ON THE DIFFEOMORPHISM TYPE OF SEIFERT FIBERED SPHERICAL 3-ORBIFOLDS

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Dedicated to Bruno Zimmermann on his 70th birthday

Abstract. It is well known that, among closed spherical Seifert three-manifolds, only lens spaces and prism manifolds admit several Seifert fibrations which are not equivalent up to diffeomorphism. Moreover the former admit infinitely many fibrations, and the latter exactly two. In this work, we analyse the non-uniqueness phenomenon for orbifold Seifert fibrations. For any closed spherical Seifert three-orbifold, we determine the number of its inequivalent fibrations. When these are in a finite number (in fact, at most three) we provide a complete list. In case of infinitely many fibrations, we describe instead an algorithmic procedure to determine whether two closed spherical Seifert orbifolds are diffeomorphic.

1. Introduction

Seifert fibered 3-manifolds were introduced by Seifert [Sei80] and are one of the cornerstones in the study of 3-dimensional manifolds (see for example [Ore72, Sco83, BMP03]). Roughly speaking a Seifert fibered 3-manifold is a fiber bundle whose fibers are circle, except that some fibers are exceptional, meaning that a tubular neighborhood is a torus which is however not fibered as a product. The presence of singular fibers is reflected by the fact that the base of the fibration is not a manifold, but a particular type of 2-dimensional orbifold, namely a surface containing cone points.

Orbifolds are a generalization of manifolds, which had been introduced in different contexts by Satake [Sat56], by Thurston [Thu97, Chapter 13] and by Haefliger [Hae84] – useful references being also [BMP03, Cho12, Dun88, Sco83]. The most standard example of an orbifold (of dimension $n$) is the quotient of a manifold $M^n$ by a group $\Gamma$ which acts properly discontinuously – but in general not freely – on $M$. If the action is not free, singular points appear in the quotient $M/\Gamma$, keeping track of the action of point stabilizers $\text{Stab}_\Gamma(x)$ on a neighborhood of a fixed point $x \in M$. More generally, an orbifold is locally the quotient of a manifold by the action of a finite group.

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Bonahon and Siebenmann \cite{BS85} generalized the definition of Seifert fibration to 3-orbifolds. This notion is actually more general than its counterpart for 3-manifolds: here fibers are in general quotients of the circle, and therefore some exceptional fibers are allowed to be intervals, which corresponds to the fact that the base of the fibration has singularities which are not of conical type. Seifert fibered 3-manifolds and 3-orbifolds are classified up to fibration-preserving diffeomorphisms by Seifert invariants, but some of them admit more than one Seifert fibration. The complete topological classification of the Seifert fibered 3-manifolds follows from the work of several authors (see \cite[Section 2.4.1]{BMP03}). In the present paper we give a classification by diffeomorphism type of compact Seifert fibered spherical 3-orbifolds.

Spherical 3-orbifolds are one of the eight classes of geometric 3-orbifolds, which had a large importance in Thurston’s geometrization program. These are locally the quotient of one of the eight Thurston’s model geometries by the properly discontinuous action of a group of isometries. In the case of manifolds, six of eight Thurston’s geometries give Seifert fibered 3-manifolds (the exceptions are hyperbolic and Sol geometries). In the orbifold setting the situation is the same with few exceptions: twelve euclidean 3-orbifolds and eighteen spherical 3-orbifolds are not fibered, see \cite[Theorem 1]{Dun88}. The eighteen spherical 3-orbifolds not admitting a Seifert fibration are analyzed by Dunbar \cite{Dun94}. On the other hand each compact Seifert fibered 3-orbifold without bad 2-suborbifold admits a geometric structure \cite[Proposition 2.13]{BMP03}; bad means that the orbifold is not globally the quotient of a manifold.

The 3-manifolds admitting multiple fibrations are either spherical, euclidean or covered by the Thurston geometry $S^2 \times \mathbb{R}$, see \cite{Sco83}. In the euclidean case there exists a unique 3-manifold admitting several inequivalent fibrations and in the $S^2 \times \mathbb{R}$ case we find two of these manifolds. The most intriguing case from this point of view is the spherical one where manifolds belonging to two important classes (lens spaces and prism manifolds) admit multiple fibrations. The situation for Seifert fibered 3-orbifolds is similar in the sense that interesting non-uniqueness phenomena mostly appear for spherical geometry; see for example \cite[Theorem 2.15]{BMP03}.

In this paper we compute, for each possible fibered spherical 3-orbifolds, the number of fibrations it admits. When the number of the fibrations is finite we describe explicitly the Seifert invariants of all fibrations of any 3-orbifold. When infinitely many fibrations occur, we describe an algorithm to decide if two sets of Seifert invariants give the same 3-orbifold. Hence we obtain a complete classification of compact Seifert fibered spherical 3-orbifold up to orientation-preserving diffeomorphism. In fact, all compact spherical fibered 3-orbifolds are orientable, and in this case the Seifert invariants are complete invariants of oriented fibered 3-orbifolds. The presentation of all possible diffeomorphisms is rather technical and summarizing it in a single statement seems impossible to us. In the present introduction we
describe the situation in terms of number of fibrations which are admitted. The complete
description of the classification can be found in Section 5.

**Theorem 1.1.** Let $\mathcal{O}$ be a compact spherical Seifert fibered 3-orbifold with base orbifold $\mathcal{B}$ and $b$ an integer greater than one.

1. If $\mathcal{B} \cong S^2(2, 2, b)$, $D^2(b)$, $\mathbb{R}P^2(b)$, $D^2(2; b)$ or $D^2(; 2, 2, b)$ then $\mathcal{O}$ admits two inequivalent fibrations with the following exceptions:
   - $(S^2(2, 2, b); 0, 0, \frac{1+b}{2}, \frac{1-b}{2})$, $(S^2(2, 2, b); 0, 0, \frac{1+b/2}{b}, \frac{1-b/2}{b})$ with $b$ even,
   - $(D^2(2); \frac{1+b}{2}, \frac{1-b}{2})$, $(D^2(2, b); \frac{1+b}{2}, \frac{1-b}{2})$ with $b$ odd,
   - $(D^2(b); \frac{1+b}{2}, \frac{1-b}{2})$, $(D^2(; 2, 2, b); \frac{1+b}{2}, \frac{1-b}{2})$ with $b$ even, $(\mathbb{R}P^2(b); \frac{1+b}{2}, \frac{1-b}{2})$ with $b$ odd,
   - $(D^2(2, b); \frac{1+b}{2}, \frac{1-b}{2})$, $(D^2(; 2, 2, b); \frac{1+b}{2}, \frac{1-b}{2})$ with $b$ odd
   and $(D^2(; 2, 2, b); \frac{1+b}{2}, \frac{1-b}{2})$ with $b$ odd which admit infinitely many fibrations.

2. If $\mathcal{B} \cong S^2(2, 3, b)$ or $D^2(; 2, 3, b)$ with $b = 3, 4, 5$ then $\mathcal{O}$ admits a unique fibration with the following exceptions:
   - $(S^2(2, 3, 3); 0, 0, \frac{1+b}{2}, \frac{1-b}{2})$, $(S^2(2, 3, 4); 0, 0, \frac{1+b/2}{b}, \frac{1-b/2}{b})$,
   - $(S^2(2, 3, 5); 0, 0, \frac{12}{5}, \frac{1-12}{5})$, $(S^2(2, 3, 3); 0, 0, \frac{12}{5}, \frac{1-12}{5})$,
   - $(D^2(2, 3, 3); \frac{1+b}{2}, \frac{1-b}{2})$, $(D^2(2, 3, 4); \frac{1+b}{2}, \frac{1-b}{2})$, $(D^2(2, 3, 5); \frac{1+b}{2}, \frac{1-b}{2})$,
   - $(D^2(2, 3, 3); \frac{1+b}{2}, \frac{1-b}{2})$, $(D^2(2, 3, 4); \frac{1+b}{2}, \frac{1-b}{2})$, $(D^2(2, 3, 5); \frac{1+b}{2}, \frac{1-b}{2})$ which admit two fibrations.

3. If $\mathcal{B}$ is a 2-sphere with at most two cone points or a 2-disk with at most two corner points, then $\mathcal{O}$ admits infinitely many fibrations.

For the notations for 2-orbifolds and for fibered 3-orbifolds we refer the reader to Sub-section 2.2 and 2.3 respectively. We remark that in Theorem 1.1 the list of exceptions sometimes contains the Seifert invariants of different fibrations of the same 3-orbifold. As already said, in Section 5 we actually list, for every spherical orbifold admitting multiple inequivalent fibrations as in Theorem 1.1, all of its other fibrations.

We discuss briefly some topological aspects relating to the theorem. A Seifert fibered 3-orbifold with base orbifold a 2-sphere with at most two cone points has a lens space as underlying topological space and the singular set is a subset of the union of the cores of the two tori giving the lens space; these orbifolds can be considered the generalization of lens spaces in the setting of orbifolds. However, we remark that there are Seifert fibered 3-orbifolds with base 2-orbifold different than a sphere with at most two cone points, whose underlying topological space is still a lens space; when this happens, the singular set do not entirely consist of a union of fibers. For instance, the other orbifolds in the third case of Theorem 1.1 (whose base 2-orbifold is a 2-disk) can be obtained by a quotient of an “orbifold
lens space” by an involution not acting freely; in this case the underlying topological space is always $S^3$, see also [Dun88].

The case $B \cong S^2(2,2,b)$ contains the classical class of prism manifolds. Prism manifolds admit two inequivalent fibrations, the second one with $B \cong \mathbb{R}P^2(b)$, see [Orl72] or [Hat07, Theorem 2.3]. We recover an explicit description of the relations between the two fibrations of prism manifolds in Case 1 of Subsection 5.2.

As a result of our analysis, we also obtain the following statement in analogy with the situation for spherical Seifert 3-manifolds:

**Theorem 1.2.** If a compact spherical Seifert fibered 3-orbifold admits several inequivalent fibrations, then its underlying topological space is either a lens space or a prism manifold.

However, differently from the manifold case, this is not a complete characterization of non-uniqueness, since there are 3-orbifolds with underlying manifold a lens space, whose fibration is unique up to diffeomorphism.

Finally we remark that the presence of one platonic group (tetrahedral, octahedral or icosahedral) or of one of their binary versions in the fundamental group of the 3-orbifold assures, with a few exceptions, the uniqueness of the fibration.

The methods we use in the paper are related to the fact that compact spherical 3-orbifolds are *globally* the quotient of the 3-sphere $S^3$ by the action of a finite group $G$ of isometries. In [MS15, MS19] we have analyzed different aspects of the classification of finite subgroups of $SO(4)$ up to conjugacy. Here we continue this kind of analysis but the additional difficulty consists in considering a classification up to “fibration-preserving conjugacy”.

**Organization of the paper.** In Section 2 we give an introduction to orbifolds of dimension 2 and 3, with special attention to the spherical case, and we recall the definition of Seifert fibration for orbifolds. In Section 3 we discuss the classification of finite subgroups of $SO(4)$. In Section 4 we analyze which groups leave invariant the Seifert fibrations of the 3-sphere and we explain the approach to the classification of finite subgroups of $SO(4)$ we adopt to get a classification of Seifert fibered spherical 3-orbifolds by diffeomorphism type. In Section 5 we explicitly present the classification by distinguishing the case of 3-orbifolds admitting finitely many inequivalent fibrations and the case of 3-orbifolds with infinitely many ones.

2. Spherical and Seifert fibered three-orbifolds

2.1. Spherical orbifolds. Let us start by recalling some notions on smooth and spherical orbifolds in any dimension. For details see [BMP03], [Rat06] or [Cho12].

**Definition 2.1.** A smooth orbifold $\mathcal{O}$ of dimension $n$ is a paracompact Hausdorff topological space $X$ endowed with an atlas $\varphi_i : U_i \to \tilde{U}_i/\Gamma_i$, where:

- The $U_i$ form an open covering of $\mathcal{O}$. 

The $\tilde{U}_i$ are open subsets of $\mathbb{R}^n$ on which the finite groups $\Gamma_i$ act smoothly and effectively.

Each $\varphi_i$ is a homeomorphism and the compositions $\varphi_j \circ \varphi_i^{-1}$ lift to diffeomorphisms $\tilde{\varphi}_{ij} : \tilde{U}_i \rightarrow \tilde{U}_j$.

Moreover, the orbifold $\mathcal{O}$ is:

- Orientable if the groups $\Gamma_i$ and the lifts $\tilde{\varphi}_{ij}$ preserve an orientation of $\mathbb{R}^n$ (and the choice of such an orientation makes $\mathcal{O}$ oriented).
- Spherical if each $\tilde{U}_i$ is endowed with a Riemannian metric $\tilde{g}_i$ of constant curvature 1 preserved by the action of the groups $\Gamma_i$ and such that each $\tilde{\varphi}_{ij}$ is an isometry from $(\tilde{U}_i, \tilde{g}_i)$ to $(\tilde{U}_j, \tilde{g}_j)$.

The topological space $X$ is called the underlying topological space of the orbifold. A diffeomorphism between orbifolds $\mathcal{O}$ and $\mathcal{O}'$ is a homeomorphism $f : X \rightarrow X'$ of the underlying topological spaces such that each composition $\varphi'_j \circ f|_{\tilde{U}_i} \circ \varphi_i^{-1}$, when defined, can be lifted to a diffeomorphism from $\tilde{U}_i$ to its image in $\tilde{U}'_j$. If $\mathcal{O}$ and $\mathcal{O}'$ are oriented, then $f$ is orientation-preserving if the lifts preserve the orientation of the $\tilde{U}_i$ and $\tilde{U}'_j$. If $\mathcal{O}$ and $\mathcal{O}'$ are spherical, $f$ is an isometry if the lifts are isometric for the Riemannian metrics $\tilde{g}_i$ and $\tilde{g}'_j$.

One can define a local group associated to every point $x$, namely the smallest possible group $\Gamma$ which gives a local chart $\varphi : U \rightarrow \tilde{U}/\Gamma$ for $x$. If the local group is trivial, then $x$ is a regular point of $\mathcal{O}$. Otherwise $x$ is a singular point. The set of regular points of $\mathcal{O}$ is a smooth manifold.

The most intuitive examples of orbifolds are produced as quotients $\mathcal{O} = M/G$, for $G$ a group acting smoothly and properly discontinuously on a manifold $M$. In this case the local group of a point $x$ in the quotient $M/G$ is the stabiliser of any of the preimages of $x$ (which is finite). An orbifold is called good if it is diffeomorphic to a quotient $M/G$ as above. Otherwise it is called bad.

**Theorem 2.2** ([Rat06, Theorem 13.3.10]). Every compact spherical orbifold is good, and is in fact isometric to a global quotient $S^n/G$ for $G < O(n + 1)$ a finite group of isometries of $S^n$.

If the spherical orbifold is orientable, then $G$ is a subgroup of $SO(n + 1)$. The following theorem of de Rham is a rigidity result for spherical orbifolds:

**Theorem 2.3** ([dR64, Rot78]). If two compact spherical orbifolds are diffeomorphic, then they are isometric.

If we consider two compact spherical orbifolds of the form (by Theorem 2.2) $\mathcal{O} = S^n/G$ and $\mathcal{O}' = S^n/G'$, then $\mathcal{O}$ and $\mathcal{O}'$ are isometric if and only if $G$ and $G'$ are conjugate in $O(n + 1)$. If moreover $\mathcal{O}$ and $\mathcal{O}'$ are orientable (i.e. $G, G' < SO(n + 1)$) and endowed with the orientation induced by the orientation of $S^n$, then $\mathcal{O}$ and $\mathcal{O}'$ have an orientation-preserving isometry if and only if $G$ and $G'$ are conjugate in $SO(n + 1)$.
Hence the classification of compact spherical orientable 3-orbifolds up to orientation-preserving diffeomorphisms amounts algebraically to the classification of finite subgroups of SO(4) up to conjugacy, which is the content of Section 3.

2.2. Two and three-dimensional orbifolds. Let us start by considering orbifolds of dimension 2. The underlying topological space turns out to be a manifold with boundary. In fact a neighborhood of any point \( x \) is modelled on \( D^2/\Gamma \) where \( \Gamma \) can be (see Figure 1):

- The trivial group, if \( x \) is a regular point;
- A cyclic group of rotations (in this case \( x \) is called cone point and is labelled with the order of \( \Gamma \));
- A group of order 2 generated by a reflection (\( x \) is called mirror reflector and is a boundary point of the underlying 2-manifold);
- A dihedral group (\( x \) is called corner reflector, is still a boundary point for the underlying manifold and is labelled with the order of the rotation subgroup of \( \Gamma \)).

![Figure 1. Local models of 2-orbifolds. From left to right, a cone point, a mirror reflector and a corner point.](image)

If \( \mathcal{O} \) is compact, its diffeomorphism type is denoted by \( X(n_1, \ldots, n_k; m_1, \ldots, m_h) \), where \( X \) is the underlying manifold with boundary, \( n_1, \ldots, n_k \) are the labels of cone points and \( m_1, \ldots, m_h \) are the labels of corner reflectors. (Labels are also called singularity indices.)

The Euler characteristic of \( \mathcal{O} \) is then defined as:

\[
\chi(\mathcal{O}) := \chi(X) - \sum_i \left( 1 - \frac{1}{n_i} \right) - \frac{1}{2} \sum_j \left( 1 - \frac{1}{m_j} \right)
\]

where \( \chi(X) \) is the Euler characteristic of the underlying manifold \( X \).

As a consequence of the discussion of Section 2.1 any compact spherical 2-orbifold is diffeomorphic to a quotient of \( S^2 \) by a finite subgroup \( G \) of O(3), and moreover the conjugacy class of \( G \) determines both the diffeomorphism and isometry type of \( S^2/G \). Starting by the orientable case, the following classical result classifies finite subgroups of SO(3):

**Lemma 2.4.** A finite subgroup of SO(3) is either a cyclic group, a dihedral group, or the rotational tetrahedral, octahedral or icosahedral group.
It follows that compact orientable spherical 2-orbifolds are (up to diffeomorphism):

\[ S^2, S^2(p, p), S^2(2, 2, p), S^2(2, 3, 3), S^2(2, 3, 4), S^2(2, 3, 5) \text{ for } p \geq 2. \]  

(1)

For non-orientable spherical orbifolds, it suffices to consider order 2 quotients of orientable ones, thus getting the list:

\[ D^2, D^2(p; ), D^2(; p, p), D^2(; 2, 2, p), D^2(; 2, 2, 2), D^2(; 2, 3, 2), D^2(; 2, 3, 3), D^2(; 2, 3, 4), D^2(; 2, 3, 5), \]

\[ \mathbb{R}P^2, \mathbb{R}P^2(p) \text{ for } p \geq 2. \]  

(2)

These are in fact all good two-dimensional orbifolds of positive Euler characteristic. The additional bad orbifolds with \( \chi(O) > 0 \) are:

\[ S^2(p, q) \text{ and } D^2(p, q) \text{ for } p \neq q. \]  

(3)

Let us now rise to dimension three. In this paper we only consider orientable 3-orbifolds. By a standard argument, any point \( x \) admits a local chart of the form \( D^3/\Gamma \) for \( \Gamma \) a finite subgroup of \( \text{SO}(3) \), and the local model is thus the cone over one of the spherical orientable 2-orbifolds listed above, listed in (1). It follows that the underlying topological space is a manifold and the singular set is a trivalent graph; the local group is cyclic in the complement of the vertices of the graph, and the edges are thus labelled with a singularity index which is the order of the cyclic group.

![Figure 2. Local models of orientable 3-orbifolds.](image)

2.3. **Seifert fibrations on three-dimensional orbifolds.** Let us turn our attention on a topological description of the quotient in terms of Seifert fibrations for orbifolds, which we now define.

**Definition 2.5.** Given a three-dimensional orientable orbifold \( O \), a Seifert fibration is a surjective map \( \pi : O \to B \) with image a two-dimensional orbifold \( B \), such that for every point \( x \in B \) there exist:

- An orbifold chart \( \varphi : U \cong \tilde{U}/\Gamma \) for \( B \) around \( x \);
- An action of \( \Gamma \) on \( S^1 \);
- An orbifold diffeomorphism \( \phi : \pi^{-1}(U) \to (\tilde{U} \times S^1)/\Gamma \), where \( \Gamma \) acts diagonally on \( \tilde{U} \times S^1 \) by preserving the orientation;
such that the following diagram

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\phi} & (\tilde{U} \times S^1)/\Gamma \\
\downarrow \pi & & \downarrow \Gamma \\
U & \overset{\varphi}{\longrightarrow} & \tilde{U}/\Gamma \\
\downarrow \varphi & & \downarrow \\
\tilde{U}/\Gamma & \longrightarrow & \tilde{U}
\end{array}
\]

is commutative, with the obvious maps on unspecified arrows.

Since the action of \( \Gamma \) is required to preserve the orientation (as a consequence of the assumption that \( O \) is supposed orientable) each element of \( \Gamma \) may either preserve both the orientation of \( \tilde{U} \) and \( S^1 \), or reverse both orientations. Observe that each fiber \( \pi^{-1}(x) \) is topologically either a simple closed curve or an interval. A fiber which projects to a regular point of \( B \) is called \textit{generic}; it is called \textit{exceptional} otherwise.

Let us now consider the local models for oriented Seifert fibered orbifolds. More details can be found in [BS85] or [Dun81].

- If the fiber \( \pi^{-1}(x) \) is generic, one can pick \( \Gamma \) the trivial group in Definition 2.5, hence \( \pi^{-1}(x) \) has a tubular neighborhood with a trivial fibration.
- If \( x \) is a cone point labelled by \( b \), the local group \( \Gamma \) is a cyclic group of order \( q \) acting by rotations on \( \tilde{U} \) and thus it needs to act on \( S^1 \) by rotations too. Hence \( \pi^{-1}(x) \) has a fibered neighborhood which is a solid torus, fibered in the usual sense of Seifert fibrations for manifolds, except that the central fiber might be singular. The \textit{local invariant} of \( \pi^{-1}(x) \) is defined as the ratio \( a/b \in \mathbb{Q}/\mathbb{Z} \), where a generator of \( \Gamma \) acts on \( \tilde{U} \) by rotation of an angle \( 2\pi/b \) and on \( S^1 \) by rotation of \( -2\pi a/b \). Up to adding integer multiples of \( b \), one can in fact choose \( a \) so that \( a/b \in [0,1) \). The index of singularity \( \pi^{-1}(x) \) is thus \( \gcd(a,b) \) (meaning that points with index of singularity 1 are regular).
- If \( x \) is a mirror reflector, thus with local group \( \mathbb{Z}_2 \) whose generator acts by reflection on \( \tilde{U} \) and thus also on \( S^1 \), the local model is topologically a 3-ball. The fiber \( \pi^{-1}(x) \) is an interval. The endpoints of all the fibers \( \pi^{-1}(x) \), as \( x \) varies in the mirror reflector, form two disjoint singular arcs of index 2.
- If \( x \) is a corner reflector, namely \( \Gamma \) is a dihedral group, by a similar argument the non-central involutions in \( \Gamma \) act by simultaneous reflection both on \( \tilde{U} \) and on \( S^1 \). The local model is again a topological 3-ball (called solid pillow) with some singular set inside. The fiber \( \pi^{-1}(x) \) is again an interval and is also singular in this case, while the preimages of the nearby mirror reflectors are intervals as in the above case. The local invariant associated to \( \pi^{-1}(x) \) is defined as the local invariant of the cyclic index two subgroup, and the singularity index is \( \gcd(a,b) \).
Together with the base orbifold and the local invariants, to a Seifert fibered orbifold is associated the Euler class \( e \) and an additional invariant \( \xi \in \{0, 1\} \) for each boundary component of the base orbifold. These invariants satisfy the relation:

\[
e + \sum_i a_i b_i^{-1} + \frac{1}{2} \left( \sum_j a_j b_j^{-1} + \sum_k \xi_k \right) \equiv 0 \mod 1 ,
\]

where the first sum involves local invariants associated to all cone points, the second sum is taken over mirror reflectors, and the third sum over boundary components.

The base orbifold together the local invariants, the Euler class and the boundary components invariants determine the Seifert fibered orbifold up to orientation-preserving and fibration-preserving diffeomorphism.

**Remark 2.6.** Let us observe that changing the orientation of the orbifold inverts the signs of local invariants and of the Euler number.

2.4. **Comparison with the classification of Seifert manifolds.** As already observed, if \( \mathcal{O} \) is a manifold (meaning that the singular set is empty), then the notion of Seifert fibration according to Definition 2.5 coincides with the usual notion of Seifert fibration for manifolds. If \( \mathcal{O} \) is not a manifold but the base orbifold \( \mathcal{B} \) is orientable, which means that the only singular points of \( \mathcal{B} \) are cone points, then the underlying topological space of \( \mathcal{O} \) is a Seifert fibered manifold, whose invariants in the sense of the previous section are simply obtained by “simplifying the fractions” \( a_i/b_i \). Namely, by replacing the invariants \( a_i \) and \( b_i \) of each cone point in the base orbifold by \( a_i/\gcd(a_i, b_i) \) and \( b_i/\gcd(a_i, b_i) \) respectively.

However, the classifying data for Seifert fibrations of manifolds are more classically defined as the following data:

\[
(X; \alpha_1/\beta_1, \ldots, \alpha_n/\beta_n) ,
\]

where \( X \) is a closed surface (possibly non-orientable) and the pairs \( (\alpha_i, \beta_i) \) are relatively prime. Namely, one assumes that the fractions \( \alpha_i/\beta_i \) cannot be simplified, but one does not take the class of \( \alpha_i/\beta_i \) in \( \mathbb{Q}/\mathbb{Z} \), or in other words, one cannot assume that \( \alpha_i/\beta_i \) is in \([0, 1)\) as we did for \( a_i/b_i \) in the previous section. The representation of a Seifert fibered manifold by the data as in (5) is not unique: two representations of the same Seifert fibered manifold differ by replacing each \( \alpha_i/\beta_i \) by \( \alpha'_i/\beta'_i = \alpha_i/\beta_i + k_i \) for \( k_i \in \mathbb{Z} \), under the constraint that \( \sum_i k_i = 0 \). It is not allowed to have \( n = 0 \), that is, no marked point in \( X \): in this case one needs to introduce an invariant \( \alpha/\beta \) with \( \beta = 1 \). This is necessary in order to determine the Euler class \( e \), see below. For instance, this permits to distinguish between \( S^3 \) and \( S^2 \times S^1 \) which both have base surface \( X = S^2 \).

For a Seifert fibered manifold \( \mathcal{M} \), let us now briefly explain how to go from the invariants introduced in Section 2.3 (by interpreting \( \mathcal{M} \) as a Seifert fibered orbifold \( \mathcal{O} \) with empty singular set) to the classical invariants of (5), and viceversa.
Given a representation of $M$ as in (5), the base orbifold $B$ has underlying topological space $X$ and $n$ cone points with indices $\beta_1, \ldots, \beta_n$. The local invariants are the classes of the ratios $\alpha_i/\beta_i$ in $\mathbb{Q}/\mathbb{Z}$. Finally, the Euler class equals $e = -\sum_i (\alpha_i/\beta_i)$. Observe that these data do not change if one changes the representation of $M$ as explained above.

Conversely, given the Seifert invariants in the sense of orbifolds, clearly $X$ is the topological surface underlying the base orbifold $B$. To determine the ratios $\alpha_i/\beta_i \in \mathbb{Q}/\mathbb{Z}$, it suffices to pick some representatives of the classes of $a_i/b_i$ in $\mathbb{Q}/\mathbb{Z}$ so that $-\sum_i (\alpha_i/\beta_i)$ coincides with the Euler class.

2.5. The spherical case. If a Seifert fibered orbifold $O$ is geometric, i.e. it is admits a metric locally modelled on one of Thurston’s eight geometry, then its geometry is detected by the Euler characteristic of the base orbifold and by the Euler number of the fibration (see [Dun88, page 71]). In particular for the spherical case we have:

**Proposition 2.7.** Let $O$ be a compact spherical orbifold and $\pi : O \rightarrow B$ be a Seifert fibration. Then

$$\chi(B) > 0 \quad \text{and} \quad e(\pi) \neq 0.$$  \hspace{1cm} (6)

Conversely, every compact Seifert fibered orbifold satisfying the conditions in (6) is spherical.

For the last part of the statement, see [BMP03, Proposition 2.13] and its proof.

Moreover, given an orbifold covering $O \rightarrow O'$, any Seifert fibration for $O'$ can be lifted to $O$. In light of Theorem 2.2 it will be very important to understand the Seifert fibrations of the three-sphere. Recalling again that, if $O$ is a manifold, then Definition 2.5 coincides with the usual Seifert fibrations for manifolds, Seifert fibrations for $S^3$ are well known (see [Sei80] or also [GL18, Proposition 5.2]): they have base orbifold $S^2(u,v)$ for $u, v \geq 1$ two coprime integers, hence only two non-generic fibers which are non-singular. The local invariants are given by (the classes modulo 1 of) $\bar{v}/u$ and $\bar{u}/v$ where $u\bar{u} + v\bar{v} = 1$, and the Euler class is $\pm 1/uv$.

If $u$ or $v$ equals 1, we mean that the corresponding point in the base orbifold is regular, and hence there is no local invariant to associate (the above formula would indeed give 0 as output). In particular, for $u = v = 1$ we obtain the Hopf fibration, which is a fiber bundle in the usual sense, since every fiber has a tubular neighborhood which is fibered as a usual product. A more concrete description of these fibrations in terms of the geometry of $S^3$ will be provided in Section 4.

3. Finite subgroups of $SO(4)$

In this section we discuss a classification of compact spherical orientable three-orbifolds up to orientation-preserving diffeomorphisms, from a group-theoretical point of view. In Section 4 we will then focus on a classification up to fibration-preserving diffeomorphisms.
3.1. Quaternion algebra and subgroups of $S^3$. As discussed in Section 2, two spherical 3-orbifolds $\mathcal{O} = S^3/G$ and $\mathcal{O} = S^3/G'$ are diffeomorphic if and only if they are isometric, and such an isometry can be lifted to isometry of $S^3$ which conjugates $G$ to $G'$. If the isometry between the orbifolds is orientation-preserving, then the lift to $S^3$ is orientation-preserving. For this reason, the classification of compact spherical orientable three-orbifolds $S^3/G$ up to orientation-preserving diffeomorphisms corresponds to the algebraic classification of finite subgroups of $SO(4)$ up to conjugation in $SO(4)$, originally due to Seifert and Threlfall ([TS31] and [TS33]), which we shall now briefly recall. For more detail, see [DV64], which we essentially follow although it must be mentioned that in Du Val’s list of finite subgroups of $SO(4)$ up to conjugation in $SO(4)$ there are three missing cases, see also [CS03] [MS15] [MS19].

Let us identify $\mathbb{R}^4$ with the quaternion algebra $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\} = \{z_1 + z_2j \mid z_1, z_2 \in \mathbb{C}\}$. Given $q = z_1 + z_2j \in \mathbb{H}$, its conjugate is $\bar{q} = z_1 - z_2j$. Thus the standard positive definite quadratic form of $\mathbb{R}^4$ is identified to $q\bar{q} = |z_1|^2 + |z_2|^2$. The three-sphere $S^3$ then corresponds to the set of unit quaternions:

$$S^3 = \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\} = \{z_1 + z_2j \mid |z_1|^2 + |z_2|^2 = 1\}.$$ 

which is thus endowed with a group structure induced from $\mathbb{H}$.

Let us now consider the group homomorphism $\Phi : S^3 \times S^3 \to SO(4)$ which associates to $(p, q) \in S^3 \times S^3$ the map $\Phi_{p,q} : \mathbb{H} \to \mathbb{H}$ defined by $\Phi_{p,q}(h) = phq^{-1}$, which is an isometry of $S^3$. It turns out that $\Phi$ is surjective with kernel $\text{Ker}(\Phi) = \{\pm(1, 1)\}$, hence $\Phi$ induces a 1-1 correspondence between finite subgroups of $SO(4)$ and finite subgroups of $S^3 \times S^3$ containing $\{\pm(1, 1)\}$. Since $(-1, -1)$ is central, two subgroups are conjugate in $SO(4)$ if and only if their preimages are conjugate in $S^3 \times S^3$. To give a classification of finite subgroups $G$ of $SO(4)$ up to conjugation, it thus suffices to classify the subgroups $\hat{G} = \Phi^{-1}(G) < S^3 \times S^3$ containing $\{\pm(1, 1)\}$, up to conjugation in $S^3 \times S^3$. The latter are uniquely determined by the 5-tuple $(L, L_K, R, R_K, \phi)$, where (if $\pi_i : S^3 \times S^3 \to S^3$ denotes the projection to the $i$-th factor):

- $L = \pi_1(\hat{G})$;
- $L_K = \pi_1((S^3 \times \{1\}) \cap \hat{G})$;
- $R = \pi_2(\hat{G})$;
- $R_K = \pi_2((\{1\} \times S^3) \cap \hat{G})$;
- $\phi : L/L_K \to R/R_K$ is a group isomorphism obtained by composing the isomorphisms of $\hat{G}/(L_K \times R_K)$ with $L/L_K$ and with $R/R_K$, induced by $\pi_1$ and $\pi_2$ respectively.

Based on [MS19] Proposition 1, two such 5-tuples $(L, L_K, R, R_K, \phi)$ and $(L', L'_K, R', R'_K, \phi')$ correspond to conjugate subgroups if and only if there exist $p, q \in S^3$ such that conjugation by $p$ (which we denote by $c_p$) maps $L$ to $L'$ and $L_K$ to $L'_K$, $c_q$ maps $R$ to $R'$ and $R_K$ to $R'_K$, and $\phi' = c_q \circ \phi \circ c_p^{-1}$.

It only remains to determine finite subgroups of $S^3$ up to conjugation. For this purpose, observe that the equatorial $S^2 = \{bi + cj + dk \mid b^2 + c^2 + d^2 = 1\}$ in $S^3$ consists of points equidistant from the north and south poles, which are identified to 1 and $-1$ in $\mathbb{H}$. Hence
Together with Lemma 2.4, one obtains that the finite subgroups of $S^3$ up to conjugation are:

- The cyclic group $C_n = \{\cos(2\alpha \pi/n) + i \sin(2\alpha \pi/n), \alpha = 0, \ldots, n - 1\}$ for $n \geq 1$.
  
  Observe that $C_n$ contains the center $-1$ if and only if $n$ is even.

- The binary dihedral group $D_{2n} = C_n \cup C_n j$ for $n \geq 3$, which is a central extension of the dihedral group by a group of order 2.

- The binary tetrahedral group $T^* = \cup_{r=0}^2 (1/2 + i/2 + j/2 + k/2)^r D_4^*$, which is a central extension of the tetrahedral group.

- The binary octahedral group $O^* = T^* \cup (\sqrt{2}/2 + \sqrt{2}/j/2) T^*$ which is a central extension of the octahedral group.

- The binary icosahedral group $I^* = \cup_{r=0}^3 (\tau^{-1}/2 + \tau j/2 + k/2)^r T^*$ which is a central extension of the icosahedral group, where $\tau = (\sqrt{5} + 1)/2$.

Observe that for $n = 2$, $D_4^* = \{\pm 1, \pm j\}$ is conjugate to $C_4 = \{\pm 1, \pm i\}$. For this reason, the groups $D_{2n}^*$ are only taken with indices $n \geq 3$. The case $n = 3$, namely $D_8^* = \{\pm 1, \pm i, \pm j, \pm k\}$, is also called quaternion group.

3.2. Listing finite subgroups of $SO(4)$. The list of finite subgroups of $SO(4)$ up to conjugacy is given, applying the above procedure and following [DV64], in Table I by means of the data determining their preimage $\hat{G} = \Phi^{-1}(G)$. In most cases, the isomorphism $\phi$ is uniquely determined up to conjugacy, hence the last entry of the 5-tuple $(L, L_K, R, R_K, \phi)$ is omitted. In Families 1 and 1' (resp. 11 and 11'), $\phi$ is a isomorphism between cyclic (resp. dihedral) groups of order $r$ (resp. $2r$), thus it is encoded by a subscript $s$ such that $\gcd(s, r) = 1$, meaning that the canonical generator of $\mathbb{Z}/r\mathbb{Z}$ is sent to $s$ times the canonical generator (and the non-central involution induced by $j$ is sent to itself, in the dihedral case).

There are other cases in which a subscript is necessary. For instance, for Family 33 the relevant isomorphism $f$ between $D_{8m}^*/C_{2m}$ and $D_{8n}^*/C_{2n}$, which are isomorphic to the dihedral group of four elements, is defined by

$$f[e^{i\pi/m}] = [j] \quad \text{and} \quad f[j] = [e^{i\pi/n}] \quad (7)$$

Then the group with data $(D_{8m}^*, C_{2m}, D_{8n}^*, C_{2n}, f)$ is not conjugate to Family 11 with $r = 2$ (in which case the isomorphism between $D_{8m}^*/C_{2m}$ and $D_{8n}^*/C_{2n}$ is trivial) unless $m = 1$ or $n = 1$. Indeed in this latter case one has $(L, L_K) = (D_8^*, C_2)$ (or the same for $(R, R_K)$) and the equivalence classes $[j] = \{\pm j\}$ and $[i] = \{\pm i\}$ are conjugate in $S^3$ (for instance by $(i + j)/\sqrt{2}$). This explains Family 33 in Table I which is one of the families missing in the original list of [DV64], with the restriction $m \neq 1$ and $n \neq 1$.

A very similar behaviour appears for Family 33', which is another additional missing family in Du Val’s original list, together with Family 34. For more details we refer to [MS19] Section 2.1.
It is finally important to remark that several families, which we will call for instance Families 2bis, 3bis and so on, are to be added to Table 1 for the following reason. When the pairs \((L, L_K)\) and \((R, R_K)\) do not coincide, the groups determined by the 5-tuples \((L, L_K, R, R_K, \phi)\) and \((R, R_K, L, L_K, \phi^{-1})\) are conjugate in \(O(4)\) by means of the orientation-reversing isometry of \(S^3\), sending each quaternion to its inverse, but not in \(SO(4)\). For example, we shall call Family 2bis the family of groups with data \((D^*_{4m}/D^*_{4m}, C_{2n}/C_{2n})\); on the other hand, there is no Family 1bis since for each group in Family 1, its conjugate by an orientation-reversing isometry is already contained in Family 1 itself; as a final example, there is no Family 20bis as switching the roles of left and right multiplication gives rise to the same group. This analysis is developed more carefully in [MS19, Section 3.2], leading also to the computation of orientation-reversing self-isometries.

4. Invariant Seifert fibrations

We will now compare the above list of conjugacy classes of finite subgroups of \(SO(4)\) with the classification of subgroups which preserve Seifert fibrations of \(S^3\), up to fibration-preserving conjugacy.

4.1. Seifert fibrations of \(S^3\) revisited. We have already introduced the Seifert fibrations of \(S^3\) in Section 2.5; let us now give a more geometric description.

The Hopf fibration can be defined by means of the action of \(S^1\) on \(S^3\) by left multiplication: \((e^{i\theta}, q) \mapsto e^{i\theta}q\) for \(q \in S^3\). The fibers of the Hopf fibration are then the orbits of this (free) action and the projection map can be written as \(\pi: S^3 \to S^2\) expressed by

\[
\pi(z_1 + z_2j) = \frac{z_1}{z_2}
\]

where we are identifying the target \(S^2\) with the Riemann sphere in the model \(\mathbb{C} \cup \{\infty\}\).

The subgroup \(\mathcal{N}\) of \(SO(4)\) which preserves the Hopf fibration, meaning that \(\gamma\) induces a diffeomorphism \(\rho(\gamma)\) of the base \(S^2\), coincides with \(\text{Norm}_{S^3 \times S^3}(S^1 \times \{1\}) = \text{Norm}_{S^3}(S^1) \times S^3\), where it is easily checked that an element \(w_1 + w_2j\) normalizes \(S^1\) if and only if \(w_1 = 0\) or \(w_2 = 0\). Hence \(\text{Norm}_{S^3}(S^1) = \{w_1 + w_2j \mid w_1 = 0 \text{ or } w_2 = 0\} = O(2)^*\) and

\[
\mathcal{N} = \text{Norm}_{S^3 \times S^3}(S^1 \times \{1\}) = O(2)^* \times S^3 .
\]

It is also useful for the following to compute the induced action of \(\mathcal{N}\) on the base \(S^2\) of the fibration. For this purpose, it suffices to observe that elements of the form \((e^{i\theta}, 0) \in \mathcal{N}\) clearly act trivially on the base; that \((j, 0)\) (and therefore all elements of the form \((e^{i\theta}j, 0)\)) acts by the antipodal map of \(S^2\); on the other hand \((0, w_1 + w_2j)\) induces the action

\[
\lambda \mapsto \frac{\overline{w_1} \lambda + \overline{w_2}}{-w_2 \lambda + w_1}
\]

on the base \(\mathbb{C} \cup \{\infty\}\). For instance, elements of the form \((0, e^{i\theta})\) act on \(S^2\) by rotations of angle \(2\theta\) fixing the poles \(0\) and \(\infty\) while \((0, j)\) is a rotation of order two fixing \(i\) and
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$G$ & \text{order of } G & \text{conditions} \\
\hline
1. & $(C_{2m}/C_{2n}, C_{2n}/C_{2n})_s$ & $2mnr$ \hspace{1cm} $\gcd(s, r) = 1$ \\
1'. & $(C_{mr}/C_{nr}, C_{nr}/C_{nr})_s$ & $(mnr)/2$ \hspace{1cm} $\gcd(2, n) = 1$ \\
& & $\gcd(2, m) = 1$ \hspace{1cm} $\gcd(2, r) = 2$ \\
2. & $(C_{2m}/C_{2m}, D_{4n}/D_{4n})$ & $4mn$ \\
3. & $(C_{4m}/C_{2m}, D_{4n}/C_{2n})$ & $4mn$ \\
4. & $(C_{4n}/C_{2m}, D_{8n}/D_{4n})$ & $8mn$ \\
5. & $(C_{2m}/C_{2m}, T^*/T^*)$ & $24m$ \\
6. & $(C_{6m}/C_{2m}, T^*/D_6^*)$ & $24m$ \\
7. & $(C_{2m}/C_{2m}, O^*/O^*)$ & $48m$ \\
8. & $(C_{4m}/C_{2m}, O^*/T^*)$ & $48m$ \\
9. & $(C_{2m}/C_{2m}, I^*/I^*)$ & $120m$ \\
10. & $(D_{4m}^*/D_{4n}, D_{4n}/D_{4n})$ & $8mn$ \hspace{1cm} $\gcd(s, r) = 1$ \\
11. & $(D_{4m}/C_{2n}, D_{4m}^*/C_{2n})_s$ & $4mnr$ \hspace{1cm} $\gcd(s, r) = 1$ \\
11'. & $(D_{2m}^*/C_{nr}, D_{2m}^*/C_{nr})_s$ & $mnr$ \hspace{1cm} $\gcd(s, r) = 1$ \\
& & $\gcd(2, n) = 1$ \hspace{1cm} $\gcd(2, m) = 1$ \hspace{1cm} $\gcd(2, r) = 2$ \\
12. & $(D_{8m}/D_{4m}, D_{8n}/D_{4n})$ & $16mn$ \\
13. & $(D_{8m}/D_{4m}, D_{4n}/C_{2n})$ & $8mn$ \\
14. & $(D_{4m}^*/D_{4m}^*, D_{4n}^*/C_{2n})$ & $48m$ \\
15. & $(D_{4m}^*/D_{4m}^*, T^*/T^*)$ & $96m$ \\
16. & $(D_{4m}^*/D_{4m}^*, O^*/O^*)$ & $48m$ \\
17. & $(D_{8m}^*/D_{4m}^*, O^*/T^*)$ & $96m$ \\
18. & $(D_{12m}^*/C_{2n}, O^*/D_6^*)$ & $48m$ \\
19. & $(D_{4m}^*/D_{4m}^*, I^*/I^*)$ & $240m$ \\
20. & $(T^*/T^*, T^*/T^*)$ & $288$ \\
21. & $(T^*/C_{2}, T^*/C_{2})$ & $24$ \\
21'. & $(T^*/C_1, T^*/C_1)$ & $12$ \\
22. & $(T^*/D_{8}^*, T^*/D_{8}^*)$ & $96$ \\
23. & $(T^*/T^*, O^*/O^*)$ & $576$ \\
24. & $(T^*/T^*, I^*/I^*)$ & $1440$ \\
25. & $(O^*/O^*, O^*/O^*)$ & $1152$ \\
26. & $(O^*/C_{2}, O^*/C_{2})$ & $48$ \\
26'. & $(O^*/C_1, O^*/C_1)_{Id}$ & $24$ \\
26'' & $(O^*/C_1, O^*/C_1)_{f}$ & $24$ \\
27. & $(O^*/D_6^*, O^*/D_6^*)$ & $192$ \\
28. & $(O^*/T^*, O^*/T^*)$ & $576$ \\
29. & $(O^*/O^*, I^*/I^*)$ & $2880$ \\
30. & $(I^*/I^*, I^*/I^*)$ & $7200$ \\
31. & $(I^*/C_{2}, I^*/C_{2})_{Id}$ & $120$ \\
31'. & $(I^*/C_1, I^*/C_1)_{Id}$ & $60$ \\
32. & $(I^*/C_{2}, I^*/C_{2})_{f}$ & $120$ \\
32'. & $(I^*/C_1, I^*/C_1)_{f}$ & $60$ \\
33. & $(D_{8m}^*/C_{2m}, D_{8n}^*/C_{2n})_f$ & $8mn$ \hspace{1cm} $m \neq 1 \hspace{1cm} n \neq 1.$ \\
33'. & $(D_{8m}^*/C_{m}, D_{8n}^*/C_{n})_f$ & $4mn$ \hspace{1cm} $\gcd(2, n) = 1$ \hspace{1cm} $\gcd(2, m) = 1$ \hspace{1cm} $m \neq 1$ \hspace{1cm} $n \neq 1.$ \\
34. & $(C_4/C_m, D_{4n}/C_n)$ & $2mn$ \hspace{1cm} $\gcd(2, n) = 1$ \hspace{1cm} $\gcd(2, m) = 1$ \\
\hline
\end{tabular}
\caption{Finite subgroups of SO(4)}
\end{table}
and switching 0 and ∞. The anti-Hopf fibration is then obtained by composing π with an orientation-reversing isometry of $S^3$. Choosing the isometry $q → q^{-1}$, we obtain the expression $(z_1 + z_2j) ← z_1/z_2$.

In general the Seifert fibrations of $S^3$ introduced in Section 2.5 can be obtained by the action of $S^1$ defined by $(e^{iθ}, z_1 + z_2j) ← (e^{iθ}z_1, e^{iθ}z_2)$ or $(e^{iθ}, z_1 + z_2j) ← (e^{-iθ}z_1, e^{iθ}z_2)$, for $q ∈ S^3$ and $u, v$ coprime integers. We call these fibrations standard and any Seifert fibration of $S^3$ can be mapped by an orientation-preserving diffeomorphism to a standard one.

The projection of a standard fibration can be written as

$$\pi(z_1 + z_2j) = \frac{z_1^u}{z_2^v} \quad \text{or} \quad \pi(z_1 + z_2j) = \frac{-z_1^u}{z_2^v}.$$ 

In the first case, the normalizer of the action consists of elements $(w_1 + w_2j, u_1 + u_2j)$ provided $w_2 = u_2 = 0$ or $w_1 = u_1 = 0$ (unless $u = v = 1$). From the definition of the $S^1$ action, it can be checked directly that the base orbifold is $S^2(u, v)$ where the two cone points are the image of the fibers $z_1 = 0$ and $z_2 = 0$, with local invariants $\bar{v}/u$ and $\bar{u}/v$ where $u\bar{u} + v\bar{v} = 1$.

If $u = v = 1$ we recover the Hopf fibration. (Indeed when $u = 1$ or $v = 1$ the point is rational and the corresponding fiber generic).

Similarly as before, it is not necessary to repeat the analysis for the fibrations of the second type, as for every pair $(u, v)$ one can pre-compose $π$ with the orientation-reversing isometry $q → q^{-1}$ to obtain a new fibration of $S^3$ which is inequivalent to the previous ones in the category of oriented Seifert fibered manifolds, but equivalent in the category of (unoriented) Seifert fibered manifolds. Observe moreover that these fibrations always have bad base orbifolds, with the only exceptions of the Hopf and anti-Hopf fibrations.

### 4.2. Seifert fibrations from Du Val’s list.

In this subsection we analyze which standard fibrations of $S^3$ are left invariant by the subgroups in the Du Val’s list. Going back to Du Val’s list of subgroups of $SO(4)$ (Table 1), the groups which preserve the Hopf fibration are those with $L = C_m$ or $L = D_{2m}$, for some $m$. That is, by Families 1 to 19, 33, 33’, 34, and moreover Families 2 bis, 3 bis, 4 bis, 13 bis and 34 bis. In [MS15], the invariants of the Seifert fibration induced in the quotient by each of these groups has been computed, and we report the results in Table 2. We omitted the results for Families 1, 1’, 11 and 11’ which have a more complicated expression (see [MS15] Tables 2 and 3).

The other fibrations of type $π(z_1 + z_2j) = z_1^u/z_2^v$ are left invariant only by groups in Family 1, 1’, 11, 11’ and the spherical orbifolds obtained as quotients by these groups have an infinite number of nonisomorphic fibrations. To get a similar analysis for the remaining fibrations it suffices to note that the orientation-reversing isometry $q → q^{-1}$ maps the fibration $π(z_1 + z_2j) = z_1^u/z_2^v$ to $π(z_1 + z_2j) = z_1^u/z_2^v$, and a group preserves a fibration $π(z_1 + z_2j) = z_1^u/z_2^v$ if and only if its conjugate by $q → q^{-1}$ preserves $π(z_1 + z_2j) = z_1^u/z_2^v$.

**Remark 4.1.** The Seifert invariants of the quotient orbifold $S^3/G$ induced by the anti-Hopf fibration can be obtained from those arising from the Hopf fibration. In fact if a group $G$...
with data \((L, L_K, R, R_K, \phi)\) preserves the anti-Hopf fibration, then the group \(G'\) given by \((R, R_K, L, L_K, \phi^{-1})\) preserves the Hopf fibration: the Seifert fibration induced by the anti-Hopf fibration on \(S^3/G\) has the same base orbifold of that induced by the Hopf fibration on \(S^3/G'\) while the numerical invariants of \(S^3/G\) are the opposite of those of \(S^3/G'\) (Remark 2.6).

4.3. Spherical orbifolds with multiple fibrations. In the previous sections we discussed which standard Seifert fibrations of \(S^3\) are left invariant by the groups in Du Val’s list. If a Seifert fibration of \(S^3\) is left invariant by a group \(G\) acting on \(S^3\), the Seifert fibration of \(S^3\) induces a Seifert fibration of the quotient orbifold. If two different standard fibrations are left invariant, the induced fibrations of the quotient orbifold are not equivalent.

Moreover, we remark that finite subgroups in Du Val’s list can leave invariant also fibrations that are not standard, which can induce additional fibrations in the quotient orbifolds.

In this section we explore this phenomenon; the following lemma proved in [MS19, Lemma 5] shows that it can occur only in some specific cases. We call a non-Hopf fibration a fibration that cannot be mapped neither to the Hopf nor to the anti-Hopf fibration.

**Lemma 4.2.** Let \(G\) be a finite subgroup of \(SO(4)\) leaving invariant a Seifert fibration \(\pi\) of \(S^3\), then one of the two following conditions is satisfied:

1. \(G\) is conjugate in \(SO(4)\) to a subgroup in Families 1, 1’, 11 or 11’ and \(\pi\) is a non-Hopf fibration;
2. there exists an orientation-preserving diffeomorphism \(f : S^3 \to S^3\) such that \(\pi \circ f\) is the Hopf or the anti-Hopf fibration and \(f^{-1}Gf\) is a subgroup of \(SO(4)\).

Lemma 4.2 has many interesting consequences.

First, if \(L\) and \(R\) are both isomorphic to \(T^*\), \(O^*\) or \(I^*\), then no fibration of \(S^3\) is preserved by the action of the group \((L, L_K, R, R_K, \phi)\).

Then, if a group \(G\) leaves invariant a non-Hopf fibration \(\pi\), the base orbifold of the fibration induced on \(S^3/G\) (which is a bad orbifold) is either a sphere with at most two cone points or a disk with at most two corner points, since it is obtained as a quotient of the bad 2-orbifold \(S^2(u, v)\) with \(u\) and \(v\) different coprime integers.

Finally, if \(G\) leaves invariant a fibration equivalent to the Hopf fibration or to the anti-Hopf fibration, we can suppose that the fibration is standard and \(G\) is a subgroup of \(SO(4)\). We remark that this does not imply that \(G\) is a subgroup of the Du Val’s list; \(G\) is conjugate by an isometry to a group in the Du Val’s list but this isometry in general does not leave invariant the fibration.

We will focus for a moment on the case of groups of isometries which leave invariant the Hopf fibration. If \(G\) leaves invariant the Hopf fibration, then \(G\) is a subgroup of \(\mathcal{N} = \text{Norm}_{S^3 \times S^3}(S^1 \times \{1\})\). Moreover, conjugation of \(G\) by elements of \(\mathcal{N}\) respects the Hopf
| group                                                                 | e     | base orbifold            | invariants                      | case   |
|----------------------------------------------------------------------|-------|--------------------------|---------------------------------|--------|
| $2. \quad (C_{2m}/C_{2mn}, D_{4n}^*/D_{4n}^*)$                       | $-\frac{m}{n}$ | $S^2(2, 2, n)$           | $\frac{m}{n}, \frac{m}{2}, \frac{m}{2}$ | n even |
| $2.\text{bis} \quad (D_{4m}^*/D_{4m}^*, C_{2n}/C_{2n})$             | $-\frac{m}{n}$ | $D^2(n;) \quad \mathbb{R}P^2(n)$ | $\frac{m}{n}, \frac{m}{n}$ | n even |
| $3. \quad (C_{4m}/C_{2m}, D_{4n}^*/C_{2n})$                          | $-\frac{m}{n}$ | $S^2(2, 2, n)$           | $\frac{m}{n}, \frac{m+1}{2}, \frac{m+1}{2}$ | n odd  |
| $3.\text{bis} \quad (D_{4m}^*/C_{2m}, C_{4n}/C_{2n})$               | $-\frac{m}{n}$ | $D^2(n;) \quad \mathbb{R}P^2(n)$ | $\frac{m}{n}, \frac{m}{n}$ | n odd  |
| $4. \quad (C_{4m}/C_{2m}, D_{8n}^*/D_{4n}^*)$                        | $-\frac{m}{2n}$ | $S^2(2, 2, 2n)$          | $\frac{m+n}{2n}, \frac{m}{2}, \frac{m+1}{2}$ | n even |
| $4.\text{bis} \quad (D_{8m}^*/D_{4m}^*, C_{4n}/C_{2n})$              | $-\frac{m}{2n}$ | $D^2(2n;) \quad \mathbb{R}P^2(n)$ | $\frac{m+n}{2n}$ | n even |
| $5. \quad (C_{2m}/C_{2m}, T^*/T^*)$                                 | $-\frac{m}{6}$  | $S^2(2, 3, 3)$           | $\frac{m}{2}, \frac{m}{3}, \frac{m}{3}$ | n even |
| $6. \quad (C_{6m}/C_{2m}, T^*/D_8^*)$                               | $-\frac{m}{6}$  | $S^2(2, 3, 3)$           | $\frac{m}{2}, \frac{m+1}{3}, \frac{m+3}{3}$ | n even |
| $7. \quad (C_{2m}/C_{2m}, O^*/O^*)$                                 | $-\frac{m}{12}$ | $S^2(2, 3, 4)$           | $\frac{m}{2}, \frac{m}{3}, \frac{m}{4}$ | n even |
| $8. \quad (C_{4m}/C_{2m}, O^*/T^*)$                                 | $-\frac{m}{12}$ | $S^2(2, 3, 4)$           | $\frac{m}{2}, \frac{m+1}{2}, \frac{m+2}{4}$ | n even |
| $9. \quad (C_{2m}/C_{2m}, T^*/I^*)$                                 | $-\frac{m}{30}$ | $S^2(2, 3, 5)$           | $\frac{m}{2}, \frac{m}{3}, \frac{m}{5}$ | n even |
| $10. \quad (D_{4m}^*/D_{4m}^*, D_{4n}^*/D_{4n}^*)$                   | $-\frac{m}{2n}$ | $D^2(2; 2, 2n)$          | $\frac{m}{n}, \frac{m}{2}, \frac{m}{2}$ | n even |
| $12. \quad (D_{8m}^*/D_{4m}^*, D_{8n}^*/D_{4n}^*)$                   | $-\frac{m}{4n}$ | $D^2(2; 2, 2n)$          | $\frac{m+n}{2n}, \frac{m}{2}, \frac{m+1}{2}$ | n odd  |
| $13. \quad (D_{8m}^*/D_{4m}^*, D_{4n}^*/C_{2n})$                     | $-\frac{m}{2n}$ | $D^2(2; 2, 2n)$          | $\frac{m+n}{2n}, \frac{m}{2}, \frac{m+1}{2}$ | n even |
| $13.\text{bis} \quad (D_{4m}^*/C_{2m}, D_{8n}^*/D_{4n}^*)$           | $-\frac{m}{2n}$ | $D^2(2; 2, 2n)$          | $\frac{m+n}{2n}, \frac{m}{2}, \frac{m}{2}$ | n even |
| $14. \quad (D_{4m}^*/D_{4m}^*, T^*/T^*)$                             | $-\frac{m}{12}$ | $D^2(3; 2) \quad \mathbb{R}P^2(2)$ | $\frac{m}{2}, \frac{m}{2}$ | n even |
| $15. \quad (D_{4m}^*/D_{4m}^*, O^*/O^*)$                             | $-\frac{m}{24}$ | $D^2(2; 2, 3, 4)$        | $\frac{m}{2}, \frac{m}{3}, \frac{m}{4}$ | n even |
| $16. \quad (D_{4m}^*/C_{2m}, O^*/T^*)$                               | $-\frac{m}{12}$ | $D^2(2; 2, 3, 3)$        | $\frac{m}{2}, \frac{m}{3}, \frac{m}{3}$ | n even |
| $17. \quad (D_{8m}^*/D_{4m}^*, O^*/T^*)$                             | $-\frac{m}{24}$ | $D^2(2; 2, 3, 4)$        | $\frac{m}{2}, \frac{m+1}{3}, \frac{m+2}{4}$ | n even |
| $18. \quad (D_{12m}^*/C_{2m}, O^*/D_8^*)$                            | $-\frac{m}{12}$ | $D^2(2; 2, 3, 3)$        | $\frac{m}{2}, \frac{m+1}{3}, \frac{m+2}{3}$ | n even |
| $19. \quad (D_{4m}^*/D_{4m}^*, I^*/I^*)$                             | $-\frac{m}{60}$ | $D^2(2; 2, 3, 5)$        | $\frac{m}{2}, \frac{m}{3}, \frac{m}{5}$ | n even |
| $33. \quad (D_{8m}^*/C_{2m}, D_{8n}^*/C_{2n})_{f}$                   | $-\frac{m}{2n}$ | $D^2(2; 2, 2n)$          | $\frac{m+n}{2n}, \frac{m}{2}, \frac{m+1}{2}$ | n odd  |
| $33.' \quad (D_{8m}^*/C_{m}, D_{8n}^*/C_{n})_{f}$                    | $-\frac{m}{2n}$ | $D^2(2; 2, 2n)$          | $\frac{m+n}{2n}, \frac{m}{2}, \frac{m+1}{2}$ | m,n odd|
| $34. \quad (C_{4m}/C_{m}, D_{4n}^*/C_{n})$                           | $-\frac{m}{2n}$ | $S^2(2, 2, n)$           | $\frac{m+n}{2n}, \frac{m}{2}, \frac{m+1}{2}$ | m,n odd|
| $34.\text{bis} \quad (D_{4m}^*/C_{m}, C_{4n}/C_{n})$                  | $-\frac{m}{2n}$ | $D^2(2n;) \quad \mathbb{R}P^2(n)$ | $\frac{m+n}{2n}$ | m,n odd|

Table 2. Computation of local invariants from Du Val’s presentation
fibration, and therefore induces in the quotient orbifold $S^3/G$ a fibration-preserving isometry. We remark that some of the conjugations used in the work of Du Val list do not have this property. Hence in order to get a classification of Seifert fibered spherical 3-orbifolds up to orientation-preserving diffeomorphism, we need to classify finite subgroups of $\mathcal{N}$, up to conjugation in $\mathcal{N}$. This will result in a new list. In the following remark, we explain the differences with respect to Du Val’s list (Table 1).

**Remark 4.3.** There are three classes of phenomena which can occur for the Hopf fibration, marking the difference with Du Val’s list.

1. The groups $(L, L_K, R, R_K, \phi)$ and $(R, R_K, L, L_K, \phi^{-1})$ are conjugate by the orientation reversing isometry $q \mapsto q^{-1}$, which maps the Hopf fibration to the anti-Hopf fibration. Hence these groups are not to be considered equivalent but for our purposes, although they are equivalent in Du Val’s list. As already explained in Section 3.2 and done in Table 2 if the two families obtained by swapping the roles of $L$ and $R$ do not coincide up to orientation-preserving conjugation, they are distinguished by the suffix “bis” added to the number used by Du Val’s list. Considering the families leaving invariant the Hopf fibration, this phenomenon is relevant for the families 2, 3, 4, 12 and 34. In these cases the “bis” families do also preserve the Hopf fibration, but the Seifert invariant of the quotient orbifolds are obviously given by different formulae, see Table 2.

2. The subgroups generated by $i$ and by $j$ are conjugate in $S^3$, but not in $O(2)^*$. In Table 1 the subgroup $D_4^* = \{\pm 1, \pm j\} < O(2)^*$ is not considered since it gives the same group as when replaced with $C_4 = \{\pm 1, \pm i\}$ up to conjugation in $SO(4)$. To classify subgroups in $\mathcal{N}$ it is thus necessary to distinguish the two cases for $L$. Observe moreover that $D_4^* = \{\pm 1, \pm j\} < O(2)^*$ and $C_4 = \{\pm 1, \pm i\}$ are not conjugate in the normalizer of the group $D_8$; this implies that a group with $(L, L_K) = (D_8, C_4)$ is not conjugate to a group with $(L, L_K) = (D_8, D_4)$.

3. When $L = D_8^*$ and $L_K = C_1$ or $C_2$, the groups of Family 33 with $m = 1$, namely $(D_8^*/C_2, D_{8n}/C_{2n})_f$ (recalling that the isomorphism $f$ is defined in Subsection 5.2 Equation (7)) is conjugate in $S^3$ to the case $r = 2$, $m = 1$ of Family 11, namely $(D_8^*/C_2, D_{8n}/C_{2n})$ (where the automorphism between $L/L_K$ and $R/R_K$ is the identity). But they are not conjugate in $\mathcal{N}$ unless $n$ also equals 1. The same occurs for Family 33’. Although in Du Val’s list Families 33 and 33’ come with the restriction that $m, n \neq 1$, we will thus consider the case $m = 1$, $n \neq 1$ as independent.

Of course the considerations of Remark 4.3 can be repeated analogously for the anti-Hopf fibration, by switching the roles of $(L, L_K)$ and $(R, R_K)$.
5. Classification by diffeomorphism type

The purpose of this section is to determine a classification of spherical fibered three-orbifolds up to orientation-preserving diffeomorphism. By Theorem 2.3, this also turns out to be a classification up to isometry. More concretely, we will provide a recipe to determine when two fibered spherical orientable three-orbifolds are diffeomorphic in terms of the invariants of their fibrations. In fact, recall from the classification theorem that two Seifert fibered orbifolds have the same base orbifold and numerical invariants if and only if there exists an orientation-preserving diffeomorphism which preserves the fibration.

5.1. General strategy. By Proposition 2.7, the possible base orbifolds for a Seifert fibration of a compact spherical orbifold necessarily have positive Euler characteristic. Hence the possible base orbifolds are listed in (1), (2) and (3).

Spherical orbifolds may admit infinitely many inequivalent fibrations. This occurs only if the underlying manifold is a lens space. In fact, lens spaces admit infinitely many Seifert fibrations in the manifold sense, and the classification of these Seifert fibrations up to diffeomorphism is well-understood (see Subsection 5.3 below). The situation is however more delicate here since orbifolds with underlying manifold a lens space may admit other fibrations which are substantially different (i.e. the base orbifold may have mirror reflectors and corner reflectors). Also, orbifolds which admit infinitely many fibration do admit one (infinitely many, in fact) with base orbifold $S^2(b_1, b_2)$ or $D^2(;b_1, b_2)$, namely, either a sphere with at most two cone points or a disc with at most two corner reflectors.

We will proceed as follows. We first consider fibrations with base orbifold $B$ not of the form $S^2(b_1, b_2)$ or $D^2(;b_1, b_2)$. We thus distinguish the various cases according to the base orbifold of the fibration $\pi : O \to B$ and pointing out in each case which other fibrations are admitted by $O$. If $O$ admits more than one fibration, it will re-appear in different cases, providing in each one the instructions to obtain the other fibrations of the orbifolds. When $O$ also admits a fibration with base $S^2(b_1, b_2)$ or $D^2(;b_1, b_2)$, we will point out one of them (there are in fact infinitely many). In this analysis we use the results contained in Table 2 (see also Table 4 of [MS15]), which compute the invariants of quotients $S^3/G$ when $G < SO(4)$ preserves the Hopf fibration (recall also the discussion of Section 4).

We then deal (Subsection 5.4) with base orbifolds of the form $S^2(b_1, b_2)$ or $D^2(;b_1, b_2)$. Given such a spherical 3-orbifold $O$, with underlying space a lens space, producing a “list” of all the (infinitely many) Seifert fibrations that $O$ admits is rather complicated and beyond the scope of this paper. For the manifold case, an algorithm to list all Seifert fibrations on a given lens space, satisfying a prescribed bound on the “complexity” is described in [GL18]. Here we will rather describe a procedure which permits (algorithmically) to determine whether two Seifert fibered orbifolds with base orbifold a sphere with at most two cone points are diffeomorphic. Then we do the same for two Seifert orbifolds with base a disc with at most two corner reflectors, and finally we point out the (few) cases in which an orbifold admits
5.2. **Finitely many fibrations.** Let us now exhibit the (orientation-preserving) diffeomorphisms between spherical orbifolds with different (but in a finite number) fibrations, in terms of their invariants.

**Case 1.** The base orbifold is $S^2(2, 2, b)$.

The families of groups giving $S^2(2, 2, b)$ as base orbifold are 2, 3, 4 and 34 with the fibration induced by the Hopf fibration and 2bis, 3bis, 4bis and 34bis with the anti-Hopf fibration. By Equation (4) we obtain that the Euler invariant can be represented by a fraction with denominator $2b$. (If $b$ is even, also by a fraction with denominator $b$.)

Let us consider the generic spherical fibered orbifold with base orbifold $S^2(2, 2, b)$:

$$ \left( S^2(2, 2, b); \frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{b}; \frac{a}{2b} \right). $$

We remark that if $a > 0$ the fibration is induced by the anti-Hopf fibration while if $a < 0$ it is induced by the Hopf fibration. Indeed each of the groups considered in this case preserves both and each quotient orbifold admits at least two non-equivalent fibrations. By using Table 2 and Equation (4) we can compute the Seifert invariants of both fibrations for each quotient orbifold (see also Remark 4.1).

- If $m_1 = m_2 = 0$ then $a$ is even, then necessarily $m_3 \equiv -a/2 \pmod{b}$. By comparing the quotient fibrations of Families 2 and 2bis, or 3 and 3bis, we get:
  $$ \left( S^2(2, 2, b); 0, 0, \frac{a/2}{b}, \frac{a/2}{b} \right) \cong \left( D^2(|a/2|); \frac{b}{a/2}, -\frac{b}{a/2}, 0 \right) $$

- If $m_1 = m_2 = 1$ then $a$ is again even, then $m_3 \equiv -a/2 \pmod{b}$ and we get analogously:
  $$ \left( S^2(2, 2, b); 1, 1, \frac{a/2}{b}, \frac{a/2}{b} \right) \cong \left( \mathbb{R}P^2(|a/2|); \frac{b}{a/2}, -\frac{b}{a/2} \right) $$

Note that this family of orbifolds contains the prism manifolds, one of the family with multiple fibrations in the manifold case, see for instance [Hat07, Theorem 2.3].

- If $m_1 \neq m_2$ we can suppose $m_1 = 0$ and $m_2 = 1$; in this case Equation (4) implies that $a$ and $b$ have the same parity, and $m_3 \equiv (a + b)/2 \pmod{b}$. From Families 4 and 34 and their bis versions we get:

  $$ \left( S^2(2, 2, b); 0, 1, \frac{a + b}{2b}, \frac{a}{2b} \right) \cong \left( D^2(|a/2|); \frac{(a + b)/2}{a}, -\frac{b}{2a}, 1 \right) $$

We remark that, here and in what follows, when the base orbifold has mirror reflectors, the invariant $\xi$ is determined by the other invariants (see [Dun81, Corollary 2.9]).
Some of these orbifolds admit further non-equivalent fibrations because $i$ and $j$ are not conjugated in $O(2)^*$ (see Phenomenon 2 described in Remark 4.3). For example in Family 2 the groups with $m = 2$ are conjugate in $SO(4)$ to groups with $m = 1$ in Family 10 but this conjugation can not be performed in $\mathcal{N} = \text{Norm}_{S^3 \times S^3}(S^1 \times \{1\})$. This implies that an isometric copy of the Hopf fibration in non-standard position is left invariant by $(C_4/C_4, D_4^*/D_4^*)$ inducing on the quotient orbifold a different Seifert fibration. A very similar situation holds also for the Family 2 bis and the anti Hopf-fibration. If $b$ is even we finally obtain that:

$$\left( S^2(2, 2, b); 0, \frac{0}{2}, \pm \frac{2}{b}, \mp \frac{1}{b} \right) \cong \left( D^2(; 2, 2, b);; \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{b}; \mp \frac{1}{2b}; 1 \right),$$

and if $b$ is odd we get:

$$\left( S^2(2, 2, b); 0, \frac{0}{2}, \pm \frac{2}{b}, \mp \frac{1}{b} \right) \cong \left( D^2(2; b); \frac{1}{2}, \pm \frac{1}{b}; \mp \frac{1}{2b}; 1 \right).$$

Hence we can conclude that $(S^2(2, 2, b); 0/2, 0/2, \pm 2/b; \mp 2/b)$ admits three different fibrations for every $b$. The analogous situation occurs for groups with $m = 1$ in Family 4 and groups of Family 13 bis with $m = 1$. So we consider $b$ even and we obtain that base of the extra fibration depends on the parity of $b/2$; in fact if $b/2$ is odd we obtain:

$$\left( S^2(2, 2, b); 0, \frac{1}{2}, \pm \frac{1 + b/2}{b}; \mp \frac{1}{b} \right) \cong \left( D^2(; 2, 2, b/2);; \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{b/2}; \mp \frac{1}{b}; 1 \right)$$

while if $b/2$ is even we get:

$$\left( S^2(2, 2, b); 0, \frac{1}{2}, \pm \frac{1 + b/2}{b}; \mp \frac{1}{b} \right) \cong \left( D^2(2; b/2); \frac{1}{2}, \pm \frac{1}{b/2}; \mp \frac{1}{b/2}; 1 \right).$$

Also these orbifolds admit three fibrations.

Finally the groups in Families 3 and 34 with $m = 1$ are conjugate in $SO(4)$ (but not in $\mathcal{N}$) to groups in Family 11 and 11'; the same holds for the groups in Families 3bis and 34bis with $n = 1$ and the anti-Hopf fibration. We can conclude that the orbifolds

$$\left( S^2(2, 2, b); 0, \frac{0}{2}, \pm \frac{1}{b}; \mp \frac{1}{b} \right) \quad \text{and} \quad \left( S^2(2, 2, b); 0, \frac{1}{2}, \pm \frac{(1 + b)/2}{b}; \mp \frac{1}{2b} \right) \text{ with } b \text{ odd}$$

admit infinitely many fibrations with base orbifold a disk with at most two corner points. These cases are treated in Subsection 5.4 since they need a different approach; here we only exhibit a single fibration with such base (whose Seifert invariants are computed by using [MST15 Tables 2 and 3]):

$$\left( S^2(2, 2, b); 0, \frac{0}{2}, \pm \frac{1}{b}; \mp \frac{1}{b} \right) \cong \left( D^2(; b, b); \pm \frac{1}{b}; \pm \frac{1}{b}; \mp \frac{1}{b}; 0 \right)$$
for the former, and

\[
\left( S^2(2, 2, b); \frac{0}{2}, \frac{1}{2}, \pm \frac{(1 + b)/2}{b}; \mp \frac{1}{2b} \right) \cong \left( D^2(; b, b); \pm \frac{(1 + b)/2}{b}, \pm \frac{(1 + b)/2}{b}; \mp \frac{1}{2b}, 1 \right)
\]

for the latter.

**Case 2.** *The base orbifold is either* \( D^2(b, ) \) *or* \( \mathbb{R}P^2(b) \).

In these cases the Families to be considered are 2, 3, 4 and 34 with the fibration induced by the anti-Hopf fibration and 2bis, 3bis, 4bis and 34bis with the Hopf fibration. Again each of these groups preserves both the Hopf and the anti-Hopf fibration of \( S^3 \). The relations between the two fibrations induced in the quotient orbifold can be deduced from the previous case, since one of the two fibrations has \( S^2(2, 2, b) \) as base orbifold.

It remains to consider the extra fibrations caused by Phenomenon 2 in Remark 4.3. First, groups of Family 2 with \( m = 2 \) are conjugated to groups of Family 10 with \( m = 1 \), but the conjugating elements are not contained in \( \mathcal{N} \); this implies that the groups of Families 2 with \( m = 2 \) leave invariant the standard Hopf fibration, the standard anti-Hopf fibration and a non-standard Hopf fibration whose invariants can be obtained considering Family 10 in Table 2. An analogous situation occurs for Family 2bis when \( n = 2 \). This implies that the orbifolds \( (D^2(2, ; b, b, b, b); \pm \frac{1}{2}, \pm \frac{1}{2}, 0) \) admit three inequivalent Seifert fibrations. Indeed, the three fibrations has been already described in the previous case:

\[
\left( D^2(2, ; \frac{1}{2}, \pm \frac{b}{2}; 0) \right) \cong \left( S^2(2, 2, b); \frac{0}{2}, \frac{0}{2}, \pm \frac{2}{b}; \mp \frac{2}{b} \right) \cong \left( D^2(2, b); \frac{1}{2}, \pm \frac{1}{b}; \mp \frac{1}{2b}, 1 \right)
\]

when \( b \) is odd and greater than 1, and

\[
\left( D^2(2, ; \frac{0}{2}, \pm \frac{b}{2}; 0) \right) \cong \left( S^2(2, 2, b); \frac{0}{2}, \frac{2}{b}; \pm \frac{2}{b} \right) \cong \left( D^2(; 2, 2, b); \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{b}; \mp \frac{1}{2b}, 1 \right)
\]

when \( b \) is even.

The groups in Families 2, 3 and 34 with \( n = 1 \) and the ones in Families 2bis, 3bis and 34bis with \( m = 1 \) are conjugate to groups in Families 1 and 1'. This implies that the following orbifolds admit infinitely many fibrations with base orbifold \( S^2 \) with at most two cone points, see Subsection 5.4 here we only exhibit a single fibration with such base:
Case 3. The base orbifold is $D^2(2, 2, b)$. The families we have to consider are 10, 12, 13, 13 bis, 33, 33'. By Equation (4) the Euler invariant can be represented by a fraction with denominator 4 and here the generic fibered orbifold is:

$$
\left( D^2(b); \pm \frac{1}{b}; \mp \frac{1}{b}; 0 \right) \cong \left( S^2(b, b); \pm \frac{2}{b}, \pm \frac{2}{b}, \mp \frac{4}{b} \right) \text{ with } b \text{ even}
$$

$$
\left( D^2(b); \pm \frac{1}{b}; \mp \frac{1}{b}; 0 \right) \cong \left( S^2(2b, 2b); \pm \frac{1 + b}{2b}, \pm \frac{1 + b}{2b}, \mp \frac{1}{b} \right) \text{ with } b \text{ odd}
$$

$$
\left( \mathbb{R}P^2(b); \pm \frac{1}{b}; \mp \frac{1}{b} \right) \cong \left( S^2(b, b); \pm \frac{2}{b}, \pm \frac{2}{b}, \mp \frac{4}{b} \right) \text{ with } b \text{ odd}
$$

$$
\left( \mathbb{R}P^2(b); \pm \frac{1}{b}; \mp \frac{1}{b} \right) \cong \left( S^2(2b, 2b); \pm \frac{1 + b}{2b}, \pm \frac{1 + b}{2b}, \mp \frac{1}{b} \right) \text{ with } b \text{ even}
$$

$$
\left( D^2(b); \pm \frac{(1 + b)/2}{b}; \mp \frac{1}{2b}; 1 \right) \cong \left( S^2(2b, 2b); \pm \frac{(1 + b)/2}{2b}, \pm \frac{(1 + 3b)/2}{2b}, \mp \frac{1}{2b} \right) \text{ with } b \text{ odd}
$$

Each of these groups preserves both the Hopf and the anti-Hopf fibration, and by using Table 2 and Remark 4.1, we compute the Seifert invariants induced by both.

- If $m_1 = m_2 = 0$ then $a$ is even, and $m_3 \equiv a/2 \pmod{b}$. This kind of fibrations in the quotient orbifold is induced by the Hopf fibration left invariant by groups in Family 10 with $m$ and $n$ even, in Family 13 with $m$ odd and $n$ even, in Family 13bis with $m$ even and $n$ odd and in Family 33 with $m$ and $n$ odd. Considering the anti-Hopf fibration this case occurs for groups in Families 10 with $m$ and $n$ even, in Family 13 with $n$ odd and $m$ even, in Family 13bis with $n$ even and $m$ odd and in Family 33 with $m$ and $n$ odd. If the fibration considered is induced by the Hopf fibration we compute the invariants of the other fibration induced by the anti-Hopf fibration and viceversa. Finally we get the following diffeomorphisms:

$$
\left( D^2(; 2, 2, b); \; \frac{0}{2}, \frac{0}{2}, -\frac{a/2}{b}, \frac{a/2}{2b}; 0 \right) \cong \left( D^2( ; 2, 2, |a/2|); \; \frac{0}{2}, \frac{0}{2}, \frac{b}{a/2}, -\frac{b}{a}; 0 \right)
$$

- If $m_1 = m_2 = 1$ then $a$ is again even, and $m_3 \equiv a/2 \pmod{b}$. Carrying out an analysis similar to the previous case which involves again Families 10, 13, 13bis and 33, we obtain the following diffeomorphisms:

$$
\left( D^2(2, 2, b); \; 1, \frac{1}{2}, -\frac{a/2}{b}, \frac{a/2}{2b}; 1 \right) \cong \left( D^2(2; |a/2|); \; \frac{0}{2}, \frac{b}{a/2}, -\frac{b}{a}; 0 \right)
$$
• If $m_1 \neq m_2$ we can suppose $m_1 = 0$ and $m_2 = 1$; in this case Equation \[14\] implies that $a$ and $b$ have the same parity, and $m_3 \equiv (a + b)/2 \pmod{2}$. Here we have to consider Families 12 and 33'. Finally we get:

$$
\left( D^2(2; 2, 2, b); \frac{0}{2}, \frac{1}{2}, \pm \frac{1}{b} \pm \frac{1}{2b}; 0 \right) \cong \left( D^2(2; 2, 2, \pm \frac{1}{b}); \frac{0}{2}, \frac{1}{2}, \pm \frac{1}{b}; 0 \right) \cong \left( D^2(2; 2, 2, b); \frac{0}{2}, \frac{1}{2}, \pm \frac{1}{b} \pm \frac{1}{2b}; 0 \right)
$$

when $b$ is even

$$
\left( D^2(2; 2, 2, b); \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{b} \pm \frac{1}{2b}; 1 \right) \cong \left( D^2(2); \frac{0}{2}, \pm \frac{1}{2}; 1 \right) \cong \left( S^2(2, 2, 2b); \frac{0}{2}, \frac{0}{2}, \pm \frac{1}{2b} \pm \frac{1}{2b} \right)
$$

when $b$ is odd. These orbifolds admit three fibrations.

Moreover, by Phenomenon 3 the following orbifolds

$$
\left( D^2(2; 2, 2, b); \frac{0}{2}, \frac{1}{2}, \pm \frac{1}{b} \pm \frac{1}{2b}; 0 \right) \text{ and } \left( D^2(2; 2, 2, b); \frac{0}{2}, \frac{1}{2}, \pm \frac{b+1}{2b}; \frac{1}{2b}; 1 \right)
$$

with $b$ odd given by groups in Family 33 with $m = 1$ and Family 33' with $m = 1$ have infinitely many fibrations with base orbifold a disk with two corner points (these groups are conjugate in $SO(4)$ to groups in Families 11 and 11'). As usual we describe here one of these fibrations and the general analysis can be found in Subsection 5.4.

$$
\left( D^2(2; 2, 2, b); \frac{0}{2}, \frac{1}{2}, \pm \frac{b+1}{2b}; \frac{1}{2b}; 0 \right) \cong \left( D^2(2; 2b, 2b); \pm \frac{b+1}{2b}, \pm \frac{b+1}{2b}; \mp \frac{1}{2b}; 1 \right)
$$

$$
\left( D^2(2; 2, 2, b); \frac{0}{2}, \frac{1}{2}, \pm \frac{b+1}{2b}; \mp \frac{1}{2b}; 1 \right) \cong \left( D^2(2; 2b, 2b); \pm \frac{b+1}{2b}, \pm \frac{b+1}{2b}; \mp \frac{1}{2b}; 1 \right)
$$

where $b$ is an odd integer.

**Case 4.** The base orbifold is $D^2(2; b)$.

The groups giving fibered quotients with base orbifold $D^2(2; b)$ are contained in Families 10, 13, 13bis, 33. Each of these groups leaves invariant both the Hopf and the anti-Hopf fibration, so the quotient orbifolds has at least two fibrations and the possible base orbifolds are $D^2(2; b)$ and $D^2(2, 2, b)$. The cases when $D^2(2, 2, b)$ appears as base orbifold of one of
the two fibrations have been already treated in the previous case. When both fibrations have $D^2(2; b)$ we obtain the following diffeomorphism:

$$\left( D^2(2; b); \frac{1}{2}; -\frac{a}{b}; \frac{a}{2b}; 1 \right) \cong \left( D^2(2; |a|); \frac{1}{2}; \frac{b}{a}; -\frac{b}{2a}; 1 \right).$$

Phenomena 2 and 3 of Remark 4.3 give extra fibrations to the following orbifolds:

- $\left( D^2(2; b); \frac{1}{2}; \pm \frac{1}{b}; \mp \frac{1}{2b}; 1 \right)$ whose fibration is induced by the Hopf fibration left invariant by groups in Families 10 and 13bis with $m = 1$, and by the anti-Hopf fibration left invariant by groups in Families 10 and 13bis with $n = 1$; these orbifolds have three fibrations and the extra fibrations (already described in the previous case) are:

$$\left( D^2(2; b); \frac{1}{2}; \pm \frac{b}{2}; 0 \right) \cong \left( S^2(2, 2, b); \frac{0}{2}; \frac{0}{2}; \pm \frac{2}{b} \mp \frac{2}{b} \right)$$

when $b$ is odd and

$$\left( D^2(2; b); \frac{1}{2}; \pm \frac{b}{2}; 1 \right) \cong \left( S^2(2, 2, 2b); \frac{0}{2}; \frac{1}{2}; \pm \frac{1 + b}{2b} \mp \frac{1}{2b} \right)$$

when $b$ is even.

- $\left( D^2(2; b); \frac{0}{2}; \pm \frac{1}{b}; \mp \frac{1}{2b}; 1 \right)$ with $b$ even, whose fibration is induced by the Hopf fibration left invariants by groups in Family 33 with $m = 1$; these orbifolds have infinitely many fibrations with base orbifold a disk with two corner points because the groups are conjugate to groups in Family 11. Here we describe only one of these fibrations:

$$\left( D^2(2; b); \frac{0}{2}; \pm \frac{1}{b}; \mp \frac{1}{2b}; 1 \right) \cong \left( D^2(2b, 2b); \frac{1 + b}{2b}; \pm \frac{1 + b}{2b}; \mp \frac{1}{2b}; 1 \right)$$

Case 5. The base orbifold is $S^2(2, 3, b)$ or $D^2(2, 3, b)$ with $b = 3, 4, 5$. 

Fibrations having these base orbifold are induced by the Hopf fibration left invariant by groups in Families 5,6,7,8,9,14,15,16,17,18,19 and by the anti-Hopf fibration left invariant by the bis-versions of the same families. In this case each group leaves invariant exactly one among the Hopf and the anti-Hopf fibration. With few exceptions due to Phenomenon 2 of Remark 4.3 these orbifolds have exactly one fibration.

The sporadic diffeomorphisms are given by the correspondences between Family 5 ($m = 2$) and 14 ($m = 1$), Family 7 ($m = 2$) and 15 ($m = 1$), Family 8 and 16 ($m = 1$), Family 9 ($m = 2$) and 19 ($m = 1$), and the analogous correspondence between the bis-versions of the groups. All the quotient orbifolds thus have 2 nonequivalent fibrations. We collect here
these diffeomorphisms:
\[\left( S^2(2, 3, 3); \frac{0}{2}, \pm \frac{2}{3}, \pm \frac{2}{3}; \mp \frac{1}{3} \right) \cong \left( D^2(3; 2); \frac{0}{3}, \frac{2}{3}, \pm \frac{1}{12}; 1 \right)\]
\[\left( S^2(2, 3, 4); \frac{0}{2}, \pm \frac{2}{3}, \pm \frac{2}{3}; \mp \frac{1}{6} \right) \cong \left( D^2(3; 2, 3); \frac{1}{2}, \frac{1}{3}, \pm \frac{1}{6}; 1 \right)\]
\[\left( S^2(2, 3, 4); \frac{0}{2}, \pm \frac{1}{3}, \pm \frac{3}{4}; \mp \frac{1}{12} \right) \cong \left( D^2(3, 3); \frac{1}{2}, \frac{1}{3}, \pm \frac{1}{12}; 1 \right)\]
\[\left( S^2(2, 3, 5); \frac{0}{2}, \pm \frac{2}{3}, \pm \frac{2}{5}; \mp \frac{1}{15} \right) \cong \left( D^2(3, 5); \frac{1}{2}, \frac{1}{3}, \pm \frac{1}{5}; \mp \frac{1}{60}; 1 \right)\]
Remark 5.2. We remark that the expressions in \cite{JN83} are slightly different to those recorded in \cite[Theorem 4.4]{JN83} and \cite[Theorem 4.4]{GL18}: this is because we are adopting a different convention here for the classification data of a Seifert fibration, following \cite{BS85} and \cite{Hat07}. To pass from our convention to that used in \cite{JN83} and \cite{GL18} one should switch the roles of the $\alpha_i$ and $\beta_i$, and there is also a sign difference.

We shall now explain more precisely necessary and sufficient conditions for two lens spaces to be (orientation-preserving) diffeomorphic. A fundamental fact is the following (proved in \cite{Bon83}, see also \cite[Theorem 10.1.12]{Mar16} and \cite[Theorem 2.5]{Hat07}):

**Proposition 5.3.** Any two Heegard surfaces of genus 1 in a lens space are isotopic.

Proposition 5.3 implies that, given any orientation-preserving diffeomorphism between two lens spaces $L(p, q)$ and $L(p', q')$, one can modify the diffeomorphism so that it maps $\partial T_1 = \partial T_2$ to $\partial T'_1 = \partial T'_2$. The diffeomorphism now either sends $T_i$ to $T_i$, or $T_1$ to $T'_2$ and $T'_2$ to $T_1$. It is classical to show that the first case occurs if and only if

$$p = p' \quad \text{and} \quad q \equiv q' \mod p \quad (10)$$

whereas the second if and only if

$$p = p' \quad \text{and} \quad qq' \equiv 1 \mod p \quad (11)$$

In light of (10), one checks easily that two equivalent presentations of the same Seifert manifold $\mathcal{M}$ (as explained in Subsection 2.4) give an equivalent outcome in Equation (9).

By putting together the two cases (10) and (11) together, one has:

**Proposition 5.4.** Two lens spaces $L(p, q)$ and $L(p', q')$ have an orientation-preserving diffeomorphism if and only if

$$p = p' \quad \text{and} \quad q^{\pm 1} \equiv q' \mod p \quad (12)$$

5.4. **Infinitely many fibrations.** Let us now get back to the diffeomorphism type of spherical orbifolds.

**Case 6.** The base orbifold is a sphere with at most two cone points.

In this case the underlying topological space of the spherical 3-orbifold is a lens space and the possible singular set is contained in the union of the cores of the tori giving the standard Heegard decomposition of the lens space. Consequently the singular set may be empty, a knot or a link with two components. Exactly as in the analogous manifold case, if an orbifold $S^3/G$ is contained in this family, then it admits infinitely many fibrations, since (a conjugate of) $G$ leaves invariant all the non-Hopf fibrations.

Let us consider $\mathcal{O}$ and $\mathcal{O}'$ with base orbifolds $S^2(b_1, b_2)$ and $S^2(b'_1, b'_2)$, namely

$$\left( S^2(b_1, b_2); \frac{a_1}{b_1}, \frac{a_2}{b_2}, e \right) \quad \left( S^2(b'_1, b'_2); \frac{a'_1}{b'_1}, \frac{a'_2}{b'_2}, e' \right), \quad (13)$$
and explain how one can decide whether $\mathcal{O}$ and $\mathcal{O}'$ are diffeomorphic.

The recipe is as follows. First compute the index of singularity of the preimages of each cone point. These are simply given by $\iota_i = \gcd(a_i, b_i)$ for $\mathcal{O}$ and $\iota'_i = \gcd(a'_i, b'_i)$, for $i = 1, 2$, where $a_i/b_i$ and $a'_i/b'_i$ are the local invariants. Then the underlying Seifert fibered manifolds $\mathcal{M}$ and $\mathcal{M}'$ are given by the same expressions as in [13], except that one needs to replace $a_i$ by $a_i/\iota_i$, $b_i$ by $b_i/\iota_i$, and similarly for $a'_i$ and $b'_i$.

Recalling the explanation given in Subsection 2.4, using also the Euler classes $\epsilon$ and $\epsilon'$, one can easily express the underlying manifolds $\mathcal{M}$ and $\mathcal{M}'$ in terms of the classical invariants for Seifert manifolds. These will be of the form:

$$\mathcal{M} \cong \left( S^2; \alpha_1/\beta_1, \alpha_2/\beta_2 \right) \quad \mathcal{M}' \cong \left( S^2; \alpha'_1/\beta'_1, \alpha'_2/\beta'_2 \right),$$

respectively, where $\beta_i = b_i/\iota_i$, $\beta'_i = b'_i/\iota'_i$, the $\alpha_i$ will be integer numbers relatively prime with $\beta_i$ and satisfying $\alpha_i/\beta_i \equiv a_i/b_i \mod 1$, and similarly for the $\alpha'_i$.

There are now various possibilities:

- If $\{\iota_1, \iota_2\}$ does not equal $\{\iota'_1, \iota'_2\}$, then $\mathcal{O}$ and $\mathcal{O}'$ are certainly not diffeomorphic, since a diffeomorphism should respect the singularity indices.

- If $\iota_1 = \iota_2 = \iota'_1 = \iota'_2$, determining whether $\mathcal{O}$ and $\mathcal{O}'$ are diffeomorphic amounts to whether their underlying manifolds $\mathcal{M}$ and $\mathcal{M}'$ are diffeomorphic. In fact, by the discussion above, a diffeomorphism between lens spaces always maps a solid torus of the standard Heegard decomposition of $\mathcal{M}$ to a solid torus of the decomposition of $\mathcal{M}'$, and one can arrange such a diffeomorphism to map the (singular, if the $\iota_i$ are not 1) cores of each torus of $\mathcal{M}$ to cores of each torus of $\mathcal{M}'$. Therefore, it suffices to compute the underlying lens spaces of $\mathcal{O}$ and $\mathcal{O}'$ (computed in the first step by applying [9] to [13], and determine whether they are diffeomorphic by checking if the classical formula [12] holds (Proposition 5.4).

- The last possibility is when $\{\iota_1, \iota_2\} = \{\iota'_1, \iota'_2\}$ but $\iota_1 \neq \iota_2$ (and thus $\iota'_1 \neq \iota'_2$). In this case one cannot directly apply the standard classification for the underlying lens spaces, since one has to take into account that the cores of the two solid tori in each Heegard decomposition have different singularity index, and the singularity index must be preserved by orbifold diffeomorphisms. Up to switching the order of cone points, let us assume $\iota_1 = \iota'_1$ and $\iota_2 = \iota'_2$. In the notation used above in this section, $\mathcal{O}$ and $\mathcal{O}'$ are diffeomorphic if and only if there is a diffeomorphism of the underlying lens spaces which maps $T_i$ to $T'_i$, for $i = 1, 2$. According to [10], this is the case if and only if $p = p'$ and $q, q'$ have the same residue modulo $p$. In conclusion, one has to use again the formulae [9] to compute $p, q, p', q'$, and check whether they satisfy [10] (instead of [12]).

Finally we remark that there are two spherical 3-orbifolds, each of which admits both infinitely many fibrations with base orbifold a 2-sphere with two cone points and infinitely
many fibrations with base orbifold a disk with two corner points. Fibrations having base orbifold a disk with two corner points will be considered in Case 7 below.

The first orbifold is the quotient by the group \((C_4/C_2, C_4/C_2)\) that can be conjugate (by an isometry which does not preserve the Hopf fibration) to the group \((D^*_4/C_2, D^*_4/C_2)\). The underlying topological space of this 3-orbifold is the 3-sphere and the singular set is the Hopf link with local group of order two. Two possible fibrations are \((S^2(2, 2); 0, 0; -1)\) and \((D^2; -1)\). The second orbifold is given by the group \((C_4/C_1, C_4/C_1)\) that can be conjugate to the group \((D^*_4/C_1, D^*_4/C_1)\). In this case the underlying topological space is the 3-sphere and the singular set is the trivial knot with local group of order two. Two possible fibrations are \((S^2(2, 2); 0, 0; -1)\) and \((D^2; -1)\).

**Case 7.** The base orbifold is a disk with at most two corner points.

Also in this case all the orbifolds admit infinitely many fibrations, since all the non-Hopf fibrations are preserved by the groups involved.

Let \(O\) be a fibered orbifold whose base orbifold is \(D^2(n_1, n_2)\). In this case we can consider the 2-fold branched covering \(O'\) of \(O\) induced by the 2-fold orbifold covering \(S^2(n_1, n_2) \to D^2(n_1, n_2)\) obtained by doubling the disk along its boundary. The orbifold \(O'\) has the same local invariants of \(O\). (It is worth remarking that the preimage of a corner reflector in the base orbifold of \(O\) will be a cone point in the base orbifold of \(O'\), but the associated numerical invariants will be the same.) The Euler class of \(O'\) is twice as that of \(O\).

Fibrations with base orbifold \(D^2(n_1, n_2)\) are admitted by orbifolds given by groups in Families 11 and 11’. These groups are generalized dihedral groups \(Z_2 \rtimes A\) where \(A\) is an Abelian normal subgroup of rank at most two. The subgroup \(A\) is the unique index two Abelian subgroup with this property (with the exception of the cases \(A \cong Z_2\) and \(A \cong Z_2 \times Z_2\)).

The subgroup \(A\) corresponds to the unique 2-fold branched covering induced by the orbifold covering \(S^2(n_1, n_2) \to D^2(n_1, n_2)\). Also in the two exceptional cases all the possible 2-fold branched coverings having \(S^2(n_1, n_2)\) as base orbifold are diffeomorphic. Hence we have a 1-1 correspondence between the orbifolds of Case 6 and the orbifolds of Case 7. We can reduce to the same procedure as in the previous case to decide whether two fibered orbifolds with base orbifolds \(D^2(n_1, n_2)\) are diffeomorphic.

Some of these orbifolds admit also a fibration whose base orbifold is not a disk with at most two corner points. We have already listed these exceptional fibrations in Cases 1-6.

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