Some properties of massless particles in arbitrary dimensions*

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Abstract

Various properties of two kinds of massless representations of the $n$-conformal (or $(n+1)$-De Sitter) group $G_n = SO_0(2,n)$ are investigated for $n \geq 2$. It is found that, for space-time dimensions $n \geq 3$, the situation is quite similar to the one of the $n = 4$ case for $S_n$-massless representations of the $n$-De Sitter group $SO_0(2,n-1)$. These representations are the restrictions of the singletons of $G_n$. The main difference is that they are not contained in the tensor product of two UIRs with the same sign of energy when $n > 4$, whereas it is the case for another kind of massless representation. Finally some examples of Gupta-Bleuler triplets are given for arbitrary spin and $n \geq 3$.

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1 Introduction

The (ladder) representations \( D(s + 1, s, \epsilon s) \), \( 2s \in \mathbb{N} \) and \( |\epsilon| = 1 \) of the universal covering \( \tilde{C}_4 = \tilde{SO}_0(2, 4) \) of the conformal group remain irreducible when restricted to the universal covering \( \tilde{P}_4 = \tilde{SO}_0(1, 3) \ltimes T_4 \) of the Poincaré group and each non trivial positive energy representation of the conformal group with that property is equivalent to one of them. However the restriction to the universal covering \( \tilde{S}_4 \) of the De Sitter group is irreducible only if \( s > 0 \); indeed one has:

\[
D(s + 1, s, \epsilon s)\big|_{\tilde{S}_4} = \begin{cases} 
D(s + 1, s) & \text{if } s > 0; \\
D(1, 0) \oplus D(2, 0) & \text{if } s = 0.
\end{cases}
\]

These representations are called massless (relatively to the De Sitter group) for a variety of reasons [2]. In the present paper we call them \( S_4 \)-massless representations of the De Sitter group \( \tilde{S}_4 = \tilde{SO}_0(2, 3) \) because, as indicated in [2, 11, 13] they satisfy the following masslessness conditions:

(a) They contract smoothly to a massles discrete helicity representation of the Poincaré group \( \tilde{P}_4 = \tilde{SO}_0(1, 3) \ltimes \mathbb{R}^4 \);

(b) Any massless discrete helicity representation \( U^P \) of the Poincaré group has a unique extension to a UIR \( \hat{U} \) (called \( C_4 \)-massless representation in this paper) of the conformal group \( \tilde{C}_4 = \tilde{SO}_0(2, 4) \). The restriction of \( \hat{U} \) to the De Sitter group is precisely one of the massless representations of \( \tilde{S}_4 \) recalled above;

(c) For spin \( s \geq 1 \) one may construct a gauge theory on the Anti-de Sitter space for massless particles, quantizable only by the use of an indefinite metric and a Gupta Bleuler triplet;

(d) The massless representations in question distinguish themselves by the fact that the physical signals propagate on the Anti-de Sitter light cone.

Other interesting representations of \( \tilde{S}_4 \) are the Dirac singletons \( D_i = D(1, \frac{1}{2}) \) and \( \text{Rac} = D(\frac{1}{2}, 0) \) (which are also \( C_3 \)-massless representations in the sense defined below). Some of their properties are:

1. Dirac singletons are, up to equivalence, the only unitary irreducible positive energy representations of \( \tilde{S}_4 \) which remain irreducible when restricted to the universal covering \( \tilde{L}_4 \) of the Lorentz group;

2. In the limit of zero curvature (of the De Sitter space \( \tilde{S}_4/\tilde{L}_4 \)) they contract to unitary irreducible representations (UIR) of \( \tilde{P}_4 \) that are trivial on the translation part \( T_4 \);

3. Let \( \chi(\mu_1) \otimes \pi(\mu_2) \) denote the IR (up to equivalence), with highest weight \( (\mu_1, \mu_2) \) of the universal covering \( \tilde{K}_4 \) of the maximal compact subgroup of the De Sitter group. Then the restriction to \( \tilde{K}_4 \) of the Dirac singletons UIRs of the De Sitter group is given by

\[
D\left(\frac{1}{2} + s, s\right)\big|_{\tilde{K}_4} = \bigoplus_{l \in \mathbb{N}} \chi(-\left[\frac{1}{2} + s + l\right]) \otimes \pi(s + l), \ s = 0 \text{ or } \frac{1}{2}.
\]
4. Finally the Dirac singletons satisfy the following \[ \mathbb{R} \otimes \mathbb{R} = \bigoplus_{s \in \mathbb{N}} D(s + 1, s); \]
\[ \mathbb{R} \otimes \mathbb{D} = \bigoplus_{s - \frac{1}{2} \in \mathbb{N}} D(s + 1, s); \]
\[ \mathbb{D} \otimes \mathbb{D} = \bigoplus_{s - 1 \in \mathbb{N}} D(s + 1, s) \oplus D(2, 0). \]

Note \[ \mathbb{R} \] that the Dirac singletons are not massless representations of the De Sitter group. But if one considers \( S_4 \) as the conformal group of the 3-dimensional Minkowski space then the Dirac singletons are massless, i.e. their restriction to the corresponding Poincaré group \( P_3 \) is irreducible \[ \mathbb{R}, \mathbb{B}, \mathbb{L}, \mathbb{P}, \mathbb{S} \]. In this case it is clear from the context what kind of masslessness is considered. However, for general \( n \), some confusion may arise. To avoid it we shall introduce a prefix to the word “massless” (see definition \[ \mathbb{R} \]), to distinguish between “conformal masslessness” and “De Sitter masslessness” in any dimension, to precise which group we are representing.

A common property to both types of massless representations is the existence of Gupta-Bleuler (GB) quantization; see for example \[ \mathbb{R}, \mathbb{B}, \mathbb{L}, \mathbb{P}, \mathbb{S} \].

The purpose of this work is to continue the study performed in \[ \mathbb{R} \] and more specifically to look for properties of masslessness (both types) which persist when the space-time dimension becomes an arbitrary integer \( n \geq 2 \). In Section \[ \mathbb{R} \] we fix the notations and recall some results. In Section \[ \mathbb{R} \] we discuss the irreducibility of a massless representation of the \( n \)-conformal group when restricted to the \((n + 1)\)-Lorentz group and its contractibility to UIRs of the \( n \)-Poincaré group. Reduction to the maximal compact subgroup of the conformal group is studied in Section \[ \mathbb{R} \]. Finally Dirac singletons and Gupta-Bleuler triplets are treated in (respectively) Sections \[ \mathbb{R}, \mathbb{R} \]. It is found that almost all the properties of massless representations in dimension \( n = 4 \) are conserved when \( n \geq 3 \); however the property that massless representations are, when \( n = 4 \), contained in the tensor product of two positive energy UIRs (of the De Sitter group) fails for general \( n \).

After a first version of this paper was written appeared a preprint \[ \mathbb{R} \] with somewhat different conclusions, based on a less-demanding notion of masslessness in higher dimensions. Since we need the definitions and results of this paper to compare both notions, we shall discuss this point at the end of the paper.

## 2 Generalities

We suppose \( n \geq 2 \). Let \( \mathbb{R}^{1,n-1} \) be the \( n \)-dimensional Minkowski space-time, \( T_n \) its group of translations, \( L_n = SO_0(1, n - 1) \) the \( n \)-Lorentz group, \( P_n = L_n \ltimes T_n \) the \( n \)-Poincaré group and \( S_n = SO_0(2, n - 1) \) the \( n \)-De Sitter group. We write \( T_n, L_n, P_n \) and \( S_n \) the corresponding Lie algebras.
Let $G_n = SO_0(2, n)$. The preceding groups may be considered as subgroups of $G_n$. Indeed let $(M_{ab})_{-1 \leq a < b \leq n}$ be a basis of the Lie algebra $\mathcal{G}_n$ of $G_n$ such that:

$$M_{ab} = -M_{ba}$$  \hspace{1cm} (1)

and

$$[M_{ab}, M_{cd}] = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}$$  \hspace{1cm} (2)

where

$$\eta = \begin{pmatrix} 1/2 & \hline -1/n \end{pmatrix}$$

We now imbed the above mentioned Lie algebras in $\mathcal{G}_n$ in the following way:

$$\mathcal{T}_n = \langle M_{-1,a} + M_{a,n}, \ 0 \leq a \leq n-1 \rangle$$

$$\mathcal{L}_n = \langle M_{\alpha\beta}, \ 0 \leq \alpha, \beta \leq n-1 \rangle$$

$$\mathcal{S}_n = \langle M_{ab}, \ -1 \leq a, b \leq n-1 \rangle.$$  \hspace{1cm} (3)

Let:

$$\mathcal{C}_n = \mathcal{T}_n \cup \mathcal{S}_n = \mathcal{S}_n + 1$$

and $\mathcal{D}_n$ (resp. $\mathcal{D}_n$) the connected subgroup of $G_n$, the Lie algebra of which is $\mathcal{T}_n$ (resp. $\mathcal{D}_n$). Then we define the $n$-conformal group of $\mathbb{R}^{1,n-1}$ as the closed subgroup $C_n$ of $G_n$ generated by $T_n, L_n, D_n$ and $\mathcal{T}_n$. $C_n, G_n$ and $S_{n+1}$ are locally isomorphic; one has, if $C_n$ denotes the Lie algebra of $C_n$:

$$C_n = G_n = S_{n+1}.$$  \hspace{1cm} (4)

Note that with our definition by “conformal group of $\mathbb{R}^{1,1}$” we mean here the group $SO_0(2, 2)/\mathbb{Z}_2$.

Let $G$ a Lie group. We denote by $\bar{G}$ be the spinorial covering of $G$ when $G$ is isomorphic to $L_3$ or $P_3$ and the universal covering otherwise. Let $U$ be a non trivial highest weight unitary representation of $\bar{C}_n$.

**Definition 1** We say that $U$ is $C_n$-massless\footnote{massless relatively to the $n$-conformal group.} whenever $U$ and $U|_{\bar{P}_n}$ are irreducible. We say that $U$ is $S_{n+1}$-massless\footnote{massless relatively to the $(n+1)$-De Sitter group.} whenever $U$ is a restriction to $\bar{S}_{n+1} \simeq \bar{C}_n$ of a $C_{n+1}$-massless representation of $\bar{C}_{n+1}$.

Note that the notions of $C_n$-massless and $S_n$-massless refer to $n$-dimensional space-times $\mathbb{R}^{1,n-1}$ ($n$-Minkowski space) and $\bar{S}_n/\bar{L}_n$ (which we call the $n$-De Sitter space, though $n$-Anti De Sitter space might be a more appropriate expression). The conformal group of both of them is $\bar{C}_n$, locally isomorphic to $\bar{SO}_0(2, n)$ while the invariance groups are
respectively \( P_n \) and \( \tilde{S}_n = \tilde{S}O_n(2, n - 1) \). For example usual massless particles in 4-Minkowski space, for the Poincaré group, are in fact \( C_4 \)-massless (under extension to the conformal group \( SO_0(2, 4)/\mathbb{Z}_2 \)) and \( S_4 \)-massless (under deformation to the De Sitter group \( SO_0(2, 3) \)); usual Dirac singletons are \( C_3 \)-massless, and their restrictions to \( SO_0(2, 2) \) (reducible in a sum of two) are \( S_3 \)-massless.

For simplicity we identify the group representation \( U \), the Lie algebra representation \( dU \) and the extension of the latter to \( U(\mathbb{C}_n) \) and denote \( I = \ker_{U(\mathbb{C}_n)}(U) \). Then one has \[3\]:

**Theorem 1**

\[
U \text{ is } C_n \text{-massless} \iff \eta^{cd}M_{ac}M_{bd} - \frac{n}{2}M_{ab} + \frac{2}{n+2}\eta_{ab}C_2 = 0 \pmod{I} \quad (3)
\]

where \( C_2 \) is the Casimir operator and where we have used the Einstein summation convention.

**Definition 2** We call the right hand side of the preceding equivalence (3) the fundamental relation (FR).

### 3 Irreducibility of \( U \) when restricted to \( \tilde{L}_{n+1} \) and its contractibility to UIRs of \( \tilde{P}_{n+1} \)

#### 3.1 Irreducibility of \( U \) when restricted to \( \tilde{L}_{n+1} \)

The following proposition is a characterization of UIRs \( U \) which remain irreducible when restricted to \( \tilde{L}_{n+1} \).

**Proposition 1** Let \((U, \mathcal{H})\) be a highest weight UIR of \( \tilde{C}_n \). Then:

\[
U \mid_{\tilde{L}_{n+1}} \text{ is irreducible} \iff U \text{ satisfies the FR.} \quad (4)
\]

**Proof.** Assume that the restriction \( U \mid_{\tilde{L}_{n+1}} \) is irreducible. Then both of the Casimir operators of \( \mathcal{G}_n \) and \( \mathcal{L}_{n+1} \) are sent to the scalars by \( U \) and \( U \mid_{\tilde{L}_{n+1}} \) respectively. It follows that the difference of these operators is also sent to the scalars and thanks to the adjoint action of \( \mathcal{G}_n \) one obtains the FR. The converse is proved in \[3\].

It easily follows:

**Corollary 1** If \( U \) is \( C_n \)-massless then \( U \mid_{\tilde{L}_{n+1}} \) is irreducible.

\(^3\text{summation on repeated indices.}\)
3.2 A contraction of $C_n$-massless representations

Consider a family $(S_\rho)_{0 < \rho \leq 1}$ of operators defined on the underlying vector space $V_n$ of $S_{n+1} = \mathcal{G}_n$ by:

\[ S_\rho(M_{\alpha\beta}) = M_{\alpha\beta}, \quad \text{if } 0 \leq \alpha, \beta \leq n \]  
\[ S_\rho(M_{-1\alpha}) = \sqrt{\rho} M_{-1\alpha}, \quad \text{if } 0 \leq \alpha \leq n. \]  

It defines a contraction of $S_{n+1}$ to $\mathcal{P}_{n+1}$. We are using here the notion of contractions of representations (on Hilbert spaces) given in [2] (see also [3]).

Let $\mathcal{G}_n^\rho$ be the Lie algebra isomorphic to $\mathcal{G}_n$ defined by the bracket:

\[ [x, y]_\rho = S_\rho^{-1} [S_\rho x, S_\rho y], \quad x, y \in V_n \]  

and let $U_\rho$ the representation of $\mathcal{G}_n^\rho$ defined on the corresponding space $\mathcal{H}$ by:

\[ U_\rho(x) = Z_\rho^{-1} \circ U(S_\rho x) \circ Z_\rho, \quad x \in V_n, \]  

where $(Z_\rho)_{0 < \rho \leq 1}$ is a continuous family of closed invertible operators of $\mathcal{H}$, $Z_1$ being the identity. We choose them here such that:

\[ Z_\rho^{-1} U(M_{\alpha\beta}) Z_\rho = U(M_{\alpha\beta}), \quad 0 \leq \alpha, \beta \leq n. \]  

Thus one has:

\[ U_\rho(M_{\alpha\beta}) = U(M_{\alpha\beta}), \quad 0 \leq \alpha, \beta \leq n \]  

and, using the FR:

\[ U_\rho(M_{-1\alpha}) U_\rho(M_{-1\beta}) = Z_\rho^{-1} U(S_\rho M_{-1\alpha}) U(S_\rho M_{-1\beta}) Z_\rho \]  
\[ = \rho Z_\rho^{-1} U(M_{-1\alpha} M_{-1\beta}) Z_\rho \]  
\[ = \rho U\left( \sum_{k=1}^{n} M_{\alpha k} M_{0 k} - \frac{n}{2} M_{0\alpha} - \frac{2}{n + 2} n_\alpha C_2 \right). \]  

Thus the operator $U_\rho(M^2_{10})$ has limit zero when $\rho \to 0$, in the sense that it sends a dense subspace to $\{0\}$ when $\rho \to 0$. It follows that the limit of $U_\rho(M_{-1\alpha}), 0 \leq \alpha \leq n$, is zero too.

If one chooses $\rho$ to be the curvature of the space $S_{n+1}/L_{n+1}$ then one can write, from what precedes:

**Proposition 2** In the limit of zero curvature the contracted $C_n$-massless representation is trivial on $T_{n+1}$, the translation part of $\tilde{\mathcal{P}}_{n+1}$.
4 Reduction of $U$ on the maximal compact subgroup of $\tilde{C}_n$

The following results are proved in [3]:

**Theorem 2** Let $\lambda = (\lambda_1, \ldots, \lambda_r)$, $r = \left\lfloor \frac{n+2}{2} \right\rfloor$ (the integer part of $\frac{n+2}{2}$), the highest weight (HW) of the $C_n$-massless representation $U$. Then there exists a real number $s$ such that:

$$\lambda_1 = -s - \frac{n-2}{2} \text{ and } \lambda_2 = \cdots = \lambda_{r-1} = |\lambda_r| = s$$

where:

$$\begin{cases} 
  s > 0 & \text{if } n = 2; \\
  2s \in \mathbb{N} & \text{if } n \text{ is even and } n \geq 4; \\
  \lambda_r \geq 0 \text{ and } s = 0 \text{ or } 1/2 & \text{if } n \text{ is odd.}
\end{cases}$$

**Proposition 3** Let $\mathfrak{t}_n = \mathfrak{so}(2) \oplus \mathfrak{so}(n)$ the maximal compact subalgebra of $C_n$ and let $\chi(\mu_1) \otimes \pi(\mu_2, \ldots, \mu_r)$ be an IR of $\mathfrak{t}_n$ with HW $\mu = (\mu_1, \ldots, \mu_r)$. Then one has:

$$U|_{\mathfrak{t}_n} = \bigoplus_{t = 0}^{\infty} \chi(-s - \frac{n-2}{2} - l) \otimes \pi(s + l, s, \ldots, s, \epsilon s),$$

where $|\epsilon| = 1$ (resp. $\epsilon = 1$) if $n$ is even (resp. odd).

Thus $C_n$-massless representations are very degenerate and are, in some sense, “singleton” representations.

5 Dirac singletons

**Definition 3** A positive (resp. negative) energy representation of $\tilde{S}_n$ is a lowest (resp. highest) weight representation. We say that $U$ and $U'$ are Dirac singletons (DS) if a $S_n$-massless representation of $\tilde{S}_n$ occurs in the reduction of the product $U \otimes U'$ and if $U$ and $U'$ have the same sign of energy.

It has been proved by M. Flato and C. Frønsdal in [10] for the $n = 4$ case that the (irreducible and unitary) representations $Di = D(1,1/2)$ and $Rac = D(1/2,0)$ are Dirac singletons and that the product $(Di \oplus Rac) \otimes (Di \oplus Rac)$ reduces to a direct sum of $S_4$-massless representations of $\tilde{SO}_0(2,3)$. It is interesting to note that $Di$ and $Rac$ are $C_3$-massless representations of $SO_3(2,3)$. Unfortunately when $n \geq 5$ things behave differently; the next proposition treats this case.
Proposition 4 Assume \( n \geq 5 \). Let \( U \) and \( U' \) be DS. Then only a finite number of \( S_n \)-massless representations of \( \tilde{S}_n \) can occur in the reduction of the product \( U \otimes U' \). Moreover \( U \) and \( U' \) can not be simultaneously unitary.

Proof. Since \( U \) and \( U' \) have the same sign of energy we can assume they are HW representations of \( \tilde{S}_n \simeq SO_0(2, n - 1) \). Let \( \lambda \) and \( \lambda' \) their respective HW. Let \( \nu = 0 \) (resp. 1) if \( n \) is even (resp. odd). Then \( n + 2 = 2r + \nu \) and the rank \( r' \) of \( S_n^C \) is given by \( r' = r - (1 - \nu) \), thus \( S_n^C \) and \( C_n^C \) have the same rank if and only if \( n \) is odd. Let \( (e_i)_{1 \leq i \leq r'} \) be the canonical basis of \( \mathbb{C}^r \) and let:

\[
\Delta_{n+1}^+ = \begin{cases} 
\{e_i \pm e_j, 1 \leq i < j \leq r'\} & \text{if } n+1 \text{ is even;} \\
\{e_i \pm e_j, 1 \leq i < j \leq r'\} \cup \{e_j, 1 \leq j \leq r'\} & \text{if } n+1 \text{ is odd.}
\end{cases}
\] (13)

Then \( \Delta_{n+1}^+ \) defines a set of positive roots for \( S_n^C \). Thus if \( U \) is a HW irreducible representation of \( \tilde{S}_n \) with HW \( \lambda = (\lambda_1, \ldots, \lambda_r) \) then a weight \( \mu \) of \( U \) has the form:

\[
\mu = \lambda - \sum_{\alpha \in \Delta_{n+1}^+} \sum_{p_\alpha \in \mathbb{N}} p_\alpha \alpha \tag{14}
\]

\[
= - \left( E + \sum_{j=2}^{r'} (q_j + p_j) + (1 - \nu)m_1 \right) e_1 + \sum_{i=2}^{r'} \left[ \lambda_i + q_i - p_i - \sum_{j=i+1}^{r'} (q_{ij} + p_{ij}) - (1 - \nu)m_i + \sum_{j=2}^{i-1} (q_{ji} - p_{ji}) \right] e_i
\]

where \( E = -\lambda_1 \) and \( (q_j), (p_j), (m_j), (q_{ij})_{i<j} \) and \( (p_{ij})_{i<j} \) are families of natural integers, such that \( \sum_{j=2}^{r'} (q_{j2} - p_{j2}) = \sum_{j=r'+1}^{r'} (q_{iir'} + p_{iir'}) = 0. \)

Let \( \sigma = (-s - r' + 2 + \nu/2, s, \ldots, s) = (-s - \frac{n-2}{2}, s, \ldots, s) \) where \( 2s \in \mathbb{N} \) (resp. \( s = 0 \) or \( 1/2 \)) if \( n \) is even (resp. odd) and let \( \Lambda \) be the set of such \( \sigma \)'s. Then each \( S_n \)-massless representation of \( \tilde{S}_n \) has at least one element of \( \Lambda \) as a HW. Indeed if \( U_0 \) is an \( S_n \)-massless representation of \( \tilde{S}_n \) then one has:

\[
n \text{even} \implies \begin{cases} 
U_0 \sim D(s + \frac{n-2}{2}, s, \ldots, s), 2s \in \mathbb{N} \text{ and } s \neq 0 \\
or \implies U_0 \sim D(\frac{n-2}{2}, 0, \ldots, 0) \oplus D(\frac{n}{2}, 0, \ldots, 0),
\end{cases}
\] (15)

\[
n \text{odd} \implies \begin{cases} 
U_0 \sim D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \oplus D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}) \\
or \implies U_0 \sim D(\frac{n-2}{2}, 0, \ldots, 0) \oplus D(\frac{n}{2}, 0, \ldots, 0). \tag{16}
\end{cases}
\]

Now assume that there exists, for an \( S_n \)-massless representation \( U_0 \), for which \( \sigma \in \Lambda \) is a HW, two DS \( U \) and \( U' \) with HW \( \lambda \) and \( \lambda' \) respectively. Then it is well known that there
exists a weight $\mu$ of $U$ such that $\sigma = \mu + \lambda'$. Since $\mu$ is given by (14) one has, for some $s$:

$$E + E' + \sum_{j=2}^{r'} (q_j + p_j) + (1 - \nu)m_1 = s + r' - 2 - \nu/2$$  \hspace{1cm} (17)

and, for each $i, 2 \leq i \leq r'$,

$$\lambda_i + \lambda_i' + q_i - p_i - \sum_{j=i+1}^{r'} (q_{ij} + p_{ij}) - (1 - \nu)m_i + \sum_{j=2}^{i-1} (q_{ji} - p_{ji}) = s. \hspace{1cm} (18)$$

Now assume $n \geq 5$. Then $r' \geq 3$ and (18) becomes, for $i = 2$:

$$\lambda_2 + \lambda_2' + q_2 = p_2 + \sum_{j=3}^{r'} (q_{2j} + p_{2j}) + (1 - \nu)m_2 + s. \hspace{1cm} (19)$$

Adding $\lambda_2 + \lambda_2' - E - E'$ to both sides of (17) and using (19), one gets:

$$\sum_{j=3}^{r'} (q_j + p_j) + 2p_2 + (1 - \nu)m_1 + (1 - \nu)m_2 + \sum_{j=3}^{r'} (q_{2j} + p_{2j}) =$$

$$= \lambda_2 - E + \lambda_2' - E' + r' - 1 - \nu/2. \hspace{1cm} (20)$$

Thus one has:

$$\sum_{j=3}^{r'} (q_j + p_j) + (1 - \nu)m_1 + p_2 \leq \lambda_2 - E + \lambda_2' - E' + r' - 1 - \nu/2 \hspace{1cm} (21)$$

Now adding $\lambda_3 + \lambda_3' + q_{23} - E - E'$ to both sides of (17) and using (18) for $i = 3$ one finds:

$$q_2 + p_2 + 2p_3 + \sum_{j=4}^{r'} (q_j + p_j) + (1 - \nu)m_1 + (1 - \nu)m_3 + \sum_{j=4}^{r'} (q_{3j} + p_{3j}) +$$

$$+ p_{23} = \lambda_3 - E + \lambda_3' - E' + r' - 1 - \nu/2 + q_{23}, \hspace{1cm} (22)$$

thus

$$q_2 \leq \lambda_3 - E + \lambda_3' - E' + r' - 1 - \nu/2 + q_{23}. \hspace{1cm} (23)$$

But from (20) one gets:

$$q_{23} \leq \lambda_2 - E + \lambda_2' - E' + r' - 1 - \nu/2, \hspace{1cm} (24)$$

so that

$$q_2 \leq \lambda_2 + \lambda_3 - 2E + \lambda_2' + \lambda_3' - 2E' + 2r' - 2 - \nu. \hspace{1cm} (25)$$
Finally one has, thanks to (21), (25) and (17)

\[ s = \sum_{j=2}^{r'} (q_j + p_j) + (1 - \nu)m_1 + E + E' - r' + 1 + \nu/2 \]  

\[ \leq 2\lambda_2 - 2E + \lambda_3 + 2\lambda_2' - 2E' + \lambda_3' + 2r' - 2 - \nu. \]

The right hand side being finite for fixed \( U \) and \( U' \), only \( S_n \)-massless representations whose parameter \( s \) satisfies (26), thus a finite number, may occur in the reduction of \( U \otimes U' \).

Now unitarity of \( U \) and \( U' \) implies \([1, 7]\):

\[ E \geq \lambda_2 + r' - 3/2 - \nu/2 \]  

and:

\[ E' \geq \lambda_2' + r' - 3/2 - \nu/2, \]

but that is not compatible with (20). Indeed the left hand side of (20) is a naturel integer whereas the right hand one satisfies:

\[ \lambda_2 - E + \lambda_2' - E' + r' - 1 - \nu/2 \leq -r' + 2 + \nu/2 \leq -1 + \nu/2 < 0. \]

**Remark 1** Unitarity of \( U \) or \( U' \) is however possible for \( n \geq 5 \). Indeed, the \( S_n \)-massless representation \( D(\frac{n-2}{2}, 0, \ldots, 0) \oplus D(\frac{3}{2}, 0, \ldots, 0) \) is contained in the tensor product of the \( C_{n-1} \)-massless representation \( U = D(\frac{n-3}{2}, 0, \ldots, 0) \), which is unitary, by the representation \( U' = D(\frac{1}{2}, 0, \ldots, 0) \oplus D(\frac{3}{2}, 0, \ldots, 0) \), which is not unitary. Another example is given by the \( S_n \)-massless representation \( D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \), the unitary \( C_{n-1} \)-massless representation \( U = D(\frac{n-2}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \) and the non unitary representation \( U' = D(\frac{1}{2}, 0, \ldots, 0) \).

Now let us look to the other values of \( n \). As seen above the case \( n = 4 \) is treated in [19], thus we examine only the cases \( n = 3 \) and \( n = 2 \).

Let \( n = 3 \). Then the De Sitter algebra \( S_3 \simeq so(2, 2) \) is isomorphic to \( so(2, 1) \oplus so(2, 1) \). The \( C_3 \)-massless representations of the conformal algebra \( C_3 \simeq so(2, 3) \) are the Rac = \( D(1/2, 0) \) and the Di = \( D(1, 1/2) \) or, more shortly, \( D(1/2+s, s) \), \( s \) being 0 or 1/2. The \( S_3 \)-massless representations of \( S_3 \) are thus \( D(1/2+s, s) \big|_{S_3} \), \( s \) = 0 or 1/2. Having in mind that an irreducible HW representation of \( so(2, 2) \) is equivalent to a tensor product (which we write \( \otimes \)) of two irreducible representations of \( so(2, 1) \) one gets:

\[ D(1/2, 0) \big|_{S_3} \sim D'(1/2, 0) \oplus D'(3/2, 0) \]  

\[ \sim D(1/4) \otimes D(1/4) \oplus D(3/4) \otimes D(3/4) \]

and

\[ D(1, 1/2) \big|_{S_3} \sim D'(1, 1/2) \oplus D'(1, -1/2) \]  

\[ \sim D(1/4) \otimes D(3/4) \oplus D(3/4) \otimes D(1/4). \]
Here we have denoted by $D'(E,j)$ (resp. $D(\alpha)$) the irreducible representation with HW $(-E,j)$ (resp. $(-\alpha)$) of $\mathfrak{so}(2,2)$ (resp. $\mathfrak{so}(2,1)$). Now a large number of UIRs $\pi$ and $\pi'$ of $\tilde{S}_3$ may have the property that $S_3$-massless representations are contained in $\pi \otimes \pi'$. To reduce that number we shall suppose that $\pi$ and $\pi'$ are $C_2$-massless, in analogy with the 4-dimensional case where the Dirac singletons $D_i$ and $Rac$ are $C_3$-massless. Then, if one assumes that $\pi$ and $\pi'$ are HW representations, each one has the form $D'(\alpha, \pm \alpha)$ where $\alpha > 0$. Thus one must consider the products:

\[ D'(\alpha, \pm \alpha) \otimes D'(\beta, \pm \beta) \tag{32} \]

or, equivalently, the products:

\[
\begin{align*}
[D(0) \boxtimes D(\alpha)] & \otimes [D(0) \boxtimes D(\beta)] \\
[D(0) \boxtimes D(\alpha)] & \otimes [D(\beta) \boxtimes D(0)] \\
[D(\alpha) \boxtimes D(0)] & \otimes [D(\beta) \boxtimes D(0)] \\
[D(\alpha) \boxtimes D(0)] & \otimes [D(0) \boxtimes D(\beta)].
\end{align*}
\]

Now using

\[ D(\alpha) \otimes D(\beta) \sim \bigoplus_{l=0}^{\infty} D(\alpha + \beta + l) \tag{34} \]

one finds

\[ D'(\alpha, \pm \alpha) \otimes D'(\beta, \pm \beta) = \bigoplus_{l=0}^{\infty} D'(\alpha + \beta + l, \pm [\alpha + \beta + l]) \tag{35} \]

\[ D'(\alpha, \pm \alpha) \otimes D'(\beta, \mp \beta) = D'(\alpha + \beta, \pm [\alpha - \beta]). \tag{36} \]

Finally it is easily seen that

\[ D'(1/4, 1/4) \otimes D'(1/4, -1/4) = D'(1/2, 0) \tag{37} \]

\[ D'(3/4, 3/4) \otimes D'(3/4, -3/4) = D'(3/2, 0) \tag{38} \]

thus

\[
\begin{align*}
[D'(1/4, 1/4) & \oplus D'(3/4, 3/4)] \otimes [D'(1/4, -1/4) \oplus D'(3/4, -3/4)] = \\
= [D'(1/2, 0) \oplus D'(3/2, 0)] & \oplus [D'(1, 1/2) \oplus D'(1, -1/2)].
\end{align*}
\]
The right hand side is a sum of $S_3$-massless representations. Using (35) and (36), one can see that this is a unique solution (up to equivalence) of the problem \( \text{Singleton} \otimes \text{Singleton} = \bigoplus S_3 \)-massless for unitary Dirac singletons, which are, here, \( D'(1/4, \pm 1/4) \oplus D'(3/4, \pm 3/4) \). They are not irreducible, but each component is irreducible on both \( S_3 \) and \( L_3 \).

Let \( n = 2 \). Then \( \mathcal{C}_2 \simeq \mathfrak{so}(2, 2), \mathcal{S}_2 \simeq \mathfrak{so}(2, 1) \) and \( \mathcal{L}_2 \simeq \mathfrak{so}(1, 1) \). A (HW) \( C_2 \)-massless representation of \( \mathcal{C}_2 \) has the form \( D'(\alpha, \pm \alpha), \alpha > 0 \). Thus the \( S_2 \)-massless representations of \( \mathcal{S}_2 \) have the form \( D'(\alpha, \pm \alpha) \big|_{S_2} \sim D(\alpha) \). Now \( C_1 \)-masslessness on \( D(\beta) \) (or irreducibility on the 2-Lorentz group \( L_2 \)) implies \( \beta = 0 \), so that, instead of the \( n = 4 \) and \( n = 3 \) cases, Dirac singletons are not compatible with \( C_1 \)-masslessness. But one has:

\[
D(\alpha/2) \otimes D(\alpha/2) \sim D(\alpha) \oplus \bigoplus_{l=0}^{\infty} D(\alpha + 1 + l).
\]

(40)

Thus \( S_2 \)-massless representations occur in the tensor product of two \( S_2 \)-massless ones.

### 6 Indecomposability. Gupta-Bleuler triplets

Gupta-Bleuler triplets are used to quantize gauge theories, in a way similar to the quantization of (4-dimensional flat) QED. This kind of quantization is done on an indefinite metric space which carries indecomposable representations, as in the Gupta-Bleuler quantization of the electromagnetic field. Let us see how it works in the case of our massless representations. If \( U_2 \) is a massless representation of \( G_n \) then it can be obtained as a component of an indecomposable representation. Indeed one can find UIRs \( U_\varepsilon, \varepsilon > 0 \), and \( U_3 \) such that \( \lim_{\varepsilon \to 0} U_\varepsilon \) is a non trivial extension \( U_2 \to U_3 \) (i.e. we have an exact sequence \( 0 \to \mathcal{H}_3 \to \mathcal{H} \to \mathcal{H}_2 \to 0 \) where \( \mathcal{H}_i \) is the carrying space of \( U_i, i = 2 \) or 3). The elements of \( \mathcal{H}_3 \), the gauge states, are obtained from those of \( \mathcal{H} \) by applying a constraint similar to the Lorentz condition in QED. The elements of \( \mathcal{H}_2 \), the physical states, are realized on the quotient \( \mathcal{H}/\mathcal{H}_3 \). Now the representation \( (U_2 + U_3, \mathcal{H}) \) has no invariant nondegenerate metric, thus covariant quantization is not possible. But if one extends the representation \( U_3 \) by \( U_2 + U_3 \) in a non trivial way \( (U_3 \to U_2 \to U_3) \) to a bigger space endowed with an invariant nondegenerate (but indefinite) Hermitian form then quantization of the gauge theory under construction becomes possible.

In the following we construct some examples of Gupta-Bleuler triplets for the massless representations when \( n \geq 3 \).

#### 6.1 Massless representations and indecomposability

Let us recall that the massless representations for \( G_n = \mathfrak{so}(2, n) \) are the \( C_n \)-massless and the \( S_{n+1} \)-massless ones. Below we write them again, according to the parity of \( n \). In analogy with 4-dimensional physics we call the parameter \( s \) the spin of the representation.
Case 1: \( n \) is even

\( C_n \)-massless representations are:

\[
D(s + \frac{n-2}{2}, s, \ldots, \pm s), \quad 2s \in \mathbb{N}
\]  

(41)

\( S_{n+1} \)-massless representations are:

\[
d(\frac{n}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \oplus d(\frac{n}{2}, \frac{1}{2}, \ldots, -\frac{1}{2}) \quad \text{for spin } \frac{1}{2}
\]  

(42)

\[
d(\frac{n-1}{2}, 0, \ldots, 0) \oplus d(\frac{n+1}{2}, 0, \ldots, 0) \quad \text{for spin } 0.
\]  

(43)

Case 2: \( n \) is odd

\( C_n \)-massless representations are:

\[
D(s + \frac{n-2}{2}, s, \ldots, s) \quad s \in \{0, \frac{1}{2}\}
\]  

(44)

\( S_{n} \)-massless representations are:

\[
d(\frac{n-1}{2}, 0, \ldots, 0) \oplus d(\frac{n+1}{2}, 0, \ldots, 0) \quad \text{for spin } 0
\]  

(45)

\[
d(\frac{n-1}{2}, s, \ldots, s), \quad 2s \in \mathbb{N} \quad \text{and} \quad s \geq \frac{1}{2}.
\]  

(46)

Some of the above irreducible representations correspond to the limit of unitarity \([1, 7]\). It is the case of the \( C_n \)-massless ones and, when \( n \) is odd, of the \( S_{n+1} \)-massless representations for which \( s \geq 1 \). Then one can look for indecomposability and Gupta-Bleuler (GB) triplets. That is what we do in the next subsections (for these representations).

In the next two subsections the cases of the representations \( D(\frac{n-2}{2}, 0, \ldots, 0) \) and \( D(\frac{1}{2} + \frac{n-2}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \) are treated without separating the \( n \) even and \( n \) odd cases, since those representations are \( C_n \)-massless for both \( n \) even and \( n \) odd. Finally, when \( s \geq 1 \) the \( C_n \)-massless \( D(s + \frac{n-2}{2}, s, \ldots, s) \) for \( n \) even and the \( S_{n+1} \)-massless \( D(s + \frac{n-1}{2}, s, \ldots, s) \) for \( n \) odd are investigated successively.

### 6.2 \( C_n \)-masslessness, spin 0

#### 6.2.1 Reduction of \( D(E_0, 0, \ldots, 0) \) to \( \mathfrak{t}_n \) and its indecomposability

Recall that \( \mathfrak{t}_n \simeq \mathfrak{so}(2) \oplus \mathfrak{so}(n) \) is the maximal compact subalgebra of \( \mathfrak{g}_n \). Let \( \mathfrak{g}_n = \mathfrak{t}_n + \mathfrak{p}_n \) the Cartan decomposition of \( \mathfrak{g}_n \). Let \( (X_{jk})_{-r \leq j, k \leq r} \) a basis \( 4 \) of \( \mathfrak{g}_n^\mathbb{C} \) such that:

\[
X_{jk} = -X_{kj}
\]  

(47)

and

\[
[X_{jk}, X_{j'k'}] = \delta_{j,-j'}X_{kk'} + \delta_{k,-k'}X_{jj'} - \delta_{j,-k'}X_{kj} - \delta_{k,-j'}X_{jk'}
\]  

(48)

\(^4\text{This basis is more appropriate to the triangular decomposition of } \mathfrak{g}_n \text{ than the } (M_{ab})_{a,b} \text{ basis.}\)
and let:

\[
\begin{align*}
n^\pm &= \{ X_{\pm j, \pm k}, (1 - \nu) \leq j, k \leq r \} + \{ X_{\pm j, \pm k}, 1 \leq j < k \leq r \}, \\
\mathfrak{h} &= \{ X_{-j,j} = H_j, 1 \leq j \leq r \}.
\end{align*}
\]

(49)

(50)

Then \( G_n^C = n^+ + \mathfrak{h} + n^- \) is a triangular decomposition of \( G_n^C \).

Let \( p^\pm = p_n^C \cap n^\pm \). Then \( G_n^C = p^+ + \mathfrak{t}_n^C + p^- \). The basis \((X_{jk})_{jk}\) is chosen such that:

\[
p^\pm = \{ X_{\pm 1,j}, -r \leq j \leq r \text{ and } |j| \neq 1 \} \\
\mathfrak{t}_n^C = \{ X_{jk}, -r \leq j, k \leq r \text{ and } |j|, |k| \neq 1 \}
\]

(51)

(52)

The root system \( \Delta_{n+2} \) is defined by the set of positive roots \( \Delta_{n+2}^+ \) which is given by (43), but with \( n + 2 \) (resp. \( r = \left\lceil \frac{n+2}{2} \right\rceil \)) instead of \( n + 1 \) (resp. \( r' = \left\lceil \frac{n+1}{2} \right\rceil \)). The new basis is also chosen such that in the decomposition \( n^\pm = \sum_{\alpha \geq 0} G_n^{(\pm \alpha)} \) the subspace \( G_n^{(e_j \pm e_k)} \) is, for \( 1 \leq j < k \leq r \), generated by \( X_{j,\pm k} \) and, if \( n \) is odd, \( G_n^{(e_j)} \) is, for \( 1 \leq j \leq r \), generated by \( X_{0,j} \). The roots which correspond to \( \mathfrak{t}_n^C \) are the compact roots and the others the noncompact ones. The set of positive compact (resp. noncompact) roots is denoted by \( \Delta_{n+2}^{c+} \) (resp. \( \Delta_{n+2}^{c-} \)).

Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) a \( \Delta_{n+2}^{c+} \)-dominant integer weight and let \( K(\lambda) \) denote the irreducible (finite dimensional) HW \( \mathfrak{t}_n \)-module. We write \( N(\lambda) \) for the induced HW \( G_n \)-module, with HW \( \lambda \), and \( L(\lambda) \) for the irreducible quotient. The HW vectors for both \( N(\lambda) \) and \( L(\lambda) \) are, for simplicity, indentified and denoted by \( v_\lambda \).

**Proposition 5** Let \( E_0 > 0 \), \( \lambda = (-E_0, 0, \ldots, 0) \), \( u_\lambda = D(E_0, 0, \ldots, 0) \), \( Z = \sum_{|k| \neq 1} X_{-1,h} X_{-1,-h} \in \mathcal{U}(G_n^C) \) and, for \( l, k \in \mathbb{N} \), \( v_{lk} = (X_{-1,2})^l Z^k v_\lambda \in N(\lambda) \). Then

\[
N(\lambda) = \bigoplus_{l, k=0}^\infty \mathcal{U}(\mathfrak{t}_n^C) v_{lk}
\]

(53)

and

\[
N(\lambda) \text{ is irreducible} \iff E_0 \notin \left\{ \frac{n}{2} - 1, \ldots, \frac{n}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor \right\}.
\]

(54)

Moreover if \( E_0 = \frac{n}{2} - j \) for some \( j \in \{1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \} \) then

\[
L(\lambda) = L\left( -\frac{n}{2} + j, 0, \ldots, 0 \right) \simeq \frac{N(\lambda)}{\bigoplus_{l, k=0}^\infty \mathcal{U}(\mathfrak{t}_n^C) v_{l,j+k}} \simeq \bigoplus_{l=0}^{\infty} \bigoplus_{k=0}^{j-1} \mathcal{U}(\mathfrak{t}_n^C) v_{lk}.
\]

(55)
Corollary 2 Let us write \( \chi(\mu_1) \otimes \pi(\mu_2, \ldots, \mu_r) \) for the irreducible representation, with \( HW \mu \), of \( \mathfrak{t}_n \) on \( K(\mu) \). Then

\[
D(E_0, \ldots, 0)|_{\mathfrak{t}_n} = \bigoplus_{l=0}^{\infty} \bigoplus_{k=0}^{\infty} \chi(-[E_0 + l + 2k]) \otimes \pi(l, 0, \ldots, 0)
\]

(56)

if \( E_0 \notin \frac{n}{2} - \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor \}, \)

\[
D(E_0, \ldots, 0)|_{\mathfrak{t}_n} = \bigoplus_{l=0}^{\infty} \bigoplus_{k=0}^{\infty} \chi(-[E_0 + l + 2k]) \otimes \pi(l, 0, \ldots, 0)
\]

(57)

if \( E_0 = \frac{n}{2} - j \) for some \( j \in \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor \}. \)

Remark 2

1. The value \( j = 1 \) corresponds to the \( C_n \)-massless case:

\[
D\left(\frac{n-2}{2}, 0, \ldots, 0\right)|_{\mathfrak{t}_n} = \bigoplus_{l=0}^{\infty} \chi(-[E_0 + l]) \otimes \pi(l, 0, \ldots, 0)
\]

which is a particular case of Proposition [3].

2. Thanks to the preceding results one can see that indecomposability arises when \( E_0 \) reaches the value \( \frac{n}{2} - j \) (we use the same notations):

\[
D(E_0, 0, \ldots, 0) \xrightarrow{E_0 \to \frac{n}{2} - j} D\left(\frac{n}{2} - j, 0, \ldots, 0\right) + D\left(\frac{n}{2} + j, 0, \ldots, 0\right).
\]

(58)

Proof of the Proposition. The \( v_{lk} \)'s are maximal vectors for \( D(E_0, 0, \ldots, 0)|_{\mathfrak{t}_n} \); indeed one has \([n^+ \cap \mathfrak{t}_n^C, X_{-1,2}] = 0 \) and \([\mathfrak{t}_n^C, Z] = 0 \), thus \( n^+ \cap \mathfrak{t}_n^C v_{lk} = 0 \). \( N(\lambda) \) is generated by the monomials \( \prod_{|j| \neq 1} X_{-1,j}^{q_j} v_\lambda \) where \( (q_j)_{|j| \neq 1} \) is a family of naturel integers and, if \( |j| \neq 1 \):

\[
X_{-1,j} v_{lk} = \begin{cases} \frac{1}{t+1} X_{-2,j} v_{l+1,k} & \text{if } |j| \neq 2, \\ v_{l+1,k} & \text{if } j = 2, \\ \frac{1}{t+1} v_{l+1,k+1} - \frac{1}{(t+1)(t+n)} \sum_{|h| \neq 1,2} X_{-2,-h} X_{-2,h} v_{l+1,k+1} & \text{if } j = -2, \end{cases}
\]

where \( v_{-1,k} = 0 \); thus one has \( p^- v_{lk} \subset U(\mathfrak{t}_n^C) v_{l-1,k+1} + U(\mathfrak{t}_n^C) v_{l+1,k} \). Since \( [p^-, \mathfrak{t}_n^C] \subset p^- \) one can conclude that

\[
N(\lambda) = U(p^-) v_\lambda \subset \bigoplus_{l,k=0}^{\infty} U(\mathfrak{t}_n^C) v_{lk}.
\]

Now \( p^+ \) = \( \{ X_{1j}, -r \leq j \leq r \text{ and } |j| \neq 1 \} \) and \( |j| \neq 1 \) implies

\[
X_{1j} v_{lk} = \delta_{j,-2l} (E_0 + 2k + l - 1) v_{l-1,k} + 2k (E_0 - \frac{n}{2} + k) X_{-1,j} v_{l,k-1},
\]

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with \( v_{-1,k} = v_{l,-1} = 0 \); thus for a maximal vector for which the weight is strictly less than \( \lambda \), necessarily proportional to some \( v_k \), one must have \( k(E_0 - \frac{n-1}{2}) = 0 \) and \( l = 0 \), i.e. \( l = 0, k \neq 0 \) and \( E_0 - \frac{n-1}{2} = 0 \). \( E_0 \) being strictly positive one has \( 1 \leq k \leq \left\lceil \frac{n-1}{2} \right\rceil \).

Finally let \( j \in \{1, \ldots, \left\lceil \frac{n-1}{2} \right\rceil \} \), \( E_0 = \frac{n}{2} - j \) and \( K_j = \bigoplus_{i=0}^{\infty} \bigoplus_{k=j}^{\infty} \mathcal{U}(\mathcal{C}_n^k) v_{jk} \). Then the relation

\[
p_n^C v_{jk} \subset \mathcal{U}(\mathcal{C}_n^k) \langle \ v_{l-1,k}; v_{l+1,k}; v_{l-1,k+1}; v_{l+1,k+1} \rangle
\]

implies \( \mathcal{U}(\mathcal{C}_n^k) K_j \subset K_j \), so that

\[
L(-\left\lceil \frac{n}{2} - j \right\rceil, 0, \ldots, 0) = N(-\left\lceil \frac{n}{2} - j \right\rceil, 0, \ldots, 0) / K_j.
\]

### 6.2.2 A Gupta-Bleuler triplet for the \( C_n \)-massless \( D(\frac{n-2}{2}, 0, \ldots, 0) \)

Using the preceding notations and results one can see that \( D(\frac{n-2}{2} + \varepsilon, 0, \ldots, 0) \) sends the operator \( \mathbb{Z} \) to zero if \( \varepsilon = 0 \) but it does not if \( \varepsilon \neq 0 \). It is precisely this fact which gives us the desired indecomposable representations. Indeed, let \( \varepsilon > 0 \) and \( E_0 = \frac{n-2}{2} + \varepsilon \). Then \( D(E_0, 0, \ldots, 0) \) is irreducible, but when \( \varepsilon \to 0 \) one obtains, from Remark 2 and for \( j = 1 \), an indecomposable representation:

\[
D(\frac{n-2}{2}, 0, \ldots, 0, \varepsilon) \to D(\frac{n-2}{2} + \varepsilon, 0, \ldots, 0, 0) \to D(\frac{n-2}{2}, 0, \ldots, 0, 0).
\]

(59)

In order to construct explicitly a Gupta-Bleuler (GB) triplet \cite{gb}, let \( \rho > 0 \) and let:

\[
H^{2,n}_\rho = \{ y, y = \sum_{a=-1}^{n} y^a e_a \in \mathbb{R}^{2,n} \text{ such that } y^2 = 1/\rho \}
\]

where \( y^2 = y^a y_a = y_1^2 + y_0^2 + y^2 \). The De Sitter space-time is the universal covering of \( H^{2,n}_\rho \). The action of \( \mathcal{G}_n \) on \( \mathcal{C}^\infty \)-functions defined on \( H^{2,n}_\rho \) is well known:

\[
U_\lambda(M_{ab}) = L_{ab} = y_a \partial_b - y_b \partial_a
\]

(60)

where \( \partial_c = \frac{\partial}{\partial y^c} \). Let \( \partial^2 = \partial^a \partial_a \) and \( \delta = y^a \partial_a \). Then one has:

\[
U_\lambda(C_2) = -\frac{1}{2} L_{ab} L^{ab} = -y^2 \partial^2 + \delta(\delta + n).
\]

(61)

Now the resolution of the Laplace-Beltrami equation on \( H^{2,n}_\rho \) is standard \cite{gb}. One finds that the following solutions form a Hilbertian basis for \( L^2(\partial^{2,n}_\rho) \), with \( d\mu(y) = \frac{1}{\rho^{-1} + y^2} \pi \text{det} n y \):

\[
\psi_{\lambda klm}^{E_0}(t, y) = \left[ \rho^{-2k+E_0-\frac{n-1}{2}} \frac{\Gamma(k + E_0 + l) \Gamma(k + 1)}{\Gamma(k + \frac{n}{2} + l) \Gamma(k + E_0 - \frac{n-1}{2})} \right]^{1/2} \times \times e^{-i(E_0+l+2k)t} \rho^{-1} y^2 - \frac{E_0 + l}{2} \left( y^2 \right)^{1/2} \times \times \rho^{l+\frac{n-1}{2}} E_0 - \frac{n}{2} \left( \rho^{-1} y^2 \right)^{1/2} Y^l_m \left( \frac{y}{\sqrt{y^2}} \right),
\]

(62)
where $P_k^{(\alpha,\beta)}$ are the Jacobi polynomials, $l = (l_2, \ldots, l_{\lfloor\frac{n+1}{2}\rfloor})$ and $m = (m_1, \ldots, m_{\lfloor\frac{n}{2}\rfloor})$ are vectors, in $\mathbb{N}^{r-1+\nu}$ and $\mathbb{N}^{r-1}$ respectively, subject to certain conditions, $l = l_2$, $Y_{lm}$ are the spherical harmonics on $S^{n-1}$ and $e^{it} = \left(\frac{y^{-1}+iy^0}{y^{-1}+iy^0}\right)^{1/2}$. The scalar product we use to normalize these functions is given by:

\[
(\psi, \psi') = \int_{\mathbb{R}^n} \frac{\overline{\psi}(y) \leftrightarrow \partial \psi'(y)}{\rho^{-1} + y^2} \, d^n y,
\]

where $\overline{\psi}(y) \leftrightarrow A \psi'(y) = A\overline{\psi}(y)\psi'(y) + \overline{\psi}(y) A\psi'(y)$. We extend the functions $\psi^{E_0}_{lm}$ to $H^{2,n}_+ = \cup_{\rho>0} H^{2,n}_\rho$ by fixing the degree of homogeneity: $\delta\psi = -E_0\psi$. Then, $\psi$ being in the kernel of $\partial^2$, one has:

\[
U_\lambda(C_2)\psi = E_0(E_0 - n)\psi
\] (64)

Let:

\[
x_{\pm j} = \begin{cases} 
\sqrt{2}(y^{-1} \pm iy^0) & \text{if } j = 1, \\
\frac{1}{\sqrt{2}}(y^{2j-1} \pm iy^{2j}) & \text{if } 2 \leq j \leq r, \\
y^n & \text{if } n \text{ is odd and } j = 0,
\end{cases}
\]

Then one has $y^2 = -\sum_{j=-r}^r x_{-j}x_j$, $\partial^2 = -\sum_{j=-r}^r \partial_{-j}\partial_j$, $\delta = \sum_{j=-r}^r x_{-j}\partial_j$ and one can choose $X_{jk}$ such that:

\[
U_\lambda(X_{jk}) = x_k\partial_j - x_j\partial_k.
\] (65)

Let $\varphi_2(y) = x_1^{-E_0}$. Then $\varphi_2$ is, up to a multiplicative constant, the maximal vector of $U_\lambda$ and $\psi^{E_0}_{lm} \in \mathcal{U}(G_n)\varphi_2$. Moreover one finds that:

\[
(Z\varphi_2)(y) = -E_0(E_0 + 1)y^2x_1^{-E_0 - 2} - 2\varepsilon E_0x_1^{-E_0 - 2},
\] (66)

thus

\[
\lim_{\varepsilon \to 0}(Z\varphi_2)(y) = -\frac{n-2}{2}y^2x_1^{-\frac{n+2}{2}}.
\] (67)

Now assume $\varepsilon = 0$ and let $\varphi_1(y) = x_1^{-\frac{n}{2}}$ and $\varphi_3(y) = y^2x_1^{-\frac{n+2}{2}}$. Then

\[
\varphi_1 \xrightarrow{\frac{1}{n}Z} \varphi_2 \xrightarrow{-\frac{2}{n-2}Z} \varphi_3
\] (68)

---

5 We use the notations of the preceding subsubsection.
where \( \overline{Z} = \sum_{j|j \neq 1} X_{1j}X_{1,-j} \) and one has:

\[
\partial^2 \varphi_2 = \partial^2 \varphi_3 = 0
\]

whereas

\[
\partial^2 \varphi_1 = \frac{1}{y^2} \varphi_3 \neq 0, \quad \text{but} \quad (\partial^2)^2 \varphi_1 = 0.
\]

Let \( cl(V) \) denotes the closure of any topological space \( V \) and let \( \mathcal{H}_i^{(0)} = cl(U(G_n^C)\varphi_i) \), \( i \) taking the value 1, 2 or 3. Then it is not difficult to prove the following.

**Proposition 6**  
1. \( \mathcal{H}_1^{(0)} \supset \mathcal{H}_2^{(0)} \supset \mathcal{H}_3^{(0)} \) and \( \mathcal{H}_i^{(0)}, i = 2 \) or 3, is a closed invariant subspace of \( \mathcal{H}_{i-1} \);

2. \( \mathcal{H}_1^{(0)}/\mathcal{H}_2^{(0)} \) and \( \mathcal{H}_3^{(0)} \) carry the IR \( D(\frac{n+2}{2}, 0, \ldots , 0) \), while \( \mathcal{H}_2^{(0)}/\mathcal{H}_3^{(0)} \) carries the \( C_n \)-massless representation \( D(\frac{n-2}{2}, 0, \ldots , 0) \).

3. 

\[
\left[ U_{\lambda}(C_2) + \frac{(n-2)(n+2)}{4} \right] \varphi_i = 0 \quad \text{if} \quad i = 2 \text{ or } 3, \quad (69)
\]

\[
\left[ U_{\lambda}(C_2) + \frac{(n-2)(n+2)}{4} \right] \varphi_1 = n \varphi_3 \neq 0, \quad \left[ U_{\lambda}(C_2) + \frac{(n-2)(n+2)}{4} \right]^2 \varphi_1 = 0.
\]

4. \( \lim_{y \to 0} \varphi(y) = 0 \forall \varphi \in \mathcal{H}_3^{(0)} \). Thus the \( C_n \)-massless \( D(\frac{n-2}{2}, 0, \ldots , 0) \) may be realized irreducibly on the cone \( Q_{2,n} = \{ y, y \in \mathbb{R}^{2,n} \text{ such that } y^2 = 0 \} \).

**Definition 4** In analogy with QED on 4-dimensional Minkowski space we call the elements of \( \mathcal{H}_s^{(0)} = \mathcal{H}_1^{(0)}/\mathcal{H}_2^{(0)} \) (resp. \( \mathcal{H}_p^{(0)} = \mathcal{H}_2^{(0)}/\mathcal{H}_3^{(0)} \), resp. \( \mathcal{H}_g^{(0)} = \mathcal{H}_3^{(0)} \)) scalar (resp. physical, resp. gauge) states.

**Remark 3** Let \( \mathcal{H}^{(0)} \) the closure of the \( G_n^C \)-module generated by \( y \mapsto x_1^{-\frac{n+2}{2}} \); it carries the IR \( D(\frac{n+2}{2}, 0, \ldots , 0) \). Let \( \partial^4 = (\partial^2)^2 \) and let us identify \( y^2 \) to the corresponding operator. Then the GB triplet

\[
D(\frac{n+2}{2}, 0, \ldots , 0) \rightarrow D(\frac{n-2}{2}, 0, \ldots , 0) \rightarrow D(\frac{n+2}{2}, 0, \ldots , 0)
\]
defined by $\varphi_1, \varphi_2$ and $\varphi_3$ may be defined by:

$$
\mathcal{H}_1^{(0)} = \{ \text{positive energy solutions } f \text{ of } \partial^4 f = 0 \text{ and } \delta f = -\frac{n-2}{2} f \} \\
\mathcal{H}_2^{(0)} = \{ f \in \mathcal{H}_1^{(0)} \text{ such that } \partial^2 f = 0 \} \\
\mathcal{H}_3^{(0)} = \{ f \in \mathcal{H}_2^{(0)} \text{ such that } f \in \mathcal{K}^{(0)} \}.
$$

Now, for $\varphi$ and $\varphi'$ in $\mathcal{H}_1^{(0)}$, define $(\varphi, \varphi')_1 = \int_{S^1 \times \mathbb{R}^n} \frac{1}{y^2} \partial^2 \varphi(y) \frac{\partial^2 \varphi'(y)}{\partial y^2} \, dt \, dy$ and $(\varphi, \varphi')_2 = \int_{S^1 \times \mathbb{R}^n} \frac{1}{y^2} \varphi(y) \partial^2 \varphi'(y) \, dt \, dy$, where $y$ belongs to some $H^{2,n}_0$ (resp. $Q^{2,n}$) in the first (resp. second) integral. Then it is not difficult to choose the constant $c$ such that the form defined by $\langle \varphi, \varphi' \rangle = (\varphi, \varphi')_1 + c(\varphi, \varphi')_2$ is an invariant non-degenerate indefinite metric such that $\langle \varphi_i, \varphi_j \rangle \neq 0$ if and only if $(i, j) \in \{(1, 3), (3, 1), (2, 2)\}$.

**Definition 5** Again in analogy with 4-dimensional Minkowskian QED, the condition $\partial^2 f = 0$, on $f \in \mathcal{H}_2^{(0)}$, which fixes the space $\mathcal{H}_2^{(0)}$ will be called Lorentz condition; the equation $\partial^4 f = 0$ will be called the dipole equation.

### 6.3 $C_n$-masslessness, spin 1/2

#### 6.3.1 Reduction on $\mathfrak{t}_n$ and indecomposability of $D(E_0, \frac{1}{2}, \ldots, \frac{1}{2})$

The following result is known; see for example [1, 7].

**Proposition 7** $D(E_0, \frac{1}{2}, \ldots, \frac{1}{2})$ is unitarizable if and only if $E_0 \geq \frac{n-1}{2}$.

Here we consider only the unitary case, i.e. $E_0 \geq \frac{n-1}{2}$.

**Proposition 8** Let $\lambda = (-E_0, \frac{1}{2}, \ldots, \frac{1}{2})$ and recall that $\nu = 0$ (resp. 1) if $n$ is even (resp. odd).

1. If $E_0 > \frac{n-1}{2}$ then $D(E_0, \frac{1}{2}, \ldots, \frac{1}{2})$ is irreducible and one has:

$$
D(E_0, \frac{1}{2}, \ldots, \frac{1}{2})|_{\mathfrak{t}_n} = \bigoplus_{l,k=0}^{\infty} \chi(-[E_0 + l + 2k]) \otimes \pi(\frac{1}{2} + l, \frac{1}{2}, \ldots, \frac{1}{2}) \oplus \bigoplus_{l,k=0}^{\infty} \chi(-[E_0 + l + 2k + 1]) \otimes \pi(\frac{1}{2} + l, \ldots, \frac{1}{2}, \nu - \frac{1}{2}).
$$

2. If $E_0 = \frac{n-1}{2}$ then $N(\lambda)$ is not simple; it contains a maximal submodule isomorphic to $L(-\frac{n+1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \nu - \frac{1}{2})$ which carries the UIR $D(\frac{n+1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \nu - \frac{1}{2})$. The irreducible one $D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is carried by the quotient.

**Proof.**
1. If $E_0 > \frac{n-1}{2}$ then $D(E_0 + \frac{1}{2}, 0, \ldots, 0) \otimes D(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) = D(E_0, \frac{1}{2}, \ldots, \frac{1}{2}) \oplus D(E_0 + 1, \frac{1}{2}, \ldots, \frac{1}{2}, \nu - \frac{1}{2})$. If we denote by $v_{\sigma}$ the maximal vector of $D(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ one finds that, for $l, k \in \mathbb{N}$, the vectors $v_{lk} \otimes v_{\sigma}$ and $v_{lk} \otimes (X_{-1, \nu-1}, v_{\sigma})$ generate a submodule (of the tensor product) isomorphic to $L(\lambda)$.

2. Now assume $E_0 = \frac{n-1}{2}$ and let $Y^\nu = \frac{1}{\nu+1} X_{-1, [\nu-1]} - \sum_{j=2}^{\nu} X_{-1, j} X_{j, [\nu-1]}$. Then one can see that $Y^\nu(v_{00} \otimes v_{\sigma})$ generates an irreducible submodule of $U(\mathcal{G}_n)(v_{00} \otimes v_{\sigma})$ isomorphic to $L(-\frac{n+1}{2}, \frac{1}{2}, \ldots, \nu - \frac{1}{2})$ while $D(-\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is carried by the quotient $U(\mathcal{G}_n)(v_{00} \otimes v_{\sigma})/U(\mathcal{G}_n)Y^\nu(v_{00} \otimes v_{\sigma})$.

### 6.3.2 A Gupta-Bleuler triplet for $D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$

Let $\varepsilon \geq 0$ such that $E_0 = \frac{n+1}{2} - \varepsilon$. Proposition 3 says that if $\varepsilon = 0$ then $Y^\nu$ is sent to 0 by $U_\lambda = D(E_0, \frac{1}{2}, \ldots, \frac{1}{2})$. Now assume $\varepsilon > 0$, then $U_\lambda$ is irreducible; but when $\varepsilon \to 0$ one obtains an indecomposable representation:

$$D\left(\frac{n-1}{2} + \varepsilon, \frac{1}{2}, \ldots, \frac{1}{2}\right) \to D\left(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right) + D\left(\frac{n+1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \nu - \frac{1}{2}\right).$$

(71)

To construct a Gupta-Bleuler triplet we need explicit realizations of the representations concerned. Let $\sigma = (\frac{1}{2}, \ldots, \frac{1}{2})$ and let, if $n$ is even, $\sigma^- = (\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$. We denote by $S_\sigma$ the irreducible spinor representation $D(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and, when $n$ is even, by $S_{\sigma^-}$ the irreducible one $D(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$. Let $S_+ = \text{carrier space of } S_\sigma$ and $S_- = \text{carrier space of } S_{\sigma^-}$ when $n$ is even (resp. $\{0\}$ when $n$ is odd). Finally let $S = S_+ \oplus S_-$ be the spinor module of $\mathcal{G}_n$.

Let $\gamma_1, \ldots, \gamma_{2r-2}$ be $2r$ matrices in $\mathfrak{gl}(S)$ such that $[\gamma_a, \gamma_b] = 2\eta_{ab}$ where $[A, B] = AB + BA$, and let $\gamma_{2r-1} \in \mathbb{C}\gamma_1 \cdots \gamma_{2r-2}$ such that $\gamma_{2r-1}^2 = -1$. Then:

$$[\gamma_a, \gamma_b] = 2\eta_{ab} \quad \forall a, b \in \{-1, \ldots, n\}.$$

The following realization of $S_\sigma$ on $S$ is well known:

$$M_{\gamma_a, \gamma_b} \to S_{\gamma_a, \gamma_b} = \frac{1}{4}[\gamma_a, \gamma_b] = \frac{1}{2}(\gamma_a \gamma_b - \eta_{ab}).$$

Later we shall need also the generators $\omega_j$ defined by:

$$\omega_{\pm j} = \begin{cases} \frac{i}{2}(\gamma_{-1} \pm i\gamma_0) & \text{if } j = 1, \\
 \frac{i}{2}(\gamma_{2j-1} \pm i\gamma_{2j}) & \text{if } 2 \leq j \leq r, \\
 \gamma_n & \text{if } n \text{ is odd and } j = 0, \end{cases}$$

Thus one has:

$$[\omega_j, \omega_k] = -2\delta_{j-k} \quad \forall j, k \in \{-r, \ldots, r\}$$

\(^6\)We identify the identity of $\mathfrak{gl}(S)$ with 1.
and the preceding realization of $S_\sigma$ may be written:

$$X_{jk} \rightarrow \frac{1}{4} [\omega_j, \omega_k] = \frac{1}{2} (\omega_j \omega_k + \delta_{j,-k}).$$

We realize $D(E_0, \frac{1}{2}, \ldots, \frac{1}{2})$ on spinor fields $\Psi : H^2_n \rightarrow S$ such that

$$\partial^2 \Psi = 0 \quad \text{and} \quad \delta \Psi = -(E_0 + \frac{1}{2}) \Psi.$$

The action of $G_n$ on spinor fields is given by $U(\lambda)(M_{ab}) = L_{ab} + S_{ab}$. Let:

$$\gamma = \sum_{a=-1}^n y^a \gamma_a = \sum_{j=-r}^r x_{-j} \omega_j \quad \text{and} \quad \partial = \sum_{a=-1}^n \partial^a \gamma_a = - \sum_{j=-r}^r \partial_{-j} \omega_j.$$

Then

$$U(\lambda(C_2)) \Psi = \left[ -y^2 \partial^2 + \delta (\delta + n + 1) + \frac{(n+1)(n+2)}{8} - \frac{\gamma \partial}{2} \right] \Psi \quad (72)$$

$$= \left[ (E_0 + \frac{1}{2})(E_0 - \frac{1}{2} - n) + \frac{(n+1)(n+2)}{8} - \frac{\gamma \partial}{2} \right] \Psi.$$

It is easy to prove the following Lemma.

**Lemma 1**

1. $\gamma$ and $\partial$ commute with the action of $G_n$;

2. $[\gamma, \partial] = 2\delta + n + 2$;

3. $-y^2 \partial^2 = \gamma \partial (\gamma \partial - 2\delta - n)$;

4. if $\varepsilon > 0$ then $(-2\varepsilon)^{-1}(\partial \gamma - 2)$ and $(-2\varepsilon)^{-1}\gamma \partial$ are projectors on the irreducible subspaces of the tensor product $L(-[E_0 + \frac{1}{2}], 0, \ldots, 0) \otimes L(\frac{1}{2}, \ldots, \frac{1}{2})$, namely the spaces $L(-E_0, \frac{1}{2}, \ldots, \frac{1}{2})$ and $L(-[E_0 + 1], \frac{1}{2}, \ldots, \frac{1}{2}, \nu - \frac{1}{2})$ respectively.

Let us consider the spinor fields $\Psi_2$ and $\Psi_3$ defined by

$$\Psi_2(y) = x_{1}^{-E_0 - \frac{1}{2}} v_\sigma \quad \text{and} \quad \Psi_3(y) = \gamma x_{1}^{-E_0 - \frac{1}{2}} \omega_{[\nu-1]} v_\sigma.$$

Then one has:

$$L(-[E_0 + \frac{1}{2}], 0, \ldots, 0) \otimes L(\frac{1}{2}, \ldots, \frac{1}{2}) \simeq U(G_n) \Psi_2 \oplus U(G_n^c) \Psi_3.$$

Moreover let $\Psi_1(y) = x_{1}^{-E_0 - \frac{1}{2}} \omega_{-1} \omega_{[\nu-1]}$. Then:

$$Y^\nu \Psi_2 = \frac{1}{2} (E_0 + \frac{1}{2}) \Psi_3 - \varepsilon \frac{1}{2} \Psi_1, \quad (73)$$

We identify $y^2$ with the function $y \mapsto y^2$, $\gamma$ with $y \mapsto \gamma$, and so on.
thus
\[ \lim_{\varepsilon \to 0} Y^\varepsilon \Phi = \frac{1}{2} (E_0 + \frac{1}{2}) \Phi. \] (74)

From now on we assume \( \varepsilon = 0 \), i.e. \( E_0 = \frac{n-1}{2} \). Then:
\[
\begin{array}{c}
\Psi_1 \xrightarrow{- \frac{1}{2} x_{1,-[\nu-1]}} \Psi_2 \xrightarrow{\Phi} \Psi_3.
\end{array}
\] (75)

Let \( \mathcal{H}_i^{(1/2)} = \text{cl}(U(\mathcal{G}^\mathbb{C}_n)\Phi) \), \( i \) being 1, 2 or 3. The next proposition is not difficult to prove.

**Proposition 9**  
1. \( \mathcal{H}_1^{(1/2)} \supset \mathcal{H}_2^{(1/2)} \supset \mathcal{H}_3^{(1/2)} \) and \( \mathcal{H}_i^{(1/2)} \), \( i = 2 \) or 3, is a closed invariant subspace of \( \mathcal{H}_i^{(1/2)} \);
2. \( \mathcal{H}_1^{(1/2)} \cap \mathcal{H}_2^{(1/2)} \) and \( \mathcal{H}_3^{(1/2)} \) carry the IR \( D(\frac{n+1}{2}, \frac{1}{2}, \ldots, \nu - \frac{1}{2}) \), while \( \mathcal{H}_2^{(1/2)} \cap \mathcal{H}_3^{(1/2)} \) carries the \( C_n \)-massless representation \( D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \);
3. \( \Phi \Psi = 0 \) if \( i = 2 \) or \( i = 3 \), \( \Psi \Phi \Psi_1 = n \Psi_3 \neq 0 \) but \( (\Phi \Phi) \Psi_1 = 0 \).
4. \( \lim_{y \to 0} (\Phi \Psi)(y) = 0 \) if \( \Psi \in \mathcal{H}_3^{(1/2)} \) and \( \lim_{y \to 0} (\Phi \Psi_2)(y) \neq 0 \). Thus the \( C_n \)-massless representation \( D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \) may be realized irreducibly on the cone \( Q^{2,n} \).

**Definition 6** The elements of the space \( \mathcal{H}_S^{(1/2)} = \mathcal{H}_1^{(1/2)} \cap \mathcal{H}_2^{(1/2)} \) (resp. \( \mathcal{H}_P^{(1/2)} = \mathcal{H}_2^{(1/2)} \cap \mathcal{H}_3^{(1/2)} \), resp. \( \mathcal{H}_G^{(1/2)} = \mathcal{H}_3^{(1/2)} \)) are called scalar (resp. physical, resp. gauge) states.

**Remark 4** Let \( \mathcal{K}^{(1/2)} \) be the closure of the \( \mathcal{G}_n^\mathbb{C} \)-module generated by the field \( y \mapsto \Phi(y) = x_1^{-\frac{n+1}{2}} \omega_{[\nu-1], \nu} \); it carries the IR \( D(\frac{n+1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}) \). Then the Gupta-Bleuler triplet
\[ D(\frac{n+1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \nu - \frac{1}{2}) \rightarrow D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \rightarrow D(\frac{n+1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \nu - \frac{1}{2}) \]
defined by \( \Psi_1, \Psi_2 \) and \( \Psi_3 \) may be redefined by:
\[ \mathcal{H}_1^{(1/2)} = \{ \text{positive energy solutions of } \partial^2 \Psi = 0, \delta \Psi = -\frac{n}{2} \Psi \text{ and } (\Phi \Phi) \Psi = 0 \}, \]
\[ \mathcal{H}_2^{(1/2)} = \{ \Psi \in \mathcal{H}_1^{(1/2)} \text{ such that } \Phi \Psi = 0 \}, \]
\[ \mathcal{H}_3^{(1/2)} = \{ \Psi \in \mathcal{H}_2^{(1/2)} \text{ such that } \Psi \in \Phi \mathcal{H}^{(1/2)} \}. \] (77)
Now, for $\Psi$ and $\Psi'$ in $\mathcal{H}_1^{(1/2)}$, define $(\Psi, \Psi'_1) = \rho^{-1} \int_{S^1 \times \mathbb{R}^n} \Psi^*(y) \mathcal{D} \Psi(y) \frac{dtdy}{\rho + 1 + y^2}$ and $(\Psi, \Psi')_2 = \int_{S^1 \times \mathbb{R}^{n-1}} (y^2) \frac{1}{2} \Psi^*(y) \Psi(y) dtd\Omega$, $y$ being in some $H^2_{\rho,n}$ (resp. $Q^2_{n}$) in the first (resp. second) integral. Again it is not difficult to choose the constant $c$ such that the form defined by $\langle \Psi, \Psi' \rangle = (\Psi, \Psi'_1)_1 + c(\Psi, \Psi')_2$ is an invariant non degenerate indefinite metric such that $\langle \Psi_i, \Psi_j \rangle \neq 0$ if and only if $(i, j) \in \{(1, 3), (3, 1), (2, 2)\}$.

**Definition 7** The equation $\mathcal{D} \Psi = 0$, which fixes the space $\mathcal{H}_2^{(1/2)}$, will be called the Lorentz condition.

### 6.4 Indecomposability and GB triplets for spin $s \geq 1$

We assume in this subsection that $s \geq 1$ and $2s \in \mathbb{N}$.

#### 6.4.1 Indecomposability of $D(E_0, s, \ldots, s, s_\nu)$

Let $\lambda = (-E_0, s, \ldots, s, s_\nu)$, where $|s_\nu| = s$ and, if $n$ is odd, $s_\nu \geq 0$.

**Proposition 10**

1. $D(E_0, s, \ldots, s, s_\nu)$ is unitarizable $\iff$ $E_0 \geq \frac{n-2+\nu}{2} + s$;

2. if $E_0 > \frac{n-2+\nu}{2} + s$ then $N(\lambda)$ is simple;

3. if $E_0 = \frac{n-2+\nu}{2} + s$ then $N(\lambda)$ contains, up to a multiplicative constant, a unique maximal vector of weight $(-E_0 - 1, s, \ldots, s, s_\nu - \frac{2\nu}{s})$; it is given by $Y_{-1, -\frac{2\nu}{s}, r, \nu}^\nu$, where

$$
Y_{-1, \pm r}^0 = 2sX_{-1, \pm r} - \sum_{j=2}^r X_{-1, j}X_{-j, \pm r},
$$

$$
Y_{-1, -r}^1 = 2sX_{-1, -r} + 2X_{-1, 0}X_{-r, 0} - \frac{2(s-1)}{2s-1} \sum_{j=2}^r X_{-1, j}X_{-j, -r} - \frac{2}{2s-1} \sum_{j=2}^r X_{-1, j}X_{-j, 0} X_{-r, 0}.
$$

Since, for $n$ even, the treatment of $U_\lambda$ is similar for both signs of $s_\nu$ we shall consider from now on that $s_\nu = s$.

**Proof of the Proposition.** For the first two items see [4, 5]. For the last one, a maximal vector of weight $(-E_0 - 1, s, \ldots, s, s - 1)$ for $n$ even has the general form

$$
v' = (aX_{-1, -r} + \sum_{j=2}^r b_j X_{-1, j}X_{-j, -r})v_\lambda,
$$

and $n^+v' = 0$ implies $b_j = -\frac{a}{2s}$ for each $j$. The same technique works for odd $n$.

**Remark 5** The situation for $s \geq 1$, for both $n$ even and $n$ odd, is more complicated than for the spin $0$ and spin $1/2$ cases. Indeed more than one submodule for $N(\lambda)$ exists when $E_0 = \frac{n-2+\nu}{2} + s$, thus it is a priori possible to construct very different examples of Gupta-Bleuler triplets $U' \rightarrow U_\lambda \rightarrow U'$ with $U'$ unitary.
A GB triplet for $D\left(\frac{n-2+\nu}{2} + s + i, s, \ldots, s, s - i\right)$, $i = 1$ or 2

Let $E_0 = \frac{n-2+\nu}{2} + s + \varepsilon, \varepsilon \geq 0$. To realize our Gupta-Bleuler triplet we need explicitly the representations $D(E_0, s, \ldots, s)$ and $D(E_0 + 1, s, \ldots, s, s - 1)$, especially for $\varepsilon = 0$. Both of them are contained in the reduction of the tensor product $D(E_0 + s, 0, \ldots, 0) \otimes D(-s, s, \ldots, s)$. The representation $S_{[2s]} = D(-s, s, \ldots, s)$ itself is contained in the tensor power $S_0^\otimes 2s$ of the irreducible spinorial representation.

We define the action of $M_{ab} \in G_n$ on a tensor $v_1 \otimes \cdots \otimes v_{2s} \in S_0^\otimes 2s$ by:

$$S_{ab}(v_1 \otimes \cdots \otimes v_{2s}) = \sum_{t=1}^{2s} v_1 \otimes \cdots \otimes \frac{1}{4}\{\gamma_a, \gamma_b\} v_t \otimes \cdots \otimes v_{2s} = \sum_{t=1}^{2s} S_{ab}(v_1 \otimes \cdots \otimes v_t \otimes \cdots \otimes v_{2s}).$$

Let $\gamma_a(t)$ be defined on the tensors of $S_0^\otimes 2s = (S_+ \oplus S_-)^\otimes 2s$ by:

$$\gamma_a(t)(v_1 \otimes \cdots \otimes v_{2s}) = v_1 \otimes \cdots \otimes \gamma_a v_t \otimes \cdots \otimes v_{2s}.$$

Then the action defined in (78) may be written more simply:

$$M_{ab} \rightarrow S_{ab} = \sum_{t=1}^{2s} S_{ab} = \sum_{t=1}^{2s} \frac{1}{4}\{\gamma_a(t), \gamma_b(t)\}. $$

Let $\text{Sym}(S_0^\otimes 2s)$ be the space of symmetric tensors in $S_0^\otimes 2s$ and let $\gamma(t)\gamma(t') = \gamma_a(t)\gamma_a(t') = -\sum_{j=-r}^{r} x_{-j}^t \omega_j^{t'} - \sum_{j=-r}^{r} \omega_j^{t'} x_{-j}^t$.

**Proposition 11** $S_{[2s]} = D(-s, s, \ldots, s)$ is realized irreducibly on the space:

$$V^S = \text{Sym}(S_0^\otimes 2s) \cap \left[ \bigcap_{t, t'} \ker (\gamma(t)\gamma(t') - \nu) \right].$$

We realize the unitary representations of interest on tensor-spinors $\Psi : H_+^{2n} \rightarrow V^S$ such that

$$\partial^2 \Psi = 0 \quad \text{and} \quad \delta \Psi = -(E_0 + s) \Psi.$$

To this effect, we define the action of $G_n$ on them by $M_{ab} \rightarrow L_{ab} + S_{ab}$. Let

$$y(t) = y^a \gamma_a(t) = \sum_{j=-r}^{r} x_{-j}^t \omega_j^{t} \quad \text{and} \quad \partial(t) = \partial^a \gamma_a(t) = -\sum_{j=-r}^{r} \partial_{-j} \omega_j^{t}$$

*Recall that $S_- = \{0\}$ for $n$ odd.*
then one has

\[ U_\lambda(C_2)\Psi = \left[-y^2 \partial^2 + \delta(\delta + n + 2s) + rs(s + r - 1 + \nu) - \sum_{t=1}^{2s} y^{(t)} \partial^{(t)}\right] \Psi \]  

\[ = \left[(E_0 + s)(E_0 - s - n) + rs(s + r - 1 + \nu) - \sum_{t=1}^{2s} y^{(t)} \partial^{(t)}\right] \Psi. \] \( (80) \)

**Lemma 2**

1. For fixed \( t, y^{(t)} \) and \( \partial^{(t)} \) satisfy the three first items of Lemma 4:

2. if \( t \neq t' \) then \( [y^{(t)}, y^{(t')}] = 0 \) and \( [\partial^{(t)}, \partial^{(t')}] = 0 \);

3. if \( t \neq t' \) then \( [\partial^{(t)}, y^{(t')}\partial^{(t')}] = \gamma^{(t)} \cdot \gamma^{(t')} (= \nu \text{ on } V^S). \)

Let us define, for non-negative integers \( k, l \) and spinors \( v_1, \ldots, v_k \), symmetric tensors in \( S^{\otimes 2s} \) by:

\[ v_1 \cdots v_k = \frac{1}{k!} \sum_{\tau \in \Sigma_k} \tau(v_1) \otimes \cdots \otimes \tau(v_k), \]

\[ v^l_k = \underbrace{v_1 \cdots v_l}_{l \text{ terms}} \]  

\( (81) \)

and let \( \Psi_1, \Psi_2 \) and \( \Psi_3 \) be defined by:

\[ \Psi_1(y) = x_1^{-E_0 - s} \left[ (\omega_{-1} \omega_{-r} v_\sigma) v_\sigma - \nu(\omega_{-1} v_\sigma)(\omega_{-r} v_\sigma) \right] v^{2s-2}_\sigma, \]

\[ \Psi_2(y) = x_1^{-E_0 - s} v^{2s}_\sigma, \]

\[ \Psi_3(y) = x_1^{-E_0 - s - 1} \left[ (y \omega_{-r} v_\sigma) v_\sigma - \nu(y v_\sigma)(\omega_{-r} v_\sigma) \right] v^{2s-2}_\sigma. \]

Then one has

\[ \mathcal{U}(G^C_n)\Psi_2 \oplus \mathcal{U}(G^C_n)\Psi_3 \subset L(-[E_0 + s], 0, \ldots, 0) \otimes L(s, \ldots, s) \]

and one finds that:

\[ Y^{\nu}_{-1, -r} \Psi_2 = s(E_0 + s)\Psi_3 - \varepsilon s \Psi_1, \]  

\( (82) \)

thus

\[ \lim_{\varepsilon \to 0} Y^{\nu}_{-1, -r} \Psi_2 = s(E_0 + s)\Psi_3. \]  

\( (83) \)

From now on we assume \( E_0 = \frac{n-2+\nu}{2} + s. \) Then:

\[ \Psi_1 \xrightarrow{-\frac{1}{2} X_{1, r}} \Psi_2 \xrightarrow{\frac{-2}{n(n-2+\nu)+4a}} Y^{\nu}_{-1, -r} \]

\[ \xrightarrow{\mathcal{H}^{(s)}} \Psi_3. \]  

Let \( \mathcal{H}^{(s)} = \text{cl}(\mathcal{U}(G^C_n)\Psi_i), \) \( i \) being equal to 1, 2 or 3. The next proposition is straightforward:
Proposition 12 1. \( \mathcal{H}_1^{(s)} \supset \mathcal{H}_2^{(s)} \supset \mathcal{H}_3^{(s)} \) and \( \mathcal{H}_i^{(s)} \), \( i = 2 \) or \( 3 \), is a closed invariant subspace of \( \mathcal{H}^{(s)}_{i-1} \).

2. \( \mathcal{H}_1^{(s)}/\mathcal{H}_2^{(s)} \) and \( \mathcal{H}_3^{(s)} \) carry the IR \( D(\frac{n+\nu}{2} + s, s, \ldots, s, s-1) \), while \( \mathcal{H}_2^{(s)}/\mathcal{H}_3^{(s)} \) carries the representation \( D(\frac{n-2+\nu}{2} + s, s, \ldots, s) \);

3. 
   \[ \mathcal{H}^{(t)} \Psi = 0 \forall t \in \{1, \ldots, 2s\} \quad \text{if } i = 2 \text{ or } i = 3; \]
   \[ \sum_{t=1}^{2s} y^{(t)} \mathcal{H}^{(t)} \Psi = (n - 2 + \nu + 4s)\Psi_3 \neq 0 \quad \text{but} \quad \left( \sum_{t=1}^{2s} y^{(t)} \mathcal{H}^{(t)} \right) \Psi_1 = 0; \]

4. \( \lim_{y \to 0} (y^{(1)} \cdots y^{(2s)} \Psi)(y) = 0 \forall \Psi \in \mathcal{H}_3^{(s)} \) and \( \lim_{y \to 0} (y^{(1)} \cdots y^{(2s)} \Psi_2)(y) \neq 0. \)
   Thus the representation \( D(\frac{n-2+\nu}{2} + s, s, \ldots, s) \) may be realized irreducibly on the cone \( Q^{2n} \).

Definition 8 The elements of the space \( \mathcal{H}_3^{(s)} = \mathcal{H}_1^{(s)}/\mathcal{H}_2^{(s)} \) (resp. \( \mathcal{H}_p^{(s)} = \mathcal{H}_2^{(s)}/\mathcal{H}_3^{(s)} \), resp. \( \mathcal{H}_G = \mathcal{H}_3^{(s)} \)) are called scalar (resp. physical, resp. gauge) states.

Let, for \( t \in \mathbb{N} \), \( \nu^t \otimes (v \wedge v') = v^{t+1} \otimes v' - v^t \otimes v \otimes v' \), and let \( \left( \tau_{(t,v')} \right)_{t \leq v} \) be the system of generators (permutations \( t \leftrightarrow t' \) if \( t \neq t' \) and identity if \( t = t' \)) of the group-algebra of \( \mathcal{G}_{2s} \). Let

\[
\begin{aligned}
   y &= \begin{cases}
      \frac{1}{2s} \left[ \sum_{1 \leq t \leq 2s} \tau_{(t,2s)} \right] y^{(2s)} & \text{if } n \text{ is even}; \\
      \frac{1}{2s(2s-1)} \left[ \sum_{1 \leq t \leq 2s-1} \tau_{(t,2s-1)} + \sum_{1 \leq t' \leq 2s-1} \tau_{(t,2s)} \tau_{(t',2s-1)} \right] \left[ y^{(2s-1)} - y^{(2s)} \right] & \text{if } n \text{ is odd}.
   \end{cases}
\end{aligned}
\]

Finally let

\[
\Phi(y) = x_1^{-\frac{n+\nu}{2} - 2s} \times \begin{cases}
   v^{3s-1}_{\sigma} \otimes \omega_r v_{\sigma} & \text{if } n \text{ is even}; \\
   v^{2s-2}_{\sigma} \otimes (v_{\sigma} \wedge \omega_r v_{\sigma}) & \text{if } n \text{ is odd}.
\end{cases}
\]

As in the cases \( s = 0 \) and \( s = 1/2 \) we have here:

Remark 6 Let \( \mathcal{K}^{(s)} \) be the closure of the simple \( \mathcal{G}_n^{(s)} \)-module generated by the field \( \Phi \); it carries the IR \( D(\frac{n+\nu}{2} + s, s, \ldots, s, s-1) \). Then the Gupta-Bleuler triplet
massless particles in arbitrary dimensions

\[ D\left(\frac{n+\nu}{2} + s, s, \ldots, s, s - 1\right) \rightarrow D\left(\frac{n-2+\nu}{2} + s, s, \ldots, s\right) \rightarrow D\left(\frac{n+\nu}{2} + s, s, \ldots, s, s - 1\right) \]
defined by \( \Psi_1, \Psi_2 \) and \( \Psi_3 \) may be redefined by:

\[ \mathcal{H}_1^{(s)} = \{ \text{positive energy solutions of } \partial^2 \Psi = 0, \delta \Psi = (-\frac{n-2+\nu}{2} - 2s) \Psi \text{ and } (\sum_{t=1}^{2s} y^{(t)}y^{(t)})^2 \Psi = 0 \}, \]

\[ \mathcal{H}_2^{(s)} = \{ \Psi \in \mathcal{H}_1^{(s)} \text{ such that } y^{(t)} \Psi = 0 \ \forall t \in \{1, \ldots, 2s\} \}, \]

\[ \mathcal{H}_3^{(s)} = \{ \Psi \in \mathcal{H}_2^{(s)} \text{ such that } \Psi \in \mathcal{Y}^{(s)} \}. \]

Now, as in spin 0 and spin 1/2 cases, one can find an invariant non degenerate form on \( \mathcal{H}_1^{(s)} \). Let \( (\Psi, \Psi')_1 = (\rho^{-1})^{2s+\frac{n}{2}} \int_{S_1 \times \mathbb{R}^n} \Psi^*(y) \sum_{i=1}^{2s} y_i \Psi'(y) \frac{d^n y}{\rho^2 + y^2} \) and \( (\Psi, \Psi')_2 = \int_{S_1 \times S_{n-1}} (y^2)^{\frac{n-2+\nu}{2}+2s} \Psi^*(y) \Psi'(y) d\alpha, \Psi \) and \( \Psi' \) being in \( \mathcal{H}_1^{(s)} \) and \( y \) belongs to some \( H^{2,n}_p \) (resp. \( Q^{2,n} \)) in the first (resp. second) integral. Again it is not difficult to choose the constant \( c \) such that the form defined by \( \langle \Psi, \Psi' \rangle = (\Psi, \Psi')_1 + c(\Psi, \Psi')_2 \) is an invariant non degenerate indefinite metric for which \( \langle \Psi_i, \Psi_j \rangle \neq 0 \) if and only if \( (i, j) \in \{(1, 3), (3, 1), (2, 2)\} \).

**Definition 9** The system of equations \( y^{(t)} \Psi = 0, t \in \{1, \ldots, 2s\} \), which fixes the space \( \mathcal{H}_2^{(s)} \), will be called the Lorentz condition.

### 6.5 Further remarks on GB triplets

The above considerations show that the true generalization of Dirac singletons from 4-dimensional De Sitter space to space-time in dimension \( n \geq 5 \) are in fact the \( C_{n-1} \)-massless representations. Indeed, though \((C_{n-1} \text{-massless}) \otimes (C_{n-1} \text{-massless})\) does not contain \( S_n \)-massless representations in general (this is true only for \( n=3 \) or 4), their restriction to the \( n \)-Lorentz group \( SO_0(1, n-1) \) is irreducible and they are, together with their conjugates, the only representations with that property. Thus they contract to UIRs of the \( n \)-Poincaré group which are trivial on the translations and their weight diagram is very degenerate, in some sense 1-dimensional.

Let us look at indecomposability and construction of gauge theories with GB quantization. The most interesting case is when \( n \geq 5 \). GB triplets are easily constructed for \( C_{n-1} \)-massless representations and for arbitrary spin \( s \) (\( 2s \in \mathbb{N} \)). But for \( S_n \)-massless representations, which represent massless particles on \( n \)-De Sitter space-time, the situation is different. Indeed, if \( n \) is even, \( S_n \)-massless representations exist for arbitrary spin \( s \), but one can construct a GB triplet, with our method, only for \( s \geq 1 \), because for \( s = 0 \) or \( 1/2 \) no indecomposability arises around the corresponding highest (or lowest) weight. Nevertheless these representations occur (once) in the tensor product of a \( C_{n-1} \)-massless representation by a non unitary one, namely \( D(\frac{n-3}{2}, 0, \ldots, 0) \otimes D(\frac{1}{2}, 0, \ldots, 0) = D(\frac{n-2}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \) for spin 0 and \( D(\frac{n-2}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \otimes D(\frac{1}{2}, 0, \ldots, 0) \) for spin 1/2, for which it seems that construction of GB triplets is possible. Thus one can hope to construct, for even \( n \geq 6 \),
a gauge theory analogous to that of $D(1,0) \oplus D(2,0)$ and $D(\frac{3}{2}, \frac{1}{2})$ with the usual Dirac singletons when $n = 4$.

Now assume $n$ is odd. Then $S_n$-massless representations exist only for spin 0 or 1/2 and, again, one cannot construct a GB triplet, but they are contained in the reduction of the same tensor products as in the even case (here the last term in the unitary factor is $\pm \frac{1}{2}$). However, unlike the latter, the representations $D(\frac{1}{2}, 0, \ldots, 0)$ and $D(\frac{3}{2}, 0, \ldots, 0)$, which are also below the unitary limit, cannot be naturally considered as quotients of extensions, while this is the case for the $C_{n-1}$-massless representations $D(\frac{n-3}{2}, 0, \ldots, 0)$ and $D(\frac{n-2}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2})$. Thus for odd $n$ only one factor in the tensor product has naturally GB triplets. One can ask the question of what would be the analogue of a gauge theory in this context.


7 Discussion

First we recall that in the $n = 2$ case the situation is drastically different from $n \geq 3$. Indeed though $C_2$-masslessness is easily defined, there is no good notion of spin and there exist infinitely many $C_2$-massless nonequivalent representations with equivalent restrictions to the 2-Poincaré group; but this is not the case of the restrictions to the 2-De Sitter group, locally isomorphic to $SO_0(2, 1)$. Note however that in this case (as is well known) the full conformal group is infinite-dimensional.

We have shown here that most properties of massless representations are, in some sense, independent of the space-time dimension $n$. But if $n \geq 3$, the property of occurring in the tensor product of two UIRs of the same energy sign is true only for $n = 3$ and $n = 4$. An interpretation is that compositeness of massless particles is not possible in De Sitter space-time with dimension $n \geq 5$. Concerning the Gupta-Bleuler quantization, it can be seen that for general $n \geq 3$ the construction of triplets works with no problem, but for a given massless representation there is no unique solution to the construction of a Gupta-Bleuler triplet.

There is however some ambiguity when one tries to generalize to the $n$-dimensional Anti-De Sitter space $\tilde{SO}_0(2, n - 1)/\tilde{SO}_0(1, n - 1)$, $n \geq 5$, the notion of masslessness. For $n \geq 5$ there is no canonical definition of a massless representation of the $n$-De Sitter group $\tilde{S}_n = SO_0(2, n)$, especially for “spin” $s \geq 1$. Indeed the rank of the compact subalgebra $so(n - 1)$ of the De Sitter algebra is $\geq 2$, instead of 1 in the $n = 4$ case. Thus there are several slightly different alternatives to describe massless particles in De Sitter world in higher dimensions, which coincide for $n = 4$. The two extreme are, for spin $s \geq 1$ ($2s \in \mathbb{N}$), $U(s) = D(s + \frac{n - 2}{2}, s, \ldots, s, \epsilon s)$, where $|\epsilon| = 1$ if $n$ is odd and $\epsilon = 1$ if $n$ is even, and $U'(s) = D(s + n - 3, s, 0, \ldots, 0)$ if $s$ is an integer or $U'(s) = D(s + n - 3, s, 1/2, \ldots, 1/2, \epsilon/2)$ if $s - 1/2$ is an integer. The former are what we call here $S_n$-massless representations for $n$ even (for $n$ odd, $s \geq 1$ there are no $S_n$-massless representations in our sense). The latter have very recently been called, when $n = 5$, massless (in the bulk) in [8, 9].

In what follows we shall compare somewhat in detail various properties of both alternatives. In order to do this we need first to look more closely at the notion of masslessness in the $n$-dimensional Minkowski space $\mathbb{R}^{1,n-1}$. On this basis we then compare the notions of Anti-De Sitter masslessness in $n \geq 5$ dimensions, also in both alternatives.

As in the 4-dimensional Minkowski space one needs, for massless representations of the $n$-Poincaré group $\tilde{P}_n$, the mass operator to be zero (and the representation non trivial). Thus the massless representation, say $U^P$, of interest must be induced by a UIR of a subgroup which is a semi-direct product of a subgroup of the $n$-Lorentz group isomorphic to the Euclidean group $\tilde{E}(n-2) = SO(n-2)\ltimes\mathbb{R}^{n-2}$ by the group of space-time translations $\mathbb{R}^{1,n-1}$. Moreover, for physical reasons, it seems reasonable to eliminate the “continuous spin” in the inducing representation, i.e. we assume that the Euclidean group part of the inducing representation is trivial on the translations subgroup $\mathbb{R}^{n-2}$. It is thus finite dimensional and essentially determined by a UIR $\pi_\lambda$ of $SO_0(n - 2)$ with HW $\lambda$.

A first problem (not appearing in the comparison between our approach and that of
is that the choice of \( \lambda \), to define a spin \( s \geq 1 \), is not unique for \( n \geq 6 \); to make this choice easier one can use the physically sensible fact that the wave equations for massless particles are invariant under the action of the \( n \)-conformal group \( \tilde{C}_n \). Thus we may add the following extension condition, always satisfied for massless representations when \( n = 4 \): \( U_P \) extends to a UIR \( \hat{U} \) of the \( n \)-conformal group. An interesting consequence of this (strong) condition is that \( \lambda \) depends now on a unique parameter \( s \) such that \( 3 \):

\[
\lambda = \begin{cases} 
(s, \ldots, s, \epsilon s), & 2s \in \mathbb{N} \text{ and } |\epsilon| = 1 \text{ if } n \text{ is even,} \\
(s, \ldots, s), & s \in \{0, 1/2\} \text{ if } n \text{ is odd.}
\end{cases}
\]

The bad news, with this condition, is that in odd dimensional space-times (and already for \( n = 5 \)), one can define naturally neither massless particles with spin \( s \geq 1 \) nor helicity. Nevertheless we call here, for uniformity of presentation, a mass zero representation of the \( n \)-Poincaré group which satisfies the extension condition a massless discrete helicity representation (MDHR).

Remark 7 In fact one may define helicity for a large class of representations of \( \tilde{P}_n \), especially when \( n \) is even. Indeed, take \( n \) even and denote by \( \varepsilon = (\varepsilon_{\mu_1 \cdots \mu_n})_{0 \leq \mu_1, \ldots, \mu_n \leq n-1} \) the completely skew-symmetric tensor such that \( \varepsilon_{01 \cdots (n-1)} = 1 \). Define the generalized Pauli-Lubanski vector by:

\[
W_\mu = -\frac{(i/2)^{r-2}(-1)^{(r-2)(r-3)}}{(r-2)!} \sum_{\nu_1, \ldots, \nu_{n-1}} \varepsilon_{\mu\nu_1 \cdots \nu_{n-1}} M^{\nu_1\nu_2} \cdots M^{\nu_{n-3}\nu_{n-2}} P_{\nu_{n-1}}
\]

where the \( M^{\nu\nu'} \)'s and the \( P^\nu \)'s stand for the generators of the \( n \)-Poincaré algebra and where \( r - 2 \) is the rank of the maximal compact subgroup \( K_{n-2} \) of \( \tilde{E}(n-2) \) (in fact, we have denoted by \( r \) the rank of the Lie algebra of the \( n \)-conformal group \( G_n \)). Then one can show easily that in a massless (UI) representation of \( \tilde{P}_n \) one has, if \( \lambda = (\lambda_3, \ldots, \lambda_r) \) is the HW of the irreducible representation \( \pi_\lambda \) of \( K_{n-2} \):

\[
W_\mu = \epsilon(\lambda_3 + r - 3) \cdots (\lambda_{r-1} + 1) |\lambda_r| P_\mu
\]

where \( \epsilon \) is the sign of \( \lambda_r \). Thus one may define naturally helicity thanks to this relation provided that \( \lambda_r \neq 0 \), in which case it could not. Let us look at two examples. The first one is when all the components of \( \lambda \) are equal to \( s \) modulo \( \epsilon \). In this case one has, in the same conditions as above:

\[
W_\mu = \epsilon(s + r - 3) \cdots (s + 1)s P_\mu.
\]

This relation not only fixes the sign of the helicity but determines also the spin \( s \). The second is when \( \lambda_3 = s \) and the other components equal to \( \sigma \) modulo \( \epsilon \) where \( \sigma \), being 0 or 1/2, is such that \( s - \sigma \) is an integer. Then one has:

\[
W_\mu = \epsilon(s + r - 3) \cdots (\sigma + 1)\sigma P_\mu.
\]
which equals 0 when \( s \) is an integer. Thus this relation, in this example, is not appropriate to define helicity for two kinds of particles (bosons and fermions) simultaneously. For these reasons and some others (for example the conformal invariance of equations) we prefer the first example to induce representations which describe massless particles in the Minkowski space-time \( M_n \).

If one drops the extension condition, for example for odd \( n \), then \( \pi_\lambda \) with \( \lambda = (s, \ldots, s, \epsilon s) \) or even \( \lambda = (s, 0, \ldots, 0) \) or \( (s, 1/2, \ldots, 1/2, \epsilon 1/2) \) and \( s \geq 1 \) may be used to induce a “massless” representation \( U^P \) in order to represent a massless particle with spin \( s \) in the \( n \)-dimensional Minkowski space.

Now consider the following masslessness conditions, analogous to the \( n = 4 \) ones:

(a) Massless representations of the \( n \)-De Sitter group \( \tilde{S}_n \) contracts smoothly to a MDHR of the \( n \)-Poincaré group \( \tilde{P}_n \);

(b) The unique extension to a UIR \( \hat{U} \) of the \( n \)-conformal group of any MDHR of the \( n \)-Poincaré group is such that \( \hat{U}|_{\tilde{S}_n} \) is precisely a massless representation of \( \tilde{S}_n \);

(c) For \( s \geq 1 \) one may construct a gauge theory on the \( n \)-dimensional Anti-De Sitter space for massless particles, quantizable by the use of an indefinite metric and a GB triplet;

(d) The massless representations are such that the physical signals propagate on the Anti-De Sitter light cone.

We define also what we shall call here a singleton property (SP):

\[ \text{Singleton} \otimes \text{Singleton} \text{ contains } \text{Massless representations}. \]

Then it was proved in [3] that representations of \( \tilde{S}_n \) which satisfy conditions (a),(b),(c) above have the form \( U_0 \) with \( U_0 \) given by (15) and (16) in Section 5. Thus there is no \( S_n \)-massless representation for \( s \geq 1 \) and \( n \) odd. This, of course, is related to what happens in flat \( n \)-dimensional space where (for \( n \) odd) the spin can be only 0 or 1/2 (for a MDHR). As a consequence, when \( s \geq 1 \), conditions (a),(b),(c) are relevant for \( U(s) \) only if \( n \) is even and not at all for \( U'(s) \) (\( n \geq 5 \)). We conjecture that a representation of the \( n \)-de Sitter group that satisfies (a),(b),(c) must satisfy also condition (d) (this will be proved in a forthcoming paper).

Unfortunately (for \( n \geq 5 \)) property (SP) is not satisfied by \( S_n \)-massless representations, i.e. by representations which satisfy (a),(b),(c), as shown in Proposition 4.

Now if one drops one (or more) of the masslessness conditions then one can define other representations of \( \tilde{S}_n \) to be massless ones, i.e. to represent massless particles on the \( n \)-dimensional Anti-de Sitter space-time. Between (a), (b) and (c) (and probably (d)) only dropping the stronger condition (b) has actually an effect because (b) implies both (a) and (c) (and probably (d)). Indeed if one drops (b) then things change radically. For example let \( D = D(E_0, \lambda_2, \ldots, \lambda_r) \) a UIR of \( \tilde{S}_n \). If the weight \( (-E_0, \lambda_2, \ldots, \lambda_r) \) reaches the limit of unitarity then usually one obtains an indecomposable representation from which one may construct a GB triplet and then a gauge theory. Thus condition (c) is still satisfied by a large number of representations. A contraction of \( D \) to a UIR \( U^P \) of \( \tilde{P}_n \) is usually possible but the contracted representation \( U^P \) is not a MDHR in general. Let
us look at the example of $U'(s)$, $s \geq 1$. From what precedes $U'(s)$ does not satisfy (b). Moreover $U'(s)$ contracts naturally to a representation $U_P^\lambda$, of the $n$-Poincaré group, for which $\pi_\lambda = \pi_{(s,0,\ldots,0)}$ or $\pi_\lambda = \pi_{(s,1/2,\ldots,1/2,\epsilon/2)}$, depending on whether $s$ or $s - 1/2$ is integer. But $U^P$ is not a MDHR. Even more, $U'(s)$ does not, in general, contract to a MDHR, for which the weight $\lambda$ of $\pi_\lambda$ must satisfy the relation (87), especially when $n$ is odd, case of which the allowed spin is 0 or 1/2.

Among the 3 masslessness conditions we have studied so far, only (c) is totally satisfied by $U'(s)$. Indeed let $\sigma = 0$ (resp. 1/2) if $s$ (resp. $s - 1/2$) is integer ($\geq 1$). Then the representation $D(s + n - 3 + \epsilon, s, \sigma, \ldots, \sigma, \epsilon \sigma)$ becomes indecomposable if $\epsilon \to 0$ (see some examples in [8] for low values of $n$) and one may construct a GB triplet

$$D(s + n - 2, s - 1, \sigma, \ldots, \sigma, \epsilon \sigma) \to D(s + n - 3, s, \sigma, \ldots, \sigma, \epsilon \sigma) \to D(s + n - 2, s - 1, \sigma, \ldots, \sigma, \epsilon \sigma).$$

Moreover, if we consider the $(C_{n-1}$-massless) representations $D\left(\frac{n-3}{2},0,\ldots,0\right)$ and $D\left(\frac{n-2}{2},1/2,\ldots,1/2,\epsilon/2\right)$ as singletons, because they have properties similar to the singletons $	ext{Rac}$ and Di (see sections 1, 3 and 4 and subsection 6.5), though they are not Dirac singletons in the sense of Definition 3, then $U'(s)$ satisfies property (SP) because the following is true:

$$D\left(\frac{n-3}{2},0,\ldots,0\right) \otimes D\left(\frac{n-3}{2},0,\ldots,0\right) \text{ contains } \bigoplus_{s=0}^{\infty} D(n - 3 + s, s, 0, \ldots, 0)$$

$$D\left(\frac{n-3}{2},0,\ldots,0\right) \otimes D\left(\frac{n-2}{2},1/2,\ldots,1/2,\epsilon/2\right) \text{ contains } \bigoplus_{s=1/2,0}^{\infty} D(n - 3 + s, s, 1/2, \ldots, 1/2, \epsilon/2).$$

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