BRIDGE NUMBERS OF KNOTS IN THE PAGE OF AN OPEN BOOK

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ABSTRACT. Given any closed, connected, orientable 3–manifold and integers $g \geq g(M), D > 0$, we show the existence of knots in $M$ whose genus $g$ bridge number is greater than $D$. These knots lie in a page of an open book decomposition of $M$, and the proof proceeds by examining the action of the map induced by the monodromy on the arc and curve complex of a page. A corollary is that there are Berge knots of arbitrarily large genus one bridge number.

1. Introduction

Let $M$ be a closed, connected, orientable 3–manifold. An open book decomposition of $M$ is a pair $(L, \pi)$ where $L \subseteq M$ is a link and $\pi: M \setminus L \to S^1$ is a surface bundle map such that the closure of each page $F^s = \pi^{-1}(s)$ is an embedded surface with boundary $L$. Each page is homeomorphic to an abstract surface $F$, and the link exterior $\tilde{M} \setminus \tilde{N}(L)$ can be identified with the quotient $F \times [0, 1]$ by a homeomorphism $\phi: F \to F$ called the monodromy. The closure of the union of two pages $\Sigma = F^{s_0} \cup F^{s_1}$ (for $s_0 \neq s_1 \in S^1$) is a Heegaard surface for $M$ because each component of the complement is homeomorphic to a handlebody $F^s \times [0, 1]$. Moreover, Berge noted that a nonseparating essential simple closed curve $K \subseteq F^s$ in one of the fibers will be primitive in both handlebodies. Such a knot is called doubly primitive, and Berge showed that doubly primitive knots in genus two Heegaard surfaces have lens space surgeries. In fact, essential simple closed curves in the Seifert surfaces of the trefoil and figure eight knots (both of which are fibered, and therefore give rise to open books of $S^3$) are two of Berge’s famous families of knots [3]. In general, a doubly primitive knot in a genus $g$ Heegaard surface will have a Dehn surgery producing a 3–manifold admitting a genus $g − 1$ Heegaard surface.

A bridge surface for a link $L \subseteq M$ is a Heegaard surface $\Sigma$ for $M$ such that the intersection of $L$ with each of the two handlebodies in the complement of $\Sigma$ is a collection of boundary parallel arcs. The bridge number of a bridge surface is the number arcs of intersection (i.e. bridges) with each handlebody.

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and the genus $g$ bridge number $b_g(L)$ of $L$ is the minimal bridge number among all the genus $g$ bridge surfaces for $L$.

In the present paper, we prove a general result for knots in the page of an open book decomposition of any closed 3–manifold.

**Theorem 1.1.** Let $(L, \pi)$ be an open book decomposition of a closed, connected, orientable 3–manifold $M$ with page $F$ such that $F$ is not a disk, annulus, or pair of pants. For any integer $D > 0$ there are infinitely many knots $K \subseteq F$ such that

$$b_g(K) > D$$

for every $g(M) \leq g \leq -\chi(F)$. The exteriors of these knots have Heegaard genus $1 - \chi(F)$, and when $M = S^3$ we may choose the knots to be hyperbolic.

Here $g(M)$ is the Heegaard genus of $M$, the minimum genus over all Heegaard surfaces for $M$. As stated earlier, knots in the fiber of a fibered link are primitive on both sides of the natural genus $1 - \chi(F)$ splitting given by two copies of the fiber. This means that the exteriors of these knots have Heegaard splittings of Hempel distance at most two. Contrast the following corollary with theorems of Minsky-Moriah-Schleimer [16] and Moore-Rathbun [7] which exhibit knots in $S^3$ and an arbitrary closed, orientable 3–manifold, respectively, that have high bridge number at many genera.

**Corollary 1.2.** Let $M$ be a closed, connected, orientable 3–manifold admitting an open book decomposition with pages of Euler characteristic $-k$, $k > 0$, which are not 3–punctured spheres. Then for any integers $g \geq k$ and $D > 0$, there are infinitely many knots $K \subseteq M$ such that

1. $K$ has a nontrivial surgery yielding a manifold of Heegaard genus at most $g$,
2. $b_{g'}(K) > D$ for every $g(M) \leq g' \leq g$, and
3. $M \setminus N(K)$ has a minimal genus Heegaard splitting of distance at most two and genus $g + 1$.

**Proof.** Every closed, connected, orientable 3–manifold admits an open book decomposition (see [10] for several proofs). The operation of stabilization, or plumbing with a Hopf band, decreases the Euler characteristic of the page by one. Therefore we may choose an open book decomposition of $M$ with pages $F$ satisfying $-\chi(F) = g$ for any given $g \geq k$. Applying Theorem 1.1, we obtain knots with large bridge number for every $g(M) \leq g' \leq g$.

As noted above, $K$ lies in a genus $1 - \chi(F)$ splitting surface $\Sigma$ so that it is primitive on both sides. By Berge’s construction [3], surgery at the surface slope yields a manifold of Heegaard genus at most $g$. Furthermore, isotoping $\Sigma$ off $K$ so that $K$ ends up in one of the two handlebodies $\Sigma$ bounds creates a Heegaard surface defining a Heegaard splitting of the knot exterior
Finally, we note that there are Berge knots with arbitrarily large genus one bridge number. This follows directly by applying Theorem 1.1 to knots in the fiber of the trefoil or figure eight knot. (See [5] for another argument.) Note that this fact has been known to Baker for some time [2].

Corollary 1.3. There are Berge knots of type VII and VII, knots which lie in the fiber of the trefoil or figure eight, respectively, with arbitrarily large genus one bridge number.

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2. Definitions

Let $M$ be a compact, connected, orientable 3–manifold whose boundary is either empty or a union of one or more tori. (Theorem 1.1 refers to the case when $\partial M = \emptyset$, but we will need the more general case for a number of steps in the argument.) We write $|X|$ to denote the number of components of a manifold $X$. For a submanifold $L$ of $M$, we write $N(L)$ for a closed regular neighborhood of $L$ in $M$. Let $F \subseteq M$ be a properly embedded surface and $L$ a properly embedded 1–manifold. We will write $F \setminus L$ to mean $F \setminus N(L)$, and similarly $M \setminus L$ to mean $M \setminus N(L)$. An essential curve in $F_L$ is a simple closed curve that does not bound a disk in $F_L$ and is not parallel to a boundary component of $F_L$. A disk $D$ embedded in $M_L$ is a compressing disk for $F_L$ if $D \cap F = \partial D$ is an essential simple closed curve in $F_L$. The surface $F$ will be called incompressible if there are no compressing disks for $F$.

An arc properly embedded in $F_L$ is called essential if it is not parallel in the surface to a subarc of $\partial F_L$. We say an incompressible surface $F_L$ is $\partial$–compressible if there is a disk $D$ in $M_L$ so that $\alpha = D \cap F_L$ is an essential arc in $F_L$, $\beta = D \cap \partial M_L$ is an arc in $\partial D$, $\partial D = \alpha \cup \beta$, and $\alpha \cap \beta = \partial \alpha = \partial \beta$. Otherwise $F_L$ is $\partial$–incompressible. When $F_L$ is incompressible, $\partial$–incompressible, not parallel to a component of $\partial M_L$, and not a sphere bounding a ball disjoint from $L$, we say that $F_L$ is essential.

The arc and curve complex $AC(F)$ of $F$, is a simplicial complex whose vertices represent isotopy classes of essential simple closed curves or arcs properly embedded in $F_l$, modulo isotopy. When $g(F) > 1$, $g(F) = 1$ and $|\partial F| > 0$, or $g(F) = 0$ and $|\partial F| > 3$, two vertices bound an edge if they have disjoint representatives in $F$. We will only be concerned with the one skeleton of $AC(F)$ in the present work. The distance $d(u, v)$ between vertices...
in $\mathcal{AC}(F)$ is the number of edges in the shortest path from $u$ to $v$. Similarly, if $U$ and $V$ are subsets of vertices of $\mathcal{AC}(F)$, $d(U,V)$ is the shortest path from a vertex of $U$ to a vertex of $V$.

Since the monodromy of a surface bundle may twist an arc around a boundary component, we need to keep track of arcs up to isotopy fixing the boundary. Choose a collection of points $m \subseteq \partial F$, one in each component of $\partial F$. Define the marked arc and curve complex of $F$, $\hat{\mathcal{AC}}(F)$, to be the simplicial complex whose vertices represent isotopy classes of essential simple closed curves or arcs properly embedded in $F$ and disjoint from $m$, modulo isotopy disjoint from $m$. As before, when $g(F) > 1$, $g(F) = 1$ and $|\partial F| > 0$, or $g(F) = 0$ and $|\partial F| > 3$, two vertices bound an edge if they have disjoint representatives in $F$. We denote the metric on $\hat{\mathcal{AC}}(F)$ by $\hat{d}$, and define $\hat{d}(U,V)$ for subsets of vertices as above.

Each isotopy class defining a vertex of $\hat{\mathcal{AC}}(F)$ is contained in an isotopy class that defines a vertex of $\mathcal{AC}(F)$, so there is a natural map $p$ from the vertices of $\hat{\mathcal{AC}}(F)$ to $\mathcal{AC}(F)$. This map takes simplices to simplices (though it may take a given simplex in $\hat{\mathcal{AC}}(F)$ to a lower dimensional simplex in $\mathcal{AC}(F)$), so $p$ extends to a simplicial map between the two simplicial complexes.

Although we do not use the following lemma in this paper, it clarifies the relationship between $\hat{\mathcal{AC}}(F)$ and the more commonly encountered $\mathcal{AC}(F)$.

**Lemma 2.1.** The map $p$ defines a quasi-isometry from $\hat{\mathcal{AC}}(F)$ to $\mathcal{AC}(F)$ with multiplicative constant one. In fact, for any pair of vertices $a,b$ in $\mathcal{AC}(F)$,

$$\hat{d}(a,b) - 2 \leq d(p(a),p(b)) \leq \hat{d}(a,b).$$

**Proof.** Let $a$ and $b$ be vertices in $\hat{\mathcal{AC}}(F)$. Because the map $p$ takes edges and vertices of $\hat{\mathcal{AC}}(F)$ to edges and vertices of $\mathcal{AC}(F)$, it is immediate that $d(p(a),p(b)) \leq \hat{d}(a,b)$.

For the second inequality, note that the map $p$ is one-to-one on the vertices of $\mathcal{AC}(F)$ represented by simple closed curves, but infinite-to-one on the vertices represented by arcs. Let $v_0, v_1, \ldots, v_n$ be a path in $\mathcal{AC}(F)$ such that $v_0 = p(a)$, $v_n = p(b)$, and $n = d(p(a),p(b))$.

If $v_1$ is a simple closed curve in $F$, then define $a_1$ to be a vertex in $\hat{\mathcal{AC}}(F)$ represented by this simple closed curve. Otherwise, choose $\alpha \subseteq F$ to be a representative of $a$. Since $p(a) = v_0$, $\alpha$ is also a representative of $v_0$. Moreover, because $v_0$ and $v_1$ cobound an edge in $\mathcal{AC}(F)$, we can choose a representative $\beta$ for $v_1$ that is disjoint from $\alpha$. In fact, we can choose $\beta$ so that its endpoints are disjoint from $m$, the marked points in the boundary of $F$. Thus $\beta$ defines a vertex $a_1$ in $\mathcal{AC}(F)$. By construction $a_0$ and $a_1$ will cobound an edge in $\hat{\mathcal{AC}}(F)$.
We can repeat this construction for each successive value $j \leq n$ to find a vertex $a_j$ such that $p(a_j) = v_j$ and $v_j, v_{j-1}$ cobound an edge in $\mathcal{AC}(F)$.

For the final vertex $v_n$, we find that $p(a_n) = v_n = p(b)$. Let $b'$ be a vertex represented by an essential simple closed curve in $F$ disjoint from the arc of simple closed curve representing $b$. Because this simple closed curve is disjoint from the boundary, it will also be disjoint from a representative for $a_n$. Thus the sequence $a, a_1, a_2, \ldots, a_n, b'$ defines a path in $\mathcal{AC}(F)$ of length $n + 2$, so $d(a, b) \leq d(p(a), p(b)) + 2$. Combining this with the previous inequality completes the proof. □

Work of Masur and Minsky [15] implies that $\mathcal{AC}(F)$ is a $\delta$–hyperbolic space in the sense of Gromov. The monodromy map $\phi$ of an open book induces an isometry of $\mathcal{AC}(F)$ which we will also denote by $\phi$. The distance between a point $x \in \mathcal{AC}(F)$ and its image is called the translation distance of $x$ under the isometry. Bachman and Schleimer [1] give a bound on the translation distance of the action of a surface bundle monodromy on the curve complex of a fiber, and we provide a similar bound for an open book monodromy acting on $\mathcal{AC}(F)$.

A crucial concept used in the present paper is that of the axis of $\phi$, $A_\phi \subseteq \mathcal{AC}(F)$, roughly the set of points of $\mathcal{AC}(F)$ that have sufficiently small translation distance under $\phi$ (see section 3). We show that when $\phi : F \to F$ is pseudo-Anosov, this axis behaves similarly to the axis of a hyperbolic isometry of hyperbolic 2–space.

Let $\Sigma$ be a genus $g$ Heegaard splitting of $M$, where $0 \leq g \leq -\chi(F)$. We identify $F$ with $F^0$ and let $K$ be a closed curve in $F$. The curve $K$ can be viewed from two perspectives: first, as a loop in $F$, it defines a vertex in the marked arc and curve complex. Second, since $F$ is embedded in $M$, $K$ defines a knot in $M$. The proof of Theorem 1.1 is based on a comparison between these two views of $K$.

3. Isometries of $\mathcal{AC}(F)$

In this section we consider $K$ as a vertex of $\mathcal{AC}(F)$ and examine its image under an isometry induced from a surface automorphism. We define the axis of such an isometry and show that points close to the axis have small translation distance and, conversely, that points far from the axis have large translation distance. Finally, we relate the distance between two vertices in $\mathcal{AC}(F)$ to the geometric intersection number of representatives of those vertices.

Let $(X, d)$ be a metric space with $X$ infinite. In our case, $X$ will be the 1–skeleton of the arc and curve complex $\mathcal{AC}(F)$, with the nonstandard convention of including points of the edges in the metric space as well as the
vertices. Recall that we have defined
\[ d(A, B) = \inf \{ d(a, b) : a \in A, b \in B \}. \]

A geodesic path from \( x \in X \) to \( y \in X \) is a map \( c \) from a closed interval \([0, l] \subseteq \mathbb{R}\) to \( X \) such that \( c(0) = x \), \( c(l) = y \), and \( d(c(t), c(t')) = |t - t'| \) for every \( t, t' \in [0, l] \). The image of \( c \) is called a geodesic segment or arc. When the choice of geodesic segment connecting two points \( x, y \in X \) does not matter, we denote it by \([x, y]\). Similarly, a geodesic ray is an isometric embedding of the interval \([0, \infty)\) to \( X \), and a geodesic line is an isometric embedding of \( \mathbb{R} \) to \( X \).

Define the Gromov product
\[ (x, y)_w = \frac{1}{2} (d(x, w) + d(y, w) - d(x, y)). \]

We say that the space \((X, d)\) is \( \delta \)-hyperbolic if triangles in \( X \) are \( \delta \)-thin: each side is contained in the \( \delta \) neighborhood of the other two. Equivalently,
\[ d(x, q) + d(y, p) \leq \max \{ d(x, y) + d(p, q), d(x, p) + d(y, q) \} + 2\delta \]
for any points \( x, y, p, q \in X \) (see [6, III.H.1.20]).

Given a closed subset \( A \subseteq X \) and a point \( p \in X \setminus A \), define a projection of \( x \) to \( A \) to be a point \( p \in A \) such that \( d(x, p) = d(x, A) \).

**Lemma 3.1.** Let \( x \in X \) and let \( p \) be a projection of \( x \) to a geodesic \( \tau \). Then for every \( q \in \tau \),
\[ (x, q)_p \leq 4\delta \]
and so
\[ d(x, q) \geq d(x, p) + d(p, q) - 8\delta. \]

**Proof.** Consider the geodesic triangle with vertices \( x, p, \) and \( q \), as on the left in Figure 1. Following [6, III.H.1.17], we may choose points \( y_1 \in [x, p] \) and \( y_2 \in [p, q] \) such that \( d(y_1, p) = (x, q)_p \) and \( d(y_1, y_2) \leq 4\delta \). We have \( d(x, y_1) = d(x, p) - d(y_1, p) \). Furthermore, \( d(x, p) \leq d(x, y_2) \) since \( p \) is a projection of \( x \) to \( \tau \). Combining these estimates and using the triangle inequality, we get
\[ d(x, p) \leq d(x, y_2) \leq d(x, y_1) + d(y_1, y_2) \leq d(x, p) - d(y_1, p) + 4\delta. \]
Therefore \( d(y_1, p) \leq 4\delta \). Finally, recall that \( d(y_1, p) = (x, q)_p \). \( \square \)

**Lemma 3.2.** Let \( p, q \in X \) be projections of points \( x, y \in X \) to a geodesic \( \tau \), and suppose that \( d(p, q) > 9\delta \). Then
\[ d(x, y) \geq d(x, p) + d(p, q) + d(q, y) - 18\delta. \]
Figure 1. Triangle and quad used in Lemma 3.1 and Lemma 3.2.

Proof. Consider the quadrilateral with vertices \( x, y, p, \) and \( q \) as on the right of Figure 1. By Lemma 3.1 we have that both \((x, q)_{p}\) and \((y, p)_{q}\) are not greater than \(4\delta\). By the definition of the Gromov product, this yields

\[
\begin{align*}
    d(x, q) &\geq d(x, p) + d(p, q) - 8\delta \\
    d(y, p) &\geq d(y, q) + d(q, p) - 8\delta.
\end{align*}
\]

Combining these two inequalities we see that

\[
\begin{align*}
    d(x, p) + d(y, q) + 2d(p, q) - 16\delta &\leq d(x, q) + d(y, p).
\end{align*}
\]

Since \( X \) is \( \delta \)-hyperbolic, we have

\[
\begin{align*}
    d(x, q) + d(y, p) \leq \max\{d(x, y) + d(p, q), d(x, p) + d(y, q)\} + 2\delta.
\end{align*}
\]

If the second argument of max is greater, we obtain using the two previous inequalities \( d(p, q) \leq 9\delta \), a contradiction. Therefore we must have

\[
\begin{align*}
    d(x, q) + d(y, p) \leq d(x, y) + d(p, q) + 2\delta,
\end{align*}
\]

and so we obtain the conclusion by using Equation 1. \( \square \)

The Gromov boundary \( \partial X \) of a metric space \( X \) is the set of equivalence classes of geodesic rays, modulo the relation that two rays are equivalent if they stay a bounded distance apart. Let \( \phi \) be an isometry of \( X \). Such isometries are classified into three types: elliptic isometries have bounded orbits, parabolic isometries fix one point in \( \partial X \), and hyperbolic isometries fix two points in \( \partial X \). Isometries of curve complexes induced from surface automorphisms are always either elliptic or hyperbolic \([8]\), and so we assume from now on that \( \phi \) has this property.

Define the translation length of \( \phi \) as

\[
|\phi| = \inf_{x \in X} d(x, \phi(x)),
\]

and the axis of \( \phi \), \( A_{\phi} \), to be the set of all points \( x \in X \) for which

\[
d(x, \phi(x)) \leq \max\{|\phi|, 10\delta\}.
\]
This definition roughly follows [9], as does the proof of Lemma 3.3 below. Note that while the term “axis” often suggests hyperbolic isometries in which the axis is a neighborhood of a bi-infinite geodesic, the same definition is valid for any isometry. In particular, we will allow \( \phi \) to be induced on \( AC(F) \) by a periodic or reducible automorphism of \( F \) as well as by a pseudo-Anosov.

Elliptic and hyperbolic isometries realize their translation lengths [11], and so \( A_\phi \) is a nonempty set. Note also that \( A_\phi \) is closed. As we will see from Corollary 3.4, \( A_\phi \) is not the whole space. To prove this we need the following Lemma, which shows that if a point \( x \) has large translation distance under \( \phi \), then \( x \) is far from \( A_\phi \), and vice versa.

**Lemma 3.3.** For every \( x \in X \),

\[
2d(x, A_\phi) + |\phi| - 18\delta \leq d(x, \phi(x)) \leq 2d(x, A_\phi) + |\phi|.
\]

**Proof.** Given \( x \in X \), the second inequality is a consequence of the triangle inequality: let \( y \) be a projection of \( x \) to \( A_\phi \), so that \( d(x, y) = d(x, A_\phi) \). Then \( d(x, y) = d(x, A_\phi) \) and \( d(y, \phi(y)) \leq |\phi| \), so

\[
d(x, \phi(x)) \leq d(x, y) + d(y, \phi(y)) + d(\phi(x), \phi(y))
\]

\[
\leq 2d(x, A_\phi) + |\phi|.
\]

To prove the first inequality, we may assume that \( x \notin A_\phi \). Let \( y \) be a projection of \( x \) to \( A_\phi \). We claim that \( [y, \phi(y)] \subseteq A_\phi \). To prove this, choose a point \( y' \in [y, \phi(y)] \). Because \( \phi \) is an isometry,

\[
d(y', \phi(y')) \leq d(y', \phi(y)) + d(\phi(y), \phi(y'))
\]

\[
= d(y, y') + d(y', \phi(y))
\]

\[
= d(y, \phi(y)).
\]

Since \( y \in A_\phi \), \( y' \) is also in this set. Finally, note that \( \phi(y) \) is a projection of \( \phi(x) \) to the geodesic segment \([y, \phi(y)]\).

We claim that \( d(y, \phi(y)) \geq 10\delta \). This is because no point \( z \neq y \) in \([x, y]\) lies in \( A_\phi \). Therefore \( d(z, \phi(z)) > \max\{|\phi|, 10\delta\} \), and taking the limit as \( z \) approaches \( y \) we obtain the claim.

By Lemma 3.2,

\[
d(x, \phi(x)) \geq d(x, y) + d(y, \phi(y)) + d(\phi(x), \phi(y)) - 18\delta
\]

\[
= 2d(x, y) + d(y, \phi(y)) - 18\delta
\]

\[
\geq 2d(x, A_\phi) + |\phi| - 18\delta.
\]

□

The following corollary is immediate:
Corollary 3.4. If $\phi : F \to F$ is not the identity then for every $C > 0$, there is a vertex $x$ in $\mathcal{AC}(F)$ such that $d(x, A\phi) > C$.

Proof. As noted in [17, Corollary 1.1], if an automorphism $\phi$ of $F$ induces an isometry of the curve complex of $F$ for which there is a universal bound on $d(x, \phi(x))$ then $\phi$ must be the identity map. As the curve complex is quasi-isometric to $\mathcal{AC}(F)$ this argument carries over directly to our case. Since we have assumed $\phi$ is not the identity, the contrapositive implies that there must be an $x$ such that $d(x, \phi(x)) > 2C + |\phi|$. Then by the second inequality in Lemma 3.3,

$$C < \frac{d(x, \phi(x)) - |\phi|}{2} \leq d(x, A\phi).$$

□

Before finishing this section, we will need a Lemma relating the geometric intersection number between two curves in $\mathcal{AC}(F)$ to their distance in the complex. An analogous result for loops in the curve complex is well known, but when arcs are involved the proof becomes slightly more complex.

Let $x$ and $y$ be properly embedded 1–manifolds in a surface $F$. Denote the geometric intersection number between $x$ and $y$, the minimum number of intersections among all 1–manifolds properly isotopic to $x$ and $y$, by $\iota(x,y)$. Since we are working in the marked arc and curve complex, we consider only isotopy disjoint from the marked points.

Lemma 3.5. Let $F$ be a connected, orientable surface of genus $g > 0$ with one or more marked boundary components. Let $x, y \in \mathcal{AC}(F)$ be represented by arcs in $F$, and assume that $\iota(x, y) > 0$. Then

$$d(x, y) \leq \iota(x, y) + 1.$$

Note that unlike the analogous result in the curve complex, the distance is bounded by the intersection number rather than the log of the intersection number.

Proof. Isotope $x$ and $y$ to minimize $|x \cap y|$. If $\iota(x, y) = 0$, the result holds. Otherwise, assume $\iota(x, y) = n > 0$ and suppose that the result holds for all arcs $\alpha, \beta \in \mathcal{AC}(F)$ with $\iota(\alpha, \beta) < n$. We will construct an arc $z \in \mathcal{AC}(F)$ with $\iota(x, z) = 0$ and $\iota(z, y) \leq n - 1$, which gives the result.

Let $a$ be a longest subarc of $y$ connecting an endpoint of $\partial y$ to a point $p$ of $x \cap y$ whose interior is disjoint from $x$. The point $p$ divides $x$ into two arcs, $x_1$ and $x_2$. We form two new properly embedded arcs $z_1 = a \cup x_1$ and $z_2 = a \cup x_2$. Note that we may isotope $z_1$ and $z_2$ to be disjoint from $x$. Furthermore, both $\iota(z_1, y)$ and $\iota(z_2, y)$ are less than $n$. We must show that one of $z_1$ and $z_2$ is an essential arc in $F$; this will be our $z$. 
Suppose that \( z_i, i = 1 \text{ or } 2 \), cobounds a disk with part of \( \partial F \). Then this disk must contain the marked point \( m \), for otherwise we could use the disk to reduce \( |x \cap y| \). Therefore \( z_j \) cannot also cobound a disk with part of \( \partial F \), where \( \{i, j\} = \{1, 2\} \). □

**Lemma 3.6.** Let \( F \) be a connected, orientable surface of genus \( g > 0 \) with one or more marked boundary components. Let \( x, y \in \mathcal{AC}(F) \) be vertices that are represented by an arc and a simple closed curve in \( F \), respectively. Then

\[
\hat{d}(x, y) \leq \iota(x, y) + 1.
\]

**Proof.** By an isotopy of \( y \) supported in a neighborhood of \( x \), we obtain a properly embedded arc \( y' \) with \( \iota(y, y') = 0 \) and \( \iota(x, y') \leq \iota(x, y) - 1 \). The result follows by applying Lemma 3.5. □

4. Surfaces compatible with the fibration

We now switch from considering vertices of \( \mathcal{AC}(F) \) as loops in an abstract surface to thinking about them as knots in a 3–manifold with an open book decomposition. We find a special surface \( S \subseteq M \) which allows us to give a bound on the translation distance of the isometry on \( \mathcal{AC}(F) \) induced by the monodromy \( \phi \). Let \( K \) be the knot defined by a loop in \( F^0 \), as before, and write \( L = K \cup L \).

Recall that we say the link \( L \) is in bridge position with respect to \( \Sigma \) if \( \Sigma \) divides \( M \) into two compression bodies \( H^1 \) and \( H^2 \) such that each arc of \( H^i \cap L \) is trivial in \( H^i \), \( i = 1, 2 \). Say that such a bridge position is minimal if \( |H^i \cap L| \) is minimized over all surfaces isotopic to \( \Sigma \). In this case we define \( b\Sigma(L) = |H^i \cap L| \).

Note that if \( \Sigma \) is in bridge position with respect to \( L \) then it is in bridge position with respect to each of \( K \) and \( L \), independently. Isotope \( \Sigma \) so that it is a minimal bridge surface with respect to \( K \). Then isotope it further so that it has minimal bridge number with respect to \( L = K \cup L \), subject to the constraint that it remains a minimal bridge surface with respect to \( K \). We call this a minimal \( K \text{-bridge position} \).

We say that \( (\Sigma, L) \) is weakly reducible if there are disjoint essential disks \( D_1 \subseteq H^1 \) and \( D_2 \subseteq H^2 \). If \( (\Sigma, L) \) is not weakly reducible, we say that it is strongly irreducible. We use a characterization of bridge surfaces, due in its original form to Hayashi and Shimokawa [12], and reformulated by Taylor and Tomova [18]. It says that weakly reducible splittings have one of several properties, or that the exterior of the link contains a special essential surface. Let \( (\Sigma, L) \) be a bridge splitting. The splitting is stabilized if there are compressing disks on either side of \( \Sigma \) that intersect exactly once. The splitting is boundary stabilized if it is obtained from a bridge splitting.
(\Sigma', L) by amalgamating with the standard splitting of a neighborhood of a boundary component of M. The splitting is perturbed if there are bridge disks (disks whose boundary consists of the union of an arc of H^i \cap L and an arc in \Sigma) on opposite sides of \Sigma meeting in a single point. Finally, we say that a component \( L_0 \) of \( L \) is removable if \( L_0 \) is isotopic to a core of either \( H^1 \) or \( H^2 \).

**Theorem 4.1.** Let \( M \) be a compact, orientable 3–manifold containing a link \( L \), and suppose that \( M_L \) is irreducible and that no sphere in \( M \) intersects \( L \) exactly once. If \( \Sigma \) is a weakly reducible bridge surface for \( L \), then one of the following holds:

1. \( \Sigma \) is stabilized, boundary stabilized, perturbed, or a component of \( L \) is removable, or
2. \( M \) contains an essential meridional surface \( S \) such that \( \chi(\Sigma_L) \leq \chi(S_L) \). Furthermore, \( S \) is a thin surface in a generalized bridge splitting obtained by untelescoping \( \Sigma \).

**Proof.** This is [18, Corollary 9.4] where we have taken \( \Gamma = \emptyset \) and \( T = L \). Note that meridional stabilization and boundary meridional stabilization cannot occur as \( \Gamma \) is empty.

We wish to give a lower bound on the bridge number of \( K \) with respect to the splitting \( \Sigma \). Note the following Lemma, which follows directly from the definitions:

**Lemma 4.2.** Suppose that \((\Sigma', L)\) is obtained by destabilizing or boundary destabilizing the \( K \)-minimal bridge splitting \((\Sigma, L)\). Then \( b_{\Sigma'}(K) \geq b_{\Sigma}(K) \). Furthermore, a minimal bridge splitting \((\Sigma, L)\) is never perturbed.

By this lemma and Theorem 4.1 we may assume that either \((\Sigma, L)\) is strongly irreducible, \( M_L \) contains an essential surface \( S_L \) of Euler characteristic greater than or equal to that of \( \Sigma_L \), a component of \( L \) is removable, or \( K \) is removable. We will show that either \( \Sigma \) or \( S \) can be isotoped to intersect the fibers \( F^* \) in a very controlled manner in all of these cases.

As noted in [18], if a component \( R \) of a link \( L \) is removable with respect to a Heegaard surface \( \Sigma \) then \( \Sigma \) can be isotoped so that \( R \) is a core of one of the compression bodies bounded by \( \Sigma \). In this case, removing an open regular neighborhood of \( R \) turns the handlebody into a compression body. Thus after the isotopy, we find a Heegaard surface \( \Sigma' \) for the complement \( M_R \) such that \( \Sigma' \) is isotopic in \( M \) to \( \Sigma \). We consider two cases: when \( R \) is a component of \( L \) and when \( R = K \).

If \( R \) is a component of \( L \) then the restriction of the bundle map \( \pi : M \setminus L \to S^1 \) to \( M_R \setminus L \) is a surface bundle such that \( K \) is contained in a fiber. We can apply Theorem 4.1 to the isotoped Heegaard surface \( \Sigma' \) for \( M_R \), which
is a bridge surface for \( \mathcal{L} \setminus R \). If \( R = K \) is the removable component, then we no longer have a surface bundle structure for \( M_R \), but we can still find a Heegaard surface \( \Sigma' \) for \( M_R \).

If a component of \( \mathcal{L} \) is again removable with respect to \( \Sigma' \) then we can repeat this process. We will eventually find a sublink \( L' \) of \( \mathcal{L} \) and a Heegaard surface \( \Sigma' \) for \( M_{L'} \) that is a bridge surface for \( \mathcal{L} \setminus L' \) in which no component is removable. Note that we may find \( L' = \mathcal{L} \), in which case \( \Sigma' \) is just a Heegaard surface and \( \mathcal{L} \setminus L' \) is empty.

If \( K \) is a component of \( L' \) then we let \( L'' = L' \setminus K \). Then \( \Sigma' \) is a Heegaard surface for \( M_{L''} \) such that \( \Sigma' \) is a strongly irreducible bridge surface for \( \mathcal{L} \setminus L'' \) and \( K \) is a core of one of the compression bodies bounded by \( \Sigma' \). Thus we have the following:

**Lemma 4.3.** If \( \Sigma \) is a bridge surface for \( \mathcal{L} \) then there is either

1. a surface \( S \subseteq M \) such that \( \partial S \cap \mathcal{L} \) is essential and \( \chi(S) \geq \chi(\Sigma) \),
2. a sublink \( L' \subseteq \mathcal{L} \) and a Heegaard surface \( \Sigma' \) for \( M_{L'} \) such that \( \Sigma' \) is a strongly irreducible bridge surface for \( \mathcal{L} \setminus L' \) and \( \chi(\Sigma') \geq \chi(\Sigma) \), or
3. a sublink \( L'' \subseteq \mathcal{L} \) and a Heegaard surface \( \Sigma' \) for \( M_{L''} \) such that \( \chi(\Sigma') \geq \chi(\Sigma) \), \( \Sigma' \) is a strongly irreducible bridge surface for \( \mathcal{L} \setminus L'' \) and \( K \) is a core of a compression body bounded by \( \Sigma' \).

Suppose that \( M_L \) contains an essential surface \( S_L \) such that \( \chi(S_L) \leq \chi(\Sigma_L) \) as in case one of Lemma 4.3. Isotope \( S \) so that \( \partial S \cap \partial F_{sL} \) is minimal for every \( s \in S^1 \). Moreover, by a general position argument, we can isotope \( S \) so that the restriction of \( \pi \) to \( S_L \) is Morse and 0 is a regular value.

**Lemma 4.4.** Let \( S_L \) be an essential meridional surface in \( M_L \) such that \( \pi|_{S_L} \) is Morse and \( \partial S_L \cap \partial F_{sL} \) is minimal for every \( s \in S^1 \). Then for every regular value \( s \) of \( \pi|_{S_L} \), each arc of \( S_L \cap F_{sL} \) is essential in both surfaces and each simple closed curve of intersection is either essential in both surfaces or trivial in both surfaces.

Note that the conclusion of the lemma applies to surfaces in the exterior of \( L \) as opposed to \( \mathcal{L} \).

**Proof.** In this proof and the sequel we often think of \( S^1 \) as the interval \( [0, 1] \) with its endpoints identified. Suppose then that \( s \in (0, 1) \). Every arc or simple closed curve of \( F_{sL} \cap S_L \) which is trivial in \( S_L \) must also be trivial in \( F_{sL} \) since \( F_{sL} \) is essential in \( M_L \). If there is a trivial simple closed curve or arc of \( F_{sL} \cap S_L \) in \( F_{sL} \) that is essential in \( S_L \), an innermost such simple closed curve or outermost such arc bounds or cobounds a disk \( D \) in \( F_{sL} \) disjoint from \( K \). Therefore \( S_L \) is compressible or \( \partial \)-compressible, a contradiction. If such a disk exists when \( s = 0 \), there must be some \( s > 0 \) for which there is a similar disk in \( F_{sL} \).
Finally, note that no arc of intersection can be trivial in both surfaces since this would define an isotopy reducing \( \partial S_L \cap \partial F_L^s \).

\[ \square \]

Suppose now that we are in case two or three of Lemma 4.3, so that \( \Sigma' \) is a Heegaard surface for \( M_{L'} \) or \( M_{L''} \). Let \( \hat{L} \) be \( L \setminus L' \) in case two or \( L \setminus L'' \) in case three. Then \( \Sigma' \) is either a strongly irreducible bridge surface for \( \hat{L} \cup K \) or \( \Sigma' \) is a strongly irreducible bridge surface for \( \hat{L} \) and \( K \) is a core of one of the compression bodies bounded by \( \Sigma' \). We will show that \( \Sigma' \) behaves in much the same way as the essential surface of Lemma 4.4.

Let \( H_- \) and \( H_+ \) be the compression bodies bounded by \( \Sigma' \) and let \( G_- \), \( G_+ \) be spines for these compression bodies. If \( \Sigma' \) is a bridge surface for \( \hat{L} \cup K \) then we can extend each of \( G_- \) and \( G_+ \) to contain a single vertex in each arc \( \hat{L} \cap H_- \) and \( \hat{L} \cap H_+ \), respectively. Otherwise, if \( K \) is a core of one of \( H_- \), \( H_+ \) then we can choose \( G_- \) and \( G_+ \) so that \( K \) is contained in one of the spines and each arc of \( \hat{L} \cap H_- \) and \( \hat{L} \cap H_+ \) contains a vertex of \( G_- \) or \( G_+ \), respectively.

Let \( h \) be a sweep-out subordinate to the bridge surface \( \Sigma' \), i.e. a function \( h: M \to I \) such that \( h^{-1}(0) = G_- \), \( h^{-1}(1) = G_+ \), \( h^{-1}(\frac{1}{2}) = \Sigma' \), and for each \( t \in (0,1) \), the level surface \( \Sigma^t = h^{-1}(t) \) is isotopic to \( \Sigma' \) by an isotopy transverse to \( \hat{L} \cup K \). Let \( H_-^t \) and \( H_+^t \) be the compression bodies bounded by \( \Sigma^t \).

Consider the map \( \Phi: M \setminus L \to S^1 \times I \) given by sending \( x \mapsto (\pi(x), h(x)) \). Each point \( (s,t) \) in the cylinder \( S^1 \times I \) represents a pair of surfaces \( F^s \) and \( \Sigma^t \), as described above. The graphic is the subset of the cylinder consisting of all points \( (s,t) \) where \( F^s \) and \( \Sigma^t \) are tangent. We may assume that \( \Phi \) is generic in the sense that it is stable (see [14] and [13]) on the complement of \( G^+ \cup G^- \), each arc \( \{s\} \times I \) contains at most one vertex of the graphic, and each circle \( S^1 \times \{t\} \) contains at most one vertex of the graphic. Vertices in the interior of the graphic are valence four (crossings) and valence two (cusps). By general position of \( G^+ \cup G^- \), the graphic is incident to the boundary of the cylinder in only a finite number of points, and each vertex in the boundary has valence one or two.

For a regular value \( (s,t) \) of \( \Phi \), say that \( F^s \) is essentially above \( \Sigma^t \) if there is a component of \( F^s \cap \Sigma^t \) that is an essential arc or circle in \( \Sigma^t \) but bounds a compressing or \( \partial \)-compressing disk for \( \Sigma^t \) contained in \( H_-^t \). (Note that this compressing disk may not be contained in \( F^s \), though it will often be parallel to a disk in \( F^s \).) Similarly, say that \( F^s \) is essentially below \( \Sigma^t \) if there is a component of \( F^s \cap \Sigma^t \) that is an essential arc or circle in \( \Sigma^t \) but bounds a compressing or \( \partial \)-compressing disk for \( \Sigma^t \) in \( H_+^t \). Let \( Q_a \) and \( Q_b \)
denote the points in $S^1 \times (0,1)$ for which $F^s$ is essentially above or below $\Sigma^t$, respectively.

In the present case $\Sigma'$ is a strongly irreducible bridge surface, and so we must have $Q_a \cap Q_b = \emptyset$ and, further, no circle $S^1 \times \{t\}$ meets both $Q_a$ and $Q_b$ for any $t \in I$ (cf. [4, Lemma 7.3]). Therefore there is a $t_0 \in I$ such that for all $s \in S^1$, $(s, t_0)$ does not meet $Q_a \cup Q_b$ and meets at most one vertex of the graphic.

Recall that a function $f: S \to S^1$ is Morse if every critical point is non-degenerate and the images in $S^1$ of any two critical points are distinct. We say that a smooth function $f$ is almost-Morse if every critical point is non-degenerate and at most two critical points are sent to the same level in $S^1$. Note that according to this definition, Morse functions are also almost-Morse.

Compare the following lemma to [4, Lemma 6.5], whose proof is similar.

**Lemma 4.5.** There is a surface $S$ isotopic to $\Sigma'$ such the restriction of $\pi$ to $S_L$ is almost-Morse and for every regular value $s \in S^1$ of $\pi|S_L$, each arc of $S_L \cap F^s_L$ is essential in both surfaces and each simple closed curve of intersection is either essential in both surfaces or trivial in both surfaces.

Furthermore, if $c_1$ and $c_2$ are two distinct critical points of $\pi|S$ with $\pi(c_1) = \pi(c_2) = s_0$, then for small $\epsilon > 0$, each arc of $F^{s_0-\epsilon}_L \cap S_L$ can be isotoped rel boundary in $F_L$ to have interior disjoint from the interior of each arc of $F^{s_0+\epsilon}_L \cap S_L$.

Again, note that this lemma applies to surfaces in the exterior of $L$, not $\mathcal{L}$.

**Proof.** As noted above, we can choose a value $t_0$ such that for all $s \in S^1$, $(s, t_0)$ does not meet $Q_a \cup Q_b$ and meets at most one vertex of the graphic. Define $S = \Sigma^{t_0}$.

The restriction of $\pi$ to $S_L$ defines a function $\pi_0: S_L \to S^1$. The critical values of $\pi_0$ are the values $s$ such that the point $(s, t_0)$ is contained in an edge of the graphic. Points in the interior of edges correspond to non-degenerate critical points, with vertices corresponding to two critical points whose images in $S^1$ coincide. (Cusp points correspond to degenerate critical points in $\pi_0$.) Because of the way we chose $t_0$, we conclude that $\pi_0$ is Morse (if there is no vertex $(s, t_0)$) or almost-Morse.

Suppose first that $s \in (0, 1)$. Note that every arc or simple closed curve of $F^s_L \cap S_L$ which is trivial in $S_L$ must also be trivial in $F^s_L$ since $F^s_L$ is essential. Suppose then that there is a trivial simple closed curve or arc of $F^s_L \cap S_L$ in $F^s_L$ that is essential in $S_L$. An innermost such simple closed curve or arc bounds or cobounds a disk $D$ in $F^s_L$. Isotope $D$ fixing $\partial D \cap S_L$ so that $|D \cap S_L|$ is minimal. An innermost disk argument shows that we may take the interior of $D$ disjoint from $S_L$, contradicting the assumption that $(s, t_0)$
does not meet \( Q_a \cup Q_b \) for any \( s \in S^1 \). If \( s = 0 \), so that \( D \) lies in \( F_L^0 \), then there must be some \( s > 0 \) for which there is a similar disk in \( F_L^s \). Therefore, for every regular value \( s \in S^1 \) of \( \pi|_{S_L} \), each arc of \( S_L \cap F_L^s \) is essential in both surfaces and each simple closed curve of intersection is either essential in both surfaces or trivial in both surfaces.

If \( \pi_0 \) is Morse (so that \( (s, t_0) \) does not meet a vertex of the graphic for any \( s \in S^1 \)), we proceed as in the proof of Lemma 4.4. We are left to show that when \( \pi_0 \) is almost-Morse with two critical points \( c_1, c_2 \) such that \( \pi(c_1) = \pi(c_2) = v \) then we can pair arcs of \( F_L^{s_0-\epsilon} \cap S_L \) with disjoint (in \( F_L \)) arcs of \( F_L^{s_0+\epsilon} \cap S_L \) for small \( \epsilon \). As noted above, this case will only occur if \( (s_0, t_0) \) is a valence-four vertex of the graphic in the intersection of the closures of \( Q_a \) and \( Q_b \). Let \( \epsilon > 0 \) be small enough so that this is the only vertex with \( t \)-value in the interval \( [t_0 - \epsilon, t_0 + \epsilon] \). Let \( s_- = s_0 - \epsilon \) and \( s_+ = s_0 + \epsilon \).

Therefore we must show that each component of \( W = F_L^{s_-} \cap S_L^{t_0} \) can be isotoped in \( F_L \) to be disjoint from each component of \( E = F_L^{s_+} \cap S_L^{t_0} \). Going from \( N = F_L^{s_0} \cap S_L^{t_0+\epsilon} \) to \( S = F_L^{s_0} \cap S_L^{t_0-\epsilon} \), we pass through two saddles of \( F_L \). This corresponds to adding two bands, \( b_1 \) and \( b_2 \), to components of \( N \) to obtain the new components of \( S \). If ends of \( b_1 \) and \( b_2 \) are adjacent to different components of \( N \) or if ends of \( b_1 \) and \( b_2 \) are adjacent to the same side of the same component, then by isotoping those components slightly we see that \( N \) can be made disjoint from \( S \) in \( S_L \). This contradicts the strong irreducibility of \( S \) and implies in particular that the critical set is connected. This in turn implies that at most 3 isotopy classes of arcs differ between \( F_L^{s_-} \cap S_L \) and \( F_L^{s_+} \cap S_L \).

Figure 2 shows two examples that violate the strong irreducibility of \( S \). In both cases the critical set is a tree. Pictured above is the surface \( S \) in a regular neighborhood \( B \subseteq M \) of this tree. The intersections \( F^s \cap B \) form a family of parallel horizontal disks. Intersections of these disks with \( S \) are shown at the critical level as well as just before and just after the critical level. The projection of the arcs into \( F \) is shown below each 3-dimensional picture.

From the previous discussion we see that an end of \( b_1 \) and an end of \( b_2 \) are adjacent to opposite sides of the same component \( \alpha \) of \( N \). If the other ends of \( b_1 \) and \( b_2 \) are adjacent to a component \( \alpha' \) of \( N \), then those ends must be adjacent to opposite sides of \( \alpha' \). If they were not, one of \( b_1 \) or \( b_2 \) would be a band with ends attached to opposite sides of a component of \( E \) or \( W \), contradicting the orientability of \( F \) and \( \Sigma \). Thus the arcs and simple closed curves that result from attaching \( b_1 \) to \( N \) are disjoint from the arcs and simple closed curves that result from attaching \( b_2 \) to \( N \) since we may push each such component slightly in the direction that the bands approach from. This shows that each component of \( E \) can be made disjoint from each
Figure 2. Two cases which violate the strong irreducibility of $S$.

The next proposition follows by putting together Lemma 4.3, Lemma 4.4, and Lemma 4.5. Note first that if $S$ is an essential surface, it is a thin surface in a generalized bridge splitting by Theorem 4.1, and so $b_\Sigma(K) \geq \frac{1}{2}|K \cap S|$. Second, the surface $S$ is obtained from $\Sigma$ by compression (possibly zero times), so $g(S) \leq g(\Sigma)$.

**Proposition 4.6.** There is a surface $S \subseteq M$ such that the following properties hold:

1. $g(S) \leq g(\Sigma)$
2. $b_\Sigma(K) \geq \frac{1}{2}|K \cap S|$, 
3. the restriction of $\pi$ to $S_L$ is Morse or almost-Morse,
4. every arc of $S_L \cap F_L^{s}$ is essential in both surfaces for regular values of $s \in S^1$ and, 
5. if $c_1$ and $c_2$ are two distinct critical points of $\pi|_{S_L}$ with $\pi(c_1) = \pi(c_2) = s_0$, then the arcs of $F_L^{s_0 - \epsilon} \cap S_L$ may be isotoped rel boundary in $F_L$ to have interiors disjoint from the interiors of $F_L^{s_0 + \epsilon} \cap S_L$ for small $\epsilon$. 
5. Distance bounds

Recall that $L$ is the binding of an open book decomposition of the closed, orientable, connected manifold $M$ with page $F$, $\phi$ is the monodromy of the open book, and $A_{\phi}$ is the axis of $\phi$ as an automorphism of $\mathcal{AC}(F)$. By Proposition 4.6 there is a “nice” surface $S$ which we will use now. Compare the following lemma with [1] Theorem 5.3, whose proof is similar.

**Lemma 5.1.** There is an arc $x$ of $F_L \cap S_L$ whose class in $\mathcal{AC}(F)$ (which we also write as $x$) satisfies

$$d(x, \phi(x)) \leq \frac{-4\chi(S_L)}{|\partial S_L \cap \partial F_L|}.$$

**Proof.** The number of arcs of $F_L^{s_{\pm}} \cap S_L$ is constant for all regular values $s \in S^1$ of $\pi|_S$, namely equal to $n = |\partial S_L \cap \partial F_L|/2$. Recall that we have arranged for $0$ to be a regular value of $\pi|_S$, and think of $S^1$ as the interval $I = [0, 1]$ with its endpoints identified. Suppose that $c \in S$ is an index-one critical point of $\pi|_S$ and that $v = \pi(c) \in I$ is the associated critical value. Choose $s_- < v < s_+$ so that $v$ is the only index-one critical value in $[s_-, s_+]$. As $v$ goes from $s_-$ to $s_+$, arcs and circles of $F_L^{s_{\pm}} \cap S_L$ are banded together by one or two bands to obtain curves isotopic to $F_L^{s_{\pm}} \cap S_L$.

Let $A_{\pm}$ be the union of the arc components of $F_L^{s_{\pm}} \cap S_L$ adjacent to a band whose other end is adjacent to either an arc of $F_L^{s_{\pm}} \cap S_L$ or to a simple closed curve of $F_L^{s_{\pm}} \cap S_L$ that is essential in $S$. The components of $A_-$ are called pre-active arcs and the components of $A_+$ are called post-active arcs. An arc that is pre- or post-active is called active. A circle component of $F_L^{s_{\pm}} \cap S_L$ to which a band or bands are adjacent is called an active circle if it is essential in $S_L$. If $A_{\pm} \neq \emptyset$, then we call $v$ an active index-one critical value and say that the index-one critical points in its preimage are active index-one critical points.

The surface $F_L$ is essential in $M_L$, so for regular values $s$, no component of $F_L^s \cap S_L$ is essential in $F_L^s$ and inessential in $S_L$. Therefore each arc component of $F_L^{s_-} \cap S_L$ that is not pre-active is isotopic in $F_L$ to an arc component of $F_L^{s_+} \cap S_L$ that is not post-active. Thus there is a bijection between the arc components of $F_L^{s_-} \cap S_L$ and $F_L^{s_+} \cap S_L$ taking $A_-$ to $A_+$ which is constant on the isotopy classes of arcs in $F_L^{s_-}$ that are not in $A_-$. 

By Proposition 4.6 there is a pairing of arcs between $F_L^{s_-} \cap S_L$ and $F_L^{s_+} \cap S_L$ such that each arc is distance one from its paired arc in $\mathcal{AC}(F)$. Therefore we may construct paths in $\mathcal{AC}(F)$ from the isotopy classes of arcs in $F_L^0 \cap S_L$ to isotopy classes of arcs in $F_L^1 \cap S_L$. Let $q_i$ be the length of the $i$-th path, and note that as $d(F_L^0 \cap S_L, F_L^1 \cap S_L)$ gives the length of the shortest path
between these two sets,
\[ n \cdot \hat{d}(F^0_L \cap S_L, F^1_L \cap S_L) \leq \sum_{i=1}^{n} q_i. \]

We wish to bound the sum \( \sum_{i=1}^{n} q_i \). By the above remarks, each time \( v \) passes through a critical value of \( \pi \), arcs and circles are banded together to obtain new arcs and circles. When \( v \) is a critical value that has a single associated critical point \( c \), at most two arcs are banded together. When \( v \) is a critical value with two associated critical points \( c_1 \) and \( c_2 \), at most three arcs are banded by Lemma 4.5. Therefore if \( V_s \) is the number of critical values that have a single associated critical point and \( V_d \) is the number of critical values that have two associated critical points, then

\[ \sum_{i=1}^{n} q_i \leq 2V_s + 3V_d. \]  

Let \( P \) be the closure of the complement of the active arcs and circles in \( S_L \), and denote its components by \( P_1, \ldots, P_m \). The boundary of each \( P_k \) is the union of arcs and simple closed curves contained in \( \partial M_L \) and active arcs and simple closed curves. Each \( P_k \) contains zero, one, or two active index-one critical points of \( \pi|_{S_L} \), and there is at most one \( P_k \) that contains two active index-one critical points. Let \( b_k \) be the number of active arcs in \( \partial P_k \), and define the index of \( P_k \) to be

\[ J(P_k) = \frac{b_k}{2} - \chi(P_k). \]

Since each active arc shows up twice in \( \partial P \) and since Euler characteristic increases by one when cutting along an arc, we have

\[ -\chi(S_L) = \sum_{k} J(P_k). \]  

Fix \( k \). By hypothesis \( P_k \) is not \( S^2 \). If \( P_k \) is a disk, its boundary cannot be an active circle or contain only a single active arc as active circles and arcs are essential in \( S \). Thus, if \( P_k \) is a disk, \( J(P_k) \geq 0 \). If \( P_k \) is not a disk, \( -\chi(P_k) \geq 0 \). It follows that if \( P_k \) does not contain an active index-one critical point, then its index is nonnegative.

Suppose \( P_k \) contains a unique active index-one critical point \( c \in P_k \) and let \( \alpha \) be a pre-active arc at \( c \). If \( \alpha \) is banded to a different pre-active arc, then \( b_k \geq 4 \) since there must be at least two pre-active arcs and two post active arcs. Then \( J(P_k) \geq 1 \). Otherwise let \( \gamma \) be the circle that is either banded to \( \alpha \) or that results from banding \( \alpha \) to itself. In this case, \( \gamma \) is essential, so \( P_k \) is not a disk. There are two active arcs in \( \partial P_k \), so \( J(P_k) \geq 1 \). It follows that if \( P_k \) contains a single active index-one critical point, then \( J(P_k) \geq 1 \).
If $P_k$ contains two active index-one critical points, they must have the same height and be connected. Let $C_-$ and $C_+$ be the number of pre-active and post-active circles, respectively, at $v$. Note that any pre- or post-active circle, along with at least one post-active arc at $v$, lies in $\partial P_k$. Furthermore, the bands must lie in $P_k$ since the bands themselves contain the index-one critical points. Thus, if $P_k$ is a disk, then $C_- = C_+ = 0$ and $b_k = 6$. If $P_k$ is not a disk, then either $C_- + C_+ \geq 1$, $b_k \geq 2$, or $b_k = 2$ and $P_k$ is not planar. In any of these cases, $J(P_k) \geq 2$. Consequently,

$$\sum_k J(P_k) \geq V_s + 2V_d.$$  

Putting together Equation 2, Equation 3, and Equation 4, we have

$$n \cdot \bar{d}(F_L^0 \cap S_L, F_L^1 \cap S_L) \leq \sum_{i=1}^{n} q_i \leq 2V_s + 3V_d \leq 2(V_s + 2V_d) \leq 2 \sum_k J(P_k) = -2\chi(S_L),$$

and since $n = |\partial F_L \cap \partial S_L|/2$ is the total number of arcs, the result follows.

\[\]


Therefore,
\[
\begin{align*}
\dot{d}(x, K) & \geq \dot{d}(K, A_\phi) - \dot{d}(x, A_\phi) \\
& > C + 4\chi(F) + 2 + \frac{1}{2}|\phi| - 9\delta \\
& = D + 2.
\end{align*}
\]

Using the inequality given by Lemma 3.6, we see that
\[
\iota(x, K) \geq \dot{d}(x, K) - 1 > D.
\]

Since \(x\) is an arc in \(S_L\), this implies that \(|K \cap S| > D\).

Recall that \(S\) is either the bridge splitting \(\Sigma\) or a thin surface in a generalized bridge splitting for \(L\) obtained by untelescoping \(\Sigma\). In either case, we have
\[
b_{\Sigma}(K) > D.
\]

Since \(\Sigma\) was arbitrary, it follows that \(b_g(K) > D\).

Finally, suppose that \(M = S^3\) and note that \(K\) is not a torus knot from the bridge number bounds above. Suppose that \(M_K\) contains an essential torus. By an isotopy minimizing \(|T \cap L|\) and keeping \(T\) disjoint from \(K\), we obtain an essential punctured torus in \(M_L\). An argument similar to the one above (with \(T\) taking the place of \(S\)) shows that \(T\) must meet \(K\). This is impossible, and so \(M_K\) is atoroidal. Therefore \(K\) is a hyperbolic knot in \(S^3\). \(\square\)

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