Status of $\Delta\alpha_{\text{had}}$ and $g - 2$

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Abstract. A brief status report on the hadronic contributions to the running of the electromagnetic coupling and the anomalous magnetic moment of the muon is given.

1. Introduction
The anomalous magnetic moment of leptons, $a_l = (g - 2)_l/2$, is one of the strongest tests of Quantum Field Theory. While the anomaly of the electron, $a_e$, is measured to 0.66 ppb accuracy and determines the fine structure constant $\alpha$ with unprecedented accuracy, the anomaly of the muon, $a_\mu$, is also sensitive to the electro-weak (EW) and strong sectors of the Standard Model (SM). Any discrepancy ($a_\mu^{\text{exp}} - a_\mu^{\text{TH}}$), if clearly established, would signal physics beyond the SM. The case of $g - 2$ of the muon crucially depends on the evaluation of the hadronic contributions, as these totally dominate the uncertainty of the SM prediction. Similarly, in the case of the running QED coupling, $\alpha(q^2)$, hadronic uncertainties limit the precision of its evaluation. At low scales, $a(q^2)$ is used as input in many QCD analyses and is also needed for the $g - 2$ predictions. At higher scales, $\alpha(M_Z^2)$ is the least well known of the EW parameters of the SM, $\{G_\mu, M_Z, \alpha(M_Z^2)\}$. It is required in EW precision fits which e.g. indirectly determine the Higgs mass. In both cases, $g - 2$ and $\alpha(q^2)$, dispersion relations are currently the only way to arrive at accurate predictions for the leading effects due to low energy vacuum polarization (VP) contributions. For this one relies on the experimentally measured hadronic cross section, $\sigma(e^+e^- \rightarrow \text{hadrons})$. In the following, recent changes and improvements of these input data and the status of $g - 2$ and $\alpha(q^2)$ will be discussed.

2. Dispersion integrals and input data
The dispersion integral for the leading order hadronic VP contributions to $g - 2$ reads

$$a_{\mu,\text{LO}}^{\text{had}} = \frac{1}{4\pi^2} \int_{m^2_\mu}^\infty ds \sigma_{\text{had}}^0(s)K(s), \quad K(s) = \frac{m^2_\mu}{3s} \cdot (0.63 \ldots 1),$$

where $K$ the well known monotonic kernel function. The superscript ‘0’ indicates that the ‘undressed’ hadronic cross section has to be used, i.e. $\sigma_{\text{had}}$ without the inclusion of VP corrections. Note the factor $1/s$ which weights $\sigma_{\text{had}}$ at lowest energies strongest. For $\alpha(q^2) = \alpha/(1 - \Delta\alpha(q^2))$, the dispersion integral

$$\Delta\alpha_{\text{had}}^{(5)}(q^2) = -\frac{g^2}{4\pi^2\alpha} P \int_{m^2_\mu}^\infty \frac{\sigma_{\text{had}}^0(s) ds}{s - q^2},$$
with $P$ denoting the principal value, provides a more evenly weighting, so that $\sigma_{\text{had}}$ in a large energy range becomes important.

Recently there have been significant improvements in the determination of $\sigma_{\text{had}}$. At intermediate energies, the experiments BES and CLEO have measured the inclusive hadronic cross section with improved accuracy. At low energies, many exclusive measurements have become available from CMD-2, SND, BaBar and KLOE.\footnote{For details we refer the reader to [1, 2].} The latter two experiments use the method of Radiative Return, where the measurement of events with (energetic) initial state photon radiation effectively makes a scan of the whole hadronic spectrum below the colliders’ fixed centre-of-mass energy possible.

### 3. $g - 2$

Most relevant for $g - 2$ have been the new analyses of the $\pi^+\pi^-$ channel from CMD-2 [3] ($0.37 < \sqrt{s} < 1.38$ GeV), SND [4] ($0.4 \ldots 1$ GeV) and KLOE [5] ($0.6 \ldots 1$ GeV). CMD-2 and the (re-analysed) SND data show a very good agreement over the whole energy range. KLOE, with the first Radiative Return analysis of this sort, provides a much needed independent cross check and further improves the accuracy. The integral (1) using their data for the $2\pi$ channel agrees very well with the evaluation based on CMD-2 and SND data. However, differences in the energy spectrum are significant and affect the data combination, see the detailed discussion in [1]. Two further KLOE analyses are under way, and the situation is expected to further improve in the near future.\footnote{See e.g. the talk by S. Müller for the KLOE Collaboration in these proceedings.}

The good agreement between CMD-2, SND and KLOE clearly discourages the use of $\tau$ spectral function data supplemented by isospin-breaking corrections. While the disagreement between the $\tau$ and the $e^+e^-$ data is currently not yet understood, the uncertainty of the former due to additional corrections from unknown differences between charged and neutral $\rho$ masses and widths is large [6]. The current consensus among the experts is hence to not use the $\tau$ data for the prediction of $g - 2$ until these issues are better understood.

To obtain the complete SM prediction of $g - 2$, one has to combine the leading order hadronic VP contributions to $a_\mu$ from (1) with the next-to-leading order VP and the so-called light-by-light (l-by-l) scattering corrections, and the much more accurately known QED and EW contributions. A discussion of these contributions is beyond this report, see e.g. [7, 8] for details. Suffice to say that all but the l-by-l contributions are very well under control and contribute only insignificantly to the uncertainty of the SM prediction. For l-by-l, different, model-dependent evaluations exist which however agree within the quoted uncertainties [9].

In figure 1 we give a comparison of recent SM evaluations with the BNL measurement of $g - 2$. With the recent advances the discrepancy has further widened to more than three standard deviations. HMNT obtain $a_\mu^{\text{EXP}} - a_\mu^{\text{TH}} = (27.6 \pm 8.1) \cdot 10^{-10}$, corresponding to $\sim 3.4\sigma$ [1], while other groups now obtain very similar results (DEHZ [10]: $3.3\sigma$, Jegerlehner [8]: $3.2\sigma$). This strengthens the indication of physics beyond the SM in $g - 2$. As studied extensively, supersymmetric extensions of the SM could well solve the discrepancy, but only in a restricted part of the parameter space. This in turn makes $g - 2$ a very valuable tool to constrain physics beyond the SM.

### 4. $\Delta\alpha$

With the dispersion integral (2) and the same data compilation for $\sigma_{\text{had}}$ as for $g - 2$, and using pQCD only for energies $\sqrt{s} > 11.09$ GeV, one can calculate $\alpha(q^2)$ (more information will be given in a forthcoming publication). In figure 2 we display our results (labelled ‘HMNT’ in the graphs) and compare with other available evaluations from Burkhardt and Pietrzyk [12] and...
from Jegerlehner\(^3\) (labelled ‘BP05’ and ‘J03’). In the space-like regime, \(\alpha(\sqrt{s} < 0)\) is a smooth function, and the (relative) differences in \(\Delta\alpha^{(5)}_{\text{had}}/\alpha\) are of about one percent and rather small, see the left panel in figure 2. In the time-like regime (right panel), \(\alpha(\sqrt{s} > 0)\) depends on the details of the hadronic resonances and rapidly varies with energy, and there appear sizeable differences between our evaluation and the parametrization available from [13], which may be relevant e.g. when used for subtracting VP effects in hadronic cross sections.

At the \(Z\) scale, we obtain \(\Delta\alpha^{(5)}_{\text{had}}(M^2_Z) = 0.02768 \pm 0.00022\) which corresponds to \(\alpha(M^2_Z)^{-1} = 128.937 \pm 0.030\). This result is compared to other recent evaluations, see table 1. Analyses which are (mostly) data driven quote larger uncertainties than evaluations which depend more on the use of perturbative QCD. The second line in the entry labelled ‘Jegerlehner ’06’ corresponds to a method which employs the use of the Adler function, \(D(s) = \frac{3\pi s^2}{2} \frac{d}{ds} \Delta\alpha(s) = -(12\pi^2)s \frac{dH(\alpha)}{dx}\), which leads to an improvement of the error as it minimizes the dependence on the input data.

For the indirect determination of the Higgs mass \(M_H\) from fits of precision observables the precise input for \(\Delta\alpha^{(5)}_{\text{had}}(M^2_Z)\) is crucial. Note that the LEP Electro-Weak Working Group uses the result of Burkhardt and Pietrzyk as default in their SM EW precision fits.\(^4\) If one uses a larger value with a smaller error, the parabola of the ‘blue-band plot’ moves further in the excluded region; both the preferred value and upper limit of \(M_H\) go down, causing an increased tension in the fit.

5. Outlook

Further improvements in the determination of \(\sigma_{\text{had}}\) are anticipated in the whole relevant energy range. New and improved measurements will come from the upgraded collider in Novosibirsk, VEP2000, and, using Radiative Return, from KLOE, BaBar and possibly BELLE. At higher energies CLEO and BES will provide more data. It is therefore realistic to significantly improve on \(\Delta\alpha\) and to bring the error of the SM determination of \(g - 2\) down by about a factor of two. This clearly makes a strong case for an updated \(g - 2\) experiment, and could establish ‘New Physics’ in \(g - 2\) beyond doubt. Already now, \(g - 2\) serves as an important constraint on physics beyond the SM.

\(^3\) Routine made available by F. Jegerlehner, for a discussion see e.g. [13].
\(^4\) For detailed information and references see their website http://lepwg.web.cern.ch/LEPEWWG/.
Table 1. Comparison of recent evaluations of $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2)$.

| Group                     | $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2)$ | remarks            |
|---------------------------|-----------------------------------------|--------------------|
| Burkhardt+Pietrzyk '05   | 0.02758 ± 0.00035                       | data driven        |
| Troconiz+Yndurain '05    | 0.02749 ± 0.00012                       | pQCD               |
| Kühn+Steinhauser '98    | 0.02775 ± 0.00017                       | pQCD               |
| Jegerlehner '06          | 0.02761 ± 0.00023                       | data driven/pQCD   |
| ($s_0^2 = (10\text{GeV})^2$) | 0.02759 ± 0.00017                      | Adler fct, pQCD    |
| HMNT '06                 | 0.02768 ± 0.00022                       | data driven        |

Figure 2. Comparison of different evaluations of $\Delta\alpha^{(5)}_{\text{had}}/\alpha$ in the space- (left panel) and time-like (right panel) regime.

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