GLOBAL SOLUTION TO THE 3-D INHOMOGENEOUS INCOMPRESSIBLE MHD SYSTEM WITH DISCONTINUOUS DENSITY

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Abstract. In this paper, we consider the Cauchy problem of the incompressible MHD system with discontinuous initial density in $\mathbb{R}^3$. We establish the global well-posedness of the MHD system if the initial data satisfies $(\rho_0, u_0, H_0) \in L^\infty(\mathbb{R}^3) \times H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ with $\frac{1}{2} < s \leq 1$ and

$$0 < \underline{\rho} \leq \rho_0 \leq \overline{\rho} < +\infty, \quad \|(u_0, H_0)\|_{H^\frac{1}{2}} \leq c,$$

for some small $c > 0$ which only depends on $\underline{\rho}, \overline{\rho}$. As a byproduct, we also get the decay estimate of the solution.

1. Introduction and the main results. Magnetohydrodynamic (MHD) system describes the interaction between the magnetic field and conductive fluid, which is a nonlinear system that couples Navier-Stokes equations with Maxwell equations. In magnetohydrodynamics, the displacement currents can be neglected in the time dependent Maxwell equations. So it becomes the following system:

$$\begin{align*}
\partial_t p + \text{div}(pu) &= 0, \\
\partial_t (pu) + \text{div}(pu \otimes u) - \Delta u + \nabla P &= \text{curl}H \times H, \\
\partial_t H - \Delta H &= \text{curl}(u \times H), \\
\text{div}u &= \text{div}H = 0, \\
(p, u, H)|_{t=0} &= (\rho_0, u_0, H_0),
\end{align*}$$

(1)

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where \( \rho \) is the density, \( u \) is the velocity field, \( H \) is the magnetic field and \( P \) is the scalar pressure. The body force \( \text{curl} H \times H = H \cdot \nabla H - \nabla \left( \frac{|H|^2}{2} \right) \) and \( \text{curl}(u \times H) = H \cdot \nabla u - u \cdot \nabla H \) if \( \text{div} u = \text{div} H = 0 \).

When \( H = 0 \), (1) turns into the well-known inhomogeneous incompressible Navier-Stokes system, which has been studied by many researchers (see [1], [2], [9], [12], [13], [14], [28], [31], [32], [35], [38]). When the density \( \rho \) is a constant, (1) reduces to be the classical MHD system which has been studied also by many researchers (see [6], [7], [10], [16], [21], [22], [29], [30], [33]). Duvaut and Lions [16] established the local well-posedness in \( H^s(\mathbb{R}^N) \), \( s \geq N \), and certified global existence of the solution with small initial data. Sermange and Temam [34] proved that the 2-D local strong solution can be extended to global and unique. With mixed partial dissipation and additional magnetic diffusion in the 2-D MHD system, Cao and Wu [7] proved that such a system is globally well-posed for any data in \( H^2(\mathbb{R}^2) \).

In a recent remarkable paper, Lin, Xu and Zhang [29] proved the global existence of smooth solution of the 2-D MHD system around the non-trivial steady state solution \( (x_2, 0) \) (see [30] for 3-D case). In [33], Ren, Wu, Xiang and Zhang got the global existence and the decay estimates of small smooth solution for the 2-D MHD equations without magnetic diffusion. There are also many results on the regularity criteria (see [10], [21], [22]).

When the density \( \rho \) is not a constant, (1) is the so called inhomogeneous incompressible MHD system. Compared with the Navier-Stokes equations, the dynamic motion of the fluid and the magnetic field interact on each other and both the hydrodynamic and electrodynamic effects in the motion are strongly coupled, the problems of MHD system are considerably more complicated. Even though, in the past several years, there are also many mathematical results related to the incompressible MHD system (see [3], [4], [8], [11], [15], [16], [18], [26], [27], [34], [36], [37], [38]). Gerbeau and Le Bris [18] and also Desjardins and Le Bris [15] researched global existence of weak solution of finite energy in \( \mathbb{R}^3 \) and in the torus \( \mathbb{T}^3 \). Chen, Tan and Wang [11] showed the local strong solution in \( H^2 \) when the initial data contain vacuum states (i.e. the initial density \( \rho \) may vanish in some open set of \( \Omega \)). Then, Huang and Wang [26] extended the local strong solution to be global in two dimensions (see also Gong and Li [19] for three dimensions).

When the initial density is away from zero and is close enough to a positive constant, Abidi and Hmidi in [3] and Abidi and Paicu in [4] obtained the global solution with small initial data in the critical Besov spaces. By critical, we mean that we want to solve the system (1) in functional spaces with invariant norms by the changes of scales which leaves (1) invariant. In the case of inhomogeneous incompressible MHD fluids, it is easy to see that the transformations: \( (\rho_\lambda, u_\lambda, H_\lambda)(t, x) = (\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda H(\lambda^2 t, \lambda x)) \) have that property, provided that the pressure term has been changed accordingly. The results in [3], [4] have been extended by Chen, Li and Xu in [8], Zhai, Li and Xu in [36], Zhai, Li and Yan in [37].

When the initial density is away from zero and is not close enough to a positive constant, Gui in [20] considered the global well-posedness of 2-D MHD equations with constant viscosity and variable conductivity for a generic family of the variations of the initial data, and established the global well-posedness of the equations in the critical spaces with constant conductivity. Zhai and Yin in [38] got the global solution in \( \mathbb{R}^3 \) by applying a new a priori estimate for an elliptic equation with non-constant coefficients. Hoff [23, 24] and Huang, Li and Xin [25] respectively proved the global existence of small energy weak solutions and global well-posedness of
small energy classical solutions of the isentropic compressible Navier-Stokes equations, where, [25] is the first for global classical solutions that may have large oscillations and can contain vacuum states. An important idea in [23], [24] and [25] is that by using an appropriate time weight, the energy estimate for space derivatives of the velocity field can be closed although the initial date has low regularity (even only in $L^2(\mathbb{R}^d)$). By using a similar idea, Chen, Zhang and Zhao in [9] obtained the Fujita-Kato solution for the 3-D inhomogeneous Navier-Stokes equations.

In this paper, we will get a similar result to [9] for the 3-D inhomogeneous compressible MHD system. We establish the global existence and uniqueness of the solution, under the condition that the initial date $(u_0, H_0)$ is small in the critical space $H^\frac{2}{s}$. We use the idea of time weight in [9], [23], [24] and [25] to deal with less regular initial velocity field and magnetic field, and the Lagrangian idea in [13], [14] to deal with the proof of the uniqueness of the solution. At last, we also get the decay estimate of the solution by the dual method as a generalization of the result in [9].

The main results of this paper are the following theorems:

**Theorem 1.1.** Let $\frac{1}{2} < s \leq 1$. Given $\underline{\rho}, \underline{\rho} \in (0, \infty)$, suppose that

\[
0 < \rho \leq \rho_0 \leq \rho < +\infty, \quad (u_0, H_0) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3),
\]

and there exists a constant $c$ depending only on $\underline{\rho}, \underline{\rho}$ such that

\[
\|(u_0, H_0)\|_{H^\frac{2}{s}} \leq c,
\]

then the system (1) has a unique global solution $(\rho, u, H)$ satisfying

\[
\rho \leq \rho(t, x) \leq \overline{\rho} \quad \text{for} \quad (t, x) \in [0, \infty) \times \mathbb{R}^3,
\]

\[
F_0(t) \leq C\|(u_0, H_0)\|_{H^s}^2 \quad \text{for} \quad t \in [0, \infty),
\]

\[
F_1(t) \leq C\|(u_0, H_0)\|_{H^s}^2 \quad \text{for} \quad t \in [0, \infty),
\]

\[
F_2(t) \leq C\|(u_0, H_0)\|_{H^s}^2 \exp\{C\|(u_0, H_0)\|_{H^s}^4\} \quad \text{for} \quad t \in [0, \infty),
\]

where the constant $C$ depending only on $\underline{\rho}, \underline{\rho}$ and

\[
F_0(t) = \int_{\mathbb{R}^3} |(\nabla u, H)(t, x)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |(\nabla H)|^2 dx dt,
\]

\[
F_1(t) = \omega(t)^{-s} \int_{\mathbb{R}^3} |(\nabla u, \nabla H)|^2 dx
+ \int_0^t \int_{\mathbb{R}^3} \omega(\tau)^{-s} |(\nabla u, H)|^2 + |(\nabla^2 u, \nabla^2 H)|^2 + |\nabla P|^2 |dx dt,
\]

\[
F_2(t) = \omega(t)^{2-s} \int_{\mathbb{R}^3} |(\nabla u, H)|^2 + |(\nabla^2 u, \nabla^2 H)|^2 + |\nabla P|^2 |dx
+ \int_0^t \int_{\mathbb{R}^3} \omega(\tau)^{2-s} |(\nabla u, \nabla H)|^2 dx dt,
\]

in which $\omega(t) \triangleq \min\{1, t\}$.

**Remark 1.** In Theorem 1, The powers to the weight $\omega(t)$ in $F_i(t)$, $i = 1, 2$ are inspired by the following characterization of Besov norm (c.f. Theorem 2.34 of [5]):

\[
C^{-1}\|u\|_{B^{\frac{2}{s}, p}_{p, q}} \leq \|\|t^r e^{A t} u\|_{L^p}\|_{L^q(\mathbb{R}^+; B^{\frac{2}{s}, p}_{p, q})} \leq C\|u\|_{B^{\frac{2}{s}, p}_{p, q}}, \quad r > 0, \quad 1 \leq p, q \leq +\infty.
\]
Which means that for \( u_0, H_0 \in H^s(\mathbb{R}^3) \) with \( s \in (0, 1) \), (5) and
\[
\nabla u_0, \nabla H_0 \in H^{s-1}(\mathbb{R}^3) \rightarrow B^{s-1}_{2,\infty}(\mathbb{R}^3), \quad \nabla^2 u_0, \nabla^2 H_0 \in B^{s-2}_{2,\infty}(\mathbb{R}^3),
\]
lead to
\[
t^{\frac{1-s}{2}} \| (e^{t \Delta} \nabla u_0, e^{t \Delta} \nabla H_0) \|_{L^2} \leq C \|(u_0, H_0)\|_{H^s},
\]
and
\[
t^{\frac{s-2}{2}} \| (e^{t \Delta} \nabla^2 u_0, e^{t \Delta} \nabla^2 H_0) \|_{L^2} \leq C \|(u_0, H_0)\|_{H^s}.
\]

**Remark 2.** The aim of the introduction of the time weight \( \omega(t) \) in Theorem 1.1 is to close the energy estimates for space derivatives of the velocity field and the magnetic field with low regular initial data \((u_0, H_0) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)\) when \( \frac{1}{2} < s \leq 1 \). Here we only consider the low regular case when \( \frac{1}{2} < s \leq 1 \), since it is much easier to deal with the higher regular case.

We also get the following decay estimate of the solution obtained in Theorem 1.1:

**Theorem 1.2.** Assume that \((u, H)\) is a solution obtained in Theorem 1.1, and the initial data \((u_0, H_0) \in L^q(\mathbb{R}^3), \ q \in \left[ \frac{6}{5}, 2 \right] \). Then the solution satisfies the following inequality
\[
\| (\nabla^k u, \nabla^k H)(t) \|_{L^2} \leq C(1 + t)^{-\frac{k}{2} - \frac{3}{2q}} \zeta(q), \quad \text{for} \quad t \geq 1, \ k = 0, 1,
\]
where \( \zeta(q) = \frac{3}{2} \left( \frac{1}{q} - \frac{1}{2} \right) \).

Throughout this paper, we use the following notations. For simplicity, we denote
\[
\int f \, dx = \int_{\mathbb{R}^3} f \, dx.
\]
For \( 1 \leq p \leq \infty \), and \( m \in \mathbb{N} \), the Sobolev spaces are defined in a standard manner:
\[
L^p \triangleq L^p(\mathbb{R}^3), \quad H^m \triangleq W^{m,2}(\mathbb{R}^3).
\]
Given a Banach space \( X \), we shall denote by \( \| (a, b) \|_X = \| a \|_X + \| b \|_X \). The rest of the paper unfolds as follows. In Section 2, we prove the global existence part of Theorem 1.1 by using the time weighted energy method; in Section 3, we complete the uniqueness of the solutions by using the Lagrangian coordinate method; in Section 4, we further prove the decay of the solution by applying the dual method.

2. **The proof of the existence part of Theorem 1.1.** In this section, we concentrate on the proof of the existence part of Theorem 1.1. Let \( j_\sigma \) be the standard Friedrich’s mollifier, and define
\[
\rho^\sigma_0 \triangleq j_\sigma * \rho_0, \quad u^\sigma_0 \triangleq j_\sigma * u_0, \quad \text{and} \quad H^\sigma_0 \triangleq j_\sigma * H_0.
\]
Choose \( \sigma \) small enough such that
\[
\rho^\sigma / 2 \leq \rho^\sigma (x) \leq 2 \rho, \quad x \in \mathbb{R}^3.
\]
With the initial data \((\rho^\sigma_0, u^\sigma_0, H^\sigma_0)\), there exists a time \( T^\sigma > 0 \) such that system (1) has a unique smooth solution \((\rho^\sigma, u^\sigma, H^\sigma)\) on \([0, T^\sigma]\). In what follows, we’ll only present the uniform estimates (4) of the smooth approximate solutions \((\rho^\sigma, u^\sigma, H^\sigma)\).

Then, the existence part of Theorem 1.1 essentially follows by (4) and a standard compactness argument. In the sequel, we omit the superscript \( \sigma \) for simplicity. And we denote by \( C \) or \( C_i \) \((i = 1, 2)\) the positive constant, which may depend on \( \overline{\rho}, \overline{\rho} \) but does not rely on the time \( T \) and the superscript \( \sigma \).
Next, we’ll use the bootstrap theory to get the uniform estimate. We suppose that the following a priori hypothesis holds:

$$\|(u, H)(t)\|_{L^3} \leq c_0, \quad t \in [0, T].$$

(6)

for some small enough \(c_0 > 0\) determined later. Next, we’ll get the estimate (4) and deduce the following refined estimate

$$\|(u, H)(t)\|_{L^3} \leq \frac{c_0}{2}, \quad t \in [0, T].$$

(7)

**Step 1.** \(L^2\) and \(H^1\) estimates without the time weight.

From the transport equation, we can easily get

$$\|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty},$$

and with (2), we have

$$\underline{\rho} \leq \rho(t, x) \leq \overline{\rho}, \quad (t, x) \in [0, T] \times \mathbb{R}^3.$$

Taking \(L^2\) inner product of (1)\_2 with \(u\), integrating by parts over \(\mathbb{R}^3\), and using the transport equation (1)\_1, we get

$$\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx + \int |\nabla u|^2 dx = \int (\text{curl} H \times H) \cdot u dx.$$  

(8)

Similarly, taking \(L^2\) inner product of (1)\_3 with \(H\), we obtain

$$\frac{1}{2} \frac{d}{dt} \int |H|^2 dx + \int |\nabla H|^2 dx = \int \text{curl}(u \times H) \cdot H dx = \int (u \times H) \cdot \text{curl} H dx = - \int (\text{curl} H \times H) \cdot u dx.$$  

(9)

By combining (8) with (9) and integrating it over \([0, t], \forall t \in [0, T]\), we have

$$\frac{1}{2} \|\sqrt{\rho} u, H(t)\|_{L^2}^2 + \int_0^t \|\nabla u, \nabla H\|_{L^2}^2 \, d\tau = \frac{1}{2} \|\sqrt{\rho_0} u_0, H_0\|_{L^2}^2,$$

(10)

which means \(F_0(t) \leq \|\sqrt{\rho_0} u_0, H_0\|_{L^2}^2\).

Taking \(L^2\) inner product of (1)\_2 with \(u_t\) and integrating by parts over \(\mathbb{R}^3\), we obtain

$$\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 dx + \int \rho |u_t|^2 dx = - \int (u \cdot \nabla) u \cdot u_t dx + \int (H \cdot \nabla) H \cdot u_t dx = - \int (u \cdot \nabla) u \cdot u_t dx + \int (H \cdot \nabla) H \cdot (\sqrt{\rho} u_t)(\sqrt{\rho})^{-1} \leq C \int \|u\| \|\nabla u\| + |H| \|\nabla H\| \sqrt{\rho} u_t dx$$

$$\leq C \|(u, H)\|_{L^6} \|\nabla u, \nabla H\|_{L^3} \sqrt{\rho} u_t \|_{L^2} \leq C \|(u, H)\|_{L^3} \left(\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla^2 u, \nabla^2 H\|_{L^2}^2\right).$$  

(11)
Similarly, taking $L^2$ inner product of (13) with $H_t$, we achieve
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla H|^2 dx + \int |H_t|^2 dx \\
\leq C \int (|H||\nabla u||H_t| + |u||\nabla H||H_t|) dx \\
\leq C \|(u, H)\|_{L^3} (\|H_t\|_{L^2}^2 + \|(\nabla^2 u, \nabla^2 H)\|_{L^2}^2) .
\] (12)

We rewrite (1) as
\[-\Delta u + \nabla P = -\rho u_t - \rho (u \cdot \nabla)u + \text{curl}H \times H,
\]
which along with the classical estimates of the Stokes equation leads to
\[
\|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2} \leq (\|\rho u_t\|_{L^2} + \|\rho (u \cdot \nabla)u\|_{L^2} + \|\text{curl}H \times H\|_{L^2}) \\
\leq C (\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^6} + \|H\|_{L^3} \|\nabla H\|_{L^6}) \\
\leq C (\|\sqrt{\rho} u_t\|_{L^2} + \|(u, H)\|_{L^3} \|(\nabla^2 u, \nabla^2 H)\|_{L^2}).
\] (13)

Similarly,
\[
\|\nabla^2 H\|_{L^2} \leq C (\|H_t\|_{L^2} + \|(H \cdot \nabla)u\|_{L^2} + \|(u \cdot \nabla) H\|_{L^2}) \\
\leq C (\|H_t\|_{L^2} + \|(u, H)\|_{L^3} \|(\nabla^2 u, \nabla^2 H)\|_{L^2}).
\] (14)

By combining (13) with (14), we obtain
\[
\|(\nabla^2 u, \nabla^2 H)\|_{L^2} + \|\nabla P\|_{L^2} \leq C_1 (\|\sqrt{\rho} u_t, H_t\|_{L^2} + \|(u, H)\|_{L^3} \|(\nabla^2 u, \nabla^2 H)\|_{L^2}),
\]
and thus
\[
\|(\nabla^2 u, \nabla^2 H)\|_{L^2} + \|\nabla P\|_{L^2} \leq C \|(\sqrt{\rho} u_t, H_t)\|_{L^2},
\] (15)

provided
\[
\sup_{0 \leq t \leq T} \|(u, H)(t)\|_{L^3} \leq \frac{1}{2C_2}.
\]

Summing up (11) and (12), we get by (15) that
\[
\frac{d}{dt} \int (|\nabla u|^2 + |\nabla H|^2) dx + \int (|u_t|^2 + |H_t|^2) dx \leq C_2 \|(u, H)\|_{L^3} \|(\sqrt{\rho} u_t, H_t)\|_{L^2}^2,
\]
which shows that
\[
\frac{d}{dt} \int (|\nabla u|^2 + |\nabla H|^2) dx + \int (|u_t|^2 + |H_t|^2) dx \leq 0,
\] (16)

provided the constant $c_0$ in (6) satisfies
\[
\sup_{0 \leq t \leq T} \|(u, H)(t)\|_{L^3} \leq c_0 \leq \min \left\{ \frac{1}{2C_1}, \frac{1}{2C_2} \right\}.
\]

For any $t \in [0, T]$, integrating (16) on $[0, t]$, we can get by (15) that,
\[
\|(\nabla u, \nabla H)(t)\|_{L^2}^2 + \int_0^t \|(\sqrt{\rho} u_t, H_t, \nabla^2 u, \nabla^2 H, \nabla P)\|_{L^2}^2 dt \\
\leq C \|(\nabla u_0, \nabla H_0)\|_{L^2}^2.
\] (17)
Step 2. $H^1$ estimate with the time weight and interpolation.

Multiplying (16) with $\omega(t)$ and integrating it on $[0, t]$, $\forall t \in [0, T]$, we get by (15) that

$$\omega(t)\|\nabla u, \nabla H(t)\|_{L^2}^2 + \int_0^t \omega(\tau)\| (\sqrt{\rho_0} u_t, H_t, \nabla^2 u, \nabla^2 H, \nabla P) \|_{L^2}^2 d\tau$$

$$\leq C \int_0^t \| (\nabla u, \nabla H)(\tau) \|_{L^2}^2 d\tau$$

$$\leq C \|(\sqrt{\rho_0} u_0, H_0)\|_{L^2}^2$$

$$\leq C \|(u_0, H_0)\|_{H^1}^2, \quad (18)$$

where we have used (10).

Noting that $(u, H)$ is under the assumption (6), and by using an interpolation argument, we will get the estimate of $F_1(t)$. Considering the following linear system

$$\begin{cases}
\rho(\nu_t + u \cdot \nabla v) - \Delta v + \nabla \tilde{P} = H \cdot \nabla B, \\
B_t + u \cdot \nabla B - \Delta B = H \cdot \nabla v, \\
div v = div B = 0, \\
(v, B)|_{t=0} = (v_0, B_0),
\end{cases}$$

where $\tilde{P}$ includes the magnetic pressure. From the proof of (17) and (18), we get

$$\| (\nabla v, \nabla B)(t) \|_{L^2}^2 + \int_0^t \| (\sqrt{\rho_0} v_t, B_t, \nabla^2 v, \nabla^2 B, \nabla P) \|_{L^2}^2 d\tau \leq C \|(v_0, \nabla B_0)\|_{L^2}^2,$$

$$\omega(t)\|\nabla v, \nabla B(t)\|_{L^2}^2 + \int_0^t \omega(\tau)\| (\sqrt{\rho_0} v_t, B_t, \nabla^2 v, \nabla^2 B, \nabla P) \|_{L^2}^2 d\tau \leq C \|(v_0, B_0)\|_{L^2}^2.$$

Similar to the analysis in [32], by the complex interpolation, we deduce that for any $\theta \in [0, 1],$

$$\omega(t)^{1-\theta}\|\nabla v, \nabla B(t)\|_{L^2}^2 + \int_0^t \omega(\tau)^{1-\theta}\| (\sqrt{\rho_0} v_t, B_t, \nabla^2 v, \nabla^2 B, \nabla P) \|_{L^2}^2 d\tau \leq C \|(v_0, B_0)\|_{H^\theta}^2,$$

which implies that for $t \in [0, T],$

$$F_1(t) \leq C \|(u_0, H_0)\|_{H^\theta}^2, \quad (19)$$

$$\|(u, H)(t)\|_{L^2} \leq C \|(u, H)(t)\|_{H^\frac{1}{2}} \leq C \|(u_0, H_0)\|_{H^\frac{1}{2}}. \quad (20)$$

By choosing $c \leq \frac{\gamma}{2\rho_0}$ in (3), we get (7). Then we complete the bootstrap arguments.

Step 3. $H^2$ estimate.

Differentiating (1) with respect to $t$, taking the $L^2$ inner product with $u_t$, and integrating by parts over $\mathbb{R}^3$, we get by (1) that

$$\frac{1}{2} \int \frac{d}{dt} \rho |u_t|^2 dx + \int |\nabla u_t|^2 dx$$

$$= \int (-2\rho (u \cdot \nabla) u_t \cdot u_t - \rho (u_t \cdot \nabla) u \cdot u_t - \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t)) dx$$

$$+ \int ((\text{curl} H \times H) \cdot u_t + (\text{curl} H \times H_t) \cdot u_t) dx. \quad (21)$$
By similar arguments as (21), from (1), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( |H_t|^2 + |\nabla H_t|^2 \right) dx \\
= \int \left( \text{curl}(u_t \times H_t) \cdot H_t + \text{curl}(u \times H_t) \cdot H_t \right) dx \\
= \int \left( (u_t \times H) \cdot \text{curl}H_t + (u \times H_t) \cdot \text{curl}H_t \right) dx \\
= \int \left( -(\text{curl}H_t \times H_t) \cdot u_t + (u \times H_t) \cdot \text{curl}H_t \right) dx.
\]
(22)

By combining (21) with (22), we have
\[
\frac{1}{2} \frac{d}{dt} \int (\rho |u_t|^2 + |H_t|^2) dx + \int (|\nabla u_t|^2 + |\nabla H_t|^2) dx \\
= \int \left( -2\rho(u \cdot \nabla)u_t \cdot u_t - \rho(u_t \cdot \nabla)u \cdot u_t - \rho u \cdot \nabla(u \cdot \nabla u \cdot u_t) \\
+ (\text{curl}H \times H_t) \cdot u_t + (u \times H_t) \cdot \text{curl}H_t \right) dx.
\]
(23)

Define
\[
\tilde{F}_2(t) \triangleq \frac{1}{2} \omega(t)^{2-s} \int (\rho |u_t|^2 + |H_t|^2) dx \\
+ \int_0^t \omega(\tau)^{2-s} \left( |\nabla u_t|^2 + |\nabla H_t|^2 \right) d\tau.
\]

Multiplying \(\omega(\tau)^{2-s}\) on both sides of (23), integrating with respect to \(\tau\) and then integrating by parts over \(\mathbb{R}^3\), we get
\[
\tilde{F}_2(t) \leq -\frac{1}{2} \int_0^t \omega(\tau)^{2-s} \left( \rho |u_t|^2 + |H_t|^2 \right) dx \\
- \int_0^t \omega(\tau)^{2-s} \int (u_t \cdot \nabla)u \cdot u_t d\tau \\
+ \int_0^t \omega(\tau)^{2-s} \int (u \times H_t) \cdot \text{curl}H_t d\tau \\
+ C \left| \int_0^t \omega(\tau)^{1-s} \left( |u_t|^2 + |H_t|^2 \right) d\tau \right|
\]
\[
\triangleq \sum_{i=1}^6 I_i.
\]
(24)

From (19), Hölder’s inequality, Sobolev inequality in [17], and by choosing \(c_0 \leq \frac{1}{16} C\), we get that
\[
I_1 \leq 2 \int_0^t \omega(\tau)^{2-s} \rho_{L^\infty} \left( \|u\|_{L^2} \|\nabla u_t\|_{L^2} \|u_t\|_{L^2} \right) d\tau \\
\leq Cc_0 \int_0^t \omega(\tau)^{2-s} \|\nabla u_t\|_{L^2}^2 d\tau \\
\leq \frac{1}{16} \tilde{F}_2(t).
\]
(25)
In addition, we deduce by Gagliardo-Nirenberg inequality and Young's inequality that

\[
I_2 \leq \int_0^t \omega(\tau)^{2-s} \|\rho\|_{L^\infty} \|u_t\|_{L^6} \|\nabla u\|_{L^2} \|u_t\|_{L^6} d\tau \\
\leq C \int_0^t \omega(\tau)^{2-s} \|\nabla u\|_{L^2} \|u_t\|_{L^6} \|\nabla u_t\|_{L^6} d\tau \\
\leq C \int_0^t \left(\omega(\tau)^{\frac{2-s}{4}} \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{1}{2}}\right) \left(\omega(\tau)^{\frac{3(2-s)}{8}} \|\nabla u_t\|_{L^6}^{\frac{3}{2}}\right) d\tau \\
\leq C \int_0^t \omega(\tau)^{2-s} \|\nabla u\|_{L^2}^4 \|\sqrt{\rho} u_t\|_{L^2}^2 d\tau + \frac{1}{16} \bar{F}_2(t) \\
\leq C \int_0^t \|\nabla u\|_{L^2}^4 \bar{F}_2(\tau) d\tau + \frac{1}{16} \bar{F}_2(t),
\]  
(26)

and

\[
I_3 = -\int_0^t \omega(\tau)^{2-s} \int (\rho(u \cdot \nabla) u \cdot \nabla u \cdot u_t \\
\quad + \rho(u \otimes u) : \nabla^2 u \cdot u_t + \rho(\nabla u) (u \cdot \nabla u_t) ) \, dx \, d\tau \\
\leq \int_0^t \omega(\tau)^{2-s} \left(\|\rho\|_{L^\infty} \|u\|_{L^6} \|\nabla u\|_{L^2} \|\nabla u\|_{L^6} \|u_t\|_{L^6} \right) \\
\quad + \|\rho\|_{L^\infty} \|u\|_{L^6}^2 \|\nabla^2 u\|_{L^2} \|u_t\|_{L^6} + \|\rho\|_{L^\infty} \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2} \right) d\tau \\
\leq \int_0^t \omega(\tau)^{2-s} \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} \|\nabla u_t\|_{L^2} d\tau \\
\leq C \int_0^t \|\nabla u\|_{L^2}^4 \bar{F}_2(\tau) d\tau + \frac{1}{16} \bar{F}_2(t).
\]  
(27)

Similarly, we get that

\[
I_4 + I_5 \leq C \int_0^t \omega(\tau)^{2-s} \int (|u| |H_i| |\nabla H_i| + |\nabla H| |H_i| |u_t|) \, dx \, d\tau \\
\leq C \int_0^t \omega(\tau)^{2-s} \left(|u| |L^6| |H_i| |L^3| |\nabla H_i|_{L^2} + |\nabla H| |L^2| |H_i| |L^3| |u_t|_{L^6}\right) \, d\tau \\
\leq C \int_0^t \omega(\tau)^{2-s} \left(|\nabla u, \nabla H| |L^2| |H_i|^{1/2} |\nabla H_i|^{1/2} |\nabla u_t, \nabla H_i|\right) |L^2| \, d\tau \\
\leq \frac{1}{32} \int_0^t \omega(\tau)^{2-s} \|\nabla u_t, \nabla H_i\|_{L^2}^2 d\tau \\
+ C \int_0^t \omega(\tau)^{2-s} \|\nabla u_t, \nabla H\|_{L^2}^2 |H_t|_{L^2} |\nabla H_t|_{L^2} \, d\tau \\
\leq \frac{1}{16} \bar{F}_2(t) + C \int_0^t \omega(\tau)^{2-s} \|\nabla u_t, \nabla H\|_{L^2}^2 |H_t|_{L^2}^2 \, d\tau \\
\leq \frac{1}{16} \bar{F}_2(t) + C \int_0^t \|\nabla u_t, \nabla H\|_{L^2}^2 \bar{F}_2(\tau) d\tau,
\]  
(28)

and

\[
I_6 \leq F_1(t) \leq C \left\|\left(\sqrt{\rho_0} u_0, H_0\right)\right\|_{H^s}^2.
\]  
(30)
Then, by substituting (25)-(30) into the summation of (24), we have
\[ F_2(t) \leq C\|{(\sqrt{\rho_0}u_0, H_0)}\|_{\dot{H}^r}^2 + C \int_0^t \|{(\nabla u, \nabla H)}\|_{L^2_x}^4 F_2(\tau) d\tau. \]
So, by Gronwall’s inequality, we obtain
\[ F_2(t) \leq C\|{(\sqrt{\rho_0}u_0, H_0)}\|_{\dot{H}^r}^2, \]
and if $t \geq 1$, then
\[ \int_1^t \|{(\nabla u, \nabla H)}\|_{L^2_x}^2 d\tau \leq \sup_{1 \leq \tau \leq t} \|{(\nabla u, \nabla H)}\|_{L^2_x}^2 \int_1^t \|{(\nabla u, \nabla H)}\|_{L^2_x}^2 d\tau \leq C\|{(u_0, H_0)}\|_{\dot{H}^r}^2 \|{(u_0, H_0)}\|_{L^2}^2 \leq C\|{(u_0, H_0)}\|_{\dot{H}^r}^4, \]
so, by substituting (32) and (33) into the summation of (31) and (15), we get
\[ F_2(t) \leq C\|{(u_0, H_0)}\|_{\dot{H}^r}^2 \exp\{C\|{(u_0, H_0)}\|_{\dot{H}^r}^4\}. \]
Since we get the uniform energy estimates, the proof of the existence part is completed.

3. The proof of the uniqueness part of Theorem 1.1.

3.1. More regularity of the solutions. In this section, the aim is to prove the uniqueness of the solutions in Theorem 1.1. Firstly, we’ll focus on some more information on the regularity of the solutions, which will be used in the proof.

**Lemma 3.1.** If $(\rho, u, H)$ is the solution of system (1) obtained in Theorem 1.1, $T \in [0, 1]$ and $\gamma \in [0, 1]$, then
\[
\int_0^T t^{1+\gamma-s} \left( \|{(\partial_t u, \partial_t H)}\|_{L^2_x}^{2\gamma} + \|{(\nabla^2 u, \nabla^2 H)}\|_{L^{\frac{6}{5}}_x}^{2\gamma} + \|{\nabla P}\|_{L^{\frac{6}{5}}_x}^{2\gamma} \right) dt \leq C,
\]
\[
\int_0^T \|{(\nabla u, \nabla H)}\|_{L^\infty} dt \leq C T^{\frac{2\gamma-1}{2\gamma}},
\]
\[
\int_0^T t^{\gamma} \|{(\nabla u, \nabla H)}\|_{L^\infty} dt \leq C T^{\frac{4\gamma-2}{4\gamma}},
\]
where the constant $C$ depends on $\|(u_0, H_0)\|_{H^\gamma}$.

Proof. (1) We prove the first inequality. By Hölder’s inequality, Gagliardo-Nirenberg inequality and (4), we have

\[
\|t^{\frac{\gamma-1}{2}} (\partial_t u, \partial_t H)\|_{L^\gamma_x} \leq C \|t^{\frac{\gamma-1}{2}} (\partial_t u, \partial_t H)\|_{L^2_x}^\gamma \quad (t^{\frac{\gamma-1}{2}} (\partial_t u, \partial_t H)) \|_{L^\gamma_x}^\gamma
\]

\[
\leq C \|(\partial_t u, \partial_t H)\|_{L^\gamma_x} \quad \|((\partial_t u, \partial_t H)) \|_{L^2_x}^\gamma \leq C (\|(u_0, H_0)\|_{H^\gamma})
\]

(34)

Similarly, we get that

\[
\|t^{\frac{\gamma-1}{2}} \|\rho u \cdot \nabla u\|_{L^\gamma_x} \leq C \|t^{\frac{\gamma-1}{2}} \|\rho u \cdot \nabla u\|_{L^2_x}^\gamma
\]

\[
\|t^{\frac{\gamma-1}{2} + \frac{2(2-s)}{3}} \|H \cdot \nabla H\|_{L^\gamma_x} \leq C \|t^{\frac{\gamma-1}{2} + \frac{2(2-s)}{3}} \|H\|_{L^\gamma_x} \quad \|\nabla H\|_{L^2_x}^\gamma
\]

(35)

(36)

Since $-\Delta u + \nabla P = -\rho \partial_1 u - \rho u \cdot \nabla u + \text{curl} H \times H$, by the $W^{2,p}$ estimate of the Stokes system and noting that $1 - s + \frac{2s(2-s)}{3} \leq 1 + \gamma - s$, if $s > \frac{1}{2}$, we have

\[
\|t^{1+\gamma-s} (\|(\partial_t u, \partial_t H)\|_{L^\gamma_x}^\gamma + \|((\nabla^2 u, \nabla^2 H))\|_{L^\gamma_x}^\gamma + \|\nabla P\|_{L^\gamma_x}^\gamma) dt \leq C (\|(u_0, H_0)\|_{H^\gamma})
\]

Then by Gagliardo-Nirenberg inequality and (4), we obtain that

\[
\|t^{1+\gamma-s} (\|(\partial_t u, \partial_t H)\|_{L^\gamma_x}^\gamma + \|((\nabla^2 u, \nabla^2 H))\|_{L^\gamma_x}^\gamma + \|\nabla P\|_{L^\gamma_x}^\gamma) dt \leq C (\|(u_0, H_0)\|_{H^\gamma}) T^{\frac{2s-1}{2}}
\]

Similarly, we get that

\[
\int_0^T t^{1+\gamma-s} \|(\nabla u, \nabla H)\|_{L^\gamma_x} dt \leq C (\|(u_0, H_0)\|_{H^\gamma}) T^{\frac{2s-1}{2}}
\]

Then the proof of Lemma 3.1 is completed.
3.2. The Lagrangian formulation. Next, we show the Lagrangian formulation which will play a vital role in the proof. Define the trajectory of $X(t, y)$ of $u(t, x)$ by

$$\partial_t X(t, y) = u(t, X(t, y)), \quad X(0, y) = y,$$

which shows the following relation between the Eulerian coordinate $x$ and the Lagrangian coordinate $y$:

$$X(t, y) = y + \int_0^t u(\tau, X(\tau, y))d\tau. \quad (37)$$

By choosing $T$ small enough, we can get from Lemma 3.1 that

$$\int_0^T \|\nabla u(t)\|_{L^{\infty}}dt \leq \frac{1}{2}, \quad (38)$$

which ensures that for $t \leq T$, $X(t, y)$ is invertible with respect to the variable $y$, and we denote its inverse mapping $Y(t, x)$.

Define $v(t, y) \overset{\Delta}{=} u(t, x) = u(t, X(t, y))$, $A(t, y) \overset{\Delta}{=} (D_y X(t, y))^{-1} = (D_x Y(t, x))$, where $(D_y X)_{ij} \overset{\Delta}{=} \partial_{y_i} X^j$. It holds that

$$\nabla_x u(t, x) = A^t(t, y)\nabla_y v(t, y), \quad \text{and} \quad \text{div}_x u(t, x) = \text{div}_y (A(t, y)v(t, y)),$$

which shows

$$\text{div}_y (A^t) = A^t : \nabla_y. \quad (39)$$

Here and in the sequel, $A^t$ denotes the transpose matrix of $A$, and $A^t : \nabla_y$ means $\text{Tr}(A^t \nabla_y)$. We denote as in [32] that

$$\nabla \overset{\Delta}{=} A^t \cdot \nabla_y, \quad \text{div} \overset{\Delta}{=} \text{div}(A^t), \quad \Delta_u \overset{\Delta}{=} \text{div}_u \nabla_u, \quad \eta(t, y) \overset{\Delta}{=} \rho(t, X(t, y)), \quad v(t, y) \overset{\Delta}{=} u(t, X(t, y)), \quad B(t, y) \overset{\Delta}{=} H(t, X(t, y)), \quad \Pi(t, y) \overset{\Delta}{=} P(t, X(t, y)). \quad (40)$$

So the Lagrangian formulation of (1) becomes

$$\begin{cases}
\partial_t \eta = 0, \\
\eta \partial_t v - \Delta_u v + \nabla_u \Pi = B \cdot \nabla_u B - \nabla_u \|B\|_2^2, \\
\partial_t B - \Delta_u B - B \cdot \nabla_u v = 0, \\
\text{div}_u v = \text{div}_u B = 0, \\
(\eta, v, B)_{t=0} = (\rho_0, u_0, H_0).
\end{cases} \quad (41)$$

Then, we transform the regularity information of the solution in the Eulerian coordinates into that in the Lagrangian coordinates.

**Lemma 3.2.** We assume that $(\rho, u, H, \nabla P)$ is the solution of system $(1)$ obtained in Theorem 1.1 and $(\eta, v, B, \nabla \Pi)$ is given by $(40)$, then for any $t \leq T$ small enough and $0 \leq \gamma \leq 1$, one has

$$\int_0^t \tau^{1+\gamma-s} \left( \|\partial_t v, \partial_t B\|_{L^{\frac{2}{\gamma}}}^2 + \|\nabla \Pi\|_{L^{\frac{2}{1-\gamma}}}^2 \right) d\tau \leq C,$$

$$\int_0^t \|\nabla v, \nabla B\|_{L^{\infty}}d\tau \leq Ct^{\frac{2-\gamma}{\gamma}},$$

$$\int_0^t \tau \|\nabla v, \nabla B\|_{L^{\infty}}^2 d\tau \leq Ct^{\frac{4-2\gamma}{\gamma}}.$$
and if $\gamma < s$, then we have
\[
\int_0^t \tau^{1+\gamma-s} \|
abla^2 v, \nabla^2 B \|^2_{L^6(\mathbb{R}^3)} \, d\tau \leq C, \quad \text{and} \quad \|
abla A\|_{L^\infty(L^{6/5}(\mathbb{R}^3))} \leq Ct^{\frac{s}{2-\gamma}}.
\]
Here the constant $C$ depends on $\|(u_0, H_0)\|_{H^s}$.

Proof. Similar to the proof of Lemma 3.3 in [32], we know by (37), (38) and Lemma 3.1 that
\[
\|\nabla y X(t, \cdot)\|_{L^\infty} \leq e^\int_0^t \|\nabla_x u(\tau)\|_{L^\infty} \, d\tau \leq e^\frac{1}{4}.
\]  
By (42) and Lemma 3.1, we can easily deduce that
\[
\int_0^t \|\nabla v, \nabla B\|_{L^\infty} \, d\tau \leq \|\nabla y X\|_{L^\infty} \int_0^t \|\nabla_x u, \nabla_x H\|_{L^\infty} \, d\tau \leq Ct^{\frac{s}{2-\gamma}},
\]
and
\[
\|\tau^{1+\gamma-s} \nabla^2 v\|_{L^2_t(L^{6/5})} \leq \|\nabla y X\|_{L^\infty} \|\tau^{1+\gamma-s} \nabla_x P\|_{L^2_t(L^{6/5})} \leq C.
\]
Noting that
\[
\partial_t v(t, x) = (\partial_t u + u \cdot \nabla u)(t, X(t, y)) \quad \text{and} \quad \partial_t B(t, x) = (\partial_t H + u \cdot \nabla H)(t, X(t, y)),
\]
then by (34)-(36), it’s easy to get
\[
\|\tau^{1+\gamma-s} (\partial_t v, \partial_t B)\|_{L^2_t(L^{6/5})} \leq \|\tau^{1+\gamma-s} (\partial_t u, \partial_t H)\|_{L^2_t(L^{6/5})} + \|\tau^{1+\gamma-s} (u \cdot \nabla u, u \cdot \nabla H)\|_{L^2_t(L^{6/5})} \leq C.
\]
From (37) and (42), we have
\[
\|\nabla^2 y X(t, y)\|_{L^{6/5}} \leq C \int_0^t \|\nabla^2 y X\|_{L^{6/5}} \|\nabla_x u\|_{L^\infty} \, d\tau + C \int_0^t \|\nabla^2 u\|_{L^{6/5}} \, d\tau,
\]
in addition, if $\gamma < s$, then we deduce by Gronwall’s inequality and Lemma 3.1 that
\[
\|\nabla^2 y X(t, y)\|_{L^{6/5}} \leq C \int_0^t \|\nabla^2 u\|_{L^{6/5}} \, d\tau,
\]
\[
\leq C \left( \int_0^t \tau^{1+\gamma-s} \|\nabla^2 u\|_{L^{6/5}}^2 \, d\tau \right)^{1/2} \left( \int_0^t \tau^{-(1+\gamma-s)} \, d\tau \right)^{1/2},
\]
\[
\leq Ct^{\frac{s}{2-\gamma}}.
\]  
(43)
Hence, by (42), (43) and Lemma 3.1, we get
\[
\|\tau^{1+\gamma-s} (\nabla^2 v, \nabla^2 B)\|_{L^2_t(L^{6/5})} \leq C.
\]
Finally, we estimate $|\nabla A|_{L^\infty(L^\frac{s}{\gamma}T)}$. Thanks to (38), for any $t \leq T$, we have

$$A(t, y) = D_x Y(t, x) = (Id + (D_y X - Id))^{-1}$$

$$= \sum_{k=0}^{\infty} (-1)^k \left( \int_0^t D_y u(\tau, X(\tau, y)) d\tau \right)^k.$$ 

By choosing $T$ in (38) small enough, noting $|\nabla u| = |Du|$, we get from (42), (43) and Lemma 3.1 that for any $t \leq T$ and $\gamma < s$,

$$\|\nabla A\|_{L^\infty(L^\frac{s}{\gamma}T)} \leq C \|\nabla^2 u\|_{L^1_\gamma(L^\frac{s}{\gamma}T)} \|\nabla_y X\|_{L^\infty(L^\infty)}^2 + C \|\nabla_x u\|_{L^1_\gamma(L^\infty)} \|\nabla^2 X\|_{L^\infty(L^\frac{s}{\gamma}T)}$$

$$\leq C t^{\frac{s}{\gamma-2}} \|\nabla^2 u\|_{L^2_\gamma(L^\frac{s}{\gamma}T)}^2 + C t^{\frac{s}{\gamma-2}}$$

$$\leq C t^{\frac{s}{\gamma-2}}.$$ 

This complete the proof of Lemma 3.2. \qed

### 3.3. The proof of the uniqueness.

Firstly, we recall the following important lemma from [14], [32].

**Lemma 3.3.** Let $\eta \in L^\infty(\mathbb{R}^d)$ be a time independent positive function, and be bounded away from zero. Let $R$ satisfy $R_i \in L^2((0, T) \times \mathbb{R}^d)$ and $\nabla \text{div} R \in L^2((0, T) \times \mathbb{R}^d)$. Then the following system

$$\begin{align*}
\eta \partial_t v - \Delta v + \nabla \Pi &= f, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
\text{div} v &= \text{div} R, \\
v|_{t=0} &= v_0,
\end{align*}$$

has a unique solution $(v, \nabla \Pi)$ such that

$$\|\nabla v\|_{L^\infty(L^2)} + \|(v_t, \nabla^2 v, \nabla \Pi)\|_{L^2_\gamma(L^2)} \leq C \left( \|\nabla v_0\|_{L^2} + \|(f, R_t)\|_{L^2_\gamma(L^2)} + \|\nabla \text{div} R\|_{L^2_\gamma(L^2)} \right),$$

where the constant $C$ depends on $\inf \eta$ and $\sup \eta$, but is independent of $T$.

Next, we will focus on the proof of the uniqueness of the solutions.

Let $(\rho_i, u_i, H_i, \nabla P_i)$, $i = 1, 2$ be two solutions of system (1) obtained in Theorem 1.1, and $(\eta_i, \nu_i, B_i, \nabla \Pi_i)$, $i = 1, 2$, be determined by (40). Denote $A_i := A(u_i)$, $i = 1, 2$, and

$$\delta v = v_2 - v_1, \quad \delta B = B_2 - B_1, \quad \delta \Pi = \Pi_2 - \Pi_1, \quad \delta A = A_2 - A_1.$$ 

Then by (40) and (41), we get that

$$\begin{align*}
\rho \partial_t \delta v - \Delta \delta v + \nabla \delta \Pi &= \delta f_1 + \delta f_2 + \delta f_3, \\
\partial_t \delta B - \Delta \delta B &= \delta f_4 + \delta f_5, \\
\text{div} \delta v &= \text{div} \delta g_1, \\
\text{div} \delta B &= \text{div} \delta g_2, \\
(\delta v, \delta B)|_{t=0} &= 0,
\end{align*}$$

where $\delta f_i, \delta g_i$ are determined by (40) and (41).
where
\[
\begin{align*}
\delta f_1 &= (I_d - A_2^1)\nabla \delta H - \delta A^1 \nabla H_1, \\
\delta f_2 &= \text{div}[-(I_d - A_2 A_2^1)\nabla \delta v + (A_2 A_2^1 - A_1 A_1^1)\nabla v_1], \\
\delta f_3 &= \delta B \cdot A_1^1 \nabla B_2 + B_1 \cdot \delta A^1 \nabla B_2 + B_1 \cdot A_1^1 \nabla \delta B \\
&\quad + A_2^1 \nabla B_2 \cdot \delta B + \delta A^1 \nabla B_2 \cdot B_1 + A_1^1 \nabla \delta B \cdot B_1, \\
\delta f_4 &= \text{div}[-(I_d - A_2 A_2^1)\nabla \delta B + (A_2 A_2^1 - A_1 A_1^1)\nabla B_1], \\
\delta f_5 &= \delta B \cdot A_2^1 \nabla v_2 + B_1 \cdot \delta A^1 \nabla v_2 + B_1 \cdot A_1^1 \nabla \delta v, \\
\delta g_1 &= (I_d - A_2)\delta v - \delta A v_1, \quad \delta g_2 = (I_d - A_2)\delta B - \delta A B_1.
\end{align*}
\]

Denote
\[
\delta F(t) \triangleq \|\nabla \delta v, \nabla \delta B\|_{L^\infty_t(L^2)} + \|\partial_t \delta v, \partial_t \delta B\|_{L^2_t(L^2)} + \|\nabla^2 \delta v, \nabla^2 \delta B\|_{L^2_t(L^2)} + \|\nabla \delta H\|_{L^2_t(L^2)}.
\]

**Lemma 3.4.** The following estimate holds:
\[
\|\delta f_1, \delta f_2, \delta f_3, \delta f_4, \delta f_5\|_{L^2_t(L^2)} + \|\nabla \text{div}\delta g_1, \nabla \text{div}\delta g_2\|_{L^2_t(L^2)} + \|\partial_t \delta g_1, \partial_t \delta g_2\|_{L^2_t(L^2)} \leq \varepsilon(t) \delta F(t),
\]
where \(\varepsilon(t)\) tends to zero as \(t\) goes to zero.

Now, we assume that Lemma 3.4 holds, then by applying Lemma 3.3 to (45), we can deduce from lemma 3.4 that
\[
\delta F(t) \leq C\|\delta f_1, \delta f_2, \delta f_3, \delta f_4, \delta f_5\|_{L^2_t(L^2)} + \|\nabla \text{div}\delta g_1, \nabla \text{div}\delta g_2\|_{L^2_t(L^2)} + \|\partial_t \delta g_1, \partial_t \delta g_2\|_{L^2_t(L^2)} \leq \varepsilon(t) \delta F(t),
\]
which ensures the uniqueness of the solutions obtained in Theorem 1.1 on a sufficiently small time interval \([0, T_1]\). Then the uniqueness on the whole time interval \([0, \infty)\) can be deduced by a bootstrap argument.

### 3.4. The proof of Lemma 3.4.

In the following, we choose \(t\) small enough so that
\[
\int_0^t \|\nabla v_i, \nabla B_i\|_{L^\infty} d\tau \leq \frac{1}{2}, \quad \text{for} \quad i = 1, 2.
\]
(46)

Denote \(\varepsilon(t)\) a function of \(t\) tending to zero as \(t \to 0\), which may be different in different lines.

**Step 1.** Estimate of \(\|\delta f_1\|_{L^2_t(L^2)}\).

By Lemma 3.2, we have
\[
\|\delta f_1\|_{L^2_t(L^2)} \leq \int_0^t \|\nabla v_2(\tau)\|_{L^\infty} d\tau \|\nabla \delta H\|_{L^2_t(L^2)} \leq C t^{\frac{2s-1}{2}} \delta F(t).
\]
(47)

We get by (44) that
\[
\delta A(t) = \left(\int_0^t D\delta v d\tau\right) \left(\sum_{k\geq 0} \sum_{0 \leq j < k} V_1^j V_2^{k-1-j}\right),
\]
(48)
where $V_i(t) \triangleq \int_0^t Du_i d\tau$, $i = 1, 2$. And by (46), Sobolev inequality and Hölder’s inequality, we obtain

$$
\|\delta A(t)\|_{L^6} \leq C\int_0^T |\nabla \delta v(\tau', \cdot)| d\tau' \|L^{\infty}(L^6) \leq C \int_0^T \|\nabla^2 \delta v\|_{L^2} d\tau \leq C \epsilon^{\frac{\gamma}{2}} F(t). \quad (49)
$$

So by (49) and choosing $\gamma = \frac{1}{2}$ in Lemma 3.2, we have

$$
\|\delta A^i \nabla \Pi_1\|_{L^2_t(L^2)} \leq \|\tau^{-\frac{1}{2}} - \delta A(\tau)\|_{L^{\infty}_t(L^6)} \|\nabla \Pi_1\|_{L^2_t(L^3)} \leq C \epsilon^{\frac{2\gamma - 1}{4}} F(t). \quad (50)
$$

According to $s > \frac{1}{2}$, (47) and (50), we get

$$
\|\delta f_1\|_{L^2_t(L^2)} \leq C\epsilon(t) F(t).
$$

**Step 2.** Estimate of $\|\delta f_2, \delta f_4\|_{L^2_t(L^2)}$.

Noting that

$$
\begin{aligned}
(A_2 A_2^t - A_1 A_1^t) \nabla v_1 &= (-\delta A(Id - A_2^t) - (Id - A_1) \delta A^t) \nabla v_1 \\
&+ (\delta A^t + \delta A) \nabla v_1,

(A_2 A_2^t - A_1 A_1^t) \nabla B_1 &= (-\delta A(Id - A_2^t) - (Id - A_1) \delta A^t) \nabla B_1 \\
&+ (\delta A^t + \delta A) \nabla B_1,
\end{aligned}
$$

we get by (48), (49), Lemma 3.2 and Hölder’s inequality that

$$
\begin{aligned}
\|\tau^{-\frac{1}{2}} \nabla \delta A\|_{L^{\infty}_t(L^2)} &\leq C \|\tau^{-\frac{1}{2}} \int_0^T |\nabla^2 \delta v| d\tau'\|_{L^{\infty}_t(L^2)} \\
&+ \|\tau^{-\frac{1}{2}} \int_0^T |\nabla \delta v| d\tau' \int_0^T (|\nabla^2 v_1| + |\nabla^2 v_2|) d\tau'\|_{L^{\infty}_t(L^2)} \\
&\leq C \left(\|\nabla^2 \delta v\|_{L^2_t(L^2)} + \|\tau^{-\frac{1}{2}} + \tau^{-\frac{1}{2}} \int_0^T |\nabla \delta v| d\tau'\|_{L^{\infty}_t(L^6)} \|\nabla^2 v_1, \nabla^2 v_2\|_{L^2_t(L^3)}\right) \\
&\leq C \epsilon(t) F(t).
\end{aligned}
$$

Then by Lemma 3.2, (49) and (51), we deduce that

$$
\begin{aligned}
\|(\text{div} [\delta A(Id - A_2^t) \nabla v_1], \text{div} [\delta A(Id - A_1) \nabla B_1])\|_{L^2_t(L^2)} &\leq \|\tau^{-\frac{1}{2}} \nabla \delta A\|_{L^{\infty}_t(L^2)} \|Id - A_2\|_{L^\infty_t(L^6)} \|\nabla^2 (v_1, \nabla B_1)\|_{L^2_t(L^\infty)} \\
&+ \|\tau^{-\frac{1}{2}} \delta A\|_{L^{\infty}_t(L^6)} \|\nabla A_2\|_{L^\infty_t(L^6)} \|\nabla^2 (v_1, \nabla B_1)\|_{L^2_t(L^\infty)} \\
&+ \|\tau^{-\frac{1}{2}} + \tau^{-\frac{1}{2}} \delta A\|_{L^{\infty}_t(L^6)} \|Id - A_2\|_{L^\infty_t(L^6)} \|\nabla^2 (v_1, \nabla B_1)\|_{L^2_t(L^3)} \\
&\leq C \epsilon(t) F(t).
\end{aligned}
$$

We can get the same estimates for $\|(Id - A_1) \delta A^t \nabla v_1 + (\delta A^t + \delta A) \nabla v_1\|$ and $\|(Id - A_1) \delta A^t \nabla B_1 + (\delta A^t + \delta A) \nabla B_1\)$. So we prove that

$$
\|(\text{div}([A_2 A_2^t - A_1 A_1^t] \nabla v_1), \text{div}([A_2 A_2^t - A_1 A_1^t] \nabla B_1))\|_{L^2_t(L^2)} \leq \epsilon(t) F(t). \quad (52)
$$

We write

$$
\begin{aligned}
(Id - A_2 A_2^t) \nabla v &= -((Id - A_2)(Id - A_2^t) + (Id - A_2) + (Id - A_2^t)) \nabla \delta v, \\
(Id - A_2 A_2^t) \nabla B &= -((Id - A_2)(Id - A_2^t) + (Id - A_2) + (Id - A_2^t)) \nabla \delta B.
\end{aligned}
$$
Then we infer by Lemma 3.2 that
\[\|(\nabla \text{div}[(Id - A_2 A_2^t) \nabla \delta v_1] + \nabla \text{div}[(Id - A_2 A_2^t) \nabla \delta B])\|_{L^2_t(L^2)}\]
\[\leq C \left( \|\nabla A_2\|_{L^\infty_t(L^2)} \|(|\nabla \delta v|, \nabla \delta B)|_{L^2_t(L^\infty)} + \|Id - A_2\|_{L^\infty_t(L^\infty)} \right) \|\nabla^2 \delta v, \nabla^2 \delta B\|_{L^2_t(L^2)}\]
\[\leq \varepsilon(t) \delta F(t). \quad (53)\]
So, by (52) and (53), we prove that
\[\|\delta f_2, \delta f_4\|_{L^2_t(L^2)} \leq \varepsilon(t) \delta F(t).\]

**Step 3.** Estimate of \(\|\delta f_3, \delta f_5\|_{L^2_t(L^2)}\).

Noting
\[\delta f_3 + \delta f_5 = \delta B \cdot A_2^t \nabla B_2 + B_1 \cdot (\Delta B_1 \nabla B_2 + B_1) \cdot A_1^t \nabla \delta B\]
\[+ A_2^t \nabla B_2 \cdot \delta B + \Delta A^t \nabla B_2 \cdot B_1 + A_1^t \nabla \delta B \cdot B_1\]
\[+ \delta B \cdot A_1^t \nabla v_2 + B_1 \cdot \Delta A^t B \nabla v_2 + B_1 \cdot A^t \nabla \delta v,\]
by (20), Gagliardo-Nirenberg inequality, Hölder’s inequality and Lemma 3.2, we get
\[\|B_1\|_{L^2_t(L^\infty)} \leq \left( \int_0^t \|B_1\|_{L^2_t}^2 \|A_1^t \nabla B_1\|_{L^2_t}^2 d\tau \right)^{1/2}\]
\[\leq \sup_{0 \leq \tau \leq t} \|B_1(\tau)\|_{L^2_t}^2 \left( \int_0^t \tau^{-2(1-s)} d\tau \right)^{1/2} \left( \int_0^t \tau^{1-s} \|A_1^t \nabla B_1\|_{L^2_t}^2 d\tau \right)^{1/2}\]
\[\leq Ct^{\frac{2s-1}{2s}}, \quad (54)\]
\[\|\tau^{-\frac{1}{2}} \delta B\|_{L^\infty_t(L^2)} \leq \|\tau^{-\frac{1}{2}} \int_0^\tau d\tau B d\tau\|_{L^\infty_t(L^2)} \leq \|\partial_\tau \delta B\|_{L^2_t(L^2)} \leq \|\partial_\tau \delta B\|_{L^2_t(L^2)}. \quad (55)\]

Then, by (20), (49), (54), (55), Lemma 3.2 and Hölder’s inequality, we find that
\[\|\delta f_3, \delta f_5\|_{L^2_t(L^2)} \leq C \left( \|\partial B \cdot A_2^t (\nabla B_2, \nabla v_2)\|_{L^2_t(L^2)} + \|B_1 \cdot \Delta A^t (\nabla B_2, \nabla v_2)\|_{L^2_t(L^2)}\right)\]
\[\leq C \left( \|\tau^{-\frac{1}{2}} \delta B\|_{L^\infty_t(L^2)} \|A_2^t \|_{L^\infty_t(L^\infty)} \|\tau^{\frac{1}{2}} (\nabla B_2, \nabla v_2)\|_{L^2_t(L^\infty)}\right)\]
\[+ C \left( \|B_1\|_{L^\infty_t(L^2)} \|\tau^{-\frac{1}{2}} \delta A\|_{L^\infty_t(L^\infty)} \|\tau^{\frac{1}{2}} (\nabla B_2, \nabla v_2)\|_{L^2_t(L^\infty)}\right)\]
\[+ C \left( \|B_1\|_{L^\infty_t(L^2)} \|A_1^t \|_{L^\infty_t(L^\infty)} \|\nabla B, \nabla v_1\|_{L^\infty_t(L^\infty)}\right)\]
\[\leq \varepsilon(t) \delta F(t).\]

**Step 4.** Estimate of \(\|\nabla (\text{div} \delta g_1, \nabla (\text{div} \delta g_2)\|_{L^2_t(L^2)}\).

By the chain rule (39), we get that
\[\|\nabla (\text{div} \delta g_1, \nabla (\text{div} \delta g_2)\|_{L^2_t(L^2)}\]
\[= \|\nabla (\text{div}[(Id - A_2) \delta v - \delta A v_1], \nabla (\text{div}[(Id - A_2) \delta B - \delta A B_1])\|_{L^2_t(L^2)}\]
\[\leq \|\nabla ((Id - A_2)^t : \nabla \delta v), \nabla ((Id - A_2)^t : \nabla \delta B)\|_{L^2_t(L^2)}\]
\[+ \|\nabla (\delta A^t : \nabla v_1), \nabla (\delta A^t : \nabla B_1)\|_{L^2_t(L^2)}].\]
From (46), (48), (51) and Lemma 3.2, we get that
\[ \| (\nabla \delta A^t : \nabla v_1), \nabla (\delta A^t : \nabla B_1) \|_{L_t^2(L^2)} \]
\[ \leq C \left( \| \nabla \delta A(\nabla v_1, \nabla B_1) \|_{L_t^2(L^2)} + \| \delta A(\nabla^2 v_1, \nabla^2 B_1) \|_{L_t^2(L^2)} \right) \]
\[ \leq C \left( \| \tau^{-\frac{1}{2}} \nabla \delta A \|_{L_t^\infty(L^2)} \| \tau^{\frac{1}{2}} (\nabla v_1, \nabla B_1) \|_{L_t^2(L^\infty)} \right. 
\left. + \| \int_0^T |\nabla \delta v| d\tau' \|_{L_t^\infty(L^2)} \right) \]
\[ \leq \varepsilon(t) \delta F(t), \quad (56) \]
and
\[ \| (\nabla ((Id - A_2)^t : \nabla \delta v), \nabla ((Id - A_2)^t : \nabla \delta B) \|_{L_t^2(L^2)} \]
\[ \leq C \| \nabla A_2 \|_{L_t^\infty(L^2)} \| (\nabla \delta v, \nabla \delta B) \|_{L_t^2(L^\infty)} \]
\[ \left. + \| \int_0^T |\nabla \delta v| d\tau' \|_{L_t^\infty(L^2)} \| (\nabla^2 \delta v, \nabla^2 \delta B) \|_{L_t^2(L^2)} \right) \]
\[ \leq \varepsilon(t) \delta F(t). \quad (57) \]
So, by (56) and (57), we obtain
\[ \| (\nabla \text{div}\delta g_1, \nabla \text{div}\delta g_2) \|_{L_t^2(L^2)} \leq \varepsilon(t) \delta F(t). \]

**Step 5.** Estimate of \( \| (\partial_t \delta g_1, \partial_t \delta g_2) \|_{L_t^2(L^2)} \).
\[ \| (\partial_t \delta g_1, \partial_t \delta g_2) \|_{L_t^2(L^2)} \]
\[ \leq \| (\partial_t ((Id - A_2)\delta v), \partial_t ((Id - A_2)\delta B)) \|_{L_t^2(L^2)} + \| (\partial_t (\delta Av_1), \partial_t (\delta AB_1)) \|_{L_t^2(L^2)} \]
By (46) and (48), we know that
\[ \| (\partial_t (\delta Av_1), \partial_t (\delta AB_1)) \|_{L_t^2(L^2)} \]
\[ \leq C \left( \| \nabla \delta v(v_1, B_1) \|_{L_t^2(L^2)} + \| \int_0^T |\nabla \delta v| d\tau' \| ((\nabla v_1, \nabla v_2))(v_1, B_1) \|_{L_t^2(L^2)} \right) \]
\[ \triangleq J_1 + J_2 + J_3. \quad (58) \]
By Lemma 3.2 and (54), one has
\[ J_1 \leq \| \nabla \delta v \|_{L_t^\infty(L^2)} \| (v_1, B_1) \|_{L_t^2(L^\infty)} \leq \varepsilon(t) \delta F(t). \quad (59) \]
and
\[ J_2 \leq \int_0^T |\nabla \delta v| d\tau' \|_{L_t^\infty(L^2)} \| \tau^{\frac{3}{2}} (\nabla v_1, \nabla v_2) \|_{L_t^2(L^\infty)} \| (v_1, B_1) \|_{L_t^2(L^\infty)} \]
\[ \leq \varepsilon(t) \delta F(t). \quad (60) \]
Also, we get by (49) and Lemma 3.2 that
\[ J_3 \leq \| \tau^{\frac{3}{2}} \delta A \|_{L_t^\infty(L^2)} \| \tau^{\frac{3}{2}} (\partial_t v_1, \partial_t B_1) \|_{L_t^2(L^2)} \leq \varepsilon(t) \delta F(t). \quad (61) \]
By substituting (59)-(61) into (58), we have
\[ \| (\partial_t (\delta Av_1), \partial_t (\delta AB_1)) \|_{L_t^2(L^2)} \leq \varepsilon(t) \delta F(t). \quad (62) \]
Similarly, by (55) and Lemma 3.2, one has
\[\| (\partial_t ((I - A_2)\delta v), \partial_t ((I - A_2)\delta B) ) \|_{L^2_t(L^2)} \]
\[\leq \| \tau^{\frac{1}{2}} \partial_t A_2 \|_{L^2_t(L^\infty)} \| \tau^{-\frac{1}{2}} (\delta v, \delta B) \|_{L^\infty(L^2)} + \| I - A_2 \|_{L^\infty(L^\infty)} \| (\partial_t \delta v, \partial_t \delta B) \|_{L^2_t(L^2)} \]
\[\leq \| \tau^{\frac{1}{2}} \nabla v_2 \|_{L^2_t(L^\infty)} \| (\partial_t \delta v, \partial_t \delta B) \|_{L^2_t(L^2)} + \| \nabla v_2 \|_{L^2_t(L^\infty)} \| (\partial_t \delta v, \partial_t \delta B) \|_{L^2_t(L^2)} \]
\[\leq \varepsilon(t) \delta F(t). \quad (63)\]

Then by (62) and (63), we obtain
\[\| (\partial_t \delta g_1, \partial_t \delta g_2) \|_{L^2_t(L^2)} \leq \varepsilon(t) \delta F(t). \]
This finishes the proof of Lemma 3.4.

4. The proof of Theorem 1.2. In this part, we are devoted to studying the decay of the solution by using the dual method to deal with the MHD system with discontinuous density.

Let \((\rho, u, H)\) be the solution obtained in Theorem 1.1. For any given \(T > 0\), let’s introduce the adjoint system of the momentum equation and the magnetic equation of (1):
\[
\begin{align*}
\rho (\partial_t \phi + u \cdot \nabla \phi) + \Delta \phi + \nabla \tilde{P} &= H \cdot \nabla \psi \quad \text{in} \quad [0, T] \times \mathbb{R}^3, \\
\partial_t \psi + u \cdot \nabla \psi + \Delta \psi &= H \cdot \nabla \phi, \\
\text{div} \phi &= \text{div} \psi = 0, \\
\phi(T, x) = u(T, x), \psi(T, x) &= H(T, x). 
\end{align*}
\]

Lemma 4.1. For \(0 \leq t \leq T\), and \(i = 0, 1\), it holds that
\[A_i(t) \leq C (\| u(T), H(T) \|_{L^2}^2), \]
where
\[
\begin{align*}
A_0(t) &= \int \frac{(\rho \phi(t, x))^2 + |\psi(t, x)|^2)}{dx} + \int_t^T \int (|\nabla \phi|^2 + |\nabla \psi|^2) dxd\tau, \\
A_1(t) &= (T - t) \int (|\nabla \phi(t, x)|^2 + |\nabla \psi(t, x)|^2) dx \\
&\quad + \int_t^T \int (T - \tau)(|\rho \phi_t|^2 + |\psi_t|^2 + |\nabla^2 \phi|^2 + |\nabla^2 \psi|^2) dxd\tau. \quad (64) 
\end{align*}
\]

Proof. Since the proof of (64) is similar to that of (10) and (18), here we omit it. \(\Box\)

Taking the \(L^2\) inner product of the momentum equation and the magnetic equation of (1) with \(\phi, \psi\), respectively, and using integration by parts, we obtain that
\[
\begin{align*}
\int (\rho u(T, x)\phi(T, x), H(T, x)\psi(T, x)) dxdx \\
&- \int (\rho u(0, x)\phi(0, x), H(0, x)\psi(0, x)) dxdx \\
&= \int_0^T u(\rho \phi_t + \rho u \cdot \nabla \phi + \Delta \phi + \nabla \tilde{P}) dxd\tau + \int_0^T H \cdot \nabla H \phi dxd\tau \\
&\quad + \int_0^T H(\psi_t + u \cdot \nabla \psi + \Delta \psi) dxd\tau + \int_0^T H \cdot \nabla u \psi dxd\tau 
\end{align*}
\]
Then the proof of Theorem 1.2 is completed by (67) and (70).

By the dual and Sobolev inequality, we get that for \( \xi = 2\zeta(q) \),

\[
\int (\rho(T,x)|u(T,x)|^2, |H(T,x)|^2)dx = \int ((\rho_0u_0)(x)\phi(0,x), H_0(x)\psi(0,x))dx \\
\leq C\|(u_0, H_0)\|_{L^q}(\|(\phi(0), \psi(0))\|_{L^{q'}} \\
\leq C\|(u_0, H_0)\|_{L^q}(\|(\phi(0), \psi(0))\|_{H^s}),
\]

(65)

where \( \frac{1}{q} + \frac{1}{q'} = 1 \), \( \xi = 3(\frac{1}{q} - \frac{1}{2}) \). In addition, by Lemma 4.1 and interpolation, we get

\[
\|(\phi(t), \psi(t))\|_{H^s} \leq C(T-t)^{-\frac{\xi}{2}}\|(u(T), H(T))\|_{L^2}.
\]

(66)

From (65) and (66), one has

\[
\|(u(T), H(T))\|_{L^2} \leq CT^{-\xi(q)}\|(u_0, H_0)\|_{L^q}.
\]

(67)

We know from the proof of (15) and (16) that

\[
\frac{d}{dt}\int |(\nabla u(t,x), \nabla H(t,x))|^2dx + c\int |(\nabla^2 u(t,x), \nabla^2 H(t,x))|^2dx \leq 0,
\]

(68)

where the constant \( c > 0 \). By (67) and interpolation, we have

\[
\|(\nabla^2 u(t), \nabla^2 H(t))\|_{L^2} \geq ct^{\xi(q)}\|(\nabla u(t), \nabla H(t))\|_{L^2}^2.
\]

(69)

Define \( U(t) = \int |(\nabla u(t,x), \nabla H(t,x))|^2dx \), then by (68) and (69), we obtain

\[
\frac{d}{dt}U(t) + c^2t^{\xi(q)}U(t)^2 \leq 0,
\]

which ensures that for \( t \geq 1 \),

\[
U(t) \leq C(1+t)^{-\xi(q)-1}.
\]

That is

\[
\|(\nabla u(t), \nabla H(t))\|_{L^2} \leq C(1+t)^{-\xi(q)-\frac{1}{2}}.
\]

(70)

Then the proof of Theorem 1.2 is completed by (67) and (70).

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