Non-abelian correction to the Poisson approximation for multi-gluon bremsstrahlung

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Abstract. We present a technique that we are developing to compute the non-abelian effects to the well-known poisson distribution from QED results to obtain the QCD momentum distribution. We first introduce the MHV method used compute scattering amplitudes in gauge theory, especially $\mathcal{N} = 4$ SYM, then we discuss the case of massless QCD. In second part we present why the irreducible representation of the symmetric group is the natural framework to obtain the non-abelian correction to the QED distribution.

1. Introduction

During the last decade, in theoretical and mathematical physics side, one of the most exciting advanced fields of research is the Maximal Helicity Violating technique (MHV) also known as the on-shell method of Quantum Field Theory (QFT). The MHV technique provides a new framework to understand symmetries and its on-shell property makes computation of scattering amplitudes simpler than the standard diagrammatic methods of QFT [1, 2, 3]. In this framework, amplitudes involving a large productions of gluon have been computed for both pure gauge theory and massless QCD.

In the experimental side, the discovery of the quark gluon plasma (QGP) at RHIC and LHC from heavy ion collisions provide a new area to understand the phase diagram of QCD [4]. The QGP is a new phase of matter, where the quarks and gluons are free to move relatively independently, and behaves nearly as a perfect fluid [5]. At low-momentum regime the physics of the QGP are best described by strong-coupling physics of the anti-de-Sitter conformal field theory (AdS/CFT) conjecture, however in the high momentum regime they are described by weak-coupling physics of perturbative QCD. In addition to the observed fluid behaviour QGP there are also evidence of jets productions from heavy ion collisions. It is clear that the jets produced evolve in time with medium and due the in-medium interaction, energy loss are expected. It is also known that jets propagating in colored matter lose energy predominantly through medium-induced emission of gluon radiation [6, 7, 8], a jet scatters off color charges in the QGP matter and radiates gluon bremsstrahlung.

In this essay, we want to introduce our idea that using the MHV technique we are able to use QED radiative corrections as a central point in order to obtain radiative gluon bremsstrahlung in a large number of emission. In another word we will add non-abelian correction to the QED result. It has been shown in the QED that the radiative corrections is trivially resummed and exponentiate into a poisson distribution, for the case of QCD many attempts using different
approximations as been done in order to be able to exponentiate the radiative corrections. Here, we want to see a smooth resummation as in QED by using the irreducible representation of symmetric group. Before we get into the detail let remind ourselves that for a given amplitude $\mathcal{M}(k_1, \ldots, k_n)$ that have $m$ bremsstrahlung gluons in the soft-collinear limit factorize as
\begin{equation}
\mathcal{M}(k_1, \ldots, k_n) \rightarrow \mathcal{M}(k_{m+1}, \ldots, k_n) J(k_1, \ldots, k_m)
\end{equation}
where $J$ is the radiative corrections also known as soft collinear factor. That function $J$ captured all information on the behaviour of the amplitude near the soft and collinear limit, and our goal is to take the corresponding correction $J_{QED}$ and find the non-abelian correction using MHV.
\begin{equation}
J_{QCD} = J_{QED} + \text{corrections.}
\end{equation}

2. Short introduction to MHV amplitudes
Scattering amplitudes are the central objects in perturbative quantum field theory to bridge between theoretical to experimental predictions. To compute scattering amplitudes it is very natural to us to use Feynman diagram which seems to be the right approach since Feynman diagrams make unitarity and locality manifest. In order to make locality manifest we introduce virtual particles, off-shell particles, but some how after we summed all relevant diagrams the final amplitude doesn’t depend on those virtual particles [9].

Complications appear when we want to compute amplitudes with large number of particles, because the number of Feynman diagrams tends to grow fast with the number of particles, for example in $g + g \rightarrow g + g + g$ we have 25 diagrams at tree level. It turns out that after simplifications and using spinor helicity representation the amplitude can be written in a very simple expression.

Starting by using the spinor representation instead of the usual four vectors representation, a null momentum vector becomes
\begin{equation}
p_\mu \rightarrow p_{\dot{a}\dot{b}} = (\sigma^\mu)_{\dot{a}\dot{b}} p_\mu = \lambda_{\dot{a}} \tilde{\lambda}_{\dot{b}},
\end{equation}
where $\lambda_{\dot{a}}$ and $\tilde{\lambda}_{\dot{b}}$ are respectively a left and right handed spinors. And since amplitudes are made up with Lorentz invariant objects, we need to introduce two Lorentz invariant products made of those new variables
\begin{equation}
\langle p, q \rangle = \epsilon_{ab} \lambda_p^a \lambda_q^b \quad \text{and} \quad [p, q] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_p^{\dot{a}} \tilde{\lambda}_q^{\dot{b}}.
\end{equation}
Here $\epsilon_{ab}$ and $\epsilon_{\dot{a}\dot{b}}$ are the antisymmetric tensors in two dimensions. The use of left or right handed spinor is now depending on the helicity $h$ of the particles, projection of the spin along the momentum of the particle. In the same way for gauge field ($h = \pm 1$) we can introduce the spinor helicity representation of the polarization vector
\begin{equation}
\varepsilon^{-}_{\dot{a}\dot{b}}(p; q) = \frac{\lambda_{\dot{a}} \tilde{\lambda}_{\dot{b}}}{[p, q]} \quad \text{and} \quad \varepsilon^{+}_{\dot{a}\dot{b}}(p; q) = \frac{\mu_{\dot{a}} \tilde{\lambda}_{\dot{b}}}{[p, q]},
\end{equation}
with $p = \lambda \tilde{\lambda}$ and $q = \mu \tilde{\mu}$. The momentum $q$ is not physical in the sense that it is part of the gauge freedom in the spinor helicity formalism, so $q \neq p$ is an arbitrary momentum. Putting all of this together and using the color-kinematic factorization, the full amplitude for the MHV
case, where we have one quark line interact with \( n + 1 \) gluons with positive helicities and one gluon with negative helicity, is given by

\[
M_n = \sum_{\sigma \in S_n} T_{a_{\sigma_1}} \cdots T_{a_{\sigma_n}} A_n(p, p'; \sigma_1, \ldots, \sigma_n),
\]

where we are summing over all permutation of gluon indices. Here the product of \( T_{a_i} \) which are the generators of \( SU(N) \) are called the color factor and \( A_n \) is called partial amplitude that contain only the kinematics of the process

\[
A_n(p, p'; \sigma_1, \ldots, \sigma_n) = \frac{\langle p, Q \rangle^3 \langle p', Q \rangle}{\langle p/p \rangle \prod_{a=1}^{n-1} \langle p, \sigma_a \rangle \langle \sigma_a, \sigma_{a+1} \rangle \langle \sigma_{a+1}, p' \rangle},
\]

were \( Q \) is the momentum of the gluon that carry a negative helicity. Now if we consider the case were all positive helicities gluons are bremsstrahlung, and by using the factorisation in near the soft-collinear limit \( \Pi \) we obtain

\[
J_{QCD} = \sum_{\sigma \in S_n} T_{a_{\sigma_1}} \cdots T_{a_{\sigma_n}} \frac{\langle p, Q \rangle \langle Q, p' \rangle}{\prod_{a=1}^{n-1} \langle p, \sigma_a \rangle \langle \sigma_a, \sigma_{a+1} \rangle \langle \sigma_{a+1}, p' \rangle}. \tag{8}
\]

### 3. Non-abelian correction

In this section, we are going to show that the projection of \( \mathbf{8} \) into a different representation will give us the right expansion around the abelian theory. First let us consider the fact that making the gauge field of QCD commute will transform it the theory into QED which consist to transform \( SU(N) \) to \( U(1) \) or \( T_{a_i} \rightarrow 1 \). Applying such transformation into \( \mathbf{8} \) gives us \( J_{QED} \) as expected \( \Pi \), which is the symmetrization of the kinematic part of \( J_{QCD} \)

\[
J_{QED} = \sum_{\sigma \in \mathcal{S}_n} S_{\sigma(1) \cdots \sigma(n)} = \prod_{I \neq Q} \frac{\langle p, p' \rangle}{\langle p, I \rangle \langle I, p' \rangle}, \tag{9}
\]

with

\[
S_{\sigma(1) \cdots \sigma(n)} = \frac{\langle p, Q \rangle \langle Q, p' \rangle}{\prod_{a=1}^{n-1} \langle p, \sigma_a \rangle \langle \sigma_a, \sigma_{a+1} \rangle \langle \sigma_{a+1}, p' \rangle}. \tag{10}
\]

That lead us to think that the irreducible representation of the permutation group \( S_n \) is the mathematical framework that we want to expand the QCD radiative correction since its fully symmetric projection is the QED correspondent.

The irreducible representation of \( S_n \) are usually represented by the Young diagrams corresponding. For example for \( S_2 \) we have the 2 irreducible decomposition one the symmetric and one antisymmetric, then we can always decompose any tensor of order 2 into symmetric and antisymmetric tensor \((A_{ij} = A_{ij}^{\text{sym}} + A_{ij}^{\text{ant}})\). In a general tensor of order \( n \), we can do a similar decomposition into the irreducible representation of \( S_n \). As presented in \( \Pi \), we are going to use projectors \( P_\alpha \) into the Young diagram correspondent such that

\[
\sum_{\alpha} P_\alpha = 1 \quad \text{and} \quad P_\alpha P_\beta = \delta_{\alpha \beta} P_\alpha. \tag{11}
\]

Here we give an example of projectors for \( n = 3 \):

\[
\begin{align*}
P_1 &= \frac{1}{6} \left[ 1 + (12) + (13) + (23) + (123) + (132) \right] \\
P_2 &= \frac{1}{6} \left[ 1 - (12) - (13) - (23) + (123) + (132) \right] \\
P_3 &= \frac{1}{3} \left[ 2 - (123) - (132) \right]
\end{align*}
\]
Using those projection into \[ \{8\} \), both for the color factor and for the kinematic part, where it act on the gluon labels \{1, \ldots, n\}. Taking \[ T_{a_1} \cdots T_{a_n} = C_{a_1 \cdots a_n} \], for the color factor, and considering the fact that \( J_{QED} \) is the symmetric part of the kinematics \[ \{9\} \), we obtain

\[
J_{QCD} = C_{a_1 \cdots a_n}^{\text{sym}} J_{QED}(1, \ldots, n) + \sum_{\alpha \neq 1} \sum_{\sigma \in S_n} C_{a_{\sigma(1)} \cdots a_{\sigma(n)}}^{(\alpha)} S_{\sigma(1) \cdots \sigma(n)}^{(\alpha)},
\]

with \( C_{\cdot}^{(\alpha)} = P_\alpha C_{\cdot} \) and similarly for \( S_{\cdot}^{(\alpha)} \) are the projected tensor into the symmetry \( \alpha \). This equation looks pretty much close to what we expected \[ \{2\} \), in the exception we have a color factor with the \( J_{QED} \) which make sense since QCD is a theory that carry colors.

4. Conclusion

During the last decade of development of the maximal helicity violating, many progress was made in the understanding of the mathematical structure of the scattering amplitude. In the mathematical physics community those developments push them to go further into deep mathematical framework (twistor theory, positive Grassmannian, Amplituhedron, ...) to understand the very fundamental aspect of QFT. However for the experimental, since the discovery of the quark gluon plasma leads the experimentalists to run many different experiments (p+p, p+Pb, Pb+Pb, ...) in order to test the QCD phase diagram at different scales and conditions.

In this talk, we introduced one of the applications of the actual massless theory into the most exiting research experiments heavy ion collision. This is also an opening into the permutation structure of multi-gluon emission. We saw that the purely symmetric structure correspond to QED, and any antisymmetric structure will be in the non-abelian correction as shown in \[ \{13\} \). In QCD, any excitation from the medium leads to a radiation even if it is a color flip, i.e. no energy loss. However in QED there is no such process as color flip; that means \( J_{QED} \) must vanish for \( p = p' \), true from \[ \{9\} \). So in the limit \( p = p' \) the QCD radiative correction is purely non-abelian and our actual work is focussed on understanding the color flip structure which may lead us to the smooth resumation of bremsstrahlung radiative corrections.

Acknowledgments

This work would have been impossible without the support of the Science Faculty PhD Fellowship of UCT. We are also grateful to the supports from SA-CERN and NRF-South Africa.

References

[1] F. Cachazo, P. Svrcek, E. Witten, “MHV Vertices And Tree Amplitudes In Gauge Theory”, Journal of High Energy Physics, Volume 2004, JHEP09(2004).
[2] Z. Bern, L. J. Dixon, D. A. Kosower, “On-shell recurrence relations for one-loop QCD amplitudes”, Phys. Rev. D 71, 105013, (2005).
[3] A. Brandhuber, B. Spence, G. Travaglini, “Tree-level formalism”, Journal of Physics A: Mathematical and Theoretical, Volume 44, Number 45, (2011).
[4] D.H. Rischke, Prog. Part. Nucl. Phys. 52 (2004) 197.
[5] G. Policastro et al., Phys. Rev. Lett. 87, 081601 (2001); P. K. Kovtun et al., ibid.94, 111601 (2005).
[6] Megan Connors, for the ALICE Collaboration, “Jet Production in p-Pb Collisions”, arXiv:1409.4655 [nucl-ex].
[7] A. Adare et al., “Centrality-dependent modification of jet-production rates in deuter-on-gold collisions at \( \sqrt{s_{NN}} = 200 \text{ GeV} \), arXiv:1509.04657 [nucl-ex] (2015).
[8] W.A. Horowitz, “Heavy Quark Production and Energy Loss”, Nuclear Physics A, Volumes 904905, Pages 186c193c (2013).
[9] M. E. Peskin, “Simplifying Multi-Jet QCD Computation”, arXiv:1101.2414 [hep-ph] (2011).
[10] K. J. Ozeren, W. J. Stirling, “MHV techniques for QED processes”, Journal of High Energy Physics, Volume 2005, JHEP11(2005).
[11] T. L. Wade, “Tensor Algebra and Young’s Symmetry Operators”, American Journal of Mathematics Vol.63, No.3 , pp.645-657, (1941).