Non-Markovian Particle Dynamics in Continuously Controlled Quantum Gases

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For a quantum gas, being subject to continuous feedback of a macroscopic observable, the single-particle dynamics is studied. Albeit feedback-induced particle correlations, it is shown that analytic solutions are obtained by formally extending the single-particle Hilbert space by an auxiliary degree of freedom. The particle’s motion is then fed by colored noise, which effectively maps quantum-statistical correlations onto the single particle. Thus, the single particle in the continuously controlled gas follows a non-Markovian trajectory in phase-space.

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As noted by Caves and Milburn \[1\], the continuous observation of an object inevitably results in an increasing uncertainty of its momentum, which may cause it to eventually escape from the region of observation. An experiment thus necessarily requires a mechanism to keep the object at fixed location in the laboratory. Such constraining forces can be covered under the general notion of feedback control.

Given limited observational capabilities, in most cases the macroscopic body is observed as a whole, its internal structure remaining unresolved. Clearly the same holds also for a continuous feedback process, where measurements and conditioned actions are usually considered as being decoupled, one might think that macroscopic observations have no effect on the internal dynamics of a system \[3\]. However, strong and high-order correlations emerge also due to measurement or feedback control of a macroscopic observable \[3\].

In this Letter we show, that substantial effects on the particles of a continuously feedback-controlled quantum gas emerge. We prove, that, despite the above mentioned problems of correlated systems, continuous feedback can be exactly solved for the single-particle dynamics. This dynamics contains effects due to particle correlations, that are formally introduced as colored noise feeding the single-particle motion.

Continuous measurement \[1\] and feedback \[5\] have been studied and experimentally realized \[6\] in the past for various systems, such as single atoms or single harmonic systems. Different from these treatments, we address here the case of a bosonic many-particle system and thus deal with the additional complexity due to particle correlations.

Consider an ideal gas where the constituents of mass \(m\) are indistinguishable, bosonic particles. In order to keep at all times the center of the cloud of particles at a predefined target position, a feedback loop is continuously applied to compensate for the motion of the center of mass of the gas. The feedback will act as a damping force and unavoidably also randomizes the motion of the center of mass.

The quantum dynamics due to the continuous feedback process has been shown \[1\] to be governed by the master equation of quantum Brownian motion \[3\],

\[
\dot{\hat{\vartheta}} = -\frac{i}{\hbar} [\hat{H}, \hat{\vartheta}] + \frac{\zeta}{2\hbar} [\hat{P}, \{\hat{X}, \hat{\vartheta}\}]
- \frac{1}{8\sigma^2} [\hat{X}, [\hat{X}, \hat{\vartheta}]] - \frac{\zeta^2\sigma^2}{2\hbar^2} [\hat{P}, [\hat{P}, \hat{\vartheta}]].
\]

Here \(\hat{H}\) is the Hamiltonian of the free evolution of the many-particle system, which we specify as linear dynamics of an ideal gas, and the additional terms are due to the feedback.

Continuous feedback is obtained as the limit of a discrete series of measurements and kicks of the center of mass \(\hat{X}\): In each discrete step \(\hat{X}\) is measured with resolution \(\sigma_0\) and then a kick is applied leading to a shift \(-\zeta_0\hat{X}\), where \(\hat{X}\) is the measured value. After that the free evolution takes place and the sequence is repeated. The continuous limit, where the average rate of feedbacks \(\gamma \to \infty\), then requires that \(\sigma_0 \to \infty\) and \(\zeta_0 \to 0\), such that \(\sigma = \sigma_0/\sqrt{\gamma}\) and \(\zeta = \zeta_0/\gamma\) remain constant.

Thus the parameters \(\sigma\) and \(\zeta\) in Eq. \(1\) describe the strengths of measurement and subsequent kick.

It should be emphasized that for the gas considered here, Eq. \(1\) describes the dynamics of the \textit{many-particle} density operator of the system \(\hat{\vartheta}\). Needless to say, that for the complete many-particle problem neither exact analytical nor numerical solutions exist. Nevertheless, it is straightforward to obtain solutions for collective variables, such as the center of mass \(\hat{X}(t)\) or the total momentum \(\hat{P}(t)\). In fact, these properties will lack any features due to the many-particle aspect, but will be identical to that of a single-particle system \(1\) of correspondingly larger mass.

The center-of-mass rms deviation of a harmonically trapped gas with trap frequency \(\omega_0\), for instance, will converge to the stationary value

\[\lim_{t \to \infty} \Delta X(t) = \Delta X_0 \sqrt{\frac{\eta + \eta^{-1}}{2}}.\]

Here \(\Delta X_0 = \sqrt{\hbar/(2M\omega_0)}\) is the ground-state width of the center of mass (total mass \(M\)) and \(\eta = (\Delta X_0)^2/(\zeta^2\sigma^2)\) denotes the ratio of spatial localization due to the potential over...
that due to the feedback. The stationary value is typically smaller than the initial value and thus indicates the gain in localization due to the feedback.

Different from a consideration of the controlled variables themselves, in this Letter we address a more subtle question. It is the question, how the single particle in the gas is affected by the continuous feedback. Various important properties of the gas, that may be derived from the single-particle behavior, for example its density profile, justify this approach.

The single-particles dynamics can in principle be deduced from Eq. (1), accompanied with the conceptual difficulties due to many-particle correlations. We proceed instead by mapping these correlations onto single-particle fluctuations, which will be shown to allow for analytical solutions.

Let us first consider the feedback in more detail in a second-quantized picture, using the bosonic matter field \( \phi(x) \) with commutator relation \([\phi(x), \phi^\dagger(x')] = \delta(x-x')\). The total momentum of particles in the gas, as used in Eq. (1), then reads

\[
\hat{P} = -i\hbar \int dx \, \hat{\phi}^\dagger(x) \frac{\partial}{\partial x} \hat{\phi}(x). \tag{3}
\]

It should be noted, that the operations of measurement and kick are typically implemented by use of external (e.g. optical) probe and control fields, that equally interact with all particles. Thus, the accessible properties of the gas are necessarily of extensive (i.e. additive) type.

The center of mass of the gas, however, is an intensive quantity: It is the ratio of summed particle positions over the particle number. In conclusion this variable cannot be accessed in the above mentioned way. Instead, a priori information on the particle number is required for rendering \( X \) quasi into an extensive quantity. Typically, the best information one can get on the particle number from some previous measurements may be its expectation value \( \langle N \rangle \). Using therefore this average particle number, the truly accessible (quasi) center-of-mass reads

\[
\hat{X} = \frac{1}{\langle N \rangle} \int dx \, \hat{\phi}^\dagger(x) \, x \, \hat{\phi}(x). \tag{4}
\]

Our goal is to gain information on the dynamics of a single particle in the gas, described by the time evolution of the reduced single-particle density matrix,

\[
\rho(x, x') = \langle \hat{\phi}^\dagger(x') \, \hat{\phi}(x) \rangle. \tag{5}
\]

Deriving the equation of motion of this density matrix from Eq. (1), an infinite hierarchy of coupled equations for particle correlations of increasing order would be obtained. This would not allow for a closed equation of motion for the single-particle density matrix. Of course, this is the central problem in many-particle physics, where only few model systems provide analytical solutions. In general one is forced to rely on approximations, such as the truncation of higher-order correlations.

In our case, the many-particle correlations do not decay with increasing order, they represent the strong correlating effects of the measurement of the collective variable \( X \). Any outcome of the measurement can be realized by a vast number of microstates compatible with the observed value \( X \). After the measurement all these microstates become parts of the highly correlated many-particle superposition state. In addition, the indistinguishability of bosonic particles leads to further quantum-statistical effects in this conditioned quantum state.

There have been solutions to continuous-feedback problems for many-particle systems, in effect however, considering distinguishable particles. In this letter we deal with bosons, requiring thus a conceptually different approach. One should note, that due to the indistinguishability, Eq. (5) is the expectable behavior of a representative particle in the gas and clearly not that of a specific (distinguished) one.

Given the above mentioned problems, we proceed by extending the Hilbert space beyond that of \( \rho(x, x') \), such as to obtain closed equations of motion. A replacement for Eq. (5) is searched for that, however, contains the complete information of \( \rho(x, x') \) and even more. A suitable candidate for such an approach is the following function

\[
W_N(x, p; X, P) = (2\pi\hbar)^{-\frac{N}{2}} \int dx' \int dX' \int dP' \int dS \times \exp\{-i [x'p + PX' + XP' + (N-1)S]/\hbar\} \times \left( \frac{\langle \hat{\phi}(x' + x') \rangle}{2} \right)^2. \tag{6}
\]

It can be interpreted as the Wigner function of a joint two-body system, conditioned on \( N \) particles being in the gas. The first body of the system is given by a single particle with phase-space variables \( x \) and \( p \). The second one is the center of mass of the “other” \( N-1 \) particles with phase-space variables \( X \) and \( P \).

Tracing Eq. (6) over the macroscopic variables \( N, X, \) and \( P \), the single-particle Wigner function, being equivalent to \( \rho(x, x') \), is obtained. However, integrating over \( x, p \) one does not obtain the true center-of-mass Wigner function. For a large average particle number \( \langle N \rangle \gg 1 \) though, the annihilation of only a single particle by the field operators in Eq. (6) can be neglected. Then Eq. (6) converges to the true two-body Wigner function of single particle and center-of-mass system.

We can show that for Eq. (6) a closed equation of motion can be derived from Eq. (1). In consequence, also the single-particle state \( \rho(x, x') \) can be obtained by tracing over the macroscopic variables \( N, X, P \). For particles of mass \( m \), being bound in a harmonic potential of frequency \( \omega_0 \), for example, \( W_N(x, p; X, P) \) obeys the following Fokker–Planck equation

\[
\begin{cases}
\partial_t + \frac{p \partial_x}{m} - m\omega_0^2 x \partial_p + \frac{P \partial_X}{M} - M\omega_0^2 X \partial_P \\
-\frac{\hbar^2}{8\sigma^2} \left( \Theta_N \partial_P + \frac{\partial_x}{\langle N \rangle} \right)^2 - \frac{\zeta^2\sigma^2}{2} (\Theta_N \partial_X + \partial_x)^2 \\
-\zeta (\Theta_N \partial_X + \partial_x) \left( X + \frac{x}{\langle N \rangle} \right) \end{cases} \tag{7}
\]
Here $\Theta_N = (N-1) \langle \dot{N} \rangle$, the total mass is $M = m \langle N \rangle$, and $N$ plays the role of a parameter. This Fokker–Planck equation is of linear type with a positive semi-definite diffusion matrix. Thus a bound analytic solution for its Green function of Gaussian type can be found [11]. In consequence, the analytic solution for $\rho(x,x')$ can be obtained in a rather straightforward way from the (analytic) solution of Eq. (7).

For getting more insight into the dynamics described by Eq. (7), we now turn to the equivalent $N$-parameterized stochastic differential equations, that read

$$dx_N = \left[ \frac{P_N}{m} - \zeta \left( \frac{x_N}{\langle N \rangle} + X_N \right) \right] dt + \zeta \sigma d\xi_1,$$

$$dp_N = -m\omega_0^2 x_N dt + \frac{h}{2\sigma\langle N \rangle} d\xi_2,$$

$$dX_N = \left[ \frac{P_N}{M} - \zeta \Theta_N \left( \frac{x_N}{\langle N \rangle} + X_N \right) \right] dt + \Theta_N \zeta \sigma d\xi_1,$$

$$dP_N = -M\omega_0^2 X_N dt + \frac{h\Theta_N}{2\sigma} d\xi_2,$$

with $\xi_1, \xi_2$ being statistically independent Wiener processes.

Besides the free evolution of the single particle and of the (quasi) center of mass, these equations also contain a feedback-induced coupling between the microscopic single-particle and the macroscopic center-of-mass degree of freedom. Thus our method of solution, that incorporates all many-particle and the macroscopic center-of-mass degree of freedom, was obtained by allowing for a coupling with an auxiliary macroscopic degree of freedom.

It is important to note, that since the drift matrix of Eqs (8)–(11) is not of normal form, it is impossible to diagonalize this set of dynamical equations. Furthermore, both systems are fed by the same noise sources, which is apparent since both macroscopic and microscopic systems are subject to the same feedback process.

When tracking the motion of both systems, i.e. microscopic and macroscopic, Eqs (8)–(11) generate Markovian trajectories in the four-dimensional phase space. The macroscopic system, however, being the center of mass of the $N-1$ “other” particles, is not a physically accessible observable. Thus it appears natural to eliminate this auxiliary degree of freedom, to obtain stochastic differential equations for the macroscopic trajectory alone. Let us perform this elimination for the case of large particle numbers:

For $\langle N \rangle \rightarrow \infty$, due to its much larger inertia, the (quasi) center-of-mass system will no longer be affected by the motion of the single particle. Thus in Eq. (11) the term proportional to $x_N/\langle N \rangle$ can be discarded, which decouples the macroscopic system. Consequently, the formal solution $X_N(t)$ is then obtained as Ornstein–Uhlenbeck process. Inserting it into Eq. (8), one is finally left with equations of motion for $x_N$ and $p_N$ alone.

The modifications of Eq. (8) due to the elimination procedure are formally done by two replacements: $X_N$ in Eq. (8) becomes the deterministic part of the solution $X_N(t)$, which now acts as an external drive. And the Wiener increment $d\xi_1(t)$ is replaced by $d\xi_N(t)$, which is now a sum of a Wiener and an Ornstein–Uhlenbeck process:

$$d\xi_N(t) = d\xi_1(t) + \frac{h}{2\sigma\langle N \rangle} d\xi_2.$$

$$d\xi_2(t) = \frac{\eta d\xi_2(t') - \alpha_N d\xi_1(t')}{\sqrt{1 - \alpha_N^2}}$$

$$+ \frac{\eta d\xi_2(t') - \alpha_N d\xi_1(t')}{\sqrt{1 - \alpha_N^2}} \sin[\Omega_N(t-t')] e^{-\Gamma_N(t-t')}.$$

Here $\Omega_N = \omega_0^2 - \Gamma_N^2$ and $\alpha_N = \Gamma_N/\omega_0$, with $\Gamma_N = \zeta \Theta_N/\alpha$ being the feedback-induced damping of the coherent oscillation in the trap potential.

To characterize this effective noise source we consider the Fourier transform of the stationary correlation function

$$S_N(\omega) = \lim_{t \rightarrow \infty} \int dt \ e^{i \omega t} \overline{\xi_N(t+\tau)\xi_N(t)}.$$

which is a symmetric spectrum depending on $\Omega_N$ only via the parameter $\alpha_N$. From Figs. 1 and 2 it is observed that only in the high-frequency limit the white-noise background $S_N(\omega) = 1$ is reached. At low frequencies several extrema appear depending on the values of $\alpha_N$ and $\eta$ (or instead $\zeta$).

For weak feedback-induced damping, $\alpha_N < 1/\sqrt{2}$, the peaks at the damped center-of-mass oscillation frequencies $\pm \Omega_N$ can be resolved under the condition

$$\eta > 1/\sqrt{2 - 4\alpha_N^2},$$

see Fig. 1. Such large values of $\eta$ represent strong feedback-induced localization of the center of mass. For smaller values of $\eta$ two peaks overlap and only a single maximum at $\omega = 0$ remains (see dashed curve in Fig. 1). A remarkable effect, however, appears as minima are formed at frequencies larger than $\omega_0$. These minima represent noise reduction below the white-noise background and are generated by self-correlation of white-noise with its modulated version, as seen from the contributions to Eq. (12).
For \( \alpha_N \geq 1/\sqrt{2} \), i.e. for large damping, the maxima at \( \pm \Omega_N \) can never be resolved, leaving the single maximum at \( \omega = 0 \). Also in this case symmetric minima at noise levels below unity appear at frequencies slightly higher than the trap frequency \( \omega_0 \), cf. Fig. 2.

This noise reduction may be understood as follows: Both the single particle and the (quasi) center of mass are subject to the same feedback-generated noise. With some time delay the center of mass transfers its noise to the single particle via the coupling, cf. Eqs. 8 and 10. Thus the effective noise seen by the single particle at the frequencies of the minima is partially compensated due to destructive phase shifts of the two paths of the noise input.

Since the noise feeding the single-particle coordinate is now colored, as seen from the previous discussion, the resulting stochastic trajectories in single-particle phase space will be non-Markovian. This feature is due to the feedback-generated strong many-particle correlations, that are now cast into an effective noise source for the single particle. Thus the coloredness of the noise 12 is a manifestation of generation of correlations between the particles.

One may interpret this also as a feedback-mediated effective interaction of the single particle with the cloud of surrounding particles, that bears a non-vanishing correlation time. The latter may be quantified by the inverse spectral width of Eq. 13. Thus the memory effects in the particle cloud can be made responsible for the non-Markovian trajectory of the single particle.

Summarizing, we have shown that a specific many-particle problem, that is dominated by strong particle correlations can be analytically solved for the reduced single-particle dynamics. The solution was obtained by first extending the Hilbert space of a single particle by an auxiliary degree of freedom to obtain analytically solvable closed equations of motion. Eliminating the auxiliary degree of freedom, it could be shown that the single particle follows a non-Markovian trajectory in phase space. Thus the many-particle correlations have been mapped onto a coloredness of the noise feeding the particle’s motion.

The specific problem, for which this solution was obtain is quite general, in that it represents typical control of a system on the macroscopic level. In conclusion even such a macroscopic intervention creates correlations and a resulting non-Markovian behavior of single particles inside the controlled system.

One may ask for a generalization of our approach: Is it generally possible to map many-particle correlations onto specific features of noise sources feeding the motion of single particles? Clearly in the presented case, the correlations were not generated by true particle interactions but by continuous feedback. Thus an immediate application to problems of interacting particles seems not obvious. Considering however, that we can interpret the solved system equally well as quantum Brownian motion of scattering, and thus interacting, particles, a potential route to a more general methodology may come into sight.

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1. C.M. Caves and G.J. Milburn, Phys. Rev. A 36, 5543 (1987).
2. E.g. V.E. Korepin, N.M. Bogoliubov, and A.G. Izergin, Quantum inverse scattering method and correlation functions (Cambridge University Press, 1993).
3. S. Wallentowitz, Phys. Rev. A 66, 032114 (2002).
4. M.B. Mensky, Phys. Rev. D 20, 384 (1979); A. Barchielli, L. Lanz, and G.M. Prosperi, Nuovo Cimento B 72, 79 (1982); C.M. Caves, Phys. Rev. D 33, 1643 (1986); 35, 1815 (1987).
5. H.M. Wiseman and G.J. Milburn, Phys. Rev. Lett. 70, 548 (1993); H.M. Wiseman, Phys. Rev. A 49, 2133 (1994).
6. J.A. Dunningham, H.M. Wiseman, and D.F. Walls, Phys. Rev. A 55, 3980 (1997); S. Mancini and P. Tombesi, ibid. 56, 2466 (1997); S. Mancini, D. Vitali, and P. Tombesi, Phys. Rev. Lett. 80, 688 (1998); D. Vitali, P. Tombesi, and G.J. Milburn, Phys. Rev. A 57, 4930 (1998); S. Mancini, D. Vitali, and P. Tombesi, iberdi 61, 053404 (2000).
7. T. Fischer et al., Phys. Rev. Lett. 88, 163002 (2002); N.V. Morrow, S.K. Dutta, and G. Raithel, ibid. 88, 093003 (2002).
8. A. Barchielli, Nuovo Cimento B 74, 113 (1983); M.B. Mensky and S. Stenholm, Phys. Lett. A 308, 243 (2003).
9. For other descriptions of Brownian motion, see e.g. A.O. Caldeira and A.J. Leggett, Physica A 121, 587 (1983); L. Diosi, Europhys. Lett. 22, 1 (1993); C.W. Gardiner and P. Zoller, Quantum noise (Springer, Berlin, 2000), 2nd edition.
10. L.K. Thomsen, S. Mancini, and H.M. Wiseman, Phys. Rev. A 65, 061801(R) (2002).
11. H. Risken, The Fokker–Planck equation (Springer, Berlin, 1996) 2nd edition.
12. M. Arndt et al., Nature 401, 680 (1999).
13. The inverse relation, e.g. having coherence in the center of mass of fullerenes despite internal degrees of freedom being excited may suggest this viewpoint too.
14. Note, that if a truly extensive quantity would be measured, this feature would appear instead in the kick operation and thus can-
not be avoided in principle.