ASYMPTOTIC PROPORTION OF ARBITRAGE POINTS IN FRACTIONAL BINARY MARKETS

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Abstract. A fractional binary market is an approximating sequence of binary models for the fractional Black-Scholes model, which Sottinen constructed by giving an analogue of the Donsker’s theorem. In a binary market the arbitrage condition can be expressed as a condition on the nodes of a binary tree. We call “arbitrage points” the points in the binary tree which verify such an arbitrage condition and “arbitrage paths” the paths in the binary tree which cross at least one arbitrage point. Using this terminology, a binary market admits arbitrage if and only if there is at least one arbitrage point in the binary tree or equivalently if there is at least one arbitrage path. Following the lines of Sottinen, who showed that the arbitrage persists in the fractional binary market, we further prove that starting from any point in the tree, we can reach an arbitrage point. This implies that, in the limit, there is an infinite number of arbitrage points. Next, we provide an in-depth analysis of the asymptotic proportion of arbitrage points at asymptotic levels and of arbitrage paths in the fractional binary market. All these results are obtained by studying a rescaled disturbed random walk. We moreover show that, when $H$ is close to 1, with probability 1 a path in the binary tree crosses an infinite number of arbitrage points. In particular, for such $H$, the asymptotic proportion of arbitrage paths is equal to 1.

1. Introduction

In the classical theory of mathematical finance a crucial role is played by the notion of arbitrage, which is the cornerstone of the option pricing theory that goes back to F. Black, R. Merton and M. Scholes [2]. Significant work has been done to develop this theory, one such contribution being given by Dzhaparidze in [6]. He extensively describes a general mathematical model for the finite binary securities market in which he gives a complete characterization of the absence of arbitrage in terms of the parameters of the model. Intuitively, a binary market is a market in which the stock price process $(S_n)_{n=0}^N$ is an adapted stochastic process with strictly positive values and such that at time $n$ the stock price evolves from $S_{n-1}$ to either $\alpha_n S_{n-1}$ or $\beta_n S_{n-1}$, where $\beta_n < \alpha_n$. The values $\alpha_n$ and $\beta_n$ depend only on the past. So there are exactly $2^n$ different possible paths for the stock price to evolve up to time $n$.

One advantage of working with binary markets is given, on one hand, by their simplicity and, on the other hand, by their flexibility to approximate more complicated models. Indeed, a typical situation that may occur is when a continuous model can be expressed as a limiting process of a sequence of binary market models. Such a construction makes sense for Black-Scholes type markets that are driven by a process, for which we dispose of a random walk approximation. Examples of this
are the fractional Brownian motion and the Rosenblatt process, as one can see in [13] and [14] respectively.

In this paper we provide an in-depth analysis of fractional binary markets, which are defined by Sottinen [13] as a sequence of binary models approximating the fractional Black-Scholes model, i.e., a Black-Scholes type model where the randomness of the risky asset comes from a fractional Brownian motion and not from the standard Brownian one. Along this work we assume the case when the Hurst parameter $H$ is strictly bigger than $1/2$. In this case, the fractional Brownian motion exhibits self-similarity and long-range dependence, properties that were observed in some empirical studies of financial time series (see [3] and [16] for a discussion on the relevance of these properties in financial modelling). For this reason these models are thought to describe real world markets in a better way, and hence their use substantially increased. However, these models admit arbitrage opportunities, since the fractional Brownian motion fails to be a semimartingale. This drawback can be corrected if, e.g., one introduces transaction costs.

In [13] Sottinen constructs the fractional binary markets by giving an analogue of the Donsker theorem, which in this case means that the fractional Brownian motion can be approximated by a “disturbed” random walk. As in the limiting case, Sottinen proves that the arbitrage opportunities appear also in the sequence of fractional binary markets. Such an arbitrage is explicitly constructed using the path information starting from time zero.

As mentioned before, the existence of arbitrage is expressed in terms of the parameters of the binary market, which can be seen as functions on a binary tree. In this way, the absence of arbitrage can be written as a family of conditions indexed by the nodes of the binary tree. We call an “arbitrage point” a node in the binary tree which does not verify the corresponding arbitrage condition. An “arbitrage path” is a path that crosses at least one arbitrage point. By [13] we know that, for each fractional binary market in the sequence, the associated set of arbitrage points is not empty and, moreover, we dispose of a lower bound for the proportion of arbitrage paths.

The aim of this paper is to study qualitative and quantitative properties of the sets of arbitrage points and paths for the fractional binary market. First, we prove that starting from any point in the binary tree we reach an arbitrage point by going enough times only up or only down. This generalizes the result of Sottinen, who showed the existence of arbitrage starting only from the root of the tree. This gives information about the structure of the set of arbitrage points and implies that its cardinal is asymptotically infinite. The main results are related with the asymptotic proportion of arbitrage points and paths. We first observe that the parameters of the fractional binary models verify a scaling property. This makes possible to characterize the proportion of arbitrage points in terms of a rescaled random walk. However, the convergence properties of the rescaled random walk, and then the asymptotic proportion of arbitrage points, cannot be obtained using standard techniques, e.g., a central limit theorem (CLT). In our approach we write the random walk as a sum of two independent random variables, which are of very different nature. The first one, representing the contribution of the first jumps, is defined by means of a null array of independent random variables and a strong convergence to zero is proved. The second part of the random walk contains the contribution of the last jumps, which inherits the “bad” properties of the initial random walk. Nevertheless, we are able to prove convergence in law by using an auxiliary sequence of random variables with the same law. This new sequence strongly converges to a random variable which is defined by means of the autocovariance functions of a fractional Brownian motion with smaller Hurst parameter. This limit provides us
with the desired characterization of the asymptotic proportion of arbitrage points. We also study the properties of this limit with respect to the Hurst parameter. On the other hand, by construction, the limit depends only on the contribution of the “last” jumps and, by the use of $0 - 1$ Kolmogorov law we are able to deduce information about the proportion of arbitrage paths. In particular, when $H$ is close to 1, this asymptotic proportion is equal to 1.

A motivation for studying the structure of the set of arbitrage points for the fractional binary markets comes from the more involved problem of characterizing the arbitrage opportunities of such markets under transaction costs. As it is mentioned by Sottinen, one may expect that the arbitrage disappears when the transaction costs are taken into account. This latter problem was treated in its most generality in [4], where a characterization of the smallest transaction cost (called “critical” transaction costs) starting from which the arbitrage is eliminated is provided. However, since the fractional binary markets are not homogeneous, i.e. the parameters of the model depend on time and space, this characterization does not give a closed-form solution. More precisely, the critical transaction cost are obtained as a solution of an optimization problem in a binary tree. The complexity of this optimization problem increases with the number of arbitrage points, and, hence, the understanding of qualitative and quantitative properties give us an insight to this more complicated problem. One can also see from our work that the arbitrage opportunities appear more frequently when the Hurst parameter increases, i.e. when the regularity of the approximated model increases.

The paper is organized as follows. In Section 2, we start recalling the notion of binary market and introduce the definitions of the above mentioned sets of arbitrage points and arbitrage paths in relation with the arbitrage conditions given in [6]. Next, we give some definitions and properties concerning the fractional Brownian motion and we present the random walk approximation given in [13]. Finally, we introduce the definition of fractional binary markets, which approximate the fractional Black-Scholes model as shown by Sottinen. In Section 3 we prove that the parameters of the fractional binary markets satisfy a scaling property, which allows us to rewrite them in terms of new rescaled ones. This procedure helps us to get rid of the dependence on the size of the fractional binary market. We end this part by providing good estimations for the rescaled parameters. In the next section, we follow the lines of [13] and, making use of the estimations given in Section 2 we show how from any point in the binary tree one can reach an arbitrage point. In the last two sections we present our main results. In Section 5, we first relate the proportion of arbitrage points at a fixed level with a rescaled random walk arising from the definition of the fractional binary markets and the previously referred scaling property. Using non-standard techniques, we show that this rescaled random walk converges in distribution. We then study the properties of this limit, which permits to provide a characterization of the asymptotic proportion of arbitrage points. This asymptotic proportion is then also studied with respect to the Hurst parameter. In the last section, we give results concerning the asymptotic proportion of arbitrage paths. In particular, we show that, for $H$ close to 1, a path in the binary tree crosses an infinite number of arbitrage points with probability 1.

2. Preliminaries

2.1. Binary markets and their arbitrage opportunities. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, P)$ be a finite filtered probability space. By a binary market we mean a market in which two assets (a bond $B$ and a stock $S$) are traded at successive times $t_0 = 0 < t_1 < \cdots < t_N$. The evolution of the bond and stock is described by:

$$B_n = (1 + r_n)B_{n-1}$$
and

\[ S_n = (a_n + (1 + X_n)) \, S_{n-1}, \quad \forall n \in \{1, \ldots, N\}, \tag{2.1} \]

where \( r_n \) and \( a_n \) are the interest rate and the drift of the stock in the time interval \([t_n, t_{n+1})\). The value of \( S \) at time 0 is given by:

\[ S_0 = s_0 = 1 + a_0 + x_0. \]

We may assume, for the sake of simplicity, that the bond plays the role of a numéraire, and, in this case, that it is equal to 1 at every time \( n \) \((r_n = 0)\). The process \((X_n)_{n=0}^N\) is an adapted stochastic process starting at \( X_0 = x_0 \) and such that, at each time \( n \), \( X_n \) can take only two possible values \( u_n \) and \( d_n \) with \( d_n < u_n \). While \( a_n \) from (2.1) is deterministic, the values of \( u_n \) and \( d_n \) may depend on the path of \( X \) up to time \( n - 1 \). The parameters \( u_n \) and \( d_n \) can be seen as real valued functions on \((-1,1)^{n-1}\) (\( u_1 \) and \( d_1 \) are constants).

We know by Proposition 3.6.2 in [6] that a binary market excludes arbitrage opportunities if and only if for all \( n \in \{1,\ldots,N\} \) and \( x \in \{-1,1\}^{n-1} \), we have:

\[ d_n(x) < -a_n < u_n(x). \tag{2.2} \]

The previous characterization of the arbitrage opportunities in a binary market motivates the following definitions. We call the following set \( N \)-binary tree:

\[ X_N = \{\tau\} \bigcup_{n=1}^{N-1} \{-1,1\}^n, \]

where \( \tau \) denotes the root of the tree. We say that a point \( x \in X_N \) is an arbitrage point for the corresponding binary market if \( x \) does not satisfy condition (2.2) (when \( x = \tau \) this means \( u_1 \leq -a_1 \) or \( d_1 \geq -a_1 \)). More precisely, given a level \( n \in \{1,\ldots,N\} \), we call the set of arbitrage points at level \( n \) the set:

\[ A_n := \{x \in \{-1,1\}^{n-1} : u_n(x) \leq -a_n \text{ or } d_n(x) \geq -a_n\}, \quad n \geq 2, \]

and \( A_1 \) is equal to \( \{\tau\} \) if \( u_1 \leq -a_1 \) or \( d_1 \geq -a_1 \) and the empty set otherwise. The set of arbitrage points is given by:

\[ A := \bigcup_{n=1}^{N} A_n \subseteq X_N. \]

In addition, we call arbitrage paths the paths in the binary tree which cross at least one arbitrage point, i.e., the elements of the set:

\[ A^p := \{(x_1, \ldots, x_{N-1}) \in \{-1,1\}^{N-1} : \exists n \in \{1,\ldots,N\}, (x_1,\ldots,x_{n-1}) \in A_n\}. \]

**Remark 2.1.** Note that using the previous definitions, we can state that a binary market admits arbitrage opportunities if and only if there is at least one arbitrage point, or equivalently, if there is at least one arbitrage path.

**2.2. Fractional Brownian motion.** We denote by \( Z^H = (Z^H_t; t \geq 0) \) a fractional Brownian motion with Hurst index \( H \in (0,1) \), i.e., a continuous centered Gaussian process with stationary centered Gaussian increments and with covariance function

\[ \text{Cov}(Z^H_t, Z^H_s) = E(Z^H_t Z^H_s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right). \]

The process \( Z^H \) is \( H \)-self-similar, which means that

\[ \{Z^H_{at}, t \geq 0\} \overset{d}{=} \{a^H Z^H_t, t \geq 0\}. \]
Along this paper we assume that $H > 1/2$. In that case, the increments of the fractional Brownian motion are positively correlated and exhibit long-range dependence. More precisely, the autocovariance function

$$\rho_H(n) := \text{Cov}(Z_H^k - Z_H^{k-1}, Z_H^{k+n} - Z_H^{k+n-1}) = \frac{1}{2} \left( (n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \right),$$

satisfies

$$\rho_H(n) \sim H(2H-1)n^{2H-2}, \quad \text{as } n \to \infty. \quad (2.3)$$

In particular, the dependence between the increments $Z_H^k - Z_H^{k-1}$ and $Z_H^{k+n} - Z_H^{k+n-1}$ decays slowly as $n \to \infty$ and

$$\sum_{n=1}^{\infty} \rho_H(n) = \infty.$$

It is a well know fact that $Z_H^t$ admits the following kernel representation with respect to the standard Brownian motion $W$ (see [5] and [11]):

$$Z_H^t = \int_0^t k_H(t, s) dW_s, \quad (2.4)$$

where:

$$k_H(t, s) := c_H \left( H - \frac{1}{2} \right) s^{\alpha-H} \int_s^t u^{H-\frac{1}{2}}(u-s)^{\frac{1}{2}-H} du,$$

and

$$c_H := \sqrt{\frac{2H \Gamma \left( \frac{3}{2} - H \right)}{\Gamma \left( H + \frac{1}{2} \right) \Gamma(2-2H)}}$$

is a normalizing constant.

In order to shorten some of the proofs, we will use from time to time the notations $C_H := c_H \left( H - \frac{1}{2} \right)$ and $\alpha := H - \frac{1}{2} \in (0, \frac{1}{2})$.

2.3. A random walk approximation of fBm. Sottinen constructs in [13] a Donsker type approximation for the fractional Brownian motion. Such approximation is based on the kernel representation (2.4) and the classical Donsker theorem for the standard Brownian motion. The construction goes as follows.

Consider a sequence $(\xi_i; i \geq 1)$ of i.i.d. random variables with $E[\xi_1] = 0$ and $E[\xi_1^2] = 1$ and define the random walk:

$$Z_{N,H}^t := \frac{1}{\sqrt{N}} \sum_{i=1}^{\lfloor Nt \rfloor} k_H^N \left( \frac{i}{N}, \frac{t}{N} \right) \xi_i, \quad t \in [0,1], \quad (2.5)$$

where $k_H^N$ is the approximation of the kernel $k_H$ given by:

$$k_H^N(t, s) := N \int_s^{\frac{\lfloor Nt \rfloor}{N}} k_H \left( \frac{\lfloor Nt \rfloor}{N}, u \right) du$$

and $[x]$ denotes the greatest integer smaller or equal than $x$. In the aforementioned paper, Sottinen proves that:

$$Z_{N,H}^t \xrightarrow{d} Z_H^t, \quad \text{as } N \to \infty. \quad (2.6)$$

This result is the key ingredient in the construction of the sequence of binary markets approximating the fractional Black-Scholes model (2.7) given in [13].
2.4. Fractional Black-Scholes model. This market model consists of two assets, that are traded continuously over the interval \([0, 1]\). The dynamics of the bond and stock are given by:

\[
\begin{align*}
    dB_t &= r(t) B_t \, dt \\
    dS^H_t &= (a(t) + \sigma \, dZ^H_t) \, S^H_t,
\end{align*}
\]

where \(\sigma > 0\) is a constant representing the volatility and \(Z^H\) is a fractional Brownian motion of Hurst parameter \(H > 1/2\). The functions \(r\) and \(a\) are deterministic and represent the interest rate and the drift of the stock, respectively. If we suppose in addition that \(r\) and \(a\) are continuously differentiable, the solutions of the problem are given by:

\[
    B_t = B_0 \exp \left( \int_0^t r(s) \, ds \right) \quad \text{and} \quad S^H_t = S_0 \exp \left( \int_0^t a(s) \, ds + \sigma Z^H_t \right),
\]

respectively (see [17]).

2.5. Fractional binary markets. Sottinen introduces in [13] the fractional binary markets as a sequence of binary markets approximating the fractional Black-Scholes model (2.7). The key point in that construction is the approximation of the fractional Brownian motion given by (2.6).

For each \(N > 1\), the \(N\)-fractional binary market is the binary market in which the bond and stock are traded at the times \(\left\{ 0, \frac{1}{N}, \ldots, \frac{N}{N}, 1 \right\}\) under the dynamics:

\[
    B_n^{(N)} = \left( 1 + r_n^{(N)} \right) B_{n-1}^{(N)} \quad \text{and} \quad S_n^{(N, H)} = \left( 1 + a_n^{(N)} + X_n^{(N, H)} \right) S_{n-1}^{(N, H)},
\]

where

\[
    r_n^{(N)} = \frac{1}{N} r(n/N), \quad a_n^{(N)} = \frac{1}{N} a(n/N),
\]

and:

\[
    X_n^{(N, H)} = \sigma \Delta Z_n^{(N, H)} = \sigma \left( Z_n^{(N, H)} - Z_{n-1/N}^{(N, H)} \right).
\]

The process \(Z^{(N, H)}\) is defined as in (2.5), but here the random variables \((\xi_i; i \geq 1)\) are supposed to be binary, i.e., \(P(\xi_1 = -1) = P(\xi_1 = 1) = 1/2\). We assume in addition that the parameters in (2.7) verify \(r = 0\), \(\sigma > 0\) and that \(a\) is continuously differentiable.

We note that \(X_n^{(N, H)}\) can be expressed as:

\[
    X_n^{(N, H)} = \sum_{i=1}^{n-1} J_n^{(N, H)}(i) \xi_i + g_n^{(N, H)} \xi_n,
\]

where, for \(1 \leq i < n \leq N\):

\[
    J_n^{(N, H)}(i) := \frac{\sigma}{\sqrt{N}} \left( k_H^{(N)} \left( \frac{n}{N}, \frac{i}{N} \right) - k_H^{(N)} \left( \frac{n-1}{N}, \frac{i}{N} \right) \right),
\]

and

\[
    g_n^{(N, H)} := \frac{\sigma}{\sqrt{N}} k_H^{(N)} \left( \frac{n}{N}, \frac{n}{N} \right).
\]

For each \(n \in \{1, \ldots, N\}\), we define the functions \(Y_n^{(N, H)} : \{-1, 1\}^{n-1} \to \mathbb{R}\) by:

\[
    Y_n^{(N, H)}(x_1, \ldots, x_{n-1}) := \sum_{i=1}^{n-1} J_n^{(N, H)}(i) x_i, \quad Y_1^{(N, H)} = 0.
\]

We shortly denote \(Y_n^{(N, H)}\) the random variable \(Y_n^{(N, H)}(\xi_1, \ldots, \xi_{n-1})\). In particular, we have the following identity:

\[
    X_n^{(N, H)} = Y_n^{(N, H)} + g_n^{(N, H)} \xi_n.
\]
More precisely, same holds for the coefficients $g_n^{(N,H)}$ to obtain the explicit behavior with respect to the variable $N$ for the other parameters, we show first some kind of scaling property, which permits

$$u_n^{(N)}(x_1,\ldots,x_{n-1}) := Y_n^{(N,H)}(x_1,\ldots,x_{n-1}) + g_n^{(N,H)},$$

and

$$d_n^{(N)}(x_1,\ldots,x_{n-1}) := Y_n^{(N,H)}(x_1,\ldots,x_{n-1}) - g_n^{(N,H)}.$$

As in Section 2.1, for any $N \geq 1$, we denote $A_n^{(N,H)}$, $A^{(N,H)}$ and $A_P^{(N,H)}$ respectively the set of arbitrage points at level $n$, the set of arbitrage points and the set of arbitrage paths associated to the $N$-fractional binary market.

**Remark 2.2.** Sottinen proves in [13] that the price processes $B^{(N)}$ and $S^{(N,H)}$ converge weakly to the corresponding price process $B$ and $S^H$ in the fractional Black-Scholes model. In addition, he proves that for $N$ large enough, the $N$-fractional binary market admits arbitrage opportunities. More precisely, it is proved that there exists $n_H > 0$ such that for all $N$ sufficiently large:

$$A_{n_H}^{(N,H)} \neq \emptyset \quad \text{and} \quad \lim_{N \to \infty} \frac{|A_P^{(N,H)}|}{2^N} \geq 2^{2-n_H} > 0.$$

3. Some previous estimations

In this section, we obtain estimations for the quantities involved in the definition of the fractional binary markets, i.e., $a_n^{(N)}$, $J_n^{(N,H)}$, $g_n^{(N,H)}$ and $Y_n^{(N,H)}$. These estimations are the starting point in order to study quantitative and qualitative properties of the fractional binary markets.

Note first that, from its definition and the continuity of the function $a$, the drift term $a_n^{(N)}$ verifies:

$$|a_n^{(N)}| \leq \frac{||a||_{\infty}}{N}, \quad n \in \{1,\ldots,N\}. \quad (3.1)$$

For the other parameters, we show first some kind of scaling property, which permits to obtain the explicit behavior with respect to the variable $N$ of these parameters. The study of their dependence on $n$ involves the estimation of some integrals.

3.1. Discrete scaling property. The random variables $Y_n^{(N,H)}$ can be expressed as weighted sums of independent Bernoulli random variables, the weights being given by the coefficients $J_n^{(N,H)}(i)$. In the next proposition, we prove that the dependence on $N$ of these coefficients appears as a multiplicative scaling factor. The same holds for the coefficients $g_n^{(N,H)}$ and as a consequence, the random variables $X_n^{(N,H)}$ verify a scaling property.

**Proposition 3.1.** For all $1 \leq n \leq N$, we have that:

$$N^H X_n^{(N,H)} = n^H X_n^{(n,H)}. \quad (3.2)$$

More precisely,

(1) For all $1 \leq i < n \leq N$, we have:

$$J_n^{(N,H)}(i) = \frac{1}{N^H} J_n^{(n,H)}(i),$$

where $J_n^{(n,H)}(i) := n^H J_n^{(n,H)}(i)$.

(2) For all $1 \leq n \leq N$, we have:

$$g_n^{(N,H)} = \frac{1}{N^H} g_n^{(n,H)},$$

where $g_n^{(n,H)} := n^H g_n^{(n,H)}$. 
In addition, the coefficients $j^H_n(i)$ and $g^H_n$ can be expressed as follows:

$$ j^H_n(i) = \sigma C^H \int_{i-1}^{i} x^{\frac{1}{2} - H} \left( \int_{0}^{1} (v + n - 1)^{H - \frac{1}{2}} (v + n - 1 - x)^{H - \frac{3}{2}} dv \right) dx, \quad (3.3) $$

and

$$ g^H_n = \sigma C^H \int_{n-1}^{n} x^{\frac{1}{2} - H} (n - x)^{H - \frac{1}{2}} \left( \int_{0}^{1} (y(n - x) + x)^{H - \frac{3}{2}} dy \right) dx. \quad (3.4) $$

Proof. Note first that (3.2) is a direct consequence of the statements (1) and (2) in the proposition.

(1) It is enough to prove that $J^N_H(i) = N^{-H} j^H_n(i)$ where $j^H_n(i)$ is defined as in (3.3) (and as a consequence $j^H_n(i) = n^H J^{(n,H)}_n(i)$). From the definition, we have that:

$$ J^N_H(i) = \sigma \sqrt{N} \int_{i-1}^{i} \left( k_H \left( \frac{n}{N}, u \right) - k_H \left( \frac{n - 1}{N}, u \right) \right) du. \quad (3.5) $$

On the other hand, we have that:

$$ k_H \left( \frac{n}{N}, u \right) - k_H \left( \frac{n - 1}{N}, u \right) = C^H u^{\frac{1}{2} - H} \int_{-\frac{1}{2}}^{\frac{n}{N - 1}} s^{H - \frac{1}{2}} (s - u)^{H - \frac{3}{2}} ds. $$

By means of the change of variable $v = Ns - n + 1$, the last identity implies that:

$$ k_H \left( \frac{n}{N}, u \right) - k_H \left( \frac{n - 1}{N}, u \right) = C^H u^{\frac{1}{2} - H} \int_{n-1}^{n} (v + n - 1)^{H - \frac{1}{2}} (v + n - 1 - Nu)^{H - \frac{3}{2}} dv. $$

The result follows by plugging this expression in (3.5) and making the change of variable $x = Nu$.

(2) As before, it is enough to prove that $g^N_H = N^{-H} g^H_n$ where $g^H_n$ is defined as in (3.4). Note first that:

$$ g^N_H = \sigma \sqrt{N} \int_{\frac{n}{N - 1}}^{\frac{n}{N}} k_H \left( \frac{n}{N}, u \right) du = \sigma \sqrt{N} \int_{n-1}^{n} k_H \left( \frac{n}{N}, \frac{x}{N} \right) dx. \quad (3.6) $$

The last identity follows from the change of variable $x = Nu$.

On the other hand, we have that:

$$ k_H \left( \frac{n}{N}, \frac{x}{N} \right) = C^H N^{H - \frac{1}{2}} x^{\frac{1}{2} - H} \int_{\frac{-1}{2}}^{\frac{n}{N}} s^{H - \frac{1}{2}} (s - \frac{x}{N})^{H - \frac{3}{2}} ds. \quad (3.7) $$
By means of the successive change of variables \( v = Ns \) and \( y = (v - x)/(n - x) \), the integral in the previous identity can be expressed in the following form:

\[
\int_{-\infty}^{\infty} s^{H-\frac{1}{2}} \left( \frac{s - x}{N} \right)^{H-\frac{1}{2}} ds = \frac{1}{N^{2H-1}} \int_{-\infty}^{\infty} \nu^{H-\frac{1}{2}} (v-x)^{H-\frac{1}{2}} dv \\
= \frac{1}{N^{2H-1}} (n-x)^{H-\frac{1}{2}} \int_{0}^{1} (y(n-x) + x)^{H-\frac{1}{2}} y^{H-\frac{1}{2}} dy.
\]

Plugging the last expression in (3.7), and using the resulting identity in (3.6), we obtain the desired result.

**Remark 3.2.** Note that the identity (3.2) in Proposition 3.1 can be expressed in terms of the jumps of the random walk \( Z^{(N,H)} \) as follows:

\[
Z_{(n/N)}^{(N,H)} - Z_{(n-1)/N}^{(N,H)} = \left( \frac{n}{N} \right)^{H} \left( Z_{1}^{(n,H)} - Z_{1-\frac{1}{2}}^{(n,H)} \right).
\]

This relation can be viewed as the analogue of the following property for the fractional Brownian motion:

\[
Z_{n/N}^{H} - Z_{(n-1)/N}^{H} = \frac{d}{n^{H}} \left( Z_{1}^{H} - Z_{1-\frac{1}{2}}^{H} \right),
\]

which follows from the self-similarity of the fractional Brownian motion. In contrast to the fractional Brownian motion, only the increments of the random walk satisfy a scaling property and not the process itself.

Inspired by the previous proposition, we define the random variables \( Y_{n}^{H} \) as:

\[
Y_{n}^{H} := \sum_{i=1}^{n-1} j_{i}^{H} \xi_{i}.
\]

**Corollary 3.3.** For all \( 1 \leq n \leq N \), the following identities hold:

\[
Y_{n}^{(N,H)} = \frac{1}{N^{H}} Y_{n}^{H}, \quad X_{n}^{(N,H)} = \frac{1}{N^{H}} (Y_{n}^{H} + g_{n}^{H} \xi_{n}),
\]

and

\[
Y_{n}^{H} + g_{n}^{H} \xi_{n} = \sigma_{n}^{H} \left( Z_{1}^{(n,H)} - Z_{1-\frac{1}{2}}^{(n,H)} \right).
\]

**Proof.** Direct from Proposition 3.1 and the definitions of \( Y_{n}^{(N,H)}, X_{n}^{(N,H)} \) and \( Y_{n}^{H} \).

### 3.2. Estimations for \( j_{n}^{H}(i) \) and \( g_{n}^{H} \)

From the above-mentioned scaling property, it is enough to obtain good estimates for the parameters \( j_{n}^{H}(i) \) and \( g_{n}^{H} \) in order to deduce good estimates for the parameters \( J_{n}^{(N,H)}(i) \) and \( g_{n}^{(N,H)} \).

**Lemma 3.4.** For all \( 1 \leq i \leq n - 1 < N \), we have:

\[
\sigma c_{H} (n-1)^{H-\frac{1}{2}} I_{n}(i) \leq j_{n}^{H}(i) \leq \sigma c_{H} n^{H-\frac{1}{2}} I_{n}(i),
\]

where:

\[
I_{n}(i) := \int_{i-1}^{i} x^{H-\frac{1}{2}} \varphi_{n}^{H}(x) dx \quad \text{and} \quad \varphi_{n}^{H}(x) := (n-x)^{H-\frac{1}{2}} - (n-1-x)^{H-\frac{1}{2}}.
\]
Proof. Since, for every \( v \in [0,1] \), we have \( n-1 \leq v+n-1 \leq n \), we deduce that:

\[
(n-1)^{H-\frac{1}{2}} \frac{c_n^H(x)}{H-\frac{1}{2}} \leq \int_0^1 (v+n-1)^{H-\frac{1}{2}} (v+n-1-x)^{H-\frac{1}{2}} dv \\
\leq n^{H-\frac{1}{2}} \frac{c_n^H(x)}{H-\frac{1}{2}}.
\]

The result is obtained by plugging the previous inequalities in \( g_n^H \).

\[ \square \]

**Lemma 3.5.** For all \( 1 < n \leq N \), we have:

\[
\frac{\sigma c_H}{H + \frac{1}{2}} \leq g_n^H \leq \frac{\sigma c_H}{H + \frac{1}{2}} \left( 1 + \frac{1}{n-1} \right)^{H - \frac{1}{2}}.
\]

**Proof.** Note first that for every \( x \in (n-1,n) \) we have:

\[
x^{H-\frac{1}{2}} \leq \int_0^1 (y(n-x)+x)^{H-\frac{1}{2}} y^{H-\frac{1}{2}} dy \leq \frac{n^{H-\frac{1}{2}}}{H-\frac{1}{2}}.
\]

Using these inequalities and \( \frac{\sigma c_H}{H + \frac{1}{2}} \), we obtain the following sequence of inequalities:

\[
\frac{\sigma c_H}{H + \frac{1}{2}} \leq g_n^H \leq \sigma c_H n^{H-\frac{1}{2}} \int_{n-1}^n x^{\frac{1}{2}} (n-x)^{H-\frac{1}{2}} dx \\
\leq \sigma c_H n^{H-\frac{1}{2}}(n-1)^{\frac{1}{2}} \frac{n}{H + \frac{1}{2}}.
\]

Which proves the desired result.

\[ \square \]

**Corollary 3.6.** For any strictly increasing sequence \( N_n \) of positive integers, we have:

\[
\lim_{n \to +\infty} N_n^H g_n^{(N_n,H)} = g_H := \frac{\sigma c_H}{H + \frac{1}{2}}.
\]

**Proof.** Direct from Proposition 3.4 and Lemma 3.5

\[ \square \]

**Remark 3.7.** The statement of Lemma 3.5 holds true for \( n = 1 \) in a slightly different way. In fact, following the same arguments used in the proof of Lemma 3.5, we obtain:

\[
\frac{\sigma c_H}{H + \frac{1}{2}} \leq g_1^H \leq \frac{\sigma c_H}{H + \frac{1}{2}} \frac{\pi(H - \frac{1}{2})}{\sin(\pi(H - \frac{1}{2}))}.
\]

4. On the structure of the set of arbitrage points

In [13], Theorem 5, the author shows that starting from the root of the binary tree and going always up we can always reach an arbitrage point. In this section, we provide a generalization of that result, establishing that starting from any point in the binary tree by going always up (or always down) we can always reach an arbitrage point. As a consequence the number of arbitrage points in the \( N \)-fractional binary markets converges to infinity when \( N \) goes to infinity.

**Proposition 4.1.** For all \( k \geq 2 \) and \( x \in \{-1,1\}^{k-1} \), there exist \( n_k(x) \geq 1 \) and \( N_k(x) \geq k + n_k(x) \) such that for all \( N \geq N_k(x) \):

\[
(x, 1_{n_k(x)}) \in A^{(N,H)}_{k+n_k(x)} \text{ and } (x, -1_{n_k(x)}) \in A^{(N,H)}_{k+n_k(x)}.
\]

where \( 1_{n_k(x)} \) denotes the vector in \( \mathbb{R}^{n_k(x)} \) with all the coordinates equal to 1. In particular,

\[
\lim_{N \to \infty} |A^{(N,H)}| = \infty.
\]
Proof. Fix $k \geq 2$. We will prove only the first statement. The proof of the second statement is analogous. Note that, it is enough to show the result for $x = -1_{k-1}$.

More precisely, we prove that $d_{n+k}^{(N)}(1_{k-1}, 1_n) \geq -d_{n+k}^{(N)}$, which is equivalent to:

$$R_n^{(N)}(k) := a_{k+n}^{(N)} + Y_{k+n}^{(N,H)}(-1_{k-1}, 1_n) - g_{n+k}^{(N,H)} \geq 0. \quad (4.1)$$

For the first term we have the upper bound (3.1). For the last term, we can use Lemma 3.5 to obtain:

$$g_{n+k}^{(N,H)} \leq \frac{c_n}{N^H}. \quad (4.2)$$

where $c_n$ is a positive constant. It remains to obtain good estimations for the second term of $R_n^{(N)}$. Note first that:

$$Y_{k+n}^{(N,H)}(-1_{k-1}, 1_n) = \frac{1}{N^H} \left( - \sum_{i=1}^{k-1} j_{k+n}^{H}(i) + \sum_{i=k}^{k+n-1} j_{k+n}^{H}(i) \right).$$

Using the upper bounds of Lemma 3.4 for $j_{k+n}(i)$, we obtain:

$$\sum_{i=1}^{k-1} j_{k+n}^{H}(i) \leq \sigma c_H (n+k)^{\alpha} \int_{0}^{x} x^{-\alpha} \varphi_{n+k}^{H}(x) dx,$$

where $\alpha = H - \frac{1}{2} \in (0, \frac{1}{2})$. Using the definition of the function $\varphi_{n+k}^{H}$ and some appropriate change of variables, we obtain:

$$\int_{0}^{x} x^{-\alpha} \varphi_{n+k}^{H}(x) dx = (n+k) \int_{0}^{(n+k)^{\alpha}} v^{-\alpha} dv - (n+k-1) \int_{0}^{(1-v)^{\alpha} v^{-\alpha} dv}$$

$$\leq \int_{0}^{(1-v)^{\alpha} v^{-\alpha} dv} \leq \frac{1}{1-\alpha} \left( \frac{k-1}{n+k-1} \right)^{1-\alpha}.$$  

Thus, for $n \geq 1$:

$$\sum_{i=1}^{k-1} j_{k+n}^{H}(i) \leq \frac{2 \sigma c_H}{1-\alpha} n^{\alpha} \left( \frac{k-1}{n+k-1} \right)^{1-\alpha}. \quad (4.3)$$

Now, using the lower bounds of Lemma 3.4 for $j_{k+n}(i)$, we have:

$$\sum_{i=k}^{k+n-1} j_{k+n}^{H}(i) \geq \sigma c_H (n+k-1)^{\alpha} \int_{k-1}^{x} x^{-\alpha} \varphi_{n+k}^{H}(x) dx$$

Proceeding as before, using an appropriate change of variables, we deduce that:

$$\int_{k-1}^{n+k-1} x^{-\alpha} \varphi_{n+k}^{H}(x) dx = (n+k) \int_{0}^{(n+k)^{\alpha}} v^{-\alpha} dv - (n+k-1) \int_{0}^{(1-v)^{\alpha} v^{-\alpha} dv}$$

$$\geq \int_{0}^{(1-v)^{\alpha} v^{-\alpha} dv} \leq \frac{1}{1-\alpha} \left( \frac{n}{n+k-1} \right)^{1+\alpha} - \frac{1}{(n+k-1)^{\alpha}}.$$
and then, for \( n \geq k \) big enough:
\[
\sum_{i=k}^{k+n-1} j_{n+k}^H (i) \geq \frac{\sigma c_H}{4(1 + \alpha)} n^\alpha.
\] (4.4)

Now, using (3.1), (4.2), (4.3) and (4.4), we obtain for \( n \) big enough:
\[
N^H R_n^N (k) \geq \sigma c_H n^\alpha \left( \frac{1}{4(1 + \alpha)} - \frac{2}{1 - \alpha} \left( \frac{k - 1}{n + k - 1} \right)^{1 - \alpha} \right) - c g - \frac{|a|_\infty}{N^{1-H}}.
\]

As a consequence, for \( n \) and \( N \) large enough, \( R_n^N (k) \geq 0 \), which proves the result.

\[\square\]

5. On the proportion of arbitrage points

In this section we give a probabilistic approach to the study of the asymptotic proportion of arbitrage points. More precisely, we identify this asymptotic behaviour with the convergence of a well-chosen sequence of random variables.

5.1. Some important relations. From the definition of the set \( A^{(N,H)}_n \), we have:
\[(x_1, ..., x_{n-1}) \in A^{(N,H)}_n \iff Y^{(N,H)}_n (x_1, ..., x_{n-1}) \notin \left( -g^{(N,H)}_n, g^{(N,H)}_n - a^{(N)}_n \right) \]
\[\iff |Y^{(N,H)}_n (x_1, ..., x_{n-1}) + a^{(N)}_n| \geq g^{(N,H)}_n.\]
Since the paths in \( \{-1, 1\}^{n-1} \) are equidistributed, we have that:
\[
P \left( |Y^{(N,H)}_n + a^{(N)}_n| \geq g^{(N,H)}_n \right) = \frac{|A^{(N,H)}_n|}{2^{n-1}}.
\] (5.1)

In a similar way, we can see that
\[
P \left( \exists n \in \{1, ..., N\} : |Y^{(N,H)}_n + a^{(N)}_n| \geq g^{(N,H)}_n \right) = \frac{|A^{(N,H)}_n|}{2^{n-1}}.
\] (5.2)

Thanks to (5.1), the limit behaviour of the random variables \( \{Y^{(H)}_n\}_{n \geq 1} \) can be related with the proportion of arbitrage points at asymptotic levels in the fractional binary markets. More precisely, for any strictly increasing sequence of positive integers \( N_n \), we have the following relation:
\[
\frac{|A^{(N_n,H)}_n|}{2^{n-1}} = P \left( |Y^{(H)}_n + a^{(N_n)}_n N^{(H)}_n| \geq g^{(H)}_n \right).
\]

Note first that, since the random variables \( \xi_i \) are independents:
\[
\text{Var}(Y^{(H)}_n) = \sum_{i=1}^{n-1} (j_n^H (i))^2.
\]

In addition, in the next lemma we state that the variances of the random variables \( Y^{(H)}_n \) are uniformly bounded.

Lemma 5.1. For all \( n > 1 \):
\[
\text{Var}(Y^{(H)}_n) \leq \sigma H \left( 1 - \frac{\sigma^2}{(H + \frac{1}{n})^2} \right).
\]

Proof. We have from Corollary 3.3,
\[
\sigma n H \left( Z^{(n,H)}_1 - Z^{(n,H)}_{1-H} \right) = Y^{(H)}_n + a^{(H)} \xi_n.
\]

The result follows by taking the variance in both sides, using Lemma 3.5 and inequality (8) in [13] (p. 347). \[\square\]
Since the random variables \((Y^H_n)_{n \geq 1}\) are sums of independent random variables, one could expect to describe the asymptotics of these random variables by means of a CLT. However, it is not difficult to see that:

\[
\lim_{n \to \infty} j^H_n (n - 1) = \frac{\sigma^2}{H} + \frac{1}{2} \left( 2H + 1 - 2 \right) > 0. \tag{5.3}
\]

As a consequence, the Lindeberg condition can not be satisfied and a CLT can not be obtained by the classical assumptions. From (5.3), we can also conclude that the triangular array \(\{ j^H_n(i) \xi_i, 1 \leq i \leq n - 1 \}\) is not a null array and then, most of the interesting results for row-independent triangular arrays are not applicable. In our approach, in the next section we split our random variable \(Y^H_n\) as a sum of two independent random variables \(\bar{Y}^H_n\) and \(\hat{Y}^H_n\). The random variables \((\bar{Y}^H_n)_{n \geq 1}\) will be defined by means of a null array of random variables and standard results will be applied in Section 5.4 to understand its behaviour. On the other hand, we see in Section 5.4 that the nature of the random variables \((\hat{Y}^H_n)_{n \geq 1}\) is quite different and non-standard techniques will be used in order to describe its asymptotics.

5.2. Splitting the random variables \(Y^H_n\). As previously announced, the random variables \((Y^H_n)_{n \geq 1}\) are composed of two independent parts with very different properties. In order to visualize clearer how to split the random variables \((Y^H_n)_{n \geq 1}\), we study the estimates of \(j^H_n(i)\) obtained in Lemma 3.4. The first step is to look at the function \(g_n : (0, n - 1) \to (0, \infty)\) defined by \(g_n(x) = x^{-\alpha} \varphi^H_n(x)\), where \(\alpha = H - \frac{1}{y} \in (0, \frac{1}{y})\). The relation between this function and the coefficients \(j^H_n(i)\) is given by Lemma 3.4.

**Lemma 5.2.** For each \(n > 1\), there exist a unique \(x_n \in (0, n - 1)\) such that the function \(g_n\) is strictly decreasing in the interval \((0, x_n)\) and strictly increasing in the interval \((x_n, n - 1)\). In addition, we have:

\[
\lim_{n \to \infty} \frac{x_n}{\alpha(n - 1)} = 1.
\]

**Proof.** Let’s define the function \(f_n : (1/(n - 1), \infty) \to (0, \infty)\) by:

\[
f_n(y) = (ny - 1) - ((n - 1)y - 1)^\alpha.
\]

Note that \(g_n(x) = f_n(1/x)\). Thus, in order to prove the first result, it is enough to show that there exists a unique \(y_n \in (1/(n - 1), \infty)\) such that \(f_n\) is strictly decreasing in \((1/(n - 1), y_n)\) and strictly increasing in \((y_n, \infty)\) and then to put \(x_n = 1/y_n\). In fact, it is straightforward to prove that:

\[
f'_n(y) = \alpha n (ny - 1)^{\alpha - 1} \left[ 1 - \frac{(n - 1)}{n} \left( 1 + \frac{1}{n - 1 - \frac{1}{y}} \right)^{1-\alpha} \right],
\]

and then, the previous assertion comes from the fact that the function in the square parenthesis is strictly increasing and equal to 0 in only one point. This point is given by the unique solution of the equation:

\[
\left( 1 + \frac{1}{n - 1 - \frac{1}{y}} \right)^{1-\alpha} = \frac{n}{n - 1}.
\]

Solving this equation, calling \(y_n\) the solution and setting \(x_n = 1/y_n\), we obtain:

\[
x_n = n - 1 - \frac{1}{\left( 1 + \frac{1}{n - 1} \right)^{1/(1-\alpha)}}.
\]
The last result in the lemma is a consequence of the fact that:

\[
\left(1 + \frac{1}{n-1}\right)^{1/\alpha} - 1 \xrightarrow{n \to \infty} \frac{1}{1 - \alpha}.
\]

An important consequence of the previous lemma is that the upper and lower bounds of \(j_n^H(i)\) given in Lemma 3.4, viewed as functions of \(i\), are first decreasing until some index \(i_n\) and increasing after that. This is exactly the statement of the following corollary.

**Corollary 5.3.** Denote \(i_n = \lfloor x_n \rfloor + 1\). The function \(I_n : \{1, ..., n - 1\} \to (0, \infty)\) given in Lemma 3.4 is decreasing in \(\{1, ..., i_n - 1\}\) and increasing in \(\{i_n + 1, ..., n - 1\}\).

**Proof.** Direct from the definitions and Lemma 5.2.

It seems natural now to split the array \(\{j_n^H(i) \xi_i; 1 \leq i \leq n - 1, n \geq 1\}\) in two parts following the monotonicity properties of the estimates of the coefficients. In the next corollary we state a useful property for the first part of the array.

**Corollary 5.4.** The triangular array given by the family of random variables

\[
\{j_n^H(i) \xi_i; 1 \leq i \leq i_n - 1, n \geq 1\}
\]

is a null array.

**Proof.** By definition of null array, we have to prove that:

\[
\lim_{n \to \infty} \sup_{1 \leq i \leq i_n - 1} E[|j_n^H(i) \xi_i| \wedge 1] = 0.
\]

Note first that:

\[
E[|j_n^H(i) \xi_i| \wedge 1] = j_n^H(i) \wedge 1.
\]

Using Corollary 5.3 and Lemma 3.4 we obtain for \(1 \leq i \leq i_n - 1\):

\[
E[|j_n^H(i) \xi_i| \wedge 1] \leq \sigma c_H n^\alpha I_n(1),
\]

and then, it is enough to prove that \(n^\alpha I_n(1) \to 0\). On the other hand, the function \(x \mapsto \phi_n^H(x) = (n - x)^\alpha - (n - 1 - x)^\alpha\) is increasing, and then:

\[
n^\alpha I_n(1) \leq \frac{n^\alpha}{1 - \alpha} [(n - 1)^\alpha - (n - 2)^\alpha] \leq \frac{n^{2\alpha}}{1 - \alpha} \left(1 + \frac{1}{n - 2}\right)^\alpha - 1 \xrightarrow{n \to \infty} 0.
\]

In the last limit, we use that \(2\alpha < 1\). The result is proved.

The previous results suggest to split \(Y_n^H\) as \(\tilde{Y}_n^H + \hat{Y}_n^H\), where:

\[
\tilde{Y}_n^H = \sum_{i=1}^{i_n-1} j_n^H(i) \xi_i \quad \text{and} \quad \hat{Y}_n^H = \sum_{i=i_n}^{n-1} j_n^H(i) \xi_i.
\]

These random variables are clearly independent and symmetric. However, as announced at the beginning of this section, their properties as well as the techniques we use to deal with them are quite different.
5.3. On the random variables $\tilde{Y}_n^H$. From Theorem 5.11 in [9] and Corollary 5.4 we know that:

$$\sum_{i=1}^{i_n-1} (j_n^H(i))^2 \xrightarrow{n \to \infty} c \geq 0 \implies \tilde{Y}_n^H \xrightarrow{d} N(0, c).$$

The degenerate case $c = 0$ means that $\tilde{Y}_n^H$ converges in distribution to 0 (and then in probability) when $n$ goes to $\infty$. This is indeed the case and we can even prove that the convergence holds in $L^2(\Omega)$.

Proposition 5.5. We have that:

$$\tilde{Y}_n^H \xrightarrow{L^2} 0.$$

Proof. We have only to prove that $\text{Var}(\tilde{Y}_n^H)$ converges to 0. Note first that, since $\phi_n^H$ is increasing, we have for $1 < i \leq i_n - 1$:

$$\int_{i-1}^{i} x^{-\alpha} \phi_n^H(x) dx \leq \frac{1}{(i-1)^{\alpha}} \phi_n^H(i_n - 1).$$

Thus, using Lemma 5.3, we obtain:

$$(j_n^H(i))^2 \leq (\sigma c_n^H)^2 \frac{n^{2\alpha}}{(i-1)^{2\alpha}} (\phi_n^H(i_n - 1))^2.$$

Using again the monotonicity of $\phi_n^H$ and Lemma 5.2 we obtain for $n$ sufficiently large:

$$(j_n^H(i))^2 \leq (\sigma c_n^H)^2 \frac{n^{2\alpha}}{(i-1)^{2\alpha}} (\phi_n^H(2\alpha(n-1)))^2.$$

On the other hand, using similar arguments as in Lemma 5.2 or Corollary 5.4, we can prove that there exists $c_{\alpha}^* > 0$ such that for any $n$ large enough:

$$\phi_n^H(2\alpha(n-1)) \leq \frac{c_{\alpha}^*}{n^{1-\alpha}}.$$

As a consequence, there is a constant $c_{\alpha}^* > 0$ such that for any $n$ large enough:

$$(j_n^H(i))^2 \leq \frac{c_{\alpha}^*}{(i-1)^{2\alpha} n^{2-4\alpha}}.$$

Since:

$$\sum_{i=1}^{\infty} \frac{1}{i^{2\alpha} i^{2-4\alpha}} = \sum_{i=1}^{\infty} \frac{1}{i^{1+1-2\alpha}} < \infty,$$

by Kronecker’s lemma (see Lemma 4.21 in [9]), we conclude that:

$$\frac{1}{n^{2-4\alpha}} \sum_{i=1}^{i_n-1} \frac{1}{i^{2\alpha}} \xrightarrow{n \to \infty} 0.$$

It follows that:

$$\text{Var}(\tilde{Y}_n^H) \leq j_n^H(1)^2 + \frac{c_{\alpha}^*}{n^{2-4\alpha}} \sum_{i=2}^{i_n-1} \frac{1}{(i-1)^{2\alpha}} \xrightarrow{n \to \infty} 0.$$

The result is proved.

Remark 5.6. We can deduce from the previous lemma that there exists a sub-sequence $\tilde{Y}_{n_k}^H$ converging a.s. to 0.
5.4. On the random variables $\hat{Y}_n^H = (\hat{Y}_n^H)_{n \geq 1}$. From the discussion in Section 5.3.1 we know that the array $\{j_n^H(i); i_n \leq i \leq n-1, n \geq 1\}$ is not a null array. As a consequence, useful results for row-independent triangular arrays, like Theorem 5.11 of [9], are not applicable. However, we know from Lemma 5.3.1 that:

$$\sup_{n \geq 1} E(|\hat{Y}_n^H|^2) \leq \sigma^2 \left(1 - \frac{C_H^2}{(H + \frac{3}{2})^2}\right) < \infty.$$ 

A classical result in analysis implies the existence of a subsequence of $(\hat{Y}_n^H)_{n \geq 1}$ which is weakly convergent in $L^2$. Unfortunately, this argument does not provide the uniqueness of the limit and one cannot go further in this direction. Next lemma provides us a stronger result than the uniform boundedness in $L^2$, namely, the convergence of the variances. This result gives also an insight about the limit candidate.

**Lemma 5.7.** Let $Z^h = (Z^h)_{t \geq 0}$ be a fractional Brownian motion with Hurst parameter $h = \frac{H}{2} + \frac{1}{4} \in (\frac{1}{2}, \frac{3}{4})$, then:

$$\text{Var}(\hat{Y}_n^H) \xrightarrow{n \to \infty} 4 \left(\frac{\sigma c_H}{H + \frac{3}{2}}\right)^2 \sum_{k=1}^{\infty} \rho_k^2(k) < \infty,$$

where:

$$\rho_k(k) = \text{Cov}(Z^h_{i_n+k+1} - Z^h_k) = \frac{1}{2} \left((k+1)^{2h} + (k-1)^{2h} - 2 k^{2h}\right).$$

**Proof.** Note first that:

$$\text{Var}(\hat{Y}_n^H) = \sum_{i=i_n}^{n-1} (j_n^H(i))^2 = \sum_{k=1}^{n-i_n} (j_n^H(n-k))^2.$$ 

On the other hand, using Lemma 5.3.1 we obtain for $1 \leq k \leq n-i_n$:

$$j_n^H(n-k) \leq \sigma c_H \left(\frac{n}{n-k-1}\right) \left((k+1)^{1+\alpha} + (k-1)^{1+\alpha} - 2k^{1+\alpha}\right)$$

and

$$j_n^H(n-k) \geq \sigma c_H \left(\frac{n-1}{n-k}\right) \left((k+1)^{1+\alpha} + (k-1)^{1+\alpha} - 2k^{1+\alpha}\right).$$

It follows that, for any $k \geq 1$:

$$\lim_{n \to \infty} j_n^H(n-k) = \sigma c_H \left(\frac{(k+1)^{1+\alpha} + (k-1)^{1+\alpha} - 2k^{1+\alpha}}{1 + \alpha}\right) = \frac{2\sigma c_H}{H + \frac{3}{2}} \rho_k(k).$$

In the same way, using Lemma 5.2 and the previous upper bound for $j_n^H(n-k)$, one can find a constant $M > 0$, such that for any $n$ sufficiently large:

$$j_n^H(n-k) \leq M \rho_k(k).$$

On the other hand, recognizing that $\rho_k(k)$ represents the covariance between $Z^h_{i_n+k+1} - Z^h_k$, we can use 2.3.3 to see that:

$$\rho_k(k) \sim h(2h-1) k^{2h-2} = \frac{(2H+1)(2H-1)}{8} k^{H-\frac{3}{2}} \text{ as } k \to \infty,$$

and conclude that $\sum_{k=1}^{\infty} \rho_k^2(k) < \infty$. We can then use the dominated convergence theorem to obtain the desired result. \qed
Since $h > \frac{1}{2}$, it is well known that the increments of $Z^h$ are positively correlated and exhibit long-range dependence. As a consequence, we have that $\sum_{k=1}^{\infty} \rho_h(k) = \infty$. In particular, the series $\sum_{k=1}^{\infty} \rho_h(k) \xi_k$ are not absolutely convergent. Nevertheless, since $\sum_{k=1}^{\infty} \rho^2_h(k) < \infty$, we know that $\sum_{k=1}^{\infty} \rho_h(k) \xi_k < \infty$ a.s. (see p. 113 in [15] or [8]).

Motivated by Lemma 5.7 and the previous discussion, we introduce the random variables $(\Upsilon^H_n)_{n \geq 1}$ and $\Upsilon^H$ as follows:

$$\Upsilon^H_n := \sum_{k=1}^{n-i_n} j^H_n (n - k) \xi_k, \quad n \in \mathbb{N}.$$  

and

$$\Upsilon^H := \frac{2\sigma c_H}{H + \frac{1}{2}} \sum_{k=1}^{\infty} \rho_h(k) \xi_k.$$  

Note that $\text{Var}(\Upsilon^H_n) = \text{Var}(\hat{\Upsilon}^H_n)$. In addition, since the random variables $(\xi_i)_{i \geq 1}$ are i.i.d, we have even that $\Upsilon^H_n \overset{d}{=} \hat{\Upsilon}^H_n$. Next proposition reinforces Lemma 5.7 with the help of the random variables $\Upsilon^H_n$, and permits to conclude the convergence in law of the random variables $\hat{\Upsilon}^H_n$.

**Proposition 5.8.** We have that:

$$\Upsilon^H_n \overset{L^2}{\underset{n \to \infty}{\longrightarrow}} \Upsilon^H.$$  

As a consequence, we have that:

$$\hat{\Upsilon}^H_n \overset{d}{\underset{n \to \infty}{\longrightarrow}} \Upsilon^H.$$  

**Proof.** Let’s denote $k_n = n - i_n$ and $d_H = \frac{2\sigma c_H}{H + \frac{1}{2}}$. It is straightforward to see that:

$$E((\Upsilon^H_n - \Upsilon^H)^2) = E \left( \left( \sum_{k=1}^{k_n} (j^H_n (n - k) - d_H \rho_h(k)) \xi_k - d_H \sum_{k=k_n+1}^{\infty} \rho_h(k) \xi_k \right)^2 \right)$$  

$$= \sum_{k=1}^{k_n} (j^H_n (n - k) - d_H \rho_h(k))^2 + d_H^2 \sum_{k=k_n+1}^{\infty} \rho^2_h(k).$$  

Note that the convergence to 0 of the second sum is guaranteed as:

$$\sum_{k=1}^{\infty} \rho^2_h(k) < \infty \Rightarrow \sum_{k=k_n+1}^{\infty} \rho^2_h(k) \overset{n \to \infty}{\longrightarrow} 0.$$  

We only have to show now that the first sum on the right-hand side also converges to 0. For this, using the estimates obtained in the proof of the previous lemma, we have that for each $k \leq k_n$:

$$j^H_n (n - k) - d_H \rho_h(k) \overset{n \to \infty}{\longrightarrow} 0$$  

and $|j^H_n (n - k) - d_H \rho_h(k)| \leq C \rho_h(k)$ for some constant $C > 0$. The result follows as an application of the dominated convergence theorem.

For the second part of the result, we use that convergence in $L^2$ implies convergence in law and that $\Upsilon^H_n \overset{d}{=} \hat{\Upsilon}^H_n$. \qed
Remark 5.9. The random variables \( \tilde{Y}_n^H \) and \( Y_n^H \) are of a different nature. Even if they have the same law, the first one depends on the “last” elements of the sequence \((\xi_i)\) and the second one on the “first” elements. A very relevant point in this respect is the following. Since the random variables \( Y_n^H \) converge in \( L^2 \) to \( Y_H \), then there is a subsequence of \( Y_n^H \) convergent to \( Y_H \) a.s. On the other hand, we will see in Section 6.1 that \( Y_n^H \) has no subsequences convergent a.s.

5.5. On the convergence of \((Y_n^H)_{n \geq 1}\) and applications. Now we have all the necessary elements to establish the main results of this section, namely, the convergence of the random variables \((Y_n^H)_{n \geq 1}\) and the characterization of the proportion of arbitrage points at asymptotic levels.

Theorem 5.10. We have that:

\[ Y_n^H \xrightarrow{d} Y_H. \]

Proof. The result follows immediately from the fact that \( Y_n^H = \tilde{Y}_n^H + \hat{Y}_n^H \) and applying the previous results. Indeed, Proposition 5.8 gives us that \( \tilde{Y}_n^H \xrightarrow{p} 0 \), and since, by Proposition 5.9, \( \hat{Y}_n^H \xrightarrow{d} Y_H \) we obtain that

\[ (\tilde{Y}_n^H, \hat{Y}_n^H) \xrightarrow{d} (Y_H, 0). \]

Applying now the continuous mapping theorem for the function \( g(x, y) = x + y \), we get that

\[ Y_n^H = \tilde{Y}_n^H + \hat{Y}_n^H = g(\tilde{Y}_n^H, \hat{Y}_n^H) \xrightarrow{d} g(Y_H, 0) = Y_H. \]

\[ \square \]

Remark 5.11. The proof of this theorem can be shortened by using the Slutski’s theorem (see Theorem 8.6.1, Chapter 8 in [12]).

Theorem 5.12. The law of \( Y_H \) is absolutely continuous with respect to the Lebesgue measure, its density \( f_H \) is symmetric, bounded, \( L^2(\mathbb{R}) \)-integrable and has non-compact support.

Proof. We claim that the characteristic function of \( Y_H \), \( F_H(u) = \mathbb{E}[e^{iuY_H}] \), decays faster than exponentially. If this is true, then \( F_H \) is in \( L^2(\mathbb{R}) \) and the law of \( Y_H \) admits a density function \( f_H \) in \( L^2(\mathbb{R}) \) (Lemma 2.1 in [1]). The relation between the \( L^2 \)-norms of \( F_H \) and \( f_H \) is given by the Plancherel’s theorem. We can also deduce that \( F_H \) is in \( L^1(\mathbb{R}) \), which implies that \( f_H \) is bounded (Corollary 5.1, Chapter 9 in [12]). The fact that \( f_H \) is symmetric comes from the symmetry of the law of \( Y_H \). The last assertion is a consequence of the uncertainty principle, which formally asserts that \( F_H \) and \( f_H \) cannot both decay too fast at infinity (see for example [7]).

Now, we turn to the proof of the claim. Note first that:

\[ E[e^{iuY_H}] = E\left[\exp\left(iud_H \sum_{k=1}^{\infty} \rho_H(k) \xi_k\right)\right] = E\left[\prod_{k=1}^{\infty} \exp(iud_H \rho_H(k) \xi_k)\right] = \prod_{k=1}^{\infty} E[\exp(iud_H \rho_H(k) \xi_k)] = \prod_{k=1}^{\infty} \cos(u \rho_H(k)), \]

where \( d_H = \frac{2 \pi e_q}{H + \frac{1}{2}}. \) The first step will be to obtain good estimates for \( \cos(u \rho_H(k)) \). We assert that for any \( x \in (0, \pi/2) \):

\[ 0 < \cos(x) \leq 1 - \frac{x^2}{\pi}. \]
In order to prove that, we consider the function \( f \) defined by 
\[
 f(x) = 1 - \frac{x^2}{
\pi} - \cos(x).
\]
Since \( f(0) = 0 \), it would be enough to prove that \( f \) is increasing in \((0, \pi/2)\). This is indeed the case, as for each \( x \in (0, \pi/2) \):
\[
 f'(x) = -\frac{2x}{\pi} + \sin(x) \geq 0,
\]
which proves our assertion (the last inequality follows from the concavity of the sinus function on \((0, \pi/2)\)).

On the other hand, since \( \rho_h(k) \sim h(2h-1)k^{2h-2} \) when \( k \) goes to infinity, we can find \( k_0 \) and \( b_H > \gamma_H > 0 \) such that, for any \( k > k_0 \):
\[
 \gamma_H^{-\beta} \leq \rho_h(k) \leq b_H^{-\beta}
\]
where \( \beta = 2 - 2h \in (1/2, 1) \). Now, for each \( u > 0 \), we define:
\[
 k(u) = k_0 \vee \inf \left\{ k \in \mathbb{N} : \frac{u b_H}{k^\beta} \leq \frac{\pi}{2} \right\}.
\]
From the definition, we have that for any \( k \geq k(u) \):
\[
 0 < \frac{u \gamma_H}{k^\beta} \leq u \rho_h(k) \leq \frac{u b_H}{k^\beta} \leq \frac{\pi}{2}. 
\]
In particular,
\[
 \prod_{k=1}^{\infty} |\cos(u \rho_h(k))| \leq \prod_{k=k(u)}^{\infty} |\cos(u \rho_h(k))| \leq \prod_{k=k(u)}^{\infty} \left( 1 - \frac{(u \rho_h(k))^2}{\pi} \right),
\]
and then:
\[
 \prod_{k=1}^{\infty} |\cos(u \rho_h(k))| \leq \prod_{k=k(u)}^{\infty} \left( 1 - \frac{(u \gamma_H)^2}{\pi k^{2\beta}} \right).
\]
On the other hand:
\[
 \ln \left( \prod_{k=k(u)}^{\infty} \left( 1 - \frac{(u \gamma_H)^2}{\pi k^{2\beta}} \right) \right) = \sum_{k=k(u)}^{\infty} \ln \left( 1 - \frac{(u \gamma_H)^2}{\pi k^{2\beta}} \right).
\]
Note that for \( u \) big enough, \( k(u) > k_0 \) and hence \( k(u) = \inf \{ k \in \mathbb{N} : \frac{u b_H}{k^\beta} \leq \frac{\pi}{2} \} \). This implies that \((k(u) - 1)^\beta \leq \frac{2u b_H}{\pi} \leq k(u)^\beta \) and then:
\[
 \sum_{k=k(u)}^{\infty} \ln \left( 1 - \frac{(u \gamma_H)^2}{\pi k^{2\beta}} \right) \leq \sum_{k=k(u)}^{\infty} \ln \left( 1 - \frac{\pi \gamma_H^2}{(2b_H)^2} \left( \frac{k(u) - 1}{k} \right)^{2\beta} \right)
\]
\[
 \leq (k(u) - 1) \int_{1}^{\infty} \ln \left( 1 - \frac{\pi \gamma_H^2}{(2b_H)^2 x^{2\beta}} \right) dx
\]
\[
 \leq -\theta_H u^{1/\beta},
\]
for some constant \( \theta_H > 0 \). Since the above construction works for any \( u > 0 \) sufficiently large, we have that:
\[
 |E[e^{iuY_H}]| \leq e^{-\theta_H (d_H u)^{1/\beta}},
\]
and by symmetry we obtain, that for \(|u|\) sufficiently large:
\[
 |E[e^{iuY_H}]| \leq e^{-\theta_H (d_H |u|)^{1/\beta}}.
\]
The claim is then proved. \(\square\)
Theorem 5.13. For any sequence $N_n \geq n$:

$$\lim_{n \to \infty} \frac{|A_{n}^{(N,H)}|}{2^{n-1}} = P(|Y_n| > g_H) > 0.$$ 

In particular:

$$\lim_{N \to \infty} \frac{|A^{(N,H)}|}{2^N - 1} = P(|Y_H| > g_H).$$

Proof. We know that

$$W_e \text{know that}$$

$$|X + Y| > \epsilon$$

by Theorem 5.13.

Now using the fact that the law of $X$ and by the continuous mapping theorem, this time applied to the function $(x, y) \mapsto \frac{x}{y}1_{y \neq 0}$ to obtain that

$$\frac{Y_H}{g_H^n} \xrightarrow{n \to \infty} \frac{Y_H}{g_H}.$$ 

Since $|a_{n}^{(N)}| \leq \frac{||a||_{N}}{N_n}$ and $\lim_{n \to \infty} g_H^n = g_H > 0$ we have that $\lim_{n \to \infty} \frac{a_{n}^{(N)} N_H}{g_H^n} = 0$ deterministically and, therefore in probability. This gives us that

$$\left(\frac{Y_H}{g_H^n} + \frac{a_{n}^{(N)} N_H}{g_H^n}\right) \xrightarrow{n \to \infty} \left(\frac{Y_H}{g_H}, 0\right)$$

and by the continuous mapping theorem, this time applied to the function $(x, y) \mapsto x + y$, we obtain that

$$\frac{Y_H}{g_H^n} + \frac{a_{n}^{(N)} N_H}{g_H^n} \xrightarrow{n \to \infty} \frac{Y_H}{g_H}.$$ 

Now using the fact that the law of $Y_H$ is absolutely continuous, and applying the Portmanteau theorem and Lemma 17.2 of [15] for the random variables $Z_n = \frac{Y_H + a_{n}^{(N)} N_H}{g_H^n}$ and $Z = \frac{2(Y_H - \epsilon)}{g_H}$, it follows

$$\lim_{n \to \infty} P\left(|Y_H + a_{n}^{(N)} N_H| \geq g_H\right) = \lim_{n \to \infty} P\left(\frac{|Y_H + a_{n}^{(N)} N_H|}{g_H^n} \geq 1\right)$$

$$= P\left(\frac{|Y_H|}{g_H} > 1\right) = P(|Y_H| > g_H).$$

Since the density $f_H$ has no compact support, $P(|Y_H| > g_H)$ is strictly positive and the proof is achieved.

For the second statement in the theorem, first note that:

$$|A^{(N,H)}| = \sum_{n=1}^{N} |A_{n}^{(N,H)}| = \sum_{n=1}^{N} P(|Y_n| + a_{n}^{(N)} N_H| \geq g_H^n) 2^{n-1}. \quad (5.4)$$

Now, fix $\epsilon > 0$ and consider $N$ sufficiently large in order to verify $||a||_{N^{-1-H}} \leq \epsilon$. For such $N$ and $n \leq N$ it is straightforward to see that:

$$P(|Y_n| \geq g_H^n + \epsilon) \leq P(|Y_n| + a_{n}^{(N)} N_H| \geq g_H^n) \leq P(|Y_n| \geq g_H^n - \epsilon),$$

and then, plugging this in (5.4), we get:

$$\sum_{n=1}^{N} P(|Y_n| \geq g_H^n + \epsilon) 2^{n-1} \leq |A^{(N,H)}| \leq \sum_{n=1}^{N} P(|Y_n| \geq g_H^n - \epsilon) 2^{n-1}. \quad (5.5)$$
Additionally, using the convergence properties of $Y^H_n$ and $g^H_n$ as in the proof of the first statement above, we get:

$$P(|Y^H_n| \geq g^H_n + \epsilon) \xrightarrow{n \to \infty} P(|Y^H| > g_H + \epsilon),$$

and

$$P(|Y^H_n| \geq g^H_n - \epsilon) \xrightarrow{n \to \infty} P(|Y^H| > g_H - \epsilon).$$

On the other hand, it is a well known fact that, the convergence of a sequence of real numbers implies the convergence of its arithmetic means to the same limit. As a consequence, if $(a_n)_{n \geq 1}$ is a sequence of real numbers converging to $a$, then:

$$\frac{1}{2^N - 1} \sum_{k=1}^{N} a_k 2^{k-1} \xrightarrow{N \to \infty} a.$$ 

Using this in (5.5):

$$P(|Y^H| > g_H + \epsilon) \leq \lim \inf_{N \to \infty} \frac{|A^{(N,H)}|}{2^N - 1} \leq \lim \sup_{N \to \infty} \frac{|A^{(N,H)}|}{2^N - 1} \leq P(|Y^H| > g_H - \epsilon).$$

The result follows by taking the limit when $\epsilon$ tends to 0.

**Remark 5.14.** Note that, when the drift is zero in the above proof, the second statement in the theorem is a direct application of the first one, as the involved sets of arbitrage points at each level don’t depend on $N$.

### 5.6. On the functions $\rho_k(.)$.

We have already established the main results of this section. Nevertheless, we would like to have more information about the quantities $P(|Y^H| > g_H)$. In order to do that, the first step will be to obtain estimates for the autocovariance functions $\rho_k(.)$ and its $\ell^2$ norm.

**Lemma 5.15.** For all $k \geq 1$, the functions $F_k : (\frac{1}{2}, \frac{1}{4}) \to \mathbb{R}^+$ defined by $F_k(h) = \rho_k(h)$ are increasing. As a consequence:

$$\lim_{h \to \frac{1}{2}^+} \sum_{k=1}^{\infty} \rho^2_k(k) = 0.$$

**Proof.** First note that the second assertion follows from the first one as an application of the monotone convergence theorem. Now we turn to the proof of the first assertion. For $k = 1$, we have $2F_1(h) = 2^{2h} - 2$ and the assertion is clearly true. For $k > 1$, we note that:

$$2F_k(h) = k^{2h} G_k(2h),$$

where for $\varepsilon \in (0, 1)$ and $x > 1$:

$$G_\varepsilon(x) = (1 + \varepsilon)^x + (1 - \varepsilon)^x - 2.$$

If we show that $G_\varepsilon$ is increasing in $(1, \infty)$, then the first assertion of the lemma follows. One can easily see that $G_\varepsilon(x) \xrightarrow{|x| \to \infty} \infty$ and:

$$G_\varepsilon'(x) = \ln(1 + \varepsilon)e^{x\ln(1 + \varepsilon)} + \ln(1 - \varepsilon)e^{x\ln(1 - \varepsilon)}.$$

In addition, $G_\varepsilon'(x_*(\varepsilon)) = 0$ if and only if:

$$x_*(\varepsilon) = \frac{\ln \left( \frac{\ln(1/(1-\varepsilon))}{\ln(1+\varepsilon)} \right)}{\ln \left( \frac{1+\varepsilon}{1-\varepsilon} \right)}.$$

It remains to prove that $G_\varepsilon'(1) > 0$ and $x_*(\varepsilon) < 1$. These are a consequence of the fact that the function $g$ given by:

$$g(x) = (1 + x)\ln(1 + x) + (1 - x)\ln(1 - x)$$
is increasing in $(0,1)$. Then, since $G'_*(1) = g(\varepsilon) > g(0) = 0$ we obtain the first assertion and that:

\[
\frac{\ln(1/(1-\varepsilon))}{\ln(1+\varepsilon)} < \frac{1+\varepsilon}{1-\varepsilon}.
\]

The second assertion follows by taking logarithm on both sides of the inequality. \qed

**Lemma 5.16.** For any $h \in (\frac{1}{2}, \frac{3}{4})$:

\[
\rho_h(k) \geq \frac{h(2h-1)}{2k^{2-2h}}, \quad k \geq 1.
\]

In particular:

\[
\sum_{k=1}^{\infty} \rho_h^2(k) \geq \frac{h^2(2h-1)^2}{4} \zeta(4-4h),
\]

where $\zeta$ is the Riemann zeta function. As a consequence:

\[
\lim_{h \to \frac{3}{4}} \sum_{k=1}^{\infty} \rho_h^2(k) = \infty.
\]

**Proof.** Let’s first prove that the function $f_h : [0,1) \to \mathbb{R}$ given by:

\[
f_h(x) = (1+x)^{2h} + (1-x)^{2h} - 2h(2h-1)x^2
\]

is positive. Note that:

\[
f'h(x) = 2h ((1+x)^{2h-1} - (1-x)^{2h-1} - (2h-1)x),
\]

and

\[
f''h(x) = 2h(2h-1) ((1+x)^{2h-2} + (1-x)^{2h-2} - 1) \geq 0.
\]

Since $f'_h$ is increasing and $f'_h(0) = 0$, we conclude that $f'_h$ is positive. Thus, $f_h$ is increasing and since $f_h(0) = 0$, the claim follows. On the other hand, we have:

\[
\rho_h(k) = \frac{k^{2h}}{2} f_h \left( \frac{1}{k} \right) + \frac{h(2h-1)}{2k^{2-2h}}.
\]

The result follows. \qed

As a consequence of the two previous results, we can deduce the following corollary.

**Corollary 5.17.** There exists $\frac{1}{2} < h_c < \frac{3}{4}$ such that:

1. For all $h \in (\frac{1}{2}, h_c)$: $\sum_{k=1}^{\infty} \rho_h^2(k) < \frac{1}{4}$.

2. For all $h \in (h_c, \frac{3}{4})$: $\sum_{k=1}^{\infty} \rho_h^2(k) > \frac{1}{4}$.

**Proof.** One can observe first that the continuity and the monotonicity of the auto-covariance functions with respect to the parameter $h$ and the monotone convergence theorem implies the continuity of the function $h \mapsto \sum_{k=1}^{\infty} \rho_h^2(k)$. The statements in the corollary follow then directly from the mean value theorem and from Lemmas 5.15 and 5.16. \qed

Another important consequence of the estimates obtained in Lemmas 5.16 and 5.15 is related with the behaviour of $\gamma_H$ when $H$ is close to $1/2$ and when $H$ is close to $1$. In the next corollary, we provide a precise result around $H = 1/2$ and a partial result around $H = 1$. 
Corollary 5.18. We have that:

$$\lim_{H \to 1} P(|Y_H| > g_H) = 0$$

and

$$\liminf_{H \to 1} P(|Y_H| > g_H) \geq \frac{1}{3}.$$ 

Proof. Thanks to the Tchebyshcheff’s inequality and Corollary 3.6, we obtain that:

$$P(|Y_H| > g_H) \leq \frac{4}{\sum_{k=1}^{\infty} \rho^2_h(k)}.$$ 

In addition, from Lemma 5.15, we have that:

$$\lim_{H \to 1} \frac{1}{2} \sum_{k=1}^{\infty} \rho^2_h(k) = 0,$$

and the first result follows.

For the second result, we use the Paley-Zigmund inequality (Lemma 4.1 in [9]) and a particular case of the Khintchine’s inequality (see [10]), to obtain for $H > 2h_c - 1/2$:

$$P(|Y_H| > g_H) \geq \frac{1}{3} \left( 1 - \frac{1}{4 \sum_{k=1}^{\infty} \rho^2_h(k)} \right)^2.$$ 

On the other hand, we know from Lemma 5.10 that:

$$\lim_{H \to 1} \sum_{k=1}^{\infty} \rho^2_h(k) = \infty.$$ 

Combining this with the previous inequality, we obtain the desired result. □

6. On the asymptotic proportion of arbitrage paths

6.1. On the Kolmogorov 0-1 law. Now, we will try to exploit the nature of the random variables $\hat{Y}_H^n$. Intuitively, in the limit, these random variables should depend only on the tail $\sigma$-field, which is defined by:

$$\mathcal{T}_\infty = \bigcap_{n \geq 1} \bigvee_{k > n} \sigma(\xi_k) = \bigcap_{n \geq 1} \sigma(\xi_k, k > n).$$ 

Since the random variables $(\xi_k)_{k \geq 1}$ are independent, we know from the Kolmogorov 0-1 law (see Theorem 3.13 in [9]) that $\mathcal{T}_\infty$ is $P$-trivial and that the $\mathcal{T}_\infty$-measurable random variables are constant. One could be tempted to guess that $Y_H$ is then constant, which is in contradiction with the fact that its variance is strictly positive. This contradiction is only apparent and the reason is that the random variables $\hat{Y}_H^n$ converge to $Y_H$ only in distribution and one can not conclude that $Y_H$ is $\mathcal{T}_\infty$-measurable. One can in particular say that there is no subsequence of $\hat{Y}_H^n$ convergent a.s. to $Y_H$. Anyhow, we can still use our naive idea in order to obtain some interesting results.

Before we state the following lemma, let’s recall some definitions. For any sequence of measurable sets $A_1, A_2, \ldots$, we define $\{A_n \text{ i.o.}\}$ and $\{A_n \text{ ult.}\}$, respectively the sets where $A_n$ happens infinitely often and where $A_n$ happens ultimately, by:

$$\{A_n \text{ i.o.}\} := \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k \quad \text{and} \quad \{A_n \text{ ult.}\} := \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k.$$
Thus, the first result is a consequence of Lemma 6.1.

Let 

\[ \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k = \bigcap_{n \geq m} \bigcup_{k \geq n} A_k. \]

and

\[ \bigcup_{n \geq 1} A_k = \bigcup_{n \geq m} A_k. \]

Lemma 6.1. Consider a sequence of positive numbers \( s_n \). For any subsequence \( n_k \), the sets \( \{ |\hat{\mathcal{Y}}_{n_k}^H| > s_{n_k} \ i.o. \} \) and \( \{ |\hat{\mathcal{Y}}_{n_k}^H| > s_{n_k} \ \text{ult.} \} \) have probability 0 or 1.

Proof. By the Kolmogorov 0-1 law, it is enough to prove that both sets belong to the tail \( \sigma \)-field. More precisely, we have to prove that for all \( m \geq 1 \), both sets belong to \( \sigma(\xi_i, i > m) \).

First, fix \( m \geq 1 \). Since \( i_k \to \infty \) and \( n_k \to \infty \) when \( k \) goes to infinity, we can find \( k_m \) such that \( i_{n_{km}} > m \). On the other hand, we have that:

\[ \{ |\hat{\mathcal{Y}}_{n_k}^H| > s_{n_k} \} \in \sigma(\xi_i, i \geq i_{n_k}). \]

Thus,

\[ \bigcup_{k \geq \ell} \{ |\hat{\mathcal{Y}}_{n_k}^H| > s_{n_k} \} \in \sigma(\xi_i, i \geq i_{n_k}) \quad \text{and} \quad \bigcap_{k \geq \ell} \{ |\hat{\mathcal{Y}}_{n_k}^H| > s_{n_k} \} \in \sigma(\xi_i, i \geq i_{n_k}). \]

We conclude that:

\[ \{ |\hat{\mathcal{Y}}_{n_k}^H| > s_{n_k} \ i.o. \} = \bigcap_{\ell \geq k_m} \bigcup_{k \geq \ell} \{ |\hat{\mathcal{Y}}_{n_k}^H| > s_{n_k} \} \subseteq \sigma(\xi_i, i > m) \]

and

\[ \{ |\hat{\mathcal{Y}}_{n_k}^H| > s_{n_k} \ \text{ult.} \} = \bigcup_{\ell \geq k_m} \bigcap_{k \geq \ell} \{ |\hat{\mathcal{Y}}_{n_k}^H| > s_{n_k} \} \subseteq \sigma(\xi_i, i > m) \]

and the proof is completed. \( \square \)

Lemma 6.2. Let \( n_k \) a strictly increasing sequence of positive integers. For any \( H < 2h_c - 1/2 \), where \( h_c \) is chosen like in Corollary 5.17, there exists \( \varepsilon > 0 \):

\[ P(|\hat{\mathcal{Y}}_{n_k}^H| > g_{n_k}^H (1 - \varepsilon) \ \text{ult.}) = 0 \]

and for any \( H > 2h_c - 1/2 \), there exists \( \delta > 0 \):

\[ P(|\hat{\mathcal{Y}}_{n_k}^H| > g_{n_k}^H (1 + \delta) \ i.o.) = 1. \]

Proof. Fix \( H < 2h_c - 1/2 \). By Corollary 5.17, we can choose \( \varepsilon \) such that

\[ \sum_{k=1}^{\infty} \rho_k^2(k) < \frac{(1 - \varepsilon)^2}{4} < \frac{1}{4}. \]

Using Fatou’s lemma and Tchebyshoff’s inequality we obtain that:

\[ P(|\hat{\mathcal{Y}}_{n_k}^H| > g_{n_k}^H (1 - \varepsilon) \ \text{ult.}) \leq \liminf_{k \to \infty} P(|\hat{\mathcal{Y}}_{n_k}^H| > g_{n_k}^H (1 - \varepsilon)) \leq \frac{4}{(1 - \varepsilon)^2} \sum_{k=1}^{\infty} \rho_k^2(k) < 1, \]

Thus, the first result is a consequence of Lemma 6.1.

Now, fix \( H > 2h_c - 1/2 \) and choose \( \delta > 0 \) such that

\[ \sum_{k=1}^{\infty} \rho_k^2(k) > \frac{(1 + \delta)^2}{4} > \frac{1}{4}. \]

Another application of Fatou’s lemma, implies that:

\[ P(|\hat{\mathcal{Y}}_{n_k}^H| > g_{n_k}^H (1 + \delta) \ i.o.) \geq \limsup_{n \to \infty} P(|\hat{\mathcal{Y}}_{n_k}^H| > g_{n_k}^H (1 + \delta)). \]
On the other hand, using Paley-Zigmund inequality and Khintchine’s inequality, we obtain that:

\[
\limsup_{n \to \infty} P(|\hat{Y}_{n_k}^H| > g_{n_k}^H (1 + \delta)) = P(|Y_H| > g_H (1 + \delta)) \\
\geq \frac{1}{3} \left( 1 - \frac{(1 + \delta)^2}{4 \sum_{k=1}^{\infty} p_k^2(k)} \right)^2 > 0,
\]

and the second statement follows from Lemma 6.1.

\[\square\]

**Theorem 6.3.** For \( H > 2h_c - 1/2 \)

\[ P(|Y_n^H| > g_n^H \text{ i.o.}) = 1 \]

and for \( H < 2h_c - 1/2 \)

\[ P(|Y_n^H| > g_n^H \text{ ult.}) = 0 \]

**Proof.** Since \( \hat{Y}_n^H \) converges to 0 in \( L^2 \), we can extract a subsequence \( \hat{Y}_{n_k}^H \) such that the convergence holds almost surely. Consider \( H > 2h_c - 1/2 \) and let \( \delta > 0 \) as in Lemma 6.2. Note that

\[
\{|\hat{Y}_{n_k}^H| > (1 + \delta)g_{n_k}^H\} \subseteq \{|\hat{Y}_{n_k}^H| > (1 + \delta)g_{n_k}^H\} \cap \{|\hat{Y}_{n_k}^H| \leq \delta g_{n_k}^H\} \\
\subseteq \{|Y_{n_k}^H| > g_{n_k}^H\} \cup \{|\hat{Y}_{n_k}^H| > \delta g_{n_k}^H\},
\]

which implies that

\[
\{|\hat{Y}_{n_k}^H| > (1 + \delta)g_{n_k}^H \text{ i.o.}\} \subseteq \{|Y_{n_k}^H| > g_{n_k}^H \text{ i.o.}\} \cup \{|\hat{Y}_{n_k}^H| > \delta g_{n_k}^H \text{ i.o.}\}.
\]

Since \( \hat{Y}_{n_k}^H \xrightarrow{a.s.} 0 \) and \( g_{n_k}^H \to g_H > 0 \), it follows that \( P(\{|\hat{Y}_{n_k}^H| > \delta g_{n_k}^H \text{ i.o.}\}) = 0 \). The desired statement is a consequence of Lemma 6.2 and of the fact that

\[
\{|Y_{n_k}^H| > g_{n_k}^H \text{ i.o.}\} \subseteq \{|Y_{n_k}^H| > g_{n_k}^H \text{ i.o.}\}.
\]

For \( H < 2h_c - 1/2 \), let \( \epsilon > 0 \) as in Lemma 6.2. Note that

\[
\{|Y_{n_k}^H| > g_{n_k}^H\} = \{|Y_{n_k}^H| > g_{n_k}^H\} \cap \{|Y_{n_k}^H| \leq \epsilon g_{n_k}^H\} \\
\subseteq \{|Y_{n_k}^H| > (1 - \epsilon)g_{n_k}^H\} \cup \{|\hat{Y}_{n_k}^H| > \epsilon g_{n_k}^H\},
\]

which implies that

\[
\{|Y_{n_k}^H| > g_{n_k}^H \text{ ult.}\} \subseteq \{|\hat{Y}_{n_k}^H| > (1 - \epsilon)g_{n_k}^H \text{ ult.}\} \cup \{|\hat{Y}_{n_k}^H| > \epsilon g_{n_k}^H \text{ i.o.}\}.
\]

Following the same reasoning as before, the desired statement is obtained using Lemma 6.2 and the fact that

\[
\{|Y_{n_k}^H| > g_{n_k}^H \text{ ult.}\} \supseteq \{|Y_{n_k}^H| > g_{n_k}^H \text{ ult.}\}.
\]

\[\square\]

**Remark 6.4.** The results in the above theorem can be viewed as follows. In the case when \( H > 2h_c - 1/2 \), with probability 1 a path in the binary tree asymptotically crosses infinite number of arbitrage points, whereas, when \( H < 2h_c - 1/2 \), with probability 1 a path in the binary tree asymptotically crosses an infinite number of non-arbitrage points. In particular, when \( H > 2h_c - 1/2 \), a path in the binary tree crosses almost surely at least one arbitrage point, which intuitively implies that the asymptotic proportion of arbitrage path is one. This is illustrated in the next corollary.
Corollary 6.5. If $H > 2h_c - 1/2$ then
\[
\lim_{N \to \infty} \frac{A_{\mathcal{E}}(N,H)}{2^{N-1}} = 1.
\]

Proof. Note that
\[
\{ |Y^H_n| > g_n^H \text{ i.o.} \} \subseteq \{ \exists \ n : \ |Y^H_n| > g_n^H \}
\]
and
\[
\{ \exists \ n : \ |Y^H_n| > g_n^H \} \subseteq \{ \exists \ n \in \{1, \ldots, N\} : \ |Y^H_n + a(N)^N| > g_n^H \text{ ult.} \}.
\]
Using (5.2), Fatou’s lemma and Theorem 6.3 we obtain
\[
\liminf_{N \to \infty} \frac{A_{\mathcal{E}}(N,H)}{2^{N-1}} \geq 1
\]
and the result follows. □

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