A Constructive Algebraic Proof of Student’s Theorem

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ABSTRACT. Student’s theorem is an important result in statistics which states that for normal population, the sample variance is independent from the sample mean and has a chi-square distribution. The existing proofs of this theorem either overly rely on advanced tools such as moment generating functions, or fail to explicitly construct an orthogonal matrix used in the proof. This paper provides an elegant explicit construction of that matrix, making the algebraic proof complete. The constructive algebraic proof proposed here is thus very suitable for being included in textbooks.

Keywords: sample variance; chi-square distribution; t-distribution; statistical education

1 STUDENT’S THEOREM

In mathematical statistics, there is a well-known theorem about the sample variance of a random sample from a normal distribution. This theorem is directly related to the discovery of the t-distribution by statistician William Sealy Gosset (1876-1937), known as “Student”, a pseudonym he used when he published his paper. Therefore, this theorem is often referred to as Student’s theorem. Let \( N(\mu, \sigma^2) \) denote the normal distribution with mean \( \mu \) and variance \( \sigma^2 \). Then the theorem reads as follows.

**Theorem 1 (Student’s Theorem).** Let \( X_1, \ldots, X_n \) be a random sample from the distribution \( N(\mu, \sigma^2) \), i.e., they all have that distribution and are mutually independent. Define the random variables

\[
\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}, \quad (1)
\]

\[
S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}. \quad (2)
\]

Then

1. \( \bar{X} \) has distribution \( N(\mu, \frac{\sigma^2}{n}) \).
2. \( \bar{X} \) and \( S^2 \) are independent.
3. \( \frac{(n-1)S^2}{\sigma^2} \) has distribution \( \chi^2(n-1) \).

This theorem is equivalent to the following version where the general normal distribution is replaced by standard normal distribution.

**Theorem 2 (Student’s Theorem, Standardized Version).** Let \( Z_1, \ldots, Z_n \) all have distribution \( N(0, 1) \) and are mutually independent. Define the random variables

\[
\bar{Z} = \frac{\sum_{i=1}^{n} Z_i}{n}, \quad (3)
\]

\[
W = \sum_{i=1}^{n} (Z_i - \bar{Z})^2. \quad (4)
\]

Then

1. \( \sqrt{n} \bar{Z} \) has distribution \( N(0, 1) \).
2. \( \bar{Z} \) and \( W \) are independent.
3. \( W \) has distribution \( \chi^2(n-1) \).

Since these two versions are equivalent, and it is easier to formulate a proof of the standardized version, in the rest of the paper the standardized version will be used when we give our proof.
2 LITERATURE PROOFS OF STUDENT’S THEOREM

To the author’s best knowledge, the original paper of Gosset is not currently available to the general public, so we do not know if it contained a proof of the above theorem. However, it is believed that even if such a “proof” did exist, it could hardly be regarded as a proof by today’s standard, because the mathematically rigorous theory of probability only began to emerge in 1930s. We therefore should look into the modern literature, mainly textbooks, for proofs of Student’s theorem. In one way or another, all the proofs rely on two important theorems of multivariate normal distribution, whose proofs require a very deep mathematical tool: moment-generating functions (m.g.f. in the sequel), or alternatively, characteristic functions. These two theorems are familiar to the majority of statistics students. They are given here as lemmas.

**Lemma 3.** Let random variables $X_1, \ldots, X_n$ have the multivariate normal distribution with mean $\mu$ and covariance matrix $\Sigma$. Let $Y = [Y_1, \ldots, Y_m]^T = AX + b$, where $A$ is an $m \times n$ full row-rank constant matrix, $X = [X_1, \ldots, X_n]^T$, and $b = [b_1, \ldots, b_m]^T$ is a constant column vector. Then $Y_1, \ldots, Y_m$ have the multivariate normal distribution with mean $A\mu + b$ and covariance matrix $A\Sigma A^T$.

**Lemma 4.** Let random variables $X_1, \ldots, X_n$ have the multivariate normal distribution with mean $\mu$ and covariance matrix $\Sigma$. Define random vectors $X$, $X_1$, and $X_2$ as

$$X^T = [X_1, \ldots, X_r, X_{r+1}, \ldots, X_n].$$

Partition $\Sigma$ as

$$\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix}.$$

Then $X_1$ and $X_2$ are independent if and only if $\Sigma_{12} = 0$.

A consequence of Lemma 4 is the following proposition, which we will use later.

**Proposition 5.** Let random variables $X_1, \ldots, X_n$ have multivariate normal distribution with covariance matrix $\Sigma$. Then $X_1, \ldots, X_n$ are mutually independent if and only if $\Sigma$ is a diagonal matrix.

After looking into a number of renowned modern statistical textbooks, which are supposed to have incorporated the latest developments in the whole literature on this subject, we found two typical proofs. They are commented below.

**Proof in [1] Section 3.6.3.** This proof first shows the independence of $\overline{X}$ and $S^2$ using Lemma 4, then it shows that $(n-1)S^2$ has distribution $\chi^2(n-1)$ using an argument that invokes m.g.f. a further time. This, we believe, is a drawback because the typical reader, who is usually only a sophomore, is not expected to have the skill of directly dealing with m.g.f. There is a similar proof in [2] Section 8.5], which we consider is somewhat less rigorous than the one in [1].

**Proof in [3] Section 7.3 and [4] Section 8.3.** This proof shows the independence and the $\chi^2(n-1)$ distribution in one single step. It defines a new vector $Y = OZ$, where $O$ is an orthogonal matrix and the first row of $O$ is $[1, \ldots, 1]/\sqrt{n}$, so that $Y_1 = \sqrt{n}Z$ and the sum of squares of the other entries of $Y$ is $W$. This proof is algebraic, without using advanced tools, and hence is much simpler and easier to understand than the proof in [1]. However, there is still a little drawback of this proof: it is nonconstructive in that it only states the existence of the orthogonal matrix $O$, without giving it specifically. While not affecting the rigor of the proof, this drawback does hurt its pedagogical value.

3 PROPOSED CONSTRUCTIVE ALGEBRAIC PROOF

We consider the proof in [3] [4] nearly perfect, and we seek to make it fully perfect by fixing its drawback we just mentioned, i.e. by explicitly constructing the $O$ matrix. In fact, in [4] page 478] the Gram-Schmidt orthogonalization method is suggested for constructing the $O$ matrix, but no hint is
given about the choice of the starting matrix. We tried that method with the starting matrix being the matrix obtained by replacing the first row of the identity matrix by \( \frac{1}{\sqrt{n}} \), and we found that the resulting orthogonal matrix is very ugly and prohibitively difficult to describe. Therefore we tend to believe that Gram-Schmidt orthogonalization is not an elegant method of construction for our purpose here. However, we finally succeeded in finding an elegant construction. Let us now illustrate it by a few base examples.

\[
O_2 = \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\end{bmatrix}.
\]

(5)

\[
O_3 = \begin{bmatrix}
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & \frac{1}{\sqrt{3}} \\
\end{bmatrix}.
\]

(6)

\[
O_4 = \begin{bmatrix}
\frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & 0 & 0 \\
\frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{4}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{4}} \\
\end{bmatrix}.
\]

(7)

\[
O_5 = \begin{bmatrix}
\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 & 0 & 0 \\
\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{5}} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{5}} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}.
\]

(8)

The general method of construction is contained in the following key lemma.

**Lemma 6.** For every integer \( n \geq 2 \), define the matrix \( O_n = [o_{ij}]_{n \times n} \) by:

For each \( i \) with \( 1 \leq i \leq n - 1 \),

\[
o_{ij} = \begin{cases} 
\frac{1}{\sqrt{i(i+1)}}, & j = i, \\
\frac{1}{\sqrt{i(i+1)}} - \frac{1}{\sqrt{(i+1)(i+2)}}, & j = i + 1, \\
0, & \text{otherwise};
\end{cases}
\]

(9)

\[
o_{nj} = \frac{1}{\sqrt{n}} \text{ for all } 1 \leq j \leq n.
\]

(10)

Then \( O_n \) is orthogonal, i.e. \( O_n O_n^T = I \).

**Proof.** Let \( P = O_n O_n^T = [p_{ij}]_{n \times n} \).

1) If \( 1 \leq i \leq n - 1 \), then \( p_{ii} = \sum_{k=1}^{n} o_{ik}^2 = \sum_{k=1}^{i} \frac{1}{i(i+1)} + \frac{1}{i+1} = 1 \).

2) \( p_{nn} = \sum_{k=1}^{n} o_{nk}^2 = \sum_{k=1}^{n} \frac{1}{k} = 1 \).

3) If \( 1 \leq i \leq n - 1 \), then \( p_{nn} = p_{ni} = \sum_{k=1}^{n} o_{ik} o_{nk} = \sum_{k=1}^{i} \frac{1}{\sqrt{i(i+1)}} + \frac{1}{\sqrt{(i+1)(i+2)}} = 0 \).

4) If \( 1 \leq i \neq j \leq n - 1 \), without loss of generality, let us assume \( i < j \), then 

\[
p_{ij} = \sum_{k=1}^{n} o_{ik} o_{jk} = \sum_{k=1}^{i+1} o_{ik} o_{jk} = \frac{1}{\sqrt{j(j+1)}} \sum_{k=1}^{i+1} o_{ik} = 0.
\]

Thus we have shown \( O_n O_n^T = I \). \( \blacksquare \)

Now for the sake of self-completeness of this paper, we here give a proof of Theorem 2. It uses the same idea as the proof in [H] page 478] except for our explicit construction of \( O \) and a few minor details.
Proof of Theorem 2. Denote \( Z = [Z_1, \ldots, Z_n]^T \). Define the random vector \( Y = [Y_1, \ldots, Y_n]^T \) by
\[
Y = O_n Z
\] (11)
where \( O_n \) is defined by (9,10).
From (10) we know that
\[
Y_n = \sum_{i=1}^{n} \frac{Z_i}{\sqrt{n}} = \sqrt{n} Z_n.
\] (12)
It is obvious that \( Y_n \) has the distribution \( N(0,1) \).
Furthermore, by Lemma 6, we have
\[
\sum_{i=1}^{n} Y_i^2 = Y^T Y = Z^T O_n^T O_n Z = Z^T Z = \sum_{i=1}^{n} Z_i^2.
\]
Theorem 2 
Therefore,
\[
\sum_{i=1}^{n-1} Y_i^2 = \sum_{i=1}^{n} Y_i^2 - Y_n^2 = \sum_{i=1}^{n} Z_i^2 - nZ^2 = \sum_{i=1}^{n} (Z_i - Z)^2.
\]
We have thus obtained the relation
\[
W = \sum_{i=1}^{n} (Z_i - \bar{Z})^2 = \sum_{i=1}^{n-1} Y_i^2.
\] (13)
By Lemma 3, \( Y_1, \ldots, Y_n \) have multivariate normal distribution with covariance matrix \( O_n I O_n^T = I \), which is diagonal, therefore by Proposition 5,
\[
Y_1, \ldots, Y_n \text{ all have the } N(0,1) \text{ distribution and are mutually independent.}
\] (14)
Since \( W \) is entirely based on \( Y_1, \ldots, Y_{n-1} \), and \( Z = \frac{Y_n}{\sqrt{n}} \), (14) implies that \( W \) and \( Z \) are independent.
Finally, (13) and (14) together imply that \( W \) has distribution \( \chi^2(n-1) \).

4 CONCLUSION
The proof proposed here of Student’s theorem is algebraic and fully constructive. To our best knowledge, such a construction has not appeared in the literature before. A constructive proof is expected to make the reader more comfortable and consequently enhance their understanding of this important result. We believe this paper to be of significant pedagogical value in statistical education, and hope the construction proposed here to be included in future textbooks.

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