On the Advantage of Irreversible Processes in Single-System Games

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Abstract—The CHSH no-signalling game studies Bell nonlocality by showcasing a gap between the win rates of classical strategies, quantum-entangled strategies, and no-signalling strategies. Similarly, the CHSH* single-system game explores the advantage of irreversible processes (which break Landauer's principle) by showcasing a gap between the win rates of classical reversible strategies, quantum reversible strategies, and irreversible strategies. The irreversible process of erasure rules supreme for the CHSH* single-system game, but this erasure advantage do not necessarily extend to every single-system game: We introduce the 32-Game, in which reversibility is irrelevant and only the distinction between classical and quantum operations matters. We turn our new insight into a more general necessary condition for the existence of erasure advantage, and showcase it by modifying the CHSH* game to make it erasure-immune, while conserving its quantum advantage. We conclude by the reverse procedure: We tune the 32-Game to make it erasure-vulnerable, and erase its quantum advantage in the process.

Index Terms—Single-system games, quantum channels, quantum state discrimination, Landauer’s principle

I. INTRODUCTION TO SINGLE-SYSTEM GAMES: CHSH*

No-signalling games (sometimes called nonlocal games) demonstrate that exploiting quantum mechanics—or more precisely, quantum entanglement—provide an advantage in certain distributed-computing tasks. The CHSH game [1, 2] is the most well-known example of such games; the RGB no-signalling game is another very simple example [3]. For both of these games, sharing quantum entanglement allows to win with better probability than using purely classical strategies, but not with probability 1 (this was first proven by Tsirelson [4]); and for both of these games, hypothetical no-signalling devices called Popescu-Rohrlich boxes [5] (PR-boxes, or non-local boxes) makes winning with certainty possible.

Single-system games do not study locality, but space-constrained computations. They were recently (re-)introduced by Henaut, Catani et al. [8], who reframed the standard CHSH game into a game, CHSH* (Fig. 1), where the two players—instead of being spatially separated—are limited to deterministically applying conditional gates on a common two-dimensional system. Henaut, Catani et al. then analyzed the best performance that can be achieved when the players are restricted to certain types of gates and found out that quantum reversible gates can do better than classical reversible gates, but cannot win with certainty—a scenario very similar to the no-signalling case, but where the advantage comes simply from the geometry of the quantum state, and not from entanglement. They also showed that erase gates were to the CHSH* single-system game what PR boxes are to the standard CHSH no-signalling game (see some of their results in the leftmost column of Tab. I).

\[
\begin{align*}
|0\rangle & \xrightarrow{A_{a}} |a\rangle \xrightarrow{B_{b}} |a \cdot b\rangle
\end{align*}
\]

Fig. 1: CHSH* is a single-system game that exhibits a behaviour analog to the CHSH no-signalling game. Quantum irreversible gates can win better than classical irreversible ones, but cannot reach perfection; while erase gates (which are for closed systems as non-physical as PR-boxes generally are) can.

| Gate Set        | CHSH* | EI-CHSH* | 32-Game | B32-Game |
|-----------------|-------|----------|---------|----------|
| Classical       | 3/4   | 3/4      | 7/9     | 4/5      |
| Classical (ir)reversible | 1     | 3/4      | 7/9     | 13/15    |
| Quantum        | 0.85  | 0.85     | 5/6     | (4/5)    |
| Quantum (ir)reversible | 1     | —        | 5/6     | 13/15    |

TABLE I: We examine whether having access to irreversible processes can improve Alice and Bob’s—whether classical or quantum—win rates in three new single-system games (rightmost columns). For CHSH and the B32-Game, it does; while for EI-CHSH and the 32-Game, it does not. (0.85 stands for Tsirelson’s bound \(1/2 + \sqrt{2}/4\). The win rate in parenthesis is conjectured.)

Intrigued by this fact, we investigate, in this work, the question further and devise a new single-system game, the
32-Game, for which irreversible processes are not always superior to reversible ones (Sec. III). We then formulate El-CHSH*, an erasure-immune variant of the CHSH* game in which irreversible processes lost their edge (Sec. III). Finally, we conclude with the opposite endeavour: We bias the input distribution of our first game—we rename it the B32-Game—to give back an advantage to irreversible gates (Sec. IV). Our results are summarized in Tab. 1.

II. PRELIMINARIES: SETS OF GATES

We study the optimal strategies for two players (Alice and Bob) that are restricted to using various types of logical gates. We ignore statistical mixture of reversible gates because—when the input distribution is fixed—drawing gates randomly following some distribution is never better than applying the best strategy from this distribution.

We study the following four sets of gates: classical reversible, classical (ir)reversible, quantum reversible, and quantum (ir)reversible. We represent all these gates as quantum channels acting on two-dimensional systems, using the formalism of quantum information theory (The books [9] and [10] are two excellent references).

If the (classical or quantum) channel is reversible, it is simply a unitary operator $U$, with $U^\dagger U = U U^\dagger = I$.

a) Classical Reversible: We say the channel is classical reversible if it acts as a permutation between classical states. A classical state is a quantum state that is diagonal in the rectilinear basis. There are only two classical reversible gates for two-dimensional systems: the identity $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the bit-flip $X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

b) Quantum Reversible: Quantum reversible channels are arbitrary unitary operators and are for this reason usually called quantum unitaries.

If a channel is not necessarily reversible—we write (ir)reversible—it can be represented by Kraus operators $\{K_i\}$ such that $\sum_i K_i^\dagger K_i = I$, and whose action on a quantum state $\rho$ is

$$C(\rho) := \sum_i K_i \rho K_i^\dagger.$$ 

c) Classical (Ir)reversible: We build the classical (ir)reversible set of gates by adding the the erasure gate to the classical reversible gates set: The erasure gate simply outputs 0 (or 1) no matter the input; it is an irreversible process. While such erasure is a classical operation, it can be seen as an amplitude-damping quantum channel of probability 1, and be represented by the two Kraus operators $K_{E1}, K_{E2} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Erasing to 1 can be obtained by erasing first to 0, and then flipping the result.

d) Quantum (Ir)reversible: Quantum (ir)reversible gates are arbitrary quantum channels and constitute the most general set of gates we will consider. Note that closed quantum systems follow Schrödinger’s equation and always evolve unitarily: Genuinely irreversible quantum gates can thus physically only happen in open systems, where an external leakage of information is possible.

e) Rectilinear Measurement: A measurement in the rectilinear basis on a state $\rho$ gives the result 1 with probability $\text{tr}(\rho |1\rangle \langle 1|)$.

III. QUANTUM ADVANTAGE BUT NO IRREVERSIBILITY ADVANTAGE: THE 32-GAME

Definition 1. The 32-Game is the single-system game defined in Fig. 2 with the inputs $a, b$ drawn uniformly at random ($\forall i, j \in \{0, 1, 2\}, \text{Pr}(a = i, b = j) = 1/9$).

$$\{0, 1, 2\} \ni a \quad b \in \{0, 1, 2\}$$

$$|0\rangle \quad A_a \quad B_b \quad \bigoplus \quad > m \equiv \delta_{ab}$$

Fig. 2: Alice and Bob must each choose 3 conditional gates. They then independently receive a random input trit and apply the corresponding gates on an initial state $|0\rangle$, which is ultimately measured in the rectilinear basis: They try to maximize the probability that this measurement yields 1 when their inputs are identical, and that it yields 0 when their inputs are different. We vary the set of operations they can choose from.

A. Classical Gates

Proposition 1. The best classical reversible strategy for the 32-Game wins it with probability 7/9.

Proof. The following strategy wins with probability 7/9 using only classical reversible gates,

$$A_a := X^a, B_b := X^{b+1}.$$ 

It loses only when the inputs are $(a = 0, b = 2)$ or $(a = 2, b = 0)$.

Its optimality is a direct corollary of Proposition 2 \hfill \Box

Definition 2. A communication strategy of a single-system game is a classical strategy where instead of applying blindly a predetermined gate depending on its input, Bob is allowed to read Alice’s (classical) output and use it to compute his own output. The communication goes from Alice to Bob and is bounded by the size of the system.

Lemma 1. The winning rate of the best communication strategy is an upper bound on the winning rate of the best (ir)classical reversible strategy.

Proof. Communication strategies can simulate any (ir)reversible-processes strategy: Reversibility is irrelevant to Alice because she knows the initial state of the system; and Bob can simulate any irreversible process given Alice’s output. \hfill \Box

Proposition 2. Classical (ir)reversible strategies cannot win the 32-Game better than classical strictly reversible strategies.
Proof. We prove that one-bit communication strategies, as described in Lemma 1, cannot win more often than with probability $7/9$, and that, therefore, neither can classical (ir)reversible strategies.

We call $x_a$ the bit that Alice sends to Bob on input $a$. Up to relabelling of the inputs, there are two different behaviours for Alice: Either $x_a = x_1 = x_2$, which is equivalent to no communication and cannot win with probability better than $2/3$, or $x_0 = x_1 = x_2$. In this case, let $y(x_a, b)$ be Bob’s output: Then $y(x_0, 0) = y(x_1, 0)$ and $y(x_0, 1) = y(x_1, 1)$, but their respective winning conditions, $\delta_{x_0} \neq \delta_{x_1}$ and $\delta_{x_0} \neq \delta_{x_2}$, are orthogonal—Bob is sure to get at least one wrong answer for each couple—and the win rate is, therefore, at most $7/9$.

B. Quantum Gates

Proposition 3. The best quantum reversible strategy for the 32-Game wins it with probability $5/6$.

Proof. If we define $R_{2\pi/3} := \left( \begin{array}{cc} \cos \pi/3 & -i \sin \pi/3 \\ -i \sin \pi/3 & \cos \pi/3 \end{array} \right)$, the following quantum reversible strategy wins with probability $5/6$:

$$A_a := X \cdot R_{2\pi/3}^a, \quad B_b := R_{2\pi/3}^b.$$  

Note that $A_a \cdot B_{b=a} = I$; $A_a \cdot B_{b \neq a} \in \{R_{2\pi/3}, R_{2\pi/3}^2\}$ and $\text{tr}(R_{2\pi/3} |0\rangle \langle 0|) = \text{tr}(R_{2\pi/3}^2 |0\rangle \langle 0|) = 1/4$.

Its optimality is a direct corollary of Proposition 4.

Definition 3. The discrimination experiment corresponding to a single-system game is the following: 1) Alice is given her random input $a$, and sends to Bob a quantum state $\rho_a$ of her choice. 2) Bob is given his random input $b$, and needs to guess whether Alice sent him $\rho_a = b$ or $\rho_a \neq b$ using any means allowed by quantum mechanics. Alice and Bob play together in trying to minimize Bob’s probability of error.

Lemma 2. The maximum winning probability for a discrimination experiment bounds from above the winning rate of the best quantum (ir)reversible strategy for the corresponding single-system game.

Proof. We prove it by contradiction: Any strategy for the single-system game can be turned into a strategy for the discrimination experiment that wins with the same probability; it thus cannot be better.

The resulting strategy for the discrimination experiment is the following: Alice sends the conditional state $\sum_i A_{a,i} |0\rangle \langle 0| A_{a,i}^\dagger$ to Bob; Bob applies the quantum channel $B_b$ and then measures in the rectilinear basis. On outcomes $0$ he outputs the guess $\rho_a = b$, and $\rho_a \neq b$ otherwise.

Proposition 4. Quantum (ir)reversible strategies cannot win the 32-Game better than quantum strictly reversible strategies.

Proof. We bound from above the probability of winning the discrimination-experiment scenario, and the conclusion then follows from Lemma 2.

The minimal-error measurement for distinguishing two arbitrary quantum states $\rho$ and $\sigma$, of respective prior probabilities $p$ and $q$, was characterized by Helstrom [11]. It gives a tight bound on the distinguishability success rate:

$$p_{\text{guess}} \leq \frac{1}{2} + \frac{1}{2} \|p \rho - q \sigma\|_1,$$

where $\|A\|_1 := \text{tr} \sqrt{AA^\dagger}$.

We apply it to the 32-Game discrimination-experiment scenario:

$$p_{\text{guess}} \leq \frac{1}{2} + \frac{1}{6} \sum_{i=0}^{2} \|\rho_i - \rho_{i+1} + \rho_{i+2}\|_1,$$

and introduce the changes of variables

$$\vec{v}_i := \vec{v}_{i} - \vec{v}_{i+1} - \vec{v}_{i+2}.$$  

Eq. (1) becomes

$$p_{\text{guess}} \leq \frac{1}{2} + \frac{1}{6} \sum_{i=0}^{2} \left\| -I + r_{ix} \sigma_x + r_{iy} \sigma_y + r_{iz} \sigma_z \right\|_{1} = \frac{1}{2} + \frac{1}{18} \sum_{i=0}^{2} \|\rho_i - \rho_{i+1} - \rho_{i+2}\|_1,$$

To evaluate $\|A_i\|_1$, we note that $-A_i$ is a hermitian matrix with eigenvalues $(1 + \| \vec{v}_i \|_2)/2$, and that, therefore,

$$\|A_i\|_1 = \text{tr} \sqrt{A_i A_i^\dagger} = (1 + \|\vec{v}_i\|_2^2 + 1 - \|\vec{v}_i\|_2^2)/2$$

$\equiv \begin{cases} \|\vec{v}_i\|_2 & \text{if } \|\vec{v}_i\|_2 \geq 1 \\ 1 & \text{if } \|\vec{v}_i\|_2 < 1. \end{cases}$

We now separate the analysis into 4 different cases.

Definition 4. We separate the strategy distributions for the 32-Game discrimination experiment using the parameter

$$D_{\max} := \# \{ i \text{ s.t. } \|A_i\|_1 = 1 \}.$$

$D_{\max}$ corresponds to the maximum number of inputs $b = i$ for which Bob could completely ignore Alice’s action and guess according to the highest prior; this is what happens effectively when for a certain input Bob uses an erasure action: He wins with conditional probability $\max(p_i, q_i) = 2/3$ no matter Alice’s behaviour.
a) Case $D_{\text{max}} = 0$: We use the Cauchy-Schwarz inequality, and then the three length constraints, to obtain (note that $\|x\|_2 = x^2$ in the dot-product notation)
\[
\sum_{i=0}^{2} \|A_i\|_1 = \|\bar{r}_0\|_2 + \|\bar{r}_1\|_2 + \|\bar{r}_2\|_2 \\
\leq \sqrt{3} \left(\|\bar{r}_0\|_2 + \|\bar{r}_1\|_2 + \|\bar{r}_2\|_2\right) \\
= \sqrt{3} (\|x_0\|^2 + \|x_1\|^2 + \|x_2\|^2) - 2(\bar{r}_0 \cdot \bar{v}_1 + \bar{v}_1 \cdot \bar{v}_2 + \bar{v}_0 \cdot \bar{v}_2) \\
\leq \sqrt{3}(\sqrt{9} - 2(\bar{r}_0 \cdot \bar{v}_1 + \bar{v}_1 \cdot \bar{v}_2 + \bar{v}_0 \cdot \bar{v}_2)).
\]

Our task is now to bound from below
\[
\bar{v}_0 \cdot \bar{v}_1 + \bar{v}_1 \cdot \bar{v}_2 + \bar{v}_0 \cdot \bar{v}_2 \quad \text{(minimize)} \\
\forall i, \bar{v}_i^2 \leq 1. \quad \text{(constraints)}
\]

We use a Lagrangian multipliers method (the $s_i$ are slack constraints):
\[
\mathcal{L}(\{\bar{v}_i, \lambda, s_i\}_{i=0}^{2}) = \bar{v}_0 \cdot \bar{v}_1 + \bar{v}_1 \cdot \bar{v}_2 + \bar{v}_0 \cdot \bar{v}_2 + \sum_{i=0}^{2} \lambda_i (\bar{v}_i^2 + s_i^2 - 1).
\]

The set of vectors minimizing our function include at least one vector that saturates the unit-length constraint without a loss of generality, and invoking the spherical symmetry, let us say that it is $\bar{v}_0 = \hat{x}$.

We pose $\nabla \mathcal{L} = 0$ and obtain 15 scalar equalities. Examples are
\[
\frac{\partial \mathcal{L}}{\partial y_0} = 0 \iff y_1 + y_2 = 0, \\
\frac{\partial \mathcal{L}}{\partial y_1} = 0 \iff y_2 + 2 \lambda_1 y_1 = 0, \\
\frac{\partial \mathcal{L}}{\partial x_0} = 0 \iff x_1 + x_2 + 2 \lambda_0 = 0, \\
\frac{\partial \mathcal{L}}{\partial x_1} = 0 \iff 1 + x_2 + 2 \lambda_1 x_1 = 0,
\]

Comparing the first two equations, and then the next two, we find that either $\lambda_0 = \lambda_1 = 1/2$, or $y_1 = y_2 = 0$. Similarly, if we were to take the analogous in the $\hat{z}$ direction of the first two equations, we would conclude that either $\lambda_0 = \lambda_1 = 1/2$, or $z_1 = z_2 = 0$. This implies that if $\lambda_0 \neq 1/2$, all vectors are co-linear (in the $\hat{z}$ direction) and the strategy is classical, but then they are of no interest since classical strategies cannot win better than $7/9$ (Prop. 2).

We thus assume $\lambda_0 = 1/2$, and go back to $\nabla \mathcal{L} = 0$. We develop the vectorial equality
\[
\frac{\partial \mathcal{L}}{\partial \bar{v}_0} = 0 \iff \bar{v}_0 + \bar{v}_1 + \bar{v}_2 = 0,
\]
square it,
\[
(\bar{v}_0 + \bar{v}_1 + \bar{v}_2)^2 = 0,
\]
and expand it, in conjunction with the length constraints, to conclude finally that
\[
\bar{v}_0 \cdot \bar{v}_1 + \bar{v}_1 \cdot \bar{v}_2 + \bar{v}_0 \cdot \bar{v}_2 = -(\bar{v}_0^2 + \bar{v}_1^2 + \bar{v}_2^2)/2 \geq -3/2.
\]

Injecting this bound back into Eq. 4 and then Eq. 2 we find that $p_{\text{guess}} \leq 5/6$ for the case $D_{\text{max}} = 0$.

The other cases do not violate this bound as the following crude inequalities show.

b) Case $D_{\text{max}} = 1$: Without losing generality, we pose $\|A_0\|_1 = 1$. We re-write Eq. 4 (it was bounded from above by 6).
\[
\sum_{i=0}^{2} \|A_i\|_1 = 1 + \|\bar{r}_1\|_2 + \|\bar{r}_2\|_2 \\
\leq 1 + \sqrt{2} \left(\|\bar{r}_0\|_2 + \|\bar{r}_1\|_2 + \|\bar{r}_2\|_2\right) \\
\leq 1 + \sqrt{2} \sqrt{12} \approx 5.9.
\]

c) Case $D_{\text{max}} = 2$: Without losing generality, we pose $\|A_0\|_1 = \|A_1\|_1 = 1$. Then Eq. 4 becomes
\[
\sum_{i=0}^{2} \|A_i\|_1 = 1 + \|\bar{r}_2\|_2 \leq 5.
\]

d) Case $D_{\text{max}} = 3$: Finally, $\sum_{i=0}^{2} \|A_i\|_1 = 3$. This proves that none of the 4 cases could allow a better probability of winning than $p_{\text{guess}} \leq 5/6$, the one achieved in Prop. 3 and which is then optimal.

IV. GENERALIZING THE INSIGHT: EI-CHSH*

A. A Necessary Condition For Erasure Advantage

**Proposition 5.** For any binary-output 2-player single-system game, classical irreversible processes might provide an advantage over classical reversible processes only if
\[
\sum_{a,b} p_{ab} |W_{ab}^{(0)} - W_{ab}^{(1)}| > 0,
\]
where $p_{ab}$ is the prior of the inputs, and $W_{ab}^{(o)} = \begin{cases} 1 & \text{if output } o \text{ wins on inputs } (a, b), \\ 0 & \text{otherwise} \end{cases}$.

**Proof.** The negation of Eq. 5 implies that for all inputs $b$ of Bob, erasing the final bit to 0, and erasing it to 1 leads to equal probabilities of winning (Bob does not know Alice’s input). This means that for any input $b$ for which Bob could have programmed an erasure, he can do whatever he wants—namely the identity, a very reversible strategy—without diminishing his winning rate. Erasing cannot, therefore, provide any advantage.

As shown by the 32-Game with uniformly distributed inputs—where $\sum_{a,b} p_{ab} |W_{ab}^{(0)} - W_{ab}^{(1)}| \neq 0$ but there is still no erasure advantage—this condition is necessary, but not sufficient.

It is open how Prop. 5 extends to the quantum case, or how to generalize it to higher-dimension outputs.
B. The Erasure-Immune CHSH* Game

We illustrate Prop. 5 by modifying the CHSH* game as to remove its erasure advantage, while keeping the quantum-unitary advantage. In the variant, Alice is given a second output that inverts the winning condition with probability 1/2.

**Definition 5.** The erasure-immune CHSH* single-system game (EL-CHSH*) is defined in Fig. 3. The inputs are selected uniformly at random.

**Proposition 6.** Classical (ir)reversible gates are not better than classical strictly reversible ones in the erasure-immune CHSH* game: They win with at most probability 3/4; while a quantum-unitary strategy can reach Tsirelson’s bound (≈ 0.85). These bounds are tight.

**Proof.** Any reversible strategy for the CHSH* game—whether classical or quantum—can be turned into a strategy for the erasure-immune CHSH* game with the same winning rate, and vice versa: Alice simply needs to apply a $X^{a_2}$ gate at the very beginning of the circuit, effectively turning, when $a_2 = 1$, the $|0\rangle$ initial state into $|1\rangle$ (this works because for two-dimensional reversible gates, the transition rates $|0\rangle \leftrightarrow |1\rangle$ of any strategy are symmetric).

The absence of advantage of classical (ir)reversible processes is a direct consequence of Prop. 5.

\[
\{0, 1\}^2 \ni a_1, a_2 \quad b \in \{0, 1\} \\
|0\rangle \xrightarrow{A_{a_1, a_2}, B_b} |f\rangle \xrightarrow{r_{a_1, b}} \text{ with } m = a_1 \cdot b \oplus a_2
\]

Fig. 3: In this erasure-immune variant of the CHSH* game, classical irreversible processes do not win better than reversible ones, but the quantum advantage remains.

V. RECOVERING THE ADVANTAGE OF IRREVERSIBLE PROCESSES: THE B32-GAME

We now do the opposite: We modify the 32-Game as to make irreversibility relevant again.

**Definition 6.** We call biased 32-Game (B32-Game) the same single-system game that is defined in Fig. 3 but with the biased input distribution

\[
\forall i, j : p_{a=i, b=j} = \begin{cases} 1/15 & \text{if } i = j, \\ 2/15 & \text{otherwise.} \end{cases}
\]

**Proposition 7.** The best classical reversible strategy for the biased 32-Game wins it with probability 4/5.

**Proof.** For each of their inputs, Alice and Bob must choose between two different classical reversible operations (I or X). This makes for a total of only $2^6$ classical reversible strategies; an exhaustive search reveals none of them win more than with probability 4/5.

We conjecture the best quantum reversible strategy achieves the same 4/5 bound.

**Proposition 8.** The best classical (ir)reversible strategy for the biased 32-Game wins it with probability 13/15.

**Proof.** The following classical (ir)reversible strategy is sufficient to win with probability $p_{\text{win}} = 1/3 + 2/3 \cdot 4/5 = 13/15$: Alice applies $X$ if $a = 0$, and $I$ otherwise; while Bob applies $I$ if $b = 0$, and erases Alice’s bit otherwise.

To show it is optimal, we prove that not even quantum (ir)reversible strategies can do better. We use the same discrimination-experiment technique as in Section III but skip over the details. Eq. 2 becomes

\[
p_{\text{guess}} \leq 1 - \frac{1}{2} + \frac{1}{30} \sum_{i=0}^{2} \left| \begin{array}{c} -3 + r_{i,x}\sigma_x + r_{i,y}\sigma_y + r_{i,z}\sigma_z \\ 2 \end{array} \right|_1.
\]

Eq. 3 becomes

\[
\|A_i\|_1 = \text{tr} A_i A_i^\dagger = \begin{cases} \|r_i\|_2 & \text{if } \|r_i\|_2 \geq 3 \\
1 & \text{if } \|r_i\|_2 < 3. \end{cases}
\]

We again analyze the four cases $D_{\text{max}} = 0, 1, 2, 3$.

a) Case $D_{\text{max}} = 0$: Eq. 4 becomes

\[
\sum_{i=0}^{2} \|A_i\|_1 = \|\vec{r}_0\|_2 + \|\vec{r}_1\|_2 + \|\vec{r}_2\|_2
\]

\[
\leq \sqrt{3} \sqrt{3 + 3 + 3} = 9;
\]

all crossed terms disappeared. This bound imply a maximal winning rate of only

\[
p_{\text{guess}} \leq 1 - \frac{1}{2} + \frac{9}{30} = 4/5,
\]

which is the same as Bob simply guessing $\rho_{a \neq b}$ without looking at what Alice sent him.

b) Case $D_{\text{max}} = 3$: $\sum_{i=0}^{2} \|A_i\|_1 = 9.$

c) Case $D_{\text{max}} = 2$: $\sum_{i=0}^{2} \|A_i\|_1 = 6 + \|\vec{r}_2\|_2 \leq 11$;

This bound corresponds to the winning-rate $p_{\text{guess}} = 13/15$ of the classical reversible strategy we mentioned previously (it uses 2 conditional-erasure gates).

d) Case $D_{\text{max}} = 1$: Finally,

\[
\sum_{i=0}^{2} \|A_i\|_1 = 3 + \|\vec{r}_1\|_2 + \|\vec{r}_2\|_2
\]

\[
\leq 3 + \sqrt{2\sqrt{2}} \approx 10.3 < 11.
\]

The reversible classical strategy is optimal.

**Corollary 1.** Quantum (ir)reversible processes cannot win the biased 32-Game better than classical (ir)reversible ones.

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