Inflationary attractor property of phantoms

Xin-He Meng\textsuperscript{1,2,3} * Peng Wang\textsuperscript{1} †

1. Department of Physics, Nankai University, Tianjin 300071, P.R.China
2. Institute of Theoretical Physics, CAS, Beijing 100080, P.R.China
3. Department of Physics, University of Arizona, Tucson, AZ 85721

There are some motivations to consider inflation driven by a phantom field. Before entering into some specific models and perform data fitting, it is important first investigating some general features that any viable inflation model should hold. The inflationary attractor property is an important one of those features. In this paper we will show that the inflationary attractor property still holds for canonical and Born-Infeld phantom fields in the standard Friedmann-Robertson-Walker cosmology, however, it does not hold for canonical and Born-Infeld phantom fields in the Randall-Sundrum II cosmology.

1. Introduction

Recent observations do not exclude, and even seem to favor, the equation of state of the dark energy $\omega < -1$\textsuperscript{[1]}. However, most of the popular candidates for dark energy, such as quintessence\textsuperscript{[2]}, K-essence\textsuperscript{[3]} and tachyonic scalar fields described by Born-Infeld (B-I) action\textsuperscript{[4]}, lead to $\omega > -1$. Thus the sort of matter or entity with equation of state $\omega < -1$, called "phantom matter", has received increasing attention recently. Such a field has a very unusual dynamics as it violates null dominate energy condition (NDEC). In the literature, there are now mainly two effective descriptions of phantom matter. First, by opposing the sign of the kinetic term in the Lagrangian of a canonical scalar field, we will get a phantom field and we call it "canonical" phantom thereafter\textsuperscript{[5]}. The negative sign of the kinetic term will cause instabilities, however, it was shown in Ref.\textsuperscript{[6]} that the instability timescale may be long enough to make it a sensible effective field theory. Thus it is sensible and interesting to study their cosmological implications\textsuperscript{[7]}. Second, by opposing the sign of the kinetic term in the Lagrangian of a B-I field, we will get the B-I phantom\textsuperscript{[8, 9]}. The interesting feature of the B-I phantom is that it admits a later time attractor solution\textsuperscript{[8]} while the ordinary B-I field does not have this important property\textsuperscript{[10]}.

Thus, it is interesting to consider the scenario of inflation driven by a phantom field\textsuperscript{[11]} and/or the density perturbation generated by the phantom field\textsuperscript{[12]}. If an ordinary inflation model gives a red spectrum, the corresponding phantom inflation model gives a blue one, which is an important character, since current WMAP data gives a blue shift spectrum $n_s \simeq 1.1$. In ordinary inflation model, only hybrid inflation can provide blue spectrum. Furthermore, in some more physical models, phantom inflation appears naturally. For example, it is well-known that the $R + R^2$ gravity can drive an inflation without an inflaton\textsuperscript{[13]}. Generally, those modified gravity models with a Lagrangian of the type $L(R)$ have two inequivalent formulations: the metric formulation (second order formulation) and the Palatini formulation (first order formulation)\textsuperscript{[15, 17, 18]}. However, due to an observation of Nima Arkani-Hamed, the Palatini formalism has fine-tuning problems as an effective quantum field theory. Specifically, matter loops will give rise to a correction to the action proportional to the Ricci scalar of the metric\textsuperscript{[16]}. In this matter loop corrected version of $R + R^2$ gravity, it was shown in Ref.\textsuperscript{[19]} that when expressed in Einstein frame, it corresponds to an Einstein-Hilbert term plus a canonical phantom field. Therefore, in this formulation, the inflation driven by $R + R^2$ gravity corresponds to a phantom inflation (It is interesting to contrast this to the fact that in the "pure" Palatini formulation, the $R + R^2$ gravity can not drive an inflation\textsuperscript{[17]}. It remains an interesting problem which one of those three formulations of modified gravity is the physical one (or all of them are not). Actually, this is the main reason motivating us to investigate the properties of phantom inflation seriously.

If inflation is to be truly predictive, the evolution when the scalar field is at some given point on the potential has to be independent of the initial conditions. Otherwise, any result, such as the amplitude of density perturbations, would depend on the unknowable initial conditions. However, the scalar wave equation is a second order equation, implying that $\dot{\phi}$ can in principle take on any value anywhere on the potential we may be, and so there certainly is not a unique solution at each point on the potential. Inflation can therefore only be predictive if the solution exhibit an attractor behavior, where the differences between solutions of different initial conditions rapidly vanish\textsuperscript{[21]} (See Sec.3.7 of Ref.\textsuperscript{[22]} for a review). The inflationary attractor property of B-I field in standard FRW cosmology is shown to be held in Ref.\textsuperscript{[23]}

In this paper, we will investigate the inflationary attractor properties of the canonical phantom and B-I phantoms in standard Friedmann-Robertson-Walker (FRW) cosmology and Randall-Sundrum II (RSII) cosmology\textsuperscript{[24]}. We
consider the later case because inflation on the brane has now comprised a vast literature and attracted lots of interests (See Ref. [13, 25] for a review and references therein). It has now become a standard activity to consider any existing inflation model in RSII cosmology. The inflationary attractor property was shown to be hold for the canonical scalar field and B-I fields in RSII cosmology in Ref. [26]. We will show that the inflationary property still holds for canonical and B-I phantom fields in standard FRW cosmology, however, it does not hold in RSII cosmology.

2. Attractor property of canonical phantom

The canonical phantom field is described by the effective Lagrangian

\[ L_{\text{phantom}} = \frac{1}{2} (\partial_{\mu} \phi)^2 - V(\phi) \] (1)

where we use the metric signature \{-, +, +, +\}. It is the negative kinetic energy term that distinguishes the phantom field from the ordinary scalar fields. In a spatially flat FRW universe model, we can assume that \( \phi \) is spatially homogeneous. The energy density and the pressure are given by

\[ \rho = -\frac{1}{2} \dot{\phi}^2 + V(\phi) \] (2)

\[ p = -\frac{1}{2} \dot{\phi}^2 - V(\phi) \] (3)

The evolution equation of the field \( \phi \) is

\[ \ddot{\phi} + 3H \dot{\phi} - V'(\phi) = 0 \] (4)

In analyzing the inflationary attractor property, we will use the Hamilton-Jacobi (H-J) formulation of the Friedmann equation [20]. In this formulation, we will view the scalar field \( \phi \) as the time variable. This requires that the \( \phi \) field does not change sign during the inflation period. Without loss of generality, we can choose \( \dot{\phi} > 0 \) in the following discussions.

2.1 Standard FRW cosmology

For a standard FRW cosmology model contained only canonical phantom fields, the Friedmann equation is

\[ H^2 = \frac{\kappa^2}{3} (-\frac{1}{2} \dot{\phi}^2 + V(\phi)) \] (5)

where \( H = \dot{a}/a \) is the Hubble parameter and \( \kappa^2 = 8\pi G \).

Differentiating Eq. (5) with respect to \( t \) and using Eq. (4) gives

\[ H' = \frac{\kappa^2}{2} \dot{\phi} \] (6)

Substituting this into Eq. (5) will give the H-J formulation of the Friedmann equation

\[ (H')^2 + \frac{3\kappa^2}{2} H^2 = \frac{\kappa^4}{2} V(\phi) \] (7)

Eqs. (6) and (7) are the Hamilton-Jacobi equations, which are more conveniently to be employed in analyzing the inflationary attractor behaviors than Eqs. (4) and (5). In this formulation, one considers \( H(\phi) \), rather than \( V(\phi) \), as the fundamental quantities. If we can solve \( H(\phi) \) from Eq. (6), by substituting into Eq. (7) we can immediately obtain \( V(\phi) \). Therefore, the Hamilton-Jacobi formalism is also very useful to obtain a large set of exact inflationary solution (See Ref. [24] for some examples) and put general constraints on the form of the potentials [27].

Supposing \( H_0(\phi) \) is any solution to Eq. (6), which can be either inflationary or non-inflationary. We consider a homogeneous perturbation \( \delta H(\phi) \) to this solution; the attractor property will be satisfied if it becomes smaller as \( \phi \) increases. Substituting \( H(\phi) = H_0(\phi) + \delta H(\phi) \) into Eq. (7) and linearizing, we find that the perturbation obeys

\[ \delta H'(\phi) = -\frac{3\kappa^2}{2} \frac{H_0(\phi)}{H_0'(\phi)} \delta H(\phi) \] (8)

which has the general solution

\[ \delta H(\phi) = \delta H(\phi_i) \exp \left[ -\frac{3\kappa^2}{2} \int_{\phi_i}^{\phi} \frac{H_0(\phi)}{H_0'(\phi)} d\phi \right] \] (9)
where \( \delta H(\phi_i) \) is the value at some initial point \( \phi_i \). Since \( H'_0 \) and \( \phi \) have the same sign, if \( H_0 \) is an inflationary solution, all linear perturbations damp at least exponentially. Note that the number of e-foldings is

\[
N \equiv \int_{t_{end}}^{t} Hdt = \frac{\kappa^2}{2} \int_{\phi_{end}}^{\phi} \frac{H(\phi)}{H'(\phi)} d\phi
\]  

Then Eq. (10) can be written as

\[
H(\phi) = H(\phi_i) \exp[-3(N_i - N)]
\]

In this form, if \( H_0 \) is an inflationary solution, it is obvious that all linear perturbations approach it at least exponentially fast as the scalar field rolls. In conclusion, the inflationary attractor property still holds for canonical phantom field in the standard FRW cosmology.

2.2 Randall-Sundrum II cosmology

For a spatially flat FRW model, the Modified Friedman equation in Randall-Sundrum II model is \[28\]

\[
H^2 = \frac{\kappa^2}{3} (\rho + \frac{\rho_2^2}{2\lambda})
\]  

where \( \lambda \) is the brane tension. When \( \rho \ll \lambda \) (the low energy limit), this reduces to the standard Friedmann equation; when \( \rho \gg \lambda \) (the high energy limit), this reduces to

\[
H^2 = \frac{\kappa^2}{6\lambda} \rho^2
\]

Assume that during inflation stage, \( V \gg \lambda \), thus the Modified Friedman equation in the high energy limit is

\[
H^2 = \frac{\kappa^2}{3} \left( -\frac{1}{2} \dot{\phi}^2 + V(\phi) \right)^2
\]

Differentiating Eq. (14) with respect to \( t \) and using Eq. (4) gives

\[
H'(\phi) = \frac{3\kappa}{\sqrt{6}\lambda} H(\phi) \dot{\phi}
\]  

Substituting this into Eq. (14) will give the H-J formulation of the Modified Friedmann equation

\[
\frac{\lambda}{3\kappa^2} (H'(\phi))^2 + \frac{\sqrt{6}\lambda}{\kappa} H = V(\phi)
\]

Analogous to the previous section, substituting \( H(\phi) = H_0(\phi) + \delta H(\phi) \) into Eq. (16) and linearizing, where \( H_0(\phi) \) is any solution to Eq. (7), we find that \( \delta H(\phi) \) obeys

\[
\delta H'(\phi) = \left( H'_0(\phi) \frac{H_0(\phi)}{H_0(\phi)} - \frac{3\sqrt{6}\kappa}{2\sqrt{\lambda}} \frac{H_0^2(\phi)}{H'_0(\phi)} \right) dH(\phi)
\]

which has with the general solution form

\[
\delta H(\phi) = \delta H(\phi_i) \exp\left( \int_{\phi_i}^{\phi} \left( \frac{H'_0(\phi)}{H_0(\phi)} - \frac{3\sqrt{6}\kappa}{2\sqrt{\lambda}} \frac{H_0^2(\phi)}{H'_0(\phi)} \right) d\phi \right)
\]

where \( \delta H(\phi_i) \) is the value at some initial point \( \phi_i \). Since \( H'_0 \) and \( d\phi \) have got the same sign, if \( H_0 \) is an inflationary solution, the integrand within the exponential term is positively definite if the condition

\[
H'_0(\phi)^2 > \frac{3\sqrt{6}\kappa}{2\sqrt{\lambda}} H_0(\phi)^4
\]

is satisfied. Using Eqs. (14) and (16), this condition can be written as

\[
\frac{3}{2} \dot{\phi}^2 > V(\phi)
\]


Thus when the initial value of the $\dot{\phi}$ satisfies Eq. (20), the perturbations will grow exponentially. In conclusion, the phantom field in RS II model does not have the inflationary attractor character.

3. Attractor property of Born-Infeld phantom

The Born-Infeld phantom is described by the effective action [8]

$$L_{BI-phantom} = -V(\phi)\sqrt{1 - (\partial_\mu \phi)^2}$$

In a spatially flat FRW universe model, we can assume that the scalar field $\phi$ is spatially homogeneous. The energy density and the pressure are given by

$$\rho = \frac{V(\phi)}{\sqrt{1 + \dot{\phi}^2}}$$

$$p = -V(\phi)\sqrt{1 + \dot{\phi}^2}$$

The evolution equation of $\phi$ is

$$\frac{\ddot{\phi}}{1 + \dot{\phi}^2} + 3H\dot{\phi} - \frac{V'(\phi)}{V(\phi)} = 0$$

3.1 Standard FRW cosmology

For a homogenous and isotropic standard FRW cosmology model the Friedmann equation is taken as

$$H^2 = \frac{\kappa^2}{3} \frac{V(\phi)}{\sqrt{1 + \dot{\phi}^2}}$$

Differentiating Eq. (25) with respect to $t$ and using Eq. (21) gives

$$H' = \frac{3}{2} H^2 \dot{\phi}$$

Substituting this into Eq. (26) will give the H-J formulation of the Friedmann equation

$$H(\phi)^2 \sqrt{1 + \frac{4}{9} \frac{H'(\phi)^2}{H(\phi)^2}} = \frac{\kappa^2}{3} V(\phi)$$

Substituting $H(\phi) = H_0(\phi) + \delta H(\phi)$ into Eq. (27) and linearizing, where $H_0(\phi)$ is any solution to Eq. (27), we find that $\delta H(\phi)$ abides by

$$\delta H'(\phi) = -\frac{9}{2} \frac{H_0(\phi)^3}{H_0'(\phi)} \delta H(\phi)$$

which has the general solution

$$\delta H(\phi) = \delta H(\phi_i) \exp \left[ -\frac{9}{2} \int_{\phi_i}^{\phi} \frac{H_0(\phi)^3}{H_0'(\phi)} d\phi \right]$$

where $\delta H(\phi_i)$ is the value at some initial point $\phi_i$. Since $H_0$ and $\phi$ have possessed the same sign, if $H_0$ is an inflationary solution, all linear perturbations are damped at least exponentially. In conclusion, the inflationary attractor property still holds for the B-I phantom field in standard FRW cosmology.

3.2 Randall-Sundrum II cosmology

The Modified Friedman equation in the high energy region can be expressed as

$$H^2 = \frac{\kappa^2}{6\lambda} \frac{V(\phi)^2}{1 + \dot{\phi}^2}$$

The solution to this equation is analyzed in Ref. [11] and it was shown analytically that the scale factor $a$ expands exponentially. Thus Born-Infeld phantom can drive an inflation.
Differentiating Eq. (30) with respect to $t$ and using Eq. (24) gives

$$H'(\phi) = 3H^2 \dot{\phi}$$  \hspace{1cm} (31)

Substituting this into Eq. (30) will give the H-J form of the Modified Friedmann equation

$$\frac{2\lambda}{3\kappa^2} \left( \frac{H'(\phi)}{H(\phi)} \right)^2 + \frac{6\lambda}{\kappa^2} H(\phi)^2 = V(\phi)^2$$  \hspace{1cm} (32)

Substituting $H(\phi) = H_0(\phi) + \delta H(\phi)$ into Eq. (32) and linearizing, where $H_0(\phi)$ is any solution to Eq. (7), we find that $\delta H(\phi)$ obeys

$$\delta H' = \left( -\frac{9H_0(\phi)^3}{H_0'(\phi)} + \frac{H_0'(\phi)}{H_0(\phi)} \right) \delta H$$ \hspace{1cm} (33)

which has the general solution

$$\delta H = \delta H_i \exp \left[ \int_{\phi_i}^{\phi} \left( -\frac{9H_0(\phi)^3}{H_0'(\phi)} + \frac{H_0'(\phi)}{H_0(\phi)} \right) d\phi \right]$$  \hspace{1cm} (34)

where $\delta H_i$ is the value at some initial point $\phi_i$. Since $H_0'$ and $d\phi$ have the same sign, the integrand within the exponential term is positive definite if the condition

$$H_0'(\phi)^2 > 9H_0(\phi)^4$$  \hspace{1cm} (35)

is met. Using Eqs. (30) and (31) this condition can be written as

$$\dot{\phi}^2 > 1$$  \hspace{1cm} (36)

Thus when the initial value of the $\dot{\phi}$ satisfies Eq. (36), the perturbations will grow exponentially. Note that the Lagrangian for a spatially homogenous ordinary B-I field reads

$$L_{B-I} = -V(\phi) \sqrt{1 - \dot{\phi}^2}$$  \hspace{1cm} (37)

Thus, the ordinary B-I field satisfies $\dot{\phi}^2 \leq 1$ a priori. For the phantom B-I field, there is no such natural bound. And we can see that this will make the B-I phantom not a good candidate for inflaton.

In conclusion, the B-I phantom in RSII model does not have the inflationary attractor property.

4. Conclusions and discussions

In this paper, we have shown that the inflationary attractor property still holds for canonical and B-I phantom fields in standard FRW cosmology model, however, it does not hold for canonical and B-I phantom fields in RSII cosmology model. Thus, subsequently, it is worth considering some specific models of a phantom inflation in the standard FRW cosmology to see whether it can accommodate the data more easily. However, from our analysis, considering the phantom inflation in RSII model is not a quite good choice. This reveals another difference between the standard FRW cosmology and RSII cosmology models. Presently, there is an extension of the RSII model by adding a Gauss-Bonnet term in the bulk action [28], it is interesting to see whether the inclusion of the Gauss-Bonnet term can restore the inflationary attractor property of canonical and B-I phantoms. And we will find that, quite interestingly, this is indeed the case [29].

Acknowledgements

We would like to thank D.Lyth, S.D.Odintsov, S.Nojiri and Y.S.Piao for helpful discussions. This work is partly supported by a China Doctoral Foundation of National Education Ministry and an ICSC-World Laboratory Scholarship.

[1] S. Perlmutter el al. Nature 404 (2000) 955; Astroph. J. 517 (1999) 565; D.N.Spergel et al, astro-ph/0302207
[2] B.Ratra and P.J.E.Peebles, Phys.Rev. D37 (1988) 3406; R.R.Caldwell, R.Dave and P.J.Steinhardt, Phys.Rev.Lett. 80 (1998) 1582;
[3] Armendariz-Picon, T.Damour and V.Mukhanov, Phys.Lett. B458 (1999) 209 hep-th/9904075; T.Chiba, T.Okabe and M.Yamaguchi, Phys.Rev. D62 (2000) 023511;
[4] A.Sen, JHEP 0204 (2002) 048; ibid, JHEP 0207 (2002) 065;
[5] R.R.Caldwell, Phys.Lett B545 (2002) 23; G.W. Gibbons, hep-th/0302199;
[6] S.M.Carroll, M.Hoffman and M.Trodden, Phys.Rev. D68 (2003) 023509 astro-ph/0301273;
[7] S.Nojiri and S.D.Odintsov, Phys.Lett. B565 (2003) 1 hep-th/0304131; S.Nojiri and S.D.Odintsov, Phys.Lett. B562 (2003) 147 hep-th/0303117; P.Singh, M.Sami and N.Dadhich, Phys.Rev. D68 (2003) 023522 hep-th/0305110; L.P.Chimento and R.Lazkoz, gr-qc/0307111; J.G.Hao, X.Z.Li, Phys. Rev. D68 (2003) 083514 hep-th/0306033; M.P.Dabrowski, T.Stachowiak and M.Szydłowski, hep-th/0307128; J.G.Hao, X.Z.Li, astro-ph/0309746; J.G.Hao, X.Z.Li, Phys.Rev. D67 (2003) 107303 gr-qc/0302100;
[8] J.G.Hao and X.Z.Li, Phys.Rev. D68 (2003) 043501 hep-th/0305207;
[9] S.Nojiri and S.D.Odintsov, Phys.Lett. B571 (2003) 1 hep-th/0306212;
[10] X.Z.Li, J.G.Hao and D.J.Liu, Chin.Phys.Lett. 19 (2002) 1584;
[11] D.J.Liu and X.Z.Li, Phys.Rev. D68 (2003) 067301 hep-th/0307239;
[12] Y.S.Piao and E.Zhou, hep-th/0308080;
[13] D.Lyth and A.Riotto, Phys.Rept. 314 (1999) 1;
[14] A.A.Starobinsky, Phys.Lett.B 91 (1980) 99;
[15] D. N. Vollick, astro-ph/0306630;
[16] E.É.Flanagan, astro-ph/0308111 ibid, gr-qc/0309015;
[17] X.H.Meng and P.Wang, astro-ph/0308284;
[18] X.H.Meng and P.Wang, Class. and Quant. Grav. 20 (2003) 4949 astro-ph/0307354; ibid, astro-ph/0308031 ibid, hep-th/0310006;
[19] X.H.Meng and P.Wang, hep-th/0310038;
[20] D.S.Salopek and J.R.Bond, Phys.Rev. D42 (1990) 3936; A.G.Muslimov, Class.Quant.Grav. 7 (1990) 231; J.E.Lidsey, Phys.Lett. B273 (1991) 42; A.R.Liddle, P.Parsons and J.D.Barrow, Phys.Rev. D50 (1994) 7222 astro-ph/9408015;
[21] D.S.Goldwirth, Phys.Lett. B243 (1990) 41; R.Brandenberger, G.Geshnizjani and S.Watson, Phys.Rev. D67 (2003) 123510 hep-th/0302222;
[22] A.R.Liddle and D.H.Lyth, Cosmological Inflation and Large Scale Structure, Cambridge University Press, 2000;
[23] Z.K.Guo, Y.S.Piao, R.G.Cai, Y.Z.Zhang, Phys.Rev. D68 (2003) 043508 hep-ph/0304236;
[24] L.Randall and R.Sundrum, Phys.Rev.Lett. 83 (1999) 4690 hep-th/9906004;
[25] J.E.Lidsey, astro-ph/0305528;
[26] Z.K.Guo, H.S.Zhang and Y.Z.Zhang, hep-ph/0309160;
[27] A.Vallinotto, E.J.Copeland, E.W.Kolb and A.R.Liddle, astro-ph/0311005;
[28] P.Binetruy, C.Deffayet, U.Ellwanger and D.Langlois, Phys.Lett. B477 (2000) 285 hep-ph/0910219;
[29] J.E.Lidsey and N.J.Nunes, Phys.Rev. D67 (2003) 103510 astro-ph/0303168; S.Nojiri and S.D.Odintsov, JHEP 07 (2000) 049; S.Nojiri, S.D.Odintsov and S.Ogushi, Int.J.Mod.Phys. A16 (2001) 5085; S.Nojiri, S.D.Odintsov and S.Ogushi, Phys.Rev. D65 (2002) 023521; J.E.Lidsey, S.Nojiri and S.D.Odintsov, JHEP 06 (2002) 026; J.E.Kim, B.Kyae and H.M.Lee, Phys.Rev. D62 (2000) 045013;
[30] X.H.Meng and P.Wang, work in preparation;