Homotopy classification of director fields on polyhedral domains with tangent and periodic boundary conditions, with applications to bi-stable post-aligned liquid crystal displays.

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Abstract

We obtain complete topological classification of inequivalent states of nematic liquid crystal in the geometry of periodic array of rectangular posts between two parallel slabs, with tangent or normal boundary conditions. This classification has applications in bi-stable post-aligned liquid crystal display design and have technological significance. Methods used in classification are those of algebraic topology and go beyond relative homotopy groups.

1 Introduction

This paper introduces an approach in the style of Eilenberg [1] to study homotopy classification problems which appear in the context of topological classification of states of nematic liquid crystal in polyhedral domain with tangent or a mixture of tangent and periodic boundary conditions.

Topological classification of liquid crystal configurations in $\mathbb{R}^3$ as well as in domains with smooth boundary and Dirichlet boundary conditions has been extensively studied – see, eg, Mermin [2], de Gennes and Prost [6], and Kléman [7]. For domains with smooth boundary with tangent or normal boundary conditions the local classification of defects is equivalent to computing relative homotopy groups $\pi_i$. However this is not sufficient for the classification of defects at vertices (or edges) of a polyhedron. We are also interested in the global classification of maps up to homotopy rather than in just local classification of defects. For the case of nematic liquid crystal in polyhedral cells $P \subset \mathbb{R}^3$ with tangent boundary conditions on faces, topologically inequivalent continuous configurations of nematic liquid crystal are described by homotopy classes of director fields (i.e. maps $P \to \mathbb{R}P^2$), which on faces of $P$ must be parallel to the face. Related homotopy classification of tangent maps $P \to S^2$ in the case when $P$ is
convex was obtained in [9] using explicit construction of homotopies (which is very tedious). Homotopy classification of [9] was subsequently used to analyze harmonic map problems in fixed homotopy classes [10]- [13] (for the case of point defects or holes in $\mathbb{R}^3$ with fixed degree, such harmonic map problems were studied in [11]).

The paper is organized as follows. In section 2 we introduce the necessary background and notations. We define similarly to [1] certain homotopy invariants of maps and pairs of maps on cells with values in appropriate homotopy groups (Definitions 2.1 and 2.3), and describe their basic properties and basic extension and homotopy classification results.

In section 3 we obtain homotopy classification of director fields on a contractible polyhedron $P$, with tangent boundary conditions on faces, using homotopy invariants and basic extension properties, but not explicit homotopies. Tangent boundary conditions on faces imply that on edges of $P$ director field must be parallel to the edge, and thus it must be discontinuous at vertices of $P$; we assume that it is continuous elsewhere. Homotopy classification of such discontinuous director fields on $P$ is equivalent to classification of continuous director fields on a truncated polyhedron $\hat{P}$ (Section 2.1).

For a pair of tangent maps $\Phi_0, \Phi_1$, we define homotopy invariants associated to edges of $\hat{P}$ with values in $\pi_1(RP^1)$, and use those invariants as coefficients in the formal chain $d^1(\Phi_0, \Phi_1)$ (Eq. (8)). If $d^1(\Phi_0, \Phi_1) = 0$, there exist homotopies on edges of $\hat{P}$ satisfying tangent boundary conditions, moreover we can construct certain homotopy invariants with values in $\pi_2(S^2)$, associated to faces of $\hat{P}$, using homotopies on edges to glue along the edges the restrictions of maps $\Phi_0, \Phi_1$ to faces. We show that such invariants do not depend from a choice of tangent edge homotopies, and only depend from $\Phi_0, \Phi_1$ (Proposition 3.2). Thus if $d^1(\Phi_0, \Phi_1) = 0$, we can define a formal chain $d^2(\Phi_0, \Phi_1)$ (18) with the above invariants as coefficients. We show (Theorem 3.1) that $\Phi_0, \Phi_1$ are homotopic if and only if $d^1(\Phi_0, \Phi_1) = 0$ and $d^2(\Phi_0, \Phi_1) = 0$.

Results of section 3 were first obtained for convex polyhedra in [9], using an equivalent formulation in terms of unit-vector fields and explicit homotopies construction. Basic extension and classification results allow to streamline homotopy classification, generalize it immediately to any contractible polyhedron, not just convex one, and apply the technique to study more complicated classification problems. Advantage of using explicit homotopies [9] is that it provides representative maps in each homotopy class.

We illustrate our method by providing homotopy classification of nematic liquid crystal states in the domain between two horizontal plates outside a periodic array of rectangular posts located on the bottom plate (section 4). Boundary conditions are tangent on the bottom plate and the post surface, and periodic with respect to integer translations in horizontal directions, with normal or tangent boundary conditions on the top plate. This problem is important due to applications in new bi-stable liquid crystal displays [18]. In [14] configurations of nematic of several simple topological types were studied in such geometry, and energy-minimising configurations were found numerically to be smooth away from the vertices. In this paper we provide complete topological classification of states continuous away from the vertices in such geometry (Theorem 4.1).
In the paper we assume that director fields are as continuous as possible, so they are continuous except at vertices where tangent boundary conditions disallow continuity. More general types of singularities can be classified using similar techniques.

In this paper we are focusing on homotopy classification problems for nematic liquid crystals. However, similar techniques may be used algorithmically for boundary-value problems for other ordered media, such as spin systems, superfluid helium-3, etc. Indeed, in our approach all the relevant information about the target space is contained in its homotopy groups.

Notations. Throughout the paper, we use symbol $\simeq$ to denote homotopic maps. We use symbol $\cong$ to denote homeomorphic spaces. If $\Phi : X \to Y$ and $W$ is the universal covering space for $Y$, we denote by $\Phi^1$ the lifted map $X \to W$; such lifted map is determined by lifting at one point. In our examples motivated by nematic liquid crystals, $Y = RP^2$, and the universal covering space is $S^2$. We denote by $\epsilon_{\alpha\beta}$ the antisymmetric tensor with $\epsilon_{12} = 1$.

2 Homotopy Invariants, Extension and Classification

In the paper we will be concerned with homotopy classification of maps of a polyhedron to $RP^2$, describing topologically inequivalent states of nematic liquid crystal, with tangent or periodic boundary conditions on faces. In this section we introduce some notations, and briefly outline a more general set-up in which it is natural to address such classification problems. We largely follow notations and terminology of Eilenberg paper [1], which is simple, very readable, and is straight to the point. A physicist-oriented introduction to the subject, with lots of pictures drawn, is an excellent review by Mermin [5]. More pedantic definitions of the main objects can be found e.g. in [3].

An open $k$-cell $\sigma^k$ in $\mathbb{R}^N$ is a subspace homeomorphic to an open $k$-disk. A finite cell complex $X$ is a space (we may consider it to be a subspace of $\mathbb{R}^N$ for large enough $N$), which can be partitioned into finitely many cells, so that for every cell $\sigma^k \subset X$ there is a continuous map $\chi_{\sigma^k}$ of a closed $k$-disk $D^k$ into $X$, which maps interior of $D^k$ homeomorphically onto $\sigma^k$, and maps $S^{k-1}$, the boundary of $D^k$ onto a union of cells of dimension less then $k$. If $K$ is a cell complex, the $n$-skeleton $K_n$ is a subcomplex consisting of all cells of dimension $\leq n$.

We denote by $E$ the closed interval $[0, 1]$. For a set $A$, we denote by $\bar{A}$ the closure of $A$, and by $A_0$ the interior of $A$.

We denote by $(\sigma^k)^*$ the $(k-1)$-sphere parameterizing the boundary of a cell $\sigma^k$. Given a map $\Phi : X \to Y$, its restriction to $(\sigma^k)^*$, denoted $\Phi|_{(\sigma^k)^*}$ is a map $\Phi \circ \chi_{\sigma^k} : S^{k-1} \to Y$. If $W \subset X$ is such that its interior $W_0$ is a cell $\sigma^k$, by $(W)^*$ we mean $(W_0)^*$.

Two maps $\Phi_0, \Phi_1 : S^n \to Y$ are (free) homotopic, if there is a continuous map $\Phi : S^n \times E \to Y$, such that $\Phi|_{(S^n \times \{m\})} = \Phi_m$, $m = 0, 1$. Given basepoints $s_0 \in S^n, y_0 \in Y$, two maps $\Phi_0, \Phi_1 : S^n \to Y, \Phi_0(s_0) = \Phi_1(s_0) = y_0$ are homotopic maps between spaces with basepoints $(S^n, s_0) \to (Y, y_0)$ if there is a map $\Phi : S^n \times E \to Y$, such that $\Phi|_{(S^n \times \{m\})} = \Phi_m$, $m = 0, 1$ and $\Phi|_{\{s_0\} \times E} = y_0$. Homotopy classes of maps $(S^n, s_0) \to (Y, y_0)$ is a group $\pi_n(Y, y_0)$. A
topological space $Y$ is called $n$-simple, if any free homotopy class of maps $S^n \rightarrow Y$ contains exactly one homotopy class of maps between spaces with basepoints $(S^n, \sigma_0) \rightarrow (Y, y_0)$. If $Y$ is $n$-simple we can write $\pi_n(Y)$ unambiguously to describe free homotopy classes $S^n \rightarrow Y$ regardless of a basepoint.

**Definition 2.1.** Let $\sigma^k$ be a k-cell, $Y$ a $k$-simple topological space, and $\Phi_0, \Phi_1$ maps $\bar{\sigma}^k \rightarrow Y$ which agree on the boundary, $\Phi_0|((\sigma^k)\cdot) = \Phi_1|((\sigma^k)\cdot)$. Let $h_+$ be a homeomorphism of the closed northern hemisphere of $S^k$ to a disk $D^k$, preserving the orientation, and $h_-$ a homeomorphism of the closed southern hemisphere of $S^k$ to $D^k$, reversing the orientation, and agreeing with $h_+$ on the equator. Let $(\Phi_0, \Phi_1, \sigma^k, Y)$ be the map $S^k \rightarrow Y$ given by $\Phi_0 \circ \chi_{\sigma^k} \circ h_+$ in the northern hemisphere, and by $\Phi_1 \circ \chi_{\sigma^k} \circ h_-$ in the southern hemisphere, see Figure 1. We define $\text{d}(\Phi_0, \Phi_1, \sigma^k, Y)$ (1)

$$d(\Phi_0, \Phi_1, \sigma^k, Y)$$

to be the element of $\pi_k(Y)$, corresponding to the map $(\Phi_0, \Phi_1, \sigma^k, Y)$. If $W_0$, the interior of $W \subset X$, is a cell $\sigma^k$, by $d(\Phi_0, \Phi_1, W, Y)$ we mean $d(\Phi_0, \Phi_1, W_0, Y)$.

![Figure 1: $(\Phi_0, \Phi_1, \sigma^k, Y)$](image)

**Definition 2.2.** Let $\sigma^k$ be a k-cell, $Y$ a $(k-1)$-simple topological space, and $\Phi$ a map $(\sigma^k)\cdot \rightarrow Y$. We denote by $c^k(\Phi, \sigma^k, Y)$ the element of $\pi_{k-1}(Y)$ corresponding to $\Phi|((\sigma^k)\cdot)$. If $W_0$, the interior of $W \subset X$, is a cell $\sigma^k$, we define $c^k(\Phi, W, Y)$ as $c^k(\Phi, W_0, Y)$.

Let us cut $(k-1)$-sphere $(\sigma^k)\cdot$ into two disks $D_{+}^{k-1}$, $D_{-}^{k-1}$. Let $h$ be a homeomorphism from $D_{+}^{k-1}$ to $D_{-}^{k-1}$, which is identity on the boundary of $D_{+}^{k-1}$. It’s clear that

$$c^k(\Phi, \sigma^k, Y) = d(\Phi|D_{+}^{k-1}, \Phi|D_{-}^{k-1} \circ h, D_{+}^{k-1}, Y).$$

(2)

In the paper we are concerned with $Y = \text{RP}^2$, since mean direction of molecules of a nematic liquid crystal is described by a director, that is a point in $\text{RP}^2$. $\text{RP}^2$ is $n$-simple for $n = 1, 3$, but not for $n = 2$ (nontrivial element of $\pi_1(\text{RP}^2)$ acts as $(-1)$ on $\pi_2(\text{RP}^2)$). Thus $d(\Phi_0, \Phi_1, \sigma^k, \text{RP}^2)$ is defined by (1) for $k = 1, 3$ but not for $k = 2$, and $c^k(\Phi, \sigma^k, \text{RP}^2)$ is defined by Definition 2.2 for $k = 2, 4$, but not for $k = 3$. We resolve this difficulty by lifting a map $S^2 \rightarrow \text{RP}^2$ to a map to
$S^2$, the universal covering space for $RP^2$, which is $n$-simple for any $n$. Indeed, by the covering homotopy lemma (see eg [15]), a map $\phi : X \to RP^2$ with $\pi_0(X) = \pi_1(X) = 0$ (in particular $X = S^2$) can be lifted to a map $\phi^\uparrow : X \to S^2$. Such lifting is uniquely determined by lifting at one point, and denote by $\pi_2(S^2)$ is a 2:1 map, there are two ways of lifting the map at one point. Given a map $(\Phi, \sigma^2, RP^2) : S^2 \to RP^2$, we lift it to a map $S^2 \to S^2$ by choosing one of the two possible liftings at one point, and denote by

$$d^\uparrow(\Phi_0, \Phi_1, \sigma^2, S^2) \equiv d(\Phi_0^\uparrow, \Phi_1^\uparrow, \sigma^2, S^2)$$

the element of $\pi_2(S^2)$ corresponding to the lifted map.

Similarly, given a map $\Phi|\sigma^3 : S^2 \to RP^2$, we lift it to a map $\Phi^\uparrow|\sigma^3 : S^2 \to S^2$. $c^\uparrow(\Phi^\uparrow, \sigma^3, S^2)$ is the element of $\pi_2(S^2)$ corresponding to the lifted map.

Due to the choice of lifting at one point, $d^\uparrow(\Phi_0, \Phi_1, \sigma^2, S^2)$, $c^\uparrow(\Phi^\uparrow, \sigma^3, S^2)$ are in general defined up to a sign, but are uniquely defined if those elements are zero.

Let $A$ be a subspace of $X$. We recall that maps $\Phi_0, \Phi_1 : X \to Y$, $\Phi_0|A = \Phi_1|A$ are homotopic relative to $A$ (denoted $\Phi_0 \simeq_{rel} \Phi_1$) if there is $\Phi : X \times E \to Y$, such that

$$\Phi|X \times \{i\} = \Phi_i, \quad i = 0, 1;$$

$$\Phi|A \times E = \Phi_0|A = \Phi_1|A.$$

Proposition 2.1.

1. a) Let $\Phi_0, \Phi_1 : \sigma^k \to Y$, where $Y$ is $k$-simple (e.g. $Y = RP^2$, $k = 1, 3$), and $\Phi_0|\sigma^k \cdot = \Phi_1|\sigma^k \cdot$. Then

$$\Phi_0 \simeq_{rel} \Phi_1 \sigma^k \Leftrightarrow d(\Phi_0, \Phi_1, \sigma^k, Y) = 0. \quad (4)$$

b) Let $\Phi_0, \Phi_1 : \sigma^2 \to RP^2$ and $\Phi_0|\sigma^2 \cdot = \Phi_1|\sigma^2 \cdot$. Then

$$\Phi_0 \simeq_{rel} \Phi_1 \sigma^2 \Leftrightarrow d(\Phi_0^\uparrow, \Phi_1^\uparrow, \sigma^2, S^2) = 0,$$

where $\Phi_0^\uparrow, \Phi_1^\uparrow$ are lifted maps to $S^2$, such that $\Phi_0^\uparrow|\sigma^2 \cdot = \Phi_1^\uparrow|\sigma^2 \cdot$.

2. Let $\Phi_0, \Phi_1, \Phi_2$ be maps $\sigma^k \to Y$, $\Phi_0|\sigma^k \cdot = \Phi_1|\sigma^k \cdot = \Phi_2|\sigma^k \cdot$, and let $Y$ be $k$-simple. Then

$$d(\Phi_0, \Phi_1, \sigma^k, Y) = -d(\Phi_1, \Phi_0, \sigma^k, Y), \quad (5)$$

$$d(\Phi_0, \Phi_1, \sigma^k, Y) + d(\Phi_1, \Phi_2, \sigma^k, Y) = d(\Phi_0, \Phi_2, \sigma^k, Y). \quad (6)$$

The proof is standard and can be found e.g. in [1].
3 Homotopy Classification of Tangent Director Fields on Polyhedra

3.1 Tangent director fields on polyhedra.

Let $P \subset \mathbb{R}^3$ be a contractible polyhedron, $K_{\text{sing}}$ the set of vertices of $P$, and $P = P \setminus K_{\text{sing}}$. We consider continuous maps $\Phi : P \to Y = \mathbb{R}P^2$, director fields on $P$. We associate to faces $\sigma_i^2$ of $\tilde{P}$ subspaces $Y_i^1 \cong \mathbb{R}P^1 = S^1 \subset \mathbb{R}P^2$ of directors parallel to the face $\sigma_i^2$. We call $\Phi$ a tangent map if the restriction of $\Phi$ to any face $\sigma_i^2$ of $P$ is a map to the subspace $Y_i^1$.

Tangent maps $P \to \mathbb{R}P^2$ can be lifted to maps to the universal covering space $S^2$, such that restrictions of the maps to any face of $P$ is a map to the great circle of $S^2$ parallel to the face. Truncated polyhedron $\hat{P}$ Truncated polyhedron $\hat{P}$ is obtained by removing from $\tilde{P}$ small 3-cells incident to vertices of $\tilde{P}$. For a vertex $v_i$ of $\tilde{P}$, let $b_i = B_{v_i, \epsilon_i} \cap \tilde{P}$ where $B_{v_i, \epsilon_i}$ is a ball with center at $v_i$, and with radius $\epsilon_i$ small enough so that $b_i$ does not intersect any edges or faces not incident to the vertex $v_i$, and $b_i \cap b_j = \emptyset, i \neq j$. Then $\hat{P} = \overline{P \setminus \cup_i b_i}$. We view $\hat{P}$ as a cell complex. 0-cells $\hat{P}_0$ in this complex are vertices of $\hat{P}$. 1-cells are edges of $\hat{P}$; they are of two kind, truncated edges $\sigma^{1t}$, which are parts of edges of $P$, and cleaved edges $\sigma^{1c}$, introduced by removing 3-cells around vertices. 2-cells are faces of $\hat{P}$; they are of two kind: truncated faces $\sigma^{2t}$, which are parts of faces of $P$, and cleaved faces $\sigma^{2c}$, introduced by removing 3-cells around vertices (Figure 2). There is just one 3-cell, $\hat{P}$ itself. A map $\hat{P} \to \mathbb{R}P^2$ is called tangent, if it is tangent on truncated faces of $\hat{P}$ (there is no restriction on cleaved faces).

![Figure 2: Truncated polyhedron $\hat{P}$. Cleaved edges $\sigma^{1c}$, faces $\sigma^{2c}$ correspond to dark regions, while truncated edges $\sigma^{1t}$, faces $\sigma^{2t}$ correspond to white regions.](image)

**Proposition 3.1.** Two tangent maps $\Phi_0, \Phi_1 : P \to \mathbb{R}P^2$ are homotopic if and only if their restrictions to $\hat{P}$ are homotopic.
For unit-vector fields, it is shown in Proposition 2.1 in [9]. Generalization to director fields is straightforward.

### 3.2 Homotopy Classification

Let $\Phi_0, \Phi_1$ be two tangent maps $\hat{P} \to Y = RP^2$. Such maps necessarily agree on truncated edges, and on vertices of $\hat{P}$, $\Phi_0|\hat{P}_0 = \Phi_1|\hat{P}_0$, due to tangent boundary conditions. Let $\sigma_i^{1c}$ be a cleaved edge, and let $\sigma_{s(i)}^{2r}$ denote the truncated face incident to $\sigma_i^{1c}$. Since $\Phi_0|\hat{P}_0 = \Phi_1|\hat{P}_0$, we can define $d(\Phi_0, \Phi_1, \sigma_i^{1c}, Y_{s(i)}^1) \in \pi_1(Y_{s(i)}^1)$ as in (1). It follows from Proposition 2.1 that

$$\Phi_0|\sigma_i^{1c} \simeq \Phi_1|\sigma_i^{1c} \text{ rel. } (\sigma_i^{1c} \cdot \Leftrightarrow d(\Phi_0, \Phi_1, \sigma_i^{1c}, Y_{s(i)}^1) = 0. \tag{7}$$

Define a formal linear combination

$$d^1(\Phi_0, \Phi_1) = \sum_i d(\Phi_0, \Phi_1, \sigma_i^{1c}, Y_{s(i)}^1)\sigma_i^{1c}. \tag{8}$$

(The summation is over cleaved edges of $\hat{P}$. Note that $\Phi_0$ and $\Phi_1$ coincide and are constant on truncated edges, thus $d(\Phi_0, \Phi_1, \sigma_i^{1r}, Y) = 0$.) It follows from Proposition 2.1 that

$$\Phi_0|\hat{P}_1 \simeq \Phi_1|\hat{P}_1 \text{ rel. } \hat{P}_0 \Leftrightarrow d^1(\Phi_0, \Phi_1) = 0. \tag{9}$$

Thus if $d^1(\Phi_0, \Phi_1) = 0$, there exist tangent homotopy $H_{\Phi_0, \Phi_1}^1: \hat{P}_1 \times E \to RP^2$, such that

\begin{align*}
H_{\Phi_0, \Phi_1}^1|\hat{P}_1 \times \{m\} & = \Phi_m|\hat{P}_1, m = 0, 1, \\
H_{\Phi_0, \Phi_1}^1(\hat{P}_1 \times \{m\} \times E) & \subset Y_{s(i)}^1 \cong S^1 \text{ on cleaved edges,} \\
H_{\Phi_0, \Phi_1}^1(\hat{P}_1 \times \{m\} \times E) & \subset \hat{P}_0 \times \{m\} \times E \supset \hat{P}_1 \times \{m\} \times E,
\end{align*}

where $q_j$ correspond to the constant director parallel to the edge $\sigma_j^{1r}$. Assume that $d^1(\Phi_0, \Phi_1) = 0$. For a face $\sigma^2$ define a map $\Phi$ on $(\sigma^2 \times E) \cdot = S^2$ by

$$\Phi|(\sigma^2 \times \{m\}) = \Phi_m, m = 0, 1;$$
$$\Phi|(\sigma_i^1 \times E) = H_{\Phi_0, \Phi_1}|(\sigma_i^1 \times E), \quad \sigma_i^1 \subset (\sigma^2)\cdot. \tag{11}$$

We can lift maps $\Phi_0$, $\Phi_1$, and $\Phi$ to maps to $S^2$, $\Phi_0^\dagger$, $\Phi_1^\dagger$, $\Phi_1$. Such lifting is uniquely determined by selecting one of the two possible values for the lifting at a point $t_p$ on a truncated edge $\sigma_p^{1r}$ of $\hat{P}$; we choose this value to be the same for $\Phi_0^\dagger$ and $\Phi_1^\dagger$. The lifted maps $\Phi_0^\dagger$ and $\Phi_1^\dagger$ will be constant and equal on $\sigma_p^{1r}$,

$$\Phi_0^\dagger|t_p = \Phi_1^\dagger|t_p = \Phi_0^\dagger|\sigma_p^{1r} = \Phi_1^\dagger|\sigma_p^{1r} = \hat{e}_p, \tag{12}$$

$\hat{e}_p \in S^2$. It then follows from (12) and $d^1(\Phi_0, \Phi_1) = 0$ that

$$\Phi_0^\dagger|\sigma_i^{1r} = \Phi_1^\dagger|\sigma_i^{1r} = \hat{e}_i. \tag{13}$$
for all truncated edges. (Unit vectors \( \hat{e}_i \) are parallel to respective \( \hat{\sigma}_i^{1t} \) and determined up to an overall sign affecting all truncated edges, corresponding to two possible liftings in (12). \( \hat{e}_i \) are called edge orientations in \([9]\).) We lift the map \( \Phi|_{(\sigma^2 \times E)} \cdot \) in (11) to a map to \( S^2 \), \( \Phi^1|_{(\sigma^2 \times E)} \cdot \) by requiring that

\[
\Phi^1(t_i \times E) = \Phi_0^1(t_i) = \Phi_1^1(t_i) = \hat{e}_i, \quad t_i \in \hat{\sigma}_i^{1t} \cap \sigma^2.
\] (14)

Such lifting defined for different faces is consistent on common edges, and defines the lifting of \( H_{\Phi_0, \Phi_1}^1 \) to a map \( H_{\Phi_0, \Phi_1}^{11} : \hat{P}_1 \times E \rightarrow S^2 \),

\[
H_{\Phi_0, \Phi_1}^{11}|(\sigma_i^1 \times E) = \Phi^1|(\sigma_i^1 \times E), \quad \hat{\sigma}_i^1 \subset (\sigma^2)^*.
\] (15)

**Proposition 3.2.** Let \( \Phi_0, \Phi_1 : \hat{P} \rightarrow RP^2 \) be tangent maps with \( d^1(\Phi_0, \Phi_1) = 0 \), \( \Phi \) is given by (11), and the lifted to \( S^2 \) maps \( \Phi_0^1, \Phi_1^1 \) are defined by (12)-(14).

1. Let \( \sigma^{2c} \) be a cleaved face.
   a) \( c^3(\Phi^1, \sigma^{2c} \times E, S^2) \) does not depend from the choice of tangent homotopy \( H_{\Phi_0, \Phi_1}^1 \) in (11).
   b) \( c^3(\Phi^1, \sigma^{2c} \times E, S^2) = 0 \) if and only if the map \( \Phi \) can be extended to a map \( H_{\Phi_0, \Phi_1}^{2c} : \hat{\sigma}^{2c} \times E \rightarrow RP^2 \).
   The latter is equivalent to \( \Phi_0|_{\hat{\sigma}^{2c}} \simeq \Phi_1|_{\hat{\sigma}^{2c}} \).

2. Let \( \sigma_i^{2r} \) be a truncated face, let \( S_i^1 \) be the great circle of \( S^2 \) parallel to \( \sigma_i^{2r} \), then

\[
c^3(\Phi^1, \sigma_i^{2r} \times E, S_i^1) \equiv 0.
\] (16)

It follows that the tangent homotopy \( H_{\Phi_0, \Phi_1}^1 \) can always be extended to homotopies on truncated faces \( H_{\Phi_0, \Phi_1}^{2r,i} : \hat{\sigma}_i^{2r} \times E \rightarrow Y_i^1 \).

**Proof.** Let \( H_{\Phi_0, \Phi_1}^1 \), \( \hat{H}_{\Phi_0, \Phi_1}^1 \) be two tangent edge homotopies, which by construction satisfy \( H_{\Phi_0, \Phi_1}^1|(\hat{P}_1 \times E) \cdot \) = \( \hat{H}_{\Phi_0, \Phi_1}^1|(\hat{P}_1 \times E) \cdot \). Let \( \tilde{\Phi} \) be given by (11), with \( H_{\Phi_0, \Phi_1}^1 \) replaced by \( \hat{H}_{\Phi_0, \Phi_1}^1 \), and \( \tilde{\Phi}^1 \), \( \hat{H}_{\Phi_0, \Phi_1}^{11} \) lifted to \( S^2 \) maps, \( \tilde{\Phi}^1|_{\sigma_i^{1t} \times E} = \hat{H}_{\Phi_0, \Phi_1}^{11}|_{\sigma_i^{1t} \times E} = \Phi^1|_{\sigma_i^{1t} \times E} = \Phi_0^1|_{\sigma_i^{1t} \times E} = \Phi_1^1|_{\sigma_i^{1t} \times E} \) on all truncated edges \( \sigma_i^{1t} \).
1a). By construction \( H^1_{\Phi_0, \Phi_1}, \tilde{H}^1_{\Phi_0, \Phi_1} \) are constant, equal maps on truncated edges. On cleaved edges, \( H^1_{\Phi_0, \Phi_1} | (\sigma_i^{1c} \times E)^\ast = \tilde{H}^1_{\Phi_0, \Phi_1} | (\sigma_i^{1c} \times E)^\ast \). It follows from (2) and (3) that
\[
c^3(\Phi^1, \sigma^{2c} \times E, S^2) - c^3(\tilde{\Phi}^1, \sigma^{2c} \times E, S^2) = \sum_{i: \sigma_i^{1c} \subset \sigma^{2c}} d \left( H^1_{\Phi_0, \Phi_1}, \tilde{H}^1_{\Phi_0, \Phi_1}, \sigma_i^{1c} \times E, S^2 \right) = 0. \tag{17}
\]
The last equality is due to the fact that on \( \sigma_i^{1c} \times E, H^1_{\Phi_0, \Phi_1} \), \( \tilde{H}^1_{\Phi_0, \Phi_1} \) are maps to \( S^1 \), the great circle of \( S^2 \) parallel to the truncated face containing \( \sigma_i^{1c} \), and \( \pi_2(S^1) = 0 \).

1b). \( (\sigma^{2c} \times E)^\ast \cong S^2 \), and if the element of \( \pi_2(S^2) \) corresponding to \( \Phi^1 : (\sigma^{2c} \times E)^\ast \rightarrow S^2 \) is zero, then \( \Phi^1 \) can be extended to a map of \( \tilde{\sigma}^{2c} \times E \cong D^3 \rightarrow S^2 \). Such an extension followed by the projection from \( S^2 \) to \( RP^2 \) provides the required homotopy.

2. \( \pi_2(S^1) = 0 \), thus a map \( \Phi^1 |(\sigma^{2t} \times E)^\ast \rightarrow S^1 \) can be always extended to a map \( \tilde{\sigma}^{2t} \times E \rightarrow S^1 \). Such an extension followed by the projection to \( Y_i \subset RP^2 \) provides the required homotopy. \( \square \)

**Definition 3.1.** Assume that \( d^1(\Phi_0, \Phi_1) = 0 \). We define \( d^2(\Phi_0, \Phi_1) \) by
\[
d^2(\Phi_0, \Phi_1) = \sum_{i} c^3(\Phi^1, \sigma^{2c}_i \times E, S^2) \sigma^{2c}_i, \tag{18}\]
where \( c^3(\Phi^1, \sigma^{2c}_i \times E, S^2) \) is defined on cleaved faces by [11]–[14]. \( d^2(\Phi_0, \Phi_1) \) does not depend from the choice of edge homotopy \( H^1_{\Phi_0, \Phi_1} \), and is defined up to an overall sign, as there are two ways of lifting in [12]–[14].

If \( d^1(\Phi_0, \Phi_1) = 0 \) and \( d^2(\Phi_0, \Phi_1) = 0 \), by Proposition 3.2 there exist a tangent homotopy
\[
H^2_{\Phi_0, \Phi_1} : \tilde{P}_2 \times E \rightarrow RP^2, H^2_{\Phi_0, \Phi_1}|\tilde{P}_2 \times \{m\} = \Phi_{m}|\tilde{P}_2, m = 0, 1, \tag{19}\]
given by \( H^2_{\Phi_0, \Phi_1}|\tilde{\sigma}^{2t} \times E = H^2_{\Phi_0, \Phi_1}|\tilde{\sigma}^{2t} \times E \), \( a = c, t (H^2_{\Phi_0, \Phi_1} \) were introduced in Proposition 3.2). Moreover, by construction
\[
H^2_{\Phi_0, \Phi_1}|\tilde{P}_1 \times E = H^1_{\Phi_0, \Phi_1}. \tag{20}\]

**Remark 3.1.** In terms of invariants in [9], \( d^1(\Phi_0, \Phi_1) = 0 \) means that kink numbers of \( \Phi_0, \Phi_1 \) are the same, and edge orientations are the same up to a simultaneous change of sign on all edges. \( d^2(\Phi_0, \Phi_1) = 0 \) means that \( \Phi_0, \Phi_1 \) have the same wrapping numbers on cleaved faces.

**Theorem 3.1.** Tangent maps \( \Phi_0, \Phi_1 \) are homotopic if and only if
\[
d^1(\Phi_0, \Phi_1) = 0, \tag{21}\]
\[
d^2(\Phi_0, \Phi_1) = 0. \tag{22}\]
proof. It is clear that \((21)\), \((22)\) are necessary. To show they are sufficient, note that from Proposition 3.2 there exist face homotopy \(H_{\Phi_0,\Phi_1}^2\). Define a map \(\Phi\) on \((\hat{P} \times E)^* \cong S^3\) by

\[
\Phi|((\hat{P} \times \{m\}) = \Phi_m, m = 0, 1, \\
\Phi|((\sigma_0^2 \times E) = H_{\Phi_0,\Phi_1}^2|\sigma_0^2 \times E = \hat{H}_{\Phi_0,\Phi_1}^{2a,1}, \sigma_0^2 \subset (\hat{P})^*.
\]

(23)

Let \(c^4 \equiv c^4(\Phi, \hat{P} \times E, RP^2)\) be the corresponding element of \(\pi_3(RP^2) = \mathbb{Z}\). If \(c^4 = 0\), \(\Phi\) can be extended to a map of \((\hat{P} \times E)\) to \(RP^2\), and thus \(\Phi_0\) and \(\Phi_1\) are homotopic. \(c^4\) depends from the maps \(\Phi_0, \Phi_1\), and face homotopy \(H_{\Phi_0,\Phi_1}^2\) and in general need not to be zero. However, we can always modify \(H_{\Phi_0,\Phi_1}^2\) on \(\sigma_0^2 \times E_0\) for a selected cleaved face \(\sigma_0^2\), so that \(c^4\) becomes zero. This process may be described briefly as gluing an \(S^3\) to a point \(p \in \sigma_0^2 \times E_0\), and taking a map on \(S^3\) such that the corresponding element of \(\pi_3(RP^2)\) equals \(-c^4\). In more detail, we first make some room for such modification, by taking a point \(p \in \sigma_0^2 \times E_0\) and expanding it to a ball \(b^3_p \subset \sigma_0^2 \times E_0\). It is clear that there is a map \(\hat{H}_{\Phi_0,\Phi_1}^{2c}\) homotopic to \(H_{\Phi_0,\Phi_1}^2\), constant on \(b^3_p\), \(\hat{H}_{\Phi_0,\Phi_1}^{2c}|b^3_p \equiv q = H_{\Phi_0,\Phi_1}^{2c}(p)\) and unchanged on the boundary, \(\hat{H}_{\Phi_0,\Phi_1}^{2c}|(\sigma_0^2 \times E)^* = H_{\Phi_0,\Phi_1}^{2c}|(\sigma_0^2 \times E)^*\). (Indeed, up to a homeomorphism we can take \(\sigma_0^2 \times E\) to be a 3-ball \(B^3\) centered at \(p\) and \(b^3_p\) a ball of smaller radius with the same center; then on the spherical shell \(B^3 \setminus b^3_p\) we take \(\hat{H}_{\Phi_0,\Phi_1}^{2c}\) to be a radially rescaled map \(H_{\Phi_0,\Phi_1}^{2c}\)). We now modify \(\hat{H}_{\Phi_0,\Phi_1}^{2c}\) on \(b^3_p\); let \(\varphi_{-n}\) be a map \(b^3_p \to RP^2\), with \((b^3_p)^* \mapsto q \in RP^2\), such that the element of \(\pi_3(RP^2)\) corresponding to this map is \(-n = -c^4\). Define \(\tilde{\Phi}\) by

\[
\tilde{\Phi}|((\sigma_0^2 \times E) \setminus b^3_p) = \hat{H}_{\Phi_0,\Phi_1}^{2c}, \\
\tilde{\Phi}|b^3_p = \varphi_{-n}, \\
\tilde{\Phi}|((\hat{P} \times E)^* \setminus (\sigma_0^2 \times E_0)) = \Phi.
\]

(24)

Then \(\tilde{c}^4 \equiv c^4(\Phi, \hat{P} \times E, RP^2) = 0\), and so \(\tilde{\Phi}\) can be extended to a map \(\hat{P} \times E \to RP^2\), providing a tangent homotopy between maps \(\Phi_0\) and \(\Phi_1\).

Condition \(d^1(\Phi_0, \Phi_1) = 0\) in \((21)\) is equivalent to \(d(\Phi_0, \Phi_1, \sigma_i^{1c}, Y^{1}) = 0\) for all cleaved edges \(\sigma_i^{1c}\). Those conditions are not independent, there is one relation \((26)\) among them per every
truncated face. Similarly, condition $d^2(\Phi_0, \Phi_1) = 0$ in (22) is equivalent to $d(\Phi^1_0, \Phi^1_1, \sigma_i^{2c}, S^2) = 0$ for all cleaved faces $\sigma_i^{2c}$; there is one overall relation (28) between those conditions.

**Proposition 3.3. (Sum rules).**

1. Let $\sigma_i^{2r}$ be a truncated face. Orient cleaved edges $\sigma_j^{1c}$ of $\sigma_i^{2r}$ consistent with $\partial \sigma_i^{2r}$. Let $\Phi_0$ and $\Phi_1$ be two tangent maps. Since restrictions of $\Phi_0$ and $\Phi_1$ to $\sigma_i^{2r}$ are continuous, tangent maps to $Y_i^1$, we have that

$$c^2(\Phi_m, \sigma_i^{2r}, Y_i^1) = 0, m = 0, 1.$$  

(25)

Using addition formula (6), we get

$$\sum_{j: \sigma_j^{1c} \in (\sigma_i^{2r})} d(\Phi_0, \Phi_1, \sigma_j^{1c}, Y_i^1) = c^2(\Phi_0, \sigma_i^{2r}, Y_i^1) - c^2(\Phi_1, \sigma_i^{2r}, Y_i^1) = 0.$$  

(26)

2. Continuity of $\Phi_0$ and $\Phi_1$ on $\hat{P}$ implies

$$c^3(\Phi_m^1, \hat{P}, S^2) = 0, m = 0, 1,$$  

(27)

where $\Phi_m^1$ is lifted to $S^2$ map. Assume that $d^1(\Phi_0, \Phi_1) = 0$. Orient cleaved faces $\{\sigma_i^{2c}\}$ consistent with $\partial \hat{P}$. Then using (2), (6), (16) we get

$$\sum_{\sigma_i^{2c} \in (\hat{P})} c^3(\Phi^1, \sigma_i^{2c} \times E, S^2) = \sum_{\sigma_i^{2c} \in (\hat{P})} c^3(\Phi^1, \sigma_i^{2c} \times E, S^2) = c^3(\Phi_0^1, \hat{P}, S^2) - c^3(\Phi_1^1, \hat{P}, S^2) = 0.$$  

(28)

4 **Topological classification of post-aligned nematic**

We now consider topological classification of a nematic liquid crystal in a domain $D$ between two horizontal plates and outside a periodic array of rectangular posts situated on the bottom plate, Figure 5. Boundary conditions are tangent on the bottom plate and the post surface, and periodic with respect to integer translations in horizontal $X,Y$ directions. We consider normal (N) or tangent (T) boundary conditions on the top plate. Thus we consider homotopy classification of maps $D \to RP^2$, i.e. director fields, satisfying the above boundary conditions, and continuous, except at the vertices of rectangular post (where boundary conditions disallow continuity).

Let $C$ be the fundamental domain of $D$ with respect to horizontal translations, i.e. a part of $D$ inside a rectangular prism with unit length and width, and $\hat{C}$ be its truncated version, $\hat{C} = C \setminus \cup_i b_i$, where $b_i = B_{i,\epsilon_i} \cap C$, and $B_{i,\epsilon_i}$ are small enough open 3-balls centered at the vertices of the rectangular post. As in Proposition 2.1, two maps $C \to RP^2$ are homotopic if and only if their restrictions to the truncated region $\hat{C}$ are homotopic.
It is clear that a periodic in $X, Y$ map $\Phi : \hat{C} \to RP^2$ can be lifted to a map $\Phi^\uparrow : \hat{C} \to S^2$, which is periodic or anti-periodic with respect to $X, Y$ translations. Indeed, extend the map a periodic map of the universal covering space of $\hat{C}$ (the truncated version of $D$), lift it to a map to $S^2$, and then restrict to $\hat{C}$. Since the composition of the resulting map with the projection to $RP^2$, identifying diametrically opposite points of $S^2$, is periodic, $\Phi^\uparrow$ is either periodic or anti-periodic.

Let $\Phi_0, \Phi_1$ be two maps $\hat{C} \to RP^2$ satisfying boundary conditions. It is clear that for $\Phi_0$ and $\Phi_1$ to be homotopic, we must have

$$d_1^P(\Phi_0, \Phi_1) = 0, \quad d_2^P(\Phi_0, \Phi_1) = 0,$$

(29)

where $d_1^P(\Phi_0, \Phi_1), d_2^P(\Phi_0, \Phi_1)$ are invariants associated with the vertices of the post and constructed as in Section 3.

Let $\mathcal{K}$ denote the surface of the prism bounding $\hat{C}$. We can assume it is a unit prism with the center at the origin. We label vertices of $\mathcal{K}$ as $\tau_{0\alpha\beta\gamma}^\pm = (\xi_1^\pm, \xi_2^\pm, \xi_3^\pm)$, $\xi_i = \pm$. Here and below let $\alpha, \beta, \gamma$ be a cyclic permutation of $1, 2, 3$. We label an edge $x_\beta = \xi_\beta^1/2$, $x_\gamma = \xi_\gamma^1/2$ as $\tau_{\alpha\beta\gamma}^1$, $\xi_\beta, \xi_\gamma = \pm$. We label a face $x_\alpha = \xi_\alpha^1/2$ as $\tau_{\alpha\beta\gamma}^2$. Our maps are not defined on the footprint of the truncated post on the bottom face of $\mathcal{K}$; this however will not play any role. In fact, due to the periodic boundary conditions and the addition formula (2), (6) we have that $c^2(\Phi_0, \tau_{3-}^1, S^1) = c^2(\Phi_1, \tau_{3-}^1, S^1) = 0$, thus restrictions of maps $\Phi_0, \Phi_1$ to the bottom plate have tangent extensions from the boundary of the truncated post footprint to its interior.

Select a point $p_0$ on a truncated edge, e.g. $p_0 \in (p_1, p_4)$ (Figure 5). Let $\gamma_- \subset \hat{C}$ be a path on the bottom plate from $p_0$ to the vertex $q_1 \equiv \tau_0^{0-\tau}$, which does not intersect any cleaved
regions, and let $\gamma_+$ be the path from $q_1$ to $q_5$ along the vertical edge $\tau^1_{-\pm}$. Let $H_0^-, H_0^+$ be paths $E \to Y^1_u \cong S^1$ in the space of directors parallel to horizontal plane such that

$$H_0^-(m) = \Phi_m(q_1) = \Phi_m(\tau^0_{\xi_1 \xi_2^-}), \quad H_0^+(m) = \Phi_m(q_5) = \Phi_m(\tau^0_{\xi_1 \xi_2^+}), m = 0, 1. \tag{30}$$

Define $\Phi : \gamma \times E \cdot \cong S^1 \to Y^1_u \cong S^1$, $(\gamma_+ \times E) \cdot \cong S^1 \to RP^2$ by

$$\begin{align*}
\Phi|\gamma_+ \times \{m\} &= \Phi_m|\gamma_+, m = 0, 1; \\
\Phi|p_0 \times E &= \Phi(p_0) = \Phi_1(p_0), \\
\Phi|q_1 \times E &= H_0^+(E), \quad \Phi|q_5 \times E = H_0^+(E). \tag{31}
\end{align*}$$

Moreover, select paths $H_0^-, H_0^+$ such that the elements of $\pi_1(S^1)$ corresponding to $\Phi|\gamma_+ \times E) \cdot$, and of $\pi_1(RP^2)$ corresponding to $\Phi|\gamma_+ \times E)$ are both trivial,

$$c^2_\gamma(\Phi_0, \Phi_1) \equiv c^2(\Phi, \gamma_+ \times E, S^1) = 0, \quad c^2_\gamma(\Phi_0, \Phi_1) \equiv c^2(\Phi, \gamma_+ \times E, RP^2) = 0. \tag{32}$$

Define map $(\Phi_0, \Phi_1) : (\tau^1_{\alpha \xi_0 \xi_1} \times E) \cdot \cong S^1 \to S^1, \alpha = 1, 2$ by

$$(\Phi_0, \Phi_1) \bigg| (\tau^1_{\alpha \xi_0 \xi_1} \times \{m\}) = \Phi_m|\tau^1_{\alpha \xi_0 \xi_1}, \quad \alpha = 1, 2, m = 0, 1; \\
(\Phi_0, \Phi_1) \bigg| (\tau^0_{\xi_1 \xi_0 \xi_1} \cdot \times E) = H_0^+(E). \tag{33}$$

Let $d(\Phi_0, \Phi_1, \tau^1_{\alpha \xi_0 \xi_1^-}), \alpha = 1, 2$ be the corresponding elements of $\pi_1(S^1)$. In fact due to periodic boundary conditions it is enough to consider only edges not related by horizontal translations. In the case of (T) boundary conditions on the top plate we define $d^{1}_{\mathcal{K}}(\Phi_0, \Phi_1)$ as

$$d^{1}_{\mathcal{K}}(\Phi_0, \Phi_1) = \sum_{\alpha = 1}^{2} \sum_{\sigma = \pm} d(\Phi_0, \Phi_1, \tau^1_{\alpha \sigma}) \tau^1_{\alpha \sigma}, \quad \tau^1_{1 \pm} \equiv \tau^1_{1 - \pm}, \quad \tau^1_{2 \pm} \equiv \tau^1_{2 \pm -}, \quad (T). \tag{34}$$

Note that $(\Phi_0, \Phi_1)|(\tau^1_{\alpha \pm} \times E) \cdot$ and $(\Phi_0, \Phi_1)|(\tau^1_{\alpha \pm} \times E) \cdot$ are free-homotopic as $S^1 \to RP^2$ maps, thus $d(\Phi_0, \Phi_1, \tau^1_{\alpha \pm}) = d(\Phi_0, \Phi_1, \tau^1_{\alpha \pm}) \mod 2$. In the case of (N) boundary conditions on the top plate $d(\Phi_0, \Phi_1, \tau^1_{\alpha \pm}) = 0, \alpha = 1, 2$, thus

$$d^{1}_{\mathcal{K}}(\Phi_0, \Phi_1) = \sum_{\alpha = 1}^{2} d(\Phi_0, \Phi_1, \tau^1_{\alpha \pm}) \tau^1_{\alpha \pm}. \tag{35}$$

Assuming $\Phi_0|\mathcal{K}_1 \simeq \Phi_1|\mathcal{K}_1 \Leftrightarrow d^{1}_{\mathcal{K}}(\Phi_0, \Phi_1) = 0 \tag{36}$

(as usual, $\mathcal{K}_n$ is the subcomplex of $\mathcal{K}$ consisting of cells of $\mathcal{K}$ of dimension $\leq n$). Thus if

$$d^{1}_{\mathcal{K}}(\Phi_0, \Phi_1) = 0, \tag{37}$$

13
there exists tangent homotopy $H_{\Phi_0, \Phi_1}^{K_1}$ periodic with respect to horizontal translations:

$$H_{\Phi_0, \Phi_1}^{K_1} : K_1 \times E \to \mathbb{R}P^2, H_{\Phi_0, \Phi_1}^{K_1}((1)_{\alpha \xi_3} \times E)^\bullet = (\Phi_0, \Phi_1), \ H_{\Phi_0, \Phi_1}^{K_1}|(\tau^1_{\alpha \xi_3} \times E) \subset Y_1 \approx S^1, \alpha = 1, 2.$$ \hfill (38)

Assume that (29), (37) are satisfied. Let $\Phi_0^\dagger, \Phi_1^\dagger, H_{\Phi_0, \Phi_1}^{K_1}$ be lifted to $S^2$ maps, periodic or antiperiodic with respect to horizontal translations and defined similarly to (12) - (14),

$$\Phi_0^\dagger(p_0) = \Phi_1^\dagger(p_0) = \hat{e}_2, \ H_{\Phi_0, \Phi_1}^{K_1}|(\tau^1_{\alpha \xi_3} \times \{m\}) = \Phi_m^\dagger|\tau^1_{\alpha \xi_3} \times \{m\} = 0, 1.$$ \hfill (39)

Define $\Phi_1^\dagger((\tau^2_{\alpha \xi_3} \times E))^\bullet$ by

$$\Phi_1^\dagger((\tau^2_{\alpha \xi_3} \times \{m\})) = \Phi_m^\dagger|\tau^2_{\alpha \xi_3}, \ m = 0, 1,$$

$$\Phi_1^\dagger((\tau^1_{\beta \xi_3} \times E)) = H_{\Phi_0, \Phi_1}^{K_1}|(\tau^1_{\beta \xi_3}), \ \Phi_1^\dagger((\tau^0_{\xi_3} \times E)) = H_{\Phi_0, \Phi_1}^{K_1}|(\tau^0_{\xi_3} \times E).$$ \hfill (40)

**Lemma 4.1.** Assume $d_1^3(K_0, \Phi_1) = 0$. Let $H_{\Phi_0, \Phi_1}^{K_1}$ and $\tilde{H}_{\Phi_0, \Phi_1}^{K_1}$ be two 1-cell homotopies as in (38), $H_{\Phi_0, \Phi_1}^{K_1}$ and $\tilde{H}_{\Phi_0, \Phi_1}^{K_1}$ their lifted to $S^2$ versions (39), and $\Phi^\dagger, \tilde{\Phi}^\dagger$ the corresponding maps $(\tau^2_{\alpha, \xi_3} \times E)^\bullet \to S^2$ defined by (40). Let $d_3^3 = d(\tilde{H}_{\Phi_0, \Phi_1}^{K_1}), H_{\Phi_0, \Phi_1}^{K_1}|(\tau^1_{3-} \times E), S^2) \in \mathbb{Z}$. Let $s_\alpha$ be an element of $\pi_1(\mathbb{R}P^2)$ corresponding to $\Phi_0|\tau^1_{\alpha \pm} \approx \Phi_1|\tau^1_{\alpha \pm}, \ a = 1, 2$. Then

$$c^3(\Phi^\dagger, \tau^2_{\alpha-} \times E, S^2) - c^3(\tilde{\Phi}^\dagger, \tau^2_{\alpha-} \times E, S^2) = \sum_{\beta=1}^2 \epsilon_{\alpha\beta}(1 - (-1)^{s_\beta})d_3^3, \ \alpha = 1, 2.$$ \hfill (41)

**Proof.** It follows from (38), (39) that $H_{\Phi_0, \Phi_1}^{K_1}|(K_1 \times E)^\bullet = \tilde{H}_{\Phi_0, \Phi_1}^{K_1}|(K_1 \times E)^\bullet$. For the lifted maps periodic/antiperiodic conditions imply

$$H_{\Phi_0, \Phi_1}^{K_1}|(\tau^1_{3\xi_3} \times E) = (-1)^{\sum_{\alpha=1}^2 \frac{1}{2}(1+\xi_3)s_\alpha} H_{\Phi_0, \Phi_1}^{K_1}|(\tau^1_{3-} \times E),$$ \hfill (42)

and the same condition for $\tilde{H}_{\Phi_0, \Phi_1}^{K_1}$. Using (2), (10), and the addition formulas (6) we get

$$c^3(\Phi^\dagger, \tau^2_{1-} \times E, S^2) - c^3(\tilde{\Phi}^\dagger, \tau^2_{1-} \times E, S^2) = d \left(\tilde{H}_{\Phi_0, \Phi_1}^{K_1}, H_{\Phi_0, \Phi_1}^{K_1}, \tau^1_{3-} \times E, S^2\right) - d \left(H_{\Phi_0, \Phi_1}^{K_1}, H_{\Phi_0, \Phi_1}^{K_1}, \tau^1_{3-} \times E, S^2\right),$$ \hfill (43)

and similarly for $c^3(\tilde{\Phi}^\dagger, \tau^2_{2-} \times E, S^2)$. Using periodic/antiperiodic conditions (42), we get (41). \hfill \Box

**Theorem 4.1.** Maps $\Phi_0, \Phi_1$ are homotopic if and only if $d_1^K(\Phi_0, \Phi_1) = 0, \ d_2^K(\Phi_0, \Phi_1) = 0$, and

1. Case (N) : $d_1^K(\Phi_0, \Phi_1) = 0, \ d_2^K(\Phi_0, \Phi_1) = 0$,
2. Case (T) : $d_1^K(\Phi_0, \Phi_1) = 0, \ d_2^K(\Phi_0, \Phi_1) = 0 \ mod \ \sum_{\alpha, \beta=1}^2 \epsilon_{\alpha\beta}(1 - (-1)^{s_\beta})\tau^2_{\alpha-}.$
**Proof.** It is clear that those conditions are necessary. Let us show that they are sufficient. Similarly to Theorem 3.1, it follows from \(d_1^P(\Phi_0, \Phi_1) = 0, \ d_2^P(\Phi_0, \Phi_1) = 0\) that there exist a surface homotopy \(H_{\Phi_0, \Phi_1}^3\) on the truncated post surface.

We will show that there exist a homotopy for maps restricted to \(K\), the surface of the prism bounding \(\hat{C}\). It follows from \(d_1^K(\Phi_0, \Phi_1) = 0\) that there exist a homotopy \(H_{\Phi_0, \Phi_1}^{K_1}\) on edges of \(K\).

In case (N), it follows from \(d_2^K(\Phi_0, \Phi_1) = 0\) and the periodic boundary conditions that there exist a homotopy \(H_{\Phi_0, \Phi_1}^{K_2}\) on the vertical faces \(\tau_{\alpha\pm}^2, \alpha = 1, 2\) of \(K\), extending edge homotopy \(H_{\Phi_0, \Phi_1}^{K_1}\).

In case (T), we can select edge homotopy \(H_{\Phi_0, \Phi_1}^{K_1}\) in such a way that we have \(d_2^K(\Phi_0, \Phi_1) = 0\), and thus there is a homotopy on vertical faces extending such \(H_{\Phi_0, \Phi_1}^{K_1}\).

On the top plate \(\tau_{3+}^2\), with \(\Phi^1\) defined in (40), we always have \(c^3(\Phi^1, \tau_{3+}^2 \times E, S^2) = 0\) due to the tangent boundary conditions, thus there is always a homotopy between restrictions of \(\Phi_0, \Phi_1\) to the top plate.

On the bottom plate, due to the periodic boundary conditions and the addition formula (2), (6) we have that \(c^2(\Phi_0, \tau_{3-}^2, S^1) = c^2(\Phi_1, \tau_{3-}^2, S^1) = 0\), thus restrictions of maps \(\Phi_0, \Phi_1\) to the bottom plate have tangent extensions from the boundary of the truncated post footprint to its interior. Thus as in the case of the top plate, there is always a homotopy between restrictions of \(\Phi_0, \Phi_1\) to the bottom plate.

The above considerations yield a homotopy between restrictions of \(\Phi_0, \Phi_1\) to the full boundary of \(\hat{C}\) (consisting of the truncated surface of the post and a part of \(K\) outside the truncated post). The proof that the surface homotopy can be extended to the rest of \(\hat{C}\) is the same as in Theorem 3.1.

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