Isometric immersions with flat normal bundle between space forms

MARCOS DAJCZER, CHRISTOS-RAENT ONTI, AND THEODOROS VLACHOS

Abstract. We investigate the behavior of the second fundamental form of an isometric immersion of a space form with negative curvature into a space form so that the extrinsic curvature is negative. If the immersion has flat normal bundle, we prove that its second fundamental form grows exponentially.

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It is a long-standing problem if the complete hyperbolic space $\mathbb{H}^n$ can be isometrically immersed in the Euclidean space $\mathbb{R}^{2n-1}$. In fact, the non-existence of such an immersion has been frequently conjectured; see Yau [13], Moore [11], and Gromov [9]. A positive answer to the conjecture would be a natural generalization to higher dimensions of the classical result from 1901 by Hilbert for the hyperbolic plane. On the one hand, Cartan [4,5] in 1920 proved that $\mathbb{H}^n$, $n \geq 3$, cannot be isometrically immersed in $\mathbb{R}^{2n-2}$ even locally. On the other hand, he showed that any local isometric immersion of $\mathbb{H}^n$ into $\mathbb{R}^{2n-1}$ has flat normal bundle and that there is an abundance of such submanifolds.

Nikolayevsky [12] proved that complete non-simply connected Riemannian manifolds of constant negative sectional curvature cannot be isometrically immersed into Euclidean space with flat normal bundle. Let $Q^m_c$ denote a complete simply connected $m$-dimensional Riemannian manifold of constant sectional curvature $c$, that is, the Euclidean space $\mathbb{R}^m$, the Euclidean sphere $S^m_c$, or the hyperbolic space $\mathbb{H}^m_c$ according to whether $c = 0$, $c > 0$, or $c < 0$, respectively. It was observed in [8] that the proof by Nikolayevsky gives, in fact, the following slightly more general result:

If there exists an isometric immersion $f: M^c_n \to Q^{n+p}_c$, $n \geq 2$ and $c < 0$, with flat normal bundle of a complete Riemannian manifold $M^c_n$ of constant sectional curvature $c$ with $c < \tilde{c}$, then $M^c_n = \mathbb{H}^n_c$. 
In view of Nikolayevsky’s result, the following weaker version of the problem discussed above has already been considered by Brander [3].

**Problem 1.** Do isometric immersions with flat normal bundle of $\mathbb{H}^n_c$ into $Q^{n+p}_\tilde{c}$ for $n \geq 2$ and $c < \tilde{c}$ exist?

In this paper, we analyze the behavior of the second fundamental form of a possible submanifold as in the problem above, and conclude that it must have exponential growth, as defined next.

Let $f : M^n \to Q^{n+p}_\tilde{c}$ be an isometric immersion of a complete non-compact Riemannian manifold $M^n$. It is said that the second fundamental form $\alpha_f : TM \times TM \to N_f M$ of $f$ has **exponential growth** if there exist $x_0 \in M^n$ and positive constants $k, \ell \in \mathbb{R}$ such that

$$\max\{\|\alpha_f(x)\| : x \in D_r(x_0)\} \geq ke^{\ell r}$$

for any $r > r_0$ for some $r_0 > 0$, where $D_r(x_0)$ denotes the closed geodesic ball of $M^n$ of radius $r$ centered at $x_0$ and $\|\alpha_f\|$ is the norm of the second fundamental form given by

$$\|\alpha_f(x)\|^2 = \sum_{i,j} \|\alpha_f(X_i, X_j)(x)\|^2$$

where $X_1, \ldots, X_n \in T_x M$ is an orthonormal basis.

**Theorem 2.** If a complete $n$-dimensional Riemannian manifold $M^n_c$, $n \geq 2$ and $c < 0$, admits an isometric immersion $f : M^n_c \to Q^{n+p}_\tilde{c}$, $c < \tilde{c}$, with flat normal bundle, then $M^n_c = \mathbb{H}^n_c$ and the second fundamental form of $f$ has exponential growth.

The above gives as corollary the result due to Bolotov [2] that there is no isometric immersion of $\mathbb{H}^n_c$ into $\mathbb{R}^{n+p}$ with mean curvature vector field of bounded length.

The conclusion of Theorem 2 does not hold if the assumption of having flat normal bundle is dropped. For instance, it was shown by Aminov [1] that the example constructed by Rozendorn of an isometric immersion of $\mathbb{H}^2_2$ in $\mathbb{R}^5$ has no flat normal bundle and that the norm of its second fundamental form is globally bounded.

The aforementioned result for codimension $p = n - 1$ due to Cartan has the following immediate consequence:

**Corollary 3.** If there exists an isometric immersion $f : \mathbb{H}^n_2 \to Q^{2n-1}_{\tilde{c}}$ with $c < \tilde{c}$, then the second fundamental form of $f$ has exponential growth.

**1. The proof.** Let $f : M^n \to Q^{n+p}_\tilde{c}$ be an isometric immersion of a Riemannian manifold $M^n$ into the space form $Q^{n+p}_\tilde{c}$. If the immersion $f$ has flat normal bundle, that is, if at any point the curvature tensor of the normal connection vanishes, then it is a standard fact (cf. [8]) that at any point $x \in M^n$, there exists a set of unique pairwise distinct normal vectors $\eta_i(x) \in N_f M(x)$, $1 \leq i \leq s(x)$, called the **principal normals** of $f$ at $x$, and an associate orthogonal splitting of the tangent space as

$$T_x M = E_{\eta_1}(x) \oplus \cdots \oplus E_{\eta_{s(x)}}(x),$$

for any $r > r_0$ for some $r_0 > 0$. The above gives as corollary the result due to Bolotov [2] that there is no isometric immersion of $\mathbb{H}^n_2$ into $\mathbb{R}^{n+p}$ with mean curvature vector field of bounded length.

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Proof. The Codazzi equation is equivalent to an open subset $U$ flat normal bundle. Since $C$ Lemma 5.

The following holds: $\alpha_f(X,Y) = \langle X,Y \rangle \eta_i$ for all $Y \in T_x M$. 

If $X_i \in \Gamma(E_{\eta_i})$, $1 \leq i \leq n$, is a unit local vector field, then the local orthonormal frame $X_1, \ldots, X_n$ diagonalizes the second fundamental form of $f$, that is, $\alpha_f(X_i, X_j) = \delta_{ij} \eta_i$, $1 \leq i, j \leq n,$

where $\delta_{ij}$ is the Kronecker delta. Such a frame is called a principal frame.

Lemma 4. The following holds:

$$\nabla_{X_i} X_j = -\lambda_i X_j (1/\lambda_i) X_i, \ 1 \leq i \neq j \leq n,$$

where $\lambda_i = 1/\sqrt{\|\eta_i\|^2 + C}$.

Proof. The Codazzi equation is equivalent to

$$\nabla_{X_j} \eta_i = \langle \nabla X, X_i, X_j \rangle (\eta_i - \eta_j)$$

and

$$\langle \nabla X, X_j, X_i \rangle (\eta_i - \eta_j) = \langle \nabla X, X_j, X_i \rangle (\eta_i - \eta_{\ell})$$

for all $1 \leq i \neq j \neq \ell \neq i \leq n$.

The vectors $\eta_i - \eta_j$ and $\eta_i - \eta_{\ell}$, $1 \leq i \neq j \neq \ell \neq i \leq n$ are linearly independent. Suppose otherwise that $\eta_i - \eta_j = \mu (\eta_i - \eta_{\ell})$. Taking the inner product with $\eta_i$ and using (1) gives $\|\eta_i\|^2 = C < 0$, a contradiction.

It now follows from (4) that

$$\nabla_{X_i} X_j = \Gamma_{ij}^i X_i + \Gamma_{ij}^j X_j, \ i \neq j,$$

where $\Gamma_{ij}^k = \langle \nabla X_i, X_j, X_k \rangle$. Since $\Gamma_{ij}^j = \langle \nabla X_i, X_j, X_j \rangle = 0$, it follows that

$$\nabla_{X_i} X_j = \Gamma_{ij}^i X_i = -\Gamma_{ji}^i X_i.$$

On the other hand, taking the inner product of (3) with $\eta_i$ and using (1), it is easily seen that $\Gamma_{ii}^j = \lambda_i X_j (1/\lambda_i)$, as we wished.

Lemma 5. For each $x_0 \in M^n_{c}$, there exists a diffeomorphism $F: U \to V$ from an open subset $U \subset \mathbb{R}^n$ endowed with coordinates $\{u_1, \ldots, u_n\}$ onto an open neighborhood $V \subset M^n_{c}$ of $x_0$ such that the tangent frame

$$\sqrt{\|\eta_i\|^2 + CF_\ast (\partial/\partial u_1), \ldots, \sqrt{\|\eta_n\|^2 + CF_\ast (\partial/\partial u_n)}}$$

where $E_{\eta_i}(x) = \{ X \in T_x M : \alpha_f(X,Y) = \langle X,Y \rangle \eta_i$ for all $Y \in T_x M \}$. The multiplicity of a principal normal $\eta_i \in N_f M(x)$ of $f$ at $x \in M^n$ is the dimension of the tangent subspace $E_{\eta_i}(x)$. If $s(x) = k$ is constant on $M^n$, then the maps $x \in M^n \mapsto \eta_i(x)$, $1 \leq i \leq k$, are smooth vector fields, called the principal normal vector fields of $f$. Moreover, also the distributions $x \in M^n \mapsto E_{\eta_i}(x)$, $1 \leq i \leq k$, are smooth.

In the sequel, let $f: M^n_{c} \to \mathbb{R}^{n+p}$, $c < \tilde{c}$, be an isometric immersion with flat normal bundle. Since $C = \tilde{c} - c > 0$, it follows from the Gauss equation that any principal normal has multiplicity one. Thus, there exist exactly $n$ non-zero principal normal vector fields $\eta_1, \ldots, \eta_n$ satisfying

$$\langle \eta_i, \eta_j \rangle = C, \ 1 \leq i \neq j \leq n.$$
is orthonormal and principal. Moreover, if \( M^n_c \) is complete and simply connected, then \( F: \mathbb{R}^n \to M^n_c \) is a diffeomorphism.

**Proof.** For the local existence, observe that Lemma 4 implies

\[
\{ \lambda_i X_i, \lambda_j X_j \} = 0, 1 \leq i \neq j \leq n.
\]

For the proof of the global part, we follow a similar argument as in the proof of [10, Theorem 3] or [8, Proposition 5.6]. Assume that \( M^n_c \) is complete and simply connected. Set \( Y_i = \lambda_i X_i \) and let \( \varphi_i(x,t), \ x \in M^n_c, \ t \in \mathbb{R}, \) be the one-parameter group of diffeomorphisms generated by \( Y_i \). Since the vector fields \( Y_i, \ 1 \leq i \leq n, \) have bounded lengths, it follows that \( \varphi_i(x,t) \) is defined for all values of \( x \) and \( t \). Thus, for any \( x \in M^n_c, \) the map \( t \mapsto \varphi_i(x,t) \) is the integral curve of \( Y_i \) with \( \varphi_i(x,0) = x \). Let \( x_0 \) be a fixed point in \( M^n_c \) and define a function \( F = F_{x_0}: \mathbb{R}^n \to M^n_c \) by

\[
F(t_1, t_2, \ldots, t_n) = \varphi_n(\varphi_{n-1}(\cdots \varphi_2(\varphi_1(x_0, t_1), t_2), \cdots), t_n).
\]

Since the Lie bracket \( [Y_1, Y_j] \) vanishes, the parameter groups \( \varphi_i \) and \( \varphi_j \) commute. This implies that

\[
F_{x_0}(t + s) = \varphi_n(\varphi_{n-1}(\cdots \varphi_2(\varphi_1(F_{x_0}(s), t_1), t_2), \cdots), t_n) = F_{F_{x_0}(s)}(t) \quad (6)
\]

where \( t = (t_1, \ldots, t_n) \) and \( s = (s_1, \ldots, s_n) \). Thus

\[
F_{s}(s)\partial_i = \frac{d}{dt} |_{t=0} F(t_1, \ldots, s_i + t, \ldots, s_n) = \frac{d}{dt} |_{t=0} \varphi_i(F(s), t) = Y_i(F(s)).
\]

We claim that \( F \) is a covering map. Then this and that \( M^n_c \) is simply connected yield that \( F \) is a diffeomorphism, which gives the proof.

Given \( x \in M^n_c, \) let \( \tilde{B}_{2\varepsilon}(0) \) be an open ball of radius \( 2\varepsilon \) centered at the origin such that \( F_\varepsilon|_{\tilde{B}_{2\varepsilon}(0)} \) is a diffeomorphism onto \( B_{2\varepsilon}(x) = F_{\varepsilon}(\tilde{B}_{2\varepsilon}(0)) \). Set \( \{ \tilde{x}_\alpha \}_{\alpha \in A} = F^{-1}(x) \) and denote by \( \tilde{B}_{2\varepsilon}(\tilde{x}_\alpha) \) the open ball of radius \( 2\varepsilon \) centered at \( \tilde{x}_\alpha \). Define a map \( \phi_\alpha: B_{2\varepsilon}(x) \to \tilde{B}_{2\varepsilon}(\tilde{x}_\alpha) \) by

\[
\phi_\alpha(y) = \tilde{x}_\alpha + F^{-1}_\varepsilon(y).
\]

From (6), we obtain

\[
F_{x_0}(\phi_\alpha(y)) = F_{x_0}(\tilde{x}_\alpha + x^{-1}_\varepsilon(y)) = F_{F_{x_0}(\tilde{x}_\alpha)}(F^{-1}_\varepsilon(y)) = F_x(F^{-1}_\varepsilon(y)) = y
\]

for all \( y \in B_{2\varepsilon}(x) \). Thus \( F_{x_0} \) is a diffeomorphism from \( \tilde{B}_{2\varepsilon}(\tilde{x}_\alpha) \) onto \( B_{2\varepsilon}(x) \) having \( \phi_\alpha \) as its inverse. In particular, this implies that \( \tilde{B}_{2\varepsilon}(\tilde{x}_\alpha) \) and \( \tilde{B}_{2\varepsilon}(\tilde{x}_\beta) \) are disjoint if \( \alpha, \beta \in A \) are distinct indices. Finally, it remains to check that if \( \tilde{y} \in F^{-1}_{\varepsilon}(B_\varepsilon(x)), \) then \( \tilde{y} \in \tilde{B}_\varepsilon(\tilde{x}_\alpha) \) for some \( \alpha \in A \). This follows from the fact that

\[
F_{x_0}(\tilde{y} - F^{-1}_\varepsilon(F_{x_0}(\tilde{y}))) = F_{F_{x_0}(\tilde{y})}(-F^{-1}_\varepsilon(F_{x_0}(\tilde{y}))) = x.
\]

For the last equality, observe from (6) that for all \( x, y \in M^n_c, \) we have \( F_x(t) = y \) if and only if \( F_y(-t) = x \).

The third fundamental form \( \text{III}_f(x) \) of \( f \) at \( x \in M^n \) is given by

\[
\text{III}_f(X, Y)(x) = \text{tr} (\alpha_f(X, \cdot), \alpha_f(Y, \cdot)), \ X, Y \in T_x M.
\]

Since \( \alpha_f \) has no kernel (that is, positive index of relative nullity), \( \text{III}_f(x) \) is a positive definite inner product.
Lemma 6. The Riemannian metric $g^0 = Cg + III_f$ is flat where $g$ is the metric of $M^n_c$. Moreover, the metric $g^0$ is complete if $g$ is complete.

Proof. In terms of the system of principal coordinates $\{u_1, \ldots, u_n\}$ given by Lemma 5, we have

$$g^0_{ij} = Cg_{ij} + III_f(\partial/\partial u_i, \partial/\partial u_j) = \frac{C}{\|\eta_i\|^2 + C\delta_{ij}} + \frac{\|\eta_i\|^2}{\|\eta_i\|^2 + C}\delta_{ij} = \delta_{ij}.$$ 

Moreover, the metric $g^0$ is complete since $g^0_{ij} > Cg_{ij}$. 

Proof of Theorem 2. By Nikolayevsky’s result, we have that $M^n_c = \mathbb{H}^n_c$. Let $F: \mathbb{R}^n \rightarrow \mathbb{H}^n_c$ be the global diffeomorphism given by Lemma 5. We endow $\mathbb{R}^n$ with the pullbacks of the two metrics considered in Lemma 6 that are still denoted by $g$ and $g^0$. Notice that $(\mathbb{R}^n, g^0)$ is the standard flat Euclidean space.

Given a smooth curve $\gamma: [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^n$, set

$$\hat{S}(\gamma) = \max_{t \in [a, b]} \|\alpha_t\|^2(F(\gamma(t))).$$

We have from (5) that

$$g_{ij} = \frac{1}{\|\eta_i\|^2 + C}\delta_{ij} \geq \frac{1}{\hat{S}(\gamma) + C}\delta_{ij}.$$ 

Then the lengths of $\gamma$ satisfy

$$L_{g^0}(\gamma) < (\hat{S}(\gamma) + C)^{1/2}L_g(\gamma).$$  \hspace{1cm} (7)

Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ and $\tilde{\gamma}: [a, b] \rightarrow \mathbb{R}^n$ be the unique Euclidean and hyperbolic geodesics, respectively, joining $\gamma(a) = \tilde{\gamma}(a)$ to $\gamma(b) = \tilde{\gamma}(b)$. From (7), we have

$$L_{g^0}(\gamma) \leq L_{g^0}(\tilde{\gamma}) < (\hat{S}(\tilde{\gamma}) + C)^{1/2}L_g(\tilde{\gamma}).$$

Thus, if $\gamma_{x,y}$ is the unique hyperbolic geodesic joining $x \neq y \in \mathbb{R}^n$, then the distances with respect to $g^0$ and $g$ satisfy

$$d_{g^0}(x, y) < (\hat{S}(\gamma_{x,y}) + C)^{1/2}d_g(x, y).$$ \hspace{1cm} (8)

Fix $x_0 \in \mathbb{R}^n$ and let $D^g(x_0)$ and $D^0_r(x_0)$ be the closed geodesic balls of radius $r > 0$ centered at $x_0$ with respect to $g$ and $g^0$, respectively.

It holds that

$$D^g_r(x_0) \subset \text{int}\left(D^0_{\psi(r)}(x_0)\right),$$ \hspace{1cm} (9)

where

$$\psi(r) = r(S(r) + C)^{1/2} \quad \text{and} \quad S(r) = \max_{x \in D^g_r(x_0)} \|\alpha_r\|^2(F(x)).$$

In fact, if $y \in D^g_r(x_0)$, we have, using (8), that

$$d_{g^0}(x_0, y) < (\hat{S}(\gamma_{x_0,y}) + C)^{1/2}d_g(x_0, y) \leq r(\hat{S}(\gamma_{x_0,y}) + C)^{1/2} \leq \psi(r).$$

Then we obtain, using (9), that the volumes of the geodesic balls satisfy
\[ \text{Vol}_g(D^g_r(x_0)) \leq \text{Vol}_g(D^g_0(x_0)) \]
\[ = \int_{D^g_0(x_0)} \prod_{i=1}^{n}(\|\eta_i\|^2 + C)^{-1/2} du_1 \wedge \cdots \wedge du_n \]
\[ < \int_{D^g_0(x_0)} C^{-n/2} du_1 \wedge \cdots \wedge du_n = C^{-n/2} \text{Vol}_g(D^g_0(x_0)) \]
\[ = r^n (1 + S(r)/C)^{n/2} \omega_n, \]
where \( \omega_n \) is the volume of the Euclidean unit \( n \)-ball. Since \( \text{Vol}_g(D^g_r(x_0)) \) is well known to grow exponentially with \( r \) (for instance, see [6]), it follows that also \( S(r) \) grows exponentially with \( r \), and thus the second fundamental form of \( f \) has exponential growth. \( \square \)

**Remark 7.** It is worth mentioning that it was shown in [7] that there is no isometric immersion with flat normal bundle of a complete Riemannian manifold \( M^n_c, c > 0, \) into \( \mathbb{Q}^{n+p}_{\tilde{c}} \) with \( c < \tilde{c} \). Notice that this follows using Lemma 5.

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