CONVEX HYPERSURFACES EXPANDING BY THE POWER GAUSS CURVATURE IN A CONE

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Abstract. We consider convex hypersurfaces with boundary which meet a strictly convex cone perpendicularly. If those hypersurfaces expand inside this cone by the $-\alpha$-th power of the Gauss curvature with $0 < \alpha < 1$, we prove that this evolution exists for all the time and the evolving hypersurfaces converge smoothly to a piece of round sphere after rescaling.

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1. Introduction

During the past decades, geometric flows have been studied intensively. Following the groundbreaking work of Huisken [16], who showed the surfaces converge, after rescaling, to round spheres for strictly convex initial surfaces moving by the mean curvature flow in $\mathbb{R}^{n+1}$, several authors started to investigate whether the same results hold true for surfaces moving by the other curvature flow. One of the famous example is the closed, strictly convex initial surfaces moving by the Gauss curvature flow in $\mathbb{R}^3$ is due to Andrews [1]. For the closed, strictly convex initial surfaces moving by the $\alpha$-Gauss curvature flow in Euclidean space with arbitrary dimension $n + 1$ with power $\alpha > \frac{1}{n+2}$ is recently solved by Brendle, Choi and Daskalopoulos [2].

Compared with the above inward flow, Gerhardt [12] and Urbas [42] independently considered outward flows, or expanding curvature flows of star-shaped closed hypersurfaces in $\mathbb{R}^{n+1}$. In particular, they studied convex hypersurfaces moving outward in $\mathbb{R}^{n+1}$ with speed $K^{-\frac{1}{n}}\nu$. They showed the flow existed for all time and converges to infinity. After a proper rescaling, the rescaled flow would converge to a sphere for $0 < \alpha \leq 1$.

It is interesting to pursue the Neumann analogue of the above results. The mean curvature flow respectively Gauss curvature flow with boundary conditions were studied in the works of Huisken [17], [21] and Schnurer and Schwetlick [40]. And inverse mean curvature flow with Neumann boundary conditions was studies in the work of Marquardt [30]. Recently, inverse Gauss curvature flow for the power $\alpha = 1$ with Neumann boundary conditions were studied by M. Sani [32]. We shall show the same results hold true for inverse Gauss curvature flow with the power $0 < \alpha < 1$.

Let $S^n$ be the sphere of radius one in $\mathbb{R}^{n+1}$. Let $\Omega \subset S^n$ be a portion of $S^n$ such that $\Sigma^n := \{(rx \in \mathbb{R}^{n+1}| r > 0, x \in \partial \Omega\}$ is the boundary of a smooth, strictly convex cone. We can prove the following statement:

**Theorem 1.1.** Let $0 < \alpha < 1$ and $M_0$ be a strictly convex hypersurface which meets $\Sigma^n$ orthogonally. That is

$$M_0 \subset \Sigma^n, \quad \langle \mu, \nu_0 \rangle|_{\partial M_0} = 0,$$

where $\nu_0$ is the outward unit normal to $M_0$ and $\mu$ is the outward unit normal vector of $\Sigma$. And assume that

$$M_0 = \text{graph}_{S^n} u_0|_{\Omega}$$

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for a positive map \( u_0 : \Omega \to \mathbb{R} \). Then 

(i) there exists a family of convex hypersurfaces \( M_t \) given by the unique embedding

\[
X(\cdot, t) : \Omega \to \mathbb{R}^{n+1}
\]

with \( X(\partial \Omega, t) \subset \Sigma^n \) for \( t \geq 0 \), satisfying the following system

\[
\begin{aligned}
\frac{d}{dt}X &= K^{-\frac{n}{2}}\nu \quad \text{in } \Omega \times (0, \infty) \\
\langle \mu(X), \nu(X) \rangle &= 0 \quad \text{on } \partial \Omega \times (0, \infty) \\
X(\cdot, 0) &= M_0 \quad \text{in } \Omega
\end{aligned}
\]

where \( K \) is the Gauss curvature of \( M_t := X(\Omega, t) \) and \( \nu \) is the unit normal vector to \( M_t \) pointing away from the center of the cone.

(ii) the leaves \( M_t \) are graphs over \( S^n \),

\[
M_t = \text{graph}_{S^n} u(\cdot, t)|_\Omega
\]

(iii) Moreover, the evolving hypersurfaces converge smoothly after rescaling, to a piece of round sphere.

The motivation to study the behavior of inverse, or expanding curvature flows has mostly been driven by their power to deduce geometric inequalities for hypersurfaces. The most prominent example is the proof of the Riemannian Penrose inequality due to Huisken and Ilmanen [18, 19] by using the inverse mean curvature flow. Another very interesting applications is the proof of some Minkowski-type inequalities and Alexandrov-Fenchel-type inequalities in Space forms and even some warped products by inverse curvature flows. For this application, see [4, 3, 5, 7, 8, 9, 10, 11, 23, 24, 25, 26, 29, 41, 34, 35, 36, 37, 38, 39, 43], etc.

The main technique employed in this paper was from the well-known paper written by Lions, Trudinger and Urbas [28], who treated the elliptic Neumann boundary problem for the equation of Monge-Ampère type. They made full use of the convexity of the domain in the second order derivative estimates. Their result was extended to the parabolic Monge-Ampère type equation with Neumann boundary problem in [41]. Their technique is also applied to the inverse Gauss curvature flow with Neumann boundary by M. Sani [32].

2. THE CORRESPONDING SCALAR EQUATION WITH NEUMANN BOUNDARY

In this section, working with coordinates on the sphere, we equivalently formulate the problem by the corresponding scalar equation with Neumann boundary. Since the initial hypersurface is convex, there exists a scalar function \( u_0 \in C^{2,\alpha}(\Omega) \) such that the \( X_0 : \Omega \to \mathbb{R}^{n+1} \) has the form \( x \mapsto (u_0(x), x) \). The hypersurface \( M_t \) given by the embedding

\[
X(\cdot, t) : \Omega \to \mathbb{R}^{n+1}
\]

at time \( t \) may be represented as a graph over \( \Omega \subset S^n \subset \mathbb{R}^{n+1} \), and then we can make ansatz

\[
X(x, t) = (u(x, t), x)
\]

for some function \( u : \Omega \times [0, T) \to \mathbb{R} \).

Lemma 2.1. Define \( p := X(x, t) \) and assume that a point on \( \Omega \) is described by local coordinates \( \xi^1, \ldots, \xi^n \), that is, \( x = (\xi^1, \ldots, \xi^n) \). Let \( \partial_i \) be the corresponding coordinate fields on \( \Omega \subset S^n \) and \( \sigma_{ij} = g_{p\nu}(\partial_i, \partial_j) \) be the metric on \( \Omega \subset S^n \). Let \( u_i = Du_i, u_{ij} = D_j D_i u, \) and \( u_{ijk} = D_k D_j D_i u \) denote the covariant derivatives of \( u \) with respect to the round metric \( g_{S^n} \) and let \( \nabla \) be the Levi-Civita connection of \( M_t \) with respect to the metric \( g \) induced from the standard metric of \( \mathbb{R}^{n+1} \). Then, the following formulas holds:
(i) The tangential vector on $M_t$ is
$$X_i = \partial_i + D_i u \partial_r,$$
and the corresponding outward unit normal vector is given by
$$\nu = \frac{1}{v} \left( \partial_r - \frac{1}{u^2} D^j u \partial_j \right),$$
where $D^j u = \sigma^{ij} D_i u$, and $v := \sqrt{1 + u^{-2} |Du|^2}$ with $Du$ the gradient of $u$.

(ii) The induced metric $g$ on $M_t$ has the form
$$g_{ij} = u^2 \sigma_{ij} + D_i u D_j u$$
and its inverse is given by
$$g^{ij} = \frac{1}{u^2} \left( \sigma^{ij} - \frac{D^i u D^j u}{v^2} \right).$$

(iii) The second fundamental form of $M_t$ is given by
$$h_{ij} = \frac{1}{v} \left( -u_{ij} + u \sigma_{ij} + 2 \frac{u_i}{u} u_j \right).$$
Thus, the Gauss curvature takes the form
$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{1}{(u^2 + |Du|^2)^2} \frac{\det(u^2 \sigma_{ij} - u w_{ij} + 2 u_i u_j)}{\det(u^2 \sigma_{ij} + u_i u_j)}.$$

Proof. This formulas can be verified by direct calculation. The details can be found in [4, 32].

Using techniques as in Ecker [6], see also [31, 30]. The problem (1.1) is reduced to solving the following scalar equation with Neumann boundary
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{v}{K^{\frac{1}{n}}} \quad \text{in } \Omega \times (0, \infty) \\
D_\mu u &= 0 \quad \text{on } \partial \Omega \times (0, \infty) \\
u(\cdot, 0) &= u_0 \quad \text{in } \Omega
\end{align*}
\]
Define a new function $\varphi(x, t) = \log u(x, t)$ and then the Gauss curvature can be rewritten as
$$K = \frac{e^{-n \varphi}}{(1 + |D\varphi|^2)^{\frac{n+2}{2}}} \frac{\det(\sigma_{ij} - \varphi_{ij} + \varphi_i \varphi_j)}{\det(\sigma_{ij})}.$$  

The evolution equation (2) can be rewrite as
$$\frac{\partial}{\partial t} \varphi = e^{(\alpha - 1) \varphi} (1 + |D\varphi|^2)^\frac{n}{2} \frac{\det(\sigma_{ij})}{\det(\sigma_{ij} - \varphi_{ij} + \varphi_i \varphi_j)} = Q(\varphi, D\varphi, D^2 \varphi),$$
where $\beta = \frac{(\alpha + 1)n}{2} + \alpha$.

Remark 2.1. In particular, the power $\alpha = 1$, the equation (1.1) is scale-invariant. In this case, the functional $Q$ in the right of the above scalar equation does not depend on $\varphi$. However, in this paper, we consider the more complex case $\alpha \neq 1$, it means that the equation (1.1) is non-scale-invariant and the functional $Q$ depends on $\varphi$. 

The convexity of the initial hypersurface \( M_0 \) means the matrix
\[
\sigma_{ij} - \varphi_{0;ij} + \varphi_{0;i}\varphi_{0;j}
\]
is positive definite up to the boundary \( \partial \Omega \). Thus, the problem (1.1) is again reduced to solving the following scalar equation with Neumann boundary
\[
\begin{cases}
\frac{\partial \varphi}{\partial t} = Q(\varphi, D\varphi, D^2\varphi) & \text{in } \Omega \times (0, T) \\
D_{\nu} \varphi = 0 & \text{on } \partial \Omega \times (0, T) \\
\varphi(\cdot, 0) = \varphi_0 & \text{in } \Omega,
\end{cases}
\]
with the matrix
\[
\sigma_{ij} - \varphi_{ij} + \varphi_{i}\varphi_{j}
\]
is positive definite up to the boundary \( \partial \Omega \). Based on the above facts and [31, 30], we can get the following existence and uniqueness for the parabolic system (1.1).

**Lemma 2.2.** Let \( X_0(\Omega) = M_0 \) be as in Theorem 1.1. Then there exist some \( T > 0 \), a unique solution \( u \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\Omega \times [0, T], \mathbb{R}^{n+1}) \), where \( \varphi(x, t) = \log u(x, t) \), of the parabolic system (2.1) with the matrix
\[
\sigma_{ij} - \varphi_{ij} + \varphi_{i}\varphi_{j}
\]
is positive definite up to the boundary \( \partial \Omega \). Therefore there exist a unique map \( \psi : \Omega \times [0, T] \rightarrow \Omega \) such that the map \( \hat{X} \) defined by
\[
\hat{X} : \Omega \times [0, T] \rightarrow \mathbb{R}^{n+1} : (x, t) \mapsto X(\psi(x, t), t)
\]
has the same regularity as stated in Theorem 1.1 and is the unique solution to the parabolic system (1.1).

Let \( T^* \) be the maximal time such that there exists some \( u \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\Omega \times [0, T^*]) \cap \mathcal{C}^{\infty}(\Omega \times (0, T^*)) \) which solves (2.1). In the following, we shall prove a priori estimates for those admissible solutions on \( [0, T] \) where \( T < T^* \).

3. \( C^0 \) estimates

To obtain \( C^0 \) estimates, the Comparison Theorem for parabolic equations must be needed.

**Lemma 3.1.** Let \( \varphi_1 \) and \( \varphi_2 \) be the two solutions of (2.1) with \( \varphi_1(x, 0) \leq \varphi_2(x, 0) \) for all \( x \in \Omega \), then we have for \( 0 < \alpha < 1 \)
\[
\varphi_1(x, t) \leq \varphi_2(x, t)
\]
holds true for all \( (x, t) \in \Omega \times [0, T] \).

**Proof.** Let \( \psi(x, t) = \varphi_1(x, t) - \varphi_2(x, t) \) and \( \varphi_s(x, t) = s\varphi_1(x, t) + (1 - s)\varphi_2(x, t) \). The main theorem of calculus implies
\[
\frac{\partial}{\partial t} \psi(x, t) = \frac{\partial}{\partial t} \varphi_1(x, t) - \frac{\partial}{\partial t} \varphi_2(x, t)
\]
\[
= \int_0^1 \frac{d}{ds} Q(\varphi_s, D\varphi_s, D^2\varphi_s) ds
\]
\[
= \int_0^1 Q^i_j(s) ds \psi_{ij} + \int_0^1 Q^k(s) ds \psi_k
\]
\[
+ (\alpha - 1) \int_0^1 Q(\varphi_s, D\varphi_s, D^2\varphi_s) ds \psi_k,
\]

\[(3.1)\]
where \(Q^{ij}(s) = \frac{\partial Q}{\partial \varphi_{ij}}(\varphi_s, D\varphi_s, D^2\varphi_s)\) and \(Q^k(s) = \frac{\partial Q}{\partial \varphi_k}(\varphi_s, D\varphi_s, D^2\varphi_s)\). Introducing the following notation
\[
a^{ij}(x, t) = \int_0^1 Q^{ij}(s)ds
\]
and
\[
b^k(x, t) = \int_0^1 Q^k(s)ds, \quad c(x, t) = (\alpha - 1) \int_0^1 Q(\varphi_s, D\varphi_s, D^2\varphi_s)ds.
\]
It follows, in view of (3.1) and of the last computations,
\[
\begin{cases}
\frac{\partial \psi}{\partial t} = a^{ij}(x, t)\psi_{ij} + b^k(x, t)\psi_k + c(x, t)\psi = 0 & \text{in } \Omega \times (0, T) \\
D\mu \psi = 0 & \text{on } \partial\Omega \times (0, T) \\
\psi(\cdot, 0) \leq 0 & \text{on } \Omega.
\end{cases}
\]
Since the set of positive definite matrices is convex, the matrix
\[
a^{ij}(x, t)
\]
is positive definite. Noticing that \(c(x, t) \leq 0\), using the parabolic maximum principle and Hopf's lemma, we can hence conclude that \(\psi(x, t)\) has to be nonpositive for all \(t \in (0, T)\). \(\square\)

Applying the above Comparison Theorem, we can compare the solution of (2.1) with its radical solution.

**Lemma 3.2.** Let \(\varphi\) be the solution of the parabolic system (2.1), then we have for \(0 < \alpha < 1\)
\[
\frac{1}{1-\alpha} \ln((1-\alpha)t + e^{(1-\alpha)\varphi_1}) \leq \varphi(x, t) \leq \frac{1}{1-\alpha} \ln((1-\alpha)t + e^{(1-\alpha)\varphi_2})
\]
where \(\varphi_1 = \inf_\Omega \varphi(\cdot, 0)\) and \(\varphi_2 = \sup_\Omega \varphi(\cdot, 0)\).

**Proof.** Let \(\varphi(x, t) = \varphi(t)\) (independent of \(x\)) be the solution of (2.1) with \(\varphi(0) = c\). In this case, the equation (2.1) reduced to an ODE
\[
\frac{d}{dt}\varphi = e^{(\alpha-1)\varphi}.
\]
Therefore,
\[
\varphi(t) = \frac{1}{1-\alpha} \ln((1-\alpha)t + e^{(1-\alpha)c}) \quad \text{for } 0 < \alpha < 1
\]
and
\[
\varphi(t) = \frac{t}{n} + c \quad \text{for } \alpha = 1.
\]
Using the above Comparison Theorem, we can obtain the results. \(\square\)

**Remark 3.1.** From (3.3), we know that \(\varphi(t) \to \infty\) in finite time if \(\alpha > 1\). Thus, if the initial hypersurface is a sphere, the flow will blow up in finite time for \(\alpha > 1\). This is the reason why we only need consider the case \(0 < \alpha < 1\).

**Corollary 3.3.** If \(\varphi\) satisfies (2.1), then we have for \(0 < \alpha < 1\)
\[
c_1 \leq u(x, t)\Theta^{-1}(t, c) \leq c_2, \quad \forall \ x \in \Omega, \ t \in [0, T],
\]
where \(\Theta(c, t) = \{(1-\alpha)t + e^{(1-\alpha)c}\}^{\frac{1}{1-\alpha}}\) and \(\varphi_1 \leq c \leq \varphi_2\). And the flow is compactly contained in \(\mathbb{R}^n\) for finite \(t\).
4. $\dot{\varphi}$ ESTIMATES

In this section, we shall show that $\dot{\varphi}(x, t)\Theta(t)^{1-\alpha}$ keep bounded during the flow.

Lemma 4.1. Let $\varphi$ be a solution of (2.1), then
\[
\min\{\inf_{\Omega} \dot{\varphi}(\cdot, 0) \cdot \Theta(0)^{1-\alpha}, 1\} \leq \varphi(x, t)\Theta(t)^{1-\alpha} \leq \max\{\sup_{\Omega} \dot{\varphi}(\cdot, 0) \cdot \Theta(0)^{1-\alpha}, 1\}.
\]

Proof. Set
\[
M(x, t) = \dot{\varphi}(x, t)\Theta(t)^{1-\alpha}.
\]
Differentiating both sides of the first evolution equation of (2.1), it is easy to get that $\dot{\varphi}$ satisfies
\[
\begin{cases}
\frac{\partial M}{\partial t} = Q^{ij} D_{ij} M + Q^k D_k M + (1 - \alpha) \Theta^{\alpha-1}(1 - M)M & \text{in } \Omega \times (0, T) \\
D_\mu M = 0 & \text{on } \partial \Omega \times (0, T) \\
M(\cdot, 0) = \dot{\varphi}_0 \cdot \Theta(0)^{1-\alpha} & \text{on } \Omega.
\end{cases}
\]

Then, we have
\[
\frac{\partial M}{\partial t} = Q^{ij} D_{ij} M + Q^k D_k M + (1 - \alpha) \Theta^{\alpha-1}(1 - M)M.
\]
If $M(x, t) \leq 1$,
\[
(1 - \alpha) \Theta^{\alpha-1}(1 - M(x, t)) \geq 0,
\]
it follows by applying Hopf’s Lemma,
\[
M(x, t) \geq \inf_{\partial^n} \dot{\varphi}(\cdot, 0) \cdot \Theta(0)^{1-\alpha}.
\]
Then,
\[
M(x, t) \geq \min\{\inf_{\partial^n} \dot{\varphi}(\cdot, 0) \cdot \Theta(0)^{1-\alpha}, 1\}.
\]
Similarly, we have
\[
M(x, t) \leq \max\{\sup_{\partial^n} \dot{\varphi}(\cdot, 0) \cdot \Theta(0)^{1-\alpha}, 1\}.
\]
Therefore, we complete our proof. \qed

5. Gradient Estimates

Lemma 5.1. Let $\varphi$ be a solution of (2.1), then we have for $0 < \alpha < 1$
\[
|D\varphi| \leq \sup_{\Omega} |D\varphi(\cdot, 0)|.
\]

Proof. Set $\psi = \frac{\sqrt{\varphi}}{2}$. By differentiating the $\psi$, we have
\[
\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t} D_m \varphi D^m \varphi
= D_m \varphi D^m \varphi
= D_m Q D^m \varphi.
\]
Then,
\[
\frac{\partial \psi}{\partial t} = Q^{ij} D_{ij} \varphi D^m \varphi + Q^k D_k \varphi D^m \varphi + (\alpha - 1)Q |D\varphi|^2.
\]
Interchanging the covariant derivatives, we have
\[ D_{ij}\psi = D_j(D_{mi}\varphi D^m\varphi) = D_{mij}\varphi D^m\varphi + D_{mi}D^m_j\varphi = (D_{ijm}\varphi + R_{limj}^m D^m\varphi + D_{mi}D^m_j\varphi). \]

Therefore, we can express \( D_{ijm}\varphi D^m\varphi \) as
\[ D_{ijm}\varphi D^m\varphi = D_{ij}\psi - R_{limj}^m D^m\varphi - D_{mi}D^m_j\varphi. \]

Then, in view of the fact \( R_{limj} = \sigma_{lm}\sigma_{ij} - \sigma_{lj}\sigma_{im} \) on \( \mathbb{S}^n \) we have
\[ \frac{\partial \psi}{\partial t} = Q^{ij}\nabla_{ij}\psi + Q^k D_k\psi - Q^{ij}(\sigma_{ij}|D\varphi|^2 - D_i\varphi D_j\varphi) - Q^{ij}D_{mi}\varphi D^m_j\varphi + (\alpha - 1)Q|D\varphi|^2. \]

Next, we shall consider the boundary condition. Choosing an orthonormal frame and \( e_1, e_2, \ldots, e_{n-1} \in T_x\Omega \) and \( e_n = \mu \). Using the Neumann boundary condition \( \nabla\mu\varphi = 0 \), we have
\[ D_\mu\psi = D_{e_n}\psi = \sum_{i=1}^n D_{in}\varphi D_i\varphi \]
\[ = D^2\varphi(e_i, e_n)D_{e_i}\varphi \]
\[ = \sum_{i=1}^{n-1} (e_ie_n\varphi - D_{ei}e_n\varphi)D_{e_i}\varphi \]
\[ = -\sum_{i=1}^{n-1} (D_{ei}e_n\varphi)D_{e_i}\varphi \]
\[ = -\sum_{i=1}^{n-1} (D_{ei}e_n, e_j)D_{e_j}\varphi D_{e_i}\varphi \]
\[ = -\sum_{i=1}^{n-1} h^{ij}_{\partial \Omega} D_{ei}\varphi D_{e_j}\varphi \]
\[ \leq 0. \]

where \( h^{ij}_{\partial \Omega} \) is the second fundamental form of \( \partial \Omega \) and it is a positive definite, since \( \Omega \) is convex.

Since the matrix \( Q^{ij} \) is positive definite, the forth and fifth terms in the right of (6.28) are non-positive. And noticing that the sixth term in the right of (6.28) is also non-positive if \( 0 < \alpha < 1 \). So we got the equation about \( \psi \) as follows:
\[
\begin{cases}
\frac{\partial \psi}{\partial t} \leq Q^{ij} D_{ij}\psi + Q^k D_k\psi & \text{in } \Omega \times (0, \infty) \\
D_\mu\psi \leq 0 & \text{on } \partial \Omega \times (0, \infty) \\
\psi(0) = \frac{|D\varphi(0)|^2}{2} & \text{in } \Omega.
\end{cases}
\]

Using the maximum principle and Hopf’s lemma, we get the gradient estimates of \( \varphi \).

Combing the gradient estimates with \( \dot{\varphi} \) estimate, we obtain

**Corollary 5.2.** If \( \varphi \) satisfies (2.1), then we have
\[ 0 < c_1 \leq det(\sigma_{ij} - \varphi_{ij} + \varphi_i\varphi_j) \leq c_2 < +\infty, \]
where \( c_1 \) and \( c_2 \) are positive constants independent of \( \varphi \).

6. \( C^2 \) Estimates

In this section, we come to the a priori estimates of second order derivative of \( \varphi \).

**Theorem 6.1.** Let \( \varphi \) be a solution of the flow (2.1) and \( 0 < \alpha < 1 \). Then, there exists \( C = C(n, M_0) \) such that

\[
|D^2 \varphi(x,t)| \leq C(n, M_0), \quad \forall (x,t) \in \overline{\Omega} \times [0, T^*).
\]

We remark that (5.3) together the \( C^1 \)-estimate (6.27), implies an upper bound on \( \varphi_{ij} \). We hence only need to control the second covariant derivatives of \( \varphi \) from below. Our proof will be divided into the following three cases.

6.1. Interior \( C^2 \)-estimate.

The main technique employed here was from M. Sani [32], but with great simplicity. For convenience, set

\[
w_{ij} = \sigma_{ij} - \varphi_{ij} + \varphi_{i}\varphi_{j},
\]

\( w^{ij} \) be the inverse of \( w_{ij} \) and \( \dot{U} = \frac{\partial U}{\partial t} \). We consider the following operator

\[
\mathcal{L}U = \dot{U} - Q^{ij}U_{ij} - Q^kU_k
\]

\[
= \dot{U} - \frac{\alpha}{n} \dot{\varphi} w^{ij}U_{ij} - \frac{2\dot{\varphi}}{n} \left( \frac{\beta}{1 + |D\varphi|^2} \sigma^{kl} - \alpha w^{kl} \right) \varphi_kU_l.
\]

where in fact

\[
Q^{ij} := \frac{\partial Q}{\partial \varphi_{ij}} = \frac{\alpha}{n} \dot{\varphi} w^{ij}
\]

and

\[
Q^k := \frac{\partial Q}{\partial \varphi_k} = \frac{2\dot{\varphi}}{n} \left( \frac{\beta}{1 + |D\varphi|^2} \sigma^{kl} - \alpha w^{kl} \right) \varphi_k.
\]

First, we prove some equalities on \( \mathbb{S}^n \) which will play an important role in later computations.

**Lemma 6.2.** The following equalities hold on \( \mathbb{S}^n \):

\[
w^{kl}w_{11;kl} - w^{kl}w_{kl,11} = -2trw^{kl}\varphi_{11} + 2(trw^{kl} - n + w^{kl}\varphi_k\varphi_l)
\]

\[
+ 2w^{kl}(-\varphi_{1k}\varphi_1 - \varphi_{k11}\varphi_1).
\]

\[
w^{kl}(w_{11;k}w_{11;l} - w_{1k;1}w_{1l;1}) = 2w^{kl}w_{11;k}\varphi_{11} - 2w_{11;1}\varphi_1 - (w_{11})^2 w^{kl}\varphi_k\varphi_l + w_{11}(\varphi_1)^2.
\]

**Proof.** Interchanging the covariant derivatives

\[
\varphi_{1k} = \varphi_{k1} + R_{11k}^{p} \varphi_p = \varphi_{k1} + \delta_{1k} \varphi_1 - \varphi_k
\]

in view of

\[
R_{ijk} = \sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk} \quad \text{on} \quad \mathbb{S}^n.
\]

Rewriting it as

\[
w_{11;k} = -\varphi_{k11} - \delta_{1k}\varphi_1 + \varphi_k + 2\varphi_1\varphi_{1k}.
\]
Then, using the Ricci identities and the fact that the covariant derivatives of the curvature tensor of unit sphere vanish, we have

\[
\varphi_{11kl} = \varphi_{k1l1} + R^p_{11k} \varphi_{pl}
\]

\[
= \varphi_{k1l1} + R^p_{11l} \varphi_{kp} + R^p_{k1l} \varphi_{pl} + R^p_{11k} \varphi_{pl}
\]

\[
= (\varphi_{k1l1} + R^p_{k1l} \varphi_{pl})_1 + R^p_{11l} \varphi_{kp} + R^p_{k1l} \varphi_{pl} + R^p_{11k} \varphi_{pl}
\]

\[
= \varphi_{kl} + R^p_{k1l} \varphi_{pl} + 2R^p_{k1l} \varphi_{pl} + R^p_{11k} \varphi_{pl}.
\]

It follows that

\[
w^{kl} w_{11;kl} = w^{kl} (-\varphi_{11kl} + (\varphi_{11} \varphi_{1})_{kl})
\]

\[
= w^{kl} w_{kl,11} + w^{kl} (-2R^s_{k1l} \varphi_{s1} - 2R^s_{11l} \varphi_{ks} + 2\varphi_{1kl} \varphi_{1} - 2\varphi_{1ll} \varphi_{l})
\]

\[
= w^{kl} w_{kl,11} - 2\text{tr} w^{kl} \varphi_{11} + 2 w^{kl} \varphi_{1l} + 2 w^{kl} (-\varphi_{1kl} \varphi_{1} - \varphi_{k1l} \varphi_{1})
\]

\[
= w^{kl} w_{kl,11} - 2\text{tr} w^{kl} \varphi_{11} + 2(\text{tr} w - n + w^{kl} \varphi_{k1l}) + 2 w^{kl} (-\varphi_{1kl} \varphi_{1} - \varphi_{k1l} \varphi_{1}).
\]

So, the equality (6.1) is obtained.

Now, we pursue the second equality. We can rewrite (6.3) as

\[
w_{1k;1} = w_{11;k} - w_{11} \varphi_{k} + w_{1k} \varphi_{1}.
\]

Thus,

\[
w^{kl}(w_{11;k} w_{11;l} - w_{1k;1} w_{1l;1})
\]

\[
= w^{kl} w_{11;k} w_{11;l} - 2 w^{kl} (w_{11;k} - w_{11} \varphi_{k} + w_{1k} \varphi_{1}) (w_{11;l} - w_{11} \varphi_{l} + w_{1l} \varphi_{1})
\]

\[
= -2 w^{kl} (w_{11;k} - w_{11} \varphi_{k} + w_{1k} \varphi_{1}) (w_{11;l} - w_{11} \varphi_{l} + w_{1l} \varphi_{1})
\]

\[
= - w^{kl} (-w_{11} \varphi_{k} + w_{1k} \varphi_{1}) (-w_{11} \varphi_{l} + w_{1l} \varphi_{1})
\]

\[
= 2 w^{kl} w_{11;k} \varphi_{l} w_{11} - 2 w_{11;1} \varphi_{1} - (w_{11})^2 w^{kl} \varphi_{k} \varphi_{l} + w_{11} (\varphi_{1})^2.
\]

\[
\square
\]

**Remark 6.1.** The equality (6.2) is also obtained in [32] with very long computations by (6.3), but our calculation is of great simplicity by rewriting (6.3) as another form (6.1).

To make progress, we need the following evolution.

**Lemma 6.3.** Under the flow (2.1), the following evolution equations hold true

\[
\mathcal{L} \left( \frac{1}{2} |D\varphi|^2 \right) = -\frac{\alpha \dot{\varphi} }{n} \left( 1 + |D\varphi|^2 \right) w^{ij} \sigma_{ij} + (1 + |D\varphi|^2) w^{ij} \varphi_{i} \varphi_{j}
\]

\[
- \Delta \varphi - |D\varphi|^2 - n \right) \right) + (\alpha - 1) \dot{\varphi} |D\varphi|^2.
\]

\[
\mathcal{L} w_{11} = -\frac{(\varphi_{1})^2}{\varphi} + \frac{\alpha}{n} \dot{\varphi} w^{kl;1} w_{kl;1}
\]

\[
+ \frac{4 \beta \dot{\varphi} }{n} \left( \frac{1}{1 + |D\varphi|^2} \right)^2 (\sigma^{kl} \varphi_{k} \varphi_{l})^2 - \frac{2 \beta \dot{\varphi} }{n} \left( \frac{1}{1 + |D\varphi|^2} \right) \varphi^{kl} \varphi_{k1} \varphi_{l1}
\]

\[
+ \frac{2 \dot{\varphi} }{n} \left( \frac{\beta}{1 + |D\varphi|^2} \right) (\varphi_{1})^2 - |D\varphi|^2 \right) - \frac{4 \alpha }{n} \dot{\varphi} w^{kl} \sigma_{1k} \varphi_{1l}
\]

\[
+ \frac{2 \alpha }{n} \dot{\varphi} \text{tr} w^{ij} (-w_{11} + 2(\varphi_{1})^2 + n) + (1 - \alpha) \dot{\varphi} (w_{11} - (\varphi_{1})^2 - 1).
\]
Using the evolution equation (2.1), we have
\[ \mathcal{L}(\gamma^i \varphi_i) = Q^{ij}(\sigma_{ij} \gamma^l - \sigma_{ij} \varphi_l) - Q^k \varphi_{i;k} + (\alpha - 1)Q \varphi_{i;k}. \]

Where \( \gamma^i : \Omega \to \mathbb{R} \) is a smooth function that does not depend on \( \varphi \).

Proof. We begin to prove the first evolution equation. Clearly,
\[ \mathcal{L}(\varphi_i) = \sigma^r \varphi_r \varphi_s - Q^{ij}(\varphi_i \varphi_j) - Q^k \varphi_{i;k} + (\alpha - 1)\varphi_{i;k}. \]

Using the evolution equation (2.1), the first term on the right of the above equation becomes
\[ \sigma^r \varphi_r \varphi_s = \sigma^r \varphi_{i} \varphi_{j} + Q^k \varphi_{i;k} + (\alpha - 1)\varphi_{i;k}. \]

Interchanging covariant derivative and inserting the Riemannian curvature tensor of the sphere \( R^m_{ijr} = \delta^m_{ir} \sigma_{ir} - \delta^m_{ri} \sigma_{ij} \), we have
\[ \varphi_{i} \varphi_{j} = \varphi_{ir} \varphi^r + R^m_{ijr} \varphi_{m} \varphi^r = \varphi_{ir} \varphi^r + (\delta^m_{ir} \sigma_{ir} - \delta^m_{ri} \sigma_{ij}) \varphi_{m} \varphi^r. \]

Thus, we obtain
\[ \mathcal{L}(\varphi_i) = Q^j \varphi_{i,j} \varphi_s + Q^{ij}(\varphi_i \varphi_j - |D\varphi|^2 \sigma_{ij}) + (\alpha - 1)\varphi_{i;k}. \]

It follows that
\[ \mathcal{L}(\varphi_i) = \frac{\alpha \dot{\varphi}}{n} \left( (1 + |D\varphi|^2)w^{ij} \sigma_{ij} + (1 + |D\varphi|^2)w^{ij} \varphi_i \varphi_j \right) - \Delta \varphi - |D\varphi|^2 - n \right) + (\alpha - 1)\dot{\varphi}|D\varphi|^2 \]

in view of
\[ w^{ij} \varphi_{ri} = w^{ij} (\sigma_{ri} - w_{ri} + \varphi_r \varphi_i) = w^{ij} \sigma_{ri} - \delta^i_r + w^{ij} \varphi_r \varphi_i. \]

Now, we prove the second evolution equation. Clearly,
\[ \mathcal{L}(w_{11}) = \dot{w}_{11} - Q^{ij} w_{11,ij} - Q^k w_{11;k}. \]

Using the evolution equation (2.1), we have
\[ \dot{w}_{11} = - \dot{\varphi}_{11} + 2 \dot{\varphi} \varphi_1 \]

\[ \dot{w}_{11} = - \left( - \frac{\alpha \dot{\varphi}}{n} w^{kl} w_{kl,11} + \frac{\alpha \dot{\varphi}}{n} \frac{\beta}{1 + |D\varphi|^2} \sigma^r \varphi_{r,1} + (\alpha - 1)\dot{\varphi}_{1} \right) + 2 \dot{\varphi} \varphi_1 \]

\[ \dot{w}_{11} = - \left( \frac{(\dot{\varphi})^2}{\varphi} + \frac{\alpha \dot{\varphi}}{n} w^{kl} w_{kl,11} + \frac{\alpha \dot{\varphi}}{n} w^{kl} w_{kl,11} + \frac{4 \beta \dot{\varphi}}{n} \frac{1}{1 + |D\varphi|^2} (\sigma^k \varphi_{1;k})^2 \right) - \frac{2 \beta \dot{\varphi}}{n} \frac{1}{1 + |D\varphi|^2} \sigma^k \varphi_{1;k+1} - (\alpha - 1)\dot{\varphi}_{11} + 2 \dot{\varphi} \varphi_1. \]
Inserting (6.1) into the above equality, we obtain

\[
\dot{w}_{11} = -\frac{(\dot{\varphi})^2}{\varphi} + \frac{\alpha}{n} \dot{\varphi} w^{kl}_{11} \dot{w}_{kl,1} + \frac{\alpha}{n} \dot{\varphi} w^{kl}_{11} \dot{w}_{11,kl} \\
+ \frac{2\alpha}{n} \dot{\varphi} w^{ij}_{11} \varphi_{11} - \frac{2\alpha}{n} \dot{\varphi} (\text{tr} w^{ij} - n + w^{ij} \varphi_{ij}) \\
+ \frac{4\beta \dot{\varphi}}{n} \frac{1}{(1 + |D\varphi|^2)^2} (\sigma^{kl} \varphi_{k\varphi_{11}})^2 - \frac{2\beta \dot{\varphi}}{n} \frac{1}{1 + |D\varphi|^2} \sigma^{kl} \varphi_{k1} \varphi_{11} \\
- \frac{2\dot{\varphi}}{n} \frac{\beta}{1 + |D\varphi|^2} \alpha w^{kl} \varphi_{l} \varphi_{k11} + 2(\dot{\varphi} - \frac{\alpha}{n} \dot{\varphi} w^{kl}_{11} \varphi_{11} + (\alpha - 1) \dot{\varphi}_{11}.
\]

(6.8)

Since

\[-2w^{kl}(\varphi_{11} + \sigma_{k1} \varphi_l - \sigma_{k1} \varphi_1) \varphi_1 = -2w^{kl}(\varphi_{k11} + \sigma_{k1} \varphi_l - \sigma_{k1} \varphi_1) \varphi_1 \]

\[= 2w^{kl}_{11} - 2w^{kl}(\varphi_{k} \varphi_{l})_{11} \varphi_1 - 2w^{kl} \sigma_{k1} \varphi_l \varphi_1 + 2w^{kl} \sigma_{k1} (\varphi_1)^2 \]

\[= 2w^{kl}_{11} - 4w^{kl} \varphi_{k11} \varphi_1 - 2w^{kl} \sigma_{k1} \varphi_l \varphi_1 + 2w^{kl} \sigma_{kl} (\varphi_1)^2, \]

the second term in the last line of (6.8) can be rewritten as

\[2(\dot{\varphi} - \frac{\alpha}{n} \dot{\varphi} w^{kl}_{11} \varphi_{11}) = 2(-\frac{\alpha \dot{\varphi}}{n} w^{kl}_{11} \dot{w}_{11,kl} + \frac{2\dot{\varphi}}{n} \frac{\beta}{1 + |D\varphi|^2} \sigma^{kl} \varphi_{k\varphi_{11}} - \frac{\alpha}{n} \dot{\varphi} w^{kl}_{11} \varphi_{11}) \varphi_1 \\
= \frac{4\dot{\varphi}}{n} \frac{\beta}{1 + |D\varphi|^2} \sigma^{kl} \varphi_{k\varphi_{11}} \varphi_1 \\
+ \frac{2\alpha}{n} \dot{\varphi} w^{kl}_{11} \varphi_{l} \varphi_{k11} = -2 \frac{\alpha}{n} \dot{\varphi} \varphi_{k11} \varphi_1 - \sigma_{k1} \varphi_1 + \sigma_{kl}(\varphi_1)^2. \]

And the first term in the last line of (6.8) can be rewritten as

\[-\frac{2\dot{\varphi}}{n} \frac{\beta}{1 + |D\varphi|^2} \sigma^{kl} - \alpha w^{kl} \varphi_l \varphi_{k11} \\
= -\frac{2\dot{\varphi}}{n} \frac{\beta}{1 + |D\varphi|^2} \sigma^{kl} - \alpha w^{kl} \varphi_l (-\delta_{1k} \varphi_1 + \varphi_k + 2 \varphi_{11} \varphi_{1k}), \]

in view of (6.3), it follows that

\[-\frac{2\dot{\varphi}}{n} \frac{\beta}{1 + |D\varphi|^2} \sigma^{kl} - \alpha w^{kl} \varphi_l (-\delta_{1k} \varphi_1 + \varphi_k + 2 \varphi_{11} \varphi_{1k}) = 2\dot{\varphi} \frac{\beta}{1 + |D\varphi|^2} \sigma^{kl} \varphi_{k11} \varphi_{11} - Q_k w_{11,k} \]

\[= -\frac{2\dot{\varphi}}{n} \frac{\beta}{1 + |D\varphi|^2} \sigma^{kl} - \alpha w^{kl} \varphi_l (-\delta_{1k} \varphi_1 + \varphi_k + 2 \varphi_{11} \varphi_{1k}). \]

Therefore,

\[\mathcal{L} w_{11} = -\frac{(\dot{\varphi})^2}{\varphi} + \frac{\alpha}{n} \dot{\varphi} w^{kl}_{11} \dot{w}_{kl,1} + \frac{2\alpha}{n} \dot{\varphi} w^{ij}_{11} \varphi_{11} - \frac{2\alpha}{n} \dot{\varphi} (\text{tr} w^{ij} - n + w^{ij} \varphi_{ij}) \\
+ \frac{4\beta \dot{\varphi}}{n} \frac{1}{(1 + |D\varphi|^2)^2} (\sigma^{kl} \varphi_{k\varphi_{11}})^2 - \frac{2\beta \dot{\varphi}}{n} \frac{1}{1 + |D\varphi|^2} \sigma^{kl} \varphi_{k1} \varphi_{11} \\
+ \frac{2\dot{\varphi}}{n} \frac{\beta}{1 + |D\varphi|^2} \left((\varphi_1)^2 - |D\varphi|^2\right) + \frac{2\alpha}{n} \dot{\varphi} w^{kl} \left(-2 \sigma_{1k} \varphi_1 + \sigma_{kl}(\varphi_1)^2 + \varphi_k \varphi_l\right) \\
+ (\alpha - 1) \dot{\varphi}_{11}, \]
which implies that

\[
\mathcal{L}w_{11} = -\left(\frac{\dot{\varphi}_1}{\varphi}\right)^2 + \frac{\alpha}{n}\dot{\varphi}_1 w^{kl}_{11} + \frac{4\beta\dot{\varphi}}{n} \frac{1}{(1 + |D\varphi|^2)^2}(\sigma^{kl}\varphi_k\varphi_{l1})^2 - \frac{2\beta\dot{\varphi}}{n} \frac{1}{1 + |D\varphi|^2} \sigma^{kl}\varphi_{k1}\varphi_{l1} \\
+ \frac{2\beta}{n} \frac{\varphi_1}{1 + |D\varphi|^2} \left((\varphi_1)^2 - |D\varphi|^2 \right) - \frac{4\alpha}{n} \dot{\varphi}_1 \sigma_{kl}\varphi_{k1}\varphi_{l1} \\
+ \frac{2\alpha}{n} \dot{\varphi}_1 \sigma_{ij}\varphi_{i1}\varphi_{j1} (-w_{11} + 2(\varphi_1)^2 + n) + (1 - \alpha)\dot{\varphi}(w_{11} - (\varphi_1)^2 - 1).
\]

(6.9)

Now, we only leave the third equality to prove. Differentiating the function \(\gamma^k\varphi_k\) twice with \(x\), we have

\[
(\varphi_k\gamma^k)_{ij} = \varphi_{ki}\gamma^k + \varphi_k\gamma^k_{,ij}
\]

and

\[
(\varphi_k\gamma^k)_{,j} = \varphi_{kij}\gamma^k + 2\varphi_{ki}\gamma^k_{,j} + \varphi_k\gamma^k_{,ij}.
\]

Differentiating \(\varphi_k\gamma^k\) with \(t\), we have

\[
(\varphi_k\gamma^k)_t = \varphi_{kt} \cdot \gamma^k
= \left(Q^{ij}\varphi_{ij1} + Q^k\varphi_{kl} + (\alpha - 1)Q\varphi_l\right)_{,t}
= \left(Q^{ij}\varphi_{ij1} + Q^i_j\varphi_{j1} + Q^k\varphi_{kl} + (\alpha - 1)Q\varphi_l\right)_{,t}.
\]

Therefore,

\[
\mathcal{L}(\varphi_k\gamma^k) = (\varphi_k\gamma^k)_t - Q^{ij}(\varphi_k\gamma^k)_{ij} - Q^k(\varphi_k\gamma^k)_{,kt}
= Q^{ij}\left(\sigma_{ji}\varphi_{j1} - \sigma_{ij}\varphi_{i1}\gamma^l\right) - Q^k\varphi_{k1}\gamma^l_{,k}
+ (\alpha - 1)Q\varphi_l\gamma^l.
\]

\[\square\]

Let \(\mu\) be a smooth extension of the outward unit normal to \(\partial\Omega\) that vanishes outside a tubular neighborhood of \(\partial\Omega\). We define for \((x, \xi, t) \in \overline{\Omega} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times [0, T]\)

\[
w'(x, \xi_1, \xi_2, t) = -\mu^i_{,j}\varphi_i((\xi_1, \mu)\xi'_2 + (\xi_2, \mu)\xi'_1),
\]

where

\[
\xi'_i = \xi_i - (\xi_i, \mu)\mu
\]

indicate the tangential component of the vector \(\xi_i\), with \(i = 1, 2\) and where \((,\) is the inner product induced by \(\sigma\). Moreover, let \(w'_i(x, t) : \overline{\Omega} \times [0, T] \to \mathbb{R}^n\), with \(1 \leq i, j \leq n\), represent the component functions

\[
w'_{ij}(x, t) = -\mu^i_{,j}\varphi_q[\sigma_{ki}\mu^k(\delta^p_j - \sigma_{ij}\mu^p) + \sigma_{kj}\mu^k(\delta^p_i - \sigma_{li}\mu^p)],
\]

of the symmetric 2-tensor filed \(w'\).

**Remark 6.2.** \(w'(x, \xi_1, \xi_2, t)\) is not an important part in the following interior estimate, but will play a great role in later oblique (non-tangential and non-normal) boundary estimate.
We define for \((x, \xi, t) \in \Omega \times \mathbb{R}^n \times [0, T]\) as done in \[32\]

\[W(x, \xi, t) = \log \left( \frac{[w_{ij}(x, t) + w'_{ij}(x, t)]\xi^i \xi^j}{\sigma_{ij}\xi^i \xi^j} + C \right) + \frac{1}{2} \lambda |D\varphi|^2,\]

where \(C\) and \(\lambda\) are constants which will be chosen later.

**Proposition 6.4.** Let \(\varphi\) be a solution of the flow \[2.1\], assume \(W\) attains its maximum in \(\Omega \times S^{n-1} \times [0, T]\) for some fixed \(T < T^*\). Then, there exists \(C = C(n, M_0)\) such that

\[C(n, M_0) \leq \varphi_{ij}\xi^i \xi^j, \quad \forall (x, \xi, t) \in \Omega \times S^{n-1} \times [0, T].\]

**Proof.** Assume \(W(x, \xi, t)\) achieves its maximum at \((x_0, \xi_0, t_0) \in \Omega \times S^{n-1} \times [0, T]\). Choose Riemannian normal coordinates at \((x_0, t_0)\) such that at this point we have \(\sigma_{ij}(x_0) = \delta_{ij}, \partial_k \sigma_{ij}(x_0) = 0\).

And we further rotate the coordinate system at \((x_0, t_0)\) such that the matrix \(w_{ij} + w'_{ij}\) is diagonal, i.e.

\[w_{ij} + w'_{ij} = (w_{ii} + w'_{ii})\delta_{ij}\]

with

\[w_{nn} + w'_{nn} \leq \cdots \leq w_{22} + w'_{22} \leq w_{11} + w'_{11}.\]

Thus, at \((x_0, t_0)\)

\[(6.10) \quad |w_{ii}| \leq w_{11} + c \quad \text{and} \quad |w_{ij}| \leq c \quad \text{for} \quad i \neq j\]

in view of the \(C^1\)-estimate \[6.27\]. Set \(\xi_1(x) = (1, 0, ..., 0)\) around a neighbor of \(x_0\). Clearly, \(\xi_1(x_0) = \xi_0\) and there holds at \((x_0, t_0)\)

\[w_{11} + w'_{11} = \sup_{\xi \in S^n} \left[ \frac{[w_{ij}(x, t) + w'_{ij}(x, t)]\xi^i \xi^j}{\sigma_{ij}\xi^i \xi^j} \right.\]

and in a neighborhood of \((x_0, t_0)\)

\[w_{11} + w'_{11} \leq \sup_{\xi \in S^n} \left[ \frac{[w_{ij}(x, t) + w'_{ij}(x, t)]\xi^i \xi^j}{\sigma_{ij}\xi^i \xi^j} \right].\]

Furthermore, it is easy to check that the covariant (at least up to the second order) and the first time derivatives of

\[\frac{[w_{ij}(x, t) + w'_{ij}(x, t)]\xi^i \xi^j}{\sigma_{ij}\xi^i \xi^j}\]

and

\[w_{11} + w'_{11}\]

do coincide at \((x_0, t_0)\) (in normal coordinate). Without loss of generality, we treat \(w_{11} + w'_{11}\) like a scalar and pretend that \(W\) is defined by

\[W(x, t) = \log(w_{11} + w'_{11} + C) + \frac{1}{2} \lambda |D\varphi|^2,\]

which achieves its maximum at \((x_0, t_0) \in \Omega \times [0, T]\). Here, noticing that we can choose \(C\) large enough satisfying

\[(6.11) \quad 0 \leq w'_{11} + C,\]

since \(w'_{11}\) is bounded by the \(C^1\) estimate \[6.27\].
In the following, we want to compute
\[ \mathcal{L}W = W - Q^{ij}W_{ij} - Q^kW_k \]
\[ = \mathcal{L}(\log(w_{11} + w'_1 + C)) + \frac{1}{2} \lambda \mathcal{L}(|D\varphi|^2). \]

First, after a simple calculation, we can rewrite the first term in the following form
\[ \mathcal{L}(\log(w_{11} + w'_1 + C)) = \frac{\mathcal{L}w_{11}}{w_{11} + w'_1 + C} + \frac{\mathcal{L}w'_1}{w_{11} + w'_1 + C} + \frac{\alpha}{n} \varphi w^{ij}(w_{11;i} + w'_{1;i})(w_{11;j} + w'_{1;j})}{(w_{11} + w'_1 + C)^2}. \]

Now, we begin to estimate \( \mathcal{L}w_{11} \) through the evolution (6.6). Using the Cauchy-Schwarz inequality, the second line of (6.6) takes the form
\[ \frac{2\beta\dot{\varphi}}{n} \left( \frac{2}{1 + |D\varphi|^2}(\sigma^{kl}\varphi_{k\varphi_{1l}})^2 - \sigma^{kl}\varphi_{k\varphi_{1l}} \right) \]
\[ \leq \frac{2\beta\dot{\varphi}}{n} \left( \frac{2}{1 + |D\varphi|^2}\sigma^{kl}\varphi_{k\varphi_{1l}} \right) \leq \frac{2\beta\dot{\varphi}}{n} \left( \frac{2}{1 + |D\varphi|^2}\sigma^{kl}\varphi_{k\varphi_{1l}} \right). \]

On the other hand,
\[ \sigma^{kl}\varphi_{k\varphi_{1l}} = \sigma^{kl}w_{k1}w_{1l} + \sigma_{11} - 2w_{11} + 2(\varphi_1)^2 - 2\varphi_1\sigma^{kl}w_{k1}\varphi_l + (\varphi_1)^2|D\varphi|^2. \]

Using (6.10), together with the \( C^1 \)-estimate (6.27), (6.12) becomes
\[ \frac{2\beta\dot{\varphi}}{n} \left( \frac{2}{1 + |D\varphi|^2}(\sigma^{kl}\varphi_{k\varphi_{1l}})^2 - \sigma^{kl}\varphi_{k\varphi_{1l}} \right) \]
\[ \leq \frac{2\beta\dot{\varphi}}{n} \left( \frac{2}{1 + |D\varphi|^2}(\sigma^{kl}\varphi_{k\varphi_{1l}}) \right) \leq \frac{2\beta\dot{\varphi}}{n} \left( \frac{2}{1 + |D\varphi|^2}(\sigma^{kl}w_{k1}w_{1l} + cw_{11} + c) \right). \]

Inserting (6.13) into \( \mathcal{L}w_{11} \), abandoning the non-positive terms and using the \( C^1 \)-estimate (6.27) again, we obtain
\[ \mathcal{L}w_{11} \leq \frac{2\beta\dot{\varphi}}{n} \left( \frac{2}{1 + |D\varphi|^2}(\sigma^{kl}w_{k1}w_{1l} + \varphi(trw^{ij} + cw_{11} + c) + \frac{\alpha}{n} \varphi w^{kl}_{1}w_{kl1} \right). \]

Next, recalling (6.5)
\[ \lambda \mathcal{L}(\frac{1}{2}|D\varphi|^2) = -\frac{\alpha}{n} \left( w_{ij}\sigma^{ij} + (1 + |D\varphi|^2)w^{ij}\varphi_{i\varphi_j} - 2|D\varphi|^2 \right) \]
\[ -\alpha\lambda\dot{\varphi}trw^{ij} + c\lambda\dot{\varphi}. \]

Then, it follows in view of (6.14) and (6.15)
\[ \mathcal{L}W \leq \frac{1}{w_{11} + w'_1 + C} \left( \frac{2\beta\dot{\varphi}}{n} \left( \frac{2}{1 + |D\varphi|^2}(\sigma^{kl}w_{k1}w_{1l} + \varphi(trw^{ij} + cw_{11} + c) + \frac{\alpha}{n} \varphi w^{kl}_{1}w_{kl1} \right) \right) \]
\[ + \frac{\mathcal{L}w'_{11}}{w_{11} + w'_1 + C} + \frac{\alpha}{n} \varphi w^{ij}(w_{11;i} + w'_{1;i})(w_{11;j} + w'_{1;j})}{(w_{11} + w'_1)^2} \]
\[ - \frac{\alpha}{n} \left( w_{ij}\sigma^{ij} + (1 + |D\varphi|^2)w^{ij}\varphi_{i\varphi_j} - |D\varphi|^2 \right) \]
\[ - \alpha\lambda\dot{\varphi}trw^{ij} + c\lambda\dot{\varphi}. \]
To make progress, we need to estimate

\[
\frac{1}{w_{11} + w_{11}^t + C} \frac{2\beta\dot{\varphi}}{n} \frac{1}{1 + |D\varphi|^2} \sigma^{kl} w_{k1} w_{l1} - \frac{\alpha\lambda\dot{\varphi}}{n} w_{ij} \sigma^{ij}
\]

\[
\leq c\dot{\varphi} \left( \frac{(w_{11})^2}{w_{11} + w_{11}^t + C} - \lambda w_{11} \right)
\]

\[
\leq c(1 - \lambda) \dot{\varphi} w_{11}
\]

in view of (6.10) and (6.11). Now, we only leave the term \( Lw'_{11} \) to estimate. Clearly, \( w'_{11} \) can be rewrite as

\[
w'_{11} = \gamma^i \varphi_i + C
\]

with \( \gamma^i : \Omega \to \mathbb{R} \) that does not depend on \( \varphi \). Recalling (6.7), we have

\[
L(w'_{11}) = Q^{ij} \left( \sigma_i \dot{\varphi} \gamma^j - \sigma_{ij} \dot{\varphi} \gamma^j \right) - Q^{k} \dot{\varphi} \sigma^j \gamma^i, k + (\alpha - 1) Q \dot{\varphi} \gamma^i.
\]

In view of

\[
w^{ij} \varphi_{lj} = w^{ij} \sigma_{lj} - \delta^j_l + w^{ij} \varphi_j \varphi_l,
\]

we obtain by the \( C^1 \)-estimate (6.27)

\[
L w'_{11} \leq c\dot{\varphi} (\operatorname{tr} w^{ij} + 1) + \frac{2\alpha\dot{\varphi}}{n} w^{ij} \varphi_i w'_{11;j}.
\]

Thus,

\[
 LW \leq \frac{\dot{\varphi}}{w_{11} + w_{11}^t + C} \left( \operatorname{tr} w^{ij} + cw_{11} + c + \frac{\alpha}{n} \dot{\varphi} w^{kl} w_{kl};1 \right)
\]

\[
+ \frac{\dot{\varphi}}{w_{11} + w_{11}^t + C} \left( c \operatorname{tr} w^{ij} + c + \frac{2\alpha\dot{\varphi}}{n} w^{ij} \varphi_i w'_{11;j} \right) + \frac{\alpha}{n} \dot{\varphi} w^{ij} \frac{(w_{11;i} + w'_{11;i})(w_{11;j} + w'_{11;j})}{(w_{11} + w_{11}^t + C)^2}
\]

\[
- \frac{\alpha}{n} \dot{\varphi} \operatorname{tr} w^{ij} + c\lambda \dot{\varphi} + c\dot{\varphi} \left( \frac{(w_{11})^2}{w_{11} + w_{11}^t + C} - \lambda w_{11} \right).
\]

The last term which we have to estimate is

\[
\frac{\alpha}{n} \dot{\varphi} \left( \frac{1}{w_{11} + w_{11}^t + C} \sigma^{kl} w_{kl};1 + w^{ij} \frac{(w_{11;i} + w'_{11;i})(w_{11;j} + w'_{11;j})}{(w_{11} + w_{11}^t + C)^2} \right).
\]

For convenience later, we set \( V = w_{11} + w_{11}^t + C \). Then,

\[
\frac{1}{V} w^{kl} w_{kl};1 + w^{ij} \frac{V_i V_j}{V^2}
\]

\[
= - \frac{1}{V} w^{pq} w^{kl} w_{pq};1 w_{kl};1 + w^{ij} \frac{V_i V_j}{V^2}
\]

\[
\leq - \frac{1}{V} w^{kl} w_{kl};1 w_{kl};1 + w^{ij} \frac{V_i V_j}{V^2}
\]

\[
= w^{ij} \frac{V_i V_j}{V w_{11}} - \frac{1}{V} w^{kl} w_{kl};1 w_{kl};1 - \frac{w_{11} + C}{V^2 w_{11}} w^{ij} V_i V_j.
\]

In view of (6.11), together with the fact that the matrix \( w^{ij} \) is positive definite, we can say

\[
- \frac{w_{11} + C}{V^2 w_{11}} w^{ij} V_i V_j \leq 0.
\]
Thus,

\[ \frac{1}{V} w^{kl} w_{kl,1} + w^{ij} \frac{V_i V_j}{V^2} \leq \frac{1}{V w_{11}} (w^{ij} V_i V_j - w^{kl} w_{11;k} w_{11,l}). \]

Recalling that

\[ w^{ij} V_i V_j = w^{ij} (w_{11;k} w_{11;l} + 2 w_{11;k} w_{11;l}' + w_{11;k}' w_{11;l}). \]

It follows from the equality (6.2)

\[ \frac{1}{V} w^{kl} w_{kl,1} + w^{ij} \frac{V_i V_j}{V^2} \]

(6.16)

\[ \leq \frac{1}{V w_{11}} (2 w^{kl} w_{11,k} \varphi_{1} w_{11} - 2 w_{11;l} \varphi_{1} - (w_{11})^2 w^{kl} \varphi_{k} \varphi_{l} + w_{11} (\varphi_{1})^2 + 2 w^{kl} w_{11,k} w_{11,l}' + w^{kl} w_{11;k} w_{11;l}). \]

Since \( W(x, t) \) achieves its maximum at \( (x_0, t_0) \in \Omega \times [0, T] \), so \( W_t = 0 \) implies

\[ W_i = \frac{V_i}{V} + \lambda \sigma^{kl} \varphi_{k} \varphi_{l} = 0. \]

Therefore,

(6.17)

\[ w_{11;l} = (-\lambda V \sigma^{kl} \varphi_{k} \varphi_{l} - w_{11;l}') \]

and

\[ w^{kl} w_{11;k} = w^{kl} (-\lambda V \sigma^{pq} \varphi_{p} \varphi_{q} - w_{11;k}'). \]

Thus, we have by the \( C^1 \)-estimate (6.27)

\[ \frac{1}{V w_{11}} (2 w^{kl} w_{11;k} \varphi_{1} w_{11} + 2 w^{kl} w_{11;k} w_{11;l}' + w^{kl} w_{11;k} w_{11;l}') \]

\[ \leq -2 \lambda (1 + |D\varphi|^2) w^{kl} \varphi_{k} \varphi_{l} + c \lambda \frac{1 + tr w^{ij}}{w_{11}} + c \lambda \]

\[ - \frac{2}{V} w^{kl} \varphi_{l} w_{11,l}' - \frac{1}{V w_{11}} w^{kl} w_{11;k} w_{11;l}' \]

\[ \leq -2 \lambda (1 + |D\varphi|^2) w^{kl} \varphi_{k} \varphi_{l} + 2 \lambda |D\varphi|^2 + c \lambda - \frac{2}{V} w^{kl} \varphi_{l} w_{11;k} \]

in view of

\[ w_{11;l}' = \varphi_{k} \gamma_{k}^{l} + \varphi_{k} \gamma_{k,l}' \cdot \]

Inserting the above equality and (5.17) into (6.16), we get at \( (x_0, t_0) \)

(6.18)

\[ \frac{1}{V} w^{kl} w_{kl,1} + w^{ij} \frac{V_i V_j}{V^2} \]

\[ \leq -2 \lambda (1 + |D\varphi|^2) w^{kl} \varphi_{k} \varphi_{l} + 2 \lambda |D\varphi|^2 + c \lambda - \frac{2}{V} w^{kl} \varphi_{l} w_{11;k} + c \lambda \frac{1 + tr w^{ij}}{w_{11}} + c \frac{1 + w_{11}}{V w_{11}} \cdot \]
Thus,
\[
\mathcal{L}W \leq \frac{\dot{\varphi}}{V} \left( \text{tr} w^{ij} + cw_{11} + c \right) + \frac{\dot{\varphi}}{V} \left( c \text{tr} w^{ij} + c + \frac{2\alpha \dot{\varphi}}{n} w^{ij} \varphi_i w'_{11;j} \right) - \frac{\alpha \lambda \dot{\varphi}}{n} \left( 1 + |D\varphi|^2 \right) w^{ij} \varphi_i \varphi_j - 2|D\varphi|^2 \\
- \alpha \lambda \dot{\varphi} \text{tr} w^{ij} + c\lambda \dot{\varphi} + c\dot{\varphi}(1-\lambda)w_{11} \\
\frac{\alpha \dot{\varphi}}{n} \left( 2(1 + |D\varphi|^2)w^{kl} \varphi_k \varphi_l + c\lambda - \frac{2}{V} w^{kl} \varphi_l w'_{11;k} + c\lambda \frac{1 + \text{tr} w^{ij}}{w_{11}} + \frac{1 + w_{11}}{V w_{11}} \right) \\
\leq \frac{\dot{\varphi}}{V} \left( \text{tr} w^{ij} + cw_{11} + c \right) + \frac{\dot{\varphi}}{V} \left( c \text{tr} w^{ij} + c \right) - \frac{\alpha \lambda \dot{\varphi}}{n} \left( 3(1 + |D\varphi|^2)w^{ij} \varphi_i \varphi_j - 4|D\varphi|^2 \right) \\
- \alpha \lambda \dot{\varphi} \text{tr} w^{ij} + c\lambda \dot{\varphi} + c\dot{\varphi}(1-\lambda)w_{11} \\
\frac{\alpha \dot{\varphi}}{n} \left( c\lambda + \frac{1 + \text{tr} w^{ij}}{w_{11}} + c \frac{1 + w_{11}}{V w_{11}} \right) \\
\leq \frac{\dot{\varphi}}{V} \left( \text{tr} w^{kl} + cw_{11} + c \right) - \alpha \lambda \dot{\varphi} \text{tr} w^{kl} + c\lambda \dot{\varphi} + c\dot{\varphi}(1-\lambda)w_{11} \\
+ \frac{\alpha \dot{\varphi}}{n} \left( c\lambda + \frac{1 + \text{tr} w^{kl}}{w_{11}} + c \frac{1 + w_{11}}{V w_{11}} \right) \\
\leq \dot{\varphi} \text{tr} w^{kl} \left( \frac{c}{w_{11}} + c - c\lambda \right) + c\dot{\varphi} \left( 1 + \lambda + \frac{1}{w_{11}} + \frac{1}{w_{11}^2} + (1-\lambda)w_{11} \right).
\]

Assume $w_{11} \geq 1$, otherwise $w_{11}$ is upper bounded. Then, choosing $\lambda > 2$ and in view of $\mathcal{L}W \geq 0$, we obtain
\[
(\lambda - 1)w_{11} \leq 1 + \lambda + \frac{1}{w_{11}} + \frac{1}{w_{11}^2},
\]
we conclude that $w_{11}$ has upper bounded. Thus, the second covariant derivatives of $\varphi$ is bounded from below. 

6.2. Double normal $C^2$ boundary estimates.

Let
\[
\tilde{\mathcal{L}}U = \dot{U} - Q^{ij} U_{ij} - \frac{2\beta}{n} \frac{\dot{\varphi}}{1 + |D\varphi|^2} \varphi^k U_k = \dot{U} - \frac{\alpha}{n} \dot{\varphi} w^{ij} U_{ij} - \frac{2\beta}{n} \frac{\dot{\varphi}}{1 + |D\varphi|^2} \varphi^k U_k
\]
and
\[
q(x) = -d(x) + \eta d^2(x),
\]
where $d$ denotes the distance to $\partial \Omega$ which is a smooth function in $\Omega_\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \delta \}$ for $\delta$ small enough and $\eta$ denotes a constant to be chosen sufficiently large. Thus, $q : \Omega_\delta \to \mathbb{R}$ is a smooth function.

To derive double normal $C^2$ boundary estimates, we need the following lemma.
Lemma 6.5. For any solution \( \varphi \) of the flow (2.1), we can choose \( \eta \) so large and \( \delta \) so small such that
\[
\tilde{L} q(x) \leq -\frac{c_3}{4} k_0 \dot{\varphi} \cdot \text{tr}(w^{ij}),
\]
where \( \min_{x \in \Omega(x,0)} \frac{\dot{\varphi}}{n} = c_3 > 0 \).

Proof. Derivative the function \( q \) twice with \( x \),
\begin{equation}
q_i(x) = -d_i(x) + 2\eta d_i(x)
\end{equation}
and
\begin{equation}
q_{ij}(x) = -d_{ij}(x) + 2\eta d_i(x)d_j(x) + 2\eta d_i(x) d_{ij}(x).
\end{equation}
For any \( x_0 \in \partial \Omega \), after a rotation of the first \( n-1 \) coordinates and remembering that \( \mu(x_0) = e_n \), we have
\[
d_{ij}(x_0) = \begin{pmatrix}
\kappa_1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & -\kappa_{n-1} \\
0 & 0 & \cdots & 0
\end{pmatrix},
\]
where there is a constant \( k_0 = k_0(\partial \Omega) > 0 \) such that \( \kappa_i \geq k_0 \) for all principle curvature \( \kappa_i, i = 1, 2, \ldots, n-1 \) of \( \partial \Omega \) and for any \( x_0 \in \partial \Omega \). Since the differential of the distance coincide with the inward normal vector \( -Dd(x_0) = \mu(x_0) = e_n \). Thus, it holds at \( x_0 \)
\[
q_{ij}(x_0) = \begin{pmatrix}
\kappa_1(1-2\eta d) & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \kappa_{n-1}(1-2\eta d) \\
0 & 0 & \cdots & 2\eta
\end{pmatrix}.
\]
Choosing \( \eta \delta \leq \frac{1}{4} \), we have
\[
w^{ij}q_{ij} \geq \frac{1}{2} k_0 (w^{11} + w^{22} + \ldots + w^{n-1 \cdot n-1}) + 2\eta w^{nn}.
\]
Again choosing \( \eta \geq \frac{1}{4} k_0 \), we have
\[
w^{ij}q_{ij} \geq \frac{1}{2} k_0 \text{tr}(w^{ij}).
\]
Using the inequality of arithmetic and geometric, we obtain
\[
w^{ij}q_{ij} \geq c(n, k_0) \eta \frac{1}{n} \left( \prod_{i=1}^{n} w^{ii} \right)^{\frac{1}{n}}
\]
The Hadamard inequality for positive definite matrices
\[
\det(w^{ij}) \leq \left( \prod_{i=1}^{n} w^{ii} \right)
\]
implies
\[
w^{ij}q_{ij} \geq c(n, k_0) \eta \frac{1}{n} \det(w^{ij})^{\frac{1}{n}}
\]
Recalling (5.3), there is a positive constant \( c_2 \) such that
\[
\det(w^{ij}) = \det^{-1}(w_{ij}) \geq \frac{1}{c_2} > 0,
\]
it follows that
\[ w_{ij}q_{ij} \geq \frac{1}{c_2} c(n, k_0) \eta \frac{1}{n}. \]

Using the $C^1$-estimate (6.27), we have
\[ \left| \frac{2\beta}{n} \frac{1}{1 + |D\varphi|^2} \varphi^k q_k \right| = \left| \frac{2\beta}{n} \frac{1}{1 + |D\varphi|^2} \varphi^k (-d_k + 2\eta dd_k) \right| \leq c_3 (1 + \eta \delta), \]
for all $(x, t) \in \Omega_\delta \times [0, T]$. Choose $\eta$ so large and $\delta$ so small such that
\[ \frac{1}{2} \frac{1}{c_2} c(n, k_0) \eta \frac{1}{n} \geq c_3 (1 + \eta \delta). \]

Thus,\[ \tilde{L} q(x) \leq -\frac{c_3}{4} k_0 \varphi tr(w_{ij}). \]

Clearly, choosing $\frac{1}{8} \leq \eta \delta \leq \frac{1}{4}$, from (6.20) and (6.21), we make sure that $q$ satisfied the following properties in $\Omega_\delta$:
\[ -\delta + \eta \delta^2 \leq q(x) \leq 0, \]
\[ \frac{1}{2} \leq |Dq| \leq 1, \]
\[ \frac{k_0}{2} \sigma_{ij} \leq D^2 q \leq (k_0 + \eta) \sigma_{ij} \]
and
\[ |D^3 q| \leq C(k_0 + \eta). \]

It is easy to see
\[ \frac{Dq}{|Dq|} = \mu \]
for unit outer normal $\mu$ on the boundary $\partial \Omega$. We consider the following function
\[ P(x, t) = D\varphi \cdot Dq + Aq(x), \]
where the constant $A$ will be choose later.

**Lemma 6.6.** For any solution $\varphi$ of the flow (2.1) in $\Omega \times [0, T]$ for some fixed $T < T^*$, we have
\[ \tilde{\mathcal{L}} P(x, t) \leq 0. \]

**Proof.** The calculation of $\tilde{\mathcal{L}} P(x, t)$ is similar to that of (6.7). We derivative this function $P(x, t)$ twice with $x$
\[ P_i = \sum_l \varphi_l q_l + \sum_l \varphi q_{li} + Aq_i \]
and
\[ P_{ij} = \sum_l \varphi_{lij} q_l + 2 \sum_l \varphi_{li} q_{lj} + \sum_l \varphi_{lij} + Aq_{ij}. \]
Differentiating \( P(x, t) \) with \( t \), we have
\[
P_t = D \varphi_t \cdot Dq
\]
\[
= \sum_l \left( Q^{ij} \varphi_{ijl} + Q^k \varphi_{kl} + (\alpha - 1)Q \varphi_l \right) q_l
\]
\[
= \sum_l \left( Q^{ij} \varphi_{ijl} + Q^j \varphi_j \sigma_{il} - Q^j \varphi_i \sigma_{lj} + Q^k \varphi_{kl} + (\alpha - 1)Q \varphi_l \right) q_l.
\]
Therefore, we have
\[
\tilde{L} P(x, t) = P_t - Q^{ij} P_{ij} - 2\beta \frac{\dot{\varphi}}{1 + |D\varphi|^2} \varphi^k P_k
\]
\[
= -2Q^{ij} \varphi_{ijl} q_j + Q^{ij} \left( \sigma_{il} \varphi_j q_l - \varphi_{ij} \varphi_l q_l \right) - 2Q^{ij} \varphi_{ijl} q_j
\]
\[
- Q^{ij} \varphi_{lqij} - 2\beta \frac{\dot{\varphi}}{1 + |D\varphi|^2} \varphi^k \varphi_{lqk}
\]
\[
- 2Q^{kl} \varphi_{lqkm} q_m + (\alpha - 1)Q \varphi_l q_l + A \mathcal{L} q(x).
\]
Since
\[
w^{ij} \varphi_{ij} = w^{ij} \sigma_{ij} - \delta^i_l + w^{ij} \varphi_j \varphi_l,
\]
we obtain by using the \( C^1 \)-estimate (6.27) and (6.22), (6.23), (6.24)
\[
\tilde{L} P(x, t) \leq C(1 + \eta) \text{tr} Q^{ij} + C(1 + \eta) \dot{\varphi} + A \mathcal{L} q(x).
\]
Using Lemma 6.5, we get
\[
\tilde{L} P(x, t) \leq C \dot{\varphi} \left( (1 + \eta - A) \text{tr} w^{ij} + (1 + \eta) \right).
\]
Recalling (5.3),
\[
\left( \frac{\text{tr}(w^{ij})}{n} \right)^n = \det(w^{ij}) = \det^{-1}(w_{ij}) \geq \frac{1}{c_2} > 0.
\]
Choosing \( A \geq \frac{c_2}{n}(1 + \eta) + \eta + 1 \), we get
\[
\tilde{L} P(x, t) \leq 0.
\]

Proposition 6.7. For \( \varphi \) be a solution of the flow (2.1) in \( \Omega \times [0, T] \) for some fixed \( T < T^* \), \( \varphi_{\mu \mu} \) is uniformly bounded from below, i.e., there exists \( C = C(n, \mu_0) \) such that
\[
- \varphi_{\mu \mu} \leq C(n, \mu_0), \quad \forall (x, t) \text{ on } \partial \Omega \times [0, T].
\]

Proof. It is easy to see from the boundary condition in the flow (2.1)
\[
P = 0 \text{ on } \partial \Omega \times [0, T].
\]
On the \( \partial \Omega_\delta \setminus \partial \Omega \times [0, T] \), we have
\[
P \leq C - A \delta \leq 0,
\]
provided \( A \geq \frac{C}{\mu} \). Applying the maximum principle, it follows that
\[
P \leq 0 \text{ in } \Omega_\delta \times [0, T].
\]
Assume \((x_0, t_0) \in \partial \Omega \times [0, T]\) is the minimum point of \(\varphi_{\mu \mu}\) on \(\partial \Omega \times [0, T]\), using the \(C^1\)-estimate \((6.27)\), we have by noticing \((6.25)\)

\[
0 \leq P_\mu(x_0, t_0) = \varphi_{i\mu} q_i + \varphi_{i\mu} A q_i + A \varphi + C + A.
\]

Therefore,

\[
-\varphi_{\mu \mu} \leq C + A.
\]

\(\square\)

6.3. Remaining \(C^2\) boundary estimates.

We have obtained the interior estimates and the double normal boundary estimates. It is easy to get the second tangential-normal derivative estimates on the boundary. We shall follow the same discussion as in Lions-Trudinger-Urbas in [28].

**Proposition 6.8.** Let \(\varphi\) be a solution of the flow \((2.21)\) in \(\Omega \times [0, T]\) for some fixed \(T < T^*\), assume \(W\) attains its maximum on the boundary of \(\Omega \times S^{n-1} \times [0, T]\). Then, there exists \(C = C(n, M_0)\) such that

\[
C(n, M_0) \leq \varphi_{ij}(x, t) \xi^i \xi^j, \quad \forall (x, \xi, t) \in \overline{\Omega} \times S^{n-1} \times [0, T].
\]

**Proof.** Assume \(W\) attains its maximum at a point \((x_0, \xi_0, t_0) \in \partial \Omega \times S^{n-1} \times [0, T]\). From Proposition 6.7 we know

\[
C(n, M_0) \leq \varphi_{\mu \mu}, \quad \forall (x, t) \in \partial \Omega \times [0, T^*].
\]

Thus, the remaining case is \(\xi_0 \neq \mu\). Without loss of generality that \(W\) attains its maximum at a point \((x_0, \xi_0, t_0) \in \partial \Omega \times S^{n-1} \times [0, T]\) with \(\xi_0 \neq \mu\). We represent \(\partial \Omega\) locally as graph \(f\) over its tangent plane at a fixed point \(x_0 \in \partial \Omega\) such that \(\Omega = \{(x^n, \widehat{x}) : x^n > f(\widehat{x})\}\) and we distinguish two cases.

(i) \(\xi_0\) is tangential: if \(\xi_0\) is tangential to \(\partial \Omega\), we differentiate the boundary condition

\[
\mu^i \varphi_i = 0
\]

with respect to tangential directions \(\xi\)

\[
\mu^i \varphi_i + \mu^i \varphi_{i\xi} + \mu^i \varphi_{i\eta} f_\xi = 0,
\]

then at \(x_0\)

\[
\mu^i \varphi_i + \mu^i \varphi_{i\xi} = 0
\]

in view of \(Df(\widehat{x}_0) = 0\), which implies

\[
|\mu^i \varphi_{i\xi}| \leq c.
\]

We differentiate the boundary condition again and we get in view of \(Df(\widehat{x}_0) = 0\)

\[
\mu^i \varphi_{i\xi} + 2\mu^i \varphi_{i\eta} + \mu^i \varphi_{i\xi\xi} + \mu^i \varphi_{i\eta\xi} + \mu^i \varphi_{i\eta\eta} f_{\xi\xi} = 0.
\]

The \(C^1\)-estimate and the double normal estimate provide at \(x_0\)

\[
\mu^i \varphi_{i\xi} \leq c
\]

and

\[
\mu^i \varphi_{i\eta\xi} \leq c
\]

in view of \(D^2 f(\widehat{x}_0) > 0\). So we obtain

\[
\varphi_{\mu \xi_0} \geq -2\mu^i \varphi_{i\xi_0} \xi_0 - c \geq 2\mu^i \xi_0 (1 + \varphi_i \varphi_{\xi_0} - \varphi_{i\xi_0}) - c \geq 2w_{ij}(x_0, t_0) \xi^i \xi^j - c,
\]
as \( \partial \Omega \) is strictly convex. On the other hand the maximality of \( W \) at \( x_0 \) gives \( 0 \leq W_\mu \).

\[
0 \leq \frac{w_{\xi_0\xi_0;\mu} + w'_{\xi_0\xi_0;\mu}}{V} + \lambda \varphi^i \varphi_{i\mu}.
\]

Since \( \partial \Omega \) is strictly convex, so

\[
\lambda \varphi^i \varphi_{i\mu} = -\lambda \varphi^i \mu^j \varphi_j \leq -c \lambda |D\varphi|^2 \leq 0,
\]

which implies that

\[
0 \leq -\varphi_{\xi_0\xi_0;\mu} + c,
\]

together with

\[
-\varphi_{\xi\xi\nu} = -\varphi_{\nu\xi\xi} + R_{\nu\xi\xi} \varphi^i,
\]

imply

\[
W(x_0, \xi_0, t_0) \leq c.
\]

So we obtain the desired estimate

\[
C(n, \Sigma_0) \leq \varphi_{ij}(x, t) \xi^i \xi^j, \quad \forall (x, \xi, t) \in \Omega \times S^{n-1} \times [0, T].
\]

(ii) \( \xi_0 \) is non-tangential: if \( \xi_0 \) is neither tangential nor normal we need the tricky choice of \[28\]. We find \( 0 < \vartheta < 1 \) and a tangential direction \( \tau \) such that

\[
\xi_0 = \vartheta \tau + \sqrt{1 - \vartheta^2} \mu.
\]

Thus,

\[
\varphi_{\xi_0\xi_0} = \vartheta^2 \varphi_{\tau\tau} + (1 - \vartheta^2) \varphi_{\mu\mu} + 2 \vartheta \sqrt{1 - \vartheta^2} \varphi_{\tau\mu}.
\]

Differential the boundary condition at a point on the boundary, we have

\[
\mu^i_{\jmath} \varphi_i = -\mu^i \varphi_{ij}.
\]

Therefore, at the boundary point

\[
w'(x, \xi_0, \xi_0, t) = -2 \mu^i_{\jmath} \varphi_i(\xi_0, \mu) \xi^i_0 = 2 \vartheta \sqrt{1 - \vartheta^2} w_{\tau\mu},
\]

and consequently,

\[
\varphi_{\xi_0\xi_0} = \vartheta^2 \varphi_{\tau\tau} + (1 - \vartheta^2) \varphi_{\mu\mu} - w'_{\xi_0\xi_0}.
\]

Thus, in view of the Neumann boundary condition,

\[
w_{\xi_0\xi_0} + w'_{\xi_0\xi_0} = 1 + \vartheta^2 \varphi_{\tau\tau} - (\vartheta^2 \varphi_{\tau\tau} + (1 - \vartheta^2) \varphi_{\mu\mu}),
\]

which means we can rewrite \( \exp(W - \frac{1}{2} \lambda |D\varphi|^2) - C \) as

\[
1 + \vartheta^2 \varphi_{\tau\tau} - (\vartheta^2 \varphi_{\tau\tau} + (1 - \vartheta^2) \varphi_{\mu\mu}),
\]

so we obtain in view of the maximality of \( W \) and the fact that \( \exp(W - \frac{1}{2} \lambda |D\varphi|^2) - C \) is independent of \( \xi \)

\[
1 + \varphi_{\tau\tau} - \varphi_{\tau\tau} \leq 1 + \vartheta^2 \varphi_{\tau\tau} - (\vartheta^2 \varphi_{\tau\tau} + (1 - \vartheta^2) \varphi_{\mu\mu}),
\]

which implies

\[
-\varphi_{\tau\tau} \leq -\varphi_{\mu\mu}.
\]

Therefore,

\[
W_{\tau\tau} \leq W_{\mu\mu} + c.
\]
So

$$W_{ξ0ξ0} ≤ c,$$

in view of (6.7). Thus, we obtain the desired estimate

$$C(n, Σ₀) ≤ φ_{ij}(x, t)ξ^iξ^j, \quad ∀(x, ξ, t) ∈ Ω \times S^{n-1} \times [0, T].$$

\[\square\]

**Theorem 6.9.** Under the hypothesis of Theorem 1.1, we conclude

$$T^* = +∞.$$

**Proof.** Recalling that φ satisfies the equation (2.1)

$$\frac{∂φ}{∂t} = Q(x, φ, Dφ, D^2φ).$$

By a simple calculation, we get

$$\frac{∂Q}{∂φ_{ij}} = \frac{α n}{n} e^{(α-1)φ} (1 + |Dφ|^2) \frac{∂φ}{det φ(σ_{kl})} w_{ij},$$

which is uniformly parabolic on finite intervals from $C^0$-estimate (3.2), $C^1$-estimate (6.27) and the estimate (5.3). Then by Krylov-Safonov estimate [20], or the results of chapter 14 in [22], we have

$$|φ|_{C^{2,α}(Ω)} ≤ C(n, M₀, T^*),$$

which implies the maximal time interval is unbounded, i.e., $T^* = +∞.$

\[\square\]

6.4. **Convergence of the rescaled flow.**

Now, we define the rescaled flow by

$$\tilde{X} = X Θ^{-1}.$$

Thus,

$$\tilde{u} = u Θ^{-1},$$

$$\tilde{φ} = φ - log Θ,$$

and the rescaled Gauss curvature

$$\tilde{K} = K Θ^n.$$

Then, the rescaled scalar curvature equation takes the form

$$\frac{∂}{∂t} \tilde{u} = v K^{-\frac{α}{n}} \tilde{u}^{α-1} - \tilde{u}^{α-1}.$$

Defining $s = s(t)$ by the relation

$$\frac{ds}{dt} = Θ^{α-1},$$

such that $s(0) = 0$ we conclude that $s$ ranges from 0 to $+∞$ and $\tilde{u}$ satisfies

$$\frac{∂}{∂s} \tilde{u} = v \tilde{K}^{-\frac{α}{n}} - \tilde{u},$$

or equivalently, with $\tilde{φ} = log \tilde{u}$

$$\frac{∂}{∂s} \tilde{φ} = v \tilde{u}^{-1} \tilde{K}^{-\frac{α}{n}} - 1 = \tilde{Q}(\tilde{φ}, D\tilde{φ}, D^2\tilde{φ}).$$

(6.26)
Since the spatial derivatives of $\tilde{\varphi}$ are identical to those of $\varphi$, \(6.26\) is a nonlinear parabolic equation with a uniformly parabolic and concave operator $\tilde{K}$. Then, similar to do in the $C^1$ estimate, we can deduce a decay estimate of $\tilde{\varphi}(., s)$:

**Lemma 6.10.** Let $\varphi$ be a solution of \((2.1)\), then we have for $0 < \alpha < 1$

\[
|D\tilde{\varphi}(x, s)| \leq \sup_{\Omega} e^{-\lambda s} |D\tilde{\varphi}(., 0)|,
\]

where $\lambda$ is a positive constant.

**Proof.** Set $\psi = \frac{|\nabla \tilde{\varphi}|^2}{2}$. Similar to do in Lemma \(5.1\), we can obtain

\[
\frac{\partial \psi}{\partial s} = \tilde{Q}^{ij} \nabla_{ij} \psi + \tilde{Q}^k D_k \psi - \tilde{Q}^{ij} (\sigma_{ij} |D\tilde{\varphi}|^2 - D_i \tilde{\varphi} D_j \tilde{\varphi})
\]

\[
- \tilde{Q}^{ij} D_m \tilde{\varphi} D^n_j \tilde{\varphi} + (\alpha - 1) \tilde{Q} |D\tilde{\varphi}|^2.
\]

the boundary condition

\[
D \mu \psi \leq 0.
\]

Using the $C^2$ estimate, we can find a positive constant $\lambda$ such that

\[
\begin{cases}
\frac{\partial \psi}{\partial s} \leq \tilde{Q}^{ij} D_{ij} \psi + \tilde{Q}^k D_k \psi - \lambda \psi & \text{in } \Omega \times (0, \infty) \\
D \mu \psi \leq 0 & \text{on } \partial \Omega \times (0, \infty) \\
\psi(., 0) = \frac{|D\tilde{\varphi}(., 0)|^2}{2} & \text{in } \Omega.
\end{cases}
\]

Using the maximum principle and Hopf lemma, we get the gradient estimates of $\tilde{\varphi}$. \(\square\)

Thus, we can apply the Krylov-Safonov estimate \[20\] and thereafter the parabolic Schauder estimate to conclude:

**Lemma 6.11.** Let $\varphi$ be a solution of the inverse Gauss curvature flow \((2.1)\). Then,

\[
\tilde{\varphi}(., s).
\]

converges in $C^\infty$ to a real number for $s \to +\infty$.

So, we have

**Theorem 6.12.** The rescaled flow with boundary which meet a strictly convex cone perpendicularly

\[
\begin{cases}
\frac{d\tilde{X}}{ds} = \tilde{K} - \tilde{\pi} \nu - \tilde{X} & \text{in } \Omega \times (0, \infty) \\
\langle \mu(\tilde{X}), \nu(\tilde{X}) \rangle = 0 & \text{on } \partial \Omega \times (0, \infty) \\
\tilde{X}(., 0) = \tilde{M}_0 & \text{in } \Omega
\end{cases}
\]

exists for all time and the leaves converge in $C^\infty$ to a piece of round sphere.
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