Pre-Hamiltonian operators related to hyperbolic equations of Liouville type

S. Ya. Startsev

Institute of Mathematics, Ufa Federal Research Centre, Russian Academy of Sciences

Abstract. This text is devoted to hyperbolic equations admitting differential operators that map any function of one independent variable into a symmetry of the corresponding equation. We use the term ‘symmetry driver’ for such operators and prove that any symmetry driver of the smallest order is pre-Hamiltonian (i.e., the image of the driver is closed with respect to the standard bracket). This allows us to prove that the composition of a symmetry driver with the Fréchet derivative of an integral is also pre-Hamiltonian (in a new set of the variables) if both the symmetry driver and the integral have the smallest orders.
1 Introduction: Pre-Hamiltonian operators

Suppose that \( u \) is a function of \( x \) and \( y \). Let \( \mathcal{F} \) denote the set of all functions depending on a finite number of the variables
\[
y, \ x, \ u_0 = u, \ u_1 = u_x, \ u_2 = u_{xx}, \ldots . \tag{1.1}
\]
Notice that (1.1) does not contain derivatives of \( u \) with respect to \( y \). Moreover, the objects defined in this Section do not employ the variable \( y \), and this variable serves here as a parameter (which is absent in the standard definitions). The origin of the parameter \( y \) will be explained in Section 2.

By \( D \) we denote the total derivative with respect to \( x \). On functions from \( \mathcal{F} \), it is defined by the formula
\[
D = \frac{\partial}{\partial x} + \sum_{i=0}^{+\infty} u_{i+1} \frac{\partial}{\partial u_i} . \tag{1.2}
\]
For any function \( a \in \mathcal{F} \), the differential operator \( a_* \) is called the Fréchet derivative of \( a \).

For any \( a \in \mathcal{F} \) we can also consider the total derivative \( \partial_a \) with respect to \( t \) in virtue (i.e. on solutions) of the evolution equation \( u_t = a \). It easy to check that \( \partial_a(h) = h_*(a) \) if \( h \in \mathcal{F} \). The commutator \( [\partial_f, \partial_g] \) of such derivatives corresponds to the Lie bracket
\[
[f, \ g] \overset{df}{=} g_*(f) - f_*(g) \tag{1.3}
\]
on \( \mathcal{F} \). Namely, it is well-known (see, for example, [1]) that
\[
[\partial_f, \partial_g] = \partial_{[f,g]} . \tag{1.4}
\]

Let \( M \) be a differential operator of the form
\[
M = \sum_{i=0}^{k} \xi_i D^i, \quad \xi_i \in \mathcal{F}, \quad k \geq 0 . \tag{1.5}
\]
The direct calculation shows that \( D(a)_* = D \circ a_* \) for any \( a \in \mathcal{F} \), where the symbol \( \circ \) denotes the composition of differential operators. Taking this into account, we see that the relation
\[
[M(a), M(b)] = M(b_*(M(a)) - a_*(M(b))) + \sum_{i=0}^{k} (D^i(b)(\xi_i)_*(M(a)) - D^i(a)(\xi_i)_*(M(b)))
\]
holds for any differential operator (1.5) and any functions \( a, b \in \mathcal{F} \). Denoting the highest of the orders of the operators \((\xi_i)_* \circ M\) by \( m \), we can rewrite this relation in the form
\[
[M(a), M(b)] - M(b_*(M(a)) - a_*(M(b))) = \sum_{i=0}^{m} \sum_{j=0}^{m} c_{ij} D^i(a) D^j(b) , \tag{1.6}
\]
\(^{1}\text{In the formal sense, just as a way to define new differentiations on } \mathcal{F}.\)
where the functions $c_{ij} \in \mathcal{F}$ do not depend on $a$ and $b$.

Generally speaking, the right-hand side of (1.6) does not belong to the image of $M$ for all $a$ and $b$. But, for example, if $M = D + u_1$, then

$$\left[ M(a), M(b) \right] = M \left( b_s(M(a)) - a_s(M(b)) + D(a)b - D(b)a \right)$$

(1.7)
for any $a, b \in \mathcal{F}$.

**Definition 1.1.** A differential operator $M$ of the form (1.5) is called *pre-Hamiltonian* if for any $a, b \in \mathcal{F}$ there exists $\vartheta \in \mathcal{F}$ such that $\left[ M(a), M(b) \right] = M(\vartheta)$.

It is known (see [1] for example) that the image $\text{Im} \mathcal{H}$ of an operator $\mathcal{H}$ satisfies the relation $\left[ \text{Im} \mathcal{H}, \text{Im} \mathcal{H} \right] \subseteq \text{Im} \mathcal{H}$ (i.e., $\text{Im} \mathcal{H}$ forms a Lie subalgebra in $\mathcal{F}$) if the operator $\mathcal{H}$ is Hamiltonian. Thus, we can consider pre-Hamiltonian operators as a generalization of Hamiltonian operators. (Note that the pre-Hamiltonian operator $D + u_1$ in the above example is not Hamiltonian because it is not skew-symmetric.)

Pre-Hamiltonian operators (under different names or without a name) were studied in several works of this century. As far as the author knows, the definition of pre-Hamiltonian operators actually arose for the first time in subsection 7.1 of [2] as an ‘experimentally observed’ remarkable property of operators associated with hyperbolic equations of the Liouville type (i.e., with Darboux integrable equations).

The present text can be considered as an addition to these ‘experimental observations’ of [2] and provides them with the proof. Namely, we prove in Section 3 that any Darboux integrable equation generates four pre-Hamiltonian operators (a couple of such operators per each of the characteristics).

## 2 Darboux integrable hyperbolic equations. Notation, definitions and known facts

### 2.1 Notation

From now on, we deal with hyperbolic equations of the form

$$u_{xy} = F(x, y, u, u_x, u_y).$$

(2.1)

If $u(x, y)$ is a solution of equation (2.1), then we can express all mixed derivatives of $u$ in terms of the variables

$$x, \quad y, \quad u_0 = \bar{u}_0 = u, \quad u_1 = u_x, \quad \bar{u}_1 = u_y, \quad u_2 = u_{xx}, \quad \bar{u}_2 = u_{yy}, \ldots.$$  

(2.2)

These variables will be considered as independent. We use notation $f \langle u \rangle$ for a function $f$ depending on a finite number of the variables (2.2). We also use the notation of Section 1 and, in particular, continue to write $f \in \mathcal{F}$ if $f \langle u \rangle$ does not depend on $\bar{u}_j$ for all $j > 0$.

---

2References to these works will be added in the next version of this text; a part of them are mentioned in subsection 3.5.2 of https://arxiv.org/abs/1711.10624
For equations (2.1) there is a duality in notation between objects related to $x$ and $y$-characteristics. If we denote an $x$-object by a symbol, then the corresponding $y$-object is denoted by the same symbol with a dash above. For example, by $D$ we denote the total derivative with respect to $x$ defined on solutions of equation (2.1), while the total derivative with respect to $y$ is denoted by $\bar{D}$. These derivatives are defined by formulas

$$D = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i} + \sum_{i=1}^{\infty} \bar{D}^{i-1}(F) \frac{\partial}{\partial \bar{u}_i},$$

$$\bar{D} = \frac{\partial}{\partial y} + \sum_{i=0}^{\infty} \bar{u}_{i+1} \frac{\partial}{\partial \bar{u}_i} + \sum_{i=1}^{\infty} \bar{D}^{i-1}(F) \frac{\partial}{\partial \bar{u}_i}.$$ 

Note that the restriction of $D$ onto $\mathcal{F}$ coincides with (1.2).

### 2.2 Darboux integrable hyperbolic equations

#### Definition 2.1.

A function $W(u)$ is called an $x$-integral of equation (2.1) if $\bar{D}(W) = 0$. Any function $W(x)$ is called a trivial $x$-integral.

If we replace $x$ with $y$ (and $\bar{D}$ with $D$) in the above definition, then we obtain the definition of $y$-integrals. Using the symmetry of formula (2.1) with respect to the interchange $x \leftrightarrow y$, we hereafter give only one of two ‘symmetric’ definitions and statements.

Differentiating the defining relation $\bar{D}(W) = 0$ with respect to the highest variable of the form $\bar{u}_i$, we obtain that any $x$-integral $W$ does not depend on $\bar{u}_i$ for all $i > 0$ and, hence, it has the form

$$W = W(x, y, u, u_1, u_2, \ldots, u_p).$$

The number $p$ is called order of the $x$-integral $W$.

**Example 2.2.** The function $w = u_2^2 - \frac{1}{2} u_1^2$ is an $x$-integral of the Liouville equation

$$u_{xy} = \exp u.$$ (2.3)

Obviously, if $w$ is an $x$-integral of equation (2.1), then the expression

$$W = S \left( x, w, D(w), \ldots, D^j(w) \right)$$ (2.4)

is also an $x$-integral of (2.1) for any function $S$ and any $j \geq 0$.

**Proposition 2.3.** Any $x$-integral $W$ can be represented in the form (2.4), where $w$ is an $x$-integral of the smallest order. In particular, $w = \phi(x, \tilde{w})$ if $\tilde{w}$ is another $x$-integral of the smallest order.

#### Definition 2.4.

An equation of the form (2.1) is said to be Darboux integrable if this equation possesses both nontrivial $x$ and $y$-integrals.

The Liouville equation (2.3) is the most known example of nonlinear Darboux integrable equation.

---

3 Notice that in different papers (even in articles of the same authors) the names for the integrals are varied: in a part of the papers, the term ‘$x$-integral’ is used to denote $y$-integrals and vice versa.
2.3 Hyperbolic equations of Liouville type

Let us introduce the functions

\[ H_1 \overset{\text{def}}{=} -D \left( \frac{\partial F}{\partial u_1} \right) + \frac{\partial F}{\partial u} \frac{\partial F}{\partial \bar{u}_1} + \frac{\partial F}{\partial u} \frac{\partial F}{\partial \bar{u}}, \quad H_0 \overset{\text{def}}{=} -\bar{D} \left( \frac{\partial F}{\partial \bar{u}_1} \right) + \frac{\partial F}{\partial u} \frac{\partial F}{\partial \bar{u}_1} + \frac{\partial F}{\partial u}. \]

Then we can define the functions \( H_i \) for \( i > 1 \) and \( i < 0 \) by the recurrent formula

\[ D\bar{D}(\log H_i) = -H_{i+1} - H_{i-1} + 2H_i, \quad i \in \mathbb{Z}. \quad (2.5) \]

The functions \( H_i \) are called Laplace invariants of equation (2.1).

Definition 2.5. Equation (2.1) is called a Liouville-type equation if \( H_r = H_{-s} \equiv 0 \) for some integers \( r \geq 1 \) and \( s \geq 0 \).

Theorem 2.6. [4, 5] An equation of the form (2.1) is a Liouville-type equation if and only if this equation admits both a non-trivial \( x \)-integrals \( W(x, y, u, u_1, u_2, \ldots, u_p) \) and a non-trivial \( y \)-integral \( \bar{W}(x, y, u, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{\bar{p}}) \). In addition, \( s < p \) and \( r \leq \bar{p} \).

Thus, Definitions 2.5 and 2.4 are equivalent for scalar hyperbolic equations.

Differentiating the defining relations \( D(W) = 0 \) and \( \bar{D}(\bar{W}) = 0 \) with respect to \( u_p \) and \( \bar{u}_p \), respectively, we obtain the following statement.

Corollary 2.7. If (2.1) is a Liouville-type equation, then the relations

\[ \frac{\partial F}{\partial u_1} = \bar{D} \log \psi(x, y, u, u_1, \ldots, u_p), \quad \frac{\partial F}{\partial \bar{u}_1} = D \log \bar{\psi}(x, y, u, \bar{u}_1, \ldots, \bar{u}_{\bar{p}}) \quad (2.6) \]

hold for some functions \( \psi \) and \( \bar{\psi} \).

3 Pre-Hamiltonian operators related to Liouville-type equations

3.1 Symmetries

An inherent feature of Darboux integrable equations is the existence of higher symmetries in both \( x \) and \( y \)-directions.

Definition 3.1. An equation of the form

\[ u_t = g(u) \quad (3.1) \]

is a symmetry of equation (2.1) if \( g \) satisfies the following relation

\[ \left( \bar{D}D - \frac{\partial F}{\partial u_1} \right)D - \frac{\partial F}{\partial u} \bar{D} + \frac{\partial F}{\partial u} \frac{\partial F}{\partial \bar{u}} \right)(g) = 0. \]

If the right-hand side of a symmetry belongs to \( \mathcal{F} \), then we call it an \( x \)-symmetry\(^4\).

\(^4\)The right-hand side of \( y \)-symmetries depends on \( x, y, u, \bar{u}_1, \bar{u}_2, \ldots \).
We often identify symmetry (3.1) with its right-hand side \( g \). The set of \( x \)-symmetries is a Lie algebra with respect to the bracket (1.3).

The following formula generates symmetries for Liouville-type equations (see, for example, Theorem 5 in [2], where this formula was proved but was given in the slightly different form (3.3)).

**Theorem 3.2.** Suppose that for equation (2.1) there exists a non-zero function \( \psi(u) \in \ker(\bar{D} - F_{u_1}) \) and \( H_r = 0 \) for some \( r > 0 \). Let us define the operator \( \mathcal{M} \) by the formula

\[
\mathcal{M} = \begin{cases} 
\frac{1}{H_1}(D - F_{u_1}) \circ \frac{1}{H_2}(D - F_{u_1}) \circ \cdots \circ \frac{1}{H_{r-1}}(D - F_{u_1}) \circ H_r \cdots H_1 \psi & \text{if } r > 1, \\
\psi & \text{if } r = 1.
\end{cases}
\]

(3.2)

Then \( u_t = \mathcal{M}(W) \) is a symmetry of the equation (2.1) for any \( x \)-integral \( W \).

The above Theorem is applicable to any Liouville-type equation because \( H_r = 0 \) by Definition 2.3 and the function \( \psi \) exists by Corollary 2.7. In the case of Liouville-type equations (as well as in other cases when both relations (2.6) hold), we can rewrite (3.2) for \( r > 1 \) in the form

\[
\mathcal{M} = \frac{\psi}{H_1} D \circ \frac{1}{H_2} D \circ \cdots \circ \frac{1}{H_{r-1}} D \circ \frac{\psi H_1 \cdots H_{r-1}}{\psi}
\]

(3.3)

found in [2].

Notice that Theorem 3.2 can be applied not only to Liouville-type equations but also to some equations (2.1) without nontrivial \( x \)-integrals. In the latter case, \( W \) is an arbitrary function of \( x \).

**Example 3.3.** We have \( H_1 = 0 \) and \( \psi = u_1 \) for any equation of the form \( u_{xy} = \eta(y, u, u_y, u_x) \). In this case, \( \mathcal{M} \) is the operator of multiplication by \( u_1 \). This operator maps arbitrary function of \( x \) to a symmetry of this equation, while the equation generically is not Darboux integrable (admits nontrivial \( x \)-integrals not for all \( \eta \) in accordance with [3, 6]).

**Definition 3.4.** An operator \( M = \sum_{i=0}^{k} \xi_i(u) D^i \), \( \xi_k \neq 0 \) is called an \( x \)-symmetry driver if \( k \geq 0 \) and \( u_t = M(W) \) is a symmetry of equation (2.1) for any \( x \)-integral \( W \). The integer \( k \) is called order of the driver \( M \).

The above definition remains applicable even if (2.1) admits only trivial \( x \)-integrals (see Example 3.3). It is clear that for a driver \( M \) the operator \( M \circ D^i \circ W \) is a driver for any \( i \geq 0 \) and any \( x \)-integral \( W \) (in particular, \( W \) can be equal to 1).

**Lemma 3.5.** The coefficients of any \( x \)-symmetry driver do not depend on \( \bar{u}_j \) for all \( j > 0 \). The leading coefficient belongs to the kernel of the operator \( \bar{D} - F_{u_1} \).

**Proof.** Let \( M = \sum_{i=0}^{k} \xi_i(u) D^i \) be an \( x \)-symmetry driver. Then

\[
(D \bar{D} - F_{u_1} D - F_{u_1} \bar{D} - F_u)(M(g)) = 0
\]

for arbitrary function \( g(x) \). Collecting the coefficients of \( g^{(i)} \) in this identity, we arrive at the following relations:

\[
(\bar{D} - F_{u_1})(\xi_k) = 0, \\
(\bar{D} - F_{u_1})(\xi_{i-1}) = (F_u + F_{u_1} D + F_{u_1} \bar{D} - D \bar{D})(\xi_i), \quad 1 \leq i \leq k.
\]

(3.4)

It follows from the first relation that \( \xi_k \) does not depend on derivatives of \( u \) with respect to \( y \), while the second one implies that \( (\xi_{i-1})_{\bar{a}_j} = 0 \) for all \( j > 0 \) if \( \xi_i \) has the same property. \( \square \)
**Corollary 3.6.** Let $M$ and $\tilde{M}$ be $x$-symmetry drivers of the smallest possible order for an equation of the form (2.1). Then $\tilde{M} = M \circ W$, where $W$ is an $x$-integral of (2.1).

**Proof.** Let $M$ and $\tilde{M}$ have order $q$ and their leading coefficients be denoted by $\xi_q$ and $\tilde{\xi}_q$, respectively. Both $\bar{D}(\log \xi_q)$ and $\bar{D}(\log \tilde{\xi}_q)$ are equal to $F_{u_1}$ by Lemma 3.5. Therefore, $\bar{D}(\log \xi_q - \log \tilde{\xi}_q) = 0$ and $\tilde{\xi}_q = W\xi_q$, where $W \in \ker \bar{D}$. This means that the operator $\tilde{M} - M \circ W$ has order less than $q$.

On the other hand, the last operator maps $x$-integrals into symmetries. But (2.1) admits no $x$-symmetry drivers of order less than $q$ by the assumption of Corollary. Hence, all the coefficients of $\tilde{M} - M \circ W$ must be equal to zero.

According to Theorem 3.2, the operator $M$ is an $x$-symmetry driver of order $r - 1$. As it is demonstrated in [6], $H_j = 0$ for some positive $j \leq r$ if (2.1) admits an $x$-symmetry driver of order $r - 1$. This means that (2.1) admits no driver of order less than $r - 1$ if $H_{r-1} \neq 0$. Therefore, $M$ is a driver of the smallest order. In addition, relation (2.6) defines the function $\psi$ up to multiplication by $x$-integrals. Hence, the operator $M$ is defined up to transformations $M \rightarrow M \circ W$, where $W$ is an $x$-integral. Comparing this with Corollary 3.6 and taking Lemma 3.5 and Theorem 3.2 into account, we obtain the following statement.

**Proposition 3.7.** Let equation (2.1) admit $x$-symmetry drivers. Then $H_r = 0$ for some $r > 0$, the kernel of $\bar{D} - F_{u_1}$ contains non-zero elements and any $x$-symmetry driver of the smallest order is defined by the formula (3.2) with some $\psi \in \ker(\bar{D} - F_{u_1})$.

The equations of Liouville type also possess differential operators that map symmetries to integrals.

**Lemma 3.8.** Let $W(x, y, u, u_1, u_2, \ldots, u_p)$ be an $x$-integral of equation (2.1). Then the differential operator $W_\ast$ maps any symmetry to some $x$-integral.

The above Lemma directly follows from Lemma 1 in [7]. Not so formal, Lemma 3.8 is valid because the total derivative with respect to $t$ in virtue of any symmetry $u_t = g$ and the operator $\bar{D}$ commute: $[\partial g, \bar{D}] = 0$.

**Corollary 3.9.** For any $x$-symmetry driver $M$ and any $x$-integral $W$ the operator

$$L = W_\ast \circ M$$

maps $x$-integrals into $x$-integrals again. After rewriting $L$ in the form $\sum \mu_i D^i$, the coefficients $\mu_i$ of this operator are $x$-integrals.

The most significant operator of this kind is the composition

$$\mathcal{L} = w_\ast \circ \mathcal{M},$$

where $w$ is a nontrivial $x$-integral of the smallest order and the operator $\mathcal{M}$ is given by (3.2) (i.e., $\mathcal{M}$ is an $x$-symmetry driver of the smallest order by Proposition 3.7).
Example 3.10. Recall that $H_0 = H_1 = \exp(u)$ and $H_2 = 0$ in the case of the Liouville equation (2.3). In this case, we can set $\psi = 1$ and the formula (3.2) gives us

$$M = \exp(-u)D \circ \exp(u) = D + u_1.$$ (3.6)

Since $H_0$ does not vanish, $w = u_2 - \frac{1}{2}u_1^2$ is an $x$-integral of the smallest order (see the last sentence of Theorem 2.6). We have $w_* = D^2 - u_1D$ and

$$\mathcal{L} = (D^2 - u_1D) \circ (D + u_1) = D^3 + 2wD + D(w).$$ (3.7)

In Section 1 we show that the operator (3.6) is pre-Hamiltonian (see (1.7)). As it is noted in [2], the operator (3.7) is also pre-Hamiltonian if we change the notation and rewrite this operator as $D^3 + 2uD + u_1$.

3.2 Main results

Theorem 3.11. Let $M$ be an $x$-symmetry driver of the smallest order for equation (2.1). Then the operator $M$ is pre-Hamiltonian and, moreover, there exist functions $\gamma_{ij} \in \ker \tilde{D}$ such that

$$[M(a), M(b)] = M \left( b_* (M(a)) - a_* (M(b)) + \sum_{i=0}^{n} \sum_{j=0}^{n} \gamma_{ij} D^i(a)D^j(b) \right),$$ (3.8)

for any $a, b \in \mathcal{F}$. In particular, for any $x$-integrals $f$ and $g$ there exists $\phi \in \ker \tilde{D}$ such that $[M(f), M(g)] = M(\phi)$.

Note that $M$ is always defined by formula (3.2) in accordance with Proposition 3.7. It should also be emphasized that we do not assume the existence of nontrivial $x$-integrals in Theorem 3.11. If nontrivial $x$-integrals are absent for equation (2.1), then this theorem means that $\gamma_{ij}$ (as well as $f, g$ and $\phi$) are functions of $x$ only.

Remark 3.12 (Preliminaries for the proof). We can rewrite equation (3.8) in the form

$$\sum_{i=0}^{m} \sum_{j=0}^{m} c_{ij} D^i(a)D^j(b) = M \left( \sum_{i=0}^{n} \sum_{j=0}^{n} \gamma_{ij} D^i(a)D^j(b) \right),$$ (3.9)

where the functions $c_{ij}$ are defined by (1.6) for $M$. The functions $a$ and $b$ are arbitrary here. Therefore, $D^i(a)$ and $D^j(b)$ can be considered as independent variables. Collecting the coefficients at these variables in the right-hand side of (3.9) and equating them to $c_{ij}$, we see that formula (3.8) is equivalent to relations between the coefficients of the operator $M$ (which, in particular, determine $c_{ij}$) and the functions $\gamma_{ij}$. These relations do not contain $a$ and $b$ in any way and therefore remain unchanged if we prove (3.9) for $a$ and $b$ belonging to an appropriate subset of $\mathcal{F}$ (such that $D^i(a)$ and $D^j(b)$ continue to play the role of independent variables when $a$ and $b$ are arbitrary elements of this subset). Thus, it is enough for the proof of Theorem 3.11 to demonstrate that (3.8) holds for the functions $a$ and $b$ of a special form. We take arbitrary $x$-integrals as such a special form for $a$ and $b$.

5I.e. a polynomial in these variables is equal to zero if and only if all the coefficients of this polynomial are zero.
For further reasoning, it is convenient to prove the following lemma first.

**Lemma 3.13.** Let $M$ be an $x$-symmetry driver of order $k$ for equation (2.1) and an expression

$$N = \sum_{i=0}^{q} \sum_{j=0}^{\ell} c_{ij} D^i(f) D^j(g), \quad c_{ij} \in \mathcal{F}, \quad q \geq k,$$

be a symmetry of (2.1) for arbitrary $x$-integrals $f$ and $g$. Then there exist $x$-integrals $\theta_j$ such that the expression

$$\tilde{N} = N - M \left( D^{q-k} \left( f \sum_{j=0}^{\ell} \theta_j D^j(g) \right) \right)$$

has the form $\sum_{i=0}^{q-1} \sum_{j=0}^{\ell} \tilde{c}_{ij} D^i(f) D^j(g)$ and is a symmetry of (2.1) for any $x$-integrals $f$ and $g$.

In simpler words, the driver $M$ allows us to reduce the order $q$ of $N$ without loosing the other property of $N$. Therefore, Lemma 3.13 remains applicable to the reduced expression $\tilde{N}$ if its order $\tilde{q}$ is greater than $k - 1$.

**Proof of Lemma.** The assumptions of the lemma imply that the operator $R = \sum_{i=0}^{q} C_i(g) D^i$, where $C_i(g) = \sum_{j=0}^{\ell} c_{ij} D^j(g)$, is an $x$-symmetry driver for any $g \in \ker \bar{D}$. According to Lemma 3.5, we have

$$(\bar{D} - F_{u_1})(C_q(g)) = \sum_{j=0}^{\ell} (\bar{D} - F_{u_j})(c_{qj}) D^j(g) = 0.$$

Since the integral $g$ is arbitrary, we get $(\bar{D} - F_{u_1})(c_{qj}) = 0$. The leading coefficient $\xi_k$ of the driver $M$ also belongs to the kernel of the operator $\bar{D} - F_{u_1}$. Therefore $c_{qj} = \theta_j \xi_k$, where $\theta_j \in \ker \bar{D}$ (see the proof of Corollary 3.6 if a more detailed explanation is needed). The last equalities imply that

$$M \left( D^{q-k} \left( f \sum_{j=0}^{\ell} \theta_j D^j(g) \right) \right) = C_q(g) D^q(f) + \ldots,$$

where the dots denote terms with $D^i(f), i < q$. Since the $M \circ D^{q-k}$ is a driver, the above expression is a symmetry for any $f, g \in \ker \bar{D}$. Subtracting this symmetry from $\tilde{N}$, we complete the proof.

**Proof of Theorem 3.11.** As $\mathcal{M}$ is an $x$-symmetry driver, the functions $\mathcal{M}(g)$ and $\mathcal{M}(f)$ are symmetries of (2.1) for any $x$-integrals $f$ and $g$. Lemma 3.8 implies that $f_*(\mathcal{M}(g))$ and $g_*(\mathcal{M}(f))$ are $x$-integrals (in particular, are zero if (2.1) admits trivial $x$-integrals only). Therefore, $\mathcal{M} \left( g_*(\mathcal{M}(f)) - f_*(\mathcal{M}(g)) \right)$ is a symmetry. In addition, $[\mathcal{M}(f), \mathcal{M}(g)]$ is also a symmetry since $x$-symmetries forms a Lie algebra with respect to the bracket (1.3). Thus, substituting $\mathcal{M}$, $f$, $g$ for $M$, $a$, $b$ in formula (1.6), we obtain that

$$\sum_{i=0}^{m} \sum_{j=0}^{m} c_{ij} D^i(f) D^j(g)$$

is an $x$-symmetry for any $x$-integrals $f$ and $g$. 9
By the assumption of Theorem 3.11, $M$ is a driver of smallest order. Let us denote this order by $k$ (recall that $k = r - 1$ in terms of Proposition 3.7). Starting from (3.10), we apply Lemma 3.13 several times and arrive at the relation

$$[M(f), M(g)] - M \left( g_*(M(f)) - f_*(M(g)) + \sum_{i=0}^{\hat{m}} \sum_{j=0}^{n} \gamma_{ij} D^i(f) D^j(g) \right) = \sum_{i=0}^{\hat{m}} \sum_{j=0}^{i} \zeta_{ij} D^i(f) D^j(g),$$

where $\gamma_{ij} \in \ker \bar{D}$ and the non-negative integer $\hat{m} < k$. If $k = 0$, then the right-hand side of the above relation is zero because we can completely absorb (3.10) into the image of $M$ by using Lemma 3.13. If $k > 0$ and there are non-zero coefficients $\zeta_{ij}$, then the right-hand side defines a driver for arbitrary $g \in \ker \bar{D}$. But the order of this driver is less than the smallest order $k$ and, therefore, all the coefficients $\zeta_{ij}$ must be equal to zero.

We have proved that the operator $M$ is pre-Hamiltonian with respect to the bracket (1.3) defined on functions from $F$.

It turns out that there exists a pre-Hamiltonian operator in a set of variables other than (1.1) if equation (2.1) admits nontrivial $x$-integrals in addition to $x$-symmetry drivers.

Let $w$ be an $x$-integral of smallest order (see Proposition 2.3). Consider the operator $L$ defined by (3.5). According to Corollary 3.9 all coefficients of $L$ are function of $x, w, w_i$ (by Proposition 2.3).

**Theorem 3.14.** The operator $L$ is pre-Hamiltonian with respect to the bracket

$$\{a, b\} = \sum_{i=0}^{+\infty} \left( \frac{\partial b}{\partial w_i} D^i(a) - \frac{\partial a}{\partial w_i} D^i(b) \right)$$

defined on functions of the variables $x, w, w_1, \ldots$.

It was already noted in [2] that the operator $L$ is pre-Hamiltonian for many (all checked) examples of Liouville-type equations. In other words, the work [2], in fact, contains the above theorem as a conjecture. By using Theorem 3.11, we can now prove Theorem 3.14. The proof below belongs to V. V. Sokolov.

**Proof.** Consider the evolution equations $u_{t_1} = M(f)$ and $u_{t_2} = M(g)$, where $f$ and $g$ are arbitrary function of the form (2.4) (i.e. $f$ and $g$ are $x$-integrals). Recall (see Section 1) that differentiations with respect to $t_1$ and $t_2$ in virtue of these equation are denoted by $\partial_{M(f)}$ and $\partial_{M(g)}$. Let us apply the commutator $[\partial_{M(f)}, \partial_{M(g)}]$ to the $x$-integral $w$ of smallest order. Taking (1.4) and Theorem 3.11 into account, we obtain

$$[\partial_{M(f)}, \partial_{M(g)}](w) = w_* ([M(f), M(g)]) = w_* (M(\phi)) = L(\phi)$$

for some $x$-integral $\phi$ (which is a function of $x, w, w_i$ by Proposition 2.3).

On the other hand, we have

$$w_{t_1} = \partial_{M(f)}(w) = L(f), \quad w_{t_2} = \partial_{M(g)}(w) = L(g).$$

As it is demonstrated in [6], equation (2.1) is a Liouville-type equation if it admits both $x$-symmetry drivers and nontrivial $x$-integrals.
This means that the differentiations $\partial_{M(f)}$ and $\partial_{M(g)}$ respectively coincide with $\partial_{L(f)}$ and $\partial_{L(g)}$ on functions of $x, w, w_i$. Since $L(f)$ and $L(g)$ are function of $x, w, w_i$ by Corollary 3.9, we have

$$ [\partial_{M(f)}, \partial_{M(g)}](w) = \partial_{M(f)}(L(g)) - \partial_{M(g)}(L(f)) = \partial_{L(f)}(L(g)) - \partial_{L(g)}(L(f)) = \{L(f), L(g)\}.$$ 

Thus, we arrive at the relation

$$ \{L(f), L(g)\} = L(\phi) \quad (3.11) $$

that holds when $w_i$ are functions from $\mathcal{F}$. Since the functions $x, w, w_1, \ldots \in \mathcal{F}$ are functionally independent, the relation (3.11) holds identically (i.e. without substituting $w_i$ with their values from $\mathcal{F}$).

\section*{Acknowledgements}

The author thanks V. V. Sokolov for many useful discussions as well as for suggestions that made the text more readable even in this very preliminary version.

\section*{References}

[1] Olver P.J., Applications of Lie Groups to Differential Equations, (2nd edn), Graduate Texts in Mathematics, 1993, 107, Springer–Verlag, New York.

[2] Zhiber A.V. and Sokolov V.V., Exactly integrable hyperbolic equations of Liouville type, Russ. Math. Surv., 2001 56(1), 61–101.

[3] Zhiber A.V., Quasilinear hyperbolic equations with an infinite-dimensional symmetry algebra, Russ AC SC Izv. Math., 1995, 45(1), 33–54.

[4] Sokolov V.V. and Zhiber A.V., On the Darboux integrable hyperbolic equation, Phys. Lett. A, 1995, 208, 303–308.

[5] Anderson I. M. and Kamran N., The variational bicomplex for hyperbolic second-order scalar partial differential equations in the plane, Duke Math. J., 1997, 87(2), 265–319.

[6] Startsev S.Ya., Differential substitutions of the Miura transformation type, Theoret. and Math. Phys., 1998, 116(3), 1001–1010.

[7] Sokolov V. V. and Startsev S. Ya., Symmetries of nonlinear hyperbolic systems of the Toda chain type, Theoret. and Math. Phys., 2008, 155(2), 802–811.