Research Article

Weakly Coupled Systems of Semilinear Damped Waves with Different Scale-Invariant Time-Dependent Dissipation Terms

Abdelhamid Mohammed Djaouti

Preparatory Year Deanship, King Faisal University, Hofuf 31982, Al-Ahsa, Saudi Arabia

Correspondence should be addressed to Abdelhamid Mohammed Djaouti; djaouti_abdelhamid@yahoo.fr

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1. Introduction

In this paper, we investigate the following weakly coupled system:

\[ u_{tt} - \Delta u + \frac{\mu_1}{1 + t} u_t = |v|^p, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \]

\[ v_{tt} - \Delta v + \frac{\mu_2}{1 + t} v_t = |u|^q, \quad v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x), \]

(1)

where \((t, x) \in [0, \infty) \times \mathbb{R}^n\) and \(\mu_1, \mu_2 > 1\) are real constants.

Let us first present some history and previous results for problems strongly related to our model. In the last decades, the Cauchy problem for wave equation

\[ u_{tt} - \Delta u = f(t, u), \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \]

(2)

where \((t, x) \in [0, \infty) \times \mathbb{R}^n\), was investigated by a lot of researchers. For \(f(t, u) = |u|^p, \quad p > 1\), the critical exponent describes the threshold between global (in time) existence of small data weak solutions and blow-up of local (in time) small data weak solutions (for more details, the readers can see [1–16]). Let us turn to the study of semilinear damped wave equation with power nonlinearity

\[ u_{tt} - \Delta u + u_t = f(t, u), \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \]

(3)

where \((t, x) \in [0, \infty) \times \mathbb{R}^n\). Matsumura in [17] proved the estimates for the homogeneous problem. Having these estimates, Nakao and Ono in [18] proved the local (in time) existence of energy solutions for \(p \leq p_{GN} = \left(\frac{n}{n-2}\right), \quad (n \geq 3)\), and the global (in time) existence using special technique.

Problem (3) was studied also in [19–24] where the Fujita exponent \(p_{Fuj} = 1 + \left(\frac{2}{n}\right)\) plays a pivotal role as critical exponent. It has been proved in [25] that the Fujita exponent remains critical if we consider in (3) the wave equation with a time-dependent effective damping \(b(t)u_t\), satisfying suitable assumptions. So, the model we have in mind is

\[ u_{tt} - \Delta u + b(t)u_t = f(t, u), \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x). \]

(4)

The dissipation term is effective with respect to classification of Wirth [26] which means that the linear estimates have the same decay rate of the corresponding heat equation \(b(t)u_t - \Delta u = 0\) (for more details about the classification, the readers can see [23, 27–30]). The set of coefficients of effective damping can be presented by functions satisfying the following assumptions:
(i) \( b \) is a positive and monotonic function with 
\[ \lim_{t \to \infty} \frac{b(t)}{t} = 0. \]
(ii) \( ((1 + t)^{2}b(t))^{-1} \in L^{1}(0, \infty). \]
(iii) \( b \in \mathbb{R}^{3}[0, \infty) \) and 
\[ |b^{(k)}(t)| \leq b(t)/(1 + t)^{k} \text{ for } k = 1, 2, 3. \]
(iv) \( 1/b \notin L^{1}(0, \infty) \) and there exists a constant \( a \in [0, 1) \) \( \) such that \( tb^{(1)}(t) \leq ab(t). \)

For the special case \( b(t) = \mu/(1 + t)^{r}, \mu > 0, |r| < 1, \) the global existence has been obtained in [31, 32]. If \( b(t) \) is a sufficiently smooth function satisfying \( \lim \sup_{|t| \to \infty} tb(t) < 1, \) then the dissipation is noneffective [33].

Many papers treated the limit where \( r = 1 \) which means the following semilinear Cauchy problem:

\[
\begin{align*}
\dot{u} & - \Delta u + \frac{\mu}{1+t}u = |u|^{p}, u(0, x) = u_{0}(x), u_{t}(0, x) = u_{1}(x), \\
\dot{v} & - \Delta v + v_{t} = |v|^{q}, v(0, x) = v_{0}(x), v_{t}(0, x) = v_{1}(x),
\end{align*}
\]

where \( (t, x) \in [0, \infty) \times \mathbb{R}^{n}. \) In 2007, Sun and Wang proved in [38] that if

\[
\frac{\max\{p, q\} + 1}{pq - 1} < \frac{n}{2},
\]

for \( n = 1 \) or \( 3, \) then the solution exists globally in time for small initial data, while if \( \frac{\max\{p, q\} + 1}{pq - 1} \geq n/2, \) then every solution having positive average value does not exist globally. In [39], the author generalized the previous results to the case where \( n = 1, 2, 3 \) and improved time decay estimates for \( n = 2. \)

In 2014, using the weighted energy method, Nishihara determined in [32] the critical exponent for any space dimension.

In [40–43], the authors studied the system of weakly coupled semilinear damped waves with time-dependent coefficients in the dissipation terms

\[
\begin{align*}
\dot{u} & - \Delta u + b(t)u_{t} = |v|^{p}, \quad \dot{v} - \Delta v + b(t)v_{t} = |u|^{q}, \\
\end{align*}
\]

\[
\begin{align*}
u(0, x) = u_{0}(x), \quad &u_{t}(0, x) = u_{1}(x), \quad v(0, x) = v_{0}(x), \quad v_{t}(0, x) = v_{1}(x).
\end{align*}
\]

Recently, in [43], the author treated problem (8) with modified nonlinearities.

Let us now turn to our main problem described in (1). In this paper, we will show how the interaction between parameters \( \mu_{1} \) and \( \mu_{2} \) influence the results obtained for the effective case, that is, the change from effective to noneffective.

The paper is organized as follows. We start by some background and previous results for single equation. After that, we will show how the interaction between parameters \( \mu_{1} \) and \( \mu_{2} \) influence the results obtained for the effective case, that is, the change from effective to noneffective.

We introduce for \( s > 0 \) and \( m \in \mathbb{R}^{3} \) the function space

\[
\mathcal{A}_{m,s} := \left( \mathcal{H}^{1} \cap L^{m} \right)^{2} \times \left( \mathcal{H}^{\max\{s-1,0\}} \cap L^{m} \right),
\]

with the norm

\[
\| (u, v) \|_{\mathcal{A}_{m,s}} := \| u \|_{\mathcal{H}^{1}} + \| u \|_{L^{m}} + \| v \|_{\mathcal{H}^{\max\{s-1,0\}}} + \| v \|_{L^{m}}.
\]

In [41], the estimates for the solution to the Cauchy problem

\[
\dot{u} - \Delta u + \frac{\mu}{1+t}u_{t} = 0, \quad u(0, x) = u_{0}(x), \quad u_{t}(0, x) = u_{1}(x)
\]

were proved for different classes of regularity of the data, low regular data, data from energy space, data from Sobolev spaces with suitable regularity, and large regular data. We summarize these results in the following theorems.

**Theorem 1.** Let us assume the data \((u_{0}, u_{1}) \in \mathcal{A}_{m,s} \) with \( s > 0. \) Then, the solution \( u \) to the Cauchy problem (11) satisfies for \( m > 1 \) the following decay estimates.

For \( s \geq 0, \)

\[
\begin{align*}
\| D^{\alpha}u(t, \cdot) \|_{L^{2}(\mathbb{R}^{n})} & \leq \| (u_{0}, u_{1}) \|_{\mathcal{A}_{m,s}} \times \begin{cases} (1 + t)^{-(2m-2)/(2m)} & \text{if } \mu > \frac{2 - m}{m}n + 2s, \\\n(1 + t)^{-(m-1)/(2m)} (1 + \log(1 + t))^{(2m-2)/(2m)} & \text{if } \mu = \frac{2 - m}{m}n + 2s, \\\n(1 + t)^{-(m-1)/2} & \text{if } \mu < \frac{2 - m}{m}n + 2s, \end{cases}
\end{align*}
\]

\[
(1 + t)^{-(m-1)/2} (1 + \log(1 + t))^{(2m-2)/(2m)}
\]

\[
(1 + t)^{-(m-1)/2}
\]

\[
(1 + t)^{-(m-1)/2}
\]
and for \( s \geq 1 \):

\[
\|D^{m-1} u_t(t, \cdot)\|_{L^1(\mathbb{R}^n)} \lesssim (1 + t)^{-s} \left\{ \begin{array}{ll}
(1 + t)^{-\left(\frac{2-m}{2}m\right)n-s} & \text{if } \mu > \frac{2-m}{m} n + 2s, \\
(1 + t)^{-\left(\frac{\mu}{2}\right)\left(1 + \log(1 + t)\right)\left(\frac{2-m}{2}m\right)n} & \text{if } \mu = \frac{2-m}{m} n + 2s, \\
(1 + t)^{-\left(\frac{\mu}{2}\right)\left(\frac{2-m}{2}m\right)n} & \text{if } \mu < \frac{2-m}{m} n + 2s.
\end{array} \right.
\]

In order to use Duhamel’s principle in the next sections, we consider the family of parameter-dependent Cauchy problems

\[
v_{tt} - \Delta v + \frac{\mu}{1 + t} v_t = 0, v(\tau, x) = \nu, v_t(\tau, x) = \nu_1(\tau, x), \quad 0 \leq \tau \leq t. \tag{14}
\]

\[
\|D^{m-1} v_t(t, \cdot)\|_{L^1(\mathbb{R}^n)} \lesssim \left\{ \begin{array}{ll}
(1 + t)^{-\left(\frac{2-m}{2}m\right)n-s} & \text{if } \mu > \frac{2-m}{m} n + 2s, \\
(1 + t)^{-\left(\frac{\mu}{2}\right)\left(1 + \log(1 + t)\right)\left(\frac{2-m}{2}m\right)n} & \text{if } \mu = \frac{2-m}{m} n + 2s, \\
(1 + t)^{-\left(\frac{\mu}{2}\right)\left(\frac{2-m}{2}m\right)n} & \text{if } \mu < \frac{2-m}{m} n + 2s.
\end{array} \right.
\]

\[\text{Theorem 2. Let us assume } v_1 \in H^{\max\{p-1, 0\}}(\mathbb{R}^m) \cap L^m(\mathbb{R}^n) \text{ with } s > 0. \text{ Then, the solution } v \text{ to the Cauchy problem (14) satisfies for } \mu > 1 \text{ the following decay estimates for } s \geq 0:\]

\[
\|D^{m-1} v(t, \cdot)\|_{L^1(\mathbb{R}^n)} \lesssim \left\{ \begin{array}{ll}
(1 + t)^{-\left(\frac{2-m}{2}m\right)n-s} & \text{if } \mu > \frac{2-m}{m} n + 2s, \\
(1 + t)^{-\left(\frac{\mu}{2}\right)\left(1 + \log(1 + t)\right)\left(\frac{2-m}{2}m\right)n} & \text{if } \mu = \frac{2-m}{m} n + 2s, \\
(1 + t)^{-\left(\frac{\mu}{2}\right)\left(\frac{2-m}{2}m\right)n} & \text{if } \mu < \frac{2-m}{m} n + 2s.
\end{array} \right.
\]

\[
\text{and for } s \geq 1:
\]

\[
\|D^{m-1} v(t, \cdot)\|_{L^1(\mathbb{R}^n)} \lesssim \left(1 + t\right)^{-\left(\frac{2-m}{2}m\right)n+\left(\frac{\mu}{2}\right)\left(\frac{2-m}{2}m\right)n-s} \left\{ \begin{array}{ll}
(1 + t)^{-\left(\frac{2-m}{2}m\right)n-s} & \text{if } \mu > \frac{2-m}{m} n + 2s, \\
(1 + t)^{-\left(\frac{\mu}{2}\right)\left(1 + \log(1 + t)\right)\left(\frac{2-m}{2}m\right)n} & \text{if } \mu = \frac{2-m}{m} n + 2s, \\
(1 + t)^{-\left(\frac{\mu}{2}\right)\left(\frac{2-m}{2}m\right)n} & \text{if } \mu < \frac{2-m}{m} n + 2s.
\end{array} \right.
\]

where \( H^s \) is the homogeneous Sobolev space.

\section{Main Results}

\subsection{Low Regular Data} In this section, we are interested in Cauchy problem (1), where the initial data are supposed to have low regularity; or in other words, the data belong to the Sobolev space \( H^s(\mathbb{R}^n) \) for \( s \in (0, 1) \), with additional regularity \( L^m(\mathbb{R}^n) \) for \( m \in [1, 2] \).

From the estimates of Theorem 1 and further considerations, we remark the existence of five cases corresponding to the value of \( \mu \). These cases are as follows:

1. \( \mu > \frac{2-m}{m} n + 2s \)
2. \( \mu = \frac{2-m}{m} n + 2s \)
3. \( \mu < \frac{2-m}{m} n + 2s \)
\[ \mu > \frac{2 - m}{m} n + 2s, \]
\[ \mu = \frac{2 - m}{m} n + 2s, \]
\[ \frac{2 - m}{m} n + 2s > \mu > \frac{2 - m}{m} n, \quad (17) \]
\[ \mu = \frac{2 - m}{m} n, \]
\[ \mu < \frac{2 - m}{m} n. \]

These cases generate for the system (1) a lot of cases corresponding to the values of \( \mu_1 \) and \( \mu_2 \). In Theorem 3, we restrict ourselves to three cases which are from our point of view are more interesting and important. The remaining cases will be treated in Remark 1.

**Lemma 1.** Let \( p \) satisfy the conditions
\[ \frac{2}{m} \leq p, \text{ if } n = 1 \text{ and } s \in \left[ \frac{1}{2}, 1 \right), \]
\[ \frac{2}{m} \leq p \leq p_{GN, s} (n), \text{ otherwise.} \quad (18) \]

Then, the following statements are valid:

(i) If
\[ M (\tau, u) = (1 + \tau)^{-\left(2 - \frac{m}{2} m n\right) n} \| u (\tau, \cdot) \|_{L^2 (\mathbb{R}^s)} + (1 + \tau)^{-\left(2 - \frac{m}{2} m n\right) n} \| D^1 u (\tau, \cdot) \|_{L^2 (\mathbb{R}^s)}, \]
\[ \| u (\tau, x) \|_{L^m (\mathbb{R}^s)} \leq (1 + \tau)^{-\left(\frac{n}{m} \right) \left(p - 1\right) M (\tau, u)^{\frac{1}{p}}}. \]

(ii) If
\[ M (\tau, u) = (1 + \tau)^{-\left(2 - \frac{m}{2} m n\right) n} \| u (\tau, \cdot) \|_{L^2 (\mathbb{R}^s)} + \| D^1 u (\tau, \cdot) \|_{L^2 (\mathbb{R}^s)}, \]
\[ \| u (\tau, x) \|_{L^m (\mathbb{R}^s)} \leq (1 + \tau)^{-\left(\frac{n}{m} \right) \left(p - 2\right) M (\tau, u)^{\frac{1}{p}}}. \]

These estimates imply

\[ \| u (\tau, x) \|_{L^m (\mathbb{R}^s)} \leq (1 + \tau)^{-\left(\frac{n}{m} \right) \left(p - 2\right) M (\tau, u)^{\frac{1}{p}}}. \]

The proof of Lemma 1 is basically concluded after using the Gagliardo–Nirenberg inequality from Proposition A.1 and the definition of the space \( M (\tau, u) \).

**Theorem 3.** Let \( n \leq 4s / (2 - m) \). The data \((u_0, u_1), (v_0, v_1)\) are supposed to belong to \( \mathcal{A}_{m,s} \times \mathcal{A}_{m,s} \), \( s \in (0, 1) \), \( m \in [1, 2) \), and \( \min \{ \mu_1; \mu_2 \} > 1 \). Moreover, let the exponents \( p \) and \( q \) of the power nonlinearities satisfy condition (18) and

\[ \min \{ p; q \} > p_{GN, s} (n) \text{ if } \min \{ \mu_1; \mu_2 \} > \frac{2 - m}{m} n + 2s, \]
\[ p > \frac{4}{\mu_2} m, \quad q > \frac{2m}{2} + \frac{m n \mu_2}{2m} \text{ if } \mu_1 > \frac{2 - m}{m} n + 2s, \mu_2 < \frac{2 - m}{m} n, \]
\[ p > \frac{4}{\mu_2} m, \quad q > \frac{2m}{2} + \frac{m n \mu_2}{2m} \text{ if } \mu_1 < \frac{2 - m}{m} n + 2s, \mu_2 > \frac{2 - m}{m} n. \]
If we consider model (1) for $n = 2$, $m = 11/10$, and $s = 9/10$, then by using (25)–(27) from Theorem 3, we obtain the following statements:

1. If $\mu_1 = 4, \mu_2 = 3$, then $\min \{ p; q \} > p_{Fuj,11/10} (2) = \frac{21}{10}$.
2. If $\mu_1 = 4, \mu_2 = 2$, then $p > \frac{53}{22} \sim 2.4, q > \frac{763}{220} \sim 3.46$.
3. If $\mu_1 = 2, \mu_2 = \frac{3}{2}$, then $p > 4, q > \frac{11}{4} = 2.75$.

Example 1. If we consider model (1) for $n = 2, m = 11/10, s = 9/10$, then by using (25)–(27) from Theorem 3, we obtain the following statements:

1. If $\mu_1 = 4, \mu_2 = 3$, then $\min \{ p; q \} > p_{Fuj,11/10} (2) = \frac{21}{10}$.
2. If $\mu_1 = 4, \mu_2 = 2$, then $p > \frac{53}{22} \sim 2.4, q > \frac{763}{220} \sim 3.46$.
3. If $\mu_1 = 2, \mu_2 = \frac{3}{2}$, then $p > 4, q > \frac{11}{4} = 2.75$.

Example 1 shows that for $s < 1$, when $\mu_1 = 4, \mu_2 = 3$, then $\min \{ p; q \} > p_{Fuj,11/10} (2) = \frac{21}{10}$. Hence, the solution satisfies for $s < 1$ the following decay estimates:

$$
\| [D^{\gamma} u(t, \cdot)] \|_{L^2(\mathbb{R}^n)} \leq \left( \| (u_0, u_1) \|_{\mathcal{D}_{m_1}} + \| (v_0, v_1) \|_{\mathcal{D}_{m_1}} \right) \times \begin{cases} (1 + t)^{-((2-2m)n-s)/2} & \text{if } \mu_1 > \frac{2-m}{m}n + 2s, \\ (1 + t)^{-(\mu_1/2)} & \text{if } \mu_1 < \frac{2-m}{m}n, \end{cases}
$$

Then, there exists a uniquely determined global (in time) Sobolev solution to (1) in

$$
(\mathcal{C}( [0, \infty), H^s(\mathbb{R}^n) ))^2.
$$

Furthermore, the solution satisfies for $s \in (0, 1)$ the following decay estimates:

$$
\| [D^{\gamma} v(t, \cdot)] \|_{L^2(\mathbb{R}^n)} \leq \left( \| (u_0, u_1) \|_{\mathcal{D}_{m_1}} + \| (v_0, v_1) \|_{\mathcal{D}_{m_1}} \right) \times \begin{cases} (1 + t)^{-((2-2m)n-s)/2} & \text{if } \mu_1 > \frac{2-m}{m}n + 2s, \\ (1 + t)^{-(\mu_1/2)} & \text{if } \mu_1 < \frac{2-m}{m}n. \end{cases}
$$

2.2. Data from Energy Space. If the data are in the energy space, then we get for $s = 1$ a similar case to the case of the previous section because the estimates for $\| [D^{\gamma} u(t, \cdot)] \|_{L^2(\mathbb{R}^n)}$ and $\| u(t, \cdot) \|_{L^2(\mathbb{R}^n)}$ coincide with those of the previous section. Moreover, we obtain the global existence in time of energy solutions. Consequently, we have the following result.

Theorem 4. Let $n \leq 4/(2-m), (u_0, u_1), (v_0, v_1) \in \mathcal{D}_{m_1} \times \mathcal{D}_{m_1}, m \in [1, 2)$, and $\min \{ \mu_1; \mu_2 \} > 1$. Moreover, let the exponents $p$ and $q$ of power nonlinearities satisfy condition (18) and

$$
\min \{ p; q \} > p_{Fuj,m}(n) \text{if } \min \{ \mu_1; \mu_2 \} > \frac{2-m}{m}n + 2,
$$

$$
p > \frac{4}{\mu_2} + \frac{2-m}{m}n, q > \frac{2m}{n} + \frac{m}{2} + \frac{nu_1}{2m} \text{if } \mu_1 > \frac{2-m}{m}n + 2, \mu_2 < \frac{2-m}{m}n, \quad (31)
$$

Then, there exists a small constant $\epsilon_0$ such that if

$$
\| (u_0, u_1) \|_{\mathcal{D}_{m_1}} + \| (v_0, v_1) \|_{\mathcal{D}_{m_1}} \leq \epsilon_0,
$$

then there exists a uniquely determined global (in time) energy solution to (1) in

$$
(\mathcal{C}( [0, \infty), H^1(\mathbb{R}^n) ) \cap \mathcal{C}^1([0, \infty), tL^2 n(\mathbb{R}^n) ))^2.
$$

Then, there exists a uniquely determined global (in time) energy solution to (1) in

$$
(\mathcal{C}( [0, \infty), H^s(\mathbb{R}^n) ))^2.
$$

(27)
Table 1: Cases of the interplay between $\mu_1$ and $\mu_2$.

| Case | Admissible range for $p$ | Admissible range for $q$ |
|------|-------------------------|-------------------------|
| (1)  | $\mu_1 > (2 - m/m)n + 2s$  $\mu_2 = (2 - m/m)n + 2s$ | $p > p_{\text{Fuj,m}}(n)$ | $q > p_{\text{Fuj,m}}(n)$ |
| (2)  | $(2 - m/m)n < \mu_2 < (2 - m/m)n + 2s$ | $p > \min\{(4m/(2 - m)n) + 1; (2 + (n/m))(2/\mu_2 - 2s + n)\}$ | $q > p_{\text{Fuj,m}}(n)$ |
| (3)  | $\mu_1 > (2 - m/m)n + 2s$  $\mu_2 = (2 - m/m)n$ | $p > 1 + (4/\mu_2)$ | $q > p_{\text{Fuj,m}}(n)$ |
| (4)  | $\mu_1 = \mu_2 = (2 - m/m)n + 2s$ | $p > p_{\text{Fuj,m}}(n)$ | $q > p_{\text{Fuj,m}}(n)$ |
| (5)  | $(2 - m/m)n < \mu_2 < (2 - m/m)n + 2s$ | $p > \min\{(4m/(2 - m)n) + 1; (2 + (n/m))(2/\mu_2 - 2s + n)\}$ | $q > p_{\text{Fuj,m}}(n)$ |
| (6)  | $\mu_1 = (2 - m/m)n + 2s$  $\mu_2 = (2 - m/m)n$ | $p > 1 + (4/\mu_2)$ | $q > p_{\text{Fuj,m}}(n)$ |
| (7)  | $(2 - m/m)n < \mu_1 < (2 - m/m)n + 2s$  $\mu_2 = (2 - m/m)n$ | $p > (4/\mu_2) + (2 - m/\mu_2 m)n$ | $q > (2m/n) + (m/2) + (nm_2/2m)$ |
| (8)  | $(2 - m/m)n < \mu_2 < (2 - m/m)n + 2s$  $\mu_2 = (2 - m/m)n$ | $p > \min\{(4m/(2 - m)n) + 1; (2 + (n/m))(2/\mu_2 - 2s + n)\}$ | $q > \min\{(4m/(2 - m)n) + 1; (2 + (n/m))(2/\mu_2 - 2s + n)\}$ |
| (9)  | $(2 - m/m)n < \mu_1 < (2 - m/m)n + 2s$  $\mu_2 = (2 - m/m)n$ | $p > 1 + (4/\mu_2)$ | $q > \min\{(4m/(2 - m)n) + 1; (2 + (n/m))(2/\mu_2 - 2s + n)\}$ |
| (10) | $(2 - m/m)n < \mu_1 < (2 - m/m)n + 2s$  $\mu_2 = (2 - m/m)n$ | $p > (4/\mu_2) + (2 - m/\mu_2 m)n$ | $q > \min\{(4m/(2 - m)n) + 1; (2 + (n/m))(2/\mu_2 - 2s + n)\}$ |
| (11) | $\mu_1 = \mu_2 = (2 - m/m)n$ | $p > 1 + (4/\mu_2)$ | $q > 1 + (4/\mu_1)$ |
| (12) | $\mu_1 = (2 - m/m)n$  $\mu_2 = (2 - m/m)n$ | $p > 1 + (4/\mu_2)$ | $q > (\mu_2/\mu_1) + (4/\mu_1)$ |
Furthermore, the solution satisfies the following decay estimates:

\[
\|u_1(t, \cdot)\|_{L^2(\mathbb{R}^n)} + \|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq \left(\|u_0, u_1\|_{H^m_\mathcal{E}} + \|v_0, v_1\|_{H^m_\mathcal{E}}\right) \times \begin{cases} (1 + t)^{-(1/2-m)n/2-1}, & \text{if } \mu_1 > \frac{2-m}{m}n + 2, \\ (1 + t)^{(\mu_1/2)}, & \text{if } \mu_1 < \frac{2-m}{m}n, \end{cases}
\]

2.3. Data from Sobolev Spaces with Suitable Regularity.

This section is devoted to the case where the data are from Sobolev spaces with suitable regularity. We will treat the same cases of the previous sections corresponding to the values of \(\mu_1\) and \(\mu_2\). In the following lemma, we will provide some estimates which are important tools in the proofs of our main results.

Lemma 2. Let \(p > \lfloor s \rfloor\) satisfy the following condition:

\[
2 < p, \quad \text{if } s \in \left[\frac{n}{2} + 1, \frac{n}{2}\right],
\]

\[
2 < p, \quad \text{if } s \in \left(\frac{n}{2}, \frac{n}{2}\right).
\]

Then, the following statements are valid.

(i) If

\[
M(t, u) = (1 + t)^{(2-m/2)n/2} \|u(t,.\|_{L^2(\mathbb{R}^n)} + (1 + t)^{(2-m/2)n+s+1} \|u_1(t,.\|_{L^2(\mathbb{R}^n + (1 + t)^{(2-m/2)n+s}} \|u_1(t,.\|_{H^m_\mathcal{E}} + (1 + t)^{(2-m/2)n+(s-1)} M(t, u)^p) + (1 + t)^{(2-m/2)n+(s-1)} M(t, u)^p,
\]

\[
\|u(t,x)\|_{L^m(\mathbb{R}^n)} \leq (1 + t)^{-(n/m)(p-1)} M(t, u)^p,
\]

\[
\|u(t,x)\|_{L^m(\mathbb{R}^n)} \leq (1 + t)^{-(n/m)p+(n/2)} M(t, u)^p,
\]

\[
\|u(t,x)\|_{H^m(\mathbb{R}^n)} \leq (1 + t)^{-(n/m)p+(n/2)-(s-1)} M(t, u)^p,
\]

\[
\|u(t,x)\|_{H^m_{\mathcal{E}}(\mathbb{R}^n)} \leq (1 + t)^{-(n/m)p+(n/2)-(s-1)} M(t, u)^p,
\]

\[
\times (1 + B(t, 0))^{-(n/m)p+(n/2)-(s-1)} \times M(t, v - \bar{v}) \leq (M(t, u)^p + M(t, \bar{u})^p)\]
These estimates imply

\[
\left( \|u\|^P_{L^\infty(\mathbb{R}^m)} + (1 + \tau)^{(2m/2m)\nu + s - 1} \|u\|^P_{H^{-1}(\mathbb{R}^m)} + (1 + \tau)^{(2m/2m)\nu + s - 2} \|u\|^P_{H^{-2}(\mathbb{R}^m)} \right) \times (1 + \tau)^{\nu} \leq (1 + \tau)^{-(\nu/2)p} M(\tau, u)^p.
\]

(ii) If

\[
M(\tau, u) = (1 + \tau)^{\nu/2} \left( \|u(\tau, \cdot)\|_{L^2(\mathbb{R}^m)} + \|u_1(\tau, \cdot)\|_{L^2(\mathbb{R}^m)} \right),
\]

then

\[
\left( \|u\|^P_{L^\infty(\mathbb{R}^m)} + (1 + \tau)^{(2m/2m)\nu + s - 1} \|u\|^P_{H^{-1}(\mathbb{R}^m)} + (1 + \tau)^{(2m/2m)\nu + s - 2} \|u\|^P_{H^{-2}(\mathbb{R}^m)} \right) \times (1 + \tau)^{\nu} \leq (1 + \tau)^{-(\nu/2)p} M(\tau, u)^p.
\]

Using the fractional chain rule from the Appendix and the definition of space \( M(\tau, u) \), one can prove the desired statements.

**Theorem 5.** Let \( n \geq 4 \). The regularity parameters \( s_1 \) and \( s_2 \) satisfy the following conditions:

\[
\begin{align*}
[s_1] &< p, [s_2] < q, \text{if } n \leq 2s_1, \\
[s_1] &< p, \quad [s_2] < q \leq 1 + \frac{2}{n - 2s_1}, \text{if } 2s_1 < n \leq 2s_2, \\
[s_1] &< p \leq 1 + \frac{2}{n - 2s_2}, \quad [s_2] < q \leq 1 + \frac{2}{n - 2s_1}, \text{if } n > 2s_2,
\end{align*}
\]

\[
\min[p, q] > p_{Fuj,m}(n), \quad \text{if } \mu_1 > \frac{2 - m}{m} n + 2s_1, \mu_2 > \frac{2 - m}{m} n + 2s_2,
\]

\[
p > \frac{2 + 2s_2}{\mu_2} + \frac{2 - m}{\mu_2} n, \quad q > \frac{2m}{n} + \frac{m}{2} + \frac{nu_2}{2m}, \quad \text{if } \mu_1 > \frac{2 - m}{m} n + 2s_1, \mu_2 < \frac{2 - m}{m} n,
\]

\[
p > \frac{\mu_1 + 2 + 2s_1}{\mu_2}, \quad q > \frac{\mu_2 + 2 + 2s_1}{\mu_1}, \quad \text{if } \max[\mu_1; \mu_2] < \frac{2 - m}{m} n.
\]

The data \((u_0, u_1)\) and \((v_0, v_1)\) are supposed to belong to \( \mathcal{A}_{m,s_1} \times \mathcal{A}_{m,s_2} \) with \( m \in [1, 2) \). Furthermore, we assume for the exponents \( p \) and \( q \) the following conditions:
Then, there exists a uniquely determined global (in time) energy solution to (1) in

\[
(\mathcal{C}([0, \infty), H^{1}(\mathbb{R}^n)) \cap H^1([0, \infty), H^{1/2}((\mathbb{R}^n))) \times (\mathcal{C}([0, \infty), H_{x}^{1/2}((\mathbb{R}^n))) \cap \mathcal{C}([0, \infty), H^{1/2-1}((\mathbb{R}^n)))).
\]

Furthermore, the solution satisfies the following decay estimates:

\[
\begin{align*}
\|u(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \left(\|u_0, v_1\|_{H^{n_1}} + \|v_0, v_1\|_{H^{n_2}}\right) \times \\
&\begin{cases} 
(1 + t)^{-\frac{2m}{m-n}n}\text{ if } \mu_1 > \frac{2 - m}{m-n} + 2s_1, \\
(1 + t)^{-\frac{\mu_1}{2}}\text{ if } \mu_1 < \frac{2 - m}{m-n},
\end{cases} \\
\|u_t(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \left(\|u_0, v_1\|_{H^{n_1}} + \|v_0, v_1\|_{H^{n_2}}\right) \times \\
&\begin{cases} 
(1 + t)^{-\frac{2m}{m-n}n}\text{ if } \mu_1 > \frac{2 - m}{m-n} + 2s_1, \\
(1 + t)^{-\frac{\mu_1}{2}}\text{ if } \mu_1 < \frac{2 - m}{m-n},
\end{cases} \\
\|D^i u(t, \cdot)\|_{L^2(\mathbb{R})} + \|D^{i-1} u_t(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \left(\|u_0, v_1\|_{H^{n_1}} + \|v_0, v_1\|_{H^{n_2}}\right) \times \\
&\begin{cases} 
(1 + t)^{-\frac{2m}{m-n}n}\text{ if } \mu_2 > \frac{2 - m}{m-n} + 2s_2, \\
(1 + t)^{-\frac{\mu_2}{2}}\text{ if } \mu_2 < \frac{2 - m}{m-n},
\end{cases} \\
\|v(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \left(\|u_0, v_1\|_{H^{n_1}} + \|v_0, v_1\|_{H^{n_2}}\right) \times \\
&\begin{cases} 
(1 + t)^{-\frac{2m}{m-n}n}\text{ if } \mu_2 > \frac{2 - m}{m-n} + 2s_2, \\
(1 + t)^{-\frac{\mu_2}{2}}\text{ if } \mu_2 < \frac{2 - m}{m-n},
\end{cases} \\
\|v_t(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \left(\|u_0, v_1\|_{H^{n_1}} + \|v_0, v_1\|_{H^{n_2}}\right) \times \\
&\begin{cases} 
(1 + t)^{-\frac{2m}{m-n}n}\text{ if } \mu_2 > \frac{2 - m}{m-n} + 2s_2, \\
(1 + t)^{-\frac{\mu_2}{2}}\text{ if } \mu_2 < \frac{2 - m}{m-n},
\end{cases} \\
\|D^i v(t, \cdot)\|_{L^2(\mathbb{R})} + \|D^{i-1} v_t(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \left(\|u_0, v_1\|_{H^{n_1}} + \|v_0, v_1\|_{H^{n_2}}\right) \times \\
&\begin{cases} 
(1 + t)^{-\frac{2m}{m-n}n}\text{ if } \mu_2 > \frac{2 - m}{m-n} + 2s_2, \\
(1 + t)^{-\frac{\mu_2}{2}}\text{ if } \mu_2 < \frac{2 - m}{m-n}.
\end{cases}
\end{align*}
\]

(48)

2.4. Large Regular Data. Comparing this section, where the data are supposed to have large regularity, with the previous section, we feel differences in the treatment only if \(\mu_1\) and \(\mu_2\) are sufficiently large. For this reason, we restrict ourselves to formulate the results without giving a proof. The proof is very similar to the proof of Theorem 5.
Lemma 3. Let \( p > s \). Then, using the rule for fractional powers from Proposition D.1 in the Appendix, the following estimates hold: if

\[
M(\tau, u) = (1 + \tau)^{(2-m/2)\mu + 1} \|u(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} + (1 + \tau)^{2-m/2} \|u_{\tau}(\tau, \cdot)\|_{L^p(\mathbb{R}^n)}
\]  

\[
\left( \|D^*_t u(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} + \|D^{s-1} u_{\tau}(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} \right),
\]

(49)

where we assume \( s^* < n/2 \).

We remark that

\[
\frac{n}{2} \left( \frac{1}{m} - \frac{1}{2} \right) p - s - \frac{s - 1}{2} \left( \frac{1}{p} - 1 \right) \leq -\frac{n}{2m} p + \frac{n}{4} 
\]

(51)

These estimates imply the estimate

\[
\left( \|u(\tau, x)^{p}\|_{L^p(\mathbb{R}^n)} + (1 + \tau)^{2-m/2} \|u(\tau, x)^{p}\|_{H^{s-1}(\mathbb{R}^n)} \right) \leq (1 + \tau)^{(2-m/2)\mu + s} M(\tau, u)^p.
\]

(52)

If

\[
M(\tau, u) = (1 + \tau)^{\mu/2} \left( \|u(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} + \|u_{\tau}(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} \right)
\]

\[
+ \|D^*_t u(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} + \|D^{s-1} u_{\tau}(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} \right),
\]

\]

then

\[
\left( \|u(\tau, x)^{p}\|_{L^p(\mathbb{R}^n)} + \|D^*_t u(\tau, x)^{p}\|_{H^{s-1}(\mathbb{R}^n)} \right) \leq (1 + \tau)^{\mu/2} M(\tau, u)^p.
\]

(53)

These estimates imply the estimate

\[
\left( \|u(\tau, x)^{p}\|_{L^p(\mathbb{R}^n)} + (1 + \tau)^{(2-m/2)\mu + 2} \|u(\tau, x)^{p}\|_{H^{s-1}(\mathbb{R}^n)} \right) \leq (1 + \tau)^{(2-m/2)\mu + s} M(\tau, u)^p.
\]

(54)

Theorem 6. Let \( n \geq 4 \). The data \( (u_0, u_1) \) and \( (v_0, v_1) \) are supposed to belong to \( \mathcal{A}_{m, s_2} \times \mathcal{A}_{m, s_2} \) with \( m \in [1, 2] \) and \( s_2 > s_1 > (n/2) + 1 \). Moreover, we assume

\[
p > s_1,
\]

\[
q > s_2
\]

(56)

where \( s_2 \in [s_1, s_1 + 1] \) and \( s_2 \leq s_2 \). Furthermore, we assume for the exponents \( p \) and \( q \) the following conditions:

\[
p > 1 + \frac{(s_1 + 1)m}{n}, q > 1 + \frac{(s_1 + 1)m}{n}, \quad \text{if } \mu_1 > \frac{2 - m}{m} n + 2s_1, \mu_2 > \frac{2 - m}{m} n + 2s_2,
\]

\[
p > \frac{2 + 2s_2}{\mu_2} + \frac{2 - m}{\mu_2 m} n, q > \frac{(2 + 2s_2)m}{n} + \frac{m + \mu_2}{2m}, \quad \text{if } \mu_1 > \frac{2 - m}{m} n + 2s_1, \mu_2 < \frac{2 - m}{m} n,
\]
\[
\rho > \frac{\mu_1}{\mu_2} + \frac{2 + 2s_2}{\mu_2},
q > \frac{\mu_2}{\mu_1} + \frac{2 + 2s_1}{\mu_1},
\text{ if } \max\{\mu_1; \mu_2\} < \frac{2 - m}{m}n.
\]

(57)

Then, there exists a uniquely determined global (in time) energy solution to (1) in

\[
\mathcal{C}([0, \infty)) \cap \mathcal{C}^1([0, \infty), H^{s_n-1}(\mathbb{R}^n)) \times \mathcal{C}^1([0, \infty), H^{s_2}(\mathbb{R}^n)) \cap \mathcal{C}^1([0, \infty), H^{s_1}(\mathbb{R}^n)).
\]

(58)

Furthermore, the solution satisfies the following decay estimates:

\[
\|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq \left\|\left\|(u_0, u_1)\|_{\mathcal{S}_{n_1}} + \|v_0, v_1\|_{\mathcal{S}_{n_2}}\right\| \times \begin{cases} (1 + t)^{-\frac{2 - m}{2m}n}, & \text{if } \mu_1 > \frac{2 - m}{m}n + 2s_1, \\ (1 + t)^{-\frac{\mu_1}{2}}, & \text{if } \mu_1 < \frac{2 - m}{m}n, \end{cases}
\]

(59)
3. Philosophy of Our Approach and Proofs

3.1. Proof of Theorem 3. Let us define the space of solutions $X(t)$ as follows:

$$X(t) = \{(u, v) \in (\mathcal{C}([0, t], H^s(R^n)) \times \mathcal{C}([0, t], H^s(R^n)))^2 \},$$

(60)

with the norm

$$\| (u, v) \|_{X(t)} = \sup_{t \in [0, t]} \| M_1 (t, u, v) + M_2 (t, u, v) \|,$$

(61)

where $M_1 (t, u, v)$ and $M_2 (t, u, v)$ will be defined in the treatment of each case. Let $N$ be the mapping on $X(t)$ which is defined by

$$N: (u, v) \in X(t) \rightarrow N(u, v) = \left( u^1 + u^2, v^1 + v^2 \right),$$

(62)

where

$$u^1 (t, x) = E_{1,0} (t, 0, x) \ast (\Delta u_0 (x) + E_{1,1} (t, 0, x) \ast (\Delta u_1 (x)),

v^1 (t, x) = \int_0^t E_{1,1} (t, \tau, x) \ast (\Delta f (\tau, v_1 (x)) d\tau,$$

$$u^2 (t, x) = E_{2,0} (t, 0, x) \ast u_0 (x) + E_{2,1} (t, 0, x) \ast u_1 (x),

v^2 (t, x) = \int_0^t E_{2,1} (t, \tau, x) \ast g (\tau, u_1) d\tau.$$

(63)

We denote by $E_{1,0} (t, 0, x)$ and $E_{1,1} (t, 0, x)$ the fundamental solutions to the Cauchy problem

$$u_{tt} - \Delta u + \frac{\mu_1}{1 + t} u_t = 0,$$

$$u (0, x) = u_0 (x),$$

(64)

$$u_t (0, x) = u_1 (x),$$

and by $E_{2,0} (t, 0, x)$ and $E_{2,1} (t, 0, x)$ the fundamental solutions to the Cauchy problem

$$v_{tt} - \Delta v + \frac{\mu_2}{1 + t} v_t = 0,$$

$$v (0, x) = v_0 (x),$$

$$v_t (0, x) = v_1 (x).$$

(65)

From Proposition E.1, the goal is to prove the following estimates:

From the definition of the norm of the solution space $X(t)$, which we will define in each case in correspondence with the main goals, we can immediately obtain

$$\| (u^1, v^1) \|_{X(t)} \leq \| (u_0, u_1) \|_{X(t)} + \| (v_0, v_1) \|_{X(t)},$$

(66)

$$\| N (u, v) - N (\tilde{u}, \tilde{v}) \|_{X(t)} \leq \| (u, v) - (\tilde{u}, \tilde{v}) \|_{X(t)} \times \left( \| (u, v) \|_{X(t)}^{p-1} + \| (\tilde{u}, \tilde{v}) \|_{X(t)}^{p-1} + \| (u, v) \|_{X(t)}^{q-1} + \| (\tilde{u}, \tilde{v}) \|_{X(t)}^{q-1} \right).$$

(67)

We choose

$$M_1 (t, u, v) = (1 + t)^{2 - m/2(n + 2s)} \| u (\tau, \cdot) \|_{L^2 (R^n)}$$

$$+ (1 + t)^{2 - m/2(n + 2s)} \| D^1 u (\tau, \cdot) \|_{L^2 (R^n)},$$

(70)

$$M_2 (t, v, \tau) = (1 + t)^{2 - m/2(n + 2s)} \| v (\tau, \cdot) \|_{L^2 (R^n)}$$

$$+ (1 + t)^{2 - m/2(n + 2s)} \| D^1 v (\tau, \cdot) \|_{L^2 (R^n)}.$$
For the first component \( \mu > (2 - m/m)n + 2s \) and estimate (21) to obtain

\[
\|D^p u^{pl}(t, \cdot)\|_{L^2(R^n)} \leq \|(u, v)\|_{X(t)}(1 + t)^{-(2 - m/2)n}.
\] (73)

For \( v^{pl} \), we use estimate (15) for \( \mu_2 > (2 - m/m)n + 2s \) and estimate (21) to obtain

\[
\|D^q v^{pl}(t, \cdot)\|_{L^2(R^n)} \leq \|(u, v)\|_{X(t)}(1 + t)^{-(2 - m/2)n}.
\] (76)

From (72) to (76), we complete the proof of (69). (ii) \( \mu_1 > (2 - m/m)n + 2s, \mu_2 < (2 - m/m)n \).

We choose

\[
M_1 (\tau, u) = (1 + \tau)^{(2 - m/2)n} \|u(\tau, \cdot)\|_{L^2(R^n)} + (1 + \tau)^{(2 - m/2)n - s} \|D^p u(\tau, \cdot)\|_{L^2(R^n)}
\]
\[
M_2 (\tau, v) = (1 + \tau)^{(2 - m/2)n} \|v(\tau, \cdot)\|_{L^2(R^n)} + \|D^p v(\tau, \cdot)\|_{L^2(R^n)}.
\] (77)

For the first component \( u^{pl} \), we use estimate (15) for \( \mu_1 > (2 - m/m)n + 2s \) and estimate (24) to obtain

\[
\|D^p u^{pl}(t, \cdot)\|_{L^2(R^n)} \leq \left( 1 + t \right)^{-(2 - m/2)n - s} \int_0^t \left( \|v(\tau, x)\|^p_{L^m(R^n)} + (1 + \tau)^{(2 - m/2)n} \|v(\tau, x)\|_{L^2(R^n)} \right) (1 + \tau) \|D^p u(\tau, \cdot)\|_{L^2(R^n)} d\tau
\]
\[
\leq \|(u, v)\|_{X(t)}(1 + t)^{-(2 - m/2)n}.
\] (78)

\[
\|D^q v^{pl}(t, \cdot)\|_{L^2(R^n)} \leq \|(u, v)\|_{X(t)}(1 + t)^{-(2 - m/2)n}.
\] (79)

where we used \( p > p_{Fuj.m}(n) \). Then, we get

\[
\|D^p u^{pl}(t, \cdot)\|_{L^2(R^n)} \leq \|(u, v)\|_{X(t)}(1 + t)^{-(2 - m/2)n}.
\] (72)

In the same way, we prove

\[
\|D^q v^{pl}(t, \cdot)\|_{L^2(R^n)} \leq \|(u, v)\|_{X(t)}(1 + t)^{-(2 - m/2)n}.
\] (75)

In the same way, we prove

\[
M_1 (\tau, u) = (1 + \tau)^{(2 - m/2)n} \|u(\tau, \cdot)\|_{L^2(R^n)} + (1 + \tau)^{(2 - m/2)n - s} \|D^p u(\tau, \cdot)\|_{L^2(R^n)},
\]
\[
M_2 (\tau, v) = (1 + \tau)^{(2 - m/2)n} \|v(\tau, \cdot)\|_{L^2(R^n)} + \|D^p v(\tau, \cdot)\|_{L^2(R^n)}.
\] (77)
where we used (26) for the exponent \( p \). Then, we get
\[
\|D^s u^\mu (t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u, v)\|_{X(t)}^p (1 + t)^{-(2 - m/2m)n - s}.
\] (79)

In the same way, we prove
\[
\|D^s v^\mu (t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u, v)\|_{X(t)}^p (1 + t)^{-(2 - m/2m)n}. \tag{80}
\]

For \( v^\mu \), we use estimate (15) for \( \mu < (2 - m/m)n \) and estimate (21) to obtain
\[
\|v^\mu (t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u, v)\|_{X(t)}^p (1 + t)^{-(2 - m/2m)n}. \tag{81}
\]

We choose
\[
M_1 (\tau, u) = (1 + \tau)^{\mu_1/2} \|u(\tau, \cdot)\|_{L^2(\mathbb{R}^n)} + \|D^s u(\tau, \cdot)\|_{L^2(\mathbb{R}^n)},
\]

\[
M_2 (\tau, v) = (1 + \tau)^{\mu_2/2} \|v(\tau, \cdot)\|_{L^2(\mathbb{R}^n)} + \|D^s v(\tau, \cdot)\|_{L^2(\mathbb{R}^n)}. \tag{82}
\]

For \( u^\mu \), we use estimate (15) for \( \mu < (2 - m/m)n \) and estimate (24) to obtain
\[
\|D^s u^\mu (t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u, v)\|_{X(t)}^p (1 + t)^{-(2 - m/2m)n}. \tag{83}
\]

\[
\|D^s v^\mu (t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u, v)\|_{X(t)}^p (1 + t)^{-(2 - m/2m)n}. \tag{84}
\]

Analogously, using (27) for \( q \), we can prove
\[
\|D^s v^\mu (t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u, v)\|_{X(t)}^p (1 + t)^{-(2 - m/2m)n - s} \tag{85}
\]

where we used \( p > (4 - 2s/\mu_2 + \mu_1/\mu_1 \) which is included in (27). Then, we get
\[
\|D^s u^\mu (t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u, v)\|_{X(t)}^p (1 + t)^{-(2 - m/2m)n - s}. \tag{86}
\]

In the same way, for \( s = 0 \), we can prove
\[
\|u^\mu (t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u, v)\|_{X(t)}^p (1 + t)^{-(\mu_2/2)}, \tag{87}
\]

where we use condition (27).
\[ N(u, v) - N(\tilde{u}, \tilde{v}) = (u^d(t, x) - \tilde{u}^d(t, x), v^d(t, x) - \tilde{v}^d(t, x)) = \left( \int_0^t E_1(t, t, \tau)^* (|v(\tau, x)|^p - |\tilde{v}(\tau, x)|^p) d\tau, \int_0^t E_1(t, t, \tau)^* (|u(\tau, x)|^q - |\tilde{u}(\tau, x)|^q) d\tau \right). \] (90)

Following the same steps and cases used to prove the first inequality (66), we can estimate

\[ \left\| |D|^{s_1} E_1(t, t, \tau)^* (|v(\tau, x)|^p - |\tilde{v}(\tau, x)|^p) d\tau \right\|_{L^2}, \]

\[ \left\| |D|^{s_1} E_1(t, t, \tau)^* (|u(\tau, x)|^q - |\tilde{u}(\tau, x)|^q) d\tau \right\|_{L^2}, \]

\[ \left\| |D|^{s_1} E_1(t, t, \tau)^* (|v(\tau, x)|^p - |\tilde{v}(\tau, x)|^p) d\tau \right\|_{L^2}, \]

\[ \left\| |D|^{s_1} E_1(t, t, \tau)^* (|u(\tau, x)|^q - |\tilde{u}(\tau, x)|^q) d\tau \right\|_{L^2}, \]

which complete the proof.

3.2. Proof of Theorem 5. To prove this theorem, we follow the same steps of the proof of Theorem 3. Then, our main goal is to prove (69) which implies (66). To prove (67), we use estimates (39) or (43) and follow the same steps used to prove previous theorem, in particular, inequality (66). We split the proof into three cases.

(i) \( \mu_1 > (2 - m/m)n + 2s_1, \mu_2 > (2 - m/m)n + 2s_2. \)

We choose

\[ M_1(\tau, u) = (1 + \tau)^{(2 - m/m)n} |u(\tau, \cdot)|_{L^2(\mathbb{R}^n)} + (1 + \tau)^{(2 - m/m)n+1} \|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \]

\[ M_2(\tau, v) = (1 + \tau)^{(2 - m/m)n} |v(\tau, \cdot)|_{L^2(\mathbb{R}^n)} + (1 + \tau)^{(2 - m/m)n+1} \|v(t, \cdot)\|_{L^2(\mathbb{R}^n)} \]

For the first component \( u^d \), we estimate the most complicate norm (from our point of view) which is

\[ \| |D|^{s_1-1} u^d(t, \cdot) \|_{L^2(\mathbb{R}^n)} \leq \int_0^t \left( |v(\tau, x)|^p \|v(\tau, \cdot)|_{L^{p}((\mathbb{R}^n), x)} + (1 + \tau)^{(2 - m/m)n+s_1-1} |v(\tau, x)|^p \|v(\tau, \cdot)|_{L^{p}((\mathbb{R}^n), x)} \right) \]

\[ (1 + \tau)^{(2 - m/m)n+s_1} \]

\[ \leq \|u(\tau, x)|_{L^p(t)} (1 + t)^{(2 - m/m)n+s_1}, \]

where we used the condition \( p > p_{Fujim}(n) \). Then, we get

\[ \| |D|^{s_1-1} u^d(t, \cdot) \|_{L^2(\mathbb{R}^n)} \leq \|u(\tau, x)|_{L^p(t)} (1 + t)^{(2 - m/m)n-s_1}. \] (94)

In the same way, we get

\[ \| |D|^{s_1-1} u^d(t, \cdot) \|_{L^2(\mathbb{R}^n)} \leq \|u(\tau, x)|_{L^p(t)} (1 + t)^{(2 - m/m)n-s_1}. \] (95)

\[ \|u^d(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq \|u(\tau, x)|_{L^p(t)} (1 + t)^{(2 - m/m)n-1}, \] (96)

\[ \|D^{s_1} u^d(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq \|u(\tau, x)|_{L^p(t)} (1 + t)^{(2 - m/m)n-s_1}. \] (97)
Analogously, for the second component $v_{p}^{| \cdot |}$, we use $q > P_{F_{i,j,m}}(n)$ to derive the estimates

\[ \left\| D^{s_{1}}v_{p}^{| \cdot |}(t, \cdot) \right\| _{L^{2}(R^{n})} \leq \left\| (u, v) \right\| _{X_{i}(t)} (1 + t)^{-(2m/2m)n-s_{1}}, \]  
\[ \left\| v_{p}^{| \cdot |}(t, \cdot) \right\| _{L^{2}(R^{n})} \leq \left\| (u, v) \right\| _{X_{i}(t)} (1 + t)^{-(2m/2m)n}, \]  
\[ \left\| v_{p}^{1}(t, \cdot) \right\| _{L^{2}(R^{n})} \leq \left\| (u, v) \right\| _{X_{i}(t)} (1 + t)^{-(2m/2m)n-1}, \]  

From (94)–(101), we get (69).

(ii) $\mu_{1} > (2 - m/m)n + 2s_{1}, \mu_{2} < (2 - m/m)n$.

We choose

\[ M_{1}(t, u, v) = (1 + t)^{(2m/2m)n} + (1 + t)^{(2m/2m)n+1}\left\| u_{t}(t, \cdot) \right\| _{L^{2}(R^{n})}, \]

\[ M_{2}(t, u, v) = (1 + t)^{(2m/2m)n+1}\left\| v_{t}(t, \cdot) \right\| _{L^{2}(R^{n})} + \left\| D^{s_{2}}v_{p}^{| \cdot |}(t, \cdot) \right\| _{L^{2}(R^{n})} + \left\| D^{s_{2}}v_{p}^{1}(t, \cdot) \right\| _{L^{2}(R^{n})}. \]

We begin to estimate the norm $\left\| D^{s_{1}}u_{p}^{| \cdot |}(t, \cdot) \right\| _{L^{2}(R^{n})}$. After using estimate (16) for $\mu_{1} > (2 - m/m)n + 2s_{1}$ and estimate (44), we have

\[ \left\| D^{s_{1}}u_{p}^{| \cdot |}(t, \cdot) \right\| _{L^{2}(R^{n})} \leq \int_{0}^{t} \left( \left\| (u, v) \right\| _{X_{i}(t)} (1 + t)^{-(2m/2m)n-s_{1}} \int_{0}^{t} (1 + r)^{-(m/2)(-2m/2m)n+s_{1}} dr \right) \]

\[ \leq \left\| (u, v) \right\| _{X_{i}(t)} (1 + t)^{-(2m/2m)n+s_{1}}, \]

where we used the condition $p > (2 + 2s_{2}/\mu_{2}) + (2 - m/\mu_{2}m)n$. Then, we get (94), and in a similar way, one can prove (95)–(97). To estimate the norm $\left\| D^{s_{1}}v_{p}^{| \cdot |}(t, \cdot) \right\| _{L^{2}(R^{n})}$, we use estimate (16) for $\mu_{2} < (2 - m/m)n$ and estimate (40). Then, it follows

\[ \left\| D^{s_{1}}v_{p}^{1}(t, \cdot) \right\| _{L^{2}(R^{n})} \leq \int_{0}^{t} \left( \left\| (u, v) \right\| _{X_{i}(t)} (1 + t)^{-(2m/2m)n-s_{1}} \int_{0}^{t} (1 + r)^{-(m/2)(2m/2m)n+s_{1}} dr \right) \]

\[ \leq \left\| (u, v) \right\| _{X_{i}(t)} (1 + t)^{-(m/2)-s_{1}}, \]

where we used the condition $q > (2 - 2s_{2}/m/n) + (m/2) + (n\mu_{2}/2m)$. We may conclude

\[ \left\| D^{s_{1}}v_{p}^{| \cdot |}(t, \cdot) \right\| _{L^{2}(R^{n})} \leq \left\| (u, v) \right\| _{X_{i}(t)} (1 + t)^{-\mu_{2}/2}. \]
Analogously, by using the condition $q > (2m/n) + (m/2) + (n\mu_2/2m)$, we can prove

\[
\left\| u^{d_l}(t, \cdot) \right\|_{L^2(R^n)} + \left\| v^{d_l}(t, \cdot) \right\|_{L^2(R^n)} + \left\| |D|^{s_2}u^{d_l}(t, \cdot) \right\|_{L^2(R^n)} \leq \| (u, v) \|_{X(t)}^p (1 + t)^{-\mu_1/2}.
\]  

(106)

In this way, we complete the proof of the second case.

(iii) max\{\mu_1; \mu_2\} < (2 - m/m)n.

For $u^{d_l}$, we use estimate (16) for $\mu_1 < (2 - m/m)n$ and estimate (44) to obtain

\[
\left\| |D|^{s_2}u^{d_l}(t, \cdot) \right\|_{L^2(R^n)} \leq \int_0^t \left( \left\| |D|^{s_2}v(t, \cdot) \right\|_{L^2(R^n)} + \left\| |D|^{s_2}u(t, \cdot) \right\|_{L^2(R^n)} + \left\| |D|^{s_2}u(t, \cdot) \right\|_{L^2(R^n)} + \left\| |D|^{s_2}u(t, \cdot) \right\|_{L^2(R^n)} \right) \cdot (1 + r)(1 + t)^{1/2} \cdot (2 - 2s_2)\partial_\tau - (\mu_2/2)\partial_\tau - s_2 \partial_\tau - s_2 \partial_\tau + \left\| u(t, \cdot) \right\|_{X(t)}^p (1 + t)^{-\mu_1/2}.
\]  

(108)

where we used the condition $p > (\mu_1/ \mu_2) + ((2 + 2s_2 - 2s_1)/ \mu_2)$. Then, in the same way, we can prove for $p > (\mu_1/ \mu_2) + ((2 + 2s_2 - 2s_1)/ \mu_2)$ the estimate

\[
\left\| |D|^{s_2}u^{d_l}(t, \cdot) \right\|_{L^2(R^n)} \leq \| (u, v) \|_{X(t)}^p (1 + t)^{-\mu_1/2}.
\]  

(109)

\[
\left\| u^{d_l}(t, \cdot) \right\|_{L^2(R^n)} + \left\| v^{d_l}(t, \cdot) \right\|_{L^2(R^n)} + \left\| |D|^{s_2}u^{d_l}(t, \cdot) \right\|_{L^2(R^n)} \leq \| (u, v) \|_{X(t)}^p (1 + t)^{-\mu_1/2}.
\]  

(110)
Proposition B.1. \( B. \) Fractional Leibniz Rule

\[ \| D^\alpha_\ast v_i (t,) \|_{L^2(\mathbb{R}^d)} + \| v_i (t,) \|_{L^2(\mathbb{R}^d)} + \| v_i (t,) \|_{L^2(\mathbb{R}^d)} + \| D^\alpha_\ast v_i (t,) \|_{L^2(\mathbb{R}^d)} \leq \|(u,v)\|_{L^2(t)} \left(1 + t\right)^{-\left(\nu/2\right)}. \] (111)

From (109)–(111), we get (69).

This completes the proof.

3.3. Proof of Theorem 6. The proof can be obtained by following the same steps of the proof of Theorem 5, but using the estimates of Lemma 3 instead of the estimates of Lemma 2. In other words, we use the rules for fractional powers instead of the fractional chain rule.

4. Concluding Remarks

In this section, we compare the results obtained previously in [40–42] with our results. Then, we consider the following weakly coupled systems of semilinear damped wave equations (see Table 2).

(i) If the values of \( \mu_1 \) and \( \mu_2 \) are sufficiently large, we remark that the obtained results for both models coincide. For this reason, we may interpret the above system for large \( \mu_1 \) and \( \mu_2 \) as an effective one.

(ii) If \( \mu_1 \) and \( \mu_2 \) are small, then we obtain quite different results for the above models. In particular, the admissible ranges for the exponents \( p \) and \( q \) determined are \( p > (\mu_1/\mu_2) + (4/\mu_2) \) and \( q > (\mu_2/\mu_1) + (4/\mu_1) \) which are different to \( F_{E_{2},m}(n) \) for the effective case. We see that the results are more restrictive for the scale-invariant case. For this reason, we should interpret the above system for small \( \mu_1 \) and \( \mu_2 \) as a noneffective one.

Appendix

A. Fractional Gagliardo–Nirenberg Inequality

Proposition A.1. Let \( 1 < p, p_0, p_1 < \infty \) and \( \kappa \in (0, \sigma) \). Then, the following fractional Gagliardo–Nirenberg inequality holds for all \( u \in L^p \cap H^k_{p_1} \):

\[ \| u \|_{H^k_{p_1}} \leq \| u \|_{H^k_{p_2}}^{1-\theta} \| u \|_{H^k_{p_1}}^\theta, \quad \text{for some } \theta \in [0,1], \quad \text{with } k_1 (1 - \theta) + k_2 \theta = \sigma. \] (A.6)

B. Fractional Leibniz Rule

Proposition B.1. Let \( \sigma > 0, \quad 1 \leq r \leq \infty \), and \( 1 < p_1, p_2, q_1, q_2 \leq \infty \) satisfying

\[ \frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{q_2} \] (B.1)

Then, it holds the following fractional Leibniz rule:

\[ \| D^\alpha (fg) \|_{L^r} \leq \| D^\alpha f \|_{L^{p_1}} \| g \|_{L^{q_1}} + \| f \|_{L^{q_1}} \| D^\alpha g \|_{L^{q_2}}, \] (B.2)

for any \( f \in H^k_{p_1} \cap L^{q_1} \) and \( g \in H^k_{q_1} \cap L^{q_2} \).
C. Fractional Chain Rule

Proposition C.1. Let $\sigma \in (0, 1)$, $1 < r, r_1, r_2 < \infty$ and $u, v \in \mathbb{R}$. If $f$ is in $\mathcal{C}^\infty$, the following inequality holds for any $\tau \in [0, 1]$ and

$$|F'(t u + (1 - \tau)v)| \leq \mu(\tau)(G(u) + G(v)),$$

(C.1)

for some continuous nonnegative function $G$ and $\mu \in L^1[0, 1]$. Then, for any $u \in \mathcal{H}^r$ such that $G(u) \in L^1$, provided that

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2},$$

(C.3)

In particular, to estimate norms like $\|u\|_{\mathcal{H}^r}$ or $\|\pm u|^{p-1}\|_{\mathcal{H}^r}$ we use the fractional chain rule and the Gagliardo–Nirenberg inequality. In this way, we may conclude

$$\|\pm u|^{p-1}\|_{\mathcal{H}^r} + \|u\|_{\mathcal{H}^r} \leq \|u\|_{\mathcal{H}^r} \|D|^{p-1}\|_{L^1},$$

(C.4)

where

$$\frac{p - 1}{q_1} + \frac{1}{q_2} = \frac{1}{2}.$$

(C.5)

Theorem C.1. Let $r \in (1, \infty)$, $p > 1$, and $s \in (0, p)$. Let $F(u)$ denote one of the functions $\|u\|^p$ or $\pm u|^{p-1}$. Then, it holds the following inequality:

$$\|F(u)\|_{\mathcal{H}^s} \leq \|u\|_{\mathcal{H}^r} \|u\|_{L^s}^{p-1},$$

(C.6)

for any $u \in \mathcal{H}^r$. The next result is a direct consequence of the previous one for the case of homogeneous Sobolev spaces.

Corollary C.1. Let $r \in (1, \infty)$, $p > 1$, and $s \in (0, p)$. Let $F(u)$ denote one of the functions $\|u\|^p$ or $\pm u|^{p-1}$. Then, it holds the following inequality:

$$\|F(u)\|_{\mathcal{H}^s} \leq \|u\|_{\mathcal{H}^r} \|u\|_{L^s}^{p-1},$$

(C.7)

for any $u \in \mathcal{H}^r$.

D. Fractional Powers

The following approach is useful to estimate the power of a given function and the product of two functions in $\mathcal{H}^s$. This approach is meaningful in the case in which we have the embedding

$$L^\infty(\mathbb{R}^n) \rightarrow \mathcal{H}^s(\mathbb{R}^n),$$

(D.1)

that is, when $s > (n/r)$.

Proposition D.1. Let $0 < 2s^* < n < 2s$. Then, for any function $f \in \mathcal{H}^{2s^*} \cap \mathcal{H}^s$, one has the estimate

$$\|f\|_{L^\infty} \leq \|f\|_{\mathcal{H}^{2s^*}} + \|f\|_{\mathcal{H}^s}.$$

(D.2)

E. Fixed Point Argument

Proposition E.1. The operator $N$ maps $X(t)$ into itself and has one and only one fixed point $u \in X(t)$ if the following inequalities hold:

$$\|Nu\|_{X(t)} \leq C_0(t)\left(\|u_0\|_{X(0)} + C_1(t)\|u\|_{X(t)}^p\right),$$

(E.1)

$$\|Nu - Nv\|_{X(t)} \leq C_2(t)\|u - v\|_{X(t)}\left(\|u\|_{X(t)}^{p-1} + \|v\|_{X(t)}^{p-1}\right),$$

(E.2)

where $C_0(t), C_1(t), C_2(t) \rightarrow 0$ for $t \rightarrow +0$ and $C_0(t), C_1(t), C_2(t) \leq C$ for all $t \in [0, \infty)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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