Z₂-graded Cayley-Hamilton trace identities in Mₙ(E)

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Abstract. We show, how the combination of the Cayley-Hamilton theorem and a certain companion matrix construction can be used to derive Z₂-graded trace identities in Mₙ(E).

1. INTRODUCTION

Throughout the paper an algebra R means a not necessarily commutative unitary algebra over a commutative ring C (or over a field K). The notation for the full n × n matrix algebra over R is Mₙ(R).

In case of char(K) = 0, Kemer’s pioneering work (see [K1],[K2]) on the T-ideals of associative K-algebras (leading to the solution of the Specht problem) revealed the importance of the identities satisfied by Mₙ(E) and Mₙ,d(E), where E = E₀ ⊕ E₁ is the naturally Z₂-graded Grassmann (exterior) algebra (over K) generated by the infinite sequence of anticommutative indeterminates (vᵢ)ᵢ≥1. We note that E is Lie nilpotent of index 2.

Let K(x₁,x₂,...,xᵢ,...) denote the polynomial K-algebra generated by the infinite sequence x₁,x₂,...,xᵢ,... of non-commuting indeterminates. The prime T-ideals of this (free associative K-) algebra are exactly the T-ideals of the identities satisfied by Mₙ(K) for n ≥ 1 (see [A]). The T-prime (or verbally prime) T-ideals are the prime T-ideals plus the T-ideals of the identities of Mₙ(E) for n ≥ 1 and of Mₙ,d(E) for 1 ≤ d ≤ n − 1, where Mₙ,d(E) is the K-subalgebra of the so-called (n,d) supermatrices in Mₙ(E). Another remarkable result is that for a sufficiently large n ≥ 1, any T-ideal contains the T-ideal of the identities satisfied by Mₙ(E).

The above mentioned three classes of T-prime (verbally prime) PI-algebras serve as basic building blocks in Kemer’s theory, where Z₂-graded identities also play an important role. Since the appearance of [K1] and [K2] considerable efforts have been concentrated on the study of the various algebraic properties of Mₙ(E) and Mₙ,d(E), see [BR], [DiV], [Do], [KT], [SSz], [Re], [S1], [S2], [V1], [V2].

Accordingly, the importance of matrices over non-commutative rings is an evidence in the theory of PI-rings, nevertheless this fact has been obvious for a long time in other branches of algebra (structure theory of semisimple rings, K-theory, quantum matrices, etc.). The Cayley-Hamilton theorem and the corresponding
trace identity play a fundamental role in proving classical results about the polynomial and trace identities of $M_n(K)$. Thus any Cayley-Hamilton type identity for $M_n(E)$ seems to be of general interest.

The main aim of our work is to show, how the combination of the classical Cayley-Hamilton theorem and a companion matrix construction (from [SSz]) can be used to derive new $\mathbb{Z}_2$-graded trace identities in $M_n(E) = M_n(E_0) \oplus M_n(E_1)$. A similar approach is used in [HSz] to obtain $\mathbb{Z}_2$-graded polynomial identities in $M_n(E)$.

2. THE APPLICATIONS OF THE CAYLEY-HAMILTON IDENTITY

The Grassmann algebra

$$E = K \langle v_1, v_2, \ldots, v_i, \ldots \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j \rangle = K \langle V \rangle$$

over a field $K$ of characteristic zero (it has been already mentioned in the introduction) generated by (the countably) infinite set $V = \{v_1, v_2, \ldots, v_i, \ldots\}$ of anticommuting indeterminates can naturally be extended as

$$F = K \langle \{w\} \cup V \rangle = K \langle w, v_1, v_2, \ldots, v_i, \ldots \rangle$$

by using a bigger set $\{w\} \cup V$ of anticommuting generators, where $w \notin V$. Now we have $v_i v_j + v_j v_i = 0$, $v_i w + w v_i = 0$ for all $1 \leq i \leq j$ and $w^2 = 0$. Since the cardinalities of $V$ and $\{w\} \cup V$ are both equal to $\aleph_0$, the $K$-algebras $E$ and $F$ are isomorphic.

If $H \in M_n(C)$ is an $n \times n$ matrix over a unitary commutative ring $C$ with $\frac{1}{n} \in C$ for all integers $m \geq 1$, then its characteristic polynomial $p_H(x) = \det(xI - H) \in C[x]$ can be written as

$$p_H(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_{n-1} x^{n-1} + \lambda_n x^n,$$

where $\lambda_n = 1$, $\text{tr}(H)$ denotes the trace of $H$ and for $0 \leq k \leq n - 1$ we have the following Faddeev-LeVerrier descending recursion (closely related to Newton’s identities concerning the elementary symmetric polynomials)

$$\lambda_k = -\frac{1}{n-k} \left( \lambda_{k+1} \text{tr}(H) + \cdots + \lambda_{n-1} \text{tr}(H^{n-k-1}) + \lambda_n \text{tr}(H^{n-k}) \right) = -\frac{1}{n-k} \left\{ \sum_{i=1}^{n-k} \lambda_{k+i} \text{tr}(H^i) \right\}.$$

2.1. Theorem. If $A \in M_n(E_0)$ and $B \in M_n(E_1)$, then a Cayley-Hamilton trace identity of the form

$$\beta_0 I_n + \sum_{k=1}^{n} \left\{ \beta_k A^k + \alpha_k (A^{k-1}B + A^{k-2}BA + \cdots + ABA^{k-2} + BA^{k-1}) \right\} = 0$$

holds, where $I_n \in M_n(E_0)$ is the identity matrix,

$$p_A(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} + \alpha_n x^n = \det(xI - A) \in E_0[x]$$

is the characteristic polynomial of $A$, $\alpha_n = 1$, $\beta_n = 0$ and for $0 \leq k \leq n - 1$ we have

$$\beta_k = \left\{ -\frac{1}{n-k} \sum_{i=1}^{n-k} \beta_{k+i} \text{tr}(A^i) \right\} + \left\{ -\frac{1}{n-k} \sum_{r+s \leq n-k-1, 0 \leq r, s} \alpha_{k+r+s+1} \text{tr}(A^r B A^s) \right\}.$$
The so-called companion matrix $A + wB$ is in $M_n(F_0)$, where $F_0$ is the even (and commutative) part of the extended Grassmann algebra $F = F_0 \oplus F_1$. Any element $\lambda \in F_0$ can be written as $\lambda = \alpha + w\beta$, where $\alpha \in E_0$ and $\beta \in E_1$ are uniquely determined by $\lambda$. Thus we have $\lambda_k = \alpha_k + w\beta_k$ for the coefficients of the characteristic polynomial

$$p_{A+wB}(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_{n-1} x^{n-1} + \lambda_n x^n = (\alpha_0 + w\beta_0) + (\alpha_1 + w\beta_1)x + \cdots + (\alpha_{n-1} + w\beta_{n-1})x^{n-1} + (\alpha_n + w\beta_n)x^n,$$

where $\alpha_n = 1$, $\beta_n = 0$ and $\alpha_k \in E_0$, $\beta_k \in E_1$ for all $0 \leq k \leq n - 1$. Using $Aw = wA$, $Bw = -wB$ and $w^2 = 0$, for the exponent $1 \leq i$ we obtain that

$$(A + wB)^i = A^i + w(A^{i-1}B + A^{i-2}BA + \cdots + ABA^{i-2} + BA^{i-1})$$

and

$$\text{tr}((A+WB)^i) = \text{tr}(A^i) + w\text{tr}(A^{i-1}B) + w\text{tr}(A^{i-2}BA) + \cdots + w\text{tr}(ABA^{i-2} + BA^{i-1}).$$

In view of the Faddeev-LeVerrier recursion, we deduce that

$$\alpha_k + w\beta_k = -\frac{1}{n-k} \sum_{i=1}^{n-k} (\alpha_{k+i} + w\beta_{k+i}) \{ \text{tr}(A^i) + w\text{tr}(A^{i-1}B) + w\text{tr}(A^{i-2}BA) + \cdots + w\text{tr}(ABA^{i-2} + BA^{i-1}) \} =$$

$$\left\{ -\frac{1}{n-k} \sum_{i=1}^{n-k} \alpha_{k+i} \text{tr}(A^i) \right\} + w \left\{ -\frac{1}{n-k} \sum_{i=1}^{n-k} \beta_{k+i} \text{tr}(A^i) \right\} +$$

$$+ w \left\{ -\frac{1}{n-k} \sum_{i=1}^{n-k} \alpha_{k+i} \{ \text{tr}(A^{i-1}B) + \text{tr}(A^{i-2}BA) + \cdots + \text{tr}(ABA^{i-2} + BA^{i-1}) \} \right\},$$

whence

$$\alpha_k = -\frac{1}{n-k} \sum_{i=1}^{n-k} \alpha_{k+i} \text{tr}(A^i)$$

and

$$\beta_k = \left\{ -\frac{1}{n-k} \sum_{i=1}^{n-k} \beta_{k+i} \text{tr}(A^i) \right\} + \left\{ -\frac{1}{n-k} \sum_{r+s \leq n-k-1, 0 \leq r, s} \alpha_{k+r+s} \text{tr}(A^r BA^s) \right\},$$

can be derived. Thus

$$p_A(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} + \alpha_n x^n = \det(xI - A) \in E_0[x]$$

is the characteristic polynomial of $A \in M_n(E_0)$.

The application of the Cayley-Hamilton theorem to $A + wB \in M_n(F_0)$ gives that

$$0 = p_{A+wB}(A + wB) = (\alpha_0 + w\beta_0) I_n + (\alpha_1 + w\beta_1)(A + wB) + \cdots + (\alpha_{n-1} + w\beta_{n-1})(A + wB)^{n-1} + (\alpha_n + w\beta_n)(A + wB)^n =$$

$$(\alpha_0 + w\beta_0) I_n + \sum_{k=1}^{n} (\alpha_k + w\beta_k) \{ A^k + w(A^{k-1}B + A^{k-2}BA + \cdots + ABA^{k-2} + BA^{k-1}) \} =$$

$$\left\{ \alpha_0 I_n + \sum_{k=1}^{n} \alpha_k A^k \right\} + w \left\{ \beta_0 I_n + \sum_{k=1}^{n} \beta_k A^k \right\} +$$
Using the fact that for $\alpha \in E_0$ and $\beta \in E_1$ the equality $\alpha + w\beta = 0$ implies $\alpha = \beta = 0$, we get

$$I_n + \sum_{k=1}^{n} \{ \beta_k A^k + \alpha_k (A^{k-1}B + A^{k-2}BA + \cdots + ABA^{k-2} + BA^{k-1}) \} = 0 \quad \square$$

2.2. Corollary. The case $n = 2$ in Theorem 2.1 gives that

$$\left\{ \frac{1}{2} \text{tr}(B)\text{tr}(A) + \frac{1}{2} \text{tr}(A)\text{tr}(B) - \frac{1}{2} \text{tr}(AB) - \frac{1}{2} \text{tr}(BA) \right\} I_2 - \text{tr}(B)A - \text{tr}(A)B + AB + BA = 0$$

holds for $A \in M_2(E_0)$ and $B \in M_2(E_1)$.

Proof. We have

$$I_n + \beta_0 I_2 + \beta_1 A + \alpha_1 B + \alpha_2 (AB + BA) = 0$$

with $\alpha_2 = 1$, $\alpha_1 = -\text{tr}(A)$, $\beta_1 = -\text{tr}(B)$ and

$$\beta_0 = -\frac{1}{2} \beta_1 \text{tr}(A) + \left( -\frac{1}{2} \right) \{ \alpha_1 \text{tr}(B) + \alpha_2 \text{tr}(AB) + \alpha_2 \text{tr}(BA) \} = \frac{1}{2} \text{tr}(B)\text{tr}(A) + \frac{1}{2} \text{tr}(A)\text{tr}(B) - \frac{1}{2} \text{tr}(AB) - \frac{1}{2} \text{tr}(BA). \quad \square$$

2.3. Theorem. If $B \in M_n(E_1)$, then

$$\delta_0 I_n + \gamma_1 B + \delta_1 B^2 + 2\gamma_2 B^3 + \delta_2 B^4 + \cdots + (n-1)\gamma_{n-1} B^{2n-3} + \delta_{n-1} B^{2n-2} + nB^{2n-1} = 0$$

is a Cayley-Hamilton trace identity of degree $2n-1$ with invertible leading coefficient and

$$p_{B^2}(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_{n-1} x^{n-1} + \gamma_n x^n = \det(xI - B^2) \in E_0[x]$$

is the characteristic polynomial of $B^2 \in M_n(E_0)$. If $n = 1$, $\delta_n = 0$ and for $0 \leq k \leq n - 1$ we have

$$\delta_k = \left\{ \frac{1}{n-k} \sum_{i=1}^{n-k} \delta_{k+i} \text{tr}(B^{2i}) \right\} + \left\{ \frac{1}{n-k} \sum_{r+s \leq n-k-1, 0 \leq r, s} \gamma_{k+r+s+1} \text{tr}(B^{2r+2s+1}) \right\}.$$

Proof. The substitution $A = B^2 \in M_n(E_0)$ in Theorem 2.1 gives that

$$0 = \delta_0 I_n + \sum_{k=1}^{n} \{ \delta_k B^{2k} + \gamma_k (B^{2k-2}B + B^{2k-4}BB^2 + \cdots + B^2 BB^{2k-4} + BB^{2k-2}) \} = \delta_0 I_n + \sum_{k=1}^{n} \{ k\gamma_k B^{2k-1} + \delta_k B^{2k} \}$$

where

$$p_{B^2}(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_{n-1} x^{n-1} + \gamma_n x^n = \det(xI - B^2) \in E_0[x]$$
is the characteristic polynomial of $B^2$, $\gamma_n = 1$, $\delta_n = 0$ and for $0 \leq k \leq n - 1$ we have
$$\delta_k = \left\{-\frac{1}{n-k} \sum_{i=1}^{n-k} \delta_{k+i} \text{tr}(B^{2i})\right\} + \left\{-\frac{1}{n-k} \sum_{r+s \leq n-k-1,0 \leq r,s} \gamma_{k+r+s+1} \text{tr}(B^{2r+2s+1})\right\}. \Box$$

2.4. Remark. The direct application of the Cayley-Hamilton theorem to $B^2 \in M_n(E_0)$ gives a monic identity for $B$ of degree $2n$ with coefficients in $E_0$. The degree of our (also can be considered as monic) identity in Theorem 2.3 is only $2n - 1$, where the even degree coefficients are in $E_1$ and the odd degree coefficients are in $E_0$.

2.5. Corollary. If $B \in M_n(E_1)$, $2 \leq n$ and $\text{tr}(B^t) = 0$ for all $1 \leq t \leq 2n-2$, then $nB^{2n-1} = \text{tr}(B^{2n-1})I_n$ is a scalar matrix. The additional condition $\text{tr}(B^{2n-1}) = 0$ implies that $B^{2n-1} = 0$.

Proof. In view of the recursions ($\gamma_n = 1$, $\delta_n = 0$, $0 \leq k \leq n - 1$)
$$\gamma_k = -\frac{1}{n-k} \sum_{i=1}^{n-k} \gamma_{k+i} \text{tr}(B^{2i})$$
and
$$\delta_k = \left\{-\frac{1}{n-k} \sum_{i=1}^{n-k} \delta_{k+i} \text{tr}(B^{2i})\right\} + \left\{-\frac{1}{n-k} \sum_{r+s \leq n-k-1,0 \leq r,s} \gamma_{k+r+s+1} \text{tr}(B^{2r+2s+1})\right\},$$
we obtain that $\gamma_1 = \cdots = \gamma_{n-1} = 0$, $\delta_1 = \cdots = \delta_{n-1} = 0$ and
$$\delta_0 = -\frac{1}{n} \sum_{r+s \leq n-1,0 \leq r,s} \gamma_{r+s+1} \text{tr}(B^{2r+2s+1}) = -\frac{1}{n} \sum_{r+s = n-1,0 \leq r,s} \gamma_{r+s+1} \text{tr}(B^{2r+2s+1}) = -\frac{1}{n} \sum_{r+s = n-1,0 \leq r,s} \gamma_n \text{tr}(B^{2n-1}) = -\text{tr}(B^{2n-1})$$
in Theorem 2.3, whence $nB^{2n-1} = \text{tr}(B^{2n-1})I_n$ follows. \Box

2.6. Remark. In Rosset’s proof of the Amitsur-Levitzki theorem ([Ro]) and in [Ksz] a similar implication was used, namely that $\text{tr}(B^2) = \text{tr}(B^4) = \cdots = \text{tr}(B^{2n}) = 0$ implies $B^{2n} = 0$.

2.7. Corollary. The case $n = 2$ in Theorem 2.3 gives that
$$\{\text{tr}(B)\text{tr}(B^2) - \text{tr}(B^3)\} I_2 - \text{tr}(B^2)B - \text{tr}(B)B^2 + 2B^3 = 0.$$ holds for $B \in M_2(E_1)$. The case $n = 3$ in Theorem 2.3 gives that
$$\left\{-\frac{1}{2} \text{tr}^2(B^2)\text{tr}(B) + \text{tr}(B^3)\text{tr}(B^2) + \frac{1}{2} \text{tr}(B^4)\text{tr}(B) - \text{tr}(B^5)\right\} I_3 +$$
$$\left\{\frac{1}{2} \text{tr}^2(B^2) - \frac{1}{2} \text{tr}(B^4)\right\} B + \{\text{tr}(B^2)\text{tr}(B) - \text{tr}(B^3)\} B^2 - 2\text{tr}(B^2)B^3 - \text{tr}(B)B^4 + 3B^5 = 0$$
holds for $B \in M_3(E_1)$. 
Proof. In case $n = 2$ we have
\[
\delta_0 I_2 + \gamma_1 B + \delta_1 B^2 + 2B^3 = 0,
\]
where
\[
p_{B^2}(x) = \gamma_0 + \gamma_1 x + \gamma_2 x^2 = \det(xI - B^2) \in E_0[x]
\]
is the characteristic polynomial of $B^2 \in M_2(E_0)$ with $\gamma_2 = 1$ and $\gamma_1 = -\text{tr}(B^2)$.
The recursion ($\delta_2 = 0, \ 0 \leq k \leq 1$)
\[
\delta_k = \left\{ -\frac{1}{2 - k} \sum_{i=1}^{2-k} \gamma_k \right\} + \left\{ -\frac{1}{2 - k} \sum_{r+s\leq 2-k-1,0\leq r,s} \gamma_{k+r+s+1} \text{tr}(B^{2r+2s+1}) \right\}
\]
gives that $\delta_1 = -\text{tr}(B)$ and
\[
\delta_0 = -\frac{1}{2} \delta_1 \text{tr}(B^2) + \left( -\frac{1}{2} \right) \{ \gamma_1 \text{tr}(B) + \gamma_2 \text{tr}(B^3) + \gamma_3 \text{tr}(B^4) \} = \text{tr}(B)\text{tr}(B^2) - \text{tr}(B^3).
\]
In case $n = 3$ we have
\[
\delta_0 I_3 + \gamma_1 B + \delta_1 B^2 + 2\gamma_2 B^3 + \delta_2 B^4 + 3B^5 = 0,
\]
where
\[
p_{B^3}(x) = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3 = \det(xI - B^2) \in E_0[x]
\]
is the characteristic polynomial of $B^2 \in M_3(E_0)$. In view of the recursion ($\gamma_3 = 1, \ 0 \leq k \leq 2$)
\[
\gamma_k = -\frac{1}{3-k} \sum_{i=1}^{3-k} \gamma_{k+i} \text{tr}(B^{2i}),
\]
we obtain that $\gamma_2 = -\text{tr}(B^2)$ and
\[
\gamma_1 = -\frac{1}{2} \sum_{i=1}^{3} \gamma_{1+i} \text{tr}(B^{2i}) = -\frac{1}{2} \{ -\text{tr}^2(B) + \text{tr}(B^4) \}.
\]
The other recursion ($\delta_3 = 0, \ 0 \leq k \leq 2$)
\[
\delta_k = \left\{ -\frac{1}{3-k} \sum_{i=1}^{3-k} \gamma_{k+i} \text{tr}(B^{2i}) \right\} + \left\{ -\frac{1}{3-k} \sum_{r+s\leq 3-k-1,0\leq r,s} \gamma_{k+r+s+1} \text{tr}(B^{2r+2s+1}) \right\}
\]
gives that $\delta_2 = -\text{tr}(B)$,
\[
\delta_1 = \left\{ -\frac{1}{2} \delta_1 \text{tr}(B^2) \right\} + \left\{ -\frac{1}{2} \sum_{r+s\leq 1,0\leq r,s} \gamma_{r+s+2} \text{tr}(B^{2r+2s+1}) \right\} =
\]
\[
\frac{1}{2} \text{tr}(B)\text{tr}(B^2) + \left\{ -\frac{1}{2} \gamma_2 \text{tr}(B) - \frac{1}{2} \gamma_3 \text{tr}(B^3) - \frac{1}{2} \gamma_3 \text{tr}(B^3) \right\} = \text{tr}(B^2)\text{tr}(B) - \text{tr}(B^3)
\]
\[
\delta_0 = \left\{ -\frac{1}{3} \sum_{i=1}^{3} \delta_i \text{tr}(B^{2i}) \right\} + \left\{ -\frac{1}{3} \sum_{r+s\leq 2,0\leq r,s} \gamma_{r+s+1} \text{tr}(B^{2r+2s+1}) \right\} =
\]
\[
\left\{ -\frac{1}{3} \delta_1 \text{tr}(B^2) - \frac{1}{3} \delta_2 \text{tr}(B^4) \right\} + \left\{ -\frac{1}{3} \right\} \{ \gamma_1 \text{tr}(B) + 2\gamma_2 \text{tr}(B^3) + 3\gamma_3 \text{tr}(B^5) \} =
\]
\[
\left\{ -\frac{1}{3} \{ \text{tr}(B^2)\text{tr}(B) - \text{tr}(B^3) \} \text{tr}(B^2) + \frac{1}{3} \text{tr}(B)\text{tr}(B^4) \right\} +
\]

\[
\left(- \frac{1}{3}\right) \left\{ - \frac{1}{2} \{ - \text{tr}^2(B^2) + \text{tr}(B^4) \} \text{tr}(B) - 2\text{tr}(B^2)\text{tr}(B^3) + 3\text{tr}(B^5) \right\} = - \frac{1}{2} \text{tr}^2(B^2)x(B) + \text{tr}(B^3)x(B^2) + \frac{1}{2} \text{tr}(B^4)x(B) - x(B^5)
\]

Thus we have
\[
\delta_0 I_3 + \gamma_1 B + \delta_1 B^2 + 2\gamma_2 B^3 + \delta_2 B^4 + 3B^5 = \left\{ - \frac{1}{2} \text{tr}^2(B^2)x(B) + \text{tr}(B^3)x(B^2) + \frac{1}{2} \text{tr}(B^4)x(B) - x(B^5) \right\} I_3 + \left\{ - \frac{1}{2} \{ - \text{tr}^2(B^2) + \text{tr}(B^4) \} \right\} B + \left\{ \text{tr}(B^2)x(B) - x(B^3) \right\} B^2 - 2\text{tr}(B^2)x(B^3) - x(B)B^4 + 3B^5. \square
\]

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