On the maximal solution of a linear system over tropical semirings

Sedighe Jamshidvand · Shaban Ghalandarzadeh · Amirhossein Amiraslani · Fateme Olia

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Abstract
Nowadays, certain problems in automata theory, manufacturing systems, telecommunication networks, parallel processing systems and traffic control are intimately linked with linear systems over tropical semirings. Due to non-invertibility of matrices—except monomial matrices—over certain semirings, we cannot generally take advantage of having the inverse of the coefficient matrix of a system to solve it. The main purpose of this paper is to introduce two methods based on the pseudo-inverse of a matrix for solving a linear system of equations over tropical semirings. To this end, under suitable conditions, we first reduce the order of the system through some row–column operational analysis. We then present a necessary and sufficient condition for the system to have a maximal solution. This solution is also obtained through a new version of Cramer’s rule for overdetermined system of equations. Finally, some illustrative examples are given to show the efficiency of the proposed methods, and Maple procedures are also included in the end.

Keywords Tropical semiring · Linear system · Row–column analysis · Pseudo-inverse · Maximal solution

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Introduction
Solving systems of linear equations is an important aspect of linear algebra. Considering solution techniques for systems of linear equations over rings and, as a special case, over fields, we intend to develop systematic methods to understand the behavior of linear systems and extend some well-known results over rings to tropical semirings. Systems of linear equations over tropical semirings find applications in various areas of engineering, computer science, optimization theory, control theory, manufacturing systems, telecommunication networks, parallel processing systems, traffic control, etc (see for example [1, 3, 5–7]).

Semirings are algebraic structures similar to rings, but subtraction and division cannot necessarily be defined for them. The notion of a semiring was first introduced by Vandiver [13] in 1934. A semiring \((S, +, \cdot, 0, 1)\) is an algebraic structure in which \((S, +)\) is a commutative monoid with an identity element 0 and \((S, \cdot)\) is a monoid with an identity element 1, connected by ringlike distributivity. The additive identity 0 is multiplicatively absorbing, and \(0 \neq 1\). Note that for convenience, we mainly consider \(S = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)\), which is a well-known tropical semiring called “max–plus algebra,” in this work. Different kinds of tropical semirings, which are isomorphic to “max–plus algebra,” are “max–times algebra,” “min–times algebra” and “min–plus algebra,” which have widely studied. (see for example [4, 5]).

Let us consider a linear system of equations \(AX = b\), over a tropical semiring \(S\), where \(A \in M_n(S)\), \(b \in S^n\) and \(X\) is an unknown vector.

To simplify the solution process, we transform the linear system to a reduced one through row–column operational analysis. We then construct a method based on the “pseudo-inverse,” \(A^*\) of matrix \(A\), with determinant \(\det_{S}(A) \in U(S)\) to...
solve the system, where \( U(S) \) is the set of the unit elements of \( S \).

It is shown that the proposed method is not limited to square matrices and can be extended to arbitrary matrices of size \( m \times n \) as well. In such cases, we try to convert the non-square system to a square one of size \( \min\{m, n\} \).

We also derive a version of Cramer’s rule to obtain the “maximal” solution of a system over tropical semirings from a new point of view. Although Cramer’s rule over max-plus algebra has been fairly exhaustively studied in [2] and [8], here we propose an approach that is more aligned with algebra has been fairly exhaustively studied in [2] and [8], an algebraic system consisting of a nonempty set \( S \) with two binary operations, addition and multiplication, such that the following conditions hold:

1. \((S, +)\) is a commutative monoid with identity element \( 0 \);
2. \((S, \cdot)\) is a monoid with identity element \( 1 \);
3. Multiplication distributes over addition from either side, that is \( a(b + c) = ab + ac \) and \((b + c)a = ba + ca\) for all \( a, b, c \in S \);
4. The neutral element of \( S \) is an absorbing element, that is \( a \cdot 0 = 0 = 0 \cdot a \) for all \( a \in S \);
5. \( 1 \neq 0 \).

A semiring is called commutative if \( a \cdot b = b \cdot a \) for all \( a, b \in S \).

**Definition 1** (See [4]) A semiring \((S, +, \cdot, 0, 1)\) is called a commutative semiring if every nonzero element of \( S \) is multiplicatively invertible.

**Definition 2** A commutative semiring \((S, +, \cdot, 0, 1)\) is called a semifield if every nonzero element of \( S \) is multiplicatively invertible.

Our focus is the tropical semiring \((\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)\), denoted by \( \mathbb{R}_{\max,+} \), that is called “max–plus algebra” whose additive and multiplicative identities are \(-\infty\) and 0, respectively. Moreover, the notation \( a - b \) in “max–plus algebra” is equivalent to \( a + (-b) \), where “-” and “+” denote the usual real numbers subtraction, addition and the typical additively inverse of the element \( b \), respectively. Note further that “max–plus algebra” is a commutative semifield.

**Definition 3** (See [4]) Let \( S \) be a semiring. A left \( S \)-semimodule is a commutative monoid \((M, +)\) with identity element \( 0_M \) for which we have a scalar multiplication function \( S \times M \rightarrow M \), denoted by \( (s, m) \mapsto sm \), which satisfies the following conditions for all \( s, s' \in S \) and \( m, m' \in M \):

1. \((ss')m = s(s'm)\);
2. \(s(m + m') = sm + sm'\);
3. \((s + s')m = sm + s'm\);
4. \(1sm = m\);
5. \(s0_M = 0_M = 0_M \).

Right semimodules over \( S \) are defined in an analogous manner.

**Definition 4** A nonempty subset \( \mathcal{N} \) of a left \( S \)-semimodule \( M \) is a subsemimodule of \( M \) if \( \mathcal{N} \) is closed under addition and scalar multiplication. Note that this implies \( 0_M \in \mathcal{N} \). Subsemimodules of right semimodules are defined analogously.

**Definition 5** Let \( M \) be a left \( S \)-semimodule and \( \{\mathcal{N}_i | i \in \Omega\} \) be a family of subsemimodules of \( M \). Then \( \bigcap_{i \in \Omega} \mathcal{N}_i \) is a subsemimodule of \( M \) which, indeed, is the largest subsemimodule of \( M \) contained in each of the \( \mathcal{N}_i \). In particular, if \( \mathcal{A} \) is a subset of a left \( S \)-semimodule \( M \), then the intersection of all subsemimodules of \( M \) containing \( \mathcal{A} \) is a subsemimodule of \( M \), called the subsemimodule generated by \( \mathcal{A} \). This subsemimodule is denoted by

\[
S\mathcal{A} = \operatorname{Span}(\mathcal{A}) = \left\{ \sum_{i=1}^{n} s_i a_i \mid s_i \in S, a_i \in \mathcal{A}, i \in \mathbb{N}, n \in \mathbb{N} \right\}.
\]
If \( A \) generates all of the semimodule \( \mathcal{M} \), then \( A \) is a set of generators for \( \mathcal{M} \). Any set of generators for \( \mathcal{M} \) contains a minimal set of generators. A left \( S \)-semimodule having a finite set of generators is finitely generated. Note that the expression \( \sum_{i=1}^{n} s_{ij} \) is a linear combination of the elements of \( A \).

**Definition 6** (See [11]) Let \( \mathcal{M} \) be a left \( S \)-semimodule. A nonempty subset, \( A \), of \( \mathcal{M} \) is called linearly independent if \( a \notin \text{Span}(A) \setminus \{a\} \) for any \( a \in A \). If \( A \) is not linearly independent, then it is called linearly dependent.

**Definition 7** The rank of a left \( S \)-semimodule \( \mathcal{M} \) is the smallest \( n \) for which there exists a set of generators of \( \mathcal{M} \) with cardinality \( n \).

It is clear that \( \text{rank}(\mathcal{M}) \) exists for any finitely generated left \( S \)-semimodule \( \mathcal{M} \). This rank need not be the same as the cardinality of a minimal set of generators for \( \mathcal{M} \), as the following example shows:

**Example 1** Let \( S \) be a semiring and \( \mathcal{R} = S \times S \) be the Cartesian product of two copies of \( S \). Then \( \{(1_s, 1_s), (0_s, 1_s), (1_s, 0_s), (0_s, 0_s)\} \) and \( \{(0_s, 1_s), (1_s, 0_s)\} \) are both minimal sets of generators for \( \mathcal{R} \), considered as a left semimodule over itself with componentwise addition and multiplication. Hence, \( \text{rank}(\mathcal{R}) = 1 \).

Now, let \( S \) be a commutative semiring. We denote the set of all \( m \times n \) matrices over \( S \) by \( M_{mxn}(S) \). For \( A \in M_{mxn}(S) \), we denote by \( a_{ij} \) and \( A^T \) the \((i, j)\)-entry of \( A \) and the transpose of \( A \), respectively. For any \( A = (a_{ij}) \in M_{mxn}(S) \), \( B = (b_{ij}) \in M_{mxn}(S) \), \( C = (c_{ij}) \in M_{nxn}(S) \), and \( \lambda \in S \), we define:

\[
A + B = (a_{ij} + b_{ij})_{mxn},
\]

\[
AC = \left( \sum_{k=1}^{n} a_{ik}c_{kj} \right)_{mn},
\]

and \( \lambda A = (\lambda a_{ij})_{mxn} \).

Clearly, \( M_{mxn}(S) \) equipped with matrix addition and matrix scalar multiplication is a left \( S \)-semimodule.

It is easy to verify that \( M_n(S) := M_{nxn}(S) \) forms a semiring with respect to the matrix addition and the matrix multiplication.

The above matrix operations over \( \text{max} - \text{plus} \) algebra can be considered as follows:

\[
A + B = (\max(a_{ij}, b_{ij}))_{mxn},
\]

\[
AC = (\max(a_{ik} + c_{kj}))_{ml},
\]

and \( \lambda A = (\lambda a_{ij})_{mxn} \).

For convenience, we can denote the scalar multiplication \( \lambda A \) by \( \lambda + A \). Moreover, \( \text{max} - \text{plus} \) algebra is a commutative semiring which implies \( \lambda + A = A + \lambda \).

**Definition 8** Let \( A, B \in M_{mxn}(S) \) such that \( A = (a_{ij}) \) and \( B = (b_{ij}) \). We say \( A \leq B \) if and only if \( a_{ij} \leq b_{ij} \) for every \( i \in m \) and \( j \in n \).

**Definition 9** A square matrix \( A \in M_n(S) \) is called a monomial matrix if every row and every column of \( A \) contains exactly one nonzero element. In fact, diagonal matrices, permutation matrices and their products are all monomial matrices.

Let \( A \in M_n(S) \), \( S_n \) be the symmetric group of degree \( n \geq 2 \), and \( A_n \) be the alternating group on \( n \) such that \( A_n = \{ \sigma | \sigma \in S_n \text{ and } \sigma \text{ is an even permutation} \} \).

The positive determinant, \( \text{det}^+(A) \), and negative determinant, \( \text{det}^-(A) \), of \( A \) are

\[
\text{det}^+(A) = \sum_{\sigma \in A_n} \prod_{i=1}^{n} a_{\sigma(i)},
\]

and

\[
\text{det}^-(A) = \sum_{\sigma \in S_n \setminus A_n} \prod_{i=1}^{n} a_{\sigma(i)}.
\]

Let \( S \) be a commutative ring, then

\[
\text{det}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \text{sgn}(\sigma)a_{\sigma(i)} = \text{det}^+(A) - \text{det}^-(A),
\]

where

\[
\text{sgn}(\sigma) = \begin{cases} 
+1 & \sigma \in A_n, \\
-1 & \text{otherwise}
\end{cases}
\]

The concept of the determinant of matrices over commutative semirings requires to define an \( \varepsilon \)-function. (See [10] for more details.)

**Definition 10** Let \( S \) be a semiring. A bijection \( \varepsilon \) on \( S \) is called an \( \varepsilon \)-function of \( S \) if \( \varepsilon(\varepsilon(a)) = a \), \( \varepsilon(a + b) = \varepsilon(a) + \varepsilon(b) \), and \( \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \) for all \( a, b \in S \). Consequently, \( \varepsilon(a)\varepsilon(b) = ab \) and \( \varepsilon(0) = 0 \). The identity mapping: \( a \mapsto a \) is an \( \varepsilon \)-function of \( S \) that is called the identity \( \varepsilon \)-function.
Remark 1 Any semiring $S$ has at least one $\varepsilon$-function since the identical mapping of $S$ is an $\varepsilon$-function of $S$. If $S$ is a ring, then the mapping $\alpha \mapsto -\alpha$, $(\alpha \in S)$ is an $\varepsilon$-function of $S$.

Definition 11 Let $S$ be a commutative semiring with an $\varepsilon$-function, $\varepsilon$, and $A \in M_n(S)$. The $\varepsilon$-determinant of $A$, denoted by $\det_\varepsilon(A)$, is defined by

$$\det_\varepsilon(A) = \sum_{\sigma \in S_n} \varepsilon(\sigma)(a_{i\sigma(1)} \cdots a_{i\sigma(n)}),$$

where $\sigma$ is the number of the inversions of the permutation $\sigma$, and $\varepsilon(\sigma)$ is defined by $\varepsilon(\sigma) = a$ and $\varepsilon(k) = \varepsilon(k-1)\varepsilon(a)$ for all positive integers $k$. Since $\varepsilon(1) = a$, $\det_\varepsilon(A)$ can be rewritten in the form of

$$\det_\varepsilon(A) = \det^\varepsilon(A) \oplus \varepsilon(\det^\varepsilon(A)).$$

In particular, for $S = \mathbb{R}_{\max,+}$ with the identity $\varepsilon$-function $\varepsilon$, we have $\det_\varepsilon(A) = \max(\det^\varepsilon(A), \det^-\varepsilon(A))$.

Definition 12 Let $S$ be a commutative semiring with $\varepsilon$-function, $\varepsilon$, and $A \in M_n(S)$. The $\varepsilon$-adjoint matrix $A$, written as $\text{adj}_\varepsilon(A)$, is defined as follows:

$$\text{adj}_\varepsilon(A) = ((\varepsilon^{i+j}\det_\varepsilon(A(ij)))_{n \times n})^\top,$$

where $A(ij)$ denotes the $(n-1) \times (n-1)$ submatrix of $A$ obtained from $A$ by removing the $i$th row and the $j$th column. It is clear that if $S$ is a commutative ring, $\varepsilon$ is the mapping: $a \mapsto -a$, $(a \in S)$, and $A \in M_n(S)$, then $\text{adj}_\varepsilon(A) = \text{adj}(A)$.

Theorem 1 (See [10]) Let $A \in M_n(S)$. We have

1. $\text{adj}_\varepsilon(A) = (\det_\varepsilon(A(i \mapsto j)))_{n \times n}$.
2. $\text{adj}_\varepsilon(A)A = (\det_\varepsilon(A(i \mapsto j)))_{n \times n}$

where $A(i \mapsto j)$ $(A(i \mapsto j))$ denotes the matrix obtained from $A$ by replacing the $j$th row (column) of $A$ by the $i$th row (column) of $A$.

Definition 13 (See [14]) Let $S$ be a semiring and $A \in M_{m \times n}(S)$. The column space of $A$ is the finitely generated right $S$-subsemimodule of $M_{m \times 1}(S)$ generated by the columns of $A$:

$$\text{Col}(A) = \{Av | v \in M_{n \times 1}(S)\}.$$

The column rank of $A$ is the rank of its column subsemimodule, which is denoted by $\text{colrank}(A)$.

Definition 14 (See [14]) Let $S$ be a semiring and $A \in M_{m \times n}(S)$. The row space of $A$ is the finitely generated left $S$-subsemimodule of $M_{1 \times n}(S)$ generated by the rows of $A$:

$$\text{Row}(A) = \{uA | u \in M_{1 \times m}(S)\}.$$

The row rank of $A$ denoted by $\text{rowrank}(A)$ is the rank of its row subsemimodule.

The following example shows that the column rank and the row rank of a matrix over an arbitrary semiring are not necessarily equal. If these two values coincide, their common value is called the rank of matrix $A$.

Example 2 Consider $A \in M_2(S)$ where $S = \mathbb{R}_{\max,+}$, as follows:

$$A = \begin{bmatrix} 3 & 6 & 5 \\ -5 & 0 & -2 \\ 4 & 1 & 6 \end{bmatrix}.$$

Clearly, $\text{rowrank}(A) = 3$, but $\text{colrank}(A) = 2$, since the third column of $A$ is a linear combination of its other columns:

$$C_3 = \max(C_1 + 2, C_2 + (-2)).$$

In this position, we analyze the system of linear equations $AX = b$ where $A \in M_{m \times n}(S)$, $b \in S^n$ and $X$ is an unknown column vector of size $n$ over tropical semiring $S = \mathbb{R}_{\max,+}$, whose $i$-th equation is

$$\max(a_{ij} + x_1, \ldots, a_{in} + x_n) = b_i.$$

Definition 15 A solution $X^*$ of the system $AX = b$ is called maximal if $X \leq X^*$ for any solution $X$.

Definition 16 Let $b \in S^n$. Then $b$ is called a regular vector if $b_i \neq -\infty$ for any $i \in m$.

Without loss of generality, we can assume that $b$ is regular in the system $AX = b$. Otherwise, let $b_i = -\infty$ for some $i \in m$. Then in the $i$-th equation of the system, we have

$$a_{ij} + x_j = -\infty$$

for any $j \in n$. As such, $x_j = -\infty$ if $a_{ij} \neq -\infty$. Consequently, the $i$-th equation can be removed from the system together with every column $A_j$ where $a_{ij} \neq -\infty$, and the corresponding $x_j$ can be set to $-\infty$.

Definition 17 Let $A \in M_n(S)$, $\lambda \in S$ and $X \in S^n$ be a regular vector such that

$$AX = \lambda X.$$

Then $\lambda$ is called an eigenvalue of $A$ and $X$ an eigenvector of $A$ associated with eigenvalue $\lambda$. Note that this definition allows an eigenvalue to be $-\infty$. Moreover, eigenvectors are allowed to contain elements equal to $-\infty$. 
Row-column operational analysis

In this section, we reduce the order of the system $AX = b$, where $A \in M_{m \times n}(X), b \in S^m$ and $X$ is an unknown column vector of size $n$, through a row-column operational analysis to obtain a reduced system. Throughout this section, the reduced matrix $A$ is denoted by $\tilde{A}$. 

Column analysis

Suppose that $C_1, \ldots, C_n$ are the columns of matrix $A$. Without loss of generality, we can assume that $C_1, \ldots, C_k$ are linearly independent and the other columns are linearly dependent on them.

\[
\begin{bmatrix}
C_1 & \cdots & C_k & \cdots & C_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_k \\
x_{k+1} \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix} b_1 \\
\vdots \\
b_m \end{bmatrix}. \tag{1}
\]

We can rewrite the system as follows:

\[
\max(C_1 + x_1, \ldots, C_k + x_k, C_{k+1} + x_1, \ldots, C_n + x_n) = \begin{bmatrix} b_1 \\
\vdots \\
b_m \end{bmatrix}.
\]

There exist scalars $\eta_{ij} \in S$ for every $1 \leq i \leq k$ and $k + 1 \leq j \leq n$, such that

\[
C_j = \max(C_1 + \eta_{1j}, \ldots, C_k + \eta_{kj}). \tag{2}
\]

By replacing (2) in (1), we have:

\[
\max(C_1 + x_1, \ldots, C_k + x_k, \max(C_1 + \eta_{1(k+1)}), \ldots, C_k + \eta_{k(k+1)}) + x_{k+1}, \ldots, \max(C_1 + \eta_{1n}, \ldots, C_k + \eta_{kn}) + x_n) = \begin{bmatrix} b_1 \\
\vdots \\
b_m \end{bmatrix}.
\]

Due to the distributivity of $\{\} + \{\}$ over $\{\}$, the following equality is obtained:

\[
\max\{C_1 + \max(x_1, \eta_{1(k+1)} + x_{k+1}, \ldots, \eta_{1n} + x_n), \\
\ldots, C_k + \max(x_k, \eta_{k(k+1)} + x_{k+1}, \ldots, \eta_{kn} + x_n)\} = \begin{bmatrix} b_1 \\
\vdots \\
b_m \end{bmatrix}.
\]

Now, we can rewrite this system as

\[
\max(C_1 + y_1, \ldots, C_k + y_k) = \begin{bmatrix} b_1 \\
\vdots \\
b_m \end{bmatrix},
\]

where

\[
y_i = \max(x_i, \eta_{i(k+1)} + x_{k+1}, \ldots, \eta_{in} + x_n), \tag{3}
\]

for every $1 \leq i \leq k$. As such, the number of variables decreases from $n$ to $k$.

We will show that the existence of solutions of the system $AX = b$ depends on the row rank of $A$. Assume that $Y^* = (y^*_1, \ldots, y^*_k)$, is the maximal solution of the system:

\[
\begin{bmatrix} C_1 & \cdots & C_k \end{bmatrix} \begin{bmatrix} y^*_1 \\
\vdots \\
y^*_k \end{bmatrix} = \begin{bmatrix} b_1 \\
\vdots \\
b_m \end{bmatrix}.
\]

Hence, equalities (3) imply the system $AX = b$ should have solutions $x_j \leq \min(y^*_1 - \eta_{1j}, \ldots, y^*_k - \eta_{kj})$ for every $k + 1 \leq j \leq n$ and $x_i = y^*_i$ for every $1 \leq i \leq k$.

Row analysis

Consider the system $AX = b$ in the form of

\[
\begin{bmatrix}
R_1 \\
\vdots \\
R_h \\
\vdots \\
R_m
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix} b_1 \\
\vdots \\
b_m \end{bmatrix}, \tag{4}
\]

where $R_i$ is the $i$th row of the matrix $A$, for every $1 \leq i \leq m$. Without loss of generality, we can assume that $R_1, \ldots, R_h$ are linearly independent rows of $A$ and the other rows $R_i, h + 1 \leq i \leq m$ are linear combinations of them. Consequently, there exist scalars $\xi_i \in S$ for every $1 \leq j \leq h$ and $h + 1 \leq i \leq m$ such that:

\[
R_i = \max(R_1 + \xi_{i1}, \ldots, R_h + \xi_{ih}), \tag{5}
\]

for every $h + 1 \leq i \leq m$. We can now rewrite the system of equations (4) as

\[
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix} b_1 \\
\vdots \\
b_m \end{bmatrix}, \text{ for any } 1 \leq i \leq m,
\]

which can become the $h$-equation system:

\[
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix} b_1 \\
\vdots \\
b_m \end{bmatrix}, \text{ for any } 1 \leq j \leq h.
\]

We now obtain the row-reduced system with $h$ equations. Note that in the process of reducing the system $AX = b$, it does not matter which of the row or column analysis is first applied to the system. This argument leads us to investigate the existence of solutions of the linear system $AX = b$. 

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Theorem 2 Let \( A \in \mathcal{M}_{m \times n}(S) \). The system \( AX = b \) has solutions if and only if its reduced system, \( \bar{A}Y = \bar{b} \), has solutions.

Proof Let \( \text{colrank}(A) = k \) and \( \text{rowrank}(A) = h \). By applying row-column analysis on the system \( AX = b \) and replacing \( (5) \) in the \( m \)-equation system \( (4) \), we conclude that

\[
b_i = R_i \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \max(b_1 + \xi_{i1}, \ldots, b_n + \xi_{ih}),
\]

(6)

for every \( h + 1 \leq i \leq m \). If equalities \( (6) \) hold for every \( h + 1 \leq i \leq m \), then we can reduce the system \( AX = b \) to the system \( \bar{A}Y = \bar{b} \), where \( \bar{A} \) is the reduced \( h \times k \) matrix obtained from \( A \), \( Y \) is an unknown vector of size \( k \), and \( \bar{b} \) is the reduced vector obtained from \( b \). Thus, the existence of solution \( AX = b \) and \( \bar{A}Y = \bar{b} \) depends on each other. \( \Box \)

Remark 2 Note that if \( \bar{b} \) is not a linear combination of any column of \( \bar{A} \), then the systems \( AX = b \) and \( \bar{A}Y = \bar{b} \) have no solutions.

Solution methods for a linear system

The only invertible matrices in tropical semirings are monomial matrices. As such, unlike in classic linear algebra, the invertibility of the coefficient matrix of a system cannot generally used as a method for solving a linear system of equations. Instead, in this section we present two methods based on the pseudo-inverse of a matrix for solving a linear system of equations over tropical semirings. The solutions obtained from these methods are guaranteed to be the maximal solution of the system.

The pseudo-inverse method

Definition 18 Let \( A \in \mathcal{M}_{n}(S) \) and \( \det_\xi(A) \in U(S) \). The pseudo-inverse of \( A \), denoted by \( A^* \), is defined by:

\[
A^* = \det_\xi(A)^{-1}\text{adj}_\xi(A).
\]

Especially, if \( S = \mathbb{R}_{\text{max,+}} \), then \( A^* = (a_{ij}^-) \) where \( a_{ij}^- = (\text{adj}_\xi(A))_{ij} - \det_\xi(A) \).

Remark 3 Let \( A \in \mathcal{M}_{n}(S) \). Then \( AA^* = ((AA^*)_{ij}) \) is a square matrix of size \( n \), such that by Theorem 1,

\[
(\text{AA}^*)_{ij} = \det_\xi(A)^{-1}(\text{Aadj}_\xi(A))_{ij} = \det_\xi(A)^{-1}\det_\xi(A_i(i \Rightarrow j)).
\]

In max-plus algebra, this becomes

\[
(\text{AA}^*)_{ij} = (\text{Aadj}_\xi(A))_{ij} - \det_\xi(A) = \det_\xi(A_i(i \Rightarrow j)) - \det_\xi(A).
\]

The matrix \( A^*A \) is defined similarly. Note further that the diagonal entries of the matrices \( \text{AA}^* \) and \( A^*A \) are 0:

\[
(\text{AA}^*)_{ii} = (\text{Aadj}_\xi(A))_{ii} - \det_\xi(A) = \det_\xi(A_i(i \Rightarrow i)) - \det_\xi(A) = 0
\]

Theorem 3 Let \( A \in \mathcal{M}_{n}(S) \) and \( b \in S^n \) be a regular vector. Then \( (\text{AA}^*)_{ij} \leq b_i - b_j \) for any \( i, j \in n \) if and only if the system \( AX = b \) has the maximal solution \( X^* = A^*b \) where

\[
X^* = (x_i^*)_{i=1}^n.
\]

Proof Suppose that \( (\text{AA}^*)_{ij} \leq b_i - b_j \) for any \( i, j \in n \). First, we show that the system \( AX = b \) has the solution \( X^* = A^*b \). Clearly, \( AX^* = AA^*b \), so for any \( i \in n \):

\[
(AX^*)_i = (\text{AA}^*)_{ii} = \max_{j=1}^n((\text{AA}^*)_{ij} + b_i) = \max_{i=1}^n((\text{AA}^*)_{ii} + b_i, \max_{i \neq j}(\text{AA}^*)_{ij} + b_i)).
\]

Since for any \( i, j \in n \), \((\text{AA}^*)_{ij} + b_i \leq b_i \) and \((\text{AA}^*)_{ii} + b_i = b_i \), we have \((AX^*)_i = b_i \). As such, \( X^* \) is a solution of the system \( AX = b \).

Now, we prove \( X^* \) is a maximal solution. \( AX^* = b \), so \( A^*AX^* = X^* \). The \( k \)th equation of the system \( A^*AX^* = X^* \) is

\[
\max((A^*A)_{i1} + x_1^*, \ldots, (A^*A)_{i2} + x_2^*, \ldots, (A^*A)_{in} + x_n^*) = x_k^*.
\]

that implies

\[
(A^*A)_{kl} + x_k^* \geq x_l^*.
\]

for any \( l \neq k \). Now, suppose that \( Y = (y_i)_{i=1}^n \) is another solution of the system \( AX = b \). This means \( AY = b \), and \( (A^*A)Y = X^* \). Without loss of generality, we can assume there exists only \( j \in n \) such that \( y_j \neq x_j^* \), i.e., \( y_i = x_i^* \) for any \( i \neq j \). The \( j \)th equation of the system \( A^*AY = X^* \) is

\[
\max((A^*A)_{j1} + x_1^*, \ldots, (A^*A)_{jj} + y_j, \ldots, (A^*A)_{jn} + x_n^*) = x_j^*.
\]

This means \( (A^*A)_{jj} + y_j \leq x_j^* \) which implies \( y_j < x_j^* \). Moreover, if all inequalities (7) for \( k = j \) are proper, then

\[
\max((A^*A)_{j1} + x_1^*, \ldots, y_j, \ldots, (A^*A)_{jn} + x_n^*) < x_j^*.
\]

Hence, \( Y \) is not the solution of the system \( AX = b \). That leads to a contradiction.

This happens if all inequalities in (7) are proper, so we can conclude that \( X^* \) is a unique solution of the system \( AX = b \). Otherwise, if some of the inequalities are not
proper, i.e., \((A^*A)_{ij} = x_j^* - x_i^*\) for some \(l \neq j\), then \(Y\) is a solution of the system \(AX = b\) such that \(Y \leq X^*\). Consequently, \(X^*\) is a maximal solution.

Conversely, suppose that \(X^* = A^{-1}b\) is a maximal solution of the system \(AX = b\). Then \(AA^{-1} = b\). That implies \((AA^{-})_{ij} \leq b_i - b_j\) for any \(i, j \in n\).

In the following example, we show that \((AA^{-})_{ij} \leq b_i - b_j\) is a sufficient condition for the system \(AX = b\) to have the maximal solution \(X^* = A^{-1}b\).

**Example 3** Let \(A \in M_3(S)\). Consider the system \(AX = b\), with:

\[
\begin{bmatrix}
1 & -6 & 2 & -5 \\
4 & 5 & 1 & -2 \\
7 & -1 & 3 & 0 \\
-2 & -9 & 0 & 5 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
= \begin{bmatrix}
2 \\
7 \\
3 \\
-4 \\
\end{bmatrix}
\]

where \(\text{det}(A) = 14\). Following Theorem 3, we should check the condition \((AA^{-})_{ij} \leq b_i - b_j\) for any \(i, j \in \{1, \ldots, 4\}\) where \((AA^{-})_{ij} = (A\text{adj}(A))_{ij} - \text{det}_i(A) = \text{det}_j(A, (i \Rightarrow j)) - \text{det}_i(A)\) (see Theorem 1). As such, \(AA^{-}\) is

\[
\begin{bmatrix}
0 & -11 & -6 & -5 \\
-1 & 0 & -3 & -2 \\
1 & -6 & 0 & 0 \\
-7 & -14 & -9 & 0 \\
\end{bmatrix}
\]

Indeed, it is easier to check \((AA^{-})_{ij} \leq b_i - b_j \leq -(AA^{-})_{ij}\) for any \(1 \leq i \leq j \leq 4\). Since these inequalities hold, for instance \((AA^{-})_{12} \leq 2 - 7 \leq -(AA^{-})_{21}\), the system \(AX = b\) has the maximal solution \(X^* = A^{-1}b\):

\[
X^* = 
\begin{bmatrix}
-6 & -13 & -7 & -7 \\
-6 & -5 & -8 & -7 \\
-2 & -13 & -8 & -7 \\
-7 & -14 & -9 & 0 \\
\end{bmatrix}
\begin{bmatrix}
2 \\
7 \\
3 \\
-4 \\
\end{bmatrix}
= 
\begin{bmatrix}
-4 \\
2 \\
0 \\
-4 \\
\end{bmatrix}
\]

The next example shows that the condition of Theorem 3 is necessary.

**Example 4** Suppose the assumptions of previous example are satisfied and consider the system \(AX = b\), where:

\[
\begin{bmatrix}
5 & 2 & 8 & 10 \\
4 & 1 & 7 & 9 \\
3 & 7 & 9 & 11 \\
-1 & 0 & 2 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
= \begin{bmatrix}
7 \\
4 \\
8 \\
1 \\
\end{bmatrix}
\]

Then the matrix \(AA^{-}\) is as follows.

\[
AA^{-} = 
\begin{bmatrix}
0 & 1 & -1 & 6 \\
-1 & 0 & -2 & 5 \\
1 & 2 & 0 & 7 \\
-6 & -5 & -7 & 0 \\
\end{bmatrix}
\]

It can be checked that \((AA^{-})_{ij} \leq b_i - b_j \leq -(AA^{-})_{ij}\) do not hold for \(i = 2\) or \(j = 2\). Therefore, \(X^* = A^{-1}b\) cannot be the solution of the system \(AX = b\), where \(X^*\) is

\[
X^* = 
\begin{bmatrix}
-5 & -4 & -6 & 1 \\
-6 & -5 & -7 & 0 \\
-8 & -7 & -9 & 2 \\
-10 & -9 & -11 & 4 \\
\end{bmatrix}
\begin{bmatrix}
7 \\
4 \\
8 \\
1 \\
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
1 \\
-1 \\
-3 \\
\end{bmatrix}
\]

If \(X^*\) is the solution of the system \(AX = b\), then in the second equation of the system we encounter a contradiction:

\[
\max(a_{21} + x_1^*, a_{22} + x_2^*, a_{23} + x_3^*, a_{24} + x_4^*) = 6 \neq b_2.
\]

**Extension of the method to non-square linear systems**

We are interested in studying the solution of a non-square linear system of equations as well. Let \(A \in M_{m \times n}(S)\) with \(m \neq n\), and \(b \in S^n\) be a regular vector. For solving the non-square system \(AX = b\) by Theorem 3, we must consider a square linear system of order \(\min(m, n)\) corresponding to it. Without loss of generality, we can assume \(A\) is a reduced matrix, i.e., the number of independent rows (columns) is \(m(n)\), respectively. Since \(m \neq n\), we have the following two cases:

1. If \(m < n\), then we consider the square linear system of order \(m\) corresponding to the system \(AX = b\). Let \(X = A^TY\) where \(Y\) is an unknown vector of size \(m\). Then the square linear system \(AA^TY = b\) is obtained from replacing \(X\) in \(AX = b\). Suppose that the conditions of Theorem 3 hold for the system \(AA^TY = b\), so the system \(AA^TY = b\) has the maximal solution \(Y^* = (AA^T)^{-}b\). If so, the system \(AX = b\) has (at least) a solution in the form of \(X = A^TY^* = A^T(AA^T)^{-}b\), which is not necessarily maximal. The matrix \(A^T(AA^T)^{-}\), denoted by \(A^†\), is called the semipseudo-inverse of matrix \(A\). Hence, the system \(AX = b\) has the solution \(X = A^†b\).

2. If \(n < m\), then we consider the square linear system of size \(n\) corresponding to the system \(AX = b\). Clearly, we have the square linear system \(A^TXA = b\) of size \(n\). Assume that the conditions of Theorem 3 hold for the system \(A^TXA = b\). If so, it has the maximal solution \(X^* = (A^T)^{-}b\). Note further that \(X^* = (A^T)^{-}b\) is not necessarily the solution of the system \(AX = b\) unless \(b\) is an eigenvector of \((A^T)^{-}\) corresponding to the eigenvalue 0, i.e., \(AX^* = A(A^T)^{-}b\).
In the next examples, we try to solve some non-square linear systems:

**Example 5** Let \( A \in M_{4 \times 5}(S) \). Consider the following system \( AX = b \):

\[
\begin{bmatrix}
-4 & 7 & 12 & -3 & 0 \\
3 & 2 & 8 & 3 & -1 \\
-9 & 1 & 6 & 0 & 2 \\
2 & 8 & -5 & 1 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= 
\begin{bmatrix}
14 \\
10 \\
8 \\
11
\end{bmatrix}.
\]

Due to the extension method, the non-square system \( AX = b \) can be converted into the following square system \( AA^T Y = b \), considering \( X = A^T Y \):

\[
\begin{bmatrix}
24 & 20 & 18 & 15 \\
20 & 16 & 14 & 10 \\
18 & 14 & 12 & 9 \\
15 & 10 & 9 & 16
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix}
= 
\begin{bmatrix}
14 \\
10 \\
8 \\
11
\end{bmatrix}.
\]

The conditions of Theorem 3 hold for the system \( AA^T Y = b \), that is \((AA^T)(AA^T)^{-1})_{ij} \leq b_i - b_j \) for any \( i, j \in \{1, \ldots, 4\} \), where \((AA^T)(AA^T)^{-1}\) is the following matrix:

\[
\begin{bmatrix}
0 & 4 & 6 & -1 \\
-4 & 0 & 2 & -5 \\
-6 & -2 & 0 & -7 \\
-9 & -5 & -3 & 0
\end{bmatrix}.
\]

As such, the system \( AA^T Y = b \) has the maximal solution \( Y^* = (AA^T)^{-1}b \):

\[
Y^* = 
\begin{bmatrix}
-24 & -20 & -18 & -25 \\
-20 & -16 & -14 & -21 \\
-18 & -14 & -12 & -19 \\
-25 & -21 & -19 & -16
\end{bmatrix}
\begin{bmatrix}
14 \\
10 \\
8 \\
11
\end{bmatrix}
= 
\begin{bmatrix}
-10 \\
-6 \\
-4 \\
-5
\end{bmatrix}.
\]

Hence, \( X = A^T Y^* \) is a solution of the non-square system \( AX = b \):

\[
X = 
\begin{bmatrix}
-3 \\
3 \\
2 \\
-3 \\
-2
\end{bmatrix},
\]

which is not necessarily maximal solution.

**Example 6** Let \( A \in M_{4 \times 5}(S) \). Consider the following non-square system \( AX = b \):

\[
\begin{bmatrix}
2 & 5 & -2 \\
1 & 4 & 3 \\
7 & 8 & 1 \\
0 & 1 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
8 \\
3 \\
5 \\
2
\end{bmatrix}.
\]

According to the second case of the extension method, the following square system \( A^T A X = A^T b \) is obtained from the non-square system \( AX = b \):

\[
\begin{bmatrix}
14 & 15 & 8 \\
16 & 19 & 9 \\
15 & 10 & 9 \\
13 & 10 & 8
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
12 \\
13 \\
6
\end{bmatrix}.
\]

Since the conditions \((A^T A)(A^T A)^{-1})_{ij} \leq (A^T b)_i - (A^T b)_j \) hold for any \( i, j \in \{1, 2, 3\} \), where \((A^T A)(A^T A)^{-1}\) is the following matrix:

\[
\begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 1 \\
-6 & -7 & 0
\end{bmatrix},
\]

by Theorem 3, the system \( A^T A X = A^T b \) has the maximal solution \( X^* = (A^T A)^{-1} A^T b \):

\[
X^* = 
\begin{bmatrix}
-14 & -15 & -14 \\
-15 & -16 & -15 \\
-14 & -15 & -8
\end{bmatrix}
\begin{bmatrix}
12 \\
13 \\
6
\end{bmatrix}
= 
\begin{bmatrix}
-2 \\
-3 \\
-2
\end{bmatrix}.
\]

while \( X^* \) is not a solution of the system \( AX = b \):

\[
AX^* = 
\begin{bmatrix}
2 & 5 & -2 \\
1 & 4 & 3 \\
7 & 8 & 1 \\
0 & 1 & 4
\end{bmatrix}
\begin{bmatrix}
-2 \\
-3 \\
-2
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
-1 \\
5
\end{bmatrix} \neq b.
\]

In this manner, we actually solve the nearest square system corresponding to the system \( AX = b \).

**New version of Cramer’s rule**

In [12], Tan developed Cramer’s rule for a system \( AX = b \) when \( A \) is an invertible matrix over a commutative semiring. Moreover, Sararunkskul proved a square matrix \( A \) over a semifield is invertible, if and only if \( A \) is a monomial matrix (see [9, Theorem 2.2]). Consequently, Cramer’s rule can be used only for monomial matrices over semifields and, as a special case, over tropical semirings.

In the next theorem, we present a new version of Cramer’s rule to determine the maximal solution of a linear system by using the pseudo-inverse of its system matrix.
Theorem 4. Let $A \in M_n(S)$, $b \in S^n$ be a regular vector, and $\det_1(A) \in U(S)$. Then the system $AX = b$ has the maximal solution

$$X^* = (d^{-1}d_1, \ldots, d^{-1}d_n)^T$$

if and only if $(AA^-)_ij \leq b_i - b_j$ for every $i, j \in \mathbb{n}$, where $d = \det_1(A)$ and $d_j = \det_1(A_{ij})$ for $j \in \mathbb{n}$ and $A_{ij}$ is the matrix formed by replacing the $jth$ column of $A$ by the column vector $b$.

Proof. By using Theorem 3, the inequalities $(AA^-)_ij \leq b_i - b_j$ for every $i, j \in \mathbb{n}$ are equivalent to have the maximal solution $X^* = A^-b$ for the system $AX = b$, so

$$X^* = A^-b$$

$$= \det_1(A)^{-1} \text{adj}_1(A)b$$

$$= \det_1(A)^{-1} \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} b_n^\top$$

$$= \det_1(A)^{-1} \begin{bmatrix} \max(A_{11} + b_1, \ldots, A_{1n} + b_n) \\ \vdots \\ \max(A_{n1} + b_1, \ldots, A_{nn} + b_n) \end{bmatrix}.$$ 

where $A_{ij} = (\text{adj}_1(A))_{ij} = \det_1(A_{ij})$. The $jth$ component of $X^*$ is:

$$x_j^* = \det_1(A)^{-1} \max(A_{1j} + b_1, \ldots, A_{nj} + b_n)$$

$$= \det_1(A)^{-1} \max(\det_1(A(1j)), \ldots, \det_1(A(nj)) + b_n)$$

$$= \det_1(A)^{-1} \det_1 \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

$$= \det_1(A)^{-1} \det_1(A_{ij})$$

$$= d^{-1}d_j,$$ 

i.e., $X^* = (d^{-1}d_1, \ldots, d^{-1}d_n)^T$.

Note that equality (8) is obtained from Laplace’s theorem for semirings [10, Theorem 3.3]. \hfill \Box

Remark 4. It should be noted that in classic linear algebra, the unique solution of the system $AX = b$, when $\det(A) \neq 0$, can be obtained from Cramer’s rule without calculating the inverse matrix of $A$. Similarly, in max-plus linear algebra, we can use the new version of Cramer’s rule to get the maximal solution of the system $AX = b$, when $(AA^-)_ij \leq b_i - b_j$ for any $i, j \in \mathbb{n}$, without computing $A$ (see Remark 3).

Example 7. Let $A \in M_3(S)$. Consider the following system $AX = b$:

$$\begin{bmatrix} 5 & 2 & 6 \\ 4 & 1 & 4 \\ 3 & 7 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 13 \end{bmatrix}.$$ 

Clearly, $(AA^-)_ij \leq b_i - b_j \leq -(AA^-)_ij$ hold for any $1 \leq i \leq j \leq 3$, where $AA^-$ is the following matrix:

$$\begin{bmatrix} 0 & 1 & -8 \\ -1 & 0 & 9 \\ 5 & 6 & 0 \end{bmatrix}.$$ 

Due to Theorem 4, the system $AX = b$ has the maximal solution $X^* = (x^*_i)_{i=1}^3$ where:

$$x^*_1 = \det_1 \begin{bmatrix} 5 & 2 & 6 \\ 4 & 1 & 4 \\ 13 & 7 & 14 \end{bmatrix} - \det_1(A) = 20 - 20 = 0,$$

$$x^*_2 = \det_1 \begin{bmatrix} 5 & 5 & 6 \\ 4 & 4 & 4 \\ 3 & 13 & 14 \end{bmatrix} - \det_1(A) = 23 - 20 = 3,$$

$$x^*_3 = \det_1 \begin{bmatrix} 5 & 2 & 5 \\ 4 & 1 & 4 \\ 3 & 7 & 13 \end{bmatrix} - \det_1(A) = 19 - 20 = -1.$$

Concluding remarks

In this paper, we studied the order reduction in systems of linear equations through a row-column analysis technique over tropical semirings in order to simplify their solution process. We presented necessary and sufficient conditions for the linear systems of equations to have a maximal solution using the pseudo-inverse of system matrices. The maximal solution of non-square linear systems was also obtained through a new version of Cramer’s rule.

Appendix

See Tables 1, 2, 3.
Table 1 Finding the determinant of a square matrix in max-plus

\[
\text{MaxPlusDet} := \text{proc} (A::\text{Matrix}) \\
\text{local} \ i, \ j, \ s, \ n, \ \text{detA}, \ \text{ind}, \ K, \ V; \\
\text{description} \ "\text{This program finds the determinant of a square matrix in max-plus.}"; \\
\text{Use LinearAlgebra in} \\
\ n := \text{ColumnDimension}(A); \\
\ V := \text{Matrix}(n); \\
\ \text{ind} := \text{Vector}(n); \\
\ \text{if} \ n = 1 \ \text{then} \\
\ V := A[1, 1]; \\
\ \text{detA} := V; \\
\ \text{ind}[1] := 1 \\
\ \text{elif} \ n = 2 \ \text{then} \\
\ V[1, 1] := A[1, 1] + A[2, 2]; \\
\ V[1, 2] := A[1, 2] + A[2, 1]; \\
\ V[2, 1] := A[1, 2] + A[2, 1]; \\
\ V[2, 2] := A[1, 1] + A[2, 2]; \\
\ \text{detA} := \text{max}(V); \\
\ \text{for} \ s \ \text{to} \ 2 \ \text{do} \\
\ K := V[s, 1 .. 2]; \\
\ \text{ind}[s] := \text{max}[\text{index}](K) \\
\ \text{end do}; \\
\ \text{else} \\
\ \text{for} \ i \ \text{to} \ n \ \text{do} \\
\ \ \text{for} \ j \ \text{to} \ n \ \text{do} \\
\ \ \ \ V[i, j] := A[i, j] + \text{op}(1, \text{MaxPlusDet}(A[[1 .. i-1, i+1 .. n], [1 .. j-1, j+1 .. n]])); \\
\ \ \ \ \text{end do}; \\
\ \ \ \ \text{detA} := \text{max}(V); \\
\ \ \ \ K := V[i, 1 .. n]; \\
\ \ \ \ \text{ind}[i] := \text{max}[\text{index}](K); \\
\ \ \ \ \text{end do}; \\
\ \text{end if}; \\
\ \text{end use}; \\
\ \text{[detA, ind, V] \\
\text{end proc:} \\

Table 2 Calculation of matrix multiplication in max-plus

\[
\text{Matmul} := \text{proc} (A::\text{Matrix}, \ B::\text{Matrix}) \\
\text{local} \ i, \ j, \ m, \ n, \ p, \ q, \ C, \ L; \\
\text{description} \ "\text{This program finds the multiplication of two matrices in max-plus.}" \\
\text{Use LinearAlgebra in} \\
\ m := \text{RowDimension}(A); \\
\ n := \text{ColumnDimension}(A); \\
\ p := \text{RowDimension}(B); \\
\ q := \text{ColumnDimension}(B); \\
\ C := \text{Matrix}(m, q); \\
\ \text{if} \ n <> p \ \text{then} \\
\ \ \text{print}(\text{"impossible");} \\
\ \ \text{break} \\
\ \text{else} \\
\ \ \text{for} \ i \ \text{to} \ m \ \text{do} \\
\ \ \ \ \text{for} \ j \ \text{to} \ q \ \text{do} \\
\ \ \ \ \ \ L := \text{seq}(A[i, k]*B[k, j], k = 1 .. n]); \\
\ \ \ \ \ \ C[i, j] := \text{max}(L) \\
\ \ \ \ \text{end do}; \\
\ \ \ \ \text{end if}; \\
\ \text{end use}; \\
\ C \\
\text{end proc:} \]
Table 3  Calculating the pseudo-inverse, \( A^{-1} \), of a square matrix, \( A \), as well as \( A A^{-1} \) in max-plus.

\[
\text{Ainv := proc (A::Matrix)} \\
\text{local n, d, H, V, B, C, E, G, Z, i, j;} \\
\text{description "This program finds a pseudo-inverse of A in max-plus.";} \\
\text{use LinearAlgebra in} \\
\text{n := ColumnDimension(A);} \\
\text{d := op(1, MaxPlusDet(A));} \\
\text{H := Matrix(1 .. n, 1 .. n, d);} \\
\text{Z := Matrix(1 .. n, 1 .. n);} \\
\text{for i to n do} \\
\text{for j to n do} \\
\text{if A[i, j] = (-1)*Float(infinity) then} \\
\text{Z[i, j] := Float(infinity);} \\
\text{else} \\
\text{Z[i, j] := A[i, j];} \\
\text{end if;} \\
\text{end do;} \\
\text{end do;} \\
\text{V := op(3, MaxPlusDet(A));} \\
\text{B := V-H-Z;} \\
\text{G := Transpose(B);} \\
\text{C := Matmul(A, G);} \\
\text{E := Matmul(G, A);} \\
\text{end use;} \\
\text{G, C} \\
\text{end proc:}
\]

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