Reversible linear differential equations
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Abstract
Let $\nabla$ be a meromorphic connection on a vector bundle over a compact Riemann surface $\Gamma$. An automorphism $\sigma : \Gamma \to \Gamma$ is called a symmetry of $\nabla$ if the pull-back bundle and the pull-back connection can be identified with $\nabla$. We study the symmetries by means of what we call the Fano Group; and, under the hypothesis that $\nabla$ has a unimodular reductive Galois group, we relate the differential Galois group, the Fano group and the symmetries by means of an exact sequence.

"the solution of an intellectual problem comes about in a way not very different from what happens when a dog carrying a stick in its mouth tries to get through a narrow door: it will go on turning its head left and right until the stick slips through" - Robert Musil

1 Introduction
In 1900, G. Fano addressed the following problem [8]: what are the consequences of algebraic relations between the solutions of a linear differential equation? The problem was apparently proposed to him by F. Klein. A particular concern was whether or not, a linear differential equation with solutions satisfying a homogeneous polynomial, can be “solved in terms of linear equations of lower order”. This has been successfully studied by M. Singer, cf. [17] and later by K.A. Nguyen, cf. [15].

Fano considered the group of projective automorphisms of the projective variety having as coordinate functions the solutions of the differential equation. This could be viewed as a primitive version of the differential Galois Group, as M. van der Put considered the differences. Here, we replace Fano’s group with one slightly smaller group: the group of projective automorphisms of the projective variety taking coordinate functions the solutions of the differential equation together with the $n-1$st derivatives (where $n$ is the order of the equation).

We treat the problem in terms of connections. Suppose we are given a ramified covering map $\phi : \Gamma' \to \Gamma$ of compact Riemann surfaces, together with a meromorphic vector bundle $E$ with a connection $\nabla$ over $\Gamma$. We can use $\phi$ to pull back the bundle and the connection, obtaining $E' = \phi^*E$ together with $\nabla' = \phi^*\nabla$. In this context one obtains [14 Appendix B] a natural injection:

$$\text{Gal}(\nabla') \to \text{Gal}(\nabla)$$
where the induced map on the Lie algebra is an isomorphism (i.e. the connected components of 1 are isomorphic). Note that, a covering transformation $\sigma \in \text{Aut}_C(\Gamma')$ of $\phi : \Gamma' \to \Gamma$ lifts to a parallel automorphism of the vector bundle $(E', \nabla')$. The automorphisms lifting to the connection are called symmetries (of the connection). A concise exposition on the symmetries is given in [5].

Conversely, starting with a connection on a meromorphic vector bundle over a Riemann surface together with a symmetry; we can consider the quotient Riemann surface with a canonically induced vector bundle with connection. This new connection has a Galois group with bigger monodromy subgroup, but with isomorphic identity component. Checking for symmetries of a given connection over the Riemann sphere is quite easy (just consider permutations of singular points). Methods for revealing them in arbitrary contexts are far from simple. For example one can consider the work by Dwork and Baldassarri [1][2].

The purpose of this paper is to explain how the difference between the Fano group and the Galois group may be explained using the symmetries of the connection, and to give a proof of this in the cases where the Galois group is reductive and unimodular and the connection is standard.

2 Examples

EXAMPLE 2.1. Consider the differential equation [18] $L(y) = 0$ given by

$$y''' + \frac{3(3z^2 - 1)}{z(z - 1)(z + 1)} y'' + \frac{221z^4 - 206z^2 + 5}{12z^2(z - 1)^2(z + 1)^2} y' + \frac{374z^6 - 673z^4 + 254z^2 + 5}{54z^2(z - 1)^2(z + 1)^3} y = 0$$

Its Picard Vessiot extension has differential Galois group $G_{54}$ of order 54. The singular points of $L(y) = 0$ are 0, 1, −1 and $\infty$, with respective exponents

$$\frac{-1}{6}, \frac{5}{6}, \frac{-2}{3} = \frac{-1}{6}, \frac{5}{6}, \frac{-2}{3} = \frac{11}{6}, \frac{17}{6}, \frac{4}{3}$$

The ramification data in 0, 1 and −1 is the same, so one can expect some kind of symmetry in between these three points. A quick glance to the equation reveals that all the coefficients of the numerator have even power of $z$, and the denominator present the same exponents for $z - 1$ and for $z + 1$. This equation admits one symmetry $z \mapsto -z$.

Now, with some computation we can see that if $X$ denotes the solution with exponent $-\frac{1}{6}$, $Y$ the one with $-\frac{5}{6}$, and $Z$ the last one, around 0, then:

$$YZ^2 + X^3 - \frac{16}{81}XY^2 = 0$$

This corresponds to an elliptic curve. The other third degree $G_{54}$-semi-invariant is given by:

$$XZ^2 + \frac{32}{162}X^3Y + \frac{256}{19683}Y^3 = \left(\frac{1}{x^3(x^2 - 1)^3}\right)^{\frac{1}{2}}$$
The vanishing $G_{54}$-invariant polynomial is then:

$$(YZ^2 + X^3 - \frac{16}{81}XY^2)(XZ^2 + \frac{32}{162}X^2Y + \frac{256}{19683}Y^3)$$

The elements of $GL_3(\mathbb{C})$ that leaves the homogeneous ideal generated by this vanishing $G_{54}$-invariant polynomial form the group $H_{216}^{GL_3(\mathbb{C})}$ (the lifting of the Hessian group $H_{216} \subseteq PSL_3(\mathbb{C})$ to $GL_3(\mathbb{C})$). The image in the group of automorphisms of $G_{54}$ of the normalizer of $G_{54}$ in $H_{216}^{GL_3(\mathbb{C})}$ corresponds to $F_{36} \subseteq PSL_3(\mathbb{C})$, which contains as a normal subgroup a copy of $PG_{54}$ (the projective version of $G_{54}$) with index 2. So we obtain the following exact sequence:

$$1 \rightarrow Z(G_{54}) \rightarrow G_{54} \rightarrow F_{36} \rightarrow \langle z \mapsto -z \rangle \rightarrow 1$$

The first term in the sequence is the center of the Galois group, the second term is the Galois group, the third term is the copy of the automorphisms of $G_{54}$ in the projective automorphisms of the projective curve described by the solutions of the equation, and the third term is the group of symmetries.

**EXAMPLE 2.2.** Consider the differential equation $L(y) = 0$ given by

$$y'' + \frac{21(z^2 - z + 1)}{25z^2(z - 1)^2}y' + \frac{21(-2z^3 + 3z^2 - 5z + 2)}{50z^3(z - 1)^3}y = 0$$

Its Picard Vessiot extension has differential Galois group $A_5$, the rotational icosahedral group of order 60. This equation admits one symmetry: $z \mapsto -z + 1$. The equation is a symmetric power of a second order linear differential equation and so its solutions satisfies the equation:

$$XY - Z^2 = 0$$

The group of automorphism obtain by conjugating $A_5$ in the subgroup of $GL_3(\mathbb{C})$ fixing the homogeneous ideal generated by this conic, is $S_5(\mathbb{C})$ (conjugation with a diagonal matrix with determinant $-1$ together with the inner automorphisms). So we obtain the following exact sequence:

$$1 \rightarrow A_5 \rightarrow S_5 \rightarrow \langle z \mapsto -z + 1 \rangle \rightarrow 1$$

where the first term is the Galois group, the second term is the automorphisms of the Galois group in the group of automorphisms of the cone of the projective curve described by the solutions of the equation, and the third term is the group of symmetries.

**EXAMPLE 2.3.** Consider the differential equation $L(y) = 0$ given by:

$$y'' - \frac{z^4 - 3z^2 - 1}{1 + z^4}y = 0$$

Two linearly independent solutions are given by:

$$Y_1 = \sqrt[4]{z^4 - 1}e^{\int \frac{1}{\sqrt[4]{z^4 - 1}}} , \quad Y_2 = \sqrt[4]{z^4 - 1}e^{-\int \frac{1}{\sqrt[4]{z^4 - 1}}}$$
So according to Kovacic algorithm, the Differential Galois group is the infinite dihedral group $D_{\infty}$. A basis of invariants of this group on $\mathbb{C}[X^i]$ is:

$$
X_{2,1} := X_1^1X_2^2 - X_2^1X_1^1 \quad \longrightarrow \quad -2
$$

$$
X_{4,1} := (X_1^1X_2^1)^2 \quad \longrightarrow \quad z^4 - 1
$$

$$
X_{4,2} := (X_2^2X_2^2)^2 \quad \longrightarrow \quad \frac{(z^6 - z^4 + 1)^2}{(z^4 - 1)^3}
$$

$$
X_{4,3} := (X_1^1X_2^2 + X_2^2X_1^1)^2 \quad \longrightarrow \quad \frac{4z^6}{z^4 - 1}
$$

$$
X_{4,4} := (X_1^1X_2^2)(X_1^1X_2^2 + X_2^2X_1^1) \quad \longrightarrow \quad 2z^8
$$

$$
X_{4,5} := (X_2^2X_2^2)(X_1^1X_2^2 + X_2^2X_1^1) \quad \longrightarrow \quad \frac{2z^3(z^6 - z^4 + 1)}{(z^4 - 1)^2}
$$

$$
X_{4,6} := X_1^1X_2^2X_2^2X_2^2 \quad \longrightarrow \quad \frac{z^6 - z^4 + 1}{z^4 - 1}
$$

Where the arrow refers to the image on the Picard-Vessiot extension under the map

$$
X_{ij} \longrightarrow Y_{ij}^{(i-1)}
$$

The homogeneous relations that vanishes under this evaluative homomorphism are given by the homogeneous ideal generated by:

$$
X_{4,4}X_{4,5} - X_{4,6}X_{4,3}
$$

$$
X_{4,6} - X_{4,1}X_{4,2}
$$

$$
X_{4,1}X_{4,2} - \frac{1}{16}(X_{4,3} - X_{2,1}^2)^2
$$

$$
X_{4,3}X_{2,1} - (X_{4,3} - 2X_{4,6})^2
$$

$$
X_{4,4} - X_{4,1}X_{4,3}
$$

$$
X_{4,5} - X_{4,2}X_{4,3}
$$

All of them invariant under the group $G_F$ generated by $D_{\infty}$ together with

$$
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
$$

and so corresponds to the automorphisms of the space

$$
Specm(\mathbb{C}[X_{2,1}, X_{4,1}, X_{4,2}, X_{4,3}, X_{4,4}, X_{4,5}, X_{4,6}])
$$

Again we have an exact sequence:

$$
1 \longrightarrow G \longrightarrow G_F \longrightarrow \langle z \mapsto -z \rangle \longrightarrow 1
$$
3 Definitions

REMARK 3.1. We will use Einstein convention for indices.

Let $\Gamma$ be a (connected) compact Riemann surface with field of meromorphic functions $K$. Let

$$\Pi : E \longrightarrow \Gamma$$

be an $n$-dimensional meromorphic vector bundle, with a meromorphic connection

$$\nabla : \mathfrak{E} \longrightarrow \Omega^1 \otimes_K \mathfrak{E},$$

where $\Omega^1$ denotes the meromorphic differential forms over $\Gamma$ and $\mathfrak{E}$ the meromorphic sections of $\Pi$. We also denote by $\mathfrak{X}_\Gamma$ the vector fields of meromorphic tangent vectors to $\Gamma$. For the definitions of meromorphic sections of holomorphic vector bundles and meromorphic connection, and their main properties, we refer to \cite[Appendix A]{13}.

There is a natural map:

$$\mathfrak{X}_\Gamma \otimes_K \Omega^1 \Gamma \longrightarrow K$$

$$v \otimes \eta \longmapsto \eta(v)$$

which canonically extends to

$$\mathfrak{X}_\Gamma \otimes_K \Omega^1 \Gamma \otimes_K \mathfrak{E} \longrightarrow \mathfrak{E}$$

$$v \otimes \eta \otimes X \longmapsto \langle v, \eta \otimes X \rangle := \eta(v)X$$

Given a meromorphic tangent vector field $v \in \mathfrak{X}_\Gamma$ we denote by $\nabla_v$ the derivation on $\mathfrak{E}$:

$$\nabla_v(X) = \langle v, \nabla X \rangle$$

Definition 3.2. Let $\sigma \in \text{Aut}_\mathbb{C}(\Gamma)$ (or equivalently $\sigma^* \in \text{Aut}_\mathbb{C}(K)$). We say that $\sigma$ is a symmetry of $\nabla$, if there is a parallel vector bundle morphism $\tilde{\sigma} : (E, \nabla) \rightarrow (E, \nabla)$ lifting $\sigma$, i.e. $\tilde{\sigma}(\nabla_v X) = \nabla_{\sigma^* v} \tilde{\sigma}(X)$ and $\sigma \circ \Pi = \Pi \circ \tilde{\sigma}$.

$$\begin{array}{ccc}
E & \xrightarrow{\tilde{\sigma}} & E \\
\Pi \downarrow & & \Pi \downarrow \\
\Gamma & \xrightarrow{\sigma} & \Gamma
\end{array}$$

The group of symmetries of $\nabla$ will be denoted by $\text{Aut}_\mathbb{C}(\Gamma)$.

REMARK 3.3. A symmetry of $\nabla$ permutes its singular points.

Let $(U, z)$ be a holomorphic chart of $\Gamma$ centered at $p \in U$, where $U \subseteq \Gamma$ is an open ball avoiding the singularities of $\nabla$. $E$ is holomorphic and trivial above $U$ and $\nabla$ induces a holomorphic connection $\nabla'$ on $\Pi^{-1}U \rightarrow U$. Take $(\Pi^{-1}U, z, y^1, \ldots, y^n)$ a chart of $E$ giving a trivialization. There exists $\Pi$ a holomorphic parallel frame $V_1, \ldots, V_n$ over $U$, i.e.

$$v^j_i(z) = y^j(V_j(z))$$
with \( v_j^i(z) \) holomorphic in \( U \) such that \( \nabla_{\bar{z}} V_j = 0 \) and \( \det(v_j^i)(z) \) does not vanish in \( U \).

**Definition 3.4.** The Fano group \( G_F \) of \( \nabla \) is the subgroup of \( GL_n(\mathbb{C}) \) fixing the homogeneous ideal in \( \mathbb{C}[X_j^i] \) generated by the \( G \)-invariant homogeneous polynomials \( P[X_j^i] \in \mathbb{C}[X_j^i] \) vanishing at \( v_j^i \).

**Definition 3.5.** The Fano group we take into consideration here differs slightly from the one considered in [15].

**Remark 3.6.** It follows directly from the definition that there is canonical inclusion of the representation of the Galois group \( G \) given by \( v_j^i \) on \( GL_n(\mathbb{C}) \) into \( G_F \). Let us make this remark more explicit.

Fix \( v \in \mathfrak{X}_\Gamma, v \neq 0 \), and \((e_1, \ldots, e_n)\) a meromorphic frame of \( \mathfrak{E} \), that is an \( n \)-tuple of meromorphic sections such that in a Zariski open subset \( W \subseteq \Gamma \), \((e_1(q), \ldots, e_n(q))\) is a basis of \( \Pi^{-1}(q) \) for any \( q \in W \). Let \( a_j^i \in K \) be such that:

\[
\nabla_v e_j = -a_j^i e_i
\]

Thus, in this frame, the equation \( \nabla_v X = 0 \) is equivalent to \( X' = AX \) where \( A = (a_j^i) \) and \( X' = v(x')e_i \) if \( X = x'e_i, x' \in K \).

We define the differential ring extension \((K[X_j^i, \frac{1}{\det}], \tilde{v})\) of \((K, v)\) into \( G_F \) by setting for \((g_j^i) \in GL_n(\mathbb{C})\):

\[
\bar{v}(X_j^i) = a_j^i X_j^i
\]

Note that we can make \( GL_n(\mathbb{C}) \) act on \( K[X_j^i, \frac{1}{\det}] \) through differential automorphisms over \((K, v)\) by setting for \((g_j^i) \in GL_n(\mathbb{C})\):

\[
(g_j^i): K[X_j^i, \frac{1}{\det}] \longrightarrow K[X_j^i, \frac{1}{\det}]
X_j^i \mapsto X'_j g_j^i
\]

A Picard-Vessiot extension of \( K \) for the matrix differential equation \( X' = AX \) is given by the quotient field of

\[
K[X_j^i, \frac{1}{\det}]/I
\]

where \( I \) is a maximal differential ideal. Since \( GL_n(\mathbb{C}) \) acts through differential automorphisms, the action permutes the maximal differential ideals. Actually, the action is transitive on the maximal differential ideals [15]. A representation of the differential Galois group of \( \nabla \) is given by [12]

\[
G = \{ (g_j^i) \in GL_n(\mathbb{C}) \mid (g_j^i) : I \mapsto I \}
\]

the stabilizer of \( I \) under this action.
Let us assume that the holomorphic chart \((U, z)\) and \(v\) are such that \(v\) restricted to \(U\) coincides with \(\frac{\partial}{\partial z}\). Now, fix an injection \(\iota : K \to \mathbb{C}[\frac{1}{z}][z]\); then, identifying \(v_j^i(z)\) with their power series expansion we have a differential homomorphism
\[
\Phi : (K[X_j^i, \frac{1}{\det}], \tilde{v}) \longrightarrow (\mathbb{C}[\frac{1}{z}][z], \frac{\partial}{\partial z})
\]
such that \(\Phi(f) = \iota(f)\) if \(f \in K\),
\[
K[X_j^i, \frac{1}{\det}] \xrightarrow{\Phi} \mathbb{C}[\frac{1}{z}][z]
\]
If we set \(I = \ker(\Phi)\), then \(I\) is a maximal differential ideal; and, the choice of \(v_j^i(z)\) induces the representation of the Galois group of \(\nabla\) by \(G\).

In order to state our theorem we need to introduce the following concepts [10] [3]:

**Definition 3.7.** Let \(P[X_j^i] \in \mathbb{C}[X_j^i]\) be a homogeneous polynomial. If \(P[v_j^i](z) = \iota(f)\) for some \(f \in K\) we say that \(f\) is a dual first integral of \(\nabla\) with degree defined by the degree of \(P\). We denote by \(K_{\nabla}\) the field generated over \(\mathbb{C}\) by the quotients of dual first integrals of same degree.

**Definition 3.8.** The connection \(\nabla\) is called standard if \(K(\nabla) = \mathbb{C}(\nabla)\).

**Remark 3.9.** As a corollary of the Lemma [5, 4] we will obtain that, under the unimodular and reductive hypothesis, the automorphisms of \(K\) over \(K_{\nabla}\) can be identified as a subgroup of the symmetries of \(\nabla\). In particular, it gives an extension to the criteria for descend obtained in [9].

**Theorem 3.10.** Let \(\nabla\) be a standard connection. If the Galois group of \(\nabla\) is unimodular and reductive; then, the sequence
\[
1 \longrightarrow Z(G) \longrightarrow G \longrightarrow Aut_{G_F}(G) \longrightarrow Aut_{\nabla}(\Gamma) \longrightarrow 1,
\]
is exact. Here \(Aut_{G_F}(G)\) denotes the image of the normalizer of \(G\) in \(G_F\) into the group of automorphisms of \(G\), where the action is conjugation; and, \(Z(G)\) is the center of \(G\).

The remain of this paper is devoted to a proof of this result. The hypothesis over the differential Galois group allows us to use the following theorem [6]:

**Theorem 3.11** (Compoint). If \(G\) is reductive and unimodular, then \(I\) is generated by the \(G\)-invariants it contains. Moreover, if \(P_1, \ldots, P_r\) is a set of homogeneous generators for the \(\mathbb{C}\)-algebra of \(G\)-invariants, with respective degrees \(n_1, \ldots, n_r\), in \(\mathbb{C}[X_j^i]\), and \(f_1, \ldots, f_r \in K\) are such that \(P_i - f_i \in I\) then \(I\) is generated over \(K[X_j^i, \frac{1}{\det}]\) by \(P_i - f_i\) where \(i \in \{1, \ldots, r\}\).
REMARK 3.12. In [6] and in [4] the statement of this theorem is reserved for $K = \mathbb{C}(z)$ and $v = \frac{\partial}{\partial z}$. But the proof in [4] carries mutatis mutandis when $\mathbb{C}(z)$ is replaced by $K$. The careful reader will noticed that F. Beukers does not make explicit on his paper the unimodular hypothesis, although it is needed to get the invariance of $\det$.

REMARK 3.13. Preserving the notation of the theorem, we obtain:

$$K_S = \mathbb{C}(f_i^{m_i}) \left\{ (i, j) \in \{1, \ldots, r\}^2 \mid m_i = m_j, f_i \neq 0 \right\}$$

Additionally, if $\nabla$ is standard, then $K_S = K$.

REMARK 3.14. It is easy to get examples to justify the unimodular hypothesis of the theorem. Indeed, consider:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 2z & 0 \\ 0 & 1 + 2z \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

a fundamental system of solutions is given by:

$$\begin{pmatrix} e^{z^2} & 0 \\ 0 & e^{z^2} \end{pmatrix}$$

so that the connection is standard, indeed $Y_2^2 / Y_1^1 = z$. The Galois group is $G_{m, \mathbb{C}}$ which is represented by:

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^* \right\}$$

The ideal of homogenous polynomials vanishing at the solutions is generated by $zX_1^1 - X_2^2$, $X_1^2$ and $X_2^1$. There is no vanishing $G$-invariant homogeneous polynomial other than 0. Whence $G_F$ is $GL_2(\mathbb{C})$, $N_{G_F}(G) = G_F$, $G_F / G = PGL_2(\mathbb{C})$ but the connection is symmetric only with respect to $z \mapsto -z$, indeed one can lift it by mapping $(y_1, y_2)$ into $(y_1, -y_2)$.

REMARK 3.15. An example justifying the requirement on $\nabla$ to be standard is a little more delicate, as when one lifts a standard equation with symmetries through a Galois covering, the symmetries are lifted together with the equation. So the way to obtain the example is by lifting and equation with symmetries, say the standard $D_{2, 4}$ equation $\mathbb{3}$ defined on the $x$-sphere, through a non-Galois covering, say through $x \mapsto z(z - 1)^2$ to the $z$-sphere.

4 A geometric construction: a covering space with covering group $Aut_{G_F}(G)$

REMARK 4.1. We keep the same notation from the previous section, we will assume, from now on, that $\nabla$ has reductive unimodular Galois group and that it is standard.
Following J.A. Weil \[19\] we will work with linear first integral (i.e. with the solutions of the adjoint system) instead of doing it with parallel sections. A linear first integral of $\nabla'_{\frac{\partial}{\partial z}}$ over $\Pi^{-1}U$ is a function $\Phi : \Pi^{-1}U \to \mathbb{C}$, of the form

$$\Phi(z, y^1, \ldots, y^n) = \frac{\partial \Phi}{\partial y^j}(z)y^j$$

where $\frac{\partial \Phi}{\partial y^j}(z)$ is holomorphic in $U$, constant on the parallel sections of $\nabla'_{\frac{\partial}{\partial z}}$, i.e. if $\nabla'_{\frac{\partial}{\partial z}}X = 0$ then $z \mapsto \Phi(X(z))$ is a constant function.

Let $\Phi^i, i \in \{1, \ldots, n\}$, be a system of linear first integrals of $\nabla'_{\frac{\partial}{\partial z}}$ over $\Pi^{-1}U$ such that $\frac{\partial \Phi^i}{\partial y^j}$ is an invertible matrix. We call such a system a fundamental system of linear first integrals.

Let $F$ be the homogeneous ideal in $\mathbb{C}[X^i_j, \frac{1}{\det G}]$ generated by the homogeneous polynomials of $\mathbb{C}[X^i_j]$ vanishing at $\frac{\partial \Phi^i}{\partial y^j}$. The ring

$$R = \mathbb{C}[X^i_j, \frac{1}{\det G}] / F$$

gives the graded algebra of the homogeneous rational linear first integrals of the symmetric products of $(E, \nabla)$.

Note that if $AX = X'$ is the matrix equation form of $\nabla, X = 0$, the adjoint system for the linear first integrals is given by $X(-A) = X'$. Indeed, if $S$ is a fundamental system of solutions for $AX = X'$; then:

$$S^{-1}S = I_n$$

whence $S^{-1}$ is a fundamental system of linear first integrals; and, $0 = I'_n = (S^{-1}S)' = (S^{-1})'S + S^{-1}S' = (S^{-1})'S + S^{-1}AS$ thus $(S^{-1})' = -S^{-1}A$. So, when we consider linear first integrals we take instead the differential ring extension $(K[X^i_j, \frac{1}{\det}], \nabla)$ of $(K, v)$, where $\nabla X^i_j = X^i_k a^k_j$, and we let $GL_n(\mathbb{C})$ act to the left.

According to Compoint’s Theorem, a maximal differential ideal of the differential ring $K[X^i_j, \frac{1}{\det}]$ giving a Picard-Vessiot extension for $\nabla, X = 0$ with Galois group represented by $G$ and linear first integrals vanishing at $F$, is uniquely determined by a $\mathbb{C}$-homomorphism $\phi : R \to K$. Indeed, under the notation of the statement of the theorem we have a map:

$$\mathbb{C}[X^i_j, \frac{1}{\det}]^G \to K$$

$$P_i \mapsto f_i$$

The ideal $F$, by definition, is contained in the kernel, so this map factors through a unique map $\phi : R \to K$. The group $H := Aut_{G_F}(G)$ describes automorphisms

\footnote{The common notation is $X' = -A^t X$ but our notation is aimed to make more transparent the interplay between both systems.}
of the rings \( \mathbb{C}[X^i_j, \frac{1}{\pi}]^G \) and \( R \). So, as \( \nabla \) is standard, the group also describes automorphisms of \( K_{\nabla} \) (cf. remark 3.13).

\[
\begin{align*}
K[X^i_j, \frac{1}{\pi}]^G & \supset K \\
K & \supset K_{\nabla} \\
(K_{\nabla})^H
\end{align*}
\]

Let us do an intermezzo to make a geometric construction that will make clear the interplay between all the groups involved.

Fix a meromorphic tangent vector field \( v \) over \( \Gamma \). Let \( p \in \Gamma \) over which \( \nabla_v \) is not singular, say \( z(p) = 0 \) and fix a frame of holomorphic parallel sections \( V_1, \ldots, V_n \) of \( \nabla_v \) around \( p \). We lift the connection to the vector bundle of frames of \( GL_nE \).

We extend \( U \subseteq \Gamma \) to the maximal open set over which the frame \((V_1, \ldots, V_n)\) can be extended holomorphically (as a multi-valued frame).

The extension of \((V_1, \ldots, V_n)\), together with its orbit under the Galois Group \( G \), defines a sub-sheaf \( \mathfrak{F} \) of \( GL_nE \mid_U \) whose sections are parallel holomorphic frames under \( \nabla_v \). The sheaf \( \mathfrak{F} \) gives rise to a regular covering of \( U \) with covering group \( G \) [Théorème 5.3.1]. Denote by \( \tilde{U} \) the covering space corresponding to the center of \( G \), \( Z(G) \).

The diagram above implies the following tower of covering spaces:

\[
\begin{align*}
\tilde{U} & \supset U \\
U & \supset U_{\nabla} \\
(U_{\nabla})^H
\end{align*}
\]

where \( U_{\nabla} \) corresponds to the space with meromorphic functions \( K_{\nabla} \). So we obtain \( \tilde{U} \) covering \( U \) with covering group \( G \), and \( \tilde{U} \) covering \( (U_{\nabla})^H \) with covering group \( H = \text{Aut}_{GF}(G) \). By Galois correspondence, the covering group of \( U \) over \( (U_{\nabla})^H \) is given by the outer automorphisms of \( G \) in \( GF \).
5 The map \( \text{Aut}_{G_F}(G) \rightarrow \text{Aut}_\nabla(\Gamma) \)

The action of the normalizer of \( G \) in \( G_F \) on \( K[X^i_1, \ldots, X^i_n] \) permutes the maximal differential ideals containing \( F \) and with Galois group represented by \( G \). Fix a maximal differential ideal \( I \) and the morphism \( \phi: R \rightarrow K \), given by Compoint’s Theorem, describing it. So that \( U \) is the variety given by \( \Gamma \setminus S \), where \( S \) is the collection of points where \( \phi(\det) \) vanishes, together with the singular points of \( \nabla \).

Now given two of these \( K \)-valued points \( \phi, \phi_0 \), in the orbit of the normalizer of \( G \) in \( G_F \), there is a \( (g^i_j) \in GL_n(\mathbb{C}) \) such that \( \phi \) is obtain by pre-composing \( \phi_0 \) with an element in the normalizer of \( G \) in \( G_F \), under the action notation, \( \phi = \phi_0 \circ (g^i_j) \).

Similarly, one can see \( (g^i_j) \) acting on \( K\nabla \). Indeed, instead of seeing \( (g^i_j) \) as acting linearly on \( G \)-invariant polynomials, one can see it as acting linearly on rational linear first integrals. We denote by \( \sigma^* \) this action on \( K\nabla \). In this fashion we obtain a commutative diagram:

\[
\begin{array}{ccc}
R & \xrightarrow{(g^i_j)} & R \\
\phi_0 \downarrow & & \phi_0 \downarrow \\
K\nabla & \xrightarrow{\sigma^*} & K\nabla
\end{array}
\]

Reversing the arrows:

\[
\begin{array}{ccc}
U & \xleftarrow{(g^i_j)^*} & U \\
\phi^* \downarrow & & \phi^* \downarrow \\
U\nabla & \xleftarrow{\sigma} & U\nabla
\end{array}
\]

**Lemma 5.1.** The automorphism \( \sigma \) (or equivalently \( (g^i_j)^* \)) is a symmetry of \( \nabla \).

**Proof:** If \( P[X^i_j] \in \mathbb{C}[X^i_j]^G \) and \( f \in K \) are such that \( \phi_0(P[X^i_j]) = f \), then

\[
\phi(P[X^i_j]) = \phi_0(P[g^i_k X^k_j]) = \sigma^* \phi_0(P[X^i_j])
\]

i.e.,

\[
\phi(P[X^i_j]) = \sigma^* f
\]

We take again \( (U, z) \) and \( (\pi^{-1}U, z, y^1, \ldots, y^n) \) holomorphic charts as before. Set \( V = \sigma(U) \) and consider a chart of \( E \) giving a vector bundle trivialization, \( (\Pi^{-1}V, w, x^1, \ldots, x^n) \), with \( w \circ \sigma = z \). Note that

\[
\sigma_* \frac{\partial}{\partial z} (w) = \frac{\partial}{\partial z} (w \circ \sigma) = \frac{\partial}{\partial z} (z) = 1
\]
Let \( p \in U \) (\( \sigma(p) \in V \)). And consider some linear first integrals \( \frac{\partial \Psi_i}{\partial x^j} \) defined by \( \phi_0 \) in \( \Pi^{-1}V \). Now, since the \( K \)-valued point given by Compoint’s Theorem is nothing else but the restriction of the evaluative homomorphism, the equalities above implies
\[
\frac{\partial \Phi_i}{\partial y^j}(p) = g_k \frac{\partial \Psi^k}{\partial x^j}(p) = \frac{\partial \Psi_i}{\partial x^j}(\sigma(p))
\]
So that
\[
\frac{\partial \Phi_i}{\partial y^j}(p) = \frac{\partial \Psi_i}{\partial x^j}(\sigma(p)) = \frac{\partial \Psi_i}{\partial y^j}(\sigma(p)) \frac{\partial x^j}{\partial y^j}(\sigma(p)) = \frac{\partial \Psi_i}{\partial x^j}(\sigma(p))
\]
But in \( U \), \( \frac{\partial \Phi_i}{\partial y^j} \) is invertible, whence:
\[
\frac{\partial x^j}{\partial y^j}(\sigma(p)) = \frac{\partial y^j}{\partial \Phi_i}(p) \frac{\partial \Psi^i}{\partial y^j}(\sigma(p)) = f^j_i(p).
\]

Letting \( G \) act on both sides of (1), it follows that \( f^j_i \) describes (by analytic extension) a meromorphic function over \( \Gamma \) (Galois correspondence). So the map
\[
U \to V
\]
described by
\[
w(z, y^1, \ldots, y^n) = \sigma(z)
\]
\[
x^j(z, y^1, \ldots, y^n) = f^j_i(z) y^i
\]
gives the transform on the fiber coordinates that lifts the covering transformation \( \sigma \) to a parallel automorphism.

Being \( \sigma_{\phi_0} \) a symmetry, it permutes the singular points. Now if there is another \( \sigma'_{\phi_0} \), with the property \( \phi^* = \phi_0^* \circ \sigma'_{\phi_0} \); then
\[
\phi^* \circ \sigma_{\phi_0}^{-1} \circ \sigma'_{\phi_0} = \phi_0^* \circ \sigma_{\phi_0} \circ \sigma_{\phi_0}^{-1} \circ \sigma'_{\phi_0} = \phi_0^* \circ \sigma'_{\phi_0} = \phi^*.
\]
This says: \( \sigma_{\phi_0}^{-1} \circ \sigma'_{\phi_0} \) is a covering transform of \( \phi^* \) and by the previous lemma it is a symmetry. In other words, considering the dual of the identity
\[
\phi^* \circ \sigma_{\phi_0}^{-1} \circ \sigma'_{\phi_0} = \phi^*,
\]
we get that \( (\sigma_{\phi_0}^{-1} \circ \sigma'_{\phi_0})^* \) is an automorphism of \( K \overline{\nabla} \) and
\[
\sigma_{\phi_0}^{-1} \circ \sigma'_{\phi_0} \in Aut_{K\overline{\nabla}}(\Gamma).
\]
Now, as \( \overline{\nabla} \) is standard, then \( (\sigma_{\phi_0}^{-1} \circ \sigma'_{\phi_0})^* \) is actually the identity. So the association
\[
\phi_0 \mapsto \sigma_{\phi_0}
\]
gives a map \( Aut_{G_F}(G) \to Aut_{\overline{\nabla}}(\Gamma) \) with kernel the inner automorphisms.
6 Right exactness

Take \( \sigma \in \text{Aut}_{\mathcal{V}}(\Gamma) \). Put \( V = \sigma(U) \) and consider a chart of \( E \) giving a vector bundle trivialization, \((\Pi^{-1}V, w, x^1, \ldots, x^n)\), with \( w \circ \sigma = z \). We will denote also by \( \nabla' \) the holomorphic connection on \( V \) induced by \( \nabla \). Since \( \sigma \) is a symmetry, there is a parallel vector bundle isomorphism

\[
\tilde{\sigma} : (\Pi^{-1}U, \nabla') \rightarrow (\Pi^{-1}V, \nabla')
\]

lifting \( \sigma \). Note that \( \tilde{\sigma}^*w = w \circ \tilde{\sigma} = (w \circ \Pi)\tilde{\sigma} = w \circ \sigma \Pi = z \circ \Pi = z \) and \( \sigma_* \frac{\partial}{\partial \sigma} = \frac{\partial}{\partial \omega} \).

The hypothesis \( \sigma \in \text{Aut}_{\mathcal{V}}(\Gamma) \) implies that \( \tilde{\sigma}^*\Psi^i \), the pullback of a fundamental system of linear first integrals \( \Psi^i \) of \( \nabla' \) over \( \Pi^{-1}V \) along \( \tilde{\sigma} \), are linear first integrals of \( \nabla' \) over \( \Pi^{-1}U \). In other words, there exist \( c^i_k \) such that

\[
c^i_k \Phi^k(z, y^1, \ldots, y^n) = \tilde{\sigma}^*\Psi^i(z, y^1, \ldots, y^n)
\]

\[
c^i_k \frac{\partial \Phi^k}{\partial y^j}(z) = \frac{\partial \Psi^i}{\partial x^j}(\sigma(z)) \tilde{\sigma}^*x^j(y^1, \ldots, y^n)
\]

Taking \( x^i \) such that \( \tilde{\sigma}^*x^j(y^1, \ldots, y^n) = y^j \) we have

\[
c^i_k \frac{\partial \Phi^k}{\partial y^j}(z) = \frac{\partial \Psi^i}{\partial x^j}(\sigma(z)) \tag{2}
\]

Set \( f \in K \) such that \( f(\sigma(q))v_{\sigma(q)} = (\sigma_*v)_{\sigma(q)} \), for every \( q \in \Gamma \). If \( AX = X' \) is the matrix equation form of \( \nabla'X = 0 \), then on \( U \) we have

\[
\nabla'X = (AX - X') \otimes dz
\]

Let \( \gamma \) be a path from \( p \in U \) to \( \sigma(p) \in V \) avoiding the singularities of \( \nabla \); then, if \( \nabla'_{\sigma^{-1}}X = 0 \), and \( X \) is analytically extended along \( \gamma \), we have

\[
\nabla'_{\sigma^{-1}}X = \langle (AX - X') \otimes dz, \sigma_* \frac{\partial}{\partial \sigma} \rangle
\]

\[
= \langle (AX - X') \otimes dz, f(z) \frac{\partial}{\partial z} \rangle
\]

\[
= f(z)\nabla'_{\sigma^{-1}}X = 0
\]

So we may take \( \Psi^i \) as the analytic extension of \( \Phi^i \) along \( \gamma \), and \( (2) \) becomes:

\[
c^i_k \frac{\partial \Phi^k}{\partial y^j}(z) = \frac{\partial \Phi^i}{\partial x^j}(\sigma(z)) \tag{3}
\]

Lemma 6.1. The matrix \((c^i_k)\) defined on \((3)\) is in the normalizer of \( G \) in \( G_F \).
Proof: Let $\Gamma_0$ be the Riemann surface obtained as the quotient space of $\Gamma$ by the group of symmetries $Aut_\nabla(\Gamma)$. By definition of symmetry, there exist a vector bundle $E_0$ and a connection $\nabla_0$ such that $(E, \nabla)$ is the pullback of $(E_0, \nabla_0)$ under the covering map induced by the action.

The projection of $\gamma$ to $\Gamma_0$, $\gamma_0$, is a closed curve, and so $(c_k)$ defines an element of the monodromie of $\nabla_0$.

Let $P[X_j^i] \in \mathbb{C}[X_j]^G$, that is, a $G$-invariant polynomial such that:

$$P[\frac{\partial \Phi^i}{\partial y^j}](z) = \iota(f)(z)$$

for some $f \in K$.

So if $f = 0$, then under analytic extension along $\gamma_0$ it is still true that $P[\psi_k \frac{\partial \Phi^k}{\partial y^j}(z)] = 0$. This implies $(c_k) \in G_F$.

On the other hand, $K$ is an extension of the field of meromorphic functions over $\Gamma_0$. So, for an arbitrary non-zero $f$, under analytic extension along $\gamma_0$, $f$ is mapped into $f_\sigma$ by a covering (a Galois) automorphism of $\Gamma$ (of $K$) over $\Gamma_0$ (over $\mathbb{C}(\Gamma_0)$). Whence $P[c_k \frac{\partial \Phi^k}{\partial y^j}](z) = \iota(f_\sigma)(z)$, and so by Galois correspondence $P[X_j^i]$ is invariant under $G(c_k)$ (the conjugate of $G$ by $(c_k)$).

A symmetric argument allows us to conclude that the invariant polynomials under $G$ and under $G(c_k)$ coincide. So, Compoint’s Theorem implies $G(c_k) = G$. This completes the proof. ⭐

The Lemma implies the map defined in section 5 is surjective, and so we get the theorem.

7 Comments

The computation involved in getting the Fano group, as well as the normalizer of the differential Galois group are quite complicated. Among other things, it requires extensive use of the Van Hoeij and Weil’s algorithm [10]. In the case where the Galois group is finite, things may be simplified. Indeed, if the chart $(U, z, y^1, \ldots, y^n)$ are taken so that the section $(z, 1, 0, \ldots, 0)$ is cyclic under $\nabla_v$, then the Picard Vessiot extension is given by $\iota(K)[\frac{\partial \Phi^i}{\partial y^j}]$. So we should be able to replace $R$ with $\mathbb{C}[X_j]^G/F_0$, where $F_0$ is the contraction of $F$. This ideal is what originally Fano worked with, cf. [17].

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