Investigation of the Seabed Profile taking into account the Multiple Scattering Approximation of Radiation

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Abstract. Based on a mathematical model of acoustic signal propagation in an oscillating medium, an inverse problem is formulated, including the definition of a function describing the deviation of the seabed level from the average given horizontal plane. Solution in the approximation of double scattering from the seabed and by volume in the case of a narrow radiation pattern of the receiving antenna is obtained.

1. Introduction

Modern methods of studying the ocean still require improvement due to the fact that each study requires a lot of time and human resources. Many research complexes are being developed to solve bathymetry problems which include monitoring of water basins and determining the bottom topography. The most popular are autonomous unmanned underwater vehicles equipped with side-scan sonar (SSS) for studying coastal waters and the World Ocean. The sonar operation is based on the periodic emission of pulsed sound parcels and the detection of reflected echoes from remote areas of the seabed. When the sonar antenna moves, an acoustic image is formed on both sides of the underwater vehicle [1 – 4].

Previously, the authors proposed to apply a phenomenological approach to the problem of determining the bottom profile by taking into account the inhomogeneity of the medium in an explicit form, namely, volume scattering. In [5] a solution to the inverse problem was obtained in the single scattering approximation with allowance for volume scattering in a medium in the form of a nonlinear differential equation. Also, in [6, 7] studies were carried out of the effect of doubly scattered radiation either from the sea bottom or in the volume on the total detected signal. As studies have shown, double interactions of radiation with a medium make an insignificant contribution. In this work, an attempt was made to obtain an equation describing the total received signal, which interacted once with the bottom surface, and then scattered in the water column and came to the receiver.

2. Formulation of the problem

The process of propagation of acoustic radiation is described by the transfer equation [5–9]:

\[
\frac{1}{c} \frac{\partial I}{\partial t} + k \cdot \nabla_r I(r, k, t) + \mu I(r, k, t) = \frac{\sigma}{2\pi} \int_{\Omega} I(r, k', t) dk' + J(r, k, t)
\] (1)
where \( \mathbf{r} \in \mathbb{R}^2, t \in [0,T] \) and wave vector \( \mathbf{k} \) belongs to the unique sphere \( \Omega = \{ \mathbf{k} \in \mathbb{R}^2; |\mathbf{k}| = 1 \} \). The function \( I(\mathbf{r}, \mathbf{k}, t) \) is interpreted as radiation intensity of wave in moment \( t \) in point \( \mathbf{r} \), propagated in the direction \( \mathbf{k} \) with constant velocity \( c \). The coefficients \( \mu \) and \( \sigma \) denote the attenuation and the scattering, correspondingly. \( J(\mathbf{r}, \mathbf{k}, t) \) describes the density of inner sources.

The process of echo signal propagation occurs in the domain \( G = \{ \mathbf{r} \in \mathbb{R}^2; r_2 > -l + u(r_1) \} \) which is the upper half-space bounded from below by the curve, \( \partial G = \{ \mathbf{y} \in \mathbb{R}^2; y_2 > -l + u(y_1) \} \) where the function \( u(y_1) \) describes the change of the ocean bottom relief.

We assume that the function \( J(\mathbf{r}, \mathbf{k}, t) \) describes a point isotropic sound source [8,9]:

\[
J(\mathbf{r}, \mathbf{k}, t) = J_0 \delta(\mathbf{r}) \delta(t),
\]

where, \( \delta \) denotes the Dirac delta function and \( J_0 \) is source power.

We complement (1) with initial and boundary conditions [5,6]:

\[
I^-(\mathbf{r}, \mathbf{k}, t)|_{t=0} = 0,
\]

\[
I^-(\mathbf{r}, \mathbf{k}, t) = 2\sigma_d \int_{\Omega_+(\mathbf{y})} |\mathbf{n}(\mathbf{r}) \cdot \mathbf{k}'| I^+(\mathbf{r}, \mathbf{k}', t) d\mathbf{k}', \quad \mathbf{r} \in \gamma, \mathbf{k} \in \Omega_-(\mathbf{r}),
\]

\[
I^\pm(\mathbf{r}, \mathbf{k}, t) = \lim_{\varepsilon \to 0} I(\mathbf{r} \pm \varepsilon \mathbf{k}, \mathbf{k}, t \pm \varepsilon/c),
\]

where, \( \mathbf{k} \in \{ \mathbf{k} \in \Omega : \text{sgn}(k_1) = \pm 1 \} \), \( \sigma_d \) denotes the constant sea bottom reflection coefficient, \( \mathbf{n}(\mathbf{r}) \) denotes the external normal to \( \partial G \).

Additional conditions at the receiver’s points:

\[
\int_{\Omega} S^\pm(\mathbf{k}) I^+(\mathbf{O}, \mathbf{k}, t) d\mathbf{k} = P_\pm(t),
\]

where, \( \mathbf{k} \in \{ \mathbf{k} \in \Omega : \text{sgn}(\mathbf{k}) = \pm 1 \} \), functions \( S^\pm(\mathbf{k}) \) characterize the radiation patterns of receiving antennas on the starboard and the portside. Functions \( P_\pm(t) \) describe measured total intensity on different sides of the antenna carrier.

The inverse problem statement: «To determine the curve \( \gamma \) by equations (1) – (5) in which \( \mu, c, \sigma_d, \sigma \) and functions \( J, P_\pm, S^\pm \) are given».

The solution of the initial-boundary problem (1), (3), (4) is deduced to the integral equation [10,11,15]:

\[
l(\mathbf{r}, \mathbf{k}, t) = \int_0^t \exp(-\mu t) J_0 \left( \mathbf{r} - \tau \mathbf{k}, \mathbf{k}, t - \frac{\tau}{c} \right) d\tau +
\]

\[
2\sigma_d \exp(-\mu d(\mathbf{r}, \mathbf{k})) \int_{\Omega+(\mathbf{r}-\mathbf{d}(\mathbf{r}, \mathbf{k}))} |\mathbf{n}(\mathbf{r} - \mathbf{d}(\mathbf{r}, \mathbf{k}) \mathbf{k}) \cdot \mathbf{k}'| \times
\]

\[
x J^+ \left( \mathbf{r} - \mathbf{d}(\mathbf{r}, \mathbf{k}) \mathbf{k}, \mathbf{k}', t - \frac{\mathbf{d}(\mathbf{r}, \mathbf{k})}{c} \right) d\mathbf{k}' +
\]

\[
+ \frac{\sigma}{2\pi} \int_0^t \exp(-\mu t') \int_{\Omega} l \left( \mathbf{r} - t' \mathbf{k}, \mathbf{k}, t - \frac{t'}{c} \right) d\mathbf{k}' dt',
\]

where \( d(\mathbf{r}, \mathbf{k}) \) denotes the distance from the point \( \mathbf{r} \in G \) in the direction \( -\mathbf{k} \) to the boundary of the domain \( G \).

For solving (6) authors construct a simple iteration method. Denote the initial approximation as \( l_0 = \int_0^t \exp(-\mu t) J_0 \left( \mathbf{r} - t' \mathbf{k}, \mathbf{k}, t - \frac{t'}{c} \right) d\tau \)

We present the solution (6) for the non-stationary radiation transfer equation in the operator form [11,13,14]:

\[
l = l_0 + (PB + ES)l, \quad l_0 = EJ
\]

(7)
The solution of the equation could be written as a Neumann series [7]:

\[
I = \sum_{n=1}^{\infty} (PB + ES)^n l_0
\]

where \((PB + ES)^n\) represents the part of the radiation flux that has experienced \(n\)-fold scattering before arrives the point \((r, k, t)\).

The solution responsible for single scattering was obtained in the articles [5], therefore, we will write a double – scattering form:

\[
(PB + ES)^2 l_0 = ((PB)^2 + PBES + ESPB + (ES)^2) l_0.
\]

Let the operators \(E, S, P, B\) be defined by the formulas:

\[
EI(r, k, t) = \int_0^\infty \exp(-\mu \tau) J_0 \left( r - \tau k, k, t - \frac{\tau}{c} \right) d\tau
\]

\[
SI(r, k, t) = \frac{\sigma}{2\pi} \int_{\Omega} I(r, k', t) d\mathbf{k}'
\]

\[
P I(r, k, t) = \exp(-\mu d(r, -k)) I \left( r - d(r, -k)k, k, t - \frac{d(r, -k)}{c} \right)
\]

\[
BI^+(r, k, t) = 2 \sigma_d \int_{\Omega_+} |n(r) \cdot \mathbf{k}'| l^+(r, k', t) d\mathbf{k}'.
\]

In (9), the first and last terms are responsible for the double scattered radiation from the boundary and in the medium, respectively, and were studied in [6,7]. \(PBES, ESPB\) characterizes the radiation that interacted with the medium and with the boundary surface.

The derivation of the signal that was scattered and reflected:

\[
PBESl(r, k, t) = \frac{\sigma_d \sigma}{\pi} \exp(-\mu d(r, -k)) \times
\]

\[
\int_{\Omega_+(r-d(r, -k)k)} |n(r-d(r, -k)k) \cdot \mathbf{k}_l| \int_0^\infty \exp(-\mu \tau) \times
\]

\[
\int_{\Omega} l \left( r - d(r, -k)k - \tau k_l, k_{II}, t - \frac{\tau}{c} - \frac{d(r, -k)}{c} \right) d\mathbf{k}_{II} d\tau d\mathbf{k}_l.
\]

Substitute the initial approximation in (10) and denote \(x_i = r_l - \tau, k_{II,i}, i = 1, 2:\)

\[
PBESl_0(r, k, t) = \frac{\sigma_d \sigma}{\pi} \exp(-\mu d(r, -k)) \int_{\Omega_+(r-d(r, -k)k)} |n(r-d(r, -k)k) \cdot \mathbf{k}_l| \times
\]

\[
\int_{\Omega_+(r-d(r, -k)k-k_l)} \int_{\Omega_+(r-d(r, -k)k-k_{II})} \frac{\exp(-\mu |r-x|)}{|r-x|} \times
\]

\[
\times \int_{\Omega} l \left( x - d(r, -k)k - \tau k_l, \frac{r-x}{|r-x|}, t - \frac{d(r, -k)}{c} - \frac{\tau}{c} - \frac{|r-x|}{c} \right) d\tau d\mathbf{k}_l.
\]

Further, we use properties of the Dirac delta function first with argument \(x\) and then \(\tau\) in (11):

\[
y = y(r, k) = r - d(r, -k)k, \quad t' = t - \frac{d(r, -k)}{c}, \quad s = \frac{\tau + |y - \tau k_l|}{c},
\]

\[
|r - x| = |y - \tau k_l| = (y - \tau k_l, y - \tau k_l)^\frac{1}{2} = (|y|^2 - 2\tau(y, k_l) + |\tau|^2)^\frac{1}{2},
\]

\[
\text{where } (PB + ES)^n\text{ represents the part of the radiation flux that has experienced } n\text{-fold scattering before arrives the point } (r, k, t).
\]
We obtain a nonlinear differential equation in the single-scattering approximation \[5\], where \(t > 2 \rho / c\) and functions \(P_\pm\) is equal to sum the single scattering radiation and functions \(Q_{2\pm}\):

\[\tau = \tau(y, k_l, s) = \frac{(cs)^2 - |y|^2}{2(cs - (y, k_l))}, \frac{ds \tau(y, k_l, s)}{ds} = \frac{c}{2} \frac{|y - cs k_l|^2}{(cs - (y, k_l))^2}.\] (12)

Given (12), the relation (11) is represented the solution of double mixed scattering in the following form:

\[PBESI_0(r, k, k_l, t) = \frac{\sigma_d}{\pi} \exp(-\mu ct) \int_{\Omega_+(r-d(r,-k)k)} |n(r - d(r,-k)k) \cdot k_l| \times \]
\[\times \chi_d(r-d(r,-k)k,k_l)(\tau) \frac{c}{ct - d(r,-k) - (y, k_l)} d k_l.\] (13)

Implement additional condition (5) and deduce a received signal: \(y = y(O, k) = -d(O, k)k\),

\[Q_{2\pm}(t) = \int_{\Omega_+(y)} S^\pm(k) PBESI_0(r, k, k_l, t) d k = \]
\[= \frac{\sigma_d}{\pi} \exp(-\mu ct) \int_{\Omega} S^\pm(k) \int_{\Omega_+(d(O,-k)k)} |n(-d(O,-k)k) \cdot k_l| \times \]
\[\times \chi_d(d(O,-k)k,-k_l)(\tau) c \frac{ct - d(O,-k) - (y, k_l)}{d k_l d k}.\] (14)

Further, we use a polar coordinates transformation and get an equation for calculating the intensity of radiation twice interacting with the medium:

\[k(\phi) = (\cos \phi \sin \phi), \quad k_l(\theta) = (\cos \theta \sin \theta), \quad \rho = \rho(\phi) = d(O, -k).\]

\[n(-d(O, -k(\phi))k(\phi)) = \left(\frac{\rho' \cos \phi - \rho \sin \phi}{\sqrt{\rho'^2 + \rho^2}}, \frac{\rho' \sin \phi + \rho \cos \phi}{\sqrt{\rho'^2 + \rho^2}}\right).\]

\[Q_{2\pm}(t) = \int_{0}^{2\pi} S^\pm(k(\phi)) \frac{\sigma_d}{\pi} \exp(-\mu ct) \int_{\Omega_+(d(O,-k)k)} \rho' \cos(\phi - \theta) + \rho \sin(\theta - \phi) \times \]
\[\times \frac{\rho'}{\sqrt{\rho'^2 + \rho^2}} \frac{c \chi_d(d(O,-k)k,-k_l)(\tau) c}{ct - \rho(1 - \cos(\phi - \theta))} d \theta d \phi \]
\[\times F(\phi, \theta, \rho) \frac{\pi}{\tau} d \phi.\] (15)

\[Q_{2\pm}(t) = \int_{0}^{\pi} S^\pm(k(\phi)) \frac{\sigma_d}{\pi} \exp(-\mu ct) \chi_d(d(O,-k)k,-k_l)(\tau) \times \]
\[\times \frac{c \chi_d(d(O,-k)k,-k_l)(\tau) c}{\sqrt{\rho'^2 + \rho^2}} d \phi.\] (16)

Equation (16) is the desired solution and will allow describing double scattering in the detected signal of the SSS receiving antenna. The derivation of the equation for ESPB is similar.

### 3. The inverse problem

Further, we consider the inverse problem, which consists in determining the curve \(y\) in polar coordinate.
\[ \gamma = \{ r_1 = -\rho \sin \phi, r_2 = \rho \cos \phi \} \]
\[ \varphi^2(\rho) = \frac{\rho^2 \left( P_\pm \left( \frac{2\rho}{c} \right) - Q \left( \rho, \varphi_\pm(\rho) \right) \right)^2}{\sigma^2 c^2 \exp(-4\mu\rho) - \rho^4 \left( P_\pm \left( \frac{2\rho}{c} \right) - Q \left( \rho, \varphi_\pm(\rho) \right) \right)} \]

where \( P_\pm \) are functions total intensity of radiation in left and right side, \( Q \) is a function that takes into account the contribution of volume scattering.

For an approximate solution of the differential equation (17), we set the initial condition, \( l \) is the depth under vehicle: \( \rho_0 = \rho(\pi) = l \).

4. Conclusions

We obtain equation (16) describing the contribution of radiation that interacted with inhomogeneities in the medium (volume scattering) and with a surface at the boundary of the medium. This addition to the modeling of the sonar signal will improve the solution of the inverse problem and, as a result, get a high-quality result when restoring the bottom surface. It is worth to note that this type of double scattering is of greater interest for practical purposes, unlike double scattering from the sea bottom and double volume scattering. Double scattering from the bottom can have a large contribution only with a very specific geometry of the boundary. And the double volume scattering itself is very small and reaches about 10% of the attenuation coefficient.

In further research we are going to conduct a quantitative and qualitative analysis of the effect of radiation taking into account double-fold interactions in the environment.

5. References

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