A generalization of the thermodynamic uncertainty relation to periodically driven systems

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Abstract
The thermodynamic uncertainty relation expresses a universal trade-off between precision and entropy production, which applies in its original formulation to current observables in steady-state systems. We generalize this relation to periodically time-dependent systems and, relatedly, to a larger class of inherently time-dependent current observables. In the context of heat engines or molecular machines, our generalization applies not only to the work performed by constant driving forces, but also to the work performed while changing energy levels. The entropic term entering the generalized uncertainty relation is the sum of local rates of entropy production, which are modified by a factor that refers to an effective time-independent probability distribution. The conventional form of the thermodynamic uncertainty relation is recovered for a time-independently driven steady state and, additionally, in the limit of fast driving. We illustrate our results for a simple model of a heat engine with two energy levels.

Keywords: current fluctuations, heat engines, entropy production

(Some figures may appear in colour only in the online journal)

Introduction
One of the main objectives of stochastic thermodynamics is to relate thermodynamic properties of a small system to the statistical fluctuations of its currents, for example the mechanical work, dissipated heat or delivered chemical output [1, 2]. A recent development in this spirit is the thermodynamic uncertainty relation (TUR)
\[ \frac{D\sigma}{J^2} \geq k_B, \quad (1) \]

which relates the relative fluctuations of a current, characterized by its diffusion coefficient \( D \) and average \( J \), to the total rate of entropy production \( \sigma \) [3, 4]. In the following, we set Boltzmann’s constant \( k_B \) to unity.

The TUR (1) applies universally to current observables of steady-state systems that can be modeled in continuous time using time-independent Markovian dynamics, either on a discrete network or in continuous space [5–7]. While this covers already a large class of systems and observables, recent efforts to push the limits of applicability of the TUR even further have been fruitful, leading to variants for, e.g. finite time [8, 9] and first-passage time fluctuations [10, 11]. However, there are various settings of stochastic systems for which a direct application of the TUR fails, calling for modifications that generalize equation (1). Such settings include the discrete-time case [12–14], ballistic transport and coherent dynamics [15], and systems in linear response with asymmetric Onsager matrices [16].

In this paper, we focus on time-periodically driven systems as a further prominent setting for which the conventional form (1) of the TUR does not hold [17]. Roughly speaking, the driving protocol itself serves here as an exact external clock that can enable the currents of the system proper to reach a precision that surpasses the limit set by its rate of entropy production. Hence, the TUR can be restored by adding the thermodynamic cost for the external driving to the entropy production of the system proper [18]. Furthermore, systems driven by time-symmetric protocols show similarities to the discrete-time case, allowing for a generalization of the TUR in which the exponential of the entropy production per period enters [13]. Recent work on large deviation theory for arbitrary periodic driving has led to bounds on the large deviation function for current fluctuations [19–21], which generalize similar bounds that imply the TUR for time-independent driving [22].

Applied to molecular motors and steady-state heat engines, the TUR yields a fundamental bound on the efficiency, which depends only on the fluctuations of measurable currents [23, 24]. However, paradigmatic models and experimental realizations of stochastic heat engines often use externally controlled, time periodic protocols [25, 26], to which the TUR in its original formulation does not apply [27]. The generalizations of the TUR following from the large deviation bounds in [21] apply to current observables that count jumps in Markovian networks, which covers for example the cycle current generated by a stochastic pump [28, 29]. Instead, the current observables most relevant for heat engines are of a different type. In particular, the work performed on the system is given by the change of the energy of the state that is currently occupied by the system. The generalized thermodynamic uncertainty relation (GTUR) we derive here applies to a broad class of current observables in periodically driven systems, which includes the currents relevant for heat engines. In this generalization, the entropy production \( \sigma \) in equation (1) is replaced by an effective entropy production, which can be larger than \( \sigma \) and which depends on a comparison between the currents in the periodic stationary state and in a time-independent state of reference. We illustrate the GTUR and an implied generalized bound on the efficiency for a simple two-level heat engine that is alternatingly coupled to two different heat baths.

Setup

We consider a Markovian dynamics on a network of states with transition rates \( k_{ij}(t) = k_{ji}(t + T) \) from state \( i \) to \( j \) that are time-dependent and periodic with period \( T \). These rates must be thermodynamically consistent and thus have to obey the local detailed balance condition.
where $\beta(t)$ is a possibly time-dependent inverse temperature, $\Delta E(t) \equiv E_j(t) - E_i(t)$ the energy difference between internal states $i$ and $j$, and $A_{ij}(t)$ a driving affinity caused, e.g., by an external non-conservative force or a chemical reaction supplied by chemostats. These transition rates define a master equation

$$\dot{p}(t) = L(t)p(t),$$

where the dot denotes a time-derivative and where the periodic matrix $L(t) = L(t + T)$ has the entries

$$L_{ij}(t) \equiv k_{ij}(t) - \delta_{ij}r_i(t).$$

The entries $p_i(t)$ of vector $p(t)$ in (3) give the probability that state $i$ is occupied at time $t$. Furthermore, $r_i(t) \equiv \sum_j k_{ji}(t)$ is the time-dependent exit rate and $\delta_{ij}$ the Kronecker delta. This periodically driven system converges for long times into a periodic stationary state $p^*(t) = p^*(t + T)$, which is the unique periodic and normalized solution of (3).

A stochastic trajectory $\mathbf{i}(\tau)$ of length $\tau$ is characterized by an occupation variable $o_i(\tau)$, which is one if state $i$ is occupied at time $\tau$ and zero, otherwise. Note that we use a notation that distinguishes the state $i$ from the trajectory $\mathbf{i}(\tau)$ by the argument. The variable $m_{ij}(\tau)$ counts the directed total number of jumps from $i$ to $j$ observed up to time $\tau$. In contrast to steady-state systems, a current can also depend on the occupation $o_i(\tau)$ and not only on jumps $m_{ij}(\tau)$. As an example, consider work that is performed while driving the energy levels $E_i(\tau)$ without an external non-conservative force, analogously to the definition of work used in the Jarzynski relation [30]. The associated time-averaged power can be expressed through the occupation variable as

$$P_i[\mathbf{i}(\tau)] \equiv -\frac{1}{\tau} \int_0^{\tau} d\tau o_i(\tau) \dot{E}_i(\tau),$$

where we use the sign convention such that $P_i$ is positive when work is delivered on average by the system. In the following, such currents that only depend on the occupation are called ‘occupation currents’, whereas currents that only depend on jumps are called ‘jump currents’. An example for a jump current is the entropy production [2]

$$\sigma[\mathbf{i}(\tau)] \equiv \frac{1}{\tau} \int_0^{\tau} d\tau \sum_{ij} m_{ij}(\tau) \ln \left( \frac{p_{ij}^R(\tau) k_{ij}(\tau)}{p_{ji}^R(\tau) k_{ji}(\tau)} \right).$$

A general current consisting of two parts, an occupation current and a jump current, reads

$$j[\mathbf{i}(\tau)] \equiv j_{\text{occ}}[\mathbf{i}(\tau)] + j_{\text{jump}}[\mathbf{i}(\tau)] = \frac{1}{\tau} \int_0^{\tau} d\tau \sum_i o_i(\tau) \dot{a}_i(\tau) + \frac{1}{\tau} \int_0^{\tau} d\tau \sum_{ij} m_{ij}(\tau) d_{ij}(\tau),$$

where $\dot{a}_i(\tau)$ is the instantaneous change of a time-periodic state variable $a_i(\tau) = a_i(\tau + T)$ and $d_{ij}(\tau) = -d_{ji}(\tau) = d_{ij}(\tau + T)$ is the increment associated with a transition from $i$ to $j$ at time $\tau$. Averages of currents sampled over one period of the periodic stationary state can be expressed as

$$J \equiv \langle j[\mathbf{i}(\tau)] \rangle = \frac{1}{T} \int_0^T d\tau \sum_i p_i^R(\tau) \dot{a}_i(\tau) + \frac{1}{T} \int_0^T d\tau \sum_{ij} p_{ij}^R(\tau) d_{ij}(\tau).$$

\[\text{(8)}\]
where $\langle \cdot \rangle$ denotes the average over all trajectories in the periodic stationary state and
\[
\dot{f}_i^p(\tau) \equiv P_i^p(\tau)k_i(\tau) - \dot{p}_i^p(\tau)k_i(\tau)
\]
denotes the periodic stationary probability current. We have used that $\langle \dot{o}_i(\tau) \rangle = \dot{p}_i^p(\tau)$ and $\langle \dot{m}_i(\tau) \rangle = p_i^p(\tau)k_i(\tau)$. The average of the power (5) delivered while the system is in state $i$ is obtained for $\dot{a}_i(\tau) = \dot{E}_i(\tau)$ and reads
\[
P_i \equiv \frac{1}{T} \int_0^T d\tau \dot{p}_i^p(\tau)\dot{E}_i(\tau).
\]
(10)
The average of the fluctuating entropy production in (6) is obtained for $d_{ij}(\tau) = \ln[p_i^p(\tau)k_{ij}(\tau)/p_j^p(\tau)k_{ji}(\tau)]$, yielding the average rate of entropy production
\[
\sigma \equiv \langle \sigma[i(\tau)] \rangle = \frac{1}{T} \int_0^T d\tau \sum_{i>j} \dot{p}_i^p(\tau) \ln \left( \frac{p_i^p(\tau)k_i(\tau)}{p_j^p(\tau)k_j(\tau)} \right).
\]
(11)
Fluctuations of currents in the ensemble of trajectories $i(\tau)$ with $0 \leq \tau \leq t$ are quantified via the scaled cumulant generating function
\[
\lambda(z) \equiv \frac{1}{t} \ln \left( e^{z\langle i(\tau) \rangle} \right),
\]
(12)
which is in the following referred to as the ‘generating function’. Its long-time limit is $\lambda(z) \equiv \lim_{t \to \infty} \lambda_t(z)$. Denoting derivatives for $z$ with $'$, the average current follows as $J = \lambda'(0)$ and the diffusion coefficient associated with that current is given by
\[
D \equiv \lim_{t \to \infty} t \left( \langle j[i(\tau)] \rangle - \langle j[i(\tau)] \rangle \right)^2/2 = \lambda''(0)/2.
\]
(13)
The calculation of these quantities using time-ordered exponentials is sketched in appendix A.

**Main result**

Our main result generalizes the TUR (1) to systems driven into a periodic stationary state and is called in the following the generalized thermodynamic uncertainty relation (GTUR). It is valid for all currents defined in (7). The GTUR reads
\[
D\sigma_{\text{eff}}/J^2 \geq 1
\]
with the effective rate of entropy production
\[
\sigma_{\text{eff}} \equiv \frac{1}{T} \int_0^T d\tau \sum_{i>j} \left( \frac{\dot{f}_{ij}^{\text{eff}}(\tau)}{\dot{f}_i^p(\tau)} \right)^2 \sigma_{ij}^p(\tau).
\]
(15)
Here,
\[
\sigma_{ij}^p(\tau) = \dot{f}_{ij}^p(\tau) \ln \left( \frac{p_i^p(\tau)k_i(\tau)}{p_j^p(\tau)k_j(\tau)} \right)
\]
(16)
is the instantaneous periodic stationary entropy production rate associated with the link $ij$. The term $\dot{f}_{ij}^{\text{eff}}(\tau)$ is an effective current
\[
\dot{f}_{ij}^{\text{eff}}(\tau) \equiv p_i^{\text{eff}}k_i(\tau) - p_j^{\text{eff}}k_j(\tau)
\]
(17)
caused by a time-independent effective density \( p_i^{\text{eff}} \). It will in general not satisfy a conservation law. The effective density is a set of free variation parameters that have to fulfill the condition \( \sum_i p_i^{\text{eff}} = 1 \). For time-independent transition rates, the effective densities can be chosen as the stationary state \( p_i \). Then, the effective currents \( j_{ij}^{\text{eff}}(\tau) \) are the stationary ones and \( \sigma^{\text{eff}} = \sigma \).

Hence, (14) assumes the conventional form of the TUR.

For an exact experimental determination of \( \sigma^{\text{eff}} \), it is necessary to extract all phase-dependent probabilities and currents from a long trajectory that spans many periods. Nonetheless, a lower bound on \( \sigma^{\text{eff}} \) can be obtained from the measurement of the diffusivity \( D \) and average \( J \) of any accessible current of the system.

We emphasize that the bound (14) has a broader applicability than two earlier generalizations of the TUR. First, it is not restricted to time-symmetric driving as the one in [13]. Second, our generalization applies not only to currents with time-independent increments, which [21] focuses on. Consequently, as we will show below, our bound on precision is non-trivial for two-level systems, where all currents with time-independent increments must vanish. Interestingly, for time-dependent increments [21], provides a variant of the TUR that replaces not only the entropy production by a modified one but also the average current \( J \). However, this modified current is liable to become zero for the most relevant currents in heat engines.

Two different choices for \( p_i^{\text{eff}} \) have an immediate physical interpretation. The first choice is defined through

\[
\left( \frac{1}{T} \int_0^T d\tau L(\tau) \right) p_i^{\text{eff}} = 0
\]

as the stationary solution of the master equation with time-averaged transition rates. If the driving frequency \( \omega \equiv 2\pi/T \) is large compared to the entries of \( L(\tau) \), the periodic stationary state \( p_i^{\text{ps}}(\tau) \) converges to this effective density \( p_i^{\text{eff}} \), see appendix B.

The second choice for the variation parameters \( p_i^{\text{eff}} \) is a simple time average over the periodic stationary state

\[
p_i^{\text{eff}} = \frac{1}{T} \int_0^T d\tau p_i^{\text{ps}}(\tau),
\]

i.e. the average fraction of the total time spent in a state \( i \) during one period.

For these two choices, the corresponding relation (14) can be regarded as a genuine generalization of the conventional TUR, which is restored for time-independent transition rates, where \( p_i^{\text{eff}} = p_i^{\text{ps}} \) holds by construction. Physically, the effective entropy production (15) may be interpreted as a modification of the actual entropy production (11). This modification is mediated by the term \( (j_{ij}^{\text{eff}}(\tau)/j_{ij}^{\text{ps}}(\tau))^2 \), which encodes the ‘distance’ from a system in a time-independent state. In particular, for zero or small affinities \( A_{ij} \) the tendency of the system to relax towards an instantaneous stationary state reduces the absolute value of \( j_{ij}^{\text{ps}}(\tau) \) with respect to that of \( j_{ij}^{\text{eff}}(\tau) \) for most times \( \tau \) and links \( ij \), such that \( \sigma^{\text{eff}} > \sigma \) holds for the vast majority of possible driving protocols. Other choices for \( p_i^{\text{eff}} \), e.g. a uniform distribution, a delta distribution, or even a choice where some of the \( p_i^{\text{eff}} \) are negative are conceivable. However, such choices do generally not yield the conventional TUR for time-independent rates and therefore lack the interpretation of \( \sigma^{\text{eff}} \) being different from \( \sigma \) as an indicator for time-dependence.

The two choices (18) and (19) become equivalent in the limiting case of large driving frequencies \( \omega \) or for linear response around a genuine non-equilibrium steady state. In leading
order, as discussed in appendix B, the periodic stationary state \( p(\tau) \) is then time-independent and solves equation (18). Consequently, the currents \( j_{ps}(\tau) \) and \( j_{eff}(\tau) \) become the same, which leads to \( \sigma_{eff} = \sigma \) and thus restores the original form (1) of the TUR. However, in those limiting cases where both currents vanish in zeroth order, in particular in linear response around an equilibrium state, \( j_{ps}(\tau) \) and \( j_{eff}(\tau) \) differ in leading order and \( \sigma_{eff} \) remains different from \( \sigma \).

**Illustration: two level heat engine**

We consider a heat engine that is coupled alternatingly to two different heat baths. It has two states with one energy periodically driven, such that

\[
E_1(t) = 0 \quad \text{and} \quad E_2(t) = E \cos(\omega t) + \epsilon_0. \tag{20}
\]

Here, \( E \) is an amplitude and \( \epsilon_0 \) an offset with respect to the energy of the first state. In the first half of the period, \( 0 \leq t < T/2 \), the temperature \( \beta(t) \) is fixed at a cold inverse temperature \( \beta_c \) and in the second half, \( T/2 \leq t < T \), it is fixed at a hot inverse temperature \( \beta_h < \beta_c \). We choose the individual rates symmetrically according to the local detailed balance condition in (2) as

\[
k_{ij}(t) = k_0 \exp(-\beta(t) \Delta_{ij} E(t)/2), \tag{21}
\]

where \( k_0 \) determines the basic time scale for particle jumps. A schematic representation of the engine is shown in figure 1.

For the analysis shown in figure 2, we vary the rate amplitude \( k_0 \) and keep all other parameters fixed. The periodic stationary distribution yielding \( \sigma \), \( \sigma_{eff} \), and \( P \) and the diffusion constants \( D \) for the respective currents are calculated numerically using the methods outlined in appendix A. The left-hand side (l.h.s) of the GTUR (14) for the two choices in (18) and (19) as well as the l.h.s of the corresponding steady state TUR (1) are shown for the power (figure 2(a)) and for the entropy production (figure 2(b)) as currents of interest.

For small \( k_0 \), i.e. in the fast driving limit \( k_{ij}(t) \ll \omega \), the periodic stationary state approaches a time-independent state, as shown in figure 2(c). Then, the two choices for the GTUR and the TUR become identical for small \( k_0 \), as explained in appendix B. Differences between the two choices for \( p_{eff} \) can be seen for larger \( k_0 \). In this regime, the choice (18) becomes better than the choice (19) for both currents. Furthermore, the TUR for power is strongly violated for
large $k_0$. Here, the GTUR does hold and becomes sharper again. For the entropy production, the GTUR is less sharp for large $k_0$ where again the TUR does not hold.

**Bound on efficiency of heat engines**

The trade-off relation between power, efficiency and constancy, derived in [24] as a consequence of the TUR, applies to steady-state heat engines, but in general not to periodically driven systems [27]. The GTUR derived here generalizes this trade-off relation and bounds the efficiency of periodically driven heat engines as we show in the following. The formally similar trade-off described in [31] applies to periodically driven engines, but does not make reference to power fluctuations.

The efficiency of a heat engine is given by

$$\eta \equiv \frac{P}{\dot{Q}_m} \leq \frac{1}{C} = 1 - \frac{\beta_h}{\beta_c},$$

(22)
where \( P \equiv \sum_i P_i \) is the total output power of the heat engine defined in (10) and \( \dot{Q}_{\text{in}} \) is the heat current flowing into the system from the hot reservoir. This efficiency is always bounded by the Carnot efficiency \( \eta_C \). Following the analogous calculations from [24], the efficiency of a periodically driven heat engine \( \eta \) is bounded due to the GTUR (14) by the stronger relation

\[
\eta \leq \tilde{\eta}^p \equiv \frac{\eta_C}{1 + P\sigma / (\beta_k D_P \sigma_{\text{eff}})} \leq \eta_C,
\]

(23)

where \( D_P \) is the diffusion coefficient (13) of the fluctuating output power.

As an example, we consider the heat engine from figure 1 and vary the rate amplitude \( k_0 \). The effective entropy production \( \sigma_{\text{eff}} \) is calculated from the choice (18) for the effective density. The quantities entering the bound (23) are shown in figure 3(a) and the efficiency \( \eta \) of the heat engine and the bound \( \tilde{\eta}^p \) are shown in figure 3(b). This bound is compared to the naive bound valid for steady-state engines, called here \( \eta^s \) and defined just as \( \tilde{\eta}^p \), but with \( \sigma_{\text{eff}} \) replaced by the true entropy production \( \sigma \). For small rate amplitudes \( k_0 \ll \omega \), where the GTUR assumes the form of the TUR, the new bound \( \tilde{\eta}^p \) based on the GTUR becomes identical to \( \eta^s \). In the regime of large rate amplitudes, \( k_0 \gg \omega \), both the efficiency and the power increase to finite limiting values, see appendix B. Since at the same time fluctuations decrease, the bound \( \tilde{\eta}^p \) becomes rather strong with the actual efficiency being only about 25% below this bound, whereas the bound \( \eta^s \) no longer holds.

**Derivation**

We now derive our main result shown in (14). For this purpose, we bound the generating function by introducing an auxiliary dynamics with path weight \( \tilde{\mathcal{P}}[i(\tau)] \). A similar formalism has been introduced in [7] for continuous degrees of freedoms. The weight of paths from the periodic stationary state is denoted by \( \mathcal{P}[i(\tau)] \), so that

\[
P^p_0 (t) = \langle o_i(t) \rangle = \sum_{i(\tau)} \mathcal{P}[i(\tau)] o_i(t), \quad P^p_0 (t) k_0 (t) = \langle m_0 (t) \rangle = \sum_{i(\tau)} \mathcal{P}[i(\tau)] m_0 (t),
\]

(24)

where the summation indicates a path integral over all trajectories \( i(\tau) \), and where the occupation variable \( o_i(t) \) and the jump variable \( m_0 (t) \) refer implicitly to these trajectories. We split up \( \mathcal{P}[i(\tau)] \) as \( \mathcal{P}[i(\tau)] = \mathcal{P}[i(\tau)] | i_0 \mathcal{P}^0_0 (0) \), where \( \mathcal{P}[i(\tau)] | i_0 \) is the path weight conditioned in the system being in state \( i_0 \) at time \( t = 0 \), which in turn is associated with the probability \( p^0_0 (0) \). Likewise, for the path weight of the auxiliary dynamics, we split \( \tilde{\mathcal{P}}[i(\tau)] = \tilde{\mathcal{P}}[i(\tau)] | i_0 \tilde{p}^0_0 (0) \), with an a priori arbitrary initial distribution \( \tilde{p}^0_0 (0) \).

The generating function in (12) for a current \( \dot{j}[i(\tau)] \) can be written in terms of \( \mathcal{P}[i(\tau)] \) and \( \tilde{\mathcal{P}}[i(\tau)] \) as

\[
\lambda_0 (z) = \frac{1}{t} \ln \left( e^{zt \dot{j}[i(\tau)]} \right) = \frac{1}{t} \ln \left( \frac{\mathcal{P}[i(\tau)]}{\mathcal{P}[i(\tau)]} e^{zt \dot{j}[i(\tau)]} \right)_{\text{aux}}
\]

\[
= \frac{1}{t} \ln \left( \exp \left( z t \dot{j}[i(\tau)] \right) - \ln \left( \frac{\tilde{\mathcal{P}}[i(\tau)] | i_0 \tilde{p}^0_0 (0)}{\mathcal{P}[i(\tau)] | i_0} \right) \right)_{\text{aux}},
\]

(25)

where \( (\cdot)_{\text{aux}} \) denotes the average over all trajectories in the auxiliary dynamics. This generating function can be bounded by using Jensen’s inequality as
The second term in (26) can be written as a Kullback–Leibler divergence between the initial distribution $p^0(0)$ of the original periodic stationary distribution and the initial distribution $\tilde{p}(0)$ of the auxiliary dynamics

$$D(\tilde{p}(0)||p^0(0)) \equiv \sum_i \tilde{p}_i(0) \ln \left( \frac{\tilde{p}_i(0)}{p^0_i(0)} \right) \geq 0. \quad (31)$$
We evaluate the third term in (26) by calculating the fraction of the two path weights for the same trajectory \(i(\tau)\)

\[
\frac{\tilde{P}[i(\tau)|i_0]}{P[i(\tau)|i_0]} = \exp \left( \int_0^\tau d\tau \sum_0 \beta_0(\tau) \ln \left( \frac{\tilde{k}_j(\tau)}{k_0(\tau)} \right) - \sum_0 a_i(\tau) (\tilde{r}_i(\tau) - r_i(\tau)) \right),
\]

(32)

where \(\tilde{r}_i(\tau) \equiv \sum \tilde{k}_j(\tau)\) are the exit rates of the auxiliary dynamics. Inserting (32) into (26) leads to terms containing averages with the path weight \(\tilde{P}[i(\tau)]\) for \(a_i(\tau)\) and \(\tilde{m}_j(\tau)\), for which we can use (27). We express the rates in terms of the current and the associated traffic as

\[
\tilde{k}_j(t) = \left( \tilde{j}_j(t) + \tilde{r}_j(t) \right) / (2\tilde{p}_j(t)).
\]

(33)

After optimizing the third term in (26) with respect to the traffic, the optimal rates read

\[
\tilde{k}_{ij}^* = \left( \tilde{j}_{ij}(t) + \sqrt{\left( \tilde{j}_{ij}(t) \right)^2 + 4\tilde{p}_j(t)\tilde{p}_k(t)k_j(t)k_k(t)} \right) / (2\tilde{p}_j(t)).
\]

(34)

Finally, using these auxiliary rates and inserting (34) into (26) leads to a bound in terms of \(\tilde{j}(t) \equiv \{\tilde{j}_j(t)\}\) and \(\tilde{p}(t) \equiv \{\tilde{p}_j(t)\}\),

\[
\lambda_i(z) \equiv z \langle j[i(\tau)] \rangle_{\text{aux}} - \frac{1}{\tau} \int_0^\tau d\tau L \left( \tilde{p}(\tau), \tilde{j}(\tau) \right) - \frac{1}{t} D \left( \tilde{p}(0) || p^0(0) \right),
\]

(35)

with

\[
L \left( \tilde{p}(\tau), \tilde{j}(\tau) \right) \equiv \sum_{i>0} \tilde{j}_i(\tau) \left( \text{arsinh} \left( \frac{\tilde{j}_i(\tau)}{a^0_j(\tau)} \right) - \text{arsinh} \left( \frac{\tilde{p}_i(\tau)}{a^0_j(\tau)} \right) \right)
\]

\[
- \left( \sqrt{(a^0_j(\tau))^2 + (\tilde{j}_i(\tau))^2} - \sqrt{(a^0_j(\tau))^2 + (\tilde{p}_i(\tau))^2} \right)^2.
\]

(36)

where

\[
\tilde{f}_i^j(\tau) \equiv \tilde{p}_i(\tau)k_j(\tau) - \tilde{p}_j(\tau)k_i(\tau), \quad a^0_j(\tau) \equiv \sqrt{4\tilde{p}_j(\tau)\tilde{p}_k(\tau)k_j(\tau)k_k(\tau)}.
\]

(37)

The densities \(\tilde{p}(t)\) and currents \(\tilde{j}(t)\) of the auxiliary dynamics must fulfill the conditions

\[
\sum_i \tilde{p}_i(t) = 1, \quad \tilde{p}_i(t) > 0 \quad \text{and} \quad \tilde{j}_i(t) = -\sum_j \tilde{j}_{ij}(t)
\]

(38)

for all \(i\) and \(t\), which guarantee that a matching set of auxiliary transition rates \(\tilde{k}_j(t)\) can be found.

Now, we choose a suitable ansatz for the densities \(\tilde{p}(t)\) and currents \(\tilde{j}(t)\) of the auxiliary dynamics. One can easily verify that the ansatz

\[
\tilde{p}_i(t) = p_i^0(t) + \epsilon \left( p_i^0(t) - p_i^\text{eff} \right), \quad \tilde{j}_i(t) = j_i^0(t) + \epsilon j_i^\text{eff}(t)
\]

(39)

with an arbitrary small optimization parameter \(\epsilon = O(z)\) fulfills the conditions (38), if \(\sum p_i^\text{eff} = 1\). Using this ansatz, one can expand (35) up to order \(O(z^2)\) for small \(z\) to obtain a local bound on the generating function after an optimization with respect to the parameter \(\epsilon\). Additionally, we restrict ourselves to observation times \(t = nT\) that are multiples of the period. Then, (35) reads up to \(O(z^2)\).
\[
\lambda_{nT}(z) \geq z \left( J + z \frac{J^2}{2\sigma^{\text{eff}}(nT)} \right) + O(z^2),
\]
\[
\tilde{\sigma}^{\text{eff}}(nT) \equiv \frac{1}{T} \int_0^T d\tau \sum_{i \neq j} \left( \frac{\tilde{J}_{ij}(\tau)}{\tilde{J}_{ij}^{\text{eff}}(\tau)} \right)^2 + \frac{1}{nT} \sum_i \frac{(p_i^{\text{ps}}(0) - p_i^{\text{eff}}(0))^2}{p_i^{\text{ps}}(0)},
\] (40)

where \( \bar{P}^i(\tau) \) is the stationary traffic and \( J \) the stationary current (8). This is our strongest and most general result, holding for finite time after \( n \) periods, small values of \( z \approx 0 \) and currents defined in (7).

Using \( D_{nT} \equiv \lambda_{nT}'(0)/2 \) as a finite-time generalization of the diffusion coefficient (13), the local quadratic bound in (40) implies an inequality on precision for an arbitrary current as
\[
2D_{nT} \tilde{\sigma}^{\text{eff}}(nT)/J^2 \geq 1.
\] (41)

Using the inequality \( \tilde{J}_{ij}^{\text{ps}}(\tau) \leq 2\tilde{J}_{ij}^{\text{ps}}(\tau)^2/\sigma^{\text{ps}}(\tau) \), one obtains the bound given in (14) with an additional term arising from the Kullback–Leibler divergence. Taking the long-time limit \( n \to \infty \), one obtains exactly the GTUR in equation (14) with the effective entropy production (15).

As an aside, we note that in the case of time-independent rates the ansatz (39) becomes \( \bar{P}_i = P_i^0, \tilde{J}_{ij} = (1 + \epsilon)J_{ij} \), where the upper index ‘\( s \)’ denotes the stationary distribution and currents. Using a quadratic bound [4, 32] on \( L(\bar{P}(\tau), \tilde{J}(\tau)) \) in (36), and performing an optimization with respect to \( \epsilon \) leads to the quadratic bound on \( \lambda(z) \), which implies the finite-time TUR [8, 9]. In the long-time limit \( t \to \infty \), this lower bound on the generating function \( \lambda(z) \) becomes equivalent to the upper bound on the large deviation function [4, 22]. In [21], such a quadratic bound on \( L(\bar{P}(\tau), \tilde{J}(\tau)) \) has led to a global bound on the large deviation function for jump currents in a periodically driven system. The corresponding local bound, though formally similar, is different from the GTUR derived here.

Unlike most variants of the TUR, the present generalization (14) is not a consequence of a simple, usually quadratic, global bound on the large deviation function or generating function. Technically, choosing small \( z \) and consequently small \( \epsilon \) in equation (40) is necessary to ensure that the specific ansatz (39) for the density is positive. From a more general perspective, we note that the fluctuations of occupation currents are always limited to a finite range that is set by those realizations of \( \sigma_i(\tau) \) that maximize or minimize \( J_{\text{occ}}[i(\tau)] \) in equation (7), which rules out the existence of any global quadratic upper bound on the large deviation function.

Finally, the local quadratic bound in (40) is valid for small enough \( z \) and also at finite time. This leads to equation (41) as a generalization of the finite-time uncertainty relation [8, 9] to periodically driven systems. Here, the Kullback–Leibler divergence (31), which leads to the second term of \( \tilde{\sigma}^{\text{eff}}(nT) \), does not vanish. This term quantifies the difference between the initial distribution of the periodic stationary state and an effective time-independent distribution. For large driving frequencies the periodic stationary state \( \bar{P}^{\text{ps}}(t) \) converges to an effective density \( \bar{P}^{\text{eff}} \), as we show in appendix B, and hence the Kullback–Leibler divergence vanishes. Moreover, the Kullback–Leibler divergence can be brought to vanish by choosing \( \bar{P}^{\text{eff}} = \bar{P}^{\text{ps}}(0) \).

Conclusion

We have generalized the TUR to time-periodically driven systems and to a larger class of current observables. This class includes the currents most relevant for periodically driven heat
engines, in particular the work associated with changing the energy level of a state occupied by the system.

Our generalization restores the ordinary form of the TUR for the special case of large driving frequencies. Hence, for large driving frequencies precision has a universal minimal cost. This is somewhat remarkable, because although the system can be described by a time-independent distribution a one-to-one mapping of a periodically driven system to a steady-state system fails at the description of currents. Thus, we extend the applicability of the TUR and the ensuing trade-off relations to heat engines driven solely by fast alterations of energy levels and temperature.

For moderate or low driving frequencies one has to compare the periodic stationary currents with the associated effective currents in a time-independent state of reference. One can then predict whether a larger entropy production or smaller currents than the effective ones are needed for a higher precision. Furthermore, due to the generalization of the TUR, one can bound the efficiency of heat engines and hence predict whether an engine is able to work close to Carnot efficiency or not. Finally, we note that in a setting where \( \beta_c = \beta_h = 1 \), the bound on efficiency for heat engines can be adapted to isothermal engines transforming work to heat or work to work.

A formulation of the GTUR for overdamped Brownian motion is straightforward, either by performing the continuum limit on a finely discretized state space or by redoing the derivation using the path weights pertaining to Langevin dynamics.

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**Appendix A. Calculation of cumulants**

The solution of the time-dependent master equation (3) for an initial distribution \( p(0) \) is formally given by the time-ordered exponential

\[
p(t) = \exp \left( \int_0^t d\tau L(\tau) \right) p(0) \equiv M(t) p(0),
\]

defining the evolution operator \( M(t) \). There exists a unique initial condition \( p^{p_0}(0) \) that corresponds to the periodic stationary state \( p^{p_0}(t) \). This initial condition can be determined using the periodicity of \( p^{p_0}(t) \), which leads to the eigenvalue equation

\[
p^{p_0}(0) = M(T) p^{p_0}(0).
\]

Hence, this initial distribution \( p^{p_0}(0) \) is the eigenvector of \( M(T) \) with eigenvalue one.

Using standard methods, as explained for example in [20], the generating function (12) for the fluctuations of a general current observable (7) in the periodic stationary state is given by

\[
\lambda_{\tau}(z) = \frac{1}{t} \ln \sum_{i,j} \mathcal{M}_{ij}(t,z) p^{p_0}_j(0), \quad \mathcal{M}(t,z) \equiv \exp \left( \int_0^t d\tau L(\tau,z) \right).
\]

with the tilted evolution operator \( \mathcal{M}(t,z) \) and the tilted generator \( L(\tau,z) \) with entries \( \mathcal{L}_{ij}(\tau,z) \equiv L_{ij}(\tau) \exp(zd_{ij}(\tau)) + \delta_{ij} \dot{a}_i(\tau) z \). In the long-time limit, the generating function follows as
\[ \lambda(z) = [\ln \text{eig}(\mathcal{M}(T, z))] / T \]  

(A.4)

with \( \text{eig}(\mathcal{M}(T, z)) \) being the maximal eigenvalue of \( \mathcal{M}(T, z) \). For the illustration of the GTUR and the TUR for the two-level heat engine, we have calculated \( \lambda(z) \) in a small region around \( z = 0 \), yielding \( D \) through numerical differentiation.

**Appendix B. Limiting cases**

In the limit of fast driving, where \( k_j(t) \ll \omega \) for all transition rates at all times, the time-ordered exponential in (A.1) can be expanded as

\[ \mathcal{M}(T) = 1 + \int_0^T d\tau \mathcal{L}(\tau) + \mathcal{O}((k/\omega)^2), \]  

(B.1)

where \( k \) stands generically for the scaling of all transition rates. The eigenvalue equation (A.2) is then given in leading order by (18). Then, the periodic stationary state \( p^{\text{st}}(\tau) \) is in leading order time-independent and given by the first choice of the effective density \( p^{\text{eff}} \), i.e. \( p^{\text{st}}(\tau) = p^{\text{eff}} + \mathcal{O}(k/\omega) \). Due to its time-independence, the leading order of \( p^{\text{st}}(\tau) \) is also captured by the second choice for \( p^{\text{eff}} \) (19). Consequently, the periodic stationary currents and the effective currents, while still being time-dependent, become equal in the leading zeroth order, i.e. \( j^{\text{st}}_i(\tau) = j^{\text{eff}}_i(\tau) + \mathcal{O}(k^2/\omega) \) and thus \( \sigma = \sigma^{\text{eff}} + \mathcal{O}(k^2/\omega) \) in equation (14).

Another special case where the original form of the TUR is restored is the one where the transition rates become time-independent and correspond to a genuine non-equilibrium steady state, which leads to non-zero stationary currents \( j^{\text{st}}_i = j^{\text{eff}}_i(\tau) \). Remarkably, this result goes beyond the classical statement of the TUR if the increments \( \dot{a}_i(\tau) \) and \( \dot{a}_i(\tau) \) are still periodically time-dependent. However, in the limiting case of time-independent transition rates that correspond to a non-driven system, the currents \( j^{\text{st}}_i(\tau) \) and \( j^{\text{eff}}_i(\tau) \) both vanish and differ in leading order, such that \( \sigma^{\text{eff}} \) does not approach \( \sigma \).

Finally, the limiting case \( k_j(t) \gg \omega \) with continuous protocols for \( E_i(\tau) \) and \( \beta(\tau) \) and in the absence of driving affinities presents the quasistatic limit, where the periodic stationary state

\[ p^{\text{st}}_i(\tau) = p^{\text{eq}}_i(\tau) + \mathcal{O}(\omega/k) \equiv \exp(-\beta(\tau)E_i(\tau))/Z(\tau) + \mathcal{O}(\omega/k) \]  

(B.2)

approaches an instantaneous equilibrium state normalized by the partition function \( Z(\tau) \). Since in the true equilibrium state corresponding to some point in time all currents would vanish, we obtain for the periodic stationary currents

\[ j^{\text{st}}_i(\tau) = j^{\text{st}}_i(\tau)k_j(\tau) - j^{\text{st}}_j(\tau)k_i(\tau) = \mathcal{O}(\omega), \]  

(B.3)

which is consistent with the condition \( \dot{p}^*_i(\tau) = -\sum_{j\neq i} j^{\text{st}}_j(\tau) = \mathcal{O}(\omega) \). Using this scaling in equation (10) yields that the power scales like \( \mathcal{O}(\omega) \). The logarithm in equation (11) scales like \( \mathcal{O}(\omega) \), such that \( \sigma = \mathcal{O}(\omega^2/k) \). Note that in figures 2 and 3, the two discontinuous jumps in \( \beta(\tau) \) come with a finite production of entropy, leading to a dominant term in \( \sigma \) that scales like \( \mathcal{O}(\omega) \) and accordingly to an efficiency \( \eta = P/(P + \sigma) \) that is finite and less than one. The effective current (17) scales in leading order like \( \mathcal{O}(k) \), leading to \( \sigma^{\text{eff}} = \mathcal{O}(k^3/\omega) \) for the effective entropy production in equation (15). Fluctuations of the power, as quantified by \( D_p \), tend to zero in the quasistatic limit, since the many jumps in any typical trajectory average the power in equation (10) to always the same value [33]. A quantitative analysis of the decay of the correlation function of the occupation observables \( o_i(\tau) \) yields that \( D_p = \mathcal{O}(\omega^2/k) \).
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