ON THE TOTAL CURVATURE OF MINIMAL ANNULI IN $\mathbb{R}^3$ AND NITSCHE’S CONJECTURE

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ABSTRACT. In this paper we prove the generalized Nitsche’s conjecture proposed by W. H. Meeks III and H. Rosenberg: For $t \geq 0$, let $P_t$ denote the horizontal plane of height $t$ over the $x_1, x_2$-plane. Suppose that $M \subset \mathbb{R}^3$ is a minimal annulus with $\partial M \subset P_0$ and that $M$ intersects every $P_t$ in a simple closed curve. Then $M$ has finite total curvature. As a consequence, we show that every properly embedded minimal surface of finite topology in $\mathbb{R}^3$ with more than one end has finite total curvature.

1. Introduction.

The geometry and topology of minimal annuli in $\mathbb{R}^3$ has attracted geometer’s attention for a long time. For example, B. Riemann described all minimal annuli in $\mathbb{R}^3$ that are foliated by the parallel circles, in terms of elliptic functions (see[9]); M. Shiffman [10] studied minimal surfaces in $\mathbb{R}^3$ bounded by two parallel convex curves. In 1962, Nitsche [8] proved that a complete minimal annulus in $\mathbb{R}^3$, which intersects every horizontal plane in a star shaped curve, is a catenoid. Then he conjectured that every minimal embedded
annulus in $\mathbb{R}^3$ is a catenoid, provided it intersects with every horizontal plane in a simple closed curve.

In recent years W. H. Meeks III and his co-workers have made much progress in understanding the structure of properly embedded minimal surface in $\mathbb{R}^3$ ([3]-[7] etc.). In particular, it was shown in [5] and [6] that Nitsche’s conjecture is closely related to the finite total curvature conjecture that every properly embedded minimal surface of finite topology in $\mathbb{R}^3$ with at least two ends has finite total curvature (conjecture 1.1 of [6]). In [6], W. H. Meeks III and H. Rosenberg reduced the finite total curvature conjecture to the following generalized Nitsche’s conjecture:

**Conjecture.** For $t \geq 0$, let $P_t$ denote the horizontal plane of height $t$ over the $x_1, x_2$-plane. Suppose that $M \subset \mathbb{R}^3$ is a minimal annulus with $\partial M \subset P_0$ and that $M$ intersects every $P_t$ in a simple closed curve. Then $M$ has finite total curvature.

The aim of this paper is to prove the generalized Nitsche’s conjecture. Our proof is based on the curvature estimate and the careful analysis of geodesic curvature and curvature of each $P_t \cap M$. The fact that each $P_t \cap M$ is an analytic curve is also fundamental in the proof.

2. **Curvature estimate.**

Let $M$ be an immersed annulus in $\mathbb{R}^3$ with the position vector $x = (x_1, x_2, x_3)$. The covariant derivatives of $\mathbb{R}^3$ and $M$ are denoted by $D$ and $\nabla$ respectively. Let $P_t$ be the plane of $x_3 = t$. From now on we suppose $\partial M \subset P_0$ and $M \cap P_t := C_t$ is a single
immersed curve, for every \( t \in (0, +\infty) \). By Sard’s theorem, for almost of all \( t \in (0, +\infty) \), \( \nabla x_3 \) never vanish along \( C_t \), and \( \frac{\nabla x_3}{|\nabla x_3|} \) is the normal vector field of \( C_t \) in \( M \).

**Lemma 1.** For almost of all \( t > 0 \), \( C_t \) is an analytic planner curve, and its geodesic curvature \( \kappa_g \) and curvature \( \kappa \) are both real analytic. Here we emphasize that \( \kappa \) is the curvature of \( C_t \) as a plane curve.

**Proof.** Suppose along \( C_t \), \( \nabla x_3 \neq 0 \). Giving a point \( p \in C_t \), we choose an isothermal coordinate \((u, v)\) in a neighbourhood of \( p \). Then \( x_3 \) is a smooth harmonic and hence real analytic function of \((u, v)\). Without lose of generality, suppose \( \frac{\partial x_3}{\partial u}(p) \neq 0 \). Then the implicit function theorem ([2] Chapter 10) reads that function \( u = u(v) \) determined by \( x_3(u, v) = t \) is a real analytic function. Therefore locally \( x|C_t = x(v) \) is real analytic; and the tangent vector of \( C_t \), say \( \frac{dx}{dv} = x_u \frac{du}{dv} + x_v \), is analytic of \( v \) and does not vanish. It follows easily that \( C_t \)'s geodesic curvature and curvature are both real analytic with respect to \( v \). Thus we complete the proof. \( \square \)

Suppose \( T \) is the unit tangent vector field of \( C_t \), such that \( \{T, \frac{\nabla x_3}{|\nabla x_3|}\} \) preserves the orientation of \( M \). We have

\[
\kappa_g = \langle D_T T, \frac{\nabla x_3}{|\nabla x_3|} \rangle, \tag{1}
\]
\[
\kappa_n = \langle D_T T, N \rangle, \tag{2}
\]

where \( N \) is the unit normal vector field of \( M \) and \( \kappa_n \) is the normal curvature of \( C_t \).

Putting \( \nabla^\perp x_3 = D x_3 - \nabla x_3 \), the projection of constant vector \( D x_3 = e_3 \) onto the
normal direction of $M$, then (1) and (2) yield
\[ |\nabla x_3| \kappa_g + \langle \nabla^\perp x_3, N \rangle \kappa_n = \langle DT, e_3 \rangle \]
\[ = -\langle T, D e_3 \rangle \]
\[ = 0. \]  \hspace{1cm} (3)

Hence we obtain
\[ |\nabla x_3|^2 \kappa_g^2 = \langle \nabla^\perp x_3, N \rangle^2 \kappa_n^2 \]
\[ = (1 - |\nabla x_3|^2) \kappa_n^2. \]  \hspace{1cm} (4)

Combining the fact $\kappa^2 = \kappa_g^2 + \kappa_n^2$ with (4) we find
\[ \kappa_g^2 = (1 - |\nabla x_3|^2) \kappa_n^2. \]  \hspace{1cm} (5)

Let $K$ denote the Gaussian curvature of $M$, we have

**Lemma 2.** If $\nabla x_3 \neq 0$ in $C_t$, then the following property holds:
\[ |\kappa| \leq \frac{\sqrt{-K}}{|\nabla x_3|}. \]  \hspace{1cm} (6)

**Proof.** Since $M$ is minimal immersed, $\sqrt{-K}$ is the absolute value of the principal curvature of $M$. If $\kappa_g = 0$, then $|\kappa| = |\kappa_n| \leq \sqrt{-K}$ and the lemma is valid. If $\kappa_g \neq 0$, we see from (4) and (5) that
\[ |\kappa| = \frac{|\kappa_g|}{\sqrt{1 - |\nabla x_3|^2}} \]
\[ = \frac{|\kappa_n|}{|\nabla x_3|} \]
\[ \leq \frac{\sqrt{-K}}{|\nabla x_3|}. \]  \hspace{1cm} (7)
which completes the proof. □

Although $\kappa$ and $\kappa_g$ might change the sign, by the analyticity we can derive from (5) that

**Lemma 3.** For almost of all $t$, along $C_t$ either $\kappa_g = \kappa \sqrt{1 - |\nabla x_3|^2}$ or $\kappa_g = -\kappa \sqrt{1 - |\nabla x_3|^2}$ holds.

**Proof.** From Lemma 1, $\kappa$ and $\kappa_g$ are both real analytic, Lemma 3 is a direct consequence of (5) and an elementary lemma given in the appendix. □

3. The proof of the generalized Nitsche’s conjecture.

In this section we wish to prove the following theorem:

**Theorem 1.** Let $M$ be an embedded minimal annulus in $\mathbb{R}^3$ such that $\partial M \subset P_0$ and that $M$ intersects every $P_t(t > 0)$ in a simple closed curve. Then $M$ has finite total curvature.

To show the theorem we need

**Lemma 4.** $\int_{C_t} |\nabla x_3|$ is independent of $t$.

**Proof.** Denote $M_t = M \cap \{0 < x_3 < t\}$. Since $x_3$ is a harmonic function, Green’s formula
yields

\[ 0 = \int_{M_t} \Delta x_3 = \int_{C_t} \langle \nabla x_3, \frac{\nabla x_3}{|\nabla x_3|} \rangle - \int_{C_0} \langle \nabla x_3, \frac{\nabla x_3}{|\nabla x_3|} \rangle = \int_{C_t} |\nabla x_3| - \int_{C_0} |\nabla x_3|, \]

this proves the lemma. □

Proof of Theorem 1. Put \( R(t) = \int_{M_t} (-K) \), where \( K \) is the Gaussian curvature of \( M \).

We have by the Gauss-Bonnet theorem

\[ R(t) = \int_{C_t} \kappa_g + \int_{C_0} \kappa_g. \quad (8) \]

Put \( c_0 = \int_{C_0} \kappa_g \). Lemma 3 and (8) lead that for almost of all \( t > 0 \),

\[ R(t) = c_0 \pm \int_{C_t} \kappa \sqrt{1 - |\nabla x_3|^2} \]
\[ = c_0 \pm \int_{C_t} (\sqrt{1 - |\nabla x_3|^2} - 1) \kappa \pm \int_{C_t} \kappa \]
\[ \leq c_0 + 2\pi + \int_{C_t} (1 - \sqrt{1 - |\nabla x_3|^2}) |\kappa|. \quad (9) \]

Here the fact that \( \int_{C_t} \kappa = \pm 2\pi \) is used in the last inequality. Then (9) and Lemma 2 imply

\[ R(t) \leq c_1 + \int_{C_t} \frac{|\nabla x_3|^2}{1 + \sqrt{1 - |\nabla x_3|^2}} |\kappa| \]
\[ \leq c_1 + \int_{C_t} |\nabla x_3| \sqrt{1 - \kappa} \]
\[ \leq c_1 + \int_{C_t} \sqrt{-K} \]
\[ \leq c_1 + \int_{C_t} \sqrt{-K}, \quad (10) \]

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where \( c_1 = c_0 + 2\pi \). By the Schwarz inequality, we derive from (10) that

\[
R(t) \leq c_1 + \left( \int_{C_t} |\nabla x_3| \right)^{\frac{1}{2}} \left( \int_{C_t} \frac{-K}{|\nabla x_3|} \right)^{\frac{1}{2}} = c_1 + c_2 \left( \int_{C_t} \frac{-K}{|\nabla x_3|} \right)^{\frac{1}{2}},
\]

(11)

where \( c_2 \) is another constant because of Lemma 4. The co-area formula([11]) leads

\[ R'(t) = \int_{C_t} \frac{-K}{|\nabla x_3|}, \]

therefore (11) implies

\[ R(t) \leq c_1 + c_2 \sqrt{R'(t)} \]

(12)

for almost of all \( t \in (0, +\infty) \).

Because \( R(t) \) is monotone non-decreasing of \( t \), if there exists a \( t_0 > 0 \) such that \( R(t) > c_1 \) when \( t \geq t_0 \), by (12) we have for almost of all \( t > t_0 \),

\[ (R(t) - c_1)^2 \leq c_2^2 R'(t), \]

or equivalently

\[ \frac{1}{c_2^2} \leq \frac{R'(t)}{(R(t) - c_1)^2}. \]

(13)

Integrating (13) from \( t_0 \) to \( t \), one has

\[
\frac{1}{c_2^2}(t - t_0) \leq \frac{1}{R(t_0) - c_1} - \frac{1}{R(t) - c_1} \leq \frac{1}{R(t_0) - c_1},
\]

which contradicts the fact that \( t \) is unbounded. Thus we have proved the theorem. \( \square \)

We notice that the embeddedness of each \( C_t \) is only employed in estimate (9). Actually we can show the following in the same way.
Theorem 2. Let $M$ be an immersed annulus in $\mathbb{R}^3$. Suppose for every $t > 0$, $M \cap P_t$ is a closed plane curve with finite rotation index, and $\partial M \subset P_0$. Then $M$ is of finite total curvature.

Theorem 1, together with Theorem 1.1 of [6], proves the finite total curvature conjecture:

Theorem 3. Every properly embedded minimal surface of finite topology in $\mathbb{R}^3$ with more than one end has finite total curvature.

4. Appendix.

Lemma 5. Let $f(t)$ and $g(t)$ be two real analytic functions in $(-1, 1)$, and $\lambda(t)$ a non-negative continuous function in $t$. Suppose $f^2 = \lambda g^2$, then in $(-1, 1)$ either $f = \sqrt{\lambda} g$ or $f = -\sqrt{\lambda} g$.

Proof. Suppose $f$ does not vanish identically, since the roots of $f$ is isolated, it suffices to prove the lemma in a neighbourhood of a root of $f$. Without lose of generality, suppose $f(0) = 0$ and $f \neq 0$ otherwise in $(-\epsilon, \epsilon)$. We express $f$ as

$$f(t) = t^n f_1(t), \quad n > 0 \text{ and } f_1(t) \neq 0.$$ 

If $g(t) = t^m g_1(t)$ with $g_1(t) \neq 0$, we see $m \leq n$ and $\lambda(t) = t^{2(n-m)} \frac{f_1^2(t)}{g_1^2(t)}$ is also real analytic in $(-\epsilon, \epsilon)$. Thus we can write $\lambda(t) = t^{2(n-m)} \lambda_1(t)$, $\lambda_1(t) \neq 0$ in $(-\epsilon, \epsilon)$, and $f_1^2(t) = \lambda_1(t) g_1^2(t)$. Therefore either $f_1(t) = \sqrt{\lambda_1(t)} g_1(t)$ or $f_1(t) = -\sqrt{\lambda_1(t)} g_1(t)$ in
(-\epsilon, \epsilon). It follows either \( f = \sqrt{\lambda} g \) or \( f = -\sqrt{\lambda} g \) in \((-\epsilon, \epsilon)\). Which complete the proof of the assertion. \( \Box \)
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