A.5)

The following relation is obtained:

$$D\delta X(t) = A(\bar{X}(t))\delta X(t)$$  \hspace{1cm} (A.6)
Where:

\[
A(\bar{X}(t)) = \begin{bmatrix}
\nabla f_1(\bar{X}(t))^T \\
\vdots \\
\vdots \\
\vdots \\
\nabla f_n(\bar{X}(t))^T \\
\end{bmatrix} = \begin{bmatrix}
\frac{\partial F(X(t))}{\partial X(t)} \bigg|_{\bar{X}(t)} \\
\end{bmatrix} \tag{A.7}
\]

Appendix A.2. The relation between $\delta X_n$ and $\delta Y_n$

\[
\delta Y_n = G(\bar{X}_n + \delta X_n) - G(\bar{X}_n) \tag{A.8}
\]

As a direct consequence of Appendix A.1, the following relationship is obtained:

\[
\delta Y_n = M(\bar{X}_n)\delta X_n + v_n \tag{A.9}
\]

Appendix B. Figure captions list

Figure 1: Target complete trajectory with manoeuvres
Figure 2: The Position RMSE is unaffected by the manoeuvres.
Figure 3: The velocity RMSE varies with manoeuvres.
Figure 4: The damping factor shows sharp peaks at start of manoeuvres.
Figure 5: The number of iterations increases during manoeuvres.

The figure numbering appears in the same order as the figures in the pdf document

Acknowledgment

The authors would like to thank Dr Norman Morrison for his contribution during the research that leads to obtaining a recursive form of GNF. Dr Morrison has been working on the GNF throughout his career and even in his retirement is enthusiastic in providing teaching and insights into the fundamentals of filter Engineering.

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