On Lazard’s Valuation
and CAD Construction

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Abstract

In 1990 Lazard proposed an improved projection operation for cylindrical algebraic decomposition (CAD). For the proof he introduced a certain notion of valuation of a multivariate Puiseux series at a point. However a gap in one of the key supporting results for the improved projection was subsequently noticed. In this report we study a more limited but rigorous concept of Lazard’s valuation: namely, we study Lazard’s valuation of a multivariate polynomial at a point. We prove some basic properties of the limited Lazard valuation and identify some relationships between valuation-invariance and order-invariance.

1 Introduction

In 1990 Lazard [18] proposed an improved projection operation for cylindrical algebraic decomposition (CAD) which is based on a certain notion of valuation of a multivariate fractional meromorphic series at a point. Inherent in [18] is the related notion of the valuation-invariance of an n-variate fractional meromorphic series in a subset of Euclidean n-space \( \mathbb{R}^n \). Lazard’s proposed approach is in contrast with that of McCallum [21, 22] which is based on the concept of the order (of vanishing) of a multivariate polynomial or analytic function at a point, and the related concept of order-invariance. However a gap in one of the key supporting results of [18] was subsequently noticed [10, 7]. This is disappointing because Lazard’s proposed approach has some advantages over other methods.

In [23] we study Lazard’s projection. It is shown there that Lazard’s projection is valid for CAD construction for so-called well-oriented polynomial sets.
The key underlying results relate to order-invariance rather than valuation-invariance; and the validity proof builds upon existing results concerning improved projection. While this is an important step forward we regard it as only a partial validation of Lazard’s approach since the method is not proved to work for non well-oriented polynomials and it does not involve valuation-invariance. However it does confirm the soundness of the intuition behind Lazard’s proposed projection.

In this report we study separately a more limited but rigorous concept of Lazard’s valuation: namely, we study Lazard’s valuation of a multivariate polynomial at a point. This study was motivated by the hope of remedying more completely the defect of [18]. Section 2 clarifies the notion and basic properties of the more limited concept of Lazard’s valuation, and identifies some relationships between valuation-invariance and order-invariance. Section 3 recalls Lazard’s main claim concerning valuation-invariance and CAD. His main claim is proved for the special case \( n = 3 \) under a slightly stronger hypothesis. Section 4 discusses some of the difficulties involved with attempting rigorously to extend the limited concept of valuation to multivariate fractional meromorphic series.

2 Definition and basic properties of Lazard’s valuation (limited)

In this section we study Lazard’s valuation [18] in a relatively special setting, namely, that of multivariate polynomials over a field. We shall clarify the notion and basic properties of this special valuation, and identify some relationships between valuation-invariance and order-invariance.

We first define the Lazard valuation in a limited way. This will allow us to provide simple, straightforward proofs of some basic properties which are sufficient for some limited uses of this concept.

We recall at the outset the standard algebraic definition of the term valuation [3, 29]. A mapping \( v : K - \{0\} \rightarrow \Gamma \) from the multiplicative group of a field \( K \) into a totally ordered abelian group (written additively) \( \Gamma \) is said to be a valuation of \( K \) if the following two conditions are satisfied:

1. \( v(fg) = v(f) + v(g) \) for all \( f \) and \( g \);
2. \( v(f + g) \geq \min\{v(f), v(g)\} \), for all \( f \) and \( g \) (with \( f + g \neq 0 \)).

By the same axioms one could define the notion of valuation of a ring [12]. (In such a case \( \Gamma \) could be a totally ordered abelian monoid.) Perhaps the simplest and most familiar example of a valuation in algebraic geometry is the order of an \( n \)-variate polynomial over a field \( k \) at a point \( a \in k^n \). That is, the mapping \( \text{ord}_a : k[x_1, \ldots, x_n] - \{0\} \rightarrow \mathbb{N} \) defined by

\[
\text{ord}_a(f) = \text{the order of } f \text{ at } a
\]
is a valuation of the ring \( k[x_1, \ldots, x_n] \). We could extend the definition of \( \text{ord}_a \)

to the multiplicative group of the rational function field \( K = k(x_1, \ldots, x_n) \),
defining \( \text{ord}_a : K - \{0\} \rightarrow \mathbb{Z} \) by

\[
\text{ord}_a(f/g) = \text{ord}_a(f) - \text{ord}_a(g).
\]

Let \( n \geq 1 \). Recall that the lexicographic order \( \leq_l \) on \( \mathbb{N}^n \) is defined by \( v = (v_1, \ldots, v_n) \leq_l (w_1, \ldots, w_n) = w \) if and only if either \( v = w \) or there is some \( i, 1 \leq i \leq n \), with \( v_j = w_j \), for all \( j \) in the range \( 1 \leq j < i \), and \( v_i < w_i \). Then \( \leq_l \) is an admissible order on \( \mathbb{N}^n \) in the sense of [6]. Indeed \( \mathbb{N}^n \), together with componentwise addition and \( \leq_l \), forms a totally ordered abelian monoid. The lexicographic order \( \leq_l \) can be defined similarly on \( \mathbb{Z}^n \), forming a totally ordered abelian group.

**Definition 1.** Let \( k \) be a field. Let \( f(x_1, \ldots, x_n) \) be a nonzero element of 
the polynomial ring \( k[x_1, \ldots, x_n] \) and let \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \). The Lazard valuation \( v_a(f) \) of \( f \) at \( a \) is the element \( (v_1, \ldots, v_n) \) of \( \mathbb{N}^n \) least (with respect to \( \leq_l \)) such that the partial derivative \( \partial^{v_1} \cdots \partial^{v_n} f/\partial x_1^{a_1} \cdots \partial x_n^{a_n} \) does not vanish at \( a \).

We remark that \( v_a(f) \) could be defined equivalently to be the element \( (v_1, \ldots, v_n) \) of \( \mathbb{N}^n \) least (with respect to \( \leq_l \)) such that \( f \) expanded about \( a \) has a term \( c(x_1 - a_1)^{v_1} \cdots (x_n - a_n)^{v_n} \) with \( c \neq 0 \). (This is compatible with Lazard’s definition in [18]; see also [23].) Notice that \( v_a(f) = (0, \ldots, 0) \) if and only if \( f(a) \neq 0 \).

We could extend the definition of \( v_a \) to the multiplicative group of the rational function field \( K = k(x_1, \ldots, x_n) \), defining \( v_a : K - \{0\} \rightarrow \mathbb{Z} \) by

\[
v_a(f/g) = v_a(f) - v_a(g).
\]

However we shall not need such an extension of the concept for the time being.

**Example 1.** Let \( n = 1 \). Then \( v_a(f) \) is the familiar order \( \text{ord}_a(f) \) of \( f \) at \( a \). Thus, for instance, if \( f(x_1) = x_1^2 - x_1^3 \) then \( v_0(f) = 2 \) and \( v_1(f) = 1 \).

Let \( n = 2 \) and \( f(x_1, x_2) = x_1 x_2 \). Then \( v_{(0,0)}(f) = (1,1) \); \( v_{(1,0)}(f) = (0,1) \); and \( v_{(0,1)}(f) = (1,0) \).

Where there is no ambiguity we shall usually omit the qualifier “Lazard” from “Lazard valuation”. We state some basic properties of the valuation \( v_a(f) \), analogues of properties of the familiar order \( \text{ord}_a(f) \). The first property is the satisfaction of the axioms.

**Proposition 1.** Let \( f \) and \( g \) be nonzero elements of \( k[x_1, \ldots, x_n] \) and let \( a \in \mathbb{N}^n \).

Theorem \( v_a(f/g) = v_a(f) + v_a(g) \) and \( v_a(f + g) \geq \min\{v_a(f), v_a(g)\} \) (if \( f + g \neq 0 \)).

**Proof.** These claims follow since \( \mathbb{N}^n \), together with componentwise addition and \( \leq_l \), forms a totally ordered abelian monoid. \( \square \)

For the remaining properties we state we shall assume that \( k = \mathbb{R} \) or \( \mathbb{C} \).
**Proposition 2.** (Upper semicontinuity of valuation) Let $f$ be a nonzero element of $k[x_1, \ldots, x_n]$ and let $a \in k^n$. Then there exists a neighbourhood $V \subset k^n$ of $a$ such that for all $b \in V$ $v_b(f) \leq v_a(f)$.

**Proof.** Let $v_a(f) = (v_1, \ldots, v_n)$. By definition of $v_a(f)$, $\partial^{v_1+\cdots+v_n} f/\partial x_1^{v_1} \cdots \partial x_n^{v_n}(a) \neq 0$. By continuity of the function $\partial^{v_1+\cdots+v_n} f/\partial x_1^{v_1} \cdots \partial x_n^{v_n}$ at $a$, there exists a neighbourhood $V \subset k^n$ of $a$ such that for all $b \in V$, $\partial^{v_1+\cdots+v_n} f/\partial x_1^{v_1} \cdots \partial x_n^{v_n}(b) \neq 0$. It follows at once by definition of the valuation that for all $b \in V$ $v_b(f) \leq v_a(f)$.

Let $f \in k[x_1, \ldots, x_n]$. We shall say that $f$ is valuation-invariant in a subset $S$ of $k^n$ if $v_a(f)$ is constant as $a$ varies in $S$.

**Proposition 3.** Let $f$ and $g$ be nonzero elements of $k[x_1, \ldots, x_n]$ and let $S \subset k^n$ be connected. Then $fg$ is valuation-invariant in $S$ if and only if both $f$ and $g$ are valuation-invariant in $S$.

**Proof.** This proposition is analogous to Lemma A.3 of [21]. Its proof, using Propositions 2.3 and 2.4, is virtually identical to that of the cited Lemma A.3.

As noted previously, the above properties are analogues of those of the familiar order $\text{ord}_a(f)$. The next lemma is in a sense also an analogue of the familiar order, and is particular to the case $n = 2$.

**Lemma 1.** Let $f(x, y) \in k[x, y]$ be primitive of positive degree in $y$ and square-free. Then for all but a finite number of points $(\alpha, \beta) \in k^2$ on the curve $f(x, y) = 0$ we have $v_{(\alpha, \beta)}(f) = (0, 1)$.

**Proof.** Denote by $R(x)$ the resultant $\text{res}_y(f, f_y)$ of $f$ and $f_y$ with respect to $y$. Then $R(x) \neq 0$ since $f$ is assumed squarefree. Let $(\alpha, \beta) \in k^2$, suppose $f(\alpha, \beta) = 0$ and assume that $v_{(\alpha, \beta)}(f) \neq (0, 1)$. Then $f_y(\alpha, \beta) = 0$. Hence $R(\alpha) = 0$. So $\alpha$ belongs to the set of roots of $R(x)$, a finite set. Now $f(\alpha, \beta) = 0$ and $f(\alpha, y) \neq 0$, since $f$ is assumed primitive. So $\beta$ belongs to the set of roots of $f(\alpha, y)$, a finite set.

Let us consider the relationship between the concepts of order-invariance and valuation-invariance for a subset $S$ of $k^n$. The concepts are the same in case $n = 1$ because order and valuation are the same for this case. For $n = 2$, order-invariance in $S$ does not imply valuation-invariance in $S$. (For consider the unit circle $S$ about the origin in $k^2$. The order of $f(x, y) = x^2 + y^2 - 1$ at every point of $S$ is 1. The valuation of $f$ at every point $(\alpha, \beta) \in S$ except $(\pm 1, 0)$ is $(0, 1)$. But $v_{(\pm 1, 0)}(f) = (0, 2)$. However for $n = 2$ we can prove the following.

**Proposition 4.** Let $f \in k[x, y]$ be nonzero and $S \subset k^2$ be connected. If $f$ is valuation-invariant in $S$ then $f$ is order-invariant in $S$. 

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Proof. Assume that \( f \) is valuation-invariant in \( S \). Write \( f \) as a product of irreducible elements \( f_i \) of \( k[x,y] \). By Proposition 2.5 each \( f_i \) is valuation-invariant in \( S \). We shall show that each \( f_i \) is order-invariant in \( S \). Take an arbitrary factor \( f_i \). If the valuation of \( f_i \) in \( S \) is \((0,0)\) then the order of \( f_i \) throughout \( S \) is 0, hence \( f_i \) is order-invariant in \( S \). So we may assume that the valuation of \( f_i \) is nonzero in \( S \), that is, that \( S \) is contained in the curve \( f_i(x,y) = 0 \). Suppose first that \( f_i \) has positive degree in \( y \). Now the conclusion is immediate in case \( S \) is a singleton, so assume that \( S \) is not a singleton. Since \( S \) is connected, \( S \) is an infinite set. By Lemma 2.6 and valuation-invariance of \( f_i \) in \( S \), we must have \( v_{(\alpha,\beta)}(f_i) = (0,1) \) for all \((\alpha, \beta) \in S\). Hence \( f_i \) is order-invariant in \( S \) (since \( \text{ord}_{f_i} = 1 \) in \( S \)). Suppose instead that \( f_i = f_i(x) \) has degree 0 in \( y \). Since \( f_i(x) \) is irreducible it has no multiple roots. Therefore \( v_{(\alpha,\beta)}(f_i) = (1,0) \) for all \((\alpha, \beta) \in S\). Hence \( f_i \) is order-invariant in \( S \) (since \( \text{ord}_{f_i} = 1 \) in \( S \)). The proof that \( f_i \) is order-invariant in \( S \) is finished. We conclude that \( f \) is order-invariant in \( S \) by Lemma A.3 of [21].

In Section 3 we shall provide an example indicating that Proposition 2.7 is not true for dimension greater than 2.

3 Valuation-invariance and CAD

Let \( A \) be a set of elements of \( \mathbb{Z}[x_1, \ldots, x_n] \). Recall that an \( A \)-invariant CAD of \( \mathbb{R}^n \) \cite{2, 4} is a partitioning of \( \mathbb{R}^n \) into connected subsets called cells compatible with the zeros of the elements of \( A \). The output of a CAD algorithm applied to \( A \) is a description of an \( A \)-invariant CAD \( D \) of \( \mathbb{R}^n \). That is, \( D \) is a decomposition of \( \mathbb{R}^n \) determined by the roots of the elements of \( A \) over the cells of some cylindrical algebraic decomposition \( D' \) of \( \mathbb{R}^{n-1} \); each element of \( A \) is sign-invariant throughout every cell of \( D \).

In this section we first prove a result which implies, roughly speaking, that many of the cells produced by a CAD algorithm applied to \( A \) are valuation-invariant with respect to each element of \( A \). Next we recall some more terminology, and the main claim, of [18]. We prove the main claim of [18] for \( n \leq 3 \) under a slightly stronger hypothesis.

Recall the fundamental concept of delineability reviewed in [23]. Delineability ensures the cylindrical arrangement of the cells in a CAD. Perhaps the most crucial part of the theory of CADs are theorems providing sufficient conditions for delineability. Now we can state a result linking delineability and valuation-invariance.

**Proposition 5.** Let \( f \in \mathbb{R}[x, x_n] \) and let \( S \) be a connected subset of \( \mathbb{R}^{n-1} \). Suppose that \( f \) is delineable on \( S \). Then \( f \) is valuation-invariant in each section (and trivially each sector) of \( f \) over \( S \).

*Proof.* Let \( \theta : S \rightarrow \mathbb{R} \) be a real root function of \( f \) on \( S \) such that \( \theta(a) \) has invariant multiplicity \( m \) as a root of \( f(a, x_n) \), as \( a \) varies in \( S \). (\( m \) exists by the second condition of delineability.) Let \( a \) be an arbitrary point of \( S \). Then
$v_{(a, \theta(a))} f = (0, \ldots, 0, m)$, since $\partial^m f / \partial x_n^m (a, \theta(a)) \neq 0$ while $\partial f / \partial x_i^k (a, \theta(a)) = 0$ for all $i$ in the range $0 \leq i < m$. We've shown that $f$ is valuation-invariant in the section of $f$ over $S$ which is the graph of $\theta$. Observe that wherever $f(a, a_n) \neq 0$ we have $v_{(a, a_n)} f = (0, \ldots, 0)$. Hence $f$ is valuation-invariant in each sector of $f$ over $S$. □

An element $f \in \mathbb{Z}[x, x_n]$ is nullified by a subset $S$ of $\mathbb{R}^{n-1}$ if $f(a, x_n) = 0$ for all points $a \in S$. In case a cell $S$ is nullified by $f$ the original CAD algorithm does not decompose the cylinder $S \times \mathbb{R}$ relative to $f$, since in such a case the whole cylinder $S \times \mathbb{R}$ is sign-invariant with respect to $f$. In [18] Lazard proposed an evaluation process for such an $f$ relative to a sample point $\alpha$ of such an $S$ which he claimed would ensure that the cylinder $S \times \mathbb{R}$ can be decomposed into cells which are valuation-invariant with respect to $f$. This technique is described in slightly more general terms as follows:

**Definition 2** (Lazard evaluation). Let $K$ be a field. (In this section $K = \mathbb{R}$ or, when explicit computation is required, $K$ is a suitable subfield of $\mathbb{R}$.) Let $n \geq 2$, $f \in K[x_1, \ldots, x_n]$ nonzero, and $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in K^{n-1}$. The Lazard evaluation $f_\alpha(x_n)$ of $f$ at $\alpha$ is defined to be the result of the following process (which determines also nonnegative integers $v_i$, with $1 \leq i \leq n - 1$):

$$f_\alpha \leftarrow f$$

For $i \leftarrow 1$ to $n - 1$ do

$$v_i \leftarrow \text{the greatest integer } v \text{ such that } (x_i - \alpha_i)^v | f_\alpha$$

$$f_\alpha \leftarrow f_\alpha / (x_i - \alpha_i)^{v_i}$$

$$f_\alpha \leftarrow f_\alpha (\alpha_i, x_{i+1}, \ldots, x_n)$$

**Remark 1.** With $K$, $n$, $f$, $\alpha$ and the $v_i$ as in the above definition of Lazard evaluation, notice that $f(\alpha, x_n) = 0$ (identically) if and only if $v_i > 0$, for some $i$ in the range $1 \leq i \leq n - 1$. With $\alpha_n \in K$ arbitrary, notice also that the integers $v_i$, with $1 \leq v_i \leq n - 1$, are the first $n - 1$ coordinates of $v_{(\alpha, \alpha_n)} (f)$. It will be handy on occasion to refer to the $(n - 1)$-tuple $(v_1, \ldots, v_{n-1})$ as the Lazard valuation of $f$ on $\alpha$.

One more definition is needed before we can state Lazard’s main claim:

**Definition 3** (Lazard delineability). Let $f \in R_n$ be nonzero and $S$ a subset of $\mathbb{R}^{n-1}$. We say that $f$ is Lazard delineable on $S$ if

1. the Lazard valuation of $f$ on $\alpha$ is the same for each point $\alpha \in S$;

2. there exist finitely many continuous functions $\theta_1 < \cdots < \theta_k$ from $S$ to $\mathbb{R}$, with $k \geq 0$, such that, for all $\alpha \in S$, the set of real roots of $f_\alpha(x_n)$ is \{\theta_1(\alpha), \ldots, \theta_k(\alpha)\}; and

3. there exist positive integers $m_1, \ldots, m_k$ such that, for all $\alpha \in S$ and all $i$, $m_i$ is the multiplicity of $\theta_i(\alpha)$ as a root of $f_\alpha(x_n)$.
We refer to the graphs of the $\theta_i$ as the Lazard sections of $f$ over $S$; the regions between successive Lazard sections, together with the region below the lowest Lazard section and that above the highest Lazard section, are called Lazard sectors.

Remark 2 (Relation between Lazard and ordinary delineability). Let $f$ and $S$ be as in the above definition of Lazard delineability. Suppose that $f(\alpha, x_n) \neq 0$ for all $\alpha \in S$. Then $f$ is Lazard delineable on $S$ if and only if $f$ is delineable on $S$ in the usual sense.

For a finite irreducible basis in $\mathbb{Z}[x_1, \ldots, x_n]$, where $n \geq 2$, recall that the Lazard projection $P_L(A)$ of $A$ is the union of the set of all leading coefficients of elements of $A$, the set of all trailing coefficients of elements of $A$, the set of all discriminants of elements of $A$, and the set of all resultants of pairs of distinct elements of $A$ [23]. Lazard’s main claim, essentially the content of his Proposition 5 and subsequent remarks, could be expressed as follows:

Let $A$ be a finite irreducible basis in $\mathbb{Z}[x_1, \ldots, x_n]$, where $n \geq 2$. Let $S$ be a connected subset of $\mathbb{R}^{n-1}$. Suppose that each element of $P_L(A)$ is valuation-invariant in $S$. Then each element of $A$ is Lazard delineable on $S$, the Lazard sections over $S$ of the elements of $A$ are pairwise disjoint, and each element of $A$ is valuation-invariant in every Lazard section and sector over $S$ so determined.

This claim concerns valuation-invariant lifting in relation to $P_L(A)$; it asserts that the condition, ‘each element of $P_L(A)$ is valuation-invariant in $S$’, is sufficient for an $A$-valuation-invariant stack in $\mathbb{R}^n$ to exist over $S$.

Theorem 1. Suppose $n \leq 3$ and $S$ is a submanifold of $\mathbb{R}^{n-1}$. Then Lazard’s main claim holds.

Proof. Suppose first that $n = 2$. By remarks in Example 2.2, the hypothesis implies that each element of $P_L(A)$ is order-invariant in $S$. Since $n = 2$ and each element of $A$ is irreducible (hence in particular primitive), no element of $A$ vanishes identically at a point of $S$. Hence by Theorem 3.1 of [23], each element of $A$ is delineable on $S$ etc. By a remark above, Lazard delineability is equivalent to ordinary delineability for $f$ on $S$ in this case. The valuation-invariance of each element of $A$ in every section and sector over $S$ determined by $A$ follows by Proposition 3.1.

Suppose second that $n = 3$. The conclusions are essentially trivial in case the dimension of $S$ is 0. So assume henceforth that the dimension of $S$ is positive. By the hypothesis and Proposition 2.7, each element of $P_L(A)$ is order-invariant in $S$. Hence by Theorem 3.1 of [23], each element of $A$ either vanishes identically on $S$ or is delineable on $S$. Since the dimension of $S$ is positive and each element of $A$ is irreducible (hence in particular primitive), no element of $A$ vanishes identically on $S$ (Lemma A.2 of [21]). Hence each element of $A$ is delineable on $S$ and (again by Theorem 3.1 of op. cit.) the sections over $S$ of the elements of

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A are pairwise disjoint. By a remark above, Lazard delineability is equivalent to ordinary delineability for f on S in this case. The valuation-invariance of each element of A in every section and sector over S determined by A follows by Proposition 3.1.

Now we present an example showing that valuation-invariance does not imply order-invariance when n > 2. Let \( f(x, y, z) = xz - y^2 \) and let S be the z-axis in \( \mathbb{R}^3 \). Now f is valuation-invariant in S, since the valuation of f at each point of S is \( (0, 2, 0) \). But f is not order-invariant in S, since ord\((0,0,0)\)f = 2 and ord\((0,0,\alpha)\)f = 1 for \( \alpha \neq 0 \). To our minds this example casts some doubt on the truth of Lazard’s main claim for \( n > 3 \). However we do not have a direct counter-example for Lazard’s main claim.

4 Extension of Lazard’s valuation: some difficulties

Let \( k \) be \( \mathbb{R} \) or \( \mathbb{C} \). In this subsection we discuss some of the problems with trying to extend the definitions and basic theory of valuations of \( k[x_1, \ldots, x_n] \), as outlined in Section 2, to larger rings and fields. We first observe that both the order and Lazard valuation could just as easily be defined in the same manner for nonzero elements of the ring of analytic functions defined in some open set \( U \subset k^n \). The basic theory (Propositions 2.3 to 2.5) carries over with very little change.

Now let \( a = (a_1, \ldots, a_n) \in k^n \). We consider the formal power series ring

\[
R = k[[x - a]] = k[[x_1 - a_1, \ldots, x_n - a_n]].
\]

We could define the order ord\(_a\)(f) of \( f \in R \) at \( a \) in the usual way. However – unless \( f \) is assumed to be convergent in some neighbourhood of \( a \) (i.e. analytic near \( a \), discussed above) – it is not in general possible to expand \( f \) about \( b \neq a \) but near \( a \). In particular analogues of Propositions 2.4 and 2.5 have no meaning in general in this case. Similar remarks apply to the Lazard valuation in this case. This elementary consideration points to the need to carefully incorporate some notion of convergence if one wishes to rectify the flawed theory relating to Lazard’s valuation for multivariate Puiseux series identified in Section 2 of [23]. Yet the convergence issue for multivariate Puiseux series is thorny – a given such series in \( x - a \) may have a slender region of convergence (neither a full nor punctured neighbourhood of \( a \)).

Next we consider extension of the valuations ord\(_a\) and \( v_a \) to the field of rational functions \( K = k(x_1, \ldots, x_n) \). As mentioned in Section 2, ord\(_a\) could be defined by the equation ord\(_a\)(f/g) = ord\(_a\)(f) − ord\(_a\)(g), for \( f \) and \( g \) nonzero elements of \( K \). Proposition 2.3 remains valid for this extension but Proposition 2.4 does not (as we could take \( n = 1, a_1 = 0, f = 1 \) and \( g = x_1 \)). However we could formulate the following analogue of Proposition 2.4:
Proposition 6. Let $f/g$ be a nonzero element of $K$, let $U \subset k^n$ be an open set throughout which $g \neq 0$, and let $a \in U$. Then there exists a neighbourhood $V \subset U$ of $a$ such that for all $b \in V \text{ord}_b(f/g) \leq \text{ord}_a(f/g)$.

A direct proof (adapting that of Proposition 2.4) could easily be given. In fact this is a special case of the analogue of Proposition 2.4 for the case of analytic functions mentioned above.

Similar remarks and an analogue of the above proposition apply to the Lazard valuation.

Now in case $k = \mathbb{C}$ there is an interesting relationship between the rational function field $K$ and the field of all $n$-variate Puiseux series $\mathbb{C}_n(x; a)$, defined in Section 2 of [23]. Indeed there is a natural embedding of $K$ into $\mathbb{C}_n(x; a)$, for every point $a$. So, with a view toward rectifying the flawed theory relating to Lazard’s valuation for multivariate Puiseux series [23], one might wish to find a way to relax the hypotheses of the above proposition (for $v_a$) without invalidating it. However it is by no means clear how to do this.

Finally we consider possible extension of the valuations and the associated basic theory to fractional power series rings. Both the order and Lazard valuation could be defined for an Abhyankar-Jung power series ring [23] such as $\mathbb{C}\{x_1, \ldots, x_{n-1}, x_n^{1/q}\}$. Moreover, it is conceivable that a suitable analogue of Proposition 2.4 could be proved for a case like this. However further extension – to some suitable subring of $\mathbb{C}(x; a)$ which includes all the desired roots, and in which one has a reasonable notion of convergence – would seem challenging in view of the difficulties noted above.

5 Conclusion, further work

Mindful of the potential benefits of Lazard’s approach to projection in [18], yet conscious of the flawed justification provided, we embarked on investigation of both the Lazard projection and valuation. In [23] we found that Lazard’s projection is valid for CAD construction for well-oriented polynomial sets. The validity proof uses order-invariance instead of valuation-invariance, and builds upon existing results [21, 22, 7] about improved projection. In this report we proved some basic properties of Lazard’s valuation in the special setting of real or complex multivariate polynomials, and identified some relationships between valuation-invariance and order-invariance.

Further work could usefully be done in a number of directions. It would be desirable to extend the CAD algorithm, with improved projection, to apply to non-well-oriented sets. It would be nice to have a more streamlined, condensed account of the theory of improved projection for CAD, which is currently scattered across several journal articles spanning nearly three decades. It would be interesting to try to pursue further the notion and theory of the Lazard valuation, which effort could yield some worthwhile algorithmic and theoretical improvements. In particular, it would be worthwhile to try to prove Lazard’s main claim (with no restriction to $n \leq 3$). Examination of the other ideas
suggested in [18] could also be fruitful.

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