Large finite group actions on surfaces: Hurwitz groups, maximal reducible and maximal handlebody groups, bounding and non-bounding actions

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Abstract. We consider large finite group-actions on surfaces and discuss and compare various notions for such actions: Hurwitz actions and Hurwitz groups; maximal reducible and completely reducible actions; bounding and geometrically bounding actions; maximal handlebody groups and maximal bounded surface groups; in particular, we discuss small simple groups of various types.

A Hurwitz group is a finite group of orientation-preserving diffeomorphisms of maximal possible order $84(g - 1)$ of a closed orientable surface of genus $g > 1$. A maximal handlebody group instead is a group of orientation-preserving diffeomorphisms of maximal possible order $12(g - 1)$ of a 3-dimensional handlebody of genus $g > 1$. Among others, we consider the question of when a Hurwitz group acting on a surface of genus $g$ contains a subgroup of maximal possible order $12(g - 1)$ extending to a handlebody (or, more generally, a maximal reducible group extending to a product with handles), and show that such Hurwitz groups are closely related to the smallest Hurwitz group $\text{PSL}_2(7)$ of order 168 acting on Klein’s quartic of genus 3. We discuss simple groups of small order which are maximal handlebody groups and, more generally, maximal reducible groups. We discuss also the problem of which Hurwitz actions bound geometrically, and in particular whether Klein’s quartic bounds geometrically: does there exist a compact hyperbolic 3-manifold with totally geodesic boundary isometric to Klein’s quartic? Finally, large bounding and non-bounding actions on surfaces of genus 2, 3 and 4 are discussed in section 3.

1. Introduction

All finite group actions in the present paper will be orientation-preserving, all manifolds will be orientable.

1.1 Hurwitz groups

By the formula of Riemann-Hurwitz, the maximum order of a finite group $H$ of diffeomorphisms of a closed surface $\Sigma$ of genus $g > 1$ is $84(g - 1)$; such a group $H$ is called a Hurwitz groups of genus $g$. The quotient orbifold $\Sigma/H$ of such a Hurwitz action is
the 2-sphere with three branch points of orders 2, 3 and 7, and the Hurwitz groups are exactly the finite quotients of its orbifold fundamental group which is isomorphic to the hyperbolic triangle group of type (2,3,7), with presentation
\[ <x, y, z \mid x^2 = y^3 = z^7 = xyz = 1 >. \]

This triangle group acts by isometries on the hyperbolic plane \( \mathbb{H}^2 \), there is an exact sequence
\[ 1 \to K \hookrightarrow (2, 3, 7) \to H \to 1, \]
and the factor group \( H \cong (2, 3, 7)/K \) acts by isometries on the hyperbolic surface \( \mathbb{H}^2/K \).

1.2 Reducible and irreducible actions; products with handles

An action of a finite group \( G \) on a surface \( \Sigma \) is reducible if there is a nontrivial simple closed curve on \( \Sigma \) which is mapped either to itself or to a disjoint curve by every element of \( G \), and otherwise irreducible. For a reducible action, one can equivariantly attach thickened 2-disks along the curve and its \( G \)-images to the boundary component \( \Sigma \times 1 \) of the product \( \Sigma \times I \) to obtain a 3-manifold which is a product with handles of genus \( g \) to which the \( G \)-action extends. Iterating the construction one finishes with a product with handles \( P \) with a \( G \)-action such that the action of each stabilizer of a boundary component of \( P \) arising from \( \Sigma \times 1 \) is irreducible.

In general, a product with handles \( P \) is defined as follows. Considering the product \( \Sigma \times I \) of a closed surface \( \Sigma \) of genus \( g \) with the unit interval, one attaches 2-handles \( D^2 \times I \) (where \( D^2 \) denotes the 2-disk) along the boundary parts \( S^1 \times I \) to a finite collection of disjoint simple closed curves on the surface \( \Sigma \times 1 \), then one closes eventually created 2-sphere boundary components by attaching 3-balls to obtain a 3-manifold \( P \) whose boundary consists of the surface \( \partial_0 P = \Sigma \times 0 \) of genus \( g \) (the outer boundary of \( P \)) and a collection of surfaces \( \partial_1 P \) (the inner boundary which may be empty). We call \( P \) a product with handles of genus \( g \) (see section 4 for an explicit example).

A \( G \)-action on a surface \( \Sigma \) is irreducible if and only if the quotient orbifold \( \Sigma/G \) is the 2-sphere with exactly three branch points. In particular, all Hurwitz actions are irreducible. By the formula of Riemann-Hurwitz, the maximal possible order of a reducible action of \( G \) on a surface of genus \( g > 1 \) is \( 12(g - 1) \), and in this case the quotient orbifold \( \Sigma/G \) is the 2-sphere with four branch points of orders 2, 2, 2 and 3 whose orbifold fundamental group is isomorphic to the hyperbolic quadrangle group of type (2,2,2,3), with presentation
\[ <x_1, x_2, x_3, x_4 \mid x_1^2 = x_2^2 = x_3^2 = x_4^3 = x_1x_2x_3x_4 = 1 >. \]

As before, there is an exact sequence
\[ 1 \to K \hookrightarrow (2, 2, 2, 3) \to G \to 1, \]
and the factor group \( G \cong (2, 2, 2, 3)/K \) acts by isometries on the hyperbolic surface \( \mathbb{H}^2/K \). Let \( n \) denote the order of the image of the element \( x_1x_2 \) in the finite group \( G \); by adding the relation \((x_1x_2)^n = 1\) to the above presentation, we obtain a presentation of the free product with amalgamation

\[
G_n = \mathbb{D}_n \ast_{\mathbb{Z}_n} (2, 3, n)
\]

where \((2, 3, n)\) denotes the triangle group of type \((2, 3, n)\) and \( \mathbb{D}_n = (2, 2, n) \) the dihedral group of order \( 2n \). So the finite groups \( G \) admitting a reducible action of maximal possible order \( 12(g - 1) \) on a surface of genus \( g > 1 \) are exactly the finite factor groups with torsionfree kernel of the quadrangle group \((2,2,2,3)\), or equivalently of one of the groups \( G_n, n \geq 2 \); we call such a group a **maximal reducible group**, and specifically a **\( G_n \)-group**. So the maximal reducible groups are exactly the \( G_n \)-groups, for some \( n \geq 2 \).

A quotient of a product with handles by a finite group action is an **orbifold product with handles**; such an orbifold can be uniformized by a function group, i.e. by a Kleinian groups with an invariant component of the regular set. An approach to orbifold product with handles and the construction of the uniformizing Kleinian groups is given in [R], using the language of finite graphs of groups.

### 1.3 Completely reducible actions, maximal handlebody groups and maximal bounded surface groups

An action of a finite group \( G \) on a surface \( \Sigma_g \) is **completely reducible** if it extends to a handlebody \( \Sigma_g \) with \( \partial \Sigma_g = \Sigma_g \). By the equivariant Dehn Lemma/loop theorem [MY], a completely reducible action is reducible, and the maximum order of a completely reducible action of a finite group \( G \) acting on a 3-dimensional handlebody \( V_g \) of genus \( g > 1 \) is again \( 12(g - 1) \) (as first observed in [Z1], prior to the appearence of the equivariant Dehn lemma/loop theorem, see also [MMZ, Theorem 7.2]); such a group \( G \) is called a **maximal handlebody group of genus \( g \)**. By [Z2] (see also [Z5], [MMZ] for the following), the maximal handlebody groups \( G \) are exactly the finite quotients with torsionfree kernel of one of the four groups

\[
G_2 = \mathbb{D}_2 \ast_{\mathbb{Z}_2} \mathbb{D}_3, \quad G_3 = \mathbb{D}_3 \ast_{\mathbb{Z}_3} \mathbb{A}_4, \quad G_4 = \mathbb{D}_4 \ast_{\mathbb{Z}_4} \mathbb{S}_4, \quad G_5 = \mathbb{D}_5 \ast_{\mathbb{Z}_5} \mathbb{A}_5
\]

where \( \mathbb{A}_4 = (2, 3, 3) \) denotes the tetrahedral group of order 12, \( \mathbb{S}_4 = (2, 3, 4) \) the octahedral group of order 24 and \( \mathbb{A}_5 = (2, 3, 5) \) the dodecahedral group of order 60; together with the dihedral groups \( \mathbb{D}_n = (2, 2, n) \) of order \( 2n \), these are exactly the finite or spherical triangle groups and, together with the cyclic groups, the finite subgroups of \( \text{SO}(3) \).

The quotient orbifold \( V_g/G \) of a maximal handlebody group \( G \) of genus \( g \) by a finite group action is a **handlebody orbifold** whose orbifold fundamental group is one of the four groups \( G_2, G_3, G_4 \) or \( G_5 \); the underlying topological space is the 3-disk, its boundary
$\partial V/G$ is the 2-sphere with four branch points of orders 2, 2, 2 and 3. The handlebody orbifolds are uniformized by virtual Schottky groups (i.e., virtually free function groups with a single component of the regular set), cf. [MMZ], [Z5].

An upper bound for the maximal order of a finite group of homeomorphisms of a compact surface with nonempty boundary (orientable or not) of algebraic genus $g$ (the rank of its free fundamental group) is also $12(g - 1)$. This can be seen by taking the product of the surface with a closed interval (twisted if the surface is nonorientable); the result is an orientable handlebody of genus $g$ to which the finite group action extends orientation-preservingly, so the upper bound for handlebodies applies, and such groups of order $12(g - 1)$ are called maximal bounded surface groups. The maximal bounded surface groups are exactly the finite quotients with torsionfree kernel of the group

$$G_2 = \mathbb{D}_2 \ast \mathbb{Z}_3$$

(cf. [MZ]). An interesting example of a maximal handlebody group which is not a maximal bounded surface group is the second Mathieu group $M_{12}$.

1.4 Hurwitz actions versus maximal reducible actions

The smallest Hurwitz group is the linear fractional group $PSL_2(7)$ of order 168, acting on Klein’s quartic $Q_3$ of genus 3. The smallest maximal handlebody group is the dihedral group $\mathbb{D}_6$ of genus 2, the second smallest the octahedral or symmetric group $S_4$ of genus 3 (see [Z5, Proposition 7]). The maximal handlebody group $S_4$ is a subgroup of index 7 in the Hurwitz group $PSL_2(7)$, and its preimage under the associated surjection $\psi : (2,3,7) \rightarrow PSL_2(7)$ is the quadrangle group $(2,2,2,3)$ (by an easy application of the formula of Riemann Hurwitz). Since the restriction $\psi : (2,2,2,3) \rightarrow S_4$ factors through one of the three groups $G_2, G_3$ or $G_4$ (in fact through all of them), the action of $S_4$ on Klein’s quartic $Q_3$ (unique up to conjugation) extends to a handlebody $V_3$ of genus 3. Our first main result is the following.

**Theorem 1.** Let $H$ be Hurwitz group of genus $g > 1$ acting on a surface $\Sigma = \Sigma_g$.

i) If the action of $H$ on $\Sigma$ has a reducible subgroup $G$ of maximal possible order $12(g - 1)$ then $\Sigma$ is a finite regular covering of Klein’s quartic and the actions of $H$ and $G$ are obtained by lifting the actions of $PSL_2(7)$ and its subgroup $S_4$ to $\Sigma$; in particular, $H$ surjects onto $PSL_2(7)$. There are infinitely many Hurwitz actions with a maximal reducible subgroup.

ii) If $H$ is simple and not isomorphic to a subgroup of the alternating group $A_{24}$ (has no subgroup of index $\leq 24$) then the action of every proper subgroup of $H$ is reducible.

In fact there seem to be very few cases of Hurwitz actions with a proper irreducible subgroup, see the Remark in section 4. Concerning part i) of Theorem 1, we note that
a lift of a subgroup $S_4$ of $\text{PSL}_2(7)$ to a finite regular covering of Klein’s quartic is always reducible but in general not completely reducible; at present we don’t know if one can obtain infinitely many completely reducible actions in this way.

**Corollary 1.** $\text{PSL}_2(7)$ is the unique simple Hurwitz group which realizes both the smallest index reducible subgroup ($S_4$ of index 7, with preimage $(2,2,2,3)$ in $(2,3,7)$) as well as the largest index irreducible subgroup ($\mathbb{Z}_7$ of index 24, with preimage $(7,7,7)$).

Considering just the types of groups and not the actions, the following holds.

**Theorem 2.** A Hurwitz group which does not surject onto $\text{PSL}_2(7)$ is also a maximal reducible group; in particular, $\text{PSL}_2(7)$ is the only simple Hurwitz group which is not a maximal reducible group.

The simple Hurwitz groups of order less than $10^6$ are of linear fractional type $\text{PSL}_2(q)$, and in addition the Janko group $J_1$ and the Hall-Janko group $J_2$ (cf. [C]). The Hurwitz groups of linear fractional type are the following:

$q = 7$;
$q = p$ prime with $p \equiv \pm 1 \text{ mod } 7$ (each with a unique Hurwitz action);
$q = p^3$ with $p \equiv \pm 2, \pm 3 \text{ mod } 7$ (each with three different Hurwitz actions).

### 1.5 Bounding and nonbounding Hurwitz actions

The action of the subgroup $S_4$ of $\text{PSL}_2(7)$ on Klein’s quartic $Q_3$ extends to a handlebody $V_3$ of genus 3; on the other hand, the action of $\text{PSL}_2(7)$ on $Q_3$ *does not bound*, i.e. does not extend to any compact 3-manifold $M$ with exactly one boundary component $\partial M = Q_3$ ([Z4, Corollary 1]): the quotient orbifold $Q_3/\text{PSL}_2(7)$ is the 2-sphere with three branch points of orders 2, 3 and 7 which does not occur as the unique boundary component of a compact 3-orbifold (since a singular axis starting in the boundary point of order 7 can end only in a dihedral point of type $D_7$ but $\text{PSL}_2(7)$ has no dihedral subgroup $D_7$).

The second smallest Hurwitz group is the linear fractional group $\text{PSL}_2(8)$ of order 504 and genus 7, and it is shown in [Z4, Corollary 2] that the action by isometries of $\text{PSL}_2(8)$ on the unique hyperbolic surface $Q_7$ *bounds geometrically*, i.e. extends to a group of isometries of a compact hyperbolic 3-manifold $M$ with totally geodesic boundary $\partial M = Q_7$. In particular, forgetting about the group action, the hyperbolic Hurwitz surface $Q_7$ bounds geometrically. This leads naturally to the following:

**Question** (see [Z6]): Does Klein’s quartic bound geometrically?

Up to conjugation by homeomorphisms, there is a unique reducible action of $S_4$ on a surface of genus 3 (cf. [B]), represented by the subgroup $S_4$ of the Hurwitz action of $\text{PSL}_2(7)$; we will show in section 3 that this $S_4$-action bounds geometrically, for some
hyperbolic structure on a surface of genus 3. But, in contrast to the irreducible Hurwitz actions, the isometric action of $S_4$ on a hyperbolic surface of genus 3 does not determine the hyperbolic structure of the surface (since the deformation space of a hyperbolic quadrangle of type (2,2,2,3) has positive dimension whereas a hyperbolic triangle of type (2,3,7) is unique up to isometry). So there are uncountably many hyperbolic structures on a surface of genus 3 such that $S_4$ acts by isometries; only countably many of these can bound geometrically, and at present we do not know if one can obtain Klein’s quartic by a construction as in section 3.

So the reducible action of $S_4$ extends both to a handlebody (in fact, $S_4$ is the second smallest maximal handlebody group), and also to a hyperbolic 3-manifold with totally geodesic boundary; this raises the following:

**Problem.** Show that every bounding action of a finite group on a closed hyperbolic surface bounds also geometrically. In particular, if an action extends to a handlebody, does it extend also to a hyperbolic 3-manifold with totally geodesic boundary?

For a proof of the following theorem, see [GZ, Corollary 3.14] (and [RZ, Proposition 2] for a proof that $\text{PSL}_2(27)$ does not bound).

**Theorem 3.** For $q \leq 1000$, the only non-bounding Hurwitz actions of linear fractional type $\text{PSL}_2(q)$ are the Hurwitz actions of the groups $\text{PSL}_2(7)$ and $\text{PSL}_2(27)$; in fact all other Hurwitz actions with $q \leq 1000$ bound geometrically, and in particular the corresponding hyperbolic Hurwitz surfaces bound geometrically.

So there is some analogy between the Hurwitz actions of $\text{PSL}_2(7)$ and $\text{PSL}_2(27)$: both do not bound, and in particular, similar as for $\text{PSL}_2(7)$, it remains open whether the Hurwitz surface of the action of $\text{PSL}_2(27)$ bounds geometrically. Supported by some strong evidence from [GZ], we state the following:

**Conjecture.** With the exceptions of $q = 7$ and 27, all Hurwitz actions of linear fractional type $\text{PSL}_2(q)$ bound geometrically.

1.6 Maximal reducible groups versus maximal handlebody groups

Examples of small maximal handlebody groups which are not maximal bounded surface groups are given in [Z3], and it is shown in [CZ] by extensive computational methods that 161 and 3781 are the two smallest genera for which there exists a maximal handlebody group but not a maximal bounded surface group (answering a question in [MZ]). In the next theorem, we confront maximal reducible groups and maximal handlebody groups; the first claim is proved in [PZ, Corollary 3.3].

**Theorem 4.** i) The linear fractional group $\text{PSL}_2(q)$ is a maximal handlebody group exactly for all $q$ different from 7, 9, 11 and $3^{2m+1}$. The group $\text{PSL}_2(27)$ is the smallest simple group which is maximal reducible but not a maximal handlebody group.
There are infinitely many maximal reducible groups which are not maximal handlebody groups.

ii) The smallest simple groups which are maximal reducible, of order less than 979,200 (the order of the symplectic group $\text{Sp}_4(4)$) and not of linear fractional type, are the second Mathieu group $M_{12}$, the Janko group $J_1$, the alternating group $A_9$ and the Hall-Janko group $J_2$. The group $M_{12}$ is a maximal handlebody group of types $G_3$, $G_4$ and $G_5$ but not a maximal bounded surface group (not of type $G_2$), $J_1$ is a maximal bounded surface group, and $A_9$ is a maximal handlebody group of type $G_3$ but not a maximal bounded surface group.

So it remains open here whether $\text{Sp}_4(4)$ is maximal reducible, and whether the maximal reducible group $J_2$ is a maximal handlebody or a maximal bounded surface group.

1.7 $G_7$-groups and bounding Hurwitz groups

An interesting class of maximal reducible groups closely related to Hurwitz groups are the $G_7$-groups,

$$G_7 = \mathbb{D}_7 \ast_{2,3,7} (2,3,7).$$

A surjection $\varphi : G_7 \to G$ with torsionfree kernel to a finite group $G$ determines a $G$-action of maximal possible order $12(g - 1)$ on a product with handles $\mathcal{P}$ of genus $g$ (obtained as the regular orbifold covering of the orbifold product with handles with fundamental group $G_7$ associated to the kernel of $\varphi$). Let $n$ be the index of the image $\varphi(2,3,7)$ of the triangle group $(2,3,7)$ in $G$.

If $n = 1$, the inner boundary $\partial_1 \mathcal{P}$ of $\mathcal{P}$ consists of a single surface of genus $g'$, with $84(g' - 1) = 12(g - 1)$, on which $G$ restricts to a Hurwitz action. For arbitrary $n$, $\partial_1 \mathcal{P}$ consists of $n$ copies of a Hurwitz surface, and the stabilizer in $G$ of each of these surfaces acts as a Hurwitz group (isomorphic to $\varphi(2,3,7)$); see section 4 for an explicit example.

We say that a Hurwitz group is bounding if it admits a bounding Hurwitz action.

**Theorem 5.** i) All bounding Hurwitz groups are $G_7$-groups. The groups $\text{PSL}_2(7)$ and $\text{PSL}_2(27)$ are the smallest non-bounding Hurwitz groups, and $\text{PSL}_2(27)$ is the smallest simple non-bounding $G_7$-group.

ii) All Hurwitz groups of linear fractional type except $\text{PSL}_2(7)$ are $G_7$-groups. The simple $G_7$-groups of order less than $10^6$ are Hurwitz groups of linear fractional type or one of the groups $\text{PSL}_2(49)$, $J_1$, $A_9$ and $J_2$. The groups $\text{PSL}_2(49)$ and $A_9$ are the smallest simple $G_7$-groups which are not Hurwitz groups.

Let $H$ be a Hurwitz group with a Hurwitz action associated to a surjection $\varphi : (2,3,7) \to H$. We denote by $[2,3,7]$ the extended triangle group generated by the reflections in the sides of a hyperbolic triangle with angles $2\pi/2$, $2\pi/3$ and $2\pi/7$ (containing the
triangle group \((2,3,7)\) as a subgroup of index 2, of orientation-preserving elements). As a consequence of [Z4, Theorem 3b] we have the following:

**Theorem 6.** Let \(H\) be a Hurwitz group with a Hurwitz action associated to a surjection \((2, 3, 7) \to H\). If the surjection extends to a surjection \([2, 3, 7] \to H\) then the Hurwitz action of \(H\) bounds geometrically; also, \(H\) is a maximal bounded surface group.

The Hurwitz groups \(H\) as in Theorem 6 are exactly the "non-orientable Hurwitz groups" (also called \(H^\ast\)-groups) of maximal possible order \(84(g - 2)\) acting on a non-orientable surface of genus \(g > 2\) (see [C2]). An interesting example of such a group is the first Janko group \(J_1\), in particular a Hurwitz action of \(J_1\) bounds geometrically (this remains open for the Hall-Janko group \(J_2\) which is also a Hurwitz group but not a surjective image of the extended triangle group \([2,3,7]\)). Since it is proved in [C3] that, for \(n \geq 168\), the alternating group \(A_n\) is a surjective image of the extended triangle group \([2,3,7]\), Theorem 6 implies:

**Corollary 2.** For each \(n \geq 168\) there is a Hurwitz-action of the alternating group \(A_n\) which bounds geometrically, in particular \(A_n\) is a bounding Hurwitz group.

The analogue remains open for the Hurwitz groups of linear fractional type \(\text{PSL}_2(q)\) (but see the stronger Conjecture in section 1.5).

**2. Proofs of Theorems 1 and 2**

In the notation of Theorem 1, the reducible group \(G\) of order \(12(g - 1)\) has index 7 in the Hurwitz group \(H\) of order \(84(g - 1)\). There is a surjection

\[
\pi : (2, 3, 7) \to H
\]

whose kernel is the universal covering group of the surface \(\Sigma_g\). Since the triangle group is a perfect group (has trivial abelianization), also the Hurwitz group \(H\) is perfect. The preimage \(\pi^{-1}(G)\) of \(G\) in the triangle group \((2,3,7)\) is a quadrangle group \((2,2,2,3)\) (by an easy application of the formula of Riemann-Hurwitz, this is the only signature of a subgroup of index 7 in the triangle group \((2,3,7)\)).

The permutation representation of \(H\) by right multiplication on the seven right cosets of \(G\) in \(H\) defines a nontrivial homomorphism

\[
\phi : H \to A_7
\]

of \(G\) to the alternating group \(A_7\) of degree 7; the image of \(\phi\) is perfect and also a Hurwitz group (the kernel of the composition \(\phi \circ \pi : (2,3,7) \to A_7\) is torsionfree since any normal subgroup of \((2,3,7)\) containing a torsion element is equal to \((2,3,7)\)). The perfect subgroups of the alternating group \(A_7\) are the simple groups \(A_5, A_6, A_7\) and
PSL\(_2(7)\) (see [CCN]). Since the only Hurwitz group among these groups is PSL\(_2(7)\) (by [C] or by a direct computation using [GAP]), we have surjections

\[ \phi : H \to \text{PSL}_2(7), \quad \phi \circ \pi : (2,3,7) \to \text{PSL}_2(7), \]

and the second one gives the unique Hurwitz action of PSL\(_2(7)\) on Klein’s quartic \(Q_3\).

The kernel \(U\) of the surjection \(\phi : H \to \text{PSL}_2(7)\) acts freely on \(\Sigma_g\) (since its preimage \(\pi^{-1}(U)\) in the triangle group \((2,3,7)\) is torsionfree), and the actions of \(H\) and \(G\) on \(\Sigma_g\) project to the actions of PSL\(_2(7)\) \(\cong H/U\) and \(S_4 \cong G/U\) on Klein’s quartic \(Q_3 = \Sigma_g/U\).

In order to obtain infinitely many examples of such Hurwitz groups, one considers finite-index characteristic subgroups of the fundamental group of Klein’s quartic and lifts the actions of PSL\(_2(7)\) and its subgroup \(S_4\) to the corresponding finite regular coverings.

This concludes the proof of part i) of Theorem 1. For the proof of part ii), assume that \(G\) is a proper subgroup of \(H\), of index \(n > 1\), which acts irreducibly on \(\Sigma\). Then the extension of \(\pi_1(\Sigma)\) associated to \(G\) is a triangle group of index \(n\) in the triangle group \((2,3,7)\). The triangle group of maximal possible index in \((2,3,7)\) is the triangle group \((7,7,7)\) of index 24 which implies \(n \leq 24\). The action by right multiplication of \(G\) on the right cosets of \(H\) in \(\Sigma\) defines a nontrivial, hence injective homomorphism of the simple group \(H\) to the alternating group \(A_n\), in particular \(H\) is isomorphic to a subgroup of \(A_{24}\).

For the proof of Theorem 2, suppose that the Hurwitz group \(H\) does not surject onto PSL\(_2(7)\). There is a surjection of the triangle group \((2,3,7)\) onto \(H\), and the image of the quadrangle subgroup \((2,2,2,3)\) of index 7 in \((2,3,7)\) has index 1 or 7 in \(H\). If it has index 7 then, by the proof of Theorem 1, \(H\) surjects onto PSL\(_2(7)\), otherwise the quadrangle group \((2,2,2,3)\) surjects onto \(H\) and \(H\) is a maximal reducible group.

Remark. The Hurwitz action of PSL\(_2(7)\) has a cyclic irreducible subgroup \(Z_7\); with this exception, every other cyclic subgroup of a Hurwitz action of a simple group is reducible. In fact, the proper triangle subgroups of the triangle group \((2,3,7)\) are the groups \((7,7,7)\), \((3,7,7)\), \((2,7,7)\) and \((3,3,7)\), of indices 24, 16, 9 and 8. The abelianizations of these triangle groups are \(Z_7 \times Z_7\), \(Z_7\), \(Z_7\) and \(Z_3\), and only \((7,7,7)\) admits a surjection with torsionfree kernel onto a cyclic group \((Z_7)\); since \((7,7,7)\) has index 24 in \((2,3,7)\), the maximal possible order of a simple Hurwitz group with an irreducible cyclic subgroup is \(7 \cdot 24 = 168\), the order of the smallest Hurwitz group PSL\(_2(7)\).
The Hurwitz action of $\text{PSL}_2(7)$ has also an irreducible subgroup $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ of index 8, with preimage $(3,3,7)$ in $(2,3,7)$. The second largest Hurwitz group $\text{PSL}_2(8)$ has an irreducible subgroup $(\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_7$ of index 9, with preimage $(3,7,7)$ in $(2,3,7)$. We believe that these three examples are the only examples of proper irreducible subgroups of a simple Hurwitz group (for example, the minimal index of a proper subgroup of the third smallest Hurwitz group $\text{PSL}_2(13)$ is already 28, and hence the subgroup is reducible, and similar for all other Hurwitz groups of linear fractional type). Here one should go over a list of the simple (Hurwitz) groups of ”small” order and consider the minimal index of a subgroup for each group (see [C3] for the Hurwitz groups among the alternating groups).

Example. The second largest Hurwitz group is the linear fractional group of order 504 and genus 7, and also the product $\text{PSL}_2(7) \times \text{PSL}_2(8)$ is a Hurwitz group acting on a unique surface of genus 1009 (by a direct computation using [GAP], see also [C]). In fact, up to isomorphism there is a unique surjection $\pi : (2,3,7) \rightarrow \text{PSL}_2(7) \times \text{PSL}_2(8)$. The preimage $\pi^{-1}(\mathbb{S}_4 \times \text{PSL}_2(8))$ has index 7 in the triangle group $(2,3,7)$ and is a quadrangle group $(2,2,2,3)$. We checked by [GAP] that, for the corresponding surjection $\pi : (2,2,2,3) \rightarrow \mathbb{S}_4 \times \text{PSL}_2(8)$, the image of the element $x_1x_2 \in (2,2,2,3)$ has order 36 and 36 is the minimal $n$ such that the surjection factors through $G_n$. Hence the action of $\mathbb{S}_4 \times \text{PSL}_2(8)$ on the surface of genus 1009 is reducible but not completely reducible. On the other hand, there are various surjections $(2,2,2,3) \rightarrow \mathbb{S}_4 \times \text{PSL}_2(8)$ with factor through $G_2$ or $G_4$, so $\mathbb{S}_4 \times \text{PSL}_2(8)$ is a maximal handlebody group.

3. Geometrically bounding actions

Up to isomorphisms there is a unique surjection of the quadrangle group $(2,2,2,3)$ onto $\mathbb{S}_4$; equivalently, up to conjugation by homeomorphisms there is a unique action of $\mathbb{S}_4$ on a surface $\Sigma_3$ of genus 3, with quotient orbifold of type $(2,2,2,3)$ (cf. [B]). There are infinitely many hyperbolic structures on the quotient orbifold $\mathbb{H}^2/(2,2,2,3)$; lifting such a hyperbolic structure to the surface $\Sigma_3$, the group $\mathbb{S}_4$ acts by isometries.

Proposition 1. The unique $\mathbb{S}_4$-action on a surface of genus 3 with quotient orbifold of type $(2,2,2,3)$ bounds geometrically, for some hyperbolic structure on the surface.

Proof. By Andreev’s theorem ([T, chapter 13], [V, p.111]) there exists a unique finite hyperbolic polyhedron $C = C(3,3,3)$ in hyperbolic 3-space $\mathbb{H}^3$ which has the combinatorial structure of a cube as follows. The four edges of a face $F$ of $C$ have right dihedral angles (hence this face is orthogonal to the four adjacent faces), the four edges connecting $F$ to the opposite face $F'$ have three right dihedral angles and one dihedral angle $\pi/3$, and the face $F'$ has two consecutive edges with right dihedral angles and two with dihedral angles $\pi/3$ such that the three edges of $C$ with angles $\pi/3$ form a segment (note that the polyhedron $C$ has no incompressible Euclidean 2-suborbifolds). Let $C$ denote the
orientation-preserving subgroup of index two in the group generated by the reflections in the five faces of $C$ different form $F$ (so $C$ is generated by rotations around the eight edges of $C$ not in the boundary of $F$); it is easy to check that there exists a surjection from $C$ to $S_4$ with torsion-free kernel.

Let $D$ denote the double of $C$ along its face $F$ (again of the combinatorial type of a cube, with five dihedral angles $\pi/3$), and by $D$ the orientation-preserving subgroup of index 2 in the Coxeter group generated by the reflections in the faces of $D$. There exists a surjection $\pi : D \to S_4$ with torsion-free kernel $K$, and hence $D/K \cong S_4$ acts on the closed hyperbolic 3-manifold $M = \mathbb{H}^3/K$. The face $F$ of $C$ lies in a hyperbolic plane on which a quadrangle subgroup $(2,2,2,3)$ of $D$ acts, and the restriction of $\pi$ to this group $(2,2,2,3)$ is also surjective. The preimage of the face $F$ of $C$ is a closed totally geodesic surface $\Sigma_3$ of genus 3 in $M$ which is invariant under the action of $S_4$. The surface $\Sigma_3$ separates $M$ into two isometric hyperbolic 3-manifolds, each with an isometric action of $S_4$, which have the hyperbolic surface $\Sigma_3$ as their common totally geodesic boundary, and hence the isometric $S_4$-action on the hyperbolic surface $\Sigma_3$ bounds geometrically, completing the proof of Proposition 1.

We don’t know if the hyperbolic surface $\Sigma_3$ is isometric to Klein’s quartic (for example, one might try to compute the lengths of the edges of the face $F$ of $Q$ and compare with the edges of the corresponding quadrangle for a subgroup $(2,2,2,3)$ of the triangle group $(2,3,7)$). Note that the construction can be modified by choosing finite cubical polyhedra $C$ with other small dihedral angles for the face $F'$, obtaining other hyperbolic surfaces $\Sigma_3$.

There is a second action of $S_4$ on a surface of genus 3 associated to a surjection $(3, 4, 4) \to S_4$ (see the list in [B] for the finite group actions on a surface of genus 3), and this action is irreducible and bounds geometrically. In fact, it is a subgroup of an action of $S_4 \times \mathbb{Z}_2$ associated to a surjection $(2, 4, 6) \to S_4 \times \mathbb{Z}_2$ which bounds geometrically since this surjection extends to a surjection of a hyperbolic tetrahedral group (the orientation-preserving subgroup of the group generated by the reflections in the four faces of the tetrahedron) associated to a hyperbolic tetrahedron with one vertex at infinity of hyperbolic type $(2,4,6)$ truncated by an orthogonal plane, one edge of singular index 3 opposite to the edge with singular index 6, and all other edges of singular index 2. The hyperbolic surface of genus 3 determined by the action is not Klein’s quartic since the orientation-preserving isometry group $\text{PSL}_2(7)$ of Klein’s quartic has no subgroup $S_4 \times \mathbb{Z}_2$. There are exactly three other finite group actions on a surface of genus 3 of order greater or equal to 48, of orders 168, 96 and again 48, and each of these three actions has a cyclic subgroup which does not bound, so we have the following:

**Proposition 2.** The unique bounding finite group action of largest possible order on a surface of genus 3 is the action of $S_4 \times \mathbb{Z}_2$ associated to a surjection $(2, 4, 6) \to S_4 \times \mathbb{Z}_2$.  

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Since the reducible $S_4$-action associated to a surjection $(2, 2, 2, 3) \to S_4$ is also a subgroup of the action of $S_4 \times \mathbb{Z}_2$, this gives another proof of Proposition 1 (however, in contrast to Proposition 1, we are sure here that this does not realize Klein's quartic).

A description of the bounding and nonbounding finite group actions on a surface of genus 2 is given in [WZ]; for $g = 2$, no irreducible action bounds, and the largest bounding action (extending to a handlebody) is by the smallest maximal handlebody group $\mathbb{D}_6$.

The situation for large group-actions on surfaces of genus 3 can be subsumed as follows (see [Z7] for some more details, and [B] for a description of the groups $D_{2,8,5}$ and $D_{2,12,5}$).

**Theorem 7.** The bounding and non-bounding finite group-actions on a surface of genus 3, of order $\geq 24$, are the following.

i) The two largest group-actions, of orders 168 and 96 are represented by surjections

$$(2, 3, 7) \to \text{PSL}_2(7) \text{ and } (2, 3, 8) \to \mathbb{D}_3 \times (\mathbb{Z}_4 \times \mathbb{Z}_4)$$

and do not bound.

ii) Two actions of order 48 associated to surjections

$$(3, 3, 4) \to \mathbb{Z}_3 \times (\mathbb{Z}_4 \times \mathbb{Z}_4) \text{ and } (2, 4, 6) \to \mathbb{Z}_2 \times S_4;$$

the first one is a subgroup of index 2 of the group of order 96 in i) and does not bound, the second one is the largest bounding group-action on a surface of genus 3; it bounds geometrically but does not extend to a handlebody.

iii) Two non-bounding actions of order 32 associated to surjections

$$(2, 4, 8) \to \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_8) \text{ and } (2, 4, 8) \to \mathbb{Z}_2 \times D_{2,8,5}.$$ 

iv) Two non-bounding actions of order 24 associated to surjections

$$(3, 3, 6) \to \text{SL}_2(3) \text{ and } (2, 4, 12) \to D_{2,12,5}.$$

v) Three bounding actions of order 24 associated to surjections

$$(2, 6, 6) \to \mathbb{Z}_2 \times A_4, \ (3, 4, 4) \to S_4 \text{ and } (2, 2, 2, 3) \to S_4;$$

these three actions are subgroups of index 2 of the geometrically bounding action of order 48 in ii). The last one is also the largest action on a surface of genus 3 which
extends to a handlebody (in fact $S_4$ is the unique maximal handlebody group of genus 3, of maximal possible order $12(g - 1)$).

Finally, for each of the non-bounding actions in i) - iv) there is already a cyclic subgroup which does not bound.

A beautiful example of a geometrically bounding action on a surface of genus 4 is the $A_5$-action described in [Z6] associated to a surjection $(2, 5, 5) \to A_5$. This is a subgroup of an action of $S_5$ associated to a surjection $(2, 4, 5) \to S_5$ which realizes the maximal order of a finite group action on a surface of genus 4 and is also geometrically bounding.

**Proposition 3.** The largest group-action on a surface of genus 4, of type $(2, 4, 5) \to S_5$, bounds geometrically. The largest action in genus 4 which extends to a handlebody is of type $(2, 2, 2, 3) \to D_3 \times D_3$.

As noted before, the first (geometrically) bounding Hurwitz actions is that of $PSL_2(8)$ on a surface of genus 7.

4. Proofs of Theorems 4 and 5

As noted before, the first claim of Theorem 4 is proved in [PZ, Corollary 3.3] (note that the case $q = 11$ was overlooked in [PZ].) In particular, $PSL_2(27)$ is not a maximal handlebody group (which can be checked easily also by a direct computation). The unique Hurwitz action of $PSL_2(27)$ is associated to a surjection $(2, 3, 7) \to PSL_2(27)$; since $PSL_2(27)$ has subgroups $D_7$, this extends to a surjection $G_7 = D_7 \ast_{Z_7} (2, 3, 7) \to PSL_2(27)$, hence $PSL_2(27)$ is a $G_7$-group.

The group $G = PSL_2(27)$ is a Hurwitz group in a unique way: up to equivalence there is a unique surjections $(2, 3, 7) \to G$; let $K$ denote the kernel of this surjection which is a surface group of genus $g = 118$. By abelianization and reduction of coordinates mod $p$, for some prime $p$, we get a surjection $K \to (\mathbb{Z}_p)^{2g}$ whose kernel is a characterestic subgroup $\tilde{K}$ of $K$, in particular normal in $(2, 3, 7)$, so we have an exact sequence

$$1 \to \tilde{K} \to (2, 3, 7) \to (\mathbb{Z}_p)^{2g} \times PSL_2(27) \to 1.$$ 

Since $G = PSL_2(27)$ has dihedral subgroups $D_7$ (of order 14), the surjection $(2, 3, 7) \to (\mathbb{Z}_p)^{2g} \times G$ extends to a surjection with torsionfree kernel of $G_7 = D_7 \ast_{Z_7} (2, 3, 7)$ to $(\mathbb{Z}_p)^{2g} \times G$, hence $(\mathbb{Z}_p)^{2g} \times G$ is a maximal reducible $G_7$-group of Hurwitz type. On the other hand, $(\mathbb{Z}_p)^{2g} \times G$ is not a maximal handlebody group for $p > |G|$ since a surjection with torsionfree kernel $G_n \to (\mathbb{Z}_p)^{2g} \times G$ would induce a surjection $G_n \to G$ with torsionfree kernel which does not exist for $2 \leq n \leq 5$ (since $G = PSL_2(27)$ is not a maximal handlebody group).

This completes the proof of Theorem 4 i). Theorem 4 ii) and Theorem 5 ii) can easily be checked by case-by-case computations (using e.g. [GAP]), going over the list of the
simple groups of small order, their maximal subgroups and the orders and conjugacy
classes of elements (see [CNN]). The first statement of Theorem 5 ii) follows from the
well-known classification of the subgroups of the linear fractional groups \( \text{PSL}_2(q) \); in
particular, with the exception of \( \text{PSL}_2(7) \), an element of order 7 in a linear fractional
Hurwitz group \( H \) lies in a dihedral subgroup \( \mathbb{D}_7 \) of order 14, and hence \( H \) is a \( G_7 \)-group.
Theorem 5 i) follows from the argument given in section 1.5 for the case of \( \text{PSL}_2(7) \).

Example. The Hurwitz group \( \text{PSL}_2(7) \) of genus 3 is not a \( G_7 \)-group since an element
of order 7 does not lie in a dihedral subgroup \( \mathbb{D}_7 \). We finish with an example of a
maximal reducible action of \( \text{PGL}_2(7) \) on a surface \( \Sigma \) of genus 29 and the induced action
on a product with handles \( \mathcal{P} \) with outer boundary \( \partial_0 \mathcal{P} = \Sigma \) whose inner boundary \( \partial_1 \mathcal{P} \)
consists of two copies of Klein’s quartic with the Hurwitz action of \( \text{PSL}_2(7) \).

The kernel of the composition of surjections

\[(2, 2, 3) \rightarrow \mathbb{D}_7 \ast_{\mathbb{Z}} (2, 3, 7) \rightarrow \text{PGL}_2(7)\]

is a surface group \( \pi_1(\Sigma) \) of genus \( g = 29 \) (note that, in contrast to \( \text{PSL}_2(7) \), \( \text{PGL}_2(7) \)
has dihedral subgroups \( \mathbb{D}_7 \)). Let \( \tilde{\mathcal{P}} \) denote the orbifold product with handles with
orbifold fundamental group \( \mathbb{D}_7 \ast_{\mathbb{Z}} (2, 3, 7) \) (cf. [R]), and let \( \mathcal{P} \) be the product with
handles of genus \( g = 29 \) which is the covering of \( \tilde{\mathcal{P}} \) associated to the kernel of the
surjection \( \mathbb{D}_7 \ast_{\mathbb{Z}} (2, 3, 7) \rightarrow \text{PGL}_2(7) \). The outer boundary \( \partial_0 \mathcal{P} \) is the surface \( \Sigma \); since
the triangle group \( (2, 3, 7) \) maps onto the subgroup \( \text{PSL}_2(7) \) of index 2 in \( \text{PGL}_2(7) \), the
inner boundary \( \partial_1 \mathcal{P} \) consists of two copies of Klein’s quartic of genus 3 with the Hurwitz
action of \( \text{PSL}_2(7) \).

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