PRODUCTS OF CONJUGACY CLASSES IN $SU(2)$

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ABSTRACT. We obtain a complete description of the conjugacy classes $C_1, \ldots, C_n$ in $SU(2)$ with the property that $C_1 \cdots C_n = SU(2)$. The basic instrument is a characterization of the conjugacy classes $C_1, \ldots, C_{n+1}$ in $SU(2)$ with $C_1 \cdots C_{n+1} \ni I$, which generalizes a result of [Je-We].

1. Introduction

The following problem was posed by D. Burago:

**Problem:** Let $G$ be a group. For which conjugacy classes $C_1, \ldots, C_n$ of $G$ is it true that the multiplication map

$$C_1 \times \cdots \times C_n \to G$$

is surjective?

We give a solution to this problem in the case $G = SU(2)$. In this case the conjugacy classes are parametrized by their eigenvalues

$$\text{diag}(e^{i\lambda}, e^{-i\lambda})$$

so they are determined by one number $\lambda \in [0, \pi]$.

Burago’s interest was primarily in discrete groups. The purpose of this note is to point out that the problem he posed is also of interest for Lie groups such as $SU(2)$, and to exhibit a solution in that case.

For more general Lie groups $G = SU(n)$ the problem could be studied by adapting results on the quantum cohomology of Grassmannians: see [Ag-Wo]. The problem is related to recent results described in the article [KLM].

2. Eigenvalues of a multiple product

For any $\lambda \in [0, \pi]$ we denote by $C(\lambda)$ the conjugacy class of the matrix

$$\text{diag}(e^{i\lambda}, e^{-i\lambda})$$
in $SU(2)$. Note that any conjugacy class in $SU(2)$ is of the form $C(\lambda)$ for a unique $\lambda \in [0, \pi]$. The following result was proved in [Je-We] (Proposition 3.1):

**Proposition 2.1.** For $\lambda_1, \lambda_2, \lambda_3 \in [0, \pi]$ we have

$$C(\lambda_1)C(\lambda_2)C(\lambda_3) \ni I$$

iff

(1) $|\lambda_1 - \lambda_2| \leq \lambda_3 \leq \min\{\lambda_1 + \lambda_2, 2\pi - (\lambda_1 + \lambda_2)\}$

Note that (1) is equivalent to

$$\lambda_1 + \lambda_2 + \lambda_3 \leq 2\pi$$

$$-\lambda_1 - \lambda_2 + \lambda_3 \leq 0$$

$$-\lambda_1 + \lambda_2 - \lambda_3 \leq 0$$

$$\lambda_1 - \lambda_2 - \lambda_3 \leq 0.$$  

The goal of this section is to prove the more general result:

**Theorem 2.2.** For $n \geq 2$ an integer and $\lambda_1, \ldots, \lambda_{n+1} \in [0, \pi]$ we have

(2) $C(\lambda_1) \ldots C(\lambda_{n+1}) \ni I$

iff the following system of inequalities holds:

a) If $n + 1 = 2k$ is an even number:

(3) $S_{2k-1}^1(\{\lambda_i\}) \leq (n - 1)\pi$, $S_{2k}^2(\{\lambda_i\}) \leq (n - 3)\pi$, \ldots, $S_{2k}^{2k-1}(\{\lambda_i\}) \leq 0$

where $S_{2k}^j(\{\lambda_i\})$ is any sum of the type $\sum_{i=1}^{n+1} \pm \lambda_i$ which contains exactly $j$ minus signs.

b) If $n + 1 = 2k + 1$ is an odd number:

(4) $S_{2k+1}^0(\{\lambda_i\}) \leq n\pi$, $S_{2k+1}^1(\{\lambda_i\}) \leq (n - 2)\pi$, \ldots, $S_{2k+1}^{2k}(\{\lambda_i\}) \leq 0$

where $S_{2k+1}^j(\{\lambda_i\})$ has the same meaning as before.

**Remarks.** 1. A more concise way to express both (3) and (4) is

$$S_{2k+1}^{n-2j}(\{\lambda_i\}) \leq 2j\pi$$

for any $0 \leq j \leq n/2$ and any sum of the type $S_{2k+1}^{n-2j}$.

2. An elementary computation involving the binomial formula shows that the number of inequalities in both (3) and (4) is

$$\binom{n+1}{0} + \binom{n+1}{2} + \ldots = \binom{n+1}{1} + \binom{n+1}{3} + \ldots = 2^n.$$
We will use induction on \( n \) to prove this theorem. In order to make the induction step we will need the following result:

**Lemma 2.3.** The condition (2) holds iff there exists \( \lambda \in [0, \pi] \) such that

\[
C(\lambda_1)\ldots C(\lambda_{n-1})C(\lambda) \ni I \tag{5}
\]

and

\[
C(\lambda)C(\lambda_n)C(\lambda_{n+1}) \ni I. \tag{6}
\]

**Proof.** The fundamental group of the \((n + 1)\)-punctured sphere \( \Sigma_{n+1} \) in two dimensions is the free group on \( n \) generators, or the group

\[
\Pi_n = \langle x_1, \ldots, x_{n+1} \mid x_1 \ldots x_{n+1} = 1 \rangle
\]

with \( n + 1 \) generators and one relation. We can form the \((n + 1)\)-punctured sphere by gluing an \( n \)-punctured sphere and a 3-punctured sphere along one of the boundary components of each. Call \( S \) the common boundary resulting from this construction and consider the fundamental groups of the two components as follows:

\[
\Pi_{n-1} = \langle x_1, \ldots, x_{n-1}, x \mid x_1 \ldots x_{n-1}x = 1 \rangle
\]

and

\[
\Pi_2 = \langle x', x_n, x_{n+1} \mid x'x_nx_{n+1} = 1 \rangle,
\]

where \( x \) and \( x' \) represent the loop \( S \) in each of the two components. From the theorem of Seifert-van Kampen, we have that

\[
\Pi_n = (\Pi_{n-1} \times \Pi_2)/\langle xx' = 1 \rangle \tag{7}
\]

Now we consider representations of these groups into \( G = SU(2) \). The condition (2) is equivalent to the existence of a representation \( \rho \) of \( \Pi_{n+1} \) such that

\[
\rho(x_i) \in C(\lambda_i)
\]

for any \( 1 \leq i \leq n + 1 \). From (5), this is equivalent to the existence of a representation \( \rho_{n-1} \) of \( \Pi_{n-1} \) which coincides with \( \rho \) on \( x_1, \ldots, x_{n-1} \), and a representation \( \rho_2 \) of \( \Pi_2 \) which coincides with \( \rho \) on \( x_n \) and \( x_{n+1} \), and such that \( \rho_{n-1} \) and \( \rho_2 \) satisfy

\[
\rho_{n-1}(x)\rho_2(x') = I.
\]

The latter equality implies that the conjugacy classes of \( \rho_{n-1}(x) \) and \( \rho_2(x') \) are equal, call them \( C(\lambda) \) (note that in \( SU(2) \) every element is conjugate to its inverse). The conditions (3) and (4) correspond respectively to the representations \( \rho_{n-1} \) and \( \rho_2 \).

\( \square \)
Proof of Theorem 2.2. Just the induction step has to be performed. We want to prove that
\[ C(\lambda_1) \ldots C(\lambda_{n+1}) \ni I \]
iff equation (3) or (4) holds. Suppose that \( n = 2k \) is an even number. Condition (3) of Lemma 2.3 is equivalent to
\[ S_n^1(\lambda_1, \ldots, \lambda_{n-1}, \lambda) \leq (n-2)\pi, \quad S_n^3(\lambda_1, \ldots, \lambda_{n-1}, \lambda) \leq (n-4)\pi, \ldots, S_n^{2k-1}(\lambda_1, \ldots, \lambda_{n-1}, \lambda) \leq 0 \]
where we have used the induction hypothesis, and condition (3) is equivalent to
\[ |\lambda_n - \lambda_{n+1}| \leq \lambda \leq \min\{\lambda_n + \lambda_{n+1}, 2\pi - (\lambda_n + \lambda_{n+1})\} \]
where we have used Proposition 2.1. By Lemma 2.3, condition (2) is equivalent to the system of inequalities obtained by considering each of the \( 2^n-1 \) inequalities from (8) and deriving from it two inequalities, as follows:

(i) if \( \lambda \) occurs with a \textit{plus} sign in that sum, replace it by \( \lambda_n - \lambda_{n+1} \) and \( -\lambda_n + \lambda_{n+1} \)
(ii) if \( \lambda \) occurs with a \textit{minus} sign in that sum, replace it by \( \lambda_n + \lambda_{n+1} \) and \( -\lambda_n - \lambda_{n+1} \),
but in the latter situation add \( 2\pi \) to the right hand side of the original inequality.

One sees that in the case (i) we replace an inequality of the type
\[ S_n^j \leq (n-j-1)\pi \]
by two different inequalities, both of the type
\[ S_n^{j+1} \leq (n-j-1)\pi. \]
In the case (ii) one again replaces an inequality of the type (10) by an inequality of the type (11) and an inequality of the type
\[ S_{n+1}^{j-1} \leq (n-j+1)\pi. \]
One obtains \( 2^n \) distinct inequalities of type (4), which means that (2) is really equivalent to
(4).

A similar argument can be used when \( n = 2k - 1 \) is an odd number. \( \square \)

Remark. The result stated in Theorem 1.2 can also be obtained from [Ag-Wo, Theorem 3.1] by using the structure of the quantum cohomology ring of \( \mathbb{C}P^1 \). More precisely, let us consider the two Schubert classes in \( H^*(\mathbb{C}P^1) \):
\[ [\sigma_1] \in H^2(\mathbb{C}P^1) \quad \text{and} \quad [\sigma_2] = 1 \in H^0(\mathbb{C}P^1). \]
The quantum cohomology ring of \( \mathbb{C}P^1 \) is
\[ QH^*(\mathbb{C}P^1) = (H^*(\mathbb{C}P^1) \otimes \mathbb{R}[q], \ast), \]
where \( q \) is a formal variable of degree 4 and \( \star \) is an \( \mathbb{R}[q] \)-linear, commutative and associative product which satisfies

\[
[\sigma_1] \star [\sigma_1] = q.
\]

Each of the \( 2^n \) inequalities indicated in Theorem 2.2 can be obtained by choosing \( i_1, \ldots, i_n \in \{1, 2\} \) and evaluating the product

\[
[\sigma_{i_1}] \star \ldots \star [\sigma_{i_n}]
\]

in \( QH^*(\mathbb{C}P^1) \). By the equation (12), this product is of the form \( q^d\sigma_k \), where \( d \) is a positive integer and \( k \in \{1, 2\} \). The inequality of the type (3) or (4) which corresponds to \( i_1, \ldots, i_n \) is

\[
\sum_{j=1}^{n} (-1)^{i_j} \lambda_j + (-1)^k \lambda_{n+1} \leq 2d\pi.
\]

3. Surjectivity of a multiple product

Our main result is

**Theorem 3.1.** We have

\[
C(\lambda_1) \ldots C(\lambda_n) = SU(2)
\]

iff for any integer \( j \) with \( 0 \leq j < n/2 \) and for any sum of the type \( S^j_n = S^j_n(\{\lambda_i\}) \) (see Thm.1.2) we have

\[
-(j - 1)\pi \leq S^j_n \leq (n - j - 1)\pi.
\]

**Proof.** The idea of the proof is that (13) holds iff (2) holds for any \( \lambda_{n+1} \in [0, \pi] \). In turn, (2) is equivalent to (3) and (4). We just have to take each inequality from (3) (respectively (4)) and make the following formal replacements in its left hand side:

(i) \( \lambda_{n+1} \) by \( \pi \)
(ii) \( -\lambda_{n+1} \) by 0.

Let us consider the case \( n = 2k - 1 \). We have to show that if we perform (i) and (ii) for each inequality contained in (2), we obtain exactly one of the following inequalities:

\[
\pi \leq S^0_n \leq (n - 1)\pi\\
0 \leq S^1_n \leq (n - 2)\pi\\
-\pi \leq S^2_n \leq (n - 3)\pi\\
-2\pi \leq S^3_n \leq (n - 4)\pi
\]

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We claim that if we label the inequalities given by (2) as [1], [3], ..., [2k − 3], [2k − 1] then each of [1] and [2k − 1] gives exactly one of (14) and (16), each of [3] and [2k − 3] gives exactly one of (17) and (18), ... and finally

- if \( k = 2p \) is even, then each of \([2p − 1]\) and \([2p + 1]\) gives exactly one of
  
  \[-(k − 1)\pi \leq S_n^k \leq (k − 2)\pi\]

  \[-(k − 2)\pi \leq S_n^{k−1} \leq (k − 1)\pi\]

- if \( k = 2p + 1 \) is odd, then each of \([2p + 1]\) gives exactly one of
  
  \[-(k − 2)\pi \leq S_n^{k−1} \leq (k − 1)\pi\]

Consider first [1] together with [2k − 1]: the only \( S_{n+1}^1 \) which contains \(-\lambda_{n+1}\) leads to

\[\lambda_1 + \ldots + \lambda_n \leq (n − 1)\pi\]

whereas the only \( S_{n+1}^{2k−1} \) which contains \(\lambda_{n+1}\) leads to

\[\lambda_1 + \ldots + \lambda_n \geq \pi.\]

The remaining inequalities of type \( S_{n+1}^1 \leq (n − 1)\pi \) lead to all possible inequalities of the type

\[S_n^1 \leq (n − 2)\pi\]

and the remaining inequalities of the type \( S_{n+1}^{2k−1} \leq 0 \) lead to all possible inequalities of the type

\[S_n^1 \geq 0.\]

The same idea applies\(^1\) to each pair \([2j + 1], [2(k − j) − 1], 0 \leq j < k/2 \) (if \( k = 2p + 1 \) is an odd number, then for \( j = p \) we have \( 2j + 1 = 2(k − j) − 1 \) and the corresponding pair reduces to just one type of inequalities).

Similar ideas can be used in the case when \( n = 2k \) is an even number.

\[\square\]

**Remark.** The system of inequalities (14) admit solutions for any \( n \geq 2 \). For \( n = 2 \) the unique solution is

\[\lambda_1 = \lambda_2 = \frac{\pi}{2};\]

For \( n \geq 3 \) there are several solutions, one of them consisting of \( \lambda_1, \lambda_2 \) given by (19) and

\[\lambda_3 = \ldots \lambda_n = 0.\]

\(^1\)If we compare the total number of inequalities we start with to the number of inequalities obtained via (i) and (ii) we “deduce” that \( \binom{n+1}{2j+1} + \binom{n+1}{n+1-(2j+1)} = 2(\binom{n+1}{2j+1} + \binom{n+1}{2j}) \). The latter equation is obviously true, by properties of Pascal’s triangle.
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