BLOW UP FOR INITIAL BOUNDARY VALUE PROBLEM OF CRITICAL SEMILINEAR WAVE EQUATION IN TWO SPACE DIMENSIONS

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

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Abstract. This paper focuses on the initial boundary value problem of semilinear wave equation in exterior domain in two space dimensions with critical power. Based on the contradiction argument, we prove that the solution will blow up in a finite time. This complements the existence result of supercritical case by Smith, Sogge and Wang [20] and blow up result of subcritical case by Li and Wang [14] in two space dimensions.

1. Introduction. We consider the initial boundary value problem of the semilinear wave equation with Dirichlet boundary condition

\[
\begin{aligned}
&u_{tt} - \Delta u = |u|^p, \quad t > 0, \quad x \in B_1^c = \mathbb{R}^2 \setminus B_1(O), \\
u(0, x) = \varepsilon f(x), \quad u_t(0, x) = \varepsilon g(x), \quad x \in B_1^c = \mathbb{R}^2 \setminus B_1(O), \\
u|_{\partial B_1(O)} = 0,
\end{aligned}
\]  

(1)

where \( p = p_c(2) \) is the positive root of the following quadratic equation

\[ p^2 - 3p - 2 = 0. \]  

(2)

And \( B_1(O) \) denotes the unit ball centered at the origin in \( \mathbb{R}^2 \), \( \varepsilon \) represents the smallness of the data.

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For semilinear wave equations with small data, we often come to Strauss conjecture, which concerns the global existence and blow up of solutions to the following Cauchy problem

$$u_{tt} - \Delta u = |u|^p, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$  \hspace{1cm} (3)

The pioneering work to this problem is due to John [8], in which the author proved that the Cauchy problem (3) in $\mathbb{R}^3$ admits a critical power $p = p_c(3) = 1 + \sqrt{2}$: when $1 < p < 1 + \sqrt{2}$ the solution blows up in a finite time while $p > 1 + \sqrt{2}$ the solution exists globally in time. Then Strauss [22] made an insightful conjecture that the Cauchy problem (3) in $\mathbb{R}^n (n \geq 2)$ with small data has global solutions when $p$ is larger than a critical power $p_c(n)$, which is the positive root of the quadratic equation

$$(n - 1)p^2 - (n + 1)p - 2 = 0. \hspace{1cm} (4)$$

Since then there is extensive literature on this conjecture, for example, Glassey [4, 5], Sideris [19], Rammaha [17], Jiao and Zhou [7], Schaeffer [18], Zhou [30], Kubo [9], Lindblad and Sogge [15], Georgiev, Lindblad and Sogge [3], Tataru [25], Takamura [23], Takamura and Wakasa [24], the authors [10]. The most difficult part of this conjecture is the case that $p = p_c(n)$ and $n \geq 4$, which was finally solved by Yordanov and Zhang [28] and Zhou [31] recently. One can find more details about the Strauss conjecture in the survey paper [26].

After this, people are trying to generalize the results to manifolds. The first direction is to consider the problem in asymptotically Euclidean non-trapping Riemannian manifolds, Schwarzschild and Kerr spacetime. We refer the reader to [1] for blow up result and to [21, 16, 27] for global existence result. The second one is to study the initial boundary value problem in exterior domain. We expect that it admits the same critical exponent as that of the Cauchy problem, based on some known results. Du et al [2] proved global existence for $p > p_c(4)$ and then Hidano et al [6] obtained the same result for $p > p_c(n)$ and $n = 3, 4$(see also Yu [29] for trapping obstacles case). The case of $p > p_c(2)$ is due to the work of Smith, Sogge and Wang [20]. In the opposite direction, Zhou and Han [32] established blow-up result and the upper bound of lifespan in the case $1 < p < p_c(n)$ and $n \geq 3$. Yu [29] gave the obstacle version of the sharp lifespan for semilinear wave equations when $p < p_c(3)$. Not much later, Li and Wang [14] showed blow-up result for $1 < p < p_c(2)$. The authors [11] studied the critical case $p = p_c(3)$ and proved the nonexistence of global solution. Very recently, the authors [12] showed blow up result and established the upper bound of the lifespan for $p = p_c(n)$ and $n \geq 5$. Thus, there are two cases still left open for initial boundary value problem in exterior domain: $p = p_c(2)$ and $p = p_c(4)$.

This paper is devoted to studying the case of $p = p_c(2)$. We aim to establish blow up result to initial boundary value problem (1) outside a unit ball. Compared to the critical Cauchy problem in $\mathbb{R}^2$, which was solved in [18], the initial boundary value problem has no explicit expression of the solution. The method used in [12] to solve the critical initial boundary value problem for $n \geq 5$ heavily relies on the fast decay rate of a special solution of the linear wave equation. In the three space dimensions case, the authors [11] found a radial, positive solution of the linear wave equation to overcome the difficulty of decay rate, while in both two and four space dimensions cases we failed to find such an exact solution. However, we find that there is big difference of the harmonic function satisfying the Dirichlet boundary condition outside the unit ball between $n = 2$ and $n \geq 3$. Indeed, we can give the formula of the harmonic functions exactly: $H(r) = \ln r$ if $n = 2$ and $H(r) = 1 - \frac{1}{r^{n-1}}$.\hfill \hfill \hfill \hfill \hfill
if $n \geq 3$. It is easy to see that they have different asymptotic behavior. Another key ingredient is that we use a special test function $\phi_q$ which is positive, homogeneous of degree $q (q > 0)$ and radially symmetric. This kind of function was first introduced by Zhou [31]. We find that we gain more decay from its derivative. By combining the behavior of $\ln r$ and decay property of the derivative of $\phi_q$, we finally obtain blow-up result for problem (1).

**Remark 1.** As mentioned above, the harmonic function in four space dimensions satisfying the Dirichlet boundary condition outside the unit ball is $1 - \frac{1}{r}$, the asymptotic behavior of which is not good enough to show blow up by using the method of this paper.

Our main result is as follows.

**Theorem 1.1.** Let $(f, g) \in H_0^1(B_1^c) \times L^2(B_1^c)$ and satisfy the Dirichlet boundary condition with $\text{supp}(f, g) \subset B_1^c \cap \{|x| < R\}$ and $R > e$. We also assume that $f(x)$ is nontrivial. Let $u(t)$ solve the initial boundary value problem (1) with $p = p_c(2)$ in $[0, T]$. Then there exists a constant $C > 0$ depending only on the initial data such that $T \leq C$.

Here and in what follows $C$ will denote various positive constants which may be different from line to line.

We arrange the paper as follows. In section 2 we introduce a special function $\phi_q$ and give some properties. The main theorem is proved in section 3.

2. **Test function.** We will use contradiction argument in the proof. This method relies on an appropriate test function. A key ingredient of the test function is a solution of the linear wave equation

$$u_{tt} - \Delta u = 0, \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$$

on $|x| \leq t$ with the special form

$$\phi_q = (t + |x|)^{-q} h \left( \frac{2|x|}{t + |x|} \right),$$

(5)

where $q = \frac{1}{2} - \frac{1}{p_c(2)}$. Direct computation shows that $h = h_q$ satisfies the ordinary differential equations

$$z(1-z)h''(z) + \left[ 1 - (q + \frac{3}{2})z \right] h'(z) - \frac{1}{2} q h(z) = 0.$$  

(6)

This type of solution was first introduced in the work [31], one can also find the details in a new book [13]. For convenience, we give some properties of this solution.

**Corollary 1.** Let $h_q$ be the function in formula (5), then we have

$$h_q(z) > 0, \ 0 \leq z < 1.$$  

(7)

Moreover, if $0 < q < \frac{1}{2}$, then $h_q(z)$ is continuous at $z = 1$ and there exists a positive constant $C$ such that

$$C^{-1} \leq h_q(z) \leq C, \ 0 \leq z \leq 1,$$

(8)

while if $q > \frac{1}{2}$, $h_q(z)$ has the following behavior

$$C^{-1} (1 - z)^{\frac{1}{2} - q} \leq h_q(z) \leq C (1 - z)^{\frac{1}{2} - q}, \ 0 \leq z \leq 1.$$  

(9)
Corollary 2. Let \( q = \frac{1}{2} - \frac{1}{p_c(2)} \), then we have
\[
C^{-1}(t + 1)^{-q} \leq \phi_q \leq C(t + 1)^{-q}.
\] (10)

Corollary 3. Let \( q = \frac{1}{2} - \frac{1}{p_c(2)} \), then we have
\[
\partial_t \phi_q(t, x) = -q\phi_{q+1}(t, x),
\] (11)
and moreover,
\[
\phi_{q+1} \leq C(t + 1)^{-\frac{1}{2}}(t + 1 - r)^{-q(\frac{p}{p} - 1)}.
\] (12)

One can find the proof of Corollary 1-Corollary 3 in [31], also we refer the reader to [13].

Let \( \Phi_q(t, x) = \phi_q(t + R + 1, x) \), we then construct the test function
\[
\tilde{\Phi}_q(t, x) = \Phi_q(t, x) \ln r.
\] (13)

Note that \( \tilde{\Phi}_q(t, x) \) satisfies the Dirichlet boundary condition on \( \partial B_1^c \).

3. Proof of the main result. In this section, we prove the main theorem.

Lemma 3.1. Let \( u \) solve the initial boundary value problem (1) and \( \phi_0(x) = \ln r \). Let the initial data satisfy the same assumption as that in Theorem 1.1. Then there exists a positive constant \( C \) such that
\[
\int_{B_1^c} \phi_0 |u|^p dx \geq C(t + R + 1)^{-\frac{p}{p} - 1} \ln(t + R + 1).
\] (14)

Proof. The proof of the above lemma can be found in [14]. We present an outline of the proof.

Step 1. As stated in [32] and [14], there exists a positive function \( w_1(t, x) \) satisfying the following boundary value problem
\[
\begin{align*}
\Delta w_1(t, x) &= w_1(t, x), \quad x \in B_1^c = \mathbb{R}^2 \setminus B_1(O), \\
 w_1(x)|_{\partial B_1(O)} &= 0, \\
 w_1(x) &\to \int_{S^1} e^{x \cdot \omega} d\omega, \quad |x| \to \infty.
\end{align*}
\] (15)

Set \( w(t, x) = e^{-t} w_1(t, x) \), then for \( p > 1 \) and \( t \geq 0 \) ([14], Lemma 2.5),
\[
\int_{B_1^c \cap |x| \leq t + R} (\ln r)^{-\frac{p}{p'}} w(t, x)^{\frac{p}{p'}} dx \leq C(t + R)^{1 - \frac{p'}{p}} (\ln(t + R))^{-\frac{p}{p'}}.
\] (16)
where \( p' = \frac{p}{p} - 1 \).

Step 2. Let \( F(t) = \int_{B_1^c} w(t, x) u(t, x) dx \), then there exists a constant \( C > 0 \) depending only on the initial data and \( w_1(x) \) such that ([32], Lemma 3.1)
\[
F(t) \geq C.
\] (17)

Step 3. By Hölder inequality, (16) and (17) we have
\[
\int_{B_1^c} \phi_0 |u|^p dx \geq \frac{|F(t)|^p}{\left( \int_{B_1^c \cap |x| \leq t + R} (\ln r)^{-\frac{p}{p'}} w(t, x)^{\frac{p}{p'}} dx \right)^{p-1}}
\geq C(t + R + 1)^{-\frac{p}{p} - 1} \ln(t + R + 1).
\] (18)
Lemma 3.2. Let \( \Phi_q(t, x) \) be as in section 2, then we have
\[
|\partial_r \Phi_q(t, x)| \leq \frac{Cr}{(t + R + 1 + r)^{q + 2}} \left( t + R + 1 + r \right)^{-q - \frac{1}{2}}. \tag{19}
\]

Proof. As in [31], \( h_q(z) \) can be taken as
\[
h_q(z) = F(q, \frac{1}{2}, 1, z), \tag{20}
\]
where \( F \) is the hypergeometric function defined by
\[
F(q, \frac{1}{2}, 1, z) = \sum_{n=0}^{\infty} \frac{(q)_n (\frac{1}{2})_n}{2^n n!} z^n, \tag{21}
\]
with
\[
\begin{align*}
(\lambda)_0 &= 1, \\
(\lambda)_k &= \lambda(\lambda + 1) \cdots (\lambda + k - 1)(k \geq 1).
\end{align*} \tag{22}
\]
Direct computation yields
\[
\partial_r \Phi_q(t, x) = (t + R + 1 + r)^{-q - 1} \left( -q h_q + (2 - z) h'_q \right) = (t + R + 1 + r)^{-q - 1} \alpha(z),
\]
where \( z = \frac{r}{t + R + 1 + r} \) and \( \alpha(z) = -q h_q + (2 - z) h'_q. \)
It is easy to get from (20), (21) and (22) that
\[
\begin{align*}
h_q(0) &= F(q, \frac{1}{2}, 1, 0) = \frac{(q)_0 (\frac{1}{2})_0}{2!(1)_0} = \frac{1}{2}, \\
h'_q(0) &= \frac{(q)_1 (\frac{1}{2})_1}{2!(1)_1} = \frac{q}{4}.
\end{align*} \tag{24}
\]
then \( \alpha(0) = 0. \) Since \( \alpha(z) \) is a power series of \( z \) and hence \( \alpha(z) \) can be written as \( z^\beta(z) \), then we gain one more decay from \( z. \) And for \( \beta(z) \), we claim that \( |\beta(z)| \leq C(1 - z)^{-(q + \frac{1}{2})} \) for \( z \in [0, 1). \) Indeed, when \( z \in [0, \frac{1}{2}] \), \( \beta(z) \) is analytical and then continuous, there exists a constant \( C > 0 \) such that \( |\beta(z)| \leq C \), which means
\[
|(1 - z)^{q + \frac{1}{2}} \beta(z)| \leq C, \quad z \in [0, \frac{1}{2}].
\]
For \( z \in [\frac{1}{2}, 1] \), \( h_q(z) \) and \( (1 - z)^{q + \frac{1}{2}} h'_q(z) \) is analytical and continuous, and so is \( (1 - z)^{q + \frac{1}{2}} \beta(z) \), then there exists a positive constant \( C > 0 \) such that
\[
|(1 - z)^{q + \frac{1}{2}} \beta(z)| \leq C, \quad z \in [\frac{1}{2}, 1],
\]
which yields
\[
|\beta(z)| \leq C(1 - z)^{-(q + \frac{1}{2})}, \quad z \in [\frac{1}{2}, 1],
\]
we then finish the claim and hence the estimate (19) holds.

Now we are in a position to prove Theorem 1.1. Easy calculation shows that
\[
\Box \Phi_q = \partial_t^2 \Phi_q - \Delta \Phi_q = -2 \partial_r \Phi_q \partial_r \phi_0. \tag{25}
\]
Multiplying the both sides of the equation in (1) with \( \Phi_q \) and then integrating over \( B_1^c \) yields
\[
\int_{B_1^c} (\Phi_q \Box u - u \Box \Phi_q) dx + \int_{B_1^c} u \Box \Phi_q dx = \int_{B_1^c} \Phi_q |u|^p dx. \tag{26}
\]
Making use of integration by parts and (25) we have
\[
\frac{d^2}{dt^2} \int_{B_i^1} \bar{\Phi}_q u dx - 2 \frac{d}{dt} \int_{B_i^1} \bar{\Phi}_q u dx + \int_{B_i^1} u \bar{\Phi}_q dx
= \frac{d^2}{dt^2} \int_{B_i^1} \bar{\Phi}_q u dx - 2 \frac{d}{dt} \int_{B_i^1} \bar{\Phi}_q u dx - 2 \int_{B_i^1} u \partial_r \Phi_q \partial_r \phi_0 dx
\]
\[
= \int_{B_i^1} \bar{\Phi}_q |u|^p dx. \tag{27}
\]

Let \( \theta = \theta(\tau) \) be a cut-off function satisfying \( 0 \leq \theta(\tau) \leq 1 \) such that
\[
\theta(\tau) = \begin{cases} 
1, & 0 \leq \tau \leq \frac{1}{2}, \\
0, & \tau \geq 1,
\end{cases}
\]
\[
\text{and } |\theta'(\tau)|, |\theta''(\tau)| \leq C. \tag{28}
\]

Multiplying both sides of (27) with \( \theta^2(\frac{t}{T}) \ln^{-2}(t + R + 1) \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( T > R + 1 \), and then integrating with \( t \) over \((0, \infty)\) yields
\[
J(T) + K(T) + H(T)
= \int_0^\infty \theta^2(\frac{t}{T}) \ln^{-2}(t + R + 1) \partial_t^2 \int_{B_i^1} \bar{\Phi}_q u dx dt
- 2 \int_0^\infty \theta^2(\frac{t}{T}) \ln^{-2}(t + R + 1) \partial_t \int_{B_i^1} \bar{\Phi}_q u dx dt
- 2 \int_0^\infty \theta^2(\frac{t}{T}) \ln^{-2}(t + R + 1) \int_{B_i^1} u \partial_r \Phi_q \partial_r \phi_0 dx dt
\]
\[
= \int_0^\infty \theta^2(\frac{t}{T}) \ln^{-2}(t + R + 1) \int_{B_i^1} \bar{\Phi}_q |u|^p dx dt
=: I(T). \tag{29}
\]

We first estimate \( I(T) \) as
\[
I(T) \geq \int_0^{\frac{T}{2}} \ln^{-2}(t + R + 1) \int_{B_i^1} \Phi_q \phi_0 |u|^p dx dt
\geq C \int_0^{\frac{T}{2}} \ln^{-2}(t + R + 1)(t + R + 1)^{-q} \int_{B_i^1} \phi_0 |u|^p dx dt
\geq C \int_0^{\frac{T}{2}} \ln^{-1}(t + R + 1)(t + R + 1)^{-q-(\frac{p}{2}-1)} dt
\geq C \left( \ln \ln \left( \frac{T}{2} + R + 1 \right) - \ln \ln (R + 1) \right), \tag{30}
\]
where we used (10) in Corollary 2 and (14) in Lemma 3.1.
By Lemma 3.2 and (10) in Corollary 2, we have

\[
\int_{B_1^T} (\partial_r \Phi_q \partial_r \phi_0) \theta^{(q+\frac{1}{2})} \Phi_q \, dx
\]

\[
\leq C(t + R + 1) \int_{B_1^T} \frac{(t + R + 1 - r)^{-p'(q+\frac{1}{2})}}{(t + R + 1 + r)^{p'(q+\frac{1}{2})}} (\ln r)^{-\frac{p'}{2}} \, dx
\]

\[
\leq C(t + R + 1)^{\frac{p'-2}{4}} \int_1^{t+R} (t + R + 1 - r)^{-p'(q+\frac{1}{2})} (\ln r)^{-\frac{p'}{2}} r \, dr
\]

\[
\leq C(t + R + 1)^{\frac{p'-2}{4}} \int_1^{t+R} (t + R + 1 - r)^{-1} (\ln r)^{-\frac{p'}{2}} r \, dr
\]

\[
\leq C(t + R + 1)^{\frac{p'-2}{4} + 1} \ln^{-\frac{p'}{2}} (t + R + 1),
\]

then the term \(H(T)\) can be estimated as

\[
H(T) \leq C \int_0^T \theta^{2p'} \left( \frac{t}{T} \right) \ln^{-2}(t + R + 1)
\]

\[
\times \left( \int_{B_1^T} (\partial_r \Phi_q \partial_r \phi_0) \theta^{(q+\frac{1}{2})} \Phi_q \, dx \right)^{\frac{1}{2}} \left( \int_{B_1^T} \Phi_q |u|^p \, dx \right)^{\frac{1}{2}} dt
\]

\[
\leq C \int_0^T \theta^{2p'} \left( \frac{t}{T} \right) (t + R + 1)^{\frac{p}{2} - \frac{1}{2} + \frac{1}{p}} \ln^{-2} \frac{1}{t + R + 1} (t + R + 1)
\]

\[
\times \left( \int_{B_1^T} \Phi_q |u|^p \, dx \right)^{\frac{1}{2}} dt
\]

\[
\leq \left( \int_0^T \ln^{-p'} (t + R + 1)^{\frac{p}{2} - \frac{3}{2} + 1} dt \right) \frac{1}{t} \ln^{-\frac{p'}{2}} (T)
\]

\[
\leq CI \frac{1}{t} (T),
\]

where we used the fact that \(\frac{p}{2} - \frac{3}{2} + 1 = -1\).

Next we deal with \(J(T)\). By integration by parts we come to

\[
J(T)
\]

\[
\leq \frac{C}{T^2} \int_0^T \theta^{2p'-2} \left( \frac{t}{T} \right) \theta'(t) \ln^{-2} (t + R + 1) \int_{B_1^T \cap \{|x| \leq t+R\}} \Phi_q |u| \, dx \, dt
\]

\[
+ \frac{C}{T^2} \int_0^T \theta^{2p'-1} \left( \frac{t}{T} \right) \theta''(t) \ln^{-2} (t + R + 1) \int_{B_1^T \cap \{|x| \leq t+R\}} \Phi_q |u| \, dx \, dt
\]

\[
+ \frac{C}{T} \int_0^T \theta^{2p'-1} \left( \frac{t}{T} \right) |\theta''(t)| \ln^{-3} (t + R + 1) t + R + 1 \int_{B_1^T \cap \{|x| \leq t+R\}} \Phi_q |u| \, dx \, dt
\]

\[
+ C \int_0^T \theta^{2p'} \left( \frac{t}{T} \right) \ln^{-4} (t + R + 1) t + R + 1 \int_{B_1^T \cap \{|x| \leq t+R\}} \Phi_q |u| \, dx \, dt
\]

\[
+ C \int_0^T \theta^{2p'} \left( \frac{t}{T} \right) \ln^{-3} (t + R + 1) (t + R + 1)^2 \int_{B_1^T \cap \{|x| \leq t+R\}} \Phi_q |u| \, dx \, dt + C
\]

\[
\leq J_1 + J_2 + J_3 + J_4 + J_5 + C,
\]
where

\[ J_1(T) = \frac{C}{T^2} \int_0^\infty \theta^{2p'-2} \left( \frac{t}{T} \right) \left( \theta' \left( \frac{t}{T} \right) \right)^2 \ln^{-2}(t + R + 1) \]
\[ \times \int_{B_1^C \cap \{|x| \leq t + R\}} \Phi_q|u| dx dt, \]

\[ J_2(T) = \frac{C}{T^2} \int_0^\infty \theta^{2p'-1} \left( \frac{t}{T} \right) |\theta'' \left( \frac{t}{T} \right)| \ln^{-2}(t + R + 1) \]
\[ \times \int_{B_1^C \cap \{|x| \leq t + R\}} \Phi_q|u| dx dt, \]

\[ J_3(T) = \frac{C}{T^2} \int_0^\infty \theta^{2p'-1} \left( \frac{t}{T} \right) \left| \theta' \left( \frac{t}{T} \right) \right| \ln^{-3}(t + R + 1) \]
\[ \times \int_{B_1^C \cap \{|x| \leq t + R\}} \Phi_q|u| dx dt, \]

\[ J_4(T) = \frac{C}{T^2} \int_0^\infty \theta^{2p'} \left( \frac{t}{T} \right) \ln^{-2}(t + R + 1) \]
\[ \times \int_{B_1^C \cap \{|x| \leq t + R\}} \Phi_q|u| dx dt, \]

\[ J_5(T) = \frac{C}{T^2} \int_0^\infty \theta^{2p'} \left( \frac{t}{T} \right) \ln^{-3}(t + R + 1) \]
\[ \times \int_{B_1^C \cap \{|x| \leq t + R\}} \Phi_q|u| dx dt. \]

It is easy to get by using (10) in Corollary 2

\[ \left( \int_{B_1^C \cap \{|x| \leq t + R\}} \Phi_q dx \right)^{\frac{1}{p'}} \leq C(t + R + 1)^{-\frac{2}{\theta}+\frac{2}{\nu}} \ln^\frac{p}{\nu}(t + R + 1), \] (34)

and hence

\[ J_1(T) \leq \frac{C}{T^2} \int_0^\infty \theta^{2p'-2} \left( \frac{t}{T} \right) \left( \theta' \left( \frac{t}{T} \right) \right)^2 \ln^{-2}(t + R + 1) \]
\[ \times \left\{ \left( \int_{B_1^C \cap \{|x| \leq t + R\}} \Phi_q|u| dx \right)^{\frac{1}{p'}} \left( \int_{B_1^C} \Phi_q|u|^{p} dx \right)^{\frac{1}{p}} dt \right\} \]
\[ \leq \frac{C}{T^2} \int_0^\infty \theta^{2p'-2} \left( \frac{t}{T} \right) \ln^{-\frac{2}{\nu}}(t + R + 1)(t + R + 1)^{-\frac{2}{\theta}+\frac{2}{\nu}} \]
\[ \times \left( \int_{B_1^C} \Phi_q|u|^{p} dx \right)^{\frac{1}{p}} dt \]
\[ \leq \frac{C}{T^2} \left\{ \int_0^T (t + R + 1)^{-\frac{2}{\theta}+\frac{2}{\nu}} \ln^{-\frac{2}{\nu}}(t + R + 1) dt \right\}^{\frac{p}{\nu}} \]
\[ \leq \frac{C}{T^2} (T + R + 1)^{-\frac{2}{\theta}+\frac{2}{\nu}} I_{\frac{1}{p}}(T) \]
\[ \leq CI_{\frac{1}{p}}(T), \] (35)
where we used the fact $-\frac{q+3}{p} = 2$ when $p = p_c(2)$. In the same way, we have

\[
\begin{align*}
J_2(T) & \leq C I^\frac{1}{p}(T), \\
J_3(T) & \leq C \left( \int_0^T (t + R + 1)^{p'-1} \ln^{p'-1}(t + R + 1) dt \right)^{\frac{1}{p}} I^\frac{1}{p}(T) \\
& \leq C \frac{T}{T + R + 1} I^\frac{1}{p}(T) \\
& \leq CI^\frac{1}{p}(T), \\
J_4(T) & \leq C \left( \int_0^T (t + R + 1)^{-1} \ln^{-2p'-1}(t + R + 1) dt \right)^{\frac{1}{p}} I^\frac{1}{p}(T) \\
& \leq CI^\frac{1}{p}(T), \\
J_5(T) & \leq C \left( \int_0^T (t + R + 1)^{-1} \ln^{-p'-1}(t + R + 1) dt \right)^{\frac{1}{p}} I^\frac{1}{p}(T) \\
& \leq CI^\frac{1}{p}(T).
\end{align*}
\]

We then get by combining (33), (35) and (36)

\[
J(T) \leq CI^\frac{1}{p}(T) + C.
\]  
(37)

The term $K(T)$ can be done by using the method parallel to that of $J(T)$. Making use of integration by parts and (11) in Corollary 3 one has

\[
\begin{align*}
K(T) & \leq C \int_0^\infty \theta^{2p'} \left( \frac{t}{T} \right) \ln^{-2}(t + R + 1) \left| \partial_t \int_{B^c_1 \cap \{ |x| \leq t + R \}} \tilde{\Phi}_{q+1} u dx \right| dt \\
& \leq C \int_0^\infty \left| \partial_t \left( \theta^{2p'} \left( \frac{t}{T} \right) \ln^{-2}(t + R + 1) \right) \right| \int_{B^c_1 \cap \{ |x| \leq t + R \}} \tilde{\Phi}_{q+1} u dx dt + C \\
& \leq C \frac{T}{T} \int_0^\infty \theta^{2p'-1} \left( \frac{t}{T} \right) \ln^{-2}(t + R + 1) \\
& \times \int_{B^c_1 \cap \{ |x| \leq t + R \}} \ln r \Phi_{q+1} u dx dt \\
& + C \int_0^\infty \theta^{2p'} \left( \frac{t}{T} \right) \ln^{-3}(t + R + 1) \left( \frac{t}{t + R + 1} \right) \int_{B^c_1 \cap \{ |x| \leq t + R \}} \ln r \Phi_{q+1} u dx dt + C \\
= & :K_1(T) + K_2(T) + C.
\end{align*}
\]  
(38)

In order to control $K_1(T)$ and $K_2(T)$, we first estimate $\int_{B^c_1 \cap \{ |x| \leq t + R \}} \ln r \Phi_{q+1} u dx$. Noting that $-\frac{q}{p'} + q - \frac{1}{2} + \frac{1}{p'} = \frac{1}{p}$ if $p = p_c(2)$ and $-p'(q + \frac{1}{2}) = -1$, then by Hölder
inequality and (12) in Corollary 3 we have
\[
\int_{B_1 \cap \{|x| \leq t+R\}} \ln r \Phi_{q+1} u \, dx \\
\leq \left( \int_{B_1 \cap \{|x| \leq t+R\}} \tilde{\Phi}_q \left( \frac{\Phi_{q+1}}{\Phi_q} \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_{B_1} \tilde{\Phi}_q |u|^p \, dx \right)^{\frac{1}{p}} \\
\leq C(t + R + 1)^{-\frac{q}{p} + q - \frac{1}{2}} \left( \int_1^{t+R} \frac{1}{(t + R + 1 - r)^{-p'(q + \frac{1}{2})}} \ln r \, dr \right)^{\frac{1}{p'}} \\
\times \left( \int_{B_1} \tilde{\Phi}_q |u|^p \, dx \right)^{\frac{1}{p}},
\]
and so
\[
K_1(T) \leq \frac{C}{T} \int_0^\infty \vartheta^{2p'-1} \left( \frac{t}{T} \right)^{t + R + 1} \ln^{-2 + \frac{q}{p'}} (t + R + 1) \\
\times \left( \int_{B_1} \tilde{\Phi}_q |u|^p \, dx \right)^{\frac{1}{p}} \, dt \\
\leq \frac{C}{T} \left( \int_0^T (t + R + 1)^{-\frac{q'}{p'}} \, dt \right)^{\frac{1}{p'}} I^{\frac{1}{p}}(t) \\
\leq C I^{\frac{1}{p}}(t),
\]
and
\[
K_2(T) \leq C \left( \int_0^T (t + R + 1)^{-1} \ln^{-p'} (t + R + 1) \, dt \right)^{\frac{1}{p'}} I^{\frac{1}{p}}(t) \\
\leq C I^{\frac{1}{p}}(t).
\]
By combining (38) and (40) we arrive at
\[
K(T) \leq C I^{\frac{1}{p}}(t) + C.
\]
From (29), (32), (37) and (41) one has
\[
I(T) \leq C I^{\frac{1}{p}}(T) + C,
\]
which means by Young's inequality
\[
I(T) \leq C,
\]
we then get a contradiction by combining (30) and (43) and hence we finish the proof of Theorem 1.1.

4. In conclusion. In this paper we obtain blow up result to initial boundary value problem for semilinear wave equation with critical power outside the unit ball in $\mathbb{R}^2$. The proof is based on contradiction argument. Noting that we may get a lifespan estimate from above by combining inequality (30) and (43). However it is far from the sharp one compared to that of the corresponding Cauchy problem. We will study the sharp lifespan estimate in the near future. Furthermore, it is interesting to consider the initial boundary value problem in more generalized exterior domain.

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