Lepton flavor models with discrete prediction of $\theta_{13}$

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Abstract
We study the lepton flavor models with the flavor symmetry $(Z_N \times Z_N \times Z_N) \rtimes Z_3$. Our models predict non-vanishing discrete values of $\theta_{13}$ as well as $\theta_{12}$ and $\theta_{23}$ depending on $N$. For certain values, our models realize the tri-bimaximal mixing angles with $\theta_{13} = 0$. For other values, our models provide with discrete deviation from the tri-bimaximal mixing angles.
1 Introduction

It is the important issue to study the origin of the flavor structure in particle physics, that is, the hierarchy of quark/lepton masses and their mixing angles. The form of lepton mixing angles is quite different from one of quark mixing angles (see e.g. for the lepton mixing [1–3]). The lepton mixing angles are large, while quark mixing angles are smaller. The tri-bimaximal mixing Ansatz in the lepton sector is quite interesting [4] and the exact tri-bimaximal mixing leads to $\theta_{13} = 0$.

Non-Abelian discrete flavor symmetry is a powerful tool to derive masses and mixing angles of quarks and leptons (see for review Refs. [5,6] and references therein). Especially, large mixing angles of leptons can naturally arise by assuming non-Abelian discrete flavor symmetries. Indeed, there are many models based on flavor symmetries to explain fermion masses and mixing.

Recently, the results of T2K and Double Chooz reported a non-vanishing value of $\theta_{13}$ [7,8]. Thus, it is important to derive the lepton mixing with $\theta_{13} \neq 0$. However, the tri-bimaximal mixing matrix is still good at a certain level. Thus, it is interesting to study a small deviation, e.g. from the tri-bimaximal mixing angles with $\theta_{13} \neq 0$. After the result of T2K is released, several attempts have been studied to to explain non-zero $U_{e3}$ with various methods [9–37]. For instance, introducing somewhat larger corrections to the model of tri-bimaximal mixing, one can obtain non-zero value for $U_{e3}$. However, such a method contains a free parameter so that it does not predict a specific value. In this paper, we build a model that predicts a special mixing pattern with deviations from the tri-bimaximal mixing angles by discrete values, which are consistent with experiments.

A simple way to derive the tri-bimaximal mixing is to assume residual symmetries for lepton mass matrices. Precisely, one can realize the tri-bimaximal mixing when there is the $Z_3$ symmetry for the charged lepton mass matrix and the $Z_2$ symmetry for the neutrino mass matrix. These $Z_3$ and $Z_2$ symmetries may be originated from larger non-Abelian flavor symmetries like $A_4$, $S_4$, etc. Thus, it is interesting to use non-Abelian discrete flavor symmetries such as $A_4$ and $S_4$ in order to derive the (exact) tri-bimaximal mixing [5,6]. To predict non-vanishing $\theta_{13}$, we propose a modified symmetry for the charged leptons. In our approach, we use the $(Z_N \times Z_N \times Z_N) \rtimes Z_3$ symmetry. With a choice of proper vacuum alignment, the charged lepton sector retains $Z_3$ and $Z_N$ symmetries while the neutrino sector has $Z_2$ symmetry, which is the subgroup of $Z_N$. Due to the additional $Z_N$ symmetry of charged lepton, we obtain non-vanishing $\theta_{13}$.

This paper is organized as follows. In section 2, we study about group-theoretical aspects of $(Z_N \times Z_N \times Z_N) \rtimes Z_3$ by giving conjugacy classes, character table, and tensor products. In section 3 we introduce the model based on $(Z_N \times Z_N \times Z_N) \rtimes Z_3$ lepton flavor symmetry with one singlet and two triplet flavon fields. In section 4 we ensure the vacuum alignment assumed in the model by two ways, that is, analyzing scalar potential and using boundary conditions in extra dimensions. In section 5, higher dimensional terms are taken into account. Section 6 is devoted to the summary.
2 \( (Z_N \times Z_N \times Z_N) \rtimes Z_3 \)

We use the group \((Z_N \times Z_N \times Z_N) \rtimes Z_3\) for the lepton flavor symmetry. Here, we study group-theoretical aspects of our group. We denote the first, second and third \(Z_N\) generators by \(a, a'\) and \(a''\), respectively, and the \(Z_3\) generator is denoted by \(b\). They satisfy the following algebraic relations:

\[
\begin{align*}
a^N &= a'^N = a''N = b^3 = e, & aa' &= a'a, & aa'' &= a''a, & a'a'' &= a''a', \\
bba^{-1} &= a^m, & bba^{-1} &= a^n, & bba^{-1} &= a^m.
\end{align*}
\]

where \(e\) denotes the identity. Group elements can be written by \(b^k a^l a'^m a''^n\) with \(\ell, m, n = 0, N - 1\) and \(k = 0, 1, 2\). When \(N/3 = \) integer, this group is isomorphic to \(\Sigma(3N^3)\) \([G]\). For \(N/3 \neq \) integer, this group corresponds to \(\Delta(3N^2) \times Z_N\). Here, we use both types of groups.

First, let us study the conjugacy classes of the elements without including \(b\), i.e. \(a^l a'^m a''^n\). These elements elements \(a^l a'^m a''^n\) satisfy the following relations:

\[
\begin{align*}
ba^l a'^m a''^n b^{-1} &= a^m a'^n a''^l, & \quad b^2a^l a'^m a''^n b^{-2} &= a^l a'^m a''^n.
\end{align*}
\]

Note that \(ba^l a'^m a''^n b^{-1} = b^2 a^l a'^m a''^n b^{-2} = a^l a'^m a''^n\). Thus, we find the following conjugacy classes of \(a^l a'^m a''^n\):

\[
\begin{align*}
C_1 : & \quad \{e\}, \\
C_{1}^{(\ell)} : & \quad \{a^l a'^m a''^n\}, \quad \text{for } \ell = 1, \ldots, N - 1, \\
C_3^{(\ell,m,n)} : & \quad \{a^l a'^m a''^n, a^m a'^n a''^l, a^n a'^l a''^m\}, \quad \text{for } \ell, m, n = 0, \ldots, N - 1.
\end{align*}
\]

The last classes \(C_3^{(\ell,m,n)}\) exclude the case with \(\ell = m = n\). The number of the classes \(C_3^{(\ell,m,n)}\) is equal to \((N^3 - N)/3\), while the number of the classes \(C_1^{(\ell)}\) is \((N - 1)\).

Next, we study the conjugacy classes of the elements including \(b\), that is \(ba^l a'^m a''^n\) and \(b^2a^l a'^m a''^n\). It is found that

\[
\begin{align*}
a^p a'^q a''^r (ba^l a'^m a''^n)a^{-p}a'^{-q}a''^{-r} &= ba^l + p a'^m + p a''^n + q - r, \\
ba^p a'^q a''^r (ba^l a'^m a''^n)a^{-p}a'^{-q}a''^{-r} b^{-1} &= ba^m + p a''^{-n} + q + r - p, \\
b^2a^p a'^q a''^r (ba^l a'^m a''^n)a^{-p}a'^{-q}a''^{-r} b^{-2} &= ba^{n+q-r} a'^l + r a'^m + p - q.
\end{align*}
\]

Note that the sum of the powers of \(a, a'\), and \(a'', \ell + m + n\), does not change. Then the conjugacy classes of \(ba^l a'^m a''^n\) are obtained as

\[
C_{N2}^{(p)} : \quad \{ba^l a'^m a''^{-p-l-m}\ell, m = 0, \ldots, N - 1\}, \quad \text{for } p = 0, \ldots, N - 1.
\]

Similarly, we obtain the conjugacy classes of \(b^2a^l a'^m a''^n\) as

\[
C_{N2}^{(p)} : \quad \{b^2a^l a'^m a''^{-p-l-m}\ell, m = 0, \ldots, N - 1\}, \quad \text{for } p = 0, \ldots, N - 1.
\]

Then, the total number of conjugacy classes is

\[
1 + (N - 1) + (N^3 - N)/3 + N + N = \frac{1}{3}N(N^2 + 8).
\]
Now, let us study representations of our group. Suppose that there are $m_n$ $n$-dimensional irreducible representations, where group elements are represented by $(n \times n)$ matrices. The identity $e$ is always represented by the $(n \times n)$ identity matrix and its character $\chi(e)$ is obtained as $\chi(e) = n$ on the $n$-dimensional representation. The orthogonality condition of characters requires $\sum_n n^2 m_n$ to be equal to the order of the group, i.e.,

$$m_1 + 4m_2 + 9m_3 + \cdots = 3N^3.$$  

(8)

In addition, the total number of irreducible representations should be equal to the total number of the conjugacy classes, i.e.,

$$m_1 + m_2 + m_3 + \cdots = \frac{1}{3}N(N^2 + 8).$$  

(9)

The solution of Eqs. (8) and (9) is obtained as $(m_1, m_2, m_3) = (3N, 0, N(N^2 - 1)/3)$. That implies that there are $3N$ singlets and $N(N^2 - 1)/3$ triplets.

It is straightforward to derive the $3N$ singlet representations, $1_k$ with $k = 0, \cdots, N-1$ and $\ell = 0, 1, 2$. Their characters for $a, a', a''$ and $b$ are obtained as $\chi_{1_k}(a) = \chi_{1_k}(a') = \chi_{1_k}(a'') = \rho^k$ and $\chi_{1_k}(b) = \omega^\ell$, where $\rho = e^{2\pi i/N}$, and $\omega = e^{2\pi i/3}$. These are shown in Table [1] where $h$ denotes the order of the element $g$ in the conjugacy class, i.e., $g^h = e$. Note that these characters are nothing but representations for $a, a', a''$ and $b$ on $1_k$.

On the other hand, we can write $a, a', a''$, and $b$ as

$$a = \begin{pmatrix} \rho^k & 0 & 0 \\ 0 & \rho^m & 0 \\ 0 & 0 & \rho^n \end{pmatrix}, \quad a' = \begin{pmatrix} \rho^n & 0 & 0 \\ 0 & \rho^k & 0 \\ 0 & 0 & \rho^m \end{pmatrix}, \quad a'' = \begin{pmatrix} \rho^m & 0 & 0 \\ 0 & \rho^n & 0 \\ 0 & 0 & \rho^l \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

(10)

on the triplet $3[\ell][m][n]$, where the notation $[\ell][m][n]$ means

$$[\ell][m][n] = (\ell, m, n), \quad (m, n, \ell), \quad \text{or} \quad (n, \ell, m).$$

(11)

Their characters are shown Table [1].

We write components of triplets as

$$3[\ell][m][n] = \begin{pmatrix} x_\ell \\ x_m \\ x_n \end{pmatrix}.$$  

(12)

Subscripts of the components describe $Z_N$ charges. The tensor product between two triplets is given by

$$\begin{pmatrix} x_\ell \\ x_m \\ x_n \end{pmatrix}_{3[\ell][m][n]} \times \begin{pmatrix} y_\ell \\ y_m \\ y_n \end{pmatrix}_{3[\ell'][m'][n']} = \begin{pmatrix} x_\ell y_\ell \\ x_m y_m \\ x_n y_n \end{pmatrix}_{3[\ell+\ell'][m+m'][n+n']} + \begin{pmatrix} x_\ell y_m' \\ x_m y_\ell \\ x_n y_n' \end{pmatrix}_{3[m+m'][n+\ell'][\ell+m']} + \begin{pmatrix} x_\ell y_n' \\ x_m y_\ell' \\ x_n y_m' \end{pmatrix}_{3[m+n'][n+\ell'][\ell+m']}.$$  

(13)

\[ \text{See e.g. [6] and references therein.} \]
If all the subscripts of $3_{[\ell+\ell'][m+m'][n+n']}$, $3_{[m+n'][n+\ell'][\ell+m']}$, or $3_{[n+m'][\ell+n'][m+\ell']}$ are the same, such a triplet can be decomposed into singlets as

$$ (x_a, x_b, x_c)_{3_{[\ell][m][n]}} = (x_a + x_b + x_c)_{1^k} + (x_a + x_b + x_c)_{1^k} + (x_a + x_b + x_c)_{1^k}. \quad (14) $$

The tensor product between singlet and triplet is obtained as

$$ \begin{pmatrix} x_l \\ x_m \\ x_n \end{pmatrix} \times \begin{pmatrix} x_l y \\ \omega x_m y \\ \omega^2 x_n y \end{pmatrix} = \begin{pmatrix} x_l y \\ \omega x_m y \\ \omega^2 x_n y \end{pmatrix}. \quad (15) $$

The tensor product between singlets is obtained as

$$ (x)_{1^k} \times (y)_{1^{k'}} = (xy)_{1^{k+k'}}. \quad (16) $$

### 3 Lepton mass and mixing

We use the group $(Z_N \times Z_N \times Z_N) \rtimes Z_3$ as the lepton flavor symmetry. We consider the case with $N/4 = \text{integer}$. We also assume the additional $Z_4$ flavor symmetry. We assign the quantum numbers of the three families of lepton doublets $(\ell_c, \ell_\mu, \ell_\tau)$ and lepton singlets $(e^c, \mu^c, \tau^c)$ and the Higgs fields $H_{u,d}$ under the flavor symmetry as shown in Table 2. These denote chiral superfields. In addition, we introduce the flavon fields, $\chi_0^\nu$, $(\chi_{1\nu}^\nu, \chi_{2\nu}^\nu, \chi_{3\nu}^\nu)$ and $(\chi_{1\nu}^\nu, \chi_{2\nu}^\nu, \chi_{3\nu}^\nu)$, which also denote chiral superfields. Their quantum numbers under $(Z_N \times Z_N \times Z_N) \rtimes Z_3$ and $Z_4$ are also shown in Table 2.

| $(Z_N^3) \rtimes Z_3$ | $e^c$ | $\mu^c$ | $\tau^c$ | $(\ell_c, \ell_\mu, \ell_\tau)$ | $H_{u,d}$ | $\chi_0^\nu$ | $(\chi_{1\nu}^\nu, \chi_{2\nu}^\nu, \chi_{3\nu}^\nu)$ | $(\chi_{1\nu}^\nu, \chi_{2\nu}^\nu, \chi_{3\nu}^\nu)$ |
|-----------------------|------|--------|--------|------------------|----------|-----------|----------------|------------------|
| $Z_4$                 | $1_{3N/4+p}$ | $1_{3N/4+q}$ | $1_{3N/4}$ | $3_{[3N/4][N/4][N/4]}$ | $1^0_{N/2}$ | $1^0_{N/2}$ | $3_{[N/2][0][0]}$ | $3_{[N-1][0][0]}$ |

Table 2: Quantum numbers of leptons, Higgs fields and flavon fields under $(Z_N \times Z_N \times Z_N) \rtimes Z_3$ and $Z_4$. Parameters $p$ and $q$ of $(Z_N \times Z_N \times Z_N) \rtimes Z_3$ charges are arbitrary integers.

The terms relevant to charged lepton masses in the superpotential are obtained as

$$ w_e = y^e e^c (\ell_c (\chi_1^\nu)^{N/2} + \ell_\mu (\chi_2^\nu)^{N/2} + \ell_\tau (\chi_3^\nu)^{N/2}) H_d (\chi_1^\nu \chi_2^\nu \chi_3^\nu)^p / \Lambda^{N/2+3p} $$

$$ + y^\mu \mu^c (\ell_c (\chi_1^\nu)^{N/2} + \ell_\mu (\chi_2^\nu)^{N/2} + \ell_\tau (\chi_3^\nu)^{N/2}) H_d (\chi_1^\nu \chi_2^\nu \chi_3^\nu)^q / \Lambda^{N/2+3q} $$

$$ + y^\tau \tau^c (\ell_c (\chi_1^\nu)^{N/2} + \ell_\mu (\chi_2^\nu)^{N/2} + \ell_\tau (\chi_3^\nu)^{N/2}) H_d / \Lambda^{N/2}. \quad (17) $$
where $\Lambda$ is the cut-off scale and $y^e, y^\mu, y^\tau$ are dimensionless couplings. Similarly, the terms relevant to neutrino masses in the superpotential are obtained as

\begin{equation}
\begin{aligned}
w_\nu = y_1^\nu (\ell_\nu \ell_\nu + \ell_\nu \mu + \ell_\tau \ell_\nu) H_u H_u \chi_0^\nu / \Lambda^2 \\
+ y_2^\nu (\ell_\nu \ell_\mu \chi_3 + \ell_\nu \ell_\tau \chi_2 + \ell_\mu \ell_\tau \chi_1) H_u H_u / \Lambda^2,
\end{aligned}
\end{equation}

where $y_{1,2}^\nu$ are dimensionless couplings.

We assume that all scalar fields develop VEVs which are written by

\begin{equation}
\begin{aligned}
\langle H_{u,d} \rangle = v_{u,d}, \quad \langle \chi_0^\nu \rangle = \alpha_\nu \Lambda, \quad \langle (\chi_1^\nu, \chi_2^\nu, \chi_3^\nu) \rangle = (\alpha_1^\nu, \alpha_2^\nu, \alpha_3^\nu) \Lambda.
\end{aligned}
\end{equation}

Furthermore, as the vacuum alignment, we assume

\begin{equation}
\begin{aligned}
\langle (\chi_1^\nu, \chi_2^\nu, \chi_3^\nu) \rangle = \alpha_\nu^r (1, 0, 0) \Lambda, \quad \langle (\chi_1^\nu, \chi_2^\nu, \chi_3^\nu) \rangle = \alpha_\nu (1, \rho, \rho') \Lambda,
\end{aligned}
\end{equation}

where $\rho$ and $\rho'$ are phases of $Z_N^r$ so that we write $\rho = e^{2\pi m/N}$, $\rho' = e^{2\pi n/N}$ with integers $m$ and $n$.

Taking the vacuum alignment, mass matrices of charged leptons and neutrinos, $M_e$ and $M_\nu$, are given by

\begin{equation}
\begin{aligned}
M_e &= v_d^{N/2} \begin{pmatrix}
y^e \alpha_\nu^3 (\rho \rho')^p & 0 & 0 \\
0 & y^\mu \alpha_\nu^3 (\rho \rho')^q & 0 \\
0 & 0 & y^\tau
\end{pmatrix} \begin{pmatrix}
1 & \omega^2 & 1 \\
1 & \omega & 1 \\
1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & \rho'
\end{pmatrix}, \\
M_\nu &= \frac{v_u^2}{\Lambda} \begin{pmatrix}
y_1^\nu \alpha_\nu & 0 & 0 \\
0 & y_1^\nu \alpha_\nu & y_2^\nu \alpha_\nu \\
0 & y_2^\nu \alpha_\nu & y_3^\nu \alpha_\nu
\end{pmatrix}.
\end{aligned}
\end{equation}

Except additional phases $\rho$ and $\rho'$, the mass matrices are the same as those of the $A_4$ model [38]. They can be easily diagonalized. The masses of charged lepton are

\begin{equation}
m_e = y^e \alpha_\nu^{N/2+3p} v_d, \quad m_\mu = y^\mu \alpha_\nu^{N/2+3q} v_d, \quad m_\mu = y^\tau \alpha_\nu^{N/2} v_d.
\end{equation}

Taking $p \approx 2q$, we can realize the mass hierarchy of charged leptons when the Yukawa couplings, $y^e, y^\mu$ and $y^\tau$, are of the same order each other. Suppose that Yukawa couplings are of $O(1)$. Then, the value of $3p$ should be taken as the same magnitude of $N$.

The mixing matrix becomes

\begin{equation}
\begin{aligned}
U_{\text{MNS}} &= \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega \\
1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & \rho^{-1} & 0 \\
0 & 0 & \rho'^{-1}
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1/\sqrt{2} & -1/\sqrt{2} \\
0 & 1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix} \\
&= \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & \omega (\rho + \omega^2 \rho')/\sqrt{2} \rho' & \omega^2 (\rho - \omega^2 \rho')/\sqrt{2} \rho' \\
1 & \omega (\rho + \omega \rho')/\sqrt{2} \rho' & \omega (\rho - \omega \rho')/\sqrt{2} \rho' \\
1 & (\rho + \rho')/\sqrt{2} \rho' & (\rho - \rho')/\sqrt{2} \rho'
\end{pmatrix}.
\end{aligned}
\end{equation}

Assuming $\rho \approx \omega^2 \rho'$, the (1, 3) element can be small enough to be fitted to experiments. Then we obtain

\begin{equation}
\begin{aligned}
|m_1| &= |y_1^\nu \alpha_\nu + y_2^\nu \alpha_\nu | v_u^2 / \Lambda, \\
|m_2| &= |y_1^\nu \alpha_\nu v_u^2 / \Lambda, \quad |m_3| = |y_1^\nu \alpha_\nu - y_2^\nu \alpha_\nu | v_u^2 / \Lambda.
\end{aligned}
\end{equation}
To explain the neutrino mass difference, we need

\[ 2|y_1^\nu \alpha_\nu||y_2^\nu \alpha_\nu'| (\cos \Delta) \frac{v^4_w}{\Lambda^2} = -\frac{1}{2} \Delta m_{\nu}^2, \quad |y_2^\nu \alpha_\nu'|^2 \frac{v^4_w}{\Lambda^2} = \frac{1}{2} \Delta m_{\nu}^2 - \Delta m_{\nu'}^2, \]  

(25)

where \( \Delta \) is the phase difference between \( y_1^\nu \alpha_\nu \) and \( y_2^\nu \alpha_\nu' \). Since experiments show \( |\Delta m_{\nu}^2|/2 > \Delta m_{\nu'}^2 \), the second equation gives \( \Delta m_{\nu}^2 > 0 \), i.e., the normal mass hierarchy. Using them, we obtain the neutrino mass spectrum as

\[ |m_1|^2 = \frac{\Delta m_{\nu}^2}{8(1 - 2r) \cos^2 \Delta} - \Delta m_{\nu'}^2, \quad |m_2|^2 = \frac{\Delta m_{\nu}^2}{8(1 - 2r) \cos^2 \Delta}, \]  

\[ |m_3|^2 = \frac{\Delta m_{\nu}^2}{8(1 - 2r) \cos^2 \Delta} + \Delta m_{\nu}^2 - \Delta m_{\nu'}^2, \]  

(26)

where \( r = \Delta m_{\nu'}^2 / \Delta m_{\nu}^2 \). Inserting the best-fit values for \( \Delta m_{\nu}^2 \) and \( \Delta m_{\nu}^2 \) and \( \cos^2 \Delta = 1 \), we obtain \( |m_1| = 16\text{meV}, |m_2| = 18\text{meV}, \) and \( |m_3| = 52\text{meV} \). The smallest value of sum of neutrino masses is 86meV.

In the usual notation, mixing angles can be expressed by

\[ \sin^2 \theta_{13} = \frac{2 + \cos(2\pi(m - n)/N) + \sqrt{3}\sin(2\pi(m - n)/N)}{6}, \]

\[ \sin^2 \theta_{12} = \frac{4 - \cos(2\pi(m - n)/N) - \sqrt{3}\sin(2\pi(m - n)/N)}{2}, \]

\[ \sin^2 \theta_{23} = 1 - \frac{4\sin^2(\pi(m - n)/N)}{4 - \cos(2\pi(m - n)/N) - \sqrt{3}\sin(2\pi(m - n)/N)}. \]

(27)

The Dirac CP phase is always vanishing. When \( N = 8 \), the smallest \( |U_{e3}| \) is obtained as

\[ |U_{e3}| = \sqrt{(2 - \sqrt{2 + \sqrt{3}})/6}, \]

(28)

which is about 0.107. In this case, we find\(^2\)

\[ \sin \theta_{12} = \frac{2}{\sqrt{8 + \sqrt{2 + \sqrt{6}}}}, \quad \sin \theta_{23} = \frac{\sqrt{5 - 3\sqrt{2} - \sqrt{3} + \sqrt{6}}}{\sqrt{7 - 3\sqrt{2} - \sqrt{3} + \sqrt{6}}}. \]

(29)

For \( N = 12 \), \( U_{e3} \) can be vanishing and mixing matrix becomes tri-bimaximal mixing.

We have calculated possible mixing angles which can be consistent in experiments up to \( N = 40 \) in Fig. 1. By fixing \( \rho \) and \( \rho' \), we can predict mixing angles. In Fig. 2, we estimate the effective mass of double beta decay \( m_{ee} = |m_1 U_{e1}^2 + m_2 U_{e2}^2 + m_3 U_{e3}^2| \). Due to Majorana phases and uncertainty of neutrino mass differences, the effective mass is predicted in continuous line with some fluctuation. For \( N = 8 \), the minimum value of the mass of double beta decay is predicted about 5.3 meV, and for other \( N \) it is about 4.6 meV.

\(^2\)These mixing pattern is obtained by another approach \([34, 36]\).
Figure 1: The left figure shows the relation between $N$ and $|U_{e3}|$ for possible values of $\rho$ and $\rho'$. The right figure shows the values of $\sin^2 \theta_{12}$, $\sin^2 \theta_{13}$, and $\sin^2 \theta_{23}$ for the region $0 \leq N \leq 40$ in the case $N/4$ is integer. Continuous line is the correlation of mixing angles in the limit of $N \to \infty$.

Figure 2: The left figure shows the relation between minimum neutrino mass and effective mass of double beta decay for $N = 8$. The right figure shows the minimum value of the effective mass of double beta decay for each $N$.

4 Vacuum alignment

Here, we discuss the vacuum alignment of flavon VEVs. We study it in two ways, one is to use the potential analysis in four-dimensional field theory and the other is to use boundary conditions in extra dimensions [39, 42].

4.1 Potential analysis

Here, we study the vacuum alignment of flavon VEVs by using four-dimensional potential analysis. The superpotential terms of flavon fields $\chi_i^\nu$ ($i = 0, 1, 2, 3$) up to four dimensional operators are

$$w_f = M_1 (\chi_0^\nu)^2 + M_2 ((\chi_1^\nu)^2 + (\chi_2^\nu)^2 + (\chi_3^\nu)^2) + \lambda_1 (\chi_0^\nu)^4 + \lambda_2 (\chi_0^\nu)^2 ((\chi_1^\nu)^2 + (\chi_2^\nu)^2 + (\chi_3^\nu)^2)
+ \lambda_3 (\chi_1^\nu)^2 + \omega (\chi_2^\nu)^2 + \omega^2 (\chi_3^\nu)^2) ((\chi_1^\nu)^2 + (\chi_2^\nu)^2 + (\chi_3^\nu)^2)
+ \lambda_4 ((\chi_1^\nu)^2 + (\chi_2^\nu)^2 + (\chi_3^\nu)^2) ((\chi_1^\nu)^2 + (\chi_2^\nu)^2 + (\chi_3^\nu)^2)
+ \lambda_5 ((\chi_1^\nu)^2 + (\chi_2^\nu)^2 + (\chi_3^\nu)^2) + \lambda_6 ((\chi_1^\nu)^4 + (\chi_2^\nu)^4 + (\chi_3^\nu)^4).$$

(30)
where \( \lambda_i = \bar{\lambda}_i / \Lambda \) \((i = 1, 2, \cdots, 6)\) with dimensionless coupling constants \( \bar{\lambda}_i \). The conditions of the potential minimum are written as

\[
\begin{align*}
2M_1 \lambda_0'' + 4\lambda_1 (\lambda_0'' + (\lambda_1')^2 + (\lambda_2')^2 + (\lambda_3')^2) &= 0, \\
2M_2 \lambda_1'' + 2\lambda_2 (\lambda_2'' + 2\lambda_3 \chi_0 (2(\lambda_1'')^2 - (\lambda_2')^2 - (\lambda_3')^2)) \\
+ 4\lambda_4 \chi_1 ((\lambda_1'')^2 + (\lambda_2')^2 + (\lambda_3')^2) + 2\lambda_5 \chi_0 ((\chi_0')^2 + (\lambda_3'^2) + 4\lambda_6 (\lambda_1')^3 &= 0, \\
2M_2 \chi_2'' + 2\lambda_2 (\chi_2'' + 2\lambda_3 \chi_0 (\chi_0'')^2 + 2\lambda_4 \chi_0 (-\chi_0'' + 2(\chi_2')^2 - (\chi_3')^2)) \\
+ 4\lambda_4 \chi_2 ((\chi_0'')^2 + (\chi_2')^2 + (\chi_3')^2) + 2\lambda_5 \chi_0 ((\chi_2')^2 + (\chi_3')^2) + 4\lambda_6 (\chi_2')^3 &= 0, \\
2M_2 \chi_3'' + 2\lambda_2 (\chi_3'' + 2\lambda_3 \chi_0 (-\chi_0'' - (\chi_2')^2 + (\chi_3')^2)) \\
+ 4\lambda_4 \chi_3 ((\chi_0'')^2 + (\chi_2')^2 + (\chi_3')^2) + 2\lambda_5 \chi_0 ((\chi_2')^2 + (\chi_3')^2) + 4\lambda_6 (\chi_3')^3 &= 0.
\end{align*}
\] (31)

There is a solution we have used as the vacuum alignment

\[
\begin{align*}
\chi_0'' &= \sqrt{\frac{\lambda_2 M_2 - 2(\lambda_3 + \lambda_4 + \lambda_6) M_1}{4\lambda_1 (\lambda_3 + \lambda_4 + \lambda_6) - \lambda_2'^2}}, \\
\chi_1'' &= \sqrt{\frac{\lambda_2 M_1 - 2\lambda_1 M_2}{4\lambda_1 (\lambda_3 + \lambda_4) - \lambda_2'^2}}, \quad \chi_2'' = 0, \quad \chi_3'' = 0.
\end{align*}
\] (32)

The leading and next leading order terms of \( \chi_i'' \) \((i = 1, 2, 3)\) in the superpotential are obtained as

\[
\begin{align*}
w'' &= m'((\chi_1')^N + (\chi_2')^N + (\chi_3')^N) \\
&\quad + \chi_1'((\chi_1')^N + \omega (\chi_2')^N + \omega^2 (\chi_3')^N)((\chi_1')^N + \omega^2 (\chi_2')^N + \omega (\chi_3')^N) \\
&\quad + \chi_2'((\chi_1')^N + (\chi_2')^N + (\chi_3')^N)((\chi_1')^N + (\chi_2')^N + (\chi_3')^N) \\
&\quad + \chi_3'((\chi_2')^N + (\chi_3')^N + (\chi_1')^N) + \chi_4'((\chi_1')^{2N} + (\chi_2')^{2N} + (\chi_3')^{2N}),
\end{align*}
\] (33)

where \( m' = m' / \Lambda^{N-3} \) and \( \lambda_i' = \bar{\lambda}_i / \Lambda^{2N-3} \) \((i = 1, 2, 3, 4)\) with dimensionless constants \( \bar{m}' \) and \( \bar{\lambda}_i' \). When \( \chi_1' \neq 0, \chi_2' \neq 0, \) and \( \chi_3' \neq 0, \) we have

\[
\begin{align*}
m' + \chi_1' (2(\chi_1')^N - (\chi_2')^N - (\chi_3')^N) + 2\lambda_2' (\chi_1')^N + (\chi_2')^N + (\chi_3')^N &= 0, \\
+ \chi_3' (\chi_2')^N + (\chi_3')^N + 2\lambda_4' (\chi_1')^N &= 0, \\
m' + \chi_1' (-\chi_1')^N + 2(\chi_2')^N - (\chi_3')^N + 2\lambda_2' (\chi_1')^N + (\chi_2')^N + (\chi_3')^N &= 0, \\
+ \chi_3' (\chi_1')^N + (\chi_2')^N + 2\lambda_4' (\chi_1')^N &= 0, \\
m' + \chi_1' (-\chi_1')^N - (\chi_2')^N + 2(\chi_3')^N + 2\lambda_2' (\chi_1')^N + (\chi_2')^N + (\chi_3')^N &= 0, \\
+ \chi_3' (\chi_1')^N + (\chi_2')^N + 2\lambda_4' (\chi_3')^N &= 0.
\end{align*}
\] (34)

Moreover, when \( 3\lambda_1' - \lambda_3' + 2\lambda_4' \neq 0, \) we have the relations

\[
(\chi_1')^N = (\chi_2')^N = (\chi_3')^N = -\frac{m'}{6\lambda_2' + 2\lambda_3' + 2\lambda_4'}. \quad (35)
\]

Then we can realize the vacuum alignment \( \chi_2 = \chi_1 \rho \) and \( \chi_3 = \chi_1 \rho' \).
There are the cross terms between $\chi^\nu$ and $\chi^\ell$. For instance, if $(\chi^\ell_i/\Lambda)^{N/2} \sim \chi^\nu_0 \sim \chi^\nu_1$, the most effective terms are

$$\Delta w_\ell = (\lambda_1^\nu(x^\nu_0)^2 + \lambda_2^\nu((x^\nu_1)^2 + (x^\nu_2)^2) + ((x^\nu_1)^N + (x^\nu_2)^N + (x^\nu_3)^N))$$

$$+ \lambda_3^\nu(x^\nu_0(x^\nu_2x^\nu_3)^{N/2} + x^\nu_1(x^\nu_2x^\nu_3)^{N/2} + x^\nu_1(x^\nu_2x^\nu_3)^{N/2})$$

$$+ \lambda_4^\nu(x^\nu_2x^\nu_3(x^\nu_1x^\nu_2)^{N/2} + x^\nu_1x^\nu_3(x^\nu_1x^\nu_2)^{N/2} + x^\nu_2x^\nu_3(x^\nu_1x^\nu_2)^{N/2}) \quad (36)$$

These terms would violate the vacuum alignment that we obtain above. We need to assume these cross terms should be suppressed sufficiently in this system.

### 4.2 Vacuum alignment on orbifold

Here, we discuss another way to realize the vacuum alignment for $(\chi^\nu_1, \chi^\nu_2, \chi^\nu_3)$ and $(\chi^\ell_1, \chi^\ell_2, \chi^\ell_3)$, using extra dimensional field theory. We study the flavor symmetry breaking by boundary conditions on the orbifold.\footnote{The orbifold models are also interesting as the origin of non-Abelian discrete flavor symmetries.}

We consider eight-dimensional field theory on the $T^2/Z_2 \times T^2/Z_3$ orbifold. The fifth and sixth dimensions, $(x^5, x^6)$, are compactified on the $T^2/Z_2$ orbifold and the seventh and eighth dimensions, $(x^7, x^8)$, are compactified on the $T^2/Z_3$ orbifold. It is useful to denote the extra-dimensional coordinates $(x^5, x^6, x^7, x^8)$ by a complex space $z = x^5 + ix^6$ and $z' = x^7 + ix^8$.

We assume that $(\chi^\nu_1, \chi^\nu_2, \chi^\nu_3)$ and $(\chi^\ell_1, \chi^\ell_2, \chi^\ell_3)$ live on $T^2/Z_2$ and $T^2/Z_3$, respectively. We have to fix the boundary conditions of these fields under the $Z_2$ twist $P$ and the $Z_3$ twist $R$, i.e.

$$\begin{pmatrix} \chi^\nu_1(-z) \\ \chi^\nu_2(-z) \\ \chi^\nu_3(-z) \end{pmatrix} = P \begin{pmatrix} \chi^\nu_1(z) \\ \chi^\nu_2(z) \\ \chi^\nu_3(z) \end{pmatrix}, \quad \begin{pmatrix} \chi^\ell_1(\omega z') \\ \chi^\ell_2(\omega z') \\ \chi^\ell_3(\omega z') \end{pmatrix} = P \begin{pmatrix} \chi^\ell_1(z') \\ \chi^\ell_2(z') \\ \chi^\ell_3(z') \end{pmatrix}, \quad (37)$$

where $\omega = e^{2\pi i/3}$.

When we take

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (38)$$

only the component $\chi^\nu_1$ has the zero-mode. Furthermore, when

$$R = \begin{pmatrix} 0 & \rho^\ell & 0 \\ 0 & 0 & \rho^m \\ \rho^n & 0 & 0 \end{pmatrix}, \quad (39)$$

where $\ell + m + n = 0 \mod N$, only the direction $(\chi^\ell_1, \chi^\ell_2, \chi^\ell_3) = \chi^\ell_1(1, \rho^i, \rho^j)$, where $i, j = 0, 1, \ldots, N-1$, has the zero-mode. If these zero modes develop their VEVs in four-dimensional effective field theory, the vacuum alignment can be realized.
5 Higher order corrections

Let us consider higher dimensional terms for charged leptons. We do not consider terms with \((\chi_1^\nu)^2\) or \((\chi_1^\nu, \chi_2^\nu, \chi_3^\nu)^2\) because it just changes the mass eigenvalues. Next-to-leading terms which modify eigenvectors are
\[
\Delta w_e = c^e(\ell_e, \ell_\mu, \ell_\tau) \mathcal{H}_d(\Delta y_1^0\chi_0^\nu(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu) + \Delta y_2^0(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu))^{N/2+\alpha}/\Lambda^{N/2+3\alpha+2} \\
+ \mu^e(\ell_e, \ell_\mu, \ell_\tau) \mathcal{H}_d(\Delta y_1^0\chi_0^\nu(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu) + \Delta y_2^0(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu))^{N/2+b}/\Lambda^{N/2+3b+2} \\
+ \tau^e(\ell_e, \ell_\mu, \ell_\tau) \mathcal{H}_d(\Delta y_1^0\chi_0^\nu(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu) + \Delta y_2^0(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu))^{N/2}/\Lambda^{N/2+2}.
\]

Compared to the leading order, corrections are of \(\mathcal{O}(\alpha_e^N\alpha_e^2)\).

Similarly, correction terms for neutrinos are
\[
\Delta w_\nu = (\ell_e, \ell_\mu, \ell_\tau)(\Delta y_1^0(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu) + \Delta y_2^0(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu))^{N/2+\alpha}/\Lambda^{N/2+3\alpha+2} \\
+ \mu_\nu(\ell_e, \ell_\mu, \ell_\tau) \mathcal{H}_d(\Delta y_1^0\chi_0^\nu(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu) + \Delta y_2^0(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu))^{N/2+b}/\Lambda^{N/2+3b+2} \\
+ \tau_\nu(\ell_e, \ell_\mu, \ell_\tau) \mathcal{H}_d(\Delta y_1^0\chi_0^\nu(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu) + \Delta y_2^0(\chi_1^\nu, \chi_2^\nu, \chi_3^\nu))^{N/2}/\Lambda^{N/2+2}.
\]

Corrections are of \(\mathcal{O}(\alpha_e^N)\) to the leading terms. This can be estimated as \((m_\tau/y^\tau\nu_\mu)^2\). If \(\tan\beta\) is not large, the corrections can be suppressed.

6 Conclusion

We have studied the models with the lepton flavor symmetry \((Z_N \times Z_N \times Z_N) \times Z_3\) in order to extend the tri-bimaximal mixing. The tri-bimaximal mixing can be derived by residual \(Z_3\) and \(Z_2\) symmetries for charged lepton and neutrino mass matrices, respectively. We extend the symmetry of charged leptons to \(Z_3\) and \(Z_N\) symmetries, then non-vanishing \(\theta_{13}\) can be obtained. For example, when we choose \(N = 8\), our model predicts \(\sin \theta_{13} \approx 0.107\), \(\sin \theta_{12} \approx 0.337\), \(\sin \theta_{23} \approx 0.651\), and \(\delta_{CP} = 0\). For larger values of \(N\), we predict other sets of mixing angles, but in any case, our model predicts \(\delta_{CP} = 0\).

We have also predicted the mass spectrum of neutrinos. Three neutrino masses are expressed by two complex parameters, so that they are correlated. In our model, only normal mass hierarchy is allowed. To compare with future experiments, the relation of effective mass of double beta decay and minimum neutrino mass are calculated.

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