NEW PROJECTION AND KORN ESTIMATES FOR A CLASS OF CONSTANT-RANK OPERATORS ON DOMAINS

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ABSTRACT. We prove that if \( \mathcal{A} : D(\mathcal{A}) \subset L^p(\Omega; V) \rightarrow L^p(\Omega; W) \) is a \( k \)th order constant-rank differential operator with maximal rank, then there exists a linear solution operator \( \mathcal{A}^{-1} \) satisfying Sobolev regularity estimates. This allows us to construct a linear projection \( T : D(T) \subset L^p(\Omega; V) \rightarrow L^p(\Omega; V) \) satisfying the estimate

\[
\|u - Tu\|_{W^{k,p}(\Omega)} \leq C\|Au\|_{L^p(\Omega)}, \quad A(Tu) = 0,
\]

for all sufficiently regular maps. The estimate generalizes Fuchs’s distance estimate for the del-var operator, to operators such as the laplacian and the divergence. On regular domains, the same estimates are (trivially) observed for operators with finite dimensional null-space. When \( \mathcal{A} \) is only assumed to be of constant rank, we are able to show that the Sobolev distance to the space of \( \mathcal{A} \)-harmonic maps is bounded by \( Au \), that is,

\[
\min_{v \in L^p(\Omega), \Delta \mathcal{A} v = 0} \|u - v\|_{W^{k,p}(\Omega)} \leq C\|Au\|_{L^p(\Omega)}.
\]

Similar projection and distance estimates involving lower-order derivatives are further discussed, as well as a plethora of examples and applications with well-known operators.

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1. Introduction and main results

In all the following, our analysis will be restricted to estimates for exponents in the range \( 1 < p < \infty \). We consider a constant coefficient \( k \)th order homogeneous linear partial differential operator \( \mathcal{A} \), acting on sufficiently regular functions \( u : \mathbb{R}^N \rightarrow V \) as

\[
\mathcal{A}u = \sum_{|\alpha|=k} A_\alpha D^\alpha u, \quad A_\alpha \in \text{Hom}(V; W),
\]
where $V, W$ are finite dimensional $\mathbb{R}$-spaces, $(\alpha_1, \ldots, \alpha_N)$ is a multi-index of non-negative integers with modulus $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $D^\alpha$ is the composition of the distributional partial derivates $\partial^\alpha_1 \circ \cdots \circ \partial^\alpha_N$. Up to a linear isomorphism, the reader may think of $V$ and $W$ as $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, in which case $A$ is precisely a system of $m \times n$ partial differential equations.

The Fourier transform establishes a one-to-one correspondence between an operator $A$ as above and its associated principal symbol map $\mathbb{A} : \mathbb{R}^N \to \text{Hom}(V; W)$, which is the $k$-homogeneous tensor-valued polynomial defined as

$$\mathbb{A}(\xi) := \sum_{|\alpha|=k} A_\alpha \xi^\alpha, \quad \xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_N^{\alpha_N} \quad (\xi \in \mathbb{R}^N).$$

Following [14], we shall consider the space

$$W^{A,p}(\Omega) := \{ u \in L^p(\Omega; V) : Au \in L^p(\Omega; W) \},$$

which is an anisotropic Sobolev space in the sense that the $p$-integrability of $D^k u$ is replaced by the $p$-integrability of $A u$.

The association between operators and their associated symbol polynomials is often deeper in the sense that certain functional properties of $A$ are transferred to certain algebraic properties of $\mathbb{A}$, and vice-versa. In this vein, a challenging problem is to find a sufficient (and necessary) algebraic condition on $\xi \mapsto \mathbb{A}(\xi)$ that guarantees the validity of the distance (Korn) estimate

$$\min_{v \in N_p(A,\Omega)} \|u - v\|_{W^{k,p}(\Omega)} \leq C \|Au\|_{L^p(\Omega)} \quad \text{for all } u \in W^{A,p}(\Omega),$$

where

$$N_p(A; \Omega) := \{ u \in L^p(\Omega; V) : Au = 0 \}$$

is the space of $A$-free $p$-integrable functions over $\Omega$. A related and stronger property is the existence of a linear (unbounded) projection

$$T : W^{A,p}(\Omega) \subset L^p(\Omega; V) \to L^p(\Omega; V),$$

with range $R(T) = N_p(A,\Omega)$, such that

$$\|u - Tu\|_{W^{k,p}(\Omega)} \leq C \|Au\|_{L^p(\Omega)} \quad \text{for all } u \in W^{A,p}(\Omega).$$

The objective of this paper is to study the validity of these and similar inequalities for a class of operators $A$ as above.

Motivation. The history of such estimates, and hence part of the motivation for this work, stems from the study of coercive $L^p$-estimates for non-elliptic operators in full-space. For $u \in C^\infty_c(\mathbb{R}^N; V)$, this is an estimate of the form

$$\inf_{v \in N_p(A,\Omega)} \|D^k(u - v)\|_{L^p(\mathbb{R}^N)} \lesssim \|u\|_{L^p(\mathbb{R}^N)} + \|Au\|_{L^p(\mathbb{R}^N)}.$$
In this vein, Schulenberger and Wilcox \cite{17, 18} introduced the \textit{constant-rank} condition
\begin{equation}
\forall \xi \in \mathbb{R}^N - \{0\}, \quad \text{rank } A(\xi) = \text{const.},
\end{equation}
which they proved to be a sufficient condition for the validity of the full-space coercive estimate (4) when $p = 2$ (in which case (3) is equivalent to (4)). Inspired by the estimates of Schulenberger and Wilcox, Kato \cite{13} and Murat \cite{14} proved that the constant-rank property further implies that the canonical linear $L^2$-projection $T_2 : L^2(\Omega; V) \to N_2(A, \Omega)$ extends to an $L^p$-bounded projection in the sense that
\[ \|T_2u\|_{L^p(\mathbb{R}^N)} \leq C\|u\|_{L^p(\mathbb{R}^N)} \quad \text{for all } u \in C^\infty_c(\mathbb{R}^N; V). \]
Later on, the ideas contained in the seminal compensated compactness paper \cite{8} of Fonseca and Müller (see also \cite{4, 15}), facilitated a proof of the \textit{projection estimate}
\begin{equation}
\|D^k(u - T_2u)\|_{L^p(\mathbb{R}^N)} \leq C\|Au\|_{W^{k,p}(\mathbb{R}^N)} \quad \text{for all } u \in C^\infty_c(\mathbb{R}^N; V). \tag{6}
\end{equation}
At this point, the experienced reader may have already devised that (6) is nothing else than a generalization of the Calderon–Zygmund estimates for elliptic operators—with the caveat that one must remove the (possibly non-trivial) null-space $N_p(A; \mathbb{R}^N)$.

Regarding the necessity of the constant-rank property, Guerra and Raita \cite{10} have recently observed that (5) is also a necessary condition for the validity projection estimate (6).\footnote{Currently, it is not known if the constant-rank condition is necessary for the validity of the distance estimate (2), when $p$ is different than 2.}

Analogous projection estimates to (3) hold if instead of considering functions defined over $\mathbb{R}^N$, one considers functions over the $N$-dimensional torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$; the crucial element of the proof being the availability of Fourier transform arguments. Proving such estimates on a domain $\Omega \subset \mathbb{R}^N$, with (possibly irregular) boundary $\partial \Omega$, turns out to be considerably more challenging than working on $\mathbb{R}^N$ or $\mathbb{T}^N$. This stems from the lack of Fourier transform, as well as the fact that, even for domains with smooth boundary, there are no suitable trace or extension operators for general constant-rank operators.

**Main results.** The goal of this paper is to study the validity of projection and distance (Korn) estimates, on a domain $\Omega \subset \mathbb{R}^N$, for maximal-rank operators (a subclass of the class of constant-rank operators). We also study the validity of other \textit{weaker} estimates for general constant-rank operators.

**Notation 1.** In all that follows $\Omega$ will denote a domain (open and bounded) subset of $\mathbb{R}^N$. Notice that we do not require $\Omega$ to be connected, unless otherwise is specified (cf. Lem. 5 and Cor. 6).
In order to state our results, it will be convenient to define the pointwise essential range of $A$,

$$W_A := \text{clos}_W \{ Au(x) : x \in \mathbb{R}^N, u \in C_c^\infty(\mathbb{R}^N; V) \},$$

which as its name suggests, is the smallest subspace of $W$ where $A$ can be defined. Since it might be easier to work with, let us remark that $W_A$ coincides with the space

$$\text{span} \{ \text{Im} A(\xi)[v] : \xi \in \mathbb{R}^N, v \in V \},$$

generated by all $A$-gradients in Fourier space. In fact, every operator $A$ as above can be defined as an operator $V$ to $W_A$, so that there is no loss of regularity in assuming that $W = W_A$.

### 1.1. Estimates for maximal-rank operators.

Our first result establishes that if the principal symbol of $A$ has maximal rank, in the sense that $A(\xi)[V] = W_A$ for all non-zero frequencies $\xi \in \mathbb{R}^N$, then a generalization of the Fonseca–Müller projection estimates hold on arbitrary sub-domains of $\mathbb{R}^N$. The precise statement is the following:

**Theorem 1.** Let $A$ be an operator of order $k$ on $\mathbb{R}^N$, from $V$ to $W$. Further assume that $A$ satisfies the maximal-rank property

$$\forall \xi \in \mathbb{R}^N - \{0\}, \quad A(\xi) : V \to W_A \text{ is onto}.$$ 

Then, there exists an unbounded linear map

$$T : W^{A,p}(\Omega) \subset L^p(\Omega; V) \to L^p(\Omega; V),$$

satisfying the following properties:

1. $T$ is a projection onto $N_p(A, \Omega)$, that is,

   $$R(T) = N_p(A, \Omega)$$

   and

   $$T(Tu) = Tu \quad \text{for all } u \in W^{A,p}(\Omega).$$

2. the estimate

   $$\|Tu\|_{W^{k,p}(\Omega)} \leq \|u\|_{L^p(\Omega)} + C(p, A, \Omega) \|Au\|_{L^p(\Omega)},$$

   and the projection estimate

   $$\|u - Tu\|_{W^{k,p}(\Omega)} \leq C(p, A, \Omega) \|Au\|_{L^p(\Omega)},$$

   hold for all $u \in W^{A,p}(\Omega)$.

**Notation 2.** In order to keep our presentation as simple as possible, we have omitted to explicitly denote that $T$ depends on $p, \Omega$ and $A$.

A direct consequence of this, is the following decomposition of $L^p(\Omega; V)$:
Corollary 2. Assume that $A$ is a maximal-rank operator of order $k$, from $V$ to $W$. Then $W^{A,p}(\Omega)$ decomposes as a topological sum (of closed subspaces)

$$W^{A,p}(\Omega) = N_p(A, \Omega) \oplus A^{-1}[L^p(\Omega; W)],$$

where $A^{-1} : L^p(\Omega; W) \rightarrow W^{k,p}(\Omega; V)$ is the solution operator constructed in Lem. 14. That is, every $u \in W^{A,p}(\Omega)$ may be uniquely decomposed as

$$u = v + w, \quad v \in N_p(A, \Omega), \quad w \in A^{-1}[L^p(\Omega; V)].$$

Moreover, in this case $Aw = Au$ on $\Omega$, and $w$ satisfies the Sobolev estimate

$$\|w\|_{W^{k-p}(\Omega)} \leq C(p, A, \Omega) \|Au\|_{L^p(\Omega)}.$$

1.1.1. Lower-order distance estimates. The projection estimates contained in Thm. 1 convey a hierarchy of lower-order distance estimates in terms of suitable negative Sobolev norms of $Au$. To make this statement precise, let $\ell > 0$ be a positive integer and let us recall that $W_{\ell,q}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the $W_{\ell,p}$-norm. Following standard notation, we write $W^{-\ell,p}(\Omega)$ to denote the dual of $W_{\ell,q}(\Omega)$, where $p^{-1} + q^{-1} = 1$. Notice that when $\Omega$ is a bounded open set, then, by Poincaré’s inequality, the classical Sobolev norm $\|\cdot\|_{W_{\ell,p}(\Omega)}$ and the homogeneous norm

$$\|\cdot\|_{\dot{W}^{\ell,p}(\Omega)} := \left( \sum_{|\alpha| = \ell} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}$$

are equivalent norms of $W_{0}^{\ell,p}(\Omega)$. Therefore, also $W^{-\ell,p}(\Omega)$ coincides with the homogeneous negative Sobolev space $\dot{W}^{-\ell,p}(\Omega)$, which is defined as the dual of $W_{0}^{\ell,p}(\Omega)$ when endowed with the homogeneous norm. In fact, their respective norms are equivalent:

$$\|\cdot\|_{W^{-\ell,p}(\Omega)} \simeq \|\cdot\|_{\dot{W}^{-\ell,p}(\Omega)}.$$

Notation 3. For completeness, we set $W^{0,p}(\Omega) = W^{0,p}(\Omega) = L^p(\Omega)$.

With these considerations in mind, we are now in position to state the following distance estimates for maximal-rank operators.

Corollary 3. Let $0 \leq r \leq k$ be an integer and let $A$ be a maximal-rank operator of order $k$, from $V$ to $W$. Then,

$$\min_{v \in N_p(A, \Omega)} \|u - v\|_{W^{k-r,p}(\Omega)} \leq C(p, r, A, \Omega) \|Au\|_{\dot{W}^{-r,p}(\Omega)}$$

for all $u \in L^p(\Omega; V)$ with $Au \in \dot{W}^{-r,p}(\Omega; W)$. 
Remark 4. The fact that all maximal-rank operators are the formal adjoint of an elliptic operator (cf. Sect. 4.6) implies that the zero-order Korn estimate
\[ \inf_{v \in N_p(A, \Omega)} \|u - v\|_{L^p(\Omega)} \leq C \|Au\|_{W^{-k, p}(\Omega)}, \]
is a direct (trivial) consequence of the extension \( W_0^{k, p}(\Omega) \hookrightarrow W^{k, p}(\mathbb{R}^N) \), the Calderón–Zygmund estimates for elliptic operators in full-space and standard duality results. However, the higher-order estimates or its corresponding projection estimate do not follow by such methods.

1.1.2. Refinements for regular domains. In general \( A^{-1} \) (and hence also \( T \)) may only be extended by a non-linear map satisfying similar Sobolev estimates. If however \( p = 2 \) or \( \Omega \) is a sufficiently regular and connected domain, then one can extend \( A^{-1} \) linearly to a solution operator on \( W^{-k, p}(\Omega; V) \). In the first case, this follows from the theory of Hilbert spaces, while in the latter case it follows from classical regularity theory for the Dirichlet problem on regular domains.

In either case, we get the following refinement:

**Lemma 5.** Assume that \( A \) is a maximal-rank operator of order \( k \), from \( V \) to \( W \). Further assume that \( p = 2 \) or that \( \Omega \) is a connected domain with smooth boundary \( \partial \Omega \), then the projection from Thm. 1 extends to a linear bounded projection
\[ \tilde{T} : L^p(\Omega; V) \to L^p(\Omega; V), \]
satisfying the classical Fonseca–Müller estimate
\[ \|u - \tilde{T}u\|_{L^p(\Omega)} \leq C(p, A, \Omega) \|Au\|_{W^{-k, p}(\Omega)} . \]
More generally, if \( 0 \leq r \leq k \) is an integer, then
\[ \|u - \tilde{T}u\|_{W^{k-r, p}(\Omega)} \leq C(p, r, A, \Omega) \|Au\|_{W^{-k, p}(\Omega)} \]
for all \( u \in L^p(\Omega; V) \) with \( Au \in \tilde{W}^{-r, p}(\Omega; W) \).

**Corollary 6.** Assume that \( A \) is a maximal-rank operator of order \( k \), from \( V \) to \( W \). Further assume that \( p = 2 \) or that \( \Omega \) is a connected domain with smooth boundary \( \partial \Omega \). Then \( L^p(\Omega; V) \) decomposes as a topological sum (of closed subspaces)
\[ L^p(\Omega; V) = N_p(A, \Omega) \oplus A^{-1}[W^{-k, p}(\Omega; W)], \]
where \( A^{-1} : L^p(\Omega; W) \to W^{k, p}(\Omega; V) \) is the extended linear solution operator from Lem. 14. More precisely, every \( u \in L^p(\Omega; V) \) may be uniquely decomposed as
\[ u = v + w, \quad v \in N_p(A, \Omega), \quad w \in A^{-1}[W^{-k, p}(\Omega; W)], \]
with
\[ \|w\|_{L^p(\Omega)} \leq C(p, A, \Omega) \|Au\|_{W^{-k, p}(\Omega)} . \]
Remark 7. These two refinements hold under significantly milder regularity assumptions on $\partial \Omega$.

1.1.3. Estimates for $L^1$-gradients. We can make use of the previous results to establish certain sub-critical Sobolev estimates when $Au$ is either representable by an integrable map or a Radon measure. To make this precise, let us first recall that the total variation of a distribution $\sigma \in \mathcal{D}'(\Omega; W)$, on a Borel subset $U \subset \Omega$, is defined as the (possibly infinite) non-negative quantity

$$|\sigma|(U) := \sup \{ \sigma[\varphi] : \varphi \in C_c^\infty(U; W), \|\varphi\|_\infty \leq 1 \}.$$

More generally, the space $\mathcal{M}_b(\Omega; W) := (C^0(\Omega; W))^*$, of finite $W$-valued Radon measures over $\Omega$, coincides with the subspace of distributions $\sigma \in \mathcal{D}'(\Omega; W)$ with finite total variation $|\sigma|(\Omega)$. Notice that if $\sigma \in L^1(\Omega; W)$, then the absolutely continuous measure $\sigma_{L^1N}$ has bounded variation on $\Omega$ and in fact $\|\sigma\|_{L^1(\Omega)} = |\sigma_{L^1N}|(\Omega)$.

Morrey’s embedding and our previous results imply the following projection estimates for functions whose $A$-gradient is a measure:

**Corollary 8.** Let $A$ be a maximal-rank operator of order $k$ on $\mathbb{R}^N$, from $V$ to $W$, and let

$$1 < q < \frac{N}{N-1}.$$ Further assume that $\partial \Omega$ is of class $C^\infty$. Then,

$$\|u - \tilde{T}u\|_{W^{k-1,q}(\Omega)} \leq C|Au|(\Omega)$$

for all $u \in L^q(\Omega; V)$, for some constant $C = C(N, q, A, \Omega)$.

Here, $\tilde{T}$ is the projection from Lem. 5 with exponent $p = q$.

1.2. Estimates for constant-rank operators. If we assume that $A$ is a constant rank operator (but not necessarily of maximal rank), we are currently only able to prove the validity of weaker distance estimates, which require one to remove a subspace of the space of all $A$-harmonic maps. To make this precise, let us introduce the generalized $A$-Laplacian operator (see Sect. 2)

$$\Delta_A := A^* \circ A.$$

The statement is the following:

**Theorem 9.** Let $0 \leq r \leq k$ be an integer and let $A$ be an operator of order $k$ on $\mathbb{R}^N$, from $V$ to $W$. Further assume that $A$ satisfies the constant-rank property

$$\forall \xi \in \mathbb{R}^N - \{0\}, \quad \text{rank } A(\xi) = \text{const.}$$
Then,
\[
\inf_{v \in N_p(A, \Omega)} \|u - v\|_{W^{k-r,p}(\Omega)} \leq C(p, r, A, \Omega) \|Au\|_{W^{-r,p}(\Omega; V)}
\]
for all \(u \in L^p(\Omega; V)\) with \(Au \in W^{-r,p}(\Omega; W)\).

1.3. Comments on the proofs. The main ingredient of the proof(s) is contained in Lemma 14, which can be labeled as an existence and regularity result for maximal-rank operators. There, we show that if \(A\) has maximal rank, then there exists a kernel \(K \in L^1_{\text{loc}}(\mathbb{R}^N; \text{Hom}(W, V))\) that acts as a true fundamental solution of \(A\), that is, \(AK = \delta_{0_W}\) on \(\mathbb{R}^N\), where \(\delta_{0_W}\) is the Dirac mass at \(0 \in W\). This gives rise to a convolution-type solution operator on \(\Omega\) for which we can show classical Sobolev estimates. Although similar solution kernels exist for general constant-rank operators (see \([6, 15, 16]\)), these, in general, satisfy the weaker identity \(A(K \ast Au) = u\) for all \(u \in C_0^\infty(\Omega; V)\). It is precisely for this step that the maximal-rank assumption is crucially used, to guarantee that there exists a convolution solution operator \(A^{-1} : L^p(\Omega; W) \rightarrow W^{k,p}(\Omega; V)\) such that, for any source \(f \in L^p(\Omega; W)\), the element \(A^{-1}[f] \in L^p(\Omega; V)\) solves the system \(Au = f\) on \(\Omega\), and satisfies the estimate
\[
\|A^{-1}[f]\|_{W^{k,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}.
\]
From this point onwards, one simply defines the projection
\[
Tu := u - A^{-1}[Au].
\]

1.4. Operators with finite dimensional null-space. For sufficiently regular domains \(\Omega \subset \mathbb{R}^N\), there are a few instances for which the Sobolev projection estimates are known to hold. For the \(k\)th order gradient operator
\[
D^k u = (D^\alpha u)^{j=1, \ldots, m}, \quad u : \Omega \rightarrow \mathbb{R}^m,
\]
the projection estimate follows from the Deny–Lions lemma
\[
\inf_{p \in P_k(\mathbb{R}^m)} \|u - p\|_{W^{k,p}(\Omega)} \leq C\|D^k u\|_{L^p(\Omega)}.
\]
Here, \(P_k(\mathbb{R}^m)\) denotes the sub-space of \(\mathbb{R}[x; \mathbb{R}^m]\) of polynomials on \(N\)-variables with \(\mathbb{R}^m\)-coefficients and order at most \(r\), which restricted to \(\Omega\) are precisely the null-space \(N(D; \Omega)\). If instead one considers the symmetric gradient operator
\[
Eu = \frac{1}{2}(Du + Du^T), \quad u : \Omega \rightarrow \mathbb{R}^N,
\]
the second version of Korn’s inequality gives
\[
\inf_{c \in \mathbb{R}^N, R \in \text{Mat}(N \times N, \text{skew})} \| u - Rx - c \|_{W^{1,p}(\Omega)} \leq C \| E u \|_{L^p(\Omega)},
\]
where \( \text{Mat}(N \times N, \text{skew}) \) is the space of skew-symmetric \( N \times N \) matrices. This is indeed a projection estimate since \( N(E; \Omega) \) coincides with the space of rigid motions on \( \Omega \). More generally, for a Lipschitz domain \( \Omega \subset \mathbb{R}^N \), the strong projection holds for a class of operators, introduced by Aronszajn \[3\] and Smith \[19, 20\], that possess a distributional finite dimensional null-space \( N(A; \mathbb{R}^N) \), that is,
\[
\dim N(A; \mathbb{R}^N) < \infty.
\]
In this vein, Smith \[19, 20\] proved that (7) is a sufficient and necessary condition for the validity of the (local) coercive inequality
\[
\| D^k u \|_{L^p(\Omega)} \leq C \left( \| u \|_{L^p(\Omega)} + \| A u \|_{L^p(\Omega)} \right).
\]
As already advanced in the introduction, a rather trivial consequence of (3) and standard Sobolev compactness arguments is that the projection estimate holds on regular domains for operators with finite dimensional null-space:

**Lemma 10.** Let \( \Omega \subset \mathbb{R}^N \) be a Lipschitz domain and let \( A \) be a homogeneous \( k \)-th order linear operator on \( \mathbb{R}^N \), from \( V \) to \( W \). Further assume that
\[
\dim N(A; \mathbb{R}^N) < \infty.
\]
Then, there exists a positive constant \( C = C(p, A, \Omega) \) and a bounded linear projection \( \pi : L^p(\Omega; V) \to L^p(\Omega; V) \) with range
\[
R(\pi) = N_p(A; \Omega),
\]
and such that
\[
\| u - \pi v \|_{W^{k,p}(\Omega)} \leq C(p, A, \Omega) \| A u \|_{L^p(\Omega)}
\]
for all \( u \in L^p(\Omega; V) \) with \( A u \in L^p(\Omega; W) \).

It is worth to emphasize that the assumption \( \dim N(A; \mathbb{R}^N) < \infty \) implies the constant-rank property (since it implies \( \ker A(\xi) = \{0\} \) for all non-zero \( \xi \in \mathbb{R}^N \)); it is in fact a very strong form of ellipticity. However, the finite dimensional null-space assumption is *by no means necessary* for the validity of (3), or (4) for that matter. Prior to this work, Fuchs \[9\] had already shown that —for a sufficiently regular domain \( \omega \) of the complex-plane \( \mathbb{C} \)— the Sobolev distance from \( u \in W^{1,p}(\omega, \mathbb{C}) \) to the space \( \text{Hol}(\omega) \) of holomorphic maps on \( \omega \) is bounded by the \( L^p \)-norm of \( \partial \bar{z} u \), that is,
\[
\inf_{v \in \text{Hol}(\omega)} \| u - v \|_{W^{1,p}(\omega)} \leq C \| \partial \bar{z} u \|_{L^p(\omega)}.
\]
Note however that \( N(\partial_{\xi}, \mathbb{C}) = \text{Hol}(\mathbb{C}) \) implies it is infinite dimensional. In this regard, the remark below and the contents of Thm. 1 provide a plethora of examples, in more general conditions, of operators with infinite dimensional distributional null-space that satisfy the projection estimates (cf. Prop. 20).

**Remark 11 (Maximal-rank vs FDN).** Coercive estimates on domains, as in (3), **fail for all operators of maximal rank**. In fact, if \( \mathcal{A} \) is non-trivial, then

\[
\mathcal{A} \text{ is a maximal-rank operator } \implies \dim N(\mathcal{A}) = \infty.
\]

This shows that the validity of strong projection estimates is independent from the validity of coercive inequalities on \( \Omega \) or the existence of suitable linear trace operators on \( \partial \Omega \) (cf. [7, 10]). The implication above follows easily from [10, Prop. 1.2], where it has been shown that (7) implies \( \mathcal{A} \) is **cancelling** (in the sense of Van Schaftingen [21]):

\[
\mathcal{W}_\mathcal{A} := \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} \text{Im} \mathcal{A}(\xi) = \{0_W\}.
\]

However, the cancellation property fails for all non-trivial maximal-rank operators because by definition these satisfy \( \mathcal{W}_\mathcal{A} = W_\mathcal{A} \neq \{0_W\} \).

2. **Set-up and preliminary results**

As already mentioned above, \( \Omega \) will always denote an arbitrary bounded and open subset of \( \mathbb{R}^N \). If \( u \in L^p(\Omega; V) \) and \( \sigma \in \mathcal{D}'(\Omega; W) \), the system

\[
\mathcal{A}u = \sigma \quad \text{on } \Omega
\]

shall always be understood in the sense of distributions, that is,

\[
\sigma[\phi] = \int_{\Omega} u \cdot \mathcal{A}^* \phi \quad \text{for all } \phi \in C^\infty_c(\Omega; W),
\]

where

\[
\mathcal{A}^* := (-1)^k \sum_{|\alpha|=k} (A_\alpha)^* D^\alpha
\]

is the formal \( L^2 \)-adjoint of \( \mathcal{A} \). In this case, we write \( \mathcal{A}u \in L^p(\Omega; W) \) provided that \( \sigma \) is representable by a \( p \)-integrable map, which we shall also identify with \( \mathcal{A}u \). With this convention, we may rigorously define the distributional null-space \( N(\mathcal{A}; \Omega) = \{ u \in \mathcal{D}'(\Omega; V) : \mathcal{A}u = 0 \} \) and its subspace

\[
N_p(\mathcal{A}; \Omega) := N(\mathcal{A}; \Omega) \cap L^p(\Omega; V) \leq L^p(\Omega; V),
\]

consisting of all \( p \)-integrable \( \mathcal{A} \)-free maps on \( \Omega \). One can see, by a simple application of the Fourier transform and a density argument, that the subspace of all \( \mathcal{A} \)-gradients in Fourier space

\[
\text{span} \{ \text{Im} \mathcal{A}(\xi)[v] : \xi \in \mathbb{R}^N, v \in V \},
\]
coincides with the point-wise essential range of $A$ given by
\[ W_A = \text{clos} \left\{ Au(x) : x \in \mathbb{R}^N, u \in C_c^\infty(\mathbb{R}^N; V) \right\}. \]

2.1. **Properties of negative Sobolev spaces.** Let us introduce the higher-order divergence operator: for a tensor $F = (F_\beta)_{|\beta|=k} \in L^p(\Omega; V)^{(N+k-1)_k}$, the $k$th order divergence operator is defined as the distributional operator
\[ \text{div}^k F := \sum_{|\beta|=k} D^\beta F_\beta. \]

With this in mind, we give short proof of a well-known representation of homogeneous negative spaces as quotient spaces of $L^p$-spaces.

**Proposition 12.** The map $S : L^p(\Omega)^{(N+k-1)_k}/N_p(\text{div}, \Omega) \rightarrow \dot{W}^{-k,p}(\Omega)$ given by
\[ S[g] = (-1)^k \text{div}^k g, \]
is a linear bijective isometry.

**Proof.** Let $q$ be the dual Hölder exponent of $p$ and consider the map linear map $H : \dot{W}^k_0(\Omega) \rightarrow L^q(\Omega)^{(N+k-1)_k}$ defined by the assignment $u \mapsto D^k u$. By definition of the homogeneous norm, $H$ defines a one-to-one isometry and therefore, by the Hahn Banach theorem, $H^* : L^p(\Omega)^{(N+k-1)_k}/\ker H^* \rightarrow \dot{W}^{-k,p}(\Omega)$ is well-defined and is a bijective isometry. By construction, $H^*$ is precisely the distributional operator $(-1)^k \text{div}^k$ and hence
\[ \ker H^* = N_p(\text{div}, \Omega), \]
as desired. □

The following refinement will be crucial to construct a linear extension of the solution operator $A^{-1}$ (see Lem. 14), to all of $W^{-k,p}(\Omega; W)$, when $p = 2$ or when $\Omega$ is a sufficiently regular and connected domain.

**Proposition 13.** Assume that either $p = 2$ or that $\Omega$ is a connected domain with smooth boundary $\partial \Omega$. Then, there exists a one-to-one linear map $S : \dot{W}^{-k,p}(\Omega) \rightarrow L^p(\Omega)^{(N+k-1)_k}$ such that
\[ \sigma = (-1)^k \text{div}^k (S\sigma) \quad \text{for all } \sigma \in \dot{W}^{-k,p}(\Omega). \]
Moreover, this map is bounded in the sense that
\[ \|\sigma\|_{\dot{W}^{-k,p}(\Omega)} \leq C(p, k, N, \Omega) \|S\sigma\|_{L^p(\Omega)}. \]

**Proof.** First, we observe that the case when $p = 2$, the assertion is a straightforward consequence of the theory of Hilbert spaces. Indeed, since every closed subspace of a Hilbert space has an orthogonal complement, the assertion follows directly from the previous proposition (in fact, in this case $S$ is an isometry). We address now the case when $\Omega$ is connected domain.
with regular boundary. In this scenario, the classical existence and regularity theory for powers of the Laplacian implies the following: for every $g \in L^p(\Omega)^{(N+k-1)}$, there exists a unique solution $u_g \in W^{k,p}_0(\Omega)$ of the equation

$$\Delta^k u := \text{div}^k(D^k u) = (-1)^k \text{div}^k(g) \quad \text{on} \; \Omega.$$ 

Moreover, the solution $u_g$ satisfies the estimate

$$\|D^k u_g\|_{L^p(\Omega)} \leq C(p, k, N, \Omega) \|g\|_{L^p(\Omega)},$$

where the constant depends both on the diameter of $\Omega$ and the regularity of $\partial \Omega$. By the previous proposition, we deduce that, for every $\sigma \in \dot{W}^{-k,p}(\Omega)$, the equation

$$\Delta^k u = \sigma \quad \text{on} \; \Omega$$

has a unique solution $u_\sigma \in W^{k,p}_0(\Omega)$ satisfying

$$\|D^k u_\sigma\|_{L^p(\Omega)} \leq C(p, k, N, \Omega) \|\sigma\|_{\dot{W}^{-k,p}(\Omega)}.$$

Since this solution unique for every such $\sigma$, it follows that the map $S : \sigma \mapsto D^k u_\sigma$ is well defined, and defines a one-to-one bounded linear map from $\dot{W}^{-k,p}(\Omega)$ to $L^p(\Omega)^{(N+k-1)}$. Moreover, by construction

$$\text{div}^k S \sigma = \Delta^k u_\sigma = \sigma \quad \text{and} \quad \|S \sigma\|_{L^p(\Omega)} \leq C(p, k, N, \Omega) \|\sigma\|_{\dot{W}^{-k,p}(\Omega)}.$$

This finishes the proof. \hfill \square

**Notation 4.** Let $r > s$ be arbitrary non-negative integers. We shall henceforth, under the isometric embedding $W^{-s,p}(\Omega) \hookrightarrow W^{-r,p}(\Omega)$, consider $W^{-s,p}(\Omega)$ as a subset of $W^{-r,p}(\Omega)$.

### 2.2. The inverse of maximal-operators.

The basis of our results consists in exploiting the existence of true fundamental solutions for maximal-rank operators. This is addressed in the following result:

**Lemma 14 (Existence and regularity).** Assume that $A$ maximal-rank operator of order $k$ on $\mathbb{R}^N$, from $V$ to $W$. Then, there exists a linear and bounded solution operator

$$A^{-1} : L^p(\Omega; W) \rightarrow W^{k,p}(\Omega; V),$$

satisfying classical Sobolev regularity estimates. That is, for every $f \in L^p(\Omega; W)$, it holds

$$A(A^{-1}[f]) = f \quad \text{on} \; \Omega$$

and

$$\|A^{-1}[f]\|_{W^{k,p}(\Omega)} \leq C(p, A, \Omega) \|f\|_{L^p(\Omega)}.$$ 

Moreover, there exists a (possibly non-linear) operator

$$\tilde{S} : \dot{W}^{-k,p}(\Omega; W) \rightarrow L^p(\Omega; V),$$

This finishes the proof. \hfill \square
which extends $A^{-1}$ and is such that
$$A(\tilde{S}\sigma) = \sigma \quad \text{on } \Omega,$$
for every $\sigma \in W^{-k,p}(\Omega; W)$. Furthermore, for every integer $0 \leq r \leq k$ and every $\sigma \in W^{-r,p}(\Omega; W)$, it holds
$$\|\tilde{S}\sigma\|_{W^{k-r,p}(\Omega)} \leq C(p, r, A, \Omega) \|\sigma\|_{W^{-r,p}(\Omega)}.$$

If additionally, $p = 2$ or $\Omega$ is a connected domain with smooth boundary $\partial \Omega$, then the solution operator $\tilde{S}$ above can be constructed as a linear extension of $A^{-1}$ (which we shall still denote by $A^{-1}$).

Proof. First, we address the existence and properties of $A^{-1}$ on $L^p(\Omega; W)$. The strategy of the proof is to exploit the regularity properties of the Moore-Penrose pseudo-inverse of the principal symbol of $A$, a technique that dates back to the work of Gustafson [11] and more recently also key for the work of Raita [16]. Let us write $k = N + \ell > 0$, where $\ell$ is an integer, so that $\ell > -N$. We recall from two these sources and the references therein that $P(\xi) := A(\xi)^\dagger (\xi \in \mathbb{R}^N - \{0\})$
defines a $-(N + \ell)$-homogeneous distribution in $C^\infty(\mathbb{R}^N - \{0\}; \text{Hom}(W; V))$. Here, for a matrix $M$, we have denoted by $M^\dagger$ its Moore–Penrose pseudo-inverse, which satisfies the fundamental algebraic identity $MM^\dagger = \text{proj}_{\text{im} M}$. By assumption, $A(\xi)$ has maximal-rank for every $\xi \in \mathbb{R}^N - \{0\}$ and by the discussion in the introduction we may hence assume without loss of generality that $\text{Im } A(\xi) = W = W_A$ for all non-zero frequencies. In particular, it follows from the universal property of the quasi-inverse that $A(\xi)P(\xi) = \text{proj}_{\text{im } A(\xi)} = \text{id}_W \in \text{Hom}(W, W)$ for all non-zero $\xi \in \mathbb{R}^N$. Moreover, by [12, Theorems 3.2.3, 3.2.4], we find that $P$ can be extended to a tempered distribution $P^* \in \mathcal{D}'(\mathbb{R}^d; \text{Hom}(W, V))$ satisfying $pP^* = (pP)^*$ (where $(\_)^*$ denotes the extension) for all homogeneous polynomials of degree $s > \ell$. It follows also that its inverse Fourier transform $K := (2\pi i)^k \mathcal{F}^{-1}(P^*)$ is a smooth homogeneous map of degree $\ell$, locally integrable on $\mathbb{R}^N$ (since $\ell > -N$), of the form
$$K(x) = |x|^\ell \Psi \left(\frac{x}{|x|}\right) - Q(x) \log |x|,$$
where $\Psi : S^{N-1} \rightarrow \text{Hom}(W, V)$ is smooth and $Q$ is a $\text{Hom}(W, V)$-valued polynomial (this follows from [12, Thms. 7.1.16-7.1.18; Eqn. (7.1.19)], when applied component-wise to the coordinates of $K$).

Now, let us write $\hat{f}$ to denote the (trivial) extension by 0 of $f$ to $L^p(\mathbb{R}^N; W)$. Since $\hat{f}$ is compactly supported (on $\mathbb{R}^N$) and $K$ is locally integrable, we may

---

2Here, since $s > \ell$, $(pP)^* \in \mathcal{D}'(\mathbb{R}^N; V \otimes W^*)$ denotes the unique extension of the homogeneous distribution $pP \in \mathcal{D}'(\mathbb{R}^N - \{0\}; V \otimes W^*)$. 

---
define a locally \( p \)-integrable map \( v := K \ast \tilde{f} \in \mathcal{S}'(\mathbb{R}^N; V) \). Note that \( K \) is, in fact, a fundamental solution of \( \mathcal{A} \) in the sense that
\[
\mathcal{A}K = \delta_{0V} \quad \text{in} \quad \mathbb{R}^N.
\]
Now, let us fix \( R > 0 \) and notice that by construction we get
\[
(9) \quad \|v\|_{L^p(B_R)} \leq C\|\tilde{f}\|_{L^p} = C\|f\|_{L^p(\Omega)},
\]
for some constant depending solely on \( N, p, R \) and \( \mathcal{A} \). Next, let \( \alpha \) be a multi-index of order \( |\alpha| = k = N + \ell \) and let \( p^\alpha(\xi) = (2\pi)^{|\alpha|}\xi^\alpha \) be the polynomial associated to the Fourier transform of \( D^\alpha \). Since the degrees of \( \mathcal{A} \) and \( p^\alpha \) are strictly larger than \( \ell \), applying the Fourier transform to the first identity gives
\[
\{A\} = F(f)
\]
and let
\[
(10) \quad \parallel\mathcal{A}^{-1}[f]\parallel_{L^p(\Omega)} \leq \|D^\alpha v\|_{L^p} \leq C\|\tilde{f}\|_{L^p} = C\|f\|_{L^p(\Omega)},
\]
for some \( C = C(N, p, \mathcal{A}). \)
To conclude, we set \( R := \text{diam} \Omega \) and observe that up to a translation \( x \mapsto x - x_0 \) with \( x_0 \in \Omega \), and iteration of the classical Poincaré inequalities on \( W^{N+\ell,p}(B_R) \) and the estimates (9)-(10) render the sought estimate
\[
(11) \quad \|A^{-1}[f]\|_{W^{k,p}(\Omega)} \leq \|v\|_{W^{N+\ell,p}(B_R)} \leq C(p, \mathcal{A}, \Omega)\|f\|_{L^p(\Omega)}.
\]
Here, we have used that both \( N \) and \( \ell \) depend on \( \mathcal{A} \).
Now that we have constructed \( \mathcal{A}^{-1} \), we may construct \( \tilde{\mathcal{A}} \) by means of the representation of negative Sobolev. Let us fix \( r \) as in the assumptions. We distinguish the following cases, which yield the second and third statements of the Lemma respectively:

1. if \( p \neq 2 \) or no regularity on \( \Omega \) is imposed, we define the non-linear map
\[
H_\rho\sigma := \arg\min \left\{ \|g\|_{L^p(\Omega)} : g \in L^p(\Omega; W)^{(N+r-1)/r}, \div^r g = \sigma \right\}
\]
(2) if \( p = 2 \) or \( \Omega \) is a simply connected regular domain, then we define \( H_r = S \), where \( S \) is the linear map from Proposition 13 for \( k = r \).

Let \( \sigma \in W^{-r,p}(\Omega; W) \) be given, so that
\[
\sigma = \text{div}^r (H_r \sigma)
\]
and
\[
\|H_r \sigma\|_{L^p(\Omega)} \leq C \|\sigma\|_{W^{-r,p}(\Omega)}.
\]

Let us define
\[
\tilde{S}_r \sigma := \sum_{|\beta|=r} D^\beta (A^{-1}((H_r \sigma)_\beta)),
\]
and notice that \( \tilde{S}_r \) is linear whenever \( H_r \) is. Moreover, the estimates we proved for \( A^{-1} \) convey the following estimate:
\[
\|\tilde{S}_r \sigma\|_{W^{k-r,p}(\Omega)} \lesssim \|D^k (A^{-1}[H_r \sigma])\|_{L^p(\Omega)} \\
\lesssim \|H_r \sigma\|_{L^p(\Omega)} \lesssim \|\sigma\|_{W^{-r,p}(\Omega)}.
\]

Lastly, we observe that \( \tilde{S}_r \) is a solution operator in the sense that
\[
A(\tilde{S}_r \sigma) = \sigma \text{ for all } \sigma \in W^{-r,p}(\Omega; W).
\]

This follows from the commutativity of distributional derivatives since (using the double-index summation convention)
\[
A(\tilde{S}_r \sigma) = D^\beta A_\alpha D^\alpha (A^{-1}[(H_r \sigma)_\beta]) = D^\beta (H_r \sigma)_\beta = \text{div}^r H_r \sigma = \sigma.
\]

3. PROOFS OF THE MAIN RESULTS

3.1. **Proof of Thm. 1.** The idea is define \( T : D(T) \subset L^p(\Omega; V) \to L^p(\Omega; V) \) as
\[
Tu := u - A^{-1}[Au], \quad D(T) = W^{A,p}(\Omega).
\]

That \( T \) is well-defined follows from the triangle inequality and the fact that \( Au \in L^p(\Omega; W) \) for all \( u \in W^{A,p}(\Omega) \). \( T \) is linear in its domain of definition due to the linearity of \( A^{-1} \) and \( A \). By construction the composition \( A \circ T \) is the zero map. Moreover, \( Tu = u \) for all \( u \in N_p(A, \Omega) \). This proves that \( R(T) = N_p(A, \Omega) \).

Notice that, by density and linearity, \( \tilde{S}_r \) extends \( \tilde{S}_m \) whenever \( k \geq r \geq m \geq 0 \). Thus, defining \( \tilde{S} := \tilde{S}_k \), the second and third statements of the lemma follow directly from (12) and (13). This completes the proof. \( \square \)
Similarly,
\[ \|u - Tu\|_{W^{k,p}(\Omega)} \leq \|A^{-1}[Au]\|_{W^{k,p}(\Omega)} \leq C\|Au\|_{L^p(\Omega)}. \]
This finishes the proof. \(\square\)

3.2. **Proof of Cor. 2.** That \(W^{A,p}(\Omega) = N_p(A,\Omega) + A^{-1}[L^p(\Omega;W)]\) follows from the fact \(N_p(A,\Omega)+W^{k,p}(\Omega;V) \subset W^{A,p}(\Omega)\) and that every \(u \in W^{A,p}(\Omega)\) can be written as
\[ u = Tu + (u - Tu) = Tu + A^{-1}[Au]. \]
Moreover \(N_p(A,\Omega) \subset W^{k,p}(\Omega)\) is closed because it coincides with the null-space of the linear map \(A : W^{A,p}(\Omega) \rightarrow L^p(\Omega;W)\). Let us verify that \(R(A^{-1})\) is closed in \(W^{k,p}(\Omega)\). Let \(g_j := A^{-1}[f_j] \in R(A^{-1})\) and assume that \(g_j \rightarrow g\) strongly in \(W^{A,p}(\Omega)\) so that \(f_j = Ag_j \rightarrow Ag\) strongly in \(L^p(\Omega;W)\). Using the continuity of \(A^{-1}\), we further obtain
\[ \|A^{-1}[Ag] - g_j\|_{W^{k,p}(\Omega)} \lesssim \|Ag - f_j\|_{L^p(\Omega;W)} \rightarrow 0, \]
which shows that \(g = A^{-1}[Ag] \in R(A^{-1})\). Since \(g_j\) was an arbitrary convergent sequence, this shows that \(R(A^{-1})\) is indeed closed in \(W^{k,p}(\Omega)\). We are left to prove that if
\[ v + A^{-1}[f] = 0, \]
for some \(v \in N_p(A,\Omega), f \in L^p(\Omega;W)\), then \(v = A^{-1}[f] = 0\). To see this, we simply apply \(A\) to this identity to get \(f = 0\). The conclusion then follows from the identity \(v = -A^{-1}[f] = 0\). \(\square\)

3.3. **Proof of Cor. 3.** Let \(\tilde{S} : L^p(\Omega;W) \rightarrow L^p(\Omega;V)\) be the (non-linear) extension of \(A^{-1}\) constructed in Lem. 14. Then, by the second statement in that lemma and writing \(u = \tilde{S}u - (u - \tilde{S}u)\), we get
\[ \min_{v \in N_p(\Omega;V)} \|u - v\|_{W^{k-r,p}(\Omega)} \leq \|\tilde{S}u\|_{W^{k-r,p}(\Omega)} \leq C(p,r,A,\Omega)\|Au\|_{W^{-r,p}(\Omega)}. \]
This finishes the proof. \(\square\)

3.4. **Proof of Lem. 5 and Cor. 6.** The proofs are analogous to the proof of Thm. 1 and Cor. 2, with the exception that, instead of \(T\), the proof for these statements appeals to the existence of a linear extension \(\tilde{T}\) of \(T\) on \(L^p(\Omega;W)\) (cf. last statement in Lem. 14).

3.5. **Proof of Cor. 8.** Let \(1 < q < N/(N-1)\) so that \(N < q' < \infty\). If \(\Omega\) is sufficiently regular, then Morrey's embedding \(W^{1,q'}_0(\Omega) \hookrightarrow C_0(\Omega)\) holds. Since the embedding is dense, we get \(M_0(\Omega;W) \hookrightarrow W^{-1,q}(\Omega;W)\), which, together with Lem. 5, gives
\[ \|u - \tilde{T}u\|_{W^{-1,q}(\Omega)} \lesssim \|Au\|_{W^{-1,q}(\Omega)} \leq C(q,N)\|Au\|_{W^{-1,q}(\Omega)}. \]
This finishes the proof. \(\square\)
3.6. **Proof of Lemma 10.** We give a contradiction argument as follows: by a standard normalization argument we may find a sequence \( u_j \in L^p(\Omega; V) \), with \( A u_j \in L^p(\Omega; W) \), and satisfying

\[
(14) \quad 1 \leq \text{dist}_{W^{k,p}(\Omega)}(u_j, N_p(A, \Omega)) \leq 2, \quad \|A u_j\|_{L^p(\Omega)} \leq \frac{1}{j}.
\]

Notice that, for each \( j \in \mathbb{N} \), the left-hand side above is always finite due to the validity of the coercive estimate (8) and an iteration of Poincaré’s classical inequality on Lipschitz domains. We may thus find another sequence, \( v_j \in N_p(A, \Omega) \), such that

\[
1 \leq \|u_j - v_j\|_{W^{k,p}(\Omega)} \leq 2, \quad \|A(u_j - v_j)\|_{L^p(\Omega)} \leq \frac{1}{j}.
\]

Since the sequence \( w_j = u_j - v_j \) is uniformly bounded in \( W^{k,p}(\Omega; V) \), passing to a further subsequence (not relabeled) we may assume that \( w_j \) converges weakly in \( W^{k,p}(\Omega; V) \) to some non-trivial \( w \in N_p(A, \Omega) \). Another application of the coercive estimate (8) yields

\[
\|D^k(w_j - w)\|_{W^{k,p}(\Omega)} \leq C \left( \|w_j - w\|_{L^p(\Omega)} + \|A u_j\|_{L^p(\Omega)} \right).
\]

Hence, by Rellich’s theorem, the right-hand side vanishes as \( j \to \infty \). This implies that \( w_j - v_j \to w \) strongly in \( W^{k,p}(\Omega; V) \) and hence

\[
\text{dist}_{W^{k,p}(\Omega)}(u_j, N_p(A, \Omega)) \leq \|u_j - v_j - w\|_{W^{k,p}(\Omega)} \to 0,
\]

which poses a contradiction with (14). This finishes the proof. \( \square \)

3.7. **Proof of the weak Korn estimates.** To work out the case when \( A \) is not a maximal-rank operator, we require to briefly discuss the dichotomy between constant-rank operators and elliptic complexes. Recently, Raita [16] (see also the simplified proof given in [5]) showed that there exists a surjective correspondence between elliptic complexes and constant-rank operators. There, the author proved that if \( A \) satisfies the constant rank property, then there exists a differential complex

\[
\mathcal{D}'(\mathbb{R}^N; V) \xrightarrow{A} \mathcal{D}'(\mathbb{R}^N; W) \xrightarrow{Q} \mathcal{D}'(\mathbb{R}^N; X)
\]

where \( Q \) is a constant-rank operator on \( \mathbb{R}^N \), from \( W \) to \( X \), of order \( k_Q \) and such that the symbol complex

\[
V \xrightarrow{A(\xi)} W \xrightarrow{Q(\xi)} X
\]

defines an elliptic complex, that is,

\[
\text{Im} A(\xi) = \ker Q(\xi) \quad \text{for all } \xi \in \mathbb{R}^N - \{0\}.
\]

The converse, that symbols of elliptic complexes define operators of constant rank, is a trivial consequence of the lower semicontinuity of the rank (and the upper semicontinuity of \( \dim \ker \)). With these considerations, it is natural to define a generalized Laplace–Beltrami operator on \( W \) by setting

\[
\Delta_W := [AA^*]^{k_Q} + [Q^*Q]^k.
\]
Remark 15. The associated annihilator $Q$ of $A$ is in general not uniquely defined, which implies that $\triangle_W$ may also not be uniquely defined.

The following observation will allow us to make use of the Lemma 14 and Theorem 1 when considering the operator $A = \triangle_W$.

**Proposition 16.** The operator $\triangle_W$ is an elliptic system (as in defined in Def. 23). In particular, $\triangle_W$ is a maximal-rank operator.

**Proof.** Let us write $\triangle = \triangle_W$. Since $\triangle$ is a homogeneous operator from $W$ to $W$, we are only left to verify that $\triangle$ is elliptic. By construction we have that $p(\xi)[w] = \langle \triangle(\xi)w, w \rangle = |A(\xi)^* [w]|^{2kQ} + |Q(\xi)[w]|^{2k}$ for all $\xi \in \mathbb{R}^N$ and $w \in W$. In particular, $p$ vanishes if and only if $A(\xi)^*[w] = 0$ and $Q(\xi)[w] = 0$. A well-known linear algebra property of exact sequences is that they split, that is, $W = \text{Im } A(\xi) \oplus \text{Im } Q(\xi)^*$ so that $w = A(\xi)[a] \oplus Q(\xi)^*[b]$. Since $\ker(M^*M) = \ker(M)$, the equality with zero may only occur provided that $\xi = 0$ or that $A(\xi)[a] = 0$ and $Q(\xi)^*[b] = 0$. Therefore, by the homogeneity of $\triangle$, we deduce that there exists a constant $C$ satisfying

$$\forall \xi \in \mathbb{R}^N, \quad |\triangle(\xi)[w]| \geq C|\xi|^{2kQ}|w| \quad (w \in W).$$

This proves that $\triangle$ is indeed an elliptic operator from $W$ to $W$, and hence it defines an elliptic system. \qed

**Proof of Theorem 9.** Let us start the proof with some basic preparations. Let us recall that the sequence

$$\mathcal{D}'(\mathbb{R}^N; V) \xrightarrow{A} \mathcal{D}'(\mathbb{R}^N; W) \xrightarrow{Q} \mathcal{D}'(\mathbb{R}^N; X)$$

defines a differential complex and therefore $A^* \circ Q^* = 0$ as distributional differential operators. In particular,

$$A^* \circ \triangle_W = A^* \circ (A \circ A^*)^{kQ}$$

also as differential operators.

Let us write $\tilde{S}_W$ to denote the (non-linear) solution operator associated to the operator $\triangle_W$ (constructed in Lemma 14). Here, we are using that $\triangle_W$ is maximal-rank operator from $W$ to $W$. Define

$$\Pi u := u - A^* (A A^*)^{kQ-1} \tilde{S}_W Au, \quad u \in L^p(\Omega; V).$$

Using the estimates for $\tilde{S}_W$, we obtain the estimate (recall that $\triangle_W$ is of order $2kQ$)

$$\|\Pi u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|\tilde{S}_W \circ A u\|_{W^{2kQ-k,p}(\Omega)} \lesssim \|u\|_{L^p(\Omega)} + \|A u\|_{W^{-k,p}(\Omega)} \lesssim \|u\|_{L^p(\Omega)}.$$
Similarly, for each integer $0 \leq r \leq k$,

$$
\| u - \Pi u \|_{W^{k-r,p}(\Omega)} \leq \| \bar{S}_W \circ Au \|_{W^{2k-2r,p}(\Omega)} \\
\lesssim \| Au \|_{W^{-r}(\Omega)}.
$$

(17)

Moreover, by construction it holds

$$
\Delta_A \Pi u = \Delta_A^* u - A^* \Delta_Q \bar{S}_W Au
$$

(16)

$$
\Delta_A^* u - A^* \triangle W \bar{S}_W Au = \Delta_A^* u - \Delta_A^* u = 0,
$$

which shows that $\text{Im} \bar{S} \leq N_p(\Delta_A; \Omega)$. The sought Korn distance estimates then follow from (17)-(18). □

4. Examples of maximal-rank operators and applications

In this section we collect examples of a number of operators satisfying the maximal-rank property, for which Thm. 1 applies.

**Notation 5.** In order to keep the statements as simple as possible, we shall henceforth use the convention that $\| \sigma \|_{L^p(\Omega)} = \infty$ whenever $\sigma \notin L^p(\Omega)$.

4.1. Divergence. The row-wise divergence, and in particular the scalar divergence when $M = 1$, defined by

$$
\text{div } u = \left( \sum_{j=1}^M \frac{\partial u_j}{\partial x_j} \right)_{i=1,\ldots,M}, \quad u : \mathbb{R}^N \to \mathbb{R}^{M \times N},
$$

is a maximal-rank operator on $\mathbb{R}^N$, from $\mathbb{R}^{M \times N}$ to $\mathbb{R}^M$.

**Proof.** It suffices to show that $\text{div}(\xi)$ has rank $M$ for all non-zero $\xi \in \mathbb{R}^N$. Indeed, $\text{div}(\xi)[v \otimes \xi] = (v \otimes \xi) \cdot \xi = v$ for every $v \in \mathbb{R}^M$ and all $\xi \in \mathbb{R}^N - \{0\}$. □

In particular, we have the following Sobolev-distance estimates:

**Proposition 17.** There exists a constant $C = C(p, N, M, \Omega)$ such that

$$
\inf_{v \in L^p(\Omega; V), \text{div } v = 0} \| u - v \|_{W^{1,p}(\Omega)} \leq C \| \text{div } u \|_{L^p(\Omega)}
$$

for all $p$-integrable tensor-fields $u : \Omega \to \mathbb{R}^{M \times N}$ with $p$-integrable divergence.

One may also consider the $k$th order divergence operator

$$
\text{div}^k U = \sum_{|\alpha| = k} D^\alpha U_\alpha, \quad U_\alpha = (U_\alpha)_{|\alpha| = k},
$$

which defines a maximal-rank $k$th order operator on $\mathbb{R}^N$, from $\mathbb{R}^{(N + k - 1)}$ to $\mathbb{R}$, for which we get the following estimate:
Proposition 18. Let \( k \) be a positive integer. Then
\[
\inf_{V \in L^p(\Omega; V)} \left\| U - V \right\|_{W^{k,p}(\Omega)} \leq C(p, k, N, M, \Omega) \left\| \text{div}^k U \right\|_{L^p(\Omega)}
\]
for all \( U \in L^p (\Omega; \mathbb{R}^{(N+k-1)}) \).

Proof. The associated principal symbol is defined for \( \xi \in \mathbb{R}^N \) and \( a \in \mathbb{R}^{(N+k-1)} \) by
\[
\text{div}^k(\xi)[a] = \sum_{|\alpha|=k} a_\alpha \xi^\alpha.
\]
In this case \( \text{div}^k(\xi) [\otimes k] = |\xi|^{2k} \), which shows that \( \text{div}^k \) is also a maximal-rank operator. \( \square \)

4.2. The Laplacian. The Laplacian operator
\[
\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i}^2, \quad u : \mathbb{R}^N \to \mathbb{R},
\]
is a second order operator on \( \mathbb{R}^N \), from \( \mathbb{R} \) to \( \mathbb{R} \). Its principal symbol is given by the map \( \xi \mapsto |\xi|^2 \), so that in particular \( \Delta \) has rank 1. In light of Thm. 1 we obtain the following Sobolev estimate in terms of the distance to the space of harmonic maps:

Proposition 19. There exists a constant \( C = C(p, N, \Omega) \) such that
\[
\inf_{v : \Omega \to \mathbb{R} \text{ harmonic}} \left\| u - v \right\|_{W^{2,p}(\Omega)} \leq C \left\| \Delta u \right\|_{L^p(\Omega)},
\]
for all \( p \)-integrable functions \( u : \Omega \to \mathbb{R} \).

A similar statement holds for the bi-Laplacian operator
\[
\Delta^2 u = \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i} \frac{\partial^2 u}{\partial x_j}, \quad u : \mathbb{R}^N \to \mathbb{R}.
\]

4.3. The Cauchy–Riemann equations. The operator
\[
Lu := \left( \frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial x_2}, \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} \right), \quad u, v : \mathbb{R}^2 \to \mathbb{R},
\]
associated to real-coefficient equations arising from the del-var operator
\[
\partial_x = \frac{1}{2}(\partial_x + i \partial_y), \quad f(x + iy) = u(x, y) + iv(x, y),
\]
conforms a first-order system on \( \mathbb{R}^2 \), from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), with associated principal symbol tensor
\[
\mathbb{L}(\xi) = \begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix} \quad (\xi \in \mathbb{R}^2).
\]
Since $\det L(\xi) = |\xi|^2$ for all $\xi \in \mathbb{R}^2$, it follows that the Cauchy–Riemann equations conform a maximal-rank system in the sense of Theorem 1. Since $f \in \operatorname{Hol}(\omega)$ if and only if $\partial \overline{f} = 0$ on $\omega$, it follows from Theorem 1 that the distance estimates for holomorphic functions proven in [9] also hold for arbitrary (possibly irregular) domains $\omega \subset \mathbb{C}$. The precise statement is the following:

**Proposition 20.** Let $\omega \subset \mathbb{C}$ be a bounded and open set. There exists a constant $C = C(p, \omega, L)$ such that

$$\inf_{g \in (\operatorname{Hol} \cap L^p)(\omega)} \|f - g\|_{W^{1,p}(\omega, \mathbb{C})} \leq C\|\partial \overline{f}\|_{L^p(\omega)},$$

for all $p$-integrable complex-functions $f : \omega \to \mathbb{C}$. Here, $W^{1,p}(\omega; \mathbb{C})$ is the space of functions $f \in L^p(\omega; \mathbb{C})$ having first-order weak complex partial derivatives in the same space.

In a similar vein, the equations corresponding to the conformal matrix inclusion $\nabla u \in K$, where

$$K := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}, \quad u : \mathbb{R}^2 \to \mathbb{R}^2,$$

conform a first order system with maximal rank, which conveys the following estimates for gradient conformal maps:

**Proposition 21.** There exists a constant $C = C(p, \Omega, K)$ such that

$$\inf_{\nabla V \in K, \ V \in L^p(\Omega; \mathbb{R}^2)} \|F - V\|_{W^{1,p}(\Omega)} \leq C\|LF\|_{L^p(\Omega)}.$$

for all vector-fields $F \in L^p(\Omega; \mathbb{R}^2)$.

### 4.4. The Laplace–Beltrami operator.
Let $\ell \in \{1, \ldots, n - 1\}$. The exterior derivative $d$ is the first-order differential operator from $\bigwedge^\ell \mathbb{R}^N$ to $\bigwedge^{\ell+1} \mathbb{R}^N$ associated to the symbol

$$d(\xi) = \xi \wedge v \quad (\xi \in \mathbb{R}^N, v \in \bigwedge^\ell \mathbb{R}^N).$$

By duality, the co-differential $\delta$ is the first-order operator from $\bigwedge^{\ell+1} \mathbb{R}^N$ to $\bigwedge^\ell \mathbb{R}^N$ associated to the symbol

$$\delta(\xi) = (-1)^\ell \star (\xi \wedge \star v),$$

where $\star$ is the Hodge-star operator acting on alternating forms. The Laplace–Beltrami operator is the 2nd order operator, from $\bigwedge^\ell \mathbb{R}^N$ to $\bigwedge^\ell \mathbb{R}^N$ defined as

$$\Delta u := \delta du + d\delta u,$$

for which we have the following estimate:
Proposition 22. There exists a constant $C = C(p, N, \ell)$ such that
\[
\inf_{\Delta w = 0, w \in L^p(\Omega:\wedge^\ell \mathbb{R}^N)} \|u - w\|_{W^{2,p}(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)}
\]
for all $p$-integrable $\ell$-form fields $u : \Omega \rightarrow \wedge^\ell \mathbb{R}^N$.

Proof. Let us fix $\xi \in \mathbb{R}^N$ a non-zero vector. First, let recall that $\delta(\xi)$ is the adjoint $d(\xi)$ so that $\Delta(\xi) = (d, \delta)^* \circ (d, \delta)$. Since $\Delta(\xi) : \wedge^\ell \mathbb{R}^N \rightarrow \wedge^\ell \mathbb{R}^N$, it suffices to observe that $(d(\xi), \delta(\xi))$ is injective, for then it follows that $\Delta(\xi)$ is onto and one can apply the the results of Thm. 1. Indeed, $a \in \ker(d(\xi), \delta(\xi))$ if and only if both $\xi \wedge a$ and $\xi \wedge (\star a)$ are zero, which implies that $a = 0$ as desired. \hfill \Box

4.5. Elliptic systems. A $k$th order operator on $\mathbb{R}^N$, from $V$ to $W$, is called elliptic provided that its principal symbol is injective for all non-zero frequencies, or equivalently, that
\[
\forall \xi \in \mathbb{R}^N, \quad |A(\xi)[v]| \geq c|\xi|^k|v|
\]
for some $c > 0$.

Definition 23 (Elliptic system). An elliptic system (not to be confused with an elliptic system of equations) is an elliptic operator with as many indeterminate coordinates as its number of linearly independent equations, that is, an elliptic operator from $V$ to $W$, with
\[
\dim(V) = \dim(W).
\]

Note that elliptic systems are maximal-rank constant-rank operators, which yields the following projection estimates for elliptic systems:

Proposition 24. Let $A$ be a $k$th order elliptic system on $\mathbb{R}^N$, from $V$ to $W$. Then,
\[
\inf_{\Delta w = 0, v \in L^p(\Omega;V)} \|u - v\|_{W^{k,p}(\Omega)} \leq C(p, A, \Omega) \|Au\|_{L^p(\Omega;W)}
\]
for all $u \in L^p(\Omega;V)$.

Example 25 (The deviatoric operator). The operator associated to the shear part of the symmetric gradient
\[
E_Du := \frac{1}{2}(Du + Du^T) - \frac{\text{div} u}{N}I_N, \quad u : \mathbb{R}^N \rightarrow \mathbb{R}^N,
\]
defines an elliptic operator on $\mathbb{R}^N$, from $\mathbb{R}^N$ to the space of real-valued $N \times N$ symmetric trace-free matrices. For $N = 2$, the space of trace-free symmetric $2 \times 2$ matrices has dimension 2 and hence $E_D$ defines an elliptic system for $N = 2$ (thus, also satisfying the projection estimate (19) on arbitrary...
domains). For $N \geq 3$, it can be shown that $E_D$ has a finite dimensional distributional null-space and therefore the projection estimate (19) also holds provided that $\Omega$ is a Lipschitz domain.

Example 26. The generalized Laplacian $\Delta_A$ associated to any elliptic operator $A$ from $V$ to $W$ defines an elliptic system from $V$ into itself. Indeed, $\Delta_A$ is also elliptic since it is $2k$-homogeneous and its principal symbol satisfies

$$\mathbb{A}(\xi)^* \circ \mathbb{A}(\xi)[a] \cdot a = |\mathbb{A}(\xi)[a]|^2 \geq C^2 |\xi|^{2k} |a|^2 \quad (\xi \in \mathbb{R}^N, a \in V).$$

Remark 27 (elliptic systems vs ADN systems). ADN systems (from $V$ to $V$) — as introduced by Agmon, Douglis and Nirenberg in [1, 2] — are maximal-rank elliptic operators. However, there exist elliptic systems $V$ to $V$ that fail to be ADN systems (take for instance the Cauchy-Riemann equations).

4.6. Adjoints of elliptic operators. Given an elliptic operator $A$ from $V$ to $W$, its associated formal adjoint $A^*$ from $W$ to $V$ defines an operator with maximal-rank principal symbol. Indeed, by definition the principal symbol of $A^*$ is given (up to a sign) by the algebraic adjoint $\mathbb{A}(\xi)^*$, and, since $\mathbb{A}(\xi)$ is injective for all non-zero $\xi \in \mathbb{R}^N$, a standard linear algebra argument implies that

$$\forall \xi \in \mathbb{R}^N - \{0\}, \quad \text{Im} \mathbb{A}(\xi)^* = \{\text{ker} \mathbb{A}(\xi)\}^\perp = \{0_V\}^\perp = V.$$  

The converse also holds: the formal adjoint of every elliptic operator defines a maximal-rank operator. In particular, the following estimates hold:

**Proposition 28.** Let $A$ be a $k^{th}$ order elliptic operator from $V$ to $W$. Then,

$$\inf_{v \in L^p(\Omega; W), \quad A^*v = 0} \|u - v\|_{W^{k,p}(\Omega)} \leq C(p, A^*, \Omega) \|A^*u\|_{L^p(\Omega)}.$$

for all $u \in L^p(\Omega; W)$.

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