A polyhedral Markov field - pushing the Arak-Surgailis construction into three dimensions

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Abstract: The purpose of the paper is to construct a polyhedral Markov field in \( \mathbb{R}^3 \) in analogy with the planar construction of the original Arak (1982) polygonal Markov field. We provide a dynamic construction of the process in terms of evolution of two-dimensional multi-edge systems tracing polyhedral boundaries of the field in three-dimensional time-space. We also give a general algorithm for simulating Gibbsian modifications of the constructed polyhedral field.

1 Introduction

The notion of a consistent polygonal Markov field has first appeared in the seminal paper by Arak (1982) who constructed an isometry-invariant process with polygonal realisations in the plane, enjoying a two-dimensional germ Markov property [the conditional behaviour

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of the field in an open bounded domain with piecewise smooth boundary depends on
the exterior configuration only through arbitrarily close neighbourhoods of the boundary]
and with one-dimensional sections coinciding in distribution with homogeneous Poisson
point processes. An attractive feature of this process is that it admits several alternative
equivalent representations. Two of them fall into the general Gibbsian framework and use,
respectively, lines and points as the basic building blocks, see Arak & Surgailis (1989),
Arak & Surgailis (1991) and Arak, Clifford & Surgailis (1993). The third representation
of the Arak-Surgailis field is available in terms of equilibrium evolution of one-dimensional
particle systems, tracing the polygonal realisations of the process in two-dimensional time-
space, see the above papers. A dynamic representation in this spirit is in fact shared by
a much richer class of planar polygonal Markov processes, as constructed and discussed in
Arak & Surgailis (1989,1991) and Arak, Clifford & Surgailis (1993).

The purpose of the present article is to construct an analogue of the Arak process in
three-dimensional polyhedral setting. We provide a plane-based Gibbsian representation
as well as a dynamic construction in terms of evolving multi-edge systems tracing polyhe-
dral boundaries of the field in three-dimensional time-space \( \mathbb{R} \times \mathbb{R}^2 \). The resulting field is
Markovian, as directly follows by the Gibbsian representation. The construction is isom-
etry invariant. We give also an explicit formula for the corresponding partition function.
However, we are not able to explicitly characterise the equilibrium initial condition for the
constructed dynamics for multi-edge systems and hence to explicitly construct a consistent
version of our polyhedral Markov field.

The results discussed above are presented Sections 2 and 3. They are complemented
with Section 4 where we provide a Metropolis type simulation algorithm for a general class
of Gibbsian modifications of the polyhedral Markov field constructed in this paper. Apart
from its theoretical interest the purpose of developing this algorithm is related to the fact
that we anticipate possible applications of polyhedral Markov fields for volumetric image segmentation, in the spirit of our papers Kluszczyński et al. (2004, 2005), where similar algorithms in two-dimensional setting are used for planar image segmentation, see also Clifford & Middleton (1989) and Clifford & Nicholls (1994).

2 Construction of the process

2.1 Preliminaries

In what follows we abuse the language granting the name polygonal face to any open connected subset of a plane in \( \mathbb{R}^3 \), with polygonal boundary. In particular, a face may well contain holes but it cannot split into several disconnected parts. For an open bounded and convex polyhedral domain \( D \subseteq \mathbb{R}^3 \) we define the family \( \Gamma_D \) of admissible polyhedral configurations in \( D \) by taking all collections \( \gamma \) of planar polygonal faces contained in \( D \) such that

(P1) the faces of \( \gamma \) do not intersect, but they may share edges,

(P2) each interior edge of \( \gamma \) (contained in \( D \)) is shared by exactly two faces,

(P3) each boundary edge of \( \gamma \) (contained in \( \partial D \)) belongs to exactly one face,

(P4) each internal vertex of \( \gamma \) (contained in \( D \)) is shared by exactly three faces and has exactly three outgoing edges,

(P5) each boundary vertex of \( \gamma \) (contained in \( \partial D \)) is shared by exactly two faces and has exactly one outgoing internal edge and exactly two outgoing boundary edges,

(P6) no two faces of \( \gamma \) are coplanar.

In other words, an admissible polyhedral configuration \( \gamma \) corresponds to a family of disjoint closed polyhedral surfaces in \( D \), possibly nested and chopped off by the boundary \( \partial D \). We
write $E_D(\gamma)$ for the collection of edges of $\gamma$ contained in $D$ (internal edges), $E_{\partial D}(\gamma)$ for the collection of edges of $\gamma$ contained in $\partial D$ (boundary edges), and $F_D(\gamma)$ for the collection of faces of $\gamma$ contained in $D$ (only internal faces are considered). For a finite collection $\{\varpi_i\}_{i=1}^n$ of planes intersecting $D$ we consider the family $\Gamma_D(\{\varpi_i\}_{i=1}^n)$ of all admissible polyhedral configurations $\gamma$ in $D$ with the additional properties that $\gamma \subseteq \bigcup_{i=1}^n \varpi_i$ and that $\gamma \cap \varpi_i$ is a single polygonal face of non-zero area, possibly with some zero-measure polygonal curves added, for each $\varpi_i$, $i = 1, 2, \ldots, n$. Let $\mu$ be the usual isometry-invariant Haar-Lebesgue measure on the space $\mathcal{P}$ of all two-dimensional planes in $\mathbb{R}^3$. One possible construction of $\mu$ goes by identifying a plane $\varpi$ with the pair $(u, \rho) \in S_2 \times \mathbb{R}_+$, with $S_2$ standing for the unit sphere in $\mathbb{R}^3$ and where $u$ is the direction of the vector orthogonal to $\varpi$ and joining it to the origin, while $\rho$ is the distance between $\varpi$ and the origin. In these terms $\mu$ arises by endowing $S_2 \times \mathbb{R}_+$ with the product of the usual surface measure on the sphere and the Lebesgue measure on $\mathbb{R}_+$. We write $\Pi$ for the Poisson plane process with intensity measure $\mu$ and for each bounded domain $D$ we let $\Pi_D$ stand for the restriction of $\Pi$ to the family $\mathcal{P}_D$ of planes intersecting $D$. The following properties of the plane process $\Pi$ will be of use in the sequel. Here and throughout, to avoid possible confusion we use $d_2x$ and $d_3x$ rather than $dx$ to denote integration respectively w.r.t. the 2- and 3-dimensional Lebesgue measure, thus explicitly indicating the dimensionality of the integration variable.

**Proposition 1** With the above notation we have:

1. **(I1)** The intersection of $\Pi$ with a given straight line $l$ in $\mathbb{R}^3$ is a homogeneous Poisson point process of intensity $\pi$,

2. **(I2)** The intersection of $\Pi$ with a given plane $\varpi$ is a Poisson process of lines $l = l(\phi, r)$ in $\varpi$ with intensity measure $\frac{1}{\pi}d\phi dr$, where $r$ stands for the distance between $l$ and a certain fixed point $\hat{0}$ in $\varpi$, while $\phi$ is the angle between some fixed line in $\varpi$ and the vector in $\varpi$ orthogonal to $l$ and joining it to $\hat{0}$,
(I3) For a given plane \( \varpi \) and \( x \in \varpi \) the probability that two planes of \( \Pi \) meet at \( [x, x + d_2 x] \subset \varpi \) is \( \frac{\pi^3}{4} d_2 x \).

(I4) For \( x \in \mathbb{R}^3 \) the probability that three planes of \( \Pi \) meet at \( [x, x + d_3 x] \subset \mathbb{R}^3 \) is \( \frac{\pi^4}{6} d_3 x \).

Indeed, the assertions (I1) and (I2) follow immediately by (3.29T) in Miles (1971) because our process \( \Pi \) coincides with \( \mathcal{B}(2\pi, 2, 3) \) there. Further, (I4) is a direct consequence of (6-1-9') in Matheron (1975) with \( a = \pi/2 \) there. Finally, (I3) follows from (I2) and (6-1-9') ibidem with \( a = \pi/2 \) there.

2.2 Gibbsian representation

For a given admissible polyhedral configuration \( \gamma \) we consider its energy

\[
\Phi_D(\gamma) = \frac{1}{2} \sum_{e \in E_D(\gamma)} (2\pi - |\angle(e)|) \ell(e) + \frac{\pi^3}{4} \sum_{f \in F_D(\gamma)} \text{Area}(f) + \frac{\pi^4}{6} \text{Vol}(D),
\]

where \( \angle(e) \) is the solid convex angle between the planes determined by the faces meeting at an internal edge \( e \), \( \ell(e) \) is the length of the edge \( e \), \( \text{Area}(f) \) stands for the area of the polygonal face \( f \) while \( \text{Vol}(D) \) denotes the volume of the domain \( D \). We define our three-dimensional polyhedral process \( A_D \) in \( D \) as the Gibbsian modification of the process induced on \( \Gamma_D \) by \( \Pi_D \) with the Hamiltonian \( \gamma \mapsto \Phi_D(\gamma) \). To be more specific, we put

\[
P(A_D \in \mathcal{F}) = \frac{\mathbb{E} \sum_{\gamma \in \Gamma_D(\Pi_D) \cap \mathcal{F}} \exp(-\Phi_D(\gamma))}{\mathbb{E} \sum_{\gamma \in \Gamma_D(\Pi_D)} \exp(-\Phi_D(\gamma))}
\]

for all \( \mathcal{F} \subseteq \Gamma_D \) Borel measurable, say, with respect to the standard Hausdorff distance topology. The finiteness of the partition function \( \mathbb{E} \sum_{\gamma \in \Gamma_D(\Pi_D)} \exp(-\Phi_D(\gamma)) \) above will be established in Section 2.4, Corollary 1.

2.3 Dynamic representation

The availability of a dynamic description in terms of equilibrium time-space evolution of one-dimensional particle systems is a crucial feature of the Arak-Surgailis process and it
underlies most of the techniques successfully used to derive explicit expressions for various characteristics of that model. It is therefore of particular importance to show that our three-dimensional model $A_D$ admits an analogous dynamic representation in terms of evolution of two-dimensional edge systems tracing the polyhedral boundaries of $A_D$ in three-dimensional time-space.

Note that a planar section of a single polygonal face as defined above coincides, for $\mu$-almost all planes in $\mathcal{P}$, with a finite collection of disjoint colinear closed segments. Motivated by this observation we shall use the name of *multi-edge* for each such collection. For the purposes of this section we shall represent $\mathbb{R}^3$ as three-dimensional time space $\mathbb{R} \times \mathbb{R}^2$ with two-dimensional spatial component; the time coordinate will be consistently denoted by $t$. All the multi-edges considered below will arise as sections of polygonal faces in $\mathbb{R} \times \mathbb{R}^2$ with purely spatial planes $\omega_t := \{t\} \times \mathbb{R}^2$. In this setting, a non-empty multi-edge given as the intersection $\varepsilon := f \cap \omega_t$ for some polygonal face $f$, $f \not\subseteq \omega_t$, can be assigned in a natural way its two-dimensional *velocity vector* lying in $\omega_t$ and defined to be the spatial component (projection onto the spatial plane $\omega_t$) of the unit normal to $\varepsilon$ lying in the plane $\text{pl}([f])$ of $f$ and contained in the *future* half-space $\omega_{t^+} := [t, +\infty) \times \mathbb{R}^2$, divided by the length of the spatial component of the unit normal to $\text{pl}([f])$. With $\varrho$ standing for the unique plane $\text{pl}([f])$ containing the face $f$ we denote by $\vec{v}[\varrho]$ the velocity vector constructed above (note that the construction does not depend on $t$). It is now easily verified that when observing the *time evolution* of the spatial plane $\omega_t$, we see the polygonal face $f$ mapped onto the time evolution of the multi-edge $\omega_t \cap f$ moving in time with the velocity vector $\vec{v}[\text{pl}(f)]$. Consequently, each finite family of disjoint and possibly nested bounded polyhedra is also mapped into the time evolution of the corresponding finite multi-edge systems. In course of the time evolution of multi-edges their components evolve as well, continuously updating their lengths according to the rule that no two edge segments can
intersect each other except when sharing a vertex, whence the endpoints of a given edge segment are determined as the meeting points with other multi-edges. This rule may lead to extinction of certain edge segments as their length reaches 0 but, on the other hand, certain segments may also get split into disjoint parts. A multi-edge dies if all its edges go extinct.

As mentioned above, we shall consider our polyhedral random field defined in a certain open bounded and convex polyhedral domain $D \subseteq \mathbb{R} \times \mathbb{R}^2$. The presence of the boundary has its effect on the dynamics, which has to be taken into account in our construction. Namely, at each time $t \in \mathbb{R}$ we observe a collection of boundary multi-edges in $\omega_t \cap \partial D$ and for each internal multi-edge in $D \cap \omega_t$ the meeting point with the boundary marks the end of its appropriate segment. The boundary multi-edges can be assigned their velocity vectors on equal rights with the internal edges. It should be noted that a boundary multi-edge has always exactly one segment due to the convexity assumption imposed on $D$.

Apart from the multi-edge extinction, the evolution of a multi-edge system, corresponding to a finite collection of disjoint and possibly nested polyhedral surfaces, comprises also multi-edge birth events. These can be divided into several groups

(IT) *Infinitesimal triangle birth* in a time-space point $x \in D \cap \omega_t$ which does not lie on any internal or boundary multi-edge in $\omega_t$. At the point $x$ three pairwise non-parallel multi-edges $e_1, e_2, e_3$ are born, moving with three different velocities $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

(IA) *Infinitesimal angle birth* in a time-space point $x$ lying on an internal or boundary edge $e_1$ moving with a certain velocity $\vec{v}_1$. At the point $x$ two multi-edges $e_2, e_3$ are born, moving with velocities $\vec{v}_2, \vec{v}_3$ and such that $\{e_1, e_2, e_3\}$ are pairwise non-parallel.

(IE) *Infinitesimal edge birth at a vertex* in a time-space point $x$ lying at the intersection of two different (either internal or boundary) edges $e_1, e_2 \subseteq (D \cup \partial D) \cap \omega_t$, moving
with velocities $\vec{v}_1$ and $\vec{v}_2$ respectively. At the point $x$ a new multi-edge $e_3$ is born, moving with velocity $\vec{v}_3$ and such that $\{e_1, e_2, e_3\}$ are pairwise non-parallel.

Observe that for certain choices of new-born multi-edges and their velocities we end up with unstable configurations where new edges die immediately upon their birth. This happens e.g. for an infinitesimal triangle with all its edges moving inward. We say that a multi-edge $e$ is stably born if it survives a positive amount of time rather than going extinct just upon its birth. The stability conditions are easily determined for all groups of birth events.

(Stability for IT) Denote by $n_i$ the unit normal to $e_i$ (in $\varpi_t$) pointing outward the infinitesimal triangle and let $\mathbf{v}[v_j, v_k]$ be the velocity vector (in $\varpi_t$) for the intersection point of $e_j$ and $e_k$, easily checked to be $av_j + bv_k$, where

$$
\begin{bmatrix}
a \\
b
\end{bmatrix} = \begin{bmatrix}
\langle v_j, v_j \rangle & \langle v_j, v_k \rangle \\
\langle v_k, v_j \rangle & \langle v_k, v_k \rangle
\end{bmatrix}^{-1} \begin{bmatrix}
|v_j|^2 \\
|v_k|^2
\end{bmatrix}.
$$

The birth is stable iff $\langle \vec{v}_i, n_i \rangle > \langle \mathbf{v}[v_j, v_k], n_i \rangle$ for all $i \neq j \neq k \neq i$. It can be easily shown that it is enough to verify this condition just for one fixed choice of $i, j, k$.

(Stability for IA) Let $n_1$ be the unit normal to $e_1$ in $\varpi_t$ pointing to the side of $e_1$ opposite to where the new angle is born and write $n_2, n_3$ for the unit normals to $e_2, e_3$ respectively, pointing outward the new-born convex angle. Once again, the birth is stable iff $\langle \vec{v}_i, n_i \rangle > \langle \mathbf{v}[v_j, v_k], n_i \rangle$ for all $i \neq j \neq k \neq i$.

(Stability for IE) Let $n_3$ be the unit normal to $e_3$ pointing inward the convex angle between $e_1$ and $e_2$. Then the birth is stable iff $\langle \vec{v}_3, n_3 \rangle > \langle \mathbf{v}[v_1, v_2], n_3 \rangle$.

In geometrical terms the stability condition for an infinitesimal triangle $e_1, e_2, e_3$ means that $e_i$ is able to escape in the direction pointed by $n_i$ from the intersection point of $e_j$ and $e_k$, whose velocity component in the direction $n_i$ is smaller than $\langle \vec{v}_i, n_i \rangle$. Clearly, otherwise
would be destroyed by other edges immediately upon its birth. The remaining stability conditions admit their geometrical interpretations along the same lines.

In the sequel, we will refer to all stable boundary birth events (involving at least one boundary edge) of either (IA) or (IE) type as to entry events. Note that (IA) or (IE) events involving internal edges only are not considered to be entry events and neither are unstable birth events. For a given admissible polyhedral configuration \( \gamma \in \Gamma_D \) we write \( \text{Entry}(\gamma) \) for the collection of entry events it determines.

Below, we construct a random dynamics on multi-edge systems conditionally on the collection \( \mathcal{E} \) of all entry events assumed to be given. This stands in contrast to the two-dimensional Arak-Surgailis construction, where boundary birth events were also governed by explicit random dynamics, and this is due to the fact that we are unable to identify the explicit equilibrium distribution for entry events in the three-dimensional case.

The random dynamics of internal birth events is given by the following rules, with \( t \) standing for the time coordinate

**Dynamics for IT** The infinitesimal triangle birth time-space sites are chosen according to a homogeneous Poisson point process of intensity \( \pi \frac{4}{6} \) in \( D \). The directions and velocities of the new-born edges are chosen according to the distribution of a typical stable vertex angle at a meeting point of three planes in \( \Pi \), where the adjective stable means here that out of the 8 vertex angles at the triple intersection point we choose the unique one which gives rise to a stable infinitesimal triangle birth in the sense of **Stability for IT**,.

**Dynamics for IA** The infinitesimal angle birth time-space sites on a face \( f \) are chosen with intensity \( \pi \frac{4}{4} \sqrt{1 + |v[pl(f)]|^2} dt d\ell d\ell' \) with \( d\ell \) standing for the edge length element on \( f \cap \varpi_i \) (note that this intensity coincides with \( \pi \frac{4}{4} \) times the area element on \( f \)). The directions and velocities of the new-born edges are chosen according to the
distribution of a typical stable vertex angle at a meeting point of \( \text{pl}(f) \) with two other planes of \( \Pi \), where the adjective stable means here that out of the 8 vertex angles at the triple intersection point we choose the unique one which gives rise to a stable infinitesimal angle birth in the sense of (Stability for IA).

(Dynamics for IE) An infinitesimal edge birth site at a vertex \( x \in \varpi_t \), at the intersection of two edges \( e_1 \) and \( e_2 \) lying respectively on faces \( f_1 \) and \( f_2 \) and moving with respective velocities \( \vec{v}_1 \) and \( \vec{v}_2 \), arises with intensity

\[
\pi \sqrt{1 + |v[v_1, v_2]|^2} d_2 x dt. \tag{3}
\]

Note that this intensity coincides with \( \pi \) times the length element on the line traced by the time-space trajectory of the intersection point \( e_1 \cap e_2 \). The direction and velocity of the new-born edge are determined according to the distribution of a typical plane \( \varpi \) of \( \Pi \) intersecting the line traced in time-space by \( e_1 \cap e_2 \). Unlike in previous cases, the stability of this birth event in the sense of (Stability for IE) is not guaranteed here and in fact it is easily seen that it holds iff the normal to \( \varpi \) pointing in time-space to the future is not contained in the solid angle between the faces \( f_1 \) and \( f_2 \) [i.e. the solid angle between the half-planes of \( \text{pl}[f_1] \) and \( \text{pl}[f_2] \) containing \( e_1 \) and \( e_2 \) respectively and meeting along \( \text{pl}[f_1] \cap \text{pl}[f_2] \)], denote this angle by \( \angle(f_1, f_2) \) for use below. The non-stable birth events resulting from the above rule have no effect on the dynamics. Thus, alternatively we can produce the birth events with intensity

\[
\pi \sqrt{1 + |v[v_1, v_2]|^2} \left( \frac{2\pi - |\angle(f_1, f_2)|}{2\pi} \right) d_2 x dt \tag{4}
\]

and let the direction and velocity of the new-born edge be determined according to the distribution of a typical plane \( \varpi \) of \( \Pi \) intersecting the line traced in time-space by \( e_1 \cap e_2 \), conditioned on yielding a stable infinitesimal edge birth event. In analogy with
the interpretation of (3) above, the intensity (4) coincides with \( \frac{1}{2}(2\pi - |\angle(f_1, f_2)|) \) times the length element on the line \( \text{pl}[f_1] \cap \text{pl}[f_2] \) traced by the time-space trajectory of the intersection point \( e_1 \cap e_2 \).

To make the above construction fully explicit, we note that the distribution of a typical vertex angle between three planes of \( \Pi \) is \( \propto |\langle n_1, n_2 \times n_3 \rangle|d\sigma(n_1)d\sigma(n_2)d\sigma(n_3) \), where \( n_i \in \mathbb{S}_2 \), \( \sigma \) is the surface measure on \( \mathbb{S}_2 \) and the planes \( \varpi_i \), \( i = 1, 2, 3 \) creating the angle are respectively chosen orthogonal to \( n_i \), \( i = 1, 2, 3 \); indeed, this follows by a minor modification of Theorem 3 in Calka (2001), specialised for \( d = 3 \). Recall that the scalar product \( |\langle n_1, n_2 \times n_3 \rangle| \) coincides with the volume of the parallelepiped spanned by the vectors \( n_1, n_2, n_3 \). Observe that the knowledge of the typical vertex angle distribution for three planes provides full knowledge of the corresponding laws with one or two planes fixed, as respectively required for (IA) and (IE).

The polyhedral process in \( D \) resulting from the above dynamic construction in presence of a collection \( \mathcal{E} \) of entry events will be denoted in the sequel by \( A_{D|\mathcal{E}} \).

### 2.4 Equivalence of representations

We now proceed to showing that both the Gibbsian representation (2) and the dynamic construction of Subsection 2.3 yield, in a sense to be specified below, the same polyhedral field. To this end, for a given collection \( \mathcal{E} \) of entry events in \( D \) we put \( \Gamma_{D|\mathcal{E}} := \{ \gamma \in \Gamma_D \mid \text{Entry}(\gamma) = \mathcal{E} \} \). Further, for a collection \( \{ \varpi_i \}_{i=1}^n \) of pairwise non-parallel planes in \( \mathbb{R}^3 \) we write \( \Gamma_{D|\mathcal{E}}(\{ \varpi \}_{i=1}^n) \) to denote the family of all admissible polyhedral configurations in \( \Gamma_{D|\mathcal{E}} \) with the additional property that \( \{ \text{pl}(f) \mid f \in F_D(\gamma) \setminus F[\mathcal{E}] \} \) coincides with \( \{ \varpi \}_{i=1}^n \), where \( F[\mathcal{E}] \) is the collection of planes arising in entry events from \( \mathcal{E} \). Define \( \kappa(D) \) to be the
measure $\mu$ of the set of planes intersecting $D$. Note that, $D$ being convex, we have

$$\kappa(D) := \mu\{\omega, \omega \cap D \neq \emptyset\} = \frac{1}{2} \sum_{e \in E(D)} |\angle(e)|\ell(e),$$  \hspace{1cm} (5)$$

see (4.2.30), (4.5.9) and (4.5.10) in Schneider (1993), whence $\kappa(D)$ is proportional to the generalised integral mean curvature functional of $D$, see ibidem.

The crucial observation in this Subsection is that, in view of Proposition 1, the form of the dynamic rules (Dynamics for IT,IA,IE) implies that, given a collection $\mathcal{E}$ of entry events in $D$, we have for all $\gamma \in \Gamma_{D|\mathcal{E}}$

$$\mathbb{P}(A_{D|\mathcal{E}} \in d\gamma) = \exp(\kappa(D))1_{\{\gamma \in \Gamma_{D|\mathcal{E}}(\Pi_D)\}} \exp(-\Phi_D(\gamma))dL(\Pi_D) \hspace{1cm} (6)$$

with $L(X)$ standing for the law of a random object $X$. Indeed, for $\gamma \in \Gamma_{D|\mathcal{E}}$, the probability differential $[dL(A_{D|\mathcal{E}})](\gamma)$ factorises into the product of the probability that all faces $f_1, \ldots, f_k$ of $F_D(\gamma) \setminus F[\mathcal{E}]$ were born in course of the evolution of the multi-edge system, which is seen to be $\prod_{i=1}^k [1_{pl[f_i]|\gamma \cap D \neq \emptyset}d\mu(pl[f_i])] = \exp(\kappa(D))[dL(A_D)]([pl[f_1], \ldots, pl[f_k]])$ by comparing the dynamic rules (Dynamics for IT,IA,IE) with Proposition 1, times the probability that no other faces were born in the evolution giving rise to $\gamma$, which is in its turn evaluated to $\exp(-\Phi_D(\gamma))$ in view of the form of the dynamic rules. This last claim is easily verified by noting that

- by (Dynamics for IT), $\exp\left(-\frac{\pi^4}{6}\text{Vol}(D)\right)$ is the probability that no extra faces were born in (IT) birth events,

- by (Dynamics for IA), $\exp\left(-\frac{\pi^2}{4} \sum_{f \in F_D(\gamma)} \text{Area}(f)\right)$ is the probability that no extra faces were born in (IA) birth events,

- by (Dynamics for IE), $\exp\left(-\frac{1}{2} \sum_{e \in E_D(\gamma)} (2\pi - |\angle(e)|)\ell(e)\right)$ is the probability that no extra faces were born in (IE) birth events, see the discussion following (4).
As an immediate consequence of (6) we come to

**Theorem 1** Given a collection \( \mathcal{E} \) of entry events in \( D \), the random polyhedral field \( A_{D|\mathcal{E}} \) coincides in distribution with the Gibbs polyhedral field \( A^*_{D|\mathcal{E}} \) given by

\[
P(A^*_{D|\mathcal{E}} \in \mathcal{F}) = \frac{\mathbb{E} \sum_{\gamma \in \Gamma_{D|\mathcal{E}}(\Pi_D) \cap \mathcal{F}} \exp(-\Phi_D(\gamma))}{\mathbb{E} \sum_{\gamma \in \Gamma_{D|\mathcal{E}}(\Pi_D)} \exp(-\Phi_D(\gamma))}
\]

for each \( \mathcal{F} \subseteq \Gamma_{D|\mathcal{E}} \) Borel measurable with respect to the usual Hausdorff topology. Moreover, for each \( \mathcal{E} \) we have

\[
\mathbb{E} \sum_{\gamma \in \Gamma_{D|\mathcal{E}}(\Pi_D)} \exp(-\Phi_D(\gamma)) = \exp(-\kappa(D)). \tag{7}
\]

It should be noted at this point that the particular form of the expression (7) for the partition function might seem to stand in an unexpected contrast to the two-dimensional formula (4.6) in Arak & Surgailis (1989). However, this difference is a matter of choice of the reference measure, which is in our case the normalised law of the Poisson plane process, while Arak & Surgailis (1989) use \( \sum_{n=0}^{\infty} \frac{1}{n!} \mu^{\otimes n} \) instead. Moreover, we introduce the volume order term \( \frac{\pi^4}{6} \text{Vol}(D) \) to our energy function, which has no equivalent in that paper. We find these choices preferable for the presentation of our setting, as leading to simpler formulae. However, should we use the reference measure and Hamiltonian analogous to those of that paper, our partition function in (7) would evaluate to \( \exp\left( \frac{\pi^4}{6} \text{Vol}(D) \right) \). The absence of \( \kappa(D) \) and a surface area order term in the exponent in this partition function expression is due to the fact that we condition on fixed collection of entry events rather than randomising it as done in Arak & Surgailis (1989).

As an obvious conclusion from Theorem 1 we get that the original polyhedral field \( A_D \) as defined by (2) admits the dynamic representation with the collection of entry events distributed according to \( \text{Entry}(A_D) \).
Corollary 1 We have

\[ \mathcal{L}(A_D) = \int \mathcal{L}(A_{D|E}) d[\mathcal{L}(\text{Entry}(A_D))](E) \]

and

\[ \mathcal{L}(A_{D|E}) = \mathcal{L}(A_D|\text{Entry}(A_D) = E). \]

Moreover,

\[ \mathbb{E} \sum_{\gamma \in \Gamma_D(\Pi_D)} \exp(-\Phi_D(\gamma)) = \exp(-\kappa(D)). \]

The use of this corollary is limited by the fact that we do not know the distribution of \( \text{Entry}(A_D) \).

3 Properties of the process

We argue first that the polyhedral fields \( A_D \) and \( A_{D|E} \) exhibit a 3-dimensional germ-Markov property in the sense specified in Corollary 2 below. For a smooth closed simple (non-self-intersecting) surface \( \sigma \) in a bounded and convex polyhedral domain \( D \), by the trace of a polyhedral configuration \( \gamma \) on \( \sigma \), denoted in the sequel by \( \gamma \cap \sigma \), we mean the intersection \( \gamma \cap \sigma \) together with the directions of normals to the face planes at intersection. This concept can be formalised in various compatible ways, yet we keep the above informal definition in hope that it does not lead to any ambiguities while allowing us to avoid unnecessary technicalities. For convenience we assume that no face of \( \gamma \) is tangent to \( c \), which can be ensured with probability 1 in view of the smoothness of \( \sigma \). In view of the Gibbsian representation (2) and by Theorem 1 we easily conclude that

Corollary 2 For each \( \sigma \) as above there exists a stochastic kernel \( A_{\text{Int}\sigma}(\cdot|\vartheta) \) such that, with \( \vartheta \) standing for a trace on \( \sigma \),

\[ \mathcal{L}_{\text{Int}\sigma}(A_D|A_D \cap \sigma = \vartheta) = \mathcal{L}_{\text{Int}\sigma}(A_{D|E}|A_{D|E} \cap \sigma = \vartheta) = A_{\text{Int}\sigma}(\cdot|\vartheta) \]  

(8)
for all bounded open and convex polyhedral domains $D \supseteq \overline{\text{Int}}\sigma$ and for each collection $\mathcal{E}$ of entry events in $D$, where $\mathcal{L}_{\text{Int}\sigma}(X)$ denotes the law of a random field $X$ restricted to $\text{Int}\sigma$ (the interior of $\sigma$).

Indeed, it is easily seen that we have for measurable $G \subseteq \Gamma_{\text{Int}\sigma}||\vartheta$

$$A_{\text{Int}\sigma}(G|\vartheta) = \frac{\mathbb{E}\sum_{\gamma \in \Gamma_{\text{Int}\sigma}||\vartheta}(\Pi_{\text{Int}\sigma})\cap G \exp(-\Phi_{\text{Int}\sigma}(\gamma))}{\mathbb{E}\sum_{\gamma \in \Gamma_{\text{Int}\sigma}||\vartheta}(\Pi_{\text{Int}\sigma}) \exp(-\Phi_{\text{Int}\sigma}(\gamma))},$$

(9)

where $\Gamma_{\text{Int}\sigma}||\vartheta := \{\gamma \in \Gamma_{\text{Int}\sigma} | \gamma \wedge \sigma = \vartheta\}$ and, for a collection $\{\varpi_i\}_{i=1}^n$ of planes in $\mathbb{R}^3$ hitting $\text{Int}\sigma$, $\Gamma_{\text{Int}\sigma}||\vartheta (\{\varpi_i\}_{i=1}^n)$ denotes the family of all polyhedral configurations $\gamma \in \Gamma_{\text{Int}\sigma}||\vartheta$ for which the set $\{\text{pl}[f] | f \in F_D(\gamma), f \cap \sigma = \emptyset\}$ coincides with $\{\varpi_i\}_{i=1}^n$.

To proceed, consider the family $\Gamma_{\mathbb{R}^3}$ of whole-space admissible polyhedral configurations, determined by $\text{(P1), (P2), (P4)}$ and $\text{(P6)}$ ($\text{(P3)}$ and $\text{(P5)}$ are meaningless in this context) and by the requirement of local finiteness (any bounded set is hit by at most a finite number of faces). It is natural to define the family $\mathcal{G}(\mathcal{A})$ of infinite volume Gibbs measures (thermodynamic limits) for $\mathcal{A}$ as the collection of all probability measures on $\Gamma_{\mathbb{R}^3}$ with the accordingly distributed random element $\mathcal{A}$ satisfying

$$\mathcal{L}_{\text{Int}\sigma} (\mathcal{A}|\mathcal{A} \wedge \sigma = \vartheta) = A_{\text{Int}\sigma}(\cdot|\vartheta)$$

(10)

for $\sigma$ ranging through the collection of all bounded smooth simple closed surfaces in $\mathbb{R}^3$.

In addition, we shall consider the family $\mathcal{G}_r(\mathcal{A})$ of isometry invariant measures in $\mathcal{G}(\mathcal{A})$. We believe that, in analogy with the results in Section 3 in Schreiber (2005), it should be possible to show that $\mathcal{G}_r(\mathcal{A}) \neq \emptyset$ by using an appropriate relative compactness argument. Moreover, we conjecture that the uniqueness of the isometry invariant thermodynamic limit $\mathcal{A}_{\mathbb{R}^3}$, as well as its coincidence with the polyhedral process traced by infinite-volume equilibrium evolution of the multi-edge system as discussed above, could possibly be established following the lines of Schreiber (2004a), where this is done for two-dimensional polygonal fields admitting dynamic representation. On the other hand, in analogy with
the two-dimensional setting, we do not expect that $A_{\mathbb{R}^3}$ be the unique element of $\mathcal{G}(A)$, see the discussion closing Section 3 in Schreiber (2005). We are working on this conjecture at present, yet we are unable to provide their formal proofs at the current stage of our research. However, should these conjectures hold true as stated, initiating the multi-edge system dynamics in a domain $D$ with the collection of entry events $\text{Entry}(A_{\mathbb{R}^3} \cap D)$, for $A_{\mathbb{R}^3}$ denoting the unique thermodynamic limit, would result in a consistent family of polyhedral fields, as constructed by Arak & Surgailis (1989) for the polygonal setting. A further essential task would be to provide a feasible description of the entry process $\text{Entry}(A_{\mathbb{R}^3} \cap D)$, which we anticipate to be of a rather complicated nature.

4 Birth site birth and death dynamics for simulating polyhedral fields

The purpose of the current section is to construct, much along the lines of Schreiber (2005), Section 2.1, a random dynamics on the space $\Gamma_D$ of admissible polyhedral configurations which leaves invariant the law of $A_{D|E}$, where the collection $E$ of entry events is to remain fixed throughout the section. This will allow us later to provide modifications of this dynamics suitable for simulation of Gibbsian modifications of $A_{D|E}$. The purpose of developing this algorithm is its envisioned application, as a component of suitable simulated annealing techniques, to volumetric image segmentation along the lines of our previous papers Kluszczyński et al. (2004,2005) where we considered the corresponding two-dimensional problem.

In the sequel, particular care is needed to distinguish between the notion of time considered in the dynamic representation of the polygonal field $A_{D|E}$ given in Subsection 2.3 above, and the notion of time to be introduced for the random dynamics on $\Gamma_{D|E}$ constructed below. To make this distinction clear we shall refer to the former as to the
representation time (r-time for short) and shall keep for it the notation \( t \), while the latter will be called the simulation time (s-time for short) and will be consequently denoted by \( s \) in the sequel.

It is convenient for our exposition below to perceive each individual infinitesimal triangle birth site ((IT)-birth site) in the dynamic representation, see \( \text{IT} \), as coming with an associated random number generator, represented for instance as an infinite sequence of i.i.d. random variables uniformly distributed on \([0, 1]\) and used to determine the subsequent moments and angles/velocities for critical events \( \text{IA} \) and \( \text{IE} \) involving multi-edges resulting from the considered \( \text{IT} \) birth event. In other words, each \( \text{IT} \)-birth site is assumed to carry a package enclosing all randomness the resulting multi-edges may possibly encounter during their evolution, and the above is just one technical possibility of how this can be achieved. We shall use the name of a birth package for an infinitesimal triangle birth site with such a random number generator attached. In these terms, it is now easily seen that the polyhedral configuration obtained in course of the dynamic construction depends deterministically on the underlying collection of birth packages.

Consider a polyhedral configuration \( \gamma \in \Gamma_{D|E} \) and a new infinitesimal triangle birth site \( x_0 \in D \) not yet present in \( \gamma \), extended to a birth package in the standard way as discussed above. Adding this birth package to the collection of birth packages determining \( \gamma \) and keeping the evolution rules of the dynamic representation (Dynamics for IT,IA,IE) results in a new configuration to be denoted by \( \gamma \oplus x_0 \). Likewise, removing an \( \text{IT} \)-birth site \( x_1 \) from a configuration \( \gamma \) in which it was present yields a new polyhedral configuration \( \gamma \ominus x_1 \).

Taking into account that the collection of the \( \text{IT} \)-birth sites for \( A_{D|E} \) is chosen according to the Poisson point process with intensity \( \frac{\pi^4}{6} \) as specified in (Dynamics for IT), we easily see that the law of \( A_{D|E} \) is invariant with respect to the following pure-jump
Markovian (IT)-birth site birth and death dynamics on \( \Gamma_{D|E} \), further denoted by (BS), with \( \gamma_s \) standing for the state at time \( s \) and with \( \gamma_0 \in \Gamma_{D|E} \).

**(BS:birth)** With intensity \( \frac{\pi}{6} ds \) set \( \gamma_{s+ds} := \gamma_s \oplus x \),

**(BS:death)** For each (IT)-birth site in \( \gamma_s \) with intensity \( 1 \cdot ds \) set \( \gamma_{s+ds} := \gamma_s \ominus x \), otherwise keep \( \gamma_{s+ds} = \gamma_s \).

In fact, more can be stated, see also Proposition 1 in Schreiber (2005)

**Theorem 2** *The distribution of the polygonal field \( \mathcal{A}_{D|E} \) is the unique invariant law of the dynamics given by (BS:birth), and (BS:death). The resulting stationary process is reversible. Moreover, for any initial distribution of \( \gamma_0 \) concentrated on \( \Gamma_{D|E} \) the laws of the random polygonal fields \( \gamma_s \) converge in total variation to the law of \( \mathcal{A}_{D|E} \) as \( s \to \infty \).*

While the invariance was discussed above and the reversibility is clear, the uniqueness and convergence statements in the above theorem require a short justification. They both follow by the observation that, in finite volume, regardless of the initial state, the process \( \gamma_s \) spends a non-null fraction of time in the state where no polyhedral faces other than those arising in \( \mathcal{E} \) are present. Indeed, this observation allows us to conclude the required uniqueness and convergence by a standard coupling argument.

**Dynamics for Gibbsian modifications** Assume that a Hamiltonian (energy function) \( \mathcal{H} \) is defined on the space \( \Gamma_D \) of admissible polyhedral configurations and it satisfies

\[
\mathcal{H}(\gamma) \geq -A \text{Vol}(D) - B
\]

for some positive constants \( A, B \). Then it is clear that the partition function

\[
Z_D[\mathcal{H}] := \mathbb{E}\exp(-\mathcal{H}(\mathcal{A}_{D|E}))
\]
is finite. Consequently, the corresponding Gibbsian modification $A_H^{D|\mathcal{E}}$ can be considered with

$$\frac{d\mathcal{L}(A_H^{D|\mathcal{E}})[\gamma]}{d\mathcal{L}(A_D|\mathcal{E})} = \frac{\exp(-\mathcal{H}(\gamma))}{Z_D[\mathcal{H}]}, \quad \gamma \in \Gamma_{D|\mathcal{E}}. \quad (12)$$

Consider the following modification of the basic (BS) dynamics:

**BS[H]:birth** With intensity $\frac{\pi^4}{6}ds$ propose the update $\delta := \gamma_s \oplus x$. Then, with probability $\min(1, \exp(\mathcal{H}(\gamma_s) - \mathcal{H}(\delta)))$ accept this update, putting $\gamma_{s+ds} := \delta$, otherwise keep $\gamma_{s+ds} := \gamma_s$.

**BS[H]:death** For each (IT)-birth site in $\gamma_s$ with intensity $1 \cdot ds$ set $\delta := \gamma_s \ominus x$. Then, with probability $\min(1, \exp(\mathcal{H}(\gamma_s) - \mathcal{H}(\delta)))$ put $\gamma_{s+ds} := \delta$, otherwise keep $\gamma_{s+ds} := \gamma_s$.

In other words, the original dynamics (BS) is used in the standard way to propose a new configuration $\delta$, which is then accepted with probability $\min(1, \exp(\mathcal{H}(\gamma_s) - \mathcal{H}(\delta)))$ and rejected otherwise. As a direct consequence of Theorem 2 and by a standard check of the detailed balance condition we get

**Theorem 3** The distribution of the polyhedral field $A_H^{D|\mathcal{E}}$ is the unique invariant law of the dynamics given by (BS[H]:birth) and (BS[H]:death). The resulting stationary process is reversible. Moreover, for any initial distribution of $\gamma_0$ concentrated on $\Gamma_{D|\mathcal{E}}$ the laws of the random polyhedral fields $\gamma_s$ converge in total variation to the law of $A_H^{D|\mathcal{E}}$ as $s \rightarrow \infty$.

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