Simple rules of functional integration in the
Schwarzian theory: SYK correlators

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Abstract: We derive the general rules of functional integration in the theories of the
Schwarzian type, and evaluate explicitly the functional integrals assigning correlation func-
tions in the SYK model.

Keywords: SYK model, Schwarzian theory, correlation functions, quasi-invariant mea-
sures

ArXiv ePrint: 1811.11863v2
1 Introduction

The appearance of a dynamics common to different physical models in different fields may indicate the existence of a hidden symmetry behind it. It has recently become clear that a quantum mechanical model of Majorana fermions with a random interaction (Sachdev-Ye-Kitaev model), the holographic description of the Jackiw-Teitelboim dilaton gravity, open string theory and some other models lead to the same effective theory with the Schwarzian action
\[
I_{Schw} = \left. -\frac{1}{\sigma^2} \int_{S^1} \left[ S_{\varphi}(t) + 2\pi^2 \left( \varphi'(t) \right)^2 \right] dt, \right. 
\]
(1.1)
where \( S^1 \) is the unit circle, and
\[
S_{\varphi}(t) = \left( \frac{\varphi''(t)}{\varphi'(t)} \right)' - \frac{1}{2} \left( \frac{\varphi''(t)}{\varphi'(t)} \right)^2
\]
(1.2)
is the Schwarzian derivative.

The list of the papers includes (but is not confined to) [1] - [46], and it will not rest here.

In some approximations, these physical models appear to be reparametrization invariant. And \( SL(2, \mathbb{R}) \)-invariant action (1.1) inherits the emergent reparametrization symmetry.
Let us remind briefly how the Schwarzian action originates in the SYK model and in 2D JT gravity. Functional integrals in the SYK model can be rewritten in terms of the bilocal bosonic functions $\tilde{G}$ and $\tilde{\Sigma}$ with the effective bosonic action $A_{eff}(\tilde{G}, \tilde{\Sigma})$

$$\int F(\tilde{G}, \tilde{\Sigma}) \exp\{-A_{eff}(\tilde{G}, \tilde{\Sigma})\} \, d\tilde{G} \, d\tilde{\Sigma}.$$ 

The saddle points of the infrared essential part of the effective bosonic action $A_{eff}$

$$G_f(t_1, t_2) = \frac{(f'(t_1)f'(t_2))^\frac{1}{4}}{|f(t_2) - f(t_1)|^\frac{1}{2}} \tag{1.3}$$

form the space $\mathcal{F}$ of one-time differentiable functions $f$ on the unit circle.

The rest (infrared inessential) part of the effective bosonic action $A_{eff}$ results (for three-time differentiable $f$) in the factor

$$\exp\left\{\frac{1}{\sigma^2} \int_{S^1} \left[ \left(\frac{f''(t)}{f'(t)}\right)' - \frac{1}{2} \left(\frac{f''(t)}{f'(t)}\right)^2 \right] \, dt \right\} \tag{1.4}$$

in the integrand.

Thus the averaged saddle point solution (or two-point SYK correlation function) has the form

$$<G_f(t_1, t_2)> = \int_{\mathcal{F}} \frac{(f'(t_1)f'(t_2))^\frac{1}{4}}{|f(t_2) - f(t_1)|^\frac{1}{2}} \exp\left\{\frac{1}{\sigma^2} \int_{S^1} S_f(t) \, dt \right\} \, df.$$ 

The action of Jackiw-Teitelboim gravity is

$$A_{JT} = \text{Topological term} - \alpha \left( \int d^2x \sqrt{g} \left( R + 2 \right) + 2 \int_{\partial} \sqrt{g} \phi K \right), \tag{1.5}$$

where $\phi$ is the dilaton field and $K$ is the extrinsic curvature. While the equation of motion for dilaton demands the space to be $AdS_2$, variations of the closed boundary curve with the proper boundary condition for $\phi$ transform the last term in the action into

$$\frac{1}{\sigma^2} \int_{S^1} S_f(t) \, dt,$$

where $t$ is a parameter of the closed boundary curve.

Although one can use the invariance of the Schwarzian theory to link it to another theory where the corresponding calculations are much simpler than in the original one [23] - [27], [50], [51], we feel that a special technique of functional integration is needed in the Schwarzian theory.

In [48], [49], we proposed to consider functional integrals in the theory with the Euclidean action (1.1) as the integrals with the measure

$$\mu_\sigma(d\phi) = \exp\left\{\frac{1}{\sigma^2} \int_{S^1} S_\phi(t) \, dt \right\} \, d\phi \tag{1.6}$$
on the group of diffeomorphisms $Diff^1_+(S^1)$.

To evaluate functional integrals assigning correlation functions, it is convenient to integrate over the group $Diff^1_+([0,1])$ (with the ends of the interval glued) instead of the integration over the group $Diff^1_+(S^1)$. After fixing a point $t = 0$ on the circle of unit length $S^1$, the integral $\int_{S^1}$ is written as $\int_0^1$.

The measure on the group of diffeomorphisms of the interval $Diff^1_+([0,1])$ ($\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(t) > 0$) $[52]-[54]

$$\mu_\sigma(d\varphi) = \frac{1}{\sqrt{\varphi'(0)\varphi'(1)}} \exp \left\{ \frac{1}{\sigma^2} \left[ \frac{\varphi''(0)}{\varphi'(0)} - \frac{\varphi''(1)}{\varphi'(1)} \right] \right\} \exp \left\{ \frac{1}{\sigma^2} \int_0^1 S_\varphi(t) \, dt \right\} \, d\varphi \quad (1.7)$$

is simplified when we glue the ends of the interval and put $\varphi'(0) = \varphi'(1)$.

Now, the integral over $Diff^1_+([0,1])$ turns into the integral over $Diff^1_+([0,1])$ as follows $[49]:$

$$\frac{1}{\sqrt{2\pi\sigma}} \int_{Diff^1_+([0,1])} F(\varphi) \, \mu_\sigma(d\varphi) = \int_{Diff^1_+([0,1])} \delta \left( \frac{\varphi'(1)}{\varphi'(0)} - 1 \right) F(\varphi) \, \mu_\sigma(d\varphi). \quad (1.8)$$

The measure $\mu_\sigma(d\varphi)$ is generated by the Wiener measure under some special substitution of variables $[52]-[54]$ (see also $[55]$). Consider a continuous function on the interval $[0, 1] \, \xi(t)$ satisfying the boundary condition $\xi(0) = 0$ ($\xi \in C_0([0, 1])$). Under the substitution

$$\varphi(t) = \int_0^t \frac{\xi(t)}{1 - e^{\xi(n)} \, d\eta}, \quad \xi(t) = \log \varphi'(t) - \log \varphi'(0), \quad (1.9)$$

the measure $\mu_\sigma(d\varphi)$ on the group $Diff^1_+([0, 1])$ turns into the Wiener measure $w_\sigma(d\xi)$ on $C_0([0, 1])$. In this way, we get the following equality of functional integrals

$$\int_{Diff^1_+([0,1])} F(\varphi) \, \mu_\sigma(d\varphi) = \int_{C_0([0,1])} F(\varphi(\xi)) \, w_\sigma(d\xi). \quad (1.10)$$

After the substitution $f(t) = -\cot \pi \varphi(t)$ in (1.4), $< G_f(t_1, t_2) >$ is rewritten in the form

$$< G(t_1, t_2) > = \int_{Diff^1_+([0,1])} \frac{(\varphi'(t_1)/\varphi'(t_2))^{\frac{3}{2}}}{\sin[\pi \varphi(t_2) - \pi \varphi(t_1)]^{\frac{1}{2}}} \exp \left\{ \frac{2\pi^2}{\sigma^2} \int_0^1 (\varphi'(t))^2 \, dt \right\} \mu_\sigma(d\varphi). \quad (1.11)$$

The aim of this paper is to present the regular calculation method for functional integrals of this kind.

In $[48], [49]$, using the quasi-invariance of the measure (1.7) we evaluated the functional integrals for the partition function and the correlation functions in the Schwarzian theory.
In this paper (sections 2 and 3), we derive the general simple rules of functional integration in the theories of the Schwarzian type, and evaluate the functional integrals assigning correlation functions in the SYK model (section 4). In section 5, we give the concluding remarks.

2 Quasi-invariance of the measure as a key to functional integration

Invariant measures analogous to the Haar measure on finite-dimensional groups do not exist for the noncompact groups $H$ [56]. However, there can exist measures that are quasi-invariant with respect to the action of a more smooth subgroup $G \subset H$. The quasi-invariance means that under the action of the subgroup $G$ the measure transforms to itself multiplied by a function $\mathcal{R}_g(h)$ (the Radon-Nikodim derivative) parametrized by the elements of the subgroup $g \in G$.

The measure (1.7) on the group of diffeomorphisms $Diff([0,1])$ is quasi-invariant with respect to the action of the subgroup $Diff([0,1])$ [52]-[54].

Thus, for functional integrals over the measure $\mu$, we have

$$\int_{Diff([0,1])} F(\varphi) \mu_\sigma(d\varphi) = \frac{1}{\sqrt{g(0)g'(1)}} \int_{Diff([0,1])} F(g(\varphi))$$

$$\times \exp \left\{ \frac{1}{\sigma^2} \left[ \frac{g''(0)}{g'(0)} \varphi'(0) - \frac{g''(1)}{g'(1)} \varphi'(1) \right] + \frac{1}{\sigma^2} \int_0^1 S_g(\varphi(t)) \left( \varphi'(t) \right)^2 dt \right\} \mu_\sigma(d\varphi). \quad (2.1)$$

In what follows, we assume the function $g$ to be

$$g(t) = g_\alpha(t) = \frac{1}{2} \left[ \frac{1}{\tan \frac{\alpha}{2}} \tan \left( \alpha(t - \frac{1}{2}) \right) + 1 \right]. \quad (2.2)$$

In this case,

$$g'_\alpha(0) = g'_\alpha(1) = \frac{\alpha}{\sin \alpha}, \quad -\frac{g''_\alpha(0)}{g'_\alpha(0)} = \frac{g''_\alpha(1)}{g'_\alpha(1)} = 2\alpha \tan \frac{\alpha}{2}, \quad S_{g_\alpha}(t) = 2\alpha^2, \quad (2.3)$$

and the equation (2.1) looks like:

$$\int_{Diff([0,1])} F(g_\alpha(\varphi)) \exp \left\{ -\frac{2\alpha}{\sigma^2} \tan \frac{\alpha}{2} (\varphi'(0) + \varphi'(1)) \right\} \exp \left\{ \frac{2\alpha^2}{\sigma^2} \int_0^1 (\varphi'(t))^2 dt \right\} \mu_\sigma(d\varphi)$$

$$= \frac{\alpha}{\sin \alpha} \int_{Diff([0,1])} F(\varphi) \mu_\sigma(d\varphi). \quad (2.4)$$

Denote

$$\Psi(\varphi) = \exp \left\{ -\frac{2\alpha}{\sigma^2} \tan \frac{\alpha}{2} (\varphi'(0) + \varphi'(1)) \right\} F(g_\alpha(\varphi)).$$

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Then
\[
F(\varphi) = \exp \left\{ \frac{2\alpha}{\sigma^2} \tan \frac{\alpha}{2} \left( (g^{-1}_\alpha(\varphi))' (0) + (g^{-1}_\alpha(\varphi))' (1) \right) \right\} \Psi \left( (g^{-1}_\alpha(\varphi)) \right). \tag{2.5}
\]

Thus the integral
\[
J^\alpha = \int_{\text{Diff}^1([0,1])} \Psi(\varphi) \exp \left\{ \frac{2\alpha^2}{\sigma^2} \int_0^1 (\varphi'(t))^2 \, dt \right\} \mu_\sigma(d\varphi)
\]

is transformed into
\[
J^\alpha = \frac{\alpha}{\sin \alpha} \times \int_{\text{Diff}^1([0,1])} \exp \left\{ \frac{2\alpha}{\sigma^2} \tan \frac{\alpha}{2} \left( (g^{-1}_\alpha(\varphi))' (0) + (g^{-1}_\alpha(\varphi))' (1) \right) \right\} \Psi \left( (g^{-1}_\alpha(\varphi)) \right) \mu_\sigma(d\varphi). \tag{2.7}
\]

For the function \( g_\alpha \) given by (2.2), the inverse function is
\[
(g^{-1}_\alpha(\varphi))(t) = \frac{1}{\alpha} \arctan \left[ \tan \frac{\alpha}{2} (2\varphi(t) - 1) \right] + \frac{1}{2},
\]
and
\[
(g^{-1}_\alpha(\varphi))'(0) = \frac{\sin \alpha}{\alpha} \varphi'(0), \quad (g^{-1}_\alpha(\varphi))'(1) = \frac{\sin \alpha}{\alpha} \varphi'(1),
\]

\[
\exp \left\{ \frac{2\alpha}{\sigma^2} \tan \frac{\alpha}{2} \left( (g^{-1}_\alpha(\varphi))' (0) + (g^{-1}_\alpha(\varphi))' (1) \right) \right\} = \exp \left\{ \frac{4 \sin^2 \frac{\alpha}{2}}{\sigma^2} (\varphi'(0) + \varphi'(1)) \right\}.
\]

For the factor
\[
\frac{(\varphi'(t_1)\varphi'(t_2))^{\frac{1}{2}}}{|\sin[\pi \varphi(t_2) - \pi \varphi(t_1)]|^{\frac{1}{2}}}
\]

in the integrand of (1.11), we have
\[
\lim_{\alpha \to 0} \int_{\text{Diff}^1([0,1])} \delta \left( \varphi'(0) - 1 \right) \frac{(\varphi'(t_1)\varphi'(t_2))^{\frac{1}{2}}}{|\varphi(t_2) - \varphi(t_1)|^{\frac{1}{2}}} \exp \left\{ \frac{4}{\sigma^2} (\varphi'(0) + \varphi'(1)) \right\} \mu_\sigma(d\varphi) + O(1)
\]

Now the functional integral for the two-point SYK correlation function (1.11) is transformed into
\[
\langle G_2(t_1, t_2) \rangle^\alpha_{\text{SYK}} = \sqrt{2\sigma} \frac{\alpha}{\sin \alpha}
\]

\[
\times \int_{\text{Diff}^1([0,1])} \delta \left( \frac{\varphi'(1)}{\varphi'(0)} - 1 \right) \frac{(\varphi'(t_1)\varphi'(t_2))^{\frac{1}{2}}}{|\varphi(t_2) - \varphi(t_1)|^{\frac{1}{2}}} \exp \left\{ \frac{4}{\sigma^2} (\varphi'(0) + \varphi'(1)) \right\} \mu_\sigma(d\varphi) + O(1).
\]

Note that the integral (2.9) is invariant under the shifts of the variable \( t \) (that is, under the choice of the point on the \( S^1 \) corresponding to the left end of the interval \( [0, 1] \)):
\[
\langle G_2(t_1, t_2) \rangle^\alpha_{\text{SYK}} = \langle G_2(0, t_2 - t_1) \rangle^\alpha_{\text{SYK}}.
\]
Generally speaking, the functional integrals (2.7) converge for $0 < \alpha < \pi$, and diverge for $\alpha = \pi$. The integral (2.9) is a typical example. The point is that the Schwarzian action is invariant under the noncompact group of linear-fractional transformations. To get the finite results for functional integrals in the Schwarzian theory, one should factor the infinite input of this noncompact group out.

Technically, it is convenient to exclude the input of the $SL(2, \mathbb{R})$ group that is a subgroup of $Diff^1_+(S^1)$, or to integrate over the quotient space $Diff^1_+(S^1)/SL(2, \mathbb{R})$.

To this end, we propose first to evaluate regularized ($\alpha < \pi$) functional integrals over the group $Diff^1_+([0, 1])$ (with the ends glued) and then normalize them to the corresponding integrals over the group $SL(2, \mathbb{R})$. Thus, we define the renormalized functional integral $J^R$ as the limit

$$J^R = \lim_{\alpha \to \pi^-} \left( \frac{J^\alpha}{\int_{SL(2, \mathbb{R})} \Psi(\varphi_z) \exp \left\{ \frac{2\alpha^2}{\sigma^2} \int_0^1 (\varphi'_z(t))^2 \, dt \right\} \, d\mu_H} \right).$$

Here, $J^\alpha$ is given by (2.6), $\varphi_z \in SL(2, \mathbb{R})$ and $d\mu_H$ is the invariant Haar measure on $SL(2, \mathbb{R})$.

The factor

$$\frac{(\varphi'(t_1)\varphi'(t_2))^\frac{1}{4}}{|\varphi(t_2) - \varphi(t_1)|^\frac{1}{2}}$$

in the integrand of (2.9) is $SL(2, \mathbb{R})$ invariant.

Therefore, for renormalized SYK correlation functions we have

$$<G>^R_{SYK} = \lim_{\alpha \to \pi^-} \frac{<G>^\alpha_{SYK}}{V^\alpha_{SL(2, \mathbb{R})}},$$

where $V^\alpha_{SL(2, \mathbb{R})}$ is the regularized volume of the group $SL(2, \mathbb{R})$ [49]:

$$V^\alpha_{SL(2, \mathbb{R})} = \int_{SL(2, \mathbb{R})} \exp \left\{ - \frac{2}{\sigma^2} \left[ \frac{\pi^2 - \alpha^2}{\sigma^2} \right] \right\} \, d\mu_H = \frac{\pi \sigma^2}{\pi^2 - \alpha^2} \exp \left\{ - \frac{2}{\sigma^2} \left( \frac{\pi^2 - \alpha^2}{\sigma^2} \right) \right\}.$$ 

Thus, the renormalization excludes the singularity $(\pi - \alpha)^{-1}$ in the functional integrals (2.7), and we get the finite results for SYK correlation functions.

3 Splitting the interval multiplies functional integrals

In the case when the functional $F$ in (1.10) has the special form

$$F(\varphi) = \Phi(\varphi(t_1); \varphi'(0), \varphi'(t_1), \varphi'(1)),$$

the equation (1.10) looks like

$$\int_{Diff^1_+([0, 1])} \Phi(\varphi(t_1); \varphi'(0), \varphi'(t_1), \varphi'(1)) \mu_\sigma(d\varphi).$$
Now the exponent of the Wiener measure is written as a product of the two measures

\begin{equation}
\Phi(\varphi(\xi(t_1)); \varphi'(0), \varphi'(\xi(t_1)), \varphi'(\xi(1))) w_\sigma(d\xi) = \int_{C_0([0,1])} \Phi(\varphi(\xi(t_1)); \varphi'(0), \varphi'(\xi(t_1)), \varphi'(\xi(1))) w_\sigma(d\xi).
\end{equation}

Due to the Markov property of the Wiener process \(\xi\), substitute \(\xi(t) = \eta_0 \left( \frac{t}{t_1} \right), \quad 0 \leq t \leq t_1; \quad \xi(t) = \eta_0(1) + \eta_1 \left( \frac{t - t_1}{1 - t_1} \right), \quad t_1 \leq t \leq 1. \tag{3.2}\)

Now the exponent of the Wiener measure is written as

\begin{equation}
\frac{1}{\sigma^2} \left( \int_0^{t_1} (\xi'(t))^2 dt + \int_{t_1}^t (\xi'(t))^2 dt \right) = \frac{1}{\sigma^2 t_1} \int_0^1 (\eta_0'(t))^2 dt + \frac{1}{\sigma^2(1-t_1)} \int_0^1 (\eta_1'(t))^2 dt.
\end{equation}

Due to the Markov property of the Wiener process \(\xi(t)\), the measure \(w_\sigma(d\xi)\) turns into the product of the two measures

\[ w_\sigma \sqrt{\tau_1} (d\eta_0) \cdot w_\sigma \sqrt{1-\tau_1} (d\eta_1). \]

To return to the integrals over the group of diffeomorphisms, define the functions \(\psi_0, \psi_1 \in \text{Diff}_+([0,1])\)

\begin{equation}
\psi_0(t) = \frac{\int_0^t e^{\eta_0(\tau)} d\tau}{\int_0^1 e^{\eta_0(\tau)} d\tau}, \quad \psi_1(t) = \frac{\int_0^t e^{\eta_1(\tau)} d\tau}{\int_0^1 e^{\eta_1(\tau)} d\tau}.
\end{equation}

In this way, we get

\begin{equation}
\int_{\text{Diff}_+([0,1])} \Phi(\varphi(t_1); \varphi'(0), \varphi'(t_1), \varphi'(1)) \mu_\sigma(d\varphi) = \int_{\text{Diff}_+([0,1])} \int_{\text{Diff}_+([0,1])} \Phi(\varphi(t_1); \varphi'(0), \varphi'(t_1), \varphi'(1)) \mu_\sigma \sqrt{\tau_1} (d\psi_0) \mu_\sigma \sqrt{1-\tau_1} (d\psi_1). \tag{3.4}\end{equation}

The functions \(\varphi\) and \(\psi\) (and their derivatives) are related by the following equations:

\begin{equation}
\varphi(t) = \frac{t_1 \psi_1'(0) \psi_0 \left( \frac{t}{t_1} \right)}{t_1 \psi_1'(0) + (1 - t_1) \psi_0'(1)}, \quad 0 \leq t \leq t_1; \tag{3.5}\end{equation}

\begin{equation}
\varphi(t) = \frac{t_1 \psi_1'(0) + (1 - t_1) \psi_0'(1) \psi_1 \left( \frac{t - t_1}{1 - t_1} \right)}{t_1 \psi_1'(0) + (1 - t_1) \psi_0'(1)}, \quad t_1 \leq t \leq 1, \tag{3.6}\end{equation}

and

\begin{equation}
\varphi'(t) = \frac{\psi_1'(0) \psi_0 \left( \frac{t}{t_1} \right)}{t_1 \psi_1'(0) + (1 - t_1) \psi_0'(1)}, \quad 0 \leq t \leq t_1; \tag{3.7}\end{equation}
\[
\varphi'(t) = \frac{\psi'_0(1) \psi'_1 \left( \frac{t-t_1}{1-t_1} \right)}{t_1 \psi'_1(0) + (1-t_1)\psi'_0(1)}, \quad t_1 \leq t \leq 1. \tag{3.8}
\]

In particular, the arguments in the integrand are

\[
\varphi(t_1) = \frac{t_1 \psi'_1(0)}{t_1 \psi'_1(0) + (1-t_1)\psi'_0(1)}; \quad \varphi'(t_1) = \frac{\psi'_0(1) \psi'_1(0)}{t_1 \psi'_1(0) + (1-t_1)\psi'_0(1)}, \tag{3.9}
\]

\[
\varphi'(0) = \frac{\psi'_0(0) \psi'_1(0)}{t_1 \psi'_1(0) + (1-t_1)\psi'_0(1)}, \quad \varphi'(1) = \frac{\psi'_0(1) \psi'_1(1)}{t_1 \psi'_1(0) + (1-t_1)\psi'_0(1)}. \tag{3.10}
\]

Note that the substitution (1.9) is nonlocal, and for diffeomorphisms \( \varphi(t) \) the Markov property is not valid.

Now we can represent all functional integrals with integrands depending on \( \varphi(t_1), \varphi'(0), \varphi'(t_1), \varphi'(1) \) in a similar way. To this end, we define the basic functional integral:

\[
E_\sigma(u, v) = \int_{\text{Diff}^1([0,1])} \delta(\varphi'(0) - u) \delta(\varphi'(1) - v) \mu_\sigma(d\varphi), \tag{3.11}
\]

and rewrite the functional integral (3.4) as

\[
\int_{\text{Diff}^1([0,1])} \int_{\text{Diff}^1([0,1])} \Phi(\varphi(t_1); \varphi'(0), \varphi'(t_1), \varphi'(1)) \mu_\sigma \sqrt{\tau_1}(d\psi_0) \mu_\sigma \sqrt{\tau_1}(d\psi_1)
\]

\[
= \int_0^{+\infty} \int_0^{+\infty} du_0 dv_0 du_1 dv_1 \Phi(\varphi_{uv}(t_1); \varphi'_{uv}(0), \varphi'_{uv}(t_1), \varphi'_{uv}(1)) E_{\sigma \sqrt{\tau_1}}(u_0, v_0) E_{\sigma \sqrt{1-\tau_1}}(u_1, v_1). \tag{3.12}
\]

Instead of the variables \((u_0, v_0, u_1, v_1)\), it is convenient to use the variables

\[
z_1 = \varphi(t_1), \quad x_0 = \varphi'(0), \quad y_1 = \varphi'(t_1), \quad x_1 = \varphi'(1).
\]

Eqs. (3.9) and (3.10) lead to

\[
u_0 = \frac{t_1}{z_1} x_0, \quad v_0 = \frac{t_1}{z_1} y_1, \quad u_1 = \frac{1-t_1}{1-z_1} y_1, \quad v_1 = \frac{1-t_1}{1-z_1} x_1,
\]

and

\[
\left| \det \frac{\partial(u_1, v_1, u_2, v_2)}{\partial(z, x_0, x_1, y_1)} \right| = \frac{t_1(1-t_1)^2}{z_1(1-z_1)^3} y_1.
\]

Thus, in this case, we get the transparent rule of the functional integration

\[
\int_{\text{Diff}^1([0,1])} \Phi(\varphi(t_1); \varphi'(0), \varphi'(t_1), \varphi'(1)) \mu_\sigma(d\varphi)
\]

\[
= [t_1(1-t_1)]^2 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} dz_1 dx_0 dy_1 dx_1
\]

\[
= [t_1(1-t_1)]^2 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} dz_1 dx_0 dy_1 dx_1
\]

\[-8-\]
\[ \times \Phi(z_1; x_0, y_1, x_1) \mathcal{E}_{\sqrt{t}} \left( \frac{t_1}{z_1}, x_0, \frac{t_1}{z_1}, y_1 \right) \mathcal{E}_{\sqrt{1-t}} \left( \frac{1-t_1}{1-z_1}, y_1, \frac{1-t_1}{1-z_1}, x_1 \right). \tag{3.13} \]

If the integrand \( \Phi \) depends on values of the function \( \varphi \) and its derivative \( \varphi' \) at other points \( t_1, t_2, \ldots t_k \) of the interval, we can continue the described procedure

\[ \varphi \rightarrow (\psi_0, \psi_1), \quad \psi_1 \rightarrow (\psi_2, \psi_3), \quad \psi_3 \rightarrow (\psi_4, \psi_5), \quad \ldots \quad \psi_{2k-3} \rightarrow (\psi_{2k-1}, \psi_{2k-1}), \]

and obtain the generalization of (3.13)

\[ \int_{Diff^+_1([0,1])} \Phi (\varphi(t_1), \ldots \varphi(t_k); \varphi'(0), \varphi'(t_1), \ldots \varphi'(t_k), \varphi'(1)) \mu_\sigma(d\varphi) \]

\[ = [t_1(t_2 - t_1) \ldots (t_k - t_{k-1})(1 - t_k)]^2 \int_0^1 dz_1 \ldots \int_0^1 dz_k \ldots \int_0^1 [z_1(z_2 - z_1) \ldots (z_k - z_{k-1})(1 - z_k)]^{-3} \]

\[ \times \int_0^\infty dx_0 \int_0^\infty y_1 dy_1 \ldots \int_0^\infty y_k dy_k \int_0^\infty dx_1 \Phi (z_1, \ldots, z_k; x_0, y_1, \ldots, y_k, x_1) \mathcal{E}_{\sqrt{t}} \left( \frac{t_1}{z_1}, x_0, \frac{t_1}{z_1}, y_1 \right) \]

\[ \times \mathcal{E}_{\sqrt{t_2-t_1}} \left( \frac{t_2 - t_1}{z_2 - z_1}, y_1, \frac{t_2 - t_1}{z_2 - z_1}, y_2 \right) \ldots \mathcal{E}_{\sqrt{t_{k-1}-t_{k-1}} \ldots} \left( \frac{t_k - t_{k-1}}{z_k - z_{k-1}}, y_{k-1}, \frac{t_k - t_{k-1}}{z_k - z_{k-1}}, y_k \right) \]

\[ \times \mathcal{E}_{\sqrt{1-t_k}} \left( \frac{1-t_k}{1-z_k}, y_k, \frac{1-t_k}{1-z_k}, x_1 \right). \tag{3.14} \]

In this case, the Jacobian of the substitution is

\[ \det \frac{\partial (u_1, v_1, \ldots, u_{k+1}, v_{k+1})}{\partial (z_1, \ldots, z_k, x_0, y_1, \ldots, y_k, x_1)} = [t_1 \cdots (1 - t_k)]^2 [z_1 \cdots (1 - z_k)]^{-3} y_1 \cdots y_k. \]

Note that (3.11) is the only one functional integral we need to evaluate. In [49], we performed the functional integration explicitly and represent the functional integral (3.11) in the form of the ordinary integral:

\[ \mathcal{E}_\sigma (u, v) = \left( \frac{2}{\pi \sigma^2} \right)^{\frac{3}{2}} \frac{1}{\sqrt{uv}} \exp \left\{ \frac{2}{\sigma^2} \left( \pi^2 - u - v \right) \right\} \]

\[ \times \int_0^\infty \exp \left\{ -\frac{2}{\sigma^2} \left( 2 \sqrt{uv} \cosh \theta + \theta^2 \right) \right\} \sin \left( \frac{4 \pi \theta}{\sigma^2} \right) \sinh(\theta) d\theta. \tag{3.15} \]

Now the functional integrals (3.13) and (3.14) are reduced to ordinary multiple integrals.
4 SYK correlation functions

The renormalized two-point SYK correlation function looks like

\[ < G_2^0 (0, t_1) > _{SYK}^{R} \]

\[ = \frac{2\pi}{\sigma^2} \pi^{-\frac{n}{2}} \int_{Diff^1([0,1])} \delta \left( \frac{\varphi'(1)}{\varphi'(0)} - 1 \right) \left( \frac{\varphi'(t_1)\varphi'(0)}{|\varphi(t_1)|^2} \right) \exp \left\{ \frac{8}{\sigma^2} \left( \varphi'(0) \right) \right\} \mu_\sigma(d\varphi) \]

\[ = \frac{2\pi}{\sigma^2} \pi^{-\frac{n}{2}} [t_1 (1 - t_1)]^2 \int_0^1 dz_1 z_1^{-3-\frac{n}{2}} (1 - z_1)^{-3} \int_0^{+\infty} dx_1 y_1 (x_0 y_1)^{1+\frac{n}{2}} \]

\times \exp \left\{ \frac{8}{\sigma^2} x_0 \right\} \mathcal{E}_{\sigma \sqrt{t_1}} \left( \frac{t_1}{z_1} x_0, \frac{t_1}{z_1} y_1 \right) \mathcal{E}_{\sigma \sqrt{1-t_1}} \left( \frac{1 - t_1}{1 - z_1} y_1, \frac{1 - t_1}{1 - z_1} x_0 \right). \tag{4.1} \]

In the similar way, we can write the time-ordered (TO) and the out-of-time-ordered (OTO) renormalized four point correlation functions:

\[ < G_4^{TO} (0, t_1; t_2, t_3) > _{SYK}^{R} \]

\[ = \frac{2}{\sigma^2} \int_{Diff^4([0,1])} \delta \left( \frac{\varphi'(1)}{\varphi'(0)} - 1 \right) \left( \frac{\varphi'(t_1)\varphi'(0)}{|\varphi(t_1)|^2} \right) \exp \left\{ \frac{8}{\sigma^2} \left( \varphi'(0) \right) \right\} \mu_\sigma(d\varphi), \tag{4.2} \]

and

\[ < G_4^{OTO} (0, t_1; t_2, t_3) > _{SYK}^{R} \]

\[ = \frac{2}{\sigma^2} \int_{Diff^4([0,1])} \delta \left( \frac{\varphi'(1)}{\varphi'(0)} - 1 \right) \left( \frac{\varphi'(t_2)\varphi'(0)}{|\varphi(t_2)|^2} \right) \exp \left\{ \frac{8}{\sigma^2} \left( \varphi'(0) \right) \right\} \mu_\sigma(d\varphi). \tag{4.3} \]

(In the both equations, we assume that 0 < t_1 < t_2 < t_3 < 1.)

In terms of the functions \( \mathcal{E} \), the four-point correlation functions look like

\[ < G_4^{TO(OTO)} (0, t_1(2); t_2(1), t_3) > _{SYK}^{R} = \frac{2}{\sigma^2} [t_1(t_2-t_1)(t_3-t_2)(1-t_3)]^2 \]

\[ \times \int_0^1 dz_1 \int_0^1 dz_2 \int_0^1 dz_3 \chi^{TO(OTO)} (z_1, z_2, z_3) \int_0^{+\infty} dx_0 \int_0^{+\infty} dy_1 \int_0^{+\infty} dy_2 \int_0^{+\infty} dy_3 [x_0 y_1 y_2 y_3]^\frac{n}{2} \]

\[ \times \exp \left\{ \frac{8}{\sigma^2} x_0 \right\} \mathcal{E}_{\sigma \sqrt{t_1}} \left( \frac{t_1}{z_1} x_0, \frac{t_1}{z_1} y_1 \right) \mathcal{E}_{\sigma \sqrt{2-t_1}} \left( \frac{t_2 - t_1}{z_2 - z_1} y_1, \frac{t_2 - t_1}{z_2 - z_1} y_2 \right) \]

\[ \times \mathcal{E}_{\sigma \sqrt{3-t_2}} \left( \frac{t_3 - t_2}{z_3 - z_2} y_2, \frac{t_3 - t_2}{z_3 - z_2} y_3 \right) \mathcal{E}_{\sigma \sqrt{1-t_3}} \left( \frac{1 - t_3}{1 - z_3} y_3, \frac{1 - t_3}{1 - z_3} x_0 \right), \tag{4.4} \]

where

\[ \chi^{TO} (z_1, z_2, z_3) = \left[ z_1 (z_2 - z_1) (z_3 - z_2) (1 - z_3) \right]^{-3} \left[ z_1 (z_3 - z_2) \right]^{-\frac{n}{2}}. \tag{4.5} \]
and
\[ \chi^{OTO}(z_1, z_2, z_3) = [z_1(z_2 - z_1)(z_3 - z_2)(1 - z_3)]^{-3} [z_2(z_3 - z_1)]^{-\frac{1}{2}}. \quad (4.6) \]

Note that the only difference between (4.5) and (4.6) is in the dependence of the integrands on the variables \( z_i \).

Neither the OTO four-point correlation function nor the TO four-point correlation function has the form of the product of two two-point correlation functions.

In appendix B, we obtain the representations for the correlation functions in terms of the ordinary integrals that can be analyzed numerically.

5 Concluding remarks

The paper completes the elaboration of the functional integrals calculus in the theories invariant under the groups of diffeomorphisms started in [48] and [49].

The general rules derived here give a straightforward scheme to evaluate functional integrals in the theories of the Schwarzian type. The great merit of the method is that it reduces various problems to the evaluation of the functional integral (3.11) only.

The functional integral (3.11)
\[ \mathcal{E}_\sigma(u, v) = \int_{Diff_1([0, 1])} \delta(\varphi'(0) - u) \delta(\varphi'(1) - v) \mu_\sigma(d\varphi) \]
is evaluated explicitly. And the result is written in the form of the ordinary integral (3.15). Therefore the evaluations of other functional integrals of the form (3.14) lead also to ordinary (multiple) integrals.

The fact that the four-point correlation functions cannot be represented in the form of the product of two two-point correlation functions is connected with the apparent non-Markov behaviour of the function \( \varphi(t) \). Although \( \xi \) is the Wiener process, the Markov property is violated by the nonlocal substitution (1.9).

This non-Markov behaviour also manifests itself in (4.1) and (4.4). The regions with \( t > t_1 \) (for \( < G_2(0, t_1) >_{SYK} \)), and with \( t > t_3 \) (for \( < G_4(0, t_1; t_2(1), t_3) >_{SYK} \)) give the nonzero inputs into the integrals for correlation functions.

If one considers \( t \) as the time variable, then one can say that the present is influenced by the future. It is a direct consequence of the fact that the ends of the interval are glued together and the theory is \( SL(2, \mathbb{R}) \) invariant.

In this paper, we obtain the correlation functions in the form that can be analyzed numerically. The analysis is planned to be the subject of another paper.

A Convolution of the \( \mathcal{E} \) functions

Let us take the integrand of (3.11) for \( \Phi \). In this case, we get the convolution rule for two \( \mathcal{E} \) functions:
\[ \mathcal{E}_\sigma(u, v) = [t_1(1 - t_1)]^2 \int_0^1 [z_1(1 - z_1)]^{-3} \int_0^{+\infty} y_1 dy_1 \]

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We would like to stress that this convolution rule has been proved by direct explicit evaluation of functional integrals.

However, to demonstrate once more that the dependence of the right-hand side of (A.1) on \( t_1 \) is fictitious, we consider the integral

\[
\int_0^\infty \int_0^\infty E_\sigma(u, u) du = \frac{1}{\sqrt{2\pi^3 \sigma^2}} \int_0^\infty d\theta \exp\left\{ -\frac{2}{\sigma^2} \left[ \theta^2 - \pi^2 \right] \right\} \sin \left( \frac{4\pi \theta}{\sigma^2} \right) \tanh \left( \frac{\theta}{2} \right). \tag{A.2}
\]

At the same time, we have from the right-hand side of (A.1)

\[
\frac{2}{\pi^3 \sigma^2} [t_1 (1 - t_1)]^{-\frac{1}{2}} \int_0^\infty \int_0^\infty d\theta_1 d\theta_2 \exp\left\{ -\frac{2}{\sigma^2} \left[ \frac{\theta_1^2 - \pi^2}{t_1} + \frac{\theta_2^2 - \pi^2}{(1 - t_1)} \right] \right\} \times \sin \left( \frac{4\pi \theta_1}{\sigma^2 t_1} \right) \sin \left( \frac{4\pi \theta_2}{\sigma^2 (1 - t_1)} \right) \sinh \theta_1 \sinh \theta_2 \int_0^1 dz_1 \left[ z_1 (1 - z_1) \right]^{-2} \times \int_0^\infty d\theta_1 d\theta_2 \exp\left\{ -\frac{2}{\sigma^2} \left[ \frac{\theta_1^2 - \pi^2}{t_1} + \frac{\theta_2^2 - \pi^2}{(1 - t_1)} \right] \right\} \sinh \theta_1 \sinh \theta_2 \times \sin \left( \frac{4\pi \theta_1}{\sigma^2 t_1} \right) \sin \left( \frac{4\pi \theta_2}{\sigma^2 (1 - t_1)} \right) \int_0^1 dz_1 \left[ \frac{b}{(b^2 - 1)^\frac{1}{2}} \arccosh b - \frac{1}{b^2 - 1} \right], \tag{A.3}
\]

where

\[ b = [(1 - z_1) \cosh \theta_1 + z_1 \cosh \theta_2]. \]

To obtain (A.3), we have re-scaled the variables

\[ \bar{u} = \frac{u}{z_1 (1 - z_1)}, \quad \bar{y}_1 = \frac{y_1}{z_1 (1 - z_1)}. \]

And then after the substitutions

\[ \bar{u} = \rho \cos^2 \omega, \quad \bar{y}_1 = \rho \sin^2 \omega, \quad \xi = \tan \omega, \tag{A.4} \]

we have used the table of integrals [57] (eq. 2.2.9.15).

The results of computations of (A.3) at ten different points of the interval \([0, 1]\) coincide with the result of computation of (A.2) with very high accuracy (e.g., the results are 0.07978 for \( \sigma = 5 \), and 0.03989 for \( \sigma = 10 \)).
B  Correlation functions in terms of ordinary integrals

In terms of the ordinary integrals, the two-point correlation function (4.1) has the form

\[ < G_2^0 (0, t_1) >^{R}_{\text{SYK}} = \frac{1}{\pi^2} \left( \frac{\sigma^2}{2\pi} \right)^{\frac{n}{2}} [t_1(1 - t_1)]^{-\frac{1}{2}} \]

\[ \times \int_0^{+\infty} \int_0^{+\infty} d\theta_1 d\theta_2 \exp \left\{ -\frac{2}{\sigma^2} \left[ \frac{\theta_1^2 - \pi^2}{t_1} + \frac{\theta_2^2 - \pi^2}{(1 - t_1)} \right] \right\} \sin \left( \frac{4\pi \theta_1}{\sigma^2 t_1} \right) \sin \left( \frac{4\pi \theta_2}{\sigma^2 (1 - t_1)} \right) \]

\[ \times \sinh(\theta_1) \sinh(\theta_2) \int_0^{+\infty} \int_0^{+\infty} dx_0 dy_1 (x_0 y_1)^{\frac{n}{2}} \]

\[ \times \exp \left\{ -(x_0 + y_1 + 2 [(1 - z_1) \cosh \theta_1 + z_1 \cosh \theta_2] \sqrt{x_0 y_1} - 4z_1(1 - z_1)x_0) \right\} . \quad (B.1) \]

After the substitutions (A.4), the integrals over \( x_0 \) and \( y_1 \) are transformed into the integral

\[ \int_0^{+\infty} \frac{2\xi^{\frac{n}{2}}}{[\xi^2 + 2b\xi + 1 - 4z_1(1 - z_1)]^{\frac{n+4}{4}}} d\xi . \quad (B.2) \]

It is convenient to use the table of integrals [57] (eq. 2.2.9.8).

\[ < G_2^0 (0, t_1) >^{R}_{\text{SYK}} = \frac{2}{\pi^2} \left( \frac{\sigma^2}{2\pi} \right)^{\frac{n}{2}} \Gamma^3 \left( \frac{n+4}{2} \right) \Gamma^{-1} \left( n + 4 \right) [t_1(1 - t_1)]^{-\frac{1}{2}} \]

\[ \times \int_0^{+\infty} \int_0^{+\infty} d\theta_1 d\theta_2 \exp \left\{ -\frac{2}{\sigma^2} \left[ \frac{\theta_1^2 - \pi^2}{t_1} + \frac{\theta_2^2 - \pi^2}{(1 - t_1)} \right] \right\} \sin \left( \frac{4\pi \theta_1}{\sigma^2 t_1} \right) \sin \left( \frac{4\pi \theta_2}{\sigma^2 (1 - t_1)} \right) \]

\[ \times \sinh(\theta_1) \sinh(\theta_2) \int_0^{1} dz_1 (1 - z_1)^{\frac{n}{2}} [1 - 4z_1(1 - z_1)]^{-\frac{n+4}{4}} \]

\[ \times F \left( \frac{n+4}{2}, \frac{n+4}{2}; \frac{n+5}{2}; \frac{1}{2} \left( 1 - \frac{b}{1 - 4z_1(1 - z_1)} \right) \right) . \quad (B.3) \]

where \( F \) is the Gaussian hypergeometric function, and

\[ b = (1 - z_1) \cosh \theta_1 + z_1 \cosh \theta_2 . \quad (B.4) \]

For \( n = 2k, \ k = 0, 1, \ldots \), we can also use eq. 2.2.9.12 of [57], and rewrite the result in the form

\[ < G_2^0 (0, t_1) >^{R}_{\text{SYK}} = (-1)^{k+1} \frac{1}{(k+1)!} \frac{1}{2\pi^2} \left( \frac{\sigma^2}{4\pi} \right)^k \Gamma(k+2) [t_1(1 - t_1)]^{-\frac{1}{2}} \]

\[ \times \int_0^{+\infty} \int_0^{+\infty} d\theta_1 d\theta_2 \exp \left\{ -\frac{2}{\sigma^2} \left[ \frac{\theta_1^2 - \pi^2}{t_1} + \frac{\theta_2^2 - \pi^2}{(1 - t_1)} \right] \right\} \sin \left( \frac{4\pi \theta_1}{\sigma^2 t_1} \right) \sin \left( \frac{4\pi \theta_2}{\sigma^2 (1 - t_1)} \right) \]

\[ \times \int_0^{+\infty} \frac{2\xi^{\frac{k}{2}}}{[\xi^2 + 2b\xi + 1 - 4z_1(1 - z_1)]^{\frac{k+2}{2}}} d\xi . \quad (B.5) \]
\[ \times \sinh(\theta_1) \sinh(\theta_2) \int_0^1 dz_1 (1 - z_1)^k \]
\[ \times \frac{\partial^{k+1}}{\partial b^{k+1}} \left( \frac{1}{\sqrt{b^2 - 1 + 4z_1(1 - z_1)}} \log \frac{b + \sqrt{b^2 - 1 + 4z_1(1 - z_1)}}{b - \sqrt{b^2 - 1 + 4z_1(1 - z_1)}} \right), \quad \text{(B.5)} \]

where \( b \) is given by (B.4).

Here, we also present the four-point correlation functions in the form that can be analyzed numerically. It is
\[
\langle G_4^{OTO}(0, t_{1(2)}; t_{2(1)}, t_3) \rangle_{SYK}^R = \left( \frac{2}{\pi^3 \sigma^2} \right)^2 [t_1(t_2 - t_1)(t_3 - t_2)(1 - t_3)]^{-\frac{1}{2}} \]
\[ \times \int_0^1 dz_1 \int_0^1 dz_2 \int_0^1 dz_3 \chi^{OTO}(z_1, z_2, z_3) \]
\[ \times \int_0^{+\infty} d\theta_1 \ldots \int_0^{+\infty} d\theta_4 \exp \left\{ -\frac{2}{\sigma^2} \left[ \frac{\theta_1^2 - \pi^2}{\tau_1} + \frac{\theta_2^2 - \pi^2}{\tau_2} + \frac{\theta_3^2 - \pi^2}{\tau_3} + \frac{\theta_4^2 - \pi^2}{1 - \tau_3} \right] \right\} \]
\[ \times \sin \left( \frac{4\pi \theta_1}{\sigma^2 \tau_1} \right) \sin \left( \frac{4\pi \theta_2}{\sigma^2 (\tau_2 - \tau_1)} \right) \sin \left( \frac{4\pi \theta_3}{\sigma^2 (\tau_3 - \tau_2)} \right) \]
\[ \times \sinh \theta_1 \sinh \theta_2 \sinh \theta_3 \sinh \theta_4 \int_0^{+\infty} dx_0 \int_0^{+\infty} dy_1 \int_0^{+\infty} dy_2 \int_0^{+\infty} dy_3 [x_0 y_1 y_2 y_3]^{\frac{1}{2}} \]
\[ \times \exp \left\{ 4x_0 - \frac{1}{z_1} [x_0 + y_1 + 2\sqrt{x_0 y_1} \cosh \theta_1] - \frac{1}{z_2 - z_1} [y_1 + y_2 + 2\sqrt{y_1 y_2} \cosh \theta_2] \right\} \]
\[ \times \exp \left\{ -\frac{1}{z_3 - z_2} [y_2 + y_3 + 2\sqrt{y_2 y_3} \cosh \theta_3] - \frac{1}{1 - z_3} [y_3 + x_0 + 2\sqrt{y_3 x_0} \cosh \theta_4] \right\}, \quad \text{(B.6)} \]

where
\[ \chi^{OTO}(z_1, z_2, z_3) = [z_1 (z_2 - z_1) (z_3 - z_2) (1 - z_3)]^{-\frac{1}{2}} [z_1 (z_3 - z_2)]^{-\frac{1}{2}}, \]
and
\[ \chi^{OTO}(z_1, z_2, z_3) = [z_2 (z_3 - z_2) (1 - z_3)]^{-\frac{1}{2}} [z_2 (z_3 - z_1)]^{-\frac{1}{2}}. \]

C Functional integrals in Liouville theory

In this appendix, we consider functional integrals of the form
\[
\int \Psi(f(\tau_1), f(\tau_2), ...) \exp \left\{ \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} S_f(\tau) \, d\tau \right\} df, \quad \text{(C.1)}
\]

where \( \mathcal{F} \) is the space of continuously differentiable functions \( f(\tau) \) on \((-\infty, +\infty)\) with arbitrary and independent of each other values \( f'(-\infty) \) and \( f'(+\infty) \).
These functional integrals determine the theory different from that considered above in the main part of the paper where the ends of the interval are glued \( (\varphi'(0) = \varphi'(1)) \).

In contrast to functional integrals of the form (1.4) that are invariant under the left action of \( SL(2, \mathbb{R}) \) group \( (g \circ f, g \in SL(2, \mathbb{R}) \) and shifts of the parameter \( t \rightarrow t + \vartheta \), integrals (C.1) are invariant under transformations \( f \rightarrow af + b \) only.

In [23], [24], [25], [28], [30], it was proposed to treat functional integrals (C.1) as functional integrals in Liouville theory.

Consider the two-point correlation function in this theory

\[
G_2^n(0, \tau) = \int_C \left( \frac{f'(0)f'(\tau)}{|f(\tau) - f(0)|^2} \right) \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} S_f(t) \, dt \, df. 
\]

After the substitution \( f(\tau) = \int_{\tau}^{\infty} \exp\{\xi(t)\} \, dt \), it looks like

\[
G_2^n(0, \tau) = \int_{C([0, \tau])} \left( \frac{e^{\frac{\xi(0)+\xi(\tau)}{2}}}{\int_0^{\tau} e^{\xi(t)} \, dt} \right)^{\frac{n}{4}} \exp \left\{ -\frac{1}{2\sigma^2} \int_{-\infty}^{+\infty} \left( \xi'(t) \right)^2 \, dt \right\} \, d\xi. 
\]

Due to the Markov property of the Wiener measure, it can be rewritten as

\[
G_2^n(0, \tau) = \int_{C([0, \tau])} \left( \frac{e^{\frac{\xi(0)+\xi(\tau)}{2}}}{\int_0^{\tau} e^{\xi(t)} \, dt} \right)^{\frac{n}{4}} \exp \left\{ -\frac{1}{2\sigma^2} \int_0^{\tau} \left( \xi'(t) \right)^2 \, dt \right\} \, d\xi. 
\] (C.2)

We would like to demonstrate that the method described in section 2 and 3 can be also used in this case.

We re-scale the time variable \( t = \tau \bar{t} \) and substitute \( \eta(\bar{t}) = \xi(t) - \xi(0) \). Now (C.2) has the form

\[
G_2^n(0, \tau) = \tau^{-\frac{n}{2}} \int_{C([0, 1])} \left( \frac{e^{\frac{\eta(1)}{2}}}{\int_0^1 e^{\eta(\bar{t})} \, d\bar{t}} \right)^{\frac{n}{4}} \exp \left\{ -\frac{1}{2\sigma^2} \tau \int_0^1 \left( \eta'(\bar{t}) \right)^2 \, d\bar{t} \right\} \, d\eta. 
\] (C.3)

In terms of the function

\[
\varphi(t) = \int_0^t \frac{e^{\eta(\bar{t})}}{\int_0^{\eta(\bar{t})} d\bar{t}},
\]

(C.3) is written as

\[
\tau^{-\frac{n}{2}} \int_{Diff^1([0, 1])} \left( \varphi'(0)\varphi'(1) \right)^{\frac{n}{2}} \mu_{\sigma \sqrt{\tau}}(d\varphi) = \tau^{-\frac{n}{2}} \int_0^{+\infty} \int_0^{+\infty} (uv)^{\frac{n}{2}} \, \mathcal{E}_{\sigma \sqrt{\tau}}(u, v) \, du \, dv. 
\] (C.4)
Integrating over $u$ and $v$ as in the appendix A, we obtain

\[
G_2^O(0, \tau) = 2\pi^{-\frac{3}{2}} \left( \frac{\sigma^2}{2} \right) \frac{\pi^{n-1}}{2} B \left( \frac{n+1}{2}, \frac{n+1}{2} \right) \tau^{-\frac{3}{2}} \int_0^{+\infty} d\theta \sin \left( \frac{4\pi \theta}{\sigma^2} \right) \sinh \theta
\]

\times \exp \left\{ -\frac{2}{\sigma^2} \left[ \theta^2 - \pi^2 \right] \right\} F \left( \frac{n+1}{2}, \frac{n+1}{2}; \frac{n+1}{2} + 1; \frac{1 - \cosh \theta}{2} \right). \quad (C.5)

In the case $n = 2$, it looks like

\[
G_2^O(0, \tau) = \left( \frac{\sigma^2}{2\pi^3} \right) \tau^{-\frac{3}{2}} \int_0^{+\infty} d\theta \exp \left\{ -\frac{2}{\sigma^2} \left[ \theta^2 - \pi^2 \right] \right\} \sin \left( \frac{4\pi \theta}{\sigma^2} \right) \frac{\theta \cosh \theta - 1}{\sinh \theta}. \quad (C.6)
\]

The asymptotic form of $G_2^O(0, \tau)$ at $\tau \to \infty$,

\[
\left( G_2^O(0, \tau) \right)_{\text{As}} = 4\pi^{-\frac{1}{2}} \left( \frac{\sigma^2}{2} \right) \frac{\pi^{n-1}}{2} B \left( \frac{n+1}{2}, \frac{n+1}{2} \right) \tau^{-\frac{3}{2}}
\]

\times \int_0^{+\infty} \theta \sinh \theta F \left( \frac{n+1}{2}, \frac{n+1}{2}; \frac{n+1}{2} + 1; \frac{1 - \cosh \theta}{2} \right) d\theta, \quad (C.7)

immediately follows from (C.5).

In the same way, we can explicitly evaluate functional integrals for other correlation functions in this theory. In particular, for $G_4^{TO}(0, \tau_1; \tau_2, \tau_3)$ we have

\[
G_4^{TO}(0, \tau_1; \tau_2, \tau_3) = \int_{\mathcal{F}} \frac{(f'/(0)f'(\tau_1))^{\frac{1}{4}}}{|f(\tau_1) - f(0)|^{\frac{1}{2}}} \frac{(f'(\tau_2)f'(\tau_3))^{\frac{1}{4}}}{|f(\tau_3) - f(\tau_2)|^{\frac{1}{2}}} \exp \left\{ \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} S_f(t) \, dt \right\} \, df
\]

\[
= G_2(0, \tau_1) G_2(\tau_2, \tau_3). \quad (C.8)
\]

The OTO four-point correlation function is given by $(0 < \tau_1 < \tau_2 < \tau_3)$

\[
G_4^{OTO}(0, \tau_1; \tau_2, \tau_3) = \int_{\mathcal{F}} \frac{(f'(0)f'(\tau_2))^{\frac{1}{4}}}{|f(\tau_2) - f(0)|^{\frac{1}{2}}} \frac{(f'(\tau_1)f'(\tau_3))^{\frac{1}{4}}}{|f(\tau_3) - f(\tau_1)|^{\frac{1}{2}}} \exp \left\{ \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} S_f(t) \, dt \right\} \, df
\]

\[
= \pi^{\frac{3}{2}} \left( \frac{\sigma^2}{2} \right)^{\frac{1}{2}} \left[ \tau_1(\tau_2 - \tau_1)(\tau_3 - \tau_2) \right]^{-\frac{1}{2}} \int_0^{+\infty} dz_1 \int_0^{+\infty} dz_2 \left[ z_1(z_2 - z_1)(1 - z_2) \right]^{-\frac{1}{2}} \left[ z_2(1 - z_1) \right]^{-\frac{1}{2}}
\]

\times \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \, dt_1 \, dt_2 \, dt_3 \exp \left\{ -\frac{2}{\sigma^2} \left[ \frac{\theta_1^2 - \pi^2}{\tau_1} + \frac{\theta_2^2 - \pi^2}{(\tau_2 - \tau_1)} + \frac{\theta_3^2 - \pi^2}{(\tau_3 - \tau_2)} \right] \right\}
\]

\times \sin \left( \frac{4\pi \theta_1}{\sigma^2 \tau_1} \right) \sin \left( \frac{4\pi \theta_2}{\sigma^2 (\tau_2 - \tau_1)} \right) \sin \left( \frac{4\pi \theta_3}{\sigma^2 (\tau_3 - \tau_2)} \right) \sin \theta_1 \sin \theta_2 \sin \theta_3
\]
\begin{align}
\times \int_0^{+\infty} dx_0 \int_0^{+\infty} dy_1 \int_0^{+\infty} dy_2 \int_0^{+\infty} dx_1 \left[ x_0 x_1 \right]^{-\frac{1}{4}} \left[ y_1 y_2 \right]^\frac{1}{4} \exp \left\{ -\frac{1}{z_1} [x_0 + y_1 + 2 \sqrt{x_0 y_1} \cosh \theta_1] \right\} \\
\times \exp \left\{ -\frac{1}{z_2 - z_1} [y_1 + y_2 + 2 \sqrt{y_1 y_2} \cosh \theta_2] - \frac{1}{1 - z_2} [y_2 + x_1 + 2 \sqrt{y_2 x_1} \cosh \theta_3] \right\} \right. 
\end{align} 

In contrast to the correlation functions in the theory considered in the main part of the paper, the correlation functions (C.2), (C.8), (C.9) have no input from the regions \( t > \tau \), or \( t > \tau_3 \).

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