SPLINE CHARACTERIZATIONS OF THE RADON-NIKODÝM
PROPERTY

MARKUS PASSENBRUNNER

Abstract. We give necessary and sufficient conditions for a Banach space
having the Radon-Nikodým property in terms of polynomial spline sequences.

1. Introduction and Preliminaries

The aim of this paper is to prove new characterizations of the Radon-Nikodým
property for Banach spaces in terms of polynomial spline sequences in the spirit of
the corresponding martingale results (see Theorem 1.2). We thereby continue the
line of research about extending martingale results to also cover (general) spline
sequences that is carried out in [11, 8, 6, 5, 7, 4]. We refer to the book [1] by J.
Diestel and J.J. Uhl for basic facts on martingales and vector measures; here, we
only give the necessary notions to define the Radon-Nikodým property below. Let
(Ω, A) be a measure space and X a Banach space. Every σ-additive map ν : A → X
is called a vector measure. The variation |ν| of ν is the set function

|ν|(E) = sup π ∑ A∈π ∥ν(A)∥X,

where the supremum is taken over all partitions π of E into a finite number of
pairwise disjoint members of A. If ν is of bounded variation, i.e., |ν|(Ω) < ∞,
the variation |ν| is σ-additive. If μ : A → [0, ∞) is a measure and ν : A → X is
a vector measure, ν is called μ-continuous if limμ(E)→0 ν(E) = 0 for all E ∈ A.
In the following, L^p_X = L^p_X(Ω, A, μ) will denote the Bochner-Lebesgue space of
integrable Bochner measurable functions f : Ω → X and if X = R, we simply write
L^p instead of L^p_R.

Definition 1.1. A Banach space X has the Radon-Nikodým property (RNP) if
for every measure space (Ω, A), for every positive measure μ on (Ω, A) and for
every μ-continuous vector measure ν of bounded variation, there exists a function
f ∈ L^1_X(Ω, A, μ) such that

ν(A) = ∫_A f dμ, A ∈ A.

Additionally, recall that a sequence (f_n) in L^1_X is uniformly integrable if the
sequence (∥f_n∥_X) is bounded in L^1 and, for any ε > 0, there exists δ > 0 such that

μ(A) < δ ⇒ sup_n ∫_A ∥f_n∥_X dμ < ε, A ∈ A.
We have the following characterization of the Radon-Nikodým property in terms of martingales, see e.g. [9] p. 50].

**Theorem 1.2.** For any \( p \in (1, \infty) \), the following statements about a Banach space \( X \) are equivalent:

(i) \( X \) has the Radon-Nikodým property (RNP),

(ii) every \( X \)-valued martingale bounded in \( L^1_X \) converges almost surely,

(iii) every uniformly integrable \( X \)-valued martingale converges almost surely and for all \( f \in L^1_X \),

(iv) every \( X \)-valued martingale bounded in \( L^p_X \) converges almost surely and in \( L^p_X \).

**Remark.** For the above equivalences, it is enough to consider \( X \)-valued martingales defined on the unit interval with respect to Lebesgue measure and the dyadic filtration (cf. [9] p. 54]).

Now, we describe the general framework that allows us to replace properties (ii)--(iv) with its spline versions.

**Definition 1.3.** A sequence of \( \sigma \)-algebras \( (\mathcal{F}_n)_{n \geq 0} \) in \([0, 1]\) is called an interval filtration if \( (\mathcal{F}_n) \) is increasing and each \( \mathcal{F}_n \) is generated by a finite partition of \([0, 1]\) into intervals of positive Lebesgue measure.

For an interval filtration \( (\mathcal{F}_n) \), we define \( \Delta_n := \{ \partial A : A \text{ atom of } \mathcal{F}_n \} \) to be the set of all endpoints of atoms in \( \mathcal{F}_n \). For a fixed positive integer \( k \), set

\[
S_n^{(k)} = \{ f \in C^{k-2}[0,1] : f \text{ is a polynomial of order } k \text{ on each atom of } \mathcal{F}_n \},
\]

where \( C^n[0,1] \) denotes the space of \( n \) times continuously differentiable, real valued functions on \([0,1]\) and the order \( k \) of a polynomial \( p \) is related to the degree \( d \) of \( p \) by the formula \( k = d + 1 \).

The finite dimensional space \( S_n^{(k)} \) admits a very special basis \( (N_i) \) of non-negative and uniformly bounded functions, called B-spline basis, that forms a partition of unity, i.e. \( \sum_i N_i(t) = 1 \) for all \( t \in [0,1] \), and the support of each \( N_i \) consists of the union of \( k \) neighboring atoms of \( \mathcal{F}_n \). If \( n \geq m \) and \( (N_i),(\tilde{N}_i) \) are the B-spline bases of \( S_n^{(k)} \) and \( S_m^{(k)} \) respectively, we can write each \( f \in S_m^{(k)} \) as \( f = \sum a_i \tilde{N}_i = \sum b_i N_i \) for some coefficients \( (a_i),(b_i) \) since \( S_m^{(k)} \subset S_n^{(k)} \). Those coefficients are related to each other in the way that each \( b_i \) is a convex combination of the coefficients \( (a_i) \). For more information on spline functions, see [10].

Additionally, we let \( P_n^{(k)} \) be the orthogonal projection operator onto \( S_n^{(k)} \) with respect to \( L^2[0,1] \) equipped with the Lebesgue measure \( |\cdot| \). Each space \( S_n^{(k)} \) is finite dimensional and B-Splines are uniformly bounded, therefore, \( P_n^{(k)} \) can be extended to \( L^1 \) and \( L^\infty_X \) satisfying \( P_n^{(k)}(f \otimes x) = (P_n^{(k)} f) \otimes x \) for all \( f \in L^1 \) and \( x \in X \), where \( f \otimes x \) denotes the function \( t \mapsto f(t)x \). Moreover, by \( S_n^{(k)} \otimes X \), we denote the space span\(\{f \otimes x : f \in S_n^{(k)}, x \in X\}\).

**Definition 1.4.** Let \( X \) be a Banach space and \( (f_n)_{n \geq 0} \) be a sequence of functions in \( L^1_X \). Then, \( (f_n) \) is an \( (X\text{-valued}) k\text{-martingale spline sequence adapted to } (\mathcal{F}_n) \), if \( (\mathcal{F}_n) \) is an interval filtration and

\[
P_n^{(k)} f_{n+1} = f_n, \quad n \geq 0.
\]
This definition resembles the definition of martingales with the conditional expectation operator replaced by $P_n^{(k)}$. For splines of order $k = 1$, i.e. piecewise constant functions, the operator $P_n^{(k)}$ even is the conditional expectation operator with respect to the $\sigma$-algebra $\mathcal{F}_n$.

Many of the results that are true for martingales (such as Doob’s inequality, the martingale convergence theorem or Burkholder’s inequality) in fact carry over to $k$-martingale spline sequences corresponding to an arbitrary interval filtration as the following two theorems show:

**Theorem 1.5.** For any positive integer $k$, any interval filtration $(\mathcal{F}_n)$ and any Banach space $X$, the following assertions are true:

(i) there exists a constant $C_k$ only depending on $k$ such that

$$ \sup_n \| P_n^{(k)} : L^1_X \to L^1_X \| \leq C_k, $$

(ii) there exists a constant $C_k$ only depending on $k$ such that for any $X$-valued $k$-martingale spline sequence $(f_n)$ and any $\lambda > 0$,

$$ |\{ \sup_n \| f_n \|_X > \lambda \} | \leq C_k \frac{\sup_n \| f_n \|_{L^1_X}}{\lambda}, $$

(iii) for all $p \in (1, \infty]$ there exists a constant $C_{p,k}$ only depending on $p$ and $k$ such that for all $X$-valued $k$-martingale spline sequence $(f_n)$,

$$ \| \sup_n \| f_n \|_X \|_{L^p} \leq C_{p,k} \sup_n \| f_n \|_{L^p_X}, $$

(iv) if $X$ has the RNP and $(f_n)$ is an $L^1_X$-bounded $k$-martingale spline sequence, $(f_n)$ converges a.s. to some $L^p_X$-function.

(i) is proved in [11] and (ii)–(iv) are proved (effectively) in [8, 5].

**Theorem 1.6 ([6]).** For all $p \in (1, \infty)$ and all positive integers $k$, scalar-valued $k$-spline-differences converge unconditionally in $L^p$, i.e. for all $f \in L^p$,

$$ \| \sum_n \pm (P_n^{(k)} - P_{n-1}^{(k)}) f \|_{L^p} \leq C_{p,k} \| f \|_{L^p}, $$

for some constant $C_{p,k}$ depending only on $p$ and $k$.

The martingale version of Theorem 1.6 is Burkholder’s inequality, which precisely holds in the vector-valued setting for UMD-spaces $X$ (by the definition of UMD-spaces). It is an open problem whether Theorem 1.6 holds for UMD-valued $k$-martingale spline sequences in this generality, but see [2] for a special case. For more information on UMD-spaces, see e.g. [9].

**Definition 1.7.** Let $X$ be a Banach space, $(\mathcal{F}_n)$ an interval filtration and $k$ a positive integer. Then, $X$ has the $(\mathcal{F}_n, k)$-martingale spline convergence property (MSCP) if all $L^1_X$-bounded $k$-martingale spline sequences adapted to $(\mathcal{F}_n)$ admit a limit almost surely.

In this work, we prove the following characterization of the Radon-Nikodým property in terms of $k$-martingale spline sequences.

**Theorem 1.8.** Let $X$ be a Banach space, $(\mathcal{F}_n)$ an interval filtration, $k$ a positive integer and $V$ the set of all accumulation points of $\cup_n \Delta_n$. Then, $(\mathcal{F}_n, k)$-MSCP characterizes RNP if and only if $|V| > 0$, i.e.,

$$ |V| > 0 \iff (X \text{ has RNP} \iff X \text{ has } (\mathcal{F}_n, k)\text{-MSCP}). $$
Proof. If \(|V| > 0\), it follows from Theorem \([1.5]^{[14]}\) that RNP implies \((\mathcal{F}_n, k)\)-MSCP for any positive integer \(k\) and any interval filtration \((\mathcal{F}_n)\). The reverse implication for \(|V| > 0\) is a consequence of Theorem \([1.10]\). We even have that if \(X\) does not have RNP, we can find a \((\mathcal{F}_n)\)-adapted \(k\)-martingale spline sequence that does not converge at all points \(t \in E\) for a subset \(E \subset V\) with \(|E| = |V|\). We simply have to choose \(E := \limsup E_n\) with \((E_n)\) being the sets from Theorem \([1.10]\).

If \(|V| = 0\), it is proved in \([5]\) that any Banach space \(X\) has \((\mathcal{F}_n, k)\)-MSCP. \(\Box\)

We also have the following spline analogue of Theorem \([1.2]\).

**Theorem 1.9.** For any positive integer \(k\) and any \(p \in (1, \infty)\), the following statements about a Banach space \(X\) are equivalent:

\((i)\) \(X\) has the Radon-Nikodým property,

\((ii)\) every \(X\)-valued \(k\)-martingale spline sequence bounded in \(L^1_X\) converges almost surely,

\((iii)\) every uniformly integrable \(X\)-valued \(k\)-martingale spline sequence converges almost surely and in \(L^p_X\),

\((iv)\) every \(X\)-valued \(k\)-martingale spline sequence bounded in \(L^p_X\) converges almost surely and in \(L^p_X\).

**Proof.** \(\circlearrowleft \Rightarrow \circlearrowright \): Theorem \([1.5][14].\)

\(\circlearrowleft \Rightarrow \circlearrowright \): clear.

\(\circlearrowright \Rightarrow \circlearrowleft \): if \((f_n)\) is a \(k\)-martingale spline sequence bounded in \(L^p_X\) for \(p > 1\), then \((f_n)\) is uniformly integrable, therefore it has a limit \(f\) (a.s. and \(L^1_X\)), which, by Fatou’s lemma, is also contained in \(L^p_X\). By Theorem \([1.5][14]\)(iii), \(\sup_n \|f_n\|_X \in L^p\) and we can apply dominated convergence to obtain \(\|f_n - f\|_{L^p_X} \to 0\).

\(\circlearrowleft \Rightarrow \circlearrowright \): follows from Theorem \([1.10]\). \(\Box\)

The rest of the article is devoted to the construction of a suitable non-RNP-valued \(k\)-martingale spline sequence, adapted to an arbitrary given filtration \((\mathcal{F}_n)\), so that the associated martingale spline differences are separated away from zero on a large set, which, more precisely, takes the following form:

**Theorem 1.10.** Let \(X\) be a Banach space without RNP, \((\mathcal{F}_n)\) an interval filtration, \(V\) the set of all accumulation points of \(\cap_n \Delta_n\) and \(k\) a positive integer.

Then, there exists a positive number \(\delta\) such that for all \(\eta \in (0, 1)\), there exists an increasing sequence of positive integers \((m_j)\), an \(L^\infty_X\)-bounded \(k\)-martingale spline sequence \((f_j)_{j \geq 0}\) adapted to \((\mathcal{F}_{m_j})\) with \(f_j \in S_k^{(m_j)} \otimes X\), and a sequence \((E_n)\) of measurable sets \(E_n \subset V\) with \(|E_n| \geq (1 - 2^{-n})|V|\) so that for all \(n \geq 1\)

\[\|f_n(t) - f_{n-1}(t)\|_X \geq \delta, \quad t \in E_n.\]

We will use the concept of dentable sets to prove Theorem \([1.10]\) and recall its definition:

**Definition 1.11.** Let \(X\) be a Banach space. A subset \(D \subset X\) is called dentable if for any \(\varepsilon > 0\) there is a point \(x \in D\) such that

\[x \notin \overline{\text{conv}}(D \setminus B(x, \varepsilon)),\]

where \(\overline{\text{conv}}\) denotes the closure of the convex hull and where \(B(x, \varepsilon) = \{y \in X : \|y - x\| < \varepsilon\}\).
Remark (cf. [11, p. 138, Theorem 10] and [9, p. 49, Lemma 2.7]). If \( D \) is a bounded non-dentable set, then, the closed convex hull \( \text{conv}(D) \) is also bounded and non-dentable. Thus, we may assume that \( D \) is convex. Moreover, we can as well assume that each \( x \in D \) can be expressed as a finite convex combination of elements in \( D \setminus B(x, \delta) \) for some \( \delta > 0 \) since if \( D \subset X \) is a convex set such that \( x \in \text{conv}(D \setminus B(x, \delta)) \) for all \( x \in D \), then, the enlarged set \( \tilde{D} = D + B(0, \eta) \) is also convex and satisfies
\[
 x \in \text{conv} \left( \tilde{D} \setminus B(x, \delta - \eta) \right), \quad x \in \tilde{D}.
\]

The reason why we are able to use the concept of dentability in the proof of Theorem 1.10 is the following geometric characterization of the RNP (see for instance [11, p. 136]).

**Theorem 1.12.** For any Banach space \( X \) we have that \( X \) has the RNP if and only if every bounded subset of \( X \) is dentable.

We record the following (special case of the) basic composition formula for determinants (see for instance [3, p. 17]):

**Lemma 1.13.** Let \( (f_i)_{i=1}^n \) and \( (g_j)_{j=1}^m \) two sequences of functions in \( L^2 \). Then,
\[
 \det \left( \int_0^1 f_i(t) g_j(t) \, dt \right)_{i,j=1}^n = \int_{0 \leq t_1 < \cdots < t_n \leq 1} \det(f_i(t_i))_{i=1}^n \cdot \det(g_j(t_j))_{j=1}^m \, d(t_1, \ldots, t_n).
\]

We also note the following simple

**Lemma 1.14.** Let \( I \subset [0, 1] \) be an interval and \( V \) an arbitrary measurable subset of \( [0, 1] \). Then, for all \( \varepsilon_1, \varepsilon_2 > 0 \), there exists a positive integer \( n \) so that for the decomposition of \( I \) into intervals \( (A_\ell)_{\ell=1}^n \) with \( \sup A_\ell \leq \inf A_{\ell+1} \) and \( n|A_\ell \cap V| = |I \cap V| \) for all \( \ell \), the index set \( \Gamma = \{ 2 \leq \ell \leq n-1 : \max(|A_{\ell-1}|, |A_\ell|, |A_{\ell+1}|) \leq \varepsilon_1 \} \) satisfies
\[
 \sum_{\ell \in \Gamma} |A_\ell \cap V| \geq (1 - \varepsilon_2)|I \cap V|.
\]

### 2. Construction of non-convergent spline sequences

In this section, we prove Theorem 1.10. In order to do that, we begin by fixing an interval filtration \( (F_n) \), the corresponding endpoints of atoms \( (\Delta_n) \) and a positive integer \( k \). For the space \( S_n^{(k)} \), we will suppress the (fixed) index \( k \) and write \( S_n \) instead. We will apply the same convention to the corresponding projection operators \( P_n = P_n^{(k)} \). We also let \( V \subset [0, 1] \) be the closed set of all accumulation points of \( \cup_n \Delta_n \).

The main step in the proof of Theorem 1.10 consists of an inductive application of the construction of a suitable martingale spline difference in the following lemma:

**Lemma 2.1.** Let \( (x_j)_{j=1}^M \) be in the Banach space \( X \), \( \bar{x} \in S_N \otimes X \) for some non-negative integer \( N \) such that \( \bar{x} = \sum_{j=1}^M \alpha_j \otimes x_j \) with \( \sum_{j=1}^M \alpha_j \equiv 1 \), \( \|x_j\| \leq 1 \), \( \alpha_j \in S_N \) having non-negative B-spline coefficients for all \( j \) and let \( I \subset [0, 1] \) be an interval so that \( |I \cap V| > 0 \).

Then, for all \( \varepsilon \in (0, 1) \), there exists a positive integer \( K \) and a function \( g \in S_K \otimes X \) with the properties
(i) $\int_I t^j g(t) \, dt = 0$ for all $j = 0, \ldots, k - 1$,
(ii) $\text{supp } g \subset \text{int } I$,
(iii) we have a splitting of the collection $\mathcal{A} = \{A \subset I : A \text{ is atom in } F_K\}$ into
$\mathcal{A}_1 \cup \mathcal{A}_2$ so that 
(a) if the functions $\alpha_j$ are all constant, then
on each $J \in \mathcal{A}_1$, $\bar{x} + g$ is constant with a value in $\cup_i \{x_i\}$,
otherwise we still have that
on each $J \in \mathcal{A}_1$, $\bar{x} + g$ is constant with a value in $\text{conv}\{x_i : 1 \leq i \leq M\}$,
(b) $|\cup_{J \in \mathcal{A}_2} J \cap V| \geq (1 - \varepsilon)|I \cap V|$,
(c) on each $J \in \mathcal{A}_2$, $\bar{x} + g = \sum \lambda t \otimes y$ for some functions $\lambda \in S_K$ having
non-negative B-spline coefficients with $\sum \lambda \equiv 1$ and $y \in \text{conv}\{x_j : 1 \leq j \leq M\}$.

Proof. The first step of the construction gives a function $g$ satisfying the desired conditions but only having mean zero instead of vanishing moments in property (i).
In the second step, we use this result to construct a function $g$ whose moments also vanish.

**Step 1:** We start with the (simpler) construction of $g$ when the functions $\alpha_j$ are not constant and condition (iii)(a) has the form that on each $J \in \mathcal{A}_1$, $\bar{x} + g$ is constant with a value in $\text{conv}\{x_i : 1 \leq i \leq M\}$.
First, decompose $I$ into intervals $(A_t)_{t=1}^n$ satisfying $n|A_t \cap V| = |I \cap V|$ with
$\sup A_t \leq \inf A_{t+1}$ and $n \geq 4/\varepsilon$. Then, choose $K \geq N$ so large that $A_1, A_2, A_{n-1}, A_n$
each contains at least $k + 1$ atoms of $F_K$. Denoting by $(N_j)$ the B-spline basis of $S_K$, we can write
$$\alpha_t = \sum_j \alpha_{t,j} N_j, \quad t = 1, \ldots, M$$
for some non-negative coefficients ($\alpha_{t,j}$). Define
$$h_t \equiv \sum_{j : \cup_{t+1}^{n} A_t \cap N_j \neq \emptyset} \alpha_{t,j} N_j.$$ 
Observe that $\text{supp } h_t \subset \text{int } I$ and $h_t \equiv \alpha_t$ on $\cup_{t=1}^{n-1} A_t$. Letting $\bar{x} = \sum \beta_{t} x_t$ for
$\beta_t = \int h_t / (\sum_j \int h_j) \in [0,1]$, we define
$$g := - \sum_{t=1}^{M} h_t \otimes x_t + \left( \sum_{j=1}^{M} h_j \right) \otimes \bar{x}.$$ 
This is a function of the desired form when defining $\mathcal{A}_1 := \{A \subset \cup_{t=2}^{n} A_t : A \text{ is atom in } F_K\}$ and $\mathcal{A}_2 := \mathcal{A} \setminus \mathcal{A}_1$ as we will now show by proving $\int g = 0$ and properties (ii), (iii). The fact that $\int g = 0$ follows from a simple calculation. Property (ii) is satisfied by the definition of the functions $h_t$. Property (iii)(a) follows from the fact that $\bar{x}(t) + g(t) = \bar{x} \in \text{conv}\{x_j : 1 \leq j \leq M\}$ for $t \in \cup_{t=2}^{n} A_t$ since $h_t \equiv \alpha_t$ on that set for any $t = 1, \ldots, M$. Since $|\cup_{t=1}^{n-1} A_t \cap V| = 4|I \cap V|/n \leq \varepsilon |I \cap V|$, (iii)(b) also follows from the construction of $\mathcal{A}_1$. Since
$$\bar{x}(t) + g(t) = \sum_{t=1}^{M} \left( \alpha_t(t) - h_t(t) \right) x_t + \left( \sum_{j=1}^{M} h_j(t) \right) \bar{x},$$
$\bar{x} \in \text{conv}\{x_j : 1 \leq j \leq M\}$, $h_t \leq \alpha_t$ and $\sum_t \alpha_t \equiv 1$, (iii)(c) is also proved.

The next step is to construct the desired function $g$ when $\alpha_j$ are assumed to be constant and (iii)(a) has the form that on each $J \in \mathcal{A}_1$, $\bar{x} + g$ is constant with a value
in $\cup_i \{ x_i \}$. Here, the idea is to construct a function of the form $g(t) = \sum f_j(t)(x_j - \bar{x})$ with $f_j \in S_K$ for some $K$ and $\int f_j \simeq C \alpha_j$ for all $j$ and some constant $C$ independent of $j$ to employ the assumption $\sum \alpha_j(x_j - \bar{x}) = 0$ implying $\int g = 0$.

We begin this construction by successively choosing parameters $\varepsilon_3 \ll \varepsilon_1 \ll \varepsilon < \varepsilon$ obeying certain given conditions depending on $\varepsilon$, $\bar{x}$, $(x_j)$, $(\alpha_j)$, $|I \cap V|$ and $|I|$.

First, set $\bar{\varepsilon} = \varepsilon |I \cap V|/(3|I|) > 0$ and

$$\varepsilon_1 = \frac{\varepsilon \bar{\varepsilon} (1 - \varepsilon/3)|I \cap V|}{72M}.$$ \hfill (2.1)

Now, we apply Lemma 1.14 with the parameters $\varepsilon_1$ and $\varepsilon_2 = \varepsilon/3$ to get a positive integer $n$ and a partition $(A_\ell)_{\ell=1}^n$ of $I$ consisting of intervals with $n|A_\ell \cap V| = |I \cap V|$ for all $\ell = 1, \ldots, n$ so that

$$\Gamma = \{ 2 \leq \ell \leq n - 1 : \max(|A_{\ell-1}|, |A_\ell|, |A_{\ell+1}|) \leq \varepsilon_1 \}$$

satisfies

$$\left( 1 - \frac{\varepsilon}{3} \right) |I \cap V| \leq \sum_{\ell \in \Gamma} |A_\ell \cap V|. \hfill (2.2)$$

Finally, we put $\varepsilon_3 = \varepsilon_1/(2n)$.

Next, for each $\ell = 1, \ldots, n$, we choose a point $p_\ell \in \text{int} A_\ell$ and an integer $K_\ell$ so that the intersection of $\text{int} A_\ell$ and the $\varepsilon_3$-neighborhood $B(p_\ell, \varepsilon_3)$ of $p_\ell$ contains at least $k + 1$ atoms of $F_{K_\ell}$ to the left as well as to the right of $p_\ell$. This is possible since $|A_\ell \cap V| = |I \cap V|/n$ and $V$ is the set of all accumulation points of $\cup_j \Delta_j$. Then set $K = \max \ell K_\ell$ and we let $u_\ell \in A_\ell$ be the leftmost point of $\Delta_K$ contained in $B(p_\ell, \varepsilon_3) \cap \text{int} A_\ell$. Similarly, let $v_\ell \in A_\ell$ be the rightmost point of $\Delta_K$ contained in $B(p_\ell, \varepsilon_3) \cap \text{int} A_\ell$. Next, for $2 \leq \ell \leq n - 1$, we put $B_\ell := (v_{\ell-1}, u_{\ell+1}) \subset A_{\ell-1} \cup A_\ell \cup A_{\ell+1}$. Observe that the construction of $u_\ell$ and $v_\ell$ implies that $B_\ell \cap B_j = \emptyset$ for all $|\ell - j| \geq 2$. Next, let $(N_i)$ be the B-spline basis of the space $S_K$ and let $((\ell(i)))_{i=1}^L$ be the increasing sequence of integers so that $\Gamma = \{ \ell(i) : 1 \leq i \leq L \}$ for $L = |\Gamma| \leq n$. We then define the set

$$\Lambda(r, s) := \left\{ j : \text{supp } N_j \cap \left( \bigcup_{i=r}^s B_{\ell(i)} \right) \neq \emptyset \right\}$$

to consist of those B-spline indices so that the support of the corresponding B-spline function intersects the set $\bigcup_{i=r}^s B_{\ell(i)}$. Observe that by (2.2),

$$\left( 1 - \frac{\varepsilon}{3} \right) |I \cap V| \leq \sum_{\ell \in \Gamma} |A_\ell \cap V| = \left| \bigcup_{\ell \in \Gamma} A_\ell \cap V \right| \leq \left| \bigcup_i B_{\ell(i)} \cap V \right| \leq \left| \bigcup_i B_{\ell(i)} \right|. \hfill (2.3)$$

Thus, the definition (2.1) of $\varepsilon_1$ in particular implies

$$72\varepsilon_1 M \leq \varepsilon \cdot \bar{\varepsilon} \cdot \left| \bigcup_i B_{\ell(i)} \right|. \hfill (2.4)$$

We continue with defining the functions $(f_j)$ contained in $S_K$ using a stopping time construction and first set $j_0 = -1$ and $C = (1 - \varepsilon/3)\left| \bigcup_i B_{\ell(0)} \right| > 0$. For $1 \leq m \leq M$, if $j_{m-1}$ is already chosen, we define $j_m$ to be the smallest integer $\leq L$ so that the function

$$f_m := \sum_{j \in \Lambda(j_{m-1}+2, j_m)} N_j \text{ satisfies } \int f_m(t) \, dt > C\alpha_m, \hfill (2.5)$$
If no such integer exists, we set \( f_m = L \) (however, we will see below that for the current choice of parameters, such an integer always exists). Additionally, we define
\[
f_{M+1} := \sum_{j \in \Lambda(j_M + 2L)} N_j.
\]

Observe that by the locality of the B-spline basis \((N_i)\), \( \text{supp} \, f_\ell \cap \text{supp} \, f_m = \emptyset \) for \( 1 \leq \ell < m \leq M + 1 \). Based on the collection of functions \((f_m)_{m=1}^{M+1}\), we will define the desired function \(g\). But before we do that, we make a few comments about \((f_m)_{m=1}^{M+1}\).

Note that for \( m = 1, \ldots, M \), by the minimality of \( f_m \),
\[
\int \sum_{j \in \Lambda(j_M + 2, j_M - 1)} N_j(t) \, dt \leq C_{m},
\]
and therefore, again by the locality of the B-splines \((N_i)\),
\[
\int f_m(t) \, dt \leq C_{m} + \sum_{j \in \Lambda(j_M, j_M)} N_j(t) \, dt \leq C_{m} + 3\varepsilon_1.
\]

Additionally, employing also the definition of \(u_\ell\) and \(v_\ell\) and the fact that the B-splines \((N_i)\) form a partition of unity,
\[
\left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right| \leq \int f_m(t) \, dt \leq \left| \bigcup_{i=\ell}^{j_M} (p_\ell(t) - 1, p_\ell(t) + 1) \right|
\]
\[
\leq \left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right| + 2\varepsilon_3.
\]

Next, we will show
\[
(1 - \varepsilon) \left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right| \leq \left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right| \leq (1 - \varepsilon/6) \left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right|.
\]

Indeed, we calculate on the one hand by \((2.7)\) and \((2.6)\)
\[
\left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right| \leq \sum_{m=1}^{M} \left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right| + \sum_{m=1}^{M} |B_{j_M+1}| \leq \sum_{m=1}^{M} \int f_m(t) \, dt + 3\varepsilon_1 M
\]
\[
\leq \sum_{m=1}^{M} (C_{m} + 3\varepsilon_1) + 3\varepsilon_1 M = C + 6\varepsilon_1 M
\]

Recalling now that \( C = (1 - \varepsilon/3) \left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right| \) and using \((2.4)\) now yields the right hand side of \((2.8)\).

On the other hand, employing \((2.7)\) and \((2.5)\),
\[
\left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right| \geq \sum_{m=1}^{M} \left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right| \geq \sum_{m=1}^{M} \left( \int f_m(t) \, dt - 2\varepsilon_3 \right)
\]
\[
\geq C \sum_{m=1}^{M} \alpha_m - 2nM\varepsilon_3 = C - 2nM\varepsilon_3.
\]

The definition of \( C = (1 - \varepsilon/3) \left| \bigcup_{i=\ell}^{j_M} B_\ell(t) \right| \) and \(\varepsilon_3 = \varepsilon_1/(2n)\), combined with \((2.4)\) gives the left hand inequality in \((2.8)\).
The inequality on the right hand side of (2.8), combined with (2.4) again, allows us to give the following lower estimate of $\int f_{M+1} dt$:

$$\int f_{M+1}(t) dt \geq \left| \sum_{i \geq JM+2} B_t(i) \right| \geq \left| \bigcup_{i \geq JM} B_t(i) \right| - 3\varepsilon_1 \geq \frac{\bar{\varepsilon}}{12} \left| \bigcup_i B_t(i) \right|.$$  

We are now ready to define the function $g \in S_K \otimes X$ as follows:

$$g = \sum_{j=1}^{M} f_j \otimes (x_j - \bar{x}) + f_{M+1} \sum_{j=1}^{M} \beta_j (x_j - \bar{x}),$$

where

$$\beta_j = \frac{C\alpha_j - \int f_j(t) dt}{\int f_{M+1}(t) dt}, \quad 1 \leq j \leq M.$$  

We proceed by proving $\int g = 0$ and properties (iii) for $g$:

The fact that $\int g = 0$ follows from a straightforward calculation using (2.11) and the assumption $\sum_{j=1}^{M} \alpha_j (x_j - \bar{x}) = 0$. (iii) follows from supp $g \subset [p_{M-1}, p_M] \subset \text{int} I$. Next, observe that by definition of $g$ and $f_1, \ldots, f_{M+1}$, on each $F_K$-atom contained in the set $B := \bigcup_{m=M} B_{m-1} \cup_{m=2} B_{1}(t)$, the function $\bar{x} + g$ is constant with a value in $\cup_i \{x_i\}$. Setting $\mathcal{A}_1 = \{A \subset B : A \text{ atom in } F_K\}$ and $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1$ now shows (iii)(a). Moreover, by (2.11), (2.3) and (2.8),

$$\left| \bigcup_{J \in \mathcal{A}_1} J \cap V \right| = \left| \bigcup_{m=1}^{M} \bigcup_{i=JM+2} B_t(i) \cap V \right| \geq \left| \bigcup_{i \geq JM} B_t(i) \cap V \right| - 3M\varepsilon_1 \geq \left| \bigcup_{i \geq JM} B_t(i) \cap V \right| - \frac{\varepsilon|I \cap V|}{24} \geq (1 - \frac{\bar{\varepsilon}}{3})|I \cap V| - \frac{\varepsilon|I \cap V|}{24}.$$  

Since $\bar{\varepsilon}|B_t(i)| \leq \bar{\varepsilon}|I| \leq \varepsilon|I \cap V|/3$ by definition of $\bar{\varepsilon}$, we conclude $|\bigcup_{J \in \mathcal{A}_1} J \cap V| \geq (1 - \varepsilon)|I \cap V|$, proving also (iii)(b). Next, we note that for $t \in \text{supp} f_j$ with $j \leq M$, we have

$$\bar{x} + g(t) = \bar{x} + f_j(t)(x_j - \bar{x}) = f_j(t)x_j + (1 - f_j(t))\bar{x}.$$  

Since $f_j(t) \in [0,1]$ and $\bar{x}$ is a convex combination of the elements $(x_j)$, we get (iii)(c) in this case. If $t \in \text{supp} f_{M+1}$, we calculate

$$(\bar{x} + g(t)) = (1 - f_{M+1}(t))\bar{x} + f_{M+1}(t)(\bar{x} + \sum_{j=1}^{M} \beta_j (x_j - \bar{x})).$$

We have by the lower estimate (2.9) for $f_{M+1}$ and by (2.6)

$$\sum_{j=1}^{M} |\beta_j| \leq \frac{12}{\bar{\varepsilon}\left| \bigcup_i B_t(i) \right|} \left( \int f_j - C\alpha_j \right) \leq \frac{12}{\bar{\varepsilon}\left| \bigcup_i B_t(i) \right|}(3\varepsilon_1 M),$$

which, by (2.4), is smaller than $\varepsilon/2$. Therefore, combining this with (2.12) yields property (iii)(c) for $t \in \text{supp} f_{M+1}$ by setting $\lambda_1 = 1 - f_{M+1}, \lambda_2 = f_{M+1}, y_1 = \bar{x}, \ldots$
Assume that \( g_R \) mean zero and properties (ii), (iii). The next step is to construct a function \( g \) so that additionally all of its moments up to order \( k \) vanish.

**Step 2:** Set \( \bar{\varepsilon} = 1 - (1 - \varepsilon)^{1/3} > 0 \). We write \( a = \inf I, b = \sup I \) and choose \( c \in I \) so that \( R := (c, b) \) satisfies \( 0 < |R \cap V| = \bar{\varepsilon}|I \cap V| \). Define \( L = I \setminus R \). Let \( (N_i) \) be the B-spline basis of \( S_{K_R} \), where we choose the integer \( K_R \) so that we can select B-spline functions \( (N_{m_i})_{i=0}^{k-1} \) that \( \text{supp} N_{m_i} \subset \text{int} R \) for any \( i = 0, \ldots, k-1 \) and \( \text{supp} N_{m_i} \cap \text{supp} N_{m_j} = \emptyset \) for \( i \neq j \). We then form the \( k \times k \)-matrix

\[
A = \left( \int_R t^i N_{m_j}(t) \, dt \right)_{i,j=0}^{k-1}.
\]

The matrix \( (t^i)_{i=0}^{k-1} \) is a Vandermonde matrix having positive determinant for \( t_0 < \cdots < t_{k-1} \). Moreover, the matrix \( (N_{m_j}(t))_{i=0}^{k-1} \) is a diagonal matrix having positive entries if \( t_i \in \text{int} \text{supp} N_{m_i} \) for \( i = 0, \ldots, k-1 \). For other choices of \( (t_i) \), the determinant of \( (N_{m_j}(t_i))_{j=0}^{k-1} \) vanishes. Therefore, Lemma 1.13 implies that \( A \neq 0 \) and \( A \) is invertible.

Next, we choose \( \varepsilon_1 = \varepsilon/(k(1 + \bar{\varepsilon})\|A^{-1}\|_\infty|L|) \) and apply Lemma 1.14 with the parameters \( \varepsilon_1, \varepsilon_2 = \bar{\varepsilon} \) and the interval \( L \) to obtain a positive integer \( n \) so that for the partition \( (A_{\ell})_{\ell=1}^n \) of \( L \) with \( n|A_{\ell} \cap V| = |L \cap V| \) and \( sup A_{\ell-1} = \inf A_{\ell} \), the set \( \Gamma = \{ 2 \leq \ell \leq n-1 : \max(|A_{\ell-1}|, |A_{\ell}|, |A_{\ell+1}|) \leq \varepsilon_1 \} \) satisfies

\[
\sum_{\ell \in \Gamma} |A_{\ell} \cap V| \geq (1 - \bar{\varepsilon})|L \cap V|.
\]

We now apply the construction of Step 1 on every set \( A_{\ell}, \ell \in \Gamma \), with the parameters \( \bar{x}, (x_j)_{j=1}^M, (\alpha_j)_{j=1}^M, \bar{\varepsilon} \) to get functions \( (g_\ell) \) with zero mean having properties (ii), (iii) with \( I \) replaced by \( A_{\ell} \). On \( L \), we define the function

\[
g(t) := \sum_{\ell \in \Gamma} g_\ell(t), \quad t \in L.
\]

Let \( z_j := \int_L t^j g(t) \, dt \) for \( j = 0, \ldots, k-1 \). Observe that, since \( \int_{A_{\ell}} g_\ell(t) \, dt = 0 \) and \( \|g_\ell\|_{L^\infty} \leq 1 + \bar{\varepsilon} \) by (iii) and \( |A_{\ell}| \leq \varepsilon_1 \), we get for all \( j = 0, \ldots, k-1 \),

\[
\|z_j\| = \left\| \sum_{\ell \in \Gamma} \int_{A_{\ell}} t^j g_\ell(t) \, dt \right\| = \left\| \sum_{\ell \in \Gamma} \int_{A_{\ell}} (t^j - (\inf A_{\ell})^j) \cdot g_\ell(t) \, dt \right\|
\]

\[
\leq j \sum_{\ell \in \Gamma} |A_{\ell}| \int_{A_{\ell}} g_\ell(t) \, dt
\]

\[
\leq j \varepsilon_1 (1 + \bar{\varepsilon})|L| \leq \bar{\varepsilon} \cdot \|A^{-1}\|^{-1}.
\]

In order to have \( \int_R t^j g(t) \, dt = 0 \) for all \( j = 0, \ldots, k-1 \), we want to define \( g \) on \( R = I \setminus L \) so that

\[
(2.13) \quad \int_R t^j g(t) \, dt = -z_j, \quad j = 0, \ldots, k-1.
\]

Assume that \( g \) on \( R \) is of the form

\[
g(t) = \sum_{i=0}^{k-1} N_{m_i}(t)w_i, \quad t \in R
\]
for some \((w_i)_{i=0}^{k-1}\) contained in \(X\). Then, (2.13) is equivalent to

\[Aw = -z\]

by writing \(w = (w_0, \ldots, w_{k-1})^T\) and \(z = (z_0, \ldots, z_{k-1})^T\). Defining \(w := -A^{-1}z\) and employing the estimate for \(\|z\|_\infty\) above, we obtain

\[
\|w\|_\infty \leq \|A^{-1}\|_\infty \|z\|_\infty \leq \delta.
\]

The definition of \(g\) immediately yields properties (i), (ii). From the application of the construction in Step 1 to each \(A_\ell, \ell \in \Gamma\), we obtained collections \(\mathcal{A}_\ell(\ell)\) of disjoint subintervals of \(A_\ell\) that are atoms in \(\mathcal{F}_{K_\ell}\) for some positive integer \(K_\ell \geq N\) satisfying that \(\tilde{x} + g_\ell\) is constant on each \(J \in \mathcal{A}_\ell(\ell)\) taking values in \(\text{conv}\{x_i : 1 \leq i \leq M\}\) and \(|\cup_{J \in \mathcal{A}_\ell(\ell)} J \cap V| \geq (1 - \tilde{\delta})|A_\ell \cap V|\). Let \(B := \cup_{\ell \in \mathcal{A}_\ell(\ell)} J\) and define \(\mathcal{A}_1\) to be the collection \(\{J \subset B : J \text{ atom in } \mathcal{F}_K\}\) where \(K := \max(\max_\ell K_\ell, K_R)\) and define \(\mathcal{A}' := \{J \subset I : J \text{ atom in } \mathcal{F}_K\}\), \(\mathcal{A}_2 := \mathcal{A} \setminus \mathcal{A}_1\).

Then, (iii)(a) is satisfied by the corresponding property of each \(g_\ell\). (iii)(b) follows from the calculation

\[
\left| \bigcup_{J \in \mathcal{A}_1} J \cap V \right| \geq (1 - \tilde{\delta})\sum_{\ell \in \Gamma} |A_\ell \cap V| \geq (1 - \tilde{\delta})^2 |L \cap V| \geq (1 - \tilde{\delta})^2 |L \cap V|.
\]

Property (iii)(c) on \(L\) is a consequence of property (iii)(c) for the functions \(g_\ell\). We can write \(\alpha_j = \sum_{\ell \in \mathcal{A}_1} \alpha_j, \ell \cdot N_{\ell}\) for some non-negative coefficients \((\alpha_j, \ell)\) that have the property \(\sum_{j=1}^M \alpha_j, \ell = 1\) for each \(\ell\). Therefore, on \(R\), we have

\[
\tilde{x}(t) + g(t) = \sum_{j=1}^M \alpha_j, \ell(t)x_j + \sum_{i=0}^{k-1} N_{\ell, m}(t)w_i = \sum_{\ell} N_{\ell}(t) \left( \sum_{j=1}^M \alpha_j, \ell x_j + \sum_{i=0}^{k-1} \delta_{\ell, m} w_i \right),
\]

which, since \(\|w\|_\infty \leq \tilde{\delta} \leq \varepsilon\) and \(\sum_{j=1}^M \alpha_j, \ell = 1\) for each \(\ell\), implies (iii)(c) on \(R\).

We now use Lemma 2.1 inductively to prove Theorem 1.10.

**Proof of Theorem 1.10.** We assume that \(X\) does not have the RNP. Then, by Theorem 1.12 the ball \(B(0, 1/2) \subset X\) contains a non-dentable convex set \(D\) satisfying

\[x \in \text{conv}(D \setminus B(x, 2\delta)), \quad x \in D\]

for some parameter \(2\delta\). Defining \(D_0 = D + B(0, \delta/2)\) and, for \(j \geq 1, D_j = D_{j-1} + B(0, 2^{-j-1}\delta)\), we use the remark after Definition 1.11 to get that all the sets \((D_j)\) are contained in \(B(0, 1)\), are convex and

\[x \in \text{conv}(D_j \setminus B(x, \delta)), \quad x \in D_j, \quad j \geq 0.\]

We will assume without restriction that \(\eta \leq \delta\).

Let \(x_{0,1} \in D_0\) arbitrary and set \(f_0 \equiv 1_{[0,1]} \otimes x_{0,1} \in S_{m_0} \otimes X\) on \(I_{0,1} := [0,1]\) for \(m_0 = 0\). By \(P_j\), we will denote the \(L^1_X\)-extension of the orthogonal projection operator onto \(S_{m_j}\), where we assume that \((m_j)_{j=1}^n\) and \((f_j)_{j=1}^n\) with \(f_j \in S_{m_j} \otimes X\) for each \(j = 1, \ldots, n\) are constructed in such a way that for all \(j = 0, \ldots, n,\)

1. \(P_{j-1} f_j = f_{j-1}\) if \(j \geq 1,\)
2. on all atoms \(I\) in \(\mathcal{F}_{m_j}\), \(f_j\) has the form

\[f_j \equiv \sum_{\ell} \lambda_\ell \otimes y_\ell, \quad \text{finite sum}\]
for functions $\lambda_\ell \in S_{m_n}$ with non-negative B-spline coefficients, $\sum \lambda_\ell \equiv 1$
and some $y_\ell \in D_{j_\ell}$.

(3) there exists a finite collection of disjoint intervals $(I_{j,i})$, that are atoms in $F_{m_n}$ so that (setting $C_j = \cup_i I_{j,i}
$
(a) for all $i$, $f_j \equiv x_{j,i} \in D_{j_\ell}$ on $I_{j,i}$,
(b) $\|f_j - f_{j-1}\|_X \geq \delta$ on $C_j \cap C_{j-1}$ if $j \geq 1$,
(c) $|C_j \cap C_{j-1} \cap V| \geq (1 - 2^{-j\eta})|V|$ if $j \geq 1$,
(d) $|C_j \cap V| \geq (1 - 2^{-j\eta})|V|$, 
(e) $|I_{j,i} \cap V| > 0$ for every $i$.

We will then perform the construction of $m_{n+1}$, $f_{n+1}$ and the collection $(I_{n+1,i})$ of atoms in $F_{m_{n+1}}$ having properties (1)–(3) for $j = n+1$. Define the collection $\mathcal{C} = \{A \text{ atom of } F_{m_n} : |A \cap V| > 0\}$. We will distinguish the two cases $B \in \mathcal{C}_1 := \{A \in \mathcal{C} : A = I_{n,i} \text{ for some } i\}$ and $B \in \mathcal{C}_2 := \mathcal{C} \setminus \mathcal{C}_1$.

CASE 1: $B \in \mathcal{C}_1$: here, $B = I_{n,i}$ for some $i$ and we use the fact that on $B$,
$f_n = x_B := x_{n,i} \in D_n$ and write

$$x_B = \sum_{\ell=1}^{M_B} \alpha_{B,\ell} x_{B,\ell}$$

with some positive numbers $(\alpha_{B,\ell})$ satisfying $\sum \alpha_{B,\ell} = 1$, some $x_{B,\ell} \in D_n$ and $\|x_B - x_{B,\ell}\| \geq \delta$ for any $\ell = 1, \ldots, M_B$. We apply Lemma 2.1 to the interval $B$ with this decomposition and with the parameter $\varepsilon = \eta_n := 2^{-n-3\eta}$. This yields a function $g_B \in S_{K_B} \otimes X$ for some positive integer $K_B$ that has the properties

(i) $\int t^i g_B(t) \, dt = 0$, $0 \leq i \leq k - 1$,
(ii) $\text{supp } g_B \subset \text{int } B$,
(iii) we have a splitting of the collection $\mathcal{A}_B = \{A \subset B : A \text{ is atom in } F_{K_B}\}$ into $\mathcal{A}_{B,1} \cup \mathcal{A}_{B,2}$ so that

(a) on each $J \in \mathcal{A}_{B,1}$, $f_n + g_B = x_B + g_B$ is constant on $J$ taking values in $\cup \{x_B, \ell\}$,
(b) $\|J \cap V\| \geq (1 - \eta_n)|B \cap V|$, 
(c) on each $J \in \mathcal{A}_{B,2}$, the function $f_n + g_B$ can be written as

$$f_n(t) + g_B(t) = x_B + g_B(t) = \sum_{\ell} \lambda_{B,\ell}(t) y_{B,\ell}$$

for some functions $\lambda_{B,\ell} \in S_{K_B}$ having non-negative B-spline coefficients with $\sum \lambda_{B,\ell} \equiv 1$ and $y_{B,\ell} \in \text{conv} \{x_{B,\ell} : 1 \leq j \leq M_B\} + B(0, \eta_n)$.

CASE 2: $B \in \mathcal{C}_2$: on $B$, $f_n$ is of the form

$$f_n(t) = \sum_{\ell=1}^{M_B} \lambda_\ell(t) y_\ell$$

for some functions $\lambda_\ell \in S_{m_n}$ having non-negative B-spline coefficients with $\sum \lambda_\ell \equiv 1$ and some $y_\ell \in D_n$. Applying Lemma 2.1 with the parameter $\eta_n = 2^{-n-3\eta}$, we obtain a function $g_B \in S_{K_B} \otimes X$ for some positive integer $K_B$ that has the properties

(i) $\int t^i g_B(t) \, dt = 0$, $0 \leq i \leq k - 1$,
(ii) $\text{supp } g_B \subset \text{int } B$,
(iii) we have a splitting of the collection $\mathcal{A}_B = \{A \subset B : A \text{ is atom in } F_{K_B}\}$ into $\mathcal{A}_{B,1} \cup \mathcal{A}_{B,2}$ so that
(a) for each $J \in \mathcal{A}_{B,1}$, $f_n + g_B$ is constant on $J$ taking values in $\text{conv}\{y_\ell : 1 \leq \ell \leq MB\}$,

(b) $| \cup_{J \in \mathcal{A}_{B,1}} J \cap V | \geq (1 - \eta_n)|B \cap V |$,

(c) for each $J \in \mathcal{A}_{B,2}$, the function $f_n + g_B$ can be written as

$$f_n(t) + g_B(t) = \sum_{\ell} \lambda_{B,\ell}(t)y_{B,\ell}$$

for some functions $\lambda_{B,\ell} \in S_{K_B}$ having non-negative B-spline coefficients with $\sum_{\ell} \lambda_{B,\ell} \equiv 1$ and $y_{B,\ell} \in \text{conv}\{y_j : 1 \leq j \leq MB\} + B(0, \eta_n)$.

Having treated those two cases, we define the index $m_{n+1} := \max\{K_B : B \in \mathcal{C}\}$ and

$$f_{n+1} = f_n + \sum_{B \in \mathcal{C}} g_B.$$

The new collection $(I_{n+1,i})$ is defined to be the decomposition of the set $\cup_{B \in \mathcal{C}} \cup_{J \in \mathcal{A}_{B,1}} J$ (from the above construction) into $\mathcal{F}_{m_{n+1}}$-atoms, after deleting those $\mathcal{F}_{m_{n+1}}$-atoms $I$ with $|I \cap V| = 0$. Since $D_n$ is convex and $\eta \leq \delta$, the corresponding function values of $f_{n+1}$ are contained in $D_n + B(0, \eta_n) \subset D_{n+1}$ and we will enumerate them as $(x_{n+1,i})$, accordingly. We additionally set $C_{n+1} := \cup_i I_{n+1,i}$.

With these definitions, we will successively show properties (1)–(3) for $j = n + 1$. Since the function $g = P_n f_{n+1} \in S_{m_n} \otimes X$ is characterized by the condition

$$\int g(t)s(t)\,dt = \int f_{n+1}(t)s(t)\,dt, \quad s \in S_{m_n},$$

property (1) for $j = n + 1$ follows if we show that $\int g_B(t)s(t)\,dt = 0$ for any $s \in S_{m_n}$ and any $B \in \mathcal{C}$. But this is a consequence of (i) for $g_B$ (in both case 1 and case 2), since $s \in S_{m_n}$ is a polynomial of order $k$ on $B$.

Property (2) now is a consequence of (iii) (again for both cases 1 and 2), we just remark once again that $D_n + B(0, \eta_n) \subset D_{n+1}$ due to $\eta \leq \delta$. Properties (3a), (3b) and (3e) are direct consequences of the construction. (3d) follows from (iii)(b) in cases 1 and 2 since

$$|C_{n+1} \cap V| = \left| \bigcup_{B \in \mathcal{C}} \bigcup_{J \in \mathcal{A}_{B,1}} J \cap V \right| = \sum_{B \in \mathcal{C}} \left| \bigcup_{J \in \mathcal{A}_{B,1}} J \cap V \right| \geq (1 - \eta_n) \sum_{B \in \mathcal{C}} |B \cap V| = (1 - \eta_n)|V|$$

and $\eta_n = 2^{-n-3}\eta$. For property (3c), we calculate

$$|C_{n+1} \cap C_n \cap V| \geq \eta_n|C_n \cap V| \geq (1 - \eta_n)(1 - 2^{-n-2}\eta)|V|$$

by (iii)(b) in case 1 and by induction hypothesis. Since $\eta_n = 2^{-n-3}\eta$, we get $(1 - \eta_n)(1 - 2^{-n-2}\eta) \geq 1 - 2^{-n+1}\eta$ and this proves (3c) for $j = n + 1$.

Finally, we note that due to (2) and (3)(c), the sequence $(m_n)$, the $k$-martingale spline sequence $(f_n)$ and the sets $E_n := C_n \cap C_{n-1} \cap V$ have the properties that are desired in the theorem. □

Acknowledgments. The author is grateful to P. F. X. Müller for many fruitful discussions related to the underlying article. This research is supported by the FWF-project Nr.P27723.
References

[1] J. Diestel and J. J. Uhl, Jr. Vector measures. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.
[2] A. Kamont and P. F. X. Müller. A martingale approach to general Franklin systems. Studia Math., 177(3):251–275, 2006.
[3] S. Karlin. Total positivity. Vol. I. Stanford University Press, Stanford, Calif, 1968.
[4] K. Keryan and M. Passenbrunner. Unconditionality of periodic orthonormal spline systems in $L^p$. preprint arXiv:1708.09294, 2017, to appear in Studia Math.
[5] P. F. X. Müller and M. Passenbrunner. Almost everywhere convergence of spline sequences. preprint, arXiv:1711.01859, 2017.
[6] M. Passenbrunner. Unconditionality of orthogonal spline systems in $L^p$. Studia Math., 222(1):51–86, 2014.
[7] M. Passenbrunner. Orthogonal projectors onto spaces of periodic splines. J. Complexity, 42:85–93, 2017.
[8] M. Passenbrunner and A. Shadrin. On almost everywhere convergence of orthogonal spline projections with arbitrary knots. J. Approx. Theory, 180:77–89, 2014.
[9] G. Pisier. Martingales in Banach spaces, volume 155 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016.
[10] L. L. Schumaker. Spline functions: basic theory. Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2007.
[11] A. Shadrin. The $L_{\infty}$-norm of the $L_2$-spline projector is bounded independently of the knot sequence: a proof of de Boor’s conjecture. Acta Math., 187(1):59–137, 2001.

Institute of Analysis, Johannes Kepler University Linz, Austria, 4040 Linz, Altenberger Strasse 69
E-mail address: markus.passenbrunner@jku.at