Is Vacuum Decay Significant in Ekpyrotic and Cyclic Models?

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It has recently been argued that bubble nucleation in ekpyrotic and cyclic cosmological scenarios can lead to unacceptable inhomogeneities unless certain constraints are satisfied. In this paper we show that this is not the case. We find that bubble nucleation is completely negligible in realistic models.

The cyclic model of the Universe\textsuperscript{1,2} is a radical alternative to standard big bang and inflationary cosmology. The cyclic picture proposes that the Universe undergoes an endless sequence of cosmic epochs which begin with expansion from a big bang and end in collapse to a big crunch. The connection between the big crunch and the ensuing big bang is presumed to occur according to the prescription recently proposed with Khoury, Ovrut, Seiberg.\textsuperscript{3} The expansion part of each cycle has a period of radiation and matter domination followed by an extended period of cosmic acceleration at low energies. The long period of acceleration is crucial in establishing the flat and vacuous initial conditions required prior to beginning the contraction phase. The density of entropy, black holes and other debris from the previous cycle is reduced to nearly zero. Subsequently, the Universe ceases to accelerate, fluctuations are generated, and the Universe heads towards a big crunch in which matter and energy are regenerated. The key ingredient is the restoration of the Universe to a pristine flat vacuum state before each big crunch, which ensures that the cycle can repeat and that the cyclic solution is an attractor.

The cyclic model can be described in terms of a scalar field $\phi$ moving back and forth in an effective potential $V(\phi)$ (see Fig. 1). Although the model can be described in terms of an ordinary quantum field in four dimensions, the model is strongly motivated by ideas developed as part of the ekpyrotic scenario,\textsuperscript{4} drawing on the braneworld picture and string theory. The scalar field represents the modulus field that determines the distance between branes. Moving from some positive value to minus infinity and back along the potential corresponds to a pair of orbifold branes being drawn together by an interbrane force, colliding (corresponding to $\phi \to -\infty$), bouncing and rebounding. The transition between big crunch and big bang corresponds to the bounce.\textsuperscript{3} The accelerating phase begins when the branes are maximally separated ($\phi = \phi_0$) once the matter density falls below the positive interbrane potential energy density. As the branes are drawn together, the accelleration rate decreases until, at $\phi = \phi_1$, the expansion begins to decelerate. As the branes draw yet further together, the potential becomes negative ($\phi < 0$), the expansion stops ($\phi = \phi_2$) and the universe begins to contract. Fluctuations are generated during the contraction phase ($\phi = \phi_3$), before the potential reaches a minimum at $\phi = \phi_{\text{min}}$.

Recently, Heyl and Loeb\textsuperscript{5} have argued that bubble nucleation during the accelerating and collapse phase, near $\phi = \phi_1$, for example, can spoil the homogeneity of the universe unless the potential obeys certain constraints. They have focused on tunneling about $\phi_1$, where the kinetic and potential energy become comparable as the field rolls to the left and where the accelerating expansion phase ends. Bubbles form whose interior corresponds to values of $\phi$ far down the potential. Once formed, the bubbles grow at the speed of light.) These bubbles destroy the homogeneity essential to the scenario. Their

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{potential_plot}
\caption{Plot of the actual analytic potential (solid line) described in Refs. 1 and 2, shown to proper scale. For the purposes of analyzing bubble nucleation, differences between the schematic plots shown in Refs. 1 and 2 and the properly scaled plot are significant. The axes are expressed in Planck units. The value of $\rho_Q$ is about $10^{-120}$ corresponding to the current dark energy (quintessence) energy density. The markings $\phi_i$ refer to points described in the text. The dotted line is the piecewise linear approximation assumed by Heyl and Loeb. Our central point is that the piece-wise linear function is an exponentially poor approximation to the slope for $\phi < \phi_1$, leading to an exponential overestimate of the nucleation rate. If the kink is moved to $\phi_4$, the piece-wise approximation is reasonable, but the decay rate is then negligible.}
\end{figure}
point is that, although the bubbles are rare events, there remain more than a billion years from when \( \phi \) passes \( \phi_1 \) until it reaches the bottom of the potential well. They estimate that this is sufficient time to have a significant amount of bubble nucleation, unless the potential well is not very deep.

In this paper we show that the approximations used by Heyl and Loeb, although motivated by schematic figures of the potentials in Refs. 1 and 2, are not valid for the actual analytic forms, as presented for either ekpyrotic or cyclic models. In particular, they estimate the bubble nucleation rate assuming a potential which can be approximated as two linear segments, one with slope zero and one with non-zero slope (Fig. 1). However, the slope of the potentials used in the ekpyrotic and cyclic models is 10\(^{-100} \) or so times smaller than in their piece-wise linear approximation for \( \phi_4 \ll \phi < \phi_1 \). This error is further amplified in the computation of the bubble nucleation rate, which is exponentially sensitive to the slope. The smaller is the slope, the more suppressed is the nucleation rate. For the potentials actually used, we will show that the bubble nucleation rate is completely negligible compared to the classical rolling rate.

The explicit model for \( V(\phi) \) which was discussed in the context of the cyclic model is of the form:\(^2\)

\[
V(\phi) = V_0(1 - e^{-c\phi})F(\phi),
\]

where, without loss of generality, we have shifted \( \phi \) so that the zero of the potential occurs at \( \phi = 0 \). The function \( F(\phi) \) is introduced to represent the vanishing of non-perturbative effects described above: \( F(\phi) \) turns off the potential rapidly as \( \phi \) goes below \( \phi_{min} \), but it approaches one for \( \phi > \phi_{min} \). For example, \( F(\phi) \) might be proportional to \( e^{-\eta/\phi} \) or \( e^{-\eta/\phi} \) (for illustration, we will assume the latter form), where \( g_\eta \propto e^{c\phi} \) for \( \gamma > 0 \).

We will choose \( c = 10 \) and \( \gamma \approx 1/8 \) so that the potential minimum corresponds to \( V(\phi_{min}) = M_0^4 = O(1) \) in Planck units, a choice which Heyl and Loeb claim is ruled out by the nucleation constraint. The constant \( V_0 \) is set roughly equal to the vacuum energy observed in today’s Universe, of order \( 10^{-120} \) in Planck units. For large \( c \), this potential has \( V''/V \ll 1 \) for \( \phi \gg 1 \) and \( V''/V \gg 1 \) for \( \phi_{min} < \phi < 0 \). These two regions account for cosmic acceleration and for ekpyrotic production of density perturbations, respectively.\(^1\) More precisely, the end of acceleration is at \( \phi_1 \approx 0.35 \). The rest of the markers in Figure 1 are scaled to the same parameters. The current value (circle) corresponds to tens of trillions of years before the big crunch. In the ekpyrotic region, the constant term is irrelevant and \( V \) may be approximated by \( -V_0 e^{-c\phi} \). The scaling solution over this range is

\[
a(t) = |t|^p, \quad V = -V_0 e^{-c\phi} = -\frac{p(1 - 3p)}{t^2}, \quad p = \frac{2}{c^2}. \tag{2}
\]

Here we chosen \( t = 0 \) to be the bounce. These expressions are used to obtain the time estimates discussed in the paper (and this agrees with Heyl and Loeb).

Heyl and Loeb focus on \( \phi \) near \( \phi_1 \), where the scaling solution first becomes valid. The Universe is expanding but at a decelerating rate. As the field roll towards more negative values, the potential energy becomes increasingly negative, the expansion stops and contraction begins. They approximate the potential as

\[
V(\phi) = -M_0^4 \frac{\phi - \phi_1}{\phi_{min} - \phi_1} \Theta(\phi_1 - \phi) \tag{3}
\]

where \( \Theta(\phi) \) is the Heaviside step function. This amounts to approximating \( V(\phi) = 0 \) for \( \phi > \phi_1 \) and \( V(\phi) = K \cdot (\phi - \phi_1) \) for \( \phi < \phi_1 \), where \( K \) is a constant chosen to be the slope of the line that connects \( \phi_1 \) to \( \phi_{min} \). We have superposed this approximate form in Figure 1. The piece-wise linear potential was introduced by Lee and Weinberg in their analysis of bubble nucleation for a rolling scalar field without a barrier.\(^7\)

There are two important aspects to the bubble nucleation problem in the piece-wise linear potential. First, as derived by Heyl and Loeb,\(^5\) gravitational effects suppress the tunneling altogether unless \( \phi \) is within a Planck distance of \( \phi_1 \), \( \eta = \phi - \phi_1 < O(1) \). Hence, for values of \( \phi \) far to the right of \( \phi_1 \) on the plateau, tunneling past \( \phi_1 \) is insignificant even though the classical rolling rate is slow. Second, as derived by Lee and Weinberg, the bubble interior corresponds to a jump in \( \phi \) in which \( \phi \) jumps past \( \phi_1 \) by an amount \( \eta \). So, the closer is \( \phi \) to \( \phi_1 \), the less is the jump. Combining the two aspects, it is clear that a good piece-wise linear approximant to \( V(\phi) \) to be used to estimate tunneling near \( \phi_1 \) should match the slope near \( \phi_1 \) within one Planck distance, \( \Delta \phi \leq 1 \). The behavior of \( V(\phi) \) for values farther away in either direction plays no role in the nucleation rate about \( \phi_1 \).

As is evident from Fig. 1, the slope of the piecewise linear approximation disagrees by an exponential factor from the actual slope of the potential for \( \phi < \phi_1 \). Within a Planck distance of \( \phi_1 \), the difference in the actual slope and the piecewise approximant is a factor of nearly \( 10^{100} \). The difference totally changes the conclusion. The bubble nucleation rate is proportional to

\[
exp \left[ \frac{32\pi^2 (\phi - \phi_1)^3}{3K} \right]. \tag{4}
\]

That is, the exponentially smaller slope of the actual potential is further exponentiated in computing the nucleation rate. Consequently, the nucleation rate is exponentially smaller than the estimate obtained by Heyl and Loeb and is completely negligible, as we suggested in our paper.\(^2\)

The argument is not complete. We have argued that the piecewise linear approximation is poor near \( \phi = \phi_1 \). However, one might worry that it is a reasonable approximation for \( \phi \) near \( \phi_4 \), where the potential becomes
Does nucleation become significant as \( \phi \) approaches \( \phi_4 \)? First, note that Heyl and Loeb’s result can be generalized to show that nucleation is insignificant if

\[
(\phi - \phi_4)^3(\phi_4 - \phi_{\text{min}}) \gg M_0^4
\]

where \(-M_0^4\) corresponds to the minimum of the potential. Here we have chosen \( M_0 \) in Planck units. In this case, where \( \phi_4 - \phi_{\text{min}} \approx 3 \), we can ignore nucleation unless \( \phi \) is within a few Planck distances of \( \phi_4 \). The stronger condition, cited above, is that gravitational effects suppress nucleation unless \( \phi \) is less than one Planck unit from \( \phi_4 \). Then, the nucleation rate is comparable to what Heyl and Loeb found. However, the time left before the field rolls to potential minimum is exponentially smaller. At \( \phi \approx \phi_1 \), the remaining time is about \( H_0^{-1} = 10^{60} \) in Planck units. For \( \phi \) within less than a Planck distance from \( \phi_4 \), the remaining time is less than one Planck time. There is not enough time left to have significant chance to nucleate because here the field is rolling too quickly.

If one considers tunneling near points between \( \phi_1 \) and \( \phi_0 \) (today’s value), the tunneling rate is more suppressed than at \( \phi_1 \) because the slope of the potential \( K \) is less. The rolling time increases as \( (V'')^{-1} \approx 1/cK \) but the tunneling time increases by \( \sim \exp(1/K) \), an exponentially greater factor.

We confess to bearing some responsibility for Heyl and Loeb’s mis-application of the piecewise linear approximation to the potential. We provided in our papers\(^1,^2\) a schematic drawing of the potential which suggests that the slope near \( \phi = \phi_1 \) (where there is a billion years before reaching the minimum) is steeper than is actually the case. This is apparent if one plots the actual potential function described in the text, as has been done in Figure 1.

The bottom line is that bubble nucleation is negligible throughout the cyclic solution and provides no significant constraint for the potentials we consider in ekpyrotic and cyclic models. However, it is good to be aware of Heyl and Loeb’s analysis if one considers more general potentials.

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