Abstract

A dynamical system is called contractive if any two solutions approach one another at an exponential rate. More precisely, the dynamics contracts lines at an exponential rate. This property implies highly ordered asymptotic behavior including entrainment to time-varying periodic vector fields and, in particular, global asymptotic stability for time-invariant vector fields. Contraction theory has found numerous applications in systems and control theory because there exist easy to verify sufficient conditions, based on matrix measures, guaranteeing contraction.

Here, we provide a geometric generalization of contraction theory called \( k \)-contraction. A dynamical system is called \( k \)-contractive if the dynamics contracts \( k \)-parallelotopes at an exponential rate. For \( k = 1 \) this reduces to standard contraction.

We describe easy to verify sufficient conditions for \( k \)-contraction based on a matrix measure of the \( k \)th additive compound of the Jacobian of the vector field. We also describe applications of the seminal work of Muldowney and Li, that can be interpreted in the framework of \( 2 \)-contraction, to systems and control theory.

Key words: Asymptotic stability, contraction analysis, matrix measures, variational equation, entrainment, compound matrices.

1 Introduction

Contraction theory provides powerful tools for analyzing the asymptotic behavior of time-varying nonlinear dynamical systems \cite{Lohmiller1998, Amintzare2014, Forni2014}. Unlike Lyapunov methods, it studies the difference between any pair of solutions rather than convergence to a specific solution. If the difference converges to zero then this implies highly ordered behavior. For example, if the state-space includes an equilibrium \( e \) then any solution is attracted to \( e \) implying global exponential asymptotic stability. More generally, if the vector field is time-varying and \( T \)-periodic and the state-space is compact and convex then there exists a unique \( T \)-periodic solution \( \gamma \) and any solution converges to \( \gamma \) \cite{Russo2014, Lohmiller1998, Forni2014}. In other words, the system entrains to the periodic excitation modeled by the time-varying vector field.

Sufficient conditions for contraction can be derived using a differential Lyapunov function \cite{Lohmiller1998, Forni2014} or by showing that some matrix measure of the Jacobian of the vector field is uniformly negative \cite{Coppe1963, Amintzare2014}. For a given matrix measure, the latter condition is easy to check. Contraction theory has found numerous applications in the field of systems and control including: control synthesis for regulation \cite{Pavlov2008} and tracking \cite{Wu2019}, observer design \cite{Lohmiller2000, Sanfelice2011, Aghamm12003, Slotine2007}, robotics \cite{Manchester2018}, multi-agents systems \cite{Russo2009}, and systems biology \cite{Margaliot2014, Russo2010}.

There is a large body of work on various generalizations of contraction theory. Examples include contraction with respect to (w.r.t.) time- and space-dependent norms, that are particularly relevant for systems whose trajectories evolve on manifolds \cite{Forni2014}. The recent paper by Jafarpour et al. \cite{Jafarpour2021} considers contraction w.r.t. a seminorm. This is closely related to partial contraction (or convergence to an invariant linear subspace) \cite{Slotine2007}. Another generalization is based on the fact that contraction guarantees strong asymptotic properties, like stability...
and entrainment, and thus it is often enough to consider systems that become contractive after some transient [Margaliot et al., 2017, 2016]. Another related line of work (Forni and Sepulchre, 2019) considers systems that are monotone w.r.t. ellipsoidal norms. These are not necessarily contractive systems, but the quadratic structure of the norm implies that they satisfy a form of partial contraction.

In this paper, we present a geometric generalization called $k$-contraction. This is motivated by the seminal work of Muldowney (1990) who used what we call here 2-contraction to derive generalizations of results of Poincaré, Bendixson, and Dulac on planar systems to higher-dimensional systems (see also Li and Muldowney (1993, 1996)). The results of Muldowney and his colleagues proved very useful in analyzing mathematical models for the spread of epidemics (see, e.g., Li and Muldowney (1995a)). Indeed, these models typically include at least two equilibrium points corresponding to the disease-free and the endemic steady-states. Thus, they cannot be contractive. However, they are sometimes 2-contractive and this can be used to analyze their asymptotic behavior.

To explain the notion of $k$-contractive systems in the simplest setting, consider a time-varying linear system. Fix $k + 1$ different initial conditions on the unit simplex, and an initial time $t_0$. The corresponding solutions define at any time $t \geq t_0$ a $k$-parallelepiped. The system is called $k$-contractive if the volume of this parallelepiped decays to zero at an exponential rate. For $k = 1$ this reduces to standard contraction. For nonlinear systems, $k$-contraction is defined by considering a $k$-parallelepiped on the tangent space (Do Carmo, 1992).

The tools needed to define and analyze $k$-contraction include the multiplicative and additive compound matrices (Muldowney, 1990). The latter also play an important role in the theory of totally positive dynamical systems (see the recent tutorial (Margaliot and Sontag, 2019) and also (Weiss and Margaliot, 2021)). These notions are not necessarily well-known in the systems and control community, and we try to provide here a self-contained exposition of $k$-contraction and its analysis using these tools.

To provide intuition, we begin with two simple linear examples. The analysis of nonlinear systems is based on studying the associated variational equation which is a linear time-varying system.

**Example 1.** Consider the LTI system
\[ \dot{x} = Ax, \quad x \in \mathbb{R}^{2 \times 2}. \] (1)

Let $x(t, x_0)$ denote the solution of (1) at time $t$ for the initial condition $x(0) = x_0$. Pick $u, v \in \mathbb{R}^2$. Consider the parallelogram $\{r_1x(t, u) + r_2x(t, v) \mid r_1, r_2 \in [0, 1]\}$. The area of this parallelogram is $|s(t)|$, where
\[ s(t) := \det\left[ x(t, u) \ x(t, v) \right]. \]

This gives
\[ s(t) = \det\left( \exp(At)u \ \exp(At)v \right) = \det(\exp(At)) \det\left[ u \ v \right] = \det(\exp(At))s(0). \]

By the Abel-Jacobi-Liouville identity (Teschl, 2012),
\[ \frac{d}{dt} \det(\exp(At)) = \text{tr}(A) \det(\exp(At)), \]
where $\text{tr}(A)$ denotes the trace of $A$, so $s(t) = \text{tr}(A)s(t)$, and $s(t) = \exp(\text{tr}(A)t)s(0)$. Summarizing, the area of the parallelogram spanned by the solutions $x(t, u)$ and $x(t, v)$ of the LTI system (1) decays to zero at an exponential rate if and only if (iff) $\text{tr}(A) < 0$. We then say that the two-dimensional system (1) is 2-contractive. The condition $\text{tr}(A) < 0$ is weaker than that needed for standard contraction, namely, that $A$ is Hurwitz (Aminzare and Sontag, 2014). On the other hand, standard contraction implies that $\text{tr}(A) < 0$, i.e., 2-contraction.

To generalize these notions to systems whose trajectories evolve on $\mathbb{R}^n$, with $n > 2$, requires the use of multiplicative and additive compound matrices. The next example demonstrates this.

**Example 2.** Consider the LTI system
\[ \dot{x} = Ax, \quad x \in \mathbb{R}^{3 \times 3}. \] (2)

Pick two arbitrary initial conditions $u, v \in \mathbb{R}^3$. Let
\[ M := \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix}. \]
Recall that the 2nd multiplicative compound matrix of $M$, denote $M^{(2)}$, is the matrix of all 2 × 2 minors of $M$ ordered lexicographically (as explained in Section 2.1 below). Specifically,
\[ M^{(2)} = \begin{bmatrix} u_1v_2 - u_2v_1 \\ u_1v_3 - u_3v_1 \\ u_2v_3 - u_3v_2 \end{bmatrix}. \]

(Up to a minus sign) the entries of the cross product $u \times v$. Thus, $|M^{(2)}|_2 = |u \times v|_2$, where $| \cdot |_2$ is the $L_2$ norm, and this implies that $|M^{(2)}|_2$ is the area of the parallelogram determined by $u, v$.

Thus, the area of the parallelogram generated by $x(t, u)$ and $x(t, v)$ is $|s(t)|_2$, where $s(t) := \left[ x(t, u) \ x(t, v) \right]^{(2)}$. Recall that the multiplicative compound satisfies $(PQ)^{(2)} = P^{(2)}Q^{(2)}$ for any $P \in \mathbb{R}^{m \times m}, Q \in \mathbb{R}^{m \times k}$. 


Thus, we have
\[ s(t) = \left[ \exp(At)u \exp(At)v \right]^{(2)} 
= (\exp(At)M)^{(2)} 
= (\exp(At))^{(2)} M^{(2)} 
= (\exp(At))^{(2)} s(0). \] (3)

It is useful to derive a differential equation for \( s(t) \). The 2nd additive compound of a square matrix \( P \) is defined as
\[ P^{[2]} := \frac{d}{d \varepsilon} (I + \varepsilon P)^{(2)}|_{\varepsilon=0} \]
\[ = \frac{d}{d \varepsilon} (\exp(\varepsilon P))^{(2)}|_{\varepsilon=0}. \]

The term additive is due to the fact that \( (P + Q)^{[2]} = P^{[2]} + Q^{[2]} \) for any \( P, Q \) in \( \mathbb{R}^{n \times n} \). Now (3) gives \( s(t) = A^{[2]}s(t) \). Thus, if \( \mu(\cdot) \) denotes the matrix measure induced by a norm \( \| \cdot \| \), and \( \mu(A^{[2]}) \leq -\eta < 0 \) then \( |s(t)| \leq \exp(-\eta t)|s(0)| \). We then say that the three-dimensional system (2) is 2-contractive.

Summarizing, \( k \)-contraction is related to the contraction of the volume of \( k \)-parallelotopes under the dynamics.

The remainder of this paper is organized as follows. The next section reviews several notions and results that are required to analyze \( k \)-contraction including compound matrices and the volume of parallelotopes. Section 3 introduces the new notion of \( k \)-contraction. Section 4 describes applications of \( k \)-contraction to several problems from systems and control theory.

2 Preliminaries

This section describes several notions that are used later on. We begin by reviewing multiplicative and additive compound matrices, and then the relation between the volume of \( k \)-parallelotopes and the multiplicative compound. We also review the spectral properties of compound matrices, and describe the relation between the stability of the LTV system \( \dot{x}(t) = A(t)x(t) \) and the stability of the associated \( k \)th compound system.

For two integers \( i, j \), with \( i \leq j \), let \([i,j] := \{i,i+1,\ldots,j\}\). Let \( Q_{k,n} \) denote the set of increasing sequences of \( k \) numbers from \([1,n] \) ordered lexicographically. For example, \( Q_{3,3} = \{(1,2),(1,3),(2,3)\} \). The lexicographic order \( \prec_{\text{le}} \) is defined as follows. If \( a,b \) are two sequences in \( Q_{k,n} \), and \( a_i [b_i] \) is the \( i \)th element of \( a [b] \) then \( a \prec_{\text{le}} b \) if \( a_j < b_j \), where \( j = \min\{i \in [1,k] : a_i \neq b_i \} \). With a slight abuse of notation we will sometimes treat a sequence in \( Q_{k,n} \) as a set.

2.1 Compound matrices

Given a matrix \( A \in \mathbb{R}^{n \times m} \) and \( k \in [1,\min\{n,m\}] \), recall that a minor of order \( k \) of \( A \) is the determinant of some \( k \times k \) submatrix of \( A \). Consider the \( \binom{n}{k} \binom{m}{k} \) minors of order \( k \) of \( A \). Each such minor is defined by a set of row indices \( \kappa_i \in Q_{k,n} \) and column indices \( \kappa_j \in Q_{k,m} \). This minor is denoted by \( A(\kappa_i | \kappa_j) \). For example, for \( A = \begin{bmatrix} 4 & 5 \\ -1 & 4 \\ 0 & 3 \end{bmatrix} \), we have \( A(\{1,3\}|\{1,2\}) = \det \begin{bmatrix} 4 & 5 \\ 0 & 3 \end{bmatrix} = 12. \)

The \( k \)th multiplicative compound matrix of \( A \), denoted \( A^{[k]} \), is the \( \binom{n}{k} \times \binom{m}{k} \) matrix that includes all the minors of order \( k \) ordered lexicographically, that is, if \( \kappa_f \) is the \( \ell \)th sequence in \( Q_{k,n} \) then the \( j \)th entry of \( A^{[k]} \) is \( A(\kappa_f | \kappa_j) \). For example, for \( n = m = 3 \) and \( k = 2 \), the matrix \( A^{(2)} \) is
\[ \begin{bmatrix} A(\{1,2\}|\{1,2\}) & A(\{1,2\}|\{1,3\}) & A(\{1,2\}|\{2,3\}) \\ A(\{1,3\}|\{1,2\}) & A(\{1,3\}|\{1,3\}) & A(\{1,3\}|\{2,3\}) \\ A(\{2,3\}|\{1,2\}) & A(\{2,3\}|\{1,3\}) & A(\{2,3\}|\{2,3\}) \end{bmatrix} \]

By definition, \( A^{(1)} = A \) and if \( A \in \mathbb{R}^{n \times n} \) then \( A^{(n)} = \det(A) \). If \( D \) is an \( n \times n \) diagonal matrix, i.e. \( D = \text{diag}(d_1, \ldots, d_n) \) then
\[ D^{(k)} = \text{diag}(d_1, \ldots, d_{k-1}, d_k, \ldots, d_n). \]

In particular, \( I^{(k)} \) is the \( r \times r \) identity matrix, with \( r := \binom{n}{k} \).

The Cauchy-Binet formula (see, e.g., [Fallat and Johnson 2011, Thm. 1.1.1]) asserts that
\[ (AB)^{(k)} = A^{(k)}B^{(k)} \] (4)
for any \( A \in \mathbb{R}^{n \times p}, B \in \mathbb{R}^{p \times m}, \) and \( k \in [1,\min\{n,p,m\}] \). When \( n = p = m = k \) this becomes the familiar formula \( \det(AB) = \det(A) \det(B) \). If \( A \) is square and non-singular then (4) implies that \( f^{(k)} = (AA^{-1})^{(k)} = A^{(k)}(A^{-1})^{(k)} \), so \( (A^{-1})^{(k)} = (A^{(k)})^{-1} \).

Note that any entry in \( A^{(k)} \) is a polynomial in the entries of \( A \). The \( k \)th additive compound matrix of \( A \in \mathbb{R}^{n \times n} \) is defined by \( A^{[k]} := \frac{d}{d \varepsilon} (I + \varepsilon A)^{(k)}|_{\varepsilon=0} \). This implies that \( A^{[k]} = A \), and that
\[ (I + \varepsilon A)^{(k)} = I + \varepsilon A^{[k]} + o(\varepsilon), \] (5)
that is, \( \varepsilon A^{[k]} \) is the first-order term in the Taylor series of \( (I + \varepsilon A)^{(k)} \).

Example 3. If \( D = \text{diag}(d_1, \ldots, d_n) \) then
\[ (I + \varepsilon D)^{(k)} = \text{diag}(\prod_{i=1}^{k}(1 + \varepsilon d_i), \ldots, \prod_{i=n-k+1}^{n}(1 + \varepsilon d_i)), \]
so (5) gives \( D^{(k)} = \text{diag}(\sum_{i=1}^{k} d_i, \ldots, \sum_{i=n-k+1}^{n} d_i). \)
Example 4. Consider the case $n = 3$ and $k = 2$. Then

\[
(I + \varepsilon A)^{(2)} = \begin{bmatrix}
1 + \varepsilon a_{11} & \varepsilon a_{12} & \varepsilon a_{13} \\
\varepsilon a_{21} & 1 + \varepsilon a_{22} & \varepsilon a_{23} \\
\varepsilon a_{31} & \varepsilon a_{32} & 1 + \varepsilon a_{33}
\end{bmatrix}
\]

\[
= I + \varepsilon \begin{bmatrix}
a_{11} + \varepsilon a_{12} & a_{23} & -a_{13} \\
a_{32} & a_{11} + \varepsilon a_{33} & a_{12} \\
-a_{31} & a_{21} & a_{22} + \varepsilon a_{33}
\end{bmatrix} + o(\varepsilon),
\]

so

\[
A^{[2]} = \frac{d}{d\varepsilon} (I + \varepsilon A)^{(2)}|_{\varepsilon = 0}
\]

\[
= \begin{bmatrix}
a_{11} + a_{22} & a_{23} & -a_{13} \\
a_{32} & a_{11} + a_{33} & a_{12} \\
-a_{31} & a_{21} & a_{22} + a_{33}
\end{bmatrix}
\]

It follows from (5) and the properties of the multiplicative compound that

\[
I + \varepsilon (A + B)^{[k]} + o(\varepsilon) = (I + \varepsilon (A + B))^{(k)}
\]

\[
= (I + \varepsilon A)^{(k)}(I + \varepsilon B)^{(k)} + o(\varepsilon)
\]

\[
= (I + \varepsilon A^{[k]} + o(\varepsilon))(I + \varepsilon B^{[k]} + o(\varepsilon))
\]

\[
= I + \varepsilon (A^{[k]} + B^{[k]}) + o(\varepsilon),
\]

so taking $\varepsilon \to 0$ and using the continuity w.r.t. $\varepsilon$ yields

\[
(A + B)^{[k]} = A^{[k]} + B^{[k]},
\]

thus justifying the term additive compound.

The matrix $A^{[k]}$ can be described explicitly in terms of the entries $a_{ij}$ of $A$.

Lemma 1 (Schwarz (1970); Fiedler (2008)).

Let $A \in \mathbb{R}^{n \times n}$. Fix $k \in [1, n]$. The entry of $A^{[k]}$ corresponding to $(i_1|\cdots|i_k|j_1, \ldots, j_k)$ is:

- $\sum_{t=1}^{k} a_{i_t j_t}$ if $i_t = j_t$ for all $t \in [1, k]$;

- $(-1)^{i_1+\cdots+i_k} a_{i_1 j_1} \cdots a_{i_k j_k}$ if all the indices in $i_t$ and $j_t$ coincide except for a single index $i_q \neq j_m$; and

- 0, otherwise.

The first case in Lemma 1 corresponds to diagonal entries of $A^{[k]}$. All the other entries of $A^{[k]}$ are either zero or an entry of $A$ multiplied by either plus or minus one. Two special cases of Lemma 1 are:

\[
A^{[1]} = A \text{ and } A^{[n]} = \text{tr}(A).
\]

Example 5. Consider the case $n = 4$, i.e., $A = \{a_{ij}\}_{i,j=1}^4$. Then Lemma 1 yields

\[
A^{[2]} = \begin{bmatrix}
a_{11} + a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\
a_{32} & a_{11} + a_{33} & a_{34} & a_{12} & 0 & a_{13} \\
-a_{31} & a_{21} & 0 & a_{22} + a_{33} & a_{34} & -a_{24} \\
-0 & a_{41} & a_{43} & a_{11} + a_{33} + a_{44} & a_{31} & a_{32} + a_{33} + a_{44}
\end{bmatrix},
\]

and

\[
A^{[3]} = \begin{bmatrix}
a_{11} + a_{22} + a_{33} & a_{34} & -a_{24} & -a_{14} & -a_{13} & a_{12} \\
a_{43} & a_{11} + a_{22} + a_{44} & -a_{23} & 0 & -a_{13} & -a_{12} \\
-a_{42} & a_{32} & a_{11} + a_{33} + a_{44} & 0 & a_{31} & -a_{12} \\
0 & a_{21} & a_{23} & a_{22} + a_{44} & a_{24} & a_{23} + a_{44}
\end{bmatrix}.
\]

The entry in the first row and third column of $A^{[3]}$ corresponds to $(\kappa_i|\kappa_j) = \{(1, 2, 3)|\{1, 3, 4\}\}$, and since $\kappa_i$ and $\kappa_j$ coincide except for the entry $j_2 = 2$ and $j_3 = 4$, this entry is $(-1)^{2+3} a_{1223} = -a_{24}$. It is useful to index compound matrices using $\kappa_i, \kappa_j$. For example, we write $A^{[k]}(\{1, 2, 3\}|\{1, 3, 4\}) = -a_{24}$.

We next review a “duality relation” of the additive compound. This is a formula relating the matrices $A^{[k]}$ and $A^{[n-k]}$ that both have dimensions $r \times r$, where $r := \binom{n}{k}$. Let $U_r \in \mathbb{R}^{r \times r}$ be the matrix with entries

\[
u_{ij} = \begin{cases} (-1)^{i+j}, & \text{if } i + j = r + 1, \\
0, & \text{otherwise.} \end{cases}
\]

For example, $U_3 = \begin{bmatrix} 0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0 \end{bmatrix}$. Note that $U_r^T = U_r^{-1}$.

Proposition 1. (Muldewa, 1992) Let $A \in \mathbb{R}^{n \times n}$. Fix $k \in [1, n - 1]$, and let $r := \binom{n}{k}$. Then

\[
(A^{[k]})^T + U_r^T A^{[n-k]} U_r = \text{tr}(A) I_r.
\]

For example, suppose that $n = 3$, and $A = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Then for $k = 2$, (8) becomes

\[
\text{diag}(\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3) + \text{diag}(\lambda_3, \lambda_2, \lambda_1) = (\lambda_1 + \lambda_2 + \lambda_3) I_3.
\]

Our next goal is to provide a clear geometric interpretation for the multiplicative compound of a matrix.

2.2 The volume of a parallelotope

Fix $k \in [1, n]$. The parallelotope generated by the vectors $x^1, \ldots, x^k \in \mathbb{R}^n$ is the set

\[
P(x^1, \ldots, x^k) := \{\sum_{i=1}^k r_i x^i | r_i \in [0, 1]\}.
\]
Note that $P$ always includes the origin, and that $P$ is the image of the unit $k$-cube under the matrix

$$X := \begin{bmatrix} x^1 & \ldots & x^k \end{bmatrix} \in \mathbb{R}^{n \times k}.$$  

The Gram matrix (see e.g. (Horn and Johnson, 2013, p. 441)) associated with $x^1, \ldots, x^k$ is the $k \times k$ symmetric matrix:

$$G(x^1, \ldots, x^k) := X^T X.$$  

Note that $G$ is positive semi-definite, and is positive definite if the vectors $x^1, \ldots, x^k$ are linearly independent. For example, for $k = 2$ we have

$$G(x^1, x^2) := \begin{bmatrix} |x^1|^2 & (x^1)^T x^2 \\ (x^2)^T x^1 & |x^2|^2 \end{bmatrix},$$

where $| \cdot |_2$ denotes the $L_2$ norm, i.e., the Euclidean norm.

The volume of $P(x^1, \ldots, x^k)$, denoted $\text{vol}(P(x^1, \ldots, x^k))$, is defined in a recursive manner.

**Definition 1.** For $k = 1$, $\text{vol}(P(x^1)) := |x^1|_2$. For any $k > 1$,

$$\text{vol}(P(x^1, \ldots, x^k)) := \text{vol}(P(x^1, \ldots, x^{k-1})) h,$$

where $h \geq 0$ is the Euclidean distance from $x^k$ to the subspace spanned by $\{x^1, \ldots, x^{k-1}\}$.

Note that $P(x^1, \ldots, x^{k-1})$ can be viewed as the $(k - 1)$-dimensional “base” of $P(x^1, \ldots, x^k)$, and $h$ is the related “height”. Hence, Definition 1 has a clear geometric interpretation, and it reduces to the standard notions of length, area, and volume if $k = 1, 2, 3$, respectively.

The next result provides a simple algebraic expression for the volume in terms of the Gram matrix.

**Proposition 2.** ([Gantmacher, 1960, Chapter IX]) The volume of $P(x^1, \ldots, x^k)$ satisfies

$$\text{vol}(P(x^1, \ldots, x^k)) = \sqrt{\det(G(x^1, \ldots, x^k))}.$$  

Note that in the special case where $k = n$, the matrix $X$ is a square matrix, and (11) gives

$$(\text{vol}(P(x^1, \ldots, x^k))^2 = \det(G(x^1, \ldots, x^n))\) = \det(X^T X)\) = (\det(X))^2\),$ 

i.e. the well-known formula

$$\text{vol}(P(x^1, \ldots, x^n)) = | \det\begin{bmatrix} x^1 & \ldots & x^n \end{bmatrix}|.$$

Fig. 1. Using the multiplicative compound to compute the area of a parametrized body defined by $\phi : \mathcal{D} \rightarrow \mathbb{R}^3$, with $\mathcal{D}$ a rectangle in $\mathbb{R}^2$.

To relate the volume of $P(x^1, \ldots, x^k)$ to the multiplicative compound, note that combining (9) and the Cauchy-Binet formula yields

$$\det(G(x^1, \ldots, x^k)) = \det(X^T X) = (X^T X)^{(k)} = (X^{(k)})^T X^{(k)}.$$ 

Since $X \in \mathbb{R}^{n \times k}$, $X^{(k)}$ is an $(n \times k)$ column vector, so $\det(G(x^1, \ldots, x^k)) = |X^{(k)}|^2$. Combining this with Prop. 2 yields the elegant formula

$$\text{vol}(P(x^1, \ldots, x^k)) = |X^{(k)}|_2.$$  

When the vectors $x^i$ depend on time and we consider asymptotic properties (e.g. convergence to zero of the volume as time goes to infinity), it is possible to use any vector norm $|X^{(k)}|$, as all norms on $\mathbb{R}^n$ are equivalent.

Based on the above analysis, the multiplicative compound can be used to compute the volume of parameterized bodies ([Mudowney, 1990]). Consider a compact set $\mathcal{D} \subset \mathbb{R}^k$ and a continuously differentiable mapping $\phi : \mathcal{D} \rightarrow \mathbb{R}^n$, with $n \geq k$. This induces the set

$$\phi(\mathcal{D}) := \{ \phi(r) \mid r \in \mathcal{D} \} \subseteq \mathbb{R}^n.$$  

Since $\mathcal{D}$ is compact and $\phi(\cdot)$ is continuous, $\phi(\mathcal{D})$ is closed. The volume of $\phi(\mathcal{D})$ is

$$\text{vol}(\phi(\mathcal{D})) = \int_{\mathcal{D}} \left| \partial \phi(r) \right| dr,$$

(14)

(where we assume that the integral exists). Fig. 1 illustrates this formula for the case $k = 2$ and $n = 3$.

### 2.3 Spectral properties of compound matrices

Recall that any $A \in \mathbb{R}^{n \times n}$ admits a Jordan canonical form ([Achieser, 1992]): there exist $T, J \in \mathbb{C}^{n \times n}$, with $T$ non-singular, such that $A = TJT^{-1}$, where $J$ has a Jordan normal form, and in particular is an upper-triangular matrix. The diagonal entries of $J$, denoted $\lambda_i$, $i = 1, \ldots, n$, are the eigenvalues of $A$. 

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**[Horn and Johnson, 2013, p. 441]**

**[Gantmacher, 1960]**

**[Mudowney, 1990]**

**[Achieser, 1992]**
Since $A(k) = T(k)J(k)(T(k))^{-1}$ and $J(k)$ is upper triangular [Muldowney, 1990], the diagonal entries of $J(k)$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$, with $1 \leq i_1 < \cdots < i_k \leq n$, and these are the eigenvalues of $A(k)$.

**Example 6.** Suppose that $Q \in \mathbb{R}^{n \times n}$ is symmetric and positive-definite. Fix $k \in [1,n]$. By definition, $(Q(k))^{T} = (Q(k))^{-1}$, so $Q(k)$ is symmetric. Every eigenvalue of $Q$ is real and positive, and since every eigenvalue of $Q(k)$ is a product of $k$ eigenvalues of $Q$, every eigenvalue of $Q(k)$ is real and positive. We conclude that $Q(k)$ is positive-definite.

Let $u^i$ denote the eigenvector of $A$ corresponding to $\lambda_i$, and let $U := \begin{bmatrix} u^{i_1} & u^{i_2} & \cdots & u^{i_k} \end{bmatrix}$. Then $AU = U \text{diag}(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k})$. If $u^{i_1}, u^{i_2}, \ldots, u^{i_k}$ are linearly independent then $U(k)$ is a nonzero vector and $A(k)U(k) = \prod_{j=1}^{k} \lambda_j U(j)$, i.e. $U(k)$ is an eigenvector of $A(k)$ corresponding to the eigenvalue $\prod_{j=1}^{k} \lambda_j$.

Similarly, $A = TJT^{-1}$ implies that $I + \varepsilon A = T(I + \varepsilon J)T^{-1}$, and thus $(I + \varepsilon A)k = T(k)(I + \varepsilon J)k(T(k))^{-1}$. By (5), $A(k) = T(k)J(k)(T(k))^{-1}$. Since $J$ is upper-triangular, so is $J(k)$, and the diagonal entries of $J(k)$ are of the form $\lambda_i, \lambda_{i_2} + \lambda_i, \ldots, \lambda_i$. According to Lemma 1. Hence, every eigenvalue of $A(k)$ is the sum of $k$ eigenvalues of $A$.

The standard tool for verifying contraction and, as we will see below also $k$-contraction, is matrix measures [Coppel, 1965, Ch. 2] (also called logarithmic norms [Strom, 1975]).

### 2.4 Matrix measures

Consider a vector norm $\cdot : \mathbb{R}^n \to \mathbb{R}_+$. The **induced matrix norm** $||| : \mathbb{R}^{n \times n} \to \mathbb{R}_+$ is $||A|| := \max_{|x|=1} |Ax|$, and the **induced matrix measure** $\mu(\cdot) : \mathbb{R}^{n \times n} \to \mathbb{R}$ is

$$
\mu(A) := \lim_{\varepsilon \to 0} \frac{||I + \varepsilon A|| - 1}{\varepsilon}.
$$

Denote the $L_1$, $L_2$, and $L_\infty$ vector norms by $|x|_1 := \sum_{i=1}^{n} |x_i|$, $|x|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$, and $|x|_\infty := \max_{i} |x_i|$. The corresponding matrix measures are [Vidyasagar, 1978]:

$$
\mu_1(A) = \max_{j} \left( a_{jj} + \sum_{i \neq j} |a_{ij}| \right),
$$

$$
\mu_2(A) = \lambda_1 \left( \frac{A + A^T}{2} \right),
$$

$$
\mu_\infty(A) = \max_{i} \left( a_{ii} + \sum_{j \neq i} |a_{ij}| \right),
$$

where $\lambda_i(S)$ denotes the i-th largest eigenvalue of the symmetric matrix $S$, that is,

$$
\lambda_1(S) \geq \lambda_2(S) \geq \cdots \geq \lambda_n(S).
$$

The matrix measures for $A(k)$ are then [Muldowney, 1990]:

$$
\mu_1(A(k)) = \max_{i} \left( \sum_{p=1}^{k} a_{i_p,i_p} + \sum_{j \neq i} (|a_{j,i_1}| + \cdots + |a_{j,i_k}|) \right),
$$

$$
\mu_2(A(k)) = \sum_{i=1}^{k} \lambda_i \left( \frac{A + A^T}{2} \right),
$$

$$
\mu_\infty(A(k)) = \max_{i} \left( \sum_{p=1}^{k} a_{i_p,i_p} + \sum_{j \neq i} (|a_{i_1,j}| + \cdots + |a_{i_k,j}|) \right),
$$

where the maximum is taken over all $k$-tuples $(i) := \{i_1, \ldots, i_k\} \in Q_{k,n}$. Note that for $k = 1$, (17) reduces to (15).

The next subsection reviews several applications of compound matrices in dynamical systems described by ODEs.

#### 2.5 Compound matrices and ODEs

Consider the LTV system:

$$
\dot{x}(t) = A(t)x(t),
$$

where $A(t)$ is a continuous matrix function. Then $x(t) = \Phi(t,t_0)x(t_0)$, where $\Phi$ is the transition matrix corresponding to (18), satisfying

$$
\frac{d}{dt} \Phi(t,t_0) = A(t)\Phi(t,t_0), \quad \Phi(t_0,t_0) = I.
$$

For the sake of simplicity, we always assume that the initial time is $t_0 = 0$ and write $\Phi(t)$ for $\Phi(t,0)$.

It is useful to know how $\Phi(k)(t) := (\Phi(t))^{(k)}$ evolves in time. Note that $\Phi(k) : \mathbb{R}_+ \to \mathbb{R}^{r \times r}$, with $r := \binom{n}{k}$. For example, Schwarz [1970] considered the following question: what conditions on $A(t)$ guarantee that every minor of $\Phi(t)$ will be positive for all $t > 0$? In other words, $\Phi(t)$ is a totally positive matrix [Fallat and Johnson, 2011] for all $t > 0$. When this holds (18) is called a totally positive differential system (TPDS). Of course, the positivity of every minor of $\Phi(t)$ is equivalent to the positivity of every entry in each of the matrices $\Phi^{(1)}(t), \Phi^{(2)}(t), \ldots, \Phi^{(n)}(t)$.

The additive compound arises naturally when studying the dynamics of the multiplicative compound $\Phi(k)(t)$. Indeed, for any $\varepsilon > 0$,

$$
\Phi^{(k)}(t + \varepsilon) = (\Phi(t) + \varepsilon A(t))\Phi(t)^{(k)} + o(\varepsilon) = (I + \varepsilon A(t))^{(k)}\Phi^{(k)}(t) + o(\varepsilon).
$$
Combining this with (5) and the fact that $\Phi(0) = I$ gives
\[
\frac{d}{dt} \Phi^{(k)}(t) = A^{[k]}(t)\Phi^{(k)}(t), \quad \Phi^{(k)}(0) = I_t,
\]
where $A^{[k]}(t) := (A(t))^{[k]}$. In other words, all the minors of order $k$ of $\Phi(t)$, stacked in the matrix $\Phi^{(k)}(t)$, also follow a linear dynamics with the matrix $A^{[k]}(t)$. Eq. (20) is sometimes called the $k$th compound equation (see e.g. Li and Muldowney (2000)).

For a constant matrix $A$, $\Phi(t) = \exp(At)$, so (20) gives
\[
\exp(A^{[k]}t) = (\exp(At))^{[k]}.
\]
For $k = n$, Lemma 1 shows that $A^{[n]} = \text{tr}(A)$, whereas $\text{det}(\exp(At))$ is the matrix that contains all the $n \times n$ minors of $\exp(At)$, that is, $\text{det}(\exp(At))$. Thus, (21) generalizes the Abel-Jacobi-Liouville identity.

It is useful to know how the $k$th compound equation (20) changes under a coordinate transformation of (18). Let $T \in \mathbb{R}^{n \times n}$ be non-singular. Then
\[
(TAT^{-1})^{[k]}(t) = \frac{d}{dt} (T(I + \varepsilon A)T^{-1})^{[k]}|_{\varepsilon=0} = \frac{d}{dt} (T(I + \varepsilon A)T^{-1})^{[k]}|_{\varepsilon=0} = T^{(k)} A^{[k]}(T^{-1})^{[k]}.
\]
In the context of systems and control theory, it is important to understand the connections between stability of an LTV system and of its associated $k$th compound equation.

2.6 Stability of an LTV system and of its $k$th compound equation

As shown in Muldowney, 1994, Corollary 3.2), under a certain boundedness assumption there is an interesting relation between the stability of (18) and the stability of $\dot{y}(t) = A^{[k]}(t)y(t)$. We state this result in a slightly modified form.

**Proposition 3.** (Muldowney, 1994) Suppose that the LTV system (18) is uniformly stable. Fix $k \in [1, n]$. Then the following two conditions are equivalent.

(a) The LTV system (18) admits an $(n - k + 1)$-dimensional linear subspace $\mathcal{X} \subseteq \mathbb{R}^n$ such that
\[
\lim_{t \to \infty} x(t, x_0) = 0 \text{ for any } x_0 \in \mathcal{X}.
\]
(b) Every solution of
\[
\dot{y}(t) = A^{[k]}(t)y(t)
\]
converges to the origin.

For the sake of completeness, we include the proof in the Appendix.

Prop. 3 implies in particular that if every solution of (24) converges to the origin then for any $\ell > k$ every solution of $\dot{y}(t) = A^{[\ell]}(t)y(t)$ also converges to the origin.

**Example 7.** Consider the simplest case, namely, the LTI system $\dot{x} = Ax$, with $A$ diagonalizable, that is, there exists a nonsingular matrix $T$ such that $TAT^{-1} = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Assume that the real part of every $\lambda_i$ is positive, so that all the solutions are bounded. If there exist $n - k + 1$ eigenvalues with a negative real part (and thus $k - 1$ eigenvalues with a zero real part) then: (1) the dynamics admits an $(n - k + 1)$-dimensional linear subspace $\mathcal{X}$ such that $\lim_{t \to \infty} x(t, a) = 0$ for any $a \in \mathcal{X}$, and (2) the sum of any $k$ eigenvalues of $A$ has a negative real part, so $\dot{y}(t) = A^{[k]}y(t)$ is asymptotically stable.

**Example 8.** Consider the LTV system $\dot{x}(t) = A(t)x(t)$ with $n = 2$ and $A(t) = \begin{bmatrix} -1 & 0 \\ -\cos(t) & 0 \end{bmatrix}$. For any $a \in \mathbb{R}^2$ the solution of this system is $x(t, a) = \Phi(t)a$, with
\[
\Phi(t) = \begin{bmatrix} \exp(-t) & 0 \\ (-1 + \exp(-t)(\cos(t) - \sin(t)))/2 & 1 \end{bmatrix}.
\]
This implies that the system is uniformly stable and that
\[
\lim_{t \to \infty} x(t, a) = \begin{bmatrix} 0 \\ a_2 - (a_1/2) \end{bmatrix}.
\]
The system is not contractive w.r.t. any norm, as not all solutions converge to the equilibrium $0$. However, $A^{[2]}(t) = \text{tr}(A(t)) = -1$ (implying as we will see below that the system is 2-contractive). In particular, for $k = 2$ Condition (b) in Prop. 3 holds. By (25), Condition (a) also holds for the 1-dimensional linear subspace $\mathcal{X} := \text{span}\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]^T$.

We are now ready to introduce the main notion studied in this paper.

3 $k$-contraction

Consider the time-varying nonlinear system:
\[
\dot{x}(t) = f(t, x),
\]
where $f : R_+ \times R^n \to R^n$. We assume throughout that the solutions evolve on a closed and convex state-space $\Omega \subseteq R^n$, and that for any initial condition $a \in \Omega$, a unique solution $x(t, a)$ exists and satisfies $x(t, a) \in \Omega$ for all $t \geq 0$. We also assume that $f$ is continuously differentiable w.r.t. its second variable, and let $J(t, x) := \frac{\partial f}{\partial x}(t, x)$ denote the Jacobian of $f(t, x)$.

Pick $a, b \in \Omega$. Let $h : [0, 1] \to \Omega$ be the line $h(r) := ra + (1-r)b$. Note that the convexity of $\Omega$ implies that $h(r) \in \Omega$ for all $r \in [0, 1]$.
If there exist a matrix measure such that

\[ \mu(J(t,z)) \leq -\eta \text{ for all } t \geq 0 \text{ and all } z \in \Omega \]  

(28)

then it is not difficult to show (Russo et al., 2010) using (27) that

\[ |x(t,a) - x(t,b)| \leq \exp(-\eta t)|a - b| \text{ for all } t \geq 0. \]

If \( \eta > 0 \) then this implies contraction.

Our goal is to generalize these ideas in the case where (28) is replaced by the more general condition \( \mu((J(t,z))[k]) \leq -\eta \) for some \( k \in [1,n] \). It turns out that this condition has a clear geometrical interpretation. To explain this, we first consider an LTV system and then proceed to explain the implications for the nonlinear system (26).

3.1 Linear time-varying systems

We begin by considering the LTV system:

\[ \dot{w}(t) = A(t)w(t). \]  

(29)

For the sake of simplicity, assume throughout that \( A(t) \) is continuous in \( t \), but the extension to the case of measurable and locally essentially bounded matrix functions is straightforward. This case is relevant, for example, when the dynamics depends on a control input.

Definition 2. Pick \( k \in [1,n] \). We say that (29) is \( k \)-contractive if there exist \( \eta > 0 \) and a vector norm \( \cdot \) such that for any \( a^1,\ldots,a^k \in \mathbb{R}^n \), the mapping \( W(t) : \mathbb{R}_+ \to \mathbb{R}^{n \times k} \) defined by \( W(t) := [w(t,a^1) \ldots w(t,a^k)] \)

satisfies

\[ |W^{(k)}(t)| \leq \exp(-\eta t)|W^{(k)}(0)|, \text{ for all } t \geq 0. \]  

(30)

In other words, under the dynamics the volume of any \( k \)-parallelotope decays to zero at an exponential rate.

Example 9. Consider (29) with \( n = 2 \) and the constant matrix \( A = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} \). Pick \( p,q \in \mathbb{R}^2 \). Then

\[
|w(t,p) w(t,q)|^{(2)} = \begin{vmatrix} \exp(3t)p_1 \exp(3t)q_1 \\ \exp(-4t)p_2 \exp(-4t)q_2 \end{vmatrix}^{(2)} = \det \begin{bmatrix} \exp(3t)p_1 \exp(3t)q_1 \\ \exp(-4t)p_2 \exp(-4t)q_2 \end{bmatrix} = \exp(-t)|p|^{(2)},
\]

so the system is 2-contractive with \( \eta = 1 \). More generally, Example 1 shows that when \( n = 2 \) and \( A \in \mathbb{R}^{2 \times 2} \) is a constant matrix then (29) is 2-contractive iff \( \text{tr}(A) < 0 \).

An important advantage of standard contraction is that it admits an easy to check sufficient condition based on matrix measures. The next result provides an easy to check sufficient condition for \( k \)-contraction of (29) in terms of \( A^{[k]}(t) \).

Theorem 4. If there exist \( \eta > 0 \) and a vector norm \( |\cdot| \), with induced matrix measure \( \mu : \mathbb{R}^{n \times n} \to \mathbb{R} \), such that

\[ \mu(A^{[k]}(t)) \leq -\eta \text{ for all } t \geq 0 \]  

(31)

then (29) is \( k \)-contractive.

Proof. For \( k = 1 \) the definition of \( k \)-contraction reduces to standard contraction, and condition (31) reduces to the standard matrix measure sufficient condition for contraction, as \( A^{[1]} = A \). Consider the case \( k > 1 \). Pick \( a^1,\ldots,a^k \in \mathbb{R}^n \). Then \( W(t) = A(t)W(t) \). Hence, \( \frac{d}{dt}W^{(k)}(t) = A^{[k]}(t)W^{(k)}(t) \). Now, using standard results on contraction, (31) implies that

\[ |W^{(k)}(t)| \leq \exp(-\eta t)|W^{(k)}(0)| \text{ for all } t \geq 0, \]

and this completes the proof.

Remark 1. Consider \( A \in \mathbb{R}^{n \times n} \). There exists some matrix measure \( \mu \) such that \( \mu(A) < 0 \) iff \( A \) is Hurwitz (Aminzare and Sontag, 2014). Hence, there exists some matrix measure \( \mu \) such that \( \mu(A^{[k]}) < 0 \) iff \( A^{[k]} \) is Hurwitz, that is, iff the sum of every \( k \) eigenvalues of \( A \) has a negative real part.

The next simple example describes an LTI system that is “on the verge” of being 2-contractive.

Example 10. Consider \( \dot{w} = Aw \) with \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

Since \( \frac{d}{dt}(x_1^2(t) + x_2^2(t)) = 0 \), the solution for any \( x(0) \) is a circle with radius \( r := \sqrt{x_1^2(t) + x_2^2(t)} = \sqrt{x_1^2(0) + x_2^2(0)} \).
Let \( W(t) := \begin{bmatrix} w(t,a^1) & w(t,a^2) \end{bmatrix} \). Then,
\[
\frac{d}{dt} W^{(2)}(t) = A^{[2]} W^{(2)}(t) = tr(A)W^{(2)}(t) = 0.
\]

This agrees with the fact that the area of the parallelopied generated by \( x(t,a^1) \) and \( x(t,a^2) \) remains constant under the flow.

We considered condition (31) in the context of the \( n \times k \) matrix \( W(t) \) and the vector \( W^{(k)}(t) \). Yet, it also has implications for the \( n \times n \) transition matrix of the LTV (29).

Let \( | \cdot | \) denote the vector norm that induces the matrix measure \( \mu \) in (31), and let \( || \cdot || \) denote the matrix norm induced by \( | \cdot | \).

**Proposition 5.** Let \( \Phi(t) \) denote the transition matrix corresponding to (29), that is,
\[
\dot{\Phi}(t) = A(t)\Phi(t), \quad \Phi(0) = I.
\]

Fix \( k \in [1,n] \). Then
\[
\exp(-\int_0^t \mu(-A^{[k]}(s)) \, ds) \leq ||\Phi^{(k)}(t)|| \leq \exp(\int_0^t \mu(A^{[k]}(s)) \, ds)
\]
for all \( t \geq 0 \). In particular, if (31) holds then
\[
||\Phi^{(k)}(t)|| \leq \exp(-\eta t), \quad \text{for all } t \geq 0.
\]

The proof follows by combining the fact that \( \dot{\Phi}^{(k)}(t) = A^{[k]}(t)\Phi^{(k)}(t), \Phi^{(k)}(0) = I \), with Coppel’s inequality.

For the particular case of the \( L_2 \) norm we have \( ||A||_2 = \sqrt{\lambda_{\text{max}}(A^T A)} \) and \( \mu_2(A) = (1/2) \lambda_{\text{max}}(A + A^T) \), so Prop. 5 gives
\[
\exp(\int_0^t \sum_{i=n-k+1}^{n} \lambda_i(A(s) + A^T(s)) \, ds) \leq \lambda_{\text{max}}((\Phi^{(k)}(t))^T \Phi^{(k)}(t)) \leq \exp(\int_0^t \sum_{i=1}^{k} \lambda_i(A(s) + A^T(s)) \, ds).
\]

This inequality has been used by Smith (1986) to bound the Hausdorff dimension of chaotic attractors of nonlinear dynamical systems.

The next result provides an inequality relating matrix measures of \( A^{[k]} \) and \( A^{[n-k]} \). Recall that if \( | \cdot | : \mathbb{R}^n \to \mathbb{R}_+ \) is a vector norm, \( \mu : \mathbb{R}^{n \times n} \to \mathbb{R}_+ \) is the induced matrix measure, and \( P \in \mathbb{R}^{n \times n} \) is non-singular then the vector norm defined by \( |x|_P := |Px| \) induces the matrix measure \( \mu_P (A) = \mu(PAP^{-1}) \).

**Proposition 6.** Let \( A \in \mathbb{R}^{n \times n} \). Fix \( k \in [1,n-1] \), and let \( r := \binom{n}{k} \). Let \( U_r \) be the matrix defined in (7). Then for any matrix measure \( \mu \), we have
\[
\mu((A^{[k]})^T) + \mu_{U_r^T} (A^{[n-k]}) \geq \text{trace}(A) \quad (32)
\]

**Proof.** Applying \( \mu \) on both sides of (8) and using the subadditivity of the matrix measure yields (32).

The next result shows that the sufficient condition for contraction w.r.t. some \( L_p \) norm, with \( p \in \{1,2,\infty\} \), induces a “graded structure”.

**Corollary 1.** If there exist \( \eta > 0 \) and \( p \in \{1,2,\infty\} \) such that
\[
\mu_p (A^{[k]}(t)) \leq -\eta \quad \text{for all } t \geq 0 \quad (33)
\]
then (29) is \( \ell \)-contractive w.r.t. the \( L_p \) norm for any \( \ell \geq k \).

**Proof.** We will prove this for the case \( p = 2 \). The proof for the cases \( p = 1 \) and \( p = \infty \) is based on similar arguments. Fix \( t \geq 0 \) and let \( A = A(t) \). For \( p = 2 \) condition (33) is \( \sum_{i=1}^{k} \lambda_i(S) \leq -\eta < 0 \), where \( S := (A + A^T)/2 \) and the eigenvalues are ordered as in (16). This implies that \( \lambda_k(S) < 0 \) and thus \( \lambda_j(S) < 0 \) for any \( j \geq k \). Hence, for any \( \ell > k \) we have
\[
\mu_2(A^{[\ell]}) = \sum_{i=1}^{\ell} \lambda_i(S) \leq \sum_{i=k+1}^{\ell} \lambda_i(S) < \mu_2(A^{[k]}) < -\eta
\]
and Theorem 4 implies that the system is \( \ell \)-contractive w.r.t. the \( L_2 \) norm.

Theorem 4 can be used to provide new sufficient conditions for \( k \)-contraction. The next two results demonstrate this.

**Proposition 7.** Suppose that \( D(t) \) is diagonal and that there exists \( k \in [1,n] \) such that the sum of every \( k \) diagonal entries of \( D(t) \) is smaller or equal to \(-\eta < 0 \) for all \( t \geq 0 \). Then \( \dot{x} = D(t)x \) is \( k \)-contractive w.r.t. the \( L_\ell \) norm for any \( \ell \in \{1,2,\infty\} \).

The proof follows from the fact that
\[
D^{[k]} = \begin{bmatrix}
d_{11} + \cdots + d_{kk} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & d_{pp} + \cdots + d_{nn}
\end{bmatrix}
\]
where \( p := n-k+1 \). Thus, \( \mu_1(D^{[k]}(t)) \leq -\eta \) for all \( t \), and since \( D^{[k]} \) is diagonal, \( \mu_1(D^{[k]}) = \mu_2(D^{[k]}) = \mu_\infty(D^{[k]}) \).

We can also derive a simple sufficient condition for \((n-1)\)-contraction in an \( n \)-dimensional system. This is based
on the following fact. For $M \in \mathbb{R}^{n \times n}$, let $\bar{M}$ denote the matrix with entries
\[ \bar{m}_{ij} := (-1)^{i+j}m_{n+1-i,n+1-j}, \quad i,j \in [1,n]. \]

Schwarz (1970) proved that if $A \in \mathbb{R}^{n \times n}$ then
\[ A^{[n-1]} = \bar{B}, \]  
(34)
where $B := \text{tr}(A)I - A^T$. For example, for $A \in \mathbb{R}^{4 \times 4}$, we have
\[ B = \begin{bmatrix}
-a_{11} & -a_{12} & -a_{13} & -a_{14} \\
-a_{21} & -a_{22} & -a_{23} & -a_{24} \\
-a_{31} & -a_{32} & -a_{33} & -a_{34} \\
-a_{41} & -a_{42} & -a_{43} & -a_{44}
\end{bmatrix}, \]
so
\[ A^{[3]} = \bar{B} = \begin{bmatrix}
a_{41}+a_{23}+a_{33} & a_{42}+a_{24}+a_{44} & -a_{24} & a_{14} \\
a_{41} & a_{42}+a_{34}+a_{44} & -a_{24} & a_{14} \\
a_{41} & a_{42} & a_{43}+a_{44} & -a_{24} \\
-a_{14} & -a_{24} & -a_{34} & -a_{44}
\end{bmatrix}, \]
(compare with (6)).

**Proposition 8.** Suppose that $A : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ satisfies
\[ \sum_{i=1}^n \left(|a_{i\ell}(t)| + a_{i\ell}(t)\right) \leq -\eta < 0, \]  
(35)
for all $\ell \in [1,n]$ and all $t \geq 0$. Then (18) is $(n-1)$-contractive w.r.t. the $L_\infty$ norm.

To show this, note that (34) implies that the sum of the entries of every row of $A^{[n-1]}(t)$, with off-diagonal terms taken with absolute value, is the expression on the left-hand side of (35) for some $\ell$, so $\mu_\infty(A^{[n-1]}(t)) \leq -\eta$ for all $t \geq 0$.

We now turn to consider $k$-contraction in nonlinear dynamical systems.

### 3.2 Nonlinear systems

Consider the time-varying nonlinear system (26). Pick $k \in [1,n-1]$. Let $S^k := \{ r \in \mathbb{R}^k | r_i \geq 0, r_1 + \cdots + r_k \leq 1 \}$ denote the unit simplex in $\mathbb{R}^k$. Pick $a^1, \ldots, a^{k+1} \in \Omega$. For $r \in S^k$, let $h(r) := (\sum_{i=1}^k r_i a^i) + (1 - \sum_{i=1}^k r_i) a^{k+1}$, i.e. a convex combination of the $a$’s, and let
\[ w^i(t,r) := \frac{\partial}{\partial r_i} x(t,h(r)), \quad i = 1, \ldots, k. \]  
(36)

Thus, $w^i(t,r)$ measures how a change in the initial condition $h(r)$ via a change in $r_i$, affects the solution at time $t$. Note that $w^i(0,r) = \frac{\partial}{\partial r_i} x(0,h(r)) = a_i - a^{k+1}$, $i = 1, \ldots, k$.

**Definition 3.** The time-varying nonlinear system (26) is called $k$-contractive if there exist $\eta > 0$ and a vector norm $\| \cdot \|$ such that for any $a^1, \ldots, a^{k+1} \in \Omega$ and any $r \in S^k$, the mapping $W : \mathbb{R}_+ \times S^k \to \mathbb{R}^{n \times k}$ defined by
\[ W(t,r) = [w^1(t,r) \cdots w^k(t,r)] \]
satisfies
\[ |W^{(k)}(t,r)| \leq \exp(-\eta t) |W^{(k)}(0,r)|, \quad \text{for all } t \geq 0. \]  
(37)

To explain the geometric meaning of this definition, pick a domain $D \subseteq S^k$. Then $k$-contraction implies that
\[ \int_D W^{(k)}(t,r) \, dr \leq \int_D |W^{(k)}(t,r)| \, dr \]
\[ = \exp(-\eta t) \int_D |W^{(k)}(0,r)| \, dr \]
\[ = \exp(-\eta t) \left| \left( a^1 - a^{k+1} \right) \cdots \left( a^k - a^{k+1} \right) \right| \int_D dr. \]  
(38)

Note that (38) is the volume of the $k$-surface $x(t,h(r))$ over the parameter space $r \in D$ (see (14)). Thus, $k$-contraction implies that this volume decays to zero at an exponential rate.

**Example 11.** Suppose that (26) is 1-contractive. Pick $a^1, a^2 \in \Omega$. Then $w^1(t,r) = \frac{\partial}{\partial r_1} x(t,ra^1 + (1-r)a^2)$, and (38) with $D = S^1 = [0,1]$ becomes
\[ \left| \int_0^1 \frac{\partial}{\partial r} x(t,ra^1 + (1-r)a^2) \, dr \right| \leq \exp(-\eta t) |a^1 - a^2|, \]
that is, $|x(t,a^1) - x(t,a^2)| \leq \exp(-\eta t) |a^1 - a^2|$. Thus, 1-contraction is just contraction.

Fig. 2 illustrates the relation between standard contraction (i.e. 1-contraction) and 2-contraction.

**Example 12.** Consider the special case where $f(t,x) = A(t)x$. Then (26) is an LTV system. Assume that $0 \in \Omega$. Then
\[ w^i(t,r) = \frac{\partial}{\partial r_i} x(t,h(r)) = \Phi(t)(a^i - a^{k+1}) = x(t,a^i) - x(t,a^{k+1}), \]
where $\Phi(\cdot)$ is the transition matrix corresponding to the linear dynamics. Taking $a^{k+1} = 0$, Eq. (37) reduces to condition (30) in Definition 2.

### 3.3 Sufficient conditions for $k$-contraction in nonlinear systems

The next result provides an easy to check sufficient condition for $k$-contraction in terms of the $k$th additive compound of the Jacobian of the vector field.

10
Fig. 2. Left: the length of the curve $\mathcal{P}^1(t)$ decays exponentially (standard contraction). Right: in 2-contraction systems the area of the surface $\mathcal{P}^2(t)$ decays exponentially.

**Theorem 9.** Suppose that there exist $\eta > 0$ and a vector norm $\| \cdot \|$ such that

$$
\mu(J^{[k]}(t,a)) \leq -\eta, \text{ for all } a \in \Omega, t \geq 0. \tag{39}
$$

Then (26) is $k$-contractive.

**Proof.** The definitions of $W(t,r)$ and $w^i(t,r)$ give

$$
\frac{d}{dt}W(t,r) = \frac{d}{dt} \frac{\partial x(t,h(r))}{\partial r} = \frac{\partial}{\partial r} f(t,x(t,h(r))) = J(t,x(t,h(r)))\frac{\partial}{\partial r} x(t,h(r)) = J(t,x(t,h(r)))W(t,r).
$$

Thus,

$$
\frac{d}{dt}W^{(k)}(t,r) = J^{[k]}(t,x(t,h(r)))W^{(k)}(t,r),
$$

and (39) implies that (37) holds for all $a^1, \ldots, a^{k+1} \in \Omega$, $r \in S^{k}$.

If (39) holds for some $L_p$ norm, with $p \in \{1, 2, \infty\}$, then arguing as in Corollary 1 shows that for any $\ell \geq k$ we have $\mu_p(J^{[\ell]}(t,a)) \leq -\eta < 0$ for all $t \geq 0$ and all $a \in \Omega$, so the nonlinear system is $\ell$-contractive w.r.t. the $L_p$ norm.

Recall that $A \in \mathbb{R}^{n \times n}$ is called Metzler if all its off-diagonal entries are non-negative. The nonlinear system (26) is called $k$-cooperative if $J^{[k]}(t,x)$ is Metzler for all $t \geq 0$ and all $x \in \Omega$ (Weiss and Margaliot, 2021). In other words, $\dot{y} = J^{[k]}y$ is a cooperative dynamical system (Smith, 1995). Since $J^{[1]} = J$, this is a generalization of cooperative systems (and in fact the case $k = n - 1$ corresponds to competitive systems (Weiss and Margaliot, 2021, Lemma 4)). The next result provides a sufficient condition for such a system to be $k$-contractive w.r.t. a scaled $L_1$ norm. For the special case $k = 1$, this is closely related to known results on contractive cooperative systems (Coogan, 2019). We use $1_q$ to denote the vector in $\mathbb{R}^q$ with all entries one.

**Proposition 10.** Suppose that (26) is $k$-cooperative. Let $r := \left(\frac{1}{q}\right)$. If there exist $\eta > 0$ and $v \in \mathbb{R}^r$, with $v_i > 0$ for all $i$, such that

$$
v^T J^{[k]}(t,x) \leq -\eta 1_r^T \text{ for all } t \geq 0 \text{ and } x \in \Omega \tag{41}
$$

then (26) is $k$-contractive w.r.t. the scaled $L_1$ norm $|x|^V := |Vx|_1$, where $V := \text{diag}(v_1, \ldots, v_r)$. Proof. Let $q^T := 1_r^TVJ^{[k]}-1$, that is, the entries of the vector $q$ are all the column sums of the matrix $VJ^{[k]}V^{-1}$. Then $q^T = v^TJ^{[k]}V^{-1}$, and (41) implies that $q^T \leq -1_r^T\eta\min\{v_i^{-1}\}$. Thus,

$$
\mu_1(VJ^{[k]}V^{-1}) = \max_{i \in [1,r]} q_i \leq -\eta \min\{v_i^{-1}\} < 0,
$$

where in the first equation we used the fact that since $J^{[k]}$ is Metzler, so is $VJ^{[k]}V^{-1}$.

**4 Applications**

Li and Muldowney derived several deep results on 2-contraction systems, although they never used this terminology (Muldowney, 1990; Li and Muldowney, 2003, 1993). These results found many applications in models from mathematical epidemiology (see e.g. Li and Muldowney (1995a)). These models typically have at least two equilibrium points, corresponding to the disease-free and endemic steady-states. Hence, they cannot be 1-contractive w.r.t. any norm. We begin with an intuitive presentation of two (somewhat simplified) results that we will use later on, referring to (Muldowney, 1990; Li and Muldowney, 1995b) for the full technical details and proofs.

**4.1 Preliminaries**

The next result is a generalization of Bendixson's criterion for the non-existence of limit cycles in planar systems.

**Theorem 11.** (Muldowney, 1990) Consider the nonlinear time-invariant system:

$$
\dot{x} = f(x), \tag{42}
$$

where $f$ is continuously differentiable, $f(0) = 0$, and $\partial f(0)/\partial x = 0$. Then $\dot{x} = f(x)$ is a competitive system if and only if $f(\cdot) > 0$ and $f'(\cdot) > 0$ on $\mathbb{R}^r$. Further, if $f$ is uniformly Lipschitz continuous, then $\dot{x} = f(x)$ is a cooperative system if and only if $f(\cdot) < 0$ and $f'(\cdot) < 0$ on $\mathbb{R}^r$.
where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^1 \). Suppose that either

\[
\mu(J^2(x)) < 0 \quad \text{for all } x \in \mathbb{R}^n,
\]

or

\[
\mu(-J^2(x)) < 0 \quad \text{for all } x \in \mathbb{R}^n.
\]

Then (42) has no non-trivial periodic solutions.

Intuitively speaking, the proof is based on the following idea. Suppose that the system admits a nontrivial periodic solution \( x(t) = x(t+T) \) with minimal period \( T > 0 \). Let \( \gamma \) denote the corresponding invariant curve. Let \( D \) denote the trace of a 2-surface whose boundary is \( \gamma \) and whose surface area is a minimum. The invariance of \( \gamma \) implies that \( x(t, \gamma) \) is also the trace of a 2-surface with boundary \( \gamma \). The 2-contraction condition (43) implies that the area of \( x(t, \gamma) \) is strictly smaller than the area of \( D \) for any \( t > 0 \). But this contradicts the definition of \( D \). Condition (44) yields the same contradiction after replacing \( t \) with \( -t \).

Note that when \( n = 2 \), Condition (43) [Condition (44)] becomes \( \text{div}(f) := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0 \) \( \text{[div}(f) > 0] \), so Thm. 11 is a generalization of Bendixson’s theorem for planar systems.

The next result provides a sufficient condition based on 2-contraction guaranteeing that an equilibrium is globally asymptotically stable.

**Theorem 12.** [Li and Muldowney, 1995a] Consider the nonlinear time-invariant system (42), where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^1 \). Assume that its trajectories evolve on a convex and compact set \( \Omega \), and that

\[
\mu(J^2(x)) < 0 \quad \text{for all } x \in \Omega.
\]

Then every solution emanating from \( \Omega \) converges to the set of equilibria. If in addition there exists a unique equilibrium \( e \in \Omega \) then every solution emanating from \( \Omega \) converges to \( e \).

The proof is based on the following argument. Recall that a point \( x_0 \in \Omega \) is called wandering for (42) if there exists a neighborhood \( U \) of \( x_0 \) and a time \( T > 0 \) such that

\[
U \cap x(t, U) = \emptyset \quad \text{for all } t \geq T.
\]

In other words, any solution emanating from \( U \) never returns to \( U \) after time \( T \). A point \( x_0 \) is called non-wandering if it is not wandering. Non-wandering points are important in analyzing the asymptotic behavior of solutions. For example, an equilibrium, and more generally, any point in an omega limit set is non-wandering. Suppose that the conditions in Thm. 11 hold. Assume that (42) admits a point \( z \in \mathbb{R}^n \) that is non-wandering and is not an equilibrium. By the Closing Lemma [Pugh, 1967], there exists a \( C^1 \) vector field \( \hat{f} \), that is arbitrarily close to \( f \) in the \( C^1 \) topology, and \( \hat{x} = \hat{f}(x) \) admits a non-trivial periodic solution. (Roughly speaking, it is possible to “close” the non-wandering trajectory into a non-trivial periodic trajectory.) But, since \( \hat{f} \) is arbitrarily close to \( f \) and \( \Omega \) is compact, \( \hat{f} \) also satisfies the 2-contraction condition in Thm. 12 and thus cannot have a non-trivial periodic solution. We conclude that any non-wandering point, and in particular any point in an omega limit set, must be an equilibrium.

The next result provides a sufficient condition for the stability of a non-trivial periodic solution.

**Theorem 13.** [Muldowney, 1994] Suppose that the nonlinear time-invariant system (42) admits a periodic solution \( \gamma(t) = \gamma(t + T) \) with minimal period \( T > 0 \). If the LTV system

\[
\dot{z} = J^2(\gamma(t))z
\]

is asymptotically stable then \( \gamma(t) \) is asymptotically orbitally stable.

**Proof.** By Floquet’s theory, the solution of

\[
\Phi(t) = J(\gamma(t))\Phi(t), \quad \Phi(0) = I,
\]

can be written as \( \Phi(t) = R(t)\exp(Lt) \), where \( R(t) = R(t+T) \) and \( L \in \mathbb{R}^{n \times n} \). The eigenvalues \( \lambda_i, i = 1, \ldots, n \), of \( L \) are called the characteristic multipliers and one of them, say \( \lambda_1 \), is zero. Then

\[
\Phi^{(2)}(t) = R^{(2)}(t)(\exp(Lt))^{(2)} = R^{(2)}(t)\exp(L^{(2)}t).
\]

where the second equation follows from (21). The eigenvalues of \( L^{(2)} \) are the sum of every pair of eigenvalues of \( L \), and since \( \lambda_1 = 0 \), every \( \lambda_i, i = 2, \ldots, n \), is an eigenvalue of \( L^{(2)} \). It follows from (46) that

\[
\dot{\Phi}^{(2)}(t) = J^{(2)}(\gamma(t))\Phi^{(2)}(t), \quad \Phi^{(2)}(0) = I.
\]

The condition in the theorem implies that \( \lim_{t \to \infty} \Phi^{(2)}(t) = 0 \). Combining this with (47) implies that all the eigenvalues of \( L^{(2)} \) have a negative real part, so in particular, the real part of \( \lambda_i, i = 2, \ldots, n \), is negative.

Standard contraction can be applied to prove that all trajectories converge to a unique equilibrium. If a dynamical system admits more than one equilibrium then it is clearly not contractive. Yet, it may be \( k \)-contractive, with \( k > 1 \), and sometimes this can be used to derive a global understanding of the dynamics. We demonstrate this using the analysis of a dynamical model that generalizes the susceptible-exposed-infectious-recovered (SEIR) model studied by Li and Muldowney (1995a). Let \( \mathbb{R}^n_+ := \{ x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \ldots, n \} \).
4.2 Global analysis of a 3D system

Consider the system:

\[
\begin{align*}
\dot{x}_1 &= -\lambda f_1(x_1, x_3) + \zeta - \zeta x_1, \\
\dot{x}_2 &= \lambda f_1(x_1, x_3) - cx_2 - \zeta x_2, \\
\dot{x}_3 &= cx_2 - f_2(x_3) - \zeta x_3,
\end{align*}
\]

(48)

where the parameters \(\lambda, \zeta, c\) are positive, and the state-space is \(\Omega := \{x \in \mathbb{R}_+^3 \mid x_1 + x_2 + x_3 \leq 1\}\). We assume that for any \(x \in \Omega\), we have

\[
\begin{align*}
f_1(x) &\geq 0, \ i \in \{1, 2\}, \\
f_1(x) &= 0 \text{ iff } x_1 x_3 = 0, \\
f_2(x_3) &= 0 \text{ iff } x_3 = 0.
\end{align*}
\]

and for any \(x \in \text{int}(\Omega)\), we have

\[
\begin{align*}
\frac{\partial}{\partial x_j} f_1(x_1, x_3) &\leq \frac{f_1(x_1, x_3)}{x_3}, \\
\frac{f_2(x_3)}{x_3} &\leq \frac{\partial}{\partial x_3} f_2(x_3).
\end{align*}
\]

(50)

In the SEIR model, \(f_1(x_1, x_3) = x_1^q x_3^p\), with \(q > 0, p \in (0, 1]\), and \(f_2(x_3) = \ell x_3\), with \(\ell > 0\), so this indeed holds.

Note that \(e^1 := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T\) is an equilibrium of (48). In the SEIR model, this corresponds to the disease-free steady-state. The next result analyzes the asymptotic behavior of (48).

**Proposition 14.** Suppose that (48) admits exactly two equilibrium points \(e^1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T\), and \(e^2 \in \text{int}(\Omega)\). If \(e^1\) is not an omega limit point for any \(x_0 \in \text{int}(\Omega)\), and \(e^2\) is locally asymptotically stable then

\[
\lim_{t \to \infty} x(t, a) = e^2, \text{ for any } a \in \text{int}(\Omega).
\]

The Jacobian of (48) is

\[
J = \begin{bmatrix}
-\lambda \frac{\partial f_1}{\partial x_1} & 0 & -\lambda \frac{\partial f_1}{\partial x_3} \\
\lambda \frac{\partial f_1}{\partial x_1} - c & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_2}{\partial x_3} \\
0 & c & -\frac{\partial f_2}{\partial x_3}
\end{bmatrix} - \zeta I,
\]

(51)

and Lemma 1 gives

\[
J^2 = \begin{bmatrix}
-\lambda \frac{\partial f_1}{\partial x_1} - c & \lambda \frac{\partial f_1}{\partial x_3} & \lambda \frac{\partial f_2}{\partial x_3} \\
c & -\lambda \frac{\partial f_1}{\partial x_3} - \frac{\partial f_2}{\partial x_3} & 0 \\
0 & \lambda \frac{\partial f_1}{\partial x_1} - c & -\frac{\partial f_2}{\partial x_3}
\end{bmatrix} - 2\zeta I.
\]

Note that \(J^2(x)\) is Metzler for any \(x \in \Omega\), and irreducible for any \(x \in \text{int}(\Omega)\). It follows from the results in [Weiss and Margaliot, 2021] that (48) is a strongly 2-cooperative system, and thus it satisfies the Poincaré-Bendixson property: a nonempty compact omega limit set which does not contain any equilibrium points is a closed orbit.

Suppose that for some \(x_0 \in \Omega\) the omega limit set \(\omega(x_0)\) does not contain any equilibrium points. Then \(\omega(x_0)\) is a periodic solution \(\gamma(t)\) of (48) with minimal period \(T > 0\). Our next goal is to use 2-contraction to show that \(\gamma\) is asymptotically orbitally stable. We require the following result.

**Lemma 2.** The periodic solution satisfies \(\gamma(t) \in \text{int}(\Omega)\) for all \(t \in [0, T)\).

Proof. We first show that \(\gamma_i(t) \neq 0\) for all \(i = 1, 2, 3\) and \(t \in [0, T)\). Note that for any \(x \in \Omega\) with \(x_1 = 0\) we have \(\dot{x}_1 > 0\), so \(\gamma_1(t) \neq 0\) for all \(t\). If \(\gamma_2(t) = 0\) for some time \(\tau\), then we must have \(\gamma_2(\tau) = \lambda f_1(\gamma_1(\tau), \gamma_3(\tau)) \leq 0\). Since \(\gamma_1(\tau) \neq 0\), \(\gamma_3(\tau) = 0\). In this case, the set \(\{x \in \Omega \mid x_2 = x_3 = 0\}\) is forward invariant for \(t \geq \tau\) and (48) implies that \(\gamma(t)\) converges to the equilibrium point \(e^1\). This contradicts the fact that \(\gamma\) is a non-trivial periodic solution, and thus \(\gamma_2(t) \neq 0\) for all \(t\). A similar argument shows that \(\gamma_3(t) \neq 0\) for all \(t\). Now suppose that for some \(\tau \in [0, T)\) we have \(\sum \gamma_i(\tau) = 1\). Then (48) gives \(\sum \gamma_i(\tau) = -f_2(\gamma_3(t)) < 0\). This implies that \(\sum \gamma_i(t) < 1\) for all \(t \in [0, T)\), and this completes the proof of the lemma.

We now show that \(\gamma\) is asymptotically orbitally stable. Consider the system:

\[
\dot{z}(t) = J^2(\gamma(t))z(t).
\]

Define \(D(t) := \text{diag}(1, \gamma_2(t)/\gamma_3(t), \gamma_2(t)/\gamma_3(t))\). This is well-defined by Lemma 2. Let \(p(t) := D(t)z(t)\). Then

\[
\dot{p} = \dot{D}z + \dot{z} = (\dot{D}D^{-1} + D\dot{J}^2(\gamma)D^{-1})p.
\]

A calculation gives \(\dot{D}D^{-1} = \text{diag}(0, 2/\gamma_3, 2/\gamma_3, 2/\gamma_3, 2/\gamma_3, 2/\gamma_3)\). Note that this implies that \(\int_0^T \dot{D}(t)D^{-1}(t)dt = 0\).
Let \( M := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), and define a scaled \( L_\infty \) norm
\[ |y|_{M,\infty} := |My|_\infty. \]
by \( |y|_{M,\infty} := |My|_\infty. \)
Then
\[ \frac{d^+}{dt^+} |p|_{M,\infty} \leq \mu_\infty(S(\gamma)) |p|_{M,\infty}, \tag{53} \]
with
\[ S := M(\dot{D}D^{-1} + DJ[2]^TD^{-1})M^{-1} \]
\[ = \dot{D}D^{-1} + MJD[2]^TD^{-1}M^{-1} \]
\[ = \dot{D}D^{-1} - 2\zeta I \]
\[ + \begin{bmatrix} -\lambda \frac{\partial f_1}{\partial x_1} - c & \frac{\gamma_2}{\gamma_3} \frac{\partial f_1}{\partial x_2} - \frac{\gamma_2}{\gamma_3} \frac{\partial f_1}{\partial x_3} & 0 \\ \frac{\gamma_2}{\gamma_3} & -\frac{\partial f_2}{\partial x_1} - \frac{\partial f_2}{\partial x_3} & \frac{\gamma_2}{\gamma_3} \\ \frac{\gamma_2}{\gamma_3} & -\frac{\partial f_1}{\partial x_1} - \frac{\partial f_1}{\partial x_3} & -\frac{\partial f_2}{\partial x_3} - \frac{\partial f_2}{\partial x_3} \end{bmatrix}. \]

Eq. (53) implies that
\[ |p(t)|_{M,\infty} \leq \exp \left( \int_0^t \mu_\infty(S(\gamma(s))) \ ds \right) |p(0)|_{M,\infty}. \tag{55} \]
By (54), \( \mu_\infty(S) = \max \{g_1, g_2\} \), with
\[ g_1 := -\lambda \frac{\partial f_1}{\partial x_1} - c + \frac{\gamma_2}{\gamma_3} \frac{\partial f_1}{\partial x_2} - 2\zeta, \]
\[ g_2 := \frac{\gamma_2}{\gamma_3} \frac{\partial f_2}{\partial x_1} + \frac{\gamma_2}{\gamma_3} \frac{\partial f_2}{\partial x_3} - 2\zeta. \tag{56} \]
Using (50) gives
\[ g_1 \leq -c + \lambda \frac{f_1}{\gamma_2} - 2\zeta. \]

The second equation of (48) gives \( \frac{\gamma_2}{\gamma_3} = \lambda \frac{f_2}{\gamma_2} - c - \zeta \), so \( g_1 \leq \frac{\gamma_2}{\gamma_3} - \zeta \). The third equation of (48) gives \( \gamma_3 \frac{\gamma_2}{\gamma_3} = \frac{\gamma_2}{\gamma_3} - \frac{f_2}{\gamma_3} - \zeta \), and combining this with (50) and (56) yields \( g_2 \leq \frac{\gamma_2}{\gamma_3} - \zeta \). We conclude that \( \mu_\infty(S) \leq \frac{\gamma_2}{\gamma_3} - \zeta \). Therefore,
\[ \int_0^t \mu_\infty(S(\gamma(s))) \ ds \leq \log \gamma_2(t) - \log \gamma_2(0) - \zeta t. \]
Since \( \log \gamma_2(t) \) is bounded for all \( t \geq 0 \), Eq. (55) implies that \( \lim_{t \to \infty} p(t) = 0 \), so \( \lim_{t \to \infty} \dot{z}(t) = 0 \). Since (52) is a linear \( T \)-periodic system, this implies asymptotic stability, and Thm. 13 implies that \( \gamma \) is asymptotically orbitally stable.

Summarizing, if for some \( x_0 \in \Omega \) we have that \( \omega(x_0) \) does not contain any equilibrium points then \( \omega(x_0) \) is a non-trivial asymptotically orbitally stable periodic solution \( \gamma \) of (48).

To complete the proof of Prop. 14, let \( B \subset \Omega \) denote the basin of attraction of \( e^2 \). Seeking a contradiction, assume that \( \text{int}(\Omega) \not\subset B \). Then \( M := (\partial B) \cap \text{int}(\Omega) \neq \emptyset \), and \( M \) is an invariant set. Thus, the closure of \( M \) includes a non empty compact omega limit set and the assumptions in the proposition imply that this omega limit set includes no equilibrium points. Thus, it includes a non-trivial periodic solution \( \gamma \), where \( \gamma \) is in the interior of \( \Omega \) and is asymptotically orbitally stable. But this is a contradiction, as \( M \) and thus \( \gamma \) is contained in the alpha limit set of \( e^2 \). This completes the proof of Prop. 14.

4.3 2-contraction in the Lotka-Volterra model
Consider the Lotka-Volterra model
\[ \dot{x}_i = x_i(b_i + \sum_{k=1}^n a_{ik}x_k), \quad i = 1, \ldots, n. \tag{57} \]

The state-variable \( x_i(t) \) represents the number of species \( i \) at time \( t \). Note that \( R^n_+ \) is an invariant set of the dynamics. This model has been widely used in mathematical ecology (Sihlak, 2007; Hoeffner and Siegmund, 1988) to study the implications of various interaction patterns between members of a population sharing a common habitat.

Eq. (57) can be written as \( \dot{x} = \text{diag}(x_1, \ldots, x_n)(b + Ax) \), where \( b := [b_1 \ldots b_n]^T \) and \( A := [a_{ij}]_{i,j=1}^n \). Thus, 0 is an equilibrium, and if \( A \) is non-singular then (\( -A^{-1}b \)) is an equilibrium.

Let \( R^n_{++} := \{ x \in R^n \mid x_i > 0, \ i \in [1, n] \} \). There exist known results on the asymptotic behaviour of this model in certain special cases. For example, if \( A \) is diagonally dominant (i.e., there exist \( d_i > 0 \) such that \( d_ia_{ii} + \sum_{j \neq i} d_j|a_{ji}| < 0 \) for \( i \in [1, n] \)) and there exists an equilibrium \( e \in R^n_{++} \) then \( \lim_{t \to \infty} x(t, a) = e \) for any \( a \in \text{int}(R^n) \) (see e.g. Li, 1998).

The quadratic terms in (57) imply that the model typically admits several equilibrium points and thus cannot be 1-contractive. Our goal is to provide a new sufficient condition for 2-contraction. To do this, let \( g_i(x) := b_i + \sum_{k=1}^n a_{ik}x_k \). Then (57) can be written as the Kolmogorov system \( \dot{x}_i = x_i g_i(x), \ i \in [1, n] \). The corresponding Jacobian is
\[ J = \text{diag}(g_1, \ldots, g_n) + \begin{bmatrix} x_1 \frac{\partial}{\partial x_1} g_1 & \cdots & x_1 \frac{\partial}{\partial x_n} g_1 \\ \vdots & \ddots & \vdots \\ x_n \frac{\partial}{\partial x_1} g_n & \cdots & x_n \frac{\partial}{\partial x_n} g_n \end{bmatrix} = \text{diag}(g_1, \ldots, g_n) + \text{diag}(x_1, \ldots, x_n)A, \]
so
\[ J[2] = \text{diag}(g_1 + g_2, g_1 + g_3, \ldots, g_{n-1} + g_n) + \text{diag}(x_1, \ldots, x_n)A[2]. \tag{58} \]
Note that the $b_i$s appear only in the first matrix on the right-hand side of (58). Using this allows to provide easy to verify sufficient conditions for 2-contraction. Recall that this implies an important asymptotic property, namely, that all bounded solutions converge to an equilibrium. The next result demonstrates this for the case $n = 3$.

**Proposition 15.** Consider (57) with $n = 3$. If

$$b_i + b_j < 0, \text{ for all } i \neq j,$$

$$\max\{a_{13} + a_{23}, a_{12} + a_{32}, a_{21} + a_{31}\} \leq 0,$$

$$2a_{11} + \max\{a_{21} + |a_{13}|, a_{31} + |a_{12}|\} \leq 0,$$

$$2a_{22} + \max\{a_{12} + |a_{23}|, a_{32} + |a_{21}|\} \leq 0,$$

$$2a_{33} + \max\{a_{13} + |a_{32}|, a_{23} + |a_{31}|\} \leq 0.$$  

Then the system is 2-contractive w.r.t. the $L_\infty$ norm.

*Proof.* For $n = 3$, Eq. (58) becomes

$$J[x] = \begin{bmatrix}
m_1 & a_{23}x_2 - a_{13}x_1 \\
a_{23}x_3 & m_2 & a_{12}x_1 \\
-a_{31}x_3 & a_{21}x_2 & m_3
\end{bmatrix},$$

where $m_1 := b_1 + b_2 + (2a_{11} + a_{21})x_1 + (2a_{22} + a_{12})x_2 + (a_{13} + a_{23})x_3$, $m_2 := b_1 + b_3 + (2a_{11} + a_{31})x_1 + (a_{12} + a_{32})x_2 + (2a_{33} + a_{13})x_3$, and $m_3 := b_2 + b_3 + (a_{21} + a_{31})x_1 + (2a_{22} + a_{32})x_2 + (2a_{33} + a_{23})x_3$.

Condition (59) ensures that

$$\mu_\infty(J[x]) \leq \max\{b_1 + b_2, b_1 + b_3, b_2 + b_3\} < 0,$$

for all $x$, so the system is 2-contractive.

**Example 13.** Consider (57) with $n = 3$, $b_1 = 1$, $b_2 = -2$, $b_3 = -2$, and $A = \begin{bmatrix} -2 & -3 & -3 \\
-1 & -2 & 0 \\
1 & 3 & -2 \end{bmatrix}$. The dynamics has three equilibrium points in $R^3_+$: the origin, $[0 \ 4 \ 5]^T$, and $[1/2 \ 0 \ 0]^T$.

Note that $A$ is not Hurwitz, so it is not diagonally dominant. In fact, the system admits unbounded solutions. However, condition (59) holds. Hence, this system is 2-contractive on $R^3_+$, so every bounded trajectory converges to an equilibrium point (see Fig. 3).

Our next application of k-contraction is to control synthesis. For a symmetric matrix $S$, we write $S \prec 0$ if $S$ is negative definite [negative semi-definite].

### 4.4 Control design in a 2-contraction system

Consider the affine nonlinear time-invariant system:

$$\dot{x} = f(x) + G(x)u,$$  

where $f : R^n \to R^n$, $G : R^n \to R^{n \times m}$ are $C^1$, and $u \in R^m$ is the control input. Let $J(x) := \frac{\partial f}{\partial x}(x)$. We assume that there exists a positive definite matrix $P \in R^{n \times n}$ such that

$$P J(x) + J^T(x) P \preceq 0 \text{ for all } x \in R^n.$$  

(61)

Note that (61) is equivalent to $\mu_2(P^{1/2} J(x) P^{-1/2}) \leq 0$ for all $x \in R^n$.

Define $V : R^n \times R_+ \to R_+$ by

$$V(a,b) := \frac{1}{2} (a - b)^T P(a - b).$$

If $G(x) \equiv G$, i.e. the input matrix is constant, and $G$ is full rank then (61) implies that the system (60) is incrementally passive [van der Schaft, 2017] w.r.t. the incremental storage function $V(x(t), a, t)$ and the output $y(x) := G^T P x$ (see also Pavlov and Marcon, 2008; Wu et al, 2019). In this case, consider the problem of steering the system’s output to a value $y(e)$ for some pre-specified $e \in R^n$. Let $\dot{x} := x - e$ and $\dot{y}(x) := y(x) - y(e)$. Then, the control design

$$u := -k G^T P \dot{x} + u^*,$$  

(62)

with $k > 0$, and $u^*$ satisfying $f(e) + Gu^* = 0$ gives

$$\dot{V}(x,e) = \dot{x}^T P A(x) \dot{x} - k \dot{y}^T \dot{y},$$  

(63)

where $A(x) := \int_0^1 J(e + s \dot{x}) ds$. Eq. (61) yields $\dot{V}(x,e) \leq -k \dot{y}^T \dot{y} \leq 0$. By LaSalle’s invariance principle, this implies that every solution of the closed-loop system (60) and (62) converges to the set $M$ which is the largest invariant set contained in $\{x \in R^n | \dot{y}(x) = 0\}$. If $M = \{e\}$ then $e$ is GAS.
The next result shows how 2-contraction allows to extend this control design method when the input matrix is allowed to be state-dependent.

**Proposition 16.** Suppose that (61) holds and also that

\[ P^{(2)} J |^2 (x) + (J |^2 (x))^T P^{(2)} < 0 \text{ for all } x \in \mathbb{R}^n, \]  

(64)

and that there exists a \( C^1 \) mapping \( \theta : \mathbb{R}^n \to \mathbb{R}^m \) such that

\[ \mu_2 \left( P^{\frac{1}{2}} \frac{\partial}{\partial x} \left( G(x) \theta(x) \right) P^{-\frac{1}{2}} \right) \leq 0 \text{ for all } x \in \mathbb{R}^n. \]  

(65)

Consider the control \( u := \theta(x) \). Then every trajectory of the closed-loop system converges to an equilibrium. If the closed-loop system admits a unique equilibrium then it is not contractive w.r.t. any norm.

**Proof.** Let \( f_i(x) := f(x) + G(x)\theta(x) \), \( J_i(x) := \frac{\partial f_i}{\partial x}(x) \), and \( g_i(x) := G(x)\theta(x) \). Recall that any matrix measure is sub-additive, i.e., \( \mu(A + B) \leq \mu(A) + \mu(B) \) for any \( A, B \in \mathbb{R}^{n \times n} \) (see e.g. [Desoer and Vidyasagar (2008)]), and combining this with (61) and (65) gives

\[
\mu_2(P^{\frac{1}{2}} J_i(x) P^{-\frac{1}{2}}) \leq \mu_2(P^{\frac{1}{2}} J(x) P^{-\frac{1}{2}}) + \mu_2(P^{\frac{1}{2}} \frac{\partial g_i}{\partial x} P^{-\frac{1}{2}}) \leq 0. 
\]

This implies that the closed-loop system is globally uniformly bounded. Hence, for any initial condition \( a \in \mathbb{R}^n \), there exists a compact set \( D = D(a) \) such that \( x(t, a) \in D \) for all \( t \geq 0 \).

By (15), (17), and (65), \( \mu_2 \left( \left( P^{\frac{1}{2}} \frac{\partial g_i}{\partial x} P^{-\frac{1}{2}} \right)^{|2|} \right) \leq \mu_2 \left( P^{\frac{1}{2}} J_i(x) P^{-\frac{1}{2}} \right) \leq 0 \). Since \( P \) is positive definite, so are \( P^{(2)} \) and \( (P^{(2)})^\frac{1}{2} \). Hence, (64) ensures that \( \mu_2((P^{(2)})^\frac{1}{2} J (x) (P^{(2)})^{-\frac{1}{2}}) < 0 \). So

\[
\mu_2((P^{\frac{1}{2}} J_i(x) P^{-\frac{1}{2}})^{|2|}) = \mu_2 \left( (P^{\frac{1}{2}} J P^{-\frac{1}{2}})^{|2|} + \left( P^{\frac{1}{2}} \frac{\partial g_i}{\partial x} P^{-\frac{1}{2}} \right)^{|2|} \right) \leq \mu_2((P^{\frac{1}{2}} J P^{-\frac{1}{2}})^{|2|}) + \mu_2 \left( P^{\frac{1}{2}} \frac{\partial g_i}{\partial x} P^{-\frac{1}{2}} \right)^{|2|} \leq \mu_2((P^{(2)})^\frac{1}{2} J (x) (P^{(2)})^{-\frac{1}{2}}) + \mu_2(P^{\frac{1}{2}} \frac{\partial g_i}{\partial x} P^{-\frac{1}{2}}) < 0.
\]

Thm. 9 implies that the closed-loop system is 2-contractive w.r.t. a scaled \( L_2 \) norm, and Thm. 12 implies that every trajectory converges to an equilibrium.

The above control design requires solving the partial differential equation (65). In certain cases, numerical algorithms can be used to design \( \theta(x) \). For example, if \( G(x) \) is a polynomial and we also parameterize \( \theta(x) \) as a polynomial, then sum of squares programming may be efficient. This approach has been used in the context of 1-contraction theory, see e.g. [Alyward et al. 2008].

5 Conclusion

Contraction theory has found numerous applications in systems and control theory. However, it is clear that this theory is too restrictive for many systems. For example, if a system admits more than one equilibrium point then it is not contractive w.r.t. any norm.

We considered a geometric generalization of contraction theory called \( k \)-contraction. For the special case \( k = 1 \) this reduces to standard contraction. An easy to check sufficient condition for \( k \)-contraction is that some matrix measure of the \( k \)th additive compound of the Jacobian is uniformly negative. In the case of \( 1 \)-contraction, it is known that under certain regularity conditions the Jacobian condition is in fact not only sufficient but also necessary for contraction (Aminzare and Sontag 2014, Prop. 3). An interesting open problem is whether this condition is also necessary for \( k \)-contraction.

We described several implications of \( k \)-contraction to the asymptotic analysis of nonlinear dynamical systems and to control synthesis. To the best of our knowledge, this is the first application of \( k \)-contraction, with \( k > 1 \), in control theory. We believe that \( k \)-contraction, with \( k > 1 \), can be used to address various system and control problems for dynamical models where standard contraction theory cannot be applied.

Standard contraction implies *entrainment* in nonlinear systems with a time-varying and periodic vector field (Russo et al. 2010, Margaliot et al. 2018). This is important in many applications. For example, synchronous generators must entrain to the frequency of the grid. Biological organisms must develop internal clocks that entrain to the 24th solar day, and so on. An important research direction is to study the implications of \( k \)-contraction in dynamical systems with a time-varying and periodic vector field.

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**Appendix**

**Proof of Prop. 3.** Let \( \Phi(t) \) be the solution of \( \dot{\Phi}(t) = A(t)\Phi(t) \), \( \Phi(0) = I_n \). Since (18) is uniformly stable, \( \Phi(t) \) is uniformly bounded. Recall that \( \dot{\Phi}(k)(t) = A^{(k)}(t)\Phi^{(k)}(t), \Phi^{(k)}(0) = I_r, \) where \( r := \binom{n}{k} \).

Suppose that Condition (a) holds. Let \( e^i \) be the \( i \)th canonical vector in \( \mathbb{R}^n \). Since \( \dim \mathcal{X} = n-k+1 \), there exist \( c_1, \ldots, c_k \in \mathbb{R} \), not all zero, such that \( \sum_{i=1}^k c_i e^i \in \mathcal{X} \).
Hence,
\[
0 = \lim_{t \to \infty} x(t, \sum_{i=1}^{k} c_i e^i) = \lim_{t \to \infty} \sum_{i=1}^{k} c_i x(t, e^i).
\]

Combining this with the uniform stability assumption implies that \(\lim_{t \to \infty} \left[ x(t, e^1) \ldots x(t, e^k) \right]^{(k)} = 0\), that is,
\[
0 = \lim_{t \to \infty} \Phi^{(k)}(t) \left[ e^1 \ldots e^k \right]^{(k)}.
\]

We conclude that the first column of \(\Phi^{(k)}(t)\) converges to zero. A similar argument shows that this holds for any column of \(\Phi^{(k)}(t)\). This shows that Condition (a) implies Condition (b).

To prove the converse implication, suppose that Condition (b) holds. Pick \(k\) vectors \(a^1, \ldots, a^k \in \mathbb{R}^n\). Define \(X(t) := \left[ x(t, a^1) \ldots x(t, a^k) \right]\). Then \(X(t) = \Phi(t)X(0)\). By uniform boundness, there exists an increasing sequence of times \(t_i\) such that \(\lim_{i \to \infty} t_i = \infty\) and \(P := \lim_{i \to \infty} X(t_i)\) exists. Since \(\dot{X}^{(k)}(t) = A^{(k)}X^{(k)}(t)\), Condition (b) implies that \(P^{(k)} = 0\), i.e., all minors of order \(k\) of \(P\) are zero. This implies that there exists \(c \in \mathbb{R}^k \setminus \{0\}\) such that
\[
0 = Pc
= \lim_{i \to \infty} \sum_{j=1}^{k} c_j x(t_i, a^j)
= \lim_{i \to \infty} x(t, \sum_{j=1}^{k} c_j a^j)
= \lim_{t \to \infty} x(t, \sum_{j=1}^{k} c_j a^j),
\]

where the last step follows from the uniform stability assumption. Summarizing, every set of \(k\) linearly independent vectors \(a^1, \ldots, a^k \in \mathbb{R}^n\) generates a vector \(\sum_{j=1}^{k} c_j a^j \neq 0\) such that \(\lim_{t \to \infty} x(t, \sum_{j=1}^{k} c_j a^j) = 0\). This proves that Condition (a) holds.

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