ON THE CANONICAL MAP OF SURFACES WITH $q \geq 6$

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Abstract. We carry out an analysis of the canonical system of a minimal complex surface $S$ of general type with irregularity $q > 0$. Using this analysis we are able to sharpen in the case $q > 0$ the well known Castelnuovo inequality $K_S^2 \geq 3p_g(S) + q(S) - 7$.

Then we turn to the study of surfaces with $p_g = 2q - 3$ and no fibration onto a curve of genus $> 1$. We prove that for $q \geq 6$ the canonical map is birational. Combining this result with the analysis of the canonical system, we also prove the inequality: $K_S^2 \geq 7\chi(S) + 2$. This improves an earlier result of the first and second author ([MP1]).

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Dedicated to Fabrizio Catanese on the occasion of his 60th birthday.

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1. INTRODUCTION

Complex surfaces of general type have been an object of study since the 19th century and nowadays their general behaviour is believed by many to be understood, but in fact there are still many open problems. In particular, little is known about the irregular surfaces, namely the surfaces that have non zero global holomorphic 1-forms. This is in part due to the fact that a fundamental tool for the study of surfaces of general type is the canonical map, which is easier to understand in the case of regular surfaces.
Here (cf. §4) we carry out an analysis of the canonical system of irregular surfaces, paying special attention to the case of surfaces without irrational pencils of genus $g > 1$, namely surfaces that have no fibration onto a curve of genus $g > 1$, and to the case where the canonical system has a fixed part. These results enable us to sharpen in the case of irregular surfaces the well known Castelnuovo inequality $K^2 \geq 3p_g + q - 7$ for a minimal surface with birational canonical map (cf. §5).

Next, we turn to surfaces with $p_g = 2q - 3$. Recall that by [Be2] an irregular surface $S$ of general type has $p_g \geq 2q - 4$, with equality holding if and only if $S$ is birational to the product of a curve of genus 2 and a curve of genus $q - 2$. Hence it seems natural to try to classify surfaces $S$ with $p_g = 2q - 3$. This is easily done under the assumption that $S$ has an irrational pencil of genus $> 1$ (cf. [MP1], [BNP]) but the matter becomes very hard if one assumes that $S$ has no such pencil. Examples of surfaces with these properties are known only for $q = 3, 4$. For $q = 3$ one has the symmetric product of a curve of genus 3 and it is known ([HP], [Pi]) that this is the only surface with $p_g = q = 3$ and no irrational pencil of genus $> 1$. A family of examples with $q = 4$ (hence $p_g = 5$) has been constructed by C. Schoen in [Sc]. Hence one is led to doubt the existence of these surfaces for $q \geq 5$. Indeed, in [MPP] it is shown that the case $q = 5$, $p_g = 7$ does not occur. However the arguments used in [MPP], besides being quite intricate, are very ad hoc and for $q \neq 3, 5$ only some general restrictions are known.

For $q \geq 4$, surfaces with $p_g = 2q - 3$ and no irrational pencil of genus $> 1$ are “generalized Lagrangian”, namely they have independent global 1-forms $\alpha_1, \ldots, \alpha_4$ such that $\alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4 = 0$. In [BNP] it is shown that a minimal generalized Lagrangian surface whose canonical system has no fixed part has $K^2 \geq 8\chi$ and in [MP1] the weaker inequality $K^2 \geq 7\chi - 1$ has been proven for all surfaces with $p_g = 2q - 3$.

Here we prove that the canonical map of surfaces with $p_g = 2q - 3$ that have no irrational pencil of genus $> 1$ is birational (Theorem 6.1). Combining this result with the improved version of the Castelnuovo inequality given in §5, we sharpen the inequality of [MP1] to $K^2 \geq 7\chi + 2$. It is our hope that these results are a step towards deciding in general of the existence of surfaces with $p_g = 2q - 3$ and no irrational pencil of genus $> 1$.

The paper is organized as follows. In §2 we recall several well known technical results that are used repeatedly in the paper. Sections 3 and 4 are the technical heart of the paper. In §3 we establish the existence of pencils of low degree on some rational surfaces, refining similar results by Reid and Xiao ([Re1], [Xi3]). (This result is essential for proving Theorem 6.1). Section 4 starts with some results on the existence of certain types of curves on an irregular surface, that are, we believe, of independent interest. Then, in order to establish the aforementioned sharpenings of Castelnuovo’s theorem, we study the quadrics through the canonical image of an irregular surface and, in addition, we give a small refinement of an inequality due to Debarre. In §5 we use the results of §4 to prove the Castelnuovo type
inequalities. Section 6 presents the results on surfaces with $p_g = 2q - 3$ and $q \geq 6$. Whilst the birationality of the canonical map for such surfaces with $q \geq 7$ is an almost immediate consequence of the results of §3, some more work is needed to show birationality for $q = 6$.

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Notation and conventions: All varieties are complex projective. A rational map $f: X \to Y$ is composed with a pencil if the image of $f$ is a curve. A linear system $|D|$ on $X$ is composed with a pencil if the map given by $|D|$ is. A surface $S$ has an irrational pencil of genus $b > 0$ if there exists a fibration $f: S \to B$, where $B$ is a curve of genus $b$. If $\Sigma$ is a singular surface we denote by $p_g(\Sigma)$ and $q(\Sigma)$ the geometric genus and the irregularity of a desingularization of $\Sigma$.

Usually a curve on a surface $S$ will mean an effective non zero divisor. We denote by $\omega_C$ the dualizing sheaf $\mathcal{O}_C(K_S + C)$ of a curve $C$ of $S$. A $(−2)$-curve on $S$ is an effective divisor $Z$ such that $Z^2 = −2$ and every irreducible component $\theta$ of $Z$ satisfies $\theta^2 = −2$ and $K_S\theta = 0$. A $(−2)$-curve $Z$ is called a $(−2)$-cycle if $\theta Z \leq 0$ for every component $\theta$ of $Z$, i.e. if $Z$ is the fundamental cycle of an A-D-E singularity in the terminology of [BPV, Ch. III, §3] or the numerical cycle of a Du Val singularity in the terminology of [Re3, Ch. IV].

If $Y$ is a connected subset of an abelian variety $A$, we denote by $< Y >$ the abelian subvariety of $A$ generated by $Y$. We denote by $\text{albdim}(X)$ the Albanese dimension of a variety $X$, namely the dimension of the image of the Albanese map of $X$.

2. Auxiliary results

In this section we collect several technical facts that will be used repeatedly in some of the proofs. Here “surface” means “smooth complex projective surface”.

2.1. Corollaries of the index theorem. We recall the following corollary of the Hodge index theorem:

**Theorem 2.1.** (see, e.g., [BPV]) Let $D, E$ be $\mathbb{Q}$-divisors on the surface $S$. If $D^2 > 0$ and $DE = 0$ then $E^2 \leq 0$ and $E^2 = 0$ if and only if $E$ is homologous to 0 in rational homology.

We will use mainly the following variations of Theorem 2.1.

**Corollary 2.2.** Let $S$ be a surface and $D$ a $\mathbb{Q}$-divisor such that $D^2 > 0$. Then for any $\mathbb{Q}$-divisor $Z$, $D^2 Z^2 - (DZ)^2 \leq 0$. 
Corollary 2.3. Let $S$ be a surface and $D$ a $\mathbb{Q}$-divisor of $S$ such that $D^2 > 0$. Then for any decomposition of $D$ as $D = A + B$ where $A, B$ are $\mathbb{Q}$-divisors, $A^2B^2 - (AB)^2 \leq 0$ and if equality holds then there exist $m, n \in \mathbb{Q}$ such that $mA$ is homologous to $nB$ in rational homology.

2.2. Properties of $m$-connected curves. We recall that by a curve we mean an effective non zero divisor on a surface and that a curve $D$ is $m$-connected if $AB \geq m$ for any decomposition $D = A + B$ with $A, B > 0$. Here we list several properties related to this notion (cf. [Re1, 3.9]).

Proposition 2.4 (see, e.g., Corollary A.2 of [CFM], also §3.9 of [Re3]). If $D$ is a 1-connected curve then $h^0(D, \mathcal{O}_D) = 1$.

Lemma 2.5. Let $S$ be minimal of general type with $K_S^2 > 1$ and let $E$ be an effective divisor of $S$ such that $E^2 = -1$ and $K_SE = 1$. Then $E$ is 1-connected, $h^0(E, \omega_E) = 1$ and $h^0(E, K_S) = 1$.

Proof. Suppose that $E$ is not 1-connected. Then there is a decomposition $E = A + B$ with $A, B > 0$ and $AB \leq 0$. Since $A^2 + 2AB + B^2 = E^2 = -1$ we have $A^2 + B^2 \geq -1$ and therefore $A^2 \geq 0$ or $B^2 \geq 0$.

On the other hand, since $K_S$ is nef, for any $0 < C < E$ one has $K_SC = 0$ or $K_SC = 1$. If $K_SC = 0$ then by the index theorem $C^2 < 0$. If $K_SC = 1$ again by the index theorem (Corollary 2.3) $C^2 \leq 0$ and by the adjunction formula $C^2$ is odd and so $C^2 \leq -1$. So we have a contradiction, that shows that $E$ is 1-connected.

For the second assertion it suffices to use that, by the 1-connectedness of $E$ and Proposition 2.4, $h^0(E, \mathcal{O}_E) = 1$, and that $p_a(E) = 1$.

For the last assertion note first that, since $p_a(E) = 1$ and $K_SE = 1$, by the Riemann-Roch theorem one has $h^0(E, K_S) = 1 + h^1(E, K_S)$.

Since $\omega_E = (K_S + E)|E$, by duality one has $h^1(E, K_S) = h^0(E, E)$.

Suppose that $h^0(E, E) \neq 0$.

If $E$ is irreducible, we have immediately a contradiction because $E^2 = -1$.

If $E$ is not irreducible there is some component $\theta$ of $E$ such that $\theta E < 0$. Then, if $h^0(E, E) \neq 0$, by [CFM, Lemma (A.1)] there is a decomposition $E = A + B$, with $A, B > 0$ where $EA \geq BA$. Since $EA = A^2 + AB$, we obtain $A^2 \geq 0$, a contradiction, because we saw above that every curve $C < E$ satisfies $C^2 < 0$.

Thus $h^0(E, E) = 0$ and $h^0(E, K_S) = 1$. $\square$

Proposition 2.6 ([M] Lemma 2.6, also [Re3] §3.9). Let $D$ be a curve on a surface $S$ such that $D^2 \geq 1$ and $D$ is nef. Then every $D' \in |D|$ is 1-connected.

Furthermore if $D' = A + B$ is a decomposition of $D$ with $A, B$ curves such that $AB = 1$, only the following possibilities can occur:

- $A^2 = -1$ or $B^2 = -1$;
- $A^2 = 0$ or $B^2 = 0$;
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- $A^2 = B^2 = 1$, $A$ and $B$ are homologous in rational homology and $D^2 = 4$.

Also if $D^2 \geq 10$, and $D' = A + B$ is a decomposition of $D'$ with $A$, $B$ curves such that $AB = 2$, only the following possibilities can occur:
- $A^2 = -2$ or $B^2 = 2$;
- $A^2 = -1$ or $B^2 = -1$;
- $A^2 = 0$ or $B^2 = 0$.

Lemma 2.7 ([CFM] Lemma A.4)). Let $D$ be an $m$-connected curve of a surface $S$ and let $D = D_1 + D_2$ with $D_1$, $D_2$ curves. Then, with $\lfloor p/2 \rfloor$ being the integer part of an integer $p$:

(i) if $D_1D_2 = m$, then $D_1$ and $D_2$ are $\lfloor (m + 1)/2 \rfloor$-connected;
(ii) if $D_1$ is chosen to be minimal subject to the condition $D_1(D - D_1) = m$, then $D_1$ is $\lfloor (m + 3)/2 \rfloor$-connected.

The following immediate consequence of Lemma 2.7 and of the 2-connectdness of the canonical divisors on minimal surfaces will be used repeatedly.

Corollary 2.8. If a canonical divisor on a minimal surface $S$ decomposes as $K_S = A + B$ where $A, B > 0$ and $AB = 2$, then both $A$ and $B$ are 1-connected.

3. Rational surfaces of small degree

The existence of pencils of low degree on ruled surfaces has been studied by M. Reid ([Re1]) and Xiao Gang ([Xi3]). In this section we prove the following refinement of their results, which is crucial in proving Theorem 6.1.

Theorem 3.1. Let $\Sigma \subset \mathbb{P}^n$ be a rational surface of degree $m$ not contained in any hyperplane and let $\eta: \Gamma \rightarrow \Sigma$ be the minimal desingularization. If the linear system $|H| := \eta^*|O_{\mathbb{P}^n}(1)|$ is complete, then:

(i) if $n \geq 9$ and $m \leq \frac{3}{2}n$, then $\Sigma$ has a pencil of curves $|L|$ such that every curve of $|L|$ spans at most a $\mathbb{P}^r$ with $r < \frac{1}{2}n$;
(ii) if $n = 8$, then $\Sigma$ has a pencil of curves $|L|$ such that every curve of $|L|$ spans at most a $\mathbb{P}^3$ for $m \leq 10$ and it has a pencil of curves of degree $\leq 4$ for $m = 11, 12$.

Proof. The proof, although long, is based on the simple classical idea of “termination of adjunction” on a rational surface. One considers the adjoint system $|D| := |K_{\Gamma} + H|$. If $\dim |D| \leq 0$, then the result follows by the classification of projective surfaces of very small degree. If $|D|$ is composed with a pencil $|L|$, then the image of $|L|$ in $\Sigma$ is a pencil of degree $< \frac{n}{2}$. If the system $|D|$ maps $\Gamma$ onto a surface, then one repeats the argument considering the second adjoint system $|K_{\Gamma} + D|$. Termination of adjunction means that this process eventually stops (in our case, it actually stops at most at the second step).
In the proof we also make repeated use of the elementary fact that a connected curve of degree $r$ spans at most a $\mathbb{P}^r$.

By [Re1] Corollary 1.1], if $4m < 6n - 81/4$, i.e., if $m < \frac{3}{2}n - 5 - 1/16$, then $\Sigma$ has a pencil of lines or conics and so the theorem is true in this case. So we are left with studying $\frac{3}{2}n - 5 \leq m \leq \frac{3}{2}n$.

Write $m = \frac{3}{2}n - \alpha$. If $m = n - 1$, then $\Sigma$ is either a cone over a rational normal curve of degree $n - 1$ or it is a rational normal scroll. In either case, it has a pencil of lines. Similarly, if $m = n$ there are two possibilities (see [Na]):

(a) $n = 8$ and $\Sigma$ is the anticanonical image of $\mathbb{P}^2$ blown up at a point $P$ or of a (possibly singular) quadric of $\mathbb{P}^3$. In either case, $\Sigma$ has a pencil of conics, corresponding in the former case to the lines through $P$ and in the latter case to the lines of a ruling of the quadric.

(b) $n = 9$ and $\Sigma$ is the anticanonical image of $\mathbb{P}^2$. In this case $\Sigma$ has a 2-dimensional system of curves of degree 3, the images of the lines of $\mathbb{P}^2$.

So we can assume that $m > n$, i.e. $\frac{1}{2}n > \alpha$.

Let $H \in |H|$ be general. The curve $H$ is smooth and irreducible by Bertini’s theorem and, by the regularity of $\Upsilon$, the system $|H|_H$ is complete. Since $|H|_H$ has dimension $n - 1$ and degree $< 2(n - 1)$, it is not special by Clifford’s theorem. So restricting $O_H$ to $H$ and taking cohomology we get $h^1(O_H) = 0$. Riemann-Roch applied to $O_H$ gives $n = \frac{3}{2}n - \alpha + 1 - g(H)$, namely $g(H) - 1 = \frac{1}{2}n - \alpha$. The adjunction formula gives $K_H = -\frac{1}{2}n - \alpha$.

We consider now the adjoint linear system $|D| := |K_H + H|$. Using the adjunction sequence for $H$, one sees that $h^0(D) = g(H) = \frac{1}{2}n - \alpha + 1$ and, because we are assuming $\frac{1}{2}n > \alpha$, $h^0(D) \geq 2$. Write $|D| = Z + |M|$, where $Z$ is the fixed part of $|D|$ and $|M|$ is the moving part.

**Step 1:** $D$ is nef. In particular, we have $D^2 \geq 0$.

Since $q(\Upsilon) = 0$, the restriction of $|D|$ to a curve $H \in |H|$ is the complete canonical system $|K_H|$. Since for a general $H$ the system $|K_H|$ is base point free, for any irreducible component $\theta$ of $Z$ we have $\theta H = 0$ and so, by the index theorem, $\theta^2 < 0$.

Let $\theta$ be an irreducible curve such that $\theta D < 0$. Since $D$ is effective, $\theta$ is a component of $Z$. Hence $\theta H = 0$, $\theta K_H < 0$, $\theta^2 < 0$, namely $\theta$ is a $-1$-curve contracted by $|H|$, against the assumption that $\Upsilon \to \Sigma$ is the minimal desingularization.

**Step 2:** If $|D|$ is composed with a pencil, then $\Sigma$ has a pencil of conics.

If $|D|$ is composed with a pencil we can write $|D| = Z + [(\frac{1}{2}n - \alpha)G]$, where $|G|$ is a pencil. Since $HZ = 0$ (cf. Step 1) and $HD = n - 2\alpha$, one has $HG = 2$ and the general $G$ is mapped by $\eta$ to a conic of $\mathbb{P}^n$. 
There are two possibilities: DG pencil of lines. Since K points K of Υ. Then 

If and maps Υ birationally onto a non degenerate surface T by η. Furthermore, if D is nef and D^2 > 0, every curve of |D| is 1-connected by Proposition 2.6, and so Z ≠ 0 if MZ > 0. Because HM = HD = H(K_Υ + H) is even (recall HZ = 0, cf. Step 1), we obtain M^2 + K_Υ M = M^2 + DM - HM = 2M^2 + MZ - HM ≡ MZ mod 2, So, if Z ≠ 0, then MZ ≥ 2 and D^2 ≥ \frac{1}{2}n - α + 1.

If D^2 = \frac{1}{2}n - α - 1, then of course |D| has no base points, whilst if D^2 = \frac{1}{2}n - α - 1, |D| has one simple base point. If this is the case, the system |D| maps Υ birationally onto a surface of minimal degree in P^{\frac{1}{2}n-α}. Hence the image of a general D ∈ |D| is a rational normal curve in P^{\frac{1}{2}n-α-1}. Since D is smooth by Bertini’s theorem, it is isomorphic to P^1. On the other hand, the restricted system |D|_D has positive dimension, it is complete since Υ is regular and it has a base point since |D| has one. Since this contradicts the theory of complete linear systems on P^1, we have proven that |D| has no base point for D^2 = \frac{1}{2}n - α.

In view of Step 2, we may assume that |D| is not composed with a pencil. We finish the proof by a case by case study, considering the various possibilities for D^2.

**Step 4:** The case D^2 = \frac{1}{2}n - α - 1.

Suppose first that \frac{1}{2}n - α - 1 ≥ 2. By Step 3, |D| has no base points and maps Υ birationally onto a non degenerate surface T of minimal degree. There are two possibilities:

(a) T is ruled by lines; or

(b) T is the Veronese surface in P^5.

In case (a), denote by |G| the moving part of the pull back to Υ of a pencil of lines. Since DG = 1, the index theorem gives G^2 = 0. It follows K_Υ G = -2, HG = 3 and therefore the curves of |G| are mapped to cubics by η.

In case (b), we have \frac{1}{2}n - α = 5, HD = 10 and we can write D = 2Δ, where Δ is the pull back of a conic contained in T. Hence HΔ = 5. This is enough to prove the statement if n ≥ 11, namely if α ≥ \frac{5}{2}. If α = 0, then n = 10, H^2 = 15. Since 4 = D^2 = 4Δ^2, we have Δ^2 = 1 and |Δ| gives a birational morphism to P^2. Since K_Υ H = -5 and D^2 = (K_Υ + H)^2 = 4, we get K_Υ^2 = -1. Hence the morphism Υ → P^2 given by |Δ| is the composition of blow ups at ten (possibly not distinct) points P_1, . . . P_{10} of P^2. Denote by E_1, . . . E_{10} the corresponding −1-curves of Υ. Then K_Υ = −3Δ + \sum_i E_i and H = D - K_Υ = 5Δ - \sum_i E_i. The
pull-back of the pencil of lines through, say, \( P_1 \) gives a pencil \(|L|\) on \( \Upsilon \) such that \( HL = 4 < 5 = \frac{1}{2}n \).

Suppose finally that \( \frac{1}{2}n - \alpha - 1 = 1 \), namely that \(|D|\) gives a birational morphism \( \Upsilon \to \mathbb{P}^2 \). Since \( HD = 4 \), the theorem is proven for \( n \geq 9 \).

Suppose \( n = 8 \), hence \( \alpha = 2 \), \( H^2 = 10 \), \( K_{\Upsilon}H = -6 \) and \( K_{\Upsilon}^2 = 3 \). The birational morphism \( \Sigma \to \mathbb{P}^2 \) given by \(|D|\) is the composition of blow ups at six (possibly infinitely near) points. So \(|H| = |D - K_{\Upsilon}|\) is the pull back of the system of plane quartics through these six points. The pull-back of the pencil of lines through one of these points gives a pencil \(|L|\) on \( \Upsilon \) such that \( HL = 3 < 4 = \frac{1}{2}n \).

**Step 5:** The case \( D^2 = \frac{1}{2}n - \alpha \).

By Step 3, the system \(|D|\) has no base points and maps \( \Upsilon \) birationally onto a rational surface \( T \) of degree \( p \) in \( \mathbb{P}^p \), where \( p = \frac{1}{2}n - \alpha \). Since the system \(|D|\) is complete, we have \( p \leq 9 \) and \( T \) is a weak Del Pezzo surface. Let \( \hat{T} \to T \) be the minimal desingularization; then \( \hat{T} \) is either an irreducible quadric of \( \mathbb{P}^3 \) \((p = 8)\) an irreducible quadric of \( \mathbb{P}^3 \) \((p = 8)\) or the blow up of \( \mathbb{P}^2 \) at \( 9 - p \) base points, and the map \( \hat{T} \to T \subset \mathbb{P}^p \) is given by the anticanonical system \( |-K_{\hat{T}}| \). The morphism \( \Upsilon \to T \) factors through a morphism \( f: \Upsilon \to \hat{T} \) such that \( D = f^*(-K_{\hat{T}}) \). For \( 3 \leq p \leq 8 \) the surface \( \hat{T} \) has a pencil of rational curves \(|G|\) of degree 2 with \( G^2 = 0 \), given in the former case by a ruling of the quadric and in the latter case by the lines through one of the blown-up points. Pulling back this pencil to \( \Upsilon \) we obtain a linear system \(|L|\) such that \( HL = 4 \). This proves the theorem for \( n \geq 9 \). For \( n = 8 \) there are two possibilities, \( p = 4, \alpha = 0, m = 12 \) and \( p = 3, \alpha = 1, m = 11 \), which correspond to the exceptions given in statement (b).

If \( p = 9 \), then the pull back of the system of lines of \( \mathbb{P}^2 \) gives a linear system \(|L|\) such that \( HL = 6 \). In this case we have \( \frac{1}{2}n - \alpha = 9 \) and so \( n \geq 18 \).

If \( p = 2 \), then \( HD = 4 \) and so if \( n \geq 9 \) the assertion is proven. We claim that \( n = 8, p = 2 \) does not occur. In fact if \( n = 8 \), then from \( \frac{1}{2}n - \alpha = 2 \) we obtain \( \alpha = 2 \), \( H^2 = 10 \) and \( K_{\Upsilon}H = -6 \). From \( D^2 = (K_{\Upsilon} + H)^2 = 2 \) we obtain \( K_{\Upsilon}^2 = 4 \) and so \( K_{\Upsilon}^2H^2 - (K_{\Upsilon}H)^2 = 40 - 36 > 0 \), contradicting the index theorem.

**Step 6:** The case \( \alpha \geq 1 \) and \( D^2 \geq \frac{1}{2}n - \alpha + 1 \).

By \( D^2 = K_{\Upsilon}^2 + \frac{1}{2}n - 3\alpha \), in this case \( K_{\Upsilon}^2 - 2\alpha \geq 1 \). By the Riemann-Roch theorem and Kawamata-Viehweg vanishing \( h^0(K_{\Upsilon} + D) = K_{\Upsilon}^2 - 2\alpha + 1 \) and so \( h^0(K_{\Upsilon} + D) \geq 2 \). On the other hand \( H(K_{\Upsilon} + D) = \frac{1}{2}n - 3\alpha \). Hence a general curve \( L \) in the moving part of \(|K_{\Upsilon} + D|\) satisfies \( HL \leq \frac{1}{2}n - 3\alpha < \frac{1}{2}n \).

**Step 7:** The case \( \alpha = 0 \) and \( D^2 \geq \frac{1}{2}n + 1 \).

In this case \( D^2 \geq \frac{1}{2}n + 1 \) implies that \( K_{\Upsilon}^2 \geq 1 \), because \( D^2 = K_{\Upsilon}^2 + \frac{1}{2}n \).

Hence \( h^0(-K_{\Upsilon}) \geq 2 \).
Let $C$ be a curve in the moving part of $|-K_Y|$. The kernel of the restriction map $H^0(H) \to H^0(C, H|_C)$ is $H^0(H - C)$. One has $h^0(H - C) \geq h^0(H + K_Y) = \frac{1}{2}n + 1$ and $h^0(H) = n + 1$. We conclude that the image via $|H|$ of $C$ spans a projective space of dimension $< \frac{1}{2}n$.

So, having covered all possible cases, we have proven the theorem. \(\Box\)

For later reference we examine more closely one of the exceptions in case (ii) of Theorem 3.1.

**Proposition 3.2.** Let $\Sigma \subset \mathbb{P}^8$ be a rational surface of degree 11 not contained in any hyperplane and let $\eta: \Upsilon \to \Sigma$ be the minimal desingularization. If the linear system $|H| := \eta^*|O_{\mathbb{P}^8}(1)|$ is complete and $\Sigma$ has no pencil of curves of degree $< 4$, then $H$ decomposes as $H = 2H' + J$, where $J$ is a non-zero effective divisor, $h^0(\Sigma, H') \geq 3$ and the linear system $|H'|$ has no fixed components.

**Proof.** A surface satisfying the hypothesis is as in Step 5 of proof of theorem 3.1. By the proof and keeping the same notation, one has that a surface of degree 11 in $\mathbb{P}^8$ that has no pencil of curves of degree $< 4$ satisfies $D^2 = 3$, $K_{\Upsilon}D = -3$ and $K_{\Upsilon}^2 = 2$. From this we have that $h^0(K_{\Upsilon} + D) \neq 0$ and that $K_{\Upsilon} + D \neq 0$. Since $\Upsilon$ is rational and $K_{\Upsilon}^2 = 2$, $h^0(-K_{\Upsilon}) \geq 3$. Then $H$ decomposes as $H = (-2K_{\Upsilon}) + (K_{\Upsilon} + D)$ and taking the moving part of $|-K_Y|$ we have the statement. \(\square\)

4. **Irregular surfaces**

In this section we collect several technical results that are needed in §5 and in §6 but are also, we believe, of independent interest.

Throughout all the section we denote by $S$ a smooth projective irregular surface, by $q > 0$ the irregularity of $S$ and by $a: S \to A := \text{Alb}(S)$ the Albanese map.

4.1. **Curves on irregular surfaces without irrational pencils.**

**Lemma 4.1.** If $D$ is an effective 1-connected divisor of $S$, then $< a(D) >$ has dimension $\leq p_a(D)$.

**Proof.** Write $D = D_{\text{red}} + A$, where $D_{\text{red}}$ is the support of $D$ and $A \geq 0$. It is easy to show that $D_{\text{red}}$ is connected. Moreover, if $A > 0$ then the decomposition sequence:

$0 \to \mathcal{O}_A(-D_{\text{red}}) \to \mathcal{O}_D \to \mathcal{O}_{D_{\text{red}}} \to 0$

shows that $p_a(D) = h^1(\mathcal{O}_D) \geq h^1(\mathcal{O}_{D_{\text{red}}}) = p_a(D_{\text{red}})$. Hence we may assume that $D$ is reduced.

We prove the statement by induction on the number $n$ of irreducible components of $D$. If $n = 1$, then there is a surjective morphism $J \to < a(D) >$
where $J$ is the Jacobian of the normalization of $D$. Since $J$ has dimension $g(D) \leq p_a(D)$, the statement follows.

To prove the inductive step, write $D = C + D_1$, where $C$ is an irreducible curve and $D_1$ is a connected effective divisor with $n - 1$ components. Since $D_1$ is connected, the decomposition sequence gives an exact sequence:

$$0 \to H^1(\mathcal{O}_C(-D_1)) \to H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{D_1}) \to 0.$$ 

Thus we have $p_a(D) = p_a(D_1) + h^1(\mathcal{O}_C(-D_1)) \geq p_a(D_1) + p_a(D)$. To complete the proof it is enough to notice that $< a(D) > = < a(D_1) > + < a(C) >$. \qed

The next lemma is a generalization of [BNP] Proposition 8.2, (a)].

**Lemma 4.2.** Let $S$ be a surface such that albdim$(S) = 2$ and let $D > 0$ be a divisor of $S$ such that one of the following conditions holds:

- $D$ is irreducible and $g(D) < q$;
- $D$ is 1-connected and $p_a(D) < q$.

Then:

(i) $D^2 \leq 0$;

(ii) if $D^2 = 0$, then there is a fibration $f : S \to B$, where $B$ is a curve of genus at least $q - p_a(D)$, and there exists an integer $m > 0$ such that $mD$ is a fibre of $f$.

**Proof.** Consider the abelian variety $A' := A/<a(D)>$ and denote by $a' : S \to A'$ the map induced by $a$. By the assumptions and by Lemma 4.1, $\text{dim} A' > 0$ and the image $Z$ of $a'$ generates $A'$ by construction.

Assume that $Z$ is a surface and let $H$ be the pull back to $S$ of a very ample line bundle of $Z$. Then $HD = 0$, $H^2 > 0$, hence $D^2 < 0$ by the index theorem. So if $D^2 \geq 0$ then $Z$ is a curve and $a'$ is composed with a pencil $f : S \to B$, where $B$ is a smooth curve. Since $B$ maps onto $Z$ and $Z$ generates $A'$, we have $g(B) \geq \text{dim} A' \geq q - p_a(D)$. Since by construction $a'$ contracts $D$ to a point and $D$ is connected, $D$ is contained in a fiber of $f$. By Zariski’s lemma one has $D^2 \leq 0$ and $D^2 = 0$ if and only if there exists an integer $m > 0$ such that $mD$ is a fiber of $f$. \qed

**Corollary 4.3.** Let $D > 0$ be a 1-connected divisor of $S$ such that $D^2 = 0$. If $b \geq 0$ is an integer such that $S$ has no irrational pencil of genus $> b$, then:

$$K_S D \geq 2(q - b) - 2.$$

4.2. **Some properties of the canonical system.** In this section we assume that the canonical system $|K_S| \neq \emptyset$. We write $p_g := p_g(S)$ and $|K_S| = |M| + Z$, where $|M|$ is the moving part and $Z$ is the fixed part. We denote by $\Sigma$ the canonical image and by $\varphi : S \to \Sigma \subset \mathbb{P}^{p_g-1}$ the canonical map.

**Lemma 4.4** ([MP2], Lemma 2.1). Let $\iota$ be an involution of $S$ such that $p_g(S/\iota) = 0$. If $q \geq 3$ then $S$ has an irrational pencil $f : S \to B$, where $g(B) \geq 2$. 
Corollary 4.5. Let $S$ be a minimal surface with $q \geq 3$. If $S$ has no irrational pencil of genus $\geq 2$ and $\Sigma$ is a surface with $p_g(\Sigma) = 0$, then $q(\Sigma) \leq 1$ and $\deg \varphi \geq 3$.

Proof. If $q(\Sigma) \geq 2$ then, by the classification of surfaces, $\Sigma$ must be a ruled surface and this is a contradiction because the pull-back of the ruling of $\Sigma$ would give an irrational pencil with base of genus $\geq 2$.

Since $p_g(\Sigma) = 0$ the canonical map of $S$ is not birational and so by Lemma 4.4 its degree must be $\geq 3$. □

The following result is essentially contained in [Xi5]:

Proposition 4.6. If $\text{albdim} \ S = 2$ and $C$ is an irreducible curve of $S$, then:

(i) if $h^0(S, C) = s \geq 2$, then $\varphi(C)$ spans at least a $\mathbb{P}^{q-2}$ and $p_a(C) \geq 2q - 3 + s$.

(ii) if $C$ is a general fiber of a fibration $f : S \to B$, with $B$ a curve of genus $b > 0$, then $\varphi(C)$ spans at least a $\mathbb{P}^{q-b-2}$.

Proof. (i) By [Xi5], if $S$ is an irregular surface of maximal Albanese dimension and $C$ is a curve of $S$ that moves in a linear system, then the image of the restriction map $H^0(S, K_S) \to H^0(C, K_S|_C)$ has dimension at least $q-1$. Passing to cohomology, the adjunction sequence for $C$ gives:

$$0 \to H^0(S, K_S) \to H^0(K_S + C) \to H^0(C, \omega_C) \to H^1(S, K_S) \to 0,$$

where exactness on the right follows by Ramanujam’s or by Kawamata-Viehweg’s vanishing. Hence we have:

$$p_a(C) = h^0(C, \omega_C) \geq h^1(S, K_S) + \dim \text{Im} r = q + \dim \text{Im} r.$$

The subspace $\text{Im} r$ contains the image of $H^0(S, K_S) \otimes H^0(S, C)$, hence it has dimension $\geq (q-1) + s - 2 = q - 3 + s$.

(ii) Also by [Xi5] (see [MP2] Proposition 2.2), given a pencil $f : S \to B$ with general fibre $C$ and such that $g(B) = b$ the image of the restriction map $H^0(S, K_S) \to H^0(C, \omega_C)$ has dimension at least $q - b - 1$. □

Following [Ko1], we define the quadric hull $\text{Quad}(S)$ of a surface of general type $S$ as the intersection of all the quadrics of $\mathbb{P}^{p_g-1}$ that contain the canonical image $\Sigma$. A component of $\text{Quad}(S)$ is said to be essential if it contains $\Sigma$; the quadric dimension $\dim \text{Quad}(S)$ is the maximum dimension of an essential component of $\text{Quad}(S)$. We quote the following:

Proposition 4.7 ([CMP], Proposition 2.4). Let $X \subset \mathbb{P}^{r+1}$ be a non-degenerate irreducible threefold and let $\gamma$ be the arithmetic genus of a general curve section of $X$. Then:

(i) if $\gamma = 0$, then $X$ is either a rational normal scroll or $X \subset \mathbb{P}^6$ is the cone over the Veronese surface in $\mathbb{P}^5$;

(ii) if $\gamma = 1$ and $X$ is not a scroll then $r \leq 9$;

(iii) if $\gamma = 2$ and $X$ is not a scroll then $r \leq 11$. 


**Proposition 4.8.** Let $S$ be a surface such that $\text{albdim} S = 2$ and $\varphi$ is birational. Then:

(i) if $p_g \geq 8$ and $q \geq 5$, then $h^0(2M) \geq 4p_g - 5$;
(ii) if $p_g \geq 12$, $q \geq 6$, then $h^0(2M) \geq 4p_g - 4$;
(iii) if $p_g \geq 14$, $q \geq 7$, then $h^0(2M) \geq 4p_g - 3$.

**Proof.** Set $r = p_g - 2$. Notice that by the Castelnuovo inequality (cf. [Be1, Remarques 5.6]) we have $\deg \Sigma \geq 3p_g - 7 \geq 3r - 1$.

It is well known (cf. [De1], [Re2], [Ba]) that $h^0(2M) \geq 4p_g - 6 = 4r + 2$.

We argue by contradiction, writing $h^0(2M) = 4r + 2 + \alpha$ and assuming that one of the following holds:

- $\alpha = 0$, $r \geq 6$ and $q \geq 5$;
- $\alpha = 1$, $r \geq 10$ and $q \geq 6$;
- $\alpha = 2$, $r \geq 12$ and $q \geq 7$.

For a non degenerate projective variety $Y \subset \mathbb{P}^{r+1}$ and $m \geq 0$ an integer, denote as usual by $h^0_Y(m)$ the Hilbert function of $Y$, namely the dimension of the image of the restriction map $H^0(O_{\mathbb{P}^{r+1}}(m)) \to H^0(O_Y(m))$. In what follows we use some basic properties of the Hilbert function, for which we refer the reader to [Ha2].

Let $C$ be a general section of the canonical image $\Sigma$ and let $Z$ be a general section of $C$. The set $Z$ consists of $\deg \Sigma \geq 3r - 1$ points in uniform position and one has:

\[(4.1) \quad 4r + 2 + \alpha = h^0(2M) \geq h_\Sigma(2) \geq r + 2 + h_C(2) \geq 2r + 3 + h_Z(2),\]

namely $h_Z(2) \leq 2r - 1 + \alpha$.

**Step 1:** $\dim \text{Quad}(S) \geq 3$

By [Ha2] Lemma 3.9 one has $h_Z(2) \geq 2r - 1$. Hence by (4.1), there are the following possibilities:

- (a) $h_Z(2) = 2r - 1$. By [Ha2] Lemma 3.9, in this case the intersection of all quadrics through $Z$ is a rational normal curve in $\mathbb{P}^{r-1}$;
- (b) $h_Z(2) = 2r$. By [Ha2] p. 109, in this case the intersection of all quadrics through $Z$ is a rational normal elliptic curve of degree $r$ in $\mathbb{P}^{r-1}$.
- (c) $h_Z(2) = 2r + 1$. Since $p_g \geq 8$, by [Ci2] Theorem 3.8 (cf. also [Pe, Proposition 4.3]), in this case the intersection of all quadrics through $Z$ is an irreducible curve of degree $r + 1$ in $\mathbb{P}^{r-1}$.

In each case, the intersection of all the quadrics of $\mathbb{P}^{r-1}$ containing $Z$ is an irreducible curve $\Gamma$. If $V$ is an essential component of Quad($S$), then Quad($S$) $\cap$ $\mathbb{P}^{r-1}$ contains $\Gamma$. Since $\mathbb{P}^{r-1} \subset \mathbb{P}^{r+1}$ is a general codimension 2 subspace, it follows that $\dim V \geq 3$.

**Step 2:** Quad($S$) has no essential component of dimension 3.

Assume for contradiction that an essential component $V$ of Quad($S$) of dimension 3 exists. Then by the proof of Step 1, the general curve section $\Gamma$ of $V$ has arithmetic genus $\leq \alpha$. Hence, in view of our assumptions on $r$ and
\(\alpha\), by Proposition \([4.7]\), \(V\) is a scroll in planes. If \(\Gamma\) is rational, as it is always the case for \(\alpha = 0\), we let \(|C|\) be the pencil of \(S\) induced by the ruling of \(V\). Since \(q \geq 5\) by assumption, we have a contradiction to Proposition \([4.6]\).

If \(\gamma\) has geometric genus \(b > 0\), we let \(B \to \Gamma\) be the normalization map and \(f: S \to B\) the fibration induced by the ruling of \(V\). Since \(b \leq \alpha\) and \(q \geq 5 + \alpha\) by assumption, we have again a contradiction to Proposition \([4.6]\).

**Step 3**: \(\dim \text{Quad}(S) \leq 3 + \alpha\).

If \(\alpha = 0\), (cf. also \([Ko1]\)), \(\text{Quad}(S)\) is a threefold by \([Ba\), Proposition 1.2\].

Consider now \(\alpha > 0\) and assume for contradiction that \(\dim \text{Quad}(S) \geq 4 + \alpha\). Since the quadrics through \(Z\) cut out a curve in \(\mathbb{P}^{r-1}\) (cf. proof of Step 1), it follows that the image of the restriction map \(\rho: H^0(\mathbb{P}^{r+1},\mathcal{I}_S(2)) \to H^0(\mathbb{P}^{r-1},\mathcal{I}_Z(2))\) is a subspace of codimension \(\geq 1 + \alpha\). Since \(Z \subset C \subset \Sigma\) are general sections and \(\Sigma\) is non degenerate, the sequences \(0 \to \mathcal{I}_{\Sigma}(1) \to \mathcal{I}_{\Sigma}(2) \to \mathcal{I}_C(2) \to 0\) and \(0 \to \mathcal{I}_C(1) \to \mathcal{I}_C(2) \to \mathcal{I}_Z(2) \to 0\) are exact. Taking cohomology, one sees that the restriction maps \(H^0(\mathbb{P}^{r+1},\mathcal{I}_{\Sigma}(2)) \to H^0(\mathbb{P}^{r-1},\mathcal{I}_S(2))\) and \(H^0(\mathbb{P}^{r},\mathcal{I}_C(2)) \to H^0(\mathbb{P}^{r-1},\mathcal{I}_Z(2))\) are injective. Hence \(\rho\), being the composition of these maps, is also injective and we get \(h^0(\mathbb{P}^{r-1},\mathcal{I}_Z(2)) \geq h^0(\mathbb{P}^{r+1},\mathcal{I}_{\Sigma}(2)) + 1 + \alpha\). Passing to the Hilbert functions, we obtain:

\[
(4.2) \quad h_{\Sigma}(2) - h_{Z}(2) = \frac{(r + 2)(r + 3)}{2} - h^0(\mathbb{P}^{r+1},\mathcal{I}_{\Sigma}(2)) - \frac{r(r + 1)}{2} + h^0(\mathbb{P}^{r-1},\mathcal{I}_Z(2)) \geq 2r + 4 + \alpha.
\]

Since \(h_{Z}(2) \geq 2r - 1\), we get \(4r + 2 + \alpha = h_{\Sigma}(2) \geq 4r + 3 + \alpha\), a contradiction.

**Step 4**: *End of proof.*

If \(\alpha = 0\), then we have a contradiction by Step 2 and Step 3.

If \(\alpha = 1\), then by Step 2 and Step 3 we have \(\dim \text{Quad}(S) = 4\). By \([Ko1\), Lemma 1.2\], we have:

\[4r + 3 = h^0(2M) \geq h_{\Sigma}(2) \geq 5p_g - 10 = 5r,\]

a contradiction since \(r \geq 4\).

If \(\alpha = 2\), then by Step 2 and Step 3 we have \(\dim \text{Quad}(S) = 4\) or \(5\). By \([Ko1\), Lemma 1.2\], we have:

\[4r + 4 = h^0(2M) \geq h_{\Sigma}(2) \geq \min\{5p_g - 10, 6p_g - 15\} = 5r,\]

and we have again a contradiction since \(r \geq 5\).

**Lemma 4.9.** Let \(S\) be a minimal surface with \(q \geq 3\) and no irrational pencil of genus \(\geq 2\) and let \(D\) is a divisor of \(S\) such that:

- \(D^2 \geq 6\), \(h^0(D) \geq 4\) and \(|D|\) has no fixed component;
- \(F := K_S - D > 0\) and \(K_SF < 2q - 4\).

Then for any effective divisor \(E\) such that \(E^2 = -1\), \(K_SE = 1\) and \(DE = 2\), \(h^0(K_S + D) \geq h^0(K_S + D - E) + 3\).
Proof. By Corollary 2.5 one has $h^0(E, \omega_E) = 1$. Using the Riemann-Roch theorem and $p_a(E) = 1$ we see that $h^0(E, D|_E) = h^1(E, D|_E) + 2$. By duality, $h^1(E, D|_E) = h^0(E, \omega_E - D|_E)$. By assumption, there exists a section $s \in H^0(S, D)$ that does not vanish on any component of $E$. The section $s$ induces an injective map $H^0(E, \omega_E - D|_E) \to H^0(E, \omega_E)$. Hence $h^1(E, D|_E) = h^0(E, \omega_E - D|_E) \leq h^0(E, \omega_E) = 1$, and we get $h^0(E, D|_E) \leq 3$, $h^0(D - E) \geq 1$. Note also that $ED = 2$ implies that $EF = -1$.

To prove the lemma it suffices to show that the restriction map $r: H^0(K_S + D) \to H^0(E, (K_S + D)|_E)$ is surjective, because by Riemann-Roch and $(K_S + D)|_E = 3$, $h^0(E, (K_S + D)|_E) \geq 3$.

The cokernel of $r$ is $H^1(K_S + D - E)$. Since $D^2 \geq 6$, we have $(D - E)^2 > 0$, hence by Ramanujam’s vanishing to prove the assertion it is enough to show that the effective divisor $D - E$ is 1-connected (Ra, cf. Bo, p.453).

Suppose for contradiction that $D - E$ is not 1-connected. Then there is a decomposition $D - E = A + B$ where $A, B$ are effective non zero divisors such that $AB \leq 0$. Since $(A + B)E = 3$ and, by the 1-connectedness of $D$, $A(B + E) \geq 1$, $B(A + E) \geq 1$ we must have $2AB + 3 \geq 2$, and so $AB = 0$. From $(A + B)E = 3$ and the 1-connectedness of every curve in $|D|$ (Proposition 2.5), we have, say, $AE = 1$ and $BE = 2$. So $A(B + E) = 1$.

Since $D$ is nef and $DF < 2q - 4$ we obtain $AF < 2q - 5$. Note also that, since $D$ is 1-connected, $A$ is also 1-connected by Lemma 2.7.

If $A^2 = 0$ then $DA = 1$ and $K_SA = 1 + FA < 2q - 4$. Since $S$ has no irrational pencils of genus $> 1$ this is a contradiction to Corollary 4.3.

So $A^2 = -1$. In this case $(A + E)^2 = 0$ and $(A + E)F = AF + EF < 2q - 6$, yielding $K_S(A + E) = (D + F)(A + E) < 2 + 2q - 6 = 2q - 4$. If $(A + E)$ is 1-connected we have again a contradiction to Corollary 4.3.

So suppose that $A + E$ is not 1-connected. Then it decomposes as $A_1 + A_2$ where $A_1A_2 \leq 0$. By 1-connectedness of $D$ and $(A + E)B = 2$ we must have $A_i(D - A_i) = 1$, for $i = 1, 2$, and so we conclude as above that $A_i$ is 1-connected and $A_i^2 \leq 0$, for $i = 1, 2$. Then from $0 = (A + E)^2 = A_1^2 + 2A_1A_2 + A_2^2$ one obtains $A_1A_2 = A_1^2 = A_2^2 = 0$. But then, since $K_S(A_1 + A_2) = K_S(A + E) < 2q - 4$, we have $K_SA_1 < 2q - 4$, contradicting Corollary 4.3.

So $D - E$ is 1-connected and therefore the Lemma is proven.

We recall the following result:

**Proposition 4.10** ([MPR], Corollary 2.7). Let $S$ be a minimal surface of general type whose canonical map is not composed with a pencil. Denote by $|M|$ the moving part and by $Z$ the fixed part of $|K_S|$. If $Z > 0$ and
$M^2 \geq 5 + K_SZ$, then

\[ K_S^2 + \chi(S) = h^0(K_S + M) + K_SZ + MZ/2 \geq h^0(2M) + K_SZ + MZ/2 + 1. \]

Furthermore, if $h^0(K_S + M) = h^0(2M) + 1$ then $|K_S + M|$ has base points and there is an effective divisor $G$ such that $GZ \geq 1$ and either $G^2 = -1$ and $MG = 0$ or $G^2 = 0$ and $MG = 1$.

Now we can show the following

**Corollary 4.11.** Let $S$ be an irregular minimal surface such that $S$ has no irrational pencils $f : S \to B$ with $g(B) \geq 2$, $q \geq 6$ and the canonical map of $S$ is not composed with a pencil. Denote by $|M|$ the moving part and by $Z$ the fixed part of $|K_S|$. If $Z > 0$, then

\[ K_S^2 + \chi(S) \geq h^0(2M) + 3. \]

Furthermore if equality holds then $Z^2 = -2$ and $K_SZ = 0$.

**Proof.** The hypothesis that $S$ has no irrational pencils $f : S \to B$ with $g(B) \geq 2$ implies that $p_g \geq 2q - 3$. Since $q \geq 6$, we have then $p_g \geq 9$ and so the hypothesis that the canonical map of $S$ is not composed with a pencil implies that $M^2 \geq 6$.

Since $K_S^2 + \chi(S) = h^0(K_S + M) + K_SZ + MZ/2$ to prove the corollary we need to study the number $m := p + K_SZ + MZ/2$, where $p := h^0(K_S + M) - h^0(2M)$. Note that $MZ$ is an even positive number by the 2-connectedness of the canonical divisors. Also, by Corollary 2.8 for any decomposition $K_S = A + B$ with $A, B > 0$ and $AB = 2$ both $A$ and $B$ are 1-connected.

We start by analyzing the case $MZ = 2$. If $MZ = 2$, then $Z$ is 1-connected by Corollary 2.8 and because $K_S$ is nef $Z^2 \geq -2$. On the other hand, by the index theorem (Corollary 2.2) and $M^2 \geq 6$, we have $Z^2 \leq 0$. Furthermore the hypothesis that $S$ has no irrational pencils $f : S \to B$ with $g(B) \geq 2$ implies that $Z^2 = 0$ does not occur, because if $Z^2 = 0$ then $K_SZ = 2$ and this is impossible by Corollary 1.3.

So we are left with the possibilities:

(i) $Z^2 = -1$, $K_SZ = 1$;
(ii) $Z^2 = -2$, $K_SZ = 0$.

Note that $K_S + M - Z = 2M$.

In the first case Lemma 4.9 gives $p \geq 3$ yielding $m \geq 5$.

In the second case suppose that $p < 3$. By Proposition 4.10 we see that $p = 1$, and that there is an effective divisor $G$ such that $GZ \geq 1$ and either $G^2 = -1$ and $MG = 0$ or $G^2 = 0$ and $MG = 1$. It is easy to check that $MZ = 2$ implies $GZ = 1$. As above $G^2 = 0$ can be excluded using the hypothesis that $S$ has no irrational pencils $f : S \to B$ with $g(B) \geq 2$. If $G^2 = -1$ we can apply Lemma 4.9 to the divisor $E = G + Z$ and obtain that $h^0(K_S + M) \geq h^0(K_S + M - E) + 3$. Since $K_S + M - E = 2M - G$, $MG = 0$ and $G$ 1-connected imply $h^0 (2M - G) \geq h^0 (2M) - 1$ we obtain $p \geq 2$ contradicting $p = 1$.  

So we proved the Corollary for the case $MZ = 2$.

Suppose now that $m = 3$ and $MZ \geq 4$. Then either $MZ = 6$, $K_SZ = 0$ and $p = 0$ and this is excluded by Corollary 4.10 or $MZ = 4$ and $p + K_SZ \leq 1$. Since $M^2 \geq 6$, the case $p = 0$ and $K_SZ = 1$ is again excluded by Corollary 4.10 and so we are left with the case $MZ = 4$, $K_SZ = 0$ and (by Corollary 4.10) $p = 1$. It is not difficult to verify that $MZ = 4$, $K_SZ = 0$ imply that $Z$ decomposes as $Z = Z_1 + Z_2$ such that $K_SZ_i = 0$, $Z_i^2 = -2$ and $Z_iM = 2$.

Since $p = 1$, by Corollary 4.10 there is an effective divisor $G$ such that either $G^2 = -1$ and $MG = 0$ or $G^2 = 0$ and $MG = 1$. Again the second possibility for $G$ can be excluded as before. In fact because $G(M - G) = 1$ the 2-connectedness of the canonical divisors implies that $GZ > 0$ and $(M - G)Z > 0$. Since $MZ = 4$, one must have $GZ \leq 3$. If $G^2 = 0$ then $K_SZ \leq 4$ and this is impossible by Corollary 4.3. If $G^2 = -1$ and $GZ = 3$ then $(G + Z)^2 = 1$ and $K_SZ(G + Z) = 3$. Since $(M - G)^2 > 0$ and $M^2 \geq 6$ and $MZ = 4$ imply $K_SZ \geq 10$, we have a contradiction to Proposition 2.6. So $G^2 = -1$ and $GZ = 1$. If $GZ_1 > 1$ then $(G + Z_1)^2 > 0$ and we find the same contradiction as above. So, say, $GZ_1 = 1$ and $GZ_2 = 0$. Then the divisor $E := G + Z_1$ satisfies $E^2 = -1$ and $K_SZ = 1$ and as before applying Lemma 4.9 we obtain $p \geq 2$, a contradiction. So if $MZ = 4$, $m \geq 4$. 

\section{Castelnuovo type inequalities}

The Castelnuovo inequality (cf. [De1 Théorème 3.2]) states that if $S$ is a minimal surface of general type such that $\varphi$ is birational then $K^2_S \geq 3p_g + q - 7$. In the case $q > 0$, in [Ba Theorem 2.1] the inequality has been improved to $K^2_S \geq 3p_g + q - 6$ under the assumption that $p_g \geq 6$.

By applying the results of [4] we are able to improve further the inequality in the case of surfaces with $q \geq 6$ (Theorem 5.1) and to sharpen it further under the assumption that $S$ has no irrational pencil and $|K_S|$ has a fixed part (Theorem 5.2). As in the previous section, $S$ denotes a smooth complex projective surface with geometric genus $p_g$ and irregularity $q$ and the canonical map of $S$ is denoted by $\varphi$: $S \to \mathbb{P}^{p_g - 1}$.

**Theorem 5.1.** Assume that $S$ is minimal and $\varphi$ is birational. Then:

(i) if $q > 0$ and $p_g \geq 6$, then $K^2_S \geq 3p_g + q - 6$;
(ii) if $q \geq 6$ and $p_g \geq 12$, then $K^2_S \geq 3p_g + q - 5$;
(iii) if $q \geq 7$ and $p_g \geq 14$, then $K^2_S \geq 3p_g + q - 4$.

**Proof.** Statement (i) is [Ba Theorem 2.1].

If albdim $S = 1$, then $K^2_S \geq 3p_g + 7q - 7$ by [Ko2 Theorem 6.1]. Hence we may assume albdim $S = 2$.

Since $S$ is minimal, by Riemann–Roch we have $K^2_S + \chi(S) = h^0(2K_S) \geq h^0(2M)$, namely $K^2_S \geq h^0(2M) - p_g + q - 1$. Hence (ii) and (iii) follow directly by Proposition 4.8.
Theorem 5.2. Assume that $S$ is minimal with no irrational pencils of genus $\geq 2$ and that $\varphi$ is birational. If the canonical system $|K_S|$ has a fixed part $Z > 0$, then:

(i) if $q \geq 6$, then $K_S^2 \geq 3p_g + q - 3$;
(ii) if $q \geq 6$ and $p_g \geq 12$, then $K_S^2 \geq 3p_g + q - 2$;
(iii) if $q \geq 7$ and $p_g \geq 14$, then $K_S^2 \geq 3p_g + q - 1$.

Furthermore, if equality holds in (i), (ii) or (ii), then $Z^2 = -2$ and $K_SZ = 0$.

Proof. By Corollary 4.11 we have $K_S^2 + \chi(S) \geq h^0(2M) + 3$, with equality holding only if $Z^2 = -2$ and $K_SZ = 0$. The result now follows immediately by Proposition 4.8 (notice that for $q \geq 6$ one has $p_g \geq 9$ by the Castelnuovo-De Franchis inequality). □

6. Surfaces with $p_g = 2q - 3$

Throughout all the section we consider a minimal surface $S$ with irregularity $q$ and geometric genus $p_g(S) = 2q - 3$. We denote by $\Sigma$ the canonical image and by $\varphi: S \to \Sigma \subset \mathbb{P}^{2q-4}$ the canonical map.

The purpose of the section is to prove the following:

Theorem 6.1. If $S$ is minimal with $q \geq 6$, $p_g = 2q - 3$ and has no irregular pencil of genus $\geq 2$, then the canonical map $\varphi$ is birational.

As a consequence, we are able to strengthen the inequalities of [MP1, Theorem 1.2] as follows:

Theorem 6.2. If $S$ is minimal with $p_g = 2q - 3$, then:

(i) if $q \geq 6$, then $K_S^2 \geq 7\chi(S) + 2$;
(ii) if $q \geq 8$, then $K_S^2 \geq 7\chi(S) + 3$;
(iii) if $q \geq 9$, then $K_S^2 \geq 7\chi(S) + 4$.

Furthermore if equality holds then the fixed part $Z$ of $|K_S|$ is a ($-2$)-cycle of type $D_n, E_6, E_7$ or $E_8$.

Proof. As explained in the proof of [MP1, Thm.1.2], we may assume that $S$ has no irrational pencil of genus $> 1$ and that $|K_S| = |M| + Z$, with the fixed part $Z > 0$.

Since in this case the canonical map $\varphi$ is birational by Theorem 6.1 we get the inequalities by applying Theorem 5.2. Again by Theorem 5.2 one has equality only if $K_SZ = 0$ and $Z^2 = -2$.

Note that for every component $\theta$ of $Z$, $M\theta \geq 0$ because $|M|$ is the moving part of $|K_S|$. Since $K_S\theta = 0$ we see that every component $\theta$ of $Z$ satisfies $\theta Z \leq 0$. So $Z$ is a ($-2$)-cycle and as such it can be of of type $A_n$, $D_n$, $E_6$, $E_7$ or $E_8$ (see, e.g., [BPV, Ch.III,§3]). However if $Z$ is of type $A_n$ then by [BNP, Theorem 5.4] one has $K_S^2 \geq 8\chi(S)$, a contradiction. □
6.1. Proof of Theorem 6.1. The proof is quite involved and requires a
detailed analysis of the case \( q = 6 \), hence we break it into several auxiliary
lemmas. The first one is of independent interest.

**Lemma 6.3.** Let \( S \) be a minimal surface of general type with \( q = 6 \) and
\( p_g = 2q - 3 = 9 \). If \( S \) has no irrational pencil of genus \( > 1 \), then
\[
K^2_S \leq 35 - r,
\]
where \( r \) is the number of irreducible curves contracted by the Albanese map
of \( S \).

**Proof.** By Noether’s formula and Hodge duality, we have \( K^2_S + h^{1,1}(S) = 52 \).
We give a lower bound for \( h^{1,1}(S) \) by using methods and results from [CP]. Let \( \Gamma_1, \ldots, \Gamma_r \)
be the irreducible curves contracted by the Albanese map \( a: S \to A. \) Since the image of \( a \) is a surface, the intersection ma-
trix \( (\Gamma_i \Gamma_j)_{i,j=1,\ldots,r} \) is negative definite, hence the classes of the \( \Gamma_i \) span an
\( r \)-dimensional subspace \( T_1 \subset H^{1,1}(S) \). Since \( T_1 \) is orthogonal to \( T_2 := a^*(H^{1,1}(A)) \subset H^{1,1}(S) \), we have
\( h^{1,1}(S) \geq r + \dim T_2 \).

Denote by \( \mathbb{H}_q \) the real vector space of \( q \times q \) Hermitian matrices and
define \( d_{q,n} \) as the maximum dimension of a subspace \( V \subset \mathbb{H}_q \) such that
every \( 0 \neq M \in V \) has rank \( \geq 2n \). By [CP, Proposition 2.2.3], one has
\( \dim T_2 \geq 30 - d_{6,2} \).

We give a rough lower bound for \( d_{6,2} \) as follows. Identify \( \mathbb{H}_5 \) with the
subspace of \( \mathbb{H}_6 \) consisting of the matrices whose last row and column are
zero. Then if \( V \subset \mathbb{H}_6 \) is a subspace such that every \( 0 \neq M \in V \) has rank
at least 4, then \( \dim V \cap \mathbb{H}_5 \leq d_{5,4} \). We have \( d_{5,4} \leq 8 \) by [CP, Proposition
2.2.2], hence using Grassmann formula we get
\[
\dim V \leq \dim \mathbb{H}_6 - \dim \mathbb{H}_5 + d_{5,4} \leq 36 - 25 + 8 = 19,
\]
which gives \( d_{6,2} \leq 19 \). Thus we get \( \dim T_2 \geq 17 \), \( h^{1,1}(S) \geq 17 + r \) and
\( K^2_S \leq 35 - r \).

\( \Box \)

The next Lemma contains the proof of Theorem 6.1 for \( q \geq 7 \).

**Lemma 6.4.** Assume that \( S \) is minimal with \( q \geq 6 \), \( p_g = 2q - 3 \) that \( \varphi \) is
not birational and that \( S \) has no irrational pencil of genus \( \geq 2 \). Then \( q = 6 \),
\( \deg \varphi = 3 \) and the canonical image \( \Sigma \subset \mathbb{P}^8 \) is a rational surface of degree
11.

**Proof.** Since by [X12] a surface of general type \( S \) whose canonical system
is composed with a pencil has \( q \leq 2 \), the canonical image \( \Sigma \) is a surface.
Hence, denoting by \( d \) be the degree of \( \varphi \) and by \( m \) the degree of \( \Sigma \), we have
\( K^2_\Sigma \geq dm \).

Assume that \( d > 1 \). By [Be1, Théorème 3.1] either \( p_g(\Sigma) = 0 \) or \( p_g(\Sigma) =
p_g(S) \) and \( \Sigma \) is a canonical surface. In the second case \( m \geq 3p_g - 7 \) and
so \( K_S^2 \geq 6p_g - 14 = 12q - 36 = 9\chi(S) + 3q - 14 \). Since \( q \geq 6 \), this is a contradiction to the Miyaoka-Yau inequality.

So \( p_g(\Sigma) = 0 \) and, by Corollary 5.3 \( d \geq 3 \) and \( q(\Sigma) \leq 1 \). Since \( d \geq 3 \), we have \( K_S^2 \geq 3m \) and therefore, since \( K_S^2 \leq 9(q - 2) \) by the Miyaoka-Yau inequality, we get

\[
(6.1) \quad m \leq 3(q - 2) = \frac{3}{2}(2q - 4).
\]

Thus \( \Sigma \) is a ruled surface by [Be1, Lemme 1.4].

Assume that \( q(\Sigma) = 1 \). By Proposition 4.6 \( \Sigma \) has no pencil of rational curves of degree \( < q - 3 \) and so by [Re1 (1.2)], \( m \geq (2(q - 3)/(q - 2))(2q - 4) = 4(q - 3) \). Since \( 4(q - 3) \leq 3(q - 2) \) iff \( q \leq 6 \), for \( q = 7 \) we have obtained a contradiction. For \( q = 6 \), the same argument gives \( m = 12 \), hence \( K_S^2 \geq 3m = 36 \), contradicting Lemma 6.3.

Hence \( q(\Sigma) = 0 \) and \( \Sigma \) is rational. By (6.1), the surface \( \Sigma \) satisfies the assumptions of Theorem 3.1. Hence, if \( q \geq 7 \) the surface \( \Sigma \) has a pencil \( |L| \) of curves such that the span of every \( L \in |L| \) has dimension \( < q - 2 \). Since this contradicts Proposition 4.6 statement (ii) is proven.

By the same arguments, if \( q = 6 \) and \( d > 1 \) then \( m = 11 \) or \( m = 12 \). By Lemma 6.3 we get \( 35 \geq K_S^2 \geq dm \geq 3m \). Hence the only possibility is \( d = 3 \) and \( m = 11 \).

\[ \square \]

**Lemma 6.5.** If \( S \) has no irrational pencil of genus \( \geq 2 \), \( q = 6 \), \( p_g = 9 \) and \( \varphi \) is not birational, then \( |K_S| \) has no fixed component and \( S \) contains no rational curves. In particular \( K_S \) is ample.

**Proof.** As usual, write \( |K_S| = |M| + Z \), where \( |M| \) is the moving part and \( Z \) is the fixed part. Since every global 2-form \( \sigma \) of \( S \) can be written \( \sigma = \alpha \wedge \beta \) for some \( \alpha, \beta \in H^0(\Omega^1_S) \) (cf. [MP1, §3]), the components of \( Z \) are the curves on which the differential of the Albanese a map drops rank.

Let \( r \) be the number of irreducible curves of \( S \) contracted by \( a \). By Lemma 6.4 and Lemma 6.3, we have:

\[
(6.2) \quad 35 - r \geq K_S^2 = K_SM + K_SZ \geq M^2 + MZ \geq MZ + 33.
\]

By the 2-connectedness of canonical divisors, if \( Z > 0 \) then \( MZ = 2 \), \( K_SZ = 0 \), \( Z^2 = -2 \). Hence every component of \( Z \) is a smooth rational curve with self-intersection \(-2 \) and \( r > 0 \), contradicting (6.2). Thus \( Z = 0 \). Furthermore since any rational curve of \( S \) would be contained in \( Z \), \( S \) has no rational curves and so \( K_S \) is ample. \[ \square \]

Finally we are in a position to show that also in the case \( q = 6 \) the canonical map is birational.

**Lemma 6.6.** Let \( S \) be a minimal surface of general type with \( q = 6 \) and \( p_g = 2q - 3 = 9 \). If \( S \) has no irregular pencil of genus \( > 1 \), then the canonical map of \( S \) is birational.
Proof. Suppose for contradiction that $\varphi$ is not birational. Then, by Lemma 6.4, $\varphi$ has degree 3 and the canonical image $\Sigma \subset \mathbb{P}^8$ is a rational surface of degree 11.

By Proposition 4.6, $\Sigma$ has no pencil of curves of degree $\leq 3$, hence by Proposition 3.2 we can write $K_S = 2D + \Gamma$ where $\Gamma$ is an effective divisor $\geq 0$ and $|D|$ is a linear system without fixed components such that $h^0(S, D) \geq 3$.

Since, by Lemmas 6.3 and 6.4, $33 \leq K_S^2 \leq 35$, $K_S$ is not divisible by 2 in $\text{Pic}(S)$. This implies that $\Gamma \neq 0$ and $\Gamma$ is also not divisible by 2 in $\text{Pic}(S)$.

Since $K_S^2 \leq 35$ and $K_S$ is nef, $K_S D \leq 17$. We claim that $K_S D \geq 16$. For contradiction suppose that $K_S D \leq 15$. By Proposition 4.6, we have $p_a(D) \geq 12$, hence $D^2 \geq 7$ by the adjunction formula. This gives a contradiction to the index theorem (Corollary 2.2), because $7 \cdot 33 = 231 > (15)^2 = 225$. So $K_S D \geq 16$.

Then we have the following possibilities:

- $K_S D = 16, K_S \Gamma = 1$ and $K_S^2 = 33$;
- $K_S D = 16, K_S \Gamma = 2$ and $K_S^2 = 34$;
- $K_S D = 16, K_S \Gamma = 3$ and $K_S^2 = 35$;
- $K_S D = 17, K_S \Gamma = 1$ and $K_S^2 = 35$;

We start by noticing that $\Gamma^2 \leq -1$. In effect, by the index theorem (Corollary 2.2), $\Gamma^2 \leq 0$. Since by the adjunction formula $\Gamma^2 \equiv K_S \Gamma$ mod 2, $\Gamma^2 = 0$ can only occur if $K_S \Gamma = 2$. But this possibility is excluded by Corollary 4.3 because $\Gamma$ is 1-connected by Corollary 2.8. The same reasoning shows that any irreducible component $\theta$ of $\Gamma$ satisfies also $\theta^2 \leq -1$. Since, by Lemma 6.5, $K_S$ is ample, $K_S \theta > 0$ for every component $\theta$ of $\Gamma$. Furthermore, since again by Lemma 6.5 there are no rational curves in $S$, any irreducible component $\theta$ of $\Gamma$ such that $K_S \theta = 1$ must satisfy $\theta^2 = -1$, whilst an irreducible component $\theta$ of $\Gamma$ such that $K_S \theta = 2$ must satisfy $\theta^2 = -2$. Similarly if $\Gamma$ is irreducible and $K_S \Gamma = 3$ then $\Gamma^2 = -3$ or $\Gamma^2 = -1$. Note that if $K_S \Gamma = 2$ and $\Gamma$ is not reducible, $\Gamma$ must be the sum of two distinct components because $\Gamma$ is not divisible by 2 in $\text{Pic}(S)$.

In conclusion:

(i) if $K_S \Gamma = 1$ then $\Gamma$ is irreducible and $\Gamma^2 = -1$;
(ii) if $K_S \Gamma = 2$, $\Gamma$ is reduced.
(iii) if $K_S \Gamma = 3$ and $\Gamma$ is not reduced then $\Gamma = 2\theta_1 + \theta_2$ where $\theta_1, \theta_2$ are smooth elliptic curves with self-intersection $-1$.

In case (i) $\Gamma^2 = -1$ and $K_S = 2D + \Gamma$ imply that $\Gamma D = 1$. Then $K_S D = 2D^2 + \Gamma D = 2D^2 + 1$ is odd and so $K_S D = 17$. This is a contradiction to the adjunction formula because then $D^2 = 8$ and $K_S D = 17$. So case (i) does not occur.

Case (iii) can be excluded in the same way, using the fact that $K_S = 2D' + \theta_2$, where $D' := D + \theta_2$, and $K_S D' = 17$.

So we are left with the cases when $K_S \Gamma \geq 2$ and $\Gamma$ is reduced. Then $K_S D = 16$ and so by the adjunction formula $D^2$ is even. From $K_S D =
2D² + ΔΓ, we conclude that ΔΓ is also even. Then the equality K_S² = 4D² + 4ΔΓ + Γ² means that Γ² ≡ K_S² mod 8.

On the other hand, since every component of Γ has geometric genus > 0 and Γ is reduced, also p_a(Γ) > 0. We have seen above that Γ² < 0 and so there are only the following possibilities:

- K_SΓ = 2 and Γ² = -2 (K_S² = 34);
- K_SΓ = 3, Γ² = -1 (K_S² = 35);
- K_SΓ = 3, Γ² = -3 (K_S² = 35).

This is a contradiction because in none of these cases Γ² ≡ K_S² mod 8.

So deg ϕ = 3 does not occur and therefore ϕ is birational.

The above Lemma concludes the proof of Theorem 6.1.

References

[Ba] M.A. Barja, Numerical bounds of canonical varieties, Osaka J. Math. 37 (2000), 701–718.

[BNP] M.A. Barja, J.C. Naranjo and G.P. Pirola, On the topological index of irregular surfaces, J. Algebraic Geom. 16 no. 3 (2007), 435–458.

[Be1] A. Beauville, L’application canonique pour les surfaces de type général, Inv. Math. 55 (1979), 121–140.

[Be2] A. Beauville, L’inégalité p_g ≥ 2q − 4 pour les surfaces de type général, appendix to [De1].

[Bo] E. Bombieri, Canonical models of surfaces of general type, Inst. Hautes Études Sci. Publ. Math. 42 (1973), 171–219.

[BPV] W. Barth, C. Peters and A. Van de Ven, Compact Complex Surfaces, Ergebnisse der Mathematik, 3. Folge, Band 4, Springer, Berlin, 1984.

[CP] A. Causin and G. Pirola, Hermitian matrices and cohomology of Kaehler varieties, Manuscripta math. 121 (2006), 157-168.

[Cil2] C. Ciliberto, Hilbert functions of finite sets of points and the genus of a curve in a projective space, in “Space curves”, Proceedings of the Rocca di Papa Conference, 1985, Springer Lecture Notes in Math., 1266, (1987), 24–73.

[CFM] C. Ciliberto, P. Francia and M. Mendes Lopes, Remarks on the bicanonical map for surfaces of general type, Math. Z. 224 (1997), 137–166.

[CMP] C. Ciliberto, M. Mendes Lopes and R. Pardini, Surfaces with K² < 3χ and finite fundamental group, Math. Res. Lett. 14, no. 6 (2007), 1081-1098.

[De1] O. Debarre, Inégalités numériques pour les surfaces de type général, with an appendix by A. Beauville, Bull. Soc. Math. France 110 3 (1982), 319–346.

[De2] O. Debarre, Tores et variétés abéliennes, S.M.F. Sciences 1999.

[HP] C.D. Hacon, R. Pardini, Surfaces with p_g = q = 3, Trans. Amer. Math. Soc. 354 no. 7 (2002), 2631-2638.

[Ha1] J. Harris, A bound on the geometric genus of projective varieties, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 8 (1981), no. 1, 35–68.

[Ha2] J. Harris, Curves in projective space, with the collaboration of D. Eisenbud, OTAN Seminar, Les Presses de l’Université de Montreal, 1982.

[Ko1] K. Konno, On the quadric hull of a canonical surface, Algebraic geometry, 217–235, de Gruyter, Berlin, 2002.

[Ko2] K. Konno, Non-hyperelliptic fibrations of small genus and certain irregular canonical surfaces, Ann. Sc. Norm. Sup. Pisa ser. IV 10 (1993), 575–595.

[LP] R. Lazarsfeld and M.Popa, BGG correspondence, cohomology of compact Kähler manifolds, and numerical inequalities, Invent. Math. 182 no.3 (2010), 605–633.
M. Mendes Lopes, *Adjoint systems on surfaces*, Boll. Un. Mat. Ital. A (7) **10** (1996), no. 1, 169–179.

M. Mendes Lopes and R. Pardini, *On surfaces with \( p_g = 2q - 3 \)*, Adv. in Geom. **10** (3) (2010), 549–555.

M. Mendes Lopes, R. Pardini, *Severi type inequalities for surfaces with ample canonical class*, to appear in Comment. Math. Helv.

M. Mendes Lopes, R. Pardini and Gian Pietro Pirola, *On surfaces of general type with \( q = 5 \)*, arXiv:1003.5991

M. Nagata, *On rational surfaces I*, Mem. Coll. Sci. Univ. Kyoto **32** (1960), 351–370.

I. Petrakiev, *A Step in Castelnuovo theory via Gröbner bases*, Journal für die reine und angewandte Mathematik, **619** (2008), 49–73.

G.P. Pirola, *Algebraic surfaces with \( p_g = q = 3 \) and no irrational pencils*, Manuscripta Math. **108** no. 2 (2002), 163–170.

C.P. Ramanujam, *Remarks on the Kodaira vanishing theorem*, Journal of the Indian Math. Soc. **36** (1972), 41–51.

M. Reid, *Surfaces of small degree*, Math. Ann. **275** (1986), 71–80.

M. Reid, *Quadrics through a canonical surface*, in “Algebraic Geometry – Hyperplane sections and related topics (L’Aquila 1988)”, Springer LNM 1417 (1990), 191–213.

M. Reid, *Chapters on algebraic surfaces*, in “Complex algebraic varieties”, J. Kollár (Ed.), IAS/Park City lecture notes series (1993 volume), AMS, Providence R.I., 1997, 1–154.

C. Schoen, *A family of surfaces constructed from genus 2 curves*, Internat. J. Math. **18** (2007), no. 5, 585–612.

G. Xiao, *L’irregularité des surfaces de type général dont le système canonique est composé d’un pinceau*, Compositio Mathematica **56** (1985), 251–257.

G. Xiao, *Algebraic surfaces with high canonical degree*, Math. Ann **274** (1986), 473–483.

G. Xiao, *Hyperelliptic surfaces of general type with \( K^2 < 4 \chi \)*, Manuscripta Math. **57** (1987), 125–148.

G. Xiao, *Irregularity of surfaces with a linear pencil*, Duke Math. J. **55** (1987), no. 3, 597–602.

G. Xiao, *Fibered algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449–466.

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