KNOT DIAGRAMS ON A PUNCTURED SPHERE AS A MODEL OF STRING FIGURES

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Abstract. A string figure is topologically a trivial knot lying on an imaginary plane orthogonal to the fingers with some crossings. The fingers prevent cancellation of these crossings. As a mathematical model of string figure we consider a knot diagram on the $xy$-plane in $xyz$-space missing some straight lines parallel to the $z$-axis. These straight lines correspond to fingers. We study minimal number of crossings of these knot diagrams under Reidemeister moves missing these lines.

1. Introduction

A string figure is topologically a trivial knot lying on an imaginary plane orthogonal to the fingers with some crossings. The fingers prevent cancellation of these crossings. As a mathematical model of string figure we consider a knot diagram on the $xy$-plane in $xyz$-space missing some straight lines parallel to the $z$-axis. These straight lines correspond to fingers. We study minimal number of crossings of these knot diagrams under Reidemeister moves missing these lines.

By the one-point compactification of the pair $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ we consider the pair $\mathbb{S}^2 \subset \mathbb{S}^3$. Namely we consider a knot $K$ in the 3-sphere $\mathbb{S}^3$ and a diagram $D$ of $K$ on the 2-sphere $\mathbb{S}^2$. Instead of straight lines parallel to the $z$-axis that restrict the deformation of $K$ we remove some open disks from $\mathbb{S}^2$ and restrict the deformation of $D$ to Reidemeister moves performed on this punctured sphere.

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The precise formulation is as follows. Let $K$ be a knot in $S^3$ and $D$ a diagram of $K$ on $S^2$. Let $P(D) \subset S^2$ be the immersed circle obtained from $D$ by forgetting the over/under crossing information at each crossing point of $D$. We denote $P(D)$ by $P$ when the choice of $D$ is clear. Sometimes $P$ does not come from $D$ and it is simply an image of a generic immersion $\varphi : \mathbb{S}^1 \to S^2$. Namely $P = \varphi(\mathbb{S}^1)$. Let $N(P) \subset S^2$ be the regular neighbourhood of $P$ in $S^2$. Let $\mathcal{R}(P)$ be the set of connected components of $S^2 \setminus N(P)$. We denote $\mathcal{R}(P)$ by $\mathcal{R}$ when the choice of $P$ is clear. Note that each element of $\mathcal{R}$ is an open disk whose closure is a closed disk. Let $S$ be a subset of $\mathcal{R}$. Set $F(S) = S^2 \setminus \bigcup_{R \in S} R$. Note that $D$ is still a diagram of $K$ on this $|S|$-punctured sphere $F(S)$. Note also that $F(\emptyset) = S^2$, $F(\mathcal{R}) = N(P)$ and if $S_1 \subset S_2$ then $F(S_1) \subset F(S_2)$. Let $c(D)$ be the number of crossing points of $D$. Let $c(D)$ be the minimal number of $C(E)$ where $E$ varies over all knot diagrams obtained from $D$ by Reidemeister moves performed on $S^2$. Namely $c(D) = c(K)$ is the minimal crossing number of the knot $K$. Let $c(D,S)$ be the minimal number of $c(E)$ where $E$ varies over all knot diagrams on $F(S)$ obtained from $D$ by Reidemeister moves performed on $F(S)$.

Example 1.1. Let $D$ be a knot diagram on $S^2$ of a trivial knot in $S^3$ as illustrated in Figure 1.2 where $S^2$ is regarded as one-point compactification of $\mathbb{R}^2$ and $D$ in $\mathbb{R}^2$ is illustrated. Let $\mathcal{R} = \{R_1, R_2, R_3, R_4, R_5\}$ as illustrated in Figure 1.2. Then we see that $D$ and $E$ in Figure 1.2 are transformed into each other by second Reidemeister move on $F(\{R_1, R_2, R_3, R_5\})$. Therefore $c(D, \{R_1, R_2, R_3, R_5\}) \leq C(E) = 1$. It follows from Theorem 2.1 in Section 2 that $c(D, \{R_1, R_3\}) \geq 1$. Since $c(D, \{R_3, R_2, R_3, R_5\}) \geq c(D, \{R_1, R_3\})$ we have $c(D, \{R_1, R_2, R_3, R_5\}) \geq 1$. Therefore we have $c(D, \{R_1, R_2, R_3, R_5\}) = 1$. By Theorem 1.3 below we have $c(D, \mathcal{R}(P)) = C(D) = 3$.

![Figure 1.2](image.png)

**Figure 1.2.** An example

By definition we have the following proposition.

**Proposition 1.2.** Let $D$ be a knot diagram on $S^2$ and $P = P(D)$.

1. $c(D, \emptyset) = c(D)$. 


(2) Let $S$ be a subset of $\mathcal{R}(P)$. Then $c(D) \leq c(D, S) \leq C(D)$.
(3) Suppose that $S_1 \subset S_2 \subset \mathcal{R}(P)$. Then $c(D, S_1) \leq c(D, S_2)$.

Namely the map $S \mapsto c(D, S)$ is an order-preserving map from the power set $2^{\mathcal{R}(P)}$ to the set of all non-negative integers $\mathbb{Z}_{\geq 0}$ where $2^{\mathcal{R}(P)}$ is partially ordered by set inclusion. Then it is natural to ask the question whether $c(D, \mathcal{R}(P)) = C(D)$ or not. We have the following affirmative answer.

**Theorem 1.3.** Let $D$ be a knot diagram on $S^2$ and $P = P(D)$. Then $c(D, \mathcal{R}(P)) = C(D)$.

We have asked the question above at some opportunities in 2019. An affirmative answer above is first given by K. Tagami. His proof of Theorem 1.3 is based on his result in [3] Corollary 4.11. After he told us his proof, we have noticed that there is a simple proof using Turaev cobracket [4] of Theorem 1.4 from which Theorem 1.3 immediately follows. Then Z. Cheng informed us that Theorem 1.4 immediately follows from [2] Theorem 4.2. Since the proof Theorem 4.2 in [2] is relatively long, we think that our proof of Theorem 1.3 based on Turaev cobracket is worth stating.

It is given in Section 2

**Theorem 1.4.** Let $\varphi : S^1 \to S^2$ be a generic immersion and $P = \varphi(S^1)$. Let $N(P)$ be a regular neighbourhood of $P$ in $S^2$. Then the minimal self-intersection number of $\varphi$ among all generic immersions homotopic to $\varphi$ on $N(P)$ is equal to the number of crossings of $\varphi$.

Suppose that $c(D) < C(D)$. Then $c(D) = c(D, \emptyset) < c(D, \mathcal{R}(P)) = C(D)$. We ask what is the smallest $S$ with $c(D) < c(D, S)$. We also ask what is the largest $S$ with $c(D) = c(D, S)$. We prepare the following definitions to be more precise.

Set $m = C(D)$. Then it is well-known that $|\mathcal{R}(P)| = m + 2$. Let $n$ be a non-negative integer with $0 \leq n \leq m + 2$. Define $c_{\text{max}}(D, n)$ (resp. $c_{\text{min}}(D, n)$) to be the maximum (resp. minimum) of $c(D, S)$ where $S$ varies over all subset of $\mathcal{R}(P)$ with $|S| = n$. By definition we have $c_{\text{max}}(D, 0) = c_{\text{min}}(D, 0) = c(D, \emptyset) = c(D)$ and $c_{\text{max}}(D, m + 2) = c_{\text{min}}(D, m + 2) = c(D, \mathcal{R}(P)) = C(D)$. Moreover we have the following proposition.

**Proposition 1.5.** Let $D$ be a knot diagram on $S^2$ with $C(D) = m$. Then we have the following inequalities.

\[
c_{\text{max}}(D, 0) = c_{\text{max}}(D, 1) \leq c_{\text{max}}(D, 2) \leq \cdots \leq c_{\text{max}}(D, m + 1) \leq c_{\text{max}}(D, m + 2) \\
\left\| \begin{array}{c|c|c|c|c|c|c}
\end{array} \right\| \\
c_{\text{min}}(D, 0) = c_{\text{min}}(D, 1) = c_{\text{min}}(D, 2) \leq \cdots \leq c_{\text{min}}(D, m + 1) \leq c_{\text{min}}(D, m + 2).
\]

A proof of Proposition 1.5 is given in Section 2. We pay attention to the inequality $c_{\text{max}}(D, 0) = c_{\text{max}}(D, 1) \leq c_{\text{max}}(D, 2)$ in Proposition 1.5. Then the next question is whether or not $c_{\text{max}}(D, 1) = c_{\text{max}}(D, 2)$. The following is a partial answer to this question.

**Theorem 1.6.** Let $D$ be a knot diagram on $S^2$ of a trivial knot with $C(D) > 0$. Then $c_{\text{max}}(D, 2) > 0$.

A proof is given in Section 2

This paper is based on the graduation thesis of the first author submitted to Waseda University. The basic idea of this paper owes to him.
Let \( \varphi : S^1 \to S^2 \) be a generic immersion and \( P = \varphi(S^1) \). Let \( C \) be the set of all crossing points of \( P \). Each connected component of \( P \setminus C \) is said to be an edge of \( P \). Let \( e \) be an edge of \( P \). Let \( u \) and \( v \) be mutually distinct connected components of \( S^2 \setminus P \) such that each of the closure of \( u \) and the closure of \( v \) contains \( e \). Let \( U \) and \( V \) be the elements of \( \mathcal{R}(P) \) with \( U \subset u \) and \( V \subset v \). Then we say that \( U \) and \( V \) are adjacent along \( e \). Let \( F(e) = F(\{U, V\}) = S^2 \setminus \{U \cup V\} \). Note that \( F(e) \) is an annulus. Let \( S \) be a subset of \( \mathcal{R} \). A loop on a surface \( F(S) \) is said to be nontrivial on \( F(S) \) if it is not null-homotopic on \( F(S) \). We give an orientation to \( S^1 \) and regard \( P = \varphi(S^1) \) as an oriented loop on \( N(P) \).

**Proof of Theorem 1.4.** Let \( H \) be the set of all free homotopy classes of nontrivial oriented loops on \( N(P) \). Let \([P]_{F(S)} \) be the free homotopy class of \( P \) on \( F(S) \) for each \( S \in \mathcal{R} \). For any edge \( e \) of \( P \) we see that \([P]_{F(e)} \) is primitive. Namely \([P]_{F(e)} \) is not a power of another class. Therefore \([P]_{N(P)} \) is also primitive. Therefore \([P]_{N(P)} \) is an element of \( H \). For each crossing point \( p \in C \) we have two loops \( P_1(p) \) and \( P_2(p) \) on \( N(P) \) as illustrated in Figure 2.1. We see that there exist edges \( d \) and \( e \) of \( P \) such that both \([P_1(p)]_{F(d)} \) and \([P_2(p)]_{F(e)} \) are primitive. Therefore both \([P_1(p)]_{N(P)} \) and \([P_2(p)]_{N(P)} \) are elements of \( H \). Let \( \mathbb{Z}H \) be the free \( \mathbb{Z} \)-module generated by \( H \). Then

\[
\tau([P]_{N(P)}) = \sum_{p \in C} ([P_1(p)]_{N(P)} \otimes [P_2(p)]_{N(P)} - [P_2(p)]_{N(P)} \otimes [P_1(p)]_{N(P)})
\]

defines the Turaev cobracket \( \tau : \mathbb{Z}H \to \mathbb{Z}H \otimes \mathbb{Z}H \). By definition we see that the minimal self-intersection number of \([P]_{N(P)} \) is greater than or equal to the half the number of terms in the linear combination \( \tau([P]_{N(P)}) \) in \( \mathbb{Z}H \otimes \mathbb{Z}H \). See for example [1]. Suppose that \( p \neq q \) or \( j \neq k \). Then we see that there is an edge \( e \) of \( P \) such that \([P_1(p)]_{F(e)} \neq [P_2(q)]_{F(e)} \). Therefore \([P_1(p)]_{N(P)} \neq [P_2(q)]_{N(P)} \). Thus we see that \([P_1(p)]_{N(P)} = [P_2(q)]_{N(P)} \) if and only if \( p = q \) and \( j = k \). Therefore no terms in the definition of \( \tau([P]_{N(P)}) \) cancel each other and the number of terms of \( \tau([P]_{N(P)}) \) is exactly the twice the number of elements of \( C \) as desired. \( \square \)

![Figure 2.1. Smoothing P at p](image)

**Figure 2.1. Smoothing P at p**

**Proof of Proposition 1.5.** By definition and by Proposition 1.2 (3) we have \( c_{\max}(D, i) \leq c_{\max}(D, i + 1) \) and \( c_{\min}(D, i) \leq c_{\min}(D, i + 1) \). By definition we have \( c_{\max}(D, i) \geq c_{\min}(D, i) \). We have already remarked that \( c_{\max}(D, 0) = c_{\min}(D, 0) \) and \( c_{\max}(D, m + 2) = c_{\min}(D, m + 2) \). Therefore it is sufficient to show \( c_{\max}(D, 0) = c_{\max}(D, 1) \) and \( c_{\min}(D, 0) = c_{\min}(D, 1) = c_{\min}(D, 2) \).

For any \( R \in \mathcal{R} \) we see that \( F(\{R\}) \) is a disk. Note that \( D \) is a diagram of \( K \) on the disk \( F(\{R\}) \). Therefore \( D \) can be deformed into a diagram \( E \) of \( K \) with \( c(E) = c(K) \) by Reidemeister moves on \( F(\{R\}) \). Thus we have \( c(D, \{R\}) = c(K) = c(D) = c(D, 0) \). Therefore \( c_{\max}(D, 0) = c_{\max}(D, 1) \) as desired. Let \( P = P(D) \). Let \( e \) be any edge of \( P \). Let \( U \) and \( V \) be the elements of \( \mathcal{R}(P) \) mutually adjacent along \( e \). Then \( F(e) = F(\{U, V\}) \) is an annulus and \( D \) is a diagram of \( K \) on \( F(e) \). Note that \( U \) and \( V \) may adjacent not only along \( e \) but also along some other edges of \( P \).
Let $k$ be the number of edges of $P$ along which $U$ and $V$ are adjacent. Then we see that $D$ is a diagram-connected sum of $k$ local knot diagrams. Therefore we see that $D$ can be transformed into a diagram $E$ of $K$ with $c(E) = c(K)$ by Reidemeister moves on $F(e)$. This means that $c(D, \{U, V\}) = c(K) = c(D) = c(D, 0)$. Thus we have $c_{\min}(D, 2) = c_{\min}(D, 0)$. Since $c_{\min}(D, 0) \leq c_{\min}(D, 1) \leq c_{\min}(D, 2)$ we have $c_{\min}(D, 0) = c_{\min}(D, 1) = c_{\min}(D, 2)$ as desired. □

Let $\varphi : S^1 \to S^2$ be a generic immersion and $P = \varphi(S^1)$. Suppose that $P$ is oriented. A locally constant map $a : S^2 \setminus P \to \mathbb{Z}$ is said to be an Alexander numbering if for each edge $e$ of $P$ the value of $a$ of a point on the left side of $e$ is always one greater than the value of $a$ of a point on the right side of $e$. See Figure 2.2 for an example.

![Figure 2.2. An example of Alexander numbering](image)

**Theorem 2.1.** Let $\varphi : S^1 \to S^2$ be a generic immersion and $P = \varphi(S^1)$. Suppose that $P$ is oriented. Let $a : S^2 \setminus P \to \mathbb{Z}$ be an Alexander numbering. Let $U$ and $V$ be elements of $R(P)$, $x \in U$ and $y \in V$. Then the minimal self-intersection number of $\varphi$ among all generic immersions homotopic to $\varphi$ on $F(\{U, V\})$ is greater than or equal to $|a(x) - a(y)| - 1$.

**Proof.** Since the case $U = V$ is trivial we may suppose $U \neq V$. Then $F(\{U, V\})$ is an annulus and $P$ is freely homotopic on $F(\{U, V\})$ to a power of the core curve of $F(\{U, V\})$. We see by the definition of Alexander numbering that the power is equal to $a(x) - a(y)$ up to sign. Therefore the curve must have at least $|a(x) - a(y)| - 1$ self-intersection points. □

**Proof of Theorem 1.6** Suppose that $P = P(D)$ is oriented. Let $a : S^2 \setminus P \to \mathbb{Z}$ be an Alexander numbering. Since $C(D) > 0$ there is a crossing point $p$ of $P$. Paying attention to a neighbourhood of $p$ we see that there are elements $U$ and $V$ of $R(P)$ such that $|a(x) - a(y)| \geq 2$ for $x \in U$ and $y \in V$. See Figure 2.3. Then by Theorem 2.1 we have $c(D, \{U, V\}) \geq 1$. Therefore $c_{\max}(D, 2) \geq c(D, \{U, V\}) \geq 1$ as desired. □

![Figure 2.3. Alexander numbering around a crossing point](image)

**3. Future directions**

After Proposition 1.5 and Theorem 1.6 we are interested in the first $i$ with $c_{\max}(D,i) < c_{\max}(D,i + 1)$, the first $i$ with $c_{\min}(D,i) < c_{\min}(D,i + 1)$, the last $i$
with $c_{\text{max}}(D,i) < c_{\text{max}}(D,i+1)$, and the last $i$ with $c_{\text{min}}(D,i) < c_{\text{min}}(D,i+1)$. We define the following to be more precise. Let $\alpha(D) = \min\{n \mid c_{\text{max}}(D,n) > c(D)\}$, $\beta(D) = \min\{n \mid c_{\text{min}}(D,n) > c(D)\}$, $\gamma(D) = \max\{n \mid c_{\text{max}}(D,n) < C(D)\}$, and $\delta(D) = \max\{n \mid c_{\text{min}}(D,n) < C(D)\}$. The decision of these numbers will be a problem. For example, the following theorem is an immediate consequence of Proposition 1.5 and Theorem 1.6.

**Theorem 3.1.** Let $D$ be a knot diagram on $S^2$ of a trivial knot with $C(D) > 0$. Then $\alpha(D) = 2$.

**Example 3.2.** Let $D_n$ be a knot diagram on a surface $F(S_n)$ as illustrated in Figure 3.1. Note that $C(D_n) = 2n$ and $|S_n| = n + 2$. By second Reidemeister moves we have $c(D_n, S_n) = 0$. By an argument using Alexander numbering we have $\beta(D_n) = n + 3$. Therefore $\beta(D_n) = \frac{C(D_n)}{2} + 3$.

![Figure 3.1. Examples](image)

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