An inverse problem of identifying the radiative coefficient in a degenerate parabolic equation

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Abstract

This work investigates an inverse problem of determining the radiative coefficient in a degenerate parabolic equation from the final overspecified data. Being different from other inverse coefficient problems in which the principle coefficients are assumed to be strictly positive definite, the mathematical model discussed in the paper belongs to the second order parabolic equations with non-negative characteristic form, namely that there exists degeneracy on the lateral boundaries of the domain. The uniqueness of the solution is obtained by the contraction mapping principle. Based on the optimal control framework, the problem is transformed into an optimization problem and the existence of the minimizer is established. After the necessary conditions which must be satisfied by the minimizer are deduced, the uniqueness and stability of the minimizer are proved. By minor modification of the cost functional and some a-priori regularity conditions imposed on the forward operator, the convergence of the minimizer for the noisy input data is obtained in the paper. The results obtained in the paper are interesting and useful, and can be extended to more general degenerate parabolic equations.

Keywords: Inverse problem, Degenerate parabolic equation, Optimal control, Existence, Uniqueness, Stability, Convergence.
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1 Introduction

In this paper, we study an inverse problem of identifying the radiative coefficient in a degenerate parabolic equation from the final overspecified data. Problems of this type have important
applications in several fields of applied science and engineering. The problem can be stated in the following form:

**Problem P**  Consider the following parabolic equation:

\[
\begin{cases}
  u_t - (a(x)u_x)_x + q(x)u = 0, & (x, t) \in Q = (0, l) \times (0, T], \\
  u|_{t=0} = \phi(x), & x \in (0, l),
\end{cases}
\]  

where \(a\) and \(\phi\) are two given smooth functions which satisfy

\[
a(0) = a(l) = 0, \quad a(x) > 0, \quad x \in (0, l),
\]

and

\[
\phi(x) \geq 0, \quad \phi(x) \neq 0, \quad x \in (0, l),
\]

and \(q(x)\) is an unknown coefficient in (1.1). In this paper, we always assume that \(a(x)\) is at least \(C^1\) continuous, i.e., \(a(x) \in C^1[0, l]\). Assume that an additional condition is given as follows:

\[
u(x, T) = g(x), \quad x \in [0, l],
\]

where \(g\) is a known function. We shall determine the functions \(u\) and \(q\) satisfying (1.1) and (1.4).

If the principle coefficient \(a(x)\) is required to be strictly positive, i.e.,

\[
a(x) \geq a_0 > 0, \quad x \in [0, l],
\]

then the equation should be rewritten as an initial-boundary value problem, e.g., the homogeneous Dirichlet boundary value problem as follows:

\[
\begin{cases}
  u_t - (a(x)u_x)_x + q(x)u = 0, & (x, t) \in Q, \\
  u|_{x=0} = u|_{x=l} = 0, \\
  u(x, 0) = \phi(x),
\end{cases}
\]

which is often referred as the classical parabolic equation. The mathematical model (1.5) arises in various physical and engineering settings. If (1.5) is used to describe the heat transfer system, the coefficient \(q(x)\) is called the radiative coefficient which is often dependent on the medium property.

Being different from the ordinary parabolic equation (1.1), system (1.1) belongs to the second order differential equations with non-negative characteristic form. The main character of such kinds of equations is degeneracy. It can be easily seen that at \(x = 0\) and \(x = l\), Eq. (1.1) degenerates into two hyperbolic equations

\[
\begin{align*}
  \frac{\partial u}{\partial t} - a'(0) \frac{\partial u}{\partial x} + q(0)u &= 0, \\
  \frac{\partial u}{\partial t} - a'(l) \frac{\partial u}{\partial x} + q(l)u &= 0.
\end{align*}
\]
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By the well known Fichera’s theory (see [32]) for degenerate parabolic equations, we know that whether or not boundary conditions should be given at the degenerate boundaries is determined by the sign of the Fichera function.

Consider the following second order equation:

\[
\sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f(x),
\]

(1.6)

where \( x = (x_1, x_2, \cdots, x_m) \in \Omega \subset \mathbb{R}^m \) and \( a_{ij} \) satisfies

\[ a_{ij} = a_{ji}, \quad i,j = 1, 2, \cdots, m, \]

and

\[ \sum_{i,j=1}^{m} a_{ij}(x)\xi_i \xi_j \geq 0, \quad \forall x \in \bar{\Omega}, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_m) \in \mathbb{R}^m. \]

Let \( n = (n_1, n_2, \cdots, n_m) \) be the unit inward normal vector of the boundary \( \partial \Omega \), and let \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), where \( \Gamma_1 \) is the non-characteristic part of \( \partial \Omega \), i.e.,

\[ \sum_{i,j=1}^{m} a_{ij}(x)n_i n_j > 0, \quad x \in \Gamma_1, \]

\[ \sum_{i,j=1}^{m} a_{ij}(x)n_i n_j = 0, \quad x \in \Gamma_2 \cup \Gamma_3. \]

Define the following Fichera function:

\[ B(x) = \sum_{i=1}^{m} \left[ b_i(x) - \sum_{j=1}^{m} \frac{\partial a_{ij}(x)}{\partial x_j} \right] n_i, \]

and on \( \Gamma_2 \) and \( \Gamma_3 \) it satisfies

\[ B(x) \begin{cases} \geq 0, & x \in \Gamma_2, \\ < 0, & x \in \Gamma_3. \end{cases} \]

Then, to guarantee the well-posedness of problem (1.6) one should give some boundary conditions on \( \Gamma_1 \cup \Gamma_3 \), while on \( \Gamma_2 \), they must not be given. For problem (1.1), by denoting \( x_1 = x \) and \( x_2 = t \) we have

\[ a_{11} = a(x), \quad a_{12} = a_{21} = a_{22} = 0, \]

\[ b_1 = a'(x), \quad b_2 = -1. \]

On the boundary \( x = 0 \), the unit inward normal vector is \((1, 0)\). By direct calculations we have

\[ B(0, t) = b_1 - \frac{d}{dx}a_{11} = 0. \]
From the Fichera’s theory we know that boundary conditions should not be given on \( x = 0 \). By analogous arguments, we can also obtain that on \( x = 1 \) and \( t = T \) boundary conditions should not be given, while on \( t = 0 \) they are indispensable. Therefore, the parabolic problem (1.1) is well defined.

In general, most physical and industrial phenomenons can be described by the classical parabolic model, such as Eq. (1.5). However, with the development of the modern financial mathematics, more and more degenerate elliptic or parabolic equations arising in derivatives pricing have to be taken into account. For example, the well known Black-Scholes equation:

\[
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S)S^2 \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} - rV = 0, \quad (S, t) \in [0, \infty) \times [0, T), \tag{1.7}
\]

is such the case, where the degenerate parabolic boundary is \( S = 0 \).

For a given coefficient \( q(x) \), the degenerate parabolic equation (1.1), which is referred as a direct problem consists of the determination of the solution from the given initial condition. It is well known that in all cases the inverse problem is ill-posed or improperly posed in the sense of Hadamard, while the direct problem is well-posed (see [29, 31]). The ill-posedness, particularly the numerical instability, is the main difficulty for problem \( P \). Since data errors in the extra condition \( g(x) \) are inevitable, arbitrarily small changes in \( g(x) \) may lead to arbitrarily large changes in \( q(x) \), which may make the obtained results meaningless (see, e.g., [19, 35]).

Inverse coefficient problems for parabolic equations are well studied in the literature. However, most of these inverse problems are governed by classical parabolic equations in which the principle coefficients are assumed to be strictly positive definite. The inverse problem of identifying the diffusion coefficient \( a(x) \) in the following parabolic equation

\[
u_t - \nabla \cdot (a(x)\nabla u) = f(x,t), \quad (x, t) \in \Omega \times (0, T)
\]

from some additional conditions has been investigated by several authors, e.g., in [14, 20, 23, 28]. In [20, 28], the output least-squares method with Tikhonov regularization is applied to the inverse problem and the numerical solution is obtained by the finite element method. The determination of \( a(x) \) with two Neumann measured data

\[
a(0)u_x(0,t) = k(t), \quad a(1)u_x(1,t) = h(t), \quad t \in [0, T]
\]

has been considered carefully in [14] by the semigroup approach. In [23], the inverse problem is reduced to a nonlinear equation and the uniqueness, as well as the conditional stability of the solution is proved.

The inverse problem of identification of the radiative coefficient \( q(x) \) in the following heat conduction equation

\[
u_t - \Delta u + q(x)u = 0, \quad (x, t) \in Q,
\]

from the final overdetermination data \( u(x,T) \) has been considered by several authors, e.g., in [8, 10, 11, 33, 38]. Moreover, treatments on the case of purely time dependent \( q = q(t) \) can
An inverse problem of identifying the radiative coefficient in a degenerate parabolic equation is found in [6, 7, 12, 13]. For the general case in which the unknown coefficient(s) depend(s) on both spatial and temporal variables, we refer the readers to the references, e.g., in [16, 26, 27, 34].

Compared with classical parabolic equations, the main difficulty for degenerate equations lies in the degeneracy of the principle coefficients which may lead to the corresponding solution has no sufficient regularity, even if the initial value and the coefficients are sufficiently smooth functions. Many effective tools, e.g., the Schauder’s type a-priori estimate which has been extensively applied in classical parabolic equations, are no longer applicable for the degenerate parabolic equations. The documents concerned with inverse degenerate problems are quite few in contrast with those dealt with non-degenerate problems. In [2], the authors investigate an inverse problem of determining the source term \( g \) in the following degenerate parabolic equation

\[
 u_t - (x^\alpha u_x)_x = g, \quad (x, t) \in (0, 1) \times (0, T),
\]

where \( \alpha \in [0, 2) \). The uniqueness and Lipschitz stability of the solution are obtained by the global Carleman estimates which is introduced in [21] in 1998. Recently, in [36] analogous methods are applied to a nonlinear inverse coefficient problem arising in the field of climate evolution, where the diffusion coefficient is assumed to vanish at both extremities of the domain. For other topics of degenerate parabolic equations, e.g., the null controllability, we may refer the reader to [3–5] and the reference therein.

The most important inverse problem in which the underlying model is degenerate may be the reconstruction of local volatility in the Black-Scholes equation (1.7). In [24, 25], the inverse problem of identifying the implied volatility \( \sigma = \sigma(S) \) from current market prices of options has been considered carefully. Based on the optimal control framework, the existence, uniqueness of \( \sigma(S) \) and a well-posed algorithm are obtained. Similar results are derived in [15], where a new extra condition, i.e., the average option premium, is assumed to be known. In [18], on the basis of the parameter-to-solution mapping, the stability and convergence of approximations for \( \sigma(S) \) are gained by Tikhonov regularization.

It should be mentioned that the degeneracy in the Black-Scholes equation can be removed by some change of variable (see [18]). However, the degeneracy in our problem can not be removed by any method, which is also the main difficulty in the paper.

To our knowledge, this paper is the first one concerning uniqueness, stability and convergence of optimal solution in inverse problem for degenerate parabolic equations such as (1.1). In this paper, we use an optimal control framework (see, e.g., [16, 17, 24, 38]) to discuss problem \( P \) mainly from the theoretical analysis angle. The outline of the manuscript is as follows: In Section 2, the uniqueness of the solution for problem \( P \) is obtained by the contraction mapping principle. In Section 3, the inverse problem \( P \) is transformed into an optimal control problem \( P1 \) and the existence of minimizer of the cost functional is proved. The necessary condition of the minimizer is established in Section 4. By assuming \( T \) is relatively small, the local uniqueness and stability of the minimizer are shown in Section 5. The convergence of the minimizer with
noisy input data is obtained in Section 6 by some \textit{a-priori} regularity conditions imposed on the forward operator. Section 7 ends this paper with concluding remarks.

2 Inverse Problem P

Let’s introduce the following function space:

\[ W^{k,\infty}(\Omega) = \{ u(x) \mid D^\alpha u \in L^\infty(\Omega), \ \forall |\alpha| \leq k \}. \]

To discuss the uniqueness of the solution, we shall first establish the weak maximum principle. We would like to consider the more general equation:

\[
\begin{aligned}
\begin{cases}
    u_t - (a(x)u_x)_x + q(x)u = f(x,t), & (x,t) \in Q = (0,l) \times (0,T], \\
    u|_{t=0} = \phi(x), & x \in (0,l),
\end{cases}
\end{aligned}
\]

Let

\[ G = (0,l) \quad \text{and} \quad G_\delta = (-\delta, l + \delta), \]

where \( \delta \) is an arbitrarily small positive constant. Assume that the functions \( a, f, \phi \) and \( q \) satisfy the following conditions:

- \( a \in W^{k+1,\infty}(G_\delta), \ f \in W^{k,\infty}(G_\delta \times (0,T)), \ \phi \in W^{k+2,\infty}(G_\delta), \ q \in W^{k,\infty}(G_\delta); \)

- \( a \geq 0, \ \forall x \in G_\delta; \)

- \( q \geq q_0 > 0. \)

\textbf{Theorem 2.1.} (see [32]) Under the above assumptions, there exists a unique solution \( u(x,t) \in W^{k,\infty}(\bar{Q}) \) to the equation (2.1).

For the extra condition (1.4) we shall assume that

\[ g(x) \in L^\infty(0,l). \]

\textbf{Remark 2.1.} The condition \( q(x) \geq q_0 > 0 \) is not essential. In fact, for the case of \( q(x) \) with lower bound, i.e., \( q(x) \geq c_0, \ c_0 < 0 \), we can make the following function transformation

\[ v(x,t) = u(x,t)e^{(c_0-1)t}. \]

One can easily check that \( v \) satisfies

\[ v_t - (a(x)v_x)_x + [q(x) - c_0 + 1]v = \tilde{f}(x,t), \]

where \( q(x) - c_0 + 1 > 0. \)
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It is known that for classical parabolic equations whose leading coefficients are assumed to be positive definite, the maximum or minimum of the solution can only be attained on the parabolic boundaries, which is known as the famous weak maximum principle. Such kind of principle is also applicable for degenerate parabolic equations. The main difference lies in that the maximum or minimum cannot be attained on the degenerate boundaries.

Denote
\[ \mathcal{L}u = u_t - (a(x)u_x)_x + q(x)u. \]

**Lemma 2.2. (weak maximum principle)** Assuming that \( u(x, t) \in C^2(Q) \cap C(\bar{Q}) \), and satisfies \( \mathcal{L}u \leq 0 \), then we assert that \( u(x, t) \) can only attain the positive maximum at the boundary
\[ \{(x, t) | \ t = 0, \ x \in [0, l] \}. \]

**Proof.** Firstly, we assume \( u(x, t) \) attains its positive maximum \( M \) at the internal point \( P_0(x_0, t_0) \in Q \), i.e.,
\[ u(x_0, t_0) = \max_Q u(x, t) = M > 0. \]

Then we have
\[
\begin{align*}
\left. \frac{\partial u}{\partial x} \right|_{P_0} &= 0, & 0 < x_0 < l, \\
\left. \frac{\partial^2 u}{\partial x^2} \right|_{P_0} &\leq 0, & 0 < x_0 < l, \\
\left. \frac{\partial u}{\partial t} \right|_{P_0} &= 0, & \text{as } t_0 < T, \\
\left. \frac{\partial u}{\partial t} \right|_{P_0} &\geq 0, & \text{as } t_0 = T.
\end{align*}
\]

Hence, we have
\[ \mathcal{L}u |_{P_0} \geq q(x_0)M \geq q_0M > 0, \]
which contradicts with the assumption of the Lemma.

Next, we illustrate that \( u(x, t) \) cannot attain its positive maximum at the degenerate boundaries
\[ \{x = 0, \ 0 < t \leq T\} \cup \{x = l, \ 0 < t \leq T\}. \]

Without loss of generality, we assume that \( u(x, t) \) attains its positive maximum \( M \) at the point \( P_1(0, t_1), 0 < t_1 \leq T \). Then we have
\[
\begin{align*}
\left. \frac{\partial u}{\partial t} \right|_{P_1} &= 0, & 0 < t_1 < T, \\
\left. \frac{\partial u}{\partial t} \right|_{P_1} &\geq 0, & t_1 = T.
\end{align*}
\]
Moreover, from
\[ a(0) = 0, \quad a(x) > 0, \quad 0 < x < l, \]
we know \( a'(0) \geq 0 \). Noting that \( u(0, t_1) = M \) is a maximum, we have
\[ u(x, t_1) \leq u(0, t_1), \quad \forall x \in (0, l), \]
which implies
\[ u_x(0, t_1) = \lim_{x \to 0^+} \frac{u(x, t_1) - u(0, t_1)}{x} \leq 0. \]
Therefore, we have
\[
\mathcal{L}u|_{P_1} = \frac{\partial u}{\partial t}_{P_1} - a(0) \frac{\partial^2 u}{\partial x^2}_{P_1} - a'(0) \frac{\partial u}{\partial x}_{P_1} + q(0)u|_{P_1} \\
\geq q(0)M \geq q_0M > 0,
\]
which also contradicts with the assumption of the Lemma.

For same arguments, we know \( u(x, t) \) cannot attain its positive maximum at the degenerate boundary \( \{x = l, \quad 0 < t \leq T\} \) either.

This completes the proof of Lemma 2.2. \( \square \)

**Corollary 2.3.** Assuming that \( u(x, t) \in C^2(Q) \cap C(\bar{Q}) \) and satisfies \( \mathcal{L}u \geq 0 \), then \( u(x, t) \) can only attain its negative minimum at the boundary
\[ \{(x, t) | \ t = 0, \ x \in [0, l]\}. \]

**Theorem 2.4.** Assuming that \( u(x, t) \in C^2(Q) \cap C(\bar{Q}) \) is the solution of (2.1), then we have for \( u(x, t) \) the following estimate
\[
\max_{\bar{Q}} |u| \leq \max \left\{ \frac{1}{q_0} \sup_{Q} |f|, \sup_{[0, l]} |\phi| \right\}. \quad (2.3)
\]

**Proof.** Let
\[
M = \left\{ \frac{1}{q_0} \sup_{Q} |f|, \sup_{[0, l]} |\phi| \right\},
\]
and \( v = M \pm u \). One can easily deduce
\[
\mathcal{L}v = \mathcal{L}M \pm \mathcal{L}u = Mq(x) \pm f(x, t) \geq 0, \quad v|_{t=0} = M \pm \phi(x) \geq 0.
\]
From Corollary 2.3, we have
\[ v(x, t) \geq 0, \quad (x, t) \in \bar{Q}. \]
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Hence

\[
\max_Q |u| \leq M.
\]

This completes the proof of Theorem 2.4.

Now, we will consider the uniqueness of the inverse problem \( P \).

Let \( \mathcal{S} \) be the following function set:

\[
\mathcal{S} = \{ q(x) | q(x) \in W^{3,\infty}(G_\delta), q(x) \geq 0 \}, \tag{2.4}
\]

and assume \( a(x), \phi(x) \) satisfy the following regularity conditions:

- \( a(x) \in W^{4,\infty}(G_\delta), \phi(x) \in W^{5,\infty}(G_\delta) \);
- \( a(x) \geq 0, \forall x \in G_\delta \).

Here we make a mini modification for the lower bound of \( q(x) \). Noting \( f(x, t) \equiv 0 \) and remark 2.1, such the change is irrelevant.

In the set \( \mathcal{S} \), we introduce the following partial order "\( \geq \)", i.e., for any \( q_1, q_2 \in \mathcal{S} \), we call \( q_1 \geq q_2 \) if and only if

\[
q_1(x) \geq q_2(x), \quad \forall x \in G_\delta.
\]

The partial order "\( \leq \)" can be defined analogously.

For any \( q \in \mathcal{S} \), we introduce a subset of \( \mathcal{S} \) denoted by \( \mathcal{S}_q \) in which all the elements are required to satisfy the partial order "\( \geq \)" or "\( \leq \)" for \( q \), i.e.,

\[
\mathcal{S}_q = \{ q_1 \in \mathcal{S} | q_1 \geq q \text{ or } q_1 \leq q \}.
\]

To obtain the uniqueness, we define the following mapping \( \mathbb{P} \) by

\[
\mathbb{P}[q] = q + \lambda (u(x, T; q) - g(x)), \tag{2.5}
\]

where \( u(x, t; q) \) is the solution Eq. (1.1) with the given coefficient \( q(x) \) and \( \lambda > 0 \) is an adjusting parameter. It can be easily seen that the existence of fixed points of \( \mathbb{P} \) is equivalent to solutions of the overposed initial value problem.

From Theorem 2.1, we know \( u(x, t) \in W^{3,\infty}(\bar{Q}) \), which by the Sobolev embedding theorem implies \( u(x, t) \in C^{2,1}(\bar{Q}) \). Then, from Theorem 2.4 we have for \( u(x, t) \) the following estimate:

\[
0 \leq u(x, t; q) \leq \|\phi\|_\infty.
\]

For any \( q, h \in \mathcal{S} \) and \( h \geq 0 \), one can easily compute the Gâteaux derivative of \( \mathbb{P} \) to obtain

\[
\mathbb{P}'[q] \cdot h = h - \lambda \dot{u}(x, T; q, h), \tag{2.6}
\]
where \( \dot{u}(x,t;q,h) \) satisfies the following degenerate parabolic equation:

\[
\begin{aligned}
\dot{u}_t - (a(x)\dot{u}_x)_x + q(x)\dot{u} &= h(x)u, \quad (x,t) \in Q, \\
\dot{u}|_{t=0} &= 0.
\end{aligned}
\] (2.7)

Noting \( h \in S \) and \( u \in W^{3,\infty}(\bar{Q}) \), from Theorem 2.1 we know that there exists a unique weak solution \( \dot{u} \in W^{3,\infty}(\bar{Q}) \) to Eq. (2.7). For Eq. (2.7), we have from Theorem 2.4 that

\[0 \leq \dot{u}(x,t;q,h) \leq v(x,t),\]

where the equality on the left if and only if \( h \equiv 0 \), and \( v(x,t) \) satisfies the following equation:

\[
\begin{aligned}
v_t - (a(x)v_x)_x + q(x)v &= h(x)\|\phi\|_{\infty}, \\
v|_{t=0} &= 0.
\end{aligned}
\] (2.8)

Therefore, for any \( \lambda > 0 \), the righthand side of (2.6) is strictly less than \( h(x) \). By choosing \( \lambda \) sufficiently small such that \( \lambda \|\phi\|_{\infty} < 2 \), we have

\[
\|P[q] \cdot h\|_{\infty} = \|h - \lambda \dot{u}\|_{\infty} < \|h\|_{\infty}.
\] (2.9)

From (2.9), we have for any \( q \in S, \ q_1 \in S_q \),

\[
\|P[q] - P[q_1]\|_{\infty} < \|q - q_1\|_{\infty},
\] (2.10)

which indicates that if the mapping \( P \) has a fixed point, then it must be unique in corresponding partial order set.

**Theorem 2.5.** If there exist a solution \( q(x) \in S \) and \( u(x,t;q) \) to (1.1)/(1.4), then the solution is unique in the set \( S_q \).

**Remark 2.2.** It is well known that the Carleman estimate is an effective tool to derive uniqueness and conditional stability for inverse problems (see [21]). But unfortunately, it fails in treating the terminal control problems such as inverse problem \( P \). To the authors’ knowledge, the uniqueness obtained in the paper is so far the best result one can expect.

### 3 Optimal Control Problem

We have obtained the uniqueness of the solution for problem \( P \) in the previous section. Now we would like to discuss the regularization of problem \( P \). Before this, we would like to discuss the forward problem (2.1) and give some basic definitions, lemmas and estimations.

**Definition 3.1.** Define \( B \) to be the closure of \( C^{\infty}_0(Q) \) under the following norm:

\[
\|u\|_B^2 = \int_Q a(x)(|u|^2 + |\nabla u|^2)dxdt, \quad u \in B.
\]
Furthermore, if \( a \) where (2.1) and satisfies the following estimate identity holds
\[
\int_Q \left( -u \frac{\partial \psi}{\partial t} + a \nabla u \cdot \nabla \psi + q u \psi \right) dx dt - \int_0^1 \phi(x) \psi(x,0) dx = \int_Q f \psi dx dt. \tag{3.1}
\]

**Remark 3.1.** Assuming \( u \in C([0,T];L^2(0,l)) \cap \mathcal{B} \), and for any \( \psi \in L^\infty((0,T);L^2(0,l)) \cap \mathcal{B} \), \( \frac{\partial \psi}{\partial t} \in L^2(Q) \), \( \psi(\cdot,T) = 0 \), the following integration estimate for \( a \) can be rewritten as
\[
\int_Q \left( \frac{\partial u}{\partial t} \psi + a \nabla u \cdot \nabla \psi + q u \psi \right) dx dt = \int_Q f \psi dx dt,
\]
where \( u \) satisfies \( u|_{t=0} = \phi(x) \) in the sense of trace.

**Theorem 3.1.** For any given \( f \in L^\infty(Q) \), \( \phi \in L^\infty(0,l) \), there exists a unique weak solution to (2.1) and satisfies the following estimate
\[
\|u\|_{L^\infty((0,T);L^2(0,l))} + \|a|\nabla u|^2\|_{L^1(Q)} \leq C \left( \|f\|_{L^2(Q)}^2 + \|\phi\|_{L^2(0,l)}^2 \right).
\]
Furthermore, if \( a|\nabla \phi|^2 \in L^1(0,l) \), then \( \frac{\partial u}{\partial t} \in L^2(Q) \) and
\[
\left\| \frac{\partial u}{\partial t} \right\|_{L^2(Q)} \leq C \left( \|f\|_{L^2(Q)} + \|\phi\|_{L^2(0,l)} + \|a|\nabla \phi|^2\|_{L^1(0,l)} \right).
\]

**Proof.** Firstly, we prove the existence. For any given \( 0 < \varepsilon < 1 \), we consider the following regularized problem
\[
\begin{align*}
\frac{\partial u_\varepsilon}{\partial t} - (a_\varepsilon(x)u_\varepsilon)_x + q(x)u_\varepsilon &= f(x,t), \quad (x,t) \in Q, \\
u_\varepsilon(0,t) &= u_\varepsilon(l,t) = 0, \\
u_\varepsilon(x,0) &= \phi(x),
\end{align*}
\tag{3.2}
\]
where
\[
a_\varepsilon(x) = a(x) + \varepsilon, \quad x \in [0,l].
\]
From the well-known theory for parabolic equations (see [30]), there exists a unique weak solution \( u_\varepsilon(x,t) \) to Eq. (3.2).

Then, we will give some apriori estimates for \( u_\varepsilon(x,t) \). Without loss of generality, we assume that \( u_\varepsilon(x,t) \) is the classical solution of (3.2). Otherwise, one can smooth the coefficients of (3.2) and then consider the solution of the approximation problem.

Multiplying on both sides of (3.2) with \( u_\varepsilon \) and integrating on \( Q_t = [0,l] \times (0,t) \), we have
\[
\int_{Q_t} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon dx dt - \int_{Q_t} (a_\varepsilon u_\varepsilon)_x u_\varepsilon dx dt + \int_{Q_t} q u_\varepsilon^2 dx dt = \int_{Q_t} f u_\varepsilon dx dt.
\]
Integration by parts, we get
\[
\int_0^t \frac{1}{2} u_\varepsilon^2 dx + \int_0^t \int_0^t a_\varepsilon |u_\varepsilon|_x^2 dx dt + \int_0^t \int_0^t q u_\varepsilon^2 dx dt
\]
\[ \leq \int_0^t \frac{1}{2} \phi^2 dx + \frac{1}{2} \int_0^t \int_0^t |u_\varepsilon|^2 dx dt + \frac{1}{2} \int_0^t \int_0^t f^2 dx dt. \] (3.3)

From (3.3) and the Gronwall inequality, we have

\[ \max_{0 < t \leq T} \int_0^t u_\varepsilon^2 dx + \int \int_a_\varepsilon |u_\varepsilon, x|^2 dx dt \leq C \left( \int_0^t \phi^2 dx + \int \int_Q f^2 dx dt \right). \]

On the other hand, if \( a|\nabla \phi|^2 \in L^1(0, l) \), then by multiplying \( \frac{\partial u_\varepsilon}{\partial t} \) on both sides of (3.2) and integrating on \( Q_t \), we obtain

\[ \int \int_{Q_t} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt - \int \int_{Q_t} (a_\varepsilon u_\varepsilon, x) \frac{\partial u_\varepsilon}{\partial t} dx dt + \int \int_{Q_t} q u_\varepsilon \frac{\partial u_\varepsilon}{\partial t} dx dt = \int \int_{Q_t} f \frac{\partial u_\varepsilon}{\partial t} dx dt. \] (3.4)

Integrating by parts, we have

\[ \int \int_{Q_t} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt + \int \int_{Q_t} \frac{q}{2} \frac{\partial}{\partial t} (u_\varepsilon^2) dx dt - \int \int_{Q_t} \frac{\partial}{\partial x} \left( a_\varepsilon \frac{\partial u_\varepsilon}{\partial x} \right) \frac{\partial u_\varepsilon}{\partial t} dx dt = \int \int_{Q_t} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt + \int \int_{Q_t} \frac{q}{2} \frac{\partial}{\partial t} (u_\varepsilon^2) dx dt + \int \int_{Q_t} a_\varepsilon \frac{\partial}{\partial x} \left( \frac{\partial u_\varepsilon}{\partial x} \right) \frac{\partial u_\varepsilon}{\partial t} dx dt = \int \int_{Q_t} f \frac{\partial u_\varepsilon}{\partial t} dx dt. \] (3.4)

From (3.4), we get

\[ \int \int_{Q_t} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt + \int_0^t a_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial x} (\cdot, t) \right|^2 dx + \int_0^t \frac{q}{2} u_\varepsilon^2 (\cdot, t) dx \leq \int_0^t a_\varepsilon \phi_\varepsilon^2 dx + \frac{1}{2} \int_0^t \int \int Q_t f^2 dx dt + \frac{1}{2} \int \int_{Q_t} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt. \] (3.5)

From (3.5), we have

\[ \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2(Q)} \leq C \left( \| f \|_{L^2(Q)} + \| \phi \|_{L^2(0, l)} + \| a_\varepsilon |\nabla \phi|^2 \|_{L^1(0, l)} \right). \]

Moreover, it follows from the maximum principle that

\[ \| u_\varepsilon \|_{L^\infty(Q)} \leq C. \]

From the estimations above, it can be derived that there exists a subsequence of \( \{ u_\varepsilon \} \) (denoted by itself) and

\[ u \in C([0, T]; L^2(0, l)), \quad \frac{\partial u}{\partial t} \in L^2(Q), \]

\[ a \in L^2(Q). \]
An inverse problem of identifying the radiative coefficient in a degenerate parabolic equation such that

\begin{align*}
  u_\varepsilon &\to u \quad \text{in} \ L^2(Q), \\
  \nabla u_\varepsilon &\to \nabla u \quad \text{in} \ L^2_{\text{loc}}(Q), \\
  \frac{\partial u_\varepsilon}{\partial t} &\to \frac{\partial u}{\partial t} \quad \text{in} \ L^2(Q), \\
  a_\varepsilon \nabla u_\varepsilon &\to a \nabla u \quad \text{in} \ L^2(Q).
\end{align*}

Letting \( u = u_\varepsilon \) in (3.1), we have

\[
\int_Q \left( -u_\varepsilon \frac{\partial \psi}{\partial t} + a \nabla u_\varepsilon \cdot \nabla \psi + q u_\varepsilon \psi \right) \, dx \, dt - \int_0^t \phi(x) \psi(x, 0) \, dx = \int_Q f \, \psi \, dx \, dt.
\]

Letting \( \varepsilon \to 0 \), one can immediately obtain

\[
\int_Q \left( -u \frac{\partial \psi}{\partial t} + a \nabla u \cdot \nabla \psi + q \psi \right) \, dx \, dt - \int_0^t \phi(x) \psi(x, 0) \, dx = \int_Q f \, \psi \, dx \, dt,
\]

which implies the existence of weak solutions.

Next, we prove the uniqueness of weak solutions. Suppose that \( u_1, u_2 \) be two solutions of (2.1) and let

\[ U(x, t) = u_1(x, t) - u_2(x, t), \quad (x, t) \in Q. \]

It can be easily seen that \( U \in C([0, T]; L^2(0, l)) \cap \mathcal{B} \), and for any \( \psi \in L^\infty((0, T); L^2(0, l)) \cap \mathcal{B} \), \( \frac{\partial \psi}{\partial t} \in L^2(Q) \), \( \psi(\cdot, T) = 0 \), the following integration identity holds

\[
\int_Q \left( -U \frac{\partial \psi}{\partial t} + a \nabla U \cdot \nabla \psi + q U \psi \right) \, dx \, dt = 0. \tag{3.6}
\]

For any given \( g \in C_0^\infty(Q) \), by the existence obtained above we know that there exists a weak solution \( v \in L^\infty((0, T); L^2(0, l)) \cap \mathcal{B} \) and \( \frac{\partial v}{\partial t} \in L^2(Q) \) for the following equation

\[
- \frac{\partial v}{\partial t} - (a(x) v_x)_x + q(x) v = g(x, t), \quad (x, t) \in Q, \\
v(x, T) = 0, \quad x \in (0, l).
\]

Letting \( \psi = v \) in (3.6), we obtain

\[
\int_Q U g \, dx \, dt = 0.
\]

Noting the arbitrariness of \( g \), we have

\[ U(x, t) = 0, \quad \text{a.e.} \ (x, t) \in Q, \]

i.e.,

\[ u_1(x, t) = u_2(x, t), \quad \text{a.e.} \ (x, t) \in Q. \]

This completes the proof of Theorem 3.1. \( \square \)
Remark 3.2. The weak solution defined above is on the whole domain $Q$. If we only consider the spatial case, we can modify the definition 3.1 as:

**Definition 3.1’**. Define $\mathcal{H}^{1}(0, l)$ to be the closure of $C_{0}^\infty(0, l)$ under the following norm:

$$
\|v\|^2_{\mathcal{H}^1} = \int_{0}^{l} a(x)(|v|^2 + |\nabla v|^2)dxdt, \quad v \in \mathcal{H}^1(0, l).
$$

For the case of $f \equiv 0$, the definition 3.2 can also be rewritten as:

**Definition 3.2’**. A function $u \in H^1((0, T); L^2(0, l)) \cap L^2((0, T); \mathcal{H}^1(0, l))$ is called the weak solution of (2.1), if $u$ satisfies

$$
u(x, 0) = \phi(x), \quad x \in (0, l), \quad (3.7)$$

and the following integration identity

$$
\int_{0}^{l} u\psi dx + \int_{0}^{l} a\nabla u \cdot \nabla \psi dx + \int_{0}^{l} qu\psi dx = 0, \quad \forall \psi \in L^2(0, l) \cap \mathcal{H}^1(0, l) \quad (3.8)
$$

holds for a.e., $t \in (0, T]$. Then, by analogously arguments, one can establish the existence, uniqueness and regularity for such kind of weak solution, which are similar to those of Theorem 3.1.

**Remark 3.3.** We recall that the principle coefficient $a(x) \in C^1[0, 1]$. Due to the degeneracy at $x = 0$ and $x = l$, from $u \in \mathcal{H}^1(0, l)$ one can only derive $u \in H^1_{loc}(0, l)$ rather than $u \in H^1(0, l)$, which is different from the case of non-degenerate. However, we may derive

$$
aux \to 0, \quad \text{as } x \to 0. \quad (3.9)
$$

In fact, if (3.3) is not true, i.e., $aux \to k$, $k \neq 0$, then we have $ux \sim \frac{k}{a(x)}$ in $B_{\delta}(0) \cap [0, l]$, where $B_{\delta}(0)$ is a ball with $\delta$-radius centered at $x = 0$. Note that

$$
a(x) = a(0) + a'(\xi)x = a'(\xi)x, \quad \xi \in [0, x], \quad x \in B_{\delta}(0) \cap [0, l].
$$

Hence,

$$
|ux|^2 \sim \frac{k^2}{a(x)} \sim \frac{k^2}{a'(\xi)x},
$$

which is contradicts with $|ux|^2 \in L^1(0, l)$. By analogous arguments, we have

$$
aux \to 0, \quad \text{as } x \to l.
$$

It should be mentioned that these conclusions are no longer valid for $a \in C^1[0, l]$. For example, let

$$
a(x) = x^\alpha(l - x)^\beta, \quad 0 < \alpha, \beta < 1. \quad (3.10)
$$

It can be easily seen that $|ux|^2 \in L^1(0, l)$ cannot guarantee $aux \to 0$, as $x$ tends to 0 or $l$. In some references, e.g., [3, 4], the case (3.10) is called the weak degeneracy and the boundary
An inverse problem of identifying the radiative coefficient in a degenerate parabolic equation is indispensable for corresponding mathematical model, e.g., we shall replace the equation (1.1) by the following initial-boundary value problem:

\[
\begin{cases}
    u_t - (a(x)u_x)_x + q(x)u = 0, & (x, t) \in Q, \\
    u|_{x=0} = u|_{x=l} = 0, \\
    u(x, 0) = \phi(x).
\end{cases}
\]

Since the inverse problem \( P \) is ill-posed, i.e., its solution depends unstably on the data, we turn to consider the following optimal control problem \( P_1 \):

Find \( \bar{q}(x) \in A \) such that:

\[
J(\bar{q}) = \min_{q \in A} J(q),
\]

where

\[
J(q) = \frac{1}{2} \int_{0}^{l} |u(x, T; q) - g(x)|^2 \, dx + \frac{N}{2} \int_{0}^{l} |\nabla q|^2 \, dx,
\]

\[
A = \{ q(x) \mid 0 < \alpha \leq q \leq \beta, \quad \|q\|_{H^1(0,l)} < \infty \},
\]

\( u(x, t; q) \) is the solution of Eq. (1.1) for a given coefficient \( q(x) \in A \), \( N \) is the regularization parameter, and \( \alpha, \beta \) are two given positive constants.

From (2.2) and Theorem 3.1, it can be easily seen that the control functional (3.12) is well-defined for any \( q \in A \).

We are now going to show the existence of minimizers to the problem (3.11). Firstly, we assert that the functional \( J(q) \) is of some continuous property in \( A \) in the following sense.

**Lemma 3.2.** For any sequence \( \{q_n\} \) in \( A \) which converges to some \( q \in A \) in \( L^1(0,l) \) as \( n \to \infty \), we have

\[
\lim_{n \to \infty} \int_{0}^{l} |u(q_n)(x,T) - g(x)|^2 \, dx = \int_{0}^{l} |u(q)(x,T) - g(x)|^2 \, dx.
\]

**Proof. Step 1:** By taking \( q = q_n \) and choosing the test function as \( u(q_n)(\cdot,t) \) in (3.8) and then integrating with respect to \( t \), we derive that

\[
\|u(q_n; t)\|_{L^2(0,t)}^2 + \int_{0}^{t} \int_{0}^{l} a|\nabla u(q_n; t)|^2 \, dx \, dt + \int_{0}^{t} \int_{0}^{l} q_n|u(q_n; t)|^2 \, dx \, dt \leq \|\phi\|_{L^2(0,l)}^2
\]

for any \( t \in (0, T] \).

From (3.13) we know that the sequence \( \{u(q_n)\} \) is uniformly bounded in the space \( L^2((0,T); H^1(0,l)) \).

So we may extract a subsequence, still denoted by \( \{u(q_n)\} \), such that

\[
u(q_n)(x,t) \rightharpoonup u^*(x,t) \in L^2((0,T); H^1(0,l)).
\]
Step 2: Prove \( u^*(x, t) = u(q)(x, t) \).

By taking \( q = q_n \) in (3.8) and multiplying both sides by a function \( \eta(t) \in C^1[0, T] \) with \( \eta(T) = 0 \), we have

\[
\int_0^t u(q_n) \psi(t) dt + \int_0^t a \nabla u(q_n) \cdot \nabla \psi(t) dt + \int_0^t q_n \psi(t) dt = 0. \tag{3.17}
\]

Then integrating with respect to \( t \), we get

\[
- \int_0^t \phi(0) \psi(t) dt = - \int_0^T \int_0^t u(q_n) \psi(t) dt dt + \int_0^T \int_0^t \eta(t) a \nabla u(q_n) \cdot \nabla \psi(t) dt dt + \int_0^T \int_0^t \eta(t) q(x) u(q_n) \psi(t) dt dt
\]

\[
+ \int_0^T \int_0^t \eta(t) (q_n - q) u(q_n) \psi(t) dt dt. \tag{3.18}
\]

Letting \( n \to \infty \) in (3.18) and using (3.16), we obtain

\[
- \int_0^t \phi(0) \psi(t) dt = - \int_0^T \int_0^t u^* \psi(t) dt dt + \int_0^T \int_0^t \eta(t) a \nabla u^* \cdot \nabla \psi(t) dt dt
\]

\[
+ \int_0^T \int_0^t \eta(t) q(x) u^* \psi(t) dt dt. \tag{3.19}
\]

By noticing that (3.19) is valid for any \( \eta(t) \in C^1[0, T] \) with \( \eta(T) = 0 \), we have

\[
\int_0^t u^* \psi dt + \int_0^t a \nabla u^* \cdot \nabla \psi dt + \int_0^t qu^* \psi dt = 0, \quad \forall \psi \in \mathcal{H}^1(0, l) \tag{3.20}
\]

and \( u^*(x, 0) = \phi(x) \).

Therefore, \( u^* = u(q) \) by the definition of \( u(q) \).

Step 3: Prove \( ||u(q_n)(\cdot, T) - g||_{L^2(0, l)} \to ||u(q)(\cdot, T) - g||_{L^2(0, l)} \) as \( n \to \infty \).

We rewrite (3.8) for \( q = q_n \) in the form

\[
\int_0^t (u(q_n) - g) \psi(t) dt + \int_0^t a \nabla (u(q_n) - g) \cdot \nabla \psi(t) dt + \int_0^t q_n (u(q_n) - g) \psi(t) dt = 0.
\]

\[
= - \int_0^t a \nabla g \cdot \nabla \psi dt - \int_0^t q_n g \psi dt. \tag{3.21}
\]

Taking \( \psi = u(q_n) - g \) in (3.21) we have

\[
\frac{1}{2} \frac{d}{dt} ||u(q_n) - g||_{L^2(0, l)}^2 + \int_0^t a |\nabla (u(q_n) - g)|^2 dt + \int_0^t q_n |u(q_n) - g|^2 dt
\]

\[
= - \int_0^t a \nabla g \cdot \nabla (u(q_n) - g) dt - \int_0^t q_n g(u(q_n) - g) dt. \tag{3.22}
\]
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Similar relations hold for \( u(q) \), namely

\[
\frac{1}{2} \frac{d}{dt} \| u(q) - g \|^2_{L^2(0,t)} + \int_0^t a|\nabla (u(q) - g)|^2 dx + \int_0^t q|u(q) - g|^2 dx
= - \int_0^t a \nabla g \cdot \nabla (u(q) - g) dx - \int_0^t q g (u(q) - g) dx. \tag{3.23}
\]

Subtracting \((3.23)\) from \((3.22)\) we obtain

\[
\left\{ \int_0^t q_n |u(q_n) - g|^2 dx - \int_0^t q |u(q) - g|^2 dx \right\} + \frac{1}{2} \frac{d}{dt} \| u(q_n) - u(q) \|^2_{L^2(0,t)}
= \int_0^t a \nabla g \cdot \nabla (u(q) - u(q_n)) dx + \int_0^t q g (u(q) - u(q_n)) dx
+ \int_0^t (q - q_n) g (u(q_n) - g) dx + \int_0^t a \nabla (u(q) - u(q_n)) \cdot \nabla (u(q) + u(q_n) - 2g) dx
- \int_0^t \frac{d}{dt} [(u(q) - g)(u(q_n) - u(q))] dx. \tag{3.24}
\]

Taking \( \psi = u(q_n) - u(q) \) in \((3.28)\) we have

\[
\int_0^t u(q)_t (u(q_n) - u(q)) dx
= \int_0^t a \nabla u(q) \cdot \nabla (u(q) - u(q_n)) dx + \int_0^t q u(q) (u(q) - u(q_n)) dx. \tag{3.25}
\]

Similarly, for \((u(q_n) - u(q))_t (u(q) - g)\) we have

\[
\int_0^t (u(q_n) - u(q))_t (u(q) - g) dx
= \int_0^t a \nabla (u(q_n) - u(q)) \cdot \nabla (g - u(q)) dx + \int_0^t q (u(q_n) - u(q)) (g - u(q)) dx
+ \int_0^t (q_n - q) u(q_n) (g - u(q)) dx. \tag{3.26}
\]

Substituting \((3.25)\) and \((3.26)\) into \((3.24)\) and after some manipulations, we derive

\[
\frac{1}{2} \frac{d}{dt} \| u(q_n) - u(q) \|^2_{L^2(0,t)} + \int_0^t a|\nabla (u(q_n) - u(q))|^2 dx
+ \left\{ \int_0^t q_n |u(q_n) - g|^2 dx - \int_0^t q |u(q) - g|^2 dx \right\}
= 2 \int_0^t q (u(q_n) - u(q)) (u(q) - g) dx + \int_0^t (q_n - q) g (u(q_n) - g) dx
+ \int_0^t (q_n - q) u(q_n) (g - u(q)) dx := A_n. \tag{3.27}
\]
Then by rewriting the third term on the left side of (3.27), we have

\[
\frac{1}{2} \frac{d}{dt} \| u(q_n) - u(q) \|_{L^2(0,t)}^2 + \int_0^t a |\nabla (u(q_n) - u(q))|^2 dx + \int_0^t q_n |u(q_n) - u(q)|^2 dx
\]

\[
= A_n + \left\{ \int_0^t (q - q_n) |u(q) - g|^2 dx - 2 \int_0^t q_n (u(q_n) - u(q)) (u(q) - g) dx \right\}
\]

\[
:= A_n + B_n. \tag{3.28}
\]

Integrating over the interval \((0, t)\) for any \(t \leq T\), we get

\[
\frac{1}{2} \| u(q_n; t) - u(q; t) \|_{L^2(0,t)}^2 \leq \int_0^T |A_n + B_n| dt. \tag{3.29}
\]

By the convergence of \(\{q_n\}\) and the weak convergence of \(\{u(q_n)\}\), one can easily get

\[
\int_0^T |A_n + B_n| dt \to 0, \quad \text{as } n \to \infty. \tag{3.30}
\]

Combining (3.29) and (3.30) we have

\[
\max_{t \in [0,T]} \| u(q_n; t) - u(q; t) \|_{L^2(0,t)} \to 0, \quad \text{as } n \to \infty. \tag{3.31}
\]

On the other hand we have from the Hölder inequality

\[
\left| \int_0^t |u(q_n)(\cdot, T) - g|^2 dx - \int_0^t |u(q)(\cdot, T) - g|^2 dx \right|
\]

\[
\leq \int_0^t \left| u(q_n)(\cdot, T) - u(q)(\cdot, T) \right| \cdot |u(q_n)(\cdot, T) + u(q)(\cdot, T) - 2g| dx
\]

\[
\leq \| u(q_n)(\cdot, T) - u(q)(\cdot, T) \|_{L^2(0,t)} \cdot \| u(q_n)(\cdot, T) + u(q)(\cdot, T) - 2g \|_{L^2(0,t)}. \tag{3.32}
\]

From (2.2), (3.15), (3.31) and (3.32) we obtain

\[
\lim_{n \to \infty} \int_0^t |u(q_n)(x, T) - g(x)|^2 dx = \int_0^t |u(q)(x, T) - g(x)|^2 dx.
\]

This completes the proof of Lemma 3.2. \(\square\)

**Theorem 3.3.** There exists a minimizer \(\bar{q} \in \mathcal{A}\) of \(J(q)\), i.e.

\[
J(\bar{q}) = \min_{q \in \mathcal{A}} J(q).
\]

**Proof.** It is obvious that \(J(q)\) is non-negative and thus \(J(q)\) has the greatest lower bound \(\inf_{q \in \mathcal{A}} J(q)\). Let \(\{q_n\}\) be a minimizing sequence, i.e.,

\[
\inf_{q \in \mathcal{A}} J(q) \leq J(q_n) \leq \inf_{q \in \mathcal{A}} J(q) + \frac{1}{n}, \quad n = 1, 2, \ldots.
\]
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By noticing that $J(q_n) \leq C$ we deduce

$$\|\nabla q_n\|_{L^2(0,l)} \leq C, \quad (3.33)$$

where $C$ is independent of $n$. Noticing the boundedness of $\{q_n\}$ and (3.33), we also have

$$\|q_n\|_{H^1(0,l)} \leq C. \quad (3.34)$$

So we can extract a subsequence, still denoted by $\{q_n\}$, such that

$$q_n(x) \rightharpoonup \bar{q}(x) \in H^1(0,l) \quad \text{as} \quad n \to \infty. \quad (3.35)$$

By the Sobolev imbedding theorem (see [1]) we obtain

$$\|q_n(x) - \bar{q}(x)\|_{L^1(0,l)} \to 0 \quad \text{as} \quad n \to \infty. \quad (3.36)$$

It can be easily seen that $\{q_n(x)\} \in A$. So we get as $n \to \infty$ that

$$q_n(x) \to \bar{q}(x) \in A \quad (3.37)$$

in $L^1(0,l)$.

Moreover, from (3.35) we have

$$\int_0^l |\nabla \bar{q}|^2 dx = \lim_{n \to \infty} \int_0^l \nabla q_n \cdot \nabla \bar{q} dx \leq \lim_{n \to \infty} \left( \int_0^l |\nabla q_n|^2 dx \right)^{\frac{1}{2}} \left( \int_0^l |\nabla \bar{q}|^2 dx \right)^{\frac{1}{2}}. \quad (3.38)$$

From Lemma 3.2 and the convergence of $\{q_n\}$, we know that there exists a subsequence of $\{q_n\}$, still denoted by $\{q_n\}$, such that

$$\lim_{n \to \infty} \int_0^l |u(q_n)(x,T) - g(x)|^2 dx = \int_0^l |u(\bar{q})(x,T) - g(x)|^2 dx. \quad (3.39)$$

From (3.37), (3.38) and (3.39), we get

$$J(\bar{q}) = \lim_{n \to \infty} \int_0^l |u(q_n)(x,T) - g(x)|^2 dx + \int_0^l |\nabla \bar{q}|^2 dx \leq \lim_{n \to \infty} J(q_n) = \inf_{q \in A} J(q). \quad (3.40)$$

Hence, $J(\bar{q}) = \min_{q \in A} J(q)$.

This completes the proof of Theorem 3.3. \hfill \Box
4 Necessary Condition

Theorem 4.1. Let \( q \) be the solution of the optimal control problem (3.11). Then there exists a triple of functions \((u, v; q)\) satisfying the following system:

\[
\begin{align*}
\left\{ \begin{array}{ll}
u_t - (au_x)_x + qu = 0, & (x, t) \in Q, \\
u|_{t=0} = \phi(x), & x \in (0, l),
\end{array} \right.
\end{align*}
\]

and

\[
\begin{align*}
v_t - (av_x)_x + qv = 0, & (x, t) \in Q, \\
v|_{t=T} = u(x, T) - g(x), & x \in (0, l),
\end{align*}
\]

and

\[
\int_0^T \int_0^l uv(q - h)dxdt - N \int_0^l \nabla q \cdot \nabla (q - h)dx \geq 0
\]

for any \( h \in A \).

Proof. For any \( h \in A, \ 0 \leq \delta \leq 1, \) we have

\[
q_\delta \equiv (1 - \delta)q + \delta h \in A.
\]

Then

\[
J_\delta \equiv J(q_\delta) = \frac{1}{2} \int_0^l [u(x, T; q_\delta) - g(x)]^2dx + \frac{N}{2} \int_0^l |\nabla q_\delta|^2dx.
\]

Let \( u_\delta \) be the solution to the equation (1.1) with given \( q = q_\delta \). Since \( q \) is an optimal solution, we have

\[
\frac{dJ_\delta}{d\delta} \bigg|_{\delta=0} = \int_0^l [u(x, T; q_{\delta = 0}) - g(x)] \frac{\partial u_\delta}{\partial \delta} \bigg|_{\delta=0} dx + N \int_0^l \nabla q \cdot \nabla (h - q)dx \geq 0.
\]

Let \( \tilde{u}_\delta \equiv \frac{\partial u_\delta}{\partial \delta} \), direct calculations lead to the following equation:

\[
\begin{align*}
&\frac{\partial}{\partial \delta}(\tilde{u}_\delta) - \frac{\partial}{\partial x}(a \frac{\partial \tilde{u}_\delta}{\partial x}) + q_\delta \tilde{u}_\delta = (q - h)u_\delta, \\
&\tilde{u}_\delta|_{t=0} = 0.
\end{align*}
\]

Let \( \xi = \tilde{u}_\delta|_{\delta=0} \), then \( \xi \) satisfies

\[
\begin{align*}
\xi_t - (a\xi_x)_x + q\xi = (q - h)u, \\
\xi|_{t=0} = 0.
\end{align*}
\]

From (3.5) we have

\[
\int_0^l [u(x, T; q) - g(x)]\xi(x, T)dx + N \int_0^l \nabla q \cdot \nabla (h - q)dx \geq 0.
\]

Let \( L\xi = \xi_t - (a\xi_x)_x + q\xi \), and suppose \( v \) is the solution of the following problem:

\[
\begin{align*}
L^*v &\equiv -v_t - (av_x)_x + qv = 0, \\
v(x, T) = u(x, T; q) - g(x).
\end{align*}
\]
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where $L^*$ is the adjoint operator of the operator $L$.

By the well known Green’s formula, we have

$$
\int_0^T \int_0^l (vL\xi - \xi L^* v) \, dx \, dt = \int_0^T \int_0^l (v\xi_t + \xi v_t) \, dx \, dt + \int_0^T \int_0^l (\xi(ax)_x - v(ax)_x) \, dx \, dt \\
= \int_0^l \xi v \bigg|_{t=0} \, dx + \int_0^T \int_0^l (a\xi v_x - av\xi_x)_x \, dx \, dt \\
= \int_0^l \xi(x,T)[u(x,T) - g(x)] \, dx,
$$

which implies

$$
\int_0^T \int_0^l vL\xi \, dx \, dt = \int_0^l \xi(x,T)[u(x,T) - g(x)] \, dx. \tag{4.11}
$$

Combining (4.8) and (4.11), one can easily obtain that

$$
\int_0^T \int_0^l uv(q - h) \, dx \, dt - N \int_0^l \nabla q \cdot \nabla (q - h) \, dx \geq 0.
$$

This completes the proof of Theorem 4.1. \square

5 Uniqueness and Stability

The optimal control problem $P_1$ is non-convex. So, in general one may not expect a unique solution. In fact, it is well known that the optimization technique is a classical tool to yield "general solution" for inverse problems without unique solution. However, we find that if the terminal time $T$ is relatively small, the minimizer of the cost functional can be proved to be local unique and stable.

Throughout this paper, if no specific illustration, $C$ will be denoted the different constants.

Lemma 5.1. Supposing $u \in H^1(0,l)$, we have for any $k \geq 0$,

$$(u - k)^+ = \sup(u - k,0) \in H^1,$$

$$(u + k)^- = \sup(-(u + k),0) \in H^1.$$

Moreover, for a.e. $x \in (0,l)$,

$$
\frac{\partial(u - k)^+}{\partial x} = \begin{cases} 
\frac{\partial u}{\partial x}, & \text{if } u > k, \\
0, & \text{if } u \leq k,
\end{cases}
$$
and

\[ \frac{\partial(u + k)^-}{\partial x} \begin{cases} 0, & \text{if } u > -k, \\
-\frac{\partial u}{\partial x}, & \text{if } u \leq -k. \end{cases} \]

**Proof.** For \( u \in \mathcal{H}^1 \), we know

\[ \int_0^l a(|u|^2 + |\nabla u|^2)dx < +\infty. \]

Noting \( a(x) > 0, \ x \in (0, l) \), we have for all \( \delta > 0 \),

\[ u \in \mathcal{H}^1(\delta, l - \delta). \]

By the definition of weak derivative (see [37]), it can be easily seen that

\[ (u - k)^+ \in \mathcal{H}^1(\delta, l - \delta) \]

and for a.e. \( x \in (\delta, l - \delta) \),

\[ \frac{\partial(u - k)^+}{\partial x} \begin{cases} \frac{\partial u}{\partial x}, & \text{if } u > k, \\
0, & \text{if } u \leq k. \end{cases} \]

Then we have

\[ \int_{\delta}^{l-\delta} a \left( (u - k)^+ \right)_x^2 dx = \int_{E_\delta} a|u_x|^2 dx, \]

where \( E_\delta = \{ x \in (\delta, l - \delta) \mid u(x) > k \} \). Since the quantity \( \int_{E_\delta} a|u_x|^2 dx \) is bounded from above \( \int_0^l a|u_x|^2 dx \) which does not depend on \( \delta \), by passing to the limit as \( \delta \to 0 \), we get

\[ \int_0^l a \left( (u - k)^+ \right)_x^2 dx \leq \int_0^l a|u_x|^2 dx < +\infty. \]

Moreover, the following inequality

\[ \int_0^l a |(u - k)^+|^2 dx \leq \int_0^l a|u|^2 dx < +\infty \]

is obvious. Hence, \( (u - k)^+ \in \mathcal{H}^1 \). Similar arguments can be applied to treat the case of \( (u + k)^- \).

This completes the proof of Lemma 5.1. \( \Box \)

Now, we can give a weak maximum principle for the weak solution of Eq. (1.1).

**Lemma 5.2.** Supposing \( \phi \in L^\infty(0, l) \cap \mathcal{H}^1(0, l) \), then we have for \( u \) the following estimate:

\[ \|u\|_\infty \leq \|\phi\|_\infty. \tag{5.1} \]

**Proof.** Let \( k = \|\phi\|_\infty \). Multiplying the equation (1.1) by \((u - k)^+\), we get from Lemma 5.1

\[ \int_0^l u_t(u - k)^+ dx + \int_0^l a \left( (u - k)^+ \right)_x^2 dx = -\int_0^l qu(u - k)^+ dx. \tag{5.2} \]
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Denoting $E = \{ x \in (0, l) \mid u(x) > k \}$, one has

$$- \int_0^l q(u - k)^+ dx = - \int_E q(u - k)^+ dx \leq 0.$$  \hspace{1cm} \text{(5.3)}

From (5.2) and (5.3), we have for all $t \in [0, T]$,

$$\frac{1}{2} \frac{d}{dt} \int_0^l |(u - k)^+|^2 dx = \int_0^l u_t(u - k)^+ dx \leq 0,$$

which implies $t \mapsto \|(u - k)^+(t)\|_{L^2}^2$ is decreasing on $[0, T]$. Since $(\phi - k)^+ \equiv 0$, we deduce that for all $t \in [0, T]$ and for a.e. $x \in (0, l)$, $u(x, t) \leq k$.

By analogous arguments for $(u + k)^-$, we can obtain that for all $t \in [0, T]$ and for a.e. $x \in (0, l)$, $u(x, t) \geq -k$.

This completes the proof of Lemma 5.2. \hfill \Box

**Lemma 5.3.** For Eq. (4.2) we have the following estimate:

$$\|v\|_\infty \leq \|u(x, T) - g(x)\|_\infty.$$  \hspace{1cm} \text{(5.4)}

**Proof.** Let $\tau = T - t$, then (4.2) is reduced to

$$\begin{cases} v_\tau - (av_x)_x + qv = 0, & (x, t) \in Q, \\ v|_{\tau=0} = u(x, T) - g(x). \end{cases}$$

The rest of the proof is similar to that of Lemma 5.2. \hfill \Box

Suppose $g_1(x)$ and $g_2(x)$ be two given functions which satisfy the condition (2.2). Let $q_1(x)$ and $q_2(x)$ be the minimizers of problem $P_1$ corresponding to $g = g_i$, $(i = 1, 2)$ respectively, and \{u_i, v_i\}, $(i = 1, 2)$ be solutions of system (4.1)/(4.2) in which $q = q_i$, $(i = 1, 2)$ respectively.

Setting

$$u_1 - u_2 = U, \quad v_1 - v_2 = V, \quad q_1 - q_2 = Q,$$

then $U$ and $V$ satisfy

$$\begin{cases} U_t - (aU_x)_x + q_1 U = -Q u_2, \\ U|_{t=0} = 0, \end{cases}$$

$$\begin{cases} -V_t - (aV_x)_x + q_1 V = -Q v_2, \\ V|_{t=T} = U(x, T) - (g_1 - g_2). \end{cases}$$  \hspace{1cm} \text{(5.5)} \hspace{1cm} \text{(5.6)}

**Lemma 5.4.** For any bounded continuous function $k(x) \in C(0, l)$, we have

$$\|k\|_\infty \leq |k(x_0)| + \sqrt{1} \|\nabla k\|_{L^2(0, l)},$$

where $x_0$ is a fixed point in $(0, l)$. 
Proof. For $0 < x < l$ we have
\[
|k(x)| \leq |k(x_0)| + \left| \int_{x_0}^{x} k' \, dx \right| \\
\leq |k(x_0)| + \left( \int_{0}^{l} \left| k' \right| \, dx \right)^{\frac{1}{2}} \left( \int_{0}^{l} |\nabla k|^2 \, dx \right)^{\frac{1}{2}}.
\]
This completes the proof of the Lemma 5.4. □

Lemma 5.5. For equation (5.5) we have the following estimate:
\[
\max_{0 \leq t \leq T} \int_{0}^{l} U^2 \, dx \, dt \leq C(\max |Q|)^2 \int_{0}^{T} \int_{0}^{l} |u_2|^2 \, dx \, dt,
\]
where $C$ is independent of $T$.

Proof. From equation (5.5) we have for $0 < t \leq T$
\[
\int_{0}^{t} \int_{0}^{l} \left( \frac{U^2}{2} \right) \, dx \, dt - \int_{0}^{t} \int_{0}^{l} (aU_x)_x U \, dx \, dt + \int_{0}^{t} \int_{0}^{l} q_1 U^2 \, dx \, dt = - \int_{0}^{t} \int_{0}^{l} u_2 QU \, dx \, dt.
\]
Integrating by parts we obtain
\[
\int_{0}^{l} \frac{U^2}{2} \bigg|_{(x,t)} \, dx + \int_{0}^{t} \int_{0}^{l} aU_x^2 \, dx \, dt - \int_{0}^{t} \int_{0}^{l} aU_xU \bigg|_{x=0}^{x=l} \, dt + \int_{0}^{t} \int_{0}^{l} q_1 U^2 \, dx \, dt
\leq \int_{0}^{t} \int_{0}^{l} U^2 \, dx \, dt + (\max |Q|)^2 \int_{0}^{t} \int_{0}^{l} |u_2|^2 \, dx \, dt,
\]
which implies
\[
\int_{0}^{l} \frac{U^2}{2} \bigg|_{(x,t)} \, dx + \int_{0}^{t} \int_{0}^{l} aU_x^2 \, dx \, dt
\leq \int_{0}^{t} \int_{0}^{l} U^2 \, dx \, dt + (\max |Q|)^2 \int_{0}^{t} \int_{0}^{l} |u_2|^2 \, dx \, dt.
\]
From the Gronwall inequality and (5.10) we have
\[
\int_{0}^{l} U^2 \, dx \, dt + \int_{0}^{T} \int_{0}^{l} aU_x^2 \, dx \, dt \leq C(\max |Q|)^2 \int_{0}^{T} \int_{0}^{l} |u_2|^2 \, dx \, dt.
\]
This completes the proof of Lemma 5.5. □

Lemma 5.6. For equation (5.6) we have the following estimate:
\[
\max_{0 \leq t \leq T} \int_{0}^{l} V^2 \, dx + \int_{0}^{T} \int_{0}^{l} a|V_x|^2 \, dx \, dt
\]
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\begin{equation}
\leq C(\max |Q|)^2 \int_0^T \int_0^l (|u_2|^2 + |v_2|^2) dx dt + C \int_0^l |g_1 - g_2|^2 dx. \tag{5.11}
\end{equation}

where \(C\) is independent of \(T\).

**Proof.** From equation (5.6) we have

\begin{equation*}
\int_t^T \int_0^l - \left( \frac{V^2}{2} \right)_t dx dt - \int_t^T \int_0^l (aV_x)_x V dx dt + \int_t^T \int_0^l q_1 V^2 dx dt = - \int_t^T \int_0^l v_2 Q V dx dt.
\end{equation*}

Integrating by parts we obtain that

\begin{align*}
\int_0^l \frac{V^2}{2} &\bigg|_{(x,t)}\ dx + \int_t^T \int_0^l a|V|^2 dx dt + \int_t^T \int_0^l q_1 V^2 dx dt \\
&\leq \int_0^l |U(x,T)|^2 dx + \int_0^l |g_1 - g_2|^2 dx - \int_t^T \int_0^l v_2 Q V dx dt \\
&\leq \int_0^l |U(x,T)|^2 dx + \int_0^l |g_1 - g_2|^2 dx + \int_t^T \int_0^l \frac{V^2}{2} dx dt \\
&\quad + \frac{1}{2}(\max |Q|)^2 \int_t^T \int_0^l |v_2|^2 dx dt. \tag{5.12}
\end{align*}

From Lemma 5.5 and (5.12) we have

\begin{align*}
\int_0^l \frac{V^2}{2} &\bigg|_{(x,t)}\ dx + \int_t^T \int_0^l a|V|^2 dx dt \\
&\leq \int_t^T \int_0^l \frac{V^2}{2} dx dt + \int_0^l |g_1 - g_2|^2 dx \\
&\quad + C(\max |Q|)^2 \int_0^T \int_0^l (|u_2|^2 + |v_2|^2) dx dt. \tag{5.13}
\end{align*}

From the Gronwall inequality we have

\begin{align*}
\max_{0 \leq t \leq T} \int_0^l V^2 dx + \int_0^T \int_0^l a|V|^2 dx dt \\
&\leq C(\max |Q|)^2 \int_0^T \int_0^l (|u_2|^2 + |v_2|^2) dx dt + C \int_0^l |g_1 - g_2|^2 dx.
\end{align*}

This completes the proof of Lemma 5.6. \(\square\)

**Theorem 5.7.** Let \(q_1(x), q_2(x)\) be the minimizers of the optimal control problem \(P1\) corresponding to \(g_1(x), g_2(x)\), respectively. If there exists a point \(x_0 \in (0, l)\) such that

\(q_1(x_0) = q_2(x_0),\)
then for relatively small $T$ we have

$$\max_{x \in (0,l)} |q_1 - q_2| \leq \frac{C T^{\frac{1}{2}}}{N^{\frac{1}{2}}} \|q_1 - q_2\|_{L^2(0,l)},$$

where the constant $C$ is independent of $T$, $l$ and $N$.

**Proof.** By taking $h = q_2$ when $q = q_1$ and taking $h = q_1$ when $q = q_2$ in (4.3), we have

$$\int_0^T \int_0^l (q_1 - q_2) u_1 v_1 dx dt - N \int_0^l \nabla q_1 \cdot \nabla (q_1 - q_2) dx \geq 0, \quad (5.14)$$

$$\int_0^T \int_0^l (q_2 - q_1) u_2 v_2 dx dt - N \int_0^l \nabla q_2 \cdot \nabla (q_2 - q_1) dx \geq 0, \quad (5.15)$$

where $\{u_i, v_i\}, (i = 1, 2)$ are solutions of system (4.1)/(4.2) with $q = q_i (i = 1, 2)$, respectively.

From (5.14) and (5.15) we have

$$N \int_0^l |\nabla (q_1 - q_2)|^2 dx \leq \int_0^T \int_0^l (u_1 v_1 - u_2 v_2)(q_1 - q_2) dx dt \leq \int_0^T \int_0^l (u_1 v_1 - u_2 v_1 + u_2 v_1 - u_2 v_2)(q_1 - q_2) dx dt \leq \int_0^T \int_0^l Q v_1 U dx dt + \int_0^T \int_0^l Q u_2 V dx dt. \quad (5.16)$$

From the assumption of Theorem 5.7, there exists a point $x_0 \in (0, l)$ such that

$$Q(x_0) = q_1(x_0) - q_2(x_0) = 0. \quad (5.17)$$

From Lemma 5.4 and (5.17) we have

$$\max_{x \in (0,l)} |Q(x)| \leq \sqrt{t} \left( \int_0^l |\nabla Q|^2 dx \right)^{\frac{1}{2}}. \quad (5.18)$$

From (5.16), (5.18) and the Young inequality, we obtain that

$$\max |Q|^2 \leq l \int_0^l |\nabla Q|^2 dx \leq \frac{l}{N} \int_0^T \int_0^l Q(U v_1 + V u_2) dx dt \leq \frac{1}{2l} \int_0^l |Q|^2 dx + \frac{T l^2}{2N^2} \int_0^T \int_0^l |U v_1 + V u_2|^2 dx dt \leq \frac{1}{2} \max |Q|^2 + \frac{T l^2}{N^2} \|v_1\|_\infty^2 \int_0^T \int_0^l U^2 dx dt + \frac{T l^2}{N^2} \|u_2\|_\infty^2 \int_0^T \int_0^l V^2 dx dt.$$
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\[
\leq \frac{1}{2} \max |Q|^2 + C \frac{T^2 l^2}{N^2} \|v_1\|_\infty^2 \left( \int_0^T \int_0^l |u_2|^2 dxdt \right) \cdot \max |Q|^2 \\
+ C \frac{T^2 l^2}{N^2} \|u_2\|_\infty^2 \cdot \left( \int_0^T \int_0^l (|u_2|^2 + |v_2|^2) dxdt \right) \cdot \max |Q|^2 \\
+ C \frac{T^2 l^2}{N^2} \int_0^l |g_1 - g_2|^2 dx, 
\]

(5.19)

where we have used estimates (5.7) and (5.11).

From Lemmas 5.2 and 5.3 we have

\[
\|v_1\|_\infty, \|v_2\|_\infty, \|u_2\|_\infty \leq C. 
\]

(5.20)

From (5.19) and (5.20) we have

\[
\max |Q|^2 \leq C \frac{T^2 l^2}{N^2} \max |Q|^2 + C \frac{T^2 l^2}{N^2} \int_0^l |g_1 - g_2|^2 dx. 
\]

(5.21)

Choose \( T << 1 \) such that

\[
C \frac{T^3 l^2}{N^2} = \frac{1}{2}. 
\]

(5.22)

Combining (5.21) and (5.22) one can easily get

\[
\max_{x \in (0,l)} |q_1 - q_2| \leq C l^\frac{1}{2} \|g_1 - g_2\|_{L^2(0,l)}. 
\]

(5.23)

This completes the proof of the Theorem 5.7.

\[ \square \]

**Remark 5.1.** It should be mentioned that the regularization parameter plays a major role in the numerical simulation of ill-posed problems. From Theorem 5.7 we can obtain that if there exists a constant \( \delta \) such that

\[
\|g_1 - g_2\| \leq \delta, \quad \text{and} \quad \frac{\delta^2}{N^\frac{3}{2}} \to 0, 
\]

then the reconstructed optimal solution is unique and stable, which is consistent with the existed results (see, for instance, [19]). Note that the estimate (5.23) is based on (5.22) from which we can see \( T = O(N^\frac{3}{2}) \). Since the parameter \( N \) is often taken to be very small, particularly in numerical computations, Theorem 5.7 is indeed the local well-posedness of the optimal solution. For more detailed discussion on the regularization parameter, we refer the readers to the references, e.g., in [9, 19].

**6 Convergence Analysis**

In this section, we would like to discuss the convergence of the optimal solution. It has been shown in previous section that the optimal solution is stable and unique, which is very important
in numerical process. However, the optimization problem is just a "modified problem" rather than the original one. Therefore, it is necessary to investigate what about the difference between the optimal solution of the optimization problem and the exact solution of the original problem.

We assume that the "real solution" \( g(x) \) is attainable, i.e., there exists a \( q^*(x) \in H^1(0,l) \) such that

\[
u(x,T;q^*) = g(x), \tag{6.1}
\]

and that an upper bound \( \delta \) for the noisy level

\[
\|g^\delta - g\|_{L^2(0,l)} \leq \delta, \tag{6.2}
\]

of the observation is known \textit{a priori}.

It should be mentioned that for terminal control problems, it is rather difficult to derive the convergence. To the authors’ knowledge, there is no any convergence results for the optimal control problem with the cost functional whose form is similar to (3.12).

In this paper, we introduce the following auxiliary control problems with observations averaged over the given terminal time interval \([T - \sigma, T] \):

\[
J_\sigma(q) = \frac{1}{2\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} |u(x,t;q) - g(x)|^2 \, dx \, dt + \frac{N}{2} \int_{0}^{l} |\nabla q|^2 \, dx. \tag{6.3}
\]

Note that as \( \sigma \to 0^+ \),

\[
\frac{1}{2\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} |u(x,t;q) - g(x)|^2 \, dx \, dt \to \int_{0}^{l} \frac{1}{2} |u(x,T;q) - g(x)|^2 \, dx,
\]

which implies \( J_\sigma(q) \to J(q) \). Analogously, instead of (6.2), we assume that for the real solution \( q^*(x) \),

\[
\frac{1}{2\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} |u(x,t;q^*) - g^\delta(x)|^2 \, dx \, dt \leq \frac{1}{2} \delta^2. \tag{6.4}
\]

Define the following forward operator \( u(q) \):

\[
u(q) : \quad \mathcal{A} \to H^1((0,T);L^2(0,l)) \cap L^2((0,T);H^1(0,l))
\]

\[
\nu(q)(x,t) = u(x,t;q(x)),
\]

where \( u(x,t;q(x)) \) is the solution of the variational problem (3.5) for \( q \in \mathcal{A} \). For any \( q \in \mathcal{A} \) and \( p \in H^1(0,l) \), one can easily deduce that the Gâteaux directional differential \( u'(q)p \) satisfies a homogeneous initial condition and solves

\[
\int_{0}^{l} (u'(q)p)_t \varphi \, dx + \int_{0}^{l} a \nabla (u'(q)p) \cdot \nabla \varphi \, dx + \int_{0}^{l} qu'(q)p \varphi \, dx = -\int_{0}^{l} pu(q) \varphi \, dx, \tag{6.5}
\]

for any \( \varphi \in L^2(0,l) \cap H^1(0,l) \). For the remainder term \( R(q) = u(p + q) - u(q) - u'(q)p \), we have the following variational characterization.
Lemma 6.1. For any \( q \in \mathcal{A} \) and \( p \in H^1(0, l) \) such that \( p + q \in \mathcal{A} \), the remainder \( R(q) = u(p + q) - u(q) - u'(q)p \) solves
\[
\int_0^l (R(q))_t \varphi dx + \int_0^l a \nabla (R(q)) \cdot \nabla \varphi dx + \int_0^l q R(q) \varphi dx = \int_0^l p \varphi (u(q) - u(q + p)) dx,
\]
for any \( \varphi \in L^2(0, l) \cap H^1(0, l) \).

**Proof.** Note that \( u(p + q) \) satisfies
\[
\int_0^l (u(p + q))_t \varphi dx + \int_0^l a \nabla (u(p + q)) \cdot \nabla \varphi dx + \int_0^l (p + q) u(p + q) \varphi dx = 0.
\]
Subtracting (6.7) from (3.8) and denoting \( W = u(p + q) - u(q) \), we obtain
\[
\int_0^l \varphi W_t dx + \int_0^l a \nabla W \cdot \nabla \varphi dx + \int_0^l q W \varphi dx = - \int_0^l pu(p + q) \varphi dx.
\]
Now (6.6) follows by subtracting (6.5) from (6.8).

This completes the proof of Lemma 6.1. \( \square \)

To obtain the convergence, we shall require some source conditions. We introduce the following linear operator \( F(q) \):
\[
F(q) : L^2((0, T); L^2(0, l)) \to L^2(0, l)
\]
\[
F(q)\Phi = - \frac{1}{\sigma} \int_{T - \sigma}^T u(q)\Phi dt, \quad \forall \Phi \in L^2((0, T); L^2(0, l)),
\]
where \( u(q) \) is the solution of (3.8). Using the equation (6.5), we immediately see that for any \( p \in H^1(0, l) \) and any \( \varphi \in L^2(0, l) \cap H^1(0, l) \), the following holds:
\[
< F(q)\varphi, p > = - \frac{1}{\sigma} \int_{T - \sigma}^T pu(q)\varphi dt
\]
\[
= \frac{1}{\sigma} \int_{T - \sigma}^T \left[ (u'(q)p)_t \varphi + a \nabla (u'(q)p) \cdot \nabla \varphi + qu'(q)p\varphi \right] dx dt,
\]
where \( < \cdot, \cdot > \) denote the scalar product in \( L^2(0, l) \). Since \( \nabla \) is a linear operator, we can define its adjoint operator \( \nabla^* \) by
\[
< \nabla^* \omega, \varphi >_{L^2(0, l)} = < \omega, \nabla \varphi >_{L^2(0, l)}, \quad \forall \omega \in H^1(0, l), \ \varphi \in H^1(0, l).
\]
It can be easily seen that if \( \varphi \in H^1_0(0, l) \), then \( \nabla^* \) is equivalent to \( \nabla \). In this paper, we will only need a weak form of \( \nabla^* \nabla \).

**Theorem 6.2.** Assume that there exists a function
\[
\varphi \in H^1_0((T - \sigma, T); L^2(0, l)) \cap L^2((T - \sigma, T); H^1(0, l))
\]
such that the following source condition holds in the weak sense:

$$F(q^*) \varphi = \nabla^* \nabla q^*$$  \hfill (6.12)

with $F(q^*)$ defined by (6.9), i.e., for any $p \in H^1(0, l)$,

$$<F(q^*) \varphi, p> = <\nabla^* \nabla q^*, p> = <\nabla q^*, \nabla p>.$$  \hfill (6.13)

Furthermore, assume that

$$\nabla \cdot (a \nabla \varphi) \in L^2((T - \sigma, T); L^2(0, l))$$  \hfill (6.14)

and $q_N^\delta$ satisfies

$$q_N^\delta(0) = q^*(0), \quad q_N^\delta(l) = q^*(l).$$  \hfill (6.15)

Then, with $N \sim \delta$, we have

$$\int_0^l |q_N^\delta - q^*|^2 \, dx \leq C\delta,$$  \hfill (6.16)

and

$$\frac{1}{\sigma} \int_{T-\sigma}^T \int_0^l |u(q_N^\delta) - u(q^*)|^2 \, dx dt \leq C\delta^2,$$  \hfill (6.17)

where $q_N^\delta$ is a minimizer of (6.3) with $g$ replaced by $g^\delta$, $u(q_N^\delta)$ is the solution of the variational problem (3.8) with $q = q_N^\delta$, and $C$ is a positive constant independent of $\delta$, $N$ and $T$.

**Proof.** Noting that $q_N^\delta$ is a minimizer of (6.3), we have

$$J_\sigma(q_N^\delta) \leq J_\sigma(q^*),$$

which implies

$$\frac{1}{2\sigma} \int_{T-\sigma}^T \int_0^l |u(q_N^\delta) - g^\delta|^2 \, dx dt + \frac{N}{2} \int_0^l |\nabla q_N^\delta|^2 \, dx \leq \frac{1}{2} \delta^2 + \frac{N}{2} \int_0^l |\nabla q^*|^2 \, dx.$$  \hfill (6.18)

From (6.18), we can derive

$$\frac{1}{2\sigma} \int_{T-\sigma}^T \int_0^l |u(q_N^\delta) - g^\delta|^2 \, dx dt + \frac{N}{2} \int_0^l |\nabla q_N^\delta - \nabla q^*|^2 \, dx \leq \frac{1}{2} \delta^2 + \frac{N}{2} \int_0^l |\nabla q^*|^2 \, dx$$

$$\leq \frac{1}{2} \delta^2 + \frac{N}{2} \int_0^l |\nabla q^*|^2 \, dx - \frac{N}{2} \int_0^l |\nabla q_N^\delta|^2 \, dx + \frac{N}{2} \int_0^l |\nabla q_N^\delta - \nabla q^*|^2 \, dx$$

$$= \frac{1}{2} \delta^2 + \frac{N}{2} \int_0^l |\nabla q^*|^2 \, dx - \frac{N}{2} \int_0^l |\nabla q_N^\delta|^2 \, dx + \frac{N}{2} \int_0^l |\nabla q_N^\delta - \nabla q^*|^2 \, dx$$

$$= \frac{1}{2} \delta^2 + \frac{N}{2} \int_0^l (\nabla q^* \cdot (q^* - q_N^\delta)) \, dx$$

$$= \frac{1}{2} \delta^2 + N \left< \nabla q^*, \nabla (q^* - q_N^\delta) \right>.$$  \hfill (6.19)

Using (6.10) and (6.13), we have for the last term in (6.19) that

$$\left< \nabla q^*, \nabla (q^* - q_N^\delta) \right>.$$
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\begin{align*}
&= \left< F(q^*) \varphi, \, q^* - q_N^\delta \right>
\leq \frac{1}{\sigma} \int_{T-\sigma}^T \int_0^l (q^* - q_N^\delta) u(q^*) \varphi \, dx \, dt
\leq \frac{1}{\sigma} \int_{T-\sigma}^T \int_0^l \left[ \left( u'(q^*) (q^* - q_N^\delta) \right)_t + a \nabla \left( u'(q^*) (q^* - q_N^\delta) \right) \cdot \nabla \varphi \right. \\
&\quad \left. + q^* u'(q^*) (q^* - q_N^\delta) \varphi \right] \, dx \, dt
\end{align*}

Using this notation, we obtain

\begin{align*}
N \left< \nabla q^*, \, \nabla \left( q^* - q_N^\delta \right) \right>
&= \frac{N}{\sigma} \int_{T-\sigma}^T \int_0^l \left[ \left( R_N^\delta \right)_t + a \nabla R_N^\delta \cdot \nabla \varphi + q^* R_N^\delta \varphi \right] \, dx \, dt \\
&\quad - \frac{N}{\sigma} \int_{T-\sigma}^T \int_0^l \left[ u(q_N^\delta) - u(q^*) \right] \varphi \, dx \, dt - \frac{N}{\sigma} \int_{T-\sigma}^T \int_0^l a \nabla \left( u(q_N^\delta) - u(q^*) \right) \cdot \nabla \varphi \, dx \, dt \\
&\quad - \frac{N}{\sigma} \int_{T-\sigma}^T \int_0^l q^* \left[ u(q_N^\delta) - u(q^*) \right] \varphi \, dx \, dt \\
&= I_1 + I_2 + I_3 + I_4.
\end{align*}

Now, we need to estimate \( I_1 - I_4 \). The main idea is to control \( I_1 - I_4 \) by the left-side item of inequality (6.19).

For \( I_1 \), we use (6.6) to get

\[ I_1 = \frac{N}{\sigma} \int_{T-\sigma}^T \int_0^l \left( q_N^\delta - q^* \right) \left[ u(q^*) - u(q_N^\delta) \right] \varphi \, dx \, dt. \] (6.23)

From (6.23) and the Hölder inequality, we have

\begin{align*}
|I_1| &\leq \frac{N}{\sigma} \int_{T-\sigma}^T \int_0^l \left| \left( q_N^\delta - q^* \right) \varphi \right| \cdot \left| u(q^*) - g^\delta \right| \, dx \, dt \\
&\quad + \frac{N}{\sigma} \int_{T-\sigma}^T \int_0^l \left| \left( q_N^\delta - q^* \right) \varphi \right| \cdot \left| g^\delta - u(q_N^\delta) \right| \, dx \, dt \\
&\leq \frac{N}{\sigma} \int_{T-\sigma}^T \left\| \left( q_N^\delta - q^* \right) \varphi \right\|_{L^2(0,l)} \cdot \left\| u(q^*) - g^\delta \right\|_{L^2(0,l)} \, dt \\
&\quad + \frac{N}{\sigma} \int_{T-\sigma}^T \left\| \left( q_N^\delta - q^* \right) \varphi \right\|_{L^2(0,l)} \cdot \left\| g^\delta - u(q_N^\delta) \right\|_{L^2(0,l)} \, dt.
\end{align*} (6.24)

Using (6.13) and the Young inequality, we obtain

\[ |I_1| \leq \frac{1}{8\sigma} \int_{T-\sigma}^T \int_0^l \left| u(q^*) - g^\delta \right|^2 \, dx \, dt + CN^2 \int_{T-\sigma}^T \int_0^l \left( q_N^\delta - q^* \right) \varphi \, dx \, dt \]
\begin{align*}
&+ \frac{1}{16\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| g^\delta - u(q_N^\delta) \right|^2 \, dx \, dt + CN^2 \int_{T-\sigma}^{T} \int_{0}^{l} \left| (q_N^\delta - q^*) \varphi \right|^2 \, dx \, dt \\
&\leq \frac{1}{8} \delta^2 + \frac{1}{16\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| g^\delta - u(q_N^\delta) \right|^2 \, dx \, dt + CN^2 \int_{T-\sigma}^{T} \int_{0}^{l} \left| \varphi \right|^2 \, dx \, dt,
\end{align*}

where we have used the assumption (6.14).

For $I_2$, using integration by parts with respect to $t$ and noticing $\varphi \in H_0^1((T-\sigma,T); L^2(0,l))$, we derive

\begin{align*}
|I_2| &= \frac{N}{\sigma} \left| \int_{T-\sigma}^{T} \int_{0}^{l} \left( u(q_N^\delta) - u(q^*) \right) \varphi \, dx \, dt \right| \\
&\leq \frac{N}{\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| \left( u(q_N^\delta) - u(q^*) \right) \varphi \right| \, dx \, dt \\
&\leq \frac{N}{\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| \left( u(q_N^\delta) - g^\delta \right) \varphi \right| \, dx \, dt + \frac{N}{\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| \left( g^\delta - u(q^*) \right) \varphi \right| \, dx \, dt \\
&\leq \frac{1}{8} \delta^2 + \frac{1}{16\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| g^\delta - u(q_N^\delta) \right|^2 \, dx \, dt + CN^2 \int_{T-\sigma}^{T} \int_{0}^{l} \left| \varphi \right|^2 \, dx \, dt,
\end{align*}

For $I_3$, using integration by parts with respect to $x$ and noticing $a(0) = a(l) = 0$, we obtain

\begin{align*}
|I_3| &= \frac{N}{\sigma} \left| \int_{T-\sigma}^{T} \int_{0}^{l} a \nabla \left( u(q_N^\delta) - u(q^*) \right) \cdot \nabla \varphi \, dx \, dt \right| \\
&= \frac{N}{\sigma} \left| \int_{T-\sigma}^{T} \left\{ a(x) \left( u(q_N^\delta) - u(q^*) \right) \frac{d\varphi}{dx} \right|_{x=0}^{x=l} \\
&\quad - \int_{0}^{l} \left( u(q_N^\delta) - u(q^*) \right) \nabla \cdot (a \nabla \varphi) \, dx \right| \, dt \right| \\
&\leq \frac{N}{\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| u(q_N^\delta) - u(q^*) \right| \cdot \left| \nabla \cdot (a \nabla \varphi) \right| \, dx \, dt \\
&\leq \frac{N}{\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| u(q_N^\delta) - g^\delta \right| \cdot \left| \nabla \cdot (a \nabla \varphi) \right| \, dx \, dt \\
&\quad + \frac{N}{\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| g^\delta - u(q^*) \right| \cdot \left| \nabla \cdot (a \nabla \varphi) \right| \, dx \, dt \\
&\leq \frac{1}{8} \delta^2 + \frac{1}{16\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| u(q_N^\delta) - g^\delta \right|^2 \, dx \, dt \\
&\quad + CN^2 \int_{T-\sigma}^{T} \int_{0}^{l} \left| \nabla \cdot (a \nabla \varphi) \right|^2 \, dx \, dt,
\end{align*}

The last term $I_4$ can be estimated similarly using the Young inequality:

\begin{align*}
|I_4| &\leq \frac{N}{\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| q^* \right| \left| u(q_N^\delta) - g^\delta \right| \left| \varphi \right| \, dx \, dt \\
&\quad + \frac{N}{\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| q^* \right| \left| g^\delta - u(q^*) \right| \left| \varphi \right| \, dx \, dt.
\end{align*}
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\[ \leq \frac{1}{8} \delta^2 + \frac{1}{16\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| u(q_N^\delta) - g^\delta \right|^2 \, dx \, dt + CN^2 \int_{T-\sigma}^{T} \int_{0}^{l} |\varphi|^2 \, dx \, dt, \quad (6.28) \]

where we have used the bound of \( q^* \).

Combining (6.19), (6.22) and (6.25)-(6.28), we obtain

\[ \begin{aligned}
& \frac{1}{2\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| u(q_N^\delta) - g^\delta \right|^2 \, dx \, dt + \frac{N}{2} \int_{0}^{l} \left| \nabla q_N^\delta - \nabla q^* \right|^2 \, dx \\
\leq \quad \frac{1}{2} \delta^2 + N \left\langle \nabla q^*, \nabla \left( q^* - q_N^\delta \right) \right\rangle \\
\leq \quad \frac{1}{2} \delta^2 + \sum_{j=1}^{4} |I_j| \\
\leq \quad \delta^2 + \frac{1}{4\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| u(q_N^\delta) - g^\delta \right|^2 \, dx \, dt \\
+ CN^2 \int_{T-\sigma}^{T} \int_{0}^{l} \left( |\varphi|^2 + |\varphi_t|^2 + |\nabla \cdot (a \nabla \varphi)|^2 \right) \, dx \, dt.
\end{aligned} \quad (6.29) \]

From (6.29) and noticing the regularity of \( \varphi \), we have

\[ \frac{1}{4\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| u(q_N^\delta) - g^\delta \right|^2 \, dx \, dt + \frac{N}{2} \int_{0}^{l} \left| \nabla q_N^\delta - \nabla q^* \right|^2 \, dx \leq \delta^2 + CN^2. \quad (6.30) \]

By choosing \( N \sim \delta \), one can easily get

\[ \frac{1}{\sigma} \int_{T-\sigma}^{T} \int_{0}^{l} \left| u(q_N^\delta) - g^\delta \right|^2 \, dx \, dt + N \int_{0}^{l} \left| \nabla q_N^\delta - \nabla q^* \right|^2 \, dx \leq C\delta^2. \quad (6.31) \]

The estimate (5.16) follows immediately from (5.31) and the Poincarè inequality.

This completes the proof of Theorem 6.2. \( \square \)

**Remark 6.1.** The motivation of replacing the cost functional (3.12) by (6.3) mainly lies in the difficulty in treating the second integration term in (6.22). In fact, if we choose the functional form (5.12), then we can deduce the second term in (6.22) (denoted by \( I_2 \)) to be \( I_2 = \frac{N}{\sigma} \int_{0}^{l} \left( u(q_N^\delta) - u(q^*) \right) \cdot (\cdot, T) \varphi \, dx \).

Since we have no any information regarding to the \( t \)-derivative of the real and approximate solution, it is quite difficult, even impossible, to control the term \( I_2 \) by the left-hand side of (6.19), and thus we cannot obtain any convergence.

## 7 Concluding Remarks

The inverse problem of identifying the coefficient in parabolic equations from some extra conditions is very important in some engineering texts and many industrial applications. Classical parabolic models are plentifully discussed and developed well, while documents dealt with degenerate parabolic models are quite few.
In this paper, we solve the inverse problem $P$ of recovering the radiative coefficient $q(x)$ in the following degenerate parabolic equation

$$u_t - (au_x)_x + q(x)u = 0$$

in an optimal control framework. Being different from other works (for example [23, 28]) which also treat with inverse radiative coefficient problems, the mathematical model discussed in the paper contains degeneracy on the lateral boundaries. Furthermore, unlike the well known Black-Scholes equation whose degeneracy can be removed by some change of variable, the degeneracy in our problem can not be removed by any method. On the basis of the optimal control framework, the existence, uniqueness, stability and convergence of the minimizer for the cost functional are established.

The paper focuses on the theoretical analysis of the 1-D inverse problem. For the multidimensional case, i.e., the determination of $q(x)$ in the following equation

$$u_t - \nabla \cdot (a(x)\nabla u) + q(x)u = 0, \quad (x,t) \in Q = \Omega \times (0,T],$$

where the principle coefficient $a(x)$ satisfies

$$a(x) \geq 0, \quad x \in \bar{\Omega}$$

and $\Omega \subset \mathbb{R}^m (m \geq 1)$ is a given bounded domain, the method proposed in the paper is also applicable.

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