The intercept of symmetric multigluon configurations in the variational approach

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Abstract

To calculate the intercept of the multigluon system in a symmetric spatial configuration a variational method is developed based on a complete system of one-gluon functions. The method is applied to two- and three-gluon cases to compare with the known results. The convergence turns out rather slow. Ways to improve results are discussed.
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1 Introduction

Much attention has recently been devoted to the perturbative "hard", or BFKL, pomeron [1], especially in relation to the study of the small $x$ behaviour of the deep inelastic scattering structure functions (see a recent review in [2]). In application to soft phenomena, the value of the pomeron intercept is of principal importance. For the BFKL pomeron it is considerably above unity:

$$\alpha_{BFKL}(0) = 1 - (3\alpha_s/\pi)E_0,$$

where the "energy" $E_0$ is equal $-4\ln 2$, and $\alpha_s$ is the (fixed) QCD coupling constant [1]. However the BFKL pomeron is only the simplest of the family of pomerons (with positive signature) and odderons (with negative signature) formed in a system of $n$ interacting reggeized gluons. Should some of them also result supercritical (with the intercept above unity), the study of the high energy behaviour of the QCD cross-sections would require summation of an arbitrary number of exchanges of all these supercritical pomerons and odderons.

It is extraordinarily difficult to obtain an explicit solution or an exact energy value for $n > 2$ interacting gluons (see, however, some ideas in [3]). For $n = 3$ (odderon) application of conformal symmetry allows to reduce calculations to an one dimensional problem [4]. Variational treatment with a simple trial wave function then gives an intercept also above unity, although lower than for the pomeron [5]:

$$\alpha_{odd}(0) = 1 - (3\alpha_s/\pi)E_{odd}, \quad E_{odd} < -0.37.$$  

For larger number $n$ of pomerons different crude approximations lead to wildly different estimates for the intercept. Developing the interaction in powers of the conformal group Casimir operator an intercept linearly rising with $n$ is obtained [6]. On the other hand, the Hartree-Fock approximation, presumably valid for large $n$, gives intercepts less than one and linearly falling with $n$ [7]. In this approximation the energy is $\epsilon$ rising linearly with $n$:

$$E_n = (n/2) \times 0.959...$$

Evidently negative energies, as witnessed by the pomeron and odderon, result from correlations. In absence of explicit solutions for $n > 2$ gluons and in view of great technical difficulties in working with conformally invariant wave functions, evident already for the odderon case, the only realistic approach for $n > 3$ gluons seems to be a direct variational one, based on some complete and simple basis of functions. Taking a finite number $N$ of these, one then computes a finite energy matrix. The intercept (minus one) is found as its smallest eigenvalue (ground state) with an opposite sign. With $N$ growing, the ground state energy goes down, so that one always obtains its upper limit. As $N \to \infty$ one is sure to
obtain an exact value, provided the basic functions form a complete set. One can have some idea of the precision for a given $N$ by calculating the energy for the two-gluon case where its exact value is known.

In this paper we give an outline of such a variational method aimed at calculating of the intercept of the system of an arbitrary number of reggeized gluons in a symmetric configuration both in the ordinary space and in the colour space. The basic functions were chosen as harmonic oscillator functions in the variable $z = \ln r^2$ (radials) multiplied by azimuthal functions for given angular momenta $l = 0, \pm 1, \pm 2, \ldots$. They retain some of the conformal invariance corresponding to the substitution $r \to 1/r$. To see the convergence, we studied the two-gluon (the BFKL pomeron) and three-gluon (odderon) cases, where the exact energy value for the former and its upper limit for the latter are known. It turns out that the convergence is rather slow. In the present series of calculations up to 201 basic states have been included for the pomeron and up to 1335 basic states for the odderon. With these basic states the resulting upper limit achieved for the two-gluon ground state energy (the BFKL pomeron) is $-1.032$, which means $\sim 53\%$ of the exact correlation energy. The upper limit obtained for the odderon ground state energy is $+0.331$. This limit lies considerably above the one obtained in [5] with a conformally invariant ansatz. From the known BFKL intercept we can study the dependence of the calculated energy on the number of basic states taken into account. Applying a similar fit to the odderon case and extrapolating to infinite number of states included gives an estimate

$$E_{\text{odd}} < -0.3 \div -0.6,$$

in agreement with [5].

2 Variational calculation of the ground state energy for $n$ reggeized gluons

As shown in [8] the transverse space and colour wave function $\psi$ of $n$ reggeized gluons in a colourless state satisfies a Schrödinger-like equation

$$H\psi = E \prod_{i=1}^{n} p_i^2 \psi, \quad (1)$$

where $p_i$ is the momentum of the $i$-th gluon. The Hamiltonian $H$ is given by a sum of pair terms

$$H = -(1/6) \sum_{i<k} T_i T_k H_{ik}. \quad (2)$$
Here $T_i$ is the colour vector of the $i$-th gluon. In a colourless state
\[\sum_{i=1}^{n} T_i = 0. \tag{3}\]

The pair Hamiltonian $H_{ik}$ acts on the wave function according to
\[H_{ik}\psi = \prod_{j=1}^{n} p_j^2 (\ln p_i^2 p_k^2 + 4C)\psi + \prod_{j=1, j\neq i,k}^{n} p_j^2 (p_i^2 \ln(r_{ik}^2/4) p_k^2 + (i \leftrightarrow k))\psi + 2(p_i + p_k)^2 \psi (r_{ik} = 0), \tag{4}\]
where $r_{ik} = r_i - r_k$ is the (transversal) distance between the gluons; $C$ is the Euler constant.

The energy eigenvalue $E$ is related to the intercept by
\[\alpha(0) = 1 - (3\alpha_s/\pi)E, \tag{5}\]
so that the rightmost singularity in the complex angular momentum plane corresponds to the ground state. As mentioned, for $n = 2$ the solution of (1) is the BFKL pomeron with
\[E_0 = -4 \ln 2, \quad \alpha_{BFKL}(0) = 1 + (\alpha_s/\pi)12 \ln 2. \tag{6}\]

No explicit solution has been found for $n > 2$.

The solution of (1) may evidently be found by a variational approach, searching the minimum value of the functional
\[\Phi = \int \prod d^2p_i \psi^* H\psi \equiv \sum_{i<k} \Phi_{ik}, \tag{7}\]
with the normalization condition
\[\int \prod d^2p_i \psi^* \prod_{i=1}^{n} p_i^2 \psi = 1. \tag{8}\]

The BFKL pomeron and odderon states are symmetric both in colour and space variables. For more gluons solutions do not presumably possess such simple symmetry properties, since Eq. (1) mixes colour and space variables. Among various solutions there certainly are those which correspond to a number of BFKL pomerons weakly interacting with each other (multipomeron cuts in the old Regge-pomeron theory). These solutions are basically symmetric neither in space nor in colour variables. We are interested in solutions of a different type, which do not reduce to BFKL pomerons and represent new pomerons with a higher intrinsic colour. While exact symmetry properties of these new states are not evident, one expects that in the variational approach they can well be approximated by states symmetric both in colour and space variables, similar to the simplest BFKL pomeron and odderon states. In any case, the energy value obtained with such a choice will give an upper limit for the exact one. According to the colour cluster separation property discussed in [7], these new pomerons
(and odderons) will also appear as subsystems for still larger number of gluons, that is, will take part in multipomeron exchanges. Such a picture was phenomenologically introduced in a model of colour string fusion [9], where it was supposed that supercritical pomerons exist for arbitrary high intrinsic colour.

As noticed in [4], for a wave function symmetric both in colour and space variables, instead of the full functional \( \Phi \) one can use any of its pair terms \( \Phi_{ik} \) in (7) with \( i \) and \( k \) fixed. Moreover, for a symmetric colour wave function,

\[
<T_iT_k> = (2/n(n-1)) \sum_{i<k} T_iT_k = -3/(n-1),
\]

(9)

where (3) and \( T^2 = 3 \) have been used. As a result, the energy \( E \) may be found via the minimal value of the functional in only spatial variables

\[
\mathcal{E} = (1/2) \int \prod_{i=1}^{n} d^2p_i \psi^* H_{12} \psi, \tag{10}
\]

where the Hamiltonian \( H_{12} \) is defined by (4) with \( ik = 12 \) and the function \( \psi \) should satisfy (8). The energy of the whole system of \( n \) gluons is determined by the minimal value \( \epsilon \) of \( \mathcal{E} \) according to

\[
E_n = (n/2)\epsilon_n. \tag{11}
\]

Actually the operator in \( \mathcal{E} \) does not depend on \( n \). The dependence on \( n \) enters only from extra arguments in \( \psi \) through the requirement of the symmetry in all arguments. The space of trial wave functions then gets smaller with rising \( n \), wherefrom one obtains

\[
\epsilon_{n+1} \geq \epsilon_n \tag{12}
\]

and in particular \( \epsilon_n \geq \epsilon_2 \). This gives a crude lower limit on the energy and an upper one on the intercept for \( n \) gluons in a symmetric state

\[
E_n \geq (n/2)E_0. \tag{13}
\]

This limit was obtained in [4] for the odderon \( n = 3 \).

For a large number of reggeized gluons one can hopefully apply the Hartree-Fock approximation and seek for the minimum of the functional \( \mathcal{E} \) on functions \( \psi \) which factorize into a product of individual gluon wave functions. For the ground state, with \( \psi \) symmetric in all gluons, all individual gluon wave functions should evidently be the same. As calculated in [7], the obtained minimal Hartree-Fock value is \( \epsilon = 0.959... \), so that the Hartree-Fock energy of the symmetric \( n \) gluon state is positive and grows linearly with \( n \), as stated in the Introduction. It means that there is little hope to expect supercritical pomerons composed of a large number of gluons. It also means that supercritical pomerons are formed exclusively due to correlations in the wave function.
For a finite number \( n \) of gluons one can expand the symmetric spatial wave function in a sum of products of individual gluon functions:

\[
\psi(r_1, \ldots, r_n) = \sum_{\alpha_1, \ldots, \alpha_n} c_{\alpha_1, \ldots, \alpha_n} \prod_{i=1}^n \psi_{\alpha_i}(r_i),
\]

where the one-gluon functions \( \psi_{\alpha_i}(r_i) \) form a discrete complete set and are orthonormalized according to (8):

\[
\int d^2r \psi_{\alpha}^* \psi_{\alpha'} = \delta_{\alpha,\alpha'}.
\]

The coefficients \( c_{\alpha_1, \ldots, \alpha_n} \) have to be symmetric in all \( \alpha \)'s by the requirement of the Bose symmetry and normalized according to

\[
\sum_{\alpha_1, \ldots, \alpha_n} |c_{\alpha_1, \ldots, \alpha_n}|^2 = 1
\]

The two-gluon Hamiltonian \( H_{12} \) acts nontrivially only on the wave functions for the gluons number one and two. So the energy functional becomes

\[
\mathcal{E} = \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} c_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}^* c_{\alpha_1', \alpha_2', \alpha_3, \alpha_4} \mathcal{E}_{\alpha_1, \alpha_2, \alpha_1', \alpha_2'},
\]

where the matrix \( \mathcal{E}_{\alpha_1, \alpha_2, \alpha_1', \alpha_2'} \) is the two-gluon energy in the basis formed by functions \( \psi_{\alpha} \). With this matrix known, the problem of minimization of the functional \( \mathcal{E} \) reduces to finding the minimal value of a quadratic form, that is, the minimal eigenvalue of the matrix \( \mathcal{E}_{\alpha_1, \alpha_2, \alpha_1', \alpha_2'} \) considered as a matrix in independent initial and final \( n \)-gluon states. The latter means that this matrix should be multiplied by unity matrices for the rest of the gluons and then symmetrized in all initial and final gluons. The procedure is quite straightforward, once the basic functions \( \psi_{\alpha} \) are chosen. It however involves a numerical evaluation of the energy matrix elements and a diagonalization of the matrix, whose dimension is rapidly growing with the number of gluons and the basic functions taken into account.

3 Two-gluon energy matrix for given angular momenta

The first task in the calculation of the energy matrix is to separate radial and angular dependence. The basic functions depend on the azymuthal angle \( \phi \) trivially:

\[
\psi_{\alpha}(r) = \psi_{k,l}(r) \exp il\phi,
\]

where \( l = 0, \pm 1, \pm 2, \ldots \) and \( k = 0, 1, 2, \ldots \) enumerates the radial functions. Thus \( \alpha = \{k, l\} \) is in fact a pair of indices. In the following, instead of \( r \), we shall use the variable \( z = \ln r^2 \) in most cases. In terms of \( z \) and \( \phi \)

\[
p^2 = -(4/r^2)(\partial_z^2 + (1/4)\partial_{\phi}^2),
\]
so that acting on the function $\psi_{k,l}(r)$

$$p^2\psi_{k,l}(r) = -(4/r^2)(\partial^2 - (1/4)l^2)\psi_{k,l}(r). \quad (20)$$

Wave functions with different values of the angular momentum are automatically orthogonal. For coinciding $l$ the normalization condition for the radial functions takes the form

$$\int dz\psi_{k,l}^*(z)(-\partial^2 + (1/4)l^2)\psi_{k',l}(z) = (1/4\pi)\delta_{kk'}. \quad (21)$$

It reduces to the standard form for functions

$$\xi_{k,l}(z) = (\partial + |l|)\psi_{k,l}(z), \quad (22)$$

which evidently satisfy

$$\int dz\xi_{k,l}^*(z)\xi_{k',l}(z) = (1/4\pi)\delta_{kk'}. \quad (23)$$

In the following we assume that the radial functions are chosen to be real.

With the angular dependence of the wave function explicitly given by (18), one can do the azimuthal integrals in the potential energy in a straightforward manner. Let $\alpha_i = \{k_i, l_i\}$ and take the transition between two gluon states $\alpha_1, \alpha_2 \rightarrow \alpha_3, \alpha_4$. Evidently the total angular momentum is conserved so that the energy matrix elements are zero unless $l_1 + l_2 = l_3 + l_4$.

According to (4) the potential energy consists of two parts, the first part $U$ given by an essentially Coulomb interaction between the gluons and the second one $Q$ given by a contact interaction, proportional to their total momentum squared. Let us begin with the Coulomb part $U$. Its two terms evidently give the same contribution due to the symmetry under the interchange of gluons 1 and 2. So we can take only one of them and drop the factor 1/2.

Denote

$$\eta_{k,l}(z) = (\partial^2 - (1/4)l^2)\psi_{k,l}(z). \quad (24)$$

Then after doing the azimuthal integration we obtain the following matrix elements for the potential energy $U$

$$U_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 16\pi^2 \int dz_1 dz_2 \eta_{\alpha_1}(z_1)\psi_{\alpha_3}(z_1)\psi_{\alpha_2}(z_2)\eta_{\alpha_4}(z_2)U_l(z_1, z_2), \quad (25)$$

where $l = |l_1 - l_3| = |l_2 - l_4|$ is the angular momentum transfer and the function $U_l(z_1, z_2)$ is given by

$$U_l = -(1/l)\exp\left(-l/2)(|z_1 - z_2|\right), \quad l \neq 0, \quad (26)$$

and

$$U_0 = \max\{z_1, z_2\}. \quad (27)$$
The contact part $Q$ involves gluonic wave functions taken at the same point, that is, with the same $r$ and $\phi$. After performing the azymuthal integration and integrating once by parts in the variable $z$ we obtain

$$Q_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 8\pi^2 \int dz((\partial + (1/2)|l_1 + l_2|)\psi_{\alpha_1}\psi_{\alpha_2})((\partial + (1/2)|l_3 + l_4|)\psi_{\alpha_3}\psi_{\alpha_4}). \quad (28)$$

One can somewhat simplify this expression by noting that

$$(\partial + (1/2)|l_1 + l_2|)\psi_{\alpha_1}\psi_{\alpha_2} = \xi_{\alpha_1}\psi_{\alpha_2} + \psi_{\alpha_1}\xi_{\alpha_2} + 2\Delta_{12}\psi_{\alpha_1}\psi_{\alpha_2}, \quad (29)$$

where $2\Delta_{12} = |l_1 + l_2| - |l_1| - |l_2|$ and similarly for the second factor in (28). Then finally

$$Q_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 8\pi^2 \int dz(\xi_{\alpha_1}\psi_{\alpha_2} + \psi_{\alpha_1}\xi_{\alpha_2} + 2\Delta_{12}\psi_{\alpha_1}\psi_{\alpha_2})(\xi_{\alpha_3}\psi_{\alpha_4} + \psi_{\alpha_3}\xi_{\alpha_4} + 2\Delta_{34}\psi_{\alpha_3}\psi_{\alpha_4}). \quad (30)$$

Passing to the kinetic energy given by the first terms in (4) we also note that the two terms give the same contribution so that we can take one of them (say, for the gluon 1) and drop the factor 1/2. For the transition of the first gluon $\alpha_1 \rightarrow \alpha_3$ the conservation of angular momentum requires that $l_1 = l_3$. The kinetic energy is easily calculated in the momentum space. So we transform the basic functions to the momentum space according to

$$\psi_{\alpha}(p) = \int (d^2 r / 2\pi)\psi_{\alpha}(r) \exp(-i pr). \quad (31)$$

The azymuthal integration leads to

$$\psi_{\alpha}(p) = (-i)^l \exp i l \phi \int r dr \psi_{k,l}(z) J_l(pr), \quad (32)$$

where $J_l$ is the Bessel function. To do the integral over $r$ it is convenient to introduce a Fourier transform of the function $\psi$ with respect to the variable $z$:

$$\psi_{k,l}(z) = \int (d\nu / \sqrt{2\pi}) \phi_{k,l}(\nu) \exp i \nu z. \quad (33)$$

Putting this representation in (33) and doing the $r$-integration we obtain

$$\psi_{k,l}(p) = (2/p^2) \exp i l \phi \int (d\nu / \sqrt{2\pi}) f_{k,l}(\nu) p^{-2i\nu}, \quad (34)$$

with

$$f_{k,l}(\nu) = (-i)^l |l| 2^{2i\nu} (|l|/2 + i\nu) \phi_{k,l}(\nu) \Gamma(|l|/2 + i\nu) / \Gamma(|l|/2 - i\nu). \quad (35)$$

With the gluon wave functions in the momentum space given by (34), both radial and azymuthal integration in $p$ are easily done. The final matrix element of the kinetic energy $T$ results as

$$T_{\alpha_1,\alpha_3} = -4\pi i \int d\nu f_{k_1,l_1}^*(\nu)(\partial/\partial \nu) f_{k_1,l_1}(\nu) \quad (36)$$
(recall that \( l_1 = l_3 \)). The differentiation gives
\[
(\partial/\partial \nu) f_{k_3,l_1}(\nu) = f_{k_3,l_1}(\nu)(2i \ln 2 + 2i \Re \psi(|l_1|/2 + i\nu) + (\partial/\partial \nu) \ln(|l|/2 + i\nu) \phi_{k,l}(\nu)). \quad (37)
\]
 Correspondingly the kinetic energy matrix element separates into terms
\[
T^{(1)}_{\alpha_1,\alpha_3} = 8\pi \int d\nu f^*_k l_1 f_{k_3,l_1}(\nu)(\ln 2 + \Re \psi(|l_1|/2 + i\nu)) \quad (38)
\]
and
\[
T^{(2)}_{\alpha_1,\alpha_3} = -4\pi i \int d\nu (|l_1|/2 + i\nu) \phi_{k_1,l_1}(\nu)^* (\partial/\partial \nu)(|l_1|/2 + i\nu) \phi_{k_3,l_1}(\nu)). \quad (39)
\]
One notes that the function \((|l|/2 + i\nu) \phi_{k,l}(\nu))\) is nothing but the Fourier transform of
\(\xi_{k,l}(z)\) with respect to \(z\). Correspondingly we denote it as
\[
(|l|/2 + i\nu) \phi_{k,l}(\nu) \equiv \xi_{k,l}(\nu). \quad (40)
\]
The part \(T^{(2)}\) can then be written as
\[
T^{(2)}_{\alpha_1,\alpha_3} = -4\pi i \int d\nu \xi_{k_1,l_1}(\nu)^* (\partial/\partial \nu) \xi_{k_3,l_1}(\nu)). \quad (41)
\]
The orthonormalization property \((23)\) transforms into the analogous property in the \(\nu\) space
\[
\int d\nu \xi^*_k l_1(\nu) \xi_{k', l_1}(\nu) = (1/4\pi) \delta_{kk'} \quad (42)
\]
Noting that \(f^*_k l_1 f_{k', l_1}(\nu) = \xi^*_k l_1(\nu) \xi_{k', l_1}(\nu)\) we observe that the term \(\ln 2\) in \((38)\) will add a constant \(2 \ln 2\) to the energy. Separating another constant term \(2\psi(1)\) we finally present the part \(T^{(1)}\) in the final form
\[
T^{(1)}_{\alpha_1,\alpha_3} = 2(\ln 2 + \psi(1)) \delta_{\alpha_1,\alpha_3} + 8\pi \int d\nu \xi^*_k l_1(\nu) \xi_{\alpha_3}(\nu)(\Re \psi(|l_1|/2 + i\nu) - \psi(1)). \quad (43)
\]
The first constant term cancels an identical one in the initial Hamiltonian \((4)\), so that we may forget about these constants and concentrate only on the resting nontrivial contributions. Using the representation
\[
\psi(x) - \psi(1) = \int_0^\infty dt (\exp(-t) - \exp(-xt))/(1 - \exp(-t)) \quad (44)
\]
and the orthonormalization property of the set \(\xi_{\alpha}\) we may cast \(T^{(1)}\) in the form
\[
T^{(1)}_{\alpha_1,\alpha_3} = 2 \int_0^\infty (dt/(\exp t - 1))(\delta_{\alpha_1,\alpha_3} - \exp(t(1 - |l_1|/2))g_{\alpha_1,\alpha_3}(t)), \quad (45)
\]
where
\[
g_{\alpha_1,\alpha_3}(t) = 4\pi \int d\nu \xi^*_k l_1(\nu) \xi_{\alpha_3}(\nu) \cos \nu t. \quad (46)
\]
Note that (44) is not valid for $\text{Re} \, x = 0$. Therefore this formula cannot be applied when the gluon orbital momentum is zero. In this case one may use

$$\psi(i\nu) + \psi(-i\nu) = \psi(1 + i\nu) + \psi(1 - i\nu),$$

which formally corresponds to changing the angular momentum to be equal to 2.

As to the second part of the kinetic energy $T^{(2)}$, we shall find out presently that it will be cancelled by a similar contribution coming from the monopole part of the Coulomb interaction for the angular momentum transfer equal to zero.

4 Monopole part of the Coulomb interaction

Most of the contributions to the energy presented in the previous section can hardly be further simplified and were used in the numerical calculations as they stand. The exception is the monopole part of the Coulomb interaction corresponding to (25) with $l = 0$ (Eq. (27)). This part contains contributions which cancel the term $T^{(2)}$ in the kinetic energy and partially the contact interaction contribution for $l = 0$. The cancellation between the monopole Coulomb interaction and the kinetic term $T^{(2)}$ is responsible for the scale invariance of the energy.

Explicitly the monopole term contribution is given by

$$U_{\alpha_1, \alpha_2; \alpha_3, \alpha_4} = 16\pi^2 \int_{-\infty}^{\infty} dz_1 \eta_{\alpha_1}(z_1) \psi_{\alpha_3}(z_1) z_1 \int_{-\infty}^{z_1} \psi_{\alpha_2}(z_2) \eta_{\alpha_4}(z_2) + (\alpha_1 \leftrightarrow \alpha_4, \alpha_2 \leftrightarrow \alpha_3).$$

(47)

Here and in the following it is assumed that $l = 0$, that is, $l_1 = l_3$ and $l_2 = l_4$. Introduce a function

$$\chi_{\alpha_2, \alpha_4}(z) = \int_{-\infty}^{z} dz' \psi_{\alpha_2}(z') \eta_{\alpha_4}(z').$$

(48)

Once integrating by parts we find

$$\chi_{\alpha_2, \alpha_4}(z) = \psi_{\alpha_2}(z) \xi_{\alpha_4}(z) - \xi_{\alpha_2, \alpha_4}(z),$$

(49)

where the function $\xi_{\alpha_2, \alpha_4}(z)$ with two indices, symmetric in these, is defined as

$$\xi_{\alpha_2, \alpha_4}(z) = \int_{-\infty}^{z} dz' \xi_{\alpha_2}(z') \xi_{\alpha_4}(z').$$

(50)

As $z \to \infty$, according to (23), $\xi_{\alpha_2, \alpha_4}(z) \to (1/4\pi) \delta_{\alpha_2, \alpha_4}$, so that

$$\chi_{\alpha_2, \alpha_4}(\infty) = -(1/4\pi) \delta_{\alpha_2, \alpha_4}.$$

Having this in mind we can rewrite (47) in the form

$$U_{\alpha_1, \alpha_2; \alpha_3, \alpha_4} = 16\pi^2 \int_{-\infty}^{\infty} dz (\chi_{\alpha_3, \alpha_1}(z) + (1/4\pi) \delta_{\alpha_1, \alpha_3})' z \chi_{\alpha_2, \alpha_4}(z) + (\alpha_1 \leftrightarrow \alpha_4, \alpha_2 \leftrightarrow \alpha_3).$$

(51)
Integrating by parts, the integral transforms into

$$-16\pi^2 \int_{-\infty}^{\infty} dz (\chi_{a_3,a_1}(z) + (1/4\pi)\delta_{a_1,a_3})(z\chi'_{a_2,a_4}(z) + \chi_{a_2,a_4}).$$  \hspace{1cm} (52)$$

The term coming from the product $\chi_{a_3,a_1}z\chi'_{a_2,a_4}$ cancels the contribution $(\alpha_1 \leftrightarrow \alpha_4, \alpha_2 \leftrightarrow \alpha_3)$ in (51) so that the monopole contribution becomes

$$U_{a_1,a_2;a_3,a_4} = -16\pi^2 \int_{-\infty}^{\infty} dz (\chi_{a_3,a_1}(z)\chi_{a_2,a_4}(z) + (1/4\pi)\delta_{a_1,a_3}(z\chi'_{a_2,a_4}(z) + \chi_{a_2,a_4}(z))). \hspace{1cm} (53)$$

Now the idea is to substitute the functions $\chi$ in (53) by the symmetric functions $\xi$ using relation (49). Take the the first term in the integrand of (53). With (49) we obtain for it

$$\chi_{a_3,a_1}\chi_{a_2,a_4} = \psi_{a_3}\xi_{a_1}\psi_{a_2}\xi_{a_4} - \psi_{a_3}\xi_{a_1}\psi_{a_2}\xi_{a_4} - \xi_{a_3,a_1}\psi_{a_2}\xi_{a_4} + \xi_{a_3,a_1}\xi_{a_2,a_4}. \hspace{1cm} (54)$$

Having in mind the subsequent symmetrization with respect to the interchange of gluons 1 and 2, we can change $\alpha_1 \leftrightarrow \alpha_2$ and $\alpha_3 \leftrightarrow \alpha_4$ in the second term. Summed with the third term it then gives

$$-\xi_{a_3,a_1}(\psi_{a_2}\xi_{a_4} + \xi_{a_2}\psi_{a_4}). \hspace{1cm} (54)$$

Recall now that $\xi_{a_2} = (\partial + (1/2)|l_2|)\psi_{a_2}$ and similarly for $\xi_{a_4}$. Integration by parts allows to substitute (54) by

$$\left(\xi_{a_3}\xi_{a_1} - |l_2|\xi_{a_3,a_1}\right)\psi_{a_2}\psi_{a_4}. \hspace{1cm} (55)$$

So finally the first term in (53) leads to the following three contributions to the monopole Coulomb energy:

$$\hat{U}_{a_1,a_2;a_3,a_4}^{(1)} = 16\pi^2 \int_{-\infty}^{\infty} dz \xi_{a_3,a_1}\xi_{a_2,a_4}, \hspace{1cm} (56)$$

$$U_{a_1,a_2;a_3,a_4}^{(2)} = 16\pi^2 |l_2| \int_{-\infty}^{\infty} dz \xi_{a_3,a_1}\psi_{a_2}\psi_{a_4}, \hspace{1cm} (57)$$

and

$$U_{a_1,a_2,a_3,a_4}^{(3)} = -16\pi^2 \int_{-\infty}^{\infty} dz (\psi_{a_3}\xi_{a_1}\psi_{a_2}\xi_{a_4} + \xi_{a_3}\xi_{a_1}\psi_{a_2}\psi_{a_4}). \hspace{1cm} (58)$$

Of these terms the first is divergent in its present form. It will receive its meaning after adding a new contributions coming from the rest of the terms in (53). For that reason we have denoted it with a tilda.

Now for the rest of the terms in (53). Changing the function $\chi$ by $\xi$ according to (49) we have under the integral

$$\xi_{a_2,a_4} + z\xi'_{a_2,a_4} = \psi_{a_2}\xi_{a_4} - \xi_{a_2,a_4} + z\psi_{a_2}(\partial - (1/2)|l_2|)\xi_{a_4}. \hspace{1cm} (59)$$

Integration by parts transforms it into

$$-\xi_{a_2,a_4} - z\xi_{a_2}\xi_{a_4}. \hspace{1cm} (59)$$
The first term can be combined with (56) to give the final part $U^{(1)}$:

$$U_{\alpha_1,\alpha_2;\alpha_3,\alpha_4}^{(1)} = 16\pi^2 \int_{-\infty}^{\infty} dz \xi_{\alpha_2,\alpha_4}((1/4\pi)\delta_{\alpha_1,\alpha_3} - \xi_{\alpha_3,\alpha_1}).$$ (60)

Now the integral converges due to the property (23). Putting here the explicit form of the functions $\xi_{\alpha_i,\alpha_k}$ and integrating over $z$ we obtain the term $U^{(1)}$ in its definitive form:

$$U_{\alpha_1,\alpha_2;\alpha_3,\alpha_4}^{(1)} = 16\pi^2 \int d\xi_1 d\xi_2 (\xi_1 - \xi_2)\xi_{\alpha_1}(\xi_1)\xi_{\alpha_3}(\xi_2)\xi_{\alpha_2}(\xi_2)\xi_{\alpha_4}(\xi_4).$$ (61)

The second term in (59) gives the last contribution to the monopole energy

$$U_{\alpha_1,\alpha_2;\alpha_3,\alpha_4}^{(4)} = 4\pi\delta_{\alpha_1,\alpha_3} \int_{-\infty}^{\infty} dz \xi_{\alpha_2,\xi_{\alpha_4}}.$$ (62)

This term cancels with the contribution $T^{(2)}$ to the kinetic energy. Indeed after the Fourier transformation to the $\nu$ space according to (33), the factor $z$ goes into $i\partial/\partial\nu$. One can then see that (62) gives exactly the contribution $T^{(2)}$, Eq. (41), with an opposite sign and with gluons 1 and 2 interchanged, which is of no importance because of the subsequent symmetrization.

The term $U^{(3)}$ cancels with the part of the contact interaction $Q$, Eq. (30), which does not contain factors $\Delta$:

$$Q_{\alpha_1,\alpha_2;\alpha_3,\alpha_4}^{(2)} = 8\pi^2 \int_{-\infty}^{\infty} dz (\psi_{\alpha_3}\xi_{\alpha_1}\psi_{\alpha_2}\xi_{\alpha_4} + \xi_{\alpha_1}\psi_{\alpha_3}\psi_{\alpha_2}\xi_{\alpha_4} + \xi_{\alpha_3}\psi_{\alpha_1}\xi_{\alpha_2}\psi_{\alpha_4} + \psi_{\alpha_3}\xi_{\alpha_1}\xi_{\alpha_2}\psi_{\alpha_4}).$$ (63)

Summed with $U^{(3)}$ this part gives

$$(Q^{(2)} + U^{(3)})_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 8\pi^2 \int_{-\infty}^{\infty} dz (\psi_{\alpha_1}\xi_{\alpha_2} - \xi_{\alpha_1}\psi_{\alpha_2})(\xi_{\alpha_3}\psi_{\alpha_4} + \psi_{\alpha_3}\xi_{\alpha_4}).$$ (64)

This expression is antisymmetric under the interchange of the gluons 1 and 2 and does not give any contribution to the energy.

So finally the only contributions which remain in the interaction for zero angular momentum transfer are $U^{(1)}$, $U^{(2)}$ and the part $Q^{(1)}$ of the contact interaction (30) which contains factors $\Delta$.

## 5 Oscillator basic functions

A natural orthonormal discrete basis for $z = \ln r^2$ varying from $-\infty$ to $+\infty$ is formed by the harmonic oscillator proper functions. Thus we choose functions $\xi_{k,l}(z)$ independent of $l$ and given by

$$\xi_k(z) = c_kH_k(z) \exp(-z^2/2),$$ (65)

where $H_k$ are the Hermite polynomials and $c_k$ are determined by the normalization condition (23) to be

$$c_k^2 = 1/(4\pi^{3/2}2^k k!).$$ (66)
The Fourier transformation to the $\nu$ space gives

$$\xi_k(\nu) = (-i)^k c_k H_k(\nu) \exp(-\nu^2/2).$$  \hspace{1cm} (67)$$

In the coordinate space the function $\eta_{k,l}(z)$ is obtained from $\xi$ by differentiation:

$$\eta_{k,l}(z) = (\partial - (1/2)|l|)\xi_k(z).$$  \hspace{1cm} (68)$$

Using the properties of $H_k$ we get

$$\eta_{k,l}(z) = 2k(c_k/c_{k-1})\xi_{k-1}(z) - (z + (1/2)|l|)\xi_k(z).$$  \hspace{1cm} (69)$$

The function $\psi_{k,l}$ is obtained from $\xi_k$ as a solution of the differential equation

$$\xi_k(z) = (\partial + (1/2)|l|)\psi_{k,l}(z),$$  \hspace{1cm} (70)$$

with a boundary condition $\psi_{k,l}(-\infty) = 0$. It is given by an integral

$$\psi_{k,l}(z) = \int_{-\infty}^{z} dz' \xi_k(z') \exp(-|l|(z-z')/2).$$  \hspace{1cm} (71)$$

For $k = 0, 1$ we find from (71):

$$\psi_{0,l}(z) = \sqrt{\pi/2} c_0 \exp((w^2 - z^2)/2)(1 - \Phi(w/\sqrt{2})),$$  \hspace{1cm} (72)$$

$$\psi_{1,l}(z) = 2(z + w)(c_1/c_0)\psi_{0,l}(z) - 2c_1 \exp(-z^2/2),$$  \hspace{1cm} (73)$$

where $w = |l|/2 - z$ and $\Phi \equiv \text{erf}$ is the error function integral. For $k > 1$ the functions $\psi$ can be found by a recurrency relation that follows from (71) upon integrating by parts:

$$\psi_{k+1,l} = |l|(c_{k+1}/c_k)\psi_{k,l} + 2k(c_{k+1}/c_{k-1})\psi_{k-1,l} - 2(c_{k+1}/c_k)\xi_k.$$  \hspace{1cm} (74)$$

With this set of functions the potential part of the energy was calculated numerically. As to the kinetic part, the function $g$ entering (45) can be found analytically. For transition $k, l \rightarrow k', l$ it is equal to zero if $k + k'$ is odd. For even $k + k' = 2s$ and $k \geq k'$

$$g_{k,k'}(t) = 4\pi^{3/2} (-1)^{d} c_k c_{k'} \exp(-t^2/4) \sum_{p=0}^{k'} 2^p p! C^p_k C^p_{k'} (-t^2)^{s-p},$$  \hspace{1cm} (75)$$

where $2d = k - k'$.

We finally note that the presented set of basic functions can be trivially generalized to include a scaling factor $a$ in the $z$ space by choosing as basic functions

$$\xi_k^{(a)}(z) = \sqrt{a}\xi_k(az).$$  \hspace{1cm} (76)$$

Other functions can then easily be found as

$$\xi_k^{(a)}(\nu) = (1/\sqrt{a})\xi_k(\nu/a),$$  \hspace{1cm} (77)$$
\[
\eta^{(a)}_{k,l}(z) = a\sqrt{a}\eta_{k,l/a}(az)
\]  
(78)

and finally

\[
\psi^{(a)}_{k}(z) = (1\sqrt{a})\psi_{k,l/a}(az).
\]  
(79)

Putting these expressions into the obtained formulas for the energy and rescaling the variables 
\(z\) or \(\nu\) we find that the new value of the potential energy is obtained by dividing all angular
momenta by \(a\) and dividing the resulting energy by \(a\). As to the kinetic energy the change
reduces to substituting \(at\) instead of \(t\) in the function \(g\), Eq. (45).

6 Results for \(n = 2, 3\) and conclusions

We applied this formalism to the cases \(n = 2\) (the BFKL pomeron) and \(n = 3\) (the
odderon) to study its convergence.

The energy matrix \(E_{\alpha_1,\alpha_2,\alpha'_1,\alpha'_2}\) has been calculated numerically for a chosen set of basic
functions described in the previous section. After proper symmetrization in two- or three-
gluon states its lowest eigenvalue has been determined, which gives an upper limit on the
exact pomeron or odderon energy according to Eq. (11). To study the minimal energy only
states with the total angular momentum equal to zero have been included.

The selected set of basic one-gluon functions is characterized by the maximal value of the
angular momentum included \(l_{\text{max}}\) and numbers of radial functions included for each wave. As
calculations show, best results are obtained when one raises \(l_{\text{max}}\) and the number of radials
in all waves simultaneously. So we present here the results for the case when the number
of radials \(r\) is the same for all angular momenta and is equal to the number of angular
momenta included \(r = l_{\text{max}} + 1\). Such a set of functions is thus characterized by a single
parameter \(r\). With a growth of \(r\) the number of states \(N\) rises very rapidly. For two gluons
\(N_2 = r(r^2 - r/2 - 1/2)\) and so rises as \(r^3\). For three gluons the rise is still steeper.

In the present calculations the number \(r\) was limited by 6 for two gluons and by 5 for
three gluons. Correspondingly the total number of basic states included was taken up to 201
for two gluons and 1335 for three gluons.

The results of the calculations of the ground state energies of the BFKL pomeron \(\epsilon_2\) and
the odderon \(\epsilon_3\) are presented in the Table for different values of \(r\). One observes that the
obtained energies are still rather far from the exact value for the pomeron and the upper
limit for the odderon obtained in [5]. Thus the convergence of the method is rather slow.
The Table also reveals that the odderon energy is essentially larger than the pomeron one for
a given \(r\). So our results confirm that, in all probability, the odderon intercept is lower than
that of the BFKL pomeron.
To be more quantitative one can estimate the precision of the variational results by comparing the calculated correlation energy with its exact value known for the BFKL pomeron. Subtracting $E_0$ from the Hartree-Fock energy we find $\epsilon_{2}^{Cor} = 3.732$. For the maximal value $r = 6$ the calculated correlation energy is 1.991, which constitutes $\sim 53\%$ of the exact value. With $r = 5$ the correlation energy is 1.871, that is $\sim 50\%$ of the exact value. If one boldly assumes that for the odderon the correlation energy calculated with $r = 5$ also constitutes 50% of the exact value, then one finds this exact value to be 1.256 and the absolute energy $\epsilon_3 = -0.3$, that is, $E_{odd} = -0.45$, which does not contradict [5].

More sophisticated estimates can be attempted by studying the dependence of the calculated energy on $r$ and extrapolating for higher values of $r$. We have chosen a 4-parameter fit

$$
\epsilon(r) = \epsilon(\infty) + a \exp(-\alpha \ln^{\beta} r).
$$

For the BFKL pomeron $\epsilon(\infty)$ is known. It turns out that the values of $\epsilon_2(r)$ given in the Table are well described by (80) with the choice of parameters

$$
a = 6.066, \quad \alpha = 0.925, \quad \beta = 0.515.
$$

According to this fit further improvement of the value for $\epsilon_2$ requires very high values of $r$. E.g., to achieve $\epsilon_2(r) = -2.0$ one has to raise $r$ to $\sim 100$. The corresponding numbers of two-gluon states are enormous and hardly possible to include.

The analogous fit for the three-gluon case determined from energy values for $r = 2, 3, 4, 5$ has the parameters

$$
\epsilon(\infty) = -0.389, \quad a = 2.472, \quad \alpha = 1.04, \quad \beta = 0.36.
$$

However this set of parameters is rather unstable: a small change in the energy values used causes rather large changes in the value of $\epsilon(\infty)$. If we take for $\beta = 1/2$, as evidently favoured by the two gluon case, then we obtain

$$
\epsilon(\infty) = -0.195, \quad a = 1.772, \quad \alpha = 0.957, \quad \beta = 0.5,
$$

with the value $\epsilon_3(2) = 0.603$ slightly smaller than the calculated one 0.605 although within the calculational errors of the order of $\pm 0.003$. In view of this we can only give a crude estimation for the upper limit for the odderon energy from our calculations,

$$
\epsilon_3 < -0.2 \div -0.4,
$$

which according to (11) translates into the estimate for $E_{odd}$

$$
E_{odd} < -0.3 \div -0.6,
$$
in agreement with [5].

To conclude, the calculations for two and three gluons show that the developed method can be applied to investigate the intercept of symmetric configurations although the convergence is slow, evidently, due to a very singular character of the gluonic wave function. It seems realistic to obtain around 50% of the correlation energy with this approach, which may serve to estimate intercepts for multigluon configurations relative to the BFKL one. We are trying to achieve better results by including more basic functions and also selecting configurations which give the dominant contribution. Calculations for $n = 4$ gluons are also in progress.

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Table

Calculated values of the ground state energy per gluon (multiplied by 2, Eq. (11)) for the pomeron ($\epsilon_2$) and odderon ($\epsilon_3$) with different numbers $r$ of radial functions and angular momenta included.

| $r$ | $\epsilon_2$ | $\epsilon_3$ |
|-----|-------------|-------------|
| 1   | 0.968       | 0.968       |
| 2   | 0.022       | 0.605       |
| 3   | −0.475      | 0.454       |
| 4   | −0.743      | 0.379       |
| 5   | −0.912      | 0.331       |
| 6   | −1.032      |             |
| $\infty$ | −2.773     |             |