Geometric dynamics on
the automorphism group of principal bundles:
geodesic flows, dual pairs and chromomorphism groups

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Abstract

We formulate Euler-Poincaré equations on the Lie group \(\text{Aut}(P)\) of automorphisms of a principal bundle \(P\). The corresponding flows are referred to as \(\text{EPAut}\) flows. We mainly focus on geodesic flows associated to Lagrangians of Kaluza-Klein type. In the special case of a trivial bundle \(P\), we identify geodesics on certain infinite-dimensional semidirect-product Lie groups that emerge naturally from the construction. This approach leads naturally to a dual pair structure containing \(\delta\)-like momentum map solutions that extend previous results on geodesic flows on the diffeomorphism group (EPDiff). In the second part, we consider incompressible flows on the Lie group \(\text{Aut}_{\text{vol}}(P)\) of volume-preserving bundle automorphisms. In this context, the dual pair construction requires the definition of chromomorphism groups, i.e. suitable Lie group extensions generalizing the quantomorphism group.

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1 Introduction

Many physical systems are described by geodesic dynamics on their Lie group configuration space. Besides finite dimensional cases, such as the rigid body, one of the most cele-
brated continuum systems is Euler’s equation for ideal incompressible fluids Arnold [1966], followed by magnetohydrodynamics in plasma physics Hattori [1994]. These geodesic fluid flows may allow for δ-like measure valued solutions, whose most famous example is the vortex filament dynamics for Euler’s vorticity equation. Another example is provided by the magnetic vortex line solutions of MHD Garcia de Andrade [2006], which are limiting cases of the magnetic flux tubes observed, for example, on the solar surface.

Then, the existence of vortex solutions in different physical contexts leads to the natural question whether these singular solutions may emerge spontaneously in general continuum descriptions. For example, Euler’s equation is not supposed to generate vortex solutions, which are usually considered as invariant solution manifolds. On the other hand, the singularity formation phenomenon (steepening) is one of the special features of certain shallow-water models such as the integrable Camassa-Holm equation (CH) Camassa and Holm [1993] and its higher dimensional version (n-CH) Holm, Marsden, and Ratiu [1998], describing geodesics on the diffeomorphism group with respect to $H^1$ metrics. Its extension to more general metrics is known as EPDiff Holm and Marsden [2004] and found applications in different contexts, from turbulence Foias et al. [2001] to imaging science Holm et al. [2004]. Remarkably, the singular solutions of these systems have been shown to be momentum maps determining collective dynamics on the space of embeddings. This momentum map turns out to be one leg of a dual pair associated to commuting lifted actions on the cotangent bundle of embeddings, Holm and Marsden [2004].

Another remarkable property of the Camassa-Holm and its n-dimensional generalization on manifolds with boundary is the smoothness of its geodesic spray in Lagrangian representation (Holm and Marsden [2004], Gay-Balmaz [2009]) which proves the local well-posedness of the equation in Sobolev spaces. Such a property was first discovered in Ebin and Marsden [1970] for ideal incompressible fluids and later extended to the averaged Euler equations with Dirichlet or Navier slip boundary conditions (Marsden, Ratiu, and Shkoller [2000], Shkoller [2000], Gay-Balmaz and Ratiu [2005]).

The integrability properties of the CH equation were extended to a two component system (CH2) Chen, Liu, and Zhang [2005]; Kuz’min [2007]. This system was interpreted in terms of shallow water dynamics by Constantin and Ivanov Constantin and Ivanov [2008], who also proved the steepening mechanism. The investigation of the geometric features of the CH2 system allowed to extend this system to any number of dimension and to anisotropic interactions Holm and Tronci [2008]. Although possessing a steepening mechanism (singularity formation), the CH2 equations do not allow for δ-like solution in both variables thereby destroying one of the peculiar features possessed by the CH equation. However, slight modifications in the CH2 Hamiltonian introduce the δ-function solutions for a modified system (MCH2). These solutions were shown to emerge spontaneously in Holm, Ó Náraigh, and Tronci [2009], where the corresponding steepening mechanism was presented. Recently, the MCH2 system has been shown to be in strict relation with the two-dimensional EPDiff equation Holm and Ivanov [2011].

All the systems mentioned above determine specific geodesic flows on certain infinite dimensional Lie groups: CH and n-CH are geodesics on the whole diffeomorphism group, while CH2 and MCH2, as well as their n dimensional generalization or their extension to anisotropic interactions, are geodesics on a semidirect product involving diffeomorphisms.
and Lie group-valued functions (i.e. the gauge group). Therefore one is led to investigate further the properties of geodesic flows on diffeomorphism groups (cf. e.g. Vizman [2008]), with particular attention towards the singular solution dynamics and its underlying geometry. Some of the momentum map properties of MCH2-type systems were found in Holm and Tronci [2008]. These suggest that more fundamental features are attached to systems on infinite-dimensional semidirect-products. This stands as the main motivation for this paper, which focuses on the investigation of the deep geometric features underlying MCH2-type systems, in terms of geodesic flows on the Lie group of bundle automorphisms. In physics, the Lie group $\text{Aut}(P)$ of automorphisms of a principal bundle $P$ is the diffeomorphism group underlying Yang-Mills charged fluids Holm and Kupershmidt [1988], Gay-Balmaz and Ratiu [2008] and this provides a physical interpretation of the geodesic flows considered here. For example, MCH2 corresponds to an Abelian Yang-Mills charged fluid, that is a fluid possessing an electric charge. When the principal connection on $P$ is used to write the explicit form of the equations, this leads to the interpretation of a charged fluid in an external magnetic field, whose vector potential is given precisely by the connection one-form.

1.1 Goal of the paper

In this paper, we investigate geodesics on the automorphism group of a principal bundle, thereby formulating what we shall call the EP$\text{Aut}$ equations ($\text{Euler-Poincaré equations on the automorphisms}$), which generalize the systems mentioned in the previous section. In particular, the principal bundle setting allows to comprise all the momentum map properties of MCH2-type systems Holm and Tronci [2008] in a single dual pair diagram. One leg of this dual pair characterizes the singular solution dynamics of EP$\text{Aut}$, whereas the other leg naturally yields Noether’s conserved quantities. In addition, the bundle setting simplifies the group actions which are involved in the reduction process, so that these actions are now simply the cotangent lifts of the left- and right- composition by the automorphism group.

In a more physical context, the present work is also motivated by the introduction of a vector potential $A$ (associated to the principal connection, denoted by $\mathcal{A}$) in MCH2-type dynamics, which arises naturally from our principal bundle approach. This quantity is intrinsic to any principal bundle: for example, in the case of a non-trivial bundle, a connection becomes necessary in order to write the equations explicitly. A similar approach was used in Gay-Balmaz and Ratiu [2008] to present the reduced Lagrangian and Hamiltonian formulations of ideal fluids interacting with Yang-Mills fields, Gibbons, Holm, and Kupershmidt [1983]. The introduction of non-trivial principal bundles in physics is motivated by their frequent appearance in common gauge theories Bleeker [1981].

Motivated by the dynamics of incompressible Yang-Mills fluids, we also consider volume-preserving fluid flows. As a result, we obtain a Yang-Mills version of Euler’s equation on the volume-preserving automorphism group $\text{Aut}_{\text{vol}}$. In this situation, an incompressible fluid carries a Yang-Mills charge under the influence of an external Yang-Mills magnetic potential (the connection one-form). The incompressibility property significantly affects the properties of the system. Then, the resulting equations may involve
the fluid vorticity $\omega = \text{curl } u$ and the natural question arises whether the geometric properties of Euler’s equation (cf. Marsden and Weinstein [1983]) can be extended to the present case. In this context, the momentum map properties deserve particular care, since they require the use of appropriate extensions of infinite dimensional Lie groups. The geometry involved is extremely rich and highly non trivial, yielding relations with the group of quantomorphisms, whose extension to the Yang-Mills setting (the chromomorphism group) becomes necessary. In the last part of the paper we shall present explicit formulas for the left and right momentum maps associated to the EP$Aut_{\text{vol}}$ system.

1.2 Summary of the main results

This paper formulates the general Euler-Poincaré equations on the $Aut(P)$ group of automorphisms on a principal $\mathcal{O}$-bundle $P$. In physical terms this construction generalizes ordinary fluid flows, such as Burgers or Camassa-Holm equations, to account for the transport of some Yang-Mills charge under the influence of a magnetic potential, given by the connection one-form on $P$.

Besides the explicit formulation of the equations, a first result is obtained in Section 2 for the case of a trivial bundle $P = M \times \mathcal{O}$, and it regards the construction of a dual pair that extends the results presented in Holm and Tronci [2008]. Section 2.4 investigates the geometric properties of the dual pair structure in terms of Clebsch variables, singular solutions and their associated conservation laws. Remarkably, these properties are all embodied in the dual pair structure, which is the unique feature of this construction.

In Section 3, the more general case of a non-trivial principal bundle is considered and the corresponding equations are presented in different fashions. The main result regards again the dual pair structure. In particular, the Clebsch representation of the fluid momentum is found to involve its own magnetic field, which in turn affects the singular solution dynamics; see Section 3.6.

Motivated by the geometric dynamics of incompressible Yang-Mills fluids, Section 4 investigates the incompressible version of EP$Aut$, thereby studying geodesic flows on the bundle automorphisms that project down to volume-preserving diffeomorphisms on the base. This immediately extends Euler’s equation for ideal fluids to account for the transport of a Yang-Mills charge.

After presenting the explicit form of the equations in different cases, Section 5 focuses on the geometry underlying this incompressible flow, thereby obtaining a new Clebsch representation of the fluid vorticity, that also encodes the presence of the advected Yang-Mills charge. Finally, Section 5.4 compares both left and right momentum maps with the corresponding expressions applying to the case of Euler’s vorticity equation Marsden and Weinstein [1983]. This construction requires the use of Lie group extensions. In particular, the left leg momentum map requires (see Section 5.4.1) the definition of new Lie groups, the chromomorphism groups, that extend the quantomorphism group to comprise Yang-Mills particle systems.
2 \textbf{EPAut} flows on a trivial principal bundle

This section presents the EP\textit{Aut} system of equations in the special case of a trivial principal bundle. The discussion below recalls background definitions of the automorphism group of principal bundles and specializes them to the case of a trivial bundle.

2.1 Generalities on bundle automorphisms

Before giving the EP\textit{Aut} equation we recall some facts concerning the Euler-Poincaré and Lie-Poisson reductions. Consider a right principal bundle $\pi : P \to M$, with structure group $\mathcal{O}$. We denote by $\Phi : \mathcal{O} \times P \to P$, \((g, p) \mapsto \Phi_g(p)\)

the associated right action. The structure group $\mathcal{O}$ is sometimes called \textit{order parameter group} in condensed matter physics. The automorphism group $\text{Aut}(P)$ of $P$ consists of all $\mathcal{O}$-equivariant diffeomorphisms of $P$, that is, we have

$$\text{Aut}(P) = \{ \varphi \in \text{Diff}(P) \mid \varphi \circ \Phi_g = \Phi_g \circ \varphi, \text{ for all } g \in \mathcal{O} \},$$

where Diff($P$) denotes the group of all diffeomorphisms of $P$. An automorphism $\varphi$ induces a diffeomorphism $\bar{\varphi}$ of the base $M$, by the condition $\pi \circ \varphi = \bar{\varphi} \circ \pi$. The Lie algebra $\mathfrak{aut}(P)$ of the automorphism group consists of $\mathcal{O}$-equivariant vector fields on $P$.

When the bundle $P$ is trivial we can write $P = M \times \mathcal{O}$. In this case the group of all automorphisms of $P$ is isomorphic to the semidirect product group

$$\text{Aut}(P) \simeq \text{Diff}(M) \ltimes \mathcal{F}(M, \mathcal{O}),$$

where $\mathcal{F}(M, \mathcal{O})$ denotes the gauge group of $\mathcal{O}$-valued functions defined on $M$, and Diff($M$) acts on $\mathcal{F}(M, \mathcal{O})$ by composition on the right. To a couple $(\eta, \chi)$ in the semidirect product, is associated the automorphism

$$(x, g) \in M \times \mathcal{O} \mapsto (\eta(x), \chi(x)g) \in M \times \mathcal{O}.$$ 

In the trivial case, the Lie algebra of the automorphism group is the semidirect product Lie algebra $\mathfrak{x}(M) \ltimes \mathcal{F}(M, \mathfrak{o})$, where $\mathfrak{x}(M)$ denotes the Lie algebra of vector fields on $M$, while $\mathfrak{o}$ denotes the Lie algebra of the structure group $\mathcal{O}$, so that $\mathcal{F}(M, \mathfrak{o})$ is the Lie algebra of $\mathfrak{o}$-valued functions defined on $M$.

2.2 The EP\textit{Aut} system on a trivial principal bundle

When $P$ is a trivial principal bundle, the Euler-Poincaré equations on the automorphism group $\text{Aut}(P)$ are given in the following proposition. (See e.g. Gay-Balmaz and Ratiu [2009] and Holm [2002] for its proof in the context of complex fluid dynamics).

\begin{proposition} \textbf{(The EP\textit{Aut} equations on a trivial principal bundle)} \end{proposition}

\begin{proof}

Let $L$ be an invariant Lagrangian $L : T\text{Aut}(M \times \mathcal{O}) \to \mathbb{R}$, so that the isomorphism $\text{Aut}(M \times \mathcal{O}) \simeq$
Diff(M)\otimes F(M,O) \text{ yields } L(\varphi, \dot{\varphi}) = L(\eta, \dot{\eta}, \chi, \dot{\chi}). \text{ Then, upon introducing the reduced variables}

\((u, \nu) = (\dot{\eta} \circ \eta^{-1}, (\dot{\chi} \chi^{-1}) \circ \eta^{-1}) \in \mathfrak{x}(M) \otimes F(M,o),\)

\text{the Euler-Poincaré equations on } Aut(M \times O) \simeq \text{Diff(M)} \otimes F(M,O) \text{ that arise from the symmetry-reduced Hamilton’s principle}

\[
\delta \int_{t_1}^{t_2} l(u, \nu) \, dt = 0,
\]

\text{are written as}

\[
\begin{aligned}
\frac{\partial}{\partial t} \frac{\delta l}{\delta u} + \mathcal{L}_u \frac{\delta l}{\delta u} + \frac{\delta l}{\delta \nu} \cdot d\nu &= 0 \\
\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} + \mathcal{L}_u \frac{\delta l}{\delta \nu} + ad^* \frac{\delta l}{\delta \nu} &= 0,
\end{aligned}
\]

where the operator \(\mathcal{L}_u\) denotes the Lie derivative acting on tensor densities.

One can similarly write the Lie-Poisson equations associated to a right-invariant Hamiltonian \(H\) defined on \(T^* Aut(P)\); see Holm and Tronci [2008].

**Remark 2.2 (Duality parings)** In this paper, by the dual of a function space \(\mathcal{F}\) we mean a space \(\mathcal{F}^*\) in nondegenerate duality with \(\mathcal{F}\) with respect to an \(L^2\) pairing \(\langle \cdot, \cdot \rangle\). We do not specify the precise regularity involved, it will be clear that different situations require different regularity properties, in order to obtain well-defined expressions. For example, in the formulation of the Euler-Poincaré equations (1), we have chosen the regular dual, that is

\[
\mathfrak{x}(M)^* \times F(M,o)^* = (\Omega^1(M) \otimes \text{Den}(M)) \times (F(M,o^*) \otimes \text{Den}(M)),
\]

whereas, the momentum maps and singular solutions will take values in the topological dual space.

**Remark 2.3 (Pairing notations)** Several notational issues emerge from the various pairings appearing in this paper. However, angle brackets (e.g. \(\langle u, v \rangle\)) and the dot pairing (e.g. \(\sigma \cdot \nu\)) will be used throughout the text depending only on convenience purposes. We hope that this will not generate confusion.

In order to write the EP\(Aut\) equation more explicitly, it is convenient to fix a Riemannian metric \(g\) on \(M\). Let \(v \in \mathfrak{x}(M)\) and \(c \in F(M,o^*)\) such that \(\delta l/\delta u = v^b \otimes \mu_M\) and \(\delta l/\delta \nu = \sigma \otimes \mu_M\), where \(\mu_M\) is the Riemannian volume. Then equation (1) can be equivalently written as

\[
\begin{aligned}
\partial_t v + \nabla_u v + \nabla u^T \cdot v + v \text{div}(u) + (\sigma \cdot d\nu)^\sharp &= 0 \\
\partial_t \sigma + d\sigma \cdot u + \sigma \text{div}(u) + ad^* \sigma &= 0,
\end{aligned}
\]

where \(\sharp\) denotes the sharp operator associated to the Riemannian metric \(g\). For \(\delta l/\delta \nu = 0\), this system reduces to an Euler-Poincaré equation on the diffeomorphisms (EPDiff); see Holm and Marsden [2004]; Holm, Marsden, and Ratiu [1998].
When the differential operators are invertible, the associated Hamiltonian is the Kaluza-Klein Lagrangian given by

$$ l(u, \nu) = \frac{1}{2} \|u\|_1^2 + \frac{1}{2} \|\nu + A \cdot u\|_2^2, $$

relative to a given one-form $A \in \Omega^1(M, \mathfrak{o})$ and to inner product norms $\|\|_1$ and $\|\|_2$ on $\mathfrak{X}(M)$ and $\mathcal{F}(M, \mathfrak{o})$, respectively. The inner products are usually given by positive symmetric differential operators $Q_1 : \mathfrak{X}(M) \to \mathfrak{X}(M)^*$ and $Q_2 : \mathcal{F}(M, \mathfrak{o}) \to \mathcal{F}(M, \mathfrak{o})^*$, that is, one defines the inner products

$$ \langle u, v \rangle_1 := \int_M u \cdot Q_1 v \quad \text{and} \quad \langle \nu, \omega \rangle_2 := \int_M \nu \cdot Q_2 \omega, $$

When these differential operators are invertible, the associated Hamiltonian $h$ on $\mathfrak{X}(M)^* \times \mathcal{F}(M, \mathfrak{o})^*$ may be found by Legendre transforming

$$ m = \frac{\delta l}{\delta u} = Q_1 u + Q_2 (A \cdot u + \nu) \cdot A \in \Omega^1(M) \otimes \text{Den}(M) $$

$$ \sigma = \frac{\delta l}{\delta \nu} = Q_2 (A \cdot u + \nu) \in \mathcal{F}(M, \mathfrak{o}^*) \otimes \text{Den}(M). $$

so that

$$ h(m, \sigma) = \frac{1}{2} \int_M (m - \sigma \cdot A) \cdot Q_1^{-1} (m - \sigma \cdot A) + \frac{1}{2} \int_M \sigma \cdot Q_2^{-1} \sigma. $$

For example, if $M = \mathbb{R}^n$ and $G_i$ denotes the Green’s function of $Q_i$, then the EP\textit{Aut} reduced Hamiltonian $h : \mathfrak{aut}(P)^* \simeq \mathfrak{X}(\mathbb{R}^n)^* \times \mathcal{F}(\mathbb{R}^n, \mathfrak{o})^* \to \mathbb{R}$ is written as

$$ h(m, \sigma) = \frac{1}{2} \int G_1(x - x') (m(x) - \sigma(x) \cdot A(x)) \cdot (m(x') - \sigma(x') \cdot A(x')) \, d^n x \, d^n x' $$

$$ + \frac{1}{2} \int G_2(x - x') \sigma(x) \cdot \sigma(x') \, d^n x \, d^n x', $$

thereby producing the Lie-Poisson equations

$$ \partial_t m + \mathcal{L}_{G_2 \ast (m - \sigma \cdot A)} m + \sigma \cdot \mathbf{d}(G_2 \ast \sigma - A \cdot G_1 \ast (m - \sigma \cdot A)) = 0 $$

$$ \partial_t \sigma + \mathcal{L}_{G_2 \ast (m - \sigma \cdot A)} \sigma + \text{ad}^*_{G_2 \ast \sigma - A \cdot G_1 \ast (m - \sigma \cdot A)} \sigma = 0. $$

In Holm and Tronci [2008], these equations were derived in the special case when $A = 0$, i.e. in the absence of external magnetic Yang-Mills fields.
Remark 2.5 (Relevant specializations) When the manifold $M$ is one dimensional, the Lie group is $\mathcal{O} = S^1$, and $A = 0$, we recover the two-component Camassa-Holm equation (CH2) [Chen, Liu, and Zhang 2005; Kuz’mín 2007] by choosing the differential operators $Q_1 = (1 - \alpha^2 \partial_x^2)$ and $Q_2 = 1$. The modified two-component system (MCH2) [Holm, Ó Náraigh, and Tronci 2009] requires $Q_1 = (1 - \alpha_1^2 \partial_x^2)$ and $Q_2 = (1 - \alpha_2^2 \partial_x^2)$. The higher dimensional and anisotropic versions studied in [Holm and Tronci 2008] are obtained by choosing an arbitrary manifold $M$, an arbitrary group $\mathcal{O}$, and $A = 0$. The corresponding choices for the differential operators are $Q_1 = (1 - \alpha^2 \Delta), Q_2 = 1$ (for $n$-CH2) and $Q_1 = (1 - \alpha_1^2 \Delta), Q_2 = (1 - \alpha_2^2 \Delta)$ (for $n$-MCH2).

Remark 2.6 (The Kelvin-Noether circulation theorem) The Kelvin-Noether theorem is a version of the Noether theorem that holds for solutions of the Euler-Poincaré equations [Holm, Marsden, and Ratiu 1998]. Let $(u, \nu)$ be a solution of the EP Aut equation (1), and let $\rho$ be a density variable satisfying the equation $\partial_t \rho + \mathcal{L}_u \rho = 0$. Then we have

$$\frac{d}{dt} \int_{c_t} 1 \frac{\delta l}{\delta u} = - \int_{c_t} 1 \frac{\delta l}{\delta \nu} \cdot d\nu,$$

where $c_t$ is a loop in $M$ moving with the fluid velocity $u$. In the special case $\mathcal{O} = S^1$, one has $\rho := \delta l/\delta \nu$ so that the right-hand side vanishes and Kelvin-Noether theorem yields circulation conservation.

2.4 Momentum maps

2.4.1 General setting: the Kaluza-Klein configuration space

In order to characterize the geometry of singular solutions, we consider their dynamics to take place in the phase space associated to a Kaluza-Klein configuration manifold

$$Q_{KK} = \text{Emb}(S, M) \times \mathcal{F}(S, \mathcal{O}) \ni (Q, \theta)$$

and we consider the left action of $(\eta, \chi) \in \text{Aut}(P) \simeq \text{Diff}(M) \otimes \mathcal{F}(M, \mathcal{O})$ defined by

$$(\eta, \chi)(Q, \theta) := (\eta \circ Q, (\chi \circ Q)\theta).$$

Since $\text{Emb}(S, M)$ is an open subset of $\mathcal{F}(S, M)$, its tangent space at $Q$ consists of vector fields $V_Q : S \to TM$ along $Q$, that is, we have $V_Q(s) \in T_{Q(s)}M$. In the same way, by fixing a volume form $d^k$s on $S$, the cotangent space at the embedding $Q$ consists of one-forms $P_Q : S \to T^*M$ along $Q$. Note that when $M = \mathbb{R}^n$ then the cotangent bundle is the product $T^* \text{Emb}(S, \mathbb{R}^n) = \text{Emb}(S, \mathbb{R}^n) \times \mathcal{F}(S, \mathbb{R}^n)$ and therefore, we can denote an element in the cotangent bundle by a pair $(Q, P)$. However, for an arbitrary manifold $M$, $T^* \text{Emb}(S, M)$ is not necessarily a product, and one should use the notation $P_Q$. The tangent space to $\mathcal{F}(S, \mathcal{O})$ at $\theta$ consists of functions $\nu_{\theta} : S \to T\mathcal{O}$ along $\theta$. Similarly, when a volume form $d^k$s is fixed on $S$, the cotangent bundle $T^*\mathcal{F}(S, \mathcal{O})$ at $\theta$ consists of functions $\kappa_{\theta} : S \to T^*\mathcal{O}$ covering $\theta$. 

9
2.4.2 Left-action momentum map and singular solutions

We now compute the momentum map associated to the action (9) cotangent lifted to $T^*Q_{KK}$. Given $(u, \nu) \in \mathfrak{X}(M) \otimes \mathcal{F}(M, \mathfrak{o})$, the infinitesimal generator associated to the left action (9) reads

$$(u, \nu)_{Q_{KK}}(Q, \theta) = (u \circ Q, (\nu \circ Q)\theta).$$

Given a Lie group $G$ acting on a configuration manifold $Q$, the momentum map $\mathcal{J} : T^*Q \to \mathfrak{g}^*$ for the cotangent lifted action to $T^*Q$ reads

$$\langle \mathcal{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle,$$

where $\alpha_q \in T^*Q$, $\xi \in \mathfrak{g}$ and $\xi_Q$ is the infinitesimal generator of the action of $\mathfrak{g}$ on $Q$ associated to the Lie algebra element $\xi$. By applying this formula to our case, $\mathfrak{g} = \mathfrak{aut}(P)$, we get the expression $J_L : T^*Q_{KK} \to \mathfrak{X}(M)^* \times \mathcal{F}(M, \mathfrak{o})^*$,

$$J_L(P_Q, \kappa_\theta) = \left( \int_S P_Q(s)\delta(x - Q(s))d^ks, \int_S \kappa_\theta(s)\theta(s)^{-1}\delta(x - Q(s))d^ks \right).$$

We shall show that this momentum map recovers the singular solution of (6) in Holm and Tronci [2008], in the case $A=0$. First, one observes that $J_L$ is invariant under the right action of $\mathcal{F}(S, \mathcal{O})$ on its cotangent bundle. Thus, $J_L$ induces a map $\widetilde{J}_L$ on the reduced space

$$T^*Q_{KK}/\mathcal{F}(S, \mathcal{O}) \simeq T^*\operatorname{Emb}(S, M) \times \mathcal{F}(S, \mathfrak{o})^*$$

that is obtained by replacing $\mu(s) = \kappa_\theta(s)\theta^{-1}(s) \in \mathcal{F}(S, \mathfrak{o})^*$ in (10). The reduced Poisson structure on $T^*\operatorname{Emb}(S, M) \times \mathcal{F}(S, \mathfrak{o})^*$ is the sum of the canonical Poisson bracket on $T^*\operatorname{Emb}(S, M)$ and the right Lie-Poisson bracket on $\mathcal{F}(S, \mathfrak{o})^*$, that is, if $M = \mathbb{R}^n$, we have

$$(11) \quad \{F, G\}_Q(P, \mu) = \int_S \left( \frac{\delta F}{\delta Q} \frac{\delta G}{\delta P} - \frac{\delta F}{\delta P} \frac{\delta G}{\delta Q} \right) d^ks + \int_S \mu \cdot \left[ \frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] d^ks.$$

Since the right action commutes with the left action of $\operatorname{Diff}(M) \otimes \mathcal{F}(M, \mathcal{O})$ on $T^*Q_{KK}$, the map $\widetilde{J}_L$ is a momentum map relative to the Poisson structure (11) and to the induced left action on the reduced space $T^*\operatorname{Emb}(S, M) \times \mathcal{F}(S, \mathfrak{o})^*$. This recovers the result of Holm and Tronci [2008], by reduction of the canonical structure. This also shows that considering $M \times \mathcal{O}$ as a trivial principal bundle is very natural in this context. Indeed, the momentum map $J_L$ is simply associated to a cotangent lifted action. For a certain class of $\text{EPAut}$ Hamiltonians $h$, such as (4), it is possible to define the collective Hamiltonian

$$H := h \circ J_L : T^*Q_{KK} \to \mathbb{R}$$

which produces collective Hamiltonian dynamics; see Holm and Marsden [2004] for the special case $\mu(s) \equiv 0$. This is for example the case for Hamiltonians associated to $H^1$ metrics.

Because of its equivariance, the map $J_L$ is also a Poisson map with respect to the canonical Poisson structure on the canonical cotangent bundle $T^*Q_{KK}$ and the right Lie-Poisson structure on $\mathfrak{X}(M)^* \times \mathcal{F}(M, \mathfrak{o})^*$. Thus, any solution $(P_Q, \kappa_\theta)$ of the canonical Hamilton’s equations on $T^*Q_{KK}$ associated to $H = h \circ J_L$ projects formally, via $J_L$,
to a (measure-valued) solution of the Lie-Poisson equation associated to \( h \). One can also use the momentum map \( \tilde{J}_L \). In this case, one has to solve Hamilton’s equations on \( T^* \text{Emb}(S, M) \times \mathcal{F}(S, \mathfrak{o})^* \), given by

\[
\begin{align*}
\dot{Q} &= \frac{\partial H}{\partial P}, \\
\dot{P} &= -\frac{\partial H}{\partial Q}, \\
\dot{\mu} &= -\text{ad}_{\delta H/\delta \mu}^* \mu,
\end{align*}
\]

when \( M = \mathbb{R}^n \).

**Remark 2.7 (Physical interpretation)** This collective Hamiltonian extends the one in Holm and Tronci [2008] to consider an external Yang-Mills magnetic field, given by the potential \( A \) through the relation \( B = dA \). This may be of interest in the gauge theory of fluids with non-Abelian interactions, which plays a fundamental role in certain areas of condensed matter physics (cf. e.g. Holm and Kupershmidt [1988]; Holm [2002]; Gay-Balmaz and Ratiu [2009]).

**Remark 2.8 (Existence of singular solutions)** The existence of the momentum map \( J_L \) does not guarantee that the latter is also a solution of the system. For example, the use of \( L^2 \) norms in the EP\textit{Aut} Lagrangian (3) prevents singularities. As a consequence, the collective Hamiltonian \( H = h \circ J_L \) may not be defined, since it requires the Hamiltonian \( h \) to be well defined on the image of the momentum map. However, the use of appropriate norms, such as \( H^1 \), allows for singular solutions and this is one of the reasons why we consider this particular case. Also, the absence of a magnetic field in the Abelian case returns the MCH2 equations (cf. Holm, Ó Náraigh, and Tronci [2009]), which are known to produce singularities in finite time. Whether this steepening phenomenon persists upon introducing a static magnetic field, this is an open question that deserves future investigation.

### 2.4.3 Right-action momentum map and Noether’s Theorem

We now consider the right action of \( (\gamma, \beta) \in \text{Aut}(P_S) = \text{Diff}(S) \otimes \mathcal{F}(S, \mathcal{O}) \) on \( Q_{KK} \) defined by

\[
(Q, \theta)(\gamma, \beta) := (Q \circ \gamma, (\theta \circ \gamma) \beta).
\]

Given \( (v, \zeta) \in \mathcal{X}(S) \otimes \mathcal{F}(S, \mathcal{O}) \), the infinitesimal generator is

\[
(v, \zeta)_{Q_{KK}}(Q, \theta) = (dQ \cdot v, d\theta \cdot v + \theta \zeta),
\]

thus, the momentum map associated to the cotangent lifted action is

\[
(13) \quad J_R : T^*Q_{KK} \to \mathcal{X}(S)^* \times \mathcal{F}(S, \mathfrak{o})^* , \quad J_R (P_Q, \kappa_\theta) = (P_Q \cdot dQ + \kappa_\theta \cdot d\theta, \theta^{-1} \kappa_\theta).
\]

which in turn identifies a Clebsch representation in the sense of Marsden and Weinstein [1983]. Since the collective Hamiltonian \( H \) on \( T^*Q_{KK} \) is invariant under the cotangent lift of the right action (12), Noether’s Theorem asserts that

\[
\frac{d}{dt} (P_Q \cdot dQ + \kappa_\theta \cdot d\theta) = 0 \quad \text{and} \quad \frac{d}{dt}(\theta^{-1} \kappa_\theta) = 0,
\]
for any solutions \((P_Q, \kappa) \in T^*Q_{KK}\) of Hamilton’s equations. To summarize the situation, we have found the following dual pair structure

\[
\begin{array}{ccc}
T^*Q_{KK} & \rightarrow & \text{aut}(M \times O)^* \\
J_L & \leftarrow & \text{aut}(S \times O)^*
\end{array}
\]

where \(\text{aut}(M \times O)^* = \mathcal{X}(M)^* \times \mathcal{F}(M, \mathfrak{o})^*\) and analogously for \(\text{aut}(S \times O)^*\). The geometric meaning of these dual pairs will be presented later in a more general setting.

**Remark 2.9 (The case \(S = M\))** When \(S = M\), then the embeddings of \(S\) into \(M\) are the diffeomorphisms of \(M\), that is, \(\text{Emb}(S, M) = \text{Diff}(M)\) and \(Q_{KK} = \text{Diff}(M) \times \mathcal{F}(M, \mathfrak{o})\). Moreover, in this case the left and right actions (9), (12) recover left and right translations on the automorphism group. Therefore, the Kaluza-Klein configuration manifold can be identified with the group \(\text{Diff}(M) \circledast \mathcal{F}(M, \mathfrak{o}) \simeq \text{Aut}(M \times O)\), and the dual pair recovers the usual body and spatial representations for mechanical systems on Lie groups. More precisely, \(J_L\) recovers the Lagrange-to-Euler map, while \(J_R\) corresponds to the conserved momentum density.

### 3 EPAut flows on non-trivial principal bundles

#### 3.1 Basic definitions

In this section we consider an arbitrary principal \(\mathcal{O}\)-bundle \(\pi : P \rightarrow M\). Recall that the Lie algebra \(\text{aut}(P)\) of the automorphism group consists of equivariant vector fields \(U\) on \(P\), that is, we have

\[T\Phi_g \circ U = U \circ \Phi_g, \quad \text{for all } g \in \mathcal{O}.
\]

In the case of non-trivial bundles, it is necessary to introduce a principal connection \(\mathcal{A}\) to split the tangent space into its vertical and horizontal subspaces. Recall that a principal connection is given by an \(\mathfrak{o}\)-valued one form \(\mathcal{A} \in \Omega^1(P, \mathfrak{o})\) such that

\[\Phi_g^* \mathcal{A} = \text{Ad}_{g^{-1}} \circ \mathcal{A} \quad \text{and} \quad \mathcal{A}(\xi_P) = \xi,
\]

where \(\xi_P\) is the infinitesimal generator associated to the Lie algebra element \(\xi\). Using a principal connection, we obtain an isomorphism

\[\text{aut}(P) \rightarrow \mathcal{X}(M) \times \mathcal{F}_\mathcal{O}(P, \mathfrak{o}), \quad U \mapsto ([U], \mathcal{A}(U)),\]

where \([U] \in \mathcal{X}(M)\) is defined by the condition \(T\pi \circ [U] = U \circ \pi\) and \(\mathcal{F}_\mathcal{O}(P, \mathfrak{o})\) is the space of \(\mathcal{O}\)-equivariant functions \(\omega : P \rightarrow \mathfrak{o}\), i.e. \(\omega \circ \Phi_g = \text{Ad}_{g^{-1}} \circ \omega\), for all \(g \in \mathcal{O}\). The inverse of the isomorphism (14) reads \((u, \omega) \mapsto \text{Hor}^\mathcal{A} u + \sigma(\omega)\), where \(\text{Hor}^\mathcal{A}\) is the horizontal-lift associated to the principal connection \(\mathcal{A}\) and \(\sigma(\omega)\) is the vertical vector field on \(P\) defined by \(\sigma(\omega)(p) := (\omega(p))_{\mathfrak{o}}(p)\). The dual isomorphism reads

\[\text{aut}(P)^* \rightarrow \mathcal{X}(M)^* \times \mathcal{F}_\mathcal{O}(P, \mathfrak{o})^*, \quad \beta \mapsto \left(\left([\text{Hor}^\mathcal{A}]^* \beta, \mathcal{A} \circ \beta\right)\right),\]

where \(\left([\text{Hor}^\mathcal{A}]^* \beta, \mathcal{A} \circ \beta\right)\) is the pair of dual forms associated to \(\beta\), obtained by pulling back \(\beta\) from \(P\) to \(M\).
where \( \text{aut}(P)^* = \Omega^1_0(P) \otimes \text{Den}(M) \) is the space of \( \mathcal{O} \)-invariant one-form densities and analogously \( \mathcal{F}_\mathcal{O}(P, \mathfrak{o})^* = \mathcal{F}_\mathcal{O}(P, \mathfrak{o}^*) \otimes \text{Den}(M) \). Here, \( \mathcal{J} : T^*P \to \mathfrak{o}^* \) denotes the cotangent bundle momentum map, \( \langle \mathcal{J}(\alpha_p), \xi \rangle = \langle \alpha_p, \xi_{T_p} \rangle \).

### 3.2 General EP\(\mathcal{A}ut\) equations on principal bundles

Given a Lagrangian \( l : \text{aut}(P) \to \mathbb{R} \), the associated EP\(\mathcal{A}ut\) equation is

\[
\frac{\partial}{\partial t} \frac{\delta l}{\delta U} + \mathcal{L}_U \frac{\delta l}{\delta U} = 0. 
\]

We now split this equation using the isomorphism (14) associated to a fixed principal connection \( \mathcal{A} \). Therefore, the Lagrangian \( l : \text{aut}(P) \to \mathbb{R} \) induces a connection dependent Lagrangian \( l^A : \mathfrak{X}(M) \times \mathcal{F}_\mathcal{O}(P, \mathfrak{o}) \to \mathbb{R} \) defined by

\[
l(U) = l^A(u, \omega), \quad u = [U], \quad \omega = \mathcal{A}(U).
\]

Using the dual isomorphism (15), the left hand side of (16) reads

\[
\frac{\partial}{\partial t} \left( \frac{\delta l^A}{\delta u}, \frac{\delta l^A}{\delta \omega} \right).
\]

In order to split the right hand side, we will need the lemma below.

Recall that, associated to the principal connection \( \mathcal{A} \), there is a covariant exterior derivative on \( \mathfrak{o} \)-valued differential forms on \( P \) defined by \( d^A = \text{hor}^* d \), i.e.

\[
d^A \omega(X_1, \ldots, X_{k+1}) = d\omega(\text{hor}(X_1), \ldots, \text{hor}(X_{k+1})),
\]

where \( \text{hor} \) is the horizontal part of a vector in \( TP \). In particular, the curvature of \( \mathcal{A} \) is given by

\[
\mathcal{B} := d^A \mathcal{A} = d\mathcal{A} + [\mathcal{A}, \mathcal{A}].
\]

**Lemma 3.1** Let \( \beta \in \text{aut}(P)^* \) and \( U \in \text{aut}(P) \), and fix a principal connection \( \mathcal{A} \) on \( P \). Then we have for \( u = [U] \) and \( \omega = \mathcal{A}(U) \):

\[
(\text{Hor}^A)^* \mathcal{L}_U \beta = \mathcal{L}_u (\text{Hor}^A)^* \beta + (\text{Hor}^A)^* (\mathcal{J} \circ \beta \cdot d\omega + \mathcal{L}_{\text{Hor}^A} u (\mathcal{J} \circ \beta \cdot \mathcal{A}))
\]

\[
\mathcal{J} \circ \mathcal{L}_U \beta = \mathcal{L}_{\text{Hor}^A} u (\mathcal{J} \circ \beta) + \text{ad}^*_{\omega} (\mathcal{J} \circ \beta).
\]

See Appendix A.1 for a proof. Applying this Lemma, we obtain the following result.

**Proposition 3.2** (EP\(\mathcal{A}ut\) equations on principal bundles) The Euler-Poincaré equations on the automorphism group of a principal bundle, relative to a Lagrangian \( l : \text{aut}(P) \to \mathbb{R} \) and a principal connection \( \mathcal{A} \), are given by

\[
\begin{align*}
\frac{\partial}{\partial t} \frac{\delta l^A}{\delta u} + \mathcal{L}_u \frac{\delta l^A}{\delta u} + (\text{Hor}^A)^* \left( \frac{\delta l^A}{\delta \omega} \cdot d\omega + \mathcal{L}_{\text{Hor}^A} u \left( \frac{\delta l^A}{\delta \omega} \cdot \mathcal{A} \right) \right) &= 0, \\
\frac{\partial}{\partial t} \frac{\delta l^A}{\delta \omega} + \mathcal{L}_{\text{Hor}^A} u \frac{\delta l^A}{\delta \omega} + \text{ad}^*_{\omega} \frac{\delta l^A}{\delta \omega} &= 0.
\end{align*}
\]
3.3 Formulation in terms of the adjoint bundle

We now rewrite the EP\textit{Aut}-equations (17) by identifying $\mathcal{F}_O(P, \mathfrak{o})$ with the space $\Gamma(\text{Ad} P)$ of sections of the adjoint bundle

$$\text{Ad} P := (P \times \mathfrak{o})/O \to M.$$  

Recall that the quotient is taken relative to the diagonal action of $O$ on $P \times \mathfrak{o}$ given by $(p, \xi) \mapsto (\Phi_g(p), \text{Ad}_{g^{-1}} \xi)$. We will denote by $[p, \xi]_O$ an element of the adjoint bundle. There is a Lie algebra isomorphism

\begin{equation}
\omega \in \mathcal{F}_O(P, \mathfrak{o}) \mapsto \tilde{\omega} \in \Gamma(\text{Ad} P)
\end{equation}

defined by

$$\tilde{\omega}(x) := [p, \omega(p)]_O,$$

where $p \in P$ is such that $\pi(p) = x$. This isomorphism extends to $k$-forms as follows. Let $\Omega^k(P, \mathfrak{o}) := \{ \alpha \in \Omega^k(P, \mathfrak{o}) \mid \Phi^*_g \alpha = \text{Ad}_{g^{-1}} \alpha \text{ and } i_{\xi^p} \alpha = 0, \text{ for all } g \in G \text{ and } \xi \in \mathfrak{o} \}$ be the space of $O$–equivariant horizontal $k$-forms on $P$, and let $\Omega^k(M, \text{Ad} P)$ be the space of $\text{Ad} P$-valued $k$-forms on $M$. Then we have the isomorphism

$$\alpha \in \Omega^k(P, \mathfrak{o}) \mapsto \tilde{\alpha} \in \Omega^k(M, \text{Ad} P), \quad \tilde{\alpha}(x)(u^1, \ldots, u^k) := [p, \alpha(p) (U^1_p, \ldots, U^k_p)]_O,$$

where $U^i_p \in T_p P$ is such that $T_p \pi(U^i_p) = u^i_x$, $i = 1, \ldots, k$. An example of $\mathcal{O}$–equivariant horizontal 2-form is provided by the curvature $\mathcal{B} = d\mathcal{A} + [\mathcal{A}, \mathcal{A}]$ to which is associated the reduced curvature form $\tilde{\mathcal{B}} \in \Omega^2(M, \text{Ad} P)$.

Given a principal connection $\mathcal{A}$ on $P$, we get a linear connection $\nabla^\mathcal{A}$ on the adjoint bundle. For $\omega \in \mathcal{F}_O(P, \mathfrak{o})$, the relation between the covariant exterior derivative $d^\mathcal{A}$ and the connection $\nabla^\mathcal{A}$ is

$$\hat{d}^\mathcal{A} \omega = \nabla^\mathcal{A} \tilde{\omega}.$$  

This formula extends the operator $\nabla^\mathcal{A}$ to $\Omega^k(M, \text{Ad} P)$. We shall use the covariant Lie derivative defined by

$$\mathcal{L}^\mathcal{A}_u \omega = i_u \nabla^\mathcal{A} \omega + \nabla^\mathcal{A} i_u \omega$$

on $\text{Ad}(P)$-valued differential forms, and related to the ordinary Lie derivative by the formula

$$\mathcal{L}^\mathcal{A}_u \tilde{\alpha} = \mathcal{L}^\mathcal{A}_{\hat{u}} \alpha = \mathcal{L}^\mathcal{A}_{\hat{u}} \tilde{\alpha}.$$  

By combining the isomorphism (14) with (18), we identify $\mathfrak{aut}(P)$ with $\mathfrak{x}(M) \times \Gamma(\text{Ad} P)$. In this case, we have the following analogue of Lemma 3.1.

**Lemma 3.3** Let $\beta \in \mathfrak{aut}(P)^*$ and $U \in \mathfrak{aut}(P)$, and fix a principal connection $\mathcal{A}$ on $P$. Then we have for $u = [U]$ and $\tilde{\omega} = \tilde{\mathcal{A}}(U)$:

$$(\text{Hor}^\mathcal{A})^* \mathcal{L}^\mathcal{A} U \beta = \mathcal{L}^\mathcal{A}_u \text{Hor}^\mathcal{A} \beta + \mathcal{J} \circ \tilde{\mathcal{A}} \cdot (\nabla^\mathcal{A} \tilde{\omega} + i_u \mathcal{B})$$

where $\mathcal{L}^\mathcal{A}$ is the covariant Lie derivative acting on $\text{Ad}^* P$-valued densities, that is, for $\mu = (\sigma \otimes \alpha)$, where $\alpha$ is a density and $\sigma$ a section of $\text{Ad}^* P \to M$, we have

$$\mathcal{L}^\mathcal{A}_v \mu = \mathcal{L}^\mathcal{A}_v (\sigma \otimes \alpha) = \nabla^\mathcal{A}_v \sigma \otimes \alpha + \sigma \text{ div}(v) \otimes \alpha.$$
We shall denote by \( \nu \) and \( \omega \) and using the formula

\[
\frac{\partial}{\partial t} \frac{\delta l^A}{\delta \nu} + \mathcal{L}_u \frac{\delta l^A}{\delta u} + \frac{\delta l^A}{\delta \omega} \cdot \left( \nabla^A \omega + i_u \mathcal{B} \right) = 0
\]

\[
\frac{\partial}{\partial t} \frac{\delta l^A}{\delta \omega} + \mathcal{L}_u \frac{\delta l^A}{\delta u} + \text{ad}^*_\omega \frac{\delta l^A}{\delta \omega} = 0.
\]

As in the trivial bundle case, it is convenient to fix a Riemannian metric \( g \) on \( M \) in order to write the equations more explicitly. Defining \( v \) and \( \zeta \) by the equalities

\[
\frac{\delta l^A}{\delta u} =: v^b \otimes \mu_M \in \Omega^1(M) \otimes \text{Den}(M)
\]

\[
\frac{\delta l^A}{\delta \omega} =: \zeta \otimes \mu_M \in \Gamma(\text{Ad}^* P) \otimes \text{Den}(M)
\]

and using the formula

\[
\mathcal{L}_u^A(\zeta \otimes \mu_M) = \nabla_u^A \zeta \otimes \mu_M + \zeta(\text{div } u) \otimes \mu_M,
\]

the EP\( \text{Aut} \) equations (19) can be rewritten as

\[
\begin{align*}
&\frac{\partial}{\partial t} v + \nabla_u v + v^T \cdot v + v \text{div } u + \left( \zeta \cdot \left( \nabla^A \omega + i_u \mathcal{B} \right) \right)^2 = 0 \\
&\frac{\partial}{\partial t} \zeta + \nabla_u^A \zeta + \zeta \text{div } u + \text{ad}^*_\omega \zeta = 0,
\end{align*}
\]

which reduce to an Euler-Poincaré equation on the diffeomorphism group (Holm, Marsden, and Ratiu [1998]) in the case when \( A = 0 \) and \( \zeta = 0 \).

**Remark 3.5 (Trivial bundles and curvature representation)** When the principal bundle is trivial, we recover the equation (1). Indeed, we can identify \( \omega \in \mathcal{F}(P, \mathfrak{o}) \) with \( \nu + A \cdot u \in \mathcal{F}(M, \mathfrak{o}) \). Here \( A \) denotes the one-form on \( M \) induced by the connection \( A \) on \( P \): we have \( \mathcal{A}(v_x, \xi_g) = \text{Ad}_{g^{-1}}(A(v_x) + \xi_g g^{-1}) \) and also

\[l^A(u, \nu + A \cdot u) = l(u, \nu).\]

Thus, \( \delta l^A/\delta \omega \) can be identified with \( \delta l/\delta \nu \) and \( \delta l^A/\delta u \) is identified with \( \delta l/\delta u - \delta l/\delta \nu \cdot A \). Similarly, the Lie derivative \( \mathcal{L}_u^A \delta l^A/\delta \omega \) can be identified with \( \mathcal{L}_u \delta l/\delta (\omega - \text{ad}^*_\mu) \delta l/\delta \nu \). We shall denote by \( B = dA + [A, A] \) the two-form on \( M \) induced by the curvature \( \mathcal{B} \). Then, the EP\( \text{Aut} \) equations (1) written in terms of the Lagrangian \( l^A = l^A(u, \omega) \) read

\[
\begin{align*}
&\frac{\partial}{\partial t} \frac{\delta l^A}{\delta u} + \mathcal{L}_u \frac{\delta l^A}{\delta u} + \frac{\delta l^A}{\delta \omega} \cdot (d\omega + [A, \omega] + i_u B) = 0 \\
&\frac{\partial}{\partial t} \frac{\delta l^A}{\delta \omega} + \mathcal{L}_u \frac{\delta l^A}{\delta u} + \text{ad}^*_\mu \frac{\delta l^A}{\delta \omega} = 0,
\end{align*}
\]

where the curvature two form \( B \) on \( M \) satisfies the relation \( i_u B = [A \cdot u, A] - d(A \cdot u) + \mathcal{L}_u A \).
3.4 Kaluza-Klein Lagrangians

Let \( g \) be a Riemannian metric on \( M \) and \( \gamma \) an Ad-invariant scalar product on \( \frak{o} \). These data, together with the principal connection \( \mathcal{A} \), define a Kaluza-Klein metric on the principal \( \frak{O} \)-bundle \( \pi : P \to M \):

\[
\kappa_p(U_p, V_p) = g_{\pi(p)}(T_p \pi(U_p), T_p \pi(V_p)) + \gamma(A_p(U_p), A_p(V_p)), \quad U_p, V_p \in T_p P.
\]

The associated \( L^2 \) Kaluza-Klein Lagrangian is obtained by integration over \( M \) and reads

\[
l(U) = \int_M g([U], [U]) \mu_M + \int_M \gamma(\mathcal{A}(U), \mathcal{A}(U)) \mu_M.
\]

Note that the last term is well defined, since the integrand is an \( \frak{O} \)-invariant function on \( P \). More generally, we can consider Kaluza-Klein Lagrangians of the form

\[
l(U) = \frac{1}{2} \|[U]\|^2_1 + \frac{1}{2} \|\mathcal{A}(U)\|^2_2,
\]

relative to inner product norms \( \| \|_1 \) and \( \| \|_2 \) on \( \frak{X}(M) \) and \( \mathcal{F}_{\frak{O}}(P, \frak{o}) \). As before, the norm \( \| \|_2 \) is usually given with the help of an Ad-invariant inner product \( \gamma \) on \( \frak{o} \), that makes possible to integrate the expression over the base manifold \( M \). See Gay-Balmaz and Ratiu [2008] for an application of the \( L^2 \) Kaluza-Klein Lagrangian to Yang-Mills fluids. In the trivial case, (22) reduces to (3) and this emphasizes the role of the vector potential \( A \) in CH2 and MCH2 dynamics.

We now compute explicitly the EP\(\text{Aut} \) equations for the Kaluza-Klein Lagrangian (22). We assume that the norms are associated to symmetric and positive definite differential operators \( Q_1 \) and \( Q_2 \) on \( TM \) and \( \text{Ad } P \), respectively. In this case, the Lagrangian \( l^A : \frak{X}(M) \times \Gamma(\text{Ad } P) \to \mathbb{R} \) reads

\[
l^A(u, \bar{\omega}) = \int_M g(Q_1 u, u) \mu_M + \int_M \bar{\gamma}(Q_2 \bar{\omega}, \bar{\omega}) \mu_M,
\]

where \( \bar{\gamma} \) is the vector bundle metric induced on \( \text{Ad } P \) by \( \gamma \). We thus obtain the EP\(\text{Aut} \) equations

\[
\begin{aligned}
\partial_t v + \nabla_u v + \nabla u^\top \cdot v + v \text{ div } u + \zeta \left( \nabla^A \bar{\omega} + i_u B \right)^\sharp &= 0 \\
\partial_t \zeta + \nabla u^\sharp \zeta + \zeta \text{ div } u + \text{ad}_u^\star \zeta &= 0,
\end{aligned}
\]

where

\[
v = Q_1 u \in \frak{X}(M) \quad \text{and} \quad \zeta = \bar{\gamma}(Q_2 \bar{\omega}, \cdot) \in \Gamma(\text{Ad}^* P),
\]

which evidently extend MCH2 dynamics to the case of a Yang-Mills fluid flow on a non-trivial principal \( \frak{O} \)-bundle with fixed connection \( \mathcal{A} \).

3.5 Kelvin-Noether circulation theorem

Let \( U \) be a solution of the EP\(\text{Aut} \) equation (16) and let \( \rho \) be a density variable on \( M \) satisfying the equation \( \partial_t \rho + \mathcal{L}_u \rho = 0 \), where \( u := [U] \). Then we have

\[
\frac{d}{dt} \oint_{\gamma_t} \frac{1}{\rho} \frac{\delta l}{\delta U} = 0,
\]
where $\gamma_t$ is a loop in the total space $P$ of the principal bundle which moves with the velocity $U$, see §7 in Gay-Balmaz and Ratiu [2008]. Given a connection $A$ on $P$, we can decompose the functional derivative as

$$\frac{\delta l}{\delta U} = \pi^* \frac{\delta l^A}{\delta u} + \frac{\delta l^A}{\delta \omega} \cdot A,$$

and the circulation reads

$$\frac{d}{dt} \left[ \oint_{c_t} \frac{1}{\rho} \frac{\delta l^A}{\delta u} + \oint_{\gamma_t} \frac{1}{\rho} \frac{\delta l^A}{\delta \omega} \cdot A \right] = 0,$$

where $c_t := \pi \circ \gamma_t$ is the closed curve in $M$ induced by $\gamma_t$. Note that the second integral takes into account the internal structure of the bundle, since $\gamma_t$ is a curve in $P$.

In the trivial bundle case the time dependent curve reads $\gamma_t = (c_t, g_t) = (\eta_t \circ c_0, (\chi_t \circ c_0) g_0)$, and a direct computation shows that formula (24) yields

$$\frac{d}{dt} \left[ \oint_{c_t} \frac{1}{\rho} \frac{\delta l}{\delta u} + \oint_{\gamma_t} \frac{1}{\rho} \frac{\delta l}{\delta \omega} \cdot \kappa^l \right] = 0,$$

where $\kappa^l(\xi_g) = \xi_g g^{-1}$ is the left Maurer-Cartan form on $O$. Note that $\frac{1}{\rho} \frac{\delta l}{\delta \nu} \cdot \kappa^l$ is interpreted as a one-form on $P = M \times O$, integrated along the curve $\gamma_t \in M \times \mathcal{O}$. In a more explicit notation, the above Kelvin-Noether theorem can be written as

$$\frac{d}{dt} \left[ \oint_{c_t} \frac{1}{\rho} \frac{\delta l}{\delta u} + \oint_{\gamma_t} \frac{1}{\rho} \frac{\delta l}{\delta \nu} \cdot dgg^{-1} \right] = 0,$$

where $\gamma_t = (c_t, g_t)$ is a loop in the trivial bundle $M \times O$.

### 3.6 Momentum maps and singular solutions

#### 3.6.1 Preliminaries on the Kaluza-Klein configuration space

Recall that in order to explain the geometric properties of the singular solutions in the case of a trivial bundle, we introduced the Kaluza-Klein configuration manifold (8). We now describe the corresponding object in the case of an arbitrary principal bundle. Let $S \subset M$ be a submanifold of $M$ and consider two principal $O$-bundles $P \rightarrow P/O = M$ and $P_S \rightarrow P_S/O = S$. We say that the map $Q : P_S \rightarrow P$ is equivariant if $Q \circ \Phi_g = \Phi_g \circ Q$, for all $g \in O$, where $\Phi_g$ denotes the $O$-action on $P_S$ or $P$. Such a $Q$ defines a unique map $Q : S \rightarrow M$ verifying the condition $\pi \circ Q = Q \circ \pi$. We now define the object that generalizes the configuration space $Q_{KK}$ (8).

**Definition 3.6** The Kaluza-Klein configuration space is defined as the following subset of equivariant maps from $P_S$ to $P$:

$$Q_{KK} = \{ Q : P_S \rightarrow P \mid Q \circ \Phi_g = \Phi_g \circ Q \text{ and } Q \in \text{Emb}(S, M) \}.$$
Thus, $Q_{KK}$ consists of equivariant mappings that projects onto embeddings. Another characterization of the Kaluza-Klein configuration space is given in the following (see Appendix A.2 for a proof of this result)

**Lemma 3.7** The Kaluza-Klein configuration space coincides with the set

$$Q_{KK} = \text{Emb}_O(P_S, P)$$

of all equivariant embeddings of $P_S$ into $P$.

The tangent space $T_QQ_{KK}$ consists of equivariant vector fields $\mathcal{V}_Q$ along $Q$, that is

$$\mathcal{V}_Q(p_s) \in T_{Q(p_s)}P, \quad \text{for all } p_s \in P_S, \quad \text{and} \quad \mathcal{V}_Q \circ \Phi_g = T\Phi_g \circ \mathcal{V}_Q.$$

Assume that the submanifold $S$ is endowed with a volume form $d^k s$. The cotangent space $T^*Q_{KK}$ can be identified with the space of equivariant one-forms $\mathcal{P}_Q : P_S \rightarrow T^*P$ along $Q$. Therefore, the contraction $\mathcal{P}_Q \cdot \mathcal{V}_Q$ defines a function on $S$ which can be integrated over $S$. We thus obtain the paring

$$\langle \mathcal{P}_Q, \mathcal{V}_Q \rangle = \int_S (\mathcal{P}_Q \cdot \mathcal{V}_Q) d^k s. \quad (27)$$

The groups $\text{Aut}(P)$ and $\text{Aut}(P_S)$ act naturally on $Q_{KK}$ by left and right composition, respectively. Note that when $P$ and $P_S$ are trivial bundles, we recover the left and right actions given in (9) and (12). As before, we denote by

$$J_L : T^*Q_{KK} \rightarrow \mathfrak{aut}(P)^* \quad \text{and} \quad J_R : T^*Q_{KK} \rightarrow \mathfrak{aut}(P_S)^*$$

the momentum maps associated to the cotangent lift of these actions. We compute below these momentum maps by introducing principal connections.

Given an equivariant map $f \in \mathcal{F}_O(P_S, o)$ and $Q \in \text{Emb}(P_S, P)$, we define the vertical vector field $\sigma_Q(f) \in T_QQ_{KK}$ by

$$\sigma_Q(f)(p_s) := (f(p_s))_P(Q(p_s)).$$

Given a vector field $\mathcal{V}_Q \in T_Q\text{Emb}(S, M)$, a connection $A$ on $P$, and $Q \in Q_{KK}$ projecting to $Q$, we define the horizontal-lift $\text{Hor}^A_Q(V_Q) \in T_QQ_{KK}$ of $V_Q$ along $Q$ by

$$\text{Hor}^A_Q(V_Q)(p_s) := \text{Hor}^A_{Q(p_s)}(V_Q(s)).$$

This defines a connection dependent isomorphism

$$\mathcal{F}_O(P_S, o) \times T_Q\text{Emb}(S, M) \rightarrow T_QQ_{KK}, \quad (f, V_Q) \mapsto \sigma_Q(f) + \text{Hor}^A_Q(V_Q),$$

with inverse given by

$$T_QQ_{KK} \rightarrow \mathcal{F}_O(P_S, o) \times T_Q\text{Emb}(S, M), \quad \mathcal{V}_Q \mapsto (A \circ \mathcal{V}_Q, [\mathcal{V}_Q]) := (f^A, V_Q),$$

where $[\mathcal{V}_Q] \in T_Q\text{Emb}(S, M)$ is defined by the condition

$$[\mathcal{V}_Q] \circ \pi = T\pi \circ \mathcal{V}_Q.$$
It will be useful to identify the space $\mathcal{F}_O(P_S, o)$ of equivariant functions with the space $\Gamma(\operatorname{Ad} P_S)$ of all sections of the adjoint vector bundle. Since $S$ is endowed with a volume form $d^k s$, the cotangent bundle $T^*_Q Q_{KK}$ can be naturally identified with the space

$$\mathcal{F}_O(P_S, o)^* \times T^*_Q \operatorname{Emb}(S, M),$$

where $\mathcal{F}_O(P_S, o)^* := \mathcal{F}_O(P_S, o^*)$ and $T^*_Q \operatorname{Emb}(S, M)$ is defined as in the preceding section. The isomorphism is

$$\mathcal{F}_O(P_S, o)^* \times T^*_Q \operatorname{Emb}(S, M) \to T^*_Q Q_{KK}, \quad (\zeta, \alpha_Q) \mapsto \zeta \cdot A + T^* \pi \cdot \alpha_Q$$

with inverse

$$T^*_Q Q_{KK} \to \mathcal{F}_O(P_S, o)^* \times T^*_Q \operatorname{Emb}(S, M), \quad \mathcal{P}_Q \mapsto \left(\mathbb{J} \circ \mathcal{P}_Q, (\operatorname{Hor}_Q^A)^* \mathcal{P}_Q\right) =: (\zeta, \mathcal{P}_Q^A),$$

where $\mathbb{J} : T^* P \to \mathfrak{o}^*$ is defined by $\langle \mathbb{J}(\alpha_p), \xi \rangle = \langle \alpha_p, \xi_p(p) \rangle$. Note that using these isomorphisms, the natural pairing (27) reads

$$(28) \quad \int_S (\zeta \cdot f^A + \mathcal{P}_Q^A \mathcal{V}_Q) d^k s.$$  

3.6.2 Left-action momentum map and singular solutions

The momentum map $\mathbf{J}_L$ associated to the cotangent lifted left action of $\mathcal{A}ut(P)$ on $T^* Q_{KK}$ takes values in $\mathfrak{a}ut(P)^* \simeq \mathfrak{x}(M)^* \times \mathcal{F}_O(P, o)^*$, where $\mathcal{F}_O(P, o)^* = \mathcal{F}_O(P, o^*) \otimes \operatorname{Den}(M)$. It will be more convenient to identify $\mathcal{F}_O(P_S, o)$ with the space $\Gamma(\operatorname{Ad} P_S)$ of sections of the adjoint bundle $\operatorname{Ad} P_S \to M$. In the same way, we will identify $\mathcal{F}_O(P_S, o)^* = \mathcal{F}_O(P_S, o^*)$ with the space $\Gamma(\operatorname{Ad} P_S)^* = \Gamma(\operatorname{Ad}^* P_S)$, and we denote by $\bar{\zeta}$ the section associated to $\zeta \in \mathcal{F}_O(P_S, o)^*$. An embedding $Q$ in $Q_{KK}$ induces naturally a map $\bar{Q} : \operatorname{Ad}^* P_S \to \operatorname{Ad}^* P$ covering $Q$.

**Proposition 3.8 (Left-action momentum map)** *With the previous notations, the momentum map associated to the cotangent lifted left action of the automorphism group $\mathcal{A}ut(P)$ on $T^* Q_{KK}$ reads*

$$\mathbf{J}_L(\mathcal{P}_Q) = \left(\int_S \mathcal{P}_Q^A(s) \delta(x - Q(s)) d^k s, \int_S \bar{Q}(\bar{\zeta}(s)) \delta(x - Q(s)) d^k s\right) \in \mathfrak{x}(M)^* \times \Gamma(\operatorname{Ad} P)^*.$$

**Proof.** Using the formula for the momentum map associated to a cotangent lifted action and formula (28), we have

$$\langle \mathbf{J}_L(\mathcal{P}_Q), U \rangle = \int_S \mathcal{P}_Q(p_s) \cdot U(Q(p_s)) d^k s = \int_S (\zeta(p_s) \cdot A(U(Q(p_s))) + \mathcal{P}_Q^A(s) \cdot [U \circ Q](s)) d^k s$$

$$= \int_S (\bar{\zeta}(s) \cdot \bar{A} \circ U \circ Q(s) + \mathcal{P}_Q^A(s) \cdot ([U] \circ Q(s))) d^k s$$

$$= \int_S \left(\bar{Q}(\bar{\zeta}(s)) \cdot \bar{A} \circ U(Q(s)) + \mathcal{P}_Q^A(s) \cdot [U](Q(s))\right) d^k s.$$
Note that in the first line, the expression $\mathcal{P}_Q(p_s)\cdot U(Q(p_s))$ only depends on $s$ and not on $p_s$, by equivariance. Thus it makes sense to integrate it on $S$. From the last equality we obtain the desired expression. □

As in the trivial case, for a certain class of Hamiltonians $h : \mathfrak{aut}(P) \to \mathbb{R}$ we can define the collective Hamiltonian $H := h \circ J_L : T^*Q_{KK} \to \mathbb{R}$. Since $J_L : T^*Q_{KK} \to \mathfrak{aut}(P)^*$ is a Poisson map, a solution of Hamilton’s equations for $H$ on $T^*Q_{KK}$ gives a (possibly measure-valued) solution of the EP$\mathcal{A}ut$ equation associated to the Hamiltonian $h$. The trivial bundle case is treated in Appendix A.3.

### 3.6.3 Right-action momentum map and Noether’s Theorem

We now compute the momentum map $J_R$ associated to the right action of $\mathcal{A}ut(P_S)$ on $T^*Q_{KK}$. The Lie algebra is denoted by $\mathfrak{aut}(P_S)$ and consists of equivariant vector fields on $P_S$. The infinitesimal generator associated to $U \in \mathfrak{aut}(P_S)$ is $dQ \circ U$. Thus, the momentum map is simply given by

$$J_R(\mathcal{P}) = \mathcal{P}_Q \cdot dQ \in \mathfrak{aut}(P_S)^*.$$  

In the following proposition we give a more concrete expression for $J_R$, by fixing principal connections $A_S$ and $\mathcal{A}$ on $P_S$ and $P$, respectively.

**Proposition 3.9 (Right-action momentum map)** The momentum map associated to the cotangent lifted right action of the automorphism group of $P_S$ on $T^*Q_{KK}$ reads

$$J_R^*(Q, P_A^A, \zeta) = (P_A^A \cdot dQ + \zeta \cdot (Q^* \mathcal{A} - A_S), \zeta) \in \mathfrak{X}(S)^* \times \mathcal{F}_\mathcal{O}(P_S, \mathfrak{o})^*,$$

where $P_A^A = (\text{Hor}_{Q})^* \mathcal{P}_Q$ and $\zeta := \mathcal{J} \circ \mathcal{P}_Q$.

**Proof.** Using the connection dependent isomorphisms

$$U \in \mathfrak{aut}(P_S) \simeq \mathfrak{X}(S) \times \mathcal{F}_\mathcal{O}(P_S, \mathfrak{o}) \ni ([U], A_S(U))$$

and

$$\mathcal{P} \in T^*_Q Q_{KK} \simeq \mathcal{F}_\mathcal{O}(P_S, \mathfrak{o})^* \times T^*_Q \text{Emb}(S, M) \ni (\zeta, P^A),$$

we find

$$\langle J_R(Q, P_A^A, \zeta), ([U], A_S(U)) \rangle = \langle J_R(Q, \mathcal{P}), U \rangle = \langle \mathcal{P}, dQ \circ U \rangle$$

$$= \langle P_A^A, dQ \circ U \rangle + \langle \zeta, \mathcal{A}(dQ \circ U) \rangle$$

$$= \langle P_A^A, dQ \circ [U] \rangle + \langle \zeta, \mathcal{A}(dQ \circ \text{Hor}^{A_S}[U] + dQ \circ (A_S(U))_{P_S}) \rangle$$

$$= \langle P_A^A, dQ \circ [U] \rangle + \langle \zeta, \mathcal{A}(dQ \circ \text{Hor}^{A_S}(U)) \rangle + \langle \zeta, \mathcal{A}(A_S(U)_{P_S} \circ Q) \rangle$$

$$= \langle P_A^A, dQ + \zeta \cdot (\mathcal{A} \circ dQ \circ \text{Hor}^{A_S}), [U] \rangle + \langle \zeta, A_S(U) \rangle, \quad (30)$$

where $\langle , \rangle$ denotes the $L^2$ pairing. Now we observe that

$$\mathcal{A} \circ dQ \circ \text{Hor}^{A_S}[U] = \mathcal{A} \circ dQ \circ U - \mathcal{A} \circ dQ \circ (A_S(U))_{P_S}$$

$$= \mathcal{A} \circ dQ \circ U - \mathcal{A} \circ (A_S(U) \circ Q)_{P} = (Q^* \mathcal{A} - A_S)(U)$$

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as required. ■

The left momentum map \( J_L \) is invariant under the cotangent lift of composition on the right by elements in \( \text{Aut}(P_S) \). Thus, the collective Hamiltonian \( H := h \circ J_L \) is also right invariant and by Noether’s theorem, we have

\[
\frac{d}{dt} \mathcal{P}_t \cdot dQ_t = 0,
\]

where \( \mathcal{P}_t \in T^*_Q, Q_{KK} \) is a solution of Hamilton’s equations. This can be written as

\[
(31) \quad \frac{d}{dt} \mathcal{P}_t^* \Theta_P = 0,
\]

where \( \Theta_P \) is the canonical one-form on \( T^*P \) and \( \mathcal{P}_t \) is interpreted as a map \( \mathcal{P}_t : S \rightarrow T^*M \) covering \( Q_t \). Introducing principal connections \( A \) and \( A_S \) on \( P \) and \( P_S \) we have the conservation laws

\[
\frac{d}{dt} \left( \mathcal{P}_t^A \cdot dQ + \zeta \cdot (Q^* A - A_S) \right) = 0 \quad \text{and} \quad \frac{d}{dt} \zeta = 0,
\]

where we recall that \( \mathcal{P}_t^A = (\text{Hor}_Q)^* \mathcal{P}_Q \) and \( \zeta := J \circ \mathcal{P}_Q \). The resulting dual pair is illustrated in the following diagram

\[
\begin{array}{ccc}
T^*\text{Emb}_O(P_S, P) & & \\
\downarrow_{J_L} & \downarrow_{J_R} & \\
\text{aut}(P)^* & & \text{aut}(P_S)^*
\end{array}
\]

Note that in the particular case \( S = M \) and \( P_S = P \), we have \( \text{Emb}_O(P, P) = \text{Aut}(P) \), since equivariant embeddings of \( P_S = P \) into \( P \) are automorphisms. In this case, \( J_L \) is the usual Lagrange-to-Euler map, and \( J_R \) is the conserved momentum density.

The trivial bundle case is treated in Appendix A.3, where it is shown that the right momentum map has the expression

\[
J_R (Q, \mathcal{P}_Q^A, \kappa_\theta) = (\mathcal{P}_Q \cdot dQ + \kappa_\theta \cdot (d\theta - \theta A_S), \theta^{-1}\kappa_\theta),
\]

which recovers (13) when \( A_S \) is the trivial connection.

**Remark 3.10 (Physical interpretation of the connection \( A_S \))** When \( O = S^1 \), the quantity \( A_S \) may be interpreted as a magnetic potential that is localized on the subbundle \( P_S \), where the whole motion takes place. For example, if we think of a trivial subbundle \( P_S = S \times O \) where \( \text{dim}S = 2 \), then the charged particles moving on \( S \) produce a current sheet carrying a magnetic field \( B_S = dA_S \) normal to the sheet. This suggestive picture is consistent with the concept that the singular solutions possess their own dynamics, independently of the properties of the ambient space. Moreover, we recall that the first component of \( J_R \) yields the Clebsch representation of a fluid variable in \( \text{aut}(P_S)^* \). In this case, the connection one-form \( A_S \) on \( S \) appears naturally in the Clebsch representation, due to the principal bundle structure of the embedded subspace \( P_S = S \times O \).
4 Incompressible EP\textit{Aut} flows

At this point of the paper, a natural question concerns the restriction of the above geodesic flows on the $\textit{Aut}(P)$ group to the volume-preserving case. This question is motivated by the fact that such construction yields a two-component Euler system in which an incompressible fluid flow transports a Yang-Mills charge, under the influence of an external magnetic potential. This picture generalizes Arnold’s well known construction of Euler’s equation as a geodesic on $\text{Diff}_{\text{vol}}(M)$ to the formulation of a geodesic fluid flow on $\text{Aut}_{\text{vol}}(P)$. In the case of a trivial bundle $P = M \times O$, one has $\text{Aut}_{\text{vol}}(P) \simeq \text{Diff}_{\text{vol}}(M) \otimes \mathcal{F}(M, O)$ and the incompressible EP\textit{Aut} equation (EP\textit{Aut}_{\text{vol}}) is then a geodesic equation on a semidirect-product Lie group. In this setting, natural questions arise about how the geometric properties of ideal fluid flows transfer to this more general situation. For example, the system can be written in terms of the vorticity $\omega = \text{curl} u$ and one can ask how the dual pair construction of Marsden and Weinstein [1983] applies to this case.

4.1 The group of volume preserving automorphisms

Let $\pi : P \to M$ be a right principal $O$-bundle and suppose that $M$ is orientable, endowed with a Riemannian metric $g$. Let $\mu_M$ be the volume form induced by $g$.

The group $\text{Aut}_{\text{vol}}(P)$ consists, by definition, of the automorphisms of the principal bundle $P$ which descend to volume preserving diffeomorphisms of the base manifold $M$ with respect to the volume form $\mu_M$. In other words, given an automorphism $\varphi \in \text{Aut}(P)$, we have by definition

$$\varphi \in \text{Aut}_{\text{vol}}(P) \iff \bar{\varphi} \in \text{Diff}_{\text{vol}}(M),$$

where $\text{Diff}_{\text{vol}}(M)$ is the group of volume preserving diffeomorphisms of $M$. Its Lie algebra, denoted by $\text{aut}_{\text{vol}}(P)$ consists of equivariant vector fields $U$ such that their projection is divergence free, that is, for $U \in \text{aut}(P)$ we have

$$U \in \text{aut}_{\text{vol}}(P) \iff [U] \in \mathfrak{X}_{\text{vol}}(M).$$

Hence, given a principal connection $\mathcal{A}$ on $P$, the Lie algebra $\text{aut}_{\text{vol}}(P)$ decomposes under the isomorphism (14) as

$$\text{aut}_{\text{vol}}(P) \to \mathfrak{X}_{\text{vol}}(M) \times \mathcal{F}_O(P, \mathfrak{o}), \quad U \mapsto ([U], \mathcal{A}(U)).$$

An inner product $\gamma$ on $\mathfrak{o}$ being also given, one defines the Kaluza-Klein Riemannian metric on $P$ as

$$\kappa(U_p, V_p) = g(T\pi(U_p), T\pi(V_p)) + \gamma(\mathcal{A}(U_p), \mathcal{A}(V_p)), \quad U_p, V_p \in T_p P.$$ 

The induced volume form $\mu_P$ on $P$ is

$$\mu_P := \pi^* \mu_M \wedge \mathcal{A}^* \text{det}_\gamma,$$

where $\mathcal{A}^* \text{det}_\gamma$ denotes the pullback by the connection $\mathcal{A} : TP \to \mathfrak{o}$ of the canonical determinant form induced by $\gamma$ on $\mathfrak{o}$, that is $\text{det}_\gamma \in \wedge^k \mathfrak{o}^*$, $k = \text{dim} \mathfrak{o}$, defined by

$$\text{det}_\gamma(\xi_1, ..., \xi_k) := \sqrt{\text{det}(\gamma(\xi_i, \xi_j))}.$$
where \( \xi_1, \ldots, \xi_k \in \mathfrak{o} \) is a positively oriented basis.

We now suppose that \( \gamma \) is \( \text{Ad} \)-invariant. In this case the Kaluza-Klein metric \( \kappa \) and the volume form \( \mu_P \) are \( \mathcal{O} \)-invariant. More general \( \mathcal{O} \)-invariant metrics on \( P \) are considered in Molitor [2009].

**Lemma 4.1** Let \( \varphi \in \text{Aut}(P) \) be a principal bundle automorphism. Then we have the equivalence

\[
\varphi \in \text{Aut}_{\text{vol}}(P) \iff \varphi^* \mu_P = \mu_P.
\]

**Proof.** The pull-back \( \mathcal{A}_\varphi := \varphi^*\mathcal{A} \) of a principal connection \( \mathcal{A} \) by an automorphism \( \varphi \) of \( P \) is a principal connection. Therefore, the \( \mathcal{O} \)-invariant form \( \mathcal{A}_\varphi \ast \det_\gamma - \mathcal{A} \ast \det_\gamma \in \Omega^k(P) \) is horizontal, so it is the pull-back \( \pi^* \alpha \) for some \( \alpha \in \Omega^k(M) \). Denoting by \( \varphi \) the diffeomorphism of \( M \) induced by \( \varphi \), we compute

\[
\varphi^* \mu_P = \varphi^*(\pi^* \mu_M \wedge \mathcal{A} \ast \det_\gamma) = \varphi^* \pi^* \mu_M \wedge \mathcal{A}_\varphi \ast \det_\gamma
\]

\[
= \pi^* \varphi^* \mu_M \wedge (\mathcal{A} \ast \det_\gamma + \pi^* \alpha) = \pi^* \varphi^* \mu_M \wedge \mathcal{A} \ast \det_\gamma.
\]

With this formula, it is clear that any \( \varphi \in \text{Aut}_{\text{vol}}(P) \) preserves the volume form \( \mu_P \).

Conversely, if \( \varphi \) preserves \( \mu_P \), then we have

\[
\pi^* \varphi^* \mu_M \wedge \mathcal{A} \ast \det_\gamma = \pi^* \mu_M \wedge \mathcal{A} \ast \det_\gamma.
\]

Plugging in the horizontal lifts of \( n = \dim M \) arbitrary vector fields \( u_1, \ldots, u_n \), we obtain

\[
\pi^*((\varphi^* \mu_M - \mu_M)(u_1, \ldots, u_n)) \mathcal{A} \ast \det_\gamma = 0,
\]

hence \( \varphi^* \mu_M = \mu_M \). \( \blacksquare \)

As a consequence, the Lie algebra \( \text{aut}_{\text{vol}}(P) \) coincides with the Lie algebra of equivariant divergence free vector fields with respect to the volume form \( \mu_P \) induced by the Kaluza-Klein metric on \( P \).

We will also need the following result.

**Lemma 4.2** Assume that \( \mathcal{O} \) is compact and consider an \( \mathcal{O} \)-invariant function \( f \) on \( P \) with compact support. Then we have the formula

\[
(33) \quad \int_P f(p) \mu_P = \text{Vol}(\mathcal{O}) \int_M f(m) \mu_M,
\]

where \( \text{Vol}(\mathcal{O}) \) denotes the Riemannian volume of \( \mathcal{O} \) relative to the bi-invariant metric induced by \( \gamma \), and \( \tilde{f} \) is the function induced on \( M \) by the formula \( \tilde{f} \circ \pi = f \).

**Proof.** It is enough to check the formula for a trivial bundle \( P = M \times \mathcal{O} \). In this case the connection 1-form is defined with a 1-form \( A \in \Omega^1(M, \mathfrak{o}) \) by \( A(v_x, \xi_g) = \text{Ad}_{g^{-1}}(A(v_x) + \kappa^l(\xi_g)) \), where \( \kappa^l(\xi_g) = \xi_g g^{-1} \in \Omega^1(\mathcal{O}, \mathfrak{o}) \) denotes the left Maurer-Cartan form on the Lie group \( \mathcal{O} \). Note that \( \mu_{\mathcal{O}} := (\kappa^l)^\ast \det_\gamma \in \Omega^k(\mathcal{O}) \) is the volume form induced by the bi-invariant Riemannian metric on \( \mathcal{O} \) associated to \( \gamma \). Let \( \pi_2 : M \times \mathcal{O} \to \mathcal{O} \) denote the projection on the second factor. Taking into account the \( \mathcal{O} \)-invariance of the inner product \( \gamma \) we obtain

\[
\mu_P = \pi^* \mu_M \wedge \mathcal{A} \ast \det_\gamma = \pi^* \pi_2^* \mu_{\mathcal{O}} = \pi^* \pi_2^* \mu_{\mathcal{O}} + \pi^* \pi_2^* \kappa^l \mu_{\mathcal{O}}.
\]

Now, formula (33) follows for every compactly supported smooth function \( f \) on \( M \). \( \blacksquare \)
4.2 Dynamics on a trivial principal bundle

If the principal bundle is trivial, then we have the group isomorphism \( \text{Aut}_{\text{vol}}(P) \cong \text{Diff}_{\text{vol}}(M) \otimes \mathcal{F}(M, \mathcal{O}) \). We now give the expression of the Euler-Poincaré equations on \( \text{Aut}_{\text{vol}}(P) \) in the case of a trivial bundle. Since a volume form \( \mu_M \) has been fixed, it is not necessary to include densities in the dual Lie algebras. The regular dual to \( \mathfrak{X}_{\text{vol}}(M) \otimes \mathcal{F}(M, \mathcal{O}) \) is identified with \( (\Omega^1(M)/d\mathcal{F}(M)) \times \mathcal{F}(M, \mathcal{O}^*) \) via the \( L^2 \)-pairing given by the volume form \( \mu_M \). Here \( \Omega^1(M)/d\mathcal{F}(M) \) denotes the space of one-forms modulo exact one-forms. As in Proposition 2.1, but this time considering a Lagrangian \( l \) on the semidirect product \( \mathfrak{X}_{\text{vol}}(M) \otimes \mathcal{F}(M, \mathcal{O}) \), we get the following

**Proposition 4.3 (The EP\text{Aut}_{\text{vol}} equations on a trivial principal bundle)** The Euler-Poincaré equations on the group of volume preserving automorphisms of a trivial principal bundle are written as

\[
\begin{align*}
\frac{\partial}{\partial t} \frac{\delta l}{\delta u} + \mathcal{L}_u \frac{\delta l}{\delta u} + \frac{\delta l}{\delta \nu} \cdot \frac{d}{d\nu} = -dp, \quad \text{div}(u) = 0 \\
\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} + d \frac{\delta l}{\delta \nu} \cdot u + \text{ad}^{\ast}_{\nu} \frac{\delta l}{\delta \nu} = 0,
\end{align*}
\]

where \( \mathcal{L}_u \) denotes the Lie-derivative of a one-form (and not a one-form density), and the pressure function \( p \) is determined by the incompressibility condition.

Here, the first equation is written on \( \Omega^1(M) \) and not on the quotient space \( \Omega^1(M)/d\mathcal{F}(M) \). As in the compressible case, one can use Kaluza-Klein Lagrangians of the form (3), relative to differential operators \( Q_1 : \mathfrak{X}(M) \to \mathfrak{X}(M) \) and \( Q_2 : \mathcal{F}(M, \mathcal{O}) \to \mathcal{F}(M, \mathcal{O}^*) \), that is,

\[
(35) \quad l(u, \nu) = \frac{1}{2} \int_M g(u, Q_1 u) \mu_M + \frac{1}{2} \int_M \langle A \cdot u + \nu, Q_2 (A \cdot u + \nu) \rangle \mu_M,
\]

where \( g \) is the given Riemannian metric on \( M \) and \( \mu_M \) is the Riemannian volume.

**Other duality pairings.** Since a Riemannian metric \( g \) is given, we can alternatively use the \( L^2 \)-pairing induced by \( g \). In this case, we can choose the dual space \( \mathfrak{X}_{\text{vol}}(M) \times \mathcal{F}(M, \mathcal{O}^*) \) and the EP\text{Aut} equations read

\[
\begin{align*}
\frac{\partial}{\partial t} \frac{\delta l}{\delta u} + \nabla u \frac{\delta l}{\delta u} + \nabla u^\top \cdot \frac{\delta l}{\delta \nu} + \left( \frac{\delta l}{\delta \nu} \cdot \frac{d}{d\nu} \right)^\sharp = -\text{grad} p, \quad \text{div}(u) = 0 \\
\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} + d \frac{\delta l}{\delta \nu} \cdot u + \text{ad}^{\ast}_{\nu} \frac{\delta l}{\delta \nu} = 0,
\end{align*}
\]

where \( \nabla \) denotes the Levi-Civita connection, \( ^\sharp \) is the sharp-operator associated to the Riemannian metric and \( \text{grad} p := (dp)^\sharp \).

Alternatively, if \( H^1(M) = \{0\} \), one can also identify the dual space to \( \mathfrak{X}_{\text{vol}}(M) \) with exact two-forms \( d\Omega^1(M) \), through the duality pairing

\[
(37) \quad \langle \omega, u \rangle = \int_M (\alpha \cdot u) \mu_M, \quad \text{where} \quad d\alpha = \omega.
\]
In this case, we get the first equation in vorticity representation as

\[
\begin{align*}
\frac{\partial}{\partial t} \chi + & \mathcal{L}_{\chi} \chi + d(\chi \wedge d\nu) = 0 \\
\frac{\partial}{\partial t} \chi \cdot u + & d\chi \wedge d\nu = 0, \\
\frac{\partial}{\partial t} \chi \cdot u + & d\chi \cdot u + \text{ad}^*_\nu \frac{\partial}{\partial \nu} = 0,
\end{align*}
\]

where the wedge product involves a contraction between \( \mathfrak{o} \) and \( \mathfrak{o}^* \)-valued forms. It is important to recall that the functional derivatives appearing above depend on the chosen pairing. For example, in (38) and with the Lagrangian (35) we have

\[
\omega := \frac{\delta l}{\delta u} = d(Q_1 u + \sigma \cdot A) \in d\Omega^1(M), \quad \sigma := \frac{\delta l}{\delta \nu} = Q_2 (A \cdot u + \nu) \in F(M, \mathfrak{o}^*).
\]

An interesting special case of the above equations comes out when \( M \) is the Euclidean space \( \mathbb{R}^3 \). In this case, the exact two-form \( \omega = d\alpha \) can be identified with the divergence free vector field \( \omega = \text{curl} \alpha \), via the relations

\[
(*)\mathcal{L}_u \omega = (*)d(i_u \omega) = (*)d(\omega \times u)^b = \text{curl}(\omega \times u) = (u \cdot \nabla) \omega - (\omega \cdot \nabla) u.
\]

In this case, the duality pairing is given by

\[
\langle \omega, u \rangle = \int_{\mathbb{R}^3} (v \cdot u) dx, \quad \text{where} \quad \text{curl} v = \omega.
\]

For example, taking the Kaluza-Klein Lagrangian (35), the corresponding EP\textit{Aut} equations are obtained by inserting the functional derivatives

\[
\omega = \frac{\delta l}{\delta u} = \text{curl}(Q_1 u) + \text{curl}(\mu \cdot A) \quad \text{and} \quad \sigma = \frac{\delta l}{\delta \nu} = Q_2 (A \cdot u + \nu)
\]

into (39). The two dimensional case is presented in Appendix B.1.

**Remark 4.4** The above equations generalize an important regularization model in fluid dynamics, which is known under the name of Euler-\( \alpha \) (Holm, Marsden, and Ratiu [1998]). This model is derived by specializing the above equations to the particular case when \( Q_1 = 1 - \alpha^2 \Delta \) and \( \nu \equiv 0 \), the latter condition being preserved by the flow. Thus, the complete system of the EP\textit{Aut}_{vol} equations generalizes the Euler-\( \alpha \) equation to the case of a Yang-Mills charged fluid moving under the influence of an external magnetic field.
4.3 Dynamics on a non trivial principal bundle

By (32) the regular dual of $\text{aut}_{\text{vol}}(P)$ can be identified with the cartesian product $(\Omega^1(M)/dF(M)) \times \Gamma(\text{Ad}^* P)$, via the $L^2$ pairing

$$\langle ([\beta], \zeta), (u, \tilde{\omega}) \rangle = \int_M (\beta(u) + \zeta \cdot \tilde{\omega}) \mu_M.$$ 

Given a Lagrangian $l : \text{aut}_{\text{vol}}(P) \to \mathbb{R}$ and a principal connection $A$, the associated EP$A_{\text{ut}_{\text{vol}}}$ equations are given in the following

**Proposition 4.5 (The general EP$A_{\text{ut}_{\text{vol}}}$ equations on principal bundles)** The Euler-Poincaré equations on the group of volume preserving automorphisms of a principal bundle are written as

$$\begin{cases}
\frac{\partial}{\partial t} \delta l A + \mathcal{L}_u \frac{\delta l A}{\delta u} + \frac{\delta l A}{\delta \tilde{\omega}} \left( \nabla^A \tilde{\omega} + i_u \tilde{B} \right) = -dp, & \text{div } u = 0 \\
\frac{\partial}{\partial t} \delta l A + \nabla^A \frac{\delta l A}{\delta \tilde{\omega}} + \text{ad}^* \frac{\delta l A}{\delta \omega} = 0,
\end{cases}$$

where $\mathcal{L}_u$ denotes the Lie derivative acting one one-forms and the pressure $p$ is determined by the incompressibility condition.

**Proof.** The equations can be obtained as in §3.2 by splitting the general formulation

$$\frac{\partial}{\partial t} \frac{\delta l}{\delta U} + \mathcal{L}_U \frac{\delta l}{\delta U} = 0,$$

and using the connection dependent isomorphism. Then we deduce (40) from (19) by simply adding the incompressibility condition and using the fixed Riemannian volume $\mu_M$ to identify one-form densities with one-forms, and sections in $\Gamma(\text{Ad}^* P) \otimes \text{Den}(M)$ with sections in $\Gamma(\text{Ad}^* P)$. Then one obtains the second equation since $\mathcal{L}_u \tilde{\omega}$ and $\nabla^A \tilde{\omega}$ coincide when applied to functions.

Alternatively, it is instructive to derive the equations directly by using the expression of the infinitesimal coadjoint action $\text{ad}^*$ on $(\Omega^1(M)/dF(M)) \times \Gamma(\text{Ad}^* P)$. In order to do this, we first need the expression of the Lie bracket on $\mathfrak{x}_{\text{vol}}(M) \times \Gamma(\text{Ad } P)$. Under the isomorphism (14), the Lie bracket is

$$\left( [u, \tilde{\omega}], (v, \tilde{\theta}) \right)_L = \left( [u, v]_L, [\tilde{\omega}, \tilde{\theta}] + \nabla^A \tilde{\omega} - \nabla^A u \tilde{\theta} + \tilde{B}(u, v) \right),$$

where $\tilde{B} \in \Omega^2(M, \text{Ad } P)$ is the reduced curvature. Then it suffices to compute the associated infinitesimal coadjoint action. This is done in the following lemma.

**Lemma 4.6** The infinitesimal coadjoint action of $\text{aut}_{\text{vol}}(P)$ on its regular dual, using the decomposition provided by a principal connection $A$, can be written as

$$\text{ad}^*_{(u, \tilde{\omega})}([\beta], \alpha) = \left( [L_u \beta + \alpha \cdot (\nabla^A \tilde{\omega} + i_u \tilde{B})], \text{ad}^* \alpha + \nabla^A \alpha \right),$$

for $(u, \tilde{\omega}) \in \mathfrak{x}_{\text{vol}}(M) \times \Gamma(\text{Ad } P)$ and $([\beta], \alpha) \in (\Omega^1(M)/dF(M)) \times \Gamma(\text{Ad}^* P)$. 

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Proof. Using the expression (41) of the bracket on $\mathfrak{aut}_\text{vol}(P)$ under the decomposition (32), we compute

$$
\langle \text{ad}^*_{(u,\bar{\omega})}(\beta), (\nu, \bar{\theta}) \rangle = \langle [\beta], [u, \nu]_L \rangle + \left\langle \alpha, [\bar{\omega}, \bar{\theta}] + \nabla^A \bar{\omega} - \nabla^A \bar{\theta} + \tilde{B}(u, \nu) \right\rangle
$$

$$
= - \int_M \beta([u, \nu]) \mu_M + \int_M \alpha \cdot (\text{ad}_{\bar{\omega}} \bar{\theta}) \mu_M + \int_M (\alpha \cdot \nabla^A \bar{\omega}) \mu_M
$$

$$
- \int_M \alpha \cdot \left( \nabla^A \bar{\omega} \right) \mu_M + \int_M \left( \alpha \cdot i_u \tilde{B} \right) (\nu) \mu_M
$$

$$
= \int_M \mathcal{L}_u \beta(\nu) \mu_M + \left\langle \alpha \cdot \left( \nabla^A \bar{\omega} + i_u \tilde{B} \right), \nu \right\rangle
$$

$$
+ \int_M \left( \text{ad}^*_{(u,\bar{\omega})} \alpha + \nabla^A \alpha \right) \cdot \bar{\theta} \mu_M
$$

$$
= \left\langle \left( \mathcal{L}_u \beta + \alpha \cdot \left( \nabla^A \bar{\omega} + i_u \tilde{B} \right) \right), \text{ad}^*_{(u,\bar{\omega})} \alpha + \nabla^A \alpha \right\rangle, (\nu, \bar{\theta}) \right\rangle,
$$

where we use that $\text{div} u = 0$. \qed

Remark 4.7 (Kaluza-Klein Lagrangians) As in the compressible case, an important class of Lagrangians is provided by the Kaluza-Klein expression (22), thus obtaining the incompressible version of (23). In the case of the $L^2$ Kaluza-Klein Lagrangian, the EP$\mathcal{A}ut_\text{vol}$ equation takes the simple form

$$
\begin{cases}
\partial_t \nu + \nabla_u \nu + \gamma \left( \bar{\omega}, i_u \tilde{B} \right) = - \text{grad} p, \quad \text{div} u = 0 \\
\partial_t \bar{\omega} + \nabla^A \bar{\omega} = 0.
\end{cases}
$$

These equations describe the motion of an ideal fluid moving in an external Yang-Mills field $\tilde{B}$, where $\bar{\omega}$ denotes the magnetic charge (Vizman [2008]). For the more general case with generic metrics $Q_1$ and $Q_2$, the EP$\mathcal{A}ut_\text{vol}$ equation generalizes the Euler-$\alpha$ model (Holm, Marsden, and Ratiu [1998]) to the case of a Yang-Mills charged fluid, whose configuration manifold is a non-trivial principal bundle. Again, the fluid moves under the influence of an external Yang-Mills magnetic field, which is given by the curvature $\tilde{B}$ as usual.

5 Clebsch variables and momentum maps

This section presents the dual pair of momentum maps underlying EP$\mathcal{A}ut_\text{vol}$ flows. This extends the dual pair structure underlying ideal incompressible fluid flows, as it was shown by Marsden and Weinstein [1983]. The next section reviews briefly the Marsden-Weinstein dual pair, whose details are given in Appendix B.2.

5.1 The Marsden-Weinstein dual pair

In the paper Marsden and Weinstein [1983], the authors present a pair of momentum maps that apply to Euler’s equation $\partial_t \omega + \text{curl}(\omega \times u) = 0$ for the fluid vorticity $\omega =$
curl \( u \). In particular, these momentum maps provide the Clebsch representation of the fluid vorticity \( \text{curl} \ u \) on the configuration manifold \( S \) in terms of canonical variables taking values in a symplectic manifold \( M \). Again, one constructs the dual pair of momentum maps

\[
\begin{array}{ccc}
\text{Emb}(S, M) & \xrightarrow{J_L} & \mathcal{F}(M)^* \\
\downarrow & & \downarrow \\
\mathcal{F}(M)^* & \xrightarrow{J_R} & \mathcal{X}_{\text{vol}}(S)^*
\end{array}
\]

whose right leg provides the Clebsch representation of the vorticity on \( S \). A rigorous construction of these momentum maps and the proof of the dual pair property were given in Gay-Balmaz and Vizman [2011].

In particular, \( S \) is a compact manifold with volume form \( \mu_S \) while \( (M, \omega) \) is an exact (and hence noncompact) symplectic manifold, with \( \omega = -d\theta \). The Marsden-Weinstein dual pair (Marsden and Weinstein [1983]) is associated to the action of the groups \( \text{Diff}_{\text{vol}}(S) \) and \( \text{Diff}_{\text{ham}}(M) \) on \( \text{Emb}(S, M) \) by composition on the right and on the left, respectively. Here \( \text{Diff}_{\text{ham}}(M) \) is the group of Hamiltonian diffeomorphisms of \( (M, \omega) \). The momentum map associated to the right action reads

\[
J_R(f) = [f^*\theta] \in \Omega^1(S)/d\mathcal{F}(S),
\]

or

\[
J_R(Q, P) = [(Q, P)^*\Theta_{\mathbb{R}^3}] = [P \cdot dQ] = \text{curl}(\nabla Q^T \cdot P).
\]

when \( M = \mathbb{R}^{2k} \) and \( S = \mathbb{R}^3 \). (Here we have used the duality pairing arguments in §4.2 to identify an exact two form \( \omega \) with a divergence-free vector field). Notice that the corresponding variables also appear in the left leg

\[
J_L(f) = \int_S \delta(n - f(x))\mu_S \in \mathcal{F}(M)^*,
\]

or, in local coordinates,

\[
J_L(Q, P) = \int_S \delta(q - Q(s))\delta(p - P(s))\mu_S \in \mathcal{F}(\mathbb{R}^{2k})^*.
\]

This is due to the fact that Clebsch variables are conjugate variables taking values in \( M \). However, in their work Gay-Balmaz and Vizman [2011] proved that the dual pair structure requires replacing the group \( \text{Diff}_{\text{ham}}(M) \) by a central extension of the type \( \text{Diff}^\times_{\text{ham}}(M) \times_B \mathbb{R} \), where \( B \) is a group two cocycle. This central extension of \( \text{Diff}^\times_{\text{ham}}(M) \) first appeared in Ismagilov, Losik, and Michor [2006], who showed how this is isomorphic to the quantomorphism group \( \text{Quant}(T^*M \times S^1) \), well known in quantization problems.

Also, if the symplectic form on \( S \) is not exact, then one needs in addition the Ismagilov extension of the group of volume preserving diffeomorphisms of \( S \). In what follows we consider only exact symplectic manifolds, so there will be no central extension needed for the right leg of the dual pair.
The purpose of this section is to present the $O$-equivariant variant of the Marsden-Weinstein dual pair, which is the dual pair of momentum maps that apply to the $\text{EPAut}_{\text{vol}}$ equation.

### 5.2 Clebsch variables for incompressible $\text{EPAut}$ flows

After the preceding review of the momentum maps underlying the Clebsch representation of Euler’s fluid vorticity, we are now ready to generalize that construction to our principal bundle setting. In particular, we shall characterize the momentum maps underlying the Clebsch representation of the $\text{EPAut}_{\text{vol}}$ system.

We recall from the Clebsch representation of Euler’s vorticity that Clebsch variables belong to a space of embeddings $S \hookrightarrow M$, where $S$ is the fluid particle configuration manifold and $M$ is a symplectic manifold. For simplicity, this section will introduce the fundamental concepts in the simple case when $S = \mathbb{R}^3$, so that the fluid moves in the ordinary physical space. In the case of $\text{EPAut}_{\text{vol}}$, the equations describe the evolution of a Yang-Mills charged fluid under the influence of an external magnetic field. Thus, it is natural to identify $M$ with the usual Yang-Mills phase space $T^*\bar{P}$, where $\bar{P}$ is a principal $O$-bundle.

As a further simplification, this section will describe the case of a trivial bundle $\bar{P} = \mathbb{R}^k \times O$ with $O \subseteq GL(n, \mathbb{R})$ being a matrix Lie group. Thus, we are led to the following definition on Clebsch variables:

$$(Q, P, \sigma, \theta) \in \text{Emb}(\mathbb{R}^3, \mathbb{R}^{2k} \times o^*) \times \mathcal{F}(\mathbb{R}^3, O),$$

where $\text{Emb}(\mathbb{R}^3, \mathbb{R}^{2k} \times o^*) \times \mathcal{F}(\mathbb{R}^3, O)$ is naturally endowed with the Poisson structure:

$$\{F, G\}(Q, P, \sigma, \theta) = \int \frac{1}{w} \left( \frac{\delta F}{\delta Q} \cdot \frac{\delta G}{\delta P} - \frac{\delta G}{\delta Q} \cdot \frac{\delta F}{\delta P} \right) \, d^3 x \,$$

$$+ \int \frac{1}{w} \left( \left\langle \sigma, \left[ \frac{\delta F}{\delta \sigma}, \frac{\delta G}{\delta \sigma} \right] \right\rangle + \left\langle \frac{\delta F}{\delta \theta}, \frac{\delta G}{\delta \sigma} \theta \right\rangle - \left\langle \frac{\delta G}{\delta \theta}, \frac{\delta F}{\delta \sigma} \theta \right\rangle \right) \, d^3 x,$$

where $w(x) \, d^3 x$ is a fixed volume form on $\mathbb{R}^3$, and the angle bracket $\langle \cdot, \cdot \rangle$ denotes the trace pairing $\langle A, B \rangle = \text{Tr}(A^T B)$. At this point, one observes that

**Theorem 5.1** The infinitesimal action of volume-preserving automorphisms $(u, \zeta) \in \text{aut}_{\text{vol}}(\mathbb{R}^3 \times O) \simeq \mathcal{X}_{\text{vol}}(\mathbb{R}^3) \otimes \mathcal{F}(\mathbb{R}^3, o)$

$$(u, \zeta)_{\text{Emb}(\mathbb{R}^3, \mathbb{R}^{2k} \times o^*) \times \mathcal{F}(\mathbb{R}^3, O)}(Q, P, \sigma, \theta) = \left( (u \cdot \nabla) Q, (u \cdot \nabla) P, (u \cdot \nabla) \sigma, (u \cdot \nabla) \theta + \theta \zeta \right)$$

yields the right-action momentum map

$$J_R(Q, P, \sigma, \theta) = \left( \text{curl}(\nabla Q^T \cdot P + \langle \sigma, \nabla \theta \theta^{-1} \rangle), \text{Ad}_o^* \sigma \right) \in \mathcal{X}_{\text{vol}}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3, o)^*.$$

See Appendix B.3 for the proof. Notice that the above result will follow very easily in the more abstract setting of the next sections.
Besides the above right-action momentum map $J_R$, one also has the momentum map $J_L$ arising from the left infinitesimal action of the Hamiltonian functions $h \in \mathcal{F}(\mathbb{R}^{2k} \times T^*O)$, similarly to the left leg of the Marsden Weinstein dual pair for ideal fluid. However, there is an important special situation that arises in the physics of Yang-Mills charged particles. Indeed, the celebrated Wong’s equations for the motion of a single Yang-Mills charge are produced by Hamiltonians in the space $\mathcal{F}(\mathbb{R}^{2k} \times T^*O)^O$ of $O$-invariant functions on $\mathbb{R}^{2k} \times T^*O$. In this context, the trivialization map $T^*O \simeq O \times o^*$ induces (by Lie-Poisson reduction) the Poisson structure on the space $\mathbb{R}^{2k} \times T^*O/O \simeq \mathbb{R}^{2k} \times o^*$ (cf. e.g. Montgomery, Marsden and Ratiu [1984]). Therefore, it is reasonable to consider a left-action momentum map arising from the infinitesimal action of the Poisson subalgebra

$$\mathcal{F}(\mathbb{R}^{2k} \times T^*O)^O = \mathcal{F}(\mathbb{R}^{2k} \times o^*) \subset \mathcal{F}(\mathbb{R}^{2k} \times T^*O).$$

Indeed, one verifies that the infinitesimal action of $\bar{h}(q,p,\zeta) \in \mathcal{F}(\mathbb{R}^{2k} \times o^*)$

$$\left( \bar{h} \right)_{\mathcal{F}(\mathbb{R}^{3,2k+T^*O})} (Q,P,\sigma,\theta) = \left( \frac{\partial \bar{h}}{\partial p}, -\frac{\partial \bar{h}}{\partial q}, -\text{ad}^\ast_{\frac{\partial \bar{h}}{\partial \theta}} \zeta, \frac{\partial \bar{h}}{\partial \zeta} \right)_{(q,p,\zeta)=(Q,P,\sigma)}$$

produces the left-action momentum map

$$J_L(Q,P,\sigma,\theta) = \int w(x) \delta(q-Q(x)) \delta(p-P(x)) \delta(\sigma-\sigma(x)) \, d^3x \in \mathcal{F}(\mathbb{R}^{2k} \times o^*).$$

In this case, the proof is straightforward and will be omitted.

**Remark 5.2 (Relations to kinetic theory and ideal fluids)** In analogy with the structures arising from Euler’s equation in two dimensions, the above expression is well known in kinetic theory, particularly the kinetic theory of Yang-Mills plasmas of interest in stellar astrophysics. Indeed, the above expression coincides with the well known Klimontovich particle solution of the Yang-Mills Vlasov equation for distributions on the reduced Yang-Mills phase space $\mathbb{R}^{2k} \times o^*$ (Gibbons, Holm, and Kupschmidt [1983]). However, the fortunate coincidence holding for ordinary Euler’s equation in two dimensions does not hold for EP$\text{Aut}_{vol}$, that is the left momentum map does not provide point vortex solutions of the 2D EP$\text{Aut}_{vol}$ system. Indeed, the latter does not have the form of a single Vlasov kinetic equation as it happens for 2D Euler.

The next section characterizes the group actions underlying the above momentum maps in the more general setting when the physical space $\mathbb{R}^3$ is replaced by a volume manifold $S$ and the trivial Yang-Mills phase space $T^k \times T^*O$ is replaced by a generic (non trivial) $O$-bundle carrying an exact invariant symplectic form.

### 5.3 Group actions and dual pair

We now present a dual pair of momentum maps in the context of the EP$\text{Aut}_{vol}$ equation, that consistently generalizes the Marsden-Weinstein dual pair. As we shall see, in the principal bundle context the groups $\text{Diff}_{vol}(S)$ and $\text{Diff}_{ham}(M) \times_B \mathbb{R}$ are naturally replaced by the automorphism groups $\text{Aut}_{vol}(P_S)$ and $\text{Aut}_{ham}(P) \times_B \mathbb{R}$ of two $O$-principal bundles $P_S$ and $P$, respectively. For the left momentum map, we will exhibit a subgroup $\text{Aut}_{ham}(P) \times_B \mathbb{R} \subset \text{Aut}_{ham}(P) \times_B \mathbb{R}$ that is more appropriate and whose Lie algebra identifies with the space $\mathcal{F}(P/O)$ endowed with the reduced Poisson bracket.
5.3.1 Geometric setting

Let $\pi : P_S \to S$ be the principal $O$-bundle of our EP $Aut_{vol}$ equation, and consider another principal $O$-bundle $P \to M$ such that $P$ carries an exact symplectic form $\omega = -d\theta$. Moreover, we assume that $\theta$ is $O$-invariant. As above, we endow $P_S$ with the $O$-invariant volume form $\mu_{P_S} = \pi^* \mu_S \wedge A^* \det_\gamma$, where $\gamma$ is an $\operatorname{Ad}$-invariant inner product on $\mathfrak{o}$ and $A$ a principal connection on $P_S$.

Consider the manifold $\operatorname{Emb}_O(P_S, P)$ of $O$-equivariant embeddings, see Appendix A.2. The tangent space $T_f \operatorname{Emb}_O(P_S, P)$ can be identified with the space of equivariant maps $u_f : P_S \to TP$ such that $\pi \circ u_f = f$. We endow the manifold $\operatorname{Emb}_O(P_S, P)$ with the symplectic form $\bar{\omega}$ as above, that is

$$\bar{\omega}(f)(u_f, v_f) := \int_{P_S} \omega(f(p))(u_f(p), v_f(p)) \mu_{P_S}.$$ 

The function under the integral is $O$-invariant, so the right hand side is an integral over $S$. Now the local triviality of the bundle $P_S \to S$ ensures the non-degeneracy of $\bar{\omega}$.

Thanks to Lemma 4.1, the group $Aut_{vol}(P_S)$ acts symplectically on $\operatorname{Emb}_O(P_S, P)$ by composition on the right. Similarly, the group of Hamiltonian automorphisms, defined by

$$Aut_{ham}(P) := Aut(P) \cap \operatorname{Diff}_{ham}(P)$$

acts symplectically on the left by composition. If the structure group $O$ of the principal bundles is the trivial group with one element, we recover the actions associated to the Marsden-Weinstein dual pair.

5.3.2 The right momentum map

The momentum map associated to the right action is

$$(47) \quad J_R(f) = [f^* \theta] \in \Omega^1_O(P_S)/d\mathcal{F}_O(P_S) = \mathfrak{aut}_{vol}(P_S)^*$$

The dual to $\mathfrak{aut}_{vol}(P_S)$ is identified here with $O$-invariant 1-forms quotiented by exact $O$-invariant 1-forms which are differentials of $O$-invariant functions.

One can write this identification in detail, using a principal connection $A$. Since a volume form is fixed, $\mathfrak{aut}(P_S)^*$ can be identified with $\Omega^1_O(P_S)$. As in (15), the dual to the isomorphism $\mathfrak{aut}(P_S) \to \mathfrak{X}(S) \times \mathcal{F}_O(P_S, \mathfrak{o})$ reads

$$(48) \quad \Omega^1_O(P_S) \to \Omega^1(S) \times \mathcal{F}_O(P_S, \mathfrak{o}^*), \quad \alpha \mapsto \begin{pmatrix} \bar{\alpha} = (\operatorname{Hor}^A)^* \alpha, \lambda = \mathcal{J} \circ \alpha \end{pmatrix},$$

where $\operatorname{Hor}^A$ denotes the horizontal-lift associated to the principal connection $A$ and $\mathcal{J} : T^*P_S \to \mathfrak{o}^*$ is the cotangent bundle momentum map. The inverse to the isomorphism (48) is

$$(\bar{\alpha}, \lambda) \mapsto \pi^* \bar{\alpha} + A^* \lambda.$$
Now the isomorphism $\text{aut}_{\text{vol}}(P_S) \to \mathfrak{x}_{\text{vol}}(S) \times \mathcal{F}_\mathcal{O}(P_S, \mathfrak{o})$, see (32), ensures the desired identification

$$\begin{align*}
\text{aut}_{\text{vol}}(P_S)^* &= \mathfrak{x}_{\text{vol}}(S)^* \times \mathcal{F}_\mathcal{O}(P_S, \mathfrak{o}^*) = (\Omega^1(S)/d\mathcal{F}(S)) \times \mathcal{F}_\mathcal{O}(P_S, \mathfrak{o}^*) \\
&= (\Omega^1(S) \times \mathcal{F}_\mathcal{O}(P_S, \mathfrak{o}^*)/(d\mathcal{F}(S) \times \{0\}) = \Omega^1_\mathcal{O}(P_S)/d\mathcal{F}_\mathcal{O}(P_S).
\end{align*}$$

As we shall see later, for trivial bundles one recovers the conclusions in theorem 5.1.

5.3.3 The left momentum map

In order to properly define the momentum map associated to the action by composition on the left, there are two difficulties to overcome. First, as above, we need to pass to the central extension of $\text{Aut}_{\text{ham}}(P)$ by the cocycle defined in Ismagilov, Losik, and Michor [2006]. This is possible since $\text{Aut}_{\text{ham}}(P)$ is a subgroup of $\text{Diff}_{\text{ham}}(P)$ on which the cocycle $B$ can be defined as in (65). Second, we note that the Lie algebra of the Hamiltonian automorphisms is given by

$$\text{aut}_{\text{ham}}(P) = \text{aut}(P) \cap \mathfrak{x}_{\text{ham}}(P),$$

that is, it consists of Hamiltonian vector fields that are $\mathcal{O}$-equivariant. Such vector fields are not necessarily associated to $\mathcal{O}$-invariant Hamiltonian functions on $P$. The latter form the Lie subalgebra

$$\overline{\text{aut}}_{\text{ham}}(P) := \{X_h \mid h \in \mathcal{F}(P)^\mathcal{O}\} \subset \text{aut}_{\text{ham}}(P),$$

since the Lie algebra bracket is $[X_h, X_k]_L = X_{\{h,k\}}$ and the Poisson bracket of two $\mathcal{O}$-invariant functions is $\mathcal{O}$-invariant, the symplectic form being $\mathcal{O}$-invariant.

The subgroup associated to this Lie subalgebra is determined in the following theorem (see Appendix B.4 for the proof).

**Theorem 5.3** Let $(P, -d\theta)$ be a connected exact symplectic manifold with $\theta \in \Omega^1_\mathcal{O}(P)$. Let $\text{Hom}(\mathcal{O}, \mathbb{R})$ be the vector space of group homomorphisms from $\mathcal{O}$ to $\mathbb{R}$ and consider the map

$$\Psi : \text{Aut}_{\text{ham}}(P) \to \text{Hom}(\mathcal{O}, \mathbb{R}),$$

defined by

$$\Psi(\varphi)(g) := F_{\varphi} - F_{\varphi} \circ \Phi_g, \quad \forall \ g \in G,$$

where $F_{\varphi} \in \mathcal{F}(P)$ is such that $dF_{\varphi} = \varphi^*\theta - \theta$. Then $\Psi$ is well-defined and is a group homomorphism.

Consider the normal subgroup $\overline{\text{Aut}}_{\text{ham}}(P) \subset \text{Aut}_{\text{ham}}(P)$ defined by

$$\overline{\text{Aut}}_{\text{ham}}(P) := \ker \Psi.$$

Then the formal Lie algebra of $\overline{\text{Aut}}_{\text{ham}}(P)$ is the space $\overline{\text{aut}}_{\text{ham}}(P)$ of Hamiltonian vector fields on $P$ associated to $\mathcal{O}$-invariant Hamiltonians.
Remark 5.4 (Special Hamiltonian automorphisms) The elements of the Lie group $\text{Aut}_\text{ham}(P)$ will be called special Hamiltonian automorphisms of the principal bundle $P$. This group is of central importance in common gauge theories: for example, they govern Wong’s equations for the dynamics of a Yang-Mills charge moving in the physical space. (Just like Hamiltonian diffeomorphisms govern canonical Hamilton’s equations).

Consider now the central extension $\text{Aut}_\text{ham}(P) \times_B \mathbb{R}$. The Lie algebra isomorphism (66) shows that the associated central extended Lie algebra can be identified with the Lie algebra $\mathcal{F}(P)^{O}$ of $O$-invariant functions on $P$, endowed with the symplectic Poisson bracket $\{f,g\}_P = \omega(X_f, X_g)$ on $P$. Equivalently, it is also identified with the Lie algebra $\mathcal{F}(M)$ of functions on $M$ with the reduced Poisson bracket $\{ \cdot, \cdot \}_M$ on $M$. Equivariant functions on $P$ are constant on the fibers, so there is an isomorphism

$$h \in \mathcal{F}(P)^{O} \rightarrow \bar{h} \in \mathcal{F}(M)$$

with inverse $\bar{h} \mapsto h = \bar{h} \circ \pi$. We recall here that the reduced Poisson bracket $\{ \cdot, \cdot \}_M$ on the quotient $M = P/O$ is uniquely determined by the formula

$$\{f,g\}_M \circ \pi = \{f \circ \pi, g \circ \pi\}_P,$$

for all $f,g \in \mathcal{F}(M)$, see §10.5 in Marsden and Ratiu [1999].

Remark 5.5 (The Vlasov chromomorphism group) The elements of the central extension $\text{Aut}_\text{ham}(P) \times_B \mathbb{R}$ will be called Vlasov chromomorphisms and this group will be denoted by $V\text{Chrom}(P)$. The name is clearly inspired by the fact that this Lie group is the configuration space underlying the collisionless Vlasov kinetic theory of interacting Yang-Mills charges moving in the physical space. As explained in Gibbons, Holm, and Kupershmidt [1983], the fluid closure of the Yang-Mills-Vlasov equation yields the equations of chromohydrodynamics for Yang-Mills plasmas, thereby providing one more reason for the name ‘chromomorphisms’.

We now consider the action of the group $V\text{Chrom}(P)$ on $\text{Emb}_O(P_S, P)$ by composition on the left. One easily checks that the associated momentum map is

$$J_L : \text{Emb}_O(P_S, P) \rightarrow \mathcal{F}(M)^* = (\mathcal{F}(P)^{O})^*, \quad \langle J_L(f), \bar{h} \rangle = \langle J_L(f), h \rangle = \int_{P_S} h(f(p)) \mu_{P_S}.$$

Since the function $p \mapsto h(f(p))$ on $P_S$ is $O$-invariant, it defines a function on $S$, and we have in fact an integral over $S$. By abuse of notation, we can write

$$J_L(f) = \int_S \delta(n - f(p)) \mu_S \in \mathcal{F}(M)^*.$$
Despite the fact that the left leg seems to be the same as the left leg of the Marsden-Weinstein dual pair, there is a key difference here, since here $M = P/O$ is not symplectic but Poisson, and the Lie bracket on $\mathcal{F}(M)$ is given by the reduced Poisson bracket $\{ , \}_M$.

The results obtained so far are summarized in the following

**Theorem 5.6 (Dual pair for \( EPA ut\) flows)** Let $\pi : P_S \rightarrow S$ be a principal $O$-bundle and consider another principal $O$-bundle $P \rightarrow M$ such that $P$ carries an exact symplectic form $\omega = - \mathbf{d}\theta$, where $\theta$ is $O$-invariant.

Then the group $Aut_{vol}(P_S)$ acts symplectically on the right on the symplectic manifold $(Emb_{O}(P_S, P), \bar{\omega})$ and admits the momentum map

$$J_R : Emb_{O}(P_S, P) \rightarrow aut_{vol}(P_S)^*, \quad J_R(f) = [f^*\theta].$$

There is a subgroup $Aut_{ham}(P) \subset Aut_{ham}(P)$ of special Hamiltonian automorphisms, whose Lie algebra consists of Hamiltonian vector fields associated to $O$-invariant Hamiltonian functions on $(P, \omega)$. The Lie algebra of $VChrom(P)$ is isomorphic to the space of functions on $M$ endowed with the reduced Poisson bracket on $M = P/O$.

The Vlasov chromomorphism group $VChrom(P)$ acts symplectically on the left on the symplectic manifold $(Emb_{O}(P_S, P), \bar{\omega})$ and admits the momentum map

$$J_L : Emb_{O}(P_S, P) \rightarrow \mathcal{F}(M)^*, \quad J_L(f) = \int_S \delta(n - f(p))\mu_S.$$

**Remark 5.7 (Equivariance, Clebsch variable, and Noether theorem)** The equivariance and invariance properties of the momentum maps still hold as in the Marsden-Weinstein dual pair, see Remark B.1. Therefore, we have the same Clebsch variables and Noether theorem interpretations. The Lie-Poisson system on $\mathcal{F}(M)^*$ is a Vlasov system whose Poisson bracket is not symplectic but is the reduced Poisson bracket on $M = P/O$. When $P = T^*\bar{P}$ and the principal bundle $\bar{P}$ is trivial, this system is related to Yang-Mills Vlasov equation. This particular case will be considered in the following section.

### 5.4 The Yang-Mills phase space

We now consider the special case when the total space $P$ of the principal bundle $P \rightarrow M$ is the cotangent bundle of another principal bundle $\bar{P} \rightarrow \bar{M}$. We endow $P = T^*\bar{P}$ with the canonical symplectic form $\Omega_P = - \mathbf{d}\Theta_P$ and we let $O$ act on $P$ by cotangent lift. Thus we have $P = T^*\bar{P}$ and $M = T^*\bar{P}/O$. This particular choice is motivated by §5.2, particularly by the dynamics of Yang-Mills-Vlasov kinetic theories, as mentioned in remark 5.2. If moreover $\bar{P}$ is trivial, that is $\bar{P} = \bar{M} \times O$, then $P = T^*\bar{P}$ is also a
trivial principal \( O \)-bundle over \( M = T^*\hat{P}/\hat{O} = T^*\hat{M} \times \mathfrak{o}^* \), since we have the equivariant diffeomorphism

\[
\rho : T^*\hat{M} \times T^*\hat{O} \to (T^*\hat{M} \times \mathfrak{o}^*) \times \mathcal{O}, \quad \rho(\alpha_q, \alpha_g) = ((\alpha_q, \alpha_g^{-1}g), g).
\]

In this case, the dual pair diagram reads

\[
\begin{array}{ccc}
\text{Emb}_\mathcal{O}(P_S, T^*\hat{M} \times T^*\mathcal{O}) & \xrightarrow{J_L} & \mathbf{F}(T^*\hat{M} \times \mathfrak{o}^*)^* \\
& \downarrow & \downarrow \\
& \mathbf{J}_R & \text{aut}_\mathcal{vol}(P_S)^*.
\end{array}
\]

The Lie bracket on \( \mathbf{F}(T^*\hat{M} \times \mathfrak{o}^*) \) is the reduced Poisson bracket given here by

\[
\{f, g\}_M = \{f, g\}_{T^*\hat{M}} + \{f, g\}_+,
\]

where the first term denotes the canonical Poisson bracket on \( T^*\hat{M} \) and the second term is the Lie-Poisson bracket on \( \mathfrak{o}^* \). In what follows, it will be easy (although not necessary) to assume that \( P_S \) is trivial, i.e. \( P_S = S \times \mathcal{O} \) and \( \text{aut}_\mathcal{vol}(P_S)^* \simeq \mathbf{x}_\mathcal{vol}(S)^* \times \mathbf{F}(S, \mathcal{O}) \).

### 5.4.1 The group of special Hamiltonian automorphisms

Note that, since the principal bundle is trivial, we can write the automorphism group of \( T^*\hat{M} \times T^*\mathcal{O} \to T^*\hat{M} \times \mathfrak{o}^* \) as a semidirect product

\[
\text{Aut}(T^*\hat{M} \times T^*\mathcal{O}) = \text{Diff}(T^*\hat{M} \times \mathfrak{o}^*) \otimes \mathbf{F}(T^*\hat{M} \times \mathfrak{o}^*, \mathcal{O}).
\]

From the general theory above, the group we need is the central extension

\[
\text{VChrom}(T^*\hat{M} \times T^*\mathcal{O}) = \overline{\text{Aut}_{\text{ham}}}(T^*\hat{M} \times T^*\mathcal{O}) \times_B \mathbb{R}.
\]

In order to describe more concretely this group, we start with the special case \( \overline{\text{Aut}_{\text{ham}}}(T^*\mathcal{O}) \). We will use the following expressions for the canonical one-form \( \Theta_\mathcal{O} \) and the Hamiltonian vector field \( X_h \), in the right trivialization \( T^*\mathcal{O} \to \mathcal{O} \times \mathfrak{o}^* \), \( \beta_g \mapsto (g, \alpha), \alpha = \beta_g^{-1} \):

\[
\left( \Theta_\mathcal{O} \right)_{(g, \alpha)}(\xi, \nu) = \langle \alpha, \xi g^{-1} \rangle, \quad X_h(g, \alpha) = \left( \frac{\delta h}{\delta \alpha} g, - \text{ad}^*_{\frac{\delta h}{\delta \alpha}} \alpha \right),
\]

see Abraham and Marsden [1978]. Note also that a \( \mathcal{O} \)-equivariant diffeomorphism \( \varphi \) of \( T^*\mathcal{O} \), expressed in the right trivialization \( \mathcal{O} \times \mathfrak{o}^* \), is of the form

\[
\varphi(g, \alpha) = (\hat{\varphi}(\alpha) g, \hat{\varphi}(\alpha)),
\]

where \( \hat{\varphi} \) is a diffeomorphism of \( \mathfrak{o}^* \) and \( \hat{\varphi} : \mathfrak{o}^* \to \mathcal{O} \) a smooth map.

**Proposition 5.8** In the right trivialization, the group \( \overline{\text{Aut}_{\text{ham}}}(T^*\mathcal{O}) \) of equivariant Hamiltonian automorphisms \( \varphi \) with the property that the \( \mathcal{O} \)-invariant 1-form \( \varphi^* \Theta_\mathcal{O} - \Theta_\mathcal{O} \) on \( \mathcal{O} \times \mathfrak{o}^* \) is the differential of an \( \mathcal{O} \)-invariant function can be expressed as

\[
\overline{\text{Aut}_{\text{ham}}}(T^*\mathcal{O}) = \left\{ \varphi \in \text{Aut}(T^*\mathcal{O}) \mid \varphi(\alpha) = \text{Ad}^*_{\hat{\varphi}(\alpha)}^{-1} \alpha \text{ and } \langle \text{id}_{\mathfrak{o}^*}, \hat{\varphi}^{-1} d\hat{\varphi} \rangle \in d\mathbf{F}(\mathfrak{o}^*) \right\},
\]

where \( \langle \text{id}_{\mathfrak{o}^*}, \hat{\varphi}^{-1} d\hat{\varphi} \rangle \) denotes the 1-form \( \langle \alpha, \hat{\varphi}(\alpha)^{-1} d\hat{\varphi}(\alpha) \rangle \).
Proof. We know that the Hamiltonian diffeomorphisms \( \varphi \) of an exact symplectic manifold with symplectic form \( -d\Theta_O \) are characterized by the condition that \( \varphi^*\Theta_O - \Theta_O \) is an exact 1-form. Using the first equality in (53) and (54), we compute

\[
(\varphi^*\Theta_O)_{(g,\alpha)}(\xi_g, \nu) = (\Theta_O)_{\varphi(g,\alpha)} (T_{(g,\alpha)} \varphi(\xi_g, \nu)) \\
= (\Theta_O)(\varphi(\alpha), \varphi(g)) (\dot{\varphi}(\alpha) \xi_g + (T_\alpha \varphi \cdot \nu) g, T_\alpha \varphi \cdot \nu) \\
= \langle \dot{\varphi}(\alpha), \varphi(\alpha) \xi_g \rangle^{-1} \varphi(\alpha)^{-1} + \langle \dot{\varphi}(\alpha), ((d_\varphi) \varphi^{-1})_\alpha \cdot \nu \rangle \\
= \langle \text{Ad}^*_{\varphi(\alpha)} \varphi(\alpha), \xi_g \rangle^{-1} + \langle \dot{\varphi}(\alpha), ((d_\varphi) \varphi^{-1})_\alpha \cdot \nu \rangle
\]

so that

(55) \( (\varphi^*\Theta_O - \Theta_O)_{(g,\alpha)}(\xi_g, \nu) = \langle \text{Ad}^*_{\varphi(\alpha)} \varphi(\alpha) - \alpha, \xi_g \rangle^{-1} + \langle \dot{\varphi}(\alpha), ((d_\varphi) \varphi^{-1})_\alpha \cdot \nu \rangle \).

Note also that an exact 1-form on \( O \times \mathfrak{o}^* \) reads

\[
\left\langle \frac{\partial h}{\partial g}, \xi_g \right\rangle^{-1} \left\langle \frac{\partial h}{\delta \alpha}, \nu \right\rangle
\]

where \( h \) is a function on \( O \times \mathfrak{o}^* \). So, the automorphism \( \varphi \) is Hamiltonian if and only if there exists a function \( h(g, \alpha) \) such that

(56) \( \text{Ad}^*_{\varphi(\alpha)} \varphi(\alpha) - \alpha = \frac{\partial h}{\partial g}(g, \alpha) \) and \( \langle \dot{\varphi}(\alpha), (d_\varphi) \varphi^{-1} \rangle_\alpha = \frac{\partial h}{\delta \alpha}(g, \alpha) \).

In the right trivialization, an invariant function on \( T^*O \) is of the form \( h(g, \alpha) = \tilde{h}(\alpha) \) with \( \tilde{h} \in \mathcal{F}(\mathfrak{o}^*) \). So, from the first equality in (56), we get the conditions \( \text{Ad}^*_{\varphi(\alpha)} \varphi(\alpha) = \alpha \) and the second equality in (56) thus reads \( \langle \alpha, \dot{\varphi}(\alpha) \rangle d_\alpha \varphi^{-1} = \frac{\partial h}{\delta \alpha} \).

Remark 5.9 (The Lie algebra of the special Hamiltonian automorphism group)

Similarly with (54), an equivariant vector field \( U \in \text{aut}(T^*O) \) reads \( U(g, \alpha) = (\tilde{U}(\alpha) g, \tilde{U}(\alpha)) \), where \( \tilde{U} : \mathfrak{o}^* \to \mathfrak{o} \) and \( \tilde{U} : \mathfrak{o}^* \to \mathfrak{o}^* \). Using Proposition 5.8, we obtain that the formal Lie algebra of \( \overline{\text{aut}_{\text{ham}}}(T^*O) \) is

\[
\overline{\text{aut}_{\text{ham}}}(T^*O) = \left\{ U \in \text{aut}(T^*O) \mid \tilde{U}(\alpha) = -\text{ad}_{\tilde{U}(\alpha)} \alpha \text{ and } \langle \text{id}_{\mathfrak{o}^*}, dU \rangle \in d\mathcal{F}(\mathfrak{o}^*) \right\}
\]

\[
= \left\{ (\tilde{U}, \tilde{U}) = \left( -\text{ad}_{\tilde{U}(\alpha)} \frac{\partial h}{\delta \alpha} \right) \mid \tilde{h} \in \mathcal{F}(\mathfrak{o}^*) \right\}
\]

\[
= \left\{ X_h \in \text{aut}_{\text{ham}}(T^*O) \mid h \in \mathcal{F}(T^*O), h(g, \alpha) = \tilde{h}(\alpha) \right\},
\]

where at the second equality, we used that fact that the 1-form \( \langle \text{id}_{\mathfrak{o}^*}, dU \rangle \) is exact if and only if there exists \( h \in \mathcal{F}(\mathfrak{o}^*) \) such that \( \tilde{U} = \frac{\partial h}{\partial \alpha} \); in the last equality we used the second expression in (53). So, consistently with Theorem 5.3, the Lie algebra \( \overline{\text{aut}_{\text{ham}}}(T^*O) \) consists of Hamiltonian vector fields associated to \( O \)-invariant Hamiltonians on \( T^*O \) (i.e. functions on \( \mathfrak{o}^* \)), hence the Lie algebra of \( \overline{\text{aut}_{\text{ham}}}(T^*O) \times_G \mathbb{R} \) can be identified with the Poisson algebra of smooth functions on \( \mathfrak{o}^* \).
Remark 5.10 (Generating functions of special Hamiltonian automorphisms) Notice that the function $S \in F(o^*)$ such that
\[
\text{d}S(\alpha) = \left\langle \text{id}_{o^*}, \varphi^{-1}\text{d}\varphi \right\rangle (\alpha) = \left\langle \alpha, \varphi^{-1}(\alpha)\text{d}\varphi(\alpha) \right\rangle
\]
is a generating function for special Hamiltonian automorphisms $\overline{\text{Aut}}_{\text{ham}}(T^*\mathcal{O})$, analogously to what happens for canonical transformations $\text{Diff}_{\text{ham}}(T^*\bar{M})$.

Remark 5.11 (The chromomorphism group) In the special case considered above, i.e. when $P = T^*\mathcal{O}$, the central extension $\overline{\text{Aut}}_{\text{ham}}(T^*\mathcal{O}) \times_T \mathbb{R}$ will be called chromomorphism group and will be denoted by $\text{Chrom}(T^*\mathcal{O})$.

In the case of a non-trivial manifold $\bar{M}$, there is an expression similar to that of Proposition 5.8 for the subgroup $\overline{\text{Aut}}_{\text{ham}}(T^*\bar{M} \times T^*\mathcal{O})$ of the automorphism group (52), namely
\[
\overline{\text{Aut}}_{\text{ham}}(T^*\bar{M} \times T^*\mathcal{O}) = \left\{ \varphi = (\bar{\varphi}, \hat{\varphi}) \in \text{Diff}(T^*\bar{M} \times o^*) \times F(T^*\bar{M} \times o^*, \mathcal{O}) \mid \hat{\varphi}_2 = \text{Ad}^*_{\hat{\varphi}^{-1}}, \right. \\
\left. \left\langle \text{id}_{o^*}, \hat{\varphi}^{-1}\text{d}\hat{\varphi} \right\rangle + \hat{\varphi}_1^*\Theta_M - p_1^*\Theta_M \in \text{d}F(T^*\bar{M} \times o^*) \right\},
\]
where $p_1$ is the projection of $T^*\bar{M} \times o^*$ onto its first component, and where we split the diffeomorphism $\hat{\varphi}$ into its two components $\hat{\varphi}_1 \in F(T^*\bar{M} \times o^*, T^*\bar{M})$ and $\hat{\varphi}_2 \in F(T^*\bar{M} \times o^*, o^*)$. The verification is left to the reader.

Let $h \in F(T^*\bar{M} \times T^*\mathcal{O})^\mathcal{O}$ and let $\bar{h} \in F(T^*\bar{M} \times o^*)$ induced via $h(\beta_q, \alpha_g) = \bar{h}(\beta_q, \alpha_g g^{-1})$ for all $\alpha \in \mathcal{O}$, as in (49). From (53), the Hamiltonian vector field $X_h$ on $T^*\bar{M} \times T^*\mathcal{O}$, pushed forward to $(T^*\bar{M} \times o^*) \times \mathcal{O}$ by the right trivialization (51), reads
\[
(\rho_*X_h)(\beta_q, \alpha, g) = \left( \left( X_{\bar{h}_\alpha}(\beta_q), -\text{ad}_{\frac{\delta h_{\beta_q}}{\delta \alpha}}^* \alpha \right), \frac{\delta h_{\beta_q}}{\delta \alpha} g \right).
\]
Here $\bar{h}_\alpha$ denotes the function on $T^*\bar{M}$ obtained from $\bar{h}$ by fixing an element $\alpha \in o^*$. Similarly, $\bar{h}_{\beta_q}$ is the function on $o^*$ obtained by fixing $\beta_q \in T^*\bar{M}$.

One can see that the Lie algebra $\overline{\text{aut}}_{\text{ham}}(T^*\bar{M} \times T^*\mathcal{O})$ of all these Hamiltonian vector fields with $\mathcal{O}$-invariant Hamiltonian function $h$ is indeed the formal Lie algebra of $\overline{\text{Aut}}_{\text{ham}}(T^*\bar{M} \times T^*\mathcal{O})$. The second condition in (57), differentiated at $(\bar{\varphi}, \hat{\varphi})$ in direction $(\bar{U}, \hat{U})$, tells us that $\left\langle \text{id}_{o^*}, \text{d}\hat{U} \right\rangle + \mathcal{L}_\hat{U}\Theta_M$ is an exact 1-form on $T^*\bar{M} \times T^*\mathcal{O}$, which is equivalent to $\hat{U} + i_{\hat{U}}\Omega_M = \text{d}\bar{h}$ for some $\bar{h} \in F(T^*\bar{M} \times T^*\mathcal{O})$. This means that $\hat{U} = X_{\bar{h}_\alpha}$ and $\hat{U} = \frac{\delta h_{\beta_q}}{\delta \alpha}$.

Remark 5.12 (The cocycle for Vlasov chromomorphisms) In the special case $P = T^*\bar{M} \times T^*\mathcal{O}$, there is a more concrete expression of the cocycle (65) when restricted to $\overline{\text{Aut}}_{\text{ham}}(P)$, namely
\[
B((\bar{\varphi}, a), (\bar{\psi}, b)) = \int_{(\beta_0, \sigma_0)} \text{d}S,
\]
where $(\beta_0, \sigma_0) \in T^*\bar{M} \times o^*$ and we have introduced the generating function $S$, such that
\[
\text{d}S = \left\langle \text{id}_{o^*}, \varphi^{-1}\text{d}\varphi \right\rangle + \varphi_1^*\Theta_M - p_1^*\Theta_M.
\]
5.4.2 Momentum maps

We now give the expression of these momentum maps in the case where both principal bundles are trivial, that is,

\[ P_S = S \times O \quad \text{and} \quad \bar{P} = \bar{M} \times O. \]

In this case, an equivariant embedding in \( \text{Emb}_O(P_S, T^*\bar{P}) = \text{Emb}_O(S \times O, T^*\bar{M} \times T^*O) \) is necessarily of the form

\[ (58) \quad f : S \times \mathcal{O} \to T^*\bar{M} \times T^*O, \quad f(x, g) = (P_Q(x), \kappa_\theta(x)g), \]

where \( P_Q : S \to T^*\bar{M}, \kappa_\theta : S \to T^*O \) with \( P_Q(x) \in T_{Q(x)}^*\bar{M} \) and \( \kappa_\theta(x) \in T_{\theta(x)}^*O \). We endow the manifold \( S \times O \) with the volume form \( \mu_{P_S} := \mu_S \wedge \mu_O \), where \( \mu_O \) is the Haar measure on the compact group \( O \), that is, \( \int_O \mu_O = 1 \) and \( \mu_O \) is \( O \)-invariant, so that \( \mu_{P_S} \) is \( O \)-invariant too. Remark that choosing the trivial connection \( A(v_x, \xi_g) = g^{-1}\xi_g \) on \( P_S = S \times O \) and the fact that \( \gamma \) is Ad-invariant, we see that \( \mu_{P_S} := \mu_S \wedge \mu_O \) and \( \mu_{P_S} = \pi^*\mu_S \wedge \mathcal{A}^* \det \gamma \) coincide, since the constant factor \( \text{Vol}(O) \) from lemma 4.2 is 1.

Right momentum map. In order to compute explicitly the momentum map associated to the action of \( \text{Aut}_{\text{vol}}(P_S) \) we shall make use of the isomorphism (48). Since the principal bundle \( P_S = S \times O \) is trivial, a principal connection 1-form \( \mathcal{A} \) is determined by a 1-form \( A \in \Omega^1(S, \mathfrak{o}) \) through \( \mathcal{A}(v_x, \xi_g) = \text{Ad}_g^{-1}(A(v_x) + \xi_g g^{-1}) \).

The space \( \mathcal{F}_O(P_S, \mathfrak{o}^*) \) can be identified with \( \mathcal{F}(S, \mathfrak{o}^*) \) via the relation

\[ \lambda(x, g) = \text{Ad}_g^* (\mathcal{A}(x)), \quad \lambda \in \mathcal{F}_O(P_S, \mathfrak{o}^*), \quad \mathcal{A} \in \mathcal{F}(S, \mathfrak{o}^*). \]

Finally, the horizontal lift of the vector \( v_x \in T_x S \) is \( (v_x, -A(v_x)g) \in T_{(x,g)}P_S \). Using these observations, the isomorphism (48) reads

\[ (59) \quad \Omega^1_O(P_S) \to \Omega^1(S) \times \mathcal{F}(S, \mathfrak{o}^*), \quad \alpha \mapsto (\mathcal{A}, \mathcal{F}), \]

where \( \mathcal{A}(v_x) = \alpha(x, e)(v_x, -A(v_x)) \) and \( \mathcal{F}(x, \xi) = \alpha(x, e)(0_x, \xi) \). This clearly induces an isomorphism

\[ \text{aut}_{\text{vol}}(P_S)^* = \Omega^1_O(P_S)/d\mathcal{F}_O(P_S) \to (\Omega^1(S)/d\mathcal{F}(S)) \times \mathcal{F}(S, \mathfrak{o}^*). \]

Note that, since the bundle is trivial, we can always choose the trivial connection \( A = 0 \).

Proposition 5.13 Suppose that \( P_S \) and \( \bar{P} \) are trivial bundles. Then the momentum map associated to the right action of \( \text{Aut}_{\text{vol}}(P_S) \) on \( \text{Emb}_O(P_S, T^*\bar{P}) \) has the expression

\[ J_R(f) = \left[ P_Q^*\Theta_{\bar{M}} + \kappa_\theta^*\Theta_O - \theta^{-1}\kappa_\theta \cdot A \right], \quad \theta^{-1}\kappa_\theta \in \left( \Omega^1(S)/d\mathcal{F}(S) \right) \times \mathcal{F}(S, \mathfrak{o}^*), \]

where the embedding \( f \) is written as

\[ f(x, g) = (P_Q(x), \kappa_\theta(x)g), \]

and \( \Theta_{\bar{M}}, \Theta_O \) are the canonical one-forms on \( T^*\bar{M}, T^*O \).
Proof. We fix a connection $\mathcal{A}$ and use the isomorphism (59). Given $\beta \in T^*\mathcal{O}$, we denote by $\ell_\beta : \mathcal{O} \to T^*\mathcal{O}$ the orbit map defined by $\ell_\beta(g) = \beta g$. Knowing that $f(x, g) = (P_Q(x), \kappa_\theta(x) g)$, the first component $\tilde{\alpha}$ of $\alpha = f^*\Theta_P$ computes

$$
\tilde{\alpha}(v_x) = (f^*\Theta_P)(v_x, -A(v_x))
= P_Q^*\Theta_M(v_x) + \kappa_\theta^*\Theta_O(v_x) - \Theta_O(T\ell_{\kappa_\theta}(A(v_x))).
$$

Using the definition of the canonical 1-form $\Theta_O$ and the identity $(T\pi \circ T\ell_{\beta})(\xi) = \pi(\beta)\xi$ for all $\xi \in \mathfrak{o}$, the last term becomes

$$
\Theta_O(T\ell_{\kappa_\theta}(A(v_x))) = \kappa_\theta(x) \cdot (T\pi \circ T\ell_{\kappa_\theta}(A(v_x)) = \kappa_\theta(x) \cdot \theta(x)(A(v_x))
$$

so $\tilde{\alpha} = P_Q^*\Theta_M + \kappa_\theta^*\Theta_O - \theta^{-1}\kappa_\theta \cdot A \in \Omega^1(S)$.

The second component of $\alpha = f^*\Theta_P$ is $\lambda = \theta^{-1}\kappa_\theta \in \mathcal{F}(S, \mathfrak{o}^*)$ because for all $x \in S$ and $\xi \in \mathfrak{o}$

$$
\lambda(x) \cdot \xi = (f^*\Theta_P(0_x, \xi)) = \Theta_O(T\ell_{\kappa_\theta}(\xi)) = \kappa_\theta(x) \cdot (T\pi \circ T\ell_{\kappa_\theta}(\xi))
= \kappa_\theta(x) \cdot \theta(x)\xi = \theta(x)^{-1}\kappa_\theta(x) \cdot \xi.
$$

This shows that

$$
J_R(f) = \left( [P_Q^*\Theta_M + \kappa_\theta^*\Theta_O - \theta^{-1}\kappa_\theta \cdot A], \theta^{-1}\kappa_\theta \right) \in \left( \Omega^1(S)/d\mathcal{F}(S) \right) \times \mathcal{F}(S, \mathfrak{o}^*)
$$

is the momentum map associated to the right action. $\blacksquare$

Remark 5.14 (Trivialized expression) For comparison with the formula obtained in Proposition 5.1, it is useful to identify the manifold $\text{Emb}_O(P_S, T^*P)$ with the product $\text{Emb}(S, T^*\tilde{M} \times \mathfrak{o}^*) \times \mathcal{F}(S, \mathcal{O})$. This is possible thanks to Lemma 3.7. More precisely, to the equivariant embedding $f : S \times \mathcal{O} \to T^*\tilde{M} \times T^*\mathcal{O}$,

$$
f(x, g) = (P_Q(x), \kappa_\theta(x) g), \quad P_Q : S \to T^*\tilde{M}, \quad \kappa_\theta : S \to T^*\mathcal{O},
$$

we associate the pair $((P_Q, \mu), \theta) \in \text{Emb}(S, T^*\tilde{M} \times \mathfrak{o}^*) \times \mathcal{F}(S, \mathcal{O})$, where

$$
\mu(x) = \kappa_\theta(x)\theta(x)^{-1} \in \mathfrak{o}^*.
$$

In order to obtain the momentum map of Proposition 5.13 in terms of the variables $(\mu, \theta)$, we introduce the right trivialization $R : \mathcal{O} \times \mathfrak{o}^* \to T^*\mathcal{O}$. $R(\theta, \mu) = \mu \theta$ and we compute the expression $R^*\Theta_O$. For $(g, \alpha) \in \mathcal{O} \times \mathfrak{o}^*$ and $(\xi_g, \nu) \in T_g\mathcal{O} \times \mathfrak{o}^*$, we have

$$
(R^*\Theta_O)(g, \alpha)(\xi_g, \nu) = (\Theta_O)_{\alpha g}(Tg\ell_\alpha(\xi_g) + V_\nu(\alpha g))
= \alpha g \cdot T\xi_g \pi(Tg\ell_\alpha(\xi_g)) = \alpha g \cdot \xi_g = \alpha \cdot \xi_g^{-1},
$$

where $V_\nu$ is the vertical vector field on $T^*\mathcal{O}$ defined by $V_\nu(\alpha g) = \frac{d}{dt}|_{t=0}((\alpha + tv)g)$. Then for $(\theta, \sigma) : S \to \mathcal{O} \times \mathfrak{o}^*$, we have

$$
(\sigma^* (R^*\Theta_O))(\nu_x) = (R^*\Theta_O)(\theta(x), \mu(x))(d\theta, d\mu) = \mu(x) \cdot (d\theta)\theta^{-1}(x).
$$

Therefore, the right momentum map from proposition 5.13 becomes

$$
J_R(P_Q, \sigma, \theta) = \left( [P_Q^*\Theta_M + (\theta, \mu)^*R^*\Theta_O - \theta^{-1}\kappa_\theta \cdot A], \theta^{-1}\kappa_\theta \right)
= \left( [P_Q^*\Theta_M + \mu \cdot (d\theta)\theta^{-1} - \text{Ad}_\theta^* \mu \cdot A], \text{Ad}_\theta^* \mu \right).
$$

For $A = 0$ and $S = \mathbb{R}^3$, we recover the expression of $J_R$ from proposition 5.1.
**Left momentum map.** From the general formula (50) above, we have
\[
J_L : \text{Emb}_\mathcal{O}(S \times \mathcal{O}, T^*\tilde{M} \times T^*\mathcal{O}) \to \mathcal{F}(T^*\tilde{M} \times T^*\mathcal{O})^\mathcal{O} = \mathcal{F}(T^*\tilde{M} \times \mathfrak{o}^*)
\]
and
\[
\langle J_L(f), h \rangle = \int_{S \times \mathcal{O}} h(f(x, g)) \mu_S \wedge \mu_\mathcal{O} = \int_{S \times \mathcal{O}} h(P_Q(x), \kappa_\theta(x) g) \mu_S \wedge \mu_\mathcal{O}
\]
\[
= \int_S \left( P_Q(x), \kappa_\theta(x) \theta(x)^{-1} \right) \mu_S.
\]
Hence, we can write the formula
\[
J_L(f) = \int_S \delta(\alpha_q - P_Q(x)) \delta \left( \mu - \kappa_\theta(x) \theta(x)^{-1} \right) \mu_S \in \mathcal{F}(T^*\tilde{M} \times \mathfrak{o}^*)^*.
\]
As we have discussed, the above momentum map is not a solution of the EP\textit{Aut}_{vol} equation. However this expression is of primary importance in the kinetic theory of Yang-Mills Vlasov plasmas, since this is nothing else than the single particle solution of the Vlasov equation, which is the base for the theory of Yang-Mills charged fluids. Indeed, in \textit{Gibbons, Holm, and Kupershmidt [1983]} it is shown that the Vlasov equation itself is directly constructed starting from the above momentum map.

### 6 Conclusions

After presenting the EP\textit{Aut} equations in the case of a trivial principal $\mathcal{O}$–bundle $P = M \times \mathcal{O}$, the paper introduced the associated left and right momentum maps which extend the well known dual pair structure for geodesic flows on the diffeomorphism group \textit{Holm and Marsden [2004]}. Thus, as a first result, the paper incorporated the momentum maps found in \textit{Holm and Tronci [2008]} in a unifying dual pair diagram, whose right leg provides the Clebsch representation of the EP\textit{Aut} system, while the left leg provides the singular $\delta$–like solutions.

In the third section, the paper extended the previous construction to consider the case of a non-trivial principal $\mathcal{O}$–bundle $P$. For this more general case, the EP\textit{Aut} equations were written also in terms of the adjoint bundle $\text{Ad}P = (P \times \mathfrak{o})/\mathcal{O}$, by using the identification between $\mathcal{O}$–equivariant maps $\mathcal{F}_\mathcal{O}(P, \mathfrak{o})$ and the space $\Gamma(\text{Ad}P)$ of sections of the adjoint bundle. In this setting, the momentum maps were defined on the space of $\mathcal{O}$–equivariant embeddings $\text{Emb}_\mathcal{O}(P_S, P)$ of the sub-bundle $P_S$ in the ambient bundle $P$. This is a crucial point because it showed for the first time how the right action momentum map (Clebsch representation) necessarily involves the connection on $P_S$. When specialized to the case of a trivial bundle, the EP\textit{Aut} dual pair showed how the singular solutions may support their own magnetic field, for example in the case of current sheets. This is a very suggestive picture that arises from the principal bundle structure of the embedded subspace.

The fourth section considered the case of an incompressible fluid flow. In this framework, one considers the group of volume-preserving automorphisms $\text{Aut}_{vol}(P)$, that is bundle automorphisms that project down to diffeomorphisms $\text{Diff}_{vol}(M)$ on the base
\[ M = P/O, \] which preserve a fixed volume form. Again, the equations were presented in both cases of a trivial and non-trivial principal bundle. This geometric construction was compared with a similar formulation that also applies to Euler’s vorticity equation, in the context of Marsden and Weinstein [1983]. Left- and right-action momentum maps were presented also for this case, along with explicit formulas. These formulas require a deep geometric construction that involves Lie group extensions. For example, the left leg momentum map required the definition of the chromomorphism group, a Yang-Mills version of the quantomorphism group from quantization theory. Indeed, new definitions of infinite-dimensional Lie groups arose naturally in this context: the special Hamiltonian automorphisms and two variants of the chromomorphism group.

\section{Appendices on EP\textit{Aut} flows}

\subsection{Proof of Lemma 3.1}

We will use the formula

\[
[U, V]_L = \sigma ([A(U), A(V)] + d^4(A(U))V - d^4(A(V))U + B(U, V)) + \text{hor}([U, V]_L),
\]

(60)

where \([., .]_L\) denotes the (left) Lie algebra bracket and \(B\) is the curvature of the connection. We refer to Lemma 5.2 in Gay-Balmaz and Ratiu [2008] for a proof of this formula.

Denoting by \(\langle ., . \rangle\) the \(L^2\)-pairing, for an arbitrary \(v \in \mathfrak{x}(M)\), we have

\[
\langle (\text{Hor}^A)^* \mathcal{L}_U \beta, v \rangle = \langle \beta, [U, \text{Hor}^A v]_L \rangle = \langle \beta, \text{hor}([U, \text{Hor}^A v]_L) + \sigma(d\omega \cdot \text{Hor}^A v + B(\text{Hor}^A u, \text{Hor}^A v)) \rangle
\]

\[
= \langle \beta, \text{Hor}^A([u, v]_L) \rangle + \langle \mathcal{J} \circ \beta, d\omega \cdot \text{Hor}^A v + A([\text{Hor}^A u, \text{Hor}^A v]_L) \rangle
\]

\[
= \langle \mathcal{L}_u (\text{Hor}^A)^* \beta, v \rangle + \langle (\text{Hor}^A)^* (\mathcal{J} \circ \beta \cdot d\omega), v \rangle
\]

\[
+ \langle (\text{Hor}^A)^* \mathcal{L}_{\text{Hor}^A u}(\mathcal{J} \circ \beta), v \rangle,
\]

thus proving the first formula.

For an arbitrary \(\theta \in \mathcal{F}_\mathcal{O}(P, \mathfrak{o})\), we have

\[
\langle \mathcal{J} \circ \mathcal{L}_U \beta, \theta \rangle = \langle \mathcal{L}_U \beta, \sigma(\theta) \rangle = \langle \beta, [U, \sigma(\theta)]_L \rangle = \langle \beta, \sigma([\omega, \theta] - d^4 \theta \cdot U) \rangle
\]

\[
= -\langle \mathcal{J} \circ \beta, d\theta \cdot U \rangle = -\langle \mathcal{J} \circ \beta, d\theta \cdot \text{Hor}^A u - [\omega, \theta] \rangle
\]

\[
= \langle \mathcal{L}_{\text{Hor}^A u}(\mathcal{J} \circ \beta), \theta \rangle + \text{ad}_{\omega}^*(\mathcal{J} \circ \beta), \theta \rangle
\]

This proves the second formula. ■

\subsection{Proof of Lemma 3.7}

Recall from §3.6 that the momentum maps of the EP\textit{Aut}-equation are defined on the cotangent bundle of the Kaluza-Klein configuration space

\[
Q_{KK} = \{ Q : P_S \to P \mid Q \circ \Phi_g = \Phi_g \circ Q \text{ and } Q \in \text{Emb}(S, M) \},
\]
where $Q : S \to M$ is defined by the condition $\pi \circ Q = Q \circ \pi$. Note that $Q_{KK}$ is a natural
generalization of the space $\Emb(S, M)$ associated to the EPDiff equations. Another
natural generalization would be the space $\Emb_O(P_S, P)$ of equivariant embeddings $Q : P_S \to P$. We now show that these spaces coincide, that is

$$Q_{KK} = \Emb_O(P_S, P).$$

(1) We first prove the inclusion $\Emb_O(P_S, P) \subset Q_{KK}$. Given $Q \in \Emb_O(P_S, P)$
we consider the induced map $Q : S \to M$. Since $Q$ is smooth, $Q$ is also smooth and
it remains to show that $Q$ is an embedding. Equivalently, we will show that $Q$ is an
injective immersion and a closed map. To prove that it is injective, choose $s_1, s_2 \in S$ such
that $Q(s_1) = Q(s_2)$. Let $p_1$ and $p_2$ be such that $\pi(p_i) = s_i, i = 1, 2$. We have

$$\pi(Q(p_1)) = Q(s_1) = Q(s_2) = \pi(Q(p_2)),$$

therefore, there exists $g \in G$ such that $Q(p_1) = \Phi_g(Q(p_2)) = Q(\Phi_g(p_2))$ by equivariance.
Thus, since $Q$ is injective, we have $p_1 = \Phi_g(p_2)$. This proves that $s_1 = s_2$. We now show
that $Q$ is an immersion. Suppose that $v_s \in TS$ is such that $T_sQ(v_s) = 0$. For $v_p \in TP_S$
such that $T\pi(v_p) = v_s$, we have

$$T\pi(TQ(v_p)) = TQ(T\pi(v_s)) = 0.$$

This proves that $TQ(v_p)$ is a vertical vector. Since $Q$ is an equivariant embedding, $v_p$
must be vertical. Thus, we obtain $v_s = T\pi(v_p) = 0$. This proves that $Q$ is an immersion.
Let $F \subset S$ be a closed subset. The preimage $F = \pi^{-1}(F) \subset P_S$ is also closed since $\pi$ is
continuous. We can write

$$Q(F) = Q(\pi(F)) = \pi(Q(F)),$$

and this space is closed since $F$ is closed, $Q$ is an embedding, and $\pi$ is a quotient map.

(2) We now show the inclusion $Q_{KK} \subset \Emb_O(P_S, P)$. We first verify that $Q \in Q_{KK}$
is injective. Given $p_1, p_2 \in P_S$, we have

$$Q(p_1) = Q(p_2) \Rightarrow Q(\pi(p_1)) = Q(\pi(p_2)) \Rightarrow \pi(p_1) = \pi(p_2)$$

by injectivity of $Q$. Therefore, there exists $g \in G$ such that $p_2 = \Phi_g(p_1)$ and we have

$$Q(p_1) = Q(p_2) = Q(\Phi_g(p_1)) = \Phi_g(Q(p_1)).$$

By the freeness of the action, we have $g = e$. This proves that $p_1 = p_2$ and that $Q$ is injective. We now prove that $Q$ is an immersion. Let $v_p \in TP_S$ such that $TQ(v_p) = 0$. The equality

$$TQ(T\pi(v_p)) = T\pi(TQ(v_p)) = 0,$$

shows that the vector $v_p$ is vertical, since $Q$ is an immersion. Thus, we can write $v_p = \xi_{rs}(p)$ for a Lie algebra element $\xi \in \mathfrak{a}$. By equivariance, we get

$$0 = TQ(\xi_{rs}(p)) = \xi_P(Q(p)).$$

This proves that $\xi = 0$ and that $v_p = 0$. We now show that $Q$ is a closed map. Let
$F \subset P_S$ be a closed subset. In order to show that $Q(F)$ is closed, we consider a sequence
(q_n) ⊂ Q(F) converging to q ∈ P and we show that q ∈ Q(F). Define y_n := π(q_n) ∈ Q(F), where F := π(F) and y := π(q). The sequence (y_n) converges to y and we have y ∈ Q(F) since Q(F) is closed. Thus, we can consider the sequence x_n := Q^{-1}(y_n) converging to x := Q^{-1}(y). Let (p_n) ∈ F ⊂ P_S be the sequence determined by the condition Q(p_n) = q_n. Let s be a local section defined on an open subset U containing x. Since π(p_n) = x_n, there exists q_{n_k} ∈ G such that p_n = Φ_{q_{n_k}}(s(x_n)), and we have q_{n_k} = Q(p_{n_k}) = Q(Φ_{q_{n_k}}(s(x_n))) = Φ_{q_{n_k}}(Q(s(x_n))). Since the sequences (q_{n_k}) and (Q(s(x_n))) converge, there exists a subsequence g_{n_k} converging to g ∈ O, by properness of the action. Thus, (g_{n_k}) converges to Φ_g(Q(s(x))) = Φ_g(Q(s(x))), and Φ_g(s(x)) ∈ F since p_{n_k} converges to Φ_g(s(x)) and F is closed. This proves that q ∈ Q(F). ■

A.3 Momentum maps on trivial bundles

If the principal bundles are trivial, then the Kaluza-Klein configuration manifold is

\[ Q_{KK} = \text{Emb}_O(S \times O, M \times O) = \text{Emb}(S, M) \times F(S, O) \]

by Lemma 3.7. Moreover, Q ∈ Q_{KK}, φ ∈ Aut(P), and ψ ∈ Aut(P_S) read

Q(s, g) = (Q(s), θ(s)g), \quad φ(x, g) = (φ(x), χ(x)g), \quad \text{and} \quad ψ(s, g) = (γ(s), β(s)g).

Thus the left and right compositions φ ◦ Q and Q ◦ ψ recover the actions (9) and (12). In the same way P_Q ∈ T_QQ_{KK} can be written P_Q(s, g) = (P_Q(s), κ_φ(s)g). Thus, ζ := J ◦ P_Q ∈ F_O(P_S, o)^* and P_Q^A := (\text{Hor}_Q)^*P_Q read

ζ(s, g) = \text{Ad}_g^*((θ^{-1}κ_φ)(s)) \quad \text{and} \quad P_Q^A(s) = P_Q(s) − (κ_φθ^{-1})(s) \cdot A(Q(s)).

Here A denotes the one-form on M induced by the connection A, that is, we have

\[ A(v_x, ξ_g) = \text{Ad}_{g^{-1}}(A(v_x) + ξ_gg^{-1}). \]

Since ξ ◦ ζ can be identified with κ_φθ^{-1}, the left momentum map reads

\[ J_L(P_Q) = \left( \int_S (P_Q(s) − κ_φ(s)θ(s)^{-1} \cdot A(Q(s))) \delta(x − Q(s))d^ks, \int_S κ_φ(s)θ(s)^{-1}δ(x − Q(s))d^ks \right) \]

(61)

∈ \mathfrak{X}(M)^* \times F(M, o)^*.

We now show that this expression agrees with formula (10). Since the bundle is trivial, a vector field U ∈ aut(P) is naturally identified with the pair (u, ν) ∈ \mathfrak{X}(M) \times F(M, o). In this case, the connection dependent isomorphism, U → (A(U), [U]) reads simply (u, ν) → (A \cdot u + ν, u). Similarly, the one-form density β ∈ aut(P)^* is identified with the pair (m, C). In this case, the isomorphism β → ((\text{Hor}_A)^*β, J ◦ β) reads simply (m, C) → (m − C \cdot A, C). This is exactly the transformation yielding (61) from (10).
Concerning the right momentum map, using the formulas
\[
\text{d}Q \left( \text{Hor}^{A_S}_{(s,g)}(v_s) \right) = \text{d}Q(v_s, -A_S(v_s)g) \\
= (\text{d}Q(v_s), \text{d}\theta(v_s)g - \theta(s)A_S(v_s)g)
\]
\[
\mathcal{A} \left( \text{d}Q \left( \text{Hor}^{A_S}_{(s,g)}(v_s) \right) \right) = \text{Ad}_{(s,g)^{-1}} \left( A(\text{d}Q(v_s)) + (\text{d}\theta(v_s)g - \theta(s)A_S(v_s)g)(\theta(s)^{-1}) \right)
= \text{Ad}_{g^{-1}} \left( \theta^{-1}A(\text{d}Q(v_s))\theta + \theta^{-1}\text{d}\theta(v_s) - A_S(v_s) \right),
\]
and the expression (30), we get
\[
J_R(Q, P^A_Q, \kappa_\theta) = \left( (P^A_Q + \kappa_\theta \theta^{-1} \cdot A \circ Q) \cdot \text{d}Q + \kappa_\theta \cdot (\text{d}\theta - \theta A_S), \theta^{-1} \kappa_\theta \right)
= (P_Q \cdot \text{d}Q + \theta^{-1} \kappa_\theta \cdot (\theta^{-1} \text{d}\theta - A_S), \theta^{-1} \kappa_\theta)
= (P_Q \cdot \text{d}Q + \kappa_\theta \cdot (\text{d}\theta - \theta A_S), \theta^{-1} \kappa_\theta).
\]
This expression recovers (13) when $A_S$ is the trivial connection.

\section*{B Appendices on incompressible EP\text{Aut} flows}

\subsection*{B.1 Two dimensional EP\text{Aut}_{vol} equations}

If the manifold $M$ has dimension two, the volume form $\mu_M$ can be thought of as a symplectic form, and a vector field $u$ is divergence free if and only if it is locally Hamiltonian. If the first cohomology of the manifold vanishes, $H^1(M) = \{0\}$, then a divergence free vector field $u$ is globally Hamiltonian and we can write $u = X_\psi$, relative to a stream function $\psi$ defined up to an additive constant. The space $\mathcal{X}_{\text{vol}}(M)$ can thus be identified with the quotient space $\mathcal{F}(M)/\mathbb{R}$ and the dual is given by functions $\varpi$ on $M$ such that $\int_M \varpi \mu_M = 0$. We denote by $\mathcal{F}(M)_0$ this space. The duality pairing is given by
\[
\langle [\psi], \varpi \rangle = \int_M \psi \varpi \mu_M, \quad [\psi] \in \mathcal{F}(M)/\mathbb{R}, \quad \varpi \in \mathcal{F}(M)_0.
\]
Note that $\varpi \mu_M$ is an exact two-form since its integral over the boundaryless manifold $M$ is zero. This description of the Lie algebra and its dual is compatible with (37) in the following sense. First, we have the isomorphisms $u = X_\psi \in \mathcal{X}_{\text{vol}}(M) \mapsto [\psi] \in \mathcal{F}(M)/\mathbb{R}$, and $\omega = \varpi \mu_M \in \text{d}\Omega^1(M) \mapsto \varpi \in \mathcal{F}(M)_0$. Second, the corresponding duality pairings verify
\[
\langle \varpi \mu_M, X_\psi \rangle = \langle \varpi, [\psi] \rangle.
\]
Consistently, the correspondent functional derivatives of a Lagrangian $l$ are related by the formula
\[
\frac{\delta l}{\delta u} = \frac{\delta l}{\delta [\psi]} \mu_M.
\]
Inserted into (38), this relation yields the system
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \frac{\delta l}{\delta [\psi]} + \left\{ \frac{\delta l}{\delta [\psi]}, \psi \right\} + \left\{ \frac{\delta l}{\delta \nu^i}, \nu^i \right\} = 0 \\
\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} + \left\{ \frac{\delta l}{\delta \nu}, \psi \right\} + \text{ad}^*_\nu \frac{\delta l}{\delta \nu} = 0,
\end{array} \right.
\]
\[
(63)
\]
where in the last term of the first equation, summation over \(i\) is assumed. At this point, an interesting analogy appears with the equations of reduced MHD in the so-called low $\beta$-limit [Morrison and Hazeltine 1984; Marsden and Morrison 1984; Holm 1985]. These equations are Lie-Poisson on the dual of the semidirect-product Lie algebra \(X_{vol}(\mathbb{R}^2) \oplus \mathcal{F}(\mathbb{R}^2, \mathbb{R})\), that is the Lie algebra of \(\mathcal{A}_{vol}(\mathbb{R}^2) \simeq \text{Diff}_{vol}(\mathbb{R}^2) \oplus \mathcal{F}(\mathbb{R}^2)\). More particularly, the low $\beta$-limit of the reduced MHD equations arises as the EP\(\mathcal{A}_{vol}\) system \((63)\) on \(\mathbb{R}^2 \times S^1\) with the Lagrangian

\[
l(\psi, \nu) = \frac{1}{2} \int (|\nabla \psi|^2 + |\nabla \nu|^2) \, dx \, dy
\]

Analogous versions involving a non-Abelian group \(\mathcal{O}\) are easily obtained.

### B.2 Group actions for the Marsden-Weinstein dual pair

Let \(S\) be a compact manifold with volume form \(\mu_S\) and let \((M, \omega)\) be an exact (and hence noncompact) symplectic manifold, with \(\omega = -d\theta\). We endow the manifold \(\text{Emb}(S, M)\) of embeddings with the symplectic form

\[
\bar{\omega}(f)(u_f, v_f) = \int_S \omega(f(x))(u_f(x), v_f(x)) \mu_S.
\]

**Right momentum map.** The infinitesimal generator associated to the right action of the group of volume preserving diffeomorphisms is given by

\[
u_{\text{Emb}}(f) = T_f \circ u, \quad u \in X_{vol}(S), \quad f \in \text{Emb}(S, M)
\]

and the associated momentum map \((43)\) reads

\[
J_R(f) = [f^* \theta] \in \Omega^1(S)/dF(S)
\]

We now check that \(J_R\) verifies the momentum map condition. Since the action of the diffeomorphisms of \(M\) on \(\text{Emb}(S, M)\) is infinitesimally transitive, any tangent vector at \(f \in \text{Emb}(S, M)\) is of the form \(v \circ f\), and it suffices to show the equality

\[
d \langle J_R, u \rangle(f)(v \circ f) = \bar{\omega}(u_{\text{Emb}}, v \circ f), \quad \text{for all } f \in \text{Emb}(S, M) \quad \text{and} \quad v \in \mathcal{X}(M).
\]

Given a curve \(\varphi_t \in \text{Diff}(M)\) with \(\frac{d}{dt}|_{t=0} \varphi_t = v\), we have

\[
\frac{d}{dt}|_{t=0} \langle J_R(\varphi_t \circ f), u \rangle = \frac{d}{dt}|_{t=0} \int_S ((\varphi_t \circ f)^* \theta)(u) \mu_S
\]

\[
= \int_S (f^* \mathcal{L}_v \theta)(u) \mu_S = -\int_S i_u f^* (i_v \omega) \mu_S
\]

\[
= \int_S \omega(Tf \circ u, v \circ f) \mu_S = (i_{u_{\text{Emb}}} \bar{\omega})(v \circ f).
\]

This proves the desired formula.
Left momentum map and the central extension of $\text{Diff}_\text{ham}(M)$. The momentum map one should obtain from the left action of Hamiltonian diffeomorphisms has the expression (44)

$$J_L(f) = \int_S \delta(n - f(x))\mu_S \in \mathcal{F}(M)^*.$$ 

Note however that there is an ambiguity in this formula since it confuses the Lie algebra of Hamiltonian vector fields with the Lie algebra of Hamiltonian functions. In order to overcome this difficulty, it is necessary to consider the action naturally induced by the prequantization central extension of the subgroup $\text{Diff}_\text{ham}(M)$ of Hamiltonian diffeomorphisms. The crucial property for us being that the Lie algebra of this central extension is isomorphic to the space of functions on $M$ endowed with the symplectic Poisson bracket. In the particular case of an exact symplectic form $\omega = -d\theta$, the prequantization central extension of $\text{Diff}_\text{ham}(M)$ is diffeomorphic to the cartesian product $\text{Diff}_\text{ham}(M) \times \mathbb{R}$ and is described by a group two cocycle $B$. We denote by $\text{Diff}_\text{ham}(M) \times_B \mathbb{R}$ this central extension. The ILM-cocycle $B$ is described in Ismagilov, Losik, and Michor [2006] and depends on the choice of a point $n_0 \in M$. It is given by

$$B(\varphi_1, \varphi_2) := \int_{n_0}^{\varphi_2(n_0)} (\theta - \varphi_1^*\theta), \quad \varphi_1, \varphi_2 \in \text{Diff}_\text{ham}(M),$$

where the integral is taken along a smooth curve connecting the point $n_0$ with the point $\varphi_2(n_0)$. It is known that the 1-form $\theta - \varphi_1^*\theta$ is exact for any Hamiltonian diffeomorphism $\varphi_1$ (McDuff and Salamon [1998]), so the value of this integral does not depend on the choice of such a curve. Moreover the cohomology class of $B$ is independent of the choice of the point $n_0$ and of the 1-form $\theta$ such that $\omega = -d\theta$, see Theorem 3.1 in Ismagilov, Losik, and Michor [2006]. As it is well-known (Kostant [1970]), the Lie algebra of the prequantization central extension is isomorphic to the space of smooth functions on $M$ endowed with the symplectic Poisson bracket. In the particular case of $\text{Diff}_\text{ham}(M) \times_B \mathbb{R}$, the Lie algebra isomorphism being given by

$$\mathfrak{x}_\text{ham}(M) \times \mathbb{R} \to \mathcal{F}(M), \quad (X_h, a) \mapsto h + a - h\theta(n_0),$$

where $h\theta = h - \theta(X_h)$, see Gay-Balmaz and Tronci [2011] for a detailed computation and further use of the ILM cocycle.

We consider the left action of $(\varphi, \alpha) \in \text{Diff}_\text{ham}(M) \times_B \mathbb{R}$ on $\text{Emb}(S, M)$ given by $(\varphi, \alpha) \cdot f := \varphi \circ f$. The infinitesimal generator associated to the Lie algebra element $h \in \mathcal{F}(M)$ reads

$$h_{\text{Emb}}(f) = X_h \circ f.$$

With this geometric setting, we obtain that the momentum mapping associated to the left action of the central extension $\text{Diff}_\text{ham}(M) \times_B \mathbb{R}$ on $\text{Emb}(S, M)$ is given by

$$J_L : \text{Emb}(S, M) \to \mathcal{F}(M)^*, \quad J_L(f) = \int_S \delta(n - f(x))\mu_S,$$

that is,

$$\langle J_L(f), h \rangle = \int_S h(f(x))\mu_S.$$
A direct computation shows that the momentum map condition
\[ d \langle J_L, h \rangle (f)(v_f) = \int_S \omega(X_h \circ f, v_f) \mu_S = i_{h_{\text{Emb}} \dot{\omega}}(v_f), \]
is verified for all \( v_f \in T_f \text{Emb}(S, M) \).

We recall below from Marsden and Weinstein [1983] some remarkable properties of these momentum maps.

**Remark B.1 (Equivariance, Clebsch variables, and Noether theorem)**

It is readily seen that the momentum maps are equivariant, since we have
\[ J_R(f \circ \eta) = \eta^* J_R(f) \quad \text{and} \quad J_L(\varphi \circ f) = \varphi^* J_L(f), \]
for all \( \eta \in \text{Diff}_{\text{vol}}(S) \) and \( \varphi \in \text{Diff}_{\text{ham}}(M) \). Therefore, these maps are Poisson and provide **Clebsch variables** for the Lie-Poisson equations on \( F(M)^* \) and \( \mathfrak{X}_{\text{vol}}(S)^* \). For appropriate choices of Hamiltonian, one obtains the Vlasov equation on \( M \) and Euler’s equation on \( S \). In particular, \( J_L \) can be interpreted as a singular Klimontovich solution of the Vlasov equation on \( F(M) \), Holm and Tronci [2009]. See Marsden et al. [1983] for a geometric treatment of the Vlasov equation in plasma physics.

One also notices that \( J_L \) is invariant under the right action of \( \text{Diff}_{\text{vol}}(S) \) and \( J_R \) is left-invariant under the action of \( \text{Diff}_{\text{ham}}(M) \times B \mathbb{R} \). Therefore, the collective Hamiltonians are invariant and, by Noether’s theorem, these momentum maps provide a conservation law for the Clebsch variables.

**Remark B.2 (Two dimensional coincidences)**

A very interesting coincidence holds for the case of Euler’s equation in two dimensions. In this case, Euler’s equation for the vorticity \( \omega \in \mathfrak{X}_{\text{vol}}(\mathbb{R}^2)^* \) has exactly the same form of a Vlasov equation on \( F(\mathbb{R}^2)^* \) and \( \mathfrak{X}_{\text{vol}}(S)^* \). For appropriate choices of Hamiltonian, one obtains the Vlasov equation on \( M \) and Euler’s equation on \( S \). In particular, \( J_L \) can be interpreted as a singular Klimontovich solution of the Vlasov equation on \( F(\mathbb{R}^2) \), Holm and Tronci [2009]. See Marsden et al. [1983] for a geometric treatment of the Vlasov equation in plasma physics.

One also notices that \( J_L \) is invariant under the right action of \( \text{Diff}_{\text{vol}}(S) \) and \( J_R \) is left-invariant under the action of \( \text{Diff}_{\text{ham}}(M) \times B \mathbb{R} \). Therefore, the collective Hamiltonians are invariant and, by Noether’s theorem, these momentum maps provide a conservation law for the Clebsch variables.

**B.3 Proof of Theorem 5.1**

We shall proof this result in the case when \( w = 1 \) in (45), since extending to arbitrary values of \( w \) is straightforward. We know from the definition of momentum map that inserting \( J_\beta = \langle J, \beta \rangle \) in the Poisson bracket (45) must return the infinitesimal generator (46), that is
\[ \{ F, \langle J_R, (u, \zeta) \rangle \} = dF \left( (u, \zeta)_{\text{Emb}(\mathbb{R}^3, \mathbb{R}^{2k} \times o^*) \times F(\mathbb{R}^3, \mathcal{O})} \right), \]
for all \( (u, \zeta) \in \mathfrak{X}_{\text{vol}}(\mathbb{R}^3) \otimes F(\mathbb{R}^3, o) \).

This proof shows that the above relation holds.
One evaluates

\[ J_{(u,\zeta)} := \langle J_R, (u, \zeta) \rangle = \int \left( \nabla Q \cdot P + \text{Tr} \left( \mu^T \nabla \theta \theta^{-1} \right) \right) \cdot u + \int \text{Tr} \left( \mu^T \text{Ad}_\theta \zeta \right) \]

Since the \((Q, P)\)-components of the infinitesimal generator are evidently recovered by a simple verification, we focus on the \((\mu, \theta)\)-components. As a first step, we compute the \(\theta\)-component, which arises from the second term in the bottom line of (45). Thus, first we calculate

\[ \frac{\delta J_{(u,\zeta)}}{\delta \mu} = u \cdot \nabla \theta \theta^{-1} + \text{Ad}_\theta \zeta \]

and therefore we obtain

\[ \int \langle \delta \frac{F}{\delta \theta}, \frac{\delta J_{(u,\zeta)}}{\delta \mu} \rangle = \int \langle \delta \frac{F}{\delta \theta}, \left( u \cdot \nabla \theta \theta^{-1} + \text{Ad}_\theta \zeta \right) \theta \rangle = \int \langle \delta \frac{F}{\delta \theta}, (u \cdot \nabla \theta + \theta \zeta) \rangle \]

which evidently coincides with the \(\theta\)-component of the Lie algebra action in (46).

We now focus on the \(\mu\)-component. As a first step, we compute

\[ \int \langle \mu, \frac{\delta F}{\delta \mu} \rangle = \int \langle \mu, \frac{\delta F}{\delta \mu}, (u \cdot \nabla \theta) \theta^{-1} + \text{Ad}_\theta \zeta \rangle \]

\[ = - \int \langle \text{ad}_u \theta^{-1} \mu, \frac{\delta F}{\delta \mu} \rangle = \int \langle \text{ad}_u \theta^{-1} \mu, \frac{\delta F}{\delta \mu} \rangle \]

The \(\theta\)-variation of \(J_{(u,\zeta)}\) is expressed as

\[ \delta J_{(u,\zeta)} = \int \text{Tr} \left( \mu^T \nabla \theta \theta^{-1} - \mu^T \nabla \theta \theta^{-1} \delta \theta \theta^{-1} \right) \cdot u + \int \langle \text{Ad}^*_\theta \left( \text{ad}_\theta \mu \right), \zeta \rangle \]

\[ = - \int \text{Tr} \left( u \cdot \nabla \theta^{-1} \mu^T \delta \theta \right) - \int \text{Tr} \left( \theta^{-1} \mu^T (u \cdot \nabla \theta) \theta^{-1} \delta \theta \right) \]

\[ = \int \text{Tr} \left( \theta^{-1} \left( \text{ad}_u^* \theta^{-1} \mu \right)^T \delta \theta \right) \]

where we have used the general formula Marsden and Ratiu [1999]

\[ \delta \left( \text{Ad}^*_\theta \mu \right) = \text{Ad}^*_\theta \left( \delta \mu - \text{ad}_{\theta^{-1}}^* \mu \right). \]

Therefore

\[ \frac{\delta J_{(u,\zeta)}}{\delta \theta} = -u \cdot \nabla \left( \mu (\theta^{-1})^T \right) - (\theta^{-1})^T \left( L_u \theta^T \mu (\theta^{-1})^T \right) - \left( \text{ad}_u^* \theta^{-1} \mu \right) (\theta^{-1})^T \]

and the last term in (45) yields

\[ \int \langle \frac{\delta J_{(u,\zeta)}}{\delta \theta}, \frac{\delta F}{\delta \mu} \rangle = - \int \text{Tr} \left( \theta \frac{\delta J_{(u,\zeta)}}{\delta \theta} \frac{\delta F}{\delta \mu} \right) \]

\[ = \int \text{Tr} \left( \theta \left( L_u \theta^{-1} \mu^T + \mu^T (L_u \theta) \theta^{-1} + \left( \text{ad}_u^* \theta^{-1} \mu \right)^T \delta \theta \right) \frac{\delta F}{\delta \mu} \right) \]

\[ = \int \text{Tr} \left( \left[ \mu^T, (L_u \theta) \theta^{-1} \right] + L_u \mu^T + \left( \text{ad}_u^* \theta^{-1} \mu \right)^T \frac{\delta F}{\delta \mu} \right) \]

\[ = \int \langle \text{ad}_u^* (L_u \theta) \theta^{-1} \mu + L_u \mu + \left( \text{ad}_u^* \theta^{-1} \mu \right)^T \frac{\delta F}{\delta \mu} \rangle. \]
In conclusion, we are left with
\[
\int \left\langle \mu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta J(u, \xi)}{\delta \mu} \right] \right\rangle - \int \left\langle \frac{\delta J(u, \xi)}{\delta \theta}, \frac{\delta F}{\delta \mu} \right\rangle = \int \left\langle L_{u\mu}, \frac{\delta F}{\delta \mu} \right\rangle
\]
which coincides with the \( \mu \)-component of the infinitesimal generator in (46). 

**B.4 Proof of Theorem 5.3**

For each Hamiltonian automorphism \( \varphi \in Aut_{ham}(P) \), the 1-form \( \varphi^* \theta - \theta \) is exact and \( \mathcal{O} \)-invariant, hence there exists a function \( F_\varphi \in \mathcal{F}(P) \) such that \( dF_\varphi = \varphi^* \theta - \theta \). Let \( \Psi : \mathcal{O} \to \mathbb{R} \) be the map defined by
\[
\Psi_\varphi(g) := F_\varphi - F_\varphi \circ \Phi_g \in \mathbb{R}, \quad \text{for all} \quad g \in \mathcal{O}.
\]

Then \( \Psi_\varphi \) is independent of the choice of the function \( F_\varphi \) and is a group homomorphism, that is, \( \Psi_\varphi(gh) = \Psi_\varphi(g) + \Psi_\varphi(h) \). We thus get a map
\[
\Psi : Aut_{ham}(P) \to \text{Hom}(\mathcal{O}, \mathbb{R}), \quad \varphi \mapsto \Psi_\varphi.
\]

To show that \( \Psi \) is a group homomorphism, we use the identity \( dF_{\varphi \circ \psi} = d(\psi^*F_\varphi + F_\psi) \) and the \( \mathcal{O} \)-equivariance of \( \psi \) to obtain, for all \( g \in \mathcal{O} \),
\[
\Psi_{\varphi \circ \psi}(g) = \psi^*F_\varphi - \Phi_g^*\psi^*F_\varphi + F_\psi - \Phi_g^*F_\psi = \psi^*\Psi_\varphi(g) + \Psi_\psi(g) = (\Psi_\varphi + \Psi_\psi)(g).
\]

This proves that \( \Psi \) is a group homomorphism, so its kernel \( \overline{Aut_{ham}}(P) := \ker \Psi \) is a normal subgroup of \( Aut_{ham}(P) \), by the first homomorphism theorem.

In order to find its Lie algebra, we compute the associated Lie algebra homomorphism
\[
\beta : \mathfrak{aut}_{ham}(P) \to \text{Hom}(\mathcal{O}, \mathbb{R}).
\]

Let \( X_h, \ h \in \mathcal{F}(P) \), be an \( \mathcal{O} \)-equivariant Hamiltonian vector field on \( P \) and let \( \varphi_t \) be a curve of Hamiltonian automorphisms of \( P \) such that \( \frac{d}{dt}\big|_{t=0} \varphi_t = X_h \). We know from (55) that there exists a function \( F_t := F_{\varphi_t} \) on \( P \) such that \( \varphi_t^* \theta - \theta = dF_t \). Then we obtain
\[
d \left( \left. \frac{d}{dt} \right|_{t=0} F_t \right) = L_{X_h} \theta = d \left( i_{X_h} \theta - h \right),
\]
which we use, together with the \( \mathcal{O} \)-invariance of the function \( i_{X_h} \theta \), to compute
\[
\beta(X_h)(g) = \left. \frac{d}{dt} \right|_{t=0} \Psi_{\varphi_t}(g) = \left. \frac{d}{dt} \right|_{t=0} \left( F_t - F_t \circ \Phi_g \right)
\]
\[
= (i_{X_h} \theta - h) - (i_{X_h} \theta - h) \circ \Phi_g = h \circ \Phi_g - h.
\]

Note that, as it should, the expression \( h \circ \Phi_g - h \) depends only on \( X_h \) and not on the chosen Hamiltonian.

In conclusion the homomorphism \( \beta(X_h) \) measures the lack of \( \mathcal{O} \)-invariance of the Hamiltonian function \( h \). This means that \( \ker \beta = \overline{\mathfrak{aut}_{ham}}(P) \), the Lie algebra of Hamiltonian vector fields with invariant Hamiltonian functions. The integrated version of this ideal is the normal subgroup \( \overline{Aut_{ham}}(P) = \ker \Psi \) of \( Aut_{ham}(P) \). 

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