Distribution Function in Center of Dark Matter Halo

Ding Ma∗
Ping He†

Institute of Theoretical Physics,
Chinese Academy of Sciences, P. O. Box 2735,
Beijing 100190, China

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N-body simulations of dark matter halos show that the density profiles of halos behave as $\rho(r) \propto r^{-\alpha(r)}$, where the density logarithmic slope $\alpha \simeq 1 \sim 1.5$ in the center and $\alpha \simeq 3 \sim 4$ in the outer parts of halos. However, some observations are not in agreement with simulations in the very central region of halos. The simulations also show that velocity dispersion anisotropy parameter $\beta \approx 0$ in the inner part of the halo and the so called "pseudo phase-space density" $\rho/\sigma^3$ behaves as a power-law in radius $r$. With these results in mind, we study the distribution function and the pseudo phase-space density $\rho/\sigma^3$ of the center of dark matter halos and find that they are closely-related.

Keywords: Dark matter halo; distribution function; pseudo phase-space density

1. Introduction

The formation and evolution of the dark matter halo which can be treated as the self-gravitational collisionless stellar system have become a challenging issue in the study of dark matter.

Thanks to the improved computational power, N-body simulations of dark matter halos become more and more accurate and important with increasing resolution. N-body simulations such as the universal NFW profile[1,2] and others[3,4,5] show that the density profiles of dark matter halos behave as $r^{-\alpha(r)}$, where $\alpha \simeq 1 \sim 1.5$ in the center and $\alpha \simeq 3 \sim 4$ in the outer parts of halos. However, the numerical inner behaviors of dark matter halos are not supported by observations[6,7,8,9,10,11,12,13]. Some work indicates that density profiles of dark matter halos might become shallower than $r \simeq 1$ in the center[14] and perhaps even tend to be a core with no cusp at all[15,16].

N-body simulations not only provide us the density profiles of halos, but also give the relevant information of the velocity space of collisionless particles in the halos. Velocity dispersion and anisotropy profiles[17,18,19,20,21,22,23] have been well described by simple analytical fits. Two interesting phenomena from simulations indicate that velocity dispersion and the density profiles of the haloes are not independent. First, Hansen and Moore[24,25] found that the density logarithmic slope $\alpha(r)$ is correlated to the velocity anisotropy which is parameterized by the anisotropy parameter

∗Email: mading@itp.ac.cn
†Email: hep@itp.ac.cn
$\beta(r)$ and they provided the empirical formula $\beta \approx 1 - 1.15(1 - \alpha/6)$. Therefore $\beta \approx 0$ (isotropic velocity dispersion) in the inner part as $\alpha \approx 1$ and $\beta \approx 0.5$ in the outer part as $\alpha \approx 3$. Second, it has been argued that the so called pseudo phase-space density follows a power law $\rho(r)/\sigma^3(r) \propto r^{-\gamma}$ with exponent $\gamma = 1.875$, where $\rho(r)$ is the density profile and $\sigma^2$ is the total velocity dispersion. Subsequent studies have confirmed that $\rho(r)/\sigma^3(r)$ is a power law in radius, but the best fitting values of the exponent $\gamma$ diverse from each other and range from $\gamma = 1.90 \pm 0.05$ to $2.19 \pm 0.03$. Because $\rho/\sigma^3$ has the same dimension as the phase-space density, $\rho/\sigma^3$ has been called pseudo phase-space density or "poor-man’s" phase-space density.

With these results in mind, much theoretical work has been done for the study of the relation between density profile behavior and pseudo phase-space density. Some authors examine this matter by solving the Jeans equation. Williams et al got a critical exponents $\gamma = 35/18$ and Dehnen and McLaughlin calculated corresponding density profiles in both isotropic and anisotropic cases. Some other work solved the Jeans equation to explore the relation between density profile and pseudo phase-space density. Recently, R. N. Henriksen considered a series expansion for a dark matter distribution function in the spherically symmetric anisotropic limit to discuss pseudo phase-space density.

In this paper, we concentrate on the center of the dark matter halo where the velocity dispersion is almost isotropic and calculate the distribution function and pseudo phase-space density where $r \to 0$ in the spherically symmetric case. In Section 2 we review the basic knowledge about distribution function which is needed for this paper. In Section 3, the distribution function, velocity dispersion and pseudo phase-space density in the center of the dark matter halo are calculated with the given density profile. We make the discussion and conclusion in Section 4.

2. General Formulae

In the center of dark matter halos, with spherically symmetric assumption, the distribution function $F(E, L)$ can be reduced to $F(E)$ as anisotropy parameter $\beta \approx 0$ (almost isotropic) when $r \to 0$. If we know the distribution function, we can calculate the density profile and total velocity dispersion profile as below:

$$\rho(\psi) = 4\pi \int_0^\psi F(E) \sqrt{2(\psi - E)} dE,$$

$$\sigma^2 = \sigma_r^2 + \sigma_T^2 = \frac{27/2 \pi M_{\text{tot}}}{\rho} \int_0^\psi F(E)(\psi - E)^{3/2} dE,$$

where the binding energy $E$ is defined as

$$E = \psi(r) - \frac{1}{2} v_r^2 - \frac{1}{2} v_T^2,$$

and $\psi(r)$ is the relative gravitational potential which can be obtained from the Poisson equation:

$$\frac{1}{r^2} \frac{d}{dr} (r^2 \frac{d\psi}{dr}) = -4\pi G \rho(r).$$

$v_r$ is the radial velocity and $v_T$ is the tangential velocity:

$$v_T = \sqrt{v_\theta^2 + v_\phi^2}.$$
Eddington provided the inversion formula of Eq. (1):

$$F(E) = \frac{1}{2\pi^2 M_{tot}} D_E^2 \int_0^E \frac{\rho(\psi)d\psi}{\sqrt{2(E-\psi)}},$$

(6)

where $D_E^2$ denotes the 2nd order differentiation operator with respect to $E$.

The anisotropy parameter mentioned in the Introduction is defined as:

$$\beta = 1 - \frac{\sigma_T^2}{2\sigma_r^2},$$

(7)

where $\sigma_T^2$ and $\sigma_r^2$ are the tangential and radial velocity dispersion. If $\beta < 0$, the velocity dispersion is tangentially anisotropic. $0 < \beta \leq 1$ correspond to the radially anisotropic cases, and $\beta = 0$, the isotropic case.

3. Distribution Function & Pseudo Phase-Space Density

In this Section, we will calculate the distribution function and pseudo phase-space density with the given density profile in the very central region of the spherically symmetric dark matter halo where the velocity dispersion is almost isotropic. Let us consider a family of density profiles with parameters $a$ and $\epsilon$ named as New generalized NFW profiles:

$$\rho = \frac{1}{(r/r_s)^{\alpha}(1 + r/r_s)^{3+\epsilon-a}}.$$  

(8)

We set the characteristic radius $r_s = 1$, the total mass $M_{tot} = 1$ and the gravitational constant $G = 1$ here. Then the density profiles reduce to

$$\rho = \frac{1}{r^{\alpha}(1 + r)^{3+\epsilon-a}},$$

(9)

$$C = \frac{\Gamma(3-a+\epsilon)}{4\pi\Gamma(3-a)\Gamma(\epsilon)}, \quad a < 3, \epsilon > 0$$

(10)

Then the relative potential $\psi(r)$ can be calculated from the Poisson Eq. (1):

$$\psi(r) = \psi_0 - 4\pi G \rho^{-a} r^{-a-2} (A_1(r) - 2A_2(r) + A_3(r))$$

$$A_1(r) = 2F_1[1-a+\epsilon, -a, 2-a, -r]$$

$$A_2(r) = 2F_1[2-a+\epsilon, -a, 2-a, -r]$$

$$A_3(r) = 2F_1[3-a+\epsilon, -a, 2-a, -r]$$

(11)

where $\psi_0 = \frac{\epsilon}{r^2}$ is the relative gravitational potential at $r = 0$ which is determined by the condition $\psi(r)|_{r=\infty} = 0$. And then, we calculate the asymptotic approximation of $\psi(r)|_{r=0}$ and $\rho(\psi)|_{r=0}$ as below:

$$\psi(r)|_{r=0} = \psi_0 - \frac{4\pi C}{6 - 5a + a^2} r^{2-a}$$

(12)
Following this observation, we can further reduce Eq. (14) in the parameter space and substitute equation (12) into equation (17) to transform

\[ \rho(\psi)|_{r \to 0} = B(\psi_0 - \psi)^{-\frac{a}{\pi \alpha}} \]

\[ B = C(\frac{6 - 5a + a^2}{4\pi C})^{-\frac{a}{\pi \alpha}} \]

From \( \psi(r)|_{r \to 0} \) and \( \rho(\psi)|_{r \to 0} \), we can get the asymptotic approximation of \( F(E)|_{r \to 0} \) when using Eq. (6)

\[ F(E)|_{r \to 0} = -\frac{B\psi^a/(a-2)[(a-2)\psi((a-2)\psi + 4E) + (4 - 4a - 3a^2)(\frac{\psi_0}{\psi} - E)^{a/2-1}E^2A_4(E)]}{4\sqrt{2}(2-a)^2E^{3/2}(E - \psi_0)^2\pi^2} \]

\[ A_4(E) = \text{$_2F_1$}[\frac{1}{2}, \frac{a}{2-a}, \frac{3}{2}, -\frac{E}{\psi_0 - E}]. \]

We should note that this result of distribution function is only valid in the very center of dark matter halo where the velocity dispersion is almost isotropic. In the outer part of the halo, no matter the velocity dispersion is isotropic or not, this result is invalid and even unphysical.

It’s easy to note that, when \( r \to 0 \), the binding energy \( E \) approaches to its maximum value \( \psi_0 \). Following this observation, we can further reduce Eq. (14) in the parameter space \( \frac{2}{3} < a < 2 \) which has covered the result of simulations. Nevertheless, it’s just in this parameter scope that \( F(E) \) in Eq. (14) is physically meaningful:

\[ F(E)|_{E \to \psi_0} = C_1(\psi_0 - E)^{-\frac{a}{\pi \alpha}}, \quad \frac{2}{3} < a < 2 \]

By using equations (2), (13) and (15), we can get the total velocity dispersion in the very center of the halo,

\[ \sigma^2(\psi)|_{\psi \to \psi_0} = \frac{2^{9/2}G_1\pi \psi^{5/2}}{5B}(\psi_0 - \psi)^{-\frac{a}{\pi \alpha}} - \frac{a-a}{\psi_0-\psi} \text{$_2F_1$}[\frac{5}{2}, \frac{1}{2}, -\frac{2}{a-2}, \frac{7}{2}, \frac{\psi}{\psi_0 - \psi}], \]

\[ 2\text{$_2F_1$}[\frac{3}{2}, \frac{1}{2}, -\frac{2}{\pi \alpha}, \frac{\psi}{\psi_0 - \psi} - \psi \to \psi_0] = \frac{5(a-2)}{4(a-1)}(\frac{\psi}{\psi_0 - \psi})^{-\frac{a}{\pi \alpha}} + \frac{15\sqrt{\pi}(-2-\frac{\psi}{\psi_0-\psi})}{8\Gamma(\frac{5}{2}, \frac{\psi}{\psi_0 - \psi})}(\frac{\psi}{\psi_0 - \psi})^{-5/2}. \]

Then, it’s not difficult to simplify equation (16) to the final result:

\[ \sigma^2(\psi)|_{\psi \to \psi_0} \propto \begin{cases} (\psi_0 - \psi), & 2 > a \geq 1, \\ (\psi_0 - \psi)^{-\frac{a}{\pi \alpha}}, & 1 > a > \frac{2}{3}. \end{cases} \]

and substitute equation (12) into equation (17) to transform \( \sigma(\psi) \) into \( \sigma(r) \)

\[ \sigma^2(r)|_{r \to 0} \propto \begin{cases} r^{2-a}, & 2 > a \geq 1, \\ r^a, & 1 > a > \frac{2}{3}. \end{cases} \]
Now, after we know the total velocity dispersion and the density profile in the very center of the halo, the pseudo phase-space density can be calculated directly,

\[
\frac{\rho(r)}{\sigma^3(r)} \propto r^{-\gamma},
\]

\[
\gamma|_{r\to0} = \begin{cases} 
3 - \frac{a}{2}, & 2 > a \geq 1, \\
\frac{5}{2}a, & 1 > a > \frac{3}{5}.
\end{cases}
\]  \hspace{1cm} (19)

The power-law behaviour of the pseudo phase-space density was first found by simulation and analyzed by subsequent theoretic work, but its physical meaning is still unclear. For this consideration, let us compare the distribution function and pseudo phase-space density in the very center of the halo.

We should note that the velocity of particles, which may stably exist in the very center of the halo, approaches 0 with \(r \to 0\). If the velocity term of the binding energy Eq. (3) or the total velocity dispersion \(\sigma^2\) is much smaller than the second term of the RHS of Eq. (12) or at most has the same asymptotic behavior as the second term of RHS of Eq. (12), this condition requires \(a \geq 1\) which can be obtained from Eq. (12) and Eq. (18), then \((\psi_0 - E)|_{r\to0} \propto r^{2-a}\). With this formula and Eq. (15), we can get

\[
F(E)|_{r\to0} \propto r^{-3+\frac{3}{2}}, \ a \geq 1.
\]  \hspace{1cm} (20)

From Eq. (19) and Eq. (20), we see that, if \(2 > a \geq 1\), the distribution function and the pseudo phase space density have the same asymptotic behavior where \(r \to 0\). This result indicates that the "poor-man’s" phase-space density is not so poor but might relate to real phase-space density at least in the very center of the halo.

4. Discussion and Conclusions

N-body simulations show us some properties of the dark matter halo, such as density profile \(\rho(r)\), anisotropy parameter \(\beta(r)\), pseudo phase-space density, and so on. Although the simulation’s resolution is limited, these information should be considered properly in the theoretic analysis in order to make theoretic work more realistic.

In this paper, we investigate the problem of the center of the halo based on the following facts: (1) the existence of universal density profiles such as NFW and Moore, and in particular the New generalized NFW profile used in our work; (2) anisotropy parameter \(\beta \approx 0\) in the center of the halo; (3) the pseudo phase space density follows a power law in radius \(r\); (4) limited resolution in the central region.

We should notice that in this paper all results are only valid in the very center of the halo. From a given density profile, we calculate the asymptotic approximation of the distribution function in the very center of the halo. Then the total velocity dispersion and the pseudo phase-space density are obtained from the distribution function. Eq. (19) shows the relation between the two parameters \(\gamma\) and \(a\). If we set \(\gamma = 1.94\) as some simulations indicate, then \(a\) should be 0.776 which is smaller than NFW’s result. Otherwise, if we set \(a \simeq 1 \sim 1.5\) to adapt NFW profile and Moore’s profile, then \(\gamma \simeq 2.25 \sim 2.5\) which is larger than the simulations’ result. This apparent contradiction between
theoretic result and simulation is not so weird since the resolution in the central region isn’t high enough to describe the very central region of the dark matter halo.

Comparing the distribution function (real phase-space density) with the pseudo phase-space density, we find that they have the same asymptotic behavior if $2 > a \geq 1$. This result indicates that the distribution function and pseudo phase-space density are closely related, even though the assertion is rather premature that the distribution function might have a power-law behavior just as the pseudo phase-space density in the whole range of radius $r$. This interesting suggestion may also give us more confidence in the study of pseudo phase-space density and some new clues to the construction of the distribution function. By the way, some authors also study the pseudo phase-space density in from $\rho/\sigma^3$. It is obvious that our conclusion in this paper is well-founded only if $n = 1$. So, from this point of view, $\rho/\sigma^3$ may be the best choice of the pseudo phase-space density.

For the intensive study of the dark matter halo, the full range of radius $r$ should be involved. We need to consider how to extend the relationship between “real” and “pseudo” phase-space density from the very central region to other regions where the velocity dispersion is anisotropic. Furthermore, reasonable knowledge and new strategies should be introduced and developed for the construction of more realistic models. First, it is admitted that dark matter halos in the real world are not spherically symmetric. Second, the physics in the center of the halo may be very complicated. Third, the halo actually is a polycrystalline system which contains the dark matter, baryonic matter and even a supermassive black hole in the center. It is not radical to say that new methods and ideas are still needed in the future research on this matter.

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