Optimal policy structure for system upgrades during an asset’s lifetime

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Abstract

We consider a continuous-time stochastic model that aims to find an optimal upgrade policy for a single system in an asset with a fixed overhaul planning. In a novel model, we include the key critical factors that may drive an asset owner to upgrade a system: increasing functionality requirements due to evolving technology, age-dependent maintenance costs, a predetermined overhaul plan, and a finite lifetime of the asset. We characterize the structure of the optimal upgrade policy through analytical results, and use these to establish an efficient solution approach based on dynamic programming. We also consider the effects of the input parameters on the optimal total costs, both analytically as well as numerically.

1 Introduction

Maintaining a satisfactory operational level for assets, i.e. capital goods, during their entire lifetime is a very challenging problem (Pierskalla and Voelker, 1976; Arts et al., 2019). Assets typically have complex designs with many different systems installed. Each system itself has to be maintained through several upgrades in order to keep up with the desired level of functionality, which continually increases with the age of the asset due to e.g. technological advancements. These maintenance activities come with costs and may also affect the availability and operations of the asset. To minimize downtime, it is decided upfront to have several so-called overhaul moments during the asset’s lifetime where the major maintenance activities of different systems can be executed in parallel. Among those maintenance activities are system upgrades, which improves the functionality and/or quality of an asset to meet the user’s requirements. In this paper, we study the planning of upgrades for all critical systems for a time horizon that equals the lifetime of the asset, where a fixed overhaul planning is set upfront for the asset.

Tangible examples can be found in many settings, such as the maritime and transport industries, or production processes that operate round-the-clock. For example, a frigate is an asset that contains many different systems, e.g. electrical, automation, radar, or weapon systems. Frigates are typically in use for over 30 years, while the incorporated systems themselves do not necessarily last several decades. Upfront, a schedule is created with designated moments (“overhauls”) where the frigate is docked and large-scale maintenance can be executed. Docking a ship is costly, and it is therefore efficient to do as much maintenance work as possible during these periods. Naturally, maintenance activities can be done between the overhaul moments as well. The costs for maintaining systems may increase over time, as difficulties can arise in finding spare parts, system expertise, and so forth.

There are many factors that play a role in the decision whether and when to execute a system upgrade. Based on extensive discussions with multiple companies from the maritime sector, we came to the following dominant factors: functionality requirements, failure frequency, age-dependent failure costs and the asset’s lifetime. We elaborate on these factors in more detail next. First, advancement of technologies leads to the availability of systems that perform much better than the older ones, which may cause
higher expectations of a system’s functionality. Also, advancement of technology may be prompted by other reasons, e.g. the desire for a more energy-sustainable system, the wish to rival or even surpass the functionality that competitors have, and more. This aspect is studied in some papers, e.g. (Nair and Hopp, 1992; Hopp and Nair, 1994), but is surprisingly often neglected in existing literature on maintenance strategies (de Jonge and Scarf, 2020).

Secondly, a well-studied aspect is the notion that system failures may occur over time. Existing literature often focuses on scenarios where failures become more likely as the system ages, corresponding to increasing failure rates, or on constant failure rates which are common for electronics. Under an age-replacement policy, a system is replaced at some planned age or upon failure. In our setting, an asset owner may wish to upgrade its system to an improved version sooner when failures become increasingly likely. There is a vast literature on age-replacement policies, and it continues to be an active field of research (Wang, 2002; Zhao et al., 2017).

Thirdly, all maintenance activities come with associated costs, which are stationary in classical variants of age-based maintenance models (Barlow and Hunter, 1960). Naturally, age-dependent costs are often necessary to model reality more closely. Several factors contribute to the need for age-dependent costs, such as the salvage values and the failure costs. Regarding the salvage values, we can observe that the residual value of the system in use typically decreases as newer and improved versions come to market. The failure costs can also be age-dependent, especially when assets have relatively long lifetimes. This is often due to obsolescence issues, i.e. spare parts become costlier or even unavailable over time since newer versions are developed and the production of older components or systems are discontinued. Moreover, trained technicians with the knowledge and skills to execute the upgrades may no longer be available. A well-chosen strategy can reduce the risk of (unexpected) obsolescence issues and their corresponding costs (IEC, 2019), such as monitoring for obsolescence, last-time-buys (Behfard et al., 2015) or partner agreements, e.g. incorporation of availability guarantees from suppliers for critical components (Tomczykowski, 2003). Nevertheless, the level of uncertainty and associated costs makes this a very challenging aspect of the problem, and can persuade the asset owner to upgrade the system to an improved version sooner.

Finally, the timing of an upgrade may be affected by the remaining lifetime of the asset. Existing literature often focuses on long-run expected average costs for systems with an infinite time horizon. However, an infinite horizon rarely applies to real-life settings: as the end of the asset’s lifetime is near, it makes little sense to execute large-scale upgrades or improvements anymore. Moreover, the total costs during the lifetime can be reduced by bringing forward (or postponing) certain maintenance activities.

In this paper, we consider a continuous-time stochastic model that aims to find an optimal upgrade policy for a single system in an asset with a fixed overhaul planning. To do so, we account for all four of the above-mentioned factors that may play a role in this decision. Our model is novel by bringing these crucial aspects together, while they are relatively under-examined in existing literature: 1) the introduction of a penalty for the functionality gap between the old system and the required level, 2) the penalization of an upgrade when it is not executed during an overhaul, 3) age-dependent cost functions and 4) accounting for a finite lifetime of the asset. For this model, we analytically characterize the structure of the optimal upgrade policy. In particular, if we do not penalize the event where upgrades are not executed during overhauls, we establish that there exists an optimal policy with equidistant upgrade times for a large class of input parameters functions. This allows for a relatively simple characterization of the optimal upgrade policies, and can be used to design an efficient solution approach based on dynamic programming for the case where the penalization is also included. Finally, we establish sensitivity results with respect to the input parameters. For example, it is intuitively clear that the optimal number of upgrades do not increase as the price for the upgrades are higher. However, surprisingly, this is not necessarily true with respect to many other input parameters such as the penalty for upgrading outside overhauls, the number of overhauls, or a more rapid increase of failure costs.

The remainder of this paper is organized as follows. In Section 2, we review related literature, and in Section 3 we introduce our model formally. In Section 4 we derive high-level insights for the structure
of the optimal policy if upgrade executions outside overhauls are not penalized. For the general case, we use the results in Section 3 to introduce a solution approach on how to calculate the optimal policy efficiently in Section 5. Finally, we derive several managerial insights in Section 6 by testing the dependency of the optimal policy with respect to the input parameters. We summarize our key findings and insights in Section 7.

2 Literature review

Our work mainly relates to two streams of literature: (i) age-based maintenance models with age-dependent costs, and (ii) upgrade/replacement decision-making by considering the effects of evolving technology. In this section, we review the related literature for both streams, and describe the contributions of our paper.

The first stream concerns age-based maintenance models. In the classical variants of these models, we consider a single system with a stochastic lifetime distribution. The policy maker can choose to execute preventive maintenance against fixed costs that are lower than the costs that are incurred if the maintenance activity would be executed upon failure (Barlow and Hunter, 1960). However, it is often the case that not all times are convenient to do maintenance on a system. Examples include systems in production lines, the aerospace or maritime industry, or other systems in industries where unplanned downs severely interrupt operations (Arts et al., 2019). In that case, the policy maker may choose to apply minimal repair until the planned maintenance moment is reached, i.e. restore the system to a state right before the failure took place any time a failure occurs.

There is a vast literature on these types of maintenance models (Wang, 2002). We point out that in these papers executing maintenance is often considered to be a replacement of the system. Initially, stationary costs are considered (Barlow and Hunter, 1960; Barlow and Proschan, 1965), but quickly also age-dependent costs were accounted for in the models. A recent paper (Sanoubar et al., 2020) shows the existence of an optimal replacement policy for an infinite time horizon that allows for a wide class of age-dependent replacement cost functions without minimal repair. In (Boland, 1982; Boland and Proschan, 1982) a setting is described where the minimal repair costs increase with the number of occurrences for both a finite and an infinite time horizon. A more general minimal repair cost function is considered in (Tilquin and Cleroux, 1975), but replacement costs are assumed to be stationary. Non-decreasing minimal repair and replacement costs are considered in (Segawa et al., 1992) for an infinite time horizon. Literature that considers a finite time horizon is scarcer; (Chun, 1992; Dagpunar and Jack, 1994) determine the optimal number of imperfect preventive maintenance for a finite horizon given that the minimal repair is made at any failure (with stationary costs).

This paper considers a continuous-time setting with non-decreasing upgrade costs, failure rate and minimal repair costs, while accounting for the finite lifetime of the asset. A major novel aspect is that not only maintenance on system level is considered, but there already exists an overhaul plan upfront for the entire asset. This means that the execution of upgrades during the overhauls is typically preferred in order to prevent disruption of asset availability. Upgrades can still be planned at moments when there is no overhaul, but this comes with a penalty that portrays the severity of the disruption. The effects of such overhaul plans are unexplored in existing literature on age-based maintenance models.

The second stream of literature relates to the effect of executing upgrades to meet (increasing) functional requirements. This aspect is studied already in a series of papers (Nair and Hopp, 1992; Hopp and Nair, 1994; Nair, 1995), where a discrete-time model is proposed in which one needs to decide whether to keep current technology or move to another version at every time step. Through a Markov Decision Problem formulation, an optimal solution can be determined through dynamic programming, and also the infinite horizon case can be solved through an efficient algorithm. Technology advancement is also considered in (Rajagopalan et al., 1998), but for an entirely different environment where technologies provide a certain capacity that is needed for operations. A continuous-time multi-component maintenance model is consid-
In this paper, we consider a continuous-time stochastic model similar as in (Mercier, 2008). We restrict to an upgrade policy for a single system, but we allow for continuously evolving technology and preventive upgrade policies (with a minimal repair strategy to deal with failures). Our underlying costs and penalty functions are age-based, and we derive structural insights for the optimal upgrade policy. In addition, we incorporate overhaul moments where large-scale maintenance is executed, and hence it can be cost-effective to execute a system upgrade at such a moment.

3 Model description

In this section, we provide a formal description of the model. This description requires some mathematical notation, for which we provide an overview in Appendix A. We consider an asset that is in operation for a finite (known) lifetime $H$. Typically, an asset has many different systems, subsystems and components in place on different indenture levels. We take a simplified view as illustrated in Figure 1. That is, an asset is composed of different systems, and in turn, every system has many different (critical) components that are needed for an operational system. The goal is to determine a cost-optimal upgrade planning on system level.

![Asset decomposition](image)

We consider a setting where the upgrade of a system may interfere with the operations of the entire asset. It is therefore efficient to plan upgrades of multiple systems together, where also other maintenance activities can be performed. We assume that there exists an overhaul plan with planned moments for the entire asset. Our goal is to provide an upgrade policy for a single system in the asset with consideration of the overhauls, i.e. the time periods at which the system is upgraded, either jointly with overhauls or separately. This yields a continuous-time model with special overhaul periods where large-scale maintenance is executed. More specifically, we assume that the asset has an overhaul plan with fixed overhaul periods. Overhaul periods are relatively short with respect to the total lifetime of the asset. Therefore, we allow for a simplification of the model by assuming that the overhaul takes no time in the planning, and hence the state of the system also has not changed after an overhaul unless an upgrade had been planned.

We denote the time between consecutive overhauls by $M_1, \ldots, M_m, M_{m+1}$, where $m$ is the number of overhauls during the asset’s lifetime. In particular, $M_1$ denotes the time of the first overhaul, and $M_{m+1}$
denotes the duration between the last overhaul and the end of the asset’s lifetime. We also allow for system upgrades to be executed at times when there is no overhaul, but we penalize these events with fixed costs \( c_d \). Note that if a system upgrade can be executed easily and has no effect on asset-level, the setting simplifies to \( c_d = 0 \). We refer to this special setting as the base case.

As mentioned in the introduction, we model several factors that drive an asset owner to upgrade a system through general costs and penalty functions. We explain this in more detail next. First, we say that a system upgrade comes with a certain upgrade costs, for which we assume the following.

**Assumption 3.1** The upgrade costs can be written as \( c_0 - v(t) \), where \( c_0 > 0 \) can be seen as the price for purchasing and installing the upgraded version of the system, \( v(t) \) corresponds to the salvage value of the current system, and \( t \geq 0 \) is the time since the current system is in use. Moreover, we assume that \( c_0 > v(0) \), and that the salvage value \( v(t) \) is non-increasing and differentiable in \( t \).

We point out that the price for a system upgrade \( c_0 \) is constant, i.e. has no dependency on time. This reflects the idea that asset owners are willing to pay a certain fixed price for the newest generation of a system, which is not uncommon for consumer goods or certain capital goods. The salvage value \( v(\cdot) \) corresponds to the possible costs or return that comes with the disposing of the current system in use, and depends only on time it has been in use. In particular, we make no assumptions on whether the salvage value is positive or negative. We do exclude the unrealistic scenarios where \( c_0 \leq v(0) \), otherwise it would be profitable to continuously do upgrades. The property that the salvage value is non-increasing describes the natural phenomenon that the disposal of an older system becomes less profitable (or costlier) over time. Also in cases where equipment is refurbished or reused, the salvage value can decrease in age since the refurbishment process may involve adjustments and additions of extra components (Sanoubar et al., 2020).

As soon as a certain system is set in place in the asset, improved versions and upgrades may come to the market. Increasing functional requirements of the current system may lead to the necessity for upgrading to the newer versions to meet the desired requirements. We model this effect through a penalty function \( c_f(t) \), where \( t \) denotes the time that the system is in use since its last upgrade. It represents the gap in functionality between the system in use and the upgraded version, see e.g. (Sols et al., 2012) for an example of such a gap function. Note that by definition there is no gap when the system in use is brand-new, i.e. \( c_f(0) = 0 \).

**Assumption 3.2** The functionality gap function \( c_f(t) \) is non-decreasing function in \( t \) with \( c_f(0) = 0 \), where \( t \) denotes the time since the last upgrade.

The system itself consists of many different (critical) components that are subject to failures. We aggregate the failures to a single failure rate for the system, where every failure comes with a certain expected cost. We consider the setting where failures do not occur less likely as time progresses, as well as that the associated costs do not lessen over time. Moreover, we assume that the repair or replacement of a component restores the total failure rate back to the value right before the failure event, also known as a minimal repair strategy (Barlow and Proschan, 1965).

**Assumption 3.3** Let \( h(t) \) denote the failure rate of the system, and \( k(t) \) the expected repair costs to bring the system back to the state just prior to the failure, where \( t \geq 0 \) denotes the time since the last upgrade. The functions \( h(t) \) and \( k(t) \) are both non-decreasing in \( t \). Repairs are immediate.

We point out that minimal repair is assumed due to the notion that the system is composed of many components, and the failure of one critical component will not (likely) affect the total failure rate of the system. The associated expected costs that comes with a failure have a general form, as long as it is non-decreasing in time. These costs are an aggregation of the actual repair and replacement costs and downtime costs, and particularly obsolescence issues for critical components. That is, the non-decreasing property of \( k(t) \) mostly stems from obsolescence issues that can arise when critical components are no longer on stock nor produced, and the repair becomes costlier as a more expensive fix needs to be applied.
Remark 3.4 For a system with many critical different components, it may hold that the failure of the system relates to the failure of one particular component by a certain probability. In this example, the expected repair costs $k(t)$ may be written as

$$k(t) = \sum_{i=1}^{I} k_i(t)p_i,$$

where $I$ is the number of (critical) components in the system, $p_i$ is the probability that the failure is due to the breakdown of component $i$, and $k_i(t)$ is the expected repair costs of a failed component $i$.

The goal is to find the policy that minimizes the total costs, which is the aggregation of all costs and penalties over the entire asset’s lifetime. To formalize this, we introduce the following notation. Under a given policy $\Pi$, let $N$ denote the number of upgrades, $T_1$ the first upgrade moment (if $N > 0$), and $T_2, \ldots, T_N$ the time between upgrade moments (if $N \geq 2$). Write $T_{N+1} = H - \sum_{i=1}^{N} T_i$ as the time between the last system upgrade (if any) and the end of the asset’s lifetime, which we also refer to as the remaining lifetime. Let a cycle be the time a single system version is in use, i.e. the time between the moment the system is installed till it is disposed off, which is either at the next upgrade moment or at the end of the asset’s lifetime. We define the cycle costs $C(T)$ as the aggregation of the salvage value, missing functionality penalty, and the expected costs that come with system failures in a cycle of length $T$, i.e.

$$C(T) = -v(T) + \int_{0}^{T} c_f(t) \, dt + \int_{0}^{T} k(t)h(t) \, dt.$$  \hspace{1cm} (3.2)

We point out that the price $c_0$ is not included in the cycle costs as the last cycle covers a disposal but not an upgrade. Under policy $\Pi$, let $S_{\Pi}$ denote the number of times a system upgrade is executed but not jointly with an overhaul, i.e.

$$S_{\Pi} = \left\{ n \leq N : \sum_{i=1}^{n} T_i \neq \sum_{i=1}^{j} M_i \text{ for all } 1 \leq j \leq m \right\}.$$  \hspace{1cm} (3.3)

Under a given policy $\Pi$, the total costs $K_{\Pi}(H)$ over a lifetime $H$ are given by

$$K_{\Pi}(H) = S_{\Pi} c_d + N c_0 + \sum_{i=1}^{N+1} C(T_i).$$  \hspace{1cm} (3.4)

The objective is to minimize the total costs, which we denote as

$$K^*(H) = \min_{\Pi} K_{\Pi}(H)$$  \hspace{1cm} (3.5)

with corresponding optimal policy $\Pi^*$ specified by the inter-upgrade times $T^*_i, i = 1, \ldots, N^*$.

If $c_d = 0$, we change the total costs variable from $K$ to $C$ to stress the fact we consider the base case. Under a given policy $\Pi$, the total costs reduces to

$$C_{\Pi}(H) = N c_0 + \sum_{i=1}^{N+1} C(T_i).$$  \hspace{1cm} (3.6)

We denote $C^*(H)$ as the minimal total costs in the base case.

A central question is whether we can derive results on what the optimal policy looks like structurally. A straightforward observation is that it can never be optimal to upgrade more than a finite number of times.
Lemma 3.5 Let $\bar{N} = (C(H) + v(0))/(c_0 - v(0)) \in [0, \infty)$. Any policy $\Pi$ with $N > \bar{N}$ upgrades satisfies $K^\Pi(H) > K^*(H)$.

A short proof of Lemma 3.5 can be found in Appendix C. As a consequence, Lemma 3.5 indicates certain scenarios where it optimal to never upgrade a system during the asset’s lifetime. This typically applies to expensive systems and with little technological advancements and which are robust to failures.

Corollary 3.6 If the upgrade costs are very high, it is never optimal to execute upgrades. More specifically, this is the case if $c_0 - 2v(0) \geq C(H)$.

Next to the upgrade costs, we can identify two extreme cases with respect to the penalty $c_d$. The first extreme case is when an upgrade disrupts the asset’s operation completely, and an extremely high penalty $c_d$ is incurred. In that case, it would only be rewarding to upgrade during overhauls, which reduces the number of possible upgrade plans significantly. One needs to only decide at the $m$ overhaul moments whether to upgrade or not, where $m$ is finite and relatively small. This can be solved through dynamic programming, which we discuss in more detail in Section 5.2.

The other extreme case concerns the base case where $c_d = 0$, i.e. no penalty is given to doing upgrades if it does not coincide with an overhaul. This corresponds to systems that can be easily maintained and upgraded without affecting the total operation mode of the asset. It turns out that the optimal policy has a simple structure for a large class of cycle cost functions. We consider this in more detail next.

4 Analysis of the base case

In this section, we assume that $c_d = 0$ and determine the optimal upgrade policy for different shapes of the cycle cost function. We use these results as building blocks to design an efficient algorithm to find the optimal policy for the general case where also positive values of the penalty $c_d \geq 0$ are allowed. Therefore, we spend significant effort to determine the structure of the optimal upgrade policy in the base case, knowing that this is directly related to determining the optimal upgrade policy for the general case as well. Unless stated otherwise, the proofs of the results in this section can be found in Appendix C.1.

Recall that the total costs expression simplifies to (3.5) in the base case for a given policy $\Pi$. In view of Lemma 3.5, we can try to derive for every $N \in \mathbb{N}_{\geq 0}$ with $N < \bar{N}$ what the corresponding optimal inter-upgrade times are. If we can answer this question, we can simply compare the total costs for a finite number of upgrades to derive the optimal upgrade policy.

Mathematically, this means that our problem can be rewritten as

$$C^*(H) = \min_{0 \leq N \leq \lfloor \bar{N} \rfloor} \left\{ Nc_0 + \min_{T_1 + \ldots + T_{N+1} = H} \left\{ \sum_{i=1}^{N+1} C(T_i) \right\} \right\}. \tag{4.1}$$

For a fixed $N$, we can therefore determine the corresponding optimal upgrade policy by deriving the inter-upgrade times that minimizes $\sum_{i=1}^{N} C(T_i) + C(H - \sum_{i=1}^{N} T_i)$, where $T_i \in [0, H]$ for all $i = 1, \ldots, N$. Over a finite interval, a minimum can only be attained at points where each partial derivative (in $T_i$) is either zero or at the boundaries. Note that for the optimal upgrade policy, it cannot hold that $T_i^* = 0$ for some $i = 1, \ldots, N$ or $\sum_{i=1}^{N} T_i^* = H$ since $c_0 > v(0)$. In other words, an optimal upgrade system must satisfy the property that every partial derivative (in $T_i$) is zero. That is, a necessary condition for optimality is given by

$$C'(T_i^*) = C' \left( H - \sum_{j=1}^{N} T_j^* \right), \quad \forall i = 1, \ldots, N^* \tag{4.2}$$
where $C'(t)$ denotes the first derivative of the cycle cost function $C(t), t \geq 0$. To find the optimal upgrade policy, we are therefore interested in the sets

$$
T^N = \left\{ (t_1, t_2, \ldots, t_N) \in \mathbb{R}_{>0}^N : \sum_{j=1}^{N} t_j < H, C'(t_i) = C'(H - \sum_{j=1}^{N} t_j), i = 1, \ldots, N \right\}, \quad N \geq 1. \quad (4.3)
$$

These sets represent the candidates for the optimal inter-upgrade times if $N \geq 1$ upgrades would be executed during the lifetime. The optimal policy is therefore contained in the set

$$
T = \bigcup_{N=0}^{\infty} T^N, \quad (4.4)
$$

where $T^0 = \emptyset$ represents the case where no upgrades are executed.

The size of the sets $T^N$ depends on the behavior of the cycle cost function. When we take a closer look to our assumptions, we observe that more information can be derived on the behavior of the cycle costs. Recall that the cycle costs are given by (3.2), and the first derivative is thus given by

$$
C'(T) = -v'(T) + c_f(T) + k(T)h(T) \geq 0.
$$

Our assumptions state that $c_f(T) + k(T)h(T)$ is non-negative and non-decreasing in $T$, and hence this gives rise to a convex term in the cycle cost function. The cycle costs are thus an aggregation of a convex non-decreasing function, and a general non-decreasing function $-v(T), T \geq 0$. Naturally, there are many settings where the salvage value also displays typical behavior, and in turn, yields a certain behavior for the cycle cost function. Therefore, we consider the special cases where the cycle costs are convex (Section 4.1), concave (Section 4.2) or S-shaped (Section 4.3). We also consider general structures for the cycle cost function for the base case in Section 4.4.

### 4.1 Convex cycle costs

In this section, we consider the case where the salvage values are such that the cycle costs are convex and non-decreasing. Special cases include:

- Salvage values that are constant over time;
- Salvage values that are linearly decreasing over time;
- Salvage value functions $v(\cdot)$ that are concave (and non-increasing) up to lifetime $H$.

An example where constant salvage values occur is when a system cannot be resold after use, or the disposal of an older version comes with some associated costs that is not affected by time. For this setting, the salvage value is constant and leads to convex cycle costs. Convex cycle costs also occur when the salvage value is linearly decreasing or non-increasing concave over time.

For many scenarios where reselling is an option, the salvage value has the following pattern. Initially, the resell value is close to the initial price. But as newer versions come to the market, this resell values decreases rapidly with increasing speed. Yet, at some point in time, the resell value start to stabilize again. Mathematically, this corresponds to the setting where the salvage value has a reversed S-shape, where $v(t)$ is concave non-increasing up to some point of inflection $x_v$. After that, the salvage value starts to stabilize, i.e. the salvage value becomes convex non-increasing, after which it is (almost) constant. This implies that the cycle costs are always convex non-decreasing on $[0, x_v]$. In particular, if the system’s lifetime $H$ is relatively
short and causes the point of inflection to satisfy \( x_v \geq H \), then the cycle costs are convex non-decreasing during the entire lifetime of the system.

If the cycle costs are convex and non-decreasing, it turns out that an optimal upgrade policy for the base model has a very simple structure. More specifically, an optimal strategy to follow is to either never upgrade, or to upgrade a finite number of times \( N^* \) with equidistant inter-upgrade times \( T_i^* = H/(N^* + 1), \ i = 1, \ldots, N^* \). Intuitively, it is clear that it is never optimal to upgrade more than a finite number of times since every upgrade always comes with a certain strictly positive cost. Equidistant inter-upgrade times is a property that comes from the convexity of the cycle costs. For example, suppose you would upgrade once at time \( T \) during the asset’s lifetime, then the total costs are given by \( c_0 + C(T) + C(H - T) \). The cycle costs are convex, and by definition, this implies that for every \( \Delta \in [0, H/2] \),

\[
c_0 + C \left( \frac{H/2 - \Delta}{2} \right) + C \left( \frac{H/2 + \Delta}{2} \right) \leq c_0 + 2C \left( \frac{H/2 - \Delta}{2} + \frac{H/2 + \Delta}{2} \right) = c_0 + 2C \left( \frac{H}{2} \right),
\]

where an equality holds if \( \Delta = 0 \). Therefore, the total costs are minimized if \( T = H/2 \). Similarly, this also holds in general when executing multiple upgrades.

**Proposition 4.1** Suppose that the cycle costs are convex. Then, it is either optimal to never upgrade, or to do so for \( N^* \in \mathbb{N} \) times with \( N^* \leq \bar{N} \). In the latter case, \( T_i^* = H/(N^* + 1), \ i = 1, \ldots, N^* \) and the minimal total costs are given by

\[
C^*(H) = \min_{0 \leq N \leq \bar{N}} \left\{ Nc_0 + (N + 1)C \left( \frac{H}{N + 1} \right) \right\}. \tag{4.5}
\]

We point out that this result already follows directly from (4.3) if the cycle costs are strictly convex. Define

\[
C^N(H) = Nc_0 + (N + 1)C \left( \frac{H}{N + 1} \right). \tag{4.6}
\]

Proposition 4.1 states that for convex cycle costs, the minimal total costs are obtained for some \( N \leq \bar{N} \) that minimizes \( C^N(H) \). In case that \( c_0 - v(0) \) is small, one may need to check a large number of values. To possibly speed up the process, we show a convexity result for \( C^N(H) \).

**Lemma 4.2** If the cycle cost function is convex, then the function \( C^N(H) \) is convex in \( N \).

Proposition 4.1 and Lemma 4.2 imply that in order to find an optimal number of upgrades \( N^* \), we need to determine the final integer \( N \) before \( C^N(H) \) increases.

**Example 4.3** Consider two settings, where in both settings \( H = 30 \) years and \( c_0 = 4 \). In setting A, let the cycle costs be given by

\[
C_A(t) = \frac{t}{3} + \frac{3}{16} \left( \frac{t}{3} \right)^2 + \frac{1}{10} t^{1.1}, \tag{4.7}
\]

and in setting B, let

\[
C_B(t) = \frac{t}{3} + \frac{3}{16} \left( \frac{t}{3} \right)^3 + \frac{1}{10} t^{1.1}. \tag{4.8}
\]

In Figure 2 we plot the cycle costs as a function of time.

Note that the cycle costs are strictly convex on \( [0, H] \) for both settings. In view of Proposition 4.1 and Lemma 4.2, we can conclude that the (unique) optimal policy is to upgrade \( N^* \) times after every \( H/(N^* + 1) \) years, where \( N^* \) is the final integer before \( C^N(H) \) increases. Using Table 1, we observe that it is optimal to execute an upgrade once \( (N^*_A = 1) \) after 15 years in setting A. In setting B, it is optimal to upgrade \( N^*_B = 4 \) times after every 6 years.
4.2 Concave cycle costs

In this section, we consider the case that the cycle cost function is concave. Due to our assumptions, a necessary condition for this case to occur is that the salvage value is convex (non-increasing) over time. Implicitly, this also means that the functionality gap and the costs due to failures should not increase too rapidly relative to the salvage values. Special instances that lead to concave cycle cost functions are

- Constant cycle cost function;
- Convex salvage value functions and \( c_f(t) + k(t)h(t) \) is constant for all \( t \in [0, H] \);

For concave cycle cost functions, it is optimal to never upgrade. Intuitively, this statement can be explained as follows. If it would be optimal to execute at least one upgrade during the lifetime, then due to the concavity of the cycle cost function, an optimal strategy would be to immediately execute all upgrades at the start of the asset’s lifetime. But since an upgrade always comes with a positive cost, this implies it is best to never upgrade at all.

**Lemma 4.4** Suppose that \( C(t) \) is concave for all \( t \in [0, H] \). Then, it is optimal to never do any upgrades.

4.3 S-shaped cycle costs

In this section, we consider the case that the cycle cost function is S-shaped, i.e. there exists a point of inflection \( x \in [0, H] \) such that the cycle costs are convex on the interval \([0, x]\) and concave on \([x, H]\). An example of a scenario where this could occur is when the salvage value function has a reversed S-shape.

| \( N \) | \( C_A^N(H) \) | \( C_B^N(H) \) |
|---|---|---|
| 0  | 32.9653 | 201.7153 |
| 1  | 27.3081 | 64.8081 |
| 2  | 28.0268 | 42.6101 |
| 3  | 30.3572 | 37.3884 |
| 4  | 33.3387 | 37.0887 |
| 5  | 36.6489 | 38.7322 |

Table 1: Total costs values for setting A and B.

Figure 2: Cycle costs for \( t \in [0, H] \).
with inflection point $x_v$, and the missing functionality penalty and expected failure costs stabilizes to a worst-case scenario at some point in time. Think of an electronic device for which upgraded versions with technological improvements appear rapidly on the market, and make older versions obsolete. If that worst-case moment occurs before the point of inflection $x_v$, we observe that the cycle costs have an S-shape.

**Remark 4.5** The point of inflection $x \in [0, H]$ does not necessarily have to be uniquely defined. For example, if the cycle costs are linearly increasing on $[a, b] \subset [0, H]$ with $a < b$, strictly convex on $[0, a]$ and strictly concave on $[b, H]$, then any $x \in [a, b]$ can be chosen as the point of inflection. Yet, in the results that are provided in this section, we require the point of inflection $x$ to be chosen in a (unique) specific manner. That is, we say that $x$ is chosen such that

$$C(x + \Delta) - C(x) < C'(x)\Delta, \quad \forall \Delta > 0. \quad \text{(4.9)}$$

For the mentioned example, this implies that the point of inflection is chosen to be equal to $b$. In the remainder of this paper, points of inflection that satisfy this property (4.9) will be referred to as satisfying the technical requirement.

It turns out that in the case of S-shaped cycle costs, there is an optimal policy that has a relatively simple structure. It has one of the following structures:

1. It is optimal to never upgrade;
2. It is optimal to upgrade $N \geq 1$ times after every $H/(N + 1)$ time units;
3. It is optimal to upgrade $N \geq 1$ times after every $T < H/(N + 1)$ time units, where $T$ differs from the time between the final upgrade and the end of the lifetime.

The reason that this holds already follows implicitly from the results in the two previous sections. Suppose there is an optimal policy where we upgrade $N = N_1 + N_2 - 1$ times, where $N_1$ and $N_2$ are the number of times we have inter-upgrade times in the interval $[0, x]$ and $(x, H]$, respectively (where we slightly abuse definitions by including the time between the last upgrade and the end of the lifetime as an inter-upgrade time). Since the cycle costs are concave on $(x, H]$, whenever $N_2 \geq 2$, we observe that there exists an optimal policy where at least one of the $N_2$ inter-upgrade times lies at the boundary of the interval $(x, H]$. This is not possible, and therefore it must hold that $N_2 \leq 1$. In particular, if $N_2 = 1$, without loss of generality we can set the corresponding inter-upgrade time as the time between the final upgrade and the end of the lifetime, i.e. as $T_{N+1}$. Moreover, there exists an optimal upgrade policy where the $N_1$ inter-upgrade times in interval $[0, x]$ equal one another (if any) since the cycle costs are convex on this interval.

In order to prove this formally, we require some additional results on the optimal policy structure. The following result shows that there exists an optimal policy where no inter-upgrade time lies strictly between $\min\{x, H/(N + 1)\}$ and $\max\{x, H/(N + 1)\}$.

**Lemma 4.6** Let the cycle costs have an S-shape with point of inflection $x \in (0, H)$ that satisfies the technical requirement. If $N^* \geq 1$, then there exists an optimal upgrade policy where for all $i = 1, \ldots, N^* + 1$ it holds that $T_i^* \notin (\min\{x, H/(N^* + 1)\}, \max\{x, H/(N^* + 1)\})$.

Secondly, it turns out that if $N^* \geq 1$, then there exists an optimal policy that always contains at least one inter-upgrade time that is at most equal to the point of inflection $x$.

**Lemma 4.7** Let the cycle costs have an S-shape with point of inflection $x \in (0, H)$, where $x$ satisfies the technical requirement. If $N^* \geq 1$, then there exists an optimal policy for which $T_i^* \leq x$ for some $i = 1, \ldots, N^* + 1$.

The previous two lemmas can be used to prove the main result of this section, which gives insight in the structure of the optimal solution in case of S-shaped cycle costs.
Proposition 4.8 Let the cycle costs have an S-shape with point of inflection \( x \in (0, H) \) satisfying the technical requirement. It is either optimal to execute no upgrades, or there exists an optimal policy with \( N^* \geq 1 \) upgrades that has one of two structures: either \( T_1^* = \ldots = T_N^* = H/(N+1) = T_{N+1}^* \), or \( T_1^* = \ldots = T_N^* \leq x < T_{N+1}^* \).

Proof of Proposition 4.8 If \( N^* = 0 \), then the proposition holds and there is nothing to prove. Therefore, let \( N^* \geq 1 \) for the remainder of the proof. That is, suppose there is an optimal policy where we upgrade \( N^* = N_1 + N_2 - 1 \geq 1 \) times, where \( N_1 \) and \( N_2 \) are the number of times we have inter-upgrade times in the interval \([0, x]\) and \([x, H]\), respectively. In view of (4.2), we observe that there exists such an optimal policy that satisfies \( T_1 = \ldots = T_{N_1} \leq x \) and \( T_{N_1+1} = \ldots = T_{N_1+N_2} > x \) (if \( N_2 > 0 \)). We point out that \( N_1 \geq 1 \) due to Lemma 4.7. To conclude the result, what remains to show is that \( N_2 \in \{0, 1\} \). To that purpose, suppose that \( N_2 \geq 2 \). Consider the policy \( \tilde{\Pi} \) with

\[
\begin{align*}
\tilde{T}_i &= T_1, \quad i = 1, \ldots, N_1, \\
\tilde{T}_j &= T_{N^*+1}, \quad j = N_1 + 1, \ldots, N^* - 1, \\
\tilde{T}_{N^*} &= x, \\
\tilde{T}_{N^*+1} &= 2T_{N^*+1} - x.
\end{align*}
\]

Note that this solution is feasible, since

\[
\sum_{i=1}^{N_1+N_2} \tilde{T}_i = N_1T_1 + (N_2 - 2)T_{N^*+1} + x + (2T_{N^*+1} - x) = N_1T_1 + N_2T_{N^*+1} = H.
\]

Due to the technical requirement, we observe that

\[
C(\tilde{\Pi})(H) = N^*c_0 + N_1C(T_1) + (N_2 - 2)C(T_{N^*+1}) + C(x) + C(2T_{N^*+1} - x) < N^*c_0 + N_1C(T_1) + N_2C(T_{N^*+1}) = C(\Pi)(H).
\]

This contradicts the hypothesis, proving our proposition.

We point out that the proof of Proposition 4.8 implies that in case of an S-shaped cycle cost function with point of inflection \( x \in (0, H) \), the candidate solutions for optimality can be simplified to the union of the set \( T^0 \) (no upgrading) and the sets

\[
\mathcal{T}^{N_1,N_2} = \{(t_1, t_2) \in \mathbb{R}^2 : 0 < t_1 \leq x < t_2 < H, N_1 \in \mathbb{N}, N_2 \in \{0, 1\}, N_1t_1 + N_2t_2 = H, C'(t_1) = C'(t_2)\},
\]

(4.10)

where \( N_1 \) is bounded due to Lemma 3.5. To illustrate how this observation can be used to determine an optimal upgrade policy, we consider the following example.

Example 4.9 Suppose that \( H = 30, c_0 \geq 1, c_f(t) + k(t)h(t) = 0 \) and

\[
C(t) = -v(t) = -1 + \left(1 + \exp^{-(x+10)}\right)^{-1}.
\]

This is an S-shaped function with \( C(t) = -v(t) \in (-1, 0) \) for all \( t \in [0, H] \) and (unique) point of inflection \( x = 10 \). Note that

\[
C'(t) = -v'(t) = \frac{\exp^{-(x-10)}}{(1 + \exp^{-(x-10)})^2},
\]

This derivative satisfies \( C'(t) = C'(20 - t) \) for all \( t \in [0, 10] \) and is strictly decreasing for \( t \geq 10 \). In particular, this implies that there exists no optimal upgrade policy where \( T_i \in (20, 30) \) for some \( i = 1, \ldots, N + 1 \). In view of (4.10), we observe that for any \( t_2 = 20 - t_1 \) with \( t_1 \in [0, 10] \) and \( N_2 = 1 \),

\[
N_1t_1 + N_2t_2 = H \quad \Rightarrow \quad t_1 = \frac{10}{N_1 - 1} = \frac{10}{N - 1}.
\]

12
Therefore, the possible candidates for the optimal upgrade policy are $T^0$ and

$$
T^{N_1,0} = \left\{ \left( \frac{30}{N+1}, 0 \right) \right\}, \quad N \in \mathbb{N}_{\geq 2}
$$

$$
T^{N_1,1} = \left\{ \left( \frac{10}{N-1}, 20 - \frac{10}{N-1} \right) \right\}, \quad N \in \mathbb{N}_{\geq 2}.
$$

We point out that any policy $\Pi \in T^{N_1,1}$ leads to total costs that satisfy $C^{\Pi}(H) \geq 0 > C(H)$ for every $c_0 \geq 1$, i.e. every policy $\Pi \in T^{N_1,1}$ leads to a higher total costs than when upgrades would never be executed and hence the optimal upgrade policy will never be contained in the set $T^{N_1,1}$. Therefore, depending on the value of $c_0$, the optimal policy is to either never upgrade, or to upgrade $N \geq 2$ times with equidistant upgrade times that equals the remaining lifetime (i.e. the time between the final upgrade and the end of the lifetime).

To illustrate that it is possible to have a (unique) upgrade policy where the remaining time between the last upgrade and the end of the lifetime differs from the (other) inter-upgrade times, we provide the following example.

**Example 4.10** Consider a setting where $H = 10$ years, and the system is not subjective to failure, i.e. $h(t) = 0$ for all $t \geq 0$. Initially, the salvage value is given by $v(0) = 0.15$. After $s = 4.9$, an upgrade comes to the market against upgrade costs $c_0 = 0.75$. This event causes the missing functionality penalty to jump to $c_f(t) = 0.15$ for all $t \geq 4.9$ (while $c_f(t) = 0$ for all $t < 4.9$). Moreover, this event leads to the salvage value to decrease rapidly till time $t = 5$, after which reselling the old system yields no return. More specifically, the salvage value is given by

$$
v(t) = \begin{cases} 
0.15 & \text{if } t \leq 4.9, \\
0.15 - 30(t - 4.9)^2 & \text{if } 4.9 \leq t \leq 4.95, \\
30(5 - t)^2 & \text{if } 4.95 \leq t \leq 5, \\
0 & \text{if } t \geq 5.
\end{cases}
$$

This leads to an S-shaped continuous cycle cost function, given by

$$
C(t) = \begin{cases} 
-0.15 & \text{if } t \leq 4.9, \\
-0.15 + 30(t - 4.9)^2 + 0.15(t - 4.9) & \text{if } 4.9 \leq t \leq 4.95, \\
-30(5 - t)^2 + 0.15(t - 4.9) & \text{if } 4.95 \leq t \leq 5, \\
0.15(t - 4.9) & \text{if } t \geq 5.
\end{cases}
$$

The point of inflection is thus given by $x = 4.95$.

To determine the optimal upgrade policy, note that $\bar{N} = (C(H) + v(0))/(c_0 - v(0)) = 1.525$ and hence it is never optimal to upgrade more than once. Since $C'(t) = 0.15$ for all $t \geq 5$ and $C'(t) > 0.15$ for all $t \in (4.9, 5)$, it is never optimal to upgrade at a time in the interval $(4.9, 5)$ by (4.12). Moreover, since the failure rate equals zero, there is no advantage to upgrading before $t = 4.9$. In view of Proposition 4.8 the optimal policy is therefore either to never upgrade, or to upgrade once at time $T_1 = 4.9$ or at time $T_1 = 5$, with corresponding total costs $C(10) = 0.765$, $c_0 + C(4.9) + C(5.1) = 0.63$ and $c_0 + 2C(5) = 0.78$, respectively. In other words, it is optimal to upgrade at time $T_1 = 4.9$ years, and the remaining time between this upgrade and the end of the lifetime is 5.1 years.

### 4.4 General cycle costs

In general, a necessary condition for optimality is given by (4.12). We point out that this observation can be used to generalize the results of the previous sections. More specifically, one can divide the lifetime of the system $[0, H]$ into intervals for which the cycle costs are convex and concave. That is, suppose $[0, H] = \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k$ for some (finite) $k \in \mathbb{N}$, where the cycle costs are convex on (closed) intervals $\mathcal{H}_i$ if $i$ is odd, and concave on (open) intervals $\mathcal{H}_i$ if $i$ is even, and $\mathcal{H}_i \cap \mathcal{H}_j = \emptyset$ if $i \neq j$. Note that these intervals can be constructed even in the extreme case when the cycle costs are concave by choosing $k = 3$ and $\mathcal{H}_1 = \{0\}$,
\( H_2 = (0, H) \) and \( H_3 = \{ H \} \). We point out that \( k \) is typically finite for instances we observe in practice, i.e. the cycle costs do not behave erratically by changing from convex to concave infinitely often on the finite interval \([0, H]\). Using the same argument as in the proof of Proposition 4.8 there is always an optimal policy that satisfies the following properties:

(i) For every \( l \leq k \) that is even, there is at most one \( i \leq N^* \) for which \( T_i^* \in H_i^c \);

(ii) if \( T_i^*, T_j^* \in H_i^c \) for some \( l \) odd, then \( T_i^* = T_j^* \).

Note that this is a generalization of the results in the previous sections. In case that the cycle costs are convex on \([0, H]\), then Property (ii) implies equidistant upgrade times. In case that the cycle costs are concave on \([0, H]\). Property (ii) implies that there can only be a single \( T_i^* \in [0, H] \), and hence it is optimal to not upgrade at all. Finally, if the cycle costs have an S-shape, then Properties (i) and (ii) imply Proposition 4.8

However, it may be clear that the calculation of the optimal upgrade policy can become cumbersome when the cycle costs behaves erratically, changing from convex to concave very often. As long as \( k \) is extremely large, the optimal inter-upgrade times can be derived efficiently for any given \( N \), and hence the overall optimal upgrade policy for the base case as well.

5 Solution approach

The results of the previous section can be used to derive a solution approach to determine the optimal policy. Note that we identified two extreme cases with respect to the penalty \( c_d \), namely the base case where an upgrade does not affect the asset’s operation, or the case where \( c_d = \infty \) and upgrades occur only during overhauls. We discuss the solution approach for both extreme cases first, after which we explain how one can combine the approaches to determine a solution approach for any value of \( c_d \).

5.1 Solution approach for the base case

As argued in Section 4.4, the optimal inter-upgrade times can be derived efficiently for any finite values of \( N \) as long as the cycle cost functions are relatively well-behaved, i.e. do not change between convex and concave too often up to lifetime \( H \). In particular, we considered three special cases in Section 4. If the cycle costs are concave, then the optimal policy is to never upgrade. If the cycle costs are convex, the optimal number of upgrades is the first integer at which \( \Delta C_N(H) = C_{N+1}(H) - C_N(H) \) is non-negative, denoted by \( N^* \). The optimal policy is then to upgrade \( N^* \) times after every \( H/(N^* + 1) \) time units. This yields Algorithm 1 as given in Appendix B. Implicitly, this solution approach is used for Example 4.3 in Section 4.1

If the cycle costs are S-shaped with point of inflection \( x \in (0, H) \), we would need to check more possibilities as there may be an optimal policy where the time between the last upgrade and the asset’s lifetime differs from the other inter-upgrade times. That is, we need to check the total costs under the policy of never upgrading, under the class of policies with equidistant upgrade times, and finally under the class of policies where the remaining lifetime differs from the other inter-upgrade times. No upgrading leads to total costs \( C(H) \). Next, Proposition 4.8 shows that if it is optimal to upgrade at least once, then there exists a policy with optimal inter-upgrade times \( T_1 = \ldots = T_N \leq x \). In the class of equidistant upgrade times, it therefore suffices to look at \( N \geq H/x - 1 \). In that case, \( C(t) \) is convex in \( t = H/(N + 1) \), and hence Lemma 4.2 follows through, i.e. \( C_N(H) \) is convex in \( N \geq H/x - 1 \). This implies that we can adopt a similar approach as for convex cycle cost function. That is, we determine the first integer \( N \geq H/x - 1 \) for which \( \Delta C_N(H) = C_{N+1}(H) - C_N(H) \) is non-negative. Finally, for the class of policies with a different remaining lifetime, note that due to the S-shape of the cycle costs and Proposition 4.8 it holds that an optimal policy

14
satisfies $T_{N+1} \in (x, H)$ and $T_i = (H - T_{N+1})/N$ for all $i = 1, \ldots, N$ if $N \geq 1$. Define

$$\hat{C}^N(H, t) := Nc_0 + NC\left(\frac{H-t}{N}\right) + C(t), \quad N \geq 1, \quad (5.1)$$

the total costs where $N$ upgrades are executed at equidistant time intervals of length $(H-t)/N$, and the remaining lifetime after the last upgrade is given by $t$. By deriving $t \geq \max\{H/(N+1), x\}$ that minimizes $\hat{C}^N(H, t)$, we inherently determine the optimal policy that minimizes the total costs for this class of policies with $N$ upgrades. Checking for a finite number of possible upgrades (see Lemma 3.5), leads to all potential optimal upgrade policies with a different remaining lifetime. We formalize this solution approach in Algorithm 2, which is given in Appendix B.

Example 5.1 Recall Example 4.10. The derived solution could also be found by executing Algorithm 2 in Appendix B.

1. The first step is to check to class of equidistant inter-upgrade times. Note that $[H/x - 1] = 2$ and $C^3(H) = 1.65 > 1.05 = C^2(H)$, this part terminates after a single step and returns $N^* = 2$, $T^* = 5$ and $C = 1.05$.

2. The next step shows that $C(10) = 0.765 < 1.05$, and hence we update our values to $N^* = 0$, $T^* = 0$ and $C = 0.765$.

3. Note that $[\bar{N}] = [(C(H) + v(0))/(c_0 - v(0))] = 1$, and hence the for loop consists only of a single step. To determine the $t^* \in [5, 5.1]$ that minimizes $\hat{C}^1(H, t)$, we observe

$$\frac{\partial \hat{C}^1(H, t)}{\partial t} = C'(H-t) + C'(t) = \begin{cases} 60t - 299.7, & \text{if } 5 \leq t \leq 5.05, \\ -60t + 306.3, & \text{if } 5.05 \leq t < 5.1, \\ 0.15, & \text{if } t = 5.1, \end{cases}$$

This function is minimized at $t^* = 5.1$, for which $\hat{C}^1(H, t^*) = 0.63 < 0.765 = C$. Therefore, we update our values to $C = 0.63$, $N^* = 1$, and $T^* = 4.9$.

4. Finally, the algorithm returns the optimal policy $N^* = 1$, $T_1^* = 4.9$, $T_2^* = 5.1$ and corresponding minimum costs $C^*(H) = 0.63$.

When the cycle costs are not convex, concave or S-shaped, a more involved analysis needs to be performed. As indicated in Section 4.4, this can still be done as long as the number of times $k$ that the cycle costs changes from convex to concave or vice versa is not too large, which is the case for most practical instances. That is, we need to find an optimal policy under the class of policies that satisfy the two properties indicated in Section 4.4. By exploiting these two properties, we can design a solution approach in a similar way as we did for the S-shaped function. However, we point that these minimization problems become more involved for larger $k$, and it is beyond the scope of this paper to move into more detail on this topic.

5.2 Solution approach when only upgrading during overhauls

The extreme case $c_d = \infty$ where upgrades are only executed during overhauls reduces the number of possible upgrade plans significantly. During the assets’ lifetime, one needs to decide only at the $m$ overhauls whether to upgrade or not, where $m$ is finite and relatively small in practice. This can be solved via dynamic programming. In essence, we condition on the first occurrence of executing an upgrade at an overhaul, if any. Suppose that this occurs at some time $T \in (0, H)$, then we split the horizon $H$ in two parts $[0, T]$ and $(T, H)$. The total costs during the first part is given by $c_0 + C(T)$, while the total costs pose an identical problem except that the time horizon has changed to a smaller value $H - T$. Since $m$ is relatively small, we can efficiently work backwards in time to determine the optimal policy. Details of this approach are given in Algorithm 3 of Appendix B.
Example 5.2 Reconsider the example as described in Example 4.3. For the additional parameters, suppose that $c_d = \infty$ and $M_i = 5$ for $i = 1, \ldots, 6$ in both settings, i.e. there is an overhaul every 5 years. In setting A, recall that the optimal policy in the base case was to upgrade after 15 years. Since this coincides with an overhaul, this policy remains the optimal choice and consequently, the corresponding minimal total costs of 27.308 are not affected. For setting B, the optimal policy does change with respect to the base model: instead of upgrading every six years, we upgrade every five years together with a planned overhaul. This leads to total costs of 38.732.

5.3 On the optimal solution for the general case

To find the optimal policy for the general case, a similar dynamic programming approach can be taken. To determine the optimal policy, we condition on the first occurrence we execute an upgrade at an overhaul (if any). This cuts the lifetime in two pieces that can be optimized separately. The expected total costs in the first piece can be determined using a strategy similar to the base model, since we conditioned that no upgrade is executed during an overhaul in this interval. The second piece of the interval is an identical problem for the general model, but with a shorter horizon length. Using dynamic programming, we can then efficiently determine the optimal policy.

The price of the upgrade at time $T$ itself is given by $c_0$. Finally, we need to consider the total costs for the second interval. However, this poses an identical problem with a shorter horizon. Since $m$ is relatively small, we can therefore use dynamic program efficiently to determine the optimal policy, see Algorithm [Algorithm 4 in Appendix B].

Example 5.3 Reconsider the example as described in Example 4.3, together with $c_d = 1.5$ and $M_i = 10$ for $i = 1, 2, 3$ in both settings. By executing Algorithm [Algorithm 4 in Appendix B] we determine the optimal policy which is given as follows. In case A, it is optimal to execute an upgrade twice after every 10 years, i.e. together with the planned overhauls. The associated costs are 28.027. In setting B, it turns out that it is optimal to upgrade once after 10 years during an overhaul. In the remaining 20 years, it is optimal to upgrade two more times at times that do not coincide with overhauls. That is, the optimal policy prescribes to upgrade at times $\{10, 16\frac{2}{3}, 23\frac{1}{3}\}$ years with associated total costs of 41.794.

6 Effects of the input parameter on the optimal total costs

We observed that in the base case, the behavior of the cycle cost function determines the structure of the optimal upgrade policy. When a penalty is incurred for upgrading while there is no overhaul, the optimal upgrade policy can be determined using dynamic programming (that uses the structural results from the
base case). The final outcome depends heavily on the values and functions of the different input parameters. In this section, we take a closer look at this notion both analytically as numerically. The proofs of the analytical results can be found in Appendix C.2.

### 6.1 The effect of the penalty \( c_d \) for upgrades outside overhaul moments

In previous sections, we already observed that there is a strong dependence on the penalty \( c_d \). Whenever this penalty is very small, the optimal policy is the same as in the base case. If the base case prescribes to upgrade only at moments that happen to not coincide with the overhaul moments, then this holds for all sufficiently small penalty values of \( c_d \). If there is a relatively high penalty \( c_d > 0 \), it would be very costly to execute upgrades outside overhaul moments, and hence it would not be optimal to do so. In fact, whenever it is optimal to only execute upgrades during overhauls for a given penalty \( \tilde{c}_d \), then the same strategy is optimal for every case with \( c_d \geq \tilde{c}_d \). This is rather intuitive: higher penalties strictly increase the total costs for any policy that executes upgrades outside the overhauls, while policies that only upgrade during overhauls yield the same total costs.

This latter notion can be generalized: as the penalty \( c_d \) increases, the optimal policy can only change to one where the total number of upgrades that are not jointly executed with overhauls is less.

**Proposition 6.1** As the penalty \( c_d \) increases, the optimal policy can only change to one where the number of upgrades outside overhauls is less.

In particular, if a policy \( \Pi^* \) prescribes to only upgrade during overhauls, then \( S^{\Pi^*} = 0 \). If this is the optimal policy for some penalty value \( \tilde{c}_d \), then Proposition 6.1 implies that the optimal policy also only upgrades during overhauls for all \( c_d \geq \tilde{c}_d \). Since for any policy that only upgrade during overhauls, the total costs do not change in \( c_d \), the minimum remains attained by upgrade strategy \( \Pi^* \).

**Corollary 6.2** Suppose that for a penalty \( \tilde{c}_d \), an optimal policy \( \Pi^* \) prescribes to upgrade only at planned overhauls. For any setting with \( c_d \geq \tilde{c}_d \) and where the other parameter settings are the same, \( \Pi^* \) is also an optimal policy.

We would like to point out that Corollary 6.2 implies scenarios where the optimal upgrade solution is insensitive to the value of \( c_d \). Whenever the base case leads to an upgrade plan where the timing of the upgrades coincide with the overhauls, Lemma 6.2 implies that this is also the optimal strategy for all \( c_d > 0 \). In particular, if it is optimal to never upgrade in the base case, then this is also the optimal strategy for positive values of \( c_d \).

**Example 6.3** To illustrate Proposition 6.1, we reconsider setting B of Example 4.3. That is, suppose that the horizon for positive values of penalty \( c_d \) is outside overhauls is less.

For any setting with \( c_d \), the total number of upgrades that are not jointly executed with overhauls is less.

- **Proposition 6.1** As the penalty \( c_d \) increases, the optimal policy can only change to one where the number of upgrades outside overhauls is less.

- **Corollary 6.2** Suppose that for a penalty \( \tilde{c}_d \), an optimal policy \( \Pi^* \) prescribes to upgrade only at planned overhauls. For any setting with \( c_d \geq \tilde{c}_d \) and where the other parameter settings are the same, \( \Pi^* \) is also an optimal policy.

**Example 6.3** To illustrate Proposition 6.1, we reconsider setting B of Example 4.3. That is, suppose that the horizon length is given by \( H = 30 \), the price is \( c_0 = 4 \), and we have cycle costs

\[
C(t) = \frac{t}{3} + \frac{3}{10} \left( \frac{t}{3} \right)^3 + \frac{1}{10} t^{1.1}. 
\]

Moreover, two overhauls are planned every 10 years, i.e. \( M_i = 10 \), \( i = 1, 2, 3 \). We consider the relation with respect to \( c_d \geq 0 \). In view of Figure C3, the optimal strategy is:

- If \( c_d \in [0, 0.29973) \), upgrade every 6 years at times \( \{6, 12, 18, 24\} \) with \( K^*(H) = 16 + 5C(6) + 4c_d; \)
- If \( c_d \in [0.29973, 1.40559) \), upgrade at times \( \{7.5, 15, 22.5\} \) with \( K^*(H) = 12 + 4C(7.5) + 3c_d; \)
- If \( c_d \in [1.40559, 1.90805) \), upgrade during one overhaul moment and two times after \( 6\frac{2}{3} \) years, i.e. at times \( \{10, 16\frac{2}{3}, 23\frac{1}{3}\} \) or \( \{6\frac{2}{3}, 13\frac{1}{3}, 20\} \) with \( K^*(H) = 12 + C(10) + 3C(6\frac{2}{3}) + 2c_d; \)
- If \( c_d \geq 1.90805 \), upgrade twice during the overhauls at times \( \{10, 20\} \) with \( K^*(H) = 8 + 3C(10). \)
6.2 The effect of the number of overhaul moments

As the number of overhauls increases, it generates more possibilities to execute upgrades without having to pay an additional penalty $c_d$. This does not necessarily imply that the number of upgrades (during overhauls) increases together with the number of overhauls. The lack of this monotonicity property can be easily explained as follows. There is an optimal upgrade policy in the base case, which is the best you can do to minimize costs if no penalty $c_d$ is incurred. This is attained in the general setting if the overhauls coincide exactly with the upgrade moments in this policy. Therefore, one cannot expect to do better than this, even if more overhauls are planned. To illustrate this notion, we consider the following example.

Example 6.4 Reconsider setting B of Example 4.3 with $H = 30$, $c_0 = 4$ and $c_d = 5$. In the base case when no penalty is incurred, the optimal policy is to upgrade $N = 4$ times (every 6 years) with minimal costs of 37.0887. This is a lower bound on the total costs for the general setting. If the overhauls are planned equidistantly, this implicitly also implies that this optimal solution is obtained whenever the number of overhaul moments is given by $m = 5k - 1$ for some $k \in \mathbb{N}$. Indeed, in Table 2 we observe that the total costs are higher when e.g. $m = 5$ than $m = 4$.

| $m$ | Optimal upgrade policy $\Pi^*$ | $K^*(H)$ |
|-----|--------------------------------|----------|
| 0   | \{7.5,15,22.5\}               | 52.3884  |
| 1   | \{7.5,15,22.5\}               | 47.3884  |
| 2   | \{10,20\}                     | 42.6101  |
| 3   | \{7.5,15,22.5\}               | 37.3884  |
| 4   | \{6,12,18,24\}                | 37.0887  |
| 5   | \{5,10,15,20,25\}             | 38.7322  |

Table 2: Optimal upgrade policy and minimal costs in case of overhaul moments $M_i = H/(m + 1)$, $i = 1, \ldots, m$. The number in bold correspond to moments that coincide with overhauls.

6.3 The effect of price $c_0$

Naturally, as the price $c_0$ increases, we can expect fewer upgrades and indeed, this intuitive notion turns out to be correct. Note that Assumption 3.1 states that we only consider values of $c_0$ such that $c_0 > v(0)$. 

Figure 3: Total costs $K^\Pi(H)$ under the four policies $\Pi_1 = \{6,12,18,24\}$, $\Pi_2 = \{7.5,15,22.5\}$, $\Pi_3 = \{10,16\frac{2}{3},23\frac{1}{3}\}$ and $\Pi_4 = \{10,20\}$. 

Lemma 6.5 The optimal number of upgrades $N^*$ is non-increasing in $c_0$.

Next, we provide an example to illustrate this result.

Example 6.6 Reconsider setting B of Example 4.3 with $H = 30$, $c_d = 5$ and two overhaul moments after every 10 years. In Table 3 we display the total costs under the optimal upgrade policy under the condition that the number of upgrades $N$ is fixed. We point out that if $c_0 = 0$, then the optimal policy is to upgrade after every 5 years with $N^* = 5$, and hence we do not consider larger values of $N$ in Table 3. We conclude that

$$N^* = \begin{cases} 
5 & \text{if } c_0 \in [0, 0.2926] \\
4 & \text{if } c_0 \in [0.2926, 0.29265] \\
2 & \text{if } c_0 \in [0.29265, 10.5944] \\
1 & \text{if } c_0 \in [10.5944, 135.907] \\
0 & \text{if } c_0 > 135.907,
\end{cases}$$

where the corresponding optimal policies can be found in Table 3.

| $N$  | $\Pi$  | $K^\Pi(H)$          |
|------|--------|----------------------|
| 0    | $\{\emptyset\}$ | 201.715             |
| 1    | $\{15\}$   | 65.8081 + $c_0$     |
| 2    | $\{10,20\}$ | 34.6101 + 2$c_0$    |
| 3    | $\{5,10,20\}$ | 34.3175 + 3$c_0$  |
| 4    | $\{5,10,15,20\}$ | 34.0248 + 4$c_0$ |
| 5    | $\{5,10,15,20,25\}$ | 33.7322 + 5$c_0$ |

Table 3: Value of $K^\Pi(H)$ under optimal upgrade policy $\Pi$ conditioned on having $N$ upgrades.

Although the optimal number of upgrades changes as $c_0$ increases, we observe that there is a rapid change from $N^* = 5$ upgrades to $N^* = 2$ in a relatively short interval ($c_0 \in [0.29, 0.30]$). Plotting the total costs in Figure 4, we observe that the total costs are extremely close for $c_0 \in [0.29, 0.30]$. For these parameter settings, it matters relatively little which policy is chosen, as all lead to similar total costs.

Figure 4: Total costs $K^\Pi(H)$ under optimal upgrade policy conditioned on having $N$ upgrades.
6.4 The effect of the cycle costs

A final aspect concerns the effect of the cycle costs, which can be seen as the combined contribution of the salvage value, the missing functionality penalty function and the failure aspect. Already in the base case, we observed the strong relation of the cycle costs with respect to what the structural properties of the optimal upgrade policy. In this section, we focus on whether we can say more about the upgrade policy whenever the cycle costs start to increase rapidly.

Intuitively, we would always upgrade before a point at which the cycle costs increase very strongly. To a certain extent, this can be made rigorous through the following result. Suppose that the cycle costs increase extremely after some time $z \in (0, H)$. If it is already optimal to upgrade multiple times with all inter-upgrade times occurring before time $z$ in an alternate setting where we linearize the cycle costs after time $z$, i.e. a setting where a less rapid increase in cycle costs is assumed after time $z$, then this must also be optimal for the original setting.

**Lemma 6.7** Consider an alternate setting with the same parameter settings, except $\tilde{C}(t) = C(z) + (t - z)C'(z)$ for all $t \geq z$ for some $z \in (0, H)$. If the optimal policy $\tilde{\Pi}^*$ for the alternate setting is to upgrade $\tilde{N}^*$ times with $\tilde{T}_i^* \leq z$ for all $i = 1, \ldots, \tilde{N}^* + 1$, then $\tilde{\Pi}^*$ is also the optimal upgrade policy for the original setting.

We point out that Lemma 6.7 may imply more information about the structure of the optimal upgrade policy. For example, suppose that there exists an alternate setting with the properties as in Lemma 6.7 for some $z \in (0, H)$, and the cycle costs $C(t)$ are convex on $[0, z]$. By construction, note that $\tilde{C}(t)$ is convex on the entire interval $[0, H]$, and hence the optimal upgrade policy $\tilde{\Pi}^*$ is one with equidistant upgrade times. In other words, there is an optimal policy where we upgrade after equidistant time intervals that are at most equal to $z$.

To conclude this section, we would like to point out that Lemma 6.7 does not imply that the number of upgrades always increases as the total cycle cost curve steepens. In fact, this is actually not true as we illustrate in the following example.

**Example 6.8** Suppose that $c_d = 0$ (base case) and $c_0 = 0.02$, and $H = 0.5$. For the cycle costs, consider the following. In setting $A$, let the cycle costs be given by $C_A(t) = t + t^2/10 = \int_0^t (1 + x/5) \, dx$, and for setting $B$, $C_B(t) = \frac{2}{3}t^{3/2} = \int_0^t \sqrt{x} \, dx$.

![Figure 5: Upgrade costs over $[0, t]$ if $t < H$.](image)

We point out that $(1 + x/5) \geq \sqrt{x}$ for all $x \in [0, H]$, and hence it also holds that $C_A(t) \geq C_B(t)$ for all $t \in [0, H]$. In particular, we see that the increase in cycle costs is steeper for setting $A$ than for setting $B$, see Figure 5. Yet, as
we see in Table 4, the total costs are minimized if \( N_A = 0 \) for setting A, and if \( N_B = 2 \) in setting B. In other words, although the total costs increase is steeper in setting A, the optimal number of upgrades is larger in setting B.

| \( N \) | \( C_A^N (H) \) | \( C_B^N (H) \) |
|-------|----------------|----------------|
| 0     | 0.525          | 0.2357         |
| 1     | 0.5325         | 0.1867         |
| 2     | 0.5483         | 0.1761         |
| 3     | 0.5663         | 0.1779         |

Table 4: Total costs values for setting A and B.

7 Conclusion

This paper considers a continuous-time stochastic model in order to determine an optimal upgrade policy for systems in an asset. This model is novel for combining several aspects that are relatively under-examined in existing literature: functionality gap, an predetermined overhaul plan, age-dependent cost functions and finite lifetime of the asset. For this model, we analytically characterize the structure of the optimal upgrade policy. For the base case, we establish that it is optimal to never upgrade if the cycle costs are concave, and to upgrade after every \( T \in (0, H] \) time units if the cycle costs are convex or S-shaped, where \( H \) is the asset’s lifetime. We also provide properties that an optimal upgrade policy satisfies for other shapes of the cycle cost function, see Section 4.4. We use these results as building blocks to design an efficient solution approach based on dynamic programming when the penalty for executing upgrades outside overhauls is non-negative.

Naturally, the optimal upgrade policy depends heavily on the input parameters. Many intuitive sensitivity results can be made rigorous. For example, the optimal number of upgrades can only decrease as the price \( c_0 \) becomes larger. A more subtle approach needs to be taken for other input parameters: as penalty \( c_d \) increases, the optimal upgrade policy can only change to one where the number of upgrades that are not executed jointly with overhauls is less. However, that does not imply that the optimal number of upgrades is non-increasing in \( c_d \). Similar for the number of overhauls: the optimal number of upgrades does not need to be non-decreasing together with the number of overhauls. Yet, adding more overhauls with respect to a former overhaul plan will ensure that the optimal number of upgrades does not increase. Finally, the number of upgrades also does not necessarily have to be non-decreasing with the steepening of the total cycle cost curve.

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## A Notation

| Variable | Meaning |
|----------|---------|
| $H$      | Lifetime of the asset |
| $c_0$    | Price of a system upgrade |
| $v(t)$   | Salvage value of the current system that has been in use for time $t$ |
| $c_f(t)$ | Penalty for missing functionality of the current system that has been in use for time $t$ |
| $h(t)$   | Failure rate of a system that has been in use for time $t$ |
| $k(t)$   | Expected repair costs for a failed system that has been in use for time $t$ |
| $c_d$    | Penalty for executing an upgrade but not during an overhaul |
| $m$      | The number of overhauls |
| $M_i$    | Time between overhaul $i - 1$ and $i$, with $M_0 = 0$ and $\sum_{i=1}^{m+1} M_i = H$ |
| $\Pi$    | The upgrade policy (specified by the upgrade times) |
| $N$      | The number of upgrades |
| $T_i$    | Time between upgrade $i - 1$ and $i$, with $T_0 = 0$ and $\sum_{i=1}^{N+1} T_i = H$ |
| $C(t)$   | Cycle costs as specified in (3.2) |
| $S^\Pi$  | The number of upgrades not jointly executed with an overhaul under policy $\Pi$ |
| $\mathcal{K}^\Pi(H)$ | The total costs over a lifetime $H$ under policy $\Pi$, as given in (3.3) |
| $C^\Pi(H)$ | The total costs over a lifetime $H$ under policy $\Pi$ in the base case ($c_d = 0$), as given in (3.5) |
| $\bar{N}$ | Upper bound for the number of upgrades under the optimal policy, given by $(C(H) + v(0))/(c_0 - v(0))$ |
| $C^N(H)$ | The total costs if $N$ upgrades are executed after every $H/(N+1)$ time, as given in (4.6) |
| $\mathcal{T}^N$ | Set of candidate solutions for the optimal inter-upgrade times with $N$ upgrades, see (4.3) |
| $\tilde{C}^N(H,t)$ | Total costs if we execute $N$ upgrades every $(H-t)/N$ time, see (5.1) |
| $\mathcal{K}^\Pi(t)$ | The total costs in $[H - t, H]$ under policy $\Pi$ |

Table 5: Overview of notation.
B Pseudo-code for solution approaches

**Algorithm 1:** Algorithm to find optimal upgrade policy in base case with convex cycle cost function

```plaintext
Require: Values $H, c_0$, and functions $v(t), c_f(t), k(t)$ and $h(t)$ for $t \in [0, H]$.
1: Set $N = 0$;
   while $C^{N+1}(H) < C^N(H)$ do
   | Set $N = N + 1$;
   end
2: return $N^* = N$ and $T^*_i = H/(N + 1), i = 1, \ldots, N^* + 1$.
```

**Algorithm 2:** Algorithm to find optimal upgrade policy and corresponding minimum costs in base case with S-shaped cycle costs

```plaintext
Require: Values $H, c_0$, and functions $v(t), c_f(t), k(t)$ and $h(t)$ for $t \in [0, H]$ (with point of inflection $x$).
1: Set $N = \lceil H/x - 1 \rceil$;
   while $C^{N+1}(H) < C^N(H)$ do
   | Set $N = N + 1$;
   end
2: Set $N^* = N$, $T^* = H/(N + 1)$ and $C = C^N(H)$.
3: if $C(H) < C$ then
4:   Set $C = C(H), N^* = 0$ and $T^* = 0$;
5: end if
6: for $1 \leq N \leq \tilde{N} = \lfloor (C(H) + v(0))/(c_0 - v(0)) \rfloor$ do
7:   Determine $t^* \in \max\{H/(N + 1), x, H\}$ that minimizes $\tilde{C}^N(H, t^*)$ as in (5.1);
8:   if $t^* \in (x, H)$ and $\tilde{C}^N(H, t^*) < C$ then
9:     Set $C = \tilde{C}^N(H, t^*), N^* = N$ and $T^* = (H - t^*)/N^*$;
10: end if
11: end for
12: return $N^*$ and $T^*_i = T^*, i = 1, \ldots, N^*, T_{N^*+1} = H - N^*T^*$ and $C^*(H) = C$.
```

For Algorithm 3 and Algorithm 4, we extend the notation for the total costs $K^\Pi(t)$ as the total costs in $(H - t, H]$ under policy $\Pi$. Let $K^*(t)$ denote the minimal costs over all upgrade policies on this interval. Moreover, we denote $\hat{K}^*(t)$ as the minimal total costs in $(H - t, H]$ among the policies where upgrades are only executed during overhaul moments.
Proof of Proposition 4.1. We need to minimize the total costs by finding the optimal policy that describes the number of upgrades \( N^* \) and corresponding \( T_1^*, \ldots, T_N^* \) such that
\[
C^*(H) = \min_{\Pi} C^{\Pi}(H).
\]  \hfill (C.1)

C \ 

Proofs

**Proof of Lemma 3.5.** Write \( \Pi_0 \) as the policy to never upgrade, and note that \( K^{\Pi_0}(H) = C(H) \). For any policy \( \Pi \) with \( N > N \), \( K^{\Pi}(H) \geq NC_0 + (N+1)(-v(0)) > N(c_0 - v(0)) - v(0) = C(H) = K^{\Pi_0}(H) \geq K^*(H) \).
Due to Lemma 3.5 we know that it is never optimal to upgrade more than $\bar{N}$ times. First, we answer the question what the optimal inter-upgrade times are if the optimal number of upgrades is given by $N^*$.

If $N^* = 0$, then there are no upgrades during the lifetime and $C^*(H) = C(H)$. Otherwise, $1 \leq N^* \leq \bar{N}$ and the policy also needs to describe the inter-upgrade times. Then,

$$C^*(H) = N^* c_0 + \min_{T_1, \ldots, T_{N^*+1} \geq 0, T_1 + \ldots + T_{N^*+1} = H} \left\{ \sum_{i=1}^{N^*+1} C(T_i) \right\}.$$  

The term in the minimization operation is the summation of $N^* + 1$ identical convex non-decreasing functions in $T_i$, under the condition that the sum of the inter-upgrade times $T_i$ is equal to $H$. By the convexity property, this is minimized if $T_1 = \ldots = T_{N^*+1} = H/(N^* + 1)$. Using this observation together with Lemma 3.5 concludes the proof.

**Proof of Lemma 4.2** To show that $C^N(H)$ is convex in $N$, we require that $\Delta^2 C^N(H) \geq 0$. Note that

$$\Delta C^N(H) = C^{N+1}(H) - C^N(H) = c_0 + C\left(\frac{H}{N+2}\right) - (N + 1) \left(C\left(\frac{H}{N+1}\right) - C\left(\frac{H}{N+2}\right)\right),$$

and hence

$$\Delta^2 C^N(H) = \Delta C^{N+1}(H) - \Delta C^N(H)$$

$$= (N + 1) \left(C\left(\frac{H}{N+1}\right) - C\left(\frac{H}{N+2}\right)\right) - (N + 3) \left(C\left(\frac{H}{N+2}\right) - C\left(\frac{H}{N+3}\right)\right).$$

Write

$$y_i = \frac{H}{N+2} + \frac{i}{(N+1)(N+2)(N+3)}, \quad z_i = \frac{H}{N+3} + \frac{i}{(N+1)(N+2)(N+3)},$$

and note that $y_{N+3} = H/(N+1)$ and $z_{N+1} = y_0 = H/(N+2)$. Therefore,

$$\Delta^2 C^N(H) = (N + 1) \sum_{i=1}^{N+3} (C(y_i) - C(y_{i-1})) - (N + 3) \sum_{j=1}^{N+1} (C(z_j) - C(z_{j-1}))$$

$$= \sum_{i=1}^{N+3} \sum_{j=1}^{N+1} ((C(y_i) - C(y_{i-1})) - (C(z_j) - C(z_{j-1})) \geq 0,$$

where the final inequality follows since every single term within the summations is non-negative due to the convexity and the non-decreasing property of the cycle cost function.

**Proof of Lemma 4.4** Suppose that there is an optimal policy $\Pi$ for which it is optimal to upgrade exactly $N \geq 1$ times. Since $C(t)$ is concave and non-decreasing in $t \geq 0$, the term $\sum_{i=1}^{N+1} C(T_i)$ is minimized if $T_1 = \ldots = T_N = 0$ and $T_{N+1} = H$. However, in that case it follows that

$$C^\Pi(H) = N(c_0 - v(0)) + C(H) > C(H),$$

where the latter corresponds to the total costs under the policy to never execute an upgrade. This contradicts the hypothesis.

**Proof of Lemma 4.6** This lemma only applies to cases where $x \neq H/(N + 1)$, because otherwise it holds that $(\min(x, H/(N^* + 1)), \max(x, H/(N^* + 1))) = \emptyset$. Therefore, we assume $x \neq H/(N + 1)$ for the remainder of the proof. Suppose that there would be an optimal strategy $\Pi$ with an inter-upgrade
time strictly between point of inflection \( \min \{x, H/(N + 1)\} \) and \( \max \{x, H/(N + 1)\} \) with \( N \geq 1 \). Due to symmetry, we can assume without loss of generality that

\[
T_{N+1} \in \left( \min \left\{ x, \frac{H}{N + 1} \right\}, \max \left\{ x, \frac{H}{N + 1} \right\} \right).
\]

Moreover, note that a necessary condition for an policy to be optimal is

\[
C'(T_i) = C'(T_{N+1}) \quad \forall i = 1, \ldots, N. \quad (C.3)
\]

Moreover, let

\[
a = \inf \{t \in [0, x] : C'(t) = C'(T_{N+1})\}, \quad c = \inf \{t \in [x, H] : C'(t) = C'(T_{N+1})\}, \quad (C.4)
\]

\[
b = \sup \{t \in [0, x] : C'(t) = C'(T_{N+1})\}, \quad d = \sup \{t \in [x, H] : C'(t) = C'(T_{N+1})\}. \quad (C.5)
\]

We point out that at least \( a \) and \( b \) or \( c \) and \( d \) are well-defined (or all). Moreover, if it is well-defined, note that \( a = b \) in case of strict convexity on \([0, x]\) and \( c = d \) in case of strict concavity on \([x, H]\). Since \( C(\cdot) \) has an S-shape with point of inflection \( x \) that satisfies the technical requirement, we observe that \([a, b] \) and \([c, d] \) contain all solutions of \((C.3)\), and \( C(\cdot) \) is linear on both intervals (if they exist). Therefore, without loss of generality, we can say that \( T_i \in [a, b] \) for all \( i = 1, \ldots, N_1 \) and \( T_i \in [c, d] \) for all \( i = N_1 + 1, \ldots, N_1 + N_2 \) where \( N_1 + N_2 = N + 1 \). Note that \( N_1 = 0 \) if \( a, b \) do not exist, and similarly, \( N_2 = 0 \) if \( c, d \) do not exist.

**Case** \( x < H/(N + 1) \): or in other words, \( x < T_{N+1} < H/(N + 1) \). In that case \( c, d \) are well-defined values and \( T_{N+1} \in [c, d] \) and \( N_2 \geq 1 \). If \( N_2 = 1 \), then \( N = N_1 \geq 1 \) and \( a, b \) must also be well-defined. Therefore, \( T_i \leq b \leq x < H/(N + 1) \) for all \( i = 1, \ldots, N \). Recalling that \( T_{N+1} < H/(N + 1) \) yields

\[
\sum_{i=1}^{N+1} T_i < \sum_{i=1}^{N+1} \frac{H}{(N + 1)} = H,
\]

which is a contradiction. Therefore, we require that \( N_2 \geq 2 \). In that case, we can construct the policy \( \tilde{\Pi} \) where

\[
\tilde{T}_i = T_i, \quad i = 1, \ldots, N - 1, \quad \tilde{T}_N = T_N + T_{N+1} - x, \quad \tilde{T}_{N+1} = x.
\]

Note that \( \tilde{T}_N \in (x, H) \). Since the cycle costs are concave and due to the technical requirement, \( C(T_N) + C(T_{N+1}) > C(\tilde{T}_N) + C(\tilde{T}_{N+1}) \), and hence

\[
C^{\tilde{\Pi}}(H) = C^{\Pi}(H) + C(\tilde{T}_N) + C(\tilde{T}_{N+1}) - C(T_N) - C(T_{N+1}) < C^{\Pi}(H). \quad (C.6)
\]

In other words, our original policy \( \Pi \) is not optimal, contradicting our hypothesis.

**Case** \( x > H/(N + 1) \): or in other words \( H/(N + 1) < T_{N+1} < x \). In that case, \( a \) and \( b \) are well-defined and \( T_{N+1} \in [a, b] \) and \( N_1 \geq 1 \). We will show that if this policy \( \Pi \) exists, then there is another policy \( \tilde{\Pi} \) that yields the same minimal total costs but has no inter-upgrade time in \((H/(N + 1), x)\). First, if \( T_i \in [a, b] \) for all \( i = 1, \ldots, N \), then let policy \( \tilde{\Pi} \) be given by \( \tilde{T}_i = H/(N + 1) \) for all \( i = 1, \ldots, N + 1 \). By construction, it holds that \( C^{\tilde{\Pi}}(H) = C^{\Pi}(H) \) since the cycle costs are linear on \([a, b]\), and hence \( \tilde{\Pi} \) is also an optimal upgrade policy.

Next, suppose that \( T_i \notin [a, b] \) for all \( i = 1, \ldots, N \), implying directly that \( c \) and \( d \) are well-defined. Without loss of generality, we can order the inter-upgrade times such that \( T_i \in [c, d] \) for \( i = 1, \ldots, l \) for some \( l \geq 1 \) and \( T_i \in [a, b] \) for \( i = l + 1, \ldots, N \) (if any). Consider the policy

\[
\tilde{T}_i = T_i, \quad i = 1, \ldots, l, \quad \tilde{T}_{N+1} = \tilde{T}_j = \frac{\sum_{i=l+1}^{N+1} T_i}{N-l+1}, \quad j = l + 1, \ldots, N.
\]

27
Note that $\tilde{T}_{N+1} \geq a$ and
\[ \tilde{T}_{N+1} = \sum_{i=t+1}^{N+1} T_i = \frac{H - \sum_{i=1}^{t} T_i}{N - t + 1} < \frac{H - tH/(N + 1)}{N - t + 1} = \frac{H}{N + 1} < x. \]
That is, policy $\Pi$ has no inter-upgrade time in $(H/(N + 1), x)$, and $C^\Pi(H) = C^\Pi(H)$ since the cycle costs are linear on $[a, b]$. ■

**Proof of Lemma 4.7** Suppose that the statement is not true, and $T_1, \ldots, T_{N+1} > x$. In view of Lemma 4.6, we observe that
\[ \sum_{i=1}^{N+1} T_i \geq \sum_{i=1}^{N+1} \frac{H}{N + 1} = H, \]
where $\sum_{i=1}^{N+1} T_i = H$ holds if and only if $T_1 = \ldots = T_{N+1} = H/(N + 1) > x$. In other words, if $T_1, \ldots, T_{N+1} > x$, then $T_1 = \ldots = T_{N+1} = H/(N + 1) > x$, and the total costs under this policy are given by $Nc_0 + (N + 1)C(H/(N + 1))$.

Consider the policy $\tilde{\Pi}$ where $\tilde{T}_1 = \ldots = \tilde{T}_N = x$ and $\tilde{T}_{N+1} = H - Nx$. Then, due to the technical requirement and the notion that we have a setting where $x < H/(N + 1)$,
\[ C^\Pi(H) = Nc_0 + NC(x) + C(H - nx) < Nc_0 + (N + 1)C(H/(N + 1)), \]
contradicting the hypothesis. ■

### C.2 Proofs for the sensitivity analysis

**Proof of Proposition 6.1** This is a direct consequence of the total cost function (3.3). Note that for any fixed policy $\Pi$, the total cost function is linear in $c_d$ with slope $S^\Pi$. Therefore, as penalty $c_d$ increases, the optimal policy can only change to one with a smaller value of $S^\Pi$. ■

**Proof of Lemma 6.5** For every policy $\Pi$, the total costs are given by (3.3). Write $\mathcal{P}_N$ as the set of all possible upgrade policies with $N$ upgrades. We observe that for every $N \in \mathbb{N}$, we can determine the upgrade times that minimize the total costs by solving
\[ K^N(H) := \min_{\Pi \in \mathcal{P}_N} S^\Pi c_d + \sum_{i=1}^{N+1} C(T_i), \]
regardless of the value of $c_0$. In particular,
\[ K^*(H) = \min_{N \in \mathbb{N}} \{ Nc_0 + K^N(H) \}. \]
We point out that
\[ \frac{\partial}{\partial c_0} (Nc_0 + K^N(H)) = N. \]
That is, it is linear in $c_0$ with slope $N$, and $N^*$ is the argument that minimizes the total costs. In conclusion, the optimal number of upgrades will never decrease. ■

**Proof of Lemma 6.7** For any general variable $X$, let $\tilde{X}$ denote the corresponding one in the alternate setting. For any policy $\Pi$, it holds that $K^{\tilde{\Pi}}(H) \geq \tilde{K}^\Pi(H)$ since $\tilde{C}(t) \leq C(t)$ for all $t \in [0, H]$. Moreover, since $\tilde{T}_i \leq z$ for all $i = 1, \ldots, \tilde{N}^* + 1$, it holds that $K^{\tilde{\Pi}^*}(H) = \tilde{K}^\Pi(H)$. In conclusion, we obtain
\[ K^{\tilde{\Pi}^*}(H) = \tilde{K}^\Pi(H) \leq \tilde{K}^\Pi(H) \leq K^{\Pi}(H), \]
for any policy $\Pi$. We can conclude that $\tilde{\Pi}^*$ is also optimal for the original setting. ■