Caputo MSM fractional differentiation of extended Mittag-Leffler function

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ABSTRACT

Recently, many researchers are interested in the investigation of an extended form of special functions like Gamma function, Beta function, Gauss hypergeometric function, Confluent hypergeometric function and Mittag-Leffler function etc. Here, in this paper, the main objective is to find the composition of Caputo MSM fractional differential of the extended form of Mittag-Leffler function in terms of extended Beta function. Further, in this sequel, some corollaries and consequences are shown that are special case of our main findings.

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1. Introduction

Integral and differential operators in fractional calculus have become research subject in recently few decades due to ability of having arbitrary order. For more recent developments in fractional integral and differential operators, we refer the reader to see Agarwal and Choi (2016), Choi and Agarwal (2014), Choi and Agarwal (2015), Gehlot (2013), Gupta and Parihar (2017), Kilbas et al. (2004), Nadir et al. (2014), Rahaman et al. (2017b), Saxena and Parmar (2017), Shishkina and Sitnik (2017), Singh (2013), Srivastava and Agarwal (2013), Srivastava et al. (2012), Suthar et al. (2017), and the references cited therein.

Now a day, a general trend is in the extensions of special functions like Gamma function, Beta function, Gauss hypergeometric function and Mittag-Leffler function etc. due to its diverse applications in many applied fields. One can consult the papers by Chaudhry et al. (1997, 2004), Luo and Raina (2013), Özarslan and Yilmaz (2014), Rahaman et al. (2017b), and Srivastava et al. (2012) containing the bibliography therein.

Srivastava et al. (2012) defined a function

$$\sum_{k=0}^{\infty} a_k k! z^k$$

where $$\Theta(z)$$ is considered to be analytical within $$|z| < R, 0 < R < \infty$$ and $$\{k_n\}$$ is a sequence of Taylor-Maclaurin coefficients and $$m_0$$ and $$\sigma$$ are constants and depend upon the bounded sequence $$\{k_n\}$$. Corresponding to the function $$\Theta(z)$$, Srivastava et al. (2012) defined extended Gamma function, extended Beta function and extended Gauss hypergeometric function respectively as

$$\frac{d\Theta(z)}{dx} = \int_0^\infty x^{-2a-1} \Theta'(x) \frac{dx}{x}$$

$$\Theta'(x) = \frac{e^{-x} x^a}{\Gamma(a)}$$

It is assumed that all the integrals existed.

Corresponding to the extended Beta function $$B_p(z)$$, Parmar (2015) defined extension of Mittag-Leffler function

$$\sum_{k=0}^{\infty} a_k k! z^k$$

where

$$z, c, y \in C; R(z) > 0, R(c) > 0, R(y) > 1; p \geq 0$$

$$\sum_{k=0}^{\infty} a_k k! z^k$$

$$\Gamma(k_c+y)$$
It is noted that generalized and extended form of Mittag-Leffler function defined in literature by Desai et al. (2016), Kilbas et al. (2004), Mittag-Leffler (1903), Mittal et al. (2016), Nadir et al. (2014), Özsarslan and Yilmaz (2014), and Rahman et al. (2017a, 2017b) are special cases of the proposed function defined in (1). Some special cases of this function are described as:

(i) When $\kappa_n = \left(\frac{\phi_n}{\sigma_n}\right)$ then the extended form of (1) takes the form

$$E^{(p,s);y}_{\xi,c}(x;p) = \sum_{k=0}^{\infty} \frac{B(p,s)(y+k,1-y;p)}{B(y,1-y)} \frac{x^k}{\Gamma(k+c)}$$

under the condition

$$\left\{\begin{array}{l}
z, c, y \in \mathbb{C}; \Re(p) > 0, \Re(s) > 0 \\
(\Re(\xi) > 0, \Re(\zeta) > 0; \Re(y) > 1; p \geq 0)\end{array}\right.$$

(ii) If we select a bounded sequence $\kappa_n = 1$, then (1) reduces to the definition of Özsarslan and Yilmaz (2014)

$$E^{y}_{\xi,c}(x;p) = \sum_{n=0}^{\infty} \frac{(y)_n x^n}{\Gamma(k+c)}$$

$$\left(\begin{array}{l}
(\xi, c, y \in \mathbb{C}; \Re(\xi) > 0, \Re(\zeta) > 0)\end{array}\right)$$

(iii) Another special case of (1) is when $\kappa_n = 1$, and $p = 0$ then (1) reduces to Prabhakar’s function (Prabhakar, 1971) of three parameters.

$$E^{y}_{\xi,0}(x;p) = \sum_{n=0}^{\infty} \frac{(y)_n x^n}{\Gamma(k+c)}k!$$

$$\left(\begin{array}{l}
(\xi, c, y \in \mathbb{C}; \Re(\xi) > 0, \Re(\zeta) > 0)\end{array}\right)$$

(iv) If we set $\alpha = \beta = 1$ then our expressions for

$$E^{(k_1\ln\ln y)c;\gamma}_{\xi,c}(x;p)$$

$$E^{(p,q);y}_{\xi,c}(x;p)$$

and

$$E^{(p,q);y}_{1,1}(x;p)$$

reduces to the extended confluent hypergeometric functions

$$E^{(k_1\ln\ln y)c;\gamma}_{\xi,0}(x;p) = \phi_p^{(k_1\ln\ln y)}/(y; 1; z)$$

$$E^{(p,q);y}_{1,1}(x;p) = \phi_p^{(p,q)}/(y; 1; z)$$

$$E^{(p,q);y}_{1,1}(x;p) = \phi_p^{(p,q)}(y; 1; z)$$

In order to establish our main results, we need definition of Fox-Wright function and the concept of Hadamard products.

**Definition:** As indicated by Pohlen (2009), Let $g(z) := \sum_{k=0}^{\infty} x_k z^k$ and $h(z) := \sum_{k=0}^{\infty} y_k z^k$ be two power series then the Hadamard product of power series is defined as

$$(g \ast h)(z) := \sum_{k=0}^{\infty} x_k y_k z^k = (h \cdot g)(z) \quad (|z| < R)$$

where

$$R = \lim_{k \to \infty} \frac{x_k y_k}{x_{k+1} y_{k+1}} \cdot \left(\lim_{k \to \infty} \frac{y_k}{y_{k+1}}\right) = R_g \cdot R_h$$

where $R_g$ and $R_h$ are the radii of convergence of two series $g(z)$ and $h(z)$ respectively. Therefore, in general, $R \geq R_g \cdot R_h$.

It is to be noted that if one of the power series is an analytical function, then the Hadamard product series is also an analytical function.

**Definition:** As considered by Samko et al. (1993), the generalized Wright’s function is defined as follows

$$p \prod_{i=1}^{n} \left(\frac{A_i}{B_i} \right) \left(\frac{\alpha_i}{\beta_i}; \gamma \right) = \sum_{k=0}^{n} \prod_{i=1}^{n} \frac{\alpha_i}{\beta_i} \cdot \frac{\gamma}{\kappa_0} \left(\sum_{i=1}^{n} \kappa_i \right)$$

where the coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ and $\beta_1, \ldots, \beta_n \in \mathbb{R}^+$ with $\sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i \geq 0$.

2. Caputo-type MSM fractional derivative formula for extended Mittag-Leffler function

Here, in this section, our main object is to establish composition of new fractional derivative formulas so called Caputo-type Marichev-Saigo-Maeda fractional operator involving the extended Mittag-Leffler function (1) which is defined by Parmar (2015). Some special cases of our main result are considered. We obtain our main goal by applying the Caputo-type MSM fractional derivative given in (8) and (9) to the proposed function (1). Thus, for this purpose, we need to recall the pair of fractional derivatives $D_{0+;\varphi}^{\alpha;\omega,\varepsilon,x,\epsilon',\eta}$ and $D_{0-;\varphi}^{\alpha;\omega,\varepsilon,x,\epsilon',\eta}$ which are defined in terms of the corresponding pairs of fractional integral operator $I_{0+;\varphi}^{\alpha;\omega,\varepsilon,x,\epsilon',\eta}$ and $I_{0-;\varphi}^{\alpha;\omega,\varepsilon,x,\epsilon',\eta}$ containing Appell function $F_3$ as a kernel.

**Definition:** As shown by Saigo and Maeda (1998), Let $\omega, \omega', \varepsilon, \varepsilon', \eta \in \mathbb{C}$ and the left-sided and right-sided Marichev-Saigo-Maeda fractional integral operators containing Appell function $F_3$ in their kernel are defined as

$$\left(\int_{0}^{\infty} x^{2\omega - 1} f(t) dt\right)$$

and

$$\left(\int_{0-}^{\infty} x^{2\omega - 1} f(t) dt\right)$$

where

$$F_3(\omega, \omega', \varepsilon, \varepsilon'; \eta; 1 - \frac{\varepsilon}{\varepsilon'}; 1 - \frac{\omega}{\omega'} f(t) dt$$

and

$$F_3(\omega, \omega', \varepsilon, \varepsilon'; \eta; 1 - \frac{\varepsilon}{\varepsilon'}; 1 - \frac{\omega}{\omega'} f(t) dt$$

are the radii of convergence of two series $g(z)$ and $h(z)$ respectively. Therefore, in general, $R \geq R_g \cdot R_h$.
Let \( \omega, \omega', \epsilon, \epsilon', \eta, \rho \in C \) and the left-sided and right-sided Caputo-type MSM fractional differential operators containing Appell function in their kernel are defined as
\[
\left( D_{0^+}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} \right) f(x) = \left( I_{0^+}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} f \right) (x) = \left( I_{0^+}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} f (\Re(\eta) + 1 - \eta + \Re(\eta) + 1) f (\Re(\eta) + 1) \right) (x)
\]  
(8)
and
\[
\left( D_{0^-}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} \right) f(x) = \left( I_{0^-}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} f \right) (x) = \left( I_{0^-}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} f (\Re(\eta) + 1 - \eta + \Re(\eta) + 1) f (\Re(\eta) + 1) \right) (x)
\]  
(9)

Here, we discuss lemma, which is essential for the establishment of our main results. This lemma provides the image of power function \( t^{\rho - 1} \) with Caputo-type Marichev-Saigo-Maeda fractional differentiation.

**Lemma:** Let \( \omega, \omega', \epsilon, \epsilon', \eta, \rho \in C \) and \( m = \Re(\eta) + 1 \)

(a) Under condition
\[
\Re(\rho) - m > \max \left\{ 0, \Re(\omega + \epsilon) \right\}
\]
then image will be
\[
\left( D_{0^+}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} t^{\rho - 1} \right) f(x) = \left( \frac{\Gamma(\rho) \Gamma'(\rho - \epsilon - \omega + m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho)} \right) \left( \frac{\Gamma(\omega + \omega' - \epsilon - \eta + \rho - m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho + m)} \right)^{x^{\omega + \omega' - \eta + \rho + 1}}
\]
(b) If condition
\[
\Re(\rho) + m > \max \left\{ \Re(\omega' + \epsilon - \eta), \Re(\omega + \omega' - \eta) + \Re(\eta) + 1 \right\}
\]
is satisfied then image will be
\[
\left( D_{0^+}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} t^{\rho - 1} \right) f(x) = \left( \frac{\Gamma(1 + \epsilon' - \rho + m)}{\Gamma(1 - \omega' + \epsilon' - \eta + \rho - m)} \right) \left( \frac{\Gamma(1 + \omega' + \epsilon' - \eta - \rho)}{\Gamma(1 - \omega' - \epsilon - \eta - \rho + m)} \right)^{x^{\omega + \omega' - \eta + \rho + 1}}
\]

**Proof:** (a) Now using the definition of left-sided Caputo-type MSM fractional differentiation operator, we have
\[
\left( D_{0^+}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} t^{\rho - 1} \right) f(x) = \left( t_{0^+}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} f (\Re(\eta) + 1 - \eta + \Re(\eta) + 1) f (\Re(\eta) + 1) \right) (x)
\]  
(10)
\[
\Gamma(\rho + m) - \Gamma(\rho - m) t^{\rho - 1} (x)
\]
\[
\left( D_{0^-}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} t^{\rho - 1} \right) f(x) = \left( \frac{\Gamma(\rho) \Gamma'(\rho - \epsilon - \omega + m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho)} \right) \left( \frac{\Gamma(\omega + \omega' - \epsilon - \eta + \rho - m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho + m)} \right)^{x^{\omega + \omega' - \eta + \rho + 1}}
\]

then we reach the required result.

(b) Analogous to above, using right-sided Caputo-type MSM differential operator, we have
\[
\left( D_{0^-}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} t^{\rho - 1} \right) f(x) = \left( \frac{\Gamma(\rho) \Gamma'(\rho - \epsilon - \omega + m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho)} \right) \left( \frac{\Gamma(\omega + \omega' - \epsilon - \eta + \rho - m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho + m)} \right)^{x^{\omega + \omega' - \eta + \rho + 1}}
\]
similarly, right-sided MSM integration result from Saigo and Maeda (1998) which is
\[
\left( D_{0^-}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} t^{\rho - 1} \right) f(x) = \left( \frac{\Gamma(\rho) \Gamma'(\rho - \epsilon - \omega + m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho)} \right) \left( \frac{\Gamma(\omega + \omega' - \epsilon - \eta + \rho - m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho + m)} \right)^{x^{\omega + \omega' - \eta + \rho + 1}}
\]
we approach the required result.

In main theorem, part (a) deals with the left-hand sided Caputo-type MSM fractional derivative and part (b) deals with the right-hand sided Caputo-type MSM fractional derivative of the extended Mittag-Leffler function (1).

**Theorem:** (a) Let \( \omega, \omega', \epsilon, \epsilon', \eta, \rho \in C, m = \Re(\eta) + 1 \) be such that
\[
\Re(\rho) - m > \max \left\{ 0, \Re(\omega + \epsilon) \right\}
\]
then
\[
\left( D_{0^+}^{\omega, \omega', \epsilon, \epsilon', \eta, \rho} t^{\rho - 1} \right) f(x) = \left( \frac{\Gamma(\rho) \Gamma'(\rho - \epsilon - \omega + m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho)} \right) \left( \frac{\Gamma(\omega + \omega' - \epsilon - \eta + \rho - m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho + m)} \right)^{x^{\omega + \omega' - \eta + \rho + 1}}
\]
\[
= x^{\omega + \omega' - \eta + \rho + 1} \left( \frac{\Gamma(\rho) \Gamma'(\rho - \epsilon - \omega + m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho)} \right) \left( \frac{\Gamma(\omega + \omega' - \epsilon - \eta + \rho - m)}{\Gamma(\omega + \omega' - \epsilon - \eta + \rho + m)} \right)^{x^{\omega + \omega' - \eta + \rho + 1}}
\]
where
\[
\Delta = \left\{ \begin{array}{ll}
(\omega, \sigma), & (\omega + \epsilon, \sigma), \\
(\omega + \omega' + \epsilon' - \eta, \rho, m, \sigma), & (\omega + \omega' + \epsilon' - \eta, \rho, m, \sigma)
\end{array} \right\}
\]
\[
\Delta' = \left\{ \begin{array}{ll}
(\omega + \epsilon - \eta, \rho, m, \sigma), & (\omega + \epsilon - \eta, \rho, m, \sigma)
\end{array} \right\}
\]

(b) Let \( \omega, \omega', \epsilon, \epsilon', \eta, \rho \in C, m = \Re(\eta) + 1 \) be such that
\[
\Re(\rho) + m > \max \left\{ \Re(-\epsilon), \Re(\omega' + \epsilon - \eta), \Re(\omega + \omega') \right\}
\]
then
Thus, with the help of a Hadamard product, last continuation expression converted into the required expression.

Number of formations of various types of Mittag-Leffler function depends upon parameters. Our main results can be deduced into numerous fractional calculus results defined by many authors. Setting $\kappa_n \rightarrow 1$ the main Theorem yields the following result and the proposed function (1) reduced to the definition of Özarslan and Yilmaz (2014).

Corollary: Let the parameters $\omega, \omega', \epsilon, \epsilon', \eta, \rho \in \mathbb{C}$, $m = \lceil \Re(\eta) \rceil + 1$ and under the stated conditions the left and the right-sided Caputo fractional differential operators of extended Mittag-Leffler function defined by Özarslan and Yilmaz (2014) is given below

$$
\left( D_{0^+}^{\omega, \omega', \epsilon, \epsilon', \eta} \left( f \left( t^{\frac{1}{\epsilon'}} \right) \right) \right) (x) = \frac{\Gamma(1 + \epsilon')}{\Gamma(1 + \epsilon')} \cdot E_{\epsilon', \eta}^{\frac{1}{\epsilon'}} \left( x^{\frac{1}{\epsilon'}} \right)
$$

and

$$
\left( D_{0^+}^{\omega, \omega', \epsilon, \epsilon', \eta} \left( f \left( t^{\frac{1}{\epsilon'}} \right) \right) \right) (x) = \frac{\Gamma(1 + \epsilon)}{\Gamma(1 + \epsilon)} \cdot E_{\epsilon, \eta}^{\frac{1}{\epsilon}} \left( x^{\frac{1}{\epsilon}} \right)
$$

where

$$
\Delta_1 = \left\{ (\rho, \sigma), (\omega - \epsilon + \rho - m, \sigma), \left( \begin{array}{c}
(\omega + \omega' - \epsilon - \eta + \rho - m, \sigma), (1, \sigma) \\
(\omega + \omega' - \eta - \rho - m, \sigma)
\end{array} \right) \right\}
$$

and

$$
\Delta_2 = \left\{ (\omega + \omega' - \epsilon - \eta - \rho - m, \sigma), (\omega + \omega' - \eta - \rho - m, \sigma) \right\}
$$
Corollary: \[ \left( D_{0+}^{\alpha, \nu, \xi} \phi_p^{(k_n \nu \lambda \mu \nu \epsilon)}(\gamma_1; x t^{\sigma}) \right)(x) = x^{\omega + \gamma - \mu + \epsilon - \eta - \rho - m, \sigma}, \right) \]

where

\[ \Delta_2 = \left\{ (1 + \epsilon - \rho + m, \sigma), (1 - \omega - \omega', + \eta - \rho, \sigma) \right\} \]

\[ \Omega_2 = \left\{ (1 + \omega' - \epsilon + \eta - \rho - m, \sigma), (1, \sigma) \right\} \]

and

\[ \Delta_2' = \left\{ (1 - \rho, \sigma), (1 - \omega - \omega' - \epsilon + \rho - m, \sigma) \right\} \]

\[ \Omega_2' = \left\{ (1 - \omega - \omega' - \epsilon + \eta - \rho + m, \sigma) \right\} \]

Remark: Several further consequences of our main result and above corollary (1) and (2) can easily be derived by selecting different values of the bounded sequence \( k_n \). If we set \( \xi = \zeta = 1 \) and \( k_n = 0 \) then the above results reduces for classical confluent hypergeometric functions. On replacing \( \xi = \zeta = 1 \) and \( k_n = 0 \), we get results for extended form of confluent hypergeometric function defined by Özarslan and Yilmaz (2014). Further, on setting \( \xi = \zeta = 1 \) and \( k_n = n/\sigma \) then the relation of the above corollary becomes for the confluent hypergeometric function defined by Chaudhry et al. (1997, 2004).

3. Further special cases

First time Rao et al. (2010) introduced and defined Caputo-type fractional derivative involving Gauss hypergeometric function in its kernel. Thus for \( \omega, \epsilon, \eta \in \mathbb{C} \) and \( xe^{\mathbb{R}^+} \) with \( \Re(\omega) > 0 \) Caputo fractional differentiation of Saigo’s operators associated with Gauss hypergeometric function are defined as

\[ \left( D_{0+}^{\omega, \epsilon, \eta} f \right)(x) = \left( (\omega + [\Re(\omega)] + 1, -[\Re(\omega)] - 1, \omega + \eta - [\Re(\omega)] - 1, \eta) \right)(x) \]

and

\[ \left( D_{0-}^{\omega, \epsilon, \eta} f \right)(x) = (-1)^{[\Re(\omega)] + 1} \cdot \left( (I_{0+}^{\omega + [\Re(\omega)] + 1, -[\Re(\omega)] - 1, \omega + \eta - [\Re(\omega)] - 1, \eta} f \right)(x) \]

Remark: Caputo-type MSM fractional differential operators are associated with Appell function \( F_2 \) reduces to the Caputo fractional differential operators associated with hypergeometric function as follows having the following relationship:

\[ \left( D_{0+}^{\alpha, \nu, \xi} \phi_p^{(k_n \nu \lambda \mu \nu \epsilon)}(\gamma_1; x t^{\sigma}) \right)(x) = \left( D_{0+}^{\alpha, \nu, \xi} \phi_p^{(k_n \nu \lambda \mu \nu \epsilon)}(\gamma_1; x t^{\sigma}) \right)(x) \]

and

\[ \left( D_{0-}^{\alpha, \nu, \xi} \phi_p^{(k_n \nu \lambda \mu \nu \epsilon)}(\gamma_1; x t^{\sigma}) \right)(x) = \left( D_{0-}^{\alpha, \nu, \xi} \phi_p^{(k_n \nu \lambda \mu \nu \epsilon)}(\gamma_1; x t^{\sigma}) \right)(x) \]

Further on setting \( \epsilon \rightarrow 0 \) in a Saigo Caputo differentiation operator reduces immediately to the Erdelyi-Kober fractional Caputo-type operator having the following relationship

\[ D_{0+}^{\alpha, \nu, \xi} = D_{0+}^{\alpha, \nu, \xi} \cdot E_{\alpha, \nu, \xi}^{(k_n \nu \lambda \mu \nu \epsilon)}(x t^{\sigma}) \]

Thus we get new fractional differential formulas of left- and right-sided Saigo and Erdelyi-Kober Caputo fractional differential operators stated in corollaries below.

Corollary: Let \( \omega, \epsilon, \eta, \rho \in \mathbb{C}, m = [\Re(\eta)] + 1 \) be such that \( \Re(\rho) - m > \max[0, -\omega - \epsilon - \eta] \). Then the left-hand sided generalized Caputo fractional differentiation \( D_{0+}^{\omega, \epsilon, \eta} \) of extended Mittag-Leffler function is given by for \( x > 0 \). Then we have the following relation:

\[ \left( D_{0+}^{\alpha, \nu, \xi} \phi_p^{(k_n \nu \lambda \mu \nu \epsilon)}(\gamma_1; x t^{\sigma}) \right)(x) = \left( D_{0+}^{\alpha, \nu, \xi} \phi_p^{(k_n \nu \lambda \mu \nu \epsilon)}(\gamma_1; x t^{\sigma}) \right)(x) \]

And under the condition \( \omega, \epsilon, \eta, \rho \in \mathbb{C}, m = [\Re(\eta)] + 1 \) such that

\[ \Re(\rho) + m > \max[\Re(\epsilon) + m, \Re(-\omega - \eta)] \]

the right-hand sided generalized Caputo fractional differentiation \( D_{0-}^{\omega, \epsilon, \eta} \) of extended Mittag-Leffler function is given by for \( x > 0 \)

\[ \left( D_{0-}^{\alpha, \nu, \xi} \phi_p^{(k_n \nu \lambda \mu \nu \epsilon)}(\gamma_1; x t^{\sigma}) \right)(x) = \left( D_{0-}^{\alpha, \nu, \xi} \phi_p^{(k_n \nu \lambda \mu \nu \epsilon)}(\gamma_1; x t^{\sigma}) \right)(x) \]

where

\[ \Omega_2 = \left\{ (1 - \omega - \omega' - \epsilon + \eta - \rho + m, \sigma) \right\} \]

Corollary: Let \( \omega, \epsilon, \eta, \rho \in \mathbb{C}, m = [\Re(\eta)] + 1 \) be such that \( \Re(\rho) - m > \max[0, -\omega - \eta] \) then the left-hand sided generalized Caputo-type Erdelyi-Kober
fractional differentiation $D^{\alpha}_{0+} (D^{\beta}_{0+} f(t)) = D^{\alpha+\beta}_{0+} f(t)$ of extended Mittag-Leffler function is given by for $z > 0$

$$D^{\alpha}_{0+} (\ell^{\beta-1} E_{\nu}^{(\nu)(\omega,\eta)} (xt^\nu)) (z) = z^{\beta-1}$$

$$\cdot E^{(\nu)(\omega,\eta)}_{\nu} (x(z)^\nu) \cdot \Psi_{1} \left[\frac{\Delta_{4}}{\Delta_{4}} \right] x^{\nu}$$

$$\Delta_{4} = (\omega + \eta + \rho - m, \sigma), (1, \sigma)$$

and under the stated conditions $\omega, \eta, \rho \epsilon C, m = [\Re(\eta)] + 1$ such that $\Re(\rho) + m > \max \{m, \Re(-\omega - \eta)\}$ then the right-hand sided generalized Caputo-type Erdelyi-Kober fractional differentiation $D^{\alpha}_{0+} (D^{\beta-\nu}_{0+} f(t))$ of extended Mittag-Leffler function is given by for $z > 0$

$$D^{\alpha}_{0+} (\ell^{\beta-1} E_{\nu}^{(\nu)(\omega,\eta)} (xt^\nu)) (z) = z^{\beta-1}$$

$$\cdot E^{(\nu)(\omega,\eta)}_{\nu} (x(z)^\nu) \cdot \Psi_{1} \left[\frac{\Omega_{4}}{\Omega_{4}} \right] x^{\nu}$$

where

$$\Omega_{4} = (1 + \omega + \eta - \rho + m, \sigma), (1, \sigma)$$

It is to be noted that several further consequences of the main Theorem and Corollaries 3-4 can easily be converted to many other known result by suitable substitutions of the parameters.

4. Conclusion

In this paper, we obtain the Caputo type MSM fractional derivative of family of Mittag-Leffler function. It is to be noted that said operator transform the required function into a function of higher order. Further, well known operators like Erdelyi-Kober and Saigo’s operators are the special case of Marichev-Saigo-Maeda fractional operator.

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