Explicit solutions of the 3–loop vacuum integral recurrence relations

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Explicit formulas for the solutions of the recurrence relations for 3–loop vacuum integrals are suggested. This formulas can be used for direct calculations and demonstrate a high efficiency. They also produce a new type of recurrence relations over the space–time dimension.

1 Introduction

Recurrence relations are powerful tools for evaluating multi–loop Feynman integrals [[1]]. They relate Feynman integrals with various degrees of their denominators. In many cases they provide the possibility to express an integral with given degrees of denominators as a linear combination of a few master integrals with some coefficients which we will call weight factors.

At two–loop level the recurrence relations are relatively simple and one can easily find and realize the corresponding recursive algorithm for the calculation of the weight factors. Nevertheless, such recursive algorithms lead to too time and memory consuming calculations because of the size of intermediate expressions grows exponentially with respect to the degrees of the denominators in the initial integral. In fact, the calculations mentioned above were made at the limits of computer capabilities.

In this work we suggest a new approach based on explicit formulas for the solutions of the recurrence relations. As an application, the case of three loop vacuum integrals with four equal mass and two massless lines is considered. The efficiency of this approach is demonstrated by calculations of previously unknown coefficients in Taylor expansion of QED photon vacuum polarization for small $q^2$.

2 General case

Let us consider the three–loop vacuum integrals with six different masses:

$$B(\underline{n}, D) \equiv B(n_1, n_2, n_3, n_4, n_5, n_6, D) = \frac{m^{2\Sigma_{i} n_{i} - 3D}}{[i\pi^{D/2}\Gamma(3 - D/2)]} \int \int \frac{d^D p d^D k d^D l}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6}}$$

(1)

where

$$D_1 = k^2 - \mu_1 m^2, \quad D_2 = l^2 - \mu_2 m^2, \quad D_3 = (p + k)^2 - \mu_3 m^2$$

$$D_4 = (p + l)^2 - \mu_4 m^2, \quad D_5 = (p + k + l)^2 - \mu_5 m^2, \quad D_6 = p^2 - \mu_6 m^2$$

Let us derive recurrence relations that result from integration by parts, by letting $(\partial / \partial p_1) \cdot p_j$ act on the integrand, with $p_{i,j} \in \{p, k, l\}$. For example, acting by $(\partial / \partial k) \cdot (p + k)$ we get

$$(D - 2n_3 - n_1 - n_5)] B(\underline{n}, D) = \{n_1 1^+ (3^- - 6^- + \mu_3 - \mu_6 + \mu_1) + 2n_3 3^+ \mu_3 + n_5 5^+ (3^- - 2^- + \mu_3 - \mu_2 + \mu_5)\} B(\underline{n}, D)$$

(2)

where $1^\pm B(n_1, \ldots) \equiv B(n_1 \pm 1, \ldots)$, etc.

Others relations can be obtained from (1) by proper permutations of the $n_i, \mu_i$ and $1^\pm$ objects.

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The common way of using these relations is step by step re-expression of the integral (3) with some values of \( n_i \) through a set of integrals with shifted values of \( n_i \), with the final goal to reduce this set to a few integrals with \( n_i \) are equal to 0 or 1, so called "master" integrals. The result can be represented as

\[
B(n_i, D) = \sum_k f^k(n_i, D)N_k(D)
\]

where the index \( k \) enumerate master integrals \( N_k(D) \) and corresponding coefficient functions \( f^k(n_i, D) \).

There are two problems on this way. First, there is no general approach to construction of such recursive procedure, that is to find proper combinations of these relations and a proper sequence of its use is the matter of art even for the case of one mass \([4]\). Second, even in cases when such procedures were constructed, they lead to very time and memory consuming calculation because of large reproduction rate at every recursion step. For example, the relation (3) expresses the integral through 7 others.

Instead, let us construct the coefficient functions \( f^k(n_i, D) \) directly as solutions of the given recurrence relations.

For that, let us diagonalize the recurrence relations with respect to \( n_i \), \( n_i^+ \) operators. We found that the recurrence relations can be represented in the following simple form

\[
\{P(x_1, \ldots, x_6) \cdot n_i I^+ - \frac{D - 4}{2} \partial_i(P(x_1, \ldots, x_6)) \}_{x_i=1+\mu_i} B(n_i, D) = 0, \quad i = 1, \ldots, 6.
\]  

(3)

where

\[
P(x_1, \ldots, x_6) = 2(x_1x_2(x_1 + x_2) + x_3x_4(x_3 + x_4) + x_5x_6(x_5 + x_6)) + x_1x_3x_6 + x_1x_4x_5 + x_2x_3x_5 + x_2x_4x_6 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6)(x_1x_2 + x_3x_4 + x_5x_6)
\]

The differential equation corresponding to (3) has the solution \( P^{D/2-2}(x_i + \mu_i) \). Let us consider "Lourent" coefficients of this function:

\[
f(n_i, D) = \frac{1}{(2\pi)^6} \oint \oint \oint \oint \oint \oint \frac{dx_1 dx_2 dx_3 dx_4 dx_5 dx_6}{x_1 x_2 x_3 x_4 x_5 x_6} P(x_1 + \mu_1, \ldots, x_6 + \mu_6)^{D/2-2}
\]  

(4)

where integral symbols denote six subsequent complex integrations with contours which will be described below. If one acts by (3) on (4), one gets up to the surface terms the same expression as acting by \( P \partial_i - (D/2 - 2)/(\partial_i P) \) on \( P^{D/2-2} \), that is zero. Then, the surface terms can be removed if we choose closed or ended in infinity point contours. For more accuracy one can consider analytical continuations of the result on \( D \) from large negative values. So (4) is the solution of the relations (3), and the different choices of the contours correspond to different independent solutions. Note, that if one chooses the contour as a small circle over zero, one get the true Lourent coefficient of the function \( P^{D/2-2} \), so this function can be called generalized generating function for the solutions of the relations (3).

Then, in accordance with the dimensional regularization rules, the integrals (4) are not equal to zero only if at least three among \( n_i \) are positive. So it is natural to construct the solutions from those that are equal to zero if the index from definite three–index set ("Taylor" indexes) is not positive. One can obtain such solutions if one chooses the contours, corresponding to these indexes, as circles over zero. In this case these three integrations can be evaluated and lead to coefficient in the common Taylor expansion in corresponding variables.

The three remaining integrations in general case lead to the sum of generalized hypergeometric seria, but for some cases of practical interest (see below) can be reduced to the finite sums of Pochhammers symbols products.

3 Example

As an example let us consider the case of integrals with four equal mass and two massless lines, that is \( \mu_1 = \mu_2 = 0, \mu_3 = \mu_4 = \mu_5 = \mu_6 = 1 \). Let us calculate the coefficient functions which corresponds to the choice of the master integrals from [3]. That is, we expand \( B(n_i) \) as
\[ B(\underline{n}, D) = N(\underline{n}, D)B(0, 0, 1, 1, 1, D) + M(\underline{n}, D)B(1, 1, 0, 0, 1, 1, D) + T(\underline{n}, D)B(0, 0, 0, 1, 1, 1, D) \]

with the following normalization conditions

\[ N(0, 0, 1, 1, 1, 1, D) = 1, \quad N(1, 1, 0, 0, 1, 1, D) = 0, \quad N(0, 0, 0, 1, 1, 1, D) = 0, \]
\[ M(0, 0, 1, 1, 1, 1, D) = 0, \quad M(1, 1, 0, 0, 1, 1, D) = 1, \quad M(0, 0, 0, 1, 1, 1, D) = 0, \]
\[ T(0, 0, 1, 1, 1, 1, D) = 0, \quad T(1, 1, 0, 0, 1, 1, D) = 0, \quad T(0, 0, 0, 1, 1, 1, D) = 1, \]

The practical rule for choosing the integration contours is: circle around zero for unity in the master integral and contour over cut for zero in the master integral.

To get \( N(\underline{n}) \) one should make first the Taylor expansion in \( x_3, x_4, x_5, x_6 \)

\[ B(n_i, D) \propto \int \int \frac{dx_1dx_2}{x_1^n x_2^n} \frac{\partial^{n_3-1} \partial^{n_5-1}}{(n_3-1)! \ldots (n_6-1)!} P(x_1, x_2, x_3 + 1, \ldots, x_6 + 1)^{D/2-2}) |x_3, \ldots, x_6 = 0 \]

The remaining integrals over \( x_1, x_2 \) are of the type

\[ \int \int \frac{dx_1dx_2}{x_1^n x_2^n} [x_1 x_2 (x_1 + x_2 - 4)]^{D/2-2} \propto (-4)^{-n_1-n_2} \frac{(D/2-1-n_1)(D/2-1-n_2)}{(3D/2-3)-n_1-n_2} \equiv N(n_1, n_2, 1, 1, 1, D) \]

where we follow the normalization \([5]\).

The case \( M(\underline{n}, D) \) is analogous. The only difference is that due to the symmetry of the task we should take the sum of the solutions with the signatures \((++ \pm \pm \pm)\) and \((+++ \pm \pm)\).

The case \( T(\underline{n}, D) \) is more complicated. The symmetry of the task assumes that one should try the following form of the solution

\[ T(n_1, n_2, n_3, n_4, n_5, n_6, D) = t(n_1, n_2, n_3, n_4, n_5, n_6, D) + t(n_1, n_2, n_4, n_3, n_6, n_5, D) + t(n_1, n_2, n_6, n_5, n_4, n_3, D) \]

where \( t(n, D) \) is non–zero only if \( n_4, n_5, n_6 > 0 \). Let us construct \( t(n, D) \) using \([6]\), keeping in mind possible mixing with \( N(n, D) \) solution. After differentiating over last three indexes the task reduces to the construction of \( t(n_1, n_2, n_3, 1, 1, 1, D) \). Let us consider the corresponding integral:

\[ \overline{t}(n_1, n_2, n_3, D) = \frac{1}{(2\pi i)^3} \int \int \int \frac{dx_1dx_2dx_3}{x_1^n x_2^n x_3^n} (x_3^2 - x_1 x_2 x_3 + x_1 x_2 (x_1 + x_2 - 4))^{D/2-2} \]

For \( n_3 < 1 \) one can calculates this integral immediately (the possible \( N(n, D) \) contribution vanish). Taking into account the normalization \([7]\) we get

\[ t(n_1, n_2, n_3 < 1, 1, 1, 1, D) = \frac{\overline{t}(n_1, n_2, n_3, D)}{\overline{t}(0, 0, 0, D)^{(-4)(n_1+n_2)(-8)^{n_3}}} \]

\[ = \frac{(-2-D)(n_1+n_3)(2-D)(n_2+n_3)}{(-4)(n_1+n_2)(-8)^{n_3}} \sum_{k=0}^{[-n_3/2]} \frac{(D-1)(-n_1-n_3-k)(D-1)(-n_2-n_3-k)(n_3-k)(n_3-k)}{(-3D/2-k)(1/2-k)(n_3-2k)!} \]

For \( n_3 > 1 \) using integration by parts for \( x_3 \) in \([5]\) (which reduces to evaluation of \((n_3-1)^{th}\) derivative of \(P^{D/2-2}\)) the \( \overline{t}(n_1, n_2, n_3, D) \) can be reduced to a set of \( \overline{t}(n_1, n_2, 1, D) \) with different \( n_1, n_2 \). Let us extract the \( t(n_1, n_2, 1, 1, 1, 1, D) \) from \( \overline{t}(n_1, n_2, 1, D) \) according to the conditions \([7]\)

\[ t(n_1, n_2, 1, 1, 1, 1, D) = \frac{1}{\overline{t}(0, 0, 0, D)} (\overline{t}(n_1, n_2, 1, D) - \overline{t}(0, 0, 1, D) N(n_1, n_2, 1, 1, 1, 1)) \]

(9)
One can calculate the \( t(n_1, n_2, 1, 1, 1, 1, D) \) by direct use of the \( \Box \) expanding it for example in series over \( D/2 - 2 \), but we found more suitable to use the recurrence relations on \( n_1, n_2 \):

\[
\begin{align*}
    t(n_1, n_2, 1, 1, 1, 1, D) &= -\frac{(D-2)^2}{4(D-3)(2n_1-D+2)} \left( -\frac{1}{2} t(n_1 - 1, n_2 - 1, 0, 1, 1, 1, D - 2) \\
    &\quad + t(n_1 - 2, n_2 - 1, 1, 1, 1, 1, D - 2) \right) \\
    &\quad - \frac{2(D-2)^2(11D-38)}{3(3D-10)(3D-8)(D-3)} N(n_1, n_2, 1, 1, 1, 1, D) \tag{10}
\end{align*}
\]

\[
\begin{align*}
    t(n_1, n_2, 1, 1, 1, 1, D) &= \frac{(n_1-n_2-1) t(n_1, n_2 + 1, 0, 1, 1, 1, D)}{(2n_1-D+2)} \\
    &\quad + \frac{(2n_2-D+4)}{(2n_1-D+2)} t(n_1 - 1, n_2 + 1, 1, 1, 1, 1, D) \tag{11}
\end{align*}
\]

With the help of \( \Box \) the \( n_1 + n_2 \) can be reduced to \(-1, 0, 1\) and with the help of \( \Box \) the \( n_1 - n_2 \) can be reduced to \(0, 1\) (note that \( t(n_1, n_2, 1, 1, 1, 1, D) = t(n_2, n_1, 1, 1, 1, 1, D) \)). Here at every recursion step the one integral reexpresses through the other one plus rational over \( D \), that is there is no "exponential reproduction". Then, the recursion acts separately on variables \( n_1 + n_2 \) and \( n_1 - n_2 \). So, although the relations \( \Box \) can be solved to explicit formulas, this "safe" variant of recursion is in this case the most effective way of calculations.

The relations \( \Box \) are the simple example of the recurrence relations with \( D \)-shifts, which can be derived in the following way. Note that if \( f^k(n_i, D) \) is a solution of \( \Box \), then \( P(I^- + \mu_i)f^k(n_i, D - 2) \) also is a solution. Hence, if \( f^k(n_i, D) \) is a complete set of solutions, then

\[
    f^k(n_i, D) = \sum_n S^k_n(D) P(I^- + \mu_i) f^n(n_i, D - 2) \tag{12}
\]

where the coefficients of mixing matrix \( S \) depends only over \( D \). For the solutions \( \Box \) the matrix \( S \) is unit matrix. On the other hand, the desire to come to some specific set of master integrals leads to nontrivial mixing matrix and for the example considered above these coefficients are

\[
\begin{align*}
    S_n &= -\frac{3}{8} \frac{(3D-8)(3D-10)}{2(D-2)^2} S_m = \frac{3}{8} \frac{(3D-8)(3D-10)}{2(D-2)^2} S_l = \frac{(D-2)^2}{4(D-3)(D-4)} S_l^\prime
\end{align*}
\]

\[
\begin{align*}
    S_l^\prime &= \frac{(11D-38)(3D-2)D}{(2D-3)(D-4)^2} \quad S_l^n = S_m^n = S_m = S_m^m = S_m^m = 0
\end{align*}
\]

To check the efficiency of this approach we evaluated, to 3 loops, the first 5 moments in the \( \alpha^q^2/4m^2 \rightarrow 0 \) expansion of the QED photon vacuum polarization

\[
    \Pi(z) = \sum_{n>0} C_n z^n + O(\alpha^4),
\]

The \( C_n \) are expressed through approximately \( 10^5 \) scalar integrals, but there is no necessary to evaluate these integrals separately. Instead, we evaluated a few integrals of \( \Box \) type, but with \( P^{D/2-2} \) produced by a long polynomial in \( x_i \). After OS mass \( \Box \) and charge \( \Box \) renormalization, we obtained the finite \( D \rightarrow 4 \) limits (the coefficients \( C_1, C_2, C_3 \) can be found in \( \Box \)):

\[
\begin{align*}
    C_4 &= \left\{ N^2 \left[ \frac{256}{69} \zeta_2 + \frac{2322821}{9496958} \zeta_3 - \frac{120586264289}{143327232000} \right] \right. \\
    &\quad + N \left[ \frac{160}{241} (1 - \frac{8}{5} \ln 2) \zeta_2 + \frac{1507351507033}{1651597200} \zeta_3 - \frac{26924066988818833}{245806202880000} \right] \right\} \frac{\alpha^3}{\pi^3} \\
    &\quad + \frac{51986}{197355} N \frac{\alpha^2}{\pi} + \frac{42}{667} N \frac{\alpha}{\pi},
\end{align*}
\]

\[
\begin{align*}
    C_5 &= \left\{ N^2 \left[ \frac{1024}{3003} \zeta_2 + \frac{1239683}{3952160} \zeta_3 - \frac{382847330943}{5055631488000} \right] \right. \\
    &\quad + N \left[ \frac{640}{1001} (1 - \frac{8}{5} \ln 2) \zeta_2 + \frac{99939943788973}{190749081600} \zeta_3 - \frac{360248170450504167133}{6083703521280000000} \right] \right\} \frac{\alpha^3}{\pi^3} \\
    &\quad + \frac{432385216}{1260653625} N \frac{\alpha^2}{\pi^2} + \frac{512}{15015} N \frac{\alpha}{\pi},
\end{align*}
\]
where we follow common practice \[10\], by allowing for \( N \) degenerate leptons. In pure QED, \( N = 1 \); formally, the powers of \( N \) serve to count the number of electron loops.

The \( N \) contribution of \( C_4 \) is in agreement with recent QCD calculations \[3\], the \( N^2 \) part of \( C_4 \) and the \( C_5 \) are new.

The bare (non-renormalized) integrals were calculated for arbitrary \( D \). Calculations for \( C_4 \) were made on PC with Pentium-75 processor by REDUCE with 24Mbyte memory, within approximately 10 CPU hours. The most difficult diagrams for \( C_5 \) were calculated on HP735 workstation.

These results demonstrates a reasonable progress in comparison with common recursive approach. For example, the common way used in \[3\] demands several CPU hours on DEC-Alpha workstation to calculate full \( D \) dependence of \( C_2 \) integrals, and further calculations became possible only after truncation in \((D/2 - 2)\). In the present approach the full \( D \) calculations for \( C_2 \) demand about 5 minutes on PC.

4 Conclusions

The new approach suggested in this work allows to produce explicit formulas \[4\] for the solutions of the recurrence relations for 3–loop vacuum integrals. This formulas can be used for direct calculations and demonstrate a high efficiency. On the other hand, they produce a new type \( D \)-shifted recurrence relations \[12\] for these integrals. Finally, we hope that simple representation \[3\] of the traditional recurrence relations which allows to obtain all these results is not intrinsic for 3–loop vacuum case and generalization for multi–loop or/and non-vacuum case is possible.

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