Some Subclasses of Linear Languages based on Nondeterministic Linear Automata

Benjamín Bedregal
Departamento de Informática e Matemática Aplicada,
Universidade Federal do Rio Grande do Norte
bedregal@dimap.ufrn.br

Abstract

In this paper we consider the class of λ-nondeterministic linear automata as a model of the class of linear languages. As usual in other automata models, λ-moves do not increase the acceptance power. The main contribution of this paper is to introduce the deterministic linear automata and even linear automata, i.e. the natural restriction of nondeterministic linear automata for the deterministic and even linear language classes, respectively. In particular, there are different, but not equivalents, proposals for the class of “deterministic” linear languages. We proved here that the class of languages accepted by the deterministic linear automata are not contained in any of the these classes and in fact they properly contain these classes. Another, contribution is the generation of an infinite hierarchy of formal languages, going from the class of languages accepted by deterministic linear automata and achieved, in the limit, the class of linear languages.

Keywords: Linear languages, nondeterministic linear automata, deterministic linear automata, deterministic linear languages, even linear languages, λ-moves and degree of explicit nondeterminism.

1 Introduction

The class of linear languages, also known by linear context-free languages, is situated in the extended Chomsky hierarchy between the classes of regular and context-free languages. Linear grammars (the more well known model for the class of linear languages) allow to have a control on matches between leftmost symbols with rightmost symbols of a substring. This capability allow that the most of the more commonly used examples of context-free languages, which are not regular languages, belong to this class, as for example the languages of palindromes and \( \{a^n b^n : n \geq 1\} \). An example of a context-free language which is not linear is \( \{a^m b^m a^n b^n : n, m \geq 1\} \) [3].

The most usual models for linear languages are the linear grammars and some normal form for them have been proposed. In terms of automata counterpart, it is possible to find at least six different models: Two-tape nondeterministic finite automata of a special type [14, 30], finite transducer [26, 30], one-turn pushdown automata [3, 12, 14, 17, 18], right-left-monotone restarting automata [21], λ-nondeterministic linear automata [5, 6] and zipper finite-state automata [31]. Despite the merits of the four first models, they are not intrinsic, in the sense that their definition includes external elements. For example, in the two-tape nondeterministic finite automata of a special type, the input is split into two tapes with the second one containing the reverse of the right part of the input string. Notice that, all the usual types of automata – e.g. finite automata, pushdown automata (PDA in short), Turing machines, etc. – assume that the input string is sequentially disposed, without any
modification, in the input tape at the start of execution that here we call natural condition. Thus, this model does not satisfy the natural condition and it is not intrinsic because the choice of how and where to divide the input and the application of the reversion is external to the model.

Analogously, the finite transducer model also presupposes that externally the input string is divided into two parts which are separated by a special symbol and where the second part is reverted. Therefore, this model also presupposes a previous knowledge of where the input must be divided. Thus, this model does not satisfy either the natural condition and it considers external agents. On the other hand, a turn in a PDA is a move which decreases the stack and which is preceded by a move that increases the stack, the one-turn PDA model (a PDA which makes at most one turn in the stack). Determining the maximal possible quantity of turns that a PDA can make (i.e. consider all possible input strings) requires an external control. Thus, one-turn PDA model is not intrinsic. Finally, the right-left-monotone restarting automata introduced in [21] is a restriction of a more general class of automata, the restarting automata introduced in [19, 20]. Since in this automata the transitions work with strings instead of symbols and they allow actions such as the reversion of strings, this model of linear language is higher leveled and also more complex than the other models. On the other hand, this model requires that at each cycle the distance from the actual place, where a rewrite takes place, to the right and left end of the tape must not increase. However the verification of that condition can not be made internally in the model, and therefore it is also a non intrinsic model.

The λ-nondeterministic linear automata, λ-NLA, which extends the notion of nondeterministic finite automata with λ-moves, λ-NFA in short, and therefore properly contains this class of automata. This automata model, in our viewpoint, does not consider external elements, it is not a subclass of automata used to model a broader class of languages, it is strongly inspired in a formal norm introduced here for linear grammars and therefore it is simpler than the previous models.

The last model, zipper finite-state automata [31], is similar to the the previous one, in the sense that it read the input string from the left and from the right side, but differently of the λ-NLA, it is made simultaneously and every movement can read more than one symbols of each side of the string.

In addition, the λ-moves in the usual classes of automata are not essential for the models. Thus, for example, λ-moves do not increase the acceptance power of finite automata (see, for example, [18]), of pushdown automata (see, for example, [14]) and of Turing machines (by Church-thesis). Analogously, in [6] it was proved that λ-moves do not increase either the acceptance power of λ-nondeterministic linear automata.

In this work we introduce the deterministic version of this automata, called deterministic linear automata, DLA in short, and compare the class of languages accepted by DLA’s with the class DL introduced by de la Higuera and Oncina in [16] and with the class of linear languages and deterministic context-free languages. We also provide a characterization of the NLA which are equivalent to some DLA. In this paper we also determine an infinite hierarchy of classes of formal languages which are among the class of languages accepted by DLA’s and the class of linear languages. Finally, we show how the class of even linear languages, introduced originally by Amar and Putzolu in [1], can be captured in the nondeterministic linear automata model.

2 Linear grammars

As usual a formal grammar $G$ is a tuple $\langle V, T, S, P \rangle$ where $V$ is a finite set of variables, $T$ is a set of terminal symbols and therefore $V \cap T = \emptyset$, $S \in V$ is the start variable and $P \subseteq (V \cup T)^+ \times (V \cup T)^*$ is the set of productions. Ordered pairs $(x, y) \in P$ are denoted by
A grammar $G = \langle V, T, S, P \rangle$ is linear, if each production in $P$ has a variable in its left side and has at most one variable in its right side, without restriction in the position of this variable. A variable $A \in V$ will be called left linear if in each production $A \rightarrow y \in P$, either $y \in T^*$ or $y = Bz$ for some $z \in T^*$ and $B \in V$. Analogously, a variable $A$ will be called right linear if in each production $A \rightarrow y \in P$ either $y \in T^*$ or $y = zB$ for some $z \in T^*$ and $B \in V$. A linear grammar $G$ is in linear normal form, in short LNF, if each variable $A \in V$ is left or right linear. Thus a variable $A$ in a linear grammar is both left and right linear, just when each production having $A$ in the left side has at the right side either a string of terminal symbols (including the empty string) or a single variable.

Notice that our linear normal form is subtly different from the usual linear normal form (e.g. see [14, 24, 32]) where it is tolerated two productions with the same variable in the left side, but with its right side containing a variable in the leftmost and rightmost positions, respectively.

**Example 1** The grammar $G = \langle \{S\}, \{a, b\}, S, P \rangle$ where $P$ is given by

$$S \rightarrow aSb \mid aSbb \mid aSbbb \mid ab \mid abb \mid abbb$$

is linear but it is not in the LNF.

On the other hand, the grammar $G = \langle \{S\}, \{a, b\}, S, P \rangle$ where $P$ is given by

$$S \rightarrow aA \mid aA \mid ab \mid abb \mid abbb$$

$$A \rightarrow Sb \mid Sbb \mid Sbbb$$

is in the LNF.

As usual, for each $u, v, w \in (V \cup T)^*$ and $A \in V$, $uAw \Rightarrow uvw$ if there is a production $A \rightarrow v \in P$. Let $\Rightarrow^*$ be the reflexive and transitive closure of $\Rightarrow$. The **language generated** by a linear grammar $G$ is

$$L(G) = \{ w \in T^* : S \Rightarrow^* w \}$$

In Example [1]

$$L(G) = \{ a^m b^n : 1 \leq m \leq n \leq 3m \} \quad (1)$$

Languages generated by linear grammars are called **linear languages**.

**Lemma 1** [6] Let $G$ be a linear grammar. Then there exists a linear grammar $G'$ in the LNF such that $L(G) = L(G')$.

**Example 2** The LNF of $G$ in Example [1] obtained following the algorithm given in the proof of Lemma [1] given in [6, Lemma 2.1] is the grammar $G' = \langle \{S, A\}, \{a, b\}, S, P' \rangle$, where $P'$ is given by

$$S \rightarrow aA \mid ab \mid abb \mid abbb$$

$$A \rightarrow Sb \mid Sbb \mid Sbbb$$

A linear grammar $G$ is in strong linear normal form, in short SLNF, if it is in LNF and the right side of each production is of the form $aA$ or $Aa$ where $a \in T \cup \{\lambda\}$ and $A \in V \cup \{\lambda\}$.

**Proposition 2** [6] Let $G$ be a linear grammar. Then there exists a linear grammar $\widehat{G}$ in SLNF such that $L(G) = L(\widehat{G})$. 

"$x \rightarrow y$. \)
Example 3  The SLNF of $G'$ in Example 2 obtained following the algorithm given in the proof of Proposition 2 given in [6, Prop. 2.1.] is the grammar $\hat{G} = \langle \{S, A, B, C, D, E, F\}, \{a, b\}, S, \hat{P} \rangle$ where $\hat{P}$ is given by

\[
S \to aA | aD \\
A \to Sb | Bb \\
B \to Sb | Cb \\
C \to Sb \\
D \to b | bE \\
E \to b | bF \\
F \to b
\]

2.1 Deterministic Linear Grammars

A grammar $G = \langle V, T, S, P \rangle$ is a deterministic linear grammar if each production in $P$ has the form $A \to aBu$ or $A \to \lambda$ and for every $a \in T$, $A, B, C \in V$ and $u, v \in T^*$, if $A \to aBu$, $A \to aCv \in P$ then $B = C$ and $u = v$.

Lemma 3  Let $G$ be a deterministic linear grammar. Then there exists a deterministic linear grammar $G'$ in the LNF such that $L(G) = L(G')$.

Proof:  For each $A \in V$ and $a \in T$ let $A_a = \{A \to aBu \in P : \text{for some} \ B \in V \text{ and} \ v \in T^+\}$. If $A_a \neq \emptyset$ then substitute in $P$ each production $A \to aBu$ by the productions $A \to aC$ and $C \to Bu$, where $C$ is a new variable and add $C$ to $V$. At this moment each production has the form

$$A \to aB, A \to Bu \text{ or } A \to u$$

for some $a \in T$, $u \in T^*$ and $A, B \in V$. If $A \in V$ is neither right linear nor left linear variable, then substitute each production $A \to Bu \in P$ by the productions $A \to C$ and $C \to Bu$, where $C$ is a new variable and add $C$ to $V$. The resulting grammar clearly is deterministic, is in LNF and is equivalent with the original. □

Proposition 4  Let $G$ be a deterministic linear grammar. Then there exists a deterministic linear grammar $\hat{G}$ in SLNF such that $L(G) = L(\hat{G})$.

Proof:  Let $G$ be a deterministic linear grammar. By the Lemma 3 we can suppose without loss of generality that $G$ is in the LNF. Now, apply the next algorithm: While exists $A \in V$ and $a \in T$ such that $A_a = \{A \to ay \in P : |y| \geq 2 \text{ or } y \in T\} \neq \emptyset$, change each $A \to ay \in A_a$ by the productions $A \to aB$ and $B \to y$, where $B$ is new and add $B$ to $V$. Lately, in analogous way, while there are $A \in V$ and $a \in T$ such that $A^a = \{A \to ya \in P : |y| \geq 2 \text{ or } y \in T\} \neq \emptyset$, change each $A \to ya \in A^a$ by the productions $A \to Ba$ and $B \to y$, where $B$ is new and add $B$ to $V$. The result is a deterministic linear grammar in SLNF equivalent with $G$. □

Example 4  Let $G = \langle \{S, A\}, \{a, b\}, S, P \rangle$ the deterministic linear grammar where $P$ is given by

\[
S \to bbS | aAb \\
A \to aaAbb | \lambda
\]

In this case $L(G) = \{b^m ab : m \text{ is even and } n \text{ is odd}\}$. Their LNF following the algorithm in the Lemma 3 is the deterministic linear grammar $G' = \langle \{S, A, B, C, D, E\}, \{a, b\}, S, P' \rangle$ where $P'$ is given by

\[
S \to bB | aC \\
A \to aD | \lambda \\
B \to bS \\
C \to Ab \\
D \to aE \\
E \to Abb
\]
Now applying the algorithm in Proposition 4 we obtain the deterministic linear grammar  
\[ \hat{G} = \langle \{ S, A, B, C, D, E, F \}, \{ a, b \}, S, \hat{P} \rangle \]  
where \( \hat{P} \) is \( P' \) by substitute the production \( E \to A b b \) for \( E \to F b \) and \( F \to A b \). Thus, clearly, \( \hat{G} \) is in SLNF.

3 \( \lambda \)-Nondeterministic linear automata

A \( \lambda \)-nondeterministic linear automata, \( \lambda \)-NLA in short, consist of two disjoint finite sets of states (\( Q_L \) and \( Q_R \)) some of which will be considered as accepting states, an input tape which is divided into cells and it is not limited at the right, each cell can hold a symbol from a finite input alphabet, two read heads and a control unit which manages the behavior of the \( \lambda \)-NLA in accordance with the current configuration. The execution of a \( \lambda \)-NLA starts with a string in the input tape, with the left read head pointing to leftmost symbol, the right read head pointing to the rightmost symbol and the current state being a state of a special set of states of \( Q_L \cup Q_R \) called set of start states. A computation step in a \( \lambda \)-NLA is made as follows: the control unit, depending on the class which is belonged the current state, uses the left or the right read head to scan a symbol from the tape, moves the left read head one cell to the right if the current state is in \( Q_L \) or moves the right read head one cell to the left if the current state is in \( Q_R \), and by making a nondeterministic choice, it changes the state choosing it from a set of possible states. The control unit of a \( \lambda \)-NLA also allows changing of state without moving the read heads, i.e. without reading an input symbol. The computation halts when a read head passes over the other read head or when there is no choice of possible actions. A string is only accepted when one read head passes over the other read head and the current state is an accepting state.

Figure 1 illustrates a schematic representation of a \( \lambda \)-NLA.

![Figure 1: Schematic representation of a \( \lambda \)-nondeterministic linear automata.](image)

Formally, a \( \lambda \)-NLA is a sextuple \( M = \langle Q_L, Q_R, \Sigma, \delta, I, F \rangle \) where \( Q_L \) and \( Q_R \) are disjoint and finite sets of states, \( \Sigma \) is a finite set of input symbols (the alphabet), \( I \subseteq Q_L \cup Q_R \) is the set of start states, \( F \subseteq Q_L \cup Q_R \) is the set of final or accepting states and \( \delta : (Q_L \cup Q_R) \times (\Sigma \cup \{ \lambda \}) \rightarrow P(Q_L \cup Q_R) \).

Analogously to finite automata, each \( \lambda \)-NLA has associated to itself a directed graph, called transition diagram. In order to distinguish the states in \( Q_L \) from the states in \( Q_R \) we use circles and squares to represent them.

Notice that, if \( M \) is a \( \lambda \)-NLA with \( Q_R = \emptyset \) (or \( Q_L = \emptyset \)), then \( \mathcal{L}(M) \) is a regular language. In fact, in this case, \( M' = \langle Q_L, \Sigma, I, \delta, F \rangle \) is a \( \lambda \)-nondeterministic finite automaton (with a set

\[1\] The use of a set of start states, although not being usual, had been used in several models of automata. For example, [10, Def.4.1] and [22, page 32] in non-deterministic finite automata and [11, page 89], [14, page 52] in transition systems.
Figure 2: Transition diagram of a λ-nondeterministic linear automata.

of start states) such that \( L(M') = L(M) \) and whose transition diagram is exactly the same as the transition diagram for \( M \). Thus, \( \lambda \)-NLA is a natural extension of \( \lambda \)-nondeterministic finite automata.

**Example 5** Figure 2 illustrates the \( \lambda \)-NLA \( M = (Q_L, Q_R, \Sigma, \delta, \{q_0\}, F) \) where

- \( Q_L = \{q_0, q_1, q_2, q_3\} \)
- \( Q_R = \{p_1, p_2, p_3, p_4\} \)
- \( \Sigma = \{a, b\} \)
- \( F = \{p_1, q_2\} \)
- \( \delta(q_0, a) = \{q_0, p_1\}, \delta(q_0, \lambda) = \{p_3\}, \delta(p_1, a) = \{p_2\}, \delta(p_2, a) = \{q_1\}, \delta(q_1, b) = \{p_1\}, \delta(p_3, b) = \{p_3, q_2\}, \delta(q_2, b) = \{q_3\}, \delta(q_3, b) = \{p_4\}, \delta(p_4, a) = \{q_2\} \) and empty for the remain (i.e. \( \delta(q_0, b) = \emptyset, \delta(p_1, b) = \emptyset, \) etc.).

An **instantaneous description** (ID) of a \( \lambda \)-NLA must record the current state, the string remaining to read and which read head is active. Thus an ID is a pair \((q, w)\) in \((Q_L \cup Q_R) \times \Sigma^*\) meaning that it remains to read the string \( w \), the current state is \( q \) and the read head which is active is the left when \( q \in Q_L \) and is the right when \( q \in Q_R \).

The symbol \( \vdash_M \) denotes a **valid move** from an ID to another ID in a \( \lambda \)-NLA \( M \). When the subscript \( M \) is clear we omit it. Thus, for each \( q \in Q_L, q' \in Q_L \cup Q_R, p \in Q_R, w \in \Sigma^* \) and \( a \in \Sigma \cup \{\lambda\} \)

\( (q, aw) \vdash (q', w) \) is possible if and only if \( q' \in \delta(q, a) \) and

\( (p, wa) \vdash (q', w) \) is possible if and only \( q' \in \delta(p, a) \)

We use \( \vdash^* \) for the reflexive and transitive closure of \( \vdash \), i.e. \( \vdash^* \) represents moves involving an arbitrary number of steps. Thus, in Example 5 we have that \( (q_0, abbaaaa) \vdash^* (p_1, baa) \), because

\[
(q_0, abbaaaa) \vdash (p_1, bbbaaa,) \vdash (p_2, bbbaaa) \vdash (q_1, bbbaa) \vdash (p_1, baa)
\]

The language accepted by a \( \lambda \)-NLA \( M \) is the set

\[
L(M) = \{w \in \Sigma^* : (q_0, w) \vdash^* (q_f, \lambda) \text{ for some } q_0 \in I \text{ and } q_f \in F\}
\]

For the case of \( \lambda \)-NLA \( M \) in Example 5

\[
L(M) = \{a^m b^n a^{2n} : m \geq 1 \text{ and } n \geq 0\} \cup \{a^k b^{2m} a^m b^n : k, m \geq 0 \text{ and } n \geq 1\}.
\]
Notice that, the halting mechanism of the NLA, despite not being explicit in its mathematical formulation, can be formalized by the ID notion as follows: when an ID \((q, \lambda)\) is achieved through a move (e.g. \((q', a) \vdash (q, \lambda)\)), we are in the situation of “a read head passes over the other read head”.

### 3.1 \(\lambda\)-NLA and linear languages

First, we will prove that each language accepted by some \(\lambda\)-NLA \(M\) is linear, i.e.

**Theorem 5** [6] Let \(M = \langle Q_L, Q_R, \Sigma, \delta, I, F \rangle\) be a \(\lambda\)-NLA. Then there exists a linear grammar \(G\) such that \(L(M) = L(G)\).

Conversely, each language generated by a linear grammar \(G\) is accepted by some \(\lambda\)-NLA.

**Theorem 6** [6] Let \(G = \langle V, T, S, P \rangle\) be a linear grammar. Then there exists a \(\lambda\)-NLA \(M\) such that \(L(M) = L(G)\).

The algorithms in the proofs (which can be found in [6]) of Theorems 5 and 6 are dual, in the sense that applying one and then the next, we obtain the same object. Obviously, for that in case of Theorem 6, the grammar must be in the slnf.

### 3.2 \(\lambda\)-moves are not necessary

A \(\lambda\)-NLA without \(\lambda\)-transitions is called nondeterministic linear automaton, in short NLA.

The \(\lambda\)-moves in the usual classes of automata are not essential for the model. Thus, for example, \(\lambda\)-moves do not increase computational power acceptance of finite automata (see, for example, [18]), of pushdown automata (see, for example, [14]) and of Turing machines (by Church-thesis). So it is reasonable to hope that NLA and \(\lambda\)-NLA have the same acceptance power. Nevertheless, when a \(\lambda\)-transition in a \(\lambda\)-NLA happens between two states of different type, it is not obvious how we can eliminate it without changing the language accepted by the automaton.

**Theorem 7** [6] Let \(M = \langle Q_L, Q_R, \Sigma, \delta, I, F \rangle\) be a \(\lambda\)-NLA. Then there exists a NLA \(M'\) such that \(L(M) = L(M')\).

### 4 Deterministic linear automata

There are several nonequivalent notions for “deterministic” linear languages (see for example [16]). The most general among those classes is DL, where a language \(L \in DL\) if it can be generated by a deterministic linear grammar. As proved in [16], DL is a proper subset of DCFL \(\cap Lin\), where DCFL is the class of deterministic context-free languages and Lin is the class of linear languages.

On the other hand, a NLA is deterministic, DLA in short, if for each \(a \in \Sigma\) and \(q \in (Q_L \cup Q_R)\), \(|\delta(q, a)| \leq 1\). Thus, in a DLA \(\delta(q, a) = \{q'\}\) or \(\delta(q, a) = \emptyset\). For that reason we consider the transition function of a DLA \(M = \langle Q_L, Q_R, \Sigma, \delta, I, F \rangle\) as a partial function from \((Q_L \cup Q_R) \times \Sigma\) into \((Q_L \cup Q_R)\).

**Theorem 8** If \(L \in DL\) then there is a DLA \(M\) such that \(L(M) = L\).

**Proof:** Let \(G\) be a deterministic linear grammar such that \(L(G) = L\). From Proposition 2 there exists a linear grammar \(\widehat{G} = \langle \overline{V}', T', \overline{S}', \overline{P}' \rangle\) in slnf such that \(L(G) = L(\widehat{G})\). By the construction, if \(A \rightarrow aB\) and \(A \rightarrow aC\) then, \(B = C\), and if \(A \rightarrow Ba\) and \(A \rightarrow Ca\) then
Corollary 9  Let $\text{DLin}$ be the class of languages accepted by some DLA. Then

$$\text{DL} \subset \text{DLin}$$

Proof:  Straightforward from Theorem 8, $\text{DL} \subseteq \text{DLin}$.

On the other hand, de la Higuera and Oncina in [16] state that the language $\{a^n b^n : n \geq 1\} \cup \{a^n c^n : n \geq 1\} \notin \text{DL}$. Nevertheless Figure 3 presents a DLA which accepts this language and therefore $\text{DL} \subset \text{DLin}$.

In the following proposition we relate the class $\text{DLin}$ with the class of deterministic context-free languages, $\text{DCFL}$ in short.

Proposition 10

$$\text{DLin} - \text{DCFL} \neq \emptyset \text{ and } \text{DCFL} - \text{DLin} \neq \emptyset$$

Proof:  The DLA in Figure 4 clearly accepts the language of palindromes over the alphabet $\{a,b\}$. But, this language is not in $\text{DCFL}$ (see [18, 23]). So, $\text{DLin} - \text{DCFL} \neq \emptyset$. Conversely, as it is well known, the language $L = \{a^m b^m a^n b^n : m \geq 1 \text{ and } n \geq 1\} \in \text{DCFL} - \text{Lin}$ and therefore $\text{DCFL} - \text{DLin} \neq \emptyset$.

Since the palindromes on the alphabet $\Sigma = \{a, b\}$ is a linear language, an immediate consequence of Proposition 10 is the following corollary.

Corollary 11

$$\text{DLin} - (\text{DCFL} \cap \text{Lin}) \neq \emptyset$$

A natural question that arises from this corollary is on the existence or not of a language in $\text{DCFL} \cap \text{Lin}$ which is not in $\text{DLin}$. We have the following conjecture:

Conjecture 12  $(\text{DCFL} \cap \text{Lin}) - \text{DLin} = \emptyset$

Case this conjecture is correct, then we will have that $(\text{DCFL} \cap \text{Lin}) \subset \text{DLin}$.

Each deterministic linear language is linear, but as it will be proved below, the converse does not hold.

Proposition 13  Let $\text{Lin}$ be the class of linear languages. Then

$$\text{DLin} \subset \text{Lin}$$
two cases:

Let $\text{Proof:}$

In case $q_0 \in Q_L$, then first $M$ must read the leftmost $a$ in $a^m b^n$. If $\delta(q_0, a) \in Q_R$ then the $a$ needs to match with one, two or three $b$'s what clearly requires a nondeterministic choice and therefore is a contradiction. If $\delta(q_0, a) = q \in Q_L$ then $M$ must read a new $a$ and again either $\delta(q, a) \in Q_R$ in which case $M$ must read one, two or three $b$'s or $\delta(q, a) \in Q_L$ in which case $M$ must read a new $a$, and so on. But in some moment $M$ should make a match between $a$ with one, two or three $b$'s.

The case of $q_0 \in Q_R$ is analogous. Therefore, $L(3) \in \text{Lin} – \text{DLin}$. 

As corollary of Prop. 13 we have that there are NLA for which there is no equivalent DLA, i.e. a DLA that accepts the same linear language. However, the next theorem provides a characterization of the NLA that are equivalent to some DLA, i.e. NLA which accept deterministic linear languages.

Given a NLA $M_N = (Q_L, Q_R, \Sigma, \delta, I, F)$, define $\bar{Q}$ as the least set containing $\{q\} : q \in I$ and such that for each $X \in \bar{Q}$ and $a \in \Sigma$ we have that $\bigcup_{q \in X} \delta(q, a) \in \bar{Q}$. Thus, $\bar{Q} \subseteq \mathcal{P}(Q_L \cup Q_R)$.

**Theorem 14** Let $L \subseteq \Sigma^*$ be a language. Then $L \in \text{DLin}$ iff there exists a NLA $M_N = (Q_L, Q_R, \Sigma, \delta, I, F)$ such that $L(M_N) = L$ and for each $X \in \bar{Q}$, $X \subseteq Q_L$ or $X \subseteq Q_R$.

**Proof:** ($\Rightarrow$) If $L \in \text{DLin}$ then there is a DLA $M_D = (Q_L, Q_R, \Sigma, \delta_D, I, F)$ such that $L(M_D) = L$. Let $M_N = (Q_L, Q_R, \Sigma, \delta, I, F)$ where $\delta(q, a) = \{\delta_D(q, a)\}$. Clearly, $M_N$ is a NLA such that $L(M_N) = L(M_D) = L$ and $Q = \{\{q\} : q \in Q\}$. Therefore, for each $X \in \bar{Q}$, $X = \{q\}$ for some $q \in Q_L \cup Q_R$ and so, trivially, $X \subseteq Q_L$ or $X \subseteq Q_R$.

($\Leftarrow$) Let $M_D = (Q_L, Q_R, \Sigma, \delta_D, I', F')$ where $Q_L = \{X \in \varphi(W) : X \subseteq Q_L\}$, $Q_R = \{X \in \varphi(W) : X \subseteq Q_R\}$, $I' = \{\{q\} : q \in I\}$, $F' = \{X \in Q : X \cap F \neq \emptyset\}$ and $\delta_D(X, a) = \bigcup_{q \in X} \delta(q, a)$ for each $a \in \Sigma$ and $X \in \bar{Q}$. Clearly, $M_D$ is DLA. Moreover, if $w \in L(M_N)$, then $(q_i, w) \models_{M_N} (q_f, \lambda)$ for some $q_i \in I$ and $q_f \in F$. Therefore, if $w = a_1 \ldots a_n$ then there are states (possibly with repetitions) $q(1) \ldots, q(n) \in Q$ such that $q(0) = q_i$, $q(n) = q_f$ and $(q(0), a_1 \ldots a_n) \models_{M_N} (q(1), a_2 \ldots a_n) \models_{M_N} \ldots \models_{M_N} (q(n-1), a_n) \models_{M_N} (q(n), \lambda)$. And in this case, clearly, we have that $(q(0), a_1 \ldots a_n) \models_{M_D} (q(1), a_2 \ldots a_n) \models_{M_D} \ldots \models_{M_D} (q(n-1), a_n) \models_{M_D} (q(n), \lambda)$ when $q(j) \models (q(j))$ for each $j = 0, \ldots, n$. Therefore, $w \in L(M_D)$.  

5 A enumerable hierarchy of linear languages

Pushdown automata, PDA in short, have a similar characteristic to linear automata: their nondeterministic version is more powerful than their deterministic version. In PDA this difference allowed to define several ways to measure nondeterminism and consequently to determine several hierarchies of classes of context-free languages varying from DCFL into (in
the limit) **CFL**, the class of Context-Free Languages \([4 15 33 36]\). From that hierarchy of classes we can establish an enumerable hierarchy of classes of linear languages in a simple way. Consider the hierarchy \(CFL(1), CFL(2), \ldots\) determined in \([1]\), in this hierarchy \(CFL(1) = \text{DCFL}, CFL(k) \subset CFL(k+1)\) for each \(k \geq 1\) and \(\lim_{k \to \infty} CFL(k) = \text{CFL}\). So, defining \(LL(k) = CFL(k) \cap \text{Lin}\) we will have an enumerable hierarchy \(LL(1) \subset LL(2) \subset \ldots\) such that, \(LL(1) = \text{DCFL} \cap \text{Lin}\) and the limit of the hierarchy is the class of linear languages, i.e. \(\lim_{k \to \infty} LL(k) = \text{Lin}\).

In the following, we will provide a enumerable hierarchy for classes of linear languages starting from \(\text{DLin}\) going, in the limits, to \(\text{Lin}\).

**Definition 15** Let \(M = \langle Q_L, Q_R, \Sigma, \delta, I, F \rangle\) be a nla. The degree of explicit nondeterminism of \(M\) is the following:

\[
N\text{deg}(M) = \left( \sum_{q \in Q_L \cup Q_R, a \in \Sigma} | \delta(q, a) | \right) - | \{(q, a) \in Q \times \Sigma : \delta(q, a) \neq \emptyset \} |
\]

Let \(\text{Lin}(k)\) be the class of linear languages which can be accepted by a nla with degree of explicit nondeterminism \(k\). Formally,

\[
\text{Lin}(k) = \{ \mathcal{L}(M) : M \text{ is an nla and } N\text{deg}(M) = k \}
\]

**Theorem 16** \(\text{Lin}(0) = \text{DLin}\), \(\text{Lin}(k-1) \subset \text{Lin}(k)\) for each \(k \in \mathbb{N}^+\) and \(\bigcup_{k \in \mathbb{N}} \text{Lin}(k) = \text{Lin}\).

**Proof:** Let \(M\) be a nla. Then \(\mathcal{L}(M) \in \text{Lin}(0)\) iff \(N\text{deg}(M) = 0\) iff \(| \delta(q, a) | \leq 1\) iff \(M\) is a dla iff \(\mathcal{L}(M) \in \text{DLin}\). Therefore \(\text{Lin}(0) = \text{DLin}\).

Let \(M = \langle Q_L, Q_R, \Sigma, \delta, I, F \rangle\) be a nla such that \(\mathcal{L}(M) \in \text{Lin}(k)\) and \(Q = \{q_1, q_2\}\) a set of states such that \(Q \cap (Q_L \cup Q_R) = \emptyset\). Then \(M' = \langle Q_L \cup Q, Q_R, \Sigma, \delta', I, F \rangle\) where for each \(a \in \Sigma\), we have that \(\delta'(q, a) = \delta(q, a)\) if \(q \in Q_L \cup Q_R\), \(\delta'(q_0, a) = Q\) and \(\delta'(q_i, a) = \emptyset\) for each \(i = 1, \ldots, k\). Clearly, \(\mathcal{L}(M') = \mathcal{L}(M)\) and \(N\text{deg}(M') = N\text{deg}(M) + 1 = k + 1\). Thus, \(\mathcal{L}(M) \in \text{Lin}(k+1)\) and therefore \(\text{Lin}(k) \subseteq \text{Lin}(k+1)\). On the other hand, for each \(k \in \mathbb{N}\), the language

\[
L(k) = \{ a^m b^n : m \leq n \leq (k+1)m \}
\]

is accepted by the nla of Figure 5 and clearly in \(L(k) \in \text{Lin}(k)\). Moreover, it is evident that it is not possible to construct another nla \(M'\), for this same language, with a lesser degree of explicit nondeterminism, i.e. such that \(N\text{deg}(M') < k\).

Let \(L\) be a linear language. Then by Theorems 6 and 7 there is a nla \(M\) such that \(\mathcal{L}(M) = L\). Let \(k = N\text{deg}(M)\) then \(L \in \text{Lin}(k)\) and therefore, \(L \in \bigcup_{k \in \mathbb{N}} \text{Lin}(k)\). So, \(\bigcup_{k \in \mathbb{N}} \text{Lin}(k) = \text{Lin}\).

**6 Even linear languages**

The class of even linear languages was introduced by Amar and Putzolu in \([1]\). This class properly contains the class of regular language and it is properly contained in the class of linear languages. An important characteristic of this class of language is that it allows a solution of the learning problem\(^3\) for some subclasses of even linear languages based on positive examples such as the class of deterministic even linear languages \([22]\).

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\(^2\)There are other hierarchies varying from \(\text{DCFL}\) into \(\text{CFL}\), but based in other machine models, e.g. based on contraction automata \([34]\) and based on restarting automata \([19 27 28]\).

\(^3\)The learning problem for a class of formal languages, is the search of “learning procedures” for acquiring grammars on the basis of exposure to evidence about languages in the class \([29]\).
Basically, a language is even linear if it is generated by an even linear grammar, i.e. a linear grammar where each production of the form $A \rightarrow uBv$ satisfies $|u| = |v|$ [22, 35]. As it is well known, each even linear grammar has a normal form where each production has either the form $A \rightarrow uBv$ or the form $A \rightarrow a$, where $|u| = |v| = 1$ and $a \in \Sigma \cup \{\lambda\}$ [22].

Let $M$ be a NLA. $M$ is an even NLA if its transition diagram is a bipartite graph with $Q_L$ and $Q_R$ as its partitions, i.e. if for each $q \in Q_L$, $p \in Q_R$ and $a \in \Sigma$, $\delta(q, a) \subseteq Q_R$ and $\delta(p, a) \subseteq Q_L$. For example, the DLA in Figure 4 is an even NLA.

### Proposition 17

Let $G$ be an even linear grammar. Then there is an even NLA $M$ such that $L(G) = L(M)$.

**Proof:** Without loss of generality we can suppose that $G = \langle V, T, S, P \rangle$ is in the even linear normal form. Let $G' = \langle V \cup V', T, S, P' \rangle$ be the grammar obtained as follows:

Start with $V' = P' = \emptyset$. For each production $A \rightarrow aBb$ in $P$ add a new variable $C$ to $V'$ and the productions $A \rightarrow aC$ and $C \rightarrow Bb$ to $P'$. Finally, add to $P'$ each production $A \rightarrow a$ in $P$. Clearly, $G'$ is equivalent to $G$ and it is in SLNF.

Now, applying the algorithm in [6] we will have an even NLA equivalent to $G$. $\square$

Conversely,

### Proposition 18

Let $M$ be an even NLA. Then there is an even linear grammar $G$ such that $L(G) = L(M)$.

**Proof:** Applying the algorithm in Theorem [5] to the even NLA $M$, we will obtain a linear grammar $G$ with three kinds of productions:

1. $q \rightarrow ap$, where $a \in \Sigma$, $q \in Q_L$ and $p \in Q_R$.
2. $p \rightarrow qa$, where $a \in \Sigma$, $q \in Q_L$ and $p \in Q_R$.
3. $q \rightarrow \lambda$, where $q \in Q_L \cup Q_R$.

Now, we construct a new grammar $G' = \langle V, T, S, P' \rangle$ from $G = \langle V, T, S, P \rangle$ as follow:

For each production $q \rightarrow ap$ in $P$ put in $P'$ the productions $q \rightarrow ax$ for each $p \rightarrow x$ in $P$. Analogously, for each production $p \rightarrow qa$ in $P$ put in $P'$ the productions $p \rightarrow xa$ for each $q \rightarrow x$ in $P$.

Clearly, $G'$ is in the formal norm for even grammar and it is equivalent to $G$. $\square$

### 7 Final remarks

Since NLA is a two-read head model which works two ways, it cannot be considered as automata model in the sense of an abstract family of automata as done by Ginsburg [13].
However, NLA can be considered as automata in the more intuitive and general notion, as for example, “An automaton is a device which recognizes or accepts certain elements of $\Sigma^*$, where $\Sigma$ is a finite alphabet” \cite{2} or “An automaton is a construayt that possesses all the indispensable features of a digital computer. It accepts inputs, produces output, may have some temporary storage, and can make decitions in transforming the input into the output” \cite{24}.

The normal form for linear grammars in section 2 is useful to turn easier the proofs in section 3 and therefore does not intend to be an alternative to the well known normal form for the linear grammars.

The contribution of this work was to provide two subclasses of NLA, namely DLA and even NLA, which model the subclasses of deterministic and even linear languages, respectively. In particular, there are different, but not equivalents, proposed for the class of “deterministic” linear languages and here we proved that DLin, i.e. the class of languages accepted by some DLA, contain all those which are originated by a restriction in the linear grammars as considered in \cite{16}. In addition, DLin is not a proper subset of DCFL $\cap$ Lin, i.e. the class of linear languages which are also deterministic context free languages. In fact, we conjecture that DCFL $\cap$ Lin $\subset$ DLin. The advantage of using the automata models introduced here (DLA and even NLA) is due to their simplicity with respect to other automata models for these classes of languages and in the case of DLA, another advantage is that the class of languages modeled by this class of automata (which are naturally deterministic) is broader than the several classes of deterministic linear languages proposed in the literature. Other minor contribution was providing a characterization of NFA which accept languages also accepted by DLA and a method to obtain this DLA and giving an enumerable hierarchy of linear languages starting by the class DLin and having as limits the class of linear languages. As a future works we can intend to prove the Conjecture 12, compare the class DLin with the classes of left-right determinitic linear languages introduced in \cite{8, 9} and study some closure property of DLin.

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