REALIZATION SPACES OF ALGEBRAIC STRUCTURES ON CHAINS

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Abstract. Given an algebraic structure on the homology of a chain complex, we define its realization space as a Kan complex whose vertices are the structures up to homotopy realizing this structure at the homology level. Our algebraic structures are parameterized by props and thus include various kinds of bialgebras. We give a general formula to compute subsets of equivalences classes of realizations as quotients of automorphism groups, and determine the higher homotopy groups via the cohomology of deformation complexes. As a motivating example, we compute subsets of equivalences classes of realizations of Poincaré duality for several examples of manifolds.

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Introduction

Motivations. Realization problems of algebraic structures appear naturally in various contexts related to topology and geometry. It consists in determining a space whose homology or homotopy groups are isomorphic to a given algebra, a chain complex equipped with an algebraic structure up to homotopy whose homology is isomorphic to a given algebra, or constructing an algebraic structure up to homotopy on a given chain complex which induces the desired structure at the homology level. In practice, one defines a moduli space of such objects (spaces, homotopy algebras or homotopy structures) and set up an obstruction theory to determine whether this moduli space is non empty (existence problem) and connected (uniqueness problem). This obstruction theoretic approach to connectedness of such moduli spaces goes back to the pioneering work of Halperin and Stasheff [17]. Such methods were successfully applied in [1] to determine the topological spaces realizing a given II-algebra (a graded group equipped with the algebraic operations existing on the homotopy groups of a pointed connected topological space). A similar approach was thoroughly developed by Goerss and Hopkins in [16] to build obstruction theories for multiplicative structures on ring spectra, proving in particular the existence
and uniqueness of $E_\infty$-structures on Lubin-Tate spectra (a non trivial improvement of the Hopkins-Miller theorem).

Realization problems also occur in the differential graded setting, and can involve algebraic structures not parameterized by operads but more general props. Moreover, to fully understand the realization problem, one has to determine the homotopy type of the corresponding moduli space. In particular, counting the connected components (the equivalence classes of realizations) when the realization is not unique is a very hard problem. Let us give two motivating examples.

The operations defining Poincaré duality on compact oriented manifolds organize into a Frobenius algebra, or equivalently a (unitary and counitary) Frobenius bialgebra. As a consequence of [25], we know that Poincaré duality can be realized at the chain level in a Frobenius bialgebra up to homotopy (a homotopy Frobenius bialgebra) which extends the $E_\infty$-structure generated by the (higher) cup products. Such realizations are not well known at present, and even an explicit notion of homotopy Frobenius bialgebra was obtained only recently in [3]. These higher operations are expected to contain geometric information about the underlying manifold.

String topology, introduced by Chas and Sullivan in [4], study the homology of loop spaces on manifolds. Due to the action of the circle on loops, one considers either the non equivariant homology or the $S^1$-equivariant homology (called the string homology). These graded spaces come with a rich algebraic structure of geometrical nature. In the non equivariant case, it forms a Batalin-Vilkovisky algebra, and in the equivariant one it forms an involutive Lie bialgebra [5]. This involutive Lie bialgebra is expected to be induced by finer operations at the chain level, organized into a homotopy involutive Lie bialgebra.

**Results in the prop setting.** Let $P$ be a prop and $X$ be a chain complex such that $H_\ast X$ forms a $P$-algebra. A $P$-algebra up to homotopy, or homotopy $P$-algebra, is an algebra over a cofibrant resolution $P_\infty$ of $P$. In a $P_\infty$-algebra structure, the relations defining the $P$-algebra structure are relaxed and are only satisfied up to a set of coherent (higher) homotopies. According to [40], such a notion is homotopy coherent with respect to the choice of a cofibrant resolution $P_\infty$. We consider two sorts of realization spaces. The first one, an analogue in the dg setting of those studied in [16], is the simplicial nerve $\mathcal{N}R^{P_\infty}$ of a category $\mathcal{R}^{P_\infty}$ whose objects are the $P_\infty$-algebras $(X, \phi)$ with a homology isomorphic to $H_\ast X$ as a $P$-algebra, and morphisms are quasi-isomorphisms of $P_\infty$-algebras. The second one is a Kan complex $\text{Real}_{P_\infty}(H_\ast X)$ whose vertices are the prop morphisms $P_\infty \to \text{End}_X$ inducing the $P$-algebra structure $P \to \text{End}_{H_\ast X}$ in homology. It is a simplicial subset of the simplicial mapping space $P_\infty \{X\}$ of maps $P_\infty \to \text{End}_X$ (the moduli space of $P_\infty$-algebra structures on $X$). We decompose $\text{Real}_{P_\infty}(H_\ast X)$ into more manageable pieces called “local realization spaces”

$$\text{Real}_{P_\infty}(H_\ast X) = \prod_{[X, \phi] \in \mathcal{N}R^{P_\infty}} P_\infty \{X\}_{[\phi]}.$$ 

For a given $\phi : P_\infty \to \text{End}_X$, the local realization space $P_\infty \{X\}_{[\phi]}$ is a Kan complex whose vertices are the $P_\infty$-algebra structures $\varphi : P_\infty \to \text{End}_X$ such that the $P_\infty$-algebras $(X, \varphi)$ and $(X, \phi)$ are related by a zigzag of quasi-isomorphisms. We introduce a variation of this realization space, denoted by $\text{Real}_{P_\infty}(H_\ast X)$, which is a Kan complex whose vertices are the prop morphisms $P_\infty \to \text{End}_{H_\ast X}$ inducing
the $P$-algebra structure $P \to \text{End}_{H_*X}$ in homology. It turns out that there is an injection of connected components

$$\pi_0 \text{Real}_{P_{\infty}}(H_*X) \hookrightarrow \pi_0 \text{Real}_{P_{\infty}}(H_*X)$$

and isomorphisms of homotopy groups

$$\pi_n(\text{Real}_{P_{\infty}}(H_*X), \phi) \cong \pi_n(\text{Real}_{P_{\infty}}(H_*X), \phi)$$

for $n > 0$. Moreover, the realization space $\text{Real}_{P_{\infty}}(H_*X)$ admits a similar decomposition in local realization spaces.

The main result of our paper reads:

**Theorem 0.1.** Let $P$ be a prop with trivial differential and $P_{\infty}$ be a cofibrant resolution of $P$. Let $X$ be a chain complex such that $H_*X$ forms a $P$-algebra.

1. The connected components of the realization space $\text{Real}_{P_{\infty}}(H_*X)$ decompose into

$$\pi_0 \text{Real}_{P_{\infty}}(H_*X) \cong \bigsqcup_{[\phi]\in \mathcal{NR}_{P_{\infty}}} \text{Aut}_{\mathbb{K}}(H_*X)/\text{Aut}_{\text{Ho}(\text{Ch}_{\mathbb{K}}^{P_{\infty}})}(H_*X, \phi)$$

where $\text{Aut}_{\mathbb{K}}(H_*X)$ stands for the group of automorphisms of $H_*X$ as a dg $\mathbb{K}$-module

and $\text{Aut}_{\text{Ho}(\text{Ch}_{\mathbb{K}}^{P_{\infty}})}(H_*X, \phi)$ stands for the group of automorphisms of $(H_*X, \phi)$ in the homotopy category of dg $P_{\infty}$-algebras.

2. For $n \geq 1$, the $n$th homotopy groups of the realization space $\text{Real}_{P_{\infty}}(H_*X)$ are given by

$$\pi_n(\text{Real}_{P_{\infty}}(H_*X), \phi) \cong H^{n-1} \text{Der}_{\phi}(P_{\infty}, \text{End}_{H_*X}).$$

3. For $n \geq 2$, we have

$$\pi_n(\text{Real}_{P_{\infty}}(H_*X), \phi) \cong \pi_n(\mathcal{NR}^{P_{\infty}}, (H_*X, \phi))$$

$$\cong \pi_{n-1}(L^H \text{wCh}_{\mathbb{K}}^{P_{\infty}}((H_*X, \phi), (H_*X, \phi)), \text{id})$$

where $L^H$ is the hammock localization functor constructed in [7].

Note that $L^H \text{wCh}_{\mathbb{K}}^{P_{\infty}}(-, -)$ is the simplicial monoid of homotopy automorphisms. When $P$ is an operad, then the $P_{\infty}$-algebras form a model category and we recover the usual simplicial monoid of self weak equivalences $\text{haut}(-)$. To put it into words, one can compute connected components of the moduli space of realizations with quotient of automorphisms groups, its higher homotopy groups at a given base point are isomorphic to the cohomology of the deformation complex of this base point, and for $n \geq 2$ we recover the homotopy groups of the 'Goerss-Hopkins' realization space and the homotopy groups of homotopy automorphisms. Moreover, if $\phi : P_{\infty} \to P \to \text{End}_{H_*X}$ is the trivial $P_{\infty}$-algebra structure on $H_*X$ induced by its $P$-algebra structure, the connected components of the local realization space associated to $\phi$ are given by

$$\pi_0 P_{\infty}\{H_*X\}_{[\phi]} \cong \text{Aut}_{\mathbb{K}}(H_*X)/\text{Aut}_{P}(H_*X).$$

The proof of Theorem 0.1 is as follows. We consider a restriction of the homotopy fiber sequence of Theorem 0.1 in [41] to local realization spaces, we explain how we can replace it with an actual Kan fibration up to weak equivalences obtained from a certain simplicial Borel construction, and we use long exact sequences arguments. The computation of higher homotopy groups follows from a correspondence between higher homotopy groups of mapping spaces and cohomology groups of deformation
complexes. The proof of this intermediate result relies on Lie theory for $L_\infty$-algebras and is fully written in [42].

We then study extensions of algebraic structures. Precisely, if we have a prop morphism $O \to P$, it implies that every $P$-algebra is in particular a $O$-algebra. The realization problem is the following. The prop morphism $O \to P$ induces a prop morphism $i: O_\infty \to P_\infty$. Let $X$ be a chain complex such that $H_*X$ is a $P$-algebra, and suppose that $X$ possesses a $O_\infty$-algebra structure. The extensions of an $O_\infty$-algebra structure into $P_\infty$-algebra structures are $(P,O)_\infty$-algebras, where

$$O_\infty \xrightarrow{i} P_\infty \xrightarrow{j} (P,O)_\infty$$

is the homotopy cofiber of $i$ in the category of dg props. Our first result in the relative setting is a generalization of the main theorem of [41]:

**Theorem 0.2.** Let $X$ be a dg $O_\infty$-algebra. Then the commutative square

$$
\begin{array}{ccc}
P_\infty\{X\} & \xrightarrow{N\chi^*} & N\text{wCh}^P_{K_\infty} \\
\downarrow & & \downarrow N\chi^* \\
\{X\} & \xrightarrow{N\chi^*} & N\text{wCh}^O_{K_\infty}
\end{array}
$$

is a homotopy pullback of simplicial sets.

We then use a simplicial Borel construction to get results similar to those of the non relative context. In particular, for local realization spaces there is a bijection

$$\pi_0(\text{Real}P_\infty(H_*X)) \cong \text{Aut}_{\text{Ho}(\text{Ch}_{K_\infty}^P)}(X,\phi \circ j \circ i)/\text{Aut}_{\text{Ho}(\text{Ch}_{K_\infty}^O)}(X,\phi \circ j).$$

**Results in the operad setting.** When $P_\infty$ is an operad, it turns out that $\text{Real}_{P_\infty}(H_*X)$ has a nice interpretation as the moduli space of $P_\infty$-algebras with underlying complex $X$ realizing the $P$-algebra structure on $H_*X$, such that $\pi_0\text{Real}_{P_\infty}(H_*X)$ is the set of $\infty$-isotopy classes of such algebras ($\infty$-morphisms of $P_\infty$-algebras which reduce to the identity on $X$). Using the obstruction theory for algebras over operads developed in [20] in terms of gamma homology, we get:

**Theorem 0.3.** Let $P$ be a $\Sigma$-cofibrant graded operad with a trivial differential. Let $H_*X$ be a $P$-algebra such that $H\Gamma^0_p(H_*X,H_*X) = H\Gamma^1_p(H_*X,H_*X) = 0$. Then

$$\pi_0(\text{Real}_{P_\infty}(H_*X)) \cong \text{Aut}_P(H_*X)/\text{Aut}_{\text{Ho}(\text{Ch}_{K_\infty}^O)}(X,\phi \circ j)$$

and for $n \geq 2$ we have

$$\pi_n(\text{Real}_{P_\infty}(H_*X),\phi) \cong \pi_n(N\mathcal{R}^P_{\infty},(H_*X,\phi))$$

$$\cong \pi_{n-1}(\text{haut}_{P_\infty}(H_\infty),\text{id})$$

$$\cong H\Gamma^0_p(H_*X,H_*X).$$

where $H_\infty$ is a cofibrant resolution of $H_*X$ in $P_\infty$-algebras.
Applications to Poincaré duality. We apply these results to determine equivalence classes of realizations of Poincaré duality on oriented compact manifolds. Let $M$ be a compact oriented manifold. Then $H_* (M; K)$ is a particular kind of Frobenius algebra, called a special symmetric Frobenius bialgebra (a structure which plays a prominent role in the algebraic counterpart of conformal field theory [14]). Let us denote by $sFrob$ the prop encoding such bialgebras, we have:

**Theorem 0.4.** (1) We have

$$\pi_0 sFrob_\infty \{H_* X\}_{[\phi]} \cong \text{Aut}_K(H_* X)/\text{Aut}_\text{Com}(H^* X)$$

where $\phi$ is the trivial $sFrob_\infty$-structure $sFrob_\infty \overset{\sim}{\to} sFrob \to \text{End}_{H_* X}$.

(2) If $H^*_\text{Com}(H_* X, H_* X, \phi \circ j \circ i) = 0$, we have

$$\pi_0 (sFrob, \text{Com}) \{H_* X\}_{[\phi]} \cong 0$$

and

$$\pi_0 \text{Real}_{(sFrob, \text{Com})} \{H_* X\} = \pi_0 NR^{(sFrob, \text{Com})}_\infty.$$

To put these observations into words, we proved the following: in general, we can compute the connected components of the local realization space of Poincaré duality on chains around the trivial homotopy structure as a quotient of its graded vector space automorphisms by its commutative algebra automorphisms. And when the zero gamma cohomology group of a certain Frobenius bialgebra structure $(H_* X, \phi)$ is zero, then the corresponding local realization space is connected.

We finally do computations for some examples of manifolds.

• (1) For the $n$-sphere $S^n$ we have

$$\pi_0 sFrob_\infty \{H_* S^n\}_{[\phi]} \cong \mathbb{Q}^*.$$

• (2) For the complex projective space $\mathbb{C}P^n$ we have

$$\pi_0 sFrob_\infty \{H_* \mathbb{C}P^n\}_{[\phi]} \cong (\mathbb{Q}^*)^{\times n}.$$

• (3) Let $S$ be a compact oriented surface of genus $g$. We have

$$\pi_0 sFrob_\infty \{H_* S\}_{[\phi]} \cong \mathbb{Q}^*.$$

• (4) For $M = \mathbb{C}P^2 \# \mathbb{C}P^2$, that is, the connected sum of two copies of $\mathbb{C}P^2$, we have

$$\pi_0 sFrob_\infty \{H_* M\}_{[\phi]} \cong GL_2(\mathbb{Q})/\text{CO}_2(\mathbb{Q}) \times \mathbb{Q}^*$$

where $\text{CO}_2(\mathbb{Q}) = \mathbb{Q}^* \ltimes O_2(\mathbb{Q})$ is the conformal orthogonal group.

**Remark 0.5.** Let us note that by [3] there is a non trivial action of the Grothendieck-Teichmüller Lie algebra on the higher homotopy groups of realization spaces of homotopy Frobenius structures.

**Organization:** Section 1 is a reminder about props and their homotopical properties, algebras over props, and moduli spaces of such structures. Section 2 is devoted to realizations spaces and the proof of Theorem 0.1. We define realization spaces and their decomposition into local realization spaces, then we show how to replace the homotopy fiber sequence in the main theorem of [41] (restricted to local realization spaces) by an actual Kan fibration obtained by a certain simplicial Borel construction (Theorem 2.8). From Theorem 2.8, Corollary 2.18 of [42] and a long exact sequence argument we get Theorem 0.1 (Theorem 2.14 in Section 2.3).
Section 3, we treat the case of algebras over operads, for which an explicit computation of connected components of the whole realization space as well as higher homotopy groups can be obtained in terms of the $\Gamma$-cohomology defined in [20].

Section 4 deals with realizations of extensions of a given algebraic structure. It extends Section 2 to the relative case by proving Theorem 0.2 and an analogue of Theorem 2.8, from which we deduce a characterization of connected components of relative realization spaces analogue to part (1) of Theorem 0.1. Finally, Section 5 focuses on an application of such results to the realization of Poincaré duality at the chain level. We prove several results about connected components of realizations of special symmetric Frobenius bialgebras, as well as extensions of a commutative algebra structure to such a bialgebra structure, and conclude with some examples.

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1. Classification spaces and moduli spaces

We denote by $\text{Ch}_K$ the category of $\mathbb{Z}$-graded chain complexes over a field $K$. All along this paper, we refer the reader to [12] or Section I of [39] for the notion of symmetric monoidal category $\mathcal{E}$ over a base symmetric monoidal category $\mathcal{C}$. Here we work in the case $\mathcal{C} = \text{Ch}_K$. We will implicitly assume throughout the paper that all small limits and small colimits exist in $\mathcal{E}$, and that it admits an internal hom bifunctor. We suppose moreover the existence of an external hom bifunctor $\text{Hom}_\mathcal{E}(-,-) : \mathcal{E}^{\text{op}} \times \mathcal{E} \to \text{Ch}_K$ satisfying an adjunction relation

$$\forall C \in \text{Ch}_K, \forall X,Y \in \mathcal{E}, \text{Mor}_\mathcal{E}(C \otimes X, Y) \cong \text{Mor}_\text{Ch}_K(C, \text{Hom}_\mathcal{E}(X,Y))$$

(so $\mathcal{E}$ is naturally an enriched category over $\text{Ch}_K$).

Example 1.1. (1) The differential graded $K$-modules $\text{Ch}_K$ form a symmetric monoidal category over itself.

(2) Let $I$ be a small category; the $I$-diagrams in $\text{Ch}_K$ form a symmetric monoidal category over $\mathcal{C}$. The external hom $\text{Hom}_{\mathcal{C}^I}(-,-) : \mathcal{C}^I \times \mathcal{C}^I \to \mathcal{C}$ is given by

$$\text{Hom}_{\mathcal{C}^I}(X,Y) = \int_{i \in I} \text{Hom}_{\mathcal{C}}(X(i), Y(i)).$$

We deal with symmetric monoidal categories equipped with a model category structure. We assume that the reader is familiar with the basics of model categories. We refer to Hirschhorn [18] and Hovey [22] for a comprehensive treatment. We just recall the axioms of symmetric monoidal model categories formalizing the interplay between the tensor and the model structure.

Definition 1.2. (1) A symmetric monoidal model category is a symmetric monoidal category $\mathcal{C}$ equipped with a model category structure such that the following axioms hold:

MM0. The unit object $1_\mathcal{C}$ of $\mathcal{C}$ is cofibrant.

MM1. The pushout-product $(i_*, j_*) : A \otimes D \oplus_A \otimes C B \otimes D \to B \otimes D$ of cofibrations $i : A \to B$ and $j : C \to D$ is a cofibration which is also acyclic as soon as $i$ or $j$ is so.

(2) Suppose that $\mathcal{C}$ is a symmetric monoidal model category. A symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$ is a symmetric monoidal model category over $\mathcal{C}$ if the axiom MM0 holds and the axiom MM1 holds for both the internal and external tensor products of $\mathcal{E}$.
Example: the category $\text{Ch}_K$ of chain complexes over a field $K$ is our main working example of symmetric monoidal model category.

1.1. On $\Sigma$-bimodules, props and algebras over a prop. Let $\mathcal{S}$ be the category having the pairs $(m, n) \in \mathbb{N}^2$ as objects together with morphisms sets such that:

$$\text{Mor}_\mathcal{S}((m, n), (p, q)) = \begin{cases} \Sigma_m^{op} \times \Sigma_n, & \text{if } (p, q) = (m, n), \\ \emptyset, & \text{otherwise.} \end{cases}$$

The (differential graded) $\Sigma$-biobjects in $\text{Ch}_K$ are the $\mathcal{S}$-diagrams in $\text{Ch}_K$. So a $\Sigma$-biobject is a double sequence $\{M((m, n)) \in \text{Ch}_K\}_{(m, n) \in \mathbb{N}^2}$ where each $M((m, n))$ is equipped with a right action of $\Sigma_m$ and a left action of $\Sigma_n$ commuting with each other.

Definition 1.3. A dg prop is a $\Sigma$-biobject endowed with associative horizontal products

$$\circ_h : P((m_1, n_1)) \otimes P((m_2, n_2)) \rightarrow P(m_1 + m_2, n_1 + n_2),$$

vertical associative composition products

$$\circ_v : P((k, n)) \otimes P((m, k)) \rightarrow P((m, n))$$

and units $1 \rightarrow P((n, n))$ neutral for both composition products. These products satisfy the exchange law

$$(f_1 \circ_h f_2) \circ_v (g_1 \circ_h g_2) = (f_1 \circ_v g_1) \circ_h (f_2 \circ_v g_2)$$

and are compatible with the actions of symmetric groups. Morphisms of props are equivariant morphisms of collections compatible with the composition products. We denote by $\mathcal{P}$ the category of props.

Let us note that Appendix A of [12] provides a construction of the free prop on a $\Sigma$-biobject. The free prop functor is left adjoint to the forgetful functor:

$$F : \mathcal{C}^\Sigma \rightleftarrows \mathcal{P} : U.$$  

The following definition shows how a given prop encodes algebraic operations on the tensor powers of an object of $\mathcal{E}$:

Definition 1.4. Let $\mathcal{E}$ be a symmetric monoidal category over $\text{Ch}_K$.

(1) The endomorphism prop of $X \in \mathcal{E}$ is given by $\text{End}_\mathcal{E}(X) = \text{Hom}_\mathcal{E}(X^\otimes m, X^\otimes n)$ where $\text{Hom}_\mathcal{E}(\cdot, \cdot)$ is the external hom bifunctor of $\mathcal{E}$.

(2) Let $P$ be a dg prop. A $P$-algebra in $\mathcal{E}$ is an object $X \in \mathcal{E}$ equipped with a prop morphism $P \rightarrow \text{End}_\mathcal{E}(X)$.

Hence any “abstract” operation of $P$ is sent to an operation on $X$, and the way abstract operations compose under the composition products of $P$ tells us the relations satisfied by the corresponding algebraic operations on $X$.

The category of $\Sigma$-biobjects $\text{Ch}_K^\Sigma$ is a diagram category over $\text{Ch}_K$, so it inherits a cofibrantly generated model category structure in which weak equivalences and fibrations are defined componentwise. The adjunction $F : \text{Ch}_K^\Sigma \rightleftarrows \mathcal{P} : U$ transfer this model category structure to the props:

Theorem 1.5. (1) (cf. [11], theorem 4.9) Suppose that $\text{char}(\mathbb{K}) > 0$. The category $P_0$ of props with non-empty inputs (or outputs) equipped with the classes of componentwise weak equivalences and componentwise fibrations forms a cofibrantly generated semi-model category.
(2) (cf. [11], theorem 4.9) Suppose that \( \text{char}(\mathbb{K}) = 0 \). Then the entire category of props inherits a full cofibrantly generated model category structure with the weak equivalences and fibrations as above.

(3) (cf. [23], theorem 1.1) Suppose that \( \text{char}(\mathbb{K}) = 0 \). Let \( C \) be a non-empty set. Then the category \( \mathcal{P}_C \) of \( C \)-colored props in forms a cofibrantly generated model category with fibrations and weak equivalences defined componentwise.

A semi-model category structure is a slightly weakened version of model category structure: the lifting axioms work only with cofibrations with a cofibrant domain, and the factorization axioms work only on a map with a cofibrant domain (see the relevant section of [12]). The notion of a semi-model category is sufficient to apply the usual constructions of homotopical algebra. A prop \( P \) has non-empty inputs if it satisfies

\[
P(0,n) = \begin{cases} 
\mathbb{K}, & \text{if } n = 0, \\
0, & \text{otherwise}. 
\end{cases}
\]

We define in a symmetric way a prop with non-empty outputs. Such props usually encode algebraic structures without unit or without counit, for instance Lie bialgebras.

1.2. Moduli spaces and relative moduli spaces of algebra structures over a prop. A moduli space of algebra structures over a prop \( P \), on a given object \( X \) of \( \mathcal{E} \), is a simplicial set whose points are the prop morphisms \( P \to \text{End}_X \). Such a moduli space can be more generally defined on diagrams of \( \mathcal{E} \). We then deal with endomorphism props of diagrams. For the sake of brevity and clarity, we refer the reader to the chapter 16 in [18] for a complete treatment of the notions of simplicial resolutions, cosimplicial resolutions and Reedy model categories.

**Definition 1.6.** Let \( M \) be a model category and let \( X \) be an object of \( M \).

1. A cosimplicial resolution of \( X \) is a cofibrant approximation to the constant cosimplicial object \( c_*X \) in the Reedy model category structure on cosimplicial objects \( M^{\Delta} \) of \( M \).

2. A simplicial resolution of \( X \) is a fibrant approximation to the constant simplicial object \( c_*X \) in the Reedy model category structure on simplicial objects \( M^{\Delta^op} \) of \( M \).

**Definition 1.7.** Let \( \mathcal{E} \) be a symmetric monoidal model category over \( Ch_{\mathbb{K}} \) and \( P \) a cofibrant prop (with non-empty inputs if \( \text{char}(\mathbb{K}) > 0 \)) in \( Ch_{\mathbb{K}} \). Let \( I \) be a small category and \( X \) a object of \( \mathcal{E} \). The moduli space of \( P \)-algebra structures on \( X \) is the simplicial set alternatively defined by

\[
P\{X\} = \text{Mor}_{\mathcal{P}_0}(P \otimes \Delta[-], \text{End}_X),
\]

where \( P \otimes \Delta[-] \) is a cosimplicial resolution of \( P \), or

\[
P\{X\} = \text{Mor}_{\mathcal{P}_0}(P, \text{End}_{\Delta X}^X[-]),
\]

where \( \text{End}_{\Delta X}^X[-] \) is a simplicial resolution of \( \text{End}_X \). Since every chain complex over a field is fibrant and cofibrant, every dg prop is fibrant: the fact that \( P \) is cofibrant
and $\text{End}_X$ is fibrant implies that these two formulae give the same moduli space up to homotopy. We get a functor

$$\begin{align*}
P_0 & \to \text{sSet} \\
P & \mapsto P\{X\}.
\end{align*}$$

We can already get two interesting properties of these moduli spaces:

**Proposition 1.8.** (1) The simplicial set $P\{X\}$ is a Kan complex and its connected components give the equivalences classes of $P$-algebra structures on $X$, i.e.

$$\pi_0 P\{X\} \cong [P, \text{End}_X]_{\text{Ho}(\mathcal{P}_0)}.$$

(2) Every weak equivalence of cofibrant props $P \xrightarrow{\sim} Q$ gives rise to a weak equivalence of fibrant simplicial sets $Q\{X\} \xrightarrow{\sim} P\{X\}$.

These properties directly follows from the properties of simplicial mapping spaces in model categories [18]. The higher simplices of these moduli spaces encode higher simplicial homotopies between homotopies. For dg props we can actually get explicit simplicial resolutions:

**Proposition 1.9.** (see [42], Proposition 2.5) Let $P$ be a prop in $\text{Ch}_K$. Let us define $P^{\Delta^*}$ by

$$P^{\Delta^*}(m,n) = P(m,n) \otimes A_{PL}(\Delta^*),$$

where $A_{PL}$ denotes Sullivan’s functor of piecewise linear forms (see [9]). Then $P^{\Delta^*}$ is a simplicial resolution of $P$ in the category of dg props.

Now let us see how to study homotopy structures with respect to a fixed one. Let $O \to P$ be a morphism of props inducing a morphism of cofibrant props $O_\infty \to P_\infty$ between the corresponding cofibrant resolutions. We can form the homotopy cofiber

$$O_\infty \to P_\infty \to (P,O)_\infty$$

in the model category of dg props and thus consider the moduli space $(P,O)_\infty\{X\}$. This moduli space encodes the homotopy classes (and higher simplicial homotopies) of the $P_\infty$-structures on $X$ "up to $O_\infty$", i.e. for a fixed $O_\infty$-structure on $X$. A rigorous way to justify this idea is the following.

**Proposition 1.10.** Let $\mathcal{M}$ be a model category, endowed with a functorial simplicial mapping space

$$\text{Map}(\_ , \_ ) = \text{Mor}_{\mathcal{M}}(\_ , (\_ )^{\Delta^*})$$

(which always exists, by existence of functorial simplicial resolutions [18]). Let $Y$ be a fibrant object of $\mathcal{M}$. Then the functor $\text{Map}(\_ , Y)$ sends cofiber sequences induced by cofibrations between cofibrant objects to fiber sequences of pointed simplicial sets.

It follows from general properties of simplicial mapping spaces for which we refer the reader to [18]. We apply this result to obtain the particular homotopy fiber

$$\begin{align*}
(P,O)_\infty\{X\} & \to P_\infty\{X\} \\
\downarrow & \downarrow \\
\{X\} & \to O_\infty\{X\}.
\end{align*}$$
Example 1.11. Our main example of interest is the following: we know that the $E_\infty$-algebra structures on the singular cochains or chains classify the rational homotopy type of the considered topological space. For a Poincaré duality space, whose chains form a Frobenius algebra (and thus a Frobenius bialgebra, see section 5), a way to understand the homotopy Frobenius structures up to the rational homotopy type of this space is to analyze $\pi_* (\text{Frob, Com}_*) \{ C_*(X;\mathbb{Q}) \}$.

1.3. Moduli spaces as homotopy fibers. We refer the reader to [41] for a proof of the following theorem. We just recall a commutative diagram which will be crucial in the remaining part of the paper. Let $P$ a cofibrant prop, and $\mathcal{N}w(E_{cf})^{\Delta[-]} \otimes P$ the bisimplicial set defined by $(\mathcal{N}w(E_{cf})^{\Delta[-]} \otimes P)_{m,n} = (\mathcal{N}w(E_{cf})^{\Delta[n]} \otimes P)_m$, where the $w$ denotes the subcategory of morphisms which are weak equivalences in $E$. Then we get a diagram

\[
\begin{array}{cccc}
P\{X\} & \xrightarrow{\text{diag}fw(E_{cf})^{\Delta[-]} \otimes P} & \xrightarrow{\sim} \text{diag}\mathcal{N}w(E_{cf})^{\Delta[-]} \otimes P & \xleftarrow{\sim} \mathcal{N}w(E_{cf})^P, \\
\text{pt} & \xrightarrow{\sim} & \mathcal{N}(fwE_{cf}) & \xrightarrow{\sim} \mathcal{N}(wE_{cf})
\end{array}
\]

where the $fw$ denotes the subcategory of morphisms which are acyclic fibrations in $E$. The crucial point here is that the left-hand commutative square of this diagram is a homotopy pullback. This diagram implies:

Theorem 1.12. (Theorem 0.1 in [41]). Let $P$ be a cofibrant prop in $\text{Ch}_K$ and $X$ a chain complex. Then the commutative square

\[
\begin{array}{cccc}
P\{X\} & \xrightarrow{N(wCh_k^P)} & \\
\{X\} & \xrightarrow{NwCh_k} & \\
\end{array}
\]

is a homotopy pullback of simplicial sets, where the right vertical map is induced by the forgetful functor.

2. Characterization of realization spaces via the Borel construction

2.1. Fibrations and the simplicial Borel construction. We denote by $sSet_*$ the category of pointed simplicial sets. First recall that any Kan fibration

\[
F \xrightarrow{id} K \xrightarrow{p} L,
\]

where we suppose $(L,l_0)$ pointed and $F$ is the fiber over this basepoint $l_0$ (the typical fiber), induces a long exact sequence

\[
\ldots \pi_{n+1}(L,l_0) \xrightarrow{\partial_2} \pi_n(F,k_0) \xrightarrow{\pi_n(p)} \pi_n(K,k_0) \xrightarrow{\partial_1} \pi_n(L,l_0) \xrightarrow{\partial_0} \ldots
\]

where $l_0 = p(k_0)$. For $n = 1$, the boundary map $\partial_1 : \pi_1(L) \to \pi_0(F)$ is a surjection which factors through a bijection $\pi_1(L)/\text{Im}(\pi_1(p)) \cong \pi_0(F)$.

Simplicial monoids are monoid objects in $sSet_*$ (or $sSet$). Let $G$ be a simplicial monoid and $K$ be a simplicial set. A left action of $G$ on $K$ is a simplicial map

\[
G_\cdot \times K_\cdot \to K_\cdot \\
(g,x) \mapsto g.x
\]
such that \( g_2(1,x) = (g_2 g_1).x \) and \( e.x = x \) where \( e \) is the neutral element of \( G \). We refer the reader to [30] for the notion of principal \( G \)-bundle in the simplicial setting.

The interesting fact for us is that one can associate to \( G \) a principal twisted cartesian product \( K \times \tau \WG \) where \( \tau \) is a twisting function and \((K \times \tau \WG)_n = K_n \times G_n\). This twisted cartesian product actually equals to the quotient \( K \times G/WG = K \times WG/(g.x,y) = (x,g.y) \) \). One can construct from the principal \( G \)-bundle \( p_G : K \times WG \rightarrow WG \) of fiber \( K \) defined by \( p_G(x,y) = p_G(x) \). Proposition 8.4 in chapter V of [30] asserts that every twisted cartesian product whose fiber is a Kan complex forms a Kan fibration, and Theorem 19.4 that \( G \)-bundles are twisted cartesian products, so a \( G \)-bundle with a Kan complex as fiber forms a Kan fibration. We refer the reader to [30] for more details about these constructions. To conclude, let us recall that there is a bijective correspondence between homotopy classes of simplicial maps \( B \rightarrow \WG \) and \( G \)-equivalence classes of \( G \)-bundles of base \( B \) with a fixed fiber.

2.2. Realization spaces and automorphisms groups. The general idea is the following. Let \( P \) be an arbitrary prop. We consider a chain complex \( X \) such that \( H_\ast X \) is equipped with a \( P \)-algebra structure. We would like to study the \( P_\infty \)-algebra structures \( \varphi : P_\infty \rightarrow End_X \) on \( X \) such that \( H_\ast (X, \varphi) \cong H_\ast X \) as \( P \)-algebras, that is the \( P_\infty \)-algebra structures realizing the \( P \)-algebra \( H_\ast X \) at the chain level.

**Assumption:** we suppose that the differential of \( P \) is trivial, since we need prop isomorphisms \( H_\ast P_\infty \cong H_\ast P \cong P \).

Any prop morphism \( \varphi : P_\infty \rightarrow End_X \) induces a prop morphism \( P \rightarrow End_{H_\ast X} \) in the following way: since \( H_\ast (-) \) is a symmetric monoidal functor on chain complexes, for every \( X \in Ch_k \) there exists a prop morphism \( \rho_X : End_X \rightarrow End_{H_\ast X} \). We thus obtain a prop morphism

\[
P_\infty \xrightarrow{\varphi} End_X \xrightarrow{\rho_X} End_{H_\ast X}
\]

and apply the homology functor aritywise

\[
H_\ast (\rho_X \circ \varphi) : P \cong H_\ast P_\infty \rightarrow H_\ast End_{H_\ast X} = End_{H_\ast X}.
\]

A way to encode these structures, their equivalences classes and higher homotopies is to construct a realization space \( Real_{P_\infty}(H_\ast X) \) which will be a certain Kan subcomplex of \( P_\infty \{X\} \). For this aim we need the following lemma:

**Lemma 2.1.** Let \( K \) be a Kan complex. Let \( L_0 \subset K_0 \) be a subset of the vertices of \( K \). There exists a unique Kan subcomplex \( L \subset K \), constructed by induction on the simplicial dimension, which parametrizes the homotopies between these vertices and their higher homotopies.

**Proof.** We define the simplices of \( L \) by induction on the simplicial dimension:

\[
L_1 = \{ k \in K_1 | d_0 k, d_1 k \in L_0 \}
\]
...\[L_n = \{k \in K_n | \forall i, d_ik \in L_{n-1}\}.
\]

By construction, the faces of \(K\) restrict to \(L\). We now check by induction on the simplicial dimension \(n\) that the degeneracies of \(K\) also restrict to \(L\). For \(n = 0\), let \(x\) be a vertex of \(L\). There is only one degeneracy \(s_0(x) \in K_1\), and by the simplicial identities we have

\[(d_0 \circ s_0)(x) = x \in L_0\]

and

\[(d_1 \circ s_0)(x) = x \in L_0,\]

so \(s_0(x) \in L_1\) since its faces belong to \(L_0\). Now, let us suppose that there exists a positive integer \(n\) such that the degeneracies \(s_j : K_{n-1} \to K_n\) restrict to maps \(s_j : L_{n-1} \to L_n\). Let \(\sigma \in L_n\) be a \(n\)-simplex of \(L\) and \(0 \leq j \leq n\) be a fixed integer. For \(i = j\) and \(i = j + 1\), we have

\[(d_i \circ s_j)(\sigma) = \sigma \in L_n.\]

For \(i < j\) we have

\[(d_i \circ s_j)(\sigma) = (s_{j-1} \circ d_i)(\sigma)\]

which belongs to \(L_n\), because by definition of \(L_n\) the \((n-1)\)-simplex \(d_i(\sigma)\) belongs to \(L_{n-1}\), and by induction hypothesis \(s_{j-1} : K_{n-1} \to K_n\) restricts to \(s_{j-1} : L_{n-1} \to L_n\). For \(i > j + 1\) we have

\[(d_i \circ s_j)(\sigma) = (s_j \circ d_{i-1})(\sigma)\]

which belongs to \(L_n\) by the same arguments. All the faces of \(s_j(\sigma)\) belongs to \(L_n\), so \(s_j(\sigma)\) belongs to \(L_{n+1}\).

Concerning the Kan condition, it is sufficient to check it in dimension 2 (it works similarly in higher dimensions, given the inductive construction of \(L\)). The horn \(\Lambda^2\) consists in two 1-simplices having one common vertex

```
  1
 /|
/ 2
/|
/ 3
```

Consider the image of \(\Lambda^2\) in \(L\) under a given simplicial map. Since \(L\) is included into a Kan complex \(K\), one can fill this horn in \(K\), choosing a 1-simplex 13 relating the simplices 1 and 3 and a 2-simplex 123 filling the triangle. Now, since 1 and 3 are in \(L\), by definition 13 also lies in \(L\). Given that 21, 23 and 13 are 1-simplices of \(L\), by construction 123 is a 2-simplex of \(L\) and the horn is filled in \(L\). \(\square\)

We apply this construction to the Kan complex \(P_\infty \{X\}\), by choosing as a set of vertices the \(P_\infty\)-algebra structures \(P_\infty \to \text{End}_X\) realizing \(P \to \text{End}_{H_*X}\). We obtain the desired Kan complex \(\text{Real}_{P_\infty}(H_*X)\).
2.2.1. Local realization spaces. The reader will notice that our definition of the realization space is not the same as the one given for instance in [16], where it is defined as the simplicial nerve of a category whose objects are the $P_\infty$-algebras with a homology isomorphic to $H_*X$ as $P$-algebras, and morphisms are quasi-isomorphisms of $P_\infty$-algebras. One the one hand, we restrict ourselves to the $P_\infty$-algebras having $X$ as underlying complex. On the other hand, according to [12], if two prop morphisms $\varphi_1, \varphi_2 : P_\infty \to End_X$ are homotopic then $(X, \varphi_1)$ and $(X, \varphi_2)$ are related by a zigzag of quasi-isomorphisms of $P_\infty$-algebras. But the converse is in general not true, so our realization space detects finer homotopical informations about the $P_\infty$-algebra structures on $X$ than the one of [16]. This difference is precisely measured by subspaces of $Real_{P_\infty}(H_*X)$ that we call local realization spaces, which also provide us a decomposition of $Real_{P_\infty}(H_*X)$ into more handable pieces.

**Definition 2.2.** Let $\phi : P_\infty \to End_X$ be a prop morphism realizing a fixed structure $P \to End_{H_*X}$. The space of local realizations of $H_*X$ at $\phi$ is the Kan subcomplex of $P_\infty\{X\}$ denoted by $P_\infty\{X\}|_{[\phi]}$ defined by

$$(P_\infty\{X\}|_{[\phi]})_0 = \{ \varphi : P_\infty \to End_X | \exists (X, \varphi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} (X, \phi) \in CH^P_* \}$$

where the zigzags are of any finite length. The higher simplices are defined by the construction explained previously.

**Lemma 2.3.** A zigzag of $P_\infty$-algebras $(X, \varphi_X) \xleftarrow{\sim} \cdot \xrightarrow{\sim} (Y, \varphi_Y)$ induces an isomorphism of $P$-algebras $H_*X \cong H_*Y$.

**Proof.** Let us consider such a zigzag $(X, \varphi_X) \xleftarrow{\sim} \cdot \xrightarrow{\sim} (Y, \varphi_Y)$. The $P_\infty$-algebra structure on the whole diagram is encoded by a prop morphism

$$P_\infty \to End_{X} \xleftarrow{\sim} \cdot \xrightarrow{\sim} Y.$$ 

For every small category $I$, the homology defines obviously a functor $H_* : CH^I_* \to CH^I_*$ which associates to a functor $F : I \to CH^I_*$ the composite $H_* \circ F$. This functor is symmetric monoidal, since the tensor product on diagrams is defined pointwise. Moreover, the category of diagrams $CH^I_*$ is a symmetric monoidal category over $CH_*$. Hence for any $F \in CH^I_*$ one can form the prop morphism $\rho_F : End_F \to End_{H_*F}$, and any prop morphism $\varphi : P_\infty \to End_F$ induces the prop morphism $H_* (\rho_F \circ \varphi) : P \to End_{H_*F}$. The conclusion follows from the fact that the endomorphism prop of $F$ that we consider in this proof, defined by using the external hom of the diagram category $CH^I_*$, is exactly the same prop as the endomorphism prop of diagrams in the sense of [12] which is used to encode $P_\infty$-algebra structures on diagrams. We apply this argument to the particular case of $(X, \varphi_X) \xleftarrow{\sim} \cdot \xrightarrow{\sim} (Y, \varphi_Y)$. 

**Corollary 2.4.** The local realization space $P_\infty\{X\}|_{[\phi]}$ is a Kan subcomplex of the realization space $Real_{P_\infty}(H_*X)$.

We could define more generally $P_\infty\{X\}|_{[\phi]}$ for any morphism $\phi$, it will be then the local realization space of $H_* (\rho_X \circ \phi)$ at $\phi$.

Now we intend to prove that moduli spaces and realization spaces decompose in a disjoint union of these local realization spaces.
Lemma 2.5. Let $\phi, \phi' : P_\infty \to \text{End}_X$ be two prop morphisms. We denote by $[X, \phi]$ and $[X, \phi']$ the weak equivalence classes respectively of $(X, \phi)$ and $(X, \phi')$ in $P_\infty$-algebras. We have

1. $P_\infty \{X\}_{[\phi]} = P_\infty \{X\}_{[\phi']}$ or $P_\infty \{X\}_{[\phi]} \cap P_\infty \{X\}_{[\phi']} = \emptyset$;
2. $P_\infty \{X\}_{[\phi]} \iff [X, \phi] = [X, \phi'].$

Proof. (1) Suppose that $P_\infty \{X\}_{[\phi]} \cap P_\infty \{X\}_{[\phi']} \neq \emptyset$. Let $\varphi$ be a common vertex of $P_\infty \{X\}_{[\phi]}$ and $P_\infty \{X\}_{[\phi']},$ there exists two zigzags $(X, \varphi) \xleftarrow{\cdot} \xrightarrow{\cdot} (X, \phi)$ and $(X, \varphi) \xleftarrow{\cdot} \xrightarrow{\cdot} (X, \phi'),$ hence a zigzag $(X, \phi) \xleftarrow{\cdot} \xrightarrow{\cdot} (X, \phi').$ This zigzag implies, by definition, that $P_\infty \{X\}_{[\phi]}$ and $P_\infty \{X\}_{[\phi']}$ share the same set of vertices. The inductive construction of $P_\infty \{X\}_{[\phi]}$ and $P_\infty \{X\}_{[\phi']}$ then implies that they share the same set of simplices in each dimension.

(2) Obvious by using (1). $\square$

Let us denote by $\mathcal{N}$ the simplicial nerve functor, $\mathcal{N} \text{wCh}_K^{P_\infty}$ the classification space of $P_\infty$-algebras with respect to quasi-isomorphisms, and $\mathcal{N} \text{R}^{P_\infty}$ a version of the realization space in the sense of [16] restricted to the complex $X$. That is, the nerve of the category $\text{R}^{P_\infty}$ whose objects are $P_\infty$-algebras $(X, \phi)$ such that $H_s(X, \phi) \cong H_s X$ as $P$-algebras and morphisms are weak equivalences of $P_\infty$-algebras. We deduce from the lemma above:

Proposition 2.6. The moduli space $P_\infty \{X\}$ decomposes into

$$\bigcup_{[X, \phi] \in \mathcal{N} \text{wCh}_K^{P_\infty}} P_\infty \{X\}_{[\phi]}$$

The realization space $\text{Real}_{P_\infty}(H_s X)$ decomposes into

$$\bigcup_{[X, \phi] \in \mathcal{N} \text{R}^{P_\infty}} P_\infty \{X\}_{[\phi]}$$

Before showing how to compute the homotopy groups of these local realization spaces, let us point out that in the definition of these spaces, we could have replaced the zigzags of quasi-isomorphisms by zigzags of acyclic fibrations. This technical point will be useful in the next proofs:

Lemma 2.7. Let $P_\infty \{X\}_{[\phi]}$ be the Kan subcomplex of the realization space defined by

$$(P_\infty \{X\}_{[\phi]})_0 = \{ \varphi : P_\infty \to \text{End}_X | \exists (X, \varphi) \xleftarrow{\cdot} \xrightarrow{\cdot} (X, \phi) \in \text{Ch}_K^{P_\infty} \}$$

where the zigzags are of any finite length. We have

$$P_\infty \{X\}_{[\phi]} = P_\infty \{X\}_{[\phi]},$$

Proof. By construction, it is sufficient to prove that these two subcomplexes share the same set of vertices. The vertices of $P_\infty \{X\}_{[\phi]}$ are included in the set of vertices of $P_\infty \{X\}_{[\phi]},$ so we just have to prove the converse inclusion. Let $\varphi$ be a vertex of $P_\infty \{X\}_{[\phi]}.$ The $P_\infty$-algebra $(X, \varphi)$ is linked to $(X, \phi)$ by a zigzag of weak equivalences of finite length. We want to prove that this zigzag can be replaced by a zigzag of acyclic fibrations. It is sufficient to prove it in two cases: the case of a direct arrow and the case of two opposite arrows. Once these two cases work, the method can be extended by induction to all zigzags.

First case. Let $f : (X, \varphi) \xrightarrow{\cdot} (X, \phi)$ be a weak equivalence of $P_\infty$-algebras. Its image under the forgetful functor admits a decomposition $X \xrightarrow{\iota} Z(X) \xrightarrow{p} X$ in
chain complexes into an acyclic cofibration followed by an acyclic fibration. Since $X$ is fibrant (like any chain complex over a field), the map $i$ admits a section $s : Z(X) \xrightarrow{\sim} X$ such that $s \circ i = id_X$, which is surjective and thus forms an acyclic fibration. We obtain the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{Y}(X) & \xrightarrow{=} & X \\
\downarrow s & & \downarrow \sim \\
Z(X) & \xrightarrow{i} & X \\
\end{array}
$$

We know that there exists a prop morphism $P_\infty \xrightarrow{i} \text{End}_{X \leftarrow X} X$, so we can apply the argument of Lemma 8.3 of [12] to get a lifting

$$
\begin{array}{ccc}
\text{End}_{\mathcal{Y}(X)} & \xrightarrow{i} & \text{End}_{X \leftarrow X} X \\
\downarrow \sim & & \downarrow \sim \\
P_\infty & \leftarrow & \text{End}_{X \leftarrow X} X
\end{array}
$$

(since $P_\infty$ is a cofibrant prop) which preserves $\varphi$ and $\phi$, hence a zigzag $(X, \varphi) \xrightarrow{i} Z(X) \xrightarrow{\sim} (X, \phi)$ of $P_\infty$-algebra implying that $\varphi \in P_\infty \{X\}_i$.

Second case. Suppose that there is a zigzag of two opposite weak equivalences $(X, \varphi) \xleftarrow{g} (X', \varphi') \xrightarrow{d} (X, \phi)$. The map of chain complexes $(g, d) : X \to X \times X$ factors through an acyclic cofibration followed by a fibration $X \xrightarrow{\sim} Z(X) \to X \times X$ hence a commutative diagram

$$
\begin{array}{ccc}
\mathcal{Y}(X) & \xrightarrow{=} & X \\
\downarrow s & & \downarrow \sim \\
Z(X) & \xrightarrow{i} & X \\
\end{array}
$$

Since there is a prop morphism $P_\infty \xrightarrow{i} \text{End}_{X \leftarrow X} X$ we can apply the same argument again to obtain a lifting $P_\infty \xrightarrow{i} \text{End}_{\mathcal{Y}(X)}$ and conclude that $\varphi \in P_\infty \{X\}_i$.

2.2.2. A characterization via homotopy automorphisms. Recall that $P_\infty \{X\}$ can be identified as a homotopy fiber

$$
\begin{array}{ccc}
P_\infty \{X\} & \xrightarrow{\text{diag} NfwCh_{p \otimes \Delta}} & \{X\} \\
\downarrow \sim & & \downarrow \sim \\
NfwCh_{p \otimes \Delta} & \sim & NfwCh_{p \otimes \Delta}
\end{array}
$$
This homotopy fiber restricts to a homotopy fiber

\[
P_{\infty}\{X\}^{f}_{[\varphi]} \longrightarrow \text{diag} NwCh_{K}^{P \otimes \Delta^{\ast}}|_{X}
\]

\[
\{X\} \longrightarrow NwCh_{K}|_{X}
\]

where \(NwCh_{K}|_{X}\) is the connected component of the chain complex \(X\) and

\[
\text{diag} NwCh_{K}^{P \otimes \Delta^{\ast}}|_{X} = \prod_{(X, \varphi)} \text{diag} NwCh_{K}^{P \otimes \Delta^{\ast}}|_{(X, \varphi)}
\]

is the union of the connected components \(\text{diag} NwCh_{K}^{P \otimes \Delta^{\ast}}|_{(X, \varphi)}\) of the \((X, \varphi)\) ranging over the acyclic fibration classes of \(P_{\infty}\)-algebras having \(X\) as underlying complex. According to the previous lemma, one gets actually the homotopy fiber

\[
P_{\infty}\{X\}^{f}_{[\varphi]} \longrightarrow \text{diag} NwCh_{K}^{P \otimes \Delta^{\ast}}|_{X}.
\]

\[
\{X\} \longrightarrow NwCh_{K}|_{X}
\]

The main goal of this section is to prove the following theorem:

**Theorem 2.8.** Let us suppose that \(X\) is a chain complex with a trivial differential (e.g. \(X\) is the homology of some chain complex). Let \(\varphi: P_{\infty} \rightarrow \text{End}_{X}\) be a prop morphism. There exists a commutative square

\[
W\text{Aut}_{K}(X) \times_{\text{Aut}_{K}(X)} P_{\infty}\{X\}^{f}_{[\varphi]} \longrightarrow \text{diag} NwCh_{K}^{P \otimes \Delta^{\ast}}|_{X}
\]

\[
\pi \downarrow \quad \downarrow \quad \downarrow
\]

\[
W\text{Aut}_{K}(X) \quad \sim \quad NwCh_{K}|_{X}
\]

where \(\pi\) is a Kan fibration obtained by the simplicial Borel construction and the horizontal maps are weak equivalences of simplicial sets.

Since \(\pi\) is obtained by applying the simplicial Borel construction to a simplicial action of \(\text{Aut}_{K}(X)\) (seen as a discrete group) on \(P_{\infty}\{X\}^{f}_{[\varphi]}\), we get:

**Proposition 2.9.** The strict pullback below is a homotopy pullback of simplicial sets:

\[
P_{\infty}\{X\}^{f}_{[\varphi]} \longrightarrow W\text{Aut}_{K}(X) \times_{\text{Aut}_{K}(X)} P_{\infty}\{X\}^{f}_{[\varphi]}.
\]

\[
\{X\} \longrightarrow W\text{Aut}_{K}(X)
\]

**Proof.** Since \(\text{Aut}_{K}(X)\) is a discrete simplicial group acting on \(P_{\infty}\{X\}^{f}_{[\varphi]}\), according to [30] we can associate to this action an \(\text{Aut}_{K}(X)\)-bundle of typical fiber \(P_{\infty}\{X\}^{f}_{[\varphi]}\) defined by \(\pi\). Moreover, \(P_{\infty}\{X\}^{f}_{[\varphi]}\) is a Kan complex so \(\pi\) is a Kan fibration. Given that the model category of simplicial sets is right proper, the fiber of \(\pi\) is a model of the homotopy fiber of \(\pi\). \(\square\)
Proof of Theorem 2.8
We first need to define the action of $\text{Aut}_K(X)$ on the local realization spaces:

**Lemma 2.10.** The group $\text{Aut}_K(X)$ acts simplicially on $P_\infty \{X\}^f_{[\varphi]}$.

**Proof.** We see $\text{Aut}_K(X)$ as a discrete simplicial group and we construct a simplicial action on the whole moduli space by conjugation in each simplicial dimension

$$\text{Aut}_K(X) \times P_\infty \{X\}_k \rightarrow P_\infty \{X\}_k$$

$$(g : X \xrightarrow{\varphi} X, \varphi : P \otimes \Delta^k \rightarrow \text{End}_X) \mapsto \varphi_g = g^{-1} \circ g^* \circ \varphi$$

where $\varphi_g(m, n) : (P \otimes \Delta^k)(m, n) \xrightarrow{\varphi} \text{Hom}_K(X^m, X^n)$.

The notation $\text{Hom}_K$ stands for the internal hom of chain complexes. The map $g^{-1} \circ g^*$ is a prop morphism:

- concerning the horizontal composition products, for any $f_1 \in \text{Hom}_K(X^{\otimes m_1}, X^{\otimes n_1})$ and $f_2 \in \text{Hom}_K(X^{\otimes m_2}, X^{\otimes n_2})$, we have

$$(g^{-1} \circ g^*)(f_1 \otimes f_2) = (g^{-1})^{\otimes n_1 + n_2} \circ f_1 \otimes f_2 \circ g^{\otimes m_1 + m_2}$$

$$= ((g^{-1})^{\otimes n_1} \circ f_1 \circ g^{\otimes m_1}) \otimes ((g^{-1})^{\otimes n_2} \circ f_2 \circ g^{\otimes m_2})$$

$$= (g^{-1} \circ g^*)(f_1) \otimes (g^{-1} \circ g^*)(f_2);$$

- concerning the vertical composition products, for any $f_1 \in \text{Hom}_K(X^{\otimes m}, X^{\otimes k})$ and $f_2 \in \text{Hom}_K(X^{\otimes m_2}, X^{\otimes n_2})$, we have

$$(g^{-1} \circ g^*)(f_2 \circ f_1) = (g^{-1})^{\otimes n} \circ f_2 \circ f_1 \circ g^{\otimes m}$$

$$= (g^{-1})^{\otimes n} \circ f_2 \circ (g \circ g^{-1})^{\otimes k} \circ f_1 \circ g^{\otimes m}$$

$$= (g^{-1})^{\otimes n} \circ f_2 \circ g^{\otimes k} \circ (g^{-1})^{\otimes k} \circ f_1 \circ g^{\otimes m}$$

$$= (g^{-1} \circ g^*)(f_2) \circ (g^{-1} \circ g^*)(f_1).$$

The compatibility with the actions of symmetric groups is trivial since it amounts to permute the factors in $g^{\otimes m}$ and $(g^{-1})^{\otimes n}$. By construction we clearly get a group action, that is $\varphi_{g_2 \circ g_1} = (\varphi_{g_1})_{g_2}$ and $\varphi_{1d} = \varphi$.

We check the compatibility of these actions defined in each simplicial dimension with the simplicial structure of $P_\infty \{X\}$. We have a commutative diagram

$$\begin{array}{ccc} P_\infty \otimes \Delta^k & \xrightarrow{\varphi} & \text{End}_X \xrightarrow{g^{-1} \circ g^*} \text{End}_X \\ \downarrow d_i & & \downarrow d_i \\ P_\infty \otimes \Delta^{k-1} & \xrightarrow{d_i \varphi} & \text{End}_X \xrightarrow{g^{-1} \circ g^*} \text{End}_X \end{array}$$

so $d_i(\varphi_g) = d_i(\varphi_g)$. The same argument holds for the degeneracies.

Finally, we have to verify that this action restricts to the local realization space $P_\infty \{X\}^f_{[\varphi]}$. We prove it inductively on the simplicial dimension $k$: since a $k$-simplex of the moduli space $P_\infty \{X\}$ lies in $P_\infty \{X\}^f_{[\varphi]}$ if and only if all its vertices lie in $P_\infty \{X\}^f_{[\varphi]}$, it is sufficient to verify the restriction of the group action for $k = 0$. Let
ϕ : \( P_\infty \rightarrow \text{End}_X \) be a vertex of \( P_\infty \{ X \}_{[\phi]} \), since \( g^{-1}_* \circ g^* \) is a prop isomorphism there is a homotopy equivalence \( \varphi_g \sim \varphi \). According to [12], this homotopy equivalence implies the existence of a zigzag of acyclic fibrations (not only weak equivalences) between \( (X, \varphi_g) \) and \( (X, \varphi) \), hence between \( (X, \varphi_g) \) and \( (X, \phi) \). Therefore \( \varphi_g \in P_\infty \{ X \}_{[\phi]} \). □

With this well defined simplicial group action we start the proof of Theorem 2.8.

First we construct the commutative diagram

\[
\begin{array}{c}
WAut_k(X) \times_{Aut_k(X)} P_\infty \{ X \}_{[\phi]} \rightarrow \text{diag} \text{Nf} \text{wCh}_{k} \text{P}^{\otimes \Delta^*}_{X} \\
\gamma \downarrow \downarrow \downarrow \\
WAut_k(X) \rightarrow \text{NwCh}_{k}|_X
\end{array}
\]

in the following way:

\[
(((f_k, \ldots, f_0), \varphi : P_\infty \otimes \Delta^k \rightarrow \text{End}_X) \rightarrow (X, f_k \cdot \varphi) \ldots \rightarrow (X, (f_k \circ \ldots \circ f_1) \cdot \varphi))
\]

\[
\pi=\text{P}_{\text{Aut}_k(X)} \rightarrow \text{forget} \rightarrow (f_k^{-1}, \ldots, f_0) \rightarrow (X \xrightarrow{f_k^{-1}} \ldots \rightarrow X)
\]

where the left vertical map is the projection associated to the Borel construction and the right vertical map forgets the \( P_\infty \otimes \Delta^k \)-algebra structure. The top horizontal map transfers the \( P_\infty \otimes \Delta^k \)-algebra structure on \( X \) along the sequence of isomorphisms given by \( f_k, \ldots, f_0 \) and the bottom horizontal map is just an inclusion. It is clear by definition of faces and degeneracies in the simplicial structures involved that these four maps are simplicial.

It remains to prove that the two horizontal maps are weak equivalences. For the bottom arrow, it follows from the work of Dwyer-Kan [8] which identifies the connected components of the classification space of a model category with the classifying complexes of homotopy automorphism and from the fact that \( W\text{haut}(X) = \overline{WAut}_k(X) \).

For the top arrow, we have a morphism of homotopy fibers

\[
\begin{array}{c}
P_\infty \{ X \}_{[\phi]} \rightarrow \text{WAut}_k(X) \times_{Aut_k(X)} P_\infty \{ X \}_{[\phi]} \rightarrow \text{WAut}_k(X) \\
\sim \downarrow \downarrow \downarrow \\
P_\infty \{ X \}_{[\phi]} \rightarrow \text{diag} \text{Nf} \text{wCh}_{k} \text{P}^{\otimes \Delta^*}_{X} \rightarrow \text{NwCh}_{k}|_X
\end{array}
\]

which induces a morphism of long exact sequences of homotopy groups. Since the two external arrows of the diagram above induce isomorphisms of homotopy groups, we conclude that the middle one is a weak equivalence by applying the five lemma.

**Remark 2.11.** (1) One can apply the same arguments to obtain similar results for \( P_\infty \{ X \} \) and \( \text{Real}_{P_\infty}(X) \).

(2) One can study for instance realization problems of diagrams of bialgebras. Let \( I \) be a small category, the category of \( I \)-diagrams \( \text{Ch}^I_k \) is equipped with the injective model category structure (pointwise cofibrations) and with the structure of a symmetric monoidal category over \( \text{Ch}_k \). These two structures are compatible,
so that $Ch^I_K$ forms a monoidal model category over $Ch_K$. We can therefore define $P$-algebras and $P_\infty$-algebras in these diagrams. The homology induces a symmetric monoidal functor $H_* : Ch^I_K \rightarrow Ch^I_K$, so for any $F \in Ch^I_K$, a morphism $P_\infty \rightarrow \text{End}_F$ induces $P \rightarrow H_* \text{End}_X \rightarrow \text{End}_{H_* X}$ and it is then meaningful to study realization problems in this setting. For instance, when $I = \{ \bullet \rightarrow \bullet \}$ one can consider realizations of morphisms of $P$-algebras.

Another way to do this is to work with the $\text{ob}(I)$-colored prop $P^I$ corresponding to the choice of a prop $P$ and a small category $I$. We have to suppose that $K$ is a field of characteristic zero, and to work within the model category of $\text{ob}(I)$-colored props defined in [23].

2.2.3. When $X$ has a non-zero differential. We recall from [12] that the endomorphism prop $\text{End}_f$ of a morphism $f : X \rightarrow Y$ can be obtained by an explicit pullback

\[
\begin{array}{ccc}
\text{End}_f & \overset{d_1}{\longrightarrow} & \text{End}_Y \\
\downarrow{d_0} & & \downarrow{f^*} \\
\text{End}_X & \overset{f_*}{\longrightarrow} & \text{Hom}_{XY}
\end{array}
\]

where the $\Sigma$-biobject $\text{Hom}_{XY}$ is defined by $\text{Hom}_{XY}(m,n) = \text{Hom}_K(X^\otimes m, X^\otimes n)$, and the maps $f^*$ and $f_*$ are defined respectively by $f^*(m,n) = - \circ f^\otimes m$ and $f_*(m,n) = f^\circ_\otimes quieter$. In this pullback square the maps $d_0$ and $d_1$ are prop morphisms. According to Lemma 7.2 of [12], if $f$ is an acyclic fibration then $d_1$ is an acyclic fibration and $d_0$ a weak equivalence. Given that every dg prop is fibrant, the zigzag of weak equivalences of fibrant props

\[
P_\infty\{X\} \xrightarrow{\sim} \text{End}_f \xrightarrow{\sim} \text{End}_Y
\]

from the pullback above gives rise by postcomposition to a zigzag of weak equivalences of Kan complexes

\[
P_\infty\{X\} \xrightarrow{\sim} P_\infty\{f\} \xrightarrow{\sim} P_\infty\{Y\}.
\]

In particular, when working over a field, there is a projection $X \rightarrow H_* X$ of a chain complex $X$ to its homology which forms a quasi-isomorphism. It implies that any $P_\infty$-algebra structure on $H_* X$ can be transferred to $X$ via this acyclic fibration (see Theorem 7.3 (1) of [12]). Then we apply the argument above to obtain a zigzag weak equivalences between $P_\infty\{X\}$ and $P_\infty\{H_* X\}$:

**Proposition 2.12.** Suppose that the moduli space $P_\infty\{H_* X\}$ is not empty. Then the moduli space $P_\infty\{X\}$ is not empty and has the same homotopy type.

When $H_* X$ is equipped with a $P$-algebra structures, it possesses at least the trivial $P_\infty$-algebra structure given by $P_\infty \sim \rightarrow P \rightarrow \text{End}_{H_* X}$ so the proposition above applies.

The homotopy equivalence above restricts to local realization spaces:

**Proposition 2.13.** Let $f : (X, \phi_X) \rightarrow (Y, \phi_Y)$ be an acyclic fibration of $P_\infty$-algebras. Then there is a zigzag of morphisms Kan complexes

\[
P_\infty\{X\}_{[\phi_X]} \leftarrow P_\infty\{f\}_{[\phi_f]} \rightarrow P_\infty\{Y\}_{[\phi_Y]}
\]
This map lifts to a map \( P^\infty \to \text{End}_f \) is the structure of \( P^\infty \)-algebra morphism of \( f \). This zigzag induces isomorphisms between the \( n^{th} \) homotopy groups for \( n > 0 \), based at any point homotopic to \( \phi \), and an injection of connected components

\[
\pi_0 P^\infty \{Y\}_{[\phi_Y]} \leftarrow \pi_0 P^\infty \{X\}_{[\phi_X]}.
\]

**Proof.** Let \( \varphi \) be a vertex of \( P^\infty \{f\}_{[\phi_f]} \). There exists a zigzag of weak equivalences

\[
(f, \varphi) \xrightarrow{\sim} \bullet \xrightarrow{\sim} (f, \phi_f) \in (Ch_{\mathbb{Z} \to \bullet})^P^\infty), \text{ that is a commutative diagram}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{f, \varphi} & X \\
\sim & \sim & \sim \\
Y & \xrightarrow{(f, \phi_f)} & Y
\end{array}
\]

This diagram induces two zigzags

\[
(d_0)_* (f, \varphi) = (X, d_0 \circ \varphi) \xrightarrow{\sim} \bullet \xrightarrow{\sim} (X, d_0 \circ \phi_f) = (d_0)_* (f, \phi_f) = (X, \phi_X)
\]

and

\[
(d_1)_* (f, \varphi) = (Y, d_1 \circ \varphi) \xrightarrow{\sim} \bullet \xrightarrow{\sim} (Y, d_1 \circ \phi_f) = (d_1)_* (f, \phi_f) = (Y, \phi_Y)
\]

so \((d_0)_* (f, \varphi) \in P^\infty \{X\}_{[\phi_X]} \) and \((d_1)_* (f, \varphi) \in P^\infty \{Y\}_{[\phi_Y]} \). Hence the zigzag

\[
P^\infty \{X\} \xrightarrow{\sim} P^\infty \{f\} \xrightarrow{\sim} P^\infty \{Y\},
\]

restricts to a zigzag

\[
P^\infty \{X\}_{[\phi_X]} \leftarrow P^\infty \{f\}_{[\phi_f]} \rightarrow P^\infty \{Y\}_{[\phi_Y]}.
\]

Since we have

\[
\pi_n (P^\infty \{X\}_{[\phi_X]}, \psi) = \pi_n (P^\infty \{X\}, \psi)
\]

for \( n > 0 \) and \( \psi \) homotopic to \( \phi \), the maps \((d_0)_* \) and \((d_1)_* \) induces isomorphisms of higher homotopy groups between the local realization spaces.

Now we want to check that \( \pi_0 (d_0)_* \) is an injection and \( \pi_0 (d_1)_* \) a bijection. The surjectivity of \( \pi_0 (d_1)_* \) follows from the lifting diagram

\[
\begin{array}{ccc}
End_f & \xrightarrow{\sim} & \text{End}_Y \\
\downarrow{d_1} & & \downarrow{d_1} \\
P^\infty & \xrightarrow{\varphi} & \text{End}_Y
\end{array}
\]

for any \( \varphi \in \infty \{Y\}_{[\phi_Y]} \), in which the lift exists because \( d_1 \) is an acyclic fibration and \( P^\infty \) is cofibrant. For the injectivity, we have to prove that given \( \varphi_f, \varphi'_f : P^\infty \to \text{End}_f \in P^\infty \{f\}_{[\phi_f]} \), if \( d_1 \circ \varphi_f \) and \( d_1 \circ \varphi'_f \) are homotopic then \( \varphi_f \) and \( \varphi'_f \) are homotopic. By Proposition 1.9, the componentwise tensor product \( \text{End}_Y \otimes A_{PL}(\Delta^1) \) gives a path object on \( \text{End}_X \), so a homotopy between \( d_1 \circ \varphi_f \) and \( d_1 \circ \varphi'_f \) is given by a map

\[
H : P^\infty \to \text{End}_Y \otimes A_{PL}(\Delta^1).
\]

This map lifts to a map

\[
\begin{array}{ccc}
End_f & \xrightarrow{\sim} & \text{End}_Y \otimes A_{PL}(\Delta^1) \\
\downarrow{d_1 \otimes A_{PL}(\Delta^1)} & & \downarrow{d_1 \otimes A_{PL}(\Delta^1)} \\
P^\infty & \xrightarrow{H} & \text{End}_Y \otimes A_{PL}(\Delta^1)
\end{array}
\]
Indeed, acyclic fibrations of props are determined componentwise in chain complexes, and the tensor product of chain complexes preserves acyclic surjections, so \( d_1 \otimes A_{PL}(\Delta^1) \) is an acyclic fibration. The lift \( H_f \) is the desired homotopy.

The injectivity of \( \pi_0(d_0)_* \) follows from the following argument. The homotopy fiber \( hofib_{\varphi_\circ}(d_0)_* \) of \( (d_0)_*: P_\infty \{f\} \xrightarrow{\sim} P_\infty \{X\} \) over a point \( \varphi: P_\infty \to End_X \) is the simplicial set generated by the lifts

![Diagram](image)

along the weak equivalence \( d_0 \). By a general homotopical algebra argument, the set of lifts along a weak equivalence is connected. Indeed, if \( \varphi_f, \varphi'_f: P_\infty \to End_f \) are two lifts of \( \varphi \), then in the homotopy category of props \( Ho(\mathcal{P}) \) we have

\[
[d_0] \circ [\varphi_f] = [d_0 \circ \varphi_f] = [d_0 \circ \varphi'_f] = [d_0] \circ [\varphi'_f]
\]

where \([ - ]\) stands for the homotopy class of a map, that is, its image under the localization functor \( \mathcal{P} \to Ho(\mathcal{P}) \). Since \( d_0 \) is a weak equivalence, the homotopy class \([d_0]\) is an isomorphism in \( Ho(\mathcal{P}) \), hence

\[
[\varphi_f] = [\varphi'_f].
\]

This means that \( \varphi_f \) and \( \varphi'_f \) belongs to the same connected component of \( hofib_{\varphi_\circ}(d_0)_* \).

Considering the exact sequence of pointed sets

\[
\pi_0 hofib_{\varphi_\circ}(d_0)_* \to \pi_0 P_\infty \{f\} \xrightarrow{\pi_0 (d_0)_*} \pi_0 P_\infty \{X\}
\]

induced by the homotopy fiber sequence

\[
\xrightarrow{hofib_{\varphi_\circ}(d_0)_*} P_\infty \{f\} \xrightarrow{\sim} P_\infty \{X\},
\]

this implies that \( \pi_0(d_0)_* \) is injective. \( \square \)

The zigzag

\[
P_\infty \{X\} \xleftarrow{\sim} P_\infty \{f\} \xrightarrow{\sim} P_\infty \{Y\}
\]

restricts to a zigzag between the realization spaces

\[
Real_{P_\infty}(H_*X) \xleftarrow{\sim} Real_{P_\infty}(H_*f) \xrightarrow{\sim} Real_{P_\infty}(H_*Y).
\]

Indeed, if \( \phi_f \) realizes \( H_*f \), then \( d_0 \circ \phi_f \) and \( d_1 \circ \phi_f \) realize respectively \( \phi_X \) and \( \phi_Y \) according to the following commutative diagrams: consider

![Diagram](image)

and apply the homology functor to obtain

![Diagram](image)
and finally

\[ P \xrightarrow{\phi_X} H_* \xrightarrow{\phi_f} \]

\[ H_* \xrightarrow{d_0} H_* \xrightarrow{f} \]

\[ \text{End}_{H_*} f \xrightarrow{H_* \phi} \text{End}_{H_*} \]

The same argument holds for \( Y \).

In particular, to understand the higher homotopy groups it is sufficient to study the realization space of \( P \rightarrow \text{End}_{H_*} X \) in \( P_\infty \rightarrow \text{End}_{H_*} X \), and the set of equivalences classes of realizations \( P_\infty \rightarrow \text{End}_{H_*} X \) injects into the set of equivalences classes of realizations \( P_\infty \rightarrow \text{End}_X \).

### 2.3. Realizations, homotopy automorphisms and obstruction theory

Our main result describes the connected components of local realization spaces as quotient of automorphisms groups and the higher homotopy groups as cohomology groups of a deformation complex. Moreover, for \( n \geq 2 \) we recover the higher homotopy groups of Goerss-Hopkins realization spaces.

**Theorem 2.14.** For every \( \phi : P_\infty \rightarrow \text{End}_{H_*} X \), the homotopy groups of the local realization space \( P_\infty \{ H_* X \}_[\phi] \) of \( H_* X \) at \( \phi \) are given by

\[ \pi_0 P_\infty \{ H_* X \}_[\phi] \cong \text{Aut}_K(H_* X) / \text{Aut}_0(Ch_P^{\infty}) \]

\[ \pi_1(P_\infty \{ H_* X \}_[\phi], \phi) \cong \ker(\pi_1(\pi)) \]

\[ = \pi_0 \text{Real}_{P_\infty}(id_{(H_* X, \phi)}) \]

\[ \cong H^0 \text{Der}_\phi(P_\infty, \text{End}_{H_*} X) \]

and for \( n \geq 2 \),

\[ \pi_n(P_\infty \{ H_* X \}_[\phi], \phi) \cong \pi_n(NR^{P_\infty}_{P_\infty}, (H_* X, \phi)) \]

\[ \cong \pi_{n-1}(L^H wCh_K^{P_\infty}((H_* X, \phi), (H_* X, \phi)), id) \]

\[ \cong H^{n-1} \text{Der}_\phi(P_\infty, \text{End}_{H_*} X) \]

where \( L^H \) is the hammock localization functor constructed in [7].

Let us just recall briefly that for any \( P_\infty \)-algebra \((X, \varphi)\), the hammocks of weight \( k \) and length \( n \), which define the \( k \)-simplices of the simplicial monoid \( L^H wCh_K^{P_\infty}((X, \varphi), (X, \varphi)) \) consist in the data of \( k - 1 \) chains of morphisms of length \( n \)

\[ X \xleftarrow{f_0} \ldots \xleftarrow{f_n} X, \]

\[ \ldots \]

\[ X \xleftarrow{f_0} \ldots \xleftarrow{f_n} X \]
organized in a commutative diagram (a “hammock”)

\[
\begin{array}{c}
X \rightarrow C_{k,1} \rightarrow C_{k,2} \rightarrow \cdots \rightarrow C_{k,n} \leftarrow C_{0,n} \\
\downarrow f_n^1 \quad \cdots \quad \downarrow f_n^i \quad \cdots \quad \downarrow f_n^{i-1} \quad \downarrow f_n^0 \\
C_{0,1} \rightarrow C_{0,2} \rightarrow \cdots \rightarrow C_{0,n} \\
\end{array}
\]

where vertical arrows and arrows going from right to left are weak equivalences. The simplicial monoid \( L^H wCh_k^{\infty}((X, \varphi), (X, \varphi)) \) encodes the homotopy automorphisms of \((X, \varphi)\). When \( P \) is an operad, then the \( P_\infty \)-algebras form a model category and we recover the usual simplicial monoid of self weak equivalences \( \text{haut}(X, \varphi) \).

**Corollary 2.15.** The connected components of the realization space \( \overline{\text{Real}}_{P_\infty}(H_* X) \) decompose into

\[
\pi_n(\overline{\text{Real}}_{P_\infty}(H_* X), \phi) \cong \prod_{[\phi] \in \pi_0 \mathbb{W} \mathbb{P}_\infty} \text{Aut}_{\mathbb{K}}(H_* X)/\text{Aut}_{H_0(\mathbb{W} \mathbb{P}_\infty)}(H_* X, \phi).
\]

For \( n \geq 1 \), we have

\[
\pi_n(\overline{\text{Real}}_{P_\infty}(H_* X), \phi) = \pi_n(\text{Real}_{P_\infty}(H_* X), \phi) = \pi_n(P_\infty \{H_* X\}_\phi, \phi) = \pi_n(P_\infty \{H_* X\}_{[\phi]}, \phi) \cong H^{n-1} \text{Der}_\phi(P_\infty, \text{End}_{H_* X})
\]

where \( P_\infty \{H_* X\}_\phi \) is the connected component of \( \phi \) in \( P_\infty \{H_* X\} \).

**Remark 2.16.** The long exact sequence associated to \( P_\infty \{H_* X\} \) tells us that there is an injection of \( \pi_1(P_\infty \{H_* X\}, \phi) \) into

\[
\pi_1(\overline{\text{Real}}_{P_\infty}(H_* X), \phi, (H_* X, \varphi), id) \cong \pi_0 L^H wCh_k^{\infty}((H_* X, \varphi), (H_* X, \varphi)) = \text{Aut}_{H_0(\mathbb{W} \mathbb{P}_\infty)}(H_* X, \varphi)
\]

and that if \( \text{Aut}(H_* X) = \{id\} \), then this injection is an isomorphism.

**Proof.** There is a long exact sequence

\[
\cdots \rightarrow \pi_{n+1}(\overline{\text{W}} \text{Aut}_\mathbb{K}(H_* X), id) \overset{\partial}{\rightarrow} \pi_n(P_\infty \{H_* X\}_{[\phi]}, \phi) \rightarrow \pi_n(\text{W} \text{Aut}_\mathbb{K}(H_* X) \times_{\text{Aut}_\mathbb{K}(H_* X)} P_\infty \{H_* X\}_{[\phi]}, (id, \phi), \pi_n(\overline{\text{W}} \text{Aut}_\mathbb{K}(H_* X), id)\ldots
\]

Given that \( \text{Aut}_\mathbb{K}(H_* X) \) is a discrete group, the higher homotopy groups of its classifying complex are

\[
\pi_1(\overline{\text{W}} \text{Aut}_\mathbb{K}(H_* X) \cong \text{Aut}_\mathbb{K}(H_* X)
\]

and

\[
\pi_n(\overline{\text{W}} \text{Aut}_\mathbb{K}(H_* X) = 0
\]

otherwise. Consequently, the group \( \pi_1(P_\infty \{X\}_{[\phi]}, \phi) \) is nothing but \( \text{ker}(\pi_1(\pi)) \), which in turn corresponds to \( \pi_0 \text{Real}_{P_\infty}(id(H_* X, \varphi)) \).
For $n \geq 2$, 
\[ \pi_n(P_\infty \{H_s X\}_{[\phi]}, \phi) \cong \pi_n((W Aut_k(H_s X) \times Aut_k(H_s X)) \ P_\infty \{H_s X\}_{[\phi]}, (id, \phi)) \]
\[ \cong \pi_n(diag N_f wCh_k^{P_\infty} \otimes \Delta^*|_{H_s X}, (H_s X, \phi)) \]
\[ \cong \pi_n(diag N_f wCh_k^{P_\infty} \otimes \Delta^*|(H_s X, \phi), (H_s X, \phi)) \]
\[ \cong \pi_n(N wCh_k^{P_\infty}|_{H_s X}, (H_s X, \phi)) \]
\[ \cong \pi_n(\mathcal{W}H_0^{P_\infty}((H_s X, \phi), (H_s X, \phi)), id) \]
where the first line is a consequence of the long exact sequence, the second line follows from the comparison of fiber sequences of Theorem 2.8, the fourth line follows from [41], and the fifth line is a result of Dwyer-Kan [8]. Recall that for any simplicial monoid $G$, the complex $\mathcal{W}G$ is a delooping of $G$, that is, we have $\Omega \mathcal{W}G = G$ where $\Omega(-)$ is the loop space pointed at the neutral element $e$. We deduce that 
\[ \pi_{n+1}(\mathcal{W}G, e) \cong \pi_n(\Omega e, \mathcal{W}G, e) \cong \pi_n(G, e), \]
hence 
\[ \pi_n(\mathcal{W}H_0^{P_\infty}((H_s X, \phi), (H_s X, \phi)), id) \cong \pi_{n-1}(L^H wCh_k^{P_\infty}((H_s X, \phi), (H_s X, \phi)), id). \]

We also get 
\[ \pi_1(W Aut_k(H_s X) \times Aut_k(H_s X)) \ P_\infty \{H_s X\}_{[\phi]}, (id, \phi)) \]
\[ \cong \pi_1(\mathcal{W}H_1^{P_\infty}((H_s X, \phi), (H_s X, \phi)), id) \]
\[ \cong \pi_0(L^H wCh_k^{P_\infty}((H_s X, \phi), (H_s X, \phi))) \]
\[ = Aut_{Ho(\mathcal{W}Ch_k^{P_\infty})}(H_s X, \phi) \]
by equivalence of fiber sequences. The image of $\pi_1(\pi)$ in $Aut_k(H_s X)$ is the subgroup of graded automorphisms which can be realized as a zigzag of quasi-isomorphisms $(H_s X, \phi) \leftarrow \bullet \rightarrow (H_s X, \phi)$ in $Ch_k^{P_\infty}$. For convenience, we will denote the quotient $Aut_k(H_s X)/Im(\pi_1(\pi))$ by $Aut_k(H_s X)/Aut_{Ho(\mathcal{W}Ch_k^{P_\infty})}(H_s X, \phi)$. For $n = 0$, the boundary 
$\partial_1 : \pi_1(W Aut_k(H_s X)) = Aut_k(H_s X) \rightarrow \pi_0 P_\infty \{X\}_{[\phi]}$ 
is a surjection which factors through a bijection 
$Aut_k(H_s X)/Aut_{Ho(\mathcal{W}Ch_k^{P_\infty})}(H_s X, \phi) \cong \pi_0 P_\infty \{X\}_{[\phi]}.$ 

To conclude, the identification of higher homotopy groups with the cohomology of deformation complexes follows from Corollary 2.18 of [42]. Although this result is originally stated for properads, its extension to props is straightforward by using the deformation complexes of algebras over props introduced in [29]. Indeed, such deformation complexes possess a filtered $L_\infty$-algebra structure to which the arguments of the proof of Theorem 2.17 in [42] apply as well. \square

A crucial question in deformation theory is to know how to relate the $P_\infty$-algebra homotopy automorphisms of $X$ with the strict $P$-algebra automorphisms of its homology $H_s X$. For this, we define a map
\[ L^H wCh_k^{P_\infty}(X, X)_0 \rightarrow Aut_P(H_s X) \]
which sends any zigzag
\[ X \xrightarrow{f_1} \bullet \xrightarrow{f_2} \ldots \xrightarrow{f_n} X \]
on the composite \( H_* f_n \circ \ldots \circ H_* f_2 \circ (H_* f_1)^{-1} \) where every arrow going to the left is reversed by taking its inverse (since it is an isomorphism). We check that this map factors through homotopy classes of vertices. It is sufficient to verify it for zigzags of length two, since the argument in any length is the same. Let \( X \xrightarrow{f_0} g_0 \xrightarrow{g_1} X \) and \( X \xrightarrow{f_1} g_1 \xrightarrow{g_2} X \) be two such zigzags. Suppose there is a homotopy relating these two vertices. Applying the homology functor \( H_*(-) \) to this diagram, we obtain \( H_* f_0 = H_* f_1 \circ H_* h \) and \( H_* g_0 = H_* g_1 \circ h \), hence

\[
H_* g_0 \circ (H_* f_0)^{-1} = H_* g_1 \circ H_* h \circ (H_* h)^{-1} \circ (H_* f_1)^{-1} = H_* g_1 \circ (H_* f_1)^{-1}.
\]

This allows us to define a map at the level of connected components

\[
\overline{H}_*: \pi_0 L^H w Ch^P_{\infty}(X, X) = Aut_{Ho(C h^P_{\infty})}(X) \to Aut_{Ho(C h^P_{\infty})}(H_* X).
\]

Questions.
1. Is this map injective, surjective?
2. How to compute \( \pi_0 NR^P_{\infty} \)?

Answering these two questions gives a complete understanding of the equivalence classes of realizations \( \pi_0 Real_{P_{\infty}}(H_* X) \).

Remark 2.17. If the differential of \( X \) is zero, then this map is surjective. Indeed, if we consider \((X, \phi) \in Ch^P_{\infty}\), and \((X, \varphi) \in Ch^P_{\infty}\), then the following diagram commutes:

\[
\begin{array}{ccc}
\text{End}_{X_0} & \xrightarrow{\pi} & \text{End}_{X_1} \\
\downarrow s \circ \varphi & & \downarrow \varphi \\
\text{End}_{X_1} & \xrightarrow{\pi} & \text{End}_{X_1}
\end{array}
\]

where \( \pi \) is the canonical projection of the endomorphism prop of a diagram onto the endomorphism prop of a subdiagram, which in this case is an isomorphism with inverse \( s \).

There is no well defined obstruction theory for algebras over a general prop. It is thus difficult to get explicit obstruction groups for the connectedness of \( \pi_0 NR^P_{\infty} \), and to know if \( \overline{H}_* \) is injective. In the case of bialgebras defined by a pair of operads with a distributive law, there exists a model category structure [38] providing a background to develop a deformation theory of these bialgebras in a future work of the author. But this does not include certain important cases like Frobenius bialgebras and more general field theories. We can however compute the local realization space of the trivial \( P_{\infty} \)-algebra structure (which is already non trivial):
Lemma 2.18. Let $P$ be a prop, and $X$ a chain complex such that $H_*X$ forms a $P$-algebra. Let $\phi : P_\infty \to P \to \text{End}_{H_*X}$ be the trivial $P_\infty$-algebra structure on $H_*X$ induced by its $P$-algebra structure. We have

$$\pi_0 P_\infty \{ H_*X \}_\phi \cong \text{Aut}_K(H_*X)/\text{Aut}_P(H_*X).$$

Proof. We already know that $\pi_0 P_\infty \{ H_*X \}_\phi \cong \text{Aut}_K(H_*X)/\text{Aut}_{H_0(C_{\infty})}(H_*X)$. We also know that $\overline{H}_* : \text{Aut}_{H_0(C_{\infty})}(H_*X) \to \text{Aut}_P(H_*X)$ is surjective. It remains to prove that $\overline{H}_*$ is injective. Let $H_*X \xrightarrow{f_0} Z_0 \xrightarrow{g_0} H_*X$ and $H_*X \xrightarrow{f_1} Z_1 \xrightarrow{g_1} H_*X$ be two vertices of $L^H wCh^p_{\infty}(X,X)$ such that $H_*g_1 \circ (H_*f_1)^{-1} = H_*g_0 \circ (H_*f_0)^{-1}$ in $\text{Aut}_P(H_*X)$. We get the commutative diagrams

$$H_*X \xrightarrow{f_i} Z_i \xrightarrow{g_i} H_*X \cong H_*X \quad \sim \quad H_*X \xrightarrow{H_*f_i} H_*Z_i \xrightarrow{H_*g_i} H_*X$$

for $i = 0,1$ in $Ch_K$ where $p_{Z_i}$ is the projection, which forms here an acyclic fibration of chain complexes. The map $f_i$ is a morphism of $P_\infty$-algebras for $H_*X$ equipped with $\phi$ and $H_*f_i$ an isomorphism of $P$-algebras. Thus an isomorphism of $P_\infty$-algebras for $H_*X$ and $H_*Z_i$ equipped with the trivial $P_\infty$-algebra structures. If we denote by $[-]$ the homotopy class of a zigzag in $L^H wCh^p_{\infty}(X,X)$, we get

$$[H_*X \xrightarrow{f_0} Z_0 \xrightarrow{g_0} H_*X] = [H_*X \xrightarrow{H_*f_0} H_*Z_0 \xrightarrow{H_*g_0} H_*X] = [H_*X \xrightarrow{H_*f_1} H_*Z_1 \xrightarrow{H_*g_1} H_*X] = [H_*X \xrightarrow{f_1} Z_0 \xrightarrow{g_1} H_*X],$$

using that the equality $H_*g_1 \circ (H_*f_1)^{-1} = H_*g_0 \circ (H_*f_0)^{-1}$ implies the existence of a commutative diagram

$$\begin{array}{ccc}
H_*X & \xrightarrow{H_*f_0} & H_*Z_0 \\
\downarrow & & \downarrow \\
H_*Z_1 & \xrightarrow{H_*g_1} & H_*X \\
\uparrow & & \uparrow \\
H_*X & \xrightarrow{H_*g_0} & H_*X \\
\end{array}$$

where the middle vertical map is defined by $(H_*f_1)^{-1} \circ H_*f_0 = (H_*g_1)^{-1} \circ H_*g_0$. □

3. The operadic case

Operads are used to parameterize various kind of algebraic structures. Fundamental examples of operads include the operad $A$ encoding associative algebras, the operad $Com$ of commutative algebras, the operad $Lie$ of Lie algebras and the operad $\text{Pois}$ of Poisson algebras. There exists several equivalent approaches for the definition of an algebra over an operad. We will use the following one which we recall for convenience:
**Definition 3.1.** Let \((P, \gamma, \iota)\) be a dg operad, where \(\gamma\) is the composition product and \(\iota\) the unit. A \(P\)-algebra is an object \(A\) of a symmetric monoidal category \(\mathcal{E}\) over \(Ch_K\) endowed with a morphism \(\gamma_A : P(A) \to A\) such that the following diagrams commute:

\[
\begin{array}{ccc}
(P \circ P)(A) & \xrightarrow{\gamma_A} & P(A) \\
\downarrow \gamma(A) & & \downarrow \gamma_A \\
P(A) & \xrightarrow{\gamma_A} & A \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\iota(A)} & P(A) \\
\downarrow \gamma_A & & \downarrow \gamma_A \\
A & = & A \\
\end{array}
\]

We will denote by \(\mathcal{E}^P\) the category of \(P\)-algebras in \(\mathcal{E}\).

For every object \(V\), we can equip \(P(V)\) with a \(P\)-algebra structure by setting \(\gamma_{P(V)} = \gamma(V) : P(P(V)) \to P(V)\). As a consequence of the definition, we thus get the free \(P\)-algebra functor:

**Proposition 3.2.** (see [26], Proposition 5.2.6) The \(P\)-algebra \((P(V), \gamma(V))\) equipped with the map \(\iota(V) : I(V) = V \to P(V)\) is the free \(P\)-algebra on \(V\).

\(P\)-algebras satisfy good homotopical properties:

**Theorem 3.3.** (see [12]) Suppose that \(\mathcal{E}\) is a cofibrantly generated symmetric monoidal model category over \(Ch_K\). The category of \(P\)-algebras in \(\mathcal{E}\) inherits a cofibrantly generated semi-model category structure such that a morphism \(f\) of \(\mathcal{E}^P\) is

(i) a weak equivalence if \(U(f)\) is a weak equivalence in \(\mathcal{E}\), where \(U\) is the forgetful functor;

(ii) a fibration if \(U(f)\) is a fibration in \(\mathcal{E}\);

(iii) a cofibration if it has the left lifting property with respect to acyclic fibrations.

We can also say that cofibrations are relative cell complexes with respect to the generating cofibrations, where the generating cofibrations and generating acyclic cofibrations are, as expected, the images of the generating (acyclic) cofibrations of \(\mathcal{E}\) under the free \(P\)-algebra functor \(P\).

The model category structure allows one to express the internal symmetries of the objects as homotopy automorphisms. In a model category \(M\), the homotopy automorphisms of an object \(X\) is the simplicial sub-monoid \(haut(X^{cf}) \subset Map(X^{cf}, X^{cf})\) of invertible connected components, where \(X^{cf}\) is a cofibrant-fibrant resolution of \(X\), i.e

\[
haut(X^{cf}) = \coprod_{\phi \in [X,X]_{Ho(M)}} Map(X^{cf}, X^{cf})_{\phi},
\]

where the \(\phi \in [X,X]_{Ho(M)}\) are the automorphisms in the homotopy category of \(M\) and \(Map(X^{cf}, X^{cf})_{\phi}\) the connected component of \(\phi\) in the standard homotopy mapping space.
In this setting, we define a map
\[ \mathcal{H} : \pi_0 \text{haut}_{P_*}(H_\infty) \to \text{Aut}_P(H_*H_\infty) = \text{Aut}_P(H_*X) \]
where \( X_\infty \to X \) is a cofibrant resolution of \( X \) in the category of \( P_\infty \)-algebras (recall that all algebras are fibrant in chain complexes over a field). According to [20] (Theorems 2.7 and 3.5), we know that obstructions to be injective or surjective lie in the \( \Gamma \)-cohomology groups of \( H_*X \) as a \( P \)-algebra:
1. if \( H\Gamma^0_p(H_*X, H_*X) = 0 \), then \( \mathcal{H} \) is injective;
2. if \( H\Gamma^1_p(H_*X, H_*X) = 0 \), then \( \mathcal{H} \) is surjective and \( \pi_0 \text{NR}^{P_\infty}(H_*X) = * \), in particular \( \pi_0 \text{NR}^{P_\infty} = * \).

Remark 3.4. If \( P \) is a \( \Sigma \)-cofibrant operad, \( A \) a \( P \)-algebra and \( M \) a \( U_P(A) \)-module (that is, a module over the enveloping algebra of \( A \)), then \( H\Gamma^0_p(A, M) = H\Gamma^0_p(A, M) \) is the usual operadic cohomology. When \( K \) is a field of characteristic zero, all operads are \( \Sigma \)-cofibrant.

Theorem 3.5. Let \( P \) be a \( \Sigma \)-cofibrant graded operad with a trivial differential. Let \( H_*X \) be a \( P \)-algebra such that \( H\Gamma^0_p(H_*X, H_*X) = H\Gamma^0_p(H_*X, H_*X) = 0 \). Then
\[ \pi_0 \text{Real}_{P_*}(H_*X) \cong \text{Aut}_K(H_*X)/\text{Aut}_P(H_*X) \]
\[ \pi_1(\text{Real}_{P_*}(H_*X), \phi) \cong \{ \text{id}_{(H_*X, \phi)} \} \]
(the realizations of the identity are all homotopic according to [20]), and for \( n \geq 2 \) we have
\[ \pi_n(\text{Real}_{P_*}(H_*X), \phi) \cong \pi_n(\text{NR}^{P_\infty}, (H_*X, \phi)) \cong \pi_{n-1}(\text{haut}_{P_*}(H_\infty), \text{id}) \cong H\Gamma^1_p(H_*X, H_*X). \]
where \( H_\infty \) is a cofibrant resolution of \( H_*X \) in \( P_\infty \)-algebras.

Note that the homotopy groups of \( P_\infty \{H_*X\} \), hence of \( P_\infty \{X\} \) by Proposition 2.12, are the same for \( n \geq 1 \).

To conclude this section, we would like to emphasize a nice interpretation of the realization space \( \text{Real}_{P_*}(H_*X) \). According to Theorem 5.2.1 of [10], left homotopies between two morphisms \( \phi_0, \phi_1 : P_\infty \to \text{End}_X \) corresponds to \( \infty \)-isotopies between the two \( P_\infty \)-algebras \( (X, \phi_0) \) and \( (X, \phi_1) \), that is, \( \infty \)-morphisms of \( P_\infty \)-algebras which reduce to the identity on \( X \). Since the source of the morphisms \( \phi_0 \) and \( \phi_1 \) is cofibrant and their target is fibrant, left homotopies between such maps are in bijection with right homotopies. The right homotopies are, in turn, the 1-simplices of our realization space \( \text{Real}_{P_*}(H_*X) \). We conclude:

Proposition 3.6. The Kan complex \( \text{Real}_{P_*}(H_*X) \) parameterizes \( P_\infty \)-algebras with underlying complex \( X \) realizing the \( P \)-algebra structure of \( H_*X \) and their \( \infty \)-isotopies.

4. Extending algebraic structures: relative realization spaces

Let \( O \to P \) be a morphism of props, and suppose that \( H_*X \) is equipped with a \( P \)-algebra structure \( P \to \text{End}_{H_*X} \). It induces an \( O \)-algebra structure \( O \to P \to \text{End}_{H_*X} \). We fix a \( O_\infty \)-algebra structure on \( X \) which realizes this \( O \)-algebra structure on \( H_*X \) (we suppose that such a structure exists). Then a natural question is to determine the realizations \( P_\infty \to \text{End}_X \) of the \( P \)-algebra structure
of \( H_\ast X \) which extend this \( O_\infty \)-algebra structure on \( X \). Interesting examples are realizations of the Poincaré duality on the chains of a manifold which extend the \( E_\infty \)-algebra structure defined by the higher cup products (which defines the Wu classes, representing the Steenrod squares on the homology).

The existence of a map \( O \to P \) implies the existence of a map \( O_\infty \to P_\infty \). When such a map forms a cofibration, Theorem A of [12] holds in the relative case:

**Theorem 4.1.** Let \( \mathcal{C} \) be a cofibrantly generated symmetric monoidal model category satisfying the monoid limit axioms (see section 6 of [12]). Let \( i : O_\infty \to P_\infty \) be a cofibration of cofibrant props in \( \mathcal{C} \). If \( (X, \varphi_X^Y) \sim (Y, \varphi_Y^X) \) is a weak equivalence of \( O_\infty \)-algebras such that \( X, Y \in \mathcal{C}^\{\text{fibrant-cofibrant}\} \) and \( Y \) is equipped with a \( P_\infty \)-algebra structure \( \varphi_Y^X \) satisfying \( \varphi_Y^X \circ i = \varphi_Y^X \), then there exists a zigzag of two opposite acyclic fibrations of \( P_\infty \)-algebras

\[
(X, \varphi_X^Y) \sim \bullet \sim (Y, \varphi_Y^X)
\]

such that \( \varphi_X^Y \circ i = \varphi_Y^X \).

**Proof.** We just have to verify that the proof of [12] can be transposed in the comma category \( (P \downarrow O_\infty) \), where \( P \) is the category of props equipped with the semi-model category structure defined in [12]. Comma categories inherit the cofibrantly generated model category structure (see [19]), and this result extends readily to the case of semi-model categories. Hence \( (P \downarrow O_\infty) \) forms a cofibrantly generated semi-model category. Now, let \( P_\infty \in (P \downarrow O_\infty) \) such that \( i : O_\infty \to P_\infty \) is a cofibration, then by definition of cofibrations and initial morphisms in comma categories, the prop \( P_\infty \) is cofibrant in \( (P \downarrow O_\infty) \). The existence of \( \varphi_X^Y \) and \( \varphi_Y^X \) ensures that \( \text{End}_X, \text{End}_Y \in (P \downarrow O_\infty) \), and the condition \( \varphi_Y^X \circ i = \varphi_Y^X \) ensures that \( \varphi_Y^X \) is a morphism of \( (P \downarrow O_\infty) \). Now all the necessary conditions are satisfied and one applies exactly the same arguments as in [12]. \( \square \)

Now let \( i : O_\infty \to P_\infty \), be any fixed prop morphism, we do not suppose that \( i \) is a cofibration anymore. The map \( i \) admits a factorization \( O_\infty \xrightarrow{i} \tilde{P}_\infty \xrightarrow{p} P_\infty \) into a cofibration \( \tilde{i} \) followed by an acyclic fibration \( p \). The pushout of \( i \) along the final morphism of \( O_\infty \) gives the cofiber of \( \tilde{i} \), which is a model for the homotopy cofiber of \( i \). Let us note \( (P, O)_\infty \) the homotopy cofiber of \( i \), which represents the operations of \( P_\infty \) which are not in \( O_\infty \). We know that \( i \) induces a map of simplicial sets \( i^* : P_\infty \{X\} \to O_\infty \{X\} \) which factors through \( P_\infty \{X\} \xrightarrow{p} P_\infty \{X\} \xrightarrow{\tilde{i}^*} O_\infty \{X\} \).

One has the chain of homotopy equivalences

\[
hofib(i^*) \sim fib(\tilde{i}^*) \sim cofib(\tilde{i})\{X\} \sim hofib(\tilde{i})\{X\} = (P, O)_\infty \{X\}
\]

relating the homotopy fiber of \( i^* \) with the moduli space of \( (P, O)_\infty \)-algebra structures on \( X \), which encodes all the extensions of \( O_\infty \)-algebra structures on \( X \) into \( P_\infty \)-algebra structures. Let us also note that, according to [39], the map \( p \) induces a weak equivalence

\[
Np^* : \text{NwCh}_{\mathbb{K}}^{P_\infty} \to \text{NwCh}_{\mathbb{K}}^{\tilde{P}_\infty}.
\]

The main purpose of this section is to prove the following results, which are adaptations to the relative case of the results obtained in the previous sections:
Theorem 4.2. Let $X$ be a dg $O_\infty$-algebra. Then the commutative square

$$
\begin{array}{ccc}
P_\infty\{X\} & \longrightarrow & NwCh^P_{\infty} \\
\downarrow & & \downarrow N_i^* \\
\{X\} & \longrightarrow & NwCh^O_{\infty}
\end{array}
$$

is a homotopy pullback of simplicial sets.

Theorem 4.3. Let us suppose that $X$ is a graded $O_\infty$-algebra with a trivial differential (e.g. $X$ is the homology of some chain complex). There exists a commutative square

$$
\begin{array}{ccc}
WL^HwCh^O_{\infty}(X,X) \times L^HwCh^O_{\infty}(P,O) & \longrightarrow & diagNwCh^P_{\infty}\otimes\Delta^*|_X \\
\downarrow \pi & & \downarrow diagN(i\otimes\Delta^*)^* \\
WL^HwCh^O_{\infty}(X,X) & \sim & NwCh^O_{\infty}|_X
\end{array}
$$

where $\pi$ is a Kan fibration obtained by the simplicial Borel construction and the horizontal maps are weak equivalences of simplicial sets.

Corollary 4.4. There is a bijection

$$
\pi_0(P,O)_\infty\{X\}_{[\phi]} \cong Aut_{Ho(Ch^O_{\infty})}(X,\phi \circ j \circ i)/Aut_{Ho(Ch^P_{\infty})}(X,\phi \circ j)
$$

Here

$$
Aut_{Ho(Ch^O_{\infty})}(X,\phi \circ j)
$$

is seen as the subgroup of automorphisms of

$$
Aut_{Ho(Ch^P_{\infty})}(X,\phi \circ j \circ i)
$$

which realizes as automorphisms of $X$ in the category of $P_\infty$-algebras. The higher homotopy groups are more difficult to compute in the relative case, since a priori the $\pi_nWL^HwCh^O_{\infty}(X)$ are non zero for $n \geq 2$. However, Corollary 2.18 of [42] provides a cohomological interpretation of the long exact sequence.

Proof of Theorem 4.2. We have a commutative diagram

$$
\begin{array}{ccc}
holim & \longrightarrow & diagNwCh^P_{\infty}\otimes\Delta^* \\
\downarrow & & \downarrow \sim \\
pt & \longrightarrow & diagNwCh^O_{\infty}\otimes\Delta^*
\end{array}
$$

We want to prove that $holim$ (the homotopy pullback of the left hand commutative square) has the homotopy type of $(P,O)_\infty\{X\}$. For this, we will prove that $holim$ has the homotopy of $fib(i^*)$, the fiber of $i^*$. We follow arguments similar to those of Rezk in [35].

We recall the variant of Quillen’s Theorem B used in [35]. Let $M^\bullet$ be a simplicial category, $N$ a category and $\pi : M^\bullet \rightarrow csN$ a simplicial functor where $csN$ is the
constant simplicial category over $N$. Then the commutative square

$$\text{diag}N(\pi \downarrow X) \longrightarrow \text{diag}NM^{ullet}$$

$$\downarrow \downarrow \downarrow \downarrow$$

$$\text{diag}N(N \downarrow X) \longrightarrow \text{NN}$$

is a homotopy pullback if each $\text{diag}N(\pi \downarrow X) \xrightarrow{\sim} \text{diag}N(\pi \downarrow X')$ induced by a morphism $X \to X'$ of $N$ is a weak equivalence. Here we consider the functor

$$\pi : fwCh_{K}P_{\infty}^\bullet \otimes \Delta^\bullet \to cs_{\bullet}fwCh_{K}P_{\infty}^\bullet$$

induced by

$$O_{\infty} \times P_{\infty} \cong P_{\infty} \otimes \Delta^0 \hookrightarrow P_{\infty} \otimes \Delta^\bullet.$$

Recall that we consider a cosimplicial frame $(-) \otimes \Delta^\bullet$ on $P_{\infty}$, which is sufficient to get a cosimplicial resolution of $P_{\infty}$ since $P_{\infty}$ is cofibrant (see Chapter 16 of [18]). We note $N(\pi \downarrow X)_{s, \bullet}$, in order to distinguish between the two simplicial degrees, the first $\bullet$ being the simplicial dimension of the nerve and the second $\bullet$ being the simplicial dimension of the simplicial category $(\pi \downarrow X)$. The key point is to establish, for every $s$, the homotopy equivalence

$$N(\pi \downarrow X)_{s, \bullet} \sim \prod_{Y_{s} \rightarrow \cdots \rightarrow Y_{0} \in \text{fwCh}_{K}O_{\infty}} (P, O)_{\infty}\{Y_{s} \rightarrow \cdots \rightarrow Y_{0}\}.$$

Once this homotopy equivalence holds, the remaining part of the proof is exactly the argument of the proof of [35]. We have

$$N(\pi \downarrow X)_{s, \bullet} = \{\pi(Y_{s}) \rightarrow \cdots \rightarrow \pi(Y_{0}) \rightarrow X\}$$

$$= \{\pi(Y_{s} \rightarrow \cdots \rightarrow Y_{0}) \rightarrow X \in \text{fwCh}_{K}O_{\infty}, Y_{s} \rightarrow \cdots \rightarrow Y_{0} \in (NfwCh_{K}P_{\infty}^\bullet \otimes \Delta^\bullet)_{s, \bullet}\}$$

$$= \prod_{Y_{s} \rightarrow \cdots \rightarrow Y_{0} \in \text{fwCh}_{K}O_{\infty}} \text{fib}(P_{\infty}\{Y_{s} \rightarrow \cdots \rightarrow Y_{0}\} \to O_{\infty}\{Y_{s} \rightarrow \cdots \rightarrow Y_{0}\})$$

$$\sim \prod_{Y_{s} \rightarrow \cdots \rightarrow Y_{0} \in \text{fwCh}_{K}O_{\infty}} (P, O)_{\infty}\{Y_{s} \rightarrow \cdots \rightarrow Y_{0}\}$$

where, in the third line, the basepoint of each $O_{\infty}\{Y_{s} \rightarrow \cdots \rightarrow Y_{0}\}$ is given by the $Y_{s} \rightarrow \cdots \rightarrow Y_{0} \in \text{fwCh}_{K}P_{\infty}^\bullet$ indexing the coproduct. \square

The homotopy fiber

$$\text{diag}NfwCh_{K}P_{\infty}^\bullet \otimes \Delta^\bullet$$

$$\downarrow$$

$$\{X\} \longrightarrow \text{diag}NfwCh_{K}O_{\infty}^\bullet \otimes \Delta^\bullet \sim \text{NfwCh}_{K}O_{\infty}^\bullet$$
restricts to a homotopy fiber
\[ (P, O)_{\infty} \{ X \}_{[\phi]} \rightarrow \text{diag} Nf \text{wCh}_{K}^{P \otimes \Delta^*}|_{X}, \]
\[ \{ X, \phi \circ j \circ i \} \rightarrow N \text{wCh}_{K}^{O_{\infty}}|_{(X, \phi \circ j \circ i)} \]
where \( N \text{wCh}_{K}^{O_{\infty}}|_{(X, \phi \circ j \circ i)} \) is the connected component of the \( O_{\infty} \)-algebra \((X, \phi \circ j \circ i)\), and
\[ \text{diag} Nf \text{wCh}_{K}^{P \otimes \Delta^*}|_{X} = \prod_{[X, \varphi]} \text{diag} Nf \text{wCh}_{K}^{P \otimes \Delta^*}|_{(X, \varphi)} \]
is the union of the connected components \( \text{diag} Nf \text{wCh}_{K}^{P \otimes \Delta^*}|_{(X, \varphi)} \) ranging over the acyclic fibration classes of \( P_{\infty} \)-algebras having \( X \) as underlying complex, and such that there exists a zigzag \((X, \varphi \circ i) \leftarrow \bullet \rightarrow (X, \phi \circ j \circ i)\) of weak equivalences of \( O_{\infty} \)-algebras. According to Lemma 2.7, one gets actually the homotopy fiber
\[ (P, O)_{\infty} \{ X \}_{[\phi]} \rightarrow \text{diag} Nf \text{wCh}_{K}^{P \otimes \Delta^*}|_{X}. \]
\[ \{ X, \phi \circ j \circ i \} \rightarrow N \text{wCh}_{K}^{O_{\infty}}|_{(X, \phi \circ j \circ i)} \]
Now let us prove Theorem 4.3:

**Proof of Theorem 4.3.** Recall that the hammocks of weight \( k \) and length \( n \), which define the \( k \)-simplices of the simplicial monoid \( L^{H} \text{wCh}_{K}^{P_{\infty}}(X, X) \), consist in the data of \( k-1 \) chains of morphisms of length \( n \)
\[ X \xrightarrow{f_{0}} \cdots \xrightarrow{f_{n}} X, \]
\[ \vdots, \]
\[ X \xrightarrow{f_{0}} \cdots \xrightarrow{f_{n}} X \]
organized in a commutative diagram (a “hammock")

We define a simplicial action of the simplicial monoid \( L^{H} \text{wCh}_{K}^{P_{\infty}}(X, X) \) on the moduli space \((P, O)_{\infty} \{ X \})\) in each simplicial dimension by a map
\[ L^{H} \text{wCh}_{K}^{P_{\infty}}(X, X)_{k} \times (P, O)_{\infty} \{ X \}_{k} \rightarrow (P, O)_{\infty} \{ X \}_{k} \]
which associates to any pair \( \{(X \xrightarrow{f_{0}} \cdots \xrightarrow{f_{n}} X)\}_{1 \leq i \leq n}, \varphi : (P, O)_{\infty} \otimes \Delta^{k} \rightarrow End_{X} \) the map \( \varphi_{\text{Tr}}(f_{0}^{1}, \ldots, f_{n}^{0}) \) defined by the conjugation action of \( \text{Tr}(f_{0}^{1}, \ldots, f_{n}^{0}) \) on \( \phi \) like
in Lemma 2.10, with \( \Pi_{\lambda}(f^1_0, \ldots, f^n_0) \) the composite of the homologies of each \( f^i_0 \) going in the right direction and the inverse of the homologies of each \( f^i_0 \) going in the left direction. We have already seen that such a map gives a well defined monoid action. Moreover, this conjugation action is compatible with the faces and degeneracies of \( (P, O, \pi) \) and the square in the proof of Theorem 2.8. The only modification is to apply the five lemma.

We thus obtain a simplicial action of \( L^H wCh^P_\infty,(X, X) \) on \( (P, O, \pi)_\infty \{X\} \). It remains to prove that this action restrict to \( (P, O, \pi)_\infty \{X\}[\phi] \). By the inductive construction of the simplices of the local realization space, it is sufficient to prove it for the vertices. We apply exactly the same argument as in the proof of Lemma 2.10.

We finally get a Kan fibration

\[
\pi : W L^H wCh^P_\infty,(X, X) \times L^H wCh^P_\infty,(X, \phi \circ j \circ i) \rightarrow W L^H wCh^P_\infty,(X, \phi \circ j \circ i)
\]

with fiber \( (P, O, \pi)_\infty \{X\}[\phi] \). We then construct a commutative square analogous to the square in the proof of Theorem 2.8. The only modification is to apply \( \Pi_{\lambda}(\cdot) \) in the definition of the upper horizontal arrow before transferring the \( P_\infty \otimes \Delta^k \)-algebra structure along the chain of isomorphisms. The lower horizontal arrow is a weak equivalence according to the work of Dwyer-Kan [8], and for the upper one we use the five lemma.

The proof of corollary 4.4 is the same as in the non relative case:

**Proof.** We use the long exact sequence associated to \( \pi \) to get a bijection

\[
\partial_1 : \pi_1 W L^H wCh^P_\infty,(X, \phi \circ j \circ i)/\operatorname{Im}(\pi_1(\pi)) \xrightarrow{\simeq} \pi_0 (P, O)_\infty \{X\}[\phi].
\]

The isomorphisms

\[
\pi_1 W L^H wCh^P_\infty,(X) \cong \pi_0 L^H wCh^P_\infty,(X) = Aut_{Ho(Ch_\infty^P)}(X)
\]

and

\[
\pi_1(W L^H wCh^P_\infty,(X, \phi \circ j \circ i)) \times L^H wCh^P_\infty,(X, \phi \circ j \circ i) \rightarrow (P, O)_\infty \{X\}[\phi], (id, \phi))
\]


\[
\cong \pi_1 W L^H wCh^P_\infty,(X)
\]

and

\[
= Aut_{Ho(Ch_\infty^P)}(X)
\]

(which follows from the equivalence of fiber sequences) allow us to conclude.

**Example 4.5.** When \( Aut_{Ho(Ch_\infty^P)}(X) \cong Aut_O(X) \) (e.g for the trivial \( O_\infty \)-algebra structure) and \( Aut_P(X) = Aut_O(X) \), then \( \pi_0 (P, O)_\infty \{X\}[\phi] = * \). We will see an important example of this condition with the realization of Poincaré duality up to a fixed rational homotopy type.

In the operadic case, we can compare by applying the obstructions criteria of [20]. A typical situation is when \( O \) is an operad which embeds into a larger algebraic structure encoded by a prop \( P \). When \( HT^O_0(X, X) = 0 \) and \( Aut_O(X) = Aut_P(X) \), then \( \pi_0 (P, O)_\infty \{X\}[\phi] = * \).
5. Realization of Poincaré Duality on chains

Poincaré duality can be realized at the level of chains in a Frobenius algebra up to homotopy, i.e. a homotopy Frobenius algebra, which is expected to contain in its higher operations some information of geometrical nature when one considers an oriented compact manifold. However, no one knows how many structures (up to homotopy) exist on the chains, e.g. if there are non equivalent realizations of Poincaré duality. In particular, how many structures extend the usual $E_\infty$ structure on chains which fixes the rational homotopy type. We provide here concrete computations of connected components of realizations spaces for a broad range of examples of formal manifolds (in the sense of Sullivan).

5.1. Frobenius algebras and their variants. We recall some definitions of dg Frobenius algebras and bialgebras, and point out how one can pass from algebras to bialgebras. We are particularly interested in special symmetric Frobenius algebras, which naturally appear on the cohomology of Poincaré duality spaces, and play also a key role in the construction of the algebraic counterpart of conformal field theories (in the appropriate tensor category, see [14]).

**Definition 5.1.** A (differential graded) Frobenius algebra is a unitary dg commutative associative algebra of finite dimension $A$ endowed with a symmetric non-degenerated bilinear form $\langle \cdot, \cdot \rangle : A \otimes A \rightarrow K$ which is invariant with respect to the product, i.e. $\langle xy, z \rangle = \langle x, yz \rangle$.

A topological instance of such kind of algebra is the cohomology ring (over a field) of a Poincaré duality space. There is also a notion of Frobenius bialgebra:

**Definition 5.2.** A differential graded Frobenius bialgebra of degree $m$ is a triple $(B, \mu, \Delta)$ such that:
- (i) $(B, \mu)$ is a dg commutative associative algebra;
- (ii) $(B, \Delta)$ is a dg cocommutative coassociative coalgebra with $\deg(\Delta) = m$;
- (iii) the map $\Delta : B \rightarrow B \otimes B$ is a morphism of left $B$-module and right $B$-module, i.e in Sweedler’s notations we get the equalities

$$
\sum_{(x, y)} (x \cdot y)_{(1)} \otimes (x \cdot y)_{(2)} = \sum_{(y)} x \cdot y_{(1)} \otimes y_{(2)}
$$

$$
= \sum_{(x)} (-1)^{|x|} x_{(1)} \otimes x_{(2)} \cdot y
$$

called the Frobenius relations.

Now we give a presentation of the properad parameterizing such structures in terms of directed graphs with a flow going from the top to the bottom. The properad $\text{Frob}^m$ of Frobenius bialgebras of degree $m$ is generated by an operation of arity $(2, 1)$ and of degree 0

$$
\sum
$$

and an operation of arity $(1, 2)$ and of degree $m$

$$
\sum
$$

which are invariant under the action of $\Sigma_2$. It is quotiented by the ideal generated by the following relations:
**Associativity and coassociativity**

\[ \begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array} \\
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\end{array} \]

and

\[ \begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array} \\
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\end{array} \]

**Frobenius relations**

\[ \begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array} \\
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\end{array} \]

and

\[ \begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array} \\
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\end{array} \]

In the unitary and counitary case, one adds a generator for the unit, a generator for the counit and the necessary compatibility relations with the product and the coproduct. We note the corresponding properad $ucFrob^m$. We refer the reader to [24] for a detailed survey about the role of these operations and relations in the classification of two-dimensional topological quantum field theories.

Definitions 5.1 and 5.2 are strongly related. Indeed, if $A$ is a Frobenius algebra, then the pairing $\langle \ , \ \rangle$ induces an isomorphism of $A$-modules $A \cong A^*$, hence a map

\[ \Delta : A \xrightarrow{\cong} A^* \xrightarrow{\mu^*} (A \otimes A)^* \cong A^* \otimes A^* \cong A \otimes A \]

which equips $A$ with a structure of Frobenius bialgebra. Conversely, one can prove that every unitary counitary Frobenius bialgebra gives rise to a Frobenius algebra, which are finally two equivalent notions. Let $B$ be a unitary counitary Frobenius bialgebra, with a product $\cdot$, a coproduct $\Delta$, a unit $\Delta$ and a counit $\Delta$. The composite $\cdot \cdot$ is a bilinear form, which is symmetric by commutativity of the product. It is invariant with respect to the product by associativity:

\[ \begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array} = \begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array} = \begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\end{array}
\end{array} \]

Let us draw the Frobenius relation:

\[ \begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array} = \begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array} = \begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\end{array}
\end{array} \]

The composite $\cdot$ defines a copairing, which is actually the dual copairing of $\cdot$ according to Frobenius relation. In particular, this bilinear form is non-degenerate and we finally obtain the desired Frobenius algebra structure. We refer the reader to [24] for a detailed survey about the role of these operations and relations in the classification of two-dimensional topological quantum field theories.
Remark 5.3. Another way to encode Frobenius algebras is to define them as algebras over the commutative cyclic operad. However, there is no known model category structure on dg cyclic operads (and explicit constructions of resolutions), so the equivalent properadic notion of unitary counitary Frobenius bialgebra is preferable for our purposes.

Finally let us define a particular sort of Frobenius algebras:

Definition 5.4. A special Frobenius algebra is a unitary and counitary Frobenius bialgebra \((B, \mu, \eta, \epsilon)\), where \(\eta\) denotes the unit and \(\epsilon\) the counit, such that \(\epsilon \circ \eta = \lambda_0 \cdot \text{id}_K\) and \(\mu \circ \Delta = \lambda_1 \cdot \text{id}_B\) for fixed \(\lambda_0, \lambda_1 \in K\). In terms of graphs it gives

\[
\begin{align*}
\bullet & = \lambda_0 \\
\bigtriangleup & = \lambda_1
\end{align*}
\]

and

\[
\begin{align*}
\bigtriangledown & = \lambda_0 \\
\bigtriangleup & = \lambda_1
\end{align*}
\]

It is said to be normalized when \(\lambda_0 = \text{dim}(B)\) and \(\lambda_1 = 1\).

We denote by \(sFrob\) the properad encoding special Frobenius bialgebras.

Example 5.5. Let \(M\) be an oriented connected closed manifold of dimension \(n\). Let \([M] \in H_n(M; \mathbb{K}) \cong H^0(M; \mathbb{K}) \cong \mathbb{K}\) be the fundamental class of \([M]\). Then the cohomology ring \(H^*(M; \mathbb{K})\) of \(M\) inherits a structure of commutative and cocommutative Frobenius bialgebra of degree \(n\) with the following data:

(i) the product is the cup product

\[
\mu : H^kM \otimes H^lM \rightarrow H^{k+l}M
\]

\[x \otimes y \mapsto x \cup y\]

(ii) the unit \(\eta : \mathbb{K} \rightarrow H^0M \cong H_nM\) sends \(1\) on the fundamental class \([M]\);

(iii) the non-degenerate pairing is given by the Poincaré duality:

\[
\beta : H^kM \otimes H^{n-k}M \rightarrow \mathbb{K}
\]

\[x \otimes y \mapsto <x \cup y, [M]>\]

i.e the evaluation of the cup product on the fundamental class;

(iv) the coproduct \(\Delta = (\mu \otimes \text{id}) \circ (\text{id} \otimes \gamma)\) where

\[
\gamma : \mathbb{K} \rightarrow \bigoplus_{k+l=n} H^kM \otimes H^lM
\]

is the dual copairing of \(\beta\), which exists since \(\beta\) is non-degenerate;

(v) the counit \(\epsilon = \langle \text{ }, [M] \rangle : H^nM \rightarrow \mathbb{K}\) i.e the evaluation on the fundamental class.

The proposition below sums up classical properties of symmetric Frobenius bialgebras:

Proposition 5.6. (1) A Frobenius algebra is symmetric and finite dimensional.

(2) A counitary Frobenius bialgebra is a Frobenius algebra, in particular it is finite dimensional.

(3) A symmetric Frobenius bialgebra which is commutative is also cocommutative.
(4) For every element \( z \) of a symmetric Frobenius bialgebra \( A \), the element \( \sum_{(z)} z^{(2)} z^{(1)} \) lies in the center of \( A \).

5.2. Computations of connected components. Special symmetric Frobenius bialgebras satisfy a strong rigidity property given by the following result of [13]:

**Theorem 5.7.** (Theorem 3.6 in [13]) If an algebra \( A \) can be endowed with a special symmetric Frobenius bialgebra structure, then this structure is the unique structure extending the algebra structure, up to normalization of the counit.

This result is available in a wide range of tensor categories, in particular chain complexes over a field. Let us denote by \( s\text{Frob} \) the properad encoding special commutative and cocommutative Frobenius bialgebras (which are in particular symmetric).

**Corollary 5.8.** If there exists \( s\text{Frob} \to \text{End}_{H^*X} \) extending a given \( \text{Com} \to \text{End}_{H^*X} \), then
\[
(s\text{Frob},\text{Com})_{\infty} \{H^*X\} \sim (s\text{Frob},\text{Com})_{\infty} \{X\} = \text{Real}_{(s\text{Frob},\text{Com})_{\infty}} \{H^*X\}.
\]

These algebras are used in conformal field theory. An important example is provided by Poincaré duality:

**Lemma 5.9.** Let \( M \) be a compact oriented manifold. Then \( H^*(M;\mathbb{K}) \) is a special symmetric Frobenius bialgebra.

**Proof.** We know that \((\epsilon \circ \eta)(1_{\mathbb{K}}) = 1_{\mathbb{K}}\) by definition, since \( \epsilon \) sends 1 on the fundamental class \([M]\) of \( M \), and \( \eta \) sends the fundamental class of \( M \) on 1. Hence \( \epsilon \circ \eta = id_{\mathbb{K}} \). Let \( \beta : H^*M \otimes H^*M \to \mathbb{K} \) be the non degenerate bilinear form associated to Poincaré duality and \( \gamma \) be its dual copairing. We know by construction that
\[
\gamma \circ \beta = \epsilon \circ \mu \Delta \circ \eta = \chi(M).id_{\mathbb{K}},
\]
where \( \chi(M) \) is the Euler characteristic of \( M \). Since \( \epsilon \) sends \([M]\) on 1, we get
\[
\mu \Delta \circ \eta = (\chi(M).id_{\mathbb{K}}).[M],
\]
in particular \((\mu \Delta \circ \eta)(1) = \chi(M).[M]\) (using also the fact that \( \mu \Delta \circ \eta \) sends \( \mathbb{K} \) in \( H^0(M) \)). Now, using the fact that \( \Delta = (\mu \otimes id) \circ (id \otimes \gamma) \), we get
\[
\mu \circ \Delta = \mu \circ (\mu \otimes id) \circ (id \otimes \gamma) = \mu \circ (id \otimes (\mu \circ \gamma)) = \mu \circ (id \otimes (\chi(M).id_{\mathbb{K}}).[M]) = (\chi(M).[M]).Id_{H^*M} = \chi(M).Id_{H^*M}.
\]
The first line is by construction of \( \Delta \), the second line follows from the associativity of \( \mu \), and the last line because \([M]\) is the unit for the product. Moreover, since \( H^*M \) is commutative and cocommutative, it is in particular symmetric. \( \square \)

We will also need the following rigidity result about automorphisms of special Frobenius bialgebras:

**Lemma 5.10.** (see [14]) Let \( A \) be a special symmetric Frobenius bialgebra. A morphism in \( \text{End}(A) \) is an algebra automorphism if and only if it is a coalgebra automorphism.
Maps between Frobenius bialgebras which are simultaneously morphisms of algebras and morphisms of coalgebras are exactly the morphisms of $sFrob$-algebras, thus automorphisms of a symmetric special Frobenius bialgebra are exactly its automorphisms as a unitary commutative algebra. Now we use:

**Lemma 5.11.** Every algebra automorphism of a unitary algebra preserves the unit.

Proof. Let $\varphi : A \rightarrow A$ be such an automorphism. We have, for every $x \in A$, $\varphi(x) = \varphi(1_A \cdot x) = \varphi(1_A) \varphi(x)$. Since $\varphi$ is bijective, for every $x \in A$ there exists $t \in A$ such that $x = \varphi(t)$, so

$$\varphi(1_A)x = \varphi(1_A)\varphi(t) = \varphi(1_A)t = \varphi(t) = x,$$

in particular

$$\varphi(1_A) = \varphi(1_A)1_A = 1_A.$$ 

□

We deduce:

**Lemma 5.12.** For every special symmetric Frobenius bialgebra $A$, we have

$$\text{Aut}_{sFrob}(A) = \text{Aut}_{Com}(A) = \text{Aut}_{Com}(A).$$

Hence

**Proposition 5.13.** (1) We have

$$\pi_0 \chi_{sFrob, Com} (H_* X)_{[\phi]} \cong \text{Aut}_K (H_* X)/\text{Aut}_{Com}(H_* X)$$

where $\phi$ is the trivial $sFrob, Com$-structure $sFrob \rightarrow sFrob \rightarrow \text{End}_{H_* X}$.

(2) We have

$$\pi_0 (sFrob, Com)_\infty (H_* X)_{[\phi]} \cong 0$$

if $H_{i_*} (H_* X, H_* X, \phi \circ j \circ i) = 0$.

To put these observations into words, we proved the following: in general, we can compute the connected components of the local realization space of Poincaré duality on chains around the trivial homotopy structure as a quotient of its graded vector space automorphisms by its commutative algebra automorphisms. And when the zero gamma cohomology group of a certain Frobenius bialgebra structure $(H_* X, \phi)$ is zero, then the corresponding local realization space is connected. We have

$$\pi_0 \text{Real}_{(sFrob, Com)} (H_* X) = \pi_0 \text{NR}_{(sFrob, Com)} (H_* X),$$

that is, the properadic homotopies contains no more informations than the chains of quasi-isomorphisms. Moreover, by Theorem 4.7 the $sFrob$-structure on $H_* X$ extending a fixed commutative structure is unique up to normalization of the counit, so

$$\text{Real}_{(sFrob, Com)} (H_* X) = (sFrob, Com)_{\infty} (H_* X)$$

and

$$\text{NR}_{(sFrob, Com)} (H_* X) = N_{WeakCh} (sFrob, Com)_{\infty}.$$ 

Let us also note that since higher homotopy groups of realization spaces of homotopy Frobenius structures are given by deformation complexes of the form

$$\text{Der}(sFrob, \text{End}_{H_* X}),$$

they admit a non trivial action of the (prounipotent) Grothendieck-Teichmüller Lie algebra as explained in [3].

To conclude this section, we do computations for some examples of manifolds.
Example 5.14. The cohomology ring of the $n$-sphere $S^n$ is $H^*S^n = \mathbb{Q}[x]/(x^2)$. A basis is given by $1, x$, so

$Aut_{\mathbb{Q}}(H^*S^n) \cong \mathbb{Q}^* \oplus \mathbb{Q}^*$.

To determine an automorphism preserving the commutative structure, we just have to fix a non zero value on $x$, so

$Aut_{\mathbb{Q}}(H^*S^n) \cong \mathbb{Q}^*$.

We deduce that

$\pi_0 sFrob_{\infty}\{H_*X\}_{[\sigma]} \cong \mathbb{Q}^*$.

Example 5.15. The cohomology ring of the complex projective space $\mathbb{C}P^n$ is $H^*\mathbb{C}P^n = \mathbb{Q}[x]/(x^{n+1})$. A basis is given by $[\mathbb{C}P^n], x, x^2, ..., x^n$, so

$Aut_{\mathbb{Q}}(H^*\mathbb{C}P^n) \cong (\mathbb{Q}^*)^{\oplus n+1}$.

To determine an automorphism preserving the commutative structure, we just have to fix a non zero value on $x$ (the unit $[\mathbb{C}P^n]$ is preserved), so

$Aut_{\mathbb{C}}(H^*\mathbb{C}P^n) \cong \mathbb{Q}^*$.

We deduce that

$\pi_0 sFrob_{\infty}\{H_*X\}_{[\sigma]} \cong (\mathbb{Q}^*)^{\oplus n}$.

Example 5.16. Let $S$ be a compact oriented surface of genus $g$. The zero and second cohomology groups of $S$ are $\mathbb{Q}$, and the first cohomology group is generated by $2g$ generators. We thus have

$Aut_{\mathbb{Q}}(H^*S) \cong \mathbb{Q}^* \oplus GL_{2g}(\mathbb{Q}) \oplus \mathbb{Q}^*$.

To determine an automorphism preserving the commutative structure, we just have to fix a non zero value on each generator, so

$Aut_{\mathbb{C}}(H^*S) \cong GL_{2g}(\mathbb{Q}) \oplus \mathbb{Q}^*$.

We deduce that

$\pi_0 sFrob_{\infty}\{H_*S\}_{[\sigma]} \cong \mathbb{Q}^*$.

Example 5.17. One constructs the cohomology ring of a connected sum of two orientable $n$-manifolds $M \sharp N$ as follows. Using a Mayer-Vietoris exact sequence argument, one checks that the zeroth cohomology group of $M \sharp N$ is generated by the fundamental class $[M \sharp N]$, the $n^{th}$ group is generated by a top cohomology class and

$H^iM \sharp N = H^iM \oplus H^iN$

for $0 < i < n$. The cup product of two classes belonging respectively to $H^iM$ and $H^iN$ is zero. The cup product of two classes belonging respectively to $H^iM$ and $H^jM$ is their cup product in $H^{i+j}M$ for $i + j < n$, and is identified with the top cohomology class of $M \sharp N$ for $i + j = n$.

Let $M = \mathbb{C}P^2 \sharp \mathbb{C}P^2$ be the connected sum of two copies of $\mathbb{C}P^2$. This is a 4-dimensional orientable closed manifold. The zeroth and fourth cohomology groups are isomorphic to $\mathbb{Q}$. For $0 < i < 4$, we have

$H^iM \cong H^i(\mathbb{C}P^2) \oplus H^i(\mathbb{C}P^2)$,

hence $H^1M = H^3M = 0$ and $H^2M \cong \mathbb{Q}x_1 \oplus \mathbb{Q}x_2$. Concerning the ring structure, the classes $x_1^2$ and $x_2^2$ are identified with the top cohomology class generating $H^4M$, hence $x_1^2 = x_2^2$, and the cup product $x_1 \cup x_2$ is zero.
The automorphisms of $H^*M$ as a graded vector space are
\[ Aut_Q(H^*M) = GL_1(Q) \times GL_2(Q) \times GL_1(Q) = Q^* \times GL_2(Q) \times Q^*. \]

The automorphisms of $H^*M$ as a graded commutative algebra fix the unit so they will be of the form
\[ Aut_{Com}(H^*M) = H \times Q^* \]
where $H$ is a certain subgroup of $GL_2(Q)$ that we now explicit. Let $f$ be a commutative algebra automorphism of $H^*M$, it induces an automorphism on $H^2M$ that we still denote by $f$. This automorphism corresponds to a certain invertible matrix
\[
\begin{pmatrix}
  a_1 & b_1 \\
  a_2 & b_2
\end{pmatrix}
\]
where $f(x_1) = a_1x_1 + a_2x_2$ and $f(x_2) = b_1x_1 + b_2x_2$. The relation $x_1^2 = x_2^2$ implies that $f(x_1)^2 = f(x_2)^2$. We have
\[
\begin{align*}
f(x_1)^2 &= (a_1x_1 + a_2x_2)^2 \\
&= a_1^2x_1^2 + 2a_1a_2x_1x_2 + a_2^2x_2^2 \\
&= (a_1^2 + a_2^2)x_1^2,
\end{align*}
\]
and
\[
f(x_2)^2 = (b_1^2 + b_2^2)x_1^2
\]
so
\[
a_1^2 + a_2^2 = b_1^2 + b_2^2.
\]

That is, the two vectors of $Q^2$ defining the columns of matrix have the same norm. The relation $x_1 \cup x_2 = 0$ implies that $f(x_1) \cup f(x_2) = 0$, that is
\[
\begin{align*}
(a_1x_1 + a_2x_2) \cup (b_1x_1 + b_2x_2) &= a_1b_1x_1^2 + (a_1b_2 + a_2b_1)x_1x_2 + a_2b_2x_2^2 \\
&= (a_1b_1 + a_2b_2)x_1^2 \\
&= 0
\end{align*}
\]
hence
\[
a_1b_1 + a_2b_2 = 0.
\]

This means that the two column vectors of the matrix are orthogonal (their scalar product is zero). This matrix is consequently orthogonal up to normalization, and we get a semi-direct product
\[
H = Q^* \ltimes O_2(Q).
\]

If we consider the cohomology with real coefficients instead of the rational one, we get the conformal orthogonal group $CO_2(R)$ (the group of conformal linear maps on $R^2$). We still denote by $CO_2(Q)$ its rational version. We have
\[
\pi_0 sFrob_{\infty} \{ H_*(CP^2 \times CP^2) \} \cong GL_2(Q)/CO_2(Q) \times Q^*.
\]
When working over $R$, by the exact sequence
\[
1 \rightarrow SL_2(R) \rightarrow GL_2(R) \stackrel{det}{\rightarrow} R^* \rightarrow 1,
\]
the general linear group is a semi-direct product
\[
GL_2(R) = R^* \ltimes SL_2(R).
\]
Moreover, the special linear group admits the Iwasawa decomposition
\[ SL_2(\mathbb{R}) = K \times A \times N. \]
The quotient \( GL_2(\mathbb{R})/CO_2(\mathbb{R}) \) is then the direct product \( A \times N \).

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