Abstract—Source localization plays a key role in many applications including radar, wireless and underwater communications. Among various localization methods, the most popular ones are Time-Of-Arrival (TOA), Time-Difference-Of-Arrival (TDOA), Angle-Of-Arrival (AOA) and Received Signal Strength (RSS) based. Since the Cramér-Rao lower bounds (CRLB) of these methods depend on the sensor geometry explicitly, sensor placement becomes a crucial issue in source localization applications. In this paper, we consider finding the optimal sensor placements for the TOA, TDOA, AOA and RSS based localization scenarios. We first unify the three localization models by a generalized problem formulation based on the CRLB-related metric. Then a unified optimization framework for optimal sensor placement (UTMOST) is developed through the combination of the alternating direction method of multipliers (ADMM) and majorization-minimization (MM) techniques. Unlike the majority of the state-of-the-art works, the proposed UTMOST neither approximates the design criterion nor considers only uncorrelated noise in the measurements. It can readily adapt to different design criteria (i.e. A, D and E-optimality) with slight modifications within the framework and yield the optimal sensor placements correspondingly. Extensive numerical experiments are performed to exhibit the efficacy and flexibility of the proposed framework.

Index Terms—Optimal sensor placement, source localization, Cramér-Rao lower bound, alternating direction method of multipliers, majorization-minimization

I. INTRODUCTION

Environment sensing via wireless sensor networks (WSNs) has been of significant research interest over the past decade, and one of their key applications is the target/source localization [1], [2]. From here on, we will use the terms target and source interchangeably. Typically, in source localization, given some potentially noisy measurements from the sensors, the position of the source is estimated based on various approaches. A variety of source localization techniques exist in the literature depending on the type of information measured and the source position recovery mechanism from the observed data. The commonly used approaches are based on time-of-arrival (TOA) [3], angle-of-arrival (AOA) [4], time-difference-of-arrival (TDOA) [5], received-signal-strength (RSS) [6] and frequency-difference-of-arrival (FDOA) [7]. Apart from the employed estimation approaches, the localization accuracy also depends on the target-sensor geometry [8], [9]. Specifically, the mean squared error (MSE) in the estimation of the source position will be a function of the sensor geometry. Therefore, optimal sensor placement is a key problem in source localization applications.

However, owing to the highly nonlinear dependence of the MSE on the geometry, it is rather challenging to arrive at the optimal sensor placements based on the MSE analysis. A viable alternative approach is to derive the Cramér-Rao lower bound (CRLB) for the source localization model and then optimize the bound with respect to the positions of the sensors. Indeed, various schemes of sensor placement have been proposed in the literature based on optimization of the CRLB matrix or the Fisher information matrix (FIM, i.e., the inverse of the CRLB matrix). The commonly used CRLB-based optimization criteria are the A-optimality (i.e., minimizing the determinant of the CRLB) and D-optimality (i.e., minimizing the trace of the CRLB) [10].

For the two-dimensional (2D) case, the optimal sensor geometries for the AOA-based model was obtained by optimizing the D-optimality criterion in [11], which are shown to be lying in an equiangular configuration. In [12], the authors have considered the problem of optimal sensor placement for AOA-localization in 3D space, they have derived optimal sensing schemes by optimizing the $A$-optimality criterion. In [13], the authors have designed optimal sensing direction by optimizing the $A$–optimality design criterion for hybrid RSS, AOA and TOA localization problem. The authors of [14] have proposed optimal sensor placement schemes for AOA-based localization problem with distance dependent noise model. In [15], the problem of optimal sensor placement for AOA model using frame theory was addressed. The authors of [16] addressed the problem of optimal access point deployment (in the context of wifi-based localization) for AOA localization model. In [17], for the TDOA measurement model, the optimal geometry of the sensors is derived to correspond to vertices of a $m$ sided regular polygon, $m$ being the number of sensors. The same approach was later extended to the 3D case as well, showing the optimal geometry as centered platonic solids (tetrahedron, cube, etc) where source is present at the center and sensors at the vertices [18]. In [19], the optimal sensor geometries were derived for the TDOA model by employed the
A-optimality criterion and considering both the centralized and decentralized pairing configuration which are distinguished by a common reference sensor. Furthermore, for the three-dimensional (3D) case, the authors in [12], [20], [21] studied the optimal sensor placement strategies by optimizing the A-optimality criterion for the TOA, AOA and RSS methodologies, in which they approximated the A-optimality criterion via a general inequality and minimized the approximated criterion with respect to the sensor positions. In [22], the optimal receiver position was determined based on A-optimality for both synchronous and asynchronous elliptic positioning (2D and 3D) that minimizes the localization error. The author in [23] proposed a framework for the optimal sensor-target geometries for different types of sensor network with Bayesian priors by taking into account the uncertainty in the prior knowledge of the target position. In [24], an optimal sensor placement strategy has been presented for received signal strength difference (RSSD) based localization problem with unknown transmitted power by maximizing the determinant of FIM. In [25], the authors studied the problem of optimal geometry analysis for TOA localization, where they employed the D-optimal design criterion to arrive at optimal geometry. In [26], a D-optimal design based optimal sensor placement strategy was explored for cognitive radar application.

Although much research has been conducted in this field, several gaps are still observed from the literature as follows:

- The majority of the works assume that the noise in the measurements are uncorrelated, which would lead to a simplified design criterion. Although it is not unnatural to assume uncorrelated noise in the measurement models, under some circumstances, the noise in some measurements could actually be correlated. For example, in ocean applications, the deployed hydrophones will be influenced by the action of the same regional swell or flows, which may cause the ADC saturation and bring the correlated measurement errors [27]. Even with the strong assumption that the sensor measurements are uncorrelated, in the case of TDOA-based source localization, the measurement noise covariance matrix will be correlated (see equation (16)). Therefore, one can not rule out the possibility of correlated measurement noise and it has to be considered while designing sensor placement strategies.

- It has been observed in the literature that various specialized algorithms/methodologies were developed to design optimal sensor placement strategies by optimizing different design criterion. There is clearly a lack of a unified framework encompassing all the design criteria (like A- and D-optimality criteria) for the various source localization methodologies. Such a framework, while being theoretically elegant, should also offer flexibility to the designer to incorporate more application-dependent constraints on the sensor locations.

- The design based on the E-optimality (i.e., minimizing the largest eigenvalue of the CRLB) [10] usually behaves substantially more reliably with respect to minimization of the variances of the parameter estimates. However, there is barely any method on designing optimal placement strategies by optimizing the E-optimality, one reason for which could be the difficulty of solving the associated optimization problem.

To address the aforementioned gaps in the literature, we propose a unified optimization framework for optimal sensor placement design in this work. The key contributions of our work are mainly as follows:

- We have formulated a general sensor placement problem to cover various commonly considered cases on this research topic. This general formulation subsumes the TOA, TDOA, AOA or RSS based model under the A-, D- or E-optimal design criterion. Therefore, by solving this general problem, it is expected that a unified solving approach can be developed.

- Based on the general problem formulation, a unified optimization-based framework has been proposed, which is to solve simpler sub-problems in an iterative manner. The A-, D- or E-optimality can be handled under the same umbrella of this framework with minor changes in the subproblems. Unlike the state-of-the-art approaches which involve handling highly nonlinear trigonometric functions directly or approximating them necessarily, our unified framework neither handles any trigonometric functions nor invokes any approximation in the associated optimization problem.

- To the best of our knowledge, the E-optimal design criterion was never considered in the literature for optimal sensor placement. Our unified framework encompasses the E-optimal design criterion. Additionally, in the data models of the three source localization methods, we do not assume that the noise in the model to be necessarily uncorrelated. Our unified framework can readily handle the case of correlated noise in the model.

- Extensive numerical simulations has been performed for designing optimal sensor placement for all three (TOA, TDOA, RSS) source localization methods for three optimal design criteria (A-, D- and E-optimal designs).

The rest of the paper is organized as follows. In Section II, we describe the system models for the TOA, TDOA, AOA and RSS based source localization, and then formulate a unified CRLB based problem of optimal sensor placement. In Section III, an optimization approach to optimal sensor placement is developed under the A-, D- and E-optimality design criteria for all the TOA, TDOA and RSS models. Section IV demonstrates numerical results. Conclusions are drawn in Section V.

Notations: $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ denote the $n$-dimensional real-valued vector space and $m \times n$ real-valued matrix space, respectively. Scalars, vectors and matrices are denoted by standard lowercase letter $a$, lower case boldface letter $\mathbf{a}$ and upper case boldface letter $\mathbf{A}$, respectively. $\mathbf{A} \succeq \mathbf{B}$ represents that $\mathbf{A} - \mathbf{B}$ is a positive semidefinite matrix. The subscripts $(\cdot)^T$, $(\cdot)^{-1}$, $(\cdot)^\frac{1}{2}$ denote the transpose, inverse and square root of a matrix, respectively. $\mathbf{Tr}(\cdot)$, $\lambda_{\max}(\cdot)$, $\det(\cdot)$ and $\|\cdot\|_F$ denote the trace, maximum eigenvalue, determinant and Frobenius norm, respectively.

\footnote{We will see that the problem formulations for RSS and AOA are essentially the same, and hence, we presents the simulations for RSS for illustration.}
norm of a matrix, respectively. \( \| \cdot \|_p \) and \( | \cdot | \) represents for the \( \ell_p \) norm of a vector and the absolute value of a scalar, respectively. \( E[\cdot] \), \( \log_{10}(\cdot) \), \( \ln(\cdot) \), \( I_m \), \( 1_m \) and \( \frac{\partial}{\partial x} \) denote the statistical expectation, base-10 logarithm, natural logarithm, \( m \times m \) identity matrix, \( m \times 1 \) vector with all elements equal to 1, and differentiation with respect to \( x \), respectively.

II. SYSTEM MODELS AND A UNIFIED PROBLEM FORMULATION

Consider the problem of locating a stationary target in the 3D space using \( m \) sensors with known locations. The sensor may be active with transmitting signals and receiving the echos (e.g. radar and sonar) or passive receiving the signal reflected or transmitted by the target (e.g. hydrophone and microphone). The static target is assumed to be located at an unknown coordinates \( \mathbf{p} \in \mathbb{R}^n \), and \( m \) stationary sensors are located at known coordinates \( \mathbf{r}_i \in \mathbb{R}^n \), \( \forall i = 1, \ldots, m \) with \( m \geq n + 1 \). Depending on the nature of the target (i.e. active or passive) and sensors (i.e. types of measurement), several localization methods can be deployed. In this section, the TOA, TDOA and RSS based localization cases will be considered. For each of them, the system model is first introduced followed by the derived CRLB. Based on these CRLBs, a general problem is formulated to unify the sensor placements of different models.

A. System Model for TOA-Based Localization

Considering the scenario of a passive target and multiple active sensors measuring the round-trip TOA, the noisy measurement at the \( i \)-th sensor is modeled as

\[
\tilde{t}_i = \frac{2 \| \mathbf{p} - \mathbf{r}_i \|_2}{c} + n_i, \quad \forall i = 1, \ldots, m,
\]

(1)

where \( \| \mathbf{p} - \mathbf{r}_i \|_2 \) is the target range from the \( i \)-th sensor, \( c \) denotes the wave propagation speed in the medium, and \( n_i \) represents the TOA measurement noise. It is usually assumed that \( n_i \) follows a Gaussian distribution [28]-[30] denoted by \( n_i \sim \mathcal{N}(0, \sigma_i^2) \). Additionally, it is worth mentioning that the design under the Gaussian CRLB yields the best performance in the worst case over a large class of distributions [31], which further validates this Gaussian assumption.

After converting the TOA measurement to the corresponding distance measurement, we have

\[
s_i = 2 \| \mathbf{p} - \mathbf{r}_i \|_2 + cn_i, \quad \forall i = 1, \ldots, m,
\]

(2)

where \( s_i = c \tilde{t}_i, \quad cn_i \sim \mathcal{N}(0, c^2 \sigma_i^2) \), and \( c \) is the speed of light.

Concatenating all measurements from the \( m \) sensors together, we have the following measurement model

\[
\mathbf{s} = 2 \mathbf{g}(\mathbf{p}) + \mathbf{n}_{\text{toa}},
\]

(3)

where \( \mathbf{s} = [s_1, \ldots, s_m]^T \) denotes the measurements from the \( m \) sensors, \( \mathbf{g}(\mathbf{p}) = [\| \mathbf{p} - \mathbf{r}_1 \|_2, \ldots, \| \mathbf{p} - \mathbf{r}_m \|_2]^T \) and \( \mathbf{n}_{\text{toa}} = [cn_1, \ldots, cn_m]^T \sim \mathcal{N}(0, \mathbf{R}_{\text{toa}}) \) with the covariance matrix \( \mathbf{R}_{\text{toa}} \) assumed to be a general positive definite matrix.

Consequently, the joint density function of the observation vector \( \mathbf{s} \) is given by

\[
p(\mathbf{s}; \mathbf{p}) = \frac{1}{(2\pi)^{m/2} \sqrt{\det(\mathbf{R}_{\text{toa}})}} \exp \left( -\frac{1}{2} (\mathbf{s} - 2 \mathbf{g}(\mathbf{p}))^T \mathbf{R}_{\text{toa}}^{-1} (\mathbf{s} - 2 \mathbf{g}(\mathbf{p})) \right) .
\]

(4)

As stated in the introduction, under some circumstances, especially when some of the sensors are of a similar nature, their inherent noise would be correlated due to the similar mechanism and hardware implementation. Thus, \( \mathbf{R}_{\text{toa}} \) will not necessarily be a diagonal matrix in general.

We denote an unbiased estimate of the true target location by \( \hat{\mathbf{p}} \), the covariance matrix of which satisfies the following well-known inequality [32]

\[
E[(\hat{\mathbf{p}} - \mathbf{p})(\hat{\mathbf{p}} - \mathbf{p})^T] \preceq \mathbf{C}(\mathbf{p}) = \mathbf{F}^{-1}(\mathbf{p}),
\]

(5)

where \( \mathbf{C}(\mathbf{p}) \) is the CRLB matrix, and \( \mathbf{F}(\mathbf{p}) \) is the Fisher information matrix (FIM) [32] given by

\[
\mathbf{F}(\mathbf{p}) = E\left( \left( \frac{\partial \ln p(\mathbf{s}; \mathbf{p})}{\partial \mathbf{p}} \right) \left( \frac{\partial \ln p(\mathbf{s}; \mathbf{p})}{\partial \mathbf{p}} \right)^T \right) \in \mathbb{R}^{n \times n}.
\]

(6)

According to (6), the FIM for the TOA based localization can be expressed as

\[
\mathbf{F}_{\text{toa}}(\mathbf{p}) = 4 \mathbf{H}^T \mathbf{R}_{\text{toa}}^{-1} \mathbf{H},
\]

(7)

and the CRLB matrix is thereby

\[
\mathbf{C}_{\text{toa}}(\mathbf{p}) = \mathbf{F}^{-1}_{\text{toa}}(\mathbf{p}) = \frac{1}{4} \left( \mathbf{H}^T \mathbf{R}_{\text{toa}}^{-1} \mathbf{H} \right)^{-1},
\]

(8)

where

\[
\mathbf{H} \triangleq \begin{bmatrix} \mathbf{h}_1^T \\ \vdots \\ \mathbf{h}_m^T \end{bmatrix} \triangleq \begin{bmatrix} (\mathbf{p} - \mathbf{r}_1)^T \\ \| \mathbf{p} - \mathbf{r}_1 \|_2 \\ \vdots \\ (\mathbf{p} - \mathbf{r}_m)^T \\ \| \mathbf{p} - \mathbf{r}_m \|_2 \end{bmatrix}.
\]

(9)

The matrix \( \mathbf{H} \) is referred to as the orientation matrix, in which each \( \mathbf{h}_i \) is a unit vector (i.e. \( \mathbf{h}_i^T \mathbf{h}_i = 1 \)) defining the orientation of the \( i \)-th sensor with respect to the target. Therefore, the CRLB essentially depends only on the orientation of sensors with respect to the target, and the ranges between the target and sensors will not affect it. In other words, designing the sensor placement is equivalent to designing the orientation of all the sensors.

Remark 1. The above parameterization of CRLB matrix in term of the unit vectors \( \{\mathbf{h}_i\} \) is very different from the common parameterization approach seen in the literature. In the current literature, the elements of \( \{\mathbf{h}_i\} \) vectors are usually specified in term of “azimuth” and “elevation” angles, and thus the resulting CRLB matrix will be a complicated function of trigonometric functions. Differently, we prefer to keep the elements of \( \{\mathbf{h}_i\} \) in Cartesian coordinates (we can easily compute the corresponding azimuth and elevation angle from \( \mathbf{h}_i \)), which enables us to develop a neat optimization method.
later. Hence, the CRLB matrix (7) will be treated as a function of the orientation matrix $H$.

**Remark 2.** The noise in the model (3) is assumed to be distance independent (distance from the target). However, when the noise is distance dependent, one can constrain that the optimal design on a predefined sphere for 3D case or circle for 2D case, which would lead to a similar constraint $(h_i^T h_i = c^2$, where $c$ denotes the radius of the sphere or the circle). Moreover, our approach in this work can also handle the case where $h_i^T h_i = c^2$ with arbitrarily predefined $c_i$ for each sensor, which can be seen clearly in Algorithm 2.

### B. System Model for TDOA-Based Localization

Consider the target to be active and each sensor receives the wave transmitted by the target. We assume that the sensors are ideally time synchronized but unsynchronized with the target’s clock. Upon receiving the wave transmitted by the target, each sensor estimates the TOA of the wave as

$$
t_i = t_0 + \frac{||p - r_i||_2}{c} + n_i, \forall i = 1, \ldots, m, \tag{10}
$$

where $t_0$ represents the unknown time at which target transmits the wave, $n_i \sim \mathcal{N}(0, \sigma_i^2)$ denotes the measurement error of the TOA of the wave, and $p$, $r_i$, and $c$ are the same as in the previous subsection. Converting the time measurements to the distance measurements, we get

$$s_i = ||p - r_i||_2 + cn_i, \forall i = 1, \ldots, m, \tag{11}
$$

where $s_i \triangleq c (t_i - t_0)$ and $cn_i \sim \mathcal{N}(0, c^2 \sigma_i^2)$.

Since $t_0$ is unknown, TOA difference or range difference can be used as an alternative for source localization. One of the sensors is set as the reference or anchor sensor, with respect to which the range difference or TOA difference is computed. Without loss of generality, considering the first sensor as the reference sensor, we can compute the range difference as follows:

$$s_{i1} = ||p - r_i||_2 - ||p - r_1||_2 + cn_i - cn_1, \forall i = 2, \ldots, m. \tag{12}
$$

Concatenating all $\{s_{i1}\}$ in a vector form, we have

$$s = Kg(p) + Kn, \tag{13}
$$

where $s = [s_{21}, \ldots, s_{m1}]^T$, $n = [cn_1, \ldots, cn_m]^T$ and

$$K \triangleq \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{14}
$$

Denoting $\eta_{tdoa} \triangleq Kn$, then (13) becomes

$$s = Kg(p) + \eta_{tdoa}, \tag{15}
$$

where $\eta_{tdoa} \sim \mathcal{N}(0, R_{tdoa})$ with

$$R_{tdoa} = \mathbb{E} \left[ \eta_{tdoa} \eta_{tdoa}^T \right] = KE \left[ nn^T \right] K^T. \tag{16}
$$

**Remark 3.** From equation (16), it can be seen clearly that $R_{tdoa}$ will not be diagonal even if $\mathbb{E} [nn^T]$ is diagonal. Thus, in the case of the TDOA based localization, the covariance matrix $R_{tdoa}$ by nature is a non-diagonal positive definite matrix.

Similarly, the FIM and CRLB for the TDOA based localization can be expressed as, respectively,

$$F_{tdoa}(p) = H^T K^{-1} R_{tdoa}^{-1} K H \tag{17}
$$

and

$$C_{tdoa}(p) = F_{tdoa}^{-1}(p) = \left( H^T K^T R_{tdoa}^{-1} K H \right)^{-1}, \tag{18}
$$

in which the CRLB matrix is also a function of the orientation matrix $H$.

### C. System Model for RSS-Based Localization

In the RSS based source localization, the target transmits some specific signal which is received by each sensor. Each sensor receives the signal and measures the RSS at its own location. In the absence of disturbance, the average power received at the $i$-th receiver is modeled as [33]

$$P_i = \frac{K_i P_t}{||p - r_i||_2^2}, \forall i = 1, \ldots, m, \tag{19}
$$

where $P_t$ denotes the transmitting power, respectively, $K_i$ accounts for all other factors which affect the received power, and $\alpha$ denotes the path loss constant ($\alpha = 2$ in case of free space). It is assumed that $K_i$, $P_t$ and $\alpha$ are known a priori obtained through calibration campaign [34], [35].

Due to shadow fading, the RSS disturbance is assumed to be log-normally distributed [29], [36]. Thus, the measured RSS at the $i$-th sensor in decibel scale is modeled as

$$10 \log_{10} P_i = 10 \log_{10} K_i + 10 \log_{10} P_t - 10 \log_{10} ||p - r_i||_2^2 + w_i, \tag{20}
$$

where the measurement error $w_i$ is now Gaussian distributed. For the notation simplicity, by converting the base-10 logarithm in (20) to natural logarithm, we have

$$\ln P_i = \ln K_i + \ln P_t - \alpha \ln ||p - r_i||_2^2 + (0.1 \ln 10) w_i, \tag{21}
$$

which can be further rewritten as

$$z_i = -\alpha \ln ||p - r_i||_2^2 + \eta_i^{rss} \tag{22}
$$

where $z_i = \ln P_i - \ln K_i - \ln P_t$, $\eta_i^{rss} = (0.1 \ln 10) w_i$, and $\eta_i^{rss} \sim \mathcal{N}(0, \sigma_i^2)$. Collecting all the measurements from the $m$ sensors, the vector matrix form is

$$z = -\alpha \varphi (p) + \eta^{rss}, \tag{23}
$$

where $z = [z_1, \ldots, z_m]^T$, $\eta^{rss} = [\eta_1^{rss}, \ldots, \eta_m^{rss}]^T \sim \mathcal{N}(0, R_{rss})$, and $\varphi (p) = [\ln ||p - r_1||_2, \ldots, \ln ||p - r_m||_2]^T$.

Similar to the TOA-based source localization, the FIM based on model (23) can be computed as
\[ F_{\text{ras}}(p) = \alpha^2 H^T D^T R_{\text{ras}}^{-1} DH \] (24)

where \( D = \text{diag}(\|p - r_1\|_2, \ldots, \|p - r_m\|_2) \) is referred to as the range matrix. Consequently, the CRLB matrix is

\[ C_{\text{ras}}(p) = F_{\text{ras}}^{-1}(p) = \frac{1}{\alpha^2} (H^T D^T R_{\text{ras}}^{-1} DH)^{-1}. \] (25)

Remark 4. Unlike the TOA and TDOA cases, the RSS based CRLB depends on both the orientation matrix \( H \) and the range matrix \( D \). Here, we would like to note that in optimal sensor-target geometry analysis, one of the key underlying assumption is that an initial estimate of the target position is known by some other means. Consequently, the sensors can be placed optimally based on the initial estimate of the target [23], which in turn can further refine the estimate of the target position. Therefore, the range matrix \( D \) is known coarsely from each sensor based on an initial estimate of the target position, and we are more interested in determining the orientation matrix \( H \) with respect to that initial target position.

Remark 5. The optimal sensor placement for AOA model can be easily included in our framework, a brief explanation on the same is as follows. From [13] and [12], the FIM for AOA–based model would be

\[ F_{\text{aoa}}(p) = H^T D^T R_{\text{aoa}}^{-1} DH, \] (26)

where

\[ H = \begin{bmatrix} -\sin \theta_1 & \cos \theta_1 \\ \vdots & \vdots \\ -\sin \theta_m & \cos \theta_m \end{bmatrix} \] (27)

and

\[ D = \begin{bmatrix} 1/d_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1/d_m \end{bmatrix} \] (28)

where \( \theta_1, \ldots, \theta_m \) are the orientation angles of the sensors with respect to the target and \( d_1, \ldots, d_m \) denote the distance between the sensors to the target. It can be seen from (26) that the FIM for AOA has the same structure as the FIM for RSS model (i.e. equation (24)) except for the differences in \( H \). In fact, we will have the similar expression for the AOA in the 3D case, and the corresponding \( H \) for AOA can also be reparameterized as matrix with unit norm rows as the norms of the rows of \( H \) would be a constant. To conclude, the unified framework proposed in our work can easily include AOA localization model.

D. A General CRLB-Based Problem Formulation

Up to this point, we have derived the three CRLB matrices for the TOA, TDOA, AOA and RSS based models, and they are listed as follows:

\[
\begin{align*}
\text{TOA} : & \quad C_{\text{toa}}(p) = \frac{1}{\sigma^2} (H^T R_{\text{toa}}^{-1} H)^{-1} \\
\text{TDOA} : & \quad C_{\text{tdoa}}(p) = (H^T K^T R_{\text{tdoa}}^{-1} K H)^{-1} \\
\text{RSS} : & \quad C_{\text{ras}}(p) = \frac{1}{\sigma^2} (H^T D^T R_{\text{ras}}^{-1} DH)^{-1} \\
\text{AOA} : & \quad C_{\text{aoa}}(p) = H^T D^T R_{\text{aoa}}^{-1} DH^{-1}.
\end{align*}
\] (29)

It is clear to see that all the three expressions are functions of the orientation matrix \( H \) and share the same structure.

Ignoring the constant scalars of \( C_{\text{toa}}(p) \), \( C_{\text{tdoa}}(p) \), \( C_{\text{ras}}(p) \) and \( C_{\text{aoa}}(p) \), a unified expression for all the three CRLBs can be defined as

\[ C(H) = (H^T \Phi^T R^{-1} \Phi H)^{-1}, \] (30)

which will be reduced to one of the expressions of (29) when \( R \) and \( \Phi \) are specified. Thus, this general expression makes it viable to unify different sensor placement problems with just a single one elegantly. Hereafter, we refer \( C(H) \) as a general CRLB matrix.

Since \( C(H) \) is a matrix, some function is required to convert the goodness of \( C(H) \) into a scalar value, which will serve as the evaluation or optimization metric. Actually, in the CRLB based design or optimization, there are many choices for this required function. Among them, the \( A- \), \( D- \) and \( E- \) optimality\(^2\) are the most widely used metrics [37], [38]. For the sake of notation simplicity, a general scalar-valued function \( f(\cdot) \) is used to represent these three optimalities, which will be specified later when solving the relevant problem. Therefore, based on the expression \( C(H) \), we have a general problem formulation for sensor placement, i.e.,

\[
\begin{align*}
\text{minimize} & \quad f(C(H)) \\
\text{subject to} & \quad h_i^T h_i = 1, \forall i = 1, \ldots, m.
\end{align*}
\] (31)

Before addressing this problem, several points which are worth to illustrate as follows:

- Problem (31) is a unified formulation in two aspects: First, \( C(H) \) can be one of the CRLBs shown in (29) for the TOA, TDOA, AOA and RSS based source localization. Second, the general function \( f(\cdot) \) can be trace \( \text{Tr}(\cdot) \), determinant \( \det(\cdot) \), or maximum eigenvalue \( \lambda_{\text{max}}(\cdot) \) for \( A-, D- \) and \( E- \)-optimal designs. Thus, any method solving problem (31) will make itself a unified approach to cover a lot of common cases in the context of sensor placement.

- Problem (31) is nonconvex in both the objective functions and constraints, which is challenging to tackle in general. Further, in the case of E-optimal design, it is non-differentiable in general due to \( f(\cdot) = \lambda_{\text{max}}(\cdot) \). Consequently, to the best of our knowledge, most of the analytic approaches can only handle some special cases (e.g. a diagonal \( R \) and only A or D-optimality). For heuristic approaches, the computational cost make themselves less appealing especially for a large-scale sensor network. Accordingly, the optimization approach to this general problem would become quite competitive.

III. A Unified Optimization Approach to Optimal Sensor Placement

A. Reformulation and The ADMM Framework

The ADMM method is a powerful optimization framework, which has been successfully applied to many

\(^2\)The \( A-, D- \) and \( E- \)-optimal designs refer to trace \( \text{Tr}(\cdot) \), determinant \( \det(\cdot) \), or maximum eigenvalue \( \lambda_{\text{max}}(\cdot) \) of the CRLB matrix, respectively.
convex and nonconvex problems. To tackle problem (31), the ADMM framework is adopted. Let $D = \{ H \in \mathbb{R}^{m \times n} \mid h^T_i h_i = 1, \forall i = 1, \ldots, m \}$ and introduce an auxiliary variable $X$ such that $\Phi H = X$, then we can rewrite (31) as

$$\begin{align*}
\min_{H \in D} & \quad f \left( (X^T R^{-1} X)^{-1} \right) \\
\text{subject to} & \quad \Phi H = X.
\end{align*}$$

Its augmented Lagrangian is formed as

$$L_\rho (X, H, G) = f \left( (X^T R^{-1} X)^{-1} \right) + \text{Tr} \left( G^T (\Phi H - X) \right) + \frac{\rho}{2} \| \Phi H - X \|^2_F,$$

where $G \in \mathbb{R}^{m \times n}$ is the Lagrangian multiplier, $\rho > 0$ is the augmented Lagrangian parameter [39].

The standard ADMM update rules for problem (32) are [39]:

$$\begin{align*}
X_{k+1} &= \arg \min_X L_\rho (X, h_k, G_k), \\
H_{k+1} &= \arg \min_{H \in D} L_\rho (X_{k+1}, H, G_k), \\
G_{k+1} &= G_k + \rho (\Phi H_{k+1} - X_{k+1}),
\end{align*}$$

The two subproblems (34a) and (34b) will be solved subsequently in the following subsections.

B. Solving the Subproblem of $X$

Given $H_k$ and $G_k$ at the $k$-th iteration, $L_\rho (X, H_k, G_k)$ can be expressed as

$$\begin{align*}
L_\rho (X, H_k, G_k) &= f \left( (X^T R^{-1} X)^{-1} \right) + \text{Tr} \left( G_k^T (\Phi H_k - X) \right) + \frac{\rho}{2} \| \Phi H_k - X \|^2_F, \\
&= f \left( (X^T R^{-1} X)^{-1} \right) + \frac{\rho}{2} \text{Tr} (X^T X) - \text{Tr} (D_k^T X) + \beta_k,
\end{align*}$$

where $\beta_k = \text{Tr} (G_k^T \Phi H_k) + \frac{\rho}{2} \text{Tr} (H_k^T \Phi^T \Phi H_k)$ and

$$D_k = G_k + \rho \Phi H_k.$$ 

Let $R^{-1/2} X = Y$, and then (35) can be written as

$$\begin{align*}
L_\rho (Y, H_k, G_k) &= f \left( (Y^T Y)^{-1} \right) + \frac{\rho}{2} \text{Tr} (Y^T R Y) - \text{Tr} (E_k^T Y) + \beta_k,
\end{align*}$$

with $E_k = R^{1/2} D_k$. Thus, the optimization problem w.r.t $X$ is equivalent to

$$\begin{align*}
\min_Y & \quad f \left( (Y^T Y)^{-1} \right) + \frac{\rho}{2} \text{Tr} (Y^T R Y) - \text{Tr} (E_k^T Y),
\end{align*}$$

where the term $f \left( (Y^T Y)^{-1} \right)$ is nonconvex and may not lead to any closed form solution. For some complicated optimization problems that cannot be handled by a single optimization technique, it has been demonstrated that MM could be incorporated to solve the subproblem [40], [41]. A brief review of the MM algorithm is as follows:

The MM (MM stands for “majorize-minimize” for minimization problem or “minimize-maximize” for maximization problem) algorithm is an iterative method to solve an optimization problem. It works by creating a surrogate function for the original objective function (at each iteration) which either majorizes or minorizes the original objective and this surrogate function is optimized instead of the original objective. Let $f(u)$ be the objective to be minimized then at $\tau$-th iteration the surrogate function $g(u | u_\tau)$ is created such that $f(u) \geq g(u | u_\tau)$ and $f(u) \leq g(u | u_\tau)$ for all $u$. The next iterate $u_{\tau+1}$ is computed as follows:

$$u_{\tau+1} = \arg \min_u g(u | u_\tau).$$

The iterations of MM monotonically decrease the objective function and more details on MM can be found in [42].

We will solve problem (38) using the majorization-minimization (MM) technique, which will lead to a double-loop algorithm finally.

At the $\tau$-th iteration of MM, the global bound of the objective function of problem (38) should be constructed, which is provided in the following lemma.

**Lemma 6.** The objective function of problem (38) is upper bounded by

$$g_L (Y) = f \left( (Y^T Y)^{-1} \right) + \frac{\rho}{2} \lambda_m (R) \text{Tr} (Y^T Y) - \text{Tr} (A_k^T \tilde{Y} Y),$$

where $\lambda_m (R)$ is the maximum eigenvalue of $R$.

$$\tilde{R} = R - \lambda_m (R) I_m,$$

and the equality holds when $Y = Y_\tau$.

**Proof:** See Appendix A.

Within the MM iterations, the next update $Y_{\tau+1}$ is computed by solving the following problem:

$$\begin{align*}
\min_Y & \quad f \left( (Y^T Y)^{-1} \right) + \frac{\rho}{2} \lambda_m (R) \text{Tr} (Y^T Y) - \text{Tr} (A_k^T \tilde{Y} Y).
\end{align*}$$

Let $Y_*$ be a minimizer of problem (38) computed by the MM iterations, then we can compute $X_{k+1}$ as

$$X_{k+1} = R_+ Y_*.$$

Note that the form of update equation for $X_{k+1}$ is the same for all TOA, TDOA and RSS based methods irrespective of the choice of $f(.)$. However, the solution step of the problem in (43) would be dependent on the choice of $f(.)$. In the following, we will discuss how to solve problem (43) when $f(.) = \text{Tr}(.)$ (for $A$-optimal design), $f(.) = \log det(.)$ (for $D$-optimal design) and $f(.) = \lambda_m(.)$ (for $E$-optimal design).

1) **Problem (43) for A-Optimal Design**: In the case of $A$-optimal design, we have $f(.) = \text{Tr}(.)$, and problem (43) becomes

$$\begin{align*}
\min_Y & \quad \text{Tr} \left( (Y^T Y)^{-1} \right) + \frac{\rho}{2} \lambda_m (R) \text{Tr} (Y^T Y) - \text{Tr} (A_k^T \tilde{Y} Y),
\end{align*}$$
where the objective function is denoted by $g_A(Y)$. Let $A_{k,r} = U\Sigma V^T$ be the singular value decomposition (SVD) of $A_{k,r}$, and then the SVD of $Y$ can be written as $Y = U\tilde{A}V^T$, where the singular value matrix $A$ is unknown. Therefore, $g_A(Y)$ in problem (45) can be seen as a function of $A$ only and can be written as

$$g_A(A) \triangleq \text{Tr} \left( (A^T A)^{-1} \right) + \frac{\rho}{2} \lambda_m(R) \text{Tr} \left( A^T A \right) - \text{Tr} \left( \Sigma^T A \right),$$

which can be further rewritten as

$$g_A(\{\gamma_i\}) = \sum_{i=1}^{n} \varphi_i^A(\gamma_i),$$

where $\varphi_i^A(\gamma_i) \triangleq \gamma_i^{-2} + \frac{\rho}{2} \lambda_m(R) \gamma_i^2 - \sigma_i \gamma_i$, and $\{\gamma_i\}_{i=1}^{n}$ and $\{\sigma_i\}_{i=1}^{n}$ are singular values of $Y$ and $A_{k,r}$, respectively.

Let $\{\hat{\gamma}_i\}_{i=1}^{n}$ be the minimizer of $g_A(\{\gamma_i\})$ in (47), then

$$\hat{\gamma}_i = \arg\min_{\gamma_i} \varphi_i^A(\gamma_i).$$

The minimizer $\hat{\gamma}_i$ can be easily computed numerically as one of the positive roots of the quartic equation

$$\frac{d\varphi_i^A(\gamma_i)}{d\gamma_i} = -2\gamma_i^{-3} + \rho \lambda_m(R) \gamma_i - \sigma_i = 0,$$

Hence, the solution of problem (45) is

$$Y_{\tau+1} = U\hat{A}V^T,$$

where $\hat{A}$ is the diagonal matrix with the elements $\{\hat{\gamma}_i\}_{i=1}^{n}$.

2) Problem (43) for D-Optimal Design: In the case of $D$-optimal design, problem (43) becomes

$$\min_Y \log\det \left( (Y^T Y)^{-1} \right) + \frac{\rho}{2} \lambda_m(R) \text{Tr} \left( Y^T Y \right) - \text{Tr} \left( A_{k,r}^T Y \right),$$

where the objective function is denoted as $g_D(Y)$.

Similar to the $A$-Optimal design case, we can write $g_D(Y)$ in term of singular value matrix $A$ as

$$g_D(A) \triangleq \log\det \left( (A^T A)^{-1} \right) + \frac{\rho}{2} \lambda_m(R) \text{Tr} \left( A^T A \right) - \text{Tr} \left( \Sigma^T A \right),$$

which can be further rewritten as

$$g_D(\{\gamma_i\}) = \sum_{i=1}^{n} \varphi_i^D(\gamma_i),$$

where $\varphi_i^D(\gamma_i) \triangleq -2 \log(\gamma_i) + 0.5 \rho \lambda_m(R) \gamma_i^2 - \sigma_i \gamma_i$.

Let $\{\hat{\gamma}_i\}_{i=1}^{n}$ be the minimizer of $g_D(A)$, then

$$\hat{\gamma}_i = \arg\min_{\gamma_i} \varphi_i^D(\gamma_i)$$

and $\hat{\gamma}_i$ is one of the positive roots of the quadratic equation

$$\frac{d\varphi_i^D(\gamma_i)}{d\gamma_i} = \rho \lambda_m(R) \gamma_i - \sigma_i - 2\gamma_i^{-1} = 0,$$

which can be computed as

$$\hat{\gamma}_i = \frac{\sigma_i + \sqrt{\sigma_i^2 + 8 \rho \lambda_m(R)}}{2 \rho \lambda_m(R)}.$$

Hence, the solution of problem (51) can be written as

$$Y_{\tau+1} = U\hat{A}V^T,$$

where matrix $\hat{A}$ is a diagonal matrix with the elements $\{\hat{\gamma}_i\}_{i=1}^{n}$.

3) Problem (43) for E-Optimal Design: In the E-optimal design case, problem (43) becomes

$$\min_Y \lambda_m \left( (Y^T Y)^{-1} \right) + \frac{\rho}{2} \lambda_m(R) \text{Tr} \left( Y^T Y \right) - \text{Tr} \left( A_{k,r}^T Y \right),$$

where the objective function is denoted by $g_E(Y)$.

Similar to the $A$ and $D$ optimal designs, we can write $g_E(Y)$ in term of the singular value matrix $A$ as

$$g_E(A) \triangleq \lambda_m \left( (A^T A)^{-1} \right) + \frac{\rho}{2} \lambda_m(R) \text{Tr} \left( A^T A \right) - \text{Tr} \left( \Sigma^T A \right),$$

which can be further rewritten as

$$g_E(\{\gamma_i\}) = \max_{1\leq i\leq n} \left\{ \frac{1}{\gamma_i} \right\} + \frac{\rho}{2} \lambda_m(R) \sum_{i=1}^{n} \gamma_i^2 - \sum_{i=1}^{n} \sigma_i \gamma_i.$$

Therefore, in case of the E-optimal design, we have to solve the following minimax problem

$$\min_{\gamma_i} \max_{1\leq i\leq n} \left\{ \frac{1}{\gamma_i} \right\} + \frac{\rho}{2} \lambda_m(R) \sum_{i=1}^{n} \gamma_i^2 - \sum_{i=1}^{n} \sigma_i \gamma_i$$

subject to $\gamma_i \geq 0, \forall i = 1, \ldots, n.$

Unlike the previous cases, we will not have any closed form solution for the nonconvex problem (61). However, through the variable transform $\gamma_i^2 = \theta_i$, problem (61) be reformulated using the epigraph form as

$$\min_{\theta_i} t + \frac{\rho}{2} \lambda_m(R) \sum_{i=1}^{n} \theta_i - \sum_{i=1}^{n} \sigma_i \sqrt{\theta_i}$$

subject to $\theta_i \geq 0, \forall i = 1, \ldots, n$

$$\frac{1}{\theta_i} \leq t, \forall i = 1, \ldots, n$$

$$t \geq 0.$$

Problem (62) is convex in $n+1$ (with $n \leq 3$) scalar variables. Thus, it can be solved efficiently by some solvers like CVX [43]. Let $\{\hat{\theta}_i\}$ be the solution of problem (62), then $\{\gamma_i\}$ can be obtained using $\gamma_i^2 = \sqrt{\theta_i}$. Hence, $Y_{\tau+1} = U\hat{A}V^T$, where matrix $\hat{A}$ is a diagonal matrix with the elements $\{\hat{\gamma}_i\}_{i=1}^{n}$.

So far, we have derived the solving methods to problem (34a) for all the $A$-, $D$- and E-optimal design criteria. The proposed algorithms are summarized in Algorithm 1.

### C. Solving the Subproblem of H

As in the next step of ADMM, we compute $H_{k+1}$, for that we write the expression for $L_p(X_{k+1}, H, G_k)$ as
Algorithm 1 Proposed method to problem (34a)

Input: \( m, n, \rho, R, H_k, G_k \)

Output: \( X_{k+1} \)

1: \( E_k = R - \frac{1}{\rho} (G_k + \rho \Phi H_k) \)
2: \( R = R - \lambda_m (R) I_m \)
3: \( Y_r = R^{-1} X_k \)
4: \( \tau = 0 \)
5: repeat
6: \( A_{k, \tau} = E_k - \tilde{\rho} Y_r \)
7: Compute the SVD \( A_{k, \tau} = \text{U} \text{diag}(\{\sigma_i\}) \text{V}^T \)
8: \( \gamma_i = \frac{\sigma_i + \sqrt{\sigma_i^2 + 8 \rho \lambda_m(R)}}{2 \rho \lambda_m(R)} \)
9: \( \hat{\gamma}_i = \sqrt{\hat{\gamma}_i} \)
10: \( \hat{Y}_r = \text{U} \text{V}^T \)
11: \( \tau \leftarrow \tau + 1 \)
12: until Convergence
13: \( X_{k+1} = R^{-1} \hat{Y}_r \)

where \( h_i \) and \( e_i^{k+1} \) are the \( i \)-th column of \( H^T \) and \((C_k)^T\), respectively. Since \( h_i^T h_i = 1 \), \( L_\rho (X_{k+1}, H, G_k) \) can be further written as

\[
L_\rho (X_{k+1}, H, G_k) = \frac{\rho}{2} \sum_{i=1}^{m} d_i^2 + \sum_{i=1}^{m} d_i e_i^{k+1} h_i^T + \alpha_{k+1}. \tag{67}
\]

The objective \( L_\rho (X_{k+1}, H, G_k) \) is separable in \( h_i \).

Therefore, the minimizer \( \hat{h}_i \) of (67) for \( H \in D \) is given by

\[
\hat{h}_i = -e_i^{k+1} / \|e_i^{k+1}\|_2, \tag{68}
\]

and \( H_{k+1} \) is computed as \( \hat{H}_{k+1} = [\hat{h}_1, \ldots, \hat{h}_m]^T \).

2) Update of \( H \) for TOA: For the TOA based model, the expression for \( H_{k+1} \) can be derived by following the same derivation of the above RSS case by setting \( \Phi = I_m \). It finally leads to the same expression as given in (68) and thus \( H_{k+1}^{toa} = [\hat{h}_1, \ldots, \hat{h}_m]^T \).

3) Update of \( H \) for TDOA: For the TDOA based model, considering \( \Phi = K \) and \( R = R_{tdoa} \), we have

\[
L_\rho (X_{k+1}, H, G_k) = \frac{\rho}{2} \text{Tr}(H^T M H) + \text{Tr}(C_{k+1}^T K H) + \alpha_{k+1}, \tag{69}
\]

where \( M = K^T K \succeq 0 \), and \( \alpha_{k+1} \) is defined by (64) with \( R = R_{tdoa} \). Minimizing (69) with respect to \( H \in D \) is not straightforward, so similar to the previous subsection, we employ MM to minimize \( L_\rho (X_{k+1}, H, G_k) \). The term \( \text{Tr}(H^T M H) \) in (69) is convex of \( H \), and we reformulate it via concave function as follows:

\[
L_\rho (X_{k+1}, H, G_k) = \frac{\rho}{2} \text{Tr}(H^T M H - \lambda_m (M) H^2 H + \lambda_{max} (M) H^2 H)
 + \text{Tr}(C_{k+1}^T K H) + \alpha_{k+1}, \tag{70}
\]

Minimizing (63) with respect to \( \Phi \) depends on the choice of \( \Phi \), which has different expressions for different localization models. In the following, we will derive the updates of \( H \) for all three models separately.

1) Update of \( H \) for RSS/AOA: For the RSS based method, we substitute \( \Phi = D \) in (63) and obtain

\[
L_\rho (X_{k+1}, H, G_k) = \frac{\rho}{2} \text{Tr}(H^T D^T D H) + \text{Tr}(C_{k+1}^T D H) + \alpha_{k+1}, \tag{65}
\]

which can be further rewritten (noting that \( D \) is diagonal matrix) as

\[
L_\rho (X_{k+1}, H, G_k) = \frac{\rho}{2} \text{Tr} \left( \sum_{i=1}^{m} d_i^2 h_i^T h_i^T \right) + \text{Tr} \left( \sum_{i=1}^{m} d_i e_i^{k+1} h_i^T \right) + \alpha_{k+1}, \tag{66}
\]
Algorithm 2 Proposed method to problem (34b)

Input: $m, n, \rho, \mathbf{R}, \mathbf{X}_{k+1}, G_k$
Output: $H_{k+1}$

1. $C_{k+1} = G_k - \rho X_{k+1}$
    (I) TOA based model:
    $\Phi = I_m$
    $\hat{h}_t = -c_t^{k+1}/\|c_t^{k+1}\|_2$

2. repeat
    $\tilde{M} = M - \lambda_m (M) I_m$
    $B_{k,t} = \rho \tilde{M}^T H_t + K^T C_{k+1}$
    $\hat{h}_{t,t} = -b_{t,t}^{k,t}/\|b_{t,t}^{k,t}\|_2$
    $t \leftarrow t + 1$
    until convergence

3. $H_{k+1} = [\hat{h}_1, \ldots, \hat{h}_m]^T$

Algorithm 3 Unified opTimization fraMework for Optimal Sensor placement (UTMOST)

Input: $m, n, \rho, \mathbf{R} \in \{R_{\text{tdoa}}, R_{\text{tdoa}}, R_{\text{rss}}\}, \Phi = \{I_m, K, D\}$
Output: $H$

1. Set $k = 0$
2. Initialize $H_k \in D$ and $G_k$
3. $X_k = \Phi H_k$
4. repeat
    5. Calculate $X_{k+1}$ via Algorithm 1
    6. Calculate $H_{k+1}$ via Algorithm 2
    7. Update $G_{k+1}$ by (34c)
    8. until convergence
9. $H_{k+1} = [\hat{h}_1, \ldots, \hat{h}_m]^T$

In Algorithm 3, we summarize the proposed unified solving approach to the general sensor placement problem (32), where $k$ denotes the index for the ADMM iterations. The indices $\tau$ and $t$ used in the Algorithms 1 and 2 respectively should not be confused with the ADMM iteration index $k$. The index $\tau$ denotes the index for the MM iterations for solving problem (43), and $t$ denotes the index for the MM iterations for computing the update $H_{k+1}$ in the case of TDOA. It is noted that the proposed approach is quite general to cover many models under different optimal design criteria, which can be seen clearly from Algorithm 1-3.

In this section, we discuss the simulation results considering various localization methodologies and different optimal design criteria.

A. Sanity and Convergence Check

First of all, we perform the sanity check or show the correctness of the proposed algorithmic framework for the special case, when matrix $R$ is diagonal with same diagonal entries and $\Phi = I_m$ (thus it becomes the TOA model), for which the analytical optimal values are known in the literature.
We initialize the proposed algorithm such that the sensors are uniformly placed with respect to target with the following initialization

\[ H_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \]  

(79)

With this initialization we can compare the gain in accuracy with the placement obtained by the proposed algorithm with respect to the considered uniform placement and moreover we can also observe how sensors change their positions with the iterations of the algorithm from initial uniform placement to achieve a final optimal configuration.

In Figure 1, we demonstrate the convergence plots and the corresponding 3D placement trajectories of the proposed method for all A-, D- and E-optimal designs. The baseline is the uniform placement (i.e. the sensors are uniformly placed w.r.t. the target), which is also set as the initial point of our proposed algorithm. First, it is clear to see that from the convergence plots, our algorithm monotonically decreases the design objective. Second, with reference to the uniform placement, the proposed method (after convergence) shows 55 – 70% improvement in terms of the design criteria, which implies further an enhancement of localization accuracy enabled by the proposed algorithm. Therefore, to obtain the maximum localization accuracy especially when the measurement noise is correlated, the sensors must be placed in their corresponding optimal configuration, which can be computed from the proposed algorithm.

Now, we apply a different sensor geometry in a 2D target localization problem to investigate the estimation improvement by designing the sensor placement. Specifically, we perform the maximum likelihood estimation (MLE) in a 2D TOA based source localization problem under different sensor placements, which includes random placement, uniform placement, and the placement via our proposed algorithm for the A-optimal design. To perform the simulation, the sensors are assumed to be located on the circumference of a circle of unit radius whose center is at origin and the target is assumed to be located at \((0.1, -0.3)\). The MLE is implemented by conducting the 2D grid search to arrive at the most probable target location and then performing the Gauss-Newton algorithm [49]. The simulated noisy TOA measurements (both uncorrelated and correlated noise case have been considered) have been generated for uniform, optimal and for any randomly selected placement. The results under different noises is provided in Table II, where the MSE and bias of the MLE are obtained using 1000 Monte Carlo simulations. We can see that the MLE and the bias of the estimates in case of optimal placement are smaller than the uniform and random placements, which is consistent to the theoretical conclusion that MLE asymptotically reach the trace of CRLB [20].

In Figure (2), we diagrammatically show how the proposed algorithmic framework finds the optimum of the design objective. We consider a total of three sensors in 2D space (i.e.

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**TABLE I**

Comparison between the theoretical and numerical optimal objective values for the TOA based model.

| \(m\) | \(f^{\text{theo}}_A\) | \(f^{\text{algo}}_A\) | \(f^{\text{theo}}_D\) | \(f^{\text{algo}}_D\) | \(f^{\text{theo}}_E\) | \(f^{\text{algo}}_E\) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|
| 5     | 1.8000         | 1.8000         | -1.5324        | -1.5324        | 0.60000        | 0.60033        |
| 10    | 0.9000         | 0.9000         | -3.6119        | -3.6119        | 0.30000        | 0.30004        |
| 15    | 0.6000         | 0.6000         | -4.8283        | -4.8283        | 0.20000        | 0.20017        |
| 20    | 0.4500         | 0.4500         | -5.6913        | -5.6913        | 0.15000        | 0.15003        |
| 25    | 0.3600         | 0.3600         | -6.3607        | -6.3607        | 0.12000        | 0.12001        |

---

[20]. When \(R = v^2I_m\) and \(\Phi = I_m\) then optimal solution of (31) for \(A\)-, \(D\)- and \(E\)-optimality criteria satisfies the following relation [48]

\[ H^T_{*}H_{*} = \frac{m}{3} I_n, \]  

(76)

where \(H_{*}\) denotes the optimal solution for the case when \(R = v^2I_m\).

The theoretical value of the objective function for \(A\)-, \(D\)- and \(E\)-optimality criteria, denoted by \(f^{\text{theo}}_A\), \(f^{\text{theo}}_D\) and \(f^{\text{theo}}_E\), and their corresponding numerical value obtained by the proposed algorithms denoted by \(f^{\text{algo}}_A\), \(f^{\text{algo}}_D\) and \(f^{\text{algo}}_E\) for the case \(R = v^2I_m\) with \(v = 1\) are listed in Table I. By applying (76), we can compute \(f^{\text{algo}}_A\), \(f^{\text{theo}}_D\) and \(f^{\text{theo}}_E\) as follows:

\[
\begin{align*}
    f^{\text{theo}}_A &= \frac{na^2}{m} \\
    f^{\text{theo}}_D &= \log \left( \frac{27v^2}{m^2} \right) \\
    f^{\text{theo}}_E &= \frac{3v^2}{m}.
\end{align*}
\]  

(77)

From Table I it is observed that the optimal value of the objective functions computed from the proposed algorithm converge to their corresponding analytical value, hence confirming the correctness of the proposed algorithmic framework.

### B. TOA Based Source Localization

In this subsection, the proposed algorithmic framework is applied to determine the optimal configuration of sensors around the target to optimize the localization accuracy for TOA-based model. Without loss of generality, we assume the target to be roughly located at the origin, that is, \(p = [0, 0, 0]^T\).

As illustrated previously, the CRLB for the TOA-based model is independent of the sensor-target distance, so the sensors are assumed to be on the unit sphere and only their optimal orientations are to be determined.

We take \(m = 6\), \(n = 3\) and the noise covariance matrix \(R_{\text{toa}}\) to be a general positive definite matrix given by

\[
R_{\text{toa}} = \begin{bmatrix}
4.88 & 3.07 & -1.73 & 1.90 & 2.63 & -1.61 \\
3.07 & 11.72 & -3.51 & 4.48 & 3.95 & 0.24 \\
-1.73 & -3.51 & 21.82 & -1.20 & 0.49 & -4.74 \\
1.90 & 4.48 & -1.20 & 3.63 & 3.71 & 1.00 \\
2.63 & 3.95 & 0.49 & 3.71 & 8.45 & 0.56 \\
-1.61 & 0.24 & -4.74 & 1.00 & 0.56 & 4.22
\end{bmatrix}
\]  

(78)
reference for measuring TDOA’s, so we will have $m - 1$ unit circle (defined by their corresponding azimuth angles $\phi_1$ and $\phi_2$) to obtain maximum localization performance. As earlier target is coarsely located at the origin and we want to determine the optimal placement of the remaining two sensors on the unit circle (defined by their corresponding azimuth angles $\phi_1$ and $\phi_2$) to obtain maximum localization performance. In figure 2a, 2b and 2c, the objective value has been plotted with $\phi_1$ and $\phi_2$ as surface plot and the path taken by the proposed algorithm to reach the optimal solution is shown on the corresponding contour plots for the A-, D- and E-optimal designs, respectively.

C. TDOA Based Source Localization

In this subsection, we perform simulations for the optimal sensor placements for the TDOA model with $m = 6, n = 3$. We assume that the first sensor is set as a reference for measuring TDOA’s, so we will have $m - 1$ TDOA measurements to localize the target. The covariance matrix $E[nn^T]$ is associated with the error in estimation of sensor-target range and assumed as diagonal matrix with $E[nn^T] = \text{diag}(0.18, 0.02, 0.46, 0.72, 0.42, 0.49)$. Consequently, the noise covariance matrix associated with the TDOA measurements is given by

$$
R_{tdoa} = K E[nn^T] K^T = 
\begin{bmatrix}
0.20 & 0.18 & 0.18 & 0.18 & 0.18 & 0.18 \\
0.18 & 0.64 & 0.18 & 0.18 & 0.18 & 0.18 \\
0.18 & 0.18 & 0.91 & 0.18 & 0.18 & 0.18 \\
0.18 & 0.18 & 0.18 & 0.60 & 0.18 & 0.18 \\
0.18 & 0.18 & 0.18 & 0.18 & 0.67 & 0.18 \\
\end{bmatrix}
$$

Similar to the TOA case, we initialize the proposed algorithm with the uniform placement as given in (79).

Figure 3 demonstrates the convergence plots and the corresponding sensor placements. The difference of the objective values between the uniform and proposed (after convergence) placements shows the $70 - 80\%$ improvement in localization accuracy obtained by the proposed algorithm when the measurement noise is correlated. In the right column of Figure 3, we can see how the sensors move from the initial uniform placement to the final optimal configuration (obtained by the proposed algorithm) for the three optimal designs, respectively.

D. RSS Based Source Localization

In this subsection, we perform simulations for the RSS model. Recall that both AOA and RSS has the same formulations and assumptions (see Remark 5 and expression (29)), the simulations of AOA will demonstrate the same patterns as RSS and thus be omitted. Unlike the TOA and TDOA cases, in the RSS-based localization, the optimal placement also depends on the sensor-target range. Assuming that the sensor-target range for each sensor is given (or roughly known), we are interested in determining the optimal configuration. Let $m = 6, n = 3$, and the sensor-target range $d_i$ (in meter) and the noise covariance matrix $R_{r_{rs}}$ are set as, respectively,

$$
[d_1, d_2, d_3, d_4, d_5, d_6] = [50, 100, 150, 200, 250, 300],
$$

TABLE II

| No. of sensors | Placement | MSE (m²) | Bias (m) |
|----------------|-----------|----------|----------|
| $m = 3$        | Random    | 0.2171   | 0.1105   |
| (Uncorrelated) | Uniform   | 0.1643   | 0.0630   |
|                | Optimal   | 0.1456   | 0.0397   |
| $m = 4$        | Random    | 0.2070   | 0.0998   |
| (Uncorrelated) | Uniform   | 0.1260   | 0.0231   |
|                | Optimal   | 0.1156   | 0.0169   |
| $m = 5$        | Random    | 31.0428  | 1.1099   |
| (Correlated)   | Uniform   | 9.2915   | 1.5758   |
|                | Optimal   | 3.7194   | 0.9507   |
| $m = 7$        | Random    | 0.9840   | 0.3654   |
| (Correlated)   | Uniform   | 1.2771   | 0.6482   |
|                | Optimal   | 0.4711   | 0.2483   |
the target), where the uniform placement (the sensors are uniformly placed w.r.t. placements of the proposed method with reference to the target) = diag (82)

\[
\begin{bmatrix}
4.88 & 3.07 & -1.73 & 1.90 & 2.63 & -1.61 \\
3.07 & 11.72 & -3.51 & 4.48 & 3.95 & 0.24 \\
-1.73 & -3.51 & 21.82 & -1.20 & 0.49 & -4.74 \\
1.90 & 4.48 & -1.20 & 3.63 & 3.71 & 1.00 \\
2.63 & 3.95 & 0.49 & 3.71 & 8.45 & 0.56 \\
-1.61 & 0.24 & -4.74 & 1.00 & 0.56 & 4.22
\end{bmatrix}
\]

Therefore, by taking \( \mathbf{D} = \text{diag}(d_1, \ldots, d_6)^{-1} \), we can use the proposed algorithm to compute the optimal configuration.

Figure 4 demonstrates the convergence plots and the sensor placements of the proposed method with reference to the uniform placement (the sensors are uniformly placed w.r.t. the target), where the \( i \)-th sensor is shown to be at distance \( d_i \) from the target in the 3D plot. The difference in objective values of the uniform placement and proposed method (after convergence) shows the 80–85% improvement (in terms of the design criteria) in localization accuracy obtained by the proposed algorithm when the measurement noise is correlated.

Next, we illustrate the impact of having imperfect knowledge of range matrix \( \mathbf{D} \). To demonstrate this we perform a simulation to compute the MLE (similar to table-II for the TOA model) for the RSS model, in which we evaluate the MSEs of the MLE method for the cases when \( \mathbf{D} \) is perfectly known and \( \mathbf{D} \) is imperfectly known. The true target position is taken as \((0.1, -0.3)\) and in case of “Coarse \( \mathbf{D} \)” we assumed that coarse target position is at \((0.0, 0.0)\) and calculate the range matrix. The MSEs of the MLE method (solved using grid search and additional Gauss-Newton step) for the case of optimal sensor placement done with the perfect knowledge of \( \mathbf{D} \) and coarsely known \( \mathbf{D} \) are shown in table III here. It can be seen from table III, the MSE in the case of coarsely known \( \mathbf{D} \) is higher than the case of perfectly known \( \mathbf{D} \), which is as expected.
plots; right: 3D view of sensor placement. Black square: target; blue circle: Figure 4. Convergence plots and corresponding sensor placements for the techniques. Within this framework, we can handle the sensor placement for all the TOA, TDOA, and RSS based source localization methods by specifying the system parameters. For each localization model, this framework can be adapted with slight modifications to design the sensor placement under all the A-, D- and E-optimality criteria. Through the numerical simulations, we have demonstrated the versatility of the unified approach by considering various placement scenarios and also the improvement of localization accuracy brought by the optimal configuration of sensors.

APPENDIX

A. Proof of Lemma 6

Proof: The objective function of problem (38) can be written as

\[ f \left( (Y^T Y)^{-1} + \frac{\rho}{2} \text{Tr} (Y^T R Y) - \text{Tr} (E_k^T Y) \right) \]

\[ = f \left( (Y^T Y)^{-1} - \text{Tr} (E_k^T Y) \right) \]

\[ + \frac{\rho}{2} \text{Tr} (Y^T R Y - \lambda_m (R) Y^T Y + \lambda_m (R) Y^T Y) \]  
(83)

\[ = f \left( (Y^T Y)^{-1} + \frac{\rho}{2} \text{Tr} (\tilde{Y}^T \tilde{R} Y) \right) \]

\[ + \frac{\rho}{2} \lambda_m (R) \text{Tr} (Y^T Y) - \text{Tr} (E_k^T Y) , \]

where \( \tilde{R} \triangleq R - \lambda_m (R) I_m \). Since \( \tilde{R} \preceq 0 \), \( \text{Tr} (Y^T \tilde{R} Y) \) is concave of \( Y \). Its first order Taylor expansion satisfies

\[ \text{Tr} (Y^T \tilde{R} Y) \leq 2 \text{Tr} (Y^T \tilde{R} Y) - \text{Tr} (Y^T \tilde{R} Y) \]  
(84)

for all \( Y \) with the equality achieved at \( Y = Y_\tau \).

Applying (84) on (83), we have the upper bound given by

\[ g_L (Y) \]

\[ = f \left( (Y^T Y)^{-1} + \frac{\rho}{2} \lambda_m (R) \text{Tr} (Y^T Y) - \text{Tr} (E_k^T Y) \right) \]

\[ + \frac{\rho}{2} \left( 2 \text{Tr} (Y^T \tilde{R} Y) - \text{Tr} (Y^T \tilde{R} Y) \right) \]

\[ = f \left( (Y^T Y)^{-1} + \frac{\rho}{2} \lambda_m (R) \text{Tr} (Y^T Y) \right) \]

\[ - \text{Tr} \left( (E_k - \rho \tilde{R} Y) Y^T \right) - \frac{\rho}{2} \text{Tr} (Y^T \tilde{R} Y) \]  
(85)

which thereby completes the proof.

V. CONCLUSIONS

In this paper, we have unified the three TOA, TDOA, AOA and RSS based sensor placement case in a generalized problem formulation based on the CRLB-related metric. For this general problem, we have developed a unified optimization approach named UTMOST based on the ADMM and MM techniques. Within this framework, we can handle the sensor placement for all the TOA, TDOA, and RSS based source localization methods by specifying the system parameters. For each localization model, this framework can be adapted with slight modifications to design the sensor placement under all the A-, D- and E-optimality criteria. Through the numerical simulations, we have demonstrated the versatility of the unified approach by considering various placement scenarios and also the improvement of localization accuracy brought by the optimal configuration of sensors.

REFERENCES

[1] I. F. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci, “A survey on sensor networks,” IEEE Communications magazine, vol. 40, no. 8, pp. 102–114, 2002.
[2] D. Li, K. D. Wong, Y. H. Hu, and A. M. Sayeed, “Detection, classification, and tracking of targets,” IEEE signal processing magazine, vol. 19, no. 2, pp. 17–29, 2002.
[3] J. Shen, A. F. Molisch, and J. Salmin, “Accurate passive location estimation using toa measurements,” IEEE Transactions on Wireless Communications, vol. 11, no. 6, pp. 2182–2192, 2012.
[4] Y. Zhu, D. Huang, and A. Jiang, “Network localization using angle of arrival,” in 2008 IEEE International Conference on Electro/Information Technology. IEEE, 2008, pp. 205–210.
[5] B. Huang, L. Xie, and Z. Yang, “Tdoa-based source localization with distance-dependent noises,” IEEE Transactions on Wireless Communications, vol. 14, no. 1, pp. 468–480, 2014.
[6] A. J. Weiss, “On the accuracy of a cellular location system based on rss measurements,” IEEE Transactions on vehicular technology, vol. 52, no. 6, pp. 1508–1518, 2003.

[7] K. Ho and W. Xu, “An accurate algebraic solution for moving source location using tdoa and fdoa measurements,” IEEE Transactions on Signal Processing, vol. 52, no. 9, pp. 2453–2463, 2004.

[8] K. Yoo, J. Chun, and C. Ryu, “Crib-based optimal radar placement for target positioning,” in 2018 International Conference on Radar (RADAR). IEEE, 2018, pp. 1–5.

[9] K. Yoo and J. Chun, “Analysis of optimal range sensor placement for tracking a moving target,” IEEE Communications Letters, vol. 24, no. 8, pp. 1700–1704, 2020.

[10] D. Ucinski, Optimal measurement methods for distributed parameter system identification, CRC press, 2004.

[11] K. Doğançay and H. Hmam, “Optimal angular sensor separation for aoa localization,” Signal Processing, vol. 88, no. 5, pp. 1248–1260, 2008.

[12] S. Xu and K. Doğançay, “Optimal sensor placement for 3-d angle-of-arrival target localization,” IEEE Transactions on Aerospace and Electronic Systems, vol. 53, no. 3, pp. 1196–1211, 2017.

[13] S. Xu, “Optimal sensor placement for target localization using hybrid rss, aoa and toa measurements,” IEEE Communications Letters, vol. 24, no. 9, pp. 1966–1970, 2020.

[14] M. Hamdollahzadeh, R. Amiri, and F. Behnia, “Optimal sensor placement for multi-source aoa localisation with distance-dependent noise model,” IET Radar, Sonar & Navigation, vol. 13, no. 6, pp. 881–891, 2019.

[15] X. Fang and J. Li, “Frame theory for optimal sensor augmentation problem of aoa localization,” IEEE Signal Processing Letters, vol. 25, no. 9, pp. 1310–1314, 2018.

[16] Y. Zheng, J. Liu, M. Sheng, S. Han, Y. Shi, and S. Valae, “Toward practical access point deployment for angle-of-arrival based localization,” IEEE Transactions on Communications, vol. 69, no. 3, pp. 2002–2014, 2020.

[17] S. Zhao, B. M. Chen, and T. H. Lee, “Optimal sensor placement for target localization and tracking in 2d and 3d,” International Journal of Control, vol. 86, no. 10, pp. 1687–1704, 2013.

[18] B. Yang and J. Scheuing, “Cramer-rao bound and optimum sensor array for source localization from time differences of arrival,” in Proceedings:(ICASSP’05). IEEE International Conference on Acoustics, Speech, and Signal Processing, 2005., vol. 4. IEEE, 2005, pp. iv–961.

[19] W. Meng, L. Xie, and W. Xiao, “Optimal tdoa sensor-pair placement with uncertainty in source location,” IEEE Transactions on Vehicular Technology, vol. 65, no. 11, pp. 9260–9271, 2016.

[20] S. Xu, Y. Ou, and W. Xu, “Optimal sensor placement for 3-d time-of-arrival target localization,” IEEE Transactions on Signal Processing, vol. 67, no. 19, pp. 5018–5031, 2019.

[21] S. Xu, Y. Ou, and W. Zheng, “Optimal sensor-target geometries for 3-d static target localization using received-signal-strength measurements,” IEEE Signal Processing Letters, vol. 26, no. 7, pp. 966–970, 2019.

[22] L. Rui and K. Ho, “Elliptic localization: Performance study and optimum receiver placement,” IEEE Transactions on Signal Processing, vol. 62, no. 18, pp. 4673–4688, 2014.

[23] N. H. Nguyen, “Optimal geometry analysis for target localization with bayesian priors,” IEEE Access, vol. 9, pp. 33419–33437, 2021.

[24] A. Heydari, M. Aghabozorgi, and M. Biguesh, “Optimal sensor placement for source localization based on rss,” Wireless Networks, vol. 26, no. 7, pp. 5151–5162, 2020.

[25] N. H. Nguyen and K. Doğançay, “Optimal geometry analysis for multistatic toe localization,” IEEE Transactions on Signal Processing, vol. 64, no. 16, pp. 4180–4193, 2016.

[26] ——, “Optimal sensor placement for doppler shift target localization,” in 2015 IEEE Radar Conference (RadarCon). IEEE, 2015, pp. 1677–1682.

[27] S. P. Robinson, P. A. Lepper, and R. A. Hazelwood, “Good practice guide for underwater noise measurement.” 2014.

[28] S. Joshi and S. Boyd, “Sensor selection via convex optimization,” IEEE Transactions on Signal Processing, vol. 57, no. 2, pp. 451–462, 2008.

[29] H. C. So and L. Lin, “Linear least squares approach for accurate received signal strength based source localization,” IEEE Transactions on signal processing, vol. 59, no. 8, pp. 4035–4040, 2011.

[30] S. P. Chepuri and G. Leus, “Sparsity-promoting sensor selection for nonlinear measurement models,” IEEE Transactions on Signal Processing, vol. 63, no. 3, pp. 684–698, 2014.

[31] P. Stoica and P. Babu, “The gaussian data assumption leads to the largest cramér-rao bound [lecture notes],” IEEE Signal Processing Magazine, vol. 28, no. 3, pp. 132–133, 2011.