On the $\xi$-Distribution of Inclusively Produced Particles in $e^+e^-$ Annihilation

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Abstract

We discuss the momentum distributions of inclusively produced particles in $e^+e^-$ annihilation. We show that the dependence of the position of the maxima of the $\xi = \ln(1/z)$ spectra on the mass of the produced particles follows naturally from the general definition of fragmentation functions when energy-momentum conservation is correctly incorporated.
$e^+e^-$ annihilation provides an excellent opportunity to study the fragmentation of quarks into hadrons. In particular, inclusive measurements of particle spectra allow us to extract fragmentation functions from such experiments and to test different theoretical models of fragmentation. Until now, fragmentation functions have not been calculated from first principles rather they have to be modeled in some way. Most of the current approaches use different algorithms, such as string and shower algorithms, and model fragmentation of a high energy quark in two phases, one of which is purely perturbative, describing the radiation and branching of the initial quarks and the other describing the subsequent non-perturbative hadronisation of the low energy quarks. Here, we follow a different approach.

Starting from the general definition of the fragmentation functions, and explicitly guaranteeing energy-momentum conservation we discuss the following interesting property of inclusive particle spectra in $e^+e^-$ annihilation. When the number of inclusively produced particles is plotted as a function of $\xi = \ln(1/z)$, (where $z$ is the momentum fraction of the fragmenting quark carried by the produced hadron) it exhibits an approximate Gaussian shape around a maximum, $\xi^\ast$. The position of the maximum depends both on the total centre of mass energy and on the mass of the produced particle [1–4]. While the shape and the energy dependence of the spectrum can be understood in perturbative QCD, as a consequence of the coherence of gluon radiation [5], the position of the maximum is a free parameter which has to be extracted from experiment. Our main purpose in this paper is to show that the dependence of location of the maximum on the mass of the produced particle follows naturally from the general definition of fragmentation functions when energy-momentum conservation is correctly incorporated.

Our starting point is the general definition of fragmentation functions [6–8]

$$\frac{1}{z}D_q(z) = \frac{1}{4} \sum_n \int \frac{d\xi}{2\pi} e^{-ip^+\xi/z} \text{Tr}\{\gamma^+ \langle 0|\bar{\psi}(0)|Pn;pp_n\rangle(Pn;pp_n)|\psi(\xi^-)|0\}. \quad (1)$$

(Here, we discuss only the twist two part of the unpolarized fragmentation functions.) $\gamma^+$ is defined as $\gamma^+ = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^3)$ and the plus components of the momenta are defined as $p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3)$. $p$ and $p_n$ are the momenta of the produced particle, $P$, and associated
hadronic system $n$. Using translational invariance to remove the $\xi^-$ dependence in the second
the matrix element and the integral representation of the delta function and projecting out
the plus components of $\psi$, we obtain

$$
\frac{1}{z} D_q(z) = \frac{1}{2\sqrt{2}} \sum_n \delta[(1/z - 1)p^+ - p_n^+]|\langle 0|\psi_+(0)|Pn; pp_n\rangle|^2. \tag{2}
$$

Here, the plus projection is defined as $\psi_+ = \frac{1}{2}\gamma^+\gamma^-\psi$. Using Eq. (2) rather than Eq. (1) has
the advantage that energy-momentum conservation is built in before any approximation is
made for the states in the matrix element. This is similar to the case of quark distributions
where the corresponding expression ensures correct support of the distribution functions as
discussed in Ref. [9].

While the detailed structure of the fragmentation function depends on the exact form of
the matrix element some general properties follow already from Eq. (2). The delta function,
for example, implies that the function, $D_q(z)/z$, peaks at

$$
z_{\text{max}} \approx \frac{M}{M + M_n}. \tag{3}
$$

Here, $M$ and $M_n$ are the mass of the produced particle and the produced system, $n$, and we
work in the rest frame of the produced particle. Here, we consider the intermediate state as a
state having a definite mass. (In general, we have to integrate over a spectrum of all possible
masses.) We see that the location of the maxima of the fragmentation function depends on
the mass of the system $n$. While the high $z$ region is dominated by the fragmentation of a
quark into the final particle and a small mass system, large mass systems contribute to the
fragmentation at lower $z$ values.

We can go a step further and eliminate the $\delta$-function in Eq. (2) by integrating over the
momentum of the unobserved state $n$.

$$
\frac{1}{z} D_q(z) = \frac{1}{2\sqrt{2}} \int_{p_{\text{min}}}^{\infty} dp_n |\langle 0|\psi_+(0)|Pn; pp_n\rangle|^2, \tag{4}
$$

with

$$
p_{\text{min}} = \frac{M^2(1-z)^2 - z^2M_n^2}{2Mz(1-z)}. \tag{5}
$$
The significance of Eqs. (4) and (5) is that $D_q(z)$ vanishes for both $z \to 1$ and $z \to 0$. Thus, fragmentation functions have the correct support. It is interesting to see how the integration region depends on the momentum fraction, $z$, for various values of the produced particle system, $M_n$. In Fig. 1, we plot $p_{\text{min}}$ for the production of protons as a function of $z$ for different values of $M_n$. $p_{\text{min}} = 0$ gives the value of $z$ at which the contribution of a given $M_n$ to the fragmentation function is largest. This gives $z_{\text{max}}$ according to Eq. (3). Further, the region of $z$ where the lower integration limit is sufficiently small that the integral will be significant, becomes narrower with increasing $M_n$. Thus, large mass states contribute to the fragmentation function at low $z$ and only in a very narrow region of $z$.

At a given centre of mass energy, there will be a maximum value for the mass of the intermediate state which can be produced in the fragmentation. This maximal mass determines the “lower” edge of the spectrum. We can use Eq. (3) to estimate the maxima of the $\xi$-distribution associated with this particular mass. This maximum determines the maximum of the fragmentation function in first approximation. Although $M_n$ is not known, it should be proportional to the available total energy $E_{CM}$. However, the precise value of $M_n$ is not needed if we are only interested in the relative position of the maxima of the $\xi$ distribution of different particles. Taking the difference of the maxima of the $\xi = \ln(1/z)$ distribution.

1Since the $\xi$-distribution is given by $d\sigma/d\xi = zd\sigma/dz \sim zD(z)$ it is proportional to $z^2$ times $D(z)/z$. Although Eq.(3) describes the location of the maximum of the distribution $D(z)/z$ we can expect that Eq.(3) is also a good approximation for the $\xi$-distribution, since the $z$ region where the lower integration limit ($p_{\text{min}}$), is sufficient small, is very narrow for large masses, $M_n$. Thus, the contributions from a given $M_n$ to the fragmentation functions are very narrow functions in $z$ for large $M_n$. Note, that the square of the matrix element in Eq. (4) must decrease faster then $1/p_n^2$ in order to give finite result for the fragmentation functions. Eq. (3) gives the maximum of the $\xi$-distribution exactly in the limiting case when the contribution of a given $M_n$ to the fragmentation function is a $\delta$-function.
distributions of two different particles, \(a\) and \(b\), the dependence on the unknown value of \(M_n\) drops out for sufficiently large \(M_n\). It follows from Eq.(3) that

\[
\Delta \xi^* = \xi_a^* - \xi_b^* \approx \ln \left( \frac{M_a + M_n}{M_b + M_n} \right) + \ln \frac{M_b}{M_a} \approx \ln \frac{M_b}{M_a} \quad (6)
\]

Thus, the difference of the maxima is determined by the logarithm of the ratio of the masses of the produced particles. Since the value of \(M_n\) for finite centre of mass energies are in general different for mesons and baryons, Eq. (3) will be only valid for the difference of the maxima of the mesons or baryons separately. We calculated the maxima of the \(\xi\) distributions using this formula and using the maxima of the \(\eta'\) and that of the proton distributions as a reference value for mesons and baryons, respectively. The results are compared to the experimental data [1–4] in Fig. 2. The location of the maxima as a function of the mass of the produced particles is reasonably well described both for mesons and baryons.

In conclusion, we have shown that the dependence of the position of the maxima of the \(\xi = \ln(1/z)\) spectra on the mass of the produced particles follows from the general definition of the fragmentation functions and from energy-momentum conservation. Our results are in remarkably good agreement with the data.

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FIG. 1. The lower integration limit, $p_{min}$, for proton production as a function of $z$ for various masses, $M_n$. 
FIG. 2. Location of the maxima of the $\xi$ distributions as a function of the particle mass. The full and open symbols represent mesons and baryons, respectively. The data are from Refs. [1–4]. The solid and dashed lines are the predictions of Eq. (6) for mesons and baryons adjusting the normalization to $\xi_\eta^*$ and to $\xi_p^*$, respectively.