Spectra of turbulence in dilute polymer solutions

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INTRODUCTION

In this paper we continue theoretical investigation of turbulence in dilute polymer solutions, started in [1,2]. As opposed to Newtonian fluids, such solutions possess additional macroscopic degrees of freedom related to the elasticity of the polymer molecules. Relaxation times of elastic stresses can be comparable with time-scales of the flow which means that the relation between the stress and the velocity gradient is non-local in time and, consequently, in space. It is a striking property of dilute polymer solutions that minute amounts of polymer can significantly modify the flow. Probably, the most famous example is the drag reduction phenomenon. Addition of long-chain polymers in concentrations as small as, say, 10^{-5} by weight, can induce a substantial reduction of the drag force needed to push a turbulent fluid through a pipe [3–5]. Another example is the elastic turbulence [6–8], which is a chaotic flow, excited in the dilute polymer solutions at low Reynolds numbers.

The reason why small amounts of polymer can significantly modify properties of the fluid is flexibility of polymer molecules. At equilibrium a polymer molecule coils up into a spongy ball of a radius R. If the number of monomers in the molecule, n, satisfying nR_0 \ll 1, an influence of equilibrium size molecules on the hydrodynamic properties of the fluid can be neglected. When placed in an inhomogeneous flow, such a molecule is deformed into an elongated structure, which can be characterized by its end-to-end distance R. If the number of monomers in a typical polymer molecule is large, the elongation R can be much larger than R_0. The influence of the molecules on the flow increases with their elongation and may become substantial when R \gg R_0.

Deformation of a polymer molecule is determined by two processes, stretching by the velocity gradients and relaxation due to elasticity of the molecule. To understand how the molecule resists the deformation by the flow, one should consider its relaxation. Recent experiments with DNA molecules indicate that the relaxation is linear in the wide region of scales R_0 \ll R \ll R_{\text{max}}, where R_{\text{max}} is the maximum molecule extension [9]. In the case of polymers, theoretical arguments and numerics presented in [10] support the linear relaxation. These results can be understood if we assume that at R \gg R_0 the role of excluded volume and hydrodynamic interactions between the monomers are negligible. Then the random walk arguments suggest that the entropy of polymer molecules is quadratic in R in the range R_0 \ll R \ll R_{\text{max}} implying linear relaxation. Whether the polymers are excited by the flow is determined by the softest relaxation mode that corresponds to the dynamics of the elongation R. In the absence of stretching, the relaxation of R is described by the equation ∂_t R = -R/τ, where τ is a relaxation time, which is expected to be R-independent at R \ll R_{\text{max}}. If the end-to-end distance R is of the order of the maximum extension, τ starts to depend on R and the dynamics of the molecule becomes nonlinear [11]. Possible statistical consequences of the non-linearity have been investigated in [12].

The behavior of the molecule in an inhomogeneous steady flow depends on the value of the Weissenberg number, Wi, defined as the product of the characteristic velocity gradient and τ. When a polymer molecule is placed in a flow, smooth at the scale R, the velocity difference between the end-points is proportional to R multiplied by the characteristic value of the velocity gradient. At Wi \ll 1 the relaxation is fast as compared to the stretching time and the polymer always relaxes to the equilibrium size, R_0. The behavior of the polymer at Wi \gtrsim 1 depends on the geometry of the flow. For purely elongational flows the molecule gets aligned along the principal stretching direction. If the velocity gradient is larger than the inverse relaxation time, i.e. Wi \gtrsim 1, the elastic response becomes too slow in comparison with the stretching and the molecule gets substantially elongated [12]. The sharp transition from the coiled state to the strongly extended state is called the coil-stretch transition. Rotation can suppress the transition and even...
damp it completely since the molecule does not always point in the stretching direction (see, e.g., [14]). For example, no coil-stretch transition occurs in the case of the shear flow, which is a particular combination of the elongational and rotational flows.

In contrast to the steady case, a polymer molecule, moving in a random flow, alternately enters regions of high and low stretching. As the intensity of the flow increases, the effect of the stretching becomes more pronounced. One can generally assert existence of the coil-stretch transition in this case. This has been first demonstrated by Lumley [14], who considered the situation where the characteristic time of variations of the velocity gradient is much larger than the inverse of the characteristic value of the gradient. He showed that if the amplitude of the velocity gradient fluctuations is large enough, the expectation value of $R^2$ grows with time, which signifies the coil-stretch transition. We have shown in [14] that the coil-stretch transition occurs in any random flow and established a general criterion for the transition. In particular, the transition occurs in the situation where the time of velocity gradient variation is of the order of the inverse of its characteristic value, which is likely to be the case for real flows. The coil-stretch transition in random flows is controlled by the parameter $\lambda_1\tau$, where $\lambda_1$ is the average logarithmic divergence rate of nearby Lagrangian trajectories, to be referred to as the principal Lyapunov exponent (which is positive for an incompressible flow [14]). The molecules are weakly stretched if $\lambda_1\tau < 1$ and strongly stretched otherwise. Therefore for random flows the parameter $\lambda_1\tau$ plays the role of the Weissenberg number.

As it is well known (see, e.g., [15]), turbulent flows in Newtonian fluids consist of chaotic eddies from a wide interval of scales, $\eta < r < L$, where $L$ is the integral scale (where the flow is excited) and $\eta$ is the viscous scale. The energy pumped at the scale $L$ cascades down to the scale $\eta$, where it dissipates. The size of the polymer molecules is usually much smaller than the viscous scale. Viscosity makes the flow smooth at scales $r < \eta$, i.e. the velocity can be approximated by linear profiles at these scales. Therefore, if $R < \eta$, then the stretching of molecules is determined by the velocity gradient, which is random in the turbulent flow. The Lyapunov exponent can be estimated as the characteristic value of the velocity gradient, which is determined by the eddies at the viscous scale $\eta$. As the Reynolds number grows, the velocity gradient increases, and so does $\lambda_1\tau$. At some value $Re_c$ of the Reynolds number the product $\lambda_1\tau$ reaches the value 1 and the coil-stretch transition occurs.

Several mechanisms can limit the polymer stretching above the coil-stretch transition. The first one is the internal non-linearity of the elasticity of the polymer molecules. If this mechanism dominates, then above the transition the molecules are stretched up to the elongation of the order of $R_{max}$. An alternative mechanism has been proposed by Tabor and de Gennes [19]. It is based on the assumption that the elongation of the polymer molecules $R$ becomes larger than the viscous length of turbulence, $\eta$, where the elastic force always wins over the stretching at a certain value of the elongation. Below, we assume that $R \ll \eta$, which seems to be reasonable for typical polymer solutions. Another limiting mechanism is the back reaction of the polymers on the flow. It is caused by the collective contribution of coherently deformed polymer molecules into the stress tensor. This elastic part of the stress grows with the molecule elongation. When it becomes of the order of the viscous stresses existing in the flow, the polymers modify the flow around them suppressing the stretching. As a result, a dynamic equilibrium is realized at a characteristic elongation, $R_{back}$. The total polymer stress is proportional to $nR^2$, so that $R_{back}$ depends on the polymer concentration $n$. We assume that the concentration is large enough for the value of $R_{back}$ to be much smaller than $R_{max}$. Probably, the condition $R_{back} \ll R_{max}$ is necessary for existence of a stationary state, because the polymer molecules, stretched up to $R_{max}$, are intensively destroyed by the flow.

Above the coil-stretch transition the back reaction modifies the small-scale properties of turbulent flows, which leads to the emergence of a new scale, $r_1 > \eta$, where energy dissipates mainly due to polymer relaxation. The scale $r_1$ plays the role of a new dissipation scale. Large-scale eddies with the sizes $r > r_1$ do not excite elastic degrees of freedom so the usual inertial energy cascade is realized at these scales. At $Re \gg Re_c$ there appears a new region of scales, $\eta_1 < r < r_1$, where elastic waves can propagate [2], which are analogous to the Alfvén waves in magnetic hydrodynamics. At the scale $\eta_1$ viscosity becomes essential, leading to the strong damping of the elastic waves.

In this work we investigate the velocity spectrum in the elastic wave range $kr_1 \gg 1$ (where $k$ is the wave vector). We show that the spectrum obey a power law. The ideas of the analysis go back to the works of Townsend [20] and Batchelor [21]. They recognized that fluctuations with scales, smaller than the smoothness scale of the flow ($\eta$ in the case of usual turbulence and $r_1$ in our case), evolve in the linear velocity profiles. For the passive scalar at large Prandtl numbers Batchelor derived the spectrum with the power-law $k^{-1}$ [21], originating from the exponential character of stretching in the linear flow. Formally, it is explained by zero scaling dimension of the advection term that implies scale invariance. This property is not broken by a linear decay term where the power-law spectrum still holds, as it was shown for a linearly decaying passive scalar by Chertkov [22]. Here we introduce a consistent theoretical scheme for the description of the small-scale fluctuations and show that though the dynamics of the small-scale fluctuations is more complicated, than in the case of the passive scalar (advection and linear decay accompanied by stretching and waves), the power-law in the spectrum still holds. The wave oscillations break the scale-invariance, but their influence on the energy balance is reduced to forcing the equipartition
of kinetic and elastic energies of small-scale fluctuations. The power law spectrum terminates at $k \sim (\nu \tau)^{-1/2}$, where viscosity overcomes stretching. The power-law interval widens as Re grows. Let us stress that this law is not related to a flux of any conserved quantity.

Another situation, where a power-law spectrum of the small-scale fluctuations is observed, is the elastic turbulence, realized in low-Reynolds polymer solutions, if the Weissenberg number $Wi$ is large enough $\delta R$. It was shown experimentally in $\delta R$ that the colli-stretch transition, occurring at increasing $Wi$, leads to a chaotic flow even though Re is small. Its existence is due to hydrodynamic instabilities caused by the presence of the elastic stresses. The velocity spectrum is observed to be power-like in a wide range of scales in this case $\delta R$. We demonstrate that, in contrast to the usual hydrodynamic turbulence, in the elastic turbulence the power velocity spectrum is not related to the energy cascade, since the main energy dissipation occurs at the largest scales. The mechanism, leading to this power spectrum is, again, similar to the linearly decaying passive scalar problem.

The structure of the paper is as follows. In Section I we introduce a system of equations describing the coupled dynamics of inertial and elastic degrees of freedom. This system is similar to the system of equations describing the magnetohydrodynamics (MHD) $\delta R$ with the important difference of a linear decay term in the equation on the “magnetic field”. In Section I we present results, concerning statistics of a passive scalar with a constant damping, embedded in a random flow. It is a prototype for the subsequent consideration. Section II is devoted to the description of the principal properties of the large Re turbulence in the presence of polymers and to the derivation of the power-law spectrum in the elastic dissipation range. In Section III we establish the power-law spectrum for the elastic turbulence. In Conclusion we summarize our results and discuss possible implications of our work for other subjects. Appendix is devoted to some details of the Lagrangian statistics.

I. BASIC RELATIONS

We study dynamics of dilute polymer solutions at scales much larger than the inter-molecular distance where the polymer solution can be regarded as a continuous medium and described by macroscopic fields. Characteristic times of considered processes are regarded to be comparable with the polymer relaxation time $\tau$. In this case, besides the usual hydrodynamic degrees of freedom, one has to take into account degrees of freedom, related to the polymer elasticity. These degrees of freedom can be described in terms of the polymer stress tensor $[1]$

We assume that the flow can be treated as incompressible, that is $\nabla \cdot v = 0$, where $v$ is the velocity of the flow. This is justified provided processes at a given scale are slow in comparison with sound oscillations at the same scale. Then the velocity dynamics can be described in terms of the following equation $[1]$

$$\partial_t v_i + (v \nabla) v_i + \nu^{-1} \nabla_i P = \nu \nabla^2 v_i + \nabla_j \Pi_{ij},$$  \hspace{1cm} (1.1)

which is a generalization of the Navier-Stokes equation to the case of viscoelastic fluids. Here $P$ is the pressure, $\nu$ is the kinematic viscosity of the solvent, $\nu$ is the fluid mass density, and $\Pi_{ij}$ is the polymer contribution into the stress tensor per unit mass.

The equation $[1]$ has to be supplemented by an equation for the polymer stress tensor $\Pi_{ij}$ $[1]$. We assume the following equation

$$\partial_t \Pi_{ij} + (v \nabla) \Pi_{ij} = \Pi_{kj} \nabla_k v_i + \Pi_{ik} \nabla_k v_j - \frac{2}{\tau} (\Pi_{ij} - \Pi_0 \delta_{ij}),$$  \hspace{1cm} (1.2)

where $\tau$ is the polymer relaxation time, and $\Pi_0$ is related to the thermal fluctuations of the polymer conformations $[2]$. Let us briefly repeat applicability conditions of the equation $[1]$, discussed in $[2]$. Linearity of the decay term in $[1]$ assumes $R \ll R_{\text{max}}$, where $R$ is the typical polymer molecule size. The equation $[1]$ implies that there is a single mode related to the polymer deformations, which is an idealization. A polymer molecule has a lot of deformational degrees of freedom, that have different relaxation times. A number of such degrees of freedom was observed in experiments with DNA $[\delta R]$.

Nevertheless, in the turbulent flows, only the mode with the largest relaxation time is strongly excited, whereas other modes are excited at most weakly. Thus, Eq. $[1]$ should be treated as the equation related to the principal mode.

The concentration of the polymer molecules $n$ enters the system $[1,2]$ only via $\Pi_0$, $\Pi_0 \propto n$ and implicitly via the assumption $R \ll R_{\text{max}}$ since $\Pi \propto n R^2$. If $n$ is inhomogeneous, then the system of equations $[1,2]$ has to be supplemented by the equation for the concentration $\partial_t n + v \nabla n = 0$ (we neglect small diffusivity of polymer molecules). In this paper we consider the case when polymer molecules are strongly extended. Then $\Pi \gg \Pi_0$, and the term with $\Pi_0$ in Eq. $[1,2]$ can be discarded. In this case any explicit dependence on the concentration of the polymer molecules $n$ drops from the system of equations $[1,2]$. Therefore the dynamics of the polymer solutions with different values of $n$ is identical in this regime as long as $n$ is large enough for the condition $R \ll R_{\text{max}}$ to be satisfied.

A. Lagrangian Description

One can establish some properties of the polymer stress tensor $\Pi$, using the Lagrangian description of a fluid. It is based on the notion of fluid particles trajectories (Lagrangian trajectories) $x(t, r)$, which are determined by the relations

$$\partial_t x = v(t, x), \quad x(t_0, r) = r.$$  \hspace{1cm} (1.3)
The point \( r \) plays the role of a Lagrangian marker. If \( \Pi_0 \) in Eq. (1.2) is neglected, then it is possible to write its solution as

\[
\Pi(t, x) = W(t, t_0)\Pi(t_0, r)W^T(t, t_0)e^{-2(t-t_0)/\tau},
\]

where the superscript \( T \) denotes a transposed matrix. Here \( W \) is the Lagrangian mapping matrix determined by the relations

\[
\partial_t W(t, t_0) = \sigma(t)W(t, t_0), \quad W(t_0, t_0) = 1, \\
\sigma_{ij}(t, r) = \nabla_j v_i[t, x(t, r)].
\]

Above \( \sigma \) is the tensor of the velocity derivatives along the Lagrangian trajectory \( x(t) \) that includes the strain tensor and local rotations. The incompressibility condition \( \nabla \cdot v = 0 \) is formulated in terms of \( \sigma \) as \( tr \sigma = 0 \). Then a consequence of Eq. (1.3) is det \( W = 1 \).

The matrix \( W \) describes deformations of infinitesimal fluid volumes. For example, the separation, \( \delta x \), between two close fluid particles, moving along the Lagrangian trajectory \( x(t) \), evolves according to

\[
\delta x(t) = W(t, t') \delta x(t').
\]

Therefore \( W_{ij}(t, t_0, r) = \delta x_i(t, r)/\partial r_j \). Now it is easy to understand a meaning of Eq. (1.4). The polymers are advected along the Lagrangian trajectories being stretched by the velocity gradient and relaxing to their equilibrium shape due to the polymer elasticity.

We now briefly describe statistical properties of the matrix \( W \), details can be found in Appendix. We represent the matrix as \( W = MA\Lambda \), while \( M \) and \( \Lambda \) are orthogonal matrices, and \( \Lambda \) is a diagonal matrix. At times much larger than the velocity gradients correlation time \( \tau_\sigma \), the main eigenvalue \( \exp(\rho_1) \) of \( \Lambda \) becomes much larger than the rest, under the condition that the set of the Lyapunov exponents \( \lambda_1 \) is non-degenerate. If a statistically steady state is realized, then the observation time is arbitrarily large, and we conclude from Eq. (1.4) that the matrix \( \Pi \) has to be uniaxial

\[
\Pi_{ik} = B_iB_k,
\]

as it was noted in [2]. This conclusion is almost self-evident once one goes back to the derivation of Eq. (1.2), recognizing that at \( R \gg R_0 \) the contribution of thermal fluctuations into \( \Pi \) is negligible, so that \( \Pi_{ij} \propto R_iR_j \) holds. We observe that the vector \( B \) characterizes the direction and the strength of the coherent molecule elongations weighted by their contribution into the stress tensor. Note, that \( B \) is defined up to sign, in analogy with the director in nematic liquid crystals. It follows from (1.3) and (1.8) that

\[
B(t, x) = \exp[-(t-t_0)/\tau]W(t, t_0)B(t_0, r).
\]

If \( t-t_0 \gg \tau \) then \( W \) in this relation can be estimated as \( e^{\rho_1} \).

There are some modifications of the \( W \) statistics with respect to a Newtonian fluid, that are imposed by the above relations. As it follows from Eq. (1.9), stationarity of the \( B \) statistics implies that \( \rho_1(t) - t/\tau \) has a stationary distribution. In particular, we conclude that the principal Lyapunov exponent \( \lambda_1 = \lim_{t \to \infty} \rho_1(t)/t \) of the flow is equal to \( 1/\tau \), independently of the Reynolds number. This means that above the coil-stretch transition the characteristic value of the velocity gradient is fixed at the scale \( 1/\tau \). The above behavior is contrasted to the Newtonian fluids for which \( \lambda_1 \) grows with increasing \( Re \) and fluctuations of \( \rho_1 - t/\tau \) grow with time. The absence of the growth of the fluctuations is related to anticorrelations in the temporal dynamics of the component \( \tilde{\sigma}_{11}(t) \) of \( \sigma_{ij} \) along \( B \). These anticorrelations show themselves in the equality \( \int dt \langle \tilde{\sigma}_{11}(0)\tilde{\sigma}_{11}(t) \rangle = 0 \) (double brackets designate an irreducible correlation function) and originate in the special interaction of the inertial and elastic degrees of freedom, explained in more detail in Section [II].

### B. Dynamic Equations for Dilute Polymer Solutions

It is convenient to rewrite the equations (1.1,1.2) in terms of the vector \( B \) thus getting rid of extra degrees of freedom. Substituting the decomposition (1.8) into Eq. (1.2), one obtains

\[
\partial_t B_i + (v\nabla)B_i = B_k\nabla_k v_i - B_i/\tau.
\]

This equation is similar to the one satisfied by the magnetic field in MHD [23], with the constant damping instead of the magnetic resistivity. The resemblance is made even stronger by noting that \( B \) has to be solenoidal. Indeed, it follows from Eq. (1.10) that

\[
\partial_t \nabla \cdot B + (v\nabla)\nabla \cdot B = -\nabla \cdot B/\tau.
\]

Therefore \( \nabla \cdot B \) monotonically decays becoming zero in the (statistically) steady state. Substituting the expression (1.5) into Eq. (1.1), and taking into account the constraint \( \nabla \cdot B = 0 \), one obtains

\[
\partial_t \nu + (v\nabla)\nu = \nu \nu^2 v - \eta^{-1} \nu \nabla P + (B\nabla)B.
\]

Now the analogy of the system (1.10,1.12) with the system describing MHD [23] at zero magnetic resistivity is almost complete. The only difference is in the damping term in Eq. (1.11).

The energy density per unit mass is given by the sum of kinetic \( v^2/2 \) and elastic \( B^2/2 \) terms. The energy balance equation, following from Eqs. (1.10,1.12), is

\[
(\partial_t + v\nabla)\left( \frac{v^2}{2} + B^2/2 \right) = (B\nabla)(B \cdot v) - \eta^{-1} \nu \nu \nabla P + \nu \nabla_i (v\nabla_i v) - \nu (\nabla \nu v_k)^2 - \tau^{-1} B^2.
\]

The energy dissipation is due to the viscous and the polymer relaxation terms. Other terms in Eq. (1.13) represent energy fluxes (in real space), they can be written as full divergences due to the constraints \( \nabla \cdot v = 0 = \nabla \cdot B \).
II. MECHANISM OF SCALE-INVARINACE:
PASSIVE SCALAR WITH LINEAR DAMPING

Before investigating statistical properties of the polymer solutions, described by the equations (1.10, 1.12), we present statistical properties of a passive scalar with constant damping at scales smaller than the smoothness scale of the flow, examined in [22]. This simple case enables one to recognize the origin of a power spectrum for passively advected fields.

The equation for the passive scalar $\theta$ in the considered case is

$$\partial_t \theta + \mathbf{v} \nabla \theta = -\theta/\tau + \phi,$$

(2.1)

where $\tau$ is the passive scalar decay time and $\phi$ is a forcing term needed to maintain the stationary state. It is assumed to be concentrated at a finite range of wavevectors near $k_f$. We have omitted the diffusive term which cannot be neglected in comparison with the constant damping (in some region of scales) provided the diffusion coefficient is small enough. Note that the constant damping of the passive scalar leads to a well-defined steady statistics even in the absence of the passive scalar diffusion.

A flow is smooth on scales smaller, than the velocity gradients correlation length $l$. The smoothness means that a velocity difference between two points can be approximated by a linear profile

$$\delta v_i = \sigma_{ij} \delta r_j,$$

(2.2)

where $\delta r_j$ is the separation between the points and $\sigma$ is a function of time. Obviously, $\sigma_{ij} = \nabla_j v_i$. For the usual turbulent velocity the correlation length $l$ is equal to the viscous scale, $l = \eta$. The smoothness of the turbulent velocity at scales less than $\eta$ was first exploited by Batchelor [21], who considered statistical properties of a passive scalar at these scales.

The linearity of the velocity difference leads to a power law for the passive scalar spectrum $E(k)$, which is defined as $\langle \theta(k)\theta(k') \rangle = (2\pi)^3 \delta(k + k')E(k)$, where $\theta(k)$ is Fourier transform of $\theta(r)$ with the wave vector $k$, angular brackets designate averaging over the statistics of $v$, and we assume homogeneity and isotropy of the statistics. Indeed, the energy transfer from $k$ to $2k$ at $kl \gg 1$ is scale-independent due to the linearity of the velocity profile. On the other hand, during the spectral transfer time the energy decay is also $k$-independent (since the damping term is scale-independent). As a result the spectral function $E(k)$ satisfies a relation $E(2k) = CE(k)$ (with a constant $C < 1$). The solution of this equation is a power-law $E(k) \propto k^{-\alpha}$ with $2^{\alpha}C = 1$.

Now we put the above consideration into a more rigorous frame. We consider the passive scalar spectrum $E(k)$ at $kl \gg 1$. The evolution of wave packets with such wavevectors is determined by the velocity gradient $\sigma$. Let us consider the evolution during a time $t_0$ and express $\theta(t_0)$ via $\theta(0)$. A value of $\theta(t_0)$ near a point $r_1$ is determined by an evolution of $\theta$ in the vicinity of the Lagrangian trajectory $\mathbf{x}(t, r_1)$. To examine this evolution, one may perform the Taylor expansion of the velocity $\mathbf{v}$ in Eq. (2.1) up to the first order in $r - \mathbf{x}$ since the homogeneous advection does not affect equal-time correlation functions due to the Galilean invariance. Then one obtains

$$\partial_t \theta + [\mathbf{u} + \sigma \cdot (r - \mathbf{x})] \nabla \theta = -\theta/\tau + \phi.$$

(2.3)

Here $\mathbf{u}(t) = \mathbf{v}(t, \mathbf{x})$ and $\sigma = \sigma(t, \mathbf{x})$ are the velocity and the velocity gradients matrix along the Lagrangian trajectory $\mathbf{x}$. Fourier transform $\theta_k$ of the field $\theta$ measured in the moving frame $\tilde{\theta}(t, r) = \theta(t, \mathbf{r} + \mathbf{x})$ satisfies

$$\partial_t \theta_k - \left( k\sigma \cdot \frac{\partial}{\partial k} \right) \theta_k = -\frac{\theta_k}{\tau} + \phi_k \exp[i\mathbf{k} \cdot \mathbf{x}].$$

(2.4)

Further we confine ourselves to wavevectors $k \gg k_f$, that is much larger than those on which the pumping $\phi$ is supported. In this case $\theta_k$ is determined by the convection from smaller wavevectors and the forcing term can be neglected. Then the equation (2.4) can be solved explicitly, and we find

$$\theta(t, \mathbf{k}) = e^{-t/\tau} \tilde{\theta}(0, \mathbf{k}W),$$

(2.5)

where $W = W(t, 0)$ is the Lagrangian mapping matrix, see Subsection A. Returning to the real space, we obtain

$$\theta(t, \mathbf{r} + \mathbf{x}) = e^{-t/\tau} \int \frac{dk}{(2\pi)^3} \exp[i\mathbf{r} \cdot \mathbf{k}] \tilde{\theta}(0, \mathbf{k}W),$$

(2.6)

The above formula is valid for $|W^{-1}\mathbf{r}| \ll l$, the condition means that the passive scalar coming to a point $r + r_1$ was all the time in the $l$-vicinity of $\mathbf{x}$ allowing the Taylor expansion for the velocity.

Let us now consider the pair correlation function of the passive scalar $f(r) = \langle \theta(t_0, \mathbf{r}_1)\theta(t, \mathbf{r}_1 + \mathbf{r}) \rangle$, defined as the spatial average over $\mathbf{r}_1$. We assume that the average $\langle \theta \rangle$ is zero (which can always be achieved by a shift of $\theta$ by a constant). The product of the fields is given by Eq. (2.3) [remind that $\mathbf{x}(t_0) = \mathbf{r}_1$] and depends on $\mathbf{r}_1$ via the argument of $W$. The average over space (over $\mathbf{r}_1$) is equivalent to the average over the velocity statistics, or over the velocity gradients statistics along the Lagrangian trajectories. If $\lambda_1 t_0 \gg 1$ then the average over the interval $0 < t < t_0$ and negative times can be done independently. Indeed, $\sigma(t)$ has a Lagrangian correlation time $\lambda_1^{-1}$. Thus velocity at negative times is correlated with $\sigma$ only at $|t| \sim \lambda_1^{-1}$ while $W(t_0)$ is not sensitive to the value of $\sigma$ there, due to $\lambda_1^{-1} \ll t_0$ (see Appendix). Therefore we can write

$$f(r) = e^{-2t_0/\tau} \int \frac{dk}{(2\pi)^3} \exp[i\mathbf{r} \cdot \mathbf{k}] \langle E[kW(t_0)] \rangle,$$

(2.7)

where $E(k)$ is the spectrum function introduced above. Noting that $E(k)$ equals the Fourier transform of the
pair-correlation function we obtain the following stationarity condition for the spectrum (we substitute \( t_0 \) by \( t \))

\[
E(k) = \langle \exp(-2t/\tau)E(kW) \rangle ,
\]

where \( W = W(t, 0) \). The equation (2.8) is applicable at \( kl \gg 1 \), as follows from the condition \( r < l \) in Eq. (2.6).

The relation (2.8) has a simple meaning. The wavevectors of small-scale fluctuations of the passive scalar evolve according to \( \mathbf{k}(t) = \mathbf{k}(0)W^{-1}(t) \) as was shown by Kraichnan [23]. Thus the energy of a fluctuation with the wavevector \( \mathbf{k} \) is equal to its energy time \( t \) ago at the wavevector \( kW(t, 0) \) multiplied by the decaying factor \( \exp(-2t/\tau) \). Note that we could equally well start directly from (2.4) to derive the equation (2.8). One can show that in the spatially homogeneous situation one can always introduce such equation for the investigation of the spectrum at \( kl \gg 1 \).

The equation (2.8) is the quantification of the heuristic arguments given in the beginning of the section taking into account that the energy transfer time is by itself a random quantity. Its solution is a power law \( E(k) \propto k^{-\alpha} \). Substituting the expression into Eq. (2.8), one gets the relation

\[
\exp(2t/\tau) = \langle |kW/k|^{-\alpha} \rangle ,
\]

which determines the exponent \( \alpha \). At \( \lambda_1 t \gg 1 \) the moments of \( |kW/k| \) behave exponentially with time. Indeed, they are roughly equal to the product of \( \lambda_1 t \) independent identically distributed random variables. Besides at these times the moments are independent of \( k/k \) due to the isotropization of \( W(t_0) \) described in Appendix. As a result the above equation has a unique physical solution examined in more detail in Appendix A3, where the inequality \( \alpha > 3 \) is established. The inequality has simple meaning that the spectrum has to decay faster than the Batchelor spectrum \( k^{-3} (k^{-1} \text{ in the spherical normalization}) \) holding at infinite \( \tau \).

Let us now extend the above results. The power law spectrum persists, even if the relaxation time \( \tau \) is \( k \)-dependent, but scales as zero power of \( k \), that is if \( \tau \) depends on the direction of \( \mathbf{k} \) only. This dependence can be regular (which makes the spectrum anisotropic) or random. Another remark is that addition of an oscillating term (with \( \omega_k \)) into the equation for \( \theta_k \),

\[
\partial_t \theta_k - (k \sigma \cdot \frac{\partial}{\partial k}) \theta_k = -\theta_k/\tau - i\omega_k(t)\theta_k ,
\]

(2.10)
does not change its spectrum even though the oscillating term breaks the scale-invariance. Indeed, let us pass from \( \theta \) to \( \bar{\theta} \), which is \( \bar{\theta} = \exp(i\bar{\varphi}_k)\theta \), with the phase, satisfying

\[
\partial_t \varphi_k - (k \sigma \cdot \frac{\partial}{\partial k}) \varphi_k = \omega_k .
\]

Then for \( \bar{\theta} \) we return to the equation (2.4), that leads to the power spectrum. It remains to note that the spectra of \( \theta \) and \( \bar{\theta} \) coincide. In other words, oscillating terms conserve energy and are, consequently, irrelevant for the energy balance.

Below we generalize the simple picture, presented in this section, to the polymer solutions.

### III. HIGH REYNOLDS FLOWS

Here we consider turbulence in dilute polymer solutions, when the Reynolds number exceeds the critical value \( \text{Re}_c \), corresponding to the coil-stretch transition. Then the polymer molecules are strongly elongated. Two different cases are possible, depending on the concentration of the polymer molecules \( n \). If it is very small, the elastic stresses are small in comparison with the viscous stresses. Then the polymers are stretched to their maximal elongation, \( R_{\max} \), and the properties of the fluid do not differ significantly from those of the pure solvent. Below we consider the second, more interesting, case, when the concentration of the polymers \( n \) is large enough, so that elastic stresses can be larger than the viscous stresses. Then the polymer back reaction substantially modifies the flow.

Whereas in the pure solvent typical velocity gradients grow unlimited as the Reynolds number increases, in polymer solutions above the coil-stretch transition the balance of inertial and elastic degrees of freedom fixes the characteristic value of the velocity gradient at \( 1/\tau \). Indeed, if the instantaneous velocity gradient exceeds \( 1/\tau \), it extends the polymers, so that the elastic stress grows and damps the gradient. On the other hand, if the velocity gradient is smaller than \( 1/\tau \), the molecules contract and their influence on the flow diminishes. Then the velocity gradients tend to grow up to the value characteristic of the pure solvent, which is larger than \( 1/\tau \). Thus the velocity gradients fluctuate near \( 1/\tau \), that explains the statistically steady state realized above the coil-stretch transition. Let us derive the condition for the existence of this steady state, related to the existence of the maximal size \( R_{\max} \) of the polymer molecules.

In the vicinity of the coil-stretch transition \( \nabla v \sim 1/\tau \) so that \( B_{\text{back}}^2 \sim \nu \nabla v \sim \nu/\tau \) as it follows from Eq. (1.12). This leads to the condition for the existence of the back reaction regime \( B_{\text{max}}^2 \gg \nu/\tau \), where \( B_{\text{max}}^2 \) is the maximal value of the elastic stress tensor achieved at \( R \sim R_{\max} \). Using estimates for the microscopic parameters, proposed in Ref. [24], one can rewrite this condition as \( n \gg (R_0 R_{\max}^2)^{-1} \). Below we will assume that there exists an interval in Re such that \( B_{\max} \) exceeds the value of \( B \) prescribed by the flow. The latter increases as Re grows so that the condition will break down at certain Re. After this happens either polymer degradation occurs or polymers start to behave as rigid bodies with size \( R_{\max} \). In the latter case the fluid becomes Newtonian again with renormalized viscosity.

We assume \( V \tau \ll L \), where \( V \) is the velocity at the turbulence integral scale. Then the gradient related to
the large eddies is smaller than \( \tau^{-1} \). Therefore, the large eddies do not excite polymers, which means that the elastic stress tensor is not correlated at these scales. Since only coherent excitations of the elastic stress tensor can influence the flow, we conclude that the elasticity is negligible for large eddies. The interaction of inertial and elastic degrees of freedom becomes essential at a scale \( r_* \), where velocity gradients are of the order of \( 1/\tau \). Here the energy starts to dissipate due to the polymer relaxation, that is the inertial cascade terminates at \( r \sim r_* \). Since the velocity gradients fluctuate near the value \( 1/\tau \), reached at \( r_* \), at \( r < r_* \) velocity difference scales linearly with the distance that is \( r_* \) the smoothness scale of the flow. Near the coil-stretch transition characteristic velocity gradient is determined by the viscous scale and is of the order of \( 1/\tau \), hence \( r_* \sim \eta \). As the Reynolds number increases, velocity fluctuations increase, so that the scale \( r_* \) grows which is different from the Newtonian fluids where the smoothness scale \( \eta \) decreases with \( \text{Re} \). The interaction of inertial and elastic stresses excited at \( r_* \), are negligible but elasticity is not, has to exist below \( r_* \) by the viscous dissipation term \( \nu/(\tau) \sim \nu/\eta \). The relation (3.1) means that a typical interval. These small-scale fluctuations, which occur on scales \( r < r_* \), are characterized by the stretching that takes place at a time scale \( \tau \), at \( r < r_* \) the differences \( \delta v \) and \( \delta B \), taken at points, separated less than \( r_* \), scale linearly with the separation according to

\[
\partial_t \delta v + (\nabla \delta v) \cdot B = 0,
\]

where \( p' = P/\rho \).

The inequalities \( \nabla v' \ll \nabla v, \nabla B' \ll \nabla B \) imply that the viscosity \( \nu \) and \( \nu \) are negligible at \( r \sim r_* \) and \( r > r_* \), respectively. The dispersion relation for the waves is \( \omega = \mathbf{k} \mathbf{B} \), where \( \omega \) is the wave frequency and \( \mathbf{k} \) is its wave vector. Therefore the group velocity of these waves is \( \mathbf{B} \) which can be estimated in accordance with Eq. (3.3) as \( \mathbf{B} \). The wave velocity fluctuates, but the fluctuations occur at times \( \sim \tau \) and are slower than the wave oscillations at \( kr_* > 1 \), showing that the waves are well-defined. There exist two mechanisms of the elastic waves attenuation: polymer relaxation and viscous dissipation. The first mechanism leads to the scale-independent attenuation \( \sim \tau^{-1} \), which is smaller than the frequency, at \( kr_* > 1 \). The second mechanism produces the attenuation \( \sim \nu k^2 \), which is much smaller than the frequency for \( k\eta_* \ll 1 \) where \( \eta_* = \nu(\tau) \). Thus the elastic waves attenuate weakly in the interval \( r_*^{-1} \gg k \gg \eta_*^{-1} \). This interval can be called the elastic waves range.

The dynamics at scales \( r \ll r_* \) is also characterized by the stretching that takes place at a time scale \( \tau \) and is slower than the wave’s oscillations. It is this dynamics that determines the velocity spectrum at \( kr_* \gg 1 \), since the wave oscillations do not influence the spectrum, like in the example presented at the end of Section II. As a result, we come to a power spectrum, which is examined in the next subsection.

### A. Power-law Spectrum in the elastic waves range

As we explained, statistical stationarity implies, that the velocity gradients fluctuate near \( 1/\tau \), the value characteristic of the scales \( r \sim r_* \). Therefore velocity gradients have to decrease with diminishing scale \( r < r_* \), which can be formulated as \( \nabla v' \ll \nabla v \). Here \( v' \) is the small-scale component of the velocity containing only harmonics with wavenumber satisfying \( kr_* \gg 1 \). The existence of elastic waves at these scales leads to equipartition of kinetic and elastic energies (see [26] and the proof below) so that \( \nabla B' \ll \nabla B \) holds too. As a result the influence of \( v' \) and \( B' \) on motions at scales \( r \sim r_* \) is negligible, that is \( v' \) and \( B' \) can be treated as passively advected and stretched by \( v \) and \( B \). Equations for \( v' \) and \( B' \) can be found by linearizing Eqs. (1.1) and  

\[
\partial_t v' + (\nabla v') \cdot B = 0,
\]

where \( p' = P/\rho \).

The inequalities \( \nabla v' \ll \nabla v, \nabla B' \ll \nabla B \) imply that at \( r \gg r_* \) the differences \( \delta v \) and \( \delta B \), taken at points, separated less than \( r_* \), scale linearly with the separation according to

\[
\delta v_1 = \sigma_{ij} \delta r_j, \quad \delta B_i = \gamma_{ij} \delta r_j,
\]

which is a generalization of Eq. (2.3). Both matrices \( \sigma_{ik} = \nabla_k v_i \) and \( \gamma_{ik} = \nabla_k B_i \) have typical values \( 1/\tau \) and correlation times of the order of \( \tau \). To investigate the statistics of the small-scale components we may use
the same scheme, as was developed in Section II for the passive scalar, expanding the equations \([2.3]\) near a Lagrangian trajectory, like in Eq. \([2.3]\). As we explained in Section II, the velocity should be expanded to the first order (because of the Galilean invariance). The need to expand \(B\) to the first order too follows from the fact that zeroth order term produces Alfven waves that do not affect energy balance of the waves of the same type as explained at the end of Section II. Passing to Fourier components (of the functions of the argument \(r-x\)), we obtain the equations
\[
\partial_t v'_k - \left( k \sigma \cdot \frac{\partial}{\partial k} \right) v'_k + \left( k \gamma \cdot \frac{\partial}{\partial k} \right) B'_k + \sigma v'_k = -ikp'_k + i(B \cdot k) B'_k + \gamma k^2 v'_k, \tag{3.4}
\]
\[
\partial_t B'_k - \left( k \sigma \cdot \frac{\partial}{\partial k} \right) B'_k + \left( k \gamma \cdot \frac{\partial}{\partial k} \right) v'_k + \gamma v'_k = i(B \cdot k) v'_k + \sigma B'_k - \frac{B'_k}{\tau}, \tag{3.5}
\]

analogous to Eq. \([2.3]\). The quantities \(\sigma, \gamma\) and \(B\) in Eqs. \([2.3][3.3]\) are measured in the Lagrangian frame (they are functions of time and the Lagrangian marker). Correlation functions of the fields \(v'\) and \(B'\) are defined as averages over volume (or, what is the same, over different Lagrangian trajectories), that is over a statistics of \(\sigma, \gamma\) and \(B\). There is a new ingredient in comparison with the consideration of Section II, which is the stretching terms like \(\sigma v'_k\). However, these terms preserve zero scaling dimension of the time-evolution operator and, consequently, they are not expected to destroy the power character of the velocity spectrum \(E(k)\).

From now on we neglect the viscous term in \([3.4]\) which is justifiable for not too large wavevectors (a criterion is explained at the end of Section II). Passing to Fourier components (of the functions of the argument \(r-x\)), we obtain the equations
\[
\partial_t v'_k - \left( k \sigma \cdot \frac{\partial}{\partial k} \right) v'_k + \left( k \gamma \cdot \frac{\partial}{\partial k} \right) B'_k + \sigma v'_k = -ikp'_k + i(B \cdot k) B'_k + \gamma k^2 v'_k, \tag{3.4}
\]
\[
\partial_t B'_k - \left( k \sigma \cdot \frac{\partial}{\partial k} \right) B'_k + \left( k \gamma \cdot \frac{\partial}{\partial k} \right) v'_k + \gamma v'_k = i(B \cdot k) v'_k + \sigma B'_k - \frac{B'_k}{\tau}, \tag{3.5}
\]

where \(\sigma \pm = \sigma \mp \gamma\). Of course, the equations \([3.6]\) are compatible with the conditions \(k \cdot g_\pm = 0\) following from the solenoidality of \(\sigma\) and \(B\). The right-hand side of \([3.6]\) contains the Alfven term \(iB \cdot k\), describing the wave oscillations. As we explained at the end of Section II, it is convenient to eliminate the oscillations, introducing the corresponding phase and amplitude \(g_\pm = a_\pm \exp[\pm i\varphi_\pm]\) with \(\varphi_\pm\) satisfying
\[
\partial_t \varphi_\pm - \left( k \sigma \cdot \frac{\partial}{\partial k} \right) \varphi_\pm = \pm (B \cdot k), \tag{3.7}
\]
\[
\varphi_\pm = \pm \int^t_0 dt' B(t') W_{\pm}(t') W_{\mp}(t) k, \tag{3.8}
\]
where \(\partial_t W_{\pm} = \sigma_\pm W_{\pm},\ W_{\pm}(0) = 1\). The equations for the amplitudes \(a_\pm\) are
\[
\begin{align*}
\partial_t a_\pm - \left( k \sigma_\pm \cdot \frac{\partial}{\partial k} \right) a_\pm & = k \left( k \sigma_\pm a_\pm \right) - \frac{a_\pm}{2\tau} \\
& + \left[ \left( \frac{1}{2\tau} - \sigma_\pm \right) a_\pm + \frac{k(k \sigma_\pm a_\pm)}{k^2} \right] \exp(\pm i\phi), \tag{3.9}
\end{align*}
\]

The above equations are, again, compatible with the constraints \(k \cdot a_\pm = 0\).

We observe that a characteristic time of the variations of the amplitudes is \(\tau\) while the phase in the last term varies by \(2\pi\) during the characteristic time \(k B \sim \tau\). Indeed, the exponent appearing in the last line of Eq. \([3.9]\) at \(t \gg \tau\) can be estimated as \(B(t) \tau k e^{t/\tau}\), where \(k e^{t/\tau}\) is the current value of the wave vector, increasing due to the stretching process. Averaging \([3.9]\) over times much larger than \((k B)^{-1}\) but much smaller than \(\tau\) (the procedure is nothing but the Bogolubov-Krylov averaging method), we find the following amplitude equation
\[
\begin{align*}
\partial_t a_\pm - \left( k \sigma_\pm \cdot \frac{\partial}{\partial k} \right) a_\pm & = \frac{k}{k} \left( k \sigma_\pm a_\pm \right) - \frac{a_\pm}{2\tau}. \tag{3.10}
\end{align*}
\]

We observe that the equations for \(a_+, a_-\) decouple. This is in accordance with the qualitative considerations of Kraichnan \([21]\) who argued that the interaction of the waves, described by the amplitudes \(a_+\) and \(a_-\), is weak because their propagation directions are reverse.

The equation \([3.10]\) has the same structure as the equation for the linearly decaying scalar, considered in Section II. The difference is in its vectorial nature and in the presence of the term directed along \(k\) that comes from the solenoidality condition \(\kappa_k = 0\). A formal solution of the equation \([3.10]\) can be written as
\[
\begin{align*}
\begin{align*}
\quad a_\pm (t, k) & = e^{-t/2\tau} M_\pm (t, q \pm) a_\pm (0, q \pm), \tag{3.11} \\
\quad \partial_t M_\pm (t, k) & = (f_\pm, f_\pm) (f_\pm, \sigma_\pm M_\pm)^T. \tag{3.12}
\end{align*}
\end{align*}
\]

where \(q \pm = k W_\pm (t),\ f_\pm = W_\mp^{-1}(t),\) and the initial condition for the matrices \(M_\pm = M_\pm (t = 0, k) = \delta_k - k_i k_i / k^2\). The term \(k_i k_i / k^2\) in the initial condition for \(M_\pm\) vanishes after contraction with solenoidal field \(a(t, k)\), that leads to the correct initial condition for \(a_\pm\).

Note that \(f_\pm M_\pm (t, k) = 0\). Indeed, \(k M_\pm (0, k) = 0\) and \(\partial_t [f_\pm M_\pm (t, k)] = 0\), as follows from Eq. \([3.12]\) and the equations \(\partial_t f_\pm = -f_\pm \sigma_\pm\).

Remind that \(B\) is defined up to a sign. Therefore all the statistical properties of the solution have to be invariant under the transformation \(B \rightarrow -B\). This transformation interchanges \(g_+\) and \(g_-\). Therefore statistical properties of \(g_+\) and \(g_-\) are identical. Particularly, the
spectra of \(g_+\) and \(g_-\) coincide, that is \(\langle g_{\pm i}(k)g_{\pm j}(k') \rangle = (2\pi)^3\delta(k+k')\delta_{ij}k_i k_j/\langle E(k) \rangle\). Thus, without any loss of generality, one can consider \(g_+\) solely. At calculating the correlation function of \(g_+\), entering the definition of \(E\), we can exploit an independence of \(W_+(t)\) and \(M_+(t)\) of \(a_\omega(0, k)\) that holds at \(t\) much larger than the correlation time \(\tau\) of \(\sigma_+\) (this is again in complete analogy with the consideration of Section II). Then the stationarity condition for \(E(k)\) reads

\[
2E(k) = \exp(-t/\tau) \langle Z(t, q_+) E(q_+) \rangle, \tag{3.13}
\]

\[
Z(t, k) = \text{tr} M_+^T(t, k) M_+(t, k) - \frac{k M_+^T(t, k) M_+(t, k) k}{k^2},
\]

where we used \(M(t, -k) = M(t, k)\). Note that \(Z = 2\). Indeed, using Eq. (3.12) and \(f_\pm M_\pm(t, k) = 0\) one easily shows that the time derivative of \(M^T(t, k) M(t, k)\) vanishes so that \(M^T(t, k) M(t, k) = \delta_{ij} - k_i k_j/k^2\) and \(Z(t, k) = 2\). Thus the equation (3.13) simplifies to

\[
E(k) = \exp(-t/\tau) \langle E(q_+) \rangle, \tag{3.14}
\]

almost identical to Eq. (2.8) established in Section II. Similarly, one can formulate an equation for the cross-correlation spectrum function \(E'(k)\) defined by \(\langle (g_{\pm i}(k)g_{\pm j}(k')) \rangle = \delta(k+k')\delta_{ij}k_i k_j/\langle E'(k) \rangle\). The fast oscillating phase does not cancel in this equation leading to the inequality \(E'(k) \leq E(k)\). It leads to the conclusion that the spectra of \(v\) and \(B\) coincide and are equal to \(E(k)/2\) each. This proves the equipartition of the elastic and the kinetic energies claimed above.

In analogy with the consideration of Section II one can establish that the solution of (3.14) is power-like \(E(k) \propto k^{-\alpha}\) where \(\alpha\) is determined implicitly by

\[
1 = \exp(-t/\tau) \left( \frac{1}{k W_+(t)/k^\alpha} \right). \tag{3.15}
\]

Again, \(\alpha > 3\) (see Appendix A). In fact a stronger inequality follows from the stationarity condition. Namely, the spectrum has to decay faster than \(k^{-5}\) \((k^{-3}\) in the spherical normalization). Otherwise \(\langle (\nabla v)^2 \rangle = \int E(k) k^2 dk\) is determined by scales smaller than \(r_\ast\), violating the condition that the gradients have to be saturated at the value \(1/\tau\) reached at \(r_\ast\). The condition \(\alpha > 5\) coincides with the applicability condition of the above consideration that uses \(\nu(x + r) - v(x) \approx \sigma r\) for \(r \ll r_\ast\). Indeed, then \(E(k) k^2 dk\) is determined by \(kr_\ast < 1\) and \(\langle (\nu(x + r) - v(r) - \sigma r)^2 \rangle \ll (\sigma r)^2\) for \(r \ll r_\ast\).

It is natural to ask, whether \(\alpha\) is a universal number, independent of \(Re\). The above analysis shows that \(\alpha\) is determined by the statistics of \(\sigma\) and \(\gamma\). Within the framework of the Kolmogorov theory the statistics is independent of the inertial interval length and can be characterized by a single parameter \(\lambda_1 = 1/\tau\). We conclude that the dimensionless quantity \(\alpha\) is a universal number in the Kolmogorov theory. In fact, due to intermittency the statistics of velocity gradients depends on the length of the inertial interval and, consequently, on the Reynolds number. The current understanding of intermittency does not allow us to estimate \(\alpha\) for a given \(Re\), yet some qualitative assertions can be formulated. Intermittency enhances the probability of large gradients which leads to faster transfer of energy to large wavelength. Therefore the velocity spectrum becomes flatter as the intermittency increases. As we established, the length of the inertial interval decreases with the growth of the Reynolds number for polymer solutions (since the lower boundary of the inertial interval \(r_\ast\) increases). Consequently, the intermittency decreases as \(Re\) grows. We conclude that \(\alpha\) should be a monotonically increasing function of \(Re\).

Now we establish the region of scales where the power spectrum exists. Its lower boundary is related to the viscous dissipation, which grows with increasing \(k\), destroying the power spectrum at large wavenumbers. Comparing the viscous term \(\nu k^2 \nu'\), the last one in Eq. (1.13), with, say, the stretching term \(\sigma \nu'\), we find, that the viscous term wins at the scale \(\sim \sqrt{\nu/\tau}\). As a result the power-law terminates at \(k \sim (\nu \tau)^{-1/2}\). For larger wave vectors the velocity spectrum diminishes faster than a power of \(k\), that is the power spectrum occurs in the interval \(r_\ast^{-1} \ll k \ll (\nu \tau)^{-1/2}\). Note that at \(Re \gg Re_c\) we have \(\sqrt{\nu/\tau} \gg \eta\) so that the power spectrum occurs in the interval, where the elastic waves are well-defined.

**IV. ELASTIC TURBULENCE**

We pass to the case of low Reynolds numbers. Then a random (chaotic) flow can be excited due to elastic instabilities, if the Weissenberg number \(\text{Wi}\) is larger than unity. This is the situation of the recently discovered “elastic turbulence” [3].

We investigate the case \(Re \ll 1\) where the substantial derivative in Eq. (1.13) can be neglected. Then the system (1.10) becomes

\[
\rho^{-1}\nabla P = (B \nabla) B + \nu \nabla^2 v, \quad \nabla \cdot v = 0, \quad \nabla \cdot B = 0. \tag{4.1}
\]

The inequality \(Re \ll 1\) implies, that the kinetic energy of the solution can be neglected in comparison with the elastic one. The dissipation of the elastic energy is, however, due to both energy dissipation mechanisms (solvent viscosity and polymer relaxation):

\[
\frac{d}{dt} \int d\mathbf{r} \frac{B^2}{2} = -\frac{1}{\tau} \int d\mathbf{r} \frac{B^2}{2} - \nu \int d\mathbf{r} (\nabla v_k)^2, \tag{4.3}
\]

as follows from Eqs. (4.1,4.2) and is in accordance with Eq. (1.13).

The system of equations (1.1,4.2) has to be complemented by the boundary conditions for the velocity, which in the absence of polymers would lead to \(\text{Wi} > 1\) so that the equilibrium state of polymers is unstable. The instability eventually leads to a chaotic, statistically
steady state maintained by the non-linear dynamics of \( B \), see Eq. \((4.2)\). Stationarity of the statistics, again, implies \( \lambda_1 = 1/\tau \) and stationarity of the \( \rho_1 - t/\tau \) statistics, as it stems from Eq. \((4.2)\). It follows that the velocity gradients are of the order of \( 1/\tau \) in the bulk. Therefore a boundary layer has to be formed, where the velocity gradient, exceeding \( 1/\tau \), at the boundary, drops to the value \( 1/\tau \) in the bulk. In the boundary layer the flow is mainly shear and the polymers are weakly stretched. The existence of the boundary layer is observed also experimentally \([27]\).

The above picture means, that instabilities lead to velocity fluctuations with scales determined by the size of the system and a characteristic gradient \( 1/\tau \). The following estimates for the values of these large-scale fluctuations hold

\[
B^2 \sim \nu \sigma \sim \frac{\nu}{\tau}, \quad v^2 \sim \frac{L^2}{\tau^2}, \quad \frac{v^2}{B^2} \sim \frac{L^2}{\nu \tau} \sim \text{Re} \ll 1, \quad (4.4)
\]

where \( L \) is the linear system size. The estimates \((4.4)\) imply that both dissipative terms in Eq. \((4.3)\) are of the same order. The correlation time of \( B \) and \( v \) is determined by the typical value of the stretching and is of the order of \( \tau \). The large-scale velocity fluctuations produce smaller scale fluctuations of \( B \) that in turn induce small-scale fluctuations of velocity. The velocity gradients become smaller when the scale decreases, since the large-scale velocity gradient is of the order of \( 1/\tau \) and the total velocity gradient is fixed at this value by the stationarity condition. This is in complete analogy with the consideration of the previous section.

Now we are going to consider statistical properties of the small-scale fields \( v' \) and \( B' \). The fields evolve passively in the large-scale fields \( v \) and \( B \). However, there is a major qualitative difference from the high-Reynolds case which is in the role of the large-scale component of \( B \). In the high-Reynolds number case (see Section \(11\)) the terms with the large-scale field \( B \) in the equations for the small-scale fields conserve energy and, consequently, do not enter the equation for the spectrum \((4.13)\). That is why the correct description of the dynamics required account of the small gradient \( \nabla_i B \) on the background of \( B \). In the elastic turbulence case the terms with \( B \) are dissipative and, consequently, one can forget about the gradient terms, as subleading ones. Thus the equations for the small-scale components of the fields, following from Eqs. \((4.1)\), \((4.2)\), are

\[
\begin{align*}
\nabla p' &= (B \nabla)B' + v \nabla^2 v', \\
\partial_t B' + (v \nabla)B' &= (B' \nabla)v + (B \nabla)v' - B'/\tau. 
\end{align*}
\]

(4.5) (4.6)

Let us stress that the above equations assume only \( B \gg B' \) (not \( \nabla B \gg \nabla B' \), which is in fact wrong for the elastic turbulence.

To analyze the above equations for the small-scale components we use the same scheme as in the previous sections. As we explained above there is no need to account for the spatial variation of \( B \) so that (for the Fourier components) the equations \((4.3)-(4.6)\) take the form

\[
v' = \frac{i(k \cdot B)}{\nu k^2} B', \quad p' = 0, \\
\partial_t B' - (k \sigma \cdot \partial) B' = \sigma B' - \frac{B'}{\tau} - \frac{(k \cdot B)^2}{\nu k^2} B'.
\]

(4.7) (4.8)

The constraints \( k \cdot B' = 0, k \cdot v' = 0 \) that follow from \( \nabla \cdot v = 0 = \nabla \cdot B \) are consistent with the equations \((4.7)-(4.8)\). The last term in the equation for \( B' \) is due to the viscous dissipation, the term is of the same order as the elastic dissipation term \( -B'/\tau \) as follows from \((4.4)\). We observe that, again, the time-evolution operator for \( B' \) has zero scaling dimension and (in accordance with the discussion at the end of Section \(11\)) one expects that the spectrum \( F(k) \) of \( B \) obeys a power-law. To demonstrate this power behavior we use a formal solution of the equation \((4.8)\)

\[
B'(t,k) = W B'(0,kW) \exp \left[ -\frac{t}{\tau} - \int_0^t dt' \xi(t') \right],
\]

(4.9)

\[
\nu \xi(t) = |B(t')n(t',t,k)|^2, \quad \partial_t n = -n \sigma + n(n \sigma n),
\]

where \( W = W(t,0), \) and \( n(t',t,k) \) is determined by the above equation with the final condition \( n_k(t,t,k) = k/k \).

Let us analyze the stress spectrum function \( F(k) \):

\[
\langle B(k')B(k') \rangle = \delta(k + k') F(k)(\delta_{ij} - k_i k_j/k^2).
\]

Assuming that \( t \) is much larger than the correlation time \( \tau \) of \( B \) and \( \sigma \), we may average independently over the velocity statistics at negative and positive times, as we did in the previous sections. Then we find

\[
2F(k) = \langle Z \exp \left[ -2 \int_0^t dt' \xi(t') \right] F(kW) \rangle,
\]

(4.10)

\[
Z = \exp(-2t/\tau) \left[ \text{tr} W W^T W - \frac{k |WW^T|^2 k}{k WW^T k} \right].
\]

(4.11)

The coefficient of \( F \) in the right-hand side of Eq. \((4.10)\) is independent of the absolute value \( k \) of \( k \). Therefore Eq. \((4.10)\) admits a power solution \( F(k) \propto k^{-\beta} \), where \( \beta \) has to be determined from the equation

\[
2 = \left( \frac{Z \exp \left[ -2 \int_0^t dt' \xi(t') \right]}{|kW(t)/|k|^\beta|} \right).
\]

(4.12)

Note that at \( t \gg \tau \) the average in Eq. \((4.12)\) is determined by the events with \( k \) directed along the eigen vector of the matrix \( W W^T \), corresponding to the smallest eigen value (in this case the denominator in the expression achieves a minimum). Then the second term in the right-hand side of Eq. \((4.11)\) can be neglected, and we obtain \( Z = \exp(2\rho_1 - 2t/\tau) \) (therefore, due to the stationarity of the \( \rho_1 - t/\tau \) probability distribution, a statistics of \( Z \) is time-independent). The positive noise \( \xi(t) \) can be considered as stationary with the correlation time \( \tau \). Indeed, it follows from Eq. \((4.9)\) that \( n(t') \) forgets its
final direction (and thus also becomes $k$-independent) at $t - t' \gg \tau$. Thus the situation is similar to the one analyzed in Sections 3A. Again, we can prove the inequality $\beta > 3$, see Appendix A3.

We are now in a position to establish the spectrum of the velocity $E(k)$. Indeed, it follows from Eq. (4.7) that $E(k) \propto k^{-\beta-2}$ and the spherically normalized spectra obey

$$E_{\text{sph}}(k) \sim v^2 L(kL)^{-\beta}, \quad F_{\text{sph}}(k) \sim B^2 L(kL)^{-2-\beta}. \quad (4.13)$$

The estimates for the values $v^2$ and $B^2$ are written in Eq. (4.4). We observe that our scheme that assumes $\nabla v' \ll \nabla v$ and $B' \ll B$ is self-consistent due to the inequality $\beta > 3$ proved above. It agrees with the experiment, where $\beta$ is $3.3 \div 3.5$.

Finally, let us discuss the question concerning the validity region of the power spectrum. Probably, it is determined by the finite diffusivity $\kappa$ of the polymer molecules which is described by the term $\kappa \nabla^2 \Pi_{ij}$ to the right-hand side of Eq. (4.2). Comparing this term with, say, the relaxation term with $\tau$, one concludes that the power spectrum terminates at $k_{\text{diff}} \sim \sqrt{\kappa / \tau}$. At smaller scales the velocity spectrum diminishes much faster due to diffusivity.

V. CONCLUSION

We have investigated properties of turbulence in dilute polymer solutions in the cases where polymer molecules are strongly stretched. We established power-law distributions of kinetic and elastic energies over scales in some regions, where these power-laws are not related to an energy or other conserved quantity cascade (in contrast to the usual turbulence). In fact, excitation of elastic degrees of freedom at any scale leads to energy dissipation since the elastic dissipation is scale-independent. However, precisely this scale-independence can lead to a scale-invariance in the dissipative intervals, where the flow can be treated as smooth. Small-scale fluctuations are relatively weak and evolve passively in the smooth flow. As a result, the evolution of fine-scale fluctuations depends trivially on the scale and power-law spectra are formed. Let us now describe the cases where the above general ideas are applicable.

The first case, we examined, is the high Reynolds number flow above the coil-stretch transition, when elastic degrees of freedom are activated. Strong interaction between the elasticity and the flow modifies the latter below the scale $r_*$ (at this scale the velocity gradients are of the order of $1/\tau$), which is the new energy dissipation scale. This scale is of the order of the Kolmogorov scale at the transition and becomes larger as Re is increased. At $r \geq r_*$, the properties of turbulence are the same as in Newtonian fluids. The energy cascades downscale from the pumping scale and dissipates due to polymer relaxation at $r \sim r_*$. The flow is smooth at $r \lesssim r_*$ with the principal Lyapunov exponent $\lambda_1$ fixed at the value $1/\tau$ by the elastic back reaction. The fluctuations in the interval of scales $\nu(r)^{-1/2} \lesssim r \leq r_*$ are elastic waves. This leads to the equipartition of the kinetic and of the elastic energies, that is the velocity spectrum $E(k)$ and the elastic spectrum $F(k)$ coincide at these scales. The smoothness of the flow at $r \lesssim r_*$ leads to the conclusion that these spectra are power-like and in the spherical normalization the equation $\sim k^{-\beta}$ at $kr_* \gtrsim 1$. The power spectra terminate at the scale $(\nu r)^{1/2}$, where the viscous dissipation overcomes stretching.

The second case, we examined, is the elastic turbulence regime. It is a chaotic state which is realized at small Reynolds numbers $\text{Re}$. The velocity gradient imposed on the system by the boundary conditions exceeds $1/\tau$ which activates polymer degrees of freedom leading to hydrodynamic instability and chaotization. Again, the power spectra $F(k)$ and $E(k)$ are power-like in this case. However there are no elastic waves that would lead to equipartition. The main energy is carried by the polymers: $E(k) \sim \text{Re} (kL)^{-2} F(k)$, where $L$ is the size of the system. The velocity spectrum $E(k)$ decays faster than $k^{-3}$ in the spherical normalization, which corresponds to the experimental data.

The above-described mechanism of forming power-law spectrum for small-scale fluctuations in a chaotic flow seems rather general to be realized for other systems. We expect it to occur in certain regimes in magnetohydrodynamics, flows in liquid crystals and low-dimensional flows on a substrate.

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APPENDIX A: LONG-TIME LAGRANGIAN STATISTICS

Let us briefly review the long-time statistical properties of the Lagrangian mapping matrix $W$, determined by Eqs. (1.3,4). We consider $W(t_+, t_-)$ at $t_+ > t_-$ and assume that $t_+ - t_-$ is much larger than the Lagrangian correlation time $\tau_r$ of the velocity derivatives matrix (4.4), where one expects a universal statistics (28). If the velocity statistics is homogeneous in time, the probability distribution of $W(t_+, t_-)$ depends on the difference $t_+ - t_-$ only. Equation (1.5) implies that at $t_+ - t_- \gg \tau_r$ the matrix $W$ is a product of a large number of independent matrices. This is the main reason for the universality of the $W$ statistics.

It is convenient to decompose the matrix $W$ as

$$W(t_+, t_-) = M \Delta N, \quad (A1)$$

where $M$ is a large number, $N$ is a random matrix with the universal statistical properties, and $\Delta$ is an uncertainty matrix.
where $\Lambda$ is a diagonal matrix, and $M$ and $N$ are orthogonal matrices \[29\]. We denote the diagonal elements of $\Lambda$ as $e^{\rho_1}$, $e^{\rho_2}$, and $e^{\rho_3}$, and assume that they are ordered: $\rho_1 > \rho_2 > \rho_3$. As a consequence of the constraint det $W = 1$ we have $\rho_1 + \rho_2 + \rho_3 = 0$. Equation (1.7) can be rewritten in terms of $\rho_i$, and the matrices $M$ and $N$. The equations for $\rho_i$ are

$$\left. \frac{\partial \rho_i}{\partial t_+} \right| = \sigma_{ii},$$  \hspace{1cm} (A2)

where $\sigma = MT^T \sigma M$ and no summation over the repeating index $i$ is implied. The matrices $M$ and $N$ satisfy $\partial_i N = \Omega_1 N$ and $\partial_i M = M \Omega_2$, where

$$(\Omega_1)_{ik} = \frac{\delta_{ik} + \delta_{ki}}{2 \sinh(\rho_i - \rho_k)}, \quad (\Omega_2)_{ik} = \frac{\delta_{ik} e^{2 \rho_k} + \delta_{ki} e^{2 \rho_i}}{e^{2 \rho_k} - e^{2 \rho_i}},$$

for $i \neq k$ and $(\Omega_1)_{ik} = (\Omega_2)_{ik} = 0$ for $i = k$. It is possible to show that the eigenvalues of $W$ repel each other, so that the inequalities $e^{\rho_1} > e^{\rho_2} > e^{\rho_3}$ are satisfied at $t_+ - t_-$ $\gg \tau_\sigma$ \[30\]. Then the matrix $\Omega_1$ tends to zero exponentially fast, i.e. $N$ is determined by times of the order of $\tau_\sigma$ in the vicinity of $t_-$. The matrix $\Omega_2$ becomes $\rho$-independent at $t_+ - t_- \gg \tau_\sigma$ and the evolution of $M$ is decoupled from that of $\rho_i$. Then the value of $M$ is determined by the time of the order of $\tau_\sigma$ at $t \approx t_+$, i.e. at $t_+ - t_- \gg \tau_\sigma$ it becomes $t_-$-independent.

The solution of Eq. (A2) is

$$\rho_i(t) = \int_{t_-}^{t_+} dt' \sigma_{ii}(t').$$  \hspace{1cm} (A3)

where the right-hand side of Eq. (A3) is an integral of a random process independent of $\rho_i$. Equation (A3) shows that the variables $\rho_i$ fluctuate around their average values $\lambda_i(t_+ - t_-)$. Here the constants $\lambda_i$ are equal to $\langle \sigma_{ii} \rangle$. They are called the Lyapunov exponents of the flow. Generally, the spectrum of the Lyapunov exponents is non-degenerate: $\lambda_1 > \lambda_2 > \lambda_3$, which is a necessary condition for the formalism to be self-consistent. The incompressibility condition ensures the identity $\lambda_1 + \lambda_2 + \lambda_3 = 0$, which implies $\lambda_1 > 0$ and $\lambda_3 < 0$. Using the relation (1.7) one can show that $\lambda_i$ is indeed the average logarithmic divergence rate of two nearby Lagrangian trajectories:

$$\langle d \ln |\delta x| / dt \rangle = \lambda_i.$$

Similarly, $\lambda_1 + \lambda_2 = -\lambda_3$ is the average logarithmic rate of the area growth.

Note that at $t_+ - t_- \gg \tau_\sigma$ the statistics of $M$, $\Lambda$ and $N$ are independent. Indeed, the values of $\rho_i$ are accumulated during the whole evolution time $t_+ - t_-$ (see (A3)) and are not sensitive both to the interval $(t_- - t_+ + \tau_\sigma)$ determining $N$ and interval $(t_+ - \tau_\sigma, t_+)$ determining $M$. Both matrices $M$ and $N$ are distributed isotropically because of the assumed isotropy of the velocity statistics (this is the isotropization of $W$ referred in the main text).

1. Lagrangian Statistics in usual Turbulent or Random Flows

Here we consider the Lagrangian statistics in the case of a usual random (turbulent or chaotic) flow, when the velocity has finite correlation time and no constraints are imposed on the flow. Then the quantity $\rho_i$ can be treated as a sum of a large number of independent random variables, provided $t_+ - t_- \gg \tau_\sigma$. It is known from the statistical mechanics (see, e.g., \[31\]) that the distribution of such quantities is given by the exponent of an extensive function. In our case the probability distribution function (PDF) of $\rho_i$ is

$$P(t, \rho_1, \rho_2, \rho_3) \propto \frac{1}{t} \exp \left[ -t S \left( \frac{\rho_1 - \lambda_1 t}{t}, \frac{\rho_3 - \lambda_3 t}{t} \right) \right] \times \delta(\rho_1 + \rho_2 + \rho_3),$$  \hspace{1cm} (A4)

where $t = t_+ - t_-$ and $\rho_1 > \rho_2 > \rho_3$ is implied \[30\]. The main exponential factor of the PDF has a self-similar form described by the function $S$, which can be called entropy function (see \[30,32,33\]). It is positive, convex and has a minimum at zero values of its arguments. The precise form of $S$ is determined by details of the velocity statistics. The PDF has a sharp maximum at $\rho_i = \lambda_i t$. In its vicinity the function $S$ has a quadratic expansion, i.e. the distribution of $\rho$ is Gaussian. However, if one is interested in the expectation values of exponential functions of $\rho_i$, they are determined by the wings of the PDF where the Gaussian approximation is invalid. This entails the use of the whole entropy function.

To average the functions of $\rho_1$ only, one can introduce the reduced probability distribution function,

$$P(t, \rho_1) \propto \frac{1}{\sqrt{t}} \exp \left[ -t S_1 \left( \frac{\rho_1 - \lambda_1 t}{t} \right) \right],$$  \hspace{1cm} (A5)

which is an integral of $P(t, \rho_1, \rho_2, \rho_3)$ over $\rho_2$ and $\rho_3$. At small $x$ the function $S_1(x)$ can be written as

$$S_1(x) \approx x^2 / (2 \Delta).$$  \hspace{1cm} (A6)

Here $\Delta = \int dt \langle \delta \sigma(t) \delta \sigma(0) \rangle$ (where double brackets designate irreducible correlation function) determines the dispersion of $\rho_1$: $\langle (\rho_1 - \lambda_1 t)^2 \rangle \approx t \Delta$. Expansion (A6) is sufficient to describe typical fluctuations of $\rho_1$, whereas the whole function $S$ is needed to describe rare events.

2. Special properties of the long-time Lagrangian statistics above the coil-stretch transition

Above the coil-stretch transition the Lagrangian statistics acquires new qualitative features caused by the polymer back reaction. They can be inferred from the equation (1.4) which leads to

$$\ln |B(t, x)| - \ln |B(t_0, r)| \approx \rho_1(t, t_0) - t - t_0 / \tau_\sigma. $$  \hspace{1cm} (A7)
The expression \( A_7 \) is correct, provided \( t - t_0 \gg \tau \) (since the polymer relaxation time \( \tau \) determines also the velocity gradients correlation time). The left-hand side of the equation \( A_3 \) has stationary statistics which leads to dramatic changes in the statistics of \( \rho_1 \). Upon averaging one finds \( \langle \rho_1 \rangle = t/\tau \) which means \( \lambda_1 = \langle \bar{\sigma}_{11} \rangle = 1/\tau \) as it was already explained in the main text. Stationarity of the dispersion of \( \rho_1 \) leads to the conclusion that \( \int_0^{t/2} dt' \langle \bar{\sigma}_{11}(0) \bar{\sigma}_{11}(t') \rangle \propto t^{-1} \) in the limit \( t \to \infty \). This is related to the anticorrelation property of \( \bar{\sigma}_{11} \), mentioned in the main text. That means that the dispersion \( \Delta \), defined by Eq. \( A_6 \), vanishes above the coil-stretch transition. In fact, this vanishing is not abrupt and occurs within \( 1/\ln(R_{\text{back}}/R_0) \) vicinity of \( R_{\text{co}} \). Finally one concludes that \( \langle \bar{\sigma}_{11}(0) \bar{\sigma}_{11}(t) \rangle \propto t^{-2} \) at large \( t \) which is again very different from a Newtonian fluid where exponential decay is observed, see Ref. [23].

3. Inequality for the exponents \( \alpha, \beta \)

Here we establish the inequality for the exponent \( \alpha \), characterizing the passive scalar spectrum, see Section 4, and the inequality for the exponent \( \beta \) appearing in the elastic turbulence problem, see Section 4.3. In both cases we investigate the solution \( \Delta \) of the equation of the type

\[
\left\langle |kW|^\alpha \exp \left[ -\int_0^t dt' y(t') \right] \right\rangle \sim k^\Delta, \tag{A8}
\]

where \( t \) is much larger than the correlation time of a random positive noise \( y(t) \) and \( \sigma \). At these times the behavior of the moments is exponential and one can define \( \exp[\int_0^t dt' y(t')]|kW|^\delta \sim k^\delta \exp[\bar{\gamma}(\delta)t] \) where \( \bar{\gamma}(\delta) \) is a convex function due to Hölder inequality. This function is strictly smaller than another convex function \( \gamma(\delta) \) defined by \( \langle |kW|^\delta \rangle \sim k^\delta \exp[\gamma(\delta)t] \).

The function \( \gamma(\delta) \) has a universal behavior which we describe now. It is convenient to write

\[
\left\langle |kW|^\delta \right\rangle = \int dk' k'^\delta \left\langle \delta (k - k'W^{-1}(t)) \right\rangle, \tag{A9}
\]

making it explicit that the wavevectors evolve according to \( k(t) = k(0)W^{-1}(t) \) [23]. Introducing \( k'W^{-1}(t) = k' \exp[\rho(t)]n \), where \( n \) is a unit vector, one finds

\[
\rho(t) = \int_0^t dt' \zeta(t'), \quad \frac{dn}{dt} = n\sigma + n\zeta, \quad \zeta \equiv n\sigma n.
\]

One observes that \( \zeta \) is independent of \( \rho(0) \) which leads to the conclusion that at \( t \gg \tau_\sigma(\sigma) \), where \( \tau_\sigma(\sigma) \) is the correlation time of \( \sigma \), the probability distribution of \( \rho \) is described by an entropy function \( S \).

It can be shown that \( \langle \rho(t) \rangle = |\lambda_3|t \), where \( |\lambda_3| \) is the lowest in the hierarchy of the Lyapunov exponents of the flow. This fact is intuitively clear as \( \lambda_3 \) determines contraction in the real space and thus stretching in \( k \)-space.

We note that \( \rho(t) \) is determined by the whole interval \( (0, t) \) while \( n(t) \) only by \( \lambda_3^{-1} \) vicinity of \( t \). As a result, \( \rho \) and \( n \) are independent at \( \lambda_3 t \gg 1 \). Since \( n \) is isotropically distributed over the unit sphere one finds \( \langle \delta(k - k'W^{-1}(t)) \rangle = \langle \delta(\rho - \ln(k/k')) \rangle \exp[-3\rho]/(4\pi k^3) \). Substituting this into \( A_9 \) and performing the integral one finds

\[
\int \frac{d\rho}{N} \exp \left[ -(\delta + 3)\rho - tS \left( \frac{\rho - |\lambda_3|t}{t} \right) \right] \sim \exp[\gamma(\delta)t],
\]

where \( N \propto \sqrt{t} \) is the normalization factor insignificant for the following considerations. It follows that besides the zero at the origin following from the definition, the function \( \gamma(\delta) \) vanishes at \( \delta = -3 \). This is a general consequence of the isotropy employed above, see also [16]. Besides, we observe that \( \gamma'(3) = \lambda_3 < 0 \). Combining these properties with the convexity of \( \gamma(\delta) \) we conclude that \( \gamma(\delta) \) is negative for \( -3 < \delta < 0 \) and positive otherwise.

To prove the inequality on \( \Delta \) appearing in Eq. \( A_8 \) one notes that both \( \gamma(\delta) \) and \( \bar{\gamma}(\delta) \) tend to \( \infty \) as \( |\delta| \to \infty \). Then it follows from \( \gamma(\delta) < \bar{\gamma}(\delta) \) that there are two solutions of Eq. \( A_8 \): one positive and one smaller than \( -3 \). Substituting \( y(t) = 2/t \) and recognizing that the exponent \( \alpha \) appearing in \( A_9 \) must be positive we conclude that \( \alpha > 3 \). Analogously, substituting \( y(t) = 2t \) we conclude that the solution of Eq. \( A_12 \) satisfies \( \beta > 3 \).

Finally, let us give an example of calculating \( \alpha \) in a limiting case. The exponent is determined by the equation

\[
\int \frac{d\rho}{N} \exp \left[ (\alpha - 3)\rho - tS \left( \frac{\rho - |\lambda_3|t}{t} \right) \right] = \exp \left[ \frac{2\gamma}{\tau} \right]. \tag{A10}
\]

Note that \( \alpha - 3 \) vanishes in the limit \( |\lambda_3|\tau \to \infty \) since in this limit the linear decay term is negligible in Eq. \( 2.1 \) and Batchelor \( k^{-3} \) spectrum must result. Therefore at large \( |\lambda_3|\tau \) the integral \( A_10 \) is determined by the maximum of the probability concentrated at \( \rho = |\lambda_3|t \) which leads to

\[
E(k) \sim k^{-3-2(|\lambda_3|\tau)^{-1}}, \quad |\lambda_3|\tau \gg 1. \tag{A11}
\]

For a general value of \( |\lambda_3|\tau \) the exponent \( \alpha \) is determined by the concrete form of the entropy function.

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