ON THE WEIGHTED $L^2$ ESTIMATE FOR THE $k$-CAUCHY-FUETER OPERATOR AND THE WEIGHTED $k$-BERGMAN KERNEL

WEI WANG

ABSTRACT. The $k$-Cauchy-Fueter operators, $k = 0, 1, \ldots$, are quaternionic counterparts of the Cauchy-Riemann operator in the theory of several complex variables. The weighted $L^2$ method to solve Cauchy-Riemann equation is applied to find the canonical solution to the non-homogeneous $k$-Cauchy-Fueter equation in a weighted $L^2$-space, by establishing the weighted $L^2$ estimate. The weighted $k$-Bergman space is the space of weighted $L^2$ integrable functions annihilated by the $k$-Cauchy-Fueter operator, as the counterpart of the Fock space of weighted $L^2$-holomorphic functions on $\mathbb{C}^n$. We introduce the $k$-Bergman orthogonal projection to this closed subspace, which can be nicely expressed in terms of the canonical solution operator, and its matrix kernel function. We also find the asymptotic decay for this matrix kernel function.

1. INTRODUCTION

The $k$-Cauchy-Fueter operators over $\mathbb{R}^{4n}$

$$\mathcal{D}_0^{(k)} : \mathcal{C}^\infty (\mathbb{R}^{4n}, \otimes^k \mathbb{C}^2) \longrightarrow \mathcal{C}^\infty (\mathbb{R}^{4n}, \otimes^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n})$$

for $k = 0, 1, \ldots$, are quaternionic counterparts of the Cauchy-Riemann operator $\mathcal{D}$ in the theory of several complex variables, where $\otimes^p \mathbb{C}^2$ is the $p$-th symmetric tensor product of $\mathbb{C}^2$. If we write a vector in the quaternionic space $\mathbb{H}^n$ as $q = (q_0, \ldots, q_{n-1})$, the usual Cauchy-Fueter operator is defined as

$$\mathcal{D} : C^1(\mathbb{H}^n, \mathbb{H}) \to C(\mathbb{H}^n, \mathbb{H}^n), \quad \mathcal{D} f = \begin{pmatrix} \overline{\partial} q_0 f \\ \vdots \\ \overline{\partial} q_{n-1} f \end{pmatrix},$$

for $f \in C^1(\mathbb{H}^n, \mathbb{H})$, where $\overline{\partial} q_l = \partial_{x_4 l+1} + i \partial_{x_4 l+2} + j \partial_{x_4 l+3} + k \partial_{x_4 l+4}$, if we write $q_l = x_{4 l+1} + x_{4 l+2} i + x_{4 l+3} j + x_{4 l+4} k \in \mathbb{H}$, $l = 0, 1, \ldots, n - 1$. It is known that the Cauchy-Fueter operator coincides with the 1-Cauchy-Fueter operator $\mathcal{D}^{(1)}$ in the quaternionic case, we have a family of operators acting on $\otimes^k \mathbb{C}^2$-valued functions, $k = 0, 1, \ldots$, because $\text{SU}(2)$ as the group of unit quaternions has a family of irreducible representations $\otimes^k \mathbb{C}^2$, while $S^1$ as the group of unit complex numbers has only one irreducible representation. The $k$-Cauchy-Fueter operators over $\mathbb{R}^4$ also have the origin in physics: they are the elliptic version of spin $k/2$ massless field operators over the Minkowski space (cf. e.g. [4] [11] [16] [17]): $\mathcal{D}_0^{(1)} \phi = 0$ corresponds to the Dirac-Weyl equation whose solutions correspond to neutrinos; $\mathcal{D}_0^{(2)} \phi = 0$ corresponds to the Maxwell equation whose solutions correspond to photons; $\mathcal{D}_0^{(3)} \phi = 0$ corresponds to the Rarita-Schwinger equation; $\mathcal{D}_0^{(4)} \phi = 0$ corresponds to linearized Einstein’s equation whose solutions correspond to weak gravitational fields; etc..
To develop the function theory of several quaternionic variables, we need to solve the non-homogeneous $k$-Cauchy-Fueter equation:

$$
\mathcal{G}^{(k)}_0 u = f,
$$

where $u$ is $\mathcal{O}^k\mathbb{C}^2$-valued and $f$ is $\mathcal{O}^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n}$-valued. Under the identification

$$
\mathcal{O}^k\mathbb{C}^2 \simeq \mathbb{C}^{k+1}, \quad \mathcal{O}^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n} \simeq \mathbb{C}^{2kn},
$$

$\mathcal{G}^{(k)}_0$ is a $2kn \times (k+1)$-matrix valued differential operator of the first order with constant coefficients. The equation (1.1) is over determined and its compatibility condition is that $f$ is $\mathcal{G}^{(k)}_1$-closed, i.e.

$$
\mathcal{G}^{(k)}_1 f = 0,
$$

where $\mathcal{G}^{(k)}_1$ is the second operator in the $k$-Cauchy-Fueter complex:

$$
0 \rightarrow C^\infty (\mathbb{R}^{4n}, \mathcal{V}_0) \xrightarrow{\mathcal{G}^{(k)}_0} C^\infty (\mathbb{R}^{4n}, \mathcal{V}_1) \xrightarrow{\mathcal{G}^{(k)}_1} C^\infty (\mathbb{R}^{4n}, \mathcal{V}_2) \xrightarrow{\mathcal{G}^{(k)}_2} \cdots,
$$

and

$$
\mathcal{V}_0 := \mathcal{O}^k\mathbb{C}^2, \quad \mathcal{V}_1 := \mathcal{O}^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n}, \quad \mathcal{V}_2 := \mathcal{O}^{k-2}\mathbb{C}^2 \otimes \mathbb{C}^{2n}.
$$

Here $\wedge^2\mathbb{C}^{2n}$ is the 2-th exterior product of $\mathbb{C}^{2n}$. These complexes play the role of Dolbeault complex in several complex variables, and are now explicitly known [21] (cf. also [1] [2] [3] [7] [8]).

The author [20] [21] solved the non-homogeneous $k$-Cauchy-Fueter equation in $L^2$-space over $\mathbb{R}^{4n}$ by using the method of classical harmonic analysis, and deduced Hartogs’ phenomenon and integral representation formulae. In this paper, the weighted $L^2$ method to solve the $\mathcal{G}$ equation on $\mathbb{C}^n$ (see e.g. [9] [12] [14] and references therein) is extended to solve the non-homogeneous $k$-Cauchy-Fueter equation (1.1). The $L^2$ method is a general method to deal with overdetermined systems of linear differential equations when we can establish the necessary $L^2$ estimate, e.g. it is applied to the Dirac operator in Clifford analysis [15]. The reason to consider the weighted $L^2$-space is as follows. $f$ is called $k$-regular if $\mathcal{G}^{(k)}_0 f = 0$ in the sense of distributions. It is known that the space of $k$-regular polynomials are infinite dimensional (cf. [13]), and such functions are $L^2$-integrable with Gaussian weight. This is similar to complex analysis, where one consider the space of $L^2$-integrable holomorphic functions with Gaussian weight, called Fock space. Without a weight, a $L^2$-integrable holomorphic (or $k$-regular) function must vanish. Given a nonnegative function $\varphi$, called a weighted function, consider the Hilbert space $L^2_\varphi(\mathbb{R}^{4n}, \mathbb{C})$ with the weighted inner product

$$
\langle u, v \rangle_\varphi := \int_{\mathbb{R}^{4n}} u\overline{v} e^{-2\varphi} dV,
$$

where $dV$ is the Lebesgue measure on $\mathbb{R}^{4n}$. For a complex linear space $\mathcal{V}$ with an inner product $\langle \cdot, \cdot \rangle$ (e.g. $\mathcal{V} = \mathcal{O}^k\mathbb{C}^2$ or $\mathcal{O}^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n}$), we define $L^2_\varphi(\mathbb{R}^{4n}, \mathcal{V})$ with the weighted inner product

$$
\langle f, g \rangle_\varphi := \int_{\mathbb{R}^{4n}} \langle f, g \rangle e^{-2\varphi} dV,
$$

and the weighted norm $\|f\|_\varphi := \langle f, f \rangle_\varphi^{\frac{1}{2}}$. The weighted $k$-Bergman space with respect to weight $\varphi = |x|^2$ is then defined as

$$
A^n_\varphi(\mathbb{R}^{4n}, \varphi) := \left\{ f \in L^2_\varphi(\mathbb{R}^{4n}, \mathbb{C}^2); \mathcal{G}^{(k)}_0 f = 0 \right\}.
$$

It is infinite dimensional [13] because $k$-regular polynomials are integrable with respect to this weight.
Theorem 1.1. Suppose that \( \varphi(x) = |x|^2 \) and \( k = 2, 3, \ldots \). Then
\[
(1.6) \quad L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0} L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1) \xrightarrow{\mathcal{D}_1} L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_2)
\]
is a complex, i.e. for any \( u \in \text{Dom}(\mathcal{D}_0) \),
\[
\mathcal{D}_0 u \in \text{Dom}(\mathcal{D}_1) \quad \text{and} \quad \mathcal{D}_1 \mathcal{D}_0 u = 0.
\]
Then if \( f \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1) \) is \( \mathcal{D}_1 \)-closed, the nonhomogeneous \( k \)-Cauchy-Fueter equation \( (1.11) \) has at most one solution \( u \in \text{Dom}(\mathcal{D}_0) \) orthogonal to \( A_2^2(\mathbb{R}^{4n}, \varphi) \). If it exists, it is called the canonical solution to the nonhomogeneous \( k \)-Cauchy-Fueter equation \( (1.11) \). Consider the associated Laplacian operator \( \Box : L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1) \rightarrow L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1) \) given by
\[
\Box := \mathcal{D}_0 \mathcal{D}_0^* + \mathcal{D}_1^* \mathcal{D}_1.
\]

Theorem 1.2. Suppose that \( \varphi(x) = |x|^2 \) and \( k = 2, 3, \ldots \). Then
(1) \( \Box \varphi \) has a bounded, self-adjoint and non-negative inverse \( N_\varphi \) such that
\[
\| N_\varphi f \|_\varphi \leq \frac{1}{4} \| f \|_\varphi, \quad \text{for any } f \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1).
\]
(2) \( \mathcal{D}_0^* N_\varphi f \) is the canonical solution operator to the nonhomogeneous \( k \)-Cauchy-Fueter equation \( (1.11) \), i.e. if \( f \in \text{Dom}(\mathcal{D}_1) \) is \( \mathcal{D}_1 \)-closed, then
\[
\mathcal{D}_0 \mathcal{D}_0^* N_\varphi f = f
\]
and \( \mathcal{D}_0^* N_\varphi f \) orthogonal to \( A_2^2(\mathbb{R}^{4n}, \varphi) \). Moreover,
\[
(1.7) \quad \| \mathcal{D}_0^* N_\varphi f \|_\varphi \leq \frac{1}{2} \| f \|_\varphi, \quad \| \mathcal{D}_1 N_\varphi f \|_\varphi \leq \frac{1}{2} \| f \|_\varphi.
\]

The key step to prove this theorem is to establish the following weighted \( L^2 \) estimate.

Theorem 1.2. Suppose that \( \varphi(x) = |x|^2 \) and \( k = 2, 3, \ldots \). Then
\[
(1.8) \quad 4 \| f \|_\varphi^2 \leq \| \mathcal{D}_0^* f \|_\varphi^2 + \| \mathcal{D}_1 f \|_\varphi^2
\]
for any \( f \in \text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1) \).

The reason we only consider the weight \( \varphi(x) = |x|^2 \) is that the weighted \( L^2 \) estimate in this case is relatively easier. On \( \mathbb{R}^{4n} \) for \( n > 1 \), the operators \( \mathcal{D}_0^* \) and \( \mathcal{D}_1^* \) are differential operators of the second order, and the weighted \( L^2 \) estimate is more difficult in these cases. While on \( \mathbb{R}^4 \), the \( k \)-Cauchy-Fueter complexes for \( k = 0, 1 \) are trivial. So we restrict to the case \( k \geq 2 \).

The weighted \( k \)-Bergman space \( A_2^2(\mathbb{R}^{4n}, \varphi) \) is a closed Hilbert subspace. We call the orthogonal projection \( P : L_\varphi^2(\mathbb{R}^{4n}, \odot^k \mathbb{C}^2) \rightarrow A_2^2(\mathbb{R}^{4n}, \varphi) \) the weighted \( k \)-Bergman projection. It can be nicely expressed in terms of the canonical solution operator as
\[
(1.9) \quad Pf = f - \mathcal{D}_0^* N_\varphi \mathcal{D}_0 f
\]
for \( f \in \text{Dom}(\mathcal{D}_0) \), as in the theory of several complex variables (cf. theorem 4.4.5 in \([5]\))

If we use the first isomorphism in \( (1.2) \), a function in \( L_\varphi^2(\mathbb{R}^{4n}, \odot^k \mathbb{C}^2) \) is \( \mathbb{C}^{k+1} \)-valued. The weighted \( k \)-Bergman projection \( P \) has a kernel \( K(x, y) \) such that the following integral formula holds
\[
(1.10) \quad f(x) = \int_{\mathbb{R}^{4n}} K(x, y) f(y) e^{-2\varphi} dV
\]
for any \( f \in A_2^2(\mathbb{R}^{4n}, \varphi) \). The kernel \( K(x, y) \) is a \( (k + 1) \times (k + 1) \)-matrix valued function, which is \( k \)-regular in variables \( x \) and anti-\( k \)-regular in variables \( y \).
The main difference between the $k$-Cauchy-Fueter complexes and Dolbeault complex in the theory of several complex variables is that there exist symmetric forms except for the exterior forms. The analysis of exterior forms is classical, while the analysis of symmetric forms is relatively new. We can handle components of a $\otimes^k \mathbb{C}^2$ or $\otimes^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n}$-valued function. Such notations are used by physicists as two-spinor notations for the massless field operators (cf. e.g. [16] [17] and references therein). They also appear in studying of quaternionic manifolds (cf. e.g. [22] and references therein).

The weighted $L^2$ estimate for the model case: $n = 1$ and $k = 2$, is obtain in section 2. The general case is proved in section 3. Based on the weighted $L^2$ estimate, Theorem 1.1 is proved in section 3. In section 4, we establish a localized a priori estimate for $\square$, and the Caccioppoli-type estimate, which hold for many systems of PDEs of the divergence form. From these estimates and the weighted $L^2$ estimate, we derive the asymptotic decay of the canonical solution $\mathcal{R}_N^* N f$ to the nonhomogeneous $k$-Cauchy-Fueter equation (1.1) when $f$ is compactly supported. Then by choosing suitable $f$ in (1.9), we find the asymptotic estimate for the weighted $k$-Bergman kernel from the asymptotic behavior of the canonical solution.

**Theorem 1.3.** Suppose that $\varphi(x) = |x|^2$ and $k = 2, 3, \ldots$. Then we have the following pointwise estimate for the weighted $k$-Bergman kernel: there exists $\varepsilon > 0$ only depending on $k, n$ such that

$$|K(x, y)| \leq C e^{(|x|^2 + |y|^2 + (|x| + |y|)^{-\varepsilon})}$$

for any $x, y \in \mathbb{R}^{4n}$ with $|x - y| > 3$, and some constant $C > 0$ only depending on $k, n, \varepsilon$.

The first estimate for the Bergman kernel of the weighted $L^2$-holomorphic functions over the complex plane $\mathbb{C}$ is due to Christ [6]. The result of Christ was extended by Delin [10] to several complex variables for strict plurisubharmonic weights. See also [9] [14] and references therein for recent results. Our estimate is a little bit weaker than the complex case because we have an extra factor $e^{\frac{1}{2}(|x| + |y|)}$. But the estimate is the same when $|y|$ is larger compared to $|x|$ (cf. Remark 1.1).

I would like to thank the referee for valuable suggestions.

2. THE WEIGHTED $L^2$ ESTIMATE IN THE MODEL CASE: $n = 1$ AND $k = 2$

2.1. **The complex vector fields $Z_{AA'}$s on $\mathbb{R}^{4n}$ and their formal adjoints.** To give the definition of the $k$-Cauchy-Fueter operator, we need the following complex vector fields

$$Z_{AA'} := \begin{pmatrix}
Z_{00'} & Z_{01'} \\
Z_{10'} & Z_{11'} \\
\vdots & \vdots \\
Z_{(2l)0'} & Z_{(2l)1'} \\
Z_{(2l+1)0'} & Z_{(2l+1)1'}
\end{pmatrix} := \begin{pmatrix}
\partial_{x_1} + i\partial_{x_2} & -\partial_{x_3} - i\partial_{x_4} \\
\partial_{x_3} - i\partial_{x_4} & \partial_{x_1} - i\partial_{x_2} \\
\vdots & \vdots \\
\partial_{x_{4l+1}} + i\partial_{x_{4l+2}} & -\partial_{x_{4l+3}} - i\partial_{x_{4l+4}} \\
\partial_{x_{4l+3}} - i\partial_{x_{4l+4}} & \partial_{x_{4l+1}} - i\partial_{x_{4l+2}}
\end{pmatrix},$$

where $A = 0, \ldots, 2n - 1$, $A' = 0', 1'$. This is motivated by the embedding of the quaternion algebra into the space of complex $2 \times 2$-matrices:

$$x_1 + ix_2 + jx_3 + kx_4 \mapsto \begin{pmatrix}
x_1 + ix_2 & -x_3 - ix_4 \\
x_3 - ix_4 & x_1 - ix_2
\end{pmatrix}.$$
We will use
\[ (\varepsilon_A B') = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\varepsilon_A B') = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]
to raise or lower primed indices, where \( (\varepsilon_A B') \) is the inverse of \( (\varepsilon_A B') \), i.e., \( \sum_{B'}=0,1' \varepsilon_A B' \varepsilon_{B'C'} = \delta_{A'C'} = \sum_{B'}=0,1' \varepsilon_{C'B'} \varepsilon_{B'A'} \). For example,
\[ Z_A' = \sum_{B'=0,1'} Z_{AB'} \varepsilon_B' A' = Z_{A0'} \varepsilon_{0'} A' + Z_{A1'} \varepsilon_{1'} A'. \]
In particular, we have \( Z_0' = Z_{A1'}, Z_1' = -Z_{A0} \) by
\[ \varepsilon_1' 0' = -\varepsilon_0' 1', \quad \varepsilon_{00'} = \varepsilon_{11'} = 0 \]
in (2.2). Then
\[ \left( \begin{array}{cccc} Z_0' & Z_1' & \cdots & \vdots \\ Z_1' & Z_1' & \cdots & \vdots \\ \vdots & \vdots & \ddots & Z_{n-1}' \\ Z_{n-1}' & Z_{n-1}' & \cdots & Z_{n-1}' \end{array} \right) = \left( \begin{array}{cccc} Z_{01'} & -Z_{00'} & \cdots & \vdots \\ Z_{11'} & -Z_{10'} & \cdots & \vdots \\ \vdots & \vdots & \ddots & Z_{n-2,1}' \\ Z_{n-1,1}' & -Z_{n-1,0} & \cdots & Z_{n-1,0} \end{array} \right). \]
We also use
\[ (\varepsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
and \( (\varepsilon_{AB}) \), the inverse of \( (\varepsilon_{AB}) \), to raise or lower unprimed indices, e.g. \( Z_A^A = \sum_{B=0}^{2n-1} Z_A B \varepsilon_{BA} \). The advantage of using raising indices is that the adjoint of \( Z_A^A \) can be written in a very simple form.

**Proposition 2.1.** (1) The formal adjoint operator \( Z_\varphi^A \) of a complex vector field \( Z \) is
\[ Z_\varphi^A = -\overline{Z} + 2\overline{Z} \varphi. \]

(2) We have
\[ Z_A^A = Z_A A', \]
and the formal adjoint operator of \( Z_A^A \) is
\[ \left( Z_A^A \right)^* = Z_A^A - 2Z_A \varphi. \]

**Proof.** (1) For a complex vector field \( Z \), we have
\[ (Z u, v)_\varphi = (u, Z_\varphi^A v)_\varphi. \]
for \( u, v \in C_0^\infty(\Omega, \mathbb{C}) \). This is because
\[ 0 = \int_\Omega Z(u \varphi e^{-2\varphi}) dV = \int_\Omega Z u \varphi e^{-2\varphi} dV + \int_\Omega u \varphi Z \varphi e^{-2\varphi} dV - 2 \int_\Omega u \varphi \cdot Z \varphi \cdot e^{-2\varphi} dV. \]
(2) By raising indices, $Z^{A'} = \sum_{B=0}^{2n-1} \sum_{B'=0',1'} Z_{BB'} e^{B} e^{B'} A'$. It is direct from definition of $Z_{AA'}$'s in (2.1) to see that
\[ Z_{00'} = Z_{11'} = Z_{00'}, \quad Z_{10'} = -Z_{01'} = Z_{10'}, \quad Z_{11'} = Z_{00'} = Z_{11'}, \cdots , \]
by (2.3) and similar relations for $e^{AB}$. Then $Z_{AA'} = Z^{A'}$. Since $(Z_{A'}^A)^\ast = -Z_{A}^A + 2Z_{A}^A$ by (1), and
\[ (Z_{A'}^A u,v) = (u, Z_{A}^A v), \]
we get (2.7). Here $Z_{B'A} = -Z_{B'} A'$ by (2.2).

We will use the notations of the following complex differential operators:
\[ \delta^A_A := Z_{AA'} - 2Z_{A}^A \varphi, \]
for $A = 0, \ldots, 2n-1$, $A' = 0', 1'$. Then we have $(Z_{A'}^A)^\ast = \delta^A_A$ and
\[ (Z_{A'}^A u,v) = (u, \delta^A_A v). \]
for $u, v \in C^1_0(\Omega, \mathbb{C})$. By taking conjugate, we also have
\[ (\delta^A_A u, v) = (u, Z_{A}^A v). \]

2. The weighted $L^2$ estimate in the model case $n = 1$ and $k = 2$. In this case,
\[ \mathcal{V}_0 := \otimes^2 \mathbb{C}^2 \cong \mathbb{C}^3, \quad \mathcal{V}_1 := \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4, \quad \mathcal{V}_2 := \wedge^2 \mathbb{C}^2 \cong \mathbb{C}^1. \]
By definition, $\otimes^2 \mathbb{C}^2$ is a subspace of $\otimes^2 \mathbb{C}^2$, and an element $f$ of $L^2_{\varphi}(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$ has 4 components $f_{00'}, f_{11'}, f_{01'}, f_{10'}$ such that $f_{10'} = f_{01'}$. Its $L^2$ inner product is induced from that of $L^2_{\varphi}(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$ by
\[ (f,g) = \sum_{A',B'=0',1'} (f_{A'B'},g_{A'B'}) = (f_{00'},g_{00'}) + 2(f_{01'},g_{01'}) + (f_{11'},g_{11'}). \]
f $\in L^2_{\varphi}(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$ has 4 components $f_{A'A}$, $A = 0, 1, A' = 0', 1'$, and
\[ (f,g) = \sum_{A=0,1} \sum_{A'=0',1'} (f_{A'A},g_{A'A}) \]
while $f \in L^2_{\varphi}(\mathbb{R}^4, \wedge^2 \mathbb{C}^2)$ has components $f_{AB}$ with $f_{AB} = -f_{BA}$, among which there is only one nontrivial (i.e. $f_{00} = f_{11} = 0$, $f_{01} = -f_{10}$), and
\[ (f,g) = \sum_{A,B=0,1} (f_{AB},g_{AB}) = 2(f_{01},g_{01}). \]

The operators in the 2-Cauchy-Fueter complex over $\mathbb{R}^4$ are given by
\[ (\mathcal{D} \phi)_{A'A} := \sum_{B'=0',1'} Z_{A'}^B \phi_{B'A'} = Z_{A'}^0 \phi_{0'A'} + Z_{A'}^1 \phi_{1'A'}, \]
for $\phi \in C^1(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$ where $A = 0, 1, A' = 0', 1'$, and
\[ (\mathcal{D} \psi)_{AB} := 2 \sum_{A'=0',1'} Z_{A'}^A \psi_{B'A'} = \sum_{A'=0',1'} (Z_{A'}^A \psi_{B'A'} - Z_{B'}^A \psi_{AA'}). \]
For any $h_{[AB]} := \frac{1}{2}(h_{AB} - h_{BA})$ is the antisymmetrisation. Here and in the sequel, we write $\psi_{AA'} := \psi_{A'A}$ for convenience. It is direct to see that

$$
(\mathcal{D}_1 \mathcal{D}_0 \phi)_{AB} = \sum_{A' = 0', A'} \left( Z^A_{A'}(\mathcal{D}_0 \phi)_{BA'} - Z^A_{A'}(\mathcal{D}_0 \phi)_{AA'} \right)
$$

(2.15)

$$
= \sum_{A', C' = B', A'} \left( Z^A_{A'} Z_{C'}^C \phi_{C'A'} - Z^A_{A'} Z_{C'}^C \phi_{C'A'} \right) = 0
$$

by relabeling indices, $\phi_{C'A'} = \phi_{A'C'}$ and the commutativity $\nabla^A_B \nabla^C_A = \nabla^C_A \nabla^A_B$, as scalar differential operators of constant complex coefficients (cf. (2.11) in [4]).

**Lemma 2.1.** (1) For any $h \in L^2(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$ and $H \in L^2(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$, we have

$$
\sum_{A', B'} (h_{A'B'}, H_{A'B'}) = \sum_{A', B'} (h_{A'B'}, H_{(A'B')}) \varphi,
$$

(2.16)

where

$$
H_{(A'B')} := \frac{1}{2}(H_{A'B'} + H_{B'A'})
$$

is the symmetrisation, i.e. $(H_{(A'B')}) \in L^2(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$.

(2) For any $h \in L^2(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$ and $H \in L^2(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$, we have

$$
\sum_{A, B} (h_{AB}, H_{AB}) = \sum_{A, B} (h_{AB}, H_{AB}) \varphi.
$$

(3) For any $h, H \in L^2(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$, we have

$$
\sum_{A, B} (h_{BA}, H_{AB}) = \sum_{A, B} (h_{AB}, H_{AB}) - 2 \sum_{A, B} (h_{AB}, H_{AB}) \varphi.
$$

**Proof.** (1) This is because

$$
\sum_{A', B'} h_{A'B'} H_{(A'B')} = \frac{1}{2} \sum_{A', B'} h_{A'B'} (H_{A'B'} + H_{B'A'}) = \sum_{A', B'} h_{A'B'} H_{A'B'}
$$

by changing indices and $h_{A'B'} = h_{B'A'}$.

(2) This is because

$$
\sum_{A, B} h_{AB} H_{AB} = \frac{1}{2} \sum_{A, B} h_{AB} (H_{AB} - H_{BA}) = \sum_{A, B} h_{AB} H_{AB}
$$

(2.19)

by changing indices and $h_{BA} = -h_{AB}$.

(3) This is because

$$
\sum_{A, B} h_{BA} H_{AB} = \sum_{A, B} h_{AB} H_{AB} + \sum_{A, B} (h_{BA} - h_{AB}) H_{AB}
$$

(2.20)

and the second term in the right hand side is $-2 \sum_{A, B} h_{AB} H_{AB} = -2 \sum_{A, B} h_{AB} H_{AB}$ by the identity (2.19).
Lemma 2.2. For \( f \in C_0^\infty(\mathbb{R}^4, \mathbb{C} \otimes \mathbb{C}^2) \), we have
\[
(\mathcal{D}_0^* f)_{A'B'} = \sum_{A=0,1} \delta^A_{(A'B')A}.
\]

Proof. For any \( g \in C_0^\infty(\mathbb{R}^4, \mathbb{C} \otimes \mathbb{C}^2) \), we have
\[
\langle \mathcal{D}_0 g, f \rangle \varphi = \sum_{A,A',B'} \left( Z_A^A Z_{A'} B' f_{B'A} \right) \varphi = \sum_{A,A',B'} \left( g_{A'B'}, \delta^A_{A'B'} f_{B'A} \right) \varphi
\]
\[
= \sum_{A,A',B'} \left( g_{A'B'}, \sum_A \delta^A_{(A'B')A} \right) \varphi = \langle g, \mathcal{D}_0^* f \rangle \varphi
\]
by using (2.10) and Lemma 2.1 (1). Here we have to symmetrise \((A'B')\) for any \( f \in \mathcal{D}_0^* \) by using (2.10) and Lemma 2.1 (1). Here we have to symmetrise \((A'B')\) in \( \sum_A \delta^A_{A'B'} f_{B'A} \) since only after symmetrisation it becomes an element of \( C_0^\infty(\mathbb{R}^4, \mathbb{C} \otimes \mathbb{C}^2) \), i.e. a \( \mathbb{C} \otimes \mathbb{C}^2 \)-valued function. \( \square \)

Theorem 2.1. Suppose that there exist a constant \( c > 0 \) such that the weight \( \varphi \) satisfies
\[
\sum_{A,B,A',B'} Z_B^A Z_{A} B' \varphi(x) \cdot (\xi_{A',A} \xi_{B,B'}) \geq c \sum_A (\xi_{A',A})^2.
\]
for any \( x \in \mathbb{R}^{4n} \) and \((\xi_{A',A}) \in \mathbb{C} \otimes \mathbb{C}^2 \). Then we have the weighted \( L^2 \) estimate
\[
c \| f \|^2_\varphi \leq \| \mathcal{D}_0^* f \|^2_\varphi + \| \mathcal{D}_1 f \|^2_\varphi,
\]
for any \( f \in \text{Dom}(\mathcal{D}_0) \cap \text{Dom}(\mathcal{D}_1) \).

Proof. By definition, we have \( \text{Dom}(\mathcal{D}_1) := \{ f \in L^2_\varphi(\mathbb{R}^{4n}, \mathfrak{Y}_1); \mathcal{D}_1 f \in L^2_\varphi(\mathbb{R}^{4n}, \mathfrak{Y}_2) \} \). Then \( \mathcal{D}_1 \) is densely-defined since \( C_0^\infty(\mathbb{R}^{4n}, \mathfrak{Y}_1) \) is contained in its domain. It is also closed since differentiation is continuous on distributions. So is \( \mathcal{D}_0^* \) as a differential operator given by (2.10). Therefore it is sufficient to show (2.22) for \( f \in C_0^\infty(\mathbb{R}^{4n}, \mathfrak{Y}_1) \). It follows from the definition of \( \mathcal{D}_0 \) in (2.13), \( \mathcal{D}_0^* \) in Lemma 2.2 and the definition of symmetrisation that
\[
2(\mathcal{D}_0^* f, \mathcal{D}_0^* f) \varphi = 2(\mathcal{D}_0^* f, f) \varphi = 2 \sum_{B,B'} \left( \sum_{A'} Z^A_A Z_{A'} B' \delta^A_{A'B'} f_{B'A} f_{B'B'} \right) \varphi
\]
\[
= \sum_{A,B,A',B'} \left( Z^A_A Z_{A'} B' \delta^A_{A'B'} f_{B'A} f_{B'B'} \right) \varphi + \sum_{A,B,A',B'} \left( Z^A_A Z_{A'} B' \delta^A_{A'B'} f_{B'A} f_{B'B'} \right) \varphi := \Sigma_0 + \Sigma_1,
\]
where
\[
\Sigma_0 = \sum_{A',B'} \left( \sum_A \delta^A_{A'B'} \sum_B \delta^B_{B'B'} \right) \varphi = \sum_{A',B'} \left\| \delta^A_{A'B'} f_{B'A} \right\|^2_\varphi \geq 0,
\]
and
\[
\Sigma_1 = \sum_{A,B,A',B'} \left\{ \left( \delta^A_{A'} Z^A_A f_{AA'}, f_{BB'} \right) + \left( \left| Z^A_A \delta^A_{A'} \right| f_{AA'}, f_{BB'} \right) \right\} \varphi
\]
\[
= \sum_{A,B,A',B'} \left\{ \left( Z^A_A f_{AA'}, Z^A_A f_{BB'} \right) + 2 \left( Z^A_A Z^A_A \varphi \cdot f_{AA'}, f_{BB'} \right) \right\} \varphi
\]
by using the formal adjoint operator (2.11), relabeling indices and using the commutator
\[
\left[ Z^A_A, \delta^A_{A'} \right] = -2Z^A_A Z^A_{A'} \varphi = 2Z^A_A Z^A_{A'} \varphi.
\]
which follows from (2.8)-(2.9) and the commutativity $Z^A_B Z^A_B = Z^A_A Z^A_A$ as scalar differential operators of constant coefficients. The first summation in the right hand side of (2.20) is equal to

\[
\sum_{A,B,A',B'} (Z^A_B f_{AA'}, Z^A_B f_{BB'}) \varphi = \sum_{A,B} \left( \sum_{A'} Z^A_B f_{AA'} \right) \left( \sum_{B'} Z^A_B f_{BB'} \right) \varphi
\]

\[
= \sum_{A,B} \left| \sum_{A'} Z^A_B f_{AA'} \right|^2 - 2 \sum_{A,B} \left| \sum_{A'} Z^A_B f_{AA'} \right| \left| \sum_{B'} Z^A_B f_{BB'} \right| \varphi
\]

\[
= \sum_{A,B} \left( \sum_{A'} Z^A_B f_{AA'} \right)^2 \varphi
\]

by applying (2.18) with $h_{BA} = \sum_{A'} Z^A_B f_{AA'}, H_{AB} = \sum_{B'} Z^A_B f_{BB'}$. Now substituting (2.25)-(2.26) into (2.24) and using the above identity, we get

\[
2 \| \mathcal{O}_0 f \|^2 + \frac{1}{2} \| \mathcal{O}_1 f \|^2 = 2 \sum_{A,B,A',B'} \left( Z^A_B \overline{Z^A_B} \varphi \cdot f_{AA'}, f_{BB'} \right) \varphi
\]

\[
+ \sum_{A',B'} \left| \sum_{A} \delta^A_A f_{BA'} \right|^2 + \sum_{A,B} \left| \sum_{A'} Z^A_B f_{BA'} \right|^2 \varphi.
\]

Now the resulting estimate follows from the assumption (2.22) for $\varphi$. □

**Remark 2.1.**

1. We do not handle the term $\Sigma_0$ in (2.25) by using commutators. Because if we do so

\[
\Sigma_0 = \sum_{A,B,A',B'} (\delta^A_A f_{AAA'}, f_{BB'}) + \left( \overline{Z^A_B} f_{AAA'}, f_{BB'} \right)
\]

\[
= \sum_{A,B,A',B'} \left( Z^A_B f_{AAA'}, Z^A_B f_{BB'} \right) \varphi + 2 \left( Z^A_B \overline{Z^A_B} \varphi \cdot f_{AAA'}, f_{BB'} \right),
\]

the first term in the right hand side above is quite difficult to control. But over $\mathbb{R}^4$ it can be controlled in terms of $\mathcal{O}_0 f$ and $\mathcal{O}_1 f$. Based on such estimates, we can solve the Neumann problem for the k-Cauchy-Fueter complexes over k-pseudoconvex domains in $\mathbb{R}^4$ (cf. [23]).

2. $\varphi = |x|^2$ satisfies the assumption (2.22) for $\varphi$ with $c = 4$ by the following Lemma 3.2.

3. **The canonical solution operator to the nonhomogeneous k-Cauchy-Fueter equation**

3.1. The weighted $L^2$ estimate in the general case. Recall that the symmetric power $\wedge k \mathbb{C}^2$ is a subspace of $\wedge^k \mathbb{C}^2$, and an element of $\wedge^k \mathbb{C}^2$ is given by a 2$^k$-tuple $(f_{A_1 \ldots A_k}) \in \wedge^k \mathbb{C}^2$ with $A_1 \ldots A_k = 0', 1'$, where $f_{A_1 \ldots A_k}$ is invariant under permutations of subscripts, i.e.

\[
f_{A_1 \ldots A_k} = f_{A_{\sigma(1)} \ldots A_{\sigma(k)}}
\]

for any $\sigma \in S_k$, the group of permutations of $k$ letters. Note that $\dim(\wedge^k \mathbb{C}^2) = k + 1$ (cf. (4.1)) while $\dim(\wedge^k \mathbb{C}^2) = 2^k$. An element of the exterior power $\wedge^k \mathbb{C}^{2n}$ is given by a tuple $(f_{AB})$ with $f_{AB} = -f_{BA}$, $A, B = 0, \ldots, 2n - 1$. An element of $\wedge^{k-1} \mathbb{C}^{2n}$ is given by a tuple $(f_{A_1 \ldots A_k}) \in \wedge^{k-1} \mathbb{C}^{2n}$, which is invariant under permutations of $A_1, \ldots, A_k$. We will use symmetrisation of primed indices

\[
f_{A_1' \ldots A_k'} = \frac{1}{k!} \sum_{\sigma \in S_k} f_{A_{\sigma(1)}' \ldots A_{\sigma(k)}'}
\]
The first two operators in \(k\)-Cauchy-Fueter complex (1.3)–(1.5) over \(\mathbb{R}^{4n}\) are given by
\[
(\mathcal{D} f)_{A''_1...A'_k} := \sum_{A'_1 = 0',1'} Z^A_{A'} f_{A'_1 A''_2...A'_k} = Z^A_{A'} f_{0'} A''_2...A'_k + Z^A_{A'} f_{1'} A''_2...A'_k,
\]
\[
(\mathcal{D} h)_{A''_1...A'_k} := 2 \sum_{A'_1 = 0',1'} Z^A_{A'} h_{A''_1 A'_2...A'_k} = \sum_{A'_1 = 0',1'} (Z^A_{A'} h_{A''_1 A'_2...A'_k} - Z^A_{A'} h_{A''_2 A'_1 A'_3...A'_k}),
\]
for \(f \in C^1(\mathbb{R}^{4n}, \mathcal{V}_0), h \in C^1(\mathbb{R}^{4n}, \mathcal{V}_1)\), where \(A, B = 0, 1, \ldots, 2n - 1, A'_2, \ldots, A'_k = 0', 1'\). Here and in the sequel, we write \(h_{A''_1 A'_2...A'_k} := h_{A''_1 A'_2...A'_k A} \) for convenience. It is direct to check that \(\mathcal{D}_1 \circ \mathcal{D}_0 = 0\) as (2.15).

The weighted inner product of \(L^2_v(\mathbb{R}^{4n}, \mathcal{V}_0)\) is induced from that of \(L^2_v(\mathbb{R}^{4n}, \otimes^k \mathbb{C}^2)\). Namely we define
\[
(f, h)_\varphi := \sum_{A = 0} \left( f_{A''_1 A'_2...A'_k} h_{A''_1 A'_2...A'_k} \right) \varphi
\]
for \(f, h \in L^2_v(\mathbb{R}^{4n}, \mathcal{V}_0)\), and \(\|f\|_{\varphi} = (f, f)_{\varphi}^{1/2}\). We define the weighted induced inner products of \(L^2_v(\mathbb{R}^{4n}, \mathcal{V}_1)\) and \(L^2_v(\mathbb{R}^{4n}, \mathcal{V}_2)\) similarly.

**Lemma 3.1.** For \(f \in C^\infty_0(\mathbb{R}^{4n}, \mathcal{V}_1)\), we have
\[
(\mathcal{D} f)_{A'_1 A''_2...A'_k} = \sum_{A = 0}^{2n - 1} \delta^A_{A'_1} f_{A''_2...A'_k} A
\]

**Proof.** For any \(g \in C^\infty_0(\mathbb{R}^{4n}, \mathcal{V}_0)\) we have
\[
(\mathcal{D} g, f)_\varphi = \sum_{A, A'_1, \ldots, A'_k} Z^A_{A'} g_{A'_1 A''_2...A'_k} (f_{A''_2...A'_k} A) \varphi = \sum_{A, A'_1, \ldots, A'_k} (g_{A'_1 A''_2...A'_k} A) \varphi (f_{A''_2...A'_k} A) \varphi
\]
by using (2.10) and symmetrisation
\[
\sum_{A'_1, \ldots, A'_k} g_{A'_1 \ldots A'_k} G_{A'_1 \ldots A'_k} \varphi = \sum_{A'_1, \ldots, A'_k} (g_{A'_1 \ldots A'_k} G_{A'_1 \ldots A'_k}) \varphi
\]
for any \(g \in L^2_v(\mathbb{R}^{4n}, \otimes^k \mathbb{C}^2)\), \(G \in L^2_v(\mathbb{R}^{4n}, \otimes^k \mathbb{C}^2)\). Here we have to symmetrise indices \((A'_1 \ldots A'_k)\) in \(\sum_A \delta^A_{A'_1} f_{A''_2...A'_k} A\) since only after symmetrisation it becomes an element of \(C^\infty_0(\mathbb{R}^{4n}, \mathcal{V}_0)\), i.e. a \(\otimes^k \mathbb{C}^2\)-valued function. (3.3) is a generalization of Lemma 2.1 (1). It holds because
\[
R.H.S. = \sum_{s_k} \sum_{A'_1, \ldots, A'_k} (g_{A'_1 \ldots A'_k} G_{A''_1 A''_2\ldots A''_{s_k}}) \varphi = \sum_{s_k} \sum_{A'_1, \ldots, A'_k} (g_{A''_1 A''_2\ldots A''_{s_k}}) \varphi
\]
by relabeling indices, which equals to L.H.S. by \(g\) symmetric in the indices, i.e. \(g_{A''_1 A''_2\ldots A''_{s_k}} = g_{A''_1 A''_2\ldots A''_{s_k}}\) for any permutation \(\sigma\).

**Proof of Theorem 1.2.** As in the model case \(n = 1, k = 2\), it is sufficient to show the weighted \(L^2\)-estimate (1.8) for \(f \in C^\infty_0(\mathbb{R}^{4n}, \otimes^k \mathbb{C}^2 \otimes \mathbb{C}^{2n})\). Recall that if \((F'_{A''_1 A'_2}) \in \otimes^k \mathbb{C}^2\) is symmetric in \(A''_1 A'_2\), then we have
\[
F_{A'_1 A''_2 A''_2} = \frac{1}{k} (F_{A''_1 A'_2 A''_2} + \cdots + F_{A''_1 A'_2 A''_2} + \cdots + F_{A''_1 A'_2 A''_2})
\]
by definition of symmetrisation \((3.1)\). Now we expand the symmetrisation to get

\[
k\langle \mathcal{D}_0^s f, \mathcal{D}_0^s f \rangle \varphi = k \langle \mathcal{D}_0^s f, f \rangle \varphi
\]

\[
= k \sum_{B, A_1', \ldots, A_k'} \left( \sum_{A_1} Z_B^{A_1'} \sum_A \delta_{(A_1', f A_2' \ldots A_k')} A, f A_2' \ldots A_k' B \right) \varphi
\]

\[
= \sum_{A, B, A_1', \ldots, A_k'} \left( Z_B^{A_1'} \delta_{A_1', f A_2' \ldots A_k'} A, f A_2' \ldots A_k' B \right) + \sum_{s=2}^k \left( Z_B^{A_1'} \delta_{A_1', f A_2' \ldots A_k'} A, f A_2' \ldots A_k' B \right) \varphi
\]

\[
= \Sigma_0 + \Sigma_1,
\]

by the adjoint operator \(\mathcal{D}_0^s\) in Lemma 3.1. Here we split the sum into the cases \(s = 1\) and \(s \geq 2\) as in the model case (cf. Remark 2.1). Note that

\[
\Sigma_0 = \sum_{A_1', \ldots, A_k'} \left( \sum_A \delta_{A_1', f A_2' \ldots A_k'} A, \sum_B \delta_{A_1', f A_2' \ldots A_k'} B \right) \varphi \geq 0
\]

by using (2.11), and

\[
\Sigma_1 = \sum_{s=2}^k \sum_{A, B, A_1', \ldots, A_k'} \left( \delta_{A_1', Z_B^{A_1'} f A_2' \ldots A_k'} A, f A_2' \ldots A_k' B \right) \varphi + \left( Z_B^{A_1'} \delta_{A_1', f A_2' \ldots A_k'} A, f A_2' \ldots A_k' B \right) \varphi
\]

\[
= \Sigma_2' + \Sigma_2''
\]

by using commutators. For the second sum,

\[
\Sigma_2' = 2 \sum_{s=2}^k \sum_{A, B, A_1', \ldots, A_k'} \left( Z_B^{A_1'} \overline{Z_A^{A_1'}} \varphi, f A_2' \ldots A_k' A, f A_2' \ldots A_k' B \right) \varphi
\]

\[
= 8 \sum_{s=2}^k \sum_{A, B, A_1', \ldots, A_k'} \left( \delta_{BA} \delta_{A_1', A_2'}, f A_2' \ldots A_k' A, f A_2' \ldots A_k' B \right) \varphi = 8(k - 1)\|f\|_2^2
\]

for \(\varphi(x) = |x|^2\) by the following Lemma 3.2 and \(f\) symmetric in the primed indices. On the other hand,

\[
\Sigma_2'' = \sum_{s=2}^k \sum_{A, B, A_1', \ldots, A_k'} \left( Z_B^{A_1'} f A_2' \ldots A_k' A, Z_A^{A_1'} f A_2' \ldots A_k' B \right) \varphi
\]

\[
= \sum_{s=2}^k \sum_{A, B, A_1', \ldots, A_k'} \left( \sum_{A_1'} Z_B^{A_1'} f A_1' \ldots A_k' A, \sum_{A_1'} Z_A^{A_1'} f A_1' \ldots A_k' B \right) \varphi
\]

\[
= (k - 1) \sum_{B_1', \ldots, B_k' = 0, 1} \sum_{A, B} \left( Z_B^{A_1'} f A_1' \ldots B_{k-1}' A, Z_A^{A_1'} f A_1' \ldots B_{k-1}' B \right) \varphi
\]

by \(f\) symmetric in the primed indices and relabelling indices. Then applying Lemma 2.1 (3) \((2.20)\) holds for \(A, B = 0, \ldots, 2n - 1\) to the right hand side with \(h_{BA} = \sum_{A'} Z_B^{A'} f A'A_1' \ldots B_{k-1}' A\) and \(H_{AB} =\)
Lemma 3.2. The estimate (1.8) follows. □

Now we get

\[ \sum_{B_1, \ldots, B_k} Z_{A}^{A'} f_{B_1^1 \ldots B_k^1} \text{ for fixed } B_1^1, \ldots, B_k^1, \]

we get

\[
\Sigma'_2 = (k - 1) \sum_{B_1, \ldots, B_k} \left\{ \left\| \sum_{A'} Z_{A}^{A'} f_{B_1^1 \ldots B_k^1} \right\|_\varphi^2 - 2 \left\| \sum_{A'} Z_{A}^{A'} f_{B_1^1 \ldots B_k^1} \right\|_\varphi^2 \right\}
\]

\[ = (k - 1) \sum_{A, B_1, \ldots, B_k} \left\| \sum_{A'} Z_{A}^{A'} f_{B_1^1 \ldots B_k^1} \right\|_\varphi^2 - \frac{k - 1}{2} \left\| \varphi \right\|_\varphi^2 .
\]

Now we get

\[ k \left\| \varphi \right\|_\varphi^2 + \frac{k - 1}{2} \left\| \varphi \right\|_\varphi^2 \geq 8(k - 1) \left\| f \right\|_\varphi^2 .
\]

The estimate (1.8) follows. □

Lemma 3.2. \( Z_{A}^{A'} Z_{A'}^{B'} |x|^2 = 4\delta_{AB} \delta_{A'B'} . \) In particular, \( \varphi = |x|^2 \) satisfies the assumption (2.22) for \( \varphi \) with \( c = 4 \).

To prove this lemma, we introduce complex linear functions

\[
(3.6) \quad (z_{AA'}) := \begin{pmatrix} z_{00'} & z_{01'} \\ z_{10'} & z_{11'} \\ \vdots & \vdots \\ z_{2(2n-1)0'} & z_{2(2n-1)1'} \\ z_{2(2+1)0'} & z_{2(2+1)1'} \end{pmatrix} = \begin{pmatrix} x_1 - ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 + ix_2 \\ \vdots & \vdots \\ x_{4l+1} - ix_{4l+2} & -x_{4l+3} + ix_{4l+4} \\ x_{4l+3} + ix_{4l+4} & x_{4l+1} + ix_{4l+2} \end{pmatrix},
\]

where \( A = 0, \ldots, 2n - 1, A' = 0', 1' . \) \( z_{AA'} \) is obtained by replacing \( \partial_{x_j} \) in \( Z_{AA'} \) in (2.1) by \( x_j \). By the following lemma, \( z_{AA'} \)'s can be viewed as independent variables and \( Z_{AA'} \)'s are derivatives with respect to these variables formally.

Lemma 3.3. \( Z_{AA'} z_{BB'} = 2\delta_{AB} \delta_{A'B'} . \)

Proof. Assume that \( A = 2l, A' = 0' . \) By (3.6), we have

\[
Z_{(2l)0'} z_{(2l)0'} = (\partial_{x_{4l+1}} + i\partial_{x_{4l+2}})(x_{4l+1} - ix_{4l+2}) = 2;
\]

\[
Z_{(2l)0'} z_{(2l+1)1'} = (\partial_{x_{4l+1}} + i\partial_{x_{4l+2}})(x_{4l+1} + ix_{4l+2}) = 0.
\]

Note that \( Z_{(2l)0'} \) is a differential operator with respect to variables \( x_{4l+1} \) and \( x_{4l+2} \), while \( z_{BB'} \) for \( BB' \neq (2l)0' \) or \( (2l+1)1' \) is independent of variables \( x_{4l+1} \) and \( x_{4l+2} \). So we get

\[
Z_{(2l)0'} z_{BB'} = 0
\]

for such \( BB' \). It is similar to check the result directly for other vectors \( Z_{(2l)1'}, Z_{(2l+1)0'} \) and \( Z_{(2l+1)1'} . \) □

Proof of Lemma 3.2. Note that \( (\partial_{x_j} \pm i\partial_{x_k}) |x|^2 = 2(x_j \pm ix_k) . \) So \( Z_{AC'} |x|^2 = 2z_{AC'} \) by definitions of \( Z_{AC'} \)'s and \( z_{AC'} \)'s in (3.6). Then we have

\[
Z_{B}^{B'} Z_{A}^{A'} |x|^2 = \sum_{D', C'} Z_{B'D'} z_{AC'} |x|^2 \cdot \varepsilon^{C'B'} \varepsilon^{D'A'} = 2 \sum_{D', C'} Z_{B'D'} z_{AC'} \cdot \varepsilon^{C'B'} \varepsilon^{D'A'} = 4 \sum_{C'} \delta_{AB} \delta_{C'D'} \cdot \varepsilon^{C'B'} \varepsilon^{D'A'} = 4 \delta_{AB} \delta_{A'B'} ,
\]
by Lemma 3.3 Here by (2.3), $\xi^{B'B'}_\xi^{C'C'} = 1$ only if $A' = B'$ and $C'$ is different from them. Otherwise, it vanishes. So for any $(\xi_{A'A}) \in \mathbb{C}^2 \otimes \mathbb{C}^2$,

$$\sum_{A,B,A',B'} Z_A^B Z_B^{B'} |x|^2 \cdot \xi_{A'B} = 4|\xi|^2.$$  

3.2. The associated Laplacian operator $\Box_\varphi$. By definition,

$$\text{Dom}(\Box_\varphi) := \{ f \in L^2(\mathbb{R}^4, \mathcal{V}_1); f \in \text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1), \mathcal{D}_0 f \in \text{Dom}(\mathcal{D}_0), \mathcal{D}_1 f \in \text{Dom}(\mathcal{D}_1^*) \}.$$  

We introduce

$$\mathcal{E}_\varphi(f, g) := \langle \mathcal{D}_0^* f, \mathcal{D}_0^* g \rangle_\varphi + \langle \mathcal{D}_1 f, \mathcal{D}_1 g \rangle_\varphi$$  

for any $f, g \in \text{Dom}(\mathcal{E}_\varphi) := \text{Dom}(\mathcal{D}_1) \cap \text{Dom}(\mathcal{D}_0^*)$. By definition of adjoint operators, we have

$$\mathcal{E}_\varphi(f, g) = \langle \Box_\varphi f, g \rangle_\varphi$$  

for any $f \in \text{Dom}(\Box_\varphi), g \in \text{Dom}(\mathcal{E}_\varphi)$.

Note that for any $F \in \text{Dom}(\mathcal{D}_0)$, we have $\mathcal{D}_1 F \in \text{Dom}(\mathcal{D}_1)$ and

$$\mathcal{D}_1 \mathcal{D}_0 F = 0.$$  

This is because $\mathcal{D}_1 \mathcal{D}_0 F = 0$ for smooth $F$ and the general result follows from the closedness of $\mathcal{D}_0$ and $\mathcal{D}_1$ as differential operators.

Proposition 3.1. The associated Laplacian operator $\Box_\varphi$ is a densely-defined, closed, self-adjoint and non-negative operator on $L^2(\mathbb{R}^4, \mathcal{V}_1)$.

Proof. It is similar to the proof of proposition 4.2.3 of [3] for $\mathcal{J}$-complex. We give the proof here for completeness.

As we mentioned before, $\mathcal{D}_0$ and $\mathcal{D}_0^*$ as differential operators are both densely-defined and closed. $\Box_\varphi$ is densely-defined in the same way. For closedness of $\Box_\varphi$, we need to show that for any $f_n \in \text{Dom}(\Box_\varphi)$ such that $f_n \rightarrow f$ in $L^2(\mathbb{R}^4, \mathcal{V}_1)$ and $\Box_\varphi f_n$ converges, we have $f \in \text{Dom}(\Box_\varphi)$ and $\Box_\varphi f_n \rightarrow \Box_\varphi f$.

Because $f_n \in \text{Dom}(\Box_\varphi)$, we have

$$\langle \Box_\varphi (f_n - f_m), f_n - f_m \rangle_\varphi = \langle \mathcal{D}_0^* \mathcal{D}_0 (f_n - f_m), f_n - f_m \rangle_\varphi + \langle \mathcal{D}_1^* \mathcal{D}_1 (f_n - f_m), f_n - f_m \rangle_\varphi$$  

$$= \| \mathcal{D}_0^* (f_n - f_m) \|^2_\varphi + \| \mathcal{D}_1 (f_n - f_m) \|^2_\varphi,$$  

and so $\mathcal{D}_0^* f_n$ and $\mathcal{D}_1 f_n$ converge in $L^2(\mathbb{R}^4, \mathcal{V}_0)$ and $L^2(\mathbb{R}^4, \mathcal{V}_1)$, respectively. It follows from the closedness of $\mathcal{D}_0^*$ and $\mathcal{D}_1$ that $f \in \text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1)$ and

$$\mathcal{D}_0^* f_n \rightarrow \mathcal{D}_0^* f, \quad \mathcal{D}_1 f_n \rightarrow \mathcal{D}_1 f.$$  

Note that $\mathcal{D}_0 \mathcal{D}_0^* f_n$ and $\mathcal{D}_1^* \mathcal{D}_1 f_n$ are orthogonal to each other by

$$\langle \mathcal{D}_0 \mathcal{D}_0^* f_n, \mathcal{D}_1^* \mathcal{D}_1 f_n \rangle_\varphi = \langle \mathcal{D}_1 \mathcal{D}_0^* f_n, \mathcal{D}_1 f_n \rangle_\varphi = 0$$  

by (3.8). So $\Box_\varphi f_n = \mathcal{D}_0 \mathcal{D}_0^* f_n + \mathcal{D}_1^* \mathcal{D}_1 f_n$ converges implies that both $\mathcal{D}_0 \mathcal{D}_0^* f_n$ and $\mathcal{D}_1^* \mathcal{D}_1 f_n$ converge. It follows from the closedness of $\mathcal{D}_0$ and $\mathcal{D}_1^*$ again that $\mathcal{D}_0^* f \in \text{Dom}(\mathcal{D}_0)$, $\mathcal{D}_1 f \in \text{Dom}(\mathcal{D}_1^*)$ and

$$\mathcal{D}_0^* f_n \rightarrow \mathcal{D}_0^* f, \quad \mathcal{D}_1^* f_n \rightarrow \mathcal{D}_1^* f.$$  

Therefore $f \in \text{Dom}(\Box_\varphi)$ and $\Box_\varphi f_n \rightarrow \Box_\varphi f$. So $\Box_\varphi$ is a closed operator.

Define

$$L_1 := \mathcal{D}_0 \mathcal{D}_0^* + \mathcal{D}_1^* \mathcal{D}_1 + I \quad \text{on} \quad \text{Dom}(\Box_\varphi).$$
It is sufficient to show that $L_1^{-1}$ is self-adjoint. By a theorem of Von Neumann (cf. §1 in Chapter 8 in [18]), $(I + \mathcal{D}_0\mathcal{R}_0)^{-1}$ and $(1 + \mathcal{D}_1\mathcal{P}_1)^{-1}$ are automatically both bounded and self-adjoint, and so is 

$$Q_1 = (I + \mathcal{D}_0\mathcal{R}_0)^{-1} + (1 + \mathcal{D}_1\mathcal{P}_1)^{-1} - I.$$  

We claim that $Q_1 = L_1^{-1}$. Since 

$$(1 + \mathcal{D}_0\mathcal{R}_0^*)^{-1} - I = (I - (I + \mathcal{D}_0\mathcal{R}_0^*))((I + \mathcal{D}_0\mathcal{R}_0^*)^{-1} = -\mathcal{D}_0\mathcal{R}_0^*(I + \mathcal{D}_0\mathcal{R}_0^*)^{-1},$$  

we see that $\mathcal{R}(I + \mathcal{D}_0\mathcal{R}_0^*)^{-1} \subset \text{Dom}(\mathcal{D}_0\mathcal{R}_0^*)$. Similarly, $\mathcal{R}(I + \mathcal{D}_1\mathcal{P}_1)^{-1} \subset \text{Dom}(\mathcal{D}_1\mathcal{P}_1)$, and so 

$$(3.10) \quad Q_1 = (I + \mathcal{D}_1\mathcal{P}_1)^{-1} - \mathcal{D}_0\mathcal{R}_0^*(I + \mathcal{D}_0\mathcal{R}_0^*)^{-1}.$$  

Since $\mathcal{D}_1\mathcal{R}_0 = 0$ by (3.8), we have $\mathcal{R}(Q_1) \subset \text{Dom}(\mathcal{D}_1\mathcal{P}_1)$ and $\mathcal{D}_1\mathcal{P}_1Q_1 = \mathcal{D}_1\mathcal{P}_1(I + \mathcal{D}_1\mathcal{P}_1)^{-1}$. Similarly $\mathcal{R}(Q_1) \subset \text{Dom}(\mathcal{D}_0\mathcal{R}_0^*)$ and $\mathcal{D}_0\mathcal{R}_0^*Q_1 = \mathcal{D}_0\mathcal{R}_0^*(I + \mathcal{D}_0\mathcal{R}_0^*)^{-1}$. Consequently, $\mathcal{R}(Q_1) \subset \text{Dom}(L_1)$ and 

$$L_1Q_1 = \mathcal{D}_1\mathcal{P}_1(I + \mathcal{D}_1\mathcal{P}_1)^{-1} + \mathcal{D}_0\mathcal{R}_0^*(I + \mathcal{D}_0\mathcal{R}_0^*)^{-1} + Q_1 = I$$  

by (3.10). This together with the injectivity of $L_1$ implies that $L_1^{-1} = Q_1$. Thus $L_1^{-1}$ is self-adjoint. So is its inverse $L_1$ (cf. §2 in Chapter 8 in [18] for this general property). 

\[\square\]

### 3.3. The canonical solution operator

**Proof of Theorem (7.7)**

1. The weighted $L^2$-estimate (1.8) implies that 

$$4\|g\|_\varphi^2 \leq \|\varphi_0 g\|_\varphi^2 + \|\varphi_1 g\|_\varphi^2 = (\varphi_0, g , g) \leq \|\varphi_0 g\|_\varphi \|g\|_\varphi,$$

for $g \in \text{Dom}(\varphi_0)$, by (3.7), i.e. 

$$4\|g\|_\varphi \leq \|\varphi_0 g\|_\varphi.$$  

Thus $\varphi_0$ is injective. This together with the self-adjointness of $\varphi_0$ by Proposition (3.7) implies the density of the range (cf. §2 in Chapter 8 in [18] for this general property). For fixed $f \in L^2_\varphi(\mathbb{R}^{4n}, \gamma_1)$, the complex anti-linear functional 

$$\lambda_f : \varphi \rightarrow (f, g)_{\varphi}$$

is then well-defined on a dense subset $\mathcal{R}(\varphi_0)$ of $L^2_\varphi(\mathbb{R}^{4n}, \gamma_1)$. It is finite since 

$$|\lambda_f(\varphi_0 g)| = |(f, g)_{\varphi}| \leq \|f\|_{\varphi} \|g\|_{\varphi} \leq \frac{1}{4}\|f\|_{\varphi} \|\varphi_0 g\|_{\varphi},$$

for any $g \in \text{Dom}(\varphi_0)$, by (3.11). So $\lambda_f$ can be uniquely extended a continuous anti-linear functional on $L^2_\varphi(\mathbb{R}^{4n}, \gamma_1)$. By the Riesz representation theorem, there exists a unique element $h \in L^2_\varphi(\mathbb{R}^{4n}, \gamma_1)$ such that $\lambda_f(F) = (h, F)_{\varphi}$ for any $F \in L^2_\varphi(\mathbb{R}^{4n}, \gamma_1)$, and $\|h\|_{\varphi} = |\lambda_f| \leq \frac{1}{4}\|f\|_{\varphi}$. In particular, we have 

$$\langle h, \varphi_0 g \rangle_{\varphi} = (f, g)_{\varphi}$$

for any $g \in \text{Dom}(\varphi_0)$. This implies that $h \in \text{Dom}(\varphi_0^*)$ and $\varphi_0 h = f$, and so $h \in \text{Dom}(\varphi_0)$ and $\varphi_0 h = f$ by self-adjointness of $\varphi_0$. We write $h = N_\varphi f$. Then $|\lambda_f| = \|h\|_{\varphi} \leq \frac{1}{4}\|f\|_{\varphi}$.

2. Since $N_\varphi f \in \text{Dom}(\varphi_0)$, we have $\mathcal{D}_0 N_\varphi f \in \text{Dom}(\mathcal{D}_0)$, $\mathcal{D}_1 N_\varphi f \in \text{Dom}(\mathcal{D}_1)$, and 

$$N_\varphi f = \mathcal{D}_0 N_\varphi f + \mathcal{D}_1 N_\varphi f$$

by $\varphi_0 N_\varphi f = f$. Because $f$ and $\mathcal{D}_0 f$ for any $F \in \text{Dom}(\mathcal{D}_0)$ are both $\mathcal{D}_1$-closed, the above identity implies $\mathcal{D}_1 N_\varphi f \in \text{Dom}(\mathcal{D}_1)$ and so $\mathcal{D}_1 N_\varphi f = 0$ by $\mathcal{D}_1$ acting in both sides. Then 

$$0 = (\mathcal{D}_1 \mathcal{D}_1^* N_\varphi f, \mathcal{D}_1 N_\varphi f) = \|\mathcal{D}_1^* N_\varphi f\|_{\varphi}^2,$$
Proof. If \( \text{Proposition 4.1.} \) \( \mathcal{Pf} \), we have \( \mathcal{D}_0^*N_\varphi f = 0 \). Moreover, we have \( \mathcal{D}_0^*N_\varphi f \perp A^2_k(\mathbb{R}^{4n}, \varphi) \) since \( (F, \mathcal{D}_0^*N_\varphi f, \varphi) = (\mathcal{D}_0 F, N_\varphi f, \varphi) = 0 \) for any \( F \in A^2_k(\mathbb{R}^{4n}, \varphi) \). The estimate (4.7) follows from

\[
\|\mathcal{D}_0^*N_\varphi f\|_{\varphi}^2 + \|\mathcal{D}_1 N_\varphi f\|_{\varphi}^2 = \langle \mathcal{D}_\varphi N_\varphi f, N_\varphi f, \varphi \rangle \leq \frac{1}{4}\|f\|_{\varphi}^2.
\]

**Corollary 3.1.** The weighted \( k \)-Bergman projection formula (1.9) holds.

**Proof.** For \( f \in \text{Dom}(\mathcal{D}_0) \), \( \mathcal{D}_0 f \) is automatically \( \mathcal{D}_1 \)-closed. Apply Theorem 7.7 to \( \mathcal{D}_0 f \) to get the canonical solution \( \mathcal{D}_0^*N_\varphi \mathcal{D}_0 f \) orthogonal to \( A^2_k(\mathbb{R}^{4n}, \varphi) \). So \( f - \mathcal{D}_0^*N_\varphi \mathcal{D}_0 f \) is an \( \mathcal{D}_0^*N_\varphi \mathcal{D}_0 f \)-orthogonal projection of \( f \) to the weighted \( k \)-Bergman space. \( \Box \)

**Remark 3.1.** As in [21], we can use Theorem 7.7 to get compactly supported solution to the non-homogeneous \( k \)-Cauchy-Fueter equation (1.7) for \( \mathcal{D}_1 \)-closed \( f \in C^1_0(\mathbb{R}^{4n}, \varphi) \), which implies Hartogs’ phenomenon for \( k \)-regular functions.

4. Decay of canonical solutions and the weighted \( k \)-Bergman kernel

4.1. The weighted \( k \)-Bergman projection and kernel. For \( f \in L^2_\varphi(\Omega, \varphi_0) \), it has \( k + 1 \) independent components \( f_{00\cdots 0}, f_{11\cdots 1}, \ldots, f_{1'1'\cdots 1'} \). We write

\[
(4.1) \quad f = \begin{pmatrix}
  f_{00\cdots 0} \\
  f_{11\cdots 1} \\
  \vdots \\
  f_{1'1'\cdots 1'}
\end{pmatrix} = \begin{pmatrix}
  f_0 \\
  f_1 \\
  \vdots \\
  f_k
\end{pmatrix},
\]

where \( f_j := f_{1'1'\cdots 1'\cdots 0} \) with \( j \) indices to be \( 1' \).

Note that for a sequence of \( k \)-regular functions \( F_n \in L^2_\varphi(\Omega, \varphi_0) \) (i.e. \( \mathcal{D}_0 F_n = 0 \)), if \( F_n \rightarrow F \) in \( L^2_\varphi(\Omega, \varphi_0) \), we have \( \mathcal{D}_0 F = 0 \) by the closedness of \( \mathcal{D}_0 \). So \( A^2_k(\mathbb{R}^{4n}, \varphi) \) is a closed subspace of \( L^2_\varphi(\Omega, \varphi_0) \). If \( \{\psi_\alpha\} \) is an orthonormal basis of the space \( A^2_k(\mathbb{R}^{4n}, \varphi) \), the weighted \( k \)-Bergman projection \( P \) can be write as \( Pf = \sum_\alpha \langle f, \psi_\alpha \rangle \varphi \psi_\alpha \).

**Proposition 4.1.** If \( f \in L^2_\varphi(\Omega, \odot^k C^2) \) is \( k \)-regular, then each component of \( f \) is harmonic.

**Proof.** It follows from

\[
(4.2) \quad \mathcal{D}_0 \mathcal{D}_0 f = \begin{pmatrix}
  \Delta & 0 & \cdots & 0 & 0 \\
  0 & 2\Delta & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 2\Delta & 0 \\
  0 & 0 & \cdots & 0 & \Delta
\end{pmatrix} \begin{pmatrix}
  f_0 \\
  f_1 \\
  \vdots \\
  f_k
\end{pmatrix}
\]

where \( \Delta := \partial^2_{z_1} + \partial^2_{z_2} + \cdots + \partial^2_{z_{4n}} \). See lemma 3.3 of [19] for this identity. \( \Box \)

By Proposition 4.1, each component of a \( k \)-regular function is smooth. So for a fixed point \( x \in \mathbb{R}^{4n} \), we can define complex linear functionals

\[
 l_j(f) = f_j(x)
\]

for \( f \in A^2_k(\mathbb{R}^{4n}, \varphi) \), \( j = 0, \ldots, k \). Since \( f_j \) is harmonic by Proposition 4.1, we see that

\[
(4.3) \quad |f_j(x)| \leq \frac{1}{|B(x, 1)|} \int_{B(x, 1)} |f_j(y)| dV(y) \leq \frac{1}{|B(x, 1)|} \|f\|_{\varphi} \left( \int_{B(x, 1)} e^{2\varphi(y)} dV(y) \right)^{1/2} \leq C_x \|f\|_{\varphi},
\]
where $C_2$ only depends on $x$, not on $f$. Consequently, linear functionals $l_j$ are bounded on $A^2_{(k_j)}(\mathbb{R}^{4n}, \varphi)$. By the Riesz representation theorem, there exists $K_j(\cdot, x) \in A^2_{(k_j)}(\mathbb{R}^{4n}, \varphi)$ such that

$$f_j(x) = \langle f, K_j(\cdot, x) \rangle_{\varphi} = \sum_{l=0}^{k} \int_{\mathbb{R}^{4n}} f_l(y) K_{jl}(y, x) e^{-2\varphi} dV.$$  

It is obvious that $\langle g, K_j(\cdot, x) \rangle_{\varphi} = 0$ for any $g \perp A^2_{(k_j)}(\mathbb{R}^{4n}, \varphi)$. So $K(x, y) = (K_{jl}(y, x))$ is the kernel of the weighted $k$-Bergman projection $P$, which is a $(k + 1) \times (k + 1)$ matrix anti-$k$-regular in $y$. Then the integral formula (1.10) holds. Since an orthogonal projection $P$ is self-adjoint on $L^2(x, y_0)$, $K$ has the Hermitian property $K(x, y) = K(y, x)$, and so $K(x, y)$ is $k$-regular in $x$.

4.2. A localized a priori estimate and Caccioppoli-type estimate. It is known that the Caccioppoli-type estimate holds for many systems of PDEs of the divergence form by establishing localized a priori estimate of the following type.

**Proposition 4.2.** There exists an absolute constant $C_0 > 0$ such that for any $f \in \text{Dom}(\square \varphi)$ and real bounded Lipschitz function $\eta$, we have estimates

$$\|\eta \partial \eta f\|_{L^2}^2 + \|\eta \partial^*_\eta f\|_{L^2}^2 \leq C_0 \left( \left| \langle \eta \varphi \rangle \cdot f \right|_{L^2}^2 + \left| \langle \eta \varphi \rangle \cdot f, \square \varphi f \rangle \right)_{\varphi}^2,$$

(4.4)

where $|dn| = \sum_{j=1}^{4n} |d\partial_{x_j}|^2$.

**Proof.** Note that $\partial^A_{A_j} (\eta f_{A_j'A_j'}) = \eta \partial^A_{A_j} f_{A_j'A_j'} + Z^A_{A_j'A_j'} \eta \cdot f_{A_j'A_j'}$ by $\partial^A_{A_j} = Z^A_{A_j} - 2Z^A_{A_j'} \varphi$ in (2.9). Then taking summation over $A$ and symmetrising $(A_1' \ldots A_k')$, we get

$$\left[ \partial^A_{A_j} f \right]_{A_1' \ldots A_k'} = \eta \left[ \partial^A_{A_j} f \right]_{A_1' \ldots A_k'} + \sum_{A=A+1}^{2n-1} Z^A_{A_j'A_j'} \eta \cdot f_{A_j'A_j'} A.$$

(4.5)

On the other hand, for fixed $A_1' \ldots A_k'$, we have

$$\left| \sum_A Z^A_{A_j'A_j'} \cdot f_{A_j'A_j'} A \right| = \frac{1}{k} \left| \sum_{s=1}^{k} \sum_A Z^A_{A_j'A_j'} \cdot f_{A_s'A_s'} A \right|$$

(4.6)

$$\leq \frac{1}{k} \sum_{s=1}^{k} \left( \sum_A \left| Z^A_{A_s'A_s'} \eta \right|^2 \right)^{\frac{1}{2}} \left( \sum_A \left| f_{A_s'A_s'} \right|^2 \right)^{\frac{1}{2}},$$

by using (3.5) and Cauchy-Schwarz inequality and $f$ symmetric in the primed indices. Note that it directly follows from definition (2.7) of $Z^A_{A'A'}$'s that

$$\sum_{A=0}^{2n-1} |Z^A_{A'A'} \eta|^2 = |dn|^2$$

for fixed $A' = 0$ or 1'. Then by raising indices, we get

$$\sum_{A=0}^{2n-1} |Z^A_{A} \eta|^2 = \sum_{A=0}^{2n-1} |Z^A_{A} \eta|^2 = \sum_{A=0}^{2n-1} |Z^A_{A} \eta|^2 = |dn|^2,$$
and so is the sum of $|Z^{A}_{1} \eta|^{2}$. Apply these to (4.6) to get

\[(4.7) \quad \sum_{A_{1}, \ldots, A_{k}} \left\| \sum_{A} Z^{A}_{(A_{1} \eta \cdot f_{A_{2}} \ldots A_{k})_{A}} \right\|^{2} \leq 2||d\eta| f||^{2}_{\varphi}.\]

Thus we get the estimate

\[\|D_{0}^{*} (\eta f)\|^{2}_{\varphi} \leq \|\eta D_{0} (f)\|^{2}_{\varphi} + 2||d\eta| f||^{2}_{\varphi},\]

by (4.5), and simultaneously,

\[(4.8) \quad \|\eta D_{0} f\|^{2}_{\varphi} \leq \|D_{0}^{*} (\eta f)\|^{2}_{\varphi} + 2||d\eta| f||^{2}_{\varphi}.\]

Note that by (4.6) again, we get

\[\|D_{0}^{*} (\eta f)\|^{2}_{\varphi} = \sum_{A_{1}, \ldots, A_{k}} \left( D_{0}^{*} (\eta f)_{A_{1}} \ldots A_{k} \sum_{A} Z^{A}_{(A_{1} \eta \cdot f_{A_{2}} \ldots A_{k})_{A}} + \eta (D_{0}^{*} f)_{A_{1}} \ldots A_{k} \right)_{\varphi} \]

\[= \sum_{A_{1}, \ldots, A_{k}} \left( D_{0}^{*} (\eta f)_{A_{1}} \ldots A_{k} \sum_{A} Z^{A}_{(A_{1} \eta \cdot f_{A_{2}} \ldots A_{k})_{A}} + (D_{0}^{*} (\eta f), \eta D_{0}^{*} f)_{\varphi} \right) \]

\[\leq \kappa \|D_{0}^{*} (\eta f)\|^{2}_{\varphi} + \frac{1}{\kappa} ||d\eta| f||^{2}_{\varphi} + (\eta f, D_{0} (\eta D_{0}^{*} f))_{\varphi} \]

by using estimates (4.6) - (4.7) and the trivial inequality $2|ab| \leq \kappa|a|^{2} + \frac{1}{\kappa}|b|^{2}$ for any $\kappa > 0$. Thus if we choose $\kappa = 1/2$, we get

\[(4.9) \quad \|D_{0}^{*} (\eta f)\|^{2}_{\varphi} \leq 4||d\eta| f||^{2}_{\varphi} + 2(\eta f, D_{0} (\eta D_{0}^{*} f))_{\varphi}.\]

But

\[|\langle \eta f, D_{0} (\eta D_{0}^{*} f) \rangle_{\varphi}| \leq |\langle \eta f, \eta D_{0} D_{0}^{*} f \rangle_{\varphi}| + \sum_{A_{1}, A_{2}, \ldots, A_{k}} \left| \left( \eta f_{A_{1}} \sum_{A} Z^{A}_{A_{1}} \eta \cdot (D_{0}^{*} f)_{A_{1}} \ldots A_{k} \right)_{\varphi} \right| \]

\[(4.10) \leq |\langle \eta^{2} f, D_{0}^{*} f \rangle_{\varphi}| + \sum_{A_{1}, A_{2}, \ldots, A_{k}} \sum_{A} \left| \left( Z^{A}_{A_{1}} \eta \cdot (D_{0}^{*} f)_{A_{1}} \ldots A_{k} \right)_{\varphi} \right| \]

\[\leq |\langle \eta^{2} f, D_{0}^{*} f \rangle_{\varphi}| + \frac{1}{\kappa} ||d\eta| f||^{2}_{\varphi} + \kappa \|D_{0} f\|^{2}_{\varphi} \]

by applying estimates similar to (4.6), (4.7) in the third inequality. Now Substitute (4.10) to (4.9) and using (4.8) to control the term $\kappa \|D_{0} f\|^{2}_{\varphi}$, we find that there exists a constant $C_{0} > 0$ such that

\[\|D_{0}^{*} (\eta f)\|^{2}_{\varphi} \leq C_{0} \left( ||d\eta| f||^{2}_{\varphi} + |\langle \eta^{2} f, D_{0} (\eta D_{0}^{*} f)_{\varphi}| \right).\]

Similarly,

\[D_{1}(\eta f)_{A_{1}A_{1} \ldots A_{k}} = \eta (D_{1} f)_{A_{1}A_{1} \ldots A_{k}} + 2 \sum_{A_{1}=0, 1} Z_{A_{1}}^{A_{1}} \eta \cdot f_{A_{1}} \ldots A_{k} \]

by definition, and so

\[\|D_{1}(\eta f)\|^{2}_{\varphi} \leq \|\eta D_{1}(f)\|^{2}_{\varphi} + 4n||d\eta| f||^{2}_{\varphi},\]

\[\|\eta D_{1} f\|^{2}_{\varphi} \leq C_{0} \left( ||d\eta| f||^{2}_{\varphi} + |\langle \eta^{2} f, D_{1}^{*} f \rangle_{\varphi}| \right).\]

The result follows. □
As a corollary, we get Caccioppoli-type estimate.

**Proposition 4.3.** Suppose that \( \varphi(x) = |x|^2 \). If \( \Box \varphi F = 0 \) on \( B(x, R) \subset \mathbb{R}^{4n} \), then for \( r < R \), we have

\[
\int_{B(x, r)} |\mathcal{D}_0^\varphi F|^2 e^{-2\varphi} dV \leq \frac{C}{(R-r)^2} \int_{B(x, R)} |F|^2 e^{-2\varphi} dV
\]

for some constant \( C \) only depending on \( n, k, R \) and \( r \).

**Proof.** Let \( \eta \) be a \( C_0^\infty(B(x, R)) \) function such that \( \eta \equiv 1 \) on \( B(x, r) \). By the localized a priori estimate \([1.2]\) in Proposition \([1.1]\) we get

\[
\|\chi_{B(x, r)} \mathcal{D}_0^\varphi F \|_\varphi^2 \leq \|\eta \mathcal{D}_0^\varphi F \|_\varphi^2 \leq C_0 \left( \|d\eta \cdot F\|_\varphi^2 + |\langle \eta^2 F, \Box \varphi \rangle \varphi| \right) = C_0 \|d\eta\|_\infty^2 \|\chi_{B(x, R)} \cdot F\|_\varphi^2
\]

since \( \Box \varphi F = 0 \) on \( \text{supp} \eta \) and \( d\eta \) is supported in \( B(x, R) \). The result follows by choosing \( \eta \). \( \square \)

### 4.3. Decay of canonical solutions and the weighted \( k \)-Bergman kernel.

**Theorem 4.1.** Suppose that \( \varphi(x) = |x|^2 \), \( k = 2, 3, \ldots \), and that \( f \in L^2_\varphi(\mathbb{R}^{4n}, \gamma_1) \) is compactly supported in \( B(y, r_0) \). Then the canonical solution \( u = \mathcal{D}_0^\varphi N_\varphi f \) has the following pointwise estimate: there exists \( \varepsilon > 0 \) only depending on \( r_0 \) and constant \( C > 0 \) only depending on \( n, k \) and \( \varepsilon \) such that

\[
|u(x)| \leq C e^{\varepsilon|x|^2 + \frac{4}{k}|x| - \varepsilon |x-y||f|_\varphi
\]

for any \( x \) such that \( |x-y| > r_0 + 2 \).

**Proof.** For the canonical solution \( u = \mathcal{D}_0^\varphi N_\varphi f \), we have \( \mathcal{D}_0 u = f \) vanishing outside of \( B(y, r_0) \). Consequently, each component of \( u \) is harmonic outside of \( B(y, r_0) \) by Proposition \([4.1]\). By the mean value formula for harmonic functions, we get

\[
|u(x)|^2 = \left| \frac{1}{|B(x, \delta)|} \int_{B(x, \delta)} u(x') dV \right|^2
\]

\[
\leq \frac{1}{|B(x, \delta)|^2} \int_{B(x, \delta)} |u(x')|^2 e^{-2|x'|^2} dV(x') \cdot \int_{B(x, \delta)} e^{2|x'|^2} dV(x')
\]

\[
\leq C_\delta e^{2|x|^2+4\delta|x|} \int_{B(x,1)} |N_\varphi f(x')|^2 e^{-2|x'|^2} dV(x')
\]

for some constant \( C_\delta > 0 \) only depending on \( n, \delta < 1 \) and any \( x \) such that \( |x-y| > r_0 + 1 \). Here in the last inequality we apply Caccioppoli-type estimate in Proposition \([4.3]\) to \( F = N_\varphi f \) with \( \Box \varphi N_\varphi f = f = 0 \) outside of \( B(y, r_0) \), and \( e^{\varepsilon |x'|^2} \leq e^{\varepsilon|x|^2+2\delta|x|+\delta^2} \) for \( x' \in B(x, \delta) \). We choose \( \delta = \frac{\varepsilon}{4} \) for \( \varepsilon \) determined later.

For fixed \( x \) outside of \( B(y, r_0) \), consider the Lipschitzian function

\[
b(x') := \min\{ |x' - y|, |x - y| \}.
\]

Let \( l : [0, \infty) \to [0, 1] \) be the Lipschitzian function vanishing on \( [0, r_0] \), equal to 1 on \( [r_0 + 1, \infty) \), and affine in between. Set \( \eta(x') = l(|x' - y|) \). Applying weighted \( L^2 \) estimate \([1.3]\) and the localized a priori estimate in Proposition \([4.4]\) to \( N_\varphi f \) with \( \eta \) replaced by \( \eta e^{\varepsilon b} \), we get

\[
\int_{\mathbb{R}^{4n}} |\eta e^{\varepsilon b} N_\varphi f(x')|^2 e^{-2\varphi} dV(x') \leq C_\varphi (\eta e^{\varepsilon b} N_\varphi f, \eta e^{\varepsilon b} N_\varphi f)
\]

\[
\leq C_0 \|d(\eta e^{\varepsilon b}) \cdot N_\varphi f\|_\varphi + C_0 \langle \eta e^{2\varepsilon b} N_\varphi f, \Box \varphi N_\varphi f \rangle \varphi
\]

\[
\leq C_0 \int_{\mathbb{R}^{4n}} \left( \|d\eta\| e^{\varepsilon b} N_\varphi f(x')|^2 + (4n\varepsilon)^2 |\eta e^{\varepsilon b} N_\varphi f(x')|^2 \right) e^{-2\varphi} dV(x')
\]
ON THE WEIGHTED \( L^2 \) ESTIMATE FOR THE \( k \)-CAUCHY-FUETER OPERATOR AND THE \( k \)-BERGMAN KERNEL

since the Lipschitzian constant of \( b \) is 1 and \( \Box_b N_\varphi f = f = 0 \) on the support of \( \eta (= B(y, r_0)^c) \). Hence if we choose \( \varepsilon \) sufficiently small (e.g. \( C_0(4n\varepsilon)^2 \leq \frac{1}{2} \)), we get

\[
\int_{\mathbb{R}^{4n}} |\eta e^{eb}N_\varphi f(x')|^2 e^{-2\varphi} dV(x') \leq 2C_0 \int_{\mathbb{R}^{4n}} ||d\eta||e^{eb}N_\varphi f(x')|^2 e^{-2\varphi} dV(x') \\
\leq C'' \int_{B(y, r_0+1)} |N_\varphi f(x')|^2 e^{-2\varphi} dV(x')
\]

for some constant \( C'' > 0 \), by \( d\eta \) supported in \( B(y, r_0 + 1) \) and \( b \) uniformly bounded on \( B(y, r_0 + 1) \) (\(|b(x')| < r_0 + 1\)). But \( b(x') \geq |x - y| - 1 \) for \( x' \in B(x, 1) \), and so the above estimate implies that

\[
\int_{B(x, 1)} |N_\varphi f(x')|^2 e^{-2\varphi} dV(x') \leq C'' e^{-2(|y - x| - 1)} \int_{B(y, r_0+1)} |N_\varphi f(x')|^2 e^{-2\varphi} dV(x').
\]

Substituting this into (4.12), we get the result by the boundedness of \( N_\varphi \) on \( L^2_\varphi(\mathbb{R}^{4n}, \mathcal{Y}_1) \) by Theorem 4.1 (1).

**Proof of Theorem 4.3.** For fixed \( y \in \mathbb{R}^{4n} \), let \( \eta_y \) be a smooth radial function supported in the ball \( B(y, \delta) \) (\( \delta < 1 \)) such that \( \int \eta_y(y') dV(y') = 1 \).

(4.13) \[ f_{y}(y') = \begin{pmatrix} \vdots & \vdots & \vdots \end{pmatrix} \in L^2_\varphi(\mathbb{R}^{4n}, \mathcal{Y}_0) \]

for fixed \( j \), where only \( j \)-th entry is nonvanishing. Note that

\[
P_{f_{y}}(x) = \int_{\mathbb{R}^{4n}} K(x, y') f_{y}(y') e^{-2|y'|^2} dV(y') = \int_{\mathbb{R}^{4n}} K(x, y') \begin{pmatrix} \vdots & \vdots & \vdots \end{pmatrix} dV(y') = \begin{pmatrix} K(x, y)\eta_{yj} \\ \vdots \\ K(x, y)\kappa_{kj} \end{pmatrix}
\]

by applying the mean value formula for harmonic functions to each component of \( K(x, \cdot) \), since \( \eta_y(\cdot) \) is constant on each sphere centered at \( y \). Hence the \( j \)-th column of \((k+1) \times (k+1)\)-matrix \( K \) is

\[
\begin{pmatrix} K(x, y)\eta_{yj} \\ \vdots \\ K(x, y)\kappa_{kj} \end{pmatrix} = P_{f_{y}}(x) = f_{y}(x) - (\mathcal{D}_0^* N_\varphi \mathcal{D}_0 f_{y})(x),
\]

by the identity (1.9). The exponential decay of the canonical solution in Theorem 4.1 implies that there exists a constant \( C > 0 \) only depending on \( \varepsilon, n, k \) such that

\[
|\mathcal{D}_0^* N_\varphi \mathcal{D}_0 f_{y}(x)| \leq C e^{c|x|+|x-y|} \mathcal{D}_0 f_{y}(x)
\]

for any \( x \) such that \(|x-y| > 3\), since \( \mathcal{D}_0 \mathcal{D}_0^* N_\varphi \mathcal{D}_0 f_{y} = \mathcal{D}_0 f_{y} \) is supported in \( B(y, 1) \). Note that \(|\mathcal{D}_0 f_{y}(y')| \leq C_3 e^{c|y'|} \mathcal{D}_0 f_{y}(y') \) for some constant \( C_3 > 0 \) depending on \( n, \delta \), by direct differentiation (4.13). It is direct to check that \( \mathcal{D}_0 f_{y}(x) \leq C_4 e^{c|x|+5\delta|x|} \mathcal{D}_0 f_{y}(x) \) for some constant \( C_4 > 0 \) depending on \( n, \delta \). The result follows by choose small \( \delta \).
Remark 4.1. Our estimate (1.11) has an extra factor $e^{\frac{2}{\epsilon}(\|x\|+|y|)}$ compared to the estimate
\[
|K(x, y)| \leq C e^{\epsilon\|x\|^2 + |y|^2 - \epsilon|x-y|},
\]
for the Bergmann kernel in complex analysis. But when $|y|$ is large compared to $|x|$, e.g. $|y| \geq 4|x|$, \[
|K(x, y)| \leq C e^{\epsilon\|x\|^2 + |y|^2 - \frac{4}{3}|y|},
\]
which has similar exponential decay with respect to the measure $e^{-|y|^2}\,dV$ as in the complex case.

References

[1] Baston, R., Quaternionic complexes, J. Geom. Phys. 8 (1992) 29-52.
[2] Bures, J., Damiano, A. and Sabadini, I., Explicit resolutions for several Fueter operators, J. Geom. Phys. 57 (2007), 765-775.
[3] Bures, J. and V. Soucek, V., Complexes of invariant differential operators in several quaternionic variables, Complex Var. Elliptic Equ. 51 (2006), no. 5-6, 463-487.
[4] Chang, D.-C., Markina, I. and Wang, W., On the Hodge-type decomposition and cohomology groups of $k$-Cauchy-Fueter complexes over domains in the quaternionic space, J. Geom. Phys. 107 (2016), 15-34.
[5] Chen, S.-C. and Shaw, M.-C., Partial differential equations in several complex variables, AMS/IP Studies in Advanced Mathematics, vol. 19, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001.
[6] Christ, M., On the $\overline{\partial}$ equation in weighted $L^2$ norms in $\mathbb{C}^1$, J. Geom. Anal. 1(3) (1991) 193-230.
[7] Colombo, F., Soucek, V. and Struppa, D., Invariant resolutions for several Fueter operators, J. Geom. Phys. 56 (2006), no. 7, 1175-1191.
[8] Colombo, F., Sabadini, I., Sommen, F. and Struppa, D., Analysis of Dirac systems and computational algebra, Progress in Mathematical Physics 39, Birkhäuser, 2004.
[9] Dall’Ara, G.M., Pointwise estimates of weighted Bergman kernels in several complex variables, Adv. in Math. 285 (2015), 1706-1740.
[10] Delin, H., Pointwise estimates for the weighted Bergman projection kernel in $\mathbb{C}^n$, using a weighted $L^2$-estimate for the $\overline{\partial}$ equation, Ann. Inst. Fourier (Grenoble) 48(4) (1998) 967-997.
[11] Eastwood, M., Penrose, R. and Wells, R., Cohomology and massless fields, Comm. Math. Phys. 78 (1980), no. 3, 305-351.
[12] Haslinger, F., The $\overline{\partial}$-Neumann Problem and Schrödinger Operators, de Gruyter Expositions in Mathematics, vol. 59, De Gruyter, Berlin, 2014.
[13] Kang, Q.-Q. and Wang, W., On Penrose integral formula and series expansion of $k$-regular functions on the quaternionic space $H^n$, J. Geom. Phys. 64 (2013), 192-208.
[14] Marzo, J. and Ortega-Cerdà, J., Pointwise estimates for the Bergman kernel of the weighted Fock space, J. Geom. Anal. 19(4) (2009) 890-910.
[15] Liu, Y., Chen, Z. H. and Pan, Y. F., A variant of Hörmander’s $L^2$ existence theorem for the Dirac operator in Clifford analysis, J. Math. Anal. Appl. 410 (2014), 39-54.
[16] Penrose, R. and Rindler, W., Spinors and Space-Time, Vol. 1, Two-spinor calculus and relativistic fields, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1984.
[17] Penrose, R. and Rindler, W., Spinors and Space-Time, Vol. 2, Spinor and twistor methods in space-time geometry, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1986.
[18] Riesz, F. and Sz.-Nagy, B., Functional analysis, Translated by Leo F. Boron, Frederick Ungar Publishing Co., New York, 1955.
[19] Wang, H. Y. and Ren, G. B., Bochner-Martinelli formula for $k$-Cauchy-Fueter operator, J. Geom. Phys. 84 (2014), 43-54.
[20] Wang, W., On non-homogeneous Cauchy-Fueter equations and Hartogs’ phenomenon in several quaternionic variables, J. Geom. Phys. 58, (2008), 1203-1210.
[21] Wang, W., The $k$-Cauchy-Fueter complexes, Penrose transformation and Hartogs’ phenomenon for quaternionic $k$-regular functions, J. Geom. Phys. 60, (2010), 513-530.
[22] Wang, W., On quaternionic complexes over unimodular quaternionic manifolds, arXiv:1610.06445 (2016).
[23] Wang, W., The Neumann problem for the $k$-Cauchy-Fueter complexes over $k$-pseudoconvex domains in $\mathbb{R}^4$ and the $L^2$ estimate, preprint.