Asymptotic analysis for a simple explicit estimator in Barndorff-Nielsen and Shephard stochastic volatility models*

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Abstract

We provide a simple explicit estimator for discretely observed Barndorff-Nielsen and Shephard models, prove rigorously consistency and asymptotic normality based on the single assumption that all moments of the stationary distribution of the variance process are finite, and give explicit expressions for the asymptotic covariance matrix.

We develop in detail the martingale estimating function approach for a bivariate model, that is not a diffusion, but admits jumps. We do not use ergodicity arguments.

We assume that both, logarithmic returns and instantaneous variance are observed on a discrete grid of fixed width, and the observation horizon tends to infinity. As the instantaneous variance is not observable in practice, our results cannot be applied immediately. Our purpose is to provide a theoretical analysis as a starting point and benchmark for further developments concerning optimal martingale estimating functions, and for theoretical and empirical investigations, that replace the variance process with a substitute, such as number or volume of trades or implied variance from option data.

KEYWORDS:
Martingale estimating functions, stochastic volatility models with jumps, consistency and asymptotic normality

1 Introduction

In [BNS01] Barndorff-Nielsen and Shephard introduced a class of stochastic volatility models in continuous time, where the instantaneous variance follows an Ornstein-Uhlenbeck type process driven by an increasing Lévy process. Those models allow flexible modelling, capture many stylized facts of financial time series, and yet are of great analytical tractability. For further information see also [BNNS02]. BNS-models, as we will call them from now on, are affine models in the sense of [DPS00] and [DFS03], where the associated Riccati type equations can be solved up to quadrature in general. In several concrete cases the integration can be performed explicitly in closed form in terms of elementary functions, see [NV03] and [Ven01].

BNS-models have been studied from various points of view in mathematical finance and related fields. In [NV03] option pricing and structure preserving martingale measures are studied.

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In [BK05, BMB05, BG05, RS06] the minimal entropy martingale measure is investigated. The papers [BKR03, Lin06] address the portfolio optimization problem. Bayesian/MCMC/computer intensive estimation is already in the seminal paper [BNS01], and in the works [RPD04, GS01, FSS01, tH03]. The papers [Jam05, Jam06] exploit the analytical tractability to develop maximum likelihood estimation using the results of [CM00, CR90] for Dirichlet processes. BNS models are also treated in the textbooks [CT04, Sch03].

Unfortunately, it seems that statistical estimation of the model is the most difficult problem, and most of the work in that area is focused on computationally intensive methods.

The contributions of the present paper are as follows: first we develop a simple and explicit estimator for BNS models. Secondly, we give rigorous proofs of its consistency and asymptotic normality. In doing so we compute explicitly the asymptotic covariance matrix and develop to that purpose formulas for arbitrary bivariate integer moments of returns and variance. Thirdly we provide a detailed application of the theory of martingale estimating functions in a non-diffusion setting, including numerical illustrations.

The literature on estimation for discretely observed diffusions is vast, a few references are [Uch04b, Uch04a, DS04, MR03, KP02, Jac02, Jac01, Sor01, BS01, Kes00, KS99a, Sor97, BS95]. In particular, the martingale estimating function approach is used, developed and studied for example in [Sø99], [Sø00], [Sø97]. In the diffusion setting the major difficulty is that the transition probabilities are not known and are difficult to compute. In contrast to that, the characteristic function of the transition probability is known in closed form for many BNS models and the transition probability can be computed with Fourier methods with high precision. Yet the model exhibits other peculiarities, see the remarks in section 2.3.

In the present paper we explore the joint distribution of logarithmic returns $X$ and the instantaneous variance $V$ supposing that both processes can be observed in discrete time. Since the joint conditional moment-generating function of $(X, V)$ is known in closed form we obtain closed form expressions for the joint conditional moments up to any desired order which yields a sequence of martingale differences. We employ then the large sample properties, in particular the strong law of large numbers for martingales and martingale central limit theorem. In this way we do not need ergodicity, mixing conditions, etc.

The remainder of the paper is organized as follows: in section 2.1 we describe the class of BNS models in continuous time and present two concrete examples, the $\Gamma$–OU and IG-OU model. In section 2.2 we introduce the quantities observed in discrete time that are used for estimation. Section 2.3 contains some remarks of particular features of the model and its estimation. In section 3 we present the estimating equations, their explicit solution which is our estimator and prove its consistency and asymptotic normality. In section 4 we present numerical illustrations. In section 5 we sketch further and alternative developments, in particular considering the issue that volatility is typically not observed in discrete time. Explicit moment calculations of any order can be found in Appendix A. In Appendix B we provide for the reader’s convenience a simple multivariate martingale central limit theorem.

\[1\] Let us mention though, that the martingale strong law and the ergodic theorem have similar proofs and can be derived from a common source, [Rao73].
2 The model

2.1 The continuous time model

2.1.1 The general setting

As in Barndorff-Nielsen and Shepard [BNS01], we assume that the price process of an asset \( S \) is defined on some filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) and is given by \( S_t = S_0 \exp(X_t) \) with \( S_0 > 0 \) a constant. The process of logarithmic returns \( X \) and the instantaneous variance process \( V \) satisfy

\[
dX(t) = (\mu + \beta V(t-))dt + \sqrt{V(t-)}dW_\theta(t) + \rho dZ_\lambda(t), \quad X(0) = 0.
\]

and

\[
dV(t) = -\lambda V(t-)dt + dZ_\lambda(t), \quad V(0) = V_0,
\]

where the parameters \( \mu, \beta, \rho \) and \( \lambda \) are real constants with \( \lambda > 0 \). The process \( W \) is a standard Brownian motion, the process \( Z \) is an increasing Lévy process, and we define \( Z_\lambda(t) = Z(\lambda t) \) for notational simplicity. Adopting the terminology introduced by Barndorff-Nielsen and Shepard, we will refer to \( Z \) as the background driving Lévy process (BDLP). The Brownian motion \( W \) and the BDLP \( Z \) are independent and \( (\mathcal{F}_t) \) is assumed to be the usual augmentation of the filtration generated by the pair \((W, Z_\lambda)\). The random variable \( V_0 \) has a self-decomposable distribution corresponding to the BDLP such that the process \( V \) is strictly stationary and

\[
E[V_0] = \zeta, \quad \text{Var}[V_0] = \eta.
\]

To shorten the notation we introduce the parameter vector

\[
\theta = (\lambda, \zeta, \eta, \mu, \beta, \rho)^T,
\]

and the bivariate process

\[
X = (X, V).
\]

If the distribution of \( V_0 \) is from a particular class \( D \) then \( X \) is called a BNS-DOU(\( \theta \)) model.

The process \((X_t, V_t)_{t \geq 0}\) is clearly Markovian.

2.1.2 The \( \Gamma \)-OU model

The \( \Gamma \)-OU model is obtained by constructing the BNS-model with stationary gamma distribution, \( V_0 \sim \Gamma(\nu, \alpha) \), where the parameters are \( \nu > 0 \) and \( \alpha > 0 \). The corresponding background driving Lévy process \( Z \) is a compound Poisson processes with intensity \( \nu \) and jumps from the exponential distribution with parameter \( \alpha \). Consequently both processes \( Z \) and \( V \) have a finite number of jumps in any finite time interval.

For the \( \Gamma \)-OU model it is more convenient to work with the parameters \( \nu \) and \( \alpha \). The connection to the generic parameters used in our general development is given by

\[
\zeta = \frac{\nu}{\alpha}, \quad \eta = \frac{\nu}{\alpha^2}.
\]

As the gamma distribution admits exponential moments we have integer moments of all orders and our Assumption II below is satisfied.
2.1.3 The IG-OU model

The IG-OU model is obtained by constructing the BNS-model with stationary inverse Gaussian distribution, $V_0 \sim (\delta, \gamma)$, with parameters $\delta > 0$ and $\gamma > 0$.

The corresponding background driving Lévy process is the sum of an $\text{IG}(\delta/2, \gamma)$ process and an independent compound Poisson process with intensity $\delta \gamma/2$ and jumps from an $\Gamma(1/2, \gamma^2/2)$ distribution. Consequently both processes $Z$ and $V$ have infinitely many jumps in any finite time interval.

For the IG-OU model it is more convenient to work with the parameters $\delta$ and $\gamma$. The connection to the generic parameters used in our general development is given by

$$\zeta = \frac{\delta}{\gamma}, \quad \eta = \frac{\delta}{\gamma^3}. \quad (7)$$

As the inverse Gaussian distribution admits exponential moments we have integer moments of all orders and our Assumption 1 below is satisfied.

2.2 Discrete observations

We observe returns and the variance process on a discrete grid of points in time,

$$0 = t_0 < t_1 < \ldots < t_n. \quad (8)$$

This implies

$$V(t_i) = V(t_{i-1}) e^{-\lambda(t_i - t_{i-1})} + \int_{t_{i-1}}^{t_i} e^{-\lambda(s-t)} dZ_\lambda(s). \quad (9)$$

Using

$$V_i := V(t_i), \quad U_i := \int_{t_{i-1}}^{t_i} e^{-\lambda(t_i - s)} dZ_\lambda(s) \quad (10)$$

we have that $(U_i)_{i \geq 1}$ is a sequence of independent random variables, and it is independent of $V_0$. If the grid is equidistant, then $(U_i)_{i \geq 1}$ are iid. Observing the returns $X$ on the grid we have

$$X(t_i) - X(t_{i-1}) = \mu(t_i - t_{i-1}) + \beta(Y(t_i) - Y(t_{i-1}))$$

$$+ \int_{t_{i-1}}^{t_i} \sqrt{V(s-t)} dW(s) + \rho(Z\lambda(t_i) - Z\lambda(t_{i-1})), \quad (11)$$

where

$$Y(t) = \int_0^t V(s-) ds$$

is the integrated variance process. This suggests introducing the discrete time quantities

$$X_i = X(t_i) - X(t_{i-1}), \quad Y_i = Y(t_i) - Y(t_{i-1}), \quad Z_i = Z\lambda(t_i) - Z\lambda(t_{i-1}) \quad (12)$$

and

$$W_i = \frac{1}{\sqrt{Y_i}} \int_{t_{i-1}}^{t_i} \sqrt{V(s-)} dW(s). \quad (13)$$

Furthermore, it is also convenient to introduce the discrete quantity

$$S_i = \frac{1}{\lambda}(Z_i - U_i). \quad (14)$$
It is not difficult to see (conditioning!) that \((W_i)_{i \geq 1}\) is an iid \(N(0, 1)\) sequence independent from all other discrete quantities. We note also that \((U_i, Z_i)_{i \geq 1}\) is a bivariate iid sequence, but \(U_i\) and \(Z_i\) are obviously dependent.

From now on, for notational simplicity, we consider the equidistant grid with 
\[
t_k = k \Delta,
\]
(15)
where \(\Delta > 0\) is fixed. This implies 
\[
V_i = \gamma V_{i-1} + U_i
\]
(16)
and 
\[
Y_i = \epsilon V_{i-1} + S_i,
\]
(17)
where 
\[
\gamma = e^{-\lambda \Delta}, \quad \epsilon = \frac{1 - \gamma}{\lambda}.
\]
(18)
Furthermore, 
\[
X_i = \mu \Delta + \beta Y_i + \sqrt{\gamma} W_i + \rho Z_i.
\]
(19)
The sequence \((X_i, V_i)_{i \geq 0}\) is clearly Markovian. From now on we assume all moments of the stationary distribution of \(V_0\) exist.

**Assumption 1** 
\[
E[V_0^n] < \infty \quad \forall n \in \mathbb{N}.
\]
(20)
In the estimating context we assume all moments are finite with respect to all probability measures \(P_\theta, \theta \in \Theta\) under consideration, where \(\Theta\) is the parameter space.

No other assumptions are made, and all conditions required for consistency and asymptotic normality of our estimator will be proven rigorously from that assumption.

**Proposition 1** We have for all \(n \in \mathbb{N}\) that 
\[
E[Z_1^n] < \infty, \quad E[U_1^n] < \infty, \quad E[S_1^n] < \infty,
\]
(21)
and 
\[
E[Y_1^n] < \infty, \quad E[W_1^n] < \infty, \quad E[X_1^n] < \infty.
\]
(22)

Consequently the expectation of any (multivariate) polynomial in \(Z_1, U_1, S_1, \sqrt{Y_1}, W_1, X_1\) exists under \(P_0\).

**Proof:** We will use repeatedly the well-known relation between the existence of moments and the differentiability of the characteristic function of a random variable, see [CT97, Theorem 8.4.1, p.295f], for example.

Let \(\phi(t)\) denote the characteristic function of \(V_0\). By assumption \(E_\theta[V_0^n] < \infty\) for all \(n \in \mathbb{N}\). Thus \(\phi(t)\) is arbitrarily many times differentiable. The law of \(V_0\) is self-decomposable, thus infinitely divisible and \(\phi(t) \neq 0\) for all \(t \in \mathbb{R}\). Thus the Fourier cumulant function \(\kappa(t) = \log \phi(t)\) is arbitrarily many times differentiable. It follows from [BNS01, equation (12)], that the characteristic function of \(Z(1)\) is \(\psi(t) = \exp(t\kappa(t))\). Thus \(\psi(t)\) is arbitrarily many times differentiable and consequently \(E[Z(1)_n] < \infty\) for all \(n \in \mathbb{N}\). As \(Z\) is a Lévy process this implies \(E[Z(\lambda)_n] < \infty\), and as \(Z_1 = Z(\lambda)\) we have shown \(E[Z_1^n] < \infty\) for all \(n \in \mathbb{N}\).

From (10) and (14) we have \(U_1 \leq Z_1\) and \(S_1 \leq \lambda^{-1} Z_1\) so \(E[U_1^n] < \infty\) and \(E[S_1^n] < \infty\) for all \(n \in \mathbb{N}\). As \(W_1\) has a standard normal distribution it follows trivially \(E[W_1^n] < \infty\) for all \(n \in \mathbb{N}\). Repeated application of the binomial resp. multinomial theorem, the Hölder and the Cauchy-Schwarz inequalities yields \(E[Y_1^n] < \infty\) and \(E[X_1^n] < \infty\) for all \(n \in \mathbb{N}\), and the final conclusion for polynomials. \(\square\)

Let us remark that, by the stationarity, the above result holds also for \(Z_i, U_i, S_i, \sqrt{Y_i}, W_i, X_i\) instead of \(Z_1, U_1, S_1, \sqrt{Y_1}, W_1, X_1\), where \(i \in \mathbb{N}\) is arbitrary.
2.3 Some remarks

Most work on estimating functions is developed for diffusions, see for example [So97] Uch04b, Uch04a, DS04, MR03, KP02, Jac02, Jac01, Sor01, BS01, Kes00, KS99a, Sør97, BS95, although it is often remarked that the results extend to Markov chains. Yet the models under consideration here display several peculiarities.

One assumption that is usually made is that the transition probabilities under $P_0$ have the same support for each $\theta$. Typically the support of the conditional distribution of $V_1$ in a BNS model given $V_0 = v$ is $(ve^{-\lambda \Delta}, +\infty)$ under $P_0$, thus depends on $\theta$. This does not affect our analysis. The experiment is not homogeneous, cf. [Str85].

If the BDLP is a compound Poisson process, as in the $\Gamma$-OU case, we have the atom of the conditional distribution of $V_1$ given $V_0 = v$ under $P_0$ at the parameter dependent position $ve^{-\lambda \Delta}$. Consequently no dominating measure exists and maximum likelihood cannot be defined in the usual way. There is an alternative definition covering that case, cf. [KW50, Ioh78], but we have not exploited that direction further. See also [NS03]. This problem does not appear with an infinite activity BDLP such as in the IG-OU model and standard maximum likelihood estimation could be studied.

The description given in sections 2.2 and 2.2 provides a BNS model for each $\theta$, but not a statistical experiment as it is taken as a starting point in section 3. The reason is that the processes $X$ and $V$ will depend on $\theta$. This can be avoided by introducing a statistical experiment generated by a BNS model. In analogy to the statistical experiment generated by a diffusion, see [SS00]. This means we take the distribution of $X$ and $V$ on the Skorohod space $(D^2, \mathcal{B}(D^2))$ under each $P_0$ as a starting point.

3 The simple explicit estimator

3.1 The simple estimating equations and their explicit solution

For estimation purposes we consider a probability space on which a parameterized family of probability measures is given:

$$(\Omega, \mathcal{F}, \{P_\theta : \theta \in \Theta\}),$$

where $\Theta = \{\theta \in \mathbb{R}^6 : \theta^1 > 0, \theta^2 > 0, \theta^3 > 0\}$. The data is generated under the true probability measure $P_{\theta_0}$ with some $\theta_0 \in \Theta$. The expectation with respect to $P_\theta$ is denoted by $E_{\theta}[^{\cdot}]$ and with respect to $P_{\theta_0}$ simply by $E[^{\cdot}]$.

We assume there is a process $X$ that is BNS-DOU$(\theta)$ under $P_\theta$. We want to find an estimator for $\theta_0$ using observations $X_1, \ldots, X_n, V_1, \ldots, V_n$. We are interested in asymptotics as $n \to \infty$. To that purpose let us consider the following martingale estimating functions:

$$
\begin{align*}
G^1_n(\theta) &= \sum_{k=1}^n [V_k - f^1(V_{k-1}, \theta)], & f^1(v, \theta) &= E_{\theta}[V_1 | V_0 = v] \\
G^2_n(\theta) &= \sum_{k=1}^n [V_k V_{k-1} - f^2(V_{k-1}, \theta)], & f^2(v, \theta) &= E_{\theta}[V_1 V_0 | V_0 = v] \\
G^3_n(\theta) &= \sum_{k=1}^n [V_k^2 - f^3(V_{k-1}, \theta)], & f^3(v, \theta) &= E_{\theta}[V_1^2 | V_0 = v] \\
G^4_n(\theta) &= \sum_{k=1}^n [X_k - f^4(V_{k-1}, \theta)], & f^4(v, \theta) &= E_{\theta}[X_1 | V_0 = v] \\
G^5_n(\theta) &= \sum_{k=1}^n [X_k V_{k-1} - f^5(V_{k-1}, \theta)], & f^5(v, \theta) &= E_{\theta}[X_1 V_0 | V_0 = v] \\
G^6_n(\theta) &= \sum_{k=1}^n [X_k V_k - f^6(V_{k-1}, \theta)], & f^6(v, \theta) &= E_{\theta}[X_1 V_1 | V_0 = v]
\end{align*}
$$

(24)
Lemma 1 We have the explicit expressions

\[ f^1(v, \theta) = \gamma_v + (1 - \gamma)\xi \]
\[ f^2(v, \theta) = \gamma_v^2 + (1 - \gamma)\xi_v \]
\[ f^3(v, \theta) = \gamma_v^2 + 2\gamma(1 - \gamma)\xi_v + (1 - \gamma)^2\xi^2_v + (1 - \gamma)\eta \]
\[ f^4(v, \theta) = \beta\xi_v + \mu\Delta + \beta(1 - \epsilon)\zeta + \rho\lambda\xi \]
\[ f^5(v, \theta) = \beta\xi_v^2 + (\mu\Delta + \beta(1 - \epsilon)\zeta + \rho\lambda\xi)\xi \]
\[ f^6(v, \theta) = \beta\gamma\xi^2 + ((\mu\Delta + \beta(1 - \epsilon)\zeta + \rho\lambda\xi)\gamma + \beta(1 - \gamma)\zeta)\xi_v + (1 - \epsilon)(1 - \gamma)\xi^2_v + \epsilon^2\eta \]

Proof: The formulas are special cases of the general moment calculations given in Appendix A. In order to demonstrate the basic idea we will prove the statements for two special and simple cases here, namely for \( f^1(v, \theta) \) and \( f^4(v, \theta) \). From (17) it follows that

\[ E_\theta[V_1|V_0 = v] = \gamma_v + E_\theta[U_1] \]

and from the stationarity of \( V \) we have

\[ E_\theta[U_1] = (1 - \gamma)E_\theta(V_0) = (1 - \gamma)\zeta. \]

Furthermore, from (19) and the fact that \( E[W_1] = 0 \), it follows that

\[ E_\theta[X_1|V_0 = v] = \mu\Delta + \beta E_\theta[Y_1|V_0 = v] + \rho E_\theta[Z_1|V_0 = v]. \]

But, from (17) we have that

\[ E_\theta[Y_1|V_0 = v] = \epsilon\xi_v + \frac{1}{\lambda} E_\theta[Z_1 - U_1] = \epsilon\xi_v + \zeta(1 - \epsilon), \]

and

\[ E_\theta[Z_1] = \lambda\zeta. \]

So, from (26) it follows that

\[ E_\theta[X_1|V_0 = v] = \mu\Delta + \beta\xi_v + \beta(1 - \epsilon)\zeta + \rho\lambda\zeta. \]

The estimator \( \hat{\theta}_n \) is obtained by solving the estimating equation \( G_n(\theta) = 0 \) and it turns out that this equation has a simple explicit solution.

Proposition 2 The estimating equation \( G_n(\hat{\theta}_n) = 0 \) admits for every \( n \geq 2 \) on the event

\[ C_n = \{ \xi_n^2 - \xi_n^1 v_n^1 > 0, v_n^2 - (v_n^1)^2 > 0 \} \]

a unique solution \( \hat{\theta}_n = (\lambda_n, \zeta_n, \eta_n, \beta_n, \rho_n, \mu_n) \) that is given by

\[ \gamma_n = (\xi_n^2 - \xi_n^1 v_n^1)/(v_n^2 - (v_n^1)^2); \]
\[ \zeta_n = (\xi_n^1 - \gamma_n v_n^1)/(1 - \gamma_n); \]
\[ \eta_n = ((\xi_n^3 - (\xi_n^1)^2) - \gamma_n^2(v_n^2 - (v_n^1)^2))/(1 - \gamma_n^2); \]
\[ \lambda_n = -\log(\gamma_n)/\Delta; \]
\[ \epsilon_n = (1 - \gamma_n)/\lambda_n; \]
\[ \beta_n = (\xi_n^5 - v_n^1 \xi_n^4)/(\epsilon_n(v_n^2 - (v_n^1)^2)); \]
\[ \rho_n = (\xi_n^6 - \xi_n^4 v_n^1 - \beta_n \epsilon_n(\eta_n(1 - \gamma_n) + \gamma_n(v_n^2 - (v_n^1)^2)))/(2(1 - \gamma_n)\eta_n); \]
\[ \mu_n = (\xi_n^4 - \beta_n \epsilon_n(v_n^2 - \zeta_n))/\Delta - (\beta_n + \lambda_n \rho_n)\zeta_n; \]
where
\[ \xi_n^1 = \frac{1}{n} \sum_{i=1}^{n} V_i \quad \xi_n^2 = \frac{1}{n} \sum_{i=1}^{n} V_i V_{i-1} \quad \xi_n^3 = \frac{1}{n} \sum_{i=1}^{n} V_i^2 \] (34)
\[ \xi_n^4 = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \xi_n^5 = \frac{1}{n} \sum_{i=1}^{n} X_i V_{i-1} \quad \xi_n^6 = \frac{1}{n} \sum_{i=1}^{n} X_i V_i \]

and
\[ \nu_n^1 = \frac{1}{n} \sum_{i=1}^{n} V_{i-1} \quad \nu_n^2 = \frac{1}{n} \sum_{i=1}^{n} V_{i-1}^2 \] (35)

Proof: The first three equations \( G_n^j(\theta) = 0 \), for \( j = 1, 2, 3 \) contain only the unknowns \( \zeta, \eta, \lambda \) and are easily solved. In fact we get a familiar estimator for the first two moments and the autocorrelation coefficient of an AR(1) process. The last three equations \( G_n^j(\theta) = 0 \), for \( j = 4, 5, 6 \) can be seen as a linear system for the unknowns \( \mu, \beta, \rho \), once the other parameters have been determined.

Remark 1 The exceptional set \( C_n \) could be simplified to
\[ C' = \{ \xi_n^2 - \xi_n^1 > 0 \} \] (36)

Since the jump times and the jump sizes of the BDLP are independent, and the former have an exponential distribution it follows that \( V_0, \ldots, V_n \) is with probability one not constant, so \( P[\nu_n^2 - (\nu_n^1)^2 > 0] = 1 \). But although it can be shown that the probability of \( C_n \) tends to zero, for finite \( n \) we have \( P[\xi_n^2 - \xi_n^1 \nu_n^1 \leq 0] > 0 \). This is the common phenomenon that sample moments do not share all properties of their theoretical counterparts. For definiteness we put \( \hat{\theta}_n = 0 \) outside \( C_n \).

3.2 Consistency

Let us investigate the consistency of the estimator from the previous section. First, we will need the following lemma.

Lemma 2 For every \( k \geq 1 \) and \( p > 0 \)
\[ \frac{V_k}{n^p} \overset{a.s.}{\longrightarrow} 0 \quad \text{as} \quad n \to \infty. \] (37)

Proof: The random variables \( \{V_n, n \geq 1\} \) are identically distributed and \( m_k = E[|V_1^k|] < \infty \) for all \( k \geq 1 \). Thus we are in the situation of [Sto74, Exercise 2.1.2(i), p.14].

Let \( k \geq 1 \) and \( \epsilon > 0 \) be arbitrarily chosen. Taking any integer \( \alpha > 1/p \) and using the Chebyshev inequality we obtain
\[ \sum_{n=1}^{\infty} P\left( \left| \frac{V_n}{n^p} \right| > \epsilon \right) \leq \sum_{n=1}^{\infty} \frac{E[|V_1^k|]^\alpha}{n^{\alpha p} \epsilon^{\alpha}} \leq \sum_{n=1}^{\infty} \frac{m_k \alpha}{n^{\alpha p} \epsilon^{\alpha}} < \infty. \] (38)

Therefore from the Borel-Cantelli lemma it follows that \( P(\limsup_n n^{-p}|V_n^k| > \epsilon) = 0 \). □

Lemma 3 We have for all \( k \in \mathbb{N} \) that
\[ \frac{1}{n} \sum_{i=1}^{n} V_i^k \overset{a.s.}{\longrightarrow} E[|V_1^k|], \] (39)
as \( n \to \infty \).
Proof: We will prove this statement by induction.

(1) $k = 1$. Let us define

$$H_i = V_i - E(V_i | V_{i-1}), \quad i \geq 1.$$ 

Obviously, $(H_i, i \geq 1)$ is a sequence of martingale differences and is therefore uncorrelated. Using expressions (10) and (16) we obtain

$$E[H_i^2] = E(V_i^2) - E[E(V_i | V_{i-1})^2]$$

$$= (1 - \gamma^2)E(V_0^2) - 2\gamma E(U_1)E(V_0) + E(U_1^2),$$

so $E[H_i^2]$ have a common bound for every $i \geq 1$. Since the assumptions of the Theorem 5.1.2 from [Chu01] are satisfied, it follows that

$$\frac{1}{n} \sum_{i=1}^{n} V_i - \frac{1}{n} \sum_{i=1}^{n} E(V_i | V_{i-1}) \xrightarrow{a.s.} 0, \quad as \quad n \rightarrow \infty.$$ 

But using again definition (16), the last expression is equivalent to

$$\frac{1}{n} \sum_{i=1}^{n} V_i + \frac{\gamma}{n} \frac{V_n - V_0}{n} - E(U_1) \xrightarrow{a.s.} 0, \quad as \quad n \rightarrow \infty.$$ 

Finally, using the result of the previous lemma and the fact that $E[U_1] = (1 - \gamma)E[V_0]$, it follows

$$\frac{1}{n} \sum_{i=1}^{n} V_i \xrightarrow{a.s.} E(V_0), \quad as \quad n \rightarrow \infty.$$ 

This completes the proof for $k = 1$.

(2) Suppose now that the statement of the theorem holds for $l \leq k - 1$, i.e. $E(V_{1}^{k-1}) < \infty$ and

$$\frac{1}{n} \sum_{i=1}^{n} V_i^{l} \xrightarrow{a.s.} E(V_0^l), \quad l \leq k - 1$$

when $n \rightarrow \infty$. For $k > 1$, and for $i \geq 1$, let

$$H_i^k := V_i^k - E[V_i^k | V_{i-1}] \quad and \quad S_n^k := \sum_{i=1}^{n} H_i^k.$$ 

Obviously, $(H_i^k, i \geq 1)$ is a sequence of martingale differences and in particular is uncorrelated. Moreover, due to the strong stationarity of the volatility sequence and relations (11) and (16) we obtain

$$E[H_i^k]^2 = E[V_i^{2k} - 2V_i^k E[V_i^k | V_{i-1}] + E[V_i^k | V_{i-1}]^2]$$

$$= E[V_i^{2k}] - E[(E[V_i^k | V_{i-1})]^2]$$

$$\leq E[V_i^{2k}] =: c_k,$$ 

$c_k$ denoting some constant that does not depend on $i$. Hence, by [Chu01] Theorem 5.1.2, p.108], it follows that $S_n^k \xrightarrow{a.s.} 0$ when $n \rightarrow \infty$, which in our case, due to the definition of $S_n^k$, is equivalent to

$$\frac{1}{n} \sum_{i=1}^{n} V_i^k - \frac{1}{n} \sum_{i=1}^{n} E[V_i^k | V_{i-1}] \xrightarrow{a.s.} 0.$$ 

(43)
Using again the definition (10) and the independency of \( U_i \) from \( V_{i-1} \), for \( i \geq 1 \), we obtain
\[
\frac{1}{n} \sum_{i=1}^{n} V_i^k - \frac{1}{n} \sum_{i=1}^{n} E[V_i^k|V_{i-1}] = \frac{1}{n} \sum_{i=1}^{n} V_i^k - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=0}^{k} \binom{k}{j} \gamma^j V_{i-1}^j E[U_1^{k-j}]
\]
\[
= \frac{1 - \gamma}{n} \sum_{i=1}^{n} V_i^k - \frac{\gamma}{n} (V_0^k - V_n^k) - \sum_{j=0}^{k-1} \binom{k}{j} E[U_1^{k-j}] \gamma^j \sum_{i=1}^{n} V_i^j.
\]

Finally, applying the assumption of the induction, Lemma 2 and the statement (43), we obtain
\[
\frac{1}{n} \sum_{i=1}^{n} V_i^k \xrightarrow{a.s.} \frac{1}{1-\gamma} \sum_{j=0}^{k-1} \binom{k}{j} \gamma^j E[U_1^{k-j}] E(V_0^j) = E(V_k),
\]
where the last equality follows calculating \( E(\gamma V_0 + U_1)^k \) using (10).

In the next lemma we extend the strong law of large numbers for \((V_i^p, i \geq 1)\) to more general sequences.

**Lemma 4** For all integers \( p, q, r \geq 0 \) we have
\[
\frac{1}{n} \sum_{i=1}^{n} X_i^p V_i^q V_i^r \xrightarrow{a.s.} E[X_i^p V_i^q V_i^r]
\]
as \( n \to \infty \).

**Proof:** Let
\[ M_i = X_i^p V_i^q V_i^r - E[X_i^p V_i^q V_i^r|V_{i-1}] \].

Obviously, \((M_i, i \geq 1)\) is a sequence of martingale differences, and in particular it is uncorrelated. It is stationary and \( E[M_i^2] < \infty \). So we can use again [Chu01, Theorem 5.12] to show
\[
\frac{1}{n} \sum_{i=1}^{n} X_i^p V_i^q V_i^r \xrightarrow{a.s.} \sum_{k=0}^{p+q} \phi_{pqk} V_i^{k+r}.
\]

The conditional expectation \( E[X_i^p V_i^q V_i^r|V_{i-1}] \) is a polynomial in \( V_{i-1} \), namely
\[
E[X_i^p V_i^q V_i^r|V_{i-1}] = \sum_{k=0}^{p+q} \phi_{pqk} V_i^{k+r}.
\]

This is shown in section A.5 in Appendix A where the coefficients \( \phi_{pqk} \) are explicitly calculated. Applying Lemma 3 yields
\[
\frac{1}{n} \sum_{i=1}^{n} E[X_i^p V_i^q V_i^r|V_{i-1}] \xrightarrow{a.s.} \sum_{k=0}^{p+q} \phi_{pqk} E[V_0^{k+r}].
\]

As we have
\[
E[X_i^p V_i^q V_i^r|V_0] = E[E[X_i^p V_i^q V_i^r|V_0]|V_0] = \sum_{k=0}^{p+q} \phi_{pqk} E[V_0^{k+r}],
\]
(45)
the proof is completed. \( \square \)
Theorem 1. We have \( P(C_n) \to 1 \) when \( n \to \infty \) and the estimator \( \hat{\theta}_n \) is consistent on \( C_n \), namely
\[
\hat{\theta}_n \xrightarrow{a.s.} \theta_0
\]
on \( C_n \) as \( n \to \infty \).

Proof: Using the results of Lemma 3 it easily follows that
\[
\xi_n^2 - \xi_n^1 v_n^1 \to \text{Cov}(V_1, V_0) > 0,
\]
so \( P(C_n) \to 1 \) as \( n \to \infty \).

Using again the results of Lemma 3 it follows that the empirical moments in (33) and (35) converge to their theoretical counterparts, \( \xi_n^i \xrightarrow{a.s.} \xi^i \) and \( v_n^i \xrightarrow{a.s.} v^i \), where
\[
\begin{align*}
\xi^1 &= \zeta, \\
\xi^2 &= \zeta^2 + \gamma \eta, \\
\xi^3 &= \zeta^2 + \eta, \\
\xi^4 &= \mu + (\beta + \lambda \rho)\zeta, \\
\xi^5 &= \mu \zeta + (\beta + \lambda \rho)\zeta^2 + \beta \epsilon \eta, \\
\xi^6 &= \mu \zeta + (\beta + \lambda \rho)\zeta^2 + (\beta + 2 \rho \lambda) \epsilon \eta.
\end{align*}
\]
Plugging the limits into (33) shows, after a short mechanical calculation, that the estimator is in fact consistent.

\[\square\]

### 3.3 Asymptotic normality

For a concise vector notation we introduce
\[
\Xi_k = (V_k, V_k V_{k-1}, V_k^2, X_k, X_k V_{k-1}, X_k V_k)^\top,
\]
and write the estimating equations in the form
\[
C_n^i(\theta) = \sum_{k=1}^n [\Xi_k^i - f^i(V_{k-1}, \theta)], \quad i = 1, \ldots, 6
\]
and \( f^i(v, \theta) \) given by (25). We write
\[
f^i(v, \theta) = \sum_{\ell=r_i}^{p_i+r_i+q_i} \phi^i_\ell(\theta) v^\ell.
\]
with
\[
p = (0, 0, 0, 1, 1, 1), \quad q = (1, 1, 1, 0, 0, 1), \quad r = (0, 1, 0, 0, 1, 0)
\]
and
\[
\begin{align*}
\phi^1_0(\theta) &= \gamma, & \phi^1_1(\theta) &= (1 - \gamma) \zeta \\
\phi^2_0(\theta) &= \gamma, & \phi^2_1(\theta) &= (1 - \gamma) \zeta \\
\phi^2_2(\theta) &= \gamma^2, & \phi^2_3(\theta) &= 2 \gamma (1 - \gamma) \zeta & \phi^3_0(\theta) &= (1 - \gamma)^2 \zeta^2 + (1 - \gamma^2) \eta \\
\phi^3_1(\theta) &= \beta \epsilon, & \phi^3_2(\theta) &= \mu + \beta (1 - \epsilon) \zeta + \rho \lambda \zeta \\
\phi^4_0(\theta) &= \beta \epsilon \gamma, & \phi^4_1(\theta) &= (\mu + \beta (1 - \epsilon) \zeta + \rho \lambda \zeta) \gamma + \beta \epsilon (1 - \gamma) \zeta & \phi^4_0(\theta) &= (1 - \epsilon) (1 - \gamma) \zeta^2 + \epsilon^2 \lambda \eta
\end{align*}
\]
We will use, that \( f^i(v, \theta) \) is a polynomial in \( v \), and that its coefficients \( \phi \) are smooth functions in \( \theta \).

We shall first prove the central limit theorem for the estimating functions.
Proposition 3 We have
\[
\frac{1}{\sqrt{n}} G_n(\theta_0) \xrightarrow{D} N(0, \Upsilon),
\] (53)
as \( n \to \infty \), where
\[
\Upsilon_{ij} = E[Cov(\Xi_1, \Xi_1^T|V_0)].
\] (54)

Proof: To show the above result, we use the multivariate martingale central limit theorem, that is recapitulated in Appendix B. To that purpose we introduce the vector martingale difference array
\[
\chi_{n,k} = \frac{1}{\sqrt{n}} \left[ \Xi_k - f(V_{k-1}, \theta) \right].
\] (55)

We have to show the two assumptions from the previous theorem. First, we prove a multivariate Lyapunov condition which implies the Lindeberg condition. From (55) it follows that
\[
\sqrt{n} \chi_{n,k} \sim p(V_{k-1}),
\] (56)
where \( q(v_0) \) is a polynomial in \( v_0 \). From Lemma 4 it thus follows
\[
\frac{1}{n} \sum_{k=1}^{\infty} q(V_{k-1}) \xrightarrow{a.s.} E[q(V_0)],
\] (57)
where the expression on the righthand side exists and is finite. Thus the first condition of the martingale central limit theorem is satisfied. In order to verify the second condition of the same theorem we consider the \((i,j)\)-th element of the matrix \( \chi_{n,k} \) which is given by
\[
\frac{1}{n} (\Xi_k^i - f^i(V_{k-1}, \theta))(\Xi_k^j - f^j(V_{k-1}, \theta)).
\] (58)
This is again a polynomial in \( V_{k-1}, V_k \) and \( X_k \) so by Lemma 4 it follows that
\[
\frac{1}{n} \sum_{k=1}^{\infty} (\Xi_k^i - f^i(V_{k-1}, \theta))(\Xi_k^j - f^j(V_{k-1}, \theta)) \xrightarrow{a.s.} E[(\Xi_k^i - f^i(V_{k-1}, \theta))(\Xi_k^j - f^j(V_{k-1}, \theta))]
\] (59)
as \( n \to \infty \).

Remark 2 A systematic method to evaluate \( \Upsilon \) is given in Appendix A and the resulting explicit expressions are given in [Pos07].

Lemma 5 We have
\[
\frac{1}{\sqrt{n}} [\xi_n - \xi] \xrightarrow{D} N(0, \Sigma),
\] (60)
where
\[
\Sigma = P^{-1} \Upsilon (P^{-1})^T
\] (61)
and
\[
P_{ij} = \delta_{ij} - \phi_1^i \delta_{ij} - \phi_2^i \delta_{ij}
\] (62)
with \( \delta_{ij} \) denoting the Kronecker delta.
\[ \frac{1}{\sqrt{n}} G_n(\theta_0) = P \sqrt{n}(\xi_n - \xi) + Q_n, \]  
\[ Q_n = \frac{1}{\sqrt{n}} [\phi_1(V_n - V_0) + \phi_2(V_n^2 - V_0^2)]. \]

In view of Lemma 2 above we see, that the remainder term \( Q_n \) goes to zero in probability as \( n \to \infty \). As \( P \) has determinant \((1 - \gamma)^2(1 + \gamma) > 0\) it is invertible, and we have

\[ \sqrt{n}(\xi_n - \xi) = P^{-1} \left( \frac{1}{\sqrt{n}} G_n(\theta_0) \right) + R_n \]

with \( R_n = -P^{-1}Q_n \) going to zero in probability as \( n \to \infty \). The expression \( P^{-1} (n^{-1/2} G_n(\theta_0)) \) is asymptotically normal with mean 0 and covariance matrix \( \Sigma \). An application of Slutsky’s Theorem proves the lemma.

Finally, we have all the ingredients for proving the following result.

**Theorem 2** The estimator

\[ \hat{\theta}_n = (\lambda_n, \zeta_n, \eta_n, \mu_n, \beta_n, \rho_n) \]

is asymptotically normal, namely

\[ \sqrt{n}[\hat{\theta}_n - \theta_0] \xrightarrow{D} N(0, T), \]

as \( n \to \infty \), where

\[ T = D\Sigma D^T \]

and \( D \) is explicitly given according to (69).

**Proof:** We observe from (33) that \( \hat{\theta}_n = g(\xi_n, \nu_n) \), where \( g \) is well defined and continuously differentiable in a neighborhood of \((\xi, \nu)\). Using the Taylor expansion in the last two variables we have \( \hat{\theta}_n = h(\xi_n) + S_n \), where \( h \) is well defined and continuously differentiable in a neighborhood of \( \xi \), and \( S_n \) goes to zero in probability in view of Lemma 2. Thus it can be neglected according to Slutsky’s Theorem. We apply the delta method, see [Leh99] for example, and compute the Jacobian matrix \( D \) with

\[ D_{ij} = \frac{\partial h_i}{\partial \xi_j}(\xi), \quad i, j = 1, \ldots, 6. \]

A lengthy elementary calculation shows that the matrix has determinant \( \lambda / (2(1 - \gamma)^2 \gamma^3) \), thus it is invertible.

**Remark 3** For comparison it is instructive to study our simple estimator in the general framework of [Sø99] where the properties of the estimator are studied without exploiting the fact that the estimating equation allows an explicit solution. This is done in [Pos07]. There the theory is extended in the case of a bivariate Markov process and Condition 2.6 of [Sø99] is proven in order to use his Corollary 2.7 and Theorem 2.8.
4 Numerical illustrations

4.1 Description of the model and its parameter values

To illustrate the results from the previous sections numerically, we consider the \( \Gamma \)-OU model from Section 2.1.2, where the variance \( V \) has a stationary gamma distribution. We use as time unit one year consisting of 250 trading days. The true parameters are

\[
\nu = 2.56, \quad \alpha = 64, \quad \lambda = 256, \quad \beta = -0.5, \quad \rho = -0.1, \quad \mu = 1.2. \tag{70}
\]

The parameters imply that there are on average 2.56 jumps per day and the jumps in the BDLP and in the volatility are exponentially distributed with mean 0.0156. The interpretation is, that typically every day two or three new pieces of information arrive and make the variance process jump. The stationary mean of the variance is 0.04. Hence, if we define instantaneous volatility to be the square root of the variance, it will fluctuate around 20% in our example. The half-life of the autocorrelation of returns is about half a day.

In our example annual log returns have (unconditional) mean 25.6% and a annual volatility 20%. Figure 1 displays a simulation of one year of daily observations from the background driving Lévy process, from the instantaneous variance process, and log returns, or more precisely, simulated realizations of \( Z_i, V_i, \) and \( X_i \) for \( i = 1, \ldots, 250. \) In [Pos07] other scenarios are considered, for example, small jumps arriving every minute, with fast decaying autocorrelation, or few jumps per year, corresponding to exceptional news with heavy impact on the variance process.

4.2 The asymptotic covariance matrix of the estimator

As our goal is an analysis of the estimator, and not an empirical study, we do not estimate the asymptotic covariance, but evaluate the explicit expression using the true parameters. Denoting the vector of asymptotic standard deviations of the estimates and the correlation matrix by \( s/\sqrt{n} \) resp. \( r \) we have

\[
\begin{bmatrix}
4.86 \\
125 \\
650 \\
7.36 \\
253 \\
0.526
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0.89 & 0.41 & 0.03 & 0.09 & -0.02 \\
0.89 & 1 & 0.4 & 0.03 & 0.09 & -0.03 \\
0.41 & 0.4 & 1 & 0.06 & 0.22 & 0 \\
0.03 & 0.03 & 0.06 & 1 & -0.75 & 0.06 \\
0.09 & 0.09 & 0.22 & -0.75 & 1 & -0.57 \\
-0.02 & -0.03 & 0 & 0.06 & -0.57 & 1
\end{bmatrix} \tag{71}
\]

4.3 Distribution of the estimates

Figure 4.3 illustrates the empirical and asymptotic distribution of the simple estimators for the \( \Gamma \)-OU model. The histograms are produced from \( m = 10000 \) replications consisting of \( n = 8000 \) observations each, corresponding to 32 years with 250 daily observations per year. We see from the graphs that in our illustration the parameters \( \nu, \alpha, \lambda, \) and \( \mu \) can be estimated quite accurately, in the sense that the usual confidence intervals yield one or two significant digits at least. The estimate for \( \rho \) is not as accurate and the accuracy for the estimate for \( \beta \) is unsatisfactory.

The bad quality of the estimator for \( \beta \) is neither surprising nor very troublesome. It has little impact on the model. The main reason for including the parameter \( \beta \) in the specification of BNS models is, for derivatives pricing: A risk-neutral BNS-model must have \( \beta = -1/2. \) In most applications working under a physical probability measure \( \beta = 0 \) can be assumed without much loss of generality or flexibility.

In ongoing work [HP06] we compare this asymptotic covariance with the covariance of the optimal quadratic estimating function.
5 Further and alternative developments

5.1 Optimal quadratic estimating functions

Our choice of estimating functions is natural, but, mathematically speaking, somewhat arbitrary. In ongoing work [HP06] we show, that the optimal quadratic estimating function based on the moments of $V_1, X_1, V_1^2, V_1 X_1, X_1^2$ can be computed explicitly, though the corresponding estimator has to be determined numerically. Our simple estimator can be used as a starting point for an iterative root-finding procedure. Consistency and asymptotic normality can be shown using the general theory as presented in [Sø99] along the lines of the present paper, although the expressions involved are slightly more complicated.

5.2 Using more integer or trigonometric moments for better efficiency

More efficient estimators than provided by the optimal quadratic estimating function can be obtained by incorporating further moments. As we have provided explicit computations for arbitrary integer moments and conditional moments, our methods can be extended to that situation. We might even have the number of moments tend to infinity with the number of observations, and obtain an estimator that is asymptotically equivalent to the maximum likelihood estimator, when the latter exists resp. can be defined, see [HHY96]. The reader might object, that very high moments are not reliable for empirical investigations. BNS-models allow also explicit computation of the characteristic function and thus of conditional and unconditional trigonometric moments $E[e^{i(\xi_k V_1 + \psi_k X_1)}]$ and $E[e^{i(\xi_k V_1 + \psi_k X_1)}|V_0]$ for arbitrary constants $\xi_k$ and $\psi_k$, that could be used instead for constructing estimating functions. See [AS02] for diffusions, [Sch05] for Lévy type processes, and [Sin01] for affine models.

5.3 Intra-day observations

Our approach is based on the explicit calculation of conditional and unconditional moments. Those calculations can be done for BNS-models on arbitrary time intervals. Hence our analysis is not restricted to a fixed time grid with the number of observation intervals tending to infinity, but could be performed also on a fixed horizon, with the number of intra-day observations increasing to infinity. The resulting estimators should then be compared to power-variation methods, cf. [Tod06].

5.4 Comparison to the generalized method of moments

We would be interested in a comparison of our results to the related generalized methods of moments. For a rigorous treatment of the latter, a precise specification of the weighting matrix is required, see [HHY96] and the references therein.

5.5 Unobserved volatility and substitutes for volatility

Finally, perhaps the biggest issue is, that the instantaneous variance is not observed in discrete time. In [Lin03] it is reported, that the number of trades is an excellent substitute for statistical purposes. This is certainly a promising starting point for an empirical analysis. For a theoretical analysis a joint model for the number prices and number of trades has to be specified.

Another direction would be to adapt the implied state method (IS-GMM) as introduced in [Pan02] to our martingale estimating function approach: We replace the unobserved $V_i$ in the estimating equations by the model-implied variance $V_i(\theta)$ that is obtained from option prices,
assuming that the dynamics are governed by BNS-models both under the physical probability measure \( P_{\theta_0} \) and a risk-neutral measure \( \tilde{P}_{\theta_0} \). The resulting estimating function will not be a martingale estimating function any more, and the bias has to be accounted for in a rigorous analysis. Nevertheless, in view of the results of Pan (2002), we are optimistic, that consistency and asymptotic normality will hold also here.

### A Explicit moment calculations

This section is about computing explicitly \( E[X^n V^m | V_0 = 0] \) and \( E[X^n V^m] \). All moments below will be given in terms of the cumulants of the stationary distribution, denoted by \( K_n \). We set

\[
\zeta = K_1, \quad \eta = K_2.
\]

If the stationary distribution is determined by the two parameters \( \zeta \) and \( \eta \) the higher cumulants are obviously functions of \( \zeta \) and \( \eta \), but the formulae hold in more general cases.

The calculations exploit the analytical tractability of the BNS-model, namely conditional Gaussianity of the logarithmic returns \( X \) and the linear structure of the OU-type process \( V \). From that it follows, and it is well-known, that univariate and multivariate cumulants can be computed easily. It remains to transform multivariate cumulants to multivariate moments, again a topic that is well-understood, and explicit expressions involve the multivariate Faà di Bruno formula, multivariate Bell polynomials and integer partitions, see for example McClelland (1987).

We have chosen to use simple recursions, that are easy to implement on a computer algebra system, in particular, since the expressions, though completely explicit and elementary, are rather lengthy when it comes to evaluating moments of order four for the asymptotic covariance matrix. For the reader’s convenience, we give the details in this appendix.

#### A.1 Preliminaries

Let us recapitulate the variables and notation from section 2.2 that are required in the following calculations. We use

\[
\gamma = e^{-\lambda \Delta}, \quad \epsilon = \frac{1 - e^{-\lambda \Delta}}{\lambda}.
\]

We have

\[
V_1 = \gamma V_0 + U_1, \quad Y_1 = \epsilon V_0 + S_1
\]

where

\[
U_1 = \int_0^{\Delta} e^{-\lambda(\Delta - s)} dZ_{\lambda s}, \quad S_1 = \int_0^{\Delta} \lambda^{-1} (1 - e^{-\lambda(\Delta - s)}) dZ_{\lambda s}.
\]

Note, that we have the simpler formula \( S_1 = (Z_1 - U_1)/\lambda \), but the integral above is sometimes notationally more convenient. We have

\[
X_1 = A_1 + \sqrt{Y_1 W_1}, \quad A_1 = \mu \Delta + \beta Y_1 + \rho Z_1.
\]

#### A.2 Stationary moments

We use the well-known recursion to compute moments from cumulants

\[
E[V^n] = \delta_n + \sum_{i=0}^{n-1} \binom{n-1}{i} K_{i+1} E[V^{n-i}].
\]

Alternatively we have \( E[V^n] = Y_n(K_1, \ldots, K_n) \), where \( Y_n(x_1, \ldots, x_n) \) denotes the complete Bell polynomials. Explicit non-recursive expressions can be given, but we do not use them.
A.3 Trivariate cumulants

From the key formula for Wiener-type integrals with Lévy process integrator, it follows that the joint cumulants of \((S_1, U_1, Z_1)\) are given by

\[
K_{nm\ell} = \lambda \epsilon_{nm}(n + m + \ell)K_{n+m+\ell},
\]

(78)

with

\[
\epsilon_{ij} = \begin{cases} 
\lambda^{-i} \left(1 + \sum_{k=1}^{i} \binom{i}{k} (-1)^k \frac{1 - \gamma^k}{k\lambda}\right) & j = 0 \\
\lambda^{-i} \left(1 - \frac{\gamma^j}{j\lambda} + \sum_{k=1}^{i} \binom{i}{k} (-1)^k \frac{1 - \gamma^k}{k\lambda}\right) & j > 0 
\end{cases}
\]

(79)

A.4 Trivariate Moments

Trivariate moments can be computed recursively from trivariate cumulants

\[
E[S_1^n U_1^m Z_1^\ell | V_0 = v] = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{k=0}^{\ell-1} \binom{n-1}{i} \binom{m-1}{j} \binom{\ell-1}{k} K_{i+j+k+1} E[S_1^{n-i} U_1^{m-j} Z_1^{\ell-k}]
\]

(80)

\[
E[S_1^n U_1^m Z_1^\ell | V_0 = v] = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{k=0}^{\ell-1} \binom{n}{i} \binom{m}{j} \binom{\ell}{k} K_{i+j+k+1} E[S_1^{n-i} U_1^{m-j} Z_1^{\ell-k}]
\]

(81)

\[
E[S_1^n U_1^m Z_1^\ell | V_0 = v] = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{k=0}^{\ell-1} \binom{n}{i} \binom{m}{j} \binom{\ell-1}{k} K_{i+j+k+1} E[S_1^{n-i} U_1^{m-j} Z_1^{\ell-k-1}]
\]

(82)

Alternatively, we can express \(E[S_1^n U_1^m Z_1^\ell | V_0 = v]\) as trivariate complete Bell polynomials \(Y_{nm\ell}\) evaluated at the trivariate cumulants of \(S_1, U_1, Z_1\), and explicit non-recursive expressions are available, but not very useful for us.

A.5 Some conditional expectations

Using (74) gives

\[
E[Y_1^n V_1^m Z_1^\ell | V_0 = v] = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \epsilon_{ij} E[S_1^{n-i} U_1^{m-j} Z_1^\ell] \cdot v^{i+j}
\]

(83)

Collecting powers of \(v\) gives

\[
E[Y_1^n V_1^m Z_1^\ell | V_0 = v] = \sum_{k=0}^{n+m} \xi_{nm\ell k} v^k
\]

(84)

with

\[
\xi_{nm\ell k} = \sum_{j=0}^{m\wedge k} \binom{n}{k-j} \binom{m}{j} \epsilon_{k-j} E[S_1^{n-k+j} U_1^{m-j} Z_1^\ell]
\]

(85)
Then using (76) and conditioning gives

\[
E[A_1^n Y_1^m V_1^\ell | V_0 = v] = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} \beta^i \rho^j \mu^{n-i-j} E[Y_1^{m+i} V_1^\ell | V_0 = v]
\]

(86)

Collecting powers of \( v \) gives

\[
E[A_1^n Y_1^m V_1^\ell | V_0 = v] = \sum_{k=0}^{n+m+\ell} \psi_{n m k} v^k
\]

(87)

with

\[
\psi_{n m k} = \sum_{i=(k-m-l)_+}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} \beta^i \rho^j \mu^{n-i-j} \xi_{m+i, \ell, j, k}
\]

(88)

Finally using (76) and the Gaussian moments gives

\[
E[A_1^n Y_1^m V_1^\ell | V_0 = v] = \sum_{i=0}^{\lceil n/2 \rceil} \binom{n}{2i} \binom{2i}{i} \frac{(2i)!}{2^i i!} E[A_1^{n-2i} Y_1^m V_1^\ell | V_0 = v]
\]

(89)

Collecting powers of \( v \) gives

\[
E[A_1^n Y_1^m V_1^\ell | V_0 = v] = \sum_{k=0}^{n+m} \phi_{n m k} v^k
\]

(90)

with

\[
\phi_{n m k} = \sum_{i=0}^{(n+m-k) \land \lfloor n/2 \rfloor} \binom{n}{2i} \binom{2i}{i} \frac{(2i)!}{2^i i!} \psi_{n-2i, i, m, k}
\]

(91)

It follows from the calculations above that \( \phi_{n m k} \) are polynomials in \( \gamma, \epsilon, \mu, \beta, \rho \).

### A.6 Some unconditional expectations

The same structure pertains for the unconditional expectations,

\[
E[Y_1^n V_1^m Z_1^\ell] = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \epsilon^i \gamma^j E[S_1^n U_1^{m-j} Z_1^\ell | E[V_0^{i+j}]
\]

(92)

then

\[
E[A_1^n Y_1^m V_1^\ell] = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} \beta^i \rho^j \mu^{n-i-j} E[Y_1^{m+i} V_1^\ell Z_1^\ell]
\]

(93)

and finally

\[
E[X_1^n V_1^m] = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} \binom{2i}{i} \frac{(2i)!}{2^i i!} E[A_1^{n-2i} Y_1^m V_1^m].
\]

(94)
B  The simple multivariate martingale central limit theorem

The following simple version of a multivariate martingale central limit theorem is certainly well-known or obvious for experts, some references are [CP05, KS99b, vZ00]. However, when looking for references, we found statements that do not exactly apply, or that are much more general (continuous time, random normalizations,...). It turned out that the elementary proof below is shorter, than an attempt to verify the assumptions and deduce the result from a more 'advanced' theorem. Yet, any concrete and precise hint for an appropriate reference would be most welcome to the authors.

**Theorem 3** Suppose \((X_{n,k})\) is a martingale difference array such that for every \(\epsilon > 0\)

\[
\sum_{k=1}^{n} E\left[\|X_{n,k}\|^2 1_{\{\|X_{n,k}\| > \epsilon\}} \mid \mathcal{F}_{k-1}\right] \xrightarrow{P} 0 \tag{95}
\]

and

\[
\sum_{k=1}^{n} \left[ X_{n,k} X_{n,k}^\top \right] \xrightarrow{P} \Upsilon \tag{96}
\]

as \(n \to \infty\). Then

\[
\sum_{k=1}^{n} X_{n,k} \xrightarrow{D} N(0, \Upsilon). \tag{97}
\]

**Proof:** We will use the Cramer-Wold device. For \(\beta \in \mathbb{R}^d\), \(\beta \neq 0\), let us define a random variable

\[
Y_{n,k} = \beta^\top X_{n,k}. \tag{98}
\]

Then we have

\[
\sum_{k=1}^{n} E\left[Y_{n,k}^2 \mid \mathcal{F}_{k-1}\right] = \sum_{k=1}^{n} E\left[(\beta^\top X_{n,k})(X_{n,k}^\top \beta) \mid \mathcal{F}_{k-1}\right] = \beta^\top \sum_{k=1}^{n} E\left[X_{n,k} X_{n,k}^\top \mid \mathcal{F}_{k-1}\right] \beta. \tag{99}
\]

From Assumption (96) it follows that the expression (99) converges to \(\beta^\top \Upsilon \beta\) and thus

\[
\sum_{k=1}^{n} E\left[Y_{n,k}^2 \mid \mathcal{F}_{k-1}\right] \xrightarrow{P} \beta^\top \Upsilon \beta \tag{100}
\]

as \(n \to \infty\). Furthermore, it holds

\[
|\beta^\top X_{n,k}| \leq \|\beta\| : \|X_{n,k}\|. \tag{101}
\]

Thus, for an arbitrary \(\epsilon > 0\) we have

\[
0 \leq \sum_{k=1}^{n} E\left[Y_{n,k}^2 1_{\{|Y_{n,k}| > \epsilon\}} \mid \mathcal{F}_{k-1}\right] = \sum_{k=1}^{n} E\left[(\beta^\top X_{n,k})^2 1_{\{\beta^\top X_{n,k} > \epsilon\}} \mid \mathcal{F}_{k-1}\right] \leq \|\beta\|^2 \sum_{k=1}^{n} E\left[\|X_{n,k}\|^2 1_{\{|\beta^\top X_{n,k}| > \epsilon\}} \mid \mathcal{F}_{k-1}\right]. \tag{102}
\]
Since for $\beta \neq 0$ the condition $|\beta^T X_{n,k}| > \epsilon$ implies
\[
\|X_{n,k}\| \geq \|\beta\|^{-1} |\beta^T X_{n,k}| > \|\beta\|^{-1} \epsilon,
\] (103)
it follows that
\[
1\{|\beta^T X_{n,k}| > \epsilon\} \leq 1\{|X_{n,k}| > \|\beta\|^{-1} \epsilon\}
\] (104)
and thus using the Assumption 95, from 102 it follows that
\[
\sum_{k=1}^{n} E\left[Y_{n,k}^2 1\{|Y_{n,k}| > \epsilon\} |\mathcal{F}_{k-1}\right] \overset{P}{\rightarrow} 0
\] (105)
as $n \to \infty$. Now, the statement follows from the univariate martingale central limit theorem from [HH80].

\textbf{Lemma 6} The conditional Lyapunov condition implies the conditional Lindeberg condition, namely, if
\[
\sum_{k=1}^{n} E[\|X_{n,k}\|^4 |\mathcal{F}_{k-1}] \longrightarrow 0
\] (106)
then
\[
\sum_{k=1}^{n} E[\|X_{n,k}\|^2 1\{|X_{n,k}| > \epsilon\} |\mathcal{F}_{k-1}] \longrightarrow 0
\] (107)
as $n \to \infty$.
\textbf{Proof:} For every $\epsilon > 0$ we have
\[
\sum_{k=1}^{n} E[\|X_{n,k}\|^4 |\mathcal{F}_{k-1}] = \sum_{k=1}^{n} E[\|X_{n,k}\|^4 1\{|X_{n,k}| > \epsilon\} |\mathcal{F}_{k-1}] + \sum_{k=1}^{n} E[\|X_{n,k}\|^4 1\{|X_{n,k}| \leq \epsilon\} |\mathcal{F}_{k-1}] \\
\geq \epsilon^2 \sum_{k=1}^{n} E[\|X_{n,k}\|^2 1\{|X_{n,k}| > \epsilon\} |\mathcal{F}_{k-1}],
\] (108)
since
\[
\sum_{k=1}^{n} E[\|X_{n,k}\|^4 1\{|X_{n,k}| \leq \epsilon\} |\mathcal{F}_{k-1}] \geq 0.
\]
From Assumption 106 the statement follows. \qed

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Figure 1: Daily observations $Z_i, V_i, X_i$. 
Figure 2: Empirical and asymptotic distribution of the simple estimators for the Γ-OU model. The histograms are produced from $m = 10000$ replications consisting of $n = 8000$ observations each, corresponding to 32 years with 250 daily observations per year. The true values are $\nu = 2.56$, $\alpha = 64$, $\lambda = 256$, $\mu = 1.2$, $\beta = -0.5$, $\rho = -0.1$. The standard deviations used for the normal curves are taken from the explicit asymptotic results, not estimated.