Boundedness and Convergence of Solutions for the String Coupled to a Nonlinear Oscillator

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Abstract

A system of equations consisting of an infinite string coupled to a nonlinear oscillator is considered. The Cauchy problem for the system with the periodic initial data is studied. The main goal is to prove the convergence of the solutions as $t \to \infty$ to a time periodic solution.

Key words and phrases: an infinite string coupled to a nonlinear oscillator, the Cauchy problem, periodic initial data, the limit amplitude principle

1 Introduction

Consider the following problem for a function $u(x,t) \in C(\mathbb{R}^2)$:

$$(\mu + m\delta(x))\ddot{u}(x,t) = \kappa u''(x,t) + \delta(x)F(u(x,t)), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}.$$ (1.1)

Here $m \geq 0$, $\mu, \kappa > 0$; $\dot{u} \equiv \partial u/\partial t$, $u' \equiv \partial u/\partial x$. The initial data (when $t = 0$) for Eq. (1.1) are assumed to be periodic, see Definition 1.5 below.

By definition, Eq. (1.1) is equivalent to the following system:

$$\mu \ddot{u}(x,t) = \kappa u''(x,t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R} \setminus \{0\},$$ (1.2)

$$m\ddot{y}(t) = F(y(t)) + \kappa [u'(0+, t) - u'(0-, t)], \quad t \in \mathbb{R},$$ (1.3)

where

$$y(t) = u(0-, t) = u(0+, t), \quad t \in \mathbb{R}.$$ (1.4)

Physically, the system describes small crosswise oscillations of an infinite string stretched parallel to the $Ox$-axis. $\mu$ is the line density of the string, $\kappa$ is its tension, $F(y)$ is an external (nonlinear, in general) force field perpendicular to $Ox$. In the case $m = 0$, the string is coupled to a spring of a rigidity $F(y)$. In the case $m > 0$, a ball of mass $m$ is attached to the string at the point $x = 0$, and the field $F(y)$ subjects the ball.

The system (1.2)–(1.4) was considered first by Lamb [3] for the linear case, i.e., when $F(y) = -ry$ with a positive constant $r$. For general nonlinear functions $F(y)$, this model was studied by Komech in the paper [9], where the transitions to stationary states were established for finite energy solutions. In the present paper, the solutions of infinite energy with space-periodic initial data are considered. Main goal is to prove that each solution
u(x, t) to the system for large times is close to a time-periodic solution (see Theorem 1.6 below).

Let us describe our assumptions on the external force \( F(y) \).

Denote by \( V(y) = -\int F(y) dy \) the potential energy of the external field, \( F(y) = -V'(y), y \in \mathbb{R} \). We assume that

\[
F(y) \in C^1(\mathbb{R}), \quad F(y) \to \mp \infty \quad \text{as} \quad y \to \pm \infty. \tag{1.5}
\]

Obviously, condition (1.5) implies that

\[
\ddot{y}(t) = \dot{F}(y), \quad y(t) \to \pm \infty \quad \text{as} \quad t \to \pm \infty.
\]

Let us introduce a class \( \mathcal{E} \) of solutions \( u(x, t) \) to Eq. (1.1) with locally finite energy.

**Definition 1.1** A function \( u(x, t) \) belongs to \( \mathcal{E} \) if \( u \in C(\mathbb{R}^2) \) and \( \dot{u}, \dot{u}' \in L^2_{\text{loc}}(\mathbb{R}^2) \), where the derivatives are understood in the sense of distributions.

For \( u(x, t) \in \mathcal{E} \), the system (1.2)–(1.3) is understood as follows (see [9]).

For \( u \in C(\mathbb{R}^2) \), Eq. (1.2) is understood in the sense of distributions in the region \( (x, t) \in \mathbb{R}^2, x \neq 0 \). Moreover, Eq. (1.2) is equivalent to the d’Alembert decomposition

\[
(a = \sqrt{\kappa/\mu})
\]

\[
\dot{u}(x, t) = f_\pm(x - at) + g_\pm(x + at), \quad \pm x > 0, \quad t \in \mathbb{R}, \tag{1.7}
\]

where \( f_\pm, g_\pm \in C(\mathbb{R}) \), since \( u(x, t) \in C(\mathbb{R}^2) \).

We now explain Eq. (1.3). Equality (1.7) implies

\[
u'(x, t) = f_\pm'(x - at) + g_\pm'(x + at), \quad \pm x > 0, \quad t \in \mathbb{R},
\]

where all derivatives are understood in the sense of distributions. For \( u(x, t) \in C(\mathbb{R}^2) \) satisfying (1.2), write

\[
u'(0 \pm, t) := f_\pm'(-at) + g_\pm'(at). \tag{1.8}
\]

Note that condition \( u(x, t) \in \mathcal{E} \) implies that \( f_\pm', g_\pm' \in L^2_{\text{loc}}(\mathbb{R}) \). The derivative \( \dot{y}(t) \) of \( y(t) \in C(\mathbb{R}) \) is understood in the sense of distributions. Moreover, for \( m \neq 0 \), Eq. (1.3) and condition (1.5) imply that \( \ddot{y}(t) = \ddot{u}(0 \pm, t) \in L^2_{\text{loc}}(\mathbb{R}) \). Hence, if \( m \neq 0 \), \( y(t) \in C^1(\mathbb{R}) \) for any solution \( u \in \mathcal{E} \).

We study the Cauchy problem for the system (1.2)–(1.3) with the initial conditions

\[
u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = u_1(x), \quad x \in \mathbb{R}, \tag{1.9}
\]

\[
\dot{y}|_{t=0} = y_1 \quad \text{(if} \quad m \neq 0). \tag{1.10}
\]

We assume that \( y_1 \in \mathbb{R} \) and the initial data \( u_0(x), u_1(x) \) belong to the space \( \mathcal{H} \).

**Definition 1.2** The pair of functions \( (u_0, u_1) \) belongs to the space \( \mathcal{H} \) if \( u_0 \in C(\mathbb{R}), u_0', u_1 \in L^2_{\text{loc}}(\mathbb{R}) \).

**Proposition 1.3** Let condition (1.6) hold and \( (u_0, u_1) \in \mathcal{H}, y_1 \in \mathbb{R} \). Then the Cauchy problem (1.2), (1.3), (1.10) has a unique solution \( u(x, t) \in \mathcal{E} \).
This proposition is proved in Section 2.

To prove the main result we impose additional conditions on the initial data \((u_0, u_1)\). At first, for an \(\omega > 0\), we introduce a class \(P^\omega\) of the space periodic functions.

**Definition 1.4** For \(\omega > 0\), we say that \(u \in P^\omega\) if \(u(x + \omega) = u(x)\) for \(\pm x > 0\).

**Definition 1.5** For \(\omega > 0\), \((u_0, u_1) \in \mathcal{H}^\omega\) if \(u_0 \in C^1(\mathbb{R})\), \(u_1 \in C(\mathbb{R})\) and \(u_0, u_1', u_1 \in P^\omega\).

In the case \(m = 0\), the main result is the following convergence theorem.

**Theorem 1.6** Let \(m = 0\), condition \((1.2)\) hold and \((u_0, u_1) \in \mathcal{H}^\omega\) for some \(\omega > 0\). Then for every solution \(u(x, t) \in \mathcal{E}\) of the Cauchy problem \((1.2)-(1.4), (1.9)\) there exists a solution \(u_p(x, t) \in \mathcal{E}\) to Eq. \((1.7)\) such that

\[
u_p(x, t + \omega/a) = u_p(x, t) \quad \text{for} \quad (x, t) \in \mathbb{R}^2 : |t| > |x|/a, \quad (1.11)
\]

and for every \(R > 0\),

\[
\int_{|x|<R} \left( |\dot{u}(x, t) - \dot{u}_p(x, t)|^2 + |u'(x, t) - u'_p(x, t)|^2 \right) dx + \max_{|x|<R} |u(x, t) - u_p(x, t)| \to 0 \quad (1.12)
\]

as \(t \to \infty\).

This theorem is proved in Section 3. The similar result holds for \(m \neq 0\) under additional restrictions on the function \(F(y)\) (see Section 3).

In Appendix B we consider Eq. \((1.1)\) for \(t > 0\) under the initial condition

\[
u(x, t)|_{t=0} = p(x + at), \quad x \in \mathbb{R}, \quad (1.13)
\]

where the function \(p(z) \in P^\omega\), \(p \in C^1(\mathbb{R})\), \(p(x) = p_0\) for \(x \leq 0\), and \(F(p_0) = 0\). In this case, the convergence \((1.12)\) holds, i.e., the solution \(u(x, t)\) of the problem \((1.2)-(1.4), (1.9)\) either is a time-periodic for \(|x| \leq at\) with period \(\omega/a\) or converges to a function \(u_p(x, t) \in \mathcal{E}\) satisfying \((1.11)\). Moreover, the function \(u_p(x, t)\) is a solution of Eq. \((1.1)\) for \(t > 0\) under the condition \(u_p(x, t)|_{t=0} = q(x + at)\). Here \(q(x) = q_0\) for \(x \leq 0\) and \(q(x) = q_0 + p(x) - p_0\) for \(x > 0\), with some point \(q_0 \in \mathbb{R}\) depending on \(p_0\).

We outline the strategy of the proof of \((1.12)\). At first, using the d’Alembert method, we reduce the problem \((1.2)-(1.4), (1.9), (1.10)\) to the study of the following Cauchy problem for the function \(y(t)\),

\[
m\ddot{y} + (2\kappa/a)\dot{y} - F(y(t)) = 2\kappa p'(at), \quad t \in \mathbb{R}, \quad (1.14)
\]

with some \(\omega\)-periodic function \(p\) (see formula \((2.5)\) below) and with the initial conditions

\[
y|_{t=0} = y_0 = u_0(0),
\]

\[
y'|_{t=0} = y_1 \quad \text{(if} \ m \neq 0). \quad (1.15)
\]

Further, for \(m = 0\), we show (see Theorem 3.1) that any solution of Eq. \((1.14)\) either \(\omega/a\)-periodic or tends to an \(\omega/a\)-periodic solution \(y_p(t)\), i.e., \(|y(t) - y_p(t)| \to 0\) as \(t \to \infty\). Finally, using the explicit formula \((2.4)\) for \(u(x, t)\) we derive the results of Theorem 1.6.
If \( m \neq 0 \), the behavior of solutions to Eq. (1.14) is more complex. If \( F(y) = -ax - by^3 \), the equation of the form (1.11) is called the Duffing equation with damping, see for example, [7–15]. Eq. (1.14) is a particular case of the generalized Liénard equations with a forcing term \( e(t) = 2\kappa p(at) \),

\[
\ddot{y} + f(y)\dot{y} + g(y) = e(t). \tag{1.16}
\]

Eq. (1.16) with \( g(y) = y \) and \( e(t) \equiv 0 \) was studied first by Liénard [16]. A class of equations of the form (1.16) has been widely investigated in the literature, see, for example, Cartwright [1], Littlewood [2], Levinson [5], Loud [7–8], Reuter [13]. We refer the reader to the survey works [4, 10, 11, 12, 14] for a detailed discussion of the results and methods concerning these equations. Some results concerning Eq. (1.14) are given in Section 3. In particular, condition (1.6) implies that for large times the pairs \((y(t), \dot{y}(t))\) (where \( y(t) \) is a solution of (1.14)) belong to a fixed bounded region of \( \mathbb{R}^2 \). Denote by \( U(t, 0) \) the solving operator to the Cauchy problem (1.14), (1.15). By the Pliss results [10–11], there exists a set \( I \subset \mathbb{R}^2 \) which is invariant w.r.t. \( U(\omega/a, 0) \). Moreover, the set \( I \) is not empty and has zero Lebesgue measure. Introduce an integral set \( S \subset \{(Y(t), t) \in \mathbb{R}^3\} \) consisting of the solutions of Eq. (1.14) with the initial values \((y_0, y_1) \in I\). Let \( S_\tau \) denote the intersection of \( S \) and the hyperplane \( t = \tau \), and \( \rho(Y, S_\tau) \) stand for the distance between a point \( Y \in \mathbb{R}^2 \) and the set \( S_\tau \). In Section 3 we check that every solution of Eq. (1.14) tends to the set \( S \) as \( t \to \infty \), i.e., \( \rho(Y(t), S_\tau) \to 0 \) as \( \tau \to \infty \). Hence the explicit formula (2.4) for the solutions \( u(x, t) \) implies that for any \( R > 0 \),

\[
\inf \left\{ \int_{|x|<R} \left( |u(x, t) - u_p(x, t)|^2 + |u'(x, t) - u'_p(x, t)|^2 \right) dx + \max_{|x|<R} |u(x, t) - u_p(x, t)| \right\} < \infty \tag{1.17}
\]

vanishes as \( t \to \infty \), where the infinitum is taken over all solutions \( u_p(x, t) \in \mathcal{E} \) of the problem (1.2)–(1.3) such that \( u_p(0\pm, t) = y_p(t) \) and \((y_p(t), \dot{y}_p(t)) \in S_\tau \).

We give additional restrictions on the function \( F(y) \) (see Examples 3.5–3.7) when the set \( I \) has a unique point and then Eq. (1.14) has a unique stable periodic solution. In this case, every solution of Eq. (1.14) tends to a \( \omega/a \)-periodic solution \( y_p(t) \) as \( t \to \infty \), and convergence (1.12) holds.

## 2 Existence of solutions

In this section we prove Proposition 1.3. The method of construction of finite energy solutions to the Cauchy problem (1.2)–(1.4), (1.9), (1.10) was given by Komech in [9]. We apply this method to the infinite energy solutions. For simplicity, we consider only the case \( t > 0 \). Substituting (1.7) into initial conditions (1.9), we have

\[
f_\pm(z) = u_0(z)/2 - 1/(2a) \int_{\pm}^{z} u_1(y) \, dy + C_\pm, \quad \text{for } \pm z > 0,
\]

\[
g_\pm(z) = u_0(z)/2 + 1/(2a) \int_{\pm}^{z} u_1(y) \, dy - C_\pm, \quad \text{for } \pm z > 0,
\]

where we can put constants \( C_\pm = 0 \). On the other hand, substituting (1.7) into the condition (1.4), we have

\[
y(t) = f_+(-at) + g_+(at) = f_-(at) + g_-(at) \quad \text{for } t \in \mathbb{R}. \quad (2.2)
\]
By (2.2), we can determinate $g_-(z)$ with $z > 0$ and $f_+(z)$ with $z < 0$ as follows:

$$g_-(z) = y(z/a) - f_-(z), \quad f_+(z) = y(z/a) - g_+(z) \quad \text{for } z > 0.$$  \hspace{1cm} (2.3)

Therefore, for $t > 0$ we obtain

$$u(x,t) = \begin{cases} 
  f_+(x - at) + g_+(x + at) & \text{for } x \geq at \\
  y(t - x/a) + g_+(x + at) - g_+(at - x) & \text{for } 0 \leq x < at \\
  y(t + x/a) + f_-(x - at) - f_-(at - x) & \text{for } -at \leq x < 0 \\
  f_-(x - at) + g_-(x + at) & \text{for } x < -at
\end{cases}$$  \hspace{1cm} (2.4)

where $f_\pm \in C(\mathbb{R}_\pm)$, $f'_\pm \in L^2_{\text{loc}}(\mathbb{R}_\pm)$ with $\mathbb{R}_\pm = \{ x \in \mathbb{R} : \pm x > 0 \}$. Moreover, by definition (1.8), we have

$$u'(0+,t) := f'_+(at) + g'_+(at) = 2g'_+(at) - \dot{y}(t)/a,
\quad u'(0-,t) := f'_-(at) + g'_-(at) = 2f'_-(at) + \dot{y}(t)/a.$$  

Hence, Eq. (1.3) writes

$$m \ddot{y}(t) = F(y(t)) + 2\kappa [g'_+(at) - f'_-(at) - \dot{y}(t)/a], \quad t > 0.$$  \hspace{1cm} (2.5)

Denote

$$p(z) := \frac{u_0(z) + u_0(-z)}{2} + \frac{1}{2a} \int_{-z}^z u_1(y) \, dy, \quad z \in \mathbb{R}.$$  \hspace{1cm} (2.6)

Therefore, $p(0) = u_0(0)$, $p'(at) = g'_+(at) - f'_-(at) \in L^2_{\text{loc}}(\mathbb{R}_+)$, and we obtain the following evolution equation for $y(t)$, $t > 0$:

$$\dot{y}(t) = (a/2\kappa)F(y(t)) + ap'(at), \quad t > 0, \quad \text{if } m = 0,
\quad m \ddot{y}(t) = F(y(t)) - (2\kappa/a)\dot{y}(t) + 2\kappa p'(at), \quad t > 0, \quad \text{if } m > 0.$$  \hspace{1cm} (2.7)

Eq. (2.1) implies the following initial condition for the function $y(t)$:

$$y(0) = f_\pm(0) + g_\pm(0) = u_0(0).$$  \hspace{1cm} (2.8)

Eqs (2.6) and (2.7) are rewritten in the equivalent integral form,

$$y(t) = \frac{a}{2\kappa} \int_0^t F(y(s)) \, ds + p(at) - p(0) + y(0), \quad t \geq 0, \quad \text{if } m = 0,$$  \hspace{1cm} (2.9)

$$m \dot{y}(t) = \int_0^t \int_0^s F(y(\tau)) \, d\tau + \frac{2\kappa}{a^2} \int_0^t (p(as) - y(s)) \, ds
\quad + m\dot{y}(0) + m\dot{y}(0)t + \frac{2\kappa}{a} (y(0) - p(0)) t, \quad t \geq 0, \quad \text{if } m > 0.$$  \hspace{1cm} (2.10)

Lemma 2.1 below implies Proposition 1.3 immediately.

**Lemma 2.1** (i) Let $m = 0$ and all assumptions of Proposition 1.3 hold. Then for any $y_0 \in \mathbb{R}$, Eq. (2.7) has a unique solution $y(t) = U(t,0)y_0 \in C(\mathbb{R}_+)$.  

(ii) Let $m > 0$. Then for any $(y_0, y_1) \in \mathbb{R}^2$, Eq. (2.7) has a unique solution $(y(t), \dot{y}(t)) = U(t,0)(y_0, y_1)$, and $y(t) \in C^1(\mathbb{R}_+)$.  

(iii) For $m \geq 0$, the following bound holds,

$$\sup_{[0, \tau]} [m|\dot{y}(t)| + |y(t)|] \leq C_1 \tau + C_2, \quad \text{for any } \tau > 0.$$  \hspace{1cm} (2.11)
Proof. We prove Lemma 2.1 only in the case when $m > 0$. For $m = 0$ the proof is similarly.

It follows from (2.10), condition (1.5) and the contraction mapping principle that for any fixed initial data $y(0+)$ and $\dot{y}(0+)$, the solution $y(t)$ to Eq. (2.10) has a unique solution on a certain interval $t \in [0, \varepsilon)$ with an $\varepsilon, \varepsilon > 0$. Let us derive an a priori estimate for $y(t)$.

This estimate will imply the existence and uniqueness of the global solution of (2.7) for any $y(0+)$ and $\dot{y}(0+)$. We multiply Eq. (2.7) by $\dot{y}(t)$. Using $\frac{d}{dt}V(y(t)) = -F(y(t))\dot{y}(t)$, we obtain

$$\frac{d}{dt} \left( \frac{m\dot{y}^2(t)}{2} + V(y(t)) \right) = 2\kappa p'(at)\dot{y}(t) - \frac{2\kappa}{a}\ddot{y}^2(t) \leq \frac{a\kappa}{2}(p'(at))^2.$$

Let us integrate this inequality and obtain

$$\frac{m\dot{y}^2(t)}{2} + V(y(t)) \leq \frac{m\dot{y}^2(0)}{2} + V(y(0)) + \frac{a\kappa}{2} \int_0^t |p'(as)|^2 ds, \quad t > 0.$$

Hence, for any $\tau > 0$, there exist constants $C_1, C_2 > 0$ such that

$$\sup_{t \in [0, \tau]} \left[ \frac{m\dot{y}^2(t)}{2} + V(y(t)) \right] \leq C_1 \tau + C_2. \quad (2.12)$$

Condition (1.6) implies the estimate (2.11). Lemma 2.1 is proved.

The following result follows from the Gronwall inequality and from an a priori estimate (2.11) (see [9]).

**Lemma 2.2** Let $m = 0$ and $y_1(t)$ and $y_2(t)$ be two solutions of Eq. (2.6) with the initial values $y_1(0)$ and $y_2(0)$, respectively. Then for every $\tau > 0$,

$$\|\dot{y}_1(t) - \dot{y}_2(t)\|_{L^2(0, \tau)} + \max_{[0, \tau]} |y_1(t) - y_2(t)| \leq C(\tau) |y_1(0) - y_2(0)|, \quad (2.13)$$

where a constant $C(\tau)$ is bounded for bounded $y_1(0), y_2(0)$. The similar result holds for Eq. (2.7) in the case $m \neq 0$.

**3 The proof of the main result**

Since $(u_0, u_1) \in \mathcal{H}$, the function $p$ defined in (2.5) has the following properties: $p \in C^1(\mathbb{R})$, $p(z \pm \omega) = p(z), \pm z > 0$. Then the function $p'(at)$ in Eqs (2.6) and (2.7) is periodic with $\omega/a$-period, and $p'(at) \in C(\mathbb{R}_+)$. 

**3.1 The string–spring system ($m = 0$)**

At first, we study the behavior of solutions to Eq. (2.6).

**Theorem 3.1** Let condition (1.5) hold. Then the following assertions are true.

(i) All solutions of Eq. (2.6) are bounded.

(ii) Eq. (2.6) has at least one $\omega/a$-periodic solution.

(iii) Any solution $y(t)$ of Eq. (2.6) either is $\omega/a$-periodic or tends to an $\omega/a$-periodic solution $y_p(t)$ as $t \to \infty$ such that for every $R > 0$,

$$\int_t^{t+R} |\dot{y}(s) - \dot{y}_p(s)|^2 ds + \sup_{s \in [t, t+R]} |y(s) - y_p(s)| \to 0 \quad \text{as} \quad t \to \infty. \quad (3.1)$$
Then \( u \) where and \( g \) the following conditions: (Remark 3.2)

Appendix A we will show that for any \( y_0 \in \mathbb{R} \) there exists the limit of \( U(n\omega/a,0)y_0 \) as \( n \to \infty \). Write \( y_0 := \lim_{n \to \infty} U(n\omega/a,0)y_0 \). Then \( y_p(t) = U(t,0)y_0 \) is the \( \omega/a \)-periodic solution of Eq. (2.6) and convergence \( (3.1) \) holds (see Appendix A).

Put \( \bar{u}_0(x) = u_0(x) - u_0(0) + y_0 \) and define functions \( \bar{f}_\pm(x) \) and \( \bar{g}_\pm(x) \) so as \( f_\pm(x) \) and \( g_\pm(x) \) in (2.1) but with \( \bar{u}_0(x) \) instead of \( u_0(x) \). Introduce a function \( u_p(x,t) \) as follows

\[
\begin{align*}
\begin{cases}
\bar{f}_+(x-at) + \bar{g}_+(x-at) & \text{for } x \geq at \\
\bar{y}_p(t-x/a) + \bar{g}_+(x-at) - \bar{g}_-(at-x) & \text{for } 0 \leq x < at \\
\bar{y}_p(t+x/a) + \bar{f}_-(x-at) - \bar{f}_-(-at-x) & \text{for } -at \leq x < 0 \\
\bar{f}_-(x-at) + \bar{g}_-(x-at) & \text{for } x < -at
\end{cases}
\end{align*}
\]

Then \( u_p(x,t) \) is the solution of (1.2)–(1.4) with the initial data \( (\bar{u}_0, u_1) \) and \( u_p(0,t) = y_p(t) \).

Since \( (\bar{u}_0, u_1) \in \mathcal{H}^\omega \), the functions \( \bar{f}_-(\pm x-at) \) and \( \bar{g}_+(\pm x+at) \) in (3.2) are \( \omega/a \)-periodic in \( t \). Then the equality (1.11) holds, and the convergence (1.12) follows from (2.4) and (3.1).

Remark 3.2 Let us consider the problem (1.1) for \( t > 0 \) with initial data (1.9), satisfying the following conditions: \( (u_0, u_1) \in \mathcal{H} \) and \( u_1 \) has a form

\[
u_1(x) = \begin{cases} 
\displaystyle a(2p'_+(x) - u'_0(x)), & x \geq 0, \\
\displaystyle a(u'_0(x) - 2p'_-(x)), & x < 0,
\end{cases}
\]

where \( p_\pm \in C^1(\mathbb{R}_\pm) \) and \( p_\pm(x) \) is \( \omega \)-periodic for \( \pm x > 0 \). Then \( f_-(z) = p_-(z) \) for \( z < 0 \) and \( g_+(z) = p_+(z) \) for \( z > 0 \). Hence, by formula (2.4), the solution \( u(x,t) \) for \( t > 0 \) has the form

\[
\begin{align*}
\begin{cases}
\bar{u}_0(x-at) - p_+(x-at) + p_+(x+at) & \text{for } x \geq at \\
\bar{y}(t-x/a) + p_+(x+at) - p_+(at-x) & \text{for } 0 < x < at \\
\bar{y}(t+x/a) + p_-(x-at) - p_-(at-x) & \text{for } -at \leq x < 0 \\
p_-(x-at) + \bar{u}_0(x+at) - p_-(x+at) & \text{for } x < -at
\end{cases}
\end{align*}
\]

where \( \bar{y}(t) \) is a solution of Eq. (2.6) with the \( \omega \)-periodic function \( p(x) := p_+(x) + p_-(x) \), \( x > 0 \), and satisfies the initial condition (2.8). Then the results of Theorems 3.1 and 1.6 hold as \( t \to +\infty \).

3.2 The string–oscillator system \( (m > 0) \)

Put \( c = 1/m \), \( k = 2\kappa/(am) = 2\sqrt{\kappa \mu/m} \). Then Eq. (2.7) is equivalent to the following system

\[
\begin{align*}
\dot{y} &= v, \\
\dot{v} &= cF(y) - kv + kapp'(at).
\end{align*}
\]

Denote by \( Y(t,Y_0,t_0) = (y(t,Y_0,t_0), \dot{y}(t,Y_0,t_0)) = U(t,t_0)Y_0 \) the solution of the Cauchy problem for the system (3.4) with the initial data

\[
Y_0 = (y, \dot{y})|_{t=t_0} = (y_0, y_1).
\]
Definition 3.3 The system is called dissipative (or D-system) if for any \((Y_0,t_0) \in \mathbb{R}^3\) there exists a \(R\), \(R > 0\), such that \(\lim_{t \to \infty} \|Y(t,Y_0,t_0)\| < R\).

Lemma 3.4 Let condition (1.5) hold. Then the following assertions hold.

(i) The system (3.4) is dissipative, and there exist constants \(M,N > 0\) such that for large time the solutions of the system (3.4) belong to a bounded set

\[
\{(y_0,y_1) \in \mathbb{R}^2 : |y_0| \leq M, |y_1| \leq N\},
\]

and \(M\) and \(N\) are independent on the parameters \(k\) and \(c\) of the system (3.4).

(ii) The system (3.4) has at least one \(\omega/a\)-periodic solution.

The item (i) of Lemma 3.4 follows from the results of Cartwright and Littlewood, Reuter and others (see [11,2,13] and the review works [14, Chapter VII], [1, Chapter XI, §4], and [12, Theorem 5.5.4]). According to the Opial theorem (see, e.g., [12, Theorem 5.3.6]) instead of condition (1.5) it suffices to assume that

\[
\lim_{y \to +\infty} F(y) < -r, \quad \lim_{y \to -\infty} F(y) > r, \quad \text{where } r = \max_{t \in \mathbb{R}} |p'(at)|.
\]

Item (i) implies item (ii) by the Brouwer Fixed Point Theorem (see [10, Chapter 1, §2]).

Introduce a mapping \(T : \mathbb{R}^2 \to \mathbb{R}^2\) as \(T = U(\omega_0,0), \omega_0 := \omega/a\). The map \(T\) is called the Poincaré transformation associated with the periodic system (3.4). Lemma 3.4 and the Pliss results (see [11, Chapter 2, §2]) imply that there exists an invariant set \(I\) w.r.t. \(T\), i.e., \(TI = I\). This set is called characteristic set of the dissipative system (3.4) or a global attractor of the diffeomorphism \(T\). The set \(I\) has the following properties (see [10–12]):

- \(I\) is closed and bounded.
- \(I\) is stable w.r.t. \(T\), i.e., for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(\rho(Y_0,I) < \delta\) then \(\rho(T^mY_0,I) < \varepsilon\) for every \(m \in \mathbb{N}\).
- For all \(Y_0 \in \mathbb{R}^2\), \(\rho(T^nY_0,I) \to 0, \ n \to \infty\).
- There exists a fixed point of the mapping \(T\) belonging to \(I\), i.e., there exists an \(\omega_0 = \omega/a\)-periodic solution (or harmonics) of the system (3.4).
- The set \(I\) has zero Lebesgue measure by Theorem 1.9 from [11].

Define a set \(S\) as

\[
S := \{(Y,t) \in \mathbb{R}^3 : Y = Y(t,Y_0,t_0), Y_0 \in I, t \in \mathbb{R}\}.
\]

The set \(S\) has the following properties:

- \(S\) is bounded and closed.
- \(S\) is \(\omega_0\)-periodic, i.e., for \((Y,t) \in S\), \((Y,t + n\omega_0) \in S\), \(\forall n \in \mathbb{N}\).
- \(S\) is invariant, i.e., if \((Y_0,t_0) \in S\), then \((Y(t,Y_0,t_0),t) \in S\) for all \(t \geq t_0\).
- \(S\) is stable, i.e., \(\forall \varepsilon > 0\) \(\exists \delta > 0\) such that if \(\rho(Y_0,S_{t_0}) < \delta\), then \(\rho(Y(t,Y_0,t_0),S_t) < \varepsilon, \forall t \geq t_0\), where \(S_t = S \cap \{t = \tau\}\).
• \( S \) is stable in whole, i.e., for all \( Y(t, Y_0, t_0) \in \mathbb{R}^2 \) we have \( \lim_{t \to \infty} \rho(Y(t, Y_0, t_0), S_t) = 0 \).

However, these properties of \( S \) do not imply, in general, the convergence \((3.1)\). Now we consider the particular case of the system \((3.4)\) when \( I \) has a unique point. Then \((3.4)\) is called the system with convergence (see \([10, \text{§7}, \text{Definition 7.1}]\)). In this case, the system \((3.4)\) has a unique stable \( \omega_0 \)-periodic solution \( Y_\mu(t) \), and any another solution \( Y(t, Y_0, t_0) \) tends to this periodic solution, i.e., \( \lim_{t \to \infty} \|Y(t, Y_0, t_0) - Y_\mu(t)\| = 0 \), and the result \((3.1)\) follows.

Below we give examples of the restrictions on the function \( F(y) \) when the system \((3.4)\) has convergence property.

**Example 3.5** Assume that

\[ F(y) = -ry \]  
with a constant \( r > 0 \).

Then by the Levinson theorem (see, e.g., \([12, \text{Theorem 5.3.2}]\)), Eq. \((2.7)\) has a unique \( \omega_0 \)-periodic solution and all other solutions tend to this periodic solution as \( t \to +\infty \).

**Example 3.6** Assume that for \( y_1 \neq y_2 \), we have

\[ k^2/2 - 1 \leq -c \frac{F(y_2) - F(y_1)}{y_2 - y_1} \leq 1, \quad 1 < k^2/2 \leq 2, \]

where \( k = \sqrt{\kappa \mu}/m \) is the constant in \((3.4)\).

Then according to the Zlamáľ theorem (see, e.g., \([12, \text{Theorem 5.3.2}]\)) all solutions tend exponentially to a unique periodic solution as \( t \to +\infty \).

**Example 3.7** (see \([10, \text{Theorem 8.4}], \[4, \text{Ch.XI, §5}] \) or \([15]\)) Assume that

\[ F \in C^2(\mathbb{R}), \quad F'(y) < 0 \quad \text{for} \quad |y| \leq M; \quad \exists \beta > 0 \quad \text{such that} \quad F(y) \text{sgn} y \leq -\beta \quad \text{for} \quad |y| \geq M, \]

with the constant \( M \) from the bound \((3.6)\). Moreover, the constant \( k \) from \((3.4)\) is enough large,

\[ k > (1/2)N \max_{|y| \leq M} (|F''(y)|/|F'(y)|), \]

where the constant \( N \) is defined in the bound \((3.6)\).

Then the system \((3.4)\) has convergence property. For instance, the function \( F(y) = -ay^3 - by \) with constants \( a, b > 0 \) satisfies these conditions.

Note that condition \((F1)\) is a particular case of \((F3)\).

**Corollary 3.8** Let condition \((F2)\) or \((F3)\) be true. Then the following assertions hold.

(i) There exists a unique \( \omega_0 \)-periodic solution \( y_\mu(t) \) of Eq. \((2.7)\), and for any another solution \( y(t) \) the convergence \((3.1)\) holds.

(ii) The convergence \((1.12)\) holds with the function \( u_p(x, t) \) satisfying \((1.11)\).

The assertion (i) follows from the results mentioned above. Now we check item (ii). Indeed, let \( u(x, t) \) be a solution of the problem \((1.2)-(1.4), (1.9), (1.10)\). Then there exists \( \lim_{n \to \infty} T^n(u_0(0), y_1) =: (\bar{y}_0, \bar{y}_1) \) and \( (\bar{y}_0, \bar{y}_1) \) is a unique point of the set \( I \). Hence \((y_p(t), \bar{y}_p(t)) = U(t, 0)(\bar{y}_0, \bar{y}_1)\) is the unique \( \omega/a \)-periodic solution of the system \((3.4)\) and convergence \((3.1)\) holds. Put \( \bar{u}_0(x) = u_0(x) - u_0(0) + \bar{y}_0 \) and define functions \( \bar{f}_\pm(x) \) and \( \bar{g}_\pm(x) \) by formulas \((2.1)\) but with \( \bar{u}_0(x) \) instead of \( u_0(x) \). Define \( u_p(x, t) \) by \((3.2)\). Then \( u_p(x, t) \) is the solution of the problem \((1.2)-(1.4)\) with the initial data \((\bar{u}_0, u_1, \bar{y}_1)\). Since \((\bar{u}_0, u_1) \in H^0, \) the functions \( \bar{f}_\pm(\pm x - at) \) and \( \bar{g}_\pm(\pm x + at) \) in \((3.2)\) are \( \omega/a \)-periodic in \( t \). Hence the equality \((1.11)\) holds, and the convergence \((1.12)\) follows from \((2.4)\) and \((3.1)\).
4 Appendix A: Proof of Theorem 3.1

For simplicity, instead of Eq. (2.6) we consider the following equation:

\[ \dot{y}(t) = F(y(t)) + P(t), \quad t > 0, \tag{4.1} \]

where \( P(t) \in C(0, +\infty) \) is a periodic function with period \( \omega_0 = \omega/a \), the function \( F(y) \) satisfies the condition (1.5). Write

\[ q = \max_{t \in \mathbb{R}} |P(t)| < \infty. \tag{4.2} \]

Denote by \( U(t, s) \), \( t \geq s \), a solving operator of the following Cauchy problem for the function \( y(t) \):

\[ \dot{y}(t) = F(y(t)) + P(t), \quad t > s, \tag{4.3} \]
\[ y|_{t=s} = y_0. \tag{4.4} \]

Then \( U(t, s) : \mathbb{R}^1 \to \mathbb{R}^1 \) transforms the initial condition \( y_0 \in \mathbb{R}^1 \) for \( t = s \) to the solution \( y(t) \) of the problem (4.3)–(4.4) in time \( t \):

\( U(t, s) : y_0 \to y(t) \equiv y(t, y_0, s) \).

Remark 4.1 (i) \( U(t, s)U(s, r) = U(t, r) \) for \( t > s > r \), (ii) \( U(s, s) = I_d \), where \( I_d \) is identity operator on \( \mathbb{R} \), (iii) \( U(t + \omega_0, s + \omega_0) = U(t, s) \), \( t, s \in \mathbb{R} \).

At first, we prove the following lemma.

Lemma 4.2 There exists a bounded interval \( B = [y_-, y_+] \subset \mathbb{R}^1 \) such that \( U(t, s) : B \subset B \), \( t \geq s \). Moreover, all solutions \( y(t) \), \( t \geq s \), to problem (4.3)–(4.4) for finite time come in the region \( B \).

Proof. It follows from (1.5) and (1.2) that there exist points \( y_- < 0 \) and \( y_+ > 0 \) such that

\[ F(y) - q > 0 \quad \text{for} \quad y \leq y_-; \]
\[ F(y) + q < 0 \quad \text{for} \quad y \geq y_. \]

Then by Eq. (4.3) we have \( F(y(t)) - q \leq \dot{y}(t) \leq F(y(t)) + q \). Therefore, if \( y(t) \geq y_+ \), then \( \dot{y}(t) < 0 \), if \( y(t) \leq y_- \), then \( \dot{y}(t) > 0 \). Lemma 4.2 is proved.

Denote by \( T := U(\omega_0, 0) \) the Poincaré transformation associated with Eq. (4.3). Since \( F \in C^1(\mathbb{R}) \), the right hand side of Eq. (1.3) is continuously differentiable on \( y \). Hence \( T \) is continuously differentiable mapping \( \mathbb{R} \) on \( \mathbb{R} \). It is easy to verify that there exists a continuously differentiable mapping \( T^{-1} = U(0, \omega_0) \), i.e., \( T \) is a diffeomorphism \( \mathbb{R} \) on \( \mathbb{R} \). It follows from Lemma 1.2 and the Brouwer Fixed Point Theorem that the mapping \( T \) has a fixed point belonging to the interval \( B \) (see [12, Theorem 2.9.1] or Massera’s theorem [16]).

Denote by \( \mathcal{Z} \) a set of fixed points of the mapping \( T \),

\[ \mathcal{Z} = \{ y \in \mathbb{R} : Ty = y \}. \]

Lemma 4.3 For every \( y_0 \in \mathbb{R} \), there exists a \( \bar{y}_0 \in \mathcal{Z} \) such that \( T^n y_0 \to \bar{y}_0 \) as \( n \to \infty \).
Proof. Since $T$ is a diffeomorphism, then $T$ is a monotone function. If $y > y_+$, then $Ty < y$, and if $y < y_-$ then $Ty > y$. Let $y_0$ be an arbitrary point. It follows from Lemma \[4.2\] that $\exists N(y_0) \in \mathbb{N}$ such that $\forall n > N(y_0)$ we have $T^n y_0 \in B$. Further, we assume that $y_0 \in B$. It is possible 3 cases:

(i) $y_0 \in (y_-, z_-)$, where $z_- = \inf_{z \in \mathcal{Z}} z$,

(ii) $y_0 \in (z_+, y_+)$, where $z_+ = \sup_{z \in \mathcal{Z}} z$,

(iii) $y_0 \in (z_i, z_{i+1})$, where $z_i, z_{i+1} \in \mathcal{Z}$ are two neighboring fixed points of $T$.

In case (i), since $z_-$ is extreme left fixed point of $T$, $Ty > y$ for all $y \in (y_-, z_-)$. Write $y_n = T^n y_0$. Obviously, $y_n > y_{n-1} > \ldots > y_0$, i.e., $y_n$ is an increasing sequence bounded above by $z_-$. Hence, the limit holds,

$$\lim_{n \to \infty} y_n = \sup_{n \in \mathbb{N}} y_n = y_s, \quad y_s \in [y_-, z_-],$$

i.e., $T^n y_0 \uparrow y_s, \; n \to \infty$. Hence, $T^{n+1} y_0 \to Ty_s = y_s, \; n \to \infty$. Therefore, $y_s$ is a fixed point of $T$. Since $z_-$ is extreme left fixed point of $T$, then $y_s = z_-.$

In case (ii), $Ty < y$, $\forall y \in (z_+, y_+)$. Hence $y_n = T^n y_0$ is a decreasing sequence bounded below by $z_+$. Moreover, $y_n \downarrow z_+$ as $n \to \infty$.

In the case (iii), we have either (a) $Ty > y$, $y \in (z_i, z_{i+1})$ or (b) $Ty < y$, $y \in (z_i, z_{i+1})$. In the case (a) (see the case (i)) $T^n y_0 \uparrow z_{i+1}$ as $n \to \infty$. In the case (b) (see the case (ii)) $T^n y_0 \downarrow z_i$ as $n \to \infty$. Theorem \[4.2\] is proved.

Proof of Theorem \[3.1\] It follows from Remark \[4.1\] that $U(n \omega_0, 0) = (U(\omega_0, 0))^n = T^n$. Theorem \[4.2\] implies that $\forall y_0 \in \mathbb{R}$, $U(n \omega_0, 0) y_0 \to y_0$ as $n \to \infty$, where $y_0$ is a fixed point of $T$. 

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For $t > 0$ we choose $n \in \mathbb{N}$ such that $t = n \omega_0 + \tau$, where $\tau \in [0, \omega_0)$. Since $\bar{y}_0$ is a fixed point of $T$, we have

$$|U(t, 0)y_0 - U(t, 0)\bar{y}_0| = |U(\tau, 0)T^n y_0 - U(\tau, 0)T^n \bar{y}_0| = |U(\tau, 0)T^n y_0 - U(\tau, 0)\bar{y}_0|.$$  

Hence, it follows from Lemma 2.2 and Lemma 4.3 that there exists a constant $C < \infty$ such that for enough large value $n \in \mathbb{N}$, we have

$$\sup_{\tau \in [0, \omega_0]} |U(\tau, 0)T^n y_0 - U(\tau, 0)\bar{y}_0| \leq C|T^n y_0 - \bar{y}_0| < \varepsilon.$$

Denote by $y(t) = U(t, 0)y_0$ and $y_p(t) = U(t, 0)\bar{y}_0$ the solutions of the problem (4.3)–(4.4) with the initial data $y_0$ and $\bar{y}_0$, respectively. Note that $y_p(t)$ is a periodic solution of Eq. (4.3), because $\bar{y}_0$ is a fixed point of $T$. Therefore, for any solution $y(t)$ of Eq. (4.3) we have $|y(t) - y_p(t)| < \varepsilon$ for enough large value $t > 0$. The convergence (3.1) follows from condition (1.5) and Eq. (4.3).

5 Appendix B: Limit amplitude principle

Here we apply the results to the following problem for a function $u(x, t) \in C(\mathbb{R}^2)$:

\[
\begin{align*}
(\mu + m\delta(x))\ddot{u}(x, t) &= ku''(x, t) + \delta(x)F(u(x, t)), \quad t > 0, \quad x \in \mathbb{R}, \quad (5.1) \\
u(x, t) \big|_{t \leq 0} &= p(x + at), \quad x \in \mathbb{R}.
\end{align*}
\]

Here $m \geq 0$, $a = \sqrt{k/\mu}$. In the case $m > 0$ we assume that condition (F2) or (F3) holds. The function $p$ from Eq. (5.2) satisfies the following conditions:

P1 $p \in C^1(\mathbb{R})$.

P2 There exist numbers $\omega > 0$ and $p_0 \in \mathbb{R}$ such that $F(p_0) = 0$ and

$$p(z + \omega) = p(z) \text{ for } z > 0, \quad p(z) = p_0 \text{ for } z \leq 0.$$

Note that the function $p(x + at)$ is a solution of Eq. (5.1) for $t < 0$. Therefore, we can consider Eq. (5.1) for $t \in \mathbb{R}$. In particular, we have

$$u_0(x) = u|_{t=0} = p(x), \quad u_1(x) = u|_{t=0} = ap'(x), \quad x \in \mathbb{R}, \quad y(0) = u_0(0) = p_0, \quad \bar{y}(0) = 0.$$

Then $f_\pm(z) = 0$ and $g_\pm(z) = p(z)$ for $\pm z > 0$. Therefore, by (2.4),

\[
u(x, t) = \begin{cases} 
p(x + at) & \text{for } x > at, \\
y(t - x/a) - p(at - x) + p(x + at) & \text{for } 0 < x < at, \\
y(t + x/a) & \text{for } -at < x < 0, \\
p_0 & \text{for } x < -at.
\end{cases}
\]

where $y(t)$ is a solution to Eq. (2.7) (or Eq. (2.6)) for $t > 0$, and $y(t) = p_0$ for $t \leq 0$. By Proposition 1.3 the Cauchy problem (5.1)–(5.2) has a unique solution $u(x, t) \in \mathcal{E}$ for every function $p \in C^1(\mathbb{R})$. 

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Let \( m \geq 0 \) and \( y_p(t) \) be the \( \omega/a \)-periodic solution of Eq. (2.6) or (2.7) with the initial data \( \bar{p}_0 = \lim_{n \to \infty} T^n p_0 \) (if \( m = 0 \)) or with the initial data \( (\bar{p}_0, \bar{y}_1) = \lim_{n \to \infty} T^n (p_0, 0) \) (if \( m > 0 \)). We extend \( y_p(t) \equiv \bar{p}_0 \) for \( t < 0 \) and define

\[
    u_p(x, t) = \begin{cases} 
    y_p(t-x/a) - p(at-x) + p(x+at) & \text{for } x > 0, \ t > 0, \\
    y_p(t+x/a) & \text{for } x < 0, \ t > 0.
    \end{cases}
\]

Then \( u_p(x, t) \in \mathcal{E} \), \( u_p(x, t) \) is the solution of Eq. (5.1) under the condition \( u_p(x, t)|_{t<0} = \bar{p}(x+at) \), where \( \bar{p}(x) = \bar{p}_0 + p(x) - p_0 \) for \( x \in \mathbb{R} \). Moreover, the identity (1.11) holds. Then convergence (1.12) follows from equality (5.3) and bound (3.1).

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