Recursive calculation of effective resistances in distance-regular networks based on Bose–Mesner algebra and Christoffel–Darboux identity

M. A. Jafarizadeh,1,2,3,a R. Sufiani,1,3,b and S. Jafarizadeh4
1Department of Theoretical Physics and Astrophysics, The University of Tabriz, Tabriz 51664, Iran
2Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-1795, Iran
3Research Institute for Fundamental Sciences, Tabriz 51664, Iran
4Department of Electrical and Computer Engineering, The University of Tabriz, Tabriz 51664, Iran

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Recently, Jafarizadeh et al. [ J. Phys. A: Math. Theor. 40, 4949 (2007)] have given a method for calculation of effective resistance (two-point resistance) on distance-regular networks, where the calculation was based on stratification introduced by Jafarizadeh and Salimi [J. Phys. A 39, 1 (2006)] and Stieltjes transform of the spectral distribution (Stieltjes function) associated with the network. Also, Jafarizadeh et al. [ J. Phys. A: Math. Theor. 40, 4949 (2007)] have shown that effective resistances between a node \( \alpha \) and all nodes \( \beta \) belonging to the same stratum with respect to \( \alpha \) \( (R_{\alpha\beta^{(n)})} \), \( \beta \) belonging to the ith stratum with respect to \( \alpha \) are the same.

In this work, an algorithm for recursive calculation of the effective resistances in an arbitrary distance-regular resistor network is provided, where the derivation of the algorithm is based on the Bose–Mesner algebra, stratification of the network, spectral techniques, and Christoffel–Darboux identity. It is shown that the effective resistance on a distance-regular network is a strictly increasing function of the shortest path distance defined on the network. In other words, the effective resistance \( R_{\alpha\beta^{(n+1)}} \) is strictly larger than \( R_{\alpha\beta^{(n)}} \). The link between effective resistance and random walks on distance-regular networks is discussed, where average commute time and its square root (called Euclidean commute time) as distance are related to effective resistance. Finally, for some important examples of finite distance-regular networks, effective resistances are calculated. © 2009 American Institute of Physics. [DOI: 10.1063/1.3077145]

I. INTRODUCTION

A classic problem in electric circuit theory studied by numerous authors over many years is the computation of the resistance between two nodes in a resistor network (see, e.g., Ref. 1). The study of electric networks was formulated by Kirchhoff2 more than 150 years ago as an instance of a linear analysis. Besides being a central problem in electric circuit theory, the computation of resistances is also relevant to a wide range of problems ranging from random walks (see Ref. 3), the theory of harmonic functions4 to lattice Green’s functions.5–9

The concepts of effective resistance (called also two-point resistance and resistance distance) and molecular structure descriptors based on it were much studied in the chemical literature.10–17 Also, the effective resistance is closely related to average first passage time and average commute time which are two important quantities in random walk models defined based on Markov chains. It is shown in Ref. 18 that computation of average commute time can be obtained from the

Electronic addresses: jafarizadeh@tabrizu.ac.ir and majafarizadeh@yahoo.com.
Electronic mail: sofiani@tabrizu.ac.ir.
pseudoinverse of the Laplacian \( L (L^{-1}) \) of the underlying network. Also, it has been shown that this quantity and its square root are distance, since it can be shown\(^{18} \) that \( L^{-1} \) is symmetric and positive semidefinite. It is therefore called the Euclidean commute time (ECT) distance. In fact, ECT distance is the same as effective resistance (effective resistance is symmetric and satisfies the triangle inequality and so is a distance metric and sometimes is called resistance distance). Therefore, any clustering algorithm (hierarchical clustering, \( k \)-means, etc.),\(^{10} \) which can be used in conjunction with ECT distance, deals with effective resistance. Also, Laplacian eigenmaps which recently proposed by Belkin and Niyogi in Ref. 20 is one of the graph-based approaches on dimensionality reduction and manifold learning. This approach which shares many similarities with other recent spectral algorithms for clustering and embedding of data, for instance, kernel principle component analysis (PCA) algorithm\(^{21} \) and spectral methods for image segmentation\(^{22} \), deals with Laplacian of the graph assigned to the data and uses effective resistance (for a unifying view of behavior of spectral embeddings and clustering algorithms, see Ref. 23). One of the most important aspects of spectral methods for clustering and embedding, including Laplacian eigenmaps, is the fact that they are all posed as eigenvalue problems. But clearly, for too large matrices, the computation by pseudoinverse becomes cumbersome.

On the other hand, the theory of association schemes, which were introduced by Bose and Nair, has its origin in the design of statistical experiments. The connection of association schemes to algebraic codes, strongly regular graphs, distance-regular graphs, design theory, etc., further intensified their study. Association schemes have since then become the fundamental, perhaps the most important objects in algebraic combinatorics. A further step in the study of association schemes was their algebraization. This formulation was done by Bose and Mesner who introduced to each association scheme a matrix algebra generated by adjacency matrices of the association scheme. This matrix algebra came to be known as adjacency algebra or Bose–Mesner algebra associated with the association scheme.

Recently, the authors have given a method for calculation of the effective resistance on distance-regular networks\(^{24} \) which are underlying networks of particular association schemes called \( P \)-polynomial association schemes, where the calculation is based on stratification introduced in Ref. 25 and Stieltjes function associated with the networks. Also, in Ref. 24 (by using the algebraic structure of these networks) it has been shown that the resistances between a node \( \alpha \) and all nodes \( \beta \) belonging to the same stratum with respect to \( \alpha \) \((R_{\alpha \beta^{(i)}})\), \( \beta \) belonging to the \( i \)th stratum with respect to \( \alpha \) are the same, and analytical formulas have been given for effective resistances \( R_{\alpha \beta^{(i)}, i=1,2,3} \) in terms of the size of the network and corresponding intersection array without any need to know the spectrum of pseudoinverse matrix \( L^{-1} \). As it can be seen from Ref. 24, obtaining explicit formulas for \( R_{\alpha \beta^{(i)}} \) with \( i>3 \) needs so cumbersome calculations. In this work, our starting point is along the same line, i.e., we consider the Laplacian matrix associated with distance-regular networks and use the algebraic structure of these networks (Bose–Mesner algebra) such as stratification and spectral techniques specially the well known Christoffel–Darboux identity\(^{26} \) from the theory of orthogonal polynomials to give a recursive formula for calculation of all of the effective resistances \( R_{\alpha \beta^{(i)}, i=1,2,\ldots,d} \) (\( d \) is diameter of the network) on the networks without any need to the spectrum of pseudoinverse \( L^{-1} \). In fact, we show that, in order to evaluate effective resistances on distance-regular networks, one needs only to know the intersection array of the networks. The other main consequence of the derived recursive formula is that the effective resistance (as a distance metric) strictly increases by increasing the shortest path distance defined on the network, i.e., \( R_{\alpha \beta^{(m+1)}}-R_{\alpha \beta^{(m)}}>0 \) for all \( m=1,2,\ldots,d-1 \).

The organization of the paper is as follows. In Sec. II, we give some preliminaries such as association schemes, distance-regular networks, stratification of these networks, and a brief review about spectral techniques used in the paper. Section III is devoted to the concept of effective resistance on regular networks and its relation with random walks. In Sec. IV (main section of the paper) we show that effective resistance increases with increasing the number of stratum and give a recursive formula for calculation of effective resistances \( R_{\alpha \beta^{(i)}} \) for \( i=1,2,\ldots,d \) in distance-regular networks, where the algorithm is based on the Bose–Mesner algebra and spectral tech-
niques specially the Christoffel–Darboux identity. The paper is ended with a brief conclusion and an appendix containing calculation of effective resistances $R_{\alpha \beta i}$, $i=1,2,\ldots,d$ on some important finite distance-regular networks.

II. PRELIMINARIES

In this section we give some preliminaries such as definitions related to association schemes, corresponding stratification, distance-regular networks, and spectral techniques used through the paper.

A. Association schemes

First we recall the definition of association schemes. For further information on association schemes, the reader is referred to Ref. 27.

Definition: Let $V$ be a set of vertices and $R_i$, $i=0,1,\ldots,d$ be nonempty relations on $V$ (i.e., subsets of $V \times V$). Then, the relations $\{R_i\}_{0 \leq i \leq d}$ on $V \times V$ satisfying the following conditions.

1. $\{R_i\}_{0 \leq i \leq d}$ is a partition of $V \times V$.
2. $R_0 = \{(\alpha, \alpha) : \alpha \in V\}$.
3. $R_i = R_i^t$ for $0 \leq i \leq d$, where $R_i^t = \{(\beta, \alpha) : (\alpha, \beta) \in R_i\}$.
4. For $(\alpha, \beta) \in R_i$, the number $p_{ij}^k = |\{\gamma \in V : (\alpha, \gamma) \in R_i \text{ and } (\gamma, \beta) \in R_j\}|$ does not depend on $(\alpha, \beta)$ but only on $i$, $j$ and $k$.

Define a symmetric association scheme of class $d$ on $V$ which is denoted by $Y = (V, \{R_i\}_{0 \leq i \leq d})$. Furthermore, if we have $p_{ij}^k = p_{ji}^k$ for all $i, j, k = 0, 2, \ldots, d$, then $Y$ is called commutative. The number $v = |V|$ of the vertices is called the order of the association scheme and $R_i$ is called $i$th relation (for examples of association schemes, see Ref. 24).

Let $Y = (V, \{R_i\}_{0 \leq i \leq d})$ be a commutative symmetric association scheme of class $d$, then the matrices $A_0, A_1, \ldots, A_d$ defined by

\[
(A_i)_{\alpha \beta} = \begin{cases} 
1 & \text{if } (\alpha, \beta) \in R_i \\
0 & \text{otherwise,}
\end{cases} \tag{2.1}
\]

are adjacency matrices of $Y$ such that

\[
A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k. \tag{2.2}
\]

From (2.2), it is seen that the adjacency matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative algebra $A$ known as Bose–Mesner algebra of $Y$. This algebra has a second basis $E_0, \ldots, E_d$ such that $E_i E_j = \delta_{ij} E_i$ and $\sum_{i=0}^{d} E_i = I$ with $E_0 = 1/vJ$ ($J$ is a $v \times v$ matrix with all-one entries). The matrices $E_i$ for $0 \leq i \leq d$ are known as primitive idempotents of $Y$. Furthermore, there are matrices $P$ and $Q$ such that the two bases of the Bose–Mesner algebras can be related to each other as follows:

\[
A_i = \sum_{j=0}^{d} P_{ij} E_j, \quad 0 \leq j \leq d,
\]

\[
E_i = \frac{1}{v} \sum_{j=0}^{d} Q_{ij} A_j, \quad 0 \leq j \leq d. \tag{2.3}
\]

Then, clearly we have
It also follows that

\[ A_j E_i = P_j E_i, \]

which indicates that \( P_{ij} \) is the \( i \)th eigenvalue of \( A_j \) and that the columns of \( E_i \) are corresponding eigenvectors. Also, \( m_i := \text{tr} E_i = v \langle \alpha | E_i | \alpha \rangle = Q_0 \) [where, we have used the fact that \( \langle \alpha | E_i | \alpha \rangle \) is independent of the choice of \( \alpha \in V \), see Eq. (2.3)] is the rank of the idempotent \( E_i \) which gives the multiplicity of the eigenvalue \( P_{ij} \) of \( A_j \) (provided that \( P_{ij} \neq P_{kj} \) for \( k \neq i \)). Clearly, we have \( m_0 = 1 \) and \( \Sigma_{i=0}^d m_i = v \) since \( \Sigma_{i=0}^d E_i = I \). It should be also noticed that, due to the tracelessness of adjacency matrices \( A_i, i \neq 0 \), and using the Eq. (2.3), one can obtain the following identity (which we will use it in obtaining the main result of the paper):

\[ v \delta_0 = \text{tr} A_1 = v \cdot \sum_{j=0}^d P_{ij} \langle \alpha | E_j | \alpha \rangle = \sum_{j=0}^d P_j m_j. \]  

Finally the underlying network of an association scheme \( \Gamma = (V, R_i) \) is an undirected connected network, where the sets \( V \) and \( R_1 \) consist of its vertices and edges, respectively. Obviously replacing \( R_1 \) with one of the other relations \( R_i, i \neq 0, 1 \), will also give us an underlying network \( \Gamma = (V, R_i) \) (not necessarily a connected network) with the same set of vertices but a new set of edges \( R_i \).

As we will see in Sec. II C, in the case of distance-regular networks, the adjacency matrices \( A_j \) are polynomials of the adjacency matrix \( A = A_1 \), i.e., \( A_j = P_j(A) \), where \( P_j \) is a polynomial of degree \( j \), then the eigenvalues \( P_{ij} \) in (2.5) are polynomials of eigenvalues \( P_{ij} = \lambda_i \) (eigenvalues of the adjacency matrix \( A \)). This indicates that in distance-regular networks, the matrix \( P \) is a polynomial transformation as

\[
P = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
P_1(\lambda_0) & P_1(\lambda_1) & \ldots & P_1(\lambda_d) \\
P_2(\lambda_0) & P_2(\lambda_1) & \ldots & P_2(\lambda_d) \\
\vdots & \vdots & \ddots & \vdots \\
P_d(\lambda_0) & P_d(\lambda_1) & \ldots & P_d(\lambda_d)
\end{pmatrix}
\]

or \( P_{ij} = P_j(\lambda_i) \).

**B. Stratifications**

For a given vertex \( \alpha \in V \), let \( \Gamma_i(\alpha) = \{ \beta \in V : (\alpha, \beta) \in R_i \} \) denotes the set of all vertices having the relation \( R_i \) with \( \alpha \). Then, the vertex set \( V \) can be written as disjoint union of \( \Gamma_i(\alpha) \) for \( i = 0, 1, 2, \ldots, d \), i.e.,

\[ V = \bigcup_{i=0}^d \Gamma_i(\alpha). \]

We fix a point \( o \in V \) as an origin of the underlying network of an association scheme, called reference vertex. Then, the relation (2.8) stratifies the network into a disjoint union of associate classes \( \Gamma_i(o) \) (called the \( i \)th stratum with respect to \( o \)). Let \( \ell^2(V) \) denote the Hilbert space of \( C \)-valued square-summable functions on \( V \). With each associate class \( \Gamma_i(o) \), we associate a unit vector in \( \ell^2(V) \) defined with...
where \(|\alpha\rangle\) denotes the eigenket of \(\alpha\)th vertex at the associate class \(\Gamma_\alpha(o)\) and \(\kappa_\alpha=|\Gamma_\alpha(o)|\) is called the \(\alpha\)th valency of the network. Now, let \(A_i\) be the adjacency matrix of the network \(\Gamma=(V,R)\). Then, for the reference state \(|\phi_0\rangle\) \((|\phi_0\rangle=|o\rangle, \text{with } o \in V \text{ as reference vertex})\), one can write

\[
A_i|\phi_0\rangle = \sum_{\beta \in \Gamma_\alpha(o)} |\beta\rangle.
\]

Then, by using (2.9) and (2.10), we obtain

\[
A_i|\phi_0\rangle = \sqrt{\kappa_i}|\phi_i\rangle.
\]

C. Distance-regular networks and orthogonal polynomials

Distance-regular networks are underlying networks of the so-called \(P\)-polynomial association schemes,\textsuperscript{27} where the relations \(R_i\) are defined based on shortest path distance. More clearly, if distance between nodes \(\alpha, \beta \in V\) denoted by \(d(\alpha,\beta)\) be the length of the shortest walk connecting \(\alpha\) and \(\beta\) (recall that a finite sequence \(\alpha_0, \alpha_1, \ldots, \alpha_n \in V\) is called a walk of length \(n\) if \(\alpha_{i-1} \sim \alpha_k\) for all \(k=1,2,\ldots,n\), where \(\alpha_{i-1} \sim \alpha_k\) means that \(\alpha_{i-1}\) is adjacent with \(\alpha_k\)), then the relations \(R_i\) in distance-regular networks are defined as \(d(\alpha,\beta) = i\) if and only if \(d(\alpha,\beta)=i\) for \(i=0,1,\ldots,d\), where \(d:=\max\{d(\alpha,\beta): \alpha, \beta \in V\}\) is called the diameter of the network.

For distance-regular networks, the nonzero intersection numbers are given by

\[
a_i=p_i^1, \quad b_i=p_{i+1,1}, \quad c_i=p_i^1,
\]

respectively. The intersection numbers (2.12) and the valencies \(\kappa_i\) with \(\kappa_i=\kappa\) \([=\deg(o)\rangle, \text{for each vertex } \alpha\rangle\) satisfy the following obvious conditions:

\[
a_i+b_i+c_i=\kappa, \quad \kappa_i=b_{i-1}=\kappa_i c_i, \quad i=1, \ldots, d,
\]

\[
\kappa_0=c_1=1, \quad b_0=\kappa_1=\kappa, \quad (c_0=b_d=0).
\]

Thus all parameters of a distance-regular network can be obtained from its intersection array \(\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}\). Then, it can be shown that the following recursion relations are satisfied:

\[
A_iA_j=b_{j-1}A_{j-1}+a_iA_j+c_{i+1}A_{i+1}, \quad i=1,2,\ldots,d-1,
\]

\[
A_iA_d=b_{d-1}A_{d-1}+(\kappa-c_d)A_d.
\]

The recursion relations (2.14) imply that

\[
A_i=P_i(A), \quad i=0,1,\ldots,d.
\]

Then, one can easily obtain the following three term recursion relations for the unit vectors: \(|\phi_i\rangle\), \(i=0,1,\ldots,d\)

\[
A|\phi_i\rangle = \beta_{i+1}|\phi_{i+1}\rangle + \alpha_i|\phi_i\rangle + \beta_i|\phi_{i-1}\rangle,
\]

where the coefficients \(\alpha_i\) and \(\beta_i\) are defined as

\[
\alpha_0=0, \quad \alpha_k=\kappa-b_k-c_k, \quad \omega_k=\beta_k^2-b_{k-1}c_k, \quad k=1, \ldots, d,
\]

(see Ref. 24 for more details).

Now, we recall some preliminary facts about spectral techniques used in the paper, where more details have been given in the previous work.\textsuperscript{24}
For any pair \((A, |\phi_0\rangle)\) of a matrix \(A\) and a vector \(|\phi_0\rangle\), one can assign a measure \(\mu\) as follows:

\[
\mu(x) = \langle \phi_0 | E(x) | \phi_0 \rangle, \tag{2.18}
\]

where \(E(x) = \Sigma_i |u_i\rangle\langle u_i|\) is the operator of projection onto the eigenspace of \(A\) corresponding to eigenvalue \(x\), i.e.,

\[
A = \int x E(x) dx.
\tag{2.19}
\]

Then, for any polynomial \(P(A)\) we have

\[
P(A) = \int P(x) E(x) dx,
\tag{2.20}
\]

where for discrete spectrum, the above integrals are replaced by summation. Therefore, using the relations (2.18) and (2.20), the expectation value of powers of adjacency matrix \(A\) over reference vector \(|\phi_0\rangle\) can be written as

\[
\langle \phi_0 | A^m | \phi_0 \rangle = \int x^m \mu(dx), \quad m = 0, 1, 2, \ldots . \tag{2.21}
\]

Obviously, the relation (2.21) implies an isomorphism from the Hilbert space of the stratification onto the closed linear span of the orthogonal polynomials with respect to the measure \(\mu\). More clearly, from orthonormality of the unit vectors \(|\phi_i\rangle\) (with \(|\phi_0\rangle\) as unit vector assigned to reference node) we have

\[
\delta_{ij} = \langle \phi_i | \phi_j \rangle = \frac{1}{\sqrt{\kappa_i \kappa_j}} \langle \phi_0 | A_i A_j | \phi_0 \rangle = \int P'_i(x) P'_j(x) \mu(dx), \tag{2.22}
\]

with \(P'_i(A) := (1/\sqrt{\kappa_i}) P_i(A)\) where, we have used the Eqs. (2.11) and (2.15) to write

\[
|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} A_i |\phi_0\rangle = \frac{1}{\sqrt{\kappa_i}} P_i(A) |\phi_0\rangle = P'_i(A) |\phi_0\rangle. \tag{2.23}
\]

Now, by substituting (2.23) in (2.16) and rescaling \(P'_k\) as \(Q_k = \beta_1 \cdots \beta_k P'_k\), the spectral distribution \(\mu\) under question will be characterized by the property of orthonormal polynomials \(\{Q_k\}\) defined recurrently by

\[
Q_0(x) = 1, \quad Q_1(x) = x,
\]

\[
xQ_k(x) = Q_{k+1}(x) + \alpha_k Q_k(x) + \beta_k^2 Q_{k-1}(x), \quad k \geq 1. \tag{2.24}
\]

1. The Christoffel–Darboux identity

In Sec. III we will treat with the calculation of the effective resistances on distance-regular networks as resistor networks, where we will use one of the most important theorems in the theory of orthogonal polynomials known as Christoffel–Darboux identity. This identity is expressed as follows.

**Theorem:** (Christoffel–Darboux identity) Let \(\{Q_n(x)\}\) satisfy 224. Then

\[
\sum_{k=1}^{n} Q_k(x) Q_k(u) = (\omega_1 \omega_2 \cdots \omega_n)^{-1} Q_{n+1}(x) Q_n(u) - Q_n(x) Q_{n+1}(u)/x - u. \tag{2.25}
\]

For the proof, the reader is referred to Ref. 26.
III. EFFECTIVE RESISTANCES ON REGULAR RESISTOR NETWORKS

For a given regular network \( \Gamma \) with \( v \) vertices and adjacency matrix \( A \), let \( r_{ij} = r_{ji} \) be the resistance of the resistor connecting vertices \( i \) and \( j \). Hence, the conductance is \( c_{ij} = r_{ij}^{-1} = c_{ji} \), so that \( c_{ij} = 0 \) if there is no resistor connecting \( i \) and \( j \). Hereafter, we will assume that all nonzero resistances are equal to 1, then the Laplacian of the network can be written as

\[
L = \kappa I - A. \tag{3.1}
\]

From Eq. (3.1) it can be seen that \( L \) has eigenvector \((1, 1, \ldots, 1)^t\) with eigenvalue of 0. Therefore, \( L \) is not invertible and so we define the pseudoinverse of \( L \) as

\[
L^{-1} = \sum_{i,h \neq 0} \lambda_i^{-1}E_i, \tag{3.2}
\]

where \( E_i \) is the operator of projection onto the eigenspace of \( L^{-1} \) corresponding to eigenvalue \( \lambda_i \neq 0 \). It has been shown that effective resistance between given nodes \( \alpha, \beta \) denoted by \( R_{\alpha\beta} \) is given by

\[
R_{\alpha\beta} = \langle \alpha | L^{-1} | \alpha \rangle + \langle \beta | L^{-1} | \beta \rangle - \langle \alpha | L^{-1} | \beta \rangle - \langle \beta | L^{-1} | \alpha \rangle. \tag{3.3}
\]

This formula may be formally derived using Kirchhoff's laws and seems to have been long known in the electrical engineering literature with it appearing in several texts such as Ref. 30.

A. Random walks and electrical networks

The computation of effective resistances is closely relevant to random walks (see Ref. 3). In Refs. 25 and 31–34, the same techniques introduced in Ref. 24 and this paper in order to evaluate the effective resistances, such as the algebraic structure of distance-regular networks (Bose–Mesner algebra), stratification, and spectral analysis methods, have been used for investigation of continuous time quantum walks on regular networks. In order to show this connection more clearly, in the following we discuss the link between effective resistance and two important quantities (average first passage time and average commute time) defined in random walks on graphs.

Let \( \Gamma \) be a complete undirected graph with \( v \) vertices numbered \( 1, 2, \ldots, v \), in which each edge \((\alpha, \beta)\) is assigned its distance \( d(\alpha, \beta) = d_{\alpha\beta} > 0 \). One can study the harmonic random walk in \( \Gamma \) with escape probability (the probability that a walk starting at \( \alpha \) reaches \( \beta \) before it returns to \( \alpha \))

\[
p_{esc}(\alpha, \beta) = \frac{1/d_{\alpha\beta}}{\sum_{\gamma \neq \alpha} 1/d_{\alpha\gamma}}. \tag{3.4}
\]

In the literature, the harmonic random walk was often studied using techniques from electrical network theory. Denote by \( H_{\alpha\beta} \) the hitting cost of the harmonic walk from \( \alpha \) to \( \beta \), defined as the expected cost (total distance) to reach \( \beta \) for the first time when started from \( \alpha \). By elementary probability, the costs of the random walks ending at vertex \( v \) satisfy the following system of equations:

\[
H_{\alpha\beta} = \sum_{\beta \neq \alpha} p_{\alpha\beta}(d_{\alpha\beta} + H_{\beta\alpha}) \quad \text{for} \quad \alpha \neq v, \tag{3.5}
\]

and \( H_{v,v} = 0 \). As noted in Ref. 3 the hitting costs have an interpretation in terms of electrical networks. We can think of \( \Gamma \) as an electrical network in which each edge \((\alpha, \beta)\) has resistance \( d_{\alpha\beta} \) [so, the Eq. (3.4) is a probabilistic interpretation of the effective conductance]. If we inject current of value \( v - 1 \) into each node and draw current of value \( v(v - 1) \) from node \( v \), then the voltages relative to node \( v \) established at the nodes satisfy the same equation as (3.5). Therefore the voltage at \( \alpha \) is equal to the hitting cost \( H_{\alpha\beta} \). Moreover, for a random walker on a network, one can assign
a quantity known as average commute time (CT) denoted by \( n(\alpha, \beta) \) which is defined as the average number of steps the random walker, starting in state \( \alpha \neq \beta \), will take before entering a given state \( \beta \) for the first time, and go back to \( \alpha \). Clearly, the average CT is symmetric and is equal to \( n(\alpha, \beta) = m(\beta | \alpha) + m(\alpha | \beta) \), where \( m(\beta | \alpha) \) is the average first passage time defined as the average number of steps the random walker, starting in state \( \alpha \), will take to enter state \( \beta \) for the first time. Note that for random walks on distance-regular graphs, we have \( m(\beta | \alpha) = m(\alpha | \beta) \) and so, \( n(\alpha, \beta) = 2m(\beta | \alpha) \). By viewing the graph as an electrical network, the average CT has an electrical equivalent,

\[
n(\alpha, \beta) = v \cdot \kappa R_{\alpha\beta}, \tag{3.6}
\]

where \( v \cdot \kappa \) is the volume of the network (the volume of a network is defined as \( \sum_{\alpha \in V} d_{\alpha} \), with \( d_{\alpha} \) as the degree of the node \( \alpha \)). Equation (3.6) indicates that average CT and effective resistance basically measure the same quantity. This quantity is sometimes called resistance distance [it has been shown that \( n(\alpha, \beta) \) is a distance measure]. Further connections between random walks and electrical networks were explored by Chandra et al. in Ref. 35.

### IV. RECURSIVE CALCULATION OF EFFECTIVE RESISTANCE ON DISTANCE-REGULAR NETWORKS BASED ON SPECTRAL ANALYSIS METHODS AND CHRISTOFFEL–DARBOUX IDENTITIY

In previous work,\(^{24}\) explicit formulas for effective resistances up to third stratum, i.e., \( R_{\alpha\beta}^{(m)} \) for \( m = 1, 2, 3 \), have been given in terms of the intersection array of the network, where the authors have employed the properties of the Stieltjes function associated with the network. Even though the method of Ref. 24 can be used to obtain analytical formulas for other resistances \( R_{\alpha\beta}^{(m)} \) for \( m > 3 \), but the needed calculations for \( m > 3 \) are so cumbersome. In this section, we use the spectral techniques and Christoffel–Darboux identity to give a recursive formula for calculating the effective resistances \( R_{\alpha\beta}^{(m)} \) for \( m = 1, 2, \ldots, d \). In addition to the preference that this formula enables us to calculate all of the effective resistances recursively, it also indicates that effective resistance on distance-regular resistor networks increases with the number of strata, i.e., \( R_{\alpha\beta}^{(m+1)} \) is strictly larger than \( R_{\alpha\beta}^{(m)} \) for \( m = 1, 2, \ldots, d-1 \).\(^{36}\)

First, it should be noticed that for distance-regular networks (in general, for underlying networks of association schemes) as resistor networks, the diagonal entries of the pseudoinverse matrix \( L^{-1} \) are independent of the node, i.e., \( L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta} \) for all \( \alpha, \beta \in V \). This is due to the fact that for these networks we have \( \langle \alpha | E_{\alpha} | \alpha \rangle = \langle \beta | E_{\beta} | \beta \rangle = m_{\alpha, \beta} \) for all \( \alpha, \beta \in V \) and so \( \langle \alpha | L^{-1} | \alpha \rangle = \langle \beta | L^{-1} | \beta \rangle \) for all \( \alpha, \beta \in V \) [see Eqs. (2.3) and (3.2)]. Therefore, from the relation (3.3) and the fact that \( L^{-1} \) is a real matrix, one can rewrite Eq. (3.3) for distance-regular networks as follows:

\[
R_{\alpha\beta} = 2(L^{-1}_{\alpha\alpha} - L^{-1}_{\alpha\beta}). \tag{4.1}
\]

As Eq. (4.1) implies, in order to evaluate effective resistance \( R_{\alpha\beta} \), we need to calculate the matrix entries \( L^{-1}_{\alpha\alpha} \) and \( L^{-1}_{\alpha\beta} \). To this end, one needs to know the spectrum of the pseudoinverse \( L^{-1} \) [see Eq. (3.2)] which is a task with high complexity for networks with large size, even with computer. In the following, we give an algebraic method such that effective resistances are calculated recursively without any knowledge about the spectrum of the pseudoinverse of Laplacian of the networks.

In order to evaluate the effective resistance between two arbitrary nodes \( \alpha, \beta \in V \), we choose one of these nodes, say \( \alpha \), as reference node and stratify the network with respect to the reference node \( \alpha \). Now, one should notice that, in distance-regular networks due to the stratification of the networks, all of the nodes belonging to the same stratum with respect to reference node, here \( \alpha \), possess the same effective resistance with respect to \( \alpha \). More clearly, for all \( \beta \in \Gamma_{m}(\alpha) \) we have
\[ L_{\alpha}^{-1} = \langle \alpha | L^{-1} | \beta \rangle = \frac{1}{\sqrt{\kappa_m}} \langle \alpha | L^{-1} | \phi_m \rangle = \frac{1}{\kappa_m} \langle \alpha | A_m L^{-1} | \alpha \rangle. \]  

Then, by using (4.1), we obtain

\[ R_{\alpha \beta} = \frac{2}{\kappa_m} \langle \kappa_m L_{\alpha}^{-1} - (A_m L_{\alpha}^{-1})_{\alpha \alpha} \rangle = \frac{2}{\kappa_m} \left\langle \alpha \left| \frac{\kappa_m - P_m(A)}{k_1 - A} \right| \alpha \right\rangle, \quad \forall \beta \in \Gamma_m(\alpha). \]

where the upper index \( m \) in \( L_{\alpha}^{-1} \) and \( R_{\alpha \beta} \) indicates that \( \beta \) belongs to the \( m \)th stratum with respect to \( \alpha \). Therefore, for each \( \alpha, \beta \in \mathcal{V} \), we need to determine that \( \beta \) belongs to which stratum with respect to \( \alpha \). Assuming that \( \beta \) belongs to the \( m \)th stratum with respect to \( \alpha \), i.e., \( \beta \in \Gamma_m(\alpha) \), we need to evaluate \( L_{\alpha}^{-1} \) and \( L_{\alpha \beta}^{-1} \). To this aim, we use the spectral techniques to write

\[ L_{\alpha \alpha}^{-1} = \left\langle \alpha \left| \frac{1}{k_1 - A} \right| \alpha \right\rangle = \sum_{j=1}^{d} \phi_j \left\langle \alpha \left| E_{\alpha} \right| \phi_j \right\rangle = \sum_{j=1}^{d} \frac{m_j}{\kappa - \lambda_j} \]

and

\[ L_{\alpha \beta}^{-1} = \left\langle \alpha \left| \frac{1}{k_1 - A} \right| \beta \right\rangle = \sum_{j=1}^{d} \phi_j \left\langle \alpha \left| E_{\alpha} \right| \phi_j \right\rangle = \sum_{j=1}^{d} \frac{m_j P_m(\lambda_j)}{\kappa - \lambda_j} \]

where we have used Eq. (2.23) to substitute \( \phi_j = P_m'(A) | \alpha \rangle \) and considered \( \lambda_0 = \kappa \) (\( \kappa \) is the eigenvalue corresponding to the idempotent \( E_0 \)). Then, by using Eq. (4.1) and the fact that \( L_{\alpha \alpha}^{-1} \) is independent of \( m \) (the number of stratum), we have

\[ R_{\alpha \beta} = 2(L_{\alpha \beta}^{-1} - L_{\alpha \beta}^{-1}) = \sum_{j=1}^{d} \frac{m_j}{\kappa - \lambda_j} \frac{P_m(\lambda_j) - P_{m+1}(\lambda_j)}{\kappa - \lambda_j} \]

\[ = \sum_{j=1}^{d} \frac{m_j P_m(\lambda_j) - P_{m+1}(\lambda_j)}{\kappa - \lambda_j} \]

\[ = \sum_{j=1}^{d} \frac{Q_{m+1}(\kappa)Q_m(\lambda_j) - Q_m(\kappa)Q_{m+1}(\lambda_j)}{\kappa - \lambda_j}. \]  

Now, from the fact that \( \{Q_l(\kappa)\} \) satisfy the three-term recursion relations (2.24), we can use the Christoffel–Darboux identity to write the right hand side of (4.6) as follows:

\[ r . h . s = \frac{2}{\sqrt{\kappa_m} \kappa_{m+1} \omega_{m+1} \cdot \omega_1 \cdots \omega_m} \sum_{j=1}^{d} m_j \left( \omega_1 \cdots \omega_m \sum_{l=0}^{m} \frac{Q_l(\kappa)Q_l(\lambda_j)}{\omega_1 \cdots \omega_l} \right) \]

\[ = \frac{2}{\sqrt{\kappa_m} \kappa_{m+1} \omega_{m+1} \cdot \omega_1 \cdots \omega_m} \sum_{j=0}^{m} \omega_j \sum_{j=1}^{d} m_j Q_l(\lambda_j) \]

\[ = \frac{2}{\sqrt{\kappa_m} \kappa_{m+1} \omega_{m+1} \cdot \omega_1 \cdots \omega_m} \sum_{j=1}^{d} m_j P_l(\lambda) \]

\[ = \frac{2}{\sqrt{\kappa_m} \kappa_{m+1} \omega_{m+1} \cdot \omega_1 \cdots \omega_m} \sum_{j=1}^{d} m_j P_l(\lambda) \]

\[ = \frac{2}{\sqrt{\kappa_m} \kappa_{m+1} \omega_{m+1} \cdot \omega_1 \cdots \omega_m} \sum_{j=1}^{d} (v \bar{d}_j - \kappa_j). \]
\[
Q_l(\lambda_j) = \frac{\beta_1 \beta_2 \cdots \beta_l}{\sqrt{|\mathcal{K}_l|}} P_l(\lambda_j),
\]
and in the forth equality we have used the following equality:
\[
Q_l(\kappa) = \frac{\beta_1 \beta_2 \cdots \beta_l}{\sqrt{|\mathcal{K}_l|}} P_l(\kappa) = \frac{\beta_1 \beta_2 \cdots \beta_l}{\sqrt{|\mathcal{K}_l|}} \kappa,
\]
to write \(Q_l(\beta_1 \beta_2 \cdots \beta_l/\omega_1 \cdots \omega_l \sqrt{|\mathcal{K}_l|} = 1\). Finally, in the last equality of (4.7), we have used the distance-regularity condition to substitute \(P_l(\lambda_j) = P_j\) and then use Eq. (2.6) [recall that \(m_0 = 1\) and \(P_0(\lambda_0) = P(\kappa = \kappa)\)]. After these simplifications, we obtain the main result of the paper as follows:
\[
R_{\alpha\beta^{(m+1)}} - R_{\alpha\beta^{(m)}} = \frac{2}{v} \kappa_m b_m \left( v - \sum_{l=0}^{m} \kappa_l \right) > 0, \quad m = 1, 2, \ldots, d - 1, \tag{4.8}
\]
where we have done the following simplification by using (2.13):
\[
\kappa_m \kappa_{m+1} \omega_{m+1} = \kappa_m \kappa_{m+1} b_m c_{m+1} = (b_m \kappa_m)^2.
\]
It should be noticed that by using the formula (4.8) we can evaluate the effective resistance between any two nodes recursively, if we know the effective resistance \(R_{\alpha\beta^{(1)}}\). In order to calculate \(R_{\alpha\beta^{(1)}}\), one can use the spectral techniques introduced in Sec. II C to write
\[
L^{-1}_{\alpha\beta^{(1)}} = \langle \alpha | \frac{1}{\kappa l - A} | \alpha \rangle = \int_{R^{(\alpha)}} \frac{d\mu(x)}{\kappa - x} = \frac{1}{v} \sum_{i=0}^{d} \frac{m_i}{\kappa - x_i}, \tag{4.9}
\]
and
\[
L^{-1}_{\alpha\beta^{(1)}} = \frac{1}{\kappa} \langle \alpha | A | \alpha \rangle = \int_{R^{(\alpha)}} \frac{d\mu(x)}{\kappa - x} = \frac{1}{v} \sum_{i=0}^{d} \frac{m_i x_i}{\kappa - x_i}. \tag{4.10}
\]
Then, by using (4.1), one can obtain
\[
R_{\alpha\beta^{(1)}} = \frac{2}{v} \sum_{i, j \neq 0} m_i (\kappa - x_j) = \frac{2}{v} \sum_{i, j \neq 0} m_i = \frac{2}{v} (v - 1). \tag{4.11}
\]
Note that, as the main result of the paper, the result (4.8) shows that effective resistance (and consequently other quantities related to effective resistance such as average first passage time and Euclidean commute time associated with a random walk) on distance-regular networks is strictly increasing function of the shortest path distance defined in Sec. II C, i.e., the nodes belonging to the farthest stratum with respect to \(\alpha\) possess the largest effective resistance with \(\alpha\). Apart from this fact, the formula (4.8) together with (4.11) gives an algebraic method for calculation of effective resistances on distance-regular networks, where one needs only to know the intersection array of the networks without any knowledge about the spectrum of the pseudoinverse of Laplacian of the networks.

### A. Example: Distance-regular networks derived from symmetric group \(S_n\)

Let \(\lambda = (\lambda_1, \ldots, \lambda_k)\) be a partition of \(n\), i.e., \(\lambda_1 + \cdots + \lambda_k = n\). We consider the subgroup \(S_k \otimes S_{n-k}\) of \(S_n\) with \(k = \lfloor n/2 \rfloor\). Then we assume the finite set \(M^n = S_n/S_k \otimes S_{n-k}\) with \(|M^n| = n! / k!(n - k)!\) as vertex set. In fact, \(M^n\) is the set of \((k - 1)\)-faces of \((n - 1)\)-simplex [recall that, the graph of an \((n-1)\)-simplex is the complete graph with \(n\) vertices denoted by \(K_n\)]. If we denote the vertex \(i\) by \(k\)-tuple \((i_1, i_2, \ldots, i_k)\), then the adjacency matrices \(A_l\), \(l = 0, 1, \ldots, k\), are defined as
\[(A_l)_{i,j} = \begin{cases} 
1 & \text{if } \delta(i,j) = l \\
0, & \text{otherwise},
\end{cases} \quad i,j \in M^k, \quad l = 0,1, \ldots, k, \quad (\text{4.12})\]

where we mean by \(\delta(i,j)\) the number of components that \(i=(i_1, \ldots, i_k)\) and \(j=(j_1, \ldots, j_k)\) are different (this is the same as Hamming distance which is defined in coding theory). The network with adjacency matrices defined by (4.12) has \(k+1\) strata such that

\[\kappa_0 = 1, \quad \kappa_l = \binom{k}{k-l} \binom{n-k}{l}, \quad l = 1,2, \ldots, m \quad (\text{4.13})\]

(clearly, we have \(v = \sum_{l=0}^{k} \kappa_l = |M_k|\)). Let the network has been stratified with respect to reference node \(|\phi_0\rangle = |i_1, i_2, \ldots, i_k\rangle\). Then, the unit vectors \(|\phi_l\rangle, i=1, \ldots, k\) are given by

\[|\phi_1\rangle = \frac{1}{\sqrt{\kappa_1}} \left( \sum_{i'_1+i_1} |i'_1,i_2, \ldots, i_k\rangle + \sum_{i'_2+i_2} |i'_1,i'_2,i_3, \ldots, i_k\rangle + \ldots + \sum_{i'_k+i_k} |i_1, \ldots, i_{k-1}, i'_k\rangle \right),\]

\[|\phi_2\rangle = \frac{1}{\sqrt{\kappa_2}} \sum_{i'_1+i_1} \sum_{i'_2+i_2} |i'_1,i'_2,i_3, \ldots, i_k\rangle, \]

\[\vdots\]

\[|\phi_k\rangle = \frac{1}{\sqrt{\kappa_k}} \sum_{i'_1+i_1} \ldots |i'_1,i'_2, \ldots, i'_k\rangle. \quad (\text{4.14})\]

The constructed network as in the above is a distance-regular network with intersection array as follows:

\[b_l = (k-l)(n-k-l), \quad l = 0,1, \ldots, k-1; \quad c_l = l^2, \quad l = 1,2, \ldots, k. \quad (\text{4.15})\]

Then, by using Eq. (2.17), one can obtain the coefficients \(\alpha_l\) and \(\omega_l\) as follows:

\[\alpha_l = l(n-2l), \quad l = 0,1, \ldots, k; \quad \omega_l = l^2(k-l+1)(n-k-l+1), \quad l = 1,2, \ldots, k. \quad (\text{4.16})\]

By using the recursion relations (2.16), one can obtain

\[A|\phi_l\rangle = (l+1) \sqrt{(k-l)(n-k-l)} |\phi_{l+1}\rangle + l(n-2l) |\phi_l\rangle + l \sqrt{(k-l+1)(n-k-l+1)} |\phi_{l-1}\rangle. \quad (\text{4.17})\]

Now, by substituting \(v=n!/(k!(n-k)!\) and using the Eqs. (4.8), (4.11), (4.13), and (4.15), one can evaluate the effective resistances recursively. For example, we obtain

\[R_{\alpha (1)} = \frac{2}{n! k(n-k)} \left\{ n! - k! (n-k)! \right\},\]

\[R_{\alpha (2)} = R_{\alpha (1)} + \frac{2}{k(k-1)(n-k)(n-k-1)} \left( 1 - \frac{k! (n-k)!}{n!} \left[ 1 + k(n-k) \right] \right) = \frac{2}{n! k(k-1)(n-k)(n-k-1)} \left\{ n! [k(n-k) - n + 2] - k! (n-k)! [2k(n-k) - n + 2] \right\}, \quad (\text{4.18})\]

the effective resistances \(R_{\alpha (m)}\) for \(m=3, 4, 5\) have been given in Appendix.
In the following, we consider the case \( k = 3 \) in details and calculate the effective resistances. In the case of \( k = 3 \), we have four strata with \( \kappa_0 = 1, \kappa_1 = \kappa = 3(n - 3), \kappa_2 = 3(n - 3)(n - 4)/2, \) and \( \kappa_3 = (n - 3)(n - 4)(n - 5)/6 \). For a given vertex \( |ijk\) with \( i < j < k \) as reference vertex, the stratification basis \( \{|\phi_i\}_{i=0,1,2,3} \) defined by (2.9) are given by

\[
|\phi_0\rangle = |ijk\rangle,
\]

\[
|\phi_1\rangle = \frac{1}{\sqrt{3(n-3)}} \left( \sum_{i' \neq j=1}^n |i'jk\rangle + \sum_{j' \neq k=1}^n |ij'k\rangle + \sum_{k' \neq k=1}^n |ijk'\rangle \right),
\]

\[
|\phi_2\rangle = \frac{1}{\sqrt{3(n-3)(n-4)/2}} \left( \sum_{i' \neq i; j' \neq j} |i'j'k\rangle + \sum_{j' \neq j; k' \neq k} |ij'k'\rangle + \sum_{i' \neq i; k' \neq k} |i'jk'\rangle \right),
\]

\[
|\phi_3\rangle = \frac{1}{\sqrt{(n-3)(n-4)(n-5)/6}} \sum_{i' \neq i; j' \neq j; k' \neq k} |i'j'k'\rangle. \quad (4.19)
\]

By using (4.1), the effective resistances between any node \( \alpha \in V \) and \( \beta \in \Gamma_m(\alpha) \) for \( m = 1, 2, 3 \) are given by

\[
R_{\alpha,\beta(1)} = \frac{2}{3(n-3)} \left[ 1 - \frac{6}{n(n-1)(n-2)} \right] = \frac{2(n(n-1)(n-2) - 6)}{3n(n-1)(n-2)(n-3)},
\]

\[
R_{\alpha,\beta(2)} = \frac{1}{3n(n-1)(n-2)(n-3)(n-4)} \left[ n(n-1)(n-2)(2n-7) - 6(5n-6) \right],
\]

\[
R_{\alpha,\beta(3)} = \frac{1}{3n(n-1)(n-2)(n-3)(n-4)(n-5)} \left[ n(n-1)(n-2)(2n^2-17n+39) - 6(11n^2-71n + 120) \right]. \quad (4.20)
\]

More examples of distance-regular networks have been given in Appendix.

V. CONCLUSION

Based on Bose–Mesner algebras corresponding to distance-regular networks, stratification, spectral techniques, and Christoffel–Darboux identity, a recursive formula for calculating effective resistances on distance-regular resistor networks was obtained such that one can evaluate the effective resistances on these networks only by knowing the corresponding intersection array, without any need to know the spectrum of the pseudoinverse of Laplacian of the networks. As an important result, it was shown that effective resistance (as a distance metric) on a distance-regular network is an increasing function of the shortest path distance defined on the network. Although we focused specifically on distance-regular networks, we hope that the introduced method might then be applied to other underlying networks of association schemes which are not distance-regular ones such as finite and infinite square lattices and underlying networks of the root lattices of type \( A_n \) particularly finite and infinite hexagonal networks \( (n=2) \) introduced in Ref. 33 by employing the Krylov-subspace Lanczos algorithm\(^{11}\) iteratively to give three-term recursion relations to the networks, where these problems are under investigation.
APPENDIX: CALCULATION OF THE EFFECTIVE RESISTANCES OVER SOME IMPORTANT FINITE DISTANCE-REGULAR NETWORKS

In this appendix, we calculate the effective resistances $R_{\alpha \beta}^{(i)}$ for $i=1,2,\ldots,d$ for some important finite distance-regular networks and give the effective resistances $R_{\alpha \beta}^{(m)}$ for $m=3,4,5$ on distance-regular networks derived from symmetric group $S_n$ discussed in Sec. IV A.

1. Collinearity network, gen. octagon $(s,1)$, $s=2,3,4$

$v = s^4 + 2s^3 + 2s^2 + 2s + 1, \quad \{b_0,b_1,b_2,b_3;c_1,c_2,c_3,c_4\} = \{2s,s,s,s;1,1,1,2\},$

$$\kappa = b_0 = 2s, \quad \kappa_2 = \frac{\kappa b_1}{c_2} = 2s^2, \quad \kappa_3 = \frac{\kappa_2 b_2}{c_3} = 2s^3, \quad \kappa_4 = \frac{\kappa_3 b_3}{c_4} = s^4.$$ Then,

$$R_{\alpha \beta}^{(1)} = \frac{s^3 + 2s^2 + 2s + 2}{s^4 + 2s^3 + 2s^2 + 2s + 1},$$

$$R_{\alpha \beta}^{(2)} = R_{\alpha \beta}^{(1)} + \frac{s^2 + 2s + 2}{s^4 + 2s^3 + 2s^2 + 2s + 1} = \frac{s^3 + 3s^2 + 4s + 4}{s^4 + 2s^3 + 2s^2 + 2s + 1},$$

$$R_{\alpha \beta}^{(3)} = R_{\alpha \beta}^{(2)} + \frac{s + 2}{s^4 + 2s^3 + 2s^2 + 2s + 1} = \frac{s^3 + 3s^2 + 5s + 6}{s^4 + 2s^3 + 2s^2 + 2s + 1},$$

$$R_{\alpha \beta}^{(4)} = R_{\alpha \beta}^{(3)} + \frac{1}{s^4 + 2s^3 + 2s^2 + 2s + 1} = \frac{s^3 + 3s^2 + 5s + 7}{s^4 + 2s^3 + 2s^2 + 2s + 1}.$$

2. Incidence network, $pg(l-1;l-1;l-1)$, $l=4;5;7;8$

$v = 2l^2, \quad \{b_0,b_1,b_2,b_3;c_1,c_2,c_3,c_4\} = \{l,l-1,l-1,l-1,l-1;l\},$

$$\kappa = l, \quad \kappa_2 = l(l-1), \quad \kappa_3 = l(l-1), \quad \kappa_4 = l-1.$$ Then,

$$R_{\alpha \beta}^{(1)} = \frac{2l^2 - 1}{l^3}, \quad R_{\alpha \beta}^{(2)} = R_{\alpha \beta}^{(1)} + \frac{2l^2 - 1 - l}{l^3(l-1)} = \frac{2l^2 - 1 - l}{l^3(l-1)},$$

$$R_{\alpha \beta}^{(3)} = R_{\alpha \beta}^{(2)} + \frac{l + 1}{l^3(l-1)} = \frac{2l^3 + 1}{l^3(l-1)}, \quad R_{\alpha \beta}^{(4)} = R_{\alpha \beta}^{(3)} + \frac{1}{l^3} = \frac{2l^2 + 1}{l^3(l-1)}.$$

3. Hadamard network

$v = 16\gamma, \quad \{b_0,b_1,b_2,b_3;c_1,c_2,c_3,c_4\} = \{4\gamma,4\gamma-1,2\gamma,1;1,2\gamma,4\gamma-1,4\gamma\}, \quad \text{where} \quad \gamma \in \mathbb{N},$

$$\kappa = 4\gamma, \quad \kappa_2 = 2(4\gamma - 1), \quad \kappa_3 = 4\gamma, \quad \kappa_4 = 1.$$ Then,

$$R_{\alpha \beta}^{(1)} = \frac{16\gamma - 1}{32\gamma^2}, \quad R_{\alpha \beta}^{(2)} = R_{\alpha \beta}^{(1)} + \frac{12\gamma - 1}{32\gamma^2(4\gamma - 1)} = \frac{8\gamma - 1}{4\gamma(4\gamma - 1)},$$
4. Distance-regular networks with classical parameters

Let $\Gamma$ denote a distance-regular network with diameter $d \geq 3$. We say $\Gamma$ has classical parameters $(d, q, \alpha, \beta)$ whenever the intersection numbers are given by

$$c_i = \left[ i \atop 1 \right] \left( 1 + \alpha \left[ i - 1 \atop 1 \right] \right), \quad i = 1, \ldots, d,$$

$$b_i = \left( \left[ d \atop 1 \right] - \left[ i \atop 1 \right] \right) \left( \beta - \alpha \left[ i \atop 1 \right] \right), \quad i = 0, 1, \ldots, d - 1,$$

where

$$\left[ i \atop 1 \right] := 1 + q + q^2 + \cdots + q^{i-1}.$$

For instance, the $d$-cube is a distance-regular network with the classical parameters $d$, $q=1$, $\alpha =0$, and $\beta=1$. As an another example of these types of networks, one can consider the network $B_d(p^n)$ which is a type of the so-called dual polar networks and is characterized by $\alpha=0$, $\beta =p^n$, and

$$b_i = \frac{q^{i+1}(p^{d-i}-1)}{q-1}, \quad i = 0, 1, \ldots, d - 1,$$

$$c_i = \frac{q^i - 1}{q-1}, \quad i = 1, 2, \ldots, d,$$

where $p$ is a prime number and $n \in \mathbb{N}$. For example, consider $d=4$ and $q=2$, then we have the network $B_4(2)$ with the number of vertices, intersection array, and valencies,

$$v = 2295, \quad \{b_0, b_1, b_2, b_3; c_1, c_2, c_3, c_4\} = \{30, 28, 24, 16; 1, 3, 7, 15\},$$

$$\kappa = 30, \quad \kappa_2 = 280, \quad \kappa_3 = 960, \quad \kappa_4 = 1024.$$
7. A distance-regular graph with

\[ u = 2 \left[ 1 + l + \frac{l(l-1)}{c} \right], \quad \{b_0, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, c_5\} = \{l, l-1, l-c, c, 1; c, l-c, l, 1, l-1\}, \]

where \( l = \gamma(\gamma^2 + 3\gamma + 1) \) and \( c = \gamma(\gamma + 1) \), \( \gamma \in \mathbb{N} \). Then,

\[ \kappa = l, \quad \kappa_2 = \frac{l(l-1)}{c}, \quad \kappa_3 = \frac{l(l-1)}{c}, \quad \kappa_4 = l, \quad \kappa_5 = 1. \]

Then,

\[ R_{\alpha\beta} = \frac{c + 2lc + 2l(l-1)}{l(c + lc + l(l-1))}. \]
Then, 
\[ R_{\alpha\beta^2} = R_{\alpha\beta^1} + \frac{c + lc + 2l(l - 1)}{l(l - 1)(c + lc + l(l - 1))} = \frac{2(c + l)}{c + lc + l(l - 1)}, \]
\[ R_{\alpha\beta^3} = R_{\alpha\beta^2} + \frac{c}{l(l - 1)(l - c)} = \frac{2l(l - 1)(l^2 - c^2) + c^2(l + 1) + cl(l - 1)}{l(l - 1)(l - c)(c + cl + l(l - 1))}, \]
\[ R_{\alpha\beta^4} = R_{\alpha\beta^3} + \frac{c(l + 1)}{l(l - 1)(c + lc + l(l - 1))} = \frac{2l^2(2l^2 - c^2 - l) + cl(l + c)}{l(l - 1)(l - c)(c + lc + l(l - 1))}, \]
\[ R_{\alpha\beta^5} = R_{\alpha\beta^4} + \frac{c}{l(c + lc + l(l - 1))} = \frac{2l^2(2l^2 - c^2 - l) + cl(3l + c - 1) + c^2}{l(l - 1)(l - c)(c + lc + l(l - 1))}. \]

8. Biggs-Smith graph

\[ v = 102, \quad \{b_0, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6, c_7\} = \{3, 2, 2, 2, 1, 1; 1, 1, 1, 1, 1, 3\}, \]
\[ \kappa = 3, \quad \kappa_2 = 6, \quad \kappa_3 = 12, \quad \kappa_4 = 24, \quad \kappa_5 = 24, \quad \kappa_6 = 24, \quad \kappa_7 = 8. \]

Then,
\[ R_{\alpha\beta^1} = \frac{101}{153}, \quad R_{\alpha\beta^2} = R_{\alpha\beta^1} + \frac{49}{153} = \frac{150}{153}, \quad R_{\alpha\beta^3} = R_{\alpha\beta^2} + \frac{23}{153} = \frac{173}{153}, \]
\[ R_{\alpha\beta^4} = R_{\alpha\beta^3} + \frac{10}{153} = \frac{183}{153}, \quad R_{\alpha\beta^5} = R_{\alpha\beta^4} + \frac{7}{153} = \frac{190}{153}, \quad R_{\alpha\beta^6} = R_{\alpha\beta^5} + \frac{4}{153} = \frac{194}{153}, \]
\[ R_{\alpha\beta^7} = R_{\alpha\beta^6} + \frac{1}{153} = \frac{195}{153}. \]

9. Foster graph

\[ v = 90, \quad \{b_0, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\} = \{3, 2, 2, 2, 1, 1; 1, 1, 1, 1, 2, 2, 2, 3\}, \]
\[ \kappa = 3, \quad \kappa_2 = 6, \quad \kappa_3 = 12, \quad \kappa_4 = 24, \quad \kappa_5 = 24, \quad \kappa_6 = 12, \quad \kappa_7 = 6, \quad \kappa_8 = 2. \]

Then,
\[ R_{\alpha\beta^1} = \frac{89}{135}, \quad R_{\alpha\beta^2} = R_{\alpha\beta^1} + \frac{43}{135} = \frac{132}{135}, \quad R_{\alpha\beta^3} = R_{\alpha\beta^2} + \frac{4}{27} = \frac{152}{135}, \]
\[ R_{\alpha\beta^4} = R_{\alpha\beta^3} + \frac{17}{270} = \frac{321}{270}, \quad R_{\alpha\beta^5} = R_{\alpha\beta^4} + \frac{11}{540} = \frac{653}{540}, \quad R_{\alpha\beta^6} = R_{\alpha\beta^5} + \frac{5}{270} = \frac{663}{540}, \]
\[ R_{\alpha\beta^7} = R_{\alpha\beta^6} + \frac{2}{135} = \frac{671}{540}, \quad R_{\alpha\beta^8} = R_{\alpha\beta^7} + \frac{1}{135} = \frac{675}{540}. \]
10. Explicit formulas for $R_{a\beta(m)}$, $m=3,4,5$ on distance-regular networks derived from symmetric group $S_n$

$$R_{a\beta(3)} = R_{a\beta(2)} + \frac{8}{k(k-1)(k-2)(n-k)(n-k-1)(n-k-2)} \left[ 1 - k! (n-k)! \left( \frac{1}{n!} + k(n-k) \right) \right]$$

$$+ \frac{k(k-1)(n-k)(n-k-1)}{k(k-1)(n-k)(n-k-1)(n-k-2)} \right] = \frac{2}{n! k(k-1)(k-2)(n-k)(n-k-1)(n-k-2)(n-k-3)} \times \{ n! [(k-3)(n-k-3)$$

$$\times[(k-2)(n-k-2)[k(n-k)+n+2]+4]\} - k! (n-k)! [(k-3)(n-k-3)[k-2](n-k-2)[k(n-k)+n+2]+4] + k(k-3)(n-k-4)(k-2)(n-k-2)+36] + k! (n-k)! [(k-2)(n-k-2)[k(n-k)+n+2]+4] + k(k-3)(n-k-4)(k-2)(n-k-2)+36] + k! (n-k)! [(k-2)(n-k-2)+9]+36] + 36)}.$$
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