Unified two-way wave equation and its symmetry properties

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SUMMARY

The two-way wave equation is a compact first-order matrix-vector differential equation. It was originally used for the analysis of elastodynamic waves in laterally invariant media. It has been extended by various authors for laterally varying media. Other authors derived a similar formalism for other wave phenomena in stratified media. This paper presents a unified treatment of the two-way wave equation for acoustic, quantum-mechanical, electromagnetic, elastodynamic, piezoelectric, poroelastic and seismoelectric waves, in most cases in a 3D inhomogeneous, anisotropic, dissipative medium. It appears that the two-way wave equation obeys unified symmetry relations for all these wave phenomena. These symmetry relations underly two-way reciprocity theorems of the convolution and correlation type which, in turn, form the basis for unified representations of the two-way wave vector and the two-way homogeneous Green’s function.

Key words: Wave propagation, Theoretical seismology, Numerical modelling, Electromagnetic theory

1 INTRODUCTION

For wave problems in layered media, the coupled basic equations for wave propagation can be organised in a compact first-order matrix-vector differential equation. In this paper this equation is called the two-way wave equation and its solution the two-way wave vector. The
adjective “two-way” is used because, unlike one-way wave equations, the two-way wave equation does not explicitly distinguish between downward and upward propagation. The two-way wave equation finds its roots in early work on the analysis of waves in laterally invariant media. Thomson (1950) introduced a matrix formalism for the analysis of elastic plane waves propagating through a stratified solid medium. Haskell (1953) used the same formalism to analyse the dispersion of surface waves in layered media. Backus (1962) used similar concepts to derive long-wave effective anisotropic parameters for stratified media. This approach has become known as Backus averaging (Mavko et al. 2009). Gilbert & Backus (1966) used the matrix equation to derive so-called propagator matrices for elastic wave problems in stratified media. Woodhouse (1974) extended the formalism for arbitrary anisotropic inhomogeneous media and used it for the study of surface waves in laterally varying layered media. Frasier (1970), Kennett et al. (1978), Frazer & Fryer (1989) and Chapman (1994) used the two-way wave equation to derive symmetry properties of reflection and transmission responses of laterally invariant media. Haines (1988), Kennett et al. (1990), Koketsu et al. (1991) and Takenaka et al. (1993) exploited the symmetry properties of the two-way wave equation to derive so-called propagation invariants for laterally varying layered media and used this for modelling of reflection and transmission responses of such media. Using the same symmetry properties, Haines & de Hoop (1996) and Wapenaar (1996b) derived reciprocity theorems and representations for the two-way wave vector.

The two-way wave equation has been used by many authors as the starting point for decomposition into one-way wave equations for coupled downgoing and upgoing waves. This has been used for example for modelling in layered media (Kennett & Kerry 1979, Kennett & Illingworth 1981), wide-angle one-way propagation in laterally variant media (Fishman & McCoy 1984, Weston 1989, Fishman 1992), reciprocity theorems for one-way wave fields (Wapenaar & Grimbergen 1996, Thomson 2015a,b), generalised Bremmer series representations for reflection data (Corones 1975, Haines & de Hoop 1996, Wapenaar 1996a, de Hoop 1996) and seismic interferometry (Wapenaar 2003).

This paper discusses the two-way wave equation and its symmetry properties for a range of wave phenomena in a unified way. This builds on earlier systematic treatments of different wave phenomena by Auld (1973), Ursin (1983), Wapenaar & Berkhout (1989), de Hoop (1995, 1996), Gangi (2000), Carcione (2007) and Mittet (2015). In section 2 we discuss the two-way wave equation in the space-frequency domain for an inhomogeneous, anisotropic, dissipative fluid. The two-way wave equation relates the vertical derivative of the two-way wave vector to an operator matrix acting on the same wave vector. This $2 \times 2$ two-way operator
matrix contains the medium parameters and lateral differential operators. We review the concept of transposed and adjoint operators and use this to discuss the symmetry properties of the acoustic two-way operator matrix. These symmetry properties appear to hold for all wave phenomena discussed in this paper. In section 3 we recast Schrödinger’s equation for quantum-mechanical waves in the same form as the acoustic two-way wave equation, containing again a $2 \times 2$ operator matrix. In sections 4 and 5 we discuss the electromagnetic and elastodynamic two-way wave equations for an inhomogeneous, anisotropic, dissipative medium. These equations contain $4 \times 4$ and $6 \times 6$ two-way operator matrices, respectively. Section 6 on piezoelectric waves, starts with two constitutive relations which account for the coupling between electromagnetic and elastodynamic waves in an inhomogeneous, anisotropic, dissipative piezoelectric medium. These constitutive relations, combined with the electromagnetic and elastodynamic equations of sections 4 and 5 are cast into a piezoelectric two-way wave equation, containing a $10 \times 10$ two-way operator matrix. In section 7 we derive the poroelastic two-way wave equation for an inhomogeneous, anisotropic, dissipative, fluid-saturated porous solid. Because it accounts for coupled waves in the fluid and the solid, the two-way operator matrix is this time an $8 \times 8$ matrix. Section 8 on seismoelectric waves, starts with two constitutive relations which account for the coupling between electromagnetic and poroelastic waves in an inhomogeneous, isotropic, dissipative, fluid-saturated porous solid. These constitutive relations, combined with the electromagnetic and poroelastic equations of sections 4 and 7 are cast into a seismoelectric two-way wave equation, containing a $12 \times 12$ two-way operator matrix. In section 9 we exploit the symmetry properties and derive unified two-way reciprocity theorems (of the convolution and correlation type), a power balance, propagation invariants, symmetry properties of the two-way Green’s function and representations for the two-way wave vector and for the two-way homogeneous Green’s function. Moreover, we indicate some applications. We end with conclusions in section 10.

2 ACOUSTIC WAVES

2.1 Basic equations for acoustic wave propagation

The basic equations for acoustic wave propagation in an inhomogeneous, anisotropic, dissipative, non-flowing fluid are the linearized equation of motion

$$\partial_i p + \rho_{ij} \partial_j v_j = f_i$$

(1)
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and the linearized stress-strain relation

\[ \partial_t v_i + \kappa \ast \partial_t p = q. \]  \hfill (2)

Here \( p = p(\mathbf{x}, t) \) and \( v_i = v_i(\mathbf{x}, t) \) represent the acoustic wave field in terms of acoustic pressure and particle velocity, respectively, both as a function of spatial position \( \mathbf{x} \) and time \( t \). The Cartesian coordinate vector \( \mathbf{x} \) is defined as \( \mathbf{x} = (x_1, x_2, x_3) \) and the positive \( x_3 \)-axis is pointing downward. The functions \( \rho_{ij} = \rho_{ij}(\mathbf{x}, t) \) and \( \kappa = \kappa(\mathbf{x}, t) \) are the medium parameters mass density and compressibility, respectively. To account for losses in a dissipative medium, they are defined as convolutional causal relaxation functions (de Hoop 1988; Carcione 2007); the convolution process is represented by the inline asterisks in equations (1) and (2). To account for anisotropy, the mass density is defined as a tensor. Although ideal fluids are by definition isotropic, inhomogeneities at the micro scale can often be represented by effective anisotropic parameters at the macro scale. For example, a periodic stratified fluid can, in the long wavelength limit, be represented by a homogeneous fluid with an effective transverse isotropic mass density tensor and an effective isotropic compressibility (Schoenberg & Sen 1983). Note that the mass density tensor is symmetric, i.e., \( \rho_{ij} = \rho_{ji} \). The functions \( f_i = f_i(\mathbf{x}, t) \) and \( q = q(\mathbf{x}, t) \) represent the sources in terms of external volume force density and volume-injection rate density, respectively. Operator \( \partial_i \) stands for the spatial differential operator \( \partial/\partial x_i \) and \( \partial_t \) for the temporal differential operator \( \partial/\partial t \). Lower-case Latin subscripts (except \( t \)) take on the values 1, 2 and 3, and Einstein’s summation convention applies to repeated subscripts.

We define the temporal Fourier transform of a space- and time-dependent quantity \( h(\mathbf{x}, t) \) as

\[ h(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} h(\mathbf{x}, t) \exp(i\omega t) dt. \]  \hfill (3)

Here \( \omega \) denotes angular frequency and \( i \) is the imaginary unit. For notational convenience, we use the same symbol (here \( h \)) for quantities in the time domain and in the frequency domain. We use equation (3) to transform equations (1) and (2) to the frequency domain. The time derivatives are thus replaced by \( -i\omega \) and the convolutions by multiplications. Hence

\[ \partial_i p - i\omega \rho_{ij} v_j = f_i, \]  \hfill (4)
\[ \partial_t v_i - i\omega \kappa p = q, \]  \hfill (5)

with \( p = p(\mathbf{x}, \omega), v_i = v_i(\mathbf{x}, \omega), \rho_{ij} = \rho_{ij}(\mathbf{x}, \omega), \kappa = \kappa(\mathbf{x}, \omega), f_i = f_i(\mathbf{x}, \omega) \) and \( q = q(\mathbf{x}, \omega) \). Note that in the frequency domain not only the wave field and source quantities are complex-valued, but in the case of a medium with losses also the medium parameters are complex-valued.
2.2 Acoustic two-way wave equation

The unified two-way wave equation expresses the derivative in the $x_3$-direction of a two-way wave vector in terms of an operator matrix acting on the same vector. For all wave phenomena considered in this paper (except for quantum-mechanical waves) we let the two-way wave vector consist of the quantities that constitute the power-flux density $j$ in the $x_3$-direction. For acoustic waves the power-flux density $j$ is defined as

$$j = \frac{1}{4}(p^*v_3 + v_3^*p),$$

where the superscript asterisk denotes complex conjugation. Hence, the acoustic two-way wave vector will contain the quantities $p$ and $v_3$. To arrive at an equation for these quantities, we need to eliminate the remaining wave field quantities, $v_1$ and $v_2$, from equations (4) and (5).

To this end, we first introduce the inverse of the mass density tensor, the so-called lightness tensor $l_{hi} = l_{hi}(x, \omega)$, via

$$l_{hi} \rho_{ij} = \delta_{hj},$$

with $\delta_{hj}$ being the Kronecker delta. On account of the symmetry of the mass density tensor and equation (7), the lightness tensor is symmetric as well, hence $l_{hi} = l_{ih}$. Applying $l_{hi}$ to both sides of equation (4), using equation (7), gives

$$l_{hi} \partial_i p - i\omega v_h = l_{hi} f_i.$$

We separate the derivatives in the $x_3$-direction from the lateral derivatives in equations (8) and (5), according to

$$\partial_3 p = l_{33}^{-1}(-l_{3\beta}\partial_\beta p + i\omega v_3 + l_{3i}f_i),$$

$$\partial_3 v_3 = i\omega \kappa p - \partial_\alpha v_\alpha + q.$$

Greek subscripts take on the values 1 and 2, and Einstein’s summation convention applies also to repeated Greek subscripts. The lateral components of the particle velocity, $v_\alpha$, need to be eliminated. From equation (8) we obtain

$$v_\alpha = \frac{1}{i\omega} (l_{\alpha\beta}\partial_\beta p + l_{\alpha3}\partial_3 p - l_{\alpha i} f_i).$$

Substituting equation (11) into equation (10), using equation (9), we obtain

$$\partial_3 v_3 = i\omega \kappa p - \frac{1}{i\omega} \partial_\alpha (l_{\alpha\beta}\partial_\beta p + l_{\alpha3}\partial_3 p - l_{\alpha i} f_i) + q$$

$$= i\omega \kappa p - \frac{1}{i\omega} \partial_\alpha \left( l_{\alpha\beta}\partial_\beta p + l_{\alpha3} l_{33}^{-1} (-l_{3\beta}\partial_\beta p + i\omega v_3 + l_{3i}f_i) - l_{\alpha i} f_i \right) + q.$$

Equations (9) and (12) can be cast in the form of the unified two-way wave equation

$$\partial_3 q = \mathcal{A} q + d,$$
with the two-way wave vector $q = q(x, \omega)$ and two-way source vector $d = d(x, \omega)$ defined as

$$q = \begin{pmatrix} p \\ v_3 \end{pmatrix}, \quad d = \begin{pmatrix} l_{3i}^{-1} l_{i3} f_i \\ \frac{1}{i\omega} \partial_\alpha ((l_{\alpha i} - l_{\alpha 3} l_{33}^{-1} l_{i3}) f_i) + q \end{pmatrix}$$

and the two-way operator matrix $A = A(x, \omega)$ defined as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

with

$$A_{11} = -l_{33}^{-1} l_{3\beta} \partial_\beta,$$

$$A_{12} = i\omega l_{33}^{-1},$$

$$A_{21} = i\omega \kappa - \frac{1}{i\omega} \partial_\alpha ((l_{\alpha \beta} - l_{\alpha 3} l_{33}^{-1} l_{3\beta}) \partial_\beta),$$

$$A_{22} = -\partial_\alpha (l_{\alpha 3} l_{33}^{-1}).$$

The dot in an operator like $\partial_\alpha (h \partial_\beta \cdot)$ indicates the position of the quantity to which this operator is applied.

For the special case of an isotropic fluid we have $l_{hi} = \frac{1}{\rho} \delta_{hi}$, with $\rho$ being the mass density of the isotropic fluid. For this situation the source vector and the operator matrix reduce to the well-known expressions

$$d = \begin{pmatrix} f_3 \\ \frac{1}{i\omega} \partial_\alpha (\frac{1}{\rho} f_\alpha) + q \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & i\omega \rho \\ i\omega \kappa - \frac{1}{i\omega} \partial_\alpha (\frac{1}{\rho} \partial_\alpha \cdot) & 0 \end{pmatrix},$$

respectively (Corones 1975; Ursin 1983; Fishman & McCoy 1984; Wapenaar & Berkhout 1989; de Hoop 1996).

### 2.3 Symmetry properties of the two-way operator matrix

For the discussion of the symmetry properties of the two-way operator matrix, we introduce transposed and adjoint operators via their integral properties

$$\int_{\mathbb{H}} (\mathcal{U} f) g d^2x = \int_{\mathbb{H}} f (\mathcal{U}^t g) d^2x$$

and

$$\int_{\mathbb{H}} (\mathcal{U} f)^* g d^2x = \int_{\mathbb{H}} f^* (\mathcal{U}^* g) d^2x,$$
where \( \mathbb{A} \) denotes an infinite horizontal integration boundary at arbitrary depth (\( x_3 \) is constant), \( f = f(\mathbf{x}) \) and \( g = g(\mathbf{x}) \) are functions with “sufficient decay” along \( \mathbb{A} \) towards infinity, and \( \mathcal{U} \) is an operator containing the lateral differential operators \( \partial_1 \) and \( \partial_2 \). Superscript \( t \) denotes the transposed operator and \( \dagger \) the adjoint. Equation (22) implies (via integration by parts) \( \partial_t^t \alpha = -\partial_t \alpha \). Moreover, \( (\mathcal{U} \mathcal{V} \mathcal{W})^t = \mathcal{W}^t \mathcal{V}^t \mathcal{U}^t \), where also \( \mathcal{V} \) and \( \mathcal{W} \) are operators containing \( \partial_1 \) and \( \partial_2 \). Equations (22) and (23) imply \( \mathcal{U}^\dagger = (\mathcal{U}^t)^* \). Using these properties and the symmetry relation \( l_{hi} = l_{ih} \), we find for the operators defined in equations (16) – (19)

\[
\begin{align*}
\mathcal{A}_{11}^t & = -\mathcal{A}_{22}, \quad (24) \\
\mathcal{A}_{12}^t & = \mathcal{A}_{12}, \quad (25) \\
\mathcal{A}_{21}^t & = \mathcal{A}_{21}. \quad (26)
\end{align*}
\]

We define adjoint acoustic medium parameters as

\[
\begin{align*}
\bar{\kappa} & = \kappa^*, \quad (27) \\
\bar{l}_{hi} & = l_{hi}^* \quad (28)
\end{align*}
\]

When a medium is dissipative, its adjoint is effectual \cite{de_Hoop1987, de_Hoop1988, Wapenaar_et_al_2001}, which means that a wave propagating through this medium gains energy (effectual media are usually associated with a computational state). Throughout this paper we assume that outside a finite domain the medium is lossless, so that (outside this domain) the adjoint medium is lossless as well. Note that, although for the acoustic situation an adjoint medium parameter is identical to its complex conjugate, this is not always the case, as is seen in some of the following sections (in particular, equations (83), (173) and (220)). For the operators in an adjoint medium we have

\[
\begin{align*}
\mathcal{A}_{11} & = \mathcal{A}_{11}^*, \quad (29) \\
\mathcal{A}_{12} & = -\mathcal{A}_{12}^*, \quad (30) \\
\mathcal{A}_{21} & = -\mathcal{A}_{21}^*, \quad (31) \\
\mathcal{A}_{22} & = \mathcal{A}_{22}^* \quad (32)
\end{align*}
\]

where the bar above an operator means that the medium parameters contained in that operator (\( \kappa \) and \( l_{hi} \)) have been replaced by their adjoints (\( \bar{\kappa} \) and \( \bar{l}_{hi} \)). Using \( \mathcal{U}^\dagger = (\mathcal{U}^t)^* \), we find
from equations (24) – (26) and (29) – (32)
\[ A_{11}^\dagger = -\bar{A}_{22}, \]  
\[ A_{12}^\dagger = -\bar{A}_{12}, \]  
\[ A_{21}^\dagger = -\bar{A}_{21}, \]  
\[ A_{22}^\dagger = -\bar{A}_{22}. \]  

Analogous to equations (22) and (23) we introduce transposed and adjoint operator matrices via
\[ \int_A (\mathbf{U} f)^t g \, d^2x = \int_A f^t (\mathbf{U}^t g) \, d^2x \]  
and
\[ \int_A (\mathbf{U} f)^\dagger g \, d^2x = \int_A f^\dagger (\mathbf{U}^\dagger g) \, d^2x, \]  
where \( f = f(x) \) and \( g = g(x) \) are vector functions, \( f^t \) is the transposed vector, \( f^\dagger \) is the complex conjugate transposed vector, and \( \mathbf{U} \) is an operator matrix containing the differential operators \( \partial_1 \) and \( \partial_2 \). Equation (37) implies that \( \mathbf{U}^t \) involves transposition of the matrix and transposition of the operators contained in the matrix. Equations (37) and (38) imply \( \mathbf{U}^\dagger = (\mathbf{U}^t)^* \). Using these properties and equations (24) – (26) and (29) – (36), we find for the acoustic two-way operator matrix \( \mathbf{A} \) defined in equations (15) – (19)
\[ \mathbf{A}^t \mathbf{N} = -\mathbf{N} \mathbf{A}, \]  
\[ \mathbf{A}^\dagger \mathbf{J} = \mathbf{J} \bar{\mathbf{A}}, \]  
\[ \mathbf{A}^\dagger \mathbf{K} = -\mathbf{K} \bar{\mathbf{A}}, \]  
with
\[ \mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]  

Equations (24) – (26), (29) – (36) and (39) – (41) appear to be unified symmetry relations for all wave phenomena considered in this paper (except that in most cases the scalar operators in equations (24) – (26) and (29) – (36) are replaced by operator matrices, and the zeros and ones in equation (42) are replaced by zero and identity matrices of appropriate size). Symmetry relations in the wavenumber-frequency domain for laterally invariant media are given in Appendix F.

Using the matrix-vector notation, the acoustic power-flux density \( j \), defined in equation (6), can be written as
\[ j = \frac{1}{2} \mathbf{q}^\dagger \mathbf{K} \mathbf{q}. \]  

(43)
This is the unified formulation of power-flux density for all wave phenomena considered in this paper (except for quantum-mechanical waves, where it is the formulation of probability current density).

3 QUANTUM-MECHANICAL WAVES

Schrödinger’s wave equation for a particle with mass $m$ in a potential $V = V(x)$ is given by (Messiah 1961; Merzbacher 1961)

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \partial_x \partial_x \psi + V \psi,$$

where $\psi = \psi(x,t)$ is the wave function and $\hbar = h/2\pi$, with $h$ Planck’s constant. We use equation (3) to transform this equation to the space-frequency domain, which means we can replace $\partial_t$ by $-i\omega$. This gives

$$\hbar \omega \psi = -\frac{\hbar^2}{2m} \partial_x \partial_x \psi + V \psi,$$

with $\psi = \psi(x,\omega)$. The probability current density $j$ in the $x_3$-direction is defined as

$$j = \frac{\hbar}{2m i} (\psi^* \partial_3 \psi - \psi \partial_3 \psi^*).$$

To comply with the unified formulation for $j$, given by equation (43), we define the quantum-mechanical two-way wave vector as

$$q = \left( \frac{\psi}{2 \hbar m i \partial_3 \psi} \right).$$

Hence, the two-way wave equation will be built from equations for the quantities $\psi$ and $\frac{2h}{mi} \partial_3 \psi$. To this end, we first recast equation (45) (using the fact that $\hbar$ and $m$ are constants) as

$$\partial_3 \left( \frac{2h}{mi} \partial_3 \psi \right) = 4i \left( \omega - \frac{V}{\hbar} \right) \psi - \frac{2h}{mi} \partial_\alpha \partial_\alpha \psi.$$

This equation, together with the trivial equation

$$\partial_3 \psi = \frac{mi}{2h} \left( \frac{2h}{mi} \partial_3 \psi \right),$$

can be cast in the form of two-way wave equation (13), with the two-way wave vector $q = q(x,\omega)$ defined in equation (47) and the two-way operator matrix $A = A(x,\omega)$ defined as

$$A = \left( \begin{array}{cc} 0 & \frac{mi}{2h} \\ 4i \left( \omega - \frac{V}{\hbar} \right) - \frac{2h}{mi} \partial_\alpha \partial_\alpha & 0 \end{array} \right).$$

We define the adjoint potential as $\bar{V} = V^*$. Note that the quantum-mechanical two-way operator matrix $\mathcal{A}$ obeys the symmetry relations formulated in equations (39) – (41).
In the frequency domain, the Maxwell equations for electromagnetic wave propagation read (Landau & Lifshitz 1960; de Hoop 1995)

\[-i\omega D_i + J_i - \epsilon_{ijk} \partial_j H_k = -J^e_i, \tag{51}\]
\[-i\omega B_k + \epsilon_{klm} \partial_l E_m = -J^m_k, \tag{52}\]

where \(E_m = E_m(x, \omega)\) is the electric field strength, \(H_k = H_k(x, \omega)\) the magnetic field strength, \(D_i = D_i(x, \omega)\) the electric flux density, \(B_k = B_k(x, \omega)\) the magnetic flux density, \(J_i = J_i(x, \omega)\) the induced electric current density, \(J^e_i = J^e_i(x, \omega)\) and \(J^m_k = J^m_k(x, \omega)\) are source functions in terms of external electric and magnetic current densities and, finally, \(\epsilon_{ijk}\) is the alternating tensor (or Levi-Civita tensor), with \(\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = -\epsilon_{213} = -\epsilon_{321} = -\epsilon_{132} = 1\) and all other elements being equal to 0. The constitutive relations for an inhomogeneous, anisotropic, dissipative medium are given by

\[D_i = \epsilon_{ik} E_k = \epsilon_{0r,ik} E_k, \tag{53}\]
\[B_k = \mu_{km} H_m = \mu_{0r,km} H_m, \tag{54}\]
\[J_i = \sigma_{ik} E_k, \tag{55}\]

where \(\epsilon_{ik} = \epsilon_{ik}(x, \omega)\), \(\mu_{km} = \mu_{km}(x, \omega)\) and \(\sigma_{ik} = \sigma_{ik}(x, \omega)\) are the permittivity, permeability and conductivity tensors, respectively. The subscripts 0 refer to the parameters in vacuum and the subscripts \(r\) denote relative parameters for the anisotropic medium. These tensors obey the symmetry relations \(\epsilon_{ik} = \epsilon_{ki}\), \(\mu_{km} = \mu_{mk}\) and \(\sigma_{ik} = \sigma_{ki}\), respectively. Substituting the constitutive relations \(53 - 55\) into Maxwell’s electromagnetic field equations \(51\) and \(52\) yields

\[-i\omega \mathcal{E}_{ik} E_k - \epsilon_{ijk} \partial_j H_k = -J^e_i, \tag{56}\]
\[-i\omega \mu_{km} H_m + \epsilon_{klm} \partial_l E_m = -J^m_k, \tag{57}\]

with

\[\mathcal{E}_{ik} = \epsilon_{ik} - \frac{\sigma_{ik}}{i\omega}. \tag{58}\]

A two-way wave equation for electromagnetic waves in an isotropic stratified medium is given by Ursin (1983) and van Stralen (1997). This has been extended for an anisotropic stratified medium by Løseth & Ursin (2007). Here we derive the two-way wave equation for electromagnetic waves in an arbitrary 3D inhomogeneous, anisotropic, dissipative medium. For electromagnetic waves the power-flux density \(j\) in the \(x_3\)-direction is defined as

\[j = \frac{1}{3}(E^*_1 H_2 - E^*_2 H_1 + H^*_2 E_1 - H^*_1 E_2). \tag{59}\]
To comply with the unified formulation for \( j \), given by equation (43), we define the electromagnetic two-way wave vector as

\[
\mathbf{q} = \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix},
\]

with

\[
\mathbf{E}_0 = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad \mathbf{H}_0 = \begin{pmatrix} H_2 \\ -H_1 \end{pmatrix}.
\]

Our aim is to derive a two-way wave equation for the wave field quantities \( \mathbf{E}_0 \) and \( \mathbf{H}_0 \). The other wave field quantities (\( \mathbf{E}_3 \) and \( \mathbf{H}_3 \)) will be eliminated. We start by rewriting equations (56) and (57) as

\[
-i\omega \mathbf{E}_1 \mathbf{E}_0 - i\omega \mathbf{E}_3 \mathbf{E}_3 + \partial_3 \mathbf{H}_0 - \partial_2 \mathbf{H}_3 = -\mathbf{J}_0^e, 
\]

\[
-i\omega \mathbf{E}_3 \mathbf{E}_0 - i\omega \mathbf{E}_33 \mathbf{E}_3 + \partial_1^t \mathbf{H}_0 = -\mathbf{J}_3^m, 
\]

\[
-i\omega \mathbf{E}_1 \mathbf{H}_0 - i\omega \mathbf{E}_3 \mathbf{H}_3 + \partial_3 \mathbf{E}_0 - \partial_1 \mathbf{E}_3 = -\mathbf{J}_0^m, 
\]

\[
-i\omega \mathbf{E}_3 \mathbf{H}_0 - i\omega \mathbf{E}_33 \mathbf{H}_3 + \partial_1^t \mathbf{E}_0 = -\mathbf{J}_3^m, 
\]

with

\[
\mathbf{E}_1 = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{12} & \mathbf{E}_{22} \end{pmatrix}, \quad \mathbf{E}_3 = \begin{pmatrix} \mathbf{E}_{13} \\ \mathbf{E}_{23} \end{pmatrix}, \quad \mathbf{\mu}_1 = \begin{pmatrix} \mu_{22} & -\mu_{12} \\ -\mu_{12} & \mu_{11} \end{pmatrix}, \quad \mathbf{\mu}_3 = \begin{pmatrix} \mu_{23} \\ -\mu_{13} \end{pmatrix},
\]

\[
\partial_1 = \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix}, \quad \partial_2 = \begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix}, \quad \mathbf{J}_0^e = \begin{pmatrix} J_1^e \\ J_2^e \end{pmatrix}, \quad \mathbf{J}_3^m = \begin{pmatrix} J_2^m \\ -J_1^m \end{pmatrix}.
\]

Note that equation (37) implies (via integration by parts)

\[
\partial_1^t = (-\partial_1, -\partial_2), \quad \partial_2^t = (-\partial_2, \partial_1).
\]

We separate the derivatives in the \( x_3 \)-direction from the lateral derivatives in equations (64) and (62), according to

\[
\partial_3 \mathbf{E}_0 = i\omega \mathbf{\mu}_1 \mathbf{H}_0 + i\omega \mathbf{\mu}_3 \mathbf{H}_3 + \partial_1 \mathbf{E}_3 - \mathbf{J}_0^m, 
\]

\[
\partial_3 \mathbf{H}_0 = i\omega \mathbf{E}_1 \mathbf{E}_0 + i\omega \mathbf{E}_3 \mathbf{E}_3 + \partial_2 \mathbf{H}_3 - \mathbf{J}_3^e.
\]

The field components \( \mathbf{E}_3 \) and \( \mathbf{H}_3 \) need to be eliminated. From equations (63) and (65) we obtain

\[
\mathbf{E}_3 = \mathbf{\varepsilon}_{33}^{-1} \left( -\mathbf{\varepsilon}_{33}^t \mathbf{E}_0 + \frac{1}{i\omega} \partial_1^t \mathbf{H}_0 + \frac{1}{i\omega} J_3^e \right),
\]

\[
\mathbf{H}_3 = \mathbf{\mu}_{33}^{-1} \left( -\mathbf{\mu}_{33}^t \mathbf{H}_0 + \frac{1}{i\omega} \partial_2^t \mathbf{E}_0 + \frac{1}{i\omega} J_3^m \right).
\]
Substituting equations (71) and (72) into equations (69) and (70) we obtain

\[
\partial_3 E_0 = \left( \mu_3 \mu_{33}^{-1} \partial_2^t - \partial_1 E_{33}^{-1} E_3^t \right) E_0 + \left( i \omega \mu_1 - i \omega \mu_3 \mu_{33}^{-1} \mu_3^t + \frac{1}{i \omega} \partial_1 E_{33}^{-1} \partial_1^t \right) H_0 \\
+ \frac{1}{i \omega} \partial_1 (E_{33}^{-1} J_3^m) - J_0^m + \mu_3 \mu_{33}^{-1} J_3^m,
\]

(73)

\[
\partial_3 H_0 = \left( i \omega E_1 - i \omega \mu_3 \mu_{33}^{-1} \mu_3^t + \frac{1}{i \omega} \partial_2 \mu_{33}^{-1} \partial_2^t \right) E_0 + \left( E_{33}^{-1} \partial_1^t - \partial_2 \mu_{33}^{-1} \mu_3^t \right) H_0 \\
- J_0^e + E_{33} \mu_3 \mu_{33}^{-1} J_3^e + \frac{1}{i \omega} \partial_2 (\mu_{33}^{-1} J_3^m).
\]

(74)

Equations (73) and (74) are now combined into the form of two-way wave equation (13), with the two-way wave vector \( \mathbf{q} = \mathbf{q}(x, \omega) \) defined in equation (60), the two-way source vector \( \mathbf{d} = \mathbf{d}(x, \omega) \) defined as

\[
\mathbf{d} = \begin{pmatrix}
\frac{1}{i \omega} \partial_1 (E_{33}^{-1} J_3^e) - J_0^m + \mu_3 \mu_{33}^{-1} J_3^m \\
-J_0^e + E_{33} \mu_3 \mu_{33}^{-1} J_3^e + \frac{1}{i \omega} \partial_2 (\mu_{33}^{-1} J_3^m)
\end{pmatrix}
\]

(75)

and the two-way operator matrix \( \mathbf{A} = \mathbf{A}(x, \omega) \) defined as

\[
\mathbf{A} = \begin{pmatrix}
\mathbf{A}_{11} & \mathbf{A}_{12} \\
\mathbf{A}_{21} & \mathbf{A}_{22}
\end{pmatrix},
\]

(76)

with

\[
\mathbf{A}_{11} = \mu_3 \mu_{33}^{-1} \partial_2^t - \partial_1 (E_{33}^{-1} E_3^t),
\]

(77)

\[
\mathbf{A}_{12} = i \omega (\mu_1 - \mu_3 \mu_{33}^{-1} \mu_3^t) + \frac{1}{i \omega} \partial_1 (E_{33}^{-1} \partial_1^t),
\]

(78)

\[
\mathbf{A}_{21} = i \omega (E_1 - E_{33} \mu_3 \mu_{33}^{-1} E_3^t) + \frac{1}{i \omega} \partial_2 (\mu_{33}^{-1} \partial_2^t),
\]

(79)

\[
\mathbf{A}_{22} = E_{33} \mu_3 \mu_{33}^{-1} \partial_1^t - \partial_2 (\mu_{33}^{-1} \mu_3^t).
\]

(80)

Note that these submatrices obey the symmetry relations formulated in equations (24) \( \text{--} \) (26).

We define adjoint electromagnetic medium parameters as

\[
\tilde{\varepsilon}_{ik} = \varepsilon_{ik}^*,
\]

(81)

\[
\tilde{\mu}_{km} = \mu_{km}^*,
\]

(82)

\[
\tilde{\sigma}_{ik} = -\sigma_{ik}^*.
\]

(83)

Note the minus sign in equation (83). From equations (58), (81) and (83) it follows that

\[
\tilde{\varepsilon}_{ik} = \varepsilon_{ik}^*.
\]

(84)

Similar relations hold for \( \varepsilon_1, \varepsilon_3, \mu_1 \) and \( \mu_3 \), which contain the parameters \( \tilde{\varepsilon}_{ik} \) and \( \mu_{km} \). It thus follows that the submatrices defined in equations (77) \( \text{--} \) (80) also obey the symmetry relations formulated in equations (29) \( \text{--} \) (36). Hence, the electromagnetic two-way operator matrix \( \mathbf{A} \)
obeys the symmetry relations formulated in equations (39) - (41). Explicit expressions for the operator submatrices in an isotropic medium are given in Appendix A.

5 ELASTODYNAMIC WAVES

In the frequency domain, the basic equations for elastodynamic wave propagation in an inhomogeneous, anisotropic, dissipative solid read (Achenbach 1973; Aki & Richards 1980; de Hoop 1995)

\[-i\omega \rho_{ij} v_j - \partial_j \tau_{ij} = f_i,\]
\[i\omega s_{klmn} \tau_{mn} - i\omega e_{kl} = h_{kl},\]

with

\[-i\omega e_{kl} = \frac{1}{2}(\partial_k v_l + \partial_l v_k),\]

where \(v_j = v_j(x, \omega)\), \(e_{kl} = e_{kl}(x, \omega)\) and \(\tau_{ij} = \tau_{ij}(x, \omega)\) are the particle velocity, strain tensor and stress tensor, respectively, associated to the elastodynamic wave field, \(\rho_{ij} = \rho_{ij}(x, \omega)\) and \(s_{klmn} = s_{klmn}(x, \omega)\) are the mass density and compliance tensors, respectively, of the medium and \(f_i = f_i(x, \omega)\) and \(h_{kl} = h_{kl}(x, \omega)\) are source functions in terms of external volume force density and deformation rate tensor, respectively. The strain, stress, mass density, compliance and deformation rate tensors obey the symmetry relations \(e_{kl} = e_{lk}\), \(\tau_{ij} = \tau_{ji}\), \(\rho_{ij} = \rho_{ji}\), \(s_{klmn} = s_{klmn} = s_{lkmn} = s_{mnkl}\) and \(h_{kl} = h_{lk}\), respectively (Aki & Richards 1980; Dahlen & Tromp 1998). We introduce the stiffness tensor \(c_{ijkl} = c_{ijkl}(x, \omega)\) as the inverse of the compliance \(s_{klmn}\), according to

\[c_{ijkl} s_{klmn} = s_{ijkl} c_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}).\]

The stiffness tensor obeys the symmetry relation \(c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}\). Multiplying all terms in equation (86) by \(c_{ijkl}\), using equations (87) and (88) as well as the symmetry relations \(\tau_{ij} = \tau_{ji}\) and \(c_{ijkl} = c_{ijlk}\), we obtain an alternative form of the stress-strain relation (86), according to

\[i\omega \tau_{ij} + c_{ijkl} \partial_l v_k = c_{ijkl} h_{kl}.\]

A two-way wave equation for elastodynamic waves in an inhomogeneous anisotropic medium is given by Woodhouse (1974). Here we review this derivation, which also serves as a starting point for the derivation of the two-way wave equations for piezoelectric waves (section 6), poroelastic waves (section 7) and seismoelectric waves (section 8). For elastodynamic waves
the power-flux density $j$ in the $x_3$-direction is defined as

$$j = \frac{1}{4}(-\tau_{3}^{*}v_{3} - v_{3}^{*}\tau_{3}). \quad (90)$$

To comply with the unified formulation for $j$, given by equation (43), we define the elastodynamic two-way wave vector as

$$q = \begin{pmatrix} -\tau_{3} \\ v \end{pmatrix}, \quad (91)$$

with $(\tau_{3})_{i} = \tau_{3i}$ and $(v)_{i} = v_{i}$. Our aim is to derive a two-way wave equation for the wave field quantities $\tau_{3}$ and $v$. The other wave field quantities $(\tau_{1} \text{ and } \tau_{2})$, with $(\tau_{j})_{i} = \tau_{ij}$ will be eliminated. We start by rewriting equations (85) and (89) as

$$-i\omega Rv - \partial_{j}\tau_{j} = f, \quad (92)$$

$$i\omega \tau_{j} + C_{jl}\partial_{l}v = C_{jl}h_{l}, \quad (93)$$

with $(R)_{ij} = \rho_{ij}$, $(C_{jl})_{ik} = c_{ijkl}$, $(f)_{i} = f_{i}$ and $(h_{l})_{k} = h_{kl}$. Note that $R' = R$ and $C_{jl}' = C_{lj}$.

We separate the derivatives in the $x_3$-direction from the lateral derivatives in equations (92) and (93), according to

$$-\partial_{3}\tau_{3} = i\omega Rv + \partial_{a}\tau_{a} + f, \quad (94)$$

$$\partial_{3}v = C_{-33}^{-1}(-i\omega \tau_{3} - C_{3\beta}\partial_{\beta}v + C_{3l}h_{l}). \quad (95)$$

The field components $\tau_{1}$ and $\tau_{2}$ need to be eliminated. From equation (93) we obtain

$$\tau_{a} = -\frac{1}{i\omega}(C_{a\beta}\partial_{\beta}v + C_{a3}\partial_{3}v - C_{al}h_{l}). \quad (96)$$

Substituting equation (95) into (96) and the result into equation (94), we obtain

$$-\partial_{3}\tau_{3} = \partial_{a}(C_{a3}C_{-33}^{-1}\tau_{3}) + i\omega Rv - \frac{1}{i\omega}\partial_{a}\left(U_{a\beta}\partial_{\beta}v - U_{al}h_{l}\right) + f, \quad (97)$$

with

$$U_{al} = C_{al} - C_{a3}C_{-33}^{-1}C_{3l}. \quad (98)$$

Note that $U_{a\beta}^{t} = U_{\beta a}$. Equations (95) and (97) are now combined into the form of two-way wave equation (13), with the two-way wave vector $\mathbf{q} = \mathbf{q}(x, \omega)$ defined in equation (91), the two-way source vector $\mathbf{d} = \mathbf{d}(x, \omega)$ defined as

$$\mathbf{d} = \begin{pmatrix} f + \frac{1}{i\omega}\partial_{a}(U_{al}h_{l}) \\ C_{-33}^{-1}C_{3l}h_{l} \end{pmatrix}. \quad (99)$$
Unified two-way wave equation

and the two-way operator matrix $\mathbf{A} = \mathbf{A}(x, \omega)$ having the form defined in equation (76), with

\begin{align}
\mathbf{A}_{11} &= -\partial_\alpha \left( C_{\alpha \beta} C_{\beta \gamma}^{-1} \right), \\
\mathbf{A}_{12} &= i\omega R - \frac{1}{i\omega} \partial_\alpha \left( \mathbf{u}_{\alpha \beta} \partial_\beta \right), \\
\mathbf{A}_{21} &= i\omega C_{\beta \gamma}^{-1}, \\
\mathbf{A}_{22} &= -C_{\alpha \beta}^{-1} C_{\gamma \delta} \partial_\delta.
\end{align}

(100) (101) (102) (103)

Note that these submatrices obey the symmetry relations formulated in equations (24)−(26).

We define adjoint elastodynamic medium parameters as

\begin{align}
\bar{c}_{ijkl} &= c_{ijkl}^*, \\
\bar{\rho}_{ij} &= \rho_{ij}^*.
\end{align}

(104) (105)

Similar relations hold for $\mathbf{C}_{jl}$ and $\mathbf{R}$, which contain the parameters $c_{ijkl}$ and $\rho_{ij}$. It thus follows that the submatrices defined in equations (100)−(103) also obey the symmetry relations formulated in equations (29)−(30). Hence, the elastodynamic two-way operator matrix $\mathbf{A}$ obeys the symmetry relations formulated in equations (39)−(41). Explicit expressions for the operator submatrices in an isotropic medium are given in Appendix B.

6 PIEZOELECTRIC WAVES

Piezoelectric waves are governed by the equations for electromagnetic waves (section 4) and elastodynamic waves (section 5), in which two of the constitutive relations need to be modified to account for the coupling between the two wave types. For piezoelectric waves, the modified constitutive relations are (Auld 1973)

\begin{align}
D_i &= \varepsilon_{ik} E_k + d_{ijk} \tau_{jk}, \\
e_{kl} &= d_{klm} E_m + s_{klmn} \tau_{mn}.
\end{align}

(106) (107)

The field quantities and medium parameters (except $d_{ijk}$) have been defined in sections 4 and 5. Note that $\varepsilon_{ik}$ in equation (106) and $s_{klmn}$ in equation (107) are parameters measured under constant stress and constant electric field, respectively. The coupling tensor $d_{ijk} = d_{ijk}(x, \omega)$ obeys the symmetry relation $d_{ijk} = d_{jik} = d_{ikj}$. Equation (106) replaces constitutive relation (53) and is substituted, together with constitutive relations (54) and (55), into the Maxwell equations (51) and (52). Equation (107) replaces stress-strain relation (86). Multiplying all terms in equation (107) by $-i\omega c_{ijkl}$, using equations (87) and (88) as well as the symmetry relations $\tau_{ij} = \tau_{ji}$ and $c_{ijkl} = c_{ijlk}$, yields $i\omega \tau_{ij} + c_{ijkl} \partial_k v_k + i\omega c_{ijkl} d_{klm} E_m = 0$. Similar as in equation (89), we introduce a source term $c_{ijkl} h_{kl}$ on the right-hand side. Taking everything
together, the basic equations for coupled electromagnetic and elastodynamic waves read

\[ -i\omega \varepsilon_{ik} E_k - \epsilon_{ijk} \partial_j H_k - i\omega d_{ijk} \tau_{jk} = -J_i^e, \]  
\[ -i\omega \mu_{km} H_m + \epsilon_{klm} \partial_l E_m = -J_m^m, \]  
\[ -i\omega \rho_{ij} v_j - \partial_j \tau_{ij} = f_i, \]  
\[ i\omega \tau_{ij} + c_{ijkl} \partial_k v_k + i\omega c_{ijkl} d_{klm} E_m = c_{ijkl} h_{kl}, \]

with \( E_{ik} = \varepsilon_{ik} - \sigma_{ik} i\omega \).

A two-way wave equation in the quasi-static approximation for 2D piezoelectric waves in an anisotropic stratified medium is given by [Honein et al. (1991), Wang & Rokhlin (2002)] and [Zhao et al. (2012)]. Here we derive the exact two-way wave equation for piezoelectric waves in an arbitrary 3D inhomogeneous, anisotropic, dissipative, piezoelectric medium. For piezoelectric waves the power-flux density \( j \) in the \( x_3 \)-direction is defined as

\[ j = \frac{1}{4} (-\tau_{i3}^* v_i + E_1^* H_2 - E_2^* H_1 - v_i^* \tau_{i3} + H_2^* E_1 - H_1^* E_2). \]

To comply with the unified formulation for \( j \), given by equation (43), we define the piezoelectric two-way wave vector as

\[ q = \begin{pmatrix} -\tau_3 \\ E_0 \\ v \\ H_0 \end{pmatrix}, \]

with \( E_0 \) and \( H_0 \) defined in section [4] and \( \tau_3 \) and \( v \) defined in section [3]. Using the notation introduced in those sections, we rewrite equations (108) – (111) as

\[ -i\omega \mathcal{E}_1 E_0 - i\omega \mathcal{E}_3 E_3 + \partial_3 H_0 - \partial_2 H_3 - i\omega D_{1k}^i \tau_{jk} = -J_0^e, \]  
\[ -i\omega \mathcal{E}_3^* E_0 - i\omega \mathcal{E}_3 E_3 + \partial_1^i H_0 - i\omega D_{3k}^i \tau_{jk} = -J_3^e, \]  
\[ -i\omega \mu_1 H_0 - i\omega \mu_3 H_3 + \partial_3 E_0 - \partial_1 E_3 = -J_0^m, \]  
\[ -i\omega \mu_3^i H_0 - i\omega \mu_3 H_3 + \partial_1^i E_0 = -J_3^m, \]  
\[ -i\omega R v - \partial_j \tau_{ij} = f, \]  
\[ i\omega \tau_{ij} + C_{jl} \partial_k v_k + i\omega C_{jl} (D_{1l} E_0 + D_{3l} E_3) = C_{jl} h_i, \]

with

\[
D_{1k} = \begin{pmatrix} d_{11k} & d_{12k} \\ d_{21k} & d_{22k} \\ d_{31k} & d_{32k} \end{pmatrix}, \quad D_{3k} = \begin{pmatrix} d_{13k} \\ d_{23k} \\ d_{33k} \end{pmatrix}.
\]

Equations (114) – (119) form the starting point for deriving a two-way wave equation for the
The two-way operator matrix $A$ and the two-way wave equation has the form of equation (13), with the two-way wave vector $q = q(x, \omega)$ defined in equation (113), the source vector $d = d(x, \omega)$ defined as

$$d = \begin{pmatrix}
    f + \frac{1}{i\omega} \partial_{\alpha} (U_{\alpha \mu} E_{\mu}) - \frac{1}{i\omega} \partial_{\alpha} ((\mathcal{E}^\prime_{33})^{-1} U_{\alpha l} D_{3l} J_{3}^e) \\
    \frac{1}{i\omega} \partial_{1} ((\mathcal{E}^\prime_{33})^{-1} (J_{3}^e - D_{3k}^t U_{k\mu} h_{1})) - J_{m}^m + \mu_3 \mu_{33}^{-1} J_{3}^m \\
    C_{33}^{-1} (C_{3m}^t E_{m} - (\mathcal{E}^\prime_{33})^{-1} C_{3l} D_{3l} J_{3}^e) \\
    -J_{0}^e + (\mathcal{E}^\prime_{33})^{-1} \mathcal{E}^\prime_{33} J_{3}^e + \frac{1}{i\omega} \partial_2 (\mu_{33}^{-1} J_{3}^m) + (D_{4k}^l)^t U_{km} E_{m}
\end{pmatrix}
$$

(121)

and the two-way operator matrix $A = A(x, \omega)$ having the form defined in equation (76), with

$$A_{11} = \begin{pmatrix} A_{11}^{11} & A_{11}^{12} \\ A_{11}^{21} & A_{11}^{22} \end{pmatrix}, \quad A_{12} = \begin{pmatrix} A_{12}^{11} & A_{12}^{12} \\ (A_{12}^{21})^t & A_{12}^{22} \end{pmatrix},
$$

(122)

$$A_{21} = \begin{pmatrix} A_{21}^{11} & A_{21}^{12} \\ (A_{21}^{21})^t & A_{21}^{22} \end{pmatrix}, \quad A_{22} = -A_{11}^t.
$$

(123)

Here

$$A_{11}^{11} = -\partial_{\alpha} ((\mathcal{C}_{33}^\prime)^t \mathcal{C}_{33}^{-1}),
$$

(124)

$$A_{11}^{12} = -\partial_{\alpha} (U_{\alpha \mu} D_{1l}^t),
$$

(125)

$$A_{11}^{21} = \partial_{1} ((\mathcal{E}^\prime_{33})^{-1} D_{3k}^t C_{k3} C_{33}^{-1}),
$$

(126)

$$A_{11}^{22} = \mu_3 \mu_{33}^{-1} \partial_1^t - \partial_1 ((\mathcal{E}^\prime_{33})^{-1} \mathcal{E}^\prime_{33}^t),
$$

(127)

$$A_{12}^{11} = i\omega R - \frac{1}{i\omega} \partial_{\alpha} (U_{\alpha \beta} \partial_{\beta}),
$$

(128)

$$A_{12}^{12} = -\frac{1}{i\omega} \partial_{\alpha} ((\mathcal{E}^\prime_{33})^{-1} U_{\alpha l} D_{3l} \partial_1^t),
$$

(129)

$$A_{12}^{22} = i\omega (\mu_1 - \mu_3 \mu_{33}^{-1} \mu_3^t) + \frac{1}{i\omega} \partial_{1} ((\mathcal{E}^\prime_{33})^{-1} \mathcal{E}^\prime_{33}^t),
$$

(130)

$$A_{21}^{11} = i\omega (\mathcal{C}_{33}^{-1} - (\mathcal{E}^\prime_{33})^{-1} C_{3l} C_{3l} D_{3l}^t D_{3k}^t C_{k3} C_{33}^{-1}),
$$

(131)

$$A_{21}^{12} = -i\omega \mathcal{C}_{33}^{-1} C_{3l} D_{3l}^t,
$$

(132)

$$A_{21}^{22} = i\omega (\mathcal{E}^\prime_{33} - (\mathcal{E}^\prime_{33})^{-1} \mathcal{E}^\prime_{33}^t) + \frac{1}{i\omega} \partial_2 (\mu_{33}^{-1} \partial_2),
$$

(133)
with

\[ U_{kl} = C_{kl} - C_{k3}C_{33}^{-1}C_{3l}, \]  
\[ \varepsilon_1' = \varepsilon_1 - D_{1k}U_{kl}D_{3l}, \]  
\[ \varepsilon_3' = \varepsilon_3 - D_{1k}U_{kl}D_{3l}, \]  
\[ \varepsilon_{33}' = \varepsilon_{33} - D_{3k}U_{kl}D_{3l}, \]  
\[ U_{am}' = U_{am} + (\varepsilon_{33}')^{-1}U_{al}D_{3d}D_{3k}U_{km}, \]  
\[ C_{3m}' = C_{3m} + (\varepsilon_{33}')^{-1}C_{3d}D_{3d}D_{3k}U_{km}, \]  
\[ D_{1l}' = D_{1l} - (\varepsilon_{33}')^{-1}D_{3l}(\varepsilon_3'). \]

Note that the submatrices \( A_{11}, A_{12}, A_{21} \) and \( A_{22} \) obey the symmetry relations formulated in equations (24) – (26). We define the adjoints of the medium parameters \( \varepsilon_{ik}, \mu_{km}, c_{ijkl} \) and \( \rho_{ij} \) via equations (84), (82), (104) and (105). Moreover,

\[ \bar{d}_{ijk} = d_{ijk}^t. \]

Similar relations hold for \( \varepsilon_1, \varepsilon_3, \mu_1, \mu_3, C_{jl}, R, D_{1k} \) and \( D_{3k} \), which contain the parameters \( \varepsilon_{ik}, \mu_{km}, c_{ijkl}, \rho_{ij} \) and \( d_{ijk} \). It thus follows that the submatrices \( A_{11}, A_{12}, A_{21} \) and \( A_{22} \) also obey the symmetry relations formulated in equations (29) – (36). Hence, the piezoelectric two-way operator matrix \( A \) obeys the symmetry relations formulated in equations (39) – (41).

7 POROELASTIC WAVES

In the frequency domain, the basic equations for poroelastic wave propagation in an inhomogeneous, anisotropic, dissipative, fluid-saturated porous solid read (Biot 1956a,b; Pride et al. 1992; Pride & Haartsen 1996)

\[-i\omega \rho_{ij}^b v_j^s - i\omega \rho_{ij}^f w_j - \partial_j \tau_{ij}^b = f_i^b, \]  
\[-i\omega \eta k_{ij} r_{ij}^f v_j^s + w_i + \frac{1}{\eta} k_{ij} \partial_j p = \frac{1}{\eta} k_{ij} f_j^f, \]  
\[i\omega \tau_{ij}^b + c_{ijkl} \partial_l v_j^s + C_{ijkl} \partial_k w_k = c_{ijkl} h_{kl}^b + C_{ijkl} q, \]  
\[-i\omega p + C_{kl} \partial_l v_k^s + M \partial_k w_k = C_{kl} h_{kl}^b + M q, \]  
}\]
Unified two-way wave equation

with

\[ w_j = \phi (v_j^f - v_s^j), \]  
\[ v_j^b = \phi v_j^f + (1 - \phi) v_j^s = v_j^s + w_j, \]  
\[ \tau_{ij}^b = \phi \tau_{ij}^f + (1 - \phi) \tau_{ij}^s = -\phi \delta_{ij} p + (1 - \phi) \tau_{ij}^s, \]  
\[ f_i^b = \phi f_i^f + (1 - \phi) f_i^s, \]  
\[ \rho_{ij}^b = \phi \rho_{ij}^f + (1 - \phi) \rho_{ij}^s. \]

Superscripts \(b\), \(f\) and \(s\) stand for bulk, fluid and solid, respectively. The wave field quantity \(v_j = v_j(x, \omega)\) is the averaged particle velocity in the bulk, fluid or solid (depending on the superscript), \(w_j = w_j(x, \omega)\) is the filtration velocity, \(\tau_{ij} = \tau_{ij}(x, \omega)\) the averaged stress in the bulk, fluid or solid and \(p = p(x, \omega)\) the averaged fluid pressure (no superscript \(f\) is needed here to indicate the fluid). The stress tensors are symmetric, i.e., \(\tau_{ij} = \tau_{ji}\). The medium parameter \(\rho_{ij} = \rho_{ij}(x, \omega)\) is the mass density of the bulk, fluid or solid (depending on the superscript). Furthermore, \(k_{ij} = k_{ij}(x, \omega)\) is the dynamic permeability tensor, \(\eta = \eta(x, \omega)\) is the fluid viscosity parameter and \(\phi = \phi(x)\) the porosity. Moreover, \(c_{ijkl} = c_{ijkl}(x, \omega)\), \(C_{ij} = C_{ij}(x, \omega)\) and \(M = M(x, \omega)\) are stiffness parameters of the porous solid. The medium parameters obey the following symmetry relations \(\rho_{ij} = \rho_{ji}, k_{ij} = k_{ji}, c_{ijkl} = c_{ijlk} = c_{klji}\) and \(C_{ij} = C_{ji}\).

The source function \(f_i = f_i(x, \omega)\) is the volume density of external force on the bulk or on the fluid. For many source types the forces on the bulk and fluid are equal but in the following they will be treated distinctly. The source functions \(h_{kl}^b = h_{kl}^b(x, \omega)\) and \(q = q(x, \omega)\) are the volume densities of external deformation rate on the bulk and volume-injection rate in the fluid (Wapenaar & Berkhout (1989); Pride (1994); Grobbe (2016)). The deformation rate tensor is symmetric, i.e., \(h_{kl}^b = h_{lk}^b\). For later convenience, we eliminate \(\partial_k w_k\) from equation (144), using equation (145). This yields

\[ \imath \omega \tau_{ij}^b + c_{ijkl}' \partial_l v_k^s + \frac{\imath \omega}{M} C_{ij} p = c_{ijkl}' h_{kl}^b, \]

with \(c_{ijkl}' = c_{ijkl}'(x, \omega)\) defined as

\[ c_{ijkl}' = c_{ijkl} - \frac{1}{M} C_{ij} C_{kl}. \]

A two-way wave equation for normal-incidence poroelastic waves in a stratified isotropic medium is given by Norris (1993) and Gurevich & Lopatnikov (1995). This has been extended for oblique-incidence poroelastic waves in a stratified anisotropic medium, separately for P-SV and SH propagation, by Gelinsky & Shapiro (1997). Here we derive the two-way wave equation for poroelastic waves in an arbitrary 3D inhomogeneous, anisotropic, dissipative, fluid-saturated porous solid. For poroelastic waves the power-flux density \(j\) in the \(x_3\)-direction
where \( \mathbf{q} \) is defined as

\[
\mathbf{q} = \begin{pmatrix} -\tau_b^3 \\ p \\ v^s \\ w_3 \end{pmatrix},
\]

with \( (\tau_b^b)_i = \tau_b^b \) and \( (v^s)_i = v^s_i \). We rewrite the equations (142), (143), (151) and (145) as

\[
\begin{align*}
-\omega \mathbf{R} v^s - \omega \mathbf{R}^f i_j w_j - \partial_j \tau_b^b &= \mathbf{f}^b, \quad (155) \\
-\frac{\omega}{\eta} k \mathbf{R}^f v^s + i_j w_j + \frac{1}{\eta} k i_j \partial_j p &= \frac{1}{\eta} k i_j f_j^f, \quad (156) \\
\omega \tau_b^b + C_{ji} \partial_i v^s + \frac{\omega}{M} c_j p &= C_{ji} h_l^b, \quad (157) \\
-\omega p + c_i^j \partial_i v^s + M \partial_k w_k &= c_i^j h_l^b + M q, \quad (158)
\end{align*}
\]

with \( (\mathbf{R})_{ij} = \rho_{ij}, (k)_{ij} = k_{ij}, (C_{ji})_{ik} = c_{ijkl}, (c_j)_{i} = C_{ij}, \mathbf{f}^b_i = f^b_i, \mathbf{h}_l^b = h_l^b \) and \( (\dot{r}_j)_i = \delta_{ij} \). Note that \( \mathbf{R}^f = \mathbf{R}, \mathbf{k}^f = \mathbf{k} \) and \( \mathbf{C}^f_{ji} = \mathbf{C}_{ij} \). Equations (155) – (158) form the starting point for deriving a two-way wave equation for the quantities \( \tau_b^b, p, v^s \) and \( w_3 \) in vector \( \mathbf{q} \). The other wave field quantities \( (\tau_1^b, \tau_2^b, w_1 \text{ and } w_2) \) need to be eliminated. The detailed derivation can be found in Appendix D. The resulting two-way wave equation has the form of equation (13), with the two-way wave vector \( \mathbf{q} = \mathbf{q}(\mathbf{x}, \omega) \) defined in equation (154), the source vector \( \mathbf{d} = \mathbf{d}(\mathbf{x}, \omega) \) defined as

\[
\mathbf{d} = \begin{pmatrix}
\frac{i \omega}{\eta} \mathbf{R}^f \alpha_i^a k (I + \frac{1}{\eta} i \beta_3^3 k^{-1} i \gamma_3^3 i \alpha_i^a k) i_j f_j^f - \frac{i \omega}{\eta} \mathbf{R}^f \alpha_i^a k \beta_3^3 f_3^f + \mathbf{f}^b + \frac{1}{\omega} \partial^a (\mathbf{U}_a^b h_l^b) \\
\frac{1}{b} (-i \beta_3^3 k^{-1} i \gamma_3^3 i \alpha_i^a k) i_j f_j^f + f_3^f \\
C_{33}^{-1} C_{3i} h_l^b \\
-\partial^a \left( \frac{1}{\eta} i \gamma_3^3 k (I + \frac{1}{\eta} i \beta_3^3 k^{-1} i \gamma_3^3 i \alpha_i^a k) i_j f_j^f - \frac{1}{\eta} i \gamma_3^3 i \alpha_i^a k \beta_3^3 f_3^f \right) + \frac{1}{\eta} \mathbf{u}_l^b h_l^b + q
\end{pmatrix}
\]

and the two-way operator matrix \( \mathbf{A} = \mathbf{A}(\mathbf{x}, \omega) \) having the form of equation (76), with sub-
matrices $A_{11}$, $A_{12}$, $A_{21}$ and $A_{22}$ having the form defined in equations (122) and (123). Here

$$A_{11}^{11} = -\partial_\alpha (C_{\alpha 3} C_{33}^{-1}),$$  \(160\)

$$A_{11}^{12} = -\frac{i\omega}{\eta} R^f \left( I - \frac{1}{b} k_i^3 i^3_i k^{-1} i^3_j k \right) ki_\beta \partial_\beta - \partial_\alpha \left( \frac{1}{M} u_{\alpha^*} \right),$$  \(161\)

$$A_{11}^{21} = 0^t,$$  \(162\)

$$A_{11}^{22} = -\frac{1}{b} i^3_i k^{-1} i^3_j ki_\beta \partial_\beta,$$  \(163\)

$$A_{12}^{11} = i\omega b - \frac{1}{i\omega} \partial_\alpha (U_{\alpha 3} \partial_\beta) - \frac{\omega^2}{\eta} R^f i_\alpha i^3_i (k - \frac{1}{b} k_i^3 i^3_i k^{-1} i^3_j k) i_\beta i^3_j R^f,$$  \(164\)

$$A_{12}^{12} = \frac{i\omega}{b} R^f k_i^3 i^3_j k^{-1} i^3_j,$$  \(165\)

$$A_{12}^{22} = -\frac{\eta}{b} i^3_i k^{-1} i^3_j,$$  \(166\)

$$A_{21}^{11} = i\omega C_{33}^{-1},$$  \(167\)

$$A_{21}^{12} = -\frac{i\omega}{M} C_{33}^{-1} c_3,$$  \(168\)

$$A_{22}^{22} = \frac{i\omega}{M^2} c_3^t C_{33}^{-1} c_3 + \frac{i\omega}{M} + \partial_\alpha \left( \frac{1}{\eta} (i^3_i k_i^3 - \frac{1}{b} i^3_i k_i^3 k^{-1} i^3_j k) i^3_j \right),$$  \(169\)

with $0$ being a zero vector and

$$U_{\alpha l} = C_{\alpha l} - C_{\alpha 3} C_{33}^{-1} C_{3 l},$$  \(170\)

$$u_l = c_l - C_{l 3} C_{33}^{-1} c_3,$$  \(171\)

$$b = 1 - i^3_i k^{-1} i_\alpha i^3_j k_i^3.$$  \(172\)

Note that the submatrices $A_{11}$, $A_{12}$, $A_{21}$ and $A_{22}$ obey the symmetry relations formulated in equations (24) – (26). We define the adjoints of the medium parameters $c_{ijkl}$ and $\rho_{ij}$ via equations (104) and (105) (where $\rho_{ij}$ can have superscript $b$ or $f$). Moreover, we define

$$\bar{k}_{ij} = -k_{ij}^*,$$  \(173\)

$$\bar{\eta} = \eta^*,$$  \(174\)

$$\bar{C}_{ij} = C_{ij}^*,$$  \(175\)

$$\bar{M} = M^*.$$  \(176\)

Note the minus sign in equation (173). Similar relations hold for $C_{jl}$, $R^b$, $R^f$, $k$ and $c_j$, which contain the parameters $c_{ijkl}^* = c_{ijkl} - \frac{1}{M} C_{ijkl} c_{kl} = \rho_{ij}^b$, $\rho_{ij}^f$, $k_{ij}$ and $C_{ij}$. It thus follows that the submatrices $A_{11}$, $A_{12}$, $A_{21}$ and $A_{22}$ also obey the symmetry relations formulated in equations (29) – (36). Hence, the poroelastic two-way operator matrix $A$ obeys the symmetry relations formulated in equations (39) – (41). Explicit expressions for the operator submatrices in an isotropic medium are given in Appendix D.
8 SEISMOELECTRIC WAVES

Seismoelectric waves are governed by the equations for electromagnetic waves (section 4) and poroelastic waves (section 7), in which two of the constitutive relations need to be modified to account for the coupling between the two wave types. In this section we consider an isotropic medium (the derivation for the anisotropic situation is disproportionately long). For seismoelectric waves, the modified constitutive relations are (Pride 1994; Pride & Haartsen 1996)

\[
J_i = \sigma E_i + L(-\partial_i p + i\omega p^f v_i^s + f_i^f),
\]

\[
w_i = LE_i + \frac{k}{\eta}(-\partial_i p + i\omega p^f v_i^s + f_i^f).
\]

The field quantities, sources and medium parameters (except \(L\)) have been defined in sections 4 and 7 (except that tensors are now replaced by scalars). \(L(x, \omega)\) accounts for the coupling between the elastodynamic and electromagnetic waves and vice versa. Note that equation (178) contains the same coupling coefficient \(L\) as equation (177) (Onsager’s reciprocity relation (Pride 1994)). Equation (177) replaces constitutive relation (55) and is substituted, together with the isotropic versions of constitutive relations (53) and (54), into the Maxwell equations (51) and (52). Equation (178) replaces the isotropic version of equation (143). Taking everything together, the basic equations for coupled electromagnetic and poroelastic waves read

\[
-i\omega p^f v_i^s - i\omega p^f w_i - \partial_j \tau_{ij}^h = f_i^h,
\]

\[
-i\omega p^f v_i^s + \eta k^{-1}(w_i - LE_i) + \partial_i p = f_i^f,
\]

\[
i\omega \tau_{ij}^b + c_{ijkl} \partial_l v_k^s + C\delta_{ij} \partial_k w_k = c_{ijkl} h_{kl}^b + C\delta_{ij} q,
\]

\[
-i\omega p + C\delta_{kl} \partial_l v_k^s + M\partial_k w_k = C\delta_{kl} h_{kl}^b + M q,
\]

\[
-i\omega \epsilon E_i + \sigma E_i + L(-\partial_i p + i\omega p^f v_i^s + f_i^f) - \epsilon_{ijk} \partial_j H_k = -J_i^e,
\]

\[
-i\omega \mu H_k + \epsilon_{klm} \partial_l E_m = -J_k^m.
\]

For the isotropic medium we have

\[
c_{ijkl} = (K_G - \frac{2}{3} G_{fr}) \delta_{ij} \delta_{kl} + G_{fr} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),
\]

where \(G_{fr}\) is the shear modulus of the framework of the grains when the fluid is absent and \(K_G\) a weighted bulk modulus. More details about these parameters are given in Appendix D. The permittivity and permeability are defined as \(\varepsilon = \varepsilon_0 \varepsilon_r\) and \(\mu = \mu_0 \mu_r\). The subscripts 0 refer to the parameters in vacuum and the subscripts r denote relative parameters. For \(\varepsilon_r\) and
where $\kappa^f$ and $\kappa^s$ are the dielectric parameters of the fluid and solid, respectively, and $\alpha_\infty$ is the tortuosity at infinite frequency. For later convenience, we eliminate the term $\partial_k w_k$ from equation (181), using equation (182). This yields

$$i\omega \tau_{ij}^b + c'_{ijkl} \partial_l v_k^s + i\omega \frac{C}{M} \delta_{ij} p = c'_{ijkl} h_{kl}^b,$$

with $c'_{ijkl} = c_{ijkl} - \frac{C^2}{M} \delta_{ij} \delta_{kl}$. (188)

Also for later convenience, we add $L$ times equation (180) to equation (183) in order to compensate for the term $L(-\partial_i p + i\omega p^f v_i^s + f_i^f)$. This yields

$$-i\omega E_i + \frac{\eta}{k} L w_i - \epsilon_{ijk} \partial_j H_k = -J_i^e,$$

with

$$E = \epsilon - \frac{1}{i\omega} \left(\sigma - \frac{\eta}{k} L^2\right).$$

(191)

A two-way wave equation for oblique-incidence seismoelectric waves in a stratified isotropic medium, separately for P-SV-TM and SH-TE propagation, is given by [Haartsen & Pride 1997], [White & Zhou 2006] and [Grobbe 2016]. Here we derive the two-way wave equation for an arbitrary 3D inhomogeneous, isotropic, dissipative, fluid-saturated porous solid. For seismoelectric waves the power-flux density $j$ in the $x_3$-direction is defined as

$$j = \frac{1}{4} (-\tau_{i3}^b v_i^s + p^s w_3 + E_1^s H_2 - E_2^s H_1 - v_i^{s*} \tau_{i3}^b + w_3^s p + H_2^* E_1 - H_1^* E_2).$$

(192)

To comply with the unified formulation for $j$, given by equation (43), we define the poroelastic two-way wave vector as

$$\mathbf{q} = \begin{pmatrix} -\tau_{3i}^b \\ p \\ E_0 \\ v^s \\ w_3 \\ H_0 \end{pmatrix},$$

(193)

with the sub-vectors defined in sections 4 and 7. We rewrite equations (179), (180), (188),
and the two-way operator matrix \( \mathbf{A} = \mathbf{A}(\mathbf{x}, \omega) \) having the form of equation (76), with

\[
\mathbf{A}_{11} = \begin{pmatrix}
\mathbf{A}_{11}^{11} & \mathbf{A}_{11}^{12} & \mathbf{A}_{11}^{13} \\
0' & 0 & 0' \\
\mathbf{O} & \mathbf{O} & \mathbf{O}
\end{pmatrix},
\mathbf{A}_{12} = \begin{pmatrix}
\mathbf{A}_{12}^{11} & \mathbf{A}_{12}^{12} & \mathbf{O} \\
(\mathbf{A}_{12}^{11})' & \mathbf{A}_{12}^{12} & \mathbf{A}_{12}^{23} \\
\mathbf{O} & (\mathbf{A}_{12}^{23})' & \mathbf{A}_{12}^{33}
\end{pmatrix},
\mathbf{A}_{21} = \begin{pmatrix}
\mathbf{A}_{21}^{11} & \mathbf{A}_{21}^{12} & \mathbf{O} \\
(\mathbf{A}_{21}^{11})' & \mathbf{A}_{21}^{12} & \mathbf{A}_{21}^{23} \\
\mathbf{O} & (\mathbf{A}_{21}^{23})' & \mathbf{A}_{21}^{33}
\end{pmatrix},
\mathbf{A}_{22} = -\mathbf{A}_{11}'.
\]
Here

\begin{align}
\mathcal{A}_{11} &= -\partial_\alpha \left( C_{\alpha 3} C_{33}^{-1} \cdot \right), \\
\mathcal{A}_{12} &= -i \omega \rho f \eta \iota_\alpha \partial_\alpha - \partial_\alpha \left( \frac{1}{M} \mathbf{u} \cdot \right), \\
\mathcal{A}_{13} &= i \omega \rho f \mathbf{L} \iota_\alpha \gamma_\alpha^t, \\
\mathcal{A}_{12} &= -\frac{1}{i \omega} \partial_\alpha \left( \mathbf{U}_{\alpha \beta} \partial_\beta \cdot \right) + i \omega \left( \rho b \mathbf{I}_3 + \omega (\rho f)^2 \frac{k}{\eta} \iota_\alpha \iota_\alpha^t \right), \\
\mathcal{A}_{12} &= i \omega \rho f \iota_3, \\
\mathcal{A}_{12} &= -\frac{\eta}{k} \left( 1 - \frac{1}{i \omega \mathbf{E} k^2} \right), \\
\mathcal{A}_{12} &= \frac{1}{i \omega \mathbf{E} k^2} \mathbf{L} \partial_1^t, \\
\mathcal{A}_{12} &= i \omega \mu \mathbf{I}_2 + \partial_1 \left( \frac{1}{i \omega \mathbf{E}} \partial_1^t \cdot \right), \\
\mathcal{A}_{11} &= i \omega C_{33}^{-1}, \\
\mathcal{A}_{12} &= -i \omega \frac{C}{M} C_{33}^{-1} \iota_3, \\
\mathcal{A}_{21} &= i \omega \frac{C}{M} C_{33}^{-1} \iota_3, \\
\mathcal{A}_{22} &= i \omega \frac{C^2}{M^2} \iota_3 \iota_3 \iota_3 + i \omega \frac{k}{M} + \partial_\beta \left( \frac{k}{\eta} \partial_\beta \cdot \right), \\
\mathcal{A}_{22} &= -\partial_\beta \left( \mathbf{L} \gamma_\beta^t \cdot \right), \\
\mathcal{A}_{22} &= \left( i \omega \mathbf{E} - \frac{\eta}{k} \mathbf{L}^2 \right) \mathbf{I}_2 + \partial_2 \left( \frac{1}{i \omega \mu} \partial_2^t \cdot \right), \\
\end{align}

where \( \mathbf{I}_3 \) is a 3 \( \times \) 3 identity matrix, \( \mathbf{I}_2 \) a 2 \( \times \) 2 identity matrix and

\begin{align}
\mathbf{U}_{\alpha l} &= C_{\alpha l} - C_{\alpha 3} C_{33}^{-1} C_{3 l}, \\
\mathbf{u}_l &= C (i_l - C_{i 3} C_{33}^{-1} i_3). \\
\end{align}

Note that the submatrices \( \mathcal{A}_{11}, \mathcal{A}_{12}, \mathcal{A}_{21} \) and \( \mathcal{A}_{22} \) obey the symmetry relations formulated in equations \( 24 \) – \( 26 \). We define the adjoints of the medium parameters \( \epsilon, \mu, \sigma, c_{ijkl}, \rho \) (with superscript \( b \) or \( f \)), \( k, \eta, C \) and \( M \) via equations \( 81 \) – \( 83 \), \( 104 \), \( 105 \), \( 173 \) – \( 176 \) for the isotropic situation. Moreover, we define

\[ \bar{\mathbf{L}} = -\mathbf{L}^* \]  

Note the minus sign in this equation. It thus follows that the submatrices \( \mathcal{A}_{11}, \mathcal{A}_{12}, \mathcal{A}_{21} \) and \( \mathcal{A}_{22} \) also obey the symmetry relations formulated in equations \( 29 \) – \( 36 \). Hence, the seismo-electric two-way operator matrix \( \mathbf{A} \) obeys the symmetry relations formulated in equations \( 39 \) – \( 41 \). Explicit expressions for the operator submatrices are given in Appendix E.
9 TWO-WAY RECIPROCITY THEOREMS AND REPRESENTATIONS

9.1 Reciprocity theorems

The symmetry relations [39] and [41] underly unified two-way reciprocity theorems and representations. We consider two wave field states \( A \) and \( B \), characterised by independent wave vectors \( \mathbf{q}_A(x, \omega) \) and \( \mathbf{q}_B(x, \omega) \), obeying two-way wave equation [13], with source vectors \( \mathbf{d}_A(x, \omega) \) and \( \mathbf{d}_B(x, \omega) \) and operator matrices \( \mathbf{A}_A(x, \omega) \) and \( \mathbf{A}_B(x, \omega) \). The subscripts \( A \) and \( B \) of these operator matrices refer to possibly different medium parameters in states \( A \) and \( B \). We consider a spatial domain \( \mathbb{D} \) enclosed by two infinite horizontal boundaries \( \partial \mathbb{D}_0 \) and \( \partial \mathbb{D}_1 \) (with \( \partial \mathbb{D}_1 \) below \( \partial \mathbb{D}_0 \)), together denoted by \( \partial \mathbb{D} \), see Figure 1. In this domain we define the interaction quantities \( \partial_3 \{ \mathbf{q}_A^t \mathbf{N} \mathbf{q}_B \} \) and \( \partial_3 \{ \mathbf{q}_A^\dagger \mathbf{K} \mathbf{q}_B \} \). Applying the product rule for differentiation, using wave equation [13] in both states, integrating the result over domain \( \mathbb{D} \) and applying the theorem of Gauss, we obtain

\[
\int_{\mathbb{D}} \left[ (\mathbf{A}_A \mathbf{q}_A)^t + \mathbf{d}_A^t \right] \mathbf{N} \mathbf{q}_B + \mathbf{q}_A^t \mathbf{N} (\mathbf{A}_B \mathbf{q}_B + \mathbf{d}_B) \, d^3x = \int_{\partial \mathbb{D}} \mathbf{q}_A^t \mathbf{N} \mathbf{q}_B n_3 d^2x \quad (221)
\]

and

\[
\int_{\mathbb{D}} \left[ (\mathbf{A}_A \mathbf{q}_A)^\dagger + \mathbf{d}_A^\dagger \right] \mathbf{K} \mathbf{q}_B + \mathbf{q}_A^\dagger \mathbf{K} (\mathbf{A}_B \mathbf{q}_B + \mathbf{d}_B) \, d^3x = \int_{\partial \mathbb{D}} \mathbf{q}_A^\dagger \mathbf{K} \mathbf{q}_B n_3 d^2x. \quad (222)
\]

Here \( n_3 \) is the vertical component of the outward pointing normal vector on \( \partial \mathbb{D} \), with \( n_3 = -1 \) at the upper boundary \( \partial \mathbb{D}_0 \) and \( n_3 = +1 \) at the lower boundary \( \partial \mathbb{D}_1 \). Using equations [37] and [38], together with the symmetry relations [39] and [41] for operator \( \mathbf{A}_A \), we obtain the following two-way reciprocity theorems [Haines & de Hoop 1996; Wapenaar 1996b]

\[
\int_{\mathbb{D}} (\mathbf{d}_A^t \mathbf{N} \mathbf{q}_B + \mathbf{q}_A^t \mathbf{N} \mathbf{d}_B) \, d^3x = \int_{\partial \mathbb{D}} \mathbf{q}_A^t \mathbf{N} \mathbf{q}_B n_3 d^2x + \int_{\mathbb{D}} \mathbf{q}_A^t \mathbf{N} (\mathbf{A}_A - \mathbf{A}_B) \mathbf{q}_B d^3x \quad (223)
\]

**Figure 1.** Configuration for the two-way reciprocity theorems, equations (223) and (224). The combination of boundaries \( \partial \mathbb{D}_0 \) and \( \partial \mathbb{D}_1 \) is called \( \partial \mathbb{D} \) in these equations.
and
\[
\int_{\mathcal{D}} (d_A^\dagger K q_B + q_A^\dagger K d_B) d^3x = \int_{\partial \mathcal{D}} q_A^\dagger K q_B n_3 d^2x + \int_{\mathcal{D}} q_A^\dagger K (\bar{A}_A - A_B) q_B d^3x. \tag{224}
\]

Equation (223) is a convolution-type reciprocity theorem (Fokkema & van den Berg 1993; de Hoop 1995) because products like \( q_A^\dagger N q_B \) in the frequency domain correspond to convolutions in the time domain. Similarly, equation (224) is a correlation-type reciprocity theorem (Bojarski 1983) because products like \( q_A^\dagger K q_B \) in the frequency domain correspond to correlations in the time domain. Because these reciprocity theorems follow from the unified symmetry relations, they hold for all wave phenomena considered in this paper. We briefly discuss some applications of these theorems.

9.2 Power balance

When the sources, medium parameters and wave fields are identical in both states, we may drop the subscripts \( A \) and \( B \). In this case equation (224) simplifies to
\[
\int_{\mathcal{D}} \frac{1}{4} (d^\dagger K q + q^\dagger K d) d^3x = \int_{\partial \mathcal{D}} \frac{1}{4} q^\dagger K q n_3 d^2x + \int_{\mathcal{D}} \frac{1}{4} q^\dagger K (\bar{A}_A - A) q d^3x. \tag{225}
\]

Note that the integrand of the boundary integral equals the power-flux density, defined in equation (43). Hence, equation (225) formulates the unified power balance. The term on the left-hand side is the power generated by the sources in \( \mathcal{D} \). The first term on the right-hand side is the power-flux through the boundary \( \partial \mathcal{D} \) and the second term on the right-hand side is the dissipated power in \( \mathcal{D} \).

9.3 Propagation invariants

When there are no sources in \( \mathcal{D} \) and the medium parameters in \( \mathcal{D} \) are equal in the two states, then in equation (223) only the boundary integral remains. This means that for this situation the quantity
\[
\int_{\mathcal{D}} q_A^\dagger N q_B d^2x \tag{226}
\]
is a unified propagation invariant (i.e., it is independent of the depth \( x_3 \) of \( A \)). When there are no sources in \( \mathcal{D} \) and the medium parameters in \( \mathcal{D} \) are each other’s adjoints in the two states, then in equation (224) only the boundary integral remains, meaning that the quantity
\[
\int_{\mathcal{D}} q_A^\dagger K q_B d^2x \tag{227}
\]
is a unified propagation invariant for this situation. These two-way propagation invariants have been extensively used in the analysis of symmetry properties of reflection and transmission
responses and for the design of efficient numerical modelling schemes (Haines 1988; Kennett et al. 1990; Koketsu et al. 1991; Takenaka et al. 1993).

9.4 Green’s functions

We introduce the two-way Green’s function \( G(x, x_A, \omega) \) as a square matrix (with the same dimensions as matrix \( \mathbf{A} \)), being the solution of the unified two-way wave equation \( (13) \) with the source vector \( \mathbf{d} \) replaced by a diagonal point-source matrix. Hence

\[
\partial_3 G = \mathbf{A} G + \mathbf{I} \delta(x - x_A),
\]

(228)

where \( \mathbf{I} \) is an identity matrix and \( x_A \) defines the position of the point source. We let \( G \) represent the forward propagating solution of equation (228), which corresponds to imposing causality in the time domain, i.e., \( G(x, x_A, t) = \mathbf{O} \) for \( t < 0 \), where \( \mathbf{O} \) is a zero matrix. We define a second Green’s function \( G(x, x_B, \omega) \) for a point source at \( x_B \), with a similar causality condition. We derive a reciprocity relation between these Green’s functions. To this end we replace \( q_A \) and \( q_B \) in reciprocity theorem (223) by \( G(x, x_A, \omega) \) and \( G(x, x_B, \omega) \), respectively. Accordingly, we replace \( \mathbf{d}_A \) and \( \mathbf{d}_B \) by \( \mathbf{I} \delta(x - x_A) \) and \( \mathbf{I} \delta(x - x_B) \), respectively. Assuming that both Green’s functions are defined in the same medium, the second integral on the right-hand side of equation (223) vanishes. We assume that \( x_A \) and \( x_B \) are both situated in \( \mathbb{D} \). When Neumann or Dirichlet boundary conditions apply on \( \partial \mathbb{D} \), or when the medium outside \( \partial \mathbb{D} \) is homogeneous, the first integral on the right-hand side of equation (223) vanishes as well. We thus obtain

\[
\mathbf{N} G(x_A, x_B, \omega) + \mathbf{G}^T(x_B, x_A, \omega) \mathbf{N} = \mathbf{O}.
\]

(229)

Using \( \mathbf{N}^{-1} = -\mathbf{N} \) we obtain

\[
G(x_A, x_B, \omega) = \mathbf{N} \mathbf{G}^T(x_B, x_A, \omega) \mathbf{N}.
\]

(230)

This is the unified source-receiver reciprocity relation for the Green’s function. For example, for the acoustic situation the Green’s function can be written as

\[
G(x, x_A, \omega) = \begin{pmatrix} G_{p,f_3} & G_{p,q} \\ G_{v_3,f_3} & G_{v_3,q} \end{pmatrix} (x, x_A, \omega),
\]

(231)

where subscripts \( p \) and \( v_3 \) stand for the observed wave quantities acoustic pressure and particle velocity at \( x \), and subscripts \( f_3 \) and \( q \) stand for the source quantities volume force and volume-injection rate at \( x_A \). Substituting this into equation (230) yields

\[
\begin{pmatrix} G_{p,f_3} & G_{p,q} \\ G_{v_3,f_3} & G_{v_3,q} \end{pmatrix} (x_A, x_B, \omega) = \begin{pmatrix} -G_{v_3,q} & G_{p,q} \\ G_{v_3,f_3} & -G_{p,f_3} \end{pmatrix} (x_B, x_A, \omega).
\]

(232)
We introduce the Green’s function of the adjoint medium, \( \bar{G}(x, x_A, \omega) \), as the forward propagating solution of the following two-way wave equation

\[
\partial^3 \bar{G} = \bar{A} \bar{G} + I \delta(x - x_A). \tag{233}
\]

Pre- and post multiplying all terms by \( J \) and subsequently using equation (40) gives

\[
\partial^3 J \bar{G} J = \bar{A}^* J \bar{G} J + JJ \delta(x - x_A). \tag{234}
\]

Taking the complex conjugate of all terms and using \( JJ = I \) gives

\[
\partial^3 J \bar{G}^* J = \bar{A} J \bar{G}^* J + I \delta(x - x_A). \tag{235}
\]

Subtracting all terms in this equation from the corresponding terms in equation (228) we obtain

\[
\partial^3 G_h(x, x_A, \omega) = \bar{A} G_h(x, x_A, \omega), \tag{236}
\]

with

\[
G_h(x, x_A, \omega) = G(x, x_A, \omega) - J \bar{G}^* J(x, x_A, \omega). \tag{237}
\]

Because \( G_h(x, x_A, \omega) \) obeys a two-way wave equation without a source term, we call it the two-way homogeneous Green’s function. The second term represents a backward propagating wave field in the adjoint medium. Using \( NJ = -JN \), it follows that the second term obeys the same source-receiver reciprocity relation as the first term (i.e., equation (230)), hence

\[
G_h(x_A, x_B, \omega) = NG_h^t(x_B, x_A, \omega)N. \tag{238}
\]

9.5 Representations

We use the convolution-type reciprocity theorem (equation (223)) to derive a representation for the two-way wave field vector \( q \). For state \( A \) we choose the Green’s function in a reference medium, hence, we replace \( q_A \) by \( G(x, x_A, \omega) \), \( d_A \) by \( I \delta(x - x_A) \), and \( A_A \) by \( A_{ref} \), where the subscript ‘ref’ denotes that the medium parameters contained in this operator matrix are the reference medium parameters. For state \( B \) we choose the wave field in the actual medium, hence, we drop the subscript \( B \) from \( q_B \), \( d_B \) and \( A_B \). Making these substitutions in equation (223), pre-multiplying all terms by \( -N \), using \( -NN = I \) and equation (230), we obtain

\[
\chi(x_A)q(x_A, \omega) = \int_D G(x_A, x, \omega) d(x, \omega) d^3x - \int_{\partial D} G(x_A, x, \omega) q(x, \omega) n_3 d^2x
+ \int_D G(x_A, x, \omega) \{ A - A_{ref} \} q(x, \omega) d^3x, \tag{239}
\]
where $\chi(x_A)$ is the characteristic function, defined as

$$\chi(x_A) = \begin{cases} 
1, & \text{for } x_A \text{ inside } D, \\
\frac{1}{2}, & \text{for } x_A \text{ on } \partial D, \\
0, & \text{for } x_A \text{ outside } D.
\end{cases}$$

(240)

Note that the left-hand side of equation (239) is the wave field vector $q$, observed at $x_A$ (assuming $x_A$ is inside $D$). The right-hand side contains, respectively, a contribution from the source distribution $d(x, \omega)$ inside $D$, a contribution from the wave field $q(x, \omega)$ at the boundary $\partial D$, and a contribution from the contrast operator $A - A_{\text{ref}}$ applied to the wave field $q(x, \omega)$ inside $D$. This unified two-way wave field representation holds for all wave phenomena considered in this paper. For example, when $A_{\text{ref}} = A$ and the domain $D$ is source free, the remaining boundary integral in equation (239) is the unified Kirchhoff-Helmholtz integral (Morse & Feshbach 1953; Born & Wolf 1965; Pao & Varatharajulu 1976; Frazer & Sen 1985; Berkhout 1985).

Next, we use the correlation-type reciprocity theorem (equation (224)) to derive a representation for the two-way homogeneous Green’s function $G_h$. For state $A$ we choose the Green’s function in the adjoint medium medium, hence, we replace $q_A$ by $G(x, x_A, \omega)$, $d_A$ by $\delta(x - x_A)$, and $A_A$ by $\tilde{A}$. For state $B$ we choose the Green’s function in the actual medium medium, hence, we replace $q_B$ by $G(x, x_B, \omega)$, $d_B$ by $\delta(x - x_B)$, and $A_B$ by $A$. Note that with these choices the contrast operator $A_B - \tilde{A}_A = A - \tilde{A}$ vanishes. Making these substitutions in equation (224), taking $x_A$ and $x_B$ both inside $D$, pre-multiplying all terms by $K$, using $KK = I$, $K = JN = -NJ$ and equations (230) and (237), we obtain

$$G_h(x_A, x_B, \omega) = \int_{\partial D} KG^\dagger(x, x_A, \omega)KG(x, x_B, \omega)n_3d^2x.$$  

(241)

This is the unified homogeneous Green’s function representation, which finds applications in optical, acoustic and seismic holography (Porter 1970; Maynard et al. 1985; Lindsey & Braun 2004), imaging and inverse scattering (Esersoy & Oristaglio 1988; Oristaglio 1989), time-reversal acoustics (Fink & Prada 2001) and Green’s function retrieval from ambient noise (Derode et al. 2003; Wapenaar 2003; Weaver & Lobkis 2004; Wapenaar et al. 2006).

10 CONCLUSIONS

We have presented a unified treatment of the two-way wave equation for acoustic, quantum-mechanical, electromagnetic, elastodynamic, piezoelectric, poroelastic and seismoelectric waves. For most cases we considered a 3D inhomogeneous, anisotropic, dissipative medium. The ex-
pressions for an isotropic medium follow immediately by making some simplifying substitutions for the isotropic medium parameters (for most cases these expressions are given in explicit form in the appendices). The two-way wave equation obeys unified symmetry relations for all the wave phenomena considered in this paper. These symmetry relations underly reciprocity theorems of the convolution and correlation type which, in turn, form the basis for unified representations of the two-way wave vector and the two-way homogeneous Green’s function.

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APPENDIX A: ELECTROMAGNETIC WAVES

The electromagnetic two-way wave equation for an inhomogeneous, anisotropic, dissipative medium is derived in section 4. For the special case of an isotropic medium we have $E_{ik} = E \delta_{ik}$ and $\mu_{km} = \mu \delta_{km}$, or

$$E_1 = \begin{pmatrix} \mathcal{E} & 0 \\ 0 & \mathcal{E} \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad E_{33} = \mathcal{E}, \quad \mu_1 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mu_{33} = \mu. \quad (A.1)$$

For this situation the two-way wave and source vectors read

$$q = \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}, \quad d = \begin{pmatrix} \frac{1}{i \omega} \partial_1 (\mathcal{E}^{-1}J_0^e) - \frac{1}{i \omega} \partial_2 (\mu^{-1}J_0^m) \\ -J_0^e + \frac{1}{i \omega} \partial_2 (\mu^{-1}J_3^m) \end{pmatrix}, \quad (A.2)$$

and the operator submatrices reduce to

$$A_{12} = i \omega \mu_1 + \frac{1}{i \omega} \partial_1 (\mathcal{E}^{-1} \partial_1 \cdot) = \begin{pmatrix} i \omega \mu - \frac{1}{i \omega} \partial_1 \left( \frac{1}{\mathcal{E}} \partial_1 \cdot \right) & -\frac{1}{i \omega} \partial_1 \left( \frac{1}{\mathcal{E}} \partial_2 \cdot \right) \\ -\frac{1}{i \omega} \partial_2 \left( \frac{1}{\mathcal{E}} \partial_1 \cdot \right) & i \omega \mu - \frac{1}{i \omega} \partial_2 \left( \frac{1}{\mathcal{E}} \partial_2 \cdot \right) \end{pmatrix}, \quad (A.3)$$

$$A_{21} = i \omega \mathcal{E}_1 + \frac{1}{i \omega} \partial_2 (\mu^{-1} \partial_2 \cdot) = \begin{pmatrix} i \omega \mathcal{E} - \frac{1}{i \omega} \partial_2 \left( \frac{1}{\mu} \partial_2 \cdot \right) & \frac{1}{i \omega} \partial_2 \left( \frac{1}{\mu} \partial_1 \cdot \right) \\ \frac{1}{i \omega} \partial_1 \left( \frac{1}{\mu} \partial_2 \cdot \right) & i \omega \mathcal{E} - \frac{1}{i \omega} \partial_1 \left( \frac{1}{\mu} \partial_1 \cdot \right) \end{pmatrix} \quad (A.4)$$

and $A_{11} = A_{22} = O$, where $O$ is a $2 \times 2$ null matrix.
APPENDIX B: ELASTODYNAMIC WAVES

The elastodynamic two-way wave equation for an inhomogeneous, anisotropic, dissipative solid is derived in section 5. For the special case of an isotropic medium we have

\begin{align*}
c_{ijkl} &= \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad \text{(B.1)} \\
\rho_{ij} &= \rho \delta_{ij}, \quad \text{(B.2)}
\end{align*}

with Lamé parameters \(\lambda = \lambda(x, \omega)\) and \(\mu = \mu(x, \omega)\) and mass density \(\rho = \rho(x, \omega)\). Hence, the mass density matrix reduces to \(\mathbf{R} = \rho \mathbf{I}\). For the stiffness matrices \(\mathbf{C}_{jl}\) we have \((\mathbf{C}_{jl})_{ik} = c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})\), hence

\begin{align*}
\mathbf{C}_{11} &= \begin{pmatrix} K_c & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, & \mathbf{C}_{12} &= \begin{pmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbf{C}_{13} &= \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}, \\
\mathbf{C}_{21} &= \begin{pmatrix} 0 & \mu & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbf{C}_{22} &= \begin{pmatrix} \mu & 0 & 0 \\ 0 & K_c & 0 \\ 0 & 0 & \mu \end{pmatrix}, & \mathbf{C}_{23} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \mu & 0 \end{pmatrix}, \\
\mathbf{C}_{31} &= \begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}, & \mathbf{C}_{32} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \lambda & 0 \end{pmatrix}, & \mathbf{C}_{33} &= \begin{pmatrix} \mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mu & K_c \end{pmatrix},
\end{align*}

with \(K_c = K_c(x, \omega) = \lambda(x, \omega) + 2\mu(x, \omega)\). For this situation the operator submatrices reduce to

\begin{align*}
\mathbf{A}_{11} &= \begin{pmatrix} 0 & 0 & -\partial_1 \left( \frac{\lambda}{\lambda + 2\mu} \right) \\ 0 & 0 & -\partial_2 \left( \frac{\lambda}{\lambda + 2\mu} \right) \\ -\partial_1 & -\partial_2 & 0 \end{pmatrix}, \\
\mathbf{A}_{12} &= \begin{pmatrix} \imath \omega - \frac{1}{\imath \omega} (\partial_1 (\nu_1 \partial_1 \cdot) + \partial_2 (\mu \partial_2 \cdot)) & -\frac{1}{\imath \omega} (\partial_2 (\mu \partial_1 \cdot) + \partial_1 (\nu_2 \partial_2 \cdot)) & 0 \\ -\frac{1}{\imath \omega} (\partial_2 (\nu_2 \partial_1 \cdot) + \partial_1 (\mu \partial_2 \cdot)) & \imath \omega - \frac{1}{\imath \omega} (\partial_1 (\mu \partial_1 \cdot) + \partial_2 (\nu_1 \partial_2 \cdot)) & 0 \\ 0 & 0 & \imath \omega \rho \end{pmatrix}.
\end{align*}
\[ A_{21} = \begin{pmatrix} \frac{i \omega}{\mu} & 0 & 0 \\ 0 & \frac{i \omega}{\mu} & 0 \\ 0 & 0 & \frac{i \omega}{\lambda + 2\mu} \end{pmatrix}, \] (B.6)

\[ A_{22} = \begin{pmatrix} 0 & 0 & -\partial_1 \\ 0 & 0 & -\partial_2 \\ -\left(\frac{\lambda}{\lambda + 2\mu}\right) \partial_1 & -\left(\frac{\lambda}{\lambda + 2\mu}\right) \partial_2 & 0 \end{pmatrix}, \] (B.7)

where

\[ \nu_1 = \nu_1(x, \omega) = 4\mu \left(\frac{\lambda + \mu}{\lambda + 2\mu}\right), \] (B.8)

\[ \nu_2 = \nu_2(x, \omega) = 2\mu \left(\frac{\lambda}{\lambda + 2\mu}\right). \] (B.9)

**APPENDIX C: PIEZOELECTRIC WAVES**

In section 6 we introduced the following equations for coupled electromagnetic and elastodynamic wave propagation in an inhomogeneous, anisotropic, dissipative, piezoelectric medium

\[ -i\omega \mathbf{E}_0 - i\omega \mathbf{E}_3 \mathbf{E}_3 + \partial_3 \mathbf{H}_0 - \partial_2 \mathbf{H}_3 - i\omega \mathbf{D}_{1k}^l \tau_k = -\mathbf{J}_0^c, \] (C.1)

\[ -i\omega \mathbf{E}_3 \mathbf{E}_0 - i\omega \mathbf{E}_3 \mathbf{E}_3 + \partial_3 \mathbf{H}_0 - i\omega \mathbf{D}_{3k}^l \tau_k = -\mathbf{J}_3^c, \] (C.2)

\[ -i\omega \mu_1 \mathbf{H}_0 - i\omega \mu_3 \mathbf{H}_3 + \partial_3 \mathbf{E}_0 - \partial_1 \mathbf{E}_3 = -\mathbf{J}_0^m, \] (C.3)

\[ -i\omega \mu_3 \mathbf{H}_0 - i\omega \mu_3 \mathbf{H}_3 + \partial_2 \mathbf{E}_0 = -\mathbf{J}_3^m, \] (C.4)

\[ -i\omega \mathbf{R} \mathbf{v} - \partial_j \tau_j = \mathbf{f}, \] (C.5)

\[ i\omega \tau_j + C_{jl} \partial_l \mathbf{v} + i\omega C_{jl} (\mathbf{D}_{1l} \mathbf{E}_0 + \mathbf{D}_{3l} \mathbf{E}_3) = \mathbf{C}_{jl} \mathbf{h}_l, \] (C.6)

with

\[ \mathbf{D}_{1k} = \begin{pmatrix} d_{11k} & d_{12k} \\ d_{21k} & d_{22k} \\ d_{31k} & d_{32k} \end{pmatrix}, \quad \mathbf{D}_{3k} = \begin{pmatrix} d_{13k} \\ d_{23k} \\ d_{33k} \end{pmatrix}. \] (C.7)

Equations (C.1) – (C.6) form the starting point for deriving a two-way wave equation for the quantities \( \tau_3, \mathbf{E}_0, \mathbf{v} \) and \( \mathbf{H}_0 \) in vector \( \mathbf{q} \). We separate the derivatives in the \( x_3 \)-direction from the lateral derivatives, according to

\[ -\partial_3 \tau_3 = i\omega \mathbf{R} \mathbf{v} + \partial_3 \tau_\alpha + \mathbf{f}, \] (C.8)

\[ \partial_3 \mathbf{E}_0 = i\omega \mu_1 \mathbf{H}_0 + i\omega \mu_3 \mathbf{H}_3 + \partial_1 \mathbf{E}_3 - \mathbf{J}_0^m, \] (C.9)

\[ \partial_3 \mathbf{v} = C_{33}^{-1} \left( -i\omega \tau_3 - C_{33} \partial_\beta \mathbf{v} - i\omega C_{3l} (\mathbf{D}_{1l} \mathbf{E}_0 + \mathbf{D}_{3l} \mathbf{E}_3) + \mathbf{C}_{3l} \mathbf{h}_l \right), \] (C.10)

\[ \partial_3 \mathbf{H}_0 = i\omega \mathbf{E}_1 \mathbf{E}_0 + i\omega \mathbf{E}_3 \mathbf{E}_3 + \partial_2 \mathbf{H}_3 + i\omega \mathbf{D}_{1k}^l \tau_k - \mathbf{J}_0^c. \] (C.11)
The field components \( \tau_\alpha, E_3 \) and \( H_3 \) need to be eliminated. We start by deriving an explicit expression for \( E_3 \). From equations (C.12) and (C.6) we obtain

\[
E_3 = \varepsilon_{33}^{-1} \left( -\varepsilon'_{33} \mathbf{E}_0 + \frac{1}{i\omega} \partial'_1 \mathbf{H}_0 \right) \quad \text{(C.12)}
\]

\[
+ \varepsilon'_{33} \mathbf{D}_{3k} \left( \frac{1}{i\omega} \mathbf{C}_{kl} \partial_1 \mathbf{v} + \mathbf{C}_{kl} (\mathbf{D}_1 \mathbf{E}_0 + \mathbf{D}_3 \mathbf{E}_3) - \frac{1}{i\omega} \mathbf{C}_{kl} \mathbf{h}_l \right) + \frac{1}{i\omega} \mathbf{J}_3'^e \left( \mathbf{E}_3 - \mathbf{D}_{3k} \mathbf{C}_{kl} \mathbf{D}_{3l} \right),
\]

\[
(C_{33} - \mathbf{D}_{3k} \mathbf{C}_{kl} \mathbf{D}_{3l}) E_3 = -\varepsilon'_3 \mathbf{E}_0 + \frac{1}{i\omega} \partial'_1 \mathbf{H}_0
\]

\[
+ \varepsilon'_{33} \mathbf{D}_{3k} \left( \frac{1}{i\omega} \mathbf{C}_{kl} \partial_1 \mathbf{v} + \mathbf{C}_{kl} (\mathbf{D}_1 \mathbf{E}_0 - \frac{1}{i\omega} \mathbf{C}_{kl} \mathbf{h}_l \right) + \frac{1}{i\omega} \mathbf{J}_3 ,
\]

\[
E_3 = \left( \varepsilon''_{33} \right)^{-1} \left( -\varepsilon'_3 \mathbf{E}_0 + \frac{1}{i\omega} \partial'_1 \mathbf{H}_0 \right)
\]

\[
+ \frac{1}{i\omega} \mathbf{D}_{3k} \left( \mathbf{C}_{k\beta} \partial_1 \mathbf{v} + \mathbf{C}_{k\beta} \partial_3 \mathbf{v} - \mathbf{C}_{kl} \mathbf{h}_l \right) + \frac{1}{i\omega} \mathbf{J}_3 ,
\]

with

\[
\varepsilon''_{33} = \varepsilon_{33} - \mathbf{D}_{3k} \mathbf{C}_{kl} \mathbf{D}_{3l},
\]

\[
(C_3')^t = \varepsilon''_{33} - \mathbf{D}_{3k} \mathbf{C}_{kl} \mathbf{D}_{3l}.
\]

The term \( \partial_3 \mathbf{v} \) is eliminated from equation (C.14) by substituting equation (C.10), hence

\[
E_3 = \left( \varepsilon''_{33} \right)^{-1} \left( -\varepsilon'_3 \mathbf{E}_0 + \frac{1}{i\omega} \partial'_1 \mathbf{H}_0 + \frac{1}{i\omega} \mathbf{D}_{3k} \left( \mathbf{C}_{k\beta} \partial_1 \mathbf{v} - \mathbf{C}_{kl} \mathbf{h}_l \right) + \frac{1}{i\omega} \mathbf{J}_3 \right),
\]

\[
\left( \varepsilon''_{33} + \mathbf{D}_{3k} \mathbf{C}_{kl} \mathbf{D}_{3l} \right) E_3 = -\varepsilon''_{33} \mathbf{E}_0 + \frac{1}{i\omega} \partial'_1 \mathbf{H}_0 + \frac{1}{i\omega} \mathbf{D}_{3k} \left( \mathbf{C}_{k\beta} \partial_1 \mathbf{v} - \mathbf{C}_{kl} \mathbf{h}_l \right) + \frac{1}{i\omega} \mathbf{J}_3
\]

\[
+ \frac{1}{i\omega} \mathbf{D}_{3k} \mathbf{C}_{kl} \mathbf{C}_{kl} \left( -\mathbf{C}_{33} \right) \left( \mathbf{D}_1 \mathbf{E}_0 + \mathbf{D}_3 \mathbf{E}_3 \right) + \mathbf{C}_{kl} \mathbf{h}_l),
\]

\[
E_3 = \left( \varepsilon''_{33} \right)^{-1} \left( -\varepsilon''_{33} \mathbf{E}_0 + \frac{1}{i\omega} \partial'_1 \mathbf{H}_0 + \frac{1}{i\omega} \mathbf{D}_{3k} \left( \mathbf{U}_{k\beta} \partial_1 \mathbf{v} - \mathbf{U}_{kl} \mathbf{h}_l \right) \right)
\]

\[
- \mathbf{D}_{3k} \mathbf{C}_{kl} \mathbf{C}_{33} \mathbf{h}_l + \frac{1}{i\omega} \mathbf{J}_3
\]

with

\[
\varepsilon''_{33} = \varepsilon_{33} + \mathbf{D}_{3k} \mathbf{C}_{kl} \mathbf{C}_{33}^{-1} \mathbf{C}_{3l} \mathbf{D}_{3l} = \varepsilon_{33} - \mathbf{D}_{3k} \mathbf{U}_{kl} \mathbf{D}_{3l},
\]

\[
(C_3')^t = \varepsilon''_{33} + \mathbf{D}_{3k} \mathbf{C}_{kl} \mathbf{D}_{3l} = \varepsilon''_{33} - \mathbf{D}_{3k} \mathbf{U}_{kl} \mathbf{D}_{3l},
\]

\[
\mathbf{U}_{kl} = \mathbf{C}_{kl} - \mathbf{C}_{kl} \mathbf{C}_{33}^{-1} \mathbf{C}_{3l}.
\]

Next, we derive an expression for \( H_3 \) from equation (C.4), according to

\[
H_3 = \mu_{33}^{-1} \left( -\mu''_{33} \mathbf{H}_0 + \frac{1}{i\omega} \partial'_2 \mathbf{E}_0 + \frac{1}{i\omega} J_3'^m \right).
\]

From equation (C.6) we obtain the following expression for \( \tau_\alpha \)

\[
-\tau_\alpha = \frac{1}{i\omega} \mathbf{C}_{\alpha\beta} \partial_1 \mathbf{v} + \frac{1}{i\omega} \mathbf{C}_{\alpha\beta} \partial_3 \mathbf{v} + \mathbf{C}_{\alpha\beta} (\mathbf{D}_1 \mathbf{E}_0 + \mathbf{D}_3 \mathbf{E}_3) - \frac{1}{i\omega} \mathbf{C}_{\alpha\beta} \mathbf{h}_l,
\]
from which $\partial_3 v$ and $E_3$ need to be eliminated. Substituting equation (C.19) into (C.10) yields
\[
\partial_3 v = C_{33}^{-1} \left( -i\omega \tau_3 - C_{33}^{\prime} \partial_{33} \tau_3 - i\omega C_{33} D_{11} E_0 + C_{33} h_l \right) - i\omega (\mathcal{E}_{33} \cdot C_{33}) \left( -i\omega \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \left( \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \right) \right) + \frac{1}{i\omega} D_{33} \left( \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \right),
\]
(C.25)

Substituting equations (C.19) and (C.26) into (C.24) gives
\[
-\tau_3 = \frac{1}{i\omega} C_{33} \cdot C_{33} \cdot I \cdot (C_{33} \cdot D_{33} \cdot D_{33} \cdot C_{33} \cdot C_{33} \cdot C_{33} \cdot C_{33}) \left( -i\omega \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \right),
\]
(C.30)

with
\[
I = I - (C_{33} \cdot D_{33} \cdot D_{33} \cdot C_{33} \cdot C_{33} \cdot C_{33} \cdot C_{33} \cdot C_{33}),
\]
(C.37)

Substituting equations (C.19) and (C.26) into (C.24) gives
\[
-\tau_3 = \frac{1}{i\omega} C_{33} \cdot C_{33} \cdot I \cdot (C_{33} \cdot D_{33} \cdot D_{33} \cdot C_{33} \cdot C_{33} \cdot C_{33} \cdot C_{33}) \left( -i\omega \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \mathcal{E}_{33} \cdot C_{33} \cdot D_{33} \cdot \partial \right),
\]
(C.31)

with
\[
S_3 = C_{33} \cdot C_{33} \cdot I \cdot (C_{33} \cdot D_{33} \cdot D_{33} \cdot C_{33} \cdot C_{33} \cdot C_{33} \cdot C_{33} \cdot C_{33}),
\]
(C.32)

Note that
\[
(U_{km})^t = U_{mk}.
\]
(C.36)
Substituting equations (C.19) and (C.23) into (C.9) yields

\[-\partial_3 \tau_3 = \partial_\alpha (S_\alpha \tau_3 + T_\alpha E_0 - \frac{1}{\imath \omega} U'_{\alpha \beta} \partial_\beta v) + \frac{1}{\imath \omega} V_\alpha \theta_1^t H_0 + \frac{1}{\imath \omega} U'_{\alpha m} h_m + \frac{1}{\imath \omega} V_\alpha J_3^m + f,\]

\[
= \partial_\alpha S_\alpha \tau_3 + \partial_\alpha T_\alpha E_0 + (\imath \omega R - \frac{1}{\imath \omega} \partial_\alpha U'_{\alpha \beta} \partial_\beta v) + \frac{1}{\imath \omega} \partial_\alpha V_\alpha \theta_1^t H_0 + \frac{1}{\imath \omega} \partial_\alpha U'_{\alpha m} h_m + \frac{1}{\imath \omega} \partial_\alpha V_\alpha J_3^m + f.
\]

Substituting equations (C.19) and (C.23) into (C.9) yields

\[
\partial_3 E_0 = \imath \omega \mu_1 H_0 + \imath \omega \mu_3 \mu_3^{-1} \left(-\mu_2^t H_0 + \frac{1}{\imath \omega} \theta_1^t E_0 + \frac{1}{\imath \omega} J_3^m\right) - J_0^m
\]

\[
+ \partial_1 \left((E'_{33})^{-1} \left(-\left(E'_{33}\right)^t E_0 + \frac{1}{\imath \omega} \theta_1^t H_0 + \frac{1}{\imath \omega} D_{3k}^t \left(U_{k \beta} \partial_\beta v - U_{k \beta} h_l\right) - D_{3k}^t C_{33}^{-1} \tau_3 + \frac{1}{\imath \omega} J_3^m\right)\right)
\]

\[
= -\partial_1 \left((E'_{33})^{-1} D_{3k}^t C_{33}^{-1} \tau_3 + \left(\mu_3 \mu_3^{-1} \theta_2 - \partial_1 \left(E'_{33}\right)^t (E'_{33})^{-1} (E'_{33})^t E_0\right)\right)
\]

\[
+ \frac{1}{\imath \omega} \partial_1 \left((E'_{33})^{-1} D_{3k}^t U_{k \beta} \partial_\beta v\right)
\]

\[
+ \left(i \omega \left(\mu_1 - \mu_3 \mu_3^{-1} \mu_3^t\right) + \frac{1}{\imath \omega} \partial_1 \left(E'_{33}\right)^t H_0\right)
\]

\[
+ \mu_3 \mu_3^{-1} J_3^m - J_0^m - \frac{1}{\imath \omega} \partial_1 \left(E'_{33}\right)^t D_{3k}^t U_{k \beta} h_l + \frac{1}{\imath \omega} \partial_1 \left(E'_{33}\right)^t J_3^m.
\]

Substituting equations (C.6), (C.19), (C.23) and (C.26) into (C.11) gives

\[
\partial_1 H_0 = \imath \omega \varepsilon_1 E_0 + \imath \omega \varepsilon_3 E_3 + \partial_2 \left(\mu_3^{-1} \left(-\mu_2^t H_0 + \frac{1}{\imath \omega} \theta_1^t E_0 + \frac{1}{\imath \omega} J_3^m\right)\right)
\]

\[-D_{1k}^t \left(C_{k \beta} \partial_\beta v + C_{33} \partial_\beta v + \imath \omega C_{33} (D_{3l}^t E_0 + D_{3l}^t E_3) - C_{33} h_l\right) - J_0^m
\]

\[
= \imath \omega \varepsilon_1 E_0 + \imath \omega \varepsilon_3 E_3 + \left(-\varepsilon_3 - D_{3k}^t C_{33} \tau_3 + \frac{1}{\imath \omega} J_3^m\right) + \partial_2 \left(\mu_3^{-1} \left(-\mu_2^t H_0 + \frac{1}{\imath \omega} \theta_1^t E_0 + \frac{1}{\imath \omega} J_3^m\right)\right)
\]

\[-D_{1k}^t \left(C_{k \beta} \partial_\beta v + C_{33} \tau_3 + \left(-i \omega \varepsilon_3 \tau_3 - \imath \omega C_{33} D_{3l}^t E_0 - C_{33} \mu_3 \partial_\beta v - \left(-\varepsilon_3\right)^t C_{33} D_{3l}^t \theta_1^t H_0\right)\right)
\]

\[-\left(-\varepsilon_3\right)^t C_{33} D_{3l}^t J_3^m + C_{33} h_m\right) + \imath \omega C_{33} D_{3l}^t E_0 - C_{33} h_l\right) - J_0^m
\]

\[
= \imath \omega \left(D_{1k}^t C_{k \beta} \tau_3 - \left(-\varepsilon_3\right)^t \varepsilon_3^t D_{3k}^t C_{33} \tau_3\right) + \left(\imath \omega \left(\varepsilon_1^t - \left(-\varepsilon_3\right)^t \varepsilon_3^t\right) + \frac{1}{\imath \omega} \partial_2 \mu_3^{-1} \theta_2\right) E_0
\]

\[+ (\varepsilon_3^{-1} \varepsilon_3^t D_{3k}^t U_{k \beta} - D_{1k}^t (C_{k \beta} - C_{33} C_{33} C_{33} C_{33}^t)) \partial_\beta v + \left(\left(-\varepsilon_3\right)^t \varepsilon_3^t - \partial_2 \mu_3^{-1} \mu_3^t\right) H_0
\]

\[-\left(\varepsilon_3^{-1} \varepsilon_3^t D_{3k}^t U_{k \beta} - D_{1k}^t (C_{k \beta} - C_{33} C_{33} C_{33} C_{33}^t)) \partial_\beta v + \left(\left(-\varepsilon_3\right)^t \varepsilon_3^t - \partial_2 \mu_3^{-1} \mu_3^t\right) H_0
\]

\[= \varepsilon_1'' = -D_{1k}^t (C_{kl} D_{ll} - C_{33} C_{33}^{-1} C_{33}^t D_{1l}) = \varepsilon_1 - D_{1k}^t U_{kl} D_{ll} - \left(-\varepsilon_3\right)^t D_{1k}^t C_{33} C_{33}^{-1} C_{33} D_{3l}^t \left(-\varepsilon_3\right)^t.
\]

(C.40)
Using the following relations

\[
D_{ik}^t C_{kl} C_{jk}^{-1} Y - (\mathcal{E}_{33}')^{-1} \mathcal{E}_{33}' D_{ik}^t C_{kl} C_{jk}^{-1} = (D_{ik}^t - (\mathcal{E}_{33}')^{-1} \mathcal{E}_{33}' D_{ik}^t) C_{kl} C_{jk}^{-1},
\]

\[
\mathcal{E}_{33}' = \mathcal{E}_{33}' - 1
\]

\[
(D_{ik}^t - (\mathcal{E}_{33}')^{-1} \mathcal{E}_{33}' D_{ik}^t) U_{km} - D_{ik}^t (C_{km} - C_{kl} C_{jk}^{-1} C_{km}) = -(D_{ik}^t - (\mathcal{E}_{33}')^{-1} \mathcal{E}_{33}' D_{ik}^t) U_{km},
\]

the unified two-way wave equation \([C.39]\) can be rewritten as

\[
\partial_3 H_0 = i\omega (D_{ik}^t - (\mathcal{E}_{33}')^{-1} \mathcal{E}_{33}' D_{ik}^t) C_{kl} C_{jk}^{-1} \tau_3
\]

\[
+ \left( i\omega (\mathcal{E}_{33}' - D_{ik}^t U_{kl} D_{jk} - \mathcal{E}_{33}'(\mathcal{E}_{33}')^{-1}(\mathcal{E}_{33}')^t) + \frac{1}{i\omega} \partial_2 \mu_{33}^{-1} \partial_2 \right) E_0
\]

\[
-(D_{ik}^t - (\mathcal{E}_{33}')^{-1} \mathcal{E}_{33}' D_{ik}^t) U_{km} \partial_\nu + \left( (\mathcal{E}_{33}')^{-1} \mathcal{E}_{33}' \partial_1^t - \partial_2 \mu_{33}^{-1} \mu_3^t \right) H_0
\]

\[
+(D_{ik}^t - (\mathcal{E}_{33}')^{-1} \mathcal{E}_{33}' D_{ik}^t) U_{km} h_m + (\mathcal{E}_{33}')^{-1} \mathcal{E}_{33}' \tau_3^e + \frac{1}{i\omega} \partial_2 \mu_{33}^{-1} J_m + J_0^e,
\]

Equations \([C.37], [C.38], [C.26]\) and \([C.44]\) can be cast in the form of two-way wave equation \([13]\), with the two-way wave vector \(q = q(x, \omega)\) and the two-way source vector \(d = d(x, \omega)\) defined as

\[
q = \begin{pmatrix} -\tau_3 \\ E_0 \\ v \\ H_0 \end{pmatrix}, \quad d = \begin{pmatrix} f + \frac{1}{i\omega} \partial_\alpha (U_{am} h_m) - \frac{1}{i\omega} \partial_\alpha ((\mathcal{E}_{33}')^{-1} U_{am} D_{33} J_3^t) \\ \frac{1}{i\omega} \partial_1 ((\mathcal{E}_{33}')^{-1} (J_3^t - D_{33}^t U_{kl} h_l)) - J_0^m + \mu_3 \mu_{33}^{-1} J_3^m \\ C_{33}^{-1} (J_3^m) h_m - (\mathcal{E}_{33}')^{-1} C_{33} D_{33} J_3^e \\ -J_0^e + (\mathcal{E}_{33}')^{-1} \mathcal{E}_{33}' J_3^e + \frac{1}{i\omega} \partial_2 (\mu_{33}^{-1} J_3^m) + (D_{ik}^t)^t U_{km} h_m \end{pmatrix}
\]

and the two-way operator matrix \(\mathcal{A} = \mathcal{A}(x, \omega)\) having the form defined in equation \([76]\), with

\[
\mathcal{A}_{11} = \begin{pmatrix} A_{11}^{11} & A_{12}^{21} \\ A_{11}^{21} & A_{11}^{22} \end{pmatrix}, \quad \mathcal{A}_{12} = \begin{pmatrix} A_{12}^{11} & A_{12}^{12} \\ A_{12}^{21} & A_{12}^{22} \end{pmatrix},
\]

\[
\mathcal{A}_{21} = \begin{pmatrix} A_{21}^{11} & A_{21}^{12} \\ A_{21}^{21} & A_{21}^{22} \end{pmatrix}, \quad \mathcal{A}_{22} = \begin{pmatrix} A_{22}^{11} & A_{22}^{12} \\ A_{22}^{21} & A_{22}^{22} \end{pmatrix},
\]

where

\[
\mathcal{A}_{11}^{11} = -\partial_\alpha (S_{\alpha \cdot}) = -\partial_\alpha \left( (C_{\alpha 3} + (\mathcal{E}_{33}')^{-1} U_{\alpha 3} D_{33}^t C_{\alpha 3}) C_{33}^{-1} \right),
\]

\[
\mathcal{A}_{11}^{12} = \partial_\alpha (T_{\alpha \cdot}) = \partial_\alpha \left( U_{\alpha l} (D_{ll} - (\mathcal{E}_{33}')^{-1} D_{33} (\mathcal{E}_{33}')^t) \right),
\]

\[
\mathcal{A}_{21}^{21} = \partial_1 ((\mathcal{E}_{33}')^{-1} D_{33} C_{\alpha 3} C_{33}^{-1}),
\]

\[
\mathcal{A}_{21}^{22} = \mu_3 \mu_{33}^{-1} \partial_2^t - \partial_1 ((\mathcal{E}_{33}')^{-1} (\mathcal{E}_{33}')^t).
\]
\[ A_{12}^{11} = i\omega R - \frac{1}{i\omega} \partial_\alpha (U'_\alpha \partial_\beta) , \] (C.52)
\[ A_{12}^{12} = \frac{1}{i\omega} \partial_\alpha (V'_\alpha \partial_1) = -\frac{1}{i\omega} \partial_\alpha \left( (\varepsilon''_{33})^{-1} U_{al'} D_{3l'} \partial_1 \right) , \] (C.53)
\[ A_{12}^{21} = \frac{1}{i\omega} \partial_1 \left( (\varepsilon''_{33})^{-1} D_{3k} U_{k\beta} \partial_\beta \right) , \] (C.54)
\[ A_{22}^{12} = i\omega (\mu_1 - \mu_3 \mu_{33}^{-1} \mu_3^F) + \frac{1}{i\omega} \partial_1 \left( (\varepsilon''_{33})^{-1} \partial_1 \right) , \] (C.55)
\[ A_{21}^{21} = i\omega (C_{33}^{-1} - (\varepsilon''_{33})^{-1} C_{33}^{-1} C_{33} D_{3l} D_{3k} C_{k3} C_{33}^{-1} ) , \] (C.56)
\[ A_{21}^{12} = -i\omega C_{33}^{-1} C_{33} D_{1l} = -i\omega C_{33}^{-1} C_{33} (D_{1l} - (\varepsilon''_{33})^{-1} D_{3l} (\varepsilon''_{33})^t) , \] (C.57)
\[ A_{21}^{21} = i\omega (\varepsilon_{1} - \varepsilon_{3} (\varepsilon''_{33})^{-1} (\varepsilon''_{3})^t - D_{1k} U_{k\beta} D_{1l}) + \frac{1}{i\omega} \partial_1 (\mu_3^{-1} \partial_2) , \] (C.58)
\[ A_{21}^{12} = -C_{33}^{-1} C_{33} \partial_\beta = -C_{33}^{-1} (C_{33} + (\varepsilon''_{33})^{-1} C_{33} D_{3l} D_{3k} U_{k\beta} \partial_\beta , \] (C.59)
\[ A_{22}^{22} = (\varepsilon''_{33})^{-1} \varepsilon''_{3} \partial_{1} - \partial_2 (\mu_3^{-1} \mu_3^F) . \] (C.60)

APPENDIX D: POROELASTIC WAVES

In section 7 we introduced the following equations for poroelastic wave propagation in an inhomogeneous, anisotropic, dissipative, fluid-saturated porous solid

\[-i\omega R^b v^s - i\omega R^f i_j w_j - \partial_j \tau^b_j = f^b , \] (D.1)
\[-i\omega k R^f v^s + i_j w_j + \frac{1}{\eta} k i_j f^f_j = \frac{1}{\eta} k i_j f^f_j , \] (D.2)
\[ i\omega \tau^b_j + C_{ji} \partial_i v^s + \frac{i\omega}{M} c_j p = C_{ji} h_{ji}^b , \] (D.3)
\[ -i\omega p + c_{ij} \partial_i v^s + M \partial_k w_k = c_{ij} h_{ji}^b + M q , \] (D.4)

with

\[ i_j = \begin{pmatrix} \delta_{1j} \\ \delta_{2j} \\ \delta_{3j} \end{pmatrix} . \] (D.5)

Equations (D.1) – (D.4) form the starting point for deriving a two-way wave equation for the quantities \( \tau^b_j , p , v^s \) and \( w_3 \) in vector \( q \). Pre-multiplying all terms in equation (D.2) by \( \eta i k^{-1} \) gives

\[-i\omega i^l R^f v^s + \eta i^l k^{-1} i_j w_j + \partial_i p = f^f_i . \] (D.6)
We separate the derivatives in the $x_3$-direction from the lateral derivatives in equations (D.1), (D.6), (D.3) and (D.4), according to

\[ -\partial_3 \tau^b_3 = i\omega R^b v^s + i\omega R^b (i_\alpha w_\alpha + i_3 w_3) + \partial_3 \tau^b_3 + f^b, \quad (D.7) \]

\[ \partial_3 p = i\omega i_3^b \frac{R^j}{M} v^s - \eta i_3^b k^{-1} (i_\alpha w_\alpha + i_3 w_3) + f_3^b, \quad (D.8) \]

\[ \partial_3 v^s = C_{-33}^b \left( -i\omega \tau_3^b - \frac{i\omega}{M} c_3 p - C_{33} \partial_3 v^s + C_{33} h_3^b \right), \quad (D.9) \]

\[ \partial_3 w_3 = \frac{i\omega}{M} p - \frac{1}{M} (c_3^b \partial_3 v^s + c_3^b \partial_3 v^s) - \partial_3 w_\alpha + \frac{1}{M} c_3^b h_3^b + q. \quad (D.10) \]

The field components $\tau^b_\alpha$ and $w_\alpha$ need to be eliminated. From equation (D.3) we obtain

\[ \tau^b_\alpha = \frac{1}{i\omega} \left( C_{\alpha \beta} \partial_\beta v^s + C_{\alpha 3} \partial_3 v^s + \frac{i\omega}{M} c_\alpha p - C_{\alpha l} h^b_l \right). \quad (D.11) \]

Pre-multiplying all terms in equation (D.2) by $i_\alpha^t$ gives

\[ w_\alpha = i_\alpha^t \left( \frac{i\omega}{\eta} R^j \frac{v^s}{v^s} - \frac{1}{\eta} k (i_\beta \partial_\beta p + i_3 \partial_3 p) + \frac{1}{\eta} k i_j f_j^f \right). \quad (D.12) \]

Using this in equation (D.8) gives

\[ \partial_3 p = i\omega i_3^b \frac{R^j}{M} v^s - \eta i_3^b k^{-1} i_3 w_3 \]

\[ - i_3^b k^{-1} i_\alpha i_\alpha^t \left( i_3^b k^t - k (i_\beta \partial_\beta p + i_3 \partial_3 p) + k i_j f_j^f \right) + f_3^b, \quad (D.13) \]

\[ (1 - i_3^b k^{-1} i_\alpha i_\alpha^t k i_3) \partial_3 p = i\omega i_3^b \frac{R^j}{M} v^s - \eta i_3^b k^{-1} i_3 w_3 - i_3^b k^{-1} i_\alpha i_\alpha^t \left( i_3^b k^t - k (i_\beta \partial_\beta p + i_3 \partial_3 p) + k i_j f_j^f \right) + f_3^b, \quad (D.14) \]

\[ \partial_3 p = \frac{1}{b} \left( i_\alpha i_3^b \frac{R^j}{M} v^s - \eta i_3^b k^{-1} i_3 w_3 - i_3^b k^{-1} i_\alpha i_\alpha^t \left( i_3^b k^t - k (i_\beta \partial_\beta p + i_3 \partial_3 p) + k i_j f_j^f \right) + f_3^b \right) \]

\[ \quad = \frac{1}{b} \left( i_3^b k^{-1} i_\alpha i_\alpha^t k \partial_\beta p + i_\alpha i_3^b (I - k^{-1} i_\gamma i_\gamma^t k) R^j \frac{v^s}{v^s} - \eta i_3^b k^{-1} i_3 w_3 \right. \]

\[ \left. - i_3^b k^{-1} i_\alpha i_\alpha^t k i_j f_j^f + f_3^b \right), \quad (D.15) \]

with

\[ b = 1 - i_3^b k^{-1} i_\alpha i_\alpha^t k i_3. \quad (D.16) \]

Substituting this into equation (D.12) gives

\[ w_\alpha = i_\alpha^t \left( \frac{i\omega}{\eta} R^j \frac{v^s}{v^s} - \frac{1}{\eta} k i_j f_j^f \right) \]

\[ - \frac{1}{\eta b} i_\alpha i_3^b \left( i_3^b k^{-1} i_\alpha i_\alpha^t k \partial_\beta p + i_\alpha i_3^b (I - k^{-1} i_\gamma i_\gamma^t k) R^j \frac{v^s}{v^s} - \eta i_3^b k^{-1} i_3 w_3 \right. \]

\[ \left. - i_3^b k^{-1} i_\alpha i_\alpha^t k i_j f_j^f + f_3^b \right), \quad (D.17) \]

\[ w_\alpha = \frac{1}{\eta} \left( i_\alpha i_3^b \partial_3 p + \frac{1}{b} i_\alpha i_3^b i_3^b \partial_3 p - \frac{1}{b} i_3^b k (I - k^{-1} i_\gamma i_\gamma^t k) R^j \frac{v^s}{v^s} \right. \]

\[ + \frac{1}{b} i_\alpha i_3^b i_3^b \partial_3 w_3 + \frac{1}{\eta} i_\alpha i_3^b (I + \frac{1}{b} i_3^b k^{-1} i_\gamma i_\gamma^t k) i_j f_j^f \]

\[ - \frac{1}{\eta b} i_\alpha i_3^b k i_3 f_3^f. \quad (D.18) \]

We are now ready to eliminate $\tau^b_\alpha$ and $w_\alpha$ from equations (D.7) – (D.10). The expression for $\partial_3 v$, equation (D.9), already has the desired form. Substituting equations (D.11) and (D.18)
Note that we obtain
\[ -\partial_3 \tau^b_3 = -\frac{i\omega}{\eta} R^b_i i^b a(i^b a_k i^b \beta + \frac{1}{b} i^b a_k i^b \gamma i^b \gamma i^b \gamma k i^b \beta) \partial_3 p \]
\[ \quad + i\omega R^b_i v^s - \frac{\omega^2}{\eta} R^b_i i^b a_k (I - \frac{1}{b} i^b a \gamma k i^b \gamma k) \]
\[ \quad + i\omega R^b_i (I + \frac{1}{b} i^b a_k i^b \gamma k i^b \gamma k) i^b \gamma k i^b \gamma k \]
\[ \quad - \frac{i\omega}{\eta b} R^b_i i^b a_k i^b \gamma k i^b \gamma k \]
\[ \quad - \frac{1}{i\omega} \partial_3 \left( C_{\alpha\beta} \partial_3 v^s + C_{\alpha\beta} \partial_3 v^s + \frac{i\omega}{M} c_\alpha p \right) + f^b + \frac{1}{i\omega} \partial_3 \left( C_{\alpha\beta} h^b \right). \]
\[ \text{D.19} \]

Upon substitution of equation \text{(D.9)} and using
\[ k^{-1} i^b a_3 k = k^{-1} (I - i^b a_3 k) k = I - k^{-1} i^b a_3 k, \]
\[ i^b a_3 k^{-1} i^b a_3 k \beta = i^b a_3 k^{-1} (I - i^b a_3 k) k \beta = -i^b a_3 k^{-1} i^b a_3 k \beta, \]
we obtain
\[ -\partial_3 \tau^b_3 = \partial_3 \left( C_{\alpha\beta} C_{\beta\alpha}^{-1} \omega^b \right) - \frac{i\omega}{\eta} R^b_i i^b a (i^b a_k i^b \beta - \frac{1}{b} i^b a_k i^b \gamma i^b \gamma i^b \gamma k i^b \beta) \partial_3 p \]
\[ \quad - \frac{1}{i\omega} \partial_3 \left( \frac{i\omega}{M} u_a p + U_{a\beta} \partial_3 v^s \right) \]
\[ \quad + i\omega R^b_i v^s - \frac{\omega^2}{\eta} R^b_i i^b a_k (I - \frac{1}{b} i^b a \gamma k i^b \gamma k) \]
\[ \quad + i\omega R^b_i (I + \frac{1}{b} i^b a_k i^b \gamma k i^b \gamma k) i^b \gamma k i^b \gamma k \]
\[ \quad - \frac{i\omega}{\eta b} R^b_i i^b a_k i^b \gamma k i^b \gamma k \]
\[ \quad + f^b + \frac{1}{i\omega} \partial_3 \left( U_{a\beta} h^b \right), \]
\[ \text{D.22} \]

with
\[ U_{a\beta} = C_{\alpha\beta} - C_{\alpha3} C_{3\beta}^{-1} C_{3\beta}, \]
\[ u_t = c_t - C_{t3} C_{33}^{-1} c_3. \]
\[ \text{D.23} \]
\[ \text{D.24} \]

Note that
\[ U_{a\beta} = U_{\beta \alpha}, \]
\[ \text{D.25} \]
on account of \[ C_{ij} = C_{ji}. \] Using equations \text{(D.20)} and \text{(D.21)} in equation \text{(D.15)}, we obtain
\[ \partial_3 p = \frac{1}{b} \left( -i^b a_3 k^{-1} i^b a_3 k \beta \partial_3 p + i\omega i^b a_3 k^{-1} i^b a_3 k \right) R^b_i v^s - \eta i^b a_3 k^{-1} i^b a_3 k \]
\[ \quad - i^b a_3 k^{-1} i^b a_3 k i^b \gamma k i^b \gamma k \]
\[ \quad + f^b + f_3 \],
\[ \text{D.26} \]

with
\[ b = 1 - i^b a_3 k^{-1} i^b a_3 k i^b a_3 i^b a_3 i^b a_3 k^{-1} i^b a_3. \]
\[ \text{D.27} \]
Substituting equations (D.9) and (D.18) into equation (D.10), we obtain
\[
\partial_3 w_3 = \frac{i\omega}{M} p - \frac{1}{M} c^b_3 \partial_3 v^s + \frac{1}{M} c^t_3 C_{33}^{-1} \left( \frac{i\omega}{M} \tau^b_3 + \frac{i\omega}{M} c^b_3 p + C_{33} \partial_3 v^s - C_{33} h^b_1 \right) \\
- \partial_\alpha \left( \frac{1}{\eta} \left( i \alpha \iota^b_\alpha k i_\beta - \frac{1}{b} \iota^b_\alpha k i_\beta i^t_\gamma i^t_\gamma k i_\beta \right) \right) \partial_\beta p + \frac{i\omega}{\eta} \iota^b_\alpha k \left( I - \frac{1}{b} i_3 \iota^t_3 k^{-1} i_3 \iota^t_3 k \right) R^f v^s \\
+ \frac{1}{b} \iota^t_\alpha k i_3 \iota^t_3 k^{-1} i_3 w_3 t + \frac{1}{\eta} \iota^b_\alpha k \left( I + \frac{1}{b} i_3 \iota^t_3 k^{-1} i_3 \iota^t_3 k \right) i_j f^f_j - \frac{1}{\eta} \iota^t_\alpha k i_3 f^f_3 + \frac{1}{M} c^b_3 h^b_1 + q.
\]

or
\[
\partial_3 w_3 = \frac{i\omega}{M} c^b_3 C_{33}^{-1} \tau^b_3 + \frac{i\omega}{M} \left( c^b_3 C_{33}^{-1} c^t_3 c^b_3 + \alpha \left( \frac{1}{\eta} \iota^b_\alpha k \right) \right) \partial_3 p \\
- \frac{1}{M} u^t_\beta \partial_3 v^s - \partial_\alpha \left( \frac{1}{\eta} \iota^b_\alpha k \left( I - \frac{1}{b} i_3 \iota^t_3 k^{-1} i_3 \iota^t_3 k \right) R^f v^s \right) \\
- \partial_\alpha \left( \frac{1}{\eta} \iota^b_\alpha k i_3 \iota^t_3 k^{-1} i_3 w_3 t + \frac{1}{\eta} \iota^b_\alpha k \left( I + \frac{1}{b} i_3 \iota^t_3 k^{-1} i_3 \iota^t_3 k \right) i_j f^f_j - \frac{1}{\eta} \iota^t_\alpha k i_3 f^f_3 \right) + \frac{1}{M} u^b_\beta h^b_1 + q.
\]

Equations (D.22), (D.26), (D.9) and (D.29) can be cast in the form of two-way wave equation (13), with the two-way wave vector \( \mathbf{q} = \mathbf{q}(x, \omega) \) and the two-way source vector \( \mathbf{d} = \mathbf{d}(x, \omega) \) defined as
\[
\begin{pmatrix}
-\tau^s \\
p \\
v^s \\
w_3
\end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix}
\frac{i\omega}{M} R^f i_\alpha \iota^t_\alpha k \left( I + \frac{1}{b} i_3 \iota^t_3 k^{-1} i_3 \iota^t_3 k \right) i_j f^f_j - \frac{i\omega}{\eta} R^f i_\alpha \iota^t_\alpha k i_3 f^f_3 + r^b + \frac{1}{\eta} \partial_\alpha \left( u_\alpha \iota^b_\alpha h^b_1 \right) \\
n \frac{1}{b} \left( -\iota^t_\alpha k^{-1} i_\alpha \iota^t_\alpha k i_j f^f_j + f^f_j \right) \\
c^t_3 C_{33}^{-1} C^b_3 h^b_1 \\
-\partial_\alpha \left( \frac{1}{\eta} \iota^b_\alpha k \left( I + \frac{1}{b} i_3 \iota^t_3 k^{-1} i_3 \iota^t_3 k \right) i_j f^f_j - \frac{1}{\eta} \iota^t_\alpha k i_3 f^f_3 \right) + \frac{1}{\eta} u^b_\beta h^b_1 + q
\end{pmatrix}
\]

and the two-way operator matrix \( \mathbf{A} = \mathbf{A}(x, \omega) \) having the form of equation (76), with
\[
\begin{align*}
\mathbf{A}_{11} &= \begin{pmatrix} \mathbf{A}^{11}_{11} & \mathbf{A}^{11}_{12} \\ \mathbf{A}^{12}_{11} & \mathbf{A}^{12}_{12} \end{pmatrix}, \\
\mathbf{A}_{21} &= \begin{pmatrix} \mathbf{A}^{11}_{21} & \mathbf{A}^{11}_{22} \\ \mathbf{A}^{12}_{21} & \mathbf{A}^{12}_{22} \end{pmatrix},
\end{align*}
\]
where, using equation (D.27),
\[
\begin{align*}
\mathbf{A}^{11}_{11} &= -\partial_\alpha \left( C_{\alpha 3} C_{33}^{-1} \cdot \right), \\
\mathbf{A}^{12}_{11} &= -\frac{i\omega}{\eta b} R^f \left( b_\alpha i_\alpha^t i_\alpha^{-t} - i_\alpha i_\alpha^t k i_3 \iota^t_\gamma i_\gamma^t k i^t_\beta \right) k i_\beta \partial_\beta - \partial_\alpha \left( \frac{1}{M} u_\alpha \cdot \right),
\end{align*}
\]

and
\[
\begin{align*}
\mathbf{A}^{11}_{12} &= -\frac{i\omega}{\eta b} R^f \left( b_\alpha i_\alpha^t i_\alpha^{-t} - i_\alpha i_\alpha^{-t} k i_3 \iota^t_\gamma i_\gamma^t k i^t_\beta \right) k i_\beta \partial_\beta - \partial_\alpha \left( \frac{1}{M} u_\alpha \cdot \right),
\end{align*}
\]

and
\( \mathcal{A}_{11}^{21} = 0^t, \quad (D.35) \)
\( \mathcal{A}_{11}^{22} = -\frac{1}{b} i_3 k^{-1} i_3\beta i_3 k, \quad (D.36) \)
\( \mathcal{A}_{12}^{11} = i\omega R - \frac{1}{i\omega} \partial_\alpha (U_{\alpha\beta} \partial_{\beta}) - \frac{\omega^2}{\eta b} R^{f} i_3 i_3 k (b I - i_3 i_3 k^{-1} i_3 i_3 k) R^{f} \)
\( = i\omega R - \frac{1}{i\omega} \partial_\alpha (U_{\alpha\beta} \partial_{\beta}) - \frac{\omega^2}{\eta b} R^{f} i_3 i_3 k (b i_3 i_3 - i_3 i_3 k^{-1} i_3 i_3 k) R^{f} \)
\( = i\omega R - \frac{1}{i\omega} \partial_\alpha (U_{\alpha\beta} \partial_{\beta}) - \frac{\omega^2}{\eta b} R^{f} i_3 i_3 k (b - \frac{1}{b} k i_3 i_3 k^{-1} i_3 i_3 k) i_3 i_3 k R^{f}, \quad (D.37) \)
\( \mathcal{A}_{12}^{12} = \frac{i\omega}{b} R^f (i_3 b + i_3 i_3 k i_3 i_3 k^{-1} i_3) = \frac{i\omega}{b} R^f (i_3 i_3 + i_3 i_3 k i_3 i_3 k^{-1} i_3), \quad (D.38) \)
\( \mathcal{A}_{12}^{21} = \frac{i\omega}{b} i_3 k^{-1} i_3 i_3 k R^{f}, \quad (D.39) \)
\( \mathcal{A}_{12}^{22} = -\frac{\eta i_3 k}{b} i_3 i_3 k, \quad (D.40) \)
\( \mathcal{A}_{11}^{21} = i\omega C_{33}^{-1}, \quad (D.41) \)
\( \mathcal{A}_{12}^{21} = -\frac{i\omega}{M} C_{33}^{-1} c_{3}, \quad (D.42) \)
\( \mathcal{A}_{21}^{21} = -\frac{i\omega}{M} c_{3} C_{33}^{-1}, \quad (D.43) \)
\( \mathcal{A}_{21}^{22} = \frac{i\omega}{M} C_{33}^{-1} c_{3} + \frac{i\omega}{M} \partial_\alpha \left( \frac{1}{\eta} (i_3 i_3 k i_3 - \frac{1}{b} i_3 i_3 k^{-1} i_3 i_3 k) \partial_{\beta} \right), \quad (D.44) \)
\( \mathcal{A}_{22}^{11} = -C_{33}^{-1} C_{33} \partial_{\beta}, \quad (D.45) \)
\( \mathcal{A}_{22}^{21} = 0, \quad (D.46) \)
\( \mathcal{A}_{22}^{22} = -\frac{1}{M} u \partial_{\beta} - \partial_\alpha \left( \frac{i\omega}{\eta} i_3 i_3 k (I - \frac{1}{b} i_3 i_3 k^{-1} i_3 i_3 k) R^f \right), \quad (D.47) \)
\( \mathcal{A}_{22}^{22} = -\partial_\alpha \left( \frac{i\omega}{\eta} i_3 i_3 k i_3 i_3 k^{-1} i_3 \right). \quad (D.48) \)

For the special case of an isotropic medium we have

\[ c_{ijkl} = (K_G - \frac{2}{3} G_{fr}) \delta_{ij} \delta_{kl} + G_{fr} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (D.49) \]
\[ C_{ij} = C \delta_{ij}, \quad (D.50) \]
\[ \rho^b_{ij} = \rho^b \delta_{ij}, \quad (D.51) \]
\[ \rho^f_{ij} = \rho^f \delta_{ij}, \quad (D.52) \]
\[ k_{ij} = k \delta_{ij}. \quad (D.53) \]

\( G_{fr} \) is the complex frequency-dependent shear modulus of the framework of the grains when
the fluid is absent. The elastic parameters $K_G$, $C$ and $M$ are given by Pride (1994)

$$K_G = \frac{K_{fr} + \phi K_f + (1 + \phi)K_s \Delta}{1 + \Delta}, \quad \text{(D.54)}$$

$$C = \frac{K_f + K_s \Delta}{1 + \Delta}, \quad \text{(D.55)}$$

$$M = \frac{1}{\phi} \frac{K_f}{1 + \Delta}, \quad \text{(D.56)}$$

$$\Delta = \frac{K_f}{\phi (K_s)^2} ((1 - \phi)K_s - K_{fr}), \quad \text{(D.57)}$$

where $K_s$ and $K_f$ are the solid and fluid compression moduli and $K_{fr}$ is the compression modulus of the framework of the grains. These parameters can be expressed in terms of Biot’s parameters $A$, $N$, $Q$ and $R$ Biot (1956a,b), according to

$$K_G - \frac{2}{3} G_{fr} = A + 2Q + R, \quad \text{(D.58)}$$

$$G_{fr} = N, \quad C = \frac{Q + R}{\phi}, \quad M = \frac{R}{\phi^2}. \quad \text{(D.59)}$$

Hence,

$$(C_{jl})_{ik} = c'_{ijkl} = c_{ijkl} - \frac{C^2}{M} \delta_{ij} \delta_{kl} = S \delta_{ij} \delta_{kl} + N (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad \text{(D.60)}$$

$$S = A - \frac{Q^2}{R}, \quad \text{(D.61)}$$

$$c_j = C i_j, \quad \text{(D.62)}$$

$$R^b = \rho^b I, \quad \text{(D.63)}$$

$$R^f = \rho^f I, \quad \text{(D.64)}$$

$$k = k I. \quad \text{(D.65)}$$

With these substitutions, we obtain for the source vector

$$d = \begin{pmatrix} i \omega \rho^f \frac{k}{\eta} i_a f^f_{al} + f^b + \frac{1}{\omega} \partial_{\alpha} (U_{al} h^b_l) \\ f^f_a \\ C^{-1}_{33} C_{33} h^b_l \\ - \partial_{\alpha} (\frac{k}{\eta} f^f_{al}) + \frac{1}{M} u^t_t h^b_t + q \end{pmatrix} \quad \text{(D.66)}$$

and for the operator submatrices

$$A^{11}_{11} = - \partial_{\alpha} (C_{\alpha 3} C^{-1}_{33}), \quad \text{(D.67)}$$

$$A^{12}_{11} = -i \omega \rho^f \frac{k}{\eta} i_{\alpha} \partial_{\beta} - \partial_{\alpha} \left( \frac{1}{M} u_{\alpha} \right), \quad \text{(D.68)}$$

$$A^{21}_{11} = 0^t, \quad \text{(D.69)}$$

$$A^{22}_{11} = 0, \quad \text{(D.70)}$$
Using this in equations (D.31) and (D.32) we obtain

\[ \mathbf{A}_{12}^{11} = \begin{pmatrix} 0 & 0 & -\partial_1 \left( \frac{\rho}{\kappa} \right) & \frac{\rho}{\kappa} \partial_1 - \partial_1 \left( \frac{2CN}{MK} \right) \\ 0 & 0 & -\partial_2 \left( \frac{\rho}{\kappa} \right) & \frac{\rho}{\kappa} \partial_2 - \partial_2 \left( \frac{2CN}{MK} \right) \\ -\partial_1 & -\partial_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \mathbf{A}_{12}^{12} = \begin{pmatrix} i\omega (\rho^b - (\rho^f)^2) - \frac{1}{\kappa} (\partial_1 (\nu_1 \partial_1 \cdot \partial_2 (N \partial_2 \cdot \cdot \cdot)) + \partial_2 (N \partial_1 \cdot \partial_1 (\nu_2 \partial_2 \cdot \cdot \cdot)) \\ -\frac{1}{\kappa} (\partial_2 (\nu_2 \partial_2 \cdot \partial_1 (N \partial_1 \cdot \cdot \cdot)) + \partial_1 (N \partial_2 \cdot \partial_2 (\nu_1 \partial_1 \cdot \cdot \cdot)) \\ 0 & 0 & 0 & i\omega \rho^b \iota \omega \rho^f \iota \omega \rho^f \end{pmatrix}, \]

\[ \mathbf{A}_{21}^{11} = \begin{pmatrix} \frac{i\omega}{\kappa} & 0 & 0 & 0 \\ 0 & \frac{i\omega}{\kappa} & 0 & 0 \\ 0 & 0 & -\frac{i\omega C}{MK} & i\omega \left( \frac{C^2}{MK} + \frac{1}{\kappa^2} \right) \partial_0 \left( \frac{1}{\kappa \rho^b} \partial_0 \right) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \mathbf{A}_{22}^{11} = \begin{pmatrix} -\frac{\rho}{\kappa} \partial_1 & 0 & 0 & -\partial_1 \\ 0 & 0 & -\partial_2 & 0 \\ 0 & 0 & -\partial_2 & 0 \\ \partial_1 \left( \frac{\rho^f}{\kappa \rho^b} \right) - \frac{2CN}{MK} \partial_1 & \partial_2 \left( \frac{\rho^f}{\kappa \rho^b} \right) - \frac{2CN}{MK} \partial_2 & 0 & 0 \end{pmatrix}. \]
where

\[ \rho^E = \rho^E(x, \omega) = -\frac{\eta}{i\omega k}, \]  
(D.87)

\[ \nu_1 = \nu_1(x, \omega) = 4N\left(\frac{S + N}{K_c}\right), \]  
(D.88)

\[ \nu_2 = \nu_2(x, \omega) = 2N\left(\frac{S}{K_c}\right), \]  
(D.89)

\[ K_c = K_c(x, \omega) = S + 2N. \]  
(D.90)

**APPENDIX E: SEISMOELECTRIC WAVES**

In section 8 we introduced the following equations for seismoelectric wave propagation in an inhomogeneous, isotropic, dissipative, fluid-saturated porous solid:

\[ -i\omega \rho^b v^s - i\omega \rho^f i_j w_j - \partial_j \tau^b_j = f^b, \]  
(E.1)

\[ -i\omega \rho^f i_j v^s + \frac{\eta}{k}(w_i - L(\gamma^f_i E_0 + \delta_3, E_3)) + \partial_i p = f^f_i, \]  
(E.2)

\[ i\omega \tau^b_j + C_{ji} \partial_l v^s + i\omega C_{ji} i_j p = C_{ji} h^b_i, \]  
(E.3)

\[ -i\omega p + C_{ji} \partial_l v^s + M \partial_k w_k = C_{ji} h^b_i + M q, \]  
(E.4)

\[ -i\omega E_0 + \frac{\eta}{k} L \gamma^c w_\alpha + \partial_3 H_0 - \partial_3 H_3 = -J^c_0, \]  
(E.5)

\[ -i\omega E_3 + \frac{\eta}{k} L w_3 + \partial_3^i H_0 = -J^c_3, \]  
(E.6)

\[ -i\omega \mu H_0 + \partial_3 E_0 - \partial_1 E_3 = -J^m_0, \]  
(E.7)

\[ -i\omega \mu H_3 + \partial_3^2 E_0 = -J^m_3, \]  
(E.8)

with

\[ i_i = \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \end{pmatrix}. \]  
(E.9)

Equations (E.1) – (E.8) form the starting point for deriving a two-way wave equation for the quantities \( \tau^b_3, p, E_0, v^s, w_3 \) and \( H_0 \) in vector \( q \). These equations and the following derivation were first published in a Ph.D. thesis [Ranada Shaw (2005)], based on notes of the author. We
separate the derivatives in the \( x_3 \)-direction from the lateral derivatives, according to

\[
-\partial_3 \tau^b_3 = i \omega \rho^b v^s + i \omega \rho^f (i_\alpha w_\alpha + i_3 w_3) + \partial_\alpha \tau^b_\alpha + f^b, \tag{E.10}
\]

\[
\partial_3 p = i \omega \rho^f i_3^b v^s - \frac{\eta}{k} (w_3 - L E_3) + f^f_3, \tag{E.11}
\]

\[
\partial_3 E_0 = i \omega \mu H_0 + \partial_1 E_3 - J^m_0, \tag{E.12}
\]

\[
\partial_3 v^s = -C^{-1}_{33} \left( i \omega \tau^b_3 + i \omega \frac{C}{M} i_3 p + C_{33} \partial_3 v^s - C_{33} h^b_3 \right), \tag{E.13}
\]

\[
\partial_3 w_3 = \frac{i \omega}{M} p - \frac{C}{M} (i_3^b \partial_\beta v^s + i_3^b \partial_3 v^s) - \partial_\beta w_\beta + \frac{C}{M} i^b_3 h^b_3 + q, \tag{E.14}
\]

\[
\partial_3 H_0 = i \omega \mathcal{E} E_0 - \frac{\eta}{k} L \gamma_\alpha w_\alpha + \partial_2 H_3 - J^s_0. \tag{E.15}
\]

The field components \( \tau^b_\alpha, \ w_\alpha, \ E_3 \) and \( H_3 \) need to be eliminated. Using equations (E.2) and (E.3), we can eliminate the terms \( i_\alpha w_\alpha \) and \( \partial_\alpha \tau^b_\alpha \) from equation (E.10), yielding

\[
-\partial_3 \tau^b_3 = i \omega \rho^b v^s - i \omega \rho^f i_\alpha \left( -i \omega \rho^\frac{k}{\eta} i_\alpha^f v^s + \frac{k}{\eta} \partial_\alpha p - L \gamma^f_\alpha E_0 \right) + i \omega \rho^f i_3 w_3
\]

\[
-\frac{1}{i \omega} \partial_\alpha \left( C_{\alpha \beta} \partial_\beta v^s + C_{\alpha 3} \partial_3 v^s + i \omega \frac{C}{M} i_3 p \right) + f^b + i \omega \rho^f \frac{k}{\eta} i_\alpha f^f_\alpha + \frac{1}{i \omega} \partial_\alpha (C_{\alpha i} h^b_i), \tag{E.16}
\]

or, upon substitution of equation (E.13),

\[
-\partial_3 \tau^b_3 = \partial_\alpha \left( C_{\alpha 3} C_{33}^{-1} \tau^b_3 \right) - i \omega \rho^f \frac{k}{\eta} i_\alpha \partial_\alpha p - \frac{1}{i \omega} \partial_\alpha \left( \frac{i \omega}{M} u_\alpha p + U_{\alpha \beta} \partial_\beta v^s \right)
\]

\[+ i \omega \left( \rho^b I_3 + i \omega (\rho^f) \frac{k}{\eta} i_\alpha^f \right) v^s + i \omega \rho^f i_3 w_3 + i \omega \rho^f L i_\alpha \gamma^f_\alpha E_0
\]

\[+ f^b + i \omega \rho^f \frac{k}{\eta} i_\alpha f^f_\alpha + \frac{1}{i \omega} \partial_\alpha (U_{\alpha i} h^b_i), \tag{E.17}
\]

with \( I_3 \) being a \( 3 \times 3 \) identity matrix and

\[
U_{\alpha l} = C_{\alpha l} - C_{\alpha 3} C_{33}^{-1} C_{3 l}, \tag{E.18}
\]

\[
u_l = C(i_l - C_{l 3} C_{33}^{-1} i_3). \tag{E.19}
\]

Note that

\[
U^t_{\alpha \beta} = U_{\beta \alpha} \tag{E.20}
\]

on account of \( C^t_{ji} = C_{ij} \). Using equation (E.6), we eliminate \( E_3 \) from equation (E.11), yielding

\[
\partial_3 p = i \omega \rho^f i_3^b v^s - \frac{\eta}{k} \left( 1 - \frac{1}{i \omega \mathcal{E} k} L^2 \right) w_3 + \frac{1}{i \omega \mathcal{E} k} L \partial_1^2 H_0 + \frac{1}{i \omega \mathcal{E} k} L J^m_3 + f^f_3. \tag{E.21}
\]

Using equation (E.6), we eliminate \( E_3 \) from equation (E.12), yielding

\[
\partial_3 E_0 = \partial_1 \left( \frac{1}{i \omega \mathcal{E} k} L w_3 \right) + i \omega \mu H_0 + \partial_1 \left( \frac{1}{i \omega \mathcal{E} k} \partial_1^2 H_0 \right) - J^m_0 + \partial_1 \left( \frac{1}{i \omega \mathcal{E} k} J^s_3 \right). \tag{E.22}
\]
Using equations (E.13) and (E.2), we eliminate the terms $\partial_3 v^*$ and $\partial_\beta w_\beta$ from equation (E.14), according to
\[
\partial_3 w_3 = C \frac{C_{33}^{-1}}{M} \left( i \omega \tau_3^b + i \omega \frac{C_{33}^{-1}}{M} i_3 p \right) + \frac{i \omega}{M} p + \partial_\beta \left( \frac{k}{\eta} \partial_\beta p - L \gamma_\beta^l \varepsilon_0 - i \omega p^l \frac{k}{\eta} \gamma_\beta^l v^* \right) - \frac{1}{M} u_\beta^l \partial_\beta v^* + q + \frac{1}{M} u_\alpha^l h_\alpha^b - \partial_\beta \left( \frac{k}{\eta} f_\beta^l \right).
\]
(E.23)

Using equations (E.8) and (E.2), we eliminate $H_3$ and $w_\alpha$ from equation (E.15), yielding
\[
\partial_3 H_0 = L \gamma_\alpha \left( \partial_\alpha p - \frac{\eta}{k} L \gamma_\alpha^l \varepsilon_0 - i \omega p^l i_\alpha^l v^* \right) + i \omega \varepsilon \varepsilon_0 + \partial_2 \left( \frac{1}{i \omega \mu} \partial_2 \varepsilon_0 \right) - J_0^\alpha + \partial_2 \left( \frac{1}{i \omega \mu} J_3^m \right) - L \gamma_\alpha f_\alpha^l.
\]
(E.24)

Equations (E.17), (E.21), (E.22), (E.13), (E.23) and (E.24) can be cast in the form of two-way wave equation (13), with the two-way wave vector $q = q(x, \omega)$ and the two-way source vector $d = d(x, \omega)$ defined as
\[
q = \begin{pmatrix} -\tau_3^b \\ p \\ E_0 \\ v^* \\ w_3 \\ H_0 \end{pmatrix}, \quad d = \begin{pmatrix} f^b + i \omega p^l \frac{k}{\eta} i_\alpha f_\alpha^l + \frac{1}{i \omega} \partial_\alpha \left( U_{\alpha i} h_\alpha^b \right) \\ \frac{1}{i \omega} \frac{\eta}{k} L J_3^e + f_3^l \\ -J_0^m + \partial_1 \left( \frac{1}{i \omega} J_3^m \right) \\ C_{33}^{-1} C_{3i} h_\alpha^b \\ q + \frac{1}{i \alpha} u_\beta^l h_\alpha^b - \partial_\beta \left( \frac{k}{\eta} f_\beta^l \right) \\ -J_0^e + \partial_2 \left( \frac{1}{i \omega \mu} J_3^m \right) - L \gamma_\alpha f_\alpha^l \end{pmatrix}
\]
(E.25)

and the two-way operator matrix $A = A(x, \omega)$ having the form of equation (76), with
\[
A_{11} = \begin{pmatrix} A_{11}^{11} & A_{11}^{12} & A_{11}^{13} \\ A_{11}^{12} & A_{11}^{22} & A_{11}^{23} \\ A_{11}^{13} & A_{11}^{23} & A_{11}^{33} \end{pmatrix}, \quad A_{12} = \begin{pmatrix} A_{12}^{11} & A_{12}^{12} & A_{12}^{13} \\ A_{12}^{12} & A_{12}^{22} & A_{12}^{23} \\ A_{12}^{13} & A_{12}^{23} & A_{12}^{33} \end{pmatrix},
\]
(E.26)
\[
A_{21} = \begin{pmatrix} A_{21}^{11} & A_{21}^{12} & A_{21}^{13} \\ A_{21}^{12} & A_{21}^{22} & A_{21}^{23} \\ A_{21}^{13} & A_{21}^{23} & A_{21}^{33} \end{pmatrix}, \quad A_{22} = \begin{pmatrix} A_{22}^{11} & A_{22}^{12} & A_{22}^{13} \\ A_{22}^{12} & A_{22}^{22} & A_{22}^{23} \\ A_{22}^{13} & A_{22}^{23} & A_{22}^{33} \end{pmatrix},
\]
(E.27)

where
\[
A_{11}^{11} = -\partial_\alpha \left( C_{\alpha 3} C_{33}^{-1} \right),
\]
(E.28)
\[
A_{11}^{12} = -i \omega p^l \frac{k}{\eta} i_\alpha \partial_\alpha - \partial_\alpha \left( \frac{1}{M} u_\alpha \right),
\]
(E.29)
\[
A_{11}^{13} = i \omega p^l L i_\alpha \gamma_\alpha^l.
\]
(E.30)
\[ A_{12}^{11} = -\frac{1}{i\omega} \partial_3 (U_{\alpha\beta} \partial_{\beta} \cdot) + i\omega \left( \rho^2 \textbf{I}_3 + i\omega (\rho^f)^2 \frac{k}{\eta} i_\alpha i_\alpha \right), \] (E.31)

\[ A_{12}^{12} = \rho \rho^f i_3, \] (E.32)

\[ A_{12}^{21} = i\omega \rho^f i_3, \] (E.33)

\[ A_{12}^{22} = -\frac{\eta}{k} \left( 1 - \frac{1}{i\omega} \frac{\eta}{k} L^2 \right), \] (E.34)

\[ A_{12}^{23} = \frac{1}{i\omega E} \eta L \theta_1, \] (E.35)

\[ A_{12}^{32} = \partial_1 \left( \frac{1}{i\omega E} \frac{\eta}{k} L \cdot \right), \] (E.36)

\[ A_{12}^{33} = i\omega I_2 + \partial_1 \left( \frac{1}{i\omega E} \theta_1 \cdot \right), \] (E.37)

\[ A_{12}^{11} = \omega C^{-1}_{-33}, \] (E.38)

\[ A_{12}^{12} = -i\omega \frac{C}{M} C^{-1}_{-33} i_3, \] (E.39)

\[ A_{12}^{21} = -i\omega \frac{C}{M} i_3^t C^{-1}_{-33}, \] (E.40)

\[ A_{12}^{22} = i\omega \frac{C^2}{M^2} i_3^t C^{-1}_{-33} + i\omega \frac{1}{M} + \partial_3 \left( \frac{k}{\eta} \partial_{\beta} \cdot \right), \] (E.41)

\[ A_{12}^{23} = -\partial_3 (L \gamma_{\beta} \cdot), \] (E.42)

\[ A_{12}^{32} = L \gamma_{\alpha} \partial_{\alpha}, \] (E.43)

\[ A_{12}^{33} = i\omega E I_2 + \partial_2 \left( \frac{1}{i\omega \mu} \theta_2 \cdot \right) - \frac{\eta}{k} L^2 \gamma_{\alpha} \gamma_{\alpha} \]

\[ = \left( i\omega E - \frac{\eta}{k} L^2 \right) I_2 + \partial_2 \left( \frac{1}{i\omega \mu} \theta_2 \cdot \right), \] (E.44)

\[ A_{22}^{11} = -C^{-1}_{-33} C_{33 \beta} \partial_{\beta}, \] (E.45)

\[ A_{22}^{12} = -\partial_{\beta} \left( i\omega \rho^f \frac{k}{\eta} i_\beta \cdot \right) - \frac{1}{M} u_{\beta} \partial_{\beta}, \] (E.46)

\[ A_{22}^{21} = -i\omega \rho^f L \gamma_{\alpha} i_\alpha, \] (E.47)

where \( I_2 \) is a \( 2 \times 2 \) identity matrix. The submatrices that have not been listed here are zero.

Note that matrices \( A_{11}, A_{12}, A_{21} \) and \( A_{22} \) can be explicitly written as

\[
A_{11} = \begin{pmatrix}
0 & 0 & -\partial_1 \left( \frac{k}{\eta} \right) & \frac{\rho^f}{\rho^e} \partial_1 & -\partial_1 \left( \frac{2CN}{MK} \right) & i\omega \rho^f L & 0 \\
0 & 0 & -\partial_2 \left( \frac{k}{\eta} \right) & \frac{\rho^f}{\rho^e} \partial_2 & -\partial_2 \left( \frac{2CN}{MK} \right) & 0 & i\omega \rho^f L \\
-\partial_1 & -\partial_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\] (E.48)
\[ A_{12} = \begin{pmatrix}
\frac{i\omega}{\rho} - \frac{\omega_i^2}{\rho E} & -\frac{1}{\rho E} (\partial_1 (\nu_1 \partial_1 \cdot) + \partial_2 (N \partial_2 \cdot)) & -\frac{1}{\rho E} (\partial_2 (N \partial_1 \cdot) + \partial_1 (\nu_2 \partial_2 \cdot)) \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{i\omega \rho^b}{\rho E} & i\omega \rho^f (1 + \frac{\rho E}{\rho E L}^2) & \frac{\rho E}{\rho E L} \partial_1 \\
\frac{i\omega \rho^f}{\rho E} & -\partial_1 (\rho E \partial_1 \cdot) & \frac{i\omega \rho^f}{\rho E} \partial_2 \\
0 & i\omega \mu - \frac{1}{\rho E} \partial_1 (\frac{1}{\rho E} \partial_1 \cdot) & -\frac{1}{\rho E} \partial_1 (\frac{1}{\rho E} \partial_2 \cdot) \\
0 & -\frac{1}{\rho E} \partial_2 (\frac{1}{\rho E} \partial_1 \cdot) & i\omega \mu - \frac{1}{\rho E} \partial_2 (\frac{1}{\rho E} \partial_2 \cdot)
\end{pmatrix}
\]

(E.49)

\[ A_{21} = \begin{pmatrix}
\frac{i\omega E}{\rho E} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{i\omega E}{\rho E} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{i\omega E}{\rho E} & -\frac{i\omega E}{\rho E} \partial_1 \partial_2 & 0 & 0 & 0 \\
0 & 0 & -\frac{i\omega E}{\rho E} \partial_1 & \frac{i\omega E}{\rho E} \partial_2 & -\partial_1 (L \cdot) & -\partial_2 (L \cdot) & 0 \\
0 & 0 & 0 & L \partial_1 & i\omega \mu - \frac{1}{\rho E} \partial_2 (\frac{1}{\rho E} \partial_2 \cdot) & -\frac{1}{\rho E} \partial_1 (\frac{1}{\rho E} \partial_1 \cdot) & 0 \\
0 & 0 & 0 & L \partial_2 & -\frac{1}{\rho E} \partial_1 (\frac{1}{\rho E} \partial_1 \cdot) & i\omega \mu - \frac{1}{\rho E} \partial_2 (\frac{1}{\rho E} \partial_2 \cdot) & 0
\end{pmatrix}
\]

(E.50)

\[ A_{22} = \begin{pmatrix}
0 & 0 & \partial_1 (\rho E \partial_1 \cdot) & \partial_2 (\rho E \partial_2 \cdot) & -\frac{i\omega \rho^f}{\rho E} \partial_1 & \frac{i\omega \rho^f}{\rho E} \partial_2 & 0 & 0 & 0 \\
0 & 0 & -\frac{i\omega \rho^f}{\rho E} \partial_1 & \frac{i\omega \rho^f}{\rho E} \partial_2 & 0 & 0 & 0 & 0 & 0 \\
\partial_1 (\rho E \partial_1 \cdot) & \partial_2 (\rho E \partial_2 \cdot) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{i\omega \rho^f}{\rho E} \partial_1 & \frac{i\omega \rho^f}{\rho E} \partial_2 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(E.51)
where

\[ E = E(x, \omega) = \varepsilon - \frac{1}{i\omega}(\sigma - \frac{\eta}{k}L^2), \quad (E.52) \]
\[ \rho^E = \rho^E(x, \omega) = -\frac{\eta}{i\omega k}, \quad (E.53) \]
\[ \nu_1 = \nu_1(x, \omega) = 4N\left(\frac{S + N}{K_c}\right), \quad (E.54) \]
\[ \nu_2 = \nu_2(x, \omega) = 2N\left(\frac{S}{K_c}\right), \quad (E.55) \]
\[ S = S(x, \omega) = A - \frac{Q^2}{R}, \quad (E.56) \]
\[ K_c = K_c(x, \omega) = S + 2N. \quad (E.57) \]

**APPENDIX F: SYMMETRY PROPERTIES IN THE WAVENUMBER-FREQUENCY DOMAIN**

Symmetry properties of the two-way operator matrix in the space-frequency domain are discussed in section 2.3. Here we discuss symmetry properties of this operator in the wavenumber-frequency domain for a laterally invariant medium. We define the spatial Fourier transform of a space- and frequency-dependent quantity \( h(x, \omega) \) as

\[ \tilde{h}(k_H, x_3, \omega) = \int_A h(x, \omega)\exp(-ik_\alpha x_\alpha)d^2x, \quad (F.1) \]

where \( k_H = (k_1, k_2) \), with \( k_1 \) and \( k_2 \) being the horizontal wavenumbers, and where \( A \) is an infinite horizontal integration boundary. We use equation (F.1) to transform equation (13) to the wavenumber-frequency domain, according to

\[ \partial_3\tilde{q} = \tilde{A}\tilde{q} + \tilde{d}. \quad (F.2) \]

Lateral derivatives \( \partial_\alpha \) are replaced by \( ik_\alpha \). Hence, for the acoustic two-way wave equation, discussed in section 2.2, the two-way wave vector \( \tilde{q} = \tilde{q}(k_H, x_3, \omega) \) and two-way source vector \( \tilde{d} = \tilde{d}(k_H, x_3, \omega) \) are defined as

\[ \tilde{q} = \begin{pmatrix} \tilde{p} \\ \tilde{v}_3 \end{pmatrix}, \quad \tilde{d} = \begin{pmatrix} \frac{k_\alpha}{\omega}(l_{a3}^{-1}l_{33}\tilde{f}_i) \\ \frac{k_\alpha k_\beta}{\omega}(l_{a3}^{-1}l_{33}l_{3\beta}) + \tilde{q} \end{pmatrix} \quad (F.3) \]

and the two-way operator matrix \( \tilde{A} = \tilde{A}(k_H, x_3, \omega) \) as

\[ \tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} = \begin{pmatrix} -ik_\alpha l_{a3}l_{33}^{-1} & i\omega l_{33}^{-1} \\ i\omega + \frac{k_\alpha k_\beta}{\omega}(l_{a3}^{-1}l_{33}l_{3\beta}) & -ik_\alpha l_{a3}l_{33}^{-1} \end{pmatrix}. \quad (F.4) \]
This matrix obeys the following symmetry relations

\[
\hat{A}^t(-k_H, x_3, \omega)N = -N\hat{A}(k_H, x_3, \omega), \quad \text{(F.5)}
\]

\[
\hat{A}^*(-k_H, x_3, \omega)J = J\hat{A}(k_H, x_3, \omega), \quad \text{(F.6)}
\]

\[
\hat{A}^t(k_H, x_3, \omega)K = -K\hat{A}(k_H, x_3, \omega). \quad \text{(F.7)}
\]

For a laterally invariant medium, equations \([\text{F.5}} - \text{F.7})\) are the Fourier transforms of equations \((39) - (41)\). Hence, equations \((\text{F.5}} - \text{F.7})\) are the unified symmetry relations for laterally invariant media for all wave phenomena considered in this paper.

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