ON THE GEODESIC FLOW ON THE GROUP OF DIFFEOMORPHISMS
OF THE CIRCLE WITH A FRACTIONAL SOBOLEV
RIGHT-INvariant METRIC

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We show that the geodesic flow on the infinite-dimensional group of diffeomorphisms of the circle,
endowed with a fractional Sobolev metric at the identity, is described by the generalized Constantin–
Lax–Majda equation with parameter \(a = -\frac{1}{2}\).

1. Introduction: The Dual Generalized Constantin–Lax–Majda Equation

The generalized Constantin–Lax–Majda equation with parameter \(a \in \mathbb{R} \cup \{+\infty\}\)

\[
\begin{aligned}
\omega_t + a \omega_x &= \omega \nu_x, \\
\nu_x(t, x) &= \mathcal{H}\omega(t, x) \\
\omega(0, x) &= \omega^0(x)
\end{aligned}
\]  

(henceforth referred to as the \(gCLM\) equation) was first derived in [17]. Here, the (spatial)
Hilbert transform of a function \(f\), denoted above by \(\mathcal{H}\), is defined via the Fourier transform
\(F\) by \(F(\mathcal{H}f)(\xi) = -\sqrt{-1} \text{sign}(\xi) F(f)(\xi), \quad \xi \in \mathbb{S}.
\) Thus, \(\mathcal{H}\) gives rise to an \(L^2(\mathbb{S})\)-isometry, and \(\mathcal{H}^2 f = -f\).

For different values of \(a \in \mathbb{R}\), the \(gCLM\) equation interpolates between several one-
dimensional model equations arising in fluid dynamics. For example, if \(a = 0\), the \(gCLM\) equation reduces
to the classical Constantin–Lax–Majda equation [10] mimicking the 3D vorticity equation. A model for the quasi-geostrophic equations [11] is obtained by setting \(a = -1\), and if \(a = 1\), we obtain the De Gregorio equation [13, 14]. For the first two cases, [10] and [11] proved that many smooth initial data give rise to solutions which blow up in finite time, while [17] provided strong numerical evidence for global existence of
8 M. Wunsch

solutions for the case $a = 1$. Moreover, if $a = \infty$ (the case of which bears close resemblance to the 2D vorticity equation) it was shown analytically in [17] that solutions exist for all times.

Let us define the dual generalized Constantin–Lax–Majda equation (gCLM* equation for short) with parameter $a^* \in \mathbb{R} \cup \{-\infty\}$ via

$$
\begin{cases}
\omega_t - v \omega_x = a^* \omega v_x, & x \in S, \ t > 0 \\
v_x(t, x) = H \omega(t, x) \\
\omega(0, x) = \omega_0(x).
\end{cases}
$$

(1.2)

It can obviously be recovered from the gCLM equation with parameter $a \in \mathbb{R}$ by $\omega(t, x) \mapsto \omega(a^* t, x)$, where $a^* = -\frac{1}{a}$ (if $a \to 0$, so that $a^* \to -\infty$, we arrive at the CLM equation in the limit).

2. The Geodesic Flow on $D(S)$ Endowed with the Right-invariant $H^\frac{1}{2}(S)$ Metric

In this paper, we observe that the gCLM* equation with parameter $a^* = 2$ describes the geodesic flow on the infinite-dimensional Lie group of orientation-preserving diffeomorphisms of the circle $S$, $D(S)$, endowed with the $H^\frac{1}{2}(S)$ fractional Sobolev right-invariant metric given at the identity by

$$
\langle u, v \rangle_{id} = \int \Lambda u \cdot v \ dx, \ u, v \in T_{id} D(S) \equiv C^\infty(S),
$$

(2.1)

where $\Lambda f$ is defined via the Fourier transform $F$:

$$
F(\Lambda f)(\xi) = \xi \text{ sign}(\xi) F(f)(\xi), \ \xi \in S.
$$

Consequently, we have the calculation rules $H \Lambda f = \Lambda H f = -f_x$, and $\Lambda f = H f_x$.

To define a smooth right-invariant Riemannian metric on $D(S)$, we extend the inner product (2.1) to each tangent space $T_{\varphi}\mathcal{D}(S)$ by right translation:

$$
\langle V, W \rangle_{\varphi} = \langle V \circ \varphi^{-1}, W \circ \varphi^{-1} \rangle_{id} \quad \text{for } V, W \in T_{\varphi}\mathcal{D}(S).
$$

(2.2)

The existence of a covariant derivative $\nabla$ preserving the inner product (2.2), which is necessary to derive the geodesic equation of the metric, is guaranteed by the following theorem.

**Theorem 2.1** [9]. Consider a non-degenerate continuous inner product $\langle \cdot, \cdot \rangle$ on $T_0 \mathcal{D}(S) \equiv C^\infty(S)$, and extend it to each tangent space $T_{\varphi}\mathcal{D}(S)$ by right translation. If there exists a bilinear operator $B : C^\infty(S) \times C^\infty(S) \to C^\infty(S)$ such that

$$
\langle B(u, v), w \rangle = \langle u, [v, w] \rangle \quad u, v, w \in C^\infty(S),
$$

(2.3)

where the commutator $[\cdot, \cdot]$ is given by $[v, w] = vw_x - v_x w$, then there is a unique Riemannian connection $\nabla$ on $\mathcal{D}(S)$ associated with the right-invariant metric $\langle \cdot, \cdot \rangle$. 

The geodesic equation (also referred to as the Euler equation) is now given as \[1,2,9\]
\[u_t = B(u, u).\] (2.4)

Let us first determine the bilinear form \(B(u, v)\) using formula (2.3):

\[\langle u, [v, w] \rangle = \int_S \Lambda uvw \, dx - \int_S \Lambda uvx \, d \xi = \int_S H(H(u)xv)Hw \, dx\]

\[= \int_S H(H(u)xv) \Lambda w \, dx + \int_S \partial_{x^{-1}} H(H(u)xv) \Lambda w \, dx,\]

hence

\[B(u, v) = H(H(u)xv) + \partial_{x^{-1}} H(H(u)xv),\]

and so the geodesic equation reads

\[u_t = H(H(u)xv) + \partial_{x^{-1}} H(H(u)xv).\]

Differentiation with respect to the space variable \(x\) now yields

\[u_{tx} = H(uH(x)x + u_xH_x) + H(u_xH_{xx}) = H(uH_{xx}) + 2H(u_xH_x).\]

Applying the Hilbert transform \(H\) to this equation, we see that

\[H(u_{tx}) = -uH_{xx} - 2u_xH_x.\]

If we set \(\omega = Hx\), then we obtain the \(gCLM^*\) equation with parameter \(a^* = 2\):

\[\omega_t = 2\omega v_x + v_\omega, \quad v_x = H\omega.\] (2.5)

Thus we have proven

**Theorem 2.2.** The generalized Constantin–Lax–Majda equation (1.1) with parameter \(a = -\frac{1}{2}\) (or equivalently, the \(gCLM^*\) equation (1.2) with parameter \(a^* = 2\)) corresponds to the equation of the geodesic flow on \(D(S)\) with respect to the right-invariant metric (2.1), (2.2).

**Remark 2.1.** It is important to point out that while we have proven that geodesics must obey the evolution prescribed by the \(gCLM^*\) equation with parameter \(a^* = 2\), we have not demonstrated the existence of geodesics on the manifold \(D(S)\) endowed with the \(H^* (S)\) right-invariant metric. This will be proven in a more detailed analysis, together with results about related topics on the \(gCLM^*\) equation and the geometry of \(D(S)\) endowed with the fractional Sobolev metric.

Let us also mention that our considerations are not in the least limited to the periodic case: appropriate conditions at infinity (cf. [5, 6]) ensuring that the diffeomorphisms approach the identity far out should facilitate the study of the case on the line \(R\).

**Remark 2.2.** Recently, there has been written a host of articles (cf. [7–9, 15, 16] and the references therein) dedicated to the study of differential geometric features of \(D(S)\) endowed...
with several right-invariant metrics, among which the $L^2(S)$, $H^1(S)$, and (the homogeneous) $\dot{H}^1(S)$ metrics attracted the greatest attention, since in these cases, the geodesic equations are re-expressions of the Burgers [3], Camassa–Holm [4], and Hunter–Saxton equations [12], respectively. Thus it is of interest to notice that also the fractional Sobolev metric can give rise to a physically meaningful equation, lying "between" the Burgers equation and the Hunter–Saxton equation, both of which have solutions which lose their initial regularity in finite time [9,18].

**Remark 2.3.** In the hierarchy of the generalized Constantin–Lax–Majda equation [17], the equation discussed lies between the 1D model equation for the quasi-geostrophic equation [11] and the original CLM equation [10]. Solutions to both equations are known to become singular in most cases: This, together with the above remark, supports the conjecture that solutions to the $g\text{CLM}^*$ equation with parameter $a^* = 2$ blow up in finite time as well. We will address this question in a forthcoming study as well.

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