The Dynamical Invariant of Open Quantum System

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The dynamical invariant, whose expectation value is constant, is generalized to open quantum system. The evolution equation of dynamical invariant (the dynamical invariant condition) is presented for Markovian dynamics. Different with the dynamical invariant for the closed quantum system, the evolution of the dynamical invariant for the open quantum system is no longer unitary, and the eigenvalues of it are time-dependent. Since any hermitian operator fulfilling dynamical invariant condition is a dynamical invariant, we propose a sort of special dynamical invariant (decoherence free dynamical invariant) in which a part of eigenvalues are still constant. The dynamical invariant in the subspace spanned by the corresponding eigenstates evolves unitarily. Via the dynamical invariant condition, the results demonstrate that this dynamical invariant exists under the circumstances of emergence of decoherence free subspaces.

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I. INTRODUCTION

The dynamical invariant, a hermitian operator with time-independent expectation value, was proposed by Lewis half a century ago[1]. This theory was designed to investigate the time evolution of dynamical system with an explicitly time-dependent Hamiltonian at the beginning[2]. Thereafter, the dynamical invariant had been successfully applied to investigate time-dependent question of quantum mechanics[3], Berry phase[4] and the connection between quantum theory and classical theory[5]. Recently, this theory has redrawn attention due to the application in "shortcuts to adiabaticity" to speed up adiabatic and slow quantum control processes[6, 7]. Such scheme is also known as "inverse engineering technology", which has been widely used in a number of quantum control and quantum information processes[8–11].

Although the dynamical invariant theory is powerful in the study of the closed quantum theory, the definition of dynamical invariant for the open quantum system is still missing. In this paper, we generalize the dynamical invariant theory into the open quantum system. The physical essence of dynamical invariant is an hermitian operator whose expectation value remains constant for any quantum state evolving with the associated Hamiltonian[1]. Based on this fact, we redefine the dynamical invariant. Different with the closed quantum system, the dynamics of the open quantum system is characterized by a variety of manners, e.g., master equation[12–14], Kraus representation[15] and so on. For the open quantum system described by Markovian master equation, we present dynamical equation for the invariant (the dynamical invariant condition). As expected, its dynamics is not unitary, and its eigenvalues are time-dependent.

The dynamical invariant is not unique for an open quantum system. Any hermitian operator which satisfies corresponding dynamical invariant condition is a dynamical invariant of this system[16, 17]. As well known, when a quantum state is prepared in the decoherence free subspace (DFS), the quantum state will evolve unitarily, once the decoherence free subspace condition is fulfilled[18–20]. Thus, whether exists a kind of dynamical invariant whose evolution is unitary or not becomes a question. What are the conditions for emerging such dynamical invariants? In this paper, we answer those questions by looking for a sort of dynamical invariant which is block-diagonal: The upper block evolves unitarily, and the lower block decouples to the upper block. Our results show that this dynamical invariant exists, if DFS presents in the dynamics of open quantum system.

This paper is structured as follows. We begin by defining the dynamical invariant and presenting the dynamical invariant condition for the Markovian master equation (Sec II A). After shortly reviewing the DFS condition in Sec II B, we propose a sort of dynamical invariant in which the part of dynamical invariant belonging to decoherence free subspace decouples to the other part (Sec II C). In Sec III, a pure dephasing model is considered to show how to determine the dynamical invariant proposed previously. Finally, Sec IV summarizes the results and presents the conclusions.
II. INVERSE ENGINEERING IN OPEN QUANTUM SYSTEM

A. Dynamical Invariant for Open Quantum System

Let us start with Lewis-Riesenfeld’s dynamical invariant theory for the closed quantum system. The dynamical invariant \( I(t) \) is defined as time-dependent hermitian operator, which satisfies von Neumann-like equation,

\[
\frac{\partial I(t)}{\partial t} + i[H(t), I(t)] = 0, \tag{1}
\]

where \( H(t) \) is a time-dependent Hamiltonian. Obviously, dynamical invariant complies with the following properties: (1) The expected value of dynamical invariant is constant. (2) The eigenvalues of dynamical invariant are constant, while the eigenstates \( |\phi_j(t)\rangle \) are time-dependent. (3) The dynamical invariant is not unique. Any time-dependent hermitian operator satisfies Eq.\((1)\) is a dynamical invariant for this closed quantum system.

As shown above, the dynamical invariant theory of the open quantum system can not be obtained from the generalization of Lewis-Riesenfeld’s theory directly. However, we may find that the essence of dynamical invariant is a conserved physical quantity. Therefore, we present the definition of dynamical invariant as follow,

**Definition 1.** The dynamical invariant \( I(t) \) of a quantum system is a time-dependent hermitian operator, where the expected value of \( I(t) \) is constant, i.e.,

\[
\frac{d}{dt} \langle I(t) \rangle = 0. \tag{2}
\]

On the one hand, when the quantum system is closed, the dynamical invariant will satisfy the von Neumann-like equation. Therefore, the definition we proposed here contains the definition proposed by Lewis. Actually, Eq.\((2)\) is just the dynamical equation of dynamical invariant, so we name the dynamical equation of dynamical invariant as dynamical invariant condition. On the other hand, once the dynamical equation of open quantum system is given, the dynamical invariant condition for the open quantum system can also be determined.

In the interest of the dynamical invariant condition to the context of open quantum system, we consider a general open quantum system described by Lindblad-Markovian master equation,

\[
\dot{\rho}(t) = -i[H(t), \rho(t)] + \mathcal{L} \rho(t),
\]

\[
\mathcal{L} \rho(t) = \sum_{\alpha} \left[ F_{\alpha} \rho(t) F_{\alpha}^\dagger - \frac{1}{2} \{ F_{\alpha}^\dagger F_{\alpha}, \rho(t) \} \right], \tag{3}
\]

where \( F_{\alpha} \) is Lindblad operator associated with the decoherence process. In the light of the definition of dynamical invariant, the expected value of dynamical invariant satisfies

\[
\frac{d}{dt} \langle I(t) \rangle = \text{Tr} \{ \dot{I}(t) \rho(t) + I(t) \dot{\rho}(t) \} = 0. \tag{4}
\]

After the consideration of the dynamical equation Eq.\((3)\), we rewrite above equation as

\[
\text{Tr} \left( \left( \dot{I}(t) + i[H(t), I(t)] \right) \right) + \frac{1}{2} \sum_{\alpha} \left( 2 F_{\alpha}^\dagger I(t) F_{\alpha} - \{ F_{\alpha}^\dagger F_{\alpha}, I(t) \} \right) \rho(t) = 0. \tag{5}
\]

Afterwards, the dynamical invariant \( I(t) \) for the open quantum system must obey

\[
\frac{\partial I(t)}{\partial t} + i[H(t), I(t)]
\]

\[
+ \sum_{\alpha} \left( F_{\alpha}^\dagger I(t) F_{\alpha} - \frac{1}{2} \{ F_{\alpha}^\dagger F_{\alpha}, I(t) \} \right) = 0. \tag{6}
\]

which is the dynamical invariant condition for the open system described by Eq.\((3)\). Different with the dynamical invariant for the closed quantum system, the evolution of dynamical invariant for the open quantum system is not unitary, but is still trace-conserved.

Since the dynamical invariant is hermitian as we assumed earlier, it can be written in its spectral decomposition

\[
I(t) = \sum_{k} \lambda_k |\psi_k(t)\rangle \langle \psi_k(t)|, \tag{7}
\]

where \( \lambda_k \) is the \( k \)-th eigenvalue of \( I(t) \) and \( |\psi_k(t)\rangle \) is corresponding eigenstate. Because the evolution of the dynamical invariant is not unitary, the eigenvalues must be time-dependent,

\[
\frac{\partial}{\partial t} \lambda_k = \langle \psi_k(t) | \sum_{\alpha} (\lambda_k F_{\alpha}^\dagger F_{\alpha} - \frac{1}{2} \{ F_{\alpha}^\dagger F_{\alpha}, I(t) \}) |\psi_k(t)\rangle. \tag{8}
\]

Note that the eigenvalues are only affected by decoherence process. Some may doubt that whether the dynamical invariant can be designed properly, so that some of the eigenvalues can be time-independent. For instance, when \( |\psi_k(t)\rangle \) is always the degenerated eigenstate of \( F_{\alpha} \) (or \( \alpha \)), the corresponding eigenvalue of dynamical invariant \( \lambda_k \) remains constant. As we know, the eigenstates mentioned above are the quantum states in decoherence free subspace. In this paper, we design a sort of dynamical invariant for the open quantum system in which part of eigenvalues are constant. Before discussing our dynamical invariant in detail, we briefly review the decoherence free subspace.

B. Decoherence Free Subspaces

According to the definition, DFS is a subspace of total Hilbert space \( \mathcal{H} \), in which the quantum state undergoes unitary evolution. In fact, there are many manners for quantum system to evolve unitarily. Here we pay our attention to the most general definition of DFSs which is inferred by the fact that the purity of quantum state
is constant if the dynamics of quantum system is unitary, i.e., $\frac{\partial \text{Tr}[\rho^2(t) \mid \partial t = 0]$. A subspace spanned by $\mathcal{H}_{\text{DFS}} := \{ |\Phi_1\rangle, |\Phi_2\rangle, \ldots, |\Phi_D\rangle \}$ is a D-dimensional decoherence free subspace, if and only if the following conditions are fulfilled: (1) All of the bases of $\mathcal{H}_{\text{DFS}}$ are the degenerated eigenstates of all Lindblad operators $F_\alpha(t)$ with common eigenvalues $c_\alpha(t)$, i.e.,

$$F_\alpha(t)|\Phi_j(t)\rangle = c_\alpha(t)|\Phi_j(t)\rangle$$

for $\forall j, \alpha$, and (2) $\mathcal{H}_{\text{DFS}}$ is invariant under the following operator

$$H_{\text{eff}} = G(t) + H(t) + \frac{i}{2} \sum_\alpha \left( c_\alpha^* F_\alpha(t) - c_\alpha(t) F_\alpha^\dagger(t) \right),$$

in which

$$G(t) = i \left( \sum_{j=1}^D |\Phi_j(t)\rangle \langle \Phi_j(t) | + \sum_{n=1}^{N-D} |\Phi_n^\dagger(t)\rangle \langle \Phi_n^\dagger(t) | \right).$$

Here we have used the notation $|\Phi_n^\dagger(t)\rangle (n = 1, \ldots, N - D)$ as the basis of complimentary subspace $\mathcal{H}^\perp$ of DFS.

Thus the Lindblad operators can be written in block form as follows:

$$F_\alpha = \left( \begin{array}{cc} c_\alpha I^D & A_\alpha \\ 0 & B_\alpha \end{array} \right)$$

where $I^D$ is the identity operator on the DFS. The upper (lower) block acts entirely inside $\mathcal{H}_{\text{DFS}} (\mathcal{H}^\perp)$; the off-diagonal block $A_\alpha$ mix $\mathcal{H}_{\text{DFS}}$ and $\mathcal{H}^\perp$. The presence of $A_\alpha$ is permitted since the DFS condition Eq. (9) gives no information about the invariant of $\mathcal{H}_{\text{DFS}}$ act by $H_{\text{eff}}$. Likewise, it is convenient to rewrite the Hamiltonian $H$ and $G$ as

$$H = \left( \begin{array}{cc} H^D & H^N \\ H^N & H^C \end{array} \right)$$

and

$$G = \left( \begin{array}{cc} G^D & G^N \\ G^N & G^C \end{array} \right)$$

with the blocks on the diagonal corresponding once again to operators restricted to $\mathcal{H}_{\text{DFS}}$ and $\mathcal{H}^\perp$. Here we do not limit the discussion on whether the DFS is time dependent (t-DFS) or not (traditional DFS). When the DFS is time independent, the basis $|\Phi_j\rangle$ and $|\Phi_n^\dagger\rangle (\forall j, n)$ are constant vectors, i.e., $G = 0$.

**C. Decoherence Free Dynamical Invariant**

Generally speaking, any hermitian operator fulfilling Eq. (10) is a dynamical invariant for open quantum system. The choice of the dynamical invariant is not unique. Within this paper’s concern, we propose a sort of dynamical invariant where a part of eigenvalues are constant. As analyzed earlier, the eigenvalues would still be constant, if the corresponding eigenstates are in DFS. Thus, we design this dynamical invariant as below,

$$I(t) = \left( \begin{array}{cc} I^D(t) & 0 \\ 0 & I^C(t) \end{array} \right)$$

where $I^D(t)$ is the dynamical invariant belongs to DFS; and $I^C(t)$ is associated with the other part of dynamical invariant which belongs to the complementary subspace. There are two key features for this DI: (1) the eigenvalues of $I^D(t)$ are constant; (2) $I^D$ decouples to $I^C$.

After placing the dynamical invariant Eq. (14) we proposed into the general dynamical invariant condition Eq. (9), the dynamical invariant $I^D$ and $I^C$ will be governed by following equations,

$$i\dot{I}^D + [G^D + H^D, I^D] = 0,$$

$$i\dot{I}^C + [G^C + H^C + \frac{i}{2} \sum_\alpha A^\dagger_\alpha A_\alpha, I^C] + \sum_\alpha A^\dagger_\alpha I^D A_\alpha - \frac{1}{2} \sum_\alpha \{ B^\dagger_\alpha B_\alpha, I^C \} - 2B^\dagger_\alpha I^C B_\alpha = 0,$$

$$I^D \left( i(G^N + H^N) - \sum_\alpha \frac{c_\alpha^*}{2} A_\alpha \right) - (i(G^N + H^N) - \sum_\alpha \frac{c_\alpha^*}{2} A_\alpha) I^C = 0.$$

in which it has been considered that the basis of DFS and complementary subspace is time-dependent. On the one hand, Eq. (17) is nothing but the decoupling condition of $I^D$ and $I^C$. If the decoupling condition fulfills, the dynamical invariant we proposed will exist and off-diagonal element of Eq. (14) will always be vanishing. It is easy to check that the decoupling condition is satisfied when the decoherence free subspace emerges: As mentioned in the DFS condition, $\mathcal{H}_{\text{DFS}}$ is invariant under the operator $H_{\text{eff}}$. In other words, the vanishing off-diagonal element of $H_{\text{eff}}$ guarantee the invariant of $\mathcal{H}_{\text{DFS}}$. Considering block form of the operators involved in $H_{\text{eff}}$, the vanishing off-diagonal element can be written as

$$i(G^N + H^N) - \sum_\alpha \frac{c_\alpha^*}{2} A_\alpha = 0.$$

Taking the DFS condition into Eq. (17), the decoupling condition is always held. Therefore, the decoupling condition is nothing but the DFS condition.

On the other hand, the evolution of $I^D$ ($I^C$) is governed by Eq. (15) (Eq. (16)). Firstly, we may observe from Eq. (15) that the evolution of $I^D$ is unitary and not related to $I^C$, which is the very dynamical invariant condition for the closed quantum system (Eq. (11)). In other words, the evolution of $I^D$ is decoherence free, so that we would like to name it as "decoherence free dynamical invariant." As a result, as soon as Eq. (9) is
placed into Eq. (8), we immediately find that $\lambda_j^D$ is constant due to $\partial \lambda_j^D/\partial t = 0$, where $\lambda_j^D$ is the $j$-th eigenvalue of $I^D$. As mentioned above, the eigenstates associated with the constant eigenvalues is a basis set of the DFS. Therefore, it is easy to design the structure of $I^D$ in accordance with the common eigenstates of the Lindblad operators. Secondly, the evolution of dynamical invariant $I^C$ is governed by Eq. (10). The dynamics of $I^C$ is similar to the Markovian-Lindblad master equation with the Lindblad operators $B_\alpha$, and the Hamiltonian $G^C + H^C + i \sum_\alpha A^\dagger_\alpha A_\alpha/2$, except that an extra term $\sum_\alpha A^\dagger_\alpha I^D A_\alpha$ is involved.

The extra term determines that the evolution of $I^C$ is not closed, but is impacted by $I^D$. In consideration of Eq. (11), the eigenvalues of $I^C$ satisfy

$$\frac{\partial}{\partial t} \lambda_n^C = \sum_j (\lambda_n^C - \lambda_j^D) \langle \psi_j | A_\alpha | \psi_n^+ \rangle^2 + \sum_m (\lambda_m^C - \lambda_n^C) \langle \psi_m^- | B_\alpha | \psi_n^- \rangle^2.$$  \hspace{1cm} (19)

Note that all of eigenvalues $\lambda_n^C$ is time-dependent. Furthermore, the evolutions of $\lambda_n^C$ are not self-determined, but affected by the eigenvalues of $I^D$.

III. THE DYNAMICAL INVARIANT FOR DEPHASING MODEL

Let us present our example now. The quantum system consists of $n$ physical qubits, which interacts collectively with a dephasing environment. The Hamiltonian of the quantum system can be written as

$$H_0 = \sum_{i<j} (g_{ij}^D(t) O_{ij}^D + B_i^x(t) O_i^z),$$  \hspace{1cm} (20)

in which $g_{ij}^D(t)$ denotes controllable coupling strength; $B_i^x(t)$ is extra magnetic field on single qubit; and

$$O_{ij}^z = \frac{\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y}{2},$$

$$O_i^z = \sum_{i=1}^n \frac{\sigma_i^z}{2}.$$  \hspace{1cm} (21)

The operators $O_{ij}^z$ is XY interaction term, where $\sigma_i^x$ and $\sigma_i^y$ are pauli operators for the $i$-th qubit. This Hamiltonian can describe a variety of quantum systems, such as trapped ions and quantum dots. The source of decoherence in quantum system considered here is dephasing. The interaction between the quantum system and the environment is described by the interaction Hamiltonian,

$$H_I = \sum_j \sigma_j^z \otimes \hat{B},$$  \hspace{1cm} (22)

where $\hat{B}$ is environment operator which affects the quantum system collectively. The reduced dynamics of quantum system is described by the following master equation,

$$\dot{\rho}(t) = -i[H_0, \rho(t)] + \mathcal{L}(\rho).$$  \hspace{1cm} (23)

Here the dephasing is characterized by

$$\mathcal{L}(\bullet) = \gamma (F \bullet F - \bullet),$$

with

$$F = \sum_j \sigma_j^z.$$  \hspace{1cm} (24)

As is well-known, this master equation is a Lindblad master equation and $F$ is the corresponding Lindblad operator. The symmetry of the master equation implies that there exists a DFS. Our aim is to find the dynamical invariant of the open quantum system which is block-diagonal.

Without loss of generality, we assume that the quantum system consists two qubits, i.e., $N = 2$. The general case can be studied by the same process as below. For a two-qubit system, the corresponding DFS is spanned by $\{|01\}, |10\rangle$ which are the degenerate eigenstates of $F$ with zero eigenvalue. The bases of the complementary subspace are chosen as $\{|00\}, |11\rangle$. According to Eq. (24), the Lindblad operator $F$ can be written in block form as Eq. (11) with $c_a = 0$, and

$$A_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, B_\alpha = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$  \hspace{1cm} (25)

The Hamiltonian $H_0$ also has a matrix representation as Eq. (12) with

$$H^D = \begin{pmatrix} 0 & g_{12}^x(t) \\ g_{12}^x(t) & 0 \end{pmatrix},$$

$$H^C = \begin{pmatrix} -B^x(t) & 0 \\ 0 & B^y(t) \end{pmatrix}, H^N = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (26)

In order to verify the dynamical invariant $I(t)$ can be expressed as Eq. (14), we must check if the decoupling condition (Eq. (17)) is satisfied. Since the DFS is time-independent, $G$ always vanishes in Eq. (17). Moreover, we have already shown that it is not only the off-diagonal elements of the Lindblad operator is trivial, but also the Hamiltonian, i.e., $A_\alpha = H^N = 0$, where $0$ denotes a matrix with zero elements. Therefore, the dynamical invariant $I(t)$ can be written in form as Eq. (14).

Next, let us present the upper and lower nonzero block elements of $I(t)$, i.e., $I^D$ and $I^C$. The evolution of $I^D$ is unitary, which is governed by Eq. (15),

$$i \partial \lambda_j^D + [H^D(t), I^D] = 0.$$  \hspace{1cm} (27)

The Hamiltonian that involves here can be rewritten in the form of the pauli matrices,

$$H^D = g_{12}^x(t) \sigma_x.$$  \hspace{1cm} (28)

Assisted by SU(2) algebra of the pauli matrices, the dynamical invariant as hermitian operator can also be expanded by the pauli matrices,

$$I^D = x^D \sigma_x + y^D \sigma_y + z^D \sigma_z.$$  \hspace{1cm} (29)
with expansion coefficients \( x^D, y^D, z^D \). Placing \( H^D \) and \( I^D \) into Eq. (15), the equations of motion for those expansion coefficients are thus given by

\[
\begin{align*}
\dot{x}^D &= 0, \\
\dot{y}^D &= 2g_{12}^D(t)z^D, \\
\dot{z}^D &= -2g_{12}^D(t)y^D.
\end{align*}
\] (31)

The solution of these equations subject to arbitrary initial conditions \( x_0^D, y_0^D, z_0^D \) can be easily obtained,

\[
\begin{align*}
x^D(t) &= x_0^D, \\
y^D(t) &= y_0^D \cos(2\Lambda(t)) + z_0^D \sin(2\Lambda(t)), \\
z^D(t) &= z_0^D \cos(2\Lambda(t)) - y_0^D \sin(2\Lambda(t)),
\end{align*}
\] (32)

where \( \Lambda(t) = \int_0^t g_{12}^D(t')dt' \). Consequently we confirm all of the parameters in dynamical invariant \( I^D \). When the initial conditions for those parameters are given, the dynamical invariant \( I^D \) can be constructed.

For the lower diagonal-block of the dynamical invariant Eq. (11), the evolution is governed by Eq. (16). Because of the vanishing off-diagonal block of \( F \), i.e., \( A_o = 0 \), the dynamical invariant \( I^C \) can be simplified into,

\[
\dot{i}C + i[H^C, I^C] - \frac{\gamma}{2} \sum_\alpha (\{B^0_\alpha \alpha, I^C\} - 2B^1_\alpha I^C B^1_\alpha) = 0,
\] (35)

where

\[
H^C = -B^z(t)\sigma_z, B_\alpha = -2\sigma_z.
\]

As an hermitian operator, \( I^C \) can be supposed as following expansion,

\[
I^C = x^C \sigma_x + y^C \sigma_y + z^C \sigma_z,
\] (36)

where \( x^C, y^C \) and \( z^C \) are time-dependent real parameters. Placing \( I^C \) in Eq. (35), the time development of those parameters is governed by,

\[
\begin{align*}
\dot{x}^C &= -2B^z(t)y^C + 8\gamma x^C, \\
\dot{y}^C &= 2B^z(t)x^C + 8\gamma y^C, \\
\dot{z}^C &= 0.
\end{align*}
\] (37)

On the one hand, as we expected, \( z^C \) is not affected by dephasing environment. On the other hand, \( x^C \) and \( y^C \) satisfy a set of coupling differential equation. Here, we may introduce new parameters as follows,

\[
\begin{align*}
x'(t) &= x^C(t) \exp(-8\gamma t), \\
y'(t) &= y^C(t) \exp(-8\gamma t),
\end{align*}
\] (40)

which fulfill

\[
\begin{align*}
\dot{x}' &= -2B^z(t)y', \\
\dot{y}' &= 2B^z(t)x'.
\end{align*}
\] (41)

The new parameters \( x' \) and \( y' \) independently satisfy the following second order differential equations,

\[
\begin{align*}
\ddot{x}' &= -\frac{B^z(t)}{B^z(t)} x' + 4B^z(t)x' = 0, \\
\ddot{y}' &= -\frac{B^z(t)}{B^z(t)} y' + 4B^z(t)y' = 0.
\end{align*}
\] (43)

And the solutions can be expressed in a general form,

\[
\delta = c_1^2 \cos(2\Theta(t)) + c_2^2 \sin(2\Theta(t)),
\] (45)

where \( \delta \in (x', y') \); \( \Theta(t) = \int_0^t B^z(t')dt' \); \( c_1^2 = \delta(0) \), \( c_2^2 = \frac{\delta(0)}{2\pi} \) are some constants related with initial condition of \( x' \) and \( y' \). Taking the solutions of \( x' \) and \( y' \) into Eq. (42) and considering the initial condition of \( I^C \), the solution of \( I^C \) can be written as a function of

\[
\begin{align*}
x^C(t) &= (x_0^C \cos(2\Theta(t)) - y_0^C \sin(2\Theta(t))) \exp(8\gamma t), \\
y^C(t) &= (y_0^C \cos(2\Theta(t)) + x_0^C \sin(2\Theta(t))) \exp(8\gamma t),
\end{align*}
\] (44)

\[
z^C(t) = z_0^C,
\] (46)

where \( x_0^C, y_0^C, z_0^C \) is initial values of \( x^C, y^C, z^C \).

By assistance from the analytic solution of \( I(t) \), we can further discuss eigenvalues and eigenstates of it. The eigenvalues of \( I^D \) and \( I^C \) have the same structure,

\[
\lambda^0_\pm = \pm \sqrt{x'^2 + y'^2 + z'^2},
\] (47)

which is associated with the eigenstate as follows,

\[
|\psi^0_\pm\rangle = \begin{pmatrix} (\lambda^0_\pm + z^0)/p^0_\pm \\ (x^0 \mp iy^0)/p^0_\pm \end{pmatrix},
\] (48)

where \( p^0_\pm = \sqrt{2\lambda^2_\pm (\lambda^0_\pm + z^0)} \) is normalized coefficient, and \( \alpha \in (D, C) \). On the one hand, with the help from Eq. (32), Eq. (33) and Eq. (34), it is easy to check that the eigenvalues of \( I^D \) are constant, even though \( y^D \) and \( z^D \) are time-dependent. Simultaneously, corresponding eigenstates are time-dependent. This result is very similar with the dynamical invariant for the closed quantum system, which attributes to the unitary dynamics of \( I^D \). On the other hand, \( I^C \) is affected by the dephasing noise. The eigenvalues of it is no more time-independent, but depend on the decoherence process. After placing the solution of \( I^C \) into Eq. (47), we obtain the following eigenvalues

\[
\lambda^C_\pm(t) = \pm \sqrt{(x_0^C(t) + y_0^C(t)) \exp(16\gamma t) + z_0^C(t)^2}.
\] (49)

This implies an exponent change of the eigenvalues. The eigenstates of \( I^C \) can also be written as a function of time,

\[
|\psi^{C_\pm}(t)\rangle = \begin{pmatrix} \frac{(\lambda^C_\pm(t) + z^0_\pm)}{p^C_\pm(t)} \\ \frac{(x_0^C(t) - iy_0^C(t))}{p^C_\pm(t)} \exp(-2i\Theta(t) + 4\gamma t) \end{pmatrix},
\] (50)
in which the normalized constant is also time-dependent
\[ p_{\pm}^C = \sqrt{\frac{2\lambda_C^2(t)}{\lambda_C^2(t) + z_0^2}}. \]
If we set \( z_0^C = 0 \), the eigenvalues and eigenstates can be further simplified,
\[ \lambda_{\pm}^C(t) = \pm \sqrt{(x_C^2 + y_C^2)} \exp(8\gamma t), \tag{51} \]
and
\[ |\psi_{\pm}^C(t)\rangle = \left( \frac{\pm \sqrt{2}/2}{\sqrt{2}/2} \exp(-2i\Theta(t)) \right), \tag{52} \]
In this special case, the dynamics of eigenvalues are dominated by the decoherence process, but the eigenstates can be only affected by the external control on the open quantum system.

\section{IV. CONCLUSION}
This work answers the question regarding what condition is required for the dynamical invariant when the quantum system couples with their surroundings. According to the fact that the expectation value of invariant is constant, the dynamical invariant condition is presented, showing that the density matrix of quantum system is not a dynamical invariant for the open quantum system. Soon after, we propose a sort of dynamical invariant, which is block-diagonal (as shown in Eq. (43)). As we present in the paper, this kind of dynamical invariant exists, if and only if the decoherence free subspace presents in the dynamics of open quantum system. The results tell us that the upper block of dynamical invariant (the dynamical invariant in DFS) satisfy a von Neumann-like equation and its eigenvalues are constant; the lower block is affected by decoherence and satisfy a Lindblad-Markovian-like master equation.

Practically speaking, dynamical invariant provide both an intuitive physical framework and a set of tools to understand and manipulate quantum state density matrices carrying information, especially for the quantum system with time-dependent Hamiltonian. These tools should prove useful in theoretical formulations of decoherence-free subspaces and further experimental developments of quantum control.

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