A DETERMINISTIC COUNTEREXAMPLE FOR HIGH DIMENSIONAL
$L^2L^\infty$ STRICHARTZ ESTIMATES FOR THE WAVE EQUATION

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Abstract. In this note we discuss the question of homogeneous $L^2L^\infty$ Strichartz estimates for the Wave equation in dimensions $n \geq 4$ raised by Fang and Wang and recently shown to fail by Guo, Li, Nakanishi and Yan using probability theory. We record a deterministic example for disproving this estimate.

1. Introduction

The question of determining the validity of the following statement was raised in [2, Remark 1] and was recently answered negatively in [6]

$$\|e^{itD}u\|_{L_t^2L_x^\infty} \lesssim \|u\|_{\dot{H}^{n-\frac{3}{2}}} \quad \text{holds for all } u \in S(\mathbb{R}^n) \text{ when } n \geq 4.$$  

The function $v = e^{itD}u$ solves the wave equation $(-\partial_t^2 + \Delta)v = 0$. Here $S(\mathbb{R}^n)$ denotes the space of Schwartz functions, while $\dot{H}^s$ denote homogeneous Sobolev spaces.

In fact [6] disproved estimates $e^{itD} : \dot{H}^{\frac{n-3}{2}} \to L_t^2L_x^\infty$ for general operators given by $a \in (0,2]$, using stable Levy processes. The case $a = 2$ corresponds to the Schrödinger equation.

In this note we record a proof of the failure of (W) using basic Fourier analysis. This argument is specific to the wave equation and does not apply to the Schrödinger case.

Theorem 1.1. The statement (W) is not true.

Estimates of $L_t^qL_x^r$ norms of solutions to dispersive equations with initial conditions in $\dot{H}^s$ spaces are known as Strichartz estimates, see [20], [24], [7], [4], [12], [5], [17], [23]. The exponents $(q,r) = (2,\bar{r})$ when $r$ is finite and minimal \footnote{i.e. $\bar{r} = \frac{2(n-1)}{n}$ for the Wave equation ($n \geq 4$) and $\bar{r} = \frac{2n}{n+1}$ for the Schrödinger eq. ($n \geq 3$).} (blue point in Figure 1 Left) are called the endpoints, and have been famously resolved in [8]. We refer to [23, Section 2.3, see "Knapp example"] for a discussion of the minimality of $\bar{r}$.

The case $r = \infty$ is usually omitted when stating Strichartz estimates in higher dimensions ($n \geq 4$ for the Wave equation and $n \geq 3$ for the Schrödinger equation), see for example [23], [10], [17], because the strongest and most useful cases occur on the blue line in Fig. 1, i.e. the range between $L_t^2L_x^r$ and $L_t^{\infty}L_x^2$. Other cases are typically obtained by Sobolev embeddings. The $L_t^2L_x^\infty$ exponents differ from the endpoints, unless $n = 3$ for the Wave equation or $n = 2$ for the Schrödinger equation, Figure 1 Right. In those low-dimension
Figure 1. Left: high dimensions. Right: $n = 3$ for W.E.

cases, counterexamples are given in [9], [15]. See Remark 1.2 below. The present case is the red point in Fig. 1 Left.

The case $r = \infty$ is generally delicate because $\dot{W}^{2,r} \not\subset L^\infty$, i.e. Sobolev embeddings fail into $L^\infty$. Thus (W) does not follow from the Keel-Tao endpoint $L^2_t L^r_x$ in [8], and instead one can only conclude from [6] estimates that are true if one replaces $L^\infty_x$ with $BMO$ or Besov spaces (or $L^2_t L^\infty_x$ estimates restricted to frequency localized functions) - see the ending of Remark 1.2.

Estimates $e^{it|D|} : \dot{H}^{\frac{n}{2} - \frac{1}{q}} \rightarrow L^q_t L^\infty_x$ for $q \in (2, \infty)$ are proved in [11], [2]. The case $q = \infty$ trivially fails at $t = 0$ due to $\dot{H}^{\frac{n}{2}} \not\subset L^\infty_x$. The case that remained was $q = 2$, which was recently treated in [6].

Remark 1.2. When $n = 3$ the estimate in (W) is disproved in [9] Remark 1 (also presented in [10]), exhibiting concentration along null rays. Using the 3d fundamental solution (i.e. Kirchhoff’s formula) they show there exists a sequence of solutions to $(-\partial_t^2 + \Delta)\phi_n = 0$ with normalized initial velocities $\|\partial_t \phi_n(0, \cdot)\|_{L^2} = 1$ and $\phi_n(0, \cdot) = 0$ such that

$$\int_0^\infty |\phi_n(t, te_1)|^2 dt \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

See also [21] Prop. 1.1 for an explicit example of such initial velocity for which the solution has $\phi(t, te_1) = \infty$ for all $t \in (1, 2)$ (presented as an adaptation of Stein’s counterexample in regards to the spherical maximal function).

Moreover, [15] shows that, when $n = 3$, (W) fails even with $L^\infty_x$ replaced by $BMO$. This is in contrast to the case $n \geq 4$ for which that variant can be obtained from the Keel-Tao endpoint [8] and the embedding $\dot{W}^{2,r} \subset BMO$.

The proof of Theorem 1.1 proceeds instead in Fourier space and must use rays of velocity $> 1$ instead of lightrays.

1.1. Notation. In what follows $\hat{f}$ denotes the Fourier transform of $f$. Operators $m(D)$, such as $e^{it|D|}$, $e^{it\Delta}$, are defined by $m(\hat{D})\hat{f}(\xi) = m(\xi)\hat{f}(\xi)$. Here $\Delta = -|D|^2$. Homogeneous Sobolev spaces are defined by $\dot{H}^s = |D|^{-s} L^2(\mathbb{R}^n)$ for $s < \frac{n}{2}$. Space-time norms are defined by

$$\|F\|_{L^q_t L^r_x} = \int_{-\infty}^{\infty} \|F(t, \cdot)\|_{L^r_x(\mathbb{R}^n)}^q dt.$$
Littlewood-Paley projections \( P_{\leq k} \) are defined by \( \widehat{P_{\leq k}} h(\xi) = \chi(\frac{\xi}{2^k}) \hat{h}(\xi) \), where \( \chi \in C_c^\infty(\mathbb{R}) \), \( \chi \geq 0 \) and \( \chi(\eta) = 1 \) for \( \eta \) near 0. For example, \( \widehat{P_{\leq k}} \delta_0(\xi) = \chi(\frac{\xi}{2^k}) \). Here \( \delta_0 \) denotes the Dirac delta measure. \( S^{n-1} \subset \mathbb{R}^n \) denotes the unit sphere. The notation \( X \lesssim Y \) means \( X \leq CY \) for an absolute constant \( 0 < C < \infty \). When \( Y \lesssim X \lesssim Y \) we denote \( X \simeq Y \).

\[ \textbf{2. Proof of Theorem 1.1} \]

Assume that \( (W) \) is true. Then, by duality, one also has

\[ (W') \quad \left\| \int_{-\infty}^{\infty} e^{-it|D|} \frac{1}{|D|^{\frac{n-1}{2}}} F(t) \, dt \right\|_{L^2} \lesssim \|F\|_{L^2_t L^1_x} \]

for all \( F \in L^2_t L^2_x \cap L^1_t \dot{H}^{\frac{n-1}{2}}_x \).

Take any \( f \in L^1_t \cap L^2_x \) with \( \|f\|_{L^2} = 1 \) such that \( \int_0^\infty |\hat{f}(\eta)|^2 \, d\eta \neq 0 \). Consider also the function \( h \in L^1_t \cap \dot{H}^{\frac{n-1}{2}}_x \) with \( \|h\|_{L^2_t} = 1 \) which remains to be defined. Fix a direction \( \omega \in S^{n-1} \) and a constant \( c > 0 \). Applying \( (W) \) with

\[ F(t, x) := f(t) h(x - ct \omega) \]

and using Plancherel’s theorem one obtains

\[ \left\| \frac{1}{|\xi|^{\frac{n-1}{2}}} \hat{h}(\xi) \int_{-\infty}^{\infty} e^{-it|\xi|} e^{-ict \omega \cdot \xi} f(t) \, dt \right\|_{L^2_\xi} \lesssim 1 \]

which implies

\[ \left\| |\xi|^{-\frac{n-1}{2}} \hat{h}(\xi) \hat{f}(|\xi| + c \omega \cdot \xi) \right\|_{L^2_\xi} \lesssim 1. \]

Consider \( h = \delta_0 \), \( \hat{h} \equiv 1 \), or rather, since \( \delta_0 \notin L^1 \cap \dot{H}^{\frac{n-1}{2}} \), take an approximation to the identity sequence \( h_k = P_{\leq k} \delta_0 \) and pass to the limit. Here \( \delta_0 \) denotes the Dirac delta measure. By the Monotone Convergence theorem one obtains

\[ \int_{\mathbb{R}^n} \frac{1}{|\xi|^{n-1}} |\hat{f}(|\xi| + c \omega \cdot \xi)|^2 \, d\xi \lesssim 1. \]

Write this integral in spherical coordinates, where

\[ |\xi| = \lambda, \quad \omega \cdot \xi = \lambda \cos \theta, \quad \theta \in [0, \pi]. \]

This implies

\[ \int_0^\infty \int_0^\pi |\hat{f}(\lambda(1 + c \cos \theta))|^2 (\sin \theta)^{n-2} \, d\theta \, d\lambda \lesssim 1. \]

\textbf{Remark 2.1.} For \( n = 3 \) one may take \( c = 1 \) and change coordinates to \( u = 1 + \cos \theta \) to obtain

\[ \int_0^\infty \int_0^2 |\hat{f}(\lambda u)|^2 \, du \, d\lambda = \int_0^2 \frac{1}{u} \, du \int_0^\infty |\hat{f}(\eta)|^2 \, d\eta \lesssim 1 \]

which gives a contradiction. One may view this as a Fourier version of the physical space construction in Remark 1.2 above, due to [9]. \( \Delta \)
For \( n \geq 4 \), choosing \( c = 1 \) does not work. Instead, take \( c = 2 \) and restrict the integral to \( \theta \in \left[ \frac{\pi}{2}, \frac{2\pi}{3} \right] \). On this interval \( \sin \theta \simeq 1 \). As before, one obtains
\[
\int_0^{\infty} |\hat{f}(\eta)|^2 \, d\eta \int_\frac{\pi}{2}^{\frac{2\pi}{3}} \frac{(\sin \theta)^{n-2}}{1 + 2 \cos \theta} \, d\theta \lesssim 1.
\]
The last integral is logarithmically divergent, being \( \simeq \int_0^1 \frac{1}{y} \, dy \), thus obtaining a contradiction again, which concludes the proof. \( \square \)

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