Gales Sufﬁce for Constructive Dimension *

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Abstract  
Supergales, generalizations of supermartingales, have been used by Lutz (2002) to define the con-
structive dimensions of individual binary sequences. Here it is shown that gales, the corresponding 
generalizations of martingales, can be equivalently used to deﬁne constructive dimension.

1 Introduction

Effective martingales have been very useful objects in theoretical computer science. Schnorr [7, 8] used 
constructive martingales to give an equivalent deﬁnition of Martin-Löf randomness [6]. Martingales com-
putable within resource bounds have been used by Lutz [2] to deﬁne various resource-bounded measures 
that have been successful in complexity theory. In all these cases, it is known that replacing the con-
structive or resource-bounded martingales with constructive or resource-bounded supermartingales results in an 
equivalent deﬁnition.

Lutz [3] recently introduced supegales and gales as natural generalizations of supermartingales and 
martingales, respectively. He showed that gales can be used to characterize classical Hausdorff dimension. 
With this as a motivation, Lutz used gales computable within resource bounds to deﬁne resource-bounded 
dimensions that work inside of complexity classes. He also showed that supegales may be used in place of 
gales to give equivalent deﬁnitions of these dimensions.

Constructive dimension [5] reﬁnes the theory of Martin-Löf randomness by assigning each individual 
binary sequence a dimension. Lutz used constructive supegales to deﬁne constructive dimension. Supega-
les were used rather than gales because he was able to show that optimal constructive supegales exist. The 
questions of whether optimal constructive gales exist and whether gales can be used to equivalently deﬁne 
constructive dimension were left open.

Therefore, martingales and supermartingales are known to give equivalent deﬁnitions for all the ap-
plications mentioned above, and gales and supegales are known to give equivalent deﬁnitions for all the 
applications mentioned above except constructive dimension. We resolve this anomaly. Here it is shown that 
constructive gales give an equivalent deﬁnition of constructive dimension. The proof is a simple and direct 
construction that uses some ideas from an earlier paper by the author [1]. As a corollary we obtain a form 
of optimal constructive gales.

2 Preliminaries

The set of natural numbers is \( \mathbb{N} = \{0, 1, 2, \ldots \} \). The set of binary strings of length \( n \in \mathbb{N} \) is \( \{0, 1\}^n \). The set 
of all ﬁnite binary strings is \( \{0, 1\}^* \). The empty string is \( \lambda \). For a language \( A \subseteq \{0, 1\}^* \), we write \( A_{=n} \) for

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the set of strings in $A$ of length $n$. For strings $w, v \in \{0, 1\}^*$, we write $w \sqsubseteq v$ if $w$ is a prefix of $v$. $C$ is the Cantor space of all infinite binary sequences. For a sequence $S \in C$, $S[0, n - 1]$ is the prefix of $S$ of length $n$.

A real number $r$ is computable if there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $|f(n) - r| \leq 2^{-n}$ for all $n \in \mathbb{N}$. A function $g : \{0, 1\}^* \to [0, \infty)$ is constructive if there is a computable function $h : \{0, 1\}^* \times \mathbb{N} \to \mathbb{Q}$ such that for all $w \in \{0, 1\}^*$, $h(w, n) \leq h(w, n + 1) < g(w)$ for all $n \in \mathbb{N}$ and $g(w) = \sup_{n \in \mathbb{N}} h(w, n)$.

3 Constructive Dimension

Constructive dimension was introduced by Lutz [5]. Here we review the basic concepts. We begin by defining suprgales and gales.

**Definition.** Let $s \in [0, \infty)$. A function $d : \{0, 1\}^* \to [0, \infty)$ is an $s$-suprgale if

$$d(w) \geq \frac{d(w0) + d(w1)}{2s} \quad (3.1)$$

for all $w \in \{0, 1\}^*$. If equality holds in (3.1) for all strings $w$, then $d$ is an $s$-gale.

Note that 1-gales are martingales and 1-suprgales are supermartingales. We are particularly interested in the success sets of suprgales and gales.

**Definition.** The success set of a suprgale $d : \{0, 1\}^* \to [0, \infty)$ is

$$S^\infty[d] = \left\{ S \in C \mid \limsup_{n \to \infty} d(S[0, n - 1]) = \infty \right\}.$$

**Notation.** For any $X \subseteq C$, we define the sets

$$\mathcal{G}_{\text{constr}}(X) = \left\{ s \mid \text{there exists a constructive } s\text{-gale } d \text{ for which } X \subseteq S^\infty[d] \right\}$$

and

$$\hat{\mathcal{G}}_{\text{constr}}(X) = \left\{ s \mid \text{there exists a constructive } s\text{-suprgale } d \text{ for which } X \subseteq S^\infty[d] \right\}$$

of nonnegative real numbers.

Constructive dimension is defined in terms of succeeding constructive suprgales.

**Definition.** For a set $X \subseteq C$, the constructive dimension of $X$ is

$$\text{cdim}(X) = \inf \hat{\mathcal{G}}_{\text{constr}}(X).$$

For a sequence $S \in C$, the constructive dimension of $S$ is

$$\text{cdim}(S) = \text{cdim}(\{S\}).$$

We now define two notions of optimality for a class of suprgales.

**Definition.** Let $d^*$ be a suprgale and let $\mathcal{D}$ be a class of suprgales.

1. We say that $d^*$ is multiplicatively optimal for $\mathcal{D}$ if for each $d \in \mathcal{D}$ there is an $\alpha > 0$ such that $d^*(w) \geq \alpha d(w)$ for all $w \in \{0, 1\}^*$.

2. We say that $d^*$ is successively optimal for $\mathcal{D}$ if for every $d \in \mathcal{D}$, $S^\infty[d] \subseteq S^\infty[d^*]$.

Lutz used Levin’s universal constructive semimeasure [9] to show that there exist multiplicatively optimal suprgales.
Theorem 3.1. (Lutz [5]) For any computable \( s \in [0, \infty) \) there is a constructive \( s \)-supergale \( d^{(s)} \) that is multiplicatively optimal for the class of constructive \( s \)-supergales.

Theorem 3.1 was used to prove the following cornerstone of constructive dimension theory.

Theorem 3.2. (Lutz [5]) For any \( X \subseteq C \),

\[
\text{cdim}(X) = \inf_{S \subseteq X} \text{cdim}(S).
\]

Remark. In [4], a conference paper preceding [5], Lutz defined constructive dimension using constructive gales. There Lutz used a false assertion about martingales to argue that there exist multiplicatively optimal constructive gales. These “optimal gales” were then used to prove Theorem 3.2. These flawed arguments were subsequently noticed and corrected in [5] by reformulating constructive dimension in terms of constructive supergales. The multiplicatively optimal supergales of Theorem 3.1 exist and Theorem 3.2 is true in the reformulation. However, Lutz left open the questions of whether there exist optimal constructive gales and whether constructive dimension can be equivalently defined using constructive gales. This paper addresses these questions.

4 The Strength of Gales

Theorem 4.1. Let \( 0 < r < t \) be computable real numbers. Then for any constructive \( r \)-supergale \( d \), there exists a constructive \( t \)-gale \( d' \) such that \( S^\infty[d] \subseteq S^\infty[d'] \).

Proof. Let \( d \) be a constructive \( r \)-supergale and assume without loss of generality that \( d(\lambda) < 1 \). Define the language \( A = \{ w \in \{0, 1\}^* \mid d(w) > 1 \} \). Observe that \( A \) is computably enumerable. For all \( n \in \mathbb{N} \),

\[
\sum_{w \in \{0, 1\}^n} d(w) \leq 2^n, \text{ so } |A_n| \leq 2^n.
\]

For each \( n \in \mathbb{N} \), define a function \( d_n^r : \{0, 1\}^* \to [0, \infty) \) by

\[
d_n^r(w) = \begin{cases} 
2^{-t(n \cdot |w|)} \cdot \left| \{ v \in A_n \mid w \subseteq v \} \right| & \text{if } |w| \leq n \\
2^{(t(n) + 1)|w|} d(w[0..n-1]) & \text{if } |w| > n.
\end{cases}
\]

Then for all \( n \), \( d_n^r(w) = 1 \) for all \( w \in A_n \).

Let \( s \in (r, t) \) be computable and define a function \( d' \) on \( \{0, 1\}^* \) by \( d' = \sum_{n=0}^\infty 2^{(s \cdot r)n} d_n^r \). Then

\[
d'(\lambda) = \sum_{n=0}^\infty 2^{s \cdot (r - 1)n} |A_n| \leq \sum_{n=0}^\infty 2^{s \cdot n} < \infty,
\]

and it follows that by induction that \( d'(w) < \infty \) for all strings \( w \). Therefore, by linearity, \( d' \) is a \( t \)-gale. Also, because the language \( A \) is computably enumerable, \( d \) is constructive.

Let \( S \in S^\infty[d] \). Then for infinitely many \( n \in \mathbb{N} \), \( S[0..n-1] \in A \). For each of these \( n \),

\[
d'(S[0..n-1]) \geq 2^{s \cdot r} d_n^r(S[0..n-1]) = 2^{s \cdot r} n,
\]

so \( S \in S^\infty[d'] \). Therefore \( S^\infty[d] \subseteq S^\infty[d'] \).

Constructive dimension may now be equivalently defined using gales instead of supergales.

Theorem 4.2. For all \( X \subseteq C \), \( \text{cdim}(X) = \inf \mathcal{G}_{\text{constr}}(X) \).

Proof. Because any gale is also a supergale, \( \mathcal{G}_{\text{constr}}(X) \subseteq \hat{\mathcal{G}}_{\text{constr}}(X) \), so \( \text{cdim}(X) = \inf \mathcal{G}_{\text{constr}}(X) \leq \inf \hat{\mathcal{G}}_{\text{constr}}(X) \).

Let \( t > r > \text{cdim}(X) \) be computable real numbers and let \( d \) be a constructive \( r \)-supergale such that \( X \subseteq S^\infty[d] \). By Theorem 4.1, there is a constructive \( t \)-gale \( d' \) such that \( X \subseteq S^\infty[d'] \subseteq S^\infty[d] \), so \( t \in \mathcal{G}_{\text{constr}}(X) \). As this holds for any computable \( t > \text{cdim}(X) \), we have \( \inf \mathcal{G}_{\text{constr}}(X) \leq \text{cdim}(X) \).
We can also state the existence of a form of optimal constructive gales.

**Corollary 4.3.** For all computable real numbers \( t > r \geq 0 \) there exists a constructive \( t \)-gale that is successively optimal for the class of constructive \( r \)-supergales.

**Proof.** Let \( d^{(r)} \) be the constructive \( r \)-supergale from Theorem 3.1 that is multiplicatively optimal for the constructive \( r \)-supergales. Theorem 4.1 provides a constructive \( t \)-gale \( d' \) that succeeds everywhere that \( d^{(r)} \) does. Therefore \( S^\infty [d] \subseteq S^\infty [d^{(r)}] \subseteq S^\infty [d'] \) for any constructive \( r \)-supergale \( d \), so the corollary is proved. \( \square \)

The optimal gales provided by Corollary 4.3 may not be technically strong as possible in two respects.

1. Lutz's optimal constructive \( r \)-supergale is multiplicatively optimal, whereas our optimal constructive \( t \)-gale is only successively optimal. Does there exist a constructive \( t \)-gale that is multiplicatively optimal for the class of constructive \( r \)-supergales?

2. Our proof seems to require the hypothesis \( t > r \). Does there exist a constructive \( r \)-gale that is successively optimal for the class of constructive \( r \)-supergales?

However, the optimality in Corollary 4.3 remains strong enough to prove Theorem 3.2.

**References**

[1] J. M. Hitchcock. Correspondence principles for effective dimensions. In *Proceedings of the 29th International Colloquium on Automata, Languages, and Programming*, 2002. To appear.

[2] J. H. Lutz. Almost everywhere high nonuniform complexity. *Journal of Computer and System Sciences*, 44:220–258, 1992.

[3] J. H. Lutz. Dimension in complexity classes. In *Proceedings of the Fifteenth Annual IEEE Conference on Computational Complexity*, pages 158–169. IEEE Computer Society Press, 2000.

[4] J. H. Lutz. Gales and the constructive dimension of individual sequences. In *Proceedings of the Twenty-Seventh International Colloquium on Automata, Languages, and Programming*, pages 902–913. Springer-Verlag, 2000.

[5] J. H. Lutz. The dimensions of individual strings and sequences. Technical Report cs.CC/0203016, ACM Computing Research Repository, 2002.

[6] P. Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.

[7] C. P. Schnorr. A unified approach to the definition of random sequences. *Mathematical Systems Theory*, 5:246–258, 1971.

[8] C. P. Schnorr. Zufälligkeit und Wahrscheinlichkeit. *Lecture Notes in Mathematics*, 218, 1971.

[9] A. K. Zvonkin and L. A. Levin. The complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms. *Russian Mathematical Surveys*, 25:83–124, 1970.