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How Complete is the Classification of $\mathcal{W}$-Symmetries?

by

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Abstract

In 2D conformal quantum field theory, we continue a systematic study of $\mathcal{W}$-algebras with two and three generators and their highest weight representations focusing mainly on rational models. We review the known facts about rational models of $\mathcal{W}(2, \delta)$-algebras. Our new rational models of $\mathcal{W}$-algebras with two generators all belong to one of the known series. The majority of $\mathcal{W}$-algebras with three generators –including the new ones constructed in this letter– can be explained as subalgebras or truncations of Casimir algebras. Nonetheless, for one solution of $\mathcal{W}(2, 4, 6)$ we reveal some features that do not fit into the pattern of Casimir algebras or orbifolds thereof. This shows that there are more $\mathcal{W}$-algebras than those predicted from Casimir algebras (or Toda field theories). However, most of the known rational conformal field theories belong to the minimal series of some Casimir algebra.

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1. Introduction

The classification of all conformal field theories which are rational with respect to some extended conformal algebra (or \( W \)-algebra) is an open problem since rational conformal field theories (RCFTs) [1] and \( W \)-algebras [2] have been discovered.

For the construction of \( W \)-algebras existing for generic value of the central charge \( c \) several methods have been developed: Casimir constructions [3], conserved currents of quantum Toda Field theories [4] and quantized Drinfeld-Sokolov reductions [5][6] (for a recent review see [7] and references therein). All these constructions seem to be equivalent and there has been some progress classifying their classical analogs [8][9] using group theoretical methods.

Direct constructions of \( W \)-algebras with given dimensions of the simple fields [10][11] revealed the existence of \( W \)-algebras which cannot be obtained directly by the above constructions. They appeared for isolated values of the central charge and the study of their highest weight representations (HWRs) [13][14] provided at least one series of RCFTs [12] where the symmetry algebra seems to be no subalgebra or truncation of some generically existing \( W \)-algebra. However, the majority of rational models are related to a generically existing algebra [7]: Some explicitly known models are minimal models of Casimir algebras \(^1\), others are extensions of the Virasoro algebra and diagonalize the non-diagonal modular invariant partition functions of the ADE-classification [15][13].

Although the rational model of \( W(2, 8) \) at \( c = -\frac{712}{7} \) belongs to the minimal series of the Casimir algebra \( WE_8 \), also a strange construction for its characters by a projection of a direct sum of tensor products of RCFTs is known [14]. Recently, for \( W(2, 8) \) at \( c = -\frac{3164}{23} \) an exceptional relation to representations of the modular group has been found [16] leading to more easily computable expressions for the characters [17] than those of the corresponding Casimir algebra \( WE_7 \). \( W(2, \delta) \)-algebras also exist for \( c = 1 - 8\delta \) and \( c = 1 - 3\delta \) with effective central charge \( \tilde{c} = c - 24h_{\text{min}} = 1 \). These models completed the classification of \( \tilde{c} = 1 \) RCFTs [12]. Already the \( c = 1 \)-classification of RCFTs [18][19] itself leads to \( W \)-algebras. It predicts a \( W(2, 4, \delta) \) for arbitrary dimension \( \delta \) and the algebras \( W(2, 16), W(2, 9, 16), W(2, 36) \) at \( c = 1 \). Finally, \( W(2, \delta) \)-algebras exist for Virasoro non-minimal values of \( c = c_{1,s} \) [20]. We verified explicitly that adding two further fields of dimension \( \delta \) restricts the relevant HWRs to a finite set for \( \delta \in \{3, 5\} \). The explicit form of the characters of the corresponding RCFTs is still not completely clear [21].

In this letter we construct further \( W(2, \delta) \)-algebras and show that they all belong to one of the above series. Thus, there is a chance that at least all rational models of \( W(2, \delta) \)-algebras are now known in principle.

For \( W \)-algebras with three and more generators less is known although direct computations have recently been pushed forward [22]. One aim of this letter is to continue a systematic study in order to understand rational models of \( W \)-algebras with three generators better. In particular, for \( W(2, 4, 6) \) there is one solution in addition to the Casimir algebra of \( B_3/C_3 \) and the bosonic projection of the \( N = 1 \) Super Virasoro algebra [23] which has not been explained yet [24]. Our study of the HWRs of this solution shows that it is neither related to a Casimir algebra nor an orbifold.

We will use the notations and methods described in [10][13][14][25].

2. New \( W \)-algebras with two generators

First, we discuss new \( W \)-algebras with one additional generator. We denote the additional simple field by ‘\( W \)’ and the corresponding dimension by \( \delta \). For \( 9 \leq \delta \leq 11 \) the Jacobi identities lead to values of \( c \) contained in the following table:

\(^1\) In the earlier version of this letter certain RCFTs had not been identified as minimal models of Casimir algebras.
The column ‘interpretation’ contains the parametrization of the central charge if the $W$-algebra is either related to the ADE-classification of modular invariant partition functions \cite{15}, or one of the parabolic algebras with $c = 1 - 8\delta$, $c = 1 - 3\delta$ \cite{12} or is related to non-minimal models at $c = c_{1,s}$ \cite{20}.

Now we are going to discuss our results for the individual algebras in more detail.

$W(2, 9), W(2, 11)$: The first algebra has already been studied in \cite{26}. The space spanned by the Jacobi polynomials has dimension 2. In the case of $W(2, 11)$ the space spanned by the Jacobi polynomials is three dimensional. For both algebras only $C_{WW}^W = 0$ is possible. $W(2, 9)$ is consistent for five values of the central charge.

For $W(2, 11)$ we are able to check two out of the three conditions if we use only fields up to dimension 18. Both conditions are satisfied for the three rational values of the central charge listed in the above table. $W(2, 10)$: The space spanned by the Jacobi polynomials has dimension 3. All three conditions are satisfied for the four values of $c$ given in the above table. A null field of dimension 18 that is linear in $W$ exists for $c = -2$. In this case $C_{WW}^W$ is non-zero so that it is important to notice that the coupling constant to this null field vanishes. Thus, the algebra is also consistent for $c = -2$.

For $W(2, 10)$ with $C_{WW}^W = 0$ A. Kliem has already derived the necessary condition $c = \frac{8}{35}$ \cite{26}. Our calculation shows that $W(2, 10)$ is indeed consistent for this value of the central charge. For $c = \frac{8}{35}$ and $c = \frac{25}{36}$ both models are Virasoro-minimal and the field $W$ can be identified with the field of dimension $h_{7,10,1,6} = 10$ resp. $h_{12,13,1,7} = 10$ in the Virasoro-minimal model.

At $c = -2$ we have $h_{1,2,5,1} = 10$. Therefore $W(2, 10)$ might be a subalgebra of $W(2, 3)$ for this value of the central charge. In order to investigate this further, we apply the Jacobi-identity-method \cite{13}\cite{14} to the HWRs of $W(2, 10)$ at $c = -2$. One obtains that one must either have

$$ h \in \left\{ 3, \frac{3}{32}, \frac{5}{32}, \frac{21}{32}, \frac{45}{32}, \frac{8}{15} \right\} $$

(1)
with corresponding fixed non-zero $W_0$-eigenvalue $w$ or the following relation among $h$ and $w$ must be satisfied:

$$w = -\frac{C_W}{5613300} (8h + 1)(8h - 3)(h - 1)h^2. \tag{2}$$

Among (1) we do indeed see $h = 3$ which should stem from the spin three field in the original symmetry algebra. (2) is very much reminiscent of the corresponding relation for $w^2$ in the normal sector of $\mathcal{W}(2,3)$. Note that the linearity of (2) designates that the field $W$ of the algebra $\mathcal{W}(2,10)$ is quadratic in the spin three field of $\mathcal{W}(2,3)$. The remaining rational $h$-values in (1) stem from the twisted sector of $\mathcal{W}(2,3)$. The additional $h$-values in (1) and the additional zeros in (2) have half-integer difference to those turning up for $\mathcal{W}(2,3)$ and are thus easily explained by a splitting of characters.

The most interesting point in these observations is that our explicit calculations for $\mathcal{W}(2,10)$ do so neatly reproduce the results for $\mathcal{W}(2,3)$ at $c = -2$ [27]. We conclude that the $(1,s)$-models are not rational with respect to the corresponding $\mathcal{W}(2,\delta)$-algebra. However, these algebras may be considered as subalgebras of $\mathcal{W}(2,\delta,\delta,\delta)$-algebras [20]. We have studied the representations of the larger algebras for $\delta \in \{3,5\}$ explicitly, revealing rational models at $c_{1,s}$. The zero modes of the additional simple fields form representations of $su(2)$ in the $L_0$-eigenspace to eigenvalue $h$. In particular, for $\mathcal{W}(2,3,3,3)$ at $c = -2$ the only admissible representations are singlets with $h \in \{0, -\frac{1}{2}\}$ and doublets with $h \in \{\frac{1}{2}, 1\}$. Note that this are precisely the zeroes of (2).

Note that the subalgebra-structure at $c = -2$ for $\mathcal{W}(2,3)$ generalizes to the whole $(1,s)$-series. In order to see this consider the fusion rules of the fields $\phi_{1,p}$ with dimension $h_{1,s;1,p}$ ($p$ odd) at $c = c_{1,s}$:

$$[\phi_{1,1}] \times [\phi_{1,p}] = [\phi_{1,p}], \quad [\phi_{1,p}] \times [\phi_{1,p}] = \sum_{q=1}^{2p-1} [\phi_{1,q}]. \tag{3}$$

$h_{1,s;1,3} = 2s - 1$ is always odd, $h_{1,s;1,5} = 3h_{1,s;1,3} + 1 = 2(3s - 1)$ always even and one has $h_{1,s;1,7} = 2h_{1,s;1,5} + 1$. Therefore the fusion rules (3) specialize for $p = 3$ and $p = 5$ to:

$$[\phi_{1,3}] \times [\phi_{1,3}] = [\phi_{1,1}], \quad [\phi_{1,5}] \times [\phi_{1,5}] = [\phi_{1,1}] + [\phi_{1,5}]. \tag{4}$$

Thus, the energy-momentum tensor $L$ (corresponding to $\phi_{1,1}$) forms a closed subalgebra with the primary field $W = \phi_{1,p}$ for $p = 3$ and $p = 5$. So, this argument predicts a $\mathcal{W}(2,h_{1,s;1,3})$ and a $\mathcal{W}(2,h_{1,s;1,5})$ at $c = c_{1,s}$. The next algebras in the second series should be $\mathcal{W}(2,16)$ at $c = -7$ and $\mathcal{W}(2,22)$ at $c = -\frac{32}{5}$. 

The construction of $\mathcal{W}(2,9)$ involves fields up to dimension 16. For $\mathcal{W}(2,10)$ one needs fields up to dimension 18. $\mathcal{W}(2,12)$ and $\mathcal{W}(2,13)$ are determined by three resp. four independent conditions from the Jacobi identities. As there are major technical problems dealing even with fields of dimension 18 which would yield only one condition we did not study these algebras.

### 3. Mixed $\mathcal{W}$-algebras with three generators

$\mathcal{W}(2,\delta_1,\delta_2)$-algebras that contain bosonic as well as fermionic fields have not been studied very intensely. We shall therefore focus on some of these ‘mixed’ algebras. Let $U$ be the fermionic field and $V$ the boson. If the dimension of $V$ is odd all coupling constants vanish. In this case a Jacobi identity of type $(U,U,V)$ cannot be satisfied and the algebra does not exist (this generalizes the observation of [10] for two additional bosonic fields with odd dimension). Thus, the only interesting cases are those where the dimension of $V$ is even. For the first few algebras with $\delta_1 = d(U) > \frac{3}{2}$ and $\delta_2 = d(V) > 2$ we obtain the following explicit results:
If the algebra $\mathcal{W}(2, \frac{5}{2}, 6)$ would exist, it would contain a $\mathcal{W}(2, \frac{5}{2})$-subalgebra which is consistent only for $c = -\frac{13}{14}$. Even for this value of the central charge the Jacobi identity involving the fermion twice and the boson once cannot be satisfied. Thus, $\mathcal{W}(2, \frac{5}{2}, 6)$ does not exist at all. This observation suggests that algebras $\mathcal{W}(2, \delta_1, \delta_2)$ with $\delta_2 \geq 2\delta_1$, $\delta_1 \geq \frac{5}{2}$ do not exist because otherwise it would have a fermionic $\mathcal{W}(2, \delta_1)$-subalgebra which are believed to be classified and none of them admits further extensions. Furthermore, it is not very plausible that the mixed Jacobi identities can be satisfied under these conditions.

The algebra $\mathcal{W}(2, \frac{5}{2}, 4)$ was already predicted by coset arguments [28] and explicitly realized in terms of five free fermions [29][30]. We obtain the same structure constants as in [29][30]. We also see that there are no other $\mathcal{W}(2, \frac{5}{2}, 4)$-algebras besides the one constructed by Ahn. $\mathcal{W}(2, \frac{5}{2}, 4)$ reduces to $\mathcal{W}(2, \frac{5}{2})$ at $c = -\frac{13}{14}$ which was the first well studied non-linear fermionic algebra [2][13].

There have been earlier studies of $\mathcal{W}(2, \frac{7}{2}, 4)$ as well [31]. Before studying Jacobi identities involving both $U$ and $V$ we obtain the same structure constants as in [31]. The mixed Jacobi identities, however, restrict $c$ to the two values given in the above table, thus completing these earlier studies.

Of course, one would like to identify the above algebras. Candidates are $N = 1$ supersymmetric models as well as subalgebras of $\mathcal{WB}(n)$, which should be distinguished from the purely bosonic $\mathcal{WB}_n$-algebras. $\mathcal{WB}(n)$ algebras can be obtained either by quantum Hamiltonian reduction of the affine Lie Super algebras $B(0, n)^{(1)} = osp(1 | 2n)^{(1)}$ [32] or by coset constructions using the non-simply laced affine Lie algebra $B_n^{(1)}$ [32][33]. The algebras $\mathcal{WB}(n)$ contain $n$ bosonic fields of dimension 2, ..., 2$n$ and one fermionic field of dimension $n + \frac{1}{2}$. The central charge for a level $k$ minimal model is given by [32][33]:

$$c = \left(n + \frac{1}{2}\right) \left(1 - \frac{2n(2n - 1)}{(k + 2n)(k + 2n - 1)}\right).$$

$\mathcal{W}(2, \frac{5}{2}, 4)$ is known to be identical to $\mathcal{WB}(2)$ [29][30]. The only candidates for Super Virasoro minimal models are $\mathcal{W}(2, \frac{7}{2}, 4)$ and $\mathcal{W}(2, \frac{5}{2}, 4)$ at $c = 1$. However, most $c$-values lie in the minimal series of $\mathcal{WB}(n)$ where $n = \delta_1 - \frac{1}{2}$. The corresponding levels $k$ are given in the above table. It is therefore natural to assume that these algebras can be interpreted as subalgebras of $\mathcal{WB}(n)$ for these specific values of the central charge. In order to give further evidence for this assumption we have studied the HWRs of these algebras examining Jacobi identities.
In this table, $\tilde{v}$ denotes the eigenvalue of $V_0(C_{UU}^{-1})$. A finite set of $h$-values has to be understood in the sense that also the corresponding values of $\tilde{v}$ are fixed.

In particular for $W(2,\frac{7}{2},4)$ and $W(2,\frac{9}{2},4)$ at $c=1$ the explicit values of $h$ show that these models are indeed not Super Virasoro minimal. These two algebras confirm the expectation that $W(2,\delta_1,4)$ exists for $c=1$. This is inferred from the construction of modular invariant partition functions in $c=1$ theories [18].

One of the two exceptions – $W(2,\frac{7}{2},6)$ at $c=\frac{561}{2}$ – is certainly not rational because $c$ lies above the bound $c=\frac{5}{2}$ for rational models with two bosons and one fermion [14]. Our explicit results are in agreement with this prediction. The result for $W(2,\frac{9}{2},6)$ at $c=-\frac{304}{5}$ also has a structure which is typical for $\mathcal{W}$-algebras which are not rational.

### 4. Representations of $\mathcal{W}(2,4,4)$ and $\mathcal{W}(2,4,5)$

In this and the next section we shall examine those bosonic $\mathcal{W}$-algebras with two additional generators which have been constructed by [10],[11]. $\mathcal{W}(2,3,4)$ has been identified as the Casimir algebra $\mathcal{WA}_3$. For the remaining $\mathcal{W}(2,\delta_1,\delta_2)$-algebras we explicitly studied HWRs. We denote the additional generators by $U$ and $V$ where the dimension of $U$ is less or equal to the dimension of $V$. The eigenvalues of $U_0$ and $V_0$ will be denoted by $u$ and $v$. Usually it is more convenient to consider the normalization independent quantities $\tilde{u} := u(C_{UU}^{-1})^{-1}$, $\tilde{v} := v(C_{VV}^{-1})^{-1}$.

The results for $\delta_1 = 4$ are listed in the following table:

| $\delta_2$ | $c$ | $h \in \{0,1,\frac{9}{11},\frac{10}{11}\}$ | $\tilde{c}$ | interpret. |
|------------|-----|-------------------------------------|-------------|-------------|
| 4          | 1   | $h \in \{0,\frac{19}{11},\frac{26}{11},\frac{27}{11}\}$ | $\frac{1}{11}$ | $\mathcal{WD}_4$, c7.8 |
| 5          | 1   | $h \in \{0,\frac{1}{11},\frac{20}{11},\frac{29}{11},\frac{30}{11}\}$ | $\frac{1}{13}$ | $\mathcal{WD}_5$, c9.10 |

Table 4: Representations of $\mathcal{W}(2,4,\delta_2)$-algebras
Here, a finite set of $h$-values has to be understood in the sense that also the eigenvalues $u$, $v$ of the additional simple fields are fixed to some discrete values. In the above table this data has not been presented, neither have the multiplicities of $h$-values been indicated.

One remarks that the values of the central charge of $\mathcal{W}(2,4,4)$ and $\mathcal{W}(2,4,5)$ lie in the minimal series of $\mathcal{WD}_n$ which is given by coprime $p$, $q$ and

$$c_{p,q} = n \left(1 - (2n-1)(2n-2)\frac{(p-q)^2}{pq}\right), \quad p, q \geq 2(n-1). \quad (6)$$

Although $c = -\frac{253}{7}$ can be parametrized for $n = 5$ according to (6), this is not a minimal model because $p = 5$ is too small. This is reflected in our explicit results which are typical for non-rational models. For all other values of the central charge our explicit calculations demonstrate rationality. Furthermore, the $h$-values can be obtained from the general formula presented in [34] for the minimal models of Casimir algebras specialized to $\mathcal{D}_n$. This supports the identification of these algebras with (sub-)algebras of $\mathcal{WD}_n$ for particular values of $c$ [20].

5. The exception: An uninterpreted generic solution for $\mathcal{W}(2,4,6)$

Finally, let us focus on the algebra $\mathcal{W}(2,4,6)$ and try to clarify some open questions about the interpretation of one solution. For this algebra there exist three solutions [24]. The first one has been identified [35][25] with the bosonic projection of the $N = 1$-Super-Virasoro algebra proposed by [23]. The existence of the second solution was not completely clear in [11] but now it is supposed to yield the Casimir algebras of $\mathcal{WB}_3$, $\mathcal{WC}_3$ [24]. The solution we are going to discuss is the third one. The coupling constants for the simple fields of this third solution to $\mathcal{W}(2,4,6)$ are determined by [11][26]:

$$C^V_{UU} = \frac{10(7c + 68)(2c - 1)(c + 50)}{9(5c^2 + 309c - 14)(c + 24)} C^U_{UU}$$

$$C^V_{VV} = \frac{9(14c + 11)(5c + 22)(c + 76)(c + 24)(c - 1)}{5(1106c^5 + 72355c^4 + 839120c^3 - 574940c^2 - 1961680c - 2510336)} C^V_{VV}$$

$$(C^U_{VV})^2 = -\frac{(5c^2 + 309c - 14)^2}{(5c + 22)(5c + 3)(c - 26)}$$

$$(C^V_{VV})^2 = -\frac{50(1106c^5 + 72355c^4 + 839120c^3 - 574940c^2 - 1961680c - 2510336)^2(c + 49)}{243(14c + 11)(7c + 68)(5c + 3)(2c - 1)(c + 76)(c + 24)^3(c - 1)(c - 26)}.$$ 

For the solution (7) we have checked the validity of all Jacobi-Identities connecting the simple additional fields.

The solution we are interested in gives rise to a null field of dimension 11 for generic value of the central charge. This is something observed so far only for orbifolds of $\mathcal{W}$-algebras. Using this null field we were able to derive one relation among the $L_0$-eigenvalue $h$, $u$ and $v$ for all values of the central charge. Thus, at most two of the three parameters $h$, $u$ and $v$ are in fact independent. This is a further hint for an orbifold-construction or something similar. If this solution for $\mathcal{W}(2,4,6)$ should indeed be an orbifold, at least some of the simple fields $\phi_i$ in the original algebra would lead to HWRs with $h_i$ equal to the dimension of $\phi_i$. A corresponding highest weight vector is given by $|h_i\rangle = \phi_i |v\rangle$. Note that this is indeed true for the representations of $\mathcal{W}(2,10)$ at $c = -2$.

The algebra $\mathcal{W}(2,4,6)$ reduces to $\mathcal{W}(2,4)$ for $c\in\{ 1, -49, -76, -\frac{11}{4} \}$ as we can see from (7). The representations of $\mathcal{W}(2,4)$ have already been studied in [14]. The models at $c = 1$ and $c = -76$...
were shown to be non-rational, $c = -49$ is not accessible to the null field method leaving $c = -\frac{11}{14}$ as the only rational model among them. Here, only $h = \frac{4}{3}$ lies in $\mathbb{Z}_{\geq 0}$, and thus is the only candidate for a field that has been projected out. Note however that at $c = -\frac{11}{14}$ the bosonic projection of the $N = 1$-Super Virasoro algebra [25] and the solution $(7)$ for $W(2, 4, 6)$ reduce to the same $W(2, 4)$ and we just see remnants of the orbifolding in the first solution. Therefore it is important to study as many additional rational models for $W(2, 4, 6)$ as possible.

For $c = -\frac{13}{11}$ there exists a second null field with dimension 8. With the aid of this null field we were able to derive two further relations among $h, u$ and $v$ such that we obtained a rational model. At $c = -\frac{10}{11}$ a null field with dimension 10 exists which also enabled us to derive a rational model. We list the $h$-values of these two rational models in the following table:

| $c$ | $h$-values | $\tilde{c}$ |
|-----|------------|------------|
| $-\frac{13}{11}$ | $0, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{7}{3}, \frac{5}{3}, \frac{9}{3}, \frac{12}{3}, \frac{16}{3}, \frac{20}{3}, \frac{24}{3}, \frac{29}{3}, \frac{36}{3}, \frac{45}{3}, \frac{60}{3}, \frac{180}{31}$ | $17$ |
| $-\frac{10}{11}$ | $0, 3, \frac{4}{3}, 5, \frac{17}{3}, \frac{1}{3}, \frac{7}{3}, \frac{9}{3}, \frac{10}{3}, \frac{18}{3}, \frac{26}{3}, \frac{1}{3}, \frac{5}{3}, \frac{8}{3}, \frac{23}{3}, \frac{5}{23}, \frac{3}{23}, 33, \frac{39}{3}, \frac{47}{3}, \frac{67}{3}, \frac{131}{3}$ | $14$ |

Table 5: $h$-values for rational models of $W(2, 4, 6)$

Most of the $h$-values have unique values for $u$ and $v$, i.e. a multiplicity of at most one. The only exception is $h = \frac{3}{3}$ at $c = -\frac{10}{11}$ which admits two different values for $u, v$ thus leading to a possible multiplicity of 2. Note that $c = -\frac{13}{15}$ appears in the minimal series of $WB_3/WC_3$ and $c = -\frac{10}{14}$ in the minimal series of $WB_4/WC_4$.

At $c = -50$ and $c = -\frac{26}{23}$ there are also null fields of dimension 8 resp. 10 but explicit calculations indicated non-rational models with several one-parameter branches of representations. We omit the explicit results for these models.

Clearly, at $c = -\frac{13}{15}$ there are neither integer nor half-integer dimensions such that we have to give up the idea of a usual orbifold. Furthermore, it is neither significant that in both models presented in Table 5 there are some representations with $h \in \mathbb{Z}_{\geq 0}$ because this is not so for $c = -\frac{11}{13}$.

One might expect that a classical analogue for this exceptional solution exists. However, it was already pointed out in [36] that this solution as well as the orbifold of the $N = 1$ Super Virasoro algebra cannot be obtained via a standard limiting procedure due to divergent structure constants. The rescaling $U' = c^{-\frac{1}{2}}U$, $V' = c^{-\frac{1}{4}}V$ might lead to a well-defined classical algebra but in the limit the commutators of $U'$ and $V'$ will not contain any central term. It is also interesting to consider the limit $c \to \infty$ of the vacuum preserving algebra (VPA) [36] for these two solutions. It is well-defined in the basis of the rescaled fields $L, U'$ and $V'$. Some of the normal ordered products containing an even number of simple fields $U', V'$ (e.g. $\mathcal{N}(U', U')$) do not decouple in the limit. Thus, it does not lead to a finite Lie algebra – in contrast to the limits of VPAs of Casimir algebras [36].

In summary, our explicit results for the solution $(7)$ show some features of an orbifold which clearly conflict with a construction similar to Casimir-algebras. Even more, our studies of the HWRs also rule out the possibility of an orbifold construction of a different generically existing $\mathcal{W}$-algebra.

6. Conclusion

Our study of $\mathcal{W}(2, \delta)$-algebras did not lead to any algebra that was not covered by a known series – except that we found the first example of an orbifold of a non-unitary $(1, s)$-algebra. It is remarkable
that sporadic non-rational cases known so far [10][14] did not turn up any more for $8 < \delta \leq 11$: Neither did we find an algebra existing for irrational values of the central charge nor did models with several one parameter branches of HWRs at rational values of $c$ turn up (compare $W(2,8)$ e.g. at $c = -\frac{1015}{2}$). Thus, there is a good chance that all series and exceptional cases presented in the introduction do really cover all models which are rational with respect to some $W(2,\delta)$-algebra.

One can explain all rational models of mixed algebras with three generators and the rational models of $W(2,4,4)$ and $W(2,4,5)$ as subalgebras or contractions of the Casimir algebras $WB(n)$, $WD_{n}$. However, we showed that one solution for $W(2,4,6)$ is neither a Casimir algebra nor an orbifold of a different generically existing algebra. Thus, it has become even more mysterious why this solution exists (there are no doubts about its existence any more). A similar phenomenon has been observed for $W(2,3,4,5)$ [22] where also generic null fields appear and a possible candidate for an orbifold construction is not known. In order to define a classical limit a similar rescaling has to be performed [22] and also the behaviour of the VPA in the limit $c \to \infty$ is identical to that of the unidentified solution to $W(2,4,6)$. Thus, these unexplained algebras may have a common origin, or even belong to a large class of yet unknown algebras.

A possibly related problem turns up in the classification of $c = 24$ conformal field theories with a single character [37]. Although 71 models of this type are admitted [37] only 40 of them have been constructed explicitly up to now [38 – 40]. Most constructions start from Niemeier lattices which yield symmetry algebras that consist of abelian currents and vertex operators with conformal dimensions greater or equal to one [40][37]. Other theories are more complicated, e.g. the symmetry algebra of the one invariant under the Monster group is a $W(2^{196884},3^{21296876})$ – a surprisingly small algebra compared to the order of the Monster. For the construction of the remaining 31 theories one probably needs new methods that could be related to unexplained $W$-algebras like $W(2,4,6)$.

In summary, we have shown that quantum $W$-algebras exist which cannot be explained as the analogue of a classical $W$-algebra 2) or subalgebra respective orbifold thereof. Certainly, exceptional $W$-algebras have to be better understood – in particular the yet uninterpreted solution for $W(2,4,6)$ – and exotic constructions on them have to be studied (compare $W(2,8)$ at $c = -\frac{712}{7}$ [14]) before one can use $W$-algebras for the classification of RCFTs.

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2) For example Casimir algebras have classical analogs.
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