Power law in the angular velocity distribution of a granular needle

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Abstract. – We show how inelastic collisions induce a power law with exponent \(-\frac{1}{3}\) in the decay of the angular velocity distribution of anisotropic particles with sufficiently small moment of inertia. We investigate this question within the Boltzmann kinetic theory for an elongated granular particle immersed in a bath. The power law persists so long as the collisions are inelastic for a large range of angular velocities provided the mass ratio of the anisotropic particle and the bath particles remains small. Suggestions for observing this peculiar feature are made.

Granular gases are systems in which macroscopic particles lose a fraction of their kinetic energy at each collision [1,2]. In the absence of energy supply, starting from an homogeneous state, the granular fluid cools down [3] and after a finite time one observes a spontaneous symmetry breaking with the formation of clusters as well as of shear waves and convection. Conversely, if an external energy supply is continuously brought to the particles, the system may reach a non-equilibrium steady state (NESS), whose properties differ significantly from those of thermal equilibrium (breakdown of the equipartition [4,5], non-gaussian statistics, modified hydrodynamics [6],...). All those characteristics are intimately related to the dissipative nature of collisions.

The studies of granular gases have focused mainly on spherical particles [7]. Recent work [8] on nearly smooth spherical grains stressed the relevance of rotational degrees of freedom for the hydrodynamic behavior. The rotational energy is ubiquitous for anisotropic particles, and one can reasonably expect that the anisotropy (generally present in real granular systems) might introduce additional effects. Only few studies have addressed up to now this question (free cooling state in a three-dimensional system of needles [9], breakdown of the equipartition between different degrees of freedom [10,11]). Experimentally, the dynamics of shaken granular dimer gas has been investigated [12].

In this letter, we show that the stationary angular velocity distribution of a thin granular needle of mass \(M\) acted upon by inelastic collisions in a bath of point masses \(m\) shows a power-law decay in a large range of angular velocities provided \(M/m \ll 1\). We relate this feature to
the properties of a rigorous solution of the Boltzmann equation in the $M/m \to 0$ limit. The region $M/m \ll 1$ studied here is opposite to the Brownian motion case $M/m \gg 1$. We discovered that the dynamics of a low mass anisotropic particle is qualitatively different. It turns out that when the ratio $M/m$ is finite but small this unphysical (zero moment of inertia) solution continues to manifest itself through a power-law decay observed in a large range of angular velocities whereas a gaussian-like decay is restored only for the high-energy tail. When the restitution coefficient is set to 1, this peculiar behavior disappears. Clearly, in real systems collisions with the ends of the elongated particle should be also considered. The analysis performed in Ref. [11] has shown that when the length of the particle is large compared to its width and also to the size of the bath particles the collisional contributions from the extremities are negligible and predictions of the simplified model considered here should apply. We also present suggestions for an experiment which would permit to observe the power-law decay of the angular distribution function.

Quite recently a class of stationary states with power-law high-energy tails (implying an infinite energy) has been found for homogeneous granular systems with the suggestion of implementing them by energy injection at large scales only [15, 16]. It should be stressed however, that the underlying mechanism is completely different from the one found here. We deal with an impurity (the needle) immersed in the granular fluid (which is in a NESS). The power law appears owing to the collisional rescaling of the distribution of the bath with the weight depending on the position of the impact point. The anisotropy of the impurity plays in this rescaling an important role. The possibility of a power law for an impurity has been already mentioned in the review on the Maxwell model [17]. However, to get this effect the same power law had to be assumed for the granular bath. Our results reveal an essentially new mechanism for a physically relevant case of a free motion between collisions.

We consider a two-dimensional system consisting of an infinitely thin homogeneous needle of length $L$ and moment of inertia $I = ML^2/12$, with the fixed center of mass. The needle undergoes inelastic collisions with the bath particles. Between collisions it rotates freely around the axis passing through its center and perpendicular to the plane of motion. The only degree of freedom of the needle is its orientation specified by a unit vector $\mathbf{u}$ that points along its axis. The rate of change of the orientation $\dot{\mathbf{u}} = \omega \mathbf{u}_\perp$ involves the angular velocity $\omega \in [-\infty, +\infty]$ and a unit vector $\mathbf{u}_\perp$ perpendicular to $\mathbf{u}$.

At a binary collision the needle is hit by a point mass of the bath at some point $\lambda \mathbf{u}$, $|\lambda| < L/2$ where $\lambda$ is the algebraic abscissa along the needle (see Fig. 1). The relative velocity $\mathbf{V}$ at the point of impact equals

$$
\mathbf{V} = \mathbf{v} - \lambda \dot{\mathbf{u}} = \mathbf{v} - \lambda \omega \mathbf{u}_\perp
$$

where $\mathbf{v}$ denotes the velocity of the bath particle.

The instantaneous binary collisions conserve the total angular momentum. Labelling the post-collisional quantities with a star and using for any vector $\mathbf{w}$ the convenient notation $w_\perp = \mathbf{w} \cdot \mathbf{u}_\perp$, $w_\parallel = \mathbf{w} \cdot \mathbf{u}$ we thus write

$$
I \omega^* + \lambda m v_\perp^* = I \omega + \lambda m v_\perp
$$

The relative velocity will be assumed to change according to the collision law

$$
V_\perp^* = -\alpha V_\perp
$$

$$
V_\parallel^* = V_\parallel
$$

involving the normal restitution coefficient $0 \leq \alpha \leq 1$. 

By combining Eqs. (1)-(4), we find the collisional change of the particle velocity
\[ v_\perp^* = v_\perp - \frac{I (1 + \alpha)V_\perp}{I + m \lambda^2}. \] (5)
and the corresponding change of angular velocity
\[ \omega^* = \omega + \frac{(1 + \alpha)V_\perp m^2 \lambda}{I + m \lambda^2}. \] (6)
The inverse transformation is obtained by replacing \( \alpha \) by \( \alpha^{-1} \).

The stationary Boltzmann equation for the angular velocity distribution \( F(\omega) \) of the needle expresses the invariance of the distribution under collisional processes. The gain term corresponding to the post-collisional angular velocity \( \omega^* \) must be exactly compensated by the loss term involving the angular velocity \( \omega \) as the pre-collisional one. The equation reads
\[ \int_{L/2}^{L/2} d\lambda \int_{-L/2}^{L/2} dv_\perp |v_\perp - \lambda \omega| \left( \frac{F(\omega^*) \Phi_B(v^*)}{\alpha^2} - F(\omega) \Phi_B(v) \right) = 0, \] (7)
where the pre-collisional velocities \( v^* \) and \( \omega^* \) follow from Eqs. (2)-(6) by switching the roles of initial and final velocities and replacing \( \alpha \) by \( \alpha^{-1} \). \( \Phi_B(v) \) is a velocity distribution describing the steady state of the bath.

The integration over the velocity component \( v_\parallel \) in Eq. (4) can be readily performed as this variable does not show in the arguments of \( F \). Putting then \( v_\perp = (\lambda \omega)y \) one finds the integral equation
\[ \int_{L/2}^{L/2} d\lambda \int_{-L/2}^{L/2} dy |y| F(\omega (1 + y)) \phi_B(\lambda \omega y) = F(\omega) \int_{L/2}^{L/2} d\lambda \int_{-L/2}^{L/2} dy |y| - 1 \phi_B(\lambda \omega y) \] (8)
with \( \phi_B(v) = \int dv_\perp |\Phi_B(v)| \)

We consider first the limit where both the mass of the needle and the coefficient of restitution \( \alpha \) tend to zero. In this case the velocity of the bath particles is not modified by collisions whereas the angular velocity of the asymptotically massless needle is reset after each collision acquiring instantaneously the value \( v_\perp/\lambda \), i.e. the ratio of the bath particle approach velocity to the abscissa of the impact point. The Boltzmann equation (8) becomes
\[ \int_{L/2}^{L/2} d\lambda \int_{-L/2}^{L/2} dy |y| F(\omega (1 + y)) \phi_B(\lambda \omega) = F(\omega) \int_{L/2}^{L/2} d\lambda \int_{-L/2}^{L/2} dy |y| - 1 \phi_B(\lambda \omega y) \] (9)
The exact solution \( F(\omega) \) of Eq. (9) is obtained by scaling the bath distribution \( \phi_B(v) \) with the weight depending on the position of the impact point
\[ F(\omega) = \int_{-L/2}^{L/2} d\lambda \left( \frac{2 \lambda}{L} \right)^2 \phi_B(\lambda \omega) \] (10)
It should be stressed that this result does not require specific assumptions about the distribution of the bath particles. Indeed, Eq. (10) implies the power law decay \( F(\omega) \propto \omega^{-3} \) for
large values of $\omega$ provided the second moment of the distribution $\phi_B(v)$ exists. The specific shape of $\phi_B(v)$ is irrelevant. We also note that for the rescaled distribution $\Omega$ the second and higher moments are not defined. It is thus natural to think that the stochastic variable $\omega$ is subject to Levy flights.

When the bath is at thermal equilibrium described by the Maxwell distribution $\phi_B(v) = \sqrt{m/2\pi T} \exp(-mv^2/2T)$, the stationary distribution takes the form

$$F(\omega) = -\frac{4}{2\pi mL\omega^2} \exp\left(-\frac{mL^2\omega^2}{8T}\right) + \frac{4T}{L^2m\omega^3} \text{erf}\left(\frac{mL^2}{8T\omega}\right)$$

(11)

The power-law is reached for $\omega > \sqrt{8T/(mL^2)}$. Fig. 2 shows the logarithm of the distribution function versus the angular velocity. The inset in Fig. 2 displays the log-log plot illustrating the crossover to the power-law behavior.

Eqs. (10,11) correspond to the maximal inelasticity $\alpha = 0$. When $M = 0$ and $0 < \alpha < 1$, the solution of the Boltzmann equation behaves asymptotically as

$$F(\omega) \sim \frac{4T(1 + \alpha)}{mL^2(1 - \alpha)\omega^3}.$$  

(12)
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Fig. 2 – Log-linear plot $F(\omega)$ of the “zero mass” needle. The inset displays the log-log plot showing the power-law decay for sufficiently large $\omega$ values for $T/(mL^2) = 1/2$.

Direct numerical integration of Eq.(8) (with $I = 0$), along the same lines as in [18], is shown in Fig.3 for various values of $\alpha$. The inset of Fig.3 displays $H(\omega) = F(\omega)\frac{mL^2(1-\alpha)}{4T(1+\alpha)}\omega^3$ versus $\omega$, indicating that the asymptotic behavior is rapidly reached. It should be noticed that for elastic collisions ($\alpha = 1$) all these effects disappear as then

$$F(\omega) = \sqrt{\frac{1}{2\pi T}} \exp\left(-\frac{I\omega^2}{2T}\right)$$

and the limit $I \to 0$ does not yield a probability distribution.

We turn now to the physically relevant case of a needle with a small but finite mass $0 < M \ll m$. An asymptotic analysis of the distribution $F(\omega)$ can be then performed on the basis of Eq. (8). Assuming that at large angular velocities the distribution function is gaussian, $F(\omega) \sim \exp(-I\omega^2/(2T))$ (an assertion well supported by numerical results [19]), one can show that $\overline{T} = (1 + \alpha)T/2$, irrespective of the mass ratio of the needle and the bath particles (it is an exact asymptotics in the Brownian motion limit $M/m \to \infty$). By means of an accurate numerical solution of the Boltzmann equation (details will be given elsewhere [19]), we could essentially confirm this asymptotic behavior inferred first by analytic arguments. In addition, in the case under consideration ($0 < M/m \ll 1$), we could identify a sub-leading multiplicative term (decreasing less rapidly than a gaussian) which depends on the mass ratio as well as on the coefficient of restitution. This combined behavior occurs in the asymptotic regime.

However, our really important conclusion is that before the very high energy asymptotics
is attained the power-law $\sim \omega^{-3}$ found for a massless needle remains present with a crossover occurring only at $\omega_c \sim \sqrt{2T/I}$. This situation is illustrated in Fig. 4 where the log-linear plot of the angular distribution functions is displayed for mass ratios $M/m = 0, 0.005, 0.01$, and the dashed curve represents the zero-mass limit. For $\omega < \omega_c$, one observes the power-law decay (see the log-log plot in the inset) whereas the gaussian behavior is recovered beyond $\omega_c$. Contrary to the zero mass case, the existence of a gaussian decay at very large values of the velocity leads to a finite granular temperature for a needle with a finite mass, and more generally all moments of the distribution functions are now well defined. In fact, one can show that the granular temperature defined as the second moment of the distribution times the moment of inertia goes to zero with the mass of the needle if $\alpha < 1$, whereas the temperature remains independent of the mass if $\alpha = 1$. This result is also a consequence of the existence of a well defined solution of the Boltzmann equation for a zero-mass needle when $\alpha < 1$. In conclusion, the behavior for $M/m \ll 1$ is qualitatively different from that observed in the Brownian motion limit involving an new collisional scaling mechanism producing the $\omega^{-3}$ power law.

Since the power-law regime corresponds to low and moderate angular velocities, it should be accessible to an experimental observation. Recent experiments on intensely vibrated granular systems using high speed photography [20, 21], image analysis techniques and particle tracking [22, 23] show that many quantities (granular temperature, velocity profiles,...) can
Fig. 4 – Log-Log plot of $F(\omega)$ for a needle with a mass $M/m = 0.005, 0.01. (dashed curve corresponds to $M = 0.0$). The inset shows the crossover between the power law and the gaussian-like asymptotics be precisely measured allowing the study of the equipartition or fluctuation theorem. Several orders of magnitude of velocities can be monitored [12,24] within the region where the power-law decay is expected. More specifically, by immersing in a two-dimensional granular gas a thin rigid needle which rotates around a perpendicular axis, it should be possible to obtain the angular velocity distribution in the relevant region of $\omega$. By using different needle lengths or different sizes of the bath particles one should observe successive crossover velocities. We expect that the collisional rescaling found here will induce the power law also in the case of anisotropic particles of a general convex shape in the region accessible to experiment.

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