A FULLY NONLINEAR FREE BOUNDARY PROBLEM ARISING FROM OPTIMAL DIVIDEND AND RISK CONTROL MODEL

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Abstract. Focusing on the problem arising from a stochastic model of risk control and dividend optimization techniques for a financial corporation, this work considers a parabolic variational inequality with gradient constraint

$$\min \left\{ v_t - \max_{0 \leq a \leq 1} \left( \frac{1}{2} \sigma^2 a^2 v_{xx} + \mu a v_x \right) + cv, v_x - 1 \right\} = 0.$$ 

Suppose the company’s performance index is the total discounted expected dividends, our objective is to choose a pair of control variables so as to maximize the company’s performance index, which is the solution to the above variational inequality under certain initial-boundary conditions. The main effort is to analyze the properties of the solution and two free boundaries arising from the above variational inequality, which we call dividend boundary and reinsurance boundary.

1. Introduction. An insurance company is a typical example of a financial corporation in the problem of optimal dividend distribution and risk control. [15, 16, 9] considered some kinds of optimal risk and dividend control problems under some conditions. For a finite time model, we solved the problems of dividend control model and risk control model in [7] and [8], respectively. In this paper, we deal with a stochastic model of risk control and dividend optimization, which is an evolutionary problem of one of the models in [15]. [15] discussed some different types of conditions imposed upon a company and different types of reinsurance available,

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however, the assumption of finite time will make the problem much more complicated.

We consider that the insurance company has an option to choose the times and amounts of dividends and the reinsurance policies before \( T \) and \( \tau \), here \( T \) is the terminal time and \( \tau \) is the bankruptcy time. The type of reinsurance in this paper is proportional reinsurance, that is, it requires the reinsurer to cover a fraction of each claim equal to the fraction of total premiums he receives from the cedent.

In recent years, there is growing interest in applying parabolic variational inequality with gradient constraint and Barenblatt equation in mathematical finance, see for example [2, 1, 3, 10, 11, 17]. However, up to date, there seems to be no results on parabolic Barenblatt equation with gradient constraint. Our primary motivation stems from the free boundaries arising from the Barenblatt equation.

Specifically, we focus on the following problem

\[
\begin{align*}
\min \{ v_t - \max_{0 \leq a \leq 1} \left( \frac{1}{2} \sigma^2 a^2 v_{xx} + \mu a v_x \right) + cv, \ v_x - 1 \} &= 0, \quad x > 0, \ 0 < t \leq T, \\
v(0, t) &= 0, \quad 0 < t \leq T, \\
v(x, 0) &= x, \quad x > 0.
\end{align*}
\]

Divide the domain \( Q_T := (0, +\infty) \times (0, T] \) into

\[
D := \{ v_x(x, t) = 1 \}, \quad ND := \{ v_x(x, t) > 1 \},
\]

which are called dividend region and non-dividend region, respectively.

Now we pay attention to the equation

\[
v_t - \max_{0 \leq a \leq 1} \left( \frac{1}{2} \sigma^2 a^2 v_{xx} + \mu a v_x \right) + cv = 0.
\]

Since \( v_x \geq 1 \), the maxima of \( a \) in \([0, 1]\) is

\[
a^*(x, t) = \arg \max_{0 \leq a \leq 1} \left( \frac{1}{2} \sigma^2 a^2 v_{xx} + \mu a v_x \right) = \begin{cases} \min \left\{-\frac{\mu}{\sigma^2 v_{xx}}, 1\right\}, & \text{if } v_{xx} < 0; \\
1, & \text{if } v_{xx} \geq 0.
\end{cases}
\]

In the case \(-\frac{\mu}{\sigma^2 v_{xx}} < 1 \) and \( v_{xx} < 0 \), then \( a^* < 1 \), which means that the insurance company shall transfer \( 1 - a^* = 1 + \frac{\mu}{\sigma^2 v_{xx}} \) ratio of incomes to reinsurance company, the corresponding region denoted by \( \mathcal{R} \), called reinsurance region. In the other case \(-\frac{\mu}{\sigma^2 v_{xx}} \geq 1 \) or \( v_{xx} \geq 0 \), then \( a^* = 1 \), which means the insurance company is not need to transfer any risk to the reinsurance company, the corresponding region denoted by \( \mathcal{NR} \). That is

\[
\mathcal{R} = \left\{-\frac{\mu}{\sigma^2 v_{xx}} < 1 \right\} \cap \{ v_{xx} < 0 \}, \quad \mathcal{NR} = \left\{-\frac{\mu}{\sigma^2 v_{xx}} \geq 1 \right\} \cup \{ v_{xx} \geq 0 \}.
\]

Thus

\[
v_t + \frac{\mu^2}{2\sigma^2 v_{xx}} v_x + cv = 0, \quad (x, t) \in \mathcal{R} \cap ND,
\]

\[
v_t - \frac{1}{2} \sigma^2 v_{xx} - \mu v_x + cv = 0, \quad (x, t) \in \mathcal{NR} \cap ND.
\]

You may see that (3) is a fully non-linear equation, and we also found that on the left boundary \( x = 0 \), there exists

\[
v_x(0+, t) = +\infty, \quad v_{xx}(0+, t) = -\infty
\]

although \( v(0, t) = 0 \). Moreover, we found that \( \frac{\mu}{v_{xx}}(0+, t) = 0 \), this means that (3) degenerates on \( x = 0 \)(and degeneration only happens on \( x = 0 \) (see Corollary 2)).
Therefore, in order to prove the existence of solution and its regularity, we should
use the method of approaching to construct non-degenerate approaching problem
whose \( v_x, v_{xx} \) are nonsingular on \( x = 0 \). However, the proof of convergence is not
easy, because we need to verify the uniform parabolic condition for the approaching
problem holds in the interior of the domain, which is highly technical.

Meanwhile, we can see the free boundary between \( R \) and \( NR \) (Reinsurance free
boundary) is the set

\[
\left\{ -\frac{\mu}{\sigma^2} v_x = 1 \right\} = \left\{ v_{xx} + \frac{\mu}{\sigma^2} v_x = 0 \right\}.
\]

In order to obtain the existence and the smoothness of this free boundary line,
the directly perception is to discuss the monotonicity and the smoothness of the
function \(-\frac{\mu}{\sigma^2} v_x\) or \( v_{xx} + \frac{\mu}{\sigma^2} v_x\), but this is not an easy task. To proceed, we deal
with it in another way, observing from the equations (3) and (4) we found that the
function

\[
W(x, t) := v_t - \frac{\mu}{2} v_x + cv
\]

has the same signs with \( v_{xx} + \frac{\mu}{\sigma^2} v_x \) whether in \( R \) or \( NR \) and it is more convenient
to analyze. Therefore, we turn to \( W(x, t) \) to research the reinsurance free boundary
line. This is an innovation point of this article.

The rest of the paper is arranged as follows. The precise formulation of the
problem and the HJB equation are given in Section 2. Section 3 shows the existence
and uniqueness of the solution to problem (1) and obtains some properties of the
solution. Section 4 characterizes the properties of the dividend boundary \( h(t) \). Section 5 shows the existence and uniqueness of the reinsurance boundary \( g(t) \),
moreover, it is a differentiable curve and \( 0 < g(t) < h(t) \). We present the optimal
strategy to the financial model in section 6. In Appendix A, we give the verification
theorem to show that the solution to the HJB equation coincides with the value
function. In Appendix B, we present the expression of solution to corresponding
steady problem. In Appendix C, we show the proof of Lemma 5.1. In Appendix D,
we confirm value function (7) is a viscosity solution of (9).

2. Formulation. In this paper, we work with a complete filtered probability space
\((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), where \((\Omega, \mathcal{F}, \mathbb{P})\) is the probability space, and \(\{\mathcal{F}_t\}_{t \geq 0}\) is the fil-
tration satisfying the usual condition. We consider that a financial corporation has
an option to choose the policies of the dividends distribution and the reinsurance
part of the claims (the cedent is required to pay simultaneously diverting part of
the premiums to a reinsurance company). Let \( R_t \) be the dynamics of the reserve
process, it satisfies

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\quad dR_s = \mu a_s dt + \sigma a_s dW_s - dL_s, \quad t \leq s \leq T, \\
R_{t^-} = x, \quad x > 0,
\end{array}
\right. \\
(a, L) \in \Pi_t.
\end{align*}
\]

where \( \mu, \sigma \) are positive constants and \( W_s \) is a standard Brownian motion, control
process \((a_s, L_s)\) is adapted to \( \mathcal{F}_s \), \( 0 \leq 1 - a_s \leq 1 \) corresponds to the reinsurance
fraction, and \( L_s \geq 0 \) is a non-decreasing, càdlàg process, representing the cumulative
dividends up to time \( s \). We denote by \( \Pi_t = \{(a_s, L_s) | s \geq t\} \) the set of all admissible
control processes from time \( t \) with \( R_{t^-} = x \).

Our objective is to choose an optimal control process so as to maximize the company’s
discounted expected dividends up to time \( T \wedge \tau \), where \( \tau \) is the bankruptcy
time, defined as
\[ \tau = \inf \{ s \geq t : R_s \leq 0 \}. \]  

With the control process \((a, L)\), we can define the company’s performance index as

\[
J_{x,t}(a, L) = \mathbb{E}\left[ \int_t^{T_{a,L}} e^{-c(s-t)} dL_s \bigg| R_{t-} = x \right] = \mathbb{E}_{x,t}\left[ \int_t^{T_{a,L}} e^{-c(s-t)} dL_s \right].
\]

And the value function is

\[
V(x, t) = \sup_{(a, L) \in \Pi} J_{x,t}(a, L) = \sup_{(a, L) \in \Pi} \mathbb{E}_{x,t}\left[ \int_t^{T_{a,L}} e^{-c(s-t)} dL_s \right]. \tag{7}
\]

Using the theory of viscosity solution in differential equations, one can obtain the following HJB equation

\[
\min \left\{ -v_t - \max_{0 \leq a \leq 1} \left( \frac{1}{2} \sigma^2 a^2 \sigma^2 v_{xx} + \mu a v_x \right) + c V, V_x - 1 \right\} = 0, \quad x > 0, \quad 0 < t < T \tag{8}
\]

in viscosity sense. (See Appendix D.)

On the other hand, when \(x = 0\), then \(\tau = t\) and \(L_t - L_{t-} = 0\), thus \(V(0, t) = 0\).

When \(t = T\), then \(V(x, T) = \max(L_T - L_{T-}) = R_T = x\). So we consider the following variational inequality problem

\[
\begin{align*}
\min \{ -v_t & - \max_{0 \leq a \leq 1} \left( \frac{1}{2} \sigma^2 a^2 \sigma^2 v_{xx} + \mu a v_x \right) + c V, V_x - 1 \} = 0, \\
V(0, t) &= 0, \quad 0 < t < T, \\
V(x, T) &= x, \quad x > 0.
\end{align*} \tag{9}
\]

3. Solvability of problem (9). For the convenience in the analysis, we analyze the backward equation (9) by studying its associated forward equation of the following form. Let \(v(x, t) = V(x, T - t)\) in (9), then \(v(x, t)\) satisfies

\[
\begin{align*}
\min \{ v_t - & \sup_{0 \leq a \leq 1} \left( \frac{1}{2} a^2 \sigma^2 v_{xx} + a \mu v_x \right) + c v, v_x - 1 \} = 0, \quad (x, t) \in (0, +\infty) \times (0, T], \\
v(0, t) &= 0, \quad 0 < t \leq T, \\
v(x, 0) &= x, \quad x > 0.
\end{align*} \tag{10}
\]

We will show that there is a constant \(x_d > 0\) (the dividend free boundary point of corresponding steady problem) such that \(v_x = 1\) when \(x \geq x_d\). So we only need to consider the above problem in the truncated domain,

\[
\begin{align*}
\min \{ v_t - & \sup_{0 \leq a \leq 1} \left( \frac{1}{2} a^2 \sigma^2 v_{xx} + a \mu v_x - c v \right), v_x - 1 \} = 0, \quad (x, t) \in Q_T, \\
v(0, t) &= 0, \quad v_x(N, t) = 1, \quad 0 < t \leq T, \\
v(x, 0) &= x, \quad 0 < x < N,
\end{align*} \tag{10}
\]

where \(Q_T = (0, N) \times (0, T]\) with \(N = x_d + 1\).

The aim of this section is to prove the existence of \(W_{p,1}^2\) solution to problem (10) and some useful estimates. The main results are shown in Theorem 3.8. Firstly, we rewrite (10) to a fully non-linear problem (11). And then we construct the penalty approximation problem (14). In order to get the solution to problem (14), we turn to problem (20), whose solution is the gradient of solution to problem (14). After discussing problem (20), we succeed in proving Theorem 3.8.
3.1. The equivalent problem (11). Note that (10) is a Barenblatt equation and it is not convenience to research it directly, we should change it to a specific fully non-linear equation. Define a function $A(z)$ as

$$A(z) = \begin{cases} -\frac{\mu}{\sigma^2} z, & -\frac{\sigma^2}{\mu} < z \leq 0; \\ 1, & z \leq -\frac{\sigma^2}{\mu} \text{ or } z > 0, \end{cases}$$

and $A(\pm\infty) := \lim_{z \to \pm\infty} A(z) = 1$. Then the optimal control variable

$$a^*(x,t) = \arg \max_{0 \leq a \leq 1} \left\{ \frac{1}{2} a^2 \sigma^2 v_{xx} + a \mu v_x \right\}$$

$$= \begin{cases} \min \left\{ -\frac{\mu}{\sigma^2} \frac{v_x}{v_{xx}}, 1 \right\}, & \text{if } v_{xx} < 0; \\ 1, & \text{if } v_{xx} \geq 0 = A\left(\frac{v_x}{v_{xx}}\right). \end{cases}$$

It is not hard to see that

$$A'(z) = \begin{cases} -\frac{\mu}{\sigma^2}, & -\frac{\sigma^2}{\mu} < z \leq 0; \\ 0, & z \leq -\frac{\sigma^2}{\mu} \text{ or } z > 0. \end{cases}$$

and $A'(\pm\infty) := \lim_{z \to \pm\infty} A'(z) = 0$. Therefore, (10) can be rewritten as

$$\begin{cases} \min \left\{ v_t - \frac{\sigma^2}{2} A^2 \left( \frac{v_x}{v_{xx}} \right) v_{xx} - \mu A \left( \frac{v_x}{v_{xx}} \right) v_x + c v, v_x - 1 \right\} = 0, & (x,t) \in Q_T, \\ v(0,t) = 0, & v_x(N,t) = 1, \quad 0 < t \leq T, \\ v(x,0) = x, & 0 < x < N. \end{cases}$$

Define

$$L^v := \frac{\sigma^2}{2} A^2 \left( \frac{v_x}{v_{xx}} \right) v_{xx} + \mu A \left( \frac{v_x}{v_{xx}} \right) v_x - cv$$

and

$$T^u := \frac{\sigma^2}{2} A^2 \left( \frac{u}{u_x} \right) u_{xx} + \mu A \left( \frac{u}{u_x} \right) u_x - cu.$$
\[ G_y(x, y) = -A' \left( \frac{x}{y} \right) \frac{x}{y^2} = \begin{cases} \frac{\mu}{\sigma^2} y & x \in (0, \frac{\sigma^2}{\mu}), \quad y \leq \frac{-\mu}{\sigma^2} x, \\ 0, & y > \frac{-\mu}{\sigma^2} x. \end{cases} \]

In particular, this fact is sufficient to establish the comparison principle associated to the operator \( \mathcal{L} \) and \( \mathcal{F} \).

### 3.2. The penalty approximation problem (14) and (20).

In order to prove the existence of a solution to problem (11), we construct the following penalty approximation problem. Suppose \( v^\varepsilon(x, t) \) satisfies

\[
\begin{align*}
  & v^\varepsilon_t - \mathcal{L}_\varepsilon v^\varepsilon + \int_0^x \beta_\varepsilon(v^\varepsilon_x - 1) \, dx = 0, \quad (x, t) \in Q_T, \\
  & v^\varepsilon(0, t) = 0, \quad v^\varepsilon_x(N, t) = 1, \quad 0 < t \leq T, \\
  & v^\varepsilon(x, 0) = x, \quad 0 < x < N,
\end{align*}
\]

where

\[
\mathcal{L}_\varepsilon v := \mathcal{L} v + \frac{\sigma^2}{2} \varepsilon^2 v_{xx} = \frac{\sigma^2}{2} \left[ A^2 \left( \frac{v_x}{v_{xx}} \right) + \varepsilon^2 \right] v_{xx} + \mu A \left( \frac{v_x}{v_{xx}} \right) v_x - cv,
\]

and penalty function \( \beta_\varepsilon(t) \) (see Figure 1) satisfies

- \( \beta_\varepsilon(\cdot) \in C^2(-\infty, +\infty), \beta_\varepsilon(t) \leq 0, \)
- \( \beta'_\varepsilon(t) \geq 0, \beta''_\varepsilon(t) \leq 0, \beta_\varepsilon(0) = -c, \)
- \( \lim_{\varepsilon \to 0} \beta_\varepsilon(t) = \begin{cases} 0, & t > 0, \\
-\infty, & t < 0. \end{cases} \)

**Figure 1.** Penalty function

Differentiate the equation in (14) w.r.t. \( x \), then we have

\[
\partial_x v^\varepsilon_t - \mathcal{T}_\varepsilon v^\varepsilon + \beta_\varepsilon(v^\varepsilon_x - 1) = 0,
\]

where

\[
\mathcal{T}_\varepsilon u := \mathcal{T} u + \frac{\sigma^2}{2} \varepsilon^2 u_{xx} = \frac{\sigma^2}{2} \left[ A^2 \left( \frac{u}{u_{xx}} \right) + \varepsilon^2 \right] u_{xx} + \mu A \left( \frac{u}{u_{xx}} \right) u_x - cu.
\]

Owing to the boundary condition on \( x = 0 \) and the equation in (14), we have

\[
\left( \frac{\sigma^2}{2} \left[ A^2 \left( \frac{v^\varepsilon_x}{v^\varepsilon_{xx}} \right) + \varepsilon^2 \right] v^\varepsilon_{xx} + \mu A \left( \frac{v^\varepsilon_x}{v^\varepsilon_{xx}} \right) v^\varepsilon_x \right)|_{x=0} = (v^\varepsilon_t + cv^\varepsilon)|_{x=0} = 0. \quad (16)
\]
We claim that for any \( \varepsilon \) small enough, the above boundary condition is equivalent to the following condition
\[
v_{x}^\varepsilon + \frac{\varepsilon \sigma^2}{\mu} v_{xx}^\varepsilon \bigg|_{x=0} = 0. \tag{17}
\]

In fact, according to the definition of \( A(\varepsilon) \), (16) implies
\[
\left( -\frac{\mu^2}{2\sigma^2} \frac{v_{xx}^\varepsilon}{v_{x}^\varepsilon} + \frac{\varepsilon^2 \sigma^2}{2} v_{xx}^\varepsilon \right)_{x=0} = 0 \quad \text{if} \quad -\frac{\mu}{\sigma^2} v_{x}^\varepsilon \bigg|_{x=0} < 1, \tag{18}
\]
\[
\left( \frac{\sigma^2}{2} (1 + \varepsilon^2) v_{xx}^\varepsilon + \mu v_{x}^\varepsilon \right)_{x=0} = 0 \quad \text{if} \quad -\frac{\mu}{\sigma^2} v_{x}^\varepsilon \bigg|_{x=0} \geq 1 \quad \text{or} \quad v_{xx}^\varepsilon \bigg|_{x=0} = 0. \tag{19}
\]

There is a contradiction in (19), which implies only (18) holds true. Since we could deduce that the solution of (14) satisfies \( v_{x}^\varepsilon \geq 1 \) and \( v_{xx}^\varepsilon \leq 0 \) (see (36)), together with (18), we obtain the boundary condition (17).

Denote \( u^\varepsilon = v_{x}^\varepsilon \), from (15), (17), \( u^\varepsilon \) satisfies
\[
\begin{cases}
  u_{t}^\varepsilon - T_{\varepsilon} u_{t}^\varepsilon + \beta_{\varepsilon}(u_{x}^\varepsilon - 1) = 0, & (x, t) \in Q_T, \\
  u_{t}^\varepsilon(0, t) + \frac{\varepsilon^2}{\mu} u_{x}^\varepsilon(0, t) = 0, & u_{t}^\varepsilon(N, t) = 1, \quad 0 < t \leq T, \\
  u_{t}^\varepsilon(x, 0) = 1, & 0 < x < N.
\end{cases} \tag{20}
\]

3.3. The solution to problem (20). Note that the boundary condition and initial condition in (20) are not consistent at \( (0, 0) \). In order to have existence of classical solution, we construct the following approximation problem
\[
\begin{cases}
  u_{t}^{\varepsilon, \delta} - T_{\varepsilon} u_{t}^{\varepsilon, \delta} + \beta_{\varepsilon}(u_{x}^{\varepsilon, \delta} - 1) = 0, & (x, t) \in Q_T, \\
  u_{t}^{\varepsilon, \delta}(0, t) + \frac{\varepsilon^2}{\mu} u_{x}^{\varepsilon, \delta}(0, t) = f_{\delta}(t), & u_{t}^{\varepsilon, \delta}(N, t) = 1, \quad 0 < t \leq T, \\
  u_{t}^{\varepsilon, \delta}(x, 0) = 1, & 0 < x < N,
\end{cases} \tag{21}
\]
where \( f_{\delta}(t) \in C^2([0, +\infty)) \) and
\[
f_{\delta}(t) = \begin{cases}
  1, & t = 0, \\
  \frac{\varepsilon}{\varepsilon - t}, & 0 \leq t < \delta, \\
  0, & t \geq \delta.
\end{cases} \tag{22}
\]

Since the boundary condition in (21) is unusual, we should establish the comparison principle associated to (21).

**Lemma 3.1.** Suppose \( u_{1}, u_{2} \in C^{2,1}(Q_T) \cap C(Q_T) \) which satisfy
\[
\begin{cases}
  \partial_{t} u_{1} - T_{\varepsilon} u_{1} + \beta_{\varepsilon}(u_{1} - 1) = \partial_{t} u_{2} - T_{\varepsilon} u_{2} + \beta_{\varepsilon}(u_{2} - 1), & (x, t) \in Q_T, \\
  u_{1}(0, t) + \frac{\varepsilon^2}{\mu} u_{1}(0, t) \geq u_{2}(0, t) + \frac{\varepsilon^2}{\mu} u_{2}(0, t), & 0 < t \leq T, \\
  u_{1}(N, t) \leq u_{2}(N, t), & 0 < t \leq T, \\
  u_{1}(x, 0) \leq u_{2}(x, 0), & 0 < x < N.
\end{cases}
\]
and \( u_{1}, u_{2} \geq 1 \), then
\[
u_{1}(x, t) \leq u_{2}(x, t), \quad (x, t) \in Q_T. \tag{23}
\]

**Proof.** Denote \( \lambda = \frac{\mu}{\varepsilon \sigma^2} \) and
\[
w_{1} = e^{\lambda x} u_{1}, \quad w_{2} = e^{\lambda x} u_{2},
\]
then we have
\[
\begin{cases}
\partial_tw_1 - e^{\lambda x}T_\varepsilon(e^{-\lambda x}w_1) + e^{\lambda x}\beta_x(e^{-\lambda x}w_1 - 1) \\
\leq \partial_tw_2 - e^{\lambda x}T_\varepsilon(e^{-\lambda x}w_2) + e^{\lambda x}\beta_x(e^{-\lambda x}w_2 - 1), \quad (x,t) \in Q_T,
\end{cases}
\]
\[
w_1x(0,t) \geq w_2x(0,t), \quad w_1(N,t) \leq w_2(N,t), \quad 0 < t \leq T,
\]
\[
w_1(x,0) \leq w_2(x,0), \quad 0 < x < N.
\]
Here we emphasize \(u_1, u_2 \geq 1\) which is sufficient to let \(A(\cdot)\) in \(T_\varepsilon(e^{-\lambda x}w_i)\) be Lipschitz continuous on \(w_i, w_{ix}, i = 1, 2\) (see Remark 1). By the comparison principle for quasi-linear equations (see [4]), we obtain

\[
w_1(x,t) \leq u_2(x,t), \quad (x,t) \in Q_T.
\]

Hence, we get (23).

**Theorem 3.2.** There exists a solution \(u^{\varepsilon, \delta} \in C^{2,1}(\overline{Q_T})\) to problem (21), moreover, \(u^{\varepsilon, \delta}\) satisfies

\[
1 \leq u^{\varepsilon, \delta} \leq Ke^{pt}\left(x + \frac{\varepsilon \sigma^2\eta}{\mu}\right)^{-\eta},
\]

where
\[
\eta = \frac{\mu^2}{2\sigma^2 + \mu^2}, \quad K = \left(L + \frac{\varepsilon \sigma^2\eta}{\mu}\right)^{\eta+1}, \quad p = \max\left\{ \frac{\mu^2(\eta + 1)}{2\sigma^2\eta}, \frac{\mu^2(\eta + 1)}{\sigma^2\eta} - c \right\}.
\]

**Proof.** The Leray-Schauder fixed point theorem (see [4, 6]) and embedding theorem (see [4]) implies the existence of \(C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{Q_T})(0 < \alpha < 1)\) solution to the problem (21). By Schauder estimation (see [14]), we have \(u^{\varepsilon, \delta} \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q_T})\).

Now we will show the estimate (24). Denote \(u_1 = 1\), then

\[
\begin{cases}
\partial_tu_1 - \frac{\sigma^2}{2}[A^2(\cdot) + \varepsilon^2]u_{1xx} - \mu A(\cdot)u_{1x} + cu_1 + \beta_x(u_1 - 1) = 0, \quad (x,t) \in Q_T,
\end{cases}
\]
\[
u_1(0,t) + \frac{\sigma^2}{\mu}u_{1x}(0,t) = 1 \geq f_3(t), \quad u_1(N,t) = 1, \quad 0 < t \leq T,
\]
\[
u_1(x,0) = 1, \quad x > 0.
\]

Here, we treat \(A(\cdot) = A\left(\frac{u^{\varepsilon, \delta}}{u_{1x}}\right)\) as a known function, similar to the proof of Lemma 3.1, we have \(u^{\varepsilon, \delta} \geq u_1 = 1\).

Let \(u_2 = Ke^{pt}\left(x + \frac{\varepsilon \sigma^2\eta}{\mu}\right)^{-\eta}\), then

\[
u_2 \geq L + \frac{\varepsilon \sigma^2\eta}{\mu} \geq 1 + \varepsilon, \quad \beta_x(u_2 - 1) = 0,
\]
\[
u_{2x} = -\eta Ke^{pt}\left(x + \frac{\varepsilon \sigma^2\eta}{\mu}\right)^{-\eta-1}, \quad \frac{u_2}{u_{2x}} = -\frac{1}{\eta} \left(x + \frac{\varepsilon \sigma^2\eta}{\mu}\right),
\]
when \(\frac{\mu}{\sigma^2}\left(x + \frac{\varepsilon \sigma^2\eta}{\mu}\right) < 1\), we know \(A\left(\frac{u_2}{u_{2x}}\right) = -\frac{\mu}{\sigma^2}\frac{u_2}{u_{2x}} = \frac{\mu}{\sigma^2\eta}\left(x + \frac{\varepsilon \sigma^2\eta}{\mu}\right)\), then

\[
\partial_tu_2 - T_\varepsilon u_2 + \beta_x(u_2 - 1)
\]
\[
= -\frac{\sigma^2}{2} \left[A^2\left(\frac{u_2}{u_{2x}}\right) + \varepsilon^2\right]u_{2xx} - \mu A\left(\frac{u_2}{u_{2x}}\right)u_{2x} + cu_2 + \beta_x(u_2 - 1)
\]
Together with Lemma 3.3, for the solution $u$ applying Lemma 3.1, we obtain

$$
\sigma^2 = Ke^{pt} \left( x + \varepsilon \sigma^2 \eta \right) - \eta [p - \frac{\mu^2}{2} \left( \frac{\mu}{\sigma^2} \right) \eta (\eta + 1) - \frac{\mu^2}{\sigma^2} \eta (-\eta) + c]
$$

$$
- \frac{\sigma^2 \varepsilon^2}{2} \eta (\eta + 1) Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta^{-2}
$$

$$
= Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta [p - \frac{\mu^2}{2} \left( \frac{\mu}{\sigma^2} \right) \eta (\eta + 1) + \frac{\mu^2}{\sigma^2} + c]
$$

$$
- \frac{\sigma^2 \varepsilon^2}{2} \eta (\eta + 1) Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta^{-2}
$$

$$
= Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta [p + \frac{\mu^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} \eta + c]
$$

$$
- \frac{\sigma^2 \varepsilon^2}{2} \eta (\eta + 1) Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta^{-2}
$$

$$
= Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta [p - \frac{\sigma^2 \varepsilon^2}{2} \eta (\eta + 1)]\left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right)^{-2}
$$

$$
\geq Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta [p - \frac{\sigma^2 \varepsilon^2}{2} \eta (\eta + 1) \left( \frac{\varepsilon \sigma^2 \eta}{\mu} \right)^{-2}]
$$

$$
= Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta [p - \frac{\mu^2 (\eta + 1)}{2\sigma^2}] \geq 0;
$$

when $\frac{\mu}{\sigma^2} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) \geq 1$, we have $A \left( \frac{u_2}{u_{2x}} \right) = 1$, then

$$
\partial_t u_2 - T \varepsilon u_2 + \beta_\varepsilon (u_2 - 1)
$$

$$
= \partial_t u_2 - \frac{\sigma^2}{2} \left[ A \left( \frac{u_2}{u_{2x}} \right) + \varepsilon^2 \right] u_{2xx} - \mu A \left( \frac{u_2}{u_{2x}} \right) u_{2x} + c u_2 + \beta_\varepsilon (u_2 - 1)
$$

$$
= Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta [p - \frac{\sigma^2}{2} (1 + \varepsilon^2) \eta (\eta + 1) \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right)^{-2}
$$

$$
+ \mu \eta \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right)^{-1} + c]
$$

$$
\geq Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta [p - \frac{\sigma^2}{2} (1 + \varepsilon^2) \eta (\eta + 1) \left( \frac{\mu}{\sigma^2 \eta} \right)^2 + c]
$$

$$
\geq Ke^{pt} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right) - \eta [p - \frac{\mu^2 (\eta + 1)}{\sigma^2 \eta} + c] \geq 0.
$$

Together with

$$
\begin{align*}
u_2(0, t) + \frac{\varepsilon \sigma^2}{\mu} u_{2x}(0, t) &= 0 \leq f_\delta(t), & u_2(N, t) \geq 1, & 0 \leq t \leq T, \\
u_2(x, 0) &\geq 1, & 0 < x < N,
\end{align*}
$$

applying Lemma 3.1, we obtain $u^{\varepsilon, \delta} \leq u_2$. \hfill \Box

Lemma 3.3. For the solution $u^{\varepsilon, \delta}(x, t)$ to problem (21), the following estimates hold,

$$
u_1^{\varepsilon, \delta}(x, t) \geq 0,$$

$$
u_2^{\varepsilon, \delta}(x, t) \leq 0,$$
Proof. We first prove (26). For any \(0 < \Delta t < T\), let \(\tilde{u}^{\varepsilon, \delta}(x, t) = u^{\varepsilon, \delta}(x, t + \Delta t)\), then both \(\tilde{u}^{\varepsilon, \delta}\) and \(u^{\varepsilon, \delta}\) satisfy the equation in (21) in the domain \(Q_{T-\Delta t}\). Moreover, due to \(f_\delta(t)\) is non-increasing, we have
\[
\frac{\partial}{\partial x} \tilde{u}^{\varepsilon, \delta}(0, t) + \frac{\varepsilon \sigma^2}{\mu} \tilde{u}^{\varepsilon, \delta}(0, t) \leq u^{\varepsilon, \delta}(0, t) + \frac{\varepsilon \sigma^2}{\mu} u^{\varepsilon, \delta}(0, t).
\]
Together with \(\tilde{u}^{\varepsilon, \delta}(N, t) = u^{\varepsilon, \delta}(N, t) = 1\), \(\tilde{u}^{\varepsilon, \delta}(x, 0) = u^{\varepsilon, \delta}(x, 0) = 1 = u^{\varepsilon, \delta}(x, 0)\), we could deduce from Lemma 3.1 that \(\tilde{u}^{\varepsilon, \delta} \geq u^{\varepsilon, \delta}\), which implies (26).

In the following, we prove (27). Differentiate the equation in (21) w.r.t. \(x\) to obtain
\[
\partial_t u^{\varepsilon, \delta}_x - \frac{\sigma^2}{2} \partial_x \left( A^2 \left( \frac{u^{\varepsilon, \delta}}{u^{\varepsilon, \delta}} \right) + \varepsilon^2 \right) \partial_x u^{\varepsilon, \delta}_x - \mu A \left( \frac{u^{\varepsilon, \delta}}{u^{\varepsilon, \delta}} \right) \partial_x u^{\varepsilon, \delta}_x - \mu A' \left( \frac{u^{\varepsilon, \delta}}{u^{\varepsilon, \delta}} \right) \partial_x u^{\varepsilon, \delta}_x = 0.
\]
Note that \(\partial_x \left( \frac{u^{\varepsilon, \delta}}{u^{\varepsilon, \delta}} \right) = 1 - \frac{u^{\varepsilon, \delta}}{(u^{\varepsilon, \delta})^2}\), so the above equation becomes
\[
\partial_t u^{\varepsilon, \delta}_x - \frac{\sigma^2}{2} \partial_x \left( A^2 \left( \frac{u^{\varepsilon, \delta}}{u^{\varepsilon, \delta}} \right) + \varepsilon^2 \right) \partial_x u^{\varepsilon, \delta}_x - \mu \left[ A \left( \frac{u^{\varepsilon, \delta}}{u^{\varepsilon, \delta}} \right) \right] - \beta_s (u^{\varepsilon, \delta} - 1) u^{\varepsilon, \delta}_x = 0.
\]
This equation is in divergence form of \(u^{\varepsilon, \delta}_x\) with bounded coefficients. Moreover, note that
\[
u^{\varepsilon, \delta}_x(0, t) = \frac{\mu}{\varepsilon \sigma^2} (f_\delta(t) - u^{\varepsilon, \delta}(0, t)),
\]
which implies (26).

From \(u^{\varepsilon, \delta} \geq 1\) and \(f_\delta \leq 1\) implies
\[
u^{\varepsilon, \delta}_x(0, t) \leq 0.
\]
Combining with
\[
u^{\varepsilon, \delta}_x(x, 0) = 0,
\]
by the maximum principle for weak solution (see [12]), we deduce that \(u^{\varepsilon, \delta}_x \leq 0\).

Corollary 1. The solution \(u^{\varepsilon, \delta}(x, t)\) to problem (21) satisfies
\[
u^{\varepsilon, \delta}_x(x, t) \geq 0, \quad (x, t) \in Q_T.
\]
Proof. We deduce from the equation in (21) and the estimates (24), (26), (27) that
\[
\frac{\sigma^2}{2} \left[ A^2 \left( \frac{u^{\varepsilon, \delta}_x}{u^{\varepsilon, \delta}} \right) + \varepsilon^2 \right] u^{\varepsilon, \delta}_x = u^{\varepsilon, \delta}_x - \mu \left( \frac{u^{\varepsilon, \delta}}{u^{\varepsilon, \delta}} \right) u^{\varepsilon, \delta}_x + \mu \varepsilon \beta_s (u^{\varepsilon, \delta} - 1) \geq 0,
\]
which implies (33).
Theorem 3.4. For any fixed $\varepsilon > 0$, problem (20) has a solution $u^\varepsilon \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T} \setminus \{(0,0)\})$, moreover,

$$1 \leq u^\varepsilon \leq Ke^{\mu t} \left( x + \frac{\varepsilon \sigma^2 \eta}{\mu} \right)^{-\eta}, \quad (34)$$

$$u^\varepsilon_t \geq 0, \quad (35)$$

$$u^\varepsilon_x \leq 0, \quad (36)$$

$$u^\varepsilon_{xx} \geq 0. \quad (37)$$

Proof. Fix $\rho > 0$, apply $C^{\alpha,2}$ interior estimate to (21) for arbitrary $\delta < \rho/2$ we have

$$|u^\varepsilon(\cdot)|_{C^{\alpha,2}(Q_T \setminus B_{\rho/2})} \leq C_\rho (|u^\varepsilon(\cdot)|_{0, Q_T} + \beta(\cdot)|_{L^p(Q_T)} + 1) \leq C_{\varepsilon, \rho},$$

where $B_r$ is the disk with center $(0,0)$ and radius $r$, $C_{\varepsilon, \rho}$ depends only on $\varepsilon, \rho$ but not on $\delta$ (see [12]).

Using (29) and the second inequality in (24), there exists a $C_{\varepsilon, \rho}$ independent of $\delta$ such that

$$u^\varepsilon_{x}(0,t) \geq -C_{\varepsilon}. \quad (38)$$

Since $u^\varepsilon_{xx} \geq 0$, we have

$$-C_{\varepsilon} \leq u^\varepsilon_{xx} \leq 0, \quad (x,t) \in \overline{Q_T} \setminus B_{\rho}.$$  

Apply $C^{\alpha,2}$ interior estimate to (28) yields

$$|u^\varepsilon(\cdot)|_{C^{\alpha,2}(Q_T \setminus B_{\rho/2})} \leq C_\rho (|u^\varepsilon(\cdot)|_{0, Q_T} + 1) \leq C_{\varepsilon, \rho}.$$  

Apply Schauder interior estimate to (21), we have

$$|u^\varepsilon(\cdot)|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T \setminus B_{\rho})} \leq C_\rho (|u^\varepsilon(\cdot)|_{0, Q_T} + \beta(\cdot)|_{L^p(Q_T)} + 1) \leq C_{\varepsilon, \rho}.$$  

We emphasize that $C_{\varepsilon, \rho}$ is independent of $\delta$, let $\delta \to 0$, there exists a subsequence of $u^\varepsilon$, we still denote it by itself, lead to a limit $u^\varepsilon \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T} \setminus \{(0,0)\})$, which is the solution to problem (20).

Finally, (34)-(37) are the results of (24),(26),(27),(33), respectively. \qed

3.4. The solution to problem (14).

Theorem 3.5. Problem (14) has a solution $v^\varepsilon \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T} \setminus \{(0,0)\})$.

Proof. Set

$$v^\varepsilon(x,t) = \int_0^x u^\varepsilon(y,t)dy,$$

where $u^\varepsilon$ is a solution to problem (20). Then

$$v^\varepsilon(0,t) = 0, \quad v^\varepsilon_x(0,t) = 0,$$

together with the boundary condition in (20) on $x = 0$, we have

$$v^\varepsilon_t - \mathcal{L}v^\varepsilon \bigg|_{x=0} = 0.$$

Integrating the equation in (20) from 0 to $x$, we get $v^\varepsilon$ satisfies the equation in (14). In addition, it is easy to verify that $v^\varepsilon$ satisfies the initial and boundary conditions in (14). \qed
3.5. Solvability of problem (11). In order to obtain uniform interior estimates to $u^\varepsilon$ independent of $\varepsilon$, we need uniform parabolic conditions associated to $\mathcal{L}_\varepsilon$ and $\mathcal{F}_\varepsilon$ inner the domain, which claims that

$$\lambda \leq A(\frac{u^\varepsilon}{u^\varepsilon_x}) \leq \Lambda \text{ in } Q \subset\subset Q_T$$

for uniform positive constants $\lambda$ and $\Lambda$. Thanks to $A(z) \leq 1$, we only need

**Theorem 3.6.** For any $[a, N] \times [0, T] \subset Q_T$ $(0 < a < N)$, there exists $\kappa > 0$ which is independent of $\varepsilon$ (but depends on $a$), such that

$$\frac{|u^\varepsilon_x|}{u^\varepsilon} \geq \kappa, \quad (x, t) \in [a, N] \times [0, T].$$

Since $u^\varepsilon \geq 1$, it is only needed to prove

**Lemma 3.7.** For any $[a, N] \times [0, T] \subset Q_T$ there exists $C > 0$ independent of $\varepsilon$ such that

$$u^\varepsilon_x(x, t) \geq -C, \quad x \geq a.$$  \quad (39)

**Proof.** Fix $t \in [0, T]$, mean value theorem claims that there exists a $\xi \in (\frac{a}{2}, a)$ such that

$$u^\varepsilon_x(\xi, t) = \frac{u^\varepsilon(a, t) - u^\varepsilon(\frac{a}{2}, t)}{a/2} \geq -\frac{2u^\varepsilon(\frac{a}{2}, t)}{a} \geq -\frac{2Ke^{pt}}{a}(\frac{a}{2} + \frac{\varepsilon \sigma^2 \eta}{\mu})^{-\eta} \geq -C,$$

where, the second inequality is due to the estimate (34). Owing to $u^\varepsilon_{xx} \geq 0$, then

$$u^\varepsilon_x(x, t) \geq u^\varepsilon_x(\xi, t) \geq -C, \quad x \geq a. \quad \square$$

**Theorem 3.8.** Problem (11) has a unique solution $v \in W^{2,1}_{p, \text{loc}}(Q_T) \cap C(Q_T)$, and $v_x \in W^{2,1}_{p, \text{loc}}(Q_T) \cap C(Q_T \cup \{t = 0\})$, moreover, we have the following estimates

$$x \leq v \leq \frac{Ke^{pt}}{\gamma} x^\gamma, \quad (40)$$

$$v_t \geq 0, \quad (41)$$

$$1 \leq v_x \leq Ke^{pt} x^{-\eta}, \quad (42)$$

$$v_{xt} \geq 0, \quad (43)$$

$$v_{xx} \leq 0, \quad (44)$$

$$v_{xxx} \geq 0. \quad (45)$$

where, $\gamma = 1 - \eta$, $K$, $p$, $\eta$ are defined as (25).

**Proof.** Denote $Q^a := [a, N] \times [0, T] \subset Q_T$ $(0 < a < N)$, by Theorem 3.6, problem (20) satisfies the uniform parabolic condition in $Q^a$, apply $C^{\alpha, 1/2}$ interior estimate to (20), we have

$$|u^\varepsilon|_{\alpha, Q^a/2} \leq C(|u^\varepsilon|_{0, Q^a/4} + |\beta^\varepsilon|_{L_p(Q^a/4)} + 1) \leq C.$$

Rewrite (28) as
\[ \partial_t u_{x, \delta} - \frac{\sigma^2}{2} \partial_x \left( A^2 \left( \frac{u_{x, \delta}}{u_{x, \delta}} \right) + \varepsilon^2 \right) = -\mu \left( A \left( \frac{u_{x, \delta}}{u_{x, \delta}} \right) - A' \left( \frac{u_{x, \delta}}{u_{x, \delta}} \right) \right) \frac{u_{x, \delta}}{u_{x, \delta}} \partial_x u_{x, \delta} \]

apply \( C_{0.2} \) estimate to obtain
\[ |u_{x, \delta}|_{1, Q_{\alpha/2}} \leq C \left( |u_{x, \delta}|_{0, Q_{\alpha/4}} + |\beta(x)|_{L_0(Q_{\alpha/4})} + 1 \right) \leq C. \]

Apply \( W^{2,1}_p \) estimate to (20), we have
\[ |u_{x, \delta}|_{W^{2,1}_p(Q_{\alpha})} \leq C \left( |u_{x, \delta}|_{L_p(Q_{\alpha/2})} + |\beta(x)|_{L_p(Q_{\alpha/2})} + 1 \right) \leq C. \]

We emphasize that the above \( C \) are independent of \( \varepsilon \). Let \( \varepsilon \to 0 \), \( u_{x, \delta} \) lead to a limit \( u \) in \( W^{2,1}_p(Q_T) \) (probably a subsequence) in the fixed domain \( Q_{\alpha} \) in the sense \( u_{x, \delta} \to u \) weakly in \( W^{2,1}_p(Q_{\alpha}) \). Moreover the Sobolev embedding theorem implies that \( u_{x, \delta} \to u \) in \( C^{1+\alpha, \frac{\alpha}{2}}(Q_{\alpha}) \).

Apply \( W^{2,1}_p \) interior estimate to (14) and we deduce that
\[ |u_{x, \delta}|_{W^{2,1}_p(Q_{\alpha})} \leq C \left( \left| \frac{u_{x, \delta}}{u_{x, \delta}} \right|_{L_p(Q_{\alpha/2})} + \int_0^\alpha \left| \frac{u_{x, \delta}}{u_{x, \delta}} \right|_{L_p(Q_{\alpha/2})} + 1 \right) \leq C. \]

Letting \( \varepsilon \to 0 \), we also have a subsequence of \( u_{x, \delta} \) lead to a limit \( v \) in \( W^{2,1}_p(Q_T) \) in the sense that \( u_{x, \delta} \to v \) weakly in \( W^{2,1}_p(Q_{\alpha}) \) and \( v_{x, \delta} \to v \) in \( C^{1+\alpha, \frac{\alpha}{2}}(Q_{\alpha}) \).

The estimate (34) implies
\[ 1 \leq u \leq Ke^{pt}x^{-\eta} \in L^1([0, N]), \]

then by the dominated convergence theorem, we have
\[ v(x, t) = \lim_{\varepsilon \to 0} v_{x, \delta}(x, t) = \lim_{\varepsilon \to 0} \int_0^x u_{x, \delta}(y, t)dy = \int_0^x \lim_{\varepsilon \to 0} u_{x, \delta}(y, t)dy = \int_0^x u(y, t)dy. \]

Therefore,
\[ v(0, t) = 0, \quad v_x(N, t) = u(N, t) = 1, \quad v(x, 0) = \int_0^x u(y, 0)dy = x. \]

In the following, we show that \( v \) satisfies the variational inequality in (11). Firstly, by (34) we have
\[ v_x = u \geq 1. \]

By the equation in (14) we have
\[ v_{x} - \mathcal{L}v \geq 0, \quad (46) \]

let \( \varepsilon \to 0 \) and we get
\[ v_{\varepsilon} - \mathcal{L}v \geq 0. \]

If there exists a point \((x, t)\) such that \( v_{x}(x, t) > 1 \), then there exists \( \varepsilon_0 \) such that for all \( \varepsilon < \varepsilon_0 \),
\[ v_{x}(y, t) \geq v_{x}(x, t) > 1 + \varepsilon_0, \quad y \leq x, \]
then the right hand side of (46) is 0 at \((x, t)\), send \( \varepsilon \) approach to 0, we have
\[ (v_{\varepsilon} - \mathcal{L}v)(x, t) = 0. \]

So \( v \) satisfies the variational inequality in (11).
Moreover, (42)-(45) follow from (34)-(37). Integrating both sides of inequality (42), (43), we obtain (40), (41).

Now we will prove the uniqueness. Suppose \( v_1, v_2 \) are two solutions of (11). Set \( \mathcal{N} = \{ \partial_x v_1 > \partial_x v_2 \} \), then

\[
\begin{align*}
\partial t v_1 - \mathcal{L} v_1 &= 0, \quad \partial t v_2 - \mathcal{L} v_2 \geq 0, \quad (x, t) \in \mathcal{N}, \\
v_1 &= v_2 = 0, \quad (x, t) \in \partial \mathcal{N} \cap \{ x = 0 \}, \\
\partial_x v_1 &= \partial_x v_2, \quad \in \partial \mathcal{N} \setminus \{ \{ x = 0 \} \cup \{ t = 0 \} \cup \{ t = T \} \}, \\
v_1 &= v_2 = x, \quad \in \partial \mathcal{N} \cap \{ t = 0 \}.
\end{align*}
\]

Comparison principle for fully nonlinear equation (see [4], red p.52, Theorem 16) claims \( v_2 \geq v_1 \) in \( \mathcal{N} \), which implies

\[\{ \partial_x v_1 > \partial_x v_2 \} \subset \{ v_2 \geq v_1 \},\]

i.e.

\[\mathcal{C} := \{ v_2 < v_1 \} \subset \{ \partial_x v_1 \leq \partial_x v_2 \}.\]

If \( \mathcal{C} \) is nonempty, with the fact that \( v_2 = v_1 \) on \( \partial \mathcal{C} \), we get \( v_1 \leq v_2 \) in \( \mathcal{C} \) which is a contradiction.

4. Properties of the dividend boundary. In this section, we get existence of the dividend free boundary to the variational problem (10)(Theorem 4.1). We also prove that it is a strictly increasing \( C^\infty \) smooth curve and obtain its upper bound and the location of starting point(Theorem 4.2, Theorem 4.4). In addition, based on Theorem 3.8, we further prove the solution to (10) belongs to \( C^{2,1}(Q_T) \)(Theorem 4.3), which is important to the verification theorem.

Now, in order to characterize the optimal dividend boundary, we define

\[\mathcal{D} = \{ v_x = 1 \} \text{ (dividend area)},\]

\[\mathcal{N} / \mathcal{D} = \{ v_x > 1 \} \text{ (non - dividend area)}.\]

**Theorem 4.1.** There exists a function \( h(t) : [0, T) \rightarrow (0, +\infty) \) such that

\[\mathcal{D} = \{ (x, t) : x \geq h(t), t \geq 0 \}.\]

Moreover, \( h(t) \) is increasing.

**Proof.** Since \( v_x \) is decreasing w.r.t. \( x \), we can define

\[h(t) := \inf \left\{ x \geq 0 : v_x(x, t) = 1 \right\}, \quad t \geq 0.\]

(48)

It is obviously that (47) holds true. The monotonicity easily follows from the fact \( v_{xt} \geq 0 \). \( \square \)

**Theorem 4.2.** The free boundary \( h(t) \) is continuous with

\[h(0) := \lim_{t \to 0} h(t) = 0.\]

Moreover, \( h(t) \) satisfies

\[h(t) > 0, \quad \forall t > 0.\]

(49)

**Proof.** We prove by contradiction. Suppose not, there exists \( t_0 \) such that

\[h(t_0-) < h(t_0+).\]

By the continuity of \( v_x \), we know \( v_x(h(t_0-), t_0) = v_x(h(t_0+), t_0) = 1 \). With the fact \( v_x \) is decreasing w.r.t. \( x \), we deduce for any \( x \in (h(t_0-), h(t_0+)) \), \( v_x(x, t_0) = 1 \), it leads to \( v_{xx}(x, t_0) = 0, x \in (h(t_0-), h(t_0+)) \). By the equation in (11), we have
$v_t(x, t_0) = \mu - cv(x, t_0), \, x \in (h(t_0^-), h(t_0^+))$, thus $v_{tx}(x, t_0) = -cv_x(x, t_0) = -c < 0, \, x \in (h(t_0^-), h(t_0^+))$, it contradicts with (43). Similarly, we can prove $h(0) = 0$.

Now we prove (49). Suppose not. Let $t_0 = \inf\{t \geq 0 : h(t) > 0\} > 0$, then we have

$$h(t) \equiv 0, \quad v_x \equiv x, \quad 0 < t < t_0.$$

Denote $\tilde{v}(x, t) = v(x, t - t_0)$, then both $\tilde{v}$ and $v$ satisfy (11), since the solution is unique, we get $\tilde{v} = v$, which implies $v \equiv x$, but this is impossible. \hfill $\square$

In order to characterize the properties of $h(t)$, we focus on the neighborhood of $(h(t), t)$. Note that $v_x \in W^{2,1}_{p, \text{loc}}(Q_T)$, Sobolev embedding theorem implies that $v_x \in C^{1+\alpha, 1+\alpha}\left((t_0, T)\right)$, then we have

$$v_x, \, v_{xx} \in C^{\alpha, \frac{\alpha}{2}}(Q_T).$$

By the definition of $D$ and (47), we know

$$v_x = 1, \quad v_{xx} = 0, \quad x > h(t).$$

So at the point $(h(t_0), t_0)$, there exists a neighborhood $B$, such that $v_{xx} + \frac{\mu}{\sigma^2}v_x > 0$, namely

$$A\left(\frac{v_x}{v_{xx}}\right) = 1, \quad (x, t) \in B. \quad (50)$$

Thus

$$\min \left\{ v_t - \frac{1}{2}\sigma^2 v_{xx} - \mu v_x + cv, \, v_x - 1 \right\} = 0, \quad (x, t) \in B. \quad (51)$$

Similar to the proof of Theorem 3.8, let $\varepsilon \rightarrow 0$ in (20), we get $u = v_x$ satisfying

$$\min \left\{ u_t - \frac{1}{2}\sigma^2 u_{xx} - \mu u_x + cu, \, u - 1 \right\} = 0, \quad (x, t) \in B. \quad (52)$$

**Theorem 4.3.** The solution to problem (11) $v_{xt} \in C(Q_T)$, moreover, $v \in C^{2,1}(Q_T)$.

**Proof.** Since $u = v_x$ satisfies (52) in the neighborhood of $(h(t), t)$, using the method in [5] we get $v_{xt} \in C(Q_T)$, together with (50), it is obvious that $v \in C^{2,1}(Q_T)$. \hfill $\square$

**Theorem 4.4.** The free boundary $h(t)$ is strictly increasing with

$$h(t) \leq x_d,$$

where $x_d$ is defined in (73), moreover, $h(t) \in C^\infty(0, T]$.

**Proof.** We first prove the strictly monotonicity. If not, there exists $t_1, t_2 \in (0, T)$ such that $h(t) = x_0, \, t \in [t_1, t_2]$. Denote $\Gamma = \{x_0\} \times [t_1, t_2]$, we have $u = 1$ and $u_x = 0$ on $\Gamma$, thus $u_t = 0, \, u_{xt} = 0$ on $\Gamma$. Suppose $B$ is the neighborhood where (52) holds, applying Hopf lemma(see [4]) for $u_t$ in $\{x \leq h(t), \, t \in (0, T)\} \cap B$, we have either $u_t \equiv 0$ in $\{x \leq h(t), \, t \in (0, T)\} \cap B$ or $u_{xt} < 0$ on $\Gamma$, but both come to contradictions.

Now we will prove (53). Consider the following steady variational inequality problem

$$\begin{cases} \min \left\{ -\sup_{0 \leq a \leq 1} \left( \frac{1}{2}\sigma^2 a^2 V'' + \mu a V' \right) + cV, \, V' - 1 \right\} = 0, \quad x > 0, \\ V(0) = 0. \end{cases} \quad (54)$$

It is the HJB equation to the infinite time model and the expression of solution $V^\infty(x)$ and free boundary points $x_d$ and $x_r$ were obtained by [15]. We will present them in Appendix B. In particular, we give

$$x_d = \inf\{x \geq 0 : V^\infty(x) = 1\}.$$
In the following, we show that
\[ v_x \leq V^\infty_x, \quad (x, t) \in Q_T. \] (55)

From the equation in (20), let \( \varepsilon \to 0 \), \( u = v_x \) satisfies the following variational inequality
\[ \min\{u_t - \mathcal{T}u, u - 1\} = 0, \quad (x, t) \in Q_T. \] (56)

In order to prove (55), we have to give a comparison principle for the above variational inequality: Suppose \( u_1, u_2 \in C^2(Q_T) \cap C(\overline{Q_T}) \) satisfy (56) with the following initial and boundary conditions
\[ u_1(0, t) \leq u_2(0, t), \quad u_1(N, t) \leq u_2(N, t), \quad 0 < t \leq T. \]
\[ u_1(x, 0) \leq u_2(x, 0), \quad 0 < x < N, \]
then \( u_1 \leq u_2 \).

In fact, denote \( N = \{u_1 > u_2\} \), if it is not empty, then we have
\[
\begin{cases}
\partial_t u_1 - \mathcal{T} u_1 = 0, & \partial_t u_2 - \mathcal{T} u_2 \geq 0, \quad (x, t) \in N, \\
u_1 \leq u_2 = 0, & (x, t) \in \partial N.
\end{cases}
\]
The comparison principle claims \( u_2 \geq u_1 \) in \( N \), thus \( N = \emptyset \).

Denote \( u^a(x, t) := v_x(x + a, t), \quad U^b(x, t) := V^\infty_x(x + b) \).

In order to prove (55), we have to show that for any \( a > 0 \), there exists \( u^a \leq U^b \) for small enough \( b > 0 \). It is easy to verify that both \( u^a \) and \( U^b \) satisfy the variational inequality (56) with the following boundary and initial conditions on \( Q^a := [0, N - a] \times (0, T] \),
\[ u^a(N - a, t) = v_x(N, t) = 1, \quad U^b(N - a, t) = V^\infty_x(N - a + b) \geq 1, \quad 0 < t \leq T, \]
\[ u^a(x, 0) = 1, \quad U^b(x, 0) = V^\infty_x(x + b) \geq 1, \quad 0 < x < N - a. \]
On the boundary \( x = 0 \), since \( V^\infty_x(0) := \lim_{x \to 0^+} V^\infty_x(x) = \infty, \quad \forall t \in (0, T] \) (see (72)), choose \( b \) small enough such that
\[ U^b(0, t) = V^\infty_x(b) \geq u^a(0, t). \]

Hence, by the comparison principle, we have
\[ u^a(x, t) \leq U^b(x, t), \]
that is
\[ v_x(x + a, t) \leq V^\infty_x(x + b) \leq V^\infty_x(x). \]

Since \( a \) is arbitrary, (55) holds, which implies \( h(t) \leq x_d \).

\[ \square \]

5. Properties of the reinsurance boundary. In this section, we obtain the existence of reinsurance free boundary, and prove that it is a \( C^1 \) smooth curve. (Theorem 5.3)

Firstly, we define
\[ \mathcal{R} = \left\{ -\frac{\mu}{\sigma^2} \frac{v_x}{v_{xx}} < 1 \right\} \text{ (reinsurance area)}, \]
\[ \mathcal{NR} = \left\{ -\frac{\mu}{\sigma^2} \frac{v_x}{v_{xx}} \geq 1 \right\} \bigcup \{v_{xx} = 0\} \text{ (non - reinsurance area)}. \]
Notice that $\mathcal{D} \subset \{v_{xx} = 0\} \subset \mathcal{NR}$, then $\mathcal{R} \subset \mathcal{ND}$, which means that the optimal reinsurance free boundary locates in $\mathcal{ND}$.

We hope to find a function $W(x, t)$ to characterize the optimal reinsurance free boundary, specifically, we hope to find a function satisfying

$$W(x, t) < 0, \quad (x, t) \in \mathcal{R}, \quad (57)$$

and

$$W(x, t) \geq 0, \quad (x, t) \in \mathcal{NR}, \quad (58)$$

Now we define

$$W(x, t) := \frac{\sigma^2}{2} A^2 \left( \frac{v_x}{v_{xx}} \right) v_{xx} + \mu \left( A \left( \frac{v_x}{v_{xx}} \right) - \frac{1}{2} \right) v_x, \quad (59)$$

when $-\frac{\mu}{\sigma^2} \frac{v_x}{v_{xx}} < 1$, i.e., $A < 1$,

$$W(x, t) = \frac{\sigma^2}{2} \left( -\frac{\mu}{\sigma^2} \frac{v_x}{v_{xx}} \right)^2 v_{xx} + \mu \left( -\frac{\mu}{\sigma^2} \frac{v_x}{v_{xx}} - \frac{1}{2} \right) v_x = \frac{\mu}{2} \left( -\frac{\mu}{\sigma^2} \frac{v_x}{v_{xx}} - 1 \right) v_x < 0;$$

when $-\frac{\mu}{\sigma^2} \frac{v_x}{v_{xx}} \geq 1$ or $v_{xx} = 0$, i.e., $A = 1$,

$$W(x, t) = \frac{\sigma^2}{2} v_{xx} + \frac{\mu}{2} v_x \geq 0.$$

Thus $W(x, t)$ satisfies (57) and (58). In addition, in view of the equation in (11), we have

$$W(x, t) = v_t - \frac{\mu}{2} v_x + cv, \quad (x, t) \in \mathcal{ND}. \quad (60)$$

Thanks to (40)–(42), we get

$$W_x = v_{xt} - \frac{\mu}{2} v_{xx} + cv_x \geq c > 0, \quad (61)$$

therefore, we can define the reinsurance free boundary as

$$g(t) = \inf\{x \geq 0 : W(x, t) \geq 0\}. \quad (62)$$

To determine the position of $g(t)$, we need the following lemma.

**Lemma 5.1.** The solution $v$ to problem (11) satisfies

$$v_x(0, t) := \lim_{x \to 0^+} v_x(x, t) = +\infty, \quad t > 0.$$
Proof. It seems difficult to prove from (11) by the method of PDE, but since we’ve got the solution to problem (11) belongs to $C^{2,1}(Q_T) \cap C(Q_T)$ in Theorem 4.3, we could apply verification theorem (which is presented in Appendix A) to prove that $V(x, t) = v(x, T - t)$ is the value function defined by (7). So we turn to prove that

$$V_\varepsilon(0, t) := \lim_{x \to 0} V_\varepsilon(x, t) = +\infty, \quad t < T$$

by the techniques of stochastic analysis, this work is given in Appendix C.

Lemma 5.2. The solution $v$ to the problem (11) satisfies

$$v_t \leq e^{-ct} \mu.$$

Proof. Differentiate the equation in (14) w.r.t. $t$, we have

$$\partial_t v_t^\varepsilon - T_\varepsilon v_t^\varepsilon = - \int_0^x \beta_\varepsilon' (v_x^\varepsilon - 1) v_t^\varepsilon dy.$$

Denote the right hand side of the above equation as $F_\varepsilon(x, t)$, thanks to (43), we know $F_\varepsilon(x, t) \leq 0$. In view of the boundary conditions in (14) we have $v_t^\varepsilon(0, t) = 0$, $v_t^\varepsilon(N, t) = 0$. Substitute $v_t^\varepsilon(x, 0) = x$ into the equation in (14) we have $v_t^\varepsilon(x, 0) = \mu$.

Denote $A\left(\frac{v_t^\varepsilon}{v_x^\varepsilon}\right) = A_\varepsilon(x, t)$, $v_t^\varepsilon$ satisfies

$$\begin{cases}
\partial_t v_t^\varepsilon - \frac{\varepsilon^2}{2} [A_\varepsilon^2 + \varepsilon^2] \partial_{xx} v_t^\varepsilon - \mu A_\varepsilon \partial_x v_t^\varepsilon + cv_t^\varepsilon = F_\varepsilon(x, t), & (x, t) \in Q_T, \\
v_t^\varepsilon(0, t) = 0, & 0 < t \leq T, \\
v_t^\varepsilon(N, t) = 0, & 0 < t \leq T, \\
v_t^\varepsilon(x, 0) = \mu, & 0 < x < N.
\end{cases}$$

(62)

Construct the following approximation problem,

$$\begin{cases}
\partial_t w^{\varepsilon, \delta} - \frac{\varepsilon^2}{2} [A_\varepsilon^2 + \varepsilon^2] \partial_{xx} w^{\varepsilon, \delta} - \mu A_\varepsilon \partial_x w^{\varepsilon, \delta} + cw^{\varepsilon, \delta} = F_\varepsilon(x, t), & (x, t) \in Q_T, \\
w^{\varepsilon, \delta}(0, t) = \mu f_\delta(t), & 0 < t \leq T, \\
w^{\varepsilon, \delta}(x, 0) = \mu, & 0 < x < N,
\end{cases}$$

(63)

where $f_\delta(t) \in C^2([0, T])$ is given in (22) and satisfies

$$f_\delta(0) = 1, \quad f'_\delta(0) = -c, \quad f_\delta(t) = 0, \quad t \geq \delta.$$

Thanks to the Hölder continuity of $F_\varepsilon(x, t)$ and $A_\varepsilon(x, t)$, (63) has a unique solution $w^{\varepsilon, \delta} \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T)$, in particular, the comparison principle implies $w^{\varepsilon, \delta} \leq e^{-ct}\mu$.

Use the technique in the proof of Theorem 3.4, we could show that when $\delta \to 0$, $w^{\varepsilon, \delta}$ converge to a function $w^\varepsilon \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T \setminus \{(0, 0)\})$, which is the solution to (62). By the uniqueness of the solution to (62), we have $w^\varepsilon = v_t^\varepsilon$, thus , $v_t^\varepsilon \leq e^{-ct}\mu$.

Letting $\varepsilon \to 0$, we get the conclusion.

Now we are able to characterize the free boundary $g(t)$ as follows.

Theorem 5.3. The free boundary $g(t)$ satisfies

$$0 < g(t) < h(t), \quad t \in (0, T].$$

Moreover, $g(t) \in C[0, T] \cap C^1(0, T]$.

Proof. For any $t > 0$, Lemma 5.1 and Lemma 5.2 imply that $\lim_{x \to 0} W(x, t) = v_t(0, t) - \frac{c}{2} v_x(0, t) + cv(0, t) = -\infty$. Owing to $v_x(h(t), t) = 1$, $v_{xx}(h(t), t) = 0$, we can deduce from (59) that $W(h(t), t) = \frac{c}{2} > 0$. By the intermediate value theorem, (64) holds.
Now we will show the continuity of $g(t)$ in $[0, T]$. If there is a $t_0 \in (0, T]$ such that
\[ x_1 = \liminf_{t \to t_0} g(t) < x_2 = \limsup_{t \to t_0} g(t). \]
Note that $W(x, t)$ is continuous, then $W(x_1, t_0) = W(x_2, t_0) = 0$, by the monotonicity of $W(x, t)$, we have
\[ W_x(x, t_0) = 0, \quad x \in (x_1, x_2), \]
which contradicts with (61), thus $g(t)$ is continuous.

Finally, we prove $g(t) \in C^1(0, T]$. By the definition of $\mathcal{N}D$ and the variational inequality in (11), we have
\[ v_t - \mathcal{L}v = 0, \quad (x, t) \in \mathcal{N}D. \]
By Schauder interior estimate, $v \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathcal{N}D)$. Differentiate both sides of the above equation w.r.t. $x$ and $t$, respectively, we get
\[ \partial_t v_x - \mathcal{T} v_x = 0, \quad \partial_t v_t - \mathcal{T} v_t = 0, \quad (x, t) \in \mathcal{N}D. \]
Similarly, we can deduce from Schauder interior estimate that $v_x, v_t \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathcal{N}D)$. Hence $W \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathcal{N}D)$. Therefore, we can differentiate $W(g(t), t) = 0$ w.r.t. $t$ to obtain
\[ W_x(g(t), t)g'(t) + W_t(g(t), t) = 0, \]
thus
\[ g'(t) = -\frac{W_t(g(t), t)}{W_x(g(t), t)}, \]
which implies $g(t) \in C^1(0, T]$. 

![Figure 3. Reinsurance free boundary](image)

At the end of this section, we show that problem (11) degenerates on the left boundary $x = 0$. To see that, we give

**Corollary 2.** For all $t \in (0, T]$, 
\begin{align*}
\frac{v_x}{v_{xx}}(0, t) &:= \lim_{x \to 0} \frac{v_x}{v_{xx}}(x, t) = 0, \quad (65) \\
v_{xx}(0, t) &:= \lim_{x \to 0} v_{xx}(x, t) = -\infty. \quad (66)
\end{align*}
Proof. From $g(t) > 0$ for all $t \in (0, T]$, we know $v$ satisfies (3) near $x = 0$. Applying Lemma 5.1 and Lemma 5.2 we have
\[
\frac{v_x}{v_{xx}} = -\frac{2\sigma^2}{\mu^2 v_x}(v_t + cv) \to 0, \quad x \to 0.
\]
Reuse Lemma 5.1, we get (66).

6. The optimal strategy. In this section, we will give an optimal strategy to the original financial model. Set
\[
h_T(s) = h(T-s), \quad t \leq s < T.
\]
Choose the following dividend policy $L_s^*$ as
\[
L_s^* - L_s^+ = R_s^* - h_T(s), \quad \text{if} \quad R_s^* > h_T(s),
\]
\[
dL_s^* = 0, \quad \text{if} \quad R_s^* < h_T(s).
\]
That is, when $R_s^* > h_T(s)$, the insurance company has to pay dividend $R_s^* - h_T(s)$ at time $s$, when $R_s^* < h_T(s)$, the optimal strategy is non-dividend. Meanwhile, take
\[
a^*(x, t) = \min \left\{ -\frac{\mu}{\sigma^2} \frac{v_x}{v_{xx}}(x, t), 1 \right\}.
\]
Choose the following risk control strategy,
\[
a^*_s = a^*(R_s^*, T-s) = \min \left\{ -\frac{\mu}{\sigma^2} \frac{v_x}{v_{xx}}(R_s^*, T-s), 1 \right\}. \tag{69}
\]
We will prove that under strategy $(a^*, L^*)$, the company can get the optimal expected discounted cumulative dividends up to terminal date. The proof is presented in Appendix A.

Appendix A. The following theorem shows that $(a^*, L^*)$ defined by (67)-(69) is the optimal strategy, and the solution of problem (9) $V(x, t)$ is the value function defined by (7).

Theorem A.1. Suppose $L_s^*$ satisfies (67) and (68), $a^*_s$ satisfies (69), and $V \in C^{2,1}(Q_T) \cap C(Q_T)$ is the solution to the problem (9), then for any $(a_s, L_s) \in \Pi_s$,
\[
V(x, t) \geq J_{x,t}(a, L). \tag{70}
\]
Moreover,
\[
V(x, t) = J_{x,t}(a^*, L^*). \tag{71}
\]
Proof. We first prove (70). For any $(a_s, L_s) \in \Pi_s$, assume $R_s$ is the solution to (5) with the control variables $(a_s, L_s)$, $\tau$ is the bankruptcy time of $R_s$ defined in (6), by Itô formula,
\[
V(x, t) = \mathbb{E}_{t,x}(e^{-c(T \wedge \tau - t)}V(R_{T \wedge \tau}, T \wedge \tau))
\]
\[
+ \mathbb{E}_{t,x} \int_t^{T \wedge \tau} e^{-c(s-t)}(-V_t - \frac{1}{2}\sigma^2 a^2 V_{xx} - \mu a V_x + cV)(R_s, s)ds
\]
\[
+ \mathbb{E}_{t,x} \int_t^{T \wedge \tau} e^{-c(s-t)}V_x(R_s, s)dL_s^x
\]
\[
- \sum_{t \leq s \leq T \wedge \tau} e^{-c(s-t)}(V(R_s, s) - V(R_{s-}, s)),
\]
where, \( L^c_s \) is continuous part of \( L_s \). The first two terms are non-negative. Meanwhile, we can deduce from the fact \( V_x \geq 1 \) that
\[
\mathbb{E}_{t,x} \int_t^{T \wedge \tau} e^{-c(s-t)} V_x(R_s, s) dL^c_s(x) \geq \mathbb{E}_{t,x} \int_t^{T \wedge \tau} e^{-c(s-t)} dL^c_s(x),
\]
\[
V(R_s, s) - V(R_s-, s) \leq R_s - R_s- = L_s - L_{s-}.
\]

Thus
\[
V(x, t) \geq \mathbb{E}_{t,x} \int_t^{T \wedge \tau} e^{-c(s-t)} dL^c_s(x) + \sum_{t \leq s \leq T \wedge \tau} e^{-c(s-t)} (L_s - L_{s-})
\]
\[
= \mathbb{E}_{t,x} \int_t^{T \wedge \tau} e^{-c(s-t)} dL^c_s(x),
\]
which is (70).

In the following, we will prove (71). Assume \( R^*_s \) is the solution to problem (5) with control variables \((a^*_s, L^*_s)\) defined in (67)–(69), \( \tau^* \) is the corresponding bankruptcy time, Itô formula gives
\[
V(x, t) = \mathbb{E}_{t,x} (e^{-c(T \wedge \tau^* - t)} V(R^*_{T \wedge \tau^*}, T \wedge \tau^*))
\]
\[
+ \mathbb{E}_{t,x} \int_t^{T \wedge \tau^*} e^{-c(s-t)} \left[ - V_t - \frac{1}{2} \sigma^2 a^2 V_{xx} - \mu a^* V_x + cV(R^*_s, s) \right] ds
\]
\[
+ \mathbb{E}_{t,x} \int_t^{T \wedge \tau^*} e^{-c(s-t)} V_x(R^*_s, s) dL^c_s(x)
\]
\[
- \mathbb{E}_{t,x} \sum_{t \leq s \leq T \wedge \tau^*} e^{-c(s-t)} (V(R^*_s, s) - V(R^*_{s-}, s)).
\]

Because of \( h_T(T) = 0 \), then \( R^*_T = 0 \), which implies the first term on the right hand side is 0. By the definition of \( L^*_s \) in (67), \( R^*_s \) will not greater than \( h_T(s) \) for \( s \geq t \), so the second term is zero. And by the definition of \( L^*_s \) in (68), we know \( dL^*_s = 1_{[R^*_s = h_T(s)]} dL^*_s, \) \( s > t \), together with the fact \( V_x = 1 \) when \( R^*_s = h_T(s) \), then
\[
V(x, t) = \mathbb{E} \int_t^{T \wedge \tau^*} e^{-c(s-t)} dL^*_s(x) - \mathbb{E} \sum_{t \leq s \leq T \wedge \tau^*} e^{-c(s-t)} (V(R^*_s, s) - V(R^*_{s-}, s)).
\]

When \( x \leq h_T(t) \), \( L^*_s \) and \( R^*_s \) are continuous, so
\[
V(x, t) = \mathbb{E} \int_t^{T \wedge \tau^*} e^{-c(s-t)} dL^*_s(x);
\]

When \( x > h_T(t) \), then \( L^*_s \) and \( R^*_s \) are only discontinuous at \( s = t \) and
\[
R^*_t - R^*_{t-} = -(L^*_t - L^*_{t-}) = h_T(t) - x.
\]

Note that \( V_x(y, t) = 1, \) \( y \in (h_T(t), x) \), then
\[
V(x, t) = \mathbb{E} \int_t^{T \wedge \tau^*} e^{-c(s-t)} dL^*_s(x) - \mathbb{E}(V(R^*_t, t) - V(R^*_{t-}, t))
\]
\[
= \mathbb{E} \int_t^{T \wedge \tau^*} e^{-c(s-t)} dL^*_s(x) + \mathbb{E}(L^*_t - L^*_{t-})
\]
\[
= \mathbb{E} \int_t^{T \wedge \tau^*} e^{-c(s-t)} dL^*_s(x).
\]
Appendix B. In this section, we introduce the solution of (54), which is the value function of infinite-time model. By the literature of [15], the solution of (54) belongs to $C^2(0, \infty)$ and it is an increasing concave function which can be expressed as:

$$V^\infty(x) = \begin{cases} 
C_1 x^\gamma, & x < x_r, \\
C_2 e^{\theta_2 x} + C_3 e^{\theta_1 x}, & x_r < x < x_d, \\
C_4 + x, & x > x_d.
\end{cases} \quad (72)$$

where $\theta_1, \theta_2 (\theta_1 < 0 < \theta_2)$ are two roots of

$$-\frac{1}{2} \sigma^2 \theta^2 - \mu \theta + c = 0,$$

and

$$\gamma = \frac{2c\sigma^2}{\mu^2 + 2c\sigma^2},$$

$$x_r = \frac{\sigma^2}{\mu} (1 - \gamma) = \frac{\sigma^2 \mu}{\mu^2 + 2c\sigma^2}, \quad x_d = x_r + \frac{1}{\theta_1 - \theta_2} \ln \left( \frac{-\theta_2}{\theta_1} \right), \quad (73)$$

$$C_1 = -2\mu \theta_1 \theta_2 \sigma^2 (e^{\theta_2 (x_d - x_r)} + e^{\theta_1 (x_d - x_r)}),$$

$$C_2 = \frac{e^{-\theta_2 x_r}}{\theta_2 (e^{\theta_2 (x_d - x_r)} + e^{\theta_1 (x_d - x_r)})},$$

$$C_3 = \frac{e^{-\theta_1 x_r}}{\theta_2 (e^{\theta_2 (x_d - x_r)} + e^{\theta_1 (x_d - x_r)})}, \quad C_4 = \frac{\mu}{c} - x_d.$$

We emphasize that $x_r, x_d$ are the corresponding reinsurance boundary and dividend boundary to (54), respectively. The optimal strategy $a^*$ can be expressed as a function of $x$

$$a^*(x) = \frac{x}{x_r} \wedge 1.$$  

Appendix C. Now we prove the function $V$ defined by (7) satisfies

$$V_x(0^+, t) = +\infty, \quad t < T. \quad (74)$$

Define

$$a^{k,m}(x) := \min \{ kx, m \}, \quad k > 0, \quad m > 0.$$  

Suppose $R_{s,T}^{k,m}$ is the solution to the following stochastic differential equation,

$$\begin{cases} 
dR_{s,T}^{k,m} = \mu a^{k,m}(R_{s,T}^{k,m}) dt + \sigma a^{k,m}(R_{s,T}^{k,m}) dW_s - d\tilde{L}_s, & t \leq s \leq T, \\
R_{t-}^{k,m} = x, & x > 0
\end{cases} \quad (75)$$

with

$$\tilde{L}_s = \begin{cases} 
0, & t \leq s < T, \\
R_{T}^{k,m}, & s = T.
\end{cases}$$  

Then when $R_{s,T}^{k,m} < m$, $s < T$, $R_{s,T}^{k,m}$ is a geometric Brownian motion, so bankruptcy will not happen, thus

$$\tau^{k,m} = \inf \{ s \geq t : R_{s,T}^{k,m} \leq 0 \} = T.$$
Thus, we have

\[ V(x,t; k, m) := \mathbb{E}_{t,x} \int_t^{k,m \wedge T} e^{-(s-t)} d\tilde{L}_s = e^{-(T-t)} \mathbb{E}_{t,x} R_T^{k,m}. \]

It is easy to deduce that for any \( \lambda > 0, \)

\[ \hat{V}(\lambda x, t; k, \lambda m) = \lambda \hat{V}(x, t; k, m). \]  

(76)

By the definition of the value function \( V, \) for any \( k > 0 \)

\[ V(x,t) \geq \hat{V}(x, t; k, 1). \]  

(77)

On the other hand, for fix \( k, \) \( a^{k,m}(x) \) satisfy the uniform Lipschitz condition, i.e. for any \( x_1, x_2 > 0, \)

\[ |a^{k,m}(x_1) - a^{k,m}(x_2)| \leq k|x_1 - x_2|, \]

and \( \lim_{m \to +\infty} a^{k,m}(x) = kx. \) Thus, when \( m \to \infty, \) the solution of (75) converges to the following geometric Brownian motion in the sense of \( L^2_T(0,T; \mathbb{R}), \)

\[ \begin{aligned}
    dR^k_s &= \mu k R^k_s \, dt + \sigma k R^k_s \, dW_s, \quad t \leq s \leq T, \\
    R^k_{t-} &= x, \quad x > 0.
\end{aligned} \]

In particular, the solution to the above SDE is

\[ R^k_t = xe^{(\mu k - \frac{\sigma^2 k^2}{2})(s-t) + \sigma k (W_s - W_t)}, \]

thus

\[ \lim_{m \to +\infty} \hat{V}(x, t; k, m) = \lim_{m \to +\infty} e^{-(T-t)} \mathbb{E}_{t,x} R_T^{k,m} = xe^{(\mu k - \sigma)(T-t)}. \]

Thanks to (77) and (76), we have

\[ \lim_{x \to 0} \frac{V(x,t)}{x} = \lim_{x \to 0} \frac{\hat{V}(x, t; k, 1)}{x} = \lim_{x \to 0} \hat{V}(1, t; k, \frac{1}{x}) = e^{(\mu k - \sigma)(T-t)} \to +\infty, \quad k \to \infty. \]

By the mean value theorem and the monotonically of \( V_x(x, t), \) we prove (74).

Appendix D. In the following, we confirm value function (7) is a viscosity solution of (9). To this end, we need to use the principle of dynamic programming(see [13]), i.e., for any stopping time \( \theta \geq t \)

\[ V(x,t) = \sup_{(y,s) \to (x,t)} \mathbb{E} \left[ \int_t^{\theta \wedge \tau \wedge T} e^{-(s-t)} dL_s + e^{-(\theta-t)} 1_{\{\theta < \tau \wedge T\}} V(R^{dx}_{\theta}, \theta) \right]. \]  

(78)

Denote

\[ V^*(x,t) = \limsup_{(y,s) \to (x,t)} V(y,s), \quad V_*(x,t) = \liminf_{(y,s) \to (x,t)} V(y,s). \]

Now we prove viscosity supersolution property. Let \((\bar{x}, \bar{t}) \in (0, +\infty) \times [0, T)\) be a test function such that

\[ 0 = (V_* - \varphi)(\bar{x}, \bar{t}) = \min_{(x,t) \in (0, +\infty) \times [0, T)} (V_* - \varphi)(x,t), \quad m \to \infty. \]  

(79)

By definition of \( V_*(\bar{x}, \bar{t}), \) there exists a sequence \((x_m, t_m)\) in \((0, +\infty) \times [0, T)\) such that

\[ (x_m, t_m) \to (\bar{x}, \bar{t}) \text{ and } V(x_m, t_m) \to V_*(\bar{x}, \bar{t}), \quad m \to \infty. \]  

(80)
By the continuity of \( \varphi \) and by (79) we also have that
\[
\gamma_m := V(x_m, t_m) - \varphi(x_m, t_m) \to 0, \quad m \to \infty.
\]
Moreover, this random variable is bounded by a constant \( \eta \).

For any fix admissible reinsurance strategy \( a \), choose \( L_s \equiv 0 \) for \( s \geq t_m \), we denote by \( R_{s_m}^{m,x_m} \) the associated controlled process. Let
\[
\tau_m = \inf\{s > t_m : R_{s_m}^{m,x_m} \leq 0\}, \quad \zeta_m = \inf\{s > t_m : |R_{s_m}^{m,x_m} - x_m| \geq \eta\}
\]
in which \( \eta > 0 \) is a fixed constant. Since \( \lim_{m \to 0} x_m = x > 0 \), we can choose \( \eta < x_m \) for \( m \) sufficiently large, so we have \( \zeta_m \leq \tau_m \). Let \( h_m \) be a positive sequence such that
\[
h_m \to 0 \quad \text{and} \quad \frac{\gamma_m}{h_m} \to 0, \quad m \to \infty.
\]
We apply (78) for \( \theta_m = (t_m + h_m) \land \zeta_m \land T \) and get
\[
V(x_m, t_m) \geq \mathbb{E}\left[e^{-c(\theta_m - t_m)}V(R_{\theta_m}^{m,x_m}, \theta_m)\right].
\]
Equation (79) implies that \( V \geq V_* \geq \varphi \), and apply (81) we get
\[
\varphi(x_m, t_m) + \gamma_m \geq \mathbb{E}\left[e^{-c(\theta_m - t_m)}\varphi(R_{\theta_m}^{m,x_m}, \theta_m)\right].
\]
Applying Itô formula to \( \varphi(R_{\theta_m}^{m,x_m}, \theta_m) \) between \( t_m \) to \( \theta_m \), we obtain
\[
\frac{\gamma_m}{h_m} + \mathbb{E}\left[\frac{1}{h_m} \int_{t_m}^{\theta_m} e^{-c(s-t_m)} \left(-\varphi_t - \frac{1}{2} \sigma^2 a^2 \varphi_{xx} - \mu a \varphi_x + c \varphi\right)(R_{s_m}^{m,x_m}, s)ds\right] \geq 0 \quad (82)
\]
after noting that the stochastic integral term cancels out by taking expectations since the integrand is bounded. By a.s. continuity of the trajectory \( R_{s_m}^{m,x_m} \) (here \( L_s \equiv 0 \)), it follows that for sufficiently large \( m(\geq N(\omega)) \), \( \tau_m(\omega) \geq t_m + h_m \), i.e., \( \theta_m(\omega) = t_m + h_m \) a.s.. Thus, by the mean value theorem, the random variable inside the expectation in (82) converges a.s. to \( \left(-\varphi_t - \frac{1}{2} \sigma^2 a^2 \varphi_{xx} - \mu a \varphi_x + c \varphi\right)(x, t) \) when \( m \) goes to infinity. Moreover, this random variable is bounded by a constant independent of \( m \), so applying dominated convergence theorem, let \( m \) goes to infinity we obtain
\[
\left(-\varphi_t - \frac{1}{2} \sigma^2 a^2 \varphi_{xx} - \mu a \varphi_x + c \varphi\right)(x, t) \geq 0.
\]
On the other hand, by the definition of \( V(x,t) \), for any \((y,t) \in (0, +\infty) \times [0, T)\) and \( \varepsilon > 0 \), there exists \((R_{s}^{x,y}, L_{s}^{x,y}, a_{s}^{x,y})\) satisfying
\[
\left\{
\begin{aligned}
&dR_{s}^{x,y} = \mu a_{s}^{x,y} dt + \sigma a_{s}^{x,y} dW_{s} - dL_{s}^{x,y}, \quad t \leq s \leq T, \\
&L_{s}^{x,y} = y,
\end{aligned}
\right.
\]
such that
\[
V(y, t) \leq \mathbb{E}_{t,y}\left[\int_{s}^{\tau^{x,y}} e^{-c(s-t)}dL_{s}^{x,y}\right] + \varepsilon,
\]
where,
\[
\tau^{x,y} = \inf\{s \geq t : R_{s}^{x,y} \leq 0\}.
\]
For \( x > y \), choose \( R_{t-} = x \) and \( a_{s} = a_{s}^{x,y} \), \( s \geq t \). Let
\[
\left\{
\begin{aligned}
&L_{t-} = L_{t-}^{x,y}, \\
&L_{s} = L_{s}^{x,y} + (x - y), \quad s \geq t,
\end{aligned}
\right.
\]
then we have $R_s = R^{x,t,y}_s$, $s \geq t$. Obviously, $(R_s, L_s, a_s)$ satisfies (5), and the corresponding bankruptcy time $\tau$ is consistent with $\tau^{x}$, so

$$V(x, t) \geq \mathbb{E} \left[ \int_t^{\tau \wedge T} e^{-c(s-t)} dL_s \mid R_{t-} = x \right]$$

$$= \mathbb{E} \left[ \int_t^{\tau \wedge T} e^{-c(s-t)} dL^{x,t,y}_s \mid R^{x,t,y}_{t-} = y \right] + (x - y) \geq V(y, t) - \varepsilon + (x - y).$$

Since $\varepsilon$ is arbitrary, we obtain

$$V(x, t) - V(y, t) \geq x - y.$$  \hspace{1cm} (83)

For fix $(\bar{x}, \bar{t}) \in (0, +\infty) \times [0, T)$, let $y < \bar{x}$, by the continuity of $\varphi$ and Equation (79), (80) and (83),

$$\varphi(\bar{x}, \bar{t}) = V_*(\bar{x}, \bar{t}) = \lim_{m \to \infty} V(x_m, t_m)$$

$$\geq \lim \inf_{m \to \infty} [V(y, t_m) + (x_m - y)]$$

$$\geq \lim \inf_{m \to \infty} [V_*(y, t_m) + (x_m - y)]$$

$$\geq \lim_{m \to \infty} [\varphi(y, t_m) + (x_m - y)] = \varphi(y, \bar{t}) + (\bar{x} - y),$$

so we have

$$\varphi_x(\bar{x}, \bar{t}) - 1 \geq 0.$$

Now, we prove viscosity subsolution property. Let $(\bar{x}, \bar{t}) \in (0, +\infty) \times [0, T)$ and let $\varphi \in C^2((0, +\infty) \times [0, T))$ be a test function such that

$$0 = (V^* - \varphi)(\bar{x}, \bar{t}) = \max_{(x, t) \in (0, +\infty) \times [0, T)} (V^* - \varphi)(x, t), \quad m \to \infty.$$  \hspace{1cm} (84)

We will show the result by contradiction. Assume on the contrary that

$$\inf_{0 \leq a \leq 1} \left( - \varphi - \frac{1}{2} \sigma^2 a^2 \varphi_{xx} - \mu a \varphi_x + c \varphi \right)(\bar{x}, \bar{t}) > 0 \quad \text{and} \quad \varphi_x(\bar{x}, \bar{t}) - 1 > 0.$$  \hspace{1cm} (85)

Then by the continuity of $\varphi$, $\varphi_x$, $\varphi_{xx}$, $\varphi_t$, there exits $\eta > 0$ and $\varepsilon > 0$ such that

$$\inf_{0 \leq a \leq 1} \left( - \varphi - \frac{1}{2} \sigma^2 a^2 \varphi_{xx} - \mu a \varphi_x + c \varphi \right)(y, t) > \varepsilon \quad \text{and} \quad \varphi_x(y, t) - 1 > \varepsilon.$$  \hspace{1cm} (85)

for all $(y, t) \in B_\eta(\bar{x}, \bar{t}) := \{(y, t) : |y - \bar{x}| + |t - \bar{t}| < \eta\}$. By definition of $V^*(\bar{x}, \bar{t})$, there exists a sequence $(x_m, t_m)$ taking value in $B_\eta(\bar{x}, \bar{t})$ such that

$$(x_m, t_m) \to (\bar{x}, \bar{t}) \quad \text{and} \quad V(x_m, t_m) \to V^*(\bar{x}, \bar{t}), \quad m \to \infty.$$  

By the continuity of $\varphi$ and by (84) we also have that

$$\gamma_m := V(x_m, t_m) - \varphi(x_m, t_m) \to 0, \quad m \to \infty, \quad m \to \infty.$$  \hspace{1cm} (86)

Let $h_m$ (let $t_m + h_m < T$) be a positive sequence such that

$$h_m \to 0 \quad \text{and} \quad \frac{\gamma_m}{h_m} \to 0, \quad m \to \infty.$$  

According to (78), we can choose $(a^m, L^m) \in \Pi_1$ associated with control process $R^{x,m}_{t_m}$ to let

$$V(x_m, t_m) - \frac{\varepsilon h_m}{4} \leq \mathbb{E} \left[ \int_{t_m}^{\theta_m} e^{-c(s-t)} dL^m_s + e^{-c(\theta_m - t_m)} V(R^m_{\theta_m}, \theta_m) \right].$$
for \( \theta_m = (t_m + h_m) \land \zeta_m, \zeta_m = \inf\{s \geq t_m : |R_s - x_m| \geq \eta'\} \), where, \( 0 < \eta' < \eta \) such that \( B_{\eta'}(x_m, t_m) \subset B_\eta(x, \bar{t}) \). Equation (84) implies that \( V \leq V^* \leq \varphi \), and apply (86) we get

\[
\varphi(x_m, t_m) + \gamma_m - \frac{\varepsilon h_m}{4} \leq E\left[ \int_{t_m}^{\theta_m} e^{-c(s-t_m)} dL_s^{m} + e^{-c(\theta_m-t_m)} \varphi(R^{t_m,x_m}_{\theta_m}, \theta_m) \right].
\]

Applying Itô formula to \( \varphi(R^{t_m,x_m}_{\theta_m}, \theta_m) \) between \( t_m \) to \( \theta_m \), we obtain

\[
\gamma_m - \frac{\varepsilon h_m}{4} \leq E\left[ \int_{t_m}^{\theta_m} e^{-c(s-t_m)} dL_s^{m} \right] - E\left[ \int_{t_m}^{\theta_m} e^{-c(s-t_m)} \left( - \varphi_t - \frac{1}{2} \sigma^2(a_m)^2 \varphi_{xx} - \mu a_m \varphi_x + c \varphi \right)(R^{t_m,x_m}_s, s) ds \right] - E\left[ \int_{t_m}^{\theta_m} e^{-c(s-t_m)} \varphi_x(R_s, s) dL_s^{m,C} \right] + E\left[ \sum_{t_m \leq s < \theta_m} e^{-c(s-t_m)} (\varphi(R_s, s) - \varphi(R_{s-}, s)) \right]
\]

after noting that the stochastic integral term cancels out by taking expectations since the integrand is bounded, where, \( L_s^{m,C} \) is continuous part of \( L_s^{m} \). Using (85) we get

\[
\gamma_m - \frac{\varepsilon h_m}{4} \leq -\varepsilon \left( E\left[ \int_{t_m}^{\theta_m} e^{-c(s-t_m)} dL_s^{m} \right] + E\left[ \int_{t_m}^{\theta_m} e^{-c(s-t_m)} ds \right] \right).
\]

Thus, we have

\[
\frac{\gamma_m - \varepsilon h_m}{h_m} - \frac{\varepsilon}{4} \leq -\varepsilon \left( \frac{1}{h_m} E\left[ \int_{t_m}^{\theta_m} e^{-c(s-t_m)} dL_s^{m} \right] + E\left[ \int_{t_m}^{\theta_m} e^{-c(s-t_m)} ds \right] \right) \]

\[
\leq -\varepsilon \left( \frac{1}{h_m} E[L^{m}_{\theta_m} - L^{m}_{t_m}] + E\left[ \frac{\theta_m - t_m}{h_m} \right] \right) \]

\[
\leq -\varepsilon \left( \frac{\eta'}{2h_m} P\left( L^{m}_{\theta_m} - L^{m}_{t_m} > \frac{\eta'}{2} \right) + E\left[ \frac{\theta_m - t_m}{h_m} \right] \chi_{\{L^{m}_{\theta_m} - L^{m}_{t_m} \leq \frac{\eta'}{2}\}} \right) \]

\[
\leq -\varepsilon \left( \frac{\eta'}{2} P\left( L^{m}_{\theta_m} - L^{m}_{t_m} > \frac{\eta'}{2} \right) + E\left[ \frac{\theta_m - t_m}{h_m} \right] \chi_{\{L^{m}_{\theta_m} - L^{m}_{t_m} \leq \frac{\eta'}{2}\}} \right). \quad (87)
\]

If we can prove that

\[
\lim_{m \to 0} E\left[ \frac{\theta_m - t_m}{h_m} \chi_{\{L^{m}_{\theta_m} - L^{m}_{t_m} \leq \frac{\eta'}{2}\}} \right] \geq \lim_{m \to 0} P\left( L^{m}_{\theta_m} - L^{m}_{t_m} \leq \frac{\eta'}{2} \right), \quad (88)
\]

then by letting \( m \) go to infinity in (87), we can get a contradiction that \( -\frac{\varepsilon}{4} \leq -\frac{\varepsilon}{2} \). So we come to prove (88).
Note that
\[ P(t_m + h_m > \zeta_m, L_{\theta_m}^m - L_{t_m}^m \leq \frac{\eta'}{2}) \leq P \left( \sup_{t_m \leq s \leq t_m + h_m} |R_{t_m}^{s,x_m,C} - x_m| \geq \frac{\eta'}{2}, L_{\theta_m}^m - L_{t_m}^m \leq \frac{\eta'}{2} \right) \]
\[ 4E \left( \sup_{t_m \leq s \leq t_m + h_m} |R_{t_m}^{s,x_m,C} - x_m|^2 \right) \leq \frac{(\eta')^2}{2} \to 0, \ m \to \infty, \]
so
\[ \lim_{m \to 0} E \left[ \frac{\theta_m - t_m}{h_m} \chi \{ L_{\theta_m}^m - L_{t_m}^m \leq \frac{\eta'}{2} \} \right] \geq \lim_{m \to 0} E \left[ \frac{\theta_m - t_m}{h_m} \chi \{ t_m + h_m \leq \zeta_m, L_{\theta_m}^m - L_{t_m}^m \leq \frac{\eta'}{2} \} \right] \]
\[ = \lim_{m \to 0} P \left( t_m + h_m \leq \zeta_m, L_{\theta_m}^m - L_{t_m}^m \leq \frac{\eta'}{2} \right) \]
\[ = \lim_{m \to 0} P \left( L_{\theta_m}^m - L_{t_m}^m \leq \frac{\eta'}{2} \right), \]
thus (88) holds true.

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