Laconic schema mappings: computing core universal solutions by means of SQL queries

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Abstract. We present a new method for computing core universal solutions in data exchange settings specified by source-to-target dependencies, by means of SQL queries. Unlike previously known algorithms, which are recursive in nature, our method can be implemented directly on top of any DBMS. Our method is based on the new notion of a laconic schema mapping. A laconic schema mapping is a schema mapping for which the canonical universal solution is the core universal solution. We give a procedure by which every schema mapping specified by FO s-t tgds can be turned into a laconic schema mapping specified by FO s-t tgds that may refer to a linear order on the domain of the source instance. We show that our results are optimal, in the sense that the linear order is necessary and the method cannot be extended to schema mapping involving target constraints.

1 Introduction

We present a new method for computing core universal solutions in data exchange settings specified by source-to-target dependencies, by means of SQL queries. Unlike previously known algorithms, which are recursive in nature, our method can be implemented directly on top of any DBMS. Our method is based on the new notion of a laconic schema mapping. A laconic schema mapping is a schema mapping for which the canonical universal solution is the core universal solution. We give a procedure by which every schema mapping specified by FO s-t tgds can be turned into a laconic schema mapping specified by FO s-t tgds that may refer to a linear order on the domain of the source instance.

Outline of the paper: In Section 2, we recall basic notions and facts about schema mappings. Section 3 explains what it means to compute a target instance by means of SQL queries, and we state our main result. Section 4 introduces the notion of laconicity, and contains some initial observations. In Section 5, we present our main result, namely a method for transforming any schema mapping specified by FO s-t tgds into a laconic schema mapping specified by FO s-t tgds assuming a linear order. In Section 6, we show that our results cannot be extended to the case with target constraints.

2 Preliminaries

In this section, we recall basic notions from data exchange and fix our notation.

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2.1 Instances and homomorphisms

Fix disjoint infinite sets of constant values Cons and null values Nulls, and let $< \subseteq$ be a linear order on Cons. We consider instances whose values are from Cons $\cup$ Nulls. We use $\text{dom}(I)$ to denote the set of values that occur in facts in the instance $I$. A homomorphism $h : I \rightarrow J$, with $I, J$ instances of the same schema, is a function $h : \text{Cons} \cup \text{Nulls} \rightarrow \text{Cons} \cup \text{Nulls}$ with $h(a) = a$ for all $a \in \text{Cons}$, such that for all relations $R$ and all tuples of (constant or null) values $(v_1, \ldots, v_n) \in R^l$, $(h(v_1), \ldots, h(v_n)) \in R^l$. Instances $I, J$ are homomorphically equivalent if there are homomorphisms $h : I \rightarrow J$ and $h' : J \rightarrow I$. An isomorphism $h : I \cong J$ is a homomorphism that is a bijection between $\text{dom}(I)$ and $\text{dom}(J)$ and that preserves truth of atomic formulas in both directions. Intuitively, nulls act as placeholders for actual (constant) values, and a homomorphism from $I$ to $J$ captures the fact that $J$ “contains more, or at least as much information” as $I$.

The fact graph of an instance $I$ is the graph whose nodes are the facts $Rv$ (with $R$ a $k$-ary relation and $v \in (\text{Cons} \cup \text{Nulls})^k$, $k \geq 0$) true in $I$, and such that there is an edge between two facts if they have a null value in common.

We will denote by CQ, UCQ, and FO the set of conjunctive queries, unions of conjunctive queries, and first-order queries, respectively, and CQ $\prec$, UCQ $\prec$, and FO$^\prec$ are defined similarly, except that the queries may refer to the linear order. Thus, unless indicated explicitly, it is assumed that queries do not refer to the linear order. For any query $q$ and instance $I$, we denote by $q(I)$ the answers of $q$ in $I$, and we denote by $q(I)_1$ the ground answers of $q$, i.e., $q(I)_1 = q(I) \cap \text{Cons}^k$ for $k$ the arity of $q$.

2.2 Schema mappings, universal solutions, and certain answers

Let $S$ and $T$ be disjoint schemas, called the source schema and the target schema. As usual in data exchange, whenever we speak of a source instance, we will mean an instance of $S$ whose values belong to Cons, and when we speak of a target instance, we will mean an instance of $T$ whose values may come from Cons $\cup$ Nulls.

A schema mapping is a triple $M = (S, T, \Sigma_{st})$, where $S$ and $T$ are the source and target schemas and $\Sigma_{st}$ is a finite set of sentences of some logical language defining a class of pairs of instances $\langle I, J \rangle$. Here, $\langle I, J \rangle$ denotes union of a source instance $I$ and a target instance $J$, which is itself an instance over the joint schema $S \cup T$, and the logical languages we consider are presented below. Two schema mappings, $M = (S, T, \Sigma_{st})$ and $M' = (S, T, \Sigma'_{st})$, are said to be logically equivalent if $\Sigma_{st}$ and $\Sigma'_{st}$ are logically equivalent, i.e., satisfied by the same pairs of instances. Given a schema mapping $M = (S, T, \Sigma_{st})$ and a source instance $I$, a solution for $I$ with respect to $M$ is a target instance $J$ such that $\langle I, J \rangle$ satisfies $\Sigma_{st}$. We denote the set of solutions for $I$ with respect to $M$ by $\text{Sol}_M(I)$, or simply $\text{Sol}(I)$ when the schema mapping is clear from the context.

The concrete logical languages that we will consider for the specification of $\Sigma_{st}$ are the following. A source-to-target tuple generating dependency (s-t tgd) is a first-order sentence of the form $\forall \bar{x}(\phi(\bar{x}) \rightarrow \exists \bar{y}. \psi(\bar{x}, \bar{y}))$, where $\phi$ is a conjunction of atomic formulas over $S$ and $\psi$ is a conjunction of atomic formulas over $T$, such that each variable in $\bar{x}$ occurs in $\phi$. A more general class of constraints called FO s-t tgds is defined analogously, except that the antecedent is allowed to be any FO query over $S$. Similarly, L s-t tgds can be defined for any query language $L$. A LAV s-t tgd, finally, is an s-t tgd in which $\phi$ is a single atomic formula. To simplify notation, we will typically drop the universal quantifiers when writing (L) s-t tgds.

Given a source instance $I$, a schema mapping $M$, and a target query $q$, we will denote by $\text{cert}_{M,q}(I)$ the set of certain answers to $q$ in $I$ with respect to $M$, i.e., the intersection $\bigcap_{J \in \text{Sol}_M(I)} q(J)$. In other words, a tuple of values is a certain answer to $q$ if it belongs to the set of answers of $q$, no matter which solution of $I$ one picks. There are two methods to compute certain answers to a conjunctive query. The first method uses universal solutions and the second uses query rewriting.

A universal solution for a source instance $I$ with respect to a schema mapping $M$ is a solution $J \in \text{Sol}_M(I)$ such that, for every $J' \in \text{Sol}_M(I)$, there is a homomorphism from $J$ to $J'$. It was shown
in [1] that the certain answers for a conjunctive target query can be obtained simply by evaluating
the query in a universal solution. Moreover, universal solutions are guaranteed to exist for schema
mappings specified by L s-t tgds, for any query language L.

**Theorem 1 ([1]).** For all schema mappings $\mathcal{M}$, source instances $I$, conjunctive queries $q$, and uni-
versal solutions $J \in \text{Sol}(I)$, $\text{certain}_{\mathcal{M}}(q)(I) = q(J)$.

**Theorem 2 ([1]).** For every schema mapping $\mathcal{M}$ specified by L s-t tgds, with L any query language,
and for every source instance $I$, there is a universal solution for $I$ with respect to $\mathcal{M}$.

The second method for computing certain answers to conjunctive queries is by rewriting the given
target query to a query over the source that directly computes the certain answers to the original
query.

**Theorem 3.** Let $L$ be any of UCQ, UCQ$^<$, FO, FO$^<$. Then for every schema mapping $\mathcal{M}$ specified
by s-t tgds and for every $L$-query $q$ over $\mathbf{T}$, one can compute in exponential time an $L$-query over $\mathbf{S}$
defining $\text{certain}_{\mathcal{M}, q}$.

There are various ways in which such certain answer queries can be obtained. One possibility is to
split up the schema mapping $\mathcal{M}$ into a composition $\mathcal{M}_1 \circ \mathcal{M}_2$, with $\mathcal{M}_1$ specified by full s-t tgds and
$\mathcal{M}_2$ specified by LAV s-t tgds, and then to successively apply the known query rewriting techniques
of MiniCon [8] and full s-ttg unfolding (cf. [7]). In [9], an alternative rewriting method was given for
the case of $L = \text{FO}^{(<)}$, which can be used to compute in polynomial time an $\text{FO}^{(<)}$ source query $q'$
defining $\text{certain}_{\mathcal{M}, q}$ over source instances whose domain contains at least two elements.

### 2.3 Core universal solutions

A source instance can have many universal solutions. Among these, the *core universal solution* plays
a special role. A target instance $J$ is said to be a core if there is no proper subinstance $J' \subseteq J$
and homomorphism $h : J \rightarrow J'$. There is equivalent definition in terms of retractions. A subinstance
$J' \subseteq J$ is called a retract of $J$ if there is a homomorphism $h : J \rightarrow J'$ such that for all $a \in \text{dom}(J')$,
h($a$) = $a$. The corresponding homomorphism $h$ is called a retraction. A retract is proper if it is a
proper subinstance of the original instance. A core of a target instance $J$ is a retract of $J$ that has
itself no proper retracts. Every (finite) target instance has a unique core, up to isomorphism. Moreover,
two instances are homomorphically equivalent iff they have isomorphic cores. It follows that, for every
schema mapping $\mathcal{M}$, every source instance has at most one core universal solution up to isomorphism.
Indeed, if the schema mapping $\mathcal{M}$ is specified by FO s-t tgds then each source instance has exactly
one core universal solution up to isomorphism [3]. We will therefore freely speak of the core universal
solution.

It has been convincingly argued that, among all universal solutions for a source instance, the core
universal solution is the preferred solution. One important reason is that the core universal solution is
the smallest universal solution: if $J$ is the core universal solution for a source instance $I$ with respect to
a schema mapping $\mathcal{M}$, and $J'$ is any other solution universal solution for $I$, i.e., one that is not a core,
then $|J| < |J'|$. Consequently, the core universal solution is the universal solution that is least expensive
to materialize. We add to this a second virtue of the core universal solution, namely that, among all
universal solutions, it is the most conservative one in terms of the answers that it assigns to conjunctive
queries with inequalities. We propose another reason to be interested in the core universal solution,
namely that it is the solution that satisfies the most dependencies. In many practical data exchange
settings, one is interested in solutions satisfying certain target dependencies. One way to obtain such
solutions is to include the relevant target dependencies in the specification of the schema mapping.
If the target dependencies satisfy certain syntactic requirements (in particular, if they form a weakly acyclic set of target tgds and target egds), then a solution satisfying these target dependencies can be obtained by means of the chase. On the other hand, sometimes it happens that the universal solution one constructs without taking into account the target dependencies happens to satisfy the target dependencies. Whether this happens depends very much on which universal solution is constructed. For example, if $M$ is the schema mapping specified by the s-t tgd $Rx \rightarrow \exists y.Syx$, $I$ is any source instance and $J$ a universal solution, then the first attribute of $S$ is not necessarily a key in $J$. However, if $J$ is the core universal solution, then it will be a key. In fact, it turns out that the core universal solution is the universal solution that maximizes the set of valid target dependencies. To make this precise, let a \textit{disjunctive target dependency} be a first-order sentence of the form $\forall x.\phi(x) \rightarrow \bigvee_i \exists y_i.\psi_i(x, y_i)$, where $\phi, \psi_i$ are conjunctions of atomic formulas over the target schema $T$ and/or equalities. Then we have:

**Theorem 4.** Let $M$ be any schema mapping, $I$ be any source instance, $J$ the core universal solution of $I$, and $J'$ any other universal solution of $I$, i.e., one that is not a core. Then

1. Every disjunctive dependency valid on $J'$ is valid on $J$, and
2. Some disjunctive dependency valid on $J$ is not valid on $J'$.

**Proof.** The first half of the result follows from the fact that $J$ is a retract of $J'$ and disjunctive dependencies are preserved when going from an instance to one of its retract. This is shown in [5] for non-disjunctive embedded dependencies, but the same argument applies to disjunctive dependencies. To prove the second half, pick fresh variables $x$, one for each value (constant or null) in the domain of $J'$, and let $\psi(x)$ be the conjunction of all facts that are true in $J'$ under the natural assignment. Consider the disjunctive dependency $\forall x.\phi(x) \rightarrow \bigvee_{i \neq j} (x_i = x_j)$. This disjunctive dependency is clearly not true in $J'$ but it is trivially true in $J$, since $J$, being a proper retract of $J'$, contains strictly fewer nulls than $J'$.

Concerning the complexity of computing core universal solutions, we have the following:

**Theorem 5 ([3]).** For fixed schema mappings specified by FO$^<$ s-t tgds, given a source instance, a core universal solution can be computed in polynomial time.

Strictly speaking, in [3] this was only shown for schema mappings specified by s-t tgds. However the same argument applies to FO$^<$ s-t tgds. In fact, this holds for richer data exchange settings, there the schema mapping specification may contain also target constraints (specifically, target egds and weakly acyclic target tgds). Moreover, several algorithms for obtaining core universal solutions in polynomial time have been proposed.

### 3 Computing universal solutions with SQL queries

There is a discrepancy between the methods for computing universal solutions commonly presented in the data exchange literature, and the methods actually employed by data exchange tools. In the data exchange literature, methods for computing universal solutions are often presented in the form of a chase procedures. In practical implementations such as Clio, on the other hand, it is common to compute universal solutions using SQL queries, thus leveraging the capabilities of existing DBMSs. We briefly review here both approaches, and explain how canonical universal solutions can be computed using SQL queries.

The simplest and most well known method for computing universal solutions is the \textit{naive chase}\footnote{There are also other, more sophisticated versions of the chase, but they will not be relevant for most of what we discuss, since we will be interested in computing solutions by means of SQL queries anyway. We will briefly mention one variant of the chase later on.}

The algorithm is described in Figure 1. For a source instance $I$ and schema mapping $M$ specified by FO$^{(\leq)}$ s-t tgds, the result of applying the naive chase is called the \textit{canonical universal solution} of $I$ with respect to $M$. Note that the result of the naive chase is unique up to isomorphism, since it depends only on the exact choice of fresh nulls. Also note that, even if two schema mappings are logically equivalent, they may assign different canonical universal solutions to a source instance. We will now...
In general, the definition of a $k$-ary function symbol. For instance, if $g$ is a binary function, then $g(x, y)$ for all tuples of constants $a$ such that $I \models \phi(a)$ do.  
return $J$.

**Fig. 1.** Naive chase method for computing universal solutions.

We first Skolemize the dependencies, and split them so that the right hand side consists of a single conjunct. In this way, we get

$$Rx_1x_2 \rightarrow \exists y.(Sx_1y \land Tx_2y)$$
$$Rx \rightarrow Sxx$$

Next, for each target relation $R$ we collect the dependencies that contain $R$ in the right hand side, and we interpret these as constituting a definition of $R$. In this way, we get the following definitions of $S$ and $T$:

$$S := \{(x_1, f(x_1, x_2)) \mid Rx_1x_2\} \cup \{(x, x) \mid Rx\}$$
$$T := \{(x_2, f(x_1, x_2)) \mid Rx_1x_2\}$$

In general, the definition of a $k$-ary target relation $R \in T$ will be of the shape

$$R := \{(t_1(x), \ldots, t_k(x)) \mid \phi(x)\} \cup \cdots \cup \{(t'_1(x'), \ldots, t'_k(x')) \mid \phi'(x')\}$$

(1)

where $t_1, \ldots, t_k, t'_1, \ldots, t'_k$ are terms and $\phi, \ldots, \phi'$ are first-order queries over the source schema. Since FO queries correspond to SQL queries, one can easily use a relational DBMS in order to compute the tuples in the relation $R$.

The general idea behind the construction of the FO queries should be clear from the example. However, giving a precise definition of what it means to compute a target instance by means of SQL queries require a bit of care. We need to assume some structure on the set of nulls $\textbf{Nulls}$. Fix a countably infinite set of function symbols of arity $n$, for each $n \geq 0$. For any set $X$, denote by $\textbf{Terms}[X]$ be the set of all terms built up from elements of $X$ using these function symbols, and denote by $\textbf{PTerms}[X] \subseteq \textbf{Terms}[X]$ the set of all proper terms, i.e., those with at least one occurrence of a function symbol. For instance, if $g$ is a unary function and $h$ is a binary function, then $h(g(x), y)$, $g(x)$ and $x$ belong to $\textbf{Terms}[\{x, y\}]$, but only the first two belong to $\textbf{PTerms}[\{x, y\}]$. It is important to distinguish between proper terms built up from constants on the one hand and constants on the other hand, as the former will be treated as nulls and the latter not. More precisely, we assume that $\textbf{PTerms}[\textbf{Cons}] \subseteq \textbf{Nulls}$. Recall that $\textbf{Cons} \cap \textbf{Nulls} = \emptyset$.

**Definition 1** (L-term interpretation). Let $L$ be any query language. An L-term interpretation $\Pi$ is a map assigning to each $k$-ary relation symbol $R \in T$ a union of expressions of the form (1) where $t_1, \ldots, t_k \in \textbf{Terms}[x]$ and $\phi(x)$ is an L-query over $S$. 

**Input:** A schema mapping $\mathcal{M} = (S, T, \Sigma_{st})$ and a source instance $I$  
**Output:** A target instance $J$ that is a universal solution for $I$ w.r.t. $\mathcal{M}$
Given a source instance $I$, an $L$-term interpretation $I$ generates an target instance $I(I)$, in the obvious way. Note that $I(I)$ may contain constants as well as nulls. Although the program specifies exactly which nulls are generated, we will consider $I(I)$ only up to isomorphism, and hence the meaning of an $L$-term interpretation does not depend on exactly which function symbols it uses.

The previous example shows

**Proposition 1.** Let $L$ be any query language. For every schema mapping specified by $L$ s-t tgds there is an $L$-term interpretation that yields for each source instance the canonical universal solution.

Incidentally, even for schema mappings specified by SO tgds, as defined in [4], FO-term interpretations can be constructed that compute the canonical universal solution. However, the above suffices for present purposes.

On the other hand,

**Proposition 2.** No FO-term interpretation yields for each source instance the core universal solution with respect to the schema mapping specified by the FO (in fact LAV) s-t tgd $R_{xy} \rightarrow \exists z.(S_{xz} \land S_{yz})$.

**Proof.** The argument uses the fact that FO formulas are invariant for automorphisms. Let $I$ be the source instance whose domain consists of the constants $a, b, c, d$, and such that $R$ is the total relation over this domain. Note that every permutation of the domain is an automorphism of $I$. Suppose for the sake of contradiction that there is an FO-term interpretation $I$ such that the $I(I)$ is the core universal solution of $I$. Then the domain of $I(I)$ consists of the constants $a, b, c, d$ and a distinct null term, call it $N_{(x,y)} \in \text{Terms}(a)$, for each pair of distinct constants $x, y \in \{a, b, c, d\}$, and $I(I)$ contains the facts $R_{x}N_{(x,y)}$ and $R_{y}N_{(x,y)}$ for each of these nulls $N_{(x,y)}$. Now consider the term $N_{(a,b)}$. We can distinguish two cases. The first case is where the term $N_{(a,b)}$ does not contain any constants as arguments. In this case, it follows from the invariance of FO formulas for automorphisms that $I(I)$ contains $R_{x}N_{(a,b)}$ for every $x \in \{a, b, c, d\}$, which is clearly not true. The second case is where $N_{(a,b)}$ contains at least one constant as an argument. If $N_{(a,b)}$ contains the constant $a$ or $b$ then let $t'$ be obtained by switching all occurrences of $a$ and $b$ in $N_{(a,b)}$, otherwise let $t'$ be obtained by switching all occurrences of $c$ and $d$ in $N_{(a,b)}$. Either way, we obtain that there is a second null, namely $t'$, which is distinct from $N_{(a,b)}$, and which stands in exactly the same relations to $a$ and $b$ as $N_{(a,b)}$ does. This again contradicts our assumption that $J$ is the core universal solution of $I$.

Things change in the presence of a linear order. We will show that every schema mapping specified by $FO^< s-t$ tgds is logically equivalent to a laconic schema mapping specified by $FO^< s-t$ tgds, i.e., one for which the canonical universal solution is always a core. In particular, given Proposition 1, this shows:

**Theorem 6.** For every schema mapping specified by $FO^< s-t$ tgds there is a $FO^< -$term interpretation that yields for each source instance the core universal solution.

In the case of the example from Proposition 2, the $FO^<$-term interpretation $I$ computing the core universal solution is given by

$$I(S) = \{(x_1, f(x_1, x_2)) \mid (Rx_1x_2 \lor Rx_2x_1) \land x_1 \leq x_2\}$$

$$\cup \{(x_2, f(x_1, x_2)) \mid (Rx_1x_2 \lor Rx_2x_1) \land x_1 \leq x_2\}$$

Furthermore, we will show that every schema mapping defined by $FO^<$ s-t tgds whose right-hand-side contains at most one atomic formula is equivalent to a laconic schema mapping specified by $FO^<$ s-t tgds, and therefore, its core universal solutions can be computed by means of an FO-term interpretation. In other words, in this case the linear order is not needed. Note that in the example from Proposition 2, the right-hand-size of the s-t tgd consists of two atomic formulas.

In the next section, we formally introduce the notion of laconicity. In Section 5, we show that every schema mapping specified by $FO^<$ s-t tgds is logically equivalent to a laconic schema mapping specified by $FO^<$ s-t tgds.
Non-laconic schema mapping | Logically equivalent laconic schema mapping
---|---
(a) $P_x \rightarrow \exists yz.R_{xyz} \land R_{xz}$ | (a') $P_x \rightarrow \exists y.R_{xy}$
(b) $P_x \rightarrow \exists yz.R_{xyz}$
$P_x \rightarrow R_{xz}$ | (b') $P_x \rightarrow R_{xx}$
(c) $R_{xy} \rightarrow S_{xy}$
$P_x \rightarrow \exists y.S_{xy}$ | (c') $R_{xy} \rightarrow S_{xy}$
$P_x \rightarrow \exists yz.R_{xyz}$ | $P_x \land \exists yz.R_{xyz} \rightarrow \exists yz.S_{xyz}$
(d) $R_{xy} \rightarrow \exists z.S_{xyz}$
$R_{xx} \rightarrow S_{xxx}$ | (d') $R_{xy} \land x \neq y \rightarrow \exists zz.S_{xyz}$
(e) $R_{xy} \rightarrow \exists z.(S_{xz} \land S_{yz})$ | (e') $(R_{xy} \lor R_{yx}) \land x \leq y \rightarrow \exists zz.(S_{xz} \land S_{yz})$

Fig. 2. Examples of non-laconic schema mappings and their laconic equivalents.

4 Laconicity

A schema mapping is laconic if the canonical universal solution of a source instance coincides with the core universal solution. In particular, for laconic schema mappings the core universal solution can be computed using any method for computing canonical universal solutions, such as the ones described in Section 3. In this section, we discuss some examples and general observations concerning laconicity, in order to make the reader familiar with the notion. In the next section we will focus on constructing laconic schema mappings. In particular, we will show there that every schema mapping specified by FO\(<s-ttgds is logically equivalent to a laconic schema mapping specified by FO\(<\_s-ttgds.

Definition 2 (Laconicity). A schema mapping is laconic if for every source instance $I$, the canonical universal solution of $I$ with respect to $M$ is a core.

Note that the definition only makes sense for schema mappings specified by FO\(<s-ttgds, because we have defined the notion of a canonical universal solution only for such schema mappings.

Examples of laconic and non-laconic schema mappings are given in Figure 2. It is easy to see that every schema mapping specified by full s-t tgds only (i.e., s-t tgds without existential quantifiers) is laconic. Indeed, in this case, the canonical universal solution does not contain any nulls, and hence is guaranteed to be a core. Thus, being specified by full s-t tgds is a sufficient condition for laconicity, although a rather uninteresting one. The following provides us with a necessary condition, which explains why the schema mapping in Figure 2(a) is not laconic. Given an s-t tgd $\forall x.(\phi \rightarrow \exists yz.\psi)$, by the canonical instance of $\psi$, we will mean the target instance whose facts are the conjuncts of $\psi$, where the $x$ variables are treated as constants and the $y$ variables as nulls.

Proposition 3. If a schema mapping $(S, T, \Sigma_{st})$ specified by s-t tgds is laconic, then for each s-t tgd $\forall x.(\phi \rightarrow \exists yz.\psi) \in \Sigma_{st}$, the canonical instance of $\psi$ is a core.

Proof. We argue by contraposition. Suppose the canonical instance $J$ of $\psi$ is not a core. Let $J' \subseteq J$ be the core of $J$ and $h : J \rightarrow J'$ the corresponding retraction.

Take any source instance $I$ in which $\phi$ is satisfied under an assignment $g$, and let $K$ be the canonical universal solution of $I$. Since $\phi$ is true in $I$ under the assignment $g$ and by the construction of the canonical universal solution, we have that $g$ extends to a homomorphism $\hat{g} : J \rightarrow K$ sending the $y$ values to disjoint nulls. In fact, we may assume without loss of generality that $\hat{g}(y_i) = y_i$ for each $y_i \in y$. Moreover, by the construction of canonical universal solutions these null values will not play any further role in subsequent steps of the chase. In particular, they do not participate in any facts of $K$ other than those in the $\hat{g}$-image of $J$. By the $\hat{g}$-image of $J$ we mean the subinstance of $K$ containing those facts that are in the image of the homomorphism $\hat{g} : J \rightarrow K$. 


Finally, let $K'$ be the subinstance of $K$ in which the $\hat{g}$-image of $J$ is replaced by the $\hat{g}$-image of $J'$. Then $h : J \rightarrow J'$ naturally extends to a homomorphism $h' : K \rightarrow K'$. Since $K'$ is a proper subinstance of $K$, we conclude that $K$ is not a core, and therefore, $\mathcal{M}$ is not laconic.

In the case of schema mapping (e) in Figure 2, the linear order is used in order to obtain a logically equivalent laconic schema mapping $(e')$. Note that the schema mapping $(e')$ is order-invariant in the sense that the set of solutions of a source instance $I$ does not depend on the interpretation of the $<$ relation in $I$, as long as it is a linear order. Still, the use of the linear order cannot be avoided, as follows from Proposition 2. What is really going on, in this example, is that the right hand side of (e) has a non-trivial automorphism (viz. the map sending $x$ to $y$ and vice versa), and the conjunct $x \leq y$ in the antecedent of $(e')$ plays, intuitively, the role of a tie-breaker, cf. Section 5.3.

Testing whether a given schema mapping is laconic is not a tractable problem:

**Proposition 4.** Testing laconicity of schema mappings specified by FO s-t tgds is undecidable. It is NP-hard already for schema mappings specified by LAV s-t tgds.

*Proof.* The first claim is proved by a reduction from the satisfiability problem for first-order logic on finite instances, which is undecidable by Trakhtenbrot’s theorem. For any first-order formula $\phi(x)$, let $\mathcal{M}_\phi$ be the schema mapping containing only one dependency, namely $\forall x (\phi(x) \rightarrow \exists y_1 y_2 (P y_1 \land P y_2))$. It is easy to see that $\mathcal{M}_\phi$ is laconic iff $\phi$ is not satisfiable.

The NP-hardness in the case of LAV mappings is proved by a reduction from the core testing problem (given a graph, is it a core), which is known to be NP-complete [6]. Consider any graph $G = (V, E)$ and let $\exists y. \phi(y)$ be the Boolean canonical conjunctive query of $G$. Let $\mathcal{M}_G$ be the schema mapping whose only dependency is $\forall x. (Px \rightarrow \exists y. (\phi(y) \land \bigwedge_i R y_i))$. Then $\mathcal{M}_G$ is laconic iff $G$ is a core. □

5 Making schema mappings laconic

In this section, we present a procedure for transforming any schema mapping $\mathcal{M}$ specified by FO$^<$ s-t tgds into a logically equivalent laconic schema mapping $\mathcal{M}'$ specified by FO$^<$ s-t tgds. To simplify the notation, throughout this section, we assume a fixed input schema mapping $\mathcal{M} = (S, T, \Sigma_{st})$, with $\Sigma_{st}$ a finite set of FO$^<$ s-t tgds. Moreover, we will assume that the FO$^<$ s-t tgds $\forall x (\phi \rightarrow \exists y. \psi) \in \Sigma_{st}$ are non-decomposable [3], meaning that the fact graph of $\exists y. \phi(x, y)$ (where the facts are the conjuncts of $\phi$ and two facts are connected if they have an existentially quantified variable in common) is connected. This assumption is harmless: every FO$^<$ s-t tgd can be decomposed into a logically equivalent finite set of non-decomposable FO$^<$ s-t tgds (with identical left-hand-sides, one for each connected component of the fact graph) in polynomial time.

The outline of the procedure for making schema mappings laconic is as follows (the items correspond to subsections of the present section):

1. Construct a finite list “fact block types”: descriptions of potential fact blocks in core universal solutions.
2. Compute for each of the fact block types a precondition: a first-order formula over the source schema that tells exactly when the core universal solution will contain a fact block of the given type.
3. If any of the fact block types has non-trivial automorphisms, add an additional side condition, consisting of a Boolean combination of formulas of the form $x_i < x_j$, in order to avoid that multiple copies of the same fact block are created in the canonical universal solution.
4. Construct the new schema mapping $\mathcal{M}' = (S, T, \Sigma_{st}^{'})$, where $\Sigma_{st}^{'}$ contains an FO$^<$ s-t tgd for each of the fact block types. The left-hand-side of the FO$^<$ s-t tgd is the conjunction of the precondition and side condition of the respective fact block type, while the right-hand-side is the fact block type itself.
We illustrate the approach by means of an example. The technical notions that we use in discussing the example will be formally defined in the next subsections.

**Example 1.** Consider the schema mapping \( M = (\{P, Q\}, \{R_1, R_2\}, \Sigma_{st}) \), where \( \Sigma_{st} \) consists of the dependencies

\[
\begin{align*}
P x &\rightarrow \exists y. R_1 x y \\
Q x &\rightarrow \exists y z u. (R_2 x y \land R_2 z y \land R_1 z u)
\end{align*}
\]

In this case, there are exactly three relevant fact block types. They are listed below, together with their preconditions.

| Fact block type | Precondition |
|-----------------|--------------|
| \( t_1(x; y) = \{R_1 x y\} \) | \( pre_{t_1}(x) = P x \) |
| \( t_2(x; y z u) = \{R_2 x y, R_2 z y, R_1 z u\} \) | \( pre_{t_2}(x) = Q x \land \neg P x \) |
| \( t_3(x; y) = \{R_2 x y\} \) | \( pre_{t_3}(x) = Q x \land P x \) |

We use the notation \( t(x; y) \) for a fact block type to indicate that the variables \( x \) stand for constants and the variables \( y \) stand for distinct nulls.

As it happens, the above fact block types have no non-trivial automorphisms. Hence, no side conditions need to be added, and \( \Sigma_{st}' \) will consist of the following FO s-t tgd:

\[
\begin{align*}
P x &\rightarrow \exists y. R_1 x y \\
Q x \land \neg P x &\rightarrow \exists y z u. (R_2 x y \land R_2 z y \land R_1 z u) \\
Q x \land P x &\rightarrow \exists y. (R_2 x y)
\end{align*}
\]

The reader may verify that in this case, the obtained schema mapping is indeed laconic. We will prove in Section 5.4 that the output of our transformation is guaranteed to be a laconic schema mapping that is logically equivalent to the input schema mapping.

We will now proceed to define all the notions appearing in this example.

### 5.1 Generating the fact block types

Recall that the fact graph of an instance is the graph whose nodes are the facts of the instance, and such that there is an edge between two facts if they have a null value in common. A fact block, or \( f \)-block for short, is an instance of a connected component of the fact graph of the instance. We know from [2] that, for any schema mapping \( M \) specified by \( \text{FO}^\leq \) s-t tgd, the size of \( f \)-blocks in core universal solutions for \( M \) is bounded. Consequently, there is a finite number of \( f \)-block types, such that every core universal solution consists of \( f \)-blocks of these types. This is a crucial observation that we will exploit in our construction.

Formally, an \( f \)-block type \( t(x; y) \) will be a finite set of atomic formulas in \( x, y \), where \( x \) and \( y \) are disjoint sets of variables. We will refer to \( x \) as the constant variables of \( t \) and \( y \) as the null variables. We say that an \( f \)-block type \( t(x; y) \) is a renaming of an \( f \)-block type \( t'(x'; y') \) if there is a bijection \( f \) between \( x \) and \( x' \) and between \( y \) and \( y' \), such that \( t' = \{R(f(v)) \mid R(v) \in t\} \). In this case, we write \( f : t \cong t' \) and we call \( f \) also a renaming. We will not distinguish between \( f \)-block types that are renamings of each other. We say that an \( f \)-block \( B \) has type \( t(x; y) \) if \( B \) can be obtained from \( t(x; y) \) by replacing constant variables by constants and null variables to distinct nulls, i.e., if \( B = t(a, N) \) for some sequence of constants \( a \) and sequence of distinct nulls \( N \). Note that we require the relevant substitution to be injective on the null variables but not necessarily on the constant variables. If a target instance \( J \) contains a block \( B = t(a, N) \) of type \( t(x; y) \) then we say that \( t(x; y) \) is realized in \( J \) at \( a \). Note that, in general, an \( f \)-block type may be realized more than once at a tuple of constants \( a \), but this will not happen if the target instance \( J \) is a core universal solution.

We are interested in the \( f \)-block types that may be realized in core universal solutions. Eventually, the schema mapping \( M' \) that we will construct from \( M \) will contain an \( \text{FO}^\leq \) s-t tgd for each relevant
f-block type. Not every f-block type as defined above can be realized. We may restrict attention to a subclass. Below, by the canonical instance of an f-block type $t(x; y)$ we will mean the instance containing the facts in $t(x; y)$, considering $x$ as constants and $y$ as nulls.

**Definition 3.** The set $\text{Types}_M$ of f-block types generated by $M$ consists of all f-block types $t(x; y)$ satisfying the following conditions:

(a) $\Sigma_{st}$ contains an FO $s$-t tgd $\forall x'(\phi(x') \rightarrow \exists y' \psi(x', y'))$ with $y \subseteq y'$, and $t(x, y)$ is the set of conjuncts of $\psi$ in which the variables $y' - y$ do not occur;
(b) The canonical instance of $t(x, y)$ is a core;
(c) The fact graph of the canonical instance of $t(x; y)$ is connected.

If some f-block types generated by $M$ are renamings of each other, we add only one of them to $\text{Types}_M$.

The main result of this subsection is:

**Proposition 5.** Let $J$ be a core universal solution of a source instance $I$ with respect to $M$. Then each f-block of $J$ has type $t(x; y)$ for some $t(x; y) \in \text{Types}_M$.

**Proof.** Let $B$ be any f-block of $J$. Since $J$ is a core universal solution, it is, up to isomorphism, an induced subinstance of the canonical universal solution $J'$ of $I$. It follows that $J'$ must have an f-block $B'$ such that $B$ is the restriction of $B'$ to domain of $J$. Since $B'$ is a connect component of the fact graph of $J'$, it must have been created in a single step during the naive chase. In other words, there is an FO $s$-t tgd

$$\forall x(\phi(x) \rightarrow \exists y. \psi(x, y))$$

and an assignment $g$ of constants to the variables $x$ and distinct nulls to the variables $y$ such that $B'$ is contained in the set of conjuncts of $\psi(g(x), g(y))$. Moreover, since we assume the FO $s$-t tgd of $M$ to be non-decomposable and $B'$ is a a connected component of the fact graph of $J$, $B'$ must be exactly the set of facts listed in $\psi(g(x), g(y))$. In other words, if we let $t(x; y)$ be the set of all facts listed in $\psi$, then $B'$ has type $t(x; y)$. Finally, let $t'(x'; y') \subseteq t(x; y)$ be the set of all facts from $t(x; y)$ containing only variables $y_i$ for which $g(y_i)$ occurs in $B$. Since $B$ is the restriction of $B'$ to the domain of $J$, we have that $B$ is of type $t'(x'; y')$. Moreover, the fact graph of the canonical instance of $J$ is connected because $B$ is connected, and the canonical instance of $t'(x'; y')$ is a core, because, if it would not be, then $B$ would not be a core either, and hence $J$ would not be a core either, which would lead to a contradiction. It follows that $t'(x'; y') \in \text{Types}_M$. $\square$

Note that $\text{Types}_M$ contains only finitely many f-block types. Still, the number is in general exponential in the size of the schema mapping, as the following example shows.

**Example 2.** Consider the schema mapping specified by the following s-t tgds:

- $P_i x \rightarrow P_i' x$
- $Q x \rightarrow \exists y_0 y_1 \ldots y_k (Rxy_0 \land \bigwedge_{1 \leq i \leq k} (Ry_i y_0 \land P_i' y_i))$ (for each $1 \leq i \leq k$)

For each $S \subseteq \{1, \ldots, k\}$, the f-block type

$$t_S(x; (y_i)_{i \in S \cup \{0\}}) = \{Rxy_0\} \cup \{Ry_i y_0, P_i' y_i \mid i \in S\}$$

belong to $\text{Types}_M$. Indeed, each of these $2^k$ f-block types is realized in the core universal solution of some source instance. The example can be modified to use a fixed schemas: replace $P_i' x$ by $Sx_{x_1} \land Sx_{x_2} \land \ldots Sx_{-1} x_i \land Sx_i x_i$.

The same example can be used to show that the smallest logically equivalent schema mapping that is laconic can be exponentially longer.
5.2 Computing the precondition of an f-block type

Recall that, to simplify notation, we assume a fixed schema mapping $M$ specified by $\text{FO}^<\text{-s-t tgds}$. The main result of this subsection is the following, which shows that whether an f-block type is realized in the core universal solution at a given sequence of constants $a$ is something that can be tested by a first-order query on the source.

**Proposition 6.** For each $t(x; y) \in \text{TYPES}_M$ there is a $\text{FO}^<$ query $\text{precon}_t(x)$ such that for every source instance $I$ with core universal solution $J$, and for every tuple of constants $a$, the following are equivalent:

1. $a \in \text{precon}_t(I)$
2. $t(x; y)$ is realized in $J$ at $a$.

**Proof.** We first define an intermediate formula $\text{precon}_t(x)$ that almost satisfies the required properties, but not quite yet. For each f-block type $t(x; y)$, let $\text{precon}_t(x)$ be the following formula:

$$
certain_M(\exists y. \bigwedge t(x) \land \bigwedge_{i \neq j} \neg certain_M(\exists y_{-i}. \bigwedge t[y_i/y_j])(x) \land \bigwedge_i \neg certain_M(\exists y_{-i}. \bigwedge t[y_i/x'])(x, x')
$$

where $y_{-i}$ stands for the sequence $y$ with $y_i$ removed, and $t[u/v]$ is the result of replacing each occurrence of $u$ by $v$ in $t$. By construction, if $\text{precon}_t(a)$ holds in $I$, then every universal solution $J$ satisfies $t(a, N)$ for some some sequence of distinct nulls $N$. Still, it may not be the case that $t(x; y)$ is realized at $a$, since it may be that that $t(a, N)$ is part of a bigger f-block. To make things more precise, we introduce the notion of an embedding. For any two f-block types, $t(x; y)$ and $t'(x'; y')$, an embedding of the first into the second is a function $h$ mapping $x$ into $x'$ and mapping $y$ injectively into $y'$, such that whenever $t$ contains an atomic formula $R(z)$, then $R(h(z))$ belongs to of $t'$. The embedding $h$ is strict if $t'$ contains an atomic formula that is not of the form $R(h(z))$ for any $R(z) \in t$. Intuitively, the existence of a strict embedding means that $t'$ describes an f-block that properly contains the f-block described by $t$.

Let $J$ be any source instance, $J$ any core universal solution, $t(x; y) \in \text{TYPES}_M$, and $a$ a sequence of constants.

**Claim 1:** If $t$ is realized in $J$ at $a$, then $a \in \text{precon}_t(I)$.

**Proof of claim:** Clearly, since $t$ is realized in $J$ at $a$ and $J$ is a universal solution, the first conjunct of $\text{precon}_t$ is satisfied. That the rest of the query is satisfied is also easily seen: otherwise $J$ would not be a core. 

End of proof of claim.

**Claim 2:** If $a \in \text{precon}_t(I)$, then either $t$ is realized in $J$ at $a$ or some f-block type $t'(x'; y') \in \text{TYPES}_M$ is realized at a tuple of constants $a'$, and there is a strict embedding $h : t \rightarrow t'$ such that $a_i = a'_i$ whenever $h(x_i) = x'_j$.

**Proof of claim:** It follows from the construction of $\text{precon}_t$, and the definition of $\text{TYPES}_M$ types, that the witnessing assignment for its truth must send all existential variables to distinct nulls, which belong to the same block. By Proposition 5, the diagram of this block is a specialization of an f-block type $t' \in \text{TYPES}_M$. It follows that $t$ is embedded in $t'$ and $a$, together with possible some additional values in $\text{Cons}$, realize $t'$. 

End of proof of claim.

We now define $\text{precon}_t(x)$ to be the following formula:

$$
\text{precon}_t(x) \land \bigwedge_{t'(x'; y') \in \text{TYPES}_M} \neg \exists x'. \left( \bigwedge h(x) = h(x_i) \land \text{precon}_{t'}(x') \right)
$$
This formula satisfies the required conditions: $a \in \text{precon}_t(I)$ iff $t(x;y)$ is realized in $J$ at $a$. The left-to-right direction follows from Claim 2, while the right-to-left direction follows from Claim 1 together with the fact that $J$ is a core. □

5.3 Computing the side conditions of an f-block type

The issue we address in this subsection, namely that of non-rigid f-block types, is best explained by an example.

Example 3. Consider again schema mapping (e) in Figure 2. This schema mapping is not laconic, because, when a source instance contains $Rab$ and $Rba$, for distinct values $a, b$, the canonical universal solutions will contain two null values $N$, each satisfying $SaN$ and $SbN$, corresponding to the two assignments \( \{ x \mapsto a, y \mapsto b \} \) and \( \{ x \mapsto b, y \mapsto a \} \). The essence of the problem is in the fact that the right-hand-side of the dependency is, in some sense, symmetric: it is a non-trivial renaming of itself, the renaming in question being \( \{ x \mapsto y, y \mapsto x \} \). According to the terminology that we will introduce below, the right-hand-side of this dependency is non-rigid. Schema mapping (e') from Figure 2 does not suffer from this problem, because it contains $x \leq y$ in the antecedent, and we are assuming $< y$ to be a linear order on the values in the source instance.

In order to formalize the intuition exhibited in the above example, we need to introduce some terminology. We say that two f-blocks, $B, B'$, are copies of each other, if there is a bijection $f$ from $\text{Cons}$ to $\text{Cons}$ and from $\text{Nulls}$ to $\text{Nulls}$ such that $f(a) = a$ for all $a \in \text{Cons}$ and $B' = \{ R(f(v_1), \ldots, f(v_k)) \mid R(v_1, \ldots, v_k) \in B \}$. In other words, $B'$ can be obtained from $B$ by renaming null values.

Definition 4. An f-block type $t(x;y)$ is rigid if for any two sequences of constants $a, a'$ and for any two sequences of distinct nulls $N, N'$, if $t(a;N)$ and $t(a';N')$ are copies of each other, then $a = a'$.

The s-t tgd from the above example is easily seen to be non-rigid. Moreover, a simple variation of the argument in the above example shows:

Proposition 7. If an f-block type $t(x;y)$ is non-rigid, then the schema mapping specified by the FO (in fact LAV) s-t tgd $\forall x (R(x) \rightarrow \exists y, \land t(x;y))$ is not laconic.

In other words, if an f-block type is non-rigid, one cannot simply use it as the right-hand-side of an s-t tgd without running the risk of non-laconicity. Fortunately, it turns out that f-block types can be made rigid by the addition of suitable side conditions. By a side condition $\Phi(x)$ we will mean a Boolean combination of formulas of the form $x_i < x_j$ or $x_i = x_j$.

Definition 5. An f-block type $t(x;y)$ is rigid relative to a side condition $\Phi(x)$ if for any two sequences of constants $a, a'$ satisfying $\Phi(a)$ and $\Phi(a')$ and for any two sequences of distinct nulls $N, N'$, if $t(a;N)$ and $t(a';N')$ are copies of each other, then $a = a'$.

Definition 6. A side-condition $\Phi(x)$ is safe for an f-block type $t(x;y)$ if for every f-block $t(a;N)$ of type $t$ there is a f-block $t(a',N')$ of type $t$ satisfying $\Phi(a')$ such that the two are copies of each other.

Intuitively, safety means that the side condition is not too strong: whenever a f-block type should be realized in a core universal solution, there will be at least one way of arranging the variables so that the side condition is satisfied. The main result of this subsection, which will be put to use in the next subsection, is the following:

Proposition 8. For every f-block type $t(x;y)$ there is a side condition $\text{sidecon}_t(x)$ such that $t(x;y)$ is rigid relative to $\text{sidecon}_t(x)$, and $\text{sidecon}_t(x)$ is safe for $t(x;y)$. 

Proof. We will construct a sequence of side conditions $\Phi_i(x)$ for $t(x; y)$, such that each $\Phi_{i+1}$ logically strictly implies $\Phi_i$, and such that $t(x; y)$ is rigid relative to $\Phi_n(x)$. Note that $n$ is necessarily bounded by a single exponential function in $|x|$. For $\Phi_0(x)$ we pick the tautology $\top$, which is trivially safe for $t(x; y)$.

Suppose that $t(x; y)$ is not rigid relative to $\Phi_i(x)$, for some $i \geq 0$. By definition, this means that there is two sequences of constants $a, a'$ satisfying $\Phi_i(a)$ and $\Phi_i(a')$ and two sequences of distinct nulls $N, N'$, such that $t(a; N)$ and $t(a'; N')$ are copies of each other, but $a$ and $a'$ are not the same sequence, i.e., they differ in some coordinate. Let $\psi(x)$ be the conjunction of all formulas of the form $x_i < x_j$ or $x_i = x_j$ that are true under the assignment sending $x$ to $a$, and let $\Phi_{i+1}(x) = \Phi_i(x) \land \neg \psi(x)$. It is clear that $\Phi_{i+1}$ is strictly stronger than $\Phi_i$. Moreover, we $\Phi_{i+1}$ is still safe for $t(x; y)$: consider any $f$-block $t(b, M)$ of type $t(x; y)$. Since $\Phi_i$ is safe for $t$, we can find a $f$-block $t(b', M')$ of type $t$ such that $\Phi_i(b')$ the two blocks are copies of each other. If $\neg \psi(b')$ holds, then in fact $\Phi_{i+1}(b')$ holds, and we are done. Otherwise, we have that $t(b', M')$ is isomorphic to $t(a, N)$ and the preimage of $t(a', N')$ under this isomorphism will be again a copy of $t(b', M')$ (and therefore also of $t(b, M)$) that satisfies $\Phi_i(b') \land \neg \psi(b')$, i.e., $\Phi_{i+1}(b')$.

Incidentally, we believe the above construction of side-conditions is not the most efficient possible, in terms of the size of the side-condition obtained. It can probably be improved.

5.4 Putting things together: constructing the laconic schema mapping

Theorem 7. For each schema mapping $\mathcal{M}$ specified by $\text{FO}$ $s$-$t$ tgds, there is laconic schema mapping $\mathcal{M}'$ specified by $\text{FO}$ $s$-$t$ tgds that is logically equivalent to $\mathcal{M}$.

Proof. We define $\mathcal{M}'$ to consist of the following $\text{FO}$ $s$-$t$ tgds. For each $t(x; y) \in \text{Types}_\mathcal{M}$, we take the $\text{FO}$ $s$-$t$ tgd

$$\forall x(\text{precon}_t(x) \land \text{sidecon}_t(x) \rightarrow \exists y. \bigwedge t(x; y))$$

In order to show that $\mathcal{M}'$ is laconic and logically equivalent to $\mathcal{M}$ (on structures where $<$ denotes a linear order), it is enough to show that, for every source instance $I$, the canonical universal solution $J$ of $I$ with respect to $\mathcal{M}'$ is a core universal solution for $I$ with respect to $\mathcal{M}$. This follows from the following three facts:

1. Every $f$-block of $J$ is a copy of an $f$-block of the core universal solution of $I$. This follows from Proposition 6.
2. Every $f$-block of the core universal solution of $I$ is a copy of an $f$-block of $J$. This follows from Proposition 5 and Proposition 6, together with the safety part of Proposition 8.
3. No two distinct $f$-blocks of $J$ are copies of each other. This follows from the rigidity part of Proposition 8 together with the fact that $\text{Types}_\mathcal{M}$ contains no two distinct $f$-block type that are renamings of each other.

Incidentally, if the side conditions are left out, then the resulting schema mapping is still logically equivalent to the original mapping $\mathcal{M}$, but it may not be laconic. It will still satisfy a weak form of laconicity: a variant of the chase defined in [1], which only fires dependencies whose right hand side is not yet satisfied, will produce the core universal solution.

6 Target constraints

In this section we consider schema mappings with target constraints and we address the question whether our main result can be extended to this setting. The answer will be negative. However, first we need to revisit our basic notions, as some subtle issues arise in the case with target dependencies.
It is clear that we cannot expect to compute core universal solutions for schema mappings with target dependencies by means of FO<term>interchange interpretations. Even for the simple schema mapping defined by the s-t tgd \( Rxy \rightarrow R'xy \) and the full target tgd \( R'xy \land R'yz \rightarrow R'xz \) computing the core universal solution means computing the transitive closure of \( R \), which we know cannot be done in FO logic even on finite ordered structures. Still, we can define a notion of laconicity for schema mappings with target dependencies. Let \( M \) be any schema mapping specified by a finite set of FO<term>s-t tgds \( \Sigma_{st} \) and a finite set of target tgds and target egds \( \Sigma_t \), and let \( I \) be a source instance. We define the canonical universal solution of \( I \) with respect to \( M \) as the target instance (if it exists) obtained by taking the canonical universal solution of \( I \) with respect to \( \Sigma_{st} \) and chasing it with the target dependencies \( \Sigma_t \). We assume a standard chase but will not make any assumptions on the chase order. Laconicity is now defined as before: a schema mapping is laconic if for each source instance, the canonical universal solution coincides with the core universal solution.

Recall that, according our main result, we have (i) every schema mapping \( M \) specified by FO<term>s-t tgds is logically equivalent to a laconic schema mapping \( M' \) specified by FO<term>s-t tgds. In particular, this implies that, (ii) for each source instance \( I \), the core universal solution for \( I \) with respect to \( M \) is the canonical universal solution for \( I \) with respect to \( M' \). For the implication from (i) to (ii) the requirement of logical equivalence turns out to be stronger than needed: it is enough that \( M \) and \( M' \) are CQ-equivalent, i.e., have the same core universal solution (possibly undefined) for each source instance [2]. While CQ-equivalence and logical equivalence coincide for schema mappings specified by FO<term>s-t tgds (as follows from the closure under target homomorphisms), the first is strictly weaker than the second in the case with target dependencies [2].

**Theorem 8.** There is a schema mapping \( M \) specified by finitely many LAV s-t tgds and full target tgds, for which there is no CQ-equivalent laconic schema mapping \( M' \) specified of FO<term>s-t tgds, target tgds and target egds.

**Proof.** (sketch) Let \( M \) be the schema mapping specified by the LAV s-t tgds

- \( Rx_1x_2 \rightarrow R'x_1x_2 \)
- \( P_ix \rightarrow \exists y.Q_iy \) for \( i \in \{1, 2, 3\} \).

and the full target tgds

- \( R'xy \land R'yz \rightarrow R'xz \)
- \( R'xx \land P_3y \rightarrow P_3y \)
- \( R'xx \land P_3y \rightarrow P_3y \)

For source instances \( I \) in which the relations \( R, P_1, P_2, P_3 \) are non-empty, the core universal solution \( J \) will have the following shape: \( J(R') \) is the transitive closure of \( I(R) \), and \( J(Q_1), J(Q_2), J(Q_3) \) are non-empty. Moreover, if \( I(R) \) contains a cycle, then \( J(Q_1) = \{N_1\}, J(Q_2) = \{N_2\} \) and \( J(Q_3) = \{N_1, N_2\} \) for distinct null values \( N_1, N_2 \), while if \( I(R) \) is acyclic, \( J(Q_1), J(Q_2) \) and \( J(Q_3) \) are disjoint singleton sets of nulls.

Suppose for the sake of contradiction that there is a CQ-equivalent laconic schema mapping \( M' \) specified by a finite set of FO<term>s-t tgds \( \Sigma_{st} \) and a finite set of target tgds and egds \( \Sigma_t \). In particular, for each source instance \( I \), the canonical universal solution of \( I \) with respect to \( M' \) is the core universal solution of \( I \) with respect to \( M \). Let \( n \) be the maximum quantifier rank of the formulas in \( \Sigma_{st} \).

**Claim 1:** There is a source instance \( I_1 \) containing a cycle, such that the canonical universal solution \( J_1 \) of \( I_1 \) with respect to \( \Sigma_{st} \) contains at least three nulls, one belonging only to \( Q_1 \), one belonging only to \( Q_2 \), and one belonging only to \( Q_3 \).

The proof of Claim 1 is based on the fact that acyclicity is not first-order definable on finite ordered structures: take any two sources instances \( I_1, I_2 \) agreeing on all FO<term>s-sentences of quantifier rank \( n \) such that \( I_1 \) contains a cycle and \( I_2 \) does not. We may even assume that \( P_1, P_2, P_3 \) are non-empty in both instances.
Let \( J_1 \) and \( J_2 \) be the canonical universal solutions of \( I_1 \) and \( I_2 \) with respect to \( \Sigma_{st} \). Then \( J_2 \) must contain at least three nulls, one belonging only to \( Q_1 \), one belonging only to \( Q_2 \) and one belonging only to \( Q_3 \). To see this, note that, first of all, \( J_2 \) must be a homomorphic pre-image of the core universal solution of \( I_2 \) with respect to \( \mathcal{M} \). Secondly, if one of the relations \( Q_1 \) a non-empty in \( J_2 \), then the crucial information that \( I_2(P_i) \) is non-empty is lost, in the sense that \( J_2 \) would be a homomorphic pre-image of the source instance that is like \( I_2 \) except that the relation \( P_i \) is empty, which implies that, the result of chasing \( J_2 \) with \( \Sigma_t \) must be homomorphically contained in the core universal solution of this modified source instance with respect to \( \mathcal{M} \), which is different from the core universal solution of \( I_2 \).

This shows that \( J_2 \) must contain at least three nulls, one belonging only to \( Q_1 \), one belonging only to \( Q_2 \) and one belonging only to \( Q_3 \). Each of these nulls must have been created by the application of a dependency from \( \Sigma_{st} \). Since \( I_1 \) and \( I_2 \) agree on all \( \text{FO}^\leq \) -sentences of quantifier rank \( n \), the left-hand-side of this dependency is also satisfied in \( I_1 \), and hence the same null is also created in the canonical universal solution of \( I_1 \).

**Claim 2:** Let \( J'_2 \) be result of chasing \( J_2 \) with \( \Sigma_t \) (assuming it exists). Then \( J'_2 \) cannot be the core universal solution of \( I_1 \) with respect to \( \mathcal{M} \).

The proof of Claim 2 is based on a monotonicity argument. More precisely, we use the fact that the left-hand-side of each target dependency is a conjunctive query, and hence is preserved under homomorphisms. Let us assume for the sake of contradiction that \( J'_1 \) is the core universal solution of \( I_1 \) with respect to \( \mathcal{M} \), which contains exactly two null values, one in \( Q_1 \cap Q_3 \) and one in \( Q_2 \cap Q_3 \). Let \( N_1, N_2, N_3 \) be null values belonging only to \( J_1(P_i) \), only to \( J_1(Q_2) \) and only to \( J_1(Q_3) \), respectively. It is easy to see that, during the chase with \( \Sigma_t \), \( N_3 \) must have been identified with \( N_1 \) or \( N_2 \) by means of a target egd \( \phi \). A monotonicity argument shows that the same target egd \( \phi \) can be used to identify the two null values in the core universal solution \( J'_1 \) (note that the target dependencies cannot refer to the linear order on the constants). This contradicts the fact that \( J'_1 \) is the end-result of the chase with \( \Sigma_t \).

We expect that similar arguments can be used to find a schema mapping \( \mathcal{M} \) specified by a finite set of LAV s-t tgds and target egds, such that there is no CQ-equivalent laconic schema mapping specified by a finite set of \( \text{FO}^\leq \) s-t tgds, target tgds and target egds.

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