Thermodynamic Properties of a Yukawa—Schwarzschild Black Hole in Noncommutative Gauge Gravity

Slimane Zaim1* and Hadjar Rezki1

1Département de Physique, Faculté des Sciences de la Matière, Université de Batna 1, Algeria

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Abstract—We construct a noncommutative gauge theory for the deformed metric corresponding to the modified structure of a gravitational field in the case of noncommutative Yukawa—Schwarzschild space-time. The thermodynamic properties and corrections to the gravitational force on the horizon of a noncommutative Yukawa—Schwarzschild black hole are analyzed.

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1. INTRODUCTION

General relativity gives rise to the concept of black holes as a real physical phenomenon. The interest in this phenomenon has dramatically increased since Hawking discovered a mechanism by which black holes can radiate [1]. This mechanism involves quantum-mechanics processes near the black hole horizon. In fact, Hawking showed that a black hole radiates a spectrum of particles which is quite analogous to thermal black body radiation. Thus Hawking radiation emerges as a nontrivial consequence of combining gravity and quantum mechanics. Mathematically, this phenomenon was treated in [1, 2]. For a given static and spherically-symmetric metric, the black hole temperature ($T_H$) is proportional to the surface gravity ($k$) according to the relation $T_H = k/(2\pi)$. This is the interpretation of Hawking radiation. This formalism can also be applied to a static and spherically symmetric noncommutative Yukawa—Schwarzschild space-time. The corrections to Newton's potential in the gravitation theory has yielded interesting results. The fact that the experimental tests of general relativity using different methods constrain these Yukawa-like corrections [3]. To deal with the Yukawa effect in gravitation theory, we incorporate an additional Yukawa term in the Schwarzschild space-time with noncommutative corrections.

Quantum mechanics is based on the notion of commutation relations between the position and momentum. We can think of a different approach in which one enforces commutation relations between position coordinates themselves as well as momentum coordinates. In analogy to the Heisenberg uncertainty relations between position and momentum, a new set of uncertainty relations appears between position coordinates themselves as well as momentum coordinates. This idea results in the concept of quantum gravity since quantifying space-time leads to quantifying gravity. Noncommutativity is mainly motivated by string theory, being a limit in the presence of a background field [1, 2, 4]. Here one uses a gauge field theory with star products and Seiberg-Witten maps. The noncommutative space-time is characterized by the coordinate operators $\hat{x}^\mu$ satisfying the following commutation relation:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu},$$

where $\theta^{\mu\nu}$ is an anti-symmetric real matrix which determines the fundamental cell discretization of space-time much in the same way as the Planck’s constant $\hbar$ discretizing phase space. The physical interpretation of this parameter is the smallest area in space that can be probed. The ordinary product between functions in noncommutative space is replaced by the $\ast$-product, where the $\ast$-product between two arbitrary functions $f(x)$ and $g(x)$ is given by

$$f(x) \ast g(x) = f(x) \cdot g(x) + \exp\left(\frac{i}{2} \sum_{ij} \theta_{ij} \partial_i \partial_j \right) f(x)g(x') \bigg|_{x=x'}.$$

In flat space time, to get noncommutative local gauge theories, but with local Lorentz violation symmetry, we must apply the symmetry under the proper twist-Poincaré algebra [5, 6]. The noncommutative gauge gravity is a theory of general relativity in curved space-time with preservation of noncommutative space-time and based partly on implementing symmetries on flat noncommutative space-time. In gravity theory the action transforms under ordinary Lorentz transformation of the ordinary fields,
since these ordinary transformations, via the Seiberg-Witten map, induce the noncommutative canonical transformations of noncommutative fields under which the noncommutative action is invariant [7–9]. Our work will be in this context.

In this work we consider corrections to the gravitational force on the horizon of a Yukawa–Schwarzschild black hole due to thermodynamic properties in order to take into account the noncommutativity for black holes. Black hole thermodynamics plays an important role in the modern universe. Black hole thermodynamics provides a real connection between gravity and quantum mechanics. Recently there has been a lot of interest in the thermodynamic properties of black holes in noncommutative space-time [10–13]. The thermodynamic properties of a Yukawa–Schwarzschild black hole in commutative space was studied in [14, 15]. In this paper we apply some of these ideas to obtain the thermodynamic properties for a Yukawa–Schwarzschild black hole in noncommutative space.

In the present study, we propose a deformed Yukawa–Schwarzschild black hole solution in noncommutative gauge theory of gravity. We apply the Bekenstein-Hawking method to compute the thermodynamic properties. The obtained qualitative results show that the noncommutativity eliminates the divergent behavior of the Hawking temperature and decreases the radius within which black holes cannot radiate.

This paper is organized as follows. In Section 2, we present the noncommutative gauge gravity metric by using Seiberg-Witten maps, following the approach of [13], and we calculate the corrected event horizon radius up to the second order in the noncommutativity parameter. In Section 3, we calculate the noncommutative temperature of the deformed Yukawa–Schwarzschild black hole, where we show that the noncommutative effects eliminate the divergent behavior of the Hawking temperature and decrease the black hole radius to a new minimum limiting value. In Section 4, we calculate the entropy of the deformed Yukawa–Schwarzschild black hole and show that the entropy modification of the black hole due to noncommutative space-time is negligible.

### 2. NONCOMMUTATIVE GAUGE GRAVITY METRIC FOR A YUKAWA–SCHWARZSCHILD BLACK HOLE

In our previous work [11], we used the tetrad formalism and the inhomogeneous local Lorentz (Poincaré) transformations and imposed the invariance of the canonical noncommutative space-time commutation relations under the generalized local inhomogeneous Lorentz transformations by gauging SO(3, 1), and by translating the enveloping algebra we constructed a theory of noncommutative general relativity. The generalized coordinate transformations are

\[ \hat{x}^\mu = \hat{x}^\mu + \hat{\xi}^\mu(\hat{x}), \]  

which are compatible with the algebra given by (1). Under the change of coordinates (3), the noncommutativity parameter \( \hat{\xi}^\mu \) satisfies the following condition:

\[ \hat{\xi}^\mu = \xi^\mu + \theta^{\mu a} \partial_a \xi^\rho \partial_\rho. \]  

Here we look for a mapping \( \phi^A \rightarrow \hat{\phi}^A \) and \( \lambda \rightarrow \hat{\lambda} \), where \( \phi^A = (\epsilon^\mu_a, \omega^{ab}_\mu) \) is a generic field, \( \epsilon^\mu_a \) and \( \omega^{ab}_\mu \) are the vierbein and the gauge fields (spin connections), respectively (the Greek and Latin indices refer to curved and tangent space-time, respectively), and \( \lambda = \lambda_P \), where \( \lambda_P \) is the local Poincaré infinitesimal Lie-valued gauge-transformation parameter, such that

\[ \hat{\phi}^A(A) + \hat{\delta}_A \phi^A(A) = \hat{\phi}^A(A + \delta_A A), \]  

with

\[ \delta_A A = d\lambda + i\lambda A - iA\lambda, \]  

\[ \hat{\delta}_A \hat{A} = d\hat{\lambda} + i\hat{\lambda} \hat{A} - i\hat{\lambda} \hat{A}, \]

and

\[ \lambda_P = \xi + \lambda_L, \quad \hat{\lambda}_P = \hat{\xi} + \hat{\lambda}_L, \]

with

\[ \xi = \xi^\mu \partial_\mu, \quad \hat{\xi} = \hat{\xi}^\mu \partial_\mu, \]

and

\[ \lambda_L = \frac{1}{2} \lambda_L^{ab} S_{ab}, \]

where \( \xi \) and \( \lambda_L \) are the local translation and Lorentz infinitesimal Lie-valued gauge transformations, respectively. The corresponding symmetry generators are denoted by \( S_{ab} \), they represent the generators of the Lorentz group and satisfy the following relation:

\[ [S_{ab}, S_{cd}] = f_{abc}^{ik} S_{ik}, \]  

with \( f_{abc}^{ik} \) the structure constants of the Lorentz algebra. The infinitesimal noncommutative transformation \( \hat{\delta}_A \) of \( \epsilon^\mu_a \) and \( \omega^{ab}_\mu \) induced by local gauge transformation are given by [7–9]:

\[ \delta_A \epsilon^\mu_a = \left[ \lambda^a_b, \epsilon^\mu_b \right] = \lambda^a_b * \epsilon^\mu_b - \epsilon^\mu_a * \lambda^a_b, \]  

\[ \delta_A \omega^{ab}_\mu = \partial_\mu \omega^{ab}_\mu + \omega^{ac}_\mu \omega^{cb}_\mu - \omega^{ab}_\mu \omega^{ac}_\mu * \omega^{cb}_\mu. \]

In accordance with the general method of gauge theories, in noncommutative space-time, we impose the infinitesimal local transformations given in Eq. (5) and Eq. (12). Using these transformations, one can
get in the second order in the noncommutative parameter $\theta^{\mu \nu}$ the Seiberg-Witten map for vierbeins $\hat{e}_\mu^a$ as $[7-9]$
\[
\hat{e}_\mu^a = e_\mu^a - \frac{i}{4} \theta^{\alpha \beta} \left( \omega^a_{\alpha \beta} \partial_\beta e_\mu^c + \left( \partial_\beta \omega^c_{\alpha \mu} + R^a_{\beta \mu} \right) e_\mu^c \right) + \frac{1}{32} \theta^{\alpha \beta} \theta^{\gamma \delta} \left( 2 \left\{ R_{\alpha \delta} R_{\beta \mu} \right\} e_\mu^c \right.
\]
\[
- \omega^a_{\alpha \gamma} \left( D_\beta R_{\alpha \delta}^d + \partial_\beta R_{\alpha \delta}^d \right) e_\alpha^d
\]
\[
- \left\{ \omega_\alpha \left( D_\beta R_{\delta \mu} + \partial_\beta R_{\delta \mu} \right) \right\} e_\alpha^d
\]
\[
- \partial_\delta \left\{ \omega_\alpha \left( \partial_\beta \omega_{\gamma} + R_{\beta \gamma} \right) \right\} e_\alpha^c
\]
\[
+ \left( \partial_\beta \omega_{\epsilon}^c + \left( R_{\beta \gamma} R_{\epsilon}^d \right) e_\beta^d \right) + O \left( \theta^3 \right) \tag{14}
\]
where
\[
R_{\beta \mu} = R_{\beta \mu}^{ac} S_{ab}, \tag{15}
\]
\[
\omega_\mu = \omega^a_{\mu b} S_{ab}, \tag{16}
\]
\[
\theta_\mu = \frac{1}{2} \left( \hat{e}_\mu^a \right) \tag{17}
\]
and where the $\hat{e}_{1a}^\mu$ is the inverse of the vierbein $\hat{e}_\mu^a$
\[
\hat{e}_\mu^a \hat{e}_{1a}^\mu = \delta_\mu^a, \tag{19}
\]
and
\[
\hat{e}_\mu^a \hat{e}_{1b}^\nu = \delta_\nu^b. \tag{20}
\]

To begin, consider a noncommutative gauge theory in which the action of pure gravity written as follows:
\[
S = \frac{1}{2 R^2} \int d^4 x (\mathcal{L}_G), \tag{21}
\]
it is invariant under the noncommutative local gauge transformation given by Eq. (7), where $\mathcal{L}_G$ is the pure gravity density defined as
\[
\mathcal{L}_G = \hat{e} \ast \hat{R}, \tag{22}
\]
with the deformed tetrad and scalar curvature given as
\[
\hat{e} = \det \left( \hat{e}_\mu^a \right) \equiv \frac{1}{4!} \hat{e}^{\mu \rho \sigma \tau} e_\rho^a \hat{e}_\sigma^b \hat{e}_\tau^c \hat{e}_\mu^d, \tag{23}
\]
\[
\hat{R} = \hat{e}_{1a}^\mu \hat{e}_b^\nu \hat{R}_{ab}. \tag{24}
\]
In the following, we consider a symmetric metric $\hat{g}_{\mu \nu}$, so that
\[
\hat{g}_{\mu \nu} = \frac{1}{2} \left( \hat{e}_\mu^b \ast \hat{e}_b^c \ast \hat{e}_\nu^a \right). \tag{25}
\]
As a consequence, the first-order expansions in the noncommutativity parameter $\theta^{\alpha \beta}$ of the scalar curvature $\hat{R}$ and the metric $\hat{g}_{\mu \nu}$ vanish. Thus $\hat{R}$ and $\hat{g}_{\mu \nu}$ can be rewritten as
\[
\hat{R} = R + R^{(2)} \ast O \left( \theta^3 \right), \tag{26}
\]
\[
\hat{g}_{\mu \nu} = g_{\mu \nu} + g_{\mu \nu}^{(2)} \ast O \left( \theta^3 \right), \tag{27}
\]
where $R^{(2)} = R_{r0}^{sa}$ was expressed explicitly in $[9]$. Here we discuss the procedure above for obtaining the leading order corrections to the Yukawa–Schwarzschild metric tensor to find that
\[
ds^2 = - \left( 1 - \frac{1}{r} \left( 1 + \beta e^{-r/\lambda} \right) \right) dt^2
\]
\[
+ \frac{1}{1 - \frac{3}{r} \left( 1 + \beta e^{-r/\lambda} \right)} dr^2
\]
\[
+ r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \tag{28}
\]
We point out that the Yukawa-like correction to the Newtonian potential is a very well established result of many different alternative theories of deformed gravity. One can assign the diagonal tetrads according to
\[
\xi_0 = \left( 1 - \frac{r_s}{r} \left( 1 + \beta e^{-r/\lambda} \right) \right)^{1/2}, 0, 0, 0 \right), \tag{29}
\]
\[
\xi_1 = \left( 0, 1 - \frac{r_s}{r} \left( 1 + \beta e^{-r/\lambda} \right)^{-1/2}, 0, 0 \right), \tag{30}
\]
\[
\xi_2 = \left( 0, 0, r, 0 \right), \tag{31}
\]
\[
\xi_3 = \left( 0, 0, 0, r \sin \theta \right). \tag{32}
\]
The nonzero spin connections that follow from the zero torsion condition are:
\[
\omega_\mu^{12} = \left( 0, - \left( 1 - \frac{r_s}{r} \left( 1 + \beta e^{-r/\lambda} \right) \right)^{1/2}, 0, 0 \right), \tag{33}
\]
\[
\omega_\mu^{13} = \left( 0, 0, - \left( 1 - \frac{r_s}{r} \left( 1 + \beta e^{-r/\lambda} \right) \right)^{1/2} \cos \theta, 0 \right), \tag{34}
\]
\[
\omega_\mu^{23} = \left( 0, 0, - \cos \theta, 0 \right). \tag{35}
\]
\[ \omega_{\mu}^{10} = \left( 0, 0, 0, -\frac{r_s}{2r^2} \left( 1 + \beta \left( 1 + \frac{r}{\lambda_0} \right) e^{-r/\lambda_0} \right) \right), \]

where \( r_s = 2M/c^2 \), with \( M \) the black hole mass, \( \beta \) is the strength of the Yukawa correction, and \( \lambda_0 \) represents the range of the Yukawa potential. (For the Earth’s orbiting satellites like LAGEOS, \( \beta_{\text{min}} = 1.38 \times 10^{-11} \) and \( \lambda_0 = 6.081 \times 10^6 \text{ m} \) [17] as well \( \beta_{\text{min}} = 3.57 \times 10^{-10} \text{ for } \lambda_0 = 2.89 \times 10^{10} \text{ m} \) [18]).

Using the Seiberg-Witten maps given in Eq. (14), we can determine the deformed Yukawa Schwarzschild metric. To find this, first we have obtained the corresponding components of the tetrad fields \( e_{\mu}^a \) given by Eqs. (14). With the definitions (2) and (25), it is possible to obtain the components of the deformed metric \( \hat{g}_{\mu\nu} \). For the spherical symmetry of the metric tensor \( g_{\mu\nu} \), we choose for the noncommutative antisymmetric metric \( \theta^{\alpha\beta} \) the following:

\[
\theta^{\alpha\beta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \theta & 0 & 0 \\
0 & -\theta & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

where \( \theta \) is a constant with the dimension of length squared, and for the dimensionless homogeneity in the noncommutative spaces, the coordinate operators must be redefined as: \( x^\mu = \hat{x}^\mu/r_0 \), where \( \Theta = \theta/r_0^2 \) (\( r_0 \) is the observed cosmic radius).

We follow the same steps as outlined in [19] and look for the noncommutative correction of the metric up to the second order in \( \Theta \). Then the nonzero components of the noncommutative tetrad fields \( \hat{e}_{\mu}^a \) are:

\[
e^1_1 = \frac{1}{\left( 1 - \frac{r}{\lambda} (1 + \beta e^{-r/\lambda}) \right)^{1/2}} \left[ 1 + \frac{\alpha \left( 1 + \beta \left( 1 + \frac{r}{\lambda} \right) e^{-r/\lambda} \right)^2}{4r \left( 1 - \frac{r}{\lambda} (1 + \beta e^{-r/\lambda}) \right) \left( 1 + \beta \left( 1 + \frac{r}{\lambda} + \frac{r^2}{2\lambda^2} \right) e^{-r/\lambda} \right)} \right] \Theta^2 + \mathcal{O}(\Theta^3),
\]

\[
e^1_2 = -\frac{i}{4} \left( 1 - \alpha \left( 1 + \beta e^{-r/\lambda} \right) \right)^{1/2} \left[ 1 + \frac{\alpha \left( 1 + \beta \left( 1 + \frac{r}{\lambda} \right) e^{-r/\lambda} \right)}{r \left( 1 - \frac{r}{\lambda} (1 + \beta e^{-r/\lambda}) \right)} \right] \Theta + \mathcal{O}(\Theta^3),
\]

\[
e^2_2 = r + \frac{1}{64r^2} \left[ 7\alpha \left( 1 + \beta \left( 1 + \frac{r}{\lambda} \right) e^{-r/\lambda} \right) - 24\alpha \left( 1 + \beta \left( 1 + \frac{r}{\lambda} + \frac{r^2}{2\lambda^2} \right) e^{-r/\lambda} \right) \right] \Theta^2 + \mathcal{O}(\Theta^3),
\]

\[
e^3_3 = r \sin \theta - \frac{i}{4} \cos \theta \Theta + \frac{\alpha}{8r^2} \left[ 2 \left( 1 + \beta \frac{r}{\lambda} e^{-r/\lambda} \right) - \beta \left( 1 + \frac{r}{\lambda} + \frac{r^2}{2\lambda^2} \right) e^{-r/\lambda} \right] \Theta^2 + \mathcal{O}(\Theta^3),
\]

\[
e^0_0 = \left( 1 - \frac{r}{\lambda} (1 + \beta e^{-r/\lambda}) \right)^{1/2} + \frac{\alpha}{8r^3} \left[ \frac{\alpha \left( 1 + \beta \left( 1 + \frac{r}{\lambda} + \frac{r^2}{2\lambda^2} \right) e^{-r/\lambda} \right)}{2r^2 \left( 1 - \frac{r}{\lambda} (1 + \beta e^{-r/\lambda}) \right)} - 1 \right] \sin \theta \Theta^2 + \mathcal{O}(\Theta^3),
\]
It is clear that, for $\Theta \to 0$, we obtain the commutative Yukawa–Schwarzschild solution. It is interesting to note that the deformed metric $\hat{g}_{\mu\nu}$ is spherically symmetric and is related to the choice of the noncommutativity parameter $\Theta$ as in (37). For such a black hole, we find that the event horizon in noncommuta-
tive space–time is where the noncommutative metric (25) satisfies the condition:

\[ \dot{g}_{00} = 0. \]  

(47)

Let us analyze the condition for the existence of an event horizon in noncommutative space-time in the cases \( \mathcal{O}(\lambda^{-2}) \) and \( \mathcal{O}(1/r^4) \). In this approximation order, we can rewrite (47) as follows:

\[
\begin{align*}
(1 + \frac{\alpha\beta}{\lambda}) r^3 &- \alpha (1 + \beta) r^2 \\
+ \frac{\alpha (1 + \beta)}{2} \left( 1 + \frac{\alpha\beta}{\lambda} \right) \Theta^2 &= 0. 
\end{align*}
\]

(48)

We take \( \delta = (1 + \alpha\beta/\lambda) \) and \( \gamma = (1 + \beta) \). Then this equation becomes:

\[
\delta r^3 - \gamma r^2 + \frac{1}{2} \gamma \delta \Theta^2 = 0. 
\]

(49)

The solution of this third–order polynomial equation is given in [20]. Then the real root of the quadratic formula (49) is the event horizon of the Yukawa–Schwarzschild black hole in noncommutative space. The real root is given by

\[
\begin{align*}
\hat{r}_{H}^{NC} &= \frac{\gamma}{3\delta} \\
\times &\left[ 1 + \frac{3}{4} \left( \frac{3\delta}{\gamma} \right)^3 \Theta^2 + i \sqrt{\frac{\gamma}{2}} \left( \frac{3\delta}{\gamma} \right)^{3/2} \Theta \\
+ &\sqrt{\frac{1}{4}} \left( \frac{3\delta}{\gamma} \right)^3 \Theta^2 - i \sqrt{\frac{\gamma}{2}} \left( \frac{3\delta}{\gamma} \right)^{3/2} \Theta \\
\right] \\
&\simeq \hat{r}_{H}^{C} \left( 1 + 3 \left( 1 + \frac{\alpha\beta}{\lambda} \right) \left( \frac{\Theta}{\hat{r}_{H}^{C}} \right)^2 \right). 
\end{align*}
\]

(50)

where \( \hat{r}_{H}^{C} = \alpha(1 + \beta)/(1 + \alpha\beta/\lambda) \) is the event horizon in commutative case when \( \Theta = 0 \). The effect of noncommutativity is a small value where the event horizon is large, which is reasonable to expect, since at large distances the space-time can be considered as a smooth classical manifold.

3. NONCOMMUTATIVE TEMPERATURE OF HAWKING RADIATION

Let us now consider the noncommutative temperature of the black hole in noncommutative space-time. The black hole in noncommutative space-time emits thermal radiation with a corrected temperature that depends on the noncommutativity parameter, we call it the noncommutative temperature. The corrected temperature of a black hole whose metric has been modified by the Yukawa potential term in noncommutative space-time is given by:

\[
\hat{T}_{H} = \left( \frac{1}{4\pi} \right) \left( \frac{d\hat{g}_{00}}{dr} \right)_{r=r_{H}^{NC}} \\
= T_{H}^{C} \left[ 1 - \frac{15}{2} \left( 1 + 3\alpha\beta \right) \left( \frac{\Theta}{r_{H}^{C}} \right)^2 \right], 
\]

(51)

where \( T_{H}^{C} = 1/ \left[ 4\pi r_{H}^{C} (1 - \alpha\beta/\lambda) \right] \) is the semiclassical temperature of the Yukawa black hole in commutative spaces. For large black holes, i.e., when \( \Theta/r_{H}^{C} \ll 1 \), one recovers the standard result [15–17] for the Hawking temperature:

\[
T_{H} = \frac{1}{4\pi r_{H}^{C} \left( 1 - \frac{\alpha\beta}{\lambda} \right)} \simeq \frac{1}{4\pi r_{H}^{C}} \left( 1 + \frac{\alpha\beta}{\lambda} \right). 
\]

(52)

We plot in Fig. 1 the Hawking temperature for various noncommutativity parameters. We see that the temperature (for dashed lines) reaches a maximum value \( T_{H}^{max} = 2.28 \times 10^{-2} \Theta \) for \( r_{H} = 4.7\Theta \), and then decreases to zero as \( r_{H} = 2.7\Theta \). The commutative temperature (solid line) has a divergent behavior which is cured by non-commutativity. The formula for \( \hat{T}_{H} \) also leads to \( \hat{T}_{H} = 0 \) for

\[
r_{H}^{C} = r_{H}^{O} = \sqrt{\frac{15}{2} \left( 1 + 3\alpha\beta \right) \Theta} \simeq 2.7\Theta. 
\]

(53)

The noncommutative temperature \( \hat{T}_{H} \) reaches its maximum for

\[
r_{H}^{C} = 3 \sqrt{\frac{5}{2} \left( 1 + 3\alpha\beta \right) \Theta} \simeq 4.7\Theta. 
\]

(54)

These results differ from what is found in the standard (commutative space) case, where \( \hat{T}_{H} \) diverges as \( r_{H}^{C} \) goes to zero.

On the thermodynamics side, in the case of noncommutative space, we see that the noncommutativity eliminates the divergent behavior of the Hawking temperature. As a result, there is a new maximum temperature that a black hole can reach before turning to zero for a new small nonzero radius \( r_{H}^{O} \). In the region \( r_{H}^{C} < r_{H}^{O} \) there is a black hole with negative \( \hat{T}_{H} \), and this has no physical sense. Thus, in commutative space, the effect of vacuum fluctuations increases the value of the Hawking temperature. On the contrary, the value of the Hawking temperature increases to a maximum value and then decreases to zero, so that there are no processes of particle–antiparticle creation on the external surface of the black hole with a radius \( r_{H}^{C} = r_{H}^{O} \). The obtained results show that the noncommutative decreases
4. ENTROPY CORRECTION ON THE NONCOMMUTATIVE HORIZON OF A BLACK HOLE

The black hole entropy depends on its event horizon area $A_H$ with the initial formula:

$$ S = \frac{A_H}{4}. \quad (55) $$

The surface area of a Yukawa black hole in noncommutative space-time in the second order in $\Theta$ is

$$ A_H = 4\pi (r_H^{NC})^2 = 4\pi \left( \frac{\alpha (1 + \beta)}{1 + \frac{\alpha \beta}{\lambda}} \right)^2 $$

With the help of Eqs. (45) and (46), the corrected noncommutative entropy becomes:

$$ S^{NC}_H = \pi \left( \frac{\alpha (1 + \beta)}{1 + \frac{\alpha \beta}{\lambda}} \right)^2 + 6\pi \left( 1 + \frac{\alpha \beta}{\lambda} \right) \Theta^2 $$

$$ = S^C_H + 6\pi \left( 1 + \frac{\alpha \beta}{\lambda} \right) \Theta^2, \quad (57) $$

where

$$ S^C_H = \pi \left( \frac{\alpha (1 + \beta)}{1 + \frac{\alpha \beta}{\lambda}} \right)^2 $$

$$ \simeq \pi \alpha^2 (1 + \beta)^2 \left( 1 - 2\frac{\alpha \beta}{\lambda} \right). \quad (58) $$

The radius within which black holes cannot radiate. These results correspond to what the authors of [21, 22] expected.
See Figure 2: a similar analysis can be performed over the noncommutativity, finding that for small values of $\Theta$ variations over $S_{H}^{NC}$ and $S_{C}^{NC}$ are negligible. This finding is consistent with other results on corrections to the black hole entropy [22–24]. This confirms our confidence in quantitative mechanical corrections to Newton’s potential and their conclusions.

5. CONCLUSION

We have studied the deformed Schwarzschild black hole using Yukawa type of correction in canonical noncommutative space-time. By using the Seiberg-Witten maps up to the second order of the non-commutativity parameter, specifically for the tetrad fields, we have obtained expressions for the corrections to the Yukawa–Schwarzschild metric tensor due to noncommutativity. Furthermore, we have analyzed the thermodynamic properties of a noncommutative Schwarzschild black hole. We obtained the corrected noncommutative temperature and entropy for these black hole solutions. We noted that the noncommutativity modifies the thermodynamic properties of the Yukawa–Schwarzschild black hole. First, we found the noncommutative temperature, and that these noncommutative corrections eliminate the divergent behavior of the Hawking temperature and lead to a new maximum temperature that the black hole can reach before turning to zero at a new minimum value for the black hole radius. The values of these quantities were predicted in [25–26].

Then, we obtained the corrections to the entropy of the Yukawa–Schwarzschild black hole due to the small noncommutativity parameter. These corrections are negligible. When the noncommutativity parameter $\Theta \to 0$, these corrections vanish, and hence we derive the ordinary results for a Yukawa Schwarzschild black hole. The obtained results are the noncommutativity corrections to the non-Newtonian correction of the Yukawa type gravitational potential, they support the idea of introducing a correction to Newton’s potential for quantum gravity as the case with the radiative correction of quantum electrodynamics (QCD) as a result of the Colombian potential modification [27, 28].

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