On Regularity Property of Retarded
Ornstein-Uhlenbeck Processes
in Hilbert Spaces

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Abstract: In this work, some regularity properties of mild solutions for a class of stochastic
linear functional differential equations driven by infinite dimensional Wiener processes are
considered. In terms of retarded fundamental solutions, we introduce a class of stochastic
convolutions which naturally arise in the solutions and investigate their Yosida approxi-
mants. By means of the retarded fundamental solutions, we find conditions under which
each mild solution permits a continuous modification. With the aid of Yosida approxima-
tion, we study two kinds of regularity properties, temporal and spatial ones, for the retarded
solution processes. By employing a factorization method, we establish a retarded version of
Burkholder-Davis-Gundy’s inequality for stochastic convolutions.

Keywords: Fundamental solution; Yosida approximation; Burkholder-Davis-Gundy’s in-
equality.

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1 Introduction

Let $H$ and $K$ be two real separable Hilbert spaces with associated inner products $\langle \cdot, \cdot \rangle_H$, $\langle \cdot, \cdot \rangle_K$ and norms $\| \cdot \|_H$, $\| \cdot \|_K$, respectively. We denote by $\mathcal{L}(K, H)$ the set of all linear bounded operators from $K$ into $H$, equipped with the usual operator norm $\| \cdot \|$ topology. When $H = K$, we denote $\mathcal{L}(H, H)$ simply by $\mathcal{L}(H)$.

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\{W(t), t \geq 0\}$ denote a $K$-valued $\mathcal{F}_t$-adapted Wiener process defined on $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with covariance operator $Q$, i.e.,

$$\mathbb{E}\langle W(t), x \rangle_K \langle W(s), y \rangle_K = (t \wedge s) \langle Qx, y \rangle_K \quad \text{for all} \quad x, y \in K,$$

where $Q$ is a linear, symmetric and nonnegative bounded operator on $K$. In particular, we shall call $W(t)$, $t \geq 0$, a $K$-valued $Q$-Wiener process with respect to $\{\mathcal{F}_t\}_{t \geq 0}$. If the trace $\text{Tr} Q < \infty$, then $W$ is a genuine Wiener process. It is possible that $\text{Tr} Q = \infty$, e.g., $Q = I$ which corresponds to a cylindrical Wiener process.

In order to define stochastic integrals with respect to the $Q$-Wiener process $W(t)$, we introduce the subspace $K_Q = \text{Ran} Q^{1/2} \subset K$, the range of $Q^{1/2}$, which is a Hilbert space endowed with the inner product

$$\langle u, v \rangle_{K_Q} = \langle Q^{-1/2}u, Q^{-1/2}v \rangle_K \quad \text{for any} \quad u, v \in K_Q.$$

Let $\mathcal{L}_2(K_Q, H)$ denote the space of all Hilbert-Schmidt operators from $K_Q$ into $H$, then $\mathcal{L}_2(K_Q, H)$ turns out to be a separable Hilbert space under the inner product

$$\langle L, P \rangle_{\mathcal{L}_2(K_Q, H)} = \text{Tr}[LP^*] \quad \text{for any} \quad L, P \in \mathcal{L}_2(K_Q, H).$$

For arbitrarily given $T \geq 0$, let $J(t, \omega)$, $t \in [0, T]$, be an $\mathcal{L}_2(K_Q, H)$-valued process. We define the following norm for arbitrary $t \in [0, T]$,

$$|J|_t := \left\{ \mathbb{E} \int_0^t \text{Tr} \left[ J(s, \omega)QJ(s, \omega)^* \right] ds \right\}^{1/2}.$$  \hfill (1.1)

In particular, we denote all $\mathcal{L}_2(K_Q, H)$-valued measurable processes $J$, adapted to the filtration $\{\mathcal{F}_t\}_{t \leq T}$, satisfying $|J|_T < \infty$ by $\mathcal{U}^2([0, T]; \mathcal{L}_2(K_Q, H))$. The stochastic integral

$$\int_0^t J(s, \omega) dW(s) \in H, t \geq 0,$$

may be defined for all $J \in \mathcal{U}^2([0, T]; \mathcal{L}_2(K_Q, H))$ by

$$\int_0^t J(s, \omega) dW(s) = L^2 - \lim_{n \to \infty} \sum_{i=1}^n \int_0^t \sqrt{\lambda_i} J(s, \omega)e_i dB^i_s; \quad t \in [0, T],$$

where $W(t) = \sum_{i=1}^\infty \sqrt{\lambda_i} B^i_t e_i$. Here $(\lambda_i \geq 0, i \in \mathbb{N})$ are the eigenvalues of $Q$ with the corresponding eigenvectors $(e_i, i \in \mathbb{N})$, and $(B^i_t, i \in \mathbb{N})$ are independent standard real-valued Brownian motions.
In this work, we shall consider the following stochastic functional differential equation on the Hilbert space $H$,

$$\begin{cases} dy(t) = Ay(t)dt + Fy(t)dt + BdW(t) & \text{for any } t \in [0, T], \\ y(0) = \phi_0 \in H, \ y_0(\theta) = y(\theta) = \phi_1(\theta) \in H, \ \theta \in [-r, 0), \end{cases}$$  \tag{1.2}

for arbitrarily given initial datum $\Phi = (\phi_0, \phi_1) \in H \times L^2([-r, 0]; H)$ where $r > 0$ is a given constant and $y_t(\theta) := y(t + \theta)$ for $\theta \in [-r, 0]$, $t \geq 0$. Here $A$ is the infinitesimal generator of a $C_0$-semigroup $e^{tA}$, $t \geq 0$, $B \in \mathcal{L}_2(KQ, H)$, and $F : L^2([-r, 0]; H) \to H$ is some linear, probably unbounded, operator to be specified later on.

If $F = 0$, the solution of (1.2) is called an Ornstein-Uhlenbeck process which is Gaussian and Markovian. There exists extensive literature on various topics such as Feller semigroups, invariant measures and so on for this process. The reader is referred to, e.g., [2], [5] and reference cited therein for a comprehensive theory and related topics. If $F \neq 0$, the solution of (1.2) is called the so-called retarded Ornstein-Uhlenbeck process in Hilbert space $H$. To my knowledge, there is little work devoted to the process of (1.2) in the existing literature, e.g., [1], [9], [10], [12] and [13] which dealt with stationary solutions of the system and related topics.

Historically, regularity problem for infinite dimensional systems is quite important and it has been investigated by many researchers, e.g., in [4], [5] for stochastic evolution equations without memory and in [6], [17] for deterministic functional differential equations among others. In this work, we are interested in the regularity property of the solution processes of (1.2). Basically, in the research of the system (1.2), one of the most important approaches is to lift the system under investigation to some expanded space, e.g., $H \times L^2([-r, 0]; H)$ or $C([-r, 0]; H)$, so as that one can consider a lifted stochastic system without memory, rather than (1.2) itself. In spite of its obvious advantages, this method introduces, however, significant mathematical difficulties in dealing with regularity problem. For instance, suppose that the operator $A$ in (1.2) generates an analytic semigroup, a condition which is frequently assumed in the investigation of regularity problem, the lifted generator of the system does not generate an analytic semigroup any more on the expanded spaces as above (cf. [12]). In this work, we shall employ a straightforward method to deal with regularity problem by developing a theory of retarded type of Green operators for the system (1.2) (cf. [9], [11]).

The organization of this paper is as follows. We shall introduce in Section 2 a class of fundamental solutions or retarded Green operators for the system (1.2) and meanwhile review useful notations, definitions and properties to be used in the work. In terms of fundamental solutions, we shall define in Section 3 the so-called retarded stochastic convolutions which naturally arise in the variation of constants formula for solutions of (1.2). By using a factorization method introduced in [4], we shall establish sufficient conditions under which there exists a continuous modification of retarded stochastic convolutions. By analogy with those in the classical semigroup theory, we shall establish in Section 4 the powerful Yosida approximations for the corresponding deterministic system, i.e., $B = 0$, of (1.2). Subsequently, Yosida approximations are applied in Section 5 to the investigation of regularity property for a class of retarded linear stochastic functional differential equations. In Section 6, we proceed to establish a version of Burkholder-Davis-Gundy’s inequality for retarded stochastic
Fundamental Solutions

Let \( r > 0 \) and we denote by \( L^2_2 = L^2([-r, 0]; H) \) the space of all \( H \)-valued equivalence classes of measurable functions \( \varphi(\theta), \theta \in [-r, 0], \) such that \( \int_{-r}^{0} \|\varphi(\theta)\|^2_H d\theta < \infty. \) We also denote by \( W^{1,2}([-r, 0]; H) \) the Sobolev space of all \( H \)-valued function \( y \) on \([-r, 0]\) such that \( y \) and its distributional derivative belong to \( L^2([-r, 0]; H). \) Let \( \mathcal{H} \) denote the product Hilbert space \( H \times L^2_2 \) with its norm and inner product defined, respectively, by

\[
\|\Phi\|_{\mathcal{H}} = (\|\phi_0\|^2_H + \|\phi_1\|^2_{L^2_2})^{1/2}, \quad \langle \Phi, \Psi \rangle_{\mathcal{H}} = \langle \phi_0, \psi_0 \rangle_H + \langle \phi_1, \psi_1 \rangle_{L^2_2}
\]

for all \( \Phi = (\phi_0, \phi_1), \Psi = (\psi_0, \psi_1) \in \mathcal{H}. \)

Let \( A : \mathcal{D}(A) \subseteq H \to H \) be the infinitesimal generator of a \( C_0 \)-semigroup \( e^{tA}, \ t \geq 0, \) on \( H \) where \( \mathcal{D}(A) \) denotes the domain of operator \( A. \) Let \( T \geq 0 \) and assume that \( F : W^{1,2}([-r, 0]; H) \to H \) is a bounded linear operator such that the map \( F \) permits a bounded linear extension \( F : L^2([-r, T]; H) \to L^2([0, T]; H) \) which is defined by \( (Fy)(t) = FY_t, \ y \in L^2([-r, T]; H), \) with \( y_\theta(t) := y(t + \theta) \) for \( \theta \in [-r, 0], t \geq 0. \) That is, there exists a real number \( M_2 > 0 \) such that

\[
\int_{-r}^{T} \| (Fy)(t) \|^2_H dt \leq M_2 \int_{-r}^{T} \| y(t) \|^2_H dt \quad \text{for any} \quad y \in L^2([-r, T]; H). \tag{2.1}
\]

Consider the following deterministic functional differential equation on \( H, \)

\[
\begin{align*}
\frac{dy(t)}{dt} &= Ay(t)dt + Fy(t)dt \quad \text{for any} \quad t > 0, \\
y(0) &= \phi_0, \ y_0 = \phi_1, \ \Phi = (\phi_0, \phi_1) \in \mathcal{H},
\end{align*} \tag{2.2}
\]

and its corresponding functional integral equation

\[
\begin{align*}
y(t) &= e^{tA}\phi_0 + \int_{0}^{t} e^{(t-s)A}FY_s ds, \quad t > 0, \\
y(0) &= \phi_0, \ y_0 = \phi_1, \ \Phi = (\phi_0, \phi_1) \in \mathcal{H}.
\end{align*} \tag{2.3}
\]

It may be shown that for any \( \Phi \in \mathcal{H}, \) the equation (2.3) has a unique solution \( y(t, \Phi) \) which is called the mild solution of (2.2). For any \( x \in H, \) we define the (retarded) fundamental solution or (retarded) Green operator \( G(t) : (-\infty, \infty) \to \mathcal{L}(H) \) of (2.3) by

\[
G(t)x = \begin{cases} 
  y(t, \Phi), & t \geq 0, \\
  0, & t < 0,
\end{cases} \tag{2.4}
\]

where \( \Phi = (x, 0), \ x \in H. \) It turns out (cf. [9]) that \( G(t), \ t \geq 0, \) is a strongly continuous one-parameter family of bounded linear operators on \( H \) such that

\[
\|G(t)\| \leq c \cdot e^{\gamma t}, \quad t \geq 0, \tag{2.5}
\]
for some constants \( c > 0 \) and \( \gamma \in \mathbb{R}^1 := (-\infty, \infty) \). On the other hand, it is easy to see that \( G(t) \) is the unique solution of the functional operator integral equation

\[
G(t) = \begin{cases} 
  e^{tA} + \int_0^t e^{(t-s)A} FG(s + \cdot) ds, & t \geq 0, \\
  O, & t < 0,
\end{cases}
\]

where \( O \) denotes the null operator on \( H \).

**Remark 2.1.** It is worth mentioning that for a particular delay operator \( F \) defined by

\[
F\varphi = \sum_{i=1}^m A_i \varphi(-r_i) + \int_{-r}^0 A_0(\theta)\varphi(\theta) d\theta, \quad \forall \varphi \in W^{1,2}([-r, 0]; H).
\]

where \( 0 \leq r_1 \leq \cdots \leq r_m \leq r \), \( A_i \in \mathcal{L}(H) \), \( i = 1, \cdots, m \), and \( A_0(\cdot) \in L^2([-r, 0]; \mathcal{L}(H)) \), a similar concept of fundamental solutions was introduced in [15].

For each function \( \varphi : [-r, 0] \to H \), we define its right extension function \( \tilde{\varphi} \) by

\[
\tilde{\varphi} : [-r, \infty) \to H, \quad \tilde{\varphi}(t) = \begin{cases} 
  \varphi(t), & -r \leq t \leq 0, \\
  0, & 0 < t < \infty.
\end{cases}
\]

By virtue of (2.8), it may be shown (cf. [9]) that the mild solution of (2.2) is represented explicitly by the variation of constants formula

\[
y(t) = G(t)\phi_0 + \int_0^t G(t-s) F(\tilde{\varphi}_1)_s ds, \quad t \geq 0,
\]

and \( y(t) = \phi_1(t), \ t \in [-r, 0) \). It is useful to introduce the so-called structure operator \( S \) defined on the space \( L^2([-r, 0]; H) \) by

\[
(S\varphi)(\theta) = F\tilde{\varphi}_{-\theta}, \quad \theta \in [-r, 0], \quad \forall \varphi(\cdot) \in W^{1,2}([-r, 0]; H).
\]

It is not difficult to show that \( S \) can be extended to a linear and bounded operator from \( L^2([-r, 0]; H) \) into itself. Moreover, the variation of constants formula for the mild solution of (2.2) may be rewritten as

\[
\begin{cases} 
  y(t) = G(t)\phi_0 + \int_{-r}^0 G(t+\theta)(S\tilde{\phi}_1)(\theta) d\theta, & t \geq 0, \\
  y(t) = \phi_1(t), & t \in [-r, 0).
\end{cases}
\]

In general, the family \( G(t), \ t \in \mathbb{R}^1 \), would no longer be a semigroup on \( H \). However, we may show that it is a “quasi-semigroup” in the sense that

\[
G(t+s)x = G(t)G(s)x + \int_{-r}^0 G(t+\theta)[SG(s + \cdot)x](\theta) d\theta \quad \text{for all} \ s, \ t \geq 0, \ x \in H.
\]
In association with the operator $S$, we may define a new operator $\tilde{S}$ on $L^2([-r,0];\mathcal{L}(H))$ by
\[
[\tilde{S}J](\theta)x = [(\tilde{S}J)x](\theta) := [S(Jx)](\theta), \quad x \in H, \quad \theta \in [-r,0], \quad a.e. \quad (2.13)
\]
for any $J(\cdot) \in L^2([-r,0];\mathcal{L}(H))$. It is shown that such an operator, still denoted by $S$, is a linear bounded operator from $L^2([-r,0];\mathcal{L}(H))$ into itself. Indeed, since the operator $S$ in (2.10) is bounded on $L^2([-r,0];H)$, it follows that for some constant $C > 0$, there is
\[
\int_{-r}^{0} \|[SJ](\theta)\|^2 d\theta = \int_{-r}^{0} \sup_{\|x\|_H \leq 1} \|[SJ](\theta)x\|^2_H d\theta = \int_{-r}^{0} \sup_{\|x\|_H \leq 1} \|[S(Jx)](\theta)\|^2_H d\theta 
\leq C \int_{-r}^{0} \sup_{\|x\|_H \leq 1} \|(Jx)(\theta)\|^2_H d\theta = C \int_{-r}^{0} \|J(\theta)x\|^2_H d\theta 
\leq C \int_{-r}^{0} \|J(\theta)\|^2 d\theta.
\]
This implies that $S$ is a linear bounded operator on $L^2([-r,0];\mathcal{L}(H))$. Moreover, on this occasion the relation (2.12) yields that
\[
G(t + s) = G(t)G(s) + \int_{-r}^{0} G(t + \theta)[SG(s + \cdot)](\theta)d\theta \quad \text{for all} \quad s, \ t \geq 0. \quad (2.14)
\]

3 Continuous Sample Paths

Let $L^2_{\mathcal{F}_0}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$ denote the space of all $\mathcal{H}$-valued mappings $\Psi(\omega) = (\psi_0(\omega), \psi_1(\cdot, \omega))$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that both $\psi_0$ and $\psi_1(\cdot)$ are $\mathcal{F}_0$-measurable for any $\theta \in [-r,0]$ and satisfy
\[
\mathbb{E}\|\Psi\|^2_\mathcal{H} = \mathbb{E}\|\psi_0\|^2_H + \mathbb{E}\|\psi_1\|^2_{L^2([-r,0];H)} < \infty.
\]
We shall be concerned about the following stochastic functional evolution equation on the Hilbert space $H$,
\[
\begin{cases}
dy(t) = [Ay(t) + Fy_t]dt + BdB(t) & \text{for any} \ t \in [0, T], \\
y(0) = \psi_0, \ y_0 = \psi_1, \ \Psi = (\psi_0, \psi_1) \in L^2_{\mathcal{F}_0}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H}),
\end{cases} \quad (3.1)
\]
where $B \in \mathcal{L}_2(K, H)$, $W(t)$ is a $K$-valued $Q$-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ and the delay operator $F$ is given as in Section 2.

For any $t \geq 0$, let $Q_t = \int_0^t G(s)BQB^*G^*(s)ds$ where $G^*(s)$ denotes the adjoint operator of $G(s)$ for any $s \geq 0$. For the problem (3.1), it was shown in [9] that for any $\Psi = (\psi_0, \psi_1) \in L^2_{\mathcal{F}_0}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$, if
\[
\text{Tr} [Q_t] = \int_0^t \text{Tr} [G(s)BQB^*G^*(s)]ds < \infty \quad \text{for any} \ t \in [0, T], \quad (3.2)
\]
then there exists a unique mild solution \( y(t, \Psi) \) of (3.1). Moreover, this solution is mean square continuous with sample paths (almost surely) in \( L^2([0, T]; H) \) for each \( T \geq 0 \) and it may be explicitly represented in terms of fundamental solutions \( G(t), t \in \mathbb{R}^1 \), by

\[
y(t, \Psi) = G(t)\psi_0 + \int_0^t G(t + \theta)S\psi_1(\theta)d\theta + \int_0^t G(t - s)BdW(s), \quad t \in [0, T],
\]

(3.3)

where \( S \) is the structure operator defined in (2.10).

The aim of this section is to show that under a slightly stronger version of (3.2), the solution \( (y(t, \Psi), t \geq 0) \), or equivalently, the retarded stochastic convolution

\[
W_B \equiv G(t) := \int_0^t G(t - s)BdW(s), \quad t \geq 0,
\]

has a version with continuous sample paths. To this end, we first establish a useful lemma.

**Lemma 3.1.** Let \( T \geq 0 \) and \( \alpha \in (0, 1/2) \). Assume that function \( z(t, s) : [0, T] \times [0, T] \to H \) is continuous in \( t \) and for any \( t \in [0, T] \), \( z(t, \cdot) \in L^m([0, T]; H) \) for some natural number \( m > 1/\alpha \), then the function

\[
l(t) = \int_0^t (t - s)^{\alpha - 1}z(t, s)ds, \quad t \in [0, T],
\]

is continuous on \([0, T]\).

**Proof.** First note that if \( z(t, s) \) is an \( H \)-valued continuous function on \([0, T] \times [0, T]\), then the function \( l(t) \) is continuous on \([0, T]\).

Now suppose that \( z(t, s) : [0, T] \times [0, T] \to H \) is continuous in \( t \) and for any \( t \in [0, T] \), \( z(t, \cdot) \in L^m([0, T]; H) \) where \( m > 1/\alpha \), then it is easy to see that there exist a family of continuous functions \( z_n(t, s) \) on \([0, T] \times [0, T]\) such that

\[
\sup_{t \in [0, T]} \int_0^t \| z(t, s) - z_n(t, s) \|^m_H ds \to 0 \quad \text{as} \quad n \to \infty.
\]

(3.4)

On the other hand, we have by virtue of Hölder inequality that

\[
\| l(t) \|^m_H \leq \left( \int_0^t (t - s)^{(\alpha - 1)m} ds \right)^{m-1} \int_0^t \| z(t, s) \|^m_H ds
\]

\[
\leq C_{\alpha, m, T} \int_0^t \| z(t, s) \|^m_H ds, \quad t \in [0, T],
\]

(3.5)

where

\[
C_{\alpha, m, T} = \left( \int_0^T s^{(\alpha - 1)m} ds \right)^{m-1} = \left( \frac{m - 1}{\alpha m - 1} \right)^{m-1} T^{\frac{2m - 1}{m}} > 0.
\]

This immediately yields that

\[
\sup_{t \in [0, T]} \| l(t) \|^m_H \leq C_{\alpha, m, T} \sup_{t \in [0, T]} \int_0^t \| z(t, s) \|^m_H ds.
\]

(3.6)
Therefore, for the function $z(t, s) : [0, T] \times [0, T] \to H$ there exist, in view of (3.4) and (3.6), a family of continuous functions $z_n(t, s)$ on $[0, T] \times [0, T]$ such that

$$\sup_{t \in [0, T]} \|l(t) - l_n(t)\|_H^m \leq C_{\alpha, m, T} \sup_{t \in [0, T]} \int_0^t \|z(t, s) - z_n(t, s)\|_H^m ds \to 0 \text{ as } n \to \infty, \quad (3.7)$$

where

$$l_n(t) = \int_0^t (t - s)^{\alpha - 1}z_n(t, s)ds, \quad t \in [0, T], \quad n \in \mathbb{N}.$$ 

Since $l_n(t)$ is continuous on $[0, T]$, (3.7) implies the continuity of $l(t)$ on $[0, T]$. The proof is thus complete. \hfill \Box

**Theorem 3.1.** Assume that for some $\alpha > 0$ and $T \geq 0$, the relation

$$\int_0^T t^{-2\alpha} Tr[G(t)BQB^*G(t)^*]dt < \infty \quad (3.8)$$

holds. Then the retarded stochastic convolution $W^B_G(t) = \int_0^t G(t-s)BdW(s)$ has a continuous modification on $[0, T]$.

**Proof.** Without loss of generality, fix a number $\alpha \in (0, 1/2)$ and note the following elementary identity

$$\int_u^t (t - s)^{\alpha - 1}(s - u)^{-\alpha}ds = \frac{\pi}{\sin \pi \alpha} \text{ for any } u \leq s \leq t \leq T. \quad (3.9)$$

By virtue of (3.9), it is easy to see that

$$W^B_G(t) = \frac{\sin \pi \alpha}{\pi} \int_0^t G(t-u)[\int_u^t (t - s)^{\alpha - 1}(s - u)^{-\alpha}ds]BdW(u). \quad (3.10)$$

In view of the well-known stochastic Fubini theorem and quasi-semigroup property (2.14) of $G(t)$, one can rewrite (3.10) for any $t \in [0, T]$ as

$$W^B_G(t) = \frac{\sin \pi \alpha}{\pi} \int_0^t (t - s)^{\alpha - 1}G(t - s + s - u)(s - u)^{-\alpha}BdW(u)ds$$

$$= \frac{\sin \pi \alpha}{\pi} \int_0^t (t - s)^{\alpha - 1}\int_0^s \left[ \int_0^\theta G(t - s + \theta)[SG(s - u + \cdot)](\theta)d\theta + G(t - s)G(s - u) \right] ds \cdot (s - u)^{-\alpha}BdW(u)ds$$

$$= \frac{\sin \pi \alpha}{\pi} \int_0^t (t - s)^{\alpha - 1} \int_0^s (s - u)^{-\alpha} \int_0^\theta G(t - s + \theta)[SG(s - u + \cdot)](\theta)Bd\theta dW(u)ds$$

$$+ \frac{\sin \pi \alpha}{\pi} \int_0^t (t - s)^{\alpha - 1}G(t - s) \int_0^s G(s - u)(s - u)^{-\alpha}BdW(u)ds$$

$$=: \frac{\sin \pi \alpha}{\pi} (I_1(t) + I_2(t)). \quad (3.11)$$
We first show the existence of a continuous modification for the term \( I_1(t) \). To this end, let us rewrite the term \( I_1(t) \) as

\[
I_1(t) = \int_0^t (t-s)^{\alpha-1} Z(t,s) ds, \quad t \in [0,T],
\]

where

\[
Z(t,s) = \int_0^s (s-u)^{-\alpha} \int_{-r}^r G(t-s+\theta)[SG(s-u+\cdot)](\theta) Bd\theta dW(u), \quad s \in [0,t],
\]

and its covariance operator is

\[
\text{Cov} Z(t,s) = \int_0^s (s-u)^{-2\alpha} \int_{-r}^r G(t-s+\theta)[SG(s-u+\cdot)](\theta) Bd\theta
\]

\[
\cdot Q \left( \int_{-r}^r G(t-s+\theta)[SG(s-u+\cdot)](\theta) Bd\theta \right)^* du, \quad s \in [0,t].
\]

(3.12)

Since \( \|G(t)\| \leq ce^{\gamma t}, c > 0, \gamma \in \mathbb{R} \), for all \( t \geq 0 \) and \( S \) is bounded on \( L^2([-r,0]; \mathcal{L}(H)) \), the relations (3.12) and (3.8), together with Hölder inequality, imply that for any \( s \in [0,t], t \leq T \),

\[
\text{Tr} \left[ \text{Cov} Z(t,s) \right] = \int_0^s (s-u)^{-2\alpha} \text{Tr} \left[ \int_{-r}^r G(t-s+\theta)[SG(s-u+\cdot)](\theta) Bd\theta \right.
\]

\[
\cdot Q \left( \int_{-r}^r G(t-s+\theta)[SG(s-u+\cdot)](\theta) Bd\theta \right)^* du
\]

\[
\leq \|S\|^2 r \int_0^s (s-u)^{-2\alpha} \text{Tr} \left[ G(s-u+\theta)BQB^*G(s-u+\theta)^* \right] d\theta du
\]

\[
\leq \|S\|^2 r \int_0^s (s-u)^{-2\alpha} \text{Tr} \left[ G(u)BQB^*G(u)^* \right] d\theta du
\]

\[
\leq \|S\|^2 r^2 \int_0^T u^{-2\alpha} \text{Tr} \left[ G(u)BQB^*G(u)^* \right] du < \infty.
\]

(3.13)

This shows that for any \( t \in [0,T], Z(t,s), s \in [0,t] \), is not only Gaussian but also has a finite trace covariance operator. Then, by using Corollary 2.17 in [5] and (3.13), one can choose a natural number \( m > 1/\alpha \) and find a number \( C_m > 0 \) such that for any \( t \in [0,T] \),

\[
\mathbb{E} \left( \int_0^T \|Z(t,s)\|^m_H ds \right) = \mathbb{E} \left( \int_0^t \|Z(t,s)\|^m_H ds \right)
\]

\[
\leq C_m \int_0^T \sup_{t \in [0,T]} \left( \text{Tr} \left[ \text{Cov} Z(t,s) \right] \right)^{m/2} ds
\]

\[
\leq C_m T \left[ \|S\|^2 r^2 \int_0^T u^{-2\alpha} \text{Tr} \left[ G(u)BQB^*G(u)^* \right] du \right]^{m/2} < \infty,
\]

(3.14)
which, together with Lemma 3.1 and the strong continuity of $G(t)$, $t \in \mathbb{R}^1$, implies that $I_1(t) = \int_0^t (t-s)^{\alpha-1}Z(t,s)ds$ has a continuous modification on $[0,T]$.

In a similar way, we can show that there exists a continuous modification of $I_2(t)$, $t \in [0,T]$. The existence of a continuous modification for $I_1(t)$ and $I_2(t)$ implies further that the stochastic convolution $W^H_G(t)$ has a continuous modification on $[0,T]$. The proof is thus complete.

**Corollary 3.1.** Let $\Psi = (\psi_0, \psi_1) \in L^2_{\mathcal{F}_0}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$. Assume that the relation (3.8) holds for some $\alpha > 0$ and $T \geq 0$, then the mild solution of Eq. (3.1) has a version with continuous sample paths on $[0,T]$.

**Proof.** The claim is immediate from Theorem 3.1, (3.3) and the fact that the fundamental solution $G(t)$, $t \geq 0$, is a strongly continuous one-parameter family of bounded linear operators on $H$.

### 4 Retarded Yosida Approximant

Suppose that $A$ is the infinitesimal generator of some strongly continuous semigroup $e^{tA}$, $t \geq 0$, on the Hilbert space $H$. Recall that we may define the following *Yosida approximants*

$$A_n := AJ_n = A(nR(n,A)) = n^2R(n,A) - nI,$$

which are bounded operators for each $n \in \rho(A)$, the resolvent set of $A$, and commute with one another. Here $J_n = nR(n,A)$ and $R(n,A)$ is the resolvent operator of $A$ for any $n \in \rho(A)$. It may be shown that

$$e^{tA}x = \lim_{n \to \infty} e^{tA_n}x \quad \text{for each} \quad x \in H,$$

and the family of operators $\{e^{tA_n}\}_{n \geq 1}$ is thus called the *Yosida approximants* of $e^{tA}$, $t \geq 0$.

In this section, we shall consider Yosida approximation for fundamental solution $G(t)$, $t \geq 0$ and meanwhile establish useful properties which will be applied in the next section to the regularity problem for stochastic functional evolution equations.

Let $\mathcal{L}_s(H)$ denote the family of all bounded linear operators on $H$, endowed with the strong operator topology, i.e., the local convex topology generated by the following seminorms

$$p_x(B) := \|Bx\|_H, \quad B \in \mathcal{L}(H), \quad x \in H.$$

Thus $J(\cdot) \in C([0,T]; \mathcal{L}_s(H))$ if and only if $J(t) \in \mathcal{L}(H)$ for each $t \in [0,T]$ and $t \to J(t)x$ is continuous for each $x \in H$. By virtue of the well-known uniform boundedness principle, it may be shown (cf. [7]) that this space is a Banach space under the norm

$$\|J\|_{\text{max}} := \sup_{t \in [0,T]} \|J(t)\|, \quad J \in C([0,T]; \mathcal{L}_s(H)).$$
On the other hand, note that for any $J(\cdot) \in C([0, T]; \mathcal{L}_s(H))$, we can extend it uniquely to obtain a mapping $\tilde{J}(\cdot)$ on $[-r, T]$ such that $\tilde{J}(t) = J(t)$ as $t \in [0, T]$ and $\tilde{J}(t) = 0$ as $t \in [-r, 0)$. We shall always identify $J(\cdot) \in C([0, T]; \mathcal{L}_s(H))$ with such an extension in the sequel when no confusion is possible.

Definition 4.1. Let $e^{tA}$, $t \geq 0$, be a strongly continuous semigroup on $H$. For any $T \geq 0$, the operator $V$ defined by

$$VJ(t)x := \int_0^t e^{(t-s)A} FJ(s + \cdot) x ds, \quad t \in [0, T], \quad x \in H,$$

on $J(\cdot) \in C([0, T]; \mathcal{L}_s(H))$ is called the retarded Volterra operator.

Lemma 4.1. The retarded Volterra operator $V$ is a bounded linear operator in the space $C([0, T]; \mathcal{L}_s(H))$. Moreover, it satisfies that for any $m \in \mathbb{N}$,

$$\|V^m\| \leq \kappa^m / m!$$

where $\kappa = MTM_2^{1/2} > 0$, $M = \sup_{t \in [0, T]} \|e^{tA}\|$ and $M_2 > 0$ is given as in (2.1).

Proof. It is clear that $V$ is a linear operator. For the boundedness, we may employ Hölder inequality and (2.1) to get that for any $J(\cdot) \in C([0, T]; \mathcal{L}_s(H))$ and $t \in [0, T]$,

$$\|VJ(t)\| \leq \int_0^t \sup_{\|x\|_H \leq 1} \|e^{(t-s)A} FJ(s + \cdot) x\|_H ds$$

$$\leq Mt^{1/2} \left( \int_0^t \sup_{\|x\|_H \leq 1} \|FJ(s + \cdot) x\|^2_H ds \right)^{1/2}$$

$$\leq Mt^{1/2} M_2^{1/2} \left( \int_{-r}^t \sup_{\|x\|_H \leq 1} \|J(s)x\|^2_H ds \right)^{1/2}$$

$$\leq MtM_2^{1/2} \|J(t)\|,$$

which implies that $\|V\| \leq \kappa$ where $\kappa = MTM_2^{1/2} > 0$.

The general form (4.1) can be easily shown by using (4.2) and implementing induction on $m \in \mathbb{N}$. The proof is thus complete.

Proposition 4.1. The fundamental solution $G(t)$, $t \in [0, \infty)$, of the equation (2.1) may be explicitly represented as

$$G(t) = \sum_{m=0}^{\infty} G(m, t), \quad t \geq 0,$$

where $G(0, t) := e^{tA}$, $t \geq 0$, and for each $x \in H$ and $m \geq 1$,

$$G(m + 1, t)x := VG(m, t)x = \int_0^t e^{(t-s)A} FG(m, s + \cdot) x ds, \quad t \geq 0.$$

Moreover, the series (4.3) converges in the operator norm uniformly with respect to $t$ on any compact interval in $\mathbb{R}_+$. 

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Proof. From Lemma 4.1 it follows that for the Volterra operator $V$, its resolvent $R(\lambda, V)$ exists at $\lambda = 1$ and is given by

$$R(1, V) = (I - V)^{-1} = \sum_{m=0}^{\infty} V^m.$$ 

On the other hand, by virtue of (2.6) we have for any $x \in H$ that

$$G(t)x = R(1, V)e^{tA}x = \sum_{m=0}^{\infty} V^m e^{tA}x, \quad t \geq 0.$$ 

Therefore, let $G(m, t) = V^m e^{tA}$, $m \geq 1$, and $G(0, t) := e^{tA}$, $t \geq 0$, then we have that

$$G(t)x = \sum_{m=0}^{\infty} G(m, t)x, \quad t \geq 0, \quad x \in H,$$

and

$$G(m + 1, t)x = VG(m, t)x = \int_{0}^{t} e^{(t-s)A} FG(m, s + \cdot)xsds, \quad m \geq 1, \quad x \in H.$$ 

Finally, the uniform convergence of (4.3) in the operator norm can be deduced from the fact that $G(m, t) = V^m e^{tA}$ and $\|V^m\| \leq \kappa^m / m!$, $m \geq 1$. The proof is complete now.

Let $A_n$ be the Yosida approximants of $A$ and consider the following deterministic functional differential equation for each $n \in \mathbb{N}$,

$$\begin{cases} 
\frac{dy(t)}{dt} = A_n y(t)dt + Fy dt \quad \text{for any} \quad t > 0, \\
y(0) = \phi_0, \quad y_0 = \phi_1, \quad \Phi = (\phi_0, \phi_1) \in \mathcal{H}. 
\end{cases} \quad (4.4)
$$

By analogy with $G(t)$, $t \geq 0$, in Section 2, we can define the corresponding fundamental solutions $G_n(t)$, $n \in \mathbb{N}$, for the equations (4.4) which is a family of strongly continuous bounded linear operators on $H$ and satisfies the following equations

$$G_n(t) = \begin{cases} 
e^{tA_n} + \int_{0}^{t} e^{(t-s)A_n} FG_n(s + \cdot)ds, \quad t \geq 0, \\
o, \quad t < 0. 
\end{cases} \quad (4.5)$$

Since $\|e^{tA_n}\| \leq M e^{\alpha t}, \quad t \geq 0$, for some $M \geq 1$, $\alpha > 0$ and large $n \in \mathbb{N}$ (see, e.g., (A.13) in [5]), we can deduce by (4.5) and the well-known Gronwall inequality that

$$\|G_n(t)\| \leq c e^{\gamma t}, \quad t \geq 0, \quad \text{for some} \quad c > 0, \quad \gamma > 0 \quad \text{and large} \quad n \in \mathbb{N}. \quad (4.6)$$

In a similar way, we can show that for each fixed $n \in \mathbb{N}$, $G_n(t)$, $t \in [0, \infty)$, is a uniformly norm continuous family in $\mathcal{L}(H)$, i.e., $G_n(t) : [0, \infty) \to \mathcal{L}(H)$, by using (4.5) and the fact that $e^{tA_n} : [0, \infty) \to \mathcal{L}(H)$ is uniformly norm continuous.
Proposition 4.2. Let $A_n$ be the Yosida approximants of $A$ and $G_n(t)$, $n \in \mathbb{N}$, be the corresponding fundamental solutions of the equations (4.4), then for any $x \in H$,

$$G_n(t)x \to G(t)x \quad \text{as} \quad n \to \infty.$$  

Moreover, this limit converges uniformly with respect to $t$ on any compact interval in $\mathbb{R}_+$.

Proof. By virtue of Proposition 4.1, the fundamental solutions $G(t)$ and $G_n(t)$, $t \in [0, \infty)$, of the equations (2.2) and (4.4) may be represented, respectively, by

$$G(t) = \sum_{m=0}^{\infty} G(m, t), \quad G_n(t) = \sum_{m=0}^{\infty} G_n(m, t), \quad t \in \mathbb{R}_+, \quad (4.7)$$

where

$$G(m, t) = V^m e^{tA}, \quad m \geq 1; \quad G(0, t) = e^{tA}, \quad t \geq 0,$$

and

$$G_n(m, t) = V^m e^{tA_n}, \quad m \geq 1; \quad G_n(0, t) = e^{tA_n}, \quad t \geq 0.$$  

Since $G_n(m, t)x \to G(m, t)x$ as $n \to \infty$ for each $m \in \mathbb{N}$ and $x \in H$, and the series in (4.7) converges in the operator norm topology uniformly with respect to $t$ on any compact interval in $\mathbb{R}_+$, it follows that $G_n(t) \to G(t)$ converges in the strong sense as $n \to \infty$ uniformly with respect to $t$ on any compact interval in $\mathbb{R}_+$. The proof is thus complete.

5 Regularity Property

In this section, we shall concern about regularity property for a class of linear functional stochastic evolution equations. To this end, we shall focus throughout this section on a specific delay operator $F$ (cf. Remark 2.1 or [9]) of (2.7) which is given by

$$F \varphi = B_1 \varphi(-r) + \int_{-r}^{0} a(\theta)B_0 \varphi(\theta)d\theta, \quad \varphi \in C([-r, 0]; H), \quad (5.1)$$

where $B_1, B_0 \in \mathcal{L}(H)$ and $a(\cdot) \in L^1([-r, 0]; \mathbb{R}^1)$. For any $B \in \mathcal{L}_2(K_Q, H)$, we intend to consider the regularity of the following stochastic functional differential equation on $H$

$$\begin{cases}
    dy(t) = Ay(t)dt + B_1 y(t-r)dt + \int_{-r}^{0} a(\theta)B_0 y(t+\theta)d\theta dt + BdW(t), \quad t > 0, \\
y(0) = \phi_0, \quad y_0 = \phi_1, \quad \Phi = (\phi_0, \phi_1) \in \mathcal{H}. 
\end{cases} \quad (5.2)$$

Lemma 5.1. Suppose that $1 \leq p < \infty$ and $a(\cdot) \in L^q([-r, 0]; \mathbb{R}^1)$, $1/p + 1/q = 1$. Then for the delay operator $F$ defined in (5.7), it permits a bounded linear extension $F : L^p([-r, T]; H) \to L^p([0, T]; H)$ which is defined by $(Fy)(t) = Fy_t$, $y \in L^p([-r, T]; H)$. That is, there exists a real number $M_p > 0$ such that

$$\int_{0}^{T} \|(Fy)(t)\|_{H}^{p} dt \leq M_p \int_{-r}^{T} \|y(t)\|_{H}^{p} dt \quad \text{for any} \quad y \in L^p([-r, T]; H) \quad (5.3)$$

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We can extend the structure operator \( F \) to the space \( L^p([-r,0];H) \). It is shown that the operator (2.13) to the space \( L^p([-r,0];H) \) is linear and bounded. On the other hand, for fixed \( T \geq 0 \) and any \( y \in L^p([-r,T];H) \), \( 1 \leq p < \infty \), one has by using Hölder inequality and Fubini’s theorem that

\[
\left( \int_0^t \| F y_s \|_{H^p}^p ds \right)^{1/p} \leq \left( \int_0^t \| B_1 \| \| y(s-r) \|_{H^p}^p ds \right)^{1/p} + \| B_0 \| \| a(\cdot) \|_{L^q([-r,0];\mathbb{R}^1)} \left( \int_0^t \int_{-r}^0 \| y(s+\theta) \|_{H^p}^p d\theta ds \right)^{1/p} \\
= \| B_1 \| \left( \int_0^t \| y(s-r) \|_{H^p}^p ds \right)^{1/p} + \| B_0 \| \| a(\cdot) \|_{L^q([-r,0];\mathbb{R}^1)} \cdot r^{1/p} \left( \int_{-r}^t \| y(s) \|_{H^p}^p ds \right)^{1/p}
\]

where \( 1/p + 1/q = 1 \). As \( W^1,p([-r,T];H) \) is dense in \( L^p([-r,T];H) \), the operator \( F \) can be extended to \( L^p([-r,T];H) \) so that (5.4) remains valid for all \( y \in L^p([-r,T];H) \) and the constant \( M_p > 0 \) in (5.3) is clearly given by

\[
M_p = \left\{ \| B_1 \| + \| B_0 \| \| a(\cdot) \|_{L^q([-r,0];\mathbb{R}^1)} \cdot r^{1/p} \right\}^p > 0.
\]

The proof is complete now. \( \square \)

**Remark 5.1.** We can extend the structure operator \( S \) introduced in (2.10) (resp. \( S \) in (2.13)) to the space \( L^p([-r,0];H) \) (resp. \( L^p([-r,0];\mathcal{L}(H)) \)), \( 1 \leq p < \infty \), by defining

\[
(S\varphi)(\theta) = F\varphi_{-\theta}, \quad \theta \in [-r,0], \text{ almost everywhere} \quad \forall \varphi(\cdot) \in L^p([-r,0];H).
\]

It is shown that the operator \( S \) can be extended to a linear and bounded operator from \( L^p([-r,0];H) \) into itself. Indeed, we have by virtue of (5.3) that

\[
\int_{-r}^0 \| S\varphi(\theta) \|_{H^p}^p d\theta = \int_{-r}^0 \| F\varphi_{-\theta} \|_{H^p}^p d\theta \leq M_p \int_{-r}^0 \| \varphi(\theta) \|_{H^p}^p d\theta \\
= M_p \int_{-r}^0 \| \varphi(\theta) \|_{H^p}^p d\theta, \quad \forall \varphi(\cdot) \in W^1,p([-r,0];H).
\]

**Proposition 5.1.** Let \( G_n(t) \) be the retarded Yosida approximants of \( G(t) \), \( B \in \mathcal{L}_2(K_Q,H) \) and \( F \) be given as in (5.1). Then for any \( T \geq 0 \), the stochastic convolutions \( W^B_G(t) \) and \( W^B_{G_n}(t) \), \( t \in [0,T] \), satisfy that

\[
\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0,T]} \| W^B_G(t) - W^B_{G_n}(t) \|_H^p = 0, \quad p > 2.
\]
Proof. Let \( \alpha \in (1/p, 1/2) \) and recall that for any \( t \in [0, T] \), we have from (3.11) that
\[
W_B^G(t) = \frac{\sin \pi \alpha}{\pi} \left\{ \int_0^t (t-s)^{\alpha-1} Y(t,s) ds + \int_0^t (t-s)^{\alpha-1} G(t-s)Z(s) ds \right\}
\]
(5.8)

where
\[
Y(t,s) = \int_s^0 (s-u)^{-\alpha} \int_{-\tau}^0 G(t-s+\theta) [SG(s-u+\cdot)](\theta)Bd\theta dW(u), \quad s \in [0, T],
\]
and
\[
Z(s) = \int_0^s G(s-u)(s-u)^{-\alpha} BdW(u), \quad s \in [0, T].
\]

In a similar way, we have for \( G_n \) that
\[
W_B^{G_n}(t) = \frac{\sin \pi \alpha}{\pi} \left\{ \int_0^t (t-s)^{\alpha-1} Y_n(t,s) ds + \int_0^t (t-s)^{\alpha-1} G_n(t-s)Z_n(s) ds \right\}
\]
(5.9)

where
\[
Y_n(t,s) = \int_s^0 (s-u)^{-\alpha} \int_{-\tau}^0 G_n(t-s+\theta) [SG_n(s-u+\cdot)](\theta)Bd\theta dW(u), \quad s \in [0, T],
\]
and
\[
Z_n(s) = \int_0^s G_n(s-u)(s-u)^{-\alpha} BdW(u), \quad s \in [0, T].
\]

Hence, we can write for any \( t \in [0, T] \) that
\[
W_B^G(t) - W_B^{G_n}(t) = \frac{\sin \pi \alpha}{\pi} \left\{ \int_0^t (t-s)^{\alpha-1} [Y(t,s) - Y_n(t,s)] ds + \int_0^t (t-s)^{\alpha-1} [G(t-s)Z(s) - G_n(t-s)Z_n(s)] ds \right\}
\]
(5.10)

=: \frac{\sin \pi \alpha}{\pi} (J_1(n,t) + J_2(n,t)).

We first show that
\[
\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0, T]} \|J_1(n,t)\|^p_H = 0.
\]

Indeed, by virtue of Hölder inequality and \( \alpha \in (1/p, 1/2) \), we have that
\[
\mathbb{E} \sup_{t \in [0, T]} \|J_1(n,t)\|^p_H \leq \left( \int_0^T s^{(\alpha-1)q} ds \right)^{p/q} \mathbb{E} \sup_{t \in [0, T]} \int_0^T ||Y(t,s) - Y_n(t,s)||_H^p ds
\]
(5.11)

\[
= \left[ \frac{1}{(\alpha-1)q+1} T^{\alpha \frac{p-1}{q}} \right]^{p/q} \mathbb{E} \sup_{t \in [0, T]} \int_0^T ||Y(t,s) - Y_n(t,s)||_H^p ds
\]

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where $1/p + 1/q = 1$. On the other hand, we have that for $s \in [0, T]$,

\[
Y(t, s) - Y_n(t, s) = \int_{-r}^{0} [G(t-s+\theta) - G_n(t-s+\theta)] I_1(\theta, s) d\theta + \int_{-r}^{0} G_n(t-s+\theta) I_2(\theta, s) d\theta,
\]

where

\[
I_1(\theta, s) = \int_{0}^{s} (s-u)^{-\alpha} [SG(s-u+\cdot)](\theta) BdW(u)
\]

and

\[
I_2(\theta, s) = \int_{0}^{s} (s-u)^{-\alpha} [S(G(s-u+\cdot) - G_n(s-u+\cdot))](\theta) BdW(u).
\]

We claim that

\[
\mathbb{E} \sup_{t \in [0, T]} \int_{0}^{T} \left\| \int_{-r}^{0} [G(t-s+\theta) - G_n(t-s+\theta)] I_1(\theta, s) d\theta \right\|_H^p ds \to 0 \text{ as } n \to \infty. \tag{5.13}
\]

Indeed, by virtue of Hölder’s inequality, Lemma 7.2 in [M, Proposition 4.2] and the fact that $\|G(t)\| \leq C(T)$, $\|G_n(t)\| \leq C(T)$, $t \in [0, T]$, for some $C = C(T) > 0$ and large $n \in \mathbb{N}$, we have by using the well-known Dominated Convergence Theorem that

\[
\mathbb{E} \sup_{t \in [0, T]} \int_{0}^{T} \left\| \int_{-r}^{0} [G(t-s+\theta) - G_n(t-s+\theta)] I_1(\theta, s) d\theta \right\|_H^p ds
\]

\[
\leq r^{1/q} \mathbb{E} \sup_{t \in [0, T]} \int_{0}^{T} \left\| \int_{-r}^{0} [G(t-s+\theta) - G_n(t-s+\theta)] I_1(\theta, s) \right\|^p d\theta ds
\]

\[
\to 0 \text{ as } n \to \infty. \tag{5.14}
\]

In a similar way, it can be also shown that

\[
\mathbb{E} \sup_{t \in [0, T]} \int_{0}^{T} \left\| \int_{-r}^{0} G_n(t-s+\theta) I_2(\theta, s) d\theta \right\|_H^p ds \to 0 \text{ as } n \to \infty.
\]

In order to show $\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0, T]} \|J_2(n, t)\|_H^p = 0$, we first notice that

\[
J_2(n, t) = \int_{0}^{t} (t-s)^{\alpha-1} [G(t-s) - G_n(t-s)] Z(s) ds
\]

\[
+ \int_{0}^{t} (t-s)^{\alpha-1} G_n(t-s)(Z(s) - Z_n(s)) ds =: I_3(n, t) + I_4(n, t). \tag{5.15}
\]

By virtue of Proposition [4.2] Hölder inequality and the well-known Dominated Convergence Theorem, there exists a constant $C_T > 0$ such that

\[
\mathbb{E} \sup_{t \in [0, T]} \|I_3(n, t)\|_H^p ds \leq C_T \mathbb{E} \sup_{t \in [0, T]} \int_{0}^{t} \left\| [G(t-s) - G_n(t-s)] Z(s) \right\|_H^p ds \to 0 \text{ as } n \to \infty.
\]
On the other hand, by a similar argument as above, we can show that

\[ \mathbb{E} \sup_{t \in [0,T]} \| I_4(n,t) \|_H^p ds \to 0 \quad \text{as} \quad n \to \infty \]

by using the Dominated Convergence Theorem. Therefore, we conclude that the limit

\[ \lim_{n \to \infty} \mathbb{E} \sup_{t \in [0,T]} \| J_2(n,t) \|_H^p = 0 \]

and the proof of this theorem is complete now. \( \square \)

In the remainder of this section, we shall apply Proposition 5.1 to Eq. (5.2) to study its regularity property. To this end, we further assume that \( A \) generates an analytic semigroup \( e^{tA}, t \geq 0, \) on \( H \) and \( \text{Tr} Q < \infty. \) It is essential to establish regularity results for the stochastic convolution \( W^B_G(t) = \int_0^t G(t-s)BdW(s), t \geq 0, \)

**Lemma 5.2.** Assume that \( a(\cdot) \) in (5.1) belongs to \( L^q([-r,0]; \mathbb{R}^1) \) for some \( q > 2. \) For any \( B \in L_2(K_Q,H), \) let

\[ Z(t) = \int_0^t G(t-s)BW(s)ds, \quad t \geq 0. \]

Then \( Z(\cdot) \in C^1([0,\infty); \mathcal{D}(A)), \) and moreover

\[ \frac{dZ(t)}{dt} = AZ(t) + B_1Z(t-r) + \int_{-r}^0 a(\theta)B_0Z(t+\theta)d\theta + BW(t) = W^B_G(t), \quad t \geq 0. \]  

(5.16)

**Proof.** By virtue of Proposition 5.1, we know that \( W^B_{G_n}(t) \to W^B_G(t) \) almost surely as \( n \to \infty \) uniformly with respect to \( t \) on any bounded intervals. For any \( n \in \rho(A), \) let us consider the following stochastic functional differential equation on the Hilbert space \( H, \)

\[ \begin{cases}
  dy(t) = A_n y(t)dt + B_1 y(t-r)dt + \int_{-r}^0 a(\theta)B_0 y(t+\theta)d\theta + BdW(t), \quad t \geq 0, \\
  y(t) = 0, \quad -r \leq t \leq 0,
\end{cases} \]

(5.17)

where \( A_n, n \in \rho(A), \) are the Yosida approximants of \( A. \) According to Theorems 4.1 and 4.2 in [9], it is known that there exists a strong solution for the equation (5.17). Moreover, the strong solution is uniquely represented by

\[ y(t) = W^B_{G_n}(t) = \int_0^t G_n(t-s)BdW(s) \quad \text{for} \quad t \geq 0, \quad n \in \rho(A), \]

and \( y(t) = 0 \) if \( t \in [-r,0]. \) Therefore, it follows that for any \( t \geq 0, \)

\[ W^B_{G_n}(t) = A_n \int_0^t W^B_{G_n}(s)ds + B_1 \int_0^t W^B_{G_n}(s-r)ds \]

\[ + \int_0^t \int_{-r}^0 a(\theta)B_0 W^B_{G_n}(s+\theta)d\theta ds + BW(t) \]

\[ = A_n \int_0^t W^B_{G_n}(s)ds + B_1 \int_0^{t-r} W^B_{G_n}(s)ds + \int_0^r a(\theta)B_0 \int_0^{t+\theta} W^B_{G_n}(s)d\theta ds + BW(t). \]  

(5.18)
Let \( Z_n(t) = \int_0^t W_G^B(s)ds, \ t \geq 0 \), then the equality (5.18) yields that for any \( t \geq 0 \),
\[
dZ_n(t) = A_nZ_n(t)dt + B_1Z_n(t - r)dt + \int_{-r}^{0} a(\theta)B_0Z_n(t + \theta)d\theta dt + BW(t)dt,
\]
which immediately implies that
\[
Z_n(t) = \int_0^t G_n(t - s)BW(s)ds, \quad t \geq 0.
\tag{5.19}
\]
In view of Proposition 4.2 and (5.19), it is easy for ones to deduce that almost surely
\[
\lim_{n \to \infty} Z_n(t) = \int_0^t G(t - s)BW(s)ds = Z(t), \quad t \geq 0,
\tag{5.20}
\]
and for any \( t \geq 0 \),
\[
\lim_{n \to \infty} (B_1Z_n(t - r) + \int_{-r}^{0} a(\theta)B_0Z_n(t + \theta)d\theta) = B_1Z(t - r) + \int_{-r}^{0} a(\theta)B_0Z(t + \theta)d\theta. \tag{5.21}
\]
Therefore, the relations (5.18), (5.20) and (5.21) together imply that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} AJ_nZ_n(t) = \lim_{n \to \infty} A_nZ_n(t) = W_G^B(t) - B_1Z(t - r) - \int_{-r}^{0} a(\theta)B_0Z(t + \theta)d\theta - BW(t),
\]
where \( J_n = nR(n, A), \ n \in \rho(A) \). Since \( J_nx \to x \) for any \( x \in H \) as \( n \to \infty \), it follows, in addition to (5.20), that for any \( t \geq 0 \),
\[
J_nZ_n(t) = J_nZ(t) + J_n(Z_n(t) - Z(t)) \to Z(t) \quad \text{as} \quad n \to \infty.
\]
Due to the closedness of operator \( A \), this implies that \( Z(t) \in \mathcal{D}(A) \) and
\[
AZ(t) = W_G^B(t) - B_1Z(t - r) - \int_{-r}^{0} a(\theta)B_0Z(t + \theta)d\theta - BW(t), \quad t \geq 0,
\]
which is exactly the second equality in (5.16). Finally, as \( Z(t) \in \mathcal{D}(A), \ t \geq 0 \), and it is known (cf. Prop. 3.2, [9]) that
\[
\frac{d}{dt} G(t)h = AG(t)h + B_1G(t - r)h + \int_{-r}^{0} a(\theta)B_0G(t + \theta)d\theta \quad \text{for any} \quad h \in \mathcal{D}(A), \ t \geq 0,
\]
we thus obtain the first equality in (5.16),
\[
\frac{dZ(t)}{dt} = AZ(t) + B_1Z(t - r) + \int_{-r}^{0} a(\theta)B_0Z(t + \theta)d\theta + BW(t) \quad \text{for any} \quad t \geq 0.
\]
The proof is now complete.
Now we are ready to present the main results in this section. Recall that the domain \( \mathcal{D}(A) \subseteq H \) is a Banach space under the graph norm \( \| h \|_{\mathcal{D}(A)} := \| h \|_H + \| Ah \|_H, \ h \in \mathcal{D}(A) \). We also introduce two families of intermediate spaces between \( \mathcal{D}(A) \) and \( H \), depending on a parameter \( \alpha \in (0, 1) \):

\[
\mathcal{D}_A(\alpha, \infty) = \left\{ h \in H : \| h \|_\alpha := \sup_{t > 0} \| t^{1-\alpha} A e^{tA} h \|_H < \infty \right\}
\]

and

\[
\mathcal{D}_A(\alpha) = \left\{ h \in H : \lim_{t \to 0} t^{1-\alpha} A e^{tA} h = 0 \right\}.
\]

Obviously, it is true that \( \mathcal{D}(A) \subseteq \mathcal{D}_A(\alpha) \subseteq \mathcal{D}_A(\alpha, \infty) \). We also know (cf. \[3\], \[8\]) that \( \mathcal{D}_A(\alpha, \infty) \) and \( \mathcal{D}_A(\alpha) \) are Banach spaces under the norm \( \| \cdot \|_H + \| \cdot \|_\alpha \).

For any \(-\infty < a \leq b < \infty\) and Banach space \( X \) equipped with the norm \( \| \cdot \|_X \), we introduce Banach space

\[
C^\alpha([a, b]; X) = \left\{ u : [a, b] \to X; \left| u \right|_\alpha := \sup_{t, s \in [a, b], t \neq s} \frac{|u(t) - u(s)|_X}{|t - s|^\alpha} < \infty \right\},
\]

under the norm \( \| u \|_{C^\alpha([a, b]; X)} = \| u \|_{C([a, b]; X)} + \| u \|_\alpha \), Banach space

\[
C^1([a, b]; X) = \left\{ u : [a, b] \to X; u, u' \in C([a, b]; X) \right\}
\]

under the norm \( \| u \|_{C^1([a, b]; X)} = \| u \|_{C([a, b]; X)} + \| u' \|_{C([a, b]; X)} \), and Banach space

\[
C^{1, \alpha}([a, b]; X) = \left\{ u : [a, b] \to X; u \in C^1([a, b]; X), u' \in C^\alpha([a, b]; X) \right\}
\]

under the norm \( \| u \|_{C^{1, \alpha}([a, b]; X)} = \| u \|_{C([a, b]; X)} + \| u' \|_{C^\alpha([a, b]; X)} \).

**Proposition 5.2.** Assume that \( a(\cdot) \) in (5.7) belongs to \( L^q([-r, 0]; \mathbb{R}^1) \) for some \( q > 2 \). Let \( T \geq 0 \) and \( B \in \mathcal{L}_2(K_Q, H) \). Then

(i) for any \( \alpha \in (0, 1/2) \), the stochastic convolution \( W^B_G(t) \) has \( \alpha \)-Hölder continuous trajectories on \([0, T]\);

(ii) for any \( \alpha \in (0, 1/2) \) and \( \gamma \in (0, \frac{1}{2} - \alpha) \), \( W^B_G(t) \in \mathcal{D}((-A)^\gamma), t \in [0, T] \), and the process \((-A)^\gamma W^B_G(\cdot)\) is \( \alpha \)-Hölder continuous.

**Proof.** Note that from Lemma 5.2 we have that \( W^B_G(t) = dZ(t)/dt \) almost surely where \( Z(t) \) is the solution to the initial value problem

\[
\begin{aligned}
\left\{
\begin{array}{l}
dZ(t)/dt = AZ(t) + B_1 Z(t - r) + \int_{-r}^{0} a(\theta) B_0 Z(t + \theta) d\theta + BW(t), \quad t \in [0, T], \\
Z(t) = 0, \quad t \in [-r, 0] \;
\end{array}
\right.
\end{aligned}

(5.22)
\]

It is known that \( W(t) \in C^\alpha([0, T]; H) \) almost surely for \( \alpha \in (0, 1/2) \) and thus the conclusion (i) follows easily.
To show (ii), first note (see, e.g., Lemma 1.1 in [14]) that there is the following inclusion relations
\[ C^{1,\alpha}([0, T]; H) \cap C^\alpha([0, T]; D(A)) \subset C^{1,\alpha-\gamma}([0, T]; D(\gamma, \infty)) \] (5.23)
for all \( \alpha \in (0, 1) \) and \( \gamma \in (0, \alpha) \). Since \( W(t) \in C^\alpha([0, T]; H) \) almost surely for any \( \alpha \in (0, 1/2) \), we have by virtue of Proposition 7.2 and (5.23) that
\[ Z(t) \in C^{1,\alpha}([0, T]; H) \cap C^\alpha([0, T]; D(A)) \subset C^{1,\alpha-\gamma}([0, T]; D(\gamma, \infty)) \]
almost surely for all \( \alpha \in (0, 1/2) \) and \( \gamma \in (0, \alpha) \). Thus \( W^B_G(t) \in C^{\alpha-\gamma}([0, T]; D(\gamma, \infty)) \).
Since \( D(\gamma, \infty) \) is included in \( D((-A)^{\gamma-\epsilon}) \) (cf. Prop. A.13 in [5]) for sufficiently small \( \epsilon > 0 \), the desired conclusion (ii) thus follows. The proof is complete now. \( \square \)

Combining Lemma 5.2 and Proposition 7.2, we obtain the following regularity results for the solution of Eq. (5.2).

**Theorem 5.1.** Let \( T \geq 0 \) and \( B \in L_2(K_Q, H) \). Assume that \( a(\cdot) \) in (5.1) belongs to \( L^q([-r, 0]; \mathbb{R}^1) \) for some \( q > 2 \), \( \phi_0 \in D(A) \) and \( \phi_1 \in C^\alpha([-r, 0]; D(A)) \), \( \alpha \in (0, 1/2) \), such that
\[ A\phi_0 + B_1\phi_1(-r) + \int_{-r}^0 a(\theta)B_0\phi_1(\theta)d\theta \in D(\alpha, \infty). \]
Then the solution \( y(t) \) of Eq. (5.2) has \( \alpha \)-Hölder continuous trajectories on \( [0, T] \). Moreover, for all \( \gamma \in (0, 1/2 - \alpha) \), the solution \( y(t) \in D((-A)^\gamma) \) and the process \( (-A)^\gamma y(t) \) is \( \alpha \)-Hölder continuous on \( t \in [0, T] \).

## 6 Burkholder-Davis-Gundy Inequality

In this section, we shall establish in Theorem 6.1 a retarded version of the well-known Burkholder-Davis-Gundy type of inequality for the stochastic convolution \( W^B_G(t) = \int_0^t G(t-s)B(s)dW(s) \) for any process \( B \in U^2([0, T]; L_2(K_Q, H)) \), \( T \geq 0 \). It is worth pointing out that a similar inequality of this kind was established in [9] under the additional restriction that \( A \) generates a pseudo contraction semigroup \( e^{tA} \), \( t \geq 0 \), in the sense that \( \|e^{tA}\| \leq e^{\alpha t} \) holds true for some constant \( \alpha \in \mathbb{R}^1 \) and all \( t \geq 0 \). By using a factorization method, it is possible for one to remove this restriction to establish a similar inequality. To see this, we first impose, motivated by Lemma 5.1, the following conditions as in [9] on the delay operator \( F \) in (2.1).

Let \( 1 \leq p < \infty \) and assume that the operator \( F \) in (2.1) permits a bounded linear extension \( F : L^p([-r, T]; H) \to L^p([0, T]; H) \) which is defined by \( (Fy)(t) = Fy_t, \ y \in L^p([-r, T]; H) \), i.e., there exists a real number \( M_p > 0 \) such that
\[ \int_0^T \|(Fy)(t)\|_H^p dt \leq M_p \int_{-r}^T \|y(t)\|_H^p dt \text{ for any } y \in L^p([-r, T]; H). \] (6.1)

As a consequence, it is valid that under the condition (6.1), the structure operator \( S \) introduced in (2.10) (resp. \( S \) in (2.13)) can be extended, similarly to (5.6), to a linear
and bounded operator from the space $L^p([-r,0]; H)$ (resp. $L^p([-r,0]; \mathcal{L}(H))$ into itself. Moreover,

$$
\int_{-r}^{0} \| S\varphi(\theta) \|^p_H d\theta \leq M_p \int_{-r}^{0} \| \varphi(\theta) \|^p_H d\theta, \quad \forall \varphi(\cdot) \in L^p([-r,0]; H).
$$

(6.2)

**Theorem 6.1. (Burkholder-Davis-Gundy Inequality)** Let $p > 2$, $T \geq 0$ and $B(t)$ be an $\mathcal{L}_2(K_Q, H)$-valued, $\mathcal{F}_t$-adapted process such that

$$
\mathbb{E} \int_0^T Tr[B(t)QB(t)^*]^{p/2} dt < \infty.
$$

(6.3)

Suppose further that the inequality (6.1) holds, then there exists a number $C = C(T) > 0$ such that

$$
\mathbb{E} \left( \sup_{t \in [0,T]} \left\| \int_0^t G(t-s)B(s)dW(s) \right\|^p_H \right) \leq C(T) \mathbb{E} \int_0^T Tr[B(t)QB(t)^*]^{p/2} dt.
$$

(6.4)

**Proof.** Let $\alpha \in (0, \frac{p-2}{2p})$ and it is known in (3.11) that

$$
W^B_G(t)
= \frac{\sin \pi \alpha}{\pi} \left\{ \int_0^t (t-s)^{\alpha-1} \int_0^s (s-u)^{-\alpha} \int_{-r}^{0} G(t-s+\theta)[SG(s-u+\cdot)](\theta)B(u)d\theta dW(u)ds 
+ \int_0^t (t-s)^{\alpha-1} G(t-s) \int_0^s G(s-u)(s-u)^{-\alpha}B(u)dW(u)ds \right\}
= \frac{\sin \pi \alpha}{\pi} (I_1(t) + I_2(t)).
$$

(6.5)

Firstly, let us estimate the term $I_1(t)$. To this end, we rewrite $I_1(t)$ as

$$
I_1(t) = \int_0^t (t-s)^{\alpha-1} Z(t,s)ds, \quad t \in [0,T],
$$

where

$$
Z(t,s) = \int_0^s (s-u)^{-\alpha} \int_{-r}^{0} G(t-s+\theta)[SG(s-u+\cdot)](\theta)B(u)d\theta dW(u), \quad s \in [0,T].
$$

Recall the well-known Young inequality: for any $p > 1$,

$$
\left| \int_0^T (u * v)(t)dt \right|^p \leq T^{p-1} \left( \int_0^T |u(t)|dt \right)^p \int_0^T |v(t)|^p dt
$$

(6.6)

holds for all $u \in L^1([0,T]; \mathbb{R}^1)$ and $v \in L^p([0,T]; \mathbb{R}^1)$ where $u * v$ is the convolution of the real-valued functions $u$ and $v$. Since $\alpha > 0$, it follows by virtue of (6.6) that

$$
\mathbb{E} \sup_{t \in [0,T]} \| I_1(t) \|^p_H \leq C_{1,T} \mathbb{E} \sup_{t \in [0,T]} \int_0^T \| Z(t,s) \|^p_H ds
$$

(6.7)
where
\[ C_{1,T} = T^{p-1} \left( \int_0^T s^{\alpha-1} ds \right)^p = \frac{1}{\alpha} T^{p+\alpha-1} > 0. \]
Since \( \|G(t)\| \leq C_{2,T} \), for some number \( C_{2,T} > 0 \), one has that for any \( t \in [0, T] \),
\[ \int_0^T \|Z(t, s)\|_H^p ds \]
\[ = \int_0^T \left\| \left( \int_{-r}^0 G(t - s + \theta) \int_0^s (s - u)^{-\alpha} [SG(s - u + \cdot)](\theta) B(u) dW(u) d\theta \right) \right\|_H^p ds \]  (6.8)
\[ \leq C_{2,T}^p \int_0^T \left( \int_{-r}^0 \left\| \int_0^s (s - u)^{-\alpha} [SG(s - u + \cdot)](\theta) B(u) dW(u) \right\|_H d\theta \right)^p ds \]
which, by Hölder inequality, immediately implies that
\[ \mathbb{E} \sup_{t \in [0, T]} \int_0^T \|Z(t, s)\|_H^p ds \]
\[ \leq r^{1/p} C_{2,T}^p \int_0^T \int_{-r}^0 \mathbb{E} \left\| \int_0^s (s - u)^{-\alpha} [SG(s - u + \cdot)](\theta) B(u) dW(u) \right\|_H^p d\theta ds \]  (6.9)
\[ = r^{1/p} C_{2,T}^p \int_0^T J(s) ds, \]
where
\[ J(s) = \int_{-r}^0 \mathbb{E} \left\| \int_0^s (s - u)^{-\alpha} [SG(s - u + \cdot)](\theta) B(u) dW(u) \right\|_H^p d\theta. \]
As for the term \( J(\cdot) \), by using Lemma 7.2 in [5] and Hölder inequality we can obtain that for some number \( C_{3,T} > 0 \) and any \( s \in [0, T] \),
\[ J(s) \]
\[ \leq C_{3,T} \int_{-r}^0 \mathbb{E} \left( \int_0^s (s - u)^{-2\alpha} Tr[SG(s - u + \cdot)(\theta) B(u) Q(SG(s - u + \cdot)(\theta) B(u))]^* d\theta \right)^{p/2} \]
\[ = C_{3,T} \int_{-r}^0 \left( \int_0^s (s - u)^{-2\alpha} d\theta \right)^{p/2} \]
\[ \cdot \mathbb{E} \left( \int_0^s Tr[SG(s - u + \cdot)(\theta) B(u) Q(SG(s - u + \cdot)(\theta) B(u))]^* d\theta \right)^{p/2} d\theta. \]  (6.10)
Since \( 0 < \alpha < \frac{p-2}{2p} \), it is easy to see that \( 1 - \frac{2\alpha p}{p-2} > 0 \) and let
\[ C_{4,T} = \left( \int_0^T (T - s)^{-2\alpha} d\theta \right)^{p/2} = \left( \frac{p - 2}{p - 2 - 2\alpha p} \cdot T^{1-\frac{2\alpha p}{p-2}} \right)^{p/2} < \infty. \]  (6.11)
Then (6.2), (6.9), (6.10) and (6.11) together imply that

$$
E \sup_{t \in [0,T]} \int_0^T \|Z(t,s)\|_{\mathcal{H}}^p ds \\
\leq r^{1-1/p}C^p_{2,T}C_3,T^2 \int_0^T \int_0^s \int_{-r}^0 \int_{-r}^0 Tr \{SG(s-u+\cdot)(\theta)B(u) \\
\cdot Q(SG(s-u+\cdot)(\theta)B(u))^*)^{p/2} d\theta duds \\
\leq r^{1-1/p}C^p_{2,T}C_3,T^2 \int_0^T \int_0^s \int_{-r}^0 \int_{-r}^0 \mathbb{E}\|[SG(s-u+\cdot)](\theta)\|^{p}Tr[B(u)QB(u)^*]^{p/2} d\theta duds \\
\leq r^{1-1/p}C^p_{2,T}C_3,T^2 M_p \int_0^T \int_0^s \int_{-r}^0 \int_{-r}^0 \mathbb{E}\|G(s-u+\theta)\|^{p}Tr[B(u)QB(u)^*]^{p/2} d\theta duds \\
\leq C_{5,T} \int_0^T Tr[B(u)QB(u)^*]^{p/2} du < \infty,
$$

(6.12)

where $C_{5,T} = r^{1-1/p}C^2_{2,T}C_3,T^2 C_4,T M_p T > 0$. Therefore, the relations (6.12) and (6.7) immediately yield that

$$
E \sup_{t \in [0,T]} \|I_1(t)\|_{\mathcal{H}}^p \leq C_{1,T}C_{5,T} \mathbb{E} \int_0^T Tr[B(u)QB(u)^*]^{p/2} du.
$$

(6.13)

In a similar way, we can show that there exists a real number $C_{6,T} > 0$ such that the following inequality holds:

$$
E \sup_{t \in [0,T]} \|I_2(t)\|_{\mathcal{H}}^p \leq C_{6,T} \mathbb{E} \int_0^T Tr[B(u)QB(u)^*]^{p/2} du.
$$

(6.14)

The inequalities (6.13) and (6.14), in addition to (6.5), imply the desired result (6.4). The proof is thus complete.

7 Appendix

In this appendix, we shall recall and establish some regularity properties for a class of deterministic functional differential equations on the Hilbert space $H$. Firstly, consider the deterministic equation without time delays

$$
\left\{ \begin{align*}
\frac{dy(t)}{dt} &= Ay(t) + f(t), & t \in [0,T], & T \geq 0, \\
y(0) &= \phi_0 \in H,
\end{align*} \right.
$$

(7.1)

where $A$ generates an analytic semigroup $e^{tA}$, $t \geq 0$ and $f \in L^1([0,T]; H)$. The following regularity result is well established and its proofs are referred to the existing literature, e.g., [8] or [16].
Proposition 7.1. Let $\alpha \in (0, 1)$ and suppose that
\[
\phi_0 \in \mathcal{D}(A), \quad f \in C^\alpha([0, T]; H), \quad A\phi_0 + f(0) \in \mathcal{D}_A(\alpha, \infty),
\]
then the function
\[
y(t) = e^{tA}\phi_0 + \int_0^t e^{(t-s)A}f(s)ds \in C^1([0, T]; H) \cap C([0, T]; \mathcal{D}(A)), \quad t \in [0, T],
\]
is the unique (classical) solution of \((7.1)\). Moreover, we have
\[
y \in C^{1, \alpha}([0, T]; H) \cap C^\alpha([0, T]; \mathcal{D}(A)), \quad t \in [0, T],
\]
and the estimate
\[
\max\{\|y\|_{C^{1, \alpha}([0, T]; H)}, \|y\|_{C^\alpha([0, T]; \mathcal{D}(A))}\} \leq C\{\|f\|_{C^\alpha([0, T]; H)} + \|A\phi_0 + f(0)\|_{\mathcal{D}_A(\alpha, \infty)} + \|\phi_0\|_H\},
\]
where $C = C(\alpha, T)$ is some positive number.

The main objective in the appendix is to establish the existence and uniqueness of (classical) solutions for a class of linear functional differential equations \((7.4)\) below.

Let $T \geq 0$ and $r > 0$ and $A$ generate an analytic semigroup $e^{tA}$, $t \geq 0$, on $H$ and $B_i \in \mathcal{L}(H)$, $i = 0, 1$. Suppose that $f \in L^1([0, T]; H)$, $a \in L^1([-r, 0]; \mathbb{R}^1)$ and $\Phi = (\phi_0, \phi_1) \in \mathcal{H}$. Consider the following linear differential equation with time delays
\[
\begin{cases}
dy(t)/dt = Ay(t) + B_1y(t - r) + \int_{-r}^{0} a(\theta)B_0y(t + \theta)d\theta + f(t), & 0 \leq t \leq T, \\
y(0) = \phi_0, \quad y(t) = \phi_1(t), & t \in [-r, 0),
\end{cases}
\]
(7.4)

Definition 7.1. A function $y : [-r, T] \to H$ which is differentiable on $[0, T]$ almost everywhere is called a (classical) solution of the initial value problem \((7.4)\) on $[-r, T]$ if $y \in C([0, T]; \mathcal{D}(A)) \cap C^1([0, T]; H)$ and the equation \((7.4)\) is verified.

Proposition 7.2. Suppose that $f \in C^\alpha([0, T]; H)$, $\phi_0 \in \mathcal{D}(A)$ and $\phi_1 \in C^\alpha([-r, 0]; \mathcal{D}(A))$ such that
\[
A\phi_0 + B_1\phi_1(-r) + \int_{-r}^{0} a(\theta)B_0\phi_1(\theta)d\theta + f(0) \in \mathcal{D}_A(\alpha, \infty),
\]
then there is a unique (classical) solution $y$ of the problem \((7.4)\) with
\[
y \in C^\alpha([0, T]; \mathcal{D}(A)) \cap C^{1, \alpha}([0, T]; H).
\]
Moreover, there exists a number $C = C(T, \alpha, a, B_0) > 0$ such that
\[
\max\{\|y\|_{C^\alpha([0, T]; \mathcal{D}(A))}, \|y\|_{C^{1, \alpha}([0, T]; H)}\} \leq C\left(\|f\|_{C^\alpha([0, T]; H)} + \|\phi_1\|_{C^\alpha([-r, 0]; \mathcal{D}(A))} + \|\phi_0\|_H \right.
\]
\[
+ \left.\|A\phi_0 + B_1\phi_1(-r) + \int_{-r}^{0} a(\theta)B_0\phi_1(\theta)d\theta + f(0)\|_{\mathcal{D}_A(\alpha, \infty)}\right).
\]
(7.5)
Proof. For any \( \delta > 0 \), consider the following closed subset of \( C^\alpha([0, \delta]; \mathcal{D}(A)) \):

\[
E = \{ \bar{y} \in C^\alpha([0, \delta]; \mathcal{D}(A)) : \bar{y}(0) = \phi_0 \}.
\]

We can associate to each \( \bar{y} \in E \) a function \( y \in C^\alpha([-r, \delta]; \mathcal{D}(A)) \) by

\[
y(t) = \begin{cases} 
\phi_1(t), & -r \leq t < 0, \\
\bar{y}(t), & t \in [0, \delta],
\end{cases}
\]

and define the mapping \( \Xi \) for any \( t \in [0, \delta] \) by

\[
(\Xi \bar{y})(t) = e^{tA} \phi_0 + \int_0^t e^{(t-s)A} \left[ B_1 \phi_1(s-r) + \int_{-r}^0 a(\theta) B_0 y(s+\theta) d\theta + f(s) \right] ds.
\]

As \( \phi_1 \in C^\alpha([-r, 0]; \mathcal{D}(A)) \) and \( B_1 \in \mathcal{L}(H) \), it is immediate that \( B_1 \phi_1(\cdot - r) \in C^\alpha([0, \delta]; H) \). On the other hand, for any \( y \in C^\alpha([-\delta, \delta]; \mathcal{D}(A)) \), it is easy to see that

\[
\int_{-r}^0 a(\theta) B_0 y(s+\theta) d\theta \in C^\alpha([0, \delta]; H), \quad s \in [0, \delta],
\]

and moreover

\[
\left\| \int_{-r}^0 a(\theta) B_0 y(\cdot + \theta) d\theta \right\|_{C^\alpha([0, \delta]; H)} \leq \|B_0\| \|a\|_{L^1([-r, 0]; \mathbb{R}^1)} \|y\|_{C^\alpha([-r, \delta]; \mathcal{D}(A))}.
\]

(7.7)

This means, in addition to \( f \in C^\alpha([0, \delta]; H) \), that the function

\[
B_1 \phi_1(\cdot - r) + \int_{-r}^0 a(\theta) B_0 y(\cdot + \theta) d\theta + f(\cdot) \in C^\alpha([0, \delta]; H).
\]

(7.8)

Taking into account the fact that \( \phi_0 \in \mathcal{D}(A) \) and

\[
A\phi_0 + B_1 \phi_1(-r) + \int_{-r}^0 a(\theta) B_0 y(\theta) d\theta + f(0) \in \mathcal{D}_A(\alpha, \infty),
\]

we deduce from Proposition 7.1 that there exists a solution of the equation (7.3) in \([0, \delta]\) if and only if there exists an element \( \bar{y} \in E \) such that

\[
\Xi \bar{y} = \bar{y},
\]

and in this case it is true that \( y \in C^{1,\alpha}([0, \delta]; H) \) by virtue of (7.2).

Next we shall prove that the map \( \Xi \) is a contraction on \( E \) for properly chosen \( \delta > 0 \). To this end, for any \( \bar{y}_i \in E \) define \( y_i, \ i = 1, 2 \), according to (7.6). Then we have for \( t \in [0, \delta] \) that

\[
\Xi \bar{y}_1(t) - \Xi \bar{y}_2(t) = \int_0^t e^{(t-s)A} \int_{-r}^0 a(\theta) B_0 (y_1(s+\theta) - y_2(s+\theta)) d\theta ds.
\]

(7.9)

By analogy with (7.7), it is not difficult to see that

\[
\int_{-r}^0 a(\theta) B_0 (y_1(\cdot + \theta) - y_2(\cdot + \theta)) d\theta \in C^\alpha([0, \delta]; H),
\]
and thus applying (7.3) to (7.9) yields that
\[
\| \Xi \bar{y}_1 - \Xi \bar{y}_2 \|_{C^\alpha([0,\delta];\mathcal{D}(A))} \leq C \left\| \int_{-\tau}^{\tau} a(\theta) B_0(y_1(\cdot + \theta) - y_2(\cdot + \theta))d\theta \right\|_{C^\alpha([0,\delta];H)}
\] (7.10)
where $C > 0$ is the constant given in (7.3). On the other hand, let $\delta \in (0, r)$ and as $y_1(t) - y_2(t) = 0$ for $t \in [-\tau, 0]$, it is easy to see that
\[
\left\| \int_{-\tau}^{\tau} a(\theta) B_0(y_1(\cdot + \theta) - y_2(\cdot + \theta))d\theta \right\|_{C^\alpha([0,\delta];H)} \leq \| B_0 \|_{L^1([-\delta,0];\mathbb{R}^n)} \| y_1 - y_2 \|_{C^\alpha([0,\delta];\mathcal{D}(A))},
\]
which, in addition to (7.10), immediately yields that
\[
\| \Xi \bar{y}_1 - \Xi \bar{y}_2 \|_{C^\alpha([0,\delta];\mathcal{D}(A))} \leq C \| B_0 \|_{L^1([-\delta,0];\mathbb{R}^n)} \| \bar{y}_1 - \bar{y}_2 \|_{C^\alpha([0,\delta];\mathcal{D}(A))}.
\] (7.11)
This implies that $\Xi$ is a contraction if we choose $\delta > 0$ small enough ($\delta < r$) such that
\[
C \| B_0 \|_{L^1([-\delta,0];\mathbb{R}^n)} < 1,
\]
and thus guarantee the existence of a unique solution $y \in C^\alpha([0, \delta]; \mathcal{D}(A))$ of (7.4) in $[0, \delta]$.

Next we shall show the estimate (7.5). Firstly, it is easy to see that for $t \in [0, \delta]$, there is
\[
\bar{y}(t) = e^{t A} \phi_0 + \int_0^t e^{(t-s) A} \left[ B_1 \phi_1(s - r) + \int_{-\tau}^{\tau} a(\theta) B_0 y(s + \theta) d\theta + f(s) \right] ds,
\]
and so, by virtue of (7.3), it follows that
\[
\| \bar{y} \|_{C^\alpha([0,\delta];\mathcal{D}(A))} \leq C \left\{ \| A \phi_1 \|_{C^\alpha([-r,\delta-r];H)} + \left\| \int_{-\tau}^{\tau} a(\theta) B_0 y(\cdot + \theta) d\theta \right\|_{C^\alpha([0,\delta];H)} + \| \phi_0 \|_{H} \right. \nonumber
\] \[+ \| f \|_{C^\alpha([0,\delta];H)} + \| A \phi_0 + B_1 \phi_1(-r) + \int_{-\tau}^{\tau} a(\theta) B_0 \phi_1(\theta) d\theta + f(0) \|_{\mathcal{D}(A(a,\infty))} \right\}.
\] (7.12)
Note that we have the estimates
\[
\| A \phi_1 \|_{C^\alpha([-r,\delta-r];H)} \leq \| \phi_1 \|_{C^\alpha([-r,0];\mathcal{D}(A))},
\] (7.13)
and
\[
\left\| \int_{-\tau}^{\tau} a(\theta) B_0 y(\cdot + \theta) \right\|_{C^\alpha([0,\delta];H)} \leq \| B_0 \| \left\{ \| a \|_{L^1([-r,0];\mathbb{R}^n)} \| \phi_1 \|_{C^\alpha([-r,0];\mathcal{D}(A))} \right. \nonumber
\] \[+ \| a \|_{L^1([-\delta,0];\mathbb{R}^n)} \| \bar{y} \|_{C^\alpha([0,\delta];\mathcal{D}(A))} \right\}.
\] (7.14)
Thus, by substituting (7.13) and (7.14) into (7.12), we obtain that
\[
\| \bar{y} \|_{C^\alpha([0,\delta];\mathcal{D}(A))} \leq C \left( 1 + \| B_0 \|_{L^1([-r,0];\mathbb{R}^n)} \right) \left\{ \| f \|_{C^\alpha([0,\delta];H)} + \| \phi_1 \|_{C^\alpha([0,\delta];\mathcal{D}(A))} + \| \phi_0 \|_{H} \right. \nonumber
\] \[+ \| A \phi_0 + B_1 \phi_1(-r) + \int_{-\tau}^{\tau} a(\theta) B_0 \phi_1(\theta) d\theta + f(0) \|_{\mathcal{D}(A(a,\infty))} \right\}.
\] (7.15)
On the other hand, let us observe that
\[
\|\bar{y}\|_{C^1([0,\delta];H)} = \|\bar{y}\|_{C([0,\delta];H)} + \|\bar{y}'\|_{C^0([0,\delta];H)} \\
\leq \|\bar{y}\|_{C^0([0,\delta];D(A))} + \left\| A\bar{y}(\cdot, \theta) + B_1\phi_1(\cdot, \theta) + \int_{-\theta}^{0} a(\theta) B_0 y(\cdot, \theta) d\theta + f(\cdot) \right\|_{C^0([0,\delta];H)} \\
\leq 2\|\bar{y}\|_{C^0([0,\delta];D(A))} + \left\| B_1\phi_1(\cdot, \theta) + \int_{-\theta}^{0} a(\theta) B_0 y(\cdot, \theta) d\theta + f(\cdot) \right\|_{C^0([0,\delta];H)}.  \tag{7.16}
\]

However, we know from (7.14) that
\[
\left\| B_1\phi_1(\cdot, \theta) + \int_{-\theta}^{0} a(\theta) B_0 y(\cdot, \theta) d\theta + f(\cdot) \right\|_{C^0([0,\delta];H)} \\
\leq \|B_1\|\|\phi_1\|_{C^0([0,\delta];D(A))} + \|B_0\|\left\{ \|a\|_{L^1([-\delta,0];\mathbb{R}^1)} \|\phi_1\|_{C^0([0,\delta];D(A))} \right\} + \|f\|_{C^0([0,\delta];H)} \tag{7.17}
\]
Hence, (7.15), (7.16) and (7.17) together yield the desired result (7.5) on [0, \delta].

The existence of a solution on the whole interval [0, T] and the associated relation (7.5) can be similarly established by repeating the above arguments on [\delta, 2\delta], [2\delta, 3\delta], \ldots, and the proof is thus complete.

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