Dispersion Parameter Extension of Precise Generalized Linear Mixed Model Asymptotics

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Abstract

We extend a recently established asymptotic normality theorem for generalized linear mixed models to include the dispersion parameter. The new results show that the maximum likelihood estimators of all model parameters have asymptotically normal distributions with asymptotically mutual independence between fixed effects, random effects covariance and dispersion parameters. The dispersion parameter maximum likelihood estimator has a particularly simple asymptotic distribution which enables straightforward valid likelihood-based inference.

Keywords: Longitudinal data analysis, maximum likelihood estimation, multilevel models, studentization.

1 Introduction

We extend the main theorem of Jiang et al. (2022) to include conditional maximum likelihood estimation of the dispersion parameter. The essence of the new findings is that the dispersion parameter maximum likelihood estimator has a simple asymptotic normal distribution, identical to that for the generalized linear model case, which is amenable to practical inference. Moreover, we establish asymptotic orthogonality between the dispersion parameter and the other model parameters.

The motivations and benefits of precise asymptotics for generalized linear mixed models are described in Section 1 of Jiang et al. (2022). As mentioned there, books such as Faraway (2016), McCulloch et al. (2008) and Stroup (2013) provide summaries and access to the large literature on generalized linear mixed models. Figure 1 provides visualization of a data set that potentially benefits from generalized linear mixed model analysis. Figure 1 data is stored in the data frame MathAchieve within the package nlme (Pinheiro et al., 2022) of the R computing environment (R Core Team, 2022).

Estimation of the dispersion parameter, denoted here by \( \phi \), was not considered by Jiang et al. (2022) for a few reasons. One is that \( \phi \) is often treated as a nuisance parameter, with the fixed and random effects being of primary interest. Another is that maximum likelihood does not apply for the most common families: Bernoulli and Poisson. Instead, for these two families, quasi-likelihood is required for the \( \phi \neq 1 \) extension. A third reason is that, even when maximum likelihood estimation is available for situations such as Gamma response models, it is common to use the simpler Pearson estimator instead. Nevertheless, maximum likelihood estimation of \( \phi \) is viable in important generalized linear mixed model contexts and a full treatment of precise asymptotics requires its inclusion.

Section 2 describes the data, generalized linear mixed models set-up and maximum likelihood estimation of the model parameters. The focal point of this article is the asymptotic
normality theorem stated in Section 3 with proof provided in supplemental material. Asymptotically valid likelihood-based inference for the dispersion parameter is discussed in Section 4. Sections 2–4 are confined to reproductive exponential families, which covers the common cases arising in generalized linear mixed model applications. The extension to general expo-
2 Model Description and Maximum Likelihood Estimation

Consider the class of two-parameter reproductive linear exponential family density functions with generic form

\[ p(y; \eta, \phi) = \exp\{\eta y - b(\eta) + c(y)\} / \phi - d(\phi) - e(y)\}h(y) \tag{1} \]

where \( \eta \) is the natural parameter and \( \phi > 0 \) is the dispersion parameter. The definition of a reproductive exponential family is given in, for example, Jørgensen (1987). Not all two-parameter linear exponential family density functions are reproductive. However, those families commonly used in applications of generalized linear and mixed models are reproductive and Sections 2–4 are restricted to this case. We discuss the general case briefly in Section 6.

Table 1 gives some explicit examples of the \( b, c, d, e \) and \( h \) functions appearing in (1). In Table 1 the following notation is used: \( I(\mathcal{P}) = 1 \) if the condition \( \mathcal{P} \) is true and \( I(\mathcal{P}) = 0 \) if \( \mathcal{P} \) is false. Theoretical results given in Blæsild & Jensen (1985) imply that the three families listed in Table 1 are the only possibilities for \( p(y; \eta, \phi) \), even if \( -d(\phi) - e(y) \) is relaxed to be a general bivariate function of \( (\phi, y) \). Therefore, without loss of generality, we can assume that \( p(y; \eta, \phi) \) is one of the three forms given by Table 1.

| family         | \( b(\eta) \) | \( c(y) \) | \( d(\phi) \) | \( e(y) \) | \( h(y) \) |
|----------------|---------------|-------------|---------------|-------------|-------------|
| Gaussian       | \( \frac{1}{2} \eta^2 \) | \( -\frac{1}{2} y^2 \) | \( \frac{1}{2} \log(\phi) \) | \( \frac{1}{2} \log(2\pi) \) | 1           |
| Gamma          | \( -\log(-\eta) \) | \( \log(y) \) | \( \log \left( \phi^{1/\phi} \Gamma(1/\phi) \right) \) | \( \log(y) \) | \( I(y > 0) \) |
| Inverse Gaussian | \( -(2\eta)^{1/2} \) | \( -1/(2y) \) | \( \frac{1}{2} \log(\phi) \) | \( \frac{1}{2} \log(2\pi y^3) \) | \( I(y > 0) \) |

Table 1: Specific two-parameter exponential families and their \( b, c, d, e \) and \( h \) functions.

In this article we study generalized linear mixed models of the form, for observations of the random triples \( \{X_{Aij}, X_{Bij}, Y_{ij}\}, 1 \leq i \leq m, 1 \leq j \leq n_i, \)

\[ Y_{ij} | X_{Aij}, X_{Bij}, U_i \text{ independent having density function (1) with natural parameter} (\beta_A^0 + U_i)^T X_{Aij} + (\beta_B^0)^T X_{Bij} \text{ such that the} U_i \text{ are independent} \]

\[ N(0, \Sigma^0) \]

random vectors.

The \( U_i \) are \( d_A \times 1 \) unobserved random effects vectors. The \( X_{Aij} \) are \( d_A \times 1 \) random vectors corresponding to predictors that are partnered by a random effect. The \( X_{Bij} \) are \( d_B \times 1 \) random vectors corresponding to predictors that have a fixed effect only. Let \( X_{ij} = (X_{Aij}, X_{Bij})^T \) denote the combined predictor vectors. We assume that the \( X_{ij} \) and \( U_i \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n_i \) are totally independent, with the \( X_{ij} \) each having the same distribution as the \( (d_A + d_B) \times 1 \) random vector \( X = (X_A^T, X_B^T)^T \) and the \( U_i \) each having the same distribution as the random vector \( U \).

For any \( \beta_A \) (\( d_A \times 1 \)), \( \beta_B \) (\( d_B \times 1 \)), \( \Sigma \) (\( d_A \times d_A \)) that is symmetric and positive definite and \( \phi > 0 \), conditional on the \( X_{ij} \) data, the log-likelihood is

\[ \ell(\beta_A, \beta_B, \Sigma, \phi) = -\frac{m}{2} \log |2\pi \Sigma| \]

\[ + \sum_{i=1}^m \sum_{j=1}^{n_i} \left\{ Y_{ij}(\beta_A^T X_{Aij} + \beta_B^T X_{Bij}) + c(Y_{ij})\right\} / \phi - d(\phi) - e(Y_{ij}) \]

\[ + \sum_{i=1}^m \log \int_{\mathbb{R}^{d_A}} \exp \left\{ \sum_{j=1}^{n_i} \left\{ Y_{ij}u^T X_{Aij} - b((\beta_A + u)^T X_{Aij} + \beta_B^T X_{Bij})\right\} / \phi - \frac{1}{2}u^T \Sigma^{-1} u \right\} du \tag{3} \]
The conditional maximum likelihood estimator of \((\beta^0_A, \beta^0_B, \Sigma^0, \phi^0)\) is
\[
(\hat{\beta}_A, \hat{\beta}_B, \hat{\Sigma}, \hat{\phi}) = \arg\max_{\beta_A, \beta_B, \Sigma, \phi} \ell(\beta_A, \beta_B, \Sigma, \phi).
\]

Even though conditional maximum likelihood provides a natural estimator for \(\phi\) based on data modeled according to (2), the relevant literature and software is such that alternative approaches are common. For the generalized linear model special case, Section 8.3.6 of McCullagh & Nelder (1989) expresses a preference for the natural moment-based estimator of \(\phi\), which is sometimes referred to as the Pearson estimator. The functions \(\text{glm}()\) and \(\text{glmer}()\) within the \(\mathbb{R}\) package \textit{lme4} (Bates et al., 2015) for generalized linear mixed models each use Pearson estimation of the dispersion parameter. In Section 2 of Cordeiro & McCullagh (1991) some alternative estimators for the dispersion parameters are proposed, motivated by bias correction and computational convenience considerations. Jo & Lee (2017) compare the efficiencies of dispersion parameter estimators for Gamma generalized linear models and recommend conditional maximum likelihood estimation compared with the Pearson and Cordeiro & McCullagh (1991) estimators.

3 Main Result

Given the addition of \(\phi\) to the set of parameters being estimated, compared with the Jiang et al. (2022) set-up, our aim is to extend the asymptotic normality result for this enlarged estimation problem. We start by repeating definitions and conditions from Section 3 Jiang of et al. (2022). Let
\[
n = \frac{1}{m} \sum_{i=1}^{m} n_i = \text{average of the within-group sample sizes},
\]
\[
\Omega_{\beta_B}(U) \equiv E \left\{ b'' \left( (\beta^0_A + U)^T X_A + (\beta^0_B)^T X_B \right) \left( X_A X_A^T, X_A X_B^T, X_B X_A^T, X_B X_B^T \right) U \right\},
\]
\[
\Lambda_{\beta_B} \equiv \left( E \left[ \text{lower right } d_B \times d_B \text{ block of } \Omega_{\beta_B}(U)^{-1} \right] \right)^{-1},
\]
and \(\|v\| \equiv (v^T v)^{1/2}\) denote the Euclidean norm of a column vector \(v\). For a symmetric matrix \(M\) let \(\lambda_{\min}(M)\) denote the smallest eigenvalue of \(M\). Also, let \(D_d\) denote the matrix of zeroes and ones such that \(D_d \text{vec}(A) = \text{vec}(A)\) for all \(d \times d\) symmetric matrices \(A\). The Moore-Penrose inverse of \(D_d\) is \(D_d^+ = (D_d^T D_d)^{-1} D_d^T\). Let \(d'\) and \(d''\) denote the first and second derivatives of the \(d\) function.

The theorem relies on the following assumptions:

(A1) The number of groups \(m\) diverges to \(\infty\).

(A2) The within-group sample sizes \(n_i\) diverge to \(\infty\) in such a way that \(n_i/n \to C_i\) for constants \(0 < C_i < \infty\), \(1 \leq i \leq m\). Also, \(n/m \to 0\) as \(m\) and \(n\) diverge.

(A3) The distribution of \(X\) is such that
\[
E \left[ \frac{\max \{ 1, \|X\| \}^8 \max \{ 1, b'' \left( (\beta_A^0 + U)^T X_A + \beta_B^0 X_B \right) \}^4 |U|}{\min \{ 1, \lambda_{\min}(E \left[ X_A X_A^T b'' \left( (\beta_A^0 + U)^T X_A + \beta_B^0 X_B \right) U \right]) \}^2} \right] < \infty \quad (4)
\]
for all \(\beta_A \in \mathbb{R}^{d_A}, \beta_B \in \mathbb{R}^{d_B}\) and \(\Sigma\) a \(d_A \times d_A\) symmetric and positive definite matrix.
Theorem 1. Assume that conditions (A1)–(A3) hold. Then

\[
\sqrt{m} \begin{bmatrix}
\hat{\beta}_A - \beta_A^0 \\
\sqrt{n}(\hat{\beta}_B - \beta_B^0) \\
\text{vech}(\hat{\Sigma} - \Sigma^0)
\end{bmatrix} \xrightarrow{D} N \left( \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
\Sigma^0 & 0 & 0 \\
0 & \phi^0 & \Lambda^0 \\
0 & \Lambda^0 & 2D_d^+(\Sigma^0 \otimes \Sigma^0)D_d^+T
\end{bmatrix} \right)
\]

A proof of Theorem 1 is given in Section S.2 of the supplemental material.

Some remarks concerning Theorem 1 are:

1. The diagonal blocks appearing in the Multivariate Normal covariance matrix of Theorem 1 correspond to asymptotic covariances of: (1) fixed effects partnered by a random effect, (2) fixed effects not partnered by a random effect, (3) random effects covariance parameters and (4) the dispersion parameter. For the first and third of these types of parameters the asymptotic variances have order \(m^{-1}\). The remaining parameters, including the dispersion parameter, have order \((mn)^{-1}\) asymptotic variances.

2. Theorem 1 reveals asymptotic orthogonality between \(\phi\) and \((\beta_A, \beta_B, \Sigma)\). This is in addition to the asymptotic orthogonality between the components of \((\beta_A, \beta_B, \Sigma)\), established by Theorem 1 Jiang et al. (2022).

3. The asymptotic distribution of \(\hat{\phi}\) is the same as that arising for the generalized linear model special case of (2) when there are no random effects. In other words, the asymptotic behaviour of \(\hat{\phi}\) is not impacted by the extension from generalized linear models to generalized linear mixed models.

4. After obtaining the \(2d'(\phi)/\phi + d''(\phi)\) expressions for the specific \(d\) functions of Table 1 and simplifying, the asymptotic variances of \(\hat{\phi}\) become:

\[
\text{Asy.Var}(\hat{\phi}) = \begin{cases}
\frac{2(\phi^0)^2}{mn} & \text{for the Gaussian and the Inverse Gaussian families,} \\
\frac{(\phi^0)^4}{\text{trigamma}(1/\phi^0) - \phi^0} & \text{for the Gamma family.}
\end{cases}
\]

5. The trigamma function has the following asymptotic expansion:

\[
\text{trigamma}(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \ldots
\]

where the coefficients are simple functions of Bernoulli numbers. It follows that

\[
\frac{(\phi^0)^4}{\text{trigamma}(1/\phi^0) - \phi^0} \approx 2(\phi^0)^2 \quad \text{for small values of } \phi^0.
\]

This connects the asymptotic variance results of (5) for low values of the dispersion parameter.

6. The moment condition in (A3) is sufficient but not necessary for Theorem 1 to hold. A recently discovered erratum has resulted in the replacement of the second moment in the second numerator factor in (4) by the fourth moment, compared with (A3) of Jiang et al. (2022). For various special cases of (1) and (2) mathematical analysis can be used.
to replace (A3) by a simpler moment condition. The Gaussian response case is such that $b' = 1$ and relatively simple arguments show that (A3) can be replaced by

$$E(\|X\|^8) < \infty \quad \text{and no entry of } X_A \text{ is the zero degenerate random variable.}$$

In this case it is clear that (A3) holds for sufficiently light-tailed $X$ distributions but fails if, for example, an entry of $X_B$ has a heavy-tailed distribution such as the $t$ distribution with a low degrees of freedom parameter. A more involved example involves Poisson responses with $X_A = 1$, corresponding to random intercepts. In Section S.3 of the supplemental material it is shown that (A3) can be replaced by the moment generating function existence condition

$$E\{\exp(t^T X_B)\} < \infty \text{ for all } t \in \mathbb{R}^{d_B}$$

which is satisfied by, for example, $X_B$ having a Multivariate Skew-Normal distribution (Azzalini & Dalla Valle, 1996). The arguments in Section S.3 of the supplemental material give a flavor of what is involved to simplify (A3). Other cases require more detailed mathematical analysis.

4 Asymptotically Valid Inference

An immediate consequence of Theorem 1 is

$$\sqrt{mn} \left\{ 2d'(\phi^0) / \phi^0 + d''(\phi^0) \right\}^{1/2} \left( \hat{\phi} - \phi^0 \right) \overset{D}{\to} N(0,1).$$

This asymptotic normality result still holds when the unknown quantities on the left-hand side are replaced by consistent estimators, often referred to as studentization. Hence an asymptotically valid 100(1 - $\alpha$)% confidence interval for $\phi^0$ is

$$\hat{\phi} \pm \Phi^{-1}(1 - \frac{1}{2}\alpha) \left[ \left\{ 2d'(\hat{\phi}) / \hat{\phi} + d''(\hat{\phi}) \right\} mn \right]^{-1/2}$$

(6)

where $\Phi$ is the $N(0,1)$ cumulative distribution function. It follows that Theorem 1 provides simple closed form asymptotically valid inference for $\phi^0$. For the specific families listed in Table 1, (6) simplifies to

$$\hat{\phi} \pm \Phi^{-1}(1 - \frac{1}{2}\alpha) \sqrt{\frac{2\hat{\phi}^2}{mn}} \quad \text{for the Gaussian and Inverse Gaussian families}$$

and

$$\hat{\phi} \pm \Phi^{-1}(1 - \frac{1}{2}\alpha) \sqrt{\frac{\hat{\phi}^4}{\{\text{trigamma}(1/\hat{\phi}) - \hat{\phi}\} mn}} \quad \text{for the Gamma family.}$$

(7)

A simulation study was run to assess the actual coverages of Theorem 1-based confidence intervals for the dispersion parameter for the $d_A = 1$ and $d_B = 3$ Gamma mixed model. In this scalar random effects case, $\beta_A$ and $\Sigma$ are replaced by the scalar parameter symbols $\beta_0$ and $\sigma^2$. The true parameter vector $(\phi_0, \beta_0, (\sigma^2)^0, \phi^0)$ had the following four settings:

setting A:  \((-2.78, -1.55, 0, 0.98, 0.25, 0.54),
setting B:  \((-4.06, -2.41, 0.16, -3.93, 0.52, 1.92),
setting C:  \((-8.55, 3.13, -7.82, -0.23, 1.27, 0.86),
setting D:  \((-14.45, 8.78, 0.41, -3.32, 1.88, 2.11)

and the $X_{B_j}$s were generated independently from the uniform distribution over the unit cube. The number of groups $m$ varied over the set $\{50, 100, 150, 200, 250, 300, 350, 400\}$ and the sample size within each group was $n$ fixed at $m/5$. For each of the possible combinations of the
true parameter vector and the sample size pair we simulated 1,000 replications. Established
generalized linear mixed model software, such the `glmer()` function within the R package
`lme4` (Bates et al., 2015), does not use maximum likelihood to estimate \( \phi \). For this \( d_A = 1 \) case
the likelihood surface (3) was relatively straightforward to evaluate exactly using univariate
quadrature and maximize using a derivative-free optimization algorithm such as the Nelder-
Mead simplex method (Nelder & Mead, 1965) with `glmer()` starting values. After obtaining
the maximum likelihood estimates, we computed 95% confidence intervals based on (7) with
\( \alpha = 0.05 \).

Figure 2 summarizes the empirical coverage results. The shaded regions around the line
segments in Figure 2 indicate plus and minus two standard errors for the sample proportions.
It is seen that the actual coverage percentages are in keeping with the advertised level of 95%
across all settings and sample sizes. There is a tendency for the empirical coverages to get
closer, on average, to 95% as \( m \) and \( n \) in increase, as expected for statistical inference based on
leading term asymptotics.

5 Analysis of the Mathematics Achievement Data

As an illustration of the asymptotically valid inference results given in the previous section
we conducted an analysis of the mathematics achievement data shown in Figure 1 using a
Gaussian response version of (2). Specifically, we considered the model

\[
\text{MathAchieve}_{ij} | U_{0i}, U_{1i} \overset{\text{ind.}}{\sim} N \left( \beta_0^0 + U_{0i} + (\beta_0^1 + U_{1i}) \text{SES}_{ij} + \beta_2^0 \text{isMale}_{ij} + \beta_3^0 \text{isMinority}_{ij}, \phi^0 \right),
\]

where \( \overset{\text{ind.}}{\sim} \) stands for “independently distributed as” and \( n_i \) is the number of students in the \( i \)th school. In (8) \( \text{MathAchieve}_{ij} \) and \( \text{SES}_{ij} \) denote, respectively, the mathematics achievement and socio-economic status scores for the \( j \)th student in the \( i \)th school. In addition

\[
\text{isMale}_{ij} = \begin{cases} 
0 & \text{if the } j \text{th student in the } i \text{th school is female,} \\
1 & \text{if the } j \text{th student in the } i \text{th school is male}
\end{cases}
\]

and \( \text{isMinority}_{ij} \) is defined similarly for minority status. The \( n_i \) values range over 14 to 67.

Table 2 shows the estimates of the fixed effects and standard deviation parameters based on conditional maximum likelihood estimation as described in Section 2. Also shown in Table 2 are approximate 95% confidence intervals based on Theorem 1 and studentization. The confidence interval for \( \beta_0^2 \) indicates a statistically significant elevation in mean mathematics achievement score for the male student population. The confidence interval for \( \beta_0^3 \) indicates a statistically significant lowering for minority students. The confidence intervals for \( \sigma_0^0 \) and \( \sigma_0^1 \) are both within the positive half-line, which indicates significant heterogeneities in the intercepts and slopes of the social economic status effects. The last row of Table 2 provides an estimate and 95% confidence interval for the within-school error standard deviation.

| parameter | estimate | 95% confid. interv. |
|-----------|----------|---------------------|
| \( \beta_0^0 \) | 12.93 | (12.55, 13.31) |
| \( \beta_0^1 \) | 2.097 | (1.875, 2.319) |
| \( \beta_0^2 \) | 1.219 | (0.9003, 1.537) |
| \( \beta_0^3 \) | -2.999 | (-3.404, -2.594) |
| \( \sigma_0^0 \) | 1.903 | (1.682, 2.101) |
| \( \sigma_0^1 \) | 0.4964 | (0.4386, 0.5481) |
| \( \sqrt{\phi^0} \) | 5.982 | (5.883, 6.079) |

Table 2: Maximum likelihood estimates and approximate 95% confidence intervals of the fixed effects parameters and standard deviation parameters for the fit of the model (8) to the mathematics achievement data.

An examination of the residuals revealed reasonable accordance with model assumptions but some heteroscedasticity. More delicate modelling, involving variance function extensions of (2) and other response families may lead to model fit improvements.

6 General Two-Parameter Exponential Family Extension

We now turn attention to general two-parameter exponential families. If the restriction to reproductive exponential families is removed then (1) should be replaced by

\[
p(y; \eta, \phi) = \exp \left[ \{ y \eta - b(\eta) + c(y) \} / \phi - \tilde{d}(y, \phi) \right] h(y)
\]

where \( \tilde{d}(y, \phi) \) is some bivariate function of \( y \) and \( \phi \) that is not necessarily additive in its arguments. Section 2.1 of Jørgensen (1987) describes a procedure for generating versions of (9) from
any distribution possessing a moment generating function. An example of a non-reproductive version of (9) given there is such that
\[ b(\eta) = \log \left( -\eta - \sqrt{\eta^2 - 1} \right), \quad c(y) = 0, \quad \tilde{d}(y, \phi) = -\log \left\{ I_{1/\phi}(y/\phi)/(y\phi) \right\} \text{ and } h(y) = I(y > 0) \]
where \( I_\nu \) denotes the modified Bessel function of the first kind with index \( \nu \) (e.g. Section 8.431 of Gradshteyn & Ryzhik, 1994).

For this more general set-up, the Theorem 1 arguments still apply with \( \tilde{d}(y, \phi) \) replacing \( d(\phi) + e(y) \) and the lower right block of the covariance matrix on the right-hand side of the Theorem 1 statement generalizes to
\[
(\phi^0)^4 \left\{ E \left( \left[ \frac{\partial^2}{\partial \psi^2} \tilde{d}(Y, \frac{1}{\psi}) \right]_{\psi = 1/\phi^0} \right) \right\}^{-1/2} \tag{10}
\]
where \( Y \) is a random variable having the same distribution as the \( Y_{ij} \)s. Method of moments estimation of the expectation over \( Y \) leads to the following extension of (6) for approximate 100(1 – \( \alpha \))% confidence intervals for \( \phi^0 \):
\[
\hat{\phi} \pm \Phi^{-1}(1 - \frac{1}{2}\alpha)\hat{\phi}^2 \left( \sum_{i=1}^m \sum_{j=1}^{n_i} \left[ \frac{\partial^2 \tilde{d}(Y_{ij}, 1/\psi)}{\partial \psi^2} \right]_{\psi = 1/\hat{\phi}} \right)^{-1/2} \tag{11}
\]
The practical relevance of (10) and (11) is much lower than for the reproductive exponential family special case and they have been included for completeness.

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 Supplement for:
 Dispersion Parameter Extension of Generalized
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S.1 Introduction

This supplement contains derivational details concerning the article’s results. In Section S.2 we provide a proof of Theorem 1. Section S.3 provides illustration of how, for a special case, the (A3) moment condition can be reduced to a simpler moment condition.

S.2 Proof of Theorem 1

In this section we provide a proof of Theorem 1. We start by setting up notation for some key quantities that arise throughout the proof, as well as useful generic mathematical notation. The main body of the proof involves asymptotic approximation of the Fisher information matrix and its inverse. Much of this was achieved in Appendix A of Jiang et al. (2022). However, the dispersion parameter extension leads to new Fisher information entries. We then apply Lemma 2 of Jiang et al. (2022) to establish an asymptotic equivalence result between the matrix square roots of two relevant approximations to the inverse Fisher information matrix. The final steps required to establish Theorem 1 are then carried out.

S.2.1 Notation

We divide the notation into two parts: (1) that for key quantities specific to the model at hand and (2) generic mathematical notation that aids the proof.

S.2.1.1 Notation for Some Key Quantities

Throughout this proof we let

$$\psi \equiv \frac{1}{\phi}$$

denote the reciprocal dispersion parameter. Working with \(\psi\), rather than \(\phi\), in Fisher information approximations involves simpler expressions in the derivation of the asymptotic joint normality result for the model parameters. The transformation from \(\psi\) to \(\phi = 1/\psi\), using the Multivariate Delta Method, is carried out after such a result is established.

For each \(1 \leq i \leq m\) and \(1 \leq j \leq n_i\) let \(X_{ij} \equiv (X_{Aij}^T, X_{Bij}^T)^T\) and \(X_i \equiv (X_{i1}, \ldots, X_{in_i})\). Let \(Y_i, 1 \leq i \leq m\), be defined analogously.

Define \(G_{Ai}\) and \(H_{AAi}\), for each \(1 \leq i \leq m\), as follows

$$G_{Ai} \equiv \sum_{j=1}^{n_i} \{Y_{ij} - b'((\beta_A + U_i)^T X_{Aij} + \beta_B^T X_{Bij})\} X_{Aij},$$

$$H_{AAi} \equiv \sum_{j=1}^{n_i} b''((\beta_A + U_i)^T X_{Aij} + \beta_B^T X_{Bij}) X_{Aij} X_{Aij}^T,$$
In a similar vein, define \( \mathcal{H}'_{\text{AAA}} \) to be the \( d_A \times d_A \times d_A \) array with \((r, s, t)\) entry equal to

\[
\sum_{j=1}^{n_j} b''''((\beta_A + U_i^j)^T X_{Aij} + \beta_B^T X_{Bij}) (X_{Aij})_r (X_{Aij})_s (X_{Aij})_t
\]

S.2.1.2 Generic Mathematical Notation

For a generic \( d \times 1 \) vector \( v \) we define \( v^\otimes 2 \equiv vv^T \). We also let \( \text{diag}(v) \) denote the \( d \times d \) diagonal matrix with the entries of \( v \) along the diagonal. For a matrix \( M \) let \( \|M\|_F = \{\text{tr}(M^T M)\}^{1/2} \) denote its Frobenius norm.

For \( f \) a smooth real-valued function of the \( d \)-variate argument \( x \equiv (x_1, \ldots, x_d) \), let \( \nabla f(x) \) denote the \( d \times 1 \) vector with \( i \)th entry \( \partial f(x)/\partial x_i \), \( \nabla^2 f(x) \) denote the \( d \times d \) matrix with \((i, j)\) entry \( \partial^2 f(x)/\partial x_i \partial x_j \) and \( \nabla^3 f(x) \) denote the \( d \times d \times d \) array with \((i, j, k)\) entry \( \partial^3 f(x)/\partial x_i \partial x_j \partial x_k \).

If \( A \) is a \( d_1 \times d_2 \times d_3 \) array and \( M \) is a \( d_1 \times d_2 \) matrix then we let

\[
A \star M \quad \text{denote the } d_3 \times 1 \text{ vector with } t \text{th entry given by } \sum_{r=1}^{d_1} \sum_{s=1}^{d_2} (A)_{rst} M_{rs}.
\]

S.2.2 Fisher Information Approximation

The Fisher information corresponding to the parameter vector

\[
(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)
\]

is a symmetric matrix having \( d_A + d_B + \frac{1}{2}d_A(d_A + 1) \) rows and columns. Appendix A of Jiang et al. (2022) provides an adequate approximation of the \((\beta_A, \beta_B, \text{vech}(\Sigma))\) diagonal block. The dispersion parameter extension requires similar approximations for the \( \psi \) diagonal block and the \((\beta_A, \beta_B, \text{vech}(\Sigma), \psi)\) off-diagonal block. The sub-blocks of the \((\beta_A, \beta_B, \text{vech}(\Sigma), \psi)\) off-diagonal block correspond to each of

\[
(\beta_A, \beta_B, \text{vech}(\Sigma), \psi), \quad (\beta_A, \beta_B, \psi) \quad \text{and} \quad (\text{vech}(\Sigma), \psi)
\]

The first of these is treated in Jiang et al. (2022). The second and third of these are treated in Sections S.2.2.5 and S.2.2.6 respectively.

S.2.2.1 Higher Order Approximation of Multivariate Integral Ratios

Our main tool for approximation of the Fisher information matrix entries for generalized linear mixed models is higher order Laplace-type approximation of multivariate integral ratios. Appendix A of Miyata (2004) provides such a result, which states that for smooth real-valued \( d \)-variate functions \( g, c \) and \( h \) we have

\[
\frac{\int_{\mathbb{R}^d} g(x) c(x) \exp\{-nh(x)\} \, dx}{\int_{\mathbb{R}^d} c(x) \exp\{-nh(x)\} \, dx} = g(x^*) + \frac{\nabla g(x^*)^T \{\nabla^2 h(x^*)\}^{-1} \nabla c(x^*)}{n c(x^*)} + \frac{\text{tr}[\{\nabla^2 h(x^*)\}^{-1} \nabla^2 g(x^*)]}{2n} - \frac{\nabla g(x^*)^T \{\nabla^3 h(x^*)\} \star \{\nabla^2 h(x^*)\}^{-1}}{2n} + O(n^{-2})
\]

where

\[
x^* \equiv \arg\min_{x \in \mathbb{R}^d} h(x).
\]
S.2.2.2 The $U_i^*$ Quantity and Its Approximation

For all required Fisher information approximations for the $i$th group, the $h$ function appearing in (S.2) corresponds to the stochastic function

$$h_i(u) = -\frac{1}{n_i} \sum_{j=1}^{n_i} \{ Y_{ij} u^T X_{Aij} - b((\beta_A + u)^T X_{Aij} + \beta_B^T X_{Bij}) \}$$

and its minimum is denoted by the random vector

$$U_i^* \equiv \arg \min_{u \in \mathbb{R}^d} h_i(u).$$

Taylor series expansion, similar to that given in Appendix A.3.1 of Jiang et al. (2022), followed by asymptotic series inversion leads to the three-term approximation:

$$U_i^* = U_i + \mathcal{H}_{AAi}^{-1} \mathcal{G}_{Ai} - \frac{1}{2} \mathcal{H}_{AAi}^{-1} \left\{ \mathcal{H}'_{AAii} \otimes \left( \mathcal{H}_{AAi}^{-1} \mathcal{G}_{Ai} \mathcal{G}_{Ai}^T \mathcal{H}_{AAi}^{-1} \right) \right\} + O_P(n^{-3/2}) 1_{d_A}. \quad (S.4)$$

S.2.2.3 The $(\beta_A, \beta_B, \text{vech}(\Sigma))$ Diagonal Block

For the case of the dispersion parameter being fixed rather than estimated, Jiang et al. (2022) derives an approximation to the Fisher information of $(\beta_A, \beta_B, \text{vech}(\Sigma))$. Appendix A.5 provides the resultant approximation. For the extension to $\phi = 1/\psi$ being estimated, this approximation corresponds to the $(\beta_A, \beta_B, \text{vech}(\Sigma))$ diagonal block of the Fisher information matrix for the extended parameter vector (S.1).

S.2.2.4 The $\psi$ Diagonal Block

The $i$th contribution to the score is

$$\frac{\partial \log \{ p_{Y_i|X_i} (Y_i|X_i) \}}{\partial \psi} = \sum_{j=1}^{n_i} \left[ Y_{ij} (\beta_A)^T X_{Aij} + (\beta_B)^T X_{Bij} + \epsilon(Y_{ij}) \right] - \frac{dd(1/\psi)}{d\psi}$$

$$+ \int_{\mathbb{R}^d_A} g_i(u) c(u) \exp\{\psi g_i(u)\} \, du$$

$$+ \int_{\mathbb{R}^d_A} c(u) \exp\{\psi g_i(u)\} \, du.$$

The derivative of the $i$th contribution to the score is

$$\frac{\partial^2 \log \{ p_{Y_i|X_i} (Y_i|X_i) \}}{\partial \psi^2} = -\frac{n_i d^2 d(1/\psi)}{d\psi^2} + Q_{1i} - Q_{2i} \quad (S.5)$$

where

$$Q_{1i} = \frac{\int_{\mathbb{R}^d_A} g_i^2(u) c(u) \exp\{-n h_i(u)\} \, du}{\int_{\mathbb{R}^d_A} c(u) \exp\{-n h_i(u)\} \, du} \quad (S.6)$$

and

$$Q_{2i} = \frac{\int_{\mathbb{R}^d_A} g_i(u) c(u) \exp\{-n h_i(u)\} \, du}{\int_{\mathbb{R}^d_A} c(u) \exp\{-n h_i(u)\} \, du} \quad (S.7)$$

with

$$c(u) = \exp\left( -\frac{1}{2} u^T \Sigma^{-1} u \right), \quad g_i(u) = \sum_{j=1}^{n_i} \{ Y_{ij} u^T X_{Aij} - b((\beta_A + u)^T X_{Aij} + \beta_B^T X_{Bij}) \}$$
and \( h_i \) is as given by (S.3). Application of (S.2) to each of (S.6) and (S.7) and use of (S.4) leads to the following three-term approximations to \( Q_{1i} \) and \( Q_{2i} \):

\[
Q_{1i} = g_i(U_i)^2 + g_i^T \mathbf{H}_{ii}^{-1} g_i(U_i) - \frac{d_i g_i(U_i)}{\psi} + O_P(n^{1/2})
\]

and

\[
Q_{2i} = g_i(U_i) + \frac{1}{2} g_i^T \mathbf{H}_{ii}^{-1} g_i(U_i) - \frac{d_i g_i(U_i)}{2\psi} + O_P(n^{-1/2}).
\]

Therefore

\[
Q_{1i} - Q_{2i}^2 = g_i(U_i)^2 + g_i^T \mathbf{H}_{ii}^{-1} g_i(U_i) - \frac{d_i g_i(U_i)}{\psi} + O_P(n^{1/2})
\]

\[
- \left\{ g_i(U_i) + \frac{1}{2} g_i^T \mathbf{H}_{ii}^{-1} g_i(U_i) - \frac{d_i g_i(U_i)}{2\psi} + O_P(n^{-1/2}) \right\}^2
\]

\[
= g_i(U_i)^2 + g_i^T \mathbf{H}_{ii}^{-1} g_i(U_i) - \frac{d_i g_i(U_i)}{\psi} + O_P(n^{1/2})
\]

\[
- \left\{ g_i(U_i) + g_i^T \mathbf{H}_{ii}^{-1} g_i(U_i) - \frac{d_i g_i(U_i)}{2\psi} \right\} + O_P(n^{1/2})
\]

\[
= O_P(n^{1/2})
\]

and we arrive at the approximation

\[
\frac{\partial^2 \log \{ p_{Y_i | X_i}(Y_i | X_i) \}}{\partial \psi^2} = - \frac{n_i d^2 d(1/\psi)}{d^2} + O_P(n^{1/2}).
\]

Therefore, the \( \psi \) diagonal block of the Fisher information is

\[
\frac{mn \left( \frac{d^2 d(1/\psi)}{d^2} \right)}{O_P(mn^{1/2})}.
\]

S.2.2.5 The \((\beta_A, \beta_B, \psi)\) Off-Diagonal Block

Let

\[
\beta = \begin{bmatrix} \beta_A \\ \beta_B \end{bmatrix}
\]

denote the full fixed effects vector. As established in the appendix of Wand (2007), the \( i \)th contribution to the partial derivative, with respect to \( \psi \), of the \( \beta \) score is

\[
\frac{\partial \nabla_{\beta} \log \{ p_{Y_i | X_i}(Y_i | X_i) \}}{\partial \psi} = \mathbf{X}_i^T \{ Y_i - E[Y_i | U_i | Y_i] \}.
\]

Noting that

\[
E[\mathbf{X}_i^T \{ Y_i - E[E(Y_i | U_i) | Y_i] \} | X_i] = \mathbf{X}_i^T \left( E(Y_i) - E[E(Y_i | U_i) | Y_i] \right)
\]

\[
= \mathbf{X}_i^T \{ E(Y_i) - E(Y_i) \} = \mathbf{O}
\]

it is apparent that the \((\beta_A, \beta_B, \psi)\) off-diagonal block of the Fisher information has all entries being exactly zero.
The $(\text{vech}(\Sigma), \psi)$ Off-Diagonal Block

The $i$th contribution to the second order partial derivative with respect to $\text{vech}(\Sigma)$ and $\psi$ is

$$\nabla_{\text{vech}(\Sigma)} \frac{\partial \log \{ p_{Y_i | X_i}(Y_i | X_i) \}}{\partial \psi} = \nabla_{\text{vech}(\Sigma)} \left[ \int_{\mathbb{R}^d_+} g_i(u) c(u) \exp\{-nh_i(u)\} \, du \right]$$

$$= \frac{1}{2} (Q_{3i} - Q_{2i} Q_{4i})$$

where

$$Q_{3i} \equiv \int_{\mathbb{R}^d_+} g_i(u) D_{d_A}^T \text{vec}(\Sigma^{-1} uu^T \Sigma^{-1}) c(u) \exp\{-nh_i(u)\} \, du$$

and

$$Q_{4i} \equiv \int_{\mathbb{R}^d_+} D_{d_A}^T \text{vec}(\Sigma^{-1} uu^T \Sigma^{-1}) c(u) \exp\{-nh_i(u)\} \, du$$

Application of (S.2) to each of (S.9) and (S.10) and use of (S.4) leads to the following approximations to $Q_{3i}$ and $Q_{4i}$:

$$Q_{3i} = D_{d_A}^T \text{vec} \left( \Sigma^{-1} (U_i U_i^T + 2H_{\Sigma_i}^{-1} G_i U_i^T) \Sigma^{-1} \right) g_i(U_i) + O_P(1) 1_{d_{A}^\text{even}}$$

and

$$Q_{4i} = D_{d_A}^T \text{vec} \left( \Sigma^{-1} (U_i U_i^T + 2H_{\Sigma_i}^{-1} G_i U_i^T) \Sigma^{-1} \right) + O_P(n^{-1}) 1_{d_{A}^\text{even}}$$

where $d_{A}^\text{even} \equiv d_A(d_A + 1)/2$. Substitution of these approximations into (S.8) then gives

$$\nabla_{\text{vech}(\Sigma)} \frac{\partial \log \{ p_{Y_i | X_i}(Y_i | X_i) \}}{\partial \psi} = \frac{1}{2} D_{d_A}^T \text{vec} \left( \Sigma^{-1} (U_i U_i^T + 2H_{\Sigma_i}^{-1} G_i U_i^T) \Sigma^{-1} \right) g_i(U_i)$$

$$- \frac{1}{2} \left( g_i(U_i) + O_P(1) \right) \left\{ D_{d_A}^T \text{vec} \left( \Sigma^{-1} (U_i U_i^T + 2H_{\Sigma_i}^{-1} G_i U_i^T) \Sigma^{-1} \right) + O_P(n^{-1}) 1_{d_{A}^\text{even}} \right\}$$

$$= O_P(1) 1_{d_{A}^\text{even}}.$$

It follows that the $(\text{vech}(\Sigma), \psi)$ off-diagonal block of the Fisher information matrix is

$$- E \left[ \sum_{i=1}^m \nabla_{\text{vech}(\Sigma)} \frac{\partial \log \{ p_{Y_i | X_i}(Y_i | X_i) \}}{\partial \psi} \right] = O_P(m) 1_{d_{A}^\text{even}}.$$
S.2.2.7 Assembly of Fisher Information Sub-Block Approximations

From the Fisher information sub-block approximations obtained in the previous six subsections, we have

\[
I \left( \beta_A, \beta_B, \text{vech}(\Sigma), \psi \right) = \begin{bmatrix}
m\Sigma^{-1} + O_P(mn^{-1})1_d^{\otimes 2} & O_P(m)1_{d_A}1_{d_b}^T & O_P(mn^{-1})1_{d_A}1_{d_b}^{T\otimes 2} & 0_{d_A} \\
O_P(m)1_{d_b}1_{d_A}^T & \frac{mn\Lambda_{A_B}^{-1}}{\phi} + o_P(mn)1_{d_b}^{\otimes 2} & O_P(m)1_{d_b}1_{d_b}^{T\otimes 2} & 0_{d_b} \\
O_P(mn^{-1})1_{d_A}1_{d_b}^T & O_P(m)1_{d_A}1_{d_b}^T & \frac{mD_d^T (\Sigma^{-1} \otimes \Sigma^{-1})D_d}{2} + O_P(mn^{-1})1_{d_A}^{\otimes 2} & O_P(m)1_{d_b}^{\otimes 2} \\
0_{d_b}^T & 0_{d_b}^T & O_P(m)1_{d_b}^T & mn \left( \frac{d^2 d(1/\psi)}{d\psi^2} \right) + O_P(mn^{1/2}) 
\end{bmatrix}
\]

S.2.3 Inverse Fisher Information Approximation

Note that

\[
I \left( \beta_A, \beta_B, \text{vech}(\Sigma), \psi \right) = \begin{bmatrix}
A_{11} & A_{12} \\
A_{12}^T & A_{22}
\end{bmatrix}
\]

where

\[
A_{11} = \begin{bmatrix}
m\Sigma^{-1} + O_P(mn^{-1})1_d^{\otimes 2} & O_P(m)1_{d_A}1_{d_b}^T & O_P(mn^{-1})1_{d_A}1_{d_b}^{T\otimes 2} \\
O_P(m)1_{d_b}1_{d_A}^T & \frac{mn\Lambda_{A_B}^{-1}}{\phi} + o_P(mn)1_{d_b}^{\otimes 2} & O_P(m)1_{d_b}1_{d_b}^{T\otimes 2} \\
O_P(mn^{-1})1_{d_A}1_{d_b}^T & O_P(m)1_{d_A}1_{d_b}^T & \frac{mD_d^T (\Sigma^{-1} \otimes \Sigma^{-1})D_d}{2} + O_P(mn^{-1})1_{d_A}^{\otimes 2}
\end{bmatrix}, \quad \text{(S.11)}
\]

\[
A_{22} \equiv mn E \left\{ \frac{d^2 d(1/\psi)}{d\psi^2} \right\} + O_P(mn^{1/2}) \quad \text{and} \quad A_{12} \equiv \begin{bmatrix} 0_{d_A}^T & 0_{d_b}^T & O_P(m)1_{d_b}^T \end{bmatrix}^T.
\]

Let

\[
I \left( \beta_A, \beta_B, \text{vech}(\Sigma), \psi \right)^{-1} = \begin{bmatrix}
A_{11} & A_{12} \\
(A_{12})^T & A_{22}
\end{bmatrix}.
\]

The upper left block of \( I \left( \beta_A, \beta_B, \text{vech}(\Sigma), \psi \right)^{-1} \) is

\[
A_{11}^{-1} = A_{11}^{-1} + \frac{A_{11}^{-1} A_{12} A_{12}^T A_{11}^{-1}}{A_{22} - A_{12} A_{11}^{-1} A_{12}^T}.
\]

The leading terms of \( A_{11}^{-1} \) are provided by equation (A27) of Jiang \textit{et al.} (2022). Since \( A_{12}^T A_{11}^{-1} A_{12} = O_P(m) \) we have

\[
A_{22} - A_{12} A_{11}^{-1} A_{12} \equiv mn \left( \frac{d^2 d(1/\psi)}{d\psi^2} \right) + O_P(mn^{1/2})
\]
and so
\[
\frac{1}{A_{22} - A_{12}^T A_{11}^{-1} A_{12}} = O_P(m^{-1}n^{-1}).
\]  
(S.13)

Also,

the lower right block of \(A_{11}^{-1} A_{12} A_{12}^T A_{11}^{-1}\) is exactly zero. Hence, in view of (S.13), the second term of (S.12) is a matrix with all entries either \(O_P(m^{-1}n^{-1})\) or zero. Consequently, \(A_{11}\) equals the right-hand side of (A27) of Jiang et al. (2022) except for a rearrangement of the entries to concur with the \((\beta_A, \beta_B, \text{vech}(\Sigma))\) parameter ordering rather than \((\beta_A, \text{vech}(\Sigma), \beta_B)\).

The lower right entry of
\[
I(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)^{-1}
\]
is
\[
A_{22} = \frac{1}{A_{22}} + \frac{A_{12}^T (A_{11} - A_{12}A_{12}^T/A_{22})^{-1} A_{12}}{A_{22}^2}.
\]  
(S.14)

Note that
\[
\frac{1}{A_{22}} = \left(\frac{d^2 d(1/\psi)}{d\psi^2}\right)^{-1} (mn)^{-1} + O_P(m^{-1}n^{-3/2}).
\]

Also

the lower right block of \(A_{12} A_{12}^T/A_{22} = O_P(mn^{-1})1_{d_A^2}\)
and all other entries of \(A_{12} A_{12}^T/A_{22}\) are exactly zero. It follows that the expression on the right-hand side of (S.11) also holds for \(A_{11} - A_{12} A_{12}^T/A_{22}\) and, using equation (A27) of Jiang et al. (2022),
\[
\frac{A_{12}^T (A_{11} - A_{12} A_{12}^T/A_{22})^{-1} A_{12}}{A_{22}^2} = O_P(m^{-1}n^{-2}).
\]

Hence, the second term on the right-hand side of (S.14) is asymptotically negligible compared with the first term and we have
\[
A_{22} = \left(\frac{d^2 d(1/\psi)}{d\psi^2}\right)^{-1} (mn)^{-1} + O_P(m^{-1}n^{-3/2}).
\]

The off-diagonal block of the inverse Fisher information matrix is
\[
A_{12} = -(A_{11} - A_{12} A_{12}^T/A_{22})^{-1} A_{12}/A_{22} = \begin{bmatrix} 0_{d_{A}^T} & 0_{d_{B}^T} & O_P(m^{-1}n^{-1})1_{d_{A}^T} \end{bmatrix}^T
\]
and so we have
\[
I(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)^{-1} = I(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)^{-1}
\]
\[
+ \frac{1}{mn} \begin{bmatrix} O_P(1)1_{d_{A}^T} & O_P(1)1_{d_{A}^T} & O_P(1)1_{d_{A}^T} & O_P(1)1_{d_{A}^T} & 0_{d_{A}} \\ O_P(1)1_{d_{A}}^T & O_P(1)1_{d_{A}}^T & O_P(1)1_{d_{A}}^T & O_P(1)1_{d_{A}}^T & 0_{d_{A}} \\ O_P(1)1_{d_{A}}^T & O_P(1)1_{d_{A}}^T & O_P(1)1_{d_{A}}^T & O_P(1)1_{d_{A}}^T & 0_{d_{A}} \\ 0_{d_{A}}^T & 0_{d_{A}}^T & 0_{d_{A}}^T & 0_{d_{A}}^T & 0_{d_{A}}^T \\ O_P(1)1_{d_{B}}^T & O_P(1)1_{d_{B}}^T & O_P(1)1_{d_{B}}^T & O_P(1)1_{d_{B}}^T & O_P(1)1_{d_{B}}^T \end{bmatrix}
\]  
(S.15)
where

\[
I(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)_{\infty}^{-1} \equiv \begin{bmatrix}
\Sigma/m & O & O & O \\
O & \phi A_{\beta B}/mn & O & O \\
O & 2D_{dA}^T (\Sigma \otimes \Sigma) D_{dA}^{+T} / m & O \\
O & O & O & O \\
\end{bmatrix}.
\]

### S.2.4 Asymptotic Equivalence of \( \{I(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)_{\infty}^{-1}\}^{1/2} \)

and \( \{I(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)_{\infty}^{-1}\}^{1/2} \)

Our aim in this subsection is to establish asymptotic equivalence between \( \{I(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)^{-1}\}_{\infty}^{1/2} \)

and \( \{I(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)^{-1}\}_{\infty}^{1/2} \) in the following sense:

\[
\left\| \{I(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)^{-1}\}_{\infty}^{1/2} \{I(\beta_A, \beta_B, \text{vech}(\Sigma), \psi)^{-1}\}_{\infty}^{1/2} - I \right\|_F \to 0. \quad (S.16)
\]

Without loss of generality, we change the ordering of the parameters from \((\beta_A, \beta_B, \text{vech}(\Sigma), \psi)\)

to \((\beta_A, \text{vech}(\Sigma), \beta_B, \psi)\) and note that

\[
I(\beta_A, \text{vech}(\Sigma), \beta_B, \psi)_{\infty}^{-1} = \frac{1}{m} \begin{bmatrix} K & O \\ O & \frac{1}{n} L \end{bmatrix}
\]

where

\[
K = \begin{bmatrix} \Sigma & O \\ O & 2D_{dA}^T (\Sigma \otimes \Sigma) D_{dA}^{+T} \end{bmatrix}
\]

and \( L \equiv \begin{bmatrix} \phi A_{\beta B} & O \\ O & \left( \frac{d^2 d(1/\psi)}{d\psi^2} \right)_{\psi=\psi_0}^{-1} \end{bmatrix} \).

Also,

\[
I(\beta_A, \text{vech}(\Sigma), \beta_B, \psi)^{-1} = \frac{1}{m} \begin{bmatrix} K + O_p(1/n)1_{dA+d_B}^{\otimes 2} & O_p(n^{-1})1_{dA+d_B}^{T} 1_{d_B+1}^{T} \\ O_p(n^{-1})1_{d_B+1}^{T} 1_{dA+d_B}^{\otimes 2} & \frac{1}{n} L + O_p(n^{-1})1_{d_B+1}^{\otimes 2} \end{bmatrix}
\]

and (S.16) follows from Lemma 2 of Jiang et al. (2022).

### S.2.5 Final Steps

Using steps analogous to those given in Appendix A.8 of Jiang et al. (2022) we have

\[
\sqrt{n} \begin{bmatrix} \tilde{\beta}_A - \beta_A^0 \\ \sqrt{n} (\tilde{\beta}_B - \beta_B^0) \\ \text{vech}(\tilde{\Sigma} - \Sigma^0) \\ \sqrt{n} (\tilde{\psi} - \psi^0) \end{bmatrix} \xrightarrow{D} N \left( \begin{bmatrix} 0 \\ 0 \\ \Sigma^0 & O & O & O \\ 0 & \phi 0 A_{\beta B} & O & O \\ O & O & 2D_{dA}^T (\Sigma^0 \otimes \Sigma^0) D_{dA}^{+T} & O \\ 0 & 0 & O & O \end{bmatrix} \right).
\]

Application of the Multivariate Delta Method (e.g. Agresti, 2013, Section 16.1.3) with the mapping

\[
(x_1, \ldots, x_{d_A+d_B+d_B^\oplus}, x_{d_A+d_B+d_B^\oplus+1}) \mapsto (x_1, \ldots, x_{d_A+d_B+d_B^\oplus}, 1/x_{d_A+d_B+d_B^\oplus+1})
\]

leads to Theorem 1.
S.3 Illustration of Simplification of (A3)

Assumption (A3) of Theorem 1 is that
\[
E \left[ \max \left( 1, \|X\| \right)^8 \max \left\{ 1, b''((\beta_A + U)^T X_A + \beta_B^T X_B) \right\}^4 |U| \right] < \infty \tag{S.17}
\]
for all \( \beta_A \in \mathbb{R}^{d_A}, \beta_B \in \mathbb{R}^{d_B} \) and \( \Sigma \) a \( d_A \times d_A \) symmetric and positive definite matrix.

In this section we prove that, for the special case:
\[
d_A = 1, \quad X_A = 1 \quad \text{and} \quad b = \exp \tag{S.18}
\]
corresponding to Poisson responses, assumption (A3) is implied by the moment generating function existence condition
\[
E\{\exp(t^T X_B)\} < \infty \quad \text{for all} \quad t \in \mathbb{R}^{d_B}. \tag{S.19}
\]
To justify the sufficiency of (S.19) first note that, for the (S.18) special case, the numerator of assumption (S.19), the expectation of (S.20) is finite which implies that (S.17) holds for the special case:
\[
\max \left\{ 1, \exp \left( \beta_A + U + \beta_B^T X_B \right) \right\}^4 |U| \]
Then application of the Cauchy-Schwartz inequality for conditional expectations gives
\[
E\left[ \max \left\{ 1, \exp \left( \beta_A + U + \beta_B^T X_B \right) \right\}^4 |U| \right] \leq \left( E\left[ \max \left\{ 1, \exp \left( \beta_A + U + \beta_B^T X_B \right) \right\}^8 |U| \right] \right)^{1/2} \left( E\left[ \max \left\{ 1, \exp \left( \beta_A + U + \beta_B^T X_B \right) \right\}^8 |U| \right] \right)^{1/2}
\leq \left[ 1 + E\left\{ \left( 1 + \|X\|^8 \right) \right\} \right]^{1/2} \left[ 1 + E\left\{ \exp \{ 8(\beta_A + U + \beta_B^T X_B) \} \right\} \exp(8U) \right]^{1/2}
\leq \left[ 1 + E\left\{ \left( 1 + \|X\|^8 \right) \right\} \right]^{1/2} \left[ 1 + \exp(8\beta_A)E\{ \exp(8\beta_B^T X_B) \} \exp(8U) \right]^{1/2}.
\]
The denominator of the random variable inside the outermost expectation of (S.17) is
\[
\min \left\{ 1, \exp \left( \beta_A + U + \beta_B^T X_B \right) |U| \right\}^2 \\
= \min \left\{ 1, \exp(2\beta_A)\left| E\{\exp(\beta_B^T X_B)\} \right|^2 \exp(2U) \right\}.
\]
Noting that, for all \( x \in \mathbb{R} \) and \( a, b > 0 \),
\[
\frac{\left\{ 1 + a \exp(8x) \right\}^{1/2}}{\min \left\{ 1, b \exp(2x) \right\}} \leq 1 + \frac{a^{1/2} \exp(2x)}{b} + a^{1/2} \exp(4x) + \frac{\exp(-2x)}{b}
\]
the random variable inside the outermost expectation of (S.17) is bounded above by
\[
\left[ 1 + E\left\{ \left( 1 + \|X\|^8 \right) \right\} \right]^{1/2} \left[ 1 + \frac{\exp(2\beta_A)\left| E\{\exp(8\beta_B^T X_B)\} \right|^{1/2} \exp(2U)}{\left| E\{\exp(\beta_B^T X_B)\} \right|^2} + \exp(4\beta_A)\left| E\{\exp(8\beta_B^T X_B)\} \right|^{1/2} \exp(4U) \right]^{1/2}
\leq \exp(-2U) + \frac{\exp(-2U)}{\exp(2\beta_A)\left| E\{\exp(\beta_B^T X_B)\} \right|^2}.
\]
Since \( U \sim N(0, (\sigma^2)^0) \) we have \( E\{\exp(tU)\} = \exp\left\{ \frac{1}{2} t^2 (\sigma^2)^0 \right\} \) for all \( t \in \mathbb{R} \). Hence, under assumption (S.19), the expectation of (S.20) is finite which implies that (S.17) holds for the (S.18) special case.
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