On \( \mu \)-Symmetric Polynomials

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Abstract. In this paper, we study functions of the roots of a univariate polynomial in which the roots have a given multiplicity structure \( \mu \). Traditionally, root functions are studied via the theory of symmetric polynomials; we extend this theory to \( \mu \)-symmetric polynomials. We were motivated by a conjecture from Becker et al. (ISSAC 2016) about the \( \mu \)-symmetry of a particular root function \( D^+(\mu) \), called D-plus. To investigate this conjecture, it was desirable to have fast algorithms for checking if a given root function is \( \mu \)-symmetric. We designed three such algorithms: one based on Gröbner bases, another based on preprocessing and reduction, and the third based on solving linear equations. We implemented them in Maple and experiments show that the latter two algorithms are significantly faster than the first.

Key words. \( \mu \)-symmetric polynomial, multiple roots, symmetric function, D-plus discriminant, gist polynomial, lift polynomial

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1. Introduction. Suppose \( P(x) \in \mathbb{Z}[x] \) is a polynomial with \( m \) distinct complex roots \( r_1, \ldots, r_m \) where \( r_i \) has multiplicity \( \mu_i \). Write \( \mu = (\mu_1, \ldots, \mu_m) \) where we may assume \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 1 \). Thus \( n = \sum_{i=1}^{m} \mu_i \) is the degree of \( P(x) \). Consider the following function of the roots

\[
D^+(P(x)) := \prod_{1 \leq i < j \leq m} (r_i - r_j)^{\mu_i + \mu_j}.
\]

Call this the D-plus root function. The form of this root function was introduced by Becker et al. in their complexity analysis of a root clustering algorithm. The origin of this paper was to try to prove that \( D^+(P(x)) \) is a rational function in the coefficients of \( P(x) \). This result is needed for obtaining an explicit upper bound on the complexity of the algorithm on integer polynomials [2]. This application is detailed in our companion paper [5].

We may write \( "D^+(\mu)" \) instead of \( D^+(P(x)) \) since the expression in terms of the roots \( r = (r_1, \ldots, r_m) \) depends only on the multiplicity structure \( \mu \). For example, if \( \mu = (2, 1) \) then

\[
D^+(P(x)) := \prod_{1 \leq i < j \leq m} (r_i - r_j)^{\mu_i + \mu_j}.
\]
Throughout the paper, we use $#$ to denote the number of elements in a set, and $|\cdot|$ to denote the length of a sequence. In particular,

$$
\sigma_{\mu} = \mu_1 \mu_2 \cdots \mu_r
$$

representing the elementary symmetric functions of the roots, and where $n \geq 0$. For our purposes, $K$ is a field of characteristic 0. We also fix three sequences of variables

$$
x = (x_1, \ldots, x_n), \quad z = (z_1, \ldots, z_r), \quad r = (r_1, \ldots, r_m)
$$

where $n \geq m \geq 1$. Intuitively, the $x_i$’s are roots (not necessarily distinct), $z_i$’s are variables representing the elementary symmetric functions of the roots, and $r_i$’s are the distinct roots.

Let $\mu$ be a partition of $n$ with $m$ parts. In other words, $\mu = (\mu_1, \ldots, \mu_m)$ where $n = \mu_1 + \cdots + \mu_m$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 1$. We denote this relation by

$$
\mu \vdash n.
$$

We call $\mu$ an $m$-partition if it has $m$ parts. A specialization $\sigma$ is any function of the form $\sigma : \{x_1, \ldots, x_n\} \rightarrow \{r_1, \ldots, r_m\}$. We say $\sigma$ is of type $\mu$ if $\#\sigma^{-1}(r_i) = \mu_i$ for $i = 1, \ldots, m$. Throughout the paper, we use $#$ to denote the number of elements in a set, and $|\cdot|$ to denote the length of a sequence. In particular, $|\mu| = |r| = m$. We say $\sigma$ is canonical if $\sigma(x_i) = r_j$.

$D^+(\mu) = (r_1 - r_2)^3$ and this turns out to be

$$
[a_1^2 - (9/2)a_0a_1a_2 + (27/2)a_0^2a_3] / a_0^3
$$

when $P(x) = \sum_{i=0}^{3} a_{3-i}x^i$. More generally, for any function $F(r) = F(r_1, \ldots, r_m)$, we ask whether evaluating $F$ at the $m$ distinct roots of a polynomial $P(x)$ with multiplicity structure $\mu$ is rational in the coefficients of $P(x)$. The Fundamental Theorem of Symmetric Functions gives a partial answer: if $F(r)$ is a symmetric polynomial then $F(r)$ is a rational function in the coefficients of $P(x)$. This result does not exploit knowledge of the multiplicity structure $\mu$ of $P(x)$. We want a natural definition of “$\mu$-symmetry” such that the following property is true: if $F(r)$ is $\mu$-symmetric, then $F(r)$ is a rational function in the coefficients of $P(x)$. When $\mu = (1, \ldots, 1)$, i.e., all the roots of $P(x)$ are simple, then a $\mu$-symmetric polynomial is just a symmetric polynomial in the usual sense. So our original goal amounts to proving that $D^+(\mu)$ is $\mu$-symmetric. It is non-trivial to check if any given root function $F$ (in particular $F = D^+(\mu)$) is $\mu$-symmetric. We will designed three algorithms for this task. Although we feel that $\mu$-symmetry is a natural concept, to our knowledge, this has not been systematically studied before.

The rest of this paper is organized as follows. In Section 2, we defined $\mu$-symmetric polynomials in terms of elementary symmetric polynomials and show some preliminary properties of such polynomials. In Section 3, we proved the $\mu$-symmetry of $D^+$ for some special $\mu$. To investigate the $\mu$-symmetry of $D^+$ in the general case, three algorithms for checking $\mu$-symmetry are given in Sections 4-6. In Section 7, we discuss how to generalize the concepts and algorithms to other generators of symmetric polynomials different from the elementary symmetric polynomials. In Section 8, we show experimental results from our Maple implementation of the three algorithms. All the Maple code can be downloaded from https://github.com/JYangMATH/mu-symmetry. We conclude in Section 9.

The $D^+$ conjecture is proved in a companion paper [5] and an application is shown by giving an explicit complexity bound for root clustering.

2. $\mu$-Symmetric Polynomials. Throughout the paper, assume $K$ is a field of characteristic 0. For our purposes, $K = \mathbb{Q}$ will do. We also fix three sequences of variables

$$
x = (x_1, \ldots, x_n), \quad z = (z_1, \ldots, z_r), \quad r = (r_1, \ldots, r_m)
$$

where $n \geq m \geq 1$. Intuitively, the $x_i$’s are roots (not necessarily distinct), $z_i$’s are variables representing the elementary symmetric functions of the roots, and $r_i$’s are the distinct roots.

Let $\mu$ be a partition of $n$ with $m$ parts. In other words, $\mu = (\mu_1, \ldots, \mu_m)$ where $n = \mu_1 + \cdots + \mu_m$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 1$. We denote this relation by

$$
\mu \vdash n.
$$

We call $\mu$ an $m$-partition if it has $m$ parts. A specialization $\sigma$ is any function of the form $\sigma : \{x_1, \ldots, x_n\} \rightarrow \{r_1, \ldots, r_m\}$. We say $\sigma$ is of type $\mu$ if $\#\sigma^{-1}(r_i) = \mu_i$ for $i = 1, \ldots, m$. Throughout the paper, we use $#$ to denote the number of elements in a set, and $|\cdot|$ to denote the length of a sequence. In particular, $|\mu| = |r| = m$. We say $\sigma$ is canonical if $\sigma(x_i) = r_j$.
and \(\sigma(x_{i+1}) = r_k\) implies \(j \leq k\). Clearly the canonical specialization of type \(\mu\) is unique, and we may denote it by \(\sigma_\mu\).

Consider the polynomial rings \(K[x]\) and \(K[r]\). Any specialization \(\sigma: \{x_1, \ldots, x_r\} \rightarrow \{r_1, \ldots, r_m\}\) can be extended naturally into a \(K\)-homomorphism

\[
\sigma: K[x] \rightarrow K[r]
\]

where \(P = P(x) \in K[x]\) is mapped to \(\sigma(P) = P(\sigma(x_1), \ldots, \sigma(x_n))\). When \(\sigma\) is understood, we may write “\(\tilde{P}\)” for the homomorphic image \(\sigma(P)\).

We denote the \(i\)-th elementary symmetric functions \((i = 1, \ldots, n)\) in \(K[x]\) by \(e_i = e_i(x)\). For instance,

\[
e_1 := \sum_{i=1}^{n} x_i,
\]

\[
e_2 := \sum_{1 \leq i < j \leq n} x_ix_j,
\]

\[
\vdots
\]

\[
e_n := \prod_{i=1}^{n} x_i.
\]

Also define \(e_0 := 1\). Typically, we write \(\bar{e}_i\) for the \(\sigma_\mu\) specialization of \(e_i\) when \(\mu\) is understood from the context; thus \(\bar{e}_i = \sigma_\mu(e_i) \in K[r]\). For instance, if \(\mu = (2, 1)\) then \(\bar{e}_1 = 2r_1 + r_2\) and \(\bar{e}_2 = r_1^2 + 2r_1r_2\).

The key definition is the following: a polynomial \(F \in K[r]\) is said to be \(\mu\)-symmetric if there is a symmetric polynomial \(\hat{F} \in K[x]\) such that \(\sigma_\mu(\hat{F}) = F\). We call \(\hat{F}\) the \(\mu\)-lift (or simply “lift”) of \(F\). If \(\hat{F} \in K[z]\) satisfies \(\hat{F}(e_1, \ldots, e_n) = \hat{F}(x)\) then we call \(\hat{F}\) the \(\mu\)-gist of \(F\).

**Remark 2.1.** (i) We may also write \((F)\wedge\) for any lift of \(F\). Note that the \(\mu\)-lift and \(\mu\)-gist of \(F\) are defined if and only if \(F\) is \(\mu\)-symmetric.

(ii) We view the \(z_i\)'s as symbolic representation of the symmetric polynomials \(e_i(x)\)'s. Moreover, we can write \(\sigma_\mu(\hat{F}(e_1, \ldots, e_n))\) as \(\hat{F}(\bar{e}_1, \ldots, \bar{e}_n)\).

(iii) Since \(\hat{F}(e_1, \ldots, e_n)\) is symmetric in \(x_1, \ldots, x_n\), we could use any specialization \(\sigma\) of type \(\mu\) instead of the canonical specialization \(\sigma_\mu\), since \(\sigma(\hat{F}(e_1, \ldots, e_n)) = \sigma_\mu(\hat{F}(e_1, \ldots, e_n))\).

(iv) Although \(\hat{F}\) and \(\hat{F}\) are mathematically equivalent, the gist concept lends itself to direct evaluation based on coefficients of \(P(x)\).

**Example 1.** Let \(\mu = (2, 1)\) and \(F(r) = 3r_1^2 + r_2^2 + 2r_1r_2\). We see that \(F(r)\) is \(\mu\)-symmetric since \(F(r) = (2r_1 + r_2)^2 - (r_1^2 + r_1r_2) = \bar{e}_2^2 - \bar{e}_2 = \sigma_\mu(e_1^2 - e_2)\). Hence lift of \(F\) is \(\hat{F} = e_1^2 - e_2 = (x_1 + x_2 + x_3)^2 - (x_1x_2 + x_1x_3 + x_2x_3)\) and its gist is \(\hat{F}(z) = z_1^2 - z_2\).

We have this consequence of the Fundamental Theorem on Symmetric Functions:

**Proposition 1.** Assume

\[
P(x) = \sum_{i=0}^{n} c_i x^{n-i} \in K[x]
\]
has $m$ distinct roots $\rho = (\rho_1, \ldots, \rho_m)$ of multiplicity $\mu = (\mu_1, \ldots, \mu_m)$.

(i) If $F \in K[r]$ is $\mu$-symmetric, then $F(\rho)$ is an element in $K$.

(ii) If $\hat{F} \in K[z]$ is the $\mu$-gist of $F$, then

$$F(\rho_1, \ldots, \rho_m) = \hat{F}(-c_1/c_0, \ldots, (-1)^n c_n/c_0).$$

Proof. Let $q = (\rho_1, \ldots, \rho_m)$.

$$F(\rho_1, \ldots, \rho_m) = \sigma_\mu(\hat{F}(x))$$

(by definition of $\mu$-symmetry)

$$= \sigma_\mu(\hat{F}(e_1, \ldots, e_n))$$

(by the Fundamental Theorem of Symmetric Functions, as $\hat{F}$ is symmetric)

$$= \hat{F}(\tau_1, \ldots, \tau_n)$$

(since $\tau_i = \sigma_\mu(e_i)$)

$$= \hat{F}(\tau_1(\rho), \ldots, \tau_n(\rho))$$

$$= \hat{F}(-c_1/c_0, \ldots, (-1)^n c_n/c_0)$$

(by Vieta’s formula for roots)

This proves the formula in (ii). The assertion of (i) follows from the fact that $\hat{F} \in K[z]$ and $c_i$’s belong to $K$. Q.E.D.

Example 2. Consider the polynomial $F(r_1, r_2)$ in Example 1. Suppose the polynomial $P(x) = c_0 x^3 + \cdots + c_3 \in K[x]$ has two distinct roots $\rho_1$ and $\rho_2$ of multiplicities 2 and 1, respectively. Then Proposition 1 says that $F(\rho_1, \rho_2) = 3\rho_1^2 + 2\rho_1 \rho_2$ is equal to

$$\hat{F}(-c_1/c_0, c_2/c_0, -c_3/c_0) = (-c_1/c_0)^2 - c_2/c_0 \in K$$

since $\hat{F}(z_1, z_2, z_3) = z_1^2 - z_2$.

It is an interesting question to prove some converse of Proposition 1. We plan to take this up in a future work.

2.1. On Lifts and the $\mu$-Ideal. We want to study the lift $\hat{F} \in K[z]$ of a $\mu$-symmetric polynomial $F \in K[r]$ of total degree $\delta$. If we write $F$ as the sum of its homogeneous parts, $F = F_0 + \cdots + F_\delta$, then $\hat{F} = \hat{F}_0 + \cdots + \hat{F}_\delta$. Hence, we may restrict $F$ to be homogeneous.

Next consider a polynomial $H(z) \in K[z]$. Suppose there is a weight function

$$\omega : \{z_1, \ldots, z_n\} \to \mathbb{N} = \{1, 2, \ldots\}$$

then for any term $t = \prod_{i=1}^n z_i^{d_i}$, its $\omega$-degree is $\sum_{i=1}^n d_\omega(z_i)$. Normally, $\omega(z_i) = 1$ for all $i$; but in this paper, we are also interested in the weight function where $\omega(z_i) = i$. For short, we simply call this $\omega$-degree of $t$ its weighted degree, denoted by $\omega$-$\deg(t)$. The weighted degree of a polynomial $H(z)$ is just the maximum weighted degree of terms in its support, denoted by $\omega$-$\deg(H)$. A polynomial $H(z)$ is said to be weighted homogeneous or $\omega$-homogeneous if all of its terms have the same weighted degree. Note that the weighted degree of a polynomial $H \in K[z]$ is the same as the degree of $H(e_1, \ldots, e_n) \in K[x]$.

The gist $\hat{F}$ of $F$ is not unique: for any gist $\hat{F}$, we can decompose it as $\hat{F} = \hat{F}_0 + \hat{F}_1$ where $\hat{F}_0$ is the weighted homogeneous part of $F$ of degree $\delta$, and $\hat{F}_1 := \hat{F} - \hat{F}_0$. Then $\hat{F}(\tau_1, \ldots, \tau_n) = F$ implies that $\hat{F}_0(\tau_1, \ldots, \tau_n) = F$ and $\hat{F}_1(\tau_1, \ldots, \tau_n) = 0$. We can always omit $\hat{F}_1$ from the gist of $F$. We shall call any polynomial $H(z) \in K[z]$ a $\mu$-constraint if $H(\tau_1, \ldots, \tau_n) = 0$. Thus, $\hat{F}_1$ is a $\mu$-constraint.
It follows that when trying to check if $F$ is $\mu$-symmetric, it is sufficient to look for gists $\hat{F}$ among weighted homogeneous polynomials of the same degree as $F$, i.e., $\delta$. But even this restriction does not guarantee uniqueness of the gist of $F$ because there could be $\mu$-constraints of weighted homogeneous degree $\deg(F)$. To illustrate this phenomenon, we consider the following example.

**Example 3.** Let $\mu = (2, 2)$. Consider the polynomial $F = r_1^3 + 2r_1^2r_2 + 2r_1r_2^2 + r_2^3$. It is easy to verify that both $\hat{F} = \frac{1}{8}e_1^3 - \frac{1}{2}e_3$ and $\hat{F}' = \frac{1}{2}e_1e_2 - \frac{3}{2}e_3$ are the lifts of $F$. Therefore, $\hat{F} = \frac{1}{8}e_1^3 - \frac{1}{2}e_3$ and $\hat{F}' = \frac{1}{2}z_1z_2 - \frac{3}{2}z_3$ are the gists of $F$. It follows that the difference

$$H = \hat{F} - \hat{F}' = \frac{1}{8}(z_1^3 + 8z_3 - 4z_1z_2)$$

is a $\mu$-constraint. We may check that

$$H(e_1, \ldots, e_4) = \frac{1}{8}(2r_1 + 2r_2)^3 + (2r_1^2r_2 + 2r_1r_2^2) - \frac{1}{2}(2r_1 + 2r_2)(r_1^2 + 4r_1r_2 + r_2^2) = 0.$$

It is easy to check that the set of all $\mu$-constraints forms an ideal in $K[z]$ which we may call the $\mu$-ideal, denoted by $J_\mu$. Note that $H(e_1, \ldots, e_n)$ is in $K[r]$ but $H$ is in $K[z]$. So we introduce an ideal in $K[z, r]$ to connect them:

$$I_\mu := \langle z_1 - e_1, \ldots, z_n - e_n \rangle.$$

Actually $J_\mu$ can be generated by $I_\mu$ as indicated by Theorem 4.3.

**Example 4.** The following set of polynomials generates the $(2, 2)$-ideal:

- $G_3 : z_1^3 - 4z_1z_2 + 8z_3$
- $G_4 : z_1^2z_2 + 2z_1z_3 - 4z_2^2 + 16z_4$
- $G_5 : z_1^2z_3 + 8z_1z_4 - 4z_2z_3$
- $G_6 : z_1^2z_4 - z_3^2$
- $G_7 : 4z_1z_2z_4 - z_1z_3^2 - 8z_3z_4$
- $G_8 : 8z_1z_2z_3z_4 - 4z_2^2z_4 + z_2z_3^2 + 16z_4^2$
- $G_9 : 8z_1z_2^2 - 4z_2z_3z_4 + z_3^3$
- $G_{10} : z_1z_3^3 - 8z_2^2z_4 + 2z_2^2z_3 + 32z_2z_3^2 + 8z_3z_4$
- $G_{11} : 16z_2^2z_4^2 - 8z_2z_3^2z_4 + z_3^4 - 64z_4^3$

We computed this by first computing the Gröbner basis of the ideal

$$\langle z_1 - e_1, z_2 - e_2, z_3 - e_3, z_4 - e_4 \rangle = \langle z_1 - (2r_1 + 2r_2), z_2 - (r_1^2 + 4r_1r_2 + r_2^2), z_3 - (2r_1^2r_2 + 2r_1r_2^2), z_4 - r_1^2r_2^2 \rangle.$$

By Theorem 4.3, the restriction of the Gröbner basis to $K[z]$ is the above set of generators.
2.2. Examples of $\mu$-symmetric Polynomials. Although $\mu$-symmetric polynomials originated from symmetric polynomials, they differ in many ways as seen in these examples.

- A $\mu$-symmetric polynomial need not be symmetric. Let $\mu = (2, 1)$ and $n = 2 + 1 = 3$. Then $2r_1 + r_2$ is $\mu$-symmetric whose lift is $e_1$, but it is not symmetric.
- A symmetric polynomial need not be $\mu$-symmetric. Consider the symmetric polynomial $F = r_1 + r_2 \in K[r_1, r_2]$. It is not $\mu$-symmetric with $\mu = (2, 1)$. If it were, then there is a linear symmetric polynomial $\hat{F} = ce_1$ such that $\sigma_\mu(\hat{F}) = r_1 + r_2$. But clearly such $\hat{F}$ does not exist.
- Symmetric polynomials can be $\mu$-symmetric. Note that $(r_1 - r_2)^2$ is obviously symmetric in $K[r_1, r_2]$. According to Lemma 2.2, it is also $\mu$-symmetric for any $\mu = (\mu_1, \mu_2)$.

In the following we will use this notation: $[n] := \{1, \ldots, n\}$, and let $\binom{[n]}{k}$ denote the set of all $k$-subsets of $[n]$. For $k = 0, \ldots, n - 2$, we may define the function

$$S^m_n = S^m_n(x) := \sum_{I \in \binom{[n]}{n-m}} \prod_{i \neq j \in I} (x_i - x_j)^2$$

called the $k$th subdiscriminant in $n$ variables. By extension, we could also define $S^0_{n-1} = 1$.

When $k = 0$, we have $S^0_n = \prod_{i \neq j \in [n]} (x_i - x_j)^2$. In applications, the $x_i$’s are roots of a polynomial $P(x)$ of degree $n$, and $S^0_n$ is the standard discriminant of $P(x)$. Clearly $S^0_n$ is a symmetric polynomial in $x$.

Lemma 2.2. Define $\Delta := \prod_{1 \leq i < j \leq m} (r_i - r_j)^2$.

(a) $\Delta$ is $\mu$-symmetric with lift given by

$$\hat{\Delta} = \frac{1}{\prod_{i=1}^{m} \mu_i} \cdot S^n_{n-m}$$

where $S^n_{n-m} \in K[x]$ is the $(n-m)$-th subdiscriminant.

(b) In particular, when $m = 2$, we have an explicit formula for the lift of $\Delta$:

$$\hat{\Delta} = \frac{(n-1)e_1^2 - 2ne_2}{\mu_1\mu_2},$$

where $n = \mu_1 + \mu_2$.

Proof. Let $\mu = (\mu_1, \ldots, \mu_m)$. Consider the $m$-th subdiscriminant $S^m_n$ in $n$ variables. We may verify that

$$\sigma_\mu(S^m_{n-m}) = \Delta \cdot \prod_{i=1}^{m} \mu_i.$$ 

This is equivalent to

$$\sigma_\mu \left( \frac{1}{\prod_{i=1}^{m} \mu_i} \cdot S^m_{n-m} \right) = \Delta.$$ 

Therefore, $\frac{1}{\prod_{i=1}^{m} \mu_i} \cdot S^m_{n-m}$ is the $\mu$-lift of $\Delta$. 
To obtain the explicit formula in the case \( m = 2 \), consider the symmetric polynomial 
\[
Q := \sum_{i<j} (x_i - x_j)^2.
\]
It is easy to check that \( Q = (n - 1)e_1^2 - 2ne_2 \). A simple calculation shows that 
\[
\sigma_{\mu}(Q) = \mu_1\mu_2(r_1 - r_2)^2.
\]
Thus, we may choose \( \hat{\Delta} = \frac{(n-1)e_1^2 - 2ne_2}{\mu_1\mu_2} \). Q.E.D.

3. Explicit Formulas for Special Cases of \( D^+ \). The following two theorems show the \( \mu \)-symmetry of some special \( D^+ \) polynomials. In other words, they confirmed our conjecture about \( D^+ \).

**Theorem 3.1.** There exists \( \hat{F}_n \in K[z] \) such that for all \( \mu \) satisfying \( \mu = (\mu_1, \mu_2) \) and \( \mu_1 + \mu_2 = n \), we have 
\[
\hat{F}_n(\tau_1, \tau_2) = D^+(\mu).
\]
More explicitly,
- \( n \) is even: 
  \[
  \hat{F}_n = \left( \frac{(n-1)z_1^2 - 2nz_2}{\mu_1\mu_2} \right)^{n/2}
  \]
- \( n \) is odd:
  \[
  \hat{F}_n = \left( \frac{(n-1)z_1^2 - 2nz_2}{\mu_1\mu_2} \right)^{n-3/2} \left( k_1z_1^3 + k_2z_1z_2 + k_3z_3 \right)
  \]
where \( k_1 = -\frac{(n-1)(n-2)}{d}, k_2 = \frac{3n(n-2)}{d}, k_3 = \frac{-3n^2}{d} \) and \( d = \mu_1\mu_2(\mu_1 - \mu_2) \).

**Proof.** From Lemma 2.2(b), we know that \((r_1 - r_2)^2\) is \( \mu \)-symmetric for arbitrary \( n \) and 
\[
(r_1 - r_2)^2 = \frac{(n - 1)e_1^2 - 2ne_2}{\mu_1\mu_2}.
\]
When \( n \) is even,
\[
D^+(\mu) = ((r_1 - r_2)^2)^{\frac{n}{2}} = \left( \frac{(n - 1)e_1^2 - 2ne_2}{\mu_1\mu_2} \right)^{\frac{n}{2}} = \hat{F}_n(\tau_1, \tau_2).
\]
Thus the case for even \( n \) is proved. It remains to prove the case for odd \( n \). First, it may be verified that 
\[
(r_1 - r_2)^3 = k_1\bar{e}_1^3 + k_2\bar{e}_1\bar{e}_2 + k_3\bar{e}_3,
\]
where
\[
k_1 = -\frac{(n - 1)(n - 2)}{d}, \quad k_2 = \frac{3n(n - 2)}{d}, \quad k_3 = \frac{-3n^2}{d} \quad \text{and} \quad d = \mu_1\mu_2(\mu_1 - \mu_2).
It follows that
\[ D^+(\mu) = \left((r_1 - r_2)^2\right)^{\frac{n-3}{2}} (r_1 - r_2)^3 \]
\[ = \left(\frac{(n-1)r_1^2 - 2nr_1}{\mu_1\mu_2}\right)^{\frac{n-3}{2}} \left(k_1r_1^2 + k_2r_1 + k_3r_3\right) \]
\[ = \left(\frac{(n-1)r_1^2 - 2nr_1}{\mu_1\mu_2}\right)^{\frac{n-3}{2}} \left(k_1r_1^2 + k_2r_1 + k_3r_3\right) \]
\[ = D_n(\tau_1, \tau_2, \tau_3) \]

where
\[ k_1 = \frac{-(n-1)(n-2)}{d}, \quad k_2 = \frac{3n(n-2)}{d}, \quad k_3 = \frac{-3n^2}{d} \quad \text{and} \quad d = \mu_1\mu_2(\mu_1 - \mu_2). \]

Q.E.D.

Another special case of $D^+(\mu)$ is where $\mu = (\mu, \mu, \ldots, \mu)$.

**Theorem 3.2.** If all $\mu_i$'s are equal to $\mu$, then $D^+(\mu)$ is $\mu$-symmetric with lift given by
\[ \hat{F}_n(x) = \left(\frac{1}{\mu^n} \cdot S_{n-m}^n\right)^\mu \]
where $S_{n-m}^n$ is given by Lemma 2.2(a).

**Proof.** Since $\mu_i = \mu$ ($1 \leq i \leq m$),
\[ D^+(\mu) = \prod_{i<j}(r_i - r_j)^{2\mu} = \left(\prod_{i<j}(r_i - r_j)^2\right)^\mu. \]

This expression for $D^+$ is $\mu$-symmetric since $\prod_{i<j}(r_i - r_j)^2$ is $\mu$-symmetric by Lemma 2.2(a). Moreover, Lemma 2.2(a) also shows that the lift of $\prod_{i<j}(r_i - r_j)^2$ is $\frac{1}{\mu^n} \cdot S_{n-m}^n$. Thus we may choose $\hat{F}_n = \left(\frac{1}{\mu^n} \cdot S_{n-m}^n\right)^\mu$.

Q.E.D.

The following example shows two ways to compute $D^+$. One is using the definition and the other is using the formula of $D^+$ in coefficients.

**Example 5.** Let $P(x) = (x^2 - x - 1)^2(x - 1) = (1, -3, 1, 3, -1, -1) \cdot (x^5, x^4, \ldots, x, 1)^T$. Then $(\rho_1, \rho_2, \rho_3) = (\phi, \tilde{\phi}, 1)$ are the roots with multiplicity $\mu = (2, 2, 1)$. Here $\phi = (1 + \sqrt{5})/2$ is the golden ratio and $\tilde{\phi} = 1 - \phi$ is its conjugate. It turns out that in this case, $D^+(\mu) = -25$ as directly computed from the formula in the roots ($\rho_1, \rho_2, \rho_3$). We can also compute it using the gist $D^+(z)$ of $D^+$, i.e., $D^+(\mu) = D^+(\tau_1, \ldots, \tau_5)$. Here is the gist of $D^+$ (which can be obtained from our algorithms below):

\[
\begin{align*}
D^+(z_1, z_2, z_3, z_4, z_5) &= \frac{10125}{4} z_5^2 - \frac{11}{2} z_1^2 z_2 z_3^2 - 3z_1^2 z_2 z_4 + 67z_1^3 z_3 z_4 - 207z_1^2 z_2 z_5 \\
&+ \frac{2517}{4} z_1^2 z_5 + 171z_1^2 z_3 z_5 - \frac{5955}{4} z_2 z_3 z_5 + 615z_1 z_4 z_5 \\
&- 184z_2 z_4^2 + 12z_5^5 z_5 + z_1^4 z_2^2 + 6z_2^2 z_3^2 + \frac{9}{2} z_1 z_3^3 + 48z_2^3 z_4 \\
&+ \frac{1737}{4} z_3^2 z_4 + \frac{277}{4} z_1^2 z_4 - 1255 z_1 z_2 z_3 z_4. 
\end{align*}
\]
According to Vieta’s formula for $n = 5$, $(c_1, \ldots, c_5) = (-c_1, c_2, -c_3, c_4, -c_5) = (3, 1, -3, -1, 1)$. Then, substituting $z_i$ by $\bar{c}_i = (-1)^i c_i$, we also obtain $D^+(2, 2, 1) = -25$.

4. Computing Gists via Gröbner Bases. In this section, we consider a Gröbner basis algorithm to compute the $\mu$-gist of a given polynomial $F \in K[r]$, or detect that it is not $\mu$-symmetric. In fact, we first generalize our concept of gist: fix an arbitrary (ordered) set $D = (d_1, \ldots, d_\ell)$, $d_i \in K[r]$. Call $D$ the basis. If $F \in K[r]$ and $\hat{F} \in K[y]$ where $y = (y_1, \ldots, y_\ell)$ are $\ell$ new variables, then $\hat{F}(y)$ is called a $D$-gist of $F$ if $F(r) = \hat{F}(d_1, \ldots, d_\ell)$. Note that if $D = (\bar{e}_1, \ldots, \bar{e}_n)$ (so $\ell = n$) then a $D$-gist is just a $\mu$-gist (after renaming $y$ to $z$).

We now give a method to compute a $D$-gist of $F$ using Gröbner bases. To this end, define the ideal $I_D := \langle v_1, \ldots, v_\ell \rangle \subseteq K[r, y]$ where $v_i := y_i - d_i$. Moreover, let $G_D$ be the Gröbner basis of $I_D$ relative to the the term ordering $\prec_{ry}$. The ordering is defined as follows:

$r^\alpha y^\beta \prec_{ry} r^\alpha' y^\beta'$

iff $r^\alpha \prec_r r^\alpha'$ or else $\alpha = \alpha'$ and $y^\beta \prec_y y^\beta'$. Here $\prec_r$ and $\prec_y$ are term orderings in $K[r]$ and $K[y]$ respectively. Note that $\prec_{ry}$ is called the lexicographic product of $\prec_r$ and $\prec_y$ in [R, §12.6]. We have two useful lemmas. The first is about the ideal $I_D$, and the second about its Gröbner basis $G_D$.

Lemma 4.1. For all $R \in K[y]$, 

$$R(y) - R(D) \in I_D.$$ 

Proof. Consider any term $y^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. Its image in the quotient ring $K[y]/I_D$ is:

$$y^\alpha + I_D = (\prod_{i=1}^\ell y_i^{\alpha_i}) + I_D$$
$$= (\prod_{i=1}^\ell (d_i + (y_i - d_i))^{\alpha_i}) + I_D$$
$$= (\prod_{i=1}^\ell d_i^{\alpha_i} + I_D) + I_D$$
$$= (\prod_{i=1}^\ell d_i^{\alpha_i} + I_D)$$
$$= D^{\alpha} + I_D.$$

Thus $y^\alpha - D^{\alpha} \in I_D$. Since $R(y) - R(D)$ is a linear combination of $y^\alpha - D^{\alpha}$’s, our lemma is proved. Q.E.D.
By a **weighted homogeneous ideal** we mean one that is generated by weighted homogeneous polynomials. The following is a generalization of Theorem 12.20, p.385, where the result is stated for homogeneous ideals.

The following is a consequence of Theorem 12.21, p.387:

**Lemma 4.2.** $\mathcal{G}_D \cap K[y]$ is a Gröbner basis for the elimination ideal $\mathcal{I}_D \cap K[y]$ with respect to the term ordering $\prec_y$.

If $R(\mathcal{D}) = 0$, then $R(y)$ is called a $\mathcal{D}$-constraint, which generalizes the concept of $\mu$-constraint. Similar to $\mu$-constraints, one may verify that all $\mathcal{D}$-constraints forms an ideal, denoted by $\mathcal{J}_D$. Then we have the following theorem.

**Theorem 4.3.** $\mathcal{J}_D = \mathcal{I}_D \cap K[y]$.

**Proof.** We will prove the theorem with the following two inclusions.

- $\mathcal{J}_D \subseteq \mathcal{I}_D \cap K[y]$.
  Consider any $R \in \mathcal{J}_D$. Then $R(\mathcal{D}) = 0$ implies $R(y) = R(y) - R(\mathcal{D}) \in \mathcal{I}_D$ by Lemma 4.1.

- $\mathcal{J}_D \supseteq \mathcal{I}_D \cap K[y]$.
  For any $R \in \mathcal{I}_D \cap K[y]$, $R \in \mathcal{I}_D$. Thus there exist $B_1, \ldots, B_n \in K[r, y]$ such that

$$R(y) = \sum_{i=1}^{n} (y_i - d_i) \cdot B_i.$$  

Substitution of $y_i = d_i$ leads to $R(d_1, \ldots, d_n) = 0$, which implies $R \in \mathcal{J}_D$. **Q.E.D.**

The following is a generalization of Proposition 4 in Cox [3, Chapter 7, Section 1] (except for claims about uniqueness):

**Theorem 4.4.** Fix the above Gröbner basis $\mathcal{G}_D$. Let $R \in K[r, y]$ be the normal form of $F \in K[r]$ relative to $\mathcal{G}_D$.

(i) If $R \in K[y]$, then $R$ is a $\mathcal{D}$-gist of $F$.

(ii) If $F$ has a $\mathcal{D}$-gist, then $R \in K[y]$.

**Proof.** In the following, we use the specialization $\sigma : y_i \mapsto d_i$ for all $i$. This induces the homomorphism $\sigma : K[r, y] \to K[r]$ taking every polynomial $f(r, y)$ in the ideal $\mathcal{I}_D$ to 0, i.e., $\sigma(f) = 0$.

(i) Since $R$ is the normal form of $F$, $F - R \in \mathcal{I}_D$. Thus $\sigma(F - R) = 0$ or $\sigma(F) = \sigma(R)$. But $F \in K[r]$ implies $\sigma(F) = F$. The assumption that $R \in K[y]$ implies that $\sigma(R) = R(\mathcal{D}) = R(d_1, \ldots, d_\ell)$. We conclude that $R$ is a $\mathcal{D}$-gist of $F$:

$$F(r) = R(\mathcal{D})$$

(ii) By assumption, $F$ has a $\mathcal{D}$-gist $\tilde{F} \in K[y]$, i.e., $\tilde{F}(\mathcal{D}) = F$. Let $\tilde{R}$ be the normal form of $F$. CLAIM: $R - \tilde{R} \in \mathcal{I}_D$. To see this, we write $R - \tilde{R}$ as a sum

$$R - \tilde{R} = (R - F) + (F - \tilde{F}) + (\tilde{F} - \tilde{R}).$$

We only need to verify that each of the three summands belong to $\mathcal{I}_D$: in part (i), we noted that $R - F \in \mathcal{I}_D$; the third summand $F - \tilde{R} \in \mathcal{I}_D$ for the same reason.
The second summand $F - \tilde{F} \in \mathcal{I}_D$ by an application of Lemma 4.1. To conclude that $R \in K[y]$, we assume (by way of contradiction) that $R \notin K[y]$. By our choice of term ordering for $\mathcal{G}_D$, we know that $\text{Lt}(R - \tilde{R}) = \text{Lt}(R)$. But $R - \tilde{R} \in \mathcal{I}_D$ implies that there is polynomial $g \in \mathcal{G}_D$ such that $\text{Lt}(g)|\text{Lt}(R)$. This contradicts the fact that $R$ is a normal form.

Q.E.D.

Now we consider the special case when $y_i - d_i$ is weighted homogeneous relative to a weight function:

$$\omega : (y, r) \rightarrow \mathbb{N}.$$ 

A set of polynomials is said to be **weighted homogeneous** or **$\omega$-homogeneous** if every polynomial in the set is $\omega$-homogeneous. Let $K_\omega[y, r]$ denote the set of all the $\omega$-homogeneous polynomials in $K[y, r]$. It is obvious that all polynomials in $K_\omega[y, r]$ of weighted degree $\delta$ form a $K$-vector space, denoted by $K_\delta^{\omega}[y, r]$ where we assume $0 \in K_\delta^{\omega}[y, r]$. Therefore, we define the weighted degree of 0 to be $\delta$ when 0 is viewed as an element in $K_\delta^{\omega}[y, r]$. When $\omega \equiv 1$, we simplify $K_\delta^{\omega}[y, r]$ into $K^{\delta}[y, r]$.

Assume $f, f' \in K_\omega[y, r]$. Then the following properties can be easily verified.

(i) $\omega\deg(f, f') = \max(\omega\deg(f), \omega\deg(f'))$.

(ii) $\omega\deg(f \pm f') \leq \max(\omega\deg(f), \omega\deg(f'))$.

(iii) $\omega\deg(f \cdot f') = \omega\deg(f) + \omega\deg(f')$.

(iv) The S-polynomial of $f$ and $f'$ is weighted homogeneous.

(v) If $\mathcal{G} \subseteq K[y, r]$ is a weighted homogeneous Gröbner basis, then the normal form of $f$ relative to $\mathcal{G}$ is weighted homogeneous of weighted degree $\omega\deg(f)$.

(vi) If $\mathcal{F} \subseteq K[y, r]$ is weighted homogeneous, so is the Gröbner basis of $\mathcal{F}$.

(vii) Let $F = \sum_{i=0}^{\delta} F_i \in K[y, r]$ where $\omega\deg(F_i) = i$ and $\mathcal{G} \subseteq K_\omega[y, r]$ is a Gröbner basis. Then the normal form of $F$ relative to $\mathcal{G}$ is the sum of the normal form of $F_i$ relative to $\mathcal{G}$.

If $\mathcal{F} \subseteq K[y, r]$ is weighted homogeneous, we say a polynomial $H \in K[y, r]$ is $\mathcal{F}$-minimal if for all $H' \in K[y, r]$,

$$H \equiv H' \pmod{\mathcal{I}_F} \quad \text{implies} \quad \omega\deg(H) \leq \omega\deg(H').$$

Then we have the following lemma.

**Lemma 4.5.** If $\mathcal{G} \subseteq K_\omega[y, r]$ is a Gröbner basis and $F$ is weighted homogeneous, then the normal form of $F$ relative to $\mathcal{G}$ is $\mathcal{G}$-minimal, i.e., for any $F' \equiv F \pmod{\mathcal{I}_G}$, $\omega\deg(F') \geq \omega\deg(F)$.

**Proof.** Note that any $F' \in K[y, r]$ can be decomposed into weighted homogeneous components, i.e., $F' = \sum_i F_i'$ where $F_i'$ is weighted homogeneous. Let $R'$ and $R_i'$ be the normal forms of $F'$ and $F_i'$ relative to $\mathcal{G}$ respectively. Then $R' = \sum_i R_i'$. Let $R$ be the normal form of $F$ relative to $\mathcal{G}$. Then there exists $i$ such that $R_i' = R$ and $R_j' = 0$ if $j \neq i$. Therefore,

$$\omega\deg(F') \geq \omega\deg(F_i') = \omega\deg(R_i') = \omega\deg(R) = \omega\deg(F).$$

The lemma is proved.

Q.E.D.
Theorem 4.6. If $R \in K[y]$ is the normal form of $F \in K[r]$ relative to $G_D$ where $D$ is weighted homogeneous and $G_D$ is the Gröbner basis of

$$I_D = \langle y_1 - d_1, \ldots, y_t - d_t \rangle,$$

then $R$ is a minimal $D$-gist of $F$.

Proof. First by Theorem 4.4, $R$ is a $D$-gist of $F$. If $D$ is weighted homogeneous, so is $G_D$. By Lemma 4.5, $R$ is $G$-minimal. Since $I_G = I_D$, $R$ is $D$-minimal by definition. Q.E.D.

Theorems 4.4 and 4.6 lead to the following algorithm after specializing $y_i - d_i$ to $z_i - \tau_i$ and $\omega$ to

$$\omega(z_i) = i, \quad \omega(r_i) = 1.$$

G-gist($F, \mu$):

Input: $F \in K^d[r]$ and $\mu = (\mu_1, \ldots, \mu_m)$.

Output: a minimal $\mu$-gist of $F$ or say "$F$ does not exist".

$B \leftarrow \{z_1 - \tau_1(r), \ldots, z_n - \tau_n(r)\}$

$ord \leftarrow \text{plex}(r_1, \ldots, r_m, z_1, \ldots, z_n)$

$G \leftarrow \text{GroebnerBasis}(B, ord)$

$R \leftarrow \text{NormalForm}(F, G, ord)$

If $\deg(R, r) > 0$ then

Return "$F$ does not exist"

Else

Return $R$

Figure 1. The $G$-gist algorithm.

Example 6. We carry out the algorithm $G$-gist for $F = 3r_1^2 + r_2^2 + 2r_1r_2$ and $\mu = (2, 1)$ as follows.

Step 1 Construct $B = \{z_1 - (2r_1 + r_2), z_2 - (r_1^2 + 2r_1r_2), z_3 - r_2^2r_2\}$.

Step 2 Compute the Gröbner basis of $B$ with the lexicographical order $z_1 < z_2 < z_3 < r_1 < r_2$ to get

$G = \{4z_1^2z_3 - z_1^2z_2 - 18z_1z_2z_3 + 4z_2^3 + 27z_3^2 + 2r_1z_2^3 + 4z_1^2z_2z_3 - z_1z_2^3 - 54r_1z_3^2 + 36z_1z_3 - 15z_2^2z_3, 6r_1z_1z_3 - 2r_1z_2^2 - 4z_1^2z_3 + z_1z_2^2 + 3z_2z_3, r_1z_1z_3 - 9r_1z_3 + 6z_1z_3 - 2z_2^2, 2r_1z_2^2 - 6r_1z_2 - z_1z_2 + 9z_3, 3r_2^2 - 2r_1z_1 + z_2, -z_1 + 2r_1 + r_2\}$.

Step 3 Compute the normal form of $F$ relative to $G$ to get $R = z_1^2 - z_2$.

Step 4 Since $\deg(R, r) = 0$, the algorithm outputs $R = z_1^2 - z_2$.

5. Computing Gists via Preprocessing Approach. In the previous section, we show how to compute $\mu$-gists using Gröbner bases. This algorithm is quite slow when $\mu \neq (1, 1, \ldots, 1)$ (see Table 2, Example F3). In the next two sections, we will introduce two methods based on an analysis of the following two $K$-vector spaces:
- $K^\delta_{\text{sym}}[x]$: the set of symmetric homogeneous polynomials of degree $\delta$ in $K[x]$  
- $K^\delta_\mu[r]$: the set of $\mu$-symmetric polynomials of degree $\delta$ in $K[r]$

The first method is based on preprocessing and reduction: we first compute a basis for $K^\delta_\mu[r]$, and then use the basis to reduce $F(r)$. The second method directly computes the $\mu$-gist of $F(r)$ by solving linear equations.

### 5.1. Structure of a $\mu$-Symmetric Polynomial Set

We first consider $K^\delta_{\text{sym}}[x]$, the symmetric homogeneous polynomials of degree $\delta$. This is a $K$-vector space. By a weak partition of an integer $k$, we mean a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ where $\sum_{i=1}^k \alpha_i = k$ and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq 0$. Thus, in contrast to an ordinary partition, a weak partition allows zero parts. Given $\delta$, if $\alpha$ is a weak partition of $\delta$ and no part $\alpha_i$ larger than $n$, we will write

$$\alpha \vdash (\delta, n).$$

Let

$$e_\alpha := \prod_{i=1}^\delta e_{\alpha_i}$$

For instance if $\delta = 4, n = 2, \alpha = (2, 1, 1, 0)$ then $e_\alpha = e_2 e_1 e_1 e_0 = e_2 e_1^2$.

Let $T(x)$ denote the set of terms of $x$, and $T^\delta(x)$ denote those terms of degree $\delta$. A typical element of $T^\delta(x)$ is $\prod_{i=1}^\delta x_i^{d_i}$ where $d_1 + \cdots + d_\delta = \delta$. We totally order the terms in $T^\delta(x)$ using the lexicographic ordering in which $x_1 < x_2 < \cdots < x_n$. Given any $F \in K(x)$, its support is $\text{Supp}(F) \subseteq T(x)$ such that $F$ can be uniquely written as

$$F = \sum_{p \in \text{Supp}(F)} c(p)p$$

where $c : \text{Supp}(F) \to K \setminus \{0\}$ denote the coefficients of $F$. Let the leading term $\text{Lt}(F)$ be equal to the $p \in \text{Supp}(F)$ which is the largest under the lexicographic ordering. For instance, $\text{Supp}(e_1) = \{x_1, \ldots, x_n\}$ and $\text{Lt}(e_1) = x_n$. Also $\text{Supp}(e_1 e_2) = \{x_i x_j x_k : 1 \leq i \neq j \leq n, 1 \leq k \leq n\}$ and $\text{Lt}(e_1 e_2) = x_n^2 x_{n-1}$. The coefficient of $\text{Lt}(F)$ in $F$ is the leading coefficient of $F$, denoted by $\text{Lc}(F)$. Call $\text{Lm}(F) := \text{Lc}(F)\text{Lt}(F)$ the leading monomial of $F$. This is well-known:

**Proposition 2.** The set $B_1 := \{e_\alpha : \alpha \vdash (\delta, n)\}$ is a $K$-basis for the vector space $K^\delta_{\text{sym}}[x]$.

**Example 7.** Let $n = 4$ and $\delta = 3$. Then $B_1 = \{e_3, e_1 e_2, e_3\}$ forms a basis of the $K$-vector space $K^3_{\text{sym}}[x]$.

Now we consider the set $K^\delta_\mu[r]$ comprising the $\mu$-symmetric functions of degree $\delta$. The map

$$\sigma_\mu : K^\delta_{\text{sym}}[x] \to K^\delta_\mu[r]$$

is an onto $K$-homomorphism. Note that $K^\delta_\mu[r]$ is a vector space which is generated by the set

$$\overline{B}_1 := \{\overline{G} : G \in B_1\}$$

where $\overline{G}$ is a short hand for writing $\sigma_\mu(G)$. It follows that there is a maximal independent set $B_2 \subseteq \overline{B}_1$ that is a basis for $K^\delta_\mu[r]$. The set $B_2$ may be a proper subset of $\overline{B}_1$, which is seen in
this example: let $\mathbf{\mu} = (2, 2)$ and $\delta = 3$. From Example 7 we have $B_1 = \{e_1^2, e_1e_2, e_3\}$. Then

$$B_1 = \{ A : \tau_1^2, B : \tau_1\tau_2, C : \tau_3 \}.$$  

We can check that $B_1$ is linearly dependent since $A + 8C = 4B$. Furthermore, it is easy to verify that any 2-subset of $B_1$ forms a basis for $K_\delta^\mu[r]$. In general, we have the following lemma.

**Proposition 3.** For all $\mathbf{\mu} = (\mu_1, \mu_2)$, $B_1 = \{\tau_1^2, \tau_2\}$ is a linearly independent set.

**Proof.** Assume there exist $k_1$ and $k_2$ such that

$$k_1\tau_1^2 + k_2\tau_2 = 0. \tag{5.2}$$

Let $\mathbf{\mu} = (\mu_1, \ldots, \mu_m)$. Then

$$\tau_1 = \sum_{i=1}^m \mu_i r_i, \quad \tau_2 = \sum_{i=1}^m \left(\frac{\mu_i}{2}\right) r_i^2 + \sum_{i<j} \mu_i\mu_j r_ir_j \tag{5.3}$$

The substitution of (5.3) into (5.2) leads to

$$\sum_{i=1}^m \left[ k_1\mu_i^2 + k_2\left(\frac{\mu_i}{2}\right) \right] r_i^2 + (2k_1 + k_2) \sum_{i<j} \mu_i\mu_j r_ir_j = 0. \tag{5.2}$$

Therefore,

$$k_1\mu_i^2 + k_2\left(\frac{\mu_i}{2}\right) = (2k_1 + k_2)\mu_i\mu_j = 0, \quad \text{for} \quad i, j = 1, \ldots, m \quad \text{where} \quad i < j.$$ 

This system has a unique solution which is $k_1 = k_2 = 0$. Thus it follows that $\tau_1^2$ and $\tau_2$ are linearly independent. \hfill Q.E.D.

From the previous discussion, we saw that the dimension of $K_\delta^\mu[r]$ may be smaller than that of $K_{\text{sym}}^\mu[x]$. There are two special cases: when $\mathbf{\mu} = (1, 1, \ldots, 1)$, $\dim(K_\delta^\mu[x]) = \dim(K_\delta^\mu[r])$; when $\mathbf{\mu} = (n)$, $\dim(K_\delta^\mu[r]) = 1$. The following table shows the dimensions of $K_{\text{sym}}^\mu[x]$ and $K_\delta^\mu[r]$ for some cases. One can see that it is quite common to have a dimension drop from the specialization $\sigma_\mu$ (these lower dimensions are underlined in the table).

### 5.2. Reduction and Canonical Sequence.

This subsection is devoted to generating the basis of the vector space $K_\delta^\mu[r]$ with which one could easily check whether a given polynomial is in this vector space or not. For this purpose, we introduce a reduction procedure and its applications. This yields a more efficient method to check for $\mu$-symmetry and to compute the gists in the affirmative case.

A set $B \subseteq K[r]$ is **linearly independent** if any non-trivial $K$-linear combination over $B$ is non-zero; otherwise, $B$ is **linearly dependent**. We say $C = (C_1, \ldots, C_t)$ is a **canonical sequence** if the set $\{C_1, \ldots, C_t\}$ is linearly independent and $\text{Lt}(C_i) \prec \text{Lt}(C_j)$ for all $i < j$. 


In this subsection, we work in the vector space $K^\delta[r]$ of all homogeneous polynomials of degree $\delta$ in $K[r]$.

We will introduce the concept of reduction. As motivation, first express any non-zero polynomial $G$ as $G = \text{Lt}(G) + R$ where $R$ is the tail of $G$ (i.e., remaining terms of $G$). In the terminology of term rewriting systems (e.g., [1] and [6, Section 12.3.4]), we then view $G$ as a rule for rewriting an arbitrary polynomial $F$ in which any occurrence of $\text{Lt}(G)$ in $\text{Supp}(F)$ is removed by an operation of the form $F' \leftarrow F - c \cdot G$, with $c \in K$ chosen to eliminate $\text{Lt}(G)$ from $\text{Supp}(F')$. For instance, consider $F = r_2^2 + 2r_1r_2 - r_1^2$ and $G = r_1r_2 + r_2^2 - r_2$ where we have underlined the leading monomials of $F$ and $G$. Here we use the above convention that $r_1 \prec r_2$. Then $F' = F - 2G = r_2^2 - 3r_1^2 + 2r_2$. We say that $F$ has been reduced by $G$ to $F' = F - 2G$. The $\text{Supp}(F')$ no longer has $r_1r_2$, but has gained other terms which are smaller in the $\prec$-ordering.

If $\text{Lt}(G) \notin \text{Supp}(F)$, we say $F$ is reduced relative to $G$. For a sequence $C$, if $F$ is reduced with relative to each $G \in C$, we say $F$ is reduced relative to $C$. Then we have this basic property:

**Proposition 4.** Let $F \neq 0$ and $C = (C_1, \ldots, C_{\ell})$ be a canonical sequence. If $F$ is reduced relative to $C$, then $\{F, C_1, \ldots, C_{\ell}\}$ is linearly independent.

**Proof.** By way of contradiction, assume $F$ is linearly dependent on $C$, say $F = \sum_{i=1}^{\ell} k_i C_i$. This implies $\text{Lt}(F) = \text{Lt}(\sum_{i=1}^{\ell} k_i C_i) \preceq \text{Lt}(C_j)$. So there is a smallest $j \leq \ell$ such that $\text{Lt}(F) \preceq \text{Lt}(C_j)$. Since $F$ is reduced relative to $C$, we have $\text{Lt}(F) \prec \text{Lt}(C_j)$. It is easy to see that this implies $k_j, k_{j+1}, \ldots, k_{\ell}$ are all zero. It follows that $j \geq 2$ (otherwise $F = \sum_{i=1}^{\ell} k_i C_i = 0$). Moreover, we have $\text{Lt}(\sum_{i=1}^{\ell} k_i C_i) \preceq \text{Lt}(C_{j-1}) \prec \text{Lt}(F)$. This contradicts the assumption $\sum_{i=1}^{\ell} k_i C_i = F$. Q.E.D.

We next introduce the reduce subroutine in Figure 2 which takes an arbitrary polynomial $F \in K[r]$ and a canonical sequence $C$ as input to produce a reduced polynomial relative to $C$.

**Example 8.** Consider $F = 3r_1^2 + 4r_1r_2 + r_2^2$ and $\mu = (2, 1)$. Given a canonical sequence $C = (r_1^2 + 2r_1r_2, 2r_1^2 + r_2^2)$ with $r_1 \prec r_2$, we proceed to compute the reduced polynomial of $F$ relative to $C$ using the above reduce algorithm.

**Step 1 Initialization.** Let $R = 0$ and $t = 2$. 

| $n$ | $\mu$ | $\delta$ | $\text{dim}(K^\delta_{\text{sym}}[x])$ | $\text{dim}(K^\delta_{\mu}[r])$ | $n$ | $\mu$ | $\delta$ | $\text{dim}(K^\delta_{\text{sym}}[x])$ | $\text{dim}(K^\delta_{\mu}[r])$ |
|-----|-------|----------|----------------------------------|-------------------------------|-----|-------|----------|----------------------------------|-------------------------------|
| 4   | (2, 1) | 2        | 3                                | 5                             | 5   | (2, 1, 1, 1) | 3        | 5                             | 5                             |
|     |       |          | 3                                | 6                             |     |          | 3        | 6                             | 10                            |
| 4   | (2, 1, 1) | 2      | 3                                | 5                             | 5   | (2, 2, 1) | 3        | 5                             | 7                             |
|     |       |          | 5                                | 6                             |     |          | 5        | 6                             | 10                            |
| 4   | (3, 1) | 3        | 3                                | 5                             | 4   | (3, 1, 1) | 3        | 5                             | 5                             |
|     |       |          | 5                                | 6                             |     |          | 5        | 7                             | 7                             |
| 4   | (2, 2) | 3        | 3                                | 5                             | 4   | (3, 2)   | 3        | 5                             | 4                             |
|     |       |          | 5                                | 6                             |     |          | 5        | 7                             | 7                             |
|     |       |          | 5                                | 6                             |     |          | 5        | 7                             | 7                             |

Table 1

Dimensions of $K^\delta_{\text{sym}}[x]$ and $K^\delta_{\mu}[r]$.
reduce$(F, C)$:

**Input:** $F \in K[r], C = (C_1, \ldots, C_\ell)$ is canonical and each $C_i \in K^\delta[r]$

**Output:** $R$ such that $F = \sum_{i=1}^\ell c_i C_i + R$ with $c_i \in K$ and $R$ is reduced relative to $C$.

Let $R \leftarrow 0$, $i \leftarrow \ell$

While $(F \neq 0$ and $i > 0)$

$p \leftarrow \text{Lt}(F)$

If $p > \text{Lt}(C_i)$ then

$R \leftarrow R + \text{Lc}(F) \cdot p$; $F \leftarrow F - \text{Lc}(F) \cdot p$

else

$F \leftarrow F - \frac{\text{Lt}(F)}{\text{Lc}(C_i)} C_i$

$i \leftarrow i - 1$

Return $R + F$

**Figure 2.** The reduce algorithm.

**Step 2** First iteration. For $F \neq 0$ and $i > 0$, $p = \text{Lt}(F) = r_2^2$ which is equal to $\text{Lt}(C_2)$. Thus $F$ is updated with $F - \frac{\text{Lt}(F)}{\text{Lc}(C_2)} C_2 = r_2^2 + 4r_1r_2$ and $i$ is updated with $i - 1 = 1$.

**Step 3** Second iteration. For $F \neq 0$ and $i > 0$, $p = \text{Lt}(F) = r_1r_2$ which is equal to $\text{Lt}(C_1)$. Thus $F$ is updated with $F - \frac{\text{Lt}(F)}{\text{Lc}(C_1)} C_1 = -r_1^2$ and $i$ is updated with $i - 1 = 0$.

**Step 4** Finalization. Since $i = 0$, the iteration stops and the algorithm outputs $R + F = -r_1^2$.

**Proposition 5.** The algorithm reduce$(F, C)$ halts and takes at most $\#\text{Supp}(F) - 1 + \sum_{i=1}^\ell \#\text{Supp}(C_i)$ loops. Moreover, this bound is tight in the worst case.

**Proof.** Let $F_1$ denote the input polynomial. The variable $F$ in the algorithm is initially equal to $F_1$. In general, let $F_j$ ($j = 1, 2, \ldots$) be the polynomial denoted by $F$ at the beginning of the $j$th iteration of the while-loop. Thus $p_j = \text{Lt}(F_j)$ is the term denoted by the variable $p$ in the $j$th iteration. Note that $F_j$ transforms to $F_{j+1}$ by losing its leading term $p_j$ or furthermore, if $i(j)$ is the current value of the variable $i$, and $p_j = \text{Lt}(C_{i(j)})$ where $C_{i(j)} \in C$, we also subtract the tail of $\frac{\text{Lt}(F_j)}{\text{Lc}(C_{i(j)})} \cdot C_{i(j)}$ from $F_{j+1}$. Thus, $\text{Supp}(F) \subseteq \text{Supp}(F_1) \cup \text{Supp}(C)$. Since $p_1 \succ p_2 \succ \cdots$ and $p_j \in \text{Supp}(F_1) \cup \text{Supp}(C)$, this proves that the algorithm halts after at most $\#\text{Supp}(F_1) + \#\text{Supp}(C)$ iterations.

Let $L$ be the actual number of iterations. We now give a refined argument to show that $L \leq \#\text{Supp}(F) - 1 + \#\text{Supp}(C)$, i.e., we can improve the previous upper bound on $L$ by one. Note that we exit the while-loop when $F = 0$ or $i = 0$ holds. There are two cases.

**CASE 1:** $F = 0$ and $i = 0$ both hold. This implies that in the previous iteration, $p_L = \text{Lt}(C_1)$, and $i$ was decremented from 1 to 0. Since $p_L$ came from $\#\text{Supp}(F_1)$ or $\#\text{Supp}(C_2, \ldots, C_\ell)$, this implies

$L \leq \#(\text{Supp}(F_1) \cup \text{Supp}(C)) \leq \#\text{Supp}(F_1) - 1 + \#\text{Supp}(C)$.

**CASE 2:** $F \neq 0$ or $i > 0$. Each iteration can be “charged” to an element of $(\text{Supp}(F_1) \cup$
Supp(C)). If \( i > 0 \), then some elements in Supp\((C_1)\) are not charged. If \( F \neq 0 \), then Supp\((F) \subseteq \) Supp\((F_1) \cup \) Supp\((C)\) also implies that some elements of Supp\((F_1) \cup \) Supp\((C)\) are not charged. Thus CASE 2 implies
\[
L \leq \#\text{Supp}(F_1) - 1 + \#\text{Supp}(C).
\]
This proves our claimed upper bound on \( L \).

To prove that this bound is tight, let \( F_1 = p_1 + q_1 + \cdots + q_s \) and \( C = (p_1, \ldots, p_k) \) with the term ordering \( p_1 < \cdots < p_k < q_1 < \cdots < q_s \). In the first \( s \) loops, since Lt\((F_1) > p_k\), \( i \) is unchanged and \( q_1, \ldots, q_s \) are removed from \( F \). In the next \( \ell - 1 \) loops, since Lt\((F_i) = p_1 < p_2 < \cdots < p_k\), \( F \) is unchanged and \( i \) will drop to 1. In the last loop, since Lt\((F_1) = p_1 = \text{Lt}(C_1)\), \( F \) will be reduced relative to \( C_1 \) to 0. So the total number of loops is \( s + \ell = \#\text{Supp}(F_1) - 1 + \sum_{i=1}^{\ell} \#\text{Supp}(C_i) \).

Q.E.D.

**Proposition 6.** (Correctness) The reduce subroutine is correct.

**Proof.** Correctness of the output \( R_s \) in the reduce subroutine amounts to two assertions.

(A1) The output \( R_s \) is reduced relative to \( C \).

(A2) \( F_1 - R_s \) is a linear combination of the polynomials in \( C \) where \( F_1 \) is the input polynomial.

To prove these assertions, assume that the while-loop terminates after the \( L \)-th iteration. Also let \( F_j \), \( R_j \) and \( i_j \) denote the values of the variables \( F \), \( R \) and \( i \) at the start of the \( j \)-th iteration (for \( j = 1, \ldots, L, L + 1 \)). Thus, \( F_1 \) is the input polynomial, \( R_1 = 0 \) and \( i_1 = \ell \). Assertion (A2) follows from the fact that in each iteration, the value of \( F + R \) does not change or it changes by a scalar multiple of some \( C_i \in C \). To see Assertion (A1), we use induction on \( j \) to conclude that \( F_j \) is reduced with respect to \( C_j := (C_{1+i_1}, C_{2+i_1}, \ldots, C_{i_1}) \), and \( R_j \) is reduced with respect to \( C \). Finally, the output \( R_s \) is equal to \( R_{L+1} + F_{L+1} \). At termination, there are two cases: either \( F_{L+1} = 0 \) (so \( R_s = R_{L+1} \)) or \( i_{L+1} = 0 \) (so \( R_s = R_{L+1} + F_{L+1} \)). In the first case, Assertion (A1) holds because \( R_s = R_{L+1} \) and \( R_{L+1} \) is reduced w.r.t. \( C \). In the second case, Assertion (A1) holds because \( F_{L+1} \) is reduced w.r.t. \( C_{L+1} = C \). Q.E.D.

**Proposition 7.** If \( C = (C_1, \ldots, C_{\ell}) \) is canonical, then reduce\((F, C) = 0 \) iff \( \{ F, C_1, \ldots, C_{\ell} \} \) is linearly dependent.

**Proof.** One direction is immediate: reduce\((F, C) = 0 \) implies that \( F \) is a linear combination of the elements of \( C \). Conversely, if reduce\((F, C) = F' \neq 0 \), then \( \{ F', C_1, \ldots, C_{\ell} \} \) is linearly independent by Proposition 4. Moreover, \( F' = F - \sum_{i=1}^{\ell} k_i C_i \) for some \( k_1', \ldots, k_{\ell}' \). By way of contradiction, assume that \( \{ F', C_1, \ldots, C_{\ell} \} \) is linearly independent, i.e., \( F = \sum_{i=1}^{\ell} k_i C_i \) for some \( k_1, \ldots, k_\ell \). It follow that \( F' = \sum_{i=1}^{\ell} (k_i - k_i') C_i \), contradicting the linear independence of \( \{ F', C_1, \ldots, C_{\ell} \} \).

Q.E.D.

This gives rise to the canonize algorithm in Figure 3 to construct a canonical sequence.

We view the sequence \( C = (C_1, \ldots, C_m) \) as a sorted list of polynomials, with Lt\((C_i) < \) Lt\((C_{i+1})\). Thus insert\((B, C)\) which inserts \( B \) into \( C \), can be implemented in \( O(\log m) \) time with suitable data structures. The overall complexity is \( O(\ell \log m) \) where \( m \) is the length of the output \( C \). Alternatively, we could initialize the input \( B \) as a priority queue can pop the polynomial \( B \in B \) with the largest Lt\((B)\). This design yields a complexity of \( O(\ell \log \ell) \) which is inferior when \( \ell \gg m \).
Step 4 Finalization

Step 2 First iteration. Let $B = r_1^2 + 2r_1r_2$. Note that $C = ()$. Thus $B' = \text{reduce}(B, C) = B$ and $C$ is updated with $(r_1^2 + 2r_1r_2)$.

Step 3 Second iteration. Let $B = 4r_1^2 + 4r_1r_2 + r_2^2$. Then carry out the reduction of $B$ relative to $C$ and we get $B' = \text{reduce}(B, C) = 2r_1^2 + r_2^2$. After inserting $B'$ into $C$, $C$ is updated with $(r_1^2 + 2r_1r_2, 2r_1^2 + r_2^2)$.

Step 4 Finalization. Now the iteration stops and the algorithm outputs $C = (r_1^2 + 2r_1r_2, 2r_1^2 + r_2^2)$.

The termination of $\text{canonize}(B)$ is immediate from the termination of $\text{reduce}(F, C)$. The correctness of the output of $\text{canonize}(B)$ comes from two facts: the returned $C$ is clearly canonical. It is also maximal because any element $B \in B$ that does not contribute to $C$ is clearly dependent on $C$.

It should be pointed out that by tracking the “quotients” of $F$ relative to $C$ in the $\text{reduce}$ algorithm and integrating the information into the $\text{canonize}$ algorithm, we can derive the relationship between $B = \{r_\alpha : \alpha \vdash (\delta, n)\}$ and $C = \text{canonize}(B)$ and write polynomials in $C$ as linear combinations of polynomials in $B$. By “quotients”, we mean the coefficients $c_i$’s in the expression $F = \sum_{i=1}^{\ell} c_i C_i + R$. When the quotient information is required, we use algorithms $\text{reduce}(F, C, 'q')$ and $\text{canonize}(B, 'Q')$ where $q$ and $Q$ represents the quotient (column) vector and quotient matrix, respectively. More explicitly,

$$F = C \cdot q + R \quad \text{and} \quad C = B \cdot Q$$

where $B$ and $C$ are viewed as row vectors. These notations will be used in the $\text{CR-gist}$ algorithm in Figure 3 of the following subsection.

5.3. Computing $\mu$-gist via Reduction. In this subsection, we use $\text{reduce}$ and $\text{canonize}$ algorithms to construct the $\text{CR-gist}$ algorithm for computing the $\mu$-gist of a polynomial.

Example 10. Consider the polynomial $F = 3r_1^2 + 4r_1r_2 + r_2^2$ and $\mu = (2, 1)$ as in Example 9. In what follows, we check whether $F$ is $\mu$-symmetric or not and compute its $\mu$-gist in the affirmative case.
Step 2 Let \( \delta = \deg(F, r) = 2 \) and \( n = \sum_{i=1}^{m} \mu_i = 3 \).

Step 3 Compute a canonical \( \mathcal{C} \) from \( \mathcal{B} \) and its quotient \( \mathcal{Q} \) relative to \( \mathcal{B} \). Then we get \( \mathcal{C} = \text{canonize}(\mathcal{B}) \)

\[ = (r_1^2 + 2r_1r_2, 2r_1^2 + r_2^2) \]  and \( \mathcal{Q} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \). The detailed computation can be found in Example 8.

Step 4 Compute \( R = \text{reduce}(F, \mathcal{C}) \) and the quotient \( q \). By the result of Example 8, \( R = -r_1^2 \neq 0 \) and \( q = (2, 1)^T \). Thus the output is “No”, which means that \( F \) is not \( \mu \)-symmetric.

If we replace \( F \) with \( F = 3r_1^2 + 2r_1r_2 + r_2^2 \), then after carrying out the same procedure as above, we will get \( R = 0 \) and \( q = (1, 1) \), which means \( F \) is \( \mu \)-symmetric and its \( \mu \)-gist is

\[ \hat{F} = (z_1^2, z_2) \cdot Q \cdot q^T = z_1^2 - z_2. \]

Since termination of the algorithm \( \text{CR-gist} \) is immediate from that of \( \text{canonize} \) and \( \text{reduce} \), we only show its correctness. Assume \( \deg(F, r) = \delta \). Recall that \( F \in K[r] \) is \( \mu \)-symmetric iff there exists a homogeneous symmetric polynomial \( \hat{F} \in K[x] \) of degree \( \delta \) such that \( \sigma_\mu(\hat{F}) = F(r) \). By Proposition 3, \( \hat{F} \) is symmetric and with degree \( \delta \) iff \( \hat{F} \in K_\text{sym}^\delta[x] \).

Thus \( F = \sigma_\mu(\hat{F}) \in K_\mu^\delta[r] \) where \( K_\mu^\delta[r] \) is a \( K \)-vector space with the basis generated by \( \mathcal{B} = \{ \tau_\alpha : \alpha \vdash (\delta, n) \} \). If \( \mathcal{C} = \text{canonize}(\mathcal{B}) \), then \( \mathcal{C} \) is the basis we want to obtain. Therefore, if \( F \) is \( \mu \)-symmetric iff \( \text{reduce}(F, \mathcal{C}) = 0 \). When \( F \) is \( \mu \)-symmetric, \( F = \mathcal{C} \cdot q = \mathcal{B} \cdot Q \cdot q \). By the definition of \( \mu \)-gist, \( \hat{F} = (z_\alpha : \alpha \vdash (\delta, n)) \cdot Q \cdot q \).

5.4. Exponential lower bound for nondeterministic reduction. In this subsection, we consider an alternative reduction process where each reduction step is non-deterministic. We prove that this version can be exponential in the worst case.

For any term \( p \), let \( \text{Coef}(F, p) \) denote the coefficient of \( p \) in \( F \). If \( p \notin \text{Supp}(F) \), then
\( \text{Coef}(F,p) = 0. \) For any polynomial \( C \), define

\[
\text{reduceStep}(F,C) \leftarrow F - \frac{\text{Coef}(F,\text{Lt}(C))}{\text{Lc}(C)} C.
\]

We call \( \text{reduceStep}(F,C) \) a \textit{C-reduction step} or a \( C \)-reduction step in case \( C \in C \). We see that \( \text{reduceStep}(F,C) = F \) iff \( \text{Lt}(C) \) does not occur in \( F \). We say the reduction is improper in this case.

Let \( \text{nreduce}(F,C) \) denote the subroutine that repeatedly transforms \( F \) by applying proper \( C \)-reduction steps to \( F \) until no more more change is possible. It returns the final value of \( F \). We call this the \text{non}deterministic \text{reduction} of \( F \).

**Proposition 8.** For any linearly independent set \( C \), we have

\[
\text{nreduce}(F,C) = \text{reduce}(F,C).
\]

Then \( \text{nreduce}(F,C) \) has \( \leq 2^\ell \) \( C \)-reduction steps where \( \ell = |C| \). Moreover, \( 2^\ell \) steps may be needed.

**Proof.** Let \( R_1 = \text{nreduce}(F,C) \) and \( R_2 = \text{reduce}(F,C) \). Then there exists \( k_1, \ldots, k_\ell \) and \( k_1', \ldots, k_\ell' \) such that

\[
F = \sum_{i=1}^{\ell} k_i C_i + R_1 = \sum_{i=1}^{\ell} k_i' C_i + R_2.
\]

It is immediate that

\[
R_1 - R_2 = \sum_{i=1}^{\ell} (k_i - k_i') C_i.
\]

If \( R_1 \neq R_2 \), there exists \( i \) such that \( k_i \neq k_i' \) and \( k_j = k_j' (j = 1, \ldots, i-1) \). Then \( \text{Lt}(R_1 - R_2) = \text{Lt}(C_i) \). This implies that \( \text{Lt}(C_i) \in \text{Supp}(R_1) \) or \( \text{Lt}(C_i) \in \text{Supp}(R_2) \). Hence \( R_1 \) or \( R_2 \) is not reduced relative to \( C \). This contradicts with the output requirements of \( \text{reduce} \) or \( \text{nreduce} \).

Let us define \( a_\ell \) to be the longest \( C \)-derivation for any \( C \) with \( \ell \) elements. CLAIM A: \( a_\ell \leq 2^\ell - 1 \). Let \( C_\ell = (C_1, \ldots, C_\ell) \) be any canonical sequence with \( \ell \) elements. Let

\[
(5.4) \quad F_0 \to F_1 \to \cdots \to F_N
\]

be any \( C_\ell \)-derivation. We must prove that \( N \leq 2^\ell - 1 \) by induction of \( \ell \). Clearly, if \( \ell = 1 \), then \( a_1 \leq 1 = 2^1 - 1 \). Next, inductively assume that \( a_{\ell-1} \leq 2^{\ell-1} - 1 \). Suppose there does not exist an \( i < N \) such that \( F_i \to F_{i+1} \) is a \( C_\ell \)-reduction step. In that case, \( (5.4) \) is a \( C_{\ell-1} \)-derivation. By induction hypothesis, \( N \leq 2^{\ell-1} - 1 < 2^\ell - 1 \), as claimed. Otherwise, we may choose the smallest \( i \) such that \( F_i \to F_{i+1} \) is a \( C_\ell \)-reduction step. Note that this implies that \( \text{Lt}(C_i) \) does not appear in the support of \( F_j \) for all \( j \geq i + 1 \). In other words, \( F_0 \to \cdots \to F_i \) and \( F_{i+1} \to \cdots \to F_N \) are both \( C_{\ell-1} \)-derivations. By induction hypothesis, both these lengths are at most \( 2a_{\ell-1} + 1 \leq 2^\ell - 1 \). Thus the length of \( (5.4) \) is at most \( 2a_{\ell-1} + 1 \leq 2^\ell - 1 \). Thus CLAIM A is proved.
The last assertion of our proposition amounts to CLAIM B: \( a_\ell \geq 2^{\ell} - 1 \). To show this claim, let \( C_\ell = (C_1, \ldots, C_\ell) \) as before. But we now choose \( C_i := \sum_{j=1}^{i} p_j \) where \( p_j \)'s are terms satisfying \( p_j < p_{j+1} \). Let us write

\[
F \overset{C_\ell}{\rightarrow} G
\]

to mean that there is a \( C \)-derivation of length \( k \) from \( F \) to \( G \). Our claim follows if we show that

\[
C_\ell \overset{C_{\ell-1}}{\rightarrow} 0.
\]

The basis is obvious: \( C_1 \overset{C_1}{\rightarrow} 0 \). Inductively, assume that

\[
(5.5) \quad C_{\ell-1} \overset{C_{\ell-1}}{\rightarrow} 0.
\]

The inductive assumption implies

\[
C_\ell = p_\ell + C_{\ell-1} \overset{C_{\ell-1}}{\rightarrow} p_\ell.
\]

Next, in one step, we have \( p_\ell \overset{C_1}{\rightarrow} -C_{\ell-1} \) and, again from the induction hypothesis,

\[
-C_{\ell-1} \overset{C_{\ell-1}}{\rightarrow} 0.
\]

Concatenating these 3 derivations, shows that \( C_\ell \overset{C_\ell}{\rightarrow} 0 \). This proves CLAIM B.

Q.E.D.

6. Computing Gists via Solving Linear Equations. In this section, we introduce a direct method to compute gist of \( F(r) \) without preprocessing. Such methods depend on the choice of basis for \( K_{\text{sym}}[r] \). Our default basis is elementary symmetric polynomials.

Our algorithm that takes as input \( F \in K[r] \) and \( \mu \), and either outputs the \( \mu \)-gist \( \hat{F} \) of \( F \) or detects that \( F \) is not \( \mu \)-symmetric. The idea is this: \( F \) is \( \mu \)-symmetric iff \( \hat{F} \) exists. The existence of \( \hat{F} \) is equivalent to the existence of a solution to a linear system of equations. More precisely, there is an polynomial identity of the form \( \hat{F}(e_1, \ldots, e_n) = F \). To turn this identity into a system of linear equations, we first construct a polynomial

\[
G(k; z) \in K[k][z]
\]

in \( z \) with indeterminate coefficients in \( k \), with homogeneous weighted degree \( \delta \) in \( z \) (see Section 2.1 for definition of weighted degree). Here \( \delta \) is the degree of \( F \). Each term is of weighted degree \( \delta \) and has the form

\[
z_\alpha := \prod_{i=1}^{\delta} z_{\alpha_i}
\]
where $\alpha = (\alpha_1, \ldots, \alpha_3)$ is a weak partition of $\delta$ with parts at most $n$, i.e., $\alpha \vdash (\delta, n)$. Then $G(k; z)$ can be written as

$$G(k; z) := \sum_{\alpha \vdash (\delta, n)} k_{\alpha} z_{\alpha} = T_n^\delta(z) \cdot k$$

where $T_n^\delta(z) := (z_{\alpha} : \alpha \vdash (\delta, n))$ and $k := (k_{\alpha} : \alpha \vdash (\delta, n))^T$ viewed as a column vector are indeterminates. Next, we plug in $\bar{\tau}_i$’s for the $z_i$’s to get

$$H(k; r) := G(k; \bar{\tau}_1, \ldots, \bar{\tau}_n)$$

viewed as a polynomial in $K[k][r]$. We then set up the equation

(6.1) $$H(k; r) = F(r)$$

to solve for the values of $k$. Note that total degree of $G$ in $k$ is 1, i.e., $\deg(G, k) = 1$. Therefore, $\deg(H, k) = 1$. Thus (6.1) amounts to solving a linear system of equations in $k$.

To illustrate this process, consider the polynomial $F = 3r_1^2 + 2r_1r_2 + r_2^2$ and $\mu = (2, 1)$.

Step 1: Assign $\delta = \deg(F, r) = 2$ and $n = \sum_{i=1}^m \mu_i = 3$.

Step 2: Since the weak partitions of 2 with parts at most 3 are $(1, 1)$ and $(2, 0)$, the terms of weighted degree 2 are $z_1^2$ and $z_2^2$.

Step 3: Construct the polynomial $G(k; z) := k_1z_1^2 + k_2z_2$ where $k = (k_1, k_2)$ are the indeterminate coefficients.

Step 4: Using $\bar{\tau}_1 = 2r_1 + r_2, \bar{\tau}_2 = r_1^2 + 2r_1r_2$, construct the polynomial

$$H(k; r) := G(k; \bar{\tau}_1, \ldots, \bar{\tau}_n) = (4k_1 + k_2)r_1^2 + (4k_1 + 2k_2)r_1r_2 + k_1r_2^2.$$  

Step 5: Extract the coefficient vector $\mathbf{Coeffs}(H, r)$ of $H(k; r)$ viewed as a polynomial in $r$. The entries of this vector are linear in $k$. Thus $H = \mathbf{Coeffs}(H, r) \cdot T^\delta(r)$ where $T^\delta(r)$ is the vector of all terms of $T(r)$ of degree $\delta$.

Step 6: Extract the coefficient vector $\mathbf{Coeffs}(F, r)$ of $F(r)$. This vector is a constant $(3, 2, 1)^T$ where $T^2(r_1, r_2) = (r_1^2, r_1r_2, r_2^2)$.

Step 7: The last two steps enables the construction of a system of linear equations, $Ak = b$:

$$H(k; r) = F(r)$$

$$\begin{align*}
(4k_1 + k_2)r_1^2 + (4k_1 + 2k_2)r_1r_2 + k_1r_2^2 &= 3r_1^2 + 2r_1r_2 + r_2^2 \\
\mathbf{Coeffs}(H, r) &= \mathbf{Coeffs}(F, r) \\
\begin{bmatrix} 4 & 1 \\ 4 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} &= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \\
A \cdot k &= b
\end{align*}$$

where the last equation is the linear system to be solved for $k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$. 

Step 8: If $Ak = b$ has no solutions, we conclude that $F$ is not $\mu$-symmetric. Otherwise, choose any solution for $k$ and plugging into $G(k;r)$, we obtain a gist of $F(r)$. Here $k = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a solution and thus the input polynomial is $(2,1)$-symmetric with gist $z_1^2 - z_2$. Note that there may be multiple solutions for $k$ because of the presence of $\mu$-constraints.

We now summarize the above procedure as the LS-gist algorithm:

```
LS-gist(F, \mu):
	Input: $F \in K^\delta[r]$ and $\mu = (\mu_1, \ldots, \mu_m)$
	Output: the $\mu$-gist of $F$ if $F$ is $\mu$-symmetric; otherwise
	return “$F$ is not $\mu$-symmetric”.
	\delta \leftarrow \deg(F, r); n \leftarrow \sum_{i=1}^{m} \mu_i
	G \leftarrow \sum_{\alpha=(-\delta,n)} k_\alpha z_\alpha
	H \leftarrow G(k; \bar{e}_1, \ldots, \bar{e}_n)
	Extract Coeffs(H, r) and Coeffs(F, r).
	Find a solution $k = k_0$ of the linear system
	Coeffs(H, r) = Coeffs(F, r).
	If $k_0$ is nondefined
		Return “$F$ is not $\mu$-symmetric”
	Else
		Return $H(k_0; r)$
```

Figure 5. The LS-gist algorithm.

The correctness of the algorithm LS-gist lies in the fact that $F$ is $\mu$-symmetric iff $F \in K^\delta_\mu[r]$ which is generated by $\{\bar{e}_\alpha : \alpha \vdash (\delta, n)\}$.

7. Gists Relative to Other Bases of $K^\delta_{\text{sym}}[x]$. In this section, We briefly sketch how to extend the above methods to computing gists relative to other bases of $K^\delta_{\text{sym}}[x]$.

The set $K_{\text{sym}}[x]$ of symmetric functions can be viewed as a $K$-algebra generated by some finite set $G$. The following are three well-known choices of $G$ with $n$ elements each:

- (Elementary symmetric polynomials) $G_e := \{e_1, \ldots, e_n\}$ where $e_i$ is the $i$-th elementary symmetric function of $x$.
- (Power-sum symmetric polynomials) $G_p := \{p_1, \ldots, p_n\}$ where $p_i = x_1^i + \cdots + x_n^i$.
- (Complete homogeneous symmetric polynomials) $G_c := \{c_1, \ldots, c_n\}$ where $c_i$ is the sum of all distinct monomials of degree $i$ in the variables $x_1, \ldots, x_n$.

For each $\delta \geq 1$, the vector space $K^\delta_{\text{sym}}[x]$ of symmetric polynomials of degree $\delta$ has a basis $B^\delta$ that corresponds to a given generator set $G$. The following are bases of $K^\delta_{\text{sym}}[x]$:

- (e-basis) $B^\delta_e := \{e_\alpha : \alpha \vdash (\delta, n)\}$ where $e_\alpha = \prod_{i=1}^{\delta} e_{\alpha_i}$ and $\alpha = (\alpha_1, \ldots, \alpha_\delta)$;
- (p-basis) $B^\delta_p := \{p_\alpha : \alpha \vdash (\delta, n)\}$ where $p_\alpha = \prod_{i=1}^{\delta} p_{\alpha_i}$;
- (c-basis) $B^\delta_c := \{c_\alpha : \alpha \vdash (\delta, n)\}$ where $c_\alpha = \prod_{i=1}^{\delta} c_{\alpha_i}$.

But $K^\delta_{\text{sym}}[x]$ can also be generated with monomial symmetric polynomials. In this case, we
use $\alpha \vdash (\delta)_n$ to denote $\alpha = (\alpha_1, \ldots, \alpha_n)$ which is a weak partition of $\delta$ with exactly $n$ parts: $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$. We also write $x^\alpha$ for the product $\prod_{i=1}^n x_i^{\alpha_i}$. This yields yet another basis for $K_{2\text{ym}}[x]$:

- **(m-basis)** $E^\delta_m := \{m_\alpha : \alpha \vdash (\delta)_n\}$ where $m_\alpha = \sum_\beta x^\beta$ where $\beta$ ranges over all permutations of $\alpha$ which are distinct.

For instance, if $\alpha = (2, 0, 0)$ then $\beta$ ranges over the set $\{(2, 0, 0), (0, 2, 0), (0, 0, 2)\}$ and $m_\alpha = x_1^2 + x_2^2 + x_3^2$.

So far, this paper has focused on the $e$-basis. But concepts and algorithms relative to the choice of this basis (e.g., the $\mu$-gist and $G$-gist) can be reformulated using the other bases. In each algorithm, there are two parameters, i.e., the generator polynomials (e.g., $e_i$ and $e_j$) and the index set (e.g., $\alpha \vdash (\delta, n)$). When using $p$-basis or $c$-basis, we only need to replace $e_i$ used by the algorithms $G$-gist, CR-gist and LS-gist by $\overline{p}_i := \sigma_i(p_i)$ or $\overline{e}_i := \sigma_i(e_i)$, respectively; when using the $m$-basis, the index set $\alpha \vdash (\delta, n)$ should be replaced by $\alpha \vdash (\delta, n)$ and $e_i$ should be replaced by $\overline{m}_\alpha := \sigma_i(m_\alpha)$. The relative performance of the algorithms using different bases will be evaluated in Section 8.

**8. Experiments.** In this section, we report some experimental results to show the effectiveness and efficiency of the two approaches presented in this paper. These experiments were performed using Maple on a Windows laptop with an Intel(R) Core(TM) i7-7660U CPU in 2.50GHz and 8GB RAM.

In Table 2, we compare the performance of the three algorithms described in this paper for checking the $\mu$-symmetry of polynomials: $G$-gist, LS-gist and CR-gist. We use a test suite of 12 polynomials of degrees ranging from 6–20 (see Table 2), with corresponding $\mu$ with $n = |\mu|$ ranging from 4–6. These polynomials are either $D^+$ polynomials or subdiscriminants, or some perturbations (to create non-$\mu$-symmetric polynomials).

| F   | $\delta$ | $\mu$   | n | Y/N | G-gist Time (sec) | LS-gist Time (sec) | speedup (G-gist/LS-gist) | canoniZE (sec) | $\mu$-gist (sec) | reduce time (sec) | total=canoniZE time + reduce time (sec) | $\mu$-gist/CR-gist (sec) |
|-----|---------|---------|---|-----|------------------|-------------------|------------------------|----------------|----------------|----------------|----------------------------------|-----------------------|
| F1  | 12      | [1,1,1,1] | 4 | Y   | 0.434            | 0.235             | 1.9                     | 0.094          | 0.000          | 0.000          | 4.8                             |                       |
| F2  | 16      | [2,1,1,1] | 4 | Y   | 0.368            | 0.315             | 1.2                     | 0.016          | 0.031          | 0.031          | 10.6                             |                       |
| F3  | 20      | [1,1,1,1,1] | 5 | Y   | 0.844            | 0.283             | 3.1                     | 0.377          | 0.031          | 0.031          | 9.0                              |                       |
| F4  | 24      | [2,1,1,1,1] | 5 | Y   | 0.714            | 0.188             | 3.7                     | 0.391          | 0.040          | 0.040          | >1478                            |                       |
| F5  | 30      | [2,1,1,1,1] | 5 | N   | 0.630            | 0.144             | 4.5                     | 0.000          | 0.014          | 0.014          | >1478                            |                       |
| F6a | 40      | [2,1,1]   | 5 | Y   | 1.845            | 0.392             | 4.7                     | 0.340          | 0.097          | 0.097          | >2176                            |                       |
| F6b | 40      | [2,1,2]   | 5 | Y   | 0.978            | 0.000             | Inf                     | 0.000          | 0.016          | 0.016          | 4.9                              |                       |
| F6c | 40      | [2,1,1]   | 5 | N   | 0.438            | 0.000             | Inf                     | 0.000          | 0.016          | 0.016          | 4.9                              |                       |
| F7  | 50      | [2,2,1]   | 5 | N   | 0.456            | 0.000             | Inf                     | 0.000          | 0.016          | 0.016          | 4.9                              |                       |
| F8  | 60      | [3,1,1]   | 6 | Y   | 0.690            | 0.360             | >1079                  | 0.158          | 0.000          | 0.000          | >2110                            |                       |
| F9  | 60      | [2,2,2]   | 6 | Y   | 0.773            | 0.000             | Inf                     | 0.000          | 0.000          | 0.000          | Inf                              |                       |

From Table 2 it is clear that LS-gist is significantly faster than G-gist. There is one anomaly in the table: for the polynomial $F_3$, G-gist is 5 times faster than LS-gist. This is when $\mu$ is $[1, \ldots, 1]$, which indicates that the ideal $I_{\mu} = \langle v_1, \ldots, v_n \rangle$ has a symmetric structure in $r$. We believe it is because the Gröbner basis of $I$ can be computed very efficiently for certain types of structures.

From Table 3 we observe that the algorithm $m$-LSgist is more efficient than the other
Table 3

Timing for computing the gists of ℓ*-symmetric polynomials with Gröbner basis method using different bases (i.e., e-Ggist, p-Ggist and c-Ggist), with canonize+reduce using different bases (i.e., e-CRGist, p-CRGist, c-CRGist and e-CRGist) and with linear system solving using different bases (i.e., e-LSgist, p-LSgist, c-LSgist and e-LSgist). The most efficient method for each case is marked with * next to the running time.

| F  | Gröbner basis method | canonize+reduce | Linear system solving |
|----|----------------------|-----------------|-----------------------|
|    | e-Ggist (sec)        | p-Ggist (sec)    | c-Ggist (sec)         |
| F1 | 0.219                | 0.344           | 0.187                 |
| F2 | 0.128                | 0.289           | 0.506                 |
| F3 | 20.9                 | 61.2            | 60.3                  |
| F4 | >3000                | >3000           | 3.19                  |
| F5 | 41.2                 | 50.0            | 0.016                 |
| F6 | 0.041                | >3000           | 0.015                 |
| F7 | 0.281                | >3000           | 0.047                 |
| F8 | >3000                | >3000           | 0.047                 |
| F9 | >3000                | >3000           | 0.047                 |
| F10| >3000                | >3000           | 1.50                  |
| F11| >3000                | >3000           | 0.047                 |
| F12| >3000                | >3000           | 0.047                 |
| F13| >3000                | >3000           | 0.047                 |

Table 4

Timing for computing the gists using e-Ggist, canonize+reduce and linear system solving with e-basis when µ and δ are fixed. Here µ = (2, 2, 1) and δ = 10.

| F  | Y/N | Grobner Basis | Normal Form | canonize | reduce | e-LSgist |
|----|-----|---------------|-------------|----------|--------|---------|
|    |     | Time (sec)    | Time (sec)  | Time (sec) | Time (sec) | Time (sec) |
| F10| Y   | 0.188         | 0.063       | 0.000     | 0.063  |
| F11| Y   | 0.203         | 0.074       | 0.016     | 0.016  |
| F12| Y   | 0.203         | 0.074       | 0.000     | 0.046  |
| F13| N   | 0.544         | 0.074       | 0.000     | 0.062  |

However, from Table 4 we see that for fixed µ and δ, once we have computed the canonical set in the preprocessing step, the time cost for reduce is small. Therefore, when evaluating the total time for several examples sharing the same canonical set, the algorithm based on canonize+reduce can be superior to the linear solving method. Although the Gröbner basis method also contains a preprocessing procedure, the time cost for computing normal forms is quite expensive and thus it is not as competitive as algorithms based on canonize+reduce and linear system solving. Furthermore, for algorithms using p-basis, the algorithm p-LSgist shows higher efficiency than p-Ggist and p-CRGist, especially for big δ and n (See F3, F4 and F7). This could be attributed to the small number of terms in the generator polynomials. In contrast, for e-basis and c-basis, the algorithms e-CRGist and c-CRGist prevail over e-LSgist and c-LSgist. The possible reason might be that many terms will get canceled when computing a canonical sequence.
9. Conclusion. We have introduced the concept of $\mu$-symmetric polynomial which generalizes the classical symmetric polynomial. Such $\mu$-symmetric functions of the roots of a polynomial can be written as a rational function in its coefficients. Our original motivation was to study a conjecture that a certain polynomial $D^+ (\mu)$ is $\mu$-symmetric. In order to explore such properties for different $\mu$’s and other root functions, we introduce three algorithms to compute the $\mu$-gist of a polynomial (or detect that no such gists exist). With the help of these algorithms, we verified the $\mu$-symmetry of $D^+$ for many specific cases. In a companion paper [5], we will prove the $\mu$-symmetry conjecture on $D^+$ and show its application in the complexity analysis of root clustering.

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