A Universal Formula for Deformation Quantization on Kähler Manifolds

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May 27, 2010

Abstract

We give an explicit local formula for any formal deformation quantization, with separation of variables, on a Kähler manifold. The formula is given in terms of differential operators, parametrized by acyclic combinatorial graphs.

1 Introduction

Among the first to systematically develop the notion of deformation quantization were Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer. In [1] and [2], they developed the notion of quantization as a deformation of the commutative algebra of classical observables through a family of non-commutative products $\ast_h$, parametrized by a real parameter $h$, and gave an independent formulation of quantum mechanics using this notion.

As opposed to other approaches to quantization, such as geometric quantization, the theory of deformation quantization does not attempt to construct a space of quantum states, but focuses the algebraic structure of the space of observables.

Much work has been done on the theory of deformation quantization, and its formal counterpart, where $h$ is interpreted as a formal parameter. In its most general context, deformation quantization is studied on Poisson manifolds. In [8], Kontsevich proves the existence of a formal deformation quantization on any Poisson manifold. Moreover, he gives a formula for a deformation quantization of any Poisson structure on $\mathbb{R}^n$. His formula describes the star product in terms of bidifferential operators parametrized by graphs and with coefficients given by integrals over appropriate configuration spaces. This bears resemblance in flavour to the construction presented in this paper, which is also based on a certain interpretation of graphs as differential operators.
Other significant constructions of star products include the geometrical construction by Fedosov in [5], where he constructs a deformation quantization on an arbitrary symplectic manifold. Moreover, we should mention the work of Schlichenmaier [10], where he uses the theory of Toeplitz operators to construct a deformation quantization on any compact Kähler manifold.

The question of existence and classification of deformation quantizations on an arbitrary symplectic manifold was solved by De Wilde and Lecomte in [4], where they show that equivalence classes of star products are classified by formal cohomology classes. On Kähler manifolds, existence and classification was addressed by Karabegov in [6], where he proves that deformation quantizations with separation of variables are classified, completely and not only up to equivalence, by closed formal (1,1)-forms, which he calls formal deformations of the Kähler form. In this paper, we shall be dealing exclusively with deformation quantizations, with separation of variables, on Kähler manifolds.

In this setting, Berezin [3] originally wrote down integral formulas for a star product, but he had to make severe assumptions on the Kähler manifold. By interpreting Berezin’s integral formulas formally, and studying their asymptotic behavior, Reshetikhin and Takhtajan [9] gave an explicit formula, in terms of Feynman graphs, for a formal deformation quantization on any Kähler manifold.

Reshetikhin and Takhtajan applied the method of stationary phase to Berezin’s integrals to obtain the asymptotic expansion, and the description in terms of Feynman graphs arises in a natural way through this approach. However, the graphs produced by the expansion of Berezin’s integrals have relations among them, expressing fundamental identities on the Kähler manifold. Moreover, the expansion produces disconnected graphs which prevent the star product from being normalized.

Using the general existence of a unit, Reshetikhin and Takhtajan defined a normalized version of the star product. The coefficients of the unit for the non-normalized star product can be determined inductively by solving the defining equations for the unit, but this approach does not yield an explicit formula for the unit in terms of Feynman graphs, and consequently such a formula for the normalized star product was not given.

The present paper grew out of an attempt to find an explicit formula for this normalized star product of Reshetikhin and Takhtajan in terms of graphs. The crucial observation is that relations among the graphs, as well as the fact that the star product is not normalized, are caused by graphs with cycles.

Given a formal deformation of the Kähler form, we present a local formula for a star product on a Kähler manifold by interpreting graphs as differential operators in a way which is very similar to [9], but we restrict attention to graphs without cycles. We show that the formula in fact defines a global deformation quantization on the
Kähler manifold, with classifying Karabegov form given by the formal deformation of the Kähler form used in the definition of the star product. Thus our construction gives a local formula for any deformation quantization, with separation of variables, on a Kähler manifold.

The main result of the paper is stated in the following theorem.

**Theorem 1.** The unique formal deformation quantization on $M$ with Karabegov form $\omega$ is given by the local formula

$$f_1 \ast f_2 = \sum_{G \in \mathcal{A}_2} \frac{1}{|\text{Aut}(G)|} \Gamma_G(f_1, f_2) \hbar^W(G),$$

for any functions $f_1$ and $f_2$ on $M$.

The various ingredients of this theorem and the formula will be introduced in the following sections, as the definitions of graphs and their partition functions are a bit more involved than what is suitable for the introduction. At this point, let us instead give an overview of the organization of the paper and point to the sections where the relevant notions are introduced.

In the next section, we introduce the notion of deformation quantization, and establish some basic notation. Moreover, we recall how the classifying Karabegov form of a star product with separation of variables is calculated. Then, we move on to describe the relevant types of graphs in section 3, where the set $\mathcal{A}_2$ of acyclic weighted graphs and the total weight $W(G)$ of a graph $G$ are also defined. The interpretation of a graph $G \in \mathcal{A}_2$ as a bidifferential operator is defined in section 4 through the partition function $\Gamma_G(f_1, f_2)$, which depends on a choice of local holomorphic coordinates and a formal deformation $\omega$ of the Kähler form. It is by no means clear that the formula in Theorem 1 defines an associative product, and we will need to rewrite the formula in terms of partition functions of graphs with more structure to prove associativity. This is done in section 5, and associativity is then proved in section 6, using only combinatorial considerations. Finally, the Karabegov form of the local product defined by the formula in Theorem 1 is calculated in section 7, and the proof of the theorem is concluded.

### 2 Deformation Quantization and Kähler Manifolds

A Poisson structure on a smooth manifold $M$ is a skew-symmetric bilinear map $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying the Jacobi identity and the Leibniz rule,

$$\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2 \{f_1, f_3\},$$
with respect to multiplication of functions.

Deformation quantization makes sense for general Poisson manifolds. Let $h$ be a formal parameter, and consider the space $C_h^\infty(M) = C^\infty(M)[[h]]$ of formal power series in $h$ with coefficients in smooth complex-valued functions on the manifold.

**Definition 1.** A formal deformation quantization of a Poisson manifold $M$, is an associative and $C[[h]]$-bilinear product on $C_h^\infty(M)$,

$$f_1 \ast f_2 = \sum_k C_k(f_1, f_2)h^k,$$

which satisfies

$$C_0(f_1, f_2) = f_1 f_2 \quad \text{and} \quad C_1(f_1, f_2) - C_1(f_2, f_1) = -i\{f_1, f_2\},$$

for any functions $f_1$ and $f_2$ on $M$.

Very often, extra conditions are imposed on a deformation quantization. For instance, the operators $C_k$ are often required to be bidifferential operators, in which case the star product is said to be differential. Moreover, we say that the star product is normalized if $1 \ast f = f \ast 1 = f$, for any function $f$, or equivalently if $C_k(1, f) = C_k(f, 1) = 0$, for $k \geq 1$.

An important source of Poisson manifolds are symplectic manifolds. Any symplectic manifold $(M, \omega)$, where $\omega \in \Omega^2(M)$ is non-degenerate and closed, has a canonical Poisson structure defined by

$$\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2}),$$

where $X_f$ denotes the Hamiltonian vector field of a function $f \in C^\infty(M)$, which is the unique vector field satisfying $df = \omega(X_f, \cdot)$.

A Kähler manifold is a symplectic manifold $(M, \omega)$ equipped with a compatible complex structure. If $J$ denotes the corresponding integrable almost complex structure, then compatibility means that

$$g(X, Y) = \omega(X, JY)$$

defines a Riemannian metric on $M$. A deformation quantization $\ast$ on a Kähler manifold is said to be with separation of variables if $f_1 \ast f_2 = f_1 f_2$, whenever $f_1$ is holomorphic or $f_2$ anti-holomorphic.

We shall be working exclusively with deformation quantizations, with separation of variables, on Kähler manifolds, so for the rest of the paper, let $M$ be an arbitrary $m$-dimensional Kähler manifold with complex structure $J$, Riemannian metric $g$ and symplectic form $\omega$. 
A formal deformation of the Kähler form $\omega_{-1}$ is a formal two-form,

$$\omega = \frac{1}{\hbar} \omega_{-1} + \omega_0 + \omega_1 \hbar + \omega_2 \hbar^2 + \cdots,$$

where each $\omega_k$ is a closed form of type $(1,1)$. Karabegov has shown that deformation quantizations with separation of variables on the Kähler manifold $M$, are parametrized by such formal deformations [6].

Let us briefly recall how the Karabegov form of a star product $\ast$ is calculated. Let $z^1, \ldots, z^m$ be local holomorphic coordinates on an open subset $U$ of $M$, and suppose that $\Psi^1, \ldots, \Psi^m$ is a set of formal functions on $U$,

$$\Psi^k = \frac{1}{\hbar} \Psi^k_{-1} + \Psi^k_0 + \Psi^k_1 \hbar + \Psi^k_2 \hbar^2 + \cdots,$$

satisfying

$$\Psi^k \ast z^l - z^l \ast \Psi^k = \delta^{kl}.$$

Then the classifying Karabegov form of $\ast$, which is a global form on $M$, is given by $\omega|_U = -i\hbar (\sum_k \Psi^k dz^k)$ on the coordinate neighborhood $U$.

For the rest of the paper, $\omega$ will denote a fixed formal deformation of the Kähler form. Also, since we shall be working a lot in local coordinates, we fix a set of holomorphic coordinates $z^1, \ldots, z^m$ on an open and contractible subset $U$ of $M$.

Choose a formal potential of the form $\omega$ on $U$, that is, choose a formal function $\Phi = \frac{1}{\hbar} \Phi_{-1} + \Phi_0 + \Phi_1 \hbar + \Phi_2 \hbar^2 + \cdots$, such that $\omega|_U = i\hbar \partial \bar{\partial} \Phi$. The existence of a potential is guaranteed by the fact that $\omega$ is closed and of type $(1,1)$.

On $U$, the Kähler metric is given by the matrix with entries

$$g_{p\bar{q}} = g \left( \frac{\partial}{\partial z^p}, \frac{\partial}{\partial \bar{z}^q} \right) = \partial^2 \Phi_{-1} \frac{\partial}{\partial z^p} \frac{\partial}{\partial \bar{z}^q}.$$

Of course this matrix is invertible, and we denote the entries of the inverse by $g^{p\bar{q}}$. With this notation, the Poisson bracket is given by

$$\{f_1, f_2\} = i \sum_{p\bar{q}} g^{p\bar{q}} \left( \frac{\partial f_1}{\partial z^p} \frac{\partial f_2}{\partial \bar{z}^q} - \frac{\partial f_1}{\partial \bar{z}^\bar{q}} \frac{\partial f_2}{\partial z^p} \right).$$

Having established the basic notions, let us define the class of graphs that we shall be working with.
3 Graphs

A directed graph consists of vertices connected by directed edges. If \( G \) is a graph, the set of vertices is denoted by \( V_G \) and the set of edges by \( E_G \). The way edges are connected to vertices is encoded by two maps \( h_G, t_G : E_G \to V_G \) specifying the head and tail of each edge.

An edge is said to be a loop if it has the same head and tail, and two edges are said to be parallel if they connect the same vertices. A cycle is a path that starts and ends at the same vertex.

We will allow parallel edges in our graphs, but not cycles. In particular, we do not allow any loops.

A graph without cycles is said to be acyclic, and must have at least one vertex, called a source, with only outgoing edges and at least one sink with only incoming edges. We will consider graphs with a distinguished set of numbered vertices, which we will call external. The rest of the vertices are called internal. The sets of external and internal vertices are denoted \( \text{Ext}(G) \) and \( \text{Int}(G) \), respectively. Only an external vertex is allowed to be a source or a sink, and we require that the first external vertex is a source and that the last is a sink.

All graphs will be weighted, in the sense that each internal vertex is assigned a weight from the subset \( \{-1, 0, 1, 2, \ldots\} \) of integers, and we shall require that vertices of weight -1 have degree at least three.

The weight of a vertex \( v \) is denoted by \( w(v) \). If \( G \) is a graph, we define the total weight of the graph by

\[
W(G) = |E_G| + \sum_{v \in \text{Int}(G)} w(v).
\]

![Figure 1: A weighted acyclic graph of total weight 29.](image)

An isomorphism of two graphs is a bijective mapping of vertices to vertices and edges to edges, preserving the way vertices are connected by edges, and preserving
the external edges and their numbering. Moreover, an isomorphism should preserve the weights on internal vertices. If \( G \) is a graph, then the set of automorphisms is denoted by \( \text{Aut}(G) \).

The set of isomorphism classes of finite, acyclic and weighted graphs with \( n \) external vertices is denoted by \( \mathcal{A}_n \). The subset of graphs with total weight \( k \) is denoted by \( \mathcal{A}_n(k) \).

4 Partition Functions

In this section, we define the partition function \( \Gamma_G(f_1, \ldots, f_n) \in C^\infty(U) \), for any graph \( G \in \mathcal{A}_n \) and any functions \( f_1, \ldots, f_n \) on \( U \).

Let us first introduce some notation. If \( f \in C^\infty(U) \) is a function, we define, for each pair of non-negative integers \( p \) and \( q \), a covariant tensor \( f^{(p,q)} \) on \( U \) of type \((p,q)\) by

\[
 f^{(p,q)}\left( \frac{\partial}{\partial z^{i_1}}, \ldots, \frac{\partial}{\partial z^{i_p}}, \frac{\partial}{\partial \bar{z}^{j_1}}, \ldots, \frac{\partial}{\partial \bar{z}^{j_q}} \right) = \frac{\partial^{p+q} f}{\partial z^{i_1} \cdots \partial z^{i_p} \partial \bar{z}^{j_1} \cdots \partial \bar{z}^{j_q}}.
\]

Assign to each vertex \( v \in V_G \), with \( p \) incoming and \( q \) outgoing edges, a tensor by the following rule. If \( v \) is the \( k \)-th external vertex, we associate the tensor \( f^{(p,q)}_k \), and if \( v \) is an internal vertex of weight \( w \), we associate the tensor \(-\Phi^{(p,q)}_w\).

Then, we define the partition function \( \Gamma_G(f_1, \ldots, f_n) \) to be the function given by contracting the tensors associated to each vertex, using the Kähler metric, as prescribed by the edges of the graph. Since the tensors are completely symmetric, this contraction is well-defined.

Notice that the partition function depends on the deformation \( \omega \) of the Kähler form, but not on choice of potential \( \Phi \). This is because every internal vertex has at least one incoming and outgoing edge, and so the potential is differentiated at least once in both a holomorphic and an anti-holomorphic direction.

Using the partition functions of graphs, we define the following formal multi-differential operator

\[
 D(f_1, \ldots, f_n) = \sum_{G \in \mathcal{A}_n} \frac{1}{|\text{Aut}(G)|} \Gamma_G(f_1, \ldots, f_n) h^{W(G)}.
\]

If we define the multi-differential operators

\[
 D_k(f_1, \ldots, f_n) = \sum_{G \in \mathcal{A}_n(k)} \frac{1}{|\text{Aut}(G)|} \Gamma_G(f_1, \ldots, f_n),
\]

then \( D \) is given by the formal power series of operators \( D = \sum_k D_k h^k \).

The first result of the paper is stated in the following theorem.
Theorem 2. The product
\[ f_1 \star f_2 = D(f_1, f_2) = \sum_k D_k(f_1, f_2) h^k \]
defines a normalized formal deformation quantization with separation of variables on the coordinate neighborhood \( U \).

Since the only graph with two external vertices and total weight zero is the graph with no edges and no internal vertices, we clearly have
\[ D_0(f_1, f_2) = f_1 f_2. \]
Moreover, there is only one graph of total weight one, namely the graph with no internal vertices and only one edge connecting the two external vertices. Therefore
\[ D_1(f_1, f_2) = \sum_{pq} g^{ap} \frac{\partial f_1}{\partial \bar{z}^q} \frac{\partial f_2}{\partial z^p}, \]
and we get that
\[ D_1(f_1, f_2) - D_1(f_2, f_1) = -i \{ f_1, f_2 \}, \]
as required of a deformation quantization.

Note, that the expression for the star product is with separation of variables, since the first external vertex has no incoming edges, and the second has no outgoing. Also, note that the star product is normalized, since any graph of total weight higher than zero must have edges, and therefore the external vertices must have degree at least one.

The only part of Theorem 2 that remains to be proved is associativity of the star product. We will prove this by combinatorial arguments involving certain modifications on graphs.

Since the size of the automorphism group of a graph does not behave well under these modifications, the expression for the star product given above is not suitable to work with. Therefore, we need to find a different expression which behaves better when modifying the graphs.

5 Alternative Expression for the Operator \( D \)

Let us be a little more explicit in writing out the partition function \( \Gamma \). To this end, we need to introduce further structure on graphs.
A labelling $l$ of a graph $G \in \mathcal{A}_n$ is an assignment of indices to the incoming and outgoing edges of each vertex of the graph. If $v$ is a vertex and $e$ is an incident edge, then the index specified by the labelling is an integer in the set $\{1, \ldots, m\}$ and is denoted by $l(v, e)$.

Figure 2: A labelled graph.

An isomorphism of labelled graphs is of course an isomorphism preserving the labels. The set of labellings of a graph $G$ is denoted by $\mathcal{L}(G)$, and the set of isomorphism classes of labelled graphs with $n$ external edges is denoted by $\mathcal{L}_n$.

Let us introduce a partition function $\Lambda^l_G(f_1, \ldots, f_n)$ of a labelled graph $G$ with labelling $l$. For notational convenience, we first define a function $F_{f_1, \ldots, f_n}: V_G \sqcup E_G \rightarrow C^\infty(U)$, which assigns a function to each vertex and edge of the graph.

Let $v$ be a vertex of $G$, with $p$ incoming and $q$ outgoing edges, and suppose that the incoming edges are labelled with indices $i_1, \ldots, i_p$, and the outgoing vertices are labelled with indices $j_1, \ldots, j_q$. If $v$ is the $k$'th external vertex, then we define

$$F_{f_1, \ldots, f_n}(v) = \frac{\partial^{p+q} f_k}{\partial z^{i_1} \cdots \partial z^{i_p} \partial \bar{z}^{j_1} \cdots \partial \bar{z}^{j_q}}.$$  

If $v$ is an internal vertex with weight $w$, then we define

$$F_{f_1, \ldots, f_n}(v) = -\frac{\partial^{p+q} \Phi_w}{\partial z^{i_1} \cdots \partial z^{i_p} \partial \bar{z}^{j_1} \cdots \partial \bar{z}^{j_q}}.$$  

Notice that this does not depend on the choice of potential, since internal vertices have at least one incoming and outgoing edge. Finally, if $e$ is an edge from $u$ to $v$, and we let $s = l(u, e)$ and $r = l(v, e)$, then we define $F_{f_1, \ldots, f_n}(e) = g^{sr}$.

Using this, we define

$$\Lambda^l_G(f_1, \ldots, f_n) = \left( \prod_{v \in V_G} F_{f_1, \ldots, f_n}(v) \right) \left( \prod_{e \in E_G} F_{f_1, \ldots, f_n}(e) \right).$$  

From the definition of $\Gamma_G$, it should be obvious that

$$\Gamma_G(f_1, \ldots, f_n) = \sum_{l \in \mathcal{L}(G)} \Lambda^l_G(f_1, \ldots, f_n).$$
Therefore, we have the following expression for $D$,

$$D(f_1, \ldots, f_n) = \sum_{G \in A_n} \sum_{l \in L(G)} \frac{1}{|\text{Aut}(G)|} \Lambda_G^l(f_1, \ldots, f_n) w^{W(G)}.$$ 

The fact that we have written out $D$ in terms on $\Lambda$ will aid us in later arguments. However, the size of the automorphism group does not behave well when modifying graphs as we shall later do. Therefore, we will need to rewrite our expression for $D$ further.

If $G$ is a graph in $A_n$, a circuit structure on $G$ is a total order, for each vertex of $G$, of the incoming as well of the outgoing edges of that vertex. This gives rise to a numbering of the incoming as well as the outgoing edges at each vertex, and if $v$ is a vertex of $G$ with an incident edge $e$, the circuit structure therefore specifies a natural number $c(v, e)$. An isomorphism of circuit graphs is an isomorphism which preserves the ordering on the incoming and outgoing edges at each vertex.

Figure 3: Different representations of a circuit graph.

Figure 3 shows two ways of representing a circuit structure graphically. The latter, with rectangular vertices, is usually preferred. This also motivates the name circuit structure, as it resembles a diagram of an electrical circuit, where a number of chips, with input and output pins, are connected by wires. This analogy is also supported by the fact that our graphs are acyclic.

The set of circuit structures on $G$ is denoted by $\mathcal{C}(G)$, and the set of isomorphism classes of circuit graphs with $n$ external vertices is denoted by $\mathcal{C}_n$.

Very often, we shall be working with graphs equipped with both a labelling and a circuit structure, and we will need to enforce a certain compatibility between the two structures.

If $G \in A_n$ is a graph equipped with a labelling $l$ and a circuit structure $c$, we say that $l$ and $c$ are compatible if for any vertex $v$ and any two edges $e$ and $e'$ incident to $v$, with the same orientation, we have that $c(v, e) \leq c(v, e')$ implies $l(v, e) \leq l(v, e')$. In other words, the incoming edges of a vertex should be labelled
ascendingly with respect to the ordering given by the circuit structure, and likewise for the outgoing edges.

Figure 4: A labelled circuit graph.

If $G$ is a graph with labelling $l$, the set of compatible circuit structures is denoted by $C(G, l)$. The set of isomorphism classes of labelled graphs with a compatible circuit structure is denoted $L_n^C$.

Given a labelled graph, the number of compatible circuit structures will be important to us. To calculate this, we will need some notation.

Recall that a multi-index is an $m$-tuple $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$. The length of $\alpha$ is defined to be $|\alpha| = \alpha_1 + \ldots + \alpha_m$, and we define $\alpha! = \alpha_1! \cdots \alpha_m!$. A labelling of a graph assigns two multi-indices to each vertex in a canonical way. More precisely, if $G$ is a graph with labelling $l$, then we have two canonically defined maps $\alpha_I, \beta_I : V_G \to \mathbb{N}_0^m$. If $v$ is a vertex of $G$, then the multi-index $\alpha_I(v)$ counts the number of occurrences of each label among the incoming edges of $v$. Similarly, the multi-index $\beta_I(v)$ counts the occurrences of each label among the outgoing edges.

Now, given the graph $G$ with labelling $l$, the number of compatible circuit structures is given by

$$C(G, l) = \prod_{v \in V_G} \alpha_I(v)! \beta_I(v)!.$$ 

Using this, we can rewrite the formula for the operator $D$ as

$$D(f_1, \ldots, f_n) = \sum_{G \in \mathcal{A}_n} \sum_{l \in \mathcal{L}(G)} \sum_{e \in \mathcal{L}(G,l)} \frac{1}{|\text{Aut}(G)| |\mathcal{C}(G,l)|} \Lambda_G^l(f_1, \ldots, f_n) \mu W(G),$$

since the circuit structure does not influence on the value of the partition function.

Suppose that $G \in \mathcal{A}_n$ is any graph with $n$ external edges. If we pick a labelling $l$ and a compatible circuit structure $c$, then $(G, l, c)$ represents an element of $L_n^C$. If we choose a different labelling $l'$ and circuit structure $c'$ on $G$, then $(G, l', d')$ represents the same isomorphism class in $L_n^C$ if and only if there exists an automorphism of $G$, which sends the labelling $l$ to $l'$ and the circuit structure $c$ to $c'$. Thus, we have proved the following proposition.
Proposition 1. The operator $D$ is given by

$$D(f_1, \ldots, f_n) = \sum_{G \in \mathcal{L}_n} \frac{1}{C(G)} \Lambda_G(f_1, \ldots, f_n) h^W(G),$$

for any functions $f_1, \ldots, f_n$.

As we shall often do when the additional structure is clear from the context, we have omitted the labelling from the notation in this proposition.

6 Associativity of the Star Product

With the alternative expression for the operator $D$, given in Proposition 1, we are ready to prove associativity of the star product. This is an immediate corollary of the following theorem.

Theorem 3. We have

$$D(f_1, D(f_2, f_3)) = D(f_1, f_2, f_3) = D(D(f_1, f_2), f_3),$$

for any functions $f_1, f_2$ and $f_3$.

We shall only prove the first equality of this theorem. The second equality follows by analogous arguments.

To prove Theorem 3, we must have a better understanding of the expression $D(f_1, D(f_2, f_3))$. Writing out this expression, we have

$$D(f_1, D(f_2, f_3)) = \sum_{G_1 \in \mathcal{L}_n} \sum_{G_2 \in \mathcal{L}_n} \frac{1}{C(G_1)C(G_2)} \Lambda_{G_1}(f_1, \Lambda_{G_2}(f_2, f_3)) h^W(G_1) h^W(G_2),$$

and we see that $\Lambda_{G_1}(f_1, \Lambda_{G_2}(f_2, f_3))$ is the crucial part to understand.

Before we prove Theorem 3, let us illustrate, with an example, how graphs in the expression for $D(f_1, D(f_2, f_3))$ arise from $D(f_1, D(f_2, f_3))$.

Example 1. Suppose that we have two graphs $G_1$ and $G_2$ in $\mathcal{L}_2^C$, as depicted in Figure 5.

We think of $G_2$ as representing a term of the inner $D$ in $D(f_1, D(f_2, f_3))$, and $G_1$ as representing a term of the outer $D$. More precisely, we let

$$\hat{f} = \Lambda_{G_2}(f_2, f_3) = \frac{\partial^2 f_1}{\partial z^1 \partial z^1} \frac{\partial^3 \Phi_0}{\partial z^1 \partial z^2 \partial z^3} \frac{\partial^4 \Phi_1}{\partial z^2 \partial z^3} \frac{\partial f_2}{\partial \bar{z}^3} g_{14} g_{41} g_{32} g_{34} g_{31},$$
and we want to calculate the partition function
\[ \Lambda_{G_1}(f_1, \hat{f}) = -\frac{\partial^2 f_1}{\partial z^2 \partial \bar{z}^4} \frac{\partial^5 \Phi_2}{\partial z^3 \partial z^5 \partial \bar{z}^3} \frac{\partial^1 \hat{f}}{\partial z^1} g^{21} g^{24} g^{32} g^{34} g^{31}. \]

Informally, we have the picture in Figure 6 in mind as a graphical representation of this expression.

Since \( \hat{f} \) is given by a product, the Leibniz rule says that \( \frac{\partial^4 \hat{f}}{\partial z^1 \partial \bar{z}^2 \partial z^3 \partial \bar{z}^4} \) is given by a sum, where each term represents a certain way of distributing the partial derivatives among the factors.

Let us focus on one such term, say the one where the first and the third partial derivative from the left hit the factor \( \frac{\partial^3 f_1}{\partial z^3} \), the second derivative hits the factor \( \frac{\partial^1 \Phi_0}{\partial z^1} \), and the fourth hits the factor \( g^{34} \). That term is then given by

\[
\frac{\partial^3 f_1}{\partial z^3} \frac{\partial^5 \Phi_0}{\partial z^1 \partial z^3} \frac{\partial^1 \Phi_1}{\partial z^1} \frac{\partial^3 f_2}{\partial z^4} g^{14} g^{24} \frac{\partial g^{34}}{\partial z^4} \frac{\partial g^{31}}{\partial \bar{z}^3}.
\]

But partial derivatives of the inverse metric can be easily expressed in terms of partial derivatives of the metric, as in

\[
\frac{\partial g^{34}}{\partial z^4} = -\sum_{pq} g^{3p} \frac{\partial g_{pq}}{\partial z^4} g^{q4} = -\sum_{pq} g^{3p} \frac{\partial^3 \Phi_{-1}}{\partial z^1 z^3} \frac{\partial^3 \Phi_{-1}}{\partial z^1 z^4} \frac{\partial^3 \Phi_{-1}}{\partial z^1 z^1} \frac{\partial g^{34}}{\partial z^4}.
\]

If we choose particular values, say \( p = 1 \) and \( q = 2 \), for the summation variables, then we arrive at

\[
-\frac{\partial^3 f_1}{\partial z^3} \frac{\partial^5 \Phi_0}{\partial z^1 \partial z^3} \frac{\partial^1 \Phi_1}{\partial z^1} \frac{\partial^3 f_2}{\partial z^4} g^{14} g^{24} g^{31} g^{21} g^{34} g^{31}.
\]
as an example of what terms in the expression for \( \partial_z^4 \hat{f} \) look like.

If we insert this into the expression for \( \Lambda_{G_1}(f_1, \hat{f}) \) above, we get an example of what terms in the expression for \( D(f_1, D(f_2, f_3)) \) look like. But this particular example can be represented graphically by \( \Lambda_G(f_1, f_2, f_3) \), where \( G \in \mathcal{L}_3^C \) is the graph shown in Figure 7.

![Figure 7: A fusion G of the two graphs G_1 and G_2.](image)

With this concrete example in mind, let us turn to more general considerations. The graph in Figure 7 is an example of a fusion of the graphs \( G_1 \) and \( G_2 \). Let us define this notion more carefully.

Let \( G_1 \) and \( G_2 \) be two graphs in \( \mathcal{L}_2^C \). A fusion of \( G_1 \) onto \( G_2 \) is a graph \( G \in \mathcal{L}_3^C \) with three external vertices, obtained through the following procedure. Cut out the second external vertex of \( G_1 \), leaving a collection of labelled edges with loose ends. Connect each of these loose ends, one at a time, to the graph \( G_2 \) in one of two possible ways. The first is to connect a loose end to one of the vertices of \( G_2 \), extend the circuit structure at the vertex, in any way compatible with the labelling, to include the newly attached edge. The second possibility is to attach a loose end to one of the edges of \( G_2 \). This is done by adding a vertex of weight -1 on the edge, choosing any labelling of the two edges incident to the new vertex, attaching the loose end to the new vertex and choosing a circuit structure at the vertex. Finally, the first and second external vertices of \( G_2 \) will be the second and third external vertex of the fusion, respectively.

Clearly, a fusion of two graphs results in a labelled circuit graph with 3 external vertices. The set of isomorphism classes of such graphs, that can be obtained from two graphs \( G_1 \) and \( G_2 \) through a fusion procedure, is denoted \( \mathcal{F}(G_1, G_2) \).

Given a labelled circuit graph \( G \in \mathcal{L}_3^C \), with three external vertices, it is natural to ask if this can be obtained as a fusion of two graphs \( G_1 \) and \( G_2 \) in \( \mathcal{L}_3^C \). Moreover, it is natural to ask how much information about the graphs \( G_1 \) and \( G_2 \) is encoded in a fusion.

Given two vertices \( u \) and \( v \) of a graph, we say that \( v \) is a successor of \( u \) if there exists a directed path from \( u \) to \( v \). A crucial observation is that when \( G_1 \) is fused
to $G_2$, any vertex in $G_2$ which is a successor of the first external vertex in $G_2$ will be a successor of the second external vertex in the fusion. Moreover, vertices that arose by attaching a loose end to an edge of $G_2$ will also be successors of the second external vertex. On the other hand, none of the vertices of $G_1$ will succeed the second external vertex in the fusion.

These observations can be used to reconstruct nearly all the information about the structure of the graphs $G_1$ and $G_2$ from a fusion of these. Moreover, as we shall see, any labelled circuit graph with three external vertices arises as a fusion.

Suppose that $G \in \mathcal{L}_3^C$ is any labelled circuit graph. We seek two labelled circuit graphs $G_1$ and $G_2$ such that $G \in \mathcal{F}(G_1,G_2)$. We can completely determine the isomorphism class of $G_2$ in $\mathcal{L}_2^C$ by the following procedure. Delete all vertices from $G$ which are not successors of the second external vertex, as well as all edges incident to at least one such vertex. The result may contain vertices of weight -1 and degree 2. These are the remnants of vertices arising during the fusion when a loose edge end is connected to an edge of $G_2$. Every such vertex is deleted and the resulting two loose ends are spliced, forgetting their labelling. Finally, the second and third external vertices are the only external vertices left, and they will be the first and second external vertices in $G_2$, respectively.

In a similar way, we can almost determine the isomorphism class of the labelled circuit graph $G_1$ by deleting all successors of the second external vertex in $G$, and all edges between two such successors, and then connect all the remaining loose edge ends to a new vertex, which will be the second external vertex of $G_1$. There is however no canonical way of telling what the circuit structure at the second external vertex should be.

To deal with this ambiguity, we define an equivalence relation on the set $\mathcal{L}_2^C$ of labelled circuit graphs with two external vertices. Consider two graphs $G$ and $G'$ in $\mathcal{L}_2^C$, with labellings $l$ and $l'$ and circuit structures $c$ and $c'$. We say that these graphs are equivalent, and we write $G \sim G'$, if there exists an isomorphism between $G$ and $G'$ which preserves the labelling at all vertices, and which preserves the circuit structure, except possibly at the second external vertex. In the discussion above, the equivalence class of the graph $G_1$ is then completely determined.

We summarize our findings in the following proposition

**Proposition 2.** For any labelled circuit graph $G \in \mathcal{L}_3^C$, there exist two labelled circuit graphs $G_1,G_2 \in \mathcal{L}_2^C$ such that $G \in \mathcal{F}(G_1,G_2)$. Moreover, the equivalence class of $G_1$ is uniquely determined by $G$, and so is the isomorphism class of $G_2$.

When calculating $D(f_1,D(f_2,f_3))$, we are basically faced with the task of calculating $\Lambda_{G_1}(f_1,\Lambda_{G_2}(f_2,f_3))$ for any two labelled circuit graphs $G_1$ and $G_2$. As illustrated in Example 1, this is given by a sum, where each term can be represented by a fusion of $G_1$ and $G_2$. 
Now suppose that an edge, incident to the second external vertex in \( G_1 \) and with label \( j \), is attached to a vertex \( v \) in \( G_2 \), and that \( v \) already has \( k \) incoming edges with label \( j \). Then, when extending the circuit structure at \( v \) to include the newly attached edge, there are \( k + 1 \) ways of placing the new edge in the ordering of the incoming edges.

Moreover, suppose that \( l \) is the labelling of \( G_1 \), and let \( u \) be the second external vertex. Then, the size of the equivalence class \([G_1]\) is given by \( \alpha_l(u)\).

These observations suffice to realize that

\[
\sum_{G \in [G_1]} \frac{1}{C(G)C(G_2)} \Lambda_G(f_1, \Lambda_{G_2}(f_2, f_3)) = \sum_{G \in \mathcal{F}(G_1, G_2)} \frac{1}{C(G)} \Lambda_G(f_1, f_2, f_3).
\]

Since \( W(G_1) + W(G_2) = W(G) \) if \( G \in \mathcal{F}(G_1, G_2) \), we can multiply the left-hand side by \( h^{W(G_1)}h^{W(G_2)} \) and the right-hand side by \( h^{W(G)} \), and sum over all graphs \( G_2 \in \mathcal{L}_2^C \) and all equivalence classes \([G_1]\) in \( \mathcal{L}_2^C / \sim \) to get

\[
D(f_1, D(f_2, f_3)) = \sum_{[G_1]} \sum_{G_2 \in \mathcal{L}_2^C} \frac{1}{C(G_1)C(G_2)} \Lambda_{G_1}(f_1, \Lambda_{G_2}(f_2, f_3))h^{W(G_1)}h^{W(G_2)}
\]

\[
\quad = \sum_{[G_1]} \sum_{G_2 \in \mathcal{L}_2^C} \sum_{G \in \mathcal{F}(G_1, G_2)} \frac{1}{C(G)} \Lambda_G(f_1, f_2, f_3)h^{W(G)}.
\]

But as \([G_1]\) runs through all equivalence classes of \( \mathcal{L}_2^C / \sim \), and \( G_2 \) runs through \( \mathcal{L}_2^C \), then Proposition 2 tells us that the sets \( \mathcal{F}(G_1, G_2) \) partition the set \( \mathcal{L}_3^C \), that is, they form a collection of disjoint sets whose union is all of \( \mathcal{L}_3^C \). Thus, we conclude that

\[
D(f_1, D(f_2, f_3)) = \sum_{G \in \mathcal{L}_3^C} \frac{1}{C(G)} \Lambda_G(f_1, f_2, f_3) = D(f_1, f_2, f_3).
\]

This proves the first equality of Theorem 3. The other equality is proved by similar methods, and therefore the theorem is proved. This also proves Theorem 2, which is an immediate corollary of Theorem 3.

### 7 Coordinate Invariance and Classification

In this section, we prove that the local star product of Theorem 2 is independent of the coordinates used in its definition. This implies that it defines a global star product on \( M \), and as we shall see, the Karabegov form of this global star product is given by \( \omega \).

The claims above will follow easily from the following theorem.
Theorem 4. The local star product $\ast$ on $U$ has Karabegov form $\omega|_U$.

Proof. We shall prove that the formal functions $\Psi^r = \partial \Phi / \partial z^r$ satisfy the relation

$$
\Psi^r \ast z^s - z^s \ast \Psi^r = \delta^{rs}.
$$

(1)

This will prove the theorem, since $\omega|_U = i \bar{\partial} \partial \Phi = -i \bar{\partial} (\sum_k \Psi^k d z^k)$.

Clearly, we have $D_0(\Psi^r, z^s) - D_0(z^s, \Psi^r) = 0$ and

$$
D_1(\Psi^r, z^s) - D_1(z^s, \Psi^r) = -i \{ \Psi^r, z^s \} = \delta^{rs},
$$

so the identity (1) is equivalent to the system of identities

$$
\sum_{l=-1}^{k-1} D_{k-l}(\Psi^r, z^s) = 0, \quad k \geq 1.
$$

To prove this, we define a modification on graphs called a budding. If $l > -1$ and $G \in A_2(k-l)$ is a graph, we define the budded graph $B(G) \in A_2(k+1)$ by the following procedure. Let $u$ denote first external vertex of $G$ and convert this into an internal vertex of weight $l$. Then add a new first external vertex and connect this to $u$ by a single edge.

![Figure 8: A budding of a graph.](image)

We had to exclude the case $l = -1$, since the first external vertex of $G$ might have degree one, in which case the budded graph would not satisfy the rule that internal vertices of weight -1 must have degree at least three. However, if we let $A_2^1(k+1)$ be the set of graphs with degree one on the first external vertex, and $A_2^{<1}(k+1)$ be the set of graphs with degree more than one on the first external vertex, then the budding construction defines a map $B: A_2^{<1}(k+1) \to A_2^1(k+1)$.

We conclude that the budding construction gives a map

$$
B: A_2^{<1}(k+1) \cup \bigcup_{l=0}^{k-1} A_2(k-l) \to A_2^1(k+1).
$$

Clearly, this map is a bijection, as the inverse map is easily constructed. Moreover, it is clear that the budding map preserves the size of the automorphism group.
Now, the crucial property of the budding map is that
\[ \Gamma_{\mathcal{B}(G)}(\Psi_{r-1}^r, z^s) = -\Gamma_G(\Psi_r^r, z^s), \]
for any graph \( G \) in the domain of \( \mathcal{B} \). Since \( B \) is a bijection, which preserves the size of the automorphism group, this implies that
\[
\sum_{l=0}^{k-1} \sum_{G \in \mathcal{A}(k-l)} \frac{1}{|\text{Aut}(G)|} \Gamma_G(\Psi_r^r, z^s) + \sum_{l=0}^{k-1} \sum_{G \in \mathcal{A}(k-l)} \frac{1}{|\text{Aut}(G)|} \Gamma_G(\Psi_r^r, z^s) + \sum_{l=0}^{k-1} \sum_{G \in \mathcal{A}(k-l)} \frac{1}{|\text{Aut}(G)|} \Gamma_G(\Psi_r^r, z^s) = 0.
\]
This proves the theorem.

Karabegov’s classification has the obvious property that restriction of a star product to an open subset corresponds to restriction of the Karabegov form. Therefore, it follows immediately that \( \star \) is the restriction of the unique star product on \( M \) with Karabegov form \( \omega \). In particular, the explicit expression given in Theorem 1 must be independent of the local coordinates used. This finishes the proof of the main result given in Theorem 1, which summarizes all of our findings.

We remark that Theorem 1 gives an explicit formula, to all orders, of the Berezin star product with trivial Karabegov form \( \frac{1}{\pi} \omega_{-1} \).

Moreover, in [7] it was shown that the Berezin-Toeplitz star product, which is defined on compact Kähler manifolds through asymptotic expansions of products of Toeplitz operators [10], is a differential star product whose opposite star product is with separation of variables and has Karabegov form given by \( -\frac{1}{k} \omega_{-1} + \rho \), where \( \rho \) denotes the Ricci form on the Kähler manifold. Using Theorem 1, we can therefore give an explicit formula for the Berezin-Toeplitz star product to all orders.

The main theorem implies that the operator \( D \) is coordinate independent when applied to two functions, and hence also when applied to three by Theorem 3. In fact we conjecture that the general formula for \( D \) is coordinate invariant and that there are relations analogous to Theorem 3, when applied to a larger collection of functions.

As a closing remark, we think it would be very interesting to use the formula presented in this paper to try to find invariant expressions for the star products in terms of covariant derivatives and global forms.
Acknowledgements

I would like to thank Jørgen Ellegaard Andersen and Nicolai Reshetikhin for bringing the paper [9] to my attention and for many enlightening discussions.
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