Weighted Hurwitz numbers, $\tau$-functions and matrix integrals

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Abstract

The basis elements spanning the Sato Grassmannian element corresponding to the KP $\tau$-function that serves as generating function for rationally weighted Hurwitz numbers are shown to be Meijer $G$-functions. Using their Mellin-Barnes integral representation the $\tau$-function, evaluated at the trace invariants of an externally coupled matrix, is expressed as a matrix integral. Using the Mellin-Barnes integral transform of an infinite product of $\Gamma$ functions, a similar matrix integral representation is given for the KP $\tau$-function that serves as generating function for quantum weighted Hurwitz numbers.

1 Hurwitz numbers: classical and weighted

The fact that KP and 2D-Toda $\tau$-functions of hypergeometric type serve as generating functions for weighted Hurwitz numbers was shown in [2–5], generalizing the case of simple (single and double) Hurwitz numbers [8, 9]. Sections 1.1 and 1.2 below, and Section 2 give a brief review of this theory, together with two illustrative examples: rational and quantum weighted Hurwitz numbers. In Section 3 it is shown how evaluation of such $\tau$-functions at the trace invariants of a finite matrix may be expressed either as a Wronskian determinant or as a matrix integral. The content of subsections 3.2–3.4 are largely drawn from [6,7], in which further details and proofs of the main results may be found.

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1.1 Geometric meaning of classical Hurwitz numbers

The Hurwitz number \(H(\mu^{(1)}, \ldots, \mu^{(k)})\) is the number of inequivalent branched \(N\)-sheeted covers \(\Gamma \to \mathbb{P}^1\) of the Riemann sphere, with \(k\) branch points \((Q_1, \ldots, Q_k)\), whose ramification profiles are given by \(k\) partitions \((\mu^{(1)}, \ldots, \mu^{(k)})\) of \(N\), normalized by dividing by the order \(|\text{aut}(\Gamma)|\) of its automorphism group. The Euler characteristic \(\chi\) and genus \(g\) of the covering curve is given by the Riemann-Hurwitz formula:

\[
\chi = 2 - 2g = 2N - d, \quad d := \sum_{i=1}^{k} \ell^*(\mu^{(i)}),
\]

where \(\ell^*(\mu) := |\mu| - \ell(\mu) = N - \ell(\mu)\) is the colength of the partition.

The Frobenius-Schur formula gives \(H(\mu^{(1)}, \ldots, \mu^{(k)})\) in terms of \(S_N\) characters:

\[
H(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{\lambda, |\lambda|=N} h^{k-2}(\lambda) \prod_{j=1}^{k} \chi_{\lambda}(\mu^{(j)}) z_{\mu^{(j)}}, \quad |\mu^{(j)}| = N,
\]

where \(h(\lambda) = \left(\det \frac{1}{(\lambda_i - i + j)!}\right)^{-1}\) is the product of the hook lengths of the partition \(\lambda = (\lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0)\), \(\chi_{\lambda}(\mu^{(j)})\) is the irreducible character of representation \(\lambda\) evaluated on the conjugacy class \(\mu^{(j)}\), and

\[
z_{\mu^{(j)}} := \prod_i i^{m_i(\mu^{(j)})}(m_i(\mu^{(j)}))!\]

is the order of the stabilizer of any element of cyc(\(\mu^{(j)}\)) (and \(m_i(\mu^{(j)}) = \#\) parts of partition \(\mu^{(j)}\) equal to \(i\))

1.2 Weighted Hurwitz numbers [2–5]

Define the weight generating function \(G(z)\), or its dual \(\widetilde{G}(z)\), as an infinite (or finite) product or sum (formal or convergent).

\[
G(z) = \prod_{i=1}^{\infty} (1 + zc_i) = 1 + \sum_{j=1}^{\infty} g_j z^j
\]

\[
\widetilde{G}(z) = \prod_{i=1}^{\infty} (1 - zc_i)^{-1} = 1 + \sum_{j=1}^{\infty} \tilde{g}_j z^j.
\]

The weight for a branched covering with ramification profiles \((\mu^{(1)}, \ldots, \mu^{(k)})\) is defined to be:

\[
\mathcal{W}_G(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{1 \leq \iota_1 < \cdots < \iota_k} \ell^*(\mu^{(1)}) \cdots \ell^*(\mu^{(k)})
\]
\[
\tilde{W}_G(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{(-1)^{\sum_{i=1}^{k} \ell^*(\mu^{(i)}) + k}}{k!} \sum_{\sigma \in S_k} \sum_{1 \leq i_1 \leq \ldots \leq i_k} c_{\nu^{(1)}}^{\ell^*(\mu^{(1)})} \cdots c_{\nu^{(k)}}^{\ell^*(\mu^{(k)})}
\]

Weighted double Hurwitz numbers \( H^d_G(\mu, \nu), \tilde{H}^d_G(\mu, \nu) \) for \( n \)-sheeted branched coverings of the Riemann sphere having a pair of unweighted branch points \((Q_0, Q_\infty)\), with ramification profiles of type \((\mu, \nu)\), and \( k \) additional weighted branch points \((Q_1, \ldots, Q_k)\) with ramification profiles \((\mu^{(1)}, \ldots, \mu^{(k)})\) are defined as:

\[
H^d_G(\mu, \nu) := \sum_{k=1}^{d} \sum_{\mu^{(1)}, \ldots, \mu^{(k)}}^{\sum_{i=1}^{k} \ell^*(\mu^{(i)}) = d} W_G(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu),
\]

\[
\tilde{H}^d_G(\mu, \nu) := \sum_{k=1}^{d} \sum_{\mu^{(1)}, \ldots, \mu^{(k)}}^{\sum_{i=1}^{k} \ell^*(\mu^{(i)}) = d} \tilde{W}_G(\mu^{(1)}, \ldots, \mu^{(k)}) \tilde{H}(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu),
\]

where \(\sum'\) denotes the sum over all partitions other than the cycle type of the identity element \((1)^n\). If \(Q_\infty\) is not a branch point; i.e. \(\nu = (1)^n\), we have a weighted single Hurwitz number

\[
H^d_G(\mu) := H^d_G(\mu, (1)^n).
\]

Two cases of particular interest are: rational weight generating functions:

\[
G_{c,d}(z) := \prod_{l=1}^{L} \frac{1 + cz}{1 - d_m z},
\]

and quantum weight generating function (quantum exponential):

\[
\tilde{G}(z) = H_q(z) := \prod_{i=0}^{\infty} (1 - q^i z)^{-1} = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n},
\]

where

\[
(q; q)_n := (1 - q)(1 - q^2) \cdots (1 - q^n)
\]

for some parameter \(q\), with \(|q| < 1\).

The corresponding rationally weighted (single) Hurwitz numbers are:

\[
H^d_{G_{c,d}}(\mu, \nu) := \sum_{1 \leq k, l \leq \nu^{(1)}, \ldots, \nu^{(l)}}^{\mu^{(1)}, \ldots, \mu^{(k)}, \nu^{(1)}, \ldots, \nu^{(l)}} W_{G_{c,d}}(\mu^{(1)}, \ldots, \mu^{(k)}; \nu^{(1)}, \ldots, \nu^{(l)})
\]

\[
\times H(\mu^{(1)}, \ldots, \mu^{(k)}, \nu^{(1)}, \ldots, \nu^{(l)}, \mu),
\]

\[
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\]
where the rational weight factor is:

$$W_{G,c,d}(\mu^{(1)}, \ldots, \mu^{(k)}; \nu^{(1)}, \ldots, \nu^{(l)}) := \frac{(-1)^{\sum_{j=1}^{l} e^{*(\nu^{(j)})}-l}}{k!^l} \sum_{\sigma \in S_k} \sum_{\sigma' \in S_l} \ell^*(\mu^{(1)}) \cdots \ell^*(\mu^{(k)}) d^*_{a_{\sigma(1)}} \cdots d^*_{a_{\sigma(k)}} \ell^*(\nu^{(1)}) \cdots \ell^*(\nu^{(l)}).$$

The quantum weighted (single) Hurwitz numbers are

$$H^d_{Hq}(\mu) := \sum_{k=1}^{d} \sum_{\mu^{(1)}, \ldots, \mu^{(k)}; \nu^{(1)}, \ldots, \nu^{(l)}, \mu^{(1)}=N} \tilde{W}_{Hq}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu), \quad (1.10)$$

where the quantum weight factor is

$$\tilde{W}_{Hq}(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{(-1)^{d-k}}{k!} \sum_{\sigma \in S_k} \prod_{j=1}^{k} \frac{1}{(1 - q^{\sum_{i=1}^{k} e^{*(\mu^{(i)})}})}.$$  

2. **Hypergeometric τ-functions as generating functions for weighted Hurwitz numbers** [2–5]

To construct a KP τ-function of hypergeometric type that serves as generating function for weighted Hurwitz numbers for a given weight generating function $G$, choose a small parameter $\beta$ and define coefficients $r^{(G,\beta)}_\lambda$ that are of content product form:

$$r^{(G,\beta)}_\lambda := \prod_{(ij) \in \lambda} r^{(G,\beta)}_{j-i} = \prod_{(ij) \in \lambda} G((j-i)\beta), \quad (2.1)$$

where

$$r^{(G,\beta)}_{j} := G(j\beta) = \frac{\rho^{(G,\beta)}_j}{\beta \rho^{(G,\beta)}_{j-1}}, \quad (2.2)$$

with

$$\rho^{(G,\beta)}_j := \beta^j \prod_{i=1}^{j} G(i\beta) =: e^{T^{G}_{(\beta)}}_j, \quad \rho_0 = 1, = \frac{\rho^{(G,\beta)}_j}{\beta \rho^{(G,\beta)}_{j-1}},$$

$$\rho^{(G,\beta)}_{-j} := \beta^{-j} \prod_{i=1}^{j-1} \frac{1}{G(-i\beta)} =: e^{T^{G}_{(\beta)}}_{-j}, \quad j = 1, 2, \ldots \, (2.3)$$

We then have [3,5]:

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Theorem 2.1 (Hypergeometric Toda $\tau$-functions associated to weight generating function $G(z)$). The double Schur function series

$$\tau(G, \beta)(t, s) := \sum_{\lambda} \beta|\lambda| r^{(G, \beta)}_{\lambda}(s) s_{\lambda}(t)$$

(2.4)

defines a 2D-Toda $\tau$-function (at lattice value $n = 0$).

We now use the Frobenius character formula

$$s_{\lambda}(t) = \sum_{\mu, |\mu| = |\lambda|} \chi_{\lambda}(\mu) \frac{p_{\mu}(t)}{z_{\mu}}, \quad s_{\lambda}(s) = \sum_{\nu, |\nu| = |\lambda|} \chi_{\lambda}(\nu) \frac{p_{\nu}(s)}{z_{\nu}}$$

(2.5)

to change the basis of Schur functions to power sum symmetric functions

$$p_{\mu}(t) := \prod_{i=1}^{\ell(\mu)} p_{i}(t), \quad p_{j}(t) = jt_{j}, \quad p_{\nu}(s) := \prod_{i=1}^{\ell(\nu)} p_{i}(s), \quad p_{j}(s) = js_{j}.$$  

(2.6)

Theorem 2.2 (Hypergeometric Toda $\tau$-functions as generating function for weighted double Hurwitz numbers [3, 5]). The $\tau$-function $\tau(G, \beta)(t, s)$ can equivalently be expressed as a double infinite series in the bases of power sum symmetric functions as follows

$$\tau(G, \beta)(t, s) = \sum_{d=0}^{\infty} \sum_{\mu, |\mu| = |\nu|} \beta|\mu|+d H_{G}^{d}(\mu, \nu) p_{\mu}(t) p_{\nu}(s).$$

(2.7)

It is thus a generating function for the numbers $H_{G}^{d}(\mu, \nu)$ of weighted $n$-fold branched coverings of the sphere, with a pair of specified branch points having ramification profiles $(\mu, \nu)$ and genus given by the Riemann-Hurwitz formula

$$2 - 2g = \ell(\mu) + \ell(\nu) - d, \quad d = \sum_{i=1}^{k} \ell^{*}(\mu^{(i)}).$$

(2.8)

Corollary 2.3 (Hypergeometric KP $\tau$-functions as generating functions for weighted single Hurwitz numbers). Set: $s = \beta^{-1}t_{0} := (\beta^{-1}, 0, 0, \ldots)$. Then the series

$$\tau(G, \beta)(t, \beta^{-1}t_{0}) := \tau(G, \beta)(t) = \sum_{\lambda} (h(\lambda))^{-1} r^{(G, \beta)}_{\lambda}(s) s_{\lambda}(t)$$



$$= \sum_{d=0}^{\infty} \sum_{\mu} \beta^{d} H_{G}^{d}(\mu) p_{\mu}(t)$$

is a KP $\tau$-function which is a generating function for weighted single numbers $H_{G}^{d}(\mu)$ for $|\mu|$-fold branched coverings of the sphere, with a branch point having ramification profile $(\mu)$ at $Q_{0}$ and genus given by the Riemann-Hurwitz formula.

$$2 - 2g = |\mu| + \ell(\mu) - d.$$  

(2.9)
3 Wronskian and matrix integral representation of $\tau^{(G,\beta)}([X])$

In [6, 7] new matrix integral representations were derived for the $\tau$-functions that serve as generating functions for rationally and quantum weighted Hurwitz numbers. The main result is that, using Laurent series and Mellin-Barnes integral representations of the adapted bases for the respective elements of the infinite Grassmannian corresponding to these cases, the $\tau$-functions may be expressed as Wronskian determinants or as matrix integrals.

3.1 Adapted basis, recursion operators, quantum spectral curve

Henceforth, we always set:

$$s = \beta^{-1}t_0 := (\beta^{-1}, 0, 0, \ldots)$$  \hspace{1cm} (3.1)

and

$$\tau^{(G,\beta)}(t) := \tau^{(G,\beta)}(t, \beta^{-1}t_0)$$  \hspace{1cm} (3.2)

is a KP $\tau$-function of hypergeometric type.

For $k \in \mathbb{Z}$, define:

$$\phi_k(x) := \frac{\beta}{2\pi i x^{k-1}} \int_{|\zeta| = \epsilon} \rho^{(G,\beta)}(\zeta) e^{\beta^{-1}x\zeta} \frac{d\zeta}{\zeta^k},$$

$$= \beta x^{1-k} \sum_{j=0}^{\infty} \frac{\rho^{(G,\beta)}_{j-k}}{j!} \left(\frac{x}{\beta}\right)^j,$$  \hspace{1cm} (3.3)

where

$$\rho^{(G,\beta)}(\zeta) := \sum_{i=-\infty}^{k-1} \rho^{(G,\beta)}_{-i-1} \zeta^i.$$  \hspace{1cm} (3.4)

Then $\{\phi_k(1/z)\}_{k \in \mathbb{N}^+}$ is a basis for the element $w^{(G,\beta)}$ of the Sato Grassmannian that determines the KP $\tau$-function $\tau^{(G,\beta)}(t)$ [1].

3.2 Quantum and classical spectral curve

**Theorem 3.1** (Quantum spectral curve and eigenvalue equations [1]). The functions $\phi_k(x)$ satisfy

$$\mathcal{L}\phi_k(x) := (xG(\beta D) - D) \phi_k(x) = (k - 1)\phi_k(x)$$  \hspace{1cm} (3.5)

where $D := x \frac{d}{dx}$ is the Euler operator.
The classical spectral curve is
\[ y = G(\beta xy). \] (3.6)

### 3.2.1 Rational weighting case

For \( G(z) = G_{c,d}(z) \), denote \( \phi_k(x) =: \phi_k^{(c,d,\beta)}(x) \). Then
\[
\zeta \prod_{l=1}^{L} (D + \frac{1}{\beta c_l}) \phi_k^{(c,d,\beta)} + (D + k - 1) \prod_{m=1}^{M} (D - 1 - \frac{1}{\beta d_m}) \phi_k^{(c,d,\beta)} = 0, \tag{3.7}
\]
where
\[
\zeta := -\kappa_{c,d}x, \quad \kappa_{c,d} := (-1)^{M} \frac{\prod_{l=1}^{L} \beta c_l}{\prod_{m=1}^{M} \beta d_m}. \tag{3.8}
\]

#### 3.2.2 Mellin-Barnes integral representation: Meijer G-functions [6][7]

It may be shown that \( \phi_k^{(c,d,\beta)} \) has the Mellin-Barnes integral representation:
\[
\phi_k^{(c,d,\beta)} = C_k^{(c,d,\beta)} \left[ \begin{array}{c}
\frac{1}{\beta c_1}, \cdots, \frac{1}{\beta c_L} \\
1 - k, 1 + \frac{1}{\beta d_1}, \cdots, 1 + \frac{1}{\beta d_M}
\end{array} \right]_{k}^{-\kappa_{c,d}x} \]
\[
= \frac{C_k^{(c,d,\beta)}}{2\pi i} \int_{C_k} \frac{\Gamma(1 - k - s) \prod_{\ell=1}^{L} \Gamma \left( s + \frac{1}{\beta c_{\ell}} \right) (-\kappa_{c,d}x)^s}{\prod_{m=1}^{M} \Gamma \left( s + \frac{1}{\beta d_m} \right)} ds, \tag{3.9}
\]
\[
\sim \frac{\beta \rho - k(c,d)}{(\kappa x)^{k-1}} L F_M \left( \begin{array}{c}
1 - k + \frac{1}{\beta c_1}, \cdots, 1 - k + \frac{1}{\beta c_L} \\
1 - k - \frac{1}{\beta d_1}, \cdots, 1 - k - \frac{1}{\beta d_M}
\end{array} \right), \tag{3.10}
\]
where
\[
C_k^{(c,d,\beta)} := \frac{\prod_{j=1}^{M} \Gamma \left( -\frac{1}{\beta d_j} \right)}{(-\beta)^{k-1} \prod_{\ell=1}^{L} \Gamma \left( \frac{1}{\beta c_{\ell}} \right)}. \tag{3.10}
\]

The contour \( C_k \) is chosen so that the poles at \( 1 - k, 2 - k, \cdots \) are to the right and the poles at \( \{ -i - \frac{1}{\beta c_{\ell}} \}_{j=1, \ldots, L}, i \in \mathbb{N}^+ \) to the left. (See Figure [I])
Figure 1: The contours of integration for the function $\phi_k^{(e,d,\beta)}$ in the case $L > M + 1$.

### 3.2.3 Quantum case expressed as Mellin-Barnes integrals \[7\]

The following is an integral representation of $\phi_k^{(H_q,\beta)}(x)$, valid for all $x \in \mathbb{C}$,

$$\phi_k^{(H_q,\beta)} = \frac{1}{2\pi i} \int_{C_k} A_{H_q,k}(s)x^s ds,$$  \hspace{1cm} (3.11)

where

$$A_{H_q,k}(z) := (-\beta)^{1-k}\Gamma(1-k-z) \prod_{m=0}^{\infty} \left( (-\beta q^{-m})^{-z} \frac{\Gamma(-\beta^{-1}q^{-m})}{\Gamma(z-\beta^{-1}q^{-m})} \right).$$  \hspace{1cm} (3.12)

The contour $C_k$ is defined as starting at $+\infty$ immediately above the real axis, proceeding to the left above the axis, winding around the poles at the integers $s = -k, -k+1, \ldots$ in a counterclockwise sense and continuing below the axis back to $+\infty$.

### 3.3 Determinantal representation of $\tau^{(G,\beta)}(t)$

If $\tau^{(G,\beta)}(t)$ is evaluated at the trace invariants of diagonal $X \in \text{Mat}^{n \times n}$

$$t_i = [X], \quad t_i = \frac{1}{i} \text{tr}X^i, \quad X := \text{diag}(x_1, \ldots, x_n),$$  \hspace{1cm} (3.13)

it is expressible as the ratio of $n \times n$ determinants

$$\tau^{(G,\beta)} ([X]) = \frac{\prod_{i=1}^{n} x_i^{n-1} \det (\phi_i(x_j))_{\leq i,j,\leq n}}{\prod_{i=1}^{n} \rho^{-i}} \Delta(x),$$  \hspace{1cm} (3.14)
where
\[ \Delta(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{n-j})_{1 \leq i, j \leq n} \] (3.15)
is the Vandermonde determinant.

### 3.3.1 Eulerian Wronskian representation

It follows from the recursion relations
\[ \beta(D + k - 1)\phi_k = \phi_{k-1}, \quad k \in \mathbb{Z}, \] (3.16)
that
\[ \tau^{(G,\beta)}([X]) = \gamma_n \left( \prod_{i=1}^{n} x_i^{n-1} \right) \frac{\det(D^{i-1}\phi_n(x_j))_{1 \leq i, j \leq n}}{\Delta(x)}, \] (3.17)
where
\[ \gamma_n := \frac{\beta^{\frac{1}{2}n(n-1)}}{\prod_{i=1}^{n} \rho_i}. \] (3.18)

### 3.4 Matrix integral representation of $\tau^{(G,\beta)}([X])$ [6,7]

#### 3.4.1 Wronskian representation: rational case

For rational weight generating functions $G = G_{c,d}$, and any $n \in \mathbb{N}^+$, let
\[ \phi_n^{(c,d,\beta)}(e^y) = \int_{C_n} A_n^{(c,d,\beta)}(s)e^{ys} ds, \]
\[ A_n^{(c,d,\beta)}(s) := \frac{G_n^{(c,d,\beta)}(1 - n - s) \prod_{i=1}^{L} \Gamma \left( s + \frac{1}{\beta c_i} \right) \left( -\kappa_{c,d} \right)^s}{2\pi i \prod_{m=1}^{M} \Gamma \left( s - \frac{1}{\beta d_m} \right)}. \]

Define the diagonal matrix $Y = \text{diag}(y_1, \ldots, y_n)$
\[ X = e^Y, \quad Y = \ln(X), \quad x_i = e^{y_i}, \quad i = 1, \ldots, n. \] (3.19)

Then $\tau^{(G_{c,d,\beta})}([X])$ becomes a ratio of Wronskian determinants
\[ \tau^{(G_{c,d,\beta})}([X]) = \gamma_n \left( \prod_{i=1}^{n} x_i^{n-1} \right) \frac{\det \left( \phi_n^{(c,d,\beta)}(e^y) \right)_{1 \leq i, j \leq n}}{\Delta(e^y)}. \] (3.20)
3.4.2 Matrix integral representation of $\tau^{(G, \beta)}([X])$: rational case

It follows [6] that

$$\tau^{(G_{c,d,\beta})}([X]) = \frac{\beta \frac{1}{2} (n-1) (\prod_{i=1}^{n} x_i^{n-1}) \Delta(\ln(x))}{(\prod_{i=1}^{n} i!) \Delta(x)} Z_{d\mu(c,d,\beta,n)}(X), \quad (3.21)$$

where

$$Z_{d\mu(c,d,\beta,n)}(X) = \int_{M \in \text{Nor}_{n \times n}^C} d\mu(c,d,\beta,n)(M) e^{\text{tr}YM}$$

and

$$d\mu(c,d,\beta,n)(M) := (\Delta(\zeta))^2 \det(A^{(c,d,\beta)}(M)) d\mu_0(U) \prod_{j=1}^{n} d\zeta_i$$

is a conjugation invariant measure on the space of normal matrices

$$M = UZU^\dagger \in \text{Nor}_{n \times n}^C, \quad U \in U(n), \quad Z = \text{diag}(\zeta_1, \ldots, \zeta_n)$$

with eigenvalues $\zeta_i \in \mathbb{C}$ supported on the contour $\mathcal{C}_n$.

3.4.3 Wronskian representation: quantum case

For quantum weight generating functions $G = H_q$, and any $n \in \mathbb{N}^+$, let

$$\phi_n^{(H_q,\beta)}(e^y) = \int_{\mathcal{C}_n} A^{(c,d,\beta)}(s)e^{ys} ds,$$

$$A_{H_q,n}(z) := (-\beta)^{1-n}\Gamma(1-n-z) \prod_{m=0}^{\infty} \left( (-\beta q^m)^{-z} \frac{\Gamma(-\beta^{-1}q^{-m})}{\Gamma(z-\beta^{-1}q^{-m})} \right).$$

Define the diagonal matrix $Y = \text{diag}(y_1, \ldots, y_n)$

$$X = e^Y, \quad Y = \ln(X), \quad x_i = e^{y_i}, \quad i = 1, \ldots, n, \quad (3.24)$$

Then $\tau^{(H_q,\beta)}([X])$ becomes a ratio of Wronskian determinants

$$\tau^{(H_q,\beta)}([X]) = \gamma_n \left( \prod_{i=1}^{n} x_i^{n-1} \right) \frac{\text{det} \left( \phi_n^{(c,d,\beta)}(1) (e^y) \right)}{\Delta(e^y)} 1 \leq i, j \leq n. \quad (3.25)$$

3.4.4 Matrix integral representation of $\tau^{(G,\beta)}([X])$: quantum case

It similarly follows [7] that

$$\tau^{(H_q,\beta)}([X]) = \beta \frac{1}{2} (n-1) (\prod_{i=1}^{n} x_i^{n-1}) \Delta(\ln(x)) \frac{Z_{d\mu_q} (\ln(X))}{(\prod_{i=1}^{n} i!) \Delta(x)}, \quad (3.26)$$
where \( Z_{d\mu(q,n)}(X) = \int_{M \in \text{Nor}_{n \times n}^C} d\mu(q,n)(M) e^{\text{tr}YM} \),
and \( d\mu(q,n)(M) := (\Delta(\zeta))^2 \det(A_{H,q,n}(M)) \)

is a conjugation invariant measure on the space of normal matrices

\[ M = U Z U^\dagger \in \text{Nor}_{n \times n}^C, \quad U \in U(n), \quad Z = \text{diag}(\zeta_1, \ldots, \zeta_n) \] (3.27)

with eigenvalues \( \zeta_i \in \mathbb{C} \) supported on the contour \( \mathcal{C}_n \).

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