TWO CONVERSE THEOREMS FOR MAASS FORMS

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Abstract. We first prove a new converse theorem for Dirichlet series of Maass type which does not assume an Euler product. The underlying idea is a geometric generalisation of Weil’s classical argument. By studying the asymptotics of hypergeometric functions, we then show that the theorem remains valid if we allow twists to have arbitrary poles.

1. Introduction

Weil’s converse theorem characterises holomorphic modular forms in terms of the analytic properties of their twisted $L$-functions [Wei67]. In this paper we will prove two direct generalisations to real-analytic automorphic forms on $\text{GL}_2(\mathbb{A}_\mathbb{Q})$. Unlike the converse theorem proved by Jacquet–Langlands [JL70], our statements do not assume an Euler product. Moreover, in the second theorem the character twists need not be holomorphic.

We will frequently encounter the local factors of completed $L$-functions. In particular, we will use following variation on the gamma function:

\[
\Gamma_R(s) = \pi^{-s/2} \Gamma(s/2),
\]

and, given a Dirichlet character $\psi \mod q$, the associated Gauss sum:

\[
\tau(\psi) = \sum_{a \mod q} \psi(a) e^{2\pi i \frac{a}{q}}.
\]

Our first theorem makes precise assumptions about the analytic properties of character twists, which are to be relaxed in the sequel.

**Theorem 1.1.** Let $N$ be a positive integer, $\chi$ be a Dirichlet character mod $N$, $\epsilon \in \{0, 1\}$, $\nu \in \mathbb{C}$ be such that $\frac{1}{4} - \nu^2 > 0$ and $a_n, b_n$ be sequences of complex numbers such that $|a_n|, |b_n| = O(n^\sigma)$ for some $\sigma \in \mathbb{R}$. For all $q$ relatively prime to $N$, primitive Dirichlet characters $\psi \mod q$ and $k$ such that $\psi(-1) = (-1)^k$, define

\[
L_f(s, \psi) = \sum_{n=1}^{\infty} \psi(n)a_n n^{-s}, \quad L_g(s, \tilde{\psi}) = \sum_{n=1}^{\infty} \overline{\psi(n)b_n} n^{-s},
\]

and,

\[
\Lambda_f(s, \psi) = \Gamma_R(s + [\epsilon + k] + \nu) \Gamma_R(s + [\epsilon + k] - \nu) L_f(s, \psi),
\]

\[
\Lambda_g(s, \tilde{\psi}) = \Gamma_R(s + [\epsilon + k] + \nu) \Gamma_R(s + [\epsilon + k] - \nu) L_g(s, \tilde{\psi}),
\]

where $[\epsilon + k] \in \{0, 1\}$ is chosen to be equal to $\epsilon + k$ modulo 2. If $\psi = 1$ is the trivial character we omit it from the notation.

Let $\mathcal{P}$ be a set of primes coprime to $N$ such that the congruence $p \equiv u \mod \nu$ has a solution $p \in \mathcal{P}$ for all $u, v \in \mathbb{Z}_{>0}$ with $(u, v) = 1$. Assume that:

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(1) For primitive characters $\psi$ of conductor $q \in P$ the functions, $\Lambda_f(s, \psi)$ and $\Lambda_g(s, \psi)$ continue to entire functions,

(2) If $\epsilon = 1$ then $\Lambda_f(s)$ and $\Lambda_g(s)$ continue to entire functions,

(3) If $\epsilon = 0$ and $\nu \neq 0$ then $\Lambda_f(s)$ and $\Lambda_g(s)$ continue to meromorphic functions on $\mathbb{C}$ with at most simple poles in the set $\{1 \pm \nu, \pm \nu\}$,

(4) If $\epsilon = 0$ and $\nu = 0$ then $\Lambda_f(s)$ and $\Lambda_g(s)$ continue to meromorphic functions on $\mathbb{C}$ with at most double poles in the set $\{0, 1\}$,

and, for all primitive characters $\psi$ of conductor $q \in P \cup \{1\}$, we have the functional equations:

\[
\Lambda_f(s, \psi) = (-1)^{\epsilon} \psi(N) \chi(q) \frac{\tau(\psi)}{\tau(\psi)} (q^2N)^{\frac{1}{2} - s} \Lambda_g(1 - s, \bar{\psi}).
\]

If $\nu \neq 0$, define

\[
f_0(z) = - \text{Res}_{s = -\nu} \Lambda_f(s) y^{\frac{1}{2} + \nu} - \text{Res}_{s = \nu} \Lambda_f(s) y^{\frac{1}{2} - \nu},
\]

\[
g_0(z) = N^{\frac{1}{2} + \nu} \text{Res}_{s = 1 + \nu} \Lambda_g(s) y^{\frac{1}{2} + \nu} + N^{\frac{1}{2} - \nu} \text{Res}_{s = 1 - \nu} \Lambda_g(s) y^{\frac{1}{2} - \nu}.
\]

If $\nu = 0$, define

\[
f_0(z) = - \text{Res}_{s = 0} \Lambda_f(s) y^{\frac{1}{2}} + \text{Res}_{s = 0} s \Lambda_f(s) y^{\frac{1}{2}} \log y,
\]

\[
g_0(z) = - \text{Res}_{s = 0} \Lambda_g(s) y^{\frac{1}{2}} + \text{Res}_{s = 0} s \Lambda_g(s) y^{\frac{1}{2}} \log y.
\]

The following series define weight 0 Maass forms on $\Gamma_0(N)$ of parity $\epsilon$, nebentypus $\chi$ (resp. $\bar{\chi}$) and eigenvalue $\frac{1}{4} - \nu^2$:

\[
f(z) = f_0(z) + \tilde{f}(z), \quad g(z) = g_0(z) + \tilde{g}(z),
\]

where

\[
\tilde{f}(z) := \sum_{n \neq 0} \frac{a_n}{2 \sqrt{|n|}} W_\nu(ny)e(nx), \quad \tilde{g}(z) := \sum_{n \neq 0} \frac{b_n}{2 \sqrt{|n|}} W_\nu(ny)e(nx),
\]

in which $W_\nu(u) = 4\sqrt{|u|} K_\nu(2\pi|u|)$ is the Whittaker function, $K_\nu(u) = \frac{1}{2} \int_0^\infty e^{-|u|(t + t^{-1})/2} t^\nu \frac{dt}{t}$ is a Bessel function, and, for $n \geq 0$, $a_{-n} = (-1)^{\epsilon} a_n$, $b_{-n} = (-1)^{\epsilon} b_n$. Furthermore $f(z) = g(-1/Nz)$ for all $z \in \mathcal{H}$.

Weil’s converse theorem follows from the assertion that if a holomorphic function $F : \mathcal{H} \to \mathbb{C}$ on the upper-half plane is invariant under an infinite order elliptic matrix in $\text{SL}_2(\mathbb{R})$, then it is constant on $\mathcal{H}$. Weil applied this to functions of the form $F = f - f|_\gamma$, where $f$ is a Fourier series and $\gamma \in \Gamma_0(N)$. Growth conditions on the Fourier coefficients of $f$ then imply that $f - f|_\gamma = 0$. The proof of Theorem 1.1 closely follows Weil’s argument with one crucial difference. The geometric interpretation of invariance under an infinite order elliptic matrix $M$ is that $F$ must be constant on hyperbolic circles centered at the fixed point of $M$ in $\mathcal{H}$, and one can find non-constant real-analytic functions $F$ with this property. The new insight in the proof of Theorem 1.1 the method of “two circles”, suggested to the authors by David Farmer. We construct a second infinite order elliptic in $\text{SL}_2(\mathbb{R})$ and prove a version of Weil’s lemma valid for real-analytic (even continuous) functions on the upper-half plane.

Our second theorem is inspired by that of Booker–Krishnamurthy [BK13], and weakens the analytic assumptions on the character twists. We note that Raghunathan proved a

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1Such a matrix can not have integral coefficients, as the elliptic matrices in $\text{SL}_2(\mathbb{Z})$ are of finite order.
version of the Hecke–Maass converse theorem allowing for polar $L$-functions in [Rag10]. Our theorem is different in that it assumes twisted functional equations rather than an Euler product, and applies to arbitrary level $N$.

**Theorem 1.2.** Let $N$, $\chi$ and $\epsilon$ be as in Theorem 1.1 and let $\nu \in \mathbb{C}\setminus\{0\}$ be such that $\frac{1}{2} - \nu^2 > 0$ and $a_n, b_n$ be sequences of complex numbers such that $|a_n|, |b_n| = O(n^\sigma)$ for some $\sigma = \frac{1}{2} + \epsilon$ with $0 < \epsilon < \frac{1}{2} - |\Re \nu|$. Let $\mathcal{P}$ be a set of primes such that $\{p \in \mathcal{P} : p \equiv u \pmod{\nu}\}$ is infinite for every $u, v \in \mathbb{Z}_{>0}$ with $(u, v) = 1$ and $p \nmid N$ for any $p \in \mathcal{P}$. For all primitive Dirichlet characters of modulus $q \in \{1\} \cup \mathcal{P}$ assume $\Lambda_f(s, \psi)$ and $\Lambda_g(s, \psi)$ continue to meromorphic functions on $\mathbb{C}$ and satisfy functional equation (1.5). If there is a non-zero polynomial $P(s) \in \mathbb{C}[s]$ such that $P(s)\Lambda_f(s)$ continues to an entire function of finite order, then $\Lambda_f(s)$ and $\Lambda_g(s)$ have meromorphic continuation to $\mathbb{C}$ with possible simple poles in the set $\{\pm \nu, 1 \pm \nu\}$, and $f(z), g(z)$ as defined by equation (1.10) define weight 0 Maass forms on $\Gamma_0(N)$ of parity $\epsilon$, nebentypus $\chi$ (resp. $\overline{\chi}$) and eigenvalue $\frac{1}{4} - \nu^2$. Furthermore $f(z) = g(-1/Nz)$ for all $z \in \mathcal{H}$.

We expect to apply Theorem 1.2 to the study of cancellation of zeros between automorphic $L$-functions. The assumptions that $\nu \neq 0$ will be removed in a follow-up paper, which should include applications to cancellation of zeros between Artin $L$-functions. The assumption that $\epsilon < \frac{1}{2} - |\Re \nu|$ is first used in Lemma 4.10 and should be removed with further work.

The proof of Theorem 1.2 works by showing the assumptions imply those of Theorem 1.1. The argument uses the asymptotics of hypergeometric functions to study a Taylor expansion for $\tilde{g}$ as defined in equation (1.11).

There should be other ways to weaken the assumptions of Theorem 1.1. For example, one could seek to relax the conditions on the set $\mathcal{P}$ of twisting moduli. In the context of modular forms, this was optimised by Diaconu–Perelli–Zaharescu [DPZ02]. In another direction Booker recently proved, again in the modular case, that one does not need an explicit formula for the twisted root numbers in functional equation (1.5), [Boo18].

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2. Preliminaries

2.1. **The Mellin transform.** The Mellin transform of a function $\phi$ on the positive real axis is given by $\mathcal{M}(\phi)(s) = \int_0^\infty \phi(t)t^{s-1}dt$. We will often use the Mellin transform shifted by $1/2$:

$$\tilde{\mathcal{M}}(\phi)(s) = \mathcal{M}(\phi)\left(s - \frac{1}{2}\right) = \int_0^\infty \phi(t)t^{s-\frac{1}{2}}\frac{dt}{t},$$

Suppose $\phi$ is of rapid decay at $\infty$ and grows like $t^{-A}$ for some real number $A$ as $t \to 0$. Then $\tilde{\mathcal{M}}(\phi)$ is holomorphic in the half plane $\Re s > A + \frac{1}{2}$. The inverse of $\tilde{\mathcal{M}}$ is given by

$$\tilde{\mathcal{M}}^{-1}(f)(t) = \frac{1}{2\pi i} \int_{\sigma - \frac{1}{2}}^{\sigma + \frac{1}{2}} f(s)t^{\frac{1}{2}-s}ds.$$
2.2. Maass forms. In this section we recall some basic facts about Maass forms. For a function \( f : \mathcal{H} \to \mathbb{C} \) and \( \gamma = (a \ b \ c \ d) \in \text{GL}_2^+(\mathbb{R}) \) we write \( f \gamma(z) = f \left( \frac{a z + b}{c z + d} \right) \). A Maass form of weight zero on \( \Gamma_0(N) \) is an eigenfunction on \( \Gamma_0(N) \) of the Laplace-Beltrami operator with eigenvalue \( \frac{1}{4} \nu^2 > 0 \), satisfying certain growth conditions and the transformation rule
\[
 f|_\gamma = \chi(d)f, \forall \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N),
\]
for a Dirichlet character \( \chi \mod N \). A (weight 0) Maass form \( f \) on \( \Gamma_0(N) \) has a Fourier expansion of the form
\[
 f(z) = f_0(y) + \sum_{n \neq 0} \frac{a_n}{2|n|} W_\nu(ny)e(nx),
\]
where \( a_n \in \mathbb{C} \ (n \neq 0) \), \( W_\nu(u) = 4\sqrt{|u|}K_\nu(2\pi|u|) \) is the Whittaker function and \( K_\nu(u) = \frac{1}{2} \int_0^\infty e^{-u(t+t^{-1})/2}t^\nu dt \) is a Bessel function. The term \( f_0(y) \) will be made explicit below. By diagonalising with respect to the involution \( \iota : z \mapsto -\overline{z} \) on \( \mathcal{H} \), we may assume that \( f \) is either even \( (a_n = a_{-n}) \) or odd \( (a_n = -a_{-n}) \). The parity \( \epsilon \in \{0,1\} \) of \( f \) is 0 if \( f \) is even and 1 if it is odd. By \( \cos^{(k)} \) we denote the \( k \)-th derivative of \( \cos \). If \( f \) has parity \( \epsilon \), then
\[
 f(z) = f_0(y) + (-i)^\epsilon \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} W_\nu(ny) \cos^{(\epsilon)}(2\pi nx),
\]
where
\[
 f_0(y) = \begin{cases} \frac{1}{2} y^{\frac{1}{2}+\nu} + a_0' y^{\frac{1}{2}-\nu}, & \epsilon = 0, \nu \neq 0, \\
 \frac{1}{2} y^{\frac{1}{2}} + a_0' \log y, & \epsilon = 0, \nu = 0, \\
 0, & \epsilon = 1,
\end{cases}
\]
for \( a_0, a_0' \in \mathbb{C} \). To a Maass form we can associate an \( L \)-function a priori defined on the right half plane \( \text{Re} \, s > 3/2 \) by the Dirichlet series
\[
 L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}.
\]
The completed \( L \)-function of \( f \) is defined as
\[
 \Lambda_f(s) = \Gamma_R(s + \epsilon + \nu) \Gamma_R(s + \epsilon - \nu) L_f(s).
\]
The following facts are proved in Appendix A.1: If \( \epsilon = 1 \), then the function \( \Lambda_f(s) \) continues to an entire function; if \( \epsilon = 0 \), then \( \Lambda_f(s) \) has meromorphic continuation to \( \mathbb{C} \) with possible simple poles in the set \( \{ \pm \nu, 1 \pm \nu \} \) if \( \nu \neq 0 \) and at most double poles in \( \{0,1\} \) if \( \nu = 0 \). Setting \( g(z) = f(-1/Nz) \) the \( L \)-functions \( \Lambda_f(s) \) and \( \Lambda_g(s) \), along with their twists, satisfy the functional equations (1.5).

3. Proof of Theorem 1.1

From now on assume that \( a_n, b_n \) are sequences as in Theorem 1.1. We define \( f, g \) and the twisted \( L \)-functions associated to \( f \) and \( g \) as in that Theorem.

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2 The Selberg eigenvalue conjecture asserts that \( \nu \) is purely imaginary.
3.1. Additive Twists. Let $q \in \mathcal{P}$ be a prime such that $(q,N) = 1$ and set $\alpha = \frac{a}{q} \in \mathbb{Q}$ for some $a \in \mathbb{Z}$. We use the following notation as in [BCK18].

**Definition 3.1.** Let $k \in \mathbb{Z}_{\geq 0}$. For $\alpha \in \mathbb{Q} \times$, the additive twists of $L_f(s)$ by $\alpha$ are

$$L_f(s, \alpha, \cos(k)) = \sum_{n=1}^{\infty} \cos(k)(2\pi n \alpha) a_n n^{-s}. \quad (3.1)$$

Up to sign, Definition 3.1 depends only on $k$ modulo 2.

**Lemma 3.2.** We have

$$L_f(s, \alpha, \cos(k)) = \frac{i^k}{q-1} \sum_{\psi \pmod{q}} \tau(\psi) \psi(a) L_f(s, \psi)$$

$$+ \begin{cases} (-1)^{k/2} \left[ L_f(s) - \frac{q}{q-1} L_f(s, \psi_0) \right], & k \text{ even,} \\ 0, & k \text{ odd,} \end{cases}$$

where the sums are over Dirichlet characters modulo $q$, and $\psi_0$ denotes the trivial Dirichlet character mod $q$.

**Proof.** This follows from

$$\cos \left( \frac{2\pi na}{q} \right) = 1 - \frac{q}{q-1} \psi_0(n) + \frac{1}{q-1} \sum_{\psi \pmod{q}} \tau(\psi) \psi(an),$$

$$\sin \left( \frac{2\pi na}{q} \right) = -\frac{i}{q-1} \sum_{\psi \pmod{q}} \tau(\psi) \psi(an).$$

Define

$$\gamma_f^{(-)}(s) = \Gamma_{\mathbb{R}}(s + [k + \epsilon] + \nu) \Gamma_{\mathbb{R}}(s + [k + \epsilon] - \nu), \quad (3.2)$$

where $(-)^k$ denotes $+$ if $k$ is even and $-$ if $k$ is odd. As explained in the introduction, for $m \in \mathbb{Z}$ we write $[m]$ for the element in $\{0,1\}$ with the same parity as $m$. We see that $\Lambda_f(s) = \gamma_f^{+}(s) L_f(s)$.

We define the completion of the additive twists by

$$\Lambda_f(s, \alpha, \cos(k)) = \gamma_f^{(-)}(s) L_f(s, \alpha, \cos(k))$$

$$\frac{i^k}{q-1} \sum_{\psi \pmod{q}} \tau(\psi) \psi(a) \Lambda_f(s, \psi) + \begin{cases} (-1)^{k/2} \left[ \Lambda_f(s) - \frac{q}{q-1} \Gamma_{\mathbb{R}}(s + [k + \epsilon] - \nu) \right], & k \text{ even,} \\ 0, & k \text{ odd.} \end{cases} \quad (3.3)$$
Proposition 3.3. The additive twists satisfy the following functional equations

\[
\Lambda_f(s, \alpha, \cos(k)) = (-1)^k \frac{i^k(q^2N)^{\frac{s}{2} \chi(q)}}{q - 1} \sum_{\psi \pmod{q}, \psi \neq \psi_0, \psi(-1)^k} \psi(Na) \tau(\psi) \Lambda_g(1 - s, \psi)
\]

\[
+ \begin{cases} (-1)^{k/2} [\Lambda_f(s) - \frac{q}{a} \Lambda_f(s, \psi_0)], & k \text{ even,} \\ 0, & k \text{ odd.} \end{cases}
\]

For the following Lemma, recall the (Gauss) hypergeometric function \( _2F_1 \) which is reviewed in appendix A.2.

Lemma 3.4 (6.699(3-4) in [GR15]). Let \( w \in \mathbb{R}, k \geq 0, \) and \( \nu \in \mathbb{C} \). One has, for \( \Re(-s \pm \nu) < \epsilon \),

\[
4 \int_0^\infty K_\nu(2y) \cos(k)(2wy) y^s \frac{dy}{y} = i^k (2w)^{\nu} \pi^s \gamma_f^{(-k) + \epsilon}(s) \ _2F_1 \left( \frac{s + \nu + [k]}{2}, \frac{s - \nu + [k]}{2}; \frac{-w^2}{4} \right).
\]

Proposition 3.5. Let \( h(z) = (-i)^\epsilon \sum_{n=1}^\infty c_n W_\nu(ny) \cos(\nu)(2\pi n/w) \) with polynomially bounded \( c_n, \epsilon \in \{0, 1\}, w \in \mathbb{R} \) and \( \alpha \in \mathbb{Q} \). Define \( \Lambda_h(s) \) and its twists just as in (1.4) and (3.1) with \( a_n \) replaced by \( c_n \). One has

\[
(3.4) \quad \int_0^\infty h(iy + wy + \alpha)y^{s - \frac{1}{2}} dy = \sum_{j \in \{0, 1\}} i^{-j} (2w)^{[j + \epsilon]} \Lambda_h(s, \alpha, \cos(j)) \ _2F_1 \left( \frac{s + \nu + [k]}{2}, \frac{s - \nu + [k]}{2}; \frac{-w^2}{4} \right).
\]

Proof. The result follows readily from the following computation which uses Lemma 3.4:

\[
\int_0^\infty h(iy + wy + \alpha)y^{s - \frac{1}{2}} dy
\]

\[
= 4(-i)^\epsilon \sum_{n=1}^\infty c_n \int_0^\infty K_\nu(2\pi ny) \cos(\nu)(2\pi n(w + \alpha))y^s \frac{dy}{y}
\]

\[
= 4(-i)^\epsilon \sum_{n=1}^\infty c_n \sum_{j \in \{0, 1\}} (-1)^j \cos(j)(2\pi n) \int_0^\infty K_\nu(2\pi ny) \cos(\epsilon + j)(2\pi nw) y^s \frac{dy}{y}
\]

\[
= 4(-i)^\epsilon \pi^{-s} \sum_{n=1}^\infty c_n n^{-s} \sum_{j \in \{0, 1\}} (-1)^j \cos(j)(2\pi n) \int_0^\infty K_\nu(2y) \cos(\epsilon + j)(2wy) y^s \frac{dy}{y}
\]

\[
= \sum_{n=1}^\infty c_n n^{-s} \sum_{j \in \{0, 1\}} i^j \cos(j)(2\pi n)(2w)^{[j + \epsilon]} \gamma_f^{(-j)}(s) \ _2F_1 \left( \frac{s + \nu + [k]}{2}, \frac{s - \nu + [k]}{2}; \frac{-w^2}{4} \right).
\]

3.2. Transformation properties from the functional equation of \( \Lambda_f \). We first show that \( f(z) = g(-1/Nz) \) follows from the functional equation of \( \Lambda_f(s) \). Define \( \tilde{f}, \tilde{g} \) by equation...

□
Let $z = wy + iy \in H$ with $w \in \mathbb{R}_{\geq 0}$. If $c > 1 + |\text{Re }\nu|$, then by Proposition 3.5 for $\alpha = 0$ we can obtain $\tilde{f}(wy + iy)$ as an inverse Mellin transform.

\begin{equation}
\tilde{f}(wy + iy) = \frac{(2w)^\epsilon}{2\pi i} \int_{(c)} \Lambda_f(s) \ _2F_1\left(\frac{s + \nu}{2}, \frac{s + \nu}{2} \left| w^2 \right. \right) y^{\frac{s - \epsilon}{2}} ds.
\end{equation}

Shifting the path of integration to the left this equals

\begin{equation}
\frac{(2w)^\epsilon}{2\pi i} \int_{(1-c)} \Lambda_f(s) \ _2F_1\left(\frac{s + \nu}{2}, \frac{s + \nu}{2} \left| w^2 \right. \right) y^{\frac{s - \epsilon}{2}} ds + H(z),
\end{equation}

where

\begin{equation}
H(z) = \begin{cases} 0, & \epsilon = 1, \\ \sum_{x \in \{1, \pm \nu, \pm \nu\}} \text{Res}_{s=x} \ _2F_1\left(\frac{s + \nu}{2}, \frac{s + \nu}{2} \left| w^2 \right. \right) \Lambda_f(s)y^{\frac{s - \epsilon}{2}}, & \epsilon = 0. \end{cases}
\end{equation}

Now we apply the functional equation to $\Lambda_f(s)$ and the Euler identity (A.11) to obtain

\begin{equation}
\tilde{f}(wy + iy) = \frac{(2w)^\epsilon}{2\pi i} \int_{(c)} \Lambda_f(1-s) \ _2F_1\left(\frac{1-s+\nu}{2}, \frac{1-s+\nu}{2} \left| w^2 \right. \right) y^{s-\frac{\epsilon}{2}} ds + H(z)
\end{equation}

\begin{align*}
= & \frac{(2w)^\epsilon}{2\pi i} \int_{(c)} (-1)^s N^{s-\frac{\nu}{2}} \Lambda_g(s)(1 + w^2)^{s-\frac{\nu}{2}} \ _2F_1\left(\frac{s+\nu}{2}, \frac{s+\nu}{2} \left| w^2 \right. \right) y^{s-\frac{\epsilon}{2}} ds + H(z) \\
= & \frac{(-2w)^\epsilon}{2\pi i} \int_{(c)} \Lambda_g(s) \ _2F_1\left(\frac{s+\nu}{2}, \frac{s+\nu}{2} \left| w^2 \right. \right) (1 + w^2) y^{s-\frac{\epsilon}{2}} ds + H(z)
\end{align*}

\begin{equation}
= \tilde{g} \left( -\frac{1}{N(wy + iy)} \right) + H(z).
\end{equation}

If $\epsilon = 1$, then this shows $f(z) = g(-1/Nz)$. Let $\epsilon = 0$ and $\nu \neq 0$. The hypergeometric functions occurring in $H(z)$ can be evaluated directly\textsuperscript{3}:

\begin{equation}
_2F_1\left(\frac{\nu}{2}, \frac{\nu}{2} \left| w^2 \right. \right) = (1 + w^2)^{-\frac{\nu}{2}}, \ _2F_1\left(\frac{\nu}{2}, \frac{0}{2} \left| w^2 \right. \right) = \ _2F_1\left(\frac{0}{2}, \frac{-\nu}{2} \left| w^2 \right. \right) = 1.
\end{equation}

So

\begin{equation}
H(z) = ((1 + w^2)y)^{-\frac{\nu}{2}} \text{Res}_{s=1+\nu} \Lambda_f(s) + ((1 + w^2)y)^{-\frac{\nu}{2}} \text{Res}_{s=1-\nu} \Lambda_f(s) + y^{\frac{\nu}{2}} \text{Res}_{s=\nu} \Lambda_f(s) + y^{\frac{\nu}{2}} \text{Res}_{s=-\nu} \Lambda_f(s)
\end{equation}

\begin{align*}
= & \text{Res}_{s=\nu} \Lambda_f(s)y^{\frac{\nu}{2}} + \text{Res}_{s=-\nu} \Lambda_f(s)y^{\frac{\nu}{2}} + N^{\frac{\nu}{2}} \text{Res}_{s=1+\nu} \Lambda_g(s) \text{Im} \left( -\frac{1}{Nz} \right)^{\frac{\nu}{2}} \\
& + N^{\frac{\nu}{2}} \text{Res}_{s=1-\nu} \Lambda_g(s) \text{Im} \left( -\frac{1}{Nz} \right)^{\frac{\nu}{2}}.
\end{align*}

\textsuperscript{3}http://dlmf.nist.gov/15.4.6
It follows that \( f(z) = g(-1/Nz) \) in this case. Now let \( \epsilon = 0 \) and \( \nu = 0 \). The first term in \( H(z) \) is

\[
\text{Res}_{s=0} 2F1 \left( \frac{\epsilon}{2}, \frac{\nu}{2} \mid -w^2 \right) \Lambda_f(s) y^{\frac{1}{2} - s} = \lim_{s \to 0} \frac{d}{ds} \left( s^2 2F1 \left( \frac{\epsilon}{2}, \frac{\nu}{2} \mid -w^2 \right) \Lambda_f(s) y^{\frac{1}{2} - s} \right) = \lim_{s \to 0} \frac{d}{ds} 2F1 \left( \frac{\epsilon}{2}, \frac{\nu}{2} \mid -w^2 \right) y^{\frac{1}{2} - s} (a_0 - sa_0 + s^2 E_0(s)),
\]

where \( E_0(s) \) is holomorphic at \( s = 0 \). To evaluate the limit, expand the hypergeometric function around \( s = 0 \):

\[
2F1 \left( \frac{\epsilon}{2}, \frac{\nu}{2} \mid -w^2 \right) = \sum_{n=0}^{\infty} \frac{(-w^2)^n}{(\frac{\nu}{2})_n n!} (\frac{s}{2})_n = 1 + s^2 F_0(s),
\]

where \( F_0(s) \) is holomorphic at \( s = 0 \). We see that

\[
\text{Res}_{s=0} 2F1 \left( \frac{\epsilon}{2}, \frac{\nu}{2} \mid -w^2 \right) \Lambda_f(s) y^{\frac{1}{2} - s} = \text{Res}_{s=0} \Lambda_f(s) y^{\frac{1}{2}} - \text{Res}_{s=0} s \Lambda_f(s) y^{\frac{1}{2}} \log y.
\]

Using the (A.11) and the functional equation of \( \Lambda_f(s) \)

\[
\text{Res}_{s=1} 2F1 \left( \frac{\epsilon}{2}, \frac{\nu}{2} \mid -w^2 \right) \Lambda_f(s) y^{\frac{1}{2} - s} = \text{Res}_{s=1} (1 + w^2)Ny^{\frac{1}{2} - s} 2F1 \left( \frac{\epsilon}{2}, \frac{\nu}{2} \mid -w^2 \right) \Lambda_g(1 - s) = -\text{Res}_{s=0} (1 + w^2)Ny^{\frac{1}{2} - s} 2F1 \left( \frac{\epsilon}{2}, \frac{\nu}{2} \mid -w^2 \right) \Lambda_g(s) = \text{Im} \left( -\frac{1}{Nz} \right)^{\frac{1}{2}} \left( -\text{Res}_{s=0} \Lambda_g(s) + \text{Res}_{s=0} s \Lambda_g(s) \log \left( \text{Im} \left( -\frac{1}{Nz} \right) \right) \right).
\]

As before we conclude \( f(z) = g(-1/Nz) \).

### 3.3. Transformation properties from twisted functional equations

In this section we deduce further transformation properties of \( f \) and \( g \) from the twisted functional equations. For a primitive Dirichlet character \( \psi \) modulo a prime \( q \) we write

\[
f_\psi(z) = \sum_{n \neq 0} \frac{\psi(n) \alpha_n}{2\sqrt{|n|}} W_\nu(n y) e(n x)
\]

and define \( g_\psi \) analogously. With the definition of Proposition \( 3.5 \) we have \( \Lambda_f(s, \psi) = \Lambda_{f_\psi}(s) \).

**Lemma 3.6.** Let \( q \in \mathcal{P} \) and let \( \alpha = a/q, \beta = b/q \) and let \( z = wy + iy \in \mathcal{H} \). With the assumptions of Theorem 1.1, one has

\[
f(z + \alpha) - f(z + \beta) = \frac{\chi(q)}{q - 1} \sum_{\psi \equiv (mod q)} \psi(-N) (\psi(a) - \psi(b)) \tau(\psi) g_\psi \left( -\frac{1}{Nq^2z} \right).
\]

**Proof.** As \( \alpha, \beta \in \mathbb{R} \), we have

\[
f(z + \alpha) - f(z + \beta) = \tilde{f}(z + \alpha) - \tilde{f}(z + \beta)
\]
Applying the inverse Mellin transform to Proposition 3.5

\[ f(z + \alpha) - f(z + \beta) = \sum_{j \in \{0, 1\}} i^{-j} (2w)^{j+\epsilon} \frac{1}{2\pi i} \int_{(c)} (\Lambda_f(s, \alpha, \cos(j)) - \Lambda_f(s, \beta, \cos(j))) \]

\[ \cdot \quad \frac{1}{2\pi i} \int_{(c)} \left( \frac{\Lambda_f(s, \alpha, \cos(j)) - \Lambda_f(s, \beta, \cos(j))}{2\pi i} \right) \cdot \quad 2F_1 \left( \frac{\frac{s-v}{2}, \frac{s-v}{2}}{2} \right) - w^2 \right) y^{\frac{1}{2} - s} ds. \]

By Lemma 3.2, \( \Lambda_f(s, \alpha, \cos(j)) - \Lambda_f(s, \beta, \cos(j)) \) is a linear combination of twists of \( \Lambda_f(s) \) by characters of conductor \( q \). Therefore, by the assumptions of Theorem 1.1, it is entire and we can shift the path of integration to the left. The integral in equation (3.10) equals

\[ \int_{(1-e)} (\Lambda_f(s, \alpha, \cos(j)) - \Lambda_f(s, \beta, \cos(j))) \cdot \quad 2F_1 \left( \frac{\frac{s-v}{2}, \frac{s-v}{2}}{2} \right) - w^2 \right) y^{\frac{1}{2} - s} ds \]

\[ = \int_{(c)} (\Lambda_f(s, \alpha, \cos(j)) - \Lambda_f(s, \beta, \cos(j))) \cdot \quad 2F_1 \left( \frac{\frac{s-v}{2}, \frac{s-v}{2}}{2} \right) - w^2 \right) y^{\frac{1}{2} - s} ds \]

\[ = (-1)^\epsilon \int_{(c)} i^{j}(q^2 N)^s \frac{\chi(q)}{q-1} \sum_{\substack{\psi \mod q \\ \psi \neq 0 \\ \psi(-1) = (-1)^j}} \psi(N)(\psi(a) - \psi(b))\tau(\psi)\Lambda_g(s, \psi) \]

\[ \cdot \quad 2F_1 \left( \frac{\frac{s-v}{2}, \frac{s-v}{2}}{2} \right) - w^2 \right) y^{\frac{1}{2} - s} ds. \]

The last line follows from the functional equation in Proposition 3.3. Applying the Euler identity (A.11) we see that \( f(z + \alpha) - f(z + \beta) \) equals

\[ (-1)^\epsilon \frac{\chi(q)}{q-1} \sum_{j \in \{0, 1\}} \sum_{\substack{\psi \mod q \\ \psi \neq 0 \\ \psi(-1) = (-1)^j}} (2w)^{j+\epsilon} \psi(N)\tau(\psi)(\psi(a) - \psi(b)) \]

\[ \cdot \quad \frac{1}{2\pi i} \int_{(c)} \Lambda_g(s, \psi) \cdot \quad 2F_1 \left( \frac{\frac{s-v}{2}, \frac{s-v}{2}}{2} \right) - w^2 \right) \left( \frac{1}{1 + w^2} Nq^2 y \right)^{\frac{1}{2} - s} ds \]

\[ = \frac{\chi(q)}{q-1} \sum_{\substack{\psi \mod q \\ \psi \neq 0}} \psi(-N)(\psi(a) - \psi(b))\tau(\psi)g_\psi \left( -\frac{1}{Nq^2 z} \right). \]

In the last line we applied the inverse Mellin transform to the equality in Proposition 3.5 for \( h = g_\psi \).

\[ \square \]

**Proposition 3.7.** If \( a_n \) (resp. \( b_n \)) satisfy the assumptions of Theorem 1.1, then,

\[ f_\psi(z) = \chi(q)\psi(-N)\frac{\tau(\psi)}{\tau(\psi)}g_\psi \left( -\frac{1}{Nq^2 z} \right) \]

for all non-principal characters \( \psi \) modulo \( q \).
Proof. By the proof of Lemma 3.2,
\[
f(z + \alpha) - f(z + \beta) = \frac{1}{q-1} \sum_{\psi \not\equiv \psi_0 \pmod{q}} \tau(\psi) (\psi(a) - \psi(b)) f_{\psi}(z).
\]
So equation (3.8) can be rearranged to
\[
\sum_{\psi \not\equiv \psi_0 \pmod{q}} \psi(a) \left( \tau(\psi) f_{\psi}(z) - \chi(q) \psi(-N) \tau(\psi) \bar{\psi} \left(-\frac{1}{Nq^2z}\right) \right)
= \sum_{\psi \not\equiv \psi_0 \pmod{q}} \psi(b) \left( \tau(\psi) f_{\psi}(z) - \chi(q) \psi(-N) \tau(\psi) \bar{\psi} \left(-\frac{1}{Nq^2z}\right) \right).
\]
This implies that the expression on the left-hand side is independent of the choice of \(a \not\equiv 0 \pmod{q}\). In other words, it is a linear combination of non-principal characters modulo \(q\) that produces a multiple of the principal character. Since the set of all characters modulo \(q\) is linearly independent, the coefficients of this linear combination must vanish. \(\square\)

Proposition 3.7 is the analogue of [Bum98, Chapter 1, equation (5.13)] for Maass forms and we continue along the lines of [Bum98, Section 1.5]. Note that for a primitive Dirichlet character modulo \(q \in \mathcal{P}\) we have
\[
 f_{\psi} = \tau(\psi)^{-1} \sum_{a \mod q} \overline{\psi(a)} f \left( \begin{array}{c} 1/aq \\ 0 \\ 1 \end{array} \right).
\]
So, using the fact that \(f|(-N^{-1}b) = g\) which was proved in the previous section
\[
\tau(\psi) f_{\psi} \left(-\frac{1}{Nq^2z}\right) = \sum_{a \mod q} \overline{\psi(a)} f \left(\begin{array}{c} q \\ -Naq \\ -\tilde{a} \end{array}\right) \left(\begin{array}{c} q \tilde{a}/q \\ 1 \\ 1 \end{array}\right) = \sum_{a \mod q} \overline{\psi(a)} g \left(\begin{array}{c} q \tilde{a}/q \\ N\tilde{a} + 1 \\ q \end{array}\right).\]
Choosing for every \(a \mod q\) an element \(\tilde{a}\) that is inverse to \(-Na \mod q\) the above equals
\[
\sum_{a \mod q} \overline{\psi(a)} g \left(\begin{array}{c} q \tilde{a}/q \\ N\tilde{a} + 1 \\ q \end{array}\right) = \psi(-N) \sum_{a \mod q} \psi(a) g \left(\begin{array}{c} q \tilde{a}/q \\ N\tilde{a} + 1 \\ q \end{array}\right) \left(\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right).
\]
In the last sum we swapped \(a\) and \(\tilde{a}\) and used that \(\overline{\psi(a)} = \psi(-N) \psi(\tilde{a})\). Proposition 3.7 states that this is also equal to \(\chi(q) \psi(-N) \tau(\psi) g_{\psi}\). So
\[
\chi(q) \sum_{a \mod q} \psi(a) g \left(\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right) = \sum_{a \mod q} \psi(a) g \left(\begin{array}{c} q \\ -N\tilde{a} \tilde{a}/q \\ q \end{array}\right) \left(\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right)
\]
This equation is true for all primitive characters \(\psi\) modulo \(q\). Taking linear combinations we see that we can replace \(\psi(a)\) above with any function \(c(a)\) on \((\mathbb{Z}/N\mathbb{Z})^\times\) that satisfies \(\sum_{a \mod q} c(a) = 0\). This is exactly the starting point of the discussion in [Bum98, p. 63-64]. We assume that \(q\) is odd. Note that for a specific \(a\) we can choose \(\tilde{a}\) so that \(s = (N\tilde{a} + 1)/q \in \mathcal{P}\) and is odd and different from \(q\). Using this equation and the analogous one with \(q\) and \(s\) replaced Bump constructs an elliptic matrix
\[
M = \left(\begin{array}{cc}
\frac{1}{2N} & 2a/q \\
-2Na/s & -3 + \frac{4}{Ds} 
\end{array}\right)
\]
of infinite order under which \( g_1 = g| (\frac{P}{N_a} \circ a) - \chi(q) g \) is invariant. Conversely we could start with a choice of \( D, s \in \mathcal{P} \), choose \( a \) and \( \bar{a} \) such that \( (\frac{q}{N_a} \circ a) \in SL_2(\mathbb{Z}) \) and follow Bump’s construction to get an elliptic operator under which \( g_1 \) is invariant. Our aim is to show that \( g_1 \) is identically zero, so in the next section we study functions that are invariant under elliptic operators.

3.4. Two Circles. Recall that a matrix \( M \in SL_2(\mathbb{R}) \) is elliptic if \( |\text{tr}(M)| < 2 \). This happens if and only if \( M \) has a unique fixed point in \( \mathcal{H} \). Weil’s lemma states that a holomorphic function on the upper half-plane invariant under an infinite order elliptic matrix is constant [Bum98, Lemma 1.5.1]. We begin by proving a geometric interpretation of this statement.

**Lemma 3.8.** Let \( h \) be a continuous function on \( \mathcal{H} \) that is invariant under an infinite order elliptic operator \( M \) with fixed point \( z_0 \in \mathcal{H} \). Then \( h \) is constant on the hyperbolic circles around \( z_0 \).

**Proof.** Let \( K = \frac{1}{\sqrt{2a+b}}\left(\begin{array}{cc} 1 & -a \\ 1 & b \end{array}\right) \) be the Cayley transform that maps the upper half plane \( \mathcal{H} \) to the open unit disk \( \mathcal{D} \) and takes \( z_0 \in \mathcal{H} \) to \( 0 \in \mathcal{D} \). The transformation \( L = KMK^{-1} \) on \( \mathbb{P}^1(\mathbb{C}) \) fixes 0 and \( \infty \) and hence has the form \( \left(\begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array}\right) \) with irrational \( \theta \), because \( M \) has infinite order. Consider the function \( \tilde{h}(z) = h(K^{-1}z) \). Since

\[
\tilde{h}(Lz) = h(K^{-1}Lz) = h(K^{-1}LK^{-1}z) = h(K^{-1}z) = \tilde{h}(z),
\]

we get \( \tilde{h}(e^{2i\pi \theta}z) = \tilde{h}(z) \) for all \( m \in \mathbb{Z} \). Since the set \( \{e^{2i\pi \theta} | m \in \mathbb{Z}\} \) is dense on the unit circle we get that \( \tilde{h}(z) = \tilde{h}(|z|) \) and hence \( \tilde{h} \) is constant on all circles \( C_r = \{re^{it} | t \in [0, 2\pi)\} \). This implies that \( h \) is constant on the preimages under the Cayley transform of the circles \( C_r \). These are exactly the hyperbolic circles around \( z_0 \).

A corollary of this is that if \( h \) is holomorphic, then it is constant on all of \( \mathcal{H} \). There are counter-examples to this in the real-analytic setting of Maass forms. Instead, we resort to the following theorem.

**Theorem 3.9.** If \( h \) is a continuous function on \( \mathcal{H} \) that is invariant under two infinite order elliptic operators with distinct fixed points in \( \mathcal{H} \) then it is constant.

**Proof.** Let \( M_1, M_2 \) be the two elliptic operators and \( z_1, z_2 \) be their fixed points in \( \mathcal{H} \). Let \( K \) be the Cayley transform that maps \( z_1 \) to 0. By the above proof \( \tilde{h}(z) = h(K^{-1}z) \) is constant on all circles around 0.

Let \( d = d_{\text{hyp}} \) be the hyperbolic distance on \( \mathcal{D} \) and \( y \in \mathcal{D} \). If \( d(0, y) \leq d(0, Kz_2) \), then the circle of radius \( d(0, y) \) around 0 has two intersection points with the line connecting 0 and \( Kz_2 \). Since this line is a geodesic we can deduce the following. One of the intersection points, \( y_1 \), is between 0 and \( Kz_2 \), satisfying \( d(y_1, Kz_2) \leq d(0, y_1) + d(y_1, Kz_2) = d(0, Kz_2) \). The other one, \( y_2 \), satisfies \( d(y_2, Kz_2) = d(y_2, 0) + d(0, Kz_2) \geq d(0, Kz_2) \). By the intermediate value theorem the circle also contains an element \( y_3 \) with \( d(y_3, Kz_2) = d(0, Kz_2) \). Since \( \tilde{h} \) is constant on circles around the origin we have \( \tilde{h}(y) = \tilde{h}(y_3) \). Now \( \tilde{h} \) is also constant around hyperbolic circles with centre \( Kz_2 \). Since \( d(y_3, Kz_2) = d(0, Kz_2) \), we obtain \( \tilde{h}(y_3) = \tilde{h}(0) \). This implies that \( \tilde{h} \) is constant on the disc of radius \( d(0, Kz_2) \).

Now suppose \( \tilde{h} \) is constant on the closed disc of radius \( r \geq d(0, Kz_2) \) around the origin. Then \( \tilde{h} \) contains a point \( y \) with \( d(y, Kz_2) = r + d(0, Kz_2) \) and the point \( Kz_2 \) with \( d(Kz_2, Kz_2) =
0. Hence \( \tilde{h} \) is also constant on the closed disc of radius \( r+d(0, Kz_2) \) around \( Kz_2 \). Repeating this process we see that \( \tilde{h} \) is constant.

\[\square\]

3.5. **Proof of Theorem 1.1.** We now combine the results of the previous sections to deduce Theorem 1.1. The essential point is the construction of two infinite order elliptic matrices.

**Proof.** Let \( \gamma = (a, b) \in \Gamma_0(N) \). The proof will follow if we can find two infinite order elliptic matrices under which \( g_1 = g \left( \frac{a}{Nc}, \frac{b}{d} \right) - \psi(a)g \) is constant. Let \( D, s \) be two odd primes in \( P \) such that \( D \equiv a \) and \( s \equiv d \) modulo \( Nc \), i.e., \( D = a - uNc \) and \( s = d - vNc \) for \( u, v \in \mathbb{Z} \). By the assumptions on \( P \) there are infinitely many such \( D \) and \( s \). Let \( r = b - av + uvNc - ud \). Then

\[ g \left( \frac{a}{Nc}, \frac{b}{d} \right) = g \left( \frac{a}{Nc}, \frac{b}{d} \right) \left( \frac{1}{0} \right) \left( \frac{1}{0} \right) = g \left( \frac{a}{Nc}, \frac{b}{d} \right) \left( \frac{1}{0} \right) . \]

By the considerations at the end of Section 3.3 we can show that the function \( g_1 = g \left( \frac{a}{Nc}, \frac{b}{d} \right) - \psi(a)g \) is invariant under the elliptic operator

\[ M_{D,s} = \left( \frac{1}{2Nc}, \frac{2Nc}{4D_s} - 3 \right) . \]

The fixed point of \( M_{D,s} \) in \( \mathcal{H} \) is given by

\[ z_1 = i \sqrt{- \frac{s^2 \left( \frac{D_s}{Nc} \right) - 1}{2} + \frac{4 r s}{D Nc} - \frac{s \left( \frac{1}{D_s} - 1 \right)}{Nc}} . \]

Let \( D' = a - u'Nc \equiv a \mod N \) be another odd prime in \( P \) as above and \( r' = -av + u'vNc + b - u'd \). We have

\[ g \left( \frac{a}{Nc}, \frac{b}{d} \right) = g \left( \frac{a}{Nc}, \frac{b}{d} \right) \left( \frac{1}{0} \right) \left( \frac{1}{0} \right) = g \left( \frac{a}{Nc}, \frac{b}{d} \right) . \]

As above we can show that \( g_1' = g \left( \frac{a}{Nc}, \frac{b}{d} \right) - \psi(a)g = g_1 \) is invariant under \( M_{D',s} \). The fixed point of \( M_{D',s} \) is

\[ z_2 = i \sqrt{- \frac{s^2 \left( \frac{D'}{Nc} \right) - 1}{2} + \frac{4 r' s}{D' Nc} - \frac{s \left( \frac{1}{D'} - 1 \right)}{Nc}} . \]

Comparing real parts we find \( z_1 \neq z_2 \) if \( D \neq D' \). Hence by Theorem 3.9 \( g_1 \) is constant. Since it converges to zero for \( z \to i \infty \), \( g_1 \) vanishes. This implies

\[ g \left( \frac{a}{Nc}, \frac{b}{d} \right) = g \left( \frac{a}{Nc}, \frac{b}{d} \right) \left( \frac{1}{0} \right) \left( \frac{1}{0} \right) = \psi(a)g \left( \frac{1}{0} \right) = \psi(a)g . \]

\[\square\]

4. **Proof of Theorem 1.2**

In this section we will prove Theorem 1.2 by showing that its assumptions imply those of Theorem 1.1. To that end, let \( f \) and \( g \) be as in Theorem 1.2. From now on we assume that \( \nu \neq 0 \), though this can be relaxed with technical modifications.

**Lemma 4.1.** For \( w \in \mathbb{R}_{\geq 0} \), the function \( \Lambda_f(s) \frac{\Gamma \left( \frac{s + i \nu}{2} \right) \frac{\Gamma \left( \frac{s + i \nu}{2} + i \epsilon \nu}{\frac{1}{2} + i \epsilon} \right)}{w^2} \) is \( O \left( (\text{Im} s)^{-M} \right) \) uniformly in \( \text{Re} s \) for every \( M > 0 \) as \( \text{Im} s \to \infty \). Its poles lie in the strip \( -\sigma \leq \text{Re} s \leq \sigma + 1 \). It is uniformly bounded on every vertical strip outside of a small neighbourhood around every pole.
Proof. From Appendix A.2, it follows that \( \Lambda(f, s) \ _2F_1 \left( \begin{array}{c} \frac{s+1+w}{2}, \frac{s+1-v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) \) decays exponentially for fixed real part \( \text{Re} \ s > \sigma + 1 \) and \( \text{Im} \ s \to \infty \). The functional equation (1.5) and the Euler identity (A.11) imply
\[
\Lambda_f(s) \ _2F_1 \left( \begin{array}{c} \frac{s+1+w}{2}, \frac{s+1-v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) - w^2 = (-1)^s(N(1+w^2))^{\frac{s}{2}} - \Lambda_g(1-s) \ _2F_1 \left( \begin{array}{c} \frac{1-s+w}{2}, \frac{1-s+v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) - w^2.
\]

Therefore, we have exponential decay for \( \text{Re} \ s < -\sigma \). The occurring hypergeometric functions are entire in \( s \). For \( P(s) \in \mathbb{C}[s] \) as in Theorem 1.2, the function \( P(s)\Lambda(f, s) \) is entire, of finite order and bounded polynomially on vertical lines that lie outside of the critical strip. By the Phragmén-Lindelöf principle it is also uniformly bounded polynomially inside the critical strip, and hence \( \Lambda(f, s) \ _2F_1 \left( \begin{array}{c} \frac{s+1+w}{2}, \frac{s+1-v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) - w^2 \) decays exponentially on these strips. \( \square \)

**Lemma 4.2.** Integrating over any circle enclosing all poles of \( \Lambda_f(s) \) we have, for any \( z \in \mathcal{H} \),
\[
\tilde{f}(z) - \tilde{g} \left( \frac{-1}{Nz} \right) = \frac{(2w)^\epsilon}{2\pi i} \int_{c} \Lambda_f(s) \ _2F_1 \left( \begin{array}{c} \frac{s+1+w}{2}, \frac{s+1-v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) - w^2 \right) y^{\frac{1}{2}-s} ds,
\]
where \( w = \text{Re} \ z/\text{Im} \ z \).

Proof. Let \( z = wy + iy \) and \( c > \sigma \). Applying the inverse Mellin transform to Proposition 3.5 with \( h = \tilde{f} \) and \( \alpha = 0 \) implies
\[
\tilde{f}(z) = \frac{(2w)^\epsilon}{2\pi i} \int_{c} \Lambda_f(s) \ _2F_1 \left( \begin{array}{c} \frac{s+1+w}{2}, \frac{s+1-v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) - w^2 \right) y^{\frac{1}{2}-s}.
\]

Similarly, making the change of variables \( s \to 1 - s \) and applying the functional equation for \( \Lambda_f \) along with the Euler identity, we see that
\[
\tilde{g} \left( \frac{-1}{Nz} \right) = \frac{(2w)^\epsilon}{2\pi i} \int_{c} \Lambda_g(s) \ _2F_1 \left( \begin{array}{c} \frac{s+1+w}{2}, \frac{s+1-v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) - w^2 \right) (Ny(1+w^2))^{\frac{s}{2}} ds
\]
\[
= \frac{(2w)^\epsilon}{2\pi i} \int_{c} \Lambda_f(s) \ _2F_1 \left( \begin{array}{c} \frac{s+1+w}{2}, \frac{s+1-v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) - w^2 \right) y^{\frac{1}{2}-s} ds.
\]
Since the integrand is meromorphic and rapidly decaying in \( s \) we can write
\[
\tilde{f}(z) - \tilde{g} \left( \frac{-1}{Nz} \right) = \frac{(2w)^\epsilon}{2\pi i} \left( \int_{c} - \int_{1-c} \right) \Lambda_f(s) \ _2F_1 \left( \begin{array}{c} \frac{s+1+w}{2}, \frac{s+1-v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) - w^2 \right) y^{\frac{1}{2}-s} ds
\]
\[
= \frac{(2w)^\epsilon}{2\pi i} \int_{1-c} \Lambda_f(s) \ _2F_1 \left( \begin{array}{c} \frac{s+1+w}{2}, \frac{s+1-v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) - w^2 \right) y^{\frac{1}{2}-s} ds.
\]

\( \square \)

Let \( \alpha \in \mathbb{Q}_{>0} \) and \( z = \alpha(1 + iy) \) for \( y > 0 \). The above Lemma now reads
\[
(4.1) \quad \tilde{f}(z) - \tilde{g} \left( \frac{-1}{Nz} \right) = \frac{(2y^{-1})^\epsilon}{2\pi i} \int_{1-c} \Lambda_f(s) \ _2F_1 \left( \begin{array}{c} \frac{s+1+w}{2}, \frac{s+1-v}{2} \\ \frac{1}{2}+\epsilon \end{array} \right) - y^{-2} \right) (\alpha y)^{\frac{1}{2}-s} ds.
\]

From now on we let \( \beta = -1/N\alpha \).
Lemma 4.3 (Lemma 2.4 in [BCK18]). Let $y \in (0, \frac{1}{2}]$. With the assumptions of Theorem 1.2, for any $\ell_0 \geq 0$

\[
\tilde{g} \left( -\frac{1}{Nz} \right) = O_{\sigma, \ell_0}(y^{2\ell_0-\sigma}) + \sum_{\alpha \in \{0,1\}} i^{-\alpha} \sum_{t=0}^{2\ell_0-1} \frac{(2\pi i N\alpha)^t}{t!} \cdot \frac{1}{2\pi i} \int_{(\sigma+1)} \Lambda_g(s+t, \beta, \cos(\alpha)) \frac{\gamma_f^{(-\epsilon)}(1-s)}{\gamma_f^{(-\epsilon)}(1-s-2[t/2])} \left( \frac{y}{N\alpha} \right)^{1/2-s} ds.
\]

Remark 4.4. Although Lemma 2.4 in [BCK18] is stated for a function $F$ that is related to a Maass form, an inspection of the proof shows that we can still apply it to $g$.

Suppose that $0 < y < 1$. By (A.15)

(4.2) \( (2y^{-1})^{\epsilon} \cdot 2F1 \left( \frac{s+\epsilon+\nu}{2}, \frac{s+\epsilon+\nu}{2} \mid y^{-2} \right) (\alpha y)^{1-s} \)

\[= \sum_{\pm} \frac{\Gamma(\mp \nu) \pi^{\mp 2} \alpha^{\pm 2-s}}{\Gamma \left( \frac{s+\epsilon+\nu}{2} \right) \Gamma \left( \frac{s-\epsilon+\nu}{2} \right)} \sum_{k=0}^{\ell_0-1} \frac{(s+\epsilon+\nu)_k (s-\epsilon+\nu+1)_k}{k! (1 \pm \nu)_k} (-1)^k y^{2k+\nu} + O_{\ell_0,\nu,s}(y^{2\ell_0}). \]

Moreover, if $s$ is in a fixed compact set then we can choose the error terms to be independent of $s$. For $\ell_0 \geq 0$, we therefore conclude, for any $0 < y < 1$,

(4.3) \( \frac{(2y^{-1})^{\epsilon}}{2\pi i} \int \Lambda_f(s) \cdot 2F1 \left( \frac{s+\epsilon+\nu}{2}, \frac{s+\epsilon+\nu}{2} \mid y^{-2} \right) (\alpha y)^{1-s} ds \)

\[= \sum_{k=0}^{\ell_0-1} \sum_{\pm} \mathcal{I}_k^{\pm}(\alpha) y^{\pm 2k+\nu} + O_{\ell_0,\nu,\sigma}(y^{2\ell_0}), \]

where the integral is taken over any circle containing all poles of $\Lambda_f(s)$ and

(4.4) \[ \mathcal{I}_k^{\pm}(\alpha) = (-1)^k \frac{\Gamma(\mp \nu) \sqrt{\pi}}{k! (1 \pm \nu)_k} \cdot \frac{1}{2\pi i} \int \Lambda_f(s) \frac{(s+\epsilon+\nu)_k (s-\epsilon+\nu+1)_k}{\Gamma \left( \frac{s+\epsilon+\nu}{2} \right) \Gamma \left( \frac{s-\epsilon+\nu}{2} \right)} \alpha^{1-s} ds. \]

Analogously to [BK13, Section 2], we make the following definition.

Definition 4.5. For $\nu \in \mathbb{C} \setminus \{0\}$ and any open interval $(a, b) \subset \mathbb{R}$, denote by $\mathcal{M}^{\nu}(a, b)$ the set of meromorphic functions which are holomorphic on $a \leq \text{Re}(s) \leq b$, except for at most simple poles in the sets $\pm \nu + \mathbb{Z}$, and bounded on \( \{ s \in \mathbb{C} : \text{Re}(s) \in [c, d], |	ext{Im}(s)| \geq 1 \} \) for each compact $[c, d] \subset (a, b)$.

We will also consider the following subsets of $\mathcal{M}^{\nu}(a, b)$:

(4.5) \[ \mathcal{M}^{\nu}_t(a, b) = \{ f \in \mathcal{M}^{\nu}(a, b) : f \text{ holomorphic at } s \in 2\mathbb{Z} + t + 1 \pm \nu \}, \ t \in \mathbb{Z}, \]

(4.6) \[ \mathcal{H}(a, b) = \{ f \in \mathcal{M}^{\nu}(a, b) : f \text{ holomorphic at } s \in \mathbb{Z} \pm \nu \}. \]
Lemma 4.6. For $\alpha \in \mathbb{Q}_{>0}$ the following function is in $\mathcal{H}(\sigma - 2\ell_0, \infty)$,

\begin{equation}
H_\alpha(s) = (N\alpha^2)^{s-\frac{1}{2}} \sum_{a \in \{0,1\}} i^{-a} \sum_{t=0}^{2\ell_0-1} \frac{(2\pi i N\alpha)^t}{t!} \Lambda_g(s + t, \beta, \cos^{(a)}) \frac{\gamma_f^{(-\nu)}(1-s)}{\gamma_f^{(-\nu)}(1-s - 2\lfloor t/2 \rfloor)} \ldots
\end{equation}

Proof. Let $\chi_{(0,1)}$ be the characteristic function of the interval $(0,1)$. By Lemmas 4.2 and 4.3 we have

\begin{equation}
F_\alpha(y) := \sum_{a \in \{0,1\}} i^{-a} \sum_{t=0}^{2\ell_0-1} \frac{(2\pi i N\alpha)^t}{t!} \ldots
\end{equation}

Hence we have that the Mellin transform $\int_0^\infty \alpha^{s-\frac{1}{2}} F_\alpha(y) y^{\frac{1}{2}} \frac{dy}{y}$ is in $\mathcal{H}(\sigma - 2\ell_0, \infty)$ and indeed this equals $H_\alpha(s)$. Each term in $H_\alpha(s)$ is the Mellin transform of the corresponding term in $\alpha^{s-\frac{1}{2}} F_\alpha(y)$. The first follows from Mellin inversion, while for the second term we use Proposition 3.5. The last term is

\begin{equation}
\alpha^{s-\frac{1}{2}} \int_0^\infty \chi(y) \sum_{k=0}^{\ell_0-1} \sum_{\pm} \frac{I_k^{\pm}(\alpha) y^{\pm v + 2k + \frac{1}{2}} y^{s-\frac{1}{2}} dy}{y} = \alpha^{s-\frac{1}{2}} \sum_{k=0}^{\ell_0-1} \sum_{\pm} \frac{I_k^{\pm}(\alpha)}{s \pm v + 2k}.
\end{equation}

Suppose $\beta = \frac{u}{v} \in \mathbb{Q}^\times$ with $(u,v) = 1$ and $u > 0$. Given $\mathcal{P}$ as in Theorem 1.2, we introduce the infinite set

\[ T_\beta := \left\{ \frac{p}{u} \in \mathbb{Q}_{>0} : p \equiv u \mod v, \quad p \in \mathcal{P} \right\}. \]

An important feature of the sets $T_\beta$ is that if $\lambda \in T_\beta$, then $\Lambda_g(s, \lambda \beta, \cos^{(j)}) = \Lambda_g(s, \beta, \cos^{(j)})$. Consider $t_0 \in \mathbb{Z}_{\geq 0}$ and any subset $T_{\beta,M} \subset T_\beta$ of cardinality $M \geq 2\ell_0 > t_0$. For each $\lambda \in T_{\beta,M}$, since the Vandermonde determinant does not vanish, there exist $c_\lambda \in \mathbb{C}$ such that

\begin{equation}
\sum_{\lambda \in T_{\beta,M}} c_\lambda \lambda^{-t} = \delta_{t_0}(t), \quad t \in \{0, 1, \ldots, 2\ell_0 - 1\}.
\end{equation}
Lemma 4.7. Let \( \alpha \in \mathbb{Q}_{>0}, t_0 \in \mathbb{Z}_{>0}, T_{\beta, M} \) of size \( M \geq 2 \ell_0 > t_0, \) and \( c_\lambda \in \mathbb{Q} \) be as in (4.10). The following function is in \( \mathcal{H}(t_0 + \sigma - 2\ell_0, \infty) \):

\[
(4.11) \quad i^{\lceil \epsilon + t_0 \rceil} \left( N \alpha^2 \right)^{s-\frac{1}{2}} \alpha^{-t_0} \frac{(2\pi i)^{t_0}}{t_0!} \Lambda_g \left( s, \beta, \cos(t) \right) \frac{\gamma_f^{(-s)}(1-s+t_0)}{\gamma_f^{(-s)}(1-s+[t_0])} - \sum_{\lambda \in T_{\beta, M}} \gamma_f^{(-s-2\ell_0-\frac{1}{2})} \sum_{k=0}^{t_0-1} \sum_{s-t_0 \pm \nu + 2k} \frac{T_k(\alpha \lambda^{-1})}{s-t_0 \pm \nu + 2k}.
\]

Proof. Recall that for every \( \lambda \in T_{\beta, M} \) we have \( \Lambda_g(s, \lambda \beta, \cos(\delta)) = \Lambda_g(s, \beta, \cos(\delta)) \). The function in (4.11) is \( \sum_{\lambda \in T_{\beta, M}} c_\lambda \lambda^{2s-2\ell_0-1} H_{\lambda^{-1}}(s-t_0) \) and hence the statement follows from Lemma 4.6. We just note that an important step in the calculation is applying (4.10) as follows:

\[
(4.12) \quad \sum_{\lambda \in T_{\beta, M}} c_\lambda \lambda^{2s-1} (N \lambda^{-1} \alpha)^{2s-\frac{1}{2}} \sum_{t \equiv a \mod 2} \frac{(2\pi i N \lambda^{-1} \alpha)^t}{t!} \cdot \Lambda_g(s+t, \lambda \beta, \cos(\delta)) \frac{\gamma_f^{(-s)}(1-s)}{\gamma_f^{(-s)}(1-s-2[t/2])} = i^{\lceil \epsilon + t_0 \rceil} \left( N \alpha^2 \right)^{s-\frac{1}{2}} \frac{(2\pi i N \alpha)^{t_0}}{t_0!} \Lambda_g(s+t_0, \beta, \cos(t)) \frac{\gamma_f^{(-s)}(1-s)}{\gamma_f^{(-s)}(1-s-2[t_0/2])}.
\]

In particular, the following function is in \( \mathcal{M}^\nu(t_0 + \sigma - M + 2, \infty) \):

\[
(4.13) \quad i^{\lceil \epsilon + t_0 \rceil} \left( N \alpha^2 \right)^{s-\frac{1}{2}} \alpha^{-t_0} \frac{(2\pi i)^{t_0}}{t_0!} \Lambda_g \left( s, \beta, \cos(t) \right) \frac{\gamma_f^{(-s)}(1-s+t_0)}{\gamma_f^{(-s)}(1-s+[t_0])} - \sum_{\lambda \in T_{\beta, M}} c_\lambda \lambda^{2s-2\ell_0-1} \Lambda_f(s-t_0, \alpha \lambda^{-1}, \cos(\delta)) .
\]

In fact, (4.13) is in \( \mathcal{M}^\nu(t_0 + \sigma - M + 2, \infty) \).

Proposition 4.8. Let \( q \in \mathcal{P} \cup \{1\} \) and let \( b \in \mathbb{Z} \) be coprime to \( Nq \). Under the assumptions of Theorem 1.2, if \( \beta = \frac{b}{Nq} \) then, for any \( \delta \in \{0,1\} \), both \( \Lambda_f(s, \beta, \cos(\delta)) \) and \( \Lambda_g(s, \beta, \cos(\delta)) \) continue to elements of \( \mathcal{M}^\nu(-\infty, \infty) \).

Proof. We will present the proof for \( \Lambda_g(s, \beta, \cos(\delta)) \). Reversing the roles of \( f \) and \( g \), one may recover the result for \( \Lambda_f(s, \beta, \cos(\delta)) \). Observe that we can replace \( b \) with \(-b\), where \( b' \in \mathcal{P} \) such that \( b' \equiv -b \mod Nq \). Indeed \( \Lambda_f(s, \beta, \cos(\delta)) = \Lambda_f(s, -\frac{b}{Nq}, \cos(\delta)) \). Therefore we may assume \( \beta < 0, \alpha = -1/N \beta > 0 \) and \(-b \in \mathcal{P} \).

Consider \( q' \in \mathcal{P} \) such that \( q' \neq q \) and \( (b, q') = 1 \). Let \( \beta = \frac{b}{Nq} \) and \( \beta' = \frac{b}{Nq} \) with \( b \in \mathcal{P} \). As \( \beta \) and \( \beta' \) have the same numerator, \( T_{\beta} \cap T_{\beta'} \) is finite. Let \( t_0 \in \mathbb{Z}_{>0} \) and choose a subset \( T_M \subset T_{\beta} \cap T_{\beta'} \) of cardinality \( M > t_0 \) and \( c_\lambda \) such that (4.10) is satisfied. By considering the
difference of equation (4.13) evaluated at \( \beta \) and \( \beta' \), we see that the following function is in \( \mathcal{M}_{t_0}^{\nu} (t_0 + \sigma - M + 2, \infty) \):

\[
\frac{\gamma_f^{(-\nu)} (1 - s + t_0)}{\gamma_f^{(-\nu)} (1 - s + [t_0])} \cdot \left( \alpha^{2s-t_0-1} \Lambda_g \left( s, \beta, \cos^{(\nu+\delta)}(t_0) \right) - \alpha^{2s-t_0-1} \Lambda_g \left( s, \beta', \cos^{(\nu+\delta)}(t_0) \right) \right).
\]

The assumption that \( a_n = O(n^\sigma) \) and the fact that the poles of \( \gamma_f^\nu(s) \) lie in the half plane \( \text{Re} \, s < |\nu| < 1 \) imply that, for all \( \lambda \in T_M \), \( \Lambda_f \left( s, \alpha \lambda^{-1}, \cos(n) \right) \) is holomorphic in the half plane \( \text{Re} \, s > \sigma + 1 \). The functional equation in Proposition 3.3 hence implies that

\[
\Lambda_f \left( s - t_0, \alpha \lambda^{-1}, \cos(\epsilon) \right) - \Lambda_f \left( s - t_0, \alpha' \lambda^{-1}, \cos(\epsilon) \right)
\]

is in \( \mathcal{H}(-\infty, t_0-\sigma) \). Note that in the case \( \epsilon = 1 \), each of the terms in (4.15) is in \( \mathcal{H}(-\infty, t_0-\sigma) \) by Proposition 3.3.

We deduce that, for every \( t_0 \in \mathbb{Z}_{\geq 0} \), the following function is in \( \mathcal{M}_{t_0}^{\nu} (t_0 + \sigma - M + 2, t_0 - \sigma) \):

\[
\frac{\gamma_f^{(-\nu)} (1 - s + t_0)}{\gamma_f^{(-\nu)} (1 - s + [t_0])} \cdot \left( \alpha^{2s-t_0-1} \Lambda_g \left( s, \beta, \cos^{(\nu+\delta)}(t_0) \right) - \alpha^{2s-t_0-1} \Lambda_g \left( s, \beta', \cos^{(\nu+\delta)}(t_0) \right) \right).
\]

Since the above does not depend on \( T_M \) anymore we can taking \( M \) arbitrarily large, we see that (4.16) is in \( \mathcal{M}_{t_0}^{\nu} (-\infty, t_0 - \sigma) \). The zeros of the quotient of gamma functions

\[
\frac{\gamma_f^{(-\nu)} (1 - s + t_0)}{\gamma_f^{(-\nu)} (1 - s + [t_0])},
\]

are contained in the set of poles of \( \gamma_f^{(-\nu)}(1 - s + [t_0]) \) which is contained in \( 2\mathbb{Z}_{\geq 0} + 1 + [t_0] + \nu \). As \( \nu \neq 0 \), the poles are all simple. Hence, dividing (4.16) by (4.17) we see that

\[
\alpha^{2s-t_0-1} \Lambda_g \left( s, \beta, \cos^{(\nu+\delta)}(t_0) \right) - \alpha^{2s-t_0-1} \Lambda_g \left( s, \beta', \cos^{(\nu+\delta)}(t_0) \right),
\]

is in \( \mathcal{M}^{\nu} (-\infty, t_0 - \sigma) \). As \( \alpha \neq \alpha' \) and we may take arbitrary \( t_0 \geq 0 \), we conclude that \( \Lambda_g \left( s, \beta, \cos^{(\nu+\delta)}(t_0) \right) \) is in \( \mathcal{M}^{\nu} (-\infty, t_0 - \sigma) \). The function \( \Lambda_g \left( s, \beta, \cos^{(\nu+\delta)}(t_0) \right) \) only depends on the parity of \( t_0 \), so again we can choose \( t_0 \) arbitrarily large, but of a fixed parity, to conclude that \( \Lambda_g \left( s, \beta, \cos^{(\nu+\delta)}(t_0) \right) \) is in \( \mathcal{M}^{\nu} (-\infty, \infty) \).

**Corollary 4.9.** Make the assumptions of Proposition 4.8. If \( \beta = b/q \) for \( q \in \{1\} \cup \mathcal{P} \) and \( (b,q) = 1 \), then, for any \( \delta \in \{0,1\} \), \( \Lambda_f \left( s, \beta, \cos(\delta) \right) \) and \( \Lambda_g \left( s, \beta, \cos(\delta) \right) \) continue to elements of \( \mathcal{M}^{\nu} (-\infty, \infty) \).

**Proof.** Let \( \alpha = -\frac{1}{N\beta} = -\frac{q}{Np} \). As in the proof of Proposition 4.11 we can make the assumption that \( \beta < 0 \) and hence \( \alpha > 0 \) and \( -b \in \mathcal{P} \). Consider \( t_0 \in \mathbb{Z}_{\geq 0}, M > t_0 \) and \( T_{\beta,M} \) a subset of \( T_{\beta} \) of cardinality \( M \) satisfying (4.10). For any \( \lambda \in T_{\beta,M} \) we have that \( \alpha \lambda^{-1} = -\frac{q}{Np} \) has the form required in Proposition 4.8, so \( \Lambda_f \left( s - t_0, \alpha \lambda^{-1}, \cos(\epsilon) \right) \) is in \( \mathcal{M}^{\nu} (-\infty, \infty) \). Then for \( t_0 \in \{0,1\} \), equation (4.13) implies that \( \Lambda_g \left( s, \beta, \cos^{(\nu+\delta)}(t_0) \right) \) is in \( \mathcal{M}^{\nu} (t_0 + \sigma - M + 2, \infty) \). Taking
M arbitrarily large, we see that both \( \Lambda_g(s, \beta, \cos(0)) \) and \( \Lambda_g(s, \beta, \cos(1)) \) are in \( \mathcal{M}^\nu(-\infty, \infty) \). Reversing the roles of \( f \) and \( g \), we draw the same conclusion for \( \Lambda_f(s, \beta, \cos(0)) \).

We now assume that \( \sigma = \frac{1}{2} + \varepsilon \), where \( 0 < \varepsilon < \frac{1}{2} - |\Re \nu| \). Under this assumption, \( \Lambda_f(s) \) and \( \Lambda_g(s) \) are holomorphic for \( \Re s > 1 + |\Re \nu| \) and so equation 1.5 implies they are also holomorphic for \( \Re s < -|\Re \nu| \). Corollary 4.9 then implies that \( \Lambda_f(s) \) and \( \Lambda_g(s) \) are holomorphic away from the set \( \{\pm \nu, 1 \pm \nu\} \), where they have at most simple poles if \( \nu \neq 0 \).

To finish the proof of Theorem 1.2 it thus suffices to show firstly that if \( \epsilon = 1 \) then \( \Lambda_f(s) \) and \( \Lambda_g(s) \) are entire, and secondly that if \( \psi \) is a primitive Dirichlet character with conductor \( q \in \mathcal{P} \) then \( \Lambda_f(s, \psi) \) and \( \Lambda_g(s, \psi) \) are entire.

**Lemma 4.10.** Assume that \( \alpha \in \mathbb{Q}_{>0} \) and \( \beta = \frac{1}{N_{\alpha}} \) are such that \( \Lambda_g(s, \beta, \cos(\delta)) \) and \( \Lambda_f(s, \alpha, \cos(\delta)) \) continue to elements of \( \mathcal{M}^\nu(-\infty, \infty) \) for \( \delta \in \{0, 1\} \).

If \( s_0 \in \mathbb{Z} \) satisfies \( s_0 < 1 \), we choose an integer \( t_0 \in \mathbb{Z}_{>1} \) such that \( |t_0| = |s_0| \) and write \( j = \frac{1}{2}(t_0 - s_0) \). Moreover, we choose a set \( T_{\beta, M} \) of size \( M \geq 2t_0 \) satisfying \( (4.10) \). Fix a sign \( \delta \in \{\pm\} \).

If \( \epsilon = 0 \), then

\[
(4.19) \sum_{\lambda \in T_{\beta, M}} c_{\lambda} \lambda^{2s_0 - 2t_0 - 1 + 2\delta \nu} \Res_{s = s_0 + \delta \nu} \Lambda_f(s - t_0, \alpha \lambda^{-1}, \cos) = \left[ e^{-|t_0|(N\alpha^2)s_0 - \frac{1}{2} + \delta \nu} \frac{(2\pi i)^{2s_0 - 2t_0 - 1 + 2\delta \nu} \gamma_f(1 - s_0 + t_0 - \delta \nu)}{\gamma_f(1 - s_0 + |t_0| - \delta \nu)} \Res_{s = s_0 + \delta \nu} \Lambda_g(s, \beta, \cos(\delta \nu)) \right. \\
+ \left. (1)^j \delta_0(s_0) \frac{1}{2} - \delta \nu j \pi \sum_{\lambda \in T_{\beta, M}} c_{\lambda} \lambda^{2s_0 - 2t_0 - 1 + 2\delta \nu} \Res_{s = s_0 + \delta \nu} \Lambda_f(s). \right]
\]

If \( \epsilon = 1 \), then

\[
(4.20) \sum_{\lambda \in T_{\beta, M}} c_{\lambda} \lambda^{2s_0 + 2\delta \nu - 2t_0 + 1} \Res_{s = s_0 + \delta \nu} \Lambda_f(s - t_0, \alpha \lambda^{-1}, \sin) = \left[ e^{-|t_0|(N\alpha^2)s_0 - \frac{1}{2} + \delta \nu} \frac{(2\pi i)^{2s_0 - 2t_0 + 1} \gamma_f(1 - s_0 + t_0 - \delta \nu)}{\gamma_f(1 - s_0 + |t_0| - \delta \nu)} \Res_{s = s_0 + \delta \nu} \Lambda_g(s, \beta, \cos(\delta \nu)) \right. \\
+ \left. (1)^j \delta_0(s_0) \frac{3}{2} - \delta \nu j \pi \sum_{\lambda \in T_{\beta, M}} c_{\lambda} \lambda^{2s_0 - 2t_0 + 2\delta \nu - 1} \Res_{s = s_0 + \delta \nu} \Lambda_f(s) \right] \\
+ \left. (1)^j \sum_{\lambda \in T_{\beta, M}} c_{\lambda} \lambda^{s_0 - t_0 + 2\delta \nu} \frac{1}{\delta \nu(1 - \delta \nu)} \Res_{s = 1 + \delta \nu} \Lambda_f(s). \right]
\]

**Proof.** As \( 0 \leq j < \ell_0 \), we have

\[
(4.21) \Res_{s = s_0 + \delta \nu} \left( \sum_{k=0}^{\ell_0 - 1} \sum_{\pm} \frac{I_k^\pm(\alpha \lambda^{-1})}{s - t_0 \pm \nu + 2k} \right) = I_j^{-\delta}(\alpha \lambda^{-1}).
\]
Since \( \text{Re} (s_0 + \delta \nu) > t_0 + \sigma - 2\ell_0 \), the residue of (4.11) at \( s = s_0 + \delta \nu \) is zero. Hence

\[
(4.22) \quad \text{Res}_{s=s_0+\delta \nu} \left( \sum_{\lambda \in T_{\beta, M}} c_{\lambda} \lambda^{2s-2t_0-1} (-i\pi)^{\epsilon} \Lambda_f \left( s - t_0, \alpha \lambda^{-1}, \cos(\epsilon) \right) \right)
= \text{Res}_{s=s_0+\delta \nu} \left( i^{[\epsilon+t_0]} (N\alpha^2)^{s-\frac{1}{2}} \alpha^{-t_0} \frac{(2\pi i)^{t_0}}{t_0!} A_g \left( s, \beta, \cos([\epsilon+t_0]) \right) \frac{\gamma_f^{(-\nu)} (1 - s + t_0)}{\gamma_f^{(-\nu)} (1 - s + [t_0])} \right)
+ \sum_{\lambda \in T_{\beta, M}} c_{\lambda} (\lambda \alpha)^{s_0-t_0+\delta \nu - \frac{1}{2}} \mathcal{I}_j^{-\delta} (\alpha \lambda^{-1}).
\]

Taking \( b = q = 1 \) in Proposition 4.8, we see that \( \Lambda_f(s), \Lambda_g(s) \in \mathcal{M}'(-\infty, \infty) \). Since the poles of \( \Lambda_g(s) \) are in the critical strip \( -\sigma < \text{Re} s < \sigma + 1 \), we have:

\[
(4.23) \quad \mathcal{I}_j^{-\delta} (\alpha \lambda^{-1}) = (-1)^j \frac{\Gamma (\nu) \sqrt{\pi}}{j! (1 - \delta \nu)_j} \cdot \frac{1}{2\pi i} \oint \Lambda_f(s) \frac{(s+\epsilon-\delta \nu)_j}{\Gamma \left( \frac{s+\epsilon+\delta \nu}{2} \right) \Gamma \left( \frac{1-s+\epsilon+\delta \nu}{2} \right)} \left( \alpha \lambda^{-1} \right)^{\frac{1}{2} - s} ds
= (-1)^j \frac{\Gamma (\nu) \sqrt{\pi}}{j! (1 - \delta \nu)_j} \sum_{\substack{p \in \mathbb{Z} \pm \nu \text{ Re } p \in [-\sigma, \sigma+1]}} \text{Res}_{s=1-p} \left( \Lambda_f(s) \frac{(s+\epsilon-\delta \nu)_j}{\Gamma \left( \frac{s+\epsilon+\delta \nu}{2} \right) \Gamma \left( \frac{1-s+\epsilon+\delta \nu}{2} \right)} \left( \alpha \lambda^{-1} \right)^{\frac{1}{2} - s} \right)
= (-1)^j \frac{\Gamma (\nu) \sqrt{\pi}}{j! (1 - \delta \nu)_j} \sum_{\substack{p \in \mathbb{Z} \pm \nu \text{ Re } p \in [-\sigma, \sigma+1]}} \left( \frac{1-p+\epsilon-\delta \nu}{2} \right)_j \left( \frac{2-p-\epsilon-\delta \nu}{2} \right)_j \left( \alpha \lambda^{-1} \right)^{\frac{1}{2} - p} \text{Res}_{s=1-p} \Lambda_f(s).
\]

The final line follows because \( \frac{(s+\epsilon+\delta \nu)_j}{\Gamma \left( \frac{s+\epsilon+\delta \nu}{2} \right) \Gamma \left( \frac{1-s+\epsilon+\delta \nu}{2} \right)} \) and \( (\alpha \lambda^{-1})^{\frac{1}{2} - s} \) are entire and the poles of \( \Lambda_f(s) \) are simple. Since we assume \( 0 < \sigma + |\text{Re} \nu| < 1 \), the only values of \( p \) that can occur in the above sum are in \( \{ \pm \nu, 1 \pm \nu \} \). The factor \( \frac{1-p+\epsilon-\delta \nu}{2} \Gamma \left( \frac{1-p+\epsilon-\delta \nu}{2} \right) \frac{2-p-\epsilon-\delta \nu}{2} \Gamma \left( \frac{2-p-\epsilon-\delta \nu}{2} \right) \) vanishes at some of these values: if \( \epsilon = 0 \), then it vanishes at \( -\delta \nu, 1 + \delta \nu \) and \( 1 - \delta \nu \). If \( \epsilon = 1 \), it vanishes at \( 1 - \delta \nu \). We conclude that

\[
(4.24) \quad \mathcal{I}_j^{-\delta} (\alpha \lambda^{-1}) = (-1)^j \frac{\Gamma (\nu) \sqrt{\pi}}{j! (1 - \delta \nu)_j} \left\{ \begin{array}{ll}
\frac{(\frac{1}{2} - \delta \nu)_j}{\Gamma (\delta \nu) \sqrt{\pi}} \alpha^{\delta \nu - \frac{1}{2}} \lambda^{\frac{1}{2} - \delta \nu} \text{Res}_{s=1-\delta \nu} \Lambda_f(s), & \epsilon = 0, \\
\sum_{p \in \{ \pm \delta \nu, 1+\delta \nu \}} \frac{(\frac{2-p-\delta \nu}{2})_j \left( \frac{1-p-\delta \nu}{2} \right)_j}{\Gamma \left( \frac{2-p-\delta \nu}{2} \right) \Gamma \left( \frac{1-p-\delta \nu}{2} \right)} \alpha^{p-\frac{1}{2}} \lambda^{\frac{1}{2} - p} \text{Res}_{s=1-p} \Lambda_f(s), & \epsilon = 1.
\end{array} \right.
\]
Assume that $\epsilon = 0$. Equation (4.22) becomes

\begin{equation}
\text{Res}_{s=s_0+\delta \nu} \left( \sum_{\lambda \in T_{\beta,2t_0}} c_\lambda \lambda^{2s-2t_0-1} \Lambda_f (s-t_0, \alpha \lambda^{-1}, \cos) \right)
= \text{Res}_{s=s_0+\delta \nu} \left( -\text{i}^{-[t_0]} (N \alpha^2)^{-s} \sum_{\alpha^{-t_0}} \frac{2\pi i}{t_0!} \Lambda_g (s, \beta, \cos([t_0])) \frac{\gamma_f (1-s+t_0)}{\gamma_f (1-s+[t_0])} \right)
+ (-1)^j \frac{(\frac{1}{2} - \delta \nu) j}{\delta \nu (1-\delta \nu) j} \lambda^{s_0-t_0-2\delta \nu-1} \text{Res}_{s=1-\delta \nu} \Lambda_f(s) \right) \left( \sum_{\lambda \in T_{\beta,2t_0}} c_\lambda \lambda^{s_0-t_0} \right).
\end{equation}

Equation 4.19 now follows from equation (4.10). Indeed, by assumption, $s_0 - t_0 \in \mathbb{Z}$ satisfies $0 > s_0 - t_0 > -2t_0$. We also note that the quotient $\gamma_f (1-s+t_0)/\gamma_f (1-s+[t_0])$ does not have a pole at $s_0 + \delta \nu$, so it does not contribute to the residue.

Now assume that $\epsilon = 1$. In this case, we have

\begin{equation}
\mathcal{I}_j^{-\delta} (\alpha \lambda^{-1}) = (-1)^j \left[ \frac{(\frac{1}{2} - \delta \nu) j}{\delta \nu (1-\delta \nu) j} \lambda^{\delta \nu-\frac{1}{2}} \lambda^{-\frac{1}{2} - \delta \nu} \text{Res}_{s=1-\delta \nu} \Lambda_f(s) \right]
\end{equation}

It follows from equation (4.22) that:

\begin{equation}
i \pi \text{Res}_{s=s_0+\delta \nu} \left( \sum_{\lambda \in T_{\beta,2t_0}} c_\lambda \lambda^{2s-2t_0-1} \Lambda_f (s-t_0, \alpha \lambda^{-1}, \sin) \right)
= \text{Res}_{s=s_0+\delta \nu} \left( -\text{i}^{-[1+t_0]} (N \alpha^2)^{-s} \sum_{\alpha^{-t_0}} \frac{2\pi i}{t_0!} \Lambda_g (s, \beta, \cos([1+t_0])) \frac{\gamma_f (1-s+t_0)}{\gamma_f (1-s+[t_0])} \right)
+ (-1)^j \frac{(\frac{1}{2} - 2\delta \nu) j}{\delta \nu (1-\delta \nu) j} \lambda^{s_0-t_0-2\delta \nu-1} \text{Res}_{s=1-\delta \nu} \Lambda_f(s)
+ (-1)^j \frac{(\frac{1}{2} - \delta \nu) j}{\delta \nu (1-\delta \nu) j} \lambda^{s_0-t_0+2\delta \nu} \text{Res}_{s=1+\delta \nu} \Lambda_f(s)
- (-1)^j \frac{(\frac{1}{2} - \delta \nu) j}{\delta \nu (1-\delta \nu) j} \lambda^{s_0-t_0+2\delta \nu} \text{Res}_{s=-\delta \nu} \Lambda_f(s).
\end{equation}
The stated equation (4.20) now follows from (4.10). Note that the last term vanishes since $-t_0 > s_0 - t_0 - 1 > -2t_0$.

\[ \Box \]

**Proposition 4.11.** Make the assumptions of Theorem 1.2. If $\epsilon = 1$, then $\Lambda_g(s)$ and $\Lambda_f(s)$ continue to entire functions on $\mathbb{C}$.

**Proof.** Since we already established that the only poles $\Lambda_f$ and $\Lambda_g$ can have are simple and in the set $\{\pm \nu, 1 \pm \nu\}$ it suffices to show that the residues of these functions vanish there.

Let $\beta = -\frac{1}{Nq}$, $T_{\beta,M}$ a set of cardinality $M$ satisfying (4.10), and $\lambda = p \in T_{\beta,2t_0}$, so that $\alpha = q$ and $\alpha \lambda^{-1} = \frac{2}{p}$. By absolute convergence of the Dirichlet series we know that $\Lambda(s - t_0, \alpha \lambda^{-1}, \sin)$ is holomorphic for $\Re(s - t_0) \geq \sigma + 1$, and so by the functional equation it is also holomorphic for $\Re(s - t_0) \leq -\sigma$. When $s_0 = 0$, for all even $t_0 \geq \sigma + 1$ equation (4.20) simplifies to:

\begin{equation}
(4.28) \quad \begin{align*}
4\pi \sum_{\lambda \in T_{\beta, M}} c_\lambda \lambda^{-t_0+2\nu} & \Rightarrow \left( \begin{array}{c}
\text{Res}_{s=1-\nu} \Lambda_f(s) \\
\text{Res}_{s=1+\nu} \Lambda_f(s)
\end{array} \right) \\
\Rightarrow & \left( \begin{array}{c}
1 + \frac{(\delta \nu)\sqrt{\pi}}{\Gamma(\delta \nu + 1/2)} \\
\frac{t_0!}{(1 - 2\delta \nu) t_0} \frac{\alpha - 2\delta \nu}{\delta \nu} \sum_{\lambda \in T_{\beta, M}} c_\lambda \lambda^{-t_0+2\nu}
\end{array} \right)
\end{align*}
\end{equation}

We used the formulas $\gamma^\frac{\nu}{2}(1 + t_0 - \delta \nu)/\gamma^\frac{\nu}{2}(1 + [t_0] - \delta \nu) = \pi^{-t_0} \left( \begin{array}{c}
\frac{1}{2} - \delta \nu \\
\frac{1}{2} - \delta \nu/t_0/2
\end{array} \right)$ and $2\lambda^0(\frac{1}{2}) t_0/2(1) t_0/2 = t_0!$ above. The left-hand side of equation (4.28) does not depend on $t_0$. Recall that $T_{\beta,M}$ and the $c_\lambda$ were chosen such that (4.10) is satisfied. We want to show that we can add an element $\lambda_0 \in T_\beta$ to $T_{\beta,M}$, so that (4.10) is still satisfied and in addition $\sum_{\lambda \in T_{\beta,M}} c_\lambda \lambda^{-t_0+2\nu}$ assumes an arbitrary value.

Equivalently, we want to find $\lambda_0 \in T_\beta$ such that the vectors $(\lambda^{-t_0+2\nu})_{\lambda \in T_{\beta, M}}$ are linearly independent of the vector $(\lambda^{-t_0+2\nu})_{\lambda \in T_{\beta, M} \cup \{\lambda_0\}}$. Consider the matrix that has these $M + 1$ vectors in $\mathbb{R}^{M+1}$ as Columns. Developing the determinant with respect to the last row we obtain an expression in $\lambda_0$ of the form

\begin{equation}
(4.29) \quad \lambda_0^{-t_0+2\nu} c + P(\lambda_0),
\end{equation}

where $c$ is a non-zero constant (the Vandermonde determinant of $T_{\beta,M}$) and $P$ is a polynomial with complex coefficients of degree $M - 1$. Since $T_\beta$ is an infinite set we can choose $\lambda_0$ arbitrarily large. Suppose the expression (4.29) vanishes for all $\lambda_0 \in T_\beta$. By comparing the growth of the two terms in (4.29) for $\lambda_0 \to \infty$, we can conclude that $P = d\lambda_0^{-t_0}$ and $\nu$ is purely imaginary. Now comparing the argument of the two terms we arrive at a contradiction. Hence there exists a $\lambda_0 \in T_\beta$ such that (4.29) is non-zero. So we can apply Lemma 4.10 with $T_{\beta, M} \cup \{\lambda_0\}$ instead of $T_{\beta, M}$ and choose coefficients $c_\lambda$ for $\lambda \in T_{\beta, M} \cup \{\lambda_0\}$ such that (4.10) and (4.28) is satisfied and $\sum_{\lambda \in T_{\beta, M} \cup \{\lambda_0\}} c_\lambda \lambda^{-t_0+2\nu} = 1$. We can also choose coefficients $c'_\lambda$ with $\sum_{\lambda \in T_{\beta, M} \cup \{\lambda_0\}} c'_\lambda \lambda^{-t_0+2\nu} = 2$ such that (4.28) is satisfied with $c_\lambda$ replaced by $c'_\lambda$. We conclude $\text{Res}_{s=1+\nu} \Lambda_f(s) = 0$.

As $\delta \in \{\pm\}$ was arbitrary, we have shown that $\text{Res}_{s=1+\nu} \Lambda_f(s) = \text{Res}_{s=1-\nu} \Lambda_f(s) = 0$. Reversing the roles of $f$ and $g$, we deduce the same for $\Lambda_g(s)$. By the functional equation, we have $\text{Res}_{s=\pm \nu} \Lambda_f(s) = \text{Res}_{s=\pm \nu} \Lambda_g(s) = 0$.

\[ \Box \]
Note that equation (4.28) also implies that
\begin{equation}
\text{Res}_{s=\delta \nu} \Lambda_g \left(s, -\frac{1}{Nq}, \sin \right) = -iN^{1-\delta \nu} \frac{\delta \nu \sqrt{\pi}}{\Gamma (\delta \nu + \frac{1}{2})} \text{Res}_{s=1-\delta \nu} \Lambda_f (s),
\end{equation}
and so the residue on the left-hand side is independent of \( q \).

**Proposition 4.12.** Make the assumptions of Theorem 1.2. If \( \psi \) is a primitive Dirichlet character with conductor \( q \in \mathcal{P} \), then \( \Lambda_f (s, \psi) \) and \( \Lambda_g (s, \psi) \) continue to elements of \( \mathcal{H}(-\infty, \infty) \).

**Proof.** For \( \psi \) as in the statement, recall that
\begin{equation}
\psi(n) = (-i)^{\text{sgn} \psi} \frac{\tau (\psi)}{q} \sum_{b \mod q} \bar{\psi}(-b) \cos (\text{sgn} \psi) \left( \frac{2 \pi b}{q} \right),
\end{equation}
and so
\begin{equation}
\Lambda_g (s, \psi) = (-i)^{\text{sgn} \psi} \frac{\tau (\psi)}{q} \sum_{b \mod q} \bar{\psi}(-b) \Lambda_g \left(s, \frac{b}{q}, \cos (\text{sgn} \psi) \right).
\end{equation}
By our assumption on \( \sigma \), we know that \( \Lambda_f (s, \psi) \) are entire for \( \Re s \geq \frac{3}{2} > 1 + \sigma \). By equation (1.5) it suffices to prove that
\begin{equation}
\text{Res}_{s=\delta \nu} \Lambda_f (s, \psi) = \text{Res}_{s=\delta \nu} \Lambda_g (s, \psi) = 0,
\end{equation}
for \( \delta \in \{ \pm \} \).

Let \( \alpha > 0 \) and \( T_{\beta, M} \) be as in Lemma 4.7. Let \( s_0 < 1 \) and choose \( t_0 > 1 \) such that \( t_0 - s_0 \) is odd. Taking the residue at \( s = s_0 + \delta \nu \) of equation (4.11) we obtain
\begin{equation}
i^{-[\epsilon + t_0]} (N \alpha^2)^{s_0 + \delta \nu - \frac{1}{2}} \alpha^{-t_0} \frac{(2 \pi i)^t_0}{t_0!} \gamma_f (1 - s_0 + t_0 - \delta \nu) \text{Res}_{s=s_0+\delta \nu} \Lambda_g \left(s, \beta, \cos (s_0) \right)
= (-i \pi)^\epsilon \sum_{\lambda \in T_{\beta, 2t_0}} c_\lambda \lambda^{2s_0 - 2t_0 + 2b \nu - 1} \text{Res}_{s=s_0+\delta \nu} \Lambda_f \left(s - t_0, \alpha \lambda^{-1}, \cos (\epsilon) \right).
\end{equation}
First assume \( \epsilon = 1 \). If \( \beta = \frac{b}{Nq} \) for \( b < 0 \) coprime to \( Nq \), then \( \alpha \lambda^{-1} = \frac{a}{p} \) for some \( p \equiv b \mod Nq \). By the functional equation for \( \Lambda_f (s - t_0, \alpha \lambda^{-1}, \sin) \), the second line of (4.34) is zero and so \( \text{Res}_{s=s_0+\delta \nu} \Lambda_g \left(s, \frac{b}{Nq}, \cos (s_0) \right) = 0 \) for all \( s_0 < 1 \). If \( \beta = \frac{b}{Nq} \) with \( b > 0 \) coprime to \( Nq \), then we choose \( b' \equiv -b \mod Nq \), so that \( \Lambda_g \left(s, \frac{b}{Nq}, \cos (s_0) \right) = \Lambda_g \left(s, \frac{-b'}{Nq}, \cos (s_0) \right) \) and so \( \text{Res}_{s=s_0+\delta \nu} \Lambda_g \left(s, \beta, \cos (s_0) \right) = 0 \) for all \( \beta \) of the form \( b/Nq \).

To obtain information on the other additive twists we use Lemma 4.10. Let \( s_0 < 1, t_0 > 1 \) such that \( [s_0] = [t_0] \). Equation (4.20) and Proposition 4.11 imply:
\begin{equation}
i^{[-1 + t_0]} (N \alpha^2)^{s_0 - \frac{1}{2} + \delta \nu} \alpha^{-t_0} \frac{(2 \pi i)^t_0}{t_0!} \gamma_f (1 - s_0 + t_0 - \delta \nu) \text{Res}_{s=s_0+\delta \nu} \Lambda_g \left(s, \beta, \cos (1 + t_0) \right)
= i \pi \sum_{\lambda \in T_{\beta, M}} c_\lambda \lambda^{2s_0 + 2b \nu - 2t_0 + 1} \text{Res}_{s=s_0+\delta \nu} \Lambda_f \left(s - t_0, \alpha \lambda^{-1}, \sin \right).
\end{equation}
As above, for $\beta = \frac{b}{Nq}$ the second line vanishes and we conclude $\text{Res}_{s=s_0+\delta\nu} \Lambda_g \left( s, \frac{b}{Nq}, \cos(1+s_0) \right) = 0$ for all $s_0 < 1$.

If $\beta = \frac{b}{q}$ for $b < 0$ coprime to $q$, then $\lambda^{-1} = \frac{q}{Np}$ for a prime $p$ congruent to $-b$ modulo $q$. By the previous paragraph the second lines of equations (4.34) and (4.35) both vanish. Therefore, equation (4.34) for $s_0 = 0$ and $t_0 = 3$ implies that $\text{Res}_{s=\delta\nu} \Lambda_g \left( s, \frac{b}{q}, \cos \right) = 0$ and equation (4.35) for $s_0 = 0$ and $t_0 = 2$ implies that $\text{Res}_{s=\delta\nu} \Lambda_g \left( s, \frac{b}{q}, \sin \right) = 0$. Reversing the roles of $f$ and $g$, we deduce the same for $\text{Res}_{s=\delta\nu} \Lambda_f \left( s, \frac{b}{q}, \cos \right)$ and $\text{Res}_{s=\delta\nu} \Lambda_f \left( s, \frac{b}{q}, \sin \right)$.

By equation (4.32), we deduce equation (4.33) as required.

Now consider $\epsilon = 0$. Let $t_0 > 1$ and $s_0 < 1$ be integers. Let $t_0 - s_0$ be odd, and $\beta, \beta' \in \mathbb{Q}_{<0}$ with the same numerator. In (4.34) we choose the set $T = T_{\beta,2t_0} = T_{\beta',2t_0}$ to be a subset of $T_{\beta} \cap T_{\beta'}$. This is possible since $T_{\beta} \cap T_{\beta'}$ is infinite. Hence (4.34) applies to the pair $\beta$ and $\alpha = -1/N\beta$ and the pair $\beta'$ and $\alpha' = -1/N\beta$. Subtracting the resulting equations from each other we obtain

$$i([-t_0]) N^{s_0+\delta\nu-t_0} \left( \frac{2\pi i t_0}{t_0!} \right) \gamma^+_{2t_0}(1-s_0 + t_0 - \delta\nu) \gamma^+_{t_0}(1-s_0 + [t_0] - \delta\nu)$$

$$\cdot \left[ \alpha^{2s_0+2\delta\nu-t_0-1} \text{Res}_{s=s_0+\delta\nu} \Lambda_g \left( s, \beta, \cos^{[t_0]} \right) - \alpha^{2s_0+2\delta\nu-t_0-1} \text{Res}_{s=s_0+\delta\nu} \Lambda_g \left( s, \beta', \cos^{[t_0]} \right) \right]$$

$$= \sum_{\lambda \in T} \lambda(s_0+\delta\nu) \left[ \Lambda_f \left( s - t_0, \alpha\lambda^{-1}, \cos \right) - \Lambda_f \left( s - t_0, \alpha' \lambda^{-1}, \cos \right) \right].$$

When $\beta = \frac{b}{Nq}$ and $\beta' = \frac{b}{Nq}$ with $b < 0$ coprime to $Nq$ the last line vanishes, since $\Lambda_f \left( s - t_0, \alpha\lambda^{-1}, \cos \right) - \Lambda_f \left( s - t_0, \alpha' \lambda^{-1}, \cos \right)$ is a linear combination of twists of $\Lambda_f$ by characters and hence holomorphic at $s_0 - t_0 + \delta\nu$, since $s_0 - t_0 < -2$. Therefore, we have

$$\alpha^{s_0+2\delta\nu-t_0-1} \text{Res}_{s=s_0+\delta\nu} \Lambda_g \left( s, \frac{b}{Nq}, \cos^{[t_0]} \right) = \alpha^{s_0+2\delta\nu-t_0-1} \text{Res}_{s=s_0+\delta\nu} \Lambda_g \left( s, \frac{b}{Nq}, \cos^{[t_0]} \right).$$

Varying $t_0$ we deduce that $\text{Res}_{s=s_0+\delta\nu} \Lambda_g \left( s, \beta, \cos^{[t_0]} \right) = \text{Res}_{s=s_0+\delta\nu} \Lambda_g \left( s, \beta', \cos^{(s_0+1)} \right) = 0$ for all $s_0 < 1$ and all $\beta$ of the form $\frac{b}{Nq}$ with $b$ coprime to $Nq$. We can omit the condition $b < 0$ by the same argument as in the case $\epsilon = 1$.

On the other hand, if $t_0 - s_0$ is even we consider the equations (4.19) for $\beta$ and $\beta'$. Subtracting the two equations from each other we note that when $\beta = \frac{b}{Nq}$ and $\beta' = \frac{b}{Nq}$, again the terms $\text{Res}_{s_0+\delta\nu} \left( \Lambda_f \left( s - t_0, \alpha\lambda^{-1}, \cos \right) - \Lambda_f \left( s - t_0, \alpha' \lambda^{-1}, \cos \right) \right)$ vanish and so we are left with

$$i([-t_0]) N^{s_0+\delta\nu-t_0} \left( \frac{2\pi i}{t_0!} \right) \gamma^+_{t_0}(1-s_0 + [t_0] - \delta\nu)$$

$$\cdot \left( \alpha^{2s_0+2\delta\nu-t_0-1} \text{Res}_{s=s_0+\delta\nu} \Lambda_g \left( s, \beta, \cos^{[t_0]} \right) - \alpha^{2s_0+2\delta\nu-t_0} \text{Res}_{s=s_0+\delta\nu} \Lambda_g \left( s, \beta', \cos^{[t_0]} \right) \right)$$

$$= (-1)^{j+1} \delta_0(s_0) \left( \frac{1}{2} - \delta\nu \right) \left( \alpha^{s_0-t_0-1+2\delta\nu} - \alpha^{s_0-t_0-1+2\delta\nu} \right) \text{Res}_{s=1-\delta\nu} \Lambda_f(s).$$
where \( j = \frac{1}{2}(t_0 - s_0) \). We see that
\[
q^{2s_0 + 2\delta \nu - 1 - t_0} \text{Res}_{s = s_0 + \delta \nu} \Lambda_g(s, \frac{b}{Nq}, \cos(s_0)) = q^{2s_0 + 2\delta \nu - 1 - t_0} \text{Res}_{s = s_0 + \delta \nu} \Lambda_g(s, \frac{b}{Nq'}, \cos(s_0))
\]
for \( s_0 < 0 \), since in that case the last line of (4.37) vanishes. Varying \( t_0 \) we see that
\[
\text{Res}_{s = s_0 + \delta \nu} \Lambda_g(s, \frac{b}{Nq}, \cos(s_0)) = 0 \text{ for } s_0 < 0.
\]

Now let \( \beta = \frac{b}{q} \) with \( b < 0 \), which implies \( \lambda^{-1}\alpha = \frac{q}{\lambda \nu} \) for \( \lambda \in T_\beta \). We first insert \( s_0 = 0 \) and \( t_0 = 3 \) into (4.34). Since we have shown above that
\[
\text{Res}_{s = \delta \nu} \Lambda_f(s - 3, \lambda^{-1}\alpha, \cos(\gamma)) = \text{Res}_{s = -3 + \delta \nu} \Lambda_f(s, \lambda^{-1}\alpha, \cos(\gamma)) = 0
\]
we see that \( \text{Res}_{s = \delta \nu} \Lambda_g(s, \frac{b}{q}, \cos) = 0 \) for all odd characters \( \psi \) of conductor \( q \). For the even twists consider (4.19) for \( s_0 = 0 \) and \( t_0 = 2 \). By our previous considerations the first line vanishes and we are left with
\[
N^{-\frac{3}{2} + \delta \nu} \frac{(2\pi i)^2 \gamma_f^+(3 - \delta \nu)}{2 \gamma_f^+(1 - \delta \nu)} \text{Res}_{s = \delta \nu} \Lambda_g(s, \beta, \cos) = \left(\frac{1}{2} - \delta \nu\right) \text{Res}_{s = 1 - \delta \nu} \Lambda_f(s).
\]

Hence \( \text{Res}_{s = \delta \nu} \Lambda_g(s, \frac{b}{q}, \cos) \) is independent of \( b \) coprime to \( q \). Again we deduce that \( \text{Res}_{s = \delta \nu} \Lambda_g(s, \psi) = 0 \) from equation (4.32). Reversing the roles of \( f \) and \( g \) and using the functional equation we finally conclude (4.33).

\[\square\]

**APPENDIX A.**

A.1. **The converse of the Converse Theorem.** In this section, for the sake of completeness, we prove that the \( L \)-function of a Maass form satisfies the conditions of Theorem 1.2. Let \( f \) be a weight 0 Maass form of level \( N \) and nebentypus \( \chi \) with Fourier expansion as in equation (2.2). The \( L \)-function of \( f \) is defined on the right half plane \( \Re s > 3/2 \) by the Dirichlet series
\[
L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}.
\]

The completed \( L \)-function of \( f \) is defined as
\[
\Lambda_f(s) = \gamma_f^+(s)L_f(s),
\]
where the \( \gamma \)-factor is defined in equation (3.2). These completions can also be given in terms of the Mellin transform (Section 2.1). In the case that \( f \) is even one has for \( \Re s > 3/2 \),
\[
\Lambda_f(s) = \int_{0}^{\infty} (f(iy) - f_0(y)) y^{s-\frac{1}{2}} \frac{dy}{y}.
\]

If \( f \) is odd, we have similarly
\[
\Lambda_f(s) = \frac{1}{2\pi i} \int_{0}^{\infty} \frac{\partial f}{\partial \nu}(iy) y^{s+\frac{1}{2}} \frac{dy}{y}.
\]

Set \( g(z) := f(-\frac{1}{Nz}) \), and write the Fourier expansion of \( g \) as
\[
g(z) = g_0(y) + 4 \sum_{n=1}^{\infty} b_n \sqrt{y} K_{\nu}(2\pi ny) (e(nx) + (-1)^{r}e(-nx)),
\]

where \( j = \frac{1}{2}(t_0 - s_0) \). We see that
\[
q^{2s_0 + 2\delta \nu - 1 - t_0} \text{Res}_{s = s_0 + \delta \nu} \Lambda_g(s, \frac{b}{Nq}, \cos(s_0)) = q^{2s_0 + 2\delta \nu - 1 - t_0} \text{Res}_{s = s_0 + \delta \nu} \Lambda_g(s, \frac{b}{Nq'}, \cos(s_0))
\]
for \( s_0 < 0 \), since in that case the last line of (4.37) vanishes. Varying \( t_0 \) we see that
\[
\text{Res}_{s = s_0 + \delta \nu} \Lambda_g(s, \frac{b}{Nq}, \cos(s_0)) = 0 \text{ for } s_0 < 0.
\]

Now let \( \beta = \frac{b}{q} \) with \( b < 0 \), which implies \( \lambda^{-1}\alpha = \frac{q}{\lambda \nu} \) for \( \lambda \in T_\beta \). We first insert \( s_0 = 0 \) and \( t_0 = 3 \) into (4.34). Since we have shown above that \( \text{Res}_{s = \delta \nu} \Lambda_f(s - 3, \lambda^{-1}\alpha, \cos(\gamma)) = \text{Res}_{s = -3 + \delta \nu} \Lambda_f(s, \lambda^{-1}\alpha, \cos(\gamma)) = 0 \) we see \( \text{Res}_{s = \delta \nu} \Lambda_g(s, \frac{b}{q}, \cos) = 0 \) for all odd characters \( \psi \) of conductor \( q \). For the even twists consider (4.19) for \( s_0 = 0 \) and \( t_0 = 2 \). By our previous considerations the first line vanishes and we are left with
\[
N^{-\frac{3}{2} + \delta \nu} \frac{(2\pi i)^2 \gamma_f^+(3 - \delta \nu)}{2 \gamma_f^+(1 - \delta \nu)} \text{Res}_{s = \delta \nu} \Lambda_g(s, \beta, \cos) = \left(\frac{1}{2} - \delta \nu\right) \text{Res}_{s = 1 - \delta \nu} \Lambda_f(s).
\]

Hence \( \text{Res}_{s = \delta \nu} \Lambda_g(s, \frac{b}{q}, \cos) \) is independent of \( b \) coprime to \( q \). Again we deduce that \( \text{Res}_{s = \delta \nu} \Lambda_g(s, \psi) = 0 \) from equation (4.32). Reversing the roles of \( f \) and \( g \) and using the functional equation we finally conclude (4.33).

\[\square\]
where, for some $b_0, b'_0 \in \mathbb{C}$,
\[
(A.5) \quad g_0(y) = \begin{cases} 
    b_0 y^{{1\over 2}+\nu} + b'_0 y^{{1\over 2}-\nu}, & \epsilon = 0, \nu \neq 0, \\
    b_0 y^{{1\over 2}} + b'_0 y^{1/2} \log y, & \epsilon = 0, \nu = 0, \\
    0, & \epsilon = 1.
\end{cases}
\]

**Proposition A.1.** Let $f$ be a weight 0 Maass form with Laplace eigenvalue $\frac{1}{4} - \nu^2 > 0$.

1. If $f$ is odd, then $\Lambda_f(s)$ can be continued to an entire function.
2. If $f$ is even $\Lambda_f(s)$ has meromorphic continuation to $\mathbb{C}$ with possible poles at $s \in \{\pm \nu, 1 \pm \nu\}$. More precisely the function $\Lambda_f(s) - H_f(s)$, where
   \[
   (A.6) \quad H_f(s) = \begin{cases} 
    -{a_0 \over s+\nu} - {a'_0 \over s-\nu} + {b_0 N^{-\nu} \over s-\nu} + {b'_0 N^{-\nu} \over s+\nu}, & \nu \neq 0, \\
    -{a_0 \over s} + {a'_0 \over s^2} + N^{1-s} \left( {b_0 \over s-1} + {b'_0 \over (s-1)^2} \right), & \nu = 0,
\end{cases}
   \]
   can be continued to an entire function. So if $\nu \neq 0$ the possible poles of $L$-functions lie at $s = \pm \nu, 1 \pm \nu$ and are simple. If $\nu = 0$ the possible poles are at $s = 0, 1$ and have order at most 2.

**Proof.** We focus on case (2); the first case is analogous but easier since an odd Maass form does not have a constant term. Let $f(z) = f_0(y) + \tilde{f}(z)$ be an even Maass form and $g(z) = g_0(y) + \tilde{g}(z)$, where the constant terms $f_0(y)$ (resp. $g_0(y)$) has the form given in equation (2.3) (resp. (A.5)). To study the poles of $\Lambda_f(f)$ we follow the approach of [Iwa91, Chapter 7]. For $\text{Re } s \gg 0$
\[
\Lambda_f(s) = \int_{1/\sqrt{N}}^{\infty} (f(iy) - f_0(y)) y^{s-1/2} dy + \int_{0}^{1/\sqrt{N}} (f(iy) - f_0(y)) y^{s-1/2} dy
\]
\[
= \int_{1/\sqrt{N}}^{\infty} (f(iy) - f_0(y)) y^{s-1/2} dy + N^{1-s} \int_{1/\sqrt{N}}^{\infty} (g(iy) - g_0(y)) y^{1-s} dy
\]
\[
+ \int_{0}^{1/\sqrt{N}} g_0 \left( {1 \over Ny} \right) y^{s-1/2} dy - \int_{0}^{1/\sqrt{N}} f_0(y) y^{s-1/2} dy.
\]

The first two integrals are entire functions in $s$ because $\tilde{f}$ and $\tilde{g}$ decay exponentially for $y \to \infty$. Therefore $\Lambda_f(s) - \tilde{H}_f(s)$ with
\[
\tilde{H}_f(s) = \int_{0}^{1/\sqrt{N}} \left( g_0 \left( {1 \over Ny} \right) - f_0(y) \right) y^{s-1/2} dy,
\]
can be continued to an entire function and $\tilde{H}_f(s)$ can be calculated directly:
\[
(A.7) \quad \tilde{H}_f(s) = \begin{cases} 
    N^{-s/2} \left( -{a_0 N^{-\nu} \over s+\nu} - {a'_0 N^{\nu} \over s-\nu} + {b_0 N^{-\nu} \over s-\nu} + {b'_0 N^{\nu} \over s+\nu} \right), & \nu \neq 0, \\
    N^{-s/2} \left( -{a_0 \over s} + {a'_0 \over s^2} + {b_0 \over s-1} \log(N) + {b'_0 \over (s-1)^2} \right), & \nu = 0,
\end{cases}
\]
For $\nu \neq 0$ the statement of the proposition follows directly, since $H_f(s) - \tilde{H}_f(s)$ has no poles. For $\nu = 0$ we use the power series expansions of $N^{-s/2}$ as $1 - s \log(N) - {1 \over 2} O(s^2)$ and
\[
N^{-1/2} \left( 1 - (s - 1) \log(N) - {1 \over 2} O((s-1)^2) \right)
\]
around $s = 1$ and $s = 2$ to conclude. \qed
Remark A.2. In the even case, the poles $s = 1 \pm \nu$ occur in the non-completed $L$-function, which, for example, may take the form $\zeta(s - \nu)\zeta(s + \nu)$. On the other hand, the pole at $s = \pm \nu$ comes from the $\Gamma$-factor. Under the Langlands correspondence, the case $\nu = 0$ is supposed to correspond to 2-dimensional even Artin representations of Galois groups\footnote{Analogously, 2-dimensional odd Artin representations should correspond to weight 1 modular forms.}, and the order of the pole at 1 corresponds to the multiplicity of the trivial representation.

Let $f$ be even. In the proof of Proposition A.1 we obtained

\[
\Lambda_f(s) = \int_{1/\sqrt{N}}^{\infty} \left( f(iy) - f_0(y) \right) y^{s-\frac{1}{2}} \frac{dy}{y} + N^{\frac{1}{2}-s} \int_{1/\sqrt{N}}^{\infty} \left( g(iy) - g_0(y) \right) y^{\frac{1}{2}-s} \frac{dy}{y} + H_f(s).
\]

From this expression it follows that

\[
\Lambda_f(s) = N^{\frac{1}{2}-s} \Lambda_g(1-s), \quad H_f(s) = N^{\frac{1}{2}-s} H(g, 1-s).
\]

If $f$ is odd, the functional equation takes a similar form. Indeed, if $z = x + iy$, then

\[
-\frac{1}{Nz} = \frac{x}{N(x^2+y^2)} + i \frac{y}{N(x^2+y^2)} \quad \text{and} \quad \frac{\partial g}{\partial x}(iy) = \frac{\partial f}{\partial x} \left( \frac{i}{Ny} \right) \cdot \left( -\frac{1}{N y^2} \right).
\]

The functional equation is

\[
\Lambda_f(s) = (2\pi i)^{-1} N^{-s-\frac{1}{2}} \int_0^{\infty} \frac{\partial f}{\partial x} \left( \frac{i}{Ny} \right) y^{-s-\frac{1}{2}} \frac{dy}{y}
\]

\[
= -(2\pi i)^{-1} N^{-s+\frac{1}{2}} \int_0^{\infty} \frac{\partial g}{\partial x}(iy) y^{-s+\frac{1}{2}} \frac{dy}{y} = -N^{-s+\frac{1}{2}} \Lambda_g(1-s).
\]

Combining these calculations, we obtain the following proposition.

**Proposition A.3.** If $f$ is a weight 0 Maass form of level $N$ and parity $\epsilon$, then

\[
\Lambda_f(s) = (-1)^\epsilon N^{\frac{1}{2}-s} \Lambda_g(1-s).
\]

If $\psi$ is a Dirichlet character modulo $q$ such that $N$ and $q$ are coprime, then the twist of $f$ by $\psi$,

\[
f_\psi(z) := \sum_{n \neq 0} \psi(n) \frac{a_n}{2\sqrt{|n|}} W_\nu(ny) e(nx)
\]

is again a Maass form. Specifically, $f_\psi$ has level $Nq^2$ and nebentypus $\chi\psi^2$. Twisting by $\psi$ changes the parity if and only if $\psi$ is odd. We also define twisted $L$-functions:

\[
L_f(s, \psi) = L_{f_\psi}(s), \quad \Lambda_f(s, \psi) = \Lambda_{f_\psi}(s).
\]

Similar to the case of holomorphic modular forms we can show

\[
\Lambda_{f_\psi} \left( \frac{1}{Nq^2 z} \right) = \frac{\tau(\psi)}{\tau(\psi)} \psi(-N) \chi(q) g_\psi^{-1}(z)
\]

from which we obtain the functional equation for the twists.
Proposition A.4. If \( N, q \) are coprime integers and \( \psi \) is a primitive Dirichlet character mod \( q \), then
\[
\Lambda_f(s, \psi) = (-1)^r \psi(N) \chi(q) \frac{\tau(\psi)}{\tau(q)} (q^2 N)^{1/2-s} \Lambda_g(1 - s, \overline{\psi}).
\]

If \( \psi \) is a primitive character mod \( q \) then it follows from (A.10) that \( \Lambda_f(s, \psi) \) continues to an entire function. On the other hand, while the twist \( f_{\psi_0} \) by the principal character \( \psi_0 \) modulo \( q \), has no constant term, the function \( f_{\psi_0}(-1/Nq^2z) \) might, so \( \Lambda_f(s, \psi_0) \) has no poles at \( \pm \nu \) but might have poles at \( 1 \pm \nu \).

A.2. Hypergeometric functions. The Mellin transforms of twisted Maass forms yielded the hypergeometric function \( {}_2F_1 \), which is defined initially on \( |z| < 1 \) by the following power series
\[
{}_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right) |z| \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n,
\]
where \( (x)_n \) is the rising Pochhammer symbol, that is,
\[
(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)} = x(x + 1) \cdots (x + n - 1).
\]
For example, \( (1)_n = n! \), \( (2)_n = (n + 1)! \) and
\[
{}_2F_1 \left( \begin{array}{c} 1, 1 \\ 2 \end{array} \right) |z| = -\frac{1}{z} \log(1 - z).
\]
The function \( {}_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right) |z| \) satisfies the so-called Euler identity:
(A.11) \[
{}_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right) |z| = (1 - z)^{c-a-b} {}_2F_1 \left( \begin{array}{c} c-a, c-b \\ c \end{array} \right) |z|.
\]
We also have [Mez08, (A.11)]
(A.12) \[
{}_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right) |z| = (1 - z)^{-a} {}_2F_1 \left( \begin{array}{c} a, c-b \\ c \end{array} \right) \frac{z}{z-1}, \text{ for } z \notin (1, \infty).
\]
Hence for \( w \in \mathbb{R} \)
(A.13) \[
{}_2F_1 \left( \begin{array}{c} \frac{i+1+i+i}{2} \frac{i+1+i+i}{2} \\ i \end{array} \right) |w^2| = \left( 1 + w^2 \right)^{-\frac{i+1+i+i}{2}} \left( 1 + w^2 \right) \left( \frac{i+1+i+i}{2} \right)^{\frac{i+1+i+i}{2}} \frac{w^2}{1 + w^2}.
\]
Define \( z = \frac{1-w^2}{1+w^2} \). We introduce some notation from [Luk69, §7.1]. Let \( z = \cosh(i\nu) \), where \( \nu \in [0, \pi) \). In this special case the sector \( \mathcal{Q} \) from [Luk69, §7.1] is just \( |\arg \lambda| \leq \pi - \delta \) for some \( \delta > 0 \). In particular \( \lambda \) lies in it if \( \text{Im} \lambda \to \infty \). Equation (8) from loc.cit. tells us how the hypergeometric function above grows when \( \text{Im} \lambda \to \infty \):
\[
{}_2F_1 \left( \begin{array}{c} a+\lambda, b-\lambda \\ c \end{array} \right) |z| \frac{1-z}{2} \sim \frac{2^{a+b-1}(1-b+\lambda)(1+e^{-i\nu})^{c-a-b-1/2}}{(\lambda \pi)^{1/2}(c-b+\lambda)(1-e^{-i\nu})^{c-1/2}} \times \left[ e^{(\lambda-b)\nu} + e^{\pm i\nu(c-1/2)-(\lambda+a)\nu} + O(1/\lambda) \right]
\]
For \( s = \sigma + it \) we set \( a = \frac{\nu+\nu}{2}, b = \frac{-\nu+\nu}{2}, c = \frac{1+2\nu}{2} \) and \( \lambda = \frac{\nu}{2} \). Recall Stirling’s formula for fixed \( \sigma \):
(A.14) \[
\Gamma(\sigma + it) = \sqrt{2\pi(it)^{\sigma-1/2}e^{-\frac{\nu}{2}\nu}} \left( \frac{|t|}{e} \right)^{it} (1 + O(|t|^{-1})) , \ |t| \to \infty.
\]
We deduce that \( (A.13) \) grows at most like \( e^{\frac{\pi}{4}t} \). Finally we conclude
\[
\Gamma \left( \frac{s + \nu + \epsilon}{2} \right) \Gamma \left( \frac{s - \nu + \epsilon}{2} \right) \ {}_2F_1 \left( \frac{s + \nu + \epsilon - \nu}{2}, \frac{s - \nu + \epsilon - \nu}{2}; w^2 \right)
\]
decays exponentially as \( |t| \to \infty \) for all \( w \in \mathbb{R} \), since the exponential term in both Gamma factors is \( e^{-\frac{\pi}{4}t} \) while the hypergeometric function has exponential term \( e^{\frac{\pi}{4}t} \) and \( v < \pi \).

In section 4 we used an explicit analytic continuation for the hypergeometric function outside its disc of convergence. When \( |z| > 1 \) and \( a - b \notin \mathbb{Z} \), one has that\(^5\):
\[
(A.15) \ {}_2F_1 \left( \frac{a}{c}, \frac{b}{c} \mid z \right) = \frac{\Gamma(b-a)\Gamma(c)(-z)^{-a}}{\Gamma(b)\Gamma(c-a)} \sum_{k=0}^{\infty} \frac{(a)_k(a-c+1)_k z^{-k}}{k!(a-b+1)_k} + \frac{\Gamma(a-b)\Gamma(c)(-z)^{-b}}{\Gamma(a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(b)_k(b-c+1)_k z^{-k}}{k!(b-a+1)_k}.
\]

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\(^5\)wolfram.com//HypergeometricFunctions/Hypergeometric2F1/02/02/0001/