THE RELATIVISTIC PARTICLE WITH CURVATURE AND TORSION OF WORLD
TRAJECTORY

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Abstract

Local symmetries of the action for a relativistic particle with curvature and torsion
of its world curve in the (2+1)-dimensional space-time are studied. With the help of the
method, worked out recently by the authors (Phys.Rev., D56, 1135, 1142 (1997)), first
the local-symmetry transformations are obtained both in the phase and configuration
space. At the classical level, the dependence of the particle mass on the parameters
of curvature and torsion and the Regge trajectory are obtained. It is shown that the
tachyonic sector can be removed by a proper gauge choice.

1 Introduction

In recent years, interest in the point relativistic particle has been revived by investigation
of the action depending on the curvature and/or torsion of the world trajectory of that
particle (see, e.g. [1] and refs. therein). This relates, on the one hand, to the development
of the string theory: the above-mentioned object can be considered as a one-dimensional
analog of the relativistic string with rigidity [2]. On the other hand, such models seem to
continue realizing an old idea of modelling the particle spin without introducing additional
spin variables [3]. Note also a connection of the indicated models with the problem of
describing planar phenomena: the high-temperature superconductivity [4] and the fractional
quantum Hall effect [5].

The present work is devoted to an investigation of the action of a massive relativis-
tic particle with curvature and torsion in (2+1) dimensions. Here there are the first- and
second-class constraints, both primary and secondary (see, e.g. [1, 6]). Therefore, already
the initial stage of constructing the theory needs a complete separation of constraints of the
first and second class and finding gauge transformations. This can be made only with the
help of the methods, recently proposed in papers [7]-[9]. An explicit form of gauge trans-
formations is required, e.g., in eliminating unphysical degrees of freedom (i.e., a reduction
of the theory), carried out before or after the quantization (the schemes of covariant and
canonical quantization).

The layout of the paper is as follows. In Sec. 2, the system under investigation is consid-
ered in the framework of the generalized Hamiltonian formalism by Dirac for theories with
higher derivatives [10], and by the method, suggested in Ref.[7], we separate completely the
constraints of the first and second class and obtain a canonical set of constraints. In Sec.
3, we obtain gauge transformations for the considered system both in the phase and config-
uration space and the Noether identity. In Sec. 4, the dependence of the particle mass on
the parameters of curvature and torsion and the Regge trajectory are obtained. It is shown that the tachyonic sector can be removed by a proper gauge choice.

2 Generalized Hamilton formalism

So, let us consider a relativistic particle with curvature and torsion of its world curve in the (2+1)-dimensional space-time. Given a parametrization of the world trajectory $x^\mu(\tau) (\mu = 0,1,2)$, the action can be written in the form

$$S = -m \int d\tau \sqrt{\dot{x}^2} - \alpha \int d\tau \sqrt{\dot{x}^2} \frac{\varepsilon_{\mu\nu\sigma} \dot{x}^\mu \cdot \dot{x}^\nu \cdot \dot{x}^\sigma}{(\dot{x}^2)^2} - \beta \int d\tau \sqrt{\dot{x}^2} \varepsilon_{\mu\nu\sigma} \dot{x}^\mu \cdot \dot{x}^\nu \cdot \dot{x}^\sigma,$$

where $\dot{x} \equiv dx(\tau)/d\tau$, $\dot{x}^2 > 0$, dimensionless parameters $\alpha$ and $\beta$ characterize the contributions of curvature ($k$) and torsion ($\kappa$), respectively, $m$ is a parameter of the mass dimension, $\varepsilon_{\mu\nu\rho}$ is a completely antisymmetric unit tensor ($\varepsilon_{012} = 1$), the Lorentz metric with signature $(+,−,−)$ is used. In the following we shall use the notation

$$\varepsilon_{\mu\nu\rho} a^\mu b^\nu c^\rho = \varepsilon(abc), \quad \varepsilon_{\mu\nu\rho} a^\mu b^\nu = \varepsilon_\mu(ab).$$

We see that the action depends on the derivatives of the particle coordinates with respect to $\tau$ up to and including the third order. Therefore, when passing to the generalized Hamiltonian formalism for theories with higher derivatives, the canonical variables are to be introduced according to the Ostrogradsky method

$$q_1 = x, \quad q_2 = \dot{x}, \quad q_3 = \ddot{x},$$

$$p_1 = -\partial L/\partial \dot{x} - \dot{p}_2, \quad p_2 = -\partial L/\partial \dot{x} - \dot{p}_3, \quad p_3 = -\partial L/\partial \ddot{x},$$

where $L$ is the Lagrangian in 1. The canonical Hamiltonian has the form

$$H_c = -p_1 \dot{x} - p_2 \dot{x} - p_3 \ddot{x} - L = -p_1 q_2 - p_2 q_3 + m \sqrt{q_2^2} + \alpha \sqrt{g} \varepsilon_{\mu\nu\rho} q_2^\mu q_3^\nu q_3^\rho.$$  

The Poisson brackets (PB) are defined in the following way

$$\{A, B\} = \sum_{a=1}^{3} \left( \frac{\partial A}{\partial q_\alpha^a} \frac{\partial B}{\partial p_\mu^a} - \frac{\partial A}{\partial p_\mu^a} \frac{\partial B}{\partial q_\alpha^a} \right).$$

From the definition of the momentum $p_3$ three primary constraints are derived

$$p_{3\mu} - \beta \sqrt{\frac{q_2^2}{g}} \varepsilon_\mu (q_2 q_3) = 0, \quad \mu = 0, 1, 2,$$

where $g = (q_2 q_3)^2 - q_2^2 q_3^2$. Making use of the orthonormal basis in a three-dimensional space-time

$$n_1^\mu = \frac{q_2^\mu}{\sqrt{q_2^2}}, \quad n_2^\mu = \frac{\sqrt{q_2^2}}{g} \left( q_3^\mu - \frac{q_2 q_3}{q_2^2} q_2^\mu \right), \quad n_3^\mu = \frac{\varepsilon_\mu (q_2 q_3)}{\sqrt{g}},$$

$$2$$
and the completeness condition \( n_i^\mu n_i^\nu - n_i^\mu n_i^\nu = g^{\mu\nu} \), it is convenient to decompose the vector \( \mathbf{n} \) in this basis and to pass from three primary constraints \( \hat{\mathbf{n}} \) to the equivalent constraints (the projections of \( \mathbf{n} \) onto the vectors \( \mathbf{\hat{\mathbf{n}}} \))

\[
\begin{align*}
\phi_1^1 &= p_3 q_2, & \phi_2^1 &= p_3 q_3, & \phi_3^1 &= \varepsilon(p_3 q_2 q_3) + \beta \sqrt{q_2^2}.
\end{align*}
\]

Determinant of the matrix of the transformation to the constraint set \( \hat{\mathbf{n}} \) equals \(-g\). When investigating self-consistency of the theory, we apply the Dirac scheme for breeding constraints. From the conditions of the time conservation of constraints \( \dot{\phi}_\alpha^1 = \{\phi_\alpha^1, H_c + u_\alpha \phi_\alpha^1\} \stackrel{\Sigma_1}{=} 0 \) \((\alpha = 1, 2, 3; u_\alpha \) are the Lagrange multipliers; \( \Sigma_1 \) means this equality to hold on the surface of the primary constraints \( \Sigma_1 \)) we obtain the secondary constraints

\[
\begin{align*}
\phi_1^2 &= p_2 q_2, & \phi_2^2 &= p_2 q_3 - \alpha \frac{\sqrt{g}}{q_2^2}, & \phi_3^2 &= \varepsilon(p_2 q_2 q_3) - \beta \frac{q_2 q_3}{\sqrt{q_2^2}}.
\end{align*}
\]

From the conditions \( \dot{\phi}_\alpha^2 \stackrel{\Sigma_2}{=} 0 \) \((\alpha = 1, 2)\) two ternary constraints are obtained

\[
\begin{align*}
\phi_1^3 &= p_1 q_2 - m \sqrt{q_2^2}, & \phi_2^3 &= p_1 q_3 - m \frac{q_2 q_3}{\sqrt{q_2^2}},
\end{align*}
\]

and the equation \( \dot{\phi}_3^2 \stackrel{\Sigma_2}{=} 0 \) serves for determining the Lagrangian multiplier \( u_3 \)

\[
u_3 = \frac{1}{\alpha \sqrt{g}} \left[ \varepsilon(p_1 q_2 q_3) - \frac{\beta g}{(q_2^2)^{3/2}} \right].
\]

Further we obtain

\[
\dot{\phi}_3^3 = \phi_2^3
\]

and from \( \dot{\phi}_2^3 \stackrel{\Sigma_3}{=} 0 \) the quaternary constraint

\[
\phi_4^3 = \frac{\sqrt{g}}{\alpha} \left[ p_1^2 - m^2 + \sqrt{g} \frac{\alpha m + \beta \sigma \sqrt{m^2 - p_1^2}}{(q_2^2)^{3/2}} \right], \quad \sigma = \text{sign} \varepsilon(p_1 q_2 q_3),
\]

from the conservation (in time) conditions of which the Lagrangian multiplier \( u_2 \)

\[
u_2 = -3 \frac{q_2 q_3}{q_2^{3/2}}
\]

is determined. In eq. 13 it is taken into account that

\[
[\varepsilon(p_1 q_2 q_3)]^2 \stackrel{\Sigma_3}{=} g(m^2 - p_1^2)
\]

So, the complete set of constraints is obtained with the following PB among them:

\[
\begin{align*}
\{\phi_1^1, \phi_2^1\} &= \phi_1^1, & \{\phi_1^1, \phi_3^1\} &= 0, & \{\phi_2^1, \phi_3^1\} &= 0, \\
\{\phi_2^2, \phi_3^2\} &= \phi_1^2, & \{\phi_1^2, \phi_3^2\} &= 0, & \{\phi_2^2, \phi_3^2\} &= \phi_1^1, \\
\{\phi_3^1, \phi_1^1\} &= 0, & \{\phi_3^1, \phi_2^1\} &= 0, & \{\phi_3^1, \phi_3^1\} &= 0, & \{\phi_3^2, \phi_1^2\} &= -\phi_1^3, \\
\end{align*}
\]
\{\phi_1^1, \phi_2^1\} = -\phi_2^2, \quad \{\phi_1^3, \phi_3^1\} = -\varepsilon (p_1 q_2 g_3), \quad \{\phi_1^3, \phi_3^3\} = 0, \\
\{\phi_2^2, \phi_1^1\} = \phi_2^1 - \phi_2^2, \quad \{\phi_2^2, \phi_2^1\} = -\phi_2^2, \quad \{\phi_2^2, \phi_3^1\} = -\phi_2^3, \\
\{\phi_2^3, \phi_1^1\} = -\phi_3^1, \quad \{\phi_2^3, \phi_2^1\} = -\phi_3^2, \quad \{\phi_2^3, \phi_3^1\} = -\varepsilon (p_1 q_2 g_3), \\
\{\phi_2^4, \phi_1^1\} = 0, \quad \{\phi_2^4, \phi_2^1\} = -\phi_2^4 + \frac{g}{(q_2^2)^{3/2}} \left( m + \frac{\beta}{\alpha} \sigma \sqrt{m^2 - p_1^4} \right), \\
\{\phi_2^5, \phi_3^1\} = 0, \quad \{\phi_2^5, \phi_2^1\} = -\phi_2^5 + 2 \frac{g}{(q_2^2)^{3/2}} \left( m + \frac{\beta}{\alpha} \sigma \sqrt{m^2 - p_1^2} \right), \\
\{\phi_3^2, \phi_3^1\} = 0, \quad \{\phi_3^2, \phi_3^1\} = 0, \quad \{\phi_3^2, \phi_3^2\} = 0, \\
\{\phi_3^3, \phi_3^1\} = \phi_3^3, \quad \{\phi_3^3, \phi_3^2\} = -\phi_3^3, \\
\{\phi_3^4, \phi_3^1\} = \alpha \sqrt{\gamma} - q_3^2 \phi_3^1 + q_2^2 \phi_2^1 + q_2 g_3 (\phi_2^1 - \phi_1^2), \quad \{\phi_2^5, \phi_3^3\} = \frac{\beta g}{(q_2^2)^{3/2}}.

We see that from three chains of the constraints the two ones are candidates for second-class constraints. Making use of the method developed in Ref. [7], let us separate the constraints into the first- and second-class ones and pass to an equivalent canonical set of constraints. For this purpose, we perform the equivalence transformation to obtain the desired structure of the second-class constraints of the canonical set and then completely to separate the first-class constraints:

$$\Psi_2^2 = \phi_2^2 + a_1 \phi_1^1 + a_2 \phi_3^3, \quad \Psi_2^3 = \phi_2^3 + a_3 \phi_3^3,$$

where the coefficients $a_1, a_2$ and $a_3$ are determined to fulfil the requirements

$$\{\Psi_2^2, \Psi_3^3\} \overset{\Sigma}{=} 0, \quad \{\Psi_2^2, \Psi_2^3\} \overset{\Sigma}{=} 0, \quad \{\Psi_2^3, \Psi_3^1\} \overset{\Sigma}{=} 0$$

($\overset{\Sigma}{=} 0$ means this equality to hold on the surface of all constraints $\Sigma$). Now we carry out the following equivalence transformation

$$\Phi_1^1 = \phi_1^1 + b_1 \phi_2^1, \quad \Phi_3^3 = \phi_1^3 + b_2 \phi_3^1 + b_3 \Psi_2^2$$

with coefficients $b_1, b_2$ and $b_3$, determined to fulfil the requirements

$$\{\Phi_2^2, \Psi_2^1\} \overset{\Sigma}{=} 0, \quad \{\Phi_2^3, \Psi_2^3\} \overset{\Sigma}{=} 0, \quad \{\Phi_2^3, \Psi_3^1\} \overset{\Sigma}{=} 0.$$

The obtained set of constraints

$$\Phi_1^1 = \phi_1^1, \quad \Phi_1^3 = \phi_1^3 + 2 \phi_1^1, \\
\Phi_3^1 = \phi_3^1 + 3 \frac{q_2 q_3}{q_2^2} \phi_2^1 + \left[ \frac{\beta}{\alpha} \sqrt{\gamma} - \frac{\varepsilon (p_1 q_2 g_3)}{\alpha \sqrt{\gamma}} \right] \phi_3^1 + \phi_2^1, \\
\Psi_2^1 = \phi_2^1, \quad \Psi_3^2 = \phi_2^2 + 3 \frac{q_2 q_3}{q_2^2} \phi_2^1 + \frac{\beta}{\alpha} \sqrt{\gamma} \phi_3^1,$$

$$\Psi_2^3 = \phi_2^3 + \frac{\varepsilon (p_1 q_2 g_3)}{\alpha \sqrt{\gamma}} \phi_3^1, \quad \Psi_2^4 = \phi_2^4, \quad \Psi_3^1 = \phi_3^1, \quad \Psi_3^3 = \phi_3^2$$

(16)
is canonical, because the only non-vanishing (on $\Sigma$) PB among them are

$$\{\Psi_2^1, \Psi_4^2\} \equiv \{\Psi_2^2, \Psi_2^2\} \equiv \frac{g}{(q_2^2)^{3/2}} \left( m + \frac{\beta}{\alpha} \sqrt{m^2 - p_1^2} \right), \quad \{\Psi_3^1, \Psi_3^2\} = -\alpha \sqrt{g}, \quad (23)$$

i.e., on $\Sigma$ the maximal partition of the set of constraints is achieved: each second-class constraint of the final set has the vanishing (on the constraint surface) PB with all the constraints of the system except one, and the first-class constraints have the vanishing PB with all the constraints. Therefore, there are three constraints of the first class $[20]$ and two chains of the second-class constraints $[21], [22]$ (second-class constraints are denoted by the letter $\Psi$).

The total Hamiltonian has now the form

$$H_T = -\Phi_1^3 + u_1 \Phi_1^1, \quad (24)$$

i.e., is a function of the first-class. Note that the secondary constraints of the first class have the following PB with the primary constraint $\Phi_1^1$

$$\{\Phi_2^1, \Phi_1^1\} = -\Phi_1^1, \quad \{\Phi_3^1, \Phi_1^1\} = -3 \frac{g_2 g_3}{q_2^2} \Phi_1^1 - \Phi_2^1. \quad (25)$$

The second PB $[25]$ are not of the form needed for obtaining local-symmetry transformations – the right-hand side must be expressed only through $\Phi_1^1$ (first-class primary constraints are to be the ideal of quasi-algebra of all the first-class constraints $[8]$). Therefore, instead of $\Phi_3^1$ in $[20]$, the equivalent constraint will be used

$$\overline{\Phi}_1^3 = \Phi_1^3 - \frac{g_2 g_3}{q_2^2} \Phi_1^2, \quad (26)$$

for which

$$\{\overline{\Phi}_1^3, \Phi_1^1\} = -2 \frac{g_2 g_3}{q_2^2} \Phi_1^1. \quad (27)$$

### 3 Local-symmetry transformations

Now we can say that, according to the theorem proved in Ref. [8], the action $[1]$ is quasi-invariant under the one-parameter quasigroup of local-symmetry transformations. We shall derive these transformations by the method developed in Ref. [8, 9], where it has been proved that for the obtained canonical set of constraints the generator of local-symmetry transformations is to be the form

$$G = \varepsilon_1^1 \Phi_1^1 + \varepsilon_1^2 \Phi_1^2 + \varepsilon_1^3 \overline{\Phi}_1^3, \quad (28)$$

where the coefficients $\varepsilon_1^{m_1}$ ($m_1 = 1, 2, 3$) are the solution of the system of equations

$$\varepsilon_1^{m_1} + \varepsilon_1^{m_1'} g_1^{m_1 m_1'} = 0, \quad m_1' = m_1 - 1, \cdots, 3 \quad (29)$$

with the functions $g_1^{m_1 m_1'}$ determined from PB

$$\{\Phi_1^{m_1}, H\} = g_1^{m_1 m_1'} \Phi_1^{m_1'}, \quad m_1' = 1, \cdots, m_1 + 1 \quad (30)$$
(here $H = H_c + u_2\Psi_1^1 + u_3\Psi_3^1$ with $u_2$ and $u_3$ given by eqs. 14 and 12). As $g_{1}^{3,3} = q_{2}q_{3}/q_{2}^{2}$, $g_{1}^{3,1} = -1$, $g_{1}^{2,2} = -q_{2}^{2}/q_{2}^{2}$, $g_{1}^{2,1} = -q_{2}q_{3}/q_{2}^{2}$, $g_{1}^{1,2} = -1$, the system of equations \[29\] for determining $\varepsilon_{1}^{m_{1}}$ becomes

\[
\begin{align*}
\varepsilon_{1}^{3} + \frac{q_{2}q_{3}}{q_{2}^{2}}\varepsilon_{1}^{3} - \varepsilon_{1}^{2} &= 0, \\
\varepsilon_{1}^{2} - \frac{q_{1}^{2}}{q_{2}^{2}}\varepsilon_{1}^{3} - \frac{q_{2}q_{3}}{q_{2}^{2}}\varepsilon_{1}^{2} - \varepsilon_{1}^{1} &= 0.
\end{align*}
\]

Denoting $\varepsilon_{1}^{2} \equiv \lambda$, we obtain

\[
\lambda_{1}^{2} = \dot{\lambda} + \frac{q_{2}q_{3}}{q_{2}^{2}}\lambda_{1}^{3}, \quad \lambda_{1}^{1} = \ddot{\lambda} + \frac{\lambda}{q_{2}^{2}} \left[ q_{2}\dot{q}_{3} - \frac{q_{2}q_{3}}{q_{2}^{2}}(2q_{2}\dot{q}_{2} + q_{2}q_{3}) \right].
\]

We see that the quantity $G$ in \[28\] depends on $\dot{q}_{3}$; therefore, for the desired local-symmetry transformations being canonical it is necessary to extend the phase space in the following way:

Define the coordinates $Q_{i} = q_{i}$ ($i = 1, 2, 3$), $Q_{4} = \dot{q}_{3}$; then their conjugate momenta, calculated in accordance with the formula of theories with higher derivatives \[10\], are $P_{i} = p_{i}$ ($i = 1, 2, 3$), $P_{4} = 0$. The generalized momentum $P_{4}$ is an extra primary constraint of the first class. In the extended phase space the total Hamiltonian is written down as

\[
H_{T} = H_{T}(Q_{i}, P_{i}) + \bar{u}P_{4},
\]

(33)

where $H_{T}$ is of the same form as in the initial phase space \[24\] and $\bar{u}$ is an arbitrary function of the time. From \[33\] we may conclude that there do not appear additional secondary constraints corresponding to $P_{4}$. The set of constraints in the extended phase space remains the same as in the initial phase space, obeys the same algebra and does not depend on new coordinates and momenta as $H_{T}$ does.

We shall seek a generator $\overline{G}$ in the extended phase space in a form analogous to that in the initial phase space. Then from the requirements of quasi-invariance of the action

\[
\overline{S} = \int_{t_1}^{t_2} d\tau \left[ P_{1}Q_{2} + P_{2}Q_{3} + P_{3}Q_{4} + P_{4}\dot{Q}_{4} - H_{T} \right]
\]

(34)

under the transformations generated by $\overline{G}$, we obtain the same relations \[32\] for determining $\varepsilon_{1}^{m_{1}}$. As a result, we find the expression for $\overline{G}$:

\[
\overline{G} = \dot{\lambda}(P_{3}Q_{2}) + 2\lambda(P_{2}Q_{2} + 2P_{3}Q_{3}) + \lambda \left( P_{1}Q_{2} + P_{2}Q_{3} + 3P_{3}Q_{3} \frac{Q_{2}Q_{3}}{Q_{2}^{2}} \right) + \frac{P_{3}Q_{2}}{Q_{2}^{2}} \left[ Q_{2}Q_{4} - 3\left( \frac{Q_{2}Q_{3}}{Q_{2}^{2}} \right)^{2} \right] - m\sqrt{Q_{2}^{2}} - \alpha \sqrt{g} \frac{Q_{2}^{2}}{Q_{2}^{2}} \left[ \frac{\beta}{\alpha} \frac{\sqrt{g}}{(Q_{2}^{2})^{3/2}} - \frac{\varepsilon(P_{1}Q_{2}Q_{3})}{\alpha \sqrt{g}} \right] \left[ \varepsilon(P_{3}Q_{2}Q_{3}) + \beta \sqrt{Q_{2}^{2}} \right] + \lambda P_{4}.
\]

(35)

Note that the obtained generator \[33\] satisfies the group property

\[
\{ \overline{G}_{1}, \overline{G}_{2} \} = \overline{G}_{3},
\]

where the transformation $\overline{G}_{3}$ \[33\] is realized by carrying out two successive transformations $\overline{G}_{1}$ and $\overline{G}_{2}$ \[33\]. Now the local-symmetry transformations of the coordinates in the extended
Here the functions $f$ and $h$ are rather cumbersome. One can verify that to within quadratic terms in $\delta Q_1$ and $\delta P_j$

$$\{Q_i + \delta Q_i, P_j + \delta P_j\} = \delta_{ij},$$

i.e., the obtained infinitesimal gauge transformations are canonical in the extended (by Ostrogradsky) phase space.

We see from (37) that the vector $P_1^\mu$ is gauge-invariant, and it is conserved also in the time evolution.

The corresponding transformations of local symmetry in the Lagrangian formalism are determined in the following way:

$$\delta x(\tau) = \{q_1(\tau), G\} \left(q_1 = x, q_2 = \dot{x}, q_3 = \ddot{x}, p_3 = -\partial L / \partial \dot{x}\right), \quad \delta \dot{x}(\tau) = \frac{d}{d\tau} \delta x(\tau).$$

We obtain

$$\delta x^\mu = -\lambda \dot{x}^\mu.$$  

(40)

One can see that the Lagrangian \[1\] is quasi-invariant under these transformations. The corresponding Euler - Lagrange equation is

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{d^2}{d\tau^2} \frac{\partial L}{\partial \dot{x}^\mu} + \frac{d^3}{d\tau^3} \frac{\partial L}{\partial \dot{x}^\mu} = 0.$$  

(41)

Under the transformations \[39\], \[40\]

$$\delta L = \frac{\partial L}{\partial \dot{x}^\mu} \delta \dot{x}^\mu + \frac{\partial L}{\partial \ddot{x}^\mu} \delta \ddot{x}^\mu + \frac{\partial L}{\partial \dddot{x}^\mu} \delta \dddot{x}^\mu = \frac{\partial L}{\partial \dot{x}^\mu} \delta x^\mu +$$

$$\frac{d}{d\tau} \left[ \left( \frac{\partial L}{\partial \ddot{x}^\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} + \frac{d^2}{d\tau^2} \frac{\partial L}{\partial \dot{x}^\mu} \right) \delta \ddot{x}^\mu + \right.$$

$$\left. \left( \frac{\partial L}{\partial \dddot{x}^\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \ddot{x}^\mu} \right) \delta \dddot{x}^\mu + \frac{\partial L}{\partial \dddot{x}^\mu} \delta \dddot{x}^\mu \right].$$

(42)

We obtain the Noether identity

$$\left( \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{d^2}{d\tau^2} \frac{\partial L}{\partial \dot{x}^\mu} + \frac{d^3}{d\tau^3} \frac{\partial L}{\partial \dot{x}^\mu} \right) \dddot{x}^\mu = 0.$$  

(43)
Then we have $\delta L = -(d/d\tau)(\lambda L)$.

In particular, curvature ($k$) and torsion ($\kappa$) are transformed under the transformations $39$ and $40$ as

$$
\delta k = -\lambda \dot{k}, \quad \delta \kappa = -\lambda \dot{\kappa};
$$

therefore, their contributions to the action $S_1$ are quasi-invariant under $39$:

$$
\delta L[k] = \alpha \frac{d}{dt}(\lambda \sqrt{\dot{x}^2}k), \quad \delta L[\kappa] = \beta \frac{d}{dt}(\lambda \sqrt{\dot{x}^2}\kappa).
$$

(45)

### 4 Spin-mass-curvature relation

Consider some physical conclusions (first on the classical level) from the obtained set of constraints and the transformations of local symmetry. We have already seen that the integral of motion $P_1$ is gauge-invariant. From invariance of the action $34$ under translations we obtain that $P_1$ is the energy-momentum vector. From invariance of the action $34$ under rotations we obtain that the conserved angular momentum tensor is

$$
M_{\mu\nu} = \sum_{a=1}^{4}(Q_{a\mu}P_{a\nu} - P_{a\mu}Q_{a\nu}).
$$

(46)

This quantity is gauge-invariant in the initial phase space, i.e. on the surface $P_4 = 0$.

The quantities $P_1$ and $M_{\mu\nu}$ make up the Lie algebra with respect to PB:

$$
\{M_{\mu\nu}, P_{1\sigma}\} = -g_{\mu\sigma}P_{1\nu} + g_{\nu\sigma}P_{1\mu},
\{M_{\mu\nu}, M_{\rho\sigma}\} = g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma}.
$$

(47)

One can construct the Casimir operators $P_1^2 = M^2$ and $W$ of the Poincaré group, where $W$ is determined as

$$
W = -S^2 = \frac{1}{2}M_{\mu\nu}M^{\mu\nu}P_1^2 - M_{\mu\sigma}M^{\mu\sigma}P_1^4.
$$

(48)

with the particle spin defined in the case $d = 2 + 1$ by

$$
S = \frac{1}{2}|P_1^2|^{-1/2}\varepsilon_{\mu\nu\rho}P_1^\mu M^{\nu\rho}.
$$

(49)

Let us obtain the dependence of the particle mass $M$ on parameters of the curvature and torsion. To this end, we solve the equation of the constraint $\Psi_2^4$ with respect to $P_1^2$:

$$
P_1^2 = M_\pm^2 = m^2 - \left[\frac{\beta k}{2} \pm \sqrt{\frac{\beta^2 k^2}{4} + \alpha m k}\right]^2,
$$

(50)

where $k = \sqrt{g/(q_2^2)^3/2}$ is the curvature. At a given value of curvature $k$, the particle has the mass either $M_+$ or $M_-$. Since we can impose up to three gauge conditions

$$
\chi_\alpha(q, p, \tau) = 0, \quad \alpha = 1, 2, 3
$$

(51)
with the requirements

\[ \det||\{\chi_\alpha, \Phi^m_1}\}|| \neq 0, \quad m = 1, 2, 3, \quad (52) \]

\[ \frac{\partial \chi_\alpha}{\partial \tau} - \{\chi_\alpha, \Phi^3_1\} + v_m\{\chi_\alpha, \Phi^m_1\} \approx 0 \quad (53) \]

\( (v_m \) are determined from eqs.53 for constructing generalized Hamiltonian \( H_E \), the appearance of the tachyonic sector \( (M^2 < 0) \) can be prevented. Indeed, choosing

\[ \chi_1 = k + c q_3^2 = 0, \quad \chi_2 = q_2 q_3 = 0, \quad \chi_2 = q_1^0 - \tau = 0, \quad (54) \]

we obtain the determinant \( \text{52} \) to be equal to \(-4c q_2^0 q_2^2 q_3^2\). Then solving the inequality \( M^2 > 0 \), one can limit desired values \( c \).

For the relation between the mass of the particle and its spin we obtain the same two-branched expression as in Ref. [11]

\[ S = \pm \alpha \sqrt{\left(\frac{m}{M}\right)^2 - \beta \frac{m}{M}} \quad (\theta = \text{sign} \rho_1^2), \quad (55) \]

\( i.e., for a fixed mass there are two world curves corresponding to different spin values. \)

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