Complex dynamics and coexistence of period-doubling and period-halving bifurcations in an integrated pest management model with nonlinear impulsive control

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Abstract
An expectation for optimal integrated pest management is that the instantaneous numbers of natural enemies released should depend on the densities of both pest and natural enemy in the field. For this, a generalised predator–prey model with nonlinear impulsive control tactics is proposed and its dynamics is investigated. The threshold conditions for the global stability of the pest-free periodic solution are obtained based on the Floquet theorem and analytic methods. Also, the sufficient conditions for permanence are given. Additionally, the problem of finding a nontrivial periodic solution is confirmed by showing the existence of a nontrivial fixed point of the model’s stroboscopic map determined by a time snapshot equal to the common impulsive period. In order to address the effects of nonlinear pulse control on the dynamics and success of pest control, a predator–prey model incorporating the Holling type II functional response function as an example is investigated. Finally, numerical simulations show that the proposed model has very complex dynamical behaviour, including period-doubling bifurcation, chaotic solutions, chaos crisis, period-halving bifurcations and periodic windows. Moreover, there exists an interesting phenomenon whereby period-doubling bifurcation and period-halving bifurcation always coexist when nonlinear impulsive controls are adopted, which makes the dynamical behaviour of the model more complicated, resulting in difficulties when designing successful pest control strategies.

Keywords: Nonlinear impulsive; Threshold condition; Bifurcation; Nontrivial periodic solution; Chaos

1 Introduction
In order to reduce losses due to insect pests, a wide range of pest control strategies are available to farmers [1, 2]. Integrated pest management (IPM) is a long-term management strategy involving a combination of biological, cultural and chemical tactics aiming to minimise economic, health and environmental risks [3, 4].
An important component of an IPM strategy is biological control, which is defined as using natural enemies such as predators and parasites to suppress pest populations and typically involves an active human role [2]. For example, one known approach of biological control is augmentation, i.e. releasing beneficial natural enemies to control insects to tolerable levels [5–7]. Chemical control is also an important method for IPM, being very useful because it can kill a significant proportion of an insect population quickly, and sometimes it provides the only feasible method for preventing economic loss. In practical application, in order to avoid or delay the development of resistance, a variety of different types of pesticides are used instead of a single one. Moreover, different pest control techniques should function together rather than be antagonistic to each other. Experience with use of IPM has shown it to be more effective than classical control methods such as biological control or chemical control applied alone [2].

With the development of the theory and application of impulsive differential equations [8, 9], mathematical models can assist in the design of IPM strategies and the study of how to control pest populations at tolerable levels. Recently, many researchers have constructed impulsive differential equations to describe IPM strategies [10–15]. However, the major assumption in previous studies is that a proportion of the pest population will be killed instantly after spraying pesticide, while simultaneously releasing a constant number of natural enemies [16–22]. In reality, every community or country has an appropriate or limited capacity for agricultural resources such as pesticides, labour force, equipment and excessive use of the pesticides will potentially cause damage to the environment. So, the methods used for releasing natural enemies and the ratios of numbers of natural enemies to be released to their current density in the field could be significantly affected by limitations to agricultural resources. In order to understand how such a resource limitation could affect an IPM strategy, some mathematical models incorporating saturation of the limited resources have been proposed and analysed recently. These have mainly focused on investigating the global dynamic and addressing how nonlinear impulsive control actions affect pest control tactics [23–26]. However, the nonlinear impulsive functions in these papers only depend on the density of the natural enemy population. A more realistic case modelled in this paper is that the instantaneous number of natural enemies to be released should depend on the densities of both pest and natural enemy densities in the field, i.e. the density dependent releasing function is a saturation function of the pest population. In this case, the lower the pest population in the field, the lower the number of the predators that should be released and vice versa.

The main purpose of this paper is to construct a generalised predator–prey model with nonlinear impulsive control to investigate the effect of limited predator releases on the outbreak of pest populations. Other generalised predator–prey models with linear or constant pulse actions have already provided some analytical techniques to deal with the global stability of periodic solutions [27–29]. In Sect. 2, by using the Floquet theorem and analytical techniques, we show that there exists a globally stable pest-free periodic solution when the period of impulsive predator releases is less than some threshold, and the condition for the permanence of the system is given in Sect. 3. Next, by employing an operator theoretic approach which reduces the existence of the nontrivial periodic solutions to a fixed point and bifurcation problem [30–33], we show that a nontrivial periodic solution appears via a supercritical bifurcation once the pest-free periodic solution loses its stability. In order to apply the main results, we chose a classic pest-natural enemy model with a
Holling type II functional response function and nonlinear impulsive control to investigate how the nonlinear pulse perturbations affect success or otherwise of the pest control, which is presented in Sect. 5. Further, numerical simulations show that the model with nonlinear impulsive actions has more complex and interesting dynamic behaviour than models without them. Finally, related biological implications are discussed.

2 The predator–prey model with nonlinear pulse control

The generalised predator–prey model we consider in this paper is as follows:

\[
\begin{align*}
\frac{dx(t)}{dt} &= rx(t)(1 - \frac{x(t)}{K}) - p(x(t))y(t), \\
\frac{dy(t)}{dt} &= cp(x(t))y(t) - Dy(t),
\end{align*}
\]

where \( x(t), y(t) \) represent the densities of prey and predator populations, respectively. \( r \) is the intrinsic growth rate of the prey, \( K \) denotes the carrying capacity of the prey, \( c \) represents the efficiency rate for predator conversion of its prey, \( D \) is the death rate of the predator. The function \( p(x) \) denotes the predator response function, and let \( F(x) = r(x(1 - \frac{\lambda}{x})) / p(x) \). In order to use our main results for as many models as possible, we made the following assumptions related to the function \( p(x) \). Let \( p(x) \) be locally Lipschitz functions such that:

(i) The functional response function satisfies \( p(x) \in C^2(0, +\infty) \), \( p(0) = 0 \), \( p'(0) > 0 \) and \( p''(x) < 0 \) for all \( x > 0 \),

(ii) The function \( F(x) \) and \( \frac{p(x)}{x} \) is upper bounded for all \( x > 0 \).

The first condition indicates that the functional response function is positive and monotonically increasing for small pest populations. The \( p(x) \) is bounded or linear, so \( \frac{p(x)}{x} \) is upper bounded for all \( x > 0 \). The pest population cannot increase infinitely once the biological control is introduced, so \( F(x) \) is upper bounded. It is assumed that the IPM strategy is applied at every time point \( nT \). Moreover, the nonlinear saturation functions or density dependent functions are employed, i.e. we choose

\[
x(t^+) = \left[ 1 - \frac{\delta x(t)}{x(t) + h} \right] x(t), \quad y(t^+) = y(t) + \frac{\lambda_1 x(t)}{1 + \theta x(t)} + \lambda_2, \quad t = nT,
\]

where \( 0 \leq \delta \leq 1 \) and \( h \geq 0 \) represent the maximal mortality rate and the half saturation constant for the pest due to chemical control, respectively. We assume that the pesticide is sprayed immediately before releases of natural enemies to minimise the chance of the pesticide killing newly released predators. \( \lambda_1 \geq 0 \) is the number of the natural enemies released, and \( \theta \geq 0 \) denotes the shape parameter. The pest and natural enemy populations are updated to \( (1 - \frac{\delta x(t)}{x(t) + h} + \lambda x(t)) \) at every discrete time point \( nT \) and \( n \in \mathbb{N} \), respectively, which is a more realistic consideration compared with previous studies [25, 26].

On the basis of the above assumptions and taking model (1) into account, the following differential equation model was formulated:

\[
\begin{align*}
\frac{dx(t)}{dt} &= rx(t)(1 - \frac{x(t)}{K}) - p(x(t))y(t), \\
\frac{dy(t)}{dt} &= cp(x(t))y(t) - Dy(t), \\
x(t^+) &= \left[ 1 - \frac{\delta x(t)}{x(t) + h} \right] x(t), \\
y(t^+) &= y(t) + \frac{\lambda_1 x(t)}{1 + \theta x(t)} + \lambda_2,
\end{align*}
\]

t \neq nT, \quad t = nT.

(2)
Before our main results, we show the positivity and boundedness of solutions of model (2), and we have the following.

**Lemma 1** The solutions of model (2) are positive and bounded.

**Proof** It is easy to show that the positive initial conditions indicate the positivity of the solutions, and $x(t) < \max\{K, x(0^+)\}$ according to $x(nT^+) < x(nT)$.

For the boundedness of the $y$-component, we denote $V(t) = cx(t) + y(t)$, then when $t \neq nT$ we obtain

$$D^+ V(t) + DV(t) = c(r + D)x(t) - \frac{crx^2(t)}{K} \leq M_0$$

with $M_0 = \frac{cK(r+D)^2}{4r}$. When $t = nT$ we have

$$V(nT^+) \leq cx(nT) + y(nT) + \frac{\lambda_1}{\theta} + \lambda_2$$

$$= V(nT) + \frac{\lambda_1}{\theta} + \lambda_2.$$

Therefore, for $t \in (nT, (n+1)T]$, we have

$$V(t) \leq V(0) \exp(-Dt) + \int_0^t M_0 \exp(-D(t-s)) \, ds$$

$$+ \sum_{0 \leq kT < t} \left(\frac{\lambda_1}{\theta} + \lambda_2\right) \exp(-D(t-kT))$$

$$= V(0) \exp(-Dt) + \frac{M_0(1 - \exp(-Dt))}{D}$$

$$+ \left(\frac{\lambda_1}{\theta} + \lambda_2\right) \frac{\exp(-D(t-T)) - \exp(-D(t-(n+1)T))}{1 - \exp(DT)}$$

$$\rightarrow \frac{M_0}{D} + \frac{\left(\frac{\lambda_1}{\theta} + \lambda_2\right) \exp(DT)}{\exp(DT) - 1}, \quad (t \to \infty).$$

Thus, $V(t)$ is uniformly ultimately bounded. According to the definition of $V(t)$, we can see that $y(t)$ is bounded. \qed

One of the main purposes of implementing IPM is to find suitable control measures to eradicate the pest population, i.e. if there exists a critical value such that pest-free periodic solution is globally stable when the period $T$ is less than the critical value. The existence and stability of the pest-free periodic solution are crucial for this point. Thus, we first consider the following subsystem:

$$\begin{cases}
\frac{dx(t)}{dt} = -Dy(t), & t \neq nT, \\
y(t^+) = y(t) + \lambda_2, & t = nT.
\end{cases} \quad (3)$$

Subsystem (3) is a simple linear growth model, so we can easily obtain the following main result.
Lemma 2 Model (3) exists with a globally stable positive periodic solution
\[ y_p(t) = \frac{\lambda}{1 - \exp(-DT)} \exp(-D(t - nT)), \quad t \in (nT, (n + 1)T]. \]

Therefore, we obtain the general expression of the unique pest-free periodic solution of model (2), i.e.
\[ (x_p(t), y_p(t)) = (0, y^* \exp(-D(t - nT))), \quad t \in (nT, (n + 1)T], \]

where \( y^* = \frac{\lambda}{1 - \exp(-DT)}. \)

In the following, we show the sufficient condition for the global stability of pest-free periodic solution \((x_p(t), y_p(t))\) of model (2) and the main results for model (2).

Theorem 1 The pest-free periodic solution \((x_p(t), y_p(t))\) is locally stable provided that
\[ T < \frac{\lambda p'(0)}{rD}, \quad (4) \]

and it is globally attractive if
\[ T < \frac{\lambda}{M_sD}, \quad (5) \]

where \( M_s = \sup_{x \geq 0} F(x). \)

Proof Firstly, we prove the local stability of the solution \((x_p(t), y_p(t))\) of model (2). Let \( \Phi(t) \) be the fundamental matrix of (2), and then \( \Phi(t) \) satisfies
\[ \Phi(T) = \begin{pmatrix} \exp(\int_0^T (r - p'(0)y_p(t)) \, dt) & 0 \\ * & \exp(\int_0^T -D dt) \end{pmatrix}, \]

where \( \Phi(0) = I \) represents the identity matrix and the term * is not calculated in the exact form for the next analyses. The linearization of the third and fourth equation of model (2) becomes
\[ \begin{pmatrix} \tilde{x}(nT^+) \\ \tilde{y}(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda_1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}(nT) \\ \tilde{y}(nT) \end{pmatrix} = B(T) \begin{pmatrix} \tilde{x}(nT) \\ \tilde{y}(nT) \end{pmatrix}. \]

Therefore, the local stability of the solution \((x_p(t), y_p(t))\) of model (2) is determined by the absolute values of both eigenvalues of the following matrix:
\[ M = B(T) \Phi(T) \]
\[ = \begin{pmatrix} \exp(\int_0^T (r - p'(0)y_p(t)) \, dt) & 0 \\ ** & \exp(-DT) \end{pmatrix}. \]

The two Floquet multipliers are as follows:
\[ |\mu_1| = \exp\left(\int_0^T (r - p'(0)y_p(t)) \, dt\right), \quad |\mu_2| = \exp(-DT) < 1. \]
All these results confirm that the pest-free solution \((x_p(t), y_p(t))\) is locally stable if and only if
\[
 rT - p'(0) \int_0^T y_p(t) \, dt < 0.
\]
Since \(\int_0^T y_p(t) \, dt = \frac{\lambda_2}{D}\), consequently the periodic solution \((x_p(t), y_p(t))\) is locally stable provided that
\[
 T < \frac{\lambda_2 p'(0)}{rD}.
\]

In the following, we prove the global attractivity of the periodic solution \((x_p(t), y_p(t))\), for which we only need to show that any solution \((x(t), y(t))\) of model (2) tends to \((x_p(t), y_p(t))\) as \(t\) goes to infinity. If \(T < \frac{\lambda_2}{M_D}\), then we can choose \(\varepsilon > 0\) sufficiently small such that
\[
 \int_0^T (M_s - (y_p(s) - \varepsilon)) \, ds < 0.
\]
From model (2) we obtain
\[
 \begin{cases}
 \frac{dy(t)}{dt} \geq -Dy(t), & t \neq nT, \\
 y(t^+) \geq y(t) + \lambda_2, & t = nT.
\end{cases}
\]
According to the theory of differential equations, we have the following comparison equation:
\[
 y(t) \geq y_p(t) - \varepsilon
\]
for \(t\) large enough. Without loss of generality, we assume that \(y(t) \geq y_p(t) - \varepsilon\) holds true for all \(t \geq 0\).

From the first equation of model (2), we get \(\frac{\delta y(t)}{dt} = F(x) - y\). Define \(G(x) = \int_0^x \frac{1}{p(s)} \, ds\) for \(\delta > 0\). Integrating any interval \([nT, (n+1)T]\), we have
\[
 G(x(n+1)T^+) = \int_\delta^{x(n+1)T^+} \frac{1}{p(s)} \, ds + \int_0^{x(n+1)T^+} \frac{1}{p(s)} \, ds
\]
\[
 \leq G(x(nT^+)) + \int_{nT}^{(n+1)T} (F(s) - (y_p(s) - \varepsilon)) \, ds
\]
\[
 \leq G(x(nT^+)) + \int_{nT}^{(n+1)T} (M_s - (y_p(s) - \varepsilon)) \, ds.
\]
For any \(t \in (lT, (l + 1)T], \ l \in Z_+\), we get
\[
 G(x(t)) - G(x_0) \leq \int_0^T (M_s - (y_p(s) - \varepsilon)) \, ds
\]
\[
 = l \int_0^T (M_s - (y_p(s) - \varepsilon)) \, ds + \int_{lT}^T (M_s - (y_p(s) - \varepsilon)) \, ds.
\]
For the boundedness of the second term of the right-hand side, it is clear that \(G(x) \to -\infty\) as \(t \to \infty\) when \(\int_0^T (M_s - (y_p(s) - \varepsilon)) \, ds < 0\), i.e. \(x(t)\) converges to 0 as \(t \to \infty\).
Next, we prove that \( y(t) \to y_p(t) \) as \( t \to \infty \). Since \( x(t) \) goes to zero as \( t \to \infty \) and assumption (i), there exists a finite time \( t_1 \) such that \( p(x) < \epsilon \) and \( \frac{\lambda_1 x(t)}{1 + \eta x(t)} < \epsilon \lambda_1 \) for all \( t > t_1 \). Therefore, for all \( t > t_1 \), we deduce that

\[
-Dy(t) \leq \frac{dy}{dt} \leq y(t)(c\epsilon - D).
\]

Thus, we consider the following equation:

\[
\begin{cases}
\frac{dx(t)}{dt} = (c\epsilon - D) z(t), & t \neq nT, \\
z(t^+) = z(t) + \epsilon \lambda_1 + \lambda_2, & t = nT.
\end{cases}
\]  

(7)

From the comparison theorem of impulsive differential equations, we have

\[
y_p(t) \leq y(t) \leq z_p(t),
\]

and \( z(t) \to z_p(t), z_p(t) \to y_p(t) \) as \( t \to \infty \). Therefore, there exists \( t_2 \) for \( \epsilon_1 \) small enough such that \( t_2 \geq t_1 > 0 \) and

\[
y_p(t) - \epsilon_1 < y(t) < z_p(t) + \epsilon_1
\]

for \( t > t_2 \). Let \( \epsilon \to 0 \), then \( y_p(t) - \epsilon_1 < y(t) < y_p(t) + \epsilon_1 \). Therefore, \( y(t) \to y_p(t) \) as \( t \to \infty \), i.e. the pest-free periodic solution of model (2) is globally asymptotically stable. \( \square \)

3 Permanent

Now, we investigate the permanence of model (2).

**Theorem 2** Model (2) is permanent provided that \( T > \frac{2y_p(0)}{\eta \theta} \) holds true.

**Proof** By Lemma 1, for simplicity, we may assume that \( x(t) < M, y(t) < M \) for \( t > 0, M > \frac{1}{\eta \theta} \). From (6), we know that \( y(t) \geq y_p(t) - \epsilon \) for \( t \) large enough and for some \( \epsilon > 0 \), so we have

\[
y(t) \geq \frac{\lambda_2 \exp(-DT)}{1 - \exp(-DT)} - \epsilon \equiv m_2
\]

(8)

for \( t \) large enough. Thus, we only need to find \( m_1 \) such that \( x(t) \geq m_1 \) for \( t \) large enough. We will do this in the following two steps.

1. Let \( m_3 > 0, \epsilon_1 > 0 \) be small enough such that \( cM_1 m_3 < D \) and \( \eta_1 \equiv \left(1 - \frac{\lambda_2}{\lambda_1} \right) \exp(rT - \frac{m_3 T}{x} - p'(0)(1 - \exp(-DT - cM_1 m_3))) + \epsilon_1 T > 1 \), where \( M_1 = \sup_{x \geq x_0} \frac{p(x)}{x} \). Suppose that \( x(t) < m_3 \) for all \( t > 0 \), then we have

\[
\frac{dy(t)}{dt} = y(t) \left( \frac{p(x)}{x} \cdot cx(t) - D \right) \leq (cM_1 m_3 - D)y(t).
\]

By Lemma 2, we have \( y(t) \leq z(t) \) and \( z(t) \to z_p(t) \) as \( t \to \infty \), where \( z(t) \) is the solution of

\[
\begin{cases}
\frac{dx(t)}{dt} = (cM_1 m_3 - D) z(t), & t \neq nT, \\
z(t^+) = z(t) + \lambda_1 m_3 + \lambda_2, & t = nT.
\end{cases}
\]

(9)
and $z_p(t) = \frac{\lambda_1 m_3 + \lambda_2}{1 - \exp(-D + cM_1 m_3)} \exp(-(D - cM_1 m_3)(t - nT)), t \in (nT, (n + 1)T]$. Therefore, there exists $T_1 > 0$ such that $y(t) \leq z(t) < z_p(t) + \varepsilon_1$.

According to assumption (i), we have $p(x) < p'(0)x$, further,

$$\begin{cases}
\frac{dx(t)}{dt} \geq x(t)(r - \frac{rM_3}{K} - p'(0)(z_p(t) + \varepsilon_1)), & t \neq nT, \\
x(t^*) \geq (1 - \frac{\delta}{h} m_3)x(t), & t = nT,
\end{cases}$$

(10)

for $t > T_1$. Let $N \in Z$, and $NT > T_1$. Integrating equation (10) on $(nT, (n + 1)T], n \geq N$, we have

$$x((n + 1)T) \geq x(nT^*) \exp\left(\int_{nT}^{(n+1)T} \left(r - \frac{rM_3}{K} - p'(0)(z_p(t) + \varepsilon_1)\right)dt\right) = x(nT)\eta.$$

Then $x((N + n)T) > x(NT)\eta^n \to \infty$ as $n \to \infty$, which contradicts the boundedness of $x(t)$. Hence there exists $t_1 > 0$ such that $x(t_1) \geq m_3$.

2. If $x(t) \geq m_3$ for all $t \geq t_1$, then our aim is obtained. Otherwise, let $t^* = \inf_{t \geq t_1} \{x(t) < m_3\}$, there are two possible cases for $t^*$.

Case (i) $t^* = n_1 T, n_1 \in Z$. Then $x(t) \geq m_3$ for $t \in [t_1, t^*]$ and $(1 - \frac{\delta M}{h + M})m_3 \leq x(t^*) = (1 - \frac{\delta m_3}{h + M})x(t^*) < m_3$. Choose $n_2, n_3 \in Z$, such that

$$n_2 T > \ln\left(\frac{\varepsilon_1}{M + \lambda_2 + \lambda_3 m_3}\right)/(-D + cM_1 m_3),$$

$$\left(1 - \frac{\delta M}{h + M}\right)^{n_2} \leq \frac{\lambda_1 m_3 + \lambda_2}{1 - \exp(-(D + cM_1 m_3))(t - t^*))} \exp((D + cM_1 m_3)(t - t^*)) = \frac{\lambda_1 m_3 + \lambda_2}{1 - \exp(-(D + cM_1 m_3))(t - t^*))},$$

where $\eta_1 = r - \frac{m_3 r}{K} - p'(0)M < 0$. Let $T' = n_2 T + n_3 T$, we show that there must be $t_2 \in (t^*, t^* + T']$ such that $x(t_2) \geq m_3$. Otherwise, considering (9) with $z(t^*) = y(t^*)$, we have

$$z(t) = \left(z(t^*) - \frac{\lambda_1 m_3 + \lambda_2}{1 - \exp(-(D + cM_1 m_3))}\right) \exp((D + cM_1 m_3)(t - t^*)) + z_p(t)$$

for $t \in (nT, (n + 1)T]$ and $n_1 \leq n \leq n_1 + n_2 + n_3$. Then

$$|z(t) - z_p(t)| < (M + \lambda_2 + \lambda_1 m_3) \exp((D + cM_1 m_3)(t - t^*)) < \varepsilon_1$$

for $t^* + n_2 T \leq t \leq t^* + T'$, which implies that (10) holds for $t^* + n_2 T \leq t \leq t^* + T'$. So, as in step 1, we have

$$x(t^* + T') \geq x(t^* + n_2 T)\eta^{n_2}.$$

Furthermore, we have

$$\begin{cases}
\frac{dx(t)}{dt} \geq x(t)(r - \frac{m_3 r}{K} - p'(0)M), & t \neq nT, \\
x(t^*) \geq (1 - \frac{\delta}{h} m_3)x(t), & t = nT,
\end{cases}$$

(11)
for \( t \in [t^*, t^* + n_2 T] \). Integrating (11) on \( t \in [t^*, t^* + n_2 T] \), we have
\[
x(t^* + n_2 T) \geq x(t^*) \left( 1 - \frac{\delta}{\bar{h} m_3} \right)^{n_2 - 1} \exp(n_2 \eta_1 T).
\]
Thus, we have
\[
x(t^* + T) \geq m_3 \left( 1 - \frac{\delta M}{h + M} \right) \left( 1 - \frac{\delta}{\bar{h} m_3} \right)^{n_2 - 1} \exp(n_2 \eta_1 T) \eta'^n > m_3,
\]
which is a contradiction.

Now, let \( \tilde{t} = \inf_{t > t^*} \{ x(t) \geq m_3 \} \), then for \( t \in (t^*, \tilde{t}) \), \( x(t) \leq m_3 \) and \( x(\tilde{t}) = m_3 \) since \( x(t) \) is left continuous and \( x(t^*) \leq x(t) \) when \( t = n'T \). For \( t \in (t^*, \tilde{t}) \), suppose \( t \in (t^* + (k - 1)T, t^* + kT) \), \( k \in Z_+ \), and \( k \leq n_2 + n_3 \), from (11) we have
\[
x(t) \geq x(t^*) \left( 1 - \frac{\delta}{\bar{h} m_3} \right)^{k - 1} \exp((k - 1)\eta_1 T) \exp(\eta_1 (t - (t^* + (k - 1)T)))
\]
\[
\geq m_3 \left( 1 - \frac{\delta M}{h + M} \right) \left( 1 - \frac{\delta}{\bar{h} m_3} \right)^{k} \exp(k\eta_1 T)
\]
\[
\geq m_3 \left( 1 - \frac{\delta M}{h + M} \right) \left( 1 - \frac{\delta}{\bar{h} m_3} \right)^{n_2 + n_3} \exp((n_2 + n_3)\eta_1 T).
\]
Let \( m'_1 = m_3 (1 - \frac{\delta M}{h + M}) (1 - \frac{\delta}{\bar{h} m_3})^{n_2 + n_3} \exp((n_2 + n_3)\eta_1 T) \), so we have \( x(t) \geq m'_1 \) for \( t \in (t^*, \tilde{t}) \).

For \( t > \tilde{t} \), the same arguments can be continued since \( x(\tilde{t}) \geq m_3 \).

Case (ii) \( t^* \neq n_1 T \), \( n_1 \in Z_+ \). Then \( x(t) \geq m_3 \) for \( t \in [t_1, t^*) \) and \( x(t^*) = m_3 \). Suppose \( t^* \in (n_1' T, (n_1' + 1)T) \), \( n_1' \in Z_+ \), there are two possible cases for \( t \in (t^*, (n_1' + 1)T) \).

Case (iia) \( x(t) < m_3 \) for \( t \in (t^*, n_1' + 1)T \). By a similar argument as in step case (i), we get that there must exist \( t'_2 \in [(n_1' + 1)T, (n_1' + 1)T + T'] \) such that \( x(t'_2) \geq m_3 \).

Let \( \tilde{t} = \inf_{t > t^*} \{ x(t) \geq m_3 \} \), then \( x(t) < m_3 \) for \( t \in (t^*, \tilde{t}) \) and \( x(\tilde{t}) = m_3 \). For \( t \in (t^*, \tilde{t}) \), suppose \( t \in (n_1' T + (k - 1)T, (n_1' + 1)T + kT) \), \( k \in Z_+ \), \( k \leq 1 + n_2 + n_3 \), we have
\[
x(t) \geq m_3 \left( 1 - \frac{\delta M}{h + M} \right) \left( 1 - \frac{\delta}{\bar{h} m_3} \right)^{k - 1} \exp(k\eta_1 T)
\]
\[
\geq m_3 \left( 1 - \frac{\delta M}{h + M} \right) \left( 1 - \frac{\delta}{\bar{h} m_3} \right)^{n_2 + n_3} \exp((n_2 + n_3 + 1)\eta_1 T).
\]
Let \( m_1 = m_3 (1 - \frac{\delta M}{h + M}) (1 - \frac{\delta}{\bar{h} m_3})^{n_2 + n_3} \exp((n_2 + n_3 + 1)\eta_1 T) \), so \( x(t) \geq m_1 \) for \( t \in (t^*, \tilde{t}) \).

For \( t > \tilde{t} \), the same arguments can be continued since \( x(\tilde{t}) \geq m_3 \).

Case (iib) There exists \( t \in (t^*, (n_1' + 1)T) \) such that \( x(t) \geq m_3 \). Let \( \tilde{t} = \inf_{t > t^*} \{ x(t) > m_3 \} \), then \( x(t) < m_3 \) for \( t \in (t^*, \tilde{t}) \) and \( x(\tilde{t}) = m_3 \). For \( t \in (t^*, \tilde{t}) \), (11) holds true, integrating on \( [t^*, \tilde{t}] \), we have
\[
x(t) \geq x(t^*) \exp(\eta_1 (t - t^*)) \geq m_3 \left( 1 - \frac{\delta M}{h + M} \right) \exp(\eta_1 T) > m_1.
\]
Hence \( x(t) \geq m_1 \) for all \( t > t_1 \). The proof is completed. \qed
Remark Define \( T^* = \frac{-\lambda_2 p'(0)}{\lambda_2} \), the pest-free periodic solution is unstable if \( T > T^* \). Therefore, \( T^* \) plays a role in a bifurcation threshold value. The interesting question is how to determine the dynamics of model (2) once the pest-free periodic solution loses its stability.

4 Threshold condition of bifurcation

In the following, we deal with problem of the bifurcation of a nontrivial periodic solution of model (2) near the pest-free periodic solution, i.e. \((x_p(t), y_p(t))\). We have used the bifurcation theory in earlier publications [34, 35]. We denote \( \Phi \) to be the solution of the first two equations of model (2), and we have \( X(t) = \Phi(t, X_0) \) where \( X(0) = (y_0^*, 0) \) and \( \Phi = (\Phi_1, \Phi_2) \). We define the mapping \( \Theta_1, \Theta_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by

\[
\Theta_1(x, y) = x + \frac{\lambda_1 y}{1 + \theta y} + \lambda_2, \quad \Theta_2(x, y) = \left(1 - \frac{\delta y}{y + \delta}ight)y
\]

and the mapping \( F_1, F_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) as follows:

\[
F_1(x, y) = cp(y)x - Dx, \quad F_2(x, y) = ry \left(1 - \frac{y}{K}\right) - p(y)x.
\]

Furthermore, let us define \( \Psi : [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by

\[
\Psi(T, X_0) = \Theta(\Phi(T, X_0)); \Psi(T, X_0) = (\Psi_1(T, X_0), \Psi_2(T, X_0)).
\]

It is clear that \( \Psi \) is the stroboscopic map associated with model (2), which is determined by the values of solutions at impulsive points 0 and \( T \), and \( T \) is the stroboscopic time snapshot.

We reduce the problem of finding a periodic solution to a fixed problem. Hence, \( X = (y_p(t), 0) \) is a periodic solution of period \( T \) for model (2) if and only if its initial value \( X_0 = X(0) \) is a fixed point for map \( \Psi(T, \cdot) \). Consequently, in order to establish the existence of nontrivial periodic solutions of model (2), we need to prove the existence of the nontrivial fixed points of \( \Psi \).

To find a nontrivial periodic solution of period \( \tau \) with initial value \( X(0) \), we need to solve the fixed point problem \( X = \Psi(\tau, u) \). Denoting \( \tau = T + \tilde{T} \), \( X = X_0 + \tilde{X} \),

\[
X_0 + \tilde{X} = \Psi(T + \tilde{T}, X_0 + \tilde{X}).
\]

Let us define

\[
N(\tilde{T}, \tilde{X}) = (N_1(\tilde{T}, \tilde{X}), N_2(\tilde{T}, \tilde{X})) = X_0 + \tilde{X} - \Psi(T + \tilde{T}, X_0 + \tilde{X}), \tag{12}
\]

if \( N(\tilde{T}, \tilde{X}) = 0 \), then \( X_0 + \tilde{X} \) is a fixed point of \( \Psi(T, \cdot) \).

According to the variational equations

\[
\frac{d}{dt}(\Phi(t, X_0)) = F(\Phi(t, X_0)),
\]

which characterise the dynamics of the first two equations in model (2), we get

\[
\frac{d}{dt}(DX(\Phi(t, X_0))) = DXF(\Phi(t, X_0))(DX(\Phi(t, X_0))). \tag{13}
\]
Then we deduce that model (2) has the particular form

\[
\frac{d}{dt} \begin{pmatrix}
\frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_1}{\partial y} \\
\frac{\partial \Phi_2}{\partial x} & \frac{\partial \Phi_2}{\partial y}
\end{pmatrix}(t, X_0) = \begin{pmatrix}
\frac{\partial F_1(\zeta(t))}{\partial x} & \frac{\partial F_1(\zeta(t))}{\partial y} \\
\frac{\partial F_2(\zeta(t))}{\partial x} & \frac{\partial F_2(\zeta(t))}{\partial y}
\end{pmatrix}(t, X_0)
\]

\[
= \begin{pmatrix}
-Dc p'(0)y_p(t) & 0 \\
-r - p'(0)y_p(t) & 0
\end{pmatrix} \begin{pmatrix}
\frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_1}{\partial y} \\
\frac{\partial \Phi_2}{\partial x} & \frac{\partial \Phi_2}{\partial y}
\end{pmatrix}(t, X_0)
\]

with initial value \(D_X(\Phi(0, X_0)) = I_2\), which is the identity matrix.

According to the initial value, it follows that

\[
\frac{\partial \Phi_1(t, X_0)}{\partial x} = \exp(-Dt),
\]

\[
\frac{\partial \Phi_1(t, X_0)}{\partial y} = cp'(0) \int_0^t \exp(-D(t - \nu))y_p(\nu) \rho(\nu) \, d\nu,
\]

\[
\frac{\partial \Phi_2(t, X_0)}{\partial x} = 0,
\]

\[
\frac{\partial \Phi_2(t, X_0)}{\partial y} = \rho(t),
\]

where \(\rho(t) = \exp \int_0^t (r - p'(0)y_p(v)) \, dv\).

We denote the derivation of \(N\) at the fixed point \((\bar{t}, \bar{X}) = (0, (0, 0))\) as

\[
D_X N(0, (0, 0)) = \begin{pmatrix}
ad' & bd' \\
cd' & dd'
\end{pmatrix} = \begin{pmatrix}
1 - \frac{\partial \theta_1}{\partial x} - \frac{\partial \theta_1}{\partial y} - \frac{\partial \theta_2}{\partial x} - \frac{\partial \theta_2}{\partial y} & 0 \\
0 & 1 - \frac{\partial \theta_2}{\partial y}
\end{pmatrix}(T, X_0).
\]

(14)

So, we obtain

\[
a'_0 = 1 - \exp(-DT) > 0,
\]

\[
b'_0 = -cp'(0) \int_0^T y_p(t) \rho(t) \, dt - \lambda_1 \rho(T) < 0,
\]

\[
c'_0 = 0
\]

and

\[
d'_0 = 1 - \exp \left( rT - p'(0) \frac{\lambda_2}{D} \right)
\]

\[
= 1 - \rho(T).
\]

It is easy to see that the necessary condition for the bifurcation of the nontrivial solution is

\[
\det[D_X N(0, (0, 0))] = 0,
\]

(15)

which can reduce to \(d'_0 = 0\) because \(D_X N(0, (0, 0))\) is an upper triangular matrix with \(a'_0 > 0\).

Further, we deduce that \(d'_0 = 0\) is equivalent to \(T = T^*\). Therefore, it remains to prove the sufficient condition for the bifurcation of nontrivial solution. We have the following main result.
Theorem 3  Assume that $T = T^*$, the pest-free periodic solution of model (2) bifurcates to a positive nontrivial periodic solution, which is supercritical if $2r DT + p''(0)K\lambda_2 \geq 0$.

Proof  According to the above notations, it is that $\dim(\ker(D_N(0, (0, 0)))) = 1$ and a basis of $\ker(D_N(0, (0, 0))) = (-\frac{b_0'}{a_0}, 1)$. Then the equation $N(\tilde{\tau}, \tilde{X}) = 0$ is equivalent to

$$N_1(\tilde{\tau}, aY_0 + zE_0) = 0, \quad N_2(\tilde{\tau}, aY_0 + zE_0) = 0,$$

where $E_0 = (1, 0), Y_0 = (-\frac{b_0'}{a_0}, 1)$ and $\tilde{X} = aY_0 + zE_0$ represents the direct summation of $\tilde{X}$.

Let us define

$$f_1(\tilde{\tau}, a, z) = N_1(\tilde{\tau}, aY_0 + zE_0), \quad f_2(\tilde{\tau}, a, z) = N_2(\tilde{\tau}, aY_0 + zE_0),$$

and have

$$\frac{\partial f_1}{\partial z}(0, 0, 0) = \frac{\partial N_1}{\partial u}(0, (0, 0)) \frac{\partial u}{\partial z} = a_0' > 0.$$

By the implicit function theorem, we solve the equation $f_1(\tilde{\tau}, a, z) = 0$ near $(0, (0, 0))$, which confirms that there exists unique continuous $z$ as a function of $(\tilde{\tau}, a)$ such that $z = z(\tilde{\tau}, a)$ and $z(0, 0) = 0$. Moreover, we have

$$f_1(\tilde{\tau}, a, z(\tilde{\tau}, a)) = N_1(\tilde{\tau}, aY_0 + z(\tilde{\tau}, a)E_0) = 0$$

and

$$\frac{\partial z}{\partial a}(0, 0) = -\left(\frac{\partial N_1(0, 0)}{\partial x}\right)^{-1} \frac{\partial N_1(0, 0)}{\partial y} + \frac{b_0'}{a_0} = 0.$$

Thus, $N(\tilde{\tau}, \tilde{X}) = 0$ if and only if

$$f_2(\tilde{\tau}, a) = N_2(\tilde{\tau}, aY_0 + z(\tilde{\tau}, a)E_0) = 0,$$

(16)

and the number of its solutions equals the number of periodic solutions of model (2).

For simplification, we denote

$$f(\tilde{\tau}, a) = f_2(\tilde{\tau}, a)$$

with $f(0, 0) = N_2(0, (0, 0)) = 0$. First, we compute the Taylor expansion of function $f$ around $(0, 0)$. For this, we compute the first order partial derivatives $\frac{\partial f}{\partial \tilde{\tau}}(0, 0)$ and $\frac{\partial f}{\partial a}(0, 0)$.

Denote $\eta(\tilde{\tau}) = T + \tilde{\tau}, \eta_1(\tilde{\tau}, a) = x_0 - \frac{b_0'}{a_0}a + z(\tilde{\tau}, a)$ and $\eta_2(\tilde{\tau}, a) = a$, then we obtain

$$\frac{\partial f(\tilde{\tau}, a)}{\partial a} = \frac{\partial}{\partial a}(\eta_2 - \Phi_2(\eta_1, \eta_2))$$

$$= 1 - \frac{\partial \Phi_2(\eta_1, \eta_2)}{\partial y} \left(\frac{\partial \Phi_2(\eta_1, \eta_2)}{\partial x} \left(\frac{b_0'}{a_0} + \frac{\partial z(\tilde{\tau}, a)}{\partial a}\right) + \frac{\partial \Phi_2(\eta_1, \eta_2)}{\partial y}\right)$$

$$= 1 - \frac{\partial \Phi_2(\eta_1, \eta_2)}{\partial y}.$$

So, we observe that $d_0' = 1 - \rho(T) = 0$ indicates that $\frac{\partial f}{\partial a}(0, 0) = 0$. 


Similarly, we know

\[
\frac{\partial f}{\partial \tau}(\bar{\tau}, a) = \frac{\partial}{\partial \tau} \left( \eta_2 - \Theta_2(\Phi(\eta, \eta_1, \eta_2)) \right) \\
= -\frac{\partial \Theta_2}{\partial y} \left( \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial \tau} + \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial x} \frac{\partial z}{\partial \tau} \right).
\]

Since \( \frac{\partial \Theta_2(\eta_1, \eta_2)}{\partial \tau} = 0 \) at \((\bar{\tau}, a) = (0, 0)\), we have \( \frac{\partial f}{\partial \tau}(0, 0) = 0 \).

Furthermore, we need to calculate the second-order partial derivatives. Let us denote \( A = \frac{\partial^2 f(0, 0)}{\partial \tau^2}, B = \frac{\partial^2 f(0, 0)}{\partial \tau \partial a}, \) and \( C = \frac{\partial^2 f(0, 0)}{\partial a^2} \), it is observed that

\[
\frac{\partial^2 f(\bar{\tau}, a)}{\partial \tau^2} = \frac{\partial}{\partial \tau} \left( \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial \tau} + \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial x} \frac{\partial z}{\partial \tau} \right) \\
= -\frac{\partial^2 \Theta_2}{\partial y^2} \left( \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial \tau} + \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial x} \frac{\partial z}{\partial \tau} \right)^2 \\
- \frac{\partial \Theta_2}{\partial y} \left( \frac{\partial^2 \Phi_2(\eta, \eta_1, \eta_2)}{\partial \tau^2} + 2 \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial x} \frac{\partial z}{\partial \tau} \right) \frac{\partial z}{\partial \tau} \\
+ \frac{\partial^2 \Phi_2(\eta, \eta_1, \eta_2)}{\partial x^2} \left( \frac{\partial z}{\partial \tau} \right)^2 + \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial x} \frac{\partial^2 z}{\partial \tau^2}.
\]

We have

\[
A = \frac{\partial^2 f(0, 0)}{\partial \tau^2} = -\frac{\partial \Phi_2(T, X_0)}{\partial \tau^2} = 0
\]

according to \( \frac{\partial \Phi_2(T, X_0)}{\partial \tau^2} = 0 \) for all \( 0 \leq t \leq T \). By the same methods as shown above, we have

\[
\frac{\partial^2 \Phi_2(T, X_0)}{\partial x \partial y} = -p'(0) \int_0^T \rho(u) \exp \left( \int_u^T (r - p'(0)y_p(v)) \, dv \right) \, du \\
= -p'(0)T < 0,
\]

\[
\frac{\partial^2 \Phi_2(T, X_0)}{\partial y^2} = -\left( \frac{2rT}{K} + \frac{\lambda_2 p''(0)}{D} \right) \\
- cp''(0) \int_0^T \left\{ \exp \left( \int_v^T (r - p'(0)y_p(\xi)) \, d\xi \right) \right\} \left\{ \int_v^T \exp(-D(v-\theta))y_p(\theta) \, d\theta \right\},
\]

\[
\frac{\partial^2 \Phi_2(T, X_0)}{\partial y \partial \tau} = \frac{\partial F_2(\xi(t))}{\partial y} \exp \left( \int_0^T \frac{\partial F_2(\xi(t))}{\partial y} \, dt \right),
\]

and

\[
\frac{\partial z(0, 0)}{\partial \tau} = -\frac{D}{a_0} y_p(T),
\]

where

\[
\frac{\partial \Phi_2(T, X_0)}{\partial \tau} = 0, \quad \frac{\partial \Phi_2(T, X_0)}{\partial x} = 0, \quad \frac{\partial \Phi_2(T, X_0)}{\partial y} = 0.
\]
Therefore, we can deduce that

\[ B = -\frac{\partial^2 \Phi_2(T, X_0)}{\partial \tau \partial y} - \frac{\partial^2 \Phi_2(T, X_0)}{\partial x \partial y} \frac{\partial^2 \Phi_2(T, X_0)}{\partial \tau} \]

and

\[ C = \frac{2b_*}{h} \left( \frac{\partial \Phi_2(T, X_0)}{\partial y} \right)^2 + 2b_0 \frac{\partial^2 \Phi_2(T, X_0)}{\partial x \partial y} - \frac{\partial^2 \Phi_2(T, X_0)}{\partial y^2}. \]

In order to determine the sign of \( B \), let \( g(t) = r - p'(0)y^* \exp(-Dt) \), then \( g'(t) = Dp'(0)y^* \times \exp(-Dt) > 0 \), which means that \( g(t) \) is strictly increasing. Further, we have

\[ \int_0^T g(t) \, dt = rT - \frac{\alpha y^*(1 - \exp(-DT))}{D} = 0, \]

which indicates that \( g(T) = r - \alpha y^* \exp(-DT) > 0 \), i.e. \( \frac{\partial^2 \Phi_2(T, X_0)}{\partial y} > 0 \). Therefore, \( B < 0 \). If \( 2rDT + p''(0)K\lambda_2 \geq 0 \), then we have \( \frac{\partial^2 \Phi_2(T, X_0)}{\partial x^2} < 0 \), which means \( C > 0 \) holds true.

Since \( A = \frac{\partial^2 \Phi_2(0,0)}{\partial \tau^2} \), \( B = \frac{\partial^2 \Phi_2(0,0)}{\partial \tau \partial x} \) and \( C = \frac{\partial^2 \Phi_2(0,0)}{\partial \tau} \), we obtain

\[ f(\bar{\tau}, a) = Ba\bar{\tau} + Ca^2 + o(\bar{\tau}, a)(\bar{\tau}^2 + a^2). \]

Therefore, we deduce that equation (16) is equivalent to

\[ 2B\bar{\tau} + Ca + \frac{1}{a} o(\bar{\tau}, a)(\bar{\tau}^2 + a^2) = 0. \]

Solving the above equation, we have \( \bar{\tau} \approx -\frac{C}{B} > 0 \), which implies that there is a supercritical bifurcation to a nontrivial periodic solution.

In order to verify our main results, we choose the Holling type II functional response function as an example in the coming section.

### 5 Application of the main results

In order to show the applications of the main results and discuss the biological implications of the threshold conditions, we choose the Holling type II functional response function for \( p(x) = \frac{\alpha x}{1 + \alpha x} \). Thus model (2) becomes the following special system:

\[
\begin{align*}
\frac{dx(t)}{dt} &= rx(t)(1 - \frac{x(t)}{K}) - \frac{\alpha x(t)}{1 + \alpha x(t)} y(t), \\
\frac{dy(t)}{dt} &= \frac{\alpha x(t)}{1 + \alpha x(t)} y(t) - Dy(t), \\
x(t^*) &= (1 - \frac{\alpha x(t)}{x(t)+\lambda_2}) x(t), \\
y(t^*) &= y(t) + \frac{\lambda_1 x(t)}{1 + \alpha x(t)} + \lambda_2
\end{align*}
\]

where \( \alpha \) and \( \omega \) are positive constants, respectively.

It follows from Lemma 2 that we obtain the pest-free periodic solution of model (17) as follows:

\[(x_p(t), y_p(t)) = (0, y^* \exp(-D(t - nT))), \quad t \in (nt, (n + 1)T)]\]
with \( y^* = \frac{\lambda_2}{1 - e^(-\lambda_1 T)} \).

For the stability of the pest-free periodic solution, we employ the main results shown in Theorem 1, so we have the following theorem.

**Theorem 4** The pest-free periodic solution of model (17) is locally stable provided that

\[
T < \frac{\alpha \lambda_2}{\lambda_2} = T^*
\]

and is globally attractive if

\[
T < \frac{\alpha \lambda_2}{\lambda_2} rD M_s,
\]

where \( M_s = \begin{cases} 
\frac{\omega}{\lambda_2}, & \omega \leq 1, \\
1, & \omega > 1.
\end{cases} \)

Based on the main results shown in Theorem 3, we can theoretically address the bifurcations for model (17), and we have the following main results.

**Theorem 5** If \( T = T^* \), then the pest-free periodic solution \((x_p(t), y_p(t))\) can bifurcate to another periodic solution, which is supercritical if \( K \omega \leq 1 \).

Based on the above discussion, the dynamics of model (17) could be very complex when \( T > T^* \). To show this, we carry out a one-dimensional bifurcation analysis numerically. Figure 1 shows that if the impulsive period \( T \) exceeds some threshold levels, then both prey and predator populations can oscillate periodically with quite different amplitudes. As parameter \( T \) increases, the \( T \)-periodic solution loses its stability, then a period-doubling bifurcation occurs at \( T \approx 8 \). As \( T \) further increases, the period-doubling bifurcations lead model (17) to chaos. After that, chaos disappears and period-halving bifurcations occur.

![Figure 1](image_url)  
**Figure 1** Bifurcation diagrams of model (17) with respect to bifurcation parameter \( T \). Parameter values are: \( r = 2.79, k = 10, \alpha = 1.2, c = 0.37, \omega_0 = 0.25, D = 0.59, \delta = 0.69, h = 0.27, \lambda_3 = 1.25, \lambda_2 = 5.2, \theta = 12.5 \)
When \( T = 14.5 \), a non-unique attractor appears, i.e. two different types of \( T \)-periodic solutions coexist (see Fig. 2). In addition, an interesting phenomenon is that 1-2-4-2-1 \( T \)-periodic solutions occur periodically when \( T > 14.5 \), i.e. the dynamics of model (17) periodically recur as \( T \) increases, which means that only decreasing the period of predator releases could bring the pest population down to small amplitudes of periodic solutions. The above results reveal that the impulsive control period \( T \) plays a crucial role in dynamics and consequently in pest control.

The results shown in Fig. 3 reveal the basins of attraction for two alternative attractors, the blue and green areas are the two different types of \( T \)-periodic solutions, respectively. These results clarify that the final stable states of model (17) depend on their initial den-
Figure 4 Bifurcation diagrams of model (17) with respect to bifurcation parameter $T$. Parameter values are:

$r = 0.69, k = 20, \alpha = 0.65, C = 0.37, \omega = 0.15, D = 0.62, \delta = 0.4, h = 0.17, \lambda_1 = 0.65, \lambda_3 = 1.2, \theta = 0.75$

Figure 5 Two coexisting attractors of model (17) with $T = 21.1$. The other parameter values are identical to those in Fig. 4. The initial conditions are: (A–C) $(x_0, y_0) = (1, 0.5)$; (D–F) $(x_0, y_0) = (2.5, 1.5)$

Sities. This reveals the initial sensitivity of the model and the initial value dependence on the pest control. It also shows the importance of frequent monitoring of pest population in the process of pest control.

To further investigate the effect of the impulsive period $T$ on the dynamical behaviour of system (17), consider the results shown in Fig. 4 which reveal that model (17) has more complex and interesting dynamic behaviour including period-doubling bifurcation, chaotic solutions, chaos crisis, period-halving bifurcations and periodic windows with increase of pulse period $T$. Meanwhile, bifurcation analyses also indicate that multiple attractors can coexist for a wide range of parameters. For example, Fig. 5 shows that a $T$-periodic solution coexists with a $6T$-periodic solution when $T = 21.1$. If we choose different initial conditions $(x_0, y_0) = (1, 0.5)$ and $(x_0, y_0) = (2.5, 1.5)$, the stable attractors are the $T$-periodic solution and $6T$ periodic solution, respectively. If we fix all other parameter values as those in Fig. 4 and Fig. 6, it further shows that the solutions with different initial values will approach different amplitudes of attractors.
In order to show how the nonlinear impulsive interventions affect the dynamics of model (17), we choose $\lambda_1$ as a bifurcation parameter and fix all other parameters as shown in Fig. 7(A) and (B). A period-doubling bifurcation occurs when $\lambda_1$ approaches 0.2, which means that a $T$-periodic solution disappears suddenly and a $2T$-periodic solution occurs at this point. As $\lambda_1$ further increases, the $2T$-periodic solution becomes unstable and the period-doubling bifurcations lead model (17) to chaotic dynamics. Following these, period-halving bifurcations result in various periodic solutions with different periods as $\lambda_1$ increases. When $\lambda_1 > 5$, there is a transition from periodicity to period-doubling bifurcation to a chaotic region with some narrow periodic windows. When $\lambda_1 \approx 13.1$, the chaos bands disappear suddenly and a $2T$-periodic solution occurs. With $\lambda_1$ in the range $[13.1, 20]$, there are period-doubling bifurcation and period-halving bifurcations again. Meanwhile, we choose $\lambda_2$ as a bifurcation parameter as shown in Fig. 7(C) and (D). With $\lambda_2$ increasing, bifurcation analyses indicate that model (17) also exhibits very complex dynamics including period-doubling bifurcation, period-halving bifurcation, multi-stability
Figure 8: Bifurcation diagrams of model (17) with respect to bifurcation parameter $\delta$. Parameter values are: $r = 2.26$, $k = 13.6$, $\alpha = 1.2$, $c = 0.37$, $\omega = 0.25$, $D = 0.59$, $\delta = 0.38$, $\lambda_1 = 1.25$, $\lambda_2 = 5.5$, $\theta = 1.65$, $T = 18$. The red results are generated from initial values $(x_0, y_0) = (2.7, 2.5)$ and the blue results are generated from initial values $(x_0, y_0) = (2.5, 2.7)$.
occur in a dualism as parameter $\delta$ increases as well. Further, when fixing initial values and varying insecticide dosages (the maximal mortality rate $\delta$ or half saturation constant $h$), the model may have very complicated and interesting dynamic behaviour, such as the same types of periodic solutions that coexist, different types of periodic solutions coexisting, periodic solutions and chaotic attractors coexisting, which implies that the low initial densities of natural enemy will lead pest populations to chaos or to pest outbreaks or to small amplitude attractors with the parameters $\delta$ and $h$ varying. Obviously, solutions with small amplitudes are biologically desirable. All these results have shown that nonlinear impulsive interventions have a substantial influence on the dynamics of model (17), which implies that the dynamics of the model are not only dependent on the selection of the
initial densities of pest and natural enemy populations, but are also closely related to the system's parameters. Choosing appropriate parameters and initial densities can help us to design control strategies and to make appropriate management decisions.

6 Conclusion

The numbers of natural enemies to be released must be dependent on the densities of pest populations in the field. To show this, we focus on a generalised predator–prey model with a nonlinear impulsive control strategy. The existence and global stability of the pest-free periodic solution were addressed in detail, and the condition for permanence of the system was given. When $T = T^*$, the trivial periodic solution loses its stability and a stable nontrivial periodic solution emerges via a supercritical bifurcation once a threshold condition is reached. To verify the main results obtained and to study the effects of parameter space on the threshold conditions, we further considered the Holling type II functional response function as an example.

One of the main purposes of this paper is to understand how nonlinear impulsive control affects the dynamics of the system, which will affect the success or failure of pest control strategies. To achieve these aims, we chose the nonlinear impulsive control parameters as bifurcation parameters, and our results indicated that nonlinear impulsive control makes the dynamical behaviour of systems change more dramatically and become more complicated. For example, bifurcation diagrams in Figs. 1 and 4 indicate that system (17) has more complex and interesting dynamic behaviour, including period-doubling bifurcation, chaotic solutions, chaos crisis, period-halving bifurcations, periodic windows and multi-stability (see Figs. 2 and 5). In particular, bifurcation diagrams reveal that the dynamic behaviour of the system depends on the initial densities of pest and natural enemy populations, which can help us to design strategies for controlling the pests by varying amounts and frequencies of insecticide applications.

Furthermore, the bifurcation diagram with respect to the maximal mortality $\delta$ and half saturation constant $h$ reveals that the models with nonlinear impulsive interventions have very complicated and interesting dynamic behaviour, such as period-doubling bifurcation and period-halving bifurcation co-occurring, which implies that the model can be dramatically affected by small changes in the initial values or in the value of a relevant control parameter. In particular, there exists an interesting phenomenon whereby period-doubling bifurcation and period-halving bifurcation coexist as parameters $\delta$ and $h$ increase, which implies that it is a very difficult task to control a pest's population as the pest population could have outbreaks in a quite complex way. So it is crucial for pest control to choose appropriate control parameters and initial values. Successful pest control depends not only on the initial densities of the pest and natural enemy populations, but also on the parameters related to the nonlinear impulsive interventions.

Our model with nonlinear impulsive interventions has more interesting dynamics and is more realistic than models with linear impulsive control, and nonlinear regulatory factors should be taken into account when considering IPM. Furthermore, it is believed that the methods developed in the generalised systems could be widely used to population dynamics in relation to glucose–insulin monitor model, tumour-immune dynamic model and epidemic model. However, it will also be more realistic if we consider delayed responses and residual effects of pesticides when these are also used as part of IPM on nonlinear impulsive control. Such future research will involve more complicated analyses of model dynamics.
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Competing interests
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Authors’ contributions
CL and TS designed the study and carried out the analysis. ST and RC contributed to writing the paper. CL performed numerical simulations. All authors read and approved the final manuscript.

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