Flat space compressible fluid as holographic dual of black hole with curved horizon

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Abstract

We consider the fluid dual of \((d+2)\)-dimensional vacuum Einstein equation either with or without a cosmological constant. The background solutions admit black hole event horizons and the spatial sections of the horizons are conformally flat. Therefore, a \(d\)-dimensional flat Euclidean space \(E^d\) is contained in the conformal class of the spatial section of the black hole horizon. A compressible, forced, stationary and viscous fluid system can be constructed on the product (Newtonian) spacetime \(\mathbb{R} \times E^d\) as the lowest order fluctuation modes around such black hole background. This construction provides the first example of holographic duality which is beyond the class of bulk/boundary correspondence.

1 Introduction

Since ‘t Hooft [1] and Susskind [2] proposed the so-called holographic principle about twenty years ago, the study of holographic dual of gravitational systems has become one of the major subjects of study in the area of high energy theories. Holography is a property which matches one system in the bulk involving gravity to another system on the boundary without gravity. The most well understood realization of holographic duality is the AdS/CFT correspondence [3–5], which establishes an equivalence between the superstring theory on \(\text{AdS}_5 \times \text{S}^5\) and the four-dimensional \(\mathcal{N} = 4\) supersymmetric Yang-Mills gauge theory on the boundary of the \(\text{AdS}_5\). However, there are accumulating evidences indicating that holography can be realized in situations which requires neither AdS in the bulk nor CFT on the boundary. In this respect, the Gravity/Fluid correspondence plays a very instructive example.

The relationship between gravity and fluid system was first known through the membrane paradigm, see [6] and [7–9] for recent applications of this approach. Later

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on, such relationship is rediscovered as the long wavelength limit of AdS/CFT correspondence [10–16]. In both approaches, the ratio of the viscosity to the entropy density of the dual fluid takes a universal value $\frac{1}{4\pi}$ [17,18]. Considerable efforts have been made to clarify that the results from the membrane paradigm and from the long wavelength limit of AdS/CFT are related by RG flow [18–21].

Renewed interests in Gravity/Fluid correspondence arise following the work [22], in which a fluid dual on a finite cutoff in a Rindler background has been constructed. This is the first successful attempt in constructing Gravity/Fluid correspondence beyond the framework of AdS/CFT. Subsequent works revealed that similar construction also works in black hole backgrounds in Einstein gravity [23–26] as well as in higher curvature gravity [27–29], and the viscosity of the dual fluid is calculated in [30,31]. Numerous examples of this correspondence was studied extensively [32–36].

Besides the membrane paradigm and the AdS/CFT approach, the Gravity/Fluid correspondence can be realized either using a boost-rescaling technique [23–29,31–36] or by the introduction of Petrov I boundary conditions [37–46], which is mathematically much simpler. In all known example cases, there are two remarkable features (or drawbacks) in the Gravity/Fluid correspondence. Firstly, just like in any other realizations of holographic duality, the boundary (or holographic screen) must be chosen such that it is an equipotential hypersurface in the bulk spacetime. This requires, in particular, that if one starts from a spherically symmetric solution of the gravitational field equations, the final fluid dual must also live in a spacetime with spherical spatial sections. Thus, if one would like to understand flat space fluid mechanics from a gravitational perspective, then the only choice seems to be starting from an AdS bulk and choose a solution with flat horizon. Secondly, the dual fluid is always incompressible due to the fact that to the lowest nontrivial order, the conservation of the Brown-York tensor on the boundary implies a divergence-free condition of the velocity field of the dual fluid.

In this work, we are aimed to realize a duality relationship between gravity solutions with spatially non-flat horizons and a compressible fluid system living in a flat, Newtonian spacetime with one less dimensions. Evidently, a flat Newtonian spacetime cannot be realized as an equipotential hypersurface in the bulk spacetime unless the gravity solution is plane symmetric. In fact, the flat Newtonian space may not be a subspace of the bulk spacetime at all. Therefore, the duality relationship as described above is highly nontrivial not only because it evades the two drawbacks of general Gravity/Fluid correspondence, but also because it provides a remarkable example of holographic duality which is beyond the class of bulk/boundary dualities.
2 Static black holes in \((d + 2)\)-dimensions

Let us start by introducing a particular class of static vacuum solutions to the Einstein equation

\[ G_{\mu\nu} = -\Lambda g_{\mu\nu} \quad (1) \]

in \((d + 2)\)-dimensions, where \(G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\) is the bulk Einstein tensor, \(\Lambda\) denotes a possible cosmological constant which may be positive, zero or negative.

Under the coordinates \(x^\mu = (t, r, x^i) \quad (i = 1, \cdots, d)\), the static solution to the vacuum Einstein equation can be described by the line element

\[ ds_{d+2}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 e^{\Phi(x^i)} \delta_{ij} dx^i dx^j, \quad f(r) = \frac{\kappa}{r^{d-1}} - \frac{2\Lambda r^2}{d(d+1)}, \quad (2) \]

where \(\kappa = 1\) if \(\Lambda \geq 0\) and \(\kappa = -1, 0, 1\) if \(\Lambda < 0\). To ensure that \((2)\) is a solution to the vacuum Einstein equation \((1)\), the function \(\Phi(x^i)\) must obey a set of complicated differential equations,

\[ \delta^{ij} \partial_j \Phi + \frac{d-2}{2} \left( 2 \partial_i^2 \Phi + \delta^{ij} \partial_j \Phi \partial_k \Phi - (\partial_i \Phi)^2 \right) + 2\kappa(d-1)e^\Phi = 0, \quad (3) \]

\[ (d-2) \left( \partial_i \partial_j \Phi - \frac{1}{2} \partial_k \Phi \partial_k \Phi \right) = 0 \quad (i \neq j). \quad (4) \]

For generic \(d\), explicit solution for the function \(\Phi(x^i)\) is guaranteed to exist because the well-known Schwarzschild-Tangherlini-(A)dS solution to the vacuum Einstein equation can be written in the form \((2)\). In particular, when \(d = 2\), the eqs. \((3)-(4)\) degenerate into the Euclideanized Liouville equation (or Laplacian equation if \(\kappa = 0\)) [47]

\[ (\partial_1^2 + \partial_2^2) \Phi + 2\kappa e^\Phi = 0. \quad (5) \]

It is evident that the line element \((2)\) possesses a black hole event horizon provided \(\omega \neq 0\). The event horizon is located at one of the zeros \(r = r_h\) of the metric function \(f(r)\), which is the largest (if \(\Lambda\) is non-positive) or the second largest (if \(\Lambda\) is positive) root of \(f(r)\). The spatial section of the horizon surface has a conformally flat metric

\[ ds_h^2 = r_h^2 e^{\Phi(x^i)} \delta_{ij} dx^i dx^j, \]

which contains the \(d\)-dimensional flat Euclidean space \(\mathbb{E}^d\) with metric \(\delta_{ij}\) in its conformal class.

In what follows, it is desirable to rewrite the line element \((2)\) in the Eddington-Finkelstein coordinates \((u, r, x^i)\) with the lightlike coordinate \(u\) defined by

\[ u = t + \int_0^r \frac{dr'}{f(r')}. \quad (6) \]

Doing so, the line element \((2)\) can be rearranged into the following form

\[ ds_{d+2}^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(r) du^2 + 2 du dr + r^2 e^{\Phi(x^i)} \delta_{ij} dx^i dx^j. \quad (7) \]
3 Hypersurface and Brown-York tensor

Now consider a \((d + 1)\)-dimensional timelike hypersurface \(\Sigma_c\) located at \(r = r_c\). The geometry of this embedding hypersurface is best characterized by its first and second fundamental forms. The first fundamental form is provided by the restriction of the bulk line element on the hypersurface, i.e.

\[
ds_{d+1}^2 = \gamma_{ab}dx^a dx^b = -f(r_c)du^2 + r_c^2 e^{\Phi} \delta_{ij} dx^i dx^j
\]

\[
= -(d\tau^0)^2 + r_c^2 e^{\Phi} \delta_{ij} dx^i dx^j
\]

\[
= -\frac{1}{\lambda^2} d\tau^2 + r_c^2 e^{\Phi} \delta_{ij} dx^i dx^j,
\]

where \(x^a = (u, x^i)\), \(\tau = \lambda x^0 = (\lambda \sqrt{f_c}) u\) and the rescaling parameter \(\lambda\) is introduced to facilitate the forthcoming analysis on the non-relativistic limit\(^*\). Here and below we use the notations \(f_c = f(r_c), f'_c = f'(r)|_{r=r_c}\), etc. The notations \(f'_h, f''_h\) will also be used, which are similar to \(f'_c, f''_c\) but with \(r_c\) replaced by \(r_h\).

One can easily promote the hypersurface tensor \(\gamma_{ab}\) to a bulk tensor \(\gamma_{\mu\nu}\) by adding a raw and a column in the \(r\)-direction which are both full of zeros. The second fundamental form is the extrinsic curvature of the hypersurface, which is defined as

\[
K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu},
\]

where

\[
n^\mu = \left( \frac{1}{\sqrt{f}}, \sqrt{f}, 0, \cdots, 0 \right)
\]

is a unit vector field which is normal to \(\Sigma_c\) at \(r = r_c\) and is written in the coordinate \((u, r, x^i)\).

The projection of the bulk Einstein equation gives rise to the so-called momentum and Hamiltonian constraints,

\[
(G_{\mu\nu} + \Lambda g_{\mu\nu}) \gamma_{\mu\nu} n^\nu|_{\Sigma_c} = 0,
\]

\[
(G_{\mu\nu} + \Lambda g_{\mu\nu}) n^\mu n^\nu|_{\Sigma_c} = 0.
\]

In terms of the two fundamental forms introduced above, these can be reformulated in the form

\[
D_a (K^a_b - \gamma^a_b K) = 0,
\]

\[
\hat{R} + K_{ab}K^{ab} - K^2 = 2\Lambda,
\]

where \(\hat{R}\) is the Ricci scalar of \(\Sigma_c\), \(D_a\) is the covariant derivative that is compatible with \(\gamma_{ab}\). Using the definition \([50]\)

\[
t_{ab} = \gamma_{ab}K - K_{ab}
\]

\(^*\)One can think of \(1/\lambda\) as the speed of light, hence \(\lambda \to 0\) corresponds to infinite light speed, i.e. the non-relativistic limit.
for the Brown-York stress energy tensor $t_{ab}$ on hypersurface, the momentum constraints (10) becomes that of the covariant divergence free condition

$$D_a t^a_b = 0$$  \hspace{1cm} (13)

for $t_{ab}$, and the Hamiltonian constraint becomes

$$\hat{R} + t^a_b t^b_a - \frac{t^2}{d} = 2\Lambda.$$  \hspace{1cm} (14)

All these are the standard material for the construction of Gravity/Fluid correspondence.

Before moving on to the construction of dual fluid, let us mention that the lightlike coordinate $u$ (and hence $\tau$) in the bulk spacetime becomes naturally timelike on the hypersurface $\Sigma_c$. The time evolution of the dual fluid will be defined with respect to this coordinate. However, the spacetime in which the dual fluid lives will not be $\Sigma_c$ (which is in general curved) but rather the product space $\mathbb{R} \times \mathbb{E}^d$ in which the first factor represents the time direction and $\mathbb{E}^d$ is the $d$-dimensional Euclidean space which lies in the conformal class of the spatial section of $\Sigma_c$. In other words, the dual fluid will be living in a $(d + 1)$-dimensional Newtonian spacetime.

### 4 Petrov I boundary condition

As usual in Gravity/Fluid correspondence, we introduce the Petrov I [48,49] boundary condition on $\Sigma_c$, i.e.

$$C_{(l)(i)(j)} = l^\mu (m_i)^\nu l^\rho (m_j)^\sigma C_{\mu \nu \rho \sigma} = 0, \hspace{1cm} (15)$$

where

$$l^2 = k^2 = 0, \; , (k,l) = 1, \; , (l,m_i) = (k,m_i) = 0, \; , (m_i,m_j) = \delta_{ij} \hspace{1cm} (16)$$

are a set of Newman-Penrose basis vector fields, and $C_{\mu \nu \rho \sigma}$ is the bulk Weyl tensor. We choose the basis vector fields to be

$$l^\mu = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{f}} (\partial_0)^\mu - n^\mu \right) = \frac{1}{\sqrt{2}} ((\partial_0)^\mu - n^\mu),$$

$$k^\mu = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{f}} (\partial_0)^\mu + n^\mu \right) = \frac{1}{\sqrt{2}} ((\partial_0)^\mu + n^\mu),$$

$$(m_i)^\mu = r^{-1} e^{-\frac{1}{2} \Phi} (\partial_i)^\mu.$$  \hspace{1cm} (17)

then the boundary condition becomes

$$C_{0\hat{a}j} + C_{0ij(n)} + C_{0ji(n)} + C_{i(n)j(n)} = 0,$$  \hspace{1cm} (18)
where \( C_{abcd}, C_{abc(n)}, C_{a(n)b(n)} \) are projections of the bulk Weyl tensor

\[
C_{abcd} = \gamma^\nu_a \gamma^\sigma_b \gamma^\tau_c \gamma^\sigma_d C_{\mu \nu \sigma \rho},
C_{abc(n)} = \gamma^\nu_a \gamma^\nu_b \gamma^n_c n^\nu C_{\mu \nu \sigma \rho},
C_{a(n)b(n)} = \gamma^\mu_a n^\nu \gamma^n_c n^\nu C_{\mu \nu \sigma \rho},
\]

and all these can be expressed in terms of the the fundamental fo rms of \( \Sigma_c \):

\[
C_{abcd} = \hat{R}_{abcd} + K_{ad}K_{bc} - K_{ac}K_{bd} - \frac{4\Lambda}{d(d+1)} \gamma_a^c \gamma_d^b, \\
C_{abc(n)} = D_a K_{bc} - D_b K_{ac}, \\
C_{a(n)b(n)} = -\hat{R}_{ab} + KK_{ab} - K_{ac}K_{c}^b + \frac{2\Lambda}{(d+1)} \gamma_{ab}.
\]

Here, of course, \( \hat{R}_{abcd} \) and \( \hat{R}_{ab} \) are the Riemann and Ricci tensors of \( \Sigma_c \).

Then we rewrite the Brown-York tensor (12) in components

\[
K^\tau_{\tau} = \frac{t}{d} - t^\tau_{\tau}, \quad K^\tau_i = -t^\tau_{\tau}, \\
K^i_j = \frac{t}{d} \delta^i_j - t^i_j, \quad K = \frac{t}{d}.
\]

Inserting these relations as well as (19) into (18), the boundary conditions finally become

\[
\frac{2}{\lambda^2} t^\tau_{\tau} t^\tau_j + \frac{t^2}{d^2} \gamma_{ij} + \frac{2\Lambda}{d} \gamma_{ij} - (t^\tau_{\tau} - 2\lambda D_{\tau}) \left( \frac{t}{d} \gamma_{ij} - t_{ij} \right) \\
- \frac{2}{\lambda} D_{(i} t^\tau_{j)} - t_{ik} t^k_{\ j} - \hat{R}_{ij} = 0,
\]

where the explicit appearance of the parameter \( \lambda \) comes from the rescaling of the coordinate \( x^0 \rightarrow \tau/\lambda \) which is made in (8).

5 Near horizon and non-relativistic limit

Now let us place the hypersurface \( \Sigma_c \) very close to the black hole event horizon at \( r = r_h \). This means that \( r_c - r_h \) is a small parameter, and we take this parameter to be \( r_c - r_h = \alpha^2 \lambda^2 \), where \( \lambda \) is the same rescaling parameter appeared in (8) and \( \alpha \) is a finite constant which must be present to balance the dimensionality. Note that such an identification implies that the near horizon limit \( \lambda \rightarrow 0 \) is simultaneously the non-relativistic limit. The near horizon nature of \( \Sigma_c \) allows us to expand \( f_c \) in power series of \( \lambda \),

\[
f(r_c) = f'(r_h)(r_c - r_h) + \frac{1}{2} f''(r_h)(r_c - r_h)^2 + \cdots \\
= f'_h \cdot (\alpha^2 \lambda^2) + \frac{1}{2} f''_h \cdot (\alpha^2 \lambda^2)^2 + \cdots,
\]
which is crucial in the following constructions.

To realize the fluid dual of the gravitational theory, it is insufficient to consider only the background metric. Rather, it is necessary to consider small fluctuations around the background solution. So, on the hypersurface $\Sigma_c$, the metric can be expanded in power series in $\lambda$,

$$\gamma_{ab} = \gamma^{(B)}_{ab} + \sum_{n=1}^{\infty} \gamma^{(n)}_{ab} \lambda^n,$$

(22)

where $\gamma^{(B)}_{ab}$ represents the background metric and $\gamma^{(n)}_{ab}$ are the fluctuation modes. Consequently, both the Ricci curvature $\hat{R}_{ab}$ and the Brown-York tensor $t^a_b$ will also be subject to fluctuations, i.e.

$$\hat{R}_{ab} = \hat{R}^{(B)}_{ab} + \sum_{n=1}^{\infty} \lambda^n \hat{R}^{(n)}_{ab},$$

(23)

$$t^a_b = t^{a(B)}_b + \sum_{n=1}^{\infty} \lambda^n t^{a(n)}_b,$$

(24)

where the superscripts $(B)$ indicate contributions from the background geometry.

In the near horizon limit, the background contributions will also depend on the parameter $\lambda$, thanks to the expansion (21). So, we need to evaluate the background values of the Brown-York and the Ricci tensors on $\Sigma_c$ and expand the results near the horizon $r = r_h$. By direct calculations, we can get

$$K^\tau = \frac{f'_c}{2 \sqrt{f_c}} , \quad K^i = 0,$$

$$K^i_j = \frac{\sqrt{f_c}}{r_c} \delta^i_j , \quad K = \frac{f'_c}{2 \sqrt{f_c}} + \frac{d \sqrt{f_c}}{r_c}.$$  

(25)

These in turn lead to the background Brown-York tensor

$$t^{(B)}_\tau = \frac{d \sqrt{f_c}}{r_c},$$

$$t^{(B)}_i = 0,$$

$$t^{(B)}_j = \left( \frac{f'_c}{2 \sqrt{f_c}} + \frac{(d-1) \sqrt{f_c}}{r_c} \right) \delta^i_j,$$

$$t^{(B)}_\tau = \frac{d}{2} \frac{f'_c}{\sqrt{f_c}} + d^2 \frac{\sqrt{f_c}}{r_c},$$

(26)

where $t^{(B)}$ is the trace of $t^{a(B)}_b$. Using (21), we have the following expansions for $\frac{\sqrt{f_c}}{r_c}$. 


and \( \frac{f'}{\sqrt{f_c}} \):

\[
\frac{\sqrt{f_c}}{r_c} = \left( \alpha \lambda \right) \frac{\sqrt{f_h}}{r_h} + \frac{1}{2} \left( \alpha \lambda \right)^3 \frac{\sqrt{f''_h}}{r_h} + \cdots ,
\]

\[
\frac{f'_c}{\sqrt{f_c}} = \frac{\sqrt{f_h}}{\alpha \lambda} + \alpha \lambda \frac{f''_h}{\sqrt{f_h}} + \cdots .
\]

(27)

Inserting the expansions in (27) into (26) and then into (24), we get

\[
t^\tau = \frac{d \alpha \lambda \sqrt{f_h}}{r_h} + \lambda t^{r(1)} + \cdots ,
\]

\[
t^i = 0 + \lambda t^{r(1)} + \cdots ,
\]

\[
t^i_j = \left( \frac{1}{2} \frac{\sqrt{f_h}}{\alpha \lambda} + \frac{\alpha \lambda f''_h}{2 \sqrt{f_h}} + \frac{(d - 1) \alpha \lambda \sqrt{f''_h}}{r_h} \right) \delta^i_j + \lambda t^{(1)} + \cdots ,
\]

\[
t = d \left( \frac{1}{2} \frac{\sqrt{f_h}}{\alpha \lambda} + \frac{\alpha \lambda f''_h}{2 \sqrt{f_h}} + \frac{1}{2} \frac{\sqrt{f_h}}{r_h} \right) + \lambda t^{(1)} + \cdots .
\]

(28)

We shall also make use of the \( ij \) components of the Ricci tensor \( \hat{R}_{ab} \) on \( \Sigma_c \). By explicit calculations, we find

\[
\hat{\Gamma}^\tau_{ab} = \hat{\Gamma}^{(B)}_{\tau b} = 0 ,
\]

\[
\hat{\Gamma}^{ij}_{ab} = \frac{1}{2} \left( \delta^k_i \partial_j \Phi + \delta^k_j \partial_i \Phi - \delta_{ij} \partial^k \Phi \right) ,
\]

where \( \hat{\Gamma}^{(B)}_{ab} \) are components of the Christoffel connection under the background geometry of \( \Sigma_c \). It is remarkable that the components of \( \hat{\Gamma}^{(B)}_{ab} \) are independent of the position of \( \Sigma_c \). Consequently, the background Ricci tensor \( \hat{R}^{(B)}_{ab} \) will also independent of \( r_c \). Explicitly, we have

\[
\hat{R}^{(B)}_{\tau a} = 0 , \quad \hat{R}^{(B)}_{ij} = \kappa (d - 1) e^\Phi \delta_{ij} ,
\]

(29)

where use have been made of the equations (3) and (4). So, in the near horizon limit, \( \hat{R}^{(B)}_{ij} \) will not develop \( \lambda \) dependences. However, since the metric \( \gamma_{ab} \) on \( \Sigma_c \) may fluctuate due to (22), the fluctuation parts \( \hat{R}^{(n)}_{ab} \) in (23) will in general be nonzero. Moreover, due to the fluctuations of the metric, the covariant derivatives such as \( D_j t^\tau_k \) will also receive fluctuating corrections which are at least \( \mathcal{O}(\lambda^1) \) because

\[
\hat{\Gamma}_{ab} = \hat{\Gamma}^{(B)}_{ab} + \mathcal{O}(\lambda^1) .
\]

(30)

Finally, substituting (28) and (23) (with (29) inserted) into (20), we get in the first nontrivial order \( \lambda^{(0)} \) the following identity,

\[
\sqrt{\frac{f_h}{\alpha}} t^{(1)}_{ij} = 2 \gamma^{ik(0)} t^{r(1)}_k t^{(1)}_j - 2 \gamma^{ik(0)} \zeta_{kj} + \sqrt{\frac{f_h}{d \alpha}} t^{(1)}_j ,
\]

(31)
where we have introduced the shorthand notations

\[ \gamma^{ik(0)} = r_h^{-2} e^{-\Phi} \delta^{ik}, \]
\[ \zeta_{kj} = \partial(k t_j^{(1)}) - \partial(k \Phi t_j^{(1)}) + \frac{1}{2} \delta_{kj} \delta^{lm} \partial_l \Phi t_m^{(1)}, \]

which are, respectively, the inverse of the near horizon background metric on \( \Sigma_c \) and the leading term in \( D_i t_j^{(1)} \):

\[ D_i t_j^{(1)} = \lambda \zeta_{ij} + O(\lambda^2). \]

Some terms which ought to appear in (31) cancels out because

\[ \frac{2\Lambda}{d} + \frac{f'_h}{r_h} - \kappa \frac{d-1}{r_h^2} = 0. \]

Besides eq. (31), which is the lowest nontrivial order of the Petrov I boundary condition (20), we also need to consider the fluctuation modes in the covariant conservation condition (13) and the Hamiltonian constraint (14). Using (28) and (30), we can evaluate the \( \tau \) component of (13), which reads

\[ D_i t_j^{(1)} = \lambda \zeta_{ij} + O(\lambda^2). \]

Therefore, at order \( \lambda^{-1} \), we get

\[ \delta^{ij} \left( \frac{d-2}{2} \partial_i \Phi \right) t_j^{(1)} = 0. \]

Similarly, we can also evaluate the spatial components of (13), which yields, at the first nontrivial order \( O(\lambda^1) \), the following equation,

\[ \partial_i t_j^{(1)} - \frac{1}{2} (t_i^{(1)} - t_j^{(1)}) \partial_i \Phi + \left( \partial_j + \frac{d}{2} \partial_j \Phi \right) t_i^{(1)} = 0. \]

The first non-vanishing order of the Hamiltonian constraint is at \( O(\lambda^0) \):

\[ t^{(1)}_\tau = -2 \gamma^{ij(0)} t_i^{(1)} t_j^{(1)}. \]

In the next section we will show that the equations (31), (34), (35) and (36) give rise to the Navier-Stokes equation of a forced, stationary and compressible fluid system.

## 6 Dual fluid in flat space

In this section we study the dual fluid equations that arise from the fluctuation modes described in the last section. For this purpose, we need to insert (31) into (35) and
simplify the result. The term $\partial_j t^{(1)}_i$ can be evaluated as follows,

$$
\partial_j t^{(1)}_i = \frac{\alpha}{\sqrt{h}} \partial_j \left( 2\gamma^{jk(0)} t^{(1)}_k t^{(1)}_i \right) - \frac{\alpha}{\sqrt{h}} \partial_j \left( 2\gamma^{jk(0)} \zeta_{ki} \right) + \frac{1}{d} \partial_i t^{(1)}, \tag{37}
$$

where

$$
\partial_j \left( 2\gamma^{jk(0)} t^{(1)}_k t^{(1)}_i \right) = -d\gamma^{jk(0)} \partial_j \Phi t^{(1)}_k t^{(1)}_i + 2\gamma^{jk(0)} t^{(1)}_j \partial_k t^{(1)}_i, \tag{38}
$$

and

$$
\partial_j \left( 2\gamma^{jk(0)} \zeta_{ki} \right) = \partial_k \left( 2\gamma^{jk(0)} \partial_i t^{(1)} \right) + \gamma^{jk(0)} \left( \partial_j \partial_k - \partial_j \partial_k \Phi - 2\partial_j \Phi \partial_k + \partial_j \Phi \partial_k \Phi \right) t^{(1)}_i. \tag{39}
$$

Inserting (38) and (39) into (37), we get

$$
\partial_j t^{(1)}_i = \frac{\alpha}{\sqrt{h}} \left( -d\gamma^{jk(0)} \partial_j \Phi t^{(1)}_k t^{(1)}_i + 2\gamma^{jk(0)} t^{(1)}_j \partial_k t^{(1)}_i - \partial_k \left( 2\gamma^{jk(0)} \partial_i t^{(1)} \right) + \frac{1}{d} \partial_i t^{(1)} \right). \tag{40}
$$

Substituting (40) as well as (31), (32) and (36) into (35), we get

$$
\partial_i t^{(1)}_i + \frac{1}{d} \partial_i t^{(1)}_i + r_h^{-2} e^{-\Phi} \delta^{jk} \left[ 2t^{(1)}_k \partial_i t^{(1)}_i - \partial_j \partial_k t^{(1)}_i - 2t^{(1)}_j t^{(1)}_k \partial_i \Phi \right]
+ \left( \partial_j \partial_k \Phi - \frac{d-4}{2} \partial_j \Phi \partial_k + \frac{d-2}{2} \partial_j \Phi \partial_k \right) t^{(1)}_i
- \partial_i \Phi \partial_i t^{(1)}_k + \frac{d-2}{2} \left( t^{(1)}_k \partial_j - \partial_j \Phi t^{(1)}_k \right) \partial_i \Phi = 0, \tag{41}
$$

where we have chosen $\alpha = \sqrt{\frac{2}{h}}$ to eliminate the constant factors such as $\frac{\alpha}{\sqrt{h}}$.

Unlike the usual construction of fluid dual, we would like to interpret eqs. (34) and (41) as the continuity and the Navier-Stokes equations respectively in a flat Euclidean space with spatial coordinates $x^i$. To achieve this, let us first rewrite (34) in the following form:

$$
\partial^i \left( e^{-\frac{2}{d}\Phi} t^{(1)}_j \right) = 0. \tag{42}
$$

Adopting the following “holographic dictionary”

$$
\rho = r_h^2 e^{\frac{4}{d}\Phi}, \quad \mu = e^{\frac{4}{d}\Phi}, \quad \nu = \frac{\mu}{\rho} = r_h^{-2} e^{-\Phi}, \tag{43}
$$

and

$$
\frac{t^{(1)}_i}{d} = \frac{p}{2\mu}, \quad \frac{t^{(1)}_i}{d} = \frac{p}{2\mu}, \tag{44}
$$

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where $\rho, \mu, v_i, p$ are respectively the density, viscosity, velocity field and the pressure of the dual fluid ($\nu$ is the kinematic viscosity), then eq. (42) becomes the continuity equation

$$\partial^i (\rho v_j) = 0,$$

and eq. (41) becomes the standard Navier-Stokes equation

$$\rho (\partial_r v_i + v^j \partial_j v_i) = -\partial_i p + \partial^i d_{ij} + f_i$$

for the velocity field of the fluid, where the symmetric traceless tensor

$$d_{ij} = \mu \left( \partial_j v_i + \partial_i v_j - \frac{2}{d} \delta_{ij} \partial^k v_k \right)$$

represents the deviatoric stress, which depends only on the derivatives of the velocity field and hence vanishes in the hydrostatic equilibrium limit, and

$$f_i = \partial^j \Phi \left( d_{ij} + \frac{d-2}{2} \rho \delta_{ij} \right) + \frac{2}{d} v^j v_j \partial_i \rho - \frac{2}{d} (v^j \partial_j \rho) v_i$$

represents a body force. It is easy to identify the last term in (48) as a linear resistance force, which is proportional to the velocity field $v_i$ and to the directional gradient of the density of the fluid. The first two terms in (48) look unusual, because the factor $\partial^j \Phi$ actually is proportional to the gradient of the logarithm of the density of the fluid. Despite the unusual form of the body force, the equations (45) and (46) constitute the complete system of equations governing the motion of a compressible, forced, stationary and viscous fluid moving in the $(d + 1)$-dimensional Newtonian spacetime $\mathbb{R} \times \mathbb{E}^d$.

7 Concluding remarks

Unlike the ordinary Gravity/Fluid correspondence in which the dual fluid always lives on an equipotential hypersurface (usually taken to be a near horizon hypersurface) and is always incompressible, we have constructed fluid dual in a Newtonian spacetime with one less dimension as compared to the gravity system. It looks striking to realize such kind of a holographic dual, because the dual system does not even live on the boundary of the gravitational system. Some of the distinguished features of our construction are summarized below:

- The holographic screen, if one prefers to speak so, is not (necessarily) a boundary of the bulk spacetime. The dual system lives in a flat Newtonian spacetime even if the black hole horizon in the bulk is curved;
- The dual fluid is compressible but stationary, i.e. the density distribution does not change with time;
• The dual system possesses a non-constant viscosity;
• The dual fluid is subject to both the surface stress and a body force, even though the gravity side is free of source.

Going through the construction process, it is clear that our result depends heavily on the fact that the “angular part” of the black hole solution possesses a conformally flat geometry. This is the case for all maximally symmetric solutions of Einstein equation in any dimension as well as all solutions of Einstein equation in the case of $d = 2$ (i.e. four dimensional spacetime) – in the latter case, the angular part is two dimensional and it is known that any two dimensional manifold is conformally flat. To be more concrete, we would like to present the explicit value of the conformal factor $e^\Phi$ in the case of arbitrary $d \geq 2$ with $\kappa = 1$. In this case, we have

$$e^\Phi(x) = \frac{1}{(1 + \frac{1}{4} \sum_i x_i^2)^2}.$$  

For the particular choice $d = 2$, $e^\Phi$ is given by the solutions of the Liouville (or Laplacian) equation (5), and there are infinitely many different solutions to such equations.

Clearly, much has been left to do following this work. The first question to be answered is whether similar construction works in the case of other background geometries or starting from other (extended) theories of gravity (ether with or without source fields). Meanwhile, we have chosen to make a near horizon expansion in the intermediate steps of the construction. Whether the near horizon condition is absolutely required is in question, because there are already a number of examples in the ordinary Gravity/Fluid correspondence in which the holographic screen is not taken as a near the horizon hypersurface but rather as a finite cutoff surface [21, 24, 42]. If all these proves to be working, then a further step will be asking whether holographic duality beyond the class of bulk/boundary correspondence can be worked out in more general settings such as Gravity/Condensed Matter Theory or Gravity/QCD correspondences etc. We hope we could have more to say following these lines shortly.

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