Efficient distillation beyond qubits

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We provide generalizations of known two-qubit entanglement distillation protocols for arbitrary Hilbert space dimensions. The protocols, which are analogues of the hashing and breeding procedures, are adapted to bipartite quantum states which are diagonal in a basis of maximally entangled states. We show that the obtained rates are optimal, and thus equal to the distillable entanglement, for a \((d−1)\) parameter family of rank deficient states. Methods to improve the rates for other states are discussed. In particular, for isotropic states it is shown that the rate can be improved such that it approaches the relative entropy of entanglement in the limit of large dimensions.

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I. INTRODUCTION

The possibility of entanglement distillation plays a crucial role in Quantum communication and Quantum information processing (cf. [1]). Together with quantum error correction it enables all the fascinating applications provided by Quantum information theory in the presence of a noisy and interacting environment. It provides a method to overcome decoherence without requiring a complete isolation from the environment.

The usual abstract way of thinking about entanglement distillation is the following: Two parties, Alice and Bob, situated at distant locations share \(n\) copies of a mixed entangled quantum state \(\rho\), which they may have obtained by sending one part of a pure maximally entangled state through a noisy quantum channel. Assume that both parties are able to perform any collective quantum operation, which merely acts locally on their part of the \(n\) copies. Moreover, Alice and Bob are connected via a classical channel, such that they can perform arbitrary many rounds of local quantum operations, where each round may depend on the measurement outcomes of all the preceding operations on both sides. The set of operations accessible in this way is called LOCC (local operations and classical communication).

It was shown in one of the seminal works from the early years of quantum information theory [2] that under the above conditions Alice and Bob can for certain two-qubit states distill a smaller number \(m\) of states \(\rho'\), which are closer to the maximally entangled state than the initial ones, from a larger number \(n \geq m\) of weaker entangled states \(\rho\). This can be done in such a way that in the limit \(n \to \infty\) the output states \(\rho'\) become maximally entangled. The asymptotic ratio \(m/n\) is then called the rate of the distillation protocol and the maximally accessible rate under all LOCC protocols is an important measure of entanglement, the distillable entanglement \(D(\rho)\). The latter quantifies in some sense the amount of useful entanglement contained in the state \(\rho\).

Up to now the hashing/breeding distillation protocol presented in [2, 3], which is adapted to Bell diagonal states of two qubits, is essentially the only protocol leading to a non-zero rate in the asymptotic limit.

The present paper is devoted to generalize these protocols to higher dimensions. We will thereby closely follow the ideas of Ref.[2, 3]. The protocols are adapted to states which are diagonal in a basis of maximally entangled states or mapped onto such states by an LOCC twirl operation. The main step is to translate the apparent quantum task into a classical problem in such a way that all the operations involved are LOCC. The following results are obtained:

- We provide an entanglement assisted distillation protocol (breeding) that works for any finite dimension \(d\). It is in this case assumed that Alice and Bob share maximally entangled states initially, which they may have obtained with a different protocol (e.g. hashing in a prime dimensional subspace) and with a smaller rate. The attained distillation rate is given by
  \[
  \log_2 d - S(\mathcal{T}(\rho)),
  \]
  where \(S(\mathcal{T}(\rho))\) is the von Neumann entropy of the twirled state \(\rho\).

- We show that a hashing protocol exists for prime dimensions \(d\), which does not require any predistilled states. The obtained rate is the same as in Eq. (1). The method can easily be generalized to dimensions which are powers of primes, if one uses a different basis of maximally entangled states.

- Both protocols, breeding and hashing, are shown to be optimal for a \((d−1)\) parameter family of low rank states. This generalizes the observation of Rains [4] for the case \(d = 2\).

- We discuss methods for improving the rates for isotropic states. In particular the projection onto local subspaces will turn out to yield the optimal rate in the limit of large dimensions.

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Despite the practical relevance and quite considerable effort in the theory of entanglement distillation, many of the basic questions are yet unanswered. So do we neither know a decidable necessary and sufficient criterion for distillability in Hilbert spaces $\mathbb{C}^d \otimes \mathbb{C}^d$ with $d > 2$, nor do we know $D(\rho)$ even for otherwise simple quantum states $\rho$. What makes the investigation of distillability so difficult is on the one hand the asymptotic limit ($n \to \infty$) and on the other hand the mathematically intractable set of LOCC operations.

However, many partial results have been obtained in various directions, and before we go into the details of the present article we want at least to briefly recall some of them:

a. Distillability: For the case of two qubits it was shown in [2] that every entangled state is distillable. A general necessary condition for the distillability of a state $\rho$ is the fact that its partial transpose $\rho^{T_A}$, defined with respect to a given product basis by $(ij)|\rho^{T_A}|kl) = (kj)|\rho|il)$, has a negative eigenvalue [5]. Except for special cases like states on $\mathbb{C}^2 \otimes \mathbb{C}^2$ [4] and Gaussian states [8] it is however unclear whether this condition is sufficient as well. There is some evidence presented in [7, 9] that this may not be the case and that there are indeed undistillable states, whose partial transpose is not positive (NPPT). However, it was shown in [10] (see also [11]) that every NPPT state becomes distillable when adding a certain bound entangled state with positive partial transpose (PPT). Moreover, this activation process was shown to require only an infinitesimal amount of entanglement contained in the additional PPT state [12].

b. Distillation protocols: Up to now, essentially the only mixed state distillation protocol leading to a non-zero rate in the asymptotic limit is the breeding/hashing protocol presented in [2, 3], which is adapted to Bell-diagonal states of two qubits. There are, however, several purification schemes which increase the purity and entanglement of a state without yielding a non-zero rate on their own. The best known examples are the recurrence protocols presented in [3, 6] (see [13] for the discussion of imperfect operations) and [14]. These were investigated also for higher dimensions in [15] and [16]. In [17] it was shown that these methods can be improved by using more than one source state per target. A systematic generalization of the local unitaries applied in the recurrence method for two qubit systems has recently been provided in [18]. A protocol which is analogous to the hashing procedure but uses PPT preserving rather than LOCC operations is described in [19].

c. Distillable Entanglement: For pure states the distillable entanglement is equal to the von Neumann entropy of the reduced state [20]. For mixed states only bounds are known, which can be calculated for some very special cases (cf. [21]). The best known upper bound for the distillable entanglement can be found in [22]. A closely related bound, which we will describe in detail later, is given by the relative entropy of entanglement [4, 23].

d. Relations to other Quantum information tasks: It was already noticed in [3] that a distillation protocol involving only one-way classical communication (like the hashing protocol) corresponds to a quantum error correcting code and vice versa. Moreover, upper bounds for the distillable entanglement allow one to obtain upper bounds for quantum channel capacities (cf. [24]). In particular, the inequality

$$D_1(\rho) \geq S(\text{tr}_B[\rho]) - S(\rho),$$

where $D_1(\rho)$ is the distillable entanglement for one-way communication protocols, implies Shannon-like formulas for quantum capacities, providing the quantum noisy coding theorem [24]. The inequality is consistent with the results we obtain in Sec. V.

An important application of distillation and purification schemes is also the possibility of factoring out an eavesdropper for secret quantum communication [14, 25]. This is known as Quantum privacy amplification.

II. PRELIMINARIES

Let us first introduce the preliminaries we need for the distillation protocols described in Sec. II and IV. We begin with characterizing the states for which the protocols are adapted and we will subsequently specify the set of required local operations.

A. Basis of maximally entangled states

Bases of maximally entangled states $\{|\Psi_{kl}\rangle, k, l \in \{0, \ldots, d - 1\} \}$ exist for any Hilbert space $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$ and correspond to an orthonormal operator basis of $d^2$ unitary $d \times d$ matrices $\{U_{kl}\}$ via

$$|\Psi_{kl}\rangle \equiv (1 \otimes U_{kl})|\Omega\rangle,$$

with $|\Omega\rangle$ being a maximally entangled state. The latter is in the following chosen to be $|\Omega\rangle = (|\Psi_{00}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle$. A general construction procedure [26] for unitary operator bases involves Latin squares and complex Hadamard matrices and the best known example constructed in this way is of the form

$$U_{kl} = \sum_{r=0}^{d-1} \eta^r |k \oplus r\rangle \langle r|, \quad \eta = \frac{2}{d^2}$$

where $\oplus$ means addition modulo $d$. The set of unitaries in [30] is orthonormal with respect to the Hilbert-Schmidt scalar product, i.e., $\text{tr} [U_{ij}^* U_{kl}] = d\delta_{ik}\delta_{jl}$, and forms a discrete Weyl system since $U_{ij}U_{kl} = \eta^{ik}U_{i\oplus k,j\oplus l}$. In the following we will solely use the maximally entangled basis from Eqs. (1.40) or tensor products thereof and denote the first index as the shift and the second as the phase index.
B. Symmetric states

Symmetric states commuting with a group of local unitaries play an important and paradigmatic role in Quantum information theory and in particular in the context of entanglement distillation. The best known examples are Werner states, Isotropic states and Bell diagonal states. The latter are convex combinations of the four maximally entangled Bell states of two qubits and contain in this case the Isotropic states, which are for two qubit systems in turn equivalent to Werner states.

In analogy to the two qubit case, we will call a state on $\mathbb{C}^d \otimes \mathbb{C}^d$ Bell diagonal if it can be written as a convex combination of maximally entangled states $P_{ij} = |\Psi_{ij}\rangle\langle\Psi_{ij}|$. The corresponding symmetry group is given by

$$G = \{U_{i,j} \otimes U_{i,-j}\},$$

which is an abelian group with the property that its commutant, which is again spanned by $G$, contains $d^2$ one-dimensional projectors $P_{ij}$. Hence, for $\{\lambda_{ij}\}$ being convex weights:

$$\rho = \sum_{ij} \lambda_{ij} P_{ij} \iff \forall g \in G : [\rho, g] = 0. \quad (6)$$

Moreover, every not symmetric state $\rho$ can be mapped onto a Bell diagonal state $T(\rho)$ by means of a discrete twirl operation

$$T(\rho) = \frac{1}{d^2} \sum_{g \in G} g^* \rho g,$$

which can be implemented by means of local operations and classical communication (LOCC).

Isotropic states, which are completely characterized by their fidelity $f := \langle \Omega | \rho | \Omega \rangle$, are a special instance of Bell diagonal states with $\lambda_{00} = f$ and $\lambda_{ij} = (1 - f)/(d^2 - 1)$ for $(i, j) \neq (0, 0)$. An Isotropic state is entangled and distillable iff $f > \frac{1}{2}$ [3].

C. Local operations and measurements

An essential ingredient for any distillation protocol acting on two qubit systems is the CNOT operation. One possibility of generalizing this operation to higher dimensions is to use a controlled shift operation (CS) [4], which acts as

$$C|j\rangle \otimes |i\rangle = |i\rangle \otimes |j \oplus i\rangle,$$

where the first tensor factor is the source and the second the target. It is readily verified that a bilateral controlled shift operation (BCS), where both parties in a bipartite system apply CS operations locally on a tensor product of two maximally entangled states, acts as

$$(C \otimes C)|\Psi_{ij}\rangle \otimes |\Psi_{kl}\rangle = |\Psi_{i,j \oplus l}\rangle \otimes |\Psi_{k \oplus s,i,l}\rangle,$$  

where the first tensor product on the l.h.s. in [4] corresponds to the Alice|Bob split, whereas the others correspond to the source|target split. Note that the target state picks up the shift index of the source, i.e., $k \mapsto k \oplus i$, while the source remains unchanged iff $l = 0$.

If we measure a target state of the form $|\Psi_{kl}\rangle$ in computational basis, then $k$ is the difference of the measurement outcomes. In this way we can obtain information about the shift index of an unknown source state, if we know the target state: We first apply a BCS operation and then measure the target state in computational basis.

An analogous transformation can be defined for the phase indices, since we can simply interchange the two indices by the following local unitary operation:

$$\left(V \otimes \overline{V}\right) P_{ij} \left(V \otimes \overline{V}\right)^* = P_{j,-i}, \quad V = \frac{1}{\sqrt{d}} \sum_{k,l} \eta^{kl} |k\rangle\langle l|.$$  

So we can define a modified bilateral controlled shift operation (mBCS) by first applying a $V \otimes \overline{V}$ transformation to the source state, followed by a BSC operation on source and target and then undoing the local operation on the source state by a $(V \otimes \overline{V})^*$ rotation. The mBCS operation then acts as

$$P_{ij} \otimes P_{kl} \mapsto P_{i \oplus l,j} \otimes P_{k \oplus j, l},$$

where the second tensor factor corresponds to the target again.

Assume now we have one target state $|\Psi_{00}\rangle$ and $n$ source states with shift indices $k_1, \ldots, k_n$ and phase indices $i_1, \ldots, i_n$. We now apply $s_a$ BCS operations and $p_a$ mBCS operations to the $a$-th source and the target. If we then measure the target state again in computational basis, the difference of the measurement outcomes will be

$$\left(\bigoplus_{a=1}^n k_a s_a\right) \oplus \left(\bigoplus_{a=1}^n i_a p_a\right) =: \langle \vec{k}, \vec{s}\rangle \oplus \langle \vec{i}, \vec{p}\rangle.$$  

This enables us to obtain some information about the distribution of the shift and phase indices of the source states without disturbing them.

In the following we will organize the shift and phase indices of a sequence of maximally entangled states in a single vector $\vec{S} := (k_1, \ldots, k_n, i_1, \ldots, i_n)$ and accordingly the multiplicities of the BCS and mBCS operations in a vector $\vec{M} := (s_1, \ldots, s_n, p_1, \ldots, p_n)$. Both $\vec{S}$ and $\vec{M}$ then belong to $\{0, \ldots, d-1\}^{2n}$, i.e., they are elements of the group $\mathbb{Z}_2^n$ with addition modulo $d$.

The sequence of local operations characterized by $\vec{M}$ followed by a measurement of an additionally required target state $P_{00}$ thus yields the information

$$\langle \vec{M}, \vec{S}\rangle := \bigoplus_{i=1}^{2n} M_i S_i = \langle \vec{k}, \vec{s}\rangle \oplus \langle \vec{i}, \vec{p}\rangle$$

about a sequence $\vec{S}$ of maximally entangled states, without disturbing the latter.
To complete the list of later on required basic local operations we have still to introduce a generalization of a $\pi/2$ rotation given by

$$u(g) := \sum_k e^{-\frac{i\pi k^2}{d}g} |k\rangle \langle k|,$$  \hspace{1cm} (14)

where $g \in \mathbb{Z}$ is an arbitrary number. We will use this within bilateral operations of the form $u(g) \otimes u(g)$ and $v(g) \otimes v(g)$, where $v(g) := V^* u(g) V$. These unitary operations act on a maximally entangled state $P_{kl}$ as

$$[u(g) \otimes u(g)] P_{kl} [u(g) \otimes u(g)]^* = P_{k,t \otimes g k},$$  \hspace{1cm} (15)

$$[v(g) \otimes v(g)] P_{kl} [v(g) \otimes v(g)]^* = P_{k \otimes g t,l}.$$  \hspace{1cm} (16)

D. Why primes are special

Measurements based on the $\mathbb{Z}_{d}^{2n}$ scalar product in Eq.(13) will play a crucial role in the entanglement distillation protocols discussed in the sequel. One of the main tasks will thereby be to choose the measurement vector $\vec{M}$ such that we obtain as much information about the sequence $\vec{S}$ as possible. In general this can be a highly non-trivial problem. However, if the dimension $d$ is a prime, then choosing $\vec{M}$ randomly turns out to be a pretty good choice. The reason why this works well only for prime dimensions is that in this case $\mathbb{Z}_{d}$ is a field, which means that it is not only an abelian group with respect to the addition modulo $d$, but also with respect to the multiplication if we exclude the zero element. That is, for $a, b, x \in \mathbb{Z}_{d}$, $a \neq 0$ every equation of the form

$$b = a \cdot x \mod d$$  \hspace{1cm} (17)

has a unique solution for $x$ if and only if $d$ is prime. This leads us to the following lemma:

**Lemma 1** Let $\vec{S} \neq \vec{S}'$ be elements of $\mathbb{Z}_{d}^{n}$ with $d$ prime. Given a scalar product $\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{m} x_{i} y_{i}$ and a uniformly distributed random vector $\vec{M} \in \mathbb{Z}_{d}^{m}$, the probability for $\langle \vec{M}, \vec{S} \rangle = \langle \vec{M}, \vec{S}' \rangle$ is equal to $1/d$.

**Proof:** We ask for the probability that $\langle \vec{S} - \vec{S}', \vec{M} \rangle = 0$. Since $\vec{S} \neq \vec{S}'$, there exists a component $x$ where $S_{x} \neq S'_{x}$ and we can write

$$0 \equiv \langle \vec{S} - \vec{S}', \vec{M} \rangle = M_{x} (S_{x} - S'_{x}) + \sum_{i \neq x} M_{i} (S_{i} - S'_{i}).$$  \hspace{1cm} (18)

Assume now that all $M_{i}, i \neq x$ are already randomly chosen. Then Eq.(18) has the form of Eq.(17) with $x = M_{x}$, $a = (S_{x} - S'_{x})$ and $b = \sum_{i \neq x} M_{i} (S'_{i} - S_{i})$. Since this equation has a unique solution for $x$ and $M_{x}$ is a uniformly distributed random variable, the probability that $M_{x}$ matches the solution is indeed $\frac{1}{d}$.

III. THE BREEDING PROTOCOL

The breeding protocol is the preliminary stage of the hashing protocol discussed in the next section. Both are adapted to the distillation of Bell diagonal states and the main idea is to transform this quantum problem into the classical problem of identifying a word given the probability distribution of the respective alphabet and a restricted set of measurements.

Assume that Alice and Bob share $n$ copies of a Bell diagonal state $\rho$, such that

$$\rho^{\otimes n} = \sum_{k_{1}...k_{n}, l_{1}...l_{n}} \lambda_{k_{1}l_{1}} \cdots \lambda_{k_{n}l_{n}} P_{k_{1}l_{1}} \otimes \cdots \otimes P_{k_{n}l_{n}}.$$  \hspace{1cm} (19)

An appropriate interpretation of Eq.(19) is to say that Alice and Bob share the state

$$P_{k_{1}l_{1}} \otimes \cdots \otimes P_{k_{n}l_{n}} := P_{\vec{S}}$$  \hspace{1cm} (20)

with probability $\lambda_{k_{1}l_{1}} \cdots \lambda_{k_{n}l_{n}}$. Note that if they knew the sequence $\vec{S} = (k_{1}, \ldots, k_{n}, l_{1}, \ldots, l_{n})$, they could apply appropriate local unitary operations in order to obtain the standard maximally entangled state $P_{00}^{\otimes n}$ and thus gain $n \log_{2} d$ ebits of entanglement.

Let us further assume that they have already a sufficiently large set of predistilled maximally entangled states $P_{00}$ and that they are able to perform all the local operations such as BCS and mBCS operations described in the previous section. In this case they can utilize the predistilled states for performing local measurements leading to the result $\langle \vec{M}, \vec{S} \rangle$ as discussed in Eq.(13) in order to obtain some information about the sequence $\vec{S}$. However, every single measurement $\vec{M}$ will destroy one of the predistilled maximally entangled states, and the task is therefore to identify the sequence $\vec{S}$ using as few measurements as possible.

At this point the quantum task of distillation is transformed into an entirely classical problem: Given an unknown vector $\vec{S}$ and the possibility of measuring functions of the form $\langle \vec{M}, \vec{S} \rangle$, how many measurements $r$ do we need to identify $\vec{S}$? Knowing $\vec{S}$ we gain $n$ maximally entangled pairs, but destroyed $r$, such that the overall gain would be $(1 - r/n) \log_{2} d$ ebits per copy.

We would need $r = 2n$ measurements to identify a completely random $\vec{S} \in \mathbb{Z}_{d}^{2n}$ with independently and uniformly distributed components. So we would destroy more entangled pairs than we gain.

Fortunately, however, the set of possible vectors $\vec{S}$ is not completely random but has a distribution depending on the spectrum of the Bell diagonal state $\rho$. In the limit of many copies there exists thus a set of *likely sequences* containing only $2^{n S(\rho)}$ different vectors $\vec{S}$ with the property that the probability that a vector $\vec{S}$ is contained within this set of likely sequences approaches one in the limit $n \to \infty$ [27]. Here $S(\rho)$ is the von Neumann entropy of the density matrix, which is equal to the Shannon entropy...
entropy of the spectrum of $\rho$, i.e.
\[ S(\rho) = - \sum_{k,l} \lambda_{kl} \log_2 \lambda_{kl}. \quad (21) \]

Now let Alice and Bob choose each measurement $\hat{M}$ randomly. Then Lemma 1 tells us that if $d$ is prime, they can reduce their list of possible sequences by a factor of $\frac{1}{d}$ after every measurement by deleting every sequence, which is not consistent with the obtained result. Since their list initially contains $2^n S(\rho)$ possible sequences, $r = n S(\rho)/\log_2 d$ measurements are required in order to identify $\vec{S}$ in the limit of large $n$. The rate obtained in this way in the asymptotic limit is a lower bound to the (entanglement assisted) distillable entanglement which is thus
\[ D(\rho) \geq \log_2 d - S(\rho). \quad (22) \]

Let us now extend the obtained result, expressed in Eq.\,(22), from prime dimensions to arbitrary ones. As we have already mentioned, Lemma 1 holds only for primes and cannot be extended. A way out of this dilemma is to use controlled shift operations between Hilbert spaces of different dimensions. The definition looks similar to Eq.\,(23):
\[ C' |i\rangle \otimes |j\rangle := |i\rangle \otimes |j \oplus i\rangle, \quad (23) \]
with the only difference that $i$ runs from 0 to $d-1$ whereas $j$ runs from 0 to $d' - 1$, where $d'$ is now supposed to be a prime with $d' \geq d$. Moreover, the addition in the second ket (target) is modulo $d'$.

The penalty of the BCS and mBCS operations based on a controlled shift operation is, that the source states after the operation will in general no longer be of the form $P_{kl}$ given by Eqs.\,(34). However, they still are, if the target state, acting on $\mathbb{C}^{d'} \otimes \mathbb{C}^{d'}$, has a phase index equal to zero. Then:
\[ (C' \otimes C') |\Psi_{ij}\rangle \otimes |\Psi_{kl}\rangle = |\Psi_{i,j}\rangle \otimes |\Psi'_{k \oplus i,0}\rangle. \quad (24) \]

In this way we can straightforwardly generalize the result of Eq.\,(22) to Bell diagonal states of arbitrary dimension, just by reshuffling the degrees of freedom of the predistilled maximally entangled states. Hence, the crucial point is, that the target states, which are in the case of the breeding protocol the predistilled resources, have prime dimensions.

IV. THE HASHING PROTOCOL

So far we have assumed that Alice and Bob initially share a set of predistilled maximally entangled states. We took into account this resource when calculating the rate, but since the additivity of the distillable entanglement is not yet decided for bipartite systems, we might expect a smaller rate for the (non entanglement assisted) distillation rate. The hashing protocol, however, achieves the same asymptotic rate as the previously discussed breeding protocol without requiring additional predistilled states. The main idea is though still the same: Alice and Bob identify the sequence $\vec{S}$ by means of LOCC operations.

The protocol, however, is now a bit more complicated since if we use one of the $n$ states characterized by $\vec{S} \in \mathbb{Z}_d^n$ as target for the BCS and mBCS operations then we have to take into account the backaction from the unknown target to the remaining source states.

We will in the following choose the $n$-th state characterized by $S_n$ and $S_{2n}$ as the target and consider the remaining $n-1$ states characterized by a vector $\vec{R} = (S_1, \ldots, S_{n-1}, S_{n+1}, \ldots, S_{2n-1})$ as source states.

In the first step of the protocol a bilateral $v(g) \otimes \bar{v}(g)$ rotation with a randomly chosen number $g \in \{0, \ldots, d-1\}$ is applied to the target state. This shifts some information about the phase index of the target state into its shift index.

Then a random vector $\vec{M} \in \mathbb{Z}_d^{2n-2}$ is chosen and the two parties perform the respective BCS and mBCS operations. After measuring the target state, the difference of the measurement outcomes, i.e., the final shift index of the target, will be
\[ \langle \vec{M}; g | \vec{S} \rangle := \langle \vec{M}; \vec{R} \rangle - \sum_{i=1}^{n-1} M_i M_{i+n-1} S_{2n} + S_n + g S_{2n}. \quad (25) \]

The first term $\langle \vec{M}; \vec{R} \rangle$ in Eq.\,(23) is the familiar outcome if the target state is $P_{00}$. The terms $- \sum_{i=1}^{n-1} M_i M_{i+n-1} S_{2n} + S_n$ correspond to the other possible targets and their backaction onto the source states, and the last term $g S_{2n}$ is due to the initial $v(g) \otimes \bar{v}(g)$ rotation.

Note that in general the outcome also depends on the order in which the different BCS and mBCS operations are carried out. For Eq.\,(23) first all BCS and then all mBCS operations were applied.

With the possibility of measuring functions of the form \[ \langle \vec{M}; g | \vec{S} \rangle \] we will now follow the line of argumentation of the previous section and show that the gain of information is (asymptotically) at least $\log_d g$ bits per measurement, if the dimension $d$ is prime. Therefore we ask again for the probability that two different sequences $\vec{S} \neq \vec{S}'$ lead to the same measurement outcome under the assumption that $\vec{M}$ and $g$ are uniformly distributed random variables.

To this end we have to distinguish between three different cases:

1. The two sequences only differ in $S_n \neq S'_n$: This is the trivial case where $\vec{S}$ and $\vec{S}'$ lead to different measurement outcomes with unit probability.

2. $\vec{S}$ and $\vec{S}'$ differ in the phase index of their target, i.e., $S_{2n} \neq S'_{2n}$: If we assume in this case that $\vec{M}$ is already chosen, then the equation $\langle \vec{M}; g | \vec{S} \rangle = \langle \vec{M}; g | \vec{S}' \rangle$
\[ \langle \vec{M}; g|\vec{S'} \rangle \text{ is of the form } b = a \cdot x \text{ with } g \text{ playing the role of } x. \] Hence, the probability that a randomly chosen \( g \) matches the right solution is \( \frac{1}{d} \), if \( d \) is prime.

3. \( \vec{S} \) and \( \vec{S'} \) have a target state with the same phase index, i.e. \( S_{2n} = S'_{2n} \) and \( \vec{R} \neq \vec{R'} \). Then Lemma 1 tells us, that a randomly chosen \( \vec{M} \) distinguishes the two sequences \( \vec{R} \neq \vec{R'} \) with probability \( \frac{1}{d} \), if \( d \) is prime.

The hashing protocol consists now of several rounds of such measurements, where each round destroys the entanglement of one of the \( n \) pairs characterized by \( \vec{S} \). The relevant part of the system after \( r \) rounds is thus described by a vector \( \vec{S}(r) \in \mathbb{Z}_d^{2(n-r)} \) and the measurements applied to \( \vec{S}(r) \) are in turn characterized by \( \vec{M}(r) \in \mathbb{Z}_d^{2(n-r-2)} \) and \( g(r) \in \mathbb{Z}_d \). Note that given the sequence of measurements, the vector \( \vec{S}(r) \) deterministically depends on \( \vec{S} = \vec{S}(0) \).

The above discussion implies now, that the probability

\[ P \left[ \vec{S}(r) \neq \vec{S}(r') \wedge \forall r^{-1}_{k=0} : (\vec{M}(r); g(r))|\vec{S}(r) - \vec{S}(r') \rangle \rangle = 0 \right] \] \hspace{1cm} (26)

that \( \vec{S}(r) \) and \( \vec{S}(r') \) have agreed on all \( r \) measurement results and thereby remain distinct, is at most \( d^{-r} \), if \( d \) is prime. Since we have initially again about \( 2^{n-S(\rho)} \) likely sequences, we need (in the limit of large \( n \)) \( r = nS(\rho)/\log_2 d \) rounds in order to identify the remaining sequence \( \vec{S}(r) \). This leads us again to the rate in Eq.(22).

What if \( d \) is not prime? Unfortunately, we cannot extend the result to other dimensions in the way we did in the case of the breeding protocol, since we do not have any target states of prime dimension with zero phase indices. However, if the dimension is a power of a prime \( d = d^p \), then we can easily extend the result for states which are diagonal with respect to a different basis of maximally entangled states, given by:

\[ P_{k_1 \ldots k_p} := \bigotimes_{i=1}^p P_{k_i} = \rho \] \hspace{1cm} (27)

where \( P_{k_i} \) is a maximally entangled state acting on \( \mathbb{C}^d \otimes \mathbb{C}^d \). The (LOCC) twirl operation, which maps an arbitrary state onto a state diagonal in this basis is given by \( \rho \mapsto T \otimes \rho \).

The hashing protocol can then be applied in the described manner to \( p n \) tensor factors of prime dimension \( d' \) and we yield in this way again the rate in Eq.(22) for states in \( d = d^p \) dimensions. Moreover, note that isotropic states are diagonal in both bases, given by Eqs.(23) and Eq.(27).

V. OPTIMALITY FOR LOW RANK STATES

It is well known from the case of two qubit systems that the hashing/breeding protocol is not optimal in general. However, for a certain \( (d-1) \) parameter family of rank deficient states, we will show that the obtained rate is equal to the relative entropy of entanglement. Since the latter is an upper bound to the distillable entanglement, this implies that the protocols are optimal in this case. For \( d = 2 \) this was first noticed by Rains in [4].

Consider mixed states of rank smaller than or equal to \( d \)

\[ \rho_\mu = \sum_{l=1}^d \mu_l |\phi_l \rangle \langle \phi_l |, \] \hspace{1cm} (28)

which are diagonal with respect to

\[ |\phi_l \rangle := |\tilde{\Psi}_{0,l} \rangle = \frac{1}{\sqrt{d}} \sum_{r=0}^{d-1} e^{\frac{2\pi i}{d} r l} |rr \rangle \] \hspace{1cm} (29)

The set of these states forms a simplex in \( \mathbb{R}^{d-1} \) and has the property that every state \( \rho_\mu \) except for the barycenter, for which \( \mu_1 = \frac{1}{d} \), is entangled. To see this, note first that max_{\{\mu_l\}} \rho_\mu is the fully entangled fraction. That is, if not all \( \mu_l \) are equal, then the fully entangled fraction is larger than \( \frac{1}{d} \) and the respective state is thus entangled. On the other hand, if \( \mu_1 = \frac{1}{d} \), then

\[ \rho_\mu = \frac{1}{d} \sum_{l=1}^d |\phi_l \rangle \langle \phi_l | \] \hspace{1cm} (30)

\[ = \frac{1}{d^2} \sum_{l,r,s} \exp \left[ \frac{2\pi i}{d} (l - r) \right] |r,r \rangle \langle s,s | \] \hspace{1cm} (31)

\[ = \frac{1}{d} \sum_{r=0}^{d-1} |r \rangle \langle r | \otimes |r \rangle \langle r | \] \hspace{1cm} (32)

is evidently a separable state, which is said to be maximally correlated. We will denote this state by \( \rho_{sep} \) in the sequel.

An upper bound to the distillable entanglement is given by the relative entropy of entanglement [23], which is defined by

\[ E_R(\rho) = \inf_{\sigma} \left( \text{tr} \left[ \rho (\log_2 \rho - \log_2 \sigma) \right] \right), \] \hspace{1cm} (33)

where the infimum is taken over all separable states \( \sigma \). Hence, if we choose \( \sigma = \rho_{sep} \), the distillable entanglement is bounded by

\[ D(\rho_\mu) \leq E_R(\rho_\mu) \leq - \text{tr} \left[ \rho \log_2 \rho_{sep} \right] - S(\rho_\mu). \] \hspace{1cm} (34)

However, \( - \text{tr} [ \rho \log_2 \rho_{sep} ] = \log_2 d \) and the rate achieved by the breeding/hashing protocol is thus optimal for every \( \rho_\mu \). Strictly speaking, the hashing protocol leads to the distillable entanglement and the breeding rate is equal to the distillable entanglement \( D' \) assisted by a
maximally entangled resource $\omega$. The latter can easily be seen by noting that
\begin{align}
D'(\rho) &= D(\rho \otimes \omega) - E_R(\omega) \\
&\leq E_R(\rho \otimes \omega) - E_R(\omega) \leq E_R(\rho).
\end{align}
(35) (36)

That is, the relative entropy of entanglement is an upper bound for the entanglement assisted distillable entanglement as well.

VI. IMPROVING THE RATES FOR ISOTROPIC STATES

For a general Bell diagonal state the introduced protocols are not optimal. In particular for states with a large entropy the rate is poor or even zero. In order to improve this, one can make use of hybrid protocols, where the first step decreases the entropy while conserving most of the entanglement, and in a second step the hashing/breeding protocol is applied. There are many ways of performing such an entropy decreasing preprocessing. We will in the following discuss two of them for the case of Isotropic states: The recurrence method which is well known for qubits $[3, 4]$ and has already been investigated by $[5, 10]$ for $d > 2$, and the projection onto local subspaces. For the latter we will show that in the limit of large dimensions the rate approaches the relative entropy of entanglement. Hence, the achieved rate is optimal in that limit.

Isotropic states, for which we will discuss both methods in more detail, have the form
\begin{equation}
\rho = f|\Omega\rangle\langle\Omega| + \frac{1 - f}{d^2 - 1}(1 - |\Omega\rangle\langle\Omega|). \tag{37}
\end{equation}

That is they are depolarized maximally entangled states depending on a single fidelity parameter $f \in [0, 1]$. The twirl operation mapping every state onto an isotropic state has the form $\rho \mapsto \int dU(U \otimes U^*)\rho(U \otimes U)^*$, i.e. it is an averaging over the unitary group $U(d)$ with respect to the Haar measure $dU$. The states in Eq. (37) are entangled (and distillable) iff $f > \frac{1}{d^2 - 1}$. [5]

A. The recurrence method

The recurrence method for higher dimensions and variations thereof have already been discussed in detail in $[3, 4]$ and $[5]$. For completeness, however, we will recall the main steps. The idea is to apply a BCS operation on two copies of the state and then to measure the target state in computational basis. The source states are kept whenever the measurement outcomes coincide, otherwise they are discarded. The remaining states may then be twirled onto isotropic states again and we can proceed either with iterating the recurrence method or applying the hashing/breeding procedure if the entropy of the remaining states is already sufficiently small, i.e. $S(\rho) < \log_2 d$. We note that, as it was already mentioned in [4] for $d = 2$, the Isotropic twirling after each step is not really necessary and in general increases the entropy and therefore decreases the rate of a subsequently applied breeding/hashing protocol. However, it is sufficient to look at the fidelity parameter $f$ in order to see that a recurrence preprocessing improves the rates such that every entangled Isotropic state becomes distillable.

Straightforward calculation shows that the fidelity $f'$ after applying the recurrence method once is given by
\begin{equation}
|f'|^2 = 1 + f\left[df(d^2 + d - 1) - 2\right] \over d^3 f^2 - 2df + d^2 + d - 1, \tag{38}
\end{equation}
and the probability for equal measurement outcomes is
\begin{equation}
p_{ee} = \frac{d^3 f^2 - 2df + d^2 + d - 1}{(d + 1)^2 d(d - 1)}. \tag{39}
\end{equation}

Note that the recurrence method alone does not lead to a non-zero rate since in every round we destroy or discard at least half of the resources (all the target states) and maximally entangled states are only obtained in the limit of infinitely many rounds. However, it holds that $f' > f$ and therefore $S(\rho') < S(\rho)$ for every entangled isotropic $\rho$. This means that every entangled isotropic state can be distilled with a non-zero rate by applying sufficiently many rounds of the recurrence protocol followed by hashing/breeding.

There are many directions in which the recurrence method can be modified or improved. One could use more than one source state [7], a different bilateral operation [8], neglect the Isotropic twirling [14], apply it to other kinds of states [16] or use target and source states with different fidelities.

In Fig.[8] we have applied the method as described above to an isotropic state of dimension $d = 7$. As expected this leads to an improvement of the rate in the region where $S(\rho) \sim \log_2 d$.

B. Projecting onto local subspaces

A surprisingly efficient method for improving the distillation rate is to project the initial state locally onto blocks of smaller dimensions and to apply the hashing/breeding protocol to these subspaces. Let $\{Q_i = \sum_k |k\rangle\langle k|\}$ be a set of $b$ projectors of dimension $q_i$ corresponding to a local projective observable with $b$ outcomes, i.e. $\sum_{i=1}^b q_i = d$ and $\sum_{i=1}^b Q_i = 1_d$. Assume further that Alice and Bob have the same observable $\{Q^A_i\} = \{Q^B_i\}$ and that they apply the respective measurement to an isotropic state $\rho$. Again the state is kept if the results of the measurements coincide and discarded otherwise. If both parties obtain the same measurement outcome $i$, which happens with a probability
\begin{equation}
p_i = \frac{q_i}{d} f + \frac{(1 - f)}{(d^2 - 1)} \left(q_i^2 - \frac{q_i}{d}\right), \tag{40}
\end{equation}

the state after the measurement \( \rho_i = (Q_i^A \otimes Q_i^B)\rho(Q_i^A \otimes Q_i^B)^* / p_i \) is again an isotropic state on \( \mathbb{C}^f \otimes \mathbb{C}^f \) with fidelity
\[
    f_i = \left[ f \frac{q_i}{d} + \frac{(1 - f)}{(d^2 - 1)(1 - q_i/d)} \right] / p_i. \tag{41}
\]
If we now apply the hashing/breeding protocol to the \( b \) blocks, the obtained rate
\[
    \sum_{i=1}^{b} p_i \max \left[ 0, \log q_i - H(\rho_i) \right] \tag{42}
\]
exceeds \( \log d - H(\rho) \) for some values of \( f \) depending on the dimensions \( \{q_i\} \) of the subspaces. If the dimensions of the blocks are about the same, then an increasing number of blocks leads to a larger (smaller) rate for small (large) \( f \). Fig.1 and Fig.2 show the obtained rates for \( d = 7 \) and the case where the two parties share 10 pairs of qubits and the overall state is isotropic. For the latter case the rate is already not too far below the relative entropy of entanglement. In fact, the content of the following subsection is to sketch that in the limit of large dimensions \( d \to \infty \) the relative distance between these quantities vanishes for all \( f \).

\section{C. The limit of large dimensions}

As we have already mentioned, an upper bound for the distillable entanglement is the \textit{relative entropy of entanglement}, i.e. the minimal distance between \( \rho \) and a separable state \( \sigma \) measured in terms of the relative entropy
\[
    S(\rho, \sigma) = \text{tr} [\rho \log \rho] - \text{tr} [\rho \log \sigma]. \tag{43}
\]
For entangled isotropic states the nearest separable state is the isotropic state with \( f = 1 / d \), and the relative entropy of entanglement is thus
\[
    E_R(\rho) = \log_2 d - (1 - f) \log_2 (d - 1) - S(f, 1 - f), \tag{44}
\]
for \( f > 1 / d \) and zero otherwise, where \( S \) is again the Shannon entropy. Note that in the limit of large dimensions this becomes after normalization
\[
    \lim_{d \to \infty} \frac{E_R(\rho)}{\log_2 d} = f. \tag{45}
\]
The same limit, however, appears for the distillation rate obtained with the block projection method described in the previous subsection.

Assume that \( d = 2^m \), i.e. a single state \( \rho \) consists of \( m \) pairs of qubits, and let the local measurements have \( m \) outcomes corresponding to subspaces of equal dimensions \( q_i = 2^m / m \). We can then estimate the probability of success and the achieved fidelity (for \( m \geq (1 - f)^{-1} \)) by
\[
    p_i \geq \tilde{p} := \frac{f - 4^{-m}}{m}, \tag{46}
\]
\[
    f_i \geq \tilde{f} := \frac{fm}{fm + 1}. \tag{47}
\]
In the limit of large \( m \), the Eqs. (46, 47) tell us that we gain \( m \) (almost) maximally entangled states of dimension \( q = 2^m / m \) with probability \( f/m \). Looking at the normalized entanglement as in Eq. (45) leads then indeed to the same limit as for the relative entropy of entanglement. Hence, we (relatively) approach the distillable entanglement in the limit of large dimensions. However, the main
reason for this is, that the relative entanglement difference of two maximally entangled states in $2^m/m$ resp. $2^m$ dimensions vanishes for $m \to \infty$. Hence, the result is not as deep as it might seem at first glance.

Nevertheless, projecting onto blocks of lower dimensions can significantly improve the rates yet for finite dimensions as shown in Figs.[3][4].

VII. CONCLUSION

Despite considerable efforts in the theory of entanglement distillation, this field of investigation provides many open questions even on a very basic level. Due to the complexity of the underlying variational problem it is hard to find good upper and lower bounds to the distillable entanglement, not to mention an explicit calculation of this measure of "useful entanglement". So far, the most important distillation protocol leading to a non-zero rate, and thus to a non-trivial lower bound to the distillable entanglement, has been adapted to Bell diagonal states of two qubits. Of course, this protocol can also be applied to higher dimensional states by either projecting down to qubits or by interpreting the state as a tensor product of qubits if possible (compare Fig.2). However, both methods will in general discard most of the entanglement.

The present article shows how to generalize the breeding and hashing protocol to higher dimensions. The obtained rates are optimal only for special cases, however, they provide an improved lower bound to the distillable entanglement in general. Both protocols consist out of two steps: first the quantum task is translated to a classical problem and then the latter is solved. The classical part of the protocol is already essentially optimal, however, some information and hence entanglement is lost during the 'translation process'.

We think that the presented results admit further generalizations towards other classes of states and may be also with respect to the restriction of the hashing protocols to dimensions which are (powers of) primes.

We hope that our work initiates further investigations concerning the distillation of entanglement, including explorations of the implications coming from recent works in the field of quantum error correcting codes.

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