We say \(e \in \omega\) where \(\omega\) is a cut-edge in \((V, \omega_t)\) if changing the state of \(e_t\) changes the number of connected components \(c(\omega_t)\) in \((V, \omega_t)\). This chain is, by design, reversible with respect to \(\pi_{G, p, q}\).

A central question in the study of Markov chains is how the mixing time—defined as the number of steps until the Markov chain is close to stationarity—grows as the size of the graph \(G\) increases. Of particular interest in the context of random-cluster and Ising/Potts dynamics is the relation of mixing times to the rich equilibrium phase transitions of the model.

We consider this question when \(G\) is a random \(\Delta\)-regular graph on \(n\) vertices. The study of spin systems and their dynamics on random graphs is quite active \([10, 12–14, 17, 18, 22, 39, 40]\). Random \(\Delta\)-regular graphs undergo an exponential slowdown at \(p_u(q, \Delta)\).

More precisely, we show that for every \(q \geq 1\), \(\Delta \geq 3\), and \(p < p_u(q, \Delta)\), with probability \(1 - o(1)\) over the choice of a random \(\Delta\)-regular graph on \(n\) vertices, the Glauber dynamics for the random-cluster model has \(O(n(\log n)^2)\) mixing time. As a corollary, we deduce fast mixing of the Swendsen–Wang dynamics for the Potts model on random \(\Delta\)-regular graphs for every \(q \geq 2\), in the tree uniqueness region. Our proof relies on a sharp bound on the “shattering time”, i.e., the number of steps required to break up any configuration into \(O(\log n)\) sized clusters. This is established by analyzing a delicate and novel iterative scheme to simultaneously reveal the underlying random graph with clusters of the Glauber dynamics configuration on it, at a given time.

Aside from its inherent interest as a model of random networks, the random-cluster model provides an elegant class of Markov Chain Monte Carlo (MCMC) algorithms for sampling from the Ising/Potts distribution. For integer \(q \geq 2\), a sample \(\omega\) from (1.1) can be transformed into one for the \(q\)-state ferromagnetic Potts model by independently assigning a random state from \(\{1, \ldots, q\}\) to each connected component of \(\omega\); see, e.g., \([16, 28]\). Random-cluster based sampling algorithms, which include the popular Swendsen–Wang algorithm \([43]\), are a widely-used alternative to the standard Ising/Potts Markov chains since the former are often efficient at “low-temperatures” (large \(p\)) where the latter suffer exponential slowdowns (see \([6, 29]\)).

Our focus here is on the Glauber dynamics of the random-cluster model. Specifically, we consider the following discrete-time Glauber dynamics chain, which we refer to as the FK-dynamics. For \(t \in \mathbb{N}\), from \(\omega_t \subseteq E\), one step of the FK-dynamics transitions to a new configuration \(\omega_{t+1} \subseteq E\) as follows:

1. Choose an edge \(e_t \in E\) uniformly at random;
2. Set \(\omega_{t+1} = \omega_t \cup \{e_t\}\) with probability \(\tilde{p} := \frac{p}{q(1 - p) + p}\) if \(e_t\) is a “cut-edge” in \((V, \omega_t)\); otherwise;
3. Otherwise set \(\omega_{t+1} = \omega_t \setminus \{e_t\}\).

We say \(e\) is a cut-edge in \((V, \omega_t)\) if changing the state of \(e_t\) changes the number of connected components \(c(\omega_t)\) in \((V, \omega_t)\). This chain is, by design, reversible with respect to \(\pi_{G, p, q}\).

1. INTRODUCTION

The random-cluster model is a random graph model, unifying the study of electrical networks, independent bond percolation, and the ferromagnetic Ising/Potts model from statistical physics \([19, 28]\). It is defined on a graph \(G = (V, E)\) and parametrized by an edge probability \(p \in (0, 1)\) and cluster weight \(q > 0\). Each configuration consists of a subgraph \(\omega \subseteq E\) (equivalently \(\omega \in \{0, 1\}^E\)) and is assigned probability

\[\pi_{G, p, q}(\omega) = \frac{1}{Z_{G, p, q}} p^{\vert\omega\vert} (1 - p)^{|E| - |\omega|} q^{c(\omega)},\]

where \(c(\omega)\) is the number of connected components in \((V, \omega)\) and \(Z_{G, p, q}\) is a normalizing constant.

As a corollary, we deduce fast mixing of the Swendsen–Wang dynamics for the Potts model on random \(\Delta\)-regular graphs for every \(q \geq 2\), in the tree uniqueness region.

ABSTRACT. We establish a near-optimal rapid mixing bound for the random-cluster Glauber dynamics on random \(\Delta\)-regular graphs for all \(q \geq 1\) and \(p < p_u(q, \Delta)\), where the threshold \(p_u(q, \Delta)\) corresponds to a uniqueness/non-uniqueness phase transition for the random-cluster model on the (infinite) \(\Delta\)-regular tree. It is expected that for \(q > 2\) this threshold is sharp, and the Glauber dynamics on random \(\Delta\)-regular graphs undergoes an exponential slowdown at \(p_u(q, \Delta)\).

Our focus here is on the Glauber dynamics of the random-cluster model. Specifically, we consider the following discrete-time Glauber dynamics chain, which we refer to as the FK-dynamics. For \(t \in \mathbb{N}\), from \(\omega_t \subseteq E\), one step of the FK-dynamics transitions to a new configuration \(\omega_{t+1} \subseteq E\) as follows:

1. Choose an edge \(e_t \in E\) uniformly at random;
2. Set \(\omega_{t+1} = \omega_t \cup \{e_t\}\) with probability \(\tilde{p} := \frac{p}{q(1 - p) + p}\) if \(e_t\) is a “cut-edge” in \((V, \omega_t)\); otherwise;
3. Otherwise set \(\omega_{t+1} = \omega_t \setminus \{e_t\}\).

We say \(e\) is a cut-edge in \((V, \omega_t)\) if changing the state of \(e_t\) changes the number of connected components \(c(\omega_t)\) in \((V, \omega_t)\). This chain is, by design, reversible with respect to \(\pi_{G, p, q}\).

Central question in the study of Markov chains is how the mixing time—defined as the number of steps until the Markov chain is close to stationarity—grows as the size of the graph \(G\) increases. Of particular interest in the context of random-cluster and Ising/Potts dynamics is the relation of mixing times to the rich equilibrium phase transitions of the model.

We consider this question when \(G\) is a random \(\Delta\)-regular graph on \(n\) vertices. The study of spin systems and their dynamics on random graphs is quite active \([10, 12–14, 17, 18, 22, 39, 40]\). Random \(\Delta\)-regular graphs
are a canonical example of graphs having exponential volume growth, with a non-trivial geometry, making them an attractive alternative to lattices or trees. More generally, the study of spin systems on random graphs yields insight into hard instances of the classical computational problems of sampling and counting, and features in the study of random constraint satisfaction problems.

The equilibrium phase transition of the random-cluster model on random $\Delta$-regular graphs is largely not understood, but it is expected to qualitatively resemble those on the $\Delta$-regular tree and the complete graph. Based on this relation, and understandings of those phase diagrams \cite{8, 30, 34, 36}, it is expected to involve three critical points $p_u(q, \Delta) \leq p_c(q, \Delta) \leq p_u^*(q, \Delta)$. Most relevant to us would be the critical threshold $p_u(q, \Delta)$ which corresponds to a uniqueness/non-uniqueness transition on the infinite $\Delta$-regular wired tree, i.e., the limit of the $\Delta$-regular trees whose leaves are externally wired to be in the same connected component. The threshold $p_u(q, \Delta)$ also matches the uniqueness threshold of the Ising/Potts model on the $\Delta$-regular tree. We mention that the threshold $p_c(q, \Delta)$ above corresponds to the equilibrium order-disorder transition, and $p_u^*(q, \Delta)$ to a second non-uniqueness/uniqueness transition on the wired tree; see \cite{22, 30, 32, 34}. When $q \in (1, 2]$ the phase transition is of second-order and these three thresholds coincide.

When $q > 2$, the phase transition on random $\Delta$-regular graphs is expected to be of first-order and $p_u(q, \Delta) < p_c(q, \Delta) < p_u^*(q, \Delta)$. In this case, the uniqueness threshold $p_u(q, \Delta)$ is conjectured to mark the onset of a metastability phenomenon, which should persist up to $p_u^*(q, \Delta)$ \cite{11}. Metastability has been linked to an exponential slowdown for both random-cluster and Potts Glauber dynamics in the mean-field setting \cite{5, 11, 21, 27}. Namely, the coexistence of ordered and disordered “metastable” sets with exponentially small boundary, generate states from which reversible Markov chains cannot easily escape (i.e., these sets have bad conductance). In analogy, it is expected that on random $\Delta$-regular graphs, for every $q > 2$, the FK-dynamics undergoes an exponential slowdown throughout the critical window $(p_u(q, \Delta), p_u^*(q, \Delta))$.

For $q$ sufficiently large, such slowdown was established in \cite{22} at $p = p_c(q, \Delta) \in (p_u(q, \Delta), p_u^*(q, \Delta))$.

Mossel and Sly \cite{40} showed that the Glauber dynamics for the Ising model (the $q = 2$ case) has optimal mixing when $p < p_u(2, \Delta)$ on any graph of maximum degree $\Delta$. Establishing $p_u(q, \Delta)$ as the sharp threshold for fast mixing of random-cluster or Potts Glauber dynamics for other values of $q$ on graphs of degree at most $\Delta$ remains completely open. In this paper, we establish near-optimal mixing for the FK-dynamics on random $\Delta$-regular graphs throughout the uniqueness regime for all real $q \geq 1$ and all $\Delta \geq 3$.

**Theorem 1.1.** Fix any $q \geq 1$, $\Delta \geq 3$, and $p < p_u(q, \Delta)$. Consider the FK-dynamics on a uniformly random $\Delta$-regular graph on $n$ vertices. With probability $1 - o(1)$ over the choice of the random graph $\mathcal{G}$, the mixing time of the FK-dynamics on $\mathcal{G}$ is $O(n (\log n)^2)$.

Our bound for the mixing time in this theorem is nearly optimal; we expect that the mixing time should be $\Omega(n \log n)$ by comparison to the coupon collector problem, as is the case for general spin systems \cite{31}.

The FK-dynamics have proven difficult to analyze with the known techniques for Markov chains for spin systems. This is due, in part, to the fact that the random-cluster model presents highly non-local interactions: in particular an update on an edge $e_t$ depends on the entire configuration $\omega_t(E \setminus \{e_t\})$ (c.f. the Ising/Potts Glauber dynamics where an update on a vertex $v$ depends only on the states of its neighbors). The only other setting where the speed of convergence of FK-dynamics is well-understood is in square subsets of the infinite 2-dimensional lattice \cite{4, 6, 23–26}. All other bounds for the FK-dynamics are obtained either indirectly, via comparison with global Markov chains using the results of \cite{44, 45} (and as a result, these bounds are off by polynomial factors), or by taking a perturbative parameter, either $p$ very small or very large, or $q$ large. This is the state of affairs even in the geometrically trivial mean-field model \cite{6, 27, 33}.

From the result in Theorem 1.1 we obtain an efficient MCMC sampling algorithm, for both the random-cluster model and the ferromagnetic Ising/Potts model on random $\Delta$-regular graphs in the uniqueness regime. The running time of the algorithm is $O(n (\log n)^2 \log(1/\delta))$, where $\delta \in (0, 1)$ is the desired accuracy for the algorithm. This an improvement over the best previously known sampling algorithm for these models (see \cite{3}); the algorithm in \cite{3} is a “weak sampler” in the sense that it outputs samples from a distribution that is close in total variation distance to the target distribution but with a fixed accuracy. (See also the recent work \cite{32} for a poly$(n)$ weak sampler for all $p \in (0, 1)$ provided $q$ is sufficiently large.)
As a corollary of Theorem 1.1 we also deduce fast mixing results for the standard Swendsen-Wang (SW) algorithm for the ferromagnetic $q$-state Potts model [43]. This is an extensively-used global-update Markov chain that in each step may update every site in a Potts configuration. This dynamics starts from a Potts configuration $\sigma_t \in \{1, \ldots, q\}^V$, moves to a “joint” spin/random-cluster configuration $(\sigma_t, \omega_t)$ by including all the monochromatic edges, and then assigns to each connected component of $(V, \omega_t)$ a uniform at random spin from $\{1, \ldots, q\}$ to obtain the Potts configuration $\sigma_{t+1}$ (see [16,43]).

**Corollary 1.2.** Fix any integer $q \geq 2$ and $\Delta \geq 3$, and let $p < p_u(q, \Delta)$. Consider the Swendsen-Wang dynamics on a uniformly random $\Delta$-regular graph on $n$ vertices. With probability $1 - o(1)$ over the choice of the random graph $\mathcal{G}$, the mixing time of the Swendsen–Wang dynamics on $\mathcal{G}$ is $O(n^2(\log n)^2)$.

Corollary 1.2 follows immediately from Theorem 1.1 and the comparison results of Ullrich [44,45]. Previously, our understanding of the speed of convergence of the SW dynamics on random $\Delta$-regular graphs was very limited. For the special case of $q = 2$, which corresponds to the Ising model, it was established in [2] that the spectral gap of the SW dynamics is $\Omega(1)$ for all $p < p_u(2, \Delta)$; this implies an $O(n)$ mixing time bound. In addition, Guo and Jerrum [29] established an $O(n^{10})$ mixing time bound for the SW dynamics that applies to any graph and any $p \in (0, 1)$. The methods in both of these works are specific to the Ising model ($q = 2$) and do not generalize to other values of $q$. Beyond the special case of $q = 2$, no sub-exponential bound was previously known for either the FK-dynamics or the SW dynamics throughout the uniqueness regime $p < p_u(q, \Delta)$.

**Proof ideas.** We comment briefly on the techniques and main innovations in our analysis next: for more details and an extended proof sketch, we refer the reader to Section 3. The main ingredient in our proof is an $O(n \log n)$ bound on the “shattering time” of the FK-dynamics (Theorem 3.1); this is the number of steps the chain requires to break up any configuration into connected components of size at most $O(\log n)$. The bound on the shattering time uses a novel and delicate iterative scheme to simultaneously reveal the underlying random graph and the connected components of the FK-dynamics configuration on it at a given time: see Definition 4.8 and Figures 4.1–4.2. While revealing procedures are a standard tool in the study of both random graphs and of the random-cluster model, their combined analysis is highly non-trivial, as the law of the random-cluster configuration at an edge depends on the global geometry of the graph. We are able to understand this joint revealing process, and deduce exponential tails on cluster sizes when $p < p_u$, not only for the random-cluster measure but also for the distribution of FK-dynamics after $O(n \log n)$ many steps. To our knowledge, this the first direct upper bound for the shattering time of the FK-dynamics in any setting. In fact, understanding the shattering time is usually the main obstacle for proving rapid mixing of the FK-dynamics on other graphs: e.g., on the complete graph, the shattering time is not known and only loose mixing time bounds (off by $\Theta(n^2)$ factors) can be derived [5].

Once the dynamics has shattered, we use standard methods (i.e., censoring and monotonicity [40, 41]) to reduce the analysis of the FK-dynamics to the analysis of localized dynamics in balls of radius $o(\sqrt{n})$ centered at each vertex, but with random boundary conditions induced by the current state outside the ball. In random $\Delta$-regular graphs, these balls are “treelike” and, after shattering, their boundary conditions are “almost free”, in that only $O(1)$ vertices in their boundaries are connected through the external configuration. This implies that the FK-dynamics mixes quickly in each of these balls. It would remain to couple the edge-values on these balls with those from the equilibrium distribution: this is typically achieved using exponential decay of correlations with the boundary (sometimes called spatial mixing) together with a union bound over the $n$ possible such balls. A delicate point is that since these balls only have radius $\Theta(\log n)$, we need sharp control on the rate of this exponential decay of influence to sustain the union bound over the $n$ balls. This bound, stated in Proposition 3.6 and proved in Section 5, may be of independent interest.

**Organization of paper.** The rest of the paper is organized as follows. In Section 2, we provide a number of preliminary definitions and notations we will use. In Section 3, we give a detailed proof overview highlighting some of the key novelties in our arguments. Our revealing procedures to bound the shattering time
are the focus of Section 4. In Section 5 we establish the sharp rate of spatial mixing on treelike graphs with sparse boundary conditions. Finally, we combine these to conclude the proof of Theorem 1.1 in Section 6.

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2. Preliminaries

In this section, we collect some standard definitions and properties that are necessary to present our proofs, and to which the reader can refer throughout. See the standard texts [9], [28], and [35] for more details on random graphs, the random-cluster model, and Markov chain mixing times, respectively.

2.1. Random $\Delta$-regular graphs. We begin by considering the underlying geometry we work on. Fix $\Delta \geq 3$ and consider the uniform measure $P_{\text{RGG}}$ over $\Delta$-regular graphs on $n$ vertices, $G = (V(G), E(G))$ with $|V(G)| = n$. (Let us always assume $n$ is such that $\Delta n$ is even, so that such a graph exists.) In order to place these graphs in the same probability space, we identify the vertices $V(G)$ with the set $\{1, ..., n\}$, and the randomness of $P_{\text{RGG}}$ will be over the edge-subset of $\{ij = ji : 1 \leq i, j \leq n\}$ taken as $E(G)$. Throughout this paper, we set $d := \Delta - 1$ for convenience.

Random graphs are treelike. A key ingredient in our proof is the fact that random $\Delta$-regular graphs are locally treelike. While this can be formalized in various ways, we use a notion that is most relevant to this paper, and applies uniformly to all vertices (as opposed to a vertex chosen uniformly at random).

For a graph $G = (V, E)$ and a vertex $v \in V(G)$, we define the ball of radius $R$ around $v$ as:

$$B_R(v) := \{w \in V(G) : d(w, v) \leq R\},$$

where $d(w, v)$ is the graph distance. For a vertex set, $B \subset V(G)$, define $E(B) = \{v, w \in B : vw \in E(G)\}$.

Definition 2.1. We say that a graph $G = (V, E)$ is $L$-Treelike if there is a set $H \subset E$ with $|H| \leq L$ such that the graph $(V, E \setminus H)$ is a tree.

Definition 2.2. We say that a $\Delta$-regular graph $G = (V(G), E(G))$ is $(L, R)$-Treelike if for every $v \in V(G)$ the subgraph $B_R(v) = (B_R(v), E(B_R(v)))$ is $L$-Treelike.

Fact 2.3. Fix any $\Delta \geq 3$. For every $\delta > 0$, there exists $L(\delta, \Delta)$ such that if $R = (1/2 - \delta) \log_d n$, we have

$$P_{\text{RGG}}(G \text{ is } (L, R)\text{-Treelike}) = 1 - o(n^{-1}).$$

Compare this to the diameter of the random graph, $\log_d n + O(\log \log n)$ [9, Section 10.3] and note that for $\frac{R}{\log_d n} > \frac{1}{2}$, there will be a diverging number of cycles in every ball $B_R(v)$.

The configuration model. Fact 2.3 is a standard fact about random $\Delta$-regular graphs; for completeness, we include a proof in Section 4.2 using the configuration model, via which much of the analysis of random $\Delta$-regular graphs proceeds. The configuration model $P_{\text{CM}}$ is a distribution over multigraphs on $n$ vertices and fixed degree distribution, which we take to be $\Delta$ for every vertex, defined as follows [7]. Give every vertex $v \in \{1, ..., n\}$ $\Delta$-half-edges and select a matching on the $\Delta n$ many half-edges uniformly at random to form the $\Delta n/2$ edges of the graph. Let $\mathcal{M}_n$ be the set of possible edges (the set of pairs of half-edges).

The configuration model is a useful tool for studying the random $\Delta$-regular graph, as the distribution $P_{\text{RGG}}$ is equal to the distribution $P_{\text{CM}}(\cdot | G \in \Gamma_{\text{RGG}})$ where $\Gamma_{\text{RGG}}$ is the event that the graph $G$ is simple (i.e., has no self-loops or multi-edges). In particular, it is standard (see e.g., [7]) that $P_{\text{CM}}(\Gamma_{\text{RGG}}) > c$ for some $c(\Delta) > 0$, and therefore for any event $\Gamma$,

$$P_{\text{RGG}}(\Gamma) = P_{\text{CM}}(\Gamma | \Gamma_{\text{RGG}}) (P_{\text{CM}}(\Gamma_{\text{RGG}}))^{-1} \leq c^{-1} P_{\text{CM}}(\Gamma).$$  \hspace{1cm} (2.1)

Refer to the book [20] for more on the configuration model. We will use (2.1), with an iterative revealing scheme of a matching of the $\Delta n$ half-edges, to analyze the random $\Delta$-regular graph.
2.2. The random-cluster model. For a graph $G = (V, E)$, recall the definition of the random-cluster model from (1.1). The definition extends to naturally to multigraphs, where $G = (V, E)$ for a multiset $E$. This is the distribution over subsets of $E$, identified with configurations $\omega : E \to \{0, 1\}$. We say an edge $e \in E$ is open or wired if $\omega(e) = 1$ and closed or free if $\omega(e) = 0$. We say two vertices are connected in $\omega$ if they are in the same connected component of the sub-graph $(V, \{ e \in E : \omega(e) = 1 \})$. For a vertex set $V \subset \mathcal{V}$, denote by $\mathcal{C}_V(\omega)$ the union of connected components (clusters) containing $v \in V$ in this sub-graph. For a configuration $\omega$ and edge set $A \subset E$, we use $\omega(A)$ for the restriction of $\omega$ to $A$.

**Boundary conditions.** To help study the random-cluster measure, we introduce boundary conditions.

**Definition 2.4.** A random-cluster boundary condition $\xi$ on $G = (V, E)$ is a partition of $V$, such that the vertices in each element of the partition are identified with one another. The random-cluster measure with boundary conditions $\xi$, denoted $\pi_{G,p,q}^\xi$, is the same as in (1.1) except the number of connected components $c(\omega) = c(\omega; \xi)$ would be counted with this vertex identification, i.e., if $v, w$ are in the same element of $\xi$, they are always counted as being in the same connected component of $\omega$ in (1.1). In this manner, the boundary condition can alternatively be seen as ghost “wirings” of the vertices in the same element of $\xi$.

The free boundary condition, $\xi = 0$, is the one whose partition consists only of singletons. For a subset $\partial V \subset V$, the wired boundary conditions on $\partial V$, denoted $\xi = 1$, are those whose partition has all vertices of $\partial V$ in the same element. For boundary conditions $\xi, \xi'$ we say that $\xi \leq \xi'$ if $\xi$ is a finer partition than $\xi'$.

**Domain Markov and FKG properties.** The domain Markov property of the random-cluster model is expressed as follows: for an edge-subset $H \subset E$, and a configuration $\eta$ on $E \setminus H$, we have

$$\pi_{G,p,q}^\xi(\{ e \in H \mid \omega(E \setminus H) = \eta \}) \leq \pi_{H,p,q}^\xi(\omega \in \cdot),$$

where the boundary conditions $(\xi, \eta)$ are those induced by the configuration $\eta$, whereby two vertices in $H$ are in the same boundary element, if they are connected through the configuration $\eta$ together with the external wirings of $\xi$. This is to say that the random-cluster measure conditioned on a configuration in $E \setminus H$ is the same as a random-cluster measure on $H$ with corresponding induced boundary conditions.

The random-cluster model at $q \geq 1$ also satisfies an important monotonicity property known as the FKG inequality. For any two increasing events $A, B \subset \{0, 1\}^E$, we have $\pi_{G,p,q}^\xi(A, B) \geq \pi_{G,p,q}^\xi(A)\pi_{G,p,q}^\xi(B)$. As a consequence of this, we have the following monotonicity in boundary conditions: for any two boundary conditions $\xi$ and $\xi'$ with $\xi \geq \xi'$, we have $\pi_{G,p,q}^\xi \succeq \pi_{G,p,q}^\xi$ where $\succeq$ denotes stochastic domination.

**Uniqueness/non-uniqueness transition on the $\Delta$-regular tree.** As the geometry of the random graph is locally treelike, its dynamical transition point should be inherited from a transition on the $\Delta$-regular tree. Throughout this paper, we denote by $T_h := T_{h,\Delta} := (V(T_h), E(T_h))$ the rooted (at $\rho$) $\Delta$-regular complete tree of depth $h$ (the root has $\Delta$ children, and all other vertices have $\Delta - 1$ children and one parent). Since the tree has depth $h < \infty$, evidently it is not actually $\Delta$-regular, and has leaves $\partial T_h = \{ w \in V(T_h) : d(\rho, w) = h \}$ (where $d(\cdot, \cdot)$ denotes graph distance); observe that

$$|V(T_h)| = 1 + \Delta \sum_{i=1}^h d^{i-1} \leq 2\Delta d^h,$$

and $|E(T_h)| = \Delta \sum_{i=1}^h d^{i-1} \leq 2\Delta d^h$, \hspace{2cm} (2.2)

and $|\partial T_h| = \Delta d^{h-1}$. The wired boundary condition “1” is the one that wires all vertices of $\partial T_h$ together.

For every $\Delta \geq 3$ and $q \geq 1$, the random-cluster measure $\pi_{T_h, p,q}^1$ is known to undergo a transition at

$$p_u(q, \Delta) := \frac{1}{1 + \inf_{y > 1} g(y)}, \hspace{2cm} \text{where} \hspace{2cm} g(y) := \frac{(y - 1)(y^{\Delta - 1} + q - 1)}{y^{\Delta - 1} - y}, \hspace{2cm} (2.3)$$

when $p < p_u(q, \Delta)$, the probability that $\rho$ is connected to $\partial T_h$ in $\omega$ goes to 0 as $h \to \infty$, whereas when $p > p_u(q, \Delta)$ this probability stays bounded away from 0 (see [3, 30]). We note that $p_u(q, \Delta)$ does not in general have a closed form.
A key fact we will use is that whenever $p < p_u(q, \Delta)$ we have that
\[ \hat{p} := \frac{p}{q(1 - p) + p} < \frac{1}{d}, \quad \text{where} \quad d := \Delta - 1; \]
recall from the definition of the FK-dynamics that $\hat{p}$ is exactly the probability that $\omega(e) = 1$ conditionally on $e$ being a cut-edge in $(\omega, \xi)$. In uniqueness, most edges in the wired tree will be cut-edges, suggesting $\hat{p}$ should be the exponential decay rate of root-to-leaf connectivity on $T_h$: the following lemma establishes this fact. Let $(1, \varnothing)$ denote the wired boundary condition on $\partial T_h$ that additionally wires the root $\rho$ to $\partial T_h$.

**Lemma 2.5.** Let $p < p_u(q, \Delta)$. There exists $C(p, q, \Delta)$ such that for every $h$ and every leaf $u \in \partial T_h$,
\[ \pi^{(1, \varnothing)}_{\hat{T}_h}(\omega : u \in C_\rho(\omega)) \leq C\hat{p}^h. \]

In particular, the probability of the root being connected to $\partial T_h$ in $\omega$ is at most $C(\hat{p}d)^h$.

**Remark 2.6.** For $q \in (1, 2]$, we have $p_u(q, \Delta) = p_c(q, \Delta)$ and as $p \uparrow p_u$, we have $\hat{p} \uparrow 1$. When $q > 2$, as $p \uparrow p_u$, $\hat{p}$ stays uniformly bounded away from 1.

Lemma 2.5 is obtained by analyzing a recursion on the wired tree. The fact that we are in the tree uniqueness region [30] implies decay to zero as $h \to \infty$ in the above probability, and a recursion for this connectivity probability was calculated in [3, Lemma 33]. Our sharp bound on the decay rate follows straightforwardly from a more careful examination of this recursion: we defer its proof to Section 5.1.

### 2.3. Markov chain mixing times.

Consider a (discrete-time) Markov chain with transition matrix $P$ on a finite state space $\Omega$, reversible with respect to an invariant distribution $\pi$; denote the chain initialized from $x_0$ by $(X_t^{x_0})_{t \geq 0}$. Its **mixing time** is given by
\[ t_{\text{mix}} = t_{\text{mix}}(1/4), \quad \text{where} \quad t_{\text{mix}}(\varepsilon) = \min \{ t : \max_{x_0 \in \Omega} \| P(X_t^{x_0} \in \cdot) - \pi \|_{TV} \leq \varepsilon \}. \]

It is standard from the triangle inequality and Markov property that the total variation distance above is sub-multiplicative and $t_{\text{mix}}(\delta) \leq t_{\text{mix}}(\delta)^{\log(2\delta^{-1})}$.

By definition of total variation distance, in order to upper bound the mixing time, it suffices to bound the coupling time of the dynamics; i.e., if we construct a coupling $\hat{P}$ of the steps of the chain such that for each $x_0, x'_0 \in \Omega$, we have $\mathbb{P}(X_t^{x_0} \neq X_t^{x'_0}) \leq 1/4$, then $t_{\text{mix}} \leq T$.

**A grand coupling for the FK-dynamics.** Recall the definition of the FK-dynamics from the introduction. Note that in the presence of boundary conditions $\xi$, the only change is that in step (2) of the FK-dynamics transitions, the status of $e$ being a cut-edge is dictated by whether its presence changes $c(\omega, \xi)$. As with the random-cluster measure, this is naturally defined on multigraphs as well.

For the FK-dynamics, there is a canonical choice of coupling known as the **grand coupling**, i.e., a simultaneous coupling of the Markov chains $(X_t^{x_0})_t$ indexed by their initial configuration $x_0 \in \{0, 1\}^E$, defined as follows. Let $(U_{e,s})_{e \in E, s \geq 0}$ be a family of i.i.d. Unif$[0, 1]$ random variables. Run the chains $(X_t^{x_0})_t$ (indexed by their initial configuration $x_0 \in \{0, 1\}^E$) simultaneously by making the same choice of edge $e_t$ to update at each time, and using the same uniform random variable $U_{e_t,t}$ to decide its next state. (See Definition 4.5 for a more explicit explanation.) A key property of this coupling is that for $q \geq 1$, it is a monotone coupling, i.e., if $\omega \leq \omega'$, we have $X_t^{\omega} \leq X_t^{\omega'}$ for all $t \geq 0$. Refer to, e.g., [28] for more details.

**Boundary condition comparisons for the FK-dynamics.** After shattering, we find that the FK chain induces sparse boundary conditions on balls of volume $o(\sqrt{n})$, i.e., boundary conditions with at most $O(1)$ non-singleton components. The following formalizes the notion that such boundary conditions are “close to free”, and allows us to compare the induced mixing time on the ball to that with free boundary conditions.

**Definition 2.7** (Definition 2.1 of [4]). For two boundary conditions (partitions) $\phi \leq \phi'$, define $D(\phi, \phi') := c(\phi) - c(\phi')$ where $c(\phi)$ is the number of components in $\phi$. For two partitions $\phi, \phi'$ that are not comparable, let $\phi''$ be the smallest partition such that $\phi'' \geq \phi$ and $\phi'' \geq \phi'$ and set $D(\phi, \phi') = c(\phi) - c(\phi'') + c(\phi') - c(\phi'')$. 
Lemma 2.8 (Lemma 2.2 of [4]). Let $G = (V, E)$ be an arbitrary graph, $p \in (0, 1)$ and $q > 0$. Let $\phi$ and $\phi'$ be two partitions of $V$ encoding two distinct external wirings on the vertices of $G$. Let $\pi^\phi_G$, $\pi^{\phi'}_G$ be the resulting random-cluster measures. Then, for all FK configurations $\omega \in \{0, 1\}^E$, we have

$$q^{-2D(\phi, \phi')} \pi^{\phi'}_G(\omega) \leq \pi^\phi_G(\omega) \leq q^{2D(\phi, \phi')} \pi^{\phi'}_G(\omega).$$

From Lemma 2.8, and the definition of the Dirichlet form

$$\mathcal{E}(f, f) := \frac{1}{2} \sum_{\omega, \omega' \in \{0, 1\}^E} \pi(\omega) P(\omega, \omega')(f(\omega) - f(\omega'))^2,$$

where $P$ is the transition matrix for the FK-dynamics and $f : \{0, 1\}^E \to \mathbb{R}$, we obtain the following.

Corollary 2.9. Let $G = (V, E)$ be an arbitrary graph, $p \in (0, 1)$ and $q > 0$. Consider the FK-dynamics on $G$ with boundary conditions $\phi$ and $\phi'$, and let $\mathcal{E}_{G\phi}$, $\mathcal{E}_{G\phi'}$ denote their Dirichlet forms, respectively. Then

$$q^{-4D(\phi, \phi')} \mathcal{E}_{G\phi'}(f, f) \leq \mathcal{E}_{G\phi}(f, f) \leq q^{4D(\phi, \phi')} \mathcal{E}_{G\phi'}(f, f), \quad \text{for all } f : \{0, 1\}^E \to \mathbb{R}.$$

Together with Corollary 2.9 and Lemma 2.8 again, this will control the change in the log-Sobolev constant (6.1), and therefore mixing time, under two boundary conditions with distance $D(\phi, \phi')$.

3. Extended proof sketch

In this section, we provide a detailed sketch of our proof of fast mixing for the FK-dynamics on $\Delta$-regular random graphs. This will serve to both outline the structure of the argument, and highlight some of the key technical difficulties encountered and the novel arguments we introduce to handle them.

3.1. Proof outline. Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be an $n$-vertex graph. Let $(X^1_{\mathcal{G}, t})_{t \geq 0}$ and $(X^0_{\mathcal{G}, t})_{t \geq 0}$ denote two instances of the FK-dynamics started from the all-wired and all-free configurations, respectively, coupled under the grand coupling. We aim to show that if $T > Cn(\log n)^2$ for a suitable constant $C > 0$, then for every vertex $v \in V(\mathcal{G})$, with probability $1 - o(n^{-1})$ the configurations at time $T$, $X^1_{\mathcal{G}, T}$ and $X^0_{\mathcal{G}, T}$, agree on the $\Delta$ edges incident $v$, collected in

$$\mathcal{N}_v := \{e \in E(\mathcal{G}) : v \in e\}. \quad (3.1)$$

A union bound over the $n$ vertices would imply that with high probability, $X^1_{\mathcal{G}, T} = X^0_{\mathcal{G}, T}$ on all of $E(\mathcal{G})$, and by monotonicity of the FK-dynamics under the grand coupling, the chain would be mixed; see Section 2.

There are two key stages to establishing this coupling on $\mathcal{N}_v$ for a fixed $v \in V(\mathcal{G})$.

I. After a burn-in period of $T_{\text{burn}} = O(n \log n)$ the chain $(X^1_{\mathcal{G}, t})$ which starts from the all-wired configuration is shattered; that is, its connected components have exponential tails on their size and, in particular, with high probability every component is of size $O(\log n)$. We further show that these clusters are sufficiently independent of one another, and sufficiently independent of the underlying random graph, so that if we fix the ball of radius $R = (\frac{1}{2} - \delta) \log_d n$ about $v$, the boundary conditions induced on that ball by $X^1_{\mathcal{G}, t}(E(\mathcal{G}) \setminus B_R(v))$ are sparse: they have only $O(\delta^{-1})$ many boundary wirings.

II. At time $T_{\text{burn}}$, we freeze the two distinct boundary conditions induced by the chains $(X^1_{\mathcal{G}, t})$ and $(X^0_{\mathcal{G}, t})$ on the boundary of $B_R(v)$, and continue running the chains inside $B_R(v)$. We show that if both boundary conditions are sparse (with up to $O(1)$ vertices wired through the boundary), and the random graph $\mathcal{G}$ is $(L, R)$-Treelike per Fact 2.3, we have that:

i. The mixing time of the FK-dynamics on $B_R(v)$ is $O(d^R \log d^R)$ (see Lemma 3.5).

ii. Once mixed, the TV-distance between the marginals the two sparse boundary conditions on $B_R(v)$ induce on $\mathcal{N}_v$ is $O(p^2 d^R)$, which for $\delta$ small enough depending on $p, q, \Delta = o(n^{-1})$ since $p < 1/d$.

We remark that to freeze the boundary conditions outside the ball $B_R(v)$ and localize the processes, we use the censoring inequality from [41], phrased for the FK-dynamics in, e.g., [26]. We expand on the key ideas in stages I–II in Sections 3.2–3.3 respectively.
3.2. Stage I: Bounding the shattering time. Here, we describe the main idea in the proof of the shattering of the FK-dynamics \((X_{G,t}^1)\) after a burn-in period of \(O(n \log n)\) steps. The analysis of this burn-in phase contains many of the key novelties of the paper. Before proceeding, let us first emphasize the subtlety of analyzing the burn-in phase up to \(p_n\) by noting the following. If we naively dominate \((X_{G,t}^1)\) by a process \((Z_t)\) where every edge is independently open with probability \(p\) and closed otherwise, after every site has been updated \((Z_t)\) will not be shattered when \(p\) is sufficiently close to the uniqueness threshold of \(p_n\).

Roughly, the intuition behind our proof of shattering after a burn-in phase goes as follows. Firstly, note that on a ball \(B_r(v)\), say for \(r = O(1)\) sufficiently large, the chain \((X_{G,t}^1)\) is stochastically dominated by an auxiliary chain on the ball \(B_r(v)\) with the all-wired boundary condition outside of \(B_r(v)\). Since the mixing time on that ball is \(O(1)\) \((r\) is constant\), for \(t\) larger than \(O(n \log n)\), the configuration \(X_{G,t}^1\) can be covered by a collection of coupled samples \(\tilde{\omega}_{B_r(v)}\) from the family of random-cluster measures \(\{\pi_{B_r(v)}\}_{v \in V(G)}\).

In theory, one could then explore the coupled configurations \(\tilde{\omega}_{B_r(v)}\) out from a fixed vertex \(v\) to bound the connected component of vertex \(v\) in \(X_{G,t}^1\). Indeed, since most \(O(1)\)-sized balls that one encounters in a random \(\Delta\)-regular graph are trees, by Lemma 2.5, the expected number of sites along \(\partial B_r(v)\) connected to \(v\) in \(\tilde{\omega}_{B_r(v)}\) is at most \(O((\tilde{d}d)^r)\) which can be made smaller than one when \(p < p_u(q, \Delta)\) and \(r\) sufficiently large. This hints at a natural comparison to a sub-critical branching process; however, there is a fundamental difficulty in the fact that not every such ball is a tree, and there are strong correlations between the short cycles of the underlying graph and the places where the overlayed random-cluster configuration on \(B_r(v)\) is more wired. A key contribution of our work is to construct a simultaneous revealing procedure for the random graph \(G \sim \mathbb{P}_{CM}\) and an overlayed random-cluster component from \(X_{G,t}^1\) in a manner that handles these dependencies and is still comparable to a sub-critical branching process when \(p < p_u\); see Definition 4.8.

The following exponential tail bound (shattering estimate) on cluster sizes of \(X_{G,t}^1\) after a burn-in phase follows from this comparison to a sub-critical (non-Markovian and size-dependent) branching process.

**Theorem 3.1.** Suppose \(p < p_u(q, \Delta)\). For every \(K\), there exists \(C(p, q, \Delta, K) > 0\) such that for every \(v \in \{1, \ldots, n\}\), with \(\mathbb{P}_{\text{RG}}\)-probability \(1 - O(e^{-k/C} + n^{-K})\), \(G\) is such that for all \(t \geq C n \log n\) and \(k \geq 1\),
\[
P(X_{G,t}^1 : |C_v(X_{G,t}^1)| \geq k) \leq C \exp(-k/C) + C n^{-K}.
\]

The same tail bound applies to \(\{\omega : |C_v(\omega)| \geq k\}\) under \(\omega \sim \pi_{G}\) by monotonicity of the FK-dynamics.

By a union bound, Theorem 3.1 implies that the largest components of \(X_{G,t}^1\) and \(\omega \sim \pi_{G}\) are of size at most \(O(\log n)\) with high probability. This theorem is proved in Section 4.

Using Theorem 3.1, together with sufficient independence of the connected component \(C_v\) of the vertex \(v\) from the geometry of \(G\) (for typical instantiations of \(G\)), we can show that the boundary condition \(X_{G,t}^1\) induces on a ball of radius \(R = (\frac{1}{2} - \delta) \log_d n\) about a vertex \(v\) is sparse with high probability. Intuitively, the probability that the cluster of any \(w \in \partial B_R(v)\) intersects \(\partial B_R(v) \setminus \{v\}\) is \(O(n^{-\frac{1}{2} - \delta})\), so that the expected number of non-trivial boundary components induced by \(X_{G,t}^1\) on the boundary of \(B_R(v)\) is \(o(1)\). We introduce the following notion of \(K\)-Sparse boundary conditions and obtain a tail bound on this quantity.

**Definition 3.2.** A random-cluster boundary condition \(\xi\) on an edge-subset \(H \subset E(G)\) is said to be \(K\)-Sparse if the number of vertices in non-trivial (non-singleton) boundary components of \(\xi\) is at most \(K\).

**Definition 3.3.** A random-cluster configuration \(\omega\) on \(G = (V(G), E(G))\) is \((K, R)\)-Sparse if, for every \(v \in V(G)\), the boundary conditions induced on \(B_R(v)\) by \(\omega(E(G) \setminus E(B_r(v)))\) are \(K\)-Sparse.

The next result asserts that the boundary of every ball about a vertex is \(O(1)\)-Sparse with high probability after the burn-in phase: this is proven in Section 4.5.

**Theorem 3.4.** Fix \(p < p_u(q, \Delta)\). There exists \(C(p, q, \Delta)\) such that for every \(t \geq C n \log n\), the following holds. For every \(\delta > 0\), if \(R := (\frac{1}{2} - \delta) \log_d n\), there exists \(K(p, q, \Delta, \delta)\) such that with \(\mathbb{P}_{\text{RG}}\)-probability \(1 - o(1)\), \(G\) is such that
\[
P(X_{G,t}^1 \text{ is } (K, R)\text{-Sparse}) \geq 1 - n^{-2}.
\]
3.3. **Stage II: Coupling from \( K \)-Sparse boundary conditions.** With Theorem 3.4 in hand, it remains to show that once the configurations \( X^1_{\hat{G},t} \) and \( X^0_{\hat{G},t} \) are \((K,R)\)-Sparse, they will couple on \( \mathcal{N}_v \) except with probability \( 1 - o(n^{-1}) \) in \( O(n(\log n)^2) \) time. As described earlier, the proof of this breaks into two parts (i) a bound on the mixing time on \( B_R(v) \) with sparse boundary conditions, and (ii) a sharp bound on the difference of marginals induced on \( \mathcal{N}_v \) by different sparse configurations outside \( B_R(v) \).

The first of these is formalized in the following lemma, whose proof follows by using Lemma 2.8 and Corollary 2.9 to relate the log-Sobolev constant on \( B_R(v) \) to that on the free tree of depth \( R \).

**Lemma 3.5.** Suppose \( B_R(v) \) is \( L \)-Treelike. Let \( \xi \) be a \( K \)-Sparse boundary condition on \( \partial B_R(v) \). For every \( p \in (0,1) \) and \( q > 0 \), the mixing time of the FK-dynamics on \( B_R(v) \) with boundary condition \( \xi \) is \( O(Rd^R) \).

With the optimal bound on the local mixing time on treelike balls, after this mixing time, the probability that two instances of the FK-dynamics on \( B_R(v) \) for distinct \( K \)-Sparse boundary conditions \( \xi \) and \( \xi' \) are not coupled on \( \mathcal{N}_v \) is given by the total variation distance between \( \pi_{B_R(v)}^{\xi} \) and \( \pi_{B_R(v)}^{\xi'} \). This entails obtaining the following sharp bounds on the correlation decay, or spatial mixing, rate under sparse boundary conditions.

**Proposition 3.6.** Consider a vertex \( v \) in a \( \Delta \)-regular graph \( G \). For each \( K \), there exists \( C(p,q,\Delta,K,L) \) such that for all \( R \), if \( B_R(v) \) is \( L \)-Treelike and \( \xi, \xi' \) are any two \( K \)-Sparse boundary conditions on \( \partial B_R(v) \),

\[
\| \pi_{B_R(v)}^{\xi}(\omega(\mathcal{N}_v) \in \cdot) - \pi_{B_R(v)}^{\xi'}(\omega(\mathcal{N}_v) \in \cdot) \|_{TV} \leq C \hat{p}^{2R}.
\]

A matching lower bound of \( \Omega(\hat{p}^{2R}) \) for the decay rate is easy to construct: see Remark 5.5.

We stress the importance of obtaining the sharp \( \hat{p}^{2R} \) rate here for the spatial mixing to support a union bound over \( n \) vertices for \( R \leq \frac{1}{2} \log q n \): any larger \( R \) would cross the threshold at which point random graphs are no longer \( O(1) \) or even sub-polynomially Treelike, and we would lose control over the mixing time on \( B_R(v) \). The proof of Proposition 3.6 uses a delicate revealing and resampling argument to show that in order for information to travel from the boundary to \( \mathcal{N}_v \) there must be two disjoint open paths from \( \mathcal{N}_v \) to non-singleton elements of \( \xi \) in \( \partial B_R(v) \). We compare this to the more traditional bound on influence by the existence of a single connection from the center of a ball to its boundary, which in our setting would yield a bound of \( \hat{p}^R \). (Such a bound by a single connectivity event is the one traditionally used on amenable graphs like \( \mathbb{Z}^2 \) to go from spatial mixing with any positive rate of exponential decay to fast mixing: see [1,6,37].)

4. **The FK-dynamics shatters quickly on random graphs**

Our goal in this section is to prove Theorem 3.1 and deduce Theorem 3.4. Denote by \( t_{\text{Mix}}(\mathcal{T}_r, (1, \bigcirc)) \) the mixing time of FK-dynamics on \( \mathcal{T}_r \) with \((1, \bigcirc)\) boundary conditions, and define the burn-in time

\[
T_{\text{Burn}} = T_{\text{Burn}}(C_0, r) := C_0 n \log n \cdot \frac{t_{\text{Mix}}(\mathcal{T}_r, (1, \bigcirc))}{|E(\mathcal{T}_r)|}.
\]

We show that for a typical \( n \)-vertex \( \Delta \)-regular random graph \( G \sim P_{\text{rrg}} \), there exist constants \( C_0 \) and \( r \) large (depending only on \( p,q,\Delta \)) such that for \( t \geq T_{\text{Burn}}(C_0, r) \) the configuration \( X^1_{G,t} \) is such that its connected components are shattered. In particular, we will conclude from this that the boundary conditions \( X^1_{G,t} \) induces on any ball of volume \( o(\sqrt{n}) \) are \( O(1) \)-Sparse per Definition 3.2.

**Definition 4.1.** For a subgraph \( H = (V(H), E(H)) \) of \( G \) and a configuration \( \omega \) on \( E(G) \), let us define \( \mathcal{V}_H(\omega) \) as the subset of vertices in \( V(H) \) in non-trivial components in the boundary condition induced on \( H \) by \( \omega(E(G) \setminus E(H)) \) (a connected component is non-trivial when it has at least two vertices).

In order to deduce the desired results, we will average over the underlying random graph. It will be useful to work with the configuration model \( P_{\text{cm}} \) (as defined in Section 2) rather than \( P_{\text{rrg}} \) as it lends itself to iterative revealing schemes. The aim of this section is to prove the following bound on the non-trivial components induced on \( B_R(v) \) for \( R = o(\sqrt{n}) \). Recall that we set \( d = \Delta - 1 \).
Proposition 4.2. Let $p, q, \Delta$ be such that $p < p_w(q, \Delta)$. Fix $\delta > 0$ and let $R = (\frac{1}{2} - \delta) \log_q n$. There exists $K(p, q, \Delta, \delta)$ such that if $G \sim P_{\text{CM}}$, with probability $1 - O(n^{-2})$, $G$ is such that for all $t \geq T_{\text{BURN}}$
\[
\sup_{v \in V(G)} P\left(X^{1}_{G,t} : |\mathcal{M}_{B_{R}(v)}(X^{1}_{G,t})| > K \right) \leq O(n^{-2}).
\]

Proposition 4.2 will straightforwardly imply Theorem 3.4 via (2.1) and a union bound over the $n$ vertices. The proof of Proposition 4.2 relies on a revealing procedure which simultaneously reveals the edges of the random graph $G \sim P_{\text{CM}}$ and of a random-cluster configuration that stochastically dominates $X^{1}_{G,t}$. In Section 4.1, we focus on the key ideas in this argument, outlining the proof of Proposition 4.2 by constructing the relevant revealing procedures for FK-dynamics clusters on random graphs.

4.1. Couplings and revealing schemes for the FK-dynamics on random graphs. In this section, we summarize the key couplings and revealing schemes for the connected components of $X^{1}_{G,t}$. These are fundamental to the proof of shattering for $X^{1}_{G,t}$ in the uniqueness region after an $O(n \log n)$ burn-in time.

4.1.1. Revealing procedure for the $\Delta$-regular configuration model. Towards introducing the joint revealing procedure for the random graph $G \sim P_{\text{CM}}$ and the configuration $X^{1}_{G,t}$, let us first recall a standard revealing procedure for random $\Delta$-regular graphs according to $P_{\text{CM}}$ on its own. This procedure is useful to proving random graph estimates for the configuration model and $\Delta$-regular random graph. It also serves as a building block for the revealing procedure of the random graph together with the FK-dynamics configuration.

The following iterative algorithm is a way to sample from the configuration model for a given degree sequence. The fact that this gives a valid sample from $P_{\text{CM}}$ is straightforward after naturally identifying samples from $P_{\text{CM}}$ with samples from the uniform distribution over matchings on $\Delta n$.

**Definition 4.3.** Assign every $v \in \{1, \ldots, n\}$, $\Delta$ half-edges. Suppose $f$ is a (possibly random) function from edge-sets $A \subset \mathcal{M}_n$ (a set of matched pairs of half-edges) to a half-edge not matched in $A$.

1. Initialize the set of exposed edges as $A_0 = \emptyset$.
2. For every $1 \leq m \leq \frac{n}{2\Delta}$, choose the un-matched half-edge $f(A_{m-1})$ and match it to a half-edge selected uniformly at random from the remaining un-matched ones to form the edge $e_m$.
3. Let $A_m := A_{m-1} \cup \{e_m\}$.

This process concludes when all half-edges are matched.

Observe, importantly, that the choice of next half-edge to match (given by the function $f$) can be adaptive, specifically, adapted to the filtration generated by $(A_0, \ldots, A_{m-1})$.

Definition 4.3 provides an adaptive sampling method from the configuration model distribution $P_{\text{CM}}$ (see e.g., [38]) and can be used to prove myriad properties of random $\Delta$-regular graphs. In particular, it yields a simple proof of Fact 2.3 that $G \sim P_{\text{RGG}}$ is $(L, R)$-Treelike for $R \leq n^{3/2 - \delta}$ and $L = O(1)$; see Section 4.2.

4.1.2. A coupling of localized FK-dynamics chains. Our goal is to simultaneously expose edges of $G \sim P_{\text{CM}}$ while revealing the FK-dynamics configuration $X^{1}_{G,t}$ at time $t$ on $G$. We show that under their joint distribution the size of the connected components of $X^{1}_{G,t}$ have exponential tails; this in turn implies that the boundary condition on $B_{r}(v)$ is $O(1)$-Sparse (see Definition 3.2).

Note that a ball of radius $O(\log n)$ about a vertex $v$ may have many cycles—indeed it may encompass the entire graph $G$—but a typical FK cluster of size $O(\log n)$ does not use most of these cycles. Thus, we expose the edges of $P_{\text{CM}}$ guided by the revealing of the random-cluster component of a vertex $v$ in $X^{1}_{G,t}$; in this way, to expose the $C_v(X^{1}_{G,t})$ we will not have to reveal much of the random graph.

There are two key difficulties to consider when constructing a joint revealing process for $(G, X^{1}_{G,t})$:

1. Under either of $X^{1}_{G,t}$ or e.g., the random-cluster measure $\pi_G$, the value $\omega(e)$ on an edge $e$ shown to belong to $E(G)$, affects the distribution of the remainder of the underlying random graph.
2. Unlike the random-cluster measure $\pi_G$, the law of $X^{1}_{G,t}$ does not satisfy any domain Markov property. Indeed, the distribution of $X^{1}_{G,t}(e)$ conditionally on some $X^{1}_{G,t}(A)$ is quite difficult to analyze.
The key to overcoming these obstructions will be to reveal the configurations of a family of FK-dynamics chains that are localized (in the sense that their distribution only depends on a small $O(1)$ sized subset of edges of the graph) and whose concatenation stochastically dominates the distribution of $X_{G,t}^1$. Let us be more precise next and explicitly construct a coupling of a family of localized FK-dynamics chains.

**Definition 4.4.** For a graph $G$ and edge subset $A \subset E(G)$, let $\partial A$ be the set of vertices in $V(A)$ that are adjacent to vertices of $V(G) \setminus V(A)$, and let $\pi_A^1$ be the random-cluster measure on $A$ with wired boundary conditions on $\partial A$. Let $(X_{A,t}^1)_{t \geq 0}$ be the FK-dynamics chain that starts from all wired on $E(G)$, censors (ignores) all updates in $(1) \setminus A$, and makes FK-dynamics updates w.r.t. $\pi_A^1$ when it updates edges in $A$.

Importantly, the wiring on $\partial A$ ensures that the law of $X_{A,t}^1$ is does not depend on $E(G) \setminus A$.

**Definition 4.5.** For any graph $G$, we can construct $(X_{A,t}^1)_{t \geq 0} = \{(X_{A,t}^1)_{t \geq 0}\}_{A \in E(G)}$, a grand monotone coupling of the ensemble of FK-dynamics chains $(X_{A,t}^1)_{t \geq 0}$ for $A \subset E(G)$ as follows:

1. Initialize $X_{A,0}^1 \equiv 1$ for all $A \subset E(G)$; i.e., the all wired configuration on $A$.
2. Let $(e_t)_{t \geq 1} = (e_1, e_2, \ldots)$ be drawn i.i.d. from $E(G)$.
3. Let $(U_{e,t})_{e \in E(G), t \geq 1}$ be a sequence of i.i.d. uniform random variables on $[0, 1]$.

For $A \subset E(G)$, construct $(X_{A,t}^1)_{t \geq 1}$ as follows: for each $t \geq 1$, set $X_{A,t}^1(e) = X_{A,t-1}^1(e)$ for $e \neq e_t$ and

$$X_{A,t}^1(e_t) = \begin{cases} 1 & \text{if } e_t \notin A; \\ 1 & \text{if } e_t \in A \text{ and } U_{e_t,t} \leq \varrho; \\ 0 & \text{if } e_t \in A \text{ and } U_{e_t,t} > \varrho; \end{cases}$$

for $\varrho = \pi_A^1(\omega(e_t) = 1 \mid \omega(A \setminus \{e_t\}) = X_{A,t-1}^1(A \setminus \{e_t\}))$; i.e., if $e_t \in A$, we resample $e_t$ given the remainder of the configuration on $A$, together with the wired boundary condition on $\partial A$, using the same uniform random variable $U_{e_t,t}$ for every $X_{A,t}^1$ such that $e_t \in A$.

As in the grand coupling for different initializations, this is a monotone coupling. In particular, we have $X_{G,t}^1 \leq X_{A,t}^1$ for all $A \subset E(G)$ and thus $X_{G,t}^1 \leq \bigcap_{A \in E(G)} X_{A,t}^1$.

A key observation for our revealing process is that for every $A$, the configuration $X_{A,t}^1$ depends only on:

1. the number of updates amongst $(e_s)_{s \leq t}$ that belong to $A$, which we denote by $\kappa_{A,t}$;
2. the choice of edges to be updated on $A$ on those $\kappa_{A,t}$ updates; we denote such set by $O_{A,\kappa_{A,t}}$; and
3. the family of uniform random variables on those edges, $(U_{e,s})_{e \in A, s \leq t}$.

With this observation in hand, we can extend this to a coupling of $(X_{A,t}^1)$ averaged over $G \sim P_{CM}$.

**Definition 4.6.** Let $P_t$ be the distribution over pairs $(G, \omega_t)$ where $\omega_t$ is a random-cluster configuration on $G$ that results by first drawing $G \sim P_{CM}$, then drawing $\omega_t \sim P(X_{G,t}^1 \in \cdot)$. Likewise, for every set $A \subset \mathcal{M}_n$, let $P_{A,t}$ be the distribution over pairs $(G, \omega_{A \cap E(G), t})$ where $\omega_{A,t} \sim P(X_{A,t}^1 \in \cdot)$. Couple, under the distribution $P$, the family of distributions $(P_{A,t})_{A \subset \mathcal{M}_n, t \geq 1}$ by selecting the same random graph $G \sim P_{CM}$ for all of them, then using the coupling of Definition 4.5 of the family $(X_{A,t}^1)_{A,t}$.

In this manner, we have constructed a monotone coupling of the family $(G, (X_{A,t}^1)_{t \geq 1})_{A \subset \mathcal{M}_n}$. Note that we use this coupling for sets $A$ which we know have $E(G) \cap A = A$, so that the averaging is only over the edges of $E(G) \setminus A$, which we earlier noted $X_{A,t}^1$ is independent of; thus the role of this coupling is only to put the random graphs with their random-cluster configurations on the same probability space. We defer detailed discussion of the properties of the coupling to Section 4.2 (after constructing the revealing procedure in the sequel) but emphasize that by construction, if $A \cap B = \emptyset$, the only dependency of $X_{A,t}^1$ and $X_{B,t}^1$ is through the distributions of the binomial random variables $\kappa_{A,t}$ and $\kappa_{B,t}$.
4.1.3. The joint revealing procedure. We now construct a revealing procedure for $G$ and a configuration $\bar{\omega}_t$ on $G$ that stochastically dominates $X^1_{G,t}$. Fix $r$ to be chosen as a large constant (depending on $p$, $q$, $\Delta$) later.

**Definition 4.7.** Given an exposed set of edges $A$ of the random graph $G \sim P_{CM}$, we define $B^\text{out}_r(v) = B^\text{out}_r(v; A)$ as the ball of radius $r$ in $E(G) \setminus \mathcal{N}_r(A)$ where $\mathcal{N}_r(A)$ is the set of edges in $A$ incident to $v$. We drop the $A$ from the notation when understood contextually.

For an edge-set $A_0 \subset \mathcal{M}_n$ revealed to be part of $E(G)$ and a vertex set $V_0 \subset V(G)$, we construct a joint iterative procedure to expose (a set containing) the connected components $C_{V_0}(X^1_{G,t}(E(G) \setminus A))$. The examples to have in mind are (1) $A_0 = \emptyset$ and $V_0 = \{v\}$ and (2) $A_0 = E(B_R(v))$ and $V_0 = \partial B_R(v)$.

Through this process we will keep track of the following variables at each step:

- $A_m$: the set of edges of the random graph that have been revealed by step $m$;
- $V_k$: the set of vertices in the $k$-th generation we want to explore out of;
- $\omega_m$: the random-cluster configuration revealed up to step $m$;
- $F_m$: elements of the filtration with respect to which the configuration $\omega_m$ on $A_m$ is measurable.

The process is defined as follows (see Figures 4.1 and 4.2 for a depiction of several steps of this process):

**Definition 4.8. Initialize:**

for each $k \geq 0$ while $V_k \neq \emptyset$

for each $v \in V_k$

1. Set $v_m = v$. Conditionally on $A_{m-1}$, reveal the edges of the random graph in $B^\text{out}_r(v_m)$ and set

$$A_m := A_{m-1} \cup E(B^\text{out}_r(v_m));$$

Let $A_m := A_m \setminus A_{m-1}$ be the set of new edges revealed in $G$.

2. Reveal the triplet $\mathcal{H}_{A_m,t} := \{\kappa_{A_m,t}, O_{A_m,\kappa_{A_m,t}}, (U_{e,s})_{e \in A_m, s \leq t}\}$ conditionally on $F_{m-1}$. Recall that $\kappa_{A_m,t}$ is the number of updates of the FK-dynamics in $A_m$, $O_{A_m,\kappa_{A_m,t}}$ is the sequence of edges to be updated in $A_m$, and $(U_{e,s})_{e \in A_m, s \leq t}$ is the family of uniform random variables used for the edges updates in $A_m$.

3. Construct the random-cluster configuration $X^1_{A_m,t}(A_m)$ from the triplet $\mathcal{H}_{A_m,t}$ by simulating the steps of the FK-dynamics on $A_m$ with wired boundary conditions. Concatenate $X^1_{A_m,t}(A_m)$ with $\omega_{m-1}$ to obtain a new configuration $\omega_m$ on $A_m \setminus A_0$.

4. Add to $V_{k+1}$ all vertices of $\partial(A_m \setminus A_0)$ that are in the component of $V_0$ in $\omega_m(A_m \setminus A_0)$ and are not in $\bigcup_{t \leq k} V_t$.

5. Let $F_m = (F_{m-1}, \mathcal{H}_{A_m,t})$ and increase $m$ by 1.

Let $\kappa_0$ be the first $k$ such that $V_k = \emptyset$ and let $m_{k_0} = \sum_{k=0}^{k_0} |V_k|$. Let $\bar{\omega}_t = \omega_t(A_{m_{k_0}} \setminus A_0)$ be the random-cluster configuration revealed when the process terminates. The key observation about the above process is that we can control the cluster of $V_0$ in $X^1_t(E(G) \setminus A_0)$ by the set $A_{m_{k_0}}$; the size of this set will then be approximately controlled by comparison to a sub-critical branching process in the following subsection.

**Observation 4.9.** The connected components of the vertices of $V_0$ in $X^1_t(E(G) \setminus A_0)$ are contained in the union of the connected components of $V_0$ in $\bar{\omega}_t(E(G) \setminus A_0)$. In particular, the number of vertices in non-trivial (i.e., non-singleton) boundary components in the boundary condition that $X^1_t(E(G) \setminus A_0)$ induces on $A_0$ is less than the number of vertices in non-trivial boundary components $\bar{\omega}_t(E(G) \setminus A_0)$ induces on $A_0$. The edges in both of these sets of connected components are subsets of the edge-set $A_{m_{k_0}} \setminus A_0$. 

Figure 4.1. Left: We initialize the process with $r = 5$ from $\mathcal{V}_0 = \{v_1\}$ (dark purple) and incident edge $\mathcal{A}_0$ (black). The process begins by revealing $A_1 = B^{\text{out}}_r(v_1)$, depicted in gray. Right: The process then reveals the configuration $X^{1}_{A_1,t}$, given by $\kappa_{A_1,t}$ steps of FK-dynamics for $\pi_1^{A_1}$, to form $\tilde{\omega}_1$ (open edges in red/pink). Vertices in $\partial A_1$ shown to be connected to $v_1$ in $X^{1}_{A_1,t}(A_1)$ are added to $\mathcal{V}_1$ (purple).

Figure 4.2. Left: Proceeding from above, in the next generation, starting from $v_2 \in \mathcal{V}_1$, reveal the edges of $B^{\text{out}}_r(v_2)$ in $\mathcal{G}$; in this case, this is not a tree, but is disjoint from $\mathcal{A}_1$, so that $A_2 = B^{\text{out}}_r(v_2)$. The configuration $X^{1}_{A_2,t}$ is generated and concatenated with $\tilde{\omega}_1$ to form $\tilde{\omega}_2$. Right: For $v_3 \in \mathcal{V}_1$, $B^{\text{out}}_r(v_3)$ is a tree, but it intersects $A_2$. As such, $A_3 = B^{\text{out}}_r(v_3) \setminus A_2$. Running the FK-dynamics on $A_3$ with all-wired boundary conditions ensures that $X^{1}_{A_3,t}$ is nonetheless independent of the configuration we had revealed in $\tilde{\omega}_2$. The light purple vertices are connected to $\mathcal{V}_0$ in $\tilde{\omega}_2$ and are added to $\mathcal{V}_2$ to form the next generation.

With Observation 4.9 in hand, we focus on obtaining the exponential tail bound of Theorem 3.1 for $C_v(\tilde{\omega}_t)$ (the component of $v$ in $\tilde{\omega}_t$) and likewise, the sparsity bound of Proposition 4.2 for $\mathcal{B}_{R(v)}(\tilde{\omega}_t)$.

4.1.4. Constructing a dominating branching process. Towards proving Proposition 4.2, we construct a (non-Markovian, size-dependent) branching process which we will show stochastically dominates the sequence $(\mathcal{V}_k)_{k \geq 0}$ of our joint revealing process. This process $(Z_k)_{k \geq 0}$ will then be shown to be sub-critical and satisfy exponential tail bounds on its total population, implying the same for the cluster of $\mathcal{V}_0$ in $\tilde{\omega}_t$.

Definition 4.10. Initialize $Z_0 = |\mathcal{V}_0|$, and let $(Z_k)_{k \geq 1}$ be the (size-dependent) branching process, which for each $k$, has progeny $(\chi_{i,k})_{i \leq Z_k}$ drawn i.i.d. from the following distribution:

1. With probability $n^{-1/2}$, let $\chi_{i,k} = |V(T_r)| \sum_{\ell \leq k} Z_\ell$ and say the progeny number $\chi_{i,k}$ is Bad.
(2) Otherwise, sample \( X_{i,k} \) from the distribution of the number of leaves in the connected component of the root under \( \pi^{(1,\Delta)}_{T_r} \) (the random-cluster measure on the \( \Delta \)-regular tree of depth \( r \) with a wired boundary condition and with the root also wired to \( \partial T_r \)).

Let \( Z_{k+1} = \sum_{i \leq Z_k} X_{i,k} \); that is, the \( i \)-th member of the \( k \)-th generation gets \( X_{i,k} \) many children.

Note that this is not a branching process in the traditional sense, since the progeny distribution is not i.i.d. and depends on the population up to that generation. Nonetheless, we will show good tail bounds on \((Z_k)_{k \geq 0}\) by dominating it by super-critical branching processes between the Bad steps.

To justify the above construction, let us formalize the relation between \((Z_k)_{k \geq 0}\) and the revealed vertices of the process in Definition 4.8, \((V_k)\). Intuitively, we want to identify vertices \( v_m \in V_k \) with those of generation \( k \) in \((Z_k)\); the progeny of \( v_m \) will then be those vertices added to \( V_{k+1} \) in step (4) of Definition 4.8. Item (1) from the progeny distribution of Definition 4.10 corresponds to situations where:

1. \( B^\text{out}_r(v_m) \) intersects \( A_{m-1} \);
2. \( B^\text{out}_r(v_m) \) is not a tree; or
3. There are an insufficient number of updates on \( B^\text{out}_r(v_m) \) for \( X^1_{A_{m},t} \) to mix.

Examples of situations (1)–(2) were depicted in Figure 4.2. Otherwise, \( X^1_{A_{m},t} \) is mixed, and is comparable to the \((1, \Delta)\)-tree of depth \( r \).

Recall, the definition of the update numbers \((\kappa_{A_{m},t})_m\) and define, for every \( t \geq T_{\text{burn}} \), the event

\[
\mathcal{E}_t = \mathcal{E}_{t}^{\infty} \quad \text{where} \quad \mathcal{E}_t^{m} := \left\{ (e_s)_{s \leq t} : \bigcap_{l=1}^{m} \{ \kappa_{A_{l},t} \leq \frac{4|E(T_r)|}{\Delta n} t \} \right\}.
\]

Standard tail estimates for binomial random variables will imply that \( \mathcal{E}_t \) holds with high probability.

Let \( m_0 = 0 \) and for each \( k \geq 0 \), let \( m_{k+1} = m_k + |V_k| \) be the total number of explored vertices before the exploration for the \((k+1)\)-th generation begins. On the event \( \mathcal{E}_t \), by construction of \((Z_k)\), and the choice of \( T_{\text{burn}} \), we are able to show the following stochastic domination.

**Lemma 4.11.** There exists \( C_0(p, q, \Delta) \) in the definition of (4.1) such that the following holds for every \( t \geq T_{\text{burn}} \). For every \( A_0, V_0 \) such that \( |A_0|, |V_0| \leq n^{1/2-\delta} \) for \( \delta > 0 \), every \( K > 0 \) fixed, and every \( \ell \geq 1 \),

\[
\left( |V_j| \mathbf{1}\{c_t^{m_j} \} \mathbf{1}\{m_{j-1} \leq n^{1/2-\Delta/2} \} \right)_{j \leq t} \preceq (Z_j)_{j \leq t}.
\]

In this manner, we will have reduced the analysis of the set of exposed vertices through the revealing process of \((\mathcal{G}, \tilde{\omega}_t)\), and thus, the clusters of \( X^1_{\mathcal{G},t} \), to the analysis of the process \((Z_k)\), which except on some rare Bad increments, is a simple branching process with supercritical progeny distribution dictated by connectivity probabilities in the wired measure \( \pi^{(1,\Delta)}_{T_r} \). We will establish the following tail estimate for \((Z_k)\).

**Lemma 4.12.** Suppose \( p < p_u(q, \Delta) \) and fix any \( \delta > 0 \), any \( M \geq 1 \), and any \( 1 \leq Z_0 \leq n^{1/2-\delta} \). There exist \( r_0(p, q, \Delta), C(p, q, \Delta, M), K_0(p, q, \Delta, M) \) such that for every \( r \geq r_0 \) fixed and every \( 0 < \lambda \leq n^{1/2-\delta} \),

\[
P\left( \sum_{k \geq 0} Z_k \geq K_0 Z_0 + \lambda \right) \leq C \exp(-\lambda/C) + C n^{-\delta M}.
\]

4.1.5. **Outline of remainder of section.** Having sketched the key revealing procedures and the way they fit together to provide the desired bounds on the clusters of \( X^1_{\mathcal{G},t} \), let us prove the various relations and bounds claimed above. In Section 4.2, we prove various key properties of the configuration model revealing process of Definition 4.3 and the coupling of Definition 4.5 that will be central to the analysis of the revealing procedure of Definition 4.8. Then in Section 4.3, we show that the size-dependent branching process \((Z_k)\) of Definition 4.10 stochastically dominates the FK process \((V_k)\) of Definition 4.8 on a high-probability event, proving Lemma 4.11. In Section 4.4, we analyze the process \((Z_k)\) by comparing its population to the sum of \( O(1) \) many sub-critical branching processes to deduce Lemma 4.12. In Section 4.5, we combine these ingredients to conclude Theorem 3.1 and Proposition 4.2, and from that Theorem 3.4.
4.2. **Key properties of the revealing procedure for \( (G, \tilde{\omega}_t) \).** In this section, we describe some of the key properties of the coupling constructed in Definition 4.5, and the revealing procedure constructed for the clusters of \( V_t \) in \( \tilde{\omega}_t \) in Definition 4.8. The following preliminary lemmas describe the law of the random graph edges and overlaying FK configurations through the revealing process.

4.2.1. **Properties of the configuration model revealing procedure.** We begin with the following lemma on the law of the random graph \( G \) conditionally on a set \( A_m \) which we have revealed to be a subset of \( E(G) \). Recall the configuration model’s revealing procedure from Definition 4.3 and say a vertex is *discovered* if at least one of its half-edges has been matched, and *exhausted* if all of its half-edges have been matched.

**Lemma 4.13.** Let \( A \) be any set of edges (pairing of half-edges) revealed to belong to \( E(G) \). For every \( r \),

\[
\sup_{v \in \{1, \ldots, n\}} P_{CM}(B^\text{out}_r(v) \cap A \neq \emptyset \text{ or } B^\text{out}_r(v) \text{ is not a tree } | A) \leq \frac{2\Delta d^r(|V(A)| + d^r)}{n - (|V(A)| + d^r)}.
\]

**Proof.** Fix any edge-set \( A \). We can sample from the conditional distribution \( P_{CM}(\cdot | A) \) by defining the adaptive scheme \( f \) in Definition 4.3 so that it first matches the half-edges belonging to \( A \), yielding the set \( A_{|A|/2} = A \) after \( |A|/2 \) steps, then setting \( f \) to do a breadth-first search (BFS) of \( B^\text{out}_r(v) \): this latter part is done by choosing \( f \) so that it first exhausts \( v \), then exhausts each of the neighbors of \( v \), and so on.

Revealing the entire set \( B^\text{out}_r(v) \) takes at most \( |E(T_r)| \) many steps beyond \( |A|/2 \). If for every \( m \in \{\lfloor |A|/2 + 1, \ldots, |A|/2 + d^r \} \) the half-edge from \( f(A_{m-1}) \) is not matched to a half-edge belonging to a vertex in \( V(A_{m-1}) \), then evidently \( B^\text{out}_r(v) \cap A = \emptyset \) and \( B^\text{out}_r(v) \) is a tree.

Since on each of these steps, the half-edge \( f(A_{m}) \) is being matched to a u.a.r. un-matched half-edge, uniformly over the at most \( |E(T_r)| \) steps it takes to reveal \( B^\text{out}_r(v) \), the probability that the half-edge it is matched to belongs to \( A_{m-1} \) is at most

\[
\frac{d(|V(A)| + d^r)}{\Delta n - (d|V(A)| + d^r)} \leq \frac{|V(A)| + d^r}{n - (|V(A)| + d^r)}.
\]

(The first inequality here uses the fact that in the BFS of \( B^\text{out}_r(v) \), there are at most \( d^r \) vertices of the ball that have been discovered but not exhausted.) Union bounding over the at most \( |E(T_r)| \) \( \leq 2\Delta d^r \) such attempts yields the desired bound. 

We can use a similar reasoning as the proof above to deduce a proof of Fact 2.3 as follows.

**Proof of Fact 2.3.** Fix any \( v \) and choose \( f \) so that the revealing scheme performs a BFS revealing of \( B_R(v) \). In order for \( B_R(v) \) to not be \( L \)-Treelike, it must be the case that for more than \( L \) different \( m \)’s in the first \( |E(T_R)| \) steps, the half-edge \( f(A_{m-1}) \) is being matched to a half-edge belonging to \( A_{m-1} \). (If there were at most \( L \) such steps, then the removal of the at-most \( L \) edges formed by those at-most \( L \) matchings in the revealing scheme, evidently leaves a tree, so that \( B_R(v) \) would be \( L \)-Treelike.) Uniformly over \( A_{m-1} \), the probability of this in the \( m \)’th step is at most \( d^{R+1} / (\Delta(n-m)) \). Summing over the at most \( |E(T_R)| \) \( \leq 2\Delta d^r \) such attempts while revealing \( B_R(v) \), we find that for every \( \ell \geq 1 \),

\[
P_{CM}(B_R(v) \text{ is not } \ell \text{-Treelike}) \leq \mathbb{P}\left( \text{Bin}(|E(T_R)|, \frac{d^R}{n - |E(T_R)|}) > \ell \right).
\]

With the choice \( R = (\frac{1}{2} - \delta) \log d n \), so that \( d^R = n^{\frac{1}{2} - \delta} \) and \( |E(T_R)| \leq 2\Delta d^R \), a Chernoff bound applied to the Poisson binomial distribution implies that the right-hand side is at most \( (Cn^{-\delta})^\ell \) for some \( C(\Delta) \) and large enough \( n \). As a consequence, choosing \( L > 2\delta^{-1} \), we would find

\[
\sup_{v \in \{1, \ldots, n\}} P_{CM}(B_R(v) \text{ is not } L \text{-Treelike}) \leq o(n^{-2}). \tag{4.3}
\]

It remains to translate this to a bound under \( P_{R\Gamma} \). This follows by the following standard comparison argument. Let \( \Gamma_{R\Gamma} \) be the event that the graph \( G \sim P_{CM} \) has no self-loops or double edges (i.e., it is a simple graph). Taking \( \Gamma = \{ G : G \text{ is not } (L, R) \text{-Treelike} \} \) in (2.1) and union bounding (4.3) over the \( n \) vertices yields the desired bound.

\[\square\]
4.2.2. Properties of the coupling of localized Markov chains. The following lemma is the key fact about the construction of the grand coupling of FK dynamics, Definition 4.5, whereby after revealing some $X_{A,t}$, we can control the influence that revealing has on $X_{B,t}$ for $A \cap B = \emptyset$. In this manner, through the revealing procedure of Definition 4.8, which reveals different localized configurations $X_{A,m,t}$ iteratively, as long as $t \geq T_{burn}$ these are each close to their respective stationary distributions of $\pi_{A,m}$, so that it is approximately a concatenation of localized FK models on treelike graphs, inducing an exponential decay of connectivities.

**Lemma 4.14.** Recall the coupling of Definitions 4.5–4.6 of the distributions $(\mathbb{P}^{1}_{A,t})_{A \subseteq \mathbb{R}^{n},t \geq 1}$. Suppose we have revealed edges in $E(G)$ showing $E(G) \cap A = A$.

1. The configuration $X_{A,t}^{1}(A)$ is measurable with respect to $\kappa_{A,t}$ (the number of edge-updates in $A$), the edges chosen to update $O_{A,\kappa_{A,t}}$, and the uniform random variables $(U_{e,s})_{e \in A,s \leq t}$ on those edges. The number $\kappa_{A,t}$ is distributed as $\text{Bin}(t,2|A|/\Delta n)$; the sequence $O_{A,\kappa_{A,t}}$ is distributed as $(\bar{e}_{j})_{j \leq \kappa_{A,t}}$ drawn i.i.d. from $A$. The values $(U_{e,s})_{e \in A,s \leq t}$ are distributed as i.i.d. $\text{Unif}[0,1]$.

2. Suppose $B$ is such that $E(G) \cap B = B$ and $A \cap B = \emptyset$. Conditionally on $\kappa_{A,t}$, $O_{A,\kappa_{A,t}}$, and $(U_{e,s})_{e \in A,s \leq t}$, the distribution of $X_{B,t}^{1}(B)$ is given as follows. The number of updates $\kappa_{B,t}$ is drawn from $\text{Bin}(t(1-\kappa_{A,t}),2|B|/\Delta n)$, and the edges chosen are distributed as $(\bar{e}_{j})_{j \leq \kappa_{B,t}}$ drawn i.i.d. amongst $B$. The random variables $(U_{e,s})_{e \in B,s \leq t}$ are distributed as i.i.d. $\text{Unif}[0,1]$.

**Proof.** Let $G$ be any graph having $E(G) \cap A = A$. We claim that uniformly over $G$, items (1)–(2) above hold. Observe first that $|E(G)| = \Delta n/2$ necessarily, and therefore uniformly over such $G$, the number of updates on edges in $A$ at time $t$ in the update sequence $(e_{s})_{s \leq t}$ is distributed as $\text{Bin}(t,2|A|/\Delta n)$. Evidently, the distribution of $O_{A,\kappa_{A,t}}$ only depends on $\kappa_{A,t}$ and not on the times these updates were; in particular, given that $e_{j} \in A$ for some $j$, the law of $e_{j}$ is clearly uniform at random on $A$. Finally, notice that for every $e$, the sequence $(U_{e,s})_{s \leq t}$ is independent of all other sources of randomness, implying the desired item (1).

Turning to item (2), we fix a $\kappa_{A,t}$, $O_{A,\kappa_{A,t}}$, and family $(U_{e,s})_{e \in A,s \leq t}$. We can condition further on the exact times of the updates in $A$, i.e., $(e_{s})_{s \leq t} \cap A$. Conditionally on that set of updates, the distribution on the remaining updates is evidently $t - \kappa_{A,t}$ i.i.d. draws from $E(G) \setminus A$. It is then clear that $\kappa_{B,t}$ counts the number of times, amongst these remaining draws, that the update is in $B$. As in item (1), the induced distribution on $O_{B,\kappa_{B,t}}$ is then the same as $\kappa_{B,t}$ i.i.d. draws from the edges of $B$. Finally, for every $e \in B$, the uniform random variables $(U_{e,s})_{s \leq t}$ are independent of all other sources of randomness. 

4.3. Domination by the modified branching process $(Z_k)$. In this section, we establish the stochastic domination of the sequence $(V_{k})_{k \geq 0}$ from Definition 4.8 by the branching process $(Z_{k})$ of Definition 4.10.

**Proof of Lemma 4.11.** We prove the desired stochastic domination by induction over $\ell$. The base case, $Z_{0} = |V_{0}|$, is by construction. Now fix $\ell \geq 1$ and suppose by way of induction that the following stochastic domination holds:

$$|V_{j}^{\ell}|1_{E_{\ell}^{m},1_{\{m_{j-1} \leq n^{1/2-\delta/2}\}}}j \leq \ell - 1 \leq (Z_{j})_{j \leq \ell - 1}.$$ 

Thus there exists a monotone coupling of the sequence on the left-hand side, such that it is below the sequence $(Z_{j})_{j \leq \ell - 1}$ in the natural element-wise ordering on the sequence. Working on that coupling, it suffices for us to then show that on the intersection $E_{\ell}^{m,1_{\{m_{\ell-1} \leq n^{1/2-\delta/2}\}}}$, for every $m \in \{m_{\ell-1}+1, \ldots, m_{\ell}\}$, the distribution of the children of $v_{m}$ is stochastically below the progeny distribution of Definition 4.10.

Observe, first of all, that for every $m \in \{m_{\ell-1}+1, \ldots, m_{\ell}\}$, on $E_{\ell}^{m,1_{\{m_{\ell-1} \leq n^{1/2-\delta/2}\}}}$, deterministically the number of children of $v_{m}$ is bounded by

$$V(|A_{m} \setminus A_{0}|) \leq |V(T_{r})|m_{\ell-1} \leq |V(T_{r})| \sum_{j \leq \ell - 1} |V_{j}| \leq |V(T_{r})| \sum_{j \leq \ell - 1} Z_{j},$$

where the last inequality is by the inductive hypothesis, and the fact that $E_{\ell}^{m,1_{\{m_{\ell-1} \leq n^{1/2-\delta/2}\}}}$ implies $E_{\ell}^{m,1_{\{m_{j-1} \leq n^{1/2-\delta/2}\}}}$ for all $j < \ell$. 


Now, for every set of revealed edges \((A_t)_{t \leq m-1}\), define the following events on \(\mathcal{F}_{m-1}\) consisting of \((\kappa_{A_t}, t)_{t \leq m-1}\), edge-values \((\bigcup_{s \leq t} \kappa_{A_t})_{t \leq m-1}\), uniform random variables \((\bigcup_{s \leq t} \kappa_{A_t})_{t \leq m-1}\):

1. Let \(\Gamma_{\text{TREE},m}\) be the event that \(B_{\text{out}}(v_m) \cap A_{m-1} = \emptyset\) and \(A_m = A_m \setminus A_{m-1}\) is a tree.

2. Let \(\Gamma_{\text{UPD},m}\) be the event that \(\kappa_{A_m,t} \geq d^*T_{\text{BURN}}/(2\Delta n)\).

We first claim that these two events each happen with probability \(1 - \frac{1}{3}n^{-1/2}\), uniformly over \((A_t)_{t \leq m-1}\) and elements of \(\mathcal{F}_{m-1}\) such that \(\varepsilon_t^m\) holds and \(m_{t-1} \leq n^{1/2-\delta/2}\). Given \(v_m\), the law of \(A_m\) is independent of \(\mathcal{F}_{m-1}\), and only depends on \(A_{m-1}\). (This can be seen from the explicit construction of the law of \(\mathcal{F}_{m-1}\) in Lemma 4.14 as independent of \(E(G) \setminus A_{m-1}\)). Notice that if \(m_{t-1} \leq n^{1/2-\delta/2}\) and \(|A_0| \leq n^{1/2-\delta}\) then \(|V(A_{m-1})| \leq (1 + |V(T_r)|)n^{1/2-\delta/2}\). As such, by Lemma 4.13, for every \(A_0, \mathcal{V}_0\) such that \(|A_0| \leq n^{1/2-\delta}\),

\[
\sup_{(A_{m-1}, \mathcal{F}_{m-1}) \in \mathcal{E}_m^m \cap \{m_{t-1} \leq n^{1/2-\delta/2}\}} \mathbb{P}_{\mathcal{CM}}(\Gamma_{\text{TREE},m} | A_{m-1}, \mathcal{F}_{m-1}) \leq \sup_{A_{m-1}: V(A_{m-1}) \leq 2|V(T_r)|n^{1/2-\delta/2}} \frac{2\Delta d^* (|V(A_{m-1})| + d^*)}{n (|V(A_{m-1})| + d^*)} \leq \frac{10\Delta^2 d^{2r} |V(T_r)|^{1/2-\delta}}{n - 5\Delta d^* |V(T_r)|^{1/2-\delta}}.
\]

Thus, for \(n\) large enough and \(r = o(\log n)\), the above is at most \(\frac{1}{3}n^{-1/2}\) as desired.

We next turn to the probability of \(\Gamma_{\text{UPD},m} \cap \Gamma_{\text{TREE},m}\). Recall from item (2) of Lemma 4.14 that conditionally on \(\mathcal{F}_{m-1}\), the distribution of \(\kappa_{A_m,t}\) is

\[
\kappa_{A_m,t} \sim \text{Bin} \left( t - \sum_{t \leq m-1} \kappa_{A_t,t}, \frac{|A_m|}{\Delta n} \right).
\]

Since we are on the event \(\varepsilon_t^m\) and thus \(\varepsilon_t^{m-1}\), we have that \(\sum_{t \leq m-1} \kappa_{A_t,t} \leq 4m|E(T_r)|t/(\Delta n)\), from which we deduce, using \(m \leq m_{t-1} \leq |V(T_r)|m_{t-1} \leq |V(T_r)|n^{1/2-\delta/2}\), that the number of trials in the binomial is at least

\[
t(1 - 16\Delta d^{2r} n^{-1/2-\delta/2}) \geq t/2,
\]

as long as \(r = o(\log n)\). Since we are on the event \(\Gamma_{\text{TREE},m}\), we have \(d^r \leq |A_m| \leq |E(T_r)| \leq 2\Delta d^r\), and we see from lower tail estimates on binomial random variables that

\[
\sup_{(A_{m-1}, \mathcal{F}_{m-1}) \in \mathcal{E}_m^{m-1} \cap \{m_{t-1} \leq n^{1/2-\delta/2}\}} \mathbb{P}(\Gamma_{\text{UPD},m} \cap \Gamma_{\text{TREE},m} | A_{m-1}, \mathcal{F}_{m-1}) \leq \mathbb{P}(\text{Bin}(T_{\text{BURN}}/2, 2d^r/(\Delta n)) \leq d^r T_{\text{BURN}}/(2\Delta n)) \leq \frac{1}{3} n^{-1/2},
\]

as long as \(C_0\) in (4.1) is sufficiently large (depending on \(r, \Delta\)).

By item (2) of Lemma 4.14, conditionally on any \((A_t)_{t \leq m-1}\) and \(\mathcal{F}_{m-1}\), and any \(A_m \in \Gamma_{\text{TREE},m}\), the conditional distribution of \(X_{A_m,t}^1(A_m)\) is equivalent (up to relabeling of edges) to that of \(\kappa_{A_m,t}\) updates of a heat-bath chain \((Y_s^1)_s\) on a subtree \(T_r\) of the complete tree \(T\) with \((1,\circ)\)-wired boundary conditions, initialized from \(Y_0^1 = 1\). Notice that the equivalent sub-tree \(T_r\) consists of some \(k \leq d\) of the children of the root, together with their complete sub-trees. In particular, the random-cluster model on \(A_m\) with wired boundary conditions is stochastically below the FK model on the corresponding subset of \(T\) with its \((1,\circ)\) boundary conditions. In particular, the number of leaves in the FK cluster of the root under \(\pi_{T_r}^{(1,\circ)}\) is stochastically below the same quantity under \(\pi_{T_r}^{(1,\circ)}\). It therefore suffices for us to show that as long as \(A_m\) is a tree disjoint from \(A_{m-1}\) and \(\kappa_{A_m,t} \geq d^r T_{\text{BURN}}/(2\Delta n)\), we have

\[
\|\mathbb{P}(Y_{(1,\circ)}^{(1,\circ)} \in \cdot) - \pi_{T_r}^{(1,\circ)}\|_{\text{TV}} \leq \frac{1}{3} n^{-1/2}.
\]
This follows as long as $C_0$ is sufficiently large (depending on $\Delta$), from the fact that
\[
\frac{d^r T_{\text{burn}}}{2\Delta n} \geq \frac{C_0 d^r}{2\Delta |E(T_r)|} \log n \cdot t_{\text{mix}}(T_r, (1, \emptyset)) \geq \frac{C_0 d^r}{2\Delta |E(T_r)|} \log n \cdot t_{\text{mix}}(T_r, (1, \emptyset)),
\]
and $|E(T_r)| \leq 2\Delta d^r$, together with the sub-multiplicativity of total-variation distance. \hfill \Box

4.4. Sub-criticality and tail bounds for the dominating branching process. We now analyze the process $(Z_k)$ of Definition 4.10, and show that it indeed is sub-critical, and satisfies good tails on its total population. For ease of notation, let $\mathcal{P}_k = \sum_{\ell \leq k} Z_\ell$ be the total population after $k$ generations.

**Proof of Lemma 4.12.** Since $(Z_k)$ is a size-dependent, non-Markov process, we cannot directly use results on branching processes to control its growth. Instead, to control the population of the process $(Z_k)$, we compare it to a sum of branching processes in the following manner. Consider the stopping generation $\kappa$ for exceeding population $\kappa_0 Z_0 + \lambda$, i.e.,
\[
\kappa = \inf \{ k : \mathcal{P}_k > \kappa_0 Z_0 + \lambda \}.
\]
Our aim is to control the probability that $\kappa < \infty$. Let $\Gamma_{M,k}$ be the event that no more than $M$ of the progeny counts $((\chi_{i,t})_{i \leq Z_\ell})_{\ell \leq k-1}$ were Bad. By a Chernoff bound for the Poisson binomial distribution,
\[
\mathbb{P}(\Gamma_{c_{M,k}}) \leq \mathbb{P}(\text{Bin}(\kappa_0 Z_0 + \lambda, n^{-\frac{1}{2}}) > M) \leq C n^{-\delta M},
\]
for some $C > 0$, as long as $n$ is sufficiently large and $Z_0, \lambda \leq n^{\frac{1}{2} - \delta}$ for $\delta > 0$.

Next consider the event that $\kappa < \infty$ on the event $\Gamma_{M,k}$. On $\Gamma_{M,k}$, we dominate the population $\mathcal{P}_k$ by the following sum of sub-critical branching processes with bounded progeny distributions.

Define $(\tilde{Z}^{(1)}_{\kappa})_k$ to be the branching process initialized at $\tilde{Z}^{(1)}_0 = Z_0$ with progeny $(\tilde{X}^{(1)}_{i,k})$, distributed i.i.d. from the distribution of the number of leaves connected to the root, in a sample from $\pi_{T_r}^{(1)}$, i.e., the distribution of $(\chi_{i,k})$ conditionally on the progeny number not being Bad. Let $\tilde{\mathcal{P}}^{(1)}_k = \sum_{\ell \leq k} \tilde{Z}^{(1)}_\ell$. For each $1 \leq j \leq M$, iteratively let $\tilde{Z}^{(j)}_k$ be an independent branching process with the same progeny distribution, initialized from $\tilde{Z}^{(j)}_0 = |V(T_r)|^{|V(T_r)|^{-1}}$, where we recall $|V(T_r)| \leq 2\Delta d^r$.

The following stochastic domination is clear by construction if we decompose the process $(Z_k)$ revealed in a breadth-first manner, into its excursions between the at most $M$ times (on the event $\Gamma_{M,k}$) when the progeny number $\chi_{i,k}$ was Bad.

**Claim 4.15.** Fix any $k \geq 1$. We have the stochastic domination
\[
\mathcal{P}_k 1\{\Gamma_{M,k}\} \preceq \sum_{j \leq M} \tilde{\mathcal{P}}^{(j)}_\infty.
\]

With this domination in hand, notice that in order for $\kappa < \infty$ while $\Gamma_{M,\kappa}$ holds, there must exist some $k \leq \kappa_0 Z_0 + \lambda$ such that $\Gamma_{M,k}$ holds and $\mathcal{P}_k \geq \kappa_0 Z_0 + \lambda$. Therefore, by a union bound,
\[
\mathbb{P}(\mathcal{P}_\infty \geq \kappa_0 Z_0 + \lambda, \Gamma_{M,k}) \leq \sum_{k \leq \kappa_0 Z_0 + \lambda} \mathbb{P}(\mathcal{P}_k \geq \kappa_0 Z_0 + \lambda, \Gamma_{M,k}) \leq (\kappa_0 Z_0 + \lambda) \mathbb{P}\left( \sum_{j \leq M} \tilde{\mathcal{P}}^{(j)}_\infty \geq \kappa_0 Z_0 + \lambda \right).
\]
We claim that if $\sum_{j \leq M} \tilde{\mathcal{P}}^{(j)}_\infty \geq \kappa_0 Z_0 + \lambda$ holds, there must exist $j \leq M$ for which $\tilde{Z}^{(j)}_0 \leq \kappa_0 Z_0 + \lambda$, and
\[
\tilde{\mathcal{P}}^{(j)}_\infty \geq |V(T_r)|^{-1} \left( \frac{\kappa_0}{M} \right)^{1/M} (\tilde{Z}^{(j)}_0 + \kappa_0^{-1} M^{-2} \lambda) =: C_{r,\Delta} \kappa_0^{1/M} (\tilde{Z}^{(j)}_0 + \kappa_0^{-1} M^{-2} \lambda).
\]
Indeed, if no such $j$ existed, as long as $K_0$ is sufficiently large, we could bound $\sum_{j \leq M} \tilde{P}^{(j)}_\infty$ by

$$\sum_{j \leq M} C_{r,\Delta, M} K_0^{1/M} (\tilde{Z}^{(j)}_0 + \lambda) \leq M \left[ \frac{K_0}{M} \tilde{Z}^{(1)}_0 + K_0^{-1} M^{-2}(1 + \cdots + \frac{K_0}{M}) \right] \leq K_0 Z_0 + \lambda.$$  

Now fix any $j \leq M$, any $\tilde{Z}^{(j)}_0$, and consider the branching process $\tilde{Z}^{(j)}_k$. This is a branching process with progeny distribution having mean $\tilde{m} = A(\hat{p}d)^r$ for some $A(p, q)$ per Lemma 2.5. Since $\hat{p} < d^{-1}$ when $p < p_u(q, \Delta)$, as long as $r$ is greater than some $r_0(p, q, \Delta)$, for $n$ sufficiently large we have $\tilde{m} < 1$, and $\tilde{Z}^{(j)}_k$ is sub-critical. Additionally, the progeny distribution of $\tilde{Z}^{(j)}_k$ is almost surely bounded by $|\partial T_r| \leq \Delta d^{-1}$.

As such, using the standard breadth-first exploration of the total population of the branching process $\tilde{Z}^{(j)}_k$ (through which $\tilde{P}^{(j)}_k$ is expressed as the random walk $\tilde{Z}^{(j)}_0 + \sum_{t \leq k} \sum_{i \leq k} (\tilde{\chi}^{(j)}_{i,k} - 1)$), we can bound

$$\mathbb{P}(\tilde{P}^{(j)}_\infty \geq N^{(j)}_\lambda) \leq \mathbb{P} \left( \sum_{i \leq N^{(j)}_\lambda} \tilde{\chi}_i > N^{(j)}_\lambda - \tilde{Z}^{(j)}_0 \right) \quad \text{for} \quad N^{(j)}_\lambda := C_{r,\Delta, M} K_0^{1/M} (\tilde{Z}^{(j)}_0 + K_0^{-1} M^{-2}) \lambda,$$

where $\tilde{\chi}_i$ are i.i.d. copies of $\chi^{(j)}_{i,k}$. Now observe that if $K_0(p, q, \Delta, M)$ is sufficiently large, the right-hand in the probability above exceeds the mean $m N^{(j)}_\lambda$ by some $c N^{(j)}_\lambda$ for $c = c(p, q, \Delta, M, K_0) > 0$. As this is a tail probability for a sum of i.i.d. random variables, by Hoeffding’s inequality, it is at most

$$\exp \left( - \left( c N^{(j)}_\lambda \right)^2 / (4 N^{(j)}_\lambda d^2) \right) \leq \exp(-c' N^{(j)}_\lambda),$$

for some $c'(r, \Delta, M) > 0$. Taking a union bound over the $M$ possible values of $j \leq M$, we altogether find

$$\sum_{k \leq K_0 Z_0 + \lambda} \mathbb{P}(P_k \geq K_0 Z_0 + \lambda, \Gamma_{M,k}) \leq (K_0 Z_0 + \lambda) M \exp \left( -c' N^{(j)}_\lambda \right).$$

It follows from this, and the definition of $N^{(j)}_\lambda$, that for some $C(p, q, \Delta, M, K_0)$ large enough,

$$\mathbb{P}(\kappa < \infty) \leq \mathbb{P}(T^*_{M,k}) + \sum_{k \leq K_0 Z_0 + \lambda} \mathbb{P}(P_k \geq K_0 Z_0 + \lambda, \Gamma_{M,k}) \leq C n^{-\delta M} + C \exp \left( - (Z_0 + \lambda) / C \right),$$

concluding the proof.

4.5. Proof of exponential tail on cluster sizes and shattering. We are now in position to conclude the proof of the exponential tail bound on clusters of $\mathcal{X}^j_{k,1}$, and use that to deduce that $X^j_{1,1}$ is $(K, R)$-Sparse, except with probability $o(n^{-2})$. We begin by using Lemmas 4.11–4.12 to prove the following tail bound on the sequence $(\mathcal{V}_k)$, which are the roots of the balls revealed through the revealing process of Definition 4.8.

**Lemma 4.16.** Fix $\delta > 0$ and consider the revealing procedure for any initial subsets $\mathcal{A}_0$ and $\mathcal{V}_0$ having $|\mathcal{A}_0|, |\mathcal{V}_0| \leq n^{\frac{1}{2} - \delta}$. For every $M \geq 1$, there exist $C(p, q, \Delta, M)$, $K_0(p, q, \Delta, M)$ and $C_0(p, q, \Delta, \delta, M)$ such that for all $t \geq T_{\text{burn}}$ and all $0 \leq \lambda \leq n^{\frac{1}{2} - \delta}$,

$$\mathbb{P} \left( \sum_{k \leq k_0} |\mathcal{V}_k| \geq K_0 |\mathcal{V}_0| + \lambda \right) \leq C \exp(-\lambda/C) + C n^{-\delta M},$$

**Proof.** Fix $K_0$ large to be chosen later, and define the following stopping generation

$$\varsigma = \inf \{ \ell \mid m_{\ell-1} > K_0 |\mathcal{V}_0| + \lambda \}.$$

Recall $\mathcal{E}_\ell$ from (4.2). Since for every $\ell \leq \varsigma$, we have from Lemma 4.11, that $\{ |\mathcal{V}_\ell| 1\{\mathcal{E}_\ell\} \}_{j \leq \ell} \preceq (Z_j)_{j \leq \ell}$, we have that if $C_0$ in (4.1) is sufficiently large, the probability of $\{ \varsigma < \infty \}$ is bounded by the probability of $\mathcal{P}_\infty = \sum_{k \geq 0} Z_k \geq K_0 Z_0 + \lambda$. By Lemma 4.11, we obtain

$$\mathbb{P} \left( \sum_{k \leq k_0} |\mathcal{V}_k| \geq K_0 |\mathcal{V}_0| + \lambda \right) \leq \mathbb{P} \left( \sum_{k \geq 0} Z_k \geq K_0 Z_0 + \lambda \right) + \mathbb{P}(\mathcal{E}_\varsigma^\uparrow).$$
Lemma 4.12 implies the existence of $r(p, q, \Delta)$ such that the first-term above is at most $C_1 \exp(-\lambda/C_1) + C_1 n^{-\delta M}$ for some $C_1(p, q, \Delta, M)$.

Next, consider $\mathbb{P}(\mathcal{E}_t^c)$. By a union bound and item (1) of Lemma 4.14, with the trivial observations that $m_k \leq n$ and $|A_m| \leq |E(T_r)| \leq 2\Delta d''$ necessarily, we get for every $t \geq T_{\text{burn}}$,

$$\mathbb{P}(\mathcal{E}_t^c) \leq n \mathbb{P}(\text{Bin}(t, \frac{2|E(T_r)|}{\Delta n}) > \frac{4|E(T_r)|}{\Delta n} t^c).$$

The above entails a deviation of at least $4td''n^{-1}$ from its mean; as such, by standard tail estimates for binomials, for every $t \geq T_{\text{burn}}$,

$$\mathbb{P}(\mathcal{E}_t^c) \leq n \exp(-td''n^{-1}), \tag{4.4}$$

which is at most $n^{-\delta M}$ for $n$ large, as long as $C_0$ in (4.1) is sufficiently large (depending on $\delta M$). The desired bound then follows up to a change of the constant $C$.

Before proceeding to prove Proposition 4.2, let us translate the tail bound of Lemma 4.16 on $\sum_k |V_k|$ to a tail bound on the FK cluster of a single vertex under $X^1_{G,t}$ and $\pi_G$. Notice that towards the proofs of Theorem 3.1 and Proposition 4.2, it suffices to show these for $t \geq T_{\text{burn}}$ for some fixed choices of $C_0, r$ in (4.1) depending on $p, q, \Delta$ (as $t_{\text{mix}}(T_r, (1, \emptyset))$ is of course independent of $n$).

**Proof of Theorem 3.1.** Fix any $v \in \{1, \ldots, n\}$, let $A_0 = \emptyset$ and let $V_0 = \{v\}$ in Definition 4.8. By Observation 4.9, for each $\mathcal{G} \sim \mathcal{P}_{CM}$, the cluster of $v$ in the configuration $X^1_{G,t}$, denoted $C_v(X^1_{G,t})$ is a subset of $C_v(\tilde{\omega}_t)$, which in turn is a subset of $V(A_{\text{mr}})$, so that

$$|C_v(X^1_{G,t})| \leq |C_v(\tilde{\omega}_t)| \leq |V(T_r)| \sum_{k \leq k_0} |V_k| \leq 2\Delta d'' \sum_{k \leq k_0} |V_k|.$$

By Lemma 4.16 and the above, we find that for each $M$, there exists $C(p, q, \Delta, M)$ such that

$$\mathbb{P}(\{(\mathcal{G}, X^1_{G,t}) : |C_v(X^1_{G,t})| \geq 2\Delta d''(1 + \lambda)\} \leq C \exp(-\lambda/C) + C n^{-\delta M}.$$

Observing that $\mathbb{P}(X^1_{G,t} \in \cdot) = \mathbb{E}_{CM}[P(X^1_{G,t} \in \cdot)]$, we can use Markov’s inequality to write

$$\mathbb{P}_{CM}\left(\mathcal{G} : P\left(X^1_{G,t} : |C_v(X^1_{G,t})| \geq 2\Delta d''(1 + \lambda) \right) \geq \sqrt{Ce^{-\lambda/C} + Cn^{-\delta M}}\right) \leq \sqrt{Ce^{-\lambda/C} + Cn^{-\delta M}}.$$

We can obtain the same bound for $\mathbb{P}_{\text{BRG}}$ by (2.1), up to a multiplicative $c(\Delta)^{-1}$ on the right-hand side. Taking $M$ such that $\delta M > 2K$ and using the fact that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$, we deduce the desired tail bound on $|C_v(X^1_{G,t})|$ up to the change of constant $C$ to $2C$. Using the monotonicity $X^1_{G,t} \geq \pi^1_G$ implies the analogous bound for $|C_v(\omega)|$ where $\omega \sim \pi_G$. \qed

We now turn to proving that for typical random graphs, the configuration $X^1_{G,t}$ is $(K, R)$-Sparse with high probability for all $t \geq T_{\text{burn}}$. This allows us to localize to treelike balls with sparse boundary conditions.

**Proof of Proposition 4.2.** Fix $v \in \{1, \ldots, n\}$ and $\delta > 0$, and let $R = (\frac{1}{2} - \delta) \log_2 n$. We apply the revealing procedure of Definition 4.8 with the choices $A_0 = E(B_R(v))$ and $V_0 = \partial B_R(v)$. Recall from Observation 4.9 that the FK-clusters of $V_0$ induced by $\tilde{\omega}_t(E(\mathcal{G}) \setminus A_0)$ (where $\tilde{\omega}_t$ was extended to be all wired off of $A_{\text{mr}} \setminus A_0$) are confined to the set $A_{\text{mr}} \setminus A_0$, and the extended configuration $\tilde{\omega}_t$ satisfies $\tilde{\omega}_t \geq X^1_{G,t}$. Thus, the sets $\mathcal{V}_{B_R(v)}(\tilde{\omega}_t)$ and in turn $\mathcal{V}_{B_R(v)}(X^1_{G,t})$, are below the number of vertices in $V_0$ that share a connected component of $A_{\text{mr}} \setminus A_0$ with another vertex of $V_0$.

Suppose that through the revealing process of Definition 4.8, for each $m$, the edges of $B_{v_m}^{\text{out}}(v_m)$ are revealed one at a time per Definition 4.3. Notice then, that $|\mathcal{V}_{B_R(v)}(A_{\text{mr}} \setminus A_0)|$ is bounded by the number of times through the revealing of $A_{\text{mr}}$, that a half-edge is matched up to a half-edge belonging to a vertex that has been discovered at that point. Throughout this process, conditionally on an exposed edge-set $A$
(and the edge-update sequence, and uniform random variables given by the filtration up to that step of the revealing, but \( E(\mathcal{G}) \setminus \mathcal{A} \) is independent of these), the law of the next half-edge to be matched is uniform amongst un-matched half-edges. Thus on any such edge-revealing, uniformly on the history of the revealing, the probability that it matches with a half-edge belonging to a discovered vertex is at most \( \frac{1}{n-2} |V_{(A_{m_k})}| \).

By a union bound, we obtain for \( \Lambda \) a sufficiently large constant (depending on \( p, q, \Delta, r \)), for all \( k \geq 1 \),

\[
\mathbb{P}\left( (\mathcal{G}, \hat{\omega}_t) : |\mathcal{M}_{v,R}(X^1_{G,t})| > k \right) \leq \mathbb{P}\left( |V_{(A_{m_k})}| > \Lambda^2 |V_0| \right) + \mathbb{P}\left( \text{Bin} \left( \Lambda^2 |V_0|, \frac{2\Lambda^2 |V_0|}{n} \right) > k \right) \\
\leq \mathbb{P}\left( \sum_{k \leq k_0} |V_k| \geq \Lambda |V_0| \right) + \mathbb{P}\left( \text{Bin} \left( \Lambda^2 |V_0|, \frac{2\Lambda^2 |V_0|}{n} \right) > k \right).
\]

By Lemma 4.16 and the fact that \( \Lambda |V_0| \leq n^{\frac{1}{2} - \delta} \) for \( n \) large, as long as \( \Lambda \) is large enough, the first term is at most \( n^{-5} \). Using the fact that \( |V_0| \leq n^{\frac{1}{2} - \delta} \), we see that the mean of the binomial is at most \( n^{-3\delta/2} \), so that by a Chernoff bound for the Poisson binomial distribution, for every \( k \geq 1 \),

\[
\mathbb{P}\left( (\mathcal{G}, X^1_{G,t}) : |\mathcal{M}_{v,R}(X^1_{G,t})| > k \right) \leq n^{-\delta k + 4}, \tag{4.5}
\]

for \( n \) large enough. Choosing \( k = K \) sufficiently large (depending on \( \delta \)), we can make the right-hand side at most \( n^{-4} \). We deduce the proposition by using Markov's inequality to write

\[
P_{CM}(G : P(X^1_{G,t} : |\mathcal{M}_{B_R(v)}(X^1_{G,t})| \geq K) > n^{-2}) \leq n^2 E_{CM}[P(X^1_{G,t} : |\mathcal{M}_{B_R(v)}(X^1_{G,t})| \geq K)],
\]

and noticing that the expectation on the right equals the probability bounded in (4.5).

We can now straightforwardly conclude the desired \((K, R)\)-sparsity of \( X^1_{G,t} \).

**Proof of Theorem 3.4.** First of all, a union bound of Proposition 4.2 over \( v \in \{1, \ldots, n\} \), with \( P_{CM} \)-probability \( 1 - O(n^{-1}) \), \( \mathcal{G} \) is such that

\[
P\left( X^1_{G,t} : \bigcup_{v \in V(\mathcal{G})} \{ |\mathcal{M}_{B_R(v)}(X^1_{G,t})| > K \} \right) \leq n^{-1}.
\]

We now translate this to a bound under \( P_{RRG} \). Taking

\[
\Gamma = \left\{ \mathcal{G} : P\left( X^1_{G,t} : \bigcup_{v \in V(\mathcal{G})} \{ |\mathcal{M}_{B_R(v)}(X^1_{G,t})| > K \} \right) > n^{-1} \right\},
\]

in (2.1), we deduce that \( P_{RRG}(\Gamma) \leq c^{-1} P_{CM}(\Gamma) \leq c^{-1} n^{-1} \) for some \( c(\Delta) > 0 \), as needed. \( \Box \)

## 5. Sharp Rates of Correlation Decay in Trees and Treelike Graphs

In this section we establish the precise exponential decay rate of influence from an \( O(1) \)-Sparse boundary condition on the root of an \( O(1) \)-Treelike ball. We recall from Section 3, that getting the right decay rate, (as opposed to e.g., using the decay rate of connectivity from the root to the boundary) is central to pushing our argument through for all \( p < p_c \). In Section 5.1, we recursively analyze the root-to-leaf connectivity rate on the wired tree and establish Lemma 2.5. In Section 5.2, we will show that influence in the random-cluster model travels through the existence of two distinct connections; thus on Treelike graphs, influence has twice the exponential decay rate of root-to-leaf connectivities on the wired tree. This will yield Proposition 3.6.
5.1. Exponential decay rate in the wired $\Delta$-regular tree. Because of its recursive structure, connectivity properties of the random-cluster measure on the wired tree can be analyzed sharply. In this section, we pursue this and show that in the uniqueness regime of $p < p_u$, the probability of a connection from the root to a leaf at depth $h$ is $O(\hat{p}^h)$, as one would have for the free tree (corresponding to i.i.d. Ber($\hat{p}$) percolation on $T_h$). We first show that the probability of a root-to-boundary connection decays exponentially in $h$.

Let $T_h$ be the complete $\Delta$-regular tree of height $h$ rooted at $\rho$. The wired “1” boundary conditions on $T_h$ are those that wire all leaves of $T_h$ (all vertices in $\partial T_h$). Define the probability

$$\varphi_h := \pi_{T_h}(\omega : C_\rho(\omega) \cap \partial T_h \neq \emptyset),$$

that the root is connected to a leaf of $T_h$. Using the recursive structure of the tree, it was shown in [3, Lemma 33] that if we define $\mu := \frac{p}{q} + 1 - p$, for every $h$, we have

$$\varphi_{h+1} = f(\varphi_h), \quad \text{where} \quad f(x) = \frac{\left(\mu + p(1 - \frac{1}{q})x\right)^d - \left(\mu - \frac{p}{q}x\right)^d}{\left(\mu + p(1 - \frac{1}{q})x\right)^d + x(q - 1)(\mu - \frac{p}{q}x)^d},$$

and for every $p < p_u(q, \Delta)$, this satisfies $\lim_{h \to \infty} \varphi_h = 0$. The following lemma establishes that this convergence is exponentially fast.

**Lemma 5.1.** Let $p < p_u(q, \Delta)$. We have $\lim_{h \to \infty} \frac{\varphi_{h+1}}{\varphi_h} = \hat{p}$. Moreover, $\varphi_h \leq (\hat{p}d)^{h+o(h)}$.

**Proof.** Consider the recursion of (5.1) for $\varphi_h$. Since $\lim_{h \to \infty} \varphi_h = 0$, if $\lim_{x \to 0} \frac{f(x)}{x}$ exists, we would have

$$\lim_{h \to \infty} \frac{\varphi_{h+1}}{\varphi_h} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{\left(\mu + p(1 - \frac{1}{q})x\right)^d - \left(\mu - \frac{p}{q}x\right)^d}{\left(\mu + p(1 - \frac{1}{q})x\right)^d + x(q - 1)(\mu - \frac{p}{q}x)^d}. \quad (5.2)$$

Since both the numerator and denominator of (5.2) are differentiable and have limit 0 as $x \to 0$, using L'Hôpital's rule we get

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{\partial_x \left[\left(\mu + p(1 - \frac{1}{q})x\right)^d - \left(\mu - \frac{p}{q}x\right)^d\right]}{\partial_x \left[\left(\mu + p(1 - \frac{1}{q})x\right)^d + x(q - 1)(\mu - \frac{p}{q}x)^d\right]} = \frac{d \hat{p} (1 - \frac{1}{q}) \mu^{d-1} + d \frac{p}{q} \mu^{d-1}}{\mu^d + (q - 1) \mu^{d-1}} = \frac{d \hat{p}}{p + q(1 - p)} = d \hat{p}.$$

Recall that for every $0 < p < p_u$, we have $0 < d \hat{p} < 1$. Thus, there exists a sequence $\{\varepsilon_h\}$ such that $\lim_{h \to \infty} \varepsilon_h = 0$ and for every $h$,

$$\varphi_h = \varphi_1 \cdot \frac{\varphi_2}{\varphi_1} \ldots \frac{\varphi_h}{\varphi_{h-1}} = \varphi_1 \cdot \prod_{i=2}^{h} (\hat{p}d + \varepsilon_i).$$

Expanding this out, we deduce the desired

$$\varphi_h = \varphi_1(\hat{p}d)^h \exp \left(\sum_{i=2}^{h} \ln \left(1 + \frac{\varepsilon_i}{\hat{p}d}\right)\right) \leq \varphi_1(\hat{p}d)^h \exp \left((\hat{p}d)^{-1} \sum_{i=2}^{h} \varepsilon_i\right) = (\hat{p}d)^{h+o(h)}. \quad \Box$$

Our aim is to now prove Lemma 2.5, bounding connectivities of the root to a single leaf.

**Proof of Lemma 2.5.** To prove Lemma 2.5, we write a recursion for the root-to-leaf connection probability. Let $\vartheta_h$ be the probability under $\pi_{1_{T_h}}$ that the root is connected to the left-most leaf of depth $h$. Let $\vartheta_h^\triangleright$ be the probability of the same event, under $\pi_{T_h}^{(1,\triangleright)}$ where we recall that the $(1, \triangleright)$ boundary conditions additionally wire the leaves of $T_h$ to the root. By monotonicity we have $\vartheta_h \leq \vartheta_h^\triangleright$ and by Lemma 2.8, we have $\vartheta_h^\triangleright \leq q^2 \vartheta_h$. Therefore,
Let \((I_i)_{i \leq \Delta}\) be the indicator function of the event that there is a root-to-boundary path going through the \(i\)-th child of the root; set \(I = \sum_{i=2}^{\Delta} I_i\). Then, we can write
\[
\vartheta_{h+1} \leq p \cdot \pi_{T_h}^I (I \geq 1) \cdot \vartheta_h + \hat{p} \vartheta_h \leq \vartheta_h \left[ pq^2 \cdot \pi_{T_h}^I (I \geq 1) + \hat{p} \right],
\]
where in the first inequality we used the fact that in order for the root to be connected to the left-most leaf, it is required that the root is connected to its left-most child. The former event occurs with probability \(p\) or \(\hat{p}\), depending on whether or not the root is connected to \(\partial T_h\) through any child besides \(w_1\).

By monotonicity, for every \(i = 2, ..., \Delta\), the law of \(I_i\) under \(\pi_{T_h}^I\) is below its law under \(\pi_{T_h}^{(1,\cdot)}\) and the same holds for \(I\). Since, by Lemma 2.8 a single external wiring may distort the distribution by at most a \(q^2\) factor, we get
\[
\pi_{T_h}^{(1,\cdot)} (I \geq 1) \leq pq^2 \varphi_h \quad \text{for all } i.
\]
Hence, under \(\pi_{T_h}^{(1,\cdot)}\), \(I\) is stochastically below \(Q\), where \(Q \sim \text{Bin}(d, \hat{p}q^2 \varphi_h)\). A union bound and Proposition 5.1 then imply
\[
\pi_{T_h}^{(1,\cdot)} (I \geq 1) \leq \mathbb{P}(Q \geq 1) \leq dpq^2 (\hat{p}d)^{h-o(h)} \leq C (\hat{p}d)^{(1-\varepsilon)h},
\]
for all \(h\); note that \(\varepsilon\) can be chosen as small as needed provided the constant \(C(p, q, \Delta, \varepsilon)\) is large enough. Thus, setting \(a = Cpq^2 \hat{p}^{-1}\) we obtain
\[
\vartheta_{h+1} \leq \hat{p} \vartheta_h \left[ 1 + a (\hat{p}d)^{(1-\varepsilon)h} \right] \leq \hat{p}^{h+1} \prod_{i=1}^{h} \left[ 1 + a (\hat{p}d)^{(1-\varepsilon)i} \right],
\]
by continuing the recursion. Now, observe that since \(\hat{p}d < 1\) when \(p < p_u\),
\[
\prod_{i=1}^{h} \left[ 1 + a (\hat{p}d)^{(1-\varepsilon)i} \right] = \exp \left[ \sum_{i=1}^{h} \log \left( 1 + a (\hat{p}d)^{(1-\varepsilon)i} \right) \right] \leq \exp \left[ a \sum_{i=1}^{h} (\hat{p}d)^{(1-\varepsilon)i} \right] \leq \exp \left[ \frac{a}{(\hat{p}d)^{1-\varepsilon}} \right].
\]
Combining the above two bounds, there exists an absolute constant \(A = A(p, q, \Delta)\) such that for every \(h\) we have \(\vartheta_h \leq Ah^h\) and thus \(\vartheta_h^{(1,\cdot)} \leq Aq^2 \hat{p}^h\). The first inequality in the lemma follows by noticing that all the leaves in \(T_h\) are equivalent, and the second follows from a union bound over the \(\Delta d^{-h}\).

\section{Exponential decay rate in \((L, R)\)-Treelike graphs.}

Let \(G = (V, E)\) be an \((L, R)\)-Treelike graph. For \(v \in V\), let \(B := B_R(v)\) denote the ball of radius \(R\) around the vertex \(v\). Recall that we use \(N_v \subseteq E\) for the set of edges incident to \(v\). For each \(1 \leq \ell \leq R\), let \(Q_{\ell} = \{ u \in B : d(u, v) > \ell \}\).

For a boundary condition \(\xi\) on \(\partial B\), recall the set \(\mathcal{W}_{B,\xi}\) of vertices in non-trivial boundary components of \(\xi\) from Definition 4.1. For any \(u \in B\) such that \(d(u, v) = \ell\), let \(u \xrightarrow{Q_{\ell}} \mathcal{W}_{B,\xi}\) denote the event that \(u\) is connected to \(\mathcal{W}_{B,\xi}\) by a path of open edges fully contained in \(Q_{\ell}\); i.e.,
\[
\{ u \xrightarrow{Q_{\ell}} \mathcal{W}_{B,\xi} \} := \{ \omega : C_u(\omega(Q_{\ell})) \cap \mathcal{W}_{B,\xi} \neq \emptyset \}.
\]
Define the event
\[
\Upsilon_{B,\xi} := \{ \omega \in \{0, 1\}^{E(B)} : \{ u \in B : d(u, v) = \ell, \; u \xrightarrow{Q_{\ell}} \mathcal{W}_{B,\xi} \} \geq 2 \text{ for all } 1 \leq \ell \leq R \}.
\]
Notice that \(\Upsilon_{B,\xi}\) is an increasing event. We claim that \(\Upsilon_{B,\xi}\) controls the propagation of influence from \(\partial B\).

\begin{lemma}
Fix a graph \(G = (V, E)\), a vertex \(v \in V\) and consider the ball \(B_R(v)\); let \(\xi \geq \tau\) denote two boundary conditions on \(\partial B_R(v) = \{ w \in B_R(v) : d(v, w) = R \}\). Then,
\[
\| \pi_{B_R(v)}^\xi (\omega(N_v) \in \cdot) - \pi_{B_R(v)}^\tau (\omega(N_v) \in \cdot) \|_{TV} \leq \pi_{B_R(v)}^\xi (\Upsilon_{B_R(v), \xi}) \cdot \pi_{B_R(v)}^\tau (\Upsilon_{B_R(v), \xi}).
\]
\end{lemma}

\begin{proof}[Proof of Lemma 5.2]
For ease of notation let \(B := B_R(v)\), \(\Upsilon_\xi = \Upsilon_{B,\xi}\) and \(\mathcal{W}_\xi = \mathcal{W}_{B,\xi}\). We construct a monotone coupling \(\mathbb{P}\) of \(\omega_\xi \sim \pi_\xi_B\) and \(\omega_\tau \sim \pi_\tau_B\). The coupling \(\mathbb{P}\) reveals the configurations \(\omega_\xi \sim \pi_\xi_B\) and \(\omega_\tau \sim \pi_\tau_B\) on \(B\) one edge at a time using i.i.d. uniform random variables \(U_e \in [0, 1]\) for each \(e \in E(B)\). The same \(U_e\) is used to reveal the values \(\omega_\xi(e)\) and \(\omega_\tau(e)\) from the corresponding conditional measures. The order in which the uniform variables are revealed is irrelevant and can be adaptive; this will allow us to
reveal the boundary components. (For more details on the process of revealing random-cluster components under the monotone coupling, see below, as well as e.g., [4, 6].)

We construct an adaptive revealing scheme that ensures that on the event $\Upsilon_\xi^c$ for the top sample $\omega^\xi$, the samples $\omega^\xi$ and $\omega^\tau$ agree on $\mathcal{N}_c$. This implies the desired result as one would then have

$$
\|\pi_B^\xi(\mathcal{N}_c) - \pi_B^\tau(\mathcal{N}_c)\|_{TV} \leq \mathbb{P}(\omega^\xi(\mathcal{N}_c) \neq \omega^\tau(\mathcal{N}_c)) \leq \pi_B^\xi(\Upsilon_{B,\xi}).
$$

We construct $\mathbb{P}$ with the following iterative scheme which proceeds level-by-level from the leaves of $B$. Recall that for each $\ell \geq 1$, we let $Q_\ell = \{u \in B : d(u, v) \geq \ell\}$ and $E(Q_\ell)$ is the set of edges with both endpoints in $Q_\ell$. At any time in the revealing process, we say that a vertex $u \in Q_\ell$ is unsaturated in $Q_\ell$ if there exists $w \in Q_\ell$ such that the edge-values $(\omega^\xi(uw), \omega^\tau(uw))$ have not been revealed. Let $(U_e)_{e \in E(B)}$ be a family of i.i.d. uniform random variables on $[0, 1]$ and reveal the configuration $\omega^\xi$ as follows:

| Definition 5.3. Initialize $\mathcal{V}_\xi = \emptyset$ and $\mathcal{E}_\xi = \emptyset$; for $i = 1, 2, \ldots, R$ do
|   while $3u \in \mathcal{V}_\xi$ such that $u$ is unsaturated in $Q_{R-i}$
|     for each vertex $w \in Q_{R-i} : uw \in E(Q_{R-i})$
|       1. Reveal $\omega^\xi(uw)$ from $\pi_B^\xi(\cdot \mid \omega(\mathcal{E}_\xi))$ using $U_{uw}$, i.e., set $\omega^\xi(uw) = \begin{cases} 1 & \text{if } \pi_B^\xi(\omega(uw) = 1 \mid \omega(\mathcal{E}_\xi)) \geq U_{uw}; \\ 0 & \text{else} \end{cases}$
|       2. Add the edge $uw$ to the set $\mathcal{E}_\xi$;
|       3. If $\omega^\xi(uw) = 1$, add the vertex $w$ to $\mathcal{V}_\xi$;
| |
| Note that we can use the same family $(U_e)_{e \in E(B)}$ in this process to generate coupled samples of $\omega^\xi$ and $\omega^\tau$. Notice that this coupling is monotone, so that because $\xi \geq \tau$, $\omega^\xi \geq \omega^\tau$ almost surely. Let $C^i_{\mathcal{B}}(\omega^\xi)$ denote the set of open edges revealed up to the $i$-th iteration of the procedure; we observe that $C^i_{\mathcal{B}}(\omega^\xi)$ is not necessarily equal to the intersection of $C^i_{\mathcal{B}}(\omega^\xi)$ with $E(Q_{R-i})$, but it is a subset of $C^i_{\mathcal{B}}(\omega^\xi) \cap E(Q_{R-i})$. Refer to Figure 5.1 for a depiction of the above revealing procedure.

Through this revealing process, we see that $\omega^\xi$ is open on the edges in the random set $C^i_{\mathcal{B}}(\omega^\xi)$ and free on the edges in its outer (edge) boundary in $Q_{R-i}$. Let $\bar{C}^i_{\mathcal{B}}(\omega^\xi)$ be the union $C^i_{\mathcal{B}}(\omega^\xi)$ with its outer (edge) boundary in $Q_{R-i}$, and note that this corresponds to the state of $\mathcal{E}_\xi$ after the $i$-th iteration. The random set $C^i_{\mathcal{B}}(\omega^\xi)$ is measurable with respect to the uniform random variables assigned to edges of $\bar{C}^i_{\mathcal{B}}(\omega^\xi)$.

For each $C^i_{\mathcal{B}}(\omega^\xi)$, let $\mathcal{V}^i(\omega^\xi)$ be the vertices in $C^i_{\mathcal{B}}(\omega^\xi)$ at distance exactly $R - i$ from $v$. Then,

$$
\mathcal{V}^i(\omega^\xi) \subseteq C^i_{\mathcal{B}}(\omega^\xi) \cap \{w : d(w, v) = R - i\}.
$$

On $\Upsilon_{B,\xi}$, there must be some $i$ for which $|C^0_{\mathcal{B}}(\omega^\xi) \cap \{w : d(w, v) = R - i\}| \leq 1$, and therefore $|\mathcal{V}^i(\omega^\xi)| \leq 1$. Let $i_0$ be the first $i$ for which $|\mathcal{V}^{i_0}(\omega^\xi)| \leq 1$, and for ease of notation set $\mathcal{V}_0 = \mathcal{V}^{i_0}(\omega^\xi)$, $\bar{C}_{\mathcal{B},0} = \bar{C}^{i_0}_{\mathcal{B}}(\omega^\xi)$ and set $\bar{C}^{i_0}_{\mathcal{B},0} = E(B) \setminus \bar{C}_{\mathcal{B},0}$. Notice the inclusion

$$
\bar{C}^{i_0}_{\mathcal{B}}(\omega^\xi) \subset \bar{C}^{i_0+1}_{\mathcal{B}}(\omega^\xi),
$$

and from that deduce that $i_0$ is measurable with respect to the uniform random variables assigned to edges of $\bar{C}^{i_0}_{\mathcal{B}}(\omega^\xi)$. By the domain Markov property, conditionally on $\omega^\xi(\bar{C}^{i_0}_{\mathcal{B},0})$, the configuration $\omega^\xi(\bar{C}^{i_0}_{\mathcal{B},0})$ is distributed according to the random-cluster distribution on $\bar{C}^{i_0}_{\mathcal{B},0}$ with boundary conditions induced by $\xi$ and $\omega^\xi(\bar{C}^{i_0}_{\mathcal{B},0})$, respectively $\tau$ and $\omega^\tau(\bar{C}^{i_0}_{\mathcal{B},0})$.

To conclude the proof, it suffices to see that because $|\mathcal{V}_0| \leq 1$, both $\omega^\xi(\bar{C}^{i_0}_{\mathcal{B},0})$ and $\omega^\tau(\bar{C}^{i_0}_{\mathcal{B},0})$ induce the free boundary conditions on $\bar{C}^{i_0}_{\mathcal{B},0}$. In that case $\omega^\xi$ and $\omega^\tau$ would agree on $\bar{C}^{i_0}_{\mathcal{B},0}$ and in particular on
Figure 5.1. Top: The ball $B_R(v)$ for $R = 5$, with a $K$-sparse boundary condition $\tau$ for $K = 4$ (left), and the free boundary condition $\xi = 0$ (right). Bottom: The configurations revealed by the procedure of Definition 5.3, showing $C_i^\tau(\omega^\tau)$ (red, left) along with its outer edge boundary in $Q_i$ (light blue), revealing the dotted line (depth $i_0$) to be the largest $i$ for which the set $V_i$ is a singleton. The vertices that would have been exposed for larger values of $i$ are colored in different colors. The coupled edge configuration $\omega^0(C_i^\tau(\omega^\tau))$ is depicted on the right (open edges in red, closed edges in blue). The exposed configurations on $C_i^\tau(\omega^\tau)$ induce free boundary conditions on $E(B) \setminus \bar{C}_{i_0}(\omega^\tau)$.

$N_v$. By monotonicity, it suffices for us to show that the boundary conditions induced by $\xi$ and $\omega^\xi(\bar{C}_{i_0},\omega^\tau)$ on $\bar{C}_{i_0}$ are free. Since the wirings of $\xi$ are only on vertices of $\mathcal{V}_\xi \subset \bar{C}_{i_0}$, the only way for the boundary conditions on $\bar{C}_{i_0}$ to be not free is if multiple vertices on its boundary are incident to open edges of $\omega^\xi(\bar{C}_{i_0},\omega^\tau)$.

By construction, the only vertices in $\bar{C}_{i_0}$ which can be incident to an open edge of $\omega^\xi(\bar{C}_{i_0},\omega^\tau)$ must be at distance exactly $R - i_0$ from $v$. By the assumption that $|V_{i_0}| \leq 1$, there can be at most one such vertex, and therefore there are no non-trivial (i.e., non-singleton) boundary components induced on $\bar{C}_{i_0}$ by the boundary condition $\xi, \omega^\xi(\bar{C}_{i_0},\omega^\tau)$, implying the desired conclusion. □

Proof of Proposition 3.6. With Lemma 5.2 in hand, it suffices for us to prove the following: there exists $C(p, q, K, L) > 0$ such that if $G = (V, E)$ is an $(L, R)$-Treelike graph and $\xi$ is a $K$-Sparse boundary condition for the $L$-Treelike ball $B := B_R(v)$ about some $v \in V$, we have

$$\pi^\xi_B(\Upsilon_{B,\xi}) \leq C p^{2R}. \quad (5.3)$$

Let $H \subset E(B)$ be a set of at most $L$ edges such that the subgraph $(V, E(B) \setminus H)$ is a tree; the existence of such a set is guaranteed by the fact that $B_R(v)$ is $L$-Treelike. Let $\mathcal{Z} = \{d_1, ..., d_k\}$ be the subset of distances (from $v$) which $H$ intersects, i.e., $\mathcal{Z} = \{1 \leq \ell < R : \exists w \in V(H) : d(w, v) = \ell\}$. See Figure 5.2 for a depiction. Observe that each edge of $H$ intersects either one or two consecutive depths in $\mathcal{Z}$. Since $B$ is $L$-Treelike, we clearly have $|\mathcal{Z}| \leq 2L$. Letting $d_0 = 0$ and $d_{k+1} = R$, for $i = 0, \ldots, k$ we define:

$$\mathcal{F}_i := \{u \in B : d_i < d(u, v) < d_{i+1}\}.$$
For each $0 \leq i \leq k$, the graph $F_i = (F_i, E(F_i))$ is a forest. For each $i$, let $T_{ij} = (T_{ij}, E(T_{ij}))$ for $j = 0, 1, \ldots$ denote the distinct connected components (subtrees) of $F_i$ so that $F_i = \bigcup_{j \geq 0} T_{ij}$. (For some $i$, this may be empty, and for other $i$, this may be a single vertex.)

Now, in order for $\Upsilon_{B,\xi}$ to hold, it must be the case that in each $F_i$, every depth $\ell$ is intersected by at least two sites in the FK cluster of $\mathcal{U}_{B,\xi}$ in $\mathcal{Q}_\ell$. Specifically, for each $i$, at distance $d_i + 1$ from $v$ there must be at least two distinct vertices connected to $\mathcal{U}_{B,\xi}$ with paths in $\mathcal{Q}_{d_i + 1}$. Thus, for each $i$ there must exist two open monotone paths (each intersecting each height in $F_i$ at exactly one vertex), $\gamma_i \subset E(T_{ij})$ and $\gamma_i' \subset E(T_{ij'})$ with $j \neq j'$ such that $\gamma_i$ (resp., $\gamma_i'$) connects the root of $T_{ij}$ (resp., $T_{ij'}$) to one of its leaves. If there are multiple such paths, choose according to some predetermined ordering, and call the sequences of paths $\Gamma = \gamma_0, \ldots, \gamma_k$ and $\Gamma' = \gamma'_0, \ldots, \gamma'_{k'}$. See Figure 5.3 for a depiction.

We enumerate over the choices of such sequences of paths and then show that for any two fixed sequences of paths, the probability that they are both open is bounded by $Cp^{2R}$ for some $C(p, q, \Delta, K, L)$. (We say that a sequence of paths is open if all of its paths are.)

In order to enumerate over the choices of sequences of paths, for each monotone path $\gamma_i$, let $x_i$ be its bottom endpoint, and define $x'_i$ for $\gamma'_i$ similarly. Since $\xi$ is $K$-Sparse, there are evidently at most $K$ many choices of $x_0$ and $K$ choices of $x'_0$. Now observe that since $\gamma_i$ is a monotone path on a tree, for each $i$, the bottom endpoint $x_i$ determines the entire path $\gamma_i$. Since these paths form parts of the connections to $\mathcal{U}_{B,\xi}$, the sequence of paths can be required to have endpoints at depths $d_i + 1 - 1$ that are either an ancestor of $x_0$, or an ancestor of $V(H)$. Here, at each height $h \notin \mathcal{Z}$ an ancestor of a vertex $u$ at height $h$ is a vertex along the geodesic from $v$ to $u$. We make the following observation.

**Claim 5.4.** If $B_R(v)$ is $L$-Treelike, if $u$ is such that $d(u, v) = h$, for every $h' < h$, $u$ has at most $2^L$ many ancestors at height $h'$.

Indeed, except along the edges in $H$, every vertex has a unique parent which is an ancestor of that vertex at one smaller depth. Thus, the geodesics of $B$ are uniquely determined by their endpoints together, possibly, with a subset of edges of $H$ traversed along the geodesic, yielding the at most $2^L$ available choices.

Returning to the enumeration over $\Gamma, \Gamma'$, the heights of the endpoints $x_i, x'_i$ are predetermined by $i$, and therefore, having chosen $x_0, x'_0$ for each $i$, there are at most $2L$ many choices of bottom end-point $x_i$, and likewise of $x'_i$, and therefore at most $2L \cdot 2^L$ many choices of $\gamma_i$ and $\gamma'_i$.

Hence, a union bound implies
\[
\pi^\xi_B(\Upsilon_{B,\xi}) \leq K^2(2L)^{2L}(2^L)^{2L} \sup_{\Gamma, \Gamma'} \pi^\xi_B(\omega(\Gamma \cup \Gamma') = 1). \tag{5.4}
\]

Now fix any two such sequences of paths $\Gamma, \Gamma'$, and consider the probability that $\omega(\Gamma \cup \Gamma') = 1$. Observe that $\Gamma$ and $\Gamma'$ are vertex-disjoint by construction. Our aim is to make the events that $\Gamma$ and $\Gamma'$ are open in $\omega$
For this, let \( \rho_i \) be the set of roots of the trees in \( F_i \). We introduce auxiliary wirings (as shown in Figures 5.2–5.3) for all vertices at depths \( \{d : \min_{i=0, \ldots, k+1} |d - d_i| \leq 1\} \). Call the resulting distribution \( \tilde{\pi}_B \); by monotonicity,

\[
\pi_{B}^{\xi}(\omega(\Gamma \cup \Gamma') = 1) \leq \tilde{\pi}_B(\omega(\Gamma \cup \Gamma') = 1). \tag{5.5}
\]

The distribution \( \tilde{\pi}_B \) is a product measure over the \( T_{ij} \)'s with boundary condition \((1, \emptyset)\) in each \( T_{ij} \) (recall that this boundary condition wires all leaves \( \partial T_{ij} \) together with the root of \( T_{ij} \)). Hence, since \( \Gamma \) and \( \Gamma' \) are such that, for each \( i \geq 0 \), \( \gamma_i \) and \( \gamma_i' \) belong to distinct subtrees \( T_{ij}, T_{ij}' \) of the forest \( F_i \), and we have

\[
\tilde{\pi}_B(\omega(\Gamma \cup \Gamma') = 1) = \prod_{i=0}^{k} \pi_{T_{ij}}^{(1,\emptyset)}(\gamma_i) \prod_{i=0}^{k} \pi_{T_{ij}'}^{(1,\emptyset)}(\gamma_i').
\]

Let \( h_i = d_{i+1} - d_i \) be the height of the trees in \( F_i \). We deduce from Lemma 2.5 that there exists a constant \( A(p, q, \Delta) > 0 \) such that uniformly over \( \Gamma, \Gamma' \),

\[
\tilde{\pi}_B(\omega(\Gamma \cup \Gamma') = 1) \leq A^{2L} \prod_{i=0}^{k} p^{2h_i} \leq A^{2L} \tilde{p}^{2(R-4L)}.
\]

Plugging this bound into (5.4)–(5.5), we obtain

\[
\pi_{\xi}^{B}(\Upsilon_{B, \xi}) \leq K^2((2L)^2)\tilde{p}^{-4}A^{2L} \tilde{p}^{2R},
\]

from which the required (5.3) follows.

\[\square\]

**Remark 5.5.** A matching lower bound of \( \Omega(p^{2R}) \) for the decay rate in Proposition 3.6 is easy to construct by e.g., taking the \( K \)-Sparse boundary conditions \( \xi \) that wires two leaves \( w_1, w_2 \) on distinct sub-trees of \( v \), and the free boundary conditions \( \xi' = 0 \) on \( T_{R} \). The event that the root is connected to \( w_1 \) and its corresponding child is connected to \( w_2 \) has probability at least \( C \tilde{p}^{2R} \) by Lemma 2.5 and the FKG inequality (see e.g., [28]). On this event, the probability that the edge incident \( v \) down towards \( w_2 \) is open is \( p \) under the boundary condition \( \xi \) and \( \tilde{p} \) under \( \xi' = 0 \).

6. PROOF OF FAST MIXING

In this section, we combine the results of Sections 4–5 to conclude the proof of Theorem 1.1. As indicated in Section 3, the analysis of Sections 4–5 reduce bounding the mixing time of the FK-dynamics on a random graph to its local mixing time on \( O(1) \)-Treelike balls of volume \( O(n^{1/2 - \delta}) \) with \( O(1) \)-Sparse boundary conditions. In Section 6.1, we bound this local mixing time via a straightforward comparison argument. Then in Section 6.2, we proceed to combine all of the above ingredients to deduce the proof of Theorem 1.1 using the censoring inequalities of [41].
6.1. Local mixing: fast mixing on treelike graphs with sparse boundary conditions. In this section we establish a mixing time bound for the FK-dynamics on \( L \)-Treelike balls with \( K \)-Sparse boundary conditions (see Definitions 2.1 and 3.2). Our goal is to prove Lemma 3.5 by comparing the mixing time on an \( L \)-Treelike ball with \( K \)-Sparse boundary to a tree with \( K \)-Sparse boundary conditions, whose log-Sobolev constant is bounded as follows by comparison to a product chain. For a Markov chain with transition matrix \( P \) and Dirichlet form \( \mathcal{E}(f, f) \), recall that the log-Sobolev constant is given by

\[
\alpha(P) = \min_{f: \text{Ent}_{\pi}[f^2]} \frac{\mathcal{E}(f, f)}{\text{Ent}_{\pi}[f^2]}, \quad \text{where} \quad \text{Ent}_{\pi}[f^2] = \mathbb{E}_{\pi}[f^2 \log f^2].
\]

(6.1)

We first note the following bound on the log-Sobolev constant on trees with sparse boundary conditions.

**Corollary 6.1.** There exists \( c(p) > 0 \) such that the following holds. For every rooted (not necessarily complete) tree \( \hat{T}_h = (V(\hat{T}_h), E(\hat{T}_h)) \) of depth \( h \) and degree at most \( \Delta \), and every \( K \)-Sparse boundary condition \( \phi \) on \( \hat{T}_h \), the log-Sobolev constant of the FK-dynamics on \( \hat{T}_h \) with boundary conditions \( \phi \) is at least \( cq^{6K}\pi(\hat{T}_h)^{-1} \).

**Proof.** Consider the FK-dynamics on \( \hat{T}_h \) under the free boundary conditions. In this case, the random-cluster measure is a \( \text{Ber}(p) \) product measure and thus the log-Sobolev constant of the FK-dynamics is \( c|E(\hat{T}_h)|^{-1} \) for some \( c(p) > 0 \); see, e.g., [15]. The result then follows from Lemma 2.8 and Corollary 2.9. \( \square \)

To move from mixing on an \( L \)-Treelike ball to mixing on a tree, the following fact will be useful.

**Fact 6.2.** Let \( G \) be a subgraph of \( G' \) such that \( V(G) = V(G') \) and \( E(G) \subset E(G') \); let \( H = E(G') \setminus E(G) \). Suppose \( \phi \) is a boundary condition on \( G, G' \) such that for every \( e \in H \), the endpoints of \( e \) are wired in \( \phi \). For every \( p \in (0, 1) \) and \( q > 0 \), let \( P_G \) and \( P_{G'} \) be the transition matrices of the FK-dynamics on \( G \) and \( G' \), respectively, with boundary conditions \( \phi \), and let \( \alpha(P_G) \) and \( \alpha(P_{G'}) \) be their log-Sobolev constants. There exists a constant \( c(p) > 0 \) such that

\[
\alpha(P_{G'}) \geq \min \left\{ \frac{|E(G)|}{|E(G)| + |H|} \cdot \alpha(P_G), \frac{c|H|}{|E(G)| + |H|} \right\}.
\]

**Proof.** The FK-dynamics on \( G' \) is a product Markov chain on \( \{0, 1\}^{E(G')} \times \{0, 1\}^H \) with stationary distribution \( \pi_{G'} = \pi_G \otimes \prod_{e \in H} \nu_e \), where \( \nu_e \) are independent \( \text{Ber}(p) \) distributions over edges in \( H \). The result then follows from the fact that the \( \pi \)-Sobolev inequality (e.g., [42, Lemma 2.2.11]). \( \square \)

We can now combine the above ingredients to deduce the bound of Lemma 3.5.

**Proof of Lemma 3.5.** Let \( B = B_R(v) \) and let \( H \subset E(B) \) be a set of at most \( L \) edges such that \( (B, E(B) \setminus H) \) is a tree. Consider the tree \( \hat{T}_R = (V(B), E(B) \setminus H) \) and let \( \phi \) be the boundary condition that includes all the connections from \( \xi \) and adds wirings between \( w \) and \( w' \) for every edge \( ww' \in H \).

Corollary 6.1 implies that the log-Sobolev constant for the FK-dynamics on \( \hat{T}_R \) with boundary condition \( \phi \) is at least \( cq^{6(K+L)}\pi(\hat{T}_R)^{-1} \) for some \( c(p) > 0 \). We then get from Fact 6.2 that the log-Sobolev constant for the FK-dynamics on \( B \) with boundary condition \( \phi \) is at least \( cq^{6(k+L+12L)}\pi(B)^{-1} \). Lemma 2.8 and Corollary 2.9 then imply that the log-Sobolev constant on \( B \) with boundary conditions \( \xi \) is at least \( cq^{6K+12L}E(B)^{-1} \). The result follows from a standard comparison between the mixing time and the inverse of the log-Sobolev constant, bounding the former by the latter times \( \log(|E(B)|) \) (see e.g., [35]). \( \square \)

6.2. Proof of Theorem 1.1. Fix \( p < p_u(q, \Delta) \), let \( \varepsilon = 1 - \hat{p}d \) (positive when \( \hat{p} < p_u \)) and fix \( \delta > 0 \) small enough (depending on \( \varepsilon, \Delta \)) such that

\[
2\delta + (1 - 2\delta) \log_d(1 - \varepsilon) < 0
\]

in which case the following is polynomially decaying in \( n \):

\[
np^{(1-2\delta)\log_d n} = n^{d(1-2\delta)\log_d n} (1-\varepsilon)^{(1-2\delta)\log_d n} = n^{2\delta} (1-\varepsilon)^{(1-2\delta)\log_d n}.
\]

(6.2)
Let $R = \left(\frac{1}{2} - \delta\right) \log_q n$ and let $K$ be a constant sufficiently large (depending on $p, q, \Delta$) that both Fact 2.3 and Theorem 3.4 hold for $(K, R)$. For each $t$, let $\Gamma_t$ be the set of $\Delta$-regular graphs on $n$ vertices having

$$\Gamma_t = \{G : G \text{ is } (K, R)\text{-Treelike} \} \cap \{G : P(X^G_{1,t} \text{ is } (K, R)\text{-Sparse}) \geq 1 - n^{-2}\}.$$  

By Fact 2.3 and Theorem 3.4, there exists $C_0(p, q, \Delta)$ such that if $T = C_0 n \log n$, then $P_{\text{RRG}}(\Gamma^c_T) \leq o(1)$. It suffices for us to prove that the mixing time of the FK-dynamics on any $G \in \Gamma_T$ is $O(n(\log n)^2)$.

Fix any $G \in \Gamma_T$ and for every configuration $\omega$ on $E(G)$, let $X_t = X^G_{\omega_t}$ be the FK-dynamics chain on $G$ initialized from $X^0 = \omega$. couple the family of chains $\{(X^G_{\omega_t})_{t \geq 0} : \omega \in \{0, 1\}^{E(G)}\}$ using the grand coupling as in Definition 4.5: recall that this is the coupling that in each step picks the same random $e \in E(G)$ to update, and the same uniform random variable $U_{e,t}$ on $[0, 1]$ to decide the next state on the edge $e$. As mentioned earlier, this coupling is monotone when $q > 1$ so that for every $t \geq 0$, if $X^0_t \leq X^0_t$, then $X^0_t \leq X^0_t$. It follows from the definition of $t_{\text{mix}}$ and monotonicity of the grand coupling (see e.g., [35]), that it suffices for us to show that there exists $\bar{C}(p, q, \Delta)$ such that if $T = T + \bar{C}(n(\log n)^2),$

$$\mathbb{P}(X^1_t \neq X^0_t) \leq \frac{1}{4},$$

By a union bound over the $n$ edge-neighborhoods $N_v$ (edges of $G$ incident $v$), this reduces to showing

$$\sup_{v \in V(G)} \mathbb{P}(X^1_t(N_v) \neq X^0_t(N_v)) \leq \frac{1}{4n}. \quad (6.3)$$

Now fix any such $v$ and consider the probability above. For ease of notation, let $B_v = B_R(v)$ and $B^c_v = E(G) \setminus B_v$. Introduce two new Markov chains $Y^1_t$ and $Y^0_t$ that are coupled via the grand coupling to $X^1_t, X^0_t$ except that they censor (ignore) all updates on edges of $B^c_v$ after time $T = C_0 n \log n$. The censoring inequality [41, Lemma 2.3] implies the stochastic relations $Y^1_t \neq X^1_t$ and $Y^0_t \neq X^0_t$ for all $t \geq 0$ and thus

$$\mathbb{P}(X^1_t(N_v) \neq X^0_t(N_v)) \leq \Delta \sup_{e \in N_v} \mathbb{P}(X^1_t(e) \neq X^0_t(e)) \leq \Delta \sup_{e \in N_v} [\mathbb{P}(Y^1_t(e) = 1) - \mathbb{P}(Y^0_t(e) = 1)].$$

Fix any $e \in N_v$ and consider the difference in probabilities on the right-hand side. Let $\mathcal{E}_T$ be the event (measurable with respect to the first $T$ steps of the Markov chain) that the boundary conditions induced by $X^1_t(B^c_v)$ are $K$-Sparse. Observe that $K$-sparsity of a boundary condition is a decreasing event, so that on $\mathcal{E}_T$, the boundary conditions induced by $X^0_t(B^c_v)$ are also $K$-Sparse. As such, for all $t \geq T$,

$$\mathbb{P}(Y^1_t(e) = 1) - \mathbb{P}(Y^0_t(e) = 1) \leq \mathbb{P}(\mathcal{E}_T) + \mathbb{E}_{\phi^1, \phi^0(0, 1)^{B^c_v}} \mathbb{P}(Y^1_t(e) = 1 | Y^1_t(B^c_v) = \phi^1) - \mathbb{P}(Y^0_t(e) = 1 | Y^0_t(B^c_v) = \phi^0). \quad (6.4)$$

Since $G \in \Gamma_T$, and $Y^1_t = X^1_t$, the first term is at most $n^{-2}$. Turning to the second term, fix any two configurations $\phi^1, \phi^0$ on $B^c_v$ such that $\phi^0 \leq \phi^1$ and $\phi^1$ (and therefore also $\phi^0$) induce $K$-Sparse boundary conditions on $B_v$, and consider the difference

$$\mathbb{P}(Y^1_t(e) = 1 | Y^1_t(B^c_v) = \phi^1) - \mathbb{P}(Y^0_t(e) = 1 | Y^0_t(B^c_v) = \phi^0) \leq \mathbb{P}(Y^1_t(e) = 1 | Y^1_t(B^c_v) = \phi^1) - \mathbb{P}(Y^0_t(e) = 1 | Y^0_t(B^c_v) = \phi^1) \left| \mathbb{P}(Y^1_t(e) = 1 | Y^1_t(B^c_v) = \phi^1) - \mathbb{P}(Y^1_t(e) = 1 | Y^1_t(B^c_v) = \phi^0) \right| \left| \mathbb{P}(Y^0_t(e) = 1 | Y^0_t(B^c_v) = \phi^0) \right|. \quad (6.5)$$

Observe that $Y^0_{T+1}(B_v)$ is distributed as a lazy FK-dynamics chain $Z^1_e$ on $B_v$ with boundary conditions induced by $\phi^1$, initialized from the random configuration $Z^0_e(B_v) = Y^1_{T}(B_v)$: the laziness is in the choice that at each step, $Z^1_e$ makes an FK-dynamics update on $B_v$ with probability $|E(B_v)|/|E(G)|$ and makes no update otherwise. The analogous statement holds for $Y^0_{T+1}(B_v)$ with respect to some lazy chain $Z^0_e$. The invariant measure of $Z^1_e$ is easily seen to be

$$\pi_{\mathcal{E}}(\omega(B_v) \in \cdot | \omega(B^c_v) = \phi^1) = \pi_{\phi^1} B_v,$$
and the analogous statement holds for $Z^0_S$.

Now let $\hat{T} = T + \hat{S}$ where $\hat{S} = \hat{C} n (\log n)^2$ for a constant $\hat{C}$ to be chosen sufficiently large depending on $p, q, \Delta$. The expected number of updates in $B_v$ between time $T$ and $T + \hat{S}$ is

$$\hat{C} n (\log n)^2 \cdot \frac{|E(B_v)|}{|E(G)|} \geq 2 \Delta^{-1} \hat{C} |E(B_v)| (\log n)^2.$$ 

Let $C_1$ be the constant from Lemma 3.5. For any $C_2$, if $\hat{C}$ is sufficiently large, a Chernoff bound implies that with probability $1 - n^{-2}$, at least $C_1 C_2 |E(B_v)| (\log(|E(B_v)|)) \log n$ updates are made in $B_v$ between times $T$ and $\hat{T}$. By $K$-sparsity of $\phi^1$, Lemma 3.5, and sub-multiplicativity of total variation distance, we have for $\hat{C}$ sufficiently large, that the term in (6.5) is bounded by

$$\|\mathbb{P}(Z_S^1(B_v) \in \cdot) - \pi_{B_v}^{\phi^{1}}\|_{TV} < 2 n^{-2}.$$ 

By the same reasoning, by $K$-sparsity of $\phi^{0}$, the same bound applies to (6.7).

Finally, since both $\phi^{1}$ and $\phi^{0}$ induce $K$-Sparse boundary conditions on $B_v$, by Proposition 3.6 there exists a constant $C(p, q, \Delta, K) > 0$ such that (6.6) is at most

$$\|\pi_{B_v}^{\phi^{1}}(\omega(N_v) \in \cdot) - \pi_{B_v}^{\phi^{0}}(\omega(N_v) \in \cdot)\|_{TV} \leq C p^{2 R},$$

which is $o(n^{-1})$ by our choice of $\delta$ and (6.2). Putting these three bounds together we see that as long as $\hat{C}$ is sufficiently large (depending on $p, q, \Delta$) the difference in (6.4) is $o(n^{-1})$, from which the bound of (6.3) follows for $n$ sufficiently large, concluding the proof.  

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