Self-Assembly of 4-sided Fractals in the Two-handed Tile Assembly Model

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Abstract. In this paper, we consider the strict self-assembly of fractals in one of the most well-studied models of tile based self-assembling systems known as the Two-handed Tile Assembly Model (2HAM). We are particularly interested in a class of fractals called discrete self-similar fractals (a class of fractals that includes the discrete Sierpinski’s carpet). We present a 2HAM system that strictly self-assembles the discrete Sierpinski’s carpet with scale factor 1. Moreover, the 2HAM system that we give lends itself to being generalized and we describe how this system can be modified to obtain a 2HAM system that strictly self-assembles one of any fractal from an infinite set of fractals which we call 4-sided fractals. The 2HAM systems we give in this paper are the first examples of systems that strictly self-assemble discrete self-similar fractals at scale factor 1 in a purely growth model of self-assembly. Finally, we give an example of a 3-sided fractal (which is not a tree fractal) that cannot be strictly self-assembled by any 2HAM system.

1 Introduction

The study of fractals has both a mathematical and a practical basis, as these recursively self-similar patterns occur in nature in the form of circulatory systems and branch patterns. Evidently many fractals found in nature are the result of a process where a simple set of rules dictating how individual basic components (such as individual molecules) interact to yield larger complexes with recursive self-similar structure. One approach to understanding this process is to model such a process with artificial self-assembling systems.

One of the first and also one of the most studied mathematical models of self-assembling systems is Winfree’s abstract Tile Assembly Model (aTAM) [15] where individual autonomous components are represented as tiles with glues on their edges. When considering the self-assembly of discrete self-similar fractals (dssf) such as the Sierpinski triangle one can consider either “strict” self-assembly, wherein a shape is made by placing tiles only within the domain of the shape, or “weak” self-assembly where a pattern representing the shape forms as part of a complex of tiles that contains specially labeled tiles corresponding to points in the shape and possibly additional tiles not corresponding to points of...
the shape. In this paper, we only consider strict self-assembly of dssf’s. Previous work (including [1][9][10][12][14]) has shown the difficulty of strict self-assembly of dssf’s in the aTAM as no nontrivial dssf has been shown to self-assemble in the strict sense. In fact, the Sierpinski’s triangle [11] and similar fractals [1] are known to be impossible to self-assemble in the aTAM; though it is possible to design systems which “approximate” the strict self-assembly of fractals [11][13]. Interestingly, it is unknown whether there exists a dssf which strictly self-assembles in the aTAM. This includes the Sierpinski’s carpet dssf.

While the aTAM models single tile attachment at a time, a more generalized model and another of the most studied models of self-assembly called the 2-Handed Assembly Model [4] (2HAM, a.k.a. Hierarchical Assembly Model) allows pairs of large assemblies to bind together. The impossibility of strictly self-assembling the Sierpinski triangle [2] has been shown; this impossibility is due in part to the “tree-like” structure of Sierpinski’s triangle. In [3] it is shown that Sierpinski’s carpet self-assembles in the 2HAM at temperature 2, but with scale factor 3. That is, instead of self-assembling a structure with tiles corresponding to the points of Sierpinski’s carpet, the structure that self-assembles contains a 3 by 3 block of tiles that corresponds to a single point of Sierpinski’s carpet. Here we show that not only does Sierpinski’s carpet self-assemble with no scale factor, but a general class of fractals, which we call the 4-sided fractals, self-assemble at temperature 2 in the 2HAM. Intuitively, 4-sided fractals are fractals that have a generator (the set of points in the first stage of the fractal) such that the generator is connected and consists of a rectangle of points and points “inside” this rectangle. Informally, a 4-sided fractal is a fractal with a generator that contains all 4 sides and one can define 0, 1, 2, and 3-sided fractals analogously. (Definitions are given in Section 2.) Moreover, we show that there exists a 3-sided fractal that cannot be strictly self-assembled by any 2HAM system at any temperature. This is especially interesting considering that 3-sided fractals are not tree fractals (a class of fractals that can be seen to not strictly self-assemble in the 2HAM with no scale factor.)

Theorem 1 implies that one of the most well-known dssf’s strictly self-assembles in one of the simplest and most studied models of self-assembly, the 2HAM. It should be noted that any dssf can strictly self-assemble in the Signal-passing Tile Assembly Model (STAM) where tiles can change state and even disassociate from an existing assembly, “breaking” an assembly into two disconnected assemblies. That is, given any dssf, there is a STAM system that strictly self-assembles this fractal. Additionally, in a model similar to the STAM, the Active Signal Tile Assembly Model [7], infinite, self-similar substitution tiling patterns which fill the plane have been shown to assemble [5]. This may be considered a testament to the power of active tiles. Here we show that it is still possible to strictly self-assemble an infinite class of fractals in the 2HAM even though tiles are not active and disassociation is not allowed. While the positive result presented here

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1 or step in the self-assembly process
2 Additionally, in [6] it is shown that a large class of fractals strictly self-assembles in the STAM even with temperature restricted to 1.
pertains only to 4-sided fractals, preliminary results discussed in Section 5 show that the techniques used here to prove this positive result may give rise to a much more general classification of which fractals strictly self-assemble in the 2HAM.

2 Preliminaries

Here we provide informal descriptions of the 2-Handed Tile Assembly Model (2HAM). For more details see [4, 5]. We also give the definition of discrete self-similar fractals similar to the definitions found in [1] and [6].

2.1 Informal description of the 2HAM

The 2HAM [4, 5] is a generalization of the abstract Tile Assembly Model (aTAM) [15] in that it allows for two assemblies, both possibly consisting of more than one tile, to attach to each other. Since we must allow that the assemblies might require translation before they can bind, we define a supertile to be the set of all translations of a τ-stable assembly, and speak of the attachment of supertiles to each other, modeling that the assemblies attach, if possible, after appropriate translation. We now give a brief, informal, sketch of the 2HAM.

Given \( V \subseteq \mathbb{Z}^2 \), the full grid graph of \( V \) is the undirected graph \( G_V = (V, E) \), such that for all \( x, y \in V \), \( \{x, y\} \in E \) iff \( ||x - y|| = 1 \), i.e., iff \( x \) and \( y \) are adjacent and the 2-dimensional integer Cartesian space.

A tile type is a unit square with each side having a glue consisting of a label (a finite string) and strength (a non-negative integer). We assume a finite set \( T \) of tile types, but an infinite number of copies of each tile type, each copy referred to as a tile. A supertile is (the set of all translations of) a positioning of tiles on the integer lattice \( \mathbb{Z}^2 \). Two adjacent tiles in a supertile interact if the glues on their abutting sides are equal and have positive strength. Each supertile induces a binding graph, a grid graph whose vertices are tiles, with an edge between two tiles if they interact. The supertile is \( \tau \)-stable if every cut of its binding graph has strength at least \( \tau \), where the weight of an edge is the strength of the glue it represents. That is, the supertile is stable if at least energy \( \tau \) is required to separate the supertile into two parts. Note that throughout this paper, we will use the term assembly interchangeably with supertile.

A (two-handed) tile assembly system (TAS) is an ordered triple \( T = (T, S, \tau) \), where \( T \) is a finite set of tile types, \( S \) is the initial state, and \( \tau \in \mathbb{N} \) is the temperature. For notational convenience we sometimes describe \( S \) as a set of supertiles, in which case we actually mean that \( S \) is a multiset of supertiles with one count of each supertile. We also assume that, in general, unless stated otherwise, the count for any single tile in the initial state is infinite. Commonly, 2HAM systems are defined as pairs \( T = (T, \tau) \), with the initial state simply consisting of an infinite number of copies of each singleton tile type of \( T \), and throughout this paper this is the notation we will use.

Given a TAS \( T = (T, \tau) \), a supertile is producible, written as \( \alpha \in \mathcal{A}[T] \), if either it is a single tile from \( T \), or it is the \( \tau \)-stable result of translating two producible assemblies without overlap. A supertile \( \alpha \) is terminal, written as \( \alpha \in \mathcal{A}\square[T] \), if for every producible supertile \( \beta \), \( \alpha \) and \( \beta \) cannot be \( \tau \)-stably
A set, or shape, \( X \) _strictly self-assembles_ if there is a TAS \( T \) for which every assembly \( \alpha \in \mathcal{A}_{\neg \{T\}} \) satisfies \( \text{dom} \, \alpha = X \). Essentially, strict self-assembly means that tiles are only placed in positions defined by the shape. This is in contrast to the notion of _weak self-assembly_ in which only specially marked tiles can and must be in the locations of \( X \) but other locations can perhaps receive tiles of other types. All results in this paper are for strict self-assembly of shapes.

### 2.2 Discrete Self-Similar Fractals

In order to state the main theorem, we need to provide a few definitions. The definition of a discrete self-similar fractals and some of the notation used here also appears in [1,6,13]. First we introduce some notation.

For \( g \in \mathbb{N} \) and \( G \subseteq \mathbb{N}^2 \), let \( l_G, r_G, b_G, \) and \( t_G \) denote the integers: \( l_G = \min_{(x,y) \in G} x, \quad r_G = \max_{(x,y) \in G} x, \quad b_G = \min_{(x,y) \in G} y, \) and \( t_G = \max_{(x,y) \in G} y \). Moreover, let \( w_G = r_G - l_G + 1 \) and \( h_G = t_G - b_G + 1 \) denote the width and height of \( G \) respectively. Finally, let \( L_G = \{(l_G, y) \mid b_G \leq y \leq t_G\}, \quad R_G = \{(r_G, y) \mid b_G \leq y \leq t_G\}, \quad T_G = \{(x, t_G) \mid l_G \leq x \leq r_G\}, \) and \( B_G = \{(x, b_G) \mid l_G \leq x \leq r_G\} \). In other words, \( L_G, R_G, T_G, \) and \( B_G \) are the left, right, top, and bottom line segments of a “bounding box” of \( G \). We also use \( \mathbb{N}_n \) to denote the subset \( \{0, \ldots, g-1\} \) of \( \mathbb{N} \). Finally, if \( A \) and \( B \) are subsets of \( \mathbb{N}^2 \) and \( (x, y) \in \mathbb{N}^2 \), then \( A + (x, y) = \{(x_a, y_a) + (x \cdot x_b, y \cdot y_b) \mid (x_a, y_a) \in A \) and \( (x_b, y_b) \in B\} \).

First we give the definition of a discrete self-similar fractal.

**Definition 1.** Let \( X \subseteq \mathbb{N}^2 \). We say that \( X \) is a **discrete self-similar fractal** (or _dssf for short_), if there is a set \( \{(0,0)\} \subseteq G \subseteq \mathbb{N}^2 \) with at least one point in every row and column, such that

1. the full grid-graph of \( G \) is connected,
2. \( w_G > 1 \) and \( h_G > 1 \),
3. \( G \subseteq \mathbb{N}_{w_G} \times \mathbb{N}_{h_G} \), and
4. \( X = \bigcup_{i=1}^{\infty} X_i \), where \( X^i \), the \( i \)-th stage of \( X \), is defined by \( X^1 = G \) and \( X^{i+1} = X^i + (w_G, h_G)G \).

Moreover, we say that \( G \) is the _generator_ of \( X \).

A connected discrete self-similar fractal is one in which every component is connected in every stage, i.e. there is only one connected component in the grid graph formed by the points of the shape.

**Definition 2.** _\([n\text{-sided fractals}]\_) Let \( n \in \{0, 1, 2, 3, 4\}, \ 1 < g \in \mathbb{N} \) and \( X \subseteq \mathbb{N}^2 \). We say that \( X \) is a **\( n \)-sided fractal** iff \( X \) is a \( g \)-discrete self-similar fractal with generator \( G \) such that:

1. the full grid graph of \( G \) is connected,
2. \( S \cap G = S \) for at least \( n \) distinct sets \( S \) in \( \{L_G, R_G, T_G, B_G\} \).

Intuitively, the second condition in Definition 2 is saying that the fractal generator contains all points of at least \( n \) of the left, right, top, and bottom line segments of a “bounding box” of \( G \). In particular, the generator of a 4-sided fractal contains all of the points along the left, right, top, and bottom “sides” of
the fractal generator. Finally, for a fractal $X$ with generator $G$, an enumeration of the points in a generator $G = \{v_i\}^{|G|}_{i=1}$, and $j \in \mathbb{N}$, the stages of $X$ are $S^1 = G$ and $S^{j+1} = S^j + (w^j_G, h^j_G)G$. For $i \in \mathbb{N}$ such that $1 \leq i \leq |G|$, we call the points of the $j + 1$ stage given by $S_j + (w^j_G, h^j_G)v_i$ the $j$th stage at position $i$.

3 Four Sided Fractals

In this section we show how to strictly self-assemble the class of 4-sided discrete self-similar fractals in the 2HAM (with scale factor of 1). The most well-known example of a 4-sided fractal is Sierpinski’s carpet. This is the first example of a non-trivial dssf shown to self-assemble in either the 2HAM (or the aTAM) with no scale factor.

**Theorem 1.** Let $X$ be a 4-sided fractal. Then, there exists a 2HAM TAS $T_X = (T, 2)$ that strictly self-assembles $X$. Moreover, if $G$ is the generator for $X$ and $|G| = g$, $|T|$ is $O(g^3)$.

We build intuition for a construction showing Theorem 1 by showing that Sierpinski’s carpet strictly self-assembles in the 2HAM at scale factor 1. We then describe the modifications needed to extend the construction for the carpet to give an algorithm for obtaining a tile set $T$ given a generator for a 4-sided fractal, $X$, such that the 2HAM TAS $(T, 2)$ strictly self-assembles $X$.

3.1 Sierpinski’s carpet construction overview

The Sierpinski’s carpet dssf is the dssf with generator $G = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 2), (2, 0), (2, 1), (2, 2)\}$. Figure 1a depicts this generator, while Figures 1b and 1c depict the 2nd and 3rd stages of the dssf respectively. We denote this carpet by $S$ and for $i \in \mathbb{N}$, we denote the $i$th stage of $S$ as $S^i$. We enumerate the points of $S^1$ as depicted in Figure 1a and use this enumeration to reference the positions of some substage within a subsequent stage of the carpet.

![Fig. 1: Three stages of Sierpinski’s carpet](image)

We now describe the tile set, $T$, that is used to strictly self-assemble $S$ in the 2HAM at temperature $\tau = 2$ at scale factor 1. Tiles that are described in this section, along with the tiles depicted in Section B, give the full tile set $T$.

**Overview of stage 2 assembly.** We begin by distinguishing between two classes of tile types called grout tile types (or grout tiles when referring to actual tiles) and initializer tile types (or initializer tiles). Informally, initializer tiles self-assemble to form 8 different supertiles, the domains of which are contained in the portion of $S^2$ depicted in Figure 2. We call these 8 supertiles $C^2_i$ for $1 \leq i \leq 8$. We will describe the glues and assembly sequences of the
8 supertiles in Section A.1. The main idea is that tiles that self-assemble \( C^2_i \) have been “hard-coded” (i.e. for any glue on the edge of some tile, there exists a single matching glue on another tile) to ensure that for each \( i \), all tiles of \( C^2_i \) self-assemble before \( C^2_i \) can be a subassembly in any other strictly larger assembly.

In other words, referring to Figure 4a, the gray tiles self-assemble one of the 8 different supertiles \( C^2_i \) before any of the the aqua tiles can attach. Figure 3 depicts \( C^2_i \) for each \( i \). Note that for each \( i \), \( C^2_i \) subassemblies may expose glues of type \( g^d \) or \( \hat{g}^d \) for \( d \) either \( n, s, e, \) or \( w \), as well as possibly \( g^k \) or \( \hat{g}^k \) for \( 1 \leq k \leq 8 \). Informally, these glues encode which position (1 through 8) each \( C^2_i \) assembly will end up in an assembly corresponding to stage 3 of Sierpinski’s carpet, where \( C^2_i \) will be in position \( i \). We now explain the purpose of these glues in more detail.

**Overview of stage 3 assembly.** For each \( i \), \( C^2_i \) exposes glues that allow for the attachment of grout tiles. In Figure 4 grout tiles are depicted in aqua.
The grout supertiles that bind to some $C^2_i$ before any other grout supertiles are called start-gadgets. See Figure 5. There are 8 different classes of grout tile types (with corresponding grout tiles) which we enumerate with 1 through 8. Let $j \in \mathbb{N}$ such that $1 \leq j \leq 8$ refer to a class of grout. Glues of grout tiles have been defined so that grout supertiles cooperatively bind to $C^2_i$ assemblies, eventually surrounding such an assembly. The grout supertiles other than start-gadgets that cooperatively bind to $C^2_i$ are called crawlers. See Section A for a detailed description of these tiles with figures. Glues labeled $g_k$ or $\hat{g}_k$ for $1 \leq k \leq 8$ are called indicator glues and these have a special purpose. A grout tile that binds to an indicator glue via a south glue (likewise for north, east, and west) of the grout tile may expose (depending on its grout class) a strength-1 glue on its north edge that will eventually take part in a cooperative binding event between $C^2_i$ and $C^2_i'$ with a sufficient number of grout tiles attached to each. The type of glue and whether or not a grout tile exposes such a glue depends on the grout tiles class. We call these glues stage-binding glues for the $3^{rd}$ stage. For each $i$, grout tiles have been defined so that they attach to $C^2_i$ and eventually bind to all indicator glues of $C^2_i$ before grout tiles can no longer bind. We let $C^2_{(i,j)}$ denote the largest supertile

![Fig. 5: Left: The supertile that starts the growth of grout for $C^2_1$. Right: The supertile that starts the growth of grout for $C^2_s$ for $s > 2$. Note that for each $s \geq 2$, only one of these supertiles can bind to tiles of $C^2_1$. Moreover, the supertile depicted on the left can bind to some $C^2_1$ iff $s = 2$, and the supertile depicted on the right can bind to some $C^2_1$ iff $s > 2$.](image)

![Fig. 6: (a) A depiction of $C^2_{(1,1)}$ with stage-binding glues $h_{1,i}$ and $\hat{h}_{1,i}$. (b) A depiction of $C^2_{(2,1)}$ with stage-binding glues $h_{1,i}$, $h_{1,i}$, $h_{2,i}$, and $\hat{h}_{2,i}$. Notice that the stage-binding glues of $C^2_{(1,1)}$ and $C^2_{(2,1)}$ allow for the cooperative binding of $C^2_{(1,0)}$ and $C^2_{(2,0)}$. Intuitively, the distance between these glues ensures proper assembly of each stage of Sierpinski’s carpet.](image)

(in terms of the subassembly relation) consisting of $C^2_i$ and grout tiles of class $j$. Figure 6 depicts $C^2_{(i,j)}$ for $i = 1$ and 2 and $j$ between 1 and 8. Moreover, for $i, j, i'$ and $j'$ between 1 and 8 (inclusive), glues are defined so that $C^2_{(i,j)}$ and
\(C^2_{(i,j)}\) can bind iff \(j = j'\). That is the grunt tiles of \(C^2_{(i,j)}\) and \(C^2_{(i',j')}\) belong to the same class.

As grunt tiles attach to each \(C^2_s\) assembly, stage-binding glues are exposed on specially designated grunt tiles so that the supertiles \(C^2_{(i,j)}\) can bind to form the portion of \(S^3\) depicted in Figure 4b. Note that stage-binding glues may be exposed before \(C^2_{(i,j)}\) completely assembles and therefore for some \(i\) and \(i'\), two subassemblies of \(C^2_{(i,j)}\) and \(C^2_{(i',j')}\) may bind to form a subassembly of an assembly, which we call \(C^3_j\), corresponding to stage 3. We define glues belonging to grunt tiles so that this does not prevent tiles from binding in locations corresponding to points of stage 2 at positions \(i\) and \(i'\) from completing assembly as a subassembly of \(C^3_j\) and note that this does not permit tiles to bind in locations outside of locations in \(\text{dom}(C^3_j)\). Therefore, we assume that each \(C^2_{(i,j)}\) completely assembles before binding to some other supertile to become a subassembly of a larger assembly. \(C^3_j\) is depicted in Figure 4b. Finally, for \(i'\) such that \(1 \leq i' \leq 8\), the glues that might allow (depending on \(i\) and \(i'\)) some supertile \(C^2_{(i,j)}\) to bind to another supertile \(C^2_{(i',j')}\) are strength 1 glues separated by a distance of \(3^{2-1} = 3\). This distance is ensured by the locations of the indicator glues and will prevent supertiles corresponding to different fractal stages from binding. Moreover, we define the grunt tiles such that the \(C^2_{(i,j)}\) supertiles bind before the “next iteration” of grunt tiles can attach. In other words, \(C^2_{(i,j)}\) supertiles bind for all \(i\) between 1 and 8 before a start-gadget can bind to the resulting assembly \(C^3_j\).

**Overview of stage s assembly for \(s \geq 4\).** For each \(j\) and all \(i\), the supertiles \(C^2_{(i,j)}\) bind to form a supertile \(C^3_j\) corresponding to a portion of \(S^3\). Just as \(i\) corresponds to the position where the \(C^2_i\) supertile will bind when \(C^3_j\) forms, \(j\) corresponds to the position where \(C^3_j\) will bind when a supertile corresponding to a portion of \(S^4\) self-assembles. This portion of \(S_1\) is essentially \(S^4\) without northernmost, southernmost, easternmost, and westernmost points, the absence of which makes room for the assembly of more grunt tiles. Informally, the position in the \(C^3_j\) supertiles of each \(C^2_i\) supertile is determined by the glue exposed by the supertiles \(C^2_i\). Moreover, the grunt class \(j\) determines the grunt tiles that will bind to \(C^3_j\), which will in turn determine the position of the \(C^3_j\) supertile in \(S^4\). Finally, just as some super tile \(C^2_{(i,j)}\) exposes some indicator glues the supertile \(C^3_{(i,j)}\) for \(i = j\) expose the same strength-1 indicator glues, only at a distance of \(3^2 = 9\) apart.

Repurposing \(i\), we now let \(C^3_j\) be denoted by \(C^3_j\). Now, for each \(i\) and \(j\) with \(1 \leq i, j \leq 8\),
the 8 different classes of grout tile types can attach to each $C^i_3$ supertile to give supertiles $C^i_3(i,j)$, and the glues of each different class of grout tiles determine where the supertiles consisting of $C^i_3(i,j)$ attach to self-assemble supertiles, $C^4_3$, corresponding to a portion of $S^4$. $C^4_3$ is depicted in Figure 7. Moreover, the glues that allow some supertile $C^i_3(i,j)$ to bind to another supertile $C^i_3(i',j)$, for some $i'$ say, are strength 1 glues separated by a distance of 9.

Repeating this process, we see that for any $i,j,s \in \mathbb{N}$ such that $1 \leq i,j \leq 8$ and $s > 2$, we can self-assemble supertiles $C^{s-1}_i$ corresponding to a portion of $S^{s-1}$ (again, we are leaving room for grout tiles), and supertiles $C^{s-1}_{(i,j)}$ corresponding to $C^{s-1}_i$ with the attachment of grout tiles all belonging to the $j^{th}$ class of grout tile types. Moreover, the supertiles $C^{s-1}_{(i,j)}$ expose strength 1 glues that are at a distance of $3s-2$ apart that allow for the stable binding of these supertiles to form a supertile $C^s_i$ corresponding to $S^s$. For $i' \in \mathbb{N}$ such that $1 \leq i' \leq 8$, since the distance between the 2 glues that allow for two supertiles $C^s_{(i,j)}$ and $C^s_{(i',j)}$ to bind is $3s-2$, one can observe that no erroneous supertiles can self-assemble. In particular, glue distances ensure that for $p,q \in \mathbb{N}$ such that $p,q > 2$, $C^p_{(i,j)}$ subassemblies can bind to some $C^q_{(i,j)}$ subassemblies iff $p = q$. Hence, one can show that each supertile is a subassembly of such a $C^p_i$ for some $n \in \mathbb{N}$, and therefore for any producible assembly $\alpha \in \mathcal{A}[\mathcal{T}]$, there exists a stage $s > 1$ such that $\alpha \subset S^s$. Moreover, one can observe that for any stage $s \geq 1$, $S^s \subset C^{s+1}_i$. Therefore, as this hierarchical growth continues indefinitely, the domain of the terminal assembly of the 2HAM TAS $\mathcal{T} = (\mathcal{T}, 2)$ is $S$. In other words, $\mathcal{T}$ strictly self-assembles $S$.

3.2 4-sided fractals construction overview

The construction that shows that any 4-sided fractal strictly self-assembles in the 2HAM at scale factor 1 (Theorem 1) is a generalization of the construction given in Section 3.1. Let $G$ be the generator for a 4-sided fractal and recall the notation of $L_G$, $R_G$, $B_G$, and $T_G$. Moreover, let $|G| = r$, let $X$ denote the dssf with generator $G$, and let $X_s$ be the $s$ stage of $X$. We will describe a tile set $T$ such that $X$ strictly self-assembles in the 2HAM system $\mathcal{T} = (\mathcal{T}, 2)$. As an example, consider the generator in Figure 8a. Stage 2 of this fractal is depicted in Figure 8b. We also choose the convention of ordering the positions in $G$ from top to bottom and left to right. This enumeration is depicted in Figure 8a.

![Fig. 8: Two stages of a 4-side fractal.](image-url)
To show Theorem 1, we first show the following lemma that follows from a modification of the construction given in Section 3.1. Intuitively, this lemma states that dssf’s with generators consisting only of points on the perimeter of a rectangle strictly self-assemble in the 2HAM.

**Lemma 1.** Let $X$ be a 4-sided fractal with generator $G$ such that $G \setminus (L_G \cup L_G \cup T_G \cup B_G) = \emptyset$. Then, there exists a 2HAM TAS $T_X = (T, 2)$ that strictly self-assembles $X$.

To show Lemma 1, we show how to modify the construction given in Section 3.1. Given a 4-sided fractal $X$ with generator that satisfies the assumptions of Lemma 1 (for example, the generator depicted in Figure 9), one can see that each step in the construction in Section 3.1 generalizes to give a tile set $T$ such that the 2HAM TAS $T_X = (T, 2)$ strictly self-assembles $X$. Details of these modifications to the Sierpinski’s carpet construction are in Section C. The basic idea for proving Lemma 1 is to “elongate” the initializer and grout supertiles given in the Sierpinski’s carpet construction and most of the remaining details are analogous.

We now give a high-level overview of the proof of Theorem 1. See Section C for more detail. To prove Theorem 1 given any 4-sided fractal $X$ with generator $G$, we consider a set that consists of the points only on the perimeter of $G$ and call this set $G'$. Then, the fractal $X'$ with generator $G'$ strictly self-assembles in the 2HAM by Lemma 1 and we denote the tile set given by this lemma by $T'$. Then, to give a tile set $T$ such that the 2HAM TAS strictly self-assembles $X$ we add additional tiles to the set $T'$ to account for points of $G$ that are not on the perimeter of $G$ and modify the tiles of $T'$ by adding strength-2 glues to particular edges of the tiles of $T'$ that allow for these additional tiles to attach. Additional initializer tiles are added to $T'$ to ensure that tiles are placed at points of $G$ that are not on the perimeter of $G$ during the self-assembly of initializer supertiles. For example, the gray tiles in Figure 10 are the tiles of an initializer supertile. Note the tiles corresponding to points of $G$ that are not on the perimeter of $G$. Additional grout tiles can be added so that as grout supertiles attach to an assembly, tiles can attach in locations corresponding to points of $G$ not on the perimeter of $G$. Moreover, to ensure that stages at a position, $p$ say, corresponding to points that are not on the perimeter of $G$ correctly assemble, additional grout tiles are added such that these
additional grout supertiles always surround an entire assembly corresponding to a stage, \( i \) say, and expose glues such that the resulting supertile will bind to an assembly that (possibly after the binding of other supertiles) corresponds to an assembly corresponding to stage \( i + 1 \). Figure 10 depicts a supertile that will bind at position 12 as a supertile corresponding to stage 3 of the dssf self-assembles. Note the glues that are exposed on tiles adjacent to red tiles. These glues will permit this supertile to bind in position 12 as a supertile corresponding to stage 3 of the dssf self-assembles. Finally, we note that with these modification to the tiles set given by Lemma 1 give a tile set that satisfies Theorem 1.

To see that the tile complexity is \( O(g^3) \), note that hard-coding initializer supertiles, corner-gadgets, crawler supertiles, and start-gadgets each require \( O(g^3) \) tile types each. For example, initializer supertiles require \( O(g^2) \) tiles to hard-code their stage 2 shape and \( g \) of them must assembly (one for each point in \( G \)). For tile types making up corner-gadgets, crawler supertiles, and start-gadgets, note that none of these supertiles consists of more than \( g \) tiles. Therefore, to hard-code one of these these supertiles (requiring \( O(g) \) types) that can bind to some \( C_i^g \) for \( i \) between 1 and 8 (requiring \( O(g) \) times more types) that also belongs to one of \( g \) classes of grout (requiring \( O(g) \) times more types), \( O(g^3) \) tile types are required. Hence in total, \( O(g^3) \) tiles types are required.

4 A 3-sided Fractal that does not Strictly Self-assemble

In this section we give a high-level sketch of the proof that there exist 3-sided fractals that do not strictly self-assemble in the 2HAM. A detailed proof is given in Section D.

**Theorem 2.** There exists a 3-sided fractal \( X \) for which there is no 2HAM TAS \( T_X = (T, \tau) \) that strictly self-assembles \( X \).

To prove Theorem 2, we consider the fractal with generator given by the points in Figure 12a. Stage 2 of this fractal is shown in Figure 12b. We refer to this fractal as \( X \) and, similar to the convention in 3.1, we refer to the \( s \)th stage of \( X \) as \( X^s \). We refer to the \( i \)th position of \( X^s \) as \( X_i^s \) where \( 1 \leq i \leq 13 \) (Figure 12a). We call the assembly \( \gamma^s_i \) for which \( \text{dom} \gamma^s_i = X_i^s \).

Consider any 2HAM TAS \( T_X = (T, \tau) \). Let \( g \) be the number of tiles in \( T_X \). Consider a producible assembly \( \alpha \) such that \( X^{g+2} \subseteq \text{dom} \alpha \), and specifically the subassembly \( \gamma_0^{g+2} \), which is an assembly \( \gamma^{g+1} \). We prove in Section D that \( \gamma^{g+1} \) contains a sequence of \( g + 1 \) strength \( \tau \) cuts consisting of rectangles.
a single glue (Figure 12b). Let $\beta_k$ be the subassembly below the $k^{th}$ cut in this sequence. Essentially, because the cuts are strength $\tau$, the $\beta_k$ subassemblies can be removed and the resulting subassemblies are still producible, so the $\beta_k$ subassemblies cannot be guaranteed to attach before any point in the construction of $\alpha$. However, since there are $g+1$ cuts and only $g$ tiles, there is some $\beta_i, \beta_j$ with $|\beta_i| > |\beta_j|$ that attach with the same glue, and if other $\beta_k$ subassemblies have not yet attached, it is possible for $\beta_i$ to bind where $\beta_j$ is needed (Figure 11). Since the domain of the resulting assembly $\not\in X$, $T_X$ does not strictly self-assemble $X$.

![Fig. 12: The assemblies that form the first two stages of $X$. Strength $\tau$ cuts are shown on Stage 2.](image)

5 Conclusion

Theorem 1 shows that any 4-sided dssf strictly self-assembles in the 2HAM at temperature 2 and with no scale factor. Theorem 2 shows that there exists a 3-sided fractal that does not strictly self-assemble in any 2HAM system at any temperature. Preliminary results seem to show that similar techniques to those described in Section 3.2 can be used to give an example of a 3-sided fractal that can strictly self-assemble in the 2HAM and though still just an early investigation, the techniques used to give a tile set that strictly self-assembles a given 4-sided fractal may be modifiable to show that a much more general class of fractals strictly self-assembles. In particular, a fractal belonging to this class can be described as having a generator with a generating cycle.

Informally, a simple cycle $C$ in $G$ (technically defined in the full-grid graph of $G$) is a generating cycle iff 1) $G$ contains 2 distinct east points of contact $p_1$ and $p_2$ with corresponding west points of contact $p_3$ and $p_4$, and 2) distinct north points of contact $p_5$ and $p_6$ with corresponding south points of contact $p_7$ and $p_8$, and 2) for $i \in \mathbb{N}$ such that $1 \leq i \leq n$, $C$ contains points $p_i'$ and paths $P_i$ from $p_i'$ to $p_{i+1}$, 3) moreover, $P_i \cap P_{i+1} = \emptyset$ for $i \in \{1, 3, 5, 7\}$. Figure 13 depicts one of the simplest generators (for a dssf which we have been calling the hashtag fractal) with a generating cycle.
References

1. K. Barth, D. Furcy, S. M. Summers, and P. Totzke. Scaled tree fractals do not strictly self-assemble. In *Unconventional Computation & Natural Computation (UCNC) 2014, University of Western Ontario, London, Ontario, Canada* July 14-18, 2014, pages 27–39, 2014.
2. S. Cannon, E. D. Demaine, M. L. Demaine, S. Eisenstat, M. J. Patitz, R. T. Schweller, S. M. Summers, and A. Winslow. Two hands are better than one (up to constant factors): Self-assembly in the 2ham vs. atam. In N. Portier and T. Wilke, editors, *STACS*, volume 20 of LIPIcs, pages 172–184. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2013.
3. C. T. Chalk, D. A. Fernandez, A. Huerta, M. A. Maldonado, R. T. Schweller, and L. Sweet. Strict self-assembly of fractals using multiple hands. *Algorithmica*, pages 1–30, 2015.
4. Q. Cheng, G. Aggarwal, M. H. Goldwasser, M.-Y. Kao, R. T. Schweller, and P. M. de Espanés. Complexities for generalized models of self-assembly. *SIAM Journal on Computing*, 34:1493–1515, 2005.
5. E. D. Demaine, M. L. Demaine, S. P. Fekete, M. Ishaque, E. Rafalin, R. T. Schweller, and D. L. Souvaine. Staged self-assembly: nanomanufacture of arbitrary shapes with O(1) glues. *Natural Computing*, 7(3):347–370, 2008.
6. J. Hendricks, M. Olsen, M. J. Patitz, T. A. Rogers, and H. Thomas. Hierarchical self-assembly of fractals with signal-passing tiles (extended abstract). In *Proceedings of the 22nd International Conference on DNA Computing and Molecular Programming (DNA 22), Ludwig-Maximilians-Universitt, Munich, Germany* September 4-8, 2016, pages 82–97.
7. N. Jonoska and D. Karpenko. Active tile self-assembly, part 1: Universality at temperature 1. *International Journal of Foundations of Computer Science*, 25(02):141–163, 2014.
8. N. Jonoska and D. Karpenko. Active tile self-assembly, part 2: Self-similar structures and structural recursion. *International Journal of Foundations of Computer Science*, 25(02):165–194, 2014.
9. S. Kautz and B. Shutters. Self-assembling rulers for approximating generalized sierpinski carpets. *Algorithmica*, 67(2):207–233, 2013.
10. S. M. Kautz and J. I. Lathrop. Self-assembly of the Sierpinski carpet and related fractals. In *Proceedings of The Fifteenth International Meeting on DNA Computing and Molecular Programming (Fayetteville, Arkansas, USA, June 8-11, 2009)*, pages 78–87, 2009.
11. J. I. Lathrop, J. H. Lutz, and S. M. Summers. Strict self-assembly of discrete Sierpinski triangles. *Theoretical Computer Science*, 410:384–405, 2009.
12. J. H. Lutz and B. Shutters. Approximate self-assembly of the sierpinski triangle. *Theory Comput. Syst.*, 51(3):372–400, 2012.
13. M. J. Patitz and S. M. Summers. Self-assembly of discrete self-similar fractals. *Natural Computing*, 1:135–172, 2010.
14. P. W. Rothemund, N. Papadakis, and E. Winfree. Algorithmic self-assembly of DNA Sierpinski triangles. *PLoS Biology*, 2(12):2041–2053, 2004.
15. E. Winfree. *Algorithmic Self-Assembly of DNA*. PhD thesis, California Institute of Technology, June 1998.
Technical Appendix

A Sierpinski’s Carpet Tile Set Details

A.1 Self-assembly of stage 2 without grout.

We now describe the tile set $T$ of tile types that give a 2HAM TAS $T = (T, 2)$ that strictly self-assembles Sierpinski’s carpet $S$. We will then show that the techniques used to strictly self-assemble $S$ can be generalized to prove Theorem 4.

First, we describe the initializer tile types. For each $i$, we define 32 unique tile types of $T$, described in Figure 3, that self-assemble the supertile $C^2_i$ corresponding to a portion of $S^2$ that will be in the $i^{th}$ position of a supertile corresponding to a portion of $S^3$.

A.2 Tile types for grout tiles

In this section we describe the tile types in $T$ that make up the grout tile types. The next section describes how these supertiles attach to assemblies $C^s_1$ and $C^s_2$ for $s \geq 2$. The cases for assemblies $C^s_i$ for $3 \leq i \leq 8$ are similar and we give depictions of these tiles in Section B.

Starting from some supertile $C^s_2$, initial growth of grout tiles begins with a supertile assembly of grout tiles that cooperatively bind to some $C^s_1$ via pairs of glues that have been specifically chosen for each supertile $C^s_1$. Figure 5 depicts such a supertile (left) that binds to a $C^s_2$ assembly and another supertile (right) that binds to a $C^s_1$ assembly for $s > 2$. We call such supertiles start-gadget supertiles as the binding of grout tiles to $C^s_i$ for $1 \leq i \leq 8$ always begins with the attachment of such a supertile.

For $s \geq 2$, after grout-initiating supertiles attach to $C^s_i$, supertiles consisting of grout tiles called crawlers can then cooperatively attach to the resulting assembly that partially surround $C^s_i$. These supertiles are depicted in Figures 14 and 15.

Referring to Figure 14, $1 \leq i \leq 8$ and from left to right: (1) For each $s > 2$, after a start-gadget supertile has attached to $C^s_1$ the supertile (bottom) and tile (top) that may bind to the south of the start-gadget. Multiple copies of these supertiles bind allowing a column of grout supertiles to attach until the south-east corner of $C^s_1$ is reached. We call these supertiles crawlers. (2) The supertiles that allow for a column of tiles to attach to the easternmost tiles of $C^s_i$ for some $s > 1$. Note that these supertiles bind via cooperation with $g_{1,i}$ and either $g^c$ or $\hat{g}^c$ glues. (3) The glues $g_i$ and $\hat{g}_i$, which we call indicator glues, bind to $\hat{g}_1$ and exposes the glue $\hat{h}_{1,i}$ that will (along with $h_{1,i}$) participate in a binding event in the formation of a successive stage $C^{s+1}_i$. We call $\hat{h}_{1,i}$ and $h_{1,i}$ stage-binding glues as they cooperate to allow for the assembly of an assembly corresponding to stage 2. (4) After the tiles containing stage-binding glues bind, these supertiles attach to the easternmost tiles of $C^s_i$. Multiple copies
of these supertiles bind allowing a column of tiles to attach until the north-east corner of $C_{s}^{1}$ is reached, where a $g_{n}$ glue will be exposed. Figure 15 depicts grout tiles that assemble along the southernmost tiles of $C_{s}^{1}$ for some $s \geq 2$.

Fig. 14: A depiction of grout tiles that bind to the easternmost tiles of a $C_{s}^{1}$ assembly. Once again, labels for unlabelled glues are “hard-coded” to enforce the assembly of each supertile shown here.

Fig. 15: Similar to the supertiles depicted in Figure 14, these supertiles will self-assemble a row of tiles that bind to the southernmost tiles of $C_{s}^{1}$ for some stage $s \geq 1$. Labels for unlabelled glues are “hard-coded” to enforce the assembly of each supertile shown here.

For some $i, j, s \in \mathbb{N}$ such that $1 \leq i, j \leq 8$ and $s \geq 2$, when all grout tiles that can bind to some $C_{s}^{i}$ have, we denote the resulting supertile by $C_{(i,j)}^{s}$. Figure 6a shows $C_{(1,i)}^{2}$ with complete grout (note that in the figure, $i$ has been repurposed to denote a grout class).

For each $j$ between 1 and 8 (inclusive), there are 8 classes of grout tiles. Figures 16, 17, 18, and 19 depict these tiles for the $i^{th}$ class of grout that binds to $C_{s}^{i}$ for some $s \geq 2$. These tiles are analogous to the grout tiles that bind to $C_{s}^{1}$ for $s \geq 2$ with two important exceptions. First, as one can verify from the figures, these grout tiles attach to the easternmost, westernmost, and southernmost tiles of $C_{s}^{2}$. Second, some of these grout tiles have special $\tilde{g}_{2,i}$ and $\hat{g}_{2,i}$ glues with the property that when $i = 1, 2, 3, \text{ or } 7$, $\tilde{g}_{2,i} = g_{n}$ and $\hat{g}_{2,i} = \hat{g}_{n}$. Moreover, when $i = 4, 5, 6, \text{ or } 8$, $\tilde{g}_{2,i} = g_{i}$ and $\hat{g}_{2,i} = \hat{g}_{i}$.
Fig. 16: In this figure, $1 \leq i \leq 8$. Left: The supertile that starts the growth of grout for $C_2^2$. Right: The supertile that starts the growth of grout for $C_2^s$ for $s > 2$. Note that for each $s \geq 2$, only one of these supertiles can bind to tiles of $C_2^s$, and the supertile depicted on the left can bind to some $C_2^s$ iff $s = 2$.

Fig. 17: These tiles and supertiles are analogous to those in Figure 14. Note the presence of the glue $\tilde{g}_{2,i}$. This glue will either be $g_n$ or $g_1$ depending on $i$.

Fig. 18: Similar to the supertiles depicted in Figure 14, these supertiles will self-assemble a row of tiles that bind to the southernmost tiles of $C_1^s$ for some stage $s \geq 1$. 
Fig. 19: These tiles and supertiles are analogous to those in Figure 15 only they bind to the westernmost tiles of some $C_{s}^{2}$ for $s \geq 2$. Note the presence of the glue $\hat{g}_{2,i}$. This glue will either be $\hat{g}_{n}$ or $\hat{g}_{i}$ depending on $i$.

Fig. 20: (a) A depiction of $C_{1}^{3}$ without grout. (b) A depiction of $C_{1}^{3}$ with complete grout. For the sake of clarity, many glues have not been depicted, but the grout tiles here have been color coded and the color for each supertile is shown around the assembly of $C_{1}^{3}$ with grout. Notice the similarities between this figure and Figure 15a In particular, note the presence and spacing of indicator glues.
For each $i$ and $j$ between 1 and 8 (inclusive), once a sufficient number of grout tiles bind to $C_{2}^{i}$ to yield $C_{2}^{i,j}$, the resulting supertiles can then bind to yield a portion of $C_{1}^{3}$, which is depicted in Figure 20a. Note the similarities between the patterns of glues along the northernmost, southernmost, westernmost, and easternmost tiles. This pattern of glues permits another iteration of grout tiles to bind. Figure 20b depicts $C_{1}^{3}$ after all grout supertiles that can attach have.

For $i$ and $k$ between 1 and 8, and $s, s' \geq 2$, note that grout tiles may bind to $C_{s}^{i}$ while $C_{s'}^{k}$ only has partial grout. Therefore, it is possible for grout tiles to “turn a corner too early” as depicted in Figure 21. In this case, notice that each supertile that binds to “turn a corner” exposes a glue that allows grout tiles to continue to self-assemble until the assembly $C_{s}^{i}$ with complete grout has self-assembled. One can verify from the definitions of the grout tiles that this glues does not permit erroneous growth.

![Diagram](image)

Fig. 21: An example where grout tiles have “turned a corner too early”. The grout tiles are shown in aqua, fucia, yellow, and orange here. Note that $C_{3}^{8}$ and $C_{7}^{2}$ only have partial grout. In this case, when a grout supertile shown in orange binds a $g_{2,i}$ is exposed that will eventually allow for grout tiles to continue to bind to $C_{4}^{1}$.

To prove that the tile set, $T$, described in Section 3.1 give a 2HAM TAS $(T, 2)$ that strictly self-assembles $S$ we make the following two observations.

1. Each supertile is a subassembly of such a $C_{s}^{i}$ for some $s \in \mathbb{N}$, and therefore for any producible assembly $\alpha \in A[\mathcal{T}]$, there exists a stage $s > 1$ such that $\text{dom} \ \alpha \subset S^{s}$ (up to translation).
2. For any stage $s \geq 1$, $S^{s} \subset C_{s+1}^{i}$ (up to translation).

Observations 1 and 2 follow by induction on $s$ and the construction details. Therefore, as this hierarchical growth continues indefinitely, the domain of the terminal assembly of the 2HAM TAS $\mathcal{T} = (T, 2)$ is $S$. In other words, $\mathcal{T}$ strictly self-assembles $S$. 
B Tiles for Sierpinski’s Carpet Construction

In this section we define the tile types for the Sierpinski’s carpet construction in full detail.

B.1 start-gadget tile types

(a) The supertiles that start the growth of grout for $C_3^6$ for $s \geq 2$.

(b) The supertiles that start the growth of grout for $C_4^6$ for $s \geq 2$.

(c) The supertiles that start the growth of grout for $C_5^6$ for $s \geq 2$.

Fig. 22: initializer tile types

(a) The supertiles that start the growth of grout for $C_6^6$ for $s \geq 2$.

(b) The supertiles that start the growth of grout for $C_7^6$ for $s \geq 2$.

(c) The supertiles that start the growth of grout for $C_8^6$ for $s \geq 2$.

Fig. 23: More initializer tile types

B.2 crawler and corner-gadget tile types
Fig. 24: The tiles and supertiles that bind to the south side of $C_{s3}$ for $s \geq 2$.

Fig. 25: The tiles and supertiles that bind to the west side of $C_{s3}$ for $s \geq 2$.

Fig. 26: The tiles and supertiles that bind to the north side of $C_{s4}$ for $s \geq 2$.

Fig. 27: The tiles and supertiles that bind to the east side of $C_{s4}$ for $s \geq 2$. 

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Fig. 28: The tiles and supertiles that bind to the south side of \( C_s^4 \) for \( s \geq 2 \).

Fig. 29: The tiles and supertiles that bind to the north side of \( C_s^5 \) for \( s \geq 2 \).

Fig. 30: The tiles and supertiles that bind to the south side of \( C_s^5 \) for \( s \geq 2 \).

Fig. 31: The tiles and supertiles that bind to the west side of \( C_s^5 \) for \( s \geq 2 \).

Fig. 32: The tiles and supertiles that bind to the north side of \( C_s^6 \) for \( s \geq 2 \).
Fig. 33: The tiles and supertiles that bind to the east side of $C_6^s$ for $s \geq 2$.

Fig. 34: The tiles and supertiles that bind to the north side of $C_7^s$ for $s \geq 2$.

Fig. 35: The tiles and supertiles that bind to the east side of $C_7^s$ for $s \geq 2$.

Fig. 36: The tiles and supertiles that bind to the west side of $C_7^s$ for $s \geq 2$. 

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The construction that shows that any 4-sided fractal strictly self-assembles in the 2HAM at scale factor 1 (Theorem 1) is a generalization of the construction given in Section 3.1. Let $G$ be the generator for a 4-sided fractal and recall the notation of $L_G$, $R_G$, $B_G$, and $T_G$. Moreover, let $|G| = r$, let $X$ denote the dssf with generator $G$, and let $X^s$ be the $s$ stage of $X$ for $s \in \mathbb{N}$. We will describe a tile set $T$ such that $X$ strictly self-assembles in the 2HAM system $T = (T, 2)$. As an example, consider the generator in Figure 39a. Stage 2 of this fractal is depicted in Figure 39b. We also choose the convention of ordering the positions in $G$ from top to bottom and left to right. This enumeration is depicted in Figure 39a.

**C Strict self-assembly of 4-sided fractals**

The construction that shows that any 4-sided fractal strictly self-assembles in the 2HAM at scale factor 1 (Theorem 1) is a generalization of the construction given in Section 3.1. Let $G$ be the generator for a 4-sided fractal and recall the notation of $L_G$, $R_G$, $B_G$, and $T_G$. Moreover, let $|G| = r$, let $X$ denote the dssf with generator $G$, and let $X^s$ be the $s$ stage of $X$ for $s \in \mathbb{N}$. We will describe a tile set $T$ such that $X$ strictly self-assembles in the 2HAM system $T = (T, 2)$. As an example, consider the generator in Figure 39a. Stage 2 of this fractal is depicted in Figure 39b. We also choose the convention of ordering the positions in $G$ from top to bottom and left to right. This enumeration is depicted in Figure 39a.

![Fig. 37: The tiles and supertiles that bind to the north side of $C_8^s$ for $s \geq 2$.](image1)

![Fig. 38: The tiles and supertiles that bind to the west side of $C_8^s$ for $s \geq 2$.](image2)

![Fig. 39: Two stages of a 4-side fractal.](image3)
Proof (Sketch proof of Lemma 1). For \( s \in \mathbb{N} \), let \( X^s \) denote the \( s^{th} \) stage of \( X \), and let \( r = |G| \). We note that the construction given in Section 3.1 generalizes in a straightforward way to give a tile \( T \) satisfying Lemma 1. For example, given the generator in Figure 40a, the modifications to the construction given in Section 3.1 are as follows. Once again, we consider two types of tiles in \( T \) which we call start-gadget tiles and grout tiles.

The start-gadget tile types for Lemma 1. Let \( X^2 \) denote the set such that a point \( x \) is \( X^2 \) iff \( x \) is in \( X^2 \) and not in the perimeter of \( X^2 \). Figure 40b depicts the points of an example \( X^2 \). start-gadget tiles of \( T \) now hard-code \( r \) different versions of \( X^2 \). For \( i \) between 1 and \( r \) (inclusive), we call these hard-coded assemblies \( T^2_i \). We note that as there is a simple path in the full grid-graph of \( X^2 \) that contains every vertex, the glues of the start-gadget tiles can be specified so that \( T^2 \) completely self-assembles prior to being a subassembly of any other producible assembly. In addition to hard-coding the shape of \( X^2 \), start-gadget tiles are specified so that once \( T^2_i \) has completely self-assembled, 1) the north edges of northernmost tiles expose a \( g_n \) or \( \hat{g}_n \). In particular the westernmost tile and every other tile from west to east exposes \( g_n \) and the remaining northernmost tiles expose a \( \hat{g}_n \), 2) the east edges of easternmost tiles expose a \( g_e \) or \( \hat{g}_e \). In particular the westernmost tile and every other tile from north to south exposes \( g_e \) and the remaining easternmost tiles expose a \( \hat{g}_e \), 3) the south edges of southernmost tiles expose a \( g_s \) or \( \hat{g}_s \). In particular the easternmost tile and every other tile from east to west exposes \( g_s \) and the remaining southernmost tiles expose a \( \hat{g}_s \), and finally, 4) the west edges of westernmost tiles expose a \( g_w \) or \( \hat{g}_w \). In particular the southernmost tile and every other tile from south to north exposes \( g_w \) and the remaining westernmost tiles expose a \( \hat{g}_w \).

In addition, edges of tiles in \( T^2_i \) in “key locations” expose special glues \( \hat{g}_i \) and \( g_i \) which we call indicator glues. At these key locations, \( g_i \) is exposed instead of a \( g_n \), \( g_s \), \( g_e \), or \( g_w \) and \( \hat{g}_i \) is exposed instead of a \( \hat{g}_n \), \( \hat{g}_s \), \( \hat{g}_e \), or \( \hat{g}_w \). In general these key locations will be the second and third westernmost (resp. northernmost, easternmost, southernmost) tile locations of the northernmost (resp. easternmost, southernmost, westernmost) tile locations. The possible locations of the tiles in \( T^2_i \) that expose these glues are shown as green squares in Figure 40b. Whether or not \( T^2_i \) exposes special glues at these key locations depend on \( i \). In particular, if the \( i^{th} \) location in \( G \) is adjacent to some another point that is north (resp. south, east, or west) of it, then, \( T^2_i \) will expose these special glues on the north (resp. south, east, or west) edges of tiles in northernmost (resp. southernmost, easternmost, or westernmost) key locations. We pick these key locations to be positions \( 2, w_G - 1, |G| - 1 \), and \( |G| - w_G + 1 \) of the stage 1 subassemblies of each \( C^i \) in positions \( 2, w_G - 1, |G| - 1 \), and \( |G| - w_G + 1 \). In Figure 40b key locations are shown in green and red. Note that indicator glues may or may not be present on an edge of a tile located in a key location and moreover, these glues will only ever belong to tiles in red locations in Figure 40b. Where to place indicator glues is determined by \( i \) and is analogous to the carpet example.
The grout tile types for Lemma 1. With the “base case” hard-coded to give $I^2_i$, we are now ready to describe grout tiles. Grout tiles will be almost identical to the grout tiles described in Section 3.1 with the exception that now the grout tiles must hard-code analogous though elongated versions of grout supertiles from Section 3.1. For example, elongated version of start-gadget supertiles that initiate the binding of grout tiles to $I^2_1$ is shown on the left in Figure 41. Grout tiles of $T$ are hard-coded to form similar “elongated” versions of grout supertiles to those described in Section 3.1. That is, grout tile types are defined so that grout supertiles assemble which are analogous to the grout supertiles which self-assemble in the Sierpinski’s carpet example. The only difference being that no these supertiles must span a distance of $w_G$ between easternmost or westernmost tiles of $I^2_i$ and must span a distance of $h_G$ between northernmost or southernmost tiles of $I^2_i$ in order to attach cooperative.

Fig. 41: An “elongated” versions of the supertiles that initiates the attachment of grout tiles to an assembly $I^2_i$ (right) or $I^2_s$ (left), where $s \geq 3$.

Now, grout tiles fall into $r$ different classes where each class corresponds to a position in $G$. For some class $j$ between 1 and $r$ (inclusive), grout tiles of class $j$ bind to $I^2_i$ for each $i$ such that $1 \leq i \leq r$. Then, grout tiles bind to the special glues of edges of tiles of $I^2_i$ in the key locations described above, the resulting supertiles, which we call $\Gamma^2_{(i,j)}$, further expose glues on edges of tiles adjacent to
tiles in key locations such that the presence of these glues enables the supertiles $\Gamma^2_{(i,j)}$ to bind and form an assembly that corresponds to the subsequent stage $X^3$. Moreover, once all $\Gamma^2_{(i,j)}$ supertiles bind another supertile (like the one depicted on the left in Figure 41) can then initiate the binding of grout tiles. Furthermore, with a straightforward modification to the construction in Section 3.1, we can enforce that such a supertile that initiates the binding of grout tiles can bind only after all $\Gamma^2_{(i,j)}$ supertiles are subassemblies of the same assembly. We call this latter assembly $\Gamma^3_j$.

As in the construction of Sierpinski’s carpet, now that we have start-gadget tiles and grout tiles we can see that the start-gadget assemble a supertiles that corresponds to $X^2$ and that grout tiles can attach to supertiles that correspond to $X^s$ for some stage $s \geq 2$ to form supertiles that bind to yield a supertile corresponding to $X^{s+1}$. Therefore, with tiles $T$, the 2HAM system $T = (T, 2)$ strictly self-assembles $X$. Therefore, Lemma 1 holds.

Now we are ready to prove Theorem 1.

C.1 Proof of Theorem 1

Let $X$ be a dssf with generator $G$ and let $r = |G|$. Moreover, let $X^t$ denote the $t^{th}$ stage of $X$. In this section, we give a sketch of the proof of Theorem 1 by describing how to modify the tile set give in the proof of Lemma 1 to show obtain a tile set $T$ such that the 2HAM TAS $T = (T, 2)$ strictly self-assembles $X$. Figure 39a gives an example of a generator $G$ where we enumerate the points of $G$ from left to right and top to bottom. Now let $G_{int} = G \setminus (L_G \cup R_G \cup T_G \cup B_G)$ (i.e. the points of $G$ that are not on the perimeter of $G$), and let $G_{bdry}$ be $G \setminus G_{int}$.

By Lemma 1 there exists a tile set $T'$ such that the 2HAM system $T' = (T', 2)$ strictly self-assembles the dssf with generator $G_{bdry}$. We will show how to modify the tile set $T'$ to obtain $T$. Once again tiles in $T$ belong to a set of start-gadget tiles or a set of grout tiles.

Self-assembly of stage 2 without grout for 4-sided fractals Let $G_1$ denote the full grid-graph of $G$ and let $G_1^-$ denote the graph obtained by removing the northernmost, southernmost, easternmost, and westernmost points from $G$. Note that it is not necessary for $G_1^-$ to be connected. For example, consider $G_1^-$ for the generator shown in Figure 39a. Also note that $G_1^-$ may be empty if $G = L_G \cup R_G \cup T_G \cup B_G$ as in the case for the Sierpinski’s carpet dssf. An example of $G_1^-$ for the generator shown in Figure 39a is shown in Figure 42a where vertices correspond to squares and there is assumed to be an edge between two vertices iff these squares abut. Now let $G$ denote the full grid-graph of $X^2$. Let $G^-$ be the (not necessarily connected) graph obtained by removing the northernmost, southernmost, easternmost, and westernmost points from $G$. For the generator given in Figure 39a, $G^-$ is shown in Figure 42b. Finally, let $G_c$ be the connected component of $G^-$ that is not equal to a connected component
Fig. 42: (a) A depiction of $G^{-1}$ for the generator in Figure 39a. (b) A depiction of $G^{-1}$ for the generator in Figure 39a.

Fig. 43: A depiction of $\Gamma^2_i$. This is the portion of the second stage of the fractal with generator in Figure 39a that is hard-coded to self-assemble. It is analogous to the second stages that assemble shown in Figure 40b for the construction for Lemma 1. of $G_1^-$ up to translation. See Figure 43 for an example of $G_c$ for the generator shown in Figure 39a.

Then, the **start-gadget** tiles of $T$ are hard-coded to self-assemble $r$ different versions of $G_c$ which we call $\Gamma^2_i$ for $1 \leq i \leq r$. Similar to the **start-gadget** tiles described in the proof of Lemma 1, each $\Gamma^2_i$ contains tiles in key locations (defined as in Lemma 1) that expose **indicator** glues that depend on the value of $i$. These **start-gadget** tiles can be thought of as being equivalent to the **start-gadget** tiles of $T'$ with additional tiles that hard-code the assembly of stage 1 subassemblies whose positions in the $\Gamma^2_i$ correspond to the point of $G_{int}$. That is, the **start-gadget** tiles of $T$ are the **start-gadget** tiles of $T'$ and tiles that have been hard-coded so that tiles will eventually be in locations that correspond to $X^1$ at “interior” positions of $G$ in $X^2$. Figure 43 depicts the locations of tiles of $\Gamma^2_i$ for the generator in Figure 39a.

**Tile types for grout tiles.** $T$ also contains **grout** tiles that serve the same purpose as the **grout** tiles described in Section 3.1. That is, there are $r$ many classes of **grout** tiles. For any $i$ and a fixed $j$ with $1 \leq i, j \leq r$, the $j$th class of **grout** tiles contains tiles that bind to $\Gamma^2_i$ and partially surround it. These **grout** tiles expose glues so that once $\Gamma^2_i$ has complete grout (or sufficient but partial grout) for each $i$ between 1 and $r$ (inclusive), the resulting assemblies, which we call $\Gamma^3_{(i,j)}$ are supertiles that can then self-assemble to yield $\Gamma^3_j$. Moreover, just as $\Gamma^2_i$ corresponds to $X^2$, $\Gamma^3_{j}$ is an assembly that corresponds to $X^3$. That is, after some translation, the domain of $\Gamma^3_{j}$ is contained in $X^3$, and $\Gamma^3_{j}$ exposes...
glues such that as assembly proceeds, $\Gamma^3_j$ will be a subassembly of an assembly that contains all of the points of $X^3$. In addition, $\Gamma^2_i$ is a subassembly of $\Gamma^3_j$ that corresponds to a stage $X^2$ at position $i$ in $X^3$ to which $\Gamma^3_j$ corresponds.

The grout tiles of $T$ consists of tiles that are equivalent to the grout tiles of $T'$ and additional tiles that hard-code the appropriate stage 1 growth that complete any subassemblies that represent $X^1$. Figure 45 gives an example of $\Gamma^2_{12}$ with complete grout. In this particular example, grout tiles have been hard-coded to place tiles in locations corresponding to $X^1$ as the grout tiles bind to the northernmost tiles of $\Gamma^2_{12}$.

![Figure 44: A depiction of $\Gamma^2_{(12,j)}$ for some $j \in \mathbb{N}$ corresponding to the $j$ class of grout. Note the glues that are exposed on tiles adjacent to tiles with indicator glues (red tiles).](image)

It should be noted that grout tiles are added for each $i$ between 1 and $r$ (inclusive) and that as in Figure 45 grout tiles may bind to some $\Gamma^2_i$ where $i$ corresponds to a point in $G_{ini}$. In this case, grout tiles have been hard-coded to completely surround $\Gamma^2_i$ and expose appropriate glues at key locations to ensure that for all $i$ and $j$ both between 1 and 8 (inclusive), once a sufficient number of grout tiles bind to each $\Gamma^2_i$, the resulting supertiles, which we again call $\Gamma^2_{(i,j)}$, can bind to yield an assembly corresponding to $X^3$. We call this latter assembly $\Gamma^3_j$.

As in the construction of Sierpinski’s carpet, now that we have start-gadget tiles and grout tiles we can see that the start-gadget assemble a supertiles that corresponds to $X^2$ and that grout tiles can attach to supertiles that correspond to $X^s$ for some stage $s \geq 2$ to form supertiles that bind to yield a supertile corresponding to $X^{s+1}$. Therefore, with tiles $T$, the 2HAM system $\mathcal{T} = (T, 2)$ strictly self-assembles $X$. Therefore, Theorem 1 holds.
In this section we prove that there exist 3-sided fractals that do not strictly self-assemble in the 2HAM.

To prove Theorem 2, we consider the fractal with generator \( G = \{ (0,0), (0,1), (0,2), (0,3), (0,4), (1,0), (1,4), (2,0), (2,2), (2,3), (2,4), (3,0), (3,4) \} \). Stages 1 and 2 of this fractal are shown in Figure 46. We refer to this fractal as \( X \) and, similar to the convention in 3.1, we refer to the \( s \)th stage of \( X \) as \( X_s \). We refer to the \( i \)th position of \( X_s \) as \( X_{s,i} \) where \( 1 \leq i \leq 13 \) (Figure 46a). We call the assembly \( \gamma_{s,i} \) for which \( \text{dom} \gamma_{s,i} = X_{s,i} \).

Consider any 2HAM TAS \( T_X = (T, \tau) \). We show that \( T_X \) does not strictly self-assemble \( X \) by showing that there is a terminal assembly \( \alpha \in A_{\square} [T_X] \) for which \( \text{dom} \alpha \neq X \) even after translation.

For every assembly \( \alpha \) such that \( X^s \subseteq \text{dom} \alpha \), there is a stage 1 subassembly \( \gamma^1 \subseteq \gamma_s \), and this stage 1 subassembly contains a strength \( \tau \) cut between \( \gamma^9_1 \) and \( \gamma^9_2 \) that separates \( \gamma^9_1, \gamma^9_2, \gamma^9_3, \) and \( \gamma^9_4 \), along with a sequence of subassemblies \( \gamma^9_{10}, \gamma^9_{11}, \gamma^9_{12}, \) and \( \gamma^9_{13}, i < s, \) from the rest of \( \gamma^s \). An example of this is shown in Figure 47b for \( s = 2 \) and in Figure 48 for \( s = 3 \). If \( s - 1 > 1 \), then \( \gamma^s_8 \) is a subassembly \( \gamma^{s-1} \) which contains a similar strength \( \tau \) cut between two tiles \( \gamma^9_1 \) and \( \gamma^9_{10} \) in the \( \gamma^1 \) subassembly directly above \( \gamma^{s-1}_{10} \). By induction this may be repeated until \( s - 1 = 1 \). Then \( \gamma^s_8 \) is a subassembly \( \gamma^1 \) which contains a single strength \( \tau \) cut between \( \gamma^9_1 \) and \( \gamma^9_{10} \) (Figure 47a). We also note that when \( s = 1 \) there is one strength \( \tau \) cut between \( \gamma^9_2 \) and \( \gamma^9_{10} \). Therefore every assembly \( \alpha \) such that \( \text{dom} \alpha \supseteq X^s \) contains a sequence of \( s \) strength \( \tau \) cuts between positions 9 and 10 of \( s \) distinct stage 1 subassemblies. An example of this for \( s = 3 \) is shown in Figure 48.

Fig. 45: A schematic picture of \( I_3^s \). Note the red tile locations where tiles with indicator glues (red tiles) will be present.
Let $g$ be the number of tiles in $T_X$. Consider a producible assembly $\alpha$ such that $X^{g+2} \subseteq \text{dom } \alpha$. Within $\alpha$ there is a subassembly $\gamma^{g+2}$, and $\gamma^{g+2}$ is a subassembly $\gamma^{g+1}$. As we have shown this subassembly contains a sequence of $g+1$ single tile strength $\tau$ cuts. Let the subassembly to the south of the cut within $\gamma^1$ be called $\beta_1$, the subassembly to the south of the cut within $\gamma^2$ be called $\beta_2$, etc., with the subassembly to the south of the cut within $\gamma^{g+1}$ called $\beta_{g+1}$ (see Figure 48). As shown above, there are at least two cuts that use the same glue. Consider two such cuts directly above $\beta_s$ and $\beta_{s'}$ with $s' > s$. Let $\alpha_2 = \alpha \setminus (\beta_s \cup \beta_{s+1} \cup \cdots \cup \beta_{s'})$. We will show that $\alpha_2, \beta_{s'} \in A[T_X]$. Additionally, we notice that between $X^{g+2}_8$ and $X^{g+2}_{12}$ there is enough room to fit an entire stage $X^{g+1}$, and since $s' \leq g+1$, erroneous binding of $\alpha_2$ and $\beta_{s'}$ cannot be prevented geometrically. Hence $\alpha_2 \cup \beta_{s'} \in A[T_X]$, and since $\text{dom } \alpha_2 \cup \beta_{s'} \not\subseteq X$, $T_X$ does not strictly self-assemble $X$.

We now show that the subassemblies $\alpha_2$ and $\beta_{s'}$ are producible. If one of $\alpha_2$ or $\beta_{s'}$ is not producible, then that one must contain a cut of strength less than $\tau$. However, since every $\beta_i$, $1 \leq i \leq g+1$, is connected to $\alpha$ by a strength $\tau$ cut between two single tiles, if $\alpha_2$ or $\beta_{s'}$ contained a cut less than strength $\tau$, then $\alpha$ would contain the same cut less than strength $\tau$, contradicting the assumption that $\alpha$ is producible. Hence $\alpha_2, \beta_{s'} \in A[T_X]$.

(a) Stage 1

(b) Stage 2

Fig. 46: $X^1$ and $X^2$
Fig. 47: Strength $\tau$ cuts in $\gamma^1$ and $\gamma^2$
Fig. 48: There are $s$ strength $\tau$ cuts within each assembly $\gamma^s$. Here $\gamma^3$ with 3 strength $\tau$ cuts is shown. The subassembly to the south of the rightmost cut is referred to as $\beta_1$, the subassembly to the south of the next rightmost cut as $\beta_2$, and the subassembly to the south of the leftmost cut as $\beta_3$. 
Fig. 49: The pieces of the fractal south of the strength $\tau$ cut cannot be guaranteed to bind before higher stages of the fractal begin to assemble. An example of this is shown for $\gamma^3$. Because of the large number of tiles most of the $\gamma^3$ subassemblies are shown as rectangles and the partially completed $\gamma^4$ subassembly is shown zoomed in on the right hand side.
Fig. 50: An example of erroneous binding within $\gamma^5$. Because of the large number of tiles some of the $\gamma^3$ subassemblies are shown as rectangles.