ETA QUOTIENTS, EISENSTEIN SERIES AND ELLIPTIC CURVES

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Abstract. We express all the newforms of weight 2 and levels 30, 33, 35, 38, 40, 42, 44, 45 as linear combinations of eta quotients and Eisenstein series, and list their corresponding strong Weil curves.

Let \( p \) denote a prime and \( E(\mathbb{Z}_p) \) denote the the group of algebraic points of an elliptic curve \( E \) over \( \mathbb{Z}_p \). We give a generating function for the order of \( E(\mathbb{Z}_p) \) for certain strong Weil curves in terms of eta quotients and Eisenstein series. We then use our generating functions to deduce congruence relations for the order of \( E(\mathbb{Z}_p) \) for those strong Weil curves.

Key words and phrases: Dedekind eta function; eta quotients; Eisenstein series; modular forms; cusp forms; newforms; Fourier series; Fourier coefficients; elliptic curves; strong Weil curves; group of algebraic points; modularity theorem.

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1. Introduction

Let \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{C} \) denote the sets of positive integers, integers, rational numbers and complex numbers, respectively. Let \( N \in \mathbb{N} \) and \( k \in \mathbb{Z} \). Let \( \Gamma_0(N) \) be the modular subgroup defined by

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1, \ c \equiv 0 \pmod{N} \right\} .
\]

We write \( M_k(\Gamma_0(N)) \) to denote the space of modular forms of weight \( k \) and level \( N \).

The Dedekind eta function \( \eta(z) \) is the holomorphic function defined on the upper half plane \( \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \) by the product formula

\[
\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).
\]

A product of the form

\[
f(z) = \prod_{1 \leq \delta \mid N} \eta^{r_\delta}(\delta z),
\]

where \( r_\delta \in \mathbb{Z} \), not all zero, is called an eta quotient. We set \( q := q(z) = e^{2\pi iz} \). We also set \( [n]f(z) := a_n \) for \( f(z) = \sum_{n \in \mathbb{Z}} a_n q^n \).
Martin and Ono \[7\] listed all the weight 2 newforms that are eta quotients, and gave their corresponding strong Weil curves. There are such eta quotients only for levels 11, 14, 15, 20, 24, 27, 32, 36, 48, 64, 80, 144.

In this paper we express all the newforms of weight 2 and levels 30, 33, 35, 38, 40, 42, 44, 45 as linear combinations of eta quotients and Eisenstein series, and give their corresponding strong Weil curves.

Let \( p \) denote a prime and \( E(\mathbb{Z}_p) \) denote the the group of algebraic points of an elliptic curve \( E \) over \( \mathbb{Z}_p \). We give a generating function for the order of \( E(\mathbb{Z}_p) \) for certain strong Weil curves in terms of eta quotients and Eisenstein series. We then use our generating functions to deduce congruence relations for the order of \( E(\mathbb{Z}_p) \) for those strong Weil curves.

2. Preliminary results

Appealing to \([8\), Theorem 1.64, p. 18\] and \([5\), Corollary 2.3, p. 37\] (see also \([6\), \([4\), \([1\) \]) one can show that

\[
\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z) \in M_2(\Gamma_0(30)),
\]

\[
\frac{\eta^3(3z)\eta^3(33z)}{\eta(z)\eta(11z)}, \frac{\eta^2(3z)\eta^2(33z)}{\eta(z)\eta(11z)}, \frac{\eta(z)(3z)\eta(11z)\eta(33z)}{\eta(z)\eta(33z)} \in M_2(\Gamma_0(33)),
\]

\[
\frac{\eta^3(5z)\eta^3(7z)}{\eta(z)\eta(35z)}, \frac{\eta^3(z)\eta^3(35z)}{\eta(z)\eta(35z)} \in M_2(\Gamma_0(35)),
\]

\[
\frac{\eta^4(2z)\eta^4(38z)}{\eta^2(z)\eta^2(19z)}, \frac{\eta^2(z)\eta^2(38z)}{\eta(z)\eta(38z)}, \frac{\eta(z)(3z)\eta(11z)\eta(38z)}{\eta(z)\eta(38z)} \in M_2(\Gamma_0(38)),
\]

\[
\frac{\eta^2(z)\eta^2(5z)\eta^2(8z)\eta^2(40z)}{\eta(z)\eta(4z)\eta(10z)\eta(20z)} \in M_2(\Gamma_0(40)),
\]

\[
\frac{\eta^2(2z)\eta^2(3z)\eta^2(14z)\eta^2(21z)}{\eta(z)\eta(6z)\eta(7z)\eta(42z)}, \frac{\eta^2(z)\eta^2(6z)\eta^2(7z)\eta^2(42z)}{\eta(z)\eta(6z)\eta(7z)\eta(42z)} \in M_2(\Gamma_0(42)),
\]

\[
\frac{\eta^3(4z)\eta^3(11z)}{\eta(z)\eta(44z)}, \frac{\eta^3(z)\eta^3(44z)}{\eta(z)\eta(44z)}, \frac{\eta(z)(3z)\eta(15z)\eta(45z)}{\eta(z)\eta(45z)} \in M_2(\Gamma_0(45)),
\]

The Eisenstein series \( L(z) \) is defined as

\[
L(z) := -\frac{1}{24} + \sum_{n>0} \sigma(n)q^n,
\]

where \( \sigma(n) = \sum_{m|n} m \) is the sum of divisors function. By \([10\), Theorem 5.8\] we have

\[
L_t(z) := L(z) - tL(tz) \in M_2(\Gamma_0(N)) \text{ for all } 0 < t | N.
\]
We note that if $f(z), g(z) \in M_2(\Gamma_0(N))$, then for all $a, b \in \mathbb{C}$, we have
\[(2.3) \quad af(z) + bg(z) \in M_2(\Gamma_0(N)).\]
We use the Sturm bound $S(N)$ to show the equality of two modular forms in the same modular space. We just need to check the equality of the first $S(N) + 1$ coefficients of their Fourier series expansions. The following theorem is a special case for $M_2(\Gamma_0(N))$, see [4, Theorem 3.13] for a general case.

**Theorem 2.1.** Let $f(z), g(z) \in M_2(\Gamma_0(N))$ with the Fourier series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n q^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n q^n.$$  

The Sturm bound $S(N)$ for the modular space $M_2(\Gamma_0(N))$ is given by

$$S(N) = \frac{N}{6} \prod_{p|N} (1 + 1/p),$$

and so if $a_n = b_n$ for all $n \leq S(N)$ then $f(z) = g(z)$.

Using Theorem 2.2 we calculate $S(N)$ for $N \in \{30, 33, 35, 38, 40, 42, 44, 45\}$ as

- $S(30) = 12$, $S(33) = 8$, $S(35) = 8$, $S(38) = 10$,
- $S(40) = 12$, $S(42) = 16$, $S(44) = 12$, $S(45) = 12$.

3. MAIN RESULTS

**Theorem 3.1.** Let $N \in \{30, 33, 35, 38, 40, 42, 44, 45\}$. In Table 3.1 below we express all the newforms $F_N(z)$ in $M_2(\Gamma_0(N))$ as linear combinations of eta quotients and Eisenstein series.

| Level | Name     | The eta quotients and Eisenstein series |
|-------|----------|----------------------------------------|
| 30    | $F_{30}(z)$ | $6\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)$ $\eta(2z)\eta(5z)\eta(6z)\eta(15z)$ $+ 2L_2(z) + L_3(z) + \frac{1}{5}L_5(z) - 2L_6(z) - \frac{2}{5}L_{10}(z)$ $- \frac{1}{5}L_{15}(z) + \frac{2}{5}L_{30}(z)$ |
| 33    | $F_{33}(z)$ | $-10\eta^3(3z)\eta^3(33z)$ $\eta(z)\eta(11z) - 6\eta^2(3z)\eta^2(33z)$ |

Table 3.1: Weight 2 newforms of levels 30, 33, 35, 38, 40, 42, 44, 45
| Level | Name | The eta quotients and Eisenstein series |
|-------|------|----------------------------------------|
| 35    | $F_{35}(z)$ | $-2\eta(z)\eta(3z)\eta(11z)\eta(33z) + \frac{1}{3}L_3(z) + L_{11}(z) - \frac{1}{3}L_{33}(z)$ |
| 38    | $F_{38A}(z)$ | $\frac{3\eta^3(5z)\eta^3(7z)}{\eta(z)\eta(35z)} - \frac{\eta^3(z)\eta^3(35z)}{\eta(5z)\eta(7z)} + \frac{4}{5}L_5(z) - \frac{5}{7}L_7(z) - \frac{73}{35}L_{35}(z)$ |
| 38    | $F_{38B}(z)$ | $\frac{3\eta^3(z)\eta^3(38z)}{\eta(z)\eta(38z)} - \frac{18\eta^4(2z)\eta^4(38z)}{\eta(z)\eta(19z)} - \frac{3\eta^3(2z)\eta^3(19z)}{7\eta(z)\eta(38z)} - \frac{9\eta^4(z)\eta^4(19z)}{28\eta^2(2z)\eta^2(38z)} + \frac{1}{7}L_2(z) + \frac{1}{7}L_{19}(z) + \frac{1}{7}L_{38}(z)$ |
| 40    | $F_{40}(z)$ | $-\frac{4\eta^2(z)\eta^2(5z)\eta^2(8z)\eta^2(40z)}{\eta(z)\eta(4z)\eta(10z)\eta(20z)} + \frac{3}{2}L_2(z) + \frac{3}{2}L_4(z) + L_5(z)$ |
| 42    | $F_{42}(z)$ | $-8\frac{\eta(2z)\eta(3z)\eta^2(7z)\eta^2(42z)}{\eta(z)\eta(6z)} - \frac{8\eta(z)\eta(6z)\eta^2(14z)\eta^2(21z)}{\eta(2z)\eta(3z)} + L_2(z) + L_6(z) + \frac{1}{7}L_7(z) - \frac{1}{7}L_{14}(z) + \frac{1}{7}L_{21}(z) - \frac{1}{7}L_{42}(z)$ |
| 44    | $F_{44}(z)$ | $3\frac{\eta^3(4z)\eta^3(11z)}{\eta(z)\eta(44z)} - 3\frac{\eta^3(z)\eta^3(44z)}{\eta(4z)\eta(11z)} + 2L_4(z)$ |
| 45    | $F_{45}(z)$ | $2\frac{\eta(3z)\eta^2(5z)\eta^2(9z)\eta(15z)}{\eta(z)\eta(45z)} - 2\frac{\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)} + L_5(z) - \frac{2}{3}L_9(z) - \frac{2}{5}L_{15}(z) - \frac{14}{15}L_{45}(z)$ |
Proof. In [2] Table 3] each newform of weight 2 and level less than 1000 has been
given by listing its Fourier coefficients for primes up to 100. Using the results from
[2] p. 25] together with [2] Table 3] we determine the first $S(N) + 1$ terms of the
Fourier series expansions of all the newforms of weight 2 and levels $N = 30, 33,$
$38, 40, 42, 44, 45$. We give them in Table 3.2 below.

Table 3.2: First $S(N) + 1$ terms of the Fourier series expansions of the newforms of weight 2 and level $N$

| $N$  | First $S(N) + 1$ terms of the newforms of level $N$                                      |
|------|-----------------------------------------------------------------------------------------|
| 30   | $q - q^2 + q^3 + q^4 - q^5 - q^6 - 4q^7 - q^8 + q^9 + q^{10} + q^{12} + O(q^{13})$,     |
| 33   | $q + q^2 - q^3 - q^4 - 2q^5 - q^6 + 4q^7 - 3q^8 + O(q^9)$,                               |
| 35   | $q + q^3 - 2q^4 - q^5 + q^7 + O(q^9)$,                                                  |
| 38A  | $q - q^2 + q^3 + q^4 - q^6 - q^7 - q^8 - 2q^9 + O(q^{11})$,                              |
| 38B  | $q + q^2 - q^3 + q^4 - 4q^5 - q^6 + 3q^7 + q^8 - 2q^9 - 4q^{10} + O(q^{11})$,            |
| 40   | $q + q^5 - 4q^7 - 3q^9 + 4q^{11} + O(q^{13})$,                                          |
| 42   | $q + q^2 - q^3 + q^4 - 2q^5 - q^6 - q^7 + q^8 + q^9 - 2q^{10} - 4q^{11} - q^{12}$       |
|      | $+ 6q^{13} - q^{14} + 2q^{15} + q^{16} + O(q^{17})$,                                   |
| 44   | $q + q^3 - 3q^5 + 2q^7 - 2q^9 - q^{11} + O(q^{13})$,                                   |
| 45   | $q + q^2 - q^4 - q^5 - 3q^8 - q^{10} + 4q^{11} + O(q^{13})$.                            |

Let us consider the function $F_{30}(z)$ from Table 3.1. By (2.1)–(2.3), we have
$F_{30}(z) \in M_2(\Gamma_0(30))$. Using MAPLE we calculate the first $S(30) + 1 = 13$ terms
of the Fourier series expansion of $F_{30}(z)$ as

$$F_{30}(z) = q - q^2 + q^3 + q^4 - q^5 - q^6 - 4q^7 - q^8 + q^9 + q^{10} + q^{12} + O(q^{13}).$$

Since the first 13 terms of the newform of $M_2(\Gamma_0(30))$ in Table 3.2 are the same as
the first 13 terms of $F_{30}(z)$ then it follows from Theorem 2.1 that $F_{30}(z)$ is equal to
the newform of level 30. The remaining cases can be proven similarly. Note that
there are two different weight 2 newforms for level 38, and following the notation
from [2] Table 3] we label them as 38A and 38B in Table 3.2, which correspond to
$F_{38A}(z)$ and $F_{38B}(z)$ in Table 3.1, respectively.

4. SOME ARITHMETIC PROPERTIES OF $|E(\mathbb{Z}_p)|$ FOR CERTAIN ELLIPTIC CURVES

We recall that $p$ denotes a prime. We first state the $a_p$ version of the Modularity
Theorem, see [3] Theorem 8.8.1], which gives a relation between $|E(\mathbb{Z}_p)|$ and the
Fourier coefficients of the corresponding newform.
Theorem 4.1. (Modularity Theorem, Version $a_p$) Let $a_1, a_2, a_3, a_4, a_6 \in \mathbb{Z}$. Let $E$ be an elliptic curve over $\mathbb{Q}$ with conductor $N$ given by
\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \]
and let
\[ E(\mathbb{Z}_p) := \{ \infty \} \cup \{(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p | y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \}. \]
Then for some newform $f \in S_2(\Gamma_0(N))$, we have
\[ [p] f_E(z) = p + 1 - |E(\mathbb{Z}_p)| \text{ for } p \nmid N. \]
We deduce the following theorem from [2, Table 1].

Theorem 4.2. Table 4.1 below is a list of elliptic curves, more specifically strong Weil curves, corresponding to the newforms given in Table 3.1.

| Newform | Strong Weil curve | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_6$ |
|---------|-------------------|-------|-------|-------|-------|-------|
| $F_{30}(z)$ | $E_{30A}$ | 1 | 0 | 1 | 1 | 2 |
| $F_{33}(z)$ | $E_{33A}$ | 1 | 1 | 0 | -11 | 0 |
| $F_{35}(z)$ | $E_{35A}$ | 0 | 1 | 1 | 9 | 1 |
| $F_{38A}(z)$ | $E_{38A}$ | 1 | 0 | 1 | 9 | 90 |
| $F_{38B}(z)$ | $E_{38B}$ | 1 | 1 | 1 | 0 | 1 |
| $F_{40}(z)$ | $E_{40A}$ | 0 | 0 | 0 | -7 | -6 |
| $F_{42}(z)$ | $E_{42A}$ | 1 | 1 | 1 | -4 | 5 |
| $F_{44}(z)$ | $E_{44A}$ | 0 | 1 | 0 | 3 | -1 |
| $F_{45}(z)$ | $E_{45A}$ | 1 | -1 | 0 | 0 | -5 |

We use Theorems 3.1, 4.1 and 4.2 to give generating functions for the group order $|E_N(\mathbb{Z}_p)|$ of elliptic curves in Table 4.1.

Theorem 4.3. Consider the elliptic curves listed in Table 4.1. We have
\[
|E_{30A}(\mathbb{Z}_p)| = -6[p] \left( \frac{\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)}{\eta(2z)\eta(5z)\eta(6z)\eta(15z)} \right) \text{ for all } p \nmid 30, \\
|E_{33A}(\mathbb{Z}_p)| = 2[p] \left( \frac{5\eta^3(3z)\eta^3(33z)}{\eta(z)\eta(11z)} + \eta(z)\eta(3z)\eta(11z)\eta(33z) \right) \text{ for all } p \nmid 33, \\
|E_{35A}(\mathbb{Z}_p)| = 3(p + 1) - [p] \left( \frac{3\eta^3(5z)\eta^3(7z)}{\eta(z)\eta(35z)} - \frac{\eta^3(z)\eta^3(35z)}{\eta(5z)\eta(7z)} \right) \text{ for all } p \nmid 35,
\]
\[ |E_{38A}(\mathbb{Z}_p)| = \frac{6}{7}(p+1) - \frac{3}{7}[p] \left( \frac{\eta^3(z)\eta^3(38z)}{\eta(2z)\eta(19z)} - \frac{6}{4}\eta^4(2z)\eta^4(38z) - \frac{3}{4}\eta^4(z)\eta^4(19z)\right) \text{ for all } p \mid 38, \]

\[ |E_{38B}(\mathbb{Z}_p)| = (p+1) - [p] \left( \frac{\eta^3(2z)\eta^3(19z)}{\eta(2z)\eta(38z)} + \frac{\eta^3(z)\eta^3(38z)}{\eta(2z)\eta(19z)} \right) \text{ for all } p \mid 38, \]

\[ |E_{40A}(\mathbb{Z}_p)| = 4[p] \left( \frac{\eta^2(z)\eta^2(5z)\eta^2(8z)\eta^2(40z)}{\eta(2z)\eta(4z)\eta(10z)\eta(20z)} \right) \text{ for all } p \mid 40, \]

\[ |E_{42A}(\mathbb{Z}_p)| = 8[p] \left( \frac{\eta(2z)\eta(3z)\eta^2(7z)\eta^2(42z)}{\eta(z)\eta(6z)} + \frac{\eta(z)\eta(6z)\eta^2(14z)\eta^2(21z)}{\eta(2z)\eta(3z)} \right) \text{ for all } p \mid 42, \]

\[ |E_{44A}(\mathbb{Z}_p)| = 3(p+1) - 3[p] \left( \frac{\eta^3(4z)\eta^3(11z)}{\eta(z)\eta(44z)} - \frac{\eta^3(z)\eta^3(44z)}{\eta(4z)\eta(11z)} \right) \text{ for all } p \mid 44, \]

\[ |E_{45A}(\mathbb{Z}_p)| = 2(p+1) - 2[p] \left( \frac{\eta(3z)\eta^2(5z)\eta^2(9z)\eta(15z)}{\eta(z)\eta(45z)} - \frac{\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)} \right) \text{ for all } p \mid 45, \]

**Proof.** We just prove the first and the last equalities as the remaining ones can be proven similarly. By Theorems 3.1, 4.1 and 4.2, for all \( p \mid 30 \), we have,

\[ |E_{30A}(\mathbb{Z}_p)| = p + 1 - [p]F_{30A}(z) \]

\[ = p + 1 - [p] \left( 6\frac{\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)}{\eta(2z)\eta(5z)\eta(6z)\eta(15z)} + \frac{2L_2(z)}{L_2(z)} + \frac{1}{5}L_5(z) - \frac{2}{5}L_6(z) - \frac{1}{5}L_{10}(z) - \frac{2}{5}L_{15}(z) + \frac{2}{5}L_{30}(z) \right) \]

\[ = p + 1 - 6[p] \left( \frac{\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)}{\eta(2z)\eta(5z)\eta(6z)\eta(15z)} \right) - \sigma(p) \]

\[ = -6[p] \left( \frac{\eta(z)\eta(3z)\eta^3(10z)\eta^3(30z)}{\eta(2z)\eta(5z)\eta(6z)\eta(15z)} \right), \]

which completes the proof of the first equality.

By Theorems 3.1, 4.1 and 4.2, for all \( p \mid 45 \), we have

\[ |E_{45A}(\mathbb{Z}_p)| = p + 1 - [p]F_{45}(z) \]

\[ = p + 1 - [p] \left( 2\frac{\eta(3z)\eta^2(5z)\eta^2(9z)\eta(15z)}{\eta(z)\eta(45z)} - \frac{2\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)} + L_5(z) - \frac{2}{3}L_9(z) - \frac{2}{5}L_{15}(z) - \frac{14}{15}L_{45}(z) \right) \]
\[ p + 1 - 2[p] \left( \frac{\eta(3z)\eta^2(5z)\eta(9z)\eta(15z)}{\eta(z)\eta(45z)} - \frac{\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)} \right) + \sigma(p) \]

\[ = 2(p + 1) - 2[p] \left( \frac{\eta(3z)\eta^2(5z)\eta^2(9z)\eta(15z)}{\eta(z)\eta(45z)} - \frac{\eta^2(z)\eta(3z)\eta(15z)\eta^2(45z)}{\eta(5z)\eta(9z)} \right), \]

which completes the proof of the last equality. □

The following congruence relations follow immediately from Theorem 4.2.

**Corollary 4.1.** We have

\[- \quad \begin{array}{ll}
|E_{30A}(Z_p)| & \equiv 0 \pmod{6} \quad \text{for all } p \nmid 30, \\
|E_{33A}(Z_p)| & \equiv 0 \pmod{2} \quad \text{for all } p \nmid 33, \\
|E_{38A}(Z_p)| & \equiv 0 \pmod{3} \quad \text{for all } p \nmid 38, \\
|E_{40A}(Z_p)| & \equiv 0 \pmod{4} \quad \text{for all } p \nmid 40, \\
|E_{42A}(Z_p)| & \equiv 0 \pmod{8} \quad \text{for all } p \nmid 42, \\
|E_{44A}(Z_p)| & \equiv 0 \pmod{3} \quad \text{for all } p \nmid 44, \\
|E_{45A}(Z_p)| & \equiv 0 \pmod{2} \quad \text{for all } p \nmid 45.
\end{array} \]

**5. Remarks**

(1) The linear combinations of eta quotients and Eisenstein series in Table 3.1 have been chosen in a way that we can prove Theorem 4.3. In Table 5.1 below we give alternative representations for the newforms of levels 33, 40, 42 for which have fewer number of eta quotients and Eisenstein series. We note that representation of the newform for level 33 in [9] is the same as the one in Table 5.1.

**Table 5.1: Alternative representations for weight 2 newforms of levels 33, 40, 42**

| Level | Name | The eta quotients and Eisenstein series |
|-------|------|-----------------------------------------|
| 33    | \( F'_{33}(q) \) = \[3\eta^2(q^3)\eta^2(q^{33}) + 3\eta(q)\eta(q^3)\eta(q^{11})\eta(q^{33}) + \eta^2(q)\eta^2(q^{11})\] |
| 40    | \( F'_{40}(q) \) = \[2\eta^2(q^2)\eta^2(q^{10}) - \eta(q^2)\eta^2(q^8)\eta^5(q^{20}) \]
| 42    | \( F'_{42}(q) \) = \[\eta^2(q^2)\eta^2(q^{3})\eta^2(q^{14})\eta^2(q^{21}) \]

\[ - \eta^2(q)\eta^2(q^8)\eta^2(q^{14})\eta^2(q^{21}) \]

\[ \eta(q)^4\eta^2(q^{14})\eta(q^{21}) \]

\[ \]
In [7] Martin and Ono represented weight 2 newforms of levels 11, 14, 15, 20, 24, 27, 32, 36, 48, 64, 80, 144 in terms of single eta quotients. Let \( F_N(z) \) be the newform(s) at level \( N \). Using the arguments from this paper we give alternative representations for the new forms of levels 11, 14, 15 in Table 5.2.

Table 5.2: Alternative representations for weight 2 newforms of levels 11, 14, 15

| Level | Name | The eta quotients and Eisenstein series |
|-------|------|-----------------------------------------|
| 11    | \( F_{11}(z) = \frac{-5\eta^4(2z)\eta^4(22z)}{\eta^2(z)\eta^2(11z)} - 4\eta^2(2z)\eta^2(22z) + \frac{1}{2}L_2(z) + L_{11}(z) - \frac{1}{2}L_{22}(z) \) |
| 14    | \( F_{14}(z) = \frac{6\eta^5(z)\eta^5(14z)}{5\eta^3(2z)\eta^3(7z)} + \frac{13}{5}L_2(z) - \frac{13}{5}L_7(z) + L_{14}(z) \) |
| 15    | \( F_{15}(z) = \frac{-4\eta^3(3z)\eta^3(15z)}{\eta(z)\eta(5z)} + \frac{1}{3}L_3(z) + L_5(z) - \frac{1}{3}L_{15}(z) \) |

Corresponding strong Weil curves with conductors 11, 14 and 15 are
- \( E_{11A} : y^2 + y = x^3 - x^2 - 10x - 20 \),
- \( E_{14A} : y^2 + xy + y = x^3 + 4x - 6 \),
- \( E_{15A} : y^2 + xy + y = x^3 + x^2 - 10x - 10 \),

respectively, see [2, Table 1] and [7]. Similar to Theorem 4.3, we obtain

\[
| E_{11A}(\mathbb{Z}_p) | = 5[p] \left( \frac{\eta^4(2z)\eta^4(22z)}{\eta^2(z)\eta^2(11z)} \right) \text{ for all } p \nmid 11,
\]

\[
| E_{14A}(\mathbb{Z}_p) | = -6[p] \left( \frac{\eta^5(z)\eta^5(14z)}{\eta^3(2z)\eta^3(7z)} \right) \text{ for all } p \nmid 14,
\]

\[
| E_{15A}(\mathbb{Z}_p) | = 4[p] \left( \frac{\eta^3(3z)\eta^3(15z)}{\eta(z)\eta(5z)} \right) \text{ for all } p \nmid 15.
\]

Thus we deduce the congruence relations

\[
| E_{11A}(\mathbb{Z}_p) | \equiv 0 \pmod{5}, \text{ for all } p \nmid 11,
\]

\[
| E_{14A}(\mathbb{Z}_p) | \equiv 0 \pmod{6}, \text{ for all } p \nmid 14,
\]

\[
| E_{15A}(\mathbb{Z}_p) | \equiv 0 \pmod{4}, \text{ for all } p \nmid 15.
\]
Numerical results indicate that generating functions for $|E_N(Z_p)|$ for a family of elliptic curves $E_N$ of conductor $N$ can be given by eta quotients and Eisenstein series. To this end, we have obtained generating functions for $|E_N(Z_p)|$ for elliptic curves $E_N$ of conductor $N$ for various $N < 100$, which will be a part of the PhD thesis of Zafer Selcuk Aygin.

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