Regularity for weak solutions to nondiagonal quasilinear degenerate elliptic systems

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Abstract. The aim of this paper is to establish regularity for weak solutions to the nondiagonal quasilinear degenerate elliptic systems related to Hörmander’s vector fields, where the coefficients are bounded with vanishing mean oscillation. We first prove $L^p(p \geq 2)$ estimates for gradients of weak solutions by using a priori estimates and a known reverse Hölder inequality, and consider regularity to the corresponding nondiagonal homogeneous degenerate elliptic systems. Then we get higher Morrey and Campanato estimates for gradients of weak solutions to original systems and Hölder estimates for weak solutions.

Key words: nondiagonal quasilinear degenerate elliptic system, Hörmander’s vector fields, $L^p$ estimate, Morrey estimate, Campanato estimate, Hölder estimate

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1 Introduction

Regularity for solutions to elliptic systems in Euclidean spaces has been studied by many authors and a lot of important conclusions were got. Campanato in [2] obtained gradient estimates for weak solutions to linear elliptic system with discontinuous coefficients. For related articles, we quote [1, 14] and the references therein.

Huang in [18] derived Morrey estimates for uniformly elliptic systems by applying Campanato’s technique. Zheng and Feng in [28] established Hölder estimates for weak solutions to quasilinear elliptic systems by reverse Hölder inequality and Dirichlet growth theorem, where the coefficients belong to $L^\infty \cap VMO$, and low terms satisfy controlled

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growth conditions. For the following second order quasilinear elliptic systems

$$-D_\alpha a_\alpha^i(x, u, Du) = a_i(x, u, Du),$$

where $a_\alpha^i(x, u, Du) = A_{ij}^{\alpha\beta}(x)D_\beta u^j + g_\alpha^i(x, u, Du)$, $A_{ij}^{\alpha\beta} \in C^{0,\alpha}$, Daněček in [4] proved Morrey and Campanato estimates with $p = 2$ for weak solutions. When the coefficients $A_{ij}^{\alpha\beta}$ belongs to $L^\infty(\Omega) \cap L^\phi(\Omega)$ (where $\phi = \frac{1}{\ln|\ln r|}$), $A_{ij}^{\alpha\beta}$ belongs to $VMO(\Omega) \cap L^\infty(\Omega)$, or $A_{ij}^{\alpha\beta}$ is bounded and belongs to Campanato spaces, Daněček and Viszus in [5, 6, 7] gave similar estimates. Chiarenza, Franciosi and Frasca ([3]) obtained $L^p$ estimate for weak solutions to divergence linear elliptic systems by representation formula.

To nondiagonal elliptic systems, Kawohl in [19] investigated Hölder continuity for bounded weak solutions to quasilinear elliptic systems if the Liouville type property for these systems is satisfied. Wiegner in [25] gained Hölder regularity for weak solutions to nondiagonal systems with low terms satisfying natural conditions. More related results also see [10, 15, 21, 22, 24, 30] and the references therein.

Regularity of degenerate elliptic systems formed by Hörmander’s vector fields ([17]) has received wide attention in recent years. Di Fazio and Fanciullo in [8] proved Morrey estimates ($p = 2$) for weak solutions to linear degenerate elliptic systems. Dong and Niu in [9] showed Morrey estimates ($p \geq 2$) for linear degenerate elliptic systems. For nonlinear systems, Xu and Zuily in [26] handled interior regularity of weak solutions to quasilinear degenerate elliptic systems with the low term satisfying the natural condition. Gao, Niu and Wang in [11] settled partial Hölder regularity for weak solutions to degenerate quasilinear elliptic systems with the coefficients belonging to $VMO \cap L^\infty$ and the low term satisfying the natural condition. We mention that those systems in [8, 9, 11, 26] are all diagonal.

To our knowledge, there is no any regularity result to nondiagonal degenerate elliptic systems. Whether do they have regularity? What is the kind of regularity if they have? These are what we will answer in this paper. Concretely, we consider the following nondiagonal quasilinear degenerate elliptic system

$$-X^*_\alpha(a_\alpha^{ij}(x, u)X_\beta u^j) = g_\alpha(x, u, Xu) - X^*_\alpha f_\alpha(x, u, Xu),$$  \hspace{1cm} (1.1)

where $X_\alpha = \sum_{k=1}^n b_{\alpha k}(x) \frac{\partial}{\partial x_k}$ ($b_{\alpha k}(x) \in C^\infty(\Omega)$) are real smooth vector fields in a neighborhood $\tilde{\Omega}$ of some bounded domain $\Omega \subset \mathbb{R}^n$ ($q \leq n$) and satisfy Hörmander’s condition of step $s$ (see Section 2), $\alpha, \beta = 1, 2, \ldots, q; i, j = 1, 2, \ldots, N; X^*_\alpha = -X_\alpha + c_\alpha(c_\alpha = -\sum_{k=1}^n \frac{\partial b_{\alpha k}}{\partial x_k} \in C^\infty(\Omega)$) is the transposed vector field of $X_\alpha$. 

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The aim is to establish higher integrability of gradients for weak solutions to (1.1), higher Morrey estimates, Hölder estimates and higher Campanato estimate.

Before stating our main results, we need several assumptions of (1.1) (the detailed description for notions of Sobolev space $W^{k,p}_X$, Morrey space $L^{p,\lambda}_X$, Campanato space $L^{p,\lambda}_X$, Hölder space $C^{0,\alpha}_X$, $BMO_X$ and $VMO_X$ see Section 2):

(H1) Let $a_{ij}^\alpha(x, u) = A^{\alpha\beta}(x)\delta_{ij} + B_{ij}^\alpha(x, u)$, where $A^{\alpha\beta}(x) \in VMO_X \cap L^\infty$, $A^{\alpha\beta}(x) = A^{\beta\alpha}(x), A^{\alpha\beta}(x)$ satisfy the ellipticity condition and $B_{ij}^\alpha(x, u)$ are bounded and measurable, that is, there exist positive constants $\lambda_0, \mu_0, \delta, 0 < \lambda_0 \leq \mu_0, 0 < \delta < 1$, such that for $a.e.$ $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$, $\lim_{R \to 0} \eta_R(A^{\alpha\beta}(x)) = 0$,

$$\lambda_0 |\xi|^2 \leq A^{\alpha\beta}(x)\xi_\alpha \xi_\beta \leq \mu_0 |\xi|^2,$$

$$|B_{ij}^\alpha(x, u)| \leq \delta \lambda_0.$$  

(H2) Let $u \in W^{1,2}_X(\Omega, \mathbb{R}^N), g_i(x, u, z)$ and $f_i^\alpha(x, u, z)$ satisfy

$$|g_i(x, u, z)| \leq g^i(x) + L|z|^{\gamma_0},$$

$$|f_i^\alpha(x, u, z)| \leq g_i^\alpha(x) + L|z|,$$

$$f_i^\alpha(x, u, z)z_\alpha \geq \gamma_1 |z|^2 - (g(x))^2,$$

where $L$ and $\gamma_1$ are positive constants, $z \in \mathbb{R}^n$, $g^i \in L^{p_0,\lambda_0}_X(\Omega), 1 \leq \gamma_0 < \frac{Q+2}{Q}$, $g_i^\alpha$ and $g \in L^{p,\lambda}_X(\Omega), p \geq 2, 0 < \lambda < Q, q_0 = \frac{Q}{Q-2}, Q$ is the local homogeneous dimension relative to $\Omega$ (see Section 2). Afterwards, briefly denote $\tilde{g} = (g^i), \tilde{g} = (g_i^\alpha)$.

If $u \in W^{1,2}_X(\Omega, \mathbb{R}^N)$ and for any $\varphi \in C^{\infty}_0(\Omega, \mathbb{R}^N)$,

$$- \int_\Omega a_{ij}^\alpha(x, u)X_\beta u\varphi_j x_\alpha \varphi^j dx = \int_\Omega (g_i(x, u, Xu)\varphi^i - f_i^\alpha(x, u, Xu)X_\alpha \varphi^i) dx,$$

we say that $u$ is a weak solution to (1.1).

The main results of this paper are the following.

**Theorem 1.1** (higher integrability of gradients for weak solutions) Let $u \in W^{1,2}_X(\Omega, \mathbb{R}^N)$ be a weak solution to (1.1), the coefficients $a_{ij}^\alpha$ satisfy (H1), $g_i$ and $f_i^\alpha$ satisfy (H2). Then there exists a positive constant $\varepsilon_0 > 0$ such that for any $p \in [2, 2 + \frac{2Q}{Q+2}\varepsilon_0], \Omega' \subset \subset \Omega,$

$$\|Xu\|_{L^p(\Omega')} \leq c \left(\|g\|_{L_X^{p,\lambda}(\Omega)} + \|\tilde{g}\|_{L_X^{p,\lambda}(\Omega)} + \|\tilde{g}\|_{L_X^{p_0,\lambda_0}(\Omega)}\right).$$

**Theorem 1.2** (higher Morrey estimates of gradients for weak solutions) Under the assumptions in Theorem 1.1, we have that for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right], X u \in L^{p,\lambda}_{X, loc}(\Omega, \mathbb{R}^N).$
Theorem 1.3 (Hölder estimate for weak solutions) Under the assumptions in Theorem 1.1, it follows that for any \( p \in \left[ 2, 2 + \frac{2Q}{Q+2} \varepsilon_0 \right) \), \( Q - p < \lambda < Q \), one has

\[ u \in C^{0,\kappa}_{X,loc}(\Omega, \mathbb{R}^N), \quad \kappa = 1 - \frac{Q - \lambda}{p}. \]

Furthermore, we make the following assumption:

(H3) Let \( u \in W^{1,2}_X(\Omega, \mathbb{R}^N) \), \( g_i(x, u, z) \) and \( f_i^\alpha(x, u, z) \) satisfy

\[ |g_i(x, u, z)| \leq g_i^i(x) + L |z|^{\gamma_0}, \]
\[ |f_i^\alpha(x, u, z_1) - f_i^\alpha(y, v, z_2)| \leq L \left( |g_i^\alpha(x) - g_i^\alpha(y)| + |z_1 - z_2| \right), \]
\[ f_i^\alpha(x, u, z) z_i^\alpha \geq \gamma_1 |z|^2 - (g(x))^2, \]

where \( x, y \in \Omega, u, v \in \mathbb{R}^N, z_1, z_2 \in \mathbb{R}^{qN} \), the selections of \( \gamma_0, L, \gamma_1, g_i, g_i^\alpha \) and \( g \), are the same as (H2).

Theorem 1.4 (Campanato estimates of gradients for weak solutions) Let \( u \in W^{1,2}_X(\Omega, \mathbb{R}^N) \) be a weak solution to (1.1), the coefficients \( a^\alpha_{ij} \) satisfy (H1), \( g_i \) and \( f_i^\alpha \) satisfy (H3).

Then for any \( p \in \left[ 2, 2 + \frac{2Q}{Q+2} \varepsilon_0 \right) \), we have

\[ Xu \in L^{p,\lambda}_{X,loc}(\Omega, \mathbb{R}^N). \]

The proof of Theorem 1.1 is based on a priori estimates for weak solutions to (1.1) and the reverse Hölder inequality in [13, 27]. In proving Theorem 1.2, several different ways are attempted and an effective route is the decomposition of (1.1) into a nondiagonal homogeneous system and a nondiagonal nonhomogeneous system. To treat two systems, we discuss regularity to the homogeneous system corresponding to (1.1):

\[ -X^\ast_\alpha(a^\alpha_{ij}(x, u)X^j_\beta u^\beta) = 0. \quad (1.2) \]

With the help of analysis to (1.2), we can confirm Theorem 1.2 and Theorem 1.4 under (H2) and (H3), respectively.

Authors in [19, 25] obtained Hölder regularity for weak solutions to elliptic systems by employing Liouville theorem. Differently from this, we prove Theorem 1.3 by combining Morrey estimates given in Theorem 1.2 and a Morrey lemma in [29].

This paper is organized as follows. In section 2, we introduce Hörmander’s vector fields, the Carnot-Carathéodory distance and some related function spaces, and then recall corresponding Sobolev-Poincaré inequality and Morrey Lemma. In section 3, we prove
Theorem 1.1 by choosing appropriate text functions and then using a priori estimates argument for weak solutions of (1.1) and the reverse Hölder inequality. Section 4 is devoted to the study of nondiagonal homogeneous degenerate elliptic system (1.2). Through dividing (1.2) into two systems which are a constant coefficients diagonal homogeneous system and a constant coefficients diagonal nonhomogeneous system, we establish relations between $L^p$ estimates over balls for gradients of weak solutions to (1.2), see Theorems 4.1 and 4.2. In section 5, we first divide (1.1) into a nondiagonal homogeneous system (5.1) and a nondiagonal nonhomogeneous system (5.2), and then prove Theorem 1.2 by applying conclusions in sections 3 and 4, and the iteration lemma. The proof of Theorem 1.3 is given by using Theorem 1.2 and the known Morrey lemma. After deducing a priori estimates for weak solutions to (5.2), we finally complete the proof of Theorem 1.4.

2 Preliminaries

For every multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_d)(1 \leq \beta_i \leq q, i = 1, \ldots, d, |\beta| = d)$, we call that $d$ is the length of the commutator $X_\beta = [X_{\beta_d}, [X_{\beta_{d-1}}, \ldots [X_{\beta_2}, \ldots X_{\beta_1}]]]$. 

Definition 2.1 Let $X_1, \ldots, X_q$ be smooth vector fields. If $\{X_\beta(x_0)\}_{|\beta| \leq s}$ spans $\mathbb{R}^n$ at every $x_0 \in \Omega \subset \mathbb{R}^n$, then we say that the system $X = (X_1, \ldots, X_q)$ satisfies Hörmander’s condition of step $s$.

By [26], we can assume that Hörmander’s vector fields $X_1, \ldots, X_q$ are free up to the order $s$.

Definition 2.2 (Carnot-Carathéodory distance) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. An absolutely continuous curve $\gamma : [0, T] \to \Omega$ is called a sub-unit curve with respect to the system $X = (X_1, \ldots, X_q)$, if $\gamma'(t)$ exists for a.e. $t \in [0, T]$ and satisfies

$$<\gamma'(t), \xi>^2 \leq \sum_{j=1}^q <X_j(\gamma(t)), \xi>^2, \quad \text{for any } \xi \in \mathbb{R}^n.$$ 

We denote the length of this curve by $l_S(\gamma) = T$. Given any $x, y \in \Omega$, let $\Phi(x, y)$ be the collection of all sub-unit curves connecting $x$ and $y$, and define the Carnot–Carathéodory distance induced by $X$ by

$$d_X(x, y) = \inf\{l_S(\gamma) : \gamma \in \Phi(x, y)\}.$$ 

With this distance, we denote a metric ball of radius $R$ centered at $x_0$ by

$$B_R(x) = B(x, R) = \{y \in \Omega : d(x, y) < R\}.$$
If one does not need to consider the center of ball, then we also write $B_R$ instead of $B(x, R)$.

It is well known that the doubling property for metric balls (see [23]) holds true, i.e., there exist positive constants $c_D$ and $R_D$, such that for any $x_0 \in \Omega$, $0 < 2R < R_D$, $B_{2R} \subset \Omega$,

$$|B(x_0, 2R)| \leq c_D |B(x_0, R)|.$$ 

So $B_R(x)$ is a homogeneous space ([13]). Furthermore, it follows that for any $R \leq R_D$ and $t \in (0, 1)$,

$$|B_{tR}| \geq c_D^{-1} t^Q |B_R|.$$ 

The number $Q = \log_2 c_D$ is called a locally homogeneous dimension relative to $\Omega$. We can assume by [23] that there exist two positive constants $c_1$ and $c_2$, such that

$$c_1 R^Q \leq |B_R| \leq c_2 R^Q.$$ \hspace{1cm} (2.1)

**Definition 2.3** (Sobolev space) Let $1 \leq p \leq +\infty$, $k$ be a positive integer. If $u \in L^p(\Omega, \mathbb{R}^N)$ satisfies

$$\|u\|_{W^{k,p}_X(\Omega, \mathbb{R}^N)} \equiv \|u\|_{L^p(\Omega, \mathbb{R}^N)} + \sum_{h=1}^k \sum_{j_1=1}^q \|X_{j_1}X_{j_2} \cdots X_{j_h}u\|_{L^p(\Omega, \mathbb{R}^N)} < +\infty,$$

then we say that $u$ belongs to the Sobolev space $W^{k,p}_X(\Omega, \mathbb{R}^N)$.

**Remark:** The space $W^{k,p}_{X,0}(\Omega, \mathbb{R}^N)$ is the closure of $C_\infty(\Omega, \mathbb{R}^N)$ in $W^{k,p}_X(\Omega, \mathbb{R}^N)$ with respect to the norm $\|u\|_{W^{k,p}_X(\Omega, \mathbb{R}^N)}$.

Denote by $d_0$ the diameter of $\Omega$.

**Definition 2.4** (Morrey space) Let $p \geq 1, \lambda \geq 0, u \in L^p_{loc}(\Omega, \mathbb{R}^N)$, if

$$\|u\|_{L^{p,\lambda}_X(\Omega, \mathbb{R}^N)} \equiv \sup_{x_0 \in \Omega, 0 < R < d_0} \left( \frac{1}{R^\lambda} \int_{\Omega \cap B(x_0, R)} |u(x)|^p dx \right)^{\frac{1}{p}} < +\infty,$$

then $u$ is said to belong to the Morrey space $L^{p,\lambda}_X(\Omega, \mathbb{R}^N)$.

**Definition 2.5** (Campanato space) Let $p \geq 1, \lambda \geq 0, u \in L^p_{loc}(\Omega, \mathbb{R}^N)$, if

$$\|u\|_{\mathcal{L}^{p,\lambda}_X(\Omega, \mathbb{R}^N)} \equiv \sup_{x_0 \in \Omega, 0 < R < d_0} \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |u(x) - u_{BR}|^p dx \right)^{\frac{1}{p}} < +\infty,$$

where $u_{BR} = \frac{1}{|B_{2R}|} \int_{B_{2R}} u(x) dx$, then we say that $u$ is in the Campanato space $\mathcal{L}^{p,\lambda}_X(\Omega, \mathbb{R}^N)$. 

**Definition 2.6** (Hölder space) Let $\kappa \in (0, 1]$. The Hölder space $C^{0,\kappa}_X(\bar{\Omega}, \mathbb{R}^N)$ is the set of functions satisfying

$$
\|u\|_{C^{0,\kappa}_X(\bar{\Omega}, \mathbb{R}^N)} \equiv \sup_{\Omega} |u| + \sup_{\Omega} \frac{|u(x) - u(y)|}{|d(x, y)|^\kappa} < +\infty.
$$

**Definition 2.7** ($BMO_X$ and $VMO_X$ spaces) Let $u \in L^1_{loc}(\Omega, \mathbb{R}^N)$. If

$$
\|u\|_{BMO_X(\Omega, \mathbb{R}^N)} \equiv \sup_{x_0 \in \Omega, 0 < R < d_0} \frac{1}{|\Omega \cap B(x_0, R)|} \int_{\Omega \cap B(x_0, R)} |u(x) - u_{B_R}|dx < +\infty,
$$

then we say that $u \in BMO_X(\Omega, \mathbb{R}^N)$ (Bounded Mean Oscillation). If $u \in BMO_X(\Omega, \mathbb{R}^N)$ and

$$
\eta_R(u) = \sup_{x_0 \in \Omega, 0 < \rho < R} \frac{1}{|\Omega \cap B(x_0, \rho)|} \int_{\Omega \cap B(x_0, \rho)} |u(x) - u_{B_{\rho}}|dx \to 0, \quad R \to 0,
$$

then we say that $u \in VMO_X(\Omega, \mathbb{R}^N)$ (Vanishing Mean Oscillation).

**Lemma 2.8** (see [16]) Let $H(\rho)$ be a nonnegative increasing function, and for any $0 < \rho < R \leq R_0 = \text{dist}(x_0, \partial \Omega)$,

$$
H(\rho) \leq A \left[ \left( \frac{\rho}{R} \right)^a + \varepsilon \right] H(R) + BR^b,
$$

where $A, a$ and $b$ are nonnegative constants with $a > b$. Then there exist positive constants $\varepsilon_1 = \varepsilon_1(A, a, b)$ and $c = c(A, a, b)$, such that for any $\varepsilon < \varepsilon_1$, it follows

$$
H(\rho) \leq c \left[ \left( \frac{\rho}{R} \right)^b H(R) + BR^b \right].
$$

**Lemma 2.9** (Sobolev–Poincaré inequality, see [12] and [20]) For any open domain $\Omega'$, $\bar{\Omega}' \subset \subset \Omega$, there exist positive constants $R_0$ and $c$, such that for any $0 < R \leq R_0$, $B_R \subset \subset \Omega$ and $u \in C_0^\infty(\overline{B_R})$, it holds

$$
\left( \frac{1}{|B_R|} \int_{B_R} |u - u_R|^{p'} dx \right)^{\frac{1}{p'}} \leq cR \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{1}{p}}, \quad (2.2)
$$

where $1 < p < Q, 1 \leq p' \leq \frac{pQ}{Q - p}, u_R = \frac{1}{|B_R|} \int_{B_R} u(x)dx$, $R_0$ and $c$ depend on $\Omega'$ and $\Omega$. In particular, if $u \in C_0^\infty(\overline{B_R})$, then

$$
\left( \frac{1}{|B_R|} \int_{B_R} |u|^{p'} dx \right)^{\frac{1}{p'}} \leq cR \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{1}{p}}. \quad (2.3)
$$

**Lemma 2.10** (Morrey lemma, see [29]) Let $u \in W^{1,p}_{X}(\Omega, \mathbb{R}^N)(p > 1)$ and for any $B_R \subset \subset \Omega$, there exists a constant $\kappa \in (0, 1)$, such that

$$
\int_{B_R} |Xu|^p dx \leq cR^{Q-p+\rho\kappa}.
$$

Then $u \in C^{0,\kappa}_X(\Omega, \mathbb{R}^N)$. 7
\section{Proof of Theorem 1.1}

The following result is valid to the homogeneous space.

**Lemma 3.1** (reverse Hölder inequality, see [13, 27]) Let $\hat{g}, \hat{f} \geq 0$ satisfy

$$\hat{g} \in L^{q}(\Omega) (q > 1), \hat{f} \in L^{q'}(\Omega) (q' > q).$$

Fix a ball $B_{R_{0}} = B(x_{0}, R_{0})$ and assume that for any $x \in B_{R_{0}}$ and $R < \frac{1}{2} \text{dist}(x, \partial B_{R_{0}})$, there exist constants $b > 1$ and $\theta \in [0, 1)$, such that

$$\frac{1}{|B_{R}|} \int_{B_{R}} \hat{g}^{q} \, dx \leq b \left[ \left( \frac{1}{|B_{4R/3}|} \int_{B_{4R/3}} \hat{g}^{q} \, dx \right)^{\frac{1}{q}} + \left( \frac{1}{|B_{4R/3}|} \int_{B_{4R/3}} \hat{f}^{q} \, dx \right)^{\frac{1}{q}} + \frac{\theta}{|B_{4R/3}|} \int_{B_{4R/3}} \hat{g}^{q} \, dx \right].$$

Then there exist constants $\varepsilon_{0} > 0$ and $c > 0$ such that for any $r \in [\hat{q}, \hat{q} + \varepsilon_{0})$, it yields $\hat{g} \in L^{\infty}_{r}(B_{R_{0}})$.

Moreover, we have that for any $B_{2R} \subset \subset \Omega$, \( \left( \left( \frac{1}{|B_{R}|} \int_{B_{R}} X u \, dx \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} \leq c \left[ \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} X u \, dx \right)^{\frac{q}{2}} \right]^{\frac{1}{q}}, \) where $c$ and $\varepsilon_{0}$ are positive constants depending only on $b, \theta, \hat{q}$, and $q'$.

**Lemma 3.2** Let the coefficients $a_{ij}^{\alpha}$ in (1.1) satisfy (H1), functions $g_{i}$ and $f_{i}^{\alpha}$ satisfy (H2). If $u \in W^{1,2}_{X}(\Omega, \mathbb{R}^{N})$ is a weak solution to (1.1), then for any $p \in \left[ 2, 2 + \frac{2Q}{Q + 2 \varepsilon_{0}} \right)$, where $\varepsilon_{0}$ is in Lemma 3.1, $B_{2R} \subset \subset \Omega$, we have

$$\left( \frac{1}{|B_{R}|} \int_{B_{R}} |X u|^{p} \, dx \right)^{\frac{1}{p}} \leq c \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |X u|^{2} \, dx \right)^{\frac{1}{p}} + R \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} \frac{|g|^{p} + |\hat{g}|^{p} \, dx}{Q} \right)^{\frac{1}{pq}} (3.1)$$

**Proof:** We will prove (3.1) with two steps.

**Step 1.** Let a cut-off function $\eta \in C_{0}^{\infty}(B_{R})$ satisfy that for any $0 < \rho < R$,

$$0 \leq \eta \leq 1 \text{ (in } B_{R}), \quad \eta = 1 \text{ (in } B_{\rho}), \quad |X \eta| \leq \frac{c}{R - \rho}.$$  

Multiplying both sides of (1.1) by the test function $\varphi = (u - u_{B_{R}})\eta^{2}$ and integrating on $B_{R}$, it yields by (H1) that

$$- \int_{B_{R}} A^{\alpha \beta} \eta^{2} X_{\beta} u^{i} X_{\alpha} u^{i} \, dx + \int_{B_{R}} \eta^{2} f_{i}^{\alpha} X_{\alpha} u^{i} \, dx$$

$$= \int_{B_{R}} B^{\alpha \beta}_{ij} \eta^{2} X_{\beta} u^{i} X_{\alpha} u^{i} \, dx + \int_{B_{R}} 2 a_{ij}^{\alpha} (x, u) \eta (u^{i} - u_{B_{R}}) X_{\beta} u^{j} X_{\alpha} \eta \, dx$$

$$+ \int_{B_{R}} (g_{i}(u^{i} - u_{B_{R}})\eta^{2} - 2 \eta (u^{i} - u_{B_{R}}) f_{i}^{\alpha} X_{\alpha} \eta) \, dx. \quad (3.2)$$
It shows by (H2), (2.2), Hölder’s inequality and Young’s inequality that

\[
\int_{B_R} |g_i| |u^i - u_R| \eta^2 dx \leq \int_{B_R} (g^i + L |Xu|^{\gamma_0}) |u^i - u_R| dx
\]

\[
\leq \left( \int_{B_R} |\tilde{g}|^{2\alpha_0} dx \right)^{\frac{Q+2}{Q\alpha_0}} \left( \int_{B_R} |u - u_R|^{2\gamma_0} dx \right)^{\frac{Q-2}{Q\alpha_0}}
\]

\[
+ L \left( \int_{B_R} |Xu|^2 dx \right)^{\frac{2}{\gamma_0}} \left( \int_{B_R} |u - u_R|^{2 - \gamma_0} dx \right)^{\frac{2 - \gamma_0}{2}}
\]

\[
\leq \left( \int_{B_R} |\tilde{g}|^{2\alpha_0} dx \right)^{\frac{1}{\gamma_0}} \left( \int_{B_R} |Xu|^2 dx \right)^{\frac{1}{2}}
\]

\[
+ L \left( \int_{B_R} |Xu|^2 dx \right)^{\frac{2}{\gamma_0}} cR |B_R|^{\frac{2 - \gamma_0}{2}} \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^2 dx \right)^{\frac{1 + \gamma_0}{2}}
\]

\[
\leq c_\varepsilon \left( \int_{B_R} |\tilde{g}|^{2\alpha_0} dx \right)^{\frac{1}{\gamma_0}} + \varepsilon \int_{B_R} |Xu|^2 dx + cR \left( \int_{B_R} |Xu|^2 dx \right)^{\frac{2}{\gamma_0}}
\]

\[
\leq c_\varepsilon \left( \int_{B_R} |\tilde{g}|^{2\alpha_0} dx \right)^{\frac{1}{\gamma_0}} + \left( \varepsilon + cR \right) \left( \int_{\Omega} |Xu|^2 dx \right)^{\frac{2}{\gamma_0}} \int_{B_R} |Xu|^2 dx
\]

\[
\leq c_\varepsilon \left( \int_{B_R} |\tilde{g}|^{2\alpha_0} dx \right)^{\frac{1}{\gamma_0}} + \left( \varepsilon + cR \right) \int_{B_R} |Xu|^2 dx.
\]

Also,

\[
\int_{B_R} \eta |u^i - u_R| |f_i^0| |X_\alpha \eta| dx
\]

\[
\leq \int_{B_R} \eta |u^i - u_R| (g_i^0 + L |X_\alpha u^i|) |X_\alpha \eta| dx
\]

\[
\leq c_\varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + c_\varepsilon \int_{B_R} \eta^2 |\tilde{g}|^2 dx + \varepsilon \int_{B_R} \eta^2 |Xu|^2 dx.
\]

Inserting (3.3) and (3.4) into (3.2), and noting (H1) and (H2), we have

\[
\lambda_0 \int_{B_R} \eta^2 |Xu|^2 dx + \gamma_1 \int_{B_R} \eta^2 |Xu|^2 dx - \int_{B_R} \eta^2 |g|^2 dx
\]

\[
\leq \delta \lambda_0 \int_{B_R} \eta^2 |Xu|^2 dx + c_\varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + 2\varepsilon \int_{B_R} \eta^2 |Xu|^2 dx
\]

\[
+ c_\varepsilon \int_{B_R} \eta^2 |\tilde{g}|^2 dx + c_\varepsilon \left( \int_{B_R} |\tilde{g}|^{2\alpha_0} dx \right)^{\frac{1}{\gamma_0}} + \left( \varepsilon + cR \right) \int_{B_R} |Xu|^2 dx.
\]

By properties on \( \eta \),

\[
\left( \lambda_0 + \gamma_1 - 2\varepsilon - \delta \lambda_0 \right) \int_{B_R} |Xu|^2 dx
\]

\[
\leq \frac{c_\varepsilon}{(R - \rho)^2} \int_{B_R} |u - u_R|^2 dx + c_\varepsilon \int_{B_R} (|g|^2 + |\tilde{g}|^2) dx
\]

\[
+ c_\varepsilon \left( \int_{B_R} |\tilde{g}|^{2\alpha_0} dx \right)^{\frac{1}{\gamma_0}} + \left( \varepsilon + cR \right) \int_{B_R} |Xu|^2 dx.
\]
Letting \( \rho = \frac{3}{4} R \) and applying (2.2), it obtains

\[
\frac{\lambda_0 + \gamma_1 - 2 \varepsilon - \delta \lambda_0}{|B_{3R/4}|} \int_{B_{3R/4}} |Xu|^2 \, dx \\
\leq c \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^{\frac{2q}{2q+2}} \, dx \right)^{\frac{2q+2}{2q}} + \frac{c}{|B_R|} \left[ \int_{B_R} (|g|^2 + |\tilde{g}|^2) \, dx + \left( \int_{B_R} |\tilde{g}|^{2q_0} \, dx \right)^{\frac{1}{q_0}} \right] \\
+ \left( \varepsilon + cR^{\frac{q_0 + 2 - q_0}{2}} \right) \frac{1}{|B_R|} \int_{B_R} |Xu|^2 \, dx.
\]

Because of \( 0 < \delta < 1 \), we can choose \( \varepsilon \) and \( R \) small enough such that \( \lambda_0 + \gamma_1 - 2 \varepsilon - \delta \lambda_0 > 0 \), and \( \theta_1 = \frac{\varepsilon + cR^{\frac{q_0 + 2 - q_0}{2}}}{\lambda_0 + \gamma_1 - 2 \varepsilon - \delta \lambda_0} \in (0, 1) \), so

\[
\frac{1}{|B_{3R/4}|} \int_{B_{3R/4}} |Xu|^2 \, dx \leq c \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^{\frac{2q}{2q+2}} \, dx \right)^{\frac{2q+2}{2q}} + \frac{\theta_1}{|B_R|} \int_{B_R} |Xu|^2 \, dx \\
+ \frac{c}{|B_R|} \left[ \int_{B_R} (|g|^2 + |\tilde{g}|^2) \, dx + \left( \int_{B_R} |\tilde{g}|^{2q_0} \, dx \right)^{\frac{1}{q_0}} \right].
\] (3.5)

**Step 2.** Setting

\[
\hat{q} = \frac{Q + 2}{Q} = \frac{1}{q_0}, \quad \hat{g} = |Xu|^{\frac{2q}{2q+2}} = |Xu|^{2q_0}
\]

and

\[
\hat{f} = \left( \int_{B_R} |\tilde{g}|^{2q_0} \, dx \right)^{1-q_0} |\tilde{g}|^{2q_0} = |Xu|^{2q_0}
\]

(3.5) can be written as

\[
\frac{1}{|B_{3R/4}|} \int_{B_{3R/4}} \hat{g}^\hat{q} \, dx \leq c \left( \frac{1}{|B_R|} \int_{B_R} \hat{g}^\hat{q} \, dx \right)^{\frac{1}{\hat{q}}} + \frac{\theta_1}{|B_R|} \int_{B_R} \hat{g}^\hat{q} \, dx + \frac{c}{|B_R|} \int_{B_R} \hat{f}^\hat{q} \, dx.
\]

By Lemma 3.1, we have \( \hat{g} \in L^\infty_{\text{loc}, r} \) for any \( r \in [\hat{q}, \hat{q} + \varepsilon_0] \) and \( B_{2R} \subset \subset \Omega \),

\[
\left( \frac{1}{|B_R|} \int_{B_R} \hat{g}^\hat{q} \, dx \right)^{\frac{1}{\hat{q}}} \leq c \left[ \left( \frac{1}{|2B_R|} \int_{2B_R} \hat{g}^{\hat{q}} \, dx \right)^{\frac{1}{\hat{q}}} + \left( \frac{1}{|2B_R|} \int_{2B_R} \hat{f}^\hat{q} \, dx \right)^{\frac{1}{\hat{q}}} \right],
\]

namely,

\[
\frac{1}{|B_R|} \int_{B_R} |Xu|^{2q_0r} \, dx \leq c \left( \frac{1}{|2B_R|} \int_{2B_R} |Xu|^2 \, dx \right)^{q_0r} \\
+ \frac{c}{|2B_R|} \left( \int_{B_R} |\tilde{g}|^{2q_0} \, dx \right)^{(1-q_0)r} \int_{2B_R} |\tilde{g}|^{2q_0r} \, dx + \frac{c}{|2B_R|} \left( \int_{B_R} |g|^2 + |\tilde{g}|^2 \right)^{q_0r} \, dx.
\] (3.6)
Denote \( p = 2q_0r \), then \( p \in \left[ 2, 2 + \frac{2Q}{Q + 2\varepsilon_0} \right) \) and

\[
\frac{1}{|B_R|} \int_{B_R} |Xu|^p \, dx \\
\leq c \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |Xu|^2 \, dx \right)^{\frac{p}{2}} + c \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |\tilde{g}|^{2q_0} \, dx \right)^{\frac{1-q_0}{q_0}} \int_{B_{2R}} |\tilde{g}|^{pq_0} \, dx \\
+ \frac{c}{|B_{2R}|} \int_{B_{2R}} \left( |g|^p + |\tilde{g}|^p \right) \, dx
\]

\[
\leq c \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |Xu|^2 \, dx \right)^{\frac{p}{2}} + \frac{c}{|B_{2R}|} \left( \int_{B_{2R}} |\tilde{g}|^{pq_0} \, dx \right)^{\frac{1-q_0}{q_0}} \int_{B_{2R}} |\tilde{g}|^{pq_0} \, dx \\
+ \frac{c}{|B_{2R}|} \int_{B_{2R}} \left( |g|^p + |\tilde{g}|^p \right) \, dx.
\]

Hence (3.1) is proved.

**Corollary 3.3** Let \( u \in W^{1,2}_X(\Omega, \mathbb{R}^N) \) be a weak solution to the homogeneous degenerate elliptic system (1.2). Then for any \( p \in \left[ 2, 2 + \frac{2Q}{Q + 2\varepsilon_0} \right) \) and \( B_{2R} \subset \subset \Omega \), it follows

\[
\left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p \, dx \right)^{\frac{1}{p}} \leq c \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |Xu|^2 \, dx \right)^{\frac{1}{2}}.
\]

**Proof of theorem 1.1:** Multiplying both sides of (1.1) by the test function \( u - u_{B_{2R}} \) and integrating on \( B_{2R} \), we have

\[
- \int_{B_{2R}} a^{\alpha \beta}_{ij}(x, u) X_\beta u^j X_\alpha (u^i - u_{B_{2R}}) \, dx = \int_{B_{2R}} (g_i (u^i - u_{B_{2R}}) - f^\alpha_\alpha X_\alpha (u^i - u_{B_{2R}})) \, dx
\]

or

\[
- \int_{B_{2R}} A^{\alpha \beta} \delta_{ij} X_\beta u^j X_\alpha u^i \, dx + \int_{B_{2R}} f^\alpha_\alpha X_\alpha u^i \, dx \\
= \int_{B_{2R}} B^{\alpha \beta}_{ij} X_\beta u^j X_\alpha u^i \, dx + \int_{B_{2R}} g_i (u^i - u_{B_{2R}}) \, dx.
\]

By (H1), (H2) and (3.3), it gives

\[
\lambda_0 \int_{B_{2R}} |Xu|^2 \, dx + \gamma_1 \int_{B_{2R}} |Xu|^2 \, dx - \int_{B_{2R}} |g|^2 \, dx \\
\leq \delta \lambda_0 \int_{B_{2R}} |Xu|^2 dx + c \int_{B_{2R}} |g_i| |u^i - u_{2R}| \, dx \\
\leq c \left( \int_{B_{2R}} |\tilde{g}|^{2q_0} \, dx \right)^{\frac{1}{q_0}} + \left( \delta \lambda_0 + \varepsilon + cR^{2+2Q-2\gamma} \right) \int_{B_{2R}} |Xu|^2 \, dx \\
\leq c \left( \int_{B_{2R}} |\tilde{g}|^{2q_0} \, dx \right)^{\frac{1}{q_0}} + \theta_2 \int_{B_{2R}} |Xu|^2 \, dx,
\]
where \( \theta_2 = \delta \lambda_0 + \varepsilon + cR^{\frac{Q+2-Q^2}{2}} \). Then

\[
(\lambda_0 + \gamma_1 - \theta_2) \int_{B_{2R}} |Xu|^2 \, dx \leq c_\varepsilon \left( \int_{B_{2R}} |\tilde{g}|^{2q_0} \, dx \right)^{\frac{1}{q_0}} + \int_{B_{2R}} |g|^2 \, dx.
\]

Since \( \gamma_0 \in \left[ 1, \frac{Q+2}{Q} \right) \), \( 0 < \delta < 1 \), we can choose \( \varepsilon \) and \( R \) small enough such that \( \lambda_0 + \gamma_1 - \theta_2 > 0 \), and derive

\[
\int_{B_{2R}} |Xu|^2 \, dx \leq c \left( \int_{B_{2R}} |\tilde{g}|^{2q_0} \, dx \right)^{\frac{1}{q_0}} + c \int_{B_{2R}} |g|^2 \, dx. \tag{3.8}
\]

It shows from Lemma 3.2 that

\[
\left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p \, dx \right)^{\frac{1}{p}} \leq c \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |\tilde{g}|^{2q_0} \, dx \right)^{\frac{1}{q_0}} + \int_{B_{2R}} |g|^2 \, dx + \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |\tilde{g}|^{p_0} \, dx \right)^{\frac{1}{p_0}}
\]

\[
+ c \left[ R \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |g|^p \, dx \right)^{\frac{1}{p}} + R \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |\tilde{g}|^{p_0} \, dx \right)^{\frac{1}{p_0}} \right] \leq c \left[ \frac{1}{|B_{2R}|} \int_{B_{2R}} |g|^p \, dx \right]^{\frac{1}{p}} + R \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |\tilde{g}|^{p_0} \, dx \right)^{\frac{1}{p_0}}.
\]

So we conclude by (2.1) that

\[
\int_{B_R} |Xu|^p \, dx \leq c \int_{B_{2R}} \left( |g|^p + |\tilde{g}|^p \right) \, dx + cR^p |B_R| \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |\tilde{g}|^{p_0} \, dx \right)^{\frac{1}{p_0}}
\]

\[
\leq c \int_{B_{2R}} \left( |g|^p + |\tilde{g}|^p \right) \, dx + cR^{p-2} \left( \int_{B_{2R}} |\tilde{g}|^{p_0} \, dx \right)^{\frac{1}{p_0}}
\]

\[
\leq cR^\lambda \left( \|g\|_{L^p_{X^\lambda}} + \|\tilde{g}\|_{L^p_{X^\lambda}} \right) + cR^{p+\lambda-2} \|\tilde{g}\|_{L^{p_0}_{X^\lambda}}
\]

\[
\leq cR^\lambda \left( \|g\|_{L^p_{X^\lambda}} + \|\tilde{g}\|_{L^p_{X^\lambda}} + \|\tilde{g}\|_{L^{p_0}_{X^\lambda}} \right). \tag{3.9}
\]

It attains the assertion.

**Corollary 3.4** If (H2) in Theorem 1.1 is replaced by (H3), then the result of Theorem 1.1 still holds.

**Proof:** Since \( u \in W^{1,2}_X(\Omega, \mathbb{R}^N) \) is a weak solution to (1.1), we see that \( u \) is also a weak solution to the following system

\[
-X^*_\alpha (a^{\alpha \beta}_{ij}(x, u)X_\beta u^j) = g_i(x, u, Xu) - X^*_\alpha \left( f^\alpha_i(x, u, Xu) - (f^\alpha_i)_{B_R} \right).
\]
As in the proof of Lemma 3.2, it follows
\[- \int_{B_R} A^{\alpha \beta} \delta_{ij} \eta^2 X_\beta u^i X_\alpha u^j \, dx + \int_{B_R} \eta^2 f_i^\alpha X_\alpha u^i \, dx \]
\[= \int_{B_R} B^{\alpha \beta}_{ij} \eta^2 X_\beta u^i X_\alpha u^j \, dx + \int_{B_R} 2a^{\alpha \beta}_{ij} (x, u) \eta (u^i - u_{B_R}) X_\beta u^j X_\alpha \eta \, dx \]
\ [+ \int_{B_R} (g_i (u^i - u_{B_R}) \eta^2 - 2\eta (u^i - u_{B_R}) (f_i^\alpha - (f_i^\alpha)_{B_R}) X_\alpha \eta) \, dx. \quad (3.10)\]

Noting (H3), it implies
\[\int_{B_R} \left| f_i^\alpha - (f_i^\alpha)_{B_R} \right|^2 \, dx \leq c \int_{B_R} \left| f_i^\alpha - \frac{1}{|B_R|} \int_{B_R} f_i^\alpha \, dy \right|^2 \, dx \]
\[\leq c \frac{1}{|B_R|} \int_{B_R} \left( \int_{B_R} \left| f_i^\alpha (x, u(x), X u(x)) - f_i^\alpha (y, u(y), X u(y)) \right| \, dy \right)^2 \, dx \]
\[\leq c \frac{1}{|B_R|} \int_{B_R} \left( \int_{B_R} \left| g_i^\alpha (x) - g_i^\alpha (y) \right|^2 \, dy + \int_{B_R} \left| X u(x) - X u(y) \right|^2 \, dy \right) \, dx \]
\[\leq c \int_{B_R} \left( \hat{\gamma} - (\hat{\gamma})_{B_R} \right)^2 \, dx + c \int_{B_R} \left| X u - (X u)_{B_R} \right|^2 \, dx \quad (3.11)\]
and
\[\int_{B_R} \eta \left| u^i - u_{B_R} \right| \left| f_i^\alpha - (f_i^\alpha)_{B_R} \right| \left| X_\alpha \eta \right| \, dx \]
\[\leq c \varepsilon \int_{B_R} \left| u - u_{B_R} \right|^2 \left| X \eta \right|^2 \, dx + c \varepsilon \int_{B_R} \eta^2 \left| f_i^\alpha - (f_i^\alpha)_{B_R} \right|^2 \, dx \]
\[\leq c \varepsilon \int_{B_R} \left| u - u_{B_R} \right|^2 \left| X \eta \right|^2 \, dx + c \varepsilon \int_{B_R} \left( \hat{\gamma} - (\hat{\gamma})_{B_R} \right)^2 \, dx + c \varepsilon \int_{B_R} \left| X u - (X u)_{B_R} \right|^2 \, dx \]
\[\leq c \varepsilon \int_{B_R} \left| u - u_{B_R} \right|^2 \left| X \eta \right|^2 \, dx + c \varepsilon \int_{B_R} \left| \hat{\gamma} \right|^2 \, dx + c \varepsilon \int_{B_R} \left| X u \right|^2 \, dx, \]
which indicates that (3.4) still holds. Now we follow the proofs of Lemma 3.2 and Theorem 1.1 to reach the result.

4 Homogeneous degenerate elliptic system

An estimate of gradient of weak solutions to (1.2) is given in Corollary 3.3. In this section, we continue to study (1.2) and establish some other useful estimates. The main results in this section are Theorem 4.1 and Theorem 4.2.

Denote
\[A = (A^{\alpha \beta}(x))_{B_R} = \frac{1}{|B_R|} \int_{B_R} A^{\alpha \beta}(x) \, dx = S^2 = SS',\]
where $S$ is a positive symmetric matrix, and denote $S(B_R) = \{ Sx : x \in B_R \}$. 

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Given a weak solution $u$ to (1.2), let $u = v + w$, here $v \in W^{1,2}_X \left( S(B_R), \mathbb{R}^n \right)$ is a weak solution to the following constant coefficients diagonal homogeneous system

\[
\begin{align*}
-X^*_\alpha \left( (A^{\alpha\beta}(x))_{B_R} \delta_{ij} X_\beta u^j \right) &= 0, \\
v - u &\in W^{1,2}_X \left( S(B_R), \mathbb{R}^n \right),
\end{align*}
\] (4.1)

and $w \in W^{1,2}_{X,0} \left( S(B_R), \mathbb{R}^n \right)$ satisfies the following constant coefficients diagonal nonhomogeneous system

\[
\begin{align*}
-X^*_\alpha \left( (A^{\alpha\beta}(x))_{B_R} \delta_{ij} X_\beta u^j \right) &= -X^*_\alpha \left( (A^{\alpha\beta}(x))_{B_R} - A^{\alpha\beta}(x) \right) \delta_{ij} X_\beta u^j + X^*_\alpha \left( B^{\alpha\beta}_{ij} X_\beta u^j \right).
\end{align*}
\] (4.2)

We start by recalling a lemma in [26].

**Lemma 4.1** Let $v_0 \in C^\infty(\Omega, \mathbb{R}^n)$, $B_R \subset \subset \Omega$, $k > \frac{Q}{2}$. Then there exist positive constants $R_0$ and $c$ such that for any $R \leq R_0$,

\[
\sup_{x \in B_{R/4}} |v_0| \leq c |B_R|^{-\frac{1}{2}} \sum_{|I| \leq k} R^{|I|} \|X_I v_0\|_{L^2(B_R)}.
\] (4.3)

**Lemma 4.2** Let $v \in W^{1,2}_X(\Omega, \mathbb{R}^n)$ be a weak solution to (4.1). Then $v \in C^\infty(\Omega)$ and it follows that for any positive integer $k$ and $S(B_R) \subset \subset \Omega$,

\[
\sum_{|I| \leq k} \int_{S(B_{R/2^k})} |S X_I v|^2 dx \leq \frac{c}{R^{2k}} \int_{S(B_R)} |S v|^2 dx.
\] (4.4)

**Proof:** Since $A = (A^{\alpha\beta}(x))_R = S^2$, it sees that (4.1) can be rewrite as

\[-X^*_\alpha \left( \delta_{ij} SX_\beta \left( S v^j \right) \right) = 0.
\]

By [26], we know that assertions hold.

**Lemma 4.3** Let $v \in W^{1,2}_X(\Omega, \mathbb{R}^n)$ be a weak solution to (4.1). Then for any $0 < \rho < R$, $S(B_R) \subset \subset \Omega$,

\[
\int_{S(B_\rho)} |S X v|^2 dx \leq c \left( \frac{\rho}{R} \right)^Q \int_{S(B_R)} |S X v|^2 dx.
\] (4.5)

**Proof:** Since $u_0(y) = v(Sy)$ satisfies $-X^*_\alpha \left( X_\beta u^j_0 \right) = 0$ in $B_R$, we have by [9] that

\[
\int_{B_\rho} |X u_0|^2 dy \leq c \left( \frac{\rho}{R} \right)^Q \int_{B_R} |X u_0|^2 dy.
\]

By the transformation $x = Sy$, it finishes the proof of (4.5).
Theorem 4.1 Let \( u \in W^{1,2}_X(\Omega, \mathbb{R}^N) \) be a weak solution to (1.2) with coefficients \( a_{ij}^{\alpha\beta} \) satisfying (H1). For any \( p \in \left[ 2, 2 + \frac{2Q}{Q+2} \varepsilon_0 \right) \), \( \frac{(p-2)Q}{p} < \mu_1 < Q \), \( 0 < \rho < R \), \( S(B_R) \subset \subset \Omega \), we have

\[
\int_{S(B_R)} |SXu|^p dx \leq c \left( \frac{\rho}{R} \right)^{2Q-\frac{2Q}{Q+2}} \int_{S(B_R)} |SXu|^p dx. \tag{4.6}
\]

Proof: If \( \frac{2}{2} \leq \rho < R \), then the conclusion is evident. In the sequel it only needs to treat the case \( 0 < \rho < \frac{2}{2} \).

First, multiplying both sides in (4.2) by \( w \) and integrating on \( S(B_R) \),

\[
\int_{S(B_R)} \left( A^{\alpha\beta}(x) \right)_{B_R} \delta_{ij} X_\beta u^j X_\alpha u^i dx
\]

\[
= \int_{S(B_R)} \left( (A^{\alpha\beta}(x))_{B_R} - A^{\alpha\beta}(x) \right) \delta_{ij} X_\beta u^j X_\alpha u^i dx
\]

\[
- \int_{S(B_R)} B^{\alpha\beta}_{ij} X_\beta u^j X_\alpha u^i dx.
\]

From (H1), Young’s inequality and Hölder’s inequality, we have

\[
\int_{S(B_R)} |SXw|^2 dx
\]

\[
\leq \frac{c_\varepsilon}{\lambda_0} \int_{S(B_R)} |(A^{\alpha\beta}(x))_{B_R} - A^{\alpha\beta}(x)|^2 |Xu|^2 dx + \frac{\varepsilon}{\lambda_0} \int_{S(B_R)} |Xw|^2 dx
\]

\[
+ \delta \int_{S(B_R)} |Xu| |Xw| dx
\]

\[
\leq \frac{c_\varepsilon}{\lambda_0} S^2 \int_{S(B_R)} |(A^{\alpha\beta}(x))_{S(B_R)} - A^{\alpha\beta}(x)|^2 |SXu|^2 dx + \frac{\varepsilon}{\lambda_0} \int_{S(B_R)} |SXu|^2 dx
\]

\[
+ \delta \int_{S(B_R)} |SXu| |SXu| dx
\]

\[
\leq \frac{c_\varepsilon}{\lambda_0} \left( \int_{S(B_R)} \left| (A^{\alpha\beta}(x))_{S(B_R)} - A^{\alpha\beta}(x) \right|^{2\frac{p}{p-2}} dx \right)^{\frac{p-2}{p}} \left( \int_{S(B_R)} |SXu|^p dx \right)^{\frac{2}{p}}
\]

\[
+ \left( \frac{\varepsilon}{\lambda_0} + \varepsilon \right) \int_{S(B_R)} |SXw|^2 dx + \left( \frac{c_\varepsilon}{\lambda_0} + c_\varepsilon \right) \int_{S(B_R)} |SXu|^2 dx.
\tag{4.7}
\]

Noting

\[
\left( \int_{S(B_R)} \left| (A^{\alpha\beta})_{S(B_R)} - A^{\alpha\beta} \right|^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}} \leq c |S(B_R)|^{\frac{2p-2}{p}} \left( \eta_{S(B_R)} (A^{\alpha\beta}) \right)^{\frac{p-2}{p}},
\]
it obtains
\[
\int_{S(B_r)} |SX w|^2 \, dx \\
\leq \frac{c}{\lambda_0} |S(B_r)|^{\frac{p-2}{p}} \left( \eta_{S(B_r)} (A^{\alpha \beta}) \right)^{\frac{p-2}{p}} \left( \int_{S(B_r)} |SX u|^p \, dx \right)^{\frac{2}{p}} + \left( \frac{c}{\lambda_0} + \varepsilon \right) \int_{S(B_r)} |SX w|^2 \, dx \\
+ \left( \frac{c}{\lambda_0} + c \right) |S(B_r)|^{\frac{p-2}{p}} \left( \int_{S(B_r)} |SX u|^p \, dx \right)^{\frac{2}{p}} \\
\leq c |S(B_r)|^{\frac{p-2}{p}} \left( \eta_{S(B_r)} (A^{\alpha \beta}) \right)^{\frac{p-2}{p}} + \left( \frac{c}{\lambda_0} + \varepsilon \right) \int_{S(B_r)} |SX u|^p \, dx.
\]

Choosing \( \varepsilon \) small enough such that \( 1 - \frac{c}{\lambda_0} - \varepsilon > 0 \), it follows
\[
\int_{S(B_r)} |SX w|^2 \, dx \leq c |S(B_r)| \left( \eta_{S(B_r)} (A^{\alpha \beta}) \right)^{\frac{p-2}{p}} + c \left( \frac{1}{|S(B_r)|} \int_{S(B_r)} |SX u|^p \, dx \right)^{\frac{2}{p}} \tag{4.8}
\]

Next by \( u = v + w \), we have
\[
\int_{S(B_{2r})} |SX u|^2 \, dx \leq 2 \int_{S(B_{2r})} |SX u|^2 \, dx + 2 \int_{S(B_{2r})} |SX w|^2 \, dx \\
\leq c \left( \frac{\rho}{R} \right)^Q \int_{S(B_r)} |SX u|^2 \, dx + c \int_{S(B_r)} |SX w|^2 \, dx \\
\leq c \left( \frac{\rho}{R} \right)^Q |S(B_r)| \left( \frac{1}{|S(B_r)|} \int_{S(B_r)} |SX u|^p \, dx \right)^{\frac{2}{p}} \\
+ c |S(B_r)| \left( \eta_{S(B_r)} (A^{\alpha \beta}) \right)^{\frac{p-2}{p}} + c \left( \frac{1}{|S(B_r)|} \int_{S(B_r)} |SX u|^p \, dx \right)^{\frac{2}{p}} \tag{4.9}
\]

Using (3.7), we have
\[
\left( \frac{1}{|S(B_r)|} \int_{S(B_r)} |SX u|^p \, dx \right)^{\frac{1}{p}} \leq c \left( \frac{1}{|S(B_{2r})|} \int_{S(B_{2r})} |X u|^2 \, dx \right)^{\frac{1}{2}}.
\]

Inserting (4.9) into the above, it gets
\[
\int_{S(B_r)} |SX u|^p \, dx \\
\leq c \left( \frac{\rho}{R} \right)^Q + \left( \eta_{S(B_r)} (A^{\alpha \beta}) \right)^{\frac{p-2}{p}} + c \left( \frac{|S(B_r)|}{|S(B_r)|} \right)^{\frac{2p}{p-2}} \int_{S(B_r)} |SX u|^p \, dx.
\]

Therefore,
\[
\left( |S(B_r)| \right)^{\frac{p-2}{p}} \int_{S(B_r)} |SX u|^p \, dx \right)^{\frac{2}{p}} \\
\leq c \left( \frac{\rho}{R} \right)^Q + \left( \eta_{S(B_r)} (A^{\alpha \beta}) \right)^{\frac{p-2}{p}} + c \left( |S(B_r)| \right)^{\frac{2p}{p-2}} \int_{S(B_r)} |SX u|^p \, dx \right)^{\frac{2}{p}}.
\]
Finally, let
\[ H(\rho) = \left| S(B_\rho) \right|^{\frac{p-2}{p}} \int_{S(B_\rho)} |SXu|^p \, dx \],
\[ H(R) = \left| S(B_R) \right|^{\frac{p-2}{p}} \int_{S(B_R)} |SXu|^p \, dx \],
where \( a = Q \), \( B = 0 \).

For any \( \mu_1, (p-2)Q < \mu_1 < Q \), let \( b = \mu_1 \), then \( a > b \). Now we apply Lemma 2.8 to reach
\[ \left( \left| S(B_\rho) \right|^{\frac{p-2}{p}} \int_{S(B_\rho)} |SXu|^p \, dx \right)^\frac{2}{p} \leq c \left( \frac{\rho}{R} \right)^{\mu_1} \left( \left| S(B_R) \right|^{\frac{p-2}{p}} \int_{S(B_R)} |SXu|^p \, dx \right)^\frac{2}{p} \]
and (4.6) is proved.

**Lemma 4.4** Let \( w \in W^{1,2}_{X,0}(\Omega, \mathbb{R}^N) \) be a weak solution to (4.2), with the coefficients \( a_{ij}^{\alpha\beta} \) satisfying (H1). Then for any \( p \in \left[ 2, 2 + \frac{2Q}{Q-2} \varepsilon_0 \right) \), \( 0 < \rho < R \), \( B(\Omega) \subset \subset \Omega \),
\[ \int_{S(B_\rho)} |SXw|^p \, dx \leq c \int_{S(B_{2R})} \left| SXu - (SXu)_{S(B_{2R})} \right|^p \, dx. \]  \hspace{1cm} (4.10)

**Proof:** Clearly, \( w \) is also a weak solution to the following system
\[ -X_\alpha^* \left( (A^{\alpha\beta}(x))_{B_R} \delta_{ij} X_{\beta} w^j \right) = -X_\alpha^* \left[ (A^{\alpha\beta}(x))_{B_R} \delta_{ij} \left( X_{\beta} u^j - (X_{\beta} u^j)_{B_{2R}} \right) \right]. \]  \hspace{1cm} (4.11)

Take the cut-off function \( \eta \in C_0^\infty(B_0) \) as in the proof of Lemma 3.2. Multiplying both sides of (4.11) by \( \varphi = (w - w_{B_R}) \eta^2 \) and integrating on \( S(B_R) \), it gets
\[ \int_{S(B_R)} (A^{\alpha\beta}(x))_{B_R} \delta_{ij} X_{\beta} w^j X_\alpha ((w^i - w_{B_R}) \eta^2) \, dx \]
\[ = \int_{S(B_R)} (A^{\alpha\beta}(x))_{B_R} \delta_{ij} \left( X_{\beta} u^j - (X_{\beta} u^j)_{B_{2R}} \right) X_\alpha ((w^i - w_{B_R}) \eta^2) \, dx, \]
i.e.,
\[ \int_{S(B_R)} (A^{\alpha\beta}(x))_{B_R} \delta_{ij} \eta^2 X_\alpha w^j X_{\beta} w^i \, dx \]
\[ = - \int_{S(B_R)} 2 (A^{\alpha\beta}(x))_{B_R} \delta_{ij} \eta (w^i - w_{B_R}) X_\alpha \eta X_{\beta} w^j \, dx \]
\[ + \int_{S(B_R)} (A^{\alpha\beta}(x))_{B_R} \delta_{ij} \eta^2 X_\alpha w^j \left( X_{\beta} u^j - (X_{\beta} u^j)_{B_{2R}} \right) \, dx \]
\[ + \int_{S(B_R)} 2 (A^{\alpha\beta}(x))_{B_R} \delta_{ij} \eta (w^i - w_{B_R}) X_\alpha \eta \left( X_{\beta} u^j - (X_{\beta} u^j)_{B_{2R}} \right) \, dx. \]

It yields by (H1) and Young’s inequality that
\[ \int_{S(B_R)} \eta^2 |SXw|^2 \, dx \leq c_\varepsilon \int_{S(B_R)} |Sw - Sw_{B_R}|^2 |X\eta|^2 \, dx \]
\[ + c_\varepsilon \int_{S(B_R)} \eta^2 |SXu - (SXu)_{B_{2R}}|^2 \, dx + 2\varepsilon \int_{S(B_R)} \eta^2 |SXw|^2 \, dx. \]
We know by choosing \( \varepsilon \) inequality (4.12) is of the form

\[
\int_{S(B)} |SXw|^2 \, dx \leq \frac{c}{(R-r)^2} \int_{S(B)} |Sw - (Sw)_{S(B)}|^2 \, dx \\
+ c \int_{S(B)} |SXu - (SXu)_{S(B)}|^2 \, dx + 2 \varepsilon \int_{S(B)} |SXw|^2 \, dx.
\]

Letting \( \rho = \frac{3}{4}R \) and using (2.2), it follows

\[
\int_{S(B_{R/4})} |SXw|^2 \, dx \\
\leq \frac{c}{R^2} \int_{S(B)} |Sw - (Sw)_{S(B)}|^2 \, dx + c \int_{S(B)} |SXu - (SXu)_{S(B^2)}|^2 \, dx \\
+ 2 \varepsilon \int_{S(B)} |SXw|^2 \, dx \\
\leq c |S(B)| \left( \frac{1}{|S(B)|} \int_{S(B)} |SXw|^\|q\|_{\frac{q}{q+2}} \, dx \right)^{\frac{q+2}{q}} \\
+ c \int_{S(B)} |SXu - (SXu)_{S(B^2)}|^2 \, dx + 2 \varepsilon \int_{S(B)} |SXw|^2 \, dx. \tag{4.12}
\]

Drawing notations \( \hat{g} = |SXw|^\|q\|_{\frac{q}{q+2}} \), \( \hat{q} = \frac{q+2}{q} \) and \( \hat{f} = |SXu - (SXu)_{S(B^2)}|^\|q\|_{\frac{q}{q+2}} \), the inequality (4.12) is of the form

\[
\frac{1}{|S(B_{3R/4})|} \int_{S(B_{3R/4})} \hat{g}^q \, dx \\
\leq c \left( \frac{1}{|S(B)|} \int_{S(B)} \hat{g} \, dx \right)^{\frac{q+2}{q}} + c \frac{1}{|S(B)|} \int_{S(B)} \hat{f} \, dx + \frac{2 \varepsilon}{|S(B)|} \int_{S(B)} \hat{g} \, dx. \tag{4.13}
\]

We know by choosing \( \varepsilon \) small enough such that \( 2 \varepsilon < 1 \) and employing Lemma 3.1 that \( \hat{g} \in L_{loc}, r \in [\hat{q}, \hat{q} + \varepsilon_0] \), and for any \( S(B_{2R}) \subset \subset \Omega \),

\[
\left( \frac{1}{|S(B)|} \int_{S(B)} |SXw|^\|q\|_{\frac{q}{q+2}} \, dx \right)^{\frac{q+2}{q}} \\
\leq c \left( \frac{1}{|S(B)|} \int_{S(B^2)} |SXw|^2 \, dx \right)^{\frac{q}{q+2}} \\
+ c \left( \frac{1}{|S(B^2)|} \int_{S(B^2)} |SXu - (SXu)_{S(B^2)}|^\|q\|_{\frac{q}{q+2}} \, dx \right)^{\frac{q}{q+2}}. \tag{4.14}
\]

Let \( p = \frac{2q}{q+2} \), then \( p \in \left[ 2, 2 + \frac{2q}{q+2} \varepsilon_0 \right] \) and we can rewrite (4.14) as

\[
\left( \frac{1}{|S(B)|} \int_{S(B)} |SXw|^p \, dx \right)^{\frac{1}{p}} \\
\leq c \left( \frac{1}{|S(B)|} \int_{S(B^2)} |SXw|^2 \, dx \right)^{\frac{1}{2}} \\
+ c \left( \frac{1}{|S(B^2)|} \int_{S(B^2)} |SXu - (SXu)_{S(B^2)}|^p \, dx \right)^{\frac{1}{p}}. \tag{4.15}
\]

On the other hand, multiplying both sides of (4.11) by \( w \) and integrating on \( S(B_{2R}) \),

\[
\int_{S(B_{2R})} (A^{\alpha \beta}(x))_{B_{2R}} \delta_{ij} X_\beta w^j X_\alpha w^i \, dx \\
= \int_{S(B_{2R})} (A^{\alpha \beta}(x))_{B_{2R}} \delta_{ij} (X_\beta w^j - (X_\beta w^j)_{B_{2R}}) X_\alpha w^i \, dx.
\]
It follows by (H1) and Young’s inequality that
\[
\int_{S(B_{2R})} |SXw|^2 \, dx \leq c\varepsilon \int_{S(B_{2R})} |SXu - (SXu)_{S(B_R)}|^2 \, dx + \varepsilon \int_{S(B_{2R})} |SXw|^2 \, dx.
\]
For \(\varepsilon\) small enough, we see
\[
\int_{S(B_{2R})} |SXw|^2 \, dx \leq c\varepsilon \int_{S(B_{2R})} |SXu - (SXu)_{S(B_R)}|^2 \, dx.
\]
Putting it into (4.15) implies
\[
\frac{1}{|S(B_R)|} \int_{S(B_R)} |SXw|^p \, dx \\
\leq c \left( \frac{1}{|S(B_{2R})|} \int_{S(B_{2R})} |SXu - (SXu)_{S(B_R)}|^2 \, dx \right)^{\frac{p}{2}} \\
+ c \frac{|S(B_{2R})|}{|S(B_R)|} \int_{S(B_{2R})} |SXu - (SXu)_{S(B_{2R})}|^p \, dx \\
\leq \frac{c}{|S(B_{2R})|} \int_{S(B_{2R})} |SXu - (SXu)_{S(B_{2R})}|^p \, dx.
\]
The proof of (4.10) is ended.

**Lemma 4.5** Let \(v \in W^{1,2}_X(\Omega, \mathbb{R}^N)\) be a weak solution to (4.1). Then for any \(p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right), 0 < \rho < R, S(B_R) \subset \subset \Omega,\) we have
\[
\int_{S(B_{\rho})} |SXv - (SXv)_{S(B_{\rho})}|^p \, dx \leq c \left( \frac{\rho}{R} \right)^{Q+p} \int_{S(B_R)} |SXv - (SXv)_{S(B_R)}|^p \, dx. \tag{4.16}
\]

**Proof:** Let \(k\) be a fixed integer such that \(k > \frac{Q}{2}.\) If \(\frac{R}{2^{k+2}} \leq \rho < R,\) then the conclusion is evident. If \(\rho < \frac{R}{2^{k+2}},\) then \(Xv\) and \(X^2v\) are also weak solutions to (4.1), so (4.3) is true.
for $X^2v$ and (4.4) is true for $Xv$. Combining these and noting (2.1), it shows

$$
\int_{S(B'_p)} |SX^2v|^p dx \leq |S(B_p)| \sup_{S(B'_{p/2k+2})} |SX^2v|^p
$$

$$
\leq c |S(B_p)| \sum_{|I| \leq k} |S(B_{R/2k})|^{-\frac{2}{p}} \rho^{p/|I|} \left( \int_{S(B'_{R/2k+2})} |SX_1X^2v|^2 dx \right)^{\frac{p}{2}}
$$

$$
\leq c |S(B_p)| |S(B_{R})|^{-\frac{2}{p}} \sum_{|I| \leq k} R^{p/|I|} R^{-2(|I|+1)} \left( \int_{S(B_{R})} |SXv|^2 dx \right)^{\frac{p}{2}}
$$

$$
\leq c |S(B_p)| |S(B_{R})|^{-\frac{2}{p}} R^{-p} |S(B_{R})|^{-\frac{2}{p}} \int_{S(B_{R})} |SXv|^p dx
$$

$$
\leq c \left( \frac{\rho}{R} \right)^Q R^{-p} \int_{S(B_{R})} |SXv|^p dx
$$

$$
\leq c \left( \frac{\rho}{R} \right)^Q R^{-p} \int_{S(B_{p})} |SXv - (SXv)_{S(B_{p})}|^p dx. \quad (4.17)
$$

Since $Xv - (Xv)_{B_R}$ is a weak solution to (4.1), we know that (4.17) is valid for $Xv - (Xv)_{B_R}$ and then

$$
\int_{S(B_p)} |SX^2v|^p dx \leq c \left( \frac{\rho}{R} \right)^Q R^{-p} \int_{S(B_{p})} |SXv - (SXv)_{S(B_{p})}|^p dx. \quad (4.18)
$$

Using (2.2) and (4.18), it follows

$$
\int_{S(B_p)} |SXv - (SXv)_{S(B_p)}|^p dx \leq c \rho^p \int_{S(B_p)} |SX^2v|^p dx
$$

$$
\leq c \left( \frac{\rho}{R} \right)^{Q+p} \int_{S(B_{p})} |SXv - (SXv)_{S(B_{p})}|^p dx
$$

and (4.16) is proved.

**Theorem 4.2** Let $u \in W^{1,2}_X(\Omega, \mathbb{R}^N)$ be a weak solution to (1.2), with the coefficients $a^{\alpha\beta}_{ij}$ satisfying (H1). Then for any $p \in \left[ 2, 2 + \frac{2Q}{Q+2+0} \right]$, $0 < \mu_2 < Q + p$, $0 < \rho < R$, $S(B_R) \subset \subset \Omega$,

$$
\int_{S(B_p)} |SXu - (SXu)_{S(B_p)}|^p dx \leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_{p})} |SXu - (SXu)_{S(B_{p})}|^p dx. \quad (4.19)
$$
**Proof:** Noting Lemma 4.4, Lemma 4.5 and \( u = v + w \), it leads to
\[
\int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p dx \\
\leq c \int_{S(B_R)} |SXv - (SXv)_{S(B_R)}|^p dx + c \int_{S(B_R)} |SXw - (SXw)_{S(B_R)}|^p dx \\
\leq c \left( \frac{p}{R} \right)^{Q+p} \int_{S(B_R)} |SXv - (SXv)_{S(B_R)}|^p dx + c \int_{S(B_R)} |SXw|^p dx \\
\leq c \left( \frac{p}{R} \right)^{Q+p} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p dx + c \int_{S(B_R)} |SXw|^p dx \\
\leq c \left( \frac{p}{R} \right)^{Q+p} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p dx \\
+ c \int_{S(B_R)} |SXw|^p dx \\
\leq c \left( \frac{p}{R} \right)^{Q+p} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p dx + c \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p dx \\
\leq \left( c \frac{p}{R}^{Q+p} + c \right) \int_{S(B_{2R})} |SXu - (SXu)_{S(B_{2R})}|^p dx.
\]
Now we use Lemma 2.8 to obtain (4.19).

5 Proofs of Theorems 1.2, 1.3 and 1.4

In order to prove Theorems 1.2, 1.3 and 1.4, we divide (1.1) into two new systems and let \( u = v + w \), where \( v \) satisfies the nondiagonal homogeneous system in \( S(B_R) \)
\[
\begin{cases}
-X^*_{\alpha} \left( a_{ij}^\alpha X^j \right) = 0, \\
v - u \in W^{1,2}_{X,0} (S(B_R), \mathbb{R}^N)
\end{cases}
\tag{5.1}
\]
and \( w \) solves the nondiagonal nonhomogeneous system in \( S(B_R) \)
\[
\begin{cases}
-X^*_{\alpha} \left( a_{ij}^\alpha X^j \right) = g_i(x, u, Xu) - X^*_{\alpha} f_i(x, u, Xu), \\
w \in W^{1,2}_{X,0} (S(B_R), \mathbb{R}^N).
\end{cases}
\tag{5.2}
\]

**Proof of Theorem 1.2:** By (3.9) replacing \( u \) by \( w \) and Theorem 4.1 replacing \( u \) by \( v \), it follows
\[
\int_{S(B_R)} |SXw|^p dx \leq c R^\lambda \left( ||Sg||_{L^p_{X,\lambda}}^p + ||S\tilde{g}||_{L^p_{\lambda}}^p + ||S\tilde{g}||_{L^{p_0,\lambda_0}}^p \right),
\]
\[
\int_{S(B_R)} |SXv|^p dx \leq c \left( \frac{p}{R} \right)^{2Q-p(Q-\mu_1)} \int_{S(B_R)} |SXw|^p dx.
\]
Using \( u = v + w \), it shows
\[
\int_{S(B_R)} |SXu|^p dx \leq c \int_{S(B_R)} |SXv|^p dx + c \int_{S(B_R)} |SXw|^p dx \\
\leq c \left( \frac{p}{R} \right)^{2Q-p(Q-\mu_1)} \int_{S(B_R)} |SXu|^p dx + c \int_{S(B_R)} |SXw|^p dx \\
\leq c \left( \frac{p}{R} \right)^{2Q-p(Q-\mu_1)} \int_{S(B_R)} |SXu|^p dx + c R^\lambda \left( ||Sg||_{L^p_{X,\lambda}}^p + ||S\tilde{g}||_{L^p_{\lambda}}^p + ||S\tilde{g}||_{L^{p_0,\lambda_0}}^p \right).
Taking $H(p) = \int_{S(B_\rho)} |SXu|^p dx$, $H(R) = \int_{S(B_R)} |SXu|^p dx$, $a = \frac{2Q-p(Q-\mu)}{2}$, $b = \lambda$, $B = c \left( \|Sg\|_{L^{p,\lambda}_X} + \|S\tilde{g}\|^p_{L^{p,\lambda}_X} + \|S\tilde{g}\|^p_{L^{p,0,\lambda_0}_X} \right)$. Then there exists $\mu_1$, $\frac{(p-2)Q+2\lambda}{p} < \mu_1 < Q$, such that $a > b$. We have by Lemma 2.8 that

$$\int_{S(B_\rho)} |SXu|^p dx \leq c \left( \frac{B}{R} \right)^\lambda \int_{S(B_R)} |SXu|^p dx + c\rho^\lambda \left( \|Sg\|_{L^{p,\lambda}_X} + \|S\tilde{g}\|^p_{L^{p,\lambda}_X} + \|S\tilde{g}\|^p_{L^{p,0,\lambda_0}_X} \right)$$

and $SXu \in L^{p,\lambda}_X(S(B_\rho), \mathbb{R}^N)$. Hence the result is proved.

**Proof of Theorem 1.3:** By Theorem 1.2, we see

$$\int_{S(B_\rho)} |SXu|^p dx \leq c\rho^\lambda.$$

Since $Q - p < \lambda < Q$, it follows by taking $\kappa = 1 - \frac{Q-\lambda}{p}$ and using Lemma 2.10 that the conclusion is true.

**Remark 5.1** Of course, we can also obtain Hölder regularity by the isomorphic relationship between the Campanato space $L^{p,N}_X(-p < X < 0)$ and the Hölder space $C^{0,\alpha}_X(\alpha = -\frac{X}{p})$ given in [8, Theorem 2.2].

Before proving Theorem 1.4, we first establish the following lemma.

**Lemma 5.1** Let $w \in W^{1,2}_{X,0}(\Omega, \mathbb{R}^N)$ be a weak solution to (5.2), the coefficients $a_{ij}^{\alpha\beta}$ in (5.2) satisfy (H1), $g_i(x,u,Xu)$ and $f_i^\alpha(x,u,Xu)$ satisfy (H3). Then for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right)$, $B_{2R} \subset\subset \Omega$, we have

$$\int_{B_0} |Xw|^p dx \leq c \int_{B_{2R}} |Xu - (Xu)_{B_{2R}}|^p dx + cR^\lambda \left( \|g\|^p_{L^{p,\lambda}_X(\Omega)} + \|\tilde{g}\|^p_{L^{p,\lambda}_X(\Omega)} + \|\tilde{g}\|^p_{L^{p,0,\lambda_0}_X(\Omega)} \right) (5.3)$$

**Proof:** Let us note that $w$ is also a weak solution to the system

$$-X_\alpha^* \left( a_{ij}^{\alpha\beta} X_\beta w^j \right) = g_i - X_\alpha^* \left( f_i^\alpha - (f_i^\alpha)_{B_{2R}} \right).$$

Multiplying both sides of the system by $w$ and integrating on $B_{2R}$,

$$- \int_{B_{2R}} A_{ij}^{\alpha\beta}(x) \delta_{ij} X_\alpha w^i X_\beta w^j dx = \int_{B_{2R}} B_{ij}^{\alpha\beta} X_\alpha w^i X_\beta w^j dx + \int_{B_{2R}} g_i w^i dx - \int_{B_{2R}} f_i^\alpha \left( f_i^\alpha - (f_i^\alpha)_{B_{2R}} \right) X_\alpha w^i dx. \quad (5.4)$$
By (H3), (2.3), Hölder’s inequality and Young’s inequality, it implies

\[
\int_{B_{2R}} |g_t| |w^i| \, dx \leq \int_{S(B_{2R})} (g^i + L|Xu|^\gamma_0) |w^i| \, dx
\]

\[
\leq \left( \int_{B_{2R}} |\tilde{g}|^{\frac{2q}{2q-2}} \, dx \right)^{\frac{2q-2}{2q}} \left( \int_{B_{2R}} |w|^{\frac{2q}{2q-2}} \, dx \right)^{\frac{2q-2}{2q}}
+ c \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |Xu|^2 \, dx \right)^{\frac{\gamma_0}{q}}
+ c \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |Xu|^2 \, dx \right)^{\frac{\gamma_0}{q}}
\]

\[
\leq \left( \int_{B_{2R}} |\tilde{g}|^{2^*} \, dx \right)^{\frac{1}{2}}
+ c \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |Xu|^2 \, dx \right)^{\frac{\gamma_0}{q}}
+ c \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |Xu|^2 \, dx \right)^{\frac{\gamma_0}{q}}
\]

\[
\leq c_\epsilon \left( \int_{B_{2R}} |\tilde{g}|^{2^*} \, dx \right)^{\frac{1}{q}} + c \int_{B_{2R}} |Xu|^2 \, dx + c \int_{B_{2R}} |\tilde{g}|^2 \, dx + c \int_{B_{2R}} |\tilde{g}| \, dx
\]

where \( \theta_3 = \delta \lambda_0 + 2\varepsilon + \varepsilon R^{Q+2-Q\gamma_0} \). By choosing \( \varepsilon \) small enough such that \( \lambda_0 - \theta_3 > 0 \) and applying (3.8), we obtain

\[
\int_{B_{2R}} |Xw|^2 \, dx
\]

\[
\leq c_\epsilon \left( \int_{B_{2R}} |\tilde{g}|^{2^*} \, dx \right)^{\frac{1}{q}} + c \int_{B_{2R}} |Xu|^2 \, dx + c \int_{B_{2R}} |\tilde{g}|^2 \, dx + c \int_{B_{2R}} |\tilde{g}| \, dx
\]

\[
\leq c \left( \int_{B_{2R}} |\tilde{g}|^{2^*} \, dx \right)^{\frac{1}{q}} + c \int_{B_{2R}} |g|^2 \, dx + c \int_{B_{2R}} |\tilde{g}|^2 \, dx + c \int_{B_{2R}} |Xu - (Xu)|_{B_{2R}}^2 \, dx
\]

\[
\leq c R^{\frac{2}{\gamma_0}} |B_{2R}|^{\frac{\gamma_0}{2}} \|g\|^{2}_{L_{\gamma_0}, \lambda_0} + c R^{\frac{2}{p}} |B_{2R}|^{\frac{p}{2}} \left( \|g\|^{2}_{L^{\infty}, \lambda} + \|\tilde{g}\|^{2}_{L^{\infty}, \lambda} \right)
\]

\[
+ c |B_{2R}|^{\frac{p}{2}} \left( \int_{B_{2R}} |Xu - (Xu)|_{B_{2R}}^p \, dx \right)^{\frac{2}{p}}.
\]
From Lemma 3.2, it yields
\[
\int_{B_R} |Xw|^p \, dx \leq c |B_R|^{-\frac{2p}{p-2}} \left( \int_{B_{2R}} |Xw|^2 \, dx \right)^{\frac{p}{2}} + c \int_{B_{2R}} (|g|^p + |\tilde{g}|^p) \, dx + c |B_R|^{-\frac{p-1}{p}} R^p \left( \int_{B_{2R}} |\tilde{g}|^{pp_0} \, dx \right)^{\frac{1}{p_0}}.
\]

Putting (5.5) into it and noting (2.1), we have
\[
\int_{B_R} |Xw|^p \, dx \leq c R^{p+\lambda-2} \|\tilde{g}\|_{L_{X}^{p_0,\lambda_0}}^p + c R^\lambda \left( \|g\|_{L_{X}^{p,\lambda}}^p + \|\tilde{g}\|_{L_{X}^{p,\lambda}}^p \right) + c \int_{B_{2R}} |Xu - (Xu)_{B_{2R}}|^p \, dx
\]
\[
\leq c R^\lambda \left( \|g\|_{L_{X}^{p,\lambda}}^p + \|\tilde{g}\|_{L_{X}^{p,\lambda}}^p \right) + c \int_{B_{2R}} |Xu - (Xu)_{B_{2R}}|^p \, dx.
\]

It completes the proof.

**Proof of Theorem 1.4:** By Lemma 5.1, it follows
\[
\int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c R^\lambda \left( \|Sg\|_{L_{X}^{p,\lambda}}^p + \|S\tilde{g}\|_{L_{X}^{p,\lambda}}^p \right) + c \int_{S(B_{2R})} |SXu - (SXu)_{S(B_{2R})}|^p \, dx.
\]

Using \( u = v + w \), Theorem 4.2 and the above inequality, we have
\[
\int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx
\]
\[
\leq c \int_{S(B_R)} |SXv - (SXv)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw - (SXw)_{S(B_R)}|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXv - (SXv)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{\mu_2} \int_{S(B_R)} |SXu - (SXu)_{S(B_R)}|^p \, dx + c \int_{S(B_R)} |SXw|^p \, dx
\]
Hence

$$SXu \in \mathcal{L}^{p,\lambda}_X(S(B_\rho),\mathbb{R}^N)$$

and the proof is finished.

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