Complete monotonicity of a ratio of gamma functions and some combinatorial inequalities for multinomial coefficients

Frédéric Ouimet
California Institute of Technology, Pasadena, 91125, USA.

ABSTRACT
For $m, n \in \mathbb{N}$, let $0 < \alpha_i, \beta_j, \lambda_{ij} \leq 1$ be such that $\sum_{j=1}^{n} \lambda_{ij} = \alpha_i, \sum_{i=1}^{m} \lambda_{ij} = \beta_j,$ and $\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j \leq 1$. We prove that the ratio of gamma functions

$$t \mapsto \prod_{i=1}^{m} \Gamma(\alpha_i t + 1) \prod_{j=1}^{n} \Gamma(\beta_j t + 1) \prod_{i=1}^{m} \prod_{j=1}^{n} \Gamma(\lambda_{ij} t + 1)$$

is logarithmically completely monotonic on $(0, \infty)$. This result complements the logarithmically complete monotonicity of multinomial probabilities shown in [F. Ouimet (2018), Complete monotonicity of multinomial probabilities and its application to Bernstein estimators on the simplex, J. Math. Anal. Appl., 466(2), 1609-1617, MR3825458], [F. Qi, D.-W. Niu, D. Lim, & B.-N. Guo (2018), Some logarithmically completely monotonic functions and inequalities for multinomial coefficients and multivariate beta functions, Preprint, 1-13, hal-01769288], and the recent survey of [F. Qi & R. P. Argawal (2019), On complete monotonicity for several classes of functions related to ratios of gamma functions, J. Inequal. Appl., Paper No. 36, 42 pp, MR3908972] on the complete monotonicity of functions related to ratios of gamma functions. As a consequence of the log-convexity, we obtain new combinatorial inequalities for multinomial coefficients.

KEYWORDS
Laplace transform; logarithmically complete monotonicity; multinomial coefficient; complete monotonicity; gamma function; digamma function; special function; combinatorial inequality.

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1. Introduction

Completely monotonic functions on $(0, \infty)$ are non-negative functions for which derivatives of all orders exist on $(0, \infty)$ and alternate in sign (starting with the negative sign). Typical examples are $t^{-1}, (1 + t)^{-1}, e^{-t},$ etc. A famous theorem of [3] shows that the set of completely monotonic functions $\phi : [0, \infty) \rightarrow \mathbb{R}$ such that $\phi(0) = 1$ coincides with the set of Laplace transforms, see e.g. Section XIII.4 of [5] for a simpler proof. For a classic introduction to the theory of Laplace transforms, we refer the reader to [10]. For a survey on the complete monotonicity of functions related to ratios of gamma functions, see [6,8].
Below are the formal definitions of complete monotonicity and logarithmically complete monotonicity that we use.

**Definition 1.1.** A function $t \mapsto g(t)$ is said to be completely monotonic on $(0, \infty)$ if $g$ has derivatives of all orders and satisfies
\[
(-1)^k g^{(k)}(t) \geq 0, \quad \text{for all } k \in \mathbb{N}_0, \ t \in (0, \infty).
\]

**Definition 1.2.** A function $t \mapsto g(t)$ is said to be logarithmically completely monotonic on $(0, \infty)$ if $(-\log g)'$ is completely monotonic on $(0, \infty)$.

It turns out that logarithmically completely monotonic functions are completely monotonic, see e.g. [4, p.83].

**Lemma 1.3.** Let $g : (0, \infty) \to (0, \infty)$. If $(-\log g)'$ is completely monotonic on $(0, \infty)$, then $g$ is completely monotonic on $(0, \infty)$.

1.1. Structure of the paper

In the next section, we state and prove our main result (Theorem 2.1), then we deduce new combinatorial inequalities in Corollary 2.2. In the appendix, the reader will find a technical inequality (and its proof) which is the key step in the proof of Theorem 2.1.

2. Main result

**Theorem 2.1.** For $m, n \in \mathbb{N}$, let $0 < \alpha_i, \beta_j, \lambda_{ij} \leq 1$ be such that
\[
\sum_{j=1}^n \lambda_{ij} = \alpha_i, \quad \sum_{i=1}^m \lambda_{ij} = \beta_j,
\]
and
\[
\sum_{i=1}^m \sum_{j=1}^n \lambda_{ij} = \sum_{i=1}^m \alpha_i = \sum_{j=1}^n \sum_{i=1}^m \lambda_{ij} = \sum_{j=1}^n \beta_j \leq 1.
\]

Then, the function
\[
g(t) = \frac{\prod_{i=1}^m \Gamma(\alpha_i t + 1) \prod_{j=1}^n \Gamma(\beta_j t + 1)}{\prod_{i=1}^m \prod_{j=1}^n \Gamma(\lambda_{ij} t + 1)}
\]
(2)
is logarithmically completely monotonic on $(0, \infty)$, where $\Gamma$ denotes the classical Euler’s gamma function, which is defined by $\Gamma(z) := \int_0^\infty u^{z-1}e^{-u}du$ for $z > 0$. In particular, Lemma 1.3 implies that $g$ is completely monotonic on $(0, \infty)$.

**Proof of Theorem 2.1.** Define $h(t) := -\log g(t)$. We have
\[
h'(t) = -\sum_{i=1}^m \alpha_i \psi(\alpha_i t + 1) - \sum_{j=1}^n \beta_j \psi(\beta_j t + 1) + \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij} \psi(\lambda_{ij} t + 1),
\]
(3)
where $\psi := (\log \Gamma)' = \Gamma'/\Gamma$ is the digamma function. Using the integral representation
\[
\psi'(z) = \int_0^\infty \frac{ue^{-(z-1)u}}{e^u - 1}du, \quad z \in (0, \infty),
\]
(4)
see [1, p.260], we obtain

\[ h''(t) = -\sum_{i=1}^{m} \alpha_i^2 \psi'(\alpha_i t + 1) - \sum_{j=1}^{n} \beta_j^2 \psi'(\beta_j t + 1) + \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}^2 \psi'(\lambda_{ij} t + 1) \]

\[ = -\sum_{i=1}^{m} \int_{0}^{\infty} \frac{\alpha_i u e^{\alpha_i u t}}{e^u - 1} \, du - \sum_{j=1}^{n} \int_{0}^{\infty} \frac{\beta_j u e^{\beta_j u t}}{e^u - 1} \, du + \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{0}^{\infty} \frac{\lambda_{ij} u e^{-\lambda_{ij} u t}}{e^u - 1} \, du \]

\[ = -\int_{0}^{\infty} se^{-st} J(\lambda_{ij}/C)(e^{s/C}) \, ds, \quad (5) \]

where \( C := \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} \) and \( J(u_{ij})(y) \) is defined in (A1). By Lemma A.1, for all \( k \in \mathbb{N} \) and \( t \in (0, \infty) \),

\[ (-1)^k h^{(k+1)}(t) = \int_{0}^{\infty} s^k e^{-st} J(\lambda_{ij}/C)(e^{s/C}) \, ds > 0. \quad (6) \]

Since \( h' \) is decreasing, we show that \( \lim_{t \to \infty} h'(t) \geq 0 \) to conclude the proof.

If we apply the recurrence formula

\[ \psi(z + 1) = \psi(z) + \frac{1}{z}, \quad z \in (0, \infty), \quad (7) \]

see [1, p.258], we obtain from (3) the representation

\[ h'(t) = \frac{-m - n + mn}{t} - \sum_{i=1}^{m} \alpha_i R(\alpha_i t) - \sum_{j=1}^{n} \beta_j R(\beta_j t) + \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} R(\lambda_{ij} t) \]

\[ - \sum_{i=1}^{m} \alpha_i \log(\alpha_i) - \sum_{j=1}^{n} \beta_j \log(\beta_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} \log(\lambda_{ij}), \quad (8) \]

where \( R(z) := \psi(z) - \log z \). Using the asymptotic formula

\[ R(z) = -\frac{1}{2z} - \frac{1}{12z^2} + O(z^{-4}), \quad \text{as } z \to \infty, \quad (9) \]

see [1, p.259], all the terms on the first line on the right-hand side of (8) converge to 0 as \( t \to \infty \). Since \( \sum_{j=1}^{n} \lambda_{ij} = \alpha_i > 0 \) and \( \sum_{j=1}^{n} \beta_j \leq 1 \), Jensen’s inequality applied to \( -\log(\cdot) \) yields

\[ \lim_{t \to \infty} h'(t) = -\sum_{i=1}^{m} \alpha_i \sum_{j=1}^{n} (\lambda_{ij}/\alpha_i) \log \left( \frac{\beta_j}{\lambda_{ij}/\alpha_i} \right) \]

\[ \geq -\sum_{i=1}^{m} \alpha_i \log \left( \sum_{j=1}^{n} \beta_j \right) \]

\[ \geq 0. \quad (10) \]

This ends the proof. \( \Box \)
In the context of Theorem 2.1, note that
\[
g(t) = \frac{\prod_{i=1}^{m} \Gamma(\alpha_i t + 1) \prod_{j=1}^{n} \Gamma(\beta_j t + 1)}{\prod_{i=1}^{m} \prod_{j=1}^{n} \Gamma(\lambda_{ij} t + 1)}
\]
\[
= \prod_{i=1}^{m} \left( \alpha_i t \right) \prod_{j=1}^{n} \left( \beta_j t \right) \prod_{i=1}^{m} \prod_{j=1}^{n} \Gamma(\lambda_{ij} t + 1).
\]

We are now ready to prove the new combinatorial inequalities.

**Corollary 2.2.** Let \( \nu \in \mathbb{N} \). For all \( k \in \{1, 2, \ldots, \nu\} \), choose \( t_k \in (0, \infty) \) and let \( \mu_k \in (0,1) \) be such that \( \sum_{k=1}^{\nu} \mu_k = 1 \). The following inequalities hold:

(a) \( g(\sum_{k=1}^{\nu} \mu_k t_k) \leq \prod_{k=1}^{\nu} g(t_k)^{\mu_k} \), where equality holds if and only if all the \( t_k \)'s are the same.

(b) \( \prod_{k=1}^{\nu} g(t_k) < g(\sum_{k=1}^{\nu} t_k) \).

(c) If \( t_1 \leq t_3 \), then \( g(t_1 + t_2)g(t_3) \leq g(t_1)g(t_2 + t_3) \), where equality holds if and only if \( t_1 = t_3 \).

Using (11), we can also write the above inequalities with multinomial coefficients:

(a') If we assume further that \( \mu_k = 1/\nu \) for all \( k \), and denote \( \bar{t} := \frac{1}{\nu} \sum_{k=1}^{\nu} t_k \), then

\[
\prod_{i=1}^{m} \left( \lambda_{11} \bar{t}, \lambda_{12} \bar{t}, \ldots, \lambda_{im} \bar{t} \right) \prod_{j=1}^{n} \left( \lambda_{1j} \bar{t}, \lambda_{2j} \bar{t}, \ldots, \lambda_{mj} \bar{t} \right)
\]
\[
\cdot \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \lambda_{ij} \frac{t_k}{\nu}, \lambda_{ij} \frac{t_k}{\nu}, \ldots, \lambda_{ij} \frac{t_k}{\nu} \right)
\]
\[
\leq \prod_{k=1}^{\nu} \left( \lambda_{11} t_k, \lambda_{12} t_k, \ldots, \lambda_{1m} t_k \right) \prod_{j=1}^{n} \left( \lambda_{1j} t_k, \lambda_{2j} t_k, \ldots, \lambda_{mj} t_k \right)
\]
\[
\cdot \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \lambda_{ij} \frac{t_k}{\nu}, \lambda_{ij} \frac{t_k}{\nu}, \ldots, \lambda_{ij} \frac{t_k}{\nu} \right)
\]
\[
\left( \lambda_{11} \bar{t}, \lambda_{12} \bar{t}, \ldots, \lambda_{1m} \bar{t} \right)
\]
\[
\left( \lambda_{1j} \bar{t}, \lambda_{2j} \bar{t}, \ldots, \lambda_{mj} \bar{t} \right)
\]
\[
\cdot \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \lambda_{ij} \frac{t_k}{\nu}, \lambda_{ij} \frac{t_k}{\nu}, \ldots, \lambda_{ij} \frac{t_k}{\nu} \right).
\]

(b')

\[
\prod_{k=1}^{\nu} \left( \lambda_{11} t_k, \lambda_{12} t_k, \ldots, \lambda_{1m} t_k \right) \prod_{j=1}^{n} \left( \lambda_{1j} t_k, \lambda_{2j} t_k, \ldots, \lambda_{mj} t_k \right)
\]
\[
\leq \prod_{i=1}^{m} \left( \lambda_{11} \sum_{k=1}^{\nu} t_k, \lambda_{12} \sum_{k=1}^{\nu} t_k, \ldots, \lambda_{1m} \sum_{k=1}^{\nu} t_k \right)
\]
\[
\cdot \prod_{j=1}^{n} \left( \lambda_{1j} \sum_{k=1}^{\nu} t_k, \lambda_{2j} \sum_{k=1}^{\nu} t_k, \ldots, \lambda_{mj} \sum_{k=1}^{\nu} t_k \right)
\]
\[
\cdot \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \lambda_{ij} \sum_{k=1}^{\nu} t_k \right)
\]
(c’) If \( t_1 \leq t_3 \), then

\[
\prod_{i=1}^{m} \left( \lambda_{i1}(t_1 + t_2), \lambda_{i2}(t_1 + t_2), \ldots, \lambda_{in}(t_1 + t_2) \right) \left( \lambda_{i1}t_3, \lambda_{i2}t_3, \ldots, \lambda_{in}t_3 \right) \\
\cdot \prod_{j=1}^{n} \left( \lambda_{1j}(t_1 + t_2), \lambda_{2j}(t_1 + t_2), \ldots, \lambda_{mj}(t_1 + t_2) \right) \left( \lambda_{1j}t_3, \lambda_{2j}t_3, \ldots, \lambda_{mj}t_3 \right) \\
\cdot \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \lambda_{ij}(t_1 + t_2 + t_3) \right) \\
\leq \prod_{i=1}^{m} \left( \lambda_{i1}t_1, \lambda_{i2}t_1, \ldots, \lambda_{in}t_1 \right) \left( \lambda_{i1}(t_2 + t_3), \lambda_{i2}(t_2 + t_3), \ldots, \lambda_{in}(t_2 + t_3) \right) \\
\cdot \prod_{j=1}^{n} \left( \lambda_{1j}t_1, \lambda_{2j}t_1, \ldots, \lambda_{mj}t_1 \right) \left( \lambda_{1j}(t_2 + t_3), \lambda_{2j}(t_2 + t_3), \ldots, \lambda_{mj}(t_2 + t_3) \right) \\
\cdot \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \lambda_{ij}(t_1 + t_2 + t_3) \right),
\]

where equality holds if and only if \( t_1 = t_3 \).

**Proof of Corollary 2.2.** By (6), \( g \) is strictly log-convex, which implies (a) by definition. Point (b) follows from Lemma 3 in [2] because \( g \) is differentiable on \([0, \infty)\), \( g(0) = 1 \) and \( g \) is (strictly) positive, (strictly) decreasing and strictly log-convex on \((0, \infty)\). Point (c) follows by adapting the proof of Corollary 3 in [2] using (6). \( \square \)

### Appendix A. A technical lemma

We needed the following key inequality in the proof of Theorem 2.1.

**Lemma A.1.** For \( m, n \in \mathbb{N} \), let \( 0 < U_i, V_j, u_{ij} \leq 1 \) be such that

\[
\sum_{j=1}^{n} u_{ij} = U_i, \quad \sum_{i=1}^{m} u_{ij} = V_j,
\]

and

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} = \sum_{i=1}^{m} U_i = \sum_{j=1}^{n} \sum_{i=1}^{m} u_{ij} = \sum_{j=1}^{n} V_j = 1.
\]

Then, for any given \( y > 1 \),

\[
J(u_{ij})(y) := \sum_{i=1}^{m} \frac{1}{y^{1/U_i} - 1} + \sum_{j=1}^{n} \frac{1}{y^{1/V_j} - 1} - \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{y^{1/u_{ij}} - 1} > 0,
\]  \( \text{(A1)} \)

where \( (u_{ij}) \) is a shorthand for the matrix \((u_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}\).
Proof.} First, we write \(-\mathcal{J}(u_{ij})(y)\) as a function of the variables \((u_{ij})_{1 \leq i \leq m-1; 1 \leq j \leq n-1}\) when viewing the \(U_i\)'s and \(V_j\)'s as fixed:

\[
-\mathcal{J}(u_{ij})(y) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{y^{1/u_{ij}} - 1} - \sum_{i=1}^{m} \frac{1}{y^{1/U_i} - 1} - \sum_{j=1}^{n} \frac{1}{y^{1/V_j} - 1}
\]

this is independent of \((u_{ij})_{1 \leq i \leq m-1; 1 \leq j \leq n-1}\); call it \(C(y)\)

\[
= \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \frac{1}{y^{1/u_{ij}} - 1} + \sum_{i=1}^{m-1} \frac{1}{y^{1/(U_i-\sum_{j=1}^{n-1} u_{ij})} - 1} + \sum_{j=1}^{n-1} \frac{1}{y^{1/V_j - \sum_{i=1}^{m-1} u_{ij}) - 1}
\]

\[
+ \frac{1}{y^{1/(V_n-\sum_{i=1}^{m-1} U_i+\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} u_{ij}) - 1}} + C(y).
\] (A2)

From the proof of Lemma 1 in [2], we know that \(\frac{\partial^2}{\partial c^2}(y^{1/c} - 1)^{-1} > 0\) for all \(c \in (0, 1)\). For convenience, here are the computations (with \(t = y^{1/c}\), and \(y > 1\) by assumption):

\[
\frac{c^2(t-1)^3}{2t(\log t)^2} \frac{\partial^2}{\partial c^2}(y^{1/c} - 1)^{-1} = \frac{t+1}{2} - \frac{t-1}{\log t} = \frac{t+1}{2 \log t} \int_1^t \frac{(s-1)^2}{s(s+1)^2} ds > 0.
\] (A3)

Therefore, everywhere in the open set

\[
\mathcal{O} = \{(u_{ij})_{1 \leq i \leq m-1; 1 \leq j \leq n-1} \in (0, \infty)^{(m-1) \times (n-1)}: \sum_{i=1}^{m-1} u_{ij} < V_j, \sum_{j=1}^{n-1} u_{ij} < U_i\},
\] (A4)

we have (for \(1 \leq k \neq k' \leq m-1\) and \(1 \leq \ell \neq \ell' \leq n-1\)):

\[
\frac{\partial^2}{\partial u_{kt} \partial u_{kt}}[-\mathcal{J}(u_{ij})(y)] = \frac{\partial^2}{\partial u_{kt} \partial u_{kt} y^{1/u_{kt}} - 1}
\]

\[
= a_{kt} + a_{kn} + a_{m\ell} + a_{mn},
\] (A5)

\[
\frac{\partial^2}{\partial u_{kt} \partial u_{k\ell'}}[-\mathcal{J}(u_{ij})(y)] = \frac{\partial^2}{\partial u_{kt} \partial u_{k\ell'} y^{1/(U_k-\sum_{j=1}^{n-1} u_{ij})} - 1}
\]

\[
+ \frac{\partial^2}{\partial u_{kt} \partial u_{k\ell'} y^{1/V_n-\sum_{i=1}^{m-1} U_i+\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} u_{ij}) - 1}
\]
where \(a_{ij} > 0\) for all \(1 \leq i \leq m, 1 \leq j \leq n\), on \(\mathcal{O}\) by (A3). In other words, the Hessian matrix of \(-J_{(u_{ij})}(y)\), as a function of the variables \((u_{ij})_{1 \leq i \leq m-1; 1 \leq j \leq n-1}\), is equal to

\[
\begin{bmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_2 & \ldots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & A_{m-1}
\end{bmatrix} + \begin{bmatrix}
B_1 & 0 & \ldots & 0 \\
0 & B_2 & \ldots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & B_{m-1}
\end{bmatrix} + \begin{bmatrix}
C & C & \ldots & C \\
C & C & \ldots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
C & \ldots & C & C
\end{bmatrix} + a_{mn}1_{(m-1)(n-1)}
\]

:= (I) + (II) + (III) + (IV). \hspace{1cm} (A9)

where \(A_i = \text{diag}((a_{ij})_{1 \leq j \leq n-1})\), \(B_i = a_{in}1_{(n-1)}\), \(C = \text{diag}((a_{mj})_{1 \leq j \leq n-1})\) and \(1_{\mu}\) denotes the \(\mu \times \mu\) matrix of ones. Since all the \(a_{ij}\)'s are positive on \(\mathcal{O}\), it is easy to verify that (I) is positive definite and (II), (III) and (IV) are positive semi-definite. Indeed, for any non-zero vector \(x \in \mathbb{R}^{(m-1)(n-1) \setminus \{0\}}\), write it as the vertical concatenation of the column vectors \((x_i)_{1 \leq i \leq m-1}\) where \(x_i := (x_{ij})_{1 \leq j \leq n-1}\), then

\[
x^\top (I) x = \sum_{i=1}^{m-1} x_i^\top A_i x_i = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} a_{ij} x_{ij}^2 > 0,
\]

\[
x^\top (II) x = \sum_{i=1}^{m-1} x_i^\top B_i x_i = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \sum_{j' = 1}^{n-1} a_{in} x_{ij} x_{ij'} = \sum_{i=1}^{m-1} a_{in} \left( \sum_{j=1}^{n-1} x_{ij} \right)^2 \geq 0,
\]

\[
x^\top (III) x = \sum_{i=1}^{m-1} \sum_{j' = 1}^{n-1} x_i^\top C x_{ij'} = \sum_{i=1}^{m-1} \sum_{j' = 1}^{n-1} \sum_{j=1}^{n-1} a_{mj} x_{ij} x_{ij'} = \sum_{j=1}^{n-1} a_{mj} \left( \sum_{i=1}^{m-1} x_{ij} \right)^2 \geq 0,
\]

\[
x^\top (IV) x = a_{mn} \left( \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} x_{ij} \right)^2 \geq 0.
\]

By linearity, this means that the Hessian matrix of \(-J_{(u_{ij})}(y)\) is positive definite. Since the second-order partial derivatives are continuous on the open and convex set \(\mathcal{O}\), it implies that \(J_{(u_{ij})}(y)\), as a function of the variables \((u_{ij})_{1 \leq i \leq m-1; 1 \leq j \leq n-1}\), is strictly concave on \(\mathcal{O}\). A strictly concave function on a convex set minimizes at the extremal points of its closure. Here, these
are the points \((u_{ij})_{1\leq i\leq m-1;1\leq j\leq n-1}\) such that \(u_{i^*j^*} = 1\) for some \(1 \leq i^* \leq m - 1\) and \(1 \leq j^* \leq n - 1\), and such that \(u_{ij} = 0\) for all other \(i \neq i^*\) and \(j \neq j^*\). It is easy to verify that \(J(u_{ij})(y) = (y - 1)^{-1} > 0\) in (A1) for any such point. Hence, \(J(u_{ij})(y) > 0\) on \(O\), which was our claim.

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