Analytical Study of a Class of Rational Difference Equations

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Abstract. We obtain the solution of the fourth order difference equation

\[ x_{n+1} = \frac{\alpha x_{n-3}}{A + Bx_{n-1}x_{n-3}} \]

with the initial conditions; \( x_{-3} = d, x_{-2} = c, x_{-1} = b, \) and \( x_0 = a \) are arbitrary nonzero real numbers, \( \alpha, A \) and \( B \) are arbitrary constants. The result is used to study the convergence of solutions, the existence of unbounded solutions and the convergence to periodic solutions. We illustrate the results by several numerical examples.

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1. Introduction

Difference equations arise in the study of the evolution of dynamical systems. Their applications to various fields are rapidly increasing. For example, they are frequently used in numerical analysis to describe the different schemes of approximation, \[6\]. Moreover, difference equations are of interest in themselves, especially in view of remarkable analogy with the theory of differential equations. According to this analogy, there are two manners to treat a difference equation: The first is to study the qualitative behavior of solutions, for examples:
Camouzis and Ladas [1] considered the dynamics of the third-order rational difference equation

\[ x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}} \]

with nonnegative parameters \( \alpha, \beta, \gamma, \delta, A, B, C, D \) and with arbitrary nonnegative initial conditions \( x_{-2}, x_{-1}, x_0 \).

Elabbasy et al. [3] studied the qualitative behavior of the difference equations

\[ x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}} \quad \text{and} \quad x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}. \]

The second manner to treat a difference equation is to find the explicit formula of solutions. Contrary to the linear case, there is no general method to find such explicit solution. However, Cinar [2] obtained the solution of the difference equations

\[ x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}} \quad \text{and} \quad x_{n+1} = \frac{ax_{n-1}}{-1 + bx_n x_{n-1}}. \]

Elsayed [4] solved the equations

\[ x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1} x_{n-3}}. \]

In this paper, we solve the following class of difference equations:

\[ x_{n+1} = \frac{\alpha x_{n-3}}{A + Bx_{n-1} x_{n-3}} \quad \text{(1)} \]

with the initial data: \( x_{-3} = d, x_{-2} = c, x_{-1} = b, \) and \( x_0 = a \) are nonzero real numbers. We use the obtained result to determine the forbidden set of initial conditions and to discuss the convergence of solutions. Depending on coefficients \( \alpha \) and \( A \), the existence of unbounded solutions and the convergence to periodic solutions are studied. Our results are confirmed by numerical examples.

2. Solution of equation (1)

The following theorem gives an analytical expression of the solution of (1).
Theorem 2.1. Let \((x_n)_{n=-3}^\infty\) be the solution of \((1)\), then, for all \(n \geq 2\)

\[
x_{4n-3} = \frac{d \alpha^n \prod_{p=0}^{n-2} \left( A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}, \quad (2)
\]

\[
x_{4n-2} = \frac{c \alpha^n \prod_{p=0}^{n-2} \left( A^{2p+2} + Bac \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+1} + Bac \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}, \quad (3)
\]

\[
x_{4n-1} = \frac{b \alpha^n \prod_{p=0}^{n-1} \left( A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-1-i} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+2} + Bac \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}, \quad (4)
\]

\[
x_{4n} = \frac{a \alpha^n \prod_{p=0}^{n-1} \left( A^{2p+1} + Bac \sum_{i=0}^{2p} A^i \alpha^{2p-1-i} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}. \quad (5)
\]

Proof. By induction, we will prove the result for \(x_{4n-3}\). For \(n = 2\), it is easy
to check that \(x_5 = \frac{d \alpha^2 \prod_{p=0}^{1} \left( A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{1} \left( A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}\). Let \(n \geq 2\). Suppose
that the result holds at the step \(n\) and let us prove the result for the step \(n+1\),

\[
x_{4(n+1)-3} = x_{4n+1} = \frac{\alpha x_{4n-3}}{A + Bx_{4n-1}x_{4n-3}}
\]

\[
= \frac{d \alpha^{n+1} \prod_{p=0}^{n-1} \left( A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)} \]

\[
\left[ A \left( A^{2n} + Bbd \sum_{i=0}^{2n-1} A^i \alpha^{2n-1-i} \right) + Bbd \alpha^2n \right]
\]

\[
= \frac{d \alpha^{n+1} \prod_{p=0}^{n-1} \left( A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)} \]

\[
\left( A^{2n+1} + Bbd \sum_{i=0}^{2n-1} A^{i+1} \alpha^{2n-1-i} + \alpha^2n \right)
\]

\[
= \frac{d \alpha^{n+1} \prod_{p=0}^{n-1} \left( A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)} \]

\[
\left( A^{2n+1} + Bbd \sum_{i=0}^{2n} A^i \alpha^{2n-i} \right)
\]

Similarly, we prove the other formulas. \(\square\)

For every \(n \in \mathbb{N}, \alpha, A, B, a, \) and \(b\), denote

\[
P_{p,a,\alpha}^{B,b,d} = \left( A^p(A - \alpha + Bbd) - Bbd\alpha^p \right).
\]

The following corollary gives a simplified analytic expression of the solution
when \(A \neq \alpha\).
Corollary 2.2. If $A \neq \alpha$, then the subsequences of the solution of (1) can be written as:

$$x_{4n-3} = \frac{d\alpha^n(A - \alpha) \prod_{p=0}^{n-2} P_{B,b,d}^{B,2p+2,A,\alpha}}{\prod_{p=0}^{n-1} P_{B,b,d}^{B,2p+1,A,\alpha}}, \quad x_{4n-2} = \frac{c\alpha^n(A - \alpha) \prod_{p=0}^{n-2} P_{B,a,c}^{B,2p+2,A,\alpha}}{\prod_{p=0}^{n-1} P_{B,a,c}^{B,2p+1,A,\alpha}},$$

$$x_{4n-1} = \frac{b\alpha^n \prod_{p=0}^{n-1} P_{B,b,d}^{B,2p+1,A,\alpha}}{\prod_{p=0}^{n-1} P_{B,b,d}^{B,2p+2,A,\alpha}}, \quad x_{4n} = \frac{a\alpha^n \prod_{p=0}^{n-1} P_{B,a,c}^{B,2p+1,A,\alpha}}{\prod_{p=0}^{n-1} P_{B,a,c}^{B,2p+2,A,\alpha}}.$$

Proof. It suffices to use the binomial identity $x^p - y^p = (x - y) \sum_{i=0}^{p-1} x^i y^{p-1-i}$ in the analytical expression of the subsequences (2), (3), (4).

If $A = \alpha$, then every solution of (1) can be expressed by using the function Gamma.

Corollary 2.3. If $A = \alpha$, then

$$x_{4n-3} = \frac{A 2^{2n-2} \Gamma^2 \left( \frac{A}{2Bbd} + n \right) \Gamma \left( \frac{A}{Bbd} + 1 \right)}{Bb \Gamma^2 \left( \frac{A}{Bbd} + 1 \right) \Gamma \left( \frac{A}{Bbd} + 2n \right)},$$

$$x_{4n-2} = \frac{A 2^{2n-2} \Gamma^2 \left( \frac{A}{2Bac} + n \right) \Gamma \left( \frac{A}{Bac} + 2n \right)}{Bb \Gamma^2 \left( \frac{A}{Bac} + 1 \right) \Gamma \left( \frac{A}{Bac} + 2n + 1 \right)},$$

$$x_{4n-1} = \frac{b \Gamma \left( \frac{A}{Bbd} + 2n + 1 \right) \Gamma^2 \left( \frac{A}{Bbd} + 1 \right)}{2^{2n} \Gamma \left( \frac{A}{Bbd} + 1 \right) \Gamma^2 \left( \frac{A}{Bbd} + n + 1 \right)},$$

$$x_{4n} = \frac{a \Gamma \left( \frac{A}{Bac} + 2n + 1 \right) \Gamma^2 \left( \frac{A}{Bac} + 1 \right)}{2^{2n} \Gamma \left( \frac{A}{Bac} + 1 \right) \Gamma^2 \left( \frac{A}{Bac} + n + 1 \right)}.$$

Proof. By (2) we have:

$$x_{4n-3} = \frac{dA^n \prod_{p=0}^{n-2} \left( A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^{2p+1} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+1} + Bbd \sum_{i=0}^{2p} A^{2p} \right)} = \frac{dA^n \prod_{p=0}^{n-2} A^{2p+1} \left( A + (2p+2) Bbd \right)}{\prod_{p=0}^{n-1} A^{2p} \left( A + (2p+1) Bbd \right)}.$$

$$x_{4n-2} = \frac{dA^n \prod_{p=0}^{n-2} A^{2p} \prod_{p=0}^{n-2} \left( A + (2p+2) Bbd \right)}{\prod_{p=0}^{n-1} A^{2p} \prod_{p=0}^{n-1} \left( A + (2p+1) Bbd \right)} = \frac{dA^n \prod_{p=0}^{n-2} \left( A + (2p+2) Bbd \right)}{\prod_{p=0}^{n-1} \left( A + (2p+1) Bbd \right)}.$$

$$x_{4n-1} = \frac{dA \prod_{p=0}^{n-2} Bbd \left( A^{2p} + 2p + 2 \right)}{\prod_{p=0}^{n-1} Bbd \prod_{p=0}^{n-1} \left( A^{2p} + 2p + 1 \right)} = \frac{A \prod_{p=0}^{n-1} \left( A^{2p+1} + 2p + 2 \right)}{Bb \prod_{p=0}^{n-1} \left( A^{2p+1} + 2p + 1 \right)}.$$

$$x_{4n} = \frac{A \prod_{p=0}^{n-1} \left( A^{2p} + 2p + 2 \right)}{Bb \prod_{p=0}^{n-1} \left( A^{2p} + 2p + 1 \right)} = \frac{A \prod_{p=0}^{n-2} A^{2p+1} \left( A + (2p+2) Bbd \right)}{\prod_{p=0}^{n-1} A^{2p} \left( A + (2p+1) Bbd \right)}.$$

Similarly, one can prove the other relations. This ended the proof.

Remark 2.4.
1. The cases $\alpha = 0$ or $B = 0$ are trivial. Then, $\alpha$ and $\beta$ are assumed to be nonzero real numbers.

2. A common hypothesis in the study of rational difference equations is the choice of positive coefficients and initial conditions. Therefore, all the solutions will be automatically well defined. This is the framework, for example, in [1]. It is a problem of great difficulty to determine the good set of initial conditions for which a solution of a rational difference equation is well defined for all $n \geq 0$. The obtention of the explicit solution helps to extend the choice of coefficients and initial conditions.

By corollary, (2.3), If $abcd \neq 0$, $A/Bbd \notin \{1\} \cup \{2n, n \in \mathbb{Z}\}$, then all the solutions of (1) are well defined.

3. Our results cover the four equations considered in Elsayed [4]. For example, the equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}x_{n-3}}$$

corresponds to the case $\alpha = A = B = 1$. The obtained expressions, in this case, are:

$$x_{4n-3} = \frac{d \prod_{i=0}^{n-1} (1 + 2ibd)}{\prod_{i=0}^{n-1} (1 + (2i + 1)bd)}, \quad x_{4n-1} = \frac{b \prod_{i=0}^{n-1} (1 + (2i + 1)bd)}{\prod_{i=0}^{n-1} (1 + (2i + 2)bd)},$$

$$x_{4n-2} = \frac{c \prod_{i=0}^{n-1} (1 + 2iac)}{\prod_{i=0}^{n-1} (1 + (2i + 1)ac)}, \quad x_{4n} = \frac{a \prod_{i=0}^{n-1} (1 + (2i + 1)ac)}{\prod_{i=0}^{n-1} (1 + (2i + 2)ac)},$$

where $\prod_{i=0}^{n-1} A_i = 1$.

According to our results, these expressions can be rewritten using the Gamma function as:

$$x_{4n-3} = \frac{2^{2n-2}\Gamma^2(\frac{1}{bd} + n)\Gamma(\frac{1}{bd})}{b\Gamma^2(\frac{1}{2bd} + 1)\Gamma(\frac{1}{bd} + 2n)}, \quad x_{4n-2} = \frac{2^{2n-2}\Gamma^2(\frac{1}{2ac} + n)\Gamma(\frac{1}{2ac})}{a\Gamma^2(\frac{1}{ac} + 1)\Gamma(\frac{1}{ac} + 2n)},$$

$$x_{4n-1} = \frac{b\Gamma(\frac{1}{bd} + 2n + 1)\Gamma^2(\frac{1}{bd} + 1)}{2^{2n}\Gamma(\frac{1}{bd} + 1)\Gamma^2(\frac{1}{2bd} + n + 1)}, \quad x_{4n} = \frac{a\Gamma(\frac{1}{ac} + 2n + 1)\Gamma^2(\frac{1}{2ac} + 1)}{2^{2n}\Gamma(\frac{1}{ac} + 1)\Gamma^2(\frac{1}{2ac} + n + 1)}.$$

3. Convergence and existence of unbounded solutions

3.1. The case $|\frac{A}{\alpha}| > 1$

Theorem 3.1. Assume that $|\frac{A}{\alpha}| > 1$, then:

1. If $A - \alpha + Bbd \neq 0$ and $A - \alpha + Bac \neq 0$, then every solution of the equation (1) converges toward zero.

2. If $A - \alpha + Bbd = 0$ and $A - \alpha + Bac = 0$, then every solution of the equation (1) converges to a number $l$ if and only if $a = b = c = d = l$. 

Proof. 1. By corollary (2.2)

\[
x_{4n-3} = \frac{d\alpha^n(A - \alpha) \prod_{p=0}^{n-2} \left( A^{2p+2}(A - \alpha + Bbd) - Bbd\alpha^{2p+2} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+1}(A - \alpha + Bbd) - Bbd\alpha^{2p+1} \right)}
\]

\[
= \frac{d\alpha^n(A - \alpha) \prod_{p=0}^{n-2} A^{2p+2}(A - \alpha + Bbd) \prod_{p=0}^{n-2} \left( 1 - \frac{Bbd}{A - \alpha + Bbd} \left( \frac{\alpha}{A} \right)^{2p+2} \right)}{\prod_{p=0}^{n-1} A^{2p+1}(A - \alpha + Bbd) \prod_{p=0}^{n-1} \left( 1 - \frac{Bbd}{A - \alpha + Bbd} \left( \frac{\alpha}{A} \right)^{2p+1} \right)}
\]

\[
= \frac{d\alpha^n(A - \alpha) A^{n-1} \prod_{p=0}^{n-2} \left( 1 - \frac{Bbd}{A - \alpha + Bbd} \left( \frac{\alpha}{A} \right)^{2p+2} \right)}{(A - \alpha + Bbd) A^{2n-1} \prod_{p=0}^{n-2} \left( 1 - \frac{Bbd}{A - \alpha + Bbd} \left( \frac{\alpha}{A} \right)^{2p+1} \right)}
\]

Denote by \( \beta = \frac{Bbd}{A - \alpha + Bbd} \) and let \((U_p)_p\) be the sequence defined by \( U_p = \frac{1 - \beta(\frac{\alpha}{A})^{2p+2}}{1 - \beta(\frac{\alpha}{A})^{2p+1}} \). Then we can write

\[
x_{4n-3} = \frac{d(\frac{\alpha}{A})^n(A - \alpha)}{(A - \alpha + Bbd) \left( 1 - \beta(\frac{\alpha}{A})^{2n-1} \right)} \prod_{p=0}^{n-2} U_p.
\]

For \( p \in \mathbb{N} \) big enough, We have either: \( U_p > 1 \) or \( 0 < U_p < 1 \). By Taylor expansion, we obtain

\[
U_p = (1 - \beta(\frac{\alpha}{A})^{2p+2})(1 + \beta(\frac{\alpha}{A})^{2p+1} + o(\frac{\alpha}{A})^{2p+1})
\]

\[
= 1 + \beta(1 - \frac{\alpha}{A})(\frac{\alpha}{A})^{2p+1} + o(\frac{\alpha}{A})^{2p+1}
\]

then \( U_p \sim 1 + \beta(1 - \frac{A}{\alpha})(\frac{\alpha}{A})^{2p+1} \) which is the general term of a convergent infinite product. We deduce that \((x_{4n-3})_n\) converges toward zero. Similarly we do for the other subsequences.

2. If \( A - \alpha + Bbd = 0 \) and \( A - \alpha + Bac = 0 \), then the subsequences \((x_{4n-3})_n, (x_{4n-1})_n\) are constant: \( x_{4n-3} \equiv d \) and \( x_{4n-1} \equiv b \). Similarly, the subsequences \((x_{4n-2})_n, (x_{4n})_n\) are constant: \( x_{4n-2} \equiv c \) and \( x_{4n} \equiv a \). Then every solution of the equation \( (\text{3.1}) \) converges to a number \( l \) if and only if \( a = b = c = d = l \).

\[ \square \]

3.2. The case \( |\frac{A}{\alpha}| = 1 \)

We distinguish two cases: \( A = \alpha \) and \( A = -\alpha \).

Theorem 3.2. If \( A = \alpha \), then every solution of equation \( (\text{3.1}) \) converges toward zero.
Proof. Denote $e = \frac{A}{Bbd}$. From the proof of corollary (2.3), we deduce:

$$x_{4n-3} = \frac{A \prod_{p=1}^{n-1} \left( \frac{A}{Bbd} + 2p \right)}{Bb \prod_{p=0}^{n-1} \left( \frac{A}{Bbd} + 2p + 1 \right)}$$

$$= \frac{A}{Bb(e+1)} \prod_{p=1}^{n-1} \frac{e + 2p}{e + 2p + 1}$$

$$= \frac{A}{Bb(e+1)} \prod_{p=1}^{n-1} \frac{\frac{e}{2p} + 1}{\frac{e}{2p} + 1}$$

Let $(W_p)_p$ be the sequence defined by $W_p = \frac{\frac{e}{2p} + 1}{\frac{e}{2p} + 1}$, it is clear that: (i) $\lim_{p \to \infty} W_p = 1$, (ii) For $p$ big enough, $0 < W_p < 1$.

By Taylor expansion, $W_p = (1 + \frac{e}{2p})(1 - \frac{e}{2p} + o(\frac{1}{p})) = 1 - \frac{1}{2p} + o(\frac{1}{p})$, which is a general term of a divergent infinite product, since for $p$ big enough, $0 < W_p < 1$, then $\lim_{n \to \infty} \prod_{p=1}^{n-1} W_p = 0$, therefore $\lim_{n \to \infty} x_{4n-3} = 0$. Similarly, one can prove that the limits of the other subsequences is zero. Hence $(x_n)_{n=-3}^{\infty}$ converges to zero.

Theorem 3.3. If $A = -\alpha$, then every solution of equation (11) is unbounded.
Proof. If we replace $\alpha$ by $-A$ in the first term of equation (2), we obtain

\[
x_{4n-3} = \frac{d(-A)^n \prod_{p=0}^{n-2} \left( A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i (-A)^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i (-A)^{2p-i} \right)}
\]

\[
= \frac{d(-A)^n \prod_{p=0}^{n-2} A^{2p+1} \left( A + Bbd \sum_{i=0}^{2p+1} (-1)^{2p+1-i} \right)}{\prod_{p=0}^{n-1} A^{2p} \left( A + Bbd \sum_{i=0}^{2p} (-1)^{2p-i} \right)}
\]

\[
= \frac{d(-A)^n \prod_{p=0}^{n-2} A^{2p+1} \left( A + Bbd \sum_{k=0}^{2p} (-1)^k \right)}{\prod_{p=0}^{n-1} A^{2p} \left( A + Bbd \sum_{k=0}^{2p} (-1)^k \right)}
\]

\[
= \frac{d(-A)^n \prod_{p=0}^{n-2} A^{2p+2}}{\prod_{p=0}^{n-1} A^{2p} \left( A + Bbd \sum_{k=0}^{2p} (-1)^k \right)}
\]

\[
= \frac{d(-A)^n A^{2n-2} \prod_{p=0}^{n-2} A^{2p}}{A^{2n-2} \prod_{p=0}^{n-2} A^{2p} \left( A + Bbd \sum_{k=0}^{2p} (-1)^k \right)}
\]

\[
= \frac{d(-A)^n}{\left( A + Bbd \right)^n}
\]

\[
= \frac{d}{(-1 - e^{-1})^n}
\]

Hence:

(i) If $|1 + e^{-1}| > 1$, then $(x_{4n-3})_n$ converges to zero.

(ii) If $|1 + e^{-1}| < 1$, then $(x_{4n-3})_n$ is not bounded and $\lim_{n \to \infty} |x_{4n-3}| = +\infty$. 
In other hand, If we replace $\alpha$ by $-A$ in the first term of equation (1), we obtain

$$x_{4n-1} = \frac{b(-A)^n \prod_{p=0}^{n-1} \left( A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i (-A)^{2p-i} \right)}{\prod_{p=0}^{n-1} \left( A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i (-A)^{2p+1-i} \right)}$$

$$= \frac{b(-A)^n \prod_{p=0}^{n-1} A^{2p} (A + Bbd \sum_{i=0}^{2p} (-1)^{2p-i})}{\prod_{p=0}^{n-1} A^{2p+1} (A + Bbd \sum_{i=0}^{2p+1} (-1)^{2p+1-i})}$$

$$= b(-A)^n \prod_{p=0}^{n-1} \left( \frac{A^{2p} (A + Bbd)}{A^{2p+2}} \right)$$

$$= b(-A)^n \prod_{p=0}^{n-1} \left( \frac{A^{2p} (A + Bbd)}{A^{2p+2}} \right)$$

$$= b(-1)^n \left( 1 + \frac{Bbd}{A} \right)^n$$

$$= b(-1 - e^{-1})^n$$

Hence:

(iii) If $|1+e^{-1}| > 1$, then $(x_{4n-1})_n$ is not bounded and $\lim_{n \to \infty} |x_{4n-3}| = +\infty$.

(iv) If $|1 + e^{-1}| < 1$, then $(x_{4n-1})_n$ converges to zero.

Combining propositions (i), (ii), (iii) and (iv), the proof is ended. □

Remark 3.4. 1. Similarly, we prove that we have either $(|x_{4n-2}|)_n$ diverges to $\infty$ and $(x_{4n})_n$ converges to zero or $(x_{4n-2})_n$ converges to zero and $(|x_{4n}|)_n$ diverges to $\infty$.

2. Note that in the case $A = -\alpha$, the initial conditions intervene in the nature of the subsequences of $(x_n)_{n=-3}^\infty$, therefore in the nature of the solution $(x_n)_{n=-3}^\infty$ itself.

3.3. The case $|\frac{A}{\alpha}| < 1$

Theorem 3.5. If $|\frac{A}{\alpha}| < 1$, then for every solution $(x_n)_{n=-3}^\infty$ of the equation (1), the subsequences $(x_{4n-3})_n$, $(x_{4n-1})_n$, $(x_{4n-2})_n$ and $(x_{4n})_n$ converge.
Proof. Let us prove that \((x_{4n-3})_n\) converge.

\[
x_{4n-3} = \frac{d\alpha^n(A - \alpha) \prod_{p=0}^{n-2} \left(A^{2p+2}(A - \alpha + Bbd) - Bbd\alpha^{2p+2} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+1}(A - \alpha + Bbd) - Bbd\alpha^{2p+1} \right)} = \frac{d\alpha^n(A - \alpha) \prod_{p=0}^{n-2} -Bbd\alpha^{2p+2} \left(- \frac{A-\alpha+Bbd}{Bbd} \left(\frac{A}{\alpha}\right)^{2p+2} + 1 \right)}{\prod_{p=0}^{n-1} -Bbd\alpha^{2p+1} \prod_{p=0}^{n-2} \left(1 - \frac{A-\alpha+Bbd}{Bbd} \left(\frac{A}{\alpha}\right)^{2p+2} \right)} = \frac{d\alpha^n(A - \alpha) \prod_{p=0}^{n-2} \alpha^{2p+1} \prod_{p=0}^{n-2} \left(1 - \frac{A-\alpha+Bbd}{Bbd} \left(\frac{A}{\alpha}\right)^{2p+2} \right)}{-Bbd \prod_{p=0}^{n-2} \alpha^{2p+1} \prod_{p=0}^{n-1} \left(1 - \frac{A-\alpha+Bbd}{Bbd} \left(\frac{A}{\alpha}\right)^{2p+2} \right)}
\]

Denote \(\gamma = \frac{A-\alpha+Bbd}{Bbd}, \lambda = \frac{A}{\alpha}\) and \(V_p = \frac{1-\gamma\lambda^{2p+2}}{1-\gamma\lambda^{2p+1}}\). Using these notations, we obtain

\[
x_{4n-3} = \frac{\alpha - A}{Bb(1 - \gamma\lambda^{2n-1})} \prod_{p=0}^{n-2} V_p
\]

Since \(|\lambda| < 1\), then the sequence \(\left(\frac{\alpha - A}{Bb(1 - \gamma\lambda^{2n-1})}\right)_n\) converges toward \(\frac{\alpha - A}{Bb}\). By the Taylor expansion, we obtain

\[
V_p = \frac{1-\gamma\lambda^{2p+2}}{1-\gamma\lambda^{2p+1}} = (1 - \gamma\lambda^{2p+2})(1 + \gamma\lambda^{2p+1} + o(\lambda^{2p+1})) = 1 + \gamma(1-\lambda)\lambda^{2p+1} + o(\lambda^{2p+1})
\]

Then \(\ln(V_p) \sim \gamma(1-\lambda)\lambda^{2p+1}\), which is the general term of a convergent series, then the sequence \((x_{4n-3})_n\) is convergent. Similarly, one can prove that the other subsequences converge.

\[\blacksquare\]

Remark 3.6. Sequences \((x_{4n-3})_n\) and \((x_{4n-1})_n\) are related by the equations:

\[
x_{4(n+1)-3} = \frac{\alpha x_{4n-3}}{A + Bx_{4n-1}x_{4n-3}}, \quad (6)
\]

and

\[
x_{4(n+1)-1} = \frac{\alpha x_{4n-1}}{A + Bx_{4(n+1)-3}x_{4n-1}}, \quad (7)
\]

Denote by \(l_3, l_2, l_1\) and \(l_0\) respectively, the limits of the subsequences \((x_{4n-3})_n\), \((x_{4n-2})_n\), \((x_{4n-1})_n\) and \((x_{4n})_n\).

Passing to the limit as \(n\) goes to infinity in the equation (6), we obtain:
\[ l_3 = \frac{\alpha l_1}{A + Bl_3}, \] 
then 
\[
(S_1) : \begin{cases} 
    l_3 = 0, \\
    \text{or} \\
    l_1 = \frac{\alpha - A}{Bl_3}.
\end{cases}
\]

Passing to the limit as \( n \) goes to infinity in the equation (7), we obtain: 
\[ l_1 = \frac{\alpha l_1}{A + Bl_3}, \] 
then 
\[
(S_2) : \begin{cases} 
    l_3 = 0, \\
    \text{or} \\
    l_1 = \frac{\alpha - A}{Bl_3}.
\end{cases}
\]

Combining systems (S_1) and (S_2), since \( \alpha \neq A \), we obtain: 
either 
\[ l_3 = l_1 = 0 \]
or 
\[
(S) : \begin{cases} 
    l_3 = \frac{\alpha - A}{Bl_3}, \\
    \text{and} \\
    l_1 = \frac{\alpha - A}{Bl_3}.
\end{cases}
\]

The proposition \( l_3 = l_1 = 0 \) contradicts the fact that the infinite product \( \prod_{p \geq 0} V_p \) converges, in fact if \( \lim_{n \to \infty} \prod_{p=0}^n V_p = 0 \), then \( \lim_{n \to \infty} \sum_{p=0}^n \ln(V_p) = -\infty \) and this contradicts the fact that the series \( \sum_{p \geq 0} \ln(V_p) \) converges. Hence the only possibility is that \( l_1 \neq 0, l_3 \neq 0 \) and (S): 
\[
\begin{cases} 
    l_3 = \frac{\alpha - A}{Bl_1}, \\
    \text{and} \\
    l_1 = \frac{\alpha - A}{Bl_3}.
\end{cases}
\]

In fact \( l_3 = \frac{\alpha - A}{Bl_1} \) is equivalent to \( l_1 = \frac{\alpha - A}{Bl_3} \), then (S) is equivalent to \( l_3 = \frac{\alpha - A}{Bl_1} \). Consider the function \( f \) defined on \( \mathbb{R}^* \) as \( f(x) = \frac{\alpha - A}{Bx} \), we have \( f \circ f = \text{Id} \) and, \( l_1 \) and \( l_3 \) are related by \( f(l_1) = l_3 \). Similarly, we prove that \( l_0 \) and \( l_2 \) are related by the relation \( f(l_0) = l_2 \).

4. Periodicity

**Definition 4.1.** A solution \((x_n)_{n=-3}^\infty \) of (1) is called periodic with period \( p \) if there exists an integer \( p \), such that 
\[ x_{n+p} = x_n, \quad \forall n \geq -3 \quad (8) \]
A solution is called periodic with prime period \( p \) if \( p \) is the smallest positive integer for which (8) holds.

In the sequel, we need the following lemma, [5], which describes when a solution of (1) converges to a periodic solution of (1).
Lemma 4.2. Let \((x_n)_{n=-3}^\infty\) be a solution of (1). Suppose there exist real numbers \(l_3, l_2, l_1, l_0\) such that
\[
\lim_{n\to+\infty} x_{4n+j} = l_j \text{ for all } j = -3, \ldots, 0
\]
Let \((y_n)_{n=-3}^\infty\) be the period-4 sequence of real numbers such that
\[
y_{4n+j} = l_j \text{ for all } j \geq -3
\]
The following statements are true:
1. \((y_n)_{n=-3}^\infty\) is a period-4 solution of (1).
2. \(\lim_{n\to+\infty} x_{4n+j} = y_j \text{ for all } j \geq -3\)

Now, the field is ready to state the following theorem:

Theorem 4.3.
1. If \(|A_\alpha| > 1, A - \alpha + Bbd \neq 0 \text{ and } A - \alpha + Bac \neq 0\), then the equation (1) has no periodic-
   \(p\) solution, for all \(p \geq 2\).
2. If \(|A_\alpha| > 1, A - \alpha + Bbd = 0 \text{ and } A - \alpha + Bac = 0\), then the equation (1) has a periodic-4 solution.
3. If \(|A_\alpha| = 1\) then the equation (1) has no periodic-
   \(p\) solution, for all \(p \geq 2\).
4. If \(|A_\alpha| < 1\) then the equation (1) has a periodic-4 solution.

Proof.
1. If \(|A_\alpha| > 1, A - \alpha + Bbd \neq 0 \text{ and } A - \alpha + Bac \neq 0\), then, by theorem (3.1), every solution of (1) converges to 0, hence, the solution is not allowed to be periodic.
2. If \(|A_\alpha| > 1, A - \alpha + Bbd = 0 \text{ and } A - \alpha + Bac = 0\), then, by theorem (3.1), \(\lim_{n\to+\infty} x_{4n-3} = d, \lim_{n\to+\infty} x_{4n-2} = c, \lim_{n\to+\infty} x_{4n-1} = b, \lim_{n\to+\infty} x_{4n} = a\). Applying the lemma (4.2), the sequence: \(d, c, b, a, d, c, b, a \ldots\) is a periodic-4 solution.
3. The case \(A = \alpha\) is similar to (1). Suppose that \(A = -\alpha\), by contradiction, assume that the equation (1) has a periodic-
   \(p\) solution \((x_n)_{n=-3}^\infty\). Let \(x_{n+1}, \ldots, x_{n+p}\) be a \(p\) consecutive values of this solutions and let \(M = \max\{x_{n+1}, \ldots, x_{n+p}\}\), it follows that for all \(k \geq -3, x_k \leq M\). Contradiction with theorem (3.3).
4. If \(|A_\alpha| < 1\) then, by theorem (3.5), there exist real numbers \(l_3, l_2, l_1, l_0\) such that
\[
\lim_{n\to+\infty} x_{4n+j} = l_j \text{ for all } j = -3, \ldots, 0
\]
Applying lemma (4.2), the sequence \(l_3, l_2, l_1, l_0, l_3, l_2, l_1, l_0 \ldots\) is a periodic-

Remark 4.4. From the previous proof, we deduce:
1. If \(|A_\alpha| > 1, A - \alpha + Bbd = 0 \text{ and } A - \alpha + Bac = 0, d = b, c = a \text{ and } b \neq c\) then (1) has a 2 prime periodic solution \(a, b, a, b \ldots\).
2. If $|\frac{A}{\alpha}| < 1$, $A - \alpha + Bbd = 0$ and $A - \alpha + Bac = 0$ then, from the proof of theorem (3.5), we deduce the values of the real numbers $l_3, l_2, l_1, l_0$, thus: $l_3 = \frac{A - \alpha}{Bb}, l_2 = \frac{A - \alpha}{Ba}, l_1 = \frac{A - \alpha}{Bd}, l_0 = \frac{A - \alpha}{Bc}$.

5. Examples

Example 1. In this example, we illustrate the case $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd \neq 0$ and $A - \alpha + Bac \neq 0$, we choose $a = 3; b = -4; c = 2; d = -1; B = 1; A = 1.05; \alpha = 1$.

We remark in the figure (1) that the solution is oscillating about zero with a decreasing amplitude. In fact, the solution has to converge to zero, According to theorem (3.1).

Example 2. In this example, we illustrate the case $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd = 0$ and $A - \alpha + Bac = 0$, we choose $a = c = 2; b = d = -2; B = -2; A = 9; \alpha = 1$.

We see, figure (2), that the obtained solution is a 2 prime periodic solution. This is coherent with remark (4.4).

Example 3. In this example, we illustrate the case $A = -\alpha$, we choose $a = 0.1; b = 0.2; c = 0.3; d = -0.4; B = 1; A = 0.5; \alpha = -0.5$

We remark, figure (3), that the solution is oscillating about zero with an increasing amplitude. By theorem (3.3), it is an unbounded solution.

Example 4. In this example, we illustrate the case $|\frac{A}{\alpha}| < 1$, $A - \alpha + Bbd \neq 0$ and $A - \alpha + Bac \neq 0$, we choose $a = -1.2; b = 0.4; c = -0.3; d = 0.9; B = 1; A = 0.64; \alpha = 1$

We obtain a 4 prime periodic solution, Figure (4). According to the remark.
Figure 2. $|\frac{A}{\alpha}| > 1$, 2 prime periodic solution

Figure 3. $A = -\alpha$, the solution is unbounded

\[ \text{the solution is the 4 prime periodic sequence} \]

\[ 0.9, 0.4, -0.3, -1.2, 0.9, \ldots \]
Figure 4. $|A| < 1$, 4 prime periodic solution

References

[1] E. Camouzis and G. Ladas, Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures, Advances in Discrete Mathematics and Applications Volume 5, Chapman & Hall/CRC (2008).

[2] C Cinar, On the difference equation $x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}$, Appl Math Comp., 158 (2004) pp:813-816.

[3] E M Elabbasy, H El-Metwalli and E M Elsayed, On the difference equation $x_{n+1} = a x_n - \frac{bx_n}{c x_n - d x_{n-1}}$, Adv. Differ. Equ., 2006 :1-10, Articla ID 82579, 2006.

[4] E M Elsayed, On the solution of some difference equations, European Journal of Pure And Applied Mathematics, 4(3):287-303, 2011.

[5] E.A.Grove and G. Ladas:Periodicities in Nonlinear Difference Equations, Advances in Discrete Mathematics and Applications Volume 4, Chapman & Hall/CRC (2005).

[6] F. Kadhi and A. Trad, Characterization and Approximation of the convex envelope of a function, Journal of Application Theory and Application, Vol 110, (2001) pp: 301-306.

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