New Identities of Hall-Littlewood Polynomials and Rogers-Ramanujan Type

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Abstract

We prove two new summation formulae of Hall-Littlewood polynomials over partitions into bounded parts and derive some new multiple $q$-identities of Rogers-Ramanujan type.

1 Introduction

The Rogers-Ramanujan identities (see \cite{B, H}) :

$$\sum_{n=0}^{\infty} \frac{q^{n^2+an}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} (1-q^n)^{-1},$$

where $a = 0$ or $1$, are among the most famous $q$-series identities in partitions and combinatorics. Since their discovery the Rogers-Ramanujan identities have been proved and generalized in various ways (see \cite{B, E, F, G} and the references cited there). In \cite{G}, by adapting a method of Macdonald for calculating partial fraction expansions of symmetric formal power series, Stembridge gave an unusual proof of Rogers-Ramanujan identities as well as fourteen other non trivial $q$-series identities of Rogers-Ramanujan type and their multiple analogs. Although it is possible to describe his proof within the setting of $q$-series, two summation formulas of Hall-Littlewood polynomials were a crucial source of inspiration for such kind of identities. One of our original motivations was to look for new multiple $q$-identities of Rogers-Ramanujan type through this approach, but we think that the new

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summation formulae of Hall-Littlewood polynomials are interesting for their own.

Throughout this paper we will use the standard notations of \(q\)-series (see, for example, [3]). Set \((x)_0 := (x; q)_0 = 1\) and for \(n \geq 1\)

\[
(x)_n := (x; q)_n = \prod_{k=1}^{n} (1 - xq^{k-1}),
\]

\[
(x)_\infty := (x; q)_\infty = \prod_{k=1}^{\infty} (1 - xq^{k-1}).
\]

For \(n \geq 0\) and \(r \geq 1\), set

\[
(a_1, \ldots, a_r; q)_n = \prod_{i=1}^{r} (a_i; q)_n, \quad (a_1, \ldots, a_r; q)_\infty = \prod_{i=1}^{r} (a_i; q)_\infty.
\]

Let \(n \geq 1\) be a fixed integer and \(S_n\) the group of permutations of the set \(\{1, 2, \ldots, n\}\). Let \(X = \{x_1, \ldots, x_n\}\) be a set of indeterminates and \(q\) a parameter. For each partition \(\lambda = (\lambda_1, \ldots, \lambda_n)\) of length \(\leq n\), if \(m_i := m_i(\lambda)\) is the multiplicity of \(i\) in \(\lambda\), then we also note \(\lambda\) by \((1^{m_1} 2^{m_2} \ldots)\). Recall that the Hall-Littlewood polynomials \(P_{\lambda}(X, q)\) are defined by [10, p.208] :

\[
P_{\lambda}(X, q) = \prod_{i \geq 1} \frac{(1 - q)^{m_i}}{q^{m_i}} \sum_{\omega \in S_n} w \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i < j} \frac{x_i - qx_j}{x_i - x_j} \right),
\]

where the factor is added to ensure the coefficient of \(x_1^{\lambda_1} \cdots x_n^{\lambda_n}\) in \(P_{\lambda}\) is 1.

For a parameter \(\alpha\) define the auxiliary function

\[
\Psi_q(X; \alpha) := \prod_{i} (1 - x_i)^{-1}(1 - \alpha x_i)^{-1} \prod_{j < k} \frac{1 - qx_j x_k}{1 - x_j x_k}.
\]

Then it is well-known [10, p. 230] that the sums of \(P_{\lambda}(X, q)\) over all partitions and even partitions are given by the following formulae :

\[
\sum_{\lambda} P_{\lambda}(X, q) = \Psi_q(X; 0), \quad (1)
\]

\[
\sum_{\lambda} P_{2\lambda}(X, q) = \Psi_q(X; -1). \quad (2)
\]
For any sequence $\xi \in \{\pm 1\}^n$ set $X^\xi = \{x_1^{\xi_1}, \ldots, x_n^{\xi_n}\}$. Then, by summing $P_\lambda$ over partitions with bounded parts, Macdonald [10, p. 232] and Stembridge [13] have respectively generalized (1) and (2) as follows:

\begin{align*}
\sum_{\lambda \leq k} P_\lambda(X, q) &= \sum_{\xi \in \{\pm 1\}^n} \Psi_q(X^\xi; 0) \prod_i x_i^{k(1-\xi_i)/2}, \\
\sum_{\lambda \leq 2k} P_\lambda(X, q) &= \sum_{\xi \in \{\pm 1\}^n} \Psi_q(X^\xi; -1) \prod_i x_i^{k(1-\xi_i)}.
\end{align*}

(3)

(4)

Now, for parameters $\alpha$, $\beta$ define another auxiliary function

$$
\Phi_q(X; \alpha, \beta) := \prod_i \frac{1-\alpha x_i}{1-\beta x_i} \prod_{j<k} \frac{1-q x_j x_k}{1-x_j x_k}.
$$

Then the following summation formulae similar to (1) and (2) for Hall-Littlewood polynomials hold true [10, p.232] :

\begin{align*}
\sum_{\lambda' \text{even}} c_\lambda(q) P_\lambda(X, q) &= \Phi_q(X; 0, 0), \\
\sum_{\lambda} d_\lambda(q) P_\lambda(X, q) &= \Phi_q(X; q, 1),
\end{align*}

(5)

(6)

where $\lambda'$ is the conjugate of $\lambda$ and

$$
c_\lambda(q) = \prod_{i \geq 1} (q; q^2)_{m_i(\lambda)/2}, \quad d_\lambda(q) = \prod_{i \geq 1} \frac{(q)_{m_i(\lambda)}}{(q^2; q^2)_{[m_i(\lambda)/2]}}.
$$

In view of the numerous applications of (3) and (4) it is natural to seek such extensions for (5) and (6). However, as remarked by Stembridge [13, p. 475], in these other cases there arise complications which render doubtful the existence of expansions as explicit as those of (3) and (4). We noticed that these complications arise if one wants to keep exactly the same coefficients $c_\lambda(q)$ and $d_\lambda(q)$ as in (5) and (6) for the sums over bounded partitions. Actually we have the following

**Theorem 1** For $k \geq 1$,

\begin{align*}
\sum_{\lambda' \text{even}} c_{\lambda,k}(q) P_\lambda(X, q) &= \sum_{\xi \in \{\pm 1\}^n} \Phi_q(X^\xi; 0, 0) \prod_i x_i^{k(1-\xi_i)/2}, \\
\sum_{\lambda \leq k} d_{\lambda,k}(q) P_\lambda(X, q) &= \sum_{\xi \in \{\pm 1\}^n} \Phi_q(X^\xi; q, 1) \prod_i x_i^{k(1-\xi_i)/2}.
\end{align*}

(7)

(8)
where
\[ c_{\lambda,k}(q) = \prod_{i=1}^{k-1} (q; q^2)_{m_i(\lambda)}, \quad d_{\lambda,k}(q) = \prod_{i=1}^{k-1} \frac{(q)_{m_i(\lambda)}}{(q^2; q^2)_{[m_i(\lambda)/2]}}, \]

**Remark.** We were led to such extensions by starting from the right-hand side instead of the left-hand side and inspired by the similar formulae corresponding to the case \( q = 0 \) of Hall-Littlewood polynomials [8], i.e., Schur functions. In the initial stage we made also the Maple tests using the package ACE [1]. In the case \( q = 0 \), the right-hand sides of (3), (4), (7) and (8) can be written as quotients of determinants and the formulae reduce to the known identities of Schur functions [8].

For any partition \( \lambda \) it will be convenient to adopt the following notation:
\[ (x)_{\lambda} := (x; q)_{\lambda} = (x)_{\lambda_1-\lambda_2}(x)_{\lambda_2-\lambda_3}\cdots, \]
and to introduce the general \( q \)-binomial coefficients
\[ \left[ \begin{array}{c} n \\lambda \\ \lambda \end{array} \right] := \frac{(q)_{n}}{(q)_{n-\lambda}(q)_{\lambda}}, \]
with the convention that \( \left[ \begin{array}{c} n \\lambda \\ \lambda \end{array} \right] = 0 \) if \( \lambda_1 > n \). If \( \lambda = (\lambda_1) \) we recover the classical \( q \)-binomial coefficient. Finally, for any partition \( \lambda \) we denote by \( l(\lambda) \) the length of \( \lambda \), i.e., the number of its positive parts, and \( n(\lambda) := \sum \frac{\lambda_i}{2} \).

The following is the key \( q \)-identity which allows to produce identities of Rogers-Ramanujan type.

**Theorem 2** For \( k \geq 1 \),
\[
\sum_{\ell(\lambda) \leq k} z^{\ell(\lambda)} q^{n(2\lambda)} (a, b; q^{-2})_{\lambda} (q^2; q^2)_{\lambda} = \frac{(z; q^2)_{\infty}}{(abzq; q^2)_{\infty}} (a, b; q^{-2})_r (aq^{2r+1}z, bq^{2r+1}z; q^2)_{\infty} (1 - zq^{4r-1}).
\] (9)

The remainder of this paper is organized as follows: in section 2 we first give multiple analogs of Rogers-Ramanujan type identities which are consequences of Theorem 2, in section 3 we give the proof of Theorem 1 and some consequences, and defer the elementary proof, i.e., without using the Hall-Littlewood polynomials, of Theorem 2 and other multiple \( q \)-series identities to section 4. In section 5 we will compare our multianalogs of Rogers-Ramanujan’s type identities with those obtained through Andrews-Bailey’s method.
2 Multiple identities of Rogers-Ramanujan type

We need the *Jacobi triple product* identity [2, p. 21]:

\[
J(x, q) := 1 + \sum_{r=1}^{\infty} (-1)^r x^r q^{r^2} (1 + q^r/x^{2r}) = (q, x, q/x; q)_{\infty}. \quad (10)
\]

For any partition \( \lambda \) set \( n_2(\lambda) = \sum \lambda_i^2 \). We derive then from Theorem 2 the following identities of Rogers-Ramanujan type.

**Theorem 3** For \( k \geq 1 \),

\[
\sum_{l(\lambda) \leq k} \frac{q^{2n_2(\lambda)}}{(q; q^2)_{\lambda_k} (q^2; q^2)_{\lambda}} = \prod_n (1 - q^n)^{-1} \quad (11)
\]

where \( n \equiv \pm (2k + 1), \pm (2k + 3), \pm 2, \pm 4, \ldots, \pm 4k \pmod{8k + 8} \);

\[
\sum_{l(\lambda) \leq k} \frac{q^{2n_2(\lambda)-2\lambda_1}}{(q; q^2)_{\lambda_k} (q^2; q^2)_{\lambda}} (1 - q^{2\lambda_1}) = \frac{(q^{2k-1}, q^{6k+9}, q^{8k+8})_{\infty}}{\prod_n (1 - q^n)} \quad (12)
\]

where \( n \equiv \pm (2k + 5), \pm 2, \ldots, \pm 4k, \pm (4k + 2) \pmod{8k + 8} \);

\[
\sum_{l(\lambda) \leq k} \frac{q^{2n_2(\lambda)-\lambda_1^2}}{(q; q^2)_{\lambda_k} (q^2; q^2)_{\lambda}} (-q; q^2)_{\lambda_1} = \frac{(-q; q^2)_{\infty} (q^{4k+2}, -q^{2k}, -q^{2k+2}; q^{4k+2})_{\infty}}{(q^2; q^2)_{\infty}}; \quad (13)
\]

\[
\sum_{l(\lambda) \leq k} \frac{q^{2n_2(\lambda)-\lambda_1^2-\lambda_1}}{(q; q^2)_{\lambda_k} (q^2; q^2)_{\lambda}} (-1; q^2)_{\lambda_1} (1 - q^{2\lambda_1}) \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} (q^{4k+2}, -q^{2k-1}, -q^{2k+3}; q^{4k+2})_{\infty}; \quad (14)
\]

\[
\sum_{l(\lambda) \leq k} \frac{q^{2n_2(\lambda)-\lambda_1^2+\lambda_1}}{(q; q^2)_{\lambda_k} (q^2; q^2)_{\lambda}} (-1; q^2)_{\lambda_1} (-q; q^2)_{\lambda_1} \frac{(-q)_{\infty}}{(q)_{\infty}} (q^{4k}, -q^{2k}, -q^{2k}; q^{4k})_{\infty}; \quad (15)
\]

\[
\sum_{l(\lambda) \leq k} \frac{q^{2n_2(\lambda)-\lambda_1^2+\lambda_1}}{(q; q^2)_{\lambda_k} (q^2; q^2)_{\lambda}} (-1; q^2)_{\lambda_1} \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} (q^{4k+2}, -q^{2k+1}, -q^{2k+1}; q^{4k+2})_{\infty}. \quad (16)
\]
Proof. Set \( z = q \) in (14),

\[
\sum_{l(\lambda) \leq k} q^{l(\lambda) + n(2\lambda)} \frac{(a, b; q^{-2})_{\lambda_k}}{(q^2; q^2)_{\lambda_k}(q; q^2)_{\lambda_k}} = (17)
\]

\[
1 + \sum_{r \geq 1} q^{2kr^2 + (\frac{r}{2})} (a, b; q^{-2})_r (aq^{2r^2}, bq^{2r^2}; q^2)_\infty (1 + q^{2r}).
\]

For (11), setting \( a = b = 0 \) in (17) we obtain

\[
\sum_{l(\lambda) \leq k} q^{2n_2(\lambda)} \frac{2n_2(\lambda)}{(q; q^2)_{\lambda_k}(q^2; q^2)_{\lambda}} = (q^2; q^2)_\infty^{-1} J(-q^{2k+1}, q^{4k+4}).
\]

The right side of (17) follows then from (14) after simple manipulations.

For (12), set \( a = 0 \) in (17) and multiply both sides by \( 1 - q^{-2} \). Identifying the coefficients of \( b \) we obtain :

\[
\sum_{l(\lambda) \leq k} q^{2n_2(\lambda) - 2\lambda_1} (q; q^2)_{\lambda_k}(q^2; q^2)_{\lambda} (1 - q^{2\lambda_1}) = (q^2; q^2)_\infty^{-1} J(-q^{2k-1}, q^{4k+4}).
\]

The result follows from (14) after simple manipulations.

Identity (13) follows from (17) with \( a = -q^{-1} \) and \( b = 0 \) and then by applying (10) with \( q \) replaced by \( q^{4k+2} \) and \( x = -q^{2k} \).

For (14), we choose \( a = -1 \) in (17) and multiply both sides by \( 1 - q^{-2} \), then identify the coefficient of \( b \). The identity follows then by applying (10) with \( q \) replaced by \( q^{4k+2} \) and \( x = -q^{2k-1} \).

Identity (15) follows from (17) by taking \( a = -q^{-1} \) and \( b = -1 \) and then applying (10) with \( q \) replaced by \( q^{4k} \) and \( x = -q^{2k} \). For (16), we choose \( a = -1 \) and \( b = 0 \) in (17). The identity follows then by applying (10) with \( q \) replaced by \( q^{4k+2} \) and \( x = -q^{2k+1} \).

When \( k = 1 \) the above six identities reduce respectively to the following Rogers-Ramanujan type identities :

\[
\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q)_{2n}} = \prod_{n=1}^{\infty} \frac{1}{1 - q^{n}}, \quad (n = \pm 2, \pm 3, \pm 4, \pm 5 \text{ (mod 16)}, (18)
\]

\[
\sum_{n=0}^{\infty} \frac{q^{2n^2 + 2n}}{(q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{1}{1 - q^{n}}, \quad (n = \pm 1, \pm 4, \pm 6, \pm 7 \text{ (mod 16)}), (19)
\]

\[
\sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(q)_{2n}} = \frac{(q^6, q^6, q^{12}; q^{12})_{\infty}}{(q)_{\infty}}, \quad (20)
\]
\[
\sum_{n=0}^{\infty} q^{n^2+n} (-q^2; q^2)_n = \frac{(q^3, q^9, q^{12}; q^{12}_\infty)}{(q)_\infty}, \quad (21)
\]

\[
1 + 2 \sum_{n \geq 1} q^n (-q)_{2n-1} = \frac{(q^4, -q^2, -q^2; q^4)_\infty}{(q)_{\infty}(q; q^2)_\infty}, \quad (22)
\]

\[
1 + 2 \sum_{n \geq 1} q^{n(n+1)} (-q^2; q^2)_{n-1} = \frac{(q^6, -q^3, -q^3; q^6)_\infty}{(q)_{\infty}(-q; q^2)_\infty}. \quad (23)
\]

Note that (18), (19), (20) and (21) are already known, they correspond to Eqs. (39), (38), (29) and (28) in Slater’s list [12], respectively, but (22) and (23) seem to be new.

3 Proof of Theorem 1 and consequences

3.1 Proof of identity (7)

For any statement \( A \) it will be convenient to use the true or false function \( \chi(A) \), which is 1 if \( A \) is true and 0 if \( A \) is false. Consider the generating function

\[
S(u) = \sum_{\lambda_0, \lambda} \chi(\lambda \text{ even}) c_{\lambda, \lambda_0}(q) P_{\lambda}(X, q) u^{\lambda_0}
\]

where the sum is over all partitions \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and the integers \( \lambda_0 \geq \lambda_1 \). Suppose \( \lambda = (\mu_1^{r_1} \mu_2^{r_2} \ldots \mu_k^{r_k}) \), where \( \mu_1 > \mu_2 > \cdots > \mu_k \geq 0 \) and \((r_1, \ldots, r_k)\) is a composition of \( n \).

Let \( S_n^\lambda \) be the set of permutations of \( S_n \) which fix \( \lambda \). Each \( w \in S_n/S_n^\lambda \) corresponds to a surjective mapping \( f : X \to \{1, 2, \ldots, k\} \) such that \( |f^{-1}(i)| = r_i \). For any subset \( Y \) of \( X \), let \( p(Y) \) denote the product of the elements of \( Y \) (in particular, \( p(\emptyset) = 1 \)). We can rewrite Hall-Littlewood functions as follows:

\[
P_{\lambda}(X, q) = \sum_f p(f^{-1}(1))^{\mu_1} \cdots p(f^{-1}(k))^{\mu_k} \prod_{f(x_i) < f(x_j)} \frac{x_i - qx_j}{x_i - x_j},
\]

summed over all surjective mappings \( f : X \to \{1, 2, \ldots, k\} \) such that \( |f^{-1}(i)| = r_i \). Furthermore, each such \( f \) determines a filtration of \( X \) :

\[
\mathcal{F}: \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = X,
\]

according to the rule \( x_i \in F_l \iff f(x_i) \leq l \) for \( 1 \leq l \leq k \). Conversely, such a filtration \( \mathcal{F} = (F_0, F_1, \ldots, F_k) \) determines a surjection \( f : X \to
\{1, 2, \ldots, k\} uniquely. Thus we can write:

\[ P_\lambda(X, q) = \sum_\mathcal{F} \pi_\mathcal{F} \prod_{1 \leq i \leq k} p(F_i \setminus F_{i-1})^{\nu_i}, \]  

(25)

summed over all the filtrations \( \mathcal{F} \) such that \( |F_i| = r_1 + r_2 + \cdots + r_i \) for \( 1 \leq i \leq k \), and

\[ \pi_\mathcal{F} = \prod_{f(x_i) < f(x_j)} \frac{x_i - q x_j}{x_i - x_j}, \]

where \( f \) is the function defined by \( \mathcal{F} \).

Now let \( \nu_i = \mu_i - \mu_{i+1} \) if \( 1 \leq i \leq k - 1 \) and \( \nu_k = \mu_k \), thus \( \nu_i > 0 \) if \( i < k \) and \( \nu_k \geq 0 \). Since the lengths of columns of \( \lambda \) are \( |F_j| = r_1 + \cdots + r_j \) with multiplicities \( \nu_j \) for \( 1 \leq j \leq k \), we have

\[ \chi(\lambda' \text{ even}) = \prod_j \chi(|F_j| \text{ even}). \]  

(26)

A filtration \( \mathcal{F} \) is called \textit{even} if \( |F_j| \) is even for \( j \geq 1 \). Furthermore, let \( \mu_0 = \lambda_0 \) and \( \nu_0 = \mu_0 - \mu_1 \) in the definition of \( S(\nu) \), so that \( \nu_0 \geq 0 \) and \( \mu_0 = \nu_0 + \nu_1 + \cdots + \nu_k \). Define \( \varphi_{2n}(q) = (1 - q)(1 - q^3) \cdots (1 - q^{2n-1}) \) and \( c_\mathcal{F}(q) = \prod_{i=1}^k \varphi_{|F_i \setminus F_{i-1}|}(q) \) for even filtrations \( \mathcal{F} \). Thus, since \( r_j = m_{\mu_j}(\lambda) \) for \( j \geq 1 \), we have

\[ c_{\lambda, \lambda_0}(q) = c_\mathcal{F}(q) \left( \chi(\nu_k = 0) \varphi_{|F_k \setminus F_{k-1}|}(q) + \chi(\nu_k \neq 0) \right)^{-1} \times \left( \chi(\nu_0 = 0) \varphi_{|F_1|}(q) + \chi(\nu_0 \neq 0) \right)^{-1}. \] 

Let \( F(X) \) be the set of filtrations of \( X \). Summarizing we obtain

\[ S(\nu) = \sum_{\mathcal{F} \in F(X)} c_\mathcal{F} \pi_\mathcal{F} \chi(\mathcal{F} \text{ even}) \sum_{\nu_1, \ldots, \nu_{k-1} > 0} u^{\nu_1} p(F_j)^{\nu_j} \]

\[ \times \sum_{\nu_0 \geq 0} \frac{u^{\nu_0} \chi(\nu_0 = 0) \varphi_{|F_1|}(q) + \chi(\nu_0 \neq 0)}{\chi(\nu_0 = 0) \varphi_{|F_1 \setminus F_{k-1}|}(q) + \chi(\nu_0 \neq 0)}. \]  

(27)

For any filtration \( \mathcal{F} \) of \( X \) set

\[ A_\mathcal{F}(X, u) = c_\mathcal{F}(q) \prod_{|F_j| \text{ even}} \left[ \frac{p(F_j) u}{1 - p(F_j) u} + \frac{\chi(F_j = X)}{\varphi_{|F_j \setminus F_{j-1}|}(q)} + \frac{\chi(F_j = \emptyset)}{\varphi_{|F_1|}(q)} \right], \]

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if \( \mathcal{F} \) is even and 0 otherwise. It follows from (27) that
\[
S(u) = \sum_{\mathcal{F} \in \mathcal{F}(X)} \pi_{\mathcal{F}}A_{\mathcal{F}}(X, u).
\]
Hence \( S(u) \) is a rational function of \( u \) with simple poles at \( 1/p(Y) \), where \( Y \) is a subset of \( X \) such that \( |Y| \) is even. We are now proceeding to compute the corresponding residue \( c(Y) \) at each pole \( u = 1/p(Y) \).

Let us start with \( c(\emptyset) \). Writing \( \lambda_0 = \lambda_1 + k \) with \( k \geq 0 \), we see that
\[
S(u) = \sum_{\lambda} \chi(\lambda' \text{ even}) c_{\lambda}(q) P_{\lambda}(X, q) u^{\lambda_1} \sum_{k \geq 0} \frac{u^k}{\chi(k = 0) \varphi_{m_{\lambda_1}}(q) + \chi(k \neq 0)}
\]
\[
= \sum_{\lambda} \chi(\lambda' \text{ even}) c_{\lambda}(q) P_{\lambda}(X, q) u^{\lambda_1} \left( \frac{u}{1-u} + \frac{1}{\varphi_{m_{\lambda_1}}(q)} \right).
\]
It follows from (5) that
\[
c(\emptyset) = [S(u)(1-u)]_{u=1} = \Phi_q(X; 0, 0).
\]
For the computations of other residues, we need some more notations. For any \( Y \subseteq X \), let \( Y' = X \setminus Y \) and \( -Y = \{x_i^{-1} : x_i \in Y\} \). Let \( Y \subseteq X \) such that \( |Y| \) is even. Then
\[
c(Y) = \left[ \sum_{\mathcal{F}} \pi_{\mathcal{F}}A_{\mathcal{F}}(X; u) (1-p(Y)u) \right]_{u=p(-Y)} \quad . \tag{28}
\]
If \( Y \notin \mathcal{F} \), the corresponding summand is equal to 0. Thus we need only to consider the following filtrations \( \mathcal{F} \):
\[
\emptyset = F_0 \subsetneq \cdots \subsetneq F_t = Y \subsetneq \cdots \subsetneq F_k = X \quad 1 \leq t \leq k.
\]
We may then split \( \mathcal{F} \) into two filtrations \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \):
\[
\mathcal{F}_1 : \emptyset \subsetneq -(Y \setminus F_{t-1}) \subsetneq \cdots \subsetneq -(Y \setminus F_1) \subsetneq -Y,
\]
\[
\mathcal{F}_2 : \emptyset \subsetneq F_{t+1} \setminus Y \subsetneq \cdots \subsetneq F_{k-1} \setminus Y \subsetneq Y'.
\]
Then, writing \( v = p(Y)u \) and \( c_{\mathcal{F}} = c_{\mathcal{F}_1} \times c_{\mathcal{F}_2} \), we have
\[
\pi_{\mathcal{F}}(X) = \pi_{\mathcal{F}_1}(-Y) \pi_{\mathcal{F}_2}(Y') \prod_{x_i \in Y, x_j \in Y'} \frac{1-q x_i^{-1} x_j}{1-x_i^{-1} x_j}.
\]
and \( A_F(X; u)(1 - p(Y)u) \) is equal to
\[
A_F(-Y; v)A_F(Y'; v)(1 - v) \left( \frac{v}{1 - v} + \frac{1}{\varphi|Y \setminus F_{i-1}|(q)} \right)^{-1} \left( \frac{v}{1 - v} + \frac{1}{\varphi|F_{i+1} \setminus Y|(q)} \right)^{-1}.
\]
Thus when \( u = p(-Y) \), i.e., \( v = 1 \),
\[
[ A_F(X; u)(1 - p(Y)u) ]_{u=p(-Y)} =
\left[ A_F(-Y; v)(1 - v)A_F(Y'; v)(1 - v) \right]_{v=1} \times \prod_{x_i \in Y, x_j \in Y'} \frac{1 - qx_i^{-1}x_j}{1 - x_i^{-1}x_j}.
\]
Using (28) and the result of \( c(\emptyset) \), which can be written
\[
\left[ \sum_{\pi_F} \pi_F A_F(X, u)(1 - u) \right]_{u=1} = \Phi_q(X; 0, 0),
\]
we get
\[
c(Y) = \Phi_q(-Y; 0, 0)\Phi_q(Y'; 0, 0) \prod_{x_i \in Y, x_j \in Y'} \frac{1 - qx_i^{-1}x_j}{1 - x_i^{-1}x_j}.
\]
Each subset \( Y \) of \( X \) can be encoded by a sequence \( \xi \in \{\pm 1\}^n \) according to the rule : \( \xi_i = 1 \) if \( x_i \not\in Y \) and \( \xi_i = -1 \) if \( x_i \in Y \). Hence
\[
c(Y) = \Phi_q(X^\xi; 0, 0).
\]
Note also that
\[
p(Y) = \prod_i x_i^{(1 - \xi_i)/2}, \quad p(-Y) = \prod_i x_i^{(\xi_i - 1)/2}.
\]
Now, extracting the coefficients of \( u^k \) in the equation :
\[
S(u) = \sum_{Y \subseteq X, |Y| \text{ even } > 0} \frac{c(Y)}{1 - p(Y)u},
\]
yields
\[
\sum_{\lambda \subseteq k, \lambda' \text{ even}} c_{\lambda,k}(q)P_\lambda(X, q) = \sum_{Y \subseteq X, |Y| \text{ even}} c(Y)p(Y)^k.
\]
Finally, substituting the value of \( c(Y) \) in the above formula we obtain (7).

**Remark.** Stembridge’s formula (4) can be derived from Macdonald’s (3) and Pieri’s formula for Hall-Littlewood polynomials. Indeed, one of Pieri’s formulas states that [10, p. 215] :

\[
P_{\mu}(X, q)e_m(X) = \sum_{\lambda} \prod_{i \geq 1} \left[ \lambda'_i - \lambda'_{i+1} \right] P_{\lambda}(X, q),
\]

where the sum is over all partitions \( \lambda \) such that \( \mu \subseteq \lambda \) with \( |\lambda/\mu| = m \) and there is at most one cell in each row of the Ferrers diagram of \( \lambda/\mu \). It follows from (29) that

\[
\sum_{\mu_1 \leq 2k} P_{\mu}(X, q) \sum_{m \geq 0} e_m(X) = \sum_{\lambda_1 \leq 2k+1} P_{\lambda}(X, q),
\]

noticing that \( \lambda \) determines in a unique way \( \mu \) even by deleting a cell in each odd part of \( \lambda \), and thus \( \left[ \lambda'_i - \lambda'_{i+1} \right] = 1 \). Finally we obtain the result, using the fact that \( \prod_i (1 + x_i q_j)^{-1} = \prod_i (1 + x_i)^{-1} \times \prod_i x_i^{1-\xi_i}/2 \). It would be interesting to give a similar proof of (7) using (3) and another Pieri formula [10, p. 218].

### 3.2 Proof of identity \((8)\)

As in the proof of \((7)\), we compute the generating function

\[
F(u) = \sum_{\lambda_0, \lambda} d_{\lambda, \lambda_0}(q) P_{\lambda}(X; q) u^{\lambda_0}
\]

where the sum is over all partitions \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and integers \( \lambda_0 \geq \lambda_1 \). For any filtration \( \mathcal{F} \) of \( X \) (cf. (24)) set

\[
d_{\lambda, \lambda_0}(q) = \prod_{i=1}^{k} \psi_{|F_i \setminus F_{i-1}|}(q), \quad \text{where} \quad \psi_n(q) = (q)_n \prod_{j=1}^{\lfloor n/2 \rfloor} (1 - q^{2j})^{-1}.
\]

Thus, as \( r_j = m_{\mu_j}(\lambda) \), \( j \geq 1 \), we have

\[
d_{\lambda, \lambda_0}(q) = d_{\lambda}(q) \left( \chi(\nu_k = 0) \psi_{|F_k \setminus F_{k-1}|}(q) + \chi(\nu_k \neq 0) \right)^{-1} \times \left( \chi(\nu_0 = 0) \psi_{|F_1|}(q) + \chi(\nu_0 \neq 0) \right)^{-1}.
\]
In view of (25) we have

\[ F(u) = \sum_{F \in \mathcal{F}(X)} \pi_F \mathcal{B}_F(X, u), \]

where

\[ \mathcal{B}_F(X, u) = d_F \prod_j \left[ \frac{p(F_j)u}{1 - p(F_j)u} + \frac{\chi(F_j = X)}{\psi_{|F_j\setminus F_j^{-1}|}(q)} + \frac{\chi(F_j = \emptyset)}{\psi_{|F_1|}(q)} \right]. \]

It follows that \( F(u) \) is a rational function of \( u \) and can be written as:

\[ F(u) = \frac{c(\emptyset)}{1 - u} + \sum_{Y \subseteq X, |Y| > 0} \frac{c(Y)}{1 - p(Y)u}. \]

Extracting the coefficient of \( u^k \) in the above identity yields

\[ \sum_{\lambda_1 \leq k} d_{\lambda,k}(q) \Pi_{\lambda}(X, q) = \sum_{Y \subseteq X} c(Y)p(Y)^k. \quad (30) \]

It remains to compute the residues. Writing \( \lambda_0 = \lambda_1 + r \) with \( r \geq 0 \), then

\[ F(u) = \sum_{\lambda} d_{\lambda}(q) \Pi_{\lambda}(X, q) u_{\lambda_1} \sum_{r \geq 0} \frac{u^r}{\chi(r = 0)\psi_{m_{\lambda_1}}(q) + \chi(r \neq 0)} \]

\[ = \sum_{\lambda} d_{\lambda}(q) \Pi_{\lambda}(X, q) u_{\lambda_1} \left( \frac{u}{1 - u} + \frac{1}{\psi_{m_{\lambda_1}}(q)} \right), \]

it follows from (30) that

\[ c(\emptyset) = (F(u)(1 - u)) \big|_{u=1} = \Phi_q(X; q, 1). \quad (31) \]

For computations of the other residues, set \( Y' = X \setminus Y \) and define, for \( Y = F_t \), the two filtrations:

\[ F_1 : \emptyset \subsetneq -(Y \setminus F_{t-1}) \subsetneq \cdots \subsetneq -(Y \setminus F_1) \subsetneq -Y, \]

\[ F_2 : \emptyset \subsetneq F_{t+1} \setminus Y \subsetneq \cdots \subsetneq F_{k-1} \setminus Y \subsetneq Y'. \]

Then, writing \( v = p(Y)u \) and \( d_F = d_{F_1} \times d_{F_2} \), we have

\[ \pi_F(X) = \pi_{F_1}(-Y)\pi_{F_2}(Y') \prod_{x_i \in Y, x_j \in Y'} \frac{1 - qx_i^{-1}x_j}{1 - x_i^{-1}x_j}, \]
and \( B_f(X; u)(1 - p(Y)u) \) can be written as

\[
B_{f_1}(-Y; v)B_{f_2}(Y'; v)(1 - v) \left( \frac{v}{1 - v} + \frac{\chi(Y = X)}{\psi|_{Y \setminus F_{t-1}}} \right) \times \left( \frac{v}{1 - v} + \frac{1}{\psi|_{F_{t+1} \setminus Y'}(q)} \right)^{-1}.
\]

Rewriting (31) as

\[
\left[ \sum_{F} \pi_f B_f(X, u)(1 - u) \right]_{u=1} = \Phi_q(X; q, 1),
\]

we get

\[
c(Y) = \left[ \sum_{F} \pi_f B_f(X; u)(1 - p(Y)u) \right]_{u=p(-Y)} = \Phi_q(-Y; q, 1)\Phi_q(Y'; q, 1) \prod_{x_i \in Y, x_j \in Y'} \frac{1 - q x_i^{-1} x_j}{1 - x_i^{-1} x_j}.
\]

Finally, the proof is completed by substituting the values of \( c(Y) \) in (30).

### 3.3 Some direct consequences on \( q \)-series

The following corollary of Theorem 1 will be useful for the proof of identities of Rogers-Ramanujan type.

**Theorem 4** For \( k \geq 1 \),

\[
\sum_{l(\lambda) \leq k} c_{(2\lambda), k}(q) z^{|\lambda|} q^{n(2\lambda)} \left[ \begin{array}{c} n \\ 2\lambda \end{array} \right] = (z; q^2)_n \sum_{r \geq 0} z^{kr} q^{(k+1)(2r)}
\]

\[
\times \left[ \begin{array}{c} n \\ 2r \end{array} \right] \frac{1 - q^{4r-1}}{(z q^{2r-1})_{n+1}}.
\]

\[
\sum_{l(\lambda) \leq k} d_{\lambda}, k(q) z^{|\lambda|} q^{n(\lambda)} \left[ \begin{array}{c} n \\ \lambda \end{array} \right] = (z^2; q^2)_n \sum_{r \geq 0} z^{kr} q^{r+(k+1)(z)}
\]

\[
\times \left[ \begin{array}{c} n \\ r \end{array} \right] \frac{(1 - q)(1 - z^2 q^{2r-1})(1 - z q^n)}{(1 - q^{r-1})(1 - q^r)(z^2 q^{r-1})_{n+1}}.
\]

**Proof.** We know [10, p. 213] that if \( x_i = z^{1/2} q^{i-1} \) (1 \( \leq i \leq n \)) then :

\[
P_X(X, q) = z^{|\lambda|/2} q^{n(\lambda)} \left[ \begin{array}{c} n \\ \lambda \end{array} \right].
\]

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Replacing $\lambda$ by $2\lambda$ and taking the conjugation in the left-hand side of (3), we obtain left-hand side of (32). On the other hand, for any $\xi \in \{\pm 1\}^n$ such that the number of $\xi_i = -1$ is $r$, $0 \leq r \leq n$, we have

$$\Phi_q(X^{\xi}; 0, 0) = \Psi_q(X^{\xi}; -1) \prod_i (1 - x_i^{2\xi_i}),$$

(35)

which is readily seen to equal 0 unless $\xi \in \{-1\}^r \times \{1\}^{n-r}$. Now, in the latter case, we have

$$\prod_{i=1}^{n} (1 - x_i^{2\xi_i}) = (-1)^r z^{-r} q^{-2(\xi)}(z; q^2)_n,$$

(36)

and [13, p. 476]:

$$\Psi_q(X^{\xi}; -1) = (-1)^r z^{-r} q^{-3(\xi)} \left[ \frac{1 - zq^{2r-1}}{q^{r-1}} \right]_{n+1}.$$  

(37)

Substituting these into the right side of (3) with $r$ replaced by $2r$ we obtain the right side of (32).

Similarly, in (3), replacing $x_i$ by $zq^{-1} (1 \leq i \leq n)$ and invoking (34) we see that the left side of (8) reduces to that of (33). On the other hand, since

$$\Phi_q(X^{\xi}; q, 1) = \Phi_q(X^{\xi}; 0, 0) \prod_{i=1}^{n} \frac{1 - qx_i^{\xi_i}}{1 - x_i^{\xi_i}},$$

by (33), this is equal to zero unless $\xi \in \{-1\}^r \times \{1\}^{n-r}$ for some $r$, $0 \leq r \leq n$. In the latter case, we have

$$\prod_{i=1}^{n} \frac{1 - qx_i^{\xi_i}}{1 - x_i^{\xi_i}} = q^{-r} \frac{1 - zq^{-1}}{1 - q^{-1}} \frac{1 - zq^n}{1 - q^{-1}},$$

(38)

and invoking (35), (36) and (37) with $z$ replaced by $z^2$,

$$\Phi_q(X^{\xi}; 0, 0) = q^{-\frac{3}{2}} \left[ \frac{n}{r} \right] (1 - z^2 q^{2r-1}) \left( \frac{z^2 q^2}{z^2 q^{2r-1}} \right)_{n+1}.$$  

(39)

Plunging these into the right side of (8) yields that of (33).  

When $n \to +\infty$, Eqs. (32) and (33) reduce respectively to:

$$\sum_{\ell(\lambda) \leq k} \frac{z^{\lambda} q^{n(2\lambda)}}{(q^2; q^2)_{\lambda}(q; q^2)_{\lambda}} = (z; q^2)_{\infty} \sum_{r \geq 0} \frac{z^{kr} q^{(k+1)(2r)}}{(q)_{2r}(zq^{2r-1})_{\infty}} (1 - zq^{4r-1}),$$

(40)
\[
\sum_{l(\lambda) \leq k} (q)_{\lambda k} \prod_{i=1}^{k-1} (q^2; q^2)_{[(\lambda_i - \lambda_{i+1})/2]} z^{l(\lambda)} q^{n(\lambda)} \leq k
\]

Furthermore, setting \( z = q \) in (40) and (41) we obtain respectively (11) and

\[
\sum_{l(\lambda) \leq k} (q)_{\lambda k} \prod_{i=1}^{k-1} (q^2; q^2)_{[(\lambda_i - \lambda_{i+1})/2]} \]

4 Elementary approach to multiple \( q \)-identities

4.1 Preliminaries

Recall [2, pp. 36-37] that the binomial formula has the following \( q \)-analog :

\[
(z)_n = \sum_{m=0}^{n} \binom{n}{m} (-1)^m z^m q^{m(m-1)/2}.
\]

Since the elementary symmetric functions \( e_r(X) \) \((0 \leq r \leq n)\) satisfy

\[
(1 + x_1 z)(1 + x_2 z) \cdots (1 + x_n z) = \sum_{r=0}^{n} e_r(X) z^r,
\]

it follows from (43) that for integers \( i \geq 0 \) and \( j \geq 1 \):

\[
e_r(q^i, q^{i+1}, \ldots, q^{i+j-1}) = q^{ir} e_r(1, q, \ldots, q^{j-1}) = q^{ij} e_r(1, q, \ldots, q^{j-1}) = q^{ij} \binom{j}{r}.
\]

The following result can be derived from the Pieri’s rule for Hall-Littlewood polynomials [10, p. 215], but our proof is elementary.

**Lemma 1** For any partition \( \mu \) such that \( \mu_1 \leq n \) there holds

\[
q^{m(\mu_2) + n(\mu)} \frac{n!}{m! \mu!} = \sum_{\lambda} q^{n(\lambda)} \frac{n!}{\lambda!} \prod_{i \geq 1} \frac{\lambda - \lambda_{i+1}}{\lambda_i - \mu_i},
\]

where the sum is over all partitions \( \lambda \) such that \( \lambda/\mu \) is an \( m \)-horizontal strip, i.e., \( \mu \subseteq \lambda \), \( |\lambda/\mu| = m \) and there is at most one cell in each column of the Ferrers diagram of \( \lambda/\mu \).
Proof. Let \( l := l(\mu) \) and \( \mu_0 = n \). Partition the set \( \{1, 2, \ldots, n\} \) into \( l + 1 \) subsets:

\[
X_i = \{ j \mid 1 \leq j \leq n \text{ and } \mu_j = i \} = \{ j \mid \mu_{i+1} + 1 \leq j \leq \mu_i \}, \quad 0 \leq i \leq l.
\]

Using (44) to extract the coefficients of \( z^m \) in the following identity:

\[
(1 + z)(1 + zq) \cdots (1 + zq^{n-1}) = \prod_{i=0}^{l} \prod_{j \in X_i} (1 + zq^{j-1}),
\]

we obtain

\[
q^{(m)}_{(2)} \left[ \begin{array}{c} n \\ m \end{array} \right] = \sum_{r} \prod_{i=0}^{l} q^{r_i, \mu_{i+1}+(\lambda_{i+1}-\mu_{i+1})} \left[ \begin{array}{c} \mu_i - \mu_{i+1} \\ \mu_i - \lambda_{i+1} \end{array} \right], \quad \text{(46)}
\]

where \( r = (r_0, r_1, \ldots, r_l) \) is a sequence of non negative integers such that \( \sum_i r_i = m \). For any such \( r \) define a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) by

\[
\lambda_i = \mu_i + r_{i-1}, \quad 1 \leq i \leq l + 1.
\]

Then \( \lambda/\mu \) is a \( m \)-horizontal strip. So (46) can be written as

\[
q^{(m)}_{(2)} \left[ \begin{array}{c} n \\ m \end{array} \right] = \sum_{\lambda} \prod_{i=0}^{l} q^{(\lambda_{i+1}-\mu_{i+1})\mu_{i+1}+(\lambda_{i+1}-\mu_{i+1})} \left[ \begin{array}{c} \mu_i - \mu_{i+1} \\ \mu_i - \lambda_{i+1} \end{array} \right], \quad \text{(47)}
\]

where the sum is over all partitions \( \lambda \) such that \( \lambda/\mu \) is a \( m \)-horizontal strip. Now, since

\[
(\lambda_{i+1} - \mu_{i+1})\mu_{i+1} + \left( \frac{\lambda_{i+1} - \mu_{i+1}}{2} \right) + \left( \frac{\mu_{i+1}}{2} \right) = \left( \frac{\lambda_{i+1}}{2} \right), \quad 0 \leq i \leq l,
\]

and \( \left[ \begin{array}{c} n \\ \mu_i - \lambda_{i+1} \end{array} \right] \prod_{i=0}^{l} \) and \( \left[ \begin{array}{c} n \\ \lambda_i - \mu_i \end{array} \right] \) are equal because they are both equal to

\[
\frac{(q)_n}{(q)_{n-\lambda_1}(q)_{\lambda_1-\mu_1}(q)_{\mu_1-\lambda_2} \cdots (q)_{\mu_l}}
\]

multiplying (47) by \( q^{n(\mu)} \left[ \begin{array}{c} n \\ \mu \end{array} \right] \) yields (45). \( \square \)

Lemma 2 There hold the following identities:

\[
\sum_{\lambda} z^{\lambda} q^{2n(\lambda)} \left[ \begin{array}{c} n \\ \lambda \end{array} \right] = \frac{1}{(z)_n}, \quad \text{(48)}
\]

\[
\sum_{\lambda} z^{\lambda} q^{n(\lambda)} \left[ \begin{array}{c} n \\ \lambda \end{array} \right] = \frac{(-z)_n}{(z^2)_n}, \quad \text{(49)}
\]

\[
\sum_{\lambda} (q, q^2)_\lambda z^{\lambda} q^{n(2\lambda)} \left[ \begin{array}{c} n \\ 2\lambda \end{array} \right] = \frac{(z; q^2)_n}{(z)_n}. \quad \text{(50)}
\]
Proof. Identity (48) is due to Hall [11] and can be proved by using the $q$-binomial identity [9]. Stembridge [13] proved (49) using the $q$-binomial identity. Now, writing
\[
\binom{z^2; q^2}{n} / \binom{z^2}{n} = \binom{z; -z}{n} / \binom{z^2}{n}
\]
and applying successively (43), (49) and (45) we obtain
\[
\binom{z^2; q^2}{n} / \binom{z^2}{n} = \sum_{\mu, m} (-1)^m z^{m+|\mu|} q^{m+n(|\mu|)} \binom{n}{m} \left[ \binom{n}{\mu} \right] \left[ \binom{n}{\lambda} \right]
\]
\[
\sum_{\lambda} z^{\lambda} q^{\lambda} \binom{n}{\lambda} \prod_{i \geq 1} (-1)^{r_i} \left[ \binom{n}{\lambda_i - \lambda_{i+1}} \right].
\]
The identity (50) follows then from
\[
\sum_{j=0}^{m} (-1)^j \binom{m}{j} = \begin{cases} (q; q^2)_n & \text{if } m = 2n, \\ 0 & \text{if } m \text{ is odd}, \end{cases}
\]
which can be proved using the $q$-binomial formula [2, p. 36].

Remark. When $n \to \infty$ the above identities reduce respectively to the following:
\[
\sum_{\lambda} z^{\lambda} q^{2n(\lambda)} (q)_\lambda = \frac{1}{(z)_\infty},
\]
\[
\sum_{\lambda} z^{\lambda} q^{n(\lambda)} (q)_\lambda = \frac{(-z)_\infty}{(z^2)_\infty},
\]
\[
\sum_{\lambda} z^{\lambda} q^{n(2\lambda)} (q^2; q^2)_\lambda = \frac{1}{(zq; q^2)_\infty}.
\]
Also (51) and (53) are actually equivalent since the later can be derived from (51) by substituting $q$ by $q^2$ and $z$ by $zq$.

The following is the $q$-Gauss sum [3, p.10] due to Heine :
\[
\begin{align*}
_2\phi_1 \left( \begin{array}{c}
a, b \\
x \end{array} ; q^2 \begin{array}{c} x \\ ab \end{array} \right) := \sum_{n=0}^{\infty} \binom{a_n b_n}{n} (q)_n \binom{x}{ab}^n = \frac{(x/a, x/b; q)_{\infty}}{(x, x/ab; q)_{\infty}}.
\end{align*}
\]

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Lemma 3 We have

\[ \sum_{\lambda} z^{\lambda} q^{n(2\lambda)} \frac{a, b; q^{-2}}{(q^2; q^2)_\lambda} = \frac{(az, bq^2; q^2)_\infty}{(zq, abzq^2; q^2)_\infty}. \] (55)

**Proof.** Substituting \( q^2 \) by \( q \) and \( z \) by \( zq \), the identity is equivalent to

\[ \sum_{\lambda} z^{\lambda} q^{2n(\lambda)} \frac{a, b; q^{-1}}{(q)_\lambda} = \frac{(az, bq^{-1}; q)_\infty}{(z, abz; q)_\infty}. \] (56)

Now, writing \( k = \lambda_1 \) and \( \mu = (\lambda_2, \lambda_3, \cdots) \), and using (48) we get

\[
\sum_{\lambda} z^{\lambda} q^{2n(\lambda)} \frac{a, b; q^{-1}}{(q)_\lambda} = \sum_{k \geq 0} z^k q^{k(k-1)} \frac{a, b; q^{-1}}{(q)_k} \sum_{\mu} z^{\mu} q^{2n(\mu)} \left[ k \right]_{\mu} = \sum_{k \geq 0} (abz)^k \frac{a^{-1}, b^{-1}; q}{(q)_k(z)_k}.
\]

Identity (56) follows then from (54).

□

**Remark.** Formula (56) was derived in [13] from a more general formula of Hall-Littlewood polynomials.

### 4.2 Elementary proof of Theorem 4

We shall only prove (32) when \( n \) is even and leave the case when \( n \) is odd and (33) to the interested reader because their proofs are very similar. Consider the generating function of the left-hand side of (32) with \( n = 2r \) :

\[
\varphi(u) = \sum_{k \geq 0} u^k \sum_{l(\lambda) \leq k} \frac{(q^2; q^2)_\lambda}{(q; q^2)_{\lambda_k}} z^{\lambda} q^{n(2\lambda)} \left[ \frac{2r}{2\lambda} \right] = \sum_{\lambda} u^{l(\lambda)} z^{\lambda} q^{n(2\lambda)} \left[ \frac{2r}{2\lambda} \right] \sum_{k \geq 0} \frac{u^k}{(q; q^2)_{\lambda_k+l(\lambda)}} = \sum_{\lambda} u^{l(\lambda)} z^{\lambda} q^{n(2\lambda)} \left[ \frac{2r}{2\lambda} \right] \left( \frac{u}{1-u} + \frac{1}{(q; q^2)_{\lambda}(\lambda)} \right). \tag{57}
\]

Now, each partition \( \lambda \) with parts bounded by \( r \) can be encoded by a pair of sequences \( \nu = (\nu_0, \nu_1, \cdots, \nu_l) \) and \( m = (m_0, \cdots, m_l) \) such that \( \lambda = (\nu_0^{m_0}, \cdots, \nu_l^{m_l}) \), where \( r = \nu_0 > \nu_1 > \cdots > \nu_l > 0 \) and \( \nu_i \) has multiplicity \( m_i \geq 1 \) for \( 1 \leq i \leq l \) and \( \nu_0 = r \) has multiplicity \( m_0 \geq 0 \). Using the notation :

\[
< \alpha > = \frac{\alpha}{1-\alpha}, \quad u_i = z^i q^{i(2i-1)} \text{ for } i \geq 0,
\]
we can then rewrite (57) as follows:

\[
\varphi(u) = \sum_{\nu} (q; q^2)^{2r} \left[ u > + \frac{1}{(q; q^2)_{\nu}} \right] \\
\times \sum_{m} \left( (u_r u)^{m_0} + \chi(m_0 = 0) \frac{1}{(q; q^2)_{r-\nu}} \right) \prod_{i=1}^{l} (u_{\nu_i} u)^{m_i} \\
= \sum_{\nu} \frac{(q)_{2r}}{(q^2; q^2)^{\nu}} B_{\nu},
\]

(58)

where the sum is over all strict partitions \(\nu = (\nu_0, \nu_1, \ldots, \nu_l)\) and

\[
B_{\nu} = \left( u > + \frac{1}{(q; q^2)_{\nu}} \right) \left( u_r u > + \frac{1}{(q; q^2)_{r-\nu}} \right) \prod_{i=1}^{l} \langle u_{\nu_i} u \rangle.
\]

So \(\varphi(u)\) is a rational fraction with simple poles at \(u_p^{-1}\) for \(0 \leq p \leq r\). Let \(b_p(z, r)\) be the corresponding residue of \(\varphi(u)\) at \(u_p^{-1}\) for \(0 \leq p \leq r\). Then, it follows from (58) that

\[
b_p(z, r) = \sum_{\nu} \frac{(q)_{2r}}{(q^2; q^2)^{\nu}} [B_{\nu}(1 - u_p u)]_{u = u_p^{-1}}.
\]

(59)

We shall first consider the cases where \(p = 0\) or \(r\). Using (57) and (50) we have

\[
b_0(z, r) = [\varphi(u)(1 - u)]_{u=1} = \frac{(z; q^2)_{2r}}{(z)_{2r}}.
\]

(60)

Now, by (58) and (59) we have

\[
b_0(z, r) = \sum_{\nu} \frac{(q)_{2r}}{(q^2; q^2)^{\nu}} \left( u_r > + \frac{1}{(q; q^2)_{r-\nu}} \right) \prod_{i=1}^{l} \langle u_{\nu_i} u \rangle,
\]

(61)

and

\[
b_r(z, r) = \sum_{\nu} \frac{(q)_{2r}}{(q^2; q^2)^{\nu}} \left( 1/u_r > + \frac{1}{(q; q^2)_{r-\nu}} \right) \prod_{i=1}^{l} \langle u_{\nu_i}/u_r \rangle,
\]

which, by setting \(\mu_i = r - \nu_{l+1-i}\) for \(1 \leq i \leq l\) and \(\mu_0 = r\), can be written as

\[
b_r(z, r) = \sum_{\mu} \frac{(q)_{2r}}{(q^2; q^2)^{\mu}} \left( 1/u_r > + \frac{1}{(q; q^2)_{r-\mu_1}} \right) \prod_{i=1}^{l} \langle u_{r-\mu_1}/u_r \rangle.
\]

(62)
Comparing (\ref{eq:62}) with (\ref{eq:61}) we see that \( b_r(z, r) \) is equal to \( b_0(z, r) \) with \( z \) replaced by \( z^{-1}q^{-2(2r-1)} \). It follows from (\ref{eq:60}) that

\[
b_r(z, r) = b_0(z^{-1}q^{-2(2r-1)}, r) = (z; q^2)^{2r}q^{r(2r-1)} \frac{1 - zq^{4r-1}}{(zq^{2r-1})^{2r+1}}.
\] (\ref{eq:63})

Consider now the case where \( 0 < p < r \). Clearly, for each partition \( \nu \), the corresponding summand in (\ref{eq:59}) is not zero only if \( \nu_j = p \) for some \( j \), \( 0 \leq j \leq r \). Furthermore, each such partition \( \nu \) can be splitted into two strict partitions \( \rho = (\rho_0, \rho_1, \ldots, \rho_{j-1}) \) and \( \sigma = (\sigma_0, \ldots, \sigma_{l-j}) \) such that \( \rho_i = \nu_i - p \) for \( 0 \leq i \leq j-1 \) and \( \sigma_s = \nu_{j+s} \) for \( 0 \leq s \leq l - j \). So we can write (\ref{eq:59}) as follows :

\[
b_p(z, r) = \left[ \frac{2r}{2p} \right] \sum_{\rho} \frac{(q)_{2r-2p}}{(q^2; q^2)^{l(\rho)}} \sum_{\sigma} \frac{(q)_{2p}}{(q^2; q^2)^{l(\sigma)}} \prod_{i=1}^{l(\rho)} \frac{1}{u_{\rho_i} + p} \prod_{i=1}^{l(\sigma)} \frac{1}{u_{\sigma_i} + p},
\]

where for \( \rho = (\rho_0, \rho_1, \ldots, \rho_l) \) with \( \rho_0 = r - p \),

\[
F_\rho(p) = \left( <u_r/u_p> + \frac{1}{(q^2; q^2)^{r-p-\rho_1}} \right) \prod_{i=1}^{l(\rho)} <u_{\rho_i+p}/u_p>,
\]

and for \( \sigma = (\sigma_0, \ldots, \sigma_l) \) with \( \sigma_0 = p \),

\[
G_\sigma(p) = \left( <1/u_p> + \frac{1}{(q^2; q^2)^{\sigma_1}} \right) \prod_{i=1}^{l(\sigma)} <u_{\sigma_i}/u_p>.
\]

Comparing with (\ref{eq:61}) and (\ref{eq:62}) and using (\ref{eq:60}) and (\ref{eq:63}) we obtain

\[
b_p(z, r) = \left[ \frac{2r}{2p} \right] b_0(zq^{4p}, r - p) \frac{b_p(z, p)}{1 - zq^{4p-1}} \frac{1 - zq^{4p-1}}{(zq^{2p-1})^{2r+1}}.
\]

Finally, extracting the coefficients of \( u^k \) in the equation

\[
\varphi(u) = \sum_{p=0}^{r} \frac{b_p(z, r)}{1 - u_p u},
\]

and using the values for \( b_p(z, r) \) we obtain (\ref{eq:32}).
4.3 Proof of Theorem 2

Consider the generating function of the left-hand side of (9):

\[ \varphi_{ab}(u) := \sum_{k \geq 0} u^k \sum_{\lambda(k) \leq k} z^{\lambda_1} q^{n(2\lambda)} (a, b; q^{2^2})_{\lambda_1} \frac{(a, b; q^{-2})_{\lambda_1}}{(q^2; q^2)_{\lambda_1}} \]

\[ = \sum_{\lambda} \sum_{k \geq 0} u^{k+l(\lambda)} z^{\lambda_1} q^{n(2\lambda)} (a, b; q^{2^2})_{\lambda_1} \frac{(a, b; q^{-2})_{\lambda_1}}{(q^2; q^2)_{\lambda_1}} \left( \frac{u}{1-u} + \frac{1}{(q^2; q^2)_{\lambda(\lambda)}} \right). \quad (64) \]

As in the proof of Theorem 4, we encode the partition \( \lambda \) in the previous sum. Let \( \nu_1, \ldots, \nu_l, \nu_{l+1} = 0 \) denote the distinct parts of \( \lambda \), so that \( \nu_1 > \cdots > \nu_l > \nu_{l+1} = 0 \) and \( \nu_i \) has multiplicity \( m_i \) for \( 1 \leq i \leq l \). Then we have

\[ \varphi_{ab}(u) = \sum_{\nu, m} \frac{(a, b; q^{-2})_{\nu_1}}{(q^2; q^2)_{\nu}} \left( \frac{u}{1-u} + \frac{1}{(q^2; q^2)_{\nu}} \right) \prod_{i=1}^{l} (u_{\nu_i} u)^{m_i} \]

\[ = \sum_{\nu} \frac{(a, b; q^{-2})_{\nu_1}}{(q^2; q^2)_{\nu}} \left( \frac{u}{1-u} + \frac{1}{(q^2; q^2)_{\nu}} \right) \prod_{i=1}^{l} < u_{\nu_i} u >. \quad (65) \]

Each of the terms in this sum, as a rational function of \( u \), has a finite set of simple poles, which may occur at the points \( u_r^{-1} \) for \( r \geq 0 \). Therefore, each term is a linear combination of partial fractions. Moreover, the sum of their expansions converges coefficientwise. So \( \varphi_{ab} \) has an expansion

\[ \varphi_{ab}(u) = \sum_{r \geq 0} \frac{c_r}{1-u z^r q_r^{(2r-1)}}, \]

where \( c_r \) denotes the formal sum of partial fraction coefficients contributed by the terms of (65). It remains to compute these residues \( c_r \) (\( r \geq 0 \)). By using (55) and (64), we get immediately

\[ c_0 = \varphi_{ab}(u)(1-u)|_{u=1} = \frac{(azq, bzq; q^2)_{\infty}}{(zq, abzq; q^2)_{\infty}}. \]

In view of (65), this yields the identity

\[ \sum_{\nu} \frac{(a, b; q^{-2})_{\nu_1}}{(q^2; q^2)_{\nu}} \prod_{i=1}^{l} < u_{\nu_i} > = \frac{(azq, bzq; q^2)_{\infty}}{(zq, abzq; q^2)_{\infty}}. \quad (66) \]
To compute the residues $c_r$ for $r > 0$, the contribution in (65) is given only by the partitions $\nu$ for which $\exists j \, | \nu_j = r$. For each such partition, we define as before $\rho_i := \nu_i - r$ for $1 \leq i < j$ and $\sigma_i := \nu_{i+j}$ for $0 \leq i \leq l - j$. We get two partitions $\rho$ and $\sigma$ with $\sigma$ bounded by $r$. Using (65), we obtain

$$c_r = \left[ \varphi_{ab}(u)(1 - u_r u) \right]_{u = u_r^{-1}}$$

$$= \sum_{\rho, \sigma} \frac{(a, b; q^{-2})_{\rho_1 + r} (q^2; q^2)_\rho (q^2; q^2)_\sigma}{(q^2; q^2)_\rho (q^2; q^2)_\sigma} \prod_{i=1}^{j-1} u_{r+\rho_i} / u_r >$$

$$\times \left( <1/u_r> + \frac{1}{(q; q^2)_{\sigma_1}} \prod_{i=1}^{l-j} u_{\sigma_i} / u_r > \right).$$

To eliminate the $\sigma$-dependence of this series, we apply (63), and this leads to

$$c_r = \sum_{\rho} \frac{(aq^{-2r}, bq^{-2r}; q^{-2})_{\rho_1} (a, b; q^{-2})_\rho (q^2; q^2)_\rho}{(q^2; q^2)_\rho (q^2; q^2)_\rho}$$

$$\times \left( z; q^2 \right)_2 q^{(2r)_2} \frac{1 - z q^{4r-1}}{(z q^{2r-1})_{2r+1}} \prod_{i=1}^{j-1} u_{r+\rho_i} >$$

$$= \frac{(a, b; q^{-2})_r (z; q^2)_2 q^{(2r)_2} \frac{1 - z q^{4r-1}}{(z q^{2r-1})_{2r+1}}}{(q^2; q^2)_\rho (q^2; q^2)_\rho} \frac{(aq^{-2r+1} + 4r, bzq^{-2r+1} + 4r; q^2)_\infty}{(z q^{4r+1}, abzq; q^2)_\infty},$$

where the last equality follows from (64) with $a, b, z$ replaced respectively by $aq^{-2r}, bq^{-2r}, zq^{4r}$. After simplification, one gets

$$c_r = q^{(2r)_2} \left( z; q^2 \right)_\infty \frac{(a, b; q^{-2})_r (aq^{2r+1} + 4r, bzq^{-2r+1} + 4r; q^2)_\infty (1 - z q^{4r-1})}{(z q^{2r-1})_\infty (abzq; q^2)_\infty},$$

which completes the proof.

## 5 Comparison with Andrews-Bailey’s method

A popular method to prove identities of Rogers-Ramanujan type is based on Bailey’s lemma (see [4, 14]). In [3] Andrews noticed that by applying iteratively Bailey’s lemma to the corresponding Bailey pair in the simple sum case one can obtain multianalog identities of Rogers-Ramanujan type almost straightforwardly. In this section we shall briefly compare our multsum analogs with those obtained through Andrews-Bailey’s approach. Recall
that a pair of sequences \((\alpha_n)_{n \geq 0}\) and \((\beta_n)_{n \geq 0}\) is a Bailey pair if they are related by the following [4, p. 25-26]:

\[
\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}. \tag{67}
\]

If \((\alpha_n, \beta_n)\) is a Bailey pair and \((\alpha'_n, \beta'_n)\) is one of the following pairs:

1. \(\alpha'_n = a^n q^{n^2} \alpha_n, \quad \beta'_n = \sum_{k \geq 0} \frac{a^k q^{k^2}}{(q)_{n-k}} \beta_k;\)
2. \(\alpha'_n = \frac{(-q^{1/2})_n}{(-aq^{1/2})_n} a^n q^{n^2/2} \alpha_n, \quad \beta'_n = \sum_{k \geq 0} \frac{(-q^{1/2})_k a^k q^{k^2/2}}{(q)_{n-k}(-aq^{1/2})_k} \beta_k;\)
3. \(\alpha'_n = a^{n^2} q^{r^2/2} \alpha_n, \quad \beta'_n = \sum_{k \geq 0} \frac{(-aq^{1/2})_k a^k q^{k^2/2}}{(q)_{n-k}(-aq^{1/2})_n} \beta_k;\)

then Bailey’s lemma [4, p. 25-26] states that \((\alpha'_n, \beta'_n)\) is also a Bailey pair. What we need here is actually the limit case of (67). In (67) substituting \((\alpha_n, \beta_n)\) by one of the above \((\alpha'_n, \beta'_n)\)'s and letting \(n \to \infty\), we obtain respectively

\[
\sum_{n \geq 0} a^n q^{n^2} \beta_n = \frac{1}{(aq)_{\infty}} \sum_{r \geq 0} a^r q^{r^2} \alpha_r, \tag{68}
\]

\[
\sum_{n \geq 0} a^n q^{n^2/2} (-q^{1/2})_n \beta_n = \frac{(-aq^{1/2})_{\infty}}{(aq)_{\infty}} \sum_{r \geq 0} \frac{(-q^{1/2})_r}{(-aq^{1/2})_r} a^r q^{r^2/2} \alpha_r, \tag{69}
\]

\[
\sum_{n \geq 0} a^{n^2} q^{r^2/2} (-aq^{1/2})_n \beta_n = \frac{(-aq^{1/2})_{\infty}}{(aq)_{\infty}} \sum_{r \geq 0} a^r q^{r^2/2} \alpha_r. \tag{70}
\]

Now, if we iterate the above process \(k\) times to a same Bailey pair [4, p.30], then (68), (69) and (70) lead respectively to

\[
\frac{1}{(aq)_{\infty}} \sum_{r \geq 0} a^{kr} q^{kr^2} \alpha_r = \sum_{l(\lambda) \leq k} a^{l|\lambda| q^{n_2(\lambda)}} (q)_{\lambda} \beta_{l\lambda}; \tag{71}
\]

\[
\frac{(-q^{1/2})_{\infty}}{(aq)_{\infty}} \sum_{r \geq 0} \left( \frac{(-q^{1/2})_r}{(-aq^{1/2})_r} \right)^k a^{kr} q^{kr^2/2} \alpha_r \tag{72}
\]

\[
= \sum_{l(\lambda) \leq k} a^{l|\lambda| q^{n_2(\lambda)/2}} (q)_{\lambda} \beta_{l\lambda},
\]
\[
\frac{(-aq)^{1/2}}{(aq)_\infty} \sum_{r \geq 0} a^{kr/2} q^{kr^2/2} \alpha_r
\]
\[
= \sum_{l(\lambda) \leq k} a^{\lambda l/2} q^{n\lambda_2(\lambda)/2} \frac{(-aq)^{1/2} \lambda_k(q) \lambda_k}{(q)_\lambda} \beta_{\lambda_k}.
\]

Slater \[11, 12\] noticed that (18) and (20) follow from (68) and (69) by choosing the pair:
\[
\alpha_n = q^{n^2}(q^{n/2} + q^{-n/2}), \quad \beta_n = \frac{1}{(q^{1/2}; q)_n},
\] with \(a = 1\) and \(q\) replaced by \(q^2\), and (19) and (21) follow from (68) and (70) by choosing the pair:
\[
\alpha_n = q^{n^2+n/2} \frac{1 + q^{n+1/2}}{1 + q^{1/2}}, \quad \beta_n = \frac{1}{(q^{3/2}; q)_n}
\] with \(a = q\) and \(q\) replaced by \(q^2\).

Now, if we choose the Bailey pair (74) in (71) and (73) with \(a = 1\) and \(q\) replaced by \(q^2\), we obtain respectively (11) and
\[
\sum_{l(\lambda) \leq k} \frac{q^{n\lambda_2(\lambda)}(-q; q^2)_\lambda_k}{(q; q^2)_\lambda_k (q^2; q^2)_\lambda} = \frac{(-q^{2k+4}, -q^{k+1}, -q^{k+3}; q^{2k+4})_\infty}{(q)_\infty (-q^2; q^2)_\infty},
\]
which is different from (13). In the same way, if we choose the pair (75) in (71) and (73) with \(a = q\) and \(q\) replaced by \(q^2\) then we obtain
\[
\sum_{l(\lambda) \leq k} \frac{q^{2\lambda l + n\lambda_2(\lambda)}(q; q^2)_\lambda_k (q^2; q^2)_\lambda}{(q^2; q^2)_\lambda} = \frac{(q^{4k+4}, -q^{4k+3}, -q; q^{4k+4})_\infty}{(q^2; q^2)_\infty},
\]
\[
\sum_{l(\lambda) \leq k} \frac{q^{\lambda l + n\lambda_2(\lambda)}(-q^2; q^2)_\lambda_k (q^2; q^2)_\lambda}{(q; q^2)_\lambda_k (q^2; q^2)_\lambda} = \frac{(q^{2k+4}, -q^{2k+3}, -q; q^{2k+4})_\infty}{(q)_\infty (-q; q^2)_\infty}
\]
which are different from (12) and (14), respectively.

So, only equation (11) coincides with that directly obtained by Andrews-Bailey’s method. It seems that some new techniques may be necessary to demonstrate all our six multisum identities of Rogers-Ramanujan type through the classical Andrews-Bailey’s method. Recently, Bressoud, Ismail and Stanton \[5\] have proved all the sixteen identities in Stembridge’s paper \[13\] by means of change of base in Bailey pairs. It would be interesting to see whether their method can be applied to our identities.
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