Comparison geometry of manifolds with boundary under lower $N$-weighted Ricci curvature bounds with $\varepsilon$-range

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Abstract

We study comparison geometry of manifolds with boundary under a lower $N$-weighted Ricci curvature bound for $N \in ]-\infty, 1[ \cup ]n, +\infty]$ with $\varepsilon$-range introduced by Lu-Minguzzi-Ohta [24]. We will conclude splitting theorems, and also comparison geometric results for inscribed radius, volume around the boundary, and smallest Dirichlet eigenvalue of the weighted $p$-Laplacian. Our results interpolate those for $N \in [n, +\infty]$ and $\varepsilon = 1$, and for $N \in ]-\infty, 1]$ and $\varepsilon = 0$ by the second named author.

Keywords: $N$-weighted Ricci curvature, Weighted mean curvature, Laplacian comparison theorem, Splitting theorem, Inscribed radius, Volume comparison theorem, Weighted $p$-Laplacian, Dirichlet eigenvalue

Mathematics Subject Classification (2020): Primary 53C21; Secondary 53C20.

1 Introduction

1.1 Background

We first recall the notion of the weighted Ricci curvature, and comparison geometry of manifolds without boundary under lower $N$-weighted Ricci curvature bounds.

Let $(M, g, f)$ be an $n$-dimensional weighted Riemannian manifold, namely, $(M, g)$ is an $n$-dimensional complete Riemannian manifold, and $f \in C^\infty(M)$. For $N \in ]-\infty, +\infty]$, the $N$-weighted Ricci curvature is defined as follows ([1], [2], [21]):

$$\text{Ric}_f^N := \text{Ric}_g + \nabla^2 f - \frac{df \otimes df}{N - n}.$$ 

Here if $N = +\infty$, the last term should be interpreted as the limit 0, and when $N = n$, we only consider a constant function $f$ such that $\text{Ric}_f^n := \text{Ric}_g$.

In the rest of this subsection, the boundary $\partial M$ of $M$ is assumed to be empty. In such a case, under a classical curvature condition

$$\text{Ric}_f^N \geq Kg \quad (1.1)$$

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for $K \in \mathbb{R}$ and $N \in [n, +\infty)$, comparison geometry has been developed by [6], [23], [32], [36], and so on (see also [7], [15], [16], [17], [18]). In recent years, comparison geometry for the complementary case of $N \in ]-\infty, n]$ has begun to be investigated (see e.g., [12], [13], [14], [22], [24], [25], [26], [28], [30], [33], [37], [38]). Wylike-Yeroshkin [38] have introduced a curvature condition

$$\operatorname{Ric}^1_f \geq (n - 1)\kappa e^{-\frac{N}{n-1}} g$$

(1.2)

for $\kappa \in \mathbb{R}$ in view of the study of weighted affine connection, and obtained a Laplacian comparison theorem, Bonnet-Myers type diameter comparison theorem, Bishop-Gromov type volume comparison theorem, and rigidity results for the equality cases. The first named author and Li [12] have provided a generalized condition

$$\operatorname{Ric}^N_f \geq (n - N)\kappa e^{-\frac{N}{n-1}} g$$

(1.3)

with $N \in ]-\infty, 1]$, and extended the comparison geometric results in [38].

Very recently, Lu-Minguzzi-Ohta [24] have introduced a new curvature condition that interpolates the conditions (1.1) with $K = (N - 1)\kappa$, (1.2) and (1.3). For $N \in ]-\infty, 1] \cup [n, +\infty]$, the notion of the $\varepsilon$-range has played a crucial role in their work:

$$\varepsilon = 0 \text{ for } N = 1, \quad \varepsilon \in ]-\varepsilon_0, \varepsilon_0[ \text{ for } N \neq 1, n, \quad \varepsilon \in \mathbb{R} \text{ for } N = n, \quad (1.4)$$

where we set

$$\varepsilon_0 := \frac{N - 1}{N - n},$$

and interpret it as the limit 1 when $N = +\infty$. In this $\varepsilon$-range, they have studied

$$\operatorname{Ric}^N_f \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)^2}{n-1}} g,$$

(1.5)

where $c = c_{N, \varepsilon} \in ]0, 1]$ is a positive constant defined by

$$c := \frac{1}{n-1} \left(1 - \varepsilon^2 \frac{N - n}{N - 1}\right)$$

(1.6)

if $N \neq 1$, and $c := (n - 1)^{-1}$ if $N = 1$. When $N \in [n, +\infty]$, $\varepsilon = 1$ with $c = (N - 1)^{-1}$, the curvature condition (1.3) is reduced to (1.1) with $K = (N - 1)\kappa$. Further, when $N = 1$ and $\varepsilon = 0$ with $c = (n - 1)^{-1}$, it covers (1.2), and when $N \in ]-\infty, 1]$ and $\varepsilon = \varepsilon_0$ with $c = (n - N)^{-1}$, it does (1.3). Under the condition (1.5), they have shown a Laplacian comparison theorem, and concluded a diameter bound of Bonnet-Myers type, and a volume bound of Bishop-Gromov type under density bounds. The authors [13] have examined rigidity phenomena for the equality cases in [24].

### 1.2 Setting

In this article, we focus on the case where the boundary $\partial M$ is non-empty. Under a lower Ricci curvature bound, Heintze-Karcher [8], Kasue [9], [10] has studied comparison
geometry of (unweighted) manifolds with boundary assuming a lower mean curvature bound for the boundary. The second named author [34], [35] has developed that of weighted manifolds with boundary extending the work of [8], [9], [10].

On a weighted Riemannian manifold with boundary $(M, g, f)$, the weighted mean curvature at $z \in \partial M$ is defined by

$$H_{f,z} := H_z + g(\nabla f, u_z),$$

where $H_z$ denotes the mean curvature at $z$ defined as the trace of the shape operator with respect to the unit inner normal vector $u_z$. For a function $\Lambda : \partial M \to \mathbb{R}$, we write $H_{f,\partial M} \geq \Lambda$ when $H_{f,z} \geq \Lambda(z)$ for all $z \in \partial M$. The second named author [34] has developed comparison geometry under a curvature condition

$$\text{Ric}^N_f \geq (N-1)\kappa g, \quad H_{f,\partial M} \geq (N-1)\lambda$$

for $\kappa, \lambda \in \mathbb{R}$ and $N \in [n, +\infty]$ that corresponds to (1.1). After that, inspired by the work of Wylie-Yeroshkin [38], the second named author [35] has also considered a condition

$$\text{Ric}^N_f \geq (n-1)\kappa e^{-\frac{4f}{n-1}} g, \quad H_{f,\partial M} \geq (n-1)\lambda e^{-\frac{2f}{n-1}}$$

for $N \in [-\infty, 1]$, and derived comparison geometric results.

The aim of this note is to establish comparison geometry under the curvature condition (1.5), which enables us to treat the results in [34], [35] in a unified way. Our setting is as follows: For $N \in [-\infty, 1] \cup [n, +\infty]$ and $\varepsilon \in \mathbb{R}$ in the range (1.4),

$$\text{Ric}^N_f \geq c^{-1} e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}},$$

where $c$ is defined as (1.6). When $N \in [n, +\infty]$ and $\varepsilon = 1$, the condition (1.9) is reduced to (1.7), and when $N \in [-\infty, 1]$ and $\varepsilon = 0$, it is reduced to (1.8). Under (1.9), we prove several splitting theorem (see Section 4), and also show comparison geometric results for inscribed radius (see Section 5), volume of metric neighborhoods of the boundary (see Section 6), and Dirichlet eigenvalue of the weighted $p$-Laplacian (see Section 7).

## 2 Preliminaries

In what follows, let $(M, g, f)$ be an $n$-dimensional weighted Riemannian manifold, and we always fix $N \in [-\infty, 1] \cup [n, +\infty]$ and $\varepsilon \in \mathbb{R}$ in the range (1.4). We basically use the same notation and terminology as in [35]. We also assume familiarity with basic concepts in geometry of manifolds with boundary such as the cut locus for the boundary (cf. [33], and Section 2 in [34], [35]).

In this section, as a preliminary, we recall a Laplacian comparison theorem for the distance function from a single point under the curvature condition (1.5) obtained by Lu-Minguzzi-Ohta [24], and a rigidity results for the equality case investigated in [13].
2.1 Laplacian comparison theorem from a single point

Let \( \text{Int} \, M \) stand for the interior of \( M \). For \( x \in \text{Int} \, M \), let \( U_x M \) be the unit tangent sphere at \( x \), which can be identified with the \((n-1)\)-dimensional unit sphere \((S^{n-1}, g_{S^{n-1}})\). Let \( \rho_x : M \to \mathbb{R} \) be the distance function from \( x \) defined as \( \rho_x(y) := d(x, y) \), where \( d \) denotes the Riemannian distance function on \( M \). For \( v \in U_x M \), let \( \gamma_v : [0, T] \to M \) be the unit speed geodesic with \( \gamma_v(0) = x \) and \( \dot{\gamma}_v(0) = v \). We define \( \tau : U_x M \to ]0, +\infty] \) by

\[
\tau(v) := \sup \{ \ t > 0 \mid \rho_x(\gamma_v(t)) = t, \gamma_v([0, t[) \subset \text{Int} \, M \}.
\]

Let \( s_{f,v} : [0, \tau(v)] \to [0, +\infty] \) denote a function defined by

\[
s_{f,v}(t) := \int_0^t e^{-\frac{2(1-\epsilon)f(\gamma_v(\xi))}{n-1}} \, d\xi.
\]

For \( \kappa \in \mathbb{R} \), we denote by \( s_\kappa(s) \) the solution of the equation \( \psi''(s) + \kappa \psi(s) = 0 \) with \( \psi(0) = 0 \) and \( \psi'(0) = 1 \). We define

\[
H_\kappa(s) := -c^{-1}s_\kappa'(s).
\]

The weighted Laplacian is defined by

\[
\Delta f = \Delta + g(\nabla f, \nabla \cdot),
\]

where \( \Delta \) is the Laplacian defined as the minus of the divergence of the gradient. The Laplacian comparison theorem in [24] can be stated as follows (see [24, Remark 3.10], and also [13, Theorem 2.3]):

**Theorem 2.1 ([24])** We assume \( \text{Ric}^N_x \geq c^{-1} \kappa e^{-\frac{4(1-\epsilon)f}{n-1}} g \). Let \( x \in \text{Int} \, M \) and \( v \in U_x M \). Then for all \( t \in ]0, \tau(v)[ \) we have

\[
\Delta f \rho_x(\gamma_v(t)) \geq H_\kappa(s_{f,v}(t)) e^{-\frac{2(1-\epsilon)f(\gamma_v(t))}{n-1}}.
\]  \hspace{1cm} (2.1)

2.2 Rigidity of Laplacian comparison from a single point

The authors [13] have shown the following (see [13, Lemma 2.8]):

**Lemma 2.2 ([13])** Under the same setting as in Theorem 2.1, assume that the equality in (2.1) holds at \( t_0 = 0, \tau(v) \). Choose an orthonormal basis \( \{e_v,i\}_{i=1}^n \) of \( T_x M \) with \( e_v,n = v \). Let \( \{Y_{v,i}\}_{i=1}^{n-1} \) and \( \{E_{v,i}\}_{i=1}^{n-1} \) be the Jacobi fields and parallel vector fields along \( \gamma_v \) with \( Y_{v,i}(0) = 0_x \), \( Y'_{v,i}(0) = e_v,i \) and \( E_{v,i}(0) = e_v,i \), respectively. Then the following properties hold on \([0, t_0]\):

(i) If \( N = n \), then \( f \) is constant, and

\[
Y_{v,i}(t) = s_{\kappa e^{-\frac{4(1-\epsilon)f}{n-1}}(t)} E_{v,i}(t);
\]
(ii) if $N \neq 1, n$, then
\[ \varepsilon = 0, \quad f(\gamma_v(t)) \equiv f(x), \quad Y_{v,i}(t) = s_n e^{-\frac{n(1-\varepsilon) f(x)}{n-1}}(t)E_{v,i}(t); \]

(iii) if $N = 1$, then
\[ \varepsilon = 0, \quad Y_{v,i}(t) = \exp \left( \frac{f(\gamma_v(t)) + f(x)}{n-1} \right) s_n(s_{f,v}(t)) E_{v,i}(t). \]

2.3 Laplacian comparison from a single point with bounded density

The authors [13] have also proven the following (see [13, Lemma 2.11], and also [24, Theorem 3.9]):

Lemma 2.3 ([13]) We assume
\[ \text{Ric}_f^N \geq c^{-1} e^{-\frac{4(1-\varepsilon) f}{n-1}} g, \quad (1-\varepsilon) f \leq (n-1) \delta \]
for $\delta \in \mathbb{R}$. Let $x \in \text{Int } M$ and $v \in U_x M$. Then for all $t \in ]0, \tau(v)[$ we have
\[ \Delta_f \rho_{x}(\gamma_v(t)) \geq H_n \left( e^{-2\delta t} \right) e^{-\frac{2(1-\varepsilon) f(\gamma_v(t))}{n-1}}. \]  \hspace{1cm} (2.2)

Remark 2.4 Assume that the equality in (2.2) holds at $t_0$. Then the equality in (2.1) holds on $]0, t_0[$, and $(1-\varepsilon) f \circ \gamma_v = (n-1) \delta$ on $[0, t_0]$ (see [13, Remark 2.12]).

3 Laplacian

In this section, we present a Laplacian comparison theorem for the distance function from the boundary, which is a key ingredient of the proof of our main theorems.

3.1 Riccati inequality

Let $\rho_{\partial M} : M \rightarrow \mathbb{R}$ be the distance function from $\partial M$ defined as $\rho_{\partial M}(x) := d(x, \partial M)$. For each $z \in \partial M$, we denote by $\gamma_z : [0, T] \rightarrow M$ the unit speed geodesic with $\gamma_z(0) = z$ and $\dot{\gamma}_z(0) = u_z$. We define a function $\tau : \partial M \rightarrow ]0, +\infty[$ as
\[ \tau(z) := \sup \{ t > 0 \mid \rho_{\partial M}(\gamma_z(t)) = t \}. \]

We first show the following Riccati inequality:

Lemma 3.1 For all $t \in ]0, \tau(z)[$ we have
\[ \left( e^{-\frac{2(1-\varepsilon) f}{n-1}} \Delta_f \rho_{\partial M}(\gamma_z(t)) \right)' \geq e^{-\frac{2(1-\varepsilon) f(\gamma_z(t))}{n-1}} \text{Ric}_f^N(\gamma_z(t)) + c e^{-\frac{2(1-\varepsilon) f(\gamma_z(t))}{n-1}} \left( e^{-\frac{2(1-\varepsilon) f}{n-1}} \Delta_f \rho_{\partial M}(\gamma_z(t)) \right)^2. \]  \hspace{1cm} (3.1)
Proof. In the case of $N = n$, the function $f$ is constant, and the desired inequality is well-known. In the case of $N = 1$ with $\varepsilon = 0$, (3.1) has been obtained in [35] (see [35, Lemma 3.1]). Hence we may assume $N \neq 1, n$.

Set $h_{f,v} := (\Delta f \rho_{\partial M}) \circ \gamma_z$ and $f_z := f \circ \gamma_z$. By applying the well-known Bochner formula to the distance function $\rho_{\partial M}$, and by the Cauchy-Schwarz inequality,

\[
0 = \text{Ric}^\infty_f(\gamma_z(t)) + \|\nabla^2 \rho_{\partial M}\|^2(\gamma_z(t)) - g(\nabla \Delta f \rho_{\partial M}, \nabla \rho_{\partial M})(\gamma_z(t))
\]

\[
\geq \text{Ric}^N_f(\gamma_z(t)) + \frac{f_z^2(t)^2}{N - n} + \frac{(h_{f,z}(t) - f_z^2(t))^2}{n - 1} - h_{f,z}(t)
\]

\[
= \text{Ric}^N_f(\gamma_z(t)) + c h_{f,z}^2(t) - e^{-2(1-\varepsilon)f_z(t)/n} \left( e^{2(1-\varepsilon)f_z(t)/n - 1} h_{f,z}(t) \right)^{1/2}
\]

\[
+ \frac{1}{n - 1} \left( \sqrt{\frac{N - 1}{N - n}} f_z'(t) - \varepsilon \sqrt{\frac{N - n}{N - 1}} h_{f,z}(t) \right)^2
\]

\[
\geq \text{Ric}^N_f(\gamma_z(t)) + c h_{f,z}^2(t) - e^{-2(1-\varepsilon)f_z(t)/n} \left( e^{2(1-\varepsilon)f_z(t)/n - 1} h_{f,z}(t) \right)^{1/2}.
\]

We arrive at the desired inequality (3.1). \hfill \square

Remark 3.2 When $N \neq 1, n$, we assume that the equality in (3.1) holds at $t_0 \in ]0, \tau(z)[$. Then the equality in the Cauchy-Schwarz inequality in (3.2) holds; in particular,

\[
\nabla^2 \rho_{\partial M} = -\frac{\Delta \rho_{\partial M}}{n - 1} g
\]

on the orthogonal complement of $\nabla \rho_{\partial M}$ in $T_{\gamma_z(t_0)}M$. Moreover,

\[
\sqrt{\frac{N - 1}{N - n}} f_z'(t_0) - \varepsilon \sqrt{\frac{N - n}{N - 1}} h_{f,z}(t_0) = 0
\]

since the equality in (3.3) holds. In particular, if $\varepsilon = 0$, then $f_z'(t_0) = 0$, and if $\varepsilon \neq 0$,

\[
h_{f,z}(t_0) = \varepsilon^{-1} \frac{N - 1}{N - n} f_z'(t_0).
\]

3.2 Laplacian comparison theorem

For $\kappa, \lambda \in \mathbb{R}$, we denote by $s_{\kappa,\lambda}(s)$ a unique solution to the Jacobi equation $\psi''(s) + \kappa \psi(s) = 0$ with $\psi(0) = 1$ and $\psi'(0) = -\lambda$. We set

\[
C_{\kappa,\lambda} := \inf \{ s > 0 \mid s_{\kappa,\lambda}(s) = 0 \}.
\]

Notice that $C_{\kappa,\lambda}$ is finite if and only if either (1) $\kappa > 0$; (2) $\kappa = 0$ and $\lambda > 0$; or (3) $\kappa < 0$ and $\lambda > \sqrt{\lvert \kappa \rvert}$, and in this case, we say that $\kappa$ and $\lambda$ satisfy the ball-condition. We also note that they satisfy the ball-condition if and only if there is a closed ball $B^n_{\kappa,\lambda}$ in the space form with constant curvature $\kappa$ whose boundary has constant mean curvature $(n - 1)\lambda$. The radius of $B^n_{\kappa,\lambda}$ is given by $C_{\kappa,\lambda}$. We set

\[
H_{\kappa,\lambda}(s) := -c^{-1} s_{\kappa,\lambda}'(s) / s_{\kappa,\lambda}(s),
\]
which enjoys the following Riccati equation:

\[ H_{k,\lambda}'(s) = c^{-1} \kappa + c H_{k,\lambda}^2(s). \]  

(3.4)

Let us define functions \( s_{f,z} : [0, \tau(z)] \to [0, \tau_f(z)] \) and \( \tau_f : \partial M \to ]0, +\infty[ \) by

\[ s_{f,z}(t) := \int_{0}^{t} e^{-\frac{2(1-e^{-\epsilon t})}{n-1}} d\xi, \quad \tau_f(z) := s_{f,z}(\tau(z)). \]

Let \( t_{f,z} : [0, \tau_f(z)] \to [0, \tau(z)] \) be the inverse function of \( t_{f,z} \). Our Laplacian comparison theorem is stated as follows:

**Theorem 3.3** Assume

\[ \text{Ric}_N^\gamma(\dot{\gamma}_z(t)) \geq c^{-1} \kappa e^{-\frac{4(1-e^{-\epsilon t})}{n-1}}, \quad H_{f,z} \geq c^{-1} \lambda e^{-\frac{2(1-e^{-\epsilon t})}{n-1}} \]

(3.5)

for all \( t \in ]0, \tau(z)[ \). Then for all \( t \in ]0, \tau(z)[ \) with \( s_{f,z}(t) \in ]0, \min\{\tau_f(z), C_{k,\lambda}\}[ \),

\[ \Delta_{f,\rhoM}(\gamma_z(t)) \geq H_{k,\lambda}(s_{f,z}(t)) e^{-\frac{2(1-e^{-\epsilon t})}{n-1}}. \]

(3.6)

**Proof.** We define \( F_z : [0, \tau(z)] \to \mathbb{R} \) and \( \hat{F}_z : [0, \tau_f(z)] \to \mathbb{R} \) by

\[ F_z := (e^{\frac{2(1-e^{-\epsilon t})}{n-1}}) \Delta_{f,\rhoM} \circ \gamma_z, \quad \hat{F}_z := F_z \circ t_{f,z}. \]

From (3.1) and the curvature assumption, for all \( s \in ]0, \tau_f(z)[ \),

\[ \hat{F}_z'(s) = F_z'(t_{f,z}(s)) e^{\frac{2(1-e^{-\epsilon t})}{n-1}}(\tau_{f,z}(s)) \]

\[ \geq \text{Ric}_N^\gamma(\dot{\gamma}_z(t_{f,z}(s))) e^{\frac{4(1-e^{-\epsilon t})}{n-1}}(\tau_{f,z}(s)) + c F_z^2(t_{f,z}(s)) \]

\[ \geq c^{-1} \kappa + c \hat{F}_z^2(s). \]

The Riccati equation (3.4) implies that for all \( s \in ]0, \min\{\tau_f(z), C_{k,\lambda}\}[ \),

\[ \hat{F}_z'(s) - H_{k,\lambda}'(s) \geq c \left( \hat{F}_z^2(s) - H_{k,\lambda}^2(s) \right). \]

Let us consider a function \( G_{k,\lambda,z} : [0, \min\{\tau_f(z), C_{k,\lambda}\}[ \to \mathbb{R} \) by

\[ G_{k,\lambda,z} := s_{k,\lambda}^2(\hat{F}_z - H_{k,\lambda}). \]

From (3.2) it follows that

\[ G_{k,\lambda,z}' = 2 s_{k,\lambda} s_{k,\lambda}'(\hat{F}_z - H_{k,\lambda}) + s_{k,\lambda}^2(\hat{F}_z' - H_{k,\lambda}') \]

\[ \geq 2 s_{k,\lambda} s_{k,\lambda}'(\hat{F}_z - H_{k,\lambda}) + c s_{k,\lambda}^2(\hat{F}_z^2 - H_{k,\lambda}^2) \]

\[ = c s_{k,\lambda}^2(\hat{F}_z - H_{k,\lambda}) \geq 0. \]

Since \( G_{k,\lambda,z}(s) \to e^{\frac{2(1-e^{-\epsilon t})}{n-1}} H_{f,z} - c^{-1} \lambda \) as \( s \to 0 \), the function \( G_{k,\lambda,z} \) is non-negative; in particular, \( \hat{F}_z \geq H_{k,\lambda} \) holds on \( ]0, \min\{\tau_f(z), C_{k,\lambda}\}[ \). This proves (3.6). \( \square \)
Remark 3.4 We assume that the equality in (3.6) holds at $t_0$. Then $G_{\kappa,\lambda,z}(s_{f,z}(t_0)) = 0$. From $G'_{\kappa,\lambda,z} \geq 0$ it follows that $G_{\kappa,\lambda,z} = 0$ on $[0, s_{f,z}(t_0)]$; in particular, the equality in (3.1) holds on $[0, t_0]$ under (3.5) (see Remark 3.2).

Remark 3.5 Theorem 3.3 has been obtained by the second named author [34] and [35] under (1.7) and (1.8), respectively (see [34, Lemma 3.3], [35, Lemma 3.3]).

Theorem 3.3 leads us to the following:

Lemma 3.6 Let $\kappa$ and $\lambda$ satisfy the ball-condition. Assume

$$\text{Ric}_f^{N} (\dot{\gamma}_z(t)) \geq c^{-1} \kappa e^{-\frac{4}{n-1}(1-\varepsilon)f(\gamma(z))}, \quad H_{f,z} \geq c^{-1} \lambda e^{-\frac{2}{n-1}(1-\varepsilon)f(z)}$$

for all $t \in ]0, \tau(z)[$. Then we have $\tau_f(z) \leq C_{\kappa,\lambda}$.

Moreover, if $(1-\varepsilon)f \circ \gamma_z \leq (n-1)\delta$ on $]0, \tau(z)[$ for $\delta \in \mathbb{R}$, then

$$\tau(z) \leq C_{\kappa \epsilon^{-4}, \lambda \epsilon^{-2}}.$$

Proof. One can prove it by using Theorem 3.3 instead of [35, Lemma 3.3] along the line of the proof of [35, Lemma 3.5]. We omit the proof. □

Remark 3.7 Due to Lemma 3.6 one can drop the restriction $s_{f,z}(t) \in ]0, \min\{\tau_f(z), C_{\kappa,\lambda}\}[$ in Theorem 3.3.

3.3 RIGIDITY OF LAPLACIAN COMPARISON

Let $A_{u_z}$ stand for the shape operator on $\partial M$ with respect to $u_z$. We examine the equality case of Theorem 3.3.

Lemma 3.8 Under the same setting as in Theorem 3.3, assume that the equality in (3.6) holds at $t_0 \in ]0, \tau(z)[$. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z \partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ and $\{E_{z,i}\}_{i=1}^{n-1}$ be the Jacobi fields and parallel vector fields along $\gamma_z$ with $Y_{z,i}(0) = e_{z,i}, Y'_{z,i}(0) = -A_{u_z} e_{z,i}$ and $E_{z,i}(0) = e_{z,i}$, respectively. Then the following hold on $[0, t_0]$:

(i) If $N = n$, then $f$ is constant, and

$$Y_{z,i}(t) = s_{\kappa \epsilon^{-N/n-1}, \lambda \epsilon^{-N/n-1}}(t) E_{z,i}(t);$$

(ii) if $N \neq 1, n$, then

$$f(\gamma_z(t)) = f(z) - \varepsilon \frac{N-n}{N-1} c^{-1} \log s_{\kappa,\lambda}(s_{f,z}(t)), \quad Y_{z,i}(t) = s_{\kappa \epsilon^{-N/n-1}}(s_{f,z}(t)) E_{z,i}(t);$$
Remarks 3.2 and 3.4). If \( \varepsilon \) possess the desired formula. Let us consider the case of well-known for the equality in (3.6) holds. From direct computations and (3.7), we deduce the second named author [35] (see [35, Lemmas 3.8 and 3.9]). We may assume \( N \neq 1, n \).

On the other hand, which is the desired property.

Proof. If \( N = n \), then \( f \) is constant by the definition, and the rigidity of Jacobi fields is well-known for \( \varepsilon = 1 \). If \( N = 1 \) with \( \varepsilon = 0 \), then the desired assertion has been proved by the second named author [35] (see [35, Lemmas 3.8 and 3.9]).

We first show the rigidity of \( f \). We set \( h_{f,\varepsilon} := (\Delta f \rho_{\partial M}) \circ \varepsilon \) and \( f_{\varepsilon} := f \circ \varepsilon \). Since the equality in (3.6) holds at \( t_0 \), the equalities in (3.1) and (3.6) also hold on \([0, t_0]\) (see Remarks 3.2 and 3.4). If \( \varepsilon = 0 \), then \( f_{\varepsilon}(t) = 0 \) for each \( t \in [0, t_0] \), and hence we already possess the desired formula. Let us consider the case of \( \varepsilon \neq 0 \). Then \( h_{f,\varepsilon}(t) \) is equal to

\[
\varepsilon^{-1} N - 1 \frac{N}{n} f_{\varepsilon}(t) = H_{\varepsilon,\lambda}(s_{f,\varepsilon}(t)) e^{-\frac{2(1-\varepsilon) f_{\varepsilon}(t)}{n-1}}.
\]

This implies

\[
f_{\varepsilon}(t) = f(z) - \varepsilon N \frac{N - n}{n - 1} c^{-1} \log s_{\varepsilon,\lambda}(s_{f,\varepsilon}(t)),
\]

which is the desired property.

We next investigate the rigidity of Jacobi fields. We see

\[
\nabla^2 \rho_{\partial M} = -\frac{\Delta \rho_{\partial M}}{n - 1} g
\]

on the orthogonal complement of \( \nabla \rho_{\partial M} \) in \( T_{\gamma_z}M \). Set

\[
\varphi := -\frac{\Delta \rho_{\partial M}}{n - 1}, \quad \varphi_z := \varphi \circ \gamma_z.
\]

The radial curvature equation (see e.g., [31, Corollary 3.2.10]) yields

\[
R(E_{z,i}, \gamma_z) \gamma_z = -(\varphi_z' + \varphi_z^2) E_{z,i}.
\]

On the other hand,

\[
\varphi_z(t) = \frac{1}{n - 1} \left( f_z'(t) - H_{\varepsilon,\lambda}(s_{f,z}(t)) e^{-\frac{2(1-\varepsilon) f_{\varepsilon}(t)}{n-1}} \right) = \frac{1}{n - 1} \left( f_z(t) + \log s_{\varepsilon,\lambda}^{-1}(s_{f,z}(t)) \right),
\]

since the equality in (3.6) holds. From direct computations and (3.7), we deduce

\[
\varphi_z'(t) + \varphi_z^2(t) = \frac{1}{n - 1} \left( \frac{f_z''(t)}{N - n} - c^{-1} \kappa e^{-\frac{4(1-\varepsilon) f_{\varepsilon}(t)}{n-1}} \right)
\]

\[
+ \frac{1}{n - 1} \left( \sqrt{\frac{N - 1}{N - n}} f_z'(t) - \varepsilon \sqrt{\frac{N - n}{N - 1}} H_{\varepsilon,\lambda}(s_{f,z}(t)) e^{-\frac{2(1-\varepsilon) f_{\varepsilon}(t)}{n-1}} \right)^2
\]

\[
= \frac{1}{n - 1} \left( f_z''(t) - \frac{f_z'(t)^2}{N - n} - c^{-1} \kappa e^{-\frac{4(1-\varepsilon) f_{\varepsilon}(t)}{n-1}} \right).
\]
where

$$F_{\kappa,\lambda,z}(t) := \exp\left(\frac{f_z(t) - f(z)}{n - 1}\right) s_{\kappa,\lambda}^{-1}(s_{f,z}(t)). \quad (3.11)$$

The equality (3.10) together with (3.9) tells us that $Y_{z,i} = F_{\kappa,\lambda,z} E_{z,i}$. Substituting (3.8) into (3.11), we arrive at

$$F_{\kappa,\lambda,z}(t) = s_{\kappa,\lambda}^{-1}(1 - \frac{\kappa - n}{\kappa - 1})(s_{f,z}(t)).$$

This completes the proof. \(\square\)

**Remark 3.9** Let us compare the rigidity phenomenon of Lemma 3.8 for $N \neq 1, n$ with that of Lemma 2.2. In such a case, the equality in (2.1) holds only when $\varepsilon = 0$. On the other hand, it is possible that the equality in (3.6) holds for every $\varepsilon$.

### 3.4 LAPLACIAN COMPARISON WITH BOUNDED DENSITY

We now investigate Laplacian comparisons under a boundedness of density. Let us say that $\kappa$ and $\lambda$ satisfy the **convex-ball-condition** if they satisfy the ball-condition and $\lambda \geq 0$. We say that $\kappa$ and $\lambda$ satisfy the **monotone-condition** when $H_{\kappa,\lambda} \geq 0$ and $H'_{\kappa,\lambda} \geq 0$ on $[0, C_{\kappa,\lambda}]$. We see that they satisfy the monotone-condition if and only if either (1) they satisfy the convex-ball-condition; or (2) $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$. For $\kappa$ and $\lambda$ satisfying the monotone-condition, if $\kappa = 0$, $\lambda = 0$, then $H_{\kappa,\lambda} = 0$ on $[0, +\infty]$; otherwise, $H_{\kappa,\lambda} > 0$ on $[0, C_{\kappa,\lambda}]$. We also say that they satisfy the **weakly-monotone-condition** if $H'_{\kappa,\lambda} \geq 0$ on $[0, C_{\kappa,\lambda}]$. Notice that they satisfy the weakly-monotone-condition if and only if either (1) $\kappa \geq 0$; or (2) $\kappa < 0$ and $|\lambda| \geq \sqrt{|\kappa|}$. In particular, if $\kappa$ and $\lambda$ satisfy the ball-condition, then they also satisfy the weakly-monotone-condition. For $\kappa$ and $\lambda$ satisfying the weakly-monotone-condition, if $\kappa \leq 0$ and $|\lambda| = \sqrt{|\kappa|}$, then $H_{\kappa,\lambda} = n - 1|\lambda|$ on $[0, +\infty]$; otherwise, $H'_{\kappa,\lambda} > 0$ on $[0, C_{\kappa,\lambda}]$. From the same calculation as in the proof of [35, Lemma 4.1], we can derive the following estimates from Theorem 3.3.

**Lemma 3.10** Let $\kappa$ and $\lambda$ satisfy the weakly-monotone-condition. We assume that

$$\text{Ric}^N_f(\gamma_z'(t)) \geq (n - 1)\kappa e^{-\frac{4(1 - \varepsilon)f(\gamma_z(t))}{n - 1}}, \quad H_{f,z} \geq (n - 1)\lambda e^{-\frac{4(1 - \varepsilon)f(\gamma_z(t))}{n - 1}}$$

for all $t \in [0, \tau(z)]$. Suppose additionally that $(1 - \varepsilon)f \circ \gamma_z \leq (n - 1)\delta$ on $[0, \tau(z)]$ for $\delta \in \mathbb{R}$. Then for all $t \in [0, \tau(z)]$ we have

$$\Delta_f \rho_{BM}(\gamma_z(t)) \geq H_{\kappa,\lambda}(e^{-2\delta} t) e^{-\frac{4(1 - \varepsilon)f(\gamma_z(t))}{n - 1}}. \quad (3.12)$$

Moreover, if $\kappa$ and $\lambda$ satisfy the monotone-condition, then

$$\Delta_f \rho_{BM}(\gamma_z(t)) \geq H_{\kappa,\lambda}(e^{-2\delta} t) e^{-2\delta}. \quad (3.13)$$

**Remark 3.11** Assume that the equality in (3.12) holds at $t_0$. Then the equality in (3.6) also holds (see Lemma 3.8). Moreover, if either (1) $\kappa > 0$; or (2) $\kappa \leq 0$ and $|\lambda| > \sqrt{|\kappa|}$, then $(1 - \varepsilon)f \circ \gamma_z = (n - 1)\delta$ on $[0, t_0]$ (cf. [35, Remark 4.2]).
Remark 3.12 Assume that the equality in (3.13) holds at $t_0$ under the monotone condition. Then the equality in (3.12) holds (see Remark 3.11). Moreover, if either (1) $\kappa$ and $\lambda$ satisfy the convex-ball-condition; or (2) $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$, then $(1 - \varepsilon)(f \circ \gamma_z)(t_0) = (n-1)\delta$ (cf. [35, Remark 4.3]).

For $p \in ]1, +\infty[$, the weighted $p$-Laplacian is defined by

$$\Delta_{f,p} := -e^f \text{div} \left( e^{-f}||\nabla \cdot||^{p-2} \nabla \cdot \right).$$

By the same calculation and argument as in the proof of [35, Lemma 4.4, Proposition 4.6], we have the following assertion:

**Proposition 3.13** Let $p \in ]1, +\infty[$. Let $\kappa$ and $\lambda$ satisfy the monotone-condition. Assume

$$\text{Ric}^N \geq c^{-1}\kappa e^{-\frac{4(1-\varepsilon)f}{n-1}}g, \quad H_{f,\partial M} \geq c^{-1}\lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}, \quad (1 - \varepsilon)f \leq (n-1)\delta$$

for $\delta \in \mathbb{R}$. Set $\rho_{\partial M,\delta} := e^{-2\delta}\rho_{\partial M}$. Let $\varphi : [0, +\infty[ \to \mathbb{R}$ be a monotone increasing smooth function. Then we have

$$\Delta_{f,p} (\varphi \circ \rho_{\partial M,\delta}) \geq -e^{-2p\delta} \left[ ((\varphi')^{p-1})' - H_{\kappa,\lambda}(\varphi')^{p-1} \right] \circ \rho_{\partial M,\delta}$$

in the distribution sense.

### 3.5 Laplacian Comparison with Radial Density

Next, we are concerned with the case where $f$ is $\partial M$-radial (i.e., there is a smooth function $\phi_f : [0, +\infty[ \to \mathbb{R}$ such that $f = \phi_f \circ \rho_{\partial M}$). Consider a function $\rho_{\partial M,f} : M \to \mathbb{R}$ defined by

$$\rho_{\partial M,f}(x) := \inf_{z \in \partial M} \int_0^{\rho_{\partial M}(x)} e^{-\frac{2(1-\varepsilon)f(\gamma_z(\xi))}{n-1}} d\xi,$$

where the infimum is taken over all foot points $z \in \partial M$ of $x$ (i.e., $d(x, z) = \rho_{\partial M}(x)$). A similar calculation to the proof of [35, Lemma 4.9, Proposition 4.10] yields the following inequality. We omit the proof.

**Proposition 3.14** Let $p \in ]1, +\infty[$. We assume

$$\text{Ric}^N \geq c^{-1}\kappa e^{-\frac{4(1-\varepsilon)f}{n-1}}g, \quad H_{f,\partial M} \geq c^{-1}\lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}.$$

Suppose that $f$ is $\partial M$-radial. Let $\varphi : [0, +\infty[ \to \mathbb{R}$ denote a monotone increasing smooth function. Then we have

$$\Delta_{\left(1 - \frac{2p(1-\varepsilon)}{n-1}\right)f,p} (\varphi \circ \rho_{\partial M,f}) \geq -e^{-\frac{2p(1-\varepsilon)}{n-1}} \left[ ((\varphi')^{p-1})' - H_{\kappa,\lambda}(\varphi')^{p-1} \right] \circ \rho_{\partial M,f}$$

in the distribution sense.
4 Splitting

Hereafter, we introduce our main theorems. Once the Laplacian comparison is established, we can prove the main results by using them along the line of the proof of the corresponding results in [35] under (1.8). We will just present their statements, and outline or omit the proof.

In this section, we show several splitting theorems. Our first main result is the following splitting theorem, which has been originally proven by Kasue [9], Croke-Kleiner [5] in the unweighted case (see [9, Theorem C], [5, Theorem 2]). We will denote by \( g_{\partial M} \) the induced Riemannian metric on \( \partial M \).

**Theorem 4.1** Let \( \kappa \leq 0 \) and \( \lambda := \sqrt{|\kappa|} \). We assume

\[
\text{Ric}^N \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}} .
\]

Suppose that \((1-\varepsilon)f\) is bounded from above. If we have \( \tau(z_0) = \infty \) for some \( z_0 \in \partial M \), then \( M \) is diffeomorphic to \([0, +\infty[ \times \partial M \), and the following properties hold:

(i) If \( N = n \), then \( f \) is constant, and

\[
g = dt^2 + s^2 \kappa e^{-\frac{4(1-\varepsilon)f}{n-1} s} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1} s} \ g_{\partial M} ;
\]

(ii) If \( N \neq 1, n \), then for any \( z \in \partial M \)

\[
f(\gamma_z(t)) = f(z) - \varepsilon \frac{N-n}{N-1} c^{-1} \log s_{\kappa,\lambda}(s_{f,z}(t)), \quad g = dt^2 + s^2 \kappa e^{-1} \left(1-\varepsilon\frac{N-n}{N-1} \right) s_{f,z}(t) \ g_{\partial M} ;
\]

(iii) If \( N = 1 \), then

\[
\varepsilon = 0, \quad g = dt^2 + \exp \left( 2 \frac{f(\gamma_z(t)) - f(z)}{n-1} \right) s^2_{\kappa,\lambda}(s_{f,z}(t)) \ g_{\partial M} .
\]

**Proof.** We only sketch the proof. If \( N = n \), then the desired statement follows from [9], [5]. If \( N = 1 \), then it has been obtained by the second named author [35] (see [35, Theorem 1.1]). Thus it is enough to discuss the case of \( N \neq 1, n \).

The proof is similar to that of \( N = 1 \) in [35]. For the statement that \( M \) is diffeomorphic to \([0, +\infty[ \times \partial M \), we can prove it only by replacing the role of [35, Lemmas 2.9 and 3.3] with Lemma 2.3 and Theorem 3.3 along the line of the proof of [35, Theorem 1.1]. Then the equality in Theorem 3.3 holds over \( M \). Lemma 3.8 (ii) leads us to the conclusion. □

For \( \kappa > 0 \) and \( \lambda < 0 \), we set

\[
D_{\kappa,\lambda} := \inf \{ s > 0 | s'_{\kappa,\lambda}(s) = 0 \} .
\]

We also obtain the following splitting theorem, which has been formulated by Kasue [9] in the unweighted setting (see [9, Theorem B]):
Theorem 4.2 Let $\kappa > 0$. We assume
\[ \text{Ric}_f^N \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}, \quad (1-\varepsilon)f \leq (n-1)\delta \]
for $\delta \in \mathbb{R}$. Let $\partial M$ be disconnected, and let $\{\partial M_i\}_{i=1,2,...}$ denote the connected components of $\partial M$. Let $\partial M_1$ be compact. Then $\lambda < 0$, and
\[ \inf_{i=2,3,...} d(\partial M_1, \partial M_i) \leq 2D_\kappa e^{-4\delta_\kappa e^{-28}}, \lambda e^{-2(1-\varepsilon)f} (n-1)\delta \]
Moreover, if the equality holds, then $M$ is diffeomorphic to $[0, 2D_\kappa e^{-4\delta_\kappa e^{-28}}] \times \partial M_1$, and
\[ (1-\varepsilon)f = (n-1)\delta, \quad g = dt^2 + s^2 \kappa e^{-4\delta_\kappa e^{-28}} (t) g_{\partial M_1}. \]

Proof. The second named author [35] has proved this result when $N = 1$ (see [35, Theorem 5.8]). Along the line of its proof, we can prove the desired assertion by using (3.12) instead of [35, (4.1)] (see also Remark 3.11). \hfill \Box

Remark 4.3 The second named author [34] has shown Theorems 4.1 and 4.2 under the condition (1.7) with $N \in [n, +\infty]$ and $\varepsilon = 1$ (see [34, Theorems 1.4 and 6.14]).

5 INSCRIBED RADIUS

We next study the inscribed radius $\text{InRad} M$ which is defined as the supremum of the distance function $\rho_{\partial M}$ from the boundary over $M$. Let us consider a conformally deformed Riemannian metric
\[ g_f := e^{-\frac{4(1-\varepsilon)f}{n-1}} g. \]
Let $\rho_{\partial M}$ and $\text{InRad}_f M$ stand for the distance function from the boundary and the inscribed radius induced from $g_f$, respectively.

Theorem 5.1 Let $\kappa$ and $\lambda$ satisfy the ball-condition. We assume
\[ \text{Ric}_f^N \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}. \]
Then we have
\[ \text{InRad}_f M \leq C_{\kappa,\lambda}. \]
If $\rho_{\partial M}^g (x_0) = C_{\kappa,\lambda}$ for some $x_0 \in M$, then $M$ is diffeomorphic to a closed ball centered at $x_0$, and the following hold:

(i) If $N = n$, then $f$ is constant, and
\[ g = dt^2 + s^2 \kappa e^{-\frac{4(1-\varepsilon)f}{n-1}} (t) g_{S^{n-1}}; \]
(ii) if $N \neq 1, n$, then $f$ is constant, and
\[ \varepsilon = 0, \quad g = dt^2 + s^2 \kappa e^{-\frac{4f}{n-1}} (t) g_{S^{n-1}}; \]
(iii) if \( N = 1 \), then \( f \) is radial with respect to \( x_0 \), and

\[
\varepsilon = 0, \quad g = dt^2 + \exp \left( \frac{2f(\gamma_v(t)) + f(x_0)}{n - 1} \right) s_k^2(s_{f,v}(t))g_{S^{n-1}},
\]

here \( \gamma_v : [0, \rho_{\partial M}(x_0)] \to M \) denotes the geodesic with \( \gamma_v(0) = x_0 \) and \( \dot{\gamma}_v(0) = v \).

**Proof.** Let us sketch the proof. This is a weighted version of the result by Kasue \cite{12} (see \cite[Theorem A]{12}). The result for \( N = n \) can be directly derived from \cite{12}. Furthermore, the second named author \cite{35} has shown Theorem 5.1 when \( N = 1 \) (see \cite[Theorem 1.2]{35}). We may assume \( N \neq 1, n \).

Similarly to Theorem 4.1, we can refer to the argument of \cite{35}. In the proof of the claim that \( M \) is diffeomorphic to a closed ball, the key point is to show the subharmonicity of the function \( \rho_{x_0} + \rho_{\partial M} \). This is done by replacing the role of \cite[Lemmas 2.8 and 3.3]{35} with Theorems 2.1 and 3.3 along the line of the proof of \cite[Theorem 1.2]{35}. Then the equality in Theorem 2.1 holds on \( M \), and Lemma 2.2 implies the desired conclusion. \( \square \)

**Remark 5.2** Under the condition (1.7) with \( N \in [n, +\infty[ \) and \( \varepsilon = 1 \), Li-Wei \cite{20} have proved Theorem 5.1 when \( \kappa = 0 \) (see \cite[Theorem 4]{20}). In \cite{19}, they also have done when \( \kappa < 0 \) (see \cite[Theorem 1.2]{19}). Finally, the second named author \cite{34} has done for all \( \kappa \) and \( \lambda \) satisfying the ball-condition (see \cite[Theorems 1.1]{34}). We also refer to \cite{4} on the work in non-smooth setting.

Under a boundedness of \( f \), we conclude the following:

**Theorem 5.3** Let \( \kappa \) and \( \lambda \) satisfy the ball-condition. We assume

\[
\text{Ric}^N_{f} \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}, \quad (1 - \varepsilon)f \leq (n - 1)\delta
\]

for \( \delta \in \mathbb{R} \). Then we have

\[
\text{InRad} M \leq C_{\kappa \lambda e^{-4\varepsilon} e^{-2\delta}}.
\]

If we have \( \rho_{\partial M}(x_0) = C_{\kappa \lambda e^{-4\varepsilon} e^{-2\delta}} \) for some \( x_0 \in M \), then \( (1 - \varepsilon)f = (n - 1)\delta \) on \( M \), and the following properties hold:

(i) If \( N = n \), then \( M \) is isometric to \( B^n_{\kappa \lambda e^{-4\varepsilon} e^{-2\delta}} \);

(ii) if \( N \neq n \), then \( \varepsilon = 0 \), and \( M \) is isometric to \( B^n_{\kappa \lambda e^{-4\varepsilon} e^{-2\delta}} \).

**Proof.** The second named author \cite{35} has proved this result when \( N = 1 \) (see \cite[Theorem 6.4]{35}). Along the line of its proof, one can prove the claim by using Lemma 2.3 and (3.12) instead of \cite[Lemma 2.9 and (4.1)]{35} (see also Remark 2.4). \( \square \)

6 Volume

This section is devoted to the study of volume comparisons.
6.1 Volume elements

For \( z \in \partial M \) and \( t \in ]0, \tau(z)[ \), let \( \theta(t, z) \) be the volume element of the \( t \)-level surface of \( \rho \partial M \) at \( \gamma_z(t) \). For \( s \in ]0, \tau_f(z)[ \), we set

\[
\theta_f(t, z) := e^{-f(\gamma_z(t))} \theta(t, z), \quad \hat{\theta}_f(s, z) := \theta_f(t_f(s), z).
\]

By Theorem 3.3 and the same calculation as in [35, Lemma 7.1], we obtain:

**Lemma 6.1** Assume

\[
\text{Ric}_N^f(\dot{\gamma}_z(t)) \geq c^{-1} \kappa e^{-\frac{2(1-\varepsilon)f(\gamma_z(t))}{n-1}}, \quad H_{f,z} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f(z)}{n-1}},
\]

for all \( t \in ]0, \tau(z)[ \). Then for all \( s_1, s_2 \in ]0, \tau_f(z)[ \) with \( s_1 \leq s_2 \)

\[
\hat{\theta}_f(s_2, z) \leq \frac{g_{\kappa, \lambda}^{c^{-1}}(s_2)}{g_{\kappa, \lambda}^{c^{-1}}(s_1)} \theta_f(s_1, z).
\]

In particular, for all \( s \in ]0, \tau_f(z)[ \) we have

\[
\hat{\theta}_f(s, z) \leq e^{-f(z)} g_{\kappa, \lambda}^{c^{-1}}(s).
\]

Furthermore, in virtue of (3.13), we have the following (cf. [35, Lemma 7.3]):

**Lemma 6.2** Let \( \kappa \) and \( \lambda \) satisfy the monotone-condition. Assume

\[
\text{Ric}_N^f(\dot{\gamma}_z(t)) \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)f(\gamma_z(t))}{n-1}}, \quad H_{f,z} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f(z)}{n-1}}, \quad (1-\varepsilon)f \circ \gamma_z \leq (n-1)\delta
\]

on \( (0, \tau(z)) \). Then for all \( t_1, t_2 \in ]0, \tau(z)[ \) with \( t_1 \leq t_2 \) we have

\[
\frac{\theta_f(t_2, z)}{\theta_f(t_1, z)} \leq \frac{g_{\kappa, \lambda}^{c^{-1}}e^{-4\delta}e^{-2\delta}(t_2)}{g_{\kappa, \lambda}^{c^{-1}}e^{-4\delta}e^{-2\delta}(t_1)}.
\]

In particular, for all \( t \in ]0, \tau(z)[ \) we have

\[
\theta_f(t, z) \leq e^{-f(z)} g_{\kappa, \lambda}^{c^{-1}}e^{-4\delta}e^{-2\delta}(t).
\]

6.2 Volume comparisons

We define

\[
m_f := e^{-f}v_g, \quad m_{f, \partial M} := e^{-f_{\partial M}}v_{g_{\partial M}},
\]

where \( v_g \) and \( v_{g_{\partial M}} \) are the Riemannian volume measure on \( M \) and \( \partial M \) induced from \( g \) and \( g_{\partial M} \), respectively. For \( r > 0 \) we set

\[
B_r(\partial M) := \{ x \in M \mid \rho_{\partial M}(x) \leq r \}, \quad B_r^f(\partial M) := \{ x \in M \mid \rho_{\partial M,f}(x) \leq r \}.
\]
We notice the following (cf. [34, Lemma 5.1], [35, Lemma 7.5]): If $\partial M$ is compact, then
\[
m_{(1+\frac{2(1-\epsilon)}{n-1})f}(B^f_r(\partial M)) = \int_{\partial M} \int_0^{\min\{r,\tau_f(z)\}} \hat{\theta}_f(s,z) \, ds \, dv_{\partial M},
\]
\[
m_f(B_r(\partial M)) = \int_{\partial M} \int_0^{\min\{r,\tau(z)\}} \theta_f(t,z) \, dt \, dv_{\partial M}.
\]
We also set
\[
S_{\kappa,\lambda}(r) := \int_0^{\min\{r,C_{\kappa,\lambda}\}} \xi^{\frac{1}{\kappa,\lambda}}(\xi) \, d\xi.
\]

Lemmas 6.1, 6.2 together with the same argument as in [35, Subsection 7.2] for $N=1$ tell us the following Heintze-Karcher type comparisons (cf. [35, Lemmas 7.6 and 7.7]):

**Proposition 6.3** Assume
\[
\text{Ric}^N_f \geq c^{-1} e^{-\frac{4(1-\epsilon)}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\epsilon)}{n-1}}.
\]
Let $\partial M$ be compact. Then for all $r > 0$ we have
\[
m_{(1+\frac{2(1-\epsilon)}{n-1})f}(B^f_r(\partial M)) \leq S_{\kappa,\lambda}(r) m_f(\partial M).
\]

**Proposition 6.4** Let $\kappa$ and $\lambda$ satisfy the monotone-condition. Assume
\[
\text{Ric}^N_f \geq c^{-1} e^{-\frac{4(1-\epsilon)}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\epsilon)}{n-1}}, \quad (1-\epsilon)f \leq (n-1)\delta
\]
for $\delta \in \mathbb{R}$. Let $\partial M$ be compact. Then for all $r > 0$ we have
\[
m_f(B_r(\partial M)) \leq S_{\kappa e^{-4\delta},\lambda e^{-2\delta}}(r) m_f(\partial M).
\]

**Remark 6.5** Under the curvature condition (1.7), Bayle [3] has stated Proposition 6.3 without proof (see [3, Theorem E.2.2], and also [27], [29]).

Also, Lemmas 6.1 and 6.2 together with the same argument as in [35, Subsection 7.3] yield the following relative volume comparisons (cf. [35, Theorems 1.3 and 7.9]):

**Proposition 6.6** Under the same setting as in Proposition 6.3 for all $r, R > 0$ with $r \leq R$ we have
\[
\frac{m_{(1+\frac{2(1-\epsilon)}{n-1})f}(B^f_r(\partial M))}{m_{(1+\frac{2(1-\epsilon)}{n-1})f}(B^f_R(\partial M))} \leq \frac{S_{\kappa,\lambda}(R)}{S_{\kappa,\lambda}(r)}.
\]

**Proposition 6.7** Under the same setting as in Proposition 6.4 for all $r, R > 0$ with $r \leq R$ we have
\[
\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} \leq \frac{S_{\kappa e^{-4\delta},\lambda e^{-2\delta}}(R)}{S_{\kappa e^{-4\delta},\lambda e^{-2\delta}}(r)}.
\]
6.3 Volume rigidity

By the same method as in the proof of [35, Theorem 7.11] concerning \( N = 1 \), we see the following rigidity result for the equality case of Propositions 6.3 and 6.6:

**Theorem 6.8** Under the same setting as in Propositions 6.3 and 6.6, if \( \kappa \) and \( \lambda \) do not satisfy the ball-condition, and if

\[
\liminf_{r \to \infty} \frac{m_{f(B_r(\partial M))}}{S_{\kappa,\lambda}(r)} \geq m_{f,\partial M}(\partial M),
\]

then \( M \) is diffeomorphic to \([0, +\infty[ \times \partial M\), and the following hold:

(i) If \( N = n \), then \( f \) is constant, and

\[
g = dt^2 + s^2_{\kappa,\lambda} \left( 1 + \frac{2(1-\varepsilon)}{n-1} \right) g_{\partial M};
\]

(ii) if \( N \neq 1, n \), then for any \( z \in \partial M \)

\[
f(\gamma_z(t)) = f(z) - \varepsilon \frac{N-n}{N-1} \log s_{\kappa,\lambda}(s_{f,z}(t)),
\]

\[
g = dt^2 + s^2_{\kappa,\lambda} \left( 1 + \frac{2(1-\varepsilon)}{n-N} \right) (s_{f,z}(t)) g_{\partial M};
\]

(iii) if \( N = 1 \), then

\[
\varepsilon = 0,
\]

\[
g = dt^2 + \exp \left( 2 \frac{f(\gamma_z(t)) - f(z)}{n-1} \right) s^2_{\kappa,\lambda}(s_{f,z}(t)) g_{\partial M}.
\]

Thanks to Theorems 4.1 and 5.3, the same argument as in the proof of [35, Theorem 7.13] leads to the following result for the equality case of Propositions 6.4 and 6.7:

**Theorem 6.9** Under the same setting as in Propositions 6.4 and 6.7, if

\[
\liminf_{r \to \infty} \frac{m_f(B_r(\partial M))}{S_{\kappa,\lambda}(r)} \geq m_{f,\partial M}(\partial M),
\]

then the following hold:

(i) if \( \kappa \) and \( \lambda \) satisfy the convex-ball-condition, then \( M \) is isometric to \( B^{n}_{\kappa e^{-4\delta}, \lambda e^{-2\delta}} \), and

\[
(1-\varepsilon)f = (n-1)\delta \text{ on } M;
\]

(ii) if \( \kappa \leq 0 \) and \( \lambda = \sqrt{|\kappa|} \), then \( M \) is diffeomorphic to \([0, +\infty[ \times \partial M\), and the following rigidity properties hold:

(a) If \( N = n \), then \( f \) is constant, and

\[
g = dt^2 + s^2_{\kappa,\lambda} \left( 1 + \frac{2(1-\varepsilon)}{n-1} \right) g_{\partial M};
\]
(b) if $N \neq 1, n$, then for any $z \in \partial M$

$$f(\gamma_z(t)) = f(z) - \varepsilon \frac{N - n}{N - 1} c^{-1} \log g_{\kappa, \lambda}(s_{f, z}(t)),
$$

$$g = dt^2 + g_{\kappa, \lambda}((s_{f, z}(t)) g_{\partial M};$$

(c) if $N = 1$, then

$$\varepsilon = 0, 
 g = dt^2 + \exp \left( \frac{2 f(\gamma_z(t)) - f(z)}{n - 1} \right) s_{\kappa, \lambda}(s_{f, z}(t)) g_{\partial M};$$

moreover, if $\kappa < 0$, then $(1 - \varepsilon) f = (n - 1) \delta$ on $M$.

7 Spectrum

In this last section, we will collect comparison geometric results on the smallest Dirichlet eigenvalue for the weighted $p$-Laplacian $\Delta_{f, p}$.

7.1 Eigenvalue Rigidity

Let $p \in ]1, +\infty[$. A real number $\nu$ is said to be a Dirichlet eigenvalue of $\Delta_{f, p}$ if there exists $\psi \in W^{1, p}_0(M, m_f) \setminus \{0\}$ such that $\Delta_{f, p} \psi = \nu |\psi|^{p-2} \psi$ in the distribution sense. For $\psi \in W^{1, p}_0(M, m_f) \setminus \{0\}$, the associated Rayleigh quotient is defined as

$$R_{f, p}(\psi) := \frac{\int_M \|\nabla \psi\|^p \, dm_f}{\int_M |\psi|^p \, dm_f}.$$  

We set

$$\nu_{f, p}(M) := \inf_{\psi} R_{f, p}(\psi),$$

where the infimum is taken over all $\psi \in W^{1, p}_0(M, m_f) \setminus \{0\}$. If $M$ is compact, then $\nu_{f, p}(M)$ coincides with the infimum of the set of all Dirichlet eigenvalues.

For $D \in ]0, C_{\kappa, \lambda}] \setminus \{+\infty\}$, let $\nu_{p, \kappa, \lambda, D}$ be the positive minimum real number $\nu$ such that there is a non-zero function $\varphi : [0, D] \to \mathbb{R}$ satisfying

$$\left( (|\varphi'(s)|^{p-2} \varphi'(s))' + (n - 1) \frac{g'_{\kappa, \lambda}(s)}{g_{\kappa, \lambda}(s)} (|\varphi'(s)|^{p-2} \varphi'(s)) \right)
+ \nu |\varphi(s)|^{p-2} \varphi(s) = 0, \quad \varphi(0) = 0, \quad \varphi'(D) = 0.$$  

We recall the notion of the model spaces introduced by Kasue [10]. We say that $\kappa$ and $\lambda$ satisfy the model-condition if the equation $s'_{\kappa, \lambda}(s) = 0$ has a positive solution. Note that $\kappa$ and $\lambda$ satisfy the model-condition if and only if either (1) $\kappa > 0$ and $\lambda < 0$; (2) $\kappa = 0$ and $\lambda = 0$; or (3) $\kappa < 0$ and $\lambda \in ]0, \sqrt{|\kappa|}[$. Let $\kappa$ and $\lambda$ satisfy the ball-condition or the model-condition, and let $M$ be compact. When they
satisfy the model-condition, we set \( D_{\kappa,\lambda}(M) \) as follows: If \( \kappa = 0 \) and \( \lambda = 0 \), then 
\[ D_{\kappa,\lambda}(M) := \text{InRad} M; \] 
otherwise, 
\[ D_{\kappa,\lambda}(M) := \inf\{s > 0 \mid s'_{\kappa,\lambda}(s) = 0\}. \] 
We say that \( M \) is a \((\kappa, \lambda)\)-equational model space if \( M \) is isometric to either (1) \( B^\kappa_{\kappa,\lambda} \) for \( \kappa \) and \( \lambda \) satisfying the ball-condition; (2) \([(0, 2D_{\kappa,\lambda}(M)] \times \partial M_1, ds^2 + s^2_{\kappa,\lambda}(s) g_{\partial M_1}) \) for \( \kappa \) and \( \lambda \) satisfying the model-condition, and for some connected component \( \partial M_1 \) of \( \partial M \); or (3) the quotient space \([(0, 2D_{\kappa,\lambda}(M)] \times \partial M, ds^2 + s^2_{\kappa,\lambda}(s) g_{\partial M}) / G_\sigma \) for \( \kappa \) and \( \lambda \) satisfying the model-condition, and for some involutive isometry \( \sigma \) of \( \partial M \) without fixed points, where 
\( G_\sigma \) denotes the isometry group on \([(0, 2D_{\kappa,\lambda}(M)] \times \partial M, ds^2 + s^2_{\kappa,\lambda}(s) g_{\partial M}) \) whose elements consist of identity and the involute isometry \( \hat{\sigma} \) defined by 
\[ \hat{\sigma}(s, z) := (2D_{\kappa,\lambda}(M) - s, \sigma(z)). \]

We now set 
\[ \text{InRad}_f M := \sup_{x \in M} \rho_{0M,f}(x). \]

Having Proposition 3.14 at hand, we can show the following comparison geometric result by the same argument as in the proof of [35] Theorem 1.4 and Lemma 8.2 for \( N = 1 \), whose unweighted version has been established by Kasue [10] for \( p = 2 \), and the second named author [34] for general \( p \) (cf. [10] Theorem 2.1 and [34] Theorem 1.6):

**Theorem 7.1** Let \( p \in ]1, +\infty[ \). Assume 
\[ \text{Ric}^N_M \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}, \quad (1 - \varepsilon) f \leq (n - 1) \delta \]
for \( \delta \in \mathbb{R} \). Let \( M \) be compact, and let \( f \) be \( \partial M \)-radial. For \( D \in ]0, C_{\kappa,\lambda}] \setminus \{+\infty\} \), suppose \( \text{InRad}_f M \leq D \). Then 
\[ \nu_{f,\partial M}(M) \geq \nu_{p,\kappa e^{-4\delta} \lambda e^{-2\delta}, D e^{2\delta}}. \]

If the equality holds, then \( M \) is a \((\kappa e^{-4\delta}, \lambda e^{-2\delta})\)-equational model space, and 
\( (1 - \varepsilon) f = (n - 1) \delta \) on \( M \).

From the same method as in the proof of [35] Lemma 8.4 and Theorem 8.8 for \( N = 1 \) with Proposition 3.13 and Theorem 4.8, we can also conclude the following:

**Theorem 7.2** Let \( p \in ]1, +\infty[ \). Let \( \kappa \) and \( \lambda \) satisfy the convex-ball-condition. Assume 
\[ \text{Ric}^N_M \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}, \quad (1 - \varepsilon) f \leq (n - 1) \delta \]
for \( \delta \in \mathbb{R} \). Let \( M \) be compact. Then 
\[ \nu_{f,\partial M}(M) \geq \nu_{0,\partial B^n_{\kappa e^{-4\delta} \lambda e^{-2\delta}}}. \]

If the equality holds, then 
\( (1 - \varepsilon) f = (n - 1) \delta \) on \( M \), and the following properties hold:

(i) If \( N = n \), then \( M \) is isometric to \( B^n_{\kappa e^{-4\delta} \lambda e^{-2\delta}} \);
(ii) if \( N \neq n \), then \( \varepsilon = 0 \), and \( M \) is isometric to \( B^n_{\kappa e^{-4\delta} \lambda e^{-2\delta}} \).
7.2 Spectrum rigidity

Let $\Omega$ be a relatively compact domain in $M$ such that its boundary is a smooth hypersurface in $M$ with $\partial \Omega \cap \partial M = \emptyset$. For the canonical measure $v_{\partial \Omega}$ on $\partial \Omega$, we set

$$m_{f, \partial \Omega} := e^{-f|_{\partial \Omega}} v_{\partial \Omega}.$$

Proposition 3.13 together with a similar calculation to the proof of [35, Lemma 8.9] for $N = 1$ yields the following volume estimate, which has been shown by Kasue [11] in the unweighted case (cf. [11, Proposition 4.1]):

**Lemma 7.3** Let $\kappa$ and $\lambda$ satisfy the monotone-condition. Assume

$$\text{Ric}^N \geq e^{-1} \kappa e^{-\frac{4(1-\epsilon)f}{(n-1)}} g, \quad H_{f, \partial M} \geq e^{-1} \lambda e^{-\frac{2(1-\epsilon)f}{(n-1)}} , \quad (1-\epsilon)f \leq (n-1)\delta \quad \text{for} \; \delta \in \mathbb{R}.$$  

Define $\rho_{\partial M, \delta} := e^{-2\delta} \rho_{\partial M}$. Let $\Omega$ be a relatively compact domain in $M$ such that $\partial \Omega$ is a smooth hypersurface in $M$ satisfying $\partial \Omega \cap \partial M = \emptyset$. Set

$$D_{\delta, 1}(\Omega) := \inf_{x \in \Omega} \rho_{\partial M, \delta}(x), \quad D_{\delta, 2}(\Omega) := \sup_{x \in \Omega} \rho_{\partial M, \delta}(x).$$

Then we have

$$m_f(\Omega) \leq e^{2\delta} \sup_{s \in [D_{\delta, 1}(\Omega), D_{\delta, 2}(\Omega)]} \frac{\int_s^{D_{\delta, 2}(\Omega)} \xi e_{\kappa, \lambda}^{-1}(\xi) \, d\xi}{\xi e_{\kappa, \lambda}^{-1}(s)} m_{f, \partial \Omega}(\partial \Omega).$$

For $D \in [0, C_{\kappa, \lambda}]$, we set

$$C(\kappa, \lambda, D) := \sup_{s \in [0, D]} \frac{\int_s^D \xi e_{\kappa, \lambda}^{-1}(\xi) \, d\xi}{\xi e_{\kappa, \lambda}^{-1}(s)}.$$

Notice that $C(\kappa, \lambda, +\infty)$ is finite if and only if $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$; in this case,

$$C(\kappa, \lambda, D) = (c^{-1} \lambda)^{-1} \left(1 - e^{-c^{-1} \lambda D}\right).$$

Lemma 7.3 and the argument in the proof of [35, Lemma 8.11] tell us the following:

**Lemma 7.4** Let $p \in [1, +\infty]$. Let $\kappa$ and $\lambda$ satisfy the monotone-condition. Assume

$$\text{Ric}^N \geq e^{-1} \kappa e^{-\frac{4(1-\epsilon)f}{(n-1)}} g, \quad H_{f, \partial M} \geq e^{-1} \lambda e^{-\frac{2(1-\epsilon)f}{(n-1)}} , \quad (1-\epsilon)f \leq (n-1)\delta \quad \text{for} \; \delta \in \mathbb{R}.$$  

For $D \in [0, C_{\kappa, \lambda}]$, suppose $\text{InRad} M \leq e^{2\delta} D$. Then we have

$$\nu_{f, p}(M) \geq \left(p e^{2\delta} C(\kappa, \lambda, D)\right)^{-p}.$$  

Due to Theorem 4.1, Lemma 7.4 and the argument in the proof of [35, Theorem 8.12] for $N = 1$ lead us to the following:
Theorem 7.5 Let $p \in ]1, +\infty[$. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. Assume
\[
\operatorname{Ric}^N_f \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}, \quad (1-\varepsilon)f \leq (n-1)\delta
\]
for $\delta \in \mathbb{R}$. Let $\partial M$ be compact. Then
\[
\nu_{f,p}(M) \geq e^{-2p\delta} \left(\frac{c^{-1} \lambda}{p}\right)^p.
\]
If the equality holds, then $M$ is diffeomorphic to $[0, +\infty[ \times \partial M$, and the following hold:

(i) If $N = n$, then $f$ is constant, and
\[
g = dt^2 + s^2 e^{\frac{4(1-\varepsilon)f}{n-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}(t)} g_{\partial M};
\]

(ii) if $N \neq 1, n$, then for any $z \in \partial M$
\[
f(\gamma_z(t)) = f(z) - \varepsilon \frac{N-n}{N-1} c^{-1} \log s_{\kappa,\lambda}(s_{f,z}(t)), \quad g = dt^2 + s^2 e^{\frac{2(1-\varepsilon)\kappa}{n-1} \log(s_{f,z}(t))} (s_{f,z}(t)) g_{\partial M};
\]

(iii) if $N = 1$, then
\[
\varepsilon = 0, \quad g = dt^2 + \exp \left(\frac{2f(\gamma_z(t)) - f(z)}{n-1}\right) s^2 e^{\frac{2(1-\varepsilon)\kappa}{n-1} \log(s_{f,z}(t))} g_{\partial M}.
\]

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