Lyapunov-based Stability of Feedback Interconnections of Negative Imaginary Systems

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Abstract: Feedback control systems using sensors and actuators such as piezoelectric sensors and actuators, micro-electro-mechanical systems (MEMS) sensors and opto-mechanical sensors, are allowing new advances in designing such high precision technologies. The negative imaginary control systems framework allows for robust control design for such high precision systems in the face of uncertainties due to unmodelled dynamics. The stability of the feedback interconnection of negative imaginary systems has been well established in the literature. However, the proofs of stability feedback interconnection which are used in some previous papers have a shortcoming due to a matrix inevitability issue. In this paper we provide a new and correct Lyapunov-based proof of one such result and show that the result is still true.

Keywords: Negative imaginary systems, Strictly negative imaginary systems, Negative imaginary lemma, Lyapunov stability.

1. INTRODUCTION

Technologies such as atomic force microscopy, nano-positioning, micro-robotics and hard disc drives require high precision and performance in controller design. Feedback control systems using sensors and actuators such as piezoelectric sensors and actuators, micro-electro-mechanical systems (MEMS) sensors and opto-mechanical sensors, D. G. Wilson and Starr (2002); Harigae et al. (2003); Bhikkaji and Moheimani (2009); Mahmood et al. (2011); Salapaka et al. (2002); van Hulzen et al. (2010); Devasia et al. (2007); Diaz et al. (2012); Ray (1978), are allowing new advances in designing such high precision technologies. These negative imaginary (NI) control systems framework allows for robust control design for such high precision systems in the face of uncertainties due to unmodelled dynamics along with sensor and actuator, Petersen and Lanzon (2010); M. A. Mabrok and Lanzon (2014). In Lanzon and Petersen (2008); Petersen and Lanzon (2010), they introduced the theory of negative imaginary (NI) systems for the robust control of flexible structures. Negative imaginary systems are defined by considering the properties of the imaginary part of the system frequency response $G(j\omega)$ and requiring the condition $\text{Re}(G(j\omega)) \geq 0$ for all $\omega \in (0, \infty)$.

The robust stability of feedback interconnections of linear time-invariant (LTI) multiple-input multiple-output negative imaginary systems has been studied in Lanzon and Petersen (2008); Petersen and Lanzon (2010); Xiong et al. (2010). It is shown that a necessary and sufficient condition for the internal stability of a positive-feedback control system (see Fig. 1) consisting of an NI plant with transfer function matrix $G(s)$ and a strictly negative imaginary (SNI) controller with transfer function matrix $H(s)$ is given by the DC gain condition $\lambda_{\text{max}}(G(0)H(0)) < 1$, is satisfied.

Fig. 1. A negative-imaginary feedback control system.

If the plant transfer function matrix $G(s)$ is NI and the controller transfer function matrix $H(s)$ is SNI, then the positive-feedback interconnection is internally stable if and only if the DC gain condition, $\lambda_{\text{max}}(G(0)H(0)) < 1$, is satisfied.
motion. Also, necessary and sufficient conditions are provided for the stability of positive feedback control systems where the plant is NI according to the new definition and the controller is strictly negative imaginary is given in M. A. Mabrok and Lanzon (2014).

The NI stability result provided in Lanzon and Petersen (2008); Petersen and Lanzon (2010) has been used in a number of practical applications Cai and Hagen (Aug. 2010); Blïkkaji and Mohaimini (2009); Mahmood et al. (2011); Ahmed and Pota (2011); Blïkkaji et al. (2012); Diaz et al. (2012). For example in Cai and Hagen (Aug. 2010), this stability result is applied to the problem of decentralized control of large vehicle platoons. In Blïkkaji and Mohaimini (2009); Mahmood et al. (2011), the NI stability result is applied to nanopositioning in an atomic force microscope. A positive position feedback control scheme based on the NI stability result provided in Lanzon and Petersen (2008); Petersen and Lanzon (2010) is used to design a novel compensation method for a coupled fuselage-rotor mode of a rotary wing unmanned aerial vehicle in Ahmed and Pota (2011). In Diaz et al. (2012), an IRC scheme based on the NI stability result is used to design an active vibration control system for the mitigation of human induced vibrations in light-weight civil engineering structures, such as floors and footbridges via proof-mass actuators. An identification algorithm which enforces the NI constraint is proposed in Blïkkaji et al. (2012) for estimating model parameters, following which an Integral resonant controller is designed for damping vibrations in flexible structures. In addition, it is shown in van der Schaft (2011) that the class of linear systems having NI transfer function matrices is closely related to the class of linear port-Hamiltonian input-output systems. Also, an extension of the NI systems theory to infinite-dimensional systems is presented in Opmee (2011).

The stability of the feedback interconnection of negative imaginary systems is established in Lanzon and Petersen (2008); Petersen and Lanzon (2010); Xiong et al. (2010); M. A. Mabrok and Lanzon (2014). However, the proofs of the stability results which used in Xiong et al. (2010); M. A. Mabrok and Lanzon (2014) have a shortcoming due to a matrix inevitability issue for the case in which the plant has poles on the imaginary axis. In this paper, we are use a Lyapunov-based stability approach to prove the result of Xiong et al. (2010) and show that the result is still true. Note that the result of S. Z. Khong and Rantzer (2015) does provide a correct proof in the case of plant poles on imaginary axis but requires an extra condition in the definition of the NI property that a certain residue matrix is positive-definite. That extra condition is not required in this paper.

2. PRELIMINARIES

In this section, we introduce the concept of negative imaginary systems and strictly negative imaginary systems which allow for poles on the imaginary axis except at the origin.

Definition 1. Lanzon and Petersen (2007, 2008); Xiong et al. (2009) A square transfer function matrix \( G(s) \) is called negative imaginary (NI) if the following conditions are satisfied:

1. \( G(s) \) has no pole at the origin and in \( \text{Re}[s] > 0 \).
2. For all \( \omega > 0 \), such that \( j\omega \) is not a pole of \( G(s) \), \( j (G(j\omega) - G(j\omega)^*) \geq 0 \).
3. If \( j\omega_0, \omega_0 \in (0, \infty) \), is a pole of \( G(j\omega) \), it is at most a simple pole and the residue matrix \( K_0 = \lim_{s \to j\omega_0} (s - j\omega_0)sG(s) \) is positive semidefinite Hermitian.

A linear time invariant system is NI if its transfer function is NI.

Definition 2. Lanzon and Petersen (2008); Petersen and Lanzon (2010); Xiong et al. (2010). A square real-rational transfer function matrix \( H(s) \) is strictly negative imaginary if and only if:

1) \( G(s) \) has no poles in \( \text{Re}[s] \geq 0 \);
2) \( j(G(j\omega) - G^*(j\omega)) > 0 \) for \( \omega \in (0, \infty) \).

Lemma 1. Xiong et al. (2009). Given a real rational strictly proper transfer function matrix \( G(s) \) with minimal state space realization \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) and, define the transfer function matrix \( \tilde{G}(s) = G(s) - D \). The transfer function matrix \( \tilde{G}(s) \) is negative imaginary if and only if,

1. \( G(s) \) has no poles at the origin.
2. The transfer function matrix \( F(s) = s\tilde{G}(s) \) is positive real.

Lemma 2. Lanzon and Petersen (2008). Let \( (A, B, C, D) \) be a minimal state-space realization of the \( m \times m \) real-rational proper transfer function matrix \( G(s) \), where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times m} \). Then \( G(s) \) is negative imaginary if and only if:

1) \( \det(A) \neq 0, D = D^T; \)
2) there exists a matrix \( P = P^T > 0, P \in \mathbb{R}^{n \times n} \), such that

\[
AP^{-1} + P^{-1}A^T \leq 0, \quad \text{and} \quad B + AP^{-1}C^T = 0
\]

Lemma 3. Xiong et al. (2010) Let \( (A, B, C, D) \) be a minimal state-space realization of the \( m \times m \) real-rational proper transfer function matrix \( G(s) \), where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times m} \). Then \( G(s) \) is strictly negative imaginary if and only if:

1) \( \det(A) \neq 0, D = D^T; \)
2) there exists a matrix \( P = P^T > 0, P \in \mathbb{R}^{n \times n} \), such that

\[
AP^{-1} + P^{-1}A^T \leq 0, \quad \text{and} \quad B + AP^{-1}C^T = 0
\]
3) the transfer function matrix \( M(s) = \begin{bmatrix} A & B \\ LPA^{-1} & 0 \end{bmatrix} \) has full column rank at \( s = j\omega \) for any \( \omega \in (0, \infty) \) where \( LT = -AP^{-1} - P^{-1}A^T \). That is, rank \( M(j\omega) = m \) for any \( \omega \in (0, \infty) \).

We will consider the positive feedback interconnection of a linear NI system with a linear SNI system and prove the internal stability of the closed-loop system. Consider a minimal state-space representation for the SNI transfer function \( H(s) \),

\[
\begin{align*}
x_1(t) &= A_1x_1(t) + B_1u_1(t), \\
y_1(t) &= C_1x_1(t) + D_1u_1(t),
\end{align*}
\]

where \( A_2 \in \mathbb{R}^{n \times n}, B_2 \in \mathbb{R}^{n \times m}, C_2 \in \mathbb{R}^{m \times n}, D_2 \in \mathbb{R}^{m \times m} \).

Also, we consider a minimal state-space representation for the NI transfer function \( G(s) \),
\[ \dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t), \quad (4) \]
\[ y_2(t) = C_2 x_2(t) + D_2 u_2(t), \quad (5) \]
where \( A_1 \in \mathbb{R}^{n \times n}, B_1 \in \mathbb{R}^{n \times m}, C_1 \in \mathbb{R}^{m \times n}, D_1 \in \mathbb{R}^{m \times m}. \)

Since \( G(s) \) is NLI, Lemma 2 implies there exist a symmetric matrix \( P_1 > 0 \), and a matrix \( L_1 \) such that
\[
\begin{align*}
A_1 P_1^{-1} + A_1^T P_1^{-1} &= L_1^T L_1 \\
B_1 + A_1 P_1^{-1} C_1^T &= 0
\end{align*}
\]
which give
\[
\begin{align*}
P_1 A_1 + A_1^T P_1 &= -P_1 L_1^T L_1 P_1 \\
B_1^T P_1 - C_1 L_1^T L_1 P_1 &= C_1 A_1 \\
C_1 B_1 + (C_1 B_1)^T &= (L_1 C_1^T)^T (L_1 C_1^T)
\end{align*}
\]
Also, since \( H(s) \) is SNI, Lemma 3 implies there exist a symmetric matrix \( P_2 > 0 \), and a matrix \( L_2 \) such that
\[
\begin{align*}
A_2 P_2^{-1} + A_2^T P_2^{-1} &= L_2^T L_2 \\
B_2 + A_2 P_2^{-1} C_2^T &= 0
\end{align*}
\]
which gives
\[
\begin{align*}
P_2 A_2 + A_2^T P_2 &= -P_2 L_2^T L_2 P_2 \\
B_2^T P_2 - C_2 L_2^T L_2 P_2 &= C_2 A_2 \\
C_2 B_2 + (C_2 B_2)^T &= (L_2 C_2^T)^T (L_2 C_2^T)
\end{align*}
\]
where
\[
W(s) := s M(s) = L_2 P_2 (sI - A_2)^{-1} B_2 - L_2 C_2^T
\]
has no zeros in the \( j\omega \)-axis except at the origin.

Lemma 4. Given negative imaginary \( G(s) \) and strictly negative imaginary \( H(s) \). Assume \( G(\infty) H(\infty) = 0 \) and \( N(\infty) \geq 0 \). Then the matrix
\[
\begin{bmatrix}
P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\
-C_2^T C_1 & P_2 - C_2^T D_1 C_2
\end{bmatrix}
\]
is positive definite if and only if \( \lambda_{\text{max}}(G(0) H(0)) < 1 \).

Proof. See also Lanzon and Petersen (2008). We have \( \lambda_{\text{max}}(G(0) H(0)) < 1 \)
\[
\begin{align*}
\Leftrightarrow & H(0)^{-1} - G(0) > 0 \\
\Leftrightarrow & H(0)^{-1} - D - C_1 P_1 C_1^T > 0 \\
\Leftrightarrow & \begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\
C_1 H(0)^{-1} - D_1 & 0
\end{bmatrix} > 0 \\
\Leftrightarrow & H(0)^{-1} - D > 0, \quad \text{and} \\
P_1 - C_1 (H(0)^{-1} - D_1)^{-1} C_1 & > 0 \\
\Leftrightarrow & \lambda_{\text{max}}[D_1 H(0)] < 1, \quad \text{and} \\
P_1 - C_1 (H(0)^{-1} - D_1)^{-1} [D_2 + (H(0) - D_2)] C_1 & > 0 \\
\Leftrightarrow & \lambda_{\text{max}}[D_1 C_2 P_2^{-1} C_2^T] < 1, \quad \text{and}
\end{align*}
\]
\[
\begin{align*}
P_1 - C_1^T D_2 C_1 - C_1^T (I - H(0) D_1)^{-1} (H(0) - D_2) C_1 & > 0 \\
\Leftrightarrow & \lambda_{\text{max}}[P_2^T C_1^T D_1 C_2 P_2^T] > 0, \quad \text{and} \\
P_1 - C_1^T D_2 C_1 - C_1^T (I - C_1 P_2 C_2^T D_1) C_2 P_2 C_2^T C_1 & > 0 \\
\Leftrightarrow & P_2 - C_2^T D_1 C_2 > 0, \quad \text{and} \\
(P_1 - C_1^T D_2 C_1 - C_1^T C_2 (P_2 - C_2^T D_1 C_2)^{-1} C_2^T C_1 & > 0 \\
\Leftrightarrow & \begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\
-C_2^T C_1 & P_2 - C_2^T D_1 C_2
\end{bmatrix} > 0.
\end{align*}
\]

3. MAIN RESULTS

In this section we shall introduce the main result regarding the internal stability of the positive feedback interconnection of \( G(s) \) and \( H(s) \).

Theorem 1. Assume \( G(s) \) is negative imaginary system and \( H(s) \) is strictly negative imaginary system such that \( G(\infty) H(\infty) = 0 \) and \( N(\infty) \geq 0 \). Also, assume that \( \lambda_{\text{max}}(G(0) H(0)) < 1 \). Then, the positive feedback interconnection of \( G(s) \) and \( H(s) \) is internally stable.

Proof. This proof follows a similar approach to the proof of Lemma 3.37 in Brogliato et al. (2007).

Let \( V_1(x_1) = x_1^T P_1 x_1 \) and \( V_2(x_2) = x_2^T P_2 x_2 \) and consider the function
\[
\begin{align*}
V(x_1, x_2) &= V_1(x_1) + V_2(x_2) - 2 y_1^T y_2 \\
&= \begin{bmatrix} x_1^T \ x_2^T \end{bmatrix} \begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\
-C_2^T C_1 & P_2 - C_2^T D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\
 x_2 \end{bmatrix}
\end{align*}
\]
as a Lyapunov candidate for the feedback system. Note that it follows from Lemma 4 that the function \( V(x_1, x_2) \) is positive definite.

Now for the closed loop system we have
\[
\begin{align*}
\hat{x}_1 &= x_1^T A_1^T x_1 + x_1^T P_1 x_1 + 2 y_1^T D_1^T x_1 + 2 y_1^T u_1 \\
\hat{x}_2 &= x_2^T A_2^T x_2 + x_2^T P_2 x_2 + 2 y_2^T D_2^T x_2 + 2 y_2^T u_2 \\
\hat{y}_1 &= y_1^T u_1 \\
\hat{y}_2 &= y_2^T u_2
\end{align*}
\]
\[
\begin{align*}
\hat{x}_1^2 &= x_1^T (A_1^T + u_1^T B_1^T) P_1 x_1 + x_1^T P_1 (A_1 x_1 + B_1 u_1) \\
&+ x_1^T A_1^T u_1 + u_1^T B_1^T P_2 x_2 + x_2^T P_2 (A_2 x_2 + B_2 u_2) \\
&- 2 (y_1^T - u_1^T D^T_1) D_1 y_1 - D_1 u_1 \\
&- 2 (y_2^T - u_2^T D^T_2) D_2 y_2 - D_2 u_2 \\
\hat{x}_2^2 &= x_2^T (A_2^T + u_2^T B_2^T) P_2 x_2 + x_2^T P_2 (A_2 x_2 + B_2 u_2) \\
&- 2 (y_1^T - u_1^T D^T_1) D_1 y_2 - D_1 u_2 \\
&- 2 (y_2^T - u_2^T D^T_2) D_2 y_2 - D_2 u_2
\end{align*}
\]
Define $\tilde{\lambda}$ where we used the equations
\[
\int \text{eigen values on the imaginary axis. The closed loop matrix}
\]
We now show that the closed loop system matrix has no
Then there exists a nonzero
\[
\text{such that}
\]
for $\omega \in \mathbb{R}$. So, we have
\[
(A_1 - j\omega I + B_1 D_2 C_1)x_1 + B_1 C_2 x_2 = 0, \quad (10)
\]
and
\[
B_2 C_1 x_1 + (A_2 - j\omega I + B_2 D_1 C_2)x_2 = 0. \quad (11)
\]
Then, we have
\[
(j\omega I - A_1)x_1 - B_1 y_2 = 0, \quad (12)
\]
and
\[
(j\omega I - A_2)x_2 - B_2 y_1 = 0. \quad (13)
\]
where we used the state-space equations (3), (5) and the equations $u_1 = y_2$ and $u_2 = y_1$.
Combining (13), (9) in a matrix equation form we get
\[
\begin{bmatrix}
A_2 - j\omega I & B_2 \\
L_2 P_2 & -L_2 C_2^T
\end{bmatrix}
\begin{bmatrix}
x_2 \\
u_2
\end{bmatrix} = 0.
\]
Since the matrix on the left has full rank for $\omega \in (0, \infty)$, it follows that $x_2 = u_2 = 0$ and hence $y_1 = y_2 = 0$. This implies $C_1 x_1 = 0$, $x_1 \neq 0$, i.e. $(A, C)$ is non-observable, which contradicts the minimality of the systems $G(s)$. Therefore, $\tilde{A}$ is semistable (i.e. $j\omega \notin \text{spec}(\tilde{A})$, for nonzero $\omega \in \mathbb{R}$).

Now assume that the matrix $\bar{A}$ has an eigenvalue at the origin ($\omega = 0$). From (12) we have
\[
C_1 x_1 = -C_1 \bar{A}^{-1}B_1 y_2 = (G(0) - D_1) y_2.
\]
This implies
\[
y_1 - D_1 u_1 = (G(0) - D_1)y_2,
\]
and then
\[
y_1 = G(0)y_2. \quad (14)
\]
Similarly, from (13) we have
\[
y_2 = H(0)y_1. \quad (15)
\]
Combining (14) and (15) we get
\[
y_1 = G(0)\overline{H(0)y_1}. \quad (16)
\]
Note that from (13), if $y_1 = 0$ we get $x_2 = 0$, since $A_2$ is asymptotically stable and hence invertible. Also, we have $y_2 = C_2 x_2 + D_2 y_2 = 0$, and from (12) we have $x_1 = 0$ since $A_1$ has no have eigenvalue at the origin. That leads to $(x_1, x_2) = 0$ which is not allowed, thus $y_1$ must be nonzero. However, (16) contradicts with the DC gain condition $\lambda_{\text{max}}(G(0)\overline{H(0)}) < 1$. Therefore, we have shown by contradiction that the closed loop system does not have eigen values on the imaginary axis. From that we conclude that the feedback interconnection of $G(s)$ and $H(s)$ is internally stable.

The next corollary shows that the result in Xiong et al. (2010) for the internal stability of positive feedback interconnections of NI systems is still true.

**Corollary 1.** Given a NI transfer function matrix $G(s)$ and a SNI transfer function matrix $H(s)$ and assume that $G(\infty)H(\infty) = 0$ and $H(\infty) \geq 0$. Then, the feedback interconnection of $G(s)$ and $H(s)$ is internally stable if and only if $\lambda_{\text{max}}(G(0)H(0)) < 1$.

**Proof.** This result follows from Theorem 1 and the necessity part of Theorem 5 of Xiong et al. (2010) for which a correct proof has already been given in Xiong et al. (2010).
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