Generalized Attracting Horseshoe in the Rössler Attractor

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Abstract

We show that there is a mildly nonlinear three-dimensional system of ordinary differential equations - realizable by a rather simple electronic circuit - capable of producing a generalized attracting horseshoe map. A system specifically designed to have a Poincaré section yielding the desired map is described, but not pursued due to its complexity, which makes the construction of a circuit realization exceedingly difficult. Instead, the generalized attracting horseshoe and its trapping region is obtained by using a carefully chosen Poincaré map of the Rössler attractor. Novel numerical techniques are employed to iterate the map of the trapping region to approximate the chaotic strange attractor contained in the generalized attracting horseshoe, and an electronic circuit is constructed to produce the map. Several potential applications of the idea of a generalized attracting horseshoe and a physical electronic circuit realization are proposed.

Keywords: Generalized Attracting Horseshoe, Strange attractors, Poincare Map

1 Introduction

The seminal work of Smale [1] showed that the existence of a horseshoe structure in the iterate space of a diffeomorphism is enough to prove it is chaotic. Often these diffeomorphisms arise from certain Poincaré maps of continuous-time chaotic strange attractors (CSA), which in turn are discrete-time CSAs. Some examples of such attractors are the Lorenz strange attractor [2], the Rössler attractor [3], and the double scroll attractor [4]. An example of a Poincaré map of the Lorenz equations is the Hénon map [5], which can be further simplified to the Lozi map [6].

In more recent years Joshi and Blackmore [7] developed an attracting horseshoe (AH) model for CSAs, which has two saddles and a sink. This, however, negates the possibility of the Hénon and Lozi maps, which have two saddles. Fortunately the attracting horseshoe

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can be modified into a generalized attracting horseshoe (GAH), which can have either one or two saddles while still being an attracting horseshoe \[8\]. This results in a quadrilateral trapping region. While extensive analysis was done in Joshi et al. \[8\], a simple concrete example seemed to be illusive.

In this investigation we implement MATLAB codes to find the necessary Poincaré map of the Rössler attractor that would admit a quadrilateral trapping region. The remainder of the paper is organized as follows; in Sec. 2 we give an overview of the algorithm with the MATLAB codes relegated to the Appendix. Once we have the tools for our numerical experiments, we first propose a carefully constructed GAH model in Sec. 3. Then, we give numerical examples of Poincaré maps of the Rössler attractor and the map of interest in Sec. 4, and real world examples in Sec. 5. Finally, we end discussions in Sec. 6 with some concluding remarks.

## 2 Poincaré map algorithm

To produce a general Poincaré section of a flow we break up the program into four parts: solving the ODE, computing a Poincaré section perpendicular to either \(x = 0\), \(y = 0\), or \(z = 0\), rotating the Poincaré section, and iterating the Poincaré map. Solving the ODE is standard through ODE45 on MATLAB, which executes a modified Runge-Kutta scheme. Once we have our solution matrix we need to approximate the values of first return maps from the discretized flow. By restricting the first return onto a Poincaré section the iterate space of Poincaré map can be visualized. This is easily done for a section perpendicular to the axes, but in order to locate a highly specialized object such as the GAH we need to be able to rotate the section. Once the desired section is found we can experiment on iterating the points of trapping region candidates.

The first major task is approximating the first return map on a Poincaré section of a flow. Much of the ideas of our initial first return map code came from that of Didier Gonze \[9\]. Once the discretized flow is found numerically a planar section for a certain value of \(x\), \(y\), or \(z\) can be defined, which in general will lie between pairs of simultaneous points. Then we may draw a line between the pair through the planar section and identify the intersecting point, which approximates a point of the first return map. This can also be done with more simultaneous points in order to get higher order approximations.

Once we can approximate a map for a section perpendicular to the axes we need to have the ability to rotate and move the map to any position. This is where our program completely diverges from that of \[9\]. While the first instinct might be to try to rotate the section, it is equivalent to rotate the flow in the opposite direction to the desired rotation of the section. Once the flow is rotated, the code for the first return map can be readily used. This gives us the ability to analyze the first return map of a general Poincaré section.

Finally, we would like to not only compute a first return map, but also compute the iterates of a Poincaré map of any system; that is, given an initial condition on an arbitrary Poincaré section can we find the subsequent iterates. To accomplish this, we solve the ODE for a given initial condition on the planar section to find the first return. Once we have the first return we record it’s location and use that as the new initial condition. This iterates the map for as many returns as desired, thereby filling in a Poincaré map. Now we have the
tools needed to run numerical experiments on GAHs.

3 A constructed GAH system

In this section, we give a brief description of the generalized attracting horseshoe (GAH) map and devise a three-dimensional nonlinear ordinary differential equation with a Poincaré section that produces it.

3.1 The GAH map

The GAH is a modification of the AH that can be represented as a geometric paradigm with either just one or two fixed points, both of which are saddles. Figure 1 shows a rendering of a $C^1$ GAH with two saddle points, which can be constructed as follows: The rectangle is first contracted vertically by a factor $0 < \lambda_v < 1/2$, then expanded horizontally by a factor $1 < \lambda_h < 2$ and then folded back into the usual horseshoe shape in such a manner that the total height and width of the horseshoe do not exceed the height and width, respectively of the trapping rectangle $Q$. Then the horseshoe is translated horizontally so that it is completely contained in $Q$. Obviously, the map $f$ defined by this construction is a smooth diffeomorphism. Clearly, there are also many other ways to obtain this geometrical configuration. For example, the map $f$ as described above is orientation-preserving, and an orientation-reversing variant can be obtained by composing it with a reflection in the horizontal axis of symmetry of the rectangle, or by composing it with a reflection in the vertical axis of symmetry followed by a composition with a half-turn. Another construction method is to use the standard Smale horseshoe that starts with a rectangle, followed by a horizontal composition with just the right scale factor or factors to move the image of $Q$ into $Q$, while preserving the expansion and contraction of the horseshoe along its length and width, respectively.

It is important to note that subrectangle $S$ with its left vertical edge through $p$, which contains the arch of the horseshoe and the keystone region $K$, plays a key role in the dynamics of the iterates of $f$. In particular, we require that the map satisfy the following additional property, which is illustrated in Fig. 2:

(★) $f$ maps the keystone region $K$ (containing a portion of the arch of the horseshoe) to the left of the fixed point $p$ and the portion of its corresponding stable manifold $W^s(p)$ containing $p$ and contained in $f(Q)$.

The definition above and (★) can be shown to lead to the conclusion that

$$\mathcal{A} := \overline{W^u(p)} = \bigcap_{n=1}^{\infty} f^n(Q),$$

where $W^u(p)$ is the unstable manifold of $p$, is a global chaotic strange attractor (CSA).

The map above can be considered to be the paradigm for a GAH, but there are many analogs. In fact, let $F : \tilde{Q} \to \tilde{Q}$ be any smooth diffeomorphism of a quadrilateral trapping region $\tilde{Q}$ possessing a horseshoe-like image with a keystone region $\tilde{K}$ containing a portion of
Figure 1: A planar GAH with two saddle points
the arch of $F(\tilde{Q})$ analogous to that shown in Fig. [1] Suppose that the map is expanding by a scale factor uniformly greater than one along the length of the horseshoe and contracting transverse to it by a scale factor uniformly less than one-half in the complement of a subset of $\tilde{Q}$ containing $\tilde{K}$. Then if $F$ satisfies an additional property analogous to (★), it maps $\tilde{K}$ into an open subset of $\tilde{Q}$ to the left of the saddle point $\tilde{p}$, and

$$ \mathcal{A} := \overline{W^u(\tilde{p})} = \bigcap_{n=1}^{\infty} F^n(\tilde{Q}) $$

is a global CSA.

![Figure 2: Local (transverse) horseshoe structure of $f^2$ near $p$](image)

3.2 A GAH producing system

We now construct an ODE in $\mathbb{R}^3$ with a Poincaré section that is a GAH. The transversal we use is the following square in the $xz$-plane defined in Cartesian and polar coordinates

$$ Q_0 := \{(x, y, z) : 0.05 \leq x \leq 1.05, y = 0, -0.5 \leq z \leq 0.5\} $$

$$ = \{(r, \theta, z) : 0.05 \leq r \leq 1.05, \theta = 0, -0.5 \leq z \leq 0.5\}. \quad (1) $$
The trick is to find a relatively simple (necessarily nonlinear) $C^1$ ODE having $Q_0$ as a transversal with an induced Poincaré first-return map $P : Q_0 \rightarrow Q_0 \supset Q_{2\pi} := P(Q_0)$ such that $P(Q_0) \subset \text{int}Q_0$ is a GAH. We chose the ODE based upon a rotation about the $z$-axis so that the square evolves into the GAH as $Q_0$ makes a full rotation. The first half of the metamorphosis takes care of the vertical squeezing and horizontal stretching, while the second half produces the folding. It is not difficult to show that the system (in cylindrical coordinates)

$$\dot{r} = \frac{2\log(1.2)\sin^2 \theta}{\pi} r, \; \dot{\theta} = 1, \; \dot{z} = \frac{2\log 5\sin^2 \theta}{\pi}(z + 0.2)$$

(2)

flows $Q_0$ to

$$Q_\pi := \{(x, y, z) : -1.26 \leq x \leq -0.06, y = 0, -0.26 \leq z \leq -0.06\}$$

(3)

which is the original square in the radial half-plane plane corresponding to $\theta = 0$ stretched by a factor of 1.2 along the $x$-axis and squeezed by a factor of $1/5$ with respect to $z = -0.2$ along the $z$-axis in the radial half plane corresponding to $\theta = \pi$. Consequently, (2) produces the first half of the desired result comprising the stretching and squeezing for $0 \leq \theta \leq \pi$.

Note that (2) can be integrated directly to obtain the following for $0 \leq \theta \leq \pi$ and initial condition $(r(0), \theta(0), z(0)) := (r_0, \theta_0, z_0)$:

$$r(t) = r(\theta) = r_0 \left[\frac{\log(1.2)}{2\pi}(2\theta - \sin 2\theta)\right],$$

$$\theta(t) = t,$$

$$z(t) = z(\theta) = -0.2 + (z_0 + 0.2) \left[\frac{\log(0.2)}{2\pi}(2\theta - \sin 2\theta)\right].$$

(4)

Now we have to attend to the folding for $\pi \leq \theta \leq 2\pi$. For this we use a rotation in planes orthogonal to a fixed circle in the $xy$-plane. In these planes corresponding to a circle of radius $c$, given as $c = 0.66$, we define Euclidean coordinates with origin $r = 0.66$, $z = 0$ and corresponding polar coordinates $(\rho, \phi)$ as

$$\rho := \sqrt{(r - 0.66)^2 + z^2} := \sqrt{r^2 + z^2},$$

(5)

where $\tilde{r} := r - 0.66 = \rho \cos \phi$ and $z := \rho \sin \phi$. Then, when $\pi \leq \theta \leq 2\pi$, we take the folding part for $\phi \geq -\pi/2$ to be

$$\dot{\tilde{r}} = \dot{r} = -2\sin^2 \theta \rho \sin \phi, \; \dot{\phi} = 1, \; \dot{z} = 2\sin^2 \theta \rho \cos \phi,$$

(6)

or equivalently

$$\dot{\tilde{r}} = \dot{r} = -2\sin^2 \theta z, \; \dot{\theta} = 1, \; \dot{z} = 2\sin^2 \theta \tilde{r}.$$  

(7)

It is easy to verify from the above that $\rho$ is constant (call it $\rho_0$) for the solutions of (6) or (7) and that the solution initially (at $t = \theta = \pi$) satisfying $(\rho, \phi) = (\rho_0, \phi_0)$ is

$$\tilde{r} = \tilde{r}(t) = \rho_0 \cos (\phi(t) + \phi_0), \; \theta = \theta(t) = t, \; z = z(t) = \rho_0 \sin (\phi(t) + \phi_0),$$

(8)

where

$$\phi(t) := (t - \pi) - \sin t \cos t.$$  

(9)
The above (6) or (7) describes the folding field for \( \pi \leq \theta \leq 2\pi \) and \(-\pi/2 \leq \phi\). In order to smoothly fill in the rest of the field, we shall use the function
\[
\psi(\tilde{r}) := \begin{cases} 
0, & \tilde{r} \leq -0.6 \\
\sin^2 \left[ \frac{\pi}{12} (\tilde{r} + 0.6) \right], & -0.6 \leq \tilde{r} \leq 0
\end{cases},
\]
which can be recast as
\[
\xi(r) := \begin{cases} 
0, & r \leq 0 \\
\sin^2 \left[ \frac{\pi}{12} (r - 0.06) \right], & 0.06 \leq r \leq 0.66
\end{cases}.
\]

We have now assembled all the elements for defining an ODE that generates a GAH Poincaré section. This ODE, which incorporates (2) and (7) and is \(\pi\)-periodic in \(\theta\), has the following form:
\[
\begin{align*}
\dot{r} &= R(r, \theta, z), \\
\dot{\theta} &= 1, \\
\dot{z} &= Z(r, \theta, z),
\end{align*}
\]
subject to the initial condition
\[
(r(0), \theta(0), z(0)) = (r_0, 0, z_0) \in Q_0,
\]
where
\[
R := \begin{cases} 
\frac{\log(1.2)\sigma(\theta)r}{\pi}, & \theta \leq \pi \\
-\sigma(\theta)z, & \pi \leq \theta \leq 2\pi \text{ and } (((r \geq 0.66) \text{ or } (z \geq 0)) = (-\pi/2 \leq \phi \leq \pi)) \\
-\xi(r)\sigma(\theta)z, & \pi \leq \theta \leq 2\pi \text{ and } (((r < 0.66) \text{ and } (z \in [-0.26, -0.06])) \\
0, & \pi \leq \theta \leq 2\pi \text{ and } (((r < 0.66) \text{ and } (z < 0) \text{ and } (z \notin [-0.26, -0.06]))
\end{cases},
\]
\[
Z := \begin{cases} 
\frac{(\log(1.2)\sigma(\theta)(z + 0.2)}{\sigma(\theta)\tilde{r}}, & 0 \leq \theta \leq \pi \\
\sigma(\theta)\tilde{r}, & \pi \leq \theta \leq 2\pi \text{ and } (((r \geq 0.66) \text{ or } (z \geq 0)) = (-\pi/2 \leq \phi \leq \pi)) \\
0, & \pi \leq \theta \leq 2\pi \text{ and } (((r < 0.66) \text{ and } (z < 0)) = (-\pi \leq \phi \leq -\pi/2))
\end{cases},
\]
and
\[
\sigma(\theta) := 2 \sin^2 \theta = 1 - \cos 2\theta.
\]

Finally, it is not difficult to show that the Poincaré section of the transversal (and trapping region) \(Q_0\) under the system (12) is a GAH with image that is simply a 180-degree rotation of the horseshoe in Fig. 1. However, it appears that the construction of an electronic circuit simulating (12) would be a rather formidable undertaking, so we selected a simpler system; namely, the Rössler attractor model, which is a mildly nonlinear three-dimensional ODE that has a straightforward circuit realization.

## 4 Poincaré maps and circuit realization of the Rössler attractor

We consider the Rössler attractor
\[
\begin{align*}
\dot{x} &= y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c),
\end{align*}
\]
where we use the parameters \(a = 0.2, b = 0.1,\) and \(c = 10\). This produces the chaotic strange attractor in Fig. 3 and it can also be realized by a rather simple electronic circuit.
4.1 The Poincaré map

One can use the algorithm in Sec. 2 to compute any Poincaré section of the attractor, however what we are particularly interested in is identifying a trapping region for a generalized attracting horseshoe. Assuming the system contains a GAH, we first look for a Poincaré section with a horseshoe-like structure as shown in Fig. 4.

Now, if we can find a trapping region around this horseshoe, we will have shown evidence for the existence of a GAH. First we identify vertices of a quadrilateral that fully encompasses the horseshoe-like structure. Then using a recursive algorithm (described in Sec. 2) we compute the first return map of those vertices on that particular Poincaré section; i.e., the first iteration of the Poincaré map of those points. If the iterates are contained within that quadrilateral, the points on the quadrilateral itself can be tested. In Fig. 5 four thousand points on the quadrilateral are iterated and it is illustrated that this first return is completely contained in the quadrilateral. While this is not a proof, the grid spacing on the quadrilateral provides compelling evidence that this may be a trapping region for the GAH.

In order to provide more compelling evidence, we compute higher order iterations of the Poincaré map in Fig. 6.

4.2 Circuit realization of the Rössler system

It happens that there are several known examples of electronic circuits realizing the Rössler attractor system. We chose the one, obtained from [10], shown in Fig. 7 with a list of components in Table 1.
Figure 4: Poincaré section \((r = 5, \theta = 2\pi/5)\) of the Rössler attractor containing a horseshoe-like structure. Plot is shown in the rotated frame.
Figure 5: The first return (blue markers) of the quadrilateral trapping region (red markers) with vertices located at \((\hat{x}, \hat{y}) = (−3.55, −27), (11.91, −6.6), (12, 0), (−8.5, 3.5)\). While the quadrilateral edges look “continuous”, it should be noted that it is in fact discretized using four thousand points, which are then mapped back to the Poincaré section \((r = 5, \theta = 2\pi/5)\). Plot is shown in the rotated frame with \(\hat{x}\) and \(\hat{y}\) denoting rotated axes.

The physical realization of the Rossler attractor circuit was constructed using summing amplifiers, integrators, and a multipliers. Due to the nature of this system, the operational amplifier must operate within ±15 volts in order to avoid clipping of the Rossler Attractor output waveform. In this circuit, resistors were used to represent constant values for parameters a and b in (14). A potentiometer was used to vary the parameter value of b in order to observe the bifurcations of the physical system. We first test the circuit on Multisim and observe the aforementioned bifurcations in Fig. 8.

Next we built the circuit and observed oscilloscope outputs as shown in Fig. 9. The Poincaré section that we chose was a particular vertical plane through the top arch of the output shown (see also Fig. 3). The acceptable planes were obtained by trial and error via varying the system parameters and rotation of the plane about a vertical axis through the apex of the arc.

5 Potential applications

One can imagine several practical applications of devices containing electronic circuit realizations of a GAH. Two, which are related to communications and intelligence gathering, immediately come to mind: First, the circuit could be embedded in a communication receiving device, and tuned to certain “static” frequencies different from those in the expected incoming messages. The strong global attracting characteristics of the circuit would separate the static from the incoming messages, thereby enhancing the receiving capabilities of
Figure 6: First five iterations of the Poincaré map (blue markers) of the quadrilateral trapping region (red markers) with vertices located at \((\hat{x}, \hat{y}) = (-3.55, -27), (11.91, -6.6), (12, 0), (-8.5, 3.5)\). While the quadrilateral edges look “continuous”, it should be noted that it is in fact discretized using four thousand points, which are then mapped back to the Poincaré section \((r = 5, \theta = 2\pi/5)\). Plot is shown in the rotated frame with \(\hat{x}\) and \(\hat{y}\) denoting rotated axes.

Figure 7: Multisim circuit diagram for Rössler attractor
Figure 8: Multisim outputs of the Rossler attractor showing a period doubling Hopf bifurcation leading to chaos.

| Type               | Quantity | Code   |
|--------------------|----------|--------|
| 10kΩ Resistor      | 11       |        |
| 100kΩ Resistor     | 3        |        |
| 390kΩ Resistor     | 1        |        |
| 56kΩ Resistor      | 1        |        |
| 560kΩ Resistor     | 1        |        |
| 5.1kΩ Resistor     | 1        |        |
| 100kΩ Potentiometer| 1        |        |
| 100nF Capacitor    | 6        |        |
| 2.2nF Capacitor    | 3        |        |
| Op-Amp             | 2        | AD633JN|
| Multiplier         | 1        | TL074CN|
Figure 9: Oscilloscope output from Rössler attractor circuit
system. In effect, the GAH circuit would filter out the static.

Secondly, a stationary or compact mobile device incorporating the GAH circuit could be used to penetrate and analyze various communication systems. Either be connecting remotely in the case of a stationary device or directly for a mobile version, the global attracting properties could be employed to extract crucial characteristics of the system to which it is connected. Moreover, the same attracting features of the GAH circuit device could be used to absorb various parts of sent messages that would render them useless, false or simply misleading.

The two rather basic applications mentioned provide just a glimpse of the possible applications of GAH circuits, most of which would probably be related to information systems, data collection and filtering. Moreover, there are more applications that could exploit the chaotic strange attractor associated with a GAH circuit. For example a GAH circuit device could be used either to control chaos, introduce chaos or adjust the fractal dimension of outputs of a variety of applicable processes based on dynamical systems.

6 Conclusions

We constructed a rather complicated nonlinear three-dimensional ordinary differential equation (ODE) having a Poincaré section that is a GAH map, but is not particularly amenable to electronic circuit realization, which was a goal of the investigation.

So, instead of the initial ODE, we selected the Rössler attractor; a mildly nonlinear three-dimensional ODE that has a reasonably simple circuit realization and can actually produce GAH maps for carefully chosen Poincaré sections. We constructed the corresponding GAH circuit and used a novel iteration procedure to generate good approximations of the chaotic strange attractors associated to the GAH maps.

Finally, in addition to the experimental and analytic aspects of our investigation, we discussed a number of potential practical applications of the GAH circuit. Most of the envisioned applications were in the realms of communication and information gathering.

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A Poincaré map codes

A.1 Attractor data

clear
t=[0,1000];
xinit=[0 1 0];  % Random initial point
%Solving system using ODE45
[t,x]=ode45(@(t,x) ODESys(x), t, xinit);
%%% Rotating flow to pick out desired Poincare section %%%
q = pi/2.5;
xtrans = [1 0 0; 0 cos(q) -sin(q); 0 sin(q) cos(q)];
ytrans = [cos(q) 0 sin(q); 0 1 0; -sin(q) 0 cos(q)];
ztrans = [cos(q) -sin(q) 0; sin(q) cos(q) 0; 0 0 1];
newdata = x * ztrans;
newerdata = [t, newdata];
save 'myfile.dat' newerdata -ascii;  % Saves flow in myfile.dat
plot3(newdata(:,1), newdata(:,2), newdata(:,3), 'LineWidth', 1);
alw = 0.75;  % AxesLineWidth
fsz = 14;  % Fontsize
xlabel('$$\hat{x}$$', 'Interpreter', 'Latex', 'fontsize', 14)
ylabel('$$\hat{y}$$', 'Interpreter', 'Latex', 'fontsize', 14)
zlabel('$$\hat{z}$$', 'Interpreter', 'Latex', 'fontsize', 14)

A.2 ODE system

function dx = ODESys(x)

%%% Rossler Equation %%%
%%% Parameters %%%
a= .2;
b=.1;
c= 10;
%%% Rossler ODEs %%%
dx1=-x(2)-x(3);
dx2=x(1)+a*x(2);
dx3=b+x(3)*(x(1)-c);
dx=[dx1;dx2;dx3];

A.3 Horseshoe-like structure

D=load('myfile.dat');  % The flow is saved in myfile.dat
var=4;  % Chooses the rotated coordinate axis for Poincare section
cut=5;  % Chooses the Poincare section in rotated space; e.g.
% cut = 5 => the Poincare section is located at
% r = 5, and theta = 2pi/5 D)

%%% Picks out other two variables %%%
for i = 2:4
if i ~= var
    var01 = i;
    i = i + 1;
else
    var02 = i;
end
break;
end

x = D(:, var); % Extract time series for variable var
xt = x(2:end-1); % Value of discretized flow on var axis
xp = x(1:end-2); % Previous time value
xm = x(3:end); % Next time value

%%% Finds which pairs of points are closest to the Poincare section %%%
k = find(((xp < cut) & (xt >= cut)) | ((xp > cut) & (xm <= cut)));

%%% Finds points on the flow intersecting with the Poincare section %%%
R = zeros(size(D));
for i = 1:length(k)
    R(i, :) = D(k(i), :);
end
plot(R(:, var01), R(:, var02), 'b.' )

%%% Title %%%

A.4 Poincaré map

function firstRet = PoincareMap(xinit) % xinit calls a particular
% ordered triplet

t = [0, 1000];
axis = 4; % Chooses the rotated coordinate axis for Poincare section
cutter = 5; % Chooses the Poincare section in rotated space; e.g.
% cut = 5 => the Poincare section is located at
% r = 5, and theta = 2pi/5
Solving system using ODE45

q = −pi/2.5; % Counter rotation to pick out the proper initial conditions
% in "ODESys"

ztrans = [cos(q) −sin(q) 0; sin(q) cos(q) 0; 0 0 1];
xinit = xinit * ztrans;
[t, x] = ode45(@(t, x) ODESys(x), t, xinit);

Rotating flow to pick out desired Poincare section

q = pi/2.5; % theta = 2pi/5
ztrans = [cos(q) −sin(q) 0; sin(q) cos(q) 0; 0 0 1];
newdata = x * ztrans;
newerdata = [t, newdata];

Extract time series for variable var
xt=x(2:end−1); % Value of discretized flow on var axis
xp=x(1:end−2); % Previous time value
xm=x(3:end); % Next time value
m = length(xt);
for i = 1:m
if((xp(i, 1)<cutter) && (xt(i, 1)>=cutter)) || ((xp(i, 1)>cutter)... && (xm(i, 1)<=cutter))
firstRet = newerdata(i, :);
break
end
end

Trapping region

D=load('myfile.dat'); % The flow is saved in myfile.dat

Vertices of quadrilateral

% A = [−3.6 −27 5; 11.91 −6.5 5; 12 0 5; −8 3 5]; % This one works
A = [−3.55 −27 5; 11.91 −6.6 5; 12 0 5; −8.5 3.5 5]; % A better one

Points between the vertices on the quadrilateral

x1 = [A(1,1):(A(2,1)−A(1,1))/1000:A(2,1)] ’;
y1 = (A(1,2)−A(2,2))/(A(1,1)−A(2,1))*(x1 − A(1,1)) + A(1,2);
z1 = ones(size(x1))*5;
x2 = [A(2,1):(A(3,1)−A(2,1))/1000:A(3,1)] ’;
y2 = (A(2,2)−A(3,2))/(A(2,1)−A(3,1))*(x2 − A(2,1)) + A(2,2);
z2 = ones(size(x2))*5;
x3 = [A(3,1):(A(4,1)−A(3,1))/1000:A(4,1)] ’;
y3 = (A(3,2)−A(4,2))/(A(3,1)−A(4,1))*(x3 − A(3,1)) + A(3,2);
z3 = ones(size(x3))*5;
x4 = [A(4,1):(A(1,1)−A(4,1))/1000:A(1,1)] ’;
y4 = (A(4,2)−A(1,2))/(A(4,1)−A(1,1))*(x4 − A(4,1)) + A(4,2);
z4 = ones(size(x4))*5;
A = [x1 y1 z1; x2 y2 z2; x3 y3 z3; x4 y4 z4];
var = 4; % Chooses the rotated coordinate axis for Poincare section
cut = 5; % Chooses the Poincare section in rotated space; e.g.
% cut = 5 => the Poincare section is located at
% r = 5, and theta = 2pi/5
%%% Picks out other two variables %%%
for i = 2:4
if i ~= var
var01 = i;
i = i + 1;
if i ~= var
var02 = i;
else
var02 = i + 1;
end
break;
end
end
x = D(:, var); % Extract time series for variable var
xt = x(2:end − 1); % Value of discretized flow on var axis
xp = x(1:end − 2); % Previous time value
xm = x(3:end); % Next time value
%%% Finds which pairs of points are closest to the Poincare section %%%
k = find(((xp < cut) & (xt >= cut)) | ((xp > cut) & (xm <= cut)));
%%% Computes iterates of A %%%
Q = zeros(length(A), 4);
parfor i = 1:length(A)
Q(i, :) = PoincareMap(A(i, :));
end
%%% Manually add in number of higher order %%%
%%% returns in order to use parfor %%%
%%% Second Returns %%%
Q1 = zeros(size(Q));
Q_spare = Q(:, 2:4);
parfor i = 1:length(Q)
Q1(i, :) = PoincareMap(Q_spare(i, :));
end
%%% Third Returns %%%
Q2 = zeros(size(Q1));
Q_spare = Q1(:, 2:4);
parfor i = 1:length(Q1)
Q2(i, :) = PoincareMap(Q_spare(i, :));
end
%%% Fourth Returns %%%
Q3 = zeros(size(Q2));
Q_spare = Q2(:, 2:4);
parfor i = 1:length(Q)
Q3(i,:) = PoincareMap(Q_spare(i,:));
end
plot(Q(:,varo1),Q(:,varo2),'b.',Q1(:,varo1),Q1(:,varo2),'b.',...
Q2(:,varo1),Q2(:,varo2),'b.',Q3(:,varo1),Q3(:,varo2),'b.');
hold on
plot([A(:,1); A(1,1)],[A(:,2); A(1,2)],'r.','MarkerSize',1)
hold off
xlabel('$$\hat{x}$$', 'Interpreter', 'Latex', 'fontsize',14)
ylabel('$$\hat{y}$$', 'Interpreter', 'Latex', 'fontsize',14)
title(sprintf('Trapping Region'))
alw = 0.75;   % AxesLineWidth
fsz = 14;     % Fontsize