The chromatic symmetric homology of every non-planar graph contains \( \mathbb{Z}_2 \)-torsion in bidegree (1,0)

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Abstract

In [4] the authors defined a categorification of Stanley’s chromatic function called the chromatic symmetric homology. In this paper we prove that, as conjectured in [2], if a graph \( G \) is non-planar, then its chromatic symmetric homology in bidegree (1,0) contains \( \mathbb{Z}_2 \)-torsion. Our proof follows a recursive argument based on Kuratowsky’s theorem.

1 Introduction

The chromatic symmetric function of a graph, defined by Stanley in [5], is a remarkable combinatorial invariant which refines the chromatic polynomial. Recently, in [4], Sazdanovic and Yip categorified this invariant by defining a new homological theory, called the chromatic symmetric homology of a graph \( G \). This construction, inspired by Khovanov’s categorification of the Jones polynomial [1], is obtained by assigning a graded representation of the symmetric group to every subgraph of \( G \), and a differential to every cover relation in the Boolean poset of subgraphs of \( G \). The chromatic symmetric homology is then defined as the homology of this chain complex; its bigraded Frobenius series \( \text{Frob}_G(q,t) \), when evaluated at \( q = t = 1 \), reduces to Stanley’s chromatic symmetric function expressed in the Schur basis.

As proved in [2], this categorification produces a truly stronger invariant: in other words, chromatic symmetric homology can distinguish couples of graphs that have the same chromatic symmetric function. Furthermore, in the same paper, the properties of chromatic symmetric homology with integer coefficients have been investigated. The authors of [2] provided examples of graphs whose chromatic symmetric homology has torsion, leaving open the following conjecture:

Conjecture 1. A graph \( G \) is non-planar if and only if its chromatic symmetric homology in bidegree (1,0) contains \( \mathbb{Z}_2 \)-torsion.

In this paper we prove a half of this conjecture, namely:

Theorem. Let \( G \) be a finite non-planar graph. Then its chromatic symmetric homology in bidegree (1,0) contains \( \mathbb{Z}_2 \)-torsion.

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Our strategy will be based on Kuratowski’s theorem: we will show that the torsion elements in the homology of the complete graph $K_5$ and of the complete bipartite graph $K_{3,3}$ are mapped to torsion elements in the homology of the graphs that are obtained from them by the operations of edge subdivision and graph inclusion, i.e. all the non-planar graphs.

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2 Computing $q$-degree zero homology

The reader can find a complete description of the construction of chromatic symmetric homology for a graph in the paper [4]. We limit ourselves to briefly recall how to compute homology in $q$-degree zero. We will denote by $C_i$ the $i$-th chain module and by $H_i$ the $i$-th homology. They correspond respectively to $C_{i,0}$ and $H_{i,0}$ in the notation of [4].

Let $G$ be a graph with $n$ vertices and $m$ edges. By Proposition 3.2 of [4], we can assume without loss of generality that, for each pair of vertices of $G$, there is at most one edge with them as endpoints. We denote by $(i, j)$ the edge incident to the vertices $i$ and $j$, and we order the set of edges $E(G)$ lexicographically. Each subset $S$ of $E(G)$ is naturally identified with a subgraph of $G$, having the same vertices as $G$ and $S$ as set of edges. The authors of [2, 3] call it a "spanning subgraph", while we will call it simply a subgraph, because the word "spanning" is often used with a different meaning in graph and matroid theory. The set of all subgraphs $G$ has a structure of Boolean lattice $B(G)$, ordered by reverse inclusion. In the Hasse diagram of $B(G)$, we direct an edge $\epsilon(F, F')$ from a subgraph $F$ to a subgraph $F'$ if and only if $F'$ can be obtained by removing an edge $e$ from $F$. The sign of $\epsilon$, $\text{sgn}(\epsilon)$, is defined as $(-1)^k$, where $k$ is the number of edges of $F$ less than $e$.

Let $F \subseteq E(G)$ a subgraph of $G$ with connected components $B_1, \ldots, B_r$. Then the module associated to it in $q$-degree zero is the permutation module

$$M_F = \text{Ind}_{\mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_r}}^{\mathfrak{S}_n} (S_{(b_1)} \otimes \cdots \otimes S_{(b_r)}),$$

where $\mathfrak{S}_n$ is the permutation group on $n$ elements and $S_{(i)}$ is the Specht module related to the partition $(i)$.

We define

$$C_i(G) = \bigoplus_{|F|=i} M_F,$$

where the sum is over the subgraphs of $G$ with $i$ edges. Therefore the $i$-th chain module $C_i(G)$ of the graph is a direct sum of $\binom{m}{i}$ permutation modules of $\mathfrak{S}_n$. 
If $\lambda$ is the partition whose parts are the sizes of the connected components of $F$, then

$$M_F \cong M_\lambda = C[\mathfrak{S}_n] \otimes C[\mathfrak{S}_\lambda] S_n.$$ 

Let $F$ and $F'$ be subgraphs of $G$ where $F' = F - e$. There is an edge map $d_{e(F,F')} : M_F \to M_{F'}$, defined in our case as the inclusion (for the general definition see [3]).

Finally, the $i$-th chain map $d_i : C_i(G) \to C_{i-1}(G)$ is defined as

$$d_i = \sum e \text{ sgn}(e) d_e,$$

where the sum is over all the edges $e$ in $B(G)$ which join a subgraph of $G$ with $i$ edges to a subgraph with $i - 1$ edges. Sometimes we will use the notation $d^G_i$, where it may not be clear which graph we are referring to.

We need to recall the following definitions from [2].

**Definition 2.** Let $F$ be a subgraph of $G$, and let $\lambda \vdash n$ be the partition whose parts are the sizes of the connected components of $F$. The numbering $T(F)$ associated to $F$ is the numbering of shape $\lambda$ such that each row consists of the elements in a connected component of $F$ arranged in increasing order, and rows of $T(F)$ having the same size are ordered so that the minimum element in each row is increasing down the first column.

Let $T = T(F)$. The $q$-degree zero permutation module $M_T$ associated to the numbering $T$ is cyclically generated by the Young symmetrizer

$$a_T = \sum_{\rho \in R(T)} \rho,$$

where $R(T) \leq \mathfrak{S}_n$ is the subgroup of permutations that permute elements within each row of $T$. We have:

$$M_F \cong M_T = C[\mathfrak{S}_n] \cdot a_T.$$

**Definition 3.** For any numberings $S$ and $T$ of shape $\lambda$, let

$$v^S_T = \sigma_T S \sigma_T a_T = b_S a_S \sigma_T S \in M_T,$$

where $a_T$ is as above,

$$b_T = \sum_{\xi \in C(T)} \text{ sgn}(\xi) \xi,$$

$C(T) \leq \mathfrak{S}_n$ is the subgroup of permutations that permute elements within each column of $T$, and $\sigma_T S \in \mathfrak{S}_n$ is such that $\sigma_T S \cdot T = S$.

Moreover, the $q$-degree zero Specht module $S_T$ associated to the numbering $T$ is cyclically generated by the Young symmetrizer $c_T = b_T a_T$:

$$S_T = C[\mathfrak{S}_n] \cdot c_T,$$

and $S_T \cong S_\lambda$.

We also recall the following result from [3], Section 7.2, Proposition 2.

**Proposition 4.** Let $F$ be a subgraph of $G$ with associated numbering $T$ of shape $\lambda$. Then

$$S_T = \text{span}\{b_S a_S \sigma_T S | S \in \text{SYT}(\lambda)\} = \text{span}\{v^S_T | S \in \text{SYT}(\lambda)\},$$

where $\text{SYT}(\lambda)$ is the set of standard Young tableaux of shape $\lambda$. 

3
2.1 Computation of $H_1(G)$

We will describe the chromatic homology in terms of Specht modules. Since each Specht module is cyclically generated, then our inclusion maps are completely determined by specifying the image of a cyclic generator for each Specht module. We now show how to achieve these computations systematically.

We restrict ourselves to Specht modules of type $\lambda = (2^k, 1^{n-k})$ for $k \geq 1$, so we will be computing

$$C_2(G)|_{S_\lambda} \xrightarrow{d_2} C_1(G)|_{S_\lambda} \xrightarrow{d_1} C_0(G)|_{S_\lambda} \to 0.$$ 

We order the edges of $G$ in lexicographic order and label these as $e_1, \ldots, e_m$. In homological degree zero, there is only one subgraph without edges. The chain group $C_0(G) = \mathbb{M}_F\emptyset \cong \mathbb{M}_{S_n}$ is the regular representation of $S_n$, where $F_\emptyset$ is the edgeless subgraph. By Corollary 1 in Section 7.3 of [3], the multiplicity of $S_\lambda$ in $C[S_n]$ is the number

$$f_\lambda = K_{\lambda, 1^n}$$

of standard Young tableaux of shape $\lambda$. We list the tableaux $Y_1(G), \ldots, Y_{f_\lambda}(G) \in \text{SYT}(\lambda)$ with respect to the following total order: if $T$ and $S$ are numberings of shape $\lambda$ such that the $i$-th row is the lowest row in which the numberings are different, the $j$-th column is the rightmost column in that row in which the numberings are different and $T(i, j) > S(i, j)$, then we say that $T > S$.

We have

$$C_0(G)|_{S_\lambda} = \bigoplus_{i=1}^{f_\lambda} \mathbb{C}[S_n] \cdot \nu_{Y_i}.$$ 

Example 5. Let $G = K_5$. We order the edges of $G$ in lexicographic order; that is,

$$(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5),$$

and label these as $e_1, \ldots, e_{10}$. The standard Young tableaux of shape $\lambda = (2^2, 1)$ listed with respect to the ordering defined earlier are

$$Y_1 = \begin{array}{ccc} 1 & 2 \\ 3 & 4 \\ 5 & \end{array}, \quad Y_2 = \begin{array}{ccc} 1 & 2 \\ 3 & 5 \\ 4 & \end{array}, \quad Y_3 = \begin{array}{ccc} 1 & 3 \\ 2 & 4 \\ 5 & \end{array}, \quad Y_4 = \begin{array}{ccc} 1 & 3 \\ 2 & 5 \\ 4 & \end{array}, \quad Y_5 = \begin{array}{ccc} 1 & 4 \\ 2 & 3 \\ 5 & \end{array}.$$ 

Then

$$C_0(K_5)|_{S_{(2^2, 1)}} = \bigoplus_{i=1}^{5} (\mathbb{C}[S_5] \cdot \nu_{Y_i}) \cong S_{(2^2, 1)}^5.$$ 

In homological degree one, there are $m$ subgraphs with exactly one edge, thus

$$C_1(G) = \bigoplus_{i=1}^{m} M_{F_{e_i}}.$$
If $F_{e_i}$ is the subgraph containing the edge $e_i = (p, q)$, then the permutation module $M_{F_{e_i}}$ has the associated numbering $T(F_{e_i})$ of shape $\mu = (2, 1^{n-2})$, and $M_{F_{e_i}} = C[\mathfrak{S}_n] \cdot (e + (pq)) \cong M_\mu$.

The multiplicity of $S_\lambda$ in $M_{F_{e_i}}$ is the number $K_{\lambda, \mu}$ of semistandard Young tableaux of shape $\lambda$ and weight $\mu$. We next obtain numberings of shape $\lambda$ that will index these $K_{\lambda, \mu}$ Specht modules $S_\lambda$, by standardizing the set SSYT$(\lambda, \mu)$ of semistandard Young tableaux of shape $\lambda$ and weight $\mu$ with respect to $T(F_{e_i})$ in the following way. For any numbering $S$, the word $w(S)$ of $S$ is obtained by reading the entries of the rows of $S$ from left to right, and from the top row to the bottom row (note that this is not the usual definition of a reading word for tableaux). So, given $Y \in$ SSYT$(\lambda, \mu)$, let $w(Y) = y_1, \ldots, y_n$ be the word of $Y$, let $w(T) = t_1, \ldots, t_n$ be the word of $T = T(F_{e_i})$ and let $\sigma$ be the permutation that orders $y_1, \ldots, y_n$ without exchanging $y_i$ and $y_j$ if $y_i = y_j$. From this we obtain a numbering $X$ of shape $\lambda$ by replacing the entry in $Y$ that corresponds to $y_k$ by $t_{\sigma(k)}$. We list the numberings $X^1_1(G), \ldots, X^K_{\lambda, \mu}(G)$ obtained using the procedure just described to SSYT$(\lambda, \mu)$ with respect to $T(F_{e_i})$. Observe that since $\mu = (2, 1^{n-2})$ and $\lambda = (2^k, 1^{n-2k})$ where $k \geq 1$, then this procedure guarantees that the first row of each numbering $X^1_1(G)$ is \[
= \[1, 1, \ldots, 1, 2, 3, \ldots, n\].\] So $v_{Y_j}^{X^1_1(G)} \in M_{F_{e_i}}$ and $C[\mathfrak{S}_n] \cdot v_{Y_j}^{X^1_1(G)} \cong S_\lambda$ for $j = 1, \ldots, K_{\lambda, \mu}$. Thus

$$C_1(G)|_{S_\lambda} = \bigoplus_{i=1}^{m} C[\mathfrak{S}_n] \cdot v_{Y_j}^{X^1_1(G)}$$

**Example 6.** Let $G = K_5$. There are 10 spanning subgraphs with exactly one edge. Furthermore, there are two semistandard Young tableaux of shape $\lambda = (2^2, 1)$ and weight $(2, 1^3)$,

$Z_1 = \begin{pmatrix} 4 & 4 \\ 2 & 3 \\ 1 & \end{pmatrix}$ \quad and \quad $Z_2 = \begin{pmatrix} 4 & 4 \\ 2 & 3 \\ \end{pmatrix}$

so the multiplicity of $S_{(2^2, 1)}$ in each $M_{F_{e_i}} \cong M_{(2^2, 1)}$ is 2. Let $X^1_1(K_5)$ and $X^2_1(K_5)$ denote the numberings which index the two copies of $S_{(2^2, 1)}$ in each $M_{F_{e_i}}$, again listed with respect to the same ordering. So

$$C_1(K_5)|_{S_{(2^2, 1)}} = \bigoplus_{i=1}^{10} (C[\mathfrak{S}_5] \cdot v_{Y_j}^{X^1_1(K_5)} \oplus C[\mathfrak{S}_5] \cdot v_{Y_j}^{X^2_1(K_5)}) \cong S_{(2^2, 1)}^{10}.$$  

Consider for instance the spanning subgraph $F_{e_1}$ of $K_5$ with the edge $e_1 = (1, 2)$ only. The numbering associated to it is

$$T(F_{e_1}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & \end{pmatrix}$$
Let $z_1, \ldots, z_4$ be the word of $Z$, $i = 1, 2$. The permutation that orders $z_1, \ldots, z_4$ is the identity and the one that orders $z_1, \ldots, z_2$ is $(45)$. Therefore we have

$$X_1^1 = \begin{array}{cc} 1 & 2 \\ 2 & 4 \\ 3 & 4 \end{array} = Y_1 \quad \text{and} \quad X_1^2 = \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} = Y_2,$$

then

$$v_{X_1^1}^{Y_1} = v_{X_1^2}^{Y_1} \quad \text{and} \quad v_{X_1^1}^{Y_1} = v_{X_1^2}^{Y_2}.$$

Lastly, we consider the chain module in homological degree two. The subgraphs of $G$ with exactly two edges have connected components of partition type $(2^2, 1^{n-4})$ or $(3, 1^{n-3})$. We are only concerned with Specht modules of type $\lambda = (2^k, 1^{n-2k})$ with $k \geq 2$ necessarily and, since $\lambda \not\sim (3, 1^{n-3})$, then, by Corollary 1 in Section 7 of \cite{[3]}, $S_\lambda$ does not appear as a summand in a permutation module isomorphic to $M_{(3, 1^{n-3})}$. Hence, we only need to consider the subgraphs with connected components of partition type $(2^2, 1^{n-4})$.

So suppose $G$ has $h$ subgraphs whose connected components has partition type $\nu = (2^2, 1^{n-4})$. List these subgraphs with respect to the lexicographic order of their edge sets. Suppose $F_{e_i, e_j}$ is the subgraph that contains the edges $e_i = (p, q)$ and $e_j = (r, s)$ with $p < r$. The permutation module $M_{F_{e_i, e_j}}$ has the associated numbering $T(F_{e_i, e_j})$ of shape $\nu$ and $M_{F_{e_i, e_j}} = C[S_n] \cdot (e + (pq))(e + (rs)) \cong M_\nu$.

Similar to the previous case for $C_1(G)$, the multiplicity of $S_\lambda$ in $M_{F_{e_i, e_j}}$ is $K_{\lambda, \nu}$. We list the numberings $W^1_{i,j}(G), \ldots, W^K_{i,j}(G)$ obtained using the procedure described above to $SSYT(\lambda, \nu)$ with respect to $T(F_{e_i, e_j})$. The procedure guarantees that the top two rows of each numbering $W$ are $\begin{array}{cc} 1 & 2 \\ 2 & 4 \\ 3 & 4 \end{array}$, so $v_{W}^{Y_1} \in M_{F_{e_i, e_j}}$ and $C[S_n] \cdot v_{W}^{Y_1} \cong S_\lambda$ for $i = 1, \ldots, K_{\lambda, \nu}$. Thus

$$C_2(G) | S_\lambda = \bigoplus_{i,j}^{K_{\lambda, \nu}} \bigoplus_{l=1}^{C[S_n]} C[S_n] \cdot v_{W}^{Y_1} \cong S_\lambda,$$

where the direct sum is over the values of $i$ and $j$ corresponding to the couples of edges which form subgraphs of type $\nu$, i.e. the non-consecutive edges.

**Example 7.** Let $G = K_5$. There are 15 spanning subgraph whose connected components have partition type $(2^2, 1)$. There is only one semistandard Young tableau of shape $\lambda = (2^2, 1)$ and weight $(2^2, 1)$:

$$\begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 \end{array},$$

so the multiplicity of $S_{(2^2, 1)}$ in each $M_{F_{e_i, e_j}} \cong M_{(2^2, 1)}$ is 1, and we let $W_{i,j}$ denote the numbering which indexes the copy of $S_{(2^2, 1)}$ in $M_{F_{e_i, e_j}}$. Therefore,
\[ C_2(K_5)|_{S_{(2,1)}} = \bigoplus (C[S_5] \cdot v_{Y_i}^Y) \cong S_{(2^2,4^1)}^{\oplus 15}, \]

where the direct sum is over the values of \(i\) and \(j\) corresponding to the couples of non-consecutive edges.

To compute the edge maps we will need the following theorem (Corollary 2.18 of [2]).

**Theorem 8.** For any numberings \(S\) of shape \(\lambda\),

\[ v^T_S = (-1)^j \sum_{U \in \Xi_{i,j}(S)} v^T_U, \]

where \(\Xi_{i,j}(S)\) is the set of all numberings \(U\) obtained from \(S\) by exchanging the first \(j\) entries in the \((i + 1)\)-th row of \(S\) with \(j\) entries in the \(i\)-th row of \(S\), preserving the order of each subset of elements.

We let \(\pi_{i,j}\) denote the operator on numberings such that

\[ \pi_{i,j}(S) = (-1)^j \sum_{U \in \Xi_{i,j}(S)} U. \]

**Example 9.** Let \(G = K_5\). We compute \(d_1(v_{X_1}^Y)\) and \(d_1(v_{X_3}^Y)\), so we consider the spanning subgraph \(F_{e_3}\) of \(K_5\) with the edge \(e_3 = (1, 4)\) only. The numbering associated to it is

\[ T(F_{e_3}) = \begin{array}{c}
1 \\
4 \\
2 \\
5
\end{array} \]

We have

\[ X_1^3 = \begin{array}{c}
1 \\
4 \\
2 \\
5
\end{array} \text{ and } X_3^3 = \begin{array}{c}
1 \\
4 \\
2 \\
5
\end{array} = Y_5. \]

We compute

\[ \pi_{1,1}(X_1^3) = \begin{array}{c}
1 \\
4 \\
3 \\
5
\end{array} \]

and

\[ \pi_{1,2}(X_3^3) = \begin{array}{c}
1 \\
3 \\
2 \\
5
\end{array} = Y_3. \]

Then, by Theorem 8, we have

\[ v_{X_1}^Y = -v_{Y_1}^Y - v_{Y_3}^Y \text{ and } v_{X_3}^Y = v_{Y_5}^Y. \]
and $d_1$ sends
\[ v_{X_3}^Y \mapsto -v_{X_1}^Y - v_{X_3}^Y \quad \text{and} \quad v_{X_3}^Y \mapsto v_{Y_3}^Y. \]

Now we compute $d_2(v_{W_1,8}^Y)$, so we have to consider the spanning subgraph $F_{e_1,e_8}$ of $K_5$ with the edges $e_1 = (1,2)$ and $e_8 = (3,4)$. The numbering associated to it is
\[
\begin{array}{cccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

There are two spanning subgraphs of $K_5$ with only one edge that can be obtained removing one edge from $F_{e_1,e_8}$, i.e. $F_{e_1}$ and $F_{e_8}$. The per-edge map $d_2(F_{e_1,e_8},F_{e_1})$ appears as a summand in $d_2$ with a minus sign; instead $d_2(F_{e_1,e_8},F_{e_8})$ appears in $d_2$ with a plus sign. Therefore, $d_2$ sends
\[ v_{W_1,8}^Y \mapsto -v_{X_1}^Y + v_{Y_1}^Y. \]

## 3 The case of non-planar graphs

In this section we will prove that if $G$ is a non-planar graph, then the chromatic symmetric homology $H_1(G;\mathbb{Z})$ contains $\mathbb{Z}_2$-torsion. We first recall two results from [2]:

**Lemma 10.** The chromatic symmetric homology $H_1(K_5;\mathbb{Z})$ contains $\mathbb{Z}_2$-torsion.

**Proof.** We compute
\[
\begin{array}{c}
C_2(K_5)|S_{(2,1)} \xrightarrow{d_2} C_1(K_5)|S_{(2,1)} \xrightarrow{d_1} C_0(K_5)|S_{(2,1)} \rightarrow 0,
\end{array}
\]
restricted to the $S_{(2,1)}$ modules.

Following the notation introduced in [2] let $g = W_{1,8} + W_{1,9} + W_{1,10} + W_{2,6} - W_{2,7} - W_{2,10} + W_{3,5} + W_{3,7} + W_{3,9} + W_{4,5} + W_{4,6} + W_{4,8} - W_{5,10} - W_{6,9} + W_{7,8} \in C_2(K_5)$ and $h = X_1^3 + X_1^{10} - X_2^3 + X_2^{10} + X_3^3 \in C_1(K_5)$. We have that $h \notin \text{im } d_2$, $d_2(g) = 2h$ and $d_1(h) = 0$, so $h$ generates $\mathbb{Z}_2$-torsion in $H_1(K_5;\mathbb{Z})$. For more details see [2], Theorem 4.1.

**Lemma 11.** The chromatic symmetric homology $H_1(K_{3,3};\mathbb{Z})$ contains $\mathbb{Z}_2$-torsion.

**Proof.** We compute
\[
\begin{array}{c}
C_2(K_{3,3})|S_{(2,1,2)} \xrightarrow{d_2} C_1(K_{3,3})|S_{(2,1,2)} \xrightarrow{d_1} C_0(K_{3,3})|S_{(2,1,2)} \rightarrow 0,
\end{array}
\]
restricted to the $S_{(2,1^2)}$ modules.

Following the notation introduced in Section 2.1 let $g' = W_{1,6} - W_{l,7} + W_{1,8} + W_{1,9} - W_{2,4} + W_{2,5} + W_{2,7} + W_{2,9} + W_{3,4} - W_{3,5} + W_{3,6} + W_{3,8} + W_{4,8} + W_{4,9} + W_{5,6} + W_{5,7} - W_{6,9} + W_{7,8} \in C_2(K_{3,3})$ and $h' = X_6^3 - X_3^3 + X_9^3 - X_9^2 \in C_1(K_{3,3})$. We have that $h' \notin \text{im} d_2, d_2(g') = 2h'$ and $d_1(h') = 0$, so $h'$ generates $\mathbb{Z}_2$-torsion in $H_1(K_{3,3}; \mathbb{Z})$. For more details see [2], Theorem 4.2.

\[ \square \]

**Theorem 12.** Let $G$ be a finite non-planar graph. Then $H_1(G; \mathbb{Z})$ contains $\mathbb{Z}_2$-torsion.

**Proof.** $G$ is non-planar, so it can be obtained recursively from $K_5$ or $K_{3,3}$ inserting vertices into edges and/or adding edges a finite number of times and/or adding isolated vertices. We assume that these operations are done one at a time, starting from the insertion of vertices into edges. In this way we get a finite sequence of graphs

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_{L+1} \subseteq G_{L+M} \subseteq \cdots \subseteq G_{L+M+N} = G,$$

where $G_0 = K_5$ or $G_0 = K_{3,3}$, $L \geq 0$ is the number of vertices added into edges, $M \geq 0$ is the number of edges added and $N \geq 0$ is the number of isolated vertices added. We also number the vertices of $K_5$ from 1 to 5 and the vertices added starting from 6, so that if the penultimate added was $i$, the last one is $i + 1$; and similarly for $K_{3,3}$.

Now we assume that $G_0 = K_5$. A quite similar argument applies if $G_0 = K_{3,3}$.

Using the notation of Lemma 10 we prove that $h$ is mapped in a $\mathbb{Z}_2$-torsion generator in $H_1(G_i; \mathbb{Z})$, for all $i > 0$.

Let’s consider first $G_1$. All the edges of $K_5$ are also edges of $G_1$, except for the edge that has been broken, let it be $(i, j)$, which is no longer an edge of $G_1$, but it has been replaced by two edges, $(i, 6)$ and $(6, j)$. To prove Lemma 10 the authors of [2] considered Specht modules correspondent to the partition $\lambda = (2^2, 1)$. Now we restrict ourselves to $S_{(2,1^2)}$, so to the partition $\lambda' = (2^2, 1^2)$, i.e. the partition $\lambda$ with an extra box at the bottom. We divide the standard Young tableaux of shape $\lambda'$ into two groups: those that are obtained simply by adding the box 6 at the bottom of the standard Young tableaux of shape $\lambda$ and those that don’t have the 6 in the last row. Similarly, of the three semistandard Young tableaux of shape $\lambda'$ and weight $(2, 1^4)$, two are obtained adding the box 5 at the bottom of the semistandard Young tableaux of shape $\lambda$ and weight $(2, 1^3)$. We match these to the $X_1^1(G_1)$’s and $X_2^2(G_1)$’s. If $i$ does not correspond to the two new edges, they are identical to $X_1^1(K_5)$ and $X_2^2(K_5)$ with the box 5 added at the bottom. Therefore, the differential $d_{G_1}^i$ acts on them in exactly the same way as $d_{K_5}^i$, since the $\pi_{i,j}$ operations don’t concern the fourth row. Since $h \in \ker d_{K_5}^i$, we also have that $h \in \ker d_{G_1}^i$.

**Remark 13.** We mean $h$ written as 1-chain in $G_1$. This can always be done using the $\pi_{i,j}$ operations and Theorem 8. For example, $X_3^3(K_5)$ with a box containing
6 added at the bottom, i.e. \( \begin{array}{c} 1 \ 2 \ 3 \\ 4 \ 5 \ 6 \end{array} \) a priori is not a 1-chain in \( G_1 \) since the edge 
\((1,5)\) is not an edge of \( G_1 \); but we have \( \begin{array}{c} 1 \ 5 \\ 2 \ 3 \\ 4 \ 6 \end{array} = \pi_{1,1} \begin{array}{c} 1 \ 5 \\ 2 \ 3 \\ 4 \ 6 \end{array} = \begin{array}{c} 2 \ 5 \\ 1 \ 3 \\ 4 \ 6 \end{array} - \begin{array}{c} 1 \ 2 \\ 3 \ 5 \\ 4 \ 6 \end{array} = -X_2^2(G_1) - X_1^2(G_1) \in C_1(G_1) \).

A similar argument applies to the \( W^a_{k,k} \)'s. Since, in \( K_5 \), \( d^G_{k} (g) = 2h \) and \( g \) can be written as 2-chain in \( G_1 \) with the \( \pi_{i,j} \) operations, we also have that \( d^G_{2} (g) = 2h \).

It remains to prove that \( h \notin \text{im} \ d^G_{1} \). If \( h \) belonged to \( \text{im} \ d^G_{1} \), since we know that \( h \notin \text{im} \ d^G_{2} \), it would be linear combination of the columns of \( d^G_{1} \) which are not in \( d^G_{2} \), but it’s not possible because \( h \) is a 1-chain in \( K_5 \).

Therefore, we have a \( \mathbb{Z}_2 \)-torsion generator in \( H_1(G_1, \mathbb{Z}) \). Now we add the vertex 7 to \( G_1 \). As before, all the edges of \( G_1 \) are also edges of \( G_2 \), except for the edge broken. We restrict ourselves to \( S_{(2,1^3)} \), so to the partition \( \lambda'' = (2^2, 1^3) \), i.e. the partition \( \lambda' \) with an extra box at bottom. We divide the standard Young tableaux of shape \( \lambda'' \) into two groups: those that are obtained simply by adding the box \( \begin{array}{c} 7 \end{array} \) at the bottom of the standard Young tableaux of shape \( \lambda' \) and those that don’t have the 7 in the last row. Similarly, of the semistandard Young tableaux of shape \( \lambda'' \) and weight \((2, 1^3)\), three are obtained adding the box \( \begin{array}{c} 8 \end{array} \) at the bottom of the semistandard Young tableaux of shape \( \lambda' \) and weight \((2, 1^4)\).

We match these to the \( X^1_1(G_2)'s \), \( X^2_1(G_2)'s \) and \( X^3_1(G_2)'s \). If \( i \) does not correspond to the two new edges, they are identical to \( X^1_1(G_1) \), \( X^2_1(G_1) \) and \( X^3_1(G_1) \) with the box \( \begin{array}{c} 7 \end{array} \) added at the bottom. Therefore, the differential \( d^G_{1} \) acts on them in exactly the same way as \( d^G_{1} \), since the \( \pi_{i,j} \) operations don’t concern the fifth row. Since \( h \in \ker d^G_{1} \), we also have that \( h \in \ker d^G_{2} \). Same thing for the \( W^a_{k,k} \)'s, therefore, since \( d^G_{2} (g) = 2h \), we also have \( d^G_{2} (g) = 2h \). Moreover, if \( h \) were trivial, we proved that \( h \notin \text{im} \ d^G_{2} \), it would be linear combination of the columns of \( d^G_{2} \) which are not in \( d^G_{1} \), but this is not possible because \( h \) is a 1-chain in \( G_1 \). So we have a \( \mathbb{Z}_2 \)-torsion generator in \( H_1(G_2, \mathbb{Z}) \).

If we add a vertex to \( G_2 \) what happens is completely analogous and so on up to \( G_{L} \). Then, to go from \( G_L \) to \( G_{L+1} \), there are two possibilities:

1) We add only an edge. In this case, the number of vertices doesn’t change, so there is no need to consider a "bigger" partition. Each edge of \( G_L \) is also an edge of \( G_{L+1} \), so \( h \in C_1(G_L) \) and \( g \in C_2(G_L) \) are also a 1-chain and a 2-chain in \( G_{L+1} \), and the differentials \( d^G_{L+1} \) and \( d^G_{2} \) act on them as \( d^G_{1} \) and \( d^G_{2} \). Moreover, \( h \) can’t be trivial because otherwise it would be trivial also in \( G_L \). Therefore, \( h \) is a \( \mathbb{Z}_2 \)-torsion generator also in \( H_1(G_{L+1}, \mathbb{Z}) \).
2) We add an edge and a vertex. In this case, we need to consider a partition with one more box at the bottom than the previous one. However, all the edges of $G_L$ are also edges of $G_{L+1}$, so all the $X_i^j(G_L)$'s with the new box added at the bottom are also $X_i^j(G_{L+1})$'s (the same for the $W_{h,k}^n(G_L)$'s).

Therefore, also in this case, $h$ is a $\mathbb{Z}_2$-torsion generator in $H_1(G_{L+1};\mathbb{Z})$.

Similarly up to $G_{L+M}$.

Finally, if we add $N$ isolated vertices, we have to consider a partition with $N$ more box at the bottom than the previous one. However, all the edges of $G_{L+M}$ are also edges of $G_{L+M+N}$, so all the $X_i^j(G_{L+M})$'s with the new boxs added at the bottom are also $X_i^j(G_{L+M+N})$'s (the same for the $W_{h,k}^n(G_{L+M})$'s). Therefore, also in this case, $h$ is a $\mathbb{Z}_2$-torsion generator in $H_1(G_{L+M+N};\mathbb{Z})$.

\[\square\]

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