The text is a scientific paper discussing Lie-algebraic deformations of Minkowski space with undeformed Poincaré algebra. These deformations interpolate between Snyder and κ-Minkowski space. The authors find realizations of noncommutative coordinates in terms of commutative coordinates and derivatives. Deformed Leibniz rule, the coproduct structure and star product are found. Special cases, particularly Snyder and κ-Minkowski in Maggiore-type realizations are discussed. The construction leads to a new class of deformed special relativity theories.

Keywords: noncommutative physics; snyder space; kappa-Minkowski space.

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1. Introduction

Noncommutative (NC) physics has become an integral part of present-day high energy physics theories. It reflects a structure of space-time which is modified in comparison to space-time structure underlying the ordinary commutative physics. This modification of space-time structure is a natural consequence of the appearance of a new fundamental length scale known as Planck length. There are two main physical contexts within which a signal for the existence of a Planck length scale appears. The first one lies within a loop quantum gravity framework in which the Planck length plays a fundamental role. There, a presence of a new fundamental length scale leads after quantization to discrete eigenvalues of the area and volume operators. It appears that in loop quantum gravity, the area and volume operators have discrete spectra, with minimal eigenvalue proportional to a square and cube of the Planck length, respectively. The second physical context where one can find a signal for the existence of a fundamental length scale comes from some observations of ultra-high energy cosmic rays which seem to contradict the usual understanding of some astrophysical processes like, for example, electron-positron production in collisions of high energy photons. It turns out that deviations observed in these processes can be explained by modifying dispersion relation in such a way as to...
incorporate the fundamental length scale \( \Lambda \). NC space-time has also been revived in the paper by Seiberg and Witten \(^4\) where NC manifold emerged in a certain low energy limit of open strings moving in the background of a two form gauge field.

As a new fundamental, observer-independent quantity, Planck length is incorporated in kinematical theory within the framework of the so called doubly special relativity theory (DSR) \(^5\). The idea that lies behind DSR is that there exist two observer-independent scales, one of velocity, identified with the speed of light, and the other of mass, which is expected to be of the order of Planck mass. It can also be considered as a semi-classical, flat space limit of quantum gravity in a similar way special relativity is a limit of general relativity and, as such, reveals a structure of space-time when the gravitational field is switched off.

Following the same line of reasoning, the symmetry algebra for doubly special relativity can be obtained by deforming the ordinary Poincaré algebra to get some kind of a quantum (Hopf) algebra, known as \( \kappa \)-Poincaré algebra \(^7,8\), so that \( \kappa \)-Poincaré algebra is in the same relation to DSR theory as the standard Poincaré algebra is related to special relativity.

\( \kappa \)-Poincaré algebra is an algebra that describes in a direct way only the energy-momentum sector of the DSR theory. Although this sector alone is insufficient to set up physical theory, the Hopf algebra structure makes it possible to extend the energy-(angular)momentum algebra to the algebra of space-time. It is shown in \(^9\) that different representations (bases) of \( \kappa \)-Poincaré algebra correspond to different DSR theory. However, the resulting space-time algebra, obtained by the extension of energy-momentum sector, is independent of the representation, i.e. energy-momentum algebra chosen \(^9,10\).

It is also shown in \(^10\) that there exists a transformation which maps \( \kappa \)-Minkowski space-time into space-time with noncommutative structure described by the algebra first introduced by Snyder \(^11\). In \(^10\) the use of Snyder algebra provided NC space-time structure of Minkowski space with undeformed Lorentz symmetry. In the same paper it is argued that the algebra introduced by Snyder provides a structure of configuration space for DSR and thus can be used to construct the second order particle Lagrangian, what would make it possible to define physical four-momenta determined by the particle dynamics. This would be significant step forward because the theoretical development in this area has been mainly kinematical so far. One such dynamical picture has been given recently in \(^12,13\) where it was shown that reparametrisation symmetry of the proposed Lagrangian, together with the appropriate change of variables and conveniently chosen gauge fixing conditions, leads to an algebra which is a combination of \( \kappa \)-Minkowski and Snyder algebra. This generalized type of algebra describing noncommutative structure of Minkowski space-time is shown to be consistent with the Magueijo-Smolin dispersion relation. This type of NC space is also considered in \(^14\). It has to be stressed that NC spaces in neither of these papers are of Lie-algebra type.

In order to fill this gap, in the present paper we unify \( \kappa \)-Minkowski and Sny-
der space in a more general NC space which is of a Lie-algebra type, with Lorentz generators and NC space-time coordinates closing the Lie algebra. In addition, it is characterized by the undeformed Poincaré algebra and deformed coalgebra. In other words, we shall consider a type of NC space which interpolates between \( \kappa \)-Minkowski space and Snyder space in a Lie-algebraic way and has all deformations contained in the coalgebraic sector. Particularly, in this paper we shall be interested in finding a coproduct for translation generators, which corresponds to a generalized momentum addition rule. First example of NC space with undeformed Poincaré algebra, but with deformed coalgebra is given by Snyder. Some other types of NC spaces are also considered within the approach in which the Poincaré algebra is undeformed and coalgebra deformed, in particular the type of NC space with \( \kappa \)-deformation. Here we present a broad class of Lie-algebra type deformations with the same property of having deformed coalgebra, but undeformed algebra. The investigations carried out in this paper are along the track of developing general techniques of calculations, applicable for a widest possible class of NC spaces and as such are a continuation of the work done in a series of previous papers. The methods used in these investigations were taken over from the Fock space analysis carried out in.

The plan of the paper is as follows. In section 2 we introduce a type of deformations of Minkowski space that have a structure of a Lie algebra and which interpolate between \( \kappa \)-type of deformations and deformations of the Snyder type. In section 3 we analyze realizations of NC space in terms of operators belonging to the undeformed Heisenberg-Weyl algebra. Section 4 is devoted to an analysis of the effects which deformations we are considering have on the coalgebraic structure of the symmetry algebra. In the same section we specialize the general results obtained to some interesting special cases, such as Snyder space and \( \kappa \)-Minkowski space.

2. Noncommutative coordinates and Poincaré algebra

We are considering a Lie algebra type of noncommutative (NC) space generated by the coordinates \( \hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1} \), satisfying the commutation relations

\[
[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) + s M_{\mu\nu},
\]

where indices \( \mu, \nu = 0, 1, \ldots, n-1 \) and \( a_0, a_1, \ldots, a_{n-1} \) are components of a four-vector \( a \) in Minkowski space whose metric signature is \( \eta_{\mu\nu} = \text{diag}(-1, 1, \cdots, 1) \). The quantities \( a_\mu \) and \( s \) are deformation parameters which measure a degree of deviation from standard commutativity. They are related to length scale characteristic for distances where it is supposed that noncommutative character of space-time becomes important. When parameter \( s \) is set to zero, noncommutativity reduces to covariant version of \( \kappa \)-deformation, while in the case that all components of a four-vector \( a \) are set to 0, we get the type of NC space considered for the first time by Snyder. The NC space of this type has been analyzed in the literature from different points of view.
The symmetry of the deformed space (1) is assumed to be described by an undeformed Poincaré algebra, which is generated by generators \( M_{\mu \nu} \) of the Lorentz algebra and generators \( D_\mu \) of translations. This means that generators \( M_{\mu \nu}, M_{\mu \nu} = -M_{\nu \mu} \), satisfy the standard, undeformed commutation relations,

\[
[M_{\mu \nu}, M_{\lambda \rho}] = \eta_{\nu \lambda} M_{\mu \rho} - \eta_{\mu \lambda} M_{\nu \rho} - \eta_{\nu \rho} M_{\mu \lambda} + \eta_{\mu \rho} M_{\nu \lambda},
\]

(2)

with the identical statement as well being true for the generators \( D_\mu \),

\[
[D_\mu, D_\nu] = 0,
\]

(3)

\[
[M_{\mu \nu}, D_\lambda] = \eta_{\nu \lambda} D_\mu - \eta_{\mu \lambda} D_\nu.
\]

(4)

The undeformed Poincaré algebra, Eqs. (2), (3) and (4) define the energy-momentum sector of the theory considered. However, full description requires space-time sector as well. Thus, it is of interest to extend the algebra (2), (3) and (4) so as to include NC coordinates \( \hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1} \), and to consider the action of Poincaré generators on NC space (1),

\[
[M_{\mu \nu}, \hat{x}_\lambda] = \hat{x}_\mu \eta_{\nu \lambda} - \hat{x}_\nu \eta_{\mu \lambda} - i (a_\mu M_{\nu \lambda} - a_\nu M_{\mu \lambda}).
\]

(5)

The main point is that commutation relations (1), (2) and (5) define a Lie algebra generated by Lorentz generators \( M_{\mu \nu} \) and \( \hat{x}_\lambda \). The necessary and sufficient condition for consistency of an extended algebra, which includes generators \( M_{\mu \nu}, D_\mu \) and \( \hat{x}_\lambda \), is that Jacobi identity holds for all combinations of the generators \( M_{\mu \nu}, D_\mu \) and \( \hat{x}_\lambda \). Particularly, the algebra generated by \( D_\mu \) and \( \hat{x}_\nu \) is a deformed Heisenberg-Weyl algebra and we require that this algebra has to be of the form,

\[
[D_\mu, \hat{x}_\nu] = \Phi_{\mu \nu}(D),
\]

(6)

where \( \Phi_{\mu \nu}(D) \) are some functions of generators \( D_\mu \), which are required to satisfy the boundary condition \( \Phi_{\mu \nu}(0) = \eta_{\mu \nu} \). This condition means that deformed NC space, together with the corresponding coordinates, reduces to ordinary commutative space in the limiting case of vanishing deformation parameters, \( a_\mu, s \to 0 \).

One certain type of noncommutativity, which interpolates between Snyder space and \( \kappa \)-Minkowski space, has already been investigated in (12,13,14) in the context of Lagrangian particle dynamics. However, in these papers algebra generated by NC coordinates and Lorentz generators is not linear and is not closed in the generators of the algebra. Thus, it is not of Lie-algebra type. In contrast to this, here we consider an algebra (1), (2), (5), which is of Lie-algebra type, that is, an algebra which is linear in all generators and Jacobi identities are satisfied for all combinations of generators of the algebra. Besides that, it is important to note that, once having relations (1) and (2), there exists only one possible choice for the commutation relation between \( M_{\mu \nu} \) and \( \hat{x}_\lambda \), which is consistent with Jacobi identities and makes Lie algebra to close, and this unique choice is given by the commutation relation (5).
In subsequent considerations we shall be interested in possible realizations of the space-time algebra \((1)\) in terms of canonical commutative space-time coordinates \(X^\mu\),

\[
[X_\mu, X_\nu] = 0, \tag{7}
\]

which, in addition, with derivatives \(D_\mu\) close the undeformed Heisenberg algebra,

\[
[D_\mu, X_\nu] = \eta_{\mu\nu}. \tag{8}
\]

From the beginning, the generators \(D_\mu\) are considered as deformed derivatives conjugated to \(\hat{x}\) through the commutation relation \((6)\). However, in the whole paper we restrict ourselves to natural choice \(17\) in which deformed derivatives are identified with the ordinary derivatives, \(D_\mu \equiv \partial / \partial X^\mu\).

Thus, our aim is to find a class of covariant \(\Phi_{\alpha\mu}(D)\) realizations,

\[
\hat{x}_\mu = X^\alpha \Phi_{\alpha\mu}(D), \tag{9}
\]

satisfying the boundary conditions \(\Phi_{\alpha\mu}(0) = \eta_{\alpha\mu}\), and commutation relations \((1)\) and \((5)\). In order to complete this task, we introduce the standard coordinate representation of the Lorentz generators \(M_{\mu\nu}\),

\[
M_{\mu\nu} = X_\mu D_\nu - X_\nu D_\mu. \tag{10}
\]

All other commutation relations, defining the extended algebra, are then automatically satisfied, as well as all Jacobi identities among \(\hat{x}_\mu\), \(M_{\mu\nu}\), and \(D_\mu\). This is assured by the construction \((9)\) and \((10)\). In the next section we turn to problem of finding an explicit \(\Phi_{\mu\nu}(D)\) realizations \((9)\).

## 3. Realizations of NC coordinates

Let us define general covariant realizations:

\[
\hat{x}_\mu = X_\mu \varphi + i(aX_\nu)(D_\mu \beta_1 + ia_\mu D^2 \beta_2) + i(XD) (a_\mu \gamma_1 + i(a^2 - s)D_\mu \gamma_2), \tag{11}
\]

where \(\varphi, \beta_i\) and \(\gamma_i\) are functions of \(A = ia_\alpha D^\alpha\) and \(B = (a^2 - s)D_\alpha D^\alpha\). We further impose the boundary condition that \(\varphi(0,0) = 1\) as the parameters of deformation \(a_\mu \to 0\) and \(s \to 0\). In this way we assure that \(\hat{x}_\mu\) reduce to ordinary commutative coordinates in the limit of vanishing deformation.

It can be shown that Eq.\((5)\) requires the following set of equations to be satisfied,

\[
\frac{\partial \varphi}{\partial A} = -1, \quad \frac{\partial \gamma_2}{\partial A} = 0, \quad \beta_1 = 1, \quad \beta_2 = 0, \quad \gamma_1 = 0. \tag{12}
\]

Besides that, the commutation relation \((11)\) leads to the additional two equations,

\[
\varphi \left( \frac{\partial \varphi}{\partial A} + 1 \right) = 0, \tag{12}
\]

\[
(a^2 - s)[2(\varphi + A) \frac{\partial \varphi}{\partial B} - \gamma_2(\varphi \frac{\partial \varphi}{\partial A} + 2B \frac{\partial \varphi}{\partial B}) + \gamma_2 \varphi] - a^2 \frac{\partial \varphi}{\partial A} - s = 0. \tag{13}
\]
The important result of this paper is that all above required conditions are solved by a general form of realization which in a compact form can be written as

\[ \hat{x}_\mu = X_\mu (-A + f(B)) + i(aX)D_\mu - (a^2 - s)(XD)D_\mu \gamma_2, \]  

where \( \gamma_2 \) is necessarily restricted to be

\[ \gamma_2 = -\frac{1 + 2 f(B) \frac{df(B)}{dB}}{f(B) - 2B \frac{df(B)}{dB}}. \]  

From the above relation we see that \( \gamma_2 \) is not an independent function, but instead is related to generally an arbitrary function \( f(B) \), which has to satisfy the boundary condition \( f(0) = 1 \). Also, it is readily seen from the realization (14) that \( \varphi \) in (11) is given by \( \varphi = -A + f(B) \). Various choices of the function \( f(B) \) lead to different realizations of NC space-time algebra (1). The particularly interesting cases are for \( f(B) = 1 \) and \( f(B) = \sqrt{1 - B} \).

It is now straightforward to show that the deformed Heisenberg-Weyl algebra (6) takes the form

\[ [D_\mu, \hat{x}_\nu] = \eta_{\mu\nu} (-A + f(B)) + ia_\mu D_\nu + (a^2 - s)D_\mu D_\nu \gamma_2 \]  

and that the Lorentz generators \( M_{\mu\nu} \) can be expressed in terms of NC coordinates as

\[ M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu) \frac{1}{\varphi}. \]  

We also point out that in the special case when realization of NC space (11) is characterized by the function \( f(B) = \sqrt{1 - B} \), there exists a unique element \( Z \) satisfying:

\[ [Z^{-1}, \hat{x}_\mu] = -ia_\mu Z^{-1} + sD_\mu, \quad [Z, D_\mu] = 0. \]  

From these two equations it follows

\[ [Z, \hat{x}_\mu] = ia_\mu Z - sD_\mu Z^2, \quad \hat{x}_\mu Z \hat{x}_\nu = \hat{x}_\nu Z \hat{x}_\mu. \]  

The element \( Z \) is a generalized shift operator and its expression in terms of \( A \) and \( B \) can be shown to have the form

\[ Z^{-1} = -A + \sqrt{1 - B}. \]  

As a consequence, the Lorentz generators can be expressed in terms of \( Z \) as

\[ M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu) Z, \]  

and one can also show that the relation

\[ [Z^{-1}, M_{\mu\nu}] = -i(a_\mu D_\nu - a_\nu D_\mu) \]  

holds. In the rest of paper we shall only be interested in the realizations determined by \( f(B) = \sqrt{1 - B} \).
4. Leibniz rule and coalgebra

The symmetry underlying deformed Minkowski space, characterized by the commutation relations (1), is the deformed Poincaré symmetry which can most conveniently be described in terms of quantum Hopf algebra. As was seen in relations (2), (3) and (4), the algebraic sector of this deformed symmetry is the same as that of undeformed Poincaré algebra. However, the action of Poincaré generators on the deformed Minkowski space is deformed, so that the whole deformation is contained in the coalgebraic sector. This means that the Leibniz rules, which describe the action of \( M_{\mu\nu} \) and \( D_{\mu} \) generators, will no more have the standard form, but instead will be deformed and will depend on a given \( \Phi_{\mu\nu} \) realization.

Generally we find that in a given \( \Phi_{\mu\nu} \) realization we can write

\[
e^{ik\hat{x}}|0\rangle = e^{iK_{\mu}(k)X^\mu}(23)
\]

and

\[
e^{ik\hat{x}}(e^{i\Phi_{\mu}(k,q)X^\mu}) = e^{iP_{\mu}(k,q)X^\mu}, (24)
\]

where \( k\hat{x} = k^\alpha X^\beta \Phi_{\beta\alpha}(D) \). In (23) we have introduced the vacuum state \( |0\rangle \equiv 1 \) as a unit element in the universal enveloping algebra understood as a module over the deformed Weyl algebra, which is generated by \( \hat{x}_\mu \) and \( D_\mu \), \( \mu = 0, 1, ..., n-1 \), and allows for infinite series in \( D_\mu \). This vacuum state is defined by

\[
\phi(X)|0\rangle \equiv \phi(X) \cdot 1 = \phi(X), (25)
\]

\[
D_\mu|0\rangle = D_\mu 1 = 0, \quad M_{\mu\nu}|0\rangle = 0, (26)
\]

with \( \phi(X) \) belonging to a space of ordinary functions in commutative coordinates. It is also understood that NC coordinates \( \hat{x}_\mu \), appearing in (23) and (24) refer to some particular realization (14), i.e. they are assumed to be represented by (14).

The quantities \( K_{\mu}(k) \) are readily identified as \( K_{\mu}(k) = P_{\mu}(k,0) \) and \( P_{\mu}(k,q) \) can be found by calculating the expression

\[
P_{\mu}(k,-iD) = e^{-ik\hat{x}}(-iD_{\mu})e^{ik\hat{x}}, (27)
\]

where it is assumed that at the end of calculation the identification \( q = -iD \) has to be made. One way to explicitly evaluate the above expression is by using the BCH expansion perturbatively, order by order. To avoid this tedious procedure, we can turn to much more elegant method to obtain the quantity \( P_{\mu}(k,-iD) \). This consists in writing the differential equation

\[
\frac{dP_{\mu}^{(t)}(k,-iD)}{dt} = \Phi_{\mu\alpha}(iP_{\mu}^{(t)}(k,-iD))k^\alpha, (28)
\]

satisfied by the family of operators \( P_{\mu}^{(t)}(k,-iD) \), defined as

\[
P_{\mu}^{(t)}(k,-iD) = e^{-itk\hat{x}}(-iD_{\mu})e^{itk\hat{x}}, \quad 0 \leq t \leq 1, (29)
\]

and parametrized with the free parameter \( t \) which belongs to the interval \( 0 \leq t \leq 1 \). The family of operators (29) represents the generalization of the quantity
\[ P_\mu(k, -iD), \] determined by (27), namely, \[ P_\mu(k, -iD) = P_{\mu}^{(1)}(k, -iD). \] Note also that solutions to differential equation (28) have to satisfy the boundary condition \( P_\mu^{(0)}(k, -iD) = -iD_\mu \equiv q_\mu. \) The function \( \Phi_{\mu \alpha}(D) \) in (28) is deduced from (14) and reads as

\[ \Phi_{\mu \alpha}(D) = \eta_{\mu \alpha}(-A + f(B)) + ia_\mu D_\alpha \gamma_2. \]  

In all subsequent considerations we shall restrict ourselves to the case where \( f(B) = \sqrt{1 - B}. \) Then we have \( \gamma_2 = 0 \) and consequently Eq. (28) reads

\[ \frac{dP_\mu^{(t)}(k, q)}{dt} = k_\mu \left[ aP^{(t)} + \sqrt{1 + (a^2 - s)(P^{(t)})^2} \right] - a_\mu kP^{(t)}, \]  

where we have used an abbreviation \( P^{(t)}_\mu \equiv P^{(t)}_\mu(k, -iD). \) The solution to differential equation (31), which obeys the required boundary conditions, looks as

\[ P^{(t)}_\mu(k, q) = q_\mu + (k_\mu Z^{-1}(q) - a_\mu(kq)) \frac{\sinh(tW)}{W} \]

\[ + \left[ (k_\mu(ak) - a_\mu k^2) Z^{-1}(q) + a_\mu(ak)(kq) - sk_\mu(kq) \right] \frac{\cosh(tW) - 1}{W^2}. \]

In the above expression we have introduced the following abbreviations,

\[ W = \sqrt{(ak)^2 - sk^2}, \]

\[ Z^{-1}(q) = (aq) + \sqrt{1 + (a^2 - s)q^2} \]

and it is understood that quantities like \((kq)\) mean the scalar product in a Minkowski space with signature \( \eta_{\mu \nu} = \text{diag}(-1, 1, \ldots, 1). \) Now that we have \( P^{(t)}_\mu(k, q), \) the required quantity \( P_\mu(k, q) \) simply follows by setting \( t = 1 \) and finally we also get

\[ K_\mu(k) = \left[ k_\mu(ak) - a_\mu k^2 \right] \frac{\cosh W - 1}{W^2} + k_\mu \frac{\sinh W}{W}. \]

Furthermore, we define the star product by the relation,

\[ e^{ikX} \star e^{iqX} \equiv e^{iK^{-1}(k)\hat{x}}(e^{iqX}) = e^{iD_\mu(k, q)X_\mu}, \]

where

\[ D_\mu(k, q) = P_\mu(K^{-1}(k), q), \]  

with \( K^{-1}(k) \) being the inverse of the transformation (33).

We claim that the function \( D_\mu(k, q) \) is related to the coproduct \( \Delta D_\mu \) for the translation generators. In particular, this relation is given by

\[ iD_\mu(-iD \otimes 1, 1 \otimes (-iD)) = \Delta D_\mu. \]

This can be seen as follows.\(^{15, 16, 17}\) We start with the general definition of the star product

\[ f \star g = m_\ast(f \otimes g), \]
where \( m^\star \) denotes deformed multiplication in the commutative algebra of smooth functions. If we take a derivative of both sides in the above definition of the star product, we get

\[
D_\mu(f \star g) = m^\star(\Delta D_\mu(f \otimes g)).
\]

By applying this to (36), we find (38).

Thus, the function \( D_\mu(k, q) \) determines the deformed Leibniz rule and the corresponding coproduct \( \Delta D_\mu \). It also gives a generalized momentum addition rule.

However, in the general case of deformation, when both parameters \( a_\mu \) and \( s \) are different from zero, it is quite a difficult task to obtain a closed form for \( \Delta D_\mu \), so we give it in a form of a series expansion up to second order in the deformation parameter \( a \),

\[
\Delta D_\mu = D_\mu \otimes 1 + 1 \otimes D_\mu - iD_\mu \otimes aD + ia_\mu D_\alpha \otimes D^\alpha - \frac{1}{2}(a^2 - s)D_\mu \otimes D^2
\]

\[- a_\mu(aD)D_\alpha \otimes D^\alpha + \frac{1}{2}a_\mu D^2 \otimes aD + \frac{1}{2}sD_\mu D_\alpha \otimes D^\alpha + O(a^3). \tag{39}
\]

Now that we have a coproduct, it is a straightforward procedure \cite{15,17} to construct a star product between arbitrary two functions \( f \) and \( g \) of commuting coordinates, generalizing in this way relation \cite{30} that holds for plane waves. Thus, the general result for the star product, valid for the NC space (1), has the form

\[
(f \star g)(X) = \lim_{Y \to X} \lim_{Z \to X} e^{X_\alpha[D^\alpha(-iD_Y, -iD_Z) - D^\rho Y - D^\sigma Z]} f(Y)g(Z). \tag{40}
\]

Although star product is a binary operation acting on the algebra of functions defined on the ordinary commutative space, it encodes features that reflect noncommutative nature of space (1). Note also that in the case when \( s \) is different from zero, the star product (40) is nonassociative.

The general results obtained so far can be specialized to some particular cases, of which Snyder space and \( \kappa \)-Minkowski space are particularly interesting.

For \( a^2 = 0 \), we have a Snyder type of noncommutativity,

\[
[\hat{x}_\mu, \hat{x}_\nu] = sM_{\mu\nu}. \tag{41}
\]

In this situation, our realization (14) reduces precisely to that obtained in \cite{29}. For a special choice when \( f(B) = 1 \), we have the realization

\[
\hat{x}_\mu = X_\mu - s(XD)D_\mu, \tag{42}
\]

which is the case that was also considered in \cite{33}. In other interesting situation when \( f(B) = \sqrt{1 - B} \), the general result (14) reduces to

\[
\hat{x}_\mu = X_\mu \sqrt{1 + sD^2}. \tag{43}
\]

This choice of \( f(B) \) is the one for which most of our results, through all over the paper, are obtained and which is the main subject of our investigations. It is also
considered by Maggiore [35,36]. For this case when \( f(B) = \sqrt{1 - B} \), the exact result for the coproduct (44) can be obtained and it is given by

\[
\triangle D_\mu = D_\mu \otimes Z^{-1} + 1 \otimes D_\mu + sD_\mu D_\alpha \frac{1}{Z^{-1} + 1} \otimes D^\alpha,
\]

where

\[
Z^{-1} = \sqrt{1 + sD^2}.
\]

Another interesting case is when the parameter \( s \) is equal to zero. This corresponds to \( \kappa \)-deformed space investigated in [7,8,9,10] and [16,17]. The general form (44) for the realizations in the case of \( \kappa \)-deformed space reduces to

\[
h_\mu = X_\mu \left( -A + \sqrt{1 - B} \right) + i(aX) D_\mu,
\]

with \( B = a^2 D^2 \), and the coproduct takes on the form

\[
\triangle D_\mu = D_\mu \otimes Z^{-1} + 1 \otimes D_\mu + ia_\mu (D_\alpha Z) \otimes D^\alpha - \frac{ia_\mu}{2} \Box Z \otimes iaD,
\]

where the generalized shift operator (20) is here specialized to

\[
Z^{-1} = -iaD + \sqrt{1 - a^2 D^2}.
\]

The result (47) is exact and is also written in a closed form, as is the coproduct (44) for the case of Snyder space.

5. Conclusion

In this paper we have investigated a Lie-algebraic type of deformations of Minkowski space and analyzed the impact these deformations have on the modification of coalgebraic structure of the symmetry algebra underlying Minkowski space. By finding a coproduct, we were able to see how coalgebra, which encodes the deformation of Minkowski space, modifies and to which extent the Leibniz rule is deformed with respect to ordinary Leibniz rule. Since the coproduct is related to the star product, we were also able to write how star product looks like on NC spaces characterized by the general class of deformations of type (1). We have also found many different classes of realizations of NC space (1) and specialized obtained results to some specific cases of particular interest.

The deformations that we have considered are characterized by the common feature that the algebraic sector of the quantum (Hopf) algebra, which is described by the Poincaré algebra, is undeformed, while, on the other hand, the corresponding coalgebraic sector is affected by deformations.

There is a vast variety of possible physical applications which could be expected to originate from the modified geometry at the Planck scale, which in turn reflects itself in a noncommutative nature of the configuration space. Which type of noncommutativity is inherent to configuration space is still not clear, but it is reasonable to expect that more wider is the class of noncommutativity taken into account,
more likely is that it will reflect true properties of geometry and relevant features at Planck scale. In particular, NC space considered in this paper is an interpolation of two types of noncommutativity, $\kappa$-Minkowski and Snyder, and as such is more likely to reflect geometry at small distances than are each of these spaces alone, at least, it includes all features of both of these two types of noncommutativity, at the same time. As was already done for $\kappa$-type noncommutativity, it would be as well interesting to investigate the effects of combined $\kappa$-Snyder noncommutativity on dispersion relations \cite{37,39,38}, black hole horizons \cite{40}, Casimir energy \cite{41} and violation of CP symmetry, the problem that is considered in \cite{42} in the context of Snyder-type of noncommutativity.

Work that still remains to be done includes an elaboration and development of methods for physical theories on NC space considered here, particularly, the calculation of coproduct for the Lorentz generators, $\Delta M_{\mu\nu}$, S-antipode, differential forms \cite{22,43,44}, Drinfeld twist \cite{45,46,47,48,49}, twisted flip operator \cite{49,50,51} and $R$-matrix \cite{52}. We shall address these issues in the forthcoming papers, together with a number of physical applications, such as the field theory for scalar fields \cite{53,54,55,56} and its twisted statistics properties, as a natural continuation of our investigations put forward in previous papers \cite{49,52}.

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