Uncertainty Quantification For Low-Rank Matrix Completion With Heterogeneous and Sub-Exponential Noise

Abstract

The problem of low-rank matrix completion with heterogeneous and sub-exponential (as opposed to homogeneous and Gaussian) noise is particularly relevant to a number of applications in modern commerce. Examples include panel sales data and data collected from web-commerce systems such as recommendation engines. An important unresolved question for this problem is characterizing the distribution of estimated matrix entries under common low-rank estimators. Such a characterization is essential to any application that requires quantification of uncertainty in these estimates and has heretofore only been available under the assumption of homogenous Gaussian noise. Here we characterize the distribution of estimated matrix entries when the observation noise is heterogeneous sub-exponential and provide, as an application, explicit formulas for this distribution when observed entries are Poisson or Binary distributed.

1. Introduction

Consider the problem of low-rank matrix completion: there exists a low-rank matrix that we seek to recover, having observed only a subset of its entries, each perturbed by additive noise. A rich stream of research over the past two decades has essentially solved this problem—there exist efficient algorithms which achieve order-optimal recovery guarantees under provably-minimal assumptions (Candès and Recht 2009, Candes and Plan 2010, Keshavan et al. 2010). Further advances have yielded (and continue to yield) algorithmic improvements (Mazumder et al. 2010, Jain et al. 2013, Tanner and Wei 2016, Dong et al. 2021), and a deeper understanding of the optimization landscape itself (Ge et al. 2016, Zhu et al. 2017).

Naturally, these algorithms have been applied in a vast array of applications, including recommendation systems, bioinformatics, network localization, and modern commerce (Su and Khoshgoftaar 2009, Natarajan and Dhillon 2014, So and Ye 2007, Amjad and Shah 2017), just to name a few. Now many of these applications require, in addition to scalability and accuracy, the ability to quantify the uncertainty of an estimator— for example, something as seemingly-straightforward as confidence intervals on the estimated entries of a matrix.

Such an uncertainty quantification procedure, analogous to existing procedures for problems like linear regression, would ideally (a) apply to a commonly-used estimator, (b) require no more additional computation than the estimator itself, and (c) be justified by a (limiting) distributional characterization. Given the volume and success of the research just described, it is perhaps surprising that this problem has been largely unsolved (see the Related Work for past progress).

Fortunately, there was a recent “breakthrough.” Applying newer techniques such as the leave-one-out technique and fine-grained entry-wise analysis (Ma et al. 2018, Ding and Chen 2020, Abbe et al. 2020), Chen et al. (2019, 2020) proposed an uncertainty quantification technique for matrix completion, which satisfies the three “ideal” conditions above, in the case of homogeneous Gaussian noise. Further progress in Xia and Yuan (2021) extended this to homogeneous sub-Gaussian noise.

Toward “Realistic” Noise: A gap still exists when we seek to apply these inferential results in practice, since many applications have more sophisticated noise models (namely, heterogeneous and sub-exponential noise). For example, in discrete panel sales data, the observation for sales at a location during a period of time is commonly modeled as Poisson with a certain
expected sales rate (Amjad and Shah 2017, Shi et al. 2014). Similarly in web-commerce systems, data indicating clicks or purchases is often binary and modeled as Bernoulli random variables (Ansari and Mela 2003, Grover and Srivastava 1987).

Thus motivated, in this work we establish the first uncertainty quantification results for matrix completion with heterogeneous and sub-exponential noise. Precisely, we characterize the distribution of recovered matrix entries from common estimators. An application of our results can already be seen in Table 1, where we have derived explicit formulas under Poisson and Binary noise, which are distinctive from the homogeneous Gaussian noise case already existing in the literature. In addition, we demonstrate the quality of our procedure through experiments on real sales data. The proof of our main result generalizes the proof framework in (Chen et al. 2019), leveraging recent results for sub-exponential matrix completion from McRae and Davenport (2019), and a new high-dimensional concentration bound (Lemma 1), which may be of independent interest.

**Related Work:** This paper is related to at least three streams of work. The first is, naturally, uncertainty quantification in matrix completion. Besides the works described above, prior approaches to this were based on either (a) converting recovery guarantees on matrix norms to confidence regions (Carpentier et al. 2015, 2018), (b) the Bayesian formulation of matrix completion (Salakhutdinov and Mnih 2008, Fazaei et al. 2014, Tanaka 2021, Alquier et al. 2015), or (c) deep-learning-based methods (Lakshminarayanan et al. 2016, Zeldes et al. 2017). The second stream relates to sub-exponential matrix completion. McRae and Davenport (2019) established guarantees on the Frobenius error $\|M - M^*\|_F$; Farias et al. (2021) established entry-wise error guarantees. This work makes a step further with an entry-wise distributional characterization of the error. Finally, there is a line of work, in multivariate linear regression, advocating the use of heteroskedasticity-robust variance estimators instead of homoskedasticity estimators, since the former are more robust to heterogeneous noise (Long and Ervin 2000, Hayes and Cai 2007, Imbens and Kolesar 2016, Cribari-Neto and Maria da Glória 2014). Our work is in the same spirit, but in the context of matrix completion.

**Notation:** The sub-exponential norm of a random variable $X$ is defined as $\|X\|_{\psi_1} := \inf\{t > 0 : \mathbb{E} \exp(|X|/t)) \leq 2\}$. For a matrix $A \in \mathbb{R}^{m \times n}$, we abbreviate $\sum_{i,j \in [m] \times [n]} A_{ij}$ as $\sum A_{ij}$ when ambiguity exists. We require a few matrix norms: $\|A\|_{2,\infty} := \max_i \sum_j A_{ij}$, $\|A\|_{\max} = \max_{i,j} |A_{ij}|$, and $\|A\|_F^2 = \sum_{i,j} A_{ij}^2$. The spectral norm is denoted $\|A\|_2$.

2. Model

Let $M^* \in \mathbb{R}^{m \times n}$ be a rank-$r$ matrix, where $m \leq n$ without loss of generality. Let $O = M^* + E$ be the realization of $M^*$ corrupted by a noise matrix $E \in \mathbb{R}^{m \times n}$. We observe $P_\Omega(O)$, which is the subset of entries of $O$ restricted to an observation set $\Omega \subset [m] \times [n]$:

$$P_\Omega(O)_{ij} = \begin{cases} O_{ij} & (i,j) \in \Omega \\ 0 & (i,j) \notin \Omega \end{cases}.$$

The matrix completion problem is to recover $M^*$ from this noisy and partial observation $P_\Omega(O)$.

Let $M^* = U^* \Sigma^* V^*\top$ be the SVD of $M^*$. Here, $\Sigma^* \in \mathbb{R}^{r \times r}$ is a diagonal matrix with singular values $\sigma_{\text{max}} = \sigma_1^* \geq \sigma_2^* \geq \ldots \geq \sigma_r^* = \sigma_{\text{min}}$; and $U^* \in \mathbb{R}^{m \times r}, V^* \in \mathbb{R}^{n \times r}$ contain the left and right-singular vectors. Let $\kappa = \sigma_{\text{max}}/\sigma_{\text{min}}$ be the condition number of $M^*$.

We will make three assumptions. The first two are, by this point, canonical in the matrix completion literature (Candes and Plan 2010, Keshavan et al. 2010, Ma et al. 2018, Abbe et al. 2020, Chen et al. 2019):

**Assumption 1** (Uniform Sampling). Each element of $[m] \times [n]$ is included in $\Omega$ independently, and with probability $p$.

**Assumption 2** (Incoherence).

$$\|U^*\|_{2,\infty} \leq \sqrt{\frac{\mu r}{m}} \quad \text{and} \quad \|V^*\|_{2,\infty} \leq \sqrt{\frac{\mu r}{n}} \quad (1)$$
Finally, our third assumption is a generalization of the independent (and often homogeneous), sub-Gaussian noise that is typically assumed in the literature (Chen et al. 2019; Xia and Yuan 2021). As described above, this generalization enables a host of practical applications, such as those arising in counting data and panel sales data (Amjad and Shah 2017; Ansari and Melo 2003).

Assumption 3 (Independent Sub-exponential Noise). The entries of $E$ are independent, mean-zero random variables with variances $\sigma_y^2$, and are also independent from $\Omega$. Furthermore, $\|E_{ij}\|_{\psi_1} \leq L$ for every $(i, j)$, where $\| \cdot \|_{\psi_1}$ is the sub-exponential norm.

3. Algorithm

In this section, we describe a “de-biased” estimator $M^d$ for $M^*$. This was originally proposed in (Chen et al. 2019), where the uncertainty quantification for $M^d$ is characterized under homogeneous, Gaussian noise. Motivated by practical applications, we study new uncertainty quantification formulas for $M^d$ under heterogeneous sub-exponential noise.

To begin, consider a natural least-square estimator for $M^*$

$$\hat{M} \triangleq \arg \min_{M^* \in \mathbb{R}^{m \times n}, \text{rank}(M^*) = r} \frac{1}{2p} \|P_{\Omega}(O - M^*)\|_F^2,$$ (2)

Here, $\hat{M}$ is the projection of $M$ into the set of rank-$r$ matrices in regard to Euclidean distance (restricted on the set $\Omega$).

Directly solving Eq. (2) turns out to be a challenge task. A popular method is to represent $M^* = X Y^T$ where $X \in \mathbb{R}^{m \times r}, Y \in \mathbb{R}^{n \times r}$ are low-rank factors, and solve the following non-convex regularized optimization problem

$$\minimize_{X \in \mathbb{R}^{m \times r}, Y \in \mathbb{R}^{n \times r}} f(X, Y) := \frac{1}{2p} \|P_{\Omega}(X Y^T - O)\|_F^2 + \frac{1}{2p} \|X\|_F^2 + \frac{\lambda}{2p} \|Y\|_F^2. \tag{3}$$

With proper initializations, simple first-order methods are often sufficient to solve Eq. (3) (Sun and Luo 2016). The regularizer $\lambda > 0$ here is used to promote additional structure properties. For example, when gradient descent is performed, a positive $\lambda$ is critical for analyzing the convergence properties and also helps to achieve a balance between $X$ and $Y$ (Chen et al. 2020).

However, the use of $\lambda$ also introduces additional bias to the estimator in Eq. (3), which has been a major obstacle to analyze the uncertainty quantification properties. (Chen et al. 2019) proposes a de-bias procedure to remove the bias brought by $\lambda$, based on the solution of Eq. (3). The algorithm is summarized below\(^1\).

Algorithm 1 Gradient Descent with De-bias

Input: $P_{\Omega}(O)$

1: Spectral initialization: $X^0 = U \sqrt{\Sigma} Y^0 = V \sqrt{\Sigma}$ where $\Sigma V^T$ is the top-$r$ partial SVD decomposition of $\frac{1}{2}P_{\Omega}(O)$.

2: Gradient updates: for $t = 0, 1, \ldots, t_s - 1$ do

$$X^{t+1} = X^t - \frac{\eta}{p} [P_{\Omega}(X^t Y^T - O) Y^T + \lambda X^T];$$

$$Y^{t+1} = Y^t - \frac{\eta}{p} [P_{\Omega}(X^t Y^T - O)^T X^t + \lambda Y^T]$$

where $\eta$ determines the learning rate.

3: De-bias:

$$X^d = X^{t_s} \left( I_r + \frac{\lambda}{p} (X^{t_s} Y^{t_s})^{-1} \right)^{1/2} \tag{5}$$

$$Y^d = Y^{t_s} \left( I_r + \frac{\lambda}{p} (Y^{t_s} Y^{t_s})^{-1} \right)^{1/2} \tag{6}$$

Output: $M^d = X^d Y^{dT}$

Steps 1 and 2 in Algorithm 1 form a typical gradient descent procedure for solving Eq. (3). The de-biasing step, i.e., Eqs. (5) and (6) in Algorithm 1 is critical for enabling the uncertainty quantification analysis.

We will use the remainder of this section (which can be skipped without loss of continuity) to provide some intuition for the peculiar form of Eqs. (5) and (6) based on first-order conditions. Consider an example with $p = 1$ (no entry is missing). Since $O$ is fully observed, let $O = U_r \Sigma_r V_r^T + U_{n-r} \Sigma_{n-r} V_{n-r}^T$ be the SVD of $O$, where $\Sigma_r$ corresponds to the largest $r$ singular values and $\Sigma_{n-r}$ corresponds to the remaining one. Then it follows that the optimal solution of Eq. (2) is $\hat{M} = U_r \Sigma_r V_r^T$ (Eckart and Young 1936).

Next, consider the regularized objective Eq. (3). We can derive that the optimal solution $(X, Y)$ for Eq. (3) has the form

$$X = U_r (\Sigma_r - \lambda I_r)^{-1/2}, \quad Y = V_r (\Sigma_r - \lambda I_r)^{-1/2}.$$

In fact, this can be verified from the first-order condi-

\(^1\)We assume $\lambda \approx \frac{n \log(n)}{\sqrt{p} t^*} \approx n^{23} \eta \approx 1/(n^6 r^3 \sigma_{max})$ throughout the paper, if not specified explicitly.
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sions, \[
\frac{\partial f(X, Y)}{\partial X} = (XY^T - O)Y + \lambda X
\]
\[
= (U_r(\Sigma_r - \lambda I)V_r^T - O)Y + \lambda X
\]
\[
= (U_{n-r}V_{n-r} - \lambda U_rV_r^T)Y + \lambda X
\]
\[
(i) \quad -\lambda U_rV_r^T V_r(\Sigma_r - \lambda I)^{-1/2} + \lambda X
\]
\[
(ii) \quad 0,
\]
where in (i) we use that \( V_{n-r}^TY = V_{n-r}^TV_r(\Sigma_r - \lambda I)^{-1/2} = 0 \), and in (ii) we use that \( V_r^TV_r = I_r \). Similarly \( \frac{\partial f(X, Y)}{\partial Y} = 0 \) also vanishes.

Then, this justifies the particular de-biased form in Eqs. (5) and (6):

\[
X^d = X \left( I_r + \lambda(\Sigma_r - \lambda I)^{-1}\right)^{1/2}
\]
\[
= X(\Sigma_r(\Sigma_r - \lambda I)^{-1})^{1/2}
\]
\[
= U_r(\Sigma_r - \lambda I)^{1/2}(\Sigma_r - \lambda I)^{-1/2}S_r^{1/2}
\]
\[
= U_rS_r^{1/2}.
\]

Similarly, \( Y^d = V_rS_r^{1/2} \). Thus \( X^dY^d^\top = U_rS_rV_r^\top \) is the desired optimal solution of Eq. (2).

4. Results

We can now state our main result: an uncertainty quantification for \( M^d \) under heterogeneous, sub-exponential noise.

**Theorem 1.** Assume \( mp \gg \kappa^4\mu^2r^2\log^3n \) and \( L\log(n)\sqrt{n/p} \ll \sigma_{\min}/\sqrt{\kappa^4\mu r}\log n \). Then for every \( (i, j) \in [m] \times [n] \), we have

\[
\sup_{t \in \mathbb{R}} \left| P \left( \frac{M^d_{ij} - M^{*}_{ij}}{s_{ij}} \leq t \right) - \Phi(t) \right| \lesssim s^{-3}_{ij}L^{2}\mu^{3}\kappa^{3}m^{2}p + \frac{L^{2}\log^{3}(n)\mu r \kappa^{5}}{p\sigma_{\min}} + \frac{L^{2}\mu^{2}r^{2}\log^{2}(n)\kappa^{4}}{pm} + \frac{1}{m^{1/2}},
\]

where \( \Phi(.) \) is the CDF of the standard Gaussian, and \( s_{ij} > 0 \) is given by

\[
s^{2}_{ij} := \sum_{l=1}^{m} \sigma^{2}_{ij} \left( \sum_{k=1}^{r} U_{ik}V_{lk}^* \right)^{2} + \sum_{l=1}^{n} \sigma^{2}_{il} \left( \sum_{k=1}^{r} V_{ik}V_{jk}^* \right)^{2}.
\]

To quickly parse this result, note that a typical scaling of the parameters would see \( m = \Theta(n) \), \( mp \gg \log^{6}(n) \), \( \mu = r = \kappa = L = O(1) \), \( \sigma_{\min} = \Omega(n) \), \( \sigma_{ij} = \Omega(1) \), and \( \|V_{ij}^*\| = \|U_{ij}^*\| = \Omega(\sqrt{1/n}) \). Then \( s_{ij} \) would then imply that

\[
\frac{M^d_{ij} - M^{*}_{ij}}{s_{ij}} \xrightarrow{p} \mathcal{N}(0, 1)
\]

where \( s_{ij} \) is defined in Eq. (7). This is precisely the type of characterization we sought at the outset. The form of \( s_{ij} \), as defined in Eq. (7), is of course critical to the characterization, and probably best understood via a few examples:

1. **Homogeneous Gaussian Noise.** First as a sanity check, when \( E_{ij} \sim \mathcal{N}(0, \sigma^2) \), Theorem 1 reduces to the same variance formula as Theorem 2 in [Chen et al., 2019]:

\[
s^{2}_{ij} = \frac{\sigma^{2}(\|U_{ij}^{*}\|^{2} + \|V_{ij}^{*}\|^{2})}{p}.
\]

2. **Poisson Noise.** When the observations are Poisson, i.e. \( O_{ij} \sim \text{Poisson}(M_{ij}^{*}) \), the variance of the noise \( E_{ij} \) is \( \sigma_{ij}^{2} = \text{Var}(O_{ij} - M_{ij}^{*}) = M_{ij}^{*} \). Then applying Theorem 1 we have that \( M_{ij}^{d} - M^{*}_{ij} \sim \mathcal{N}(0, s^{2}_{ij}) \) where

\[
s^{2}_{ij} = \sum_{l=1}^{m} M_{ij}^{*} \left( \sum_{k=1}^{r} U_{ik}V_{lk}^{*} \right)^{2} + \sum_{l=1}^{n} M_{il}^{*} \left( \sum_{k=1}^{r} V_{ik}V_{jk}^{*} \right)^{2}.
\]

A special case is when \( r = 1 \) and \( M^{*} = \sigma_{1}u^{*}v^{*\top} \), for which we have

\[
s^{2}_{ij} = \sum_{l=1}^{m} M_{ij}^{*} (u_{il}^{*}v_{lj}^{*})^{2} + \sum_{l=1}^{n} M_{il}^{*} (u_{ij}^{*}v_{lj}^{*})^{2}.
\]

3. **Binary Noise.** Finally, binary observations occur frequently in applications. For example, in a recommender system or e-commerce platform, \( O_{ij} \in \{0, 1\} \) can represent whether the \( i \)-th user viewed (or purchased) the \( j \)-th item (or product) [Ansari and Mela, 2003, Grover and Srivastava, 1987, Farias and Li, 2019]. A common noise model for such observations is to assume the \( O_{ij} \) are Bernoulli random variables with mean \( M_{ij}^{*} \), i.e., \( O_{ij} \sim \text{Ber}(M_{ij}^{*}) \).

With such binary observations, the variance of the noise \( E_{ij} \) is \( \sigma_{ij}^{2} = \text{Var}(O_{ij} - M_{ij}^{*}) = M_{ij}^{*}(1 - M_{ij}^{*}) \). Then \( s_{ij} \) takes the form

\[
s^{2}_{ij} = \sum_{l=1}^{m} M_{ij}^{*}(1 - M_{ij}^{*}) \left( \sum_{k=1}^{r} U_{ik}V_{lk}^{*} \right)^{2} + \sum_{l=1}^{n} M_{il}^{*}(1 - M_{il}^{*}) \left( \sum_{k=1}^{r} V_{ik}V_{jk}^{*} \right)^{2}.
\]
When $r = 1$ and $M^* = \sigma_i u^i v^T$, we have
\[
\begin{align*}
\mathbb{E}_{ij}^2 &= \sum_{i=1}^m \sigma_i u_i^T v_j^T (1 - \sigma_i u_i^T v_j^T) (u_i^T u_i)^2 \\
&\quad + \sum_{i=1}^n \sigma_i u_i^T v_i^T (1 - \sigma_i u_i^T v_i^T) (v_i^T v_i)^2 \\
&= M^*_ij (u_i^T u_i) + M^*_ij (v_i^T v_i).
\end{align*}
\]

Empirical Inference: In practice, the underlying $U^*$ and $V^*$ are not known, and thus $s_{ij}$ cannot be computed exactly. We propose the use of the corresponding empirical estimators to estimate $s_{ij}$ for the purposes of inference. Let $M^d = U^d \Sigma^d V^d$ be the SVD of $M^d$. For example, in the Poisson noise scenario, we would use the following empirical estimator for $s_{ij}$:
\[
\hat{s}_{ij}^2 = \frac{\sum_{i=1}^m \sigma_i u_i^T v_j^T (1 - \sigma_i u_i^T v_j^T) (u_i^T u_i)^2 + \sum_{i=1}^n \sigma_i u_i^T v_i^T (1 - \sigma_i u_i^T v_i^T) (v_i^T v_i)^2}{\mathbb{E}_{ij}^2}.
\]

In cases where $\sigma_{kl}$ is also unknown, we let $\tilde{E}_{ij} = O_{ij} - M^d_{ij}$ be the empirical estimator for the noise. This procedure (i.e. the use of empirical estimators) can be justified via the following result:

**Corollary 1.** Follow the settings in Theorem 1. Assume that $\|a\|_{\ell_2} \leq \hat{a}$ and $\|b\|_{\ell_2} \leq \hat{b}$.

\[
\begin{align*}
s_{ij} &\geq L^2 \mu^2 r^2 \kappa^5 \log^4(n) \left( \frac{1}{\sigma_{\min}^p} + \frac{1}{m} + \frac{1}{m^{2/3} r^{1/3}} \right).
\end{align*}
\]

Let
\[
\begin{align*}
\hat{s}_{ij}^2 &= \frac{\sum_{i=1}^m \sigma_i u_i^T v_j (1 - \sigma_i u_i^T v_j) (u_i^T u_i)^2}{\mathbb{E}_{ij}^2} \\
&\quad + \frac{\sum_{i=1}^n \sigma_i u_i^T v_i (1 - \sigma_i u_i^T v_i) (v_i^T v_i)^2}{\mathbb{E}_{ij}^2}.
\end{align*}
\]

be the empirical estimator of $s_{ij}$. Then under the same assumptions made in Theorem 1, we have that
\[
\sup_{t \in \mathbb{R}} \left| P \left\{ \frac{M^d_{ij} - M^*_ij}{\hat{s}_{ij}} \leq t \right\} - \Phi(t) \right| = o(1).
\]

Additional justification for this procedure is given as experiments later on.

Aside: When $s_{ij} \approx 0$. Curious readers may note that $s_{ij}$ may be too small for Theorem 1 and Corollary 1 to apply. In this case, although the Gaussian approximation in Theorem 1 does not hold, an entry-wise error bound still holds, and may be sufficient for many applications (see the Appendix for details):
\[
|\hat{M}_{ij}^d - M^*_ij| \lesssim \kappa \mu r L \sqrt{\log(n)}.
\]

An uncertainty characterization when $s_{ij} \approx 0$ involves a second-order error analysis and remains an open question.

5. Proof Overview

In this section, we present the proof framework of Theorem 1 (see details in Appendix A). In order to extend to heterogeneous sub-exponential noise from homogeneous Gaussian, we generalize the proof of (Chen et al. 2019) with the help of recent sub-exponential matrix completion results (McRae and Davenport 2019) and a sub-exponential variant of matrix Bernstein inequality (Lemma B).

Similar to Chen et al. (2019), our proof is based on the leave-one-out technique that has been recently used for providing breakthrough bounds for entry-wise analysis in matrix completion problems (dated back to Ma et al. (2018), also see Ding and Chen (2020), Abbe et al. (2020), Chen et al. (2020)).

We establish the following key results to characterize the decomposition of low-rank factors ($X^d, Y^d$), as a heterogeneous sub-exponential generalization of Theorem 5 in Chen et al. (2019).

**Theorem 2.** Assume $mp \gg \kappa^4 m^2 r^2 \log^3 n$ and $L \log(n) \frac{a}{\sqrt{p}} \ll \frac{c_{\min}}{\sqrt{\kappa^4 \mu \log n}}$. There exists a rotation matrix $\tilde{X} \in \mathbb{O}^{n \times r}$ and $\Phi_X \in \mathbb{R}^{n \times r}, \Phi_Y \in \mathbb{R}^{n \times r}$ such that the following holds with probability $1 - O(n^{-10})$,
\[
\begin{align*}
X^d \tilde{X}^d - X^* &= \frac{1}{p} P_{1}(E) Y^T Y^{-1} + \Phi_X \approx \\
Y^d \tilde{X}^d - Y^* &= \frac{1}{p} P_{1}(E) X^T X^T X^{-1} + \Phi_Y
\end{align*}
\]

where
\[
\begin{align*}
\max \left\{ \|\Phi_X\|_{\ell_\infty} \right\} \lesssim L \log n \left( \frac{\log n}{\sigma_{\min}} \right) \frac{\kappa^4 \mu r^2 \log n}{p} + \frac{\kappa^7 \mu^3 r^3 \log^2 n}{mp}.
\end{align*}
\]

**Proof.** At a high level, the proof of Theorem 2 follows a similar proof of Theorem 5 in Chen et al. (2019), but with replacements that employ more fine-grained analyses of $E$ for whenever the Gaussianity of $E$ is used in Chen et al. (2019). These analyses aim to address the sub-exponentiality and heterogeneity of $E$, with the help of the following two lemmas.
Lemma 1. Given \( k \) independent random matrices \( X_1, X_2, \ldots, X_k \) with \( E[X_i] = 0 \). Let
\[
V := \max \left( \left\| \sum_{i=1}^{k} E[X_i X_i^T] \right\|, \left\| \sum_{i=1}^{k} E[X_i^T X_i] \right\| \right).
\]
Suppose \( \| X_i \|_{\psi_1} \leq B \) for \( i \in [k] \). Then,
\[
\| X_1 + X_2 + \ldots + X_k \| \lesssim \sqrt{V \log(k(m_1 + m_2))} + B \log(k(m_1 + m_2)) \log(k)
\]
with probability \( 1 - O(k^{-c}) \) for any constant \( c \).

Lemma 2. Suppose \( E \in \mathbb{R}^{m \times n} (m \leq n) \) whose entries are independent and centered. Suppose \( \| E_{ij} \|_{\psi_1} \leq L \) for any \((i, j) \in [m] \times [n] \). Let \( \Omega \subseteq [m] \times [n] \) be the subset of indices where each index \((i, j) \) is included in \( \Omega \) independently with probability \( p \). Suppose \( np \geq c_0 \log^3 n \) for some sufficient large constant \( c_0 \), then, with probability \( 1 - O(n^{-11}) \),
\[
\left\| \frac{1}{p} P_{\Omega}(E) \right\| \leq CL \sqrt{\frac{n}{p}}
\]

Here, Lemma 1 is a generalization of matrix Bernstein inequality in Theorem 6.1.1 of [Tropp et al. 2013]. Lemma 2 is an implication of Lemma 4 in [McRae and Davenport 2019].

Equipped with Lemmas 1 and 2, the desired bounds for sub-exponential \( E \) can be established. Following we provide an example of using Lemma 1 to bound \( \| X^* P_{\Omega}(E) Y^* \| \), which is critical for obtaining the bounds in Theorem 2.

To begin note that
\[
X^* P_{\Omega}(E) Y^* = \sum_{k=1}^{m} \sum_{l=1}^{n} X_{k,l}^* Y_{l,*}^* \delta_{k,l} E_{k,l}
\]
where \( \delta_{k,l} \sim \text{Ber}(p) \) indicates whether \((k, l) \in \Omega \). Let \( A_{k,l} := X_{k,l}^* Y_{l,*}^* \delta_{k,l} E_{k,l} \) for \( k \in [m], l \in [n] \). Then,
\[
\left\| X^* P_{\Omega}(E) Y^* \right\| = \max_{k,l} \| A_{k,l} \|.
\]

Note that \( A_{k,l} \in \mathbb{R}^{r \times r} \) are independent zero-mean random matrices and we aim to invoke Lemma 1 to bound \( \| \sum_{k,l} A_{k,l} \| \). Let
\[
V := \max \left( \left\| \sum_{k,l} E[A_{k,l} A_{k,l}^T] \right\|, \left\| \sum_{k,l} E[A_{k,l}^T A_{k,l}] \right\| \right).
\]
\[
B := \max_{k,l} \| A_{k,l} \|_{\psi_1}.
\]

Note that
\[
\left\| \sum_{k,l} E[A_{k,l} A_{k,l}^T] \right\| \leq \sum_{k,l} \left\| E[A_{k,l} A_{k,l}^T] \right\| \leq \sum_{k,l} \sigma_{k,l}^2 \| X_{k,l}^* \|^2 \| Y_{l,*}^* \|^2
\]
\[
\leq 2L^2 p \| X^* \|^2 \| Y^* \|^2
\]
where in (i) we use the fact that \( E(x^2) \leq 2 \|x\|^2_{\psi_1} \) for an sub-exponential zero-mean random variable \( x \). Similarly, the bounds can be established for \( \| \sum_{k,l} E[A_{k,l}^T A_{k,l}] \| \). Hence \( V \leq 2L^2 p \| X^* \|^2 \| Y^* \|^2 \).

Then consider
\[
B := \max_{k,l} \| A_{k,l} \|_{\psi_1} \leq \max_{k,l} \| E_{k,l} \delta_{k,l} \|_{\psi_1} \| X^* \|_{2,\infty} \| Y^* \|_{2,\infty}
\]
\[
\leq L \| X^* \|_{2,\infty} \| Y^* \|_{2,\infty}
\]
where (i) we use that \( \| E_{k,l} \delta_{k,l} \|_{\psi_1} \leq \| E_{k,l} \|_{\psi_1} \leq L \). Then apply Lemma 1 with probability \( 1 - O(n^{-11}) \), we obtain the desired bound for \( \| X^* P_{\Omega}(E) Y^* \| \)
\[
\left\| \sum_{k,l} A_{k,l} \right\| \lesssim \sqrt{V \log(n)} + B \log^2(n)
\]
\[
\lesssim \sqrt{pL \sigma_{\max} \log(n)} + \frac{\mu r L \sigma_{\max} \log^2(n)}{n}
\]
\[
\lesssim \sqrt{pL \sigma_{\max} \log(n)}
\]
where in (i) we use that \( \| X^* \|_{2,\infty} = \| Y^* \|_{2,\infty} \leq \sigma_{\max} \) and the incoherence condition Eq. 11, in (ii) we use that \( mp \gg k^4 \mu^2 r^2 \log^3 n \).

To establish a similar bound for \( \| X^* P_{\Omega}(E) Y^* \| \), [Chen et al. 2019] uses the Gaussianity of \( E \), which is not applicable here.

Similar to this example, we apply Lemma 1 and Lemma 2 with more fine grained analyses to address the sub-exponentiality and heterogeneity of \( E \). See Appendix A.3 for full details.

}\]
where in (i) we ignore the second-order error term $(X^dH^d - X^*)(Y^dH^d - Y^*)^\top$. Note that Theorem 2 implies that

$$\begin{align*}
X^dH^d - X^* &\approx \frac{1}{p} P_{ij}(E)Y^* (Y^*^\top Y^*)^{-1}
+ \frac{1}{p} X^* (X^*^\top X^*)^{-1} X^*^\top P_{ij}(E).
\end{align*}$$

Plug this into the decomposition of $M^d - M^*$, we have

$$\begin{align*}
M^d - M^* &\approx \frac{1}{p} P_{ij}(E)Y^* (Y^*^\top Y^*)^{-1} Y^*^\top
+ \frac{1}{p} X^* (X^*^\top X^*)^{-1} X^*^\top P_{ij}(E)
= \frac{1}{p} P_{ij}(E) V^* V^*^\top + \frac{1}{p} U^* U^*^\top P_{ij}(E).
\end{align*}$$

The results of Theorem 1 then follow from the above approximation and the use of Berry-Esseen type of inequalities. See Appendix A.5 for full details.

6. Experiments

We evaluate the results in Theorem 1 for synthetic data under multiple settings. We then compare the performances of various uncertainty quantification formulas in real data.

Synthetic Data. We generate an ensemble of instances. Each instance consists of a few parameters: (i) $(m, n)$: the size of $M^*$; (ii) $r$: the rank of $M^*$; (iii) $p$: the probability of an entry being observed; (iv) $\bar{M}^*$: the entry-wise mean of $M^*$ ($\bar{M}^* = \frac{1}{mn} \sum_{ij} M^*_{ij}$).

Given $(m, n, r, p, \bar{M}^*)$, we follow the typical procedures of generating random non-negative low-rank matrices in [Cemgil 2008; Farias et al. 2021]. Each instance is generated in two steps: (i) Generate $M^*$: let $U^* \in \mathbb{R}^{m \times r}, V^* \in \mathbb{R}^{n \times r}$ be random matrices with independent entries from Gamma(2,1). Set $M^* = k U^* V^*^\top$ where $k \in \mathbb{R}$ is picked such that $\frac{1}{mn} \sum_{ij} M^*_{ij} = \bar{M}^*$. (ii) Generate $P_{ij}(O)$: then $O_{ij} = \text{Poisson}(M^*_{ij})$ and entries in $\Omega$ is sampled independently with probability $p$.

We first verify the entry-wise distributional characterization $M^d_{ij} - M^*_ij \sim \mathcal{N}(0, s_{ij}^2)$ where $s_{ij}$ is specified in Eq. (7). See a demonstration of the Gaussian approximality of the empirical distribution $(M^d_{ij} - M^*_ij)/s_{ij}$ in Fig. 1. Given an instance, we compute the coverage rate (the percentage of coverage of entries) that corresponds to the 95% confidence interval, where an “coverage” of an entry $(i, j)$ occurs if

$$M^d_{ij} \in [M^*_ij - 1.96s_{ij}, M^*_ij + 1.96s_{ij}].$$

The average coverage rates under different settings are shown in Table 2. The closeness of the results (ranging from 91% - 95%) to the “true” coverage rate 95% suggests the applicability of inference based on our variance formula. The trends in Table 2 are also consistent with the intuition: the performance starts to degrade when $r$ increases, $p$ decreases, and the noise to signal ratio increases (decrease of $M^*$).

![Figure 1: Empirical distribution of $(M^d_{11} - M^*_{11})/s_{11}$ with $m = n = 300, r = 2, p = 0.6, and \bar{M}^* = 20$.](image)

| $(r, p, \bar{M}^*)$ | Coverage Rate |
|---------------------|---------------|
| $(3, 0.3, 5)$       | 0.936 (±0.003) |
| $(3, 0.3, 20)$      | 0.945 (±0.004) |
| $(3, 0.6, 5)$       | 0.947 (±0.003) |
| $(3, 0.6, 20)$      | 0.949 (±0.003) |
| $(6, 0.3, 5)$       | 0.910 (±0.002) |
| $(6, 0.3, 20)$      | 0.934 (±0.002) |
| $(6, 0.6, 5)$       | 0.934 (±0.003) |
| $(6, 0.6, 20)$      | 0.943 (±0.003) |

Table 2: Coverage rates for different $(r, p, \bar{M}^*)$ with $m = n = 500$. The empirical mean and empirical standard deviation are reported over 100 instances.

Real Data. Next, we study a real dataset consisting of daily sales for 1115 units with 942 days [Rossmann 2021]. To compare different uncertainty quantification formulas, we consider the coverage rate maximization task that aims to maximize the coverage rate given the total interval length constraint.

Coverage Rate Maximization. In particular, given a uncertainty quantification formula, suppose one can provide an interval predictor $[a_{ij}, b_{ij}]$ for each entry $(i, j)$ in a set $\Omega$. The “coverage” of $(i, j)$ occurs if $M_{ij} \in [a_{ij}, b_{ij}]$ where $M_{ij}$ is the true value of entry $(i, j)$. The task aims to maximize the coverage rate given that the total length of intervals $\sum_{(i, j) \in \Omega} b_{ij} - a_{ij}$.

---

2Here, we focus on the results of Poisson noise, where the results under the binary noise are similar in the experiments.
The empirical variance of Gaussian noise is computed by (Chen et al. 2019)
with the homogeneous Gaussian noise assumption (Theorem 2 in [Chen et al. (2019)], or by our Theorem 1, capable of addressing the heterogeneous sub-exponential noise. Note that both results in Chen et al. (2019) and our Theorem 1 predict that $M^d_{ij} \sim N(M^s_{ij}, s^2_{ij})$. With this distributional assumption, we tackle Eq. (11) by a greedy algorithm that achieves the maximal expected coverage rate with the budget constraint. Specifically, with given $\{M^d_{ij}, s_{ij}\}$, we provide the interval predictors $[a_{ij}, b_{ij}]$ by solving the following problem:

$$\begin{align*}
\text{maximize} & \quad \sum_{(i,j) \in \Omega} \mathbb{I}(M_{ij} \in [a_{ij}, b_{ij}]) \\
\text{subject to} & \quad \sum_{(i,j) \in \Omega} b_{ij} - a_{ij} \leq \alpha
\end{align*}$$

(11)

We are interested in comparing the performances of the above task using different variance predictors $s_{ij}$, either provided by Eq. (9) with the homogeneous Gaussian noise assumption (Theorem 2 in Chen et al. (2019)), or by our Theorem 1, capable of addressing the heterogeneous sub-exponential noise. Note that both results in Chen et al. (2019) and our Theorem 1 predict that $M^d_{ij} \sim N(M^s_{ij}, s^2_{ij})$. With this distributional assumption, we tackle Eq. (11) by a greedy algorithm that achieves the maximal expected coverage rate with the budget constraint. Specifically, with given $\{M^d_{ij}, s_{ij}\}$, we provide the interval predictors $[a_{ij}, b_{ij}]$ by solving the following problem:

$$\begin{align*}
\text{maximize} & \quad \sum_{(i,j) \in \Omega} \mathbb{E}_{M_{ij} \sim N(M^d_{ij}, s^2_{ij})}(M_{ij} \in [a_{ij}, b_{ij}]) \\
\text{subject to} & \quad \sum_{(i,j) \in \Omega} b_{ij} - a_{ij} \leq \alpha
\end{align*}$$

Experiment Results. In the experiment, the low-rankness of the dataset is verified and the “true” rank, as well as the “true” underlying matrix $M$, is predetermined through the spectrum of singular value decomposition.

We split the entries uniformly into a training set $\Omega$ (with probability $p$) and a test set $\tilde{\Omega}$. We use the observations in $\Omega$ to learn $M^d$ with Algorithm 1. Let $M^d = U^d \Sigma^d V^d \top$ be the SVD of $M^d$.

The empirical variance $s^2_{ij}^{\text{Gaussian}}$ for homogeneous Gaussian noise is computed by (Chen et al. 2019)

$$(s^2_{ij}^{\text{Gaussian}})^2 = \frac{\hat{\sigma}^2(\|U^d_{ik}\|^2 + \|V^d_{ij}\|^2)}{p}$$

where $\hat{\sigma}^2 := \sum_{(i,j) \in \Omega}(O_{ij} - M^s_{ij})^2/|\Omega|$ is the empirical estimator for the noise variance.

We then compute the empirical variance $s^2_{ij}^{\text{Poisson}}$ for Poisson noise

$$(s^2_{ij}^{\text{Poisson}})^2 = \frac{\sum_{i=1}^m M^d_{ij} \left( \sum_{k=1}^r U^d_{ik} U^d_{ik} \right) ^2 + \sum_{i=1}^m M^d_{il} \left( \sum_{k=1}^r V^d_{ik} V^d_{ik} \right) ^2}{p}$$

Given $M^d, s^2_{ij}^{\text{Poisson}}$ and $s^2_{ij}^{\text{Gaussian}}$, the coverage rate maximization task is evaluated in the test set $\tilde{\Omega}$. The results for various budgets $\alpha$ are reported in Fig. 2. The Poisson noise formula shows a higher coverage rate than the homogeneous Gaussian formula, as the former is more robust to addressing heterogeneous noises in sales data. This improvement tends to vanish with more presences of missing entries, which might be due to the degrading accuracy of matrix completion and variance estimation when $p$ decreases.

![Figure 2: Coverage rates for difference variance formulas corresponding to the total-interval-length budget.](image)

7. Conclusion

We solved the uncertainty quantification problem for matrix completion with heterogeneous and sub-exponential noise. The error variance of a common estimator was determined and the asymptotical normality with inference results were established. The explicit formulas for various scenarios such as Poisson noise and Binary noise were analyzed. Experimental results showed significant improvements of our new uncertainty quantification formulas over existing ones.

One exciting direction for further work is in assuming less restrictive $\Omega$. As in most of the matrix completion literature, we made the uniform sampling assumption for $\Omega$, which may not be applicable in some practical applications. The study of uncertainty quantification for matrix completion with non-uniform sampling pat-
terns is especially valuable, given the recent progress on deterministic matrix completion, e.g., (Chatterjee 2020).

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A. Proof

A.1. Leave-one-out sequences

Similar to Abbe et al. (2020), Chen et al. (2020, 2019), Ma et al. (2018), we employ the leave-one-out techniques. In particular, let \( O^{(j)} \in \mathbb{R}^{m \times n}, l = 1, 2, \ldots, m \) be

\[
O^{(j)}_{ik} := \begin{cases} 
O_{ik} & i \neq j \\
M_{ik} & i = j.
\end{cases}
\]

The \( O^{(j)} \) is obtained from \( O \) by replacing the \( j \)-th row with corresponding entries of \( M^* \). Let \( \Omega^{(j)} = \Omega \cup \{(j, k), k = 1, 2, \ldots, n\} \). Consider the observation \( P_{\Omega^{(j)}}(O^{(j)}) \), i.e., one observes the \( j \)-th row of \( M^* \), in additional to \( P_{\Omega}(O) \). Consider the non-convex objective function \( f^{(j)} \) associated with \( P_{\Omega^{(j)}}(O^{(j)}) \), denoted by

\[
f^{(j)}(X, Y) := \frac{1}{2p} \left\| P_{\Omega^{(j)}}(XY^T - O^{(j)}) \right\|_F^2 + \frac{\lambda}{2p} \| X \|_F^2 + \frac{\lambda}{2p} \| Y \|_F^2.
\]

The gradient descent procedure associated with \( f^{(j)} \), similar to Algorithm 1 is denoted below.

Algorithm 2 Gradient Descent with Leave-one-out

**Input:** \( P_{\Omega^{(j)}}(O^{(j)}) \)

1. **Spectral initialization:** \( X^{0,(j)} = U \sqrt{\Sigma}, Y^{0,(j)} = V \sqrt{\Sigma} \) where \( U \Sigma V \) is the top-\( r \) partial SVD decomposition of \( \frac{1}{p}P_{\Omega^{(j)}}(O^{(j)}) \).

2. **Gradient updates:** for \( t = 0, 1, \ldots, t_* - 1 \) do

\[
\begin{align*}
X^{t+1,(j)} &= X^{t,(j)} - \frac{\eta}{p} [P_{\Omega^{(j)}}(X^{t,(j)}Y^{t,(j)}^T - O^{(j)})Y^{t,(j)} + \lambda X^{t,(j)}] ; \\
Y^{t+1,(j)} &= Y^{t,(j)} - \frac{\eta}{p} [P_{\Omega^{(j)}}(X^{t,(j)}Y^{t,(j)}^T - O^{(j)})^TX^{t,(j)} + \lambda Y^{t,(j)}]
\end{align*}
\]

where \( \eta \) determines the learning rate.

3. **De-bias:**

\[
\begin{align*}
X^{d,(j)} &= X^{t_*(j)} \left( I_r + \frac{\lambda}{p} \left( X^{t_*(j)}^TX^{t_*(j)} \right)^{-1} \right)^{1/2} \\
Y^{d,(j)} &= Y^{t_*(j)} \left( I_r + \frac{\lambda}{p} \left( Y^{t_*(j)}^TY^{t_*(j)} \right)^{-1} \right)^{1/2}
\end{align*}
\]

Similar to the definition associated with rows, for \( j = m + 1, m + 2, \ldots, m + n \), let \( O^{(j)} \) be

\[
O^{(j)}_{ik} := \begin{cases} 
O_{ik} & k \neq j - m \\
M_{ik}^* & k = j - m.
\end{cases}
\]

The \( O^{(j)} \) is obtained from \( O \) by replacing the \( (j - m) \)-th column with corresponding entries of \( M^* \). Then \( f^{(j)}, X^{t,(j)}, Y^{t,(j)} \), for \( j = m + 1, m + 2, \ldots, m + n \), can be denoted accordingly.

A.2. Preliminaries for coarse-grained error guarantees

Before proceeding, we introduce a set of results that control the estimation error of various variables in a “coarse-grained” sense. This enables us to further provide finer bounds in the next sections. The proof of the following results are based on a reduction from centered sub-exponential random variables to centered sub-Gaussian random variables (see Lemma 6) and then an invoke of results in Chen et al. (2020) (also see Section A.2 in Chen et al. (2019)).

We assume \( n = m \) in the proof. The generalization is straightforward and we omit it for ease of presentation.
For ease of notation, when there is no ambiguity, let \((X, Y) = (X^t, Y^t), (X^{(j)}, Y^{(j)}) = (X^{t,(j)}, Y^{t,(j)})\) and 

\[
F := \begin{bmatrix} X \\ Y \end{bmatrix}, \quad F^d := \begin{bmatrix} X^d \\ Y^d \end{bmatrix}, \quad F^{(j)} := \begin{bmatrix} X^{(j)} \\ Y^{(j)} \end{bmatrix}, \quad F^{d,(j)} := \begin{bmatrix} X^{d,(j)} \\ Y^{d,(j)} \end{bmatrix}.
\]

We also denote the associated rotations accordingly below.

\[
H := \arg \min_{R \in \Omega^{d \times r}} \|FR - F^*\|_F^2, \quad H^{(j)} := \arg \min_{R \in \Omega^{d \times r}} \|F^{(j)}R - F^*\|_F^2, \quad R^{(j)} := \arg \min_{R \in \Omega^{d \times r}} \|F^{(j)}R - FH\|_F^2, \quad H^{d,(j)} := \arg \min_{R \in \Omega^{d \times r}} \|F^{d,(j)}R - F^*\|_F^2.
\]

Then, we have the following claims.

**Lemma 3.** Suppose

\[
n^2p \gg \kappa^4 \mu^2 r^2 n \log^3 n \quad \text{and} \quad L \log(n) \sqrt{\frac{n}{p}} \ll \frac{\sigma_{\min}}{\sqrt{\kappa^4 \mu r \log n}}.
\]

Suppose \(\lambda = O(L \log(n)\sqrt{np})\). With probability at least \(1 - 1/O(n^{10})\), we have the following set of results simultaneously.

1. **The bounds for \((X, Y)\).**

\[
\|FH - F^*\|_F \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n}{p}} \|X^*\|_F, \quad \|FH - F^*\| \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n}{p}} \|X^*\|, \quad \|FH - F^*\|_{2,\infty} \lesssim \frac{\kappa L \log(n)}{\sigma_{\min}} \sqrt{\frac{n \log(n)}{p}} \|F^*\|_{2,\infty}.
\]

Furthermore,

\[
\|\nabla f(X, Y)\|_F \lesssim \frac{1}{n^5} L \log(n) \sqrt{\frac{n}{p}} \sigma_{\min}, \quad \|X^T X - Y^T Y\|_F \lesssim \frac{1}{n^5} L \log(n) \sqrt{\frac{n}{p}} \sigma_{\max}, \quad \left\|\frac{1}{p} \Phi(XY^T - X^*Y^{*T}) - (XY^T - X^*Y^{*T})\right\| \lesssim L \sqrt{\frac{n \log(n)}{np}} \left(\frac{\kappa^4 \mu^2 r^2 \log^3(n)}{np}\right).
\]

2. **The bounds for debiased estimator \((X^d, Y^d)\).**

\[
\|F^d H - F^*\| \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n}{p}} \|X^*\|, \quad \|F^d H^d - F^*\| \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n}{p}} \|X^*\| \quad (15a), \quad \|F^d H^d - F^*\|_F \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n}{p}} \|X^*\|_F, \quad \|F^d H^d - F^*\|_{2,\infty} \lesssim \frac{\kappa L \log(n)}{\sigma_{\min}} \sqrt{\frac{n \log(n)}{p}} \|F^*\|_{2,\infty} (15b), \quad \|X^d X^d - Y^d Y^d\| \lesssim \frac{\kappa L \log(n)}{n^5} \sqrt{\frac{n}{p}} \sigma_{\max}.
\]
3. The bounds for the leave-one-out estimator \((X^{(j)}, Y^{(j)})\).

\[
\|F^{(j)} R^{(j)} - FH\|_F \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n \log(n)}{p}} \|F^*\|_{2,\infty}
\]

\[
\|F^{(j)} H^{(j)} - FH\|_F \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n \log(n)}{p}} \|F^*\|_{2,\infty}
\]

\[
\|F^{(j)} H^{(j)} - F^*\|_{2,\infty} \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n \log(n)}{p}} \|F^*\|_{2,\infty}
\]

\[
\|F^{(j)} R^{(j)} - F^*\|_{2,\infty} \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n \log(n)}{p}} \|F^*\|_{2,\infty}.
\]

4. The bounds for the leave-one-out version of the debiased estimator \((X^{d,(j)}, Y^{d,(j)})\).

\[
\|F^{d,(j)} H^{d,(j)} - F^*\|_{2,\infty} \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n \log(n)}{p}} \|F^*\|_{2,\infty}
\]

\[
\|F^{d,(j)} H^{d,(j)} - F^d\|_{2,\infty} \lesssim \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{n \log(n)}{p}} \|F^*\|_{2,\infty}.
\]

5. Additional bounds.

\[
\sigma_r(F) \geq 0.5 \sqrt{\sigma_{\min}}, \quad \|F\| \leq 2 \|X^*\|, \quad \|F\|_F \leq 2 \|X^*\|_F, \quad \|F\|_{2,\infty} \leq 2 \|F^*\|_{2,\infty}
\]

\[
\sigma_r(F^d) \geq 0.5 \sqrt{\sigma_{\min}}, \quad \|F^d\| \leq 2 \|X^*\|, \quad \|F^d\|_F \leq 2 \|X^*\|_F, \quad \|F^d\|_{2,\infty} \leq 2 \|F^*\|_{2,\infty}
\]

\[
\sigma_r(F^{(j)}) \geq 0.5 \sqrt{\sigma_{\min}}, \quad \|F^{(j)}\| \leq 2 \|X^*\|, \quad \|F^{(j)}\|_F \leq 2 \|X^*\|_F, \quad \|F^{(j)}\|_{2,\infty} \leq 2 \|F^*\|_{2,\infty}
\]

\[
\sigma_r(F^{d,(j)}) \geq 0.5 \sqrt{\sigma_{\min}}, \quad \|F^{d,(j)}\| \leq 2 \|X^*\|, \quad \|F^{d,(j)}\|_F \leq 2 \|X^*\|_F, \quad \|F^{d,(j)}\|_{2,\infty} \leq 2 \|F^*\|_{2,\infty}
\]

**Proof.** The above bounds for sub-Gaussian noise have been shown in Chen et al. (2019). To generalize these bounds to sub-exponential noise, we observe that for any sub-exponential zero-mean random variable \(X\) with \(\|X\|_{\psi_1} \leq L\), one can construct a sub-Gaussian zero-mean random variable \(Y\) with \(\|Y\|_{\psi_2} \lesssim L \log(n)\), and \(Y\) is extremely close to \(X\) where \(P(X \neq Y) = 1/poly(n)\) (see Lemma 3). Then, for a noise matrix \(E \in \mathbb{R}^{m \times n}\) with independent sub-exponential entries \(\|E_{ij}\|_{\psi_1} \leq L\), we can construct \(E' \in \mathbb{R}^{m \times n}\) with independent sub-Gaussian entries \(\|E'_{ij}\|_{\psi_2} \lesssim L \log(n)\) and \(P(E \neq E') = 1/poly(n)\). Then one can employ the results in Section A.2 of Chen et al. (2019) for \(E'\) (with an \(O(\log(n))\) increase of sub-Gaussian norm) to provide bounds for \(E\), which completes the proof.

\[\square\]

A.3. Characterization of low-rank factors and proof of Theorem 2

We will prove Theorem 2 based on a decomposition of \(X^d H^d - X^*\) and \(Y^d H^d - Y^*\). Due to the symmetry, we focus on \(X^d H^d - X^*\). From Chen et al. (2019), we have the following characterization by direct algebra.

**Lemma 4** (Eq. (5.11) Chen et al. (2019)). Let \(\tilde{X}^d = X^d H^d, \tilde{Y}^d = Y^d H^d\). Then

\[
X^d H^d - X^* = \frac{1}{p} P_{\Omega}(E) Y^* (Y^*^T Y^*)^{-1} + \Phi_X
\]
Lemma 5. prove the following lemma.

Then, to show Theorem 2, it boils down to bound $\|\Phi_X\|_{2,\infty}$. Since $\Phi_X = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$, it is sufficient to prove the following lemma.

**Lemma 5.** With probability $1 - O(n^{-10})$, we have

$$
\|\Phi_1\|_{2,\infty} \lesssim \frac{L \log(n)}{\sqrt{p \sigma_{\min}}} \frac{L \log(n) \sqrt{\kappa^3 \mu \rho n \log n}}{p} \quad (21)
$$

$$
\|\Phi_2\|_{2,\infty} \lesssim \frac{L \log(n)}{\sqrt{p \sigma_{\min}}} \left( L \log(n) \frac{\sqrt{\kappa^3 \mu \rho n \log n}}{p} + \sqrt{\frac{\kappa^7 \mu^3 \rho^3 \log n}{np}} \right) \quad (22)
$$

$$
\|\Phi_3\|_{2,\infty} \lesssim \frac{L \log(n) \sqrt{\kappa^5 \mu^3 \rho^3 \log^2 n}}{np} \quad (23)
$$

$$
\|\Phi_4\|_{2,\infty} \lesssim \frac{L \log(n)}{\sqrt{p \sigma_{\min}}} \frac{1}{n^5} \quad (24)
$$

### A.4. Proof of Lemma 5

Equipped with Lemma 3, the proof of Lemma 5 follows a similar framework for the proof of Lemma 5 to Lemma 8 in (Chen et al. 2019), where the differences are highlighted below. We extend the homogeneous Gaussian noise in (Chen et al. 2019) to the heteroskedastic and subexponential noise, based on the results from sub-exponential matrix completion (McRae and Davenport 2019) and a subexponential variant of Matrix Bernstein inequality (see Lemma 3).

#### A.4.1. Proof of Eq. (21)

By triangle inequality, for any row of $\Phi_1$, we have

$$
\|e_j^\top \Phi_1\|_2 \leq \left\| e_j^\top \frac{1}{p} P_{\Omega}(E) \left[ \hat{Y}^{d,(j)(Y^T Y^*)^{-1}} - Y^* (Y^* T Y^*)^{-1} \right] \right\|_{\alpha_1} + \left\| e_j^\top \frac{1}{p} P_{\Omega}(E) \left[ \tilde{Y}^{d,(j)(Y^T Y^*)^{-1}} - \hat{Y}^{d,(j)(Y^T Y^*)^{-1}} \right] \right\|_{\alpha_2}
$$

where $\tilde{Y}^{d,(j)} = Y^{d,(j)} H^{d,(j)}$, $j = 1, 2, \ldots, n$.

1. Next, we control $\alpha_1$. Let $\Delta(j) = \tilde{Y}^{d,(j)(Y^T Y^*)^{-1}} - Y^* (Y^* T Y^*)^{-1}$ ($\Delta(j) \in R^d \times r$). Then

$$
\alpha_1 = \left\| e_j^\top \frac{1}{p} P_{\Omega}(E) \Delta(j) \right\| _2.
$$
Furthermore, we have the following claim.

**Claim 1.** With probability $1 - O(n^{-11})$,

$$
\| \Delta^{(j)} \| \lesssim \frac{1}{\sigma_{\min}} \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{k^3 n}{p}}
$$

$$
\| \Delta^{(j)} \|_{2,\infty} \lesssim \frac{1}{\sigma_{\min}} \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{k^5 \mu r \log(n)}{p}}.
$$

**Proof.** The result can be derived from Lemma 3 (see the similar derivation for Claim 2 in Chen et al. [2019]).

In order to control $\alpha_1$, let $z = e_j^T \frac{1}{p} P_\Omega(E) \Delta^{(j)}$. Hence $\| z \|_2 = \alpha_1$. Note that in Chen et al. [2019], $z$ is a Gaussian random vector due to the Gaussianity of $E$ (conditioned on $\Omega$ and $\Delta^{(j)}$), therefore $\| z \|_2$ can be easily bounded. Instead, here we need to fine-tune the bound for $z$ using the subexponential property of $E$. In particular, for $l \in [r]$, we have

$$
z_l = \frac{1}{p} \sum_{k=1}^n \delta_{j,k} E_{j,k} \Delta_{k,l}^{(j)}
$$

where $\delta_{j,k} \in \{0,1\}$ indicates whether $(j,k) \in \Omega$. Note that $E_{j,k}$ and $\delta_{j,k}$ is independent from $\Delta^{(j)}$ by the construction of $Y^{d,(j)}$. Next, construct $A_k \in \mathbb{R}^r$, $k = 1,2,\ldots,n$ where $A_{k,l} = E_{j,k} \delta_{j,k} \Delta_{k,l}^{(j)}$ (note that $z = \frac{1}{p} \sum_{k=1}^n A_k$).

Then, condition on $\Delta^{(j)}$, note that $\delta_{j,k}$ and $E_{j,k}$, $k = 1,2,\ldots,n$, are independent Bernoulli and sub-exponential random variables respectively. We are able to apply a variant of Matrix Bernstein inequality (Lemma 1) to $A_k$, $k = 1,2,\ldots,n$. In particular, note that

$$
V := \max_{1 \leq k \leq n} \left\| A_k \right\|_{\psi_1} \leq L \left\| \Delta^{(j)} \right\|_{2,\infty}
$$

$$
B := \max_{1 \leq k \leq n} \left\| A_k \right\| \lesssim L \left\| \Delta^{(j)} \right\|_{2,\infty}
$$

Applying Lemma 1 to $A_k$, with probability $1 - O(n^{-11})$, we have

$$
\left\| \sum_{k=1}^n A_k \right\| \lesssim \sqrt{V \log(n)} + B \log^2(n)
$$

$$
\lesssim L \sqrt{1} \left\| \Delta^{(j)} \right\| \sqrt{\log(n)} + L \left\| \Delta^{(j)} \right\|_{2,\infty} \log^2(n)
$$

$$
\lesssim L \sqrt{1} \frac{1}{\sigma_{\min}} \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{k^3 n \log(n)}{p}} + L \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{k^5 \mu r \log(n)}{p}} \log^2(n)
$$

$$
\lesssim L \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{k^5 \mu r \log(n)}{p}} \log^2(n)
$$

where we have utilized $n^2 p \gg k^4 \mu^2 r^2 n \log^3 n$ in the last inequality. This provides a desired bound for $\alpha_1$.

$$
\alpha_1 = \| z \|_2 = \frac{1}{p} \left\| \sum_{k=1}^n A_k \right\| \lesssim \frac{L \log(n)}{p} \sqrt{\frac{k^3 \mu r \log^3(n)}{p}}
$$

(25)

2. Next, we control $\alpha_2$. Note that

$$
\alpha_2 = \left\| \frac{1}{p} P_\Omega(E) \left[ \tilde{Y}^{d,(j)} \tilde{Y}^{d,(j)} - \tilde{Y}^{d,(j)} \tilde{Y}^{d,(j)} \right] \right\|_2
$$

$$
\leq \frac{1}{p} \left\| P_\Omega(E) \right\| \left\| \tilde{Y}^{d,(j)} \tilde{Y}^{d,(j)} - \tilde{Y}^{d,(j)} \tilde{Y}^{d,(j)} \right\|_2
$$
Different from Chen et al. (2019), we need to bound $\|P_\Omega(E)\|$ where $E$ is a sub-exponential instead of Gaussian matrix. We obtain the bound by invoking a recent result on sub-exponential matrices from McRae and Davenport (2019). In particular, by Lemma 2 we have, with probability $1 - O(n^{-1})$,

$$\frac{1}{p} \|P_\Omega(E)\| \preceq L \sqrt{n}.$$

Then, we have

\[
\alpha_2 \leq L \sqrt{\frac{n}{p}} \left\| \tilde{Y}^d (\tilde{Y}^d)^{-1} - \tilde{Y}^d (\tilde{Y}^d)^{-1} \right\|
\]

\[
\overset{(i)}{\leq} L \sqrt{\frac{n}{p}} \max \left( \|\tilde{Y}^d (\tilde{Y}^d)^{-1}\|^2, \|\tilde{Y}^d (\tilde{Y}^d)^{-1}\| \right)
\]

\[
\overset{(ii)}{\leq} L \sqrt{\frac{n}{p}} \frac{1}{\sigma_{\min}} \|\tilde{Y}^d - \tilde{Y}^d\|
\]

\[
\overset{(iii)}{\leq} L \sqrt{\frac{n}{p}} \frac{1}{\sigma_{\min}} \mu_r \sigma(n) \sqrt{\frac{\mu_r \sigma_{\max}}{\sigma_{\min}}}
\]

where, similarly as Chen et al. (2019), (i) is due to Lemma 10 (the perturbation bound for pseudo-inverse), (ii) is due to Eqs. 18a and 18b, and (iii) is due to Eq. 17a and $\|F^*\|_{2,\infty} \leq \sqrt{\frac{\mu_r \sigma_{\max}}{n}}$.

Then, combining the control of $\alpha_1$ and $\alpha_2$, we have

$$\left\| e_j^T \Phi_1 \right\|_2 \leq \alpha_1 + \alpha_2 \overset{\text{(ii)}}{\leq} \frac{1}{\sigma_{\min}} \sqrt{\kappa_1 \mu_r n} + L \sqrt{\frac{n}{p}} \frac{1}{\sigma_{\min}} \sqrt{\kappa_1 \mu_r \sigma(n)} \frac{\kappa_1 \mu_r \sigma(n)}{p}
$$

$$\overset{\text{(iii)}}{\leq} \frac{\kappa_1 \mu_r \sigma(n)}{\sqrt{p} \sigma_{\min}} \frac{\kappa_1 \mu_r \sigma(n)}{p}.$$

This establishes the bound Eq. (21) by taking the maximum over $j \in [n]$ and the union bound.

### A.4.2. Proof of Eq. (22)

Note that $I_r = \tilde{Y}^d (\tilde{Y}^d)^{-1}$ and $\tilde{Y}^d$.

We have

\[
\left\| e_j^T \Phi_2 \right\| = \left\| e_j^T X^* (Y^d)^T - \tilde{Y}^d \right\| \overset{(i)}{\leq} \left\| X^* \right\|_{2,\infty} \|Y^d\| \left\| (\tilde{Y}^d)^{-1} \right\|
\]\n
$$\overset{(ii)}{\leq} \sqrt{\frac{\kappa_1 \mu_r}{n}} \frac{1}{\sigma_{\min}} \left\| (Y^d)^\top \tilde{Y}^d \right\|$$

where (i) is due to $\|X^*\|_{2,\infty} \leq \sqrt{\frac{\mu_r \sigma_{\max}}{n}}$ and Eq. 18a. To control $\|\tilde{Y}^d Y^d\|$, we have the following claim.

**Claim 2.**

$$\left\| (Y^d)^\top \tilde{Y}^d \right\| \overset{(i)}{\leq} \frac{1}{\sigma_{\min}} \left\| H^d \right\| \overset{(ii)}{\leq} \frac{1}{\sigma_{\min}} \frac{1}{p} \left\| P_\Omega(E) Y^* \right\| + \frac{1}{\sigma_{\min}} \alpha_2 + \kappa_3 + \frac{1}{\sigma_{\min}} \alpha_4$$

(29)
where

\[
\alpha_2 \lessapprox \sigma_{\text{max}} L \log(n) \sqrt{\frac{n}{p}} \sqrt{\frac{\kappa^4 \mu^2 r^2 \log(n)}{np}}
\]

\[
\alpha_3 \lessapprox \left( \frac{L \log(n)}{\sigma_{\text{min}}} \right) \sqrt{\frac{n}{p}} \sigma_{\text{max}}
\]

\[
\alpha_4 \lessapprox \frac{\kappa}{n^3} L \log(n) \sqrt{\frac{n}{p}} \sigma_{\text{max}}^2.
\]

**Proof.** This can be derived based on Lemma 8 following a similar derivation as Section D.3 in (Chen et al., 2019),

What remains to control is \(\|X^T \frac{1}{p} P_{\Omega}(E)Y^*\|\), where the subexponential property of \(E\) needs to be addressed. Note that

\[
\left\| X^T \frac{1}{p} P_{\Omega}(E)Y^* \right\| \leq \left\| X^T \frac{1}{p} P_{\Omega}(E)Y^* \right\| + \left\| X - X^* \right\| \left\| \frac{1}{p} P_{\Omega}(E) \right\| \left\| Y^* \right\|
\]

\[
\lessapprox \left( X^T \frac{1}{p} P_{\Omega}(E)Y^* \right) \lessapprox \left( \frac{L \log(n)}{\sigma_{\text{min}}} \right) \sqrt{\frac{n}{p}} \sigma_{\text{max}} \left\| \frac{1}{p} P_{\Omega}(E) \right\| \lessapprox \left( X^T \frac{1}{p} P_{\Omega}(E)Y^* \right) + L^2 \log(n) \frac{\kappa}{p^2}.
\]

where (i) is due to Eq. (29) and \(\|X^*\| = \sqrt{\sigma_{\text{max}}}, \|Y^*\| = \sqrt{\sigma_{\text{max}}^2}\) and (ii) is due to Lemma 2.

In order to bound \(\|X^T \frac{1}{p} P_{\Omega}(E)Y^*\|\), we will invoke the subexponential version of Matrix Bernstein inequality (Lemma 1). To begin, note that

\[
X^T \frac{1}{p} P_{\Omega}(E)Y^* = \frac{1}{p} \sum_{k=1}^{n} \sum_{l=1}^{n} X^*_k, Y^*_l E_{k,l}
\]

where \(E_{k,l} \sim \text{Ber}(p)\) indicates whether \((k, l) \in \Omega\). Then, let \(A_{k,l} := X^*_k, Y^*_l E_{k,l}\) for \(k \in [n], l \in [n] \) \((A_{k,l} \in \mathbb{R}^{r \times r})\). Note that

\[
V := \max \left( \left\| \sum_{k,l} E[A_{k,l}, A^T_{k,l}] \right\|, \left\| \sum_{k,l} E[A^T_{k,l}, A_{k,l}] \right\| \right) \leq pL^2 \|X^*\|_F^2 \|Y^*\|_F^2
\]

\[
B := \max_{k,l} \|A_{k,l}\|_F \leq L \|X^*\|_{2,\infty} \|Y^*\|_{2,\infty}
\]

Then apply Lemma 1 with probability \(1 - O(n^{-1})\), we have

\[
\left\| \sum_{k,l} A_{k,l} \right\| \lessapprox \sqrt{V \log(n)} + B \log^2(n)
\]

\[
\lessapprox \sqrt{pL \sigma_{\text{max}} r \sqrt{L \log(n)}} + \frac{\mu \log^3(n)}{n}
\]

(i)

\[
\lessapprox \sqrt{pL \sigma_{\text{max}} r \sqrt{L \log(n)}}
\]

where in (i) we use that \(n^2 p \gg \kappa^4 \mu^2 r^2 \log^3(n)\). This then implies

\[
\left\| X^T \frac{1}{p} P_{\Omega}(E)Y^* \right\| \leq \frac{1}{p} \left\| \sum_{k,l} A_{k,l} \right\| \lessapprox \frac{n}{\sqrt{p}} L \sigma_{\text{max}} \sqrt{\log(n)}.
\]
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By Theorem 2, we have
\[
\| (Y^* - \tilde{Y}^d) \tilde{Y}^d \| \leq \frac{1}{\sigma_{\min}} \alpha_1 + \frac{1}{\sigma_{\min}} \alpha_2 + \kappa \alpha_3 + \frac{1}{\sigma_{\min}} \alpha_4
\]
\[
\leq \frac{1}{\sigma_{\min}} \left( \frac{r}{\sqrt{p}} L^2 \sigma_{\max} \sqrt{\log(n)} + L^2 \log(n) \frac{n^2}{p} \right)
\]
\[
+ \frac{1}{\sigma_{\min}} \sigma_{\max} L \log(n) \sqrt{\frac{n}{p}} \sqrt{\kappa^4 \mu^2 r^2 \log(n)}
\]
\[
+ \kappa \left( \frac{L \log(n)}{\sigma_{\min}} \frac{n}{p} \right)^2 \sigma_{\max} + \frac{1}{\sigma_{\min}} \kappa \frac{L \log(n)}{\sigma_{\min}} \frac{n}{p} \sqrt{\frac{n}{p}} \sigma_{\max}^2
\]
\[
\approx \kappa \left( \frac{L \log(n)}{\sigma_{\min}} \frac{n}{p} \right)^2 \sigma_{\max} + \kappa \left( \frac{L \log(n)}{\sigma_{\min}} \frac{n}{p} \right) \sqrt{\frac{n}{p}} \sigma_{\max}^2
\].

Plug this back into Eq. (23) and take the maximum over \( 1 \leq j \leq n \), we finish the proof.
\[
\| \Phi_2 \|_{2, \infty} \leq \frac{1}{\sqrt{\sigma_{\min}}} \left( \kappa \left( \frac{L \log(n)}{\sigma_{\min}} \frac{n}{p} \right)^2 \sigma_{\max} + \kappa \left( \frac{L \log(n)}{\sigma_{\min}} \frac{n}{p} \right) \sqrt{\frac{n}{p}} \sigma_{\max}^2 \right)
\]
\[
\lesssim L \log(n) \left( \kappa \left( \frac{L \log(n)}{\sigma_{\min}} \frac{n}{p} \right)^2 \sigma_{\max} + \kappa \left( \frac{L \log(n)}{\sigma_{\min}} \frac{n}{p} \right) \sqrt{\frac{n}{p}} \sigma_{\max}^2 \right).
\]

The proof of Eq. (23) and Eq. (24) follow the similar derivations of section D.4 and D.5 in Chen et al. (2019). We omit the proof for brevity.

A.5. Proof of Theorem 1

Next, we provide a proof of Theorem 1 based on Theorem 2. Note that
\[
M_i^d - M_i^d = (X^d H^d H^d T Y^d T)_{ij} - (X^* Y^* T)_{ij}
\]
\[
= e_i^T X^* (Y^d H^d - Y^*)^T e_j + e_i^T (X^d H^d - X^*) Y^* T e_j + e_i^T (X^d H^d - X^*) (Y^d H^d - Y^*)^T e_j
\]
\[
= \frac{1}{p} \sum_{j=1}^n \sum_{k=1}^r \delta_{ij} E_{ij} \left( \sum_{k=1}^r V_{ik} V_{jk}^* \right) + \frac{1}{p} \sum_{j=1}^n \sum_{k=1}^r \delta_{ij} E_{ij} \left( \sum_{k=1}^r U_{ik} U_{jk}^* \right) + \delta_{ij} E_{ij} \left( \sum_{k=1}^r U_{ik} U_{jk}^* \right)
\]
\[
+ (\delta_{ij} E_{ij} - \delta_{ij} E_{ij}^*) \left( \sum_{k=1}^r U_{ik} U_{jk}^* \right) + e_i^T \Phi_X Y^* T e_j + e_i^T X^* \Phi_Y e_j + e_i^T \Delta_X \Delta_Y e_j
\]
where in (i) we use the notation in Theorem 2 and \( \Delta_X := X^d H^d - X^* \) and \( \Delta_Y := Y^d H^d - Y^* \), in (ii) we introduce the exogenous variables \( \delta_{ij} \sim Ber(p) \), \( E_{ij} \overset{d}{=} E_{ij} \).

By Theorem 2 we have
\[
| e_i^T \Phi_X Y^* T e_j | \leq \| \Phi_X \|_{2, \infty} \| Y^* \|
\]
\[
\lesssim \| Y^* \| \sqrt{\kappa \left( \frac{L \log(n)}{\sigma_{\min}} \frac{n}{p} \right) \left( \frac{L \log(n)}{\sigma_{\min}} \frac{n}{p} \right) \sqrt{\frac{n}{p}} \sigma_{\max}^2 \}.\]
By symmetry, we can obtain the similar bound for \( e_i^\top X^* \Phi^\top e_j \). Also note that, by Eq. (15b), we have

\[
|e_i^\top \Delta X \Delta Y^e_j| \leq \|\Delta X\|_{2,\infty} \|\Delta Y\|_{2,\infty}
\]

\[
\lesssim \left( \frac{L \log(n)}{\kappa \log(n)} \right)^{\frac{1}{2}} \left( \frac{n \log(n)}{p} \|F^*\|_{2,\infty} \right)^{\frac{2}{2}}
\]

\[
= \left( \frac{L \log(n)}{\sqrt{\sigma_{\min}}} \right)^{\frac{1}{2}} \left( \frac{\kappa^3 \mu r \log(n)}{p} \right)
\]

This then implies, with probability \( 1 - O(n^{-10}) \),

\[
|\epsilon_{ij}^{(2)}| \lesssim \left( \|V_{j, i}\| + \|U_{i, i}\| \right) \frac{L \log(n)}{\sqrt{p}} \left( \frac{L \log(n)}{\sigma_{\min}} \sqrt{\frac{\kappa^{10} \mu r n \log(n)}{p}} + \sqrt{\frac{\kappa^3 \mu r^3 \log^2(n)}{np}} \right)
\]

\[+ \left( \frac{L \log(n)}{\sqrt{\sigma_{\min}}} \right)^{\frac{1}{2}} \left( \frac{\kappa^3 \mu r \log(n)}{p} \right)^{\frac{1}{2}} + \frac{L \log(n) \mu r}{n}
\]

\[
\lesssim \frac{L^2 \log^3(n) \mu r^5}{p \sigma_{\min}} + \frac{L \mu r^2 \log^2(n) \kappa^4}{pn}
\]

\[D_{ij}\]

Next, we analyze \( \epsilon_{ij}^{(1)} \). Note that one can view \( \epsilon_{ij}^{(1)} = \sum_{l=1}^{2n} x_l \) as the sum of independent zero-mean random variables \( x_l \). Hence, one can apply the Berry-Esseen type of bounds. Consider \( s_{ij}^2 = \sum_l E[x_l^2], \rho = \sum_l E[x_l^3] \). Then

\[
s_{ij}^2 := \frac{1}{p} \sum_{l=1}^{n} \sigma_{il}^2 \left( \sum_{k=1}^{r} V_{il}^* V_{jk}^* \right)^2 + \left( \frac{1}{p} \sum_{l=1}^{n} \sigma_{il}^2 \left( \sum_{k=1}^{r} U_{ik}^* U_{jk}^* \right)^2 \right)
\]

\[\rho := \frac{1}{p} \sum_{l=1}^{n} E \left( E_{il}^3 \right) \left( \sum_{k=1}^{r} V_{il}^* V_{jk}^* \right)^3 + \left( \frac{1}{p} \sum_{l=1}^{n} E \left( E_{il}^3 \right) \left( \sum_{k=1}^{r} U_{ik}^* U_{jk}^* \right)^3 \right)
\]

\[
\lesssim \frac{L^2}{p} \frac{n \left( \frac{\mu r}{n} \right)^3}{n^3}.
\]

By Esseen (1942), we have

\[
\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{\epsilon_{ij}^{(1)}}{s_{ij}} \leq t \right) - \Phi(t) \right| \lesssim s_{ij}^{-3} \rho \lesssim s_{ij}^{-3} \frac{L^2 \mu r^3}{n^2 p}.
\]  

(32)

Next, for any \( t \in \mathbb{R} \), we want to bound \( \mathbb{P} \left( \frac{\epsilon_{ij}^{(1)} + D_{ij}}{s_{ij}} \leq t \right) \). It is easy to check that

\[
\mathbb{P} \left( \frac{\epsilon_{ij}^{(1)} + D_{ij}}{s_{ij}} \leq t \right) - \mathbb{P} \left( |\epsilon_{ij}^{(2)}| > D_{ij} \right) \leq \mathbb{P} \left( \frac{\epsilon_{ij}^{(1)} + \epsilon_{ij}^{(2)}}{s_{ij}} \leq t \right) \leq \mathbb{P} \left( \frac{\epsilon_{ij}^{(1)} - D_{ij}}{s_{ij}} \leq t \right) + \mathbb{P} \left( |\epsilon_{ij}^{(2)}| > D_{ij} \right).
\]
Hence,

\[
\begin{align*}
|P\left\{\frac{\epsilon_{ij}^{(1)} + \epsilon_{ij}^{(2)}}{s_{ij}} - \Phi(t)\right\}| & \leq |P\left\{\frac{\epsilon_{ij}^{(1)} + D_{ij}}{s_{ij}} - \Phi(t)\right\}| + O(1/n^{10}) \\
& \leq |P\left\{\frac{\epsilon_{ij}^{(1)} + D_{ij}}{s_{ij}} - \Phi(t)\right\}| + |\Phi(t + D_{ij}/s_{ij}) - \Phi(t - D_{ij}/s_{ij})| + O(1/n^{10}) \\
& \lessapprox s_{ij}^{-3}L_4^2\mu r^3 + |\Phi(t + D_{ij}/s_{ij}) - \Phi(t - D_{ij}/s_{ij})| + O(1/n^{10}) \\
& \lessapprox s_{ij}^{-3}L_4^2\mu r^3 + D_{ij}/s_{ij} + O(1/n^{10})
\end{align*}
\]

where (i) is due to Eq. (32), and (ii) is due to the property of the standard Gaussian distribution \(\Phi(\cdot)\). This completes the proof.

Note that from the Bernstein inequality, one can also verify that

\[
|\epsilon_{ij}^{(1)}| \lessapprox \frac{\mu r L \log(\sqrt{n})}{\sqrt{p}}
\]

with probability \(1 - O(n^{-c})\). This also implies an entry-wise error bound for \(M^d - M^*\).

\[
|M_{ij}^d - M_{ij}^*| \leq |\epsilon_{ij}^{(1)}| + |\epsilon_{ij}^{(2)}| \leq \frac{\mu r L \log(\sqrt{n})}{\sqrt{p}}
\]

\[
\lessapprox \frac{L_4^2\mu r^3(n)\mu_{\min}^5}{n^{2p}} + \frac{L_4^2\mu^2r^2\log^2(n)\kappa^4}{pn}
\]

\[
\lessapprox \frac{\mu r L \log(\sqrt{n})}{\sqrt{p}}
\]

A.6. Proof of Corollary [1]

To begin, since \(\sigma_{il} = \Theta(1)\), one can verify that

\[
s_{ij}^2 \gtrapprox \frac{\|U_{il}^*\|^2 + \|V_{lj}^*\|^2}{p}.
\]

We want to provide a bound for \(|s_{ij}^2 - \hat{s}_{ij}^2|\). To begin, with probability \(1 - O(n^{-c})\), consider

\[
\left|\sum_{l \in [n]} \frac{1}{p} (E_{il}^2 - \sigma_{il}^2p) \left(\sum_{k=1}^r V_{ik}^* V_{lk}^*\right)^2\right| \lessapprox \frac{1}{p} \left|\sum_{l \in [n]} L_4^4 p \|V_{lj}^*\|^2 \frac{\mu_{\min}^2}{n^2} \log(n)\right|
\]

\[
\lessapprox L^2 \frac{1}{\sqrt{p}} \|V_{lj}^*\|^2 \frac{\mu_{\min}}{\sqrt{n}} \log(n)
\]

\[
\lessapprox s_{ij}^2 \frac{L^2 \mu_{\min}}{\sqrt{n}} \log(n).
\]
by the independence of \( E, \delta \) and concentration bounds. Next, with probability \( 1 - O(n^{-c}) \), note that

\[
\left| \sum_{t \in [n]} \frac{1}{p} (E^2_{d|t} \delta_{it} - \hat{E}^2_{d|t} \delta_{it}) \left( \sum_{k=1}^{r} V_{ik}^d V_{ik}^* \right) \right|^2 \lesssim \frac{np \log(n) \mu \gamma L^2 \log^5(n) \mu \gamma}{\sqrt{np} n} \| V_j \|^2
\]

where (i) is due to Eq. (35). Then, consider,

\[
\left| \sum_{t \in [n]} \frac{1}{p} (E^2_{d|t} \delta_{it} - \hat{E}^2_{d|t} \delta_{it}) \left( \sum_{k=1}^{r} V_{jk}^d V_{ik}^* - V_{jk}^* V_{ik}^d \right) \right|^2 \lesssim np \log(n) \mu \gamma L^2 \log^5(n) \mu \gamma \left\| U^d - U^* \right\|_{2,\infty} \| U^* \|_{2,\infty}
\]

\[
\lesssim n \log^2(n) L^2 \left( \frac{\kappa L \log(n) \mu r}{\sigma_{\min}} \right) \frac{\sqrt{\log(n) \mu r}}{\sqrt{np} n} \frac{\sqrt{lp}}{\sqrt{lp}} \lesssim s_{ij} \lesssim \frac{s_{ij}^2}{\log^3 n}.
\]

The above together implies that

\[
|s_{ij}^2 - \hat{s}_{ij}^2| \lesssim \frac{s_{ij}^2}{\log^3 n}.
\]

Let \( \Delta = \frac{M_{ij} - M_{ij}^*}{s_{ij}} - \frac{M_{ij} - M_{ij}^*}{s_{ij}} \). Then, we can obtain

\[
|\Delta| \lesssim \left| \frac{M_{ij} - M_{ij}^*}{s_{ij}} \right| \frac{s_{ij}^2 - \hat{s}_{ij}^2}{s_{ij}^2} = o(1).
\]

We the finish the proof by the following.

\[
P \left( \frac{M_{ij}^d - M_{ij}^*}{s_{ij}} \leq t \right) = P \left( \frac{M_{ij}^d - M_{ij}^*}{s_{ij}} \leq t - \Delta \right)
\]

\[
= \Phi(t - \Delta) + o(1)
\]

\[
(\ ^{(i)} \ ) \Phi(t) + \Delta' + o(1)
\]

\[
= \Phi(t) + o(1)
\]

where in (i), \( |\Delta'| \leq |\Delta| \).

**B. Technical Lemmas**

**Lemma 6.** Suppose \( X \) is a zero-mean subexponential random variable with \( \| X \|_{\psi_1} \leq L \). Then there exists a zero-mean bounded random variable \( Y \) such that

\[
P(X \neq Y) \leq c_1/n^{10}
\]

and \( |Y| \leq c_2 L \log n \) where \( c_1, c_2 \) are two constants.
Proof. Our idea for proving this lemma has two steps: (i) Truncation: we construct a random variable \( Y' = X \cdot 1 \{ |X| \leq k_1 \} \) for some \( k_1 \) and show that \( E[Y'] \approx 0 \). (ii) Slight modification: we then construct \( Y \) from \( Y' \) by slightly modifying the distribution to guarantee \( E[Y] = 0 \).

Without loss of generality, suppose \( X \) is continuous. Let the density function of \( X \) be \( f_X \). By continuity, there exists \( k_1 \) such that

\[
P(|X| > k_1) = \frac{2}{n^{10}}.
\]  

(36)

Let \( Y' = X \cdot 1 \{ X \leq k_1 \} \). We aim to provide a bound on \( E[Y'] \).

Since \( X \) is centered and subexponential, \( P(|X| > t) \leq 2 \exp(-tC/L) \) for some constant \( C \). Therefore,

\[
\frac{2}{n^{10}} \leq 2 \exp(-k_1 C/L) \implies k_1 \leq \frac{10L \log(n)}{C}.
\]

Let \( k_2 = \frac{10L \log(n)}{C} \). Note that

\[
E[X] = \int_{-\infty}^{\infty} xf_X(x)dx
\]

\[
= \int_{0}^{\infty} x (f_X(x) - f_X(-x)) dx
\]

\[
= \int_{0}^{k_1} x (f_X(x) - f_X(-x)) dx + \int_{k_1}^{k_2} x (f_X(x) - f_X(-x)) dx + \int_{k_2}^{\infty} x (f_X(x) - f_X(-x)) dx.
\]

Note that \( E[Y'] = \int_{0}^{k_1} x (f_X(x) - f_X(-x)) dx \). Since \( E[X] = 0 \), then

\[
|E[Y']| \leq \left| \int_{k_1}^{k_2} x (f_X(x) - f_X(-x)) dx \right| + \left| \int_{k_2}^{\infty} x (f_X(x) - f_X(-x)) dx \right|
\]

(37)

\[
\leq k_2 \left( \int_{k_1}^{k_2} (f_X(x) + f_X(-x)) dx \right) + \left( \int_{k_2}^{\infty} x (f_X(x) - f_X(-x)) dx \right)
\]

(38)

\[
\leq k_2 \left( \int_{k_1}^{\infty} (f_X(x) + f_X(-x)) dx \right) + \left( \int_{k_2}^{\infty} (x - k_2) (f_X(x) + f_X(-x)) dx \right)
\]

(39)

\[
\leq k_2 \mathbb{P}(|X| > k_1) + \int_{k_2}^{\infty} (x - k_2) (f_X(x) + f_X(-x)) dx.
\]

(40)

We need to use the subexponential property to bound \( \int_{k_2}^{\infty} (x - k_2) (f_X(x) + f_X(-x)) dx \). In particular,

\[
\int_{k_2}^{\infty} (x - k_2) (f_X(x) + f_X(-x)) dx = \int_{x=k_2}^{\infty} (f_X(x) + f_X(-x)) \left( \int_{t=0}^{\infty} 1 \{ t + k_2 \leq x \} dt \right) dx
\]

(41)

\[
= \int_{t=0}^{\infty} \left( \int_{x=k_2}^{\infty} 1 \{ t + k_2 \leq x \} (f_X(x) + f_X(-x)) dx \right) dt
\]

(42)

\[
= \int_{t=0}^{\infty} \left( \int_{x=t+k_2}^{\infty} (f_X(x) + f_X(-x)) dx \right) dt
\]

(43)

\[
= \int_{t=0}^{\infty} \mathbb{P}(|X| > t + k_2) dt
\]

(44)

\[
\leq \int_{t=0}^{\infty} 2 \exp(-tC/L) dt
\]

(45)

\[
= \frac{2}{C} \left[ \exp(-tC/L) \right]_{t=k_2}^{\infty}
\]

(46)

\[
= \frac{2}{C} \exp(-k_2C/L).
\]

(47)
Combining Eqs. (36), (40) and (47), we have

\[ |E[Y']| \leq \frac{2k_2}{n^{10}} + 2 \frac{L}{C} \exp(-k_2C/L) \] (48)
\[ \leq \frac{2k_2}{n^{10}} + 2 \frac{L}{C} \frac{1}{n^{10}} \] (49)
\[ \leq \frac{3k_2}{n^{10}}. \] (50)

Let

\[ p \equiv \frac{|E[Y']|}{3k_2} / \frac{1}{n^{10}}. \] (51)

Then \( p \in [0, 1] \) by Eq. (50).

Now the zero-mean random variable \( Y \) can be constructed. Let \( Z \sim \text{Ber}(p) \) be independent from \( X \). Denote

\[ Y \triangleq \begin{cases} X & |X| \leq k_1 \\ -\text{sign}(E[Y'])3k_2 & |X| > k_1 \text{ and } Z = 1 \\ 0 & \text{otherwise} \end{cases} \] (52)

Then

\[ E[Y] = \left( \int_{-k_1}^{k_1} x (f_X(x) - f_X(-x)) \, dx \right) + \mathbb{P}(|X| > k_1 \text{ and } Z = 1)(-3\text{sign}(E[Y'])k_2) \]
\[ = E[Y'] - \frac{1}{n^{10}}p(3k_2)\text{sign}(E[Y']) \]
\[ = 0. \]

Also, \( |Y| \leq 3k_2 = 30L \log(n) \) is bounded. Furthermore, with probability \( 1 - \frac{2}{n^{10}} \), \( |X| \leq k_1 \) and \( Y = X \). \( \square \)

**Lemma 7** (Bernstein’s inequality (Theorem 2.8.4, Vershynin (2018))). Let \( X_1, X_2, \ldots, X_N \) be independent zero-mean random variables, such that \( |X_i| \leq K \) for all \( i \). Then, for every \( t \geq 0 \), we have

\[ \mathbb{P} \left( \left| \sum_{i=1}^{N} X_i \right| \geq t \right) \leq 2 \exp \left( -\frac{t^2/2}{\sigma^2 + Kt/3} \right) \] (53)

here \( \sigma^2 = \sum_{i=1}^{N} \mathbb{E} \left( X_i^2 \right) \) is the variance of the sum.

**Lemma 8.** Let \( X_1, X_2, \ldots, X_n \) be independent zero-mean random variables, such that \( \|X_i\|_{\psi_1} \leq L \) for all \( i \). Then, with probability \( 1 - O(n^{-c}) \) for any constant \( c \), we have

\[ \sum_{i=1}^{n} X_i \lesssim \sigma \sqrt{\log(n)} + L \log^2(n) \] (54)

here \( \sigma^2 = \sum_{i=1}^{n} \mathbb{E} \left( X_i^2 \right) \) is the variance of the sum.

**Proof.** Let \( Y_i = X_i1(\{|X_i| \leq B\}) \) be the truncated version of \( X_i \). Then \( \text{Var}(Y_i) \leq E[Y_i^2] \leq E[X_i^2] \). Furthermore,

\[ |E(Y_i)| \leq \left| \int_{B}^{\infty} X_i df(X_i) + \int_{-\infty}^{-B} X_i df(X_i) \right| \] (55)
\[ \leq B P(|X_i| > B) + \int_{B}^{\infty} P(|X_i| > B) dX \] (56)
\[ \leq Be^{-B/C} + CLe^{-B/C} \] (57)
where \( C \) is a constant. By Lemma 7, we have
\[
\mathbb{P} \left( \left| \sum_{i=1}^{N} (Y_i - \mathbb{E}(Y_i)) \right| \geq t \right) \leq 2 \exp \left( -\frac{t^2/2}{\sum_{i=1}^{N} \text{Var}(Y_i) + Bt/3} \right) \leq 2 \exp \left( -\frac{t^2/2}{\sigma^2 + Bt/3} \right). \tag{58}
\]

Then, with probability \( 1 - O(n^{-c}) \) for some constant \( c \),
\[
\left| \sum_{i=1}^{N} (Y_i - \mathbb{E}(Y_i)) \right| \lesssim \sigma \sqrt{\log(n)} + B \log(n). \tag{59}
\]

Take \( B = L \log(n)C' \) for a proper constant \( C' \), by Eq. (57), we have
\[
\left| \sum_{i=1}^{N} Y_i \right| \lesssim \sigma \sqrt{\log(n)} + L \log^2(n). \tag{60}
\]

By the union bound on the event \( |X_i| \leq B \) for all \( i \), we can conclude that, with probability \( 1 - O(n^{-c}) \),
\[
\left| \sum_{i=1}^{N} X_i \right| \lesssim \sigma \sqrt{\log(n)} + L \log^2(n). \tag{61}
\]

\[\square\]

**Lemma 9** (Matrix Bernstein inequality, Theorem 6.1.1, Tropp et al. (2015)). Given \( n \) independent random \( m_1 \times m_2 \) matrices \( X_1, X_2, \ldots, X_n \) with \( \mathbb{E}[X_i] = 0 \). Let
\[
V \triangleq \max \left( \left\| \sum_{i=1}^{n} \mathbb{E}[X_iX_i^\top] \right\|, \left\| \sum_{i=1}^{n} \mathbb{E}[X_i^\top X_i] \right\| \right). \tag{62}
\]

Suppose \( \|X_i\| \leq B \) for \( i \in [n] \). For all \( t \geq 0 \),
\[
\mathbb{P} \left( \|X_1 + X_2 + \ldots + X_n\| \geq t \right) \leq (m_1 + m_2) \exp \left\{ -\frac{t^2/2}{V + Bt/3} \right\}. \tag{63}
\]

**Lemma 10** (Perturbation of pseudo-inverses (Theorem 3.3, Stewart (1977))). Let \( A^{-1} \) and \( B^{-1} \) be the pseudo-inverse (Moore-Penrose inverse) of two matrices \( A \) and \( B \), respectively. Then
\[
\|B^{-1} - A^{-1}\| \leq 3 \max \left( \|A^{-1}\|^2, \|B^{-1}\|^2 \right) \|B - A\|. \tag{64}
\]

**Lemma 11** (Lemma 4, McRae and Davenport (2019)). Let \( X \in \mathbb{R}^{m \times n} \) be a random matrix whose entries are independent and centered, and suppose that for some \( v, t_0 > 0 \), we have, for all \( t \geq t_0 \),
\[
\mathbb{P}(|X_{ij}| \geq t) \leq 2e^{-t/v}.
\]

Let \( \epsilon \in (0, 1/2) \), and let
\[
K = \max\{t_0, v \log \frac{2mn}{\epsilon} \}.
\]

Then
\[
\mathbb{P} \left( \|X\| \geq 2\sigma + \frac{\epsilon v}{\sqrt{mn}} + t \right) \leq \max(m, n) \exp \left( -\frac{t^2}{C_0^2K^2} \right) + \epsilon
\]
where \( C_0 \) is a constant and
\[
\sigma = \max_i \sqrt{\sum_j \mathbb{E}(X_{ij}^2)} + \max_j \sqrt{\sum_i \mathbb{E}(X_{ij}^2)}.
\]
B.1. Proof of Lemma 1 and Lemma 2

Proof of Lemma 1. Let $Y_i = X_i 1\{\|X_i\| \leq B\}$ be the truncated version of $X_i$. We have,

\[
\|E(Y_i)\| \leq \left\| \int X_i 1\{\|X_i\| > B\} df(X_i) \right\|
\leq \left\| \int X_i 1\{\|X_i\| > B\} df(X_i) \right\|
\leq B(P(\|X_i\| > B) + \int_B^\infty P(\|X_i\| > t)dt)
\leq Be^{-B/CL} + CLe^{-B/CL}
\]

where (i) is due to the convexity of $\|\cdot\|$ and (ii) is due to the subexponential property of $\|X_i\|$ and $C$ is a constant.

Meanwhile, we have

\[
\left\| \sum_{i=1}^n E\left((Y_i - E(Y_i))(Y_i - E(Y_i))^T\right) \right\| = \left\| \sum_{i=1}^n E\left(Y_i Y_i^T\right) - E(Y_i) E(Y_i)^T \right\|
\leq \left\| \sum_{i=1}^n E\left(Y_i Y_i^T\right) \right\|
\leq \left\| \sum_{i=1}^n E\left(X_i X_i^T\right) \right\|
\leq \left\| \sum_{i=1}^n E\left(X_i X_i^T\right) \right\| \leq V
\]

where (i) is due to the positive-semidefinite property of $E(Y_i) E(Y_i)^T$ and $E(Y_i Y_i^T) - E(Y_i) E(Y_i)^T$, (ii) is due to the positive-semidefinite property of $E(X_i X_i^T 1\{\|X_i\| > B\})$ and $E(Y_i Y_i^T)$. Similarly, $\|\sum_{i=1}^n E\left((Y_i - E(Y_i))(Y_i - E(Y_i))^T\right)\| \leq V$.

Then, by Lemma 1, we have

\[
P\left( \left\| \sum_{i=1}^N (Y_i - E(Y_i)) \right\| \geq t \right) \leq 2 \exp\left( -\frac{t^2/2}{V + 2Bt/3} \right).
\]

Then, with probability $1 - O(n^{-c})$ for some constant $c$,

\[
\left\| \sum_{i=1}^N (Y_i - E(Y_i)) \right\| \lesssim \sqrt{V \log(n(m_1 + m_2))} + B \log(n(m_1 + m_2)).
\]

Take $B = L \log(n)C'$ for a proper constant $C'$, by Eq. (57), we have

\[
\left\| \sum_{i=1}^N Y_i \right\| \lesssim \sqrt{V \log(n)} + L \log^2(n) + nL \log(n)O(n^{-C'/C})
\lesssim \sqrt{V \log(n(m_1 + m_2))} + L \log(n(m_1 + m_2)) \log(n).
\]

By the union bound on the event $\|X_i\| \leq B$ for all $i$, we can conclude that, with probability $1 - O(n^{-c'})$ for some constant $c'$,

\[
\left\| \sum_{i=1}^N X_i \right\| \lesssim \sqrt{V \log(n(m_1 + m_2))} + L \log(n(m_1 + m_2)) \log(n).
\]
Proof of Lemma 2. We invoke Lemma 11 to prove this result. Let $C_1, C_2, C_3$ be constants. Let $X = P_{\Omega}(E)$ with $\epsilon = \frac{1}{n}$, $v = C_1 L$, $K = C_1 L \log(n)$. It is easy to verify that $E[X^2_{ij}] \leq p L^2$. Therefore, $\sigma \leq C_2 \sqrt{n p L}$. Take $t = C_3 \sqrt{n p L}$. By Lemma 11

$$\mathbb{P} \left( \| P_{\Omega}(E) \| \geq 2 C_3 \sqrt{n p L} + \frac{L}{n^{11}} \right) \leq C_4 n e^{-n p / \log^2(n)} + \frac{1}{n^{11}}.$$ 

Given that $np \geq c_0 \log^3 n$, we have, with probability $1 - O(n^{-11})$,

$$\| P_{\Omega}(E) \| \lesssim L \sqrt{np}.$$