Existence of solutions for nonlinear fractional integro-differential equations

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Abstract
In this paper, by means of the Krasnoselskii fixed point theorem, the existence of solutions for a boundary value problem of nonlinear sequential fractional integro-differential equations are investigated. Two examples are given to illustrate our results.

MSC: Primary 34B40; secondary 34B15

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1 Introduction
Fractional differential equations have attracted much attention and have been the focus of many studies due mainly to their varied applications in many fields of science and engineering. In other words, fractional differential equations are widely used to describe many important phenomena in various fields such as physics, biophysics, chemistry, biology, control theory, economy and so on; see [14, 19, 23, 29, 33]. For an extensive literature in the study of fractional differential equations, we refer the reader to [2, 11, 15, 16, 18, 20, 21, 24, 26, 30, 32]. However, it should be noted that in recent years, there have been many works related to fractional integro-differential equations, see [1, 3, 4, 6, 8, 12, 17, 22, 28, 29] and the references therein. For some interesting and considerable applied works, we refer to [5, 7, 9, 10].

In [13], Baleanu et al. studied the existence and uniqueness of solutions for the multiterm nonlinear fractional integro-differential equation

\[
\begin{cases}
\frac{d^\alpha}{dt^\alpha}u(t) = f(t, u(t), \varphi u(t), \psi u(t)), \quad (0 < t < 1), \\
u(0) + au(1) = 0 \quad \text{and} \quad u'(0) - bu'(1) = 0.
\end{cases}
\]

where \(1 < \alpha < 2, 0 < \beta_i < 1, \alpha - \beta_i \geq 1, a, b \neq -1, f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}\) is continuous, and for the mappings \(\varphi, \lambda : [0, 1] \times [0, 1] \to [0, \infty)\) with the property

\[
sup_{t \in [0,1]} \left( \int_0^t \gamma(t,s) \, ds \right) < \infty \quad \text{and} \quad sup_{t \in [0,1]} \left( \int_0^t \lambda(t,s) \, ds \right) < \infty,
\]

the maps \(\varphi\) and \(\psi\) being defined by \((\varphi u)(t) = \int_0^t \gamma(t,s) \, ds\) and \((\psi u)(t) = \int_0^t \lambda(t,s) \, ds\).
In [31], Wang et al. proved the existence and uniqueness of positive solutions for the following fractional integro-differential equation:

\[
\begin{align*}
\frac{d^\alpha}{dt^\alpha} u(t) + f(t, u(t), \phi u(t), \psi u(t)) &= 0 \quad (0 < t < 1), \\
u(0) &= b_0, \\
u'(0) &= b_1, \ldots, u^{(n-3)}(0) &= b_{n-3}, \\
u^{(n-1)}(0) &= b_{n-1}, \quad u(1) = \mu \int_0^1 u(s) \, ds,
\end{align*}
\]

where \(n - 1 < \alpha \leq n, -1, n \geq 3, b_i \geq 0 (i = 1, 2, \ldots, n - 3, n - 1), \frac{d^\alpha}{dt^\alpha} u(t) \) is the Caputo fractional derivative, \( f : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( (\phi u)(t) = \int_0^t \gamma(t, s) u(s) \, ds, \) \( (\psi u)(t) = \int_0^t \lambda(t, s) u(s) \, ds. \)

Motivated by the previous results, we discuss in this paper the existence of solutions for the following nonlinear sequential fractional boundary value problem:

\[
\begin{align*}
\frac{d^\alpha}{dt^\alpha} (\frac{d^\beta}{dt^\beta} u)(t) &= f(t, u(t), \phi u(t), \psi u(t)) \quad (0 < t < 1), \\
u(1) &= u(0) = u'(1) = 0,
\end{align*}
\]

where \(1 < \alpha \leq 2, 0 < \beta \leq 1, f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) is continuous and \( \gamma, \lambda : [0, 1] \times [0, 1] \rightarrow [0, +\infty) \) are such that \( \sup_{t \in [0, 1]} (\int_0^1 \lambda(t, s) \, ds) < \infty \) and \( \sup_{t \in [0, 1]} (\int_0^1 \gamma(t, s) \, ds) < \infty. \)

2 Preliminaries

For convenience, in this section we recall some basic definitions and properties of the fractional calculus theory and auxiliary lemmas which will be used throughout this paper, see [23, 25, 27].

**Definition 2.1** The Caputo fractional derivative of order \(\alpha > 0\) of a continuous function \(u : (0, \infty) \rightarrow \mathbb{R}\) is defined by

\[
\frac{d^\alpha}{dt^\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) \, ds,
\]

provided the right-hand side is pointwise defined on \((0, \infty)\).

**Definition 2.2** The Riemann–Liouville fractional integral of order \(\alpha > 0\) of a continuous function \(u : (0, \infty) \rightarrow \mathbb{R}\) is defined by

\[
I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \, ds,
\]

provided the right-hand side is pointwise defined on \((0, \infty)\).

**Lemma 2.1** If \(\alpha > 0\), then the differential equation \(\frac{d^\alpha}{dt^\alpha} u(t) = 0\) has a unique solution given by

\[
u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},
\]

where \(c_i \in \mathbb{R}, i = 0, 1, \ldots, n - 1 \) (\(n\) is the smallest integer such that \(n \geq \alpha\)).
Lemma 2.2 For $\alpha > 0$, let $u \in C^n[0,1]$. Then

$$I^\alpha (cDu)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}.$$  

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$ (n is the smallest integer such that $n \geq \alpha$).

Lemma 2.3 If $y \in C[0,1]$, then the boundary value problem

$$\begin{cases} cD^\alpha (cD^\beta u)(t) = y(t), & 0 < t < 1, 1 < \alpha \leq 2 \text{ and } 0 < \beta \leq 1, \\ u(1) = u(0) = u'(1) = 0 \end{cases} \quad (2.1)$$

has the unique solution given by

$$u(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\beta+\alpha-1} y(s) \, ds$$

$$+ \frac{t^\beta}{\Gamma(\alpha + \beta)} (-\beta - 1 + \beta t) \int_0^1 (1-s)^{\beta+\alpha-1} y(s) \, ds$$

$$+ \frac{t^\beta}{\Gamma(\alpha + \beta - 1)} (1-t) \int_0^1 (1-s)^{\alpha+\beta-2} y(s) \, ds \quad (2.2)$$

Proof In view of Lemma 2.2, FBVP (2.1) is equivalent to the following integral equation:

$$u(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\beta+\alpha-1} y(s) \, ds + \frac{t^\beta}{\beta \Gamma(\beta)} c_0 + \frac{t^{\beta-1}}{\beta(\beta + 1) \Gamma(\beta)} c_1 + c_2. \quad (2.3)$$

Differentiating both sides of (2.3), we get

$$u'(t) = \frac{1}{\Gamma(\alpha + \beta - 1)} \int_0^t (t-s)^{\alpha+\beta-2} y(s) \, ds + \frac{t^{\beta-1}}{\beta \Gamma(\beta)} c_0 + \frac{t^\beta}{\beta \Gamma(\beta)} c_1.$$

Using the boundary conditions $u(1) = u(0) = u'(1) = 0$, we obtain

$$c_0 = -\left( \frac{\beta(\beta+1) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \right) \int_0^1 (1-s)^{\beta+\alpha-1} y(s) \, ds + \frac{\beta \Gamma(\beta)}{\Gamma(\alpha + \beta - 1)} \int_0^1 (1-s)^{\alpha+\beta-2} y(s) \, ds,$$

$$c_1 = \frac{\beta^2(\beta+1) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\beta+\alpha-1} y(s) \, ds - \frac{\beta(\beta+1) \Gamma(\beta)}{\Gamma(\alpha + \beta - 1)} \int_0^1 (1-s)^{\alpha+\beta-2} y(s) \, ds,$$

$$c_2 = 0.$$

Substituting the values of $c_0, c_1, c_2$ in (2.3) we obtain (2.2). This completes the proof. \hspace{1cm} \Box$

3 Main results

Theorem 3.1 (Krasnoselskii fixed point theorem) Let $X$ be a closed convex and nonempty subset of a Banach space $E$. Let $A$ and $B$ be two operators such that

1. $Ax + By \in X$, whenever $x, y \in X$;
2. $A$ is compact and continuous;
3. $B$ is a contraction.

Then there exists $z \in X$ such that $z = Az + Bz$. 
Let $X = C(I)$ be the space of all continuous real-valued functions on $I = [0, 1]$ endowed with the norm $\|u\| = \max_{t \in I} |u(t)|$.

**Theorem 3.2** Assume that $\alpha + \beta - 2 \geq 0$ and there exists a nonnegative function $\theta(t) \in L^1(0, 1)$ such that

$$
|f(t, x, y, z) - f(t, x', y', z')| \leq \theta(t)(|x - x'| + |y - y'| + |z - z'|)
$$

(3.1)

for all $t \in [0, 1]$ and $t, x, y, z, t', y', z' \in \mathbb{R}$. Then problem (1.1) has at least one solution on $X$ whenever

$$
\frac{(1 + \gamma_0 + \lambda_0)(\alpha + 2 \beta + 1)\theta^*}{\Gamma(\alpha + \beta)} < 1,
$$

(3.2)

where $\gamma_0 = \sup_{t \in I} |f(t, 0, 0, 0)|$, $\lambda_0 = \sup_{t \in I} \int_0^1 \lambda(t, s) ds$, and $\theta^* = \int_0^1 \theta(s) ds$.

**Proof** Choose

$$
R \geq \frac{\sigma(\alpha + 2 \beta + 1)}{1 - \theta^*(1 + \lambda_0 + \gamma_0)(\alpha + 2 \beta + 1)}
$$

and set $\sigma = \max\{f(t, 0, 0, 0) : t \in I\}$. Consider the set $B_R = \{u \in X : \|u\| \leq R\}$, then $B_R$ is a closed, bounded, and convex set of $X$. We define the operators $A$ and $B$ on $X$ as

$$
Au(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\beta+\alpha-1} f(s,u(s),\varphi u(s),\psi u(s)) \, ds,
$$

$$
Bu(t) = \frac{t^\beta}{\Gamma(\alpha + \beta)} (-\beta - 1 + \beta t) \int_0^1 (1-s)^{\beta+\alpha-1} f(s,u(s),\varphi u(s),\psi u(s)) \, ds
$$

$$
+ \frac{t^\beta(\alpha + \beta - 1)}{\Gamma(\alpha + \beta)} (1-t) \int_0^1 (1-s)^{\beta+\alpha-2} f(s,u(s),\varphi u(s),\psi u(s)) \, ds.
$$

For any $u \in B_R$ and $t \in I$, we get with the help of inequality (3.1)

$$
|Au(t)| \leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\beta+\alpha-1} |f(s,u(s),\varphi u(s),\psi u(s))| \, ds
$$

$$
\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\beta+\alpha-1} |f(s,u(s),\varphi u(s),\psi u(s)) - f(s,0,0,0)| \, ds
$$

$$
+ \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\beta+\alpha-1} |f(s,0,0,0)| \, ds
$$

$$
\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\beta+\alpha-1} \theta(s) \left( |u(s)| + |\varphi u(s)| + |\psi u(s)| \right) \, ds
$$

$$
+ \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\beta+\alpha-1} |f(s,0,0,0)| \, ds
$$

$$
\leq \frac{(1 + \lambda_0 + \gamma_0)\|u\|}{\Gamma(\alpha + \beta)} \int_0^1 \theta(s) \, ds
$$

$$
+ \frac{\sigma}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\beta+\alpha-1} \, ds
$$

$$
\leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)} \|u\| + \frac{\sigma}{\Gamma(\alpha + \beta)}.
$$
Hence, we get

\[ \|Au\| \leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)} \|u\| + \frac{\sigma}{\Gamma(\alpha + \beta)}. \] (3.3)

Similarly, we estimate \( \|Bv\| \). Let \( v \in B_R \) and \( t \in I \), then

\[
\|Bv(t)\| \leq \frac{t^\beta}{\Gamma(\alpha + \beta)} (\beta + 1 - \beta t) \int_0^1 (1 - s)^{\beta + \alpha - 1} |v(s)| \, ds
+ \frac{t^\beta (\alpha + \beta - 1)}{\Gamma(\alpha + \beta)} (1 - t) \int_0^1 (1 - s)^{\alpha + \beta - 2} |v(s)| \, ds
\leq \frac{\beta + 1}{\Gamma(\alpha + \beta)} \int_0^1 (1 - s)^{\beta + \alpha - 1} \theta(s) (\|v\| + |\varphi(v)| + |\psi(v)|) \, ds
+ \frac{\beta + 1}{\Gamma(\alpha + \beta)} \int_0^1 (1 - s)^{\beta + \alpha - 1} |f(t, 0, 0, 0)| \, ds
+ \frac{\alpha + \beta - 1}{\Gamma(\alpha + \beta)} \int_0^1 (1 - s)^{\alpha + \beta - 2} |v(s)| \, ds
+ \frac{\alpha + \beta - 1}{\Gamma(\alpha + \beta)} \int_0^1 (1 - s)^{\alpha + \beta - 2} |f(t, 0, 0, 0)| \, ds
\leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)} (\beta + 1) \|v\| + \frac{\sigma (\beta + 1)}{\Gamma(\alpha + \beta)}
+ \frac{\alpha + \beta - 1}{\Gamma(\alpha + \beta)} \theta^*(1 + \lambda_0 + \gamma_0) \|v\| + \frac{\sigma (\alpha + \beta - 1)}{\Gamma(\alpha + \beta)}
= \frac{\theta^*(1 + \lambda_0 + \gamma_0)(\alpha + 2\beta)}{\Gamma(\alpha + \beta)} \|v\| + \frac{\sigma (\alpha + 2\beta)}{\Gamma(\alpha + \beta)}
= \frac{(\alpha + 2\beta)}{\Gamma(\alpha + \beta)} (\theta^*(1 + \lambda_0 + \gamma_0) \|v\| + \sigma).
\]

Hence, we get

\[ \|Bv\| \leq \frac{(\alpha + 2\beta)}{\Gamma(\alpha + \beta)} (\theta^*(1 + \lambda_0 + \gamma_0) \|v\| + \sigma). \] (3.4)

Taking estimates (3.3) and (3.4) into account, we get for any \( u, v \in B_R \) and \( t \in I \),

\[
\|Au + Bv\| \leq \|Au\| + \|Bv\|
\leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)} \|u\| + \frac{\sigma}{\Gamma(\alpha + \beta)}
+ \frac{(\alpha + 2\beta)}{\Gamma(\alpha + \beta)} \theta^*(1 + \lambda_0 + \gamma_0) \|v\| + \sigma
\leq R \frac{\theta^*(1 + \lambda_0 + \gamma_0)(\alpha + 2\beta + 1)}{\Gamma(\alpha + \beta)}
+ \frac{\sigma (\alpha + 2\beta + 1)}{\Gamma(\alpha + \beta)},
\]
continuous. Also set \( L \) then have yields \( \parallel Au + BV \parallel \leq R \).

Now, we prove that \( B \) is a contraction. Let \( v, u \in B_R \) and \( t \in I \). Then, thanks to (3.1), it yields

\[
|Bu(t) - Bv(t)| \leq \frac{t^\beta(1 + \beta t)}{\Gamma(\alpha + \beta)} \times \int_0^1 (1 - s)^{\beta + \alpha - 1} |f(s, u(s), \varphi u(s), \psi u(s)) - f(s, v(s), \varphi v(s), \psi v(s))| \, ds
\]

\[
+ \frac{t^\beta(1 - t)}{\Gamma(\alpha + \beta - 1)} \times \int_0^1 (1 - s)^{\beta + \alpha - 2} |f(s, u(s), \varphi u(s), \psi u(s)) - f(s, v(s), \varphi v(s), \psi v(s))| \, ds
\]

\[
\leq \frac{\beta + 1}{\Gamma(\alpha + \beta)} \int_0^1 (1 - s)^{\beta + \alpha - 1} \theta(s) (|u(s) - v(s)| + |\varphi u(s) - \varphi v(s)| + |\psi u(s) - \psi v(s)|) \, ds
\]

\[
+ \frac{1}{\Gamma(\alpha + \beta - 1)} \int_0^1 (1 - s)^{\beta + \alpha - 2} \theta(s) (|u(s) - v(s)| + |\varphi u(s) - \varphi v(s)| + |\psi u(s) - \psi v(s)|) \, ds
\]

\[
\leq \frac{(\beta + 1)(1 + \gamma_0 + \lambda_0)}{\Gamma(\alpha + \beta)} \|u - v\| \int_0^1 \theta(s) \, ds
\]

\[
+ \frac{(1 + \gamma_0 + \lambda_0)(\alpha + \beta - 1)}{\Gamma(\alpha + \beta)} \|u - v\| \int_0^1 \theta(s) \, ds
\]

\[
\leq \frac{(1 + \gamma_0 + \lambda_0)(\alpha + 2\beta)}{\Gamma(\alpha + \beta)} \|u - v\|,
\]

thus

\[
\|Bu - Bv\| \leq \frac{(1 + \gamma_0 + \lambda_0)(\alpha + 2\beta)}{\Gamma(\alpha + \beta)} \|u - v\|,
\]

so by (3.2) we conclude that \( B \) is a contraction.

Let us prove that \( A \) is compact and continuous. The continuity of \( f \) implies that \( A \) is continuous. Also \( A \) is uniformly bounded on \( B_R \), indeed, from (3.3) we have

\[
\|Au\| \leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)} \|u\| + \frac{\sigma}{\Gamma(\alpha + \beta)} \leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)} R + \frac{\sigma}{\Gamma(\alpha + \beta)}.
\]

Set \( L = \max_{0 \leq t \leq 1} \{ |f(s, u(s), \varphi u(s), \psi u(s))| : u \in B_R \} \). Let \( u \in B_R \), \( t_1, t_2 \in I \), with \( t_1 \leq t_2 \). We have

\[
|Au(t_2) - Au(t_1)|
\]

\[
= \frac{1}{\Gamma(\alpha + \beta)} \left| \int_0^{t_2} (t_2 - s)^{\beta + \alpha - 1} f(s, u(s), \varphi u(s), \psi u(s)) \, ds \right|
\]

\[
\leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)} \|u\|= \frac{\sigma}{\Gamma(\alpha + \beta)} \leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)} R + \frac{\sigma}{\Gamma(\alpha + \beta)}.
\]
Therefore, by Theorem 3.2, the problem has at least one solution in $\mathbb{B}$. So the operator $A$ is completely continuous. Thus, by Theorem 3.1, problem (1.1) has at least one solution in $\mathbb{B}$. The proof is complete.

Example 3.1 We consider the boundary value problem (1.1) with $f(t,x_1,x_2,x_3) = \frac{1}{4} t^2 e^{-t} \times \sum_{i=1}^{3} \frac{1}{1+x_i^2(0)}$, $\alpha = \frac{9}{5}$, $\beta = \frac{3}{5}$. Also we have $f(t,0,0,0) = \frac{3}{4} t^2 e^{-t}$ thus $\sigma = 0.31$. Let $\lambda(t,s) = \gamma(t,s) = ts$, so that $\gamma_0 = \lambda_0 = \frac{1}{2}$. Moreover, we can verify that condition (3.1) is satisfied

$$|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)|$$

$$\leq \frac{1}{4} t^2 e^{-t} \sum_{i=1}^{3} \left( \frac{1}{1+x_i^2} - \frac{1}{1+y_i^2} \right)$$

$$\leq \frac{1}{4} t^2 e^{-t} \sum_{i=1}^{3} \frac{|x_i - y_i||x_i + y_i|}{(1+x_i^2)(1+y_i^2)}$$

$$\leq \frac{1}{4} t^2 e^{-t} \sum_{i=1}^{3} |x_i - y_i|,$$

so $\theta(t) = \frac{1}{4} t^2 e^{-t}$ and $\theta^* = \frac{0.37894}{4}$. Also, condition (3.2) holds:

$$\frac{(1 + \gamma_0 + \lambda_0)(\alpha + 2\beta + 1)\theta^*}{\Gamma(\alpha + \beta)} = 0.61013 < 1.$$

Therefore, by Theorem 3.2, the problem has at least one solution in $\mathbb{B}$ with

$$R \geq \frac{\sigma (\alpha + 2\beta + 1)}{1 - \theta^*(1 + \lambda_0 + \gamma_0)(\alpha + 2\beta + 1)} = 5.1214$$

Example 3.2 Consider the boundary value problem (1.1) with

$$f(t,x,y,z) = 10^{-2} \left( t \sin x + e^x \sin 2y + \frac{1 + t^2}{1 + z^2} \right),$$
\(\alpha = 1.3, \beta = 0.4\). Then \(f(t, 0, 0, 0) = 10^{-2}(1 + t^2)\), thus \(\sigma = 0.02\). Let \(\lambda(t, s) = e^{-s}, \gamma(t, s) = (t - s)^\beta\), thus \(\gamma_0 = 1.7183, \lambda_0 = 0.71429\). Condition (3.1) is satisfied, in fact,

\[
\left| f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3) \right| \leq 0.02e^t \sum_{i=1}^{3} |x_i - y_i|.
\]

We choose \(\theta(t) = 0.02e^t\) then \(\theta^* = 3.4366 \times 10^{-2}\). We check condition (3.2):

\[
\frac{(1 + \gamma_0 + \lambda_0)(\alpha + 2\beta + 1)}{\Gamma(\alpha + \beta)} \theta^* = 0.40246 < 1.
\]

We conclude, by Theorem 3.2, that the problem has at least one solution in \(B_R\) with

\[
R \geq \frac{\sigma (\alpha + 2\beta + 1)}{1 - \theta^*(1 + \lambda_0 + \gamma_0)(\alpha + 2\beta + 1)} = 9.7744 \times 10^{-2}.
\]

**Conclusion.** In the present work, we have studied the existence of solutions for a fractional sequential boundary value problem. To demonstrate the existence results, we transformed the posed problem into a sum of a contraction and a compact operator, then we applied the Krasnoselskii’s fixed point theorem. We ended the article with some numerical examples illustrating the obtain results.

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