Quantum stochastic behaviour in cold Fermi gases: Phonon propagation

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We examine the effect of quantum fluctuations in a tunable cold Fermi gas on the propagation of phonons. We show that these fluctuations can be interpreted as inducing a stochastic (acoustic) space-time. The variation in times of flight induced by this stochastic behaviour can be significant in the transition region between BEC and BCS regimes.

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In this work we examine sound-cone fluctuations in cold condensates induced by quantum fluctuations, which have a direct bearing on phonon propagation. Several papers in the last few years have discussed wave propagation in random media, from light-cone fluctuations induced by quantum gravity effects and non-linear dielectrics to phonons in random fluids. In fact, these latter papers are considered as analogue gravity exemplars of the former. In all cases fluctuations in the dynamical nature of the density fluctuations into acoustic waves in the diatom density. We shall see that, for typical cold gas values the variation in times of flight induced by this stochastic behaviour can be significant in the transition region between BEC and BCS regimes.

We adopt the notation of our earlier work in describing a cold (T = 0) Fermi gas, tunable through a narrow Feshbach resonance, by the action (ℏ = 1)\[ S = \int dt d^3x\left\{ \sum \psi_\sigma^* (x) \left[ i \partial_t + \frac{\nabla^2}{2m} + \mu \right] \psi_\sigma (x) + \varphi^* (x) \left[ i \partial_t + \frac{\nabla^2}{2m} + 2\mu - \nu \right] \varphi (x) - g \left[ \varphi^* (x) \psi_\uparrow (x) \psi_\downarrow (x) + \varphi (x) \psi_\uparrow^* (x) \psi_\downarrow^* (x) \right] \right\} \] (1)

for fermion fields \( \psi_\sigma \) with spin label \( \sigma = (\uparrow, \downarrow) \). The diatomic field \( \varphi \) describes the bound-state (Feshbach) resonance with tunable binding energy \( \nu \) and mass \( M = 2m \), and -g \( \varphi (x) = g|\varphi (x)| e^{i\varphi (x)} \) represents the condensate.

As a result of spontaneous symmetry breaking the condensate acquires a non-zero \( |\varphi| = |\varphi_0| \). We expand in the derivatives of \( \theta \) and the small fluctuations in the condensate density \( \delta \varphi = |\varphi| - |\varphi_0| \), immediately preserving the Galilean invariance of the system. Immediate Galilean invariants of the theory are the density fluctuation \( \delta |\varphi| \) itself, \( G(\theta) = \hat{\theta} + (\nabla \theta)^2/4m \), and \( D_{\theta} (\delta |\varphi|, \theta) = (\delta |\varphi|) + \nabla \theta \cdot \nabla (\delta |\varphi|)/2m \). \( D_t \) is the comoving time derivative in the condensate with fluid velocity \( \nabla \theta/2m \).

The action \( S \) is quadratic in the fermion fields. On integrating them out and changing variables to \( \theta \) and \( \delta |\varphi| \), the local Galilean invariant effective density for the long-wavelength, low-frequency condensate is of the form\[ S_{\text{eff}} [\theta, \epsilon] = \int d^4x \left[ \frac{N_0}{4} G^2 (\theta, \epsilon) - \frac{1}{2} \rho_0 G (\theta, \epsilon) \right. \]

\[ \left. - \alpha e G (\theta, \epsilon) + \frac{1}{4} \eta D_\theta^2 (\epsilon, \theta) - \frac{1}{4} M^2 \epsilon^2 \right] \] (2)
in which \( \epsilon = \kappa^{-1} |\delta |\varphi| \) is a dimensionless rescaled condensate fluctuation. The scale factor is chosen so that on extending \( G(\theta) \) to \( G(\theta, \epsilon) = \hat{\theta} + (\nabla \theta)^2/4m + (\nabla \epsilon)^2/4m \), \( \epsilon \) has the same coefficients as \( \theta \) in its spatial derivatives \( \nabla \). The coefficients \( \alpha, \eta \), etc. are known functions of the scattering length and hence of the external magnetic field used to tune the condensate from the BCS to BEC regimes. For \( \epsilon = 0 \),

\[ S_{\text{eff}} [\theta, 0] \equiv S_0 [\theta] = \int d^4x \left[ \frac{N_0}{4} G^2 (\theta) - \frac{1}{2} \rho_0 G (\theta) \right] \] (3)
is the canonical BCS action. We expand \( S_{\text{eff}} [\theta, \epsilon] \) in powers of \( \epsilon \) as

\[ S_{\text{eff}} [\theta, \epsilon] = S_0 [\theta] - \alpha \int d^4x \epsilon G (\theta) \] \[ + \frac{1}{4} \int d^4x [\eta \epsilon^2 - \rho_0 (\nabla \theta)^2/2m - M^2 \epsilon^2] \] (4)

The semiclassical mean-field results above do not take the dynamical nature of the density fluctuations into account. The coarse-graining of the \( \epsilon \) field will introduce stochasticity in the acoustic metric of the \( \theta \) field via its Langevin equation.

We proceed by constructing the closed time-path (CTP) effective action, adapted from\[ S_{\text{CTP}} [\theta^+, \epsilon^+; \theta^-, \epsilon^-] = S_0 [\theta^+] - \alpha \int d^4x \epsilon^+ G (\theta^+ \} \] (5)

\[ + \frac{1}{4} \int d^4x [\eta \epsilon^+^2 - \rho_0 (\nabla \theta^+)^2/2m - M^2 \epsilon^+^2] - \{ \rightarrow \} \]
where $\pm$ denote integration on the upper and lower contours of the path respectively. We then integrate out the $\epsilon$ field (e.g. see [9]). The outcome is an effective non-local action for dynamical phonons,
\[
S_{\text{eff}}[\theta^+; \theta^-] = S_0[\theta^+] - S_0[\theta^-] + \Delta S[\theta^+; \theta^-]
\]
where $S_0[\theta]$ is given in [6] and
\[
\Delta S[\theta^+; \theta^-] = \frac{\alpha^2}{2} \int \int d^4x_1 d^4x_2 \left\{ G(\theta^+(x_1))D_{12}^+ G(\theta^+(x_2)) - G(\theta^+(x_1))D_{12}^- G(\theta^-(x_2)) - G(\theta^-(x_1))D_{12}^+ G(\theta^+(x_2)) + G(\theta^-(x_1))D_{12}^- G(\theta^-(x_2)) \right\}.
\]
In [8], the $D_{ij}^{\pm\pm} \equiv i\epsilon^{\pm\pm}(x_i)\epsilon^{\pm\pm}(x_j)$ are given in terms of $\epsilon$ correlators as
\[
D_{12}^{++} = \theta(t_1 - t_2) \langle \epsilon(x_1)\epsilon(x_2) \rangle + \theta(t_2 - t_1) \langle \epsilon(x_2)\epsilon(x_1) \rangle
D_{12}^{+-} = \theta(t_1 - t_2) \langle \epsilon(x_2)\epsilon(x_1) \rangle - \theta(t_2 - t_1) \langle \epsilon(x_1)\epsilon(x_2) \rangle
D_{12}^{-+} = \langle \epsilon(x_2)\epsilon(x_1) \rangle = D_{21}^{++}.
\]
We recover the semiclassical phonon field $\theta$ and the fluctuating field $R$ about it through the decomposition
\[
\theta^\pm(x) = \theta(x) \pm R/2.
\]
For the purpose of wave propagation we need only retain the terms linear in $\theta$ in the stochastic equations and linear and quadratic terms in $R$ in [8]. Quadratic terms in $R$ are then linearised by the introduction of noise $\xi$ enabling us to write $S_{\text{eff}}[\theta^+; \theta^-] \equiv S_{\text{eff}}[\theta; R]$ in the form
\[
S_{\text{eff}}[\theta; R] = \int d^4x R(x) L(\theta, \partial\theta, \xi, \bar{\xi}).
\]
In the path integral $R$ is now understood as a Lagrange multiplier to the Langevin equation for $\theta$,
\[
L(\theta, \partial\theta, \xi) = 0
\]
describing the propagation of phonons in a stochastic background provided by the noise.
Specifically, $G(\theta^\pm) \approx \theta \pm D_t R/2$, whence
\[
S_0[\theta^+] - S_0[\theta^-] = -\int d^4x R(x) \left[ \frac{N_0}{2} \ddot{\theta} - \frac{\rho_0}{4m} \nabla^2\theta \right]
\]
at the relevant order.

The second term in [8] splits into two terms,
\[
\Delta S[\theta; R] = \Delta S^{(1)}[\theta; R] + \Delta S^{(2)}[\theta; R]
\]
linear and quadratic in $R$ respectively. From the definitions [6], on integrating by parts,
\[
\Delta S^{(1)}[\theta; R] = -\alpha^2 \int d^4x R(x) \int d^4x' \partial_i D_R(x-x') \dot{\theta}(x')
\]
where $D_R$ is the retarded propagator
\[
D_R(x-x') = i\theta(t-t') \langle [\epsilon(x), \epsilon(x')] \rangle.
\]
It is the quadratic $\Delta S^{(2)}[\theta; R]$ which induces noise into the system. We linearise $\Delta S^{(2)}[\theta; R]$ in $R$ by writing exp($i\Delta S^{(2)}[\theta; R]$) in terms of noise $\xi$ as
\[
e^{i\Delta S^{(2)}[\theta; R]} = \int D\xi P[\xi] e^{i\alpha \int d^4x(D_t R(x))\xi(x)}
\]
where
\[
P[\xi] = e^{-\frac{1}{2} \int d^4x_1 d^4x_2 \xi(x_1)D_{ij}^{-\pm} \xi(x_2)},
\]
determines the noise distribution through the Hadamard correlator
\[
\langle \xi(x)\xi(x') \rangle = D_H(x-x') = \frac{1}{2} \langle \{\epsilon(x), \epsilon(x') \} \rangle.
\]
Specifically,
\[
D_H(x-x') = \int \frac{d^3k}{(2\pi)^3} \frac{\cos[\omega_k(t-t')]}{\omega_k\eta} e^{-ik\cdot(x-x')}
\]
in which the dispersion relation of the $\epsilon$ field is determined by $\omega_k = \sqrt{\rho_0 k^2/2m\eta + M^2/\eta}$.

The overall coefficient of $R$ in the integrand of $S_{\text{eff}}[\theta; R]$ follows directly from [12], [14] and [16], on using the integration by parts rule $(D_t R(x))\xi(x) \rightarrow -R(x)\xi + \nabla.(\xi\nabla\theta)/2m(x)$ = $-R(x)(D_t \xi + \xi\nabla\theta)/2m(x)$. The resulting Langevin equation is then
\[
\frac{N_0}{2} \ddot{\theta}(x) - \left( \frac{\rho_0}{4m} - \frac{\alpha}{2m} \right) \nabla^2\theta + \alpha^2 \int d^4x' \partial_i D_R(x-x') \dot{\theta}(x') = -\alpha D_t \xi(x).
\]
What is crucial for our subsequent discussion is the multiplicative noise term $\xi\nabla^2\theta$, a consequence of the Galilean invariance enforcing covariant derivatives. In the acoustic limit, in which we set $\omega = k = 0$ in [15],
\[
D_R(x-x') = (2/M^2)\delta^4(x-x'),
\]
to reproduce the mean value speed of sound, as derived elsewhere [6] [7]:
\[
e^2 = \frac{\rho_0/2m}{N_0 + 4\alpha^2/M^2}.
\]
More generally [6] [7] we have a ‘rainbow’ of sound speeds $c_k$, according to the wavelength $k$, of the form
\[
c_k^2 \approx c_0^2[1 + k^2/K^2],
\]
where
\[
K^2 = \frac{M^4}{4\alpha^2 c_s^6} \left[ 1 - \frac{c_s^2 \eta}{\rho_0/2m} \right].
\]
The sound waves or phonons of interest satisfy \( k < K \).

To see the relation to stochastic space-time we consider the acoustic approximation discussed above, for which \( (20) \) becomes

\[
(N_0 + 4\alpha^2/M^2)\theta(x) - (\rho_0/2m - \alpha\xi/m)\nabla^2\theta = -2\alpha D_s\xi(x).
\]

(25)

In terms of the speed of sound \( c \) of \( (22) \), this becomes

\[
\theta(x) = c^2(1 - 2\alpha\xi/\rho_0)\nabla^2\theta = -4m(\alpha/\rho_0)c^2D_s\xi(x).
\]

(26)

The outcome of this analysis is that we can interpret \( c_\xi \),

\[
c_\xi^2 = c^2(1 - 2\alpha\xi/\rho_0)
\]

(27)
as a stochastic speed of sound. See Fig.1 for an exemplary plot of \( c^2/v_F^2 \) for the condensate parameters given later.

Behaviour of the form of \( (27) \) is the starting point for the stochastic analysis of the papers of \( [1, 4] \). However, whereas these authors argue for stochastic behaviour on empirical grounds, in our case we see from \( (18) \) that the noise \( \xi \) is essentially the condensate fluctuation \( \epsilon \), as we would expect, with known properties.

The effect of the noise will be to induce stochastic perturbations on the acoustic metric \( [1, 4] \). This effect can be tested from a variation in time travel of the phonons or sound waves between a source and a detector. For a spatially homogeneous static condensate its operator-valued acoustic metric can be taken as

\[
dt^2 - c_\xi^{-2}dx^2 = 0,
\]

(28)

written as

\[
c^2dt^2 - (1 + 2\alpha/\rho_0\xi)dx^2 = 0.
\]

(29)

We follow the analysis of \( [1, 2] \). In conventional formalism the sound wave propagates along the sound cone determined by the null-geodesic:

\[
c^2dt^2 = dx^2 + h_{ij}dx^idx^j,
\]

(30)

where \( h_{ij} = (2\alpha/\rho_0)\xi\delta_{ij} \). Thus,

\[
dt = \frac{1}{c}\sqrt{1 + h_{ij}n^in^j} \; dr \approx \frac{1}{c}\left(1 + 1/2h_{ij}n^in^j\right) dr,
\]

(31)

where \( dr = d|x| \) and \( n^i = dx^i/dr \) is a unit vector along the direction of the sound wave propagation. If the spatial separation between the source and the detector is \( r \), then the travel time can be expressed as

\[
T = \int_0^T dt \approx \int_0^r \frac{dr}{\left[1 + 1/2h_{ij}n^in^j\right]},
\]

(32)

where the local velocity \( c \) is evaluated on the unperturbed path of the waves \( r(t) \). With \( \{h_{ij}\} = 0 \), the variation of the travel time is given by

\[
(\Delta T)^2 = \langle T^2 \rangle - \langle T \rangle^2
\]

(33)

\[
= \frac{1}{4} \int_0^T dt_1 \int_0^T dt_2 n^i n^j n^m \langle h_{ij}(r(t_1), t_1) h_{lm}(r(t_2), t_2) \rangle.
\]

Analogies with quantum gravity \( [1] \) are clear, as discussed in \( [2, 4] \), but this is not the goal of this paper. We assume that the wave propagates along the \( z \)-direction. Then, if \( z(t) = c_st \),

\[
(\Delta T)^2 = \left(\frac{\alpha}{\rho_0}\right)^2 \int_0^T dt_1 \int_0^T dt_2 \xi(z(t_1), t_1) \xi(z(t_2), t_2))
\]

(34)

From the expression for \( D_H \) in \( (18) \) we are able to compute \( \Delta T/T \).

Eqn. \( (34) \) is our key result, but before we can estimate \( \Delta T/T \) we need to list the basic attributes of the parameters in the model. [See \( [6, 7] \) for more detail.] We find that \( 0 \leq \alpha/\rho_0 \leq 1 \) increases as we pass from the deep BCS regime \( (\alpha/\rho_0 \approx 0) \) to the deep BEC regime \( (\alpha/\rho_0 \approx 1) \). On the contrary, \( \eta, N_0 \) and \( M^2 \) go from finite values to zero as we go from BCS to BEC regimes. As a result of \( M^2 \) vanishing \( c_\xi^2 \) falls off from \( v_F^2/3 \) in the BCS regime \( [5] \) to zero in the deep BEC regime \( [6, 7] \).

Reintroducing \( h \), we take the typical choice of arrival time as the correlation time \( T = h/2mc^2 \) for the phonon to travel a typical correlation length of the system, \( l = h/2mc \). When the phonons comprise a wavepacket with a central momentum \( k_0 \) and width \( \Delta k_0 = k_0 \pm \Delta k_0 \) will be chosen to be smaller than \( K \) of \( (24) \) such that they all experience the same sound speed \( c \).

To be concrete, consider a cold \( ^6\text{Li} \) condensate of \( 3 \times 10^5 \) atoms tuned by the narrow resonance at \( H_0 = 543.25 \text{GHz} \), discussed in some detail in \( [10] \) and used by us elsewhere \([6, 7]\). We take the achievable number density \( \rho_0 = k^2_F/3\pi^2 \approx 3 \times 10^{12} \text{cm}^{-3} \), where \( \epsilon_F \approx 7 \times 10^{-11} \text{eV} \).

In terms of the dimensionless coupling \( \bar{g} \), where \( g^2 = (64c^4_F/3k^2_F)^2 \bar{g} \) \([5]\), \( ^6\text{Li} \) at the density above corresponds to \( \bar{g}^2 \lesssim 1 \). Since the travel distance \( l \) is much shorter than the typical size of the condensate \( L \approx 100/k_F \), to obtain an approximate analytical expression of \( \Delta T/T \), it is a sufficiently good approximation to take account only of momenta smaller than the inverse of the travel distance but larger than the inverse of the size of the condensate, a natural choice of the infrared momentum cutoff in \( (34) \). They give coherent contributions so that the spatial dependence in \( (34) \) can be safely ignored.

Straightforward substitution by this momentum-truncated \( D_H \) in \( (19) \) gives

\[
\left(\frac{\Delta T}{T}\right)^2 \approx \frac{\alpha^2}{\rho_0^2} \int_0^T dt_1 \int_0^T dt_2 \int_{1/L}^{1/l} d^3k \cos[\omega_k(t_1 - t_2)]/\omega_k\eta.
\]

(35)

As it stands \( (35) \) is not very transparent, but a good analytic guide to the behaviour of \( (\Delta T/T)^2 \) is provided by the saddle-point approximation to \( (35) \),

\[
\left(\frac{\Delta T}{T}\right)^2 \approx C \left(\frac{\alpha}{\rho_0}\right)^2 \sin^2(\sqrt{3}MT/2\eta^{1/2})/\eta T^2\pi^2(\rho_0/2mn)^{3/2}.
\]

(36)
where $C = \ln(x + \sqrt{1 + x^2}) - x/\sqrt{1 + x^2}; x = (\rho_0/2m)^{1/2}/\bar{M} = (2m\rho_0)^{1/2}c/M\hbar$.

Equivalently, we can think of the fluctuations in flight times as fluctuations in sound speed $c$ over correlation lengths,

$$\frac{(\Delta T/T)^2}{(\Delta c/c)^2}$$

With $T = \hbar/2mc^2$, the time to traverse a correlation length of condensate is now a function of $k_FaS$, as are $\eta$ and $M$. We see from (36) that in the deep BEC regime $\Delta T/T \to 0$ since $MT/\eta^{1/2} \approx 0$. Also, $\Delta T/T \to 0$ in the deep BCS regime since $\alpha \approx 0$, even though $MT/\eta^{1/2} \ll 1$. The effective the approximation (36) is seen in Fig.1, where we contrast it to the full solution (35) for $(\Delta T/T)^2$ over the whole range from the BCS to the BEC regimes for a typical value of the coupling $\bar{g} = 0.9$, using the detailed values of $\alpha, \eta$ and $M$ from [3, 7].

$\Delta T/T$ achieves its maximum value $\Delta T/T \approx 0.16$ near the crossover regime at $1/k_Fa_S \approx -0.8$. At the peak the travel distance $l = \hbar/2mc \approx 1.67/k_F \ll L \approx 100/k_F$ is smaller than the size of the condensate. Further, at the peak $K \approx 0.6k_F$ as seen from Fig.1. If we take the central momentum from the speed of sound there [8, 7] as $k_0 \approx 0.3k_F$ we can accommodate a spread $\Delta k_0$ somewhat less than $0.3k_F$ while still permitting $k_0 \pm \Delta k_0 < K$.

In our model the speed of sound [22] vanishes in the BEC regime because of the absence of direct diatomic self-interactions in the Lagrangian density in [1], but the qualitative behaviour shown in Fig.1 does not rely on this fact. Suppose, as in [11], we include such a term

$$L(\phi) = -u_B|\phi(x)|^4/4$$

in the integrand of [11]. The effect in $S_{eff}(\theta, \epsilon)$ of $(1)$ is just to replace $M^2$ by $M^2 = M^2 + 6u_B\kappa^2|\phi_0|^2$ in all results following [5]. The term linear in $\epsilon$, which corresponds to making the replacement $\alpha G_0 \to \alpha G_0 + u_B\kappa|\phi|^3$ in [5] has no effect, since it always contributes to total derivatives in the calculations which follow.] This leaves $\epsilon^2$ unchanged in the BCS regime because $\alpha \approx 0$ there but, since $\kappa^2|\phi_0|^2 \neq 0$ it permits $\epsilon^2$ to tend to a non-zero limit in the deep BEC regime. Eq. (35) uses $\epsilon^2$ implicitly in our choice of $T$ as the time to traverse a correlation length. However, the vanishing of $\alpha$ in the deep BCS regime and the vanishing of $\eta$ in the deep BEC regime are sufficient for fluctuations to have no effect there. In the intermediate regime there will be an effect. The details will depend on parameter choice but qualitatively the behaviour shown in Fig.1 will persist. We stress that the behaviour described above is a consequence of quantum fluctuations and not thermal fluctuations. In Fig.1 the effect of quantum fluctuations looks significant but whether the effect is measurable, given the other experimental uncertainties, is unclear at the present time.

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