Thermodynamic descriptions of Polytropic gas and its viscous type as the dark energy candidates

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In this paper, at first, we focus on a FRW universe in which the dark energy candidate satisfies the Polytropic equation of state and study thermodynamics of dark energy. Bearing the thermal fluctuation theorem in mind, we establish a relation between the thermal fluctuation of system and mutual interaction between the dark energy and dark matter. Generalization to a viscous Polytropic gas is also investigated. We point to a condition for decaying dark energy candidate into the dark matter needed for alleviating coincidence problem. The effects of dark energy candidates and their interactions with other parts of cosmos on the horizon entropy as well as the second law of thermodynamics are also addressed. Our study signals us to a correction term besides the Bekenstein entropy which carries the information of the dark energy candidate, its interaction with other parts of cosmos and its viscosity.

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I. INTRODUCTION

The dark sectors of cosmos, at least including dark energy and dark matter, are mysterious puzzles for the current theoretical physics. Among these sectors, dark energy is assumed as the generator of the current accelerating phase, and in fact, there are various models introduced for getting a description for the nature of dark energy. It seems that observations permit an interaction between dark sectors of cosmos which increases the hopes to solve the coincidence problem if it leads to decay dark energy into the dark matter.

The mutual interaction between the dark sectors of cosmos may lead to thermal fluctuations into the thermodynamic properties of dark energy candidate, and therefore, one may get expressions for the mutual interaction between the dark sectors of cosmos in various models by investigating the thermal history of universe. Based on this approach, the entropy of interacting dark energy can be expressed as a function of entropy of non-interacting dark energy and its derivatives. The root of this approach comes back to the thermal fluctuation theorem, and indeed, thermal fluctuations theorem is valid in all area of physics such as gravitational systems. It is also shown that this view leads to find thermal fluctuations of a Chaplygin gas model of dark energy, by knowing its mutual interaction form with dark matter, up to the desired order of thermal fluctuations. Finally, we should note here that, in this approach, it is assumed that dark energy candidate satisfies the first law of thermodynamics and moreover, since the major part of cosmos is dark, non-dark sectors of cosmos are neglected from the Friedmann equations.

In another approach, by applying the unified first law of thermodynamics on the trapping or apparent horizon of FRW universe, some authors show that ghost dark energy model of dark energy and its generalization may modify the Bekenstein entropy in the Einstein general relativity framework. More attempts, in which the effects of various models of dark energy and their interactions with other parts of cosmos on the horizon entropy are investigated, can be found in Refs. Such modifications also affect the quality of validity of the second law of thermodynamics and its generalized form. The unknown origin of dark energy candidate is the backbone of this approach permitting authors to consider the possibility of affecting the horizon entropy by the dark energy candidate. Since all sources which constitute the baryon content of the cosmos have contributions to the horizon entropy, unlike the previous approach, one should consider all possible baryonic sources which fill cosmos.

Nowadays, the possibility of modelling the universe expansion history by introducing a candidate for dominated fluid, which satisfies

\[ p = K \rho^{1+\frac{1}{n}}, \]  

where \( p \) and \( \rho \) are the corresponding pressure and energy density, respectively, is investigated. In this equation \( n \)}
and $K$ denote the Polytropic index and Polytropic constant, respectively. It is also shown that, even in the absence of a cosmic background, such fluid may showcase a transition from a pressureless state to a state with $\omega = \frac{p}{\rho} = -1$, where $\omega$ is the state parameter, and vice versa. It seems that its modification and viscous forms may also use to describe the universe expansion. Indeed, this equation of state, known as the Polytropic equation of state, is too familiar in thermodynamics and also for astrophysicists. In addition, some authors make use of the Polytropic equation of state to describe dark energy and thus the current phase of universe expansion as well as to study the thermodynamics of universe in various theories of gravity. More studies, including the numerous properties of a universe in which the Polytropic gas and its various possible modified forms (as the candidates of dark energy) interact with dark matter can be found in .

Based on the above discussions, it is worthy to study the thermodynamics of mutual interaction between the Polytropic gas and its viscous form, as the dark energy candidates, and dark matter in cosmos in order to establish a relation between the interaction term and the thermal history of universe by considering the thermal fluctuation theorem. Moreover, it is also valuable to study the effects of the dark energy candidate and its interaction with other parts of cosmos on the horizon entropy and availability of the second law of thermodynamics.

The paper is organized as follows. In the next section, we consider a universe filled by dark matter and a dark energy which satisfies the Polytropic equation of state and find a relation for the entropy of dark energy, whenever the dark sectors do not interact with each other. In continue, we generalize our study to a case in which dark sectors interact mutually, and by taking into account the thermal fluctuation theorem, we establish a relation between the mutual interaction and thermal fluctuation of system. Generalization to a universe in which a viscous Polytropic gas plays the role of dark energy candidate is presented in section (III). Finally, we derive a condition for mutual interaction between dark sectors which is independent of the equation of state of the dark energy candidate, and should be obeyed by coupling constants of interaction in order to decay dark energy to dark matter. Section (IV), includes the effects of a Polytropic dark energy model and its interaction with other parts of cosmos on the horizon entropy which leads to a correction term for the horizon entropy. In addition, the quality of validity of the second law of thermodynamics is also investigated. Moreover, we also study the effects of a viscous Polytropic dark energy model and its mutual interaction with other parts of cosmos on the horizon entropy in the forth chapter. Our study points two new terms besides the Bekenstein entropy due to the Polytropic dark energy model and its viscosity. The validity of the second law of thermodynamics is also investigated for this model. Section (V) is devoted to a summary and concluding remarks. Throughout this paper we set $c = G = \hbar = 1$ for sake of simplicity.

## II. THERMODYNAMIC DESCRIPTION OF MUTUAL INTERACTION BETWEEN NON-VISCOUS POLYTROPIC GAS AND DARK MATTER

Consider a FRW universe with metric

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right],$$

where $a(t)$ and $k$ denote the scale factor and the curvature parameter, respectively. In this metric, $k = -1, 0, 1$ denotes the open, flat and closed universes, respectively \([65-67]\). The apparent horizon of the FRW universe, as the marginally trapped surface which plays the role of the causal boundary, is evaluated by

$$\partial_\alpha \zeta \partial^\alpha \zeta = 0 \Rightarrow r_A,$$

where $\zeta = a(t) r \text{ [65-67]}$. By simple calculations, One gets

$$\tilde{r}_A = a(t) r_A = \frac{1}{\sqrt{H^2 + \frac{k}{a(t)^2}}},$$

for the apparent horizon radii of the FRW universe \([65-69]\).

Since the major part of cosmos is dark, including dark energy and dark matter \([1]\), we may write the Friedman equation as

$$H^2 = \frac{8\pi}{3} (\rho_m + \rho_D),$$

where $\rho_m$ and $\rho_D$ denote the density profile of a pressureless dark matter and a Polytropic gas as the dark energy candidate, respectively. We also consider a flat FRW universe in which $k = 0 \text{ [1]}$. In a universe, in which the dark sectors do not interact with each other, the energy-momentum conservation law implies

$$\dot{\rho}_m + 3H\rho_m = 0$$

(6)
and

\[ \dot{\rho}_D + 3H(\rho_D + p_D) = 0, \quad (7) \]

where \( p_D \) is the dark energy pressure, and follows Eq. (1). Defining \( \Omega_i = \frac{\rho_i}{\rho_c} \) (where \( \rho_c = \frac{3H^2}{8\pi} \) and \( \rho_i \) refers to \( \rho_m \) and \( \rho_D \)) as a dimensionless parameter, we have the following equation as the Friedman equation

\[ \Omega_D + \Omega_m = 1. \quad (8) \]

For future calculation we also define \( u = \frac{\rho_m}{\rho_D} \) leading to

\[ u = \frac{1 - \Omega_D}{\Omega_D}, \quad (9) \]

where we have used Eq. (8) to obtain this equation.

### A. Thermodynamic description of non-interacting Polytropic gas

Here, we are going to find a relation for the entropy changes of a dark energy candidate, obeying the Polytropic equation of state (1). In order to achieve this goal, taking into account the view in which dark energy candidates satisfy the first law of thermodynamics

\[ TdS_D = dE_D + p_D dV. \quad (10) \]

In this equation \( S_D \) is the entropy of dark energy candidate, \( V_D = \frac{4}{3}\pi r_A^3 \) and \( E_D = \rho_D V \) are the volume of the flat FRW universe and the energy of dark energy, respectively. The temperature of fluids confined by the apparent horizon is equal to the Cai-Kim temperature \[ T = \frac{1}{2\pi r_A}. \quad (11) \]

Bearing Eq. (11) in mind, this equation leads to

\[ T = \frac{H}{2\pi}. \quad (12) \]

for a flat background (\( k = 0 \)). Since

\[ dE_D = \rho_D dV + V d\rho_D, \quad (13) \]

and

\[ dV = 4\pi(\dot{r}_A)^2 d\dot{r}_A = -4\pi H^{-3} dH, \quad (14) \]

Eq. (10) becomes

\[ dS_D = \frac{2\pi}{H} ((\rho_D + p_D) dV + V d\rho_D), \quad (15) \]

which yields

\[ \frac{dS_D^0}{dH_0} = \frac{2\pi}{H_0^3} (1 + K(\rho_D^0)^{1/n}) \left( -\frac{3}{2} \Omega_D^0 + \frac{1}{\rho_D^0} + K(\rho_D^0)^{1/n} \right), \quad (16) \]

where we have used Eq. (7) to get the above equation and we have also used the following equation

\[ \frac{d\rho_D^0}{dH_0} = \frac{-3H_0(\rho_D^0 + K(\rho_D^0)^{1+\frac{1}{n}})}{-4\pi(1 + u^0 + K(\rho_D^0)^{\frac{1}{n}})}. \quad (17) \]

In order to obtain this equation, use Eqs. (5) and (7) to get

\[ \dot{H}_0 = -4\pi p_D^0 (1 + u^0 + K(\rho_D^0)^{\frac{1}{n}}) \quad (18) \]
which is Rechaudhury equation and

$$\dot{\rho}_D^0 = -3H_0\dot{\rho}_D^0(\rho_D^0 + K(\rho_D^0)^{\frac{1}{n+1}}),$$

(19)

respectively. Now, by inserting these results into the

$$d\rho_D^0 = \frac{\dot{\rho}_D^0}{H_0}dH_0$$

(20)

equation one can reach Eq. (17). In the above formulas, the subscript/superscript (0) points to a universe in which dark sectors do not interact with each other. Loosely speaking, Eq. (16) gives us a relation for the entropy changes of dark energy candidate, satisfying the Polytropic equation of state [11], in the absence of an interaction with dark matter candidate.

**B. Thermodynamic description of interacting Polytropic gas**

For a universe in which dark sectors interact with each other, the energy-momentum conservation law leads to

$$\dot{\rho}_m + 3H\rho_m = Q,$$

(21)

and

$$\dot{\rho}_D + 3H(\rho_D + p_D) = -Q,$$

(22)

where $Q$ denotes the mutual interaction between the dark sectors of cosmos. Therefore, $Q > 0$ and $Q < 0$ mean dark energy is decaying into dark matter and vice versa, respectively. In this manner, using Eqs. (22) and (5) to get

$$d\rho_D = \frac{-Q - 3H(\rho_D + K(\rho_D)^{\frac{1}{n+1}})}{-4\pi(1 + u + K\rho_D^0)}dH.$$  

(23)

Now, following approach which leads to Eq. (16), to reach

$$\frac{dS_D}{dH} = \frac{2\pi}{H^3(1 + K\rho_D^0)} \left[ -\frac{3}{2} \Omega_D + \frac{1}{\Omega_D + K\rho_D^0} + \frac{Q}{3H(\Omega_D + K\rho_D^0)} \right],$$

(24)

as a relation for the entropy changes in an interacting universe. It is apparent that, as an appropriate limit, in the absence of interaction term ($Q$), this relation converges to the previous result obtained in Eq. (16). It is also useful to note here that in such a universe the Hubble parameter is $H$ which differs from the Hubble parameter of the previous non-interacting universe ($H_0$). Since an interaction between dark sectors is allowed from observational point of view which may lead to solve the coincidence problem [9–23], various interactions between the Polytropic model of dark energy and dark matter has been studied in vast articles [55, 57, 59, 61, 64].

**C. Thermodynamic description of mutual interaction**

In this subsection, we want to find a thermodynamical description for a mutual interaction between the dark sectors of cosmos. In fact, a weak interaction may leave fluctuations into the thermal properties of systems [29]. Due to this hypothesis, the entropy of interacting dark energy ($S_D$) can be expressed as the non-interacting system parameters as [18, 24, 28, 33]:

$$S_D = S_D^0 + S_D^1 + S_D^2$$

(25)

where $S_D^0$ is the entropy of non-interacting dark energy candidate which obeys Eq. (16). Moreover, $S_D^1$ is evaluated as

$$S_D^1 = -\frac{1}{2} \ln CT_0^2,$$  

(26)
in which

\[ T_0 = \frac{H_0}{2\pi} \]  

and

\[ C = T_0 \frac{\partial S_D^0}{\partial T_0} = 2\pi T_0 \frac{dS_D^0}{dH_0} \]

are the temperature and dimensionless heat capacity of non-interacting dark energy candidate, respectively. \( S_D^2 \) also includes higher order fluctuations [18, 24–30, 35]. After some algebra we get

\[ C = \frac{2\pi}{4\pi^2 T_0^2} \left(1 + K(\rho_D)^{\frac{1}{n}}\right) \left\{-\frac{3}{2} \Omega_D^0 + \frac{1}{\Omega_D^0 + K(\rho_D^{\frac{1}{n}})}\right\}, \]

which leads to

\[ S_D^1 = -\frac{1}{2} \ln \left[ \left(1 + K(\rho_D)^{\frac{1}{n}}\right) \left\{-\frac{3}{2} \Omega_D^0 + \frac{1}{\Omega_D^0 + K(\rho_D^{\frac{1}{n}})}\right\} \right], \]  

and

\[ \frac{dS_D^1}{dH_0} = \frac{dS_D^{(1)}}{d\rho_D} \frac{d\rho_D}{dH_0} \]

where we have used Eqs. (28) and (26) to obtain these relations. It is also useful to mention here that \( \frac{d\rho_D}{dH_0} \) can be found in Eq. (17). Now, using Eq. (25) to reach

\[ \frac{dS_D^2}{dH_0} = \frac{dS_D^0}{dH_0} \frac{dH_0}{dH} + \frac{dS_D^1}{dH_0} \frac{dH_0}{dH}, \]

in which \( \frac{dH_0}{dH} \) can be evaluated as

\[ \frac{dH_0}{dH} = \frac{H_0}{H} = \frac{\rho_D^0(1 + u^0 + K(\rho_D)^{\frac{1}{n}})}{\rho_D(1 + u + K\rho_D^{\frac{1}{n}})}. \]

Finally, by inserting Eqs. (16), (24) and (31) into Eq. (32) and using Eq. (17) together with Eq. (33), one can find a relation for the \( S_D^2 \) term. Therefore, fluctuation theorem helps us establish a relation between thermal fluctuations of dark energy entropy and the mutual interaction between the dark sides of cosmos. Since the \( S_D^2 \) term is negligible, one may neglect from this term and write [18, 24, 30, 35]

\[ \frac{dS_D^2}{dH} \approx \left( \frac{dS_D^0}{dH_0} + \frac{dS_D^1}{dH_0} \right) \frac{dH_0}{dH}, \]

in which \( \frac{dH_0}{dH} \) follows Eq. (33). Now, by inserting Eqs. (16), (24) and (31) into this equation one may find a relation for \( Q \) up to the first order of fluctuations. Roughly speaking, this relation helps us get a thermodynamic interpretation for the mutual interaction between the dark sectors of cosmos, up to the first order of fluctuations.
III. THERMODYNAMIC DESCRIPTION OF MUTUAL INTERACTION BETWEEN VISCOUS POLYTROPIC GAS AND DARK MATTER

In this section we generalize our debates, expressed in the previous section, to a viscous Polytropic gas which satisfies the equation of state, where $\xi$ is viscous coefficient \[55\]. Bearing this equation as well as the Friedman equation (5) in mind, since for a non-interacting viscous Polytropic gas Eqs. (6) and (7) are valid, by following the approaches yield Eqs. (16) and (17) one gets

\[
dS_D^0 = \frac{2\pi}{H_0^3} \left\{ \frac{1 + K(\rho_D^0)^{\frac{1}{n}}}{(\Omega_D^0)^{-1} + K(\rho_D^0)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D^0}} - \frac{3}{2} \Omega_D^0 \left( 1 + K(\rho_D^0)^{\frac{1}{n}} + \frac{3\xi H}{\rho_D^0} \right) \right\} dH_0, \tag{36}
\]

and

\[
\frac{d\rho_D}{dH_0} = \frac{-3H_0 \left( 1 + K(\rho_D^0)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D^0} \right)}{-4\pi \left\{ \frac{1}{\Omega_D^0} + K(\rho_D^0)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D^0} \right\}}. \tag{37}
\]

respectively. It is obvious that these equations cover previous results (Eqs. (16) and (17)), in the appropriate limit $\xi \rightarrow 0$.

**Thermodynamic description of interacting viscous Polytropic gas**

When there is an interaction, we have \[55\]

\[
\dot{\rho}_m + 3H\rho_m = Q, \tag{38}
\]

and

\[
\dot{\rho}_D + 3H(\rho_D + p_D) = -Q, \tag{39}
\]

where $Q$ again denotes the mutual interaction between the dark sectors of cosmos. Now, by using Eqs. (5) and (35) we get

\[
\dot{H} = -4\pi \rho_D \left( 1 + u + K \rho_D^{\frac{1}{n}} - \frac{3\xi H}{\rho_D} \right), \tag{40}
\]

and

\[
\dot{\rho}_D = -3H\rho_D \left\{ 1 + K(\rho_D)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D} \right\} - Q, \tag{41}
\]

respectively. By combining these equations with each other, one gets

\[
\frac{d\rho_D}{dH} = \frac{3H\rho_D \left\{ 1 + K(\rho_D)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D} \right\} + Q}{4\pi \rho_D \left\{ \frac{1}{\Omega_D^0} + K(\rho_D)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D} \right\}}. \tag{42}
\]

Bearing the approach which leads to Eq. (16) in mind, by some calculations we get

\[
\frac{dS_D}{dH} = \frac{2\pi}{H^3} \left\{ \frac{1 + K(\rho_D)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D}}{(\Omega_D)^{-1} + K(\rho_D)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D}} - \frac{3}{2} \Omega_D \left( 1 + K(\rho_D)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D} \right) \right\} dH_0, \tag{43}
\]

\[
+ \frac{Q}{3H\rho_D \left( 1 + u + K(\rho_D)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D} \right) (1 + K(\rho_D)^{\frac{1}{n}})} \left\{ \frac{1}{\Omega_D^0} + K(\rho_D)^{\frac{1}{n}} - \frac{3\xi H}{\rho_D} \right\},
\]

\]
as a relation for the entropy changes of an interacting viscous Polytropic dark energy model confined by the apparent horizon. It is obvious that in the $Q \to 0$ limit Eq. (36) is covered, while Eq. (16) is governed by taking the $Q \to 0$ and $\xi \to 0$ limits simultaneously. It is also useful to mention that the result of non-viscous interacting case (24) is obtainable by imposing the $\xi \to 0$ limit to this equation. Using Eqs. (28) and (29) to get

$$C = \frac{1}{2} \pi T_0 \left\{ \frac{1 + K(\rho_D^0)^{\frac{\nu}{\rho_D}} - \frac{3\xi H_0}{\rho_D}}{(\Omega_0^{\nu})^{-1} + K(\rho_D^0)^{\frac{\nu}{\rho_D}}} - \frac{3}{2} \Omega_0^0 \left( 1 + K(\rho_D^0)^{\frac{\nu}{\rho_D}} - \frac{3\xi H_0}{\rho_D} \right) \right\},$$

(44)

and

$$S_D^1 = -\frac{1}{2} \ln \left\{ \frac{1}{2\pi} \left( \frac{1 + K(\rho_D^0)^{\frac{\nu}{\rho_D}} - \frac{3\xi H_0}{\rho_D}}{(\Omega_0^{\nu})^{-1} + K(\rho_D^0)^{\frac{\nu}{\rho_D}}} - \frac{3}{2} \Omega_0^0 \left( 1 + K(\rho_D^0)^{\frac{\nu}{\rho_D}} - \frac{3\xi H_0}{\rho_D} \right) \right) \right\},$$

(45)

for the dimensionless heat capacity and the first order term of thermal fluctuations of dark energy candidate, respectively. The latter leads to

$$\frac{dS_D^1}{dH_0} = \frac{d\rho_D^0}{dH_0} \times \frac{-1}{2} \times \left\{ \frac{K(\rho_D^0)^{\frac{\nu}{\rho_D}}} {n(\rho_D^0)^{\frac{\nu}{\rho_D}}} + \frac{3\xi H_0}{\rho_D} \right\}$$

$$\frac{1 + K(\rho_D^0)^{\frac{\nu}{\rho_D}} - \frac{3\xi H_0}{\rho_D}} {n(\rho_D^0)^{\frac{\nu}{\rho_D}}} - \frac{3}{2} \Omega_0^0 \left( 1 + K(\rho_D^0)^{\frac{\nu}{\rho_D}} - \frac{3\xi H_0}{\rho_D} \right),$$

(46)

where $\frac{d\rho_D^0}{dH_0}$ can be found in Eq. (37). Bearing the thermal fluctuations theory (26) in mind, by using Eqs. (10), (18) together with (30) we get (31)

$$\frac{dS_D^2}{dH_0} = \frac{dS_D}{dH_0} \frac{dH_0}{dH} = \frac{dS_D}{dH_0} \frac{dH_0}{dH},$$

(47)

where

$$\frac{dH_0}{dH} = \frac{\dot{H}_0}{H} = \frac{\rho_D^0 (1 + u^0 + K(\rho_D^0)^{\frac{\nu}{\rho_D}} - \frac{3\xi H_0}{\rho_D})}{1 + u + K(\rho_D^0)^{\frac{\nu}{\rho_D}} - \frac{3\xi H_0}{\rho_D}},$$

(48)

for the higher order terms of thermal fluctuations of the dark energy candidate entropy and an expression for the relation between the Hubble parameters in the interacting and non-interacting universes, respectively. Moreover, if one only considers the $S_D^0$ and $S_D^1$ terms she can find a relation for the mutual interaction between dark sectors up to the first order of thermal fluctuations, by inserting Eqs. (40), (43), (36) and (48) into relation (24, 28, 31)

$$\frac{dS}{dH} \approx \left( \frac{dS_D^0}{dH_0} + \frac{dS_D^1}{dH_0} \right) \frac{dH_0}{dH},$$

(49)

In the above equations, the subscript and superscript (0) point to the non-interacting viscous case. Therefore, by relating the mutual interaction between dark sectors of cosmos to the thermal fluctuation theory, we could find a relation for the mutual interaction up to the first order of fluctuations together with a relation for the higher order of fluctuations ($S_D^1$).
Coincidence problem and mutual interaction

The most general form of the mutual interaction between the dark energy candidate, either satisfying the Polytropic equation state (Equation 1) or its viscous form (Equation 35), and the pressureless dark matter can be written as \[\text{(50)}\]

\[ Q = 3b_1 H \rho_D + 3b_2 H \rho_m, \]

where \(b_1\) and \(b_2\) are coupling constants of interaction. In order to solve the coincidence problem, interaction term should satisfy the \(Q > 0\) condition \[\text{(28, 31)}\] which leads to

\[ \frac{b_1}{b_2} > - \frac{1 - \Omega_D}{\Omega_D} = - u, \]

(51)

where we have used the \(\Omega_i\) and \(u\) definitions together with Eq. (9) to obtain this equation. It is worth to mention here that since in the current era of universe expansion the density of dark energy is approximately 3 times larger than that of dark matter \[\text{(1)}, u \approx \frac{1}{3}\] which means that the \(b_1 > - \frac{b_2}{3}\) condition should be met by the interaction coupling constants.

IV. DARK ENERGY CANDIDATE MAY MODIFY THE HORIZON ENTROPY

Here, we are going to study the thermodynamics of apparent horizon in a universe in which the dark energy candidate obeys either the Polytropic equation of state (Equation 1) or its viscous form (Equation 35). In order to achieve this goal, we try to find an expression for the horizon entropy by bearing in mind this fact that since the nature of DE candidate may be not similar to other parts of cosmos, it may affect the horizon entropy \[\text{(33–37)}\]. We also use the Cal-Kim approach in which the volume changes of universe in the infinitesimal time \(dt\) can be neglected \((dV \approx 0)\) \[\text{(70)}\]. Finally, we point to the second law of thermodynamics and thus a required condition for availability of the second law of thermodynamics \[\text{(72)}\].

A. Non-interacting and non-viscous Polytropic model

Since we want to calculate the entropy of horizon, we take into account the various parts of cosmos constitutes. Therefore, consider a FRW universe with Friedman equation

\[ \frac{1}{r^2} = \frac{8 \pi}{3} (\rho + \rho_d), \]

(52)

in which we have used Eq. (4), and

\[ \rho = \rho_{cdm} + \rho_{wdm} + \ldots \]

(53)

includes the energy density of everything filling the universe, such as the cold and worm dark matters and etc., except the dark energy candidate. In the non-interacting universe the energy-momentum conservation law implies

\[ \dot{\rho} + 3H(\rho + p) = 0, \]

(54)

and

\[ \dot{\rho}_D + 3H(\rho_D + p_D) = 0, \]

(55)

where \(p\) and \(p_D\) denote the corresponding pressure. The projection of total energy-momentum tensor \((T_{\mu\nu})\) on the normal direction of the two-dimensional sphere with radii \(\zeta\) and the energy flux crossing this sphere read as \[\text{(71)}\]

\[ \psi_a = T^b_a \partial_b \zeta + W \partial_a \zeta, \]

(56)

and

\[ \delta Q^m = -AH\zeta(\frac{\rho + p}{2})dt + Aa(\frac{\rho + p}{2})dr, \]

(57)
respectively. In this equation, \( \delta Q^m \) denotes the energy amount crossing two-dimensional hypersurface with radii \( \zeta \) during the universe expansion. Since \( \zeta = ar \), the latter can be rewritten as

\[
\delta Q^m = - \frac{3V(\rho + p)H}{2} dt + \frac{A(\rho + p)}{2} (d\zeta - \zeta H dt).
\]  

(58)

In the Cai-Kim approach to get the horizon entropy \([70]\), the temperature of fluids confined by the apparent horizon reads Eq. (11) and the \( d\zeta \approx 0 \) approximation is used in the infinitesimal time \( dt \). Therefore, Eq. (58) takes the

\[
\delta Q^m = -(\frac{3V}{2} + \frac{A\zeta}{2})(H(\rho + p)dt),
\]  

(59)

form. Moreover, since \( A\zeta = 3V \), we obtain

\[
\delta Q^m = -3VH(\rho + p)dt = Vd\rho,
\]  

(60)

where we have used Eq. (54) to get the last equality. Bearing the Clausius relation in mind \([70]\), we get

\[
dS_A \equiv -\frac{\delta Q^m}{T} = -\frac{V}{T}d\rho.
\]  

(61)

Now, taking differentiation from Eq. (52), and inserting the results into this equation to reach

\[
dS_A = (2\pi \tilde{r}_A + \frac{8\pi^2}{3} \tilde{r}_A^4 \rho_D')(d\tilde{r}_A),
\]  

(62)

where \( \rho_D' = \frac{d\rho_D}{\tilde{r}_A} \). After integration we get

\[
S_A = \frac{A}{4} - 8\pi^2 \int \frac{H\rho_D(1 + K\rho_D^{1/3})}{(H^2 + \frac{1}{\sigma^2})^2} dt,
\]  

(63)

where we have used Eq. (1) and \( A = 4\pi \tilde{r}_A^2 \) together with Eqs. (1) and (55) to obtain this equation. We have also set the integration constant \( (S_0) \) to zero. This equation is in full agreement with previous study about the effect of a dark energy candidate with varying density profile on the horizon entropy in the FLRW universe with curvature \( k \) \([37]\). For a flat background \( (k = 0) \), this equation is moderated to

\[
S_A = \frac{A}{4} - 8\pi^2 \int \frac{\rho_D(1 + K\rho_D^{1/3})}{H^3} dt,
\]  

(64)

which is in full agreement with previous studies about the effects of dark energy candidate with varying energy density on the horizon entropy in a flat background \([35]\).

Second law of thermodynamics

The second law of thermodynamics states that the horizon entropy as the total entropy of a gravitational system should obey the \( \frac{dS_A}{dt} \geq 0 \) condition \([72]\). By combining Eqs. (55) and (60), we reach

\[
\frac{dS_A}{dt} = \frac{3V(\rho + p)}{T}.
\]  

(65)

Therefore the second law of thermodynamics is satisfied whenever, the \( \rho + p \geq 0 \) condition is met. The same result is previously reported for some other dark energy candidates \([35, 37]\).

B. Interacting non-viscous Polytropic model

Bearing Eqs. (52) and (53) in mind, whiles the cosmos sectors interact with each other, the energy-momentum conservation law is written as

\[
\dot{\rho} + 3H(\rho + p) = Q,
\]  

(66)
\[ \dot{\rho}_D + 3H(\rho_D + p_D) = -Q. \] 
\eqn{10}

Now, taking into account the approach of section \([IV\ A]\) and using \eqn{66} to get
\[ \delta Q_m = V(d\rho - Q dt), \]
\eqn{10}

where \(\delta Q^m\) is again the energy amount crossing the boundary. Finally, using Eq. \eqn{52} together with Clausius equation \((TdS_A = -\delta Q^{m})\) to get
\[ dS_A = 2\pi \tilde{r}_A d\tilde{r}_A + \frac{8\pi^2}{3} \tilde{r}_A^3 (d\rho_D + Q dt), \]
\eqn{10}

which leads to
\[ S_A = \frac{A}{4} - \frac{8\pi^2}{3} \int \tilde{r}_A^3 (d\rho_D + Q dt). \]
\eqn{10}

This equation shows that how the mutual interaction between the Polytropic gas (as the dark energy candidate) and other parts of cosmos affects the horizon entropy, which is fully consistent with previous studies \[35, 37\]. By using Eq. \eqn{67}, this equation can be rewritten as
\[ S_A = \frac{A}{4} - 8\pi^2 \int \frac{H\rho_D(1 + K\rho_D^{1/3})}{(H^2 + \frac{k}{a^2})^2} dt, \]
\eqn{10}

which is similar to the previous result \eqn{63}. Indeed, we should note here that although this equation is similar to Eq. \eqn{63} but due to the interaction term \((Q)\) they differ from each other. The identical result is obtained for other models of dark energy \[35, 37\].

### Second law of thermodynamics

In order to investigate the validity of second law of thermodynamics, combine Eqs. \eqn{66}, \eqn{68} and Clausius relation to obtain
\[ \frac{dS_A}{dt} = 3HV \frac{\rho}{T}(\rho + p), \]
\eqn{10}

which claims that, just the same as the non-interacting case, the second law of thermodynamics is met if the \(\rho + p \geq 0\) condition is satisfied. This result is again in accordance with previous studies about other models of dark energy \[35, 37\].

### C. Non-interacting viscous Polytropic model

In order to get an expression for the horizon entropy, since dark energy candidate does not interact with other parts of cosmos, following the approach of Sec. \([IV\ A]\) to reach
\[ dS_A = 2\pi \tilde{r}_A d\tilde{r}_A + \frac{8\pi^2}{3} \tilde{r}_A^3 d\rho_D. \]
\eqn{10}

It is useful to note here that although, since there is no interaction between the cosmos sectors, this result is similar to Eq. \eqn{69}, but due to the viscosity, the result of this equation differs from that of \eqn{69}. In order to make the difference between this equation and \eqn{69} clear and find the viscosity effect on the horizon entropy, bearing Eq. \eqn{35} in mind, and using Eq. \eqn{55} to get
\[ S_A = \frac{A}{4} - 8\pi^2 \int \frac{H\rho_D(1 + K\rho_D^{1/3})}{(H^2 + \frac{k}{a^2})^2} dt - 24\pi^2 \int \frac{\xi H^2}{(H^2 + \frac{k}{a^2})^2} dt. \]
\eqn{10}

Therefore, the third term at the RHS of this equation is a criterion for the effect of the viscosity of dark energy candidate on the horizon entropy. Since the third term at the RHS of this equation depends only on the viscosity coefficient \(\xi\) and the spacetime parameter, including the Hubble parameter, scale factor and the curvature constant \(k\), this term is identical for all dark energy candidates with the viscosity modification \(-3\xi H\) to the pressure of dark energy candidate. It is apparent that, in the \(\xi \to 0\) limit, the result of Sec. \([IV\ A]\), as a desired result, is obtainable. It is also easy to check that the second law of thermodynamics is satisfied if the \(\rho + p \geq 0\) condition is met.
D. Interacting viscous Polytropic model

In the interacting cosmos case, by following the approach used in Sec. IVB, we get
\[ dS_A = 2\pi \tilde{r}_A d\tilde{r}_A + \frac{8\pi^2}{3} \rho_D^2 (d\rho_D + Q dt), \]
which leads to
\[ S_A = \frac{A}{4} - 8\pi^2 \int \frac{H\rho_D (1 + K\rho_D^2)}{(H^2 + K\rho_D^2)^2} dt - 24\pi^2 \int \frac{\xi H^2}{(H^2 + \frac{\rho_D}{\rho_B})^2} dt, \]
where we have used Eqs. (35) and (39) to obtain this equation. It is useful to note here that although this equation is similar to Eq. (74), but they are completely different from each other. It is due to this fact that due to the interaction \( Q \), the profile density \( \rho_D \) and pressure \( p_D \) of dark energy candidate differ from their values in a non-interacting universe \( (Q = 0) \). It is also easy to check that the second law of thermodynamics is met whenever the \( \rho + p \geq 0 \) condition is satisfied. Finally, we should mention that the results of non-viscous and non-interacting cosmoses are recovered by inserting the \( \xi = 0 \) and \( Q = 0 \) conditions, respectively.

V. SUMMARY AND CONCLUDING REMARKS

Throughout this paper, we tried to investigate the various thermodynamical aspects of a dark energy candidate which either satisfies the Polytropic equation of state \( 1 \) or its viscous type \( 35 \). We also adopted the Cai-Kim temperature \( 12 \) to the horizon and fields which fill the universe \( 70 \). Moreover, since the major part of cosmos is dark and the current dynamics of universe is completely compatible with its dark sectors contents, we omit the contribution of non-dark parts to the Friedmann equation in the second and third chapters. In section (II), we considered a situation in which dark energy candidate, enclosed by apparent horizon, satisfies the Polytropic equation of state \( 1 \) as well as the first law of thermodynamics. In continue, we could get a relation for the entropy changes of the dark energy candidate \( 16 \), whiles dark sectors do not interact with each other. In addition, we generalized our calculations to a case in which there is a mutual interaction between dark sectors, and found a relation for the entropy changes \( 24 \). Thereinafter, by using the thermal fluctuation theorem \( 25 \), we could get a relation for higher order thermal fluctuations of the entropy of dark energy candidate \( 42 \). Finally, we showed that if one neglects the higher order terms, she can get a relation for the mutual interaction between the dark sectors of cosmos based on the thermal fluctuations of dark energy candidate \( 31 \). Indeed, we established a relation between the mutual interaction between the dark sectors of cosmos and thermal fluctuations of dark energy, which leads to a thermodynamical interpretation for the mutual interaction between the dark sectors of cosmos, meaning that one may find a relation for such interaction by following the thermal history of dark energy candidate. In section (III), we have generalized our debates to a universe in which dark energy candidate satisfies the viscous Polytropic equation of state \( 35 \), and followed the recipe used in the second section to get the thermodynamical description for the non-interacting viscous Polytropic dark energy \( 30 \), interacting viscous Polytropic dark energy \( 13 \) as well as the higher order thermal fluctuations \( 47 \) and mutual interaction between the dark sectors \( 19 \). We have also found a relation between the Hubble parameters in interacting and non-interacting universes for both of the viscous \( 48 \) and non-viscous Polytropic \( 33 \) models of dark energy. Finally, we have considered the general form of mutual interaction between dark sectors and got a condition which should be satisfied by the coupling constant of interaction term in order to alleviate the coincidence problem \( 51 \). In the forth chapter, we have taken into account the view in which the dark energy candidate may modify the horizon entropy \( 33 \) as well as the Cai-Kim approaches \( 70 \) to get the Friedmann equations by applying the thermodynamics laws on the apparent horizon. Since all of the baryonic contents of cosmos may have contribution to the total entropy and thus the horizon entropy, unlike the second and third sections, we have tired to consider all of the possible sources, which fill the cosmos, in the Friedmann equation in our study in section four. Our study shows that horizon entropy may be modified by dark energy candidate and its interaction with other parts of cosmos. As it is clear from our calculations, the contribution of dark energy candidate to the horizon entropy, irrespective of the existence or absence of mutual interaction between the dark sectors, can be written in the \(-8\pi^2 \int \frac{H(\rho + p)}{(H^2 + \rho_B)} dt \) form where the dark energy candidate either satisfies the Polytropic equation of state \( 1 \) or its viscous type \( 35 \). Loosely speaking, it is a term besides the Bekenstein relation for the horizon entropy, and is in full agreement with the previous studies including the effects of various dark energy models on the horizon entropy in vast theories of gravity \( 33 \) \( 37 \). It is also useful to note here that the third term in the RHS of Eqs. (73) and (70) point to the probable effects of viscosity of dark energy candidate on the horizon entropy in both interacting and non-interacting cosmoses. Finally,
we investigated the validity of the second law of thermodynamics for the corrected entropy and find that whenever the $\rho + p \geq 0$ condition is met, the second law of thermodynamics is also obeyed. The latter remark is independent of the existence or absence of mutual interaction between the dark sectors and the equation of state satisfied by the dark energy candidate.

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