Measuring quantum entanglement without prior state reconstruction

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It is shown that, despite strong nonlinearity, entanglement of formation of two-qubit state can be measured without prior state reconstruction. Collective measurements on small number of copies are provided that allow to determine quantum concurrence via estimation of only four parameters. It is also pointed out that another entanglement measure based on so called negativity can also be measured in similar way. The result is related to general problem: what kind of information can be extracted efficiently from unknown quantum state?

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Two-qubit state entanglement is well characterised. It is the only case where analytical formula for entanglement of formation $E_f$ [1] has been provided [?] with help of so called concurrence (see [2] for review). The simple necessary and sufficient conditions for the presence of entanglement in two qubit state is known involving so called positive partial transpose (PPT) map. However the question is how to detect the presence of entanglement in unknown state possibly efficiently i. e. with minimal number of estimated parameters. As one knows PPT test is represented by nonphysical operation. Concurrence and entanglement of formation are highly nonlinear functions too. Apparently it seems that they require full prior state reconstruction in general. The same might be expected to hold for any other entanglement measures [1] (see also [10]). Indeed direct detection of entanglement of formation function [1] has succeeded only for pure two-qubit states and relied on very special property of the function in that case. Even then, however, the procedure requires estimation of more than one observable [8].

Quite recently, using formula of the best structural physical approximations of hermitian nonphysical maps [1], it has been shown [9] how to detect violation of PPT separability test (or any positive map separability test) experimentally without any prior knowledge about quantum state (for the analysis of the interferometric scheme see [11]).

In this work we provide a protocol detecting concurrence of unknown state by estimation of four parameters in collective measurements of small (not more than eight) number of copies. We also point out how to estimate computable entanglement measure [1] based on negativity [11] in a similar way. For unknown state this gives quadratic gain in number of parameters if compared to quantum tomography. The latter protocol is not restricted to qubits but is valid for any $d \otimes d'$ systems.

We discuss both presented protocols as far as number of involved copies are concerned. In particular to estimate two-qubit entanglement of formation one needs slightly more number of copies than in state reconstruction scheme. It is however different for “computable” measure [8] where not only number of parameters estimated but also number of pairs involved is less than what state reconstruction requires. Subsequently we shall describe the protocols and conclude with some discussion.

Measuring two-qubit entanglement of formation - Let us start with general formula for entanglement of formation $E_f$ of given bipartite state $\rho$ [1]:

$$E_f(\rho) = \min_{\{p_i, \psi_i\}} \sum p_i S(Tr_A(\rho_{i\psi}))$$

where $S(\sigma) = -Tr(\sigma \log \sigma)$ is von Neumann entropy of state $\sigma$ (counted in bits) and the minimum is taken over all ensembles $\{p_i, \psi_i\}$ such that $\sum p_i |\psi_i\rangle \langle \psi_i| = \rho$. For two qubits the entanglement of formation has been explicitly calculated [2] and it amounts to:

$$E_f(\rho) = h(\frac{1 + \sqrt{1 - C(\rho)^2}}{2})$$

(1)

where Shannon binary information is $h(x) = -x \log_2(x) - (1 - x) \log_2(1 - x)$. The important state function called concurrence [?] involves four real monotonically decreasing numbers $\{\lambda_i\}$ which are eigenvalues of (nonhermitian) matrix $\tilde{\rho}$ with

$$\tilde{\rho} = \Sigma \sigma^i \Sigma, \quad \Sigma = \sigma_y \otimes \sigma_y$$

(2)

where Pauli matrices act on Alice and Bob qubit respectively (recall that $\rho$ is two qubit state) and star stands for complex conjugate. Obviously neither $C$ nor $E_f$ is measurable in usual quantum mechanical sense i. e. neither of them is an observable. Indeed this is forbidden by the very fundamental laws of quantum mechanics: both functions are nonlinear in state parameters while only linear operations can be performed on single copy. Still one can try to estimate them in another way - involving access to several copies. Such approach succeeded for pure two-qubit states [10] based, however, on the very special property of any pure state ie. the dependence of its entanglement on eigenvalues of reduced density matrix.

We would have $C$ and $E_f$ determined for arbitrary given $\rho$ if only we knew the four numbers: $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. They come from rather complicated nonlinear function of $\rho$. Surprisingly for any unknown $\rho$ they can be physically measured by estimating only four parameters
the concept of physical approximation of PPT operation
parameter in binary POVM. The map (5) is called
structural

Finally we estimate mean values
of observables

Moreover, as we shall see below,
each \( \langle M_k \rangle \) can be determined via estimation of single parameter in binary POVM.

To see how the above scheme works we need to recall the concept of physical approximation of PPT operation
in lab [13]. Namely the tracepreserving map

\[
\Lambda_{XY} = \frac{d}{d^3+1} I_X \otimes I_Y + \frac{1}{d^3+1} \Pi_X \otimes T_Y,
\]

is physically implementable on any bipartite \( d \otimes d \) state \( \sigma_{XY} \) defined on \( \mathcal{H}_X \otimes \mathcal{H}_Y \). Thus \( \Lambda_{XY} \) corresponds to what in literature is called quantum channel. In formula \([8]\) \( I_X, I_Y \) stand for identity matrices corresponding to subsystem \( X, Y \) respectively. The map \( \Pi \) is just identity map on subsystem \( X \) while \( T \) stands for transposition on second subsystem \( Y \). The partial transposition \( \Pi_X \otimes T_Y \) simply transposes indices corresponding to the second subsystem. The map \( \Pi \) is called structural physical approximation of the nonphysical PPT map \( \Pi \otimes T \) [11][12]. In particular one can define in similar way such approximation for transposition map \( T \). For two qubits the latter coincides with previously introduced universal-NOT gate (see [14]). Subsequently detailed description of the scheme \([\Pi]\) will be presented in two steps.

**Step I. Action of channels \( \Lambda_k \) -** We define quantum channel \( \Lambda_k \) as being composed of two physical operations: (i) the map \([\Pi]\) with \( d = d_k \equiv 4^k \) and spaces \( \mathcal{H}_X \) (\( \mathcal{H}_Y \)) describing all the copies on odd (even) position in the \( k \)-th group and (ii) the subsequent action of unitary operation \( \Sigma \) on each “even” copy. For instance in the second group \( \varrho \otimes \varrho \otimes \varrho \otimes \varrho \) \( (k = 2) \) the first and the third copy corresponds to subsystem \( X \), while the remaining two represent subsystem \( Y \). Only copies belonging to \( Y \) will be subject to subsequent action (ii) of unitary operation \( \Sigma \).

Let the channels \( \Lambda_k \) act on the states according to the scheme \([\Pi]\). As a result we get the following four states

\[
\varrho_k = \frac{d_k}{d_k^2+1} I_X \otimes I_Y + \frac{1}{d_k^2+1} (\varrho \otimes \varrho)^{\otimes k}, \quad k = 1, ..., 4.
\]

Up to the maximal noise part, these states can be considered as “proportional” to the matrices \( (\varrho \otimes \varrho)^{\otimes k} \). In fact they differ from the matrices by shrinking factor \( \frac{1}{d_k^2+1} \) at the “Bloch”. This property is similar to what happens in the theory of approximate cloning or in the case of universal NOT gate \([\Pi]\).

**Step II. Measurement of moments \( \langle M_k \rangle \) -** We shall apply the approach \([14][15]\) providing spectrum state estimation with help of observables constructed from shift operators \( V_n \) \([\Pi]\). They have the following property

\[
\text{Tr}(V_{(r)} A_1 \otimes \ldots \otimes A_n) = \text{Tr}(A_1 A_2 \ldots A_n)
\]

Combining the above property with the fact that all moments \( \sum (\lambda_i)^k \) are real one can see that the latter are just by mean values of the following observables

\[
M_k = \frac{d_k^2 + 1}{2} (V_{(2k)} + V_{(2k)}^\dagger) - d_k^2 I
\]

when each of them is calculated on state \( \varrho_k \) (see the scheme \([\Pi]\)). Finally, the numbers \( \lambda_k \) can be calculated uniquely from the moments. This can be shown in analogy to the analysis of Ref. \([11]\) where state spectrum estimation was provided.

Now an important point is that each mean \( \langle M_k \rangle \) can be detected (up to rescaling) as a single parameter in special binary POVM: following Ref. \([13]\) each observable \( M_k \) can be encoded into some ancilla and then its mean value can be reproduced form measurement of elementary binary observable (Pauli matrix \( \sigma_z \)) \([7]\). This concludes the description of the protocol.

The presented method is parametrically efficient - it requires much less parameters to be measured than state reconstruction does (four instead of fifteen). It is not so as long as number of copies is concerned: one “round” of quantum tomography requires 15 copies while here (see \([\Pi]\)) we require \( 2 + 4 + 6 + 8 = 20 \) copies in each round of the protocol. However the factor \( r = r_x \cdot r_y \equiv \text{“number of parameters \times number of copies”} \) is here superior \((r = 80)\) than for the state reconstruction schemes \((r = 165)\). Even more striking is the fact that while the number
of copies consumed in one round of the experiment is increased only by 33% (from 15 to 20), the number of parameters is decreased almost four times (from 15 to 4). It suggests that the optimal method of entanglement estimation should exists that consumes the same number of copies as state reconstruction but requires less number of parameters (5 or 6). Note that the present method is useful when we are allowed to use small number of apparatus with fixed architecture. One can also anticipate the existence of trade-off between \( r_p \) and \( r_c \) involved in estimation of any entanglement. The optimisation of this trade-off is an interesting open problem.

Note that so far we have asked “How much entanglement is in the system?” The above measurement protocol not only answers but, obviously, solves also the less detailed problem: “Is there any entanglement in the system?”. The above measurement protocol answers quantitative “how much” question. Below we shall provide protocol for such a scenario basing on the fact [13] that if \( \otimes 2 \) state \( \varrho \) is entangled then the smallest eigenvalue of the matrix

\[
\gamma \equiv \Sigma \varrho_{ij} \Sigma \varrho_{kj}
\]

is proportional to \( \frac{C(\varrho^T)}{d^2} \). The modified protocol must involve the map \( \Lambda = \varrho_{ij} \otimes \varrho_{kj} \otimes \varrho_{kl} \otimes \varrho_{lm} \) to produce the new states \( \gamma_k \) with Bloch vector “proportional” to \( (\Sigma \varrho_{ij} \otimes \varrho_{kj}) \otimes \varrho_{kl} \) (that play the role of states \( \varrho_k \) form Step I). Then one should estimate eigenvalues of \( \gamma_k \) in the same way as in Step II and infer \( C(\varrho) \) following result of Ref. [19]. The above protocol answers quantitative “how much” question provided that observers has qualitative knowledge that entanglement is present. The latter must be established before and to this end physical application [12] of PPT test can be used. Due to its binary character (“Yes-No” answer) it requires less measurement precision than the qualitative stage described above. We can use it to minimise measurement effort: if PPT test does not reveal any entanglement we just abandon the second stage.

**Multilevel systems and computable entanglement measure.** - For general \( d \otimes d' \) systems (they can be called multilevel systems) there is no analytical formula for entanglement of formation. However at least one nontrivial entanglement measure can be calculated analytically. This is so called “computable” entanglement measure [1]:

\[
E_c(\varrho) = \log_2 ||\varrho^{T_B}||
\]  

(9)

where \( \varrho^{T_B} = [I \otimes T](\varrho) \) and \( || \cdot || \) stands for trace norm which for hermitian operator means sum of moduli of its eigenvalues. In particular the quantity \( ||\varrho^{T_B}|| - 1 \) introduced to quantify entanglement [11] and after normalisation \( N = (||\varrho^{T_B}|| - 1)/2 \) called negativity has been shown [10] to be entanglement monotone [2]. The measure [1] is weaker than entanglement of formation, it “detects” however all free ie. distillable entanglement of bipartite systems. To estimate the measure we must experimen-

\[
||\varrho^{T_B}|| = \sum_i |\lambda_i^c|,
\]  

(10)

where \( \{\lambda_i^c\} \) are eigenvalues of partially transposed \( \varrho \) symbolised by \( \varrho^{T_B} \). The above formula is crucial for subsequent analysis.

We now apply scheme of Ref. [12] making however full use of its output rather than focusing on minimal eigenvalue. For any bipartite \( d \otimes d' \) state \( \varrho \) the scheme consists of application the map \( \Lambda \) which leads to the new quantum state \( \varrho_1 = \Lambda_X Y(\varrho) \) with eigenvalues \( \{\lambda_i\} \). Each single \( \lambda_i \) one eigenvalue \( \lambda_i^c \) of \( \varrho^{T_B} \). Indeed they are related by the following affine function:

\[
\lambda_i = \frac{d}{d^3 + 1} \frac{1}{d^3 + 1} \lambda_i^c.
\]  

(11)

Now the key point is that \( \{\lambda_i\} \) can be measured experimentally via measurement of \( r_p = d^2 - 1 \) observables (see [11,13,12]). It is much less than \( d^4 - 1 \) required for quantum tomography. From results of the measurements one can calculate all parameters \( \lambda_i^c = (d^3 + 1)\lambda_i - d \) and substitute them to the formula (9) which provides the needed value of the corresponding measure.

The whole scenario can be immediately generalised to any \( d \otimes d' \) system with \( d \neq d' \). Only the parameters in [11] will change slightly and this can be easily calculated with help of structural approximation formula [11].

It is interesting that in the present scenario one has also gain in number of copies involved Indeed we have \( r_c = 2 + 3 + \ldots + d^2 = (d^4 + d^2 - 2)/2 \) instead of \( r_c = d^4 - 1 \).  

**Discussion.** - In conclusion we have provided the first nontrivial protocols of entanglement estimation that require no prior knowledge about the states. The first presented protocol allows to detect two-qubit entanglement of formation with help of joint measurements on small number of systems. In one round the method requires 33% of copies more than state reconstruction (20 instead of 15). However it needs almost four times less parameters (4 instead of 15) than quantum tomography does. We have also shown how to estimate experimentally computable measure of entanglement \( E_c \) based on partial transpose operation. Applying the protocol of [12] we pointed out that the measure can be determined with help of estimation of only \( d^2 - 1 \) parameters instead of \( d^4 - 1 \) ones required for state estimation. The scheme provides also gain in the number of copies consumed: \( r_c = (d^4 + d^2 - 2)/2 \) instead of \( r_c = d^4 - 1 \) due to reconstruction schemes.

It would be interesting to find similar protocol determining some parameters of the best separable approximation of two-qubit quantum states (for instance entangled part of Lewenstein-Sanpera decomposition [21]). It involves the question how to find experimentally eigenvector corresponding to the least negative eigenvalue of par-
tially transposed density matrix $\rho^T_Y$. To get the eigenvector with minimal measurement cost would be particularly interesting because it plays crucial role in the only universal protocol of two-qubit entanglement distillation \[24\]. In the above context basic questions naturally arise: (i) whether it is possible to estimate eigenvector corresponding to extreme eigenvalue of given density matrix without prior reconstruction of the latter? (ii) if so, how efficiently can it be done? Any possible solution of these problems must take into account fundamental restrictions on quantum "nonlinear operations" \[11\,23\].

In any case the present results lead naturally to another interesting question: is it possible to estimate efficiently other entanglement measures (like relative entropy of entanglement \[25\]) or - more generally - correlation measures like the one from Ref. \[24\] based of von Neumann entropy? For pure states it can be done as all measures are equivalent to the entropy of the reduced density matrices which at the same time is proportional to index of correlation $I = S_A + S_B - S_{AB}$. For mixed states the latter serves rather as a measure of global (quantum + classical) correlations \[24\]. Following the present analysis the latter can be efficiently measured without state reconstruction in general. At present one can not develop the method for the relative entropy of entanglement or distillable entanglement because existence of analytical (or partially analytical) formulas for those measures is still an open issue.

Furthermore from the point of view of general quantum information theory very important problem is: what kind of information (whatever it means) can be extracted from unknown quantum state at small measurement cost? Here we have considered extraction of information represented by special parameter - entanglement measure. It is closely related to the following problem what kind of measurement information can be extracted from the state in the protocols distroying as little quantum information as possible? Indeed measurement of small number of parameters can be always interpreted as an incomplete von Neumann measurement which destroys quantum information less than the complete measurement does. An illustrative example is a detection of spectrum without state reconstruction via Young diagrams projections proposed in Ref. \[24\] (for alternative spectrum detection see \[13\]). The above issues are crucial for some quite fundamental processes like universal quantum compression. For example it has been shown \[24\] that quantum source can be compressed efficiently if only one knowns single parameter represented by von Neumann entropy of the source. The entropy can be estimated without prior knowledge about the state: the measurement effort (see \[13\], cf. \[24\]) is the same as in the estimation of the computable measure considered above. Finally, we have considered one of the basic problems in general entanglement theory: "how one can detect the amount of entanglement experimentally?". This question concerns not only entanglement itself: any answer to it says something about properties of information contained in unknown quantum states.

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