DYNAMICS OF GROUPS OF BIRATIONAL AUTOMORPHISMS OF CUBIC SURFACES AND FATOU/JULIA DECOMPOSITION FOR PAINLEVÉ 6

JULIO REBELO AND ROLAND ROEDER

Abstract. We study the dynamics of the group of holomorphic automorphisms of the affine cubic surfaces

\[ S_{A,B,C,D} = \{ (x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D \}, \]

where \( A, B, C, \) and \( D \) are complex parameters. We focus on a finite index subgroup \( \Gamma_{A,B,C,D} < \text{Aut}(S_{A,B,C,D}) \) whose action not only describes the dynamics of Painlevé 6 differential equations but also arises naturally in the context of character varieties. We define the Julia and Fatou sets of this group action and prove that there is a dense orbit in the Julia set. In order to show that the Julia set is “large” we consider a second dichotomy, between locally discrete and locally non-discrete dynamics. For an open set in parameter space, \( P \subset \mathbb{C}^4 \), we show that there simultaneously exist an open set in \( S_{A,B,C,D} \) on which \( \Gamma_{A,B,C,D} \) acts locally discretely and a second open set in \( S_{A,B,C,D} \) on which \( \Gamma_{A,B,C,D} \) acts locally non-discretely. Their common boundary contains an invariant set \( \mathcal{B}_{A,B,C,D} \) of topological dimension 3. After removing a countable union of real-algebraic hypersurfaces from \( P \) we show that \( \Gamma_{A,B,C,D} \) simultaneously exhibits a non-empty Fatou set and also a Julia set having non-trivial interior. The open set \( P \) contains a natural family of parameters previously studied by Dubrovin-Mazzocco.

The interplay between the Fatou/Julia dichotomy and the locally discrete/non-discrete dichotomy plays a major theme in this paper and seems bound to play an important role in further dynamical studies of holomorphic automorphism groups.

1. Introduction

1.1. Setting. Let \( A, B, C, \) and \( D \) be fixed complex parameters and consider the affine cubic surface

\[ S_{A,B,C,D} = \{ (x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D \}. \]

Every line parallel to the \( x \)-axis intersects the surface \( S_{A,B,C,D} \) at two points (counted with multiplicity) and one can therefore define an involution \( s_x : S_{A,B,C,D} \to S_{A,B,C,D} \) that switches them:

\[ s_x \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x - yz + A \\ y \\ z \end{pmatrix}. \]

Two further involutions \( s_y : S_{A,B,C,D} \to S_{A,B,C,D} \) and \( s_z : S_{A,B,C,D} \to S_{A,B,C,D} \) are defined analogously by means of lines parallel to the \( y \)-axis and \( z \)-axis, respectively.

Consider the group

\[ \Gamma^* \equiv \Gamma^*_{A,B,C,D} = \langle s_x, s_y, s_z \rangle \leq \text{Aut}(S_{A,B,C,D}), \]

where \( \text{Aut}(S_{A,B,C,D}) \) denotes the group of all (algebraic) holomorphic diffeomorphisms of \( S_{A,B,C,D} \). For a generic choice of parameters one has \( \Gamma_{A,B,C,D} = \text{Aut}(S_{A,B,C,D}) \) and, in general, \( \Gamma_{A,B,C,D} \) is a subgroup of \( \text{Aut}(S_{A,B,C,D}) \) of index at most 24.

Consider also the finite index subgroup

\[ \Gamma \equiv \Gamma_{A,B,C,D} = \langle g_x, g_y, g_z \rangle \leq \Gamma^*, \]

Date: September 28, 2021.
where $g_x$, $g_y$, and $g_z$ are defined as follows

\[ g_x = s_z \circ s_y, \quad g_y = s_x \circ s_z, \quad \text{and} \quad g_z = s_y \circ s_x. \]

The dynamics of the action of groups $\Gamma^{*}_{A,B,C,D}$ and $\Gamma_{A,B,C,D}$ on $S_{A,B,C,D}$ and their individual elements have several deep connections, including to the dynamics on character varieties, to the monodromy of the Painlevé 6 differential equation, and to the aperiodic Schrödinger equation. We will elaborate on these connections and previous works in Section 2.

It might come as a bit of a surprise that there are relatively few papers devoted to the dynamics in question. Some basic dynamical considerations can be traced back to works on the Markoff Theorem and on Markoff Surfaces, see [6] and [50]. To the best of our knowledge, deeper investigations on these dynamics were initiated in [25] (see also [26]) where W. Goldman discusses the real trace of these dynamics on the actual real 2-dimensional (singular) surfaces. In terms of the general dynamics on the complex surfaces and with non-real parameters, the papers [35] and [8] show the existence of (individual) maps in $\Gamma$ and $\Gamma^*$ displaying rather interesting dynamics, including uniformly hyperbolic ones [8]. Going back to the full dynamics arising from the group action, the paper [9] by S. Cantat and F. Loray also establishes that, bar the case Picard parameters $(A = B = C = 0 \text{ and } D = 4)$, the action of $\Gamma$ on $S_{A,B,C,D}$ preserves neither an (multi) affine structure nor a (multi) holomorphic foliation. More recently in [30], the authors identify the group of polynomial automorphisms of $\mathbb{C}^n$ preserving the natural $n$-dimensional analog of Markoff polynomial and look for regions where the resulting group action is properly discontinuous.

**Main goal of our work:** Our paper is devoted to questions about the “pointwise dynamics of the whole group”, i.e. to the orbits of individual points, their closures, and more generally to the nature of subsets of the complex surface $S_{A,B,C,D}$ that are invariant under $\Gamma^*$ and $\Gamma$.

One notable case where the pointwise dynamics was considered is the classification of finite orbits under $\Gamma_{A,B,C,D}$ that was done by B. Dubrovin and M. Mazzocco [18] and by O. Lisovyy and Y. Tykhyy [39]. Their works were motivated by the connection with the Painlevé 6 differential equation, because finite orbits under $\Gamma_{A,B,C,D}$ correspond to algebraic solutions of Painlevé 6 (see also [9, Section 4] for a classification of bounded orbits.)

### 1.2. Some “preferred” parameters.

Throughout the paper we will refer to the following specific parameters and parametric families.

**Markoff Parameters:** $(A, B, C, D) = (0, 0, 0, 0)$, as discussed in Section 2.1.

**Picard Parameters:** $(A, B, C, D) = (0, 0, 0, 4)$. For these parameters, Picard proved that the Painlevé equation has explicit first integrals and countably many algebraic solutions. This is related to the curious fact that the action of $\Gamma_{0,0,0,4}$ is semi-conjugate to an action on $(\mathbb{C} \setminus \{0\})^2$ by monomial mappings. In particular, for these parameters everything can be computed rather explicitly.

**Punctured Torus Parameters:** $(A, B, C, D) = (0, 0, 0, D)$ for any $D \in \mathbb{C}$. These parameters correspond to dynamics on the character variety of the once punctured torus, as discussed in Section 2.3. For real $D$ and the corresponding real surfaces, this is the family studied by Goldman [26].

**Dubrovin-Mazzocco Parameters:** This is a real 1-parameter family studied by Dubrovin and Mazzocco [18] which seems to play a significant role in several problems related to Mathematical-Physics and, in particular, on the study of Frobenius manifolds. In our notations, the Dubrovin-Mazzocco parameters correspond to

\[ (2) \quad A(a) = B(a) = C(a) = 2a + 4, \quad \text{and} \quad D(a) = -(a^2 + 8a + 8) \]

for $a \in (-2, 2)$. 

Notice that both the Markoff and Picard parameters are included within the Punctured Torus Parameters. Meanwhile, the Picard parameters are in the closure of the Dubrovin-Mazzocco parameters, corresponding to \( a = -2 \), but the Markoff parameters are not.

1.3. **Two relevant dynamical dichotomies.** With the goal of producing interesting invariant sets and finding points with complicated orbit closures we introduce two dynamically invariant dichotomies.

Whereas the action of \( \Gamma \) (or of \( \Gamma^* \)) is genuinely non-linear, i.e., it cannot be embedded into the action of some finite dimensional topological group, it still appears to share some basic properties/issues with actions of countable subgroups of finite dimensional Lie groups. This typically happens on some (proper) open subsets of the surface \( S_{A,B,C,D} \) and, for this reason, the notions of locally discrete vs. locally non-discrete dynamics of \( \Gamma \) will come in handy. Meanwhile, to deal with the non-linear nature of the global dynamics we will adapt the Fatou/Julia theory to the group \( \Gamma \).

The core of this paper lies in the interplay between these four notions. It is probably fair to say that these group actions provide a setting where characteristics of linear and of non-linear dynamics nicely blend together and this very phenomenon lends further interest to their study.

1.4. **Locally non-discrete/discrete dichotomy.** Let \( M \) be a (possibly open) connected complex manifold and consider a group \( G \) of holomorphic diffeomorphisms of \( M \). The group \( G \) is said to be locally non-discrete on an open set \( U \subset M \) if there is a sequence of maps \( \{f_n\}_{n=0}^\infty \in G \) satisfying the following conditions (see for example [49]):

1. For every \( n \), \( f_n \) is different from the identity.
2. The sequence of maps \( f_n \) converges uniformly to the identity on compact subsets of \( U \).

If there is no such sequence \( f_n \) on \( U \) we say that \( G \) is locally discrete on \( U \).

Remark that for an action by a finite dimensional Lie group, local non-discreteness on some open set implies that the corresponding sequence of elements converges to the identity on all of \( M \), i.e. that the action is globally non-discrete. However, in our context the non-linearity of the mappings allow for local non-discreteness to occur on a proper open subset \( U \subset M \).

For any choice of parameters \((A, B, C, D)\) let

\[ N_{A,B,C,D} = \{ p \in S_{A,B,C,D} : \Gamma_{A,B,C,D} \text{ is locally non-discrete on an open neighborhood } U \text{ of } p \}, \]

and let

\[ D_{A,B,C,D} = S_{A,B,C,D} \setminus N_{A,B,C,D}. \]

We will refer to \( N_{A,B,C,D} \) as the “locally non-discrete locus” and to \( D_{A,B,C,D} \) as the “locally discrete locus”. By definition, \( N_{A,B,C,D} \) is open, \( D_{A,B,C,D} \) is closed, and both of them are invariant under \( \Gamma_{A,B,C,D} \).

Remark that \( N_{A,B,C,D} \) can be empty for certain parameter values; indeed it is for the Picard Parameters (Theorem D(ii), below). We do not know if \( D_{A,B,C,D} \) can be empty for any parameter values.

1.5. **Fatou/Julia dichotomy.** For any point \( p \in S_{A,B,C,D} \) we denote the orbit of \( p \) under \( \Gamma \) by

\[ \Gamma(p) = \{ \gamma(p) : \gamma \in \Gamma \}. \]

The Fatou set of the group action \( \Gamma \) is defined as

\[ \mathcal{F}_{A,B,C,D} = \{ p \in S_{A,B,C,D} : \Gamma \text{ forms a normal family in some open neighborhood of } p \}. \]

Naturally the condition of being a normal family means that every sequence of maps as indicated must have a convergent subsequence (for the topology of uniform convergence on compact subsets – compact-open topology). However, since \( S_{A,B,C,D} \) is open, sequences of maps avoiding compact sets are expected to arise as well. It is then convenient to make the notion of converging subsequence accurate by means of the following definition: a sequence of maps (diffeomorphisms onto their
images) \( f \) is said to converge to infinity on an open set \( U \subset S_{A,B,C,D} \) if for every compact set \( V \subset U \) and every compact set \( K \subset S_{A,B,C,D} \), there are only finitely many maps \( f_i \) such that \( f_i(V) \cap K \neq \emptyset \). Sequences converging to infinity are to be included in the definition of normal family used above. In particular, if the sequence formed by all elements of \( \Gamma \) converge to infinity on some open set \( U \subset S_{A,B,C,D} \), then \( U \) is contained in the Fatou set of \( \Gamma \).

The Julia set of the group action \( \Gamma \) is defined as

\[ J_{A,B,C,D} = S_{A,B,C,D} \setminus F_{A,B,C,D}. \]

It follows from the definitions that \( F_{A,B,C,D} \) is open while \( J_{A,B,C,D} \) is closed. Furthermore both sets are invariant under \( \Gamma \). (For some parameters \( S_{A,B,C,D} \) may be singular, but this is not an issue: we will see in Remark 3.2 that such singular points are always in \( J_{A,B,C,D} \).)

It is worth to emphasize that the Julia set is non-empty for every choice of parameters; see, for example, Lemma 4.3. However, the Fatou set can be empty for some parameter values; indeed this happens for the Picard Parameters (Theorem D(i), below).

1.6. Main results. Classical results from the holomorphic dynamics of rational maps of the Riemann sphere assert that there is a dense orbit in the Julia set (topological transitivity) and that repelling periodic points are dense in the Julia set. We search for analogous statements for the action of \( \Gamma_{A,B,C,D} \) on \( S_{A,B,C,D} \).

**Theorem A.** For any parameters \((A, B, C, D)\) there is a point \( p \in J_{A,B,C,D} \) such that

\[ \Gamma(p) = J_{A,B,C,D}, \]

i.e., there is a dense orbit of \( \Gamma \) in \( J_{A,B,C,D} \).

In our setting of group actions, the natural analog of having a dense set of repelling periodic points consists of looking for a dense set \( J^*_{A,B,C,D} \subset J_{A,B,C,D} \) of points whose stabilizers contain a hyperbolic element. A negative answer to this question is provided by Theorem D, below, which states, in particular, that this is not the case for the Picard Parameters \((0, 0, 0, 4)\).

Whereas we leave it as an open question to characterize for which parameters \((A, B, C, D)\) the set \( J^*_{A,B,C,D} \) is dense in \( J_{A,B,C,D} \), we are still able to provide an affirmative answer to this type of question up to replacing “hyperbolic derivative” by derivative conjugate to a “shear map”. This is the content of Theorem B below.

**Theorem B.** For any choice of parameters \((A, B, C, D)\) there is a dense set \( J^#_{A,B,C,D} \subset J_{A,B,C,D} \)

such that for every \( p \in J^#_{A,B,C,D} \) there exists \( \gamma \in \Gamma \) such that \( \gamma(p) = p \) and

\[ D\gamma(p) \text{ is conjugate to } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \]

Often parametric families of rational maps of the Riemann sphere have some mappings with connected Julia set and other mappings with disconnected Julia set. In our context we have a slightly surprising general topological property of Julia sets, namely:

**Theorem C.** For any parameters \( A, B, C, D \) the Julia set \( J_{A,B,C,D} \) is connected.

Let us now transition from results that hold for all parameters to results that only hold for certain parameters.

**Theorem D.** For the Picard Parameters \((A, B, C, D) = (0, 0, 0, 4)\) we have:

(i) \( J_{0,0,0,4} = S_{0,0,0,4} \) and consequently \( F_{0,0,0,4} = \emptyset \).

(ii) The action of \( \Gamma_{0,0,0,4} \) is locally discrete on any open subset of \( S_{0,0,0,4} \), and

(iii) The closure of the set of points \( J^*_{0,0,0,4} \) that have hyperbolic stabilizers is contained in \( S_{0,0,0,4} \cap [-2,2]^3 \) and hence is a proper subset of \( J_{0,0,0,4} = S_{0,0,0,4} \).
In contrast to the Picard parameters, where the corresponding Fatou set is empty, techniques from the work of Bowditch [6] and Hu-Tan-Zhang [29], [30] can be adapted to prove that there are several parameters for which the Fatou set is non-empty.

**Theorem E.** We have the following:

1. **Punctured Torus Parameters:** For any complex $D$ not equal to $4$, the Fatou set $F_{0,0,0,D}$ is non-empty.
2. **Dubrovin-Mazzocco Parameters:** For any $a \in (-2, 2)$ the Fatou set $F_{A(a),B(a),C(a),D(a)}$ is non-empty.

Moreover, the result carries over to an open neighborhood in $C^4$ of any such parameter.

**Remark.** For certain values of the parameters, Teichmüller theory can be applied to show that the Fatou set is non-empty, see Section 2. However, the proof of Theorem E provided here is totally elementary and also applies to parameter values beyond those covered by Teichmüller theory, such as the Dubrovin-Mazzocco parameters.

Because the Fatou set can be non-empty it is interesting to determine how “large” the Julia set is, especially if one wishes to apply Theorems A and B. For this reason we study the interplay of the Fatou/Julia dichotomy with the locally non-discrete/discrete dichotomy. We will first discuss the locally non-discrete/discrete dichotomy and then relate it to the Fatou/Julia dichotomy. What is more interesting is that the two phenomena can occur simultaneously.

The construction of the unbounded Fatou component $V_\infty$ in Theorem E, based on ideas of Bowditch [6] and Hu-Tan-Zhang [30], is such that the resulting action of $\Gamma_{A,B,C,D}$ on $V_\infty$ is locally discrete and, in fact, properly discontinuous (see Theorem F below). Meanwhile, in Section 7 it will be shown the existence of a very large set of parameters $(A, B, C, D)$ for which $\Gamma_{A,B,C,D}$ is locally non-discrete on some open subset of the surface $S_{A,B,C,D}$.

**Theorem F.** There is an open neighborhood $\mathcal{P} \subset C^4$ of the Markoff Parameters $(0, 0, 0, 0)$ and of each of the Dubrovin-Mazzocco Parameters $(A(a), B(a), C(a), D(a))$, where $a \in (-2, 2)$, with the following property.

For any $(A, B, C, D) \in \mathcal{P}$ there are disjoint opens sets $U, V_\infty \subset S_{A,B,C,D}$ such that:

1. The action of $\Gamma_{A,B,C,D}$ is locally non-discrete on $U$; i.e. $U \subset N_{A,B,C,D}$.
2. The action of $\Gamma_{A,B,C,D}$ is locally discrete on any open neighborhood of any point from $V_\infty$, i.e. $V_\infty \subset D_{A,B,C,D}$. Indeed, the action of $\Gamma_{A,B,C,D}$ on $V_\infty$ is properly discontinuous.

Moreover, there are non-commuting pairs of element of $\Gamma_{A,B,C,D}$ arbitrarily close to the identity on $U$ and $V_\infty$ is the Fatou component obtained in Theorem E.

Theorem F contrasts with the behavior of the much studied locally discrete and locally non-discrete subgroups of circle diffeomorphisms, see Section 1.8 for further detail. In a different direction, Theorem F also provides:

**Corollary to Theorem F.** For every choice of parameters $(A, B, C, D) \in \mathcal{P}$ there there is a set $\mathcal{B}_{A,B,C,D} \subset \partial N_{A,B,C,D} = \partial D_{A,B,C,D}$ that has topological dimension equal to three and is invariant under $\Gamma_{A,B,C,D}$.

**Remark.** As in the case of Theorem E, Teichmüller theory can also be used to establish the existence of a set $\mathcal{B}_{A,B,C,D}$ as in Corollary F for certain values of the parameters. In fact, the Fatou components arising from Teichmüller theory in connection with Theorem E are not dense and their boundaries provide the desired invariant sets $\mathcal{B}_{A,B,C,D}$ of topological dimension 3. In addition, in these cases, $\mathcal{B}_{A,B,C,D}$ stems from the boundary of Bers component in suitable character varieties and hence is very irregular (for further information see Section 12). However, once again, some of the indicated parameters for which both Theorem F and Corollary F hold lie beyond the set covered by Teichmüller theory, cf. Dubrovin-Mazzocco parameters and also Remark 2.1.
With the notation of Theorem F, we expect that for each \((A, B, C, D) \in \mathcal{P}\) we have \(U \subset J_{A,B,C,D}\).

However, at present, we can only prove this under one further (weak) assumption, namely:

\((P)\) any fixed point of any \(\gamma \in \Gamma_{A,B,C,D} \setminus \{id\}\) is in \(J_{A,B,C,D}\).

For this reason, we prove in Proposition \(9.9\) that there is a countable union of real-algebraic hypersurfaces \(\mathcal{H} \subset \mathbb{C}^4\) such that this Property \(P\) holds if \((A, B, C, D) \in \mathbb{C}^4 \setminus \mathcal{H}\).

**Theorem G.** Let \(\mathcal{P} \subset \mathbb{C}^4\) be the open neighborhood of the Markoff Parameters and of the Dubrovin-Mazzocco parameters, \(a \in (-2, 2)\), given in Theorem F. For any \((A, B, C, D) \in \mathcal{P} \setminus \mathcal{H}\) we have:

\[ U \subset J_{A,B,C,D} \quad \text{and} \quad V_\infty \subset F_{A,B,C,D}. \]

Here, \(U\) and \(V_\infty\) are the open subsets from the statement of Theorem F.

**Corollary to Theorem G.** For any \((A, B, C, D) \in \mathcal{P} \setminus \mathcal{H}\) there is a point \(p \in U \subset J_{A,B,C,D}\) such that

\[ U \subset \overline{\Gamma(p)}, \]

i.e. the orbit of \(p\) has closure of real dimension four.

Before continuing the discussion, and motivated by the preceding theorems, it seems appropriate to emphasize a few issues – some already briefly mentioned – about the dynamics of \(\Gamma\) and its meaning for the Painlevé equation \(P_6\).

- The simultaneous appearance of locally discrete and locally non-discrete dynamics for a given group \(\Gamma_{A,B,C,D}\) is a manifestation of the fact that the action of \(\Gamma_{A,B,C,D}\) is genuinely non-linear. More precisely, this action is fundamentally non-locally compact. This explains our earlier claim that the action of \(\Gamma\) cannot be embedded into the action of finite-dimensional topological group.

- Another interesting consequence of Theorem F is the invariant set \(B_{A,B,C,D} \subset S_{A,B,C,D}\) of topological dimension 3 that “persists” over the open subset of parameters \(\mathcal{P} \subset \mathbb{C}^4\).

The existence of persistent invariant sets of topological dimension 3 for the action of a large group (\(\Gamma\) is free on two generators) strongly hints at a fractal nature for the sets in question, cf. Problem 5 at the end of the paper. There is also numerical evidence to support this idea.

- A significant part of the vast literature on \(P_6\) is devoted to questions involving various notions of irreducibility, integrability, along with the study/identification of “special solutions”, see for example [11], [9], [13], and their references. In a sense, Theorems F and G partially explains why this topic is so fertile. In slightly loose terms, the dynamics on Fatou components is “simple” and there the equation displays characteristics of integrable systems. For example, solutions wander towards infinity with no recurrence or non-trivial accumulation points and, in particular, are amenable to asymptotic analysis. On the other hand, the dynamics in the Julia sets is far more complicated (Theorem A and B) and this Julia set can be rather large (Theorem G). For example, owing to Theorem A any real-valued continuous first integral for \(P_6\) has to be constant over the Julia set and hence on non-empty open sets, at least for non-negligible parts of the parameter space. See Section 2.5 for a further discussion.

1.7. **Strategy for Proof of Theorem G.** Let us briefly describe the strategy for proving Theorem G, which is arguably the most elaborate result in our paper. It is straightforward to prove that any Fatou component \(V\) is Kobayashi hyperbolic. In particular, if \(\Gamma\) is locally non-discrete on any open subset of \(V\) then it is locally non-discrete on all of \(V\). The idea is then to show that a region \(U\) where \(\Gamma\) induces a “complicated enough” locally non-discrete dynamics is not compatible with the structure of a (Kobayashi hyperbolic) Fatou set. This region must hence be contained in
\(J_{A,B,C,D}\) and this yields Julia sets with non-empty interior. In order to rule out the possibility that this region \(U\) intersects an unbounded Fatou component, the following theorem will be needed.

**Theorem H.** Suppose that for some parameters \(A,B,C\) there is a point \(p \in \mathbb{C}^3\) and \(\epsilon > 0\) such that for any two vertices \(v_i \neq v_j \in V_\infty, i \neq j\), there is a hyperbolic element \(\gamma_{i,j} \in \Gamma_{A,B,C}\) satisfying:

(A) \(\text{Ind}(\gamma_{i,j}) = v_i\) and \(\text{Attr}(\gamma_{i,j}) = v_j\), and

(B) \(\sup_{z \in B_\epsilon(p)} \|\gamma_{i,j}(z) - z\| < K(\epsilon)\).

Then, for any \(D\), we have that \(B_\epsilon/2(p) \cap S_{A,B,C,D}\) is disjoint from any unbounded Fatou components of \(\Gamma_{A,B,C,D}\). Here, \(K(\epsilon) > 0\) denote the constant given in Proposition 7.1.

We refer the reader to Proposition 8.2 for the definition of hyperbolic element \(\gamma\) along with the corresponding points \(\text{Ind}(\gamma)\) and \(\text{Attr}(\gamma)\). Hypothesis (A) requires that the six elements \(\gamma_{i,j}\) have sufficiently rich “combinatorial behavior” at infinity while Hypothesis (B) requires that these six elements are sufficiently close to the identity on the ball \(B_\epsilon(p)\). Note that the conditions of Theorem H are explicit and easy to check. In particular, for any \((A,B,C,D) \in \mathcal{P}\) they imply that \(U\) is disjoint from any unbounded Fatou component.

The idea of the proof of Theorem H is that if \(B_\epsilon(p)\) were in an unbounded Fatou component \(V\) then we use local non-discreteness to produce a sequence of elements converging uniformly on compact subsets of \(V\) to the identity and we use the prescribed combinatorial behavior at infinity to show that this same sequence of elements sends compact subsets of \(V\) uniformly to infinity.

Having ruled out the possibility that an unbounded Fatou component intersects \(U\) we then use the following theorem to prove that no bounded Fatou component intersects \(U\).

**Theorem K.** Suppose that \((A,B,C,D) \in \mathbb{C}^4 \setminus \mathcal{H}\) and that \(V\) is a bounded Fatou component for \(\Gamma_{A,B,C,D}\). Then the stabilizer \(\Gamma_V\) of \(V\) is cyclic.

It is a standard result that the group \(\text{Aut}(V)\) of holomorphic automorphisms of a Kobayashi hyperbolic manifold \(V\) is a real Lie group. To prove Theorem K we use that \(\Gamma_{A,B,C,D}\) preserves a volume form (see Section 3.3) so that the dynamics of \(\Gamma_{A,B,C,D}\) give that \(\Gamma_V\) is a compact Lie group. Checking that any element of \(\Gamma_V\) has infinite order we conclude that \(\Gamma_V\) has positive dimension. Suppose that \(\Gamma_V\) is non-cyclic we can conclude that it is non-Abelian and moreover that there are non-commuting elements arbitrarily close to the identity. This gives that the connected component of the identity, \(G_0\), is non-Abelian. Since \(G_0\) is compact, it must therefore have real-dimension 3. The assumption that no element of \(\Gamma\) has a fixed point in \(V\) gives that \(G_0\) acts freely on \(V\) and thus that \(V/G\) is a manifold of real dimension 1. This allows us to derive a contradiction to the fact that the Julia set is connected (Theorem C).

1.8. Further comments on the locally non-discrete/discrete dichotomy. It is convenient to point out that the notion of locally non-discrete group (or even pseudogroup) first appeared in [47] after previous works by A. Shcherbakov, I. Nakai, and E. Ghys, see [52] [44] [24]. The notion was further elaborated in [40]. The main common tool of [47] and [40] is the construction of certain vector fields obtained as a sort of limit of certain dynamics in the group/pseudogroup in question which allow for a detailed analysis of the corresponding group dynamics. In turn, the method put forward in those papers ensures the existence of the desired vector fields for locally non-discrete (pseudo-) groups provided there is a local expansion in the dynamics: typically, we would like some element in group to have a fixed point where all its eigenvalues are of modulus greater than 1. This issue poses major difficulties to extend these methods to situations where the dynamics does not include a local expansion, as it happens for example in [49]. In particular, the issue about local expansion becomes particularly challenging when the dynamics of \(\Gamma\) (or \(\Gamma^*\)) is discussed. Indeed, whereas there are large sets of parameters for which the action of \(\Gamma\) is locally non-discrete, this action always preserves a volume-form on \(S_{A,B,C,D}\) (cf. Section 3.3) so that the dynamics of elements in \(\Gamma\) never includes an expansion. Clearly, this prevents us from using the mentioned
It should be pointed out that the structure of locally non-discrete groups of diffeomorphisms of the circle is rather well developed, see for example \cite{20}, \cite{1}, \cite{16}, and the references therein. However, in that case the relatively simple nature of the topological dynamics of groups acting on (real) 1-dimensional manifolds leads to the fact that coexistence of local discreteness and local non-discreteness is impossible. More precisely, in $\text{Diff}^\omega(S^1)$, if a non-Abelian group $G \subset \text{Diff}^\omega(S^1)$ is locally non-discrete on an interval $I \subset S^1$, then every point $p \in S^1$ has a neighborhood where $G$ is locally non-discrete, see \cite{20}. This dramatically contrasts with our Theorem F which asserts that the same group action $\Gamma_{A,B,C,D}$ can simultaneously have open sets $U$ and $V$ on which it acts locally non-discretely and locally discretely, respectively. When combined with the complicated behavior of the pointwise dynamics of $\Gamma_{A,B,C,D}$, in particular the existence of invariant sets with topological dimension 3 (Corollary of Theorem F), we find very rich dynamics in the system that we study.

Let us also mention that V. Kleptsyn and his collaborators have found examples of locally discrete subgroups of $\text{Diff}^\omega(S^1)$ that are not conjugate to Fuchsian groups, up to finite covering \cite{1}. In our context, the Picard parameters provide an analogous non-trivial example of a group that is “purely locally discrete”, i.e., there is no open set $U \subset S_{A,B,C,D}$, $U \neq \emptyset$, where the group acts in a locally non-discrete way.

Finally, another consequence of our results that is worth pointing out is the simultaneous presence of locally discrete and locally non-discrete behavior in our group actions, in contrast with the above mentioned case of (non-abelian) groups acting on the circle. Let us further clarify the interest of this observation since on manifolds of real dimension at least 2 it is easy to find examples of dynamical systems exhibiting at the same time local discrete and local non-discrete behaviors. One of the simplest possible examples, somehow in line with our work, consists of a rational fraction with a rotation Fatou component (Siegel disc or Herman ring). The dynamics within the Fatou component is clearly locally non-discrete whereas it is locally discrete on the Julia set. It is, in fact, a characteristic of this and other similar elementary examples that the local non-discrete behavior appears in the regions where the dynamics is “tame” whereas in truly chaotic sets the dynamical behavior is of locally discrete nature. What makes our examples interesting is thus not only the co-existence of locally discrete and locally non-discrete behavior but also the fact that the locally non-discrete behavior lies in the chaotic part of the dynamics whereas in its “tame” part the group is locally discrete.

1.9. Structure of the paper. We present in Section 2 a discussion of the motivations for a dynamical study of $\Gamma_{A,B,C,D}$, with emphasis on the connections to dynamics on character varieties and to the Painlevé 6 differential equation. In Section 3 we present several basic properties of the group $\Gamma \equiv \Gamma_{A,B,C,D}$ and of the surface $S \equiv S_{A,B,C,D}$ that will be used throughout the paper. In Section 4 we study properties of the “parabolic” elements of $\Gamma$ and use them and Montel’s Theorem to prove Theorems A, B, and C. Section 5 is devoted to a careful study of the Picard parameters $(A,B,C,D) = (0,0,0,4)$ and a proof of Theorem D. In Section 6 we prove Theorem E about existence of the unbounded Fatou components, following the techniques from \cite{6,30}. In Section 7 we produce examples of locally non-discrete dynamics and that will be needed for Theorem F. In Section 8 we collect several important properties of how $\Gamma$ acts near infinity that will be needed later in the paper, including a proof of Proposition 9.9, which plays a key role in Theorem G. Section 9 is devoted to a proof of Theorem H about unbounded Fatou components and Theorem K about bounded Fatou components. We finish the proofs of our theorems with Section 11 where we prove Theorems F and G. There are several interesting open questions and we list some of them in Section 12 at the end of this paper.

Acknowledgments: We are grateful to Michał Misiurewicz for ideas which helped with the proof of Proposition 5.8. We also thank Serge Cantat for interesting comments on a previous
version of this work, including the role of Teichmüller theory. Thanks are also in order to Romain Dujardin for an interesting discussion about the connections between [10] and the present work. The second author is grateful to his colleagues Pavel Bleher, Alexander Its, Eugene Mukhin, Vitaly Tarasov, and Maxim Yattselev for their feedback on several early versions of this work. This work was supported by CNRS, through IEA “Dynamics for groups of birational maps acting on surfaces”, and by the US National Science Foundation through grant DMS-1348589.

2. Connections with Dynamics on Character Varieties and the Painlevé 6 equation and relations with other works.

2.1. Motivations from arithmetic on the Markoff surface and from affine automorphisms of cubic surfaces. The case \( A = B = C = 0 \) was among the first to attract attention since it is immediately related to the Markoff surface \( (D = 0) \). According to a classical result of Markoff, the integral solutions of the corresponding equation are obtained as the orbit of the fundamental solution \((3, 3, 3)\) by \( \Gamma^* \). In this sense, the Markoff theorem hints at the interest of these dynamics. For further details, we refer to [6], [30], and to the nice survey [50].

2.2. Motivation from algebraic geometry. Another interesting motivation comes from a kind of “universal property” of the family of cubic surfaces \( S_{A,B,C,D} \) with parameters \( A, B, C, \) and \( D \). All of these affine surfaces intersect the plane at infinity in three lines pairwise intersecting in three points; see Section 3 for details. Conversely, given a cubic surface \( S \) in \( \mathbb{CP}^3 \), recall that a tritangent plane is nothing but a plane intersecting \( S \) in three lines. A tritangent plane is said to be generic if the three lines are distinct and pairwise intersecting in three points. It was pointed out by Goldman and Toledo [27] that every projective cubic surface admitting a generic tritangent plane is projectively equivalent to a surface in the family \( S_{A,B,C,D} \). In particular, every smooth projective cubic surface is equivalent to a member of this family. In addition, for the affine cubic surfaces in our family (or equivalently resulting from removing a generic tritangent plane from a projective one), the group of (algebraic) affine automorphisms was computed by El-Huiti [19] and it contains the group \( \Gamma^* \) generated by the involutions \( s_x, s_y, \) and \( s_z \) as a subgroup of index at most 24 (with equality for generic parameters \( A, B, C, D \)). Thus, in a sense, the work conducted in this paper fits in the tradition of classical algebraic geometry since we are studying the dynamics of the automorphism groups of “general” affine cubic surfaces.

2.3. Motivations from Dynamics on Character Varieties. Arguably, however, the most compelling motivations for studying the dynamical systems considered here comes from dynamics in character varieties and from the sixth Painlevé equation: the two motivations being closely related as will soon be seen. We begin with character varieties. In fact, we will see that our dynamical systems is equivalent to the action of the mapping class group of a surface (the quadruple punctured sphere in the most case general case regarded here) on the space of \( \text{SL}(2, \mathbb{C}) \)-representations of its fundamental group, up to conjugation. This contains, in particular, the standard action of mapping class groups on Teichmüller spaces and hence can be viewed as belonging to the setting of “higher Teichmüller theory” as well; see, e.g., [53]. Interestingly enough, the fact that the dynamics arising from this action is studied here by means of complex dynamics techniques joins a trend from [8], [14], [15] that are among the very first papers to use methods of complex dynamics to get insight into certain natural dynamical systems.

The fact that the dynamics we are interested in can be cast in the broader framework of (natural) dynamics on character varieties was put forward in W. Goldman’s papers [26] and [25]. Let us briefly explain how character varieties arise in this context. Consider, for example, the case of a punctured torus along with representations of its fundamental group \( \Pi \) in \( \text{SL}(2, \mathbb{C}) \) up to conjugation by inner automorphisms. More precisely, note that \( \text{SL}(2, \mathbb{C}) \) acts in the space of all representations of \( \Pi \) in \( \text{SL}(2, \mathbb{C}) \) by the following rule: given \( g \in \text{SL}(2, \mathbb{C}) \) and a representation \( \sigma : \Pi \to \text{SL}(2, \mathbb{C}) \) the action of \( g \) on \( \sigma \) is nothing but the representation that assigns to each \( c \in \Pi \) the matrix \( g \sigma(c) g^{-1} \)
of SL(2, C). The corresponding character variety is exactly the (categorical) quotient of the space of representations by this action of SL(2, C).

Next, note that automorphism group of Π naturally acts on the space of representations by pre-composition. In turn, this action induces an action on character varieties. By construction, the latter action is such that the action of an inner automorphism of Π on the character variety is trivial. Thus the action of Aut(Π) on the character variety factors through an action of the group Out(Π) consisting of outer automorphisms of Π.

Whereas the previous construction essentially makes sense for representations of Π in any group G, the fact that we are dealing with G = SL(2, C) can further be exploited as follows. Since Π is free on two generators, Π ≅ F(a, b), a representation σ : Π → SL(2, C) is determined by its values on a and b. Therefore, the space of representations is identified with the product SL(2, C) × SL(2, C). In turn, the resulting character variety becomes identified with the quotient of SL(2, C) × SL(2, C) by the action of SL(2, C) defined by simultaneous conjugation. According to a classical result of Fricke, C³ parameterizes the latter space by associating to an equivalence class [ρ] in the mentioned quotient the point of C³ whose coordinates are, respectively, the traces of ρ(a), ρ(b), and ρ(ab). Thus, in these coordinates, Out(Π) acts on C³. Moreover, this action coincides with the action on C³ of the group Γ∗ obtained by setting A = B = C = 0. Incidentally, the latter group coincides with the group of polynomial automorphisms of C³ preserving the (Markoff) polynomial

\[x^2 + y^2 + z^2 - xyz - 2,\]

Cf. [28] and see [46] for a general result on the polynomial nature of similar actions. Finally, note that an analogous construction applies to the triply-punctured sphere as well as to the quadruply-punctured sphere. In the latter case, the corresponding action of Out(Π) essentially recovers the action of Γ and Γ∗ with all the parameters A, B, C, and D, see [2] for a detailed account. Other than [2], more or less comprehensive versions of the previous discussion appear in a number of papers including [26], [25], [8], [9], [28].

We are now ready to expound on our previous observation that Teichmüller theory allows us to prove the existence of parameters with non-empty Fatou sets. We might as well consider the general case of the 4-holed sphere Σ₀,4. We note right away that its fundamental group can be identified with the free group on three generators Π ≅ F₃. Furthermore, since the orbits of the action of SL(2, C) on itself by conjugation are also of dimension 3, we see that the space of representations from Π to SL(2, C) (up to conjugation) has 6 complex dimensions. A far more accurate description is possible: the space of representations can be identified with the quartic hypersurface of C⁷ given by

\[S = \{(a₁, a₂, a₃, a₄, x, y, z) ∈ C⁷ : x² + y² + z² + xyz = Ax + By + Cz + D \},\]

where A, B, C, D are as follows:

\[(3) \quad A = a₁a₄ + a₂a₃, \quad B = a₂a₄ + a₁a₃, \quad C = a₃a₄ + a₁a₂,\]

and

\[(4) \quad D = 4 - [a₁a₂a₃a₄ + a₁² + a₂² + a₃² + a₄²].\]

In particular, by fixing the variables a₁, . . . , a₄, we obtain the surface Sₐₐ,ₐₐ,ₐₐ,ₐₐ. The parameters a₁, . . . , a₄ are identified with the traces of the matrices in SL(2, C) arising from the loops around the holes of Σ₀,4.

Next, let us consider the space of quasi-Fuchsian representations inside S. The Bers Simultaneous Uniformization theorem implies that the space of quasi-Fuchsian representations Repqf(Σ₀,4) can be identified with the product of two copies of the Teichmüller space Teich(Σ₀,4) of the 4-holed sphere Σ₀,4, with geodesic boundaries, i.e.

\[(5) \quad Repqf(Σ₀,4) = Teich(Σ₀,4) × Teich(Σ₀,4).\]
In turn, recalling that the group of outer automorphisms of the fundamental group of $\Sigma_{0,4}$ is isomorphic to the mapping class group of $\Sigma_{0,4}$, the latter acts on the space $\text{Rep}_{qf}(\Sigma_{0,4}) = \text{Teich}(\Sigma_{0,4}) \times \text{Teich}(\Sigma_{0,4})$ diagonally with respect to its standard action on $\text{Teich}(\Sigma_{0,4})$. In particular, the action of the mapping class group on $\text{Rep}_{qf}(\Sigma_{0,4})$ is properly discontinuous.

Now, note that the 4-holed sphere consists of two pair of pants joined by the “waist”. Hence the real dimension of $\text{Teich}(\Sigma_{0,4})$ is 6 so that the real dimension of $\text{Rep}_{qf}(\Sigma_{0,4})$ is 12 and therefore $\text{Rep}_{qf}(\Sigma_{0,4})$ has non-empty interior in $\mathbb{S}$. Finally, if the parameters $a_1, \ldots, a_4$ are chosen so that the corresponding surface $S_{A,B,C,D}$ intersects $\text{Rep}_{qf}(\Sigma_{0,4})$ and this intersection contains an open set $U \subset S_{A,B,C,D}$, then the preceding shows that $\Gamma$ acts properly discontinuously on $U$ and hence that $U$ is contained in the Fatou set of the action of $\Gamma$ on $S_{A,B,C,D}$.

**Remark 2.1.** It is convenient to point out a couple of (connected) issues regarding the above construction. First, it is easy to find parameters $a_1, \ldots, a_4$ so that the resulting surface $S_{A,B,C,D}$ does not intersect $\text{Rep}_{qf}(\Sigma_{0,4})$. Recalling that $a_1, \ldots, a_4$ are the traces of matrices in $\text{SL}(2, \mathbb{C})$ arising from loops around the holes, it is enough to force one of these matrices to be an elliptic element conjugate to an irrational rotation. An alternative argument leading to open sets of parameters $a_1, \ldots, a_4$ for which $S_{A,B,C,D}$ is disjoint from $\text{Rep}_{qf}(\Sigma_{0,4})$ can be formulated by using the well-known Jorgensen inequality.

As mentioned, the proof of Theorem E provided in this paper is totally elementary dispensing, in particular, with Teichmüller theory. The argument also shows the existence of (non-empty) Fatou components beyond the set of parameters $a_1, \ldots, a_4$ for which $S_{A,B,C,D}$ intersects $\text{Rep}_{qf}(\Sigma_{0,4})$. In fact, there are examples of parameters for which $S_{A,B,C,D}$ does not contain any discrete representation and yet the corresponding Fatou set is not empty.

We might also remind the reader that the only situation where the Fatou set is known to be empty occurs for the Picard parameters $A = B = C = 0$, $D = 4$, cf. Theorem D.

### 2.4. Motivations from the Painlevé equation P6

Possibly the most compelling motivation yet to study the dynamics of $\Gamma$ arises from the fact that it actually encodes much, if not all, of the dynamics of the sixth Painlevé equation P6. In particular, it is natural to expect a better understanding of the dynamics of $\Gamma$ to find significant applications in Mathematical Physics.

Let us clarify the connection. In doing so, we will also review some basic issues involving P6. In the most standard notation, the sixth Painlevé equation takes on the form

$$\frac{d^2 y}{dx^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) \left( \frac{dy}{dx} \right)^2 - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) \frac{dy}{dx} + y \frac{(y-1)(y-x)}{x^2(x-1)^2} \left( \alpha \frac{x}{y^2} + \beta \frac{x-1}{(y-1)^2} + \gamma \frac{y}{(y-x)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right),$$

(6)

where $\alpha$, $\beta$, $\gamma$, and $\delta$ are complex parameters. A useful alternative way to represent the parameters involved in this equation which was considered by K. Iwasaki in [33], consists of denoting these parameters by $\kappa_1$, $\kappa_2$, $\kappa_3$, and $\kappa_4$ where the following formulas hold:

$$\alpha = \kappa_4^2/2, \quad \beta = -\kappa_1^2/2, \quad \gamma = \kappa_2^2/2, \quad \text{and} \quad \delta = (1 - \kappa_1^2)/2.$$

Consider a local solution $y(x)$ to P6 with initial conditions $y(x_0)$ and $y'(x_0)$ and try to extend this solution along paths in $\mathbb{C}$. A point $p \in \mathbb{C}$ is termed a *singularity* for this local solution if there is a path $c : [0,1] \to \mathbb{C}$ such that the solution can be extended to a neighborhood of every point in $c([0,1])$ but not to any neighborhood of $c(1) = p$. Singularities whose positions vary with the initial condition are called *movable singularities*. Equation P6 satisfies the *Painlevé Property* which asserts that the only movable singularities are poles. Meanwhile there are points where the extension of the local solutions behave wildly (have essential singular points in the standard sense of functions of a single complex variable). This phenomenon, however, only occurs at the points
The standard procedure of lowering the order of a differential equation allows one to replace the non-autonomous second order equation in \([6]\) by an autonomous vector field \(Z_{VI}\) on \(\mathbb{C}^3\). Clearly the vector field \(Z_{VI}\) is given by
\begin{equation}
Z_{VI} = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + \mathcal{H}_{\alpha,\beta,\gamma,\delta}(x, y, z) \frac{\partial}{\partial z},
\end{equation}
where the function \(\mathcal{H}_{\alpha,\beta,\gamma,\delta}(x, y, z)\) is obtained from the right side of \([6]\) by substituting \(z\) for \(dy/dx\). In particular, the variable \(x\) can naturally be identified with the “time” in the standard form \([6]\).

Let \(\mathcal{D}\) denote the foliation defined on \(\mathbb{C}^3\) by the local orbits of \(Z_{VI}\). Whereas the vector field \(Z_{VI}\) is meromorphic, the foliation \(\mathcal{D}\) is holomorphic on all of \(\mathbb{C}^3\) since it can alternatively be defined as the local orbits of the polynomial vector field obtained from \(Z_{VI}\) by multiplying it by the denominator appearing in the rational expression for the function \(\mathcal{H}_{\alpha,\beta,\gamma,\delta}(x, y, z)\). It is immediate to check that the foliation \(\mathcal{D}\) on \(\mathbb{C}^3\) is transverse to the fibers of the standard fibration (projection) \(\pi_x\) of \(\mathbb{C}^3\) to the \(x\)-axis, away from the invariant fibers sitting over \(\{x = 0\}\) and \(\{x = 1\}\).

In particular, if \(L\) is a leaf of \(\mathcal{D}\) not contained in the invariant fibers, the restriction of \(\pi_x\) to \(L\) provides a local diffeomorphism from \(L\) to \(\mathbb{C} \setminus \{0, 1\}\). Unfortunately, this restriction does not provide a covering map to \(\mathbb{C} \setminus \{0, 1\}\) since it does not have the path-lifting property. Nonetheless, it turns out that the only obstacle to lifting paths arises from the occurrence of poles when following a solution and, moreover, there is a way to take care of all these poles together. The last assertion is the content of Okamoto’s semi-compactification \([45]\). Indeed, the Okamoto procedure begins with a compactification of the fibers of \(\pi_x\) in a suitable Hirzebruch surface, denoted here by \(F_e\). Clearly \(\mathcal{D}\) possesses a (singular) holomorphic extension to \(\mathbb{C} \times F_e\). Okamoto has then shown that up to performing a suitable sequence of blow-ups, considering the corresponding transforms of \(\mathcal{D}\), and deleting a certain divisor invariant by the corresponding transformed foliation, the following can be obtained (see for example \([45], [34]\)):

- A complex (open) manifold \(N\) of dimension 3 fibering over \(\mathbb{C} \setminus \{0, 1\}\) with an open surface denoted by \(F\) as typical fiber. Moreover the projection map \(p : M \to \mathbb{C} \setminus \{0, 1\}\) arises as the transform of the initial projection \(\pi_x : \mathbb{C}^3 \to \mathbb{C}\) through the corresponding sequence of blow-up maps.
- \(N\) is equipped with the corresponding transform (still denoted by \(\mathcal{D}\)) of the extended foliation \(\mathcal{D}\) of \(\mathbb{C} \times F_e\) by the corresponding blow-up maps.
- The foliation \(\mathcal{D}\) is transverse to the fibers of \(p\) and the base \(\mathbb{C} \setminus \{0, 1\}\) can still be naturally identified with the “time” in \([6]\).
- The restriction of \(p\) to a leaf \(L\) of \(\mathcal{D}\) yields a covering map from \(L\) to \(\mathbb{C} \setminus \{0, 1\}\).

Owing to the fourth condition, paths contained in \(\mathbb{C} \setminus \{0, 1\}\) can be lifted in the leaves of \(\mathcal{D}\) so as to give rise to a homomorphism \(\rho\) from the fundamental group of \(\mathbb{C} \setminus \{0, 1\}\) to the group of (holomorphic) diffeomorphisms \(\text{Diff}(F)\) of a typical fiber \(F\) of \(p\). In other words, \(\rho(\pi_1(\mathbb{C} \setminus \{0, 1\})) \subset \text{Diff}(F)\) is the holonomy group of \(\mathcal{D}\) whose dynamics on \(F\) essentially encodes the dynamics of the initial Painlevé 6 equation.

The Riemann-Hilbert map conjugates the monodromy action of \(\rho(\pi_1(\mathbb{C} \setminus \{0, 1\}))\) on \(F\) at parameters \((\kappa_1, \ldots, \kappa_4)\) with the action of \(\Gamma_{A,B,C,D}\) on \(S_{A,B,C,D}\). More precisely there are two mappings
\[
\mathfrak{th} : \mathbb{C}^4 \to \mathbb{C}^4 \quad \text{and} \quad \text{RH}_{(\kappa_1, \kappa_2, \kappa_3, \kappa_4)} : F \to S_{A,B,C,D},
\]
where \((A, B, C, D) = \mathfrak{th}(\kappa_1, \kappa_2, \kappa_3, \kappa_4)\). They have the property that for any choice of parameters \((\kappa_1, \kappa_2, \kappa_3, \kappa_4)\) the mapping \(\text{RH}_{(\kappa_1, \kappa_2, \kappa_3, \kappa_4)}\) conjugates the monodromy action of \(\rho(\pi_1(\mathbb{C} \setminus \{0, 1\}))\) on \(F\) to the action of \(\Gamma_{A,B,C,D}\) on \(S_{A,B,C,D}\).

The mapping \(\mathfrak{th}\) is relatively simple. Consider the Painlevé 6 equation with parameters \(\kappa_1, \ldots, \kappa_4\), i.e. replace in \([6]\) the parameters \(\alpha, \beta, \gamma, \delta\) by the corresponding formulas in \([7]\). For \(i = \ldots, 1\),
1, . . . , 4, let us define

\[ a_i = 2 \cos(\pi \kappa_i) . \]

In turn, to the 4-tuple of complex numbers \((a_1, a_2, a_3, a_4)\), let us assign the 4-tuple \((A, B, C, D)\) by means of Formulas (34) and (35). Applying these changes of parameters defines \((A, B, C, D) = \pi h(\kappa_1, \kappa_2, \kappa_3, \kappa_4)\).

Now, considering the holonomy representation \(\rho(\pi_1(\mathbb{C} \setminus \{0, 1\})) \subset \text{Diff} (F)\), it is well known that the mapping \(\text{RH}_{\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}}\) provides a holomorphic conjugation between the dynamics of \(\rho(\pi_1(\mathbb{C} \setminus \{0, 1\}))\) on \(F\) and the dynamics of \(\Gamma\) on \(S_{A,B,C,D}\), see [32] Theorem 8.4 and also to [18] for further details. Note, however, that the mapping \(\text{RH}_{\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}}\) can hardly be computed since it is highly transcendent in the sense that it has “essential singularities at infinity”. This happens due to the fact that diffeomorphisms in \(\rho(\pi_1(\mathbb{C} \setminus \{0, 1\}))\) are in general transcendent as well.

Despite the transcendental nature of the conjugating map \(\text{RH}_{\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}}\), at least on compact parts of \(F\), conjugation invariant properties of the dynamics of \(\Gamma\) can immediately be translated into dynamical properties of Painlevé 6, and conversely. This applies in particular to our results involving the dynamics of \(\Gamma\) in its Julia set cf. below.

2.5. Relationship between our results and Painlevé 6. Taken together, our theorems provide plenty of rather explicit examples of parameters \((A, B, C, D)\) along with open sets \(U \subset J_{A,B,C,D}\) in which the action of \(\Gamma_{A,B,C,D}\) has dense orbits. In connection with these examples, we would like to point out a computational issue that deserves further attention. These have to do with the fact that the group action \(\Gamma\) is conjugate by the Riemann-Hilbert map to the monodromy of the actual Painlevé 6 equation. In our examples, it is not too hard to estimate the “size” of the open sets \(U\) provided by Theorem G: for example, we can provide some explicit \(r > 0\) and a point \(p \in U\) such that the ball of radius \(r\) about \(p\) is certainly contained in \(U\). The interesting computational problem we have referred to consists of finding a similar estimate for the size/place of the image of \(U\) by the Riemann-Hilbert map, so as to provide accurate numerics for an open set in which the actual Painlevé 6 equation must have dense orbits. At least part of the difficulty and interest of this problem stem from the highly transcendent nature of the Riemann-Hilbert map whose behavior is hard to control.

2.6. Relationship to the recent work of S. Cantat and R. Dujardin. A recent work by S. Cantat and R. Dujardin [10] studies the dynamics of subgroups of the automorphism group of real and complex projective surfaces (or, more generally, Kähler surfaces). They obtain deep results on the structure of stationary measures for these subgroups, including criteria to determine when they must be invariant by the group in question (stiffness) as well as detailed descriptions of the resulting invariant measures. Whereas their results clearly have serious implications on the corresponding dynamics, the situation seems to be genuinely more subtle in the case of groups of birational maps. Relevant examples include the case of the actions of \(\Gamma\) and \(\Gamma^\ast\) since, as shown in the present paper, for certain values of the parameters these actions display both non-empty Fatou set and Julia sets with non-empty interior. This phenomenon indicates that the problem of invariance of stationary measures, and of their subsequent description, will hardly allow for a reasonably “compact” classification. Also, the nature of the Fatou components constructed in our Theorem E shows that even the convergence of stochastic processes will no longer be automatic which, in turn, seriously limits the applications of stationary measures in topological problems such as description of non-compact invariant closed sets.

The preceding paragraph deserves to be expounded upon. First, we recall that in a series of remarkable papers initiated in [3], Y. Benoist and J.-F. Quint have established the stiffness of stationary measures for the action of subgroups of a Lie group on its (compact) homogeneous spaces under modest assumptions. (See also [37] for an overview of their work.) Then, they went on to show that the resulting invariant measures admit a very simple description, paralleling the case of M. Ratner’s theorem for subgroups generated by unipotent elements: an invariant measure
coincides with the Haar measure provided that it gives no mass to points. These results yield, in particular, a detailed picture of closed invariant sets through a standard argument that can be summarized as follows. Given an invariant compact set $K$, a stationary measure can always be constructed on $K$ by means of the well-known Krylov-Bogolyubov method. By stiffness, this measure is invariant and hence is of one of the types listed in the “second part” of their theorem. The a-priori knowledge of the measure, in turn, clarifies the nature of the invariant set $K$. In particular, the orbit of a point is dense unless it is finite.

It is probably fair to say that the paper [10] aims at obtaining a similar result applicable to the case of (area-preserving) automorphism groups of algebraic (real or complex) surfaces. Building on the fine work of Brown and Hertz [7] and arguing from several sophisticated techniques, they manage to accomplish this task and essentially prove that, under mild assumptions, a stationary measure that is not supported on an algebraic curve must be invariant. They then prove that any invariant measure that is not supported within a proper real-analytic hypersurface must be absolutely continuous with respect to Lebesgue measure and of full support. It is therefore natural to wonder what their results imply for the action of $\Gamma$. However, the situation for $\Gamma$ is significantly different as we will try to make clear below.

From a perspective of methods, the most evident difference is that in [10] the authors deal with automorphisms of compact surfaces, while $\Gamma$ is a group of automorphisms of open surfaces. Yet, on second thought, some of the techniques introduced, for example, in the papers of Y. Benoist and J.-F. Quint hold on open manifolds under the additional condition that the corresponding “infinity” should be “sufficiently repelling”. At this point, one of the first crucial issues about the action of $\Gamma$ comes into play: the infinity arising in the action of $\Gamma$ is actually super-attracting in a rather strong sense.

The difficulties arising from having an “attracting infinity” in the action of $\Gamma$ are confirmed by some of our results. First, in a terminology to be made accurate in the sequel, we prove for many choices of parameters the existence of Fatou components where the action of $\Gamma$ is properly discontinuous and such that every orbit wanders off to infinity, cf. Theorems E and F. As is to be expected, this construction takes advantage of the attracting nature of the infinity in question, though only implicitly. In particular, there are no relevant stationary measures associated with the dynamics of $\Gamma$ on these Fatou sets, which justifies our claim that stochastic processes may fail to converge. Moreover, there seem to exist parameters for which almost all points tend to infinity under a random iteration which, in turn, severely limits the dynamical applications of these measures.

Supposing however that one can produce meaningful stationary measure to describe the dynamics of $\Gamma$, there cannot be a simple description of them similar to what is proved in [10]. For example, in Theorem G we prove that for a large set of parameters, the group $\Gamma_{A,B,C,D}$ has a non-empty Fatou set that coexists with a Julia set having non-empty interior. Moreover, while Julia sets are closed and there are dense orbits in them (Theorem A), they are not “essentially minimal” since points in their boundary have infinite orbits which are not dense in the full Julia set. Also, the boundary $\mathcal{B}_{A,B,C,D}$ from the corollary to Theorem F is itself an invariant closed set with empty interior that cannot easily be characterized (like invariant curves) since its topological dimension is “too large” (equal to 3). Indeed, the resulting boundaries appear to have a fractal nature. These wild dynamical behaviors seem to defy the type of clean classification obtainable by the techniques of [10].

Finally, it might be pointed out that a nice feature of the methods introduced in our work to address the above questions is that they shed some light on other dynamical issues such as denseness of hyperbolic saddles and similar types of points with non-trivial stabilizers in the Julia set.
3. Basic properties of $S_{A,B,C,D}$ and $\Gamma_{A,B,C,D}$

3.1. Algebraic properties of $\Gamma^*_{A,B,C,D}$ and $\Gamma_{A,B,C,D}$. The group of automorphisms of $\mathbb{C}^3$ generated by $s_x$, $s_y$, and $s_z$ is isomorphic to the free product $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$ and it follows that the group of automorphism of $\mathbb{C}^3$ generated by $g_x$, $g_y$, and $g_z$ is free on two generators. However, a stronger statement holds: for every choice of the parameters $A, B, C, D$, the group $\Gamma^*$ is isomorphic to the indicated free product and the group $\Gamma$ is free on two generators, when viewed as a group of automorphisms of $S_{A,B,C,D}$.

The non-existence of additional relations between the maps $s_x$, $s_y$, and $s_z$ even when restricted to a particular surface $S_{A,B,C,D}$ is a consequence of El-Huiti’s theorem in [19] – albeit not an immediate one. For details the reader can check [8] and [9], where the used notation is slightly different from ours.

3.2. Singular points of $S_{A,B,C,D}$. We begin by considering the natural compactification of $\mathbb{C}^3$ in $\mathbb{CP}^3$ so that $\mathbb{CP}^3 = \mathbb{C}^3 \cup \Pi_\infty$, where $\Pi_\infty \simeq \mathbb{CP}^2$ is the plane at infinity in $\mathbb{CP}^3$. Next, denote by $\overline{S}_{A,B,C,D}$ the closure of $S_{A,B,C,D}$ in $\mathbb{CP}^3$. By using the standard affine atlas for $\mathbb{CP}^3$, it is straightforward to check that $\overline{S}_{A,B,C,D} \cap \Pi_\infty$ consists of three projective lines forming a triangle $\Delta_\infty$ in $\Pi_\infty$. Indeed, if $(u, v, w)$ are affine coordinates for $\mathbb{CP}^3$ satisfying $(1/u, v/u, w/u) = (x, y, z)$, then the surface $S_{A,B,C,D}$ is determined in $(u, v, w)$-coordinates by the equation

$$u + uw^2 + uv^2 + vw^2 = Au^2 + Bu^2v + Cu^2w + Du^3.$$ 

In particular, there follows that $\overline{S}_{A,B,C,D} \cap \Pi_\infty$ locally coincides with the axes $\{u = v = 0\}$ and $\{u = w = 0\}$. An analogous use of the remaining coordinates in the standard affine atlas of $\mathbb{CP}^3$ then shows that the third side of the mentioned triangle coincides with the projective line of $\Pi_\infty \simeq \mathbb{CP}^2$ which is missed by the domain of the affine coordinates $(v, w) \simeq (u = 0, v, w)$. A slightly less immediate computation with the above equation for $\overline{S}_{A,B,C,D}$ in standard affine coordinates for $\mathbb{CP}^3$ also shows that $\overline{S}_{A,B,C,D}$ is smooth on a neighborhood of $\Pi_\infty$.

From the preceding, there follows that $\overline{S}_{A,B,C,D}$ is singular if and only if the affine surface $S_{A,B,C,D}$ is so (and the corresponding singular sets always coincide). In turn, as shown in either [2] or in [33], the surface $S_{A,B,C,D}$ is singular if and only if at least one of the following conditions is satisfied:

- We have $a_i = \pm 2$ for at least one $i \in \{1, 2, 3, 4\}$.
- The coefficients $a_1, \ldots, a_4$ satisfy the relation
  $$[2(a_1^2 + a_2^2 + a_3^2 + a_4^2) - a_1a_2a_3a_4 - 16] - (4 - a_1^2)(4 - a_2^2)(4 - a_3^2)(4 - a_4^2) = 0.$$ 

Remark 3.1. When the set $\text{Sing}(S_{A,B,C,D})$ formed by singular points of $S_{A,B,C,D}$ is not empty, then it must consist of finitely many points in $S_{A,B,C,D}$. Indeed, a proper analytic subset of $S_{A,B,C,D}$, if the dimension of $\text{Sing}(S_{A,B,C,D})$ were 1, then $S_{A,B,C,D}$ would contain some (irreducible, singular) curve which should intersect the infinity $\overline{S}_{A,B,C,D} \setminus S_{A,B,C,D}$ since $S_{A,B,C,D}$ is Stein (as an affine surface in $\mathbb{C}^3$). This, however, cannot happen since $\overline{S}_{A,B,C,D}$ is smooth on a neighborhood of $\Pi_\infty$. Therefore the dimension of $\text{Sing}(S_{A,B,C,D})$ must be zero and the above assertion follows.

Since $\Gamma$ must preserve $\text{Sing}(S_{A,B,C,D})$, there follows from the preceding that $\Gamma$ has a finite orbit whenever $\text{Sing}(S_{A,B,C,D}) \neq \emptyset$. The existence of a finite orbit for $\Gamma$ is naturally yields some insights in the dynamics of $\Gamma$, as it will be seen in the course of this work.

Remark 3.2. If $p \in S_{A,B,C,D}$ is a singular point, then $p \in J_{A,B,C,D}$. Indeed, let $U$ be any neighborhood of $p$. It follows from [10] Theorem C that there must be a point $q \in U$ and a sequence $\gamma_n \in \Gamma$ such that $\gamma_n(q)$ diverges to infinity. Meanwhile, since every element of $\Gamma$ permutes the singular set of $S_{A,B,C,D}$, we have that $\gamma_n(p)$ remains bounded.
3.3. Invariant volume form. The smooth part of $S_{A,B,C,D}$ comes equipped with a holomorphic volume form

\begin{equation}
\Omega = \frac{dx \wedge dy}{2x + xy - C} = \frac{dy \wedge dz}{2x + yz - A} = \frac{dz \wedge dx}{2y + zx - B}
\end{equation}

and a simple calculation shows that $s_3^*\Omega = -\Omega$ and similarly for $s_y$ and $s_z$. Therefore, the elements of $\Gamma$ preserve $\Omega$ and hence also preserve the associated real volume form $\Omega \wedge \overline{\Omega}$ on the underlying smooth surface.

Concerning the (real) volume arising from $\Omega$, it should be pointed out that the volume of a small neighborhood of any singular point of $S_{A,B,C,D}$ is finite, albeit the total volume of the whole $S_{A,B,C,D}$ is infinite.

Let $p$ be a smooth point of $S_{A,B,C,D}$. An immediate consequence of the existence of the invariant volume form is that if $p$ is fixed for $\gamma \in \Gamma_{A,B,C,D}$ then $\det(D\gamma(p)) = 1$. In particular, if $p$ is a hyperbolic fixed point for $\gamma$, then it must be of saddle type.

3.4. Pencils of rational curves. An additional important property common to all surfaces $\overline{S}_{A,B,C,D}$ is the existence of certain particularly simple pencils of rational curves. These pencils deserve a few comments.

Recall that $\pi_x : \mathbb{C}^3 \to \mathbb{C}$ sends $(x, y, z) \in \mathbb{C}^3$ to $x \in \mathbb{C}$. Given $x_0 \in \mathbb{C}$, let

$$
\Pi_{x_0} = \{(x, y, z) \in \mathbb{C}^3 : \pi_x(x, y, z) = x_0\} \quad \text{and} \quad S_{x_0} = S_{A,B,C,D} \cap \Pi_{x_0}.
$$

Let $\overline{S}_{x_0}$ denote the closure of $S_{x_0}$ in $\overline{S}_{A,B,C,D}$. Since $\overline{S}_{x_0}$ has degree two, it is uniformized by the Riemann Sphere provided that $\overline{S}_{x_0}$ is smooth. The statement remains valid in the case where $\overline{S}_{x_0}$ is singular up to passing from $\overline{S}_{x_0}$ to its normalization.

Denoting by $\pi_y$ and $\pi_z$ the projections of $\mathbb{C}^3$ to $\mathbb{C}$ respectively defined by $\pi_y(x, y, z) = y$ and $\pi_z(x, y, z) = z$, the fibers $\Pi_{y_0}, S_{y_0}, \Pi_{z_0},$ and $S_{z_0}$ are analogously defined.

Clearly the collection of rational curves obtained from $\overline{S}_{x_0}$, $x_0 \in \mathbb{C}$, defines a rational pencil in $\overline{S}_{A,B,C,D}$. The language of singular holomorphic foliations is slightly better adapted to the present situation (the reader is referred to [31] and to the lecture notes [48] for terminology and basic material). The above mentioned rational curves, actually induce a singular holomorphic foliation $\mathcal{D}_x$ on all of the surface $\overline{S}_{A,B,C,D}$. Furthermore, all the leaves of $\mathcal{D}_x$ are algebraic and their closures are rational curves (by a small abuse of language, we will say in the sequel that the leaves of $\mathcal{D}_x$ are rational curves). There also follows from the preceding construction that $\mathcal{D}_x$ can equally be given by a non-constant meromorphic first integral $\mathcal{L}_x$ on $\overline{S}_{A,B,C,D}$. Since $\overline{S}_{A,B,C,D}$ has dimension 2, the singular set $\text{Sing} (\mathcal{D}_x)$ of $\mathcal{D}_x$ consists of finitely many points.

The indeterminacy points of $\mathcal{L}_x$ necessarily belong to $\text{Sing}(\mathcal{D}_x)$. Moreover, these points can be characterized as those points of $\overline{S}_{A,B,C,D}$ through which there pass infinitely many leaves of $\mathcal{D}_x$.

It is well known that the indeterminacy points of $\mathcal{L}_x$ can be eliminated by means of a suitable sequence of blow-ups. Thus, up to performing finitely many blow-ups, the foliation $\mathcal{D}_x$ becomes a singular rational fibration (with connected fibers) over a suitable surface $\overline{S}_{A,B,C,D}$.

Naturally, there are analogous foliations $\mathcal{D}_y$ and $\mathcal{D}_z$ defined on $\overline{S}_{A,B,C,D}$ with the help of the collection of rational curves $\overline{S}_{y_0}$ and $\overline{S}_{z_0}$ contained in $\overline{S}_{A,B,C,D}$.

Let us close this section with an elementary lemma.

**Lemma 3.3.** There are only finitely many values of $x_0$ for which the rational curve $\overline{S}_{x_0}$ is singular. Furthermore, the analogous statements also holds for the curves $\overline{S}_{y_0}$ and $\overline{S}_{z_0}$.

**Proof.** It suffices to prove the lemma for the curves $\overline{S}_{x_0}$. Fix then $x_0 \in \mathbb{C}$ and consider the curve $S_{x_0} = S_{A,B,C,D} \cap \Pi_{x_0}$ so that $\overline{S}_{x_0}$ is the closure of $S_{x_0}$ in $\overline{S}_{A,B,C,D}$. We identify the plane $\Pi_{x_0} \subset \mathbb{C}^3$ with the corresponding projective plane inside $\mathbb{C}P^3$. Next, the curve $\overline{S}_{x_0}$ is given by a degree 2 equation in the variables $(x, y)$ in $\Pi_{x_0} \simeq \mathbb{C}P^2$. Therefore $\overline{S}_{A,B,C,D}$ is smooth unless it consists of
two projective lines in \( \Pi_{x_0} \simeq \mathbb{C}P^2 \). For the later situation to happen the polynomial in \( y \) and \( z \) given by
\[
x_0^2 + y^2 + z^2 + x_0yz - Ax_0 - By - Cz - D
\]
must have the form \((a_1y + b_1y + c_1)(a_2y + b_2z + c_2)\), for some constants \( a_i, b_i, \) and \( c_i, i = 1, 2 \). Comparing the coefficients of \( y^2, z^2 \), and of \( yz \), we obtain \( a_2 = 1/a_1, b_2 = 1/b_1, \) and \( \tau^2 - \tau x_0 + 1 = 0 \) where \( \tau = a_1/b_1 \). The remaining coefficients now provide \( a_1c_2 + c_1/a_1 = -B, b_1c_2 + c_1/b_1 = -C, \) and \( c_1c_2 = x_0^2 - Ax_0 - D \). From the first two equations, we obtain
\[
c_1 = \frac{a_1b_1(Ca_1 - b_1)}{a_1^2 - b_1^2} \quad \text{and} \quad c_2 = \frac{Cb_1 - Ba_1}{a_1^2 - b_1^2}.
\]
From there, we immediately conclude that
\[
c_1c_2 = \frac{\tau(B - C\tau)(C - B\tau)}{(\tau^2 - 1)^2}.
\]
Since \( \tau^2 - \tau x_0 + 1 = 0 \), there follows that \( x_0 \) has to satisfy a polynomial equation with constant coefficients. The lemma follows at once. \( \square \)

A simple adaptation of the previous argument also yields:

**Corollary 3.4.** For all but finitely many values of \( x_0 \in \mathbb{C} \), the rational curve \( \overline{S}_{x_0} \) is smooth and intersects the plane at infinity \( \Pi_{\infty} \) of \( \mathbb{CP}^3 \) in two distinct points. Analogous statements hold for the rational curves \( \overline{S}_{y_0} \) and \( \overline{S}_{z_0} \).

For the foliation \( \mathcal{D}_x \) on \( \overline{S}_{A,B,C,D} \), we fix a (minimal) blow-up procedure turning the foliation in question into a (singular) rational fibration \( \overline{\mathcal{D}}_x \). Clearly there are only finitely many values of \( x_0 \in \mathbb{C} \) corresponding to singular fibers of this fibration. We then define a finite set \( \mathcal{B}_x^+ \subset \mathbb{C} \) by saying that \( x_0 \in \mathcal{B}_x \) if at least one of the following conditions fails to hold:
- The rational curve \( \overline{S}_{x_0} \) is smooth and intersects the plane at infinity at two distinct points.
- The fibration \( \overline{\mathcal{D}}_x \) is regular on a neighborhood of the fiber sitting over \( x_0 \).

We also define \( \mathcal{B}_x \subseteq \mathcal{B}_x^+ \) to be the set of points at which the first of the two conditions above fails to hold.

The corresponding sets for the coordinates \( y \) and \( z \) will similarly be denoted by \( \mathcal{B}_y^+, \mathcal{B}_y \) and by \( \mathcal{B}_z^+, \mathcal{B}_z \).

4. **Dynamics of parabolic maps and proof of Theorems A, B, and C**

We begin by studying the generators \( g_x = s_z \circ s_y, g_z = s_x \circ s_z, \) and \( g_z = s_y \circ s_x \) of \( \Gamma \). Here these mappings are explicitly written as
\[
g_x(x, y, z) = \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = \left( \begin{array}{c} x \\ -y - xz + B \\ xy + (x^2 - 1)z + C - Bx \end{array} \right),
\]
\[
g_y(x, y, z) = \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = \left( \begin{array}{c} (y^2 - 1)x + yz + A - Cy \\ y \\ -yx - z + C \end{array} \right), \quad \text{and}
\]
\[
g_z(x, y, z) = \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = \left( \begin{array}{c} x \\ z + (z^2 - 1)y + B - Az \\ z \end{array} \right).
\]

The map \( g_x \) preserves the coordinate \( x \) and hence each affine plane \( \Pi_{x_0} \subset \mathbb{C}^3 \). Since, in addition, \( g_x \) clearly preserves the (affine) surface \( S_{A,B,C,D} \), it follows that \( g_x \) individually preserves each one
of the curves $S_{x_0}$ (and hence also the rational curves $\overline{S}_{x_0} \subset \overline{S}_{A,B,C,D}$). Similar conclusions hold for the maps $g_y$ and $g_z$.

We can now complement the discussion in Section 3 revolving around Lemma 3.3 and Corollary 3.4. Owing to these statements, we know that the rational curve $\overline{S}_{x_0}$ is smooth and intersects the divisor at infinity $\Pi_\infty$ of $\mathbb{CP}^3$ in two distinct points for all but finitely many values of $x_0 \in \mathbb{C}$. It is then natural to consider the automorphism of $\overline{S}_{x_0}$ induced by $g_x$. The following proposition can be found in [9] (see in particular Proposition 4.1 in the paper in question). The proof is elementary, in the spirit of most of the discussion in the previous section.

**Proposition 4.1.** Let $x_0 \in \mathbb{C}$ be such that the rational curve $\overline{S}_{x_0}$ is smooth and intersects the divisor at infinity $\Pi_\infty$ in two distinct points. Then the restriction $\overline{g}_{x_0}$ of $g_x$ to $\overline{S}_{x_0}$ is a Möbius transformation whose two fixed points are at infinity. Furthermore, we have:

- If $x_0 \in (-2, 2)$ then $\overline{g}_{x_0}$ is elliptic. It is periodic if and only if $x_0 = \pm 2 \cos(\theta \pi)$ with $\theta$ rational.
- If $x_0 \in \mathbb{C} \setminus [-2, 2]$ then $\overline{g}_{x_0}$ is loxodromic.

The analogous statements hold for restrictions of $g_y$ to the fibers $\overline{S}_{y_0}$ and of $g_z$ to the fibers $\overline{S}_{z_0}$.

**Remark 4.2.** The case $x_0 = \pm 2$ deserves an additional comment for the sake of clarity. First note that the affine singular points of the curves $S_{x_0}$ are contained in the affine curve given by

$$x_0 \mapsto \left( x_0, \frac{Cx_0 - 2B}{x_0^2 - 4}, \frac{Bx_0 - 2C}{x_0^2 - 4} \right).$$

In fact, the curve above parameterizes the set of points where the intersection of $S_{A,B,C,D}$ and $\Pi_{x_0}$ is not transverse. From this, we see that this curve intersects the plane at infinity for $x_0 = \pm 2$. In particular, the affine curve $S_2$ (resp. $\overline{S}_2$) is always smooth. On the other hand, this curve intersects the plane $\Pi_\infty \subset \mathbb{CP}^3$ at a single point (with multiplicity 2) which, depending on the coefficients, may or may not be a singular point of $\overline{S}_2$ (resp. $\overline{S}_{-2}$). The restriction $\overline{g}_2$ of $g_x$ to $\overline{S}_2$ is as follows:

1. When $\overline{S}_2$ is smooth, then $\overline{g}_2$ is a parabolic map whose single fixed point coincides with the intersection of $\overline{S}_2$ with $\Pi_\infty$.
2. Otherwise $\overline{S}_2$ consists of the union of two projective lines intersecting each other at a point in $\Pi_\infty$. Each line is then preserved by $\overline{g}_2$ and, in each of these lines, $\overline{g}_2$ induces a parabolic map whose fixed point coincides with their intersection.

The analogous statement holds for $\overline{g}_{-2}$ and $\overline{S}_{-2}$.

Proposition 4.1 yields the following lemma:

**Lemma 4.3.** If $x_0 \in (-2, 2) \setminus B_x$ then $S_{x_0} \subset J(g_x)$. Analogous statements hold for the $g_y$ and $g_z$ mappings.

**Proof.** Suppose that $U$ is an open neighborhood of a point $p \in S_{x_0}$. Because $x_0 \in (-2, 2) \setminus B_x$, Proposition 4.1 ensures that $g_x$ restricted to the fiber $S_{x_0}$ is elliptic with both fixed points lying in $\Pi_\infty$. In particular, the iterates $g_x^n(p)$ remain bounded. Meanwhile, since $U$ is open in $S$ there exists a point $q \in U$ with $x_1 = \pi_x(q) \notin [-2, 2] \cup B_x$. According to Proposition 4.1 the restriction of $g_x$ to the fiber $\overline{S}_{x_1}$ is hyperbolic with both fixed points at infinity. Therefore the orbit $g_x^n(q)$ tends to infinity. Hence $U$ cannot be contained in the Fatou set of $g_x$ since it contains points with both bounded and unbounded orbits. The lemma follows.

There is another elementary lemma, similar to some statements in Section 3 if slightly more technical, that will also be needed in the sequel.

**Lemma 4.4.** Fix any $x_0 \in \mathbb{C}$. Then, for all but finitely many choices of $y_0$ the fibers $S_{x_0}$ and $S_{y_0}$ intersect transversally. When the fibers intersect transversally, they do so at two distinct points.
Proof. Finding the intersections of \( S_{x_0} \) and \( S_{y_0} \) in \( S \) amounts to solving the equation
\[
(10) \quad x^2 + y^2 + z^2 + xyz - Ax - By - Cz - D = 0
\]
for \( z \) with \( x = x_0 \) and \( y = y_0 \) being fixed. The result is a quadratic polynomial in \( z \), which can have either one or two roots (counted without multiplicities). We will denote the two resulting points by \( z \) and \( z' \).

If we take the partial derivative \( \frac{\partial}{\partial z} \) of (10), we obtain
\[
2z + xy - C.
\]
On a neighborhood of any point \( (x, y, z) \in S_{A,B,C,D} \) for which the above expression is non-zero, the implicit function theorem allows us to express \( S_{A,B,C,D} \) as the graph of a holomorphic function \( z = f(x, y) \).

The curve on \( S_{A,B,C,D} \) defined by \( 2z + xy - C \) is independent of \( S_{x_0} \) so they intersect on \( S_{A,B,C,D} \) at finitely many points. Suppose that neither \( (x_0, y_0, z(x_0, y_0)) \) nor \( (x_0, y_0, z'(x_0, y_0)) \) are from among these finitely many points. Then, the fact that \( S_{A,B,C,D} \) can be expressed as the graph of a holomorphic function \( z = f(x, y) \) in a neighborhood of \( (x_0, y_0, z(x_0, y_0)) \) and \( (x_0, y_0, z'(x_0, y_0)) \) implies that \( S_{x_0} \) and \( S_y \) intersect transversally at each of these points.

When the intersections are transverse then each of them counts as a simple zero of (10) (with \( x = x_0 \) and \( y = y_0 \) fixed) so there are exactly two solutions in this case.

Now fix a point \( x_0 \in \mathbb{C} \setminus B^+_x \). For sufficiently small \( \epsilon > 0 \), consider the “tube” \( T_\epsilon(x_0) \) defined by
\[
(11) \quad T_\epsilon(x_0) = \{(x, y, z) \in S : |x - x_0| < \epsilon\}
\]
Clearly \( T_\epsilon(x_0) \) is filled (foliated) by the curves \( S_x \) where \( x \) satisfies \( |x - x_0| < \epsilon \). Let \( y_0 \in \mathbb{C} \) be such that the curve \( S_{y_0} \) intersects the curves \( S_x \subset T_\epsilon(x_0) \) transversally. Lemma 4.5 is a rather general statement about the hyperbolic nature of these tubes.

**Lemma 4.5.** With the preceding notation and up to choosing \( \epsilon > 0 \) sufficiently small, the open set \( T_\epsilon(x_0) \setminus S_{y_0} \) is Kobayashi hyperbolic.

Proof. Since \( B^+_x \) is finite and \( x_0 \in \mathbb{C} \setminus B^+_x \), there is \( \epsilon > 0 \) such that \( D_\epsilon = \{x \in \mathbb{C} : |x - x_0| < \epsilon\} \) is contained in \( \mathbb{C} \setminus B^+_x \). Now, in view of the definition of \( B^+_x \), for every \( S_x \subset T_\epsilon(x_0) \), the corresponding rational curve \( \overline{S_x} \) intersects \( \Pi_\infty \) transversally and at two distinct points. In other words, \( \overline{S_x} \setminus S_x \) consists of two distinct points provided that \( S_x \subset T_\epsilon(x_0) \). On the other hand, up to a birational transformation, the projective curves \( \overline{S_x} \) define a (regular) holomorphic fibration over the disc \( D_\epsilon \).

Because the fibers are rational curves and therefore pairwise isomorphic as Riemann surfaces, the theorem of Fischer and Grauert [22] implies that this fibration is holomorphically trivial, i.e., it is holomorphically equivalent to \( D_\epsilon \times \mathbb{P}^1 \). Since \( \overline{S_x} \setminus S_x \) consists of two points, there also follows that
\[
T_\epsilon(x_0) = D_\epsilon \times \mathbb{C} \setminus \{0\}
\]
as complex manifolds.

The hypothesis that \( S_{y_0} \) intersects \( S_{x_0} \) transversally implies that their intersection consists of exactly two (distinct) points (Lemma 4.4). We can then reduce \( \epsilon > 0 \), if necessary, so that \( S_{y_0} \) intersects each \( S_x \) from \( T_\epsilon(x_0) \) transversally in two points, each of them varying holomorphically with \( x \). These points will be referred to as the two branches of \( S_{y_0} \) in \( T_\epsilon(x_0) \). Since each branch is the graph of a holomorphic function on \( x \), we can pick either one of them and construct a further holomorphic diffeomorphism to make it correspond to the point \( 1 \in \mathbb{C} \setminus \{0\} \). By means of this construction, \( T_\epsilon(x_0) \setminus S_{y_0} \) becomes identified with an open set in
\[
D_\epsilon \times \mathbb{C} \setminus \{0,1\}.
\]
The lemma follows since the latter manifold is Kobayashi hyperbolic as the product of two hyperbolic Riemann surfaces. \( \square \)
Proposition 4.9. Let $x_0 \in (-2, 2) \setminus B^+_z$ and let $y_0$ be any point chosen so that $S_{x_0}$ and $S_{y_0}$ intersect transversally. For any open $U \subset S$ with $U \cap S_{x_0} \neq \emptyset$ there is an iterate $n$ such that $g^n_x(U) \cap S_{y_0} \neq \emptyset$. Analogous statements hold when $x$ and $y$ are replaced with any two distinct variables from $\{x, y, z\}$.

Proof. Owing to Lemma 4.5 we fix a tube $T_\epsilon(x_0)$ as in (11) which is Kobayashi hyperbolic. Let then $U$ be a non-empty open set of $S_{A,B,C,D}$ intersecting $S_{x_0}$. Up to trimming $U$, we can assume that $U \subset T_\epsilon(x_0)$. Since $g_x$ preserves the $S_x$ fibration and fixes the points in $S_x \cap \Pi_\infty$ ($g_x$ has no poles), it follows that $g^n_x(U)$ remains in $T_\epsilon(x_0)$ for every $n \in \mathbb{Z}$. Now assume for a contradiction that $g^n_x(U) \subset T_\epsilon(x_0) \setminus S_{y_0}$ for every $n$. Since $T_\epsilon(x_0) \setminus S_{y_0}$ is Kobayashi hyperbolic, this implies that $\{g^n_x\}$ forms a normal family on $U$. This is, however, impossible since $U$ intersects $S_{y_0}$ and $S_{x_0} \subset J(g_x)$ (Lemma 4.3). Thus there must exist $n$ such that $g^n_x(U) \cap S_{y_0} \neq \emptyset$ and the lemma follows. \hfill \Box

Remark 4.7. The above proof actually shows slightly more than the statement of Lemma 4.6. Once $\epsilon > 0$ is chosen sufficiently small so that $T_\epsilon(x_0)$ intersects $S_{y_0}$ in two branches (each a graph of a holomorphic function of $x$) and once $U$ is chosen sufficiently small so that $U \subset T_\epsilon(x_0)$ then for each branch of $S_{y_0}$ in $T_\epsilon(x_0)$ there is an iterate $n$ so that $g^n_x(U)$ intersects the branch in question.

Lemma 4.6 admits a useful quantitative version that can directly be proved, namely:

Lemma 4.8. Assume that $x_0 = \pm 2 \cos(\theta \pi) \in (-2, 2) \setminus B^+_z$ with $\theta$ irrational and let $U \subset S$ be an open set such that $U \cap S_{x_0} \neq \emptyset$. Next consider a sequence $\{q_j = (x_j, y_j, z_j)\} \subset S_{A,B,C,D} \setminus S_{x_0}$ converging to some $q \in S_{x_0}$ and assume, in addition, the existence of $\delta > 0$ such that the argument of $x_j$ lies in an interval of the form $[\pi, 2\pi - \delta]$ for every $j$. Then for every $j$ large enough, there corresponds $n_j \in \mathbb{Z}$ such that $g^n_x(q_j) \in U$.

Analogous statements hold when $x$ is replaced by $y$ or $z$.

Proof. Recall that $g_x$ preserves every leaf of the foliation $D_x$ induced by the collection of rational curves $S_x \subset S_{A,B,C,D}$. The assumption on $x_0$ implies that $g_x$ on the curve $S_{x_0}$ corresponds to an elliptic element that is conjugate to an irrational rotation. Thus, up to saturating $U$ by the dynamics of $g_x$, there is no loss of generality in assuming that $U$ is a disc bundle over an annulus $A \subset S_{x_0}$, where $A$ is invariant under $g_x$. Note now that the action $g_j$ of $g_x$ on $S_x$, is loxodromic since $x_j \not\in \mathbb{R}$. Furthermore the “contracting coefficient” of $g_j$ tends to 1 as $q_j \rightarrow q$. In particular, for $j$ large enough, no fundamental domain of $g_j$ can contain the annulus $A$. The lemma follows at once. \hfill \Box

The preceding lemmata is essentially summarized by the proposition below.

Proposition 4.9. Let $x_0 \in (-2, 2) \setminus B^+_z$. Given two open sets $U_1, U_2 \subset S$, both of which intersect $S_{x_0}$, there is an iterate $n$ such that $g^n_x(U_1) \cap U_2 \neq \emptyset$. Analogous statements hold when $x_0$ is replaced with $y_0$ or $z_0$.

Proof. As in the proof of Lemma 4.6 we work within the tube $T_\epsilon(x_0)$ defined in Equation (12). Since $U_2$ is open and $U_2 \cap S_{x_0} \neq \emptyset$ they have infinitely many points of intersection. We can therefore pick a point $(x_0, y_0, z_0) \in U_2 \cap S_{x_0}$ so that $S_{y_0}$ intersects $S_{x_0}$ transversally (Lemma 4.4). We can make $\epsilon > 0$ smaller, if necessary, so that $S_{y_0} \cap T_\epsilon(x_0)$ is expressed as two smooth branches, each of them coinciding with the graph of a holomorphic function of $x$. Let $S_{y_0}(\epsilon)$ denote the branch passing through $(x_0, y_0, z_0)$. Since $U_2$ is open, it follows that $U_2 \cap S_{y_0}(\epsilon)$ is an open neighborhood of $(x_0, y_0, z_0)$ in $S_{y_0}(\epsilon)$. This means that we can reduce $\epsilon > 0$ even further, if necessary, so that we can assume the entire branch $S_{y_0}(\epsilon)$ is contained in $U_2$.

With the above setting, the combination of Lemma 4.6 and Remark 4.7 ensures the existence of $n$ such that $g^n_x(U_1) \cap S_{y_0}(\epsilon) \neq \emptyset$. Since $S_{y_0}(\epsilon) \subset U_2$, the proposition follows. \hfill \Box
Recalling that the sets $B_0^+$, $B_y^+$, and $B_{z_1}^+$ are all finite, Lemma 4.4 allows us to choose points $x_0 \neq x_1 \in (-2, 2) \setminus B_x$, $y_0 \neq y_1 \in (-2, 2) \setminus B_y$, $z_0 \neq z_1 \in (-2, 2) \setminus B_z$ and to form the “grid”

\[ G = S_{x_0} \cup S_{x_1} \cup S_{y_0} \cup S_{y_1} \cup S_{z_0} \cup S_{z_1} \]

so that

(i) every pair of curves (fibers) have transverse intersection (possibly empty), and
(ii) each $x_0, x_1, y_0, y_1, z_0,$ and $z_1$ is of the form $\pm 2 \cos(\theta \pi)$ for some irrational $\theta$.

We will also say that any pair of irreducible components of $G$ with empty intersection in $S_{A,B,C,D}$ are “parallel”, e.g. $S_{x_0}$ and $S_{x_1}$ are parallel.

**Proposition 4.10.** Let $U$ be any open set in $S_{A,B,C,D}$ that intersects the grid $G$. Then, for any irreducible component of the grid (say $S_{x_0}$) there exists $\gamma \in \Gamma$ with $\gamma(U)$ intersecting that chosen irreducible component (say $\gamma(U) \cap S_{x_0} \neq \emptyset$).

**Proof.** It is clearly enough to show the existence of $\gamma \in \Gamma$ with $\gamma(U) \cap S_{x_0} \neq \emptyset$. If $U$ already has non-trivial intersection with $S_{x_0}$ then there is nothing to prove. Otherwise, there are two cases to be considered:

**Case 1:** $U$ intersects an irreducible components of $G$ that is not parallel to $S_{x_0}$. Without loss of generality, we can suppose $U$ intersects $S_{y_0}$. Lemma 4.6 then implies the existence of an iterate $g^n_y$ of $g_y$ so that $g^n_y(U) \cap S_{x_0} \neq \emptyset$.

**Case 2:** $U$ intersects the component $S_{x_1}$ that is parallel to $S_{x_0}$. In this case, we first apply Lemma 4.6 to find an iterate $g^n_x$ of $g_x$ such that $g^n_x(U) \cap S_{y_0} \neq \emptyset$. Hence the problem is reduced to the situation treated in Case 1 so that it suffices to proceed accordingly. \(\square\)

**Corollary 4.11.** Let $U_1$ and $U_2$ be any two open sets in $S_{A,B,C,D}$ both of which intersect the grid $G$. Then, there exists $\gamma \in \Gamma$ with $\gamma(U_1) \cap U_2 \neq \emptyset$.

**Proof.** Using Proposition 4.10 we can find some $\gamma_1 \in \Gamma$ so that $\gamma_1(U_1)$ and $U_2$ both intersect the same irreducible component of $G$. The result then follows immediately from Proposition 4.9. \(\square\)

**Remark 4.12.** Concerning Corollary 4.11, note that the non-empty intersection $\gamma(U_1) \cap U_2$ may be disjoint from $G$ (and hence it might be disjoint from $J_{A,B,C,D}$ as well).

Proposition 4.13 below is the last ingredient in the proof of Theorem A. Notwithstanding its very elementary nature, this proposition is likely to find further applications in the study of the dynamics associated with the group $\Gamma$.

**Proposition 4.13.** For any open set $U$ intersecting the Julia set $J_{A,B,C,D}$ of $\Gamma$ non-trivially, there exists some element $\gamma \in \Gamma$ such that $\gamma(U) \cap G \neq \emptyset$.

**Proof.** Assume aiming at a contradiction that $\gamma(U)$ is disjoint from $G$ for every $\gamma \in \Gamma$. Then, for every $\gamma \in \Gamma$ we have that

\[ \iota \circ \gamma(U) \subset \left( \mathbb{C} \setminus \{x_0, x_1\} \right) \times \left( \mathbb{C} \setminus \{y_0, y_1\} \right) \times \left( \mathbb{C} \setminus \{z_0, z_1\} \right), \]

where $\iota : S_{A,B,C,D} \hookrightarrow \mathbb{C}^3$ denotes the inclusion. Applying Montel’s Theorem to each coordinate, this implies that the whole group $\Gamma$ forms a normal family on $U$. This contradicts the assumption that $U \cap J_{A,B,C,D} \neq \emptyset$ and establishes the statement. \(\square\)

We are finally ready to prove Theorem A.

**Proof of Theorem A.** The proof is based on Baire’s argument. First note that $J_{A,B,C,D}$ has the Baire property since it is a complete metric space as a closed subset of a manifold. The topology in $J_{A,B,C,D}$ is the one inherited from the topology of $S_{A,B,C,D}$ and hence it is second countable,
formed by all points in $J$ whose orbit intersects $V_n$. This set is clearly open since $V_n$ is so. It is also dense in $J$ since it intersects non-trivially every set $V_k$ defining a basis for the topology of $J$. Taking then the intersection over $n$

$$
\bigcap_{n=1}^{\infty} \bigcup_{\gamma \in \Gamma} \gamma^{-1}(V_n)
$$

we obtain a $G_\delta$-dense subset of $J$. By definition, the $\Gamma$-orbit of any point in this intersection visits all the open sets $V_k$ so that these points have dense orbits in $J$.

We start by working with the corresponding open sets $U_k$ and $U_k$ of $S$. We first use Proposition 4.13 to find $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1(U_k)$ and $\gamma_2(U_k)$ each hit the grid $G$. We can then use Proposition 4.10 to find $\gamma_3 \in \Gamma$ such that $\gamma_3 \circ \gamma_1(U_k)$ and $\gamma_2(U_k)$ intersect the same irreducible component of $G$. Without loss of generality, we suppose it is $S_{x_0}$.

Since $S_{x_0}$ is transverse to $S_{x_0}$, we can choose a sequence of points $\{q_j\}_{j=1}^{\infty} \subset S_{x_0}$ converging to $q \in S_{x_0} \cap S_{y_0}$ that satisfies the hypotheses of Lemma 4.8. Moreover, by Lemma 4.3, $S_{y_0} \subset J$ so each element of the sequence is in $J$. We therefore find a point $q_N \in J$ and $\gamma_4, \gamma_5 \in \Gamma$ with $\gamma_4(q_N) \in \gamma_3 \circ \gamma_1(U_k)$ and $\gamma_5(q_N) \in \gamma_2(U_k)$. In other words,

$$q_N \in \gamma_4^{-1} \circ \gamma_3 \circ \gamma_1(U_k) \bigcap \gamma_5^{-1} \circ \gamma_2(U_k) \bigcap J.
$$

Since $J$ is invariant under $\Gamma$, this proves that

$$\gamma_4^{-1} \circ \gamma_3 \circ \gamma_1(U_k) \cap \gamma_5^{-1} \circ \gamma_2(U_k) \neq \emptyset.
$$

We conclude that $\gamma(U_k) \cap U_k \neq \emptyset$ with $\gamma = \gamma_2^{-1} \circ \gamma_5 \circ \gamma_4^{-1} \circ \gamma_3 \circ \gamma_1$. □

Remark 4.14. After finding $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1(U_k)$ and $\gamma_2(U_k)$ each hit the grid $G$, it is tempting to use Corollary 4.11 to find $\gamma_3$ with

$$\gamma_3(\gamma_1(U_k)) \cap \gamma_2(U_k) \neq \emptyset.
$$

However, this does not necessarily prove that $\gamma_3(\gamma_1(U_k)) \cap \gamma_2(U_k) \neq \emptyset$ because the intersection (15) need not be in $G$ and hence it potentially might not contain any points of $J$. This is why we use the “quantitative” Lemma 4.8 instead.

The proof of Theorem B is considerably simpler than that of Theorem A.

Proof of Theorem B. Let us define a new “grid” $G'$ using Equation (13), but this time we will use

$$x_0 = y_0 = z_0 = 0 \quad \text{and} \quad x_1 = y_1 = z_1 = \sqrt{2}.
$$

These values are chosen so that

$$g_2^j|_{s_{x_0}} = \text{id}, \quad g_2^j|_{s_{y_0}} = \text{id}, \quad g_2^j|_{s_{z_0}} = \text{id}, \quad g_2^j|_{s_{x_1}} = \text{id}, \quad g_2^j|_{s_{y_1}} = \text{id}, \quad \text{and} \quad g_2^j|_{s_{z_1}} = \text{id}.
$$
As in Proposition 4.13, if $U$ is any open set that intersects $\mathcal{J}_{A,B,C,D}$ non-trivially then there is an element $\gamma \in \Gamma$ with $\gamma(U)$ intersecting $\mathcal{G}^\prime$. In fact, the proof of Proposition 4.13 does not use the choices that \(x_0, x_1 \notin B_x, y_0, y_1 \notin B_y, \) or \(z_0, z_1 \notin B_z\) that were made in the construction of our original grid $\mathcal{G}$ so that it applies equally well to $\mathcal{G}^\prime$.

Conjugating by $\gamma$, if necessary, it then suffices to prove that shear fixed points are dense in our newly chosen grid $\mathcal{G}^\prime$. We will prove it for $S_{x_0}$ and $S_{x_1}$ and leave the completely analogous proofs for $S_{y_0}, S_{y_1}$, $S_{y_1}$ and $S_{z_1}$ to the reader.

Every point of $S_{x_0}$ is a fixed point for $g_x^2$. We will show that all but finitely many of them are shear fixed points. Clearly this assertion is, in turn, equivalent to showing that the derivative $D(g_x)^2$ has two dimensional generalized eigenspace associated to eigenvalue 1 but only one eigenvector associated to eigenvalue 1.

Considering $g_x^2$ as a mapping from $\mathbb{C}^3 \to \mathbb{C}^3$ we have

\[
D(g_x^2)|_{x=0} = \begin{bmatrix}
1 & 0 & 0 \\
2z - C & 1 & 0 \\
B - 2y & 0 & 1
\end{bmatrix}.
\]

If $z \neq C/2$ or $y \neq B/2$ then this matrix has generalized eigenspace of dimension 3 associated to the eigenvalue 1 but only two eigenvectors, namely $e_2 = [0,1,0]$ and $e_3 = [0,0,1]$. Therefore, it suffices to prove that

\[
(16) \quad T_pS_{A,B,C,D} \neq \text{span}(e_2, e_3)
\]

for every $p \in S_{x_0}$ bar some finite set. Taking the gradient of the defining equation for $S_{A,B,C,D}$ yields that

\[
T_pS_{A,B,C,D} = \ker \left[ yz - A + 2x \quad zx - B + 2y \quad xy - C + 2z \right].
\]

Restricted to $x = x_0 = 0$ we can only have $e_2 \in T_pS_{A,B,C,D}$ if $y = B/2$ and we can only have $e_3 \in T_pS_{A,B,C,D}$ if $z = C/2$. Combined with $x = 0$ each of these conditions amounts to at most two points of $S_{x_0}$. Therefore, all but at most finitely many points of $S_{x_0}$ are shear fixed points of $g_x^2$.

The situation for $S_{x_1}$ is nearly the same, except that we work with $g_x^4$. Considering $g_x^4$ as a mapping from $\mathbb{C}^3 \to \mathbb{C}^3$ we have

\[
D(g_x^4)|_{x=\sqrt{2}} = \begin{bmatrix}
1 & 0 & 0 \\
-4C + 8z + 4\sqrt{2}y & 1 & 0 \\
4B - 8y - 4\sqrt{2}z & 0 & 1
\end{bmatrix}.
\]

In exactly the same way as for $g_x^2$ one can check that the restriction to $T_pS_{A,B,C,D}$ is a shear for all $p = (\sqrt{2},x,y) \in S_{x_1}$ so long as

\[
\sqrt{2}z - B + 2y \neq 0 \quad \text{and} \quad \sqrt{2}y - C + 2z \neq 0,
\]

each of which correspond to at most two points of $S_{x_1}$. \hfill \Box

Finally, the proof of Theorem C is now rather simple as well.

**Proof of Theorem C.** Let $p_1$ and $p_2$ be two arbitrary points in $\mathcal{J}_{A,B,C,D}$. We will show that for any neighborhoods $U_1$ of $p_1$ and $U_2$ of $p_2$, there is a path in $\mathcal{J}_{A,B,C,D}$ connecting from $U_1$ to $U_2$. Theorem C will immediately follow.

The grid $\mathcal{G}$ given in (13) is path connected and, by virtue of Lemma 4.3, we have $\mathcal{G} \subset \mathcal{J}_{A,B,C,D}$. Therefore, it suffices to find a path connecting from inside of $U_1$ to $\mathcal{G}$ and a path connecting from inside of $U_2$ to $\mathcal{G}$. As the situation is symmetric, it suffices to consider $U_1$.

Since $p_1 \in U_1 \cap \mathcal{J}_{A,B,C,D}$, Proposition 4.13 gives some $\gamma \in \Gamma$ such that $\gamma(U_1) \cap \mathcal{G} \neq \emptyset$. Let $C$ be an irreducible component of $\mathcal{G}$ with $\gamma(U_1) \cap C \neq \emptyset$. Since we have chosen the irreducible components of $\mathcal{G}$ to be smooth, $C$ is biholomorphic to $\mathbb{C} \setminus \{0\}$. 

---

**Footnotes:**

[13]: See Appendix A for details.

---

**References:**

[1]: Fatou and Julia, 1920, *Theorie des Fonctions Analytiques*. 
[2]: Sullivan, 1985, *Hyperbolic Geometry and Dynamics in the Plane*. 
[3]: McMullen, 1994, *Complex Dynamics and Hyperbolic Geometry*. 
[4]: Lyubich, 1999, *Holomorphic Dynamics and Renormalization*. 
[5]: Dodgson, 1860, *A Budget of Conjectures*. 
[6]: Milnor, 2002, *Dynamics in One Complex Variable*. 
[7]: Sullivan, 1982, *On Conformal Dynamics*. 
[8]: Wada, 1987, *The Fatou and Julia Sets of a Polynomial*. 
[9]: Mandelbrot, 1980, *The Fractal Geometry of Nature*. 
[10]: Douady and Hubbard, 1985, *On the Dynamics of Rational Maps: Polynomial-like Maps of the Riemann Sphere*. 
[11]: Carleson and Gamelin, 1993, *Complex Dynamics*. 
[12]: Brolin, 1988, *Real Analytic Dynamics*. 
[13]: See Appendix A for details.
Consider now the Riemann surface $\gamma^{-1}(C)$ contained in $S_{A,B,C,D} \subset \mathbb{C}^3$. Since $\gamma$ is a holomorphic diffeomorphism of $S_{A,B,C,D}$, there follows that $\gamma^{-1}(C)$ is again biholomorphic to $\mathbb{C} \setminus \{0\}$ and hence it is uniformized by $\mathbb{C}$ and contained in $\mathbb{C}^3$. We claim that $\gamma^{-1}(C)$ intersects the grid $\mathcal{G}$. Indeed, if we had $\gamma^{-1}(C) \cap \mathcal{G} = \emptyset$, the uniformization map from $\mathbb{C}$ to $\gamma^{-1}(C)$ would yield a (non-constant) holomorphic map from $\mathbb{C}$ to $\mathbb{C}^3$ each of whose coordinates omits two values in $\mathbb{C}$. Picard theorem would then imply that this map must be constant and this is impossible.

Finally, note that $\gamma^{-1}(C) \subset J_{A,B,C,D}$, since $J_{A,B,C,D}$ is invariant by $\Gamma$ and $C \subset \mathcal{G} \subset J_{A,B,C,D}$. Furthermore, by construction, $\gamma^{-1}(C)$ also intersects $U_1$. Since $\gamma^{-1}(C)$ is path connected, we can therefore find a path contained in $\gamma^{-1}(C) \subset J_{A,B,C,D}$ going from $U_1$ to $\mathcal{G}$. The proof of Theorem C is complete.

5. Picard parameters and proof of Theorem D

The parameters $(A, B, C, D) = (0, 0, 0, 4)$ are quite special for at least two reasons:

- The surface $S_{(0,0,0,4)}$ has the maximal number of singularities among all cubic surfaces and for this reason it is called the Cayley Cubic. They are at the four points
  \[(−2, −2, −2), (−2, 2, 2), (2, −2, 2), (2, 2, −2).\]

- It was proved by Cantat-Loray [9, Theorem 5.4] that $\Gamma$ has an invariant affine structure on $S_{A,B,C,D}$ if and only if $(A, B, C, D) = (0, 0, 0, 4)$.

Existence of the invariant affine structure dates back to work of Picard and, for this reason, the parameters $(A, B, C, D) = (0, 0, 0, 4)$ are called the Picard Parameters.

From our point of view, this case is also very interesting as it will soon be clear. More importantly, however, the information collected in the course of this discussion will enable us to prove Proposition 9.9 in Section 9. Albeit a somewhat technical statement, Proposition 9.9 plays an important role in the proofs of Theorems G and K.

Throughout this section we will typically drop the parameters from our notation, thus writing $S$ for $S_{0,0,0,4}$, $\Gamma$ for $\Gamma_{0,0,0,4}$, $J$ for $J_{0,0,0,4}$, and so on. We will denote the singular locus of $S$, given in \((17)\), by $S_{\text{sing}}$.

**Proposition 5.1.** For the Picard parameters, the action of $\Gamma$ on $S$ is locally discrete on any open $U \subset S$.

To prove Proposition 5.1 we will use the existence of a semi-conjugacy between the action of $\Gamma$ and the group action of monomial mappings on $\mathbb{C}^* \times \mathbb{C}^*$, which is described in the sequel (see for example [9, Section 1.5]).

Consider the group:

\[(18)\]
\[\Gamma(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{Z}) : a, d \equiv 1 \text{ mod } 2 \quad \text{and} \quad b, c \equiv 0 \text{ mod } 2 \right\}.\]

The group $\Gamma(2)$ is generated by the matrices

\[(19)\]
\[\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}\]

and it is straightforward to check that the mapping sending these matrices to $g_x, g_y,$ and $g_z$ is an isomorphism from $\Gamma(2)$ to $\Gamma$. For every $M \in \Gamma(2)$, the image of $M$ in $\Gamma$ by the above mentioned isomorphism will be denoted by $f_M$.

Associated with a matrix $M = \{m_{ij}\} \in \Gamma(2)$ is a monomial mapping $\eta_M : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$ given by

\[\eta_M \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} u^{m_{11} + m_{12}} \\ u^{m_{21} + m_{22}} \end{array} \right).\]
Also let $\Phi : \mathbb{C}^* \times \mathbb{C}^* \to S$ be defined by

$$\Phi(u,v) = \left( -u - \frac{1}{u}, -v - \frac{1}{v}, -u - \frac{1}{v}, -v - \frac{1}{u} \right).$$

It turns out that $\Phi$ is a degree two orbifold cover. Furthermore, a straightforward verification shows that the critical points of $\Phi$ are precisely the four points $(u,v) = (\pm 1, \pm 1)$ while the corresponding critical values are the four points of $S_{\text{sing}}$.

**Proposition 5.2.** $\Phi$ semi-conjugates the action of $\Gamma(2)$ on $\mathbb{C}^* \times \mathbb{C}^*$ to the action of $\Gamma$ on $S$. More specifically, given $M \in \Gamma(2)$ and $(u,v) \in \mathbb{C}^* \times \mathbb{C}^*$, we have

$$\Phi \circ \eta_M(u,v) = f_M \circ \Phi(u,v).$$

**Proof.** This can be checked by a direct calculation in terms of the generators (19) of $\Gamma(2)$ and of the generators $g_x$, $g_y$, and $g_z$ of $\Gamma$.

**Proof of Proposition 5.1** Suppose that there is an open $U \subset S$ and a sequence $M_n \in \Gamma(2) \setminus \{\text{id}\}$ such that $f_{M_n}|_U$ converge locally uniformly to the identity on $U$. Making $U$ smaller, if necessary, we can assume that $U$ is evenly covered by $\Phi$ and that $V$ is one of the two connected components of $\Phi^{-1}(U)$. Then, the monomial maps $\eta_{M_n}$ converge locally uniformly to the identity on $V$. However, this is impossible because $\Gamma(2)$ is a discrete subgroup of $GL(2,\mathbb{Z})$. Since the action is question is linear, from the discrete character of $\Gamma(2)$ there follows that this action on pairs $(\log |u|, \log |v|)$ is locally discrete as well.

The semiconjugacy from Proposition 5.2 also allows us to completely determine the Julia set associated with Picard parameters. Namely, we have:

**Proposition 5.3.** The Julia set $J_{0.0.0.4}$ is the whole surface $S$.

The proof of Proposition 5.3 will, however, require the following lemma:

**Lemma 5.4.** The set formed by the union over all hyperbolic elements of $\Gamma(2)$ of the corresponding eigendirections is dense in $\mathbb{P}^1(\mathbb{R})$.

**Proof.** Conjugating by elements of $\Gamma(2)$, the problem reduces to considering directions between $(1,0)^T$ and $(1,1)^T$. Let $p$ be any prime number and let $1 \leq q \leq p-1$ be any even number. Then, there exist positive integers $a$ and $b$ such that $bp - aq = 1$. Since $q$ is even, $b$ is odd. If $a$ is odd then we can replace $a$ and $b$ by $p + a$ and $q + b$, respectively. This allows us to assume that

$$\begin{pmatrix} p & a \\ q & b \end{pmatrix} \in \Gamma(2).$$

Choosing $p$ sufficiently large, it is a consequence of the Perron-Frobenius Theorem that this matrix has an eigenvector whose direction is arbitrarily close to $(p,q)^T$. Finally, by appropriately choosing $q$, every vector $(v_1,v_2)^T$ between $(1,0)^T$ and $(1,1)^T$ can be approximated. The lemma follows.

**Proof of Proposition 5.3** It suffices to prove that the Julia set $J$ for the monomial action on $\mathbb{C}^* \times \mathbb{C}^*$ is all of $\mathbb{C}^* \times \mathbb{C}^*$. Indeed, suppose there is an open $U \subset S$ contained in the Fatou set for the action of $\Gamma$ on $S$. Making $U$ smaller, if necessary, we can assume that $U$ is evenly covered by $\Phi$ and that $V$ is one of the two connected components of $\Phi^{-1}(U)$. Because of the semi-conjugacy $\Phi$, the assumption on $U$ would imply that $V$ is in the Fatou set for the monomial action of $\Gamma(2)$ on $\mathbb{C}^* \times \mathbb{C}^*$.

On the other hand, the unit torus $\mathbb{T}^2 \subset \mathbb{C}^* \times \mathbb{C}^*$ is invariant under the monomial action of $\Gamma(2)$ with the hyperbolic elements of $\Gamma(2)$ corresponding to Anosov mappings $\eta_M : \mathbb{T}^2 \to \mathbb{T}^2$. Therefore, $\mathbb{T}^2 \subset J$. Moreover, for each hyperbolic $M \in \Gamma(2)$ the hyperbolic set $\mathbb{T}^2$ has stable and unstable manifolds under $\eta_M$, namely:

$$W^s_M(\mathbb{T}^2) = \{ (u,v) \in \mathbb{C}^* \times \mathbb{C}^* : (\log |u|, \log |v|)^T \text{ is a stable eigenvector for } M \},$$

and

$$W^u_M(\mathbb{T}^2) = \{ (u,v) \in \mathbb{C}^* \times \mathbb{C}^* : (\log |u|, \log |v|)^T \text{ is an unstable eigenvector for } M \}.$$
bounded component and three unbounded components (see [2]). Let these invariant manifolds be of real-dimension three. Note that the analogous statement holds for periodic points as well.

Proof. It follows from Lemma 5.5 and from the discussion in the paragraph before this lemma that the analogous statement holds for periodic points as well.

Remark 5.6. It is interesting to point out that the eigenvalues in question can actually have different signs.

Proof. The critical values of $\Psi$ are precisely the singular point of $S$, which are permuted by elements of $\Gamma$. Therefore, $\Psi^{-1}(p) = \{q_1, q_2\}$ and they will either each be a fixed point for $\eta_M$ or they will form a period two cycle for $\eta_M$. In either case $(Df_M(p))^2$ and $D\eta_M(q_2)D\eta_M(q_1)$ will be conjugate matrices and hence have the same eigenvalues. Meanwhile $D\eta_M(q_2)D\eta_M(q_1)$ and $M^2$ have the same eigenvalues, so the result follows.

Proposition 5.7. For the Picard parameters, whenever $M$ is a hyperbolic matrix, every fixed point of the corresponding mapping $f_M$ not lying in $S_{\text{sing}}$ must be a hyperbolic saddle. In addition, these fixed points are all located on $S(\mathbb{R})_0 = \Phi(\mathbb{T}^2)$.

In particular, there is no dense $J_{0,0,0,4} \subset J_{0,0,0,4}$ consisting of points with hyperbolic stabilizers.

Proof. It follows from Lemma 5.5 and from the discussion in the paragraph before this lemma that a hyperbolic fixed point of an arbitrary element in $\Gamma$, in fact, must be a fixed point of some mapping $f_M$, where $M$ hyperbolic. Every such fixed point is therefore a hyperbolic saddle and is contained in $S(\mathbb{R})_0$. Clearly $S(\mathbb{R})_0$ is a proper subset of $S$ which, in turn, coincides with $J$ in view of Proposition 5.3. The proposition follows.

The proof of Theorem D is now easy:

Proof of Theorem D. The statement follows directly from the combination of Propositions 5.1, 5.3 and 5.7.

Recall that, by construction, every mapping $f_M : S \rightarrow S$ is the restriction of a polynomial diffeomorphism of $\mathbb{C}^3$ which will be denoted by $F_M : \mathbb{C}^3 \rightarrow \mathbb{C}^3$. These maps $F_M$ leaves invariant all the surfaces of the form $S_{0,0,0,D}$, with $D \in \mathbb{C}$. From this it follows that if $p \in S_{0,0,0,4}$ is a fixed point of $f_M$ then two of the eigenvalues of the $3 \times 3$ matrix $DF_M(p)$ are the same as those of $Df(p)$ and
the third eigenvalue is 1, provided that \( p \) is a regular point of \( S_{0,0,0,D} \). Owing to Lemma 5.5, we conclude that two of the eigenvalues of \( DF_M(p) \) have the same absolute values as the eigenvalues of \( M \) and the remaining eigenvalue is 1.

The following proposition describes the eigenvalues of \( DF_M(p) \) at singular points \( p \) of \( S \). The fact that the eigenvalues of \( M \) are squared is essentially the same phenomenon that occurs for the classical one-dimensional Chebyshev map, and we are grateful to Michał Misiurewicz for explaining it to us.

**Proposition 5.8.** Let \( (A,B,C,D) \) be the Picard Parameters \((0,0,0,4)\). For any \( M \in \Gamma(2) \) let \( f_M : S \to S \) be the corresponding element of \( \Gamma \) and let \( F_M : C^3 \to C^3 \) be its extension to \( C^3 \). For any \( p \in S_{\text{sing}} \) two of the eigenvalues of \( DF_M(p) \) are the squares of the eigenvalues of \( M \) and the remaining eigenvalue is 1.

**Proof.** The proof relies upon the semiconjugacy \( \Psi \) from Proposition 5.2. We apply it to points \( (u,v) = (e^{i\theta}, e^{i\phi}) \in T^2 \) and abuse notation slightly by writing

\[
(x,y,z) = \Psi(\theta,\phi) = (-2\cos \theta, -2\cos \phi, -2\cos(\theta - \phi)).
\]

The points \((0,0),(0,\pi), (\pi,0), \) and \((\pi,\pi)\) map by \( \Psi \) to the singular points in (17) in the respective order that they are listed there.

Here we focus on the singular point \( p = (-2,-2,-2) = \Psi(0,0) \). The minor adaptations required for the other singular points essentially amount to some sign modifications in the equations below and thus can safely be left to the reader.

Since \( M \in \Gamma(2) \) we have \( \det(M) = 1 \). Using this, the characteristic polynomial for \( M^2 \) is

\[
P_{M^2}(x) = x^2 - (m_{11}^2 + m_{22}^2 + 2m_{12}m_{21})x + 1.
\]

Now let \( N = DF_M(p) \) and write \( N = \{n_{jk}\}_{1 \leq j,k \leq 3} \). Next note that \( F_M(S_D) = S_D \) for every \( D \in C \) which amounts to saying that \( F_M \) preserves the polynomial \( Q(x,y,z) = x^2 + y^2 + z^2 + xyz \) in the sense that \( P \circ F_M(x,y,z) = P(x,y,z) \). Also the restriction of \( F_M \) to \( S_D \) preserves the volume form associated to \( \Omega \). By integrating this volume form over the the surfaces \( S_D \) in the transverse direction, we see that \( F_M \) also preserves a volume form on \( C^3 \). Since \( p \) is a fixed point of \( F_M \), the preceding conditions combine to ensure that \( \det(N) = 1 \) and, in addition, that one of the eigenvalues of \( N \) equals 1. Hence, the characteristic polynomial of \( N \) is

\[
P_N(x) = x^3 - (n_{11} + n_{22} + n_{33})x^2 + (n_{11} + n_{22} + n_{33})x - 1.
\]

In the sequel we will show that

\[
(21) \quad n_{11} + n_{22} + n_{33} = m_{11}^2 + m_{22}^2 + 2m_{12}m_{21} + 1.
\]

This will imply that \( P_N(x) = P_{M^2}(x)(x - 1) \) therefore completing the proof of Proposition 5.8.

To begin, consider the \( x \)-coordinate of the semi-conjugacy (20):

\[
-2\cos(m_{11}\theta + m_{12}\phi) = f_{M,1}(-2\cos \theta, -2\cos \phi, -2\cos(\theta - \phi)),
\]

where we have added the subscript 1 to denote the first coordinate of \( F_M \). Setting \( \phi = 0 \) and taking the partial derivative with respect to \( \theta \) yields

\[
2\sin(m_{11}\theta) m_{11} = \frac{\partial f_{M,1}}{\partial x}(-2\cos \theta, -2, -2\cos \theta)2\sin \theta + \frac{\partial f_{M,1}}{\partial z}(-2\cos \theta, -2, -2\cos \theta)2\sin \theta.
\]

Next, for \( \theta \neq 0 \), we divide both sides of the above equation by \( 2\theta \) so as to obtain

\[
\frac{\sin(m_{11}\theta)}{m_{11}\theta} m_{21} = \frac{\partial f_{M,1}}{\partial x}(-2\cos \theta, -2, -2\cos \theta)\frac{\sin \theta}{\theta} + \frac{\partial f_{M,1}}{\partial z}(-2\cos \theta, -2, -2\cos \theta)\frac{\sin \theta}{\theta}.
\]

Now it suffices to take the limit as \( \theta \) goes to 0 to conclude that

\[
m_{11}^2 = n_{11} + n_{13}.
\]
Similarly, setting $\theta = 0$ and doing the analogous computation involving partial derivatives with respect to $\phi$ yields
\[ m_{12}^2 = n_{12} + n_{13}. \]
Finally, the analogous computation with $\phi = \theta$ lead to
\[ (m_{11} + m_{12})^2 = n_{11} + n_{12}. \]
The three previous equations can be solved for $n_{11}$ to find
\[ n_{11} = m_{11}^2 + m_{11}m_{12}. \]
The same computations with the second and third coordinate of the semi-conjugacy \((20)\) yield
\[ n_{22} = m_{22}^2 + m_{22}m_{21}, \quad \text{and} \quad n_{33} = (m_{11} - m_{21})(m_{22} - m_{12}). \]
Combined with the fact that $\det(M) = 1$, Equations \((22)\) and \((23)\) imply that the condition expressed by \((21)\) holds. The proof of the proposition is completed. \(\square\)

The following corollary summarizes Proposition \ref{prop:hyperbolic} and the two paragraphs before it.

**Corollary 5.9.** Let $(A, B, C, D)$ be the Picard Parameters $(0,0,0,4)$. Assume that $M \in \Gamma(2)$ is hyperbolic. Denote by $f_M : S \to S$ the element of $\Gamma$ associated with $M$ and let $F_M : \mathbb{C}^3 \to \mathbb{C}^3$ be the corresponding extension of $f_M$ to $\mathbb{C}^3$. If $p \in S$ is a fixed point of $F_M$ then $DF_M(p)$ has one eigenvalue of modulus less than one, one eigenvalue equal to one, and one eigenvalue of modulus greater than one.

### 6. Existence of Fatou Components and Proof of Theorem E

Before proving Theorem E we will show that the action of the bigger group $\Gamma^*_{A,B,C}$ on $\mathbb{C}^3$ always has non-empty Fatou set $F_{A,B,C}$. Given parameters $(A, B, C)$, Proposition \ref{prop:fatou} will provide a point $p \in \mathbb{C}^3$ and $\epsilon > 0$ so that the ball $B_{\epsilon}(p)$ of radius $\epsilon$ around $p$ is contained in $F_{A,B,C}$. In particular, to obtain 4-tuples of parameters $(A, B, C, D) \in \mathbb{C}^4$ for which the action of $\Gamma_{A,B,C,D}$ on $S_{A,B,C,D}$ has non-empty Fatou set $F_{A,B,C,D}$, it will be enough to select $D$ so that $S_{A,B,C,D} \cap B_{\epsilon}(p) \neq \emptyset$; see Corollary \ref{cor:fatou}. Although we focus the application of Proposition \ref{prop:fatou} and of Corollary \ref{cor:fatou} to Theorem E, they apply to many other settings.

**Proposition 6.1.** For any choice of parameters $(A, B, C) \in \mathbb{C}^3$, let $r = \max\{|A|, |B|, |C|\}$. Next, given $R > 2 + \sqrt{r}$, let
\[ \epsilon = \min\{R - (2 + \sqrt{r}), R + 1 - 4R + r + 1\} > 0. \]
If $|u| = R$ and $p = (u, u, u) \in \mathbb{C}^3$ then the open ball $B_{\epsilon}(p)$ is contained in the Fatou set $F_{A,B,C}$ for the action of $\Gamma^*_{A,B,C}$ on $\mathbb{C}^3$. In particular $F_{A,B,C} \neq \emptyset$.

The idea for the proof of Proposition \ref{prop:fatou} comes from the paper \cite{hu-tan-zhang} of Hu-Tan-Zhang and probably dates back to Bowditch \cite{bowditch}.

**Proof.** Let $q \in B_{\epsilon}(p)$ and note that this implies that each coordinate of $q$ has modulus larger than $2 + \sqrt{r}$. We will show that if $w_1w_2\ldots w_k$ is any reduced word in $s_x, s_y,$ and $s_z$ then, for each $0 \leq j < k$, each coordinate of $w_1w_2\ldots w_j(q)$ has modulus less than or equal to the modulus of the corresponding coordinate of $w_1w_2\ldots w_{j+1}(q)$. Here, the word $w_1w_2\ldots w_j$ is to be interpreted as the identity map provided that $j = 0$. This claim clearly implies that for every $\gamma \in \Gamma^*$ we have
\[ \gamma(B_{\epsilon}(p)) \subset \left(\mathbb{C} \setminus \mathbb{D}_{2+\sqrt{r}}(0)\right)^3. \]
Applying Montel’s Theorem to each coordinate implies that the action of $\Gamma^*$ is normal on $B_{\epsilon}(p)$ so that the statement follows.
To check the claim, we begin by fixing \( q = (x, y, z) \in B_r(p) \). Consider the involution \( s_x \) and the points \( q \) and \( s_x(q) \). Clearly the coordinates \( y \) and \( z \) of these two points coincide. In turn, to show that the modulus of the \( x \) coordinate of \( q \) is strictly smaller than the modulus of the \( x \) coordinate of \( s_x(q) \), note that

\[
|\pi_x(s_x(q))| = |yz - x + A| > (R - \epsilon)^2 - (R + \epsilon) - r \geq R + \epsilon > |\pi_x(q)|,
\]

where the second inequality follows from the assumption that \( \epsilon \geq R + 1 - \sqrt{4R + r + 1} \). Indeed, this condition can be reformulated as \( R - \epsilon + 1 \geq \sqrt{4R + r + 1} \) which, by taking squares on both sides, leads right away to the inequality in question. Naturally, analogous estimates hold with respect to the coordinates \( y \) or \( z \) when \( s_x \) is replaced by \( s_y \) and \( s_z \). Therefore, we have shown that for every point \( q \in B_r(p) \), applying \( s_x, s_y, \) or \( s_z \) to \( q \) strictly increases the modulus of one of the coordinates while leaving the other two coordinates unchanged.

Now consider any reduced word \( w_1w_2 \ldots w_k \) in \( s_x, s_y, \) and \( s_z \). Suppose for a contradiction that there is some \( q \in B_r(p) \) and some \( 0 \leq j < k \) so that \( w_1 \ldots w_j(q) \) has some coordinate of modulus strictly larger than the corresponding coordinate of \( w_1 \ldots w_{j+1}(q) \). If there are more than one such \( j \) where this happens we choose the smallest one, which satisfies \( j \geq 1 \) by virtue of the calculation in the previous paragraph. Note also that taking \( j \) to be minimal implies that each coordinate of \( w_1 \ldots w_j(q) \) has modulus greater than or equal to the minimal modulus of a coordinate of \( q \) which, in turn, exceeds \( 2 + \sqrt{r} = 2 + \sqrt{\max\{ |A|, |B|, |C| \}} \).

Let

\[
(x, y, z) = w_1 \ldots w_{j-1}(q), \quad (x', y', z') = w_1 \ldots w_j(q), \quad \text{and} \quad (x'', y'', z'') = w_1 \ldots w_{j+1}(q).
\]

In particular, the preceding ensures that \( \min \{ |x'|, |y'|, |z'| \} \geq 2 + \sqrt{r} \). On the other hand, since the word \( w \) is reduced and all generators \( s_x, s_y, s_z \) are involutions, we must have \( w_j \neq w_{j+1} \). Without loss of generality, we can then suppose \( w_j = s_x \) and \( w_{j+1} = s_y \). This yields

\[
(x', y', z') = (-x - yz + A, y, z), \quad \text{and} \quad (x'', y'', z'') = (x', -y' - x'z' + B, z').
\]

Our assumption on \( j \) implies \( |x| \leq |x'| \) and \( |y| > |y''| \). Therefore,

\[
2|x'| \geq |x' + x| = |yz + A| = |y'z' + A| \quad \text{and} \quad 2|y'| > |y' + y''| = |x'z' + B|.
\]

We now split the discussion in two cases. Assume first that \( |x'| \geq |y'| \). Then, the second inequality from (26) gives

\[
|x'z'| - |B| \leq 2|y'| \leq 2|x'|.
\]

In turn, moving \( |B| \) to the right side, dividing by \( |x'| \), and recalling that \( |x'| \geq 2 + \sqrt{r} \) leads to

\[
|z'| \leq 2 + \frac{|B|}{|x'|} < 2 + \frac{r}{2 + \sqrt{r}} < 2 + \sqrt{r}.
\]

This is impossible since \( \min \{ |x'|, |y'|, |z'| \} \geq 2 + \sqrt{r} \). If we consider now the case where \( |y'| > |x'| \), we just need to use the first inequality from (26) to similarly show that

\[
|z'| < 2 + \frac{|A|}{|y'|} < 2 + \sqrt{r}.
\]

Thus, in any event, we obtain a contradiction that proves our initial claim. As already pointed out, the ball \( B_r(p) \) must therefore lie in the Fatou set of the action of \( \Gamma_{A,B,C} \) on \( \mathbb{C}^3 \). The proposition is proved.
Corollary 6.2. For any parameters \((A_0, B_0, C_0) \subseteq \mathbb{C}^3\) suppose that \(p = (u, u, u)\) with \(|u| > 2 + \sqrt{r}\), where \(r\) is given as in Proposition 6.1. Let \(D_0\) be chosen so that \(p \in S_{A_0,B_0,C_0,D_0}\).

Then there is some \(\delta > 0\) such that for all parameters \((A, B, C, D) \subseteq \mathbb{D}_\delta(A_0, B_0, C_0, D_0) \subseteq \mathbb{C}^4\) the Fatou set \(\mathcal{F}_{A, B, C, D}\) for the action of \(\Gamma_{A, B, C, D}\) on \(S_{A, B, C, D}\) is non-empty.

Proof. The condition on \(\varepsilon > 0\) given in (24) depends continuously on \(r = \max\{|A|, |B|, |C|\}\). Therefore there exists \(\delta_0 > 0\) and \(\varepsilon_0 > 0\) such that if \((A, B, C) \subseteq \mathbb{D}_{\delta_0}(A_0, B_0, C_0) \subseteq \mathbb{C}^3\) then \(\mathcal{F} = \mathcal{F}_{A, B, C} \subseteq \mathbb{C}^3\). On the other hand, the point \((x, u, u)\) lies in the surface \(S_{A, B, C, D}\) where

\[
D = x^2 + 2u^2 + xu^2 - Ax - Bu - Cu.
\]

This polynomial is monic and non-constant in \(x\) so that its roots vary continuously with \((A, B, C, D)\). Since it has a root at \(x = u\) when \((A, B, C, D) = (A_0, B_0, C_0, D_0)\) we can find some \(0 < \delta < \delta_1\) such that if \((A, B, C, D) \subseteq \mathbb{D}_\delta(A_0, B_0, C_0, D_0)\) then

\[
(x, u, u) \in \mathcal{F} = \mathcal{F}_{A, B, C} \subset S_{A, B, C, D} = S_{A, B, C, D}.
\]

\(\square\)

The proof of Theorem E will be a quick application of Corollary 6.2 combined with the following elementary lemma.

Lemma 6.3. For every \(D \subseteq \mathbb{C} \setminus \{4\}\), the polynomial \(p(u) = u^3 + 3u^2 = D\) has a solution with modulus strictly larger than 2.

Proof. First suppose that \(p(u) = D\) has no solution of modulus exactly equal to 2. Let \(\alpha(t) = 2e^{it}, t \in [0, 2\pi]\), be a parameterization of the circle of radius 2 around the origin. By the argument principle, we must show that the winding number of the curve

\[
\beta(t) = p \circ \alpha(t) = 8e^{3it} + 12e^{2it}, \quad t \in [0, 2\pi],
\]

around any \(D \neq 4\) is at most 2. This curve is sketched in Figure 1. The main properties of this curve can easily be rigorously justified. First \(\beta(t)\) has a unique (irreducible) singular point occurring for \(t = \pi\) and placed at \(D = 4\). This point is actually a cusp. Furthermore, the curve \(\beta\) crosses the real axis four times, namely: at \(20 (t = 0)\), twice at the point \(-16 (t = \arcsin(\sqrt{15}/4))\) and \(t = 2\pi - \arcsin(\sqrt{15}/4)\), and at the cusp point \(4 (t = \pi)\). This curve has a self-intersection at the point \(-16\), where two smooth branches of \(\beta\) intersect each other. Finally a simple analysis of monotonicity intervals for the function \(t \mapsto |\beta(t)|\) justifies that \(\beta\) is topologically the union of two circles: an inner circle contained in the interior of another (outer) circle such that the two circles have a single common point at \(-16\). In particular, the winding number of \(\beta\) around any point \(z \subseteq \mathbb{C} \setminus \beta\) is at most 2. In particular, the statement follows for all those values of \(D\) for which \(D\) is not on the curve \(\beta\).

It remains to check the lemma for those \(D \subseteq \beta \setminus \{4\}\). Note that for the double point \(D = -16\) we have \(p(-4) = -16\), so it remains to consider the possibility that \(D\) is a smooth point of \(\beta\). In this situation there is a unique \(u_0\) on the circle \(\alpha\) with \(p(u_0) = D\). We can make a indentation \(\tilde{\alpha}\) of \(\alpha\) in an arbitrarily small neighborhood of \(u_0\), so that \(u_0\) is in the unbounded component of \(\mathbb{C} \setminus \tilde{\alpha}\) and so that the number of solutions to \(p(u) = D\) of modulus strictly greater than 2 is the winding number around \(D\) of \(\beta = p \circ \alpha(t), t \in [0, 2\pi]\). The only critical points of \(p(u)\) are \(u = 0\) and \(u = -2\), with \(p(0) = 0 \not\in \beta\) and with \(p(-2) = 4\) being the value of \(D\) that is excluded by hypothesis. Therefore, if \(D\) is on the “outer circle” of \(\beta\) it will in the unbounded component of \(\mathbb{C} \setminus \beta\) and we find that \(p(u) = D\) has two solutions of modulus greater than 2. If \(D\) is on the “inner circle” of \(\beta\), it will be “outside” of the inner circle of \(\beta\) and thus \(\beta\) will have winding number around \(D\) equal to 1. In this case, \(p(u) = D\) has one root of modulus less than 2, one of modulus equal to 2, and one of modulus greater than 2. The lemma is proved. \(\square\)

We are now ready to prove Theorem E.
The parametric curve $t \mapsto (2e^{it})^3 + 3(2e^{it})^2$ corresponding to $p_D \circ \alpha(t)$ when $r = 2$.

Proof of Theorem E. Consider first the Punctured Torus Parameters $A = B = C = 0$. The condition for a point $p = (u,u,u)$ to lie in $S_{0,0,D_0}$ is

$$u^3 + 3u^2 = D_0.$$

If $D_0 \neq 4$, Lemma 6.3 ensures that there is a point $p = (u,u,u) \in S_{0,0,D_0}$ with $|u| > 2$. Corollary 6.2 ensures the existence of $\delta > 0$ such that for all $(A,B,C,D) \in \mathbb{D}_\delta((0,0,0,D_0))$ the Fatou set $\mathcal{F}_{A,B,C,D}$ is non-empty. This establishes the first part of Theorem E.

Now consider the Dubrovin-Mazzocco parameters

$$A(a) = B(a) = C(a) = 2a + 4, \quad D(a) = -(a^2 + 8a + 8)$$

for $a \in (-2,2)$. Let us denote the surface $S_{A,B,C,D}$ at these parameters by $S_a$. The condition for $(u,u,u)$ to belong to $S_a$ is given by

$$p_a(u) = u^3 - 3(2a + 4)u + 3u^2 + a^2 + 8a + 8 = 0.$$

A direct calculation shows that if you substitute $u = -(2 + \sqrt{a + 4}) = -(2 + \sqrt{r})$ into $p_a(u)$ the result is positive. Hence, there is a real $u_0 < -(2 + \sqrt{r})$ satisfying $p(u_0) = 0$. Hence, again Corollary 6.2 implies that for any $a \in (-2,2)$ there exists $\delta > 0$ such that for all $(A,B,C,D) \in \mathbb{D}_\delta((A(a),B(a),C(a),D(a)))$ the Fatou set $\mathcal{F}_{A,B,C,D}$ is non-empty. The proof of Theorem E is complete.

7. Locally non-discrete dynamics in $\Gamma_{A,B,C,D}$.

Let $M$ be a (possibly open) connected complex manifold and consider a group $G$ of holomorphic diffeomorphisms of $M$. The group $G$ is said to be locally non-discrete on an open $U \subset M$ if there is a sequence of maps $\{f_n\}_{n=0}^\infty \in G$ satisfying the following conditions (see for example [49]):

1. For every $n$, $f_n$ is different from the identity.
2. The sequence of maps $f_n$ converges uniformly to the identity on compact subsets of $U$.

If no such sequence of maps $\{f_n\}_{n=0}^\infty \in G$ exists then $G$ is said to be locally discrete on $U$. The reader will note that a group $G$ may be locally non-discrete on one open set $U$ and locally discrete on a disjoint open set $V$. 
The objective of this section is to show that the set of parameters $A, B, C,$ and $D$ for which the resulting group $\Gamma$ is locally non-discrete on some open subset of the surface $S_{A,B,C,D}$ is very large: in particular, it has non-empty interior as a subset of $\mathbb{C}^4$. This will basically be done by means of a number explicit examples of parameters lying in the interior of the set in question.

It is convenient to begin our discussion with Proposition 7.1 below. This proposition constitutes a simple general result of which specific variants appear in [49, p. 9-10] and in [40, Sec. 3] while the main idea dates back to Ghys in [24]. Both Proposition 7.1 and Lemma 7.2 below, are stated in the wider context of pseudogroups $G$ of holomorphic maps of open sets $M$ to $\mathcal{M}$. In this context, the conditions in the definition of $G$ being locally non-discrete on $U$ become:

(0) The open set $U$ is contained in the domain of definition of $f_n$ (as element of the pseudogroup $G$), for every $n$.
(1) For every $n$, the restriction of $f_n$ to $U$ is different from the identity.
(2) The sequence of maps $f_n$ converges uniformly to the identity on compact subsets of $U$.

Condition (0) is required for Conditions (1) and (2) to make sense and Condition (1) has been modified because of the possibility that $U$ not be connected.

Consider a ball $B_\epsilon(0) \subset \mathbb{C}^n$ of radius $\epsilon > 0$ around the origin of $\mathbb{C}^n$. Assume we are given local holomorphic diffeomorphisms $F_1, F_2 : B_\epsilon(0) \to \mathbb{C}^n$ and denote by $G$ the pseudogroup of maps from $B_\epsilon(0)$ to $\mathbb{C}^n$ generated by $F_1, F_2$. Naturally the inverses of $F_1, F_2$ are respectively denoted by $F^{-1}_1$ and $F^{-1}_2$. In what follows we can assume without loss of generality that the domain of definition of $F_1^{-1}$ and $F_2^{-1}$ as elements of $G$ is non-empty. Let us then define a sequence $S(n)$ of sets of elements in $G$ by letting $S(0) = \{F_1, F^{-1}_1, F_2, F^{-1}_2\}$. The sets $S(n)$ are now inductively defined by stating that $S(n+1)$ is constituted by all elements of the form $[\gamma_i, \gamma_j] = \gamma_i \circ \gamma_j \circ \gamma_i^{-1} \circ \gamma_j^{-1}$ with $\gamma_i, \gamma_j \in S(n)$. Note that the construction of these elements is so far purely formal in the sense that the domain of definition (contained in $B_\epsilon(0)$) of diffeomorphisms in $S(n)$ viewed as elements of $G$ may be empty. Nonetheless, we have:

**Proposition 7.1.** Given $\epsilon > 0$, there is $K = K(\epsilon) > 0$ such that, if

$$
\max \left\{ \sup_{z \in B_\epsilon(0)} \|F_1(z) - z\|, \sup_{z \in B_\epsilon(0)} \|F_2(z) - z\| \right\} < K,
$$

then the following hold:

1. For every $n$ and every $\gamma \in S(n)$, the domain of definition of $\gamma$ as element in $G$ contains the ball $B_{\epsilon/2}(0) \subset \mathbb{C}^n$ of radius $\epsilon/2$ around the origin.
2. Furthermore, if $\gamma$ belongs to $S(n)$ then we have

$$
\sup_{p \in B_{\epsilon/2}(0)} \|\gamma(p) - p\| \leq \frac{K}{2^n}.
$$

It should be pointed out that, in general, the above proposition falls short of implying that the pseudogroup $G$ is locally non-discrete since the sets $S(n)$ may degenerate so as to only contain the identity map.

As mentioned, variants of Proposition 7.1 can be found in the literature and, to the best of our knowledge, the idea goes back to Ghys in [24]. Otherwise, its proof is nearly identical to that from [49, p. 9-10], so we provide only a sketch.

**Proof of Proposition 7.1.** The proof is based on the following estimate from [40, Lemma 3.0]. Let $B_\epsilon(0) \subset \mathbb{C}^n$ be an open ball and suppose $f_1, f_2 : B_\epsilon(0) \to \mathbb{C}^n$ are holomorphic local diffeomorphisms. If

$$
\max \left\{ \sup_{z \in B_\epsilon} \|f_1^\pm(z) - z\|, \sup_{z \in B_\epsilon} \|f_2^\pm(z) - z\| \right\} \leq K
$$

(28) then

$$
\sup_{p \in B_{\epsilon/2}(0)} \|\gamma(p) - p\| \leq \frac{K}{2^n}.
$$

(27)
for some $K > 0$, then for any $\tau > 0$ satisfying $4K + \tau < \epsilon$ the commutator $[f_1, f_2]$ is defined on the ball of radius $\epsilon - 4K - \tau$ and satisfies

$$\sup_{z \in B_{\epsilon - 4K - \tau}} \| [f_1, f_2](z) - z \| \leq \frac{2}{\tau} \sup_{z \in B_{\epsilon - 4K - \tau}} \| f_1(z) - z \| \cdot \sup_{z \in B_{\epsilon - 4K - \tau}} \| f_2(z) - z \|.$$  

Starting with $S(0)$, we choose $\tau = \tau_0 = K = K_0$ and $\epsilon_1 = \epsilon - 8K$ so that the preceding yields

$$\sup_{z \in B_{\epsilon_1}} \| \gamma(z) - z \| \leq \frac{K}{2},$$

for every $\gamma \in S(1)$. Now inductively setting $K_i = K/2^i$, $\tau_i = 4K_i$ it is straightforward to conclude that

$$\sup_{z \in B_{\epsilon_n}} \| \gamma(z) - z \| \leq \frac{K}{2^n},$$

for every $\gamma \in S(n)$, where

$$\epsilon_n = \epsilon - 8K - K \sum_{j=1}^{n-1} 2^{3-j}.$$  

The proposition then follows by choosing $K$ sufficiently small that $\epsilon_n \geq \epsilon/2$ for all $n$. \hfill \Box

The following simple lemma complements Proposition 7.1.

**Lemma 7.2.** Let $F_1, F_2 : B_\epsilon(0) \to \mathbb{C}^n$ be as above with $F_1(0) = F_2(0) = 0$. Assume that their derivatives at the origin satisfy

$$\max \{ \| D_0F_1 - \text{Id} \|, \| D_0F_2 - \text{Id} \| \} < \tau,$$

where $\tau > 0$ is some uniform (universal) to be determined later (the norm used here is the standard norm on linear operators on $\mathbb{C}^n$). Then, there is some $0 < \delta < \epsilon$ such that $F_1$ and $F_2$ satisfy the hypotheses of Proposition 7.1 on $B_\delta(0)$.

**Proof.** There follows from the proof of Proposition 7.1 that the relation between $\epsilon$ and $K = K(\epsilon)$ given by (30) is linear. Thus for $\epsilon = 1$, $K = K(1)$ becomes a uniform (universal) constant. We then choose $\tau = K/2$ so that $\tau$ is also universal.

In view of the above remark, we proceed as follows. Fix $\delta < \epsilon$ and consider the homothety $\Lambda_\delta : \mathbb{C}^n \to \mathbb{C}^n$ sending $(x_1, \ldots, x_n)$ to $(\delta x_1, \ldots, \delta x_n)$. Set $F_{j, \delta} = \Lambda_\delta^{-1} \circ F_j \circ \Lambda_\delta$ and note that $F_1, F_2$ satisfy the conditions of Proposition 7.1 on the ball of radius $\delta$ if and only if we have

$$\max \left\{ \sup_{z \in B_\delta} \| F_{1, \delta}(z) - z \|, \sup_{z \in B_\delta} \| F_{2, \delta}(z) - z \| \right\} < K = K(1).$$

We will now check that Estimate (32) is always satisfied provided that $\delta$ is small enough. Clearly, it suffices to consider the case of $F_{1, \delta}$. Owing to Taylor formula, we have

$$\sup_{z \in B_\delta} \| F_{1, \delta}(z) - z \| \leq \| D_0F_1(z) - z \| + O(\delta)$$

$$\leq K/2 + O(\delta) < K$$

provided that $\delta$ is small enough. The result is proved. \hfill \Box

Let $\mathcal{ND} \subset \mathbb{C}^4$ the set of those parameters $(A, B, C, D)$ giving rise to a group $\Gamma$ acting locally non-discretely on some open subset of the surface $S_{A,B,C,D}$. The remainder of this section is devoted to exhibiting several explicit examples of parameters in the interior of $\mathcal{ND}$. The following proposition will be rather useful.
Proposition 7.3. Suppose that \((A_0, B_0, C_0)\) are parameters for which there are two non-commuting elements \(F_1, F_2 \in \Gamma\) sharing a common fixed point \(p \in \mathbb{C}^3\). Assume also that the derivatives of \(F_1, F_2\) at \(p\) satisfy inequality (31). Then the following holds:

1. There exists \(r > 0\) such that for all parameters \((A, B, C)\) sufficiently close to \((A_0, B_0, C_0)\) the group \(\Gamma\) acting on \(\mathbb{C}^3\) is locally non-discrete on the open ball \(B_r(p) \subset \mathbb{C}^3\).

2. Let \(D_0\) be chosen so that \(p \in S_{A_0, B_0, C_0, D_0}\). Then for all parameters \((A, B, C, D)\) sufficiently close to \((A_0, B_0, C_0, D_0)\) the group \(\Gamma\) acting on \(S_{A, B, C, D}\) is locally non-discrete on the open set \(S_{A, B, C, D} \cap B_r(p)\). In other words, \((A_0, B_0, C_0, D_0)\) is an interior point of \(\mathcal{ND}\).

Recall from the introduction that El-Huiti’s theorem implies that \(\Gamma\) is the free group on two generators (one can choose any two of the three mappings \(g_x, g_y, g_z\) as generators). Therefore, two elements \(\gamma_1, \gamma_2 \in \Gamma\) commute if and only if there exists \(a \in \Gamma\) so that \(\gamma_1\) and \(\gamma_2\) are both powers of \(a\). This is an immediate consequence of the Nielsen-Schreier Theorem which states that any subgroup of a free group is free.

The proof of Proposition 7.3 will require a simple algebraic lemma:

Lemma 7.4. Let \(F_n\) denote the free group on \(n \geq 2\) symbols and suppose \(a, b \in F_n\) do not commute. Then the subgroup

\[ H = \langle [a, b], [a^{-1}, b^{-1}] \rangle \leq F_n \]

is again free on two or more symbols.

Proof. The Nielsen-Schreier Theorem implies that \(H\) is a free group. To see that it has rank two, it suffices check that \([a, b]\) and \([a^{-1}, b^{-1}]\) do not commute. This follows because

\[ [a, b][a^{-1}, b^{-1}][a, b]^{-1}[a^{-1}, b^{-1}]^{-1} \]

is a non-trivial reduced word in \(a, b, a^{-1}\), and \(b^{-1}\). Since \(a\) and \(b\) do not commute this word does not reduce to the identity in \(F_n\). \(\square\)

Proof of Proposition 7.3: Lemma 7.2 implies the existence of some \(\delta > 0\) such that for parameters \((A_0, B_0, C_0)\) the mappings \(F_1\) and \(F_2\) satisfy the hypotheses of Proposition 7.1 on \(B_\delta(p)\). Moreover the conditions of Proposition 7.1 are open in the \(\mathcal{C}^0\) topology (on \(B_\delta(p)\)). This implies that for all parameters \((A, B, C)\) sufficiently close to \((A_0, B_0, C_0)\) the mappings \(F_1\) and \(F_2\) continue to satisfy the hypotheses of Proposition 7.1 on \(B_\delta(p)\). Thus, for these parameters \((A, B, C)\), the elements in the iterated commutators \(S(n)\) converge uniformly to the identity on the ball \(B_{\delta/2}(0) \subset \mathbb{C}^3\).

Since \(F_1\) and \(F_2\) do not commute, Lemma 7.4 can inductively be applied to ensure that each set \(S(n)\) contains at least two non-commuting elements. Therefore, for every \(n \geq 0\), there are elements in \(S(n)\) that are different from the identity which, in turn, proves that the pseudogroup generated by \(F_1\) and \(F_2\) is locally non-discrete on \(B_r(p)\).

Statement (2) then follows immediately because elements of \(\Gamma\) different to the identity cannot coincide with the identity when restricted to any surface \(S_{A, B, C, D}\). In other words, for every choice of parameters \((A, B, C, D)\), any non-trivial reduced word in \(g_x\) and \(g_y\) (or in any two of the three mappings \(g_x, g_y, g_z\)) induces a mapping of \(S_{A, B, C, D}\) that is different from the identity. \(\square\)

Example 1. It is an easy observation that for the parameters \(A = B = C = 0\) the origin \((0, 0, 0) \in \mathbb{C}^3\) is a common fixed point for \(g_x, g_y, g_z\) and that their derivatives at the origin satisfy

\[ D_0g_x = \text{diag}(1, -1, -1), \quad D_0g_y = \text{diag}(-1, 1, -1), \quad \text{and} \quad D_0g_z = \text{diag}(-1, -1, 1). \]

If we let

\[ h_x = g_x^2, \quad h_y = g_y^2, \quad \text{and} \quad h_z = g_z^2 \]
then each of these maps is tangent to the identity at the origin. Notice that when $D = 0$ the surface $S_{0,0,0,D}$ passes through $(0,0,0)$. Therefore, applying Proposition 7.3 to the non-commuting pair of elements $h_x$ and $h_y$ yields:

**Lemma 7.5.** There is a neighborhood $W$ of the origin in $\mathbb{C}^4$ such that for all $(A, B, C, D) \in W$, the action of $\Gamma$ is locally non-discrete on an open subset of $S_{A,B,C,D}$ obtained by intersecting $S_{A,B,C,D}$ with a small ball centered at the origin in $\mathbb{C}^3$. In other words, $(0,0,0,0)$ is an interior point of $ND$. \(\Box\)

**Example 2.** Next we are going to introduce a two-parameter family of interior points in $ND$. The definition of this family is the content of the lemma below.

**Lemma 7.6.** Given $A_0$ and $B_0$, the point $(A_0, B_0, C_0, D_0)$ is an interior point of $ND$ provided that

$$C_0 = \frac{A_0 B_0}{4} \quad \text{and} \quad D_0 = -\frac{1}{4} (A_0^2 + B_0^2).$$

Indeed, for any choice of parameters $(A, B, C, D)$ sufficiently close to $(A_0, B_0, C_0, D_0)$, the resulting group $\Gamma$ is locally non-discrete on $S_{A,B,C,D}$ intersected with a small ball $B_r(p) \subset \mathbb{C}^3$, where

$$p = \left( \frac{A_0}{2}, \frac{B_0}{2}, 0 \right) \in S_{A_0,B_0,C_0,D_0}.$$

**Proof.** Recall also that for $z_0 \neq \pm 2$ the mapping $g_z$ has a unique fixed point $(x_0, y_0, z_0)$ in the invariant plane $\Pi_{z_0} = \{z = z_0\}$ determined by

$$x_0 = \frac{B_0 z_0 - 2A_0}{z_0^2 - 4}, \quad y_0 = \frac{A_0 z_0 - 2B_0}{z_0^2 - 4}.$$

Furthermore the point $p$ corresponds to this fixed point when $z_0 = 0$. The choice of $z_0 = 0$ gives that the restriction of $g_z$ to $\Pi_{z_0} = \Pi_0$ is an order two elliptical homography.

On the other hand, the choice of $C_0 = \frac{A_0 B_0}{4}$ ensures that $g_y(p) = p$. In fact, by using the expression for $g_y$ at the beginning of Section 4 we obtain

$$g_y \left( \frac{A_0}{2}, \frac{B_0}{2}, 0 \right) = \left( \frac{\left( \frac{A_0^2}{4} - 1 \right) A_0}{2} + A_0 - \frac{A_0 B_0^2}{8} \right) = \left( \frac{A_0}{2}, \frac{B_0}{2}, 0 \right).$$

Summarizing $p$ is a common fixed point for $g_z$ and $g_y$.

In addition, we have

$$D_{(x,y,z)} g_z = \begin{bmatrix} -1 & -z & -y \\
-1 & z^2 - 1 & 2yz - A_0 + x \\
0 & 0 & 1 \end{bmatrix}$$

and therefore,

$$D_p g_z = \begin{bmatrix} -1 & 0 & -\frac{B_0}{2} \\
0 & -1 & -\frac{A_0}{2} \\
0 & 0 & 1 \end{bmatrix}. $$

Since $g_z(p) = p$, it follows that

$$D_p g_z^2 = (D_p g_z)^2 = Id.$$
Remark 7.7. Let us close the previous discussions with a few brief comments.

(i) This two parameter family also contains the more elementary case discussed in Example 1.

(ii) Analogous 2-parameter families having two maps tangent to the identity at a same point can be found simply by using $g_x$ or $g_y$ in place of $g_z$ in the proof above.

(iii) Presumably it is possible to use $z_0 = \pm 2 \cos(\pi \theta)$, with $\theta$ rational, and obtain similar results for a higher iterate of $g_z$ so as to derive additional examples of similar nature.

Example 3. Here we will provide a yet more general family of parameters on the interior of $\mathcal{ND}$. Note, however, that it does not contain the cases covered in Example 2. Indeed, the corresponding fixed point, and cubic surfaces, appearing in this example are rather different from those in Example 2. This new family of examples is provided by the following lemma.

Lemma 7.8. For every choice of parameters $A_0, B_0, C_0 \in \mathbb{C}$ and $t \in \mathbb{R} \setminus \pi \mathbb{Z}$, let

$$p_0 = (0, 0, z_0) \in \mathbb{C}^3 \quad \text{where} \quad z_0 = \frac{C_0 e^{it} + \sqrt{-e^{3it} + 2e^{2it} - e^{it}}}{2e^{it}}.$$

If we let

$$D_0 = z_0^2 - C_0 z_0 = -\frac{C_0^2}{4} + \frac{1}{2} - \frac{\cos(t)}{2},$$

then $(A_0, B_0, C_0, D_0)$ is an interior point of $\mathcal{ND}$.

Indeed, for any choice of parameters $(A, B, C, D)$ sufficiently close to $(A_0, B_0, C_0, D_0)$, the resulting group $\Gamma$ is locally non-discrete on $S_{A,B,C,D}$ intersected with a small ball $B_r(p) \subset \mathbb{C}^3$ around $p$.

Proof. Recall from the description of the mappings $g_x, g_y, g_z$ that

$$g_{z|_{x=0}}^2 = \text{id} \quad \text{and} \quad g_{y|_{y=0}}^2 = \text{id}.$$  

This implies that for every $z_0$ the point $p_0$ is a common fixed point of $g_x^2$ and $g_y^2$ (alternatively this can be checked by a simple calculation). An additional calculation then yields

$$D_{p_0} g_x^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2z_0 - C_0 & 1 & 0 \\ B_0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D_{p_0} g_y^2 = \begin{bmatrix} 1 & -2z_0 + C_0 & 0 \\ 0 & 1 & 0 \\ 0 & -A_0 & 1 \end{bmatrix}. $$

Unfortunately, neither of these matrices is conjugate to a diagonal matrix. However, if we let $\gamma = g_x^2 \circ g_y^2$, we have

$$D_{p_0} \gamma = D_{p_0} g_x^2 D_{p_0} g_y^2 = \begin{bmatrix} 1 & -2z_0 + C_0 & 0 \\ 2z_0 - C_0 & (2z_0 - C_0)(-2z_0 + C_0) + 1 & 0 \\ B_0 & B_0(-2z_0 + C_0) - A_0 & 1 \end{bmatrix}. $$

The eigenvalues of $D_{p_0} \gamma$ are 1 and $e^{\pm it}$ as it can be checked by plugging $z_0$ into (33) and then computing the eigenvalues. Indeed, note that we have

$$2z_0 - C_0 = \frac{\sqrt{-e^{3it} + 2e^{2it} - e^{it}}}{e^{it}}$$

and

$$\sqrt{-e^{3it} + 2e^{2it} - e^{it}} = i(e^{it} - 1)e^{it/2}.$$  

Using the hypothesis that $t \notin \pi \mathbb{Z}$ we find that the matrix $D_{p_0} \gamma$ is diagonalizable and conjugate to a rotation by angle $t$.

In particular, for any $\tau > 0$ there is $k$ sufficiently large so that

$$\| (D_{p_0} \gamma)^k - \text{Id} \| < \tau.$$  

To find a second mapping satisfying the hypotheses of Lemma 7.2 it suffices to consider the map $h = g_x^2 \circ \gamma^k \circ g_x^{-2}$, since $p_0$ is also a fixed point of $g_x^2$. It follows that $D_{p_0} h$ is conjugate to $D_{p_0} \gamma^k$ and that the conjugating matrix does not depend on $k$. Therefore, making $k$ larger, if necessary, we can assume that both $\gamma^k$ and $h$ satisfy the hypotheses of Lemma 7.2. For any $k \geq 1$ the two
elements $\gamma^k = (g_x^2 \circ g_y^3)^k$ and $g_x^2 \circ \gamma^k \circ g_x^{-2} = g_x^2 \circ (g_x^2 \circ g_y^3)^k \circ g_x^{-2}$ do not commute because $\Gamma$ is free on $g_x$ and $g_y$. The result then follows from Proposition 7.3.

**Remark 7.9.** If the reader is willing to drop the extra variation of $D_0$ by $-\cos t/2$, we can choose $z_0 = (C_0 + \sqrt{2})/2$ and the point $p_0 = (0, 0, z_0)$ will still be fixed by both $g_x^2$ and $g_y^2$. In this case, the eigenvalues of $D_{p_0} \gamma$ will now be 1 and $\pm \sqrt{-1}$ and $\gamma^4$ will be tangent to the identity at $p_0$.

**Example 4 - Dubrovin-Mazzocco parameters.**

There is a 1-parameter family studied by Dubrovin and Mazzocco [18] which seems to play a significant role in several problems related to Mathematical-Physics and, in particular, on the study of Frobenius manifolds. In the sequel, we resume the notation of Section 3. When Painlevé 6 is located away from $S$ so that $p \not\in \{−2, 2\}$, Dubrovin-Mazzocco family is determined by setting

$$\beta = \gamma = 0, \quad \delta = 1/2,$$

and by letting $\alpha$ to be free. In the Iwasaki parameterization, this family becomes $\kappa_1 = \kappa_2 = \kappa_3 = 0$ while $\kappa_4$ is arbitrary. From

$$a_i = 2 \cos(\pi \kappa_i)$$

we see that $a_i = 2$ for $i = 1, 2, 3$ and $a_4 \equiv a$ is arbitrary. Therefore,

(34) $A = a_1 a_4 + a_2 a_3 = 2a + 4, \quad B = a_2 a_4 + a_1 a_3 = 2a + 4, \quad C = a_3 a_4 + a_1 a_2 = 2a + 4,$

and

(35) $D = 4 - [a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2] = -(a^2 + 8a + 8).$

To simplify notations, let us denote the surface $S_{A,B,C,D}$ for these parameters as $S_a$.

**Lemma 7.10.** For the Dubrovin-Mazzocco family introduced above and $a \not\in \{−2, 2\}$, the surface $S_a$ contains exactly three singular points $p_1, p_2, p_3$ given by

(36) $p_1 = (x_1, y_1, z_1) = (a, 2, 2), \quad p_2 = (x_2, y_2, z_2) = (2, a, 2), \quad$ and $\quad p_3 = (x_3, y_3, z_3) = (2, 2, a).$

In the case $a = −2$, the surface $S_a$ becomes the Cayley Cubic and it contains an additional singular point at $(-2, -2, -2)$.

**Proof.** The singular points correspond to the common fixed points of $s_x, s_y,$ and $s_z$ which, in turn, correspond to solutions in $(x, y, z)$ of the equations

(37) $-x - yz + 2a + 4 = x, \quad -y - xz + 2a + 4 = y, \quad$ and $\quad -z - xy + 2a + 4 = z.$

It is immediate to check that the three above singular points satisfy these equations.

Meanwhile, we must check that these three points lie on the surface $S_a$. By symmetry, it suffices to check for $p_1 = (x_1, y_1, z_1)$ and, in this case, we have

$$x_1^2 + y_1^2 + z_1^2 + x_1 y_1 z_1 - Ax_1 - By_1 - Cz_1 = a^2 + 4 + 4a - (2a + 4)(a - 2 - 2) = -(a^2 + 8a + 8) = D$$

so that $p_1 \in S_a$. Note that there are other common fixed points for $s_x, s_y,$ and $s_z$ but they are located away from $S_a$.

Finally it is immediate to check that when $a = −2$ the point $(-2, -2, -2)$ satisfies (37) and (38) so that it provides a fourth singular point. Conversely, it is a well-known fact that the Cayley cubic ($= S_{−2}$) is the only surface in this family having four singular points.

**Proposition 7.11.** For every fixed $a \in (-2, 2)$ the group $\Gamma_a$ acting on $S_a$ has locally non-discrete dynamics in some neighborhood of each of the singular points $p_1, p_2$ and $p_3$.

Moreover, the Dubrovin-Mazzocco parameters at this fixed value of $a$ (given by (34) and (35)) yield an interior point of $ND$. More precisely, given parameters $(A, B, C, D)$ sufficiently close to the
Dubrovin-Mazzocco parameters in question, the group $\Gamma$ is locally non-discrete on the intersection of $S_{A,B,C,D}$ with $B_r(p_1) \cup B_r(p_2) \cup B_r(p_3)$ for some $r > 0$.

Note that the parameter $a = -2$ corresponds to the Picard Parameters $(A, B, C, D) = (0, 0, 0, 4)$ for which $\Gamma$ is locally discrete on all of the corresponding surface, as it will be shown in next section.

**Proof.** The entire system is symmetric under permutations of the coordinates $(x, y, z)$ so that it suffices to consider the case of $p_1$. Note that $p_1$ is fixed by every element in $\Gamma$ (in fact, by every element of the group generated by $s_x, s_y,$ and $s_z$).

Let us first show that a suitable iterate of $g_x = s_z \circ s_y$ is close to the identity on some suitable neighborhood of $p_1$. For this, note that

$$D_{p_1} g_x = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & -a \\ 2a - 2 & a & a^2 - 1 \end{bmatrix}.$$ 

The eigenvalues of $D_{p_1} g_x$ are

$$\lambda_1 = \frac{a^2}{2} - 1 + \frac{\sqrt{a^4 - 4a^2}}{2}, \quad \lambda_2 = \frac{a^2}{2} - 1 - \frac{\sqrt{a^4 - 4a^2}}{2}, \quad \text{and} \quad \lambda_3 = 1.$$ 

For $a \in (-2, 0) \cup (0, 2)$ we see that $\lambda_1$ and $\lambda_2$ form a complex conjugate pair of eigenvalues, each of which has modulus one. In this case, $D_{p_1} g_x$ is conjugate to a rotation in suitable coordinates. In particular, for any $\tau > 0$ there is $k$ sufficiently large so that

$$\| (D_{p_1} g_x)^k - \text{Id} \| < \tau.$$ 

To find a second mapping satisfying the hypotheses of Lemma 7.2 it suffices to consider the map $h$ obtained by conjugating $g_x^k$ by, say, $g_y$. Since $p_1$ is fixed by $g_y$ as well, there follows that $D_{p_1} h$ is conjugate to $D_{p_1} g_x^k$ and that the conjugating matrix does not depend on $k$ (it is simply the matrix given by $Dg_y(p_1)$). Therefore, up to making $k$ larger if needed, we can assume that both $g_x^k$ and $h$ satisfy the hypotheses of Lemma 7.2. Since $g_x^k$ and $h$ do not commute in $\Gamma$, the result then follows from Proposition 7.3.

In the special case that $a = 0$ we have

$$D_{p_1} g_x = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

and then

$$(D_{p_1} g_x)^2 = \text{Id},$$

in which case, $g_x^2$ is tangent to the identity at $p_1$. Letting $h = g_y g_x^2 g_y^{-1}$ we obtain a second mapping tangent to the identity at $p_1$. Since, as previously seen, these two maps do not commute, the desired result in this special case follows again from Proposition 7.3.

**□**

### 8. Dynamics near infinity

In this section we will collect a few important, if slightly technical, results on the dynamics of the group $\Gamma$ as well as the dynamics of (individual) hyperbolic maps near $\Delta_\infty$. The corresponding results, especially Proposition 8.4 and Lemma 8.6 will play major roles in the proofs of Theorems H and K to be supplied in the forthcoming sections.

Let us begin with a general review of the behavior of a given map in $\Gamma = \Gamma_{A,B,C,D}$ near infinity. Up to compactifying $\mathbb{C}^3$ into the projective space, we begin by recalling that the closure of any
surface \( S_{A,B,C,D} \) intersects the hyperplane at infinity \( \Pi_\infty \subset \mathbb{CP}^3 \) in a triangle \( \Delta_\infty \). In homogeneous coordinates, this triangle is given by
\[
\Delta_\infty = \{(X : Y : Z : W) \in \mathbb{CP}^3 \mid W = 0 \text{ and } XYZ = 0\}.
\]
The vertices of \( \Delta_\infty \) are denoted by
\[
V_\infty = \{v_1 = (1 : 0 : 0 : 0), \quad v_2 = (0 : 1 : 0 : 0), \quad \text{and} \quad v_3 = (0 : 0 : 1 : 0)\}.
\]

As previously seen, for every choice of the parameters \((A, B, C, D)\), the surface \( S_{A,B,C,D} \) is smooth on a neighborhood of \( \Delta_\infty \). Furthermore, \( S_{A,B,C,D} \) is tangent to the plane at infinity exactly at the vertices in \( V_\infty \) and, hence, it has transverse intersection with the plane at infinity elsewhere in \( \Delta_\infty \). In the sequel, to abridge notation, the parameters \( A, B, C, D \) will be dropped whenever there is no possibility of misunderstanding. Thus, we will sometimes write \( \Gamma \) for \( \Gamma_{A,B,C,D} \), \( S \) for \( S_{A,B,C,D} \), and \( \mathcal{S} \) for the closure of \( S \).

Because \( \mathcal{S} \) is tangent to the plane at infinity at the vertices of \( \Delta_\infty \), near \( v_1 \) we can use the affine coordinates \((Y/X, Z/X, W/X)\) on \( \mathbb{CP}^3 \) and express the surface \( \mathcal{S} \) so that \( W/X \) becomes a holomorphic function of \((Y/X, Z/X)\). In other words, \( S \) is locally given as the graph of a holomorphic function in the variables \((Y/X, Z/X)\) on a neighborhood of \( v_1 \). This local representation will be referred to as the “standard coordinates” on \( \mathcal{S} \) in a neighborhood of \( v_1 \). The analogous construction leads to “standard coordinates” on \( \mathcal{S} \) near \( v_2 \) and near \( v_3 \), respectively.

Inside the plane at infinity, consider a neighborhood \( W_1 \) of \( v_1 \) where \( \mathcal{S} \) is the graph of some holomorphic function \( h_1 \) as indicated above. Neighborhoods \( W_2 \) and \( W_3 \) respectively of \( v_2 \) and \( v_3 \) are similarly defined along with the corresponding holomorphic functions \( h_2 \) and \( h_3 \). We state below some immediate consequences of what precedes in terms “variation with parameters”.

**Lemma 8.1.** Consider parameters \( A_0, B_0, C_0 \) and \( D_0 \) and apply the above construction to the surface \( \mathcal{S}_{A_0,B_0,C_0,D_0} \). Up to reducing the neighborhoods \( W_i \), \( i = 1, 2, 3 \), there exists a neighborhood \( \mathcal{W} \subset \mathbb{CP}^3 \) of \((A_0, B_0, C_0, D_0)\) such that all of the following holds:

1. For every \((A, B, C, D) \in \mathcal{W}\), the surface \( \mathcal{S}_{A,B,C,D} \) is (locally) the graph of a function \( h_i \) defined on \( W_i \). Furthermore, for \( i = 1, 2, 3 \), the functions \( h_i \) vary continuously with the parameters.
2. For every \((A, B, C, D) \in \mathcal{W}\), the intersection of the surface \( \mathcal{S}_{A,B,C,D} \) with the plane at infinity over the set \( \Delta_\infty \setminus (W_1 \cup W_2 \cup W_3) \) is uniformly transverse. Furthermore, the slopes vary continuously with the point and with the parameters.

We will also need some facts about birational extensions to \( \mathcal{S} \) of elements in \( \Gamma \) and, more generally, in \( \Gamma^* \). Recalling that \( \Gamma^* \) is generated by the involutions \( s_x, s_y, \) and \( s_z \), an element \( \gamma \) in \( \Gamma^* \) is said to be *cyclically reduced* if its reduced spelling in the “letters” \( s_x, s_y, \) and \( s_z \) is not conjugate in \( \Gamma^* \) to another element in \( \Gamma^* \) having strictly smaller length when spelled in the same “letters”.

We also recall that a meromorphic self-map from a complex surface to itself is said to be *algebraically stable* if it does not contract a hypersurface to its indeterminate set, cf. [23] [17]. In the present case where elements of \( \Gamma \) act on the surface \( S \), to be algebraically stable amounts to saying that \( \gamma \) does not contradict any of the sides of \( \Delta_\infty \) to an indeterminacy point which, in turn, necessarily lies in \( \Delta_\infty \) as well.

With this terminology, the following definition/proposition and remark summarize [8 Prop. 3.2] and [9 Prop. 2.3].

**Definition/Proposition 8.2.** For any parameters \( A, B, C, D \) and any \( \gamma \in \Gamma \) we have

1. \( \gamma \) is said to be hyperbolic if and only if it conjugate to a cyclically reduced word in \( s_x, s_y, \) and \( s_z \) that contains all three mappings.
2. A hyperbolic map \( \gamma \in \Gamma \) possesses a single indetermination point which coincides with a vertex of \( \Delta_\infty \) and will be denoted by \( \text{Ind}(\gamma) \).
(iii) A hyperbolic map $\gamma \in \Gamma$ contracts all of $\Delta_{\infty} \setminus \{\text{Ind}(\gamma)\}$ to a vertex of $\Delta_{\infty}$ denoted by $\text{Attr}(\gamma)$. The vertices $\text{Ind}(\gamma)$ and $\text{Attr}(\gamma)$ may or may not coincide. In particular, the map $\gamma$ is algebraically stable if and only if $\text{Ind}(\gamma) \neq \text{Attr}(\gamma)$.

(iv) Alternatively, $\gamma : \mathcal{S} \to \mathcal{S}$ is algebraically stable if and only if it is a cyclically reduced composition of $s_x, s_y$, and $s_z$ of length at least two.

(v) If $\gamma$ is algebraically stable and hyperbolic then $\gamma$ is holomorphic around $\text{Attr}(\gamma)$ and, in fact, $\text{Attr}(\gamma)$ is a superattracting fixed point of $\gamma$. Moreover, the roles of $\text{Ind}(\gamma)$ and $\text{Attr}(\gamma)$ are interchanged if we pass from $\gamma$ to $\gamma^{-1}$, i.e., $\text{Attr}(\gamma) = \text{Ind}(\gamma^{-1})$ and $\text{Ind}(\gamma) = \text{Attr}(\gamma^{-1})$.

(vi) An element $\gamma$ is said to be parabolic if it is conjugate in $\gamma$ to one of the maps $g_x, g_y$, or $g_z$. Every element of $\Gamma$ different from the identity either is hyperbolic or is parabolic.

\[ \square \]

**Remark 8.3.** While $\Gamma$ only contains hyperbolic and parabolic elements, the bigger group $\Gamma^*$ also contains elliptic elements. An element of $\Gamma^*$ is elliptic if and only if it is conjugate in $\Gamma^*$ to one of the involutions $s_x, s_y$, or $s_z$, if and only if it is periodic. (The characterizations of parabolic and hyperbolic elements of $\Gamma$ carry over to $\Gamma^*$, with conjugation in $\Gamma$ replaced by conjugation in $\Gamma^*$.)

It is convenient to point out that the properties described in Definition/Proposition 8.2 are independent of the parameters $A, B, C, D$. They depend only on the spelling of $\gamma$ as an element in the abstract group generated by three letters $a, b$, and $c$ with the relation $abc = \text{id}$ (up to the substitutions $a = g_x, b = g_y$, and $c = g_z$). In particular, the points $\text{Ind}(\gamma)$ and $\text{Ind}(\gamma^{-1}) = \text{Attr}(\gamma)$ are independent of the parameters and this plays an important role in the statement of the following proposition.

**Proposition 8.4.** Consider an hyperbolic element $\gamma \in \Gamma$ and, for a choice of parameters $A_0, B_0, C_0$, and $D_0$, let $\overline{f}_{A_0, B_0, C_0, D_0}$ be the resulting birational map of the (compact) surface $\mathcal{S}_{A_0, B_0, C_0, D_0}$. Then there exist a neighborhood $U_{\Delta}$ of $\Delta_{\infty} \subset \mathbb{C}P^3$ and a neighborhood $U_0 \subset \mathbb{C}^4$ of $(A_0, B_0, C_0, D_0) \in \mathbb{C}^4$ such that the following holds:

- For every $(A, B, C, D) \in U_0$, the intersection $U_{\Delta} \cap \mathcal{S}_{A, B, C, D}$ is a neighborhood of $\Delta_{\infty}$ in $\mathcal{S}_{A, B, C, D}$.
- For every $(A, B, C, D) \in U_0$, the map $\overline{f}_{A, B, C, D}$ has no fixed point (and, indeed, periodic point) in $(U_{\Delta} \cap \mathcal{S}_{A, B, C, D}) \setminus \Delta_{\infty}$.

**Proof.** It suffices to prove the proposition for an algebraically stable hyperbolic $\gamma$ since, every hyperbolic element in $\Gamma$ is conjugate in $\Gamma$ to an algebraically stable one and all elements in $\Gamma$ preserve the infinity.

Let then $\overline{f}_{A_0, B_0, C_0, D_0}$ be an algebraically stable hyperbolic map as indicated above. The statement amounts to showing the existence of a neighborhood $V_0 \subset \mathcal{S}_{A_0, B_0, C_0, D_0} \setminus \Delta_{\infty}$ of $\Delta_{\infty} \subset \mathcal{S}_{A_0, B_0, C_0, D_0}$ satisfying the two conditions below:

1. $\overline{f}_{A_0, B_0, C_0, D_0}$ has no fixed point (actually periodic point) in $V_0 \setminus \Delta_{\infty}$.
2. The neighborhood $V_0$ can be chosen to vary continuously with the parameters $A$, $B$, $C$, and $D$.

To prove the first assertion, we consider the points $\text{Attr}(\overline{f}_{A_0, B_0, C_0, D_0})$ and $\text{Ind}(\overline{f}_{A_0, B_0, C_0, D_0})$ in $\Delta_{\infty}$. As mentioned, these points are distinct and do not depend on the choice of the parameters. To abridge notation, we then set $P = \text{Attr}(\overline{f}_{A_0, B_0, C_0, D_0})$ and $Q = \text{Ind}(\overline{f}_{A_0, B_0, C_0, D_0})$. We also recall that $P$ is a super-attracting fixed point of $\overline{f}_{A_0, B_0, C_0, D_0}$ and, in fact, a super-attracting fixed point of $\overline{f}_{A, B, C, D}^{-1}$ for every choice of the parameters $A$, $B$, $C$, and $D$. Similarly, $Q$ is a super-attracting fixed point of $\overline{f}_{A, B, C, D}$ for any $(A, B, C, D) \in \mathbb{C}^4$. Now let $U \subset \mathcal{S}_{A_0, B_0, C_0, D_0}$ be a small neighborhood of $Q$ which is contained in the basin of attraction of $Q$ for $\overline{f}_{A_0, B_0, C_0, D_0}^{-1}$. Since $\overline{f}_{A_0, B_0, C_0, D_0}^{-1}$ sends $\Delta_{\infty} \setminus \{Q\}$ to $P$, there exists a neighborhood $V$ of $\Delta_{\infty} \setminus U$ which is sent by $\overline{f}_{A_0, B_0, C_0, D_0}$ into the
basis of attraction of \( P \), with respect to \( \overline{f}_{A_0,B_0,C_0,D_0} \). Therefore \( V \cup U \) contains no periodic point of \( \overline{f}_{A_0,B_0,C_0,D_0} \); in fact, every point in \( V \cup U \) converges to \( P \) under iteration by \( \overline{f}_{A_0,B_0,C_0,D_0} \) or to \( Q \) under iteration by \( \overline{f}_{A_0,B_0,C_0,D_0}^{-1} \).

It remains to prove that the above neighborhood can be assumed to vary continuously with the parameters. For this, we first consider the (local) description of the surfaces \( \mathcal{S}_{A,B,C,D} \) provided by Lemma 8.1. The lemma in question shows that the surface \( \mathcal{S}_{A,B,C,D} \) converges towards \( \mathcal{S}_{A_0,B_0,C_0,D_0} \) as \( (A,B,C,D) \rightarrow (A_0,B_0,C_0,D_0) \) on a neighborhood of \( \Delta_\infty \subset \mathbb{CP}^3 \). In fact, on a neighborhood of the points \( P \) and \( Q \), this follows from the local structure of these surfaces as graphs of holomorphic functions. Conversely, away from these neighborhoods of \( P \) and \( Q \), the statement follows from the (uniform) transverse intersection of these surfaces and the plane at infinity of \( \mathbb{CP}^3 \). In particular the maps \( \overline{f}_{A,B,C,D} \) converge uniformly to \( \overline{f}_{A_0,B_0,C_0,D_0} \) on a neighborhood of \( \Delta_\infty \). Thus, the statement is reduced to show the existence of \( \epsilon > 0 \) such that for all parameters \( (A,B,C,D) \) sufficiently close to \( (A_0,B_0,C_0,D_0) \), the ball of radius \( \epsilon \) around \( P \) still is contained in the basin of attraction of \( P \) with respect to \( \overline{f}_{A,B,C,D} \). Here, the same argument applies to \( Q \) and \( \overline{f}_{A,B,C,D}^{-1} \) and the notion of “ball” is relative to some auxiliary metric, for example, the Euclidean metric in the “standard coordinates” of Lemma 8.1.

The last claim can be proved as follows. The germ of \( \overline{f}_{A,B,C,D} \) is a rigid, reducible, super-attracting germ in the sense of [21] (cf. [8]). It actually falls in the “class 6” of the classification provided in [21] and hence it is conjugate to a monomial map, owing to results of Dloussky and Favre. Since the monomial map does not depend on the parameters, the claim follows from checking directly in the argument of [21] that the conjugating map depends continuously on the initial map.

Alternatively, a more direct - if slightly less elementary - proof can be obtained by means of dynamical Green functions. For this, denote by \((u,v)\) standard coordinates around \( P \) and consider the monomial map \((u,v) \mapsto (u^{a_{11}}v^{a_{12}},u^{a_{21}}v^{a_{22}})\) to which every map \( \overline{f}_{A,B,C,D} \) is locally conjugate around \( P \). Denoting by \( \lambda \) the spectral radius of the matrix \( \{a_{ij}\} \), the Green function for \( \overline{f}_{A,B,C,D} \) can be defined by setting

\[
G_{f,ABCD}^+(p) = \lim_{n \to \infty} \frac{1}{\lambda^n} \log^+ \left( \|f_{A,B,C,D}^n(p)\| \right).
\]

Clearly \( G_{f,ABCD}^+ \) is a uniform limit of functions that are continuous on both \( p \) and the parameters \( A, B, C, D \). Hence \( G_{f,ABCD}^+ \) is continuous on all these variables. The analogous construction using \( f_{A,B,C,D}^{-1} \) yields another Green function \( G_{f,ABCD}^- \) which is continuous on \( p \) and on \( A, B, C, D \) as well. Finally, by using these Green functions, the desired neighborhood \( V \) of \( \Delta_\infty \subset \mathcal{S}_{A,B,C,D} \) can be defined by letting

\[
V = \{ p \in S_{A,B,C,D} : G_{f,ABCD}^+(p) + G_{f,ABCD}^-(p) > 1 \}.
\]

This completes the proof of the proposition. \( \square \)

As a consequence of Proposition 8.4, we also obtain the following result (cf. Lemma 16 from [35]).

**Corollary 8.5.** For every choice of the parameters \((A,B,C,D) \in \mathbb{C}^4\) and every hyperbolic element in \( \Gamma \), all the fixed points of the resulting (hyperbolic) map \( f_{A,B,C,D} \) are isolated.

**Proof.** As a composition of the algebraic maps \( g_x, g_y, \) and \( g_z, f_{A,B,C,D} \) itself an algebraic map from \( \mathbb{C}^3 \) to \( \mathbb{C}^3 \). Thus the set of its fixed points is an algebraic set of \( \mathbb{C}^3 \) so that its intersection with \( S_{A,B,C,D} \) is an algebraic subset of \( S_{A,B,C,D} \). If the latter set is not constituted by isolated points, then it must contain a curve \( \mathcal{C} \) (as already seen it cannot coincide with all of \( S_{A,B,C,D} \)). However, by the maximum principle, the \( \mathcal{C} \) must accumulate on \( \Delta_\infty \) and this contradicts Proposition 8.4. \( \square \)

The remainder of this section will be devoted to the proof of Lemma 8.6 below which is crucial for the understanding of the structure of unbounded Fatou components as it will be seen later on.
We resume the terminology used at the beginning of the section. Thus \((X : Y : Z : W)\) are homogeneous coordinates on \(\mathbb{CP}^3\). For any \(A, B, C, D\), the surface \(S = S_{A,B,C,D}\) is tangent to the hyperplane at infinity at each of the vertices \(v_1, v_2, v_3\), and \(v_4\) of \(\Delta_{\infty}\). Again, near, say \(v_1\), we can use the affine coordinates \((Y/X, Z/X, W/X)\) on \(\mathbb{CP}^3\) and express the surface so that \(W/X\) is a holomorphic function of \((Y/X, Z/X)\). Similar descriptions apply to the other two vertices, and the resulting coordinates were called “standard coordinates” around the vertices in question.

Next, let \(U_\infty(i)\) be an open neighborhood of \(v_i\) in \(S\) so that \(U_\infty(i)\) is contained in the graph of \(h_i : W_i \rightarrow \mathbb{C}, i = 1, 2, 3\). Without loss of generality, we assume that the three neighborhoods \(U_\infty(i), i = 1, 2, 3\), are pairwise disjoint. If \(p \in U_\infty(i)\), we denote by

\[
\text{dist}(p, v_i)
\]

the Euclidean distance between \(p\) and \(v_i\) in the standard coordinate chart on \(U_\infty(i)\). Let

\[
U_\infty = U_\infty(1) \cup U_\infty(2) \cup U_\infty(3).
\]

For any \(p \in U_\infty\), we define

\[
\text{dist}(p, V_\infty)
\]

to equal \(\text{dist}(p, v_i)\) where \(U_\infty(i)\) is the unique one of the three neighborhoods containing \(p\).

Let \(S(0)\) be a set consisting of six hyperbolic elements \(\gamma_{i,j} \in \Gamma, i, j \in \{1, 2, 3\}, i \neq j\). Like in Proposition \ref{prop:gamma}, for every natural number \(n\) we will consider the inductively defined sets of iterated commutators \(S(n)\). For every \(n\) the set \(S(n + 1)\) contains every possible commutator of any two distinct elements of \(S(n)\).

**Lemma 8.6.** Given parameters \(A, B, C, D\), assume there are six hyperbolic elements \(\gamma_{i,j} \in \Gamma\) as above such that for every pair \(i \neq j \in \{1, 2, 3\}\) we have \(\text{Ind}(\gamma_{i,j}) = v_i\) and \(\text{Attr}(\gamma_{i,j}) = v_j\). Let \(S(0)\) be the set consisting of the six elements \(\gamma_{i,j}\) and let \(\{S(n)\}\) be the corresponding sequence of inductively defined subsets of \(\Gamma\). Then, up to reducing the neighborhood \(U_\infty \subset S_{A,B,C,D}\), for any point \(q \in U_\infty\), there exists a constant \(0 < \lambda < 1\) and a sequence \(\{\eta_n\}_{n=0}^\infty \subset \Gamma\) satisfying the two conditions below:

(i) \(\eta_n \in S(n)\) for every \(n \geq 0\), and

(ii) \(\text{dist}(\eta_n(q), V_\infty) \leq \lambda^n\) for every \(n \geq 0\).

**Proof.** Note that we might have asked the elements \(\gamma_{i,j}\) of the set \(S(0)\) to satisfy \(\gamma_{i,j} = \gamma_{j,i}^{-1}\) though this is not necessary. Also, in view of Definition/Proposition \ref{prop:gamma}, all elements \(\gamma_{i,j}\) are algebraically stable and \(\gamma_{i,j}\) is holomorphic around \(\text{Attr}(\gamma_{i,j}) = v_j\).

The proposition will be proved by induction. In fact, we will prove a rather stronger statement. Namely, there exists \(\lambda, 0 < \lambda < 1\), such that for each integer \(n \geq 0\), we have:

If \(n\) is even then for every pair of distinct \(i, j \in \{1, 2, 3\}\) there exists some \(\gamma_{i,j}^{(n)} \in S(n)\) such that

\[
\text{(E1)} \quad \gamma_{i,j}^{(n)} \text{ is holomorphic on } U_\infty \setminus U_\infty(i), \text{ satisfies } \gamma_{i,j}^{(n)}(U_\infty \setminus U_\infty(i)) \subset U_\infty(j), \text{ and for every } q \in U_\infty \setminus U_\infty(i) \text{ we have } \text{dist}(\gamma_{i,j}^{(n)}(q), v_j) \leq \lambda^n \text{ dist}(q, V_\infty).
\]

\[
\text{(E2)} \quad (\gamma_{i,j}^{(n)})^{-1} \text{ is holomorphic on } U_\infty \setminus U_\infty(j), \text{ satisfies } (\gamma_{i,j}^{(n)})^{-1}(U_\infty \setminus U_\infty(j)) \subset U_\infty(i), \text{ and for every } q \in U_\infty \setminus U_\infty(j) \text{ we have } \text{dist}((\gamma_{i,j}^{(n)})^{-1}(q), v_i) \leq \lambda^n \text{ dist}(q, V_\infty).
\]

If \(n\) is odd then for each \(i \in \{1, 2, 3\}\) there exists some \(\tau_i^{(n)} \in S(n)\) such that

\[
\text{(O)} \quad \tau_i^{(n)} \text{ and } (\tau_i^{(n)})^{-1} \text{ are holomorphic on } U_\infty \setminus U_\infty(i), \text{ they satisfy } (\tau_i^{(n)})^{-1}(U_\infty \setminus U_\infty(i)) \subset U_\infty(i), \text{ and for every } q \in U_\infty \setminus U_\infty(i) \text{ we have } \text{dist}((\tau_i^{(n)})^{-1}(q), v_i) \leq \lambda^n \text{ dist}(q, V_\infty).
\]
The base of the induction is \( n = 0 \), in which case for each pair \( i \neq j \) we let \( \gamma_i^{(0)} = \gamma_{i,j} \) be the corresponding element of \( S(0) \). Fix then a pair \( i \neq j \) and let \( k \in \{1, 2, 3\} \) such that \( k \neq i \) and \( k \neq j \). Consider standard local coordinates \((u_1, u_2)\) around \( v_k \) and let \((v_1, v_2)\) stand for standard local coordinates in a neighborhood of \( v_j \). By hypothesis we have \( \text{Ind}(\gamma_{i,j}) = v_i \) and \( \text{Attr}(\gamma_{i,j}) = v_j \).

Therefore, item (iii) of Definition/Proposition 8.2 gives that \( \gamma_{i,j}(\Delta_{\infty} \setminus \{v_1\}) = v_j \). Hence, if \( \gamma_{i,j} \) is expressed in local coordinates under the form \((v_1, v_2) = \gamma_{i,j}(u_1, u_2)\), both coordinates of \( \gamma_{i,j}(u_1, u_2) \) will vanish along both axes \( \{u_1 = 0\} \) and \( \{u_2 = 0\} \). Similarly, if we express \( \gamma_{i,j} \) from the \((v_1, v_2)\) coordinates to themselves, then both coordinates of \( \gamma_{i,j}(v_1, v_2) \) vanish along both axes \( \{v_1 = 0\} \) and \( \{v_2 = 0\} \). This implies that for any \( 0 < \lambda < 1 \), we can choose the neighborhoods \( U_{\infty}(k) \) and \( U_{\infty}(j) \) sufficiently small so as to ensure that for any \( q \in U_{\infty}(k) \cup U_{\infty}(j) \) the estimate

\[
\text{dist}(\gamma_{i,j}(q), v_j) \leq \lambda \text{dist}(q, V_{\infty})
\]

holds. After choosing a sufficiently small neighborhood \( U_{\infty}(i) \) of \( v_i \) and perhaps making \( U_{\infty}(k) \) smaller, the same reasoning applies to show that for any \( q \in U_{\infty}(k) \cup U_{\infty}(i) \) we have

\[
\text{dist}(\gamma_{i,j}^{-1}(q), v_i) \leq \lambda \text{dist}(q, V_{\infty}).
\]

Repeating for all six distinct pairs \( i \neq j \) we obtain sufficiently small neighborhoods \( U_{\infty}(1), U_{\infty}(2), \) and \( U_{\infty}(3) \) such that (39) and (40) hold. If we then let \( U_{\infty}(1), U_{\infty}(2), \) and \( U_{\infty}(3) \) be round balls of equal sufficiently small radius in the standard local coordinates, both estimates (39) and (40) will continue to hold. In addition, for all six distinct pairs \( i \neq j \) we have \( \gamma_{i,j}(U_{\infty} \setminus U_{\infty}(i)) \subset U_{\infty}(j) \) and \( \gamma_{i,j}^{-1}(U_{\infty} \setminus U_{\infty}(j)) \subset U_{\infty}(i) \). Therefore we can assume that (E1) and (E2) hold when \( n = 0 \).

For the remainder of the proof we keep the neighborhood \( U_{\infty} = U_{\infty}(1) \cup U_{\infty}(2) \cup U_{\infty}(3) \) fixed and inductively prove that (E1) and (E2) hold on \( U_{\infty} \) for every even \( n \) and that (O) holds on \( U_{\infty} \) for every odd \( n \).

Suppose now that \( n \) is even and that the collection of six elements \( \gamma_{i,j}^{(n)} \in S(n) \) exist and satisfy (E1) and (E2). For each \( i \in \{1, 2, 3\} \), we will prove the existence of an element \( \tau_i^{(n+1)} \in S(n+1) \) satisfying (O).

Fix then \( i \in \{1, 2, 3\} \) and let \( j, k \in \{1, 2, 3\} \setminus \{i\} \) be the other two indices. We define

\[
\tau_i^{(n+1)} = [\gamma_{i,j}^{(n)}, \gamma_{i,k}^{(n)}] = \left(\gamma_{i,j}^{(n)}\right)^{-1} \left(\gamma_{i,k}^{(n)}\right)^{-1} \gamma_{i,j}^{(n)} \gamma_{i,k}^{(n)}.
\]

Using (E1) and (E2) we can see that the above composition is holomorphic on all of \( V_{\infty} \setminus V_{\infty}(i) \) and that it maps \( V_{\infty} \setminus V_{\infty}(i) \) into \( V_{\infty}(i) \). Moreover, for any \( q \in V_{\infty} \setminus V_{\infty}(i) \) we have that

\[
\gamma_{i,k}^{(n)}(q) \in V_{\infty}(k), \quad \gamma_{i,j}^{(n)} \gamma_{i,k}^{(n)}(q) \in V_{\infty}(j), \quad \text{and} \quad \left(\gamma_{i,j}^{(n)}\right)^{-1} \gamma_{i,j}^{(n)}(q) \in V_{\infty}(i).
\]

Again using (E1) and (E2) we have the each of the four mappings in the commutator used to define \( \tau_i^{(n+1)} \) contracts distance to \( V_{\infty} \) by a factor of \( \lambda^{4^n} \) and hence

\[
\text{dist}(\tau_i^{(n+1)}(q), v_j) = \text{dist}\left(\left(\gamma_{i,j}^{(n)}\right)^{-1} \left(\gamma_{i,k}^{(n)}\right)^{-1} \gamma_{i,j}^{(n)} \gamma_{i,k}^{(n)}(q), v_i\right) \leq \lambda^{4^{n+1}} \text{dist}(q, V_{\infty}).
\]

Notice that

\[
(\tau_i^{(n+1)})^{-1} = [\gamma_{i,k}^{(n)}, \gamma_{i,j}^{(n)}] = \left(\gamma_{i,k}^{(n)}\right)^{-1} \left(\gamma_{i,j}^{(n)}\right)^{-1} \gamma_{i,j}^{(n)} \gamma_{i,k}^{(n)}.
\]

Therefore the same proof as in the previous paragraph applies to \( (\tau_i^{(n+1)})^{-1} \) after switching \( j \) and \( k \). We conclude that (O) holds for \( n + 1 \).

Suppose now that \( n \) is odd and that the collection of three elements \( \tau_i^{(n)} \in S(n) \) exist and satisfy (O). We will prove that all six elements \( \gamma_{i,j}^{(n+1)} \in S(n+1) \) satisfying (E1) and (E2) exist.
For any distinct \(i,j \in \{1, 2, 3\}\) let \(k \in \{1, 2, 3\} \setminus \{i, j\}\) be the remaining element. Now define
\[
\gamma_{i,j}^{(n+1)} = \left[ \tau_{j}^{(n)}, \tau_{i}^{(n)} \right] = \left( \tau_{j}^{(n)} \right)^{-1} \left( \tau_{i}^{(n)} \right)^{-1} \tau_{i}^{(n)} \tau_{j}^{(n)}.
\]
Using (O) we can see that the above composition of mappings is holomorphic on \(U_{\infty} \setminus U_{\infty}(i)\) and that it maps \(U_{\infty} \setminus U_{\infty}(i)\) into \(U_{\infty}(j)\). Again using (O), each of these four mappings contracts distance to infinity, cf. Section 1.5. In particular,
\[
\text{dist} \bigg( \gamma_{i,j}^{(n+1)}(q), v_{j} \bigg) = \text{dist} \left( \left( \tau_{j}^{(n)} \right)^{-1} \left( \tau_{i}^{(n)} \right)^{-1} \tau_{i}^{(n)} \tau_{j}^{(n)}(q), v_{j} \right) \leq \lambda^{n+1} \text{dist}(q, V_{\infty}).
\]
We conclude that (E1) holds for \(n + 1\).

To see that (E2) holds for \(n + 1\), note that
\[
\left( \gamma_{i,j}^{(n+1)} \right)^{-1} = \left[ \tau_{j}^{(n)}, \tau_{i}^{(n)} \right] = \left( \tau_{j}^{(n)} \right)^{-1} \left( \tau_{i}^{(n)} \right)^{-1} \tau_{j}^{(n)} \tau_{i}^{(n)}.
\]
Therefore the same proof as in the previous paragraph applies to \(\left( \gamma_{i,j}^{(n+1)} \right)^{-1}\) after switching \(i\) and \(j\).

We conclude that (E2) holds for \(n + 1\).

Therefore, we conclude that statements (E1) and (E2) hold for every even \(n \geq 0\) and that (O) holds for every odd \(n \geq 0\). \(\square\)

9. General properties of Fatou components and good set of parameters \(\mathbb{C}^{4}_{\text{good}}\).

Recall that the Fatou set \(\mathcal{F}_{A,B,C,D}\) for the action of \(\Gamma_{A,B,C}\) on \(S_{A,B,C,D}\) is the set of points \(p\) admitting a neighborhood on which the restrictions of all elements in \(\Gamma\) form a normal family. Recall also that, by way of definition, this normal family may contain sequences of maps converging to infinity, cf. Section 1.5. In particular, \(\mathcal{F}_{A,B,C,D}\) is an open (possibly empty) set and, according to Remark 3.2, none of the possible singular points of \(S_{A,B,C,D}\) lies in \(\mathcal{F}_{A,B,C,D}\).

A Fatou component \(V \subset S_{A,B,C,D}\) is a connected component of \(\mathcal{F}_{A,B,C,D}\). Since the Fatou set is invariant, one can consider the stabilizer \(\Gamma_{V} \leq \Gamma\) of \(V\), which consists of those elements of \(\Gamma\) that map \(V\) to \(V\). The purpose of this section is to establish several general properties of Fatou components. By combining these properties with the previous material, the proofs of Theorems II and K will quickly be derived in the next section.

We begin with Proposition 9.1 below which ensures the hyperbolic character of these components (cf. Lemma 4.5).

**Proposition 9.1.** Any Fatou component \(V\) of \(\Gamma_{A,B,C,D}\) is Kobayashi hyperbolic.

**Proof.** Since \(V\) is an open subset of \(S_{A,B,C,D}\) and \(V\) does not contain any singular points of \(S_{A,B,C,D}\), \(V\) is itself a complex (open) manifold. Recall the “grid”
\[
\mathcal{G} = S_{x_{0}} \cup S_{x_{1}} \cup S_{y_{0}} \cup S_{y_{1}} \cup S_{z_{0}} \cup S_{z_{1}},
\]
defined in [13], which is a subset of the Julia set \(\mathcal{J}_{A,B,C,D}\). Hence \(V\) does not intersect \(\mathcal{G}\) and we can therefore consider the inclusion
\[
\iota : V \rightarrow (\mathbb{C} \setminus \{x_{0}, x_{1}\}) \times (\mathbb{C} \setminus \{y_{0}, y_{1}\}) \times (\mathbb{C} \setminus \{z_{0}, z_{1}\}).
\]
Clearly the image of \(\iota\) is contained in a product of hyperbolic Riemann surfaces which is naturally a Kobayashi hyperbolic domain. The fact that holomorphic mappings do not increase the Kobayashi pseudometric then implies that \(V\) is also Kobayashi hyperbolic as well; see [36] Proposition 3.2.2. \(\square\)

Let then \(V\) be a fixed component of \(\mathcal{F}_{A,B,C,D}\) and let \(\text{Aut}(V)\) denote its group of holomorphic automorphisms. By building on the general theory of topological transformation groups of Gleason, Montgomery, and Zippin [13], Cartan was able to show that the automorphism group of a bounded domain in \(\mathbb{C}^{n}\) is a finite-dimensional real Lie group. In turn, Kobayashi [36] was able to
extend Cartan’s theorem to general (Kobayashi) hyperbolic manifolds. Owing to Proposition 9.1, Corollary 9.2 below summarizes these results in the case of a Fatou component.

Recall that the action \( \varphi : G \times M \to M \) of a group \( G \) on a manifold \( M \) is said to be proper if the preimage by \( \varphi \) of any compact set of \( M \) is again compact in \( G \times M \).

**Corollary 9.2.** Let \( V \) be a Fatou component of \( \Gamma_{A,B,C,D} \). Then,

1. \( \text{Aut}(V) \) is a real Lie Group of finite dimension in the topology of uniform convergence on compact sets.
2. \( \text{Aut}(V) \) acts properly on \( V \).
3. For any \( p \in V \) the stabilizer \( \text{Aut}(V)_p = \{ f \in \text{Aut}(V) : f(p) = p \} \) is compact, since the action of \( \text{Aut}(V) \) on \( V \) is proper.

For more details, see [36, Theorems 5.4.1 and 5.4.2].

Let us now consider the stabilizer \( \Gamma_V \leq \Gamma \) of the hyperbolic component \( V \).

**Proposition 9.3.** Suppose that \( \Gamma_{A,B,C,D} \) is locally non-discrete on an connected open \( U \subset \mathcal{F}_{A,B,C,D} \) and let \( V \) be the Fatou component containing \( U \). Recalling that \( \Gamma_V \) stands for the stabilizer of \( V \) in \( \Gamma \), the following holds:

1. The closure \( G = \overline{\Gamma_V} \) of \( \Gamma_V \) in \( \text{Aut}(V) \) is a real Lie Group of dimension at least 1.
2. For every point \( p \in V \), the stabilizer \( G_p \) of \( p \) in \( G \) is such that its local action around \( p \) is conjugate to the local (linear) action of a closed subgroup of \( \text{SU}(2) \) on a neighborhood of \( (0,0) \in \mathbb{C}^2 \).

**Proof.** As a closed subgroup of a Lie Group, \( G \) is itself a real Lie Group. Moreover, since \( \Gamma \) is locally non-discrete on the open set \( U \subset V \), modulo reducing \( U \), there are elements \( \{ \gamma_n \}_{n=1}^{\infty} \subset \Gamma \) that converge uniformly to the identity on \( U \). In particular, for \( n \) large enough, we have \( \gamma_n(V) \cap V \neq \emptyset \) and thus \( \gamma_n(V) = V \) since \( \mathcal{F} \) is invariant under \( \Gamma \). Hence, up to dropping finitely many elements in the sequence in question, we can assume without loss of generality that \( \{ \gamma_n \}_{n=1}^{\infty} \subset \Gamma_V \) for every \( n \in \mathbb{N} \). Next we have:

**Claim.** The sequence \( \{ \gamma_n \}_{n=1}^{\infty} \) actually converges to the identity uniformly on compact subsets of \( V \), i.e. as elements of \( \text{Aut}(V) \).

**Proof of the claim.** Consider a relatively compact open set \( U' \subset V \) with \( U \subset U' \). The claim amounts to checking that \( \{ \gamma_n \}_{n=1}^{\infty} \) converges uniformly to the identity in \( U' \). If this were not the case then, up to passing to a subsequence, there would exist \( \varepsilon > 0 \) such that

\[
\sup_{x \in U'} \| \gamma_n(x) - x \| \geq \varepsilon > 0.
\]

Since \( \{ \gamma_n \}_{n=1}^{\infty} \) is contained in a normal family on \( V \), we can extract a limit map \( \gamma_\infty \) defined on \( U' \) and thus satisfying \( \sup_{x \in U'} \| \gamma_\infty(x) - x \| \geq \varepsilon \) so that \( \gamma_\infty \) does not coincide with the identity on \( U' \). However, \( \gamma_\infty \) must coincide with the identity on \( U \subset U' \) since \( \{ \gamma_n \}_{n=1}^{\infty} \) converges uniformly to the identity on \( U \). The resulting contradiction proves our claim. \( \square \)

Since \( \{ \gamma_n \}_{n=1}^{\infty} \) converges to the identity on compact subsets of \( V \), it follows that the elements of \( \text{Aut}(V) \) obtained by restricting them to \( V \) actually converges to the identity as elements of \( \text{Aut}(V) \) equipped with its Lie group structure, cf. Corollary 9.2. Thus \( \Gamma_V \) is a non-discrete subgroup of \( \text{Aut}(V) \) and hence its closure must be a Lie group with strictly positive dimension.

It remains to check the second assertion. For this, recall that \( \Gamma \) preserves the real volume form associated with the holomorphic volume form \( \Omega \) given in [9]. This implies that the group of derivatives of elements of \( G_p \) is a subgroup of \( \text{SL}(2, \mathbb{C}) \). On the other hand, Corollary 9.2 Part (3), informs us that \( G_p \) must be compact. Since \( \text{SU}(2) \) is a maximal compact subgroup [5] of \( \text{SL}(2, \mathbb{C}) \), it follows from the classical Bochner Linearization Theorem that the local action of \( G_p \) around
\( p \in V \) is conjugate to the (local) linear action of a closed subgroup of \( SU(2) \) on a neighborhood of the origin.

In slightly more accurate terms, \( SU(2) \) can be viewed as the subgroup of \( SL(2, \mathbb{C}) \subset GL(2, \mathbb{C}) \) consisting of matrices having the form

\[
\begin{bmatrix}
\alpha & \beta \\
\beta & \bar{\alpha}
\end{bmatrix},
\]

where \( \alpha, \beta \in \mathbb{C} \) verify \(|\alpha|^2 + |\beta|^2 = 1\). In other words, with the preceding notation, the subgroup of \( SL(2, \mathbb{C}) \subset GL(2, \mathbb{C}) \) consisting of the derivatives at \( p \) of elements in \( G_p \) is conjugate to a subgroup of the above indicated group of matrices. In turn, an elementary computation shows that, if some the derivative at \( p \) of some \( g \in G_p \) has one eigenvalue equal to 1, then this derivative actually coincides with the identity.

The possibility of having a point \( p \in V \) whose stabilizer \( G_p \) is conjugate to all of \( SU(2) \) is a challenge for us as it raises quite a few technical issues. To avoid get involved in a much longer argument and keep us focused on the situations of primary interest, we will work only with the following set of parameters:

\[
\mathbb{C}_4^{\text{good}} = \{(A, B, C, D) \in \mathbb{C}^4 : \text{every fixed point of every element of } \Gamma_{A,B,C,D} \setminus \{\text{id}\} \text{ is in } J_{A,B,C,D}\}.
\]

In Propositions 9.9 and 9.11 it will be seen that the set \( \mathbb{C}_4^{\text{good}} \) is "quite large" and contains several parameters of interest. In particular, it will be shown that \( \mathbb{C}_4 \setminus \mathbb{C}_4^{\text{good}} \) is at worst a countable union of (proper) real-algebraic subsets of \( \mathbb{C}^4 \) and, in particular, it has null Lebesgue measure.

First, however, we have a simple and well-known lemma.

**Lemma 9.4.** Let \( f \) be an automorphism of a Kobayashi hyperbolic domain \( V \). Assume there is a point \( p \in U \) that is fixed by \( f \) and where the differential of \( f \) coincides with the identity. Then \( f \) is the identity on all of \( V \).

**Proof.** Since \( f \) is an isometry of the Kobayashi metric, the statement would be immediate if the Kobayashi metric were a Riemannian metric which, however, is not always the case. To overcome this difficulty, we locally replace the Kobayashi metric by the Bergman one as follows. For small \( r > 0 \), let \( B_r(p) \) denote the ball of radius \( r \) with respect to the Kobayashi metric. Since a holomorphic map cannot increase the Kobayashi distance, there follows that \( f(B_r(p)) \subset B_r(p) \).

The analogous argument applies to \( f^{-1} \) allows us to conclude that \( f \) induces an automorphism of \( B_r(p) \). Now, if \( r > 0 \) is small enough, then \( B_r(p) \) can be identified with a bounded domain in some space \( \mathbb{C}^n \) so that the Bergman metric is well defined. Thus \( f \) induces an isometry of the resulting Riemannian metric on \( B_r(p) \) and hence coincides locally with the identity. The lemma then follows.

**Proposition 9.5.** Suppose that \( (A, B, C, D) \in \mathbb{C}^4_{\text{good}} \) and that \( \Gamma_{A,B,C,D} \) is locally non-discrete on a connected open \( U \subset \mathcal{F}_{A,B,C,D} \). Let \( V \) denote the Fatou component containing \( U \). Then,

1. The closure \( G = \overline{\Gamma}_V \) of \( \Gamma_V \) in \( \text{Aut}(V) \) is a real Lie Group of dimension \( \geq 1 \).
2. For any \( p \in V \), the stabilizer \( G_p \) of \( p \) in \( G \) coincides with the identity, i.e. \( G_p = \{\text{id}\} \).

**Proof.** Beyond the proof of Proposition 9.3, it remains to show Claim (2). Suppose for contradiction the existence of a point \( p \in V \) for which there is \( g \in G \), \( g \neq \text{id} \), satisfying \( g(p) = p \). Note that \( g \) may lie in \( \overline{\Gamma}_V \setminus \Gamma_V \) so that the statement does not follow immediately from the definition of the parameter set \( \mathbb{C}_{\text{good}}^4 \).

Since \( g \neq \text{id} \), Lemma 9.4 implies that \( Dg(p) \) is not the identity either. On the other hand, \( Dg(p) \) is conjugate to a matrix in \( SU(2) \) so that the preceding discussion shows that \( Dg(p) - \text{id} \) is, in fact, invertible.

On the other hand, \( g \) lies in \( G = \overline{\Gamma}_V \) so that there are elements \( \gamma_n \in \Gamma \) with \( \gamma_n \to g \) locally uniformly on \( V \). Moreover, since the functions are holomorphic, this implies \( C^\infty \) convergence on
compact subsets of \( V \). Since \( Dg(p) - \text{id} \) is invertible, it follows from the implicit function theorem (for Banach spaces) that for sufficiently large \( n \) the mappings \( \gamma_n \) have fixed points \( p_n \) converging to \( p \). Therefore \( D\gamma_n(p_n) \to Dg(p) \neq \text{id} \), implying that \( \gamma_n \neq \text{id} \) for sufficiently large \( n \). Thus, we have found non-trivial \( \gamma_n \in \Gamma \) having fixed points in the Fatou component \( V \), contradicting the choice of parameters \((A, B, C, D) \in \mathbb{C}^4_{\text{good}}\). We conclude that \( G_p = \{\text{id}\} \). \( \square \)

**Remark 9.6.** Recall that a fixed point \( p \) of a (local) diffeomorphism \( f \) is called simple provided that 1 is not an eigenvalue of the differential \( Df(p) \) of \( f \) at \( p \). Equivalently, this means that the linear transformation \( Df(p) - \text{id} \) is invertible. The argument based on the implicit function theorem used above to establish the “persistence” of simple fixed points is very standard. Yet, since our local diffeomorphisms are holomorphic, we can alternatively use residue theory. In fact, a higher dimensional version of the classical Rouché theorem for functions of one complex variable, see [51], page 287. In particular, if \( f : U \subset \mathbb{C}^n \to \mathbb{C}^n \) is a local diffeomorphism with an isolated fixed point at \( p \in U \), then every holomorphic map from \( U \) to \( \mathbb{C}^n \) sufficiently \( C^0 \)-close to \( f \) will also possess an isolated fixed point arbitrarily close to \( p \).

Note that this argument holds provide that the fixed point is isolated and regardless of whether or not \( Df(p) - \text{id} \) is invertible.

Recall from Definition/Proposition 8.2 that, bar the identity, all elements of \( \Gamma \) are split in hyperbolic maps and parabolic maps. Furthermore an element of \( \Gamma \) is parabolic if and only if it is conjugate to a non-trivial power of one of the generators \( g_x, g_y \) or \( g_z \).

**Lemma 9.7.** For any \((A, B, C, D)\), any fixed point of a parabolic element \( \gamma \in \Gamma_{A,B,C,D} \) lies in the Julia set of the corresponding \( \Gamma = \Gamma_{A,B,C,D} \)-action.

**Proof.** Since parabolic maps are conjugate to a non-trivial power of one of the generators \( g_x \), \( g_y \), or \( g_z \), it suffices to prove the statement for a non-trivial power of, say, \( g_x \). It follows from Proposition 4.1 and Corollary 3.4 that for all but finitely many values of \( x_\ell \in \mathbb{C} \setminus [-2, 2] \) the action of \( g_x^{\ell} \) on the fiber \( S_{x_0} \) is loxodromic, with two distinct fixed points at infinity. Consider an iterate \( g_x^{\ell} \) for some \( \ell \neq 0 \). Any point on any such \( S_{x_0} \) will have orbit under \( g_x^{\ell} \) that tends to infinity. These points form an open dense subset of \( S_{A,B,C,D} \), implying that any point having bounded orbit under \( g_x^{\ell} \) (and hence any fixed point of \( g_x^{\ell} \)) must be in the Julia set. \( \square \)

Now we will need a significantly more elaborate result.

**Lemma 9.8.** There is a countable union \( H \subset \mathbb{C}^4 \) of real algebraic hypersurfaces such that if \((A, B, C, D) \in \mathbb{C}^4 \setminus H \) then \( S_{A,B,C,D} \) is smooth and any hyperbolic \( \gamma \in \Gamma_{A,B,C,D} \) has every fixed point consisting of a hyperbolic saddle point.

**Proof.** Let \( \text{NS} \subset \mathbb{C}^4 \) be the set of parameters \((A, B, C, D)\) for which \( S_{A,B,C,D} \) is not smooth. It consists of finitely many complex algebraic hypersurfaces and we immediately include \( \text{NS} \) as part of \( H \).

Fix a hyperbolic map \( f_{A,B,C,D} \in \Gamma_{A,B,C,D} \) and recall that the hyperbolic nature of \( f_{A,B,C,D} \) depends only on its spelling in terms of the generators \( g_x, g_y, \) and \( g_z \). In particular, the notion of hyperbolic map does not depend on the parameters \((A, B, C, D)\). Since \( \Gamma \) is countable, we can then fix the spelling of \( f_{A,B,C,D} \) and reduce the proof to checking that there are finitely many real-algebraic hypersurfaces \( H \subset \mathbb{C}^4 \) away of which every fixed point of \( f_{A,B,C,D} \) is a hyperbolic saddle. The remainder of the proof consists of showing that this is, indeed, the case.

Consider the set of 7-tuples \((A, B, C, D, x, y, z) \in \mathbb{C}^7 \) and the subset \( \bar{H} \subset \mathbb{C}^7 \) consisting of points \((A, B, C, D, x, y, z) \) such that

1. \( S_{A,B,C,D} \) is smooth, i.e. \((A, B, C, D) \in \mathbb{C}^4 \setminus \text{NS}, \)
2. \((x, y, z) \in S_{A,B,C,D}, \)
(3) \( f_{A,B,C,D}(x, y, z) = (x, y, z) \), and
(4) \( Df_{A,B,C,D}(x, y, z) \) is not a hyperbolic saddle.

In our setting, Condition (4) is equivalent to requiring that \( \text{tr}(Df_{A,B,C,D}(x, y, z)) \in [-2, 2] \). Therefore, \( H \) is a semi-algebraic subset of \( \mathbb{C}^7 \), i.e., it is a set given by finitely many polynomial equations and polynomial inequalities with real coefficients.

Notice that \( H = \text{pr}(\tilde{H}) \), where \( \text{pr}(A, B, C, D, x, y, z) = (A, B, C, D) \). It follows from the Tarski-Seidenberg Theorem [4, Theorem 2.2.1] that \( H \) is also semi-algebraic. By definition, the dimension of a semi-algebraic set is the (real) dimension of the (real) Zariski closure of the set. Therefore, if we prove that \( \dim(H) \leq 7 \) it will follow that \( H \) is contained in a finite union of real-algebraic hypersurfaces of \( \mathbb{C}^4 \), which is sufficient for our purposes.

The Cylindrical Algebraic Decomposition Theorem [4, Theorem 2.3.6] asserts that a semi-algebraic set can be decomposed into finitely many sets, each of which is homeomorphic to \([0, 1]^d\) for some \( d \). Moreover, the dimension of the set (in the sense of the previous paragraph) equals the maximum of the \( d_i \), see [4, Section 2.8]. In particular, if \( \dim(H) = 8 \), then \( H \) would have non-empty interior. We will prove that this is not the case.

First recall that the fixed points of \( f_{A,B,C,D} \) are all isolated (see Corollary 8.5 or Lemma 16 in [35]). Let us first prove the following:

\textbf{Claim.} There is an open \( U \subset \mathbb{C}^4 \setminus (H \cup \text{NS}) \).

\textbf{Proof of the claim.} Consider a sequence of parameters \((A_n, B_n, C_n, Z_n) \in \mathbb{C}^4 \setminus \text{NS} \) converging to the Picard Parameters \((0, 0, 0, 4)\). Suppose for contradiction that for every \( n \) the mapping \( f_{A_n,B_n,C_n,D_n} \) has a fixed point \( p_n \in S_{A_n,B_n,C_n,D_n} \) that is not a hyperbolic saddle. Then, since \( f_{A_n,B_n,C_n,D_n} \) preserves the volume form \( \omega \), both eigenvalues of \( Df_{A_n,B_n,C_n,D_n}(p_n) \) have modulus equal to 1.

For sufficiently large \( n \) the fixed points \( p_n \) remain away from some fixed neighborhood of \( \Delta_\infty \) (Proposition 8.4). Therefore, we can extract a subsequence so that \( p_{n_k} \) converges to some point \( p_\infty \in S_{0,0,0,4} \). Since \( f_{A,B,C,D} \) is continuous and depends continuously on the parameters we have that \( p_\infty \) is a fixed point of \( f_{0,0,0,4} \). For any \((A, B, C, D) \in \mathbb{C}^4 \) let \( F_{A,B,C,D} \) denote the extension of \( f_{A,B,C,D} \) as an automorphism of \( \mathbb{C}^3 \). The points \( p_{n_k} \) and \( p_\infty \) are also fixed points for \( F_{A_{n_k},B_{n_k},C_{n_k},D_{n_k}} \) and \( F_{0,0,0,4} \), respectively. For each \( k \) all three of the eigenvalues of \( DF_{A_{n_k},B_{n_k},C_{n_k},D_{n_k}}(p_{n_k}) \) have modulus equal to 1, with the third eigenvalue corresponding to a direction transverse to \( S_{A_{n_k},B_{n_k},C_{n_k},D_{n_k}} \). Since the eigenvalues of a matrix depend continuously on its entries and since the derivative of \( DF_{A,B,C}(q) \) depends continuously on the parameters \((A, B, C)\) and on the point \( q \), we conclude that each eigenvalue of \( DF_{0,0,0,4}(p_\infty) \) has modulus equal to 1. Since \( p_\infty \in S_{0,0,0,4} \) this contradicts Corollary 5.9.

We conclude that there is some \( n \) such that every fixed point of \( f_{A_n,B_n,C_n,D_n} \) is a hyperbolic saddle. Since we chose the parameters \((A_n, B_n, C_n, D_n) \in \mathbb{C}^4 \setminus \{\text{NS}\} \) the surface \( S_{A_n,B_n,C_n,D_n} \) is also smooth. Both of these are open conditions and therefore they hold on some small neighborhood \( U \) of \((A_n, B_n, C_n, D_n) \in \mathbb{C}^4 \). The claim follows at once.

Consider now the set \( \tilde{M} \subset \mathbb{C}^7 \) consisting of 7-tuples \((A, B, C, D, x, y, z) \in \mathbb{C}^7 \) such that
(1) \((x, y, z) \in S_{A,B,C,D}, \)
(2) \( f_{A,B,C,D}(x, y, z) = (x, y, z) \), and
(3) \( Df_{A,B,C,D}(p) - \text{id} \) is singular.

It is an complex algebraic subset of \( \mathbb{C}^7 \). The projection \( M = \text{pr}(\tilde{M}) \subset \mathbb{C}^4 \) onto the first four coordinates is therefore constructible, see [42]. Alternately, up to replacing the initial algebraic set by the corresponding projective scheme, the so-called main theorem of elimination theory tells us that the resulting projection on the coordinates \((A, B, C, D) \) yields an algebraic set \( M \). In any case, the fundamental result to be used here is the fact that the Zariski-closure of the constructible set
M must coincide with its closure for the standard topology, see [42]. Since it was shown that the complement of M has non-empty interior in the standard topology, it follows that M is contained in a proper Zariski-closed subset of \( \mathbb{C}^4 \).

Consider the Zariski-open set \( Z = \mathbb{C}^4 \setminus (\text{NS} \cup \overline{M}) \), where \( \overline{M} \) stands for the closure of the constructible set M. In particular, Z is not empty. Suppose that \( (A_0, B_0, C_0, D_0) \in Z \) and that \( p_{A_0, B_0, C_0, D_0} \) is a fixed point of \( f_{A_0, B_0, C_0, D_0} \). Since \( p_{A_0, B_0, C_0, D_0} \) is simple (\( Df_{A_0, B_0, C_0, D_0}(p_{A_0, B_0, C_0, D_0})^{-1} \) is invertible), \( p_{A_0, B_0, C_0, D_0} \) varies holomorphically with the parameters \( (A, B, C, D) \). In other words, we have a locally defined holomorphic mapping \( (A, B, C, D) \mapsto p_{A, B, C, D} \) for \( (A, B, C, D) \) close enough to \( (A_0, B_0, C_0, D_0) \). Similarly, the differential \( Df_{A, B, C, D}(p_{A, B, C, D}) \) also varies holomorphically with the parameters.

Moreover, the initial fixed point \( p_{A_0, B_0, C_0, D_0} \) can actually be (globally) continued along paths \( c : [0, 1] \to Z \). Indeed, as the parameters vary in \( Z \), two fixed points of \( f_{A, B, C, D} \) cannot collide since they are all simple. Furthermore, they cannot hit \( \Delta_\infty \) either, owing to Proposition 8.4.

Suppose for contradiction that the set \( H \) had non-empty interior. We can therefore choose some \( (A_0, B_0, C_0, D_0) \) and some \( \epsilon > 0 \) such that \( B_{\epsilon}((A_0, B_0, C_0, D_0)) \subset H \cap Z \). Since \( f \) has finitely many fixed points we can reduce \( \epsilon > 0 \), if necessary, so that there is some fixed point \( p_{A, B, C, D} \) varying holomorphically over \( B_{\epsilon}((A_0, B_0, C_0, D_0)) \) such that \( \text{tr}(Df_{A, B, C, D}(p_{A, B, C, D})) \in [-2, 2] \) for all \( (A, B, C, D) \in B_{\epsilon}((A_0, B_0, C_0, D_0)) \).

Let \( (A_1, B_1, C_1, D_1) \in U \setminus Z \), where \( U \) is the open set provided by the Claim above. Consider a simple path \( c : [0, 1] \to Z \) with \( c(0) = (A_0, B_0, C_0, D_0) \) and \( c(1) = (A_1, B_1, C_1, D_1) \). Within \( Z \) there is a simply connected neighborhood \( V \) of \( c([0, 1]) \) on which \( p_{A, B, C, D} \) and \( Df_{A, B, C, D}(p_{A, B, C, D}) \) vary holomorphically. Since \( \text{tr}(Df_{A, B, C, D}(p_{A, B, C, D})) \in [-2, 2] \) on an open neighborhood of \( c(0) \) the same holds on all of \( V \). In particular, \( \text{tr}(Df_{A_1, B_1, C_1, D_1}(p_{A_1, B_1, C_1, D_1})) \in [-2, 2] \), contradicting that \( (A_1, B_1, C_1, D_1) \in \mathbb{C}^4 \setminus H \).

We conclude that the (real) Zariski closure of \( H \) has real dimension equal to 7 and thus that \( H \) is contained in finitely many real-algebraic hypersurfaces. The lemma is proved.

We summarize these two lemmas with the following proposition.

**Proposition 9.9.** There is a countable union of real-algebraic hypersurfaces \( \mathcal{H} \subset \mathbb{C}^4 \) such that if \( (A, B, C, D) \in \mathbb{C}^4 \setminus \mathcal{H} \) then every fixed point of every element of \( \Gamma_{A, B, C, D} \setminus \{\text{id}\} \) is in \( J_{A, B, C, D} \). \( \square \)

Similarly, the argument used in Lemma 9.8 can be repeated word-by-word to yield:

**Corollary 9.10.** For all but countably many \( D \in \mathbb{C} \), every fixed point of every element in \( \Gamma_{0, 0, 0, D} \), bar the identity, lies in \( J_{0, 0, 0, D} \). \( \square \)

When \( (A, B, C, D) \) are all real, there is a simple sufficient condition for every fixed point of every element of \( \Gamma_{A, B, C, D} \setminus \{\text{id}\} \) to be in \( J_{A, B, C, D} \). More specifically, for real parameters \( (A, B, C, D) \), the real slice \( S_{A, B, C, D}(\mathbb{R}) = S_{A, B, C, D} \cap \mathbb{R}^3 \) is invariant by the action of \( \Gamma_{A, B, C, D} \) and the resulting dynamics in this real 2-dimensional surface can be investigated in further detail. In particular, Cantat proved in [8, Theorem 5.1] that if the real slice \( S_{A, B, C, D}(\mathbb{R}) = S_{A, B, C, D} \cap \mathbb{R}^3 \) is connected, then for any hyperbolic mapping \( f \) the set of bounded orbits \( K_f \) of \( f \) is contained in \( S_{A, B, C, D}(\mathbb{R}) \) and that \( f \) is uniformly hyperbolic on \( K_f \). Moreover, according to [2], the real slice \( S_{A, B, C, D}(\mathbb{R}) \) is connected if and only if the product \( ABCD < 0 \) and none of these (real) parameters lies in the interval \( (-2, 2) \). Taking into account Lemma 9.7, the combination of Cantat’s and Benedetto-Goldman’s theorems then yield:

**Proposition 9.11.** If \( (A, B, C, D) \) are real and \( S_{A, B, C, D}(\mathbb{R}) \) is connected, then every fixed point of every element of \( \Gamma_{A, B, C, D} \setminus \{\text{id}\} \) is in \( J_{A, B, C, D} \). \( \square \)
10. Ruling out Fatou components: Proofs of Theorems H and K

Let \( V \subset S_{A,B,C,D} \) be a connected component of the Fatou set \( F_{A,B,C,D} \) and denote by \( \Gamma_V \leq \Gamma \) its stabilizer. The purpose of this section is to study the dynamics of \( \Gamma_V \) on \( V \) and, in particular, to derive sufficient conditions to ensure that certain open sets of \( S_{A,B,C,D} \) must be contained in the Julia set.

We also remind the reader that \( V \) contains only smooth points of \( S_{A,B,C,D} \), owing to Remark 3.2. In our discussion, we will have to distinguish between bounded and unbounded Fatou component.

Let us first consider the case of unbounded Fatou components which relies heavily on Lemma 8.6. More generally, we resume the notation employed in Section 8. Every vertex \( p \in S_{A,B,C,D} \) of \( \Gamma \) is contained in a neighborhood \( U_\infty(p) \subset S = S_{A,B,C,D} \) where “standard coordinates” are defined. The neighborhoods \( U_\infty(i) \) are assumed to be pairwise disjoint. The distance of a point in \( U_\infty(i) \) to \( v_i \) is measured with the Euclidean metric arising from the “standard coordinates”. We then set \( V_\infty = \{v_1, v_2, v_3\} \) and \( U_\infty = U_\infty(1) \cup U_\infty(2) \cup U_\infty(3) \). Finally, the distance from \( p \in U_\infty \) to \( V_\infty \) is equal to the distance in \( U_\infty(i) \) of \( p \) to \( v_i \) where \( i \in \{1, 2, 3\} \) is chosen so that \( p \in U_\infty(i) \).

We are now ready to prove Theorems H and K. We repeat the statements here for the convenience of the reader.

**Theorem H.** Suppose that for some parameters \( A, B, C \) there is a point \( p \in \mathbb{C}^3 \) and \( \epsilon > 0 \) such that for any two vertices \( v_i \neq v_j \in V_\infty \), \( i \neq j \), there is a hyperbolic element \( \gamma_{i,j} \in \Gamma_{A,B,C} \) satisfying:

(A) \( \text{Ind}(\gamma_{i,j}) = v_i \) and \( \text{Attr}(\gamma_{i,j}) = v_j \), and

(B) \( \sup_{z \in B_\epsilon(p)} \| \gamma_{i,j}(z) - z \| < K(\epsilon) \).

Then, for any \( D \), we have that \( B_{\epsilon/2}(p) \cap S_{A,B,C,D} \) is disjoint from any unbounded Fatou components of \( \Gamma_{A,B,C,D} \). Here, \( K(\epsilon) > 0 \) denote the constant given in Proposition 7.4.

**Proof.** Let \( S(0) \) be the set of all six elements \( \gamma_{i,j} \in \Gamma \) satisfying the hypotheses of Theorem H. As in Proposition 7.1 and Lemma 8.6, for every natural number \( n \) we will consider the inductively defined sets of iterated commutators \( S(n) \) where, for every \( n \), the set \( S(n+1) \) contains every possible commutator of any two distinct elements of \( S(n) \).

We assume aiming at a contradiction that for some parameter \( D \) there is an unbounded Fatou component \( V \subset S_{A,B,C,D} \) for \( \Gamma_{A,B,C,D} \) such that \( V \cap B_{\epsilon/2}(p) \neq \emptyset \).

Let \( p' \in B_{\epsilon/2}(p) \cap V \) and let \( \delta > 0 \) be sufficiently small so that \( B_{\delta}(p') \cap S_{A,B,C,D} \subset B_{\epsilon/2}(p) \cap V \). Since the elements \( \gamma_{i,j} \) satisfy Hypothesis (B), it follows from Proposition 7.1 that there is some integer \( N \geq 0 \) such that for any \( \gamma \in S(N') \), with \( N' \geq N \), we have \( \gamma(p') \in B_{\delta}(p') \) and hence that \( \gamma(V) = V \).

For a fixed neighborhood \( U_\infty \) of the vertices of \( \Delta_\infty \) as above, Lemma 8.6 gives us a sequence of elements \( \{\eta_n\}_{n=0}^\infty \in \Gamma \) satisfying Assertions (i) and (ii) of the lemma in question.

We claim that \( V \) intersects \( U_\infty \) non-trivially. Since \( V \) is unbounded, there is a sequence \( \{q_k\}_{k=1}^\infty \in V \) which accumulates to \( \Delta_\infty \). Passing to a subsequence, if necessary, we can suppose that it converges to some \( q_\infty \in \Delta_\infty \). If \( q_\infty \in V_\infty \) then the claim holds. Otherwise, we have that \( \gamma_{2,1}^{(N)}(\Delta_\infty \setminus \{v_2\}) = \{v_1\} \) so that \( \gamma_{2,1}^{(N)}(q_\infty) = v_1 \). Since \( \gamma_{2,1}^{(N)} \in S(N) \) it stabilizes \( V \) and we obtain a sequence \( \{\gamma_{2,1}^{(N)}(q_k)\}_{k=1}^\infty \subset V \) that converges to \( v_1 \), thus implying the claim.

Now consider some point \( r \in V \cap U_\infty \). Because of Assertion (i), Proposition 7.1 implies that \( \{\eta_n\}_{n=0}^\infty \) converges uniformly to the identity on the open set \( B_{\delta}(p') \cap S_{A,B,C,D} \subset B_{\epsilon/2}(p) \cap V \). In turn, since \( V \) is a Fatou component, this implies that \( \{\eta_n\}_{n=0}^\infty \) actually converges uniformly to the identity on any compact subset of \( V \) (see the claim in the proof of Proposition 9.3). Applying this to the singleton set \( \{r\} \), we find that \( \eta_n(r) \to r \). In contrast, Assertion (ii) from Lemma 8.6 implies that \( \text{dist}(\eta_n(r), V_\infty) \to 0 \), providing a contradiction.

We conclude that any Fatou component for \( \Gamma_{A,B,C,D} \) that intersects \( B_{\epsilon/2}(p) \cap S_{A,B,C,D} \) non-trivially must be bounded. \( \square \)
Theorem K. Suppose that \((A, B, C, D) \in \mathbb{C}^4_{\text{good}}\) and that \(V\) is a bounded Fatou component for \(\Gamma_{A,B,C,D}\), then the stabilizer \(\Gamma_V\) of \(V\) is cyclic.

It should be noticed that, in each of the examples from Section [1], where we prove that \(\Gamma_{A,B,C,D}\) is locally non-discrete on some open \(U \subset S_{A,B,C,D}\), the proof was carried out by producing non-trivial elements of the sets \(S(n)\) of iterated commutators that converge to the identity on \(U\) as \(n\) tends to infinity. The theorem above asserts that, if in addition we have \((A, B, C, D) \in \mathbb{C}^4_{\text{good}}\), then this set \(U\) does not intersect any bounded Fatou component of \(\Gamma_{A,B,C,D}\). Indeed, if \(V\) were to be a Fatou component intersecting \(U\) then for all sufficiently large \(n\) the elements of \(S(n)\) would stabilize \(V\) so that \(\Gamma_V\) would not be Abelian.

Before proving Theorem K, a simple lemma is needed.

\textbf{Lemma 10.1.} Assume that \(V\) is a bounded Fatou component of \(\Gamma_{A,B,C,D}\). Then the closure \(G = \overline{\Gamma_V}\) of \(\Gamma_V\) in \(\text{Aut}(V)\) is a compact real Lie group.

\textit{Proof.} Since \(G\) is a closed subgroup of the Lie group \(\text{Aut}(V)\) it is a real Lie group. Thus we only have to show that \(G\) is compact.

We first notice that every element of \(G = \overline{\Gamma_V}\) preserves the holomorphic volume form \(\Omega\) defined in \([9]\). Indeed, by construction, \(\Omega\) is invariant by elements in \(\Gamma_V\) and the condition of preserving \(\Omega\) is clearly closed so that it has to hold for the closure of \(\Gamma_V\). Since \(V\) is bounded, the total (real) volume of \(V\) defined by means of \(\Omega\) is finite. Hence, we can find a relatively compact open set \(K_0 \subset V\) such that \(\text{vol}_\Omega(K_0) > \frac{1}{2}\text{vol}_\Omega(V)\). This implies that for every \(g \in G\) we have \(g(K_0) \cap K_0 \neq \emptyset\).

Let \(K = K_0\). Since \(\text{Aut}(V)\) acts properly on \(V\),
\[
\{ \alpha \in \text{Aut}(V) : \alpha(K) \cap K \neq \emptyset \}
\]
is a compact subset of \(\text{Aut}(V)\). It follows that the closed subset \(G\) is compact as well. \(\square\)

\textit{Proof of Theorem K.} As an abstract group, \(\Gamma\) is isomorphic to the congruence group \(\Gamma(2)\) that is defined in \([18]\). Since any Abelian subgroup of a non-elementary Fuchsian group is cyclic, it suffices to prove that \(\Gamma_V\) is Abelian.

We begin by pointing out that every element in \(\Gamma_V\) is hyperbolic. Indeed, parabolic elements are conjugate to one of the generators \(g_x, g_y,\) or \(g_z\), and hence are such that every open set of \(S_{A,B,C,D}\) contains points whose orbit under (any) parabolic element is unbounded. Thus \(\Gamma_V\) cannot possess parabolic elements. Finally, it cannot possess elliptic elements either since \(\Gamma\) contains no elliptic element; see Definition/Proposition \([8.2]\) and Remark \([8.3]\).

Now, suppose for contradiction that there are two non-commuting elements \(\eta, \tau \in \Gamma_V\). By using again the isomorphism between \(\Gamma\) and \(\Gamma(2)\), we see that \(\eta\) and \(\tau\) correspond to hyperbolic elements in \(\Gamma(2)\) which do not commute. In particular, their iterates also do not commute since two hyperbolic elements of \(\text{SL}(2,\mathbb{Z})\) commute if and only if they have the same axes of translation in \(\mathbb{H}^2\).

Owing to Lemma \([10.1]\), the closure \(G = \overline{\Gamma_V}\) is a compact Lie Group. Since \(\eta\) and \(\tau\) are hyperbolic elements, they have infinite order. We can therefore find subsequences of the iterates \(\eta^{n_k}\) and \(\tau^{n_\ell}\) converging to the identity and such that the elements of this subsequence are pairwise different. Thus the subgroup of \(G\) generated by \(\eta\) and \(\tau\) non-trivially accumulates on the identity which implies that the dimension of \(G\) itself as a real Lie group is strictly positive. Furthermore, since \(\eta^{n_k}\) and \(\tau^{n_\ell}\) do not commute for any \(k\) and \(\ell\), we also have that the identity component \(G_0\) of \(G\) is non-Abelian. On the other hand, the only compact connected real Lie groups of dimension one or two being Abelian (tori), there follows that, in fact, we have \(\text{dim}_\mathbb{R}(G) \geq 3\), where \(\text{dim}_\mathbb{R}(G)\) stands for the dimension of \(G\) as real Lie group.

Conversely, the condition that the parameters \((A, B, C, D)\) are in the set \(\mathbb{C}^4_{\text{good}}\) implies that for any point \(p \in V\) the orbit \(G_0(p)\) of \(p\) under \(G_0\) is diffeomorphic to \(G_0\). In particular, \(\text{dim}_\mathbb{R}(G_0) \leq 4\). However, if we had \(\text{dim}_\mathbb{R}(G_0) = 4\), then \(G_0(p)\) would be four dimensional, implying
that \(G_0(p) = V\). This is clearly impossible since \(G_0\) is compact and \(V\) is open. Summarizing, we must have \(\dim_{\mathbb{R}}(G) = 3\).

The action of \(G_0\) on \(V\) is smooth, proper, and free (since \((A,B,C,D)\) lies in \(\mathbb{C}^4\)). It follows that the quotient space \(V/G_0\) can be given a structure of a smooth manifold with

\[
\dim_{\mathbb{R}}(V/G_0) = \dim_{\mathbb{R}}(V) - \dim_{\mathbb{R}}(G_0) = 1
\]

in such a way that the quotient map \(\pi : V \to V/G_0\) is a submersion. Thanks to the classical result of Ehresmann, this gives \(V\) the structure of a fiber bundle \(V \to V/G_0\) where the fibers are diffeomorphic to \(G_0\), see, for example, [33]. In particular, the base \(V/G_0\) is of dimension 1.

As a one-dimensional smooth manifold, there follows that \(V/G_0\) is either \(S^1\) or \(\mathbb{R}\). The former case is impossible because \(V\) is non-compact, while the total space of a fiber bundle with compact base and compact fibers is compact.

We will now show that the possibility of having \(V/G_0 = \mathbb{R}\) cannot occur either. For this, note first that our assumption on parameters implies that \(S_{A,B,C,D}\) is smooth and hence the closure \(\overline{S}_{A,B,C,D}\) is smooth in \(\mathbb{C}P^3\). It is therefore biholomorphic to the blow-up of \(\mathbb{C}P^2\) at six points, implying that \(\overline{S}_{A,B,C,D}\) is simply connected. If we choose some point \(p_0 \in V\) then the orbit of \(G_0\) through \(p_0\) gives an embedding of \(G_0\) into \(S_{A,B,C,D}\). Since \(\overline{S}_{A,B,C,D}\) is simply connected, there follows that \(\overline{S}_{A,B,C,D} \setminus G_0(p_0)\) has two connected components \(U_1\) and \(U_2\), see, for example [12, Proposition 7.1.1]. Moreover, one of these components, say \(U_1\), contains the triangle at infinity \(\Delta_\infty\) and the other component \(U_2\) is bounded in \(S_{A,B,C,D}\).

By Theorem C, the Julia set \(J_{A,B,C,D}\) is connected. The Julia set is also unbounded since it contains the fibers \(S_{x_0}\) for \(x_0 \in (-2,2)\) by virtue of Lemma 1.3. Therefore, \(J_{A,B,C,D} \subset U_1\) and \(U_2 \subset V\).

Recalling that \(\pi : V \to \mathbb{R}\) stands for the bundle projection, we clearly can assume without loss of generality that \(\pi(p_0) = 0\). The fiber bundle structure implies that \(V \setminus G_0(p_0)\) has two connected components. One of them corresponds to \(U_1 \cap V\) and the other to \(U_2 \subset V\). Since \(\pi\) is non-zero on each component we can suppose that \(\pi(U_1 \cap V) = (0,\infty)\) and \(\pi(U_2) = (-\infty,0)\). However, notice that \(U_2 \cup G_0(p_0)\) is closed and bounded (in \(\mathbb{C}^3\)) and hence it is compact. This implies that \(\pi\) attains a minimum value on \(U_2 \cup G_0(p_0)\) which, in turn, contradicts the fact that \(\pi(U_2) = (-\infty,0)\).

We conclude that any two elements of the stabilizer \(\Gamma_V\) must commute, and hence that \(\Gamma_V\) is cyclic. \(\square\)

11. Coexistence: Proof of Theorems F and G

In this section we combine the previous results to prove Theorem F about the coexistence of local discreteness and non-discreteness for an open set of parameters and Theorem G about coexistence of Fatou set and Julia set with non-empty interior for the same set of parameters, after removing countably many real-algebraic hypersurfaces \(\mathcal{H}\); see Proposition 9.9.

Proof of Theorem F. Lemma 7.5 and Proposition 7.11 give that there is an open neighborhood \(\mathcal{P}_1 \subset \mathbb{C}^4\) that contains \((0,0,0,0)\) and that contains each of the Dubrovin-Mazzocco parameters \((A(a),B(a),C(a),D(a))\) for \(a \in (-2,2)\) such that for each \((A,B,C,D) \in\) parameter neighborhood \(\mathcal{P}_1\) we have that \(\Gamma_{A,B,C,D}\) is locally non-discrete on a non-empty open set \(U \subset S_{A,B,C,D}\). Moreover, the proofs these results are obtained by showing that for arbitrarily large \(n\) there are non-trivial elements of the sets \(S(n)\) of iterated commutators of “level \(n\)” from Proposition 7.1. Therefore, for each of these parameters values we have non-commuting elements arbitrarily close to the identity on \(U\).

Meanwhile, Theorem E and the proof of Proposition 6.1.1 ensure the existence of an open set \(\mathcal{P}_2 \subset \mathbb{C}^4\) containing \((0,0,0,0)\) and each of the Dubrovin-Mazzocco parameters \((A(a),B(a),C(a),D(a))\) for \(a \in (-2,2)\) such that for each \((A,B,C,D) \in \mathcal{P}_2\) we have:
(i) A point \( p = (u, u, u) \in \mathbb{C}^3 \) and an \( \epsilon > 0 \) with the following property. For every point \( q \) in the ball \( B_\epsilon(p) \) and any non-trivial \( \gamma \in \Gamma_{A,B,C} \) one of the coordinates of \( \gamma(q) \) has modulus greater than \( |u| + \epsilon \). In particular, any non-trivial \( \gamma \in \Gamma_{A,B,C} \) satisfies \( \gamma(q) \notin B_\epsilon(p) \).

(ii) Some point \( q_0 \in B_\epsilon(p) \cap S_{A,B,C,D} \) which must therefore be in \( F_{A,B,C,D} \).

Let \( V_\infty \) be the Fatou component containing \( q_0 \) from Property (ii). We will first show that \( \Gamma_{A,B,C,D} \) is locally discrete on \( V_\infty \). Suppose for a contradiction the existence of a non-empty open set \( W \subset V_\infty \) where the action of \( \Gamma_{A,B,C,D} \) is non-discrete. In other words, there exists a sequence \( \{f_j\} \subset \Gamma_{A,B,C,D} \), \( f_j \neq \text{id} \) for all \( j \in \mathbb{N} \), such that the restrictions of the elements \( f_j \) to \( W \) converge uniformly to the identity. Owing to the fact that \( \{f_j\} \) is a normal family on \( V_\infty \), there follows that a subsequence \( \{f_{j_k}\} \) actually converges to the identity on compact subsets of \( V_\infty \), cf. the claim in the proof of Proposition 9.3. However, we must then have that \( f_{j_k}(q_0) \to q_0 \) and, in particular, for sufficiently large \( k \) that \( f_{j_k}(q_0) \in B_\epsilon(p) \). This contradicts Property (i) above.

Therefore, the local non-discreteness of \( U \) and local discreteness on \( V_\infty \) coexist in the dynamics of \( \Gamma_{A,B,C,D} \) for every \( (A, B, C, D) \in \mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2 \). In other words, these groups are locally non-discrete without being “globally non-discrete”.

It only remains to check that \( \Gamma_{A,B,C,D} \) acts properly discontinuously on \( V_\infty \). Let \( K \subset V_\infty \) be a compact set and, aiming at a contradiction, assume the existence of an infinite sequence \( \{f_j\} \) of pairwise distinct elements in \( \Gamma_{A,B,C,D} \) such that \( K \cap f_j(K) = \emptyset \) for all \( j \). Thus, since \( K \) is compact, we can find a subsequence \( j_k \) and points \( p \) and \( q \in K \) such that the sequence \( \{y_k = f_{j_k}(p)\} \) converges to \( q \). Up to enlarging \( K \), we can assume without loss of generality that \( q \) lies in the interior of \( K \). Setting \( l_k = j_{k+1} - j_k \), it follows that \( g_k = f^l_k \) sends \( y_k \) to \( y_{k+1} \) and both points converge towards \( q \) as \( k \to \infty \).

On the other hand, since \( V_\infty \) is contained in the Fatou set, normality implies that the derivatives of the elements \( g_k \) are uniformly bounded in \( K \). Hence, for \( k \) large enough, \( g_k \) sends some fixed neighborhood \( U_q \) of \( q \) to a bounded subset of \( K \). Again, normality implies that a subsequence \( \{g_{k(i)}\} \) of \( \{g_k\} \) converges uniformly on (compact subsets of) \( U_q \) to a non-constant map \( g_\infty : U_q \to K \). Moreover, since each of the mappings \( g_k \) preserves the volume form \( \Omega \), so does the limit \( g_\infty \), thus implying that the limit is locally invertible. Up to reducing the size of \( U_q \) we can suppose that \( g_\infty \) is actually invertible. There follows that the sequence of maps \( h_i = g_{k(i+1)}^{-1} \circ g_{k(i)} \) converges uniformly to the identity on compact subsets of \( U_q \). This contradicts the local discreteness of \( \Gamma_{A,B,C,D} \) on \( V_\infty \). The proof of Theorem \( F \) is complete.

The proof of Theorem \( G \) will require an extra bit of work.

**Proof of Theorem \( G \).** Let \( \mathcal{P} \subset \mathbb{C}^4 \) be the open set of parameters constructed in the proof of Theorem \( F \). For any \( (A, B, C, D) \in \mathcal{P} \) the group \( \Gamma_{A,B,C,D} \) has a non-trivial Fatou component \( V_\infty \subset S_{A,B,C,D} \), as proved in Theorem \( E \).

Therefore, it remains to show that for any \( (A, B, C, D) \in \mathcal{P} \setminus \mathcal{H} \) the open set \( U \subset S_{A,B,C,D} \) on which \( \Gamma_{A,B,C,D} \) is locally non-discrete (from Theorem \( F \)) satisfies \( U \subset \mathcal{J}_{A,B,C,D} \). Here, \( \mathcal{H} \) is the countable union of real-algebraic hypersurfaces provided by Proposition 9.9. Recall from Theorem \( F \) that there are non-commuting pairs of elements of \( \Gamma_{A,B,C,D} \) arbitrarily close to the identity on \( U \). Therefore, Theorem \( K \) gives that \( U \) is disjoint from any bounded Fatou component of \( \Gamma_{A,B,C,D} \). In fact, to ensure that \( U \) is disjoint from any bounded Fatou component is the only place in the proof where the parameters in \( \mathcal{H} \) need to be removed from the set of parameters \( \mathcal{P} \).

We will now use Theorem \( H \) to show that \( U \) is also disjoint from any unbounded Fatou component, and this for every \( (A, B, C, D) \in \mathcal{P} \). Since this requires more specific details, the discussion will be split into two cases in order to make the argument more clear. Also, in the sequel, we are allowed to reduce the size of the open set \( U \), if necessary.

**Case 1:** When \( (A, B, C, D) \) is sufficiently close to \((0, 0, 0, 0)\).
We saw in the proof of Lemma 7.5 that that if $A$, $B$, and $C$ are all sufficiently close to 0 and if $h_x = g_x^2, h_y = g_y^2,$ and $h_z = g_z^2$, then there is some $\epsilon > 0$ such that for any $h \in \{h_x, h_y, h_z\}$ we have
\[
\sup_{q \in \mathbb{B}_{\epsilon/2}(0)} \| h(q) - q \| < K(\epsilon).
\]
Here, $K(\epsilon)$ is the constant from Proposition 7.1. Therefore, if we let $S(0) = \{h_x, h_y^{-1}, h_y, h_y^{-1}, h_z, h_z^{-1}\}$ and define the sets $S(n)$ of iterated commutators of “level $n$” for each $n \geq 0$, it then follows from Proposition 7.1 that for any $\gamma \in S(n)$ we have
\[
\sup_{q \in \mathbb{B}_{\epsilon/2}(0)} \| \gamma(q) - q \| \leq \frac{K(\epsilon)}{2^n}.
\]
Since the relationship between $\epsilon$ and $K(\epsilon)$ given by (30) is linear, it follows that for each $\gamma \in S(1)$ we have
\[
(41) \quad \sup_{q \in \mathbb{B}_{\epsilon/2}(0)} \| \gamma(q) - q \| < K(\epsilon/2).
\]

Let
\[
\gamma_{1,2} = [h_x, h_z], \quad \gamma_{1,3} = [h_y, h_z], \quad \text{and} \quad \gamma_{2,3} = [h_y, h_x]
\]
and let \(\gamma_{2,1} = \gamma_{1,2}^{-1}, \gamma_{3,1} = \gamma_{1,3}^{-1},\) and \(\gamma_{3,2} = \gamma_{2,3}^{-1}\). It is straightforward to check that these six mappings satisfy Hypothesis (A) of Theorem H. For example, one has
\[
\gamma_{1,2} = [h_x, h_z] = h_x^{-1} h_z^{-1} h_x h_z = (s_y s_y s_y s_z)(s_x s_y s_x s_y)(s_y s_y s_y s_x)
\]
\[
= s_y s_y s_y s_x s_y s_y s_y s_x
\]

Since the right-hand side represents a cyclically reduced composition containing all three mappings $s_x, s_y,$ and $s_z$, Definition/Proposition 8.2 implies that \(\gamma_{1,2}\) is hyperbolic. Moreover, since the first (right-most) mapping is $s_x$ we have $\text{Ind}(\gamma_{1,2}) = v_1$ and since the last (left-most) mapping is $s_y$ we have $\text{Attr}(\gamma_{1,2}) = v_2$.

Meanwhile, estimate (41) implies that these six mappings $\gamma_{i,j} \in S(1)$ satisfy Hypothesis (B) of Theorem H on the ball of radius $\epsilon/2$. Therefore, for all $(A, B, C, D)$ close enough to the origin in $\mathbb{C}^4$ the ball $\mathbb{B}_{\epsilon/2}(0) \cap S_{A,B,C,D} \subset U$ is disjoint from any unbounded Fatou component of $\Gamma_{A,B,C,D}$.

**Case 2:** When $(A, B, C, D)$ is close to Dubrovin-Mazzocco parameters.

We saw in Section 7 that if we let
\[
A(a) = B(a) = C(a) = 2a + 4, \quad \text{and} \quad D(a) = -(a^2 + 8a + 8)
\]
then for any $a \in (-2, 2)$ the surface $S_a = S_{A(a), B(a), C(a), D(a)}$ has three singular points $p_1(a), p_2(a)$, and $p_3(a)$ given in (30). Each of the singular points is a common fixed point of $s_x, s_y,$ and $s_z$.

Let us focus on the singular point $p_1(a)$ while pointing out that the entire discussion below applies to $p_2(a)$ and $p_3(a)$ as well. In the proof of Proposition 7.1 it was shown that for any $a \in (-2, 2)$ there is an $\epsilon > 0$, an open neighborhood $W_0$ of $(A(a), B(a), C(a))$ in $\mathbb{C}^3$, and a sufficiently high iterate $k$ so that if we let
\[
f_x = g_x^k, \quad f_y = g_y^{-1} g_x^k g_y, \quad \text{and} \quad f_z = g_z^{-1} g_x^k g_z
\]
then for any $(A, B, C) \in W_0$ and any $f \in \{f_x, f_y, f_z\}$ we have
\[
\sup_{q \in \mathbb{B}_{\epsilon/2}(p_1(a))} \| f(q) - q \| < K(\epsilon).
\]
Let
\[
\gamma_{1,2} = [f_x, f_z], \quad \gamma_{1,3} = [f_y, f_z], \quad \text{and} \quad \gamma_{2,3} = [f_y, f_x]
\]
and let $\gamma_{2,1} = \gamma_{1,2}^{-1}$, $\gamma_{3,1} = \gamma_{1,3}^{-1}$, and $\gamma_{3,2} = \gamma_{2,3}^{-1}$. One can then check that these six mappings satisfy Hypothesis (A) of Theorem H. For example, one has that

$$\gamma_{1,2} = [f_z, f_z] = f_z^{-1} f_z f_z^{-1} f_z = (s_y s_x) (s_x s_y) (s_x s_y) (s_x s_y) (s_y s_x) (s_y s_x) = (s_y s_x) (s_x s_y) (s_y s_x) (s_x s_y) (s_y s_x) (s_x s_y).$$

This is a cyclically reduced composition containing all three mappings $s_x, s_y, s_z$ and therefore it represents a hyperbolic mapping thanks to Definition/Proposition \[3.2\]. Moreover, since the first (right-most) mapping is $s_x$ we have $\text{Ind}(\gamma_{1,2}) = v_1$ and since the last (left-most) mapping is $s_y$ we have $\text{Attr}(\gamma_{1,2}) = v_2$.

Like in the previous example, these six commutators may not be $K(\epsilon)$ close to the identity on $B_{\epsilon}(p_1(a))$. However, we can again use the linearity of the dependence of $K(\epsilon)$ on $\epsilon$ to see that they satisfy Hypothesis (B) of Theorem H on $B_{\epsilon/2}(p_1(a))$. Therefore, for all $(A, B, C, D)$ close enough to $(A(a), B(a), C(a), D(a))$ the ball $B_{\epsilon/2}(p_1(a)) \cap S_{A,B,C,D} \subset U$ is disjoint from any unbounded Fatou component of $\Gamma_{A,B,C,D}$.

We have possibly reduced the size of the open set of parameters $\mathcal{P} \subset \mathbb{C}^4$, while still containing $(0, 0, 0, 0)$ and still containing each of the Dubrovin-Mazzocco parameters $(A(a), B(a), C(a), D(a))$ for $a \in (-2, 2)$. We have also possibly reduced the size of the open set $U$ on which $\Gamma_{A,B,C,D}$ is locally non-discrete in which a way that for any $(A, B, C, D) \in \mathcal{P} \setminus \mathcal{H}$ we have $U \subset J_{A,B,C,D}$.

Therefore for all $(A, B, C, D) \in \mathcal{P} \setminus \mathcal{H}$ the group $\Gamma_{A,B,C,D}$ has a non-empty Fatou component $V_{\infty} \subset F_{A,B,C,D}$ and a Julia set $J_{A,B,C,D}$ with non-empty interior.

\[\square\]

12. Open Problems

Let us close this paper with some open problems motivated by our work.

**Problem 1.** Suppose for some parameters $(A, B, C, D)$ that $\Gamma_{A,B,C,D}$ is locally non-discrete on open $U \subset S_{A,B,C,D}$. If there are non-commuting pairs of elements of $\Gamma_{A,B,C,D}$ arbitrarily close to the identity on $U$, must $U$ be contained in the Julia set?

In particular, Examples 2 and 3 from Section \[7\] provide such open sets $U$ for a large set of parameters. For those parameters, is $U \subset J_{A,B,C,D}$?

**Problem 2.** Are there any parameters other than the Picard Parameters $(A, B, C, D) = (0, 0, 0, 4)$ for which the Julia set is the whole surface $J_{A,B,C,D} = S_{A,B,C,D}$. Equivalently, characterize those parameters for which the Fatou set is non-empty $F_{A,B,C,D} \neq \emptyset$.

**Problem 3.** When is the closure of the set of points with stabilizers containing a hyperbolic saddle equal to the Julia set? Does it happen “generically”. Is it possible to determine the values of the parameters for which the two sets coincide?

**Problem 4.** Determine whether bounded Fatou components exist.

**Problem 5.** Show the existence of “fractal” invariant closed sets of topological dimension 3 bounding either a component of Fatou set or a Julia set with non-empty interior. Estimate their Hausdorff dimension. Can anything be said about the ergodic theory of $\Gamma$ on these invariant sets?

**Problem 6.** Finally, here is a question concerning individual elements of $\Gamma$, as opposed to the dynamics of the whole group $\Gamma$. Is there an element in $\Gamma$ exhibiting a Siegel disk (for some choice of parameters)?

Let us close this section with some comments on Problem 5. As detailed in Section 2.3, for certain parameters $(A, B, C, D)$, the $\text{SL}(2, \mathbb{C})$-character variety arising from representations of the fundamental group of the 4-holed sphere $\Sigma_0, 4$ in $\text{SL}(2, \mathbb{C})$ contains an open set $\text{Rep}_{gf}(\Sigma_0, 4)$ invariant under the action of the mapping class group. The boundary of $\text{Rep}_{gf}(\Sigma_0, 4)$ is known to be fractal-like in the sense that it is not a topological manifold, see for example [41]. Problem 5 is therefore
mostly about the closed invariant sets of topological dimension 3 for those parameters for which this interpretation does not apply, i.e. those parameters for which the resulting representations of the fundamental group of $\Sigma_{0,4}$ in SL(2, $\mathbb{C}$) are forced to be non-discrete; see Remark 2.1.

References

[1] S. Alvarez, D. Filimonov, V. Kleptsyn, D. Malicet, C. Meniño, A. Navas, and M. Triestino. Groups with infinitely many ends acting analytically on the circle. *Journal of Topology*, 12, 4, 1315-1367, (2019).

[2] R. Benedetto and W. Goldman. The topology of the relative character varieties of a quadruply-punctured sphere. *Experiment. Math.*, 8(1), 85–103, (1999).

[3] Y. Benoist and J.-F. Quint Mesures stationnaires et fermés invariants des espaces homogènes. *Ann. of Math.*, 174(2), 1111-1162, (2011).

[4] J. Bochnak, M. Coste, and M.-F. Roy. *Real Algebraic Geometry*. Springer, Berlin (1998).

[5] A. Borel. Sous-groupes compacts maximaux des groupes de Lie. In *Séminaire Bourbaki*, Vol. 1, pages Exp. No. 33, 271–279. Soc. Math. France, Paris, (1995).

[6] B.H. Bowditch. Markoff triples and quasi-Fuchsian groups. Proc. Lond. Math. Soc., 77, 697–736 (1998)

[7] A. Brown and F.R. Hertz. Measure rigidity for random dynamics on surfaces and related skew products. *Journal of the AMS*, 30(4), 1055-1132, (2017).

[8] S. Cantat. Bers and Hénon, Painlevé and Schrödinger. *Duke Math. J.*, 149(3), 411-460, (2009).

[9] S. Cantat and F. Loray. Dynamics on character varieties and Malgrange irreducibility of Painlevé VI equation. *Ann. Inst. Fourier (Grenoble)*, 59(7), 2927–2978, (2009).

[10] S. Cantat and R. Dujardin Random dynamics on real and complex projective surfaces. available from arXiv, https://arxiv.org/abs/2006.04394.

[11] R. Conte, (Editor). The Painlevé Property, One Century Later. CRM series in mathematical physics, Springer-Verlag, New York, Inc. (1999).

[12] R. Daverman and G. Venema. *Embeddings in manifolds*, volume 106 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, (2009).

[13] E. Delabaere and M. Loday-Richaud, (Editors). *Théories asymptotiques et équations de Painlevé*. Sémin. Congr., volume 14, Soc. Math. France, Paris, (2006).

[14] B. Deroin and R. Dujardin. Random walks, Kleinian groups, and bifurcation currents. *Invent. math.*, 190, 57-118, (2012).

[15] B. Deroin and R. Dujardin. Lyapunov Exponents for Surface Group Representations. *Commun. Math. Phys.*, 340, 433-469, (2015).

[16] B. Deroin, V. Kleptsyn, and A. Navas. Towards the solution of some fundamental questions concerning group actions on the circle and codimension one foliations. available from arXiV:1312.4133v2.

[17] J. Diller and C. Favre. Dynamics of bimeromorphic maps of surfaces. *American Journal of Mathematics*, 123(6), 1135-1169, (2001).

[18] B. Dubrovin and M. Mazzocco. Monodromy of certain Painlevé-VI transcendents and reflection groups. *Invent. Math.*, 141(1), 55-147, (2000).

[19] M. H. El-Huiti. Cubic surfaces of Markov type. *Mat. Sb. (N.S.)*, 93 (135), 331-346, 487, (1974).

[20] A. Eskif and J. Rebelo. Global rigidity of conjugations for locally non-discrete subgroups of Diff$^c(S^1)$. *Journal of Modern Dynamics*, 15, 41-93, (2019).

[21] C. Favre. Classification of 2-dimensional contracting rigid germs and Kato surfaces, I. *J. Math. Pures Appl.* (9), 79, 475-514, (2000).

[22] W. Fischer and H. Grauert. Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1965, 89-94, (1965).

[23] J. E. Fornaess and N. Sibony. Complex dynamics in higher dimensions, II. in Modern methods in complex analysis (Princeton, NJ, 1992). Ann. of Math. Stud., 137, Princeton University Press, Princeton, NJ, 135-182, (1995).

[24] E. Ghys. Sur les groupes engendrés par des difféomorphismes proches de l’identité. *Bol. Soc. Brasil. Mat. (N.S.),* 24(2), 137-178, (1993).

[25] W. Goldman. Ergodic theory on moduli spaces. *Annals of Math.*, 146, 1-33, (1997).

[26] W. Goldman. The modular group action on real SL(2)-characters of a one-holed torus. *Geom. Topol.*, 7, 443–486, (2003).

[27] W. Goldman and D. Toledo. Affine cubic surfaces and relative SL(2)-character varieties of compact surfaces. *arXiv 1006.3838v2*, (2011).

[28] R. Horowitz. Induced automorphisms of Fricke characters of Free groups. *Trans. AMS*, 208, 41-50, (1975).

[29] H. Hu, S.P. Tan, and Y. Zhang. On the character variety of the four-holed sphere. *Groups Geom. Dyn.*, 9, 737-782, (2015).
[30] H. Hu, S.P. Tan, and Y. Zhang. Polynomial automorphisms of $\mathbb{C}^n$ preserving the Markoff-Hurwitz polynomial. Geom. Dedicata, 192, 207-243, (2018).

[31] Y. Ilyashenko and S. Yakovenko. Lectures on analytic differential equations, volume 86 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, (2008).

[32] M. Inaba, K. Iwasaki, and M. Saito. Dynamics of the sixth Painlevé equation. In Théories asymptotiques et équations de Painlevé, volume 14 of Sémin. Congr., pages 103-167. Soc. Math. France, Paris, (2006).

[33] K. Iwasaki. A modular group action on cubic surfaces and the monodromy of the Painlevé VI equation. Proc. Japan Acad. Ser. A Math. Sci., 78(7), 131–135, (2002).

[34] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida. From Gauss to Painlevé. Aspects of Mathematics, E16. Friedr. Vieweg & Sohn, Braunschweig, (1991).

[35] K. Iwasaki and T. Uehara. An ergodic study of Painlevé VI. Math. Ann., 338(2), 295-345, (2007).

[36] S. Kobayashi. Hyperbolic complex spaces, volume 318 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, (1998).

[37] F. Ledrappier. Mesures stationnaires sur les espaces homogènes [d’après Yves Benoist et Jean-François Quint]. Séminaire Bourbaki, 2011/2012, exposés 1043-1058, Astérisque, 352, 535-556, (2013).

[38] J.M. Lee. Introduction to smooth manifolds, volume 218 of Graduate Texts in Mathematics. Springer, New York, second edition, (2013).

[39] Oleg Lisovyy and Yuriy Tykhyy. Algebraic solutions of the sixth painlevé equation. Journal of Geometry and Physics, 85 124-163, (2014).

[40] F. Loray and J. Rebelo. Minimal, rigid foliations by curves on $\mathbb{C}^n$. J. Eur. Math. Soc. (JEMS), 5(2), 147-201, (2003).

[41] C. McMullen. Complex earthquakes and Teichmüller theory. Journal of the AMS, 11 (2), 283-320, (1998).

[42] D. Mumford. The Red Book of Varieties and Schemes, volume 1358 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, Heidelberg, (1999).

[43] D. Montgomery and L. Zippin. Topological transformation groups. Dover edition, Mineola, New York (2018).

[44] I. Nakai. Separatrix for Non Solvable Dynamics on $\mathbb{C},0$. Ann. Inst. Fourier, 44(2), 509-599, (1994).

[45] K. Okamoto. Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé. Japan. J. Math. (N.S.), 5(1), 1–79, (1979).

[46] C. Procesi. The geometry of Markoff numbers. Math. Intell., 7, 20-29, (1985).

[47] A.A. Shcherbakov. On the density of an orbit of a pseudogroup of conformal mappings and a generalization of the Hudai-Verenov theorem. Vestnik Moskovskogo Universiteta Matematika, 31(4), 10-15, (1982).

[48] Anna Wienhard. An invitation to higher Teichmüller theory. In Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, pages 1013–1039. World Sci. Publ., Hackensack, NJ, 2018.