Some types of numeral systems and their modeling

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Abstract
In this article, the operator approach to modelling numeral systems is introduced. This approach can be useful for coding information and providing computer protection. Certain examples of such numeral systems are considered. In addition, the pseudo-binary representation is investigated. A description of further investigations of the author of this article is given.

Keywords s-adic representation · Coding information · Pseudo-s-adic representation · Lebesgue measure · Permutations of samples ordered with reiterations of $k$ numbers from \{0, 1, ..., s-1\}

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1 Introduction

In the period of the dynamic development of science, of engineering, and of technology, the necessity for the dynamic development of encoding and information protection has arisen. It is extremely important for protecting computer networks. Modeling of generalizations of well-known numeral systems is an important tool in this case. For non-cracking the coding mechanism, new numeral systems should be difficult and should be systematically improved by a certain generalization or constructing a chain of sequential generalizations. In addition, various systems of real number encodings is an effective tool for modeling and studying “pathological” mathematical objects (for example, the notion of “pathology” in mathematics is...
described in [24]). Such mathematical objects (e.g., fractal sets, singular functions, non-differentiable, or nowhere monotonic functions, etc.) have the applied importance and the interdisciplinary character. Examples of using or applying pathological mathematical objects in one or several fields of science can be easily founded in a number of literatures, even on Wikipedia (for example, see [1, 3, 7, 11, 23, 25, 26]). As a drawback, it is possible that the applied importance of such mathematical object can be justified only with time in one or another field of science.

One can note that now in science there exists the tendency of modeling and studying numeral systems defined in terms of alternating expansions of real numbers under the condition that numeral systems defined in terms of corresponding positive expansions of real numbers are well-known. Let us consider several examples:

- The notion of the $s$-adic numeral system with a fractional base (more known as the $\beta$-expansion) was introduced by A. Rényi in [10] in 1957. However, the nega-$s$-adic numeral system with a fractional base (or $(-\beta)$-expansion) was introduced in [5] in 2009. Later, the last-mentioned numeral system was investigated by a number of researchers from Western Europe, USA, and East Asia: P. Ambrož, K. Dajani, D. Dombek, S. Elizalde, Ch. Frougny, S. Ito, Ch. Kalle, V. Komorník, A. Ch. Lai, L. Liao, P. Loreti, Z. Masáková, K. Moore, E. Pelantová, T. Sadahiro, W. Steiner, T. Vávra and other scientists.

- In 1883, in the paper [8], the German mathematician J. Lüroth introduced an expansion of a real number in the form of a special series (now these series are called Lüroth series). However, in 1990, S. Kalpazidou, A. Knopfmacher, J. Knopfmacher introduced alternating Lüroth series in the paper [6].

Such examples also exist for other expansions, e.g., for the case of positive and alternating Engel series.

Finally, let us remark that there exist a number of research devoted to modeling and investigations of different numeral systems and pathological mathematical objects (for example, see surveys in [1, 3, 4, 9, 22], etc.).

The present paper\footnote{The present research was presented as the preprint arXiv:1907.10534 in 2019.} is devoted to some new numeral systems. Their geometry is a generalization of geometries of certain numeral systems with positive or negative base. Also, the special attention is given to further investigations of the author of this paper.

Let $s > 1$ be a fixed positive integer. It is well known that any number $x \in [0, 1]$ can be represented by the following form

$$\sum_{n=1}^{\infty} \frac{z_n}{s^n} \equiv \Lambda_{s_1s_2...}^s = x,$$

where $z_n \in A_s \equiv \{0, 1, \ldots, s - 1\}$. The last-mentioned representation is called the $s$-adic representation of $x$. Also, any number $x \in \left[ -\frac{s}{s+1}, \frac{1}{s+1} \right]$ can be represented by the nega-$s$-adic representation.
\[
\begin{align*}
\Delta_{x_1, x_2, \ldots, x_n}^{-s} &= \sum_{n=1}^{\infty} \frac{\alpha_n}{(-s)^n}, \\
\text{where } \alpha_n & \in A_s.
\end{align*}
\]

It is easy to see that the following relationship between the s-adic and nega-s-adic representations holds:

\[
\Delta_{x_1, x_2, \ldots, x_n}^{-s} + \sum_{k=1}^{\infty} \frac{s-1}{s^{2k-1}} \equiv \sum_{k=1}^{\infty} \frac{\alpha_{2k}}{s^{2k}} + \sum_{k=1}^{\infty} \frac{s-1-\alpha_{2k-1}}{s^{2k-1}} \equiv \Delta_{[s-1-x_1][s-1-x_2] \ldots [s-1-x_{2k-1}][x_{2k} \ldots]^*}^s.
\]

(1)

Let \(\mathbb{N}_B\) be a fixed subset of positive integers, \(B = (b_n)\) be a fixed increasing sequence of all elements of \(\mathbb{N}_B\), and

\[
\rho_n = \begin{cases} 
1 & \text{if } n \in \mathbb{N}_B \\
2 & \text{if } n \not\in \mathbb{N}_B.
\end{cases}
\]

Let us consider the quasi-nega-s-adic representation described in [15]:

\[
x = \Delta_{x_1, x_2, \ldots, x_n}^{(\pm s, \mathbb{N}_B)} \equiv \sum_{n=1}^{\infty} \frac{(-1)^{\rho_n} \alpha_n}{s^n}.
\]

(2)

Here \(\alpha_n \in A_s\) and \(x \in [a_0', a_0^{''}]\), where \(a_0^{''} = 1 + a_0'\) and

\[
a_0' = -\sum_{n=1}^{\infty} \frac{s-1}{s^{b_n}}.
\]

Note that the term “nega” is used in the present article since corresponding encodings of real numbers are numeral systems with a negative base.

**Theorem 1** Any number \(x \in [a_0', a_0^{''}]\) can be represented in form (2).

Note that the following relationship holds:

\[
\Delta_{x_1, x_2, \ldots, x_n}^{(\pm s, \mathbb{N}_B)} + \sum_{n=1}^{\infty} \frac{s-1}{s^{b_n}} \equiv \Delta_{x_1, x_2, \ldots, x_n}^s + \sum_{n=1}^{\infty} \frac{s-1}{s^{b_n}} = x \in [0, 1],
\]

(3)

where

\[
\alpha_n' = \begin{cases} 
-1-\alpha_n & \text{if } n \in \mathbb{N}_B \\
\alpha_n & \text{if } n \not\in \mathbb{N}_B.
\end{cases}
\]

Also, we can write

\[
\Delta_{x_1, x_2, \ldots, x_n}^{(\pm s, \mathbb{N}_B)} \equiv \Delta^s_{\theta(x_1) \theta(x_2) \ldots \theta(x_n)} - \sum_{n=1}^{\infty} \frac{s-1}{s^{b_n}},
\]

where \(\theta(x_n) = \alpha_n'\).
One can generalize this idea and model representations related with the s-adic representation by certain number converters (converters of combinations of numbers). There exist numeral systems constructed by certain mixings points in known numeral systems. In the present article, the operator approach to modeling numeral systems is introduced. It is shown that there exist real number representations, that are modified s-adic representations in which relations between digits in the old and new representations are defined by certain operators $\theta$ of changes of digits in s-adic representations.

Let us consider the following operators (converters of k-digit combinations): $\theta_{k,i}(x_{km+1}, x_{km+2}, \ldots, x_{(m+1)k}) = (\beta_{km+1}, \beta_{km+2}, \ldots, \beta_{(m+1)k})$, where numbers $k$, $i$ are fixed for an operator $\theta_{k,i}$, $m = 0, 1, \ldots$. Here $\theta_{k,i}(x_1, x_2, \ldots, x_k)$, where $x_j \in A_s$ for $j = 1, k$, is a some function of $k$ variables (it is the bijective correspondence) such that the set

$$A^k_s = A_s \times A_s \times \cdots \times A_s$$

is its domain of definition and range of values. That is, each combination $(x_1, x_2, \ldots, x_k)$ of $k$ s-adic digits is assigned to the single combination $\theta_{k,i}(x_1, x_2, \ldots, x_k)$ of $k$ s-adic digits, i.e.,

$$\begin{align*}
(\beta_1, \beta_2, \ldots, \beta_k) &= \theta_{k,i}(x_1, x_2, \ldots, x_k), \\
(\beta_{k+1}, \beta_{k+2}, \ldots, \beta_{2k}) &= \theta_{k,i}(x_{k+1}, x_{k+2}, \ldots, x_{2k}), \\
&\vdots \\
(\beta_{km+1}, \beta_{km+2}, \ldots, \beta_{(m+1)k}) &= \theta_{k,i}(x_{km+1}, x_{km+2}, \ldots, x_{(m+1)k}), \\
&\vdots
\end{align*}$$

for $m = 0, 1, \ldots$, some fixed numbers $k \in \mathbb{N}$ and $i \in \{0, 1, \ldots, (s^k)! - 1\}$, where $\mathbb{N}$ is the set of all positive integers. Note that $i = 0, (s^k)! - 1$ since the number of all operators (converters) $\theta_{k,i}$ for a fixed number $k$ is equal to the number of all permutations of elements from the set of all samples ordered with reiterations of $k$ numbers from $A_s$.

Let us consider certain examples of $\theta_{k,i}$.

**Example 1** Suppose that $s = k = 2$. Let us consider the operator $\theta_{2,i}$ defined by the following table
Here \( m = 0, 1, \ldots \), \( \beta_{2m+1}\beta_{2(m+1)} = \theta_{2,i}(\alpha_{2m+1}, \alpha_{2(m+1)}) \), and \( i \in \{0, 1, \ldots, 23\} \).

For example, a number having a binary representation of the form \( \Delta^2_{1110011001(11)} \) (symbols in parentheses mean the period in the representation of a number) will have the following representation in terms of a new modified representation:

\[
\Delta^2_{0100111001(01)} = \theta_{2,i}(\Delta^2_{1110011001(11)}).
\]

That is,

\[
f_{\theta_{2,i}} : \Delta^2_{1110011001(11)} \to \Delta^2_{0100111001(01)}.
\]

Note that \( \Delta^2_{\beta_1\beta_2\ldots\beta_n\ldots} \) is a formal denotation of \( \Delta^2_{0\beta_2(i,\beta_2)\beta_2(i,\beta_3)\beta_2(i,\beta_4)\ldots} \).

**Example 2** Suppose that \( s = 7, k = 1 \), and \( \beta_n = \theta_{1,i}(\alpha_n) \), where fixed \( i \in \{0, 1, \ldots, 7! - 1\} \) and

| \( x_n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| \( \beta_n \) | 3 | 5 | 6 | 4 | 0 | 2 | 1 |

It is easy to see that for an operator defined by the last-mentioned table and for all \( n, t \in \mathbb{N}, x_n \in \{0, 1, \ldots, 6\} \), the condition

\[
\underbrace{\theta_{1,i} \circ \theta_{1,i} \circ \ldots \circ \theta_{1,i}(x_n)}_{12t} = x_n
\]

holds since the following system of conditions is true for this operator:

\[
\begin{cases}
\theta_{1,i} \circ \theta_{1,i} \circ \theta_{1,i}(x) = x & \text{whenever } x \in \{0, 3, 4\} \\
\theta_{1,i} \circ \theta_{1,i} \circ \theta_{1,i}(x) = x & \text{whenever } x \in \{1, 2, 5, 6\}.
\end{cases}
\]

For example,

\[
f_{\theta_{1,i}} : \Delta^7_{3455142(1)} \to \Delta^7_{4022506(5)}.
\]

**Remark 1** Assume that for any fixed numbers \( k \in \mathbb{N} \) and \( 2 \geq s \in \mathbb{N} \) the operators \( \theta_{k,0} \) and \( \theta_{k,(s^k)_{-1}} \) are following:
\[
\begin{align*}
\theta_{k,0} \left( x_{kn+1}, x_{kn+2}, \ldots, x_{(m+1)k} \right) &= \left( x_{km+1}, x_{km+2}, \ldots, x_{(m+1)k} \right), \\
\theta_{k,(x')!-1} \left( x_{kn+1}, x_{kn+2}, \ldots, x_{(m+1)k} \right) &= (s - 1 - x_{kn+1}, s - 1 - x_{kn+2}, \ldots, s - 1 - x_{(m+1)k}),
\end{align*}
\]

where \( m = 0, 1, 2, 3, \ldots \).

2 General definitions

Let \( s > 1 \) be a fixed positive integer and \((k_n)\) be a fixed sequence, where \( k_n \in \mathbb{N} \). Let us consider the following matrix of operators

\[
\Theta_{s,(k_n)} = \begin{pmatrix}
\theta_{k_1,0} & \theta_{k_1,1} & \cdots & \theta_{k_1,i} & \cdots & \cdots & \theta_{k_1,(x')!-1} \\
\theta_{k_2,0} & \theta_{k_2,1} & \cdots & \theta_{k_2,i} & \cdots & \cdots & \theta_{k_2,(s^2)!-1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\theta_{k_n,0} & \theta_{k_n,1} & \cdots & \theta_{k_n,i} & \cdots & \cdots & \theta_{k_n,(s^n)!-1} \\
& & & & & & \vdots \\
& & & & & & \vdots \\
& & & & & & \vdots
\end{pmatrix}
\]

(4)

Here \( i = \overline{0, (s^n)!-1} \). Clearly, in this matrix a number of rows is equal to infinity and numbers of elements in different rows can be different (finite numbers or infinity).

From matrix (4) we generate a sequence \((\theta_{k_n,i_n})\) of operators. So, we obtain a representation \( \Delta_{j_1,j_2,\ldots,j_n} \) related with the s-adic representation by the following rule:

\[
\Delta_{j_1,j_2,\ldots,j_n} \left( x_{k_n+1}x_{k_n+2}\ldots x_{(m+1)k} \right) \equiv \Delta_{s,0} x_{k_1,i_1} x_{k_2,i_2} x_{k_3,i_3} \ldots x_{k_n,i_n} x_{kn+1} x_{kn+2} \ldots x_{(m+1)k}.
\]

(5)

Definition 1 A representation \( \Delta_{j_1,j_2,\ldots,j_n} \) whose relation with the s-adic representation is described by equality (5), is called the pseudo-s-adic representation of numbers from \([0,1]\).

Remark 2 The approach described in the present article for modeling numeral systems can be used for the case of the s-adic numeral system with a fractional base \( s > 1 \) (such representation of real numbers was introduced in [10]) or for the case of the nega-s-adic numeral system with a fractional base \((-s) < -1\) (this representation of real numbers is introduced in [5]). Also, the approach can be applied to numeral systems with a negative (integer or fractional) base and/or with a variable alphabet (we have a variable alphabet for a certain representation \( \Delta_{i_1,i_2,\ldots,i_n} \), where \( i_n \in A_n \), when there exists a finite or infinite number of pairs \( l \neq m \) such that \( |A_l| \neq |A_m| \)). It is easy to see that a basis of a numeral system with a variable alphabet is a sequence or a matrix.

In the next articles of the author of the present article, the approach will be used and investigated for the cases of all these representations of real numbers.
Note that examples of representations with a variable alphabet are representations of real numbers by positive [2] or alternating [17, 19] Cantor series, the nega-$\mathbb{Q}$-[16] or $\mathbb{Q}_{\mathbb{N}}$-representation [21], etc.

**Example 3** Suppose we have the representation of real numbers from $[0, 1]$ by positive Cantor series, i.e.,

$$x = \Delta^Q_{e_1 q_2 \ldots q_n} \equiv \sum_{n=1}^{\infty} \frac{e_n}{q_1 q_2 \ldots q_n},$$

where $Q = (q_n)$ is a fixed sequence of positive integers, $q_n > 1$ for all $n \in \mathbb{N}$, and $e_n \in \{0, 1, \ldots, q_n - 1\}$. In this case, matrix (4) is following:

$$\Theta_{Q,k_n} = \begin{pmatrix}
\theta_{k_1,0} & \theta_{k_1,1} & \ldots & \theta_{k_1,i} & \ldots & \theta_{k_1,q_{k_1}+q_{k_2}+\ldots+q_{k_n}-1} \\
\theta_{k_2,0} & \theta_{k_2,1} & \ldots & \theta_{k_2,i} & \ldots & \theta_{k_2,q_{k_1}+q_{k_2}+\ldots+q_{k_n}-1} \\
\vdots & \vdots & \ddots & \vdots & \ldots & \theta_{k_3,q_{k_1}+q_{k_2}+\ldots+q_{k_n}-1} \\
\theta_{k_n,0} & \theta_{k_n,1} & \ldots & \theta_{k_n,i} & \ldots & \theta_{k_n,q_{k_1}+q_{k_2}+\ldots+q_{k_n}-1} \\
\end{pmatrix}$$

Also, here $\theta_{k_n,0}$ is the identical operator and

$$\theta_{k_n,q_{k_1}+q_{k_2}+\ldots+q_{k_n}-1}(e_{k_n-1+1}, e_{k_n-1+2}, \ldots, e_{k_n}) = (q_{k_n-1+1} - e_{k_n-1+1}, \ldots, q_{k_n} - 1 - e_{k_n}).$$

This article is an initial article in the series of papers of the author of the present article devoted to the investigation of the operator approach for modeling, studying, and applying different new numeral systems (positive and alternating expansions of real numbers with a finite, infinite, or variable alphabet, etc.).

### 3 One example of the pseudo-ternary representation

Suppose $s = 3$ and $k_n = const = 1$. Then there exist $6 = 3!$ operators $\theta_{1,i}$ (see the following table).

| $x_n$   | $\theta_{1,0}(x_n)$ | $\theta_{1,1}(x_n)$ | $\theta_{1,2}(x_n)$ | $\theta_{1,3}(x_n)$ | $\theta_{1,4}(x_n)$ | $\theta_{1,5}(x_n)$ |
|---------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| $0$     | 0                   | 1                   | 2                   | 0                   | 1                   | 0                   |
| $1$     | 0                   | 1                   | 2                   | 0                   | 1                   | 0                   |
| $2$     | 0                   | 2                   | 1                   | 0                   | 1                   | 0                   |
| $3$     | 1                   | 0                   | 2                   | 0                   | 1                   | 0                   |
| $4$     | 1                   | 2                   | 0                   | 1                   | 0                   | 0                   |
Let us consider the case of $\theta_{1,1}$. To simplify mathematical notations, let us denote $\theta_{1,1}(x_n)$ as $\theta(x_n)$. Let us investigate the pseudo-ternary representation (or the $3^\prime$-representation) generated by the operator $\theta(x_n)$, i.e.,

$$\left[0, 1\right] \ni x = \Delta^{(3, \theta)}_{\beta_1\beta_2...\beta_n} = \Delta^{3^\prime}_{\beta_1\beta_2...\beta_n} \equiv \Delta^{3}_{0(\beta_1)\theta(\beta_2)...\theta(\beta_n)} \equiv \sum_{n=1}^{\infty} \frac{\theta(x_n)}{3^n} \quad (6)$$

Note that

$$\Delta^{3}_{x_1x_2...x_n} \equiv \Delta^{3^\prime}_{\theta(\beta_1)\theta(\beta_2)...\theta(\beta_n)}$$

since $\theta(\theta(x_n)) = \theta(\beta_n) = x_n$ holds for any $n \in \mathbb{N}$. Also, one can represent the operator $\theta$ by the Lagrange polynomial, i.e.,

$$\theta(x_n) = \frac{7x_n - 3x_n^2}{2}.$$

However, in terms of the Lagrange polynomial

$$\theta(\theta(x_n)) \neq \frac{7x_n - 3x_n^2}{2}.$$

So we shall not use the Lagrange polynomial for representing such operators.

Notice that in the case of the s-adic representation there exists countable set of numbers having two different s-adic representations. These numbers are numbers of the form

$$\Delta^{s}_{x_1x_2...x_n}(0) = \Delta^{s}_{x_1x_2...x_n-1}(s-1) \equiv x_n \neq 0,$$

and are called $s$-adic rational. The other numbers in $[0, 1]$ are called $s$-adic irrational.

Let us remark that in the case of the $3^\prime$-representation we obtain

$$\Delta^{3^\prime}_{x_1x_2...x_n-1}(0) = \Delta^{3^\prime}_{x_1x_2...x_n-1}(2) \equiv x_n \neq 0.$$

Similarly, these numbers are called $3^\prime$-rational and the other numbers having the unique $3^\prime$-representation are called $3^\prime$-irrational.

**Lemma 1** Any number $x \in [0, 1]$ can be represented in form (6).

Every $3^\prime$-irrational number has the unique $3^\prime$-representation. Every $3^\prime$-rational number has two different $3^\prime$-representations.

**Proof** Let us consider the function $y = f(x)$, where

$$y = f(x) = f\left(\Delta^{3}_{x_1x_2...x_n} \right) = \Delta^{3}_{\theta(x_1)\theta(x_2)...\theta(x_n)} \quad (7)$$

The last-mentioned function and a class of functions containing this function were investigated (see [18, 20]). This function is continuous at ternary-irrational points, and ternary-rational points are points of discontinuity of the function. Also, the
function is non-differentiable and nowhere monotone.

Note also that our statement follows from properties of the function and the well-posedness of the definition of this function.

**Remark 3** (On certain outgoing problems) Certain outgoing problems follow from the last lemma. Let us consider these problems introducing in the present article.

In [10], the $f$-expansion of real numbers was considered:

$$x = f(e_1 + f(e_2 + f(e_3 + \cdots ))) \equiv \Delta^f_{e_1 e_2 \ldots e_n} \ldots ,$$

where $f$ is a fixed function having certain properties. Notice that it is proved that such function is increasing. A number of researches devoted to different cases of the $f$-expansion (for example, see [10, 13, 14], etc.).

For example, if $f(x) = \frac{x}{\beta}$, where $\beta > 1$, then

$$x = \sum_{n=1}^{\infty} \frac{e_n}{\beta^n} \equiv \Delta^f_{e_1 e_2 \ldots e_n} \ldots .$$

The last-mentioned expansion (see [10]) of real numbers is called the $\beta$-expansion.

If $f(x) = -\frac{x}{\beta}$, where $(-\beta) < -1$, then

$$x = \sum_{n=1}^{\infty} \frac{e_n}{(-\beta)^n} \equiv \Delta^{-\beta}_{e_1 e_2 \ldots e_n} \ldots .$$

The last-mentioned expansion of real numbers is called the $(-\beta)$-expansion and was introduced in [5].

**Problem 1** Let us note that we can obtain our case of the pseudo-ternary representation by the following way. Suppose that $f(x_n) = \frac{\theta(x_n)}{s}$, where $1 < s \in \mathbb{N}$ and $\theta = \theta_{1,1}$. Then the obtained $f$-expansion is following:

$$\Delta^3_{\theta(x_1) \theta(x_2) \ldots \theta(x_n)} \ldots = \Delta^3_{\beta_1 \beta_2 \ldots \beta_n} \ldots .$$

**Hypothesis** On can model the $f$-expansion such that all points except points from no more countable set have the unique representation when $f$ is determined at any point and continuous almost everywhere (having jump discontinuities) on the domain. Also, $f$ can be non-monotone or non-differentiable.

In addition, a general technique representing the pseudo-$s$-adic representation by the $f$-expansion is unknown.

Let us note that we can write the pseudo-ternary representation in our case by the following way:

$$\Delta^3_{\beta_1 \beta_2 \ldots \beta_n} \ldots = f(x) = \Delta^f_{\beta_1 \beta_2 \ldots \beta_n} \ldots = f\left(\Delta^3_{x_1 x_2 \ldots x_n} \ldots \right),$$

where $x = \Delta^3_{x_1 x_2 \ldots x_n} \ldots$, $f$ is defined by equality (7), and
\[ \Delta_1^{f(x)} = f \left( \Delta_2^{\theta(x_1)} \theta(x_2) \cdots \theta(x_n) \cdots \right). \]

**Problem 2** Let \((f_n)\) be a fixed sequence of certain functions. One can define the \((f_n)\)-expansion of real numbers by the following way:

\[ x = f_1(e_1 + f_2(e_2 + f_3(e_3 + \cdots))) = \Delta_{e_1e_2\ldots}^{f_n}. \]

This representation is unknown. However, one can describe several examples of known representations, which can be represented by the \((f_n)\)-representation.

**Example 4** If \(f_n(x) = \frac{x}{q_n^n}\), where \((q_n)\) is a fixed sequence of positive integers and \(q_n > 1\) for any \(n \in \mathbb{N}\), then we obtain the representation of real numbers by positive Cantor series [2].

**Problem 3** Suppose that \(f_n(x) = \frac{x}{q_n^n}\), where \((q_n)\) is a fixed sequence of numbers for which one of the following conditions holds: \(q_n > 1\) or \(q_n \geq 2\). Then we obtain the encoding of real numbers by positive Cantor series (with a fractional base). Such positive and alternating expansions of real numbers are unknown. Note that we can define such representations by the condition \(e_n \in \{0, 1, \ldots, [q_n]\}\), \(e_n \in \{0, 1, \ldots, [q_n - 1]\}\), or \(e_n \in \{0, 1, \ldots, [q_n] - 1\}\). Here \([a]\) is the integer part of \(a\). The cases of these conditions are interesting for future investigations.

Note that it is easy to see that one can model the known \(Q_s^T\), \(Q_s^T\), \(\tilde{Q}\) and other representations by analogy with this case.

**Example 5** (Quasi-nega-representations) Let us have

\[ f_n(x) = \frac{x}{\beta(-1)^{\rho_n} + \rho_{n-1}} \]

(here \(\beta > 1\)) and a fixed sequence \((f_n)\) of functions

\[ f_n(x) = \frac{x}{q_n(-1)^{\rho_{n-1}} + \rho_n} \]

(here \((q_n)\) is a fixed sequence of positive integers such that \(q_n > 1\)), where

\[ \rho_n = \begin{cases} 1 & \text{if } n \in \mathbb{N}_B \\ 2 & \text{if } n \notin \mathbb{N}_B, \end{cases} \]

\(\mathbb{N}_B\) is a fixed subset of positive integers, \(B = (b_n)\) is a fixed increasing sequence of all elements of \(\mathbb{N}_B\), and \(\rho_0 = 0\). Then we get the following two quasi-nega-expansions with a fractional (in a general case) base:
\[
x = \sum_{n=1}^{\infty} \frac{(-1)^{\rho_n}}{\beta^n}, \quad x = \sum_{n=1}^{\infty} \frac{(-1)^{\rho_n} \rho_n}{q_1 q_2 \cdots q_n}.
\]

Note that we can model the quasi-nega-\(Q\)-expansion by analogy.

**Hypothesis 1** Properties of \(f\) and of \((f_n)\) that are sufficient for modeling the \(f\) and \((f_n)\)-representations coincide.

These problems and new representations of real numbers will be considered and investigated in the next papers of the author of the present article.

**Definition 2** A set of the form

\[
\Delta_{c_1 c_2 \ldots c_n}^3 = \left\{ x : x = \Delta_{c_1 c_2 \ldots c_n}^3, \beta_t \in \{0, 1, 2\}, t > n \right\},
\]

where \(c_1, c_2, \ldots, c_n\) is an ordered tuple of integers such that \(c_j \in \{0, 1, 2\}\) for \(j = 1, n\), is called a cylinder \(\Delta_{c_1 c_2 \ldots c_n}^3\) of rank \(n\) with base \(c_1 c_2 \ldots c_n\).

**Lemma 2** Cylinders \(\Delta_{c_1 c_2 \ldots c_n}^3\) have the following properties:

1. A cylinder \(\Delta_{c_1 c_2 \ldots c_n}^3\) is a closed interval and

\[
\Delta_{c_1 c_2 \ldots c_n}^3 = \left[ \Delta_{c_1 c_2 \ldots c_n}(0), \Delta_{c_1 c_2 \ldots c_n}(2) \right].
\]

2. \(\left| \Delta_{c_1 c_2 \ldots c_n}^3 \right| = \frac{1}{3^n}\).

3. \(\Delta_{c_1 c_2 \ldots c_n}^3 \subseteq \Delta_{c_1 c_2 \ldots c_n}^3\).

4. \(\Delta_{c_1 c_2 \ldots c_n}^3 = \bigcup_{c=0}^{2} \Delta_{c_1 c_2 \ldots c_n}^3\).

5. Cylinders \(\Delta_{c_1 c_2 \ldots c_n}^3\) are left-to-right situated.

**Proof** Let us consider the map

\[
f(\Delta_{x_1 x_2 \ldots x_n}^3) = \Delta_{\theta(x_1) \theta(x_2) \ldots \theta(x_n)}^3.
\]

It is known that a cylinder \(\Delta_{b_1 b_2 \ldots b_n}^3\) is a closed interval and \(f\) (see [18, 20]) is continuous at ternary irrational points, and ternary rational points are points of discontinuity of this function.

Let us prove that conditions \(f(x_0) \in f(\Delta_{b_1 b_2 \ldots b_n}^3)\) and \(f(\Delta_{b_1 b_2 \ldots b_n}^3) \subseteq f(\Delta_{b_1 b_2 \ldots b_n}^3)\) hold for any \(x_0 \in \Delta_{b_1 b_2 \ldots b_n}^3\).

It is easy to see that...
\[ f(x_0) = f\left( \Delta^3_{b_1b_2...b_n}(x_0)x_{a+2}(x_0) \right) \]
\[ = \Delta^3_{\theta(b_1)\theta(b_2)...\theta(b_n)\theta(x_{a+2})} \]
\[ = \Delta^3_{c_1c_2...c_n} \in \Delta^3_{c_1c_2...c_n}. \quad (8) \]

Here \( \theta(b_j) = c_j \) for all \( j = 1, n \). Note that
\[ f\left( \inf \Delta^3_{b_1b_2...b_n} \right) = \inf \Delta^3_{c_1c_2...c_n} = \Delta^3_{c_1c_2...c_n(0)}. \]

However,
\[ f\left( \sup \Delta^3_{b_1b_2...b_n} \right) \neq f\left( \Delta^3_{b_1b_2...b_n(1)} \right) = \sup \Delta^3_{c_1c_2...c_n} = \Delta^3_{c_1c_2...c_n(2)}. \]

From relationship (8) and the following equalities
\[ f\left( \inf \Delta^3_{b_1b_2...b_n} \right) = \Delta^3_{c_1c_2...c_n(0)} \in \Delta^3_{c_1c_2...c_n(0)}, \]
\[ f\left( \sup \Delta^3_{b_1b_2...b_n} \right) = \Delta^3_{c_1c_2...c_n(1)} \in \Delta^3_{c_1c_2...c_n(1)}, \]
where
\[ f\left( \inf \Delta^3_{b_1b_2...b_n} \right) = \inf \Delta^3_{c_1c_2...c_n} \]
and
\[ f\left( \sup \Delta^3_{b_1b_2...b_n} \right) < \sup \Delta^3_{c_1c_2...c_n}, \]
it follows that
\[ \inf \Delta^3_{c_1c_2...c_n} \geq \inf \Delta^3_{c_1c_2...c_n}, \]
\[ \sup \Delta^3_{c_1c_2...c_n} < \sup \Delta^3_{c_1c_2...c_n}. \]

So,
\[ \Delta^3_{c_1c_2...c_n} \subset \Delta^3_{c_1c_2...c_n} = \bigcup_{j=0}^{2} \Delta^3_{c_1c_2...c_n}. \]

Let us prove that a cylinder \( \Delta^3_{c_1c_2...c_n} \) is a segment.

Suppose that \( x \in \Delta^3_{c_1c_2...c_n} \). That is,
\[ x = \sum_{j=1}^{n} \frac{c_j}{3^j} + \frac{1}{3^n} \sum_{l=1}^{\infty} \frac{\beta_{n+l}}{3^l}, \quad \beta_{n+l} \in \{0, 1, 2\}. \]

Whence
Let a segment

\[ x' = \sum_{j=1}^{n} \frac{c_j}{3^j} \leq x \leq \sum_{j=1}^{n} \frac{c_j}{3^j} + \frac{1}{3^n} = x''. \]

So \( x \in [x',x''] \supseteq \Delta^3_{c_1 c_2 \ldots c_n} \). Since

\[ x' = \sum_{j=1}^{n} \frac{c_j}{3^j} + \frac{1}{3^n} \inf \sum_{l=1}^{\infty} \beta_{n+l} \frac{1}{3^l} \]

and

\[ x'' = \sum_{j=1}^{n} \frac{c_j}{3^j} + \frac{1}{3^n} \sup \sum_{l=1}^{\infty} \beta_{n+l} \frac{1}{3^l}, \]

we obtain \( x, x', x'' \in \Delta^3_{c_1 c_2 \ldots c_n} \).

Now let us prove that the 5th property is true. Consider the differences:

\[
\inf \Delta^3_{c_1 c_2 \ldots c_{n-1} 2} - \sup \Delta^3_{c_1 c_2 \ldots c_{n-1} 1} = \Delta^3_{c_1 c_2 \ldots c_{n-1} 2(0)} - \Delta^3_{c_1 c_2 \ldots c_{n-1} 1(2)} = 0,
\]

\[
\inf \Delta^3_{c_1 c_2 \ldots c_{n-1} 1} - \sup \Delta^3_{c_1 c_2 \ldots c_{n-1} 0} = \Delta^3_{c_1 c_2 \ldots c_{n-1} 1(0)} - \Delta^3_{c_1 c_2 \ldots c_{n-1} 0(2)} = 0.
\]

\[ \square \]

**Lemma 3** The map \( f \) does not preserve a distance between points.

**Proof** Let us find points \( x_1, x_2 \in [0,1] \) such that \( |f(x_2) - f(x_1)| \neq |x_2 - x_1| \). The statement follows from the existence of jump discontinuities of \( f \). Really, for example, suppose that \( x_1 = \Delta^3_{1(0)} \) and \( x_2 = \Delta^3_{1(1)} \). Then

\[ |f(x_2) - f(x_1)| = |\Delta^3_{2(0)} - \Delta^3_{2(0)}| = \frac{2}{9} \neq |x_2 - x_1| = \frac{1}{9}. \]

In the next section, this statement for a general case of the pseudo-binary representation will be proved in detail. \[ \square \]

**Theorem 2** The map

\[ f : \Delta^3_{x_1 x_2 \ldots x_n} \rightarrow \Delta^3_{0(x_1)0(x_2)\ldots0(x_n)} \]

preserves the Lebesgue measure of an arbitrary interval (segment).

**Proof** Let a segment \([a', a''] \subset [0,1] \) be a certain cylinder \( \Delta^3_{b_1 b_2 \ldots b_n} \). Then from Lemma 2 it follows that the Lebesgue measure of the image and the preimage are equal:

\[ |\Delta^3_{b_1 b_2 \ldots b_n}| = |\Delta^3_{c_1 c_2 \ldots c_n}| = \frac{1}{3^n}. \]

Let \([a', a''] \subset [0,1] \) be a segment that are not a certain cylinder \( \Delta^3_{b_1 b_2 \ldots b_n} \). Then there exists \( \varepsilon \)-covering of this segment by cylinders \( \Delta^3_{b_1 b_2 \ldots b_n} \) of rank \( k \) such that

\[ \square \]
\[ [a', a''] \subseteq \bigcup_k \Delta_{b_1b_2...b_k}^3 \]

and

\[
\lim_{k \to \infty} \lambda \left( \bigcup_k \Delta_{b_1b_2...b_k}^3 \right) = \hat{\lambda} \left( [a', a''] \right).
\]

Also, there exists \( \varepsilon \)-covering of this segment by cylinders \( \Delta_{b_1b_2...b_m}^3 \) of rank \( m \) such that

\[ [a', a''] \supseteq \bigcup_m \Delta_{b_1b_2...b_m}^3 \]

and

\[
\lim_{m \to \infty} \lambda \left( \bigcup_m \Delta_{b_1b_2...b_m}^3 \right) = \hat{\lambda} \left( [a', a''] \right).
\]

Here \( \hat{\lambda}(\cdot) \) is the Lebesgue measure of a set and \( \varepsilon \) is an arbitrary small positive number. That is,

\[
\lambda \left( \bigcup_m \Delta_{b_1b_2...b_m}^3 \right) + \varepsilon \leq \hat{\lambda} \left( [a', a''] \right) \leq \lambda \left( \bigcup_k \Delta_{b_1b_2...b_k}^3 \right) - \varepsilon.
\]

However,

\[
\hat{\lambda} \left( \bigcup_m \Delta_{b_1b_2...b_m}^3 \right) = \sum_k \lambda \left( \Delta_{b_1b_2...b_k}^3 \right), \quad \hat{\lambda} \left( \bigcup_k \Delta_{b_1b_2...b_k}^3 \right) = \sum_m \lambda \left( \Delta_{b_1b_2...b_m}^3 \right).
\]

We have

\[
\lim_{m \to \infty} \sum_m \hat{\lambda} \left( \Delta_{b_1b_2...b_m}^3 \right) = \hat{\lambda} \left( [a', a''] \right) = \lim_{k \to \infty} \sum_k \hat{\lambda} \left( \Delta_{b_1b_2...b_k}^3 \right).
\]

Since from Lemma 2 it follows that the condition

\[
\hat{\lambda} \left( \Delta_{b_1b_2...b_n}^3 \right) = \hat{\lambda} \left( f \left( \Delta_{b_1b_2...b_n}^3 \right) \right)
\]

holds for any \( n \in \mathbb{N} \), we get

\[
\hat{\lambda} \left( [a', a'''] \right) = \lim_{m \to \infty} \sum_m \hat{\lambda} \left( f \left( \Delta_{b_1b_2...b_m}^3 \right) \right)
\]

\[
= \lim_{k \to \infty} \sum_k \hat{\lambda} \left( f \left( \Delta_{b_1b_2...b_k}^3 \right) \right) = \hat{\lambda} \left( f \left( [a', a''] \right) \right).
\]

\[\square\]
For example, let us consider the image of the segment \([\frac{2}{27}, \frac{4}{27}]\) under the map \(f\). Since

\[
\left[\frac{2}{27}, \frac{4}{27}\right] = \Delta^3_{002} \cup \Delta^3_{010} = \left[\Delta^3_{002(0)}, \Delta^3_{002(2)}\right] \cup \left[\Delta^3_{010(0)}, \Delta^3_{010(2)}\right],
\]

we obtain

\[
f(\Delta^3_{002} \cup \Delta^3_{010}) = \Delta^3_{001} \cup \Delta^3_{020}.
\]

But

\[
\frac{2}{27} = \Delta^3_{002(0)} = \Delta^3_{001(2)}, \quad \Delta^3_{001(2)} \not\subset \Delta^3_{002}
\]

and

\[
\frac{4}{27} = \Delta^3_{010(2)} = \Delta^3_{011(0)}, \quad \Delta^3_{011(0)} \not\subset \Delta^3_{010}.
\]

So,

\[
f\left(\left[\frac{2}{27}, \frac{4}{27}\right]\right) = \Delta^3_{001} \cup \Delta^3_{020} \cup \{\Delta^3_{022(0)}, \Delta^3_{002(1)}\}.
\]

### 4 The pseudo-binary representations

Note that the case when \(s = 2\) for the pseudo-s-adic representation is interesting for consideration since this representation (the pseudo-binary representation) can be used in computer science by analogy with the classical binary representation. It can be useful for coding information, protection of information, providing computer protection, etc. Certain theoretic aspects of such applications will be discussed in the further papers of the author of the present article.

Let us remark that in our case there exist only two one-digit converters (operators), i.e.,

\[
\theta_{1,0}(x_n) = \theta_0(x_n) = \begin{cases} 
0 & \text{if } x_n = 0 \\
1 & \text{if } x_n = 1
\end{cases}
\]

and
\[ \theta_{1,1}(x_n) = \theta_1(x_n) = \begin{cases} 1 & \text{if } x_n = 0, \\ 0 & \text{if } x_n = 1. \end{cases} \]

In other words, \( \theta_0(x_n) = x_n \) and \( \theta_1(x_n) = 1 - x_n \).

Let \((k_n)\) be a fixed sequence of positive integers. Then we obtain the matrix

\[
\Theta_{2,(k_n)} = \begin{pmatrix}
\theta_{k_1,0} & \theta_{k_1,1} & \cdots & \theta_{k_1,i} & \cdots & \cdots & \theta_{k_1,(2^{k_1})^{-1}} \\
\theta_{k_2,0} & \theta_{k_2,1} & \cdots & \theta_{k_2,i} & \cdots & \cdots & \theta_{k_2,(2^{k_2})^{-1}} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\theta_{k_n,0} & \theta_{k_n,1} & \cdots & \theta_{k_n,i} & \cdots & \cdots & \theta_{k_n,(2^{k_n})^{-1}} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\end{pmatrix}
\]

(9)

Here for a fixed number \(k_n\) there exist \((2^{k_n})!\) \(k_n\)-digit converters (operators) and \(i = 0, (s^{k_n})! - 1\). In this matrix, a number of rows is equal to infinity and a number of elements in the \(n\)th row equals \((2^{k_n})!\).

Let us choose elements \(\theta_{k_n,i_n}\) and generate a sequence \((\theta_{k_n,i_n})\). Let us investigate the representation of real numbers from \([0, 1]\) that is related with the binary representation by the following rule:

\[
y = \Delta^{(2,(\theta_{k_n,i_n}))}_{\beta_1,\beta_2,\ldots,\beta_n} \equiv \Delta^2_{\theta_{k_1,i_1}(x_1,x_2,\ldots,x_{k_1})} \theta_{k_2,i_1}(x_{k_1+1},x_{k_1+2},\ldots,x_{k_1+k_2}), \ldots, \theta_{k_n,i_1}(x_{k_1+\ldots+k_{n-1}+1},\ldots,x_{k_1+\ldots+k_{n-1}+k_n}), \ldots \equiv \sum_{j=1}^{\infty} \beta_j 2^{-j},
\]

where

\[
(\beta_1, \beta_2, \ldots, \beta_{k_1}) = \theta_{k_1,i_1}(x_1, x_2, \ldots, x_{k_1}),
\]

\[
(\beta_{k_1+1}, \beta_{k_1+2}, \ldots, \beta_{k_1+k_2}) = \theta_{k_2,i_2}(x_{k_1+1}, x_{k_1+2}, \ldots, x_{k_1+k_2}),
\]

\[
\ldots, \ldots, \ldots, \ldots,
\]

\[
(\beta_{k_1+\ldots+k_{n-1}+1}, \ldots, \beta_{k_1+\ldots+k_{n-1}+k_n}) = \theta_{k_n,i_n}(x_{k_1+\ldots+k_{n-1}+1}, \ldots, x_{k_1+\ldots+k_{n-1}+k_n}),
\]

\[
\ldots, \ldots, \ldots, \ldots,
\]

That is, for any \([0, 1] \ni x = \Delta^2_{x_1,x_2,\ldots,x_n}\), the relationship between the binary and pseudo-binary representations is following:

\[
f^{(2)}_y : x = \Delta^2_{x_1,x_2,\ldots,x_n} \rightarrow \Delta^{(2,(\theta_{k_n,i_n}))}_{\beta_1,\beta_2,\ldots,\beta_n} = f^{(2)}_y(x) = y.
\]

(11)

Note that the condition
\[ \theta_{k_n,i_n} \circ \theta_{k_n,i_n} \circ \ldots \circ \theta_{k_n,i_n} (x_{k_1 + \ldots + k_{n-1} + 1}, \ldots, x_{k_1 + \ldots + k_n}) = (x_{k_1 + \ldots + k_{n-1} + 1}, \ldots, x_{k_1 + \ldots + k_n}) \]

holds for \( 1 \leq t \leq 2^{k_n} \). It depends on the definition of \( \theta_{k_n,i_n} \). Here \( k_n, i_n \) are fixed numbers.

**Theorem 3**  
Any number \( x \in [0, 1] \) can be represented by form (10).

**Proof**  
Suppose that \( (\theta_{k_n,i_n}) \) is a fixed sequence of operators (converters) from matrix (9). Here

\[
\theta_{k_n,0}(x_{k_1 + \ldots + k_{n-1} + 1}, \ldots, x_{k_1 + \ldots + k_n}) = (x_{k_1 + \ldots + k_{n-1} + 1}, \ldots, x_{k_1 + \ldots + k_n})
\]

and

\[
\theta_{k_n,2^{k_n}-1}(x_{k_1 + \ldots + k_{n-1} + 1}, \ldots, x_{k_1 + \ldots + k_n}) = (1 - x_{k_1 + \ldots + k_{n-1} + 1}, \ldots, 1 - x_{k_1 + \ldots + k_n}),
\]

where \( k_{n-1} = 0 \) for \( n = 1 \).

A value of any operator \( \theta_{k_n,i_n}, n = 1, 2, \ldots \), is defined by the table

| \( x_{k_1 + \ldots + k_{n-1} + 1} \ldots x_{k_1 + \ldots + k_n} \) | \( \beta_{k_1 + \ldots + k_{n-1} + 1} \ldots \beta_{k_1 + \ldots + k_n} \) | \( \theta_{k_n,i_n}(00\ldots00) \) | \( \theta_{k_n,i_n}(00\ldots01) \) | \( \theta_{k_n,i_n}(01\ldots01) \) | \( \theta_{k_n,i_n}(11\ldots11) \) |
| --- | --- | --- | --- | --- |
| 00 \ldots 00 | \( k_n \) | \( k_n \) | \( k_n \) | \( k_n \) |
| 00 \ldots 01 | \( k_n \) | \( k_n \) | \( k_n \) | \( k_n \) |
| \ldots | \( k_n \) | \( k_n \) | \( k_n \) | \( k_n \) |
| 11 \ldots 11 | \( k_n \) | \( k_n \) | \( k_n \) | \( k_n \) |

Let us consider map (11).

A number \( x \in [0, 1] \) of the form

\[ x = \Delta^2_{x_1 x_2 \ldots x_{n-1} x_n}(0) = \Delta^2_{x_1 x_2 \ldots x_{n-1} \lfloor x_n \rfloor}[x_n - 1](1) \]

is called *binary rational*. The other numbers have the unique binary representation and are called *binary irrational*. Also, numbers of the form

\[ \Delta_{\beta_1 \beta_2 \ldots \beta_{n-1} \beta_n}(2(\theta_{k_n,i_n})) = \Delta_{\beta_1 \beta_2 \ldots \beta_{n-1} \beta_n}[\beta_n - 1](1) \]

are called *pseudo-binary rational*. The other numbers are called *pseudo-binary irrational*. 
If a number $x_0$ is binary rational, then for $x_0 = x_1 = \Delta^2_{z_1 \ldots z_m(0)}$ there exists a number $t$ such that

$$f'_2(x_1) = f'_2\left(\Delta^2_{z_1 \ldots z_m(0)}\right)$$

$$= \Delta^2_{\theta_{k_i,t_0}(x_1, \ldots, x_m)} \theta_{k_{i-1},t_0} (z_{k_1 + \cdots + k_{i-1} + 1}, \ldots, z_m, 0, \ldots, 0) \theta_{k_{i+1},t_1} (0, \ldots, 0) \ldots$$

$$= y_1. \text{ Clearly, the last-mentioned number can be a pseudo-binary irrational number and is a pseudo-binary rational number whenever there exists } n_0 \text{ such that for all } l \geq n_0 \text{ the following condition holds:}$$

$$\theta_{k_{i+l},t_{i+l}} (0, 0, \ldots, 0) = (0, 0, \ldots, 0)$$

(12)

or

$$\theta_{k_{i+l},t_{i+l}} (0, 0, \ldots, 0) = (1, 1, \ldots, 1).$$

(13)

By analogy, we get

$$f'_2(x_2) = f'_2\left(\Delta^2_{z_1 \ldots z_m-1[z_m-1]}\right)$$

$$= \Delta^2_{\theta_{k_i,t_1}(x_1, \ldots, x_m) \theta_{k_{i-1},t_1} (z_{k_1 + \cdots + k_{i-1} + 1}, \ldots, z_m, 1, 1, \ldots, 1) \theta_{k_{i+1},t_1} (1, \ldots, 1) \ldots$$

$$= y_2, \text{ where } \omega = k_1 + \cdots + k_{l-1}. \text{ Whence } y_2 \text{ is a pseudo-binary rational number whenever there exists } n_0 \text{ such that for all } l \geq n_0 \text{ one of conditions (12), (13) is true.}$$

Note that $y_1 = y_2$ whenever for all $j \in \mathbb{N}$.
Some types of numeral systems and their modeling

Here \( j = 1, 2, \ldots \), and \( n_0 \) is a some fixed number.

If \( x_0 = \Delta^2_{d_1,d_2,\ldots,d_n} \) is a binary-irrational number, then \( y_0 = f_{2^r}(x_0) = \Delta^{(2,(\theta_{d_1,d_2,\ldots,d_n}))}_{(1)} \) can be pseudo-binary rational (whenever its representation contains a period \((0)\) or \((1)\)) or
pseudo-binary irrational. It depends on a sequence \( \{\theta_{k_n,i_n}\} \).

In the general case, let us choose any \( x', x'' \in [0, 1] \).

If \( x' \neq x'' \), then:

- \( y' = f_{2'}(x') = f_{2'}(x'') = y'' \) whenever only \( y' = y'' \) is pseudo-binary rational. The set of all pseudo-binary rational numbers is a countable set;
- in the other case, \( y' \neq y'' \) when \( x', x'' \) are binary irrational numbers and \( y', y'' \) are pseudo-binary irrational numbers or different pseudo-binary rational numbers.

If \( x' = x'' \) (the case when \( x', x'' \) are binary rational), then \( f_{2'}(x') = f_{2'}(x'') \) or \( f_{2'}(x') = f_{2'}(x'') \) by analogy.

Since

\[
\inf_{x \in [0,1]} f_{2'}(x) = 0, \quad \sup_{x \in [0,1]} f_{2'}(x) = 1,
\]

and the sets of all binary rational and pseudo-binary rational numbers are countable sets, we obtain that the set of all numbers having the unique pseudo-binary representation is a set of full Lebesgue measure and the map \( f_{2'} \) determines a numeral system.

**Corollary 1** Any map \( f_{2'} \) is a bijective mapping on the set

\[
[0, 1] \setminus \left( S_Q \cup S'_Q \right),
\]

where \( S_Q \) is the set of all binary rational points and

\[
S'_Q \equiv \{ x : f_{2'}(x) \text{ is pseudo-binary rational} \}.
\]

**Corollary 2** The set

\[
\{ x : x = f_{2'}^{-1}(y) \}
\]

is an one- or two-element set for any \( y \in [0, 1] \).

**Lemma 4** A map \( f_{2'} \) is continuous at binary irrational points.

According to a sequence \( \{\theta_{k_n,i_n}\} \), a certain binary rational point is a point of discontinuity of \( f_{2'} \) or \( f_{2'} \) is continuous at this point.

**Proof** Let \( x_0 \) be a binary irrational number. Then there exists positive integer \( n_0 \) such that the following system of conditions holds for \( x_0 = \Delta_1/2 \ldots \Delta_n/2 \ldots \) and \( x = \Delta'_1/2 \ldots \Delta'_n/2 \ldots \):
\[
\begin{aligned}
\{ x_r = \gamma_r \quad &\text{for } r = 1, n_0 - 1 \\
\overline{x}_{n_0} \neq \overline{\gamma}_{n_0}. \}
\end{aligned}
\]

From this system, it follows that the conditions \( n_0 \to \infty \) and \( x \to x_0 \) are equivalent. In addition, for

\[
f'_2(x_0) = f'_2\left(\Delta^2_{x_1x_2...}x_0...\right) = \Delta^{(2, (\theta_{k_1k_2}))}_{\beta_1\beta_2...\beta_n...}
\]

and

\[
f'_2(x) = f'_2\left(\Delta^2_{\gamma_1\gamma_2...\gamma_n...}\right) = \Delta^{(2, (\theta_{k_1k_2}))}_{\delta_1\delta_2...\delta_n...},
\]

we get

\[
|f'_2(x) - f'_2(x_0)| = \left| \sum_{n=1}^{\infty} \delta_n - \beta_n \right| \leq \sum_{l=m_0}^{\infty} |\delta_l - \beta_l| \leq \sum_{l=m_0}^{\infty} \frac{1}{2^l} = \frac{1}{2^{m_0-1}} \to 0 \text{ as } m_0 \to \infty.
\]

Here \( m_0 \) is a number such that \( \delta_j(x) = \beta_j(x_0) \) for all \( j = 1, m_0 - 1 \).

So \( f'_2 \) is continuous at binary irrational points.

In the case when \( x_0 \) is binary rational, i.e.,

\[
x_0 = \Delta^2_{x_1...x_m(0)} = \Delta^2_{x_1...x_m-1|x_m-1}(1)
\]

it is clear that

\[
\lim_{x \to x_0-0} f'_2(x) = f'_2\left(\Delta^2_{x_1...x_m-1|x_m-1}(1)\right)
\]

and

\[
\lim_{x \to x_0+0} f'_2(x) = f'_2\left(\Delta^2_{x_1...x_m-1x_m(0)}\right).
\]

Then:

- If \( \lim_{x \to x_0-0} f'_2(x) \neq \lim_{x \to x_0+0} f'_2(x) \), then \( x_0 \) is a point of discontinuity;
- If \( \lim_{x \to x_0-0} f'_2(x) = \lim_{x \to x_0+0} f'_2(x) \), then \( f'_2 \) is continuous at \( x_0 \).

\[\square\]

**Definition 3** A set of the form

\[
\Delta^{(2, (\theta_{k_1k_2}...))}_{c_1c_2...c_{k_1+k_2+...+k_n}} \equiv \left\{ z : z = \Delta^{(2, (\theta_{k_1k_2}...))}_{c_1c_2...c_{k_1+k_2+...+k_n}\beta_{k_1+k_2+...+k_{l+1}}...}\right\},
\]

where \( c_1, \ldots, c_{k_1+k_2+...+k_n} \) is a fixed tuple of the binary digits, \( l = 1, 2, \ldots, \) and \( \beta_{k_1+k_2+...+k_{l+1}} \in \{0, 1\} \) such that...
\((\beta_{k_1+k_2+\ldots+k_{n-1}+1}, \ldots, \beta_{k_1+k_2+\ldots+k_{n-1}+1}) = \theta_{k_1+i_{n-1}}(\alpha_{k_1+k_2+\ldots+k_{n-1}+1}, \ldots, \alpha_{k_1+k_2+\ldots+k_{n-1}+1})\),

is called a pseudo-binary cylinder of rank \(k_1 + k_2 + \ldots + k_n\) with base \(c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}\).

**Remark 4** The following considerations are useful for proving the next lemma on properties of pseudo-binary cylinders.

Let us have a binary cylinder \(\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2\) of rank \(k_1 + \ldots + k_n\) with base \(b_1b_2\ldots b_{k_1+\ldots+k_n}\). Here \(b_1, b_2, \ldots, b_{k_1+\ldots+k_n}\) is a fixed tuple of the binary digits. That is,

\(\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2 = \{x : x = \Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2 a_{k_1+\ldots+k_n+1}x_{k_1+\ldots+k_n+2}\ldots\},\)

where \(a_j \in \{0, 1\}\) for all \(\forall j > k_1 + k_2 + \ldots + k_n\). Whence

\[\Delta_{c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}}^2 = f'_2 \left(\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2 \right) = \Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2 \theta_{k_1+i_{k_2+\ldots+k_n}}(b_1b_2\ldots b_{k_1+\ldots+k_n+1}\ldots b_{k_1+\ldots+k_n}),\]

where \((c_{k_1+\ldots+k_n+1}, \ldots, c_{k_1+\ldots+k_j}) = \theta_{k_1+i_{k_2+\ldots+k_n}}(b_1b_2\ldots b_{k_1+\ldots+k_n+1}\ldots b_{k_1+\ldots+k_n})\) for all \(j = 1, n\).

Since for any \(\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2\) the condition

\[\lambda \left(\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2 \right) = \frac{1}{2^n}\]

holds (a cylinder \(\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2\) is a closed interval, the Lebesgue measure \(\lambda \left(\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2 \right)\) of \(\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2\) and the diameter \(\left|\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2\right| = \sup \Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2 - \inf \Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2\) are equal), we obtain

\[\lambda \left(\Delta_{c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}}^2 \right) = \left|\Delta_{c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}}^2 \right| = \sup \Delta_{c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}}^2 \left(\Delta_{c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}}^2 \right) - \inf \Delta_{c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}}^2 \left(\Delta_{c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}}^2 \right) = \left|\Delta_{c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}}^2 \right| = \frac{1}{2^{k_1+k_2+\ldots+k_n}}.\]

So,

\[\lambda \left(\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2 \right) = \lambda \left(\Delta_{c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}}^2 \right).\]

Also, note that for any \(x \in \Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2\)

\[f'_2(x) \in f'_2 \left(\Delta_{b_1b_2\ldots b_{k_1+\ldots+k_n}}^2 \right) = \Delta_{c_1c_2\ldots c_{k_1+k_2+\ldots+k_n}}^2,\]

and
\[ \inf \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}} \leq f^{(2, (\theta_{k_1, \ldots, k_n}))}_{2'}(x) \leq \sup \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}}. \]

In addition, if \( x \) is the endpoint of \( \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{b_{k_1, \ldots, b_{k_1 + k_2 + \cdots + k_n}}} \), then

\[
v = \lim_{x \to x_0^+} f^{(2, (\theta_{k_1, \ldots, k_n}))}_{2'}(x) - \lim_{x \to x_0^-} f^{(2, (\theta_{k_1, \ldots, k_n}))}_{2'}(x) \leq \frac{1}{2^{k_1 + k_2 + \cdots + k_{n-1}}}. \]

This condition follows from the definition of \( f^{(2, (\theta_{k_1, \ldots, k_n}))}_{2'} \), the last-mentioned properties of this map, and properties of cylinders.

**Lemma 5** Cylinders \( \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}} \) have the following properties:

1. A cylinder \( \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}} \) is a closed interval and
   \[ \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}} = \left[ \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}}(0), \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}}(1) \right]. \]

2. \( \lambda \left( \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}}(0) \right) = \frac{1}{2^{k_1 + k_2 + \cdots + k_n}}. \]

3. \( \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n + 1}} \subseteq \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n + 1}} \).

4. \( x = \bigcap_{n=1}^{\infty} \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}} \).

5. Cylinders \( \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}} \) situated by the rule:

\[
f^{(2, (\theta_{k_1, \ldots, k_n}))}_{2'}:
\begin{align*}
(0, 0, \ldots, 0, 0) & \rightarrow \theta_{k_1, \ldots, k_n}(0, 0, \ldots, 0, 0) \\
(0, 0, \ldots, 0, 1) & \rightarrow \theta_{k_1, \ldots, k_n}(0, 0, \ldots, 0, 1) \\
\cdots \cdots \cdots \\
(1, 1, \ldots, 1, 1) & \rightarrow \theta_{k_1, \ldots, k_n}(1, 1, \ldots, 1, 1)
\end{align*}
\]

**Proof** Suppose \( x \in \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}} \). Then

\[ x' = \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}}(0) \leq x \leq \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}}(1) = x''. \]

Hence \( x \in [x', x''] \) and \( \Delta^{(2, (\theta_{k_1, \ldots, k_n}))}_{c_1, c_2, \ldots, c_{k_1 + k_2 + \cdots + k_n}} \subseteq [x', x''] \). Since
and

$$\Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}(0)} = \sum_{j=1}^{k_{1}+...+k_{n}} \frac{c_{j}}{2^{j}} + \frac{1}{2^{k_{1}+k_{2}+...+k_{n}}} \inf_{t=k_{1}+...+k_{n}+1} \sum_{t=1}^{\infty} \frac{\beta_{t}}{2^{t}}$$

we obtain that $x, x', x'' \in \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}}$. So a cylinder $\Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}}$ is a closed interval.

The second property follows from the last property and remark.

Let us prove the third property. Let $m$ be an arbitrary fixed positive integer. Let us show that the system of conditions

$$\left\{\begin{array}{l}
\inf \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}+1}} \geq \inf \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}}\\
\sup \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}+1}} \leq \sup \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}}
\end{array}\right.$$  

is true. Let us consider the difference

$$\inf \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}+1}} - \inf \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}}
= \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}+1}(0)} - \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}(0)} \geq 0$$

and

$$\sup \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}+1}} - \sup \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}}
= \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}+1}(1)} - \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}(1)} \geq 0.$$  

The 4th property. From the last-mentioned proved properties it follows that

$$\Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}}(0)} \subset \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}}(0)} \subset \ldots \subset \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}(1)} \subset \ldots.$$  

From Cantor’s intersection theorem it follows that

$$\bigcap_{n=1}^{\infty} \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{k_{1}+k_{2}+...+k_{n}}} = x = \Delta^{(2,(\theta_{t,k,n}))}_{c_{1}c_{2}...c_{}\ldots}.$$  

The 5th property follows from the definition of $f_{2}$. \hfill $\square$

Remark 5 As for the last-mentioned property, let us remark that properties of the pseudo-s-adic numeral system can be describe in terms of the s-adic numeral system or in terms of such modified numeral systems. Formulations of certain properties depend on such consideration. In particular, these properties are representations of numbers having two different representations and situating cylinders of rank $n$. We
can see these properties by the definition of the relationship between the s-adic and pseudo-s-adic representation. Now we give example for the case of the nega-binary representation. Also, note that in this article properties of the modified s-adic representations are more investigated in terms of the s-adic (the binary or ternary) representations.

**Example 6** The nega-binary representation is a representation of the form

\[
\left[ \frac{-2}{3} \mid \frac{1}{3} \right] \ni x = \Delta_{\beta_1,\beta_2,\ldots,\beta_n}^{-2} \equiv -\frac{\beta_1}{2} + \frac{\beta_2}{2^2} - \frac{\beta_3}{2^3} + \ldots + \frac{\beta_n}{(-2)^n} + \ldots, \quad \beta_n \in \{0, 1\}.
\]

However,

\[
\Delta_{\beta_1,\beta_2,\ldots,\beta_n}^{-2} + \Delta_{(10)}^2 = \Delta_{[1-\beta_1][1-\beta_2][1-\beta_3\ldots][1-\beta_n]}^2 \equiv \Delta_{x_1x_2\ldots x_n}^2 \in [0, 1].
\]

In terms of the binary representation,

\[
x_1 = \Delta_{x_1x_2\ldots x_n-1x_n}^2(0) = \Delta_{x_1x_2\ldots x_n-1[x_n-1]}^2(1) = x_2,
\]

where \(x_n \neq 0\) and \(x_n = 1\) (since \(x_n \in \{0, 1\}\)). But in terms of the nega-binary representation

\[
\beta_n = \begin{cases} 1 - x_n & \text{if } n \text{ is odd} \\ x_n & \text{if } n \text{ is even}. \end{cases}
\]

Whence

\[
\Delta_{\beta_1,\beta_2,\ldots,\beta_{n-1}\beta_n}^{-2}(10) = \Delta_{\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_{n-1}[1-\beta_n]}^2(01), \quad \beta_n = 1.
\]

Similarly, cylinders \(\Delta_{\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_n}^2\) are left-to-right situated for any \(n \in \mathbb{N}\). But cylinders \(\Delta_{[1-\beta_1][1-\beta_2][1-\beta_3\ldots][1-\beta_n]}^{-2}\) are left-to-right situated and cylinders \(\Delta_{[1-\beta_1][1-\beta_2][1-\beta_3\ldots][1-\beta_n]}^2\) are right-to-left situated. In addition, the map \(f_{2'}\) can be defined by the following table

| \(x_{2m+1}x_{2(m+1)}\) | \(00\) | \(01\) | \(10\) | \(11\) |
|-----------------|--------|--------|--------|--------|
| \(\beta_{2m+1}\beta_{2(m+1)}\) | \(10\) | \(11\) | \(00\) | \(01\) |

**Theorem 4** A map \(f_{2'}\) has the following properties (characteristics):

- Suppose \([a, b]\) is a closed interval. Then the set \(f_{2'}([a, b])\) is a closed interval or a union of closed intervals with a finite set of isolated points, or a union of closed intervals.
- A map \(f_{2'}\) preserves the Lebesgue measure of intervals.
- A map \(f_{2'}\), except for the cases when \(f_{2'}(x) = x\) or \(f_{2'}(x) = 1 - x\), does not preserve distance between points on \([0,1]\).
Proof Let \([a, b]\) be a certain closed interval. Then from the last lemma it follows that \(f_2([a, b])\) is a closed interval whenever \([a, b]\) is a certain binary cylinder \(\Delta^2_{b_1 b_2 \ldots b_{k_1 + \ldots + k_n}}\).

Let us consider the case when \([a, b]\) is not a binary cylinder. Then there exist coverings \([a, b]\) by binary cylinders \(\Delta^2_{b_1 b_2 \ldots b_{k_1 + \ldots + k_n}}\) such that

\[
[a, b] \subseteq \bigcup_l \Delta^2_{b_1 b_2 \ldots b_{k_1 + \ldots + k_l}};
\]

\[
[a, b] \supseteq \bigcup_m \Delta^2_{b_1 b_2 \ldots b_{k_1 + \ldots + k_m}};
\]

and

\[
\lim_{l \to \infty} \left( \bigcup_l \Delta^2_{b_1 b_2 \ldots b_{k_1 + \ldots + k_l}} \right) = \lim_{m \to \infty} \left( \bigcup_m \Delta^2_{b_1 b_2 \ldots b_{k_1 + \ldots + k_m}} \right) = \lambda([a, b]).
\]

Since \(f_2^2 \left( \Delta^2_{b_1 b_2 \ldots b_{k_1 + \ldots + k_n}} \right)\) is a closed interval,

\[
\lambda \left( f_2^2 \left( \Delta^2_{b_1 b_2 \ldots b_{k_1 + \ldots + k_n}} \right) \right) = \lambda \left( \Delta^2_{b_1 b_2 \ldots b_{k_1 + \ldots + k_n}} \right),
\]

and an arbitrary binary rational point can be a jump discontinuity of \(f_2^2\), we obtain that the first and the second properties are true.

To prove the third property, let us consider mapping two adjacent cylinders of rank \(k_1 + k_2 + \ldots + k_n\) under the action of \(f_2^2\). If such two adjacent cylinders map to non-adjacent cylinders under \(f_2^2\), then one can assume that a distance between points of these cylinders is not preserved under the map \(f_2^2\).

It is easy to see that all pairwise adjacent cylinders are pairwise adjacent under \(f_2^2\) whenever \((\theta_{k_n, i_n}) = const = \theta_{k_n, 0}\) or \((\theta_{k_n, i_n}) = const = \theta_{k_n, (2^{kn})!-1}\). That is, when \(f_2^2(x) = x\) or \(f_2^2(x) = 1 - x\). In the other case, there exist at least one pair of adjacent cylinders such that their images under \(f_2^2\) are not adjacent cylinders. That is, from the existence of jump discontinuities of \(f_2^2\) it follows that our statement is true.

The next lemma follow from the definition of \(f_2^2\), the existence of jump discontinuities of \(f_2^2\), and from the placement of adjacent cylinders of the same rank.

**Lemma 6** A map \(f_2^2\):

- is strictly monotonic whenever \(f_2^2 = x\) or \(f_2^2 = 1 - x\); that is, when for all positive integers \(n\) the following conditions hold: \(\theta_{k_n, i_n} = const\) and \(\theta_{k_n, i_n} = \theta_{k_n, 0}\) or \(\theta_{k_n, i_n} = \theta_{k_n, (2^{kn})!-1}\).
- has intervals of the monotonicity whenever almost all elements of \((\theta_{k_n, i_n})\) are equal only \(\theta_{k_n, 0}\) or \(\theta_{k_n, (2^{kn})!-1}\).
- is not monotonic in another case.
5 Certain peculiarities of $f_{2'}$

Suppose $\eta$ is a random variable defined by the following form

$$\eta = \Delta_{\xi_1, \xi_2, \cdots, \xi_n},$$

where digits $\xi_n$ are random and taking the values 0, 1 with probabilities $p_0, p_1$. Here $p_0 + p_1 = 1$. That is, $\xi_n$ are independent, and $P\{\xi_n = i_n\} = p_{i_n}$, where $i_n \in \{0, 1\}$. Then the distribution function $F_\eta$ of the random variable $\eta$ has the form

$$F_\eta(x) = \left\{ \begin{array}{ll}
0 & \text{if } x < 0 \\
\sum_{n=2}^{\infty} \left( a_{\beta_n}(x) \prod_{j=1}^{n-1} p_{\beta_j(x)} \right) & \text{if } 0 \leq x < 1 \\
1 & \text{if } x \geq 1,
\end{array} \right.$$  

where $x = \Delta_{\beta_1, \beta_2, \cdots, \beta_n}$. It follows from the equality $F_\eta = P\{\eta < x\}$.

Note that one can consider the case when $P\{\xi_n = i_n\} = p_{i_n, n}$, where for any $n \in \mathbb{N}$ $p_{0, n} + p_{1, n} = 1$. Then

$$F_\eta(x) = \left\{ \begin{array}{ll}
0 & \text{if } x < 0 \\
\sum_{n=2}^{\infty} \left( a_{\beta_n, 1}(x) \prod_{j=1}^{n-1} p_{\beta_j(x)} \right) & \text{if } 0 \leq x < 1 \\
1 & \text{if } x \geq 1.
\end{array} \right.$$  

Note also that a function of the form

$$f_D(x) = a_{\beta_1}(x) + \sum_{n=2}^{\infty} \left( a_{\beta_n}(x) \prod_{j=1}^{n-1} p_{\beta_j(x)} \right),$$

where $x = \Delta_{\xi_1, \xi_2, \cdots, \xi_n}$ and

$$(\beta_{k_1+\cdots+k_{n-1}+1}, \cdots, \beta_{k_1+\cdots+k_n}) = \theta_{k_n, i_n}(\alpha_{k_1+\cdots+k_{n-1}+1}, \cdots, \alpha_{k_1+\cdots+k_n}),$$

$n = 1, 2, \ldots$, and $k_0 = 0$, is a generalization of the Salem ([12]) function. In addition,

$$f_D = F_\eta \circ f_{2'}. $$

One of the next papers of the author of the present article will be devoted to the investigation of $F_\eta, f_D, f_{2'}$, and of their generalizations.

Hypothesis 2 The function $F_\eta$ is singular.

Now let us consider integral properties of $f_{2'}$. For the well-posedness of $f_{2'}$, we shall not consider the binary representation, which has the period (1) (without the case of the representation of the number 1).

Theorem 5 The Lebesgue integral of the function $f_{2'}$ is equal to $\frac{1}{2}$.
Proof Since $0 \leq f_2'(x) \leq 1$, we choose

$$E_n = \{x : y_{n-1} \leq f_2'(x) \leq y_n\} = \Delta_{b_1 k_2 \cdots b_{k_1 + k_2 + \cdots + k_n}}^2,$$

where

$$\lambda(E_n) = \frac{1}{2^{k_1 + k_2 + \cdots + k_n}}$$

and

$$T = \{0, \Delta_{c_1 c_2 \cdots c_{k_1} (0)}^{2, (0)}, \Delta_{c_1 c_2 \cdots c_{k_1 + k_2} (0)}^{2, (0)}, \ldots\}.$$

Clearly, $\overline{y_n} \equiv y_{n-1} \in [y_{n-1}, y_n)$ and the conditions $\lambda(E_n) \to 0$ and $n \to \infty$ are equivalent. Then

$$I = \lim_{n \to \infty} \left( \sum_{\text{all } c \in \{0,1\}} \Delta_{c_1 c_2 \cdots c_{k_1 + k_2 + \cdots + k_n} (0)}^{2, (0)} \right) = \lim_{n \to \infty} \frac{2^{k_1 + \cdots + k_n} - 1}{2^{k_1 + \cdots + k_n}} = \frac{1}{2}.$$

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