Learning Hash Function through Codewords

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Abstract—In this paper, we propose a novel hash learning approach that has the following main distinguishing features, when compared to past frameworks. First, the codewords are utilized in the Hamming space as ancillary techniques to accomplish its hash learning task. These codewords, which are inferred from the data, attempt to capture grouping aspects of the data’s hash codes. Furthermore, the proposed framework is capable of addressing supervised, unsupervised and, even, semi-supervised hash learning scenarios. Additionally, the framework adopts a regularization term over the codewords, which automatically chooses the codewords for the problem. To efficiently solve the problem, one Block Coordinate Descent algorithm is showcased in the paper. We also show that one step of the algorithms can be casted into several Support Vector Machine problems which enables our algorithms to utilize efficient software package. For the regularization term, a closed form solution of the proximal operator is provided in the paper. A series of comparative experiments focused on content-based image retrieval highlights its performance advantages.

Index Terms—Hash Function Learning, Codewords, Block Coordinate Descent, SVM, Subgradient, Proximal Methods.

INTRODUCTION

With the eruptive growth of Internet data including images, music, documents and videos, content-based image retrieval (CBIR) has drawn lots of attention over the last few years [1]. Given a query sample from a user, a typical CBIR system retrieves samples stored in a database that are most similar to the query sample. The similarity is evaluated in terms of a pre-specified distance metric and the retrieved samples are the nearest neighbors of the query sample w.r.t. this metric. However, in some practical settings, exhaustively comparing the query sample with every sample in the database may be computationally expensive. Furthermore, most CBIR frameworks may be obstructed by the sheer size of each sample; for instance, visual descriptors of an image or a video may contain thousands of features. Additionally, storage of these high-dimensional data also presents a challenge.

Substantial effort has been invested in designing hash functions transforming the original data into compact binary codes to reap the benefits of a potentially fast similarity search. For example, when compact binary codes in Hamming space used, approximate nearest neighbors (ANN) [2] search was shown to achieve sub-liner searching time. Storage of the binary code is, obviously, also much more efficient. Furthermore, hash functions are typically designed to preserve certain similarity qualities between the data in the Hamming space.

Existing popular hashing approaches can be divided into two categories: data-independent and data-dependent. While the former category designs the hash function based on a non data-driven approach, the latter category, by inferring from data, can be further clustered into supervised, unsupervised and semi-supervised learning tasks.

In this paper, we propose a novel hash function learning approach [1], dubbed "Supervised Hash Learning ("SHL") (" stands for all three learning paradigms), which exhibits the following advantages: first, the method uses a set of Hamming space codewords, that are learned during training, to capture the intrinsic similarities between the data’s hash codes, so that same-class data are grouped together. Unlabeled data also contribute to the adjustment of codewords leveraging from the inter-sample dissimilarities of their generated hash codes, as measured by the Hamming distance metric. Additionally, a regularization term is utilized in our framework to move the codewords which represent the same class closer to each other. When some codewords collapse to one single codeword, our framework achieves automatic selection of the codewords. Due to these codeword-specific characteristics, a major advantage offered by our framework is that it can engage supervised, unsupervised and, even, semi-supervised hash learning tasks using a single formulation. Obviously, the latter ability readily allows the framework to perform transductive hash learning. Note that our framework can be viewed as an Error-Correction Codes (ECOC) method. Readers can refer to [4] and [5] for more details of ECOC.

In Sec. 3, we provide "SHL’s formulation, which is mainly motivated by an attempt to minimize the within-group Hamming distances in the code space between a group’s codeword and the hash codes of data that either should be similar (because of similar labels), or are de-facto similar (due to particular state of the hash functions). With regards to the hash functions, "SHL adopts a kernel-based approach. A new regularization term over codewords is also introduced for "SHL in its formulation. The aforementioned motivation eventually leads to a minimization problem over

1. A preliminary version of the work presented here has appeared in [3].
the codewords as well as over the Reproducing Kernel Hilbert Space (RKHS) vectors defining the hash functions. A quite noteworthy aspect of the resulting formulation is that the minimizations over the latter parameters leads to a set of Support Vector Machine (SVM) problems, according to which each SVM generates a single bit of a sample’s hash code. In lieu of choosing a fixed, arbitrary kernel function, we use a simple Multiple Kernel Learning (MKL) approach (e.g., see [6]) to infer a good kernel from the data.

Next, in Sec. 4, an efficient Majorization-Minimization (MM) algorithm is showcased that can be used to optimize *SHL’s framework via the Block Coordinate Descent (BCD) approaches. To train *SHL, the first block optimization amounts to training a set of SVM, which can be efficiently accomplished by using, for example, LIBSVM [7]. The second block optimization step addresses the MKL parameters. The third block involves solving a problem with the non-smooth regularization over codewords, which is optimized by Proximal Subgradient Descent (PSD). The second and third blocks are computationally fast thanks to closed-form solutions. When confronted with a huge data set, kernel related problem has computational limitation. In this work, a version of our algorithm for big data, which is based on the software LIBSKYLRK [8], is also presented.

Finally, in Sec. 6 we demonstrate the capabilities of *SHL on a series of comparative experiments. The section emphasizes on supervised hash learning problems in the context of CBIR. Additionally, we also apply the semi-supervised version of our framework on the foreground/background interactive image segmentation problems. Remarkably, when compared to other hashing methods on supervised learning hash tasks, *SHL exhibits the best retrieval accuracy in all the datasets we considered. Some clues to the method’s superior performance are provided in Sec. 5.

2 RELATED WORK

As mentioned in Sec. 1, hashing methods can be divided into two categories: data-independent and data-dependent. The former category designs the hashing approaches without the necessity to infer from the data. For instance, in [9], Locality Sensitive Hashing (LSH) randomly projects and thresholds data into the Hamming space to generate binary codes. Data samples, which are closely located (in terms of Euclidean distances in the data’s native space), are likely to have similar binary codes. Additionally, the authors of [10] proposed a method for ANN search through using a learned Mahalanobis metric combined with LSH. [11] introduces an encoding scheme based on random projections, in which the expected Hamming distance between two binary codes of the vectors is related to the value of a shift-invariant kernel.

On the other hand, data-dependent methods can, in turn, be grouped into supervised, unsupervised and semi-supervised learning paradigms.

The majority of work in data-dependent hashing approaches has been studied so far following the supervised learning scenario. For example, Semantic Hashing [12] designs the hash function using a Restricted Boltzmann Machine (RBM). Binary Reconstructive Embedding (BRE), proposed in [13], tries to minimize a cost function measuring the difference between the original metric distances and the reconstructed distances in the Hamming space. In [14], through learning the hash functions from pairwise side information, Minimal Loss Hashing (MLH) formulated the hashing problems based on a bound inspired by the theory of structural Support Vector Machines [15], [16] addresses the scenario, where a small portion of sample pairs are manually labeled as similar or dissimilar and proposes the Label-regularized Max-margin Partition algorithm. Moreover, Self-Taught Hashing [17] first identifies binary codes for given documents via unsupervised learning; next, classifiers are trained to predict codes for query documents. Additionally, in [18], Fisher Linear Discriminant Analysis (LDA) was employed to embed the original data to a lower dimensional space and hash codes are obtained subsequently via thresholding. Boosting-based Hashing is used in [19] and [20], in which a set of weak hash functions are learned according to the boosting framework. In [21], the hash functions are learned from triplets of side information; their method is designed to preserve the relative comparison relationship from the triplets and is optimized using column generation. Furthermore, Kernel Supervised Hashing (KSH) [22] introduces a kernel-based hashing method, which seems to exhibit remarkable experimental results. Their method utilizes the equivalence between optimizing the code inner products and the Hamming distance. [23] proposes boosted decision trees for achieving non-linearity in hashing, which is fast to train. Their method employs an efficient GraphCut based block search approach. In [24], a supervised hash learning method for image retrieval is designed, in which their method automatically learns a good image representation tailored as well as several hash functions. Latent factor hashing, proposed in [25], learns similarity preserving compact binary codes based on a latent factor model. Finally, [26] combines structural Support Vector Machines with hashing methods to directly optimize over multivariate performance measure such as Area Under Curve (AUC).

Several approaches have also been proposed for unsupervised hashing: With the assumption of a uniform data distribution, Spectral Hashing (SPH) [27] designs the hash functions by utilizing spectral graph analysis. In [28], a new regularization is introduced to control the mismatch between the Hamming codes and the low-dimensional data representation. This new regularizer helps the methods better cope with the data sampled from a nonlinear manifold. Anchor Graph Hashing (AGH) [29] uses a small-size anchor graph to approximate low-rank adjacency matrices that leads to computational savings. Moreover, [30] proposed a projection learning method for error correction. Also, in [31], the authors introduce Iterative Quantization, which tries to learn an orthogonal rotation matrix so that the quantization error of mapping the data to the vertices of the binary hypercube is minimized. [32]’s idea is to decompose the feature space into a subspace shared by the hash functions. Then they design an objective function combining spectral embedding loss, binary quantization loss and shared subspace contribution. Finally, [33] presents an unsupervised hashing model based on graph model. Their method tries to preserve the neighborhood structure of massive data in a discrete code space.

As for semi-supervised hashing, there are a few ap-
proaches proposed: Semi-Supervised Hashing (SSH) in [30] and [34] minimizes an empirical error using labeled data; in order to avoid over-fitting, the framework also includes an information theoretic regularizer that utilizes both labeled and unlabeled data. Another method, semi-supervised tag hashing [35], incorporates tag information into training hash function by exploring the correlation between tags and hash bits. In [36], the authors introduce a hashing method integrating multiple modalities. Besides, semi-supervised information is also incorporated in the framework and a sequential learning scenario is adopted.

Finally, Let us note here that Self-Taught Hashing (STH) [17] employs SVMs to generate hash codes. However, STH differs significantly from *SHL; its unsupervised and supervised learning stages are completely decoupled, while our framework uses a single cost function that simultaneously integrates both labeled and unlabeled samples to reduce the distortion measure

$$E(\omega) \triangleq \sum_{n \in \mathbb{N}_L} \min_{l} d(\mathbf{h}(x_n), \mathbf{h}_{l,n}) + \sum_{n \in \mathbb{N}_U} \min_{c,s} d(\mathbf{h}(x_n), \mathbf{c,s})$$

where $d$ is the Hamming distance defined as $d(\mathbf{h}, \mathbf{h}') = \sum_b [h_b \neq h'_b]$. Note that for each sample, one best codeword of each class will be selected to represent it. However, the distortion $E$ is difficult to directly minimize. As it will be illustrated further below, an upper bound $\bar{E}$ of $E$ will be optimized instead.

In particular, for a hash code produced by *SHL, it holds that $d(\mathbf{h}(x), \mathbf{h}) = \sum_b [h_b f_b(x) < 0]$. If one defines $\tilde{d}(\mathbf{f}, \mathbf{h}) = \sum_b [1 - f_b(x)]$, where $|u|\varepsilon = \max \{0, u\}$ is the hinge function, then $\tilde{d}(\mathbf{f}, \mathbf{h})$ holds for every $\mathbf{f} \in \mathbb{R}^B$ and any $\mathbf{h} \in \mathbb{H}^B$. Based on this latter fact, it holds that

$$\bar{E}(\omega) \leq E(\omega) \triangleq \sum_{c} \sum_{s} \min_{n} \tilde{d}(\mathbf{f}(x_n), \mathbf{h}_{c,s})$$

where

$$\gamma_{c,s} = \begin{cases} [c = l_n] \ [s = \arg \min_{s'} \tilde{d}(\mathbf{f}(x_n), \mathbf{h}_{l,n,s'})] \ [n \in \mathbb{N}_L] \\ [(c, s) = \arg \min_{c,s'} \tilde{d}(\mathbf{f}(x_n), \mathbf{h}_{c,s'})] \ [n \in \mathbb{N}_U] \end{cases}$$

It turns out that $\bar{E}$, which constitutes the model’s loss function, can be efficiently minimized by a three-step algorithm, which delineated in the next section.

## 4 Learning Algorithm

### 4.1 Algorithm for *SHL

The next proposition allows us to minimize $\bar{E}$ as defined in Eq. (2) via a MM approach [37] and [38].

**Proposition 1.** For any *SHL parameter values $\omega$ and $\omega'$, it holds that

$$E(\omega) \leq \bar{E}(\omega|\omega') \triangleq \sum_c \sum_s \sum_n \gamma_{c,s} \tilde{d}(\mathbf{f}(x_n), \mathbf{h}_{c,s})$$

where the primed quantities are evaluated on $\omega'$ and
Proposition 2. Minimizing $E(\cdot | \omega)$ with respect to the Hilbert space vectors, the offsets $\beta_b$ and the MKL weights $\theta_b$, while regarding the codeword parameters as constant, one obtains the following B independent, equivalent problems:

\[
\inf_{w_{b,m} \in \mathbb{H}_m, m \in \mathbb{M}} \sum_{n} \lambda_1 \sum_{c} \sum_{s} \sum_{n} \gamma'_{c,n,s} \left[ 1 - \mu_{c,s} f_b(x_n) \right] + \sum_{m} \frac{\|w_{b,m}\|_2^2}{\theta_{b,m}} + \lambda_2 \sum_{c} \sum_{i,j \in S} \|\mu_{c,i} - \mu_{c,j}\|_2 \quad b \in \mathbb{B} \tag{6}
\]

where $f_b(x) = \sum_{m} (w_{b,m} \phi_m(x))$ and $\lambda_1 > 0$ is a regularization constant.

The proof of this proposition hinges on replacing the (independent) constraints of the Hilbert space vectors with equivalent regularization terms and, finally, performing the substitution $w_{b,m} \rightarrow \sqrt{\theta_{b,m}} w_{b,m}$ as typically done in such MKL formulations (e.g. see [6]). The third term in Prob. (6) pushes codewords representing the same class closer to each other. With an appropriate value of $\lambda_2$, this regularization helps *SHL automatically select the codewords.

Note that Prob. (6) is jointly convex with respect to all variables under consideration and, under closer scrutiny, one may recognize it as a binary MKL SVM training problem, which will become more apparent shortly.

First block minimization: By considering $w_{b,m}$ and $\beta_b$ for each $b$ as a single block, instead of directly minimizing Prob. (6), one can instead maximize the following problem:

Proposition 3. The dual form of Prob. (6) takes the form of

\[
\sup_{\alpha_b \in \Omega_b} \alpha_b^T 1_{NCS} - \frac{1}{2} \alpha_b^T D_b (1_{CS} 1_{CS}^T) \otimes K_b D_b \alpha_b \quad b \in \mathbb{B} \tag{7}
\]

where $1_b$ stands for the all ones vector of $b$ elements ($b \in \mathbb{N}_b$), $\mu_b \triangleq [\mu_{1,b} \ldots \mu_{C,b}]^T$, $D_b \triangleq \text{diag} \{\mu_b \otimes 1_N\}$, $K_b \triangleq \sum_m \theta_{b,m} K_m$, where $K_m$ is the data's $m^{th}$ kernel matrix, $\Omega_b \triangleq \{ \alpha \in \mathbb{R}^{NC} : \alpha_b^T (\mu_b \otimes 1_N) = 0, 0 \leq \alpha_b \leq 1 \gamma' \}$ and $\gamma' \triangleq [\gamma'_{1,1}, \ldots, \gamma'_{1,N,1}, \gamma'_{1,N,2}, \ldots, \gamma'_{2,N,S}, \ldots, \gamma'_{C,N,S}]^T$.

The detailed proof is provided in Appendix A. Given that $\gamma'_{c,n,s} \in \{0,1\}$, one can easily now recognize that Prob. (7) is a SVM training problem, which can be conveniently solved using software packages such as LIBSVM. After solving it, obviously one can compute the quantities $(w_{b,m}, \phi_m(x))_{\mathbb{H}_m, b}$ and $\|w_{b,m}\|_2^{2 \gamma}$, which are required in the next step.

When dealing with large scale data sets, the sequential solver LIBSVM may encounter the memory bottleneck because of the kernel matrix computation. Therefore a parallel software package is necessary for big data problems. LIBSKYLARK [8], which utilizes random features [39] to approximate kernel matrix and Alternating Direction Method of Multipliers (ADMM) [40] to parallelize the algorithm, proves to be an efficient solver for large scale SVM problem. LIBSKYLARK achieves impressive acceleration when solving SVM compared to LIBSVM in [8]. Experiments over large data sets are also conducted in Sec. 6.

Second block minimization: Having optimized the SVM parameters, one can now optimize the cost function of Prob. (6) with respect to the MKL parameters $\theta_b$ as a single block using the closed-form solution mentioned in Prop. 2 of [6] for $p > 1$, which is given below

\[
\theta_{b,m} = \frac{\|w_{b,m}\|_2^{2 \gamma}}{\left( \sum_{m'} \|w_{b,m'}\|_2^{2 \gamma} \right)^{\frac{1}{p}}} \quad m \in \mathbb{M}, b \in \mathbb{B}. \tag{8}
\]

Third block minimization: we need to optimize this problem due to the new regularization introduced:

\[
\inf_{\gamma'_{c,n,s} \in \mathbb{H}} \sum_{n} \sum_{c} \sum_{s} \gamma'_{c,n,s} \left[ 1 - \mu_{c,s} f_b(x_n) \right] + \lambda_2 \sum_{c} \sum_{i,j \in S} \|\mu_{c,i} - \mu_{c,j}\|_2 \quad b \in \mathbb{B} \tag{9}
\]

Here, $c \in \mathbb{N}_C, b \in \mathbb{N}_B, s \in \mathbb{S}_S$.

First of all, we relax $\mu$ to continuous values, similar to relaxing the hashcode as continuous when computing the hinge loss. Eq. (9) follows the formulation $l(x) + h(x)$, which can be solved by proximal methods [41]. Since both the terms (hinge loss and regularization) are convex and non-smooth, we employ PSD method in a similar fashion as in [42], [43] and [44].

The proximal subgradient descent is

\[
x^{k+1} := \text{prox}_{\eta h} (x^k - \eta \partial l(x^k)) \tag{10}
\]

where $\eta$ is the step length and $\partial l$ is the subgradient of the function. Here the proximal operator $\text{prox}$ is defined as:

\[
\text{prox}_{\eta h} (v) \triangleq \arg \min_v \left( h(x) + \frac{1}{2\eta} \|x - v\|^2 \right) \tag{11}
\]

To obtain proximal operator, one needs to solve Eq. (11). In our problem setting, the regularization is the sum of many non smooth $L_2$ norms, whose closed form proximal
operator is not obvious to achieve. Based on the conclusion from [45], the proximal operator of sums of the functions can be approximated by sums of the proximal operator of the individual function, i.e. \( \text{prox}_h \approx \sum \text{prox}_h \). Thus, all we need is the closed form proximal operator for one individual norm in Eq. (9), i.e. \( \sum_{i,j \in S} |\mu_i - \mu_j|^2 \). Let us concatenate all codewords as \( \mu = [\mu_1^T, \ldots, \mu_S^T] \in \mathbb{R}^{BS} \). Moreover, a vector is defined as \( \omega \triangleq [0, \ldots, 1, \ldots, -1, \ldots, 0] \in \mathbb{R}^S \), where the value for index \( i \) is set to 1 and -1 for index \( j \). With the definition of a matrix \( U \triangleq \omega \otimes I \in \mathbb{R}^{B \times BS} \), the regularization can be reformulated as \( h(\mu) = \|U \mu\|^2 \), whose proximal operator will be given in the following proposition:

**Proposition 4.** Given the norm as: \( h(\mu) = \|U \mu\|^2 \). Following the definition in Eq. (11), the proximal operator of this norm:

\[
\text{prox}_{\eta h}(v) = \left\{ \begin{array}{ll}
\mu_1 = v_1 \\
\vdots \\
\mu_i = \alpha_1 v_i + \alpha_2 v_j \\
\vdots \\
\mu_j = \alpha_2 v_i + \alpha_1 v_j \\
\vdots \\
\mu_S = v_S 
\end{array} \right.
\]

where \( \alpha_1 = 1 - \alpha_2 \) and \( \alpha_2 = \min\{\frac{\|v_i - v_j\|^2}{\|v_i - v_j\|^2}, 2\}\), \( v = [v_1^T, \ldots, v_S^T]^T \), which is the input vector for proximal operators in Eq. (11).

The detailed proof of Prop. 4 is showcased in Appendix B.

Note that, if we consider only one codeword for each class, Prob. (9) can be simplified without the regularization:

\[
\inf_{\mu_i \in H} \sum_n \sum_c \gamma_{c,n} \left[ 1 - \mu_c f_b(x_n) \right]_+
\]

Prob. (13) can be optimized by mere substitution.

On balance, as summarized in Algorithm 1, for each bit, the algorithm to \(*\text{SHL}\) consists of one SVM optimization and one MKL update. For the third step, we evaluate the proximal operator for each regularization and compute the summation to do the PSD to optimize codewords. Note that accelerated proximal gradient descent [46] is utilized here. \( \gamma_{c,n,s} \) is then updated according to the current estimate of the parameters. This algorithm, as shown in Algorithm 1, continues running until reaching the convergence2. Based on LIBSVM, which provides \( O(N^3) \) complexity [47], our algorithm offers the complexity \( O(BN^3) \) per iteration, where \( B \) is the code length and \( N \) is the number of instances.

### 5 Insights to Generalization Performance

The superior performance of \(*\text{SHL}\) over other state-of-the-art hash function learning approaches featured in the next section can be explained to some extent by noticing that

2. MATLAB® code of \(*\text{SHL}\)'s algorithm will be made publicly accessible, upon this manuscript’s acceptance by the journal.

\[
\hat{R}_Q(G) \triangleq \mathbb{E}_\sigma \left\{ \sup_{Q \in (\mathcal{F}, \mathcal{G})} \frac{1}{N} \sum_{n=1}^{N} \sigma_n o(z_n) \right\}
\]

where \( \mathbb{E}_\sigma \{ \cdot \} \triangleq \mathbb{E}_{\sigma_1} \{ \cdots \mathbb{E}_{\sigma_N} \{ \cdots \cdots \} \} \) and \( \{ \sigma_n \}_{n=1}^{N} \) are iid Rademacher RV. In the rest of the section, on condition on \( Q \) will be omitted for simplicity. Additionally, the Rademacher Complexity (RC) of \( G \) for a sample size \( N \) is defined as:

\[
R_N(G) \triangleq \mathbb{E}_{Q \sim \mathcal{D}^N} \left\{ \hat{R}_Q(G) \right\}
\]

We also need the following two lemmas before showing our final concentration results.

**Lemma 1.** Let \( \mathcal{Z} \) be an arbitrary set, \( \mathcal{F} \triangleq \{ f : z \mapsto f(z) \in \mathbb{R}^B, z \in \mathcal{Z} \} \), \( \mathcal{P} : \mathbb{R}^B \to \mathcal{R} \) be \( L \)-Lipschitz continuous w.r.t \( \| \cdot \|_1 \). Also, define \( \Psi \circ \mathcal{F} \triangleq \{ g : z \mapsto \Psi(f(z)) \} \) and \( \mathcal{F}_1 \triangleq \{ h : z \mapsto \| f(z) \|_1, f \in \mathcal{F} \} \). Then

\[
\hat{R}_Q \left( \mathcal{F} \circ \Psi \right) \leq L \hat{R}_Q \left( \mathcal{F}_1 \right)
\]
**Lemma 2.** Let $\mathcal{Z}$ be an arbitrary set. Define $\tilde{\mathcal{F}} \triangleq \{ f : z \mapsto f(z) \in \mathbb{R}^B, z \in \mathcal{Z} \}, \| \tilde{\mathcal{F}} \|_1 \triangleq \{ h : z \mapsto \| f(z) \|_1, f \in \tilde{\mathcal{F}} \}$ and $1^T \tilde{\mathcal{F}} \triangleq \{ g : z \mapsto 1^T f(z), f \in \tilde{\mathcal{F}} \}$. Let’s further assume that if $f(z) \triangleq [f_1(z), ..., f_B(z)]^T \in \tilde{\mathcal{F}}$ for $z \in \mathcal{Z}$, then also $[\pm f_1(z), ..., \pm f_B(z)] \in \tilde{\mathcal{F}}$ for any combination of signs. Then:

$$\tilde{\mathbb{R}}_Q(\tilde{\mathcal{F}}) \leq \tilde{\mathbb{R}}_Q(1^T \tilde{\mathcal{F}})$$

where $Q$ is a set of $N$ samples drawn from $\mathcal{Z}$.

The detailed proof is provided in Appendix C for Lemma 1 and in Appendix D for Lemma 2.

To show the main theoretical result of our paper with the help of the previous lemmas, we will consider the sets of functions

$$\mathcal{F} \triangleq \{ f : x \mapsto [f_1(x), ..., f_B(x)]^T, f_b \in \mathcal{F}_b \in \mathbb{N}_B \}$$

$$\mathcal{F} \triangleq \{ f : x \mapsto \langle w, \phi_\theta(x) \rangle_{\mathcal{H}_\theta} + \beta, \beta \in \mathbb{R}, w \in \Omega_w(\theta) \}$$

where $\Omega_w(\theta) \triangleq \{w \in \mathcal{H}_\theta : \| w \|_{\mathcal{H}_\theta} \leq R \}$ for some $R \geq 0$ and $\Omega_\theta \triangleq \{ \theta \in \mathbb{R}^M : \theta \geq 0, \| \theta \|_p \leq 1 \}$ for some $p \geq 1$.

**Theorem 3.** Assume reproducing kernels of $\{ \mathcal{H}_m \}_{m=1}^M$ s.t. $k_m(x, x') \leq r^2, \forall x, x' \in \mathcal{X}$ and a set $Q$ of iid samples $Q = \{(x_1, l_1), ..., (x_N, l_N)\}$. Then for $\rho > 0$ independent of $Q$, for any $f \in \mathcal{F}$, any $\mu \in \mathcal{M}$, where $\mathcal{M} \triangleq \{ \mu : \mathcal{N}_C \rightarrow \mathbb{R}^B \}$ and any $0 < \delta < 1$, with probability $1 - \delta$, it holds that:

$$er(f, \mu) \leq \hat{er}(f, \mu) + \frac{2R}{\rho} \sqrt{rM} \frac{\sqrt{p'}}{N^2} + \sqrt{\frac{\log(\frac{1}{\delta})}{2N}}$$

where $er(f, \mu) \triangleq \frac{1}{N} \mathbb{E}\{d(\text{sgn}(f(x), \mu(l))) \}, \mu \in \mathcal{N}_C$ is the true label of $x \in \mathcal{X}, \hat{er}(f, \mu) \triangleq \frac{1}{N_B} \sum_{n,b} \Phi_p(f_b(x_n)\mu_{n,b})$, where $\Phi_p(u) \triangleq \min\{1, \max\{0, 1 - \frac{u}{p} \}\}$ and $p' \triangleq \frac{p}{p^2}$. The detailed proofs for Theorem 3 can be found in Appendix E.

**6 EXPERIMENTS**

**6.1 Supervised Hash Learning Results**

In this section, we compare *SHL to other state-of-the-art hashing algorithms:

- Kernel Supervised Learning (KSH)

3. http://www.ee.columbia.edu/ln/dvmm/downloads/WeiKSHCode/dlcode.html
Fig. 3. The top $k$ retrieval results and Precision-Recall curve on Mnist dataset over *SHL and 6 other hashing algorithms. (view in color)

Fig. 4. The top $k$ retrieval results and Precision-Recall curve on CIFAR-10 dataset over *SHL and 6 other hashing algorithms. (view in color)

- Binary Reconstructive Embedding (BRE)$^4$ [13].
- single-layer Anchor Graph Hashing (1-AGH) and its two-layer version (2-AGH)$^5$ [29].
- Spectral Hashing (SPH)$^6$ [27].
- Locality-Sensitive Hashing (LSH) [9].

Five datasets, which are widely utilized in other hashing papers as benchmarks, were considered:

- **Pendigits**: a digit dataset (10,992 samples, 256 features, 10 classes) of 44 writers from the UCI Repository$^7$. In our experiment, we randomly choose 3,000 for training and the rest for testing.
- **USPS**: a digit dataset also from the UCI Repository, is numeric data from the scanning of handwritten digits from envelopes by the U.S. Postal Service. Among the dataset (9,298 samples, 256 features, 10 classes), 3000 were used for training and others for testing.
- **Mnist**: a hand written digit dataset which contains 70,000 samples, 784 features and 10 classes. The digits have been size-normalized and centered.

For all the algorithms used, average performances over 5 runs are reported in terms of the following two criteria: (i) retrieval precision of $k$-closest hash codes of training samples; we used $k = \{10, 15, \ldots, 50\}$. (ii) Precision-Recall (PR) curve, where retrieval precision and recall are computed for hash codes within a Hamming radius of $r \in \mathbb{N}_B$.

The following *SHL settings were used: SVM’s parameter $\lambda_1$ was set to 1000; for MKL, 11 kernels were considered: 1 normalized linear kernel, 1 normalized polynomial kernel

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4. http://web.cse.ohio-state.edu/~kulis/bre/bre.tar.gz
5. http://www.ee.columbia.edu/~kulis/bre/bre.tar.gz
6. http://web.cse.ohio-state.edu/~kulis/bre/bre.tar.gz
7. http://archive.ics.uci.edu/ml/
8. http://yann.lecun.com/exdb/mnist/
9. http://www.cs.toronto.edu/~kriz/cifar.html
10. http://groups.csail.mit.edu/vision/TinyImages/
11. http://pascallin.ecs.soton.ac.uk/challenges/VOC/
and 9 Gaussian kernels. For the polynomial kernel, the bias was set to 1.0 and its degree was chosen as 2. For the bandwidth $\sigma$ of the Gaussian kernels the following values were used: $\{2^{-7}, 2^{-5}, 2^{-3}, 2^{-1}, 1, 2, 4, 8, 16\}$. Regarding the MKL constraint set, a value of $p = 2$ was chosen. $\lambda_2$ was set to 2000 for Pendigits, USPS and PASCAL07 and $\lambda_2 = 6000$ for the rest of the datasets.

For the remaining approaches, namely KSH, SPH, AGH, BRE, parameter values were used according to recommendations found in their respective references. All obtained results are reported in Fig. 1 through Fig. 5.

We clearly observe that *SHL performs the best among all the algorithms considered. For all the datasets, *SHL achieves the highest top-10 retrieval precision. Especially for non-digits datasets (CIFAR-10, PASCAL07), *SHL achieves significantly better results. As for the PR-curve, *SHL also obtains the largest areas under the curve. Although impressive results have been reported in [22] for KSH, in our experiments, *SHL outperforms it across all datasets. Moreover, we observe that supervised hash learning algorithms, except BRE, perform better than unsupervised variants. BRE may need a longer bit length to achieve better performance like in Fig. 1 and Fig. 3. Additionally, it is worth mentioning that *SHL performs impressively with short bit length across all the datasets.

AGH also yields good results, compared with other unsupervised hashing algorithms, because it utilizes anchor points as side information to generate hash codes. With the exception of *SHL and KSH, the remaining approaches exhibit poor performance for the non-digits datasets we considered (CIFAR-10 and PASCAL07).

When varying the top-$k$ number between 10 and 50, once again with the exception of *SHL and KSH, the performance of the remaining approaches deteriorated in terms of top-$k$ retrieval precision. KSH performs slightly worse, when $s$ increases, while *SHL’s performance remain robust for CIFAR-10 and PASCAL07. It is worth mentioning that the two-layer AGH exhibits better robustness than its single-layer version for datasets involving images of digits. Finally, Fig. 6 and Fig. 7 show some qualitative results for the CIFAR-10 and Mnist datasets. In conclusion, it seems that both *SHL’s performances are superior at every code length we considered.

6.2 Transductive Hash Learning Results
As a proof of concept, in this section, we report a performance comparison of *SHL, when used in an inductive versus a transductive [49] mode. Note that, to the best of our knowledge, apart from our method, there are no other hash learning approaches to date that can accommodate transductive hash learning. For illustration purposes, we used the Vowel and Letter datasets from UCI Repository. We randomly chose 330 training and 220 test samples for the Vowel and 300 training and 200 test samples for the Letter. Each scenario was run 20 times and the code length ($B$) varied from 4 to 15 bits. The results are shown in Fig. 8 and reveal the potential merits of the transductive *SHL learning mode across a range of code lengths.

6.3 Image Segmentation
Besides content-based image retrieval, the proposed *SHL can also be utilized in the other applications, for example, the foreground/background interactive image segmentation [50], where the images are partially labeled as foreground and background by users. In *SHL, while foreground and background are represented by two codewords, the rest of the pixels can be labeled in semi-supervised learning scenario. In this section, we show the interactive image segmentation results using *SHL on the dataset introduced in [51]. The hash code length is 5 and the rest of the parameters settings follow the previous section. For each pixel, the RGB values are used as features. The results are shown in Fig. 9. We notice that, provided with partially labeled information, *SHL successfully segment the foreground object from the background. Especially in (e), although all the flower pots share the same color, *SHL only highlight the labeled one and its plant. Additionally, in some images, like (c) and (f), shaded areas fail to be segmented. In these cases, more pixel features may be necessary for better results.

6.4 *SHL for Large Data Set
Large data sets require a huge kernel matrix which can not be fit into the memory of one single machine. Thus,
7 CONCLUSIONS

In this paper, we considered a novel hash learning framework, namely *Supervised Hash Learning (*SHL). The method has the following main advantages: first, its Majorization-Minimization (MM)/Block Coordinate Descent (BCD) training algorithm is efficient and simple to implement. Secondly, the framework is able to address supervised, unsupervised, or even semi-supervised learning tasks. Additionally, after introducing a regularization over the multiple codewords, we also provide the Proximal Subgradient Descent (PSD) method to solve this regularization.

In order to show the merits of the methods, we performed a series of experiments involving 5 benchmark datasets. In experiments that were conducted, a comparison between our methods and the other 6 state-of-the-art hashing methods shows *SHL to be highly competitive. Moreover, we also give results on transductive learning scenario. Additionally, another application based on our framework, interactive image segmentation, is also showcased in the experimental section. Finally, we introduce *SHL can also solve problems containing a huge number of samples.

APPENDIX A

PROOF OF Prop. 3

Proof. By replacing hinge function in Prob. (6), we got the following problem for the first block minimization:
Fig. 7. Qualitative results on Mnist. Query image is “4”. The remaining 15 images for each row were retrieved using 45-bit binary codes generated by different hashing algorithms. Red box indicates wrong retrieval results.

TABLE 1
Top-10 retrieval results and running time for *SHL for various data sets. Here, running time secs means seconds, mins means minutes and hours means hours.

| DATA SETS | NODES | TRAINING | TESTING | BITS | ACCURACY | TIME |
|-----------|-------|----------|---------|------|----------|------|
| USPS      | 2     | 7291     | 2007    | 5    | 0.916    | 45.82 secs |
|           |       |          |         | 25   | 0.936    | 227.17 secs / 3.78 mins |
|           |       |          |         | 45   | 0.941    | 408.66 secs / 6.90 mins |
| Mnist     | 2     | 60K      | 10K     | 5    | 0.826    | 1690.46 secs / 28.11 mins |
|           |       |          |         | 25   | 0.960    | 8406.04 secs / 140.08 mins / 2.33 hours |
|           |       |          |         | 45   | 0.969    | 4253.28 secs / 251.85 mins / 4.20 hours |
| Mnist     | 10    | 60K      | 10K     | 5    | 0.839    | 487.88 secs / 8.13 mins |
|           |       |          |         | 25   | 0.962    | 2361.07 secs / 39.35 mins |
|           |       |          |         | 45   | 0.964    | 4253.28 secs / 70.89 mins / 1.18 hours |
| Mnist1M   | 10    | 1M       | 100K    | 5    | 0.824    | 1918.38 secs / 131.91 mins / 2.20 hours |
|           |       |          |         | 25   | 0.927    | 28066.37 secs / 467.57 mins / 7.80 hours |
|           |       |          |         | 45   | 0.934665 | 48574.96 secs / 809.58 mins / 13.49 hours |

\[
\min_{w_{b,m}, \beta_b, \xi_{b,c,n,s}^b} \lambda_1 \sum_c \sum_s \sum_n \gamma_{c,n,s}^b \xi_{c,n,s}^b + \frac{1}{2} \sum_m \| w_{b,m} \|^2_{\theta_{b,m}} \\
\text{s.t. } \xi_{c,n,s}^b \geq 0 \\
\xi_{c,n,s}^b \geq 1 - \left( \sum_m \langle w_{b,m}, \phi_m(x) \rangle_{\theta_{b,m}} + \beta_b \right) \mu_{c,s} 
\]  

(21)

First of all, after considering Representer Theorem [52], we have:

\[
w_{b,m} = \theta_{b,m} \sum_n \eta_{b,n} \phi_m(x_n) 
\]  

(22)

where \( n \) is index of the training samples. By defining \( \xi_b \in \mathbb{R}^{NCS} \) to be the vector containing all \( \xi_{c,n,s}^b \)'s, \( \eta_b \triangleq [\eta_{b,1,1}, \eta_{b,2,1}, \ldots, \eta_{b,N}^b]^T \in \mathbb{R}^N \) and \( \mu_b \triangleq [\mu_{1,1,1,1}, \mu_{1,1,1,2}, \ldots, \mu_{C,S}^b]^T \in \mathbb{R}^{CS} \), the vectorized version of Prob. (21) with Eq. (22):

\[
\min_{\eta_b, \xi_b, \beta_b} \lambda_1 \gamma \xi_b + \frac{1}{2} \eta_b^T \mathbf{K}_b \eta_b \\
\text{s.t. } \xi_b \geq 0 \\
\xi_b \geq 1_{NCS} - (\mu_b \otimes \mathbf{K}_b) \eta_b - (\mu_b \otimes 1_N) \beta_b 
\]  

(23)

Where \( \gamma \) and \( \mathbf{K}_b \) are defined in Prop. 3. Take the Lagrangian \( \mathcal{L} \) and its derivatives, we have the following relations, here \( \alpha_b \) and \( \zeta_b \) are Lagrangian multipliers for the two constraints:
Fig. 8. Accuracy results between Inductive and Transductive Learning.

\[ \frac{\partial L}{\partial \xi_b} = 0 \Rightarrow \begin{cases} \zeta_b = \lambda_1 \gamma' - \alpha_b \\ 0 \leq \alpha_b \leq \lambda_1 \gamma' \end{cases} \]  \hspace{1cm} (24)

\[ \frac{\partial L}{\partial \beta_b} = 0 \Rightarrow \alpha_b^T (\mu_b \otimes 1_N) = 0 \]  \hspace{1cm} (25)

\[ \frac{\partial L}{\partial \eta_b} = 0 \Rightarrow \exists \eta^{-1} \Rightarrow \eta_b = K_b^{-1} (\mu_b \otimes K_b)^T \alpha_b \]  \hspace{1cm} (26)

Substitute Eq. (24), Eq. (25) and Eq. (26) back into \( L \), meanwhile, we notice the quartic term becomes:

\[ \begin{align*}
(\mu_b \otimes K_b)K_b^{-1}(\mu_b^T \otimes K_b) \\
= (\mu_b \otimes K_b)(1 \otimes K_b^{-1})(\mu_b^T \otimes K_b) \\
= (\mu_b \otimes I_{N \times N})(\mu_b^T \otimes K_b) \\
= (\mu_b \mu_b^T) \otimes K_b
\end{align*} \]  \hspace{1cm} (27)

Eq. (27) can be further derived:

\[ \begin{align*}
(\mu_b \mu_b^T) \otimes K_b \\
= (\text{diag} (\mu_b \otimes 1_N) 1_C^T) \otimes K_b \\
= (\text{diag} (\mu_b \otimes 1_N) \text{diag} (\mu_b)) \otimes [I_N \otimes K_b] \\
= (\text{diag} (\mu_b \otimes 1_N)) [I_N \otimes \text{diag} (\mu_b) \otimes K_b] \\
= D_b[(1_C 1_C^T) \otimes K_b] \\
= D_b[(1_C 1_C^T) \otimes K_b] \\
= D_b \rightarrow (\mu_b \mu_b^T) \otimes K_b
\end{align*} \]  \hspace{1cm} (28)

The first equality comes from \( \text{diag} (u) 1 = u \) for some vector \( u \). The third equality is the mixed-product property of Kronecker product. This relation \( \text{diag} (u \otimes 1) = \text{diag} (u) \otimes I \) gives the fourth equality. \( D_b \) is defined in Prop. 3.

After considering Eq. (27) and Eq. (28), we get the final dual form as shown in Prop. 3.

**APPENDIX B**

**PROOF OF PROP. 4**

Proof. With the definitions of proximal operator Eq. (11), we have the following problem to minimize over:

\[ P(\mu) = \eta \|U \mu\|_2 + \frac{1}{2}\|v - \mu\|_2^2 \]

\[ = \eta \|U \mu\|_2 + \frac{1}{2}\|v_1 - \mu_1\|_2^2 + \cdots + \frac{1}{2}\|v_S - \mu_S\|_2^2 \]  \hspace{1cm} (29)

The second equality follows from the definitions of the vectors \( \mu \) and \( v \). Since \( L_2 \) norm is non differentiable at point 0, we optimize Eq. (29) in two cases.

**Case 1:** when \( \mu_i \neq \mu_j \), we take the gradients for each individual \( \mu_i \) to \( \mu_S \):

\[ \begin{align*}
\frac{\partial P(\mu)}{\partial \mu_i} &= \mu_i - v_i = 0 \\
\frac{\partial P(\mu)}{\partial \mu_j} &= \eta \frac{\mu_i - \mu_j}{\|\mu_i - \mu_j\|_2} + \mu_j - v_j = 0 \\
\frac{\partial P(\mu)}{\partial \mu_S} &= \mu_S - v_S = 0
\end{align*} \]  \hspace{1cm} (30)

Solve the linear equations with \( \mu_i \) and \( \mu_j \):

\[ \begin{align*}
\mu_i &= \frac{1}{2\tau + 1} [(1 + \tau) v_i + \tau v_j] \\
\mu_j &= \frac{1}{2\tau + 1} [\tau v_i + (1 + \tau) v_j]
\end{align*} \]  \hspace{1cm} (31)

where \( \tau \triangleq \eta / \delta \) and \( \delta \triangleq \|\mu_i - \mu_j\|_2 \). Now we have the following derivations:
Fig. 9. Foreground/Background interactive image segmentation. The left column contains the original images. The middle column includes labeled pixels. The right column shows the results of the segmentation.

\[
\begin{align*}
\mu_i - \mu_j &= \frac{1}{2\tau + 1} (v_i - v_j) \\
\Rightarrow \|\mu_i - \mu_j\|_2 &= \frac{1}{2\tau + 1} \|v_i - v_j\|_2 \\
\Rightarrow \delta &= \frac{\|v_i - v_j\|_2}{2\tau + 1} \\
\Rightarrow \delta &= \|\mu_i - \mu_j\|_2 = \|v_i - v_j\|_2 - 2\eta 
\end{align*}
\]

Plug \(\tau\) and Eq. (32) into Eq. (31), we achieve the results for \(\mu_i\) and \(\mu_j\):

\[
\begin{align*}
\mu_i &= \alpha_1 v_i + \alpha_2 v_j \\
\mu_j &= \alpha_2 v_i + \alpha_1 v_j
\end{align*}
\]

Here \(\alpha_1 = 1 - \alpha_2\) and \(\alpha_2 = \frac{\eta}{\|v_i - v_j\|_2}\). Additionally, Eq. (32) is larger than 0 which gives the following condition for Case 1: \(0 < \eta \leq \frac{\|v_i - v_j\|_2}{2}\).

Case 2: when \(\mu_i = \mu_j\), \(\mu_i\) and \(\mu_j\) are represented as:
Additionally, we define the following:

\[ \hat{\Psi}(\dot{\sigma}) = \frac{1}{N} \mathbb{E}_\sigma \left\{ \sup_{f \in F} \sum_n \sigma_n \Psi(f(z_n)) \right\} = \frac{1}{N} \mathbb{E}_{\sigma_{N-1}} \left\{ \mathbb{E}_{\sigma_N} \left\{ \sup_{f \in F} [u(f) + \sigma_N \Psi(f(z_n))] \right\} \right\} = \frac{1}{N} \mathbb{E}_{\sigma_{N-1}} \{ A(\sigma_{N-1}) \} \]

where \( u(f) \triangleq \sum_{n=1}^{N-1} \sigma_n \Psi(f(z_n)) \) and \( A(\sigma_{N-1}) \triangleq \mathbb{E}_{\sigma_N} \left\{ \sup_{f \in F} [u(f) + \sigma_N \Psi(f(z_n))] \right\} \).

Expanding the expectation, we get:

\[ A(\sigma_{N-1}) = \frac{1}{2} [ \sup_{f \in F} [u(f) + \Psi(f(z_n))] \right] + \sup_{f \in F} [u(f) - \Psi(f(z_n))] \]

Additionally, we define the following: \( \hat{\dot{B}}(f) \triangleq u(f) + \Psi(f(z_n)) \) and \( \dot{B}(f) \triangleq u(f) - \Psi(f(z_n)) \).

From the superium’s definition, we have that \( \forall \epsilon > 0 \), there are \( \hat{f} \) and \( \tilde{f} \) in \( F \) such that:

\[ \sup_{f \in F} \hat{B}(f) \geq \hat{\dot{B}}(\hat{f}) \geq (1 - \epsilon) \sup_{f \in F} \hat{B}(f) \]

\[ \sup_{f \in F} \tilde{B}(f) \geq \tilde{\dot{B}}(\tilde{f}) \geq (1 - \epsilon) \sup_{f \in F} \tilde{B}(f) \]

From Eq. (37) and Eq. (38), for any \( \epsilon > 0 \), we have:

\[ (1 - \epsilon)A(\sigma_{N-1}) \leq \frac{1}{2} \left[ \hat{B}(\hat{f}) + \hat{\dot{B}}(\hat{f}) \right] = \frac{1}{2} [u(\hat{f}) + u(\tilde{f}) + \Psi(f(z_n)) - \Psi(\tilde{f}(z_n))] \]

Since \( \Psi \) is \( L \)-Lipschitz continuous w.r.t the \( \| \cdot \|_1 \) norm, it holds that:

\[ \Psi(\tilde{f}(z_n)) - \Psi(\dot{f}(z_n)) \leq L \left\| \tilde{f}(z_n) - \dot{f}(z_n) \right\|_1 = L \sum_{b=1}^B |\tilde{f}_b(z_n) - \dot{f}_b(z_n)| = L \sum_{b=1}^B q_b \left( \tilde{f}_b(z_n) - \dot{f}_b(z_n) \right) \]

where \( q_b \triangleq \text{sgn}(\tilde{f}_b(z_n) - \dot{f}_b(z_n)) \). From Eq. (39) and Eq. (40), we obtain:

\[ (1 - \epsilon)A(\sigma_{N-1}) \leq \frac{1}{2} \left[ u(\hat{f}) + u(\tilde{f}) + L \sum_{b=1}^B q_b (\dot{f}_b(z_n) - \tilde{f}_b(z_n)) \right] = \frac{1}{2} \left[ (u(\hat{f}) + L \sum_{b=1}^B q_b \tilde{f}_b(z_n)) + (u(\tilde{f}) - L \sum_{b=1}^B q_b \dot{f}_b(z_n)) \right] \]

By the definition of superium, Eq. (41) is bounded by:

\[ (41) \leq \sup_{q \in \mathcal{H}_B} \frac{1}{2} \left[ \left( u(\hat{f}) + L \sum_{b=1}^B q_b \tilde{f}_b(z_n) \right) + \left( u(\tilde{f}) - L \sum_{b=1}^B q_b \dot{f}_b(z_n) \right) \right] \]

With the help of Eq. (37) and Eq. (38), Eq. (42) is bounded:

\[ (42) \leq \sup_{q \in \mathcal{H}_B} \sup_{f \in F} \left[ u(f) + L \sum_{b=1}^B q_b f_b(z_n) \right] \]

Since Eq. (43) holds for every \( \epsilon > 0 \), we have that:

\[ A(\sigma_{N-1}) \leq \mathbb{E}_{\sigma_N} \left\{ \sup_{f \in F} [u(f) + \sigma_N L \left\| f(z_n) \right\|_1] \right\} \]
APPENDIX D

PROOF OF LEMMA 2

Proof. Let \( \Psi(\cdot) \triangleq ||\cdot||_1 \), Utilizing the similar technique in Lemma 1, by defining \( u(f) \triangleq \sum_{n=1}^{N-1} \sigma_n \Psi(f(z_n)) \), we have:

\[
\mathbb{R}_Q(\frac{1}{N} \mathbb{E}_{\sigma_{N-1}} \{A(\sigma_{N-1})\}) = \frac{1}{N} \mathbb{E}_{\sigma_{N-1}} \{A(\sigma_{N-1})\}
\]

Here \( A(\sigma_{N-1}) = \mathbb{E}_{\sigma_N} \left\{ \sup_{f \in \mathcal{F}} \left[ u(f) + \sigma_N \Psi(f(z_N)) \right] \right\} \). Similarly, by defining \( \tilde{B}(f) \triangleq u(f) + \Psi(f(z_N)) \) and \( \tilde{B}(f) \triangleq u(f) - \Psi(f(z_N)) \), we have for any \( \epsilon > 0 \):

\[
(1 - \epsilon)A(\sigma_{N-1}) \leq \frac{1}{2} \left[ \tilde{B}(f) + \tilde{B}(\bar{f}) \right] = \frac{1}{2} \left[ u(f) + u(\bar{f}) + \Psi(f(z_N)) - \Psi(\bar{f}(z_N)) \right]
\]

By the reverse triangle inequality and \(|\cdot|’s 1 - \) Lipschitz property:

\[
\Psi(f(z_N)) - \Psi(\bar{f}(z_N)) = \sum_{b=1}^{B} \left[ |f_b(z_N)| - |\bar{f}_b(z_N)| \right] \leq \sum_{b=1}^{B} sgn(f_b(z_N) - \bar{f}_b(z_N))(f_b(z_N) - \bar{f}_b(z_N))
\]

With the definition of \( q_b \triangleq sgn(\bar{f}_b(z_N) - f_b(z_N)) \), we combine Eq. (46) and Eq. (47):

\[
(1 - \epsilon)A(\sigma_{N-1}) \leq \frac{1}{2} \left[ u(f) + u(\bar{f}) + \sum_{b=1}^{B} q_b(f_b(z_N) - \bar{f}_b(z_N)) \right] = \frac{1}{2} \left[ u(f) + \sum_{b=1}^{B} q_b f_b(z_N) \right] - \frac{1}{2} \left[ u(\bar{f}) - \sum_{b=1}^{B} q_b \bar{f}_b(z_N) \right]
\]

For \( b \in \mathbb{N}_B \), define \( q_b' \triangleq q_b \) if \( q_b \neq 0 \) and \( q_b' \triangleq 1 \) otherwise. Also, define \( \tilde{f}(\cdot) \triangleq q_b' \tilde{f}(\cdot) \), then we have \( f(\cdot) = q_b' \tilde{f}(\cdot) \) and :

\[
u(f) + \sum_{b=1}^{B} q_b \tilde{f}_b(z_N) = u(f) + \sum_{b=1}^{B} q_b \tilde{f}_b(z_N) = u(\tilde{f}) + \sum_{b=1}^{B} \tilde{f}_b(z_N) \leq \sup_{f \in \mathcal{F}} \left[ u(f) + \sum_{b=1}^{B} f_b(z_N) \right]
\]

The above derivation is based on the fact that if \( \{f_1(z), ..., f_B(z)\}^T \in \mathcal{F} \), then we also have \( \{\pm f_1(z), ..., \pm f_B(z)\}^T \in \mathcal{F} \) for \( z \in \mathcal{Z} \).

Using a similar rationale, we can show that:

\[
u(f) - \sum_{b=1}^{B} q_b \tilde{f}_b(z_N) \leq \sup_{f \in \mathcal{F}} \left[ u(f) - \sum_{b=1}^{B} f_b(z_N) \right] \]

Combine Eq. (48), Eq. (49) and Eq. (50):

\[
(1 - \epsilon)A(\sigma_{N-1}) \leq \frac{1}{2} \left[ \sup_{f \in \mathcal{F}} \left[ u(f) + \sum_{b=1}^{B} f_b(z_N) \right] + \sup_{f \in \mathcal{F}} \left[ u(f) - \sum_{b=1}^{B} f_b(z_N) \right] \right]
\]

Since Eq. (51) holds for every \( \epsilon > 0 \), we have that:

\[
A(\sigma_{N-1}) \leq \mathbb{E}_{\sigma_N} \left\{ \sup_{f \in \mathcal{F}} \left[ u(f) + \sigma_N \sum_{b=1}^{B} f_b(z_N) \right] \right\}
\]

Repeating this process for the remaining \( \sigma_n \) will eventually yield the result of this lemma.

APPENDIX E

PROOF OF THEOREM 3

Proof. Consider the function spaces:

\[
\mathcal{G} \triangleq \{ g : (x, l) \mapsto [\mu_1(f_1(x)), ..., \mu_B(f_B(x))]^T, \mu \in \mathcal{M}, f : \mathcal{X} \mapsto \mathbb{R}^B \}
\]

\[
\Psi \circ \mathcal{G} \triangleq \{ \Psi(g(\cdot)) : (x, l) \mapsto \frac{1}{B} \sum_{b=1}^{B} \Phi_{\rho}(g_b(x, l)), g \in \mathcal{G} \}
\]

Notice that, since \( \Phi_{\rho}(u) \in [0, 1] \) for all \( u \in \mathbb{R} \), also \( \Psi(g(x, l)) \in [0, 1] \) for all \( g \in \mathcal{G}, x \in \mathbb{R} \) and \( l \in \mathbb{N}_N \). Hence, from Theorem 3.1 of [5], for fixed (independent of \( Q \)) \( \rho > 0 \) and for any \( \delta > 0 \) and any \( g \in \mathcal{G} \), with probability at least \( 1 - \delta \), it holds that:

\[
\mathbb{E}\{\Psi(g(x, l))\} \leq \mathbb{E}_Q \{\Psi(g(x, l))\} + 2\mathbb{R}_N(\Psi \circ \mathcal{G}) + \sqrt{\frac{\ln \frac{1}{\delta}}{2N}}
\]

Where we define \( \forall h : \mathcal{X} \times \mathbb{N}_C \mapsto \mathbb{R} \) and \( \mathbb{E}_Q \{h(x, l)\} \triangleq \frac{1}{N} \sum_{n=1}^{N} h(x_n, l_n) \). Since \( u < 0 \leq \Phi_{\rho}(u) \) for all \( u \in \mathbb{R} \), \( \rho > 0 \), it holds that:

\[
\frac{1}{B} d(\operatorname{sgn} f(x), \mu(l)) = \frac{1}{B} \sum_{b=1}^{B} [\mu_b(l)f_b(x)] < 0 \leq \frac{1}{B} \sum_{b=1}^{B} \Phi_{\rho}(\mu_b(l)f_b(x)) = \Psi(g(x, l))
\]

Thus, we have the following:
Due to Eq. (54), with probability at least $1 - \delta$, Eq. (53) becomes now:

$$er(f, \mu) \leq \bar{E}_Q \{ \Psi(g(x, l)) \} + 2\mathcal{R}_N(\Psi \circ \mathcal{G}) + \sqrt{\frac{\ln \frac{1}{\delta}}{2N}} \tag{55}$$

Now due to the fact that $\Psi(\cdot)$ is $\frac{1}{B\rho}$ - Lipschitz continuous w.r.t $\|\cdot\|_1$ and Lemma 1, we have:

$$\tilde{R}_Q(\Psi \circ \mathcal{G}) \leq \frac{1}{B\rho} \mathcal{R}_Q(\|\mathcal{G}\|_1)$$

Also, since $\mu : \mathcal{N}_G \mapsto \mathbb{H}^B$, we have $\tilde{R}_Q(\|\mathcal{G}\|_1) = \tilde{R}_Q(\|\mathcal{F}\|_1)$, where $\mathcal{F}$ is defined in Eq. (18), we get:

$$\tilde{R}_Q(\Psi \circ \mathcal{G}) \leq \frac{1}{B\rho} \mathcal{R}_Q(\|\mathcal{F}\|_1) \tag{56}$$

Now due to Lemma 2, we have $\mathcal{R}_Q(\|\mathcal{F}\|_1) \leq \mathcal{R}_Q(1^T \mathcal{F})$, by taking expectations on both sides w.r.t $Q$, the above inequality becomes:

$$\mathcal{R}_N(\Psi \circ \mathcal{G}) \leq \frac{1}{B\rho} \mathcal{R}_N(1^T \mathcal{F})$$

Substitute Eq. (56) into Eq. (55):

$$er(f, \mu) \leq \bar{E}_Q \{ \Psi(g(x, l)) \} + \frac{2}{B\rho} \mathcal{R}_N(1^T \mathcal{F}) + \sqrt{\frac{\ln \frac{1}{\delta}}{2N}} \tag{57}$$

From the optimization problem in Eq. (6), we note that *SHL is utilizing the hypothesis spaces defined in Eq. (18) and Eq. (19). Note the fact that each hash function of *SHL is determined by the data independent of the others.

By considering the Representer Theorem [52], we have $w = \sum_{n=1}^N \alpha_n \Phi(x_n)$, $\alpha \in \mathbb{R}^N$, which implies: $f(x) = \sum_{n=1}^N \alpha_n \Phi(x_n) + \beta$ and $\|w\|_H^2 = \alpha^T K_{\theta} \alpha$. Here $K_{\theta}$ is the kernel matrix of the training data.

Hence, $\mathcal{F}$ can be re-expressed as:

$$\mathcal{F} = \left\{ f : x \mapsto \sum_{n} \alpha_n \Phi(x_n) + \beta, \beta \in \mathbb{R} \right\}$$

where $\Omega_\alpha(\theta) \triangleq \{ \alpha \in \mathbb{R}^N : \alpha^T K_{\theta} \alpha \leq R^2 \}$.

First of all, let’s upper bound the Rademacher Complexity of *SHL’s hypothesis space:

$$\tilde{R}_Q(1^T \mathcal{F}) = \frac{1}{N} E_{\sigma} \left\{ \sup_{f \in \mathcal{F}} \sum_{n} \sigma_n \sum_{b=1}^B f_b(x_n) \right\} = \frac{1}{N} E_{\sigma} \left\{ \sup_{f \in \mathcal{F}, b \in \mathbb{N}_B} \sum_{b} \sum_{n} \sigma_n f_b(x_n) \right\} = \sum_{b} \frac{1}{N} E_{\sigma} \left\{ \sup_{f \in \mathcal{F}} \sum_{n} \sigma_n f_b(x_n) \right\} = B \tilde{R}_Q(\mathcal{F}) \tag{58}$$

Next, we will upper bound $\tilde{R}_Q(\mathcal{F})$:

$$\tilde{R}_Q(\mathcal{F}) = \frac{1}{N} E_{\sigma} \left\{ \sup_{f \in \mathcal{F}} \sum_{n} \sigma_n f(x_n) \right\} = E_{\sigma} \left\{ \sup_{\alpha \in \Omega_\alpha(\theta)} \alpha^T K_{\theta} \alpha + \sup_{\beta \in \mathbb{R}} \sum_{n} \sigma_n \beta \right\} \leq \frac{R}{N} E_{\sigma} \left\{ \sqrt{\sup_{\theta \in \Omega_\theta} \theta^T \mathcal{U}} \right\} \tag{59}$$

where $\mathcal{U} \in \mathbb{R}^M$ such that $u_m \triangleq \sigma^T K_{\theta} \alpha$. The above inequality holds because of Cauchy-Schwarz inequality. Additionally, $E_{\sigma} \left\{ \sup_{\beta \in \mathbb{R}} \sum_{n} \sigma_n \beta \right\} = 0$ since $\beta$ is bounded.

By the definition of the dual norm, if $p' \triangleq \frac{p}{p-1}$, we have:

$$\sup_{\theta \in \Omega_\theta} \theta^T u = \|u\|_{p'}$$

Thus, Eq. (59) becomes:

$$\tilde{R}_Q(\mathcal{F}) \leq \frac{R}{N} E_{\sigma} \left\{ \sqrt{\|u\|_{p'}} \right\} = \frac{R}{N} E_{\sigma} \left\{ \left( \sum_{m=1}^M (\sigma^T K_{\theta} \sigma)^p \right)^{\frac{1}{p'}} \right\} \leq \frac{R}{N} \left( \sum_{m} E_{\sigma} \left\{ (\sigma^T K_{\theta} \sigma)^p \right\} \right)^{\frac{1}{p'}} \tag{60}$$

The above inequality holds because of Jensen’s Inequality. By the Lemma 5 from [53], the above expression is upper bounded by:

$$\tilde{R}_Q(\mathcal{F}) \leq \frac{R}{N} \left( \sum_{m} \left( p' \right)^{\frac{p}{p'}} \left( \text{trace} \{ K_m \} \right)^{\frac{p'}{p}} \right)^{\frac{1}{p'}} = \frac{R}{N} \left( p' \right)^{\frac{p}{p'}} \left( \sum_{m} \left( \text{trace} \{ K_m \} \right)^{\frac{p'}{p}} \right)^{\frac{1}{p'}} \quad \text{by Eq. (61)}$$

Since $k_m(x, x') \leq r^2, \forall m \in \mathbb{N}_M, x \in \mathcal{X}$:

$$\text{trace} \{ K_m \} \leq N r^2 \Rightarrow \left( \text{trace} \{ K_m \} \right)^{\frac{p'}{p}} \leq N^{\frac{p'}{p}} r^{p'} \Rightarrow \sum_{m} \left( \text{trace} \{ K_m \} \right)^{\frac{p'}{p}} \leq M^{\frac{p'}{p}} r^{p'} \tag{62}$$

Thus, combine Eq. (61) and Eq. (62), we have:

$$\tilde{R}_Q(\mathcal{F}) \leq \frac{R}{N} \left( p' \right)^{\frac{p}{p'}} \left( M^{\frac{p'}{p}} r^{p'} \right)^{\frac{1}{p'}} = R \left( \frac{p'}{p-1} \right)^{\frac{1}{p'}} \sqrt{r M^\frac{1}{p'}} \tag{63}$$

Combine Eq. (58) and Eq. (63):

$$\tilde{R}_Q(1^T \mathcal{F}) \leq BR \left( \frac{p'}{p-1} \right)^{\frac{1}{p'}} \sqrt{r M^\frac{1}{p'}} \Rightarrow \mathcal{R}_N(1^T \mathcal{F}) \leq BR \sqrt{r M^\frac{1}{p'}} \sqrt{\frac{p'}{N^3}} \tag{64}$$
Finally, combine Eq. (57) and Eq. (64), one can generate the bound provided in Theorem 3.

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