Gravitational Collapse of an Infinite Dust Cylinder

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Abstract

We examine the gravitational collapse of an infinite cylindrical distribution of time like dust. In order to simplify the calculation we make an assumption that the axial and azimuthal metric functions are equal. It is shown that the resulting solution describes homogeneous collapse. We show that the interior metric can be matched to a time dependent exterior. We also discuss the nature of the singularity in the matter region and show that it is covered.

1 Introduction

While there have been many analytical studies of spherical gravitational collapse, there have been only few investigations of non-spherical collapse, the obvious reason being the complexity of Einstein equations. Thus most of our present understanding of cosmic censorship and the nature of singularities stems from studies of spherical collapse. Given the impact that the results of these studies have had on our understanding of the end states of classical collapse, it is of interest to ask what may emerge from studies of gravitational collapse with different topologies.

A very useful study of the collapse of an infinite dust cylinder was initiated by Thorne \cite{1}. This is described by the metric

\[ ds^2 = e^{(G-P)} (dt^2 - dr^2) - e^P dz^2 - \alpha^2 e^{-P} d\phi^2. \]  

(1)

The metric functions depend on \( t \) and \( r \). This form of the metric can be inferred by first writing down the static Weyl axisymmetric metric for an infinite line Newtonian source

\[ ds^2 = e^P dt^2 - \alpha^2 e^{-P} d\phi^2 - e^{(G-P)} (dr^2 + dz^2) \]  

(2)

in which the metric functions depend only on \( r \) and \( z \). The transformation \( t \rightarrow iz, z \rightarrow it \) in this metric then yields the metric \cite{1}. It is invariant under Lorentz transformations in the \((t, r)\) plane and can be used to represent the collapse of a cylindrical dust shell. Apostolatos and Thorne \cite{2}

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used the above metric to show that rotation can halt the collapse of the cylinder. The formation of a naked singularity for the infinite cylindrical shell collapse was established by Echiverria [3]. Naked singularities for counter-rotating dust shells were shown to occur, by Goncalves and Jhingan [4] and by Nolan [5], and an exact dynamical solution was given by Pereira and Wang [6]. The collapse of infinite cylindrical dust clouds has been studied also by Chiba [7], by Senovilla and Vera [8] and by Bondi [9].

Although the collapse of an infinite cylinder may not constitute a “realistic” collapse scenario, it can simulate the collapse of a finite “bar” or spindle-like matter distribution very near the central regions of the spindle. In this sense the collapse of an infinite cylinder is of astrophysical interest. More importantly, however, the problem probes the structure of the general theory of relativity. Our analysis in the present paper is of an infinite cylindrical cloud of time-like dust and is motivated by what is known for spherical time-like dust collapse, i.e., the Tolman-Bondi spacetime. We make one assumption, namely that the azimuthal and axial metric components are equal to each other.

2 Collapse of an infinite dust cylinder

Consider an infinite cylindrical cloud of pressure-less, inhomogeneous, time-like dust described by the stress tensor

\[ T_{\mu\nu} = \epsilon(x)u_\mu u_\nu, \]  

where \( u^2 = -1 \). We first set up the metric for the interior of this infinite dust cylinder, using comoving coordinates \((t, r, z, \phi)\). Assuming that \( z \) represents the axis of symmetry of the cylinder, the metric functions depend only on \( t \) and \( r \). The dust cloud is assumed to extend up to some radius \( r_s \); matching to a vacuum exterior will be discussed in Section 5. As a consequence of axisymmetry the functions \( g_{tt}, g_{t\phi}, g_{r\phi} \) all vanish. Moreover, for an infinite cylinder, invariance under \( z \rightarrow -z \) implies that \( g_{tz} \) and \( g_{rz} \) also vanish. We can use the freedom of coordinate transformations from the pair \((t, r)\) to a new pair \((t', r')\) to set two functions to zero: these are \( g_{tr} \) and the radial velocity \( u_r \). As a result of the latter condition, \( r \) is determined to be a comoving coordinate. Since the matter is dust, the metric can be comoving and synchronous, so that \( g_{00} = -1 \). The metric for an infinite dust cylinder can hence be written in terms of three unknown functions of two variables as

\[ ds^2 = -dt^2 + L^2(t, r)dr^2 + M^2(t, r)dz^2 + B^2(t, r)d\phi^2. \]  

The energy-density \( \epsilon(t, r) \) being the only non-zero component of the energy-momentum tensor, Einstein’s equations for this spacetime are (dot and prime denote derivatives w.r.t. \( t \) and \( r \), respectively):

\[ G_{rr} = 0 : \frac{\dot{M}}{M} + \frac{\dot{B}}{B} + \frac{\ddot{M}}{M} - \frac{B'M'}{BML^2}, \]  

\[ G_{zz} = 0 : \frac{\dot{B}}{B} + \frac{\dot{L}}{L} + \frac{\ddot{B}}{B} - \frac{B''L - B'L'}{BL^3}. \]
\[ G_{\phi\phi} = 0 : \frac{\dot{M}}{M} + \frac{\dot{L}}{L} + \frac{\dot{M} \dot{L}}{ML} = \frac{M''L - M'L'}{ML^3}, \quad (7) \]

\[ G_{tr} = 0 : \frac{\dot{M}'}{M} + \frac{\dot{B}'}{B} = \frac{\dot{L}}{L} \left( \frac{M'}{M} + \frac{B'}{B} \right), \quad (8) \]

\[ G_{tt} = \epsilon(t, r) : -\frac{B''L - B'L'}{BL^3} - \frac{M''L - M'L'}{ML^3} - \frac{B'M'}{BL^2} + \frac{\dot{B} \dot{L}}{BL} + \frac{\dot{M} \dot{L}}{ML} + \frac{\dot{B} \dot{M}}{BM} = \epsilon. \quad (9) \]

The conservation of the energy-momentum tensor gives the relation

\[ \epsilon(t, r) = \frac{2\psi(r)}{LMB}. \quad (10) \]

where \( \psi(r) \) is an integration function, to be determined by the initial data. We also note that adding equations (5), (6) and (7), and using (9) and (10) gives the relation

\[ \frac{\dot{M}}{M} + \frac{\dot{B}}{B} + \frac{\dot{L}}{L} = -\frac{\psi(r)}{LMB}. \quad (11) \]

Instead of working with the five Einstein equations (5)-(9) we will work with the equivalent set given by the equations (5), (8), (10), (11) and the difference (6)−(7).

We now assume that \( B(t, r) \equiv rM(t, r) \). For a finite axisymmetric object like a spheroid this would probably not be allowed by Einstein equations but, as we see below, the equations are self-consistent when this assumption is made for an infinite cylinder. The study of this system may be thought of as a prelude to examining the most general case, when \( B \) and \( M \) are not related. The physical meaning of the assumption is that the object shrinks at the same rate along the radial direction and the axis, so that its ‘prolateness’ or ‘oblateness’ does not change with time. More importantly, we show that this assumption implies that the matter distribution is homogeneous. It is not the most general homogeneous solution, but a special case.

With the above assumption the metric (11) becomes

\[ ds^2 = -dt^2 + L^2(t, r)dr^2 + M^2(t, r)\left[ dz^2 + r^2d\phi^2 \right] \quad (12) \]

and Einstein’s equations for this metric are as follows: eqn. (5) becomes

\[ 2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} = \frac{1}{L^2} \left( \frac{B'^2}{B^2} - \frac{B'}{Br} \right), \quad (13) \]

while eqn. (11) reduces to

\[ 2\frac{\ddot{B}}{B} + \frac{\dot{L}}{L} = -\frac{\psi(r)}{rLM^2}. \quad (14) \]

These two are the dynamical equations, while the remaining three equations are constraints. Eqn. (8) now becomes

\[ \frac{2\dot{M}'}{M} + \frac{\dot{M}}{rM} = \frac{\dot{L}}{L} \left( \frac{2M'}{M} + \frac{1}{r} \right), \quad (15) \]

whereas the difference (10)−(7) gives

\[ \frac{2M'}{M} = \frac{L'}{L}. \quad (16) \]
and the conservation equation (10) now is
\[ \epsilon(t, r) = \frac{2\psi(r)}{rLM^2}. \]  
(17)

Eqns. (13)-(17) are the Einstein equations for the metric (12). Eqn. (16) can be solved to give
\[ L(t, r) = h(t)M^2(t, r) \]  
(18)

where \( h(t) \) is an arbitrary function of time. Eqn. (15) implies that
\[ (\sqrt{r}M)' = g(r)L(t, r) \]  
(19)

where \( g(r) \) is an integration function. Using (19) and (18) in the first dynamical equation (13) gives
\[ 2\ddot{B} + \dot{B}^2 = r\frac{g^2(r)}{h} - \frac{r^2}{4h^2B^2}. \]  
(20)

while the second dynamical equation becomes
\[ B\ddot{h} + 4\dot{B}\dot{h} = -\frac{r^3\psi(r)}{B^3} + \frac{r^2}{2hB^3} - \frac{2rg^2h}{B}. \]  
(21)

### 3 Exact solution for a class of homogeneous dust collapse

We choose the scaling \( B(0, r) = r \) at time \( t = 0 \), when collapse is assumed to begin. This implies \( M(0, r) = 1 \). We also choose \( h(0) = 1 \), which from (18) implies \( L(0, r) = 1 \). From Eqn. (19), written at \( t = 0 \), we get the relation \( g(r) = 1/\sqrt{r} \). Then, solving (19) yields
\[ \frac{1}{M} = h(t) + K(t)/\sqrt{r}. \]  
(22)

In this equation, \( h(t), K(t) \) are functions of time only. Now the differential equations (20) and (21) should be valid at all \( r \) (including \( r = 0 \)). Substituting \( B = rM \) in equation (20) gives
\[ 2\ddot{M} + M^2 = \frac{1}{4r^2} - \frac{1}{4r^2h^2M^2} \]
\[ = \frac{h^2 - (h + K\sqrt{r})^2}{4r^2h^2} \]  
(23)

The \( 1/r^2 \) term in the denominator makes the equation invalid for \( r = 0 \). This can be avoided by setting \( K(t) = 0 \), which implies \( h = 1/M \) and hence that \( M \) is a function only of time. Similarly equation (21) gives
\[ M\ddot{h} + 4\dot{M}\dot{h} = -\frac{\psi(r)}{rM^3} + \frac{1}{2hr^2M^3} - \frac{h}{2Mr^2}. \]  
(24)

The sum of the last two terms vanishes when we set \( K(t) = 0 \). There is a \( 1/r \) in the denominator of the coefficient of \( \psi(r) \). Since the left hand side is independent of \( r \), we are constrained to choose \( \psi(r) = k_1r \), where \( k_1 \) is a constant. Substituting for \( \psi \) in equation (17) gives
\[ \epsilon(t, r) = \frac{2k_1}{hM^4} = 2k_1h^3. \]  
(25)
Since $h$ is a function of time only, this means that the density distribution is homogeneous. Explicit solution for the metric can be obtained by solving the equation

$$2\ddot{M}M + \dot{M}^2 = 0$$

(26)

which gives

$$M = (c_1 t + c_2)^{2/3}$$

(27)

Substituting the value of $M$ in equation (24) gives the relation between $k_1$ and $c_1$;

$$k_1 = 2c_2/c_1^{2/3}$$

Also, with $h(0) = 1$, and assuming $c_1$ to be negative, we can write

$$M = c_2/(3c_1)(t_0 - t)^{2/3}$$

(28)

where $t_0 = 1/|c_1|$. The metric can now be written as

$$ds^2 = -dt^2 + k^2(t_0 - t)^{4/3}(dr^2 + dz^2 + r^2 d\phi^2)$$

(29)

where $k$ is a constant. At time $t = t_0$ there occurs a curvature singularity. All the shells become singular at the same time $t_0$. This is similar to the spherical Oppenheimer-Snyder dust collapse.

4 Comparison with spherical dust collapse

Let us compare this system of equations with those for spherical dust collapse, described by the Tolman-Bondi spacetime:

$$ds^2 = -dt^2 + L^2(t, r)dr^2 + R^2(t, r) [d\theta^2 + \sin^2 \theta d\phi^2].$$

(30)

The Einstein equations for this metric are:

$$G_{rr} = 0 : \frac{2\ddot{R}}{R} + \frac{\ddot{R}^2}{R^2} = \frac{1}{R^2} \left( \frac{R'^2}{L^2} - 1 \right),$$

(31)

$$G_{\theta\theta} = G_{\phi\phi} = 0 : \frac{\ddot{L}}{L} + \frac{\dot{R}}{R} + \frac{\dot{L}}{L} = \frac{R'' L - R' L'}{R L^3},$$

(32)

$$G_{tt} = 0 : \frac{\dot{R}}{R} = \frac{\dot{L}}{L},$$

(33)

$$G_{tt} = \epsilon : -\frac{R'' L - R' L'}{R L^3} - \frac{R'^2}{2L^2R^2} + \frac{\ddot{R}}{2R^2} + \frac{\dot{R}}{R} + \frac{1}{2R^2} = \epsilon$$

(34)

The conservation equation is

$$\epsilon(t, r) = \frac{\psi(r)}{LR^2}.$$  

(35)

We also note that adding equations (31) and (32) and using (34) gives

$$\frac{2\ddot{R}}{R} + \frac{\dot{L}}{L} = -\frac{\psi(r)}{LR^2}.$$  

(36)
These equations should be compared with those for the cylinder: Eqn. (31) should be compared with Eqn. (13), Eqn. (32) should be compared with Eqn. (6), Eqn. (33) should be compared with Eqn. (15), Eqn. (34) should be compared with Eqn. (9), Eqn. (35) should be compared with Eqn. (17), and Eqn. (36) should be compared with Eqn. (14).

Thus, although there is similarity in the forms of the metrics (12) and (30), the cylindrical metric (12) describes homogeneous collapse, whereas the spherical metric (30) describes general inhomogeneous collapse. This is because of the additional Einstein equation (16) which is not there in the spherical case.

5 Matching the interior cylindrical metric to a static exterior

The metric (29) is valid in the interior of a collapsing dust cloud. The exterior is vacuum. The above metric will be a solution of Einstein equations if it can be successfully matched to a static or a time dependent exterior. It can be easily guessed that the interior metric might not match to a static exterior, since the quadrupole tensor for the matter distribution is non-zero, and the collapsing system emits gravitational waves.

To verify explicitly the impossibility of static exterior vacuum solution, we assume a static exterior metric of the form

$$ds^2 = -e^A dT^2 + e^B dR^2 + e^C dZ^2 + R^2 d\Phi^2$$

(37)

where $A, B, C$ are functions of $R$. Only the diagonal components of the Einstein tensor survive.

$$G_{00} = \frac{e^{A-B}(-2C' - RC'' + B'(2 + RC') - 2RC'' + 2R'A'' - 2A'R' + 2A''R')}{4R}$$

(38)

$$G_{11} = \frac{2C' + A'(2 + RC')}{4R}$$

(39)

$$G_{22} = \frac{e^{C-B}(RA'^2 - 2B' + A'(2 - RB') + 2RA'')}{4R}$$

(40)

$$G_{33} = \frac{e^{-B}R^2(A'^2 - B'C' + C'^2 + A'(-B' + C') + 2A'' + 2C'')}{4}$$

(41)

All the components equal zero in vacuum. Solving the Einstein equations gives

$$C = l_1 \ln \left( \frac{R}{R_0} \right) + l_2.$$  

(42)

$$A = \frac{-2l_1 \ln \left( \frac{R}{R_0} \right)}{2 + l_1} + l_3.$$  

(43)

$$B = \frac{l_1^2 \ln \left( \frac{R}{R_0} \right)}{2 + l_1} + l_4.$$  

(44)
$R_0$ is a length scale which, as seen below, gets fixed by the matching. We can get rid of $l_2$ and $l_3$ by rescaling $T$ and $Z$. So there are two unknown constants. (see also [9]).

In the interior let the outermost shell be labelled by $r = r_s$. The coefficient of $d\Phi^2$ is interpreted as the square of the radius. The expression $r_sk(t_0 - t)^{2/3}$ represents the evolution of the radius of the outer boundary of the collapsing dust with time. Further, it can be assumed that $\Phi$ and $Z$ represent the same coordinates in both interior and exterior, i.e. $\Phi = \phi$ and $Z = z$. Consider the hypersurface $r = r_s$. On this hypersurface, the interior metric is given by

$$ds^2 = -dt^2 + k^2(t_0 - t)^{4/3}(dr^2 + dz^2 + r_s^2d\phi^2). \quad (45)$$

The exterior is given by (37).

The metric coefficients of $\phi$ and $z$ should be same in both the interior and exterior. Equating the coefficient of $d\phi^2$ we get

$$R_s^2 = r_s^2k^2(t_0 - t)^{4/3} \quad (46)$$

The subscript $s$ in $R_s$ is to indicate the hypersurface in the exterior coordinates. Similarly equating the coefficients of $dZ^2$ for both interior and exterior gives

$$e^{l_1lnR_s/R_0} = k^2(t_0 - t)^{4/3} \quad (47)$$

which implies $l_1 = 2$ and $R_0 = r_s$. So the exterior metric on the hypersurface is

$$ds^2 = \frac{r_s}{R_s}dT^2 + \frac{cR_s}{r_s}dR^2 + \left(\frac{R_s}{r_s}\right)^2 dZ^2 + r_s^2d\phi^2 \quad (48)$$

where $c$ is a constant to be determined by matching the second fundamental form. Matching the remaining components of the metric on the hypersurface yields

$$-dt^2 = -\frac{r_s}{R_s}dT^2 + \frac{cR_s}{r_s}dR^2. \quad (49)$$

This can be used to obtain the relation between the interior and exterior time. The second fundamental form [10] is given by

$$\Phi = (-n_{\mu\nu}dx^\mu dx^\nu) \quad (50)$$

and the extrinsic curvature is given by

$$K_{\mu\nu} = n_{\mu;\nu} \quad (51)$$

where $n_{\mu}$ is the unit normal to the hypersurface. For the interior metric given by (45) the hypersurface is given by $r = r_s$. Calculating $K_{\phi}^{\phi}$ yields

$$K_{\phi}^{\phi} = \frac{1}{R_s} \quad (52)$$

where $R_s$ is given by (46). Similarly for the exterior metric $K_{\phi}^{\phi}$ can be computed. The normal to the hypersurface $R_s$ can be obtained by differentiating (46),

$$dR + \frac{(2r_s\sqrt{k(t_0 - t)^{-1/3}}d\tau)}{3}dT = 0. \quad (53)$$
Using the above equation and evaluating $K_{\phi}^\phi$ gives

$$K_{\phi}^\phi = \frac{n^R}{R_s}$$ \hspace{1cm} (54)

where $n^R$ is the contravariant component in the $R$ direction. By equating $K_{\phi}^\phi$ for both interior and exterior we obtain the value for $c$,

$$c = \frac{1}{R_s - \frac{4r^2}{9}}$$ \hspace{1cm} (55)

c was initially assumed to be a constant. In equation (55) $c$ is dependent on $R_s$ which is given by (46). So $c$ is not a constant as assumed but is time dependent. Hence there is no static exterior for the metric given by (45).

6 Matching with time dependent exterior

In this section it will be shown that the interior metric (45) can be matched to a time dependent exterior. Let us assume an exterior of the form,

$$ds^2 = e^{(G-P)}(dR^2 - dT^2) - e^P dz^2 - \alpha^2 e^{-P} d\phi^2,$$ \hspace{1cm} (56)

where $G, P, \alpha$ are functions of $R, T$. The Einstein equations are

$$G_{TT} = \frac{-\alpha(P^2 + \dot{P}^2) + 2(G'\alpha' - 2\alpha'' + \dot{G}\dot{\alpha})}{4\alpha} = 0$$ \hspace{1cm} (57)

$$G_{TR} = \frac{\alpha'\dot{G} - \alpha P'\dot{P} + G'\dot{\alpha} - 2\alpha'}{2\alpha} = 0$$ \hspace{1cm} (58)

$$G_{RR} = \frac{-\alpha(P^2 + \dot{P}^2) + 2(G'\alpha' - 2\alpha' + \dot{G}\dot{\alpha})}{4\alpha} = 0$$ \hspace{1cm} (59)

$$G_{ZZ} = \frac{e^{-G+2P}(\alpha(P^2 + 2G'' - 4P'' - \dot{P}^2 - 2\ddot{G} + 4\dot{P}) + 4(-P'\alpha' + \alpha'' + \dot{P}\dot{\alpha} - \ddot{\alpha}))}{4\alpha} = 0$$ \hspace{1cm} (60)

$$G_{\phi\phi} = \frac{e^{-G}\alpha^2(P^2 + 2G'' - \dot{P}^2 - 2\ddot{G})}{4} = 0$$ \hspace{1cm} (61)

Here prime and dot denote partial derivative with respect to $R$ and $T$ respectively. In vacuum all the above expressions equal zero. Subtracting equations (57) and (59) gives

$$\alpha'' - \ddot{\alpha} = 0$$ \hspace{1cm} (62)

Equation (61) gives

$$G'' - \ddot{G} = \frac{\dot{P}^2 - P'^2}{2}$$ \hspace{1cm} (63)
Substituting equations (62) and (63) in equation (60) gives

\[ P'' - \ddot{P} + \frac{P'\alpha' - \dot{P}\dot{\alpha}}{\alpha} = 0 \]  

(64)

The remaining two equations can be written as

\[ \dot{G}\alpha' - P'\dot{P}\alpha + G'\dot{\alpha} - 2\dot{\alpha}' = 0 \]  

(65)

\[-\alpha(P'^2 + \dot{P}^2) + 2(G'\alpha' - 2\alpha'' + \dot{G}\dot{\alpha}) = 0. \]  

(66)

The equations (62), (63), (64) are second order partial differential equations in \( T \) and \( R \). The outer boundary of the collapsing matter (hypersurface \( r = r_s \)) is a priori unknown in the exterior coordinates. It is of the form \( f_1(T, R) = \text{const} \). To solve completely the partial differential equation (62) (second order in both \( T \) and \( R \)), we need to know the value on the hypersurface and also the first derivative in the direction perpendicular to the hypersurface. We assume that \( Z \) and \( \Phi \) represent the same coordinates in both the interior and exterior. So on the hypersurface the metric components have to match. Equating coefficient of \( d\phi^2 \) for both metrics we get

\[ \frac{\alpha^2}{e^P} = \frac{r_s^2k^2}{4/3}(t_0 - t)^4 \]  

(67)

Similarly equating coefficient of \( dZ^2 \) we get.

\[ e^P = k^2(t_0 - t)^{4/3} \]  

(68)

Multiplying both we get

\[ \alpha = k^2r_s(t_0 - t)^{4/3} \]  

(69)

To solve the equations of the type \( \alpha'' - \ddot{\alpha} = 0 \), we already know the value on the hypersurface given by (69). We need one more arbitrary function to provide information about the value of the derivative in the perpendicular direction. Similarly we know the value of \( P \) on the hypersurface (68). To solve the equation (64) we need one more arbitrary function. For solving equation (63) we need two arbitrary functions since we do not know the value of \( G \) on the hypersurface. These four arbitrary functions along with the initial hypersurface \( f_1(T, R) \) makes five arbitrary functions. The matching will be possible if the five arbitrary functions are enough to satisfy all the constraints which are required during matching. The relation between the interior time and the exterior coordinates can be obtained by the equation similar to equation (49)

\[ -dt^2 = e^{G-P}(-dT^2 + dR^2) \]  

(70)

Equations (65) and (66) act as constraints and can be used to eliminate two of the five arbitrary functions. Now matching the second fundamental form (10)

\[ \Phi = K_{tt}dt^2 + K_{zz}dz^2 + K_{\phi\phi}d\phi^2 \]  

(71)

gives at most three independent constraints. There are three arbitrary functions left which can take care of the constraints offered by the second fundamental form. So in principle the interior metric (45) can be matched to the time dependent exterior given by (56).
7 Nature of the singularity

The question of interest is to find out whether the curvature singularity resulting in the matter interior from the gravitational collapse of the dust cloud is naked or covered. This question can be addressed by looking at the null geodesic expansion $\Theta$ for the radial null geodesics. The non-zero components of the tangent to the null geodesic are $K^r(t, r)$ and $K^t(t, r)$. The calculation of $\Theta$ for the cylindrical metric (12) proceeds exactly as for the spherical case discussed in [11] and we get the result

$$\theta = 2K^r \frac{Z'}{Z} \left[ 1 + \frac{\dot{B}}{\sqrt{r g(r)}} \right] = 2K^r \frac{Z'}{Z} \left[ 1 + 2h(0)\dot{B} \right] = \frac{K^r}{r} \left[ 1 + 2r\dot{M} \right]$$

where $Z \equiv B/\sqrt{r} = \sqrt{r}M(t)$, $Z'/Z$ is positive, because of equation (19), and $K^r$ is positive because we are examining outgoing geodesics.

$K^r$ and $K^t$ can be evaluated explicitly from the geodesic equations

$$\frac{dK^t}{dk} = -\frac{\dot{M}}{M} (K^t)^2, \quad \frac{dK^t}{dk} = -2\dot{M} (K^r)^2$$

Here, $k$ is the affine parameter and we have used the relation $dt/dr = M(t) = K^t/K^r$ for outgoing geodesics.

The explicit solution is however not necessary for our purpose. As the singularity is approached, i.e., as $t$ tends to $t_0$, $\dot{M}$ tends to $-\infty$, and hence so does the geodesic expansion $\theta$, showing that the singularity is covered. The expression for $\theta$ is in fact identical to that for the spherical marginally bound homogeneous dust solution [Oppenheimer-Snyder] for which the metric is

$$ds^2 = -dt^2 + S^2(t, r) \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

and the scale factor $S(t)$ goes as $(t_0 - t)^{2/3}$. Comparison of this metric with the cylindrical homogeneous metric (29) makes it clear that the geodesic expansion $\theta$ behaves identically in the two cases.

It has earlier been demonstrated [1] that the collapse of an infinite dust cylinder results in a naked singularity, in accord with the hoop conjecture. How is our result above consistent with these earlier findings? A possible explanation is that there is a singular Cauchy horizon in the exterior, as discussed recently by Tod and Mena [12]. It should be noted that our analysis here does not provide information about the formation of a singularity, or otherwise, in the exterior spacetime.

8 Conclusion

In this paper we have considered the collapse of an infinite cylindrical cloud of time-like homogeneous and pressureless dust subject to the condition that the axial and azimuthal metric functions are equal. We have explicitly shown that the resulting solution is homogeneous and that the collapse terminates in a curvature singularity. We have shown that there exists a vacuum exterior and it can be demonstrated that the singularity forming in the matter region is covered.
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