New face of multifractality: Multi-branched left-sidedness and phase transitions in multifractality of interevent times

Jaroslaw Klamut and Ryszard Kutner
Faculty of Physics, University of Warsaw, Pasteur Str. 5, PL-02093 Warsaw, Poland

Tomasz Gubiec
Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215 USA and
Faculty of Physics, University of Warsaw, Pasteur Str. 5, PL-02093 Warsaw, Poland

Zbigniew R. Struzik
University of Tokyo, Bunkyo-ku, Tokyo 113-8655, Japan and
Advanced Center for Computing and Communication, RIKEN, 2-1 Hirosawa, Wako 351-0198, Saitama, Japan

We develop an extended multifractal analysis based on the Legendre-Fenchel transform rather than the routinely used Legendre transform. We apply this analysis to studying time series consisting of inter-event times. As a result, we discern the non-monotonic behavior of the generalized Hurst exponent – the fundamental exponent studied by us – and hence a multi-branched left-sided spectrum of dimensions. This kind of multifractality is a direct result of the non-monotonic behavior of the generalized Hurst exponent and is not caused by non-analytic behavior as has been previously suggested. We examine the main thermodynamic consequences of the existence of this type of multifractality related to the thermal stable, metastable, and unstable phases within a hierarchy of fluctuations, and also to the first and second order phase transitions between them.

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I. INTRODUCTION

The concept of extended scale invariance, referred to as multifractality, has become a routinely applied but still intensively developed methodology for studying both complex systems [1–6] and nonlinear (e.g. chaotic with a low degree of freedom) dynamical ones [7]. This is an inspiring and rapidly evolving approach to nonlinear science in many different fields stretching far beyond traditional physics [8].

The direct inspiration of the present work is our earlier results presented in papers [9, 10]. In these publications we found the left-sided multifractality on financial markets of small, medium, and large capitalizations as a direct result of a non-analytic behavior of the Rényi exponent. We indicated that a broad distribution of interevent times is responsible for the existence of left-sided multifractality. In the present work, we suggest that not only a broad distribution but primarily nonlinear long-term auto-correlations bear responsibility for the multifractality observed.

Attention was first drawn to the existence of left-sided multifractality (generated by the binomial cascade which produces singularity in the Rényi exponent or stretched exponential decay of the smallest coarse-grained probability) by Mandelbrot and coauthors [11, 12]. An interesting breakdown of multifractality in diffusion-limited aggregation was discovered by Blumenfeld and Aharony [13]. They found strongly asymmetric spectra of singularity depending on the size of the growing aggregate in DLA, showing a clear tilt to the left as a signature of phase transition to non-multifractality. Earlier, the multifractals with the right part of spectrum of singularities not well defined (caused by a phase transition) was mimicked by a random version of the paradigmatic two-scale Cantor set and also in the context of DLA [14–17] (and refs therein).

In recent years much effort has been devoted to the reliable identification of the multifractality in real data coming from various fields such as geophysics [18], in seismology [19] and hierarchical cascades of stresses in earthquake pattern [20] [21], atmospheric science and climatology (e.g., turbulent phenomena) [22] [23], financial markets [24] [25], neuroscience [26] (e.g., neuron spiking [27]), cardio-science or cardio-physics [28] (e.g., physiology of the human heart) [29] and refs. therein, and further work investigating complexity in heart rate [30] [31] and physiology [32]. However, the identification of multifractality is still a challenge, because there are many circumstances
in which an apparent (spurious) multifractality appears.

Recognizing true multifractality is all the more difficult because we are not sure that all sources of multifractality have been discovered to date [33] and because one has to deal with physical multifractality of limited range; also, the limited amount of empirical data available is a serious technical challenge. These last two hurdles are finite size effects, which sometimes manages to disarm by finite size scaling.

The spontaneous volatility clustering (present even for a single realization of the Poisson random walk in the finite time range) can hinder the identification of true, significant, and stable multifractality. There are also other difficulties with this identification, especially when nonlinear properties of time series are studied. A spurious multifractality can also arise as a result of very slow crossover phenomenon on finite time scales [24]. In addition, the pollution of a multifractal signal with noise (white or colored) as well as the presence of short memory or periodicity can significantly change properties of the multifractal signal.

Unfortunately, the physical origin of multifractality is, in fact, rarely identified. Only two sources of true multifractality have been known to date [33]: (i) presence in the system of broad distributions and/or (ii) long-term/range correlations. However, there is a widespread belief that some stochastic or deterministic mixture of monofractals should produce multifractals [7, 15, 24]. All of them are able to produce cascades that lie at the heart of multifractality.

Incontrovertibly, the situation is complicated. Nevertheless, we demonstrate, by studying the time series of interevent times, that the extraction of true multifractality is possible in this important case. Notably, the multifractality of the series of interevent times is poorly and only occasionally researched, although it comes from a key role of the dependence between inter-event times. As a part of this work, we attempt to fill this deficiency, even more striking because inter-event or waiting times are an essential element of the popular continuous time random walk formalism [35, 37].

Financial markets fluctuate, sometimes strongly by increasing the risk level in order to maximize profit. This finds its reflection in the interevent times’ patterns acting as a direct reflection of the systems’ activities – their various properties were studied in the last decade [1, 9, 10, 33–43]. Among them, the key observation is that quite often the dependence between waiting times dominates that between spatial increments [44] defining the process, which cannot be considered as renewal [45]. As without examining the role of interevent times, we are not able to describe the dynamics of financial markets, these studies are still at an early stage of development. This situation is the motivation and inspiration for our work, emphasizing the above-mentioned key role of inter-event times.

In this work, we study fluctuations of mean interevent times and their dependencies by relying on their absolute central moments and autocorrelations of fluctuations’ absolute values. In the case of financial markets, the fluctuations are (generally speaking) a consequence of the double-auction mechanism (represented by the book of orders) [16, 45]. The approach allows you to order the fluctuations according to the degree of their corresponding moments (cf. the Lyapunov inequality in ref. [49]). This is essential in a multiscaling analysis in many branches of science.

Our approach is based on the fluctuation function, similarly to canonical multifractal detrended fluctuation analysis (MF-DFA). We correctly take into account the normalized partition function. This partition function is built on the basis of the escort probability or normalized fluctuation function. Such an approach is crucial in correctly reading from empirical data the generalized scaling exponents. We have demonstrated that the multifractality obtained is real and not apparent – the latter forced mainly by finite size effect. Moreover, we obtained multi-branched left-sided multifractality, where first and second order phase transitions exist together with thermal stable and unstable phases as well. Nevertheless, it is still a challenge to find microscale physical mechanisms underlying this multifractality. We expect this to play a significant role in the future analysis of real-time series of different origins e.g. geophysical, medical, and financial.

In addition to the above, the non-monotonic behavior of the generalized Hurst exponent producing turning points in the behavior of the Hölder exponent is directly responsible for the multi-branched left-sided spectrum of dimensions and for the first and second order phase transitions, together with thermal stable and unstable multifractal phases. To find this multi-branched spectrum on the financial market (in an alternative way to that used in papers [9, 10]) and perform an analysis, the use of the Legendre-Fenchel transform is required. This transform is a generalization of the Legendre transform commonly used to extract usual (single-branched) multifractality from empirical data.

The paper is organized as follows. In addition to the present section, where we give the motivation of our work and its goal, indicating a possibility of extension of our approach to research areas far beyond the social sciences, it consists of Sec. [11 where the ex-
tension of the canonical MF-DFA is developed and applied to the description of poorly exploited
the empirical time series of inter-event times. In Sec. [II] we reveal the existence of the first and second order
phase transitions in this type of multifractality and examine the main thermodynamic consequences. Fi-
ally, in Sec. [IV] we discuss key results of the work, indicate their importance and summarize the whole
work.

II. NORMALIZED MULTIFRACTAL DETRENDED FLUCTUATION ANALYSIS

As is well known, multifractality occurs where fluctuations and/or dependences occur in many differ-
ent spatial and/or temporal scales under different scaling laws i.e. defined by various scaling ex-
ponents, which create a multiscaling phenomenon. For example, multifractality can be caused by the
long-term dependence (e.g. temporal nonlinear long-
term autocorrelations) or/and some broad distri-
butions, leading to the hierarchical organization
of many scales. The identification of multifractality in
empirical data requires caution, not only due to the
finite size effect [50] and crashes [51] (i.e., strong nonstationarities) but also because of the presence of
spurious [52] and/or corrupted multifractality [53].
Fortunately, because multifractality is sensitive to
these effects (or contaminations), they can be prop-
erly identified and eliminated or at least minimized.

The main purpose of this work is the analysis of
multifractality generated by the non-monotonic be-
havior of the generalized Hurst exponent. This non-
monotonicity results in left-sided multifractality and
a multi-branch spectrum of dimensions. The central
role in the analysis is, therefore, the generalized Leg-
dre transform called the Legendre-Fenchel trans-
form, which is also referred to as the contact trans-
form. It must be said that the nonmonotonic q-
dependence of the generalized Hurst exponent was
already observed in both true and spurious multi-
fractality contexts in [54] and refs. therein.

In this paper, we develop the normalized multi-
fractal detrended fluctuation analysis ready for the
analysis of both stationary and nonstationary de-
trended time series. This means that we allow that
after detrending, time series may still contain some
higher-order nonstationarities, which we call weak
nonstationarities. In our case, volatility clustering
of fluctuations is such a weak nonstationary phe-
nomenon generating real multifractality.

Our approach combines the statistical-physical
analysis of weakly nonstationary states, based on the
generalized statistical-mechanical partition function,
with that based on the multiscale fluctuation func-
tion. In general, we are travelling from moments of
arbitrary orders through the partition function to
multifractality. This is due to the consistent defini-
tion of escort probability introduced herein, which
is more proper than the non-normalized and some-
times even negative one given by Eq. (12) in ref.
[33]. We are dealing only with the analysis of de-
trended absolute values, i.e., that bereft of the di-
ichotomous noise. The motivation is that this type
of nonlinear quantities can be long-term autocorre-
lated as opposed to the (usual) bilinear autocorre-
lations. In our case, the autocorrelations which are
studied point to the existence of a distinct antiper-
sistent structure of fluctuations behind them. Per-
haps, this structure reflects the fact that after the
period of high market activity there is period of sig-
nificantly lesser activity and so on in an alternating
fashion, leading to the effect of volatility clustering.

A. Intraday fluctuations of interevent times

The intraday autocorrelation of the absolute addi-
tively detrended profile is defined for a single typical
day (or replica) ν and an arbitrary timescale s,
\[
F^2(j; \nu, s) = \frac{1}{s - j} \sum_{i=1}^{s-j} |U_\nu(i) - y_\nu(i)| \cdot |U_\nu(i + j) - y_\nu(i + j)|, \quad \nu = 1, \ldots, N_d; (1)
\]
where \( j = 0, \ldots, s - 1 \), defines time-step distance
or number of time windows of length Δ (see the
schematic plot in Fig. [I] for details; in the original
work [33] this length was marked by \( s \)) between
both absolute deviations (detrended fluctu-
ations) \( |U_\nu - y_\nu| \) present at day \( \nu \) at time steps \( i \)
and \( i + j \) (1 ≤ i ≤ s numbers the current time win-
dow, \( y_\nu \) is the detrending polynomial, and variable
\( U_\nu \) is defined below), \( s \) is the total number of time
windows within an arbitrary replica \( \nu \) (the same for
each one), defining the daily time scale, 1 ≤ \( \nu \) ≤ \( N_d \),
and \( N_d \) is the number of trading days (replicas).

Note that
\[
y_\nu(i) = \sum_{m=0}^{M} A_m^\nu i^m, \quad M \geq 0, \quad (2)
\]
where in all our further considerations we assume
\( M = 3 \) as it is the lowest order of the polynomial,
which enables us to reproduce an inflection point
present in the majority of empirical profiles, \( Y_s \), as
a result of common (intraday) ‘lunch effect’.
FIG. 1. Schematic diagram defining interevent times and their single-window (local) means. Apparently, this mean for \( i \)th time window, \( [i, i+1[ \), \( 1 \leq i \leq s \), (each of length \( \Delta \)) is given by the time average \( \overline{\Delta t}_{i} = \frac{1}{n_{i}} \sum_{i=1}^{n_{i}} \Delta t_{i,j} \) of inter-event times, \( \Delta t_{i,j} \), where \( n_{i} \geq 1 \) is the number of inter-event times associated with \( i \)th time window. We define the inter-event time \( \Delta t_{i,j} \) as associated with \( j \)th time window \( [i, i+1[ \), when at least the left border of \( \Delta t_{i,j} \) belongs to it. By using the ensemble average we can write \( \overline{\Delta t}_{i} \approx \Delta / n_{i} \), which is also used in the main text where we focus on ranges of \( s \geq 1 \) (keeping simultaneously \( n_{i} \geq 1 \)). For the Warsaw Stock Exchange, the duration of a single trading session \( T = 28200 \) [sec].

Apparently, for \( j = 0 \) the detrended autocorrelation function (also called in this case the detrended self-correlation function) becomes a detrended fluctuation function. Therefore, we can introduce the notation \( F^{2}(\nu, s) \) \( \stackrel{\text{def}}{=} F^{2}(j = 0; \nu, s) \).

The single-day (\( \nu \)-day) profile \( U_{\nu} \) is defined by the corresponding difference between subsequent multi-day profiles, \( Y_{s} \). This difference equals the cumulation of single-window mean inter-event times, \( \overline{\Delta t}_{i} \), shown in plot (a) in Fig. 2 (in our considerations we deal, in fact, with, \( \overline{\Delta t}_{i} \leq \Delta \)). That is, we use definitions,

\[
U_{\nu}(i) = Y_{i}[\nu \cdot (s + i)] - Y_{i}[\nu \cdot s] = \sum_{i'=1}^{i} \overline{\Delta t}_{i'},
\]

where \( Y_{\nu}[0] = 0 \) and for the first replica we have \( U_{\nu=1}(i) = Y_{\nu=1}[i] = \sum_{i'=1}^{i} \overline{\Delta t}_{i'} \). Apparently, Eq. (3) makes it possible to determine recurrently the multi-day profile,

\[
Y_{i}[\nu \cdot (s + i)] = \sum_{\nu' = 1}^{\nu - 1} \sum_{i'=1}^{s} \overline{\Delta t}_{\nu'} + \sum_{i'=1}^{i} \overline{\Delta t}_{\nu'}, \quad \nu \geq 2.
\]

Eq. (3) can be interpreted in terms of directed (climbing) random walk – see the monotonically increasing broken curve drawn on plot (b) in Fig. 2.

We use for all plots in this work tick-by-tick transaction data for KGHM (copper and silver production), one of the most liquid stock on the Warsaw Stock Exchange, from 3rd January 2011 to 14th July 2017 (1620 trading days). Fig. 2 is intended to show the daily structure of empirical data (including detrended data). Notably, a typical intraday pattern of a single-day inter-event time means, \( \overline{\Delta t}_{i} \), of transactions falling into a single time window of day \( \nu \) is shown vs single time window number \( i \) in the plot (a). The data bursts, that is spikes or explosions protruding no less than the standard deviation (solid strongly oscillating curve) above the average value (solid weakly oscillating curve) are well seen. Typically, local clusters of spikes around their local maximums are well visible. These clusters are separated by those of high system activity, where the shortest lengths of inter-event times can be observed not only close to lunchtime. We observe, more or less every 100 (= 20 x \( \Delta = 5 \) ) minutes, spikes of locally longest lengths, that is, a decrease in the activity of market investors four times a day (twice before noon and twice after). Such a long-term pattern constitutes a source of generalized volatility clustering effect within detrended mean inter-event time series, and hence their multifractality.

All plots in Fig. 2 are prepared, for instance, for typical time window of length \( \Delta = 300 \) [sec] hence, the daily total number of time windows \( s = 28200/300 = 94 \) (as the duration of a daily stock market session equals 7 [h] 50 [min] = 28200 [sec]). It is worth mentioning that the mean number of transactions within a single time window \( \Delta = 300 \) [sec] is about 20 (as the empirical mean time distance between subsequent transactions equals approximately 15 [sec]).

Detrended, weakly non-stationary \( \nu \)-day profile \( U_{\nu}(i) - y_{\nu}(i) \), is shown in plot Fig. 2(c), and its square in plot Fig. 2(d). We say weakly because it is a remnant of removing the main non-stationarity, which is the trend, that is the component of the lowest frequency. The oscillatory (or waving) behavior in the former plot can be interpreted as a reminiscence of the volatility clustering effect. This is present even for the structure which appears as an average over the statistical ensemble of days (see Fig.
The volatility clustering effect means that a series of transactions occurring with higher frequencies are preceded by much less frequent transactions and vice versa. Thus the volatility clustering effect is essentially extended for clusters of inter-event times.

Behavior analogous to that in plot Fig. 2(d) is shown in plots Fig. 2(e) – Fig. 2(g), where autocorrelators, $|U_{\nu}(i) - y_{\nu}(i)| |U_{\nu}(i + j) - y_{\nu}(i + j)|$, are shown vs $i$ for chosen typical $j$ values. Apparently, how much similar are autocorrelators’ behavior for several fixed values of $j$ to the square of the detrended profile (shown in plot Fig. 2(d)), suggesting that autocorrelators are, in fact, under control of fluctuations’ structure (that is $j = 0$). It is striking how stable their patterns are against $j$, even for its large values. Besides the above-mentioned plots, we also consider plot Fig. 2(h), showing the autocorrelation function defined by Eq. 1. This result is particularly interesting as it shows an oscillatory (and not only relaxing) character of the one-day autocorrelation function. Hence, we found fluctuation induced patterns of inter-event times based on a possible long-term dependence between local fluctuations (see Fig. 3 for details), which constitutes the main interest of this work.

Note that plots in Fig. 3 present key quantities...
FIG. 3. Characteristic averaged quantities (for instance, for \( s = 94 \)), where \( \langle \ldots \rangle \) means the averaged over a statistical ensemble of \( N_d \) days. (a) The average empirical (detrended) quantity \( \langle U(i) − y(i) \rangle \) vs time window number \( 1 \leq i \leq s \). Its typical antipersistent structure is clearly seen. This is a very significant result proving that not only the single realization (that is for a single day), shown in Fig. 2(c), has an antipersistent structure. For other values of \( s \), analogous structures were obtained. The sharp increase in the curve at the end of the range is the well-known Runge effect of fitting by the polynomial (here denoted by \( y \)). (b) The empirical (not normalized) distribution of deviations \( U−y \) collected from all \( N_d \) days, is presented in lin-log scale. Apparently, the empirical distribution of these deviations strongly differs from the corresponding (much narrower) Poisson one. Moreover, a systematic deviation from the initial exponential distribution (solid straight lines) is well seen for \( |U−y| > 200 \). Randomizing these (empirical) deviations from its distribution corresponds to the shuffling of interevent times. (c) The averaged autocorrelation function \( \langle F^2(j; s) \rangle = \frac{1}{N_d} \sum_{i=1}^{N_d} F^2(j; \nu, s) \) vs \( j \) (solid curve). It seems that it is a very slowly converging (waving) function as it is roughly approximated by the (shifted) power-law, \( \langle F^2(j; s) \rangle = A/(\alpha+j)^{\alpha}+const \), where fitted power-law exponent \( \alpha = 0.49\pm0.43 \) is definitely smaller than 1, fitted amplitude \( A = 1049\pm387 \), while background parameter \( const = 1519\pm234 > (U−y)^2 = 1190 \); hence, shift parameter \( a = \left( \frac{A}{\alpha+j} \right)^{\frac{1}{\alpha}} = 1.0\pm0.90 \), where \( (U−y)^2 \) was obtained independently. Note that the upper dashed-dotted horizontal line represents \( \langle (U−y)^2 \rangle \), while the bottom one \( \langle U−y \rangle^2 \), both obtained with high accuracy. The dotted horizontal line represents the above given \( const \). Perhaps its location (above \( \langle U−y \rangle^2 \) ) is due to the existence of a structure, e.g., such as shown in Fig. 2(a). (d) Solid curve shows \( j \)-dependence of usual autocorrelation function \( \langle F^2(j; s) \rangle = \frac{1}{N_d} \sum_{\nu=1}^{N_d} F^2(j; \nu, s) \), where \( F^2(j; \nu, s) \) is defined by an expression analogous to Eq. (1) in which there are ordinary differences and not their absolute values. It seems that this function is oscillating rapidly, it disappears quickly in comparison with \( \langle F^2(j; s) \rangle \). The \( j \)-dependence of autocorrelation function \( \langle F^2(j; s) \rangle \) (solid curve) for the empirical weakly non-stationary time series (multiplicatively standardized here by their standard deviations – see the plot in Fig. 2(a)) was shown in plot (e). Analogously to the plot (c), the power-law fit is represented here by the dashed curve, where fit parameters are: \( A = 7.28 \pm 12.42 \), \( a = 3.25 \pm 2.15 \), \( \alpha = 1.39 \pm 0.70 \), \( const = 2.46 \pm 0.05 \) resulting in analogous conclusions (even though the exponent here seems to be greater than 1.0). The decaying of the autocorrelation function is shown in the plot (f) for the Poisson process as a function of \( j \). There the exponential function \( A \exp(-\alpha j) + const \) is fitted to empirical data (dashed curve), where \( A = 12.06 \pm 0.08 \), \( a = 0.496 \pm 0.031 \), \( const = 23.12 \pm 0.08 \). Unfortunately, the range of empirical data is too short (limited to half a trading day in plots (c) – (f)) to say anything more definite, although we can suppose that very slow convergence (shown in plots (c) and (e)) and existing the structure manifested in \( const > (U−y)^2 \) are, indeed, the main causes of the multifractality considered in this paper.

averaged over the statistical ensemble of days, whose individual realizations are given in the corresponding plots in Fig. 2. It seems that the autocorrelation function \( \langle F^2(j; s) \rangle \) slowly relaxes. This results from its construction on the basis of absolute values of deviations (fluctuations), which are always non-negative. It contains some information about the existence of a long-term antipersistent structure which
makes the autocorrelation non-vanishing quantity. 
We have grounds, however, to suppose that the anti-persistent (quasi-periodic) fluctuation structure is the result of the existence of a long-term dependence or correlations between fluctuations – they are the reason for the creation of this structure and not the other way round. Thus the weakly non-stationary structure is produced.

There is also a complementary interesting aspect shown in Fig. 3(b), where the empirical histogram of deviations $U - y$ is compared with the one obtained from the Poisson distribution (by drawing the number of transactions in each time window $\Delta$, separately). You can see a huge widening of the empirical histogram in relation to the one obtained from the Poisson distribution. Notably, for this distribution, a number of transactions in each time interval $i$ of the same length $\Delta$ is drawn, where the only control parameter, i.e. the mean number of transactions, is taken from empirical data the same for the whole multi-day time series. On this basis the local (fluctuating) mean time of interevent transactions, is taken from empirical data the same

Thus, the source of multifractality can in our case be not only nonlinear long-term autocorrelations of absolute deviations $|U - y|$ but also the broadened distribution of deviations $U - y$.

One can say that the results presented in the Figs. 2 and 3 are the starting point of the following considerations.

**B. Generalized partition function**

An escort probability (that is, escorting the fluctuations) specifies the chance of occurrence of a certain fluctuation value for a given day $\nu$ within scale $s$. This probability can be constructed in the form,

$$ p(\nu, s) = \frac{[F^2(\nu, s)]^{1/2}}{\text{Norm}(s)}, $$

$$ \text{Norm}(s) = \sum_{\nu=1}^{N_d} [F^2(\nu, s)]^{1/2} $$

(5)

that is, based on the fluctuation function defined by Eq. (4). Hence, the mean value, $\langle p(s) \rangle = 1/N_d \sum_{\nu=1}^{N_d} p(\nu, s) = 1/N_d$, is fixed (as a result of normalization). An even more refined approach based on a $q$-zooming escort probability has been shown in [57].

The generalized $q$-dependent (statistical-mechanic) partition function can be defined as usual by the sum,

$$ Z_q(s) \overset{\text{def}}{=} \sum_{\nu=1}^{N_d} |p(\nu, s)|^q. $$

Assuming, the scaling hypothesis central to our work for fluctuations in the form,

$$ \sum_{\nu=1}^{N_d} [F^2(\nu, s)]^{q/2} \approx N_d A_q s^{qh(q)}, $$

(7)

where prefactor $A_q$, and the generalized Hurst exponent $h(q)$ are $s$-independent, one derives from Eq. (6)

$$ Z_q(s) \approx \frac{1}{N_d^{q-1}} A_q^{rel} s^{qh^{rel}(q)}, $$

(8)

where the relative (or reduced) prefactor $A_q^{rel} = A_q/(A_q=1)^q$ and the relative (or reduced) generalized Hurst exponent $h^{rel}(q) = h(q) - h(q = 1)$.

Note that scaling hypothesis (7) allowed to present $\text{Norm}(s)$ (given by the second equality in Eq. (5)) in the form

$$ \text{Norm}(s) = \sum_{\nu=1}^{N_d} [F^2(\nu, s)]^{1/2} \approx N_d A_q=1 s^{h(q=1)} $$

(9)

used, indeed, to obtain Eq. (8).

Because we are considering a statistical ensemble consisting of $N_d$ replicas, from Eq. (6) we have $Z_{q=0}(s) = N_d \leftrightarrow A_{q=0} = 1$

Finally, we can write Eq. (8) in useful forms

$$ Z_q(s) \approx \frac{1}{N_d^{q-1}} A_q^{rel} s^{\tau^{rel}(q)} = Z_{q}^{\text{lin}}(s) \tilde{Z}_q(s), $$

(10)

where

$$ Z_{q}^{\text{lin}}(s) = \frac{1}{N_d^{q-1}} A_q^{rel} s^{-(q-1)D(q=0)}, $$

$$ \tilde{Z}_q(s) = s^{\tau(q)}, $$

(11)

and the relative (or reduced) scaling exponent $\tau^{rel}(q) = qh^{rel}(q) = (q - 1)D^{rel}(q)$, and the relative (or reduced) Rényi dimension $D^{rel}(q) = D(q) - D(q = 0)$, while $D(q)$ is Rényi dimension defined, as usual, by scaling exponent $\tau(q)$,

$$ \tau(q) = (q - 1)D(q), $$

(12)

where

$$ \tau(q) = qh(q) - D(q = 0), $$

(13)

here and above we assume $D(q = 0) = h(q = 1)$ for self-consistency. Thanks to this, not only $\tau^{rel}$
but also \( \tau \) can be expressed in the required form by means of \( D^{rel} \) and \( D \), respectively. Hence, we have
\[
\tau^{rel}(q) = \tau(q) - (q - 1)D(q = 0) \]
vanishing at \( q = 1 \) and \( q = 0 \). Therefore, all the relative quantities defined above (and indexed by ‘rel’) disappear in \( q = 1 \) or \( q = 0 \), which justifies their relative character.

Moreover, partial partition functions \( Z_{q}^{lin} \) and \( \tilde{Z}_{q} \) are normalized separately, and the factorization given by Eq. (10) (up to multiplicative prefactor and additive exponents) is unique. These partition functions represent statistically independent monofractal and multifractal structures, respectively. We pay attention to the latter one.

There are several characteristic values \( q \) of which two \((q = 1 \text{ and } q = 0)\) are considered in this section. For \( q \to 1 \) one can write the expansion,
\[
\tau(q) \approx (q - 1)[h(q = 1) + q \frac{dh(q)}{dq}]_{q=1} + \frac{1}{2}q(q - 1)\frac{d^2h(q)}{dq^2} \bigg|_{q=1},
\]
(14)
based on the expansion of \( h(q) \) at \( q = 1 \), where the expression in square brackets is indeed,
\[
D(q) \approx D(q = 0) + q \frac{dh(q)}{dq} \bigg|_{q=1} + \frac{1}{2}q(q - 1)\frac{d^2h(q)}{dq^2} \bigg|_{q=1}
\]
(15)
or equivalently
\[
D^{rel}(q) \approx q \left[ \frac{dh(q)}{dq} \bigg|_{q=1} + \frac{1}{2}q(q - 1)\frac{d^2h(q)}{dq^2} \bigg|_{q=1} \right],
\]
(16)
Expansion in Eq. (16) emphasizes that \( D^{rel}(q) \) depends (in the vicinity of \( q = 1 \)) on the successive derivatives of the generalized Hurst exponent at \( q = 1 \) as parameters.

For instance, combining Eqs. (6) with (10), we obtain an expression,
\[
D^{rel}(q = 1) = \frac{1}{\ln s} \sum_{\nu=1}^{N_{d}} p(\nu, s) \ln p(\nu, s) = \frac{1}{\ln s} \ln p(\nu, s) \ln I_{q=1}(s),
\]
(17)
where \( I_{q=1}(s) \) can be identified with the Shannon information (within the scale of \( s \)).

Thus we define two families of \( q \)-dependent quantities: relative and non-relative (usual) ones. For example, the Rényi dimensions (i.e. \( D^{rel}(q) \) and \( D(q) \)), play a different role for \( q = 0 \) than in the canonical approach to multifractality.

In the canonical approach, one can read directly from the scaling relation for the partition function that \( D(q = 0) \) is the fractal dimension of the substrate (i.e. the support of the measure or function). Since in our approach, the Rényi dimensions enter in a relative way into the generalized partition function, such a diagnosis does not take place. Therefore, \( D(q = 0) \) does not have to be a fractal dimension of the substrate, and the \( D^{rel}(q = 0) \) even vanishes. For this reason, the \( D(q) \) family should rather be called pseudo Rényi dimensions, while \( D^{rel}(q) \) family the relative or reduced one (despite the fact that for \( q \neq 0 \) both families have usual interpretations).

C. Legendre-Fenchel transformation, and multi-branched left-sided multifractality

In this section, we show that although the spectrum of dimensions built on \( \tau \) and \( \tau^{rel} \) have the same shape, their locations within the multifractal coordinates are different. We prove that the proper location has the spectrum of dimensions built on the scaling exponent of fluctuations \( \tau \).

We obtain directly from Eq. (7),
\[
\ln F_{q}(s) \equiv h(q) \ln s + q^{-1} \ln A_{q},
\]
(18)
where \( q \)-dispersion \( F_{q}(s) \) is defined
\[
\left\{ N_{d}^{-1} \sum_{\nu=1}^{N_{d}} \left[ F^{2}(\nu, s) \right]^{q/2} \right\}^{1/q}.
\]
Using the dependence of \( F_{q}(s) \) on the scale \( s \) (see Fig. 4 for details) for values of \( q \) from its wide range (that is \(-10.0 \leq q \leq 10.0\)), we have determined both
the generalized Hurst exponent \( h(q) \) and related signatures of multifractality such as Rényi scaling exponent \( \tau(q) \), Rényi dimensions \( D(q) \), the coarse Hölder exponent \( \alpha(q) \) and multifractal spectrum \( f(\alpha) \) (see plots (a), (c), (d), (e), (f), respectively in Fig. 5) as well as significant prefactor \( B(q) \equiv q^{-1} \ln A(q) \) (see plot (b) in Fig. 5) related to reduced Rényi information.

It is worth emphasizing that aggregating events into time intervals of the same length \( \Delta \) is the one in which the scaling effect is observed, herein on \( F_q(s) \) vs \( s \equiv T/\Delta \), where \( T = 7h \) 50min and for all values of \( q \) we have a common range 3min 55s \( \leq \Delta \leq 19\)min 35s. For this range of \( s \), the measure \( \chi^2 \) per degree of freedom reaches the smallest value. Only slightly larger is this quantity when the left border of \( s \) is assumed to be \( s = 10 = \Delta = 47\)min, while the right one is \( s = 150 = \Delta = 3\)min 8s.

Finally, having scaling exponent \( \tau(q) \), the spectrum of dimensions can be found by using the Legendre-Fenchel transformation instead of Legendre transform, although formally identical to the Legendre transform, are its generalization. The Legendre transform is limited here only to the main branch of spectrum \( f \) defined by its contact relations: (i) \( f(\alpha(q = 1)) = \alpha(q = 1) \) and (ii) \( df/df|_{\alpha(q = 1)} = 1 \). The inset plot shown in Fig. 6(a) illustrates this contact character. This is emphasized by a dashed straight line with directional coefficient (slope) of 1.0 tangent to spectra of singularities at the point \( \{\alpha(q = 1), f(\alpha(q = 1))\} \). Breaking the contact character of the Legendre transformation results in the wrong location of the spectrum of singularities if it exists.

Put more generally, the given contact relations above \( (for q = 1) \) provide unambiguous location of the full multi-branched spectrum of dimensions obtained using the Legendre-Fenchel transformation. Our multi-branched multifractal contains a single contact point which means that we are dealing here with a single multifractal. Fig. 6(a) and (b) shows a significant result because it offers a necessary (not sufficient) requirement for finding true multifractality in empirical time series.

Thanks to the above, we can clarify the key term ‘multi-branched left-sided multifractality’. We say that we are dealing with this type of multifractality if its main branch (that is, the branch that meets the condition of contact) is fully determined only by the positive values of \( q \).

**III. FIRST AND SECOND ORDER PHASE TRANSITIONS**

From Eq. 19 one can obtain very useful expression for the specific heat of the multifractal structure (10) (and refs. therein) in the form,

\[
c(q) = \frac{d\alpha(q)}{d(1/q)} = -q^2 \frac{d\alpha(q)}{dq};
\]

its \( q \)-dependence is shown in Fig. 6(c).

Only two regions are visible in which the system is thermally stable, i.e. fulfilling inequality \( c(q) \geq 0 \). One of them is located between vertical dashed lines \( a \) and \( c \) or points \( A_1 \) and \( C \), defining the \( q \)-range of the main branch of left-sided spectrum of dimensions shown in plots (a) and (b). The second, the side thermally stable left-sided spectra of dimensions, is limited to the range of \( q \) preceding vertical dashed line \( b \) or point \( B_1 \). For \( q \) ranging between points \( B_2, A_1 \) and after point \( C \), we deal with thermally unstable phases. In turning (bifurcation) points \( B_1, A_1 \), and \( C \), there are phase transitions of the second order between thermally stable and unstable phases, which is consistent with specific heat
vanishing there. Between points $X_1$ and $X_2$, located in thermally stable phases, the first order (discontinuous) phase transition occurs.

To prove the above given statements concerning the order of phase transitions, we study the behavior of the first, $df/d\alpha$, and second, $d^2f/d\alpha^2$, derivatives vs $\alpha$, based on the result presented in Fig. 5(f). Using the Taylor expansion of $\alpha(q)$ function in the vicinity of its local extremes we obtain,

\[
\alpha(q) \approx \alpha(q_{extr}) + \frac{1}{2} (q - q_{extr})^2 \frac{d^2\alpha}{dq^2} \big|_{q=q_{extr}},
\]

where $q_{extr}$ is a $q$-position of the local extreme or turning point of $\alpha(q)$ function. There are three such local extrema: one maximum $A_1$ and two minima $B_1, C$.

Inverting Eq. (22) and using the first equation in (24), after simple algebraic calculations, we obtain useful two-branched formulas,

\[
\frac{df}{d\alpha} \approx \pm \sqrt{2 \left| \frac{\alpha - \alpha_s}{\alpha_s} \right| + q_{extr}},
\]

\[
\frac{d^2f}{d\alpha^2} \approx \pm \frac{1}{\sqrt{2 |\alpha_s|}} \frac{1}{\sqrt{|\alpha - \alpha_s|}},
\]

where we use the abbreviated notation: $\alpha_s = \alpha(q_{extr})$ and $\alpha_s^* = \frac{d^2f}{dq^2} |_{q=q_{extr}}$. Apparently, spectrum of dimensions $f$ has singularities of the second order at its turning points (see plots (a) and (b) in Fig. 6 for details).

Moreover, by substituting the formula given by Eq. (22) to Eq. (21), we obtain

\[
c(q) \approx -q^2(q - q_{extr})\alpha_s^*,
\]

i.e. that it linearly vanishes at turning points, which can be considered to be spinodal decomposition points.

Additionally, at the point of intersection of branches, marked twice by $X_1, X_2$, the first order phase transition is present.

Fig. 7 shows the behavior of the first $(df/d\alpha)$ and second $(d^2f/d\alpha^2)$ order derivatives of spectrum of dimensions $(f)$ versus Hölder exponent $(\alpha)$. In combination with the plots (a) and (b) in Fig. 6 this allows us to define phase transitions at points $A_1, B_1$, and $C$ and at a point having the double mark $X_1, X_2$.

The Ehrenfest like classification of phase transitions which we use is based on the spectrum of dimensions $f$, which can be treated as the analogon of

FIG. 5. The $q$-dependence of key empirical characteristics of multifractality. The nonlinear dependence of these characteristics on $q$ is well seen. Plots (a) – (f) present dependence on $q$ of the generalized Hurst exponent $h(q)$, its spread $\Delta h(q) = h(-q) - h(q)$, Rényi scaling exponent $\tau(q)$, Rényi dimensions $D(q)$, and the coarse Hölder exponent $\alpha(q)$, respectively. The vertical dashed lines visible in all plots define the range of positive values of $q$ between the maximum and the minimum of function $\alpha(q)$ vs $q$ shown in plot (f). The tangent dashed straight line visible in plot (c) has been fitted to the linear section of curve $B(q)$ vs $q$. Additional thin solid olive curves present on all plots were obtained from the time series generated by the Poisson distribution. Apparently, their variations are negligible which means that the influence of a finite size effect on a time series with a size equal to the empirical one is negligible.
entropy \cite{7,10}. Our classification is only inspired by the canonical Ehrenfest, because the latter classification uses the chemical potential and not the entropy, although both are the thermodynamic potentials.

Apparently, \( f \) and \( df/d\alpha \) are continuous functions of \( \alpha \) as opposed to \( d^2 f/d\alpha^2 \). All these functions are multi-branched but only the second derivative has separated branches. These branches are divergent to \( \pm \infty \) just in points \( A_1, B_1, \) and \( C \) according to the power-law with exponent equal to \( -1/2 \). This means that in these points, there are identical phase transitions of the second order (i.e. belonging to the same universality class), which confirms the behavior of specific heat in these points given by Eq. \cite{24}. At the second order phase transition points of specific heats, susceptibilities or other appropriate order parameters either diverge (obeying a non-trivial scaling law) or go to zero, which happens in our case.

Let us note that black curves \( (X_1, A_1) \) and \( (B_1, X_2) \) define the thermally metastable phases, while the blue curve \( (A_1, B_1) \) defines the unstable mixture of phases. If the system is located in this latter phase, it will spontaneously evolve towards a state, which favors either higher fluctuations (above point \( A_1 \)) or smaller fluctuations (below point \( B_1 \)). The probability of choosing one of these two options depends on how closely the state of the system is located near the edge of the phase (point \( A_1 \) favors large fluctuations while \( B_1 \) favors a small fluctuation). Speaking in sociological terms, in this mixture phase region, the members’ moods/opinions are divided and the victory of one of them may lead either to a permanent increase in the diversity of the members’ moods/opinions of the system or to their lasting calm.

For the second unstable phase (also defined by the blue curve \( (C, D_1) \)) a simplified interpretation should be developed as it is not placed between two metastable phases, although \( d^2 f/d\alpha^2 \) diverges at transition point \( C \) in the same way as at points \( A_1 \) and \( B_1 \). We can only say that the system left alone in this phase will spontaneously evolve into the stable phase.

IV. CONCLUDING REMARKS

Our work is part of mainstream research on the problem of long-term memory, long-term dependence, and long-term correlations in time series \cite{61}. By using the Legendre-Fenchel transform we have examined, the resulting multi-branched left-sided true multifractal properties of time series of inter-event times. We have chosen inter-event times for our research because they are a key measure of the activity of any system (not necessarily complex), research into which is only at the initial stage. The relationships between inter-event times form the foundation of other dynamic relationships occurring in an evolving system.

Our research focuses on the search for multifractality because it is the most general, as of yet, characterization of time series at a macro scale, enabling the study of their universal properties from an ex-

\[ f(\alpha) \]

\[ c(q) \]

\[ A_1 \]

\[ B_1 \]

\[ X_1 \]

\[ X_2 \]

\[ D_1 \]

\[ D_2 \]
FIG. 7. The illustration of the Ehrenfest-like classification of phase transitions. The first (black solid curve) and second (four separated red solid curves) derivatives of $f$ over $\alpha$ showing three two-branched second order singularities of $f$ vs $\alpha$. Three dashed vertical straight lines (vertical asymptotics) $c, b$, and $a$ are located, at $\alpha$ coordinates of singularities. The main branch of derivative $df/d\alpha$ is represented by the black curve $(C,D_2,B_2,X_1,A_1)$ also containing the inflection point $IP_3$. The corresponding red curve of the second derivative $d^2 f/d\alpha^2$ contains the replica of the inflection point $IP_3$ which diverges at asymptotics $c$ and $a$. This curve is singular at turning points: the left one at $\alpha$ coordinate of point $C$ and the right one at $\alpha$ coordinate of point $A_1$. The other three singular curves (also in red) are associated with three side branches of the first derivative $df/d\alpha$. The first of them (ending at point $D_1$), has its local minimum at replica of inflection point $IP_4$, and it is bound to branch $(C,D_1)$ of the first derivative containing the inflection point $IP_4$. This branch is thermally unstable (see plot in Fig. 6(c) for details). The second, having its local minimum at the replica of inflection point $IP_2$ (also marked by $IP_2$), is bound to branch $(A_1,IP_2,B_1)$ (blue curve) of the first order derivative, where points $A_1$ and $B_1$ can be considered as spinodal decomposition points – there is a thermally unstable territory between them (see again plot in Fig. 6(c) for details). The third singular solid curve, having its local maximum at a replica of inflection point $IP_1$ (also denoted by $IP_1$), is bound to branch $(B_1,IP_1,A_2)$. Of course, all branches of the first derivative are associated with the corresponding branches of spectrum of dimensions, $f$ vs $\alpha$, seen clearly in plot in Fig. 6(a).

Extended point of view (allowing their classification by using their singularity spectra). However, deriving a microscopic model from knowledge of the multifractal structure of series of interevent times is still under consideration. A step in this direction has been proposed in [10], where the surrogate model was the CTRW with waiting-time distribution weighted by stretched exponential i.e. defined by some super-statistics. This is an approach sufficient to describe multifractality generated by a broadened distribution, but in the case of multifractality generated by long-term autocorrelations of interevent times, it is still a major challenge.

As is known, the search for true/real multifractality first requires the resolution of the role of at least the main factors: (i) main non-stationarity, (ii) finite size effect, and (iii) broadened distribution leading to true and/or spurious multifractality. Point (i) was solved by the detrending procedure described in Sec. II A while points (ii) and (iii) are vividly illustrated in Fig. 8, where the Rényi scaling exponent $\tau(q)$ is presented for three different cases.
The blue (almost) linearly increasing solid curve is obtained from the test Poisson distribution. For the Poisson distribution, a number of transactions in each time interval \( i \) are drawn, and on this basis the local mean time of interevent times, \( \Delta t_i \), is determined (see Fig. 1 for detailed analysis). These local mean times create a time series with a length equal to the whole empirical time series of interevent times. The presence of possible spurious multifractality here is caused only by the finite length of the time series of interevent times of (almost) the same length as the empirical time series. Apparently, the spurious multifractality of the Poisson time series caused only by finite size effect is negligible in this case. Therefore, the influence of the finite size effect on real multifractals is also negligible.

The red solid curve indicates time series of interevent times drawn from the distribution of empirical inter-event times (cf. plot (b) on Fig. 3) that is, it reflects the kind of shuffled empirical data. Thanks to this approach, long-term autocorrelations were removed from the drawn time series of length equal (with a good approximation) to the empirical time series. As you can see, in this case too, we deal with almost linear dependence if we take into account the standard deviation leading to the corridor limited by dotted curves.

Only the black solid curve obtained from empirical time series of inter-event times presented by using our NMF-DFA, is sufficiently nonlinear (cf. also plot (d) in Fig. 5) to have a chance of generating multifractality.

Thus we show, by using our NMF-DFA developed here, that an empirical series of the inter-event times gives a true multifractal located far beyond the finite size component and other multifractal pollutions. We suggest that the true multifractality found here is caused by the long-term autocorrelations between absolute values of detrended inter-event time profiles (see Fig. 3(c) for details). These autocorrelations create some true antipersistent structure of fluctuations’ clusters of the inter-event times, seen clearly in Fig. 3 and Fig. 5, defining the volatility clustering effect. It is interesting that intraday empirical data are sufficient to detect true multifractality, despite the fact that the autocorrelations of the interevent times mentioned are long-term, stretching for many days.

A peculiar characteristics of our multifractal is the presence of negative spectra of dimensions, in the vicinity of the turning point \( C \), which could be justified by the appearance of events that occur exceptionally rarely (see 10 for some suggestions).

This work is based on two main pillars. First of all, on the NMF-DFA approach constructed in the work, which was inspired by the canonical MF-DFA. Thanks to this approach, it has been proved that the time series of inter-event times can have a multi-branched left-sided multifractal character. Secondly, the work proves that this type of multifractality can lead to phase transformations of the first and second orders.

For the traditional multifractality, the phase transition of the first order disappear, which reduced the area of metastable and unstable phases to zero. This means that traditional multifractality corresponds to critical or supercritical states of the system. Thanks to this, we better understand why long-term correlations play a key role in building of multifractality. In this situation, the details of microscopic models leading to this type of multifractality do not play a significant role. From this point of view, the multifractality presented in this work is subcritical, where stable, metastable and unstable phases are still present. From the level of this work traditional multifractality can be treated only as one of the elements of full classification. Thanks to this, the concept of multifractality has been broadened substantially.

This paper is an extension of our earlier works [9, 10]. However, a crucial question arises about the next extension of the NMF-DFA formalism to the autocorrelations of ordinary fluctuations, and not only their absolute values as they are here.
This would, however, require the transition to complex scaling exponents, e.g. the complex generalized Hurst exponent, which would go significantly beyond the scope of this work.

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