ON PARSEVAL WAVELET FRAMES VIA MULTIRESOLUTION ANALYSES

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ABSTRACT. We give a characterization of all Parseval wavelet frames arising from a given frame multiresolution analysis. As a consequence, we obtain a description of all Parseval wavelet frames associated with a frame multiresolution analysis. These results are based on a version of Unitary Extension Principle and Oblique Extension Principle with the assumption that the origin is a point of approximate continuity of the Fourier transform of the involved refinable functions.

1. INTRODUCTION

We are interested in the study of methods for constructing tight wavelet frames. In this paper, we emphasis on tight wavelet frames can be constructed via multiresolution analyses and extension principles. Mallat [37] and Meyer [38] introduced the definition of multiresolution analysis (MRA) as a general method for constructing wavelets. In order to construct wavelet frames, the requirements on the definition of MRA were weakened. In this sense, the notion of a frame multiresolution analysis (FMRA) was formulated by Benedetto and Li [4] as a natural extension of MRA. Furthermore, a generalized multiresolution analysis was first introduced by Baggett, Medina and Merril [2] and Papadakis [40] independently, see also the paper by de Boor, DeVore and A. Ron [11]. Gripenberg [24] and Hernández and Weiss [30] proved a characterization of all orthonormal wavelets associated to a multiresolution analysis in terms of its wavelet dimension function. Any orthonormal wavelet is associated with a generalized multiresolution analysis was proved by Papadakis [41]. In the paper by Kim, Kim and Lim [34] (see also Kim, Kim, Lee and Lim [33]), characterizations of the Riesz wavelets which are associated with a multiresolution analysis were proved. A generalization of these results are given by Bownik and Garrigós [9]. Note that characterizations of biorthogonal wavelets from biorthogonal multiresolution analysis are proved in [34], [9] (see also Calogero and Garrigós [13]). Zalik [49] introduced the notion of Riesz wavelet obtained by a multiresolution analysis. Moreover, he gave necessary and sufficient conditions on a given Riesz wavelet to be obtained by a multiresolution analysis. Bownik [7] studied both the notion of Riesz basis associated with a generalized multiresolution analysis and Riesz basis obtained by a generalized multiresolution analysis and proved that these two notions are equivalent. Characterizations of Parseval wavelet

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frames associated with generalized multiresolution analysis are proved by Baggett, Medina and Merrill [2] and by Bakič [3].

A slight different point of view for constructing wavelet frames was first proposed in the Unitary Extension Principle (UEP) by Ron and Shen [45] (see also [44]). The UEP leads to explicit constructions of tight wavelet frames generated by a refinable function. A more flexible way for constructing wavelet frames is the so called Oblique Extension Principle (OEP). The OEP was introduced by Daubechies, Han, Ron and Shen [21]. These extension principles have been developed by Benedetto and Treiber [5], Chui, He and Stöckler [16], Chui, He, Stöckler and Sun [17], Han [27], [26], Stavropoulos [47] and Atreas, Melas and Stavropoulos [1]. Observe that in these papers, there are also proved that the obtained sufficient conditions are also necessary.

Extensive studies on multiresolution analysis and extension principles are enclosed, for instance, in [15].

Here, we give a solution to the problem of characterizing all Parseval wavelet frames arising from a fix frame multiresolution analysis. As a consequence, we obtain a new description of all Parseval wavelet frames associated with a frame multiresolution analysis. The proofs of these results are based on the characterization of the scaling functions in [32] and on a version of Unitary Extension Principle and Oblique Extension Principle with no regularity at the origin on the modulus of the Fourier transform of the involved refinable functions. In particular, we invoke the notion of approximate continuity. Furthermore, we characterize all Parseval wavelet frames can be constructed via Oblique Extension Principles. We observe that the origin must be a point of density for the support of the Fourier transform of a refinable function used for constructing Parseval wavelet frames via OEP.

Versions of the UEP and OEP without any extra condition on refinable functions where first proved by Han [27], [26], were the contexts is in distribution spaces. The assumption at the origin of the Fourier transform of involved refinable functions is a limit in sense of distributions. At the end of this manuscript we will see that the condition used by Han and the condition used here are of different nature.

Although the results we present here are new in the classical context, i.e., in \( L^2(\mathbb{R}) \) with the dyadic dilation, we consider functions in \( L^2(\mathbb{R}^n) \), \( n \geq 1 \), and the dilation is given by \( A : \mathbb{R}^n \to \mathbb{R}^n \), a fix expansive linear map such that \( A(\mathbb{Z}^n) \subset \mathbb{Z}^n \).

Before formulating our results let us introduce some notation and definitions.

The sets of strictly positive integers, integers, real and complex numbers will be denoted by \( \mathbb{N} \), \( \mathbb{Z} \), \( \mathbb{R} \) and \( \mathbb{C} \) respectively. \( \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \), \( n \geq 1 \), and with some abuse of the notation we consider also that \( \mathbb{T}^n \) is the unit cube \([0,1]^n\).

We will denote \( B_r(y) = \{ x \in \mathbb{R}^n : |x - y| < r \} \), and will write \( B_r \) if \( y \) is the origin. If \( A : \mathbb{R}^n \to \mathbb{R}^n \) is a linear map, \( A^* \) will mean the adjoint of \( A \). With some abuse in the notation, if we write \( A \) we also mean the corresponding matrix respect to the canonical basis. Moreover \( d_A = |\det A| \). For a Lebesgue measurable set \( E \subset \mathbb{R}^n \), \( E^c = \mathbb{R}^n \setminus E \) and the Lebesgue measure of \( E \) in \( \mathbb{R}^n \) will be denoted by \( |E|_n \). If \( x \in \mathbb{R}^n \) then \( x + E = \{ x + y : \text{ for } y \in E \} \). We will denote \( A(E) = \{ x \in \mathbb{R}^n : x = A(t) \text{ for } t \in E \} \) and the volume of \( E \) changes under \( A \) according to \( |AE|_n = d_A |E|_n \). The characteristic function of a set \( E \subset \mathbb{R}^n \) will be denoted by \( \chi_E \), i.e., \( \chi_E(x) = 1 \) if \( x \in E \) and \( \chi_E(x) = 0 \) otherwise.
If we write $L^p(\mathbb{R}^n)$, $n \geq 1$, $1 \leq p \leq \infty$, we mean the usual Lebesgue space. If we write $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ we mean the linear space of all measurable functions such that $f \chi_K \in L^p(\mathbb{R}^n)$ for every $K \subset \mathbb{R}^n$ compact sets.

If we take $f \in L^p(\mathbb{T}^n)$ we will understand that $f$ is defined on the whole space $\mathbb{R}^n$ as a $\mathbb{Z}^n$-periodic function.

A linear map $A$ is called expansive if all (complex) eigenvalues of $A$ have absolute value greater than 1. If $A$ is invertible, we consider the operator $D_A$ on $L^2(\mathbb{R}^n)$ defined by $D_A f(t) = d_A^{1/2} f(At)$. The translation of a function $f \in L^2(\mathbb{R}^n)$ by $b \in \mathbb{R}^n$ will be denoted by $\tau_b f(t) = f(t - b)$. For a subspace $S$ of $L^2(\mathbb{R}^d)$,

$$D_A S = \{ D_A^\ell f : f \in S \}, \quad \text{and} \quad \tau_b S = \{ \tau_b f : f \in S \}.$$

If we write $D_A^{-1}$ we mean the operator $D_{A^{-1}}$, $D_A^\ell$ is the identity map and $D_A^\ell$, $\ell \in \mathbb{N}$, is the $\ell$-th composition of the operator $D_A$ with itself.

If $A : \mathbb{R}^n \to \mathbb{R}^n$ is an expansive linear invertible map such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$, then the quotient groups $\mathbb{Z}^n/A(\mathbb{Z}^n)$ and $A^{-1}(\mathbb{Z}^n)/\mathbb{Z}^n$ are well defined. We will denote by $\Omega_A \subset \mathbb{Z}^n$ and $\Gamma_A \subset A^{-1}(\mathbb{Z}^n)$ a full collection of representatives of the cosets of $\mathbb{Z}^n/A(\mathbb{Z}^n)$ and $A^{-1}(\mathbb{Z}^n)/\mathbb{Z}^n$ respectively. Recall that there are exactly $d_A$ cosets of $\mathbb{Z}^n/A(\mathbb{Z}^n)$ (see [25] and [18, p.109]). Thus there are exactly $d_A$ cosets of $A^{-1}(\mathbb{Z}^n)/\mathbb{Z}^n$.

For a given $\phi \in L^2(\mathbb{R}^n)$, set

$$[\phi, \phi](t) = \sum_{k \in \mathbb{Z}^n} |\phi(t + k)|^2$$

and denote

$$\mathcal{N}_\phi = \{ t \in \mathbb{R}^n : [\phi, \phi](t) = 0 \}.$$

For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ the support of $f$ is defined to be supp$(f) = \{ t \in \mathbb{R}^n ; f(t) \neq 0 \}$.

The sets are defined modulo a null measurable set and we will understand some equations as almost everywhere on $\mathbb{R}^n$ or $\mathbb{T}^n$. Moreover, in order to shorten the notation, we will consider 0/0 = 0 or $0(1/0) = 0$ in some expressions where such an indeterminacy appears.

The theory of frames was introduced by Duffin and Schaeffer [22]. A sequence $\{ \phi_n \}_{n=1}^\infty$ of elements in a separable Hilbert space $\mathbb{H}$ is a frame for $\mathbb{H}$ if there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|h\|^2 \leq \sum_{n=1}^\infty |\langle h, \phi_n \rangle|^2 \leq C_2 \|h\|^2, \quad \forall h \in \mathbb{H},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{H}$. The constants $C_1$ and $C_2$ are called frame bounds. The definition implies that a frame is a complete sequence of elements of $\mathbb{H}$. A frame $\{ \phi_n \}_{n=1}^\infty$ is tight if we may choose $C_1 = C_2$, and if in fact $C_1 = C_2 = 1$, we will call the frame a Parseval frame. A sequence $\{ h_n \}_{n=1}^\infty$ of elements in a Hilbert space $\mathbb{H}$ is a frame sequence if it is a frame for $\text{span}\{h_n\}_{n=1}^\infty$.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an expansive linear map such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$. Let $\Psi = \{ \psi_1, \ldots, \psi_N \} \subset L^2(\mathbb{R}^n)$ be a set of functions, we call

\begin{align*}
(1) \quad X_\Phi & : = \{ D_A^\ell \tau_k \psi : j \in \mathbb{Z}, \ k \in \mathbb{Z}^d, \ 1 \leq \ell \leq N \}; \\
(2) \quad X_\Psi & : = \{ D_A^\ell \tau_k \psi : j \geq 0, \ k \in \mathbb{Z}^d, \ 1 \leq \ell \leq N \} \\
& \cup \{ d_A^{1/2} \tau_k D_A^\ell \psi : j < 0, \ k \in \mathbb{Z}^d, \ 1 \leq \ell \leq N \}
\end{align*}
the affine system (resp. quasi-affine system) generated by $\Psi$. The set of functions $\Psi$ is called a \textit{wavelet frame} associated to $A$, if the affine system (1) is a frame for $L^2(\mathbb{R}^n)$. In this case, the affine system (1) is usually called \textit{affine frame}. When the context is clear we do not write “associated to $A$”. If this affine system is a tight frame for $L^2(\mathbb{R}^n)$ then $\Psi$ is called a \textit{tight wavelet frame} or \textit{tight framelet}. In particular, a tight wavelet frame with frame bounds equal to 1 is usually called a \textit{Parseval wavelet frame} or a \textit{Parseval framelet}. The functions $\psi_\ell, \ell = 1, \ldots, N$ are called the \textit{generators} of the wavelet frame. If the quasi-affine system (2) is a frame for $L^2(\mathbb{R}^n)$, then $X_0^2$ is usually called \textit{quasi-affine frame}.

Given a linear invertible map $A$ as above, by a frame multiresolution analysis associated to the dilatation $A$ (A-FMRA) we mean a sequence of closed subspaces $V_j, j \in \mathbb{Z}$, of the Hilbert space $L^2(\mathbb{R}^n)$ that satisfies the following conditions:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$;
(ii) $f \in V_j$ if and only if $D_A f \in V_{j+1}$ for every $j \in \mathbb{Z}$;
(iii) $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^n)$;
(iv) There exists a function $\phi \in V_0$, that is called \textit{scaling function}, such that

$$
V_0 = \text{span}\{\tau_k \phi : \ k \in \mathbb{Z}^n\}.
$$

According to Lemma E bellow, the condition (iv) can be replaced by

(iv') the system $\{\tau_k \phi : \ k \in \mathbb{Z}^n\}$ is a (Parseval) frame for $V_0$.

When the system $\{\tau_k \phi : \ k \in \mathbb{Z}^n\}$ is an orthonormal basis for $V_0$, then the A-FMRA is called an orthonormal multiresolution analysis or simply a multiresolution analysis (A-MRA).

One of the possible ways for constructing an A-FMRA is to start with a scaling function $\phi \in L^2(\mathbb{R}^n)$. A function $\phi \in L^2(\mathbb{R}^n)$ generates an A-FMRA if $V_0 = \text{span}\{\tau_k \phi : \ k \in \mathbb{Z}^n\}$ and the subspaces

$$
V_j = \text{span}\{D_A \tau_k \phi : \ k \in \mathbb{Z}^n\}, \ j \in \mathbb{Z}
$$

of the Hilbert space $L^2(\mathbb{R}^n)$ satisfy the conditions (i) and (iii).

If $\phi \in L^2(\mathbb{R}^n)$ generates an A-FMRA, then $\phi \in V_0 \subset V_1$. Thus,

$$
\phi = \sum_{k \in \mathbb{Z}^n} a_k D_A \tau_k \phi, \quad a_k \in \mathbb{C},
$$

where the convergence is in $L^2(\mathbb{R}^n)$. Taking the Fourier transform, we can write

$$
\hat{\phi}(A^* t) = H(t) \hat{\phi}(t) \quad \text{a.e. on} \ \mathbb{R}^n
$$

where $H$ is a $\mathbb{Z}^n$-periodic measurable function. A function that satisfies an equality as (3) is called refinable.

Given a multiresolution analysis, the following describes a standard procedure for constructing wavelet frames. If $\{V_j\}_{j \in \mathbb{Z}} \subset L^2(\mathbb{R}^n)$ is an A-FMRA, we denote by $W_j$ the orthogonal complement of $V_j$ in $V_{j+1}$. Thus, by condition (i), we have $V_{j+1} = W_j \oplus V_j$. Moreover, condition (iii) implies that $V_{j+1} = \oplus_{k \in \mathbb{Z}^d} W_k$ and $L^2(\mathbb{R}^d) = \oplus_{j \in \mathbb{Z}} W_j$. Observe that if $\{\tau_k \psi_\ell : \ k \in \mathbb{Z}^d, \ 1 \leq \ell \leq N\}$ is a Parseval frame for $V_0$, then the system $\{D_A \tau_k \psi_\ell : \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^d, \ 1 \leq \ell \leq N\}$ is a Parseval frame for $L^2(\mathbb{R}^n)$.

**Definition 1.** Let $\{V_j\}_{j \in \mathbb{Z}} \subset L^2(\mathbb{R}^n)$ be an A-FMRA. A Parseval wavelet frame $\Psi = \{\psi_1, \ldots, \psi_N\}$ for $L^2(\mathbb{R}^n)$ is said to be associated with $\{V_j\}_{j \in \mathbb{Z}}$ if $\{\tau_k \psi_\ell : \ k \in \mathbb{Z}^d, \ 1 \leq \ell \leq N\}$ is a Parseval frame for $W_0 = V_1 \oplus V_0$. We say that a
A slight more flexible type of Parseval wavelet frames is the following.

**Definition 2.** Let \( \{ V_j \}_{j \in \mathbb{Z}} \subset L^2(\mathbb{R}^n) \) be an A-FMRA. We say that \( \Psi = \{ \psi_1, \ldots, \psi_N \} \subset L^2(\mathbb{R}^n) \) is a Parseval wavelet frame arising from \( \{ V_j \}_{j \in \mathbb{Z}} \) if \( \Psi \subset V_1 \) and the associated affine system \( \{ \Psi = \{ \psi_1, \ldots, \psi_N \} \subset L^2(\mathbb{R}^n) \) is an A-FMRA based wavelet frame if \( \Psi \) is Parseval wavelet frame arising from some A-FMRA.

A key tool in the study of wavelet frames is the Fourier transform. Here, our convention is that if \( f \in L^1(\mathbb{R}^n) \),
\[
\hat{f}(\mathbf{x}) := \int_{\mathbb{R}^n} f(\mathbf{t}) e^{-2\pi i \mathbf{x} \cdot \mathbf{t}} d\mathbf{t},
\]
where \( \mathbf{t} \cdot \mathbf{x} \) denotes the usual inner product of vectors \( \mathbf{t} \) and \( \mathbf{x} \) in \( \mathbb{R}^n \). The definition of Fourier transform is extended as usual form to functions in \( L^2(\mathbb{R}^n) \).

The following definitions were introduced in \([19]\).

**Definition 3.** Let \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be an expansive linear map. It is said that \( \mathbf{x} \in \mathbb{R}^n \) is a point of \( A \)-density for a set \( E \subset \mathbb{R}^n \), \( |E|_n > 0 \), if for any \( r > 0 \)
\[
\lim_{j \rightarrow \infty} \frac{|E \cap (A^{-j} B_r + \mathbf{x})|_n}{|A^{-j} B_r|_n} = 1.
\]

**Definition 4.** Let \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be an expansive linear map. Let \( f : \mathbb{R}^n \rightarrow \mathbb{C} \) be a measurable function. It is said that \( \mathbf{x} \in \mathbb{R}^n \) is a point of \( A \)-approximate continuity of the function \( f \) if there exists \( E \subset \mathbb{R}^n \), \( |E|_n > 0 \), such that \( \mathbf{x} \) is a point of \( A \)-density for the set \( E \) and
\[
\lim_{y \rightarrow x} f(y) = f(x).
\]

**Definition 5.** Let \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be an expansive linear map. A measurable function \( f : \mathbb{R}^n \rightarrow \mathbb{C} \) is said to be \( A \)-locally nonzero at a point \( \mathbf{x} \in \mathbb{R}^n \) if for any \( \varepsilon, r > 0 \) there exists \( j \in \mathbb{N} \) such that
\[
| \{ \mathbf{y} \in A^{-j} B_r + \mathbf{x} : f(\mathbf{y}) = 0 \} |_n < \varepsilon |A^{-j} B_r|_n.
\]

Observe that if \( A = aI \), where \( a > 1 \) and \( I \) is the identity map on \( \mathbb{R}^n \), the definition of a point of \( A \)-approximate continuity coincides with the well-known definition of approximate continuity (cf. \([39]\), \([12]\)).

This paper is structured as follows. In Section 2 we write the main results of this paper, i.e., a characterization of all Parseval wavelet frames can be constructed via Oblique Extension Principle and a characterization of all Parseval wavelet frames arising from a fix frame multiresolution analysis. In Section 3 we collect some well known results that we will use along this paper. The proofs of the main results of this paper will be postponed until Section 4.

2. Main Result

We write the main results of this paper. In what follows, we fix \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) an expansive linear map such that \( A(\mathbb{Z}^n) \subset \mathbb{Z}^n \). Moreover let us fix \( \Gamma_A = \{ p \}_{k \geq 0} \), where \( p_0 = 0 \), a full collection of representatives of the cosets of \((A^*)^{-1} \mathbb{Z}^n / \mathbb{Z}^n \). In
order to short the notation, if we write a wavelet frame we mean a wavelet frame associated to the dilation $A$.

The following result characterize all Parseval wavelet frames can be constructed via Oblique Extension Principle.

**Theorem 1.** Let $\phi \in L^2(\mathbb{R}^n)$ such that

$$\hat{\phi}(A^*t) = H_0(t)\hat{\phi}(t), \quad \text{a.e.,}$$

where $H_0 \in L^\infty(\mathbb{T}^n)$. Let $H_1, \ldots, H_N \in L^\infty(\mathbb{T}^n)$ and define $\psi_1, \ldots, \psi_N \in L^2(\mathbb{R}^n)$ by

$$\hat{\psi}_\ell(A^*t) = H_\ell(t)\hat{\phi}(t) \quad \text{a.e.,} \quad \ell = 1, \ldots, N.$$ 

Then the following are equivalent:

i) The set of functions $\{\psi_\ell : \ell = 1, \ldots, N\}$ is a Parseval wavelet frame for $L^2(\mathbb{R}^n)$. 

ii) There exists $S$, a non-negative $\mathbb{Z}^n$-periodic measurable function such that $\sqrt{S}|\hat{\phi}| \in L^2(\mathbb{R}^n)$ and also

(a) the origin is a point of $A^*$-approximate continuity for $S|\hat{\phi}|^2$, provided $S(0)|\hat{\phi}(0)|^2 = 1$;

(b)

$$S(A^*t)|H_0(t)|^2 + \sum_{\ell=1}^N |H_\ell(t)|^2 = S(t) \quad \text{a.e.} \quad t \in \mathbb{R}^n \setminus N_{\hat{\phi}};$$

(c) the equality

$$S(A^*t)H_0(t)\overline{H_0(t+p_k)} + \sum_{\ell=1}^N H_\ell(t)\overline{H_\ell(t+p_k)} = 0$$

holds for a.e. $t \in \mathbb{R}^n \setminus N_{\hat{\phi}}$ and for any $p_k$, $k = 1, \ldots, d_A - 1$, such that $t + p_k \in \mathbb{R}^n \setminus N_{\hat{\phi}}$.

The following result gives a characterization of all Parseval wavelet frames arising from a fix generalized multiresolution analysis.

**Theorem 2.** Let $\{V_j\}_{j \in \mathbb{Z}} \subset L^2(\mathbb{R}^n)$ be an $A$-FMRA with a scaling function $\phi$. Let $\varphi \in L^2(\mathbb{R}^n)$ defined by $\hat{\varphi}(t) = \hat{\phi}(t)/|\hat{\phi},\hat{\phi}|^{1/2}(t)$. Let $\Psi = \{\psi_1, \ldots, \psi_N\} \subset L^2(\mathbb{R}^n)$. The following are equivalent.

i) The set of functions $\Psi$ is a Parseval wavelet frame arising from $\{V_j\}_{j \in \mathbb{Z}}$.

ii) There exist $H_0, H_1, \ldots, H_N \in L^\infty(\mathbb{T}^n)$ such that

a) $$\hat{\varphi}(A^*t) = H_0(t)\hat{\varphi}(t), \quad \text{a.e.,}$$

and

b) there exists a non-negative $S \in L^\infty(\mathbb{T}^n)$ such that the origin is a point of $A^*$-approximate continuity for $S$ if we set $S(0) = 1$, and the equalities in (6) and (7) are satisfied.

A consequence of Theorem 2 is the following characterization of all Parseval wavelet frames associated with a fix frame multiresolution analysis.
Corollary 1. Let \( \{V_j\}_{j \in \mathbb{Z}} \subset L^2(\mathbb{R}^n) \) be an A-FMRA with a scaling function \( \phi \). Let \( \varphi \in L^2(\mathbb{R}^n) \) defined by \( \widehat{\varphi}(t) = \widehat{\phi}(t)/[\widehat{\phi}, \widehat{\phi}]^{1/2}(t) \). Let \( \Psi = \{\psi_1, \ldots, \psi_N\} \subset L^2(\mathbb{R}^n) \). The following are equivalent.

\begin{enumerate} 
\item The set of functions \( \Psi \) is a Parseval wavelet frame associated with \( \{V_j\}_{j \in \mathbb{Z}} \).
\item The condition ii) in Theorem 2 holds and also
\[
\sum_{k=0}^{d_A-1} H_0(t + p_k)H(t + p_k) = 0, \quad \text{for a.e. } t \in \mathbb{R}^n \setminus N\widehat{\varphi}, \quad \ell = \{1, 2, \ldots, N\}.
\]
\end{enumerate}

3. Background

We collect well known results on tight wavelet frames and scaling functions we will use to prove our main results in this paper.

We need the following characterization of Parseval wavelet frame for \( L^2(\mathbb{R}^n) \) proved in [8]. Other versions appeared in [23], [45] and [29].

Theorem A. Suppose \( \Psi = \{\psi_\ell : \ell = 1, \ldots, N\} \subset L^2(\mathbb{R}^n) \). The following are equivalent:

\begin{enumerate} 
\item I) The set of functions \( \Psi \) is a Parseval wavelet frame for \( L^2(\mathbb{R}^n) \).
\item II) 
\[
\sum_{\ell=1}^{N} \sum_{j \in \mathbb{Z}} |\widehat{\psi}(A^j t)|^2 = 1, \quad \text{a.e. and}
\]
\[
\sum_{\ell=1}^{N} \sum_{j=0}^{\infty} \widehat{\psi}(A^j t)\widehat{\psi}(A^j (t + q)) = 0, \quad \text{a.e., } q \in \mathbb{Z}^n \setminus A^n \mathbb{Z}^n.
\]
\end{enumerate}

The following is Theorem 2 (ii) in [18].

Theorem B. Let \( \Psi \) be a finite set of functions in \( L^2(\mathbb{R}^n) \). Then \( X_\Psi \) is an affine frame if and only if \( X_\Psi^a \) is a quasi-affine frame. Furthermore, their lower and upper exact frame bounds are equal.

In [6], [4], [35] and [11] (see also [15]), different versions of a characterization of the functions in \( L^2(\mathbb{R}) \) whose integer translates generate a frame sequence were given. Here we only write a version on \( L^2(\mathbb{R}^n) \) because the proof is completely similar to the case \( L^2(\mathbb{R}) \).

Theorem C. Let \( \phi \in L^2(\mathbb{R}^n) \). The system \( \{\tau_k \phi : k \in \mathbb{Z}^n\} \) is a frame sequence with frame bounds \( C \) and \( D \) if and only if

\[
C \leq |\widehat{\phi}, \widehat{\phi}|(t) \leq D \quad \text{a.e. on } \mathbb{T}^n \setminus N\widehat{\phi}.
\]

The following two results can be found in [10] and [11].

Lemma D. Let \( \phi \in L^2(\mathbb{R}^n) \) and \( V_0 = \text{span}\{\tau_k \phi : k \in \mathbb{Z}^n\} \) and let \( V_j := D_A^j V_0 \), \( j \in \mathbb{Z} \). A function \( f \in L^2(\mathbb{R}^n) \) is in \( V_j \) if and only if there exists \( H \), an \( \mathbb{Z}^n \)-periodic measurable function, such that \( \widehat{f}(A^n t) = H(t)\widehat{\phi}(t) \) a.e.

Lemma E. Let \( \phi \in L^2(\mathbb{R}^n) \) and \( V = \text{span}\{\tau_k \phi : k \in \mathbb{Z}^n\} \). Then the system \( \{\tau_k \varphi : k \in \mathbb{Z}^n\} \) is a Parseval frame for \( V \), where \( \varphi = \phi/|\widehat{\phi}, \widehat{\phi}|^2 \).

Different versions of the following lemma appeared in various publications (cf. [11], [4], [15], [32]).
Lemma F. Let $\phi \in L^2(\mathbb{R}^n)$ and assume that $\{\tau_k \phi : k \in \mathbb{Z}^n\}$ is a frame sequence in $L^2(\mathbb{R}^n)$. If the subspaces $V_j, j \in \mathbb{Z}$, are defined by \((3)\) then the following conditions are equivalent:

\(a)\) \quad \forall j \in \mathbb{Z}, \quad V_j \subset V_{j+1};

\(b)\) \quad V_0 \subset V_1;

\(c)\) \quad There exists a function $H \in L^\infty(\mathbb{T}^n)$ such that

$$\hat{\phi}(A^* \mathbf{t}) = H(\mathbf{t})\hat{\phi}(\mathbf{t}) \quad a.e. \quad \mathbb{R}^n.$$

The following proposition appeared in \([26]\) (cf. \([31], [20], [10]\)).

Proposition G. Let $\phi \in L^2(\mathbb{R}^d)$. Then for each $f \in L^2(\mathbb{R}^d)$ we have

$$\lim_{j \to -\infty} \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^j \tau_k \phi \rangle|^2 = 0.$$

In a more general context, the following theorem was proved in \([32]\). That theorem is formulated here in a modified form in order to indicate the essential result we need in this paper.

Theorem H. Let $V_j, j \in \mathbb{Z}$, be a sequence of closed subspaces in $L^2(\mathbb{R}^n)$ satisfying the conditions (i), (ii) and (iv) with scaling function $\phi$. Then the following conditions are equivalent:

\((A)\) \quad $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^n)$

\((B)\) \quad The function $\hat{\phi}$ is $A^*$-locally nonzero at the origin;

\((C)\) \quad The origin is a point of $A^*$-approximate continuity of the function $|\hat{\phi}|^2/|\hat{\phi},\hat{\phi}|$, provided that $|\hat{\phi}(0)|^2/|\hat{\phi},\hat{\phi}(0)| = 1$.

Note that Lemma F and Theorem H together characterize all the functions $\phi \in L^2(\mathbb{R}^n)$ that generate an $A$-FMRA.

The following proposition is a slight different version of Proposition 6 in \([46]\).

Proposition I. Let $H_0 \in L^\infty(\mathbb{T}^n)$ such that $|H_0(\mathbf{t})| \leq 1$ a.e.. Let $\phi \in L^2(\mathbb{R}^n)$ such that the origin is a point of $A^*$-approximate continuity of $|\hat{\phi}|$, provided $|\hat{\phi}(0)| = 1$, and

$$\hat{\phi}(A^* \mathbf{t}) = H_0(\mathbf{t})\hat{\phi}(\mathbf{t}), \quad a.e. \quad \mathbb{R}^n.$$

Let $\hat{H}_0(\mathbf{t}) = H_0(\mathbf{t})$ if $\mathbf{t} \in \mathbb{R}^n \setminus N_{\hat{\phi}}$ and $\hat{H}_0(\mathbf{t}) = 0$ if $\mathbf{t} \in N_{\hat{\phi}}$. Then

$$|\hat{\phi}(\mathbf{t})| = \prod_{j=1}^{\infty} |\hat{H}_0((A^*)^{-j} \mathbf{t})|, \quad a.e. \quad \mathbb{R}^n.$$

We need the following auxiliary result on points of approximate continuity proved in \([46]\).

Lemma J. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an expansive linear map. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function such that for a point $\mathbf{y} \in \mathbb{R}^n$ we have

$$\lim_{j \to -\infty} f(A^{-j} \mathbf{x} + \mathbf{y}) = f(\mathbf{y}) \quad a.e. \quad \mathbb{R}^n,$$

then the point $\mathbf{y}$ is a point of $A$-approximate continuity of $f$.

The following technical result is proved in \([19]\). Note that the equality (ii) in the following lemma does not appear in the original result but it is an immediate consequence of the proof of (i).

Lemma K. Let $g \in L^2(\mathbb{T}^n)$, let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a fixed linear invertible map such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and let $\hat{A} : \mathbb{T}^n \to \mathbb{T}^n$ be the induced endomorphism. Let
\[ \Gamma_A = \{ q_i \}_{i=0}^{d_A-1} \text{ be a full collection of representatives of the cosets of } A^{-1}({\mathbb Z}^n)/{\mathbb Z}^n. \]

Then
\begin{align*}
( i) & \int_{\mathbb{R}^n} g(\hat{t}) dt = \int_{\mathbb{R}^n} g(t) dt, \\
( ii) & \int_{[0,1]^n} g(t) dt = d_A^{-1} \int_{[0,1]^n} \sum_{i=0}^{d_A-1} g(A^{-1}t + q_i) dt.
\end{align*}

4. Proofs of the main results

4.1. Proof of Theorem 1. To prove \( ii \) implies \( i \) in Theorem 1 we need the following results.

**Lemma 1.** Let \( \phi \in L^2(\mathbb{R}^n) \) such that \( |\hat{\phi}(t)| \leq 1 \text{ a.e. and the origin is a point of } \mathcal{A}^* \)-approximate continuity of \( \hat{\phi} \), provided \( |\hat{\phi}(0)| = 1 \). Let \( f \in L^2(\mathbb{R}^n) \) such that \( \hat{f} \) is continuous and compactly supported. Then, for any \( \varepsilon > 0 \) there exists \( J \in \mathbb{N} \) such that
\begin{equation}
(1 - \varepsilon) \|f\|_2^2 \leq \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^j \tau_k \phi \rangle|^2 \leq \|f\|_2^2, \quad \forall j \geq J.
\end{equation}

**Proof:** Let \( f \in L^2(\mathbb{R}^n) \) such that \( \hat{f} \) is continuous and \( \text{supp}(\hat{f}) \subset B_R \) for a fix \( R > 0 \). Let \( K > 0 \) such that \( |\hat{f}(t)| \leq K \) for every \( t \in \mathbb{R}^n \). By Parseval’s formula,
\begin{align*}
\sum_{k \in \mathbb{Z}^n} |\langle f, D_A^j \tau_k \phi \rangle|^2 &= \sum_{k \in \mathbb{Z}^n} |\langle \hat{f}, D_A^j \tau_k \phi \rangle|^2 = \sum_{k \in \mathbb{Z}^n} |\langle D_A^j \hat{f}, \tau_k \phi \rangle|^2 \\
&= \sum_{k \in \mathbb{Z}^n} \left| \int_{(A^*)^{-j}B_R} D_A^j \hat{f}(t) \overline{\phi(t)} e^{2\pi i k \cdot t} dt \right|^2.
\end{align*}

Since \( \mathcal{A}^* \) is expansive, there exists \( j_0 \in \mathbb{N} \) such that if \( j \geq j_0 \), then \( (A^*)^{-j}B_R \subset [-1/2,1/2]^d \). For those \( j \), the sum over \( k \in \mathbb{Z}^n \) in (10) may be interpreted as the sum of the square of the modulus of the \( -k \)-th Fourier coefficients of the function \( D_{A^j} \hat{f}(t) \phi(t) \). Thus
\begin{equation}
\sum_{k \in \mathbb{Z}^n} |\langle f, D_A^j \tau_k \phi \rangle|^2 = \int_{(A^*)^{-j}B_R} |D_{A^j} \hat{f}(t)|^2 |\phi(t)|^2 dt \quad \forall j \geq j_0.
\end{equation}

Since \( |\hat{\phi}(t)| \leq 1 \text{ a.e.}, the right inequality of (9) follows. Now, we prove the left inequality of (9).

Let \( 0 < \varepsilon < 1 \) and take the set \( \Lambda_\varepsilon = \{ t \in \mathbb{R}^n : |\hat{\phi}(t)| \leq 1 - \frac{\varepsilon}{2} \} \). Since \( |\hat{\phi}(0)| = 1 \), \( |\hat{\phi}(t)| \leq 1 \text{ a.e. and the origin is a point of } \mathcal{A}^* \)-approximate continuity of \( \hat{\phi} \), then
\begin{equation}
\lim_{j \to \infty} \frac{|(A^*)^{-j}B_R \cap \Lambda_\varepsilon|_n}{|(A^*)^{-j}B_R|_n} = 1.
\end{equation}

This implies that there exists \( J \geq j_0 \) such that if \( j \geq J \), we have
\begin{equation}
|(A^*)^{-j} (B_R \cap A^j \Lambda_\varepsilon)|_n = |((A^*)^{-j}B_R) \cap \Lambda_\varepsilon|_n < \frac{\varepsilon}{2K^2 d_A^2} \|f\|_2^2.
\end{equation}

Thus, if \( j \geq J \)
\begin{equation}
|B_R \cap A^j \Lambda_\varepsilon|_n < \frac{\varepsilon}{2K^2 d_A^2} \|f\|_2^2.
\end{equation}
According to (11), if \( j \geq J \) we obtain
\[
\sum_{k \in \mathbb{Z}^n} |\langle f, D_A^j \tau_k \phi \rangle|^2 \geq (1 - \frac{\varepsilon}{2}) \int_{(A^*)^{-j}B_R \cap \Lambda_e} |D_A\hat{f}(t)|^2 dt
\]
\[
= (1 - \frac{\varepsilon}{2}) \|f\|_2^2 - (1 - \frac{\varepsilon}{2}) \int_{(A^*)^{-j}B_R \cap \Lambda_e} |\hat{f}(t)|^2 dt
\]
\[
\geq (1 - \frac{\varepsilon}{2}) \|f\|_2^2 - \int_{B_R \cap A^{j+1}\Lambda_e} |\hat{f}(t)|^2 dt.
\]
Furthermore, by the inequality (12), if \( j \geq J \) we have
\[
\sum_{k \in \mathbb{Z}^n} |\langle f, D_A^j \tau_k \phi \rangle|^2 \geq (1 - \frac{\varepsilon}{2}) \|f\|_2^2 - K^2 |B_R \cap A^j \Lambda_e|_n
\]
\[
\geq (1 - \frac{\varepsilon}{2}) \|f\|_2^2 - K^2 \frac{\varepsilon}{2K^2} \|f\|_2^2 = (1 - \varepsilon) \|f\|_2^2.
\]
This finishes the proof. \( \square \)

We need the following

**Lemma 2.** Let \( \phi \in L^2(\mathbb{R}^n) \) such that \( |\hat{\phi}(t)| \leq 1 \) a.e. and

\[
\hat{\phi}(A^s t) = H_0(t) \hat{\phi}(t), \quad a.e.,
\]

where \( H_0 \in L^\infty(\mathbb{T}^n) \). Let \( H_1, \ldots, H_N \in L^\infty(\mathbb{T}^n) \) and define \( \psi_1, \ldots, \psi_N \in L^2(\mathbb{R}^n) \) by (5). Assume that

\[
|H_0(t)|^2 + \sum_{\ell=1}^N |H_\ell(t)|^2 = 1 \quad \text{a.e.} \ t \in \mathbb{R}^n \setminus N_\phi;
\]

and the equality

\[
H_0(t)H_0(t + p_k) + \sum_{\ell=1}^N H_\ell(t)H_\ell(t + p_k) = 0
\]
holds for a.e. \( t \in \mathbb{R}^n \setminus N_\phi \) and for any \( p_k, k = 1, \ldots, d_A - 1 \), such that \( t + p_k \in \mathbb{R}^n \setminus N_\phi \). Then, for all \( j \in \mathbb{Z} \) and for any \( f \in L^2(\mathbb{R}^n) \) such that \( \hat{f} \) is continuous and compactly supported, we have

\[
\sum_{k \in \mathbb{Z}^n} |\langle f, D_A^j \tau_k \phi \rangle|^2 = \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^{j-1} \tau_k \phi \rangle|^2 + \sum_{\ell=1}^N \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^{j-1} \tau_k \psi_\ell \rangle|^2.
\]

**Proof.** Let \( f \in L^2(\mathbb{R}^n) \) such that \( \hat{f} \) is continuous and compactly supported. Fix a \( j \in \mathbb{Z} \), \( k \in \mathbb{Z}^n \) and \( \ell \in \{1, \ldots, N\} \). Using the Parseval’s equality, the change of variable \( A^{s+j}t = s \) and (10), we can write

\[
\langle f, D_A^{j-1} \tau_k \psi_\ell \rangle = \langle \hat{f}, D_A^{j-1} \tau_k \psi_\ell \rangle
\]
\[
= \int_{\mathbb{R}^n} d_A^{j/2} \hat{f}(A^{s+j}s)d_A^{j/2} \tilde{\psi}(A^s s)e^{2\pi i k \cdot A^s s} ds
\]
\[
= \sum_{m \in \mathbb{Z}^n} \int_{[0,1]^n - m} d_A^{j/2} \hat{f}(A^{s+j}s)d_A^{j/2} \tilde{H}_\ell(s) \hat{\psi}(s)e^{2\pi i k \cdot A^s s} ds.
\]
Since $\hat{f}$ has compact support, the above sum only involves a finite number of $m$'s. Moreover, bearing in mind that $H_\ell$ is a $\mathbb{Z}^n$-periodic function and $A^*(\mathbb{Z}^n) \subset \mathbb{Z}^n$, we have

$$\langle f, D_A^{-1} \tau_k \psi_\ell \rangle = \int_{[0,1)^n} d_A^{1/2} H_\ell(s) \left( \sum_{m \in \mathbb{Z}^n} \tau_m \left( (D_A^\dagger \hat{f}) \hat{\phi} \right)(s) \right) e^{2\pi i k \cdot A^* s} ds.$$

According to our hypotheses, $\hat{f}$, $H_\ell$ and $\hat{\phi}$ are bounded, then the function into the above integral is in $L^2(T^n)$. By (ii) in Lemma K, we have

$$\langle f, D_A^{-1} \tau_k \psi_\ell \rangle = \int_{[0,1)^n} d_A^{-1/2} \sum_{k=0}^{d_A-1} H_\ell((A^*)^{-1} s + p_k)$$

$$\times \left( \sum_{m \in \mathbb{Z}^n} \tau_m \left( (D_A^\dagger \hat{f}) \hat{\phi} \right)((A^*)^{-1} s + p_k) \right) e^{2\pi i k \cdot s} ds$$

which is the $-k$-th Fourier coefficient of the function

$$d_A^{-1/2} \sum_{k=0}^{d_A-1} H_\ell((A^*)^{-1} s + p_k) \left( \sum_{m \in \mathbb{Z}^n} \tau_m \left( (D_A^\dagger \hat{f}) \hat{\phi} \right)((A^*)^{-1} s + p_k) \right)^2.$$

With analogous computation for $\langle f, D_A^{-1} \tau_k \phi \rangle$, we get

$$\sum_{k \in \mathbb{Z}^n} |\langle f, D_A^{-1} \tau_k \phi \rangle|^2 + \sum_{\ell=1}^{N} \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^{-1} \tau_k \psi_\ell \rangle|^2$$

$$= \left( \int_{[0,1)^n} \sum_{k=0}^{d_A-1} H_\ell((A^*)^{-1} s + p_k) \right)$$

$$\times \left( \sum_{m \in \mathbb{Z}^n} \tau_m \left( (D_A^\dagger \hat{f}) \hat{\phi} \right)((A^*)^{-1} s + p_k) \right)^2 ds.$$
Bearing in mind the functions inside the integral are $\mathbb{Z}^n$-periodic, we obtain

$$\sum_{k \in \mathbb{Z}^n} |\langle f, D_A^{-1} \tau_k \phi \rangle|^2 + \sum_{\ell = 1}^N \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^{-1} \tau_k \psi_\ell \rangle|^2$$

$$= d_A^{-1} \int_{[0,1)^n} \sum_{k = 0}^{d_A-1} \sum_{\ell = 0}^{d_A-1} |H_\ell((A^*)^{-1} s + p_k)|^2$$

$$\times \left| \sum_{m \in \mathbb{Z}^n} \tau_m \left( (D_A^j \hat{f} \hat{\phi}) \left( (A^*)^{-1} s + p_k \right) \right) \right|^2 ds$$

$$+ d_A^{-1} \int_{[0,1)^n} \sum_{k = 0}^{d_A-1} \sum_{a = 1}^{d_A-1} \left( \sum_{\ell = 0}^{N} H_\ell((A^*)^{-1} s + p_k) \bar{H}_\ell((A^*)^{-1} s + p_k + p_a) \right)$$

$$\times \left( \sum_{m \in \mathbb{Z}^n} \tau_m \left( (D_A^j \hat{f} \hat{\phi}) \left( (A^*)^{-1} s + p_k \right) \right) \right)$$

$$\times \left( \sum_{b \in \mathbb{Z}^n} \tau_b \left( (D_A^j \hat{f} \hat{\phi}) \left( (A^*)^{-1} s + p_k + p_a \right) \right) \right) ds.$$ 

By (13) and (14) we obtain

$$\sum_{k \in \mathbb{Z}^n} |\langle f, D_A^{-1} \tau_k \phi \rangle|^2 + \sum_{\ell = 1}^N \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^{-1} \tau_k \psi_\ell \rangle|^2$$

$$= d_A^{-1} \int_{[0,1)^n} \sum_{k = 0}^{d_A-1} \sum_{m \in \mathbb{Z}^n} \tau_m \left( (D_A^j \hat{f} \hat{\phi}) \left( (A^*)^{-1} s + p_k \right) \right)^2 ds.$$ 

Since the sum over $m$ is finite and the functions $\hat{f}$ and $\hat{\phi}$ are bounded, $\left| \sum_{m \in \mathbb{Z}^n} \tau_m \left( (D_A^j \hat{f} \hat{\phi}) \right)^2 \right|$ is in $L^2(\mathbb{T}^n)$. Thus, by (ii) in Lemma K we obtain

$$\sum_{k \in \mathbb{Z}^n} |\langle f, D_A^{-1} \tau_k \phi \rangle|^2 + \sum_{\ell = 1}^N \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^{-1} \tau_k \psi_\ell \rangle|^2$$

$$= \int_{[0,1)^n} \sum_{m \in \mathbb{Z}^n} \tau_m \left( (D_A^j \hat{f} \hat{\phi}) \left( s \right) \right)^2 ds = \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^j \tau_k \phi \rangle|^2,$$

as we wanted to prove. 

We are ready to prove the following version of Unitary Extension Principle.

**Theorem 3** (Unitary Extension Principle). Let $\phi \in L^2(\mathbb{R}^n)$ such that the origin is a point of $A^*$-approximate continuity of $|\hat{\phi}|$, provided $|\hat{\phi}(0)| = 1$, and

$$\hat{\phi}(A^* t) = H_0(t) \hat{\phi}(t), \ a.e.,$$

where $H_0 \in L^\infty(\mathbb{T}^n)$. Let $H_1, \ldots, H_N \in L^\infty(\mathbb{T}^n)$ such that (13) and (14) hold. If $\psi_1, \ldots, \psi_N \in L^2(\mathbb{R}^n)$ are defined by (5), then $\{\psi_\ell : \ell = 1, \ldots, N\}$ is a Parseval wavelet frame for $L^2(\mathbb{R}^n)$. 

Proof. By \( \|H_0(t)\| \leq 1 \) a.e. Thus, according to Proposition I, we have that \( |\hat{\phi}(t)| \leq 1 \) a.e.

Let \( \varepsilon > 0 \) be given and let \( f \in L^2(\mathbb{R}^n) \) such that \( \hat{f} \) is continuous and compactly supported. For \( j_0 \in \mathbb{Z} \), Lemma [2] shows that

\[
\sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{j_0} \tau_k \phi \rangle|^2 = \sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{j_0-1} \tau_k \phi \rangle|^2 + \sum_{\ell=1}^N \sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{j_0-1} \tau_k \psi_\ell \rangle|^2.
\]

Repeating the argument for \( j_0 = j_0 - 1, j_0 - 2, \ldots \), it follows that if \( j < j_0 \) we obtain

\[
\sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{j_0} \tau_k \phi \rangle|^2 = \sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{j_0-1} \tau_k \phi \rangle|^2 + \sum_{\ell=1}^{j_0-1} \sum_{m=j}^{j_0} \sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{m} \tau_k \psi_\ell \rangle|^2.
\]

Then, by Lemma [4] there exists \( J \in \mathbb{N} \) such that if \( j_0 \geq J \) and \( j < j_0 \) we have

\[
(1 - \varepsilon)\|f\|^2 \leq \sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{j_0} \tau_k \phi \rangle|^2 + \sum_{\ell=1}^{j_0-1} \sum_{m=j}^{j_0} \sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{m} \tau_k \psi_\ell \rangle|^2 \leq \|f\|^2.
\]

By Proposition G we know that

\[
\lim_{j \to -\infty} \sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{j_0} \tau_k \phi \rangle|^2 = 0.
\]

Therefore, letting \( j \to -\infty \) in (10), for \( j_0 \geq J \) we have

\[
(1 - \varepsilon)\|f\|^2 \leq \sum_{\ell=1}^{j_0-1} \sum_{m=-\infty}^{j_0} \sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{m} \tau_k \psi_\ell \rangle|^2 \leq \|f\|^2.
\]

In addition, letting \( j_0 \to \infty \),

\[
(1 - \varepsilon)\|f\|^2 \leq \sum_{\ell=1}^N \sum_{m=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{m} \tau_k \psi_\ell \rangle|^2 \leq \|f\|^2.
\]

Since \( \varepsilon > 0 \) is arbitrary, we have

\[
\sum_{\ell=1}^N \sum_{m=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^n} |\langle f, D_{j_0}^{m} \tau_k \psi_\ell \rangle|^2 = \|f\|^2.
\]

The proof is finished by a density argument. \( \square \)

A more flexible way for constructing wavelet frames is the following result.

**Theorem 4** (Oblique Extension Principle). Let \( \phi \in L^2(\mathbb{R}^n) \) such that

\[
\hat{\phi}(A^*t) = H_0(t)\hat{\phi}(t), \quad a.e.,
\]

where \( H_0 \in L^\infty(\mathbb{T}^n) \). Let \( H_1, \ldots, H_N \in L^\infty(\mathbb{T}^n) \) and define \( \psi_1, \ldots, \psi_N \in L^2(\mathbb{R}^n) \) by [5]. Assume that there exists \( S \) a non-negative \( \mathbb{Z}^n \)-periodic measurable function such that \( \sqrt{S}\hat{\phi} \in L^2(\mathbb{R}^n) \), the origin is a point of \( A^* \)-approximate continuity of \( S\hat{\phi}^2 \), provided \( S(0)\hat{\phi}(0)^2 = 1 \). Moreover (10) and (11) hold. Then the set of functions \( \{\psi_\ell : \ell = 1, \ldots, N\} \) is a Parseval wavelet frame for \( L^2(\mathbb{R}^n) \).
Proof. Assume that the conditions of Theorem 3 are satisfied. Define \( \varphi \in L^2(\mathbb{R}^n) \) by
\[
\varphi(t) = \sqrt{S(t)} \hat{\varphi}(t)
\]
and define \( Q_0, Q_1, \ldots, Q_N \), \( \mathbb{Z}^n \)-periodic measurable functions, by
\[
Q_0(t) = \sqrt{S(A^*t) / S(t)} H_0(t), \quad Q_\ell(t) = \sqrt{1 / S(t)} H_\ell(t), \quad \ell = 1, \ldots, N.
\]
Observe that according to (6), we have \( A^*t \) a.e., \( \varphi \) is a Parseval wavelet frame for \( L^2(\mathbb{R}^n) \) by hypothesis. Moreover, by (7), bearing in mind that \( S \) is a finite \( \mathbb{Z}^n \)-periodic measurable function, by the observation that \( |\varphi|^2 \) if we set \( |\hat{\varphi}(0)|^2 = 1 \) by hypothesis.

Second,
\[
\varphi(A^*t) = \sqrt{S(A^*t)} \hat{\varphi}(A^*t) = \sqrt{S(A^*t) H_0(t)} \hat{\varphi}(t) = Q_0(t) \hat{\varphi}(t) \quad \text{a.e.}
\]

Third, by the definition of \( Q_\ell \) and (6), we obtain
\[
\sum_{\ell=0}^N |Q_\ell(t)|^2 = \frac{S(A^*t)}{S(t)} |H_0(t)|^2 + \sum_{\ell=1}^N \frac{1}{S(t)} |H_\ell(t)|^2 = 1 \quad \text{a.e. } t \in \mathbb{R}^n \setminus \mathcal{N}_\varphi.
\]

Moreover, by (7), bearing in mind that \( S \) is a \( \mathbb{Z}^n \)-periodic functions and \( A^*t \), we also have
\[
\sum_{\ell=0}^N Q_\ell(t) Q_\ell(t + p_k) = \frac{S(A^*t)}{S(t)S(t + p_k)} H_0(t) H_0(t + p_k) + \sum_{\ell=1}^N \frac{1}{S(t)S(t + p_k)} H_\ell(t) H_\ell(t + p_k) = 0,
\]
for a.e. \( t \in \mathbb{R}^n \setminus \mathcal{N}_\varphi \) and for any \( p_k, k = 1, \ldots, d_A - 1 \), such that \( t + p_k \in \mathbb{R}^n \setminus \mathcal{N}_\varphi \).

Since we have seen the hypotheses of Theorem 3 we conclude that the set functions \( \{ \hat{\psi}_\ell : \ell = 1, \ldots, N \} \), where
\[
\hat{\psi}_\ell(A^*t) = Q_\ell(t) \hat{\varphi}(t), \quad \text{a.e.}
\]
is a Parseval wavelet frame for \( L^2(\mathbb{R}^n) \). Indeed, by the observation that
\[
\hat{\psi}_\ell(A^*t) = Q_\ell(t) \hat{\varphi}(t) = \sqrt{1 / S(t)} H_\ell(t) \sqrt{S(t)} \hat{\varphi}(t) = \hat{\psi}_\ell(A^*t), \quad \text{a.e.}
\]
the proof is completed. \( \square \)

A key tool to prove i) implies ii) in Theorem 4 is the fundamental function. Let \( \phi \in L^2(\mathbb{R}^n) \) such that
\[
\hat{\phi}(A^*t) = H_0(t) \hat{\phi}(t), \quad \text{a.e.,}
\]
where \( H_0 \) is a finite a.e., \( \mathbb{Z}^n \)-periodic measurable function such that \( H_0(t) = 0 \) a.e. \( t \in \mathcal{N}_\varphi \). Moreover, let \( \psi_1, \ldots, \psi_N \in L^2(\mathbb{R}^n) \) defined by
\[
\hat{\psi}_\ell(A^*t) = H_\ell(t) \hat{\phi}(t) \quad \text{a.e.,} \quad \ell = 1, \ldots, N,
\]
where $H_\ell$ is a finite a.e., $\mathbb{Z}^n$-periodic measurable function such that $H_\ell(t) = 0$ a.e. $t \in \mathcal{N}_\varphi$. The following

\begin{equation}
\Theta(t) = \sum_{m=0}^{\infty} \sum_{\ell=1}^{N} |H_\ell(A^m t)|^2 \prod_{k=0}^{m-1} |H_0(A^k t)|^2,
\end{equation}

with the convention $\prod_{k=0}^{m-1} |Q_0(A^k t)|^2 = 1$, is usually called fundamental function associated to $H_0, \ldots, H_N$. Note that fundamental functions were introduced in [45]. Assuming that $\{\psi_1, \ldots, \psi_N\}$ is a Parseval wavelet frame, we will focus on properties $\Theta$, for instance, we will see that $\Theta$ is finite a.e. measurable function.

**Proof of Theorem 7.** That $ii)$ implies $i)$ follows by Oblique Extension Principle.

We prove $i)$ implies $ii)$. Consider $Q_\ell$, $\ell = 0, \ldots, N$, defined as $Q_\ell(t) = H_\ell(t)$, a.e. on $t \in \mathbb{R}^n \setminus \mathcal{N}_\varphi$ and $Q_\ell(t) = 0$, a.e. on $t \in \mathcal{N}_\varphi$.

Observe that

\begin{equation}
\hat{\phi}(A^\ast t) = Q_0(t) \hat{\phi}(t), \quad \text{a.e.,}
\end{equation}

and

\begin{equation}
\hat{\psi}_\ell(A^\ast t) = Q_\ell(t) \hat{\phi}(t) \quad \text{a.e.,} \quad \ell = 1, \ldots, N.
\end{equation}

Let $\Theta$ be the fundamental function associated to $Q_0, \ldots, Q_N$. We will see that $\Theta$ is a non-negative $\mathbb{Z}^n$-periodic measurable function satisfying $\sqrt{\Theta} \hat{\phi} \in L^2(\mathbb{R}^n)$, (a), (b) and (c) of $ii)$ if we consider $\Theta$ instead of $S$ in those conditions.

First, we show that $\Theta$ is a non-negative, $\mathbb{Z}^n$-periodic measurable function such that $\sqrt{\Theta} \hat{\phi} \in L^2(\mathbb{R}^n)$. Let $f \in L^2(\mathbb{R}^n)$. Since $\{\psi_\ell : \ell = 1, \ldots, N\}$ is a Parseval wavelet frame for $L^2(\mathbb{R}^n)$ and by Theorem B, we have

\[
\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \sum_{\ell=1}^{N} |\langle f, d_A^{-j/2} \tau_k D_A^{-j} \psi_\ell \rangle|^2 \leq \|f\|_2^2,
\]

or equivalently

\[
\int_{[0,1]^n} |\sum_{k \in \mathbb{Z}^n} f(s-k) \overline{\psi_\ell(A^s(s-k))}|^2 \, ds \leq \|f\|_2^2.
\]

Bearing in mind that $Q_\ell$ is $\mathbb{Z}^n$-periodic, $A^\ast(\mathbb{Z}^n) \subset \mathbb{Z}^n$, [21] and [20], we obtain

\[
\int_{[0,1]^n} \Theta(t) |\sum_{k \in \mathbb{Z}^n} \hat{f}(t-k) \hat{\phi}(t-k)|^2 \, dt \leq \|f\|_2^2.
\]

Therefore, $\Theta \sum_{k \in \mathbb{Z}^n} |\hat{\phi}(\cdot - k)|^2$ is in $L^\infty(\mathbb{T}^n)$. This implies that $\sqrt{\Theta} \hat{\phi} \in L^2(\mathbb{R}^n)$ and $\Theta$ is finite a.e.. The measurability of $\Theta$ holds because it is defined as the pointwise limit of measurable functions a.e.. The $\mathbb{Z}^n$-periodicity of $\Theta$ follows by its definition.
We check (a) in \( ii \). According to Theorem A, the definition of \( \widehat{\psi}_t \) and the refinement equation associated to \( \phi \),

\[
1 = \sum_{j \in \mathbb{Z}} \sum_{\ell = 1}^{N} |\widehat{\psi}_t(A^j t)|^2 = \lim_{j \to -\infty} \sum_{j = j+1}^{\infty} \sum_{\ell = 1}^{N} |\widehat{\psi}_t(A^j t)|^2
\]

\[
= \lim_{j \to -\infty} \sum_{j = j}^{\infty} \sum_{\ell = 1}^{N} |\widehat{\psi}_t(A^j t)|^2 \prod_{k=0}^{j-1} |Q_0(A^k t)|^2 |\widehat{\phi}(A^j t)|^2
\]

Thus, the origin is a point of \( A^* \)-approximate continuity of \( \Theta(\widehat{\phi})^2 \), if we set \( \Theta(0)|\widehat{\phi}(0)|^2 = 1 \), follows by Lemma J.

To prove the condition (b) in \( ii \), observe that from the definition of \( \Theta \), we have

\[
\Theta(t) = \sum_{m=0}^{\infty} \sum_{\ell = 1}^{N} |Q_\ell(A^m t)|^2 \prod_{k=0}^{m-1} |Q_0(A^k t)|^2
\]

\[
= \Theta(A^* t)|Q_0(t)|^2 + \sum_{\ell = 1}^{N} |Q_\ell(t)|^2.
\]

Since \( Q_\ell(t) = H_\ell(t) \) a.e. on \( t \in \mathbb{R}^n \setminus \mathcal{N}_\phi \), we conclude that \( \Theta \) satisfies the condition (b).

We see (c) in \( ii \). Let \( k \in \{1, \ldots, d_A - 1 \} \). For almost \( (A^*)^{-1} t \) and \( (A^*)^{-1} t + p_k \) points that are in \( \mathbb{R}^n \setminus \mathcal{N}_\phi \), then there exist \( k_1, k_2 \in \mathbb{Z}^n \) such that \( \widehat{\phi}(A^*)^{-1} t + k_1 \neq \widehat{\phi}(A^*)^{-1} t + k_2 + p_k \). Let \( q = A^*(k_2 - k_1) + A^* p_k \) and observe that \( q \in \mathbb{Z}^n \setminus A^* \mathbb{Z}^n \). We call \( (A^*)^{-1} s = (A^*)^{-1} t + k_1 \). By Theorem A, the definition of \( \psi_t \), the refinement equation associated to \( \phi \) and the definition of the fundamental function \( \Theta \), we have

\[
0 = \sum_{\ell = 1}^{N} \sum_{j = 0}^{\infty} \widehat{\psi}_t(A^j s) \overline{\widehat{\psi}_t(A^j (s + q))}
\]

\[
= \widehat{\phi}(A^*)^{-1} s \overline{\widehat{\phi}(A^*)^{-1} s + (A^*)^{-1} q} \times \left( \Theta(s) Q_0((A^*)^{-1} s) Q_0((A^*)^{-1} s + (A^*)^{-1} q) + \sum_{\ell = 1}^{N} Q_\ell((A^*)^{-1} s) Q_\ell((A^*)^{-1} s + (A^*)^{-1} q) \right).
\]

Bearing in mind that \( Q_\ell(t) = H_\ell(t) \) a.e. on \( t \in \mathbb{R}^n \setminus \mathcal{N}_\phi \) and they are \( \mathbb{Z}^n \)-periodic functions, we have

\[
0 = \Theta(t) H_0((A^*)^{-1} t) H_0((A^*)^{-1} t + p_k) + \sum_{\ell = 1}^{N} H_\ell((A^*)^{-1} t) H_\ell((A^*)^{-1} t + p_k).
\]

Hence, the condition (c) in \( ii \) of Theorem 1 follows. Thus the proof is finished. \( \square \)

We note that if \( \Psi = \{ \psi_\ell : \ell = 1, \ldots, N \} \), a Parseval wavelet frame for \( L^2(\mathbb{R}^n) \), is constructed under the conditions in Theorem 1, then \( \Psi \) is an \( A \)-FMRA based Parseval wavelet frame. First, we have that the function \( \varphi \) defined by (17) is
refinable and \( \hat{\phi} \) is \( A^* \)-locally nonzero at the origin. Thus according to Lemma F and Theorem H, \( \{ U_j := \text{supp}(d_A^{j/2} \varphi(A^j \cdot \cdot \cdot - k)) : k \in \mathbb{Z}^n \} \}_{j \in \mathbb{Z}} \) is an \( A \)-FMRA. Finally, by the proof of Theorem 4 and Lemma C, \( \Psi \subset U_1 \) follows. This proves our assertion.

We also note that if \( \phi \) is a refinable functions involved in Theorem 1, observe that the origin must be an \( A^* \)-density point for the support of \( \hat{\phi} \). Otherwise the origin is not able to be a point of \( A^* \)-approximate continuity for \( S|\hat{\varphi}|^2 \) when we set \( S(0)|\hat{\phi}(0)|^2 = 1 \).

### 4.2. Proof of Theorem 2 and Corollary 1

Using Theorem 1 and results on multiresolution analyses we prove Theorem 2.

**Proof of Theorem 2.** We see that i) implies ii). According to Lemma D and Lemma E, there exist \( H_0, H_1, \ldots, H_N \) some \( \mathbb{Z}^n \)-periodic measurable functions such that

\[
\hat{\varphi}(A^* t) = H_0(t) \hat{\varphi}(t) \quad \text{a.e., and} \quad \hat{\psi}(A^* t) = H_\ell(t) \hat{\varphi}(t), \quad \ell = 1, \ldots, N \quad \text{a.e.}
\]

because \( \{ \psi_1, \ldots, \psi_N \} \subset V_1 \). Since \( \{ \varphi(-k) : k \in \mathbb{Z}^n \} \) is a Parseval frame for \( V_0 \), \( H_0 \) can be taken in \( L^\infty(\mathbb{T}^n) \) by Lemma F. Indeed, it can be assumed that \( H_\ell(t) = 0, \ell = 0, 1, \ldots, N, \text{ a.e. on } \mathbb{N}_0 \).

Let \( \Theta \) be the fundamental function associated to \( H_0, H_1, \ldots, H_N \) defined as in [19]. Since \( \Psi \) is a Parseval frame for \( L^2(\mathbb{R}^n) \), we have already seen that \( \Theta \) satisfies the condition ii) in Theorem 1 if we consider \( \Theta \) instead of \( S \). It remains to see that \( \Theta, H_1, \ldots, H_N \) are in \( L^\infty(\mathbb{T}^n) \), and the origin is a point of \( A^* \)-approximate continuity for \( \Theta \) is we set \( \hat{\Theta}(0) = 1 \).

We have seen that \( \Theta(t) \sum_{k \in \mathbb{Z}^n} |\hat{\varphi}(t-k)|^2 \in L^\infty(\mathbb{T}^n) \) in the proof of Theorem 1. Furthermore, since \( \sum_{k \in \mathbb{Z}^n} |\hat{\varphi}(t-k)|^2 = 1 \text{ a.e. } t \in \mathbb{T}^n \setminus \mathbb{N}_0 \) and bearing in mind that \( \Theta(t) = 0 \text{ a.e. on } \mathbb{N}_0 \), we conclude that \( \Theta \) is in \( L^\infty(\mathbb{T}^n) \). Hence \( H_1, \ldots, H_N \) are in \( L^\infty(\mathbb{T}^n) \).

By ii) in Theorem 1 we know that the origin is a point of \( A^* \)-approximate continuity for \( \Theta |\hat{\varphi}|^2 \) is we set \( \Theta(0)|\hat{\varphi}(0)|^2 = 1 \). By Theorem H, the origin is a point of \( A^* \)-approximate continuity for \( \hat{\varphi} \) we set \( \hat{\varphi}(0) = 1 \). Hence, the origin is a point of \( A^* \)-approximate continuity for \( \Theta \) is we set \( \Theta(0) = 1 \) follows.

We check that ii) implies i). By Lemma D, \( \Psi \subset V_1 \). The origin is a point of \( A^* \)-approximate continuity for \( S|\hat{\varphi}|^2 \), if we set \( S(0)|\hat{\varphi}(0)|^2 = 1 \), because the origin is a point of \( A^* \)-approximate continuity for \( S \) and \( |\hat{\varphi}|^2 \), if we set \( S(0) = 1 \) and \( |\hat{\varphi}(0)|^2 = 1 \) respectively. We conclude that \( \Psi \) is a Parseval wavelet frame for \( L^2(\mathbb{R}^n) \) by Theorem 1. Hence the proof is finished.

To prove Corollary 1 we need the following condition of orthogonality.

**Lemma 3.** Let \( \varphi \in L^2(\mathbb{R}^n) \) such that \( \{ \tau_k \varphi : k \in \mathbb{Z}^n \} \) is a Parseval frame for \( V \), a subspace of \( L^2(\mathbb{R}^n) \). Assume that there exists \( H_0 \in L^\infty(\mathbb{T}^n) \) such that

\[
\hat{\varphi}(A^* t) = H_0(t) \hat{\varphi}(t), \quad \text{a.e. on } \mathbb{R}^n.
\]

Let \( f \in D_A(V) \) such that

\[
\hat{f}(A^* t) = H_f(t) \hat{\varphi}(t), \quad \text{a.e. on } \mathbb{R}^n,
\]
for some function $H_f \in L^2(\mathbb{T}^n)$. Then $f$ is orthogonal to the subspace $V$ if and only if

$$
(24) \quad \sum_{k=0}^{d_A-1} H_f(t + p_k) H_0(t + p_k) = 0, \quad \text{for a.e. } t \in \mathbb{R}^n \setminus \mathcal{N}_\varphi.
$$

Proof. Let $k \in \mathbb{Z}^n$. We compute the following. By Parseval’s equality,

$$
\langle f, \tau_k \varphi \rangle = \langle \hat{f}, \tau_k \hat{\varphi} \rangle = \int_{\mathbb{R}^n} \hat{f}(t) \overline{\phi(t)} e^{2\pi i k \cdot t} dt = \sum_{m \in \mathbb{Z}^n} \int_{[0,1]^n - m} D_A \cdot \hat{f}(s) D_A \cdot \overline{\phi(s)} e^{2\pi i k \cdot (A^* s)} ds.
$$

Since $\hat{f}$ and $\hat{\varphi}$ are in $L^2(\mathbb{R}^n)$ and $A^*(\mathbb{Z}^n) \subset \mathbb{Z}^n$, we have

$$
\langle f, \tau_k \varphi \rangle = \int_{[0,1]^n} \sum_{m \in \mathbb{Z}^n} \tau_m \left( D_A \cdot \hat{f}(t) D_A \cdot \overline{\phi(t)} \right) e^{2\pi i k \cdot (A^* t)} dt.
$$

By $(22)$ and $(23)$,

$$
\langle f, \tau_k \varphi \rangle = d_A \int_{[0,1]^n} \sum_{m \in \mathbb{Z}^n} \tau_m \left( D_A \cdot \hat{f}(t) \right) \overline{\tau_m \left( D_A \cdot \phi(t) \right)} e^{2\pi i k \cdot (A^* t)} dt = \int_{[0,1]^n} \sum_{k=0}^{d_A-1} H_f(t + p_k) H_0(t + p_k) e^{2\pi i k \cdot (A^* t)} dt,
$$

where the last equality holds according to Theorem C. Bearing in mind that $H_0$, $\chi_{\mathbb{R}^n \setminus \mathcal{N}_\varphi}$, and $e^{2\pi i k \cdot (A^* t)}$ are bounded and $\mathbb{Z}^n$-periodic functions and $H_f \in L^2(\mathbb{T}^n)$, (ii) in Lemma K implies that

$$
(25) \quad \langle f, \tau_k \varphi \rangle = \int_{[0,1]^n} \sum_{k=0}^{d_A-1} H_f((A^*)^{-1} t + p_k) H_0((A^*)^{-1} t + p_k) \chi_{\mathbb{R}^n \setminus \mathcal{N}_\varphi}((A^*)^{-1} t) e^{2\pi i k \cdot t} dt.
$$

We see the necessity condition. By hypotheses we know that $f$ is orthogonal to $\tau_k \varphi$, $\forall k \in \mathbb{Z}^n$. Thus, $(25)$ gives that

$$
\left( \sum_{k=0}^{d_A-1} H_f((A^*)^{-1} t + p_k) H_0((A^*)^{-1} t + p_k) \right) \chi_{\mathbb{R}^n \setminus \mathcal{N}_\varphi}((A^*)^{-1} t) = 0 \quad \text{a.e.,}
$$

which is the condition $(24)$.

To see the sufficient condition, observe that $f$ is orthogonal to $V$ if and only if $f$ is orthogonal to all the generators of $V$, i.e. $\tau_k \varphi$, $\forall k \in \mathbb{Z}^n$. Therefore the proof is finished directly from $(24)$ and $(25)$.

Proof of Corollary 1. By Theorem 2, Lemma 3 and the definition of $W_j$, $j \in \mathbb{Z}$, the statements follow.

In these last paragraphs, we will see that the condition at the origin of the Fourier Transform of an involved refinable function in Extension Principles assumed by Han \cite{c9}, \cite{c10} is not equivalent to the condition used in this note.
The linear space of all compactly supported \(C^\infty(\mathbb{R}^n)\) (test) functions with the usual topology will be denoted by \(D(\mathbb{R}^n)\). For \(g \in D(\mathbb{R}^n)\) and \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\), with some abuse in the notation, we will use

\[
\langle f, g \rangle = \int_{\mathbb{R}^n} f(t)g(t)\,dt.
\]

According to our context, the condition used by Han may be written as follows. Given \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\), the following identity holds in the sense of distributions:

\[
(26) \quad \lim_{j \to \infty} |f(A^{-j}t)| = 1,
\]

more precisely,

\[
\lim_{j \to \infty} \langle |f(A^{-j}t)|, g \rangle = \langle 1, g \rangle, \quad \forall g \in D(\mathbb{R}^n).
\]

Let \(f\) be the function in \(L^1_{\text{loc}}(\mathbb{R})\) defined by

\[
f(x) = \chi_{[-1,1]}(x) + \sum_{\ell=0}^{\infty} 2^{\ell+2} \chi_{(2^{-\ell-1},2^{-\ell-1}+2^{-2\ell-2})}(x),
\]

We will see that the origin is a point of approximate continuity of \(f\) but (26) is not satisfied with \(A = 2\).

Denote \(F = [-1,1] \setminus \bigcup_{\ell=0}^{\infty}(2^{-\ell-1},2^{-\ell-1}+2^{-2\ell-2})\). Since

\[
\lim_{j \to \infty} \frac{|2^{-j}[-1,1] \cap F|}{|2^{-j}[-1,1]|} = 1 - \lim_{j \to \infty} \frac{|\bigcup_{\ell=0}^{\infty}(2^{-\ell-1},2^{-\ell-1}+2^{-2\ell-2})|}{|2^{-j}[-1,1]|} = 1,
\]

the origin is a point of density for the set \(F\). It follows rapidly that the origin is a point of approximate continuity of \(f\).

Now we see that the function \(f\) does not satisfy the condition (26) with \(A = 2\). Take \(g \in D(\mathbb{R})\) such that \(g\) is non negative, with value 1 on the interval \([-1,1]\), supported on \([-2,2]\), increasing on \([-2, -1]\) and decreasing on \([1, 2]\). We have the following inequalities

\[
\lim_{j \to \infty} \langle |f(2^{-j}t)|, g \rangle \geq \lim_{j \to \infty} \int_{-1}^{1} f(2^{-j}t)\,dt = \lim_{j \to \infty} 2^j \int_{-2^{-j}}^{2^{-j}} f(y)\,dy
\]

\[
= 2 + \lim_{j \to \infty} 2^j \int_{-2^{-j}}^{2^{-j}} \sum_{\ell=0}^{\infty} 2^{\ell+2} \chi_{(2^{-\ell-1},2^{-\ell-1}+2^{-2\ell-2})}(y)\,dy
\]

\[
= 2 + \sum_{k=0}^{\infty} 2^{k} = 4 > \langle 1, g \rangle.
\]

This implies that (26) does not hold.

References

[1] N. Atreas, A. Melas, T. Stavropoulos; Affine dual frames and extension principles. Appl. Comput. Harmon. Anal., 36, No. 1, 51–62 (2014).
[2] L.W. Baggett, H.A. Medina, K.D. Merrill; Generalized multi-resolution analyses and a construction procedure for all wavelet sets in \(\mathbb{R}^n\), J. Fourier Anal. Appl., 5 (1999), 563–573.
[3] D. Bakic; Semi-orthogonal Parseval frame wavelets and generalized multiresolution analyses, Appl. Comput. Harmon. Anal. 21 (2006), no. 3, 281–304.
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[4] J.J. Benedetto, S. Li; The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harmon. Anal. 5 (1998), 389–427.

[5] J.J. Benedetto, O.M. Treiber; Wavelet frames: Multiresolution analysis and extension principles, L. Debnath (Ed.), Wavelet Transforms and TimeFrequency Signal Analysis, Birkhäuser (2001), pp. 336.

[6] J.J. Benedetto, D.F. Walnut; Gabor frames for $L^2$ and related spaces, Wavelets: mathematics and applications, 97–162, Stud. Adv. Math., CRC, BocaRaton, FL, 1994.

[7] M. Bownik; Riesz wavelets and generalized multiresolution analyses, Appl. Comput. Harmon. Anal. 14 (2003), no. 3, 181-194.

[8] M. Bownik; A characterization of affine dual frames in $L^2(\mathbb{R}^n)$, Appl. Comput. Harmon. Anal. 8 (2000), no. 2, 203-221.

[9] M. Bownik, G. Garrigós; Biorthogonal wavelets, MRA’s and shift-invariant spaces, Studia Math. 160 (2004), no. 3, 231-248.

[10] C. de Boor, R.A. DeVore, A. Ron; Approximation from shift-invariant subspaces of $L^2(\mathbb{R}^d)$, Trans. Amer. Math. Soc. 341 (1994), no. 2, 787–806.

[11] C. de Boor, R.A. DeVore, A. Ron; On the construction of multivariate (pre)wavelets, Constructive Approximation 9 (1993), No.2-3, 123–166.

[12] A. Bruckner; Differentiation of real functions, Lecture Notes in Mathematics, 659. Springer, Berlin, 1978.

[13] A. Calogero, G. Garrigós; A characterization of wavelet families arising from biorthogonal MRA’s of multiplicity $d$, J. Geom. Anal. 11 (2001), no. 2, 187-217.

[14] P.G. Casazza, O. Christensen, N.J. Kalton; Frames of translates, Collect. Math. 52 (2001), 35–54.

[15] O. Christensen; An introduction to frames and Riesz bases, Birkhäuser, Boston, 2003.

[16] C.K. Chui, W. He, J. Stöckler; Compactly supported tight and sibling frames with maximum vanishing moments, Appl. Comput. Harmonic Anal. 13 (2002), 1–46.

[17] C.K. Chui, W. He, J. Stöckler, Q. Sun; Compactly supported tight affine frames with integer dilations and maximum vanishing moments, Adv. Comput. Math. 18 (2003), no. 2-4, 159–187.

[18] C.K. Chui, X. Shi, J. Stöckler; Affine frames, quasi-affine frames, and their duals, Adv. Comput. Math. 8 (1998), no. 1-2, 1–17.

[19] P. Cifuentes, K.S. Kazarian, A. San Antolín; Characterization of scaling functions in a multiresolution analysis, Proc. Amer. Math. Soc. 133 (2005), no. 4, 1013–1023.

[20] I. Daubechies, Ten lectures on wavelets, SIAM, Philadelphia, 1992.

[21] I. Daubechies, B. Han, A. Ron, Z.W. Shen; Framelets: MRA–based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14 (2003), 1–46.

[22] R. Duffin, A. Schaeffer; A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952), 341–366.

[23] M. Frazier, G. Garrigós, W. Wang, G. Weiss; A characterization of functions that generate wavelet and related expansion, Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996), J. Fourier Anal. Appl. 3 (1997), Special Issue, 883-906.

[24] G. Gripenberg; A necessary and sufficient condition for the existence of a father wavelet, Studia Math. 114 (1995), no. 3, 207-226.

[25] K. Gröchenig, W.R. Madych; Multiresolution analysis, Haar bases and self-similar tilings of $R^n$, IEEE Trans. Inform. Theory, 38(2), (1992) 556–568.

[26] B. Han; Nonhomogeneous wavelet systems in high dimensions, App. Comput. Harmon. Anal. 32 (2012), no. 2, 169–196.

[27] B. Han; Pairs of frequency-based nonhomogeneous dual wavelet frames in the distribution space, Appl. Comput. Harmon. Anal. 29 (2010), no. 3, 330–353.

[28] B. Han; Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix, Approximation theory, wavelets and numerical analysis, (Chattanooga, TN, 2001). J. Comput. Appl. Math. 155 (2003), no. 1, 43–67.

[29] B. Han; On dual wavelet tight frames, Applied Comput. Harmon. Anal. 4 (1997), 380–413.

[30] E. Hernández, G. Weiss; A first course on Wavelets, CRC Press Inc., 1996.

[31] R.Q. Jia, Z. Shen; Multiresolution and Wavelets, Proc. Edinburgh Math. Soc. (2) 37 (1994), no. 2, 271–300.
[32] K.S. Kazarian, A. San Antolín; \textit{Characterization of scaling functions in a frame multiresolution analysis in $H^2$}, Topics in classical analysis and applications in honor of Daniel Waterman. Hackensack, NJ: World Scientific. 118–140 (2008).

[33] H.O. Kim, R.Y. Kim, Y.H. Lee, J.K. Lim; \textit{On Riesz wavelets associated with multiresolution analyses}, Appl. Comput. Harmon. Anal. 13 (2002), no. 2, 138-150.

[34] H.O. Kim, R.Y. Kim, J.K. Lim; \textit{Characterizations of biorthogonal wavelets which are associated with biorthogonal multiresolution analyses}, Appl. Comput. Harmon. Anal. 11 (2001), no. 2, 263-272.

[35] H.O. Kim, J.K. Lim; \textit{Frame multiresolution analysis}, Commun. Korean Math. Soc. 15 (2000), 285–308.

[36] M.–J. Lai; \textit{Construction of multivariate compactly supported orthonormal wavelets}, Adv. Comput. Math. 25 (2006), 41–56.

[37] S. Mallat; \textit{Multiresolution approximations and wavelet orthonormal bases for $L^2(R)$}, Trans. Amer. Math. Soc. 315 (1989), 69–87.

[38] Y. Meyer; \textit{Ondelettes et opérateurs. I}, Hermann, Paris, 1996 [English Translation: Wavelets and operators, Cambridge University Press, 1992]

[39] I.P. Nathanson; \textit{Theory of functions of a real variable}, London, vol. II, 1960.

[40] M. Papadakis; Frames of translates in abstract Hilbert spaces and the generalized frame multiresolution analysis, Trends in approximation theory (Nashville, TN, 2000), 353-362, Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, 2001.

[41] M. Papadakis; \textit{On the dimension function of orthonormal wavelets}, Proc. Amer. Math. Soc. 128 (2000), no. 7, 2043-2049.

[42] A. Ron, Z.W. Shen; \textit{Construction of compactly supported affine frames in $L^2(R^d)$}, in: K.S. Lau (Ed.), Advances in Wavelets, Springer–Verlag, New York, 1998, pp. 27–49.

[43] A. Ron, Z.W. Shen; \textit{Compactly supported tight affine frames in $L^2(R^d)$}, Math. Comp. 67 (1998), 191–207.

[44] A. Ron, Z.W Shen; \textit{Affine systems in $L^2(R^d)$. II. Dual systems}, Dedicated to the memory of Richard J. Duffin. J. Fourier Anal. Appl. 3 (1997), no. 5, 617.637.

[45] A. Ron, Z.W. Shen; \textit{Affine systems in $L^2(R^d)$: The analysis of the analysis operator}, J. Funct. Anal. 148 (1997), 408–447.

[46] A. San Antolín, \textit{Characterization of low pass filters in a multiresolution analysis}, Studia Math. 190 (2009), 99–116.

[47] T. Stavropoulos; \textit{The geometry of extension principles}, Houston J. Math. 38, no. 3, 833-853 (2012).

[48] P. Wojtaszczyk; \textit{A mathematical introduction to wavelets}, London Math. Soc., Student Texts 37, 1997.

[49] R.A. Zalik; \textit{Riesz bases and multiresolution analyses}, Appl. Comput. Harmon. Anal. 7 (1999), no. 3, 315-331.

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