Improved Concentration Bounds for Conditional Value-at-Risk and Cumulative Prospect Theory using Wasserstein distance

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Abstract

Known finite-sample concentration bounds for the Wasserstein distance between the empirical and true distribution of a random variable are used to derive a two-sided concentration bound for the error between the true conditional value-at-risk (CVaR) of a (possibly unbounded) random variable and a standard estimate of its CVaR computed from an i.i.d. sample. The bound applies under fairly general assumptions on the random variable, and improves upon previous bounds which were either one sided, or applied only to bounded random variables. Specializations of the bound to sub-Gaussian and sub-exponential random variables are also derived. A similar procedure is followed to derive concentration bounds for the error between the true and estimated Cumulative Prospect Theory (CPT) value of a random variable, in cases where the random variable is bounded or sub-Gaussian. These bounds are shown to match a known bound in the bounded case, and improve upon the known bound in the sub-Gaussian case. The usefulness of the bounds is illustrated through an algorithm, and corresponding regret bound for a stochastic bandit problem, where the underlying risk measure to be optimized is CVaR.

1 Introduction

Conditional Value-at-Risk (CVaR) and cumulative prospect theory (CPT) value are two popular risk measures. The former is popular in financial applications, where it is necessary to minimize the worst-case losses, say in a portfolio optimization context. The latter is a risk measure, proposed by Tversky and Kahnemann, that is useful for modeling human preferences. The implicit assumption in risk-sensitive optimization is that expected value is not an appealing objective in several practical applications, and it is necessary to incorporate some notion risk in the optimization process. The reader is referred to extensive literature on risk-sensitive optimization, in particular, the shortcomings of the expected value - cf. [Allais, 1953, Ellsberg, 1961, Kahneman and Tversky, 1979, Rockafellar and Urvases, 2000].

In practical applications, the information about the underlying distribution is typically unavailable. However, one can often obtain samples from the distribution, and the aim is to estimate the chosen risk measure using these samples. We consider this problem of estimation, with CVaR and CPT-value as the risk measures. For both the risk measures, we employ well-known estimators, that are already available in the literature. Our goal is to derive concentration bounds for empirical CVaR and CPT-values, and we achieve this purpose by using a novel technique that relates the estimation error (for both CVaR and CPT-value) to the distance between the empirical and true distributions, as qualified by the Wasserstein distance. While concentration bounds for bounding the Wasserstein distance between any two given distributions are already known, it is interesting to know that one can relate the estimation error for a risk measure to the Wasserstein distance, and use this fact to derive concentration results for empirical CVaR and CPT-values. We summarize our contributions below, which apply when the underlying distribution has a bounded exponential moment, or a higher-order moment. Sub-Gaussian distributions are a popular class that satisfy the former condition, while the latter includes sub-exponential distributions.

(1) For the case of CVaR, we provide a two-sided concentration bound for both classes of distributions mentioned above. In particular, for the special case of sub-Gaussian distributions, our tail bound is of the order
There exist \( \beta > 0 \) and \( \gamma > 0 \) such that \( \mathbb{E}(\exp(\gamma|X|^\beta)) < T < \infty \).

(C2) There exists \( \beta > 0 \) such that \( \mathbb{E}(|X|^\beta) < T < \infty \).

We next define sub-Gaussian and sub-exponential r.v.s, which are two popular classes of unbounded r.v.s, that satisfy assumptions (C1) and (C2), respectively.
Definition 2. A r.v. $X$ is sub-Gaussian if there exists a $\sigma > 0$ such that
\[
\mathbb{E}(\exp(\lambda X)) \leq \exp\left(\frac{\lambda^2 \sigma^2}{2}\right) \text{ for any } \lambda \in \mathbb{R}.
\]
A sub-Gaussian r.v. $X$ satisfies (see items (II) and (IV) in Theorem 2.1 of [Wainwright, 2015] for a proof)
\[
\mathbb{E}\left(\exp\left(\frac{X^2}{4\sigma^2}\right)\right) \leq \sqrt{2}, \text{ and } P(X > \eta) \leq 8 \exp(-\frac{\eta^2}{4\sigma^2}), \text{ for } \eta \geq 0.
\] (3)

The first bound above implies that sub-Gaussian r.v.s satisfy (C1) with $\beta = 2$, $\gamma = \frac{1}{4\sigma^2}$ and $\top = \sqrt{2}$. In particular, bounded r.v.s are sub-Gaussian, and satisfy (C1) with $\beta = 2$.

Definition 3. Given $\sigma > 0$, a r.v. $X$ is $\sigma$ sub-exponential if there exist non-negative parameters $\sigma$ and $b$ such that
\[
\mathbb{E}(\exp(\lambda X)) \leq \exp\left(\frac{\lambda^2 \sigma^2}{2}\right) \text{ for any } |\lambda| < \frac{1}{b}.
\]
A sub-exponential r.v. $X$ satisfies (see items (III) and (IV) in Theorem 2.2 of [Wainwright, 2015] for a proof)
\[
\sup_{k \geq 2} \left(\frac{\mathbb{E}(X^k)}{k!}\right)^{1/k} < \infty, \text{ and } \exists k_1, k_2 > 0 \text{ such that } P(X > \eta) \leq k_1 \exp(-k_2\eta), \forall \eta \geq 0.
\] (4)

The bound (4) implies that sub-exponential r.v.s satisfy (C2) for integer values of $\beta \geq 2$.

The following result from Fournier and Guillin [2015] bounds the Wasserstein distance between the empirical distribution function (EDF) of an i.i.d. sample and the underlying CDF from which the sample is drawn. Recall that, given $X_1, \ldots, X_n$ i.i.d. samples from the distribution $F$ of a r.v. $X$, EDF $F_n$ is defined by
\[
F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_i \leq x\}, \text{ for any } x \in \mathbb{R}.
\] (5)

Lemma 2. (Wasserstein distance bound) Let $X$ be a r.v. with CDF $F$. Suppose that either (i) $X$ satisfies (C1) with $\beta > 1$, or (ii) $X$ satisfies (C2) with $\beta > 2$. Then, for any $\epsilon \geq 0$, we have
\[
P(W_1(F_n, F) > \epsilon) \leq B(n, \epsilon),
\]
where, under (i),
\[
B(n, \epsilon) = C \left(\exp\left(-c\epsilon^2\right) \mathbb{I}\{\epsilon \leq 1\} + \exp\left(-c\epsilon^\beta\right) \mathbb{I}\{\epsilon > 1\}\right),
\]
for some $C, c$ that depend on the parameters $\beta, \gamma$ and $\top$ specified in (C1); and under (ii),
\[
B(n, \epsilon) = C \left(\exp\left(-c\epsilon^2\right) \mathbb{I}\{\epsilon \leq 1\} + n (n\epsilon)^{-(\beta-\eta)/p} \mathbb{I}\{\epsilon > 1\}\right).
\]
where $\eta$ could be chosen arbitrarily from $(0, \beta)$, while $C, c$ depend on the parameters $\beta, \eta$ and $\top$ specified in (C2).
Proof. The lemma is a direct application of Theorem 2 in [Fournier and Guillin, 2015].

3 CVaR estimation
We now introduce the notion of CVaR, a risk measure that is popular in financial applications.

Definition 4. The CVaR at level $\alpha \in (0, 1)$ for a r.v $X$ is defined by
\[
\text{CVaR}_\alpha(X) = \inf_{\xi} \left\{ \xi + \frac{1}{1-\alpha} \mathbb{E}(X - \xi)^+ \right\}, \text{ where } (y)^+ = \max(y, 0).
\]
It is well known (see [Rockafellar and Uryasev, 2000]) that the infimum in the definition of CVaR above is achieved for \( \xi = \text{VaR}_\alpha(X) \), where \( \text{VaR}_\alpha(X) = \inf\{x : P(X \leq x) \geq \alpha\} \) is the value-at-risk of the random variable \( X \) at confidence level \( \alpha \). Thus CVaR may also be written alternatively as given, for instance, in [Kolla et al. 2019]. In the special case where \( X \) has a continuous distribution, \( \text{CVaR}_\alpha(X) \) equals the expectation of \( X \) conditioned on the event that \( X \) exceeds \( \text{VaR}_\alpha(X) \).

All our results below pertain to i.i.d. samples \( X_1, \ldots, X_n \) drawn from the distribution of \( X \). Following [Brown 2007], we estimate \( \text{CVaR}_\alpha(X) \) from such a sample by

\[
\text{CVaR}_\alpha = \inf_\xi \left\{ \xi + \frac{1}{n(1 - \alpha)} \sum_{i=1}^n (X_i - \xi)^+ \right\}.
\]

(6)

We now provide a concentration bound for the empirical CVaR estimate (6), by relating the estimation error \( |\text{CVaR}_n - \text{CVaR}_\alpha(X)| \) to the Wasserstein distance between the true and empirical distribution functions, and subsequently invoking Lemma 2 that bounds the Wasserstein distance between any two distributions.

**Proposition 1.** Suppose \( X \) either satisfies (C1) for some \( \beta > 1 \) or satisfies (C2) for some \( \beta > 2 \). Under (C1), for any \( \epsilon > 0 \), we have

\[
P(\{|\text{CVaR}_n - \text{CVaR}_\alpha(X)| > \epsilon\}) \leq C \left( \exp \left( -cn(1 - \alpha)^2 \epsilon^2 \right) + \exp \left( -cn(1 - \alpha)^\beta \epsilon^3 \right) \right) \cdot (\epsilon \leq 1) + \exp \left( -cn(1 - \alpha)^\beta \epsilon^3 \right) \cdot (\epsilon > 1).
\]

Under (C2), for any \( \epsilon > 0 \), we have

\[
P(\{|\text{CVaR}_n - \text{CVaR}_\alpha(X)| > \epsilon\}) \leq C \left( \exp \left( -cn(1 - \alpha)^2 \epsilon^2 \right) + n(n(1 - \alpha)^{-\beta - \eta}) \right) \cdot (\epsilon \leq 1) + n(n(1 - \alpha)^{-\beta - \eta}) \cdot (\epsilon > 1).
\]

In the above, the constants \( C, c \) and \( \eta \) are as in Lemma 2.

**Proof.** See Section 5.1.

The following corollary, which specializes Proposition 1 to sub-Gaussian random r.v.s., is immediate, as sub-Gaussian random variables satisfy (C1) with \( \beta = 2 \).

**Corollary 1.** For a sub-Gaussian r.v. \( X \) with parameter \( \sigma \), we have that

\[
P(\{|\text{CVaR}_n - \text{CVaR}_\alpha(X)| > \epsilon\}) \leq 2C \exp \left( -cn(1 - \alpha)^2 \epsilon^2 \right), \quad \text{for any } \epsilon \geq 0,
\]

where \( C, c \) are constants that depend on \( \sigma \).

In terms of dependence on \( n \) and \( \epsilon \), the tail bound above is better than the one-sided concentration bound in [Kolla et al. 2019]. In fact, the dependence on \( n \) and \( \epsilon \) matches that in the case of bounded distributions (cf. [Brown 2007, Wang and Gao 2010]).

The case of sub-exponential distributions can be handled by specializing the second result in Proposition 1.

In particular, observing that sub-exponential distributions satisfy (C2) for any \( \beta \geq 2 \), and Proposition 1 requires \( \beta > 2 \) in case (ii), we obtain the following bound:

**Corollary 2.** For a sub-exponential r.v. \( X \), for any \( \epsilon \geq 0 \), we have

\[
P(\{|\text{CVaR}_n - \text{CVaR}_\alpha(X)| > \epsilon\}) \leq C \left[ \exp \left( -cn(1 - \alpha)^2 \epsilon^2 \right) \cdot (\epsilon \leq 1) + n(n(1 - \alpha)^{-\eta} \epsilon^{-3} \cdot (\epsilon > 1) \right],
\]

where \( C, c \) are universal constants, and \( \eta \) is as in Lemma 2.

For small deviations, i.e., \( \epsilon \leq 1 \), the bound above is satisfactory, as the tail decay matches that of a Gaussian r.v. with constant variance. On the other hand, for large \( \epsilon \), the second term exhibits polynomial decay. The latter polynomial term is not an artifact of our analysis, and instead, it relates to the rate obtained in case (ii) of Lemma 2. Sub-exponential distributions satisfy an exponential moment bound with \( \beta = 1 \), and for this case, the authors in [Fournier and Guillin, 2015] remark that they were not able to obtain a satisfactory concentration result. Thus, closing this gap remains as interesting future work.
4 CPT-value estimation

For any r.v. $X$, the CPT-value is defined as

$$C(X) = \int_0^\infty w^+ (\mathbb{P} (u^+(X) > z)) dz - \int_0^\infty w^- (\mathbb{P} (u^-(X) > z)) dz,$$

(7)

Let us deconstruct the above definition. $u^+, u^- : \mathbb{R} \to \mathbb{R}_+$ are the utility functions that are assumed to be continuous, with $u^+(x) = 0$ when $x \leq 0$ and increasing otherwise, and with $u^-(x) = 0$ when $x \geq 0$ and decreasing otherwise. The utility functions capture the human inclination to play safe with gains and take risks with losses—see Fig 1. Second, $w^+, w^- : [0, 1] \to [0, 1]$ are weight functions, which are assumed to be continuous, non-decreasing and satisfy $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$. The weight functions $w^+, w^-$ capture the human inclination to view probabilities in a non-linear fashion. Tversky and Kahneman [1992], Barberis [2013] (see Fig 2 from Tversky and Kahneman [1992]) recommend the following choices for $w^+$ and $w^-$, based on inference from experiments involving human subjects:

$$w^+(p) = \frac{p^{0.61}}{(p^{0.61} + (1-p)^{0.61})^{5/6}}, \text{ and } w^-(p) = \frac{p^{0.69}}{(p^{0.69} + (1-p)^{0.69})^{5/6}}.$$  

(8)

![Utility function](image1.png)

![Weight function](image2.png)

**Figure 1:** Utility function  
**Figure 2:** Weight function

We now recall CPT-value estimation proposed in [Prashanth et al. 2016]. Let $X_i$, $i = 1, \ldots, n$ denote $n$ samples from the distribution of $X$. The EDF for $u^+(X)$ and $u^-(X)$, for any given real-valued functions $u^+$ and $u^-$, is defined as follows: $F^+_n(x) = \frac{1}{n} \sum_{i=1}^n I \{u^+(X_i) \leq x\}$, $F^-_n(x) = \frac{1}{n} \sum_{i=1}^n I \{u^-(X_i) \leq x\}$. Using EDFs, the CPT-value is estimated as follows:

$$C_n = \int_0^\infty w^+(1 - F^+_n(x)) dx - \int_0^\infty w^- (1 - F^-_n(x)) dx.$$  

(9)

Notice that we have substituted the complementary EDF $\left(1 - F^+_n(x)\right)$ (resp. $\left(1 - F^-_n(x)\right)$) for $\mathbb{P} (u^+(X) > x)$ (resp. $\mathbb{P} (u^-(X) > x)$) in (7), and then performed an integration of the weight function composed with the complementary EDF. The first and second integral, say $C^+_n$ and $C^-_n$, in (9) can be easily computed using the order statistics $\{X_{(1)}, \ldots, X_{(n)}\}$ as follows:

$$C^+_n = \sum_{i=1}^n u^+(X_{(i)}) \left[ w^+ \left( \frac{n+1-i}{n} \right) - w^+ \left( \frac{n-i}{n} \right) \right], \text{ and } C^-_n = \sum_{i=1}^n u^-(X_{(i)}) \left[ w^- \left( \frac{i}{n} \right) - w^- \left( \frac{i-1}{n} \right) \right].$$

The reader is referred to Section III of [Prashanth et al. 2016] for further details.

For the purpose of analysis, as in [Cheng et al. 2018], we make the following assumption:

**C3** The weight functions $w^+, w^-$ are $L$-Hölder continuous of order $\alpha \in (0, 1)$ for some constant $L > 0$. 

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Note that a structural assumption on the weight functions is necessary to ensure that the CPT-value is well-defined. Furthermore, the weight functions recommended by Tversky and Kahneman [1992] are Hölder continuous.

In this paper, we are interested in deriving a concentration bound for the estimator in [9]. To put things in context, in [Cheng et al. 2018], the authors derive a concentration bound assuming that the underlying distribution has bounded support. We are able to provide a matching bound for the case of distributions with bounded support, using a proof technique that relates the estimation error $|C_n - C(X)|$ to the Wasserstein distance between the empirical and true CDF, and this is the content of the proposition below.

**Proposition 2. (CPT concentration for bounded r.v.s)** Let $X_1, \ldots, X_n$ be i.i.d. samples of a r.v. $X$ that is bounded a.s. in $[-T_1, T_2]$, where $T_1, T_2 \geq 0$, and at least one of $T_1, T_2$ is positive. Let $T \triangleq \max\{u^+(T_2), u^-(-T_1)\}$. Then, under (C3), we have

$$\Pr(|C_n - C(X)| > \epsilon) \leq 2B \left(n, \left[\frac{\epsilon}{2LT^{1-\alpha}}\right]^{1/\alpha}\right), \text{ for any } \epsilon \geq 0$$

where $B(\cdot, \cdot)$ is as given in i) of Lemma 2 with $\beta = 2$.

From the form for $B(\cdot, \cdot)$ in Lemma 2, it is apparent that $|C_n - C(X)| < \epsilon$ with probability $1 - \delta$, if the number of samples $n$ is of the order $O\left(1/\epsilon^{2/\alpha} \log\left(\frac{1}{\delta}\right)\right)$, for any $\delta \in (0, 1)$. Interestingly, the bound obtained above using a proof technique based on concentration of Wasserstein distance matches the bound obtained in [Cheng et al. 2018] using the DKW inequality. Further, the bound cannot be improved (in a minimax sense) any further, as shown in Proposition 5 of [Cheng et al., 2018].

**Proof.** See Section 5.2

Next, we provide a CPT concentration result for the case when the underlying r.v. is unbounded, but either sub-Gaussian or sub-exponential.

**Proposition 3. (CPT concentration for sub-Gaussian/sub-exponential r.v.s)** Let $X_1, \ldots, X_n$ be i.i.d. samples from the distribution of $X$. Fix $\epsilon \geq 0$.

(i) Suppose that $u^+(X)$ and $u^-(X)$ are sub-Gaussian with parameter $\sigma$. Then, for all $n \geq n_0 \triangleq \left(\frac{\sigma^2}{\alpha} \ln\frac{22L^2\sigma^2}{\alpha\epsilon}\right)\frac{1}{\alpha}$,

$$\Pr(|C_n - C(X)| > \epsilon) \leq 2C \exp\left(-\frac{c}{(2L)^{2/\alpha}}n^{\alpha/2}\epsilon^{2/\alpha}\right) + 2\exp\left(-n^\alpha\right), \quad (10)$$

where $C, c$ are as in Proposition 7

(ii) Suppose that $u^+(X)$ and $u^-(X)$ are sub-exponential r.v.s. Then, for all $n \geq n_0 \triangleq \left(\frac{1}{\alpha k_2} \ln\frac{2Lk_1}{\alpha\epsilon}\right)^{1/\alpha}$,

$$\Pr(|C_n - C(X)| > \epsilon) \leq C \exp\left(-\frac{c}{(2L)^{2/\alpha}}n^{\alpha/2}\epsilon^{(2-\alpha)/\alpha}\right) + nk_1 \exp\left(-k_1n^{\alpha/(2-\alpha)}\right), \quad (11)$$

where $k_1, k_2$ are given by the equivalent characterization of sub-exponential distributions in [4], and $C, c$ are as in Proposition 7

**Proof.** See Section 5.3

The bound obtained above features a better dependence on $n$, as compared to that in Proposition 3 of [Cheng et al. 2018]. In particular, the factor involving $n$ inside the exponential terms above is $n^\alpha$, while the corresponding factor is $n^{\alpha/(2+\alpha)}$ in [Cheng et al. 2018].

**Proof.** See Section 5.3
5 Convergence proofs

5.1 Proof of Proposition 1

Proof. Consider the event $A = \{ W_i(F_n, F) \leq (1 - \alpha)\epsilon \}$, where $F_n$ is as defined in 5. Lemma 2 provides a lower bound on $P(A)$ depending on whether the r.v.s satisfy (C1) or (C2). In particular, we have

$$P(A) \geq 1 - B(n, (1 - \alpha)\epsilon),$$

(12)

where $B(\cdot, \cdot)$ is as defined in Lemma 2.

Applying Lemma 1, we have on the event $A$,

$$\left| \int_{\mathbb{R}} f(x)dF(x) - \int_{\mathbb{R}} f(x)dF_n(x) \right| \leq (1 - \alpha)\epsilon,$$

(13)

for any 1-Lipschitz function $f : \mathbb{R} \rightarrow \mathbb{R}$.

Choose $\xi \in \mathbb{R}$ arbitrarily and let $g_\xi(x) = (1 - \alpha)\xi + (x - \xi)^+$. Then,

$$\int_{\mathbb{R}} g_\xi(x)dF(x) = (1 - \alpha)\xi + \mathbb{E}(X - \xi)^+ \triangleq D(\xi),$$

and

$$\int_{\mathbb{R}} g_\xi(x)dF_n(x) = (1 - \alpha)\xi + \frac{1}{n} \sum_{i=1}^{n} (X_i - \xi)^+ \triangleq D_n(\xi).$$

Observing that $g_\xi$ is 1-Lipschitz in $x$ for every $\xi \in \mathbb{R}$ and using (13), we obtain

$$|D(\xi) - D_n(\xi)| \leq (1 - \alpha)\epsilon, \text{ on } A, \text{ for any } \xi \in \mathbb{R}.$$

Choose $m > 0$ arbitrarily, and let $\xi_1, \xi_2 \in \mathbb{R}$ be such that

$$D(\xi_1) \leq \inf_\xi D(\xi) + \frac{1}{m}, \text{ and } D_n(\xi_2) \leq \inf_\xi D_n(\xi) + \frac{1}{m}.$$

Then, on the event $A$, we have

$$-(1 - \alpha)\epsilon - \frac{1}{m} \leq D(\xi_1) - D_n(\xi_1) - \frac{1}{m} \leq \inf_\xi D(\xi) - \inf_\xi D_n(\xi) \leq D(\xi_2) - D_n(\xi_2) + \frac{1}{m} \leq (1 - \alpha)\epsilon + \frac{1}{m}.$$

Since the chain of inequalities above hold for any $m > 0$, we conclude that

$$\left| \inf_\xi D(\xi) - \inf_\xi D_n(\xi) \right| \leq (1 - \alpha)\epsilon, \text{ on } A.$$

(14)

Notice that, by definition, $\inf_\xi D(\xi) = (1 - \alpha)\text{CVaR}_\alpha(X)$ and $\inf_\xi D_n(\xi) = (1 - \alpha)\text{CVaR}_n$. Thus,

$$|\text{CVaR}_\alpha(X) - \text{CVaR}_n| \leq \epsilon, \text{ on the event } A.$$

The main claim now follows by using the bound on $P(A)$ in (12). □

5.2 Proof of Proposition 2

Proof. Let

$$\Delta^+_n = \int_0^\infty w^+(\mathbb{P}(u^+(X) > z))dz - \int_0^\infty w^+(1 - \hat{F}_n^+(z))dz.$$

(15)
The quantity above is the difference between the first integral in CPT-value estimate (9) and the first integral in the CPT-value (7). Using (C3), we have
\[
|\Delta_n^+| \leq L \int_0^\infty |F^+(z) - \hat{F}_n^+(z)|^\alpha \, dz,
\] (16)
where \(F^+(\cdot)\) is the CDF of the r.v. \(u^+(X)\).

Recall that the r.v. \(u^+(X)\) is bounded a.s. in \([0, u^+(T_2)]\) by our assumptions on \(u^+\) and \(X\). Applying Jensen’s inequality to the concave \(x \mapsto x^\alpha\) after normalizing the Lebesgue measure on the interval \([0, u^+(T_2)]\), we obtain
\[
\frac{1}{u^+(T_2)} \int_0^{u^+(T_2)} |F^+(z) - \hat{F}_n^+(z)|^\alpha \, dz \leq \left[ \frac{1}{u^+(T_2)} \int_0^{u^+(T_2)} |F^+(z) - \hat{F}_n^+(z)| \, dz \right]^\alpha.
\]

Applying the second equality in Lemma[1] to the CDFs \(F^+\) and \(\hat{F}_n^+\) gives
\[
\int_0^{u^+(T_2)} |F^+(z) - \hat{F}_n^+(z)|^\alpha \, dz \leq [W_1(F^+, \hat{F}_n^+)]^\alpha [u^+(T_2)]^{1-\alpha}.
\]
Using the bound obtained above in (16), we obtain
\[
|\Delta_n^+| \leq L [W_1(F^+, \hat{F}_n^+)]^\alpha [u^+(T_2)]^{1-\alpha}.
\]

Next, for any \(\epsilon > 0\), consider the event \(A = \{W_1(F^+, \hat{F}_n^+) \leq \epsilon/(2L[u^+(T_2)]^{1-\alpha})\} \}. Then, from Lemma[2]
\[
\mathbb{P}(A) \geq 1 - B(n, \epsilon/(2L[u^+(T_2)]^{1-\alpha}))^{1/\alpha},
\]
where \(B\) is as given in Lemma[2]. On the event \(A\), we have \(|\Delta_n^+| \leq \epsilon/2\).

Along similar lines, letting \(\Delta_n^- = \int_0^{u^+(T_2)} w^- (\mathbb{P}(u^+(X) > z)) \, dz - \int_0^{u^+(T_2)} w^- (1 - \hat{F}_n^- (z)) \, dz\), it is easy to infer that
\[
|\Delta_n^-| \leq \epsilon/2 \text{ on the set } A' = \{W_1(F^-, \hat{F}_n^-) \leq \epsilon/(2L[u^-(T_1)]^{1-\alpha})\}^{1/\alpha},
\] (17)
where \(F^- (\cdot)\) is the CDF of \(u^- (X)\). The main claim follows by using triangle inequality, that is,
\[
\mathbb{P}(|C_n - C(X)| > \epsilon) \leq \mathbb{P}(|\Delta_n^+| > \epsilon/2) + \mathbb{P}(|\Delta_n^-| > \epsilon/2)
\leq [1 - \mathbb{P}(A)] + [1 - \mathbb{P}(A')]
\leq B(n, \epsilon/(2L[u^+(T_2)]^{1-\alpha}))^{1/\alpha} + B(n, \epsilon/(2L[u^-(T_1)]^{1-\alpha}))^{1/\alpha}
\leq 2B(n, \epsilon/(2L^{1-\alpha}))^{1/\alpha}.
\]
This completes the proof. \(\square\)

5.3 Proof of Proposition[3]

Proof. We prove the claim first for the sub-Gaussian case (i). Let \(\Delta_n^+\) be as defined in (15) in the proof of Proposition[2]. Then, we have the following inequality for some positive \(\tau\) to be specified later:
\[
\mathbb{P}(|\Delta_n^+| > \epsilon) \leq \mathbb{P}(|\Delta_n^+| > \epsilon, u^+(X_n(n)) \leq n^\tau) + \mathbb{P}(u^+(X_n(n)) > n^\tau),
\] (18)
where \(X_n(n)\) is the largest among \(X_1, \ldots, X_n\). The second term on the RHS above can be bounded using the fact that \(u^+(X_i)\) are sub-Gaussian, as follows:
\[
\mathbb{P}(u^+(X_{[n]}) > n^\tau) = 1 - \mathbb{P}(u^+(X_{[n]}) \leq n^\tau) = 1 - (\mathbb{P}(u^+(X_i) \leq n^\tau))^n
\leq 1 - \left(1 - 8ne^{-n^\tau/(2\sigma^2)}\right)^n \leq 1 - \left(1 - 8n\epsilon^{-n^\tau/(2\sigma^2)}\right) = 8n\epsilon^{-n^\tau/(2\sigma^2)},
\] (19)
where the first inequality above follows from the fact that \( \mathbb{P}(Z \geq \eta) \leq 8 \exp(-\eta^2/(2\sigma^2)) \) for any sub-Gaussian r.v. \( Z \) with parameter \( \sigma > 0 \). The second inequality above follows by using a Taylor series expansion of the exponential function.

For handling the first term in (18), we start with the following observation:

\[
\int_{n^\tau}^\infty w \left( \mathbb{P} \left( u^+(X) > z \right) \right) dz \leq L \int_{n^\tau}^\infty \left( \mathbb{P} \left( u^+(X) > z \right) \right)^\alpha dz \leq 8L \int_{n^\tau}^\infty \frac{z}{n\tau} \exp\left( -\frac{\alpha z^2}{2\sigma^2} \right) dz
\]

where we used the following facts: (i) \( w \) is Hölder continuous; (ii) \( w(0) = 0 \); and (iii) a tail bound for the sub-Gaussian r.v. \( u^+(X) \).

Then, for all \( n \geq n_0 \), we have

\[
\mathbb{P} \left( |\Delta^+_n| > \epsilon, u^+ (X_{[n]}) \leq n^\tau \right) \leq \mathbb{P} \left( \left| \int_{0}^{n^\tau} w^+ (1 - F^+(z))dz - \int_{0}^{n^\tau} w^+ (1 - \tilde{F}^+ (z))dz \right| > \epsilon/2 \right)
\]

\[
\leq B \left( n, \frac{\epsilon}{2L n^{\tau(1-\alpha)}} \right)^{1/\alpha}, \quad (20)
\]

where the final inequality follows by applying Proposition\( \[2\] \) to the r.v. \( Z = u^+(X)\mathbb{1}\{u^+(X) \leq n^\tau\} \) which takes values in the bounded interval \([0, n^\tau]\). It is easy to infer that \( B(n, \epsilon') = C \exp(-c n^\epsilon) \), for any \( \epsilon' \geq 0 \), and hence, we have

\[
\mathbb{P} \left( |\Delta^+_n| > \epsilon, u^+ (X_{[n]}) \leq n^\tau \right) \leq C \exp \left( -c' n^{1-\frac{2(1-\alpha)}{\alpha}} \epsilon^{2/\alpha} \right), \quad \text{where} \quad c' = \frac{c}{(2L)^{2/\alpha}}. \quad (21)
\]

Substituting the bounds obtained in (19) and (21) into (18), we obtain

\[
\mathbb{P} \left( |\Delta^+_n| > \epsilon \right) \leq C \exp \left( -c' n^{1-\frac{2(1-\alpha)}{\alpha}} \epsilon^{2/\alpha} \right) + n e^{-n^\tau/(2\sigma^2)}. \quad (22)
\]

The inequality above holds for any \( \tau \). Equating the exponents of \( n \) in the two exponential, a choice that is optimal upto constant factors is \( 2\tau = 1 - 2\tau(1-\alpha)/\alpha \) leading to \( \tau = \frac{\alpha}{2} \). For this choice of \( \tau \), we have

\[
\mathbb{P} \left( |\Delta^+_n| > \epsilon \right) \leq C \exp \left( -c' n^\alpha \epsilon^{2/\alpha} \right) + n \exp \left( -n^{\alpha/(2\sigma^2)} \right). \quad (23)
\]

The main claims by inferring a bound similar to the above for the second integrals in \( C_n \) and \( C(X) \) and then, using a triangle inequality as in the proof of Proposition\( \[2\] \).

For the case when the underlying distribution is sub-exponential, the bound on the first term on the RHS of (18), i.e., the inequality in (21) holds. On the other hand, the second term in (18) as well as the expression for \( n_0 \) are different. In particular,

\[
\int_{n^\tau}^\infty w \left( \mathbb{P} \left( u^+(X) > z \right) \right) dz \leq L \int_{n^\tau}^\infty \left( \mathbb{P} \left( u^+(X) > z \right) \right)^\alpha dz \leq L k_1 \int_{n^\tau}^\infty \exp(-\alpha k_2 z) dz
\]

\[
= \frac{L k_1}{\alpha k_2} \exp(-\alpha k_2 n^\tau) \leq \frac{\epsilon}{2} \quad \text{for} \quad n \geq n_0,
\]

where we used the fact that, for any sub-exponential r.v. \( Z \), there exists \( k_1, k_2 > 0 \) such that \( \mathbb{P}(Z \geq \eta) \leq k_1 \exp(-k_2 \eta) \), for any \( \eta \geq 0 \). Next, the second term in (18) can be bounded as follows: \( \mathbb{P}(u^+(X_{[n]}) > n^\tau) \leq n k_1 e^{-k_2 n^\tau} \), leading to the following overall bound:

\[
\mathbb{P}(|\Delta^+_n| > \epsilon) \leq C \exp \left( -c' n^{1-\frac{2(1-\alpha)}{\alpha}} \epsilon^{2/\alpha} \right) + n k_1 e^{-k_2 n^\tau}.
\]

Equalizing the exponents of \( n \) in the two terms on the RHS above, we obtain \( \tau = \frac{\alpha}{2(\alpha - 1)} \), and for this choice of \( \tau \), we obtain

\[
\mathbb{P}(|\Delta^+_n| > \epsilon) \leq C \exp \left( -c' n^\alpha \epsilon^{(2-\alpha)/\alpha} \right) + n k_1 \exp \left( -k_2 n^\alpha/(2-\alpha) \right), \quad (24)
\]

completing the proof.
6 CVaR-sensitive bandits

The concentration bound for CVaR estimation in Proposition 1 opens avenues for bandit applications. We illustrate this claim by using the regret minimization framework in a stochastic \( K \)-armed bandit problem, with an objective based on CVaR. While CVaR optimization has been considered in a bandit setting in the literature (cf. Galichet et al. [2013]), the underlying arms’ distributions were assumed to have bounded support. We relax this assumption, and consider the case of sub-Gaussian distributions for the \( K \) arms. The one-sided concentration result derived recently for CVaR estimation in [Kolla et al., 2019] is not enough to derive a CVaR-sensitive bandit algorithm, and establish regret bounds. Moreover, unlike [Kolla et al., 2019], our bound in Proposition 1 has the optimal dependence on number of samples \( n \) and accuracy \( \epsilon \), allowing the design of a bandit algorithm, which can be shown to have a \( \tilde{O}\left(\sqrt{K n}\right) \) regret bound, where \( \tilde{O}(\cdot) \) is similar to the \( O(\cdot) \), except that log factors are ignored.

Suppose we are given \( K \) arms with unknown distributions \( P_i, i = 1, \ldots, K \). The interaction of the bandit algorithm with the environment proceeds, over \( n \) rounds, as follows: (i) select an arm \( I_t \in \{1, \ldots, K\} \); (ii) observe a sample cost from the distribution \( P_{I_t} \) corresponding to the arm \( I_t \). Let \( \text{CVaR}_n(i) \) denote the CVaR, with confidence \( \alpha \in (0, 1) \), of the distribution \( P_i \) corresponding to arm \( i \), for \( i = 1, \ldots, K \). Let \( \text{CVaR}_n = \min_{i=1, \ldots, K} \text{CVaR}_n(i) \) denote the lowest CVaR among the \( K \) distributions, and \( \Delta_i = (\text{CVaR}_n(i) - \text{CVaR}_n) \) denote the gap in CVaR values of arm \( i \) and that of the best arm.

The classic objective in a bandit problem is to find the arm with the lowest expected value. We consider an alternative formulation, where the goal is to find the arm with the lowest CVaR. Using the notion of regret, this objective is formalized as follows:

\[
R_n = \sum_{i=1}^{K} \text{CVaR}_n(i) T_i(n) - n \text{CVaR}_n = \sum_{i=1}^{K} T_i(n) \Delta_i,
\]

where \( T_i(n) = \sum_{t=1}^{n} 1 \{ I_t = i \} \) is the number of pulls of arm \( i \) up to time instant \( n \).

Next, we present a straightforward adaptation of the well-known UCB algorithm [Auer et al., 2002] to handle an objective based on CVaR.

**CVaR-LCB algorithm**

Initialization: Play each arm once,

For \( t = K + 1, \ldots, n \), repeat

1. Play arm \( I_t = \arg \min_{i=1, \ldots, K} \left( \text{LCB}_t(i) = \text{CVaR}_n(t - 1) - \frac{2}{(1 - \alpha)^2 \sqrt{T_i(t - 1)}} \right) \), where \( \text{CVaR}_n(t - 1) \) is the empirical CVaR for arm \( i \) computed using (6) from \( T_i(t - 1) \) samples, and \( C \) is a constant that depends on the sub-Gaussianity parameter \( \sigma \) (see Corollary 1).

2. Observe sample \( X_{I_t} \) from the distribution \( P_{I_t} \) corresponding to the arm \( I_t \).

The result below bounds the regret of CVaR-LUCB algorithm, and the proof follows the technique used for establishing the regret bound of the regular UCB algorithm in [Auer et al., 2002].

**Theorem 1.** For a \( K \)-armed stochastic bandit problem where the arms’ distributions are sub-Gaussian with parameter \( \sigma = 1 \), the regret \( R_n \) of CVaR-LCB satisfies

\[
\mathbb{E} R_n \leq \sum_{\{i: \Delta_i > 0\}} \frac{16 \log(Cn)}{(1 - \alpha)^2 \Delta_i} + K \left( 1 + \frac{\pi^2}{3} \right) \Delta_i.
\]

Further, \( R_n \) satisfies the following bound that does not scale inversely with the gaps:

\[
\mathbb{E} R_n \leq \frac{8}{(1 - \alpha)^2} \sqrt{K n \log(Cn)} + \left( \frac{\pi^2}{3} + 1 \right) \sum_i \Delta_i.
\]

**Proof.** See Appendix A.
7 Conclusions

We used finite sample bounds from [Fournier and Guillin 2015] for the Wasserstein distance between the empirical and true distribution of a random variable to derive a two-sided concentration bound for the error between the true and empirical CVaR of a random variable. Our bound holds for random variables that either have finite exponential moment, or finite higher-order moment, and specializes nicely to sub-Gaussian and sub-exponential random variables. The bound further improves upon previous similar results, which either gave one-sided bounds, or applied only to bounded random variables. We also used the result from [Fournier and Guillin 2015] to provide a novel Wasserstein-distance-based proof of the concentration bound from [Cheng et al. 2018] for the error between the true and estimated CPT value of a bounded random variable. This bound was further extended to apply to unbounded but sub-Gaussian random variables. To illustrate the usefulness of our concentration bounds, we used our CVaR concentration bound to provide a regret-bound analysis for an algorithm for a bandit problem where the risk measure to be optimized is CVaR.

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### A Proof of Theorem 1

*Proof*. The proof follows by using arguments analogous to that in the proof of Theorem 1 in [Auer et al., 2002](http://www.stat.berkeley.edu/~mjwain/stat210b/Chap2_TailBounds_Jan22_2015.pdf). For the sake of completeness, we provide the complete proof.

Let 1 denote the optimal arm, without loss of generality. We bound the number of pulls $T_i(n)$ of any suboptimal arm $i \neq 1$. Fix a round $t \in \{1, \ldots, n\}$ and suppose that a sub-optimal arm $i$ is pulled in this round. Then, we have

$$
\text{CVaR}_i(t-1) - \frac{2}{(1-\alpha)} \sqrt{\frac{\log(C_t)}{T_i(t-1)}} \leq \text{CVaR}_1(t-1) - \frac{2}{(1-\alpha)} \sqrt{\frac{\log(C_t)}{T_1(t-1)}}.
$$

(25)

The LCB-value of arm $i$ can be larger than that of 1 only if one of the following three conditions holds:

1. **CVaR$_1$ is outside the confidence interval:**

   $$
   \text{CVaR}_1(t-1) - \frac{2}{(1-\alpha)} \sqrt{\frac{\log(C_t)}{T_1(t-1)}} > \text{CVaR}_\alpha(1),
   $$

   (26)

2. **CVaR$_i$ is outside the confidence interval:**

   $$
   \text{CVaR}_i(t-1) + \frac{2}{(1-\alpha)} \sqrt{\frac{\log(C_t)}{T_i(t-1)}} < \text{CVaR}_\alpha(i),
   $$

   (27)

3. **Gap $\Delta_i$ is small:** If we negate the two conditions above and use (25), then we obtain

   \[
   \text{CVaR}_\alpha(i) - \frac{4}{(1-\alpha)} \sqrt{\frac{\log(C_t)}{T_i(t-1)}} \leq \text{CVaR}_i(t-1) - \frac{2}{(1-\alpha)} \sqrt{\frac{\log(C_t)}{T_1(t-1)}} \\
   \leq \text{CVaR}_1(t-1) - \frac{2}{(1-\alpha)} \sqrt{\frac{\log(C_t)}{T_1(t-1)}} \leq \text{CVaR}_\alpha(1) \\
   \Rightarrow \quad \Delta_i < \frac{4}{(1-\alpha)} \sqrt{\frac{\log(C_t)}{T_i(t-1)}} \quad \text{or} \quad T_i(t-1) \leq \frac{16 \log(C_t)}{(1-\alpha)^2 \Delta_i^2}
   \]

(28)
Let \( u = \frac{16 \log(C_n)}{(1 - \alpha)^2 \Delta^2} + 1 \). When \( T_i(t - 1) \geq u \), i.e., when the condition in (28) does not hold, then either (i) arm \( i \) is not pulled at time \( t \), or (ii) (26) or (27) occurs. Thus, we have

\[
T_i(n) = 1 + \sum_{t=K+1}^{n} I \{ I_t = i \}
\]

\[
\leq u + \sum_{t=u+1}^{n} I \{ I_t = i; T_i(t - 1) \geq u \}
\]

\[
\leq u + \sum_{t=u+1}^{n} I \left\{ \text{CVaR}_i(t - 1) - \frac{2}{(1 - \alpha)} \sqrt{\frac{\log(C_t)}{T_i(t - 1)}} \leq \text{CVaR}_1(t - 1) - \frac{2}{(1 - \alpha)} \sqrt{\frac{\log(C_t)}{T_1(t - 1)}}; T_i(t - 1) \geq u \right\}
\]

\[
\leq u + \sum_{t=u+1}^{n} I \left\{ \text{CVaR}_i(s_i) - \frac{2}{(1 - \alpha)} \sqrt{\frac{\log(C_t)}{s_i}} \leq \text{CVaR}_1(s) - \frac{2}{(1 - \alpha)} \sqrt{\frac{\log(C_t)}{s}} \right\}
\]

\[
\leq u + \sum_{t=1}^{n} \sum_{s=1}^{t-1} \sum_{s_i=u}^{t-1} I \left\{ \text{CVaR}_i(1) < \text{CVaR}_1(s) - \frac{2}{(1 - \alpha)} \sqrt{\frac{\log(C_t)}{s}} \right\}
\]

or \( \text{CVaR}_i(i) > \text{CVaR}_i(s_i) + \frac{2}{(1 - \alpha)} \sqrt{\frac{\log(C_t)}{s_i}} \) occurs \}.

Using Proposition 1, we can bound the probability of occurrence of each of the two events inside the indicator on the RHS of the final display above as follows:

\[
P \left( \text{CVaR}_i(1) < \text{CVaR}_1(s) - \frac{2}{(1 - \alpha)} \sqrt{\frac{\log(C_t)}{s}} \right) \leq \frac{1}{t^4}, \quad \text{and}
\]

\[
P \left( \text{CVaR}_i(i) > \text{CVaR}_i(s_i) + \frac{2}{(1 - \alpha)} \sqrt{\frac{\log(C_t)}{s_i}} \right) \leq \frac{1}{t^4}.
\]

Plugging the bounds on the events above and taking expectations on \( T_i(n) \) related inequality above, we obtain

\[
E[T_i(n)] \leq u + \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{s_i=u}^{t-1} \frac{1}{t^4} \leq u + 2 \sum_{t=1}^{\infty} \frac{1}{t^2} \leq u + \frac{n^2}{3}.
\]  \( (29) \)

The preceding analysis together with the fact that \( E R_n = \sum_{i=1}^{K} \Delta_i E[T_i(n)] \) leads to the first regret bound presented in the theorem.

For inferring the second bound on the regret, i.e., the bound that does not scale inversely with the gaps, observe
that

$$E R_n = \sum_i \Delta_i \mathbb{E}[T_i(n)] = \sum_{i: \Delta_i \leq \lambda} \Delta_i \mathbb{E}[T_i(n)] + \sum_{i: \Delta_i \geq \lambda} \Delta_i \mathbb{E}[T_i(n)], \text{ for } \lambda > 0$$

$$\leq n\lambda + \sum_{i: \Delta_i \geq \lambda} \left( \frac{16 \log(Cn)}{(1 - \alpha)^2 \Delta_i} + \Delta_i \left( \frac{\pi^2}{3} + 1 \right) \right), \text{ (Using (29) and } \sum_{i: \Delta_i \leq \lambda} \mathbb{E}[T_i(n)] \leq n)$$

$$\leq n\lambda + \left( \frac{16K \log(Cn)}{(1 - \alpha)^2 \lambda} \right) + \left( \frac{\pi^2}{3} + 1 \right) \sum_i \Delta_i,$$

$$\leq \frac{8}{(1 - \alpha)} \sqrt{Kn \log(Cn)} + \left( \frac{\pi^2}{3} + 1 \right) \sum_i \Delta_i, \quad \text{ (Using } \lambda = \frac{8\sqrt{K \log(Cn)}}{(1 - \alpha)}).$$