Rota—Baxter operators on a sum of fields
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Abstract

We count the number of all Rota—Baxter operators on a finite direct sum $A = F \oplus F \oplus \ldots \oplus F$ of fields and count all of them up to conjugation with an automorphism. We also study Rota—Baxter operators on $A$ corresponding to a decomposition of $A$ into a direct vector space sum of two subalgebras. We show that every algebra structure induced on $A$ by a Rota—Baxter of nonzero weight is isomorphic to $A$.

Keywords: Rota—Baxter operator, (un)labeled rooted tree, 2-coloring, subtree acyclic digraph, transitive digraph.

1 Introduction

Given an algebra $A$ and a scalar $\lambda \in F$, where $F$ is a ground field, a linear operator $R: A \to A$ is called a Rota—Baxter operator (RB-operator, for short) on $A$ of weight $\lambda$ if the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)$$

holds for any $x, y \in A$. The algebra $A$ is called Rota—Baxter algebra (RB-algebra).

G. Baxter in 1960 introduced the notion of Rota—Baxter operator [3] as natural generalization of by parts integration formula. In 1960–1970s such operators were studied by G.-C.Rota [19], P. Cartier [10], J. Miller [17], F. Atkinson [2] and others.

In 1980s, the deep connection between constant solutions of the classical Yang—Baxter equation from mathematical physics and RB-operators on a semisimple finite-dimensional Lie algebra was discovered by A. Belavin and V. Drinfel’d [1] and M. Semenov-Tyan-Shanskii [20].

About different connections of Rota—Baxter operators with symmetric polynomials, quantum field renormalization, Loday algebras, shuffle algebra see in the monograph [14] written by L. Guo in 2012.

In the paper, we study Rota—Baxter operators on a finite direct sum $A = F \oplus F \oplus \ldots \oplus F$ of $n$ copies of a field $F$. We continue investigations fulfilled by S. de Braganca in 1975 [6] and by H. An and C. Bai in 2008 [1]. Since all RB-operators on $A$ of weight zero are trivial [12], i.e., equal to 0, we study only RB-operators on $A$ of nonzero weight $\lambda$.

In §2, we formulate some preliminaries about RB-operators, including splitting RB-operators which are projections on a subalgebra $A_1$ parallel to another one $A_2$ provided the direct vector space sum decomposition $A = A_1 \oplus A_2$.

In §3, we show that RB-operators on $A$ of nonzero weight $\lambda$ are in bijection with 2-colored transitive subtree acyclic digraphs (subtree acyclic digraphs were defined by F. Harary et al. in 1992 [15]) or equivalently with labeled rooted trees on $n + 1$ vertices.
with 2-colored non-root vertices. For the last, we apply the result of R. Castelo and A. Siebes [11]. Thus, the number of all RB-operators on $A$ of nonzero weight $\lambda$ equals $2^n(n+1)^{n-1}$. With the help of the bijection, we show that splitting RB-operators on $A$ of nonzero weight $\lambda$ are in one-to-one correspondence with labeled rooted trees on $n+1$ vertices with properly 2-colored non-root vertices. We also study the number of all RB-operators and all splitting RB-operators on $A$ up to conjugation with an automorphism of $A$.

In 2012, D. Burde et al. initiated to study so called post-Lie algebra structures [7]. One of the questions arisen in the area [7, 8, 9] is the following one: starting with a semisimple Lie algebra endowed RB-operator of weight 1 what kind of Lie algebras we will get under the new Lie bracket $[R(x), y] + [x, R(y)] + [x, y]$? Such problems could be stated not only for Lie algebras but also for associative or commutative ones. In §4, we show that every algebra structure induced on a finite direct sum $A$ of fields by a Rota–Baxter operator of nonzero weight is isomorphic to $A$ itself.

2 Preliminaries

Trivial RB-operators of weight $\lambda$ are zero operator and $-\lambda \text{id}$.

**Statement 1** [14]. Given an RB-operator $R$ of weight $\lambda$,

a) the operator $-R - \lambda \text{id}$ is an RB-operator of weight $\lambda$,

b) the operator $\lambda^{-1}R$ is an RB-operator of weight 1, provided $\lambda \neq 0$.

Given an algebra $A$, let us define a map $\phi$ on the set of all RB-operators on $A$ as $\phi(R) = -R - \lambda(R)\text{id}$. It is clear that $\phi^2$ coincides with the identity map.

**Statement 2** [5]. Given an algebra $A$, an RB-operator $R$ on $A$ of weight $\lambda$, and $\psi \in \text{Aut}(A)$, the operator $R^{(\psi)} = \psi^{-1}R\psi$ is an RB-operator on $A$ of weight $\lambda$.

**Statement 3** [14]. Let an algebra $A$ to split as a vector space into the direct sum of two subalgebras $A_1$ and $A_2$. An operator $R$ defined as

$$R(a_1 + a_2) = -\lambda a_2, \quad a_1 \in A_1, \ a_2 \in A_2,$$

is RB-operator on $A$ of weight $\lambda$.

Let us call an RB-operator from Statement 3 as splitting RB-operator with subalgebras $A_1, A_2$. Note that the set of all splitting RB-operators on an algebra $A$ is in bijection with all decompositions $A$ into a direct sum of two subalgebras $A_1, A_2$.

**Remark 1.** Given an algebra $A$, let $R$ be a splitting RB-operator on $A$ of weight $\lambda$ with subalgebras $A_1, A_2$. Hence, $\phi(R)$ is an RB-operator of weight $\lambda$ and

$$\phi(R)(a_1 + a_2) = -\lambda a_1, \quad a_1 \in A_1, \ a_2 \in A_2.$$ 

So $\phi(R)$ is splitting RB-operator with the same subalgebras $A_1, A_2$.

**Lemma 1** [5]. Let $A$ be a unital algebra, $R$ be an RB-operator on $A$ of nonzero weight $\lambda$. If $R(1) \in F$, then $R$ is splitting.

We call an RB-operator $R$ satisfying the conditions of Lemma 1 as inner-splitting one.
Lemma 2. Let $A = A_1 \oplus A_2$ be an algebra, $R$ be an RB-operator on $A$ of weight $\lambda$. Then the induced linear map $P: A_1 \to A_1$ defined by the formula $P(x_1 + x_2) = \Pr_{A_1}(R(x_1))$, $x_1 \in A_1$, $x_2 \in A_2$, is an RB-operator on $A_1$ of weight $\lambda$.

3 RB-operators on a sum of fields

Statement 4. Let $A = Fe_1 \oplus Fe_2 \oplus \ldots \oplus Fe_n$ be a direct sum of copies of a field $F$. A linear operator $R(e_i) = \sum_{k=1}^{n} r_{ik} e_k$, $r_{ik} \in F$, is an RB-operator on $A$ of weight 1 if and only if the following conditions are satisfied:

(SF1) $r_{ii} = 0$ and $r_{ik} \in \{0, 1\}$ or $r_{ii} = -1$ and $r_{ik} \in \{0, -1\}$ for all $k \neq i$;

(SF2) if $r_{ik} = r_{ki} = 0$ for $i \neq k$, then $r_{il} r_{kl} = 0$ for all $l \notin \{i, k\}$;

(SF3) if $r_{ik} \neq 0$ for $i \neq k$, then $r_{ki} = 0$ and $r_{kl} = 0$ or $r_{il} = r_{ik}$ for all $l \notin \{i, k\}$.

Example. Let $A = Fe_1 \oplus Fe_2 \oplus \ldots \oplus Fe_n$. For $n = 2$, we have 12 cases $\{0, -\text{id}\}$. For $n = 1$, we have only two RB-operators $\{0, -\text{id}\}$. For $n = 2$, we have 12 cases $\{0, -\text{id}\}$.

Remark 2. It follows from (SF3) that $r_{ik} r_{ki} = 0$ for all $i \neq k$. In [1], the statement of Statement 4 was formulated with this equality and (SF1) but without (SF2) and the general version of (SF3). That’s why the formulation in [1] seems to be not complete.

Remark 3. The sum of fields in Statement 4 can be infinite.

In advance, we will identify an RB-operator on $A$ with its matrix.

Let us calculate the number of different RB-operators of nonzero weight $\lambda$ on $A = Fe_1 \oplus Fe_2 \oplus \ldots \oplus Fe_n$. By Statement 1a, we may assume that $\lambda = 1$. For $n = 1$, we have only two RB-operators $\{0, -\text{id}\}$. For $n = 2$ we have 12 cases $\{0, -\text{id}\}$.

$$
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}, \begin{pmatrix}
-1 & 0 \\
0 & -1 \\
0 & 1 \\
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix}, \begin{pmatrix}
-1 & 0 \\
-1 & -1 \\
0 & 0 \\
\end{pmatrix}, \begin{pmatrix}
-1 & 0 \\
0 & 0 \\
0 & 1 \\
\end{pmatrix}, \begin{pmatrix}
-1 & 0 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & -1 \\
0 & 0 \\
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix}.
$$

Here we identify an RB-operator with its matrix $R \in M_2(F)$ by the rule $R(e_i) = \sum_{k=1}^{n} r_{ik} e_k$.

For $n = 3$, we have $8 \cdot 16 = 128$ variants $\{0, -\text{id}\}$.
For $n = 4$, computer can help to state that there are exactly 2000 RB-operators of weight 1 on $A$. Thus, we get the first four terms from the sequence $A097629$ [18].

**Theorem 1.** Let $A = F e_1 \oplus F e_2 \oplus \ldots \oplus F e_n$ be a direct sum of copies of a field $F$. The number of different RB-operators on $A$ of nonzero weight $\lambda$ equals $2^n(n + 1)^{\lambda - 1}$.

**Proof.** Let $R$ be an RB-operator on $A$ of weight $\lambda$. We may assume that $\lambda = 1$. We follow the previous notations. We have $2^n$ variants to choose the values of the elements $r_{ii}, i = 1, \ldots, n$. The choice of any of them, say $r_{ii}$, influences only on the possible signs of all elements $r_{ik}, k \neq i$. So, we may put $r_{ii} = 0$ for all $i$ and fix the factor $2^n$ for the answer.

Now, we want to construct a directed graph $G$ on $n$ vertices by any matrix $R = (r_{ij})_{i,j=1}^n$ with chosen $r_{ii} = 0$. We consider the matrix $R$ as the adjacency matrix of a directed graph $G$. Let us interpretate conditions (SF2) and (SF3) in terms of digraphs. Firstly, we rewrite (SF3) as two conditions:

(SF3a) if $r_{ik} \neq 0$ for $i \neq k$, then $r_{ki} = 0$;
(SF3b) if $r_{ik} \neq 0$ for $i \neq k$, then $r_{kl} = 0$ or $r_{il} = r_{ik}$ for all $l \notin \{i, k\}$.

The condition (SF3a) says that if we have an edge between two vertices $i \neq k$, then the direction of such edge is well-defined, so, it is a correctness of getting a digraph by the matrix $R$. In graph theory, the condition (SF3b) is called transitivity, i.e., if have edges $(i, k) \in E$ and $(k, l) \in E$, then we have an edge $(i, l) \in E$.

Secondly, we read the condition (SF2) in terms of digraphs in such way: there are no in $G$ induced subgraphs isomorphic to $H$ with $V(H) = \{i, k, l\}$ and $E(H) = \{(i, l), (k, l)\}$ (see Pict. 1). In [11] the subgraph $H$ was called immorality, thus, a digraph without immoralities is called moral digraph [16].

![Picture 1](image)

**Picture 1.** The forbidden induced subgraph $H$ on three vertices $\{i, k, l\}$ due to (SF2)

We may reformulate our problem of counting the number $N$ of different RB-operators on $A$ of nonzero weight $\lambda$ in such way: What is the number of all transitive moral transitive digraphs on $n$ vertices? In terms of [11], the last is the same as the number of
all moral TDAGs on \( n \) vertices, here TDAG is the abbreviation for Transitive Directed Acyclic Graph (we are interested on transitive digraphs which are surely acyclic). In the graph-theoretic context, moral DAGs are known as subtree acyclic digraphs \[15\]. Thus, 

\[
N/2^n = \#\{\text{moral TDAGs on } n \text{ vertices}\} = \#\{\text{transitive subtree acyclic digraphs on } n \text{ vertices}\}. \tag{3}
\]

In \[11\], the authors constructed a bijection between the set of moral TDAGs on \( n \) vertices and the set of labeled rooted trees on \( n + 1 \) vertices as follows (see Pict. 2). Define the function \( f(i) \) for a vertex \( i \) by induction. For a source \( i \) (i.e., such a vertex \( i \) that there are no edges \((j, i)\) in a digraph), we put \( f(i) = 0 \). For a not-source vertex \( j \), we may find the unique source \( i \) such that there exists a directed path \( p \) from \( i \) to \( j \). So, we define \( f(j) \) as the length of \( p \). Now, we construct a labeled rooted tree \( T = (U, F) \) by a moral TDAG \( G = G(V, E) \):

\[
U = V \cup \{0\}, \quad F = \{(0, i) \mid f(i) = 0\} \cup \{(i, j) \mid (i, j) \in E, f(i) = f(j) - 1\}.
\]

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (1,0) {2};
\node (3) at (2,2) {3};
\node (4) at (1,-2) {4};
\node (5) at (-2,0) {5};
\draw[->] (1) -- (2);
\draw[->] (1) -- (3);
\draw[->] (2) -- (3);
\draw[->] (2) -- (4);
\draw[->] (5) -- (2);
\node (0) at (3,0) {0};
\node (1) at (4,0) {1};
\node (2) at (5,0) {2};
\node (3) at (6,2) {3};
\node (4) at (5,-2) {4};
\node (5) at (4,0) {5};
\draw[->] (0) -- (1);
\draw[->] (1) -- (2);
\draw[->] (2) -- (3);
\draw[->] (2) -- (4);
\end{tikzpicture}
\end{center}

**Picture 2.** The corresponding graph \( G \) and tree \( T \) to the RB-operator \( R(e_1) = e_2 + e_3 + e_4, R(e_2) = -e_2 - e_3 - e_4, R(e_3) = -e_3, R(e_4) = 0, R(e_5) = -e_5 \).

Applying the above constructed correspondence, the number of moral TDAGs on \( n \) vertices equals \((n + 1)^{n-1}\) by the Cayley theorem, and so \( N = 2^n(n + 1)^{n-1} \). Theorem is proved.

Below we will apply the easy fact that \( \text{Aut}(A) \cong S_n \). It could be derived, e.g., from the Molin—Wedderburn—Artin theory, in particular from the uniqueness up to a rearrangement of summands of decomposition of a semisimple finite-dimensional associative algebra into a finite direct sum of simple ones.

**Corollary 1** \[6\]. Let \( A = Fe_1 \oplus Fe_2 \oplus \ldots \oplus Fe_n \) be a direct sum of copies of a field \( F \) and \( R \) be an RB-operator on \( A \) of nonzero weight 1. There exists an automorphism \( \psi \) of \( A \) such that the matrix of the operator \( R^{(\psi)} \) in the basis \( e_1, \ldots, e_n \) is an upper-triangular matrix with entries \( r_{ij} \in \{0, \pm 1\} \) and \( r_{ii} \in \{0, -1\} \).
Proof. As we did in the proof of Theorem 1, we define by $R$ a labeled rooted tree $T$. Define $t = \max\{f(i) \mid i \in V(T)\}$ and $k_j = \#\{i \mid f(i) = j\}$. We may reorder indexes $1, 2, \ldots, n$ by action of a permutation from $S_n \cong \text{Aut}(A)$ in a way such that

$$f(1) = \ldots = f(k_0) = 0,\quad f(k_0 + 1) = \ldots = f(k_0 + k_1) = 1,\ldots\quad f(n - k_t + 1) = \ldots = f(n) = t.$$

Due to the definition of $T$, we get the upper-triangular matrix. The restrictions on the values of elements immediately follow from Statement 4.

**Corollary 2.** There is a bijection between the set of RB-operators of nonzero weight $\lambda$ on $Fe_1 \oplus Fe_2 \oplus \ldots \oplus Fe_n$ and

a) the set of 2-colored subtree acyclic digraphs on $n$ vertices;

b) the set of labeled rooted trees on $n + 1$ vertices with 2-colored non-root vertices.

Now, we want to compute the number $r_n$ of RB-operators of nonzero weight $\lambda$ on $A = Fe_1 \oplus \ldots \oplus Fe_n$ which lie in different orbits under the action of the automorphism group $\text{Aut}(A) \cong S_n$. The group $\text{Aut}(A)$ acts on the set of RB-operators of weight $\lambda$ in the way described in Statement 2, $\psi: R \rightarrow R^{(\psi)} = \psi^{-1} R \psi$.

In a light of Corollary 2b, we may interpretate the number $r_n$ as the number of unlabeled rooted trees on $n + 1$ vertices with 2-colored non-root vertices. It is exactly the sequence A000151 [18], the first eight values are 2, 7, 26, 107, 458, 2058, 9498, 44947 etc. Let us fix that in advance we will use two colors: white and black, white color corresponds to the case $r_{ii} = 0$ and black color corresponds to $r_{ii} = -\lambda$. Considering the rooted tree $T$ with $n + 1$ vertices, we may assume that the root is colored in the third color, say grey.

Note that the map $\phi$ acts on a labeled (or unlabeled) rooted tree $T$ on $n + 1$ vertices with 2-colored non-root vertices as follows. The $\phi$ interchanges a color in every non-root vertex.

Let us describe splitting RB-operators of nonzero weight $\lambda$ on $A$.

**Theorem 2.** An RB-operator $R$ of nonzero weight $\lambda$ on $A = Fe_1 \oplus \ldots \oplus Fe_n$ is splitting if and only if the corresponding (labeled) rooted tree $T = T(R)$ on $n + 1$ vertices is properly colored.

Proof. Without loss of generality, we put $\lambda = 1$. For simplicity, let us consider the graph $T' = T \setminus \{\text{root}\}$, which is a forest in general case.

Let us prove the statement by induction on $n$. For $n = 1$, we have either $R = 0$ (the only non-root vertex is white) or $R = -\lambda \text{id}$ (the only non-root vertex is black), both RB-operators are splitting with subalgebras $F$ and $(0)$.

Suppose that we have proved Theorem 2 for all natural numbers less than $n$. Let a graph $T'$ with $n$ vertices be disconnected, denote by $T_1, \ldots, T_k$ the connected components of $T'$. So, $A = A_1 \oplus \ldots \oplus A_k$ for $A_s = \text{Span}\{e_j \mid j \in V(T_s)\}$. Define $R_s$ as the induced RB-operator $R|_{A_s}$ (see Lemma 2). By the definition, $R$ is splitting if and only if $A =$
ker(R) + ker(R + id) or equivalently $A_s = \ker(R_s) + \ker(R_s + \text{id})$, $s = 1, \ldots, k$. By the induction hypothesis, we have such decomposition for every $s$ if and only if the coloring of $T_s$ is proper.

Now consider the case when $T'$ is connected. We may assume that $e_1$ corresponds to the vertex 1, the only source in $G$, and $\{2, \ldots, k\}$ is the set of all vertices of $G$ with the value of $f(x)$ equal to 1. We also define $T_s$ for $s = 2, \ldots, k$ as the connected component of $T' \setminus \{1\}$ which contains the vertex $s$. Note that $R$ induces the RB-operator of weight $\lambda$ on the subalgebra $A_s = \text{Span}\{e_j | j \in V(T_s)\}$ for all $s$ by Lemma 2.

The condition of $R$ to be splitting is equivalent to the condition

$$\text{rank}(R) + \text{rank}(R + \text{id}) = n. \quad (4)$$

Analysing the $e_1$-coordinate, we have

$$n = \text{rank}(R) + \text{rank}(R + \text{id}) \geq 1 + \text{rank}(R') + \text{rank}(R' + \text{id})$$

for $R'$, the induced RB-operator on the subalgebra $\text{Span}\{e_j | j \geq 2\}$. Thus, $\text{rank}(R') + \text{rank}(R' + \text{id}) = n - 1$, i.e. $R'$ is splitting or equivalently $R|_{A_s}$ is splitting for every $s = 2, \ldots, k$. By the induction hypothesis, the graph $T' \setminus \{1\}$ is properly 2-colored. It remains to prove that the vertices 2, $\ldots$, $k$ are colored in the same color and the vertex 1 is colored in another one.

Up to the action of $\phi$, which preserves the splitting structure of an RB-operator (see Remark 1), we may assume that the vertex 1 is colored in white. Since we know that $\text{rank}(R + \text{id}) = \text{rank}(R' + \text{id}) + 1$, we have to state the equality $\text{rank}(R) = \text{rank}(R')$. So, the condition (4) is fulfilled if and only if the first row $(0, 1, 1, \ldots, 1)$ of the matrix $R$ is linearly expressed via other rows. By the definition of the matrix $R$, the vertices 2, $\ldots$, $k$ have to be colored in black. Theorem is proved.

**Corollary 3.** An RB-operator $R$ of nonzero weight $\lambda$ on $A = F e_1 \oplus \ldots \oplus F e_n$ is inner-splitting if and only if in $T = T(R)$ all vertices with even value of $f$ are colored in one color and all vertices with odd value of $f$ are colored in another color.

**Proof.** Up to $\phi$, we may assume that $R(1) = 0$. Thus, any vertex with the value of $f(x)$ equal to 0 has to be colored in white. By Theorem 2, $T' = T \setminus \{\text{root}\}$ is properly 2-colored, so, all vertices with the value of $f(x)$ equal to 1 are colored in black, all vertices with the value of $f(x)$ equal to 2 are colored in white and so on.

Now, we collect all our knowledges about all RB-operators (in Table 1) and all nonisomorphic RB-operators (in Table 2) of nonzero weight on a sum of fields $A = F e_1 \oplus F e_2 \oplus \ldots \oplus F e_n$.

We have noticed that the first values of number of splitting RB-operators coincides with the sequence A007830 [18] (in labeled case) and coincides with the sequence A000106 [18] (in unlabeled case). Actually it should be proven for all $n$.

**Remark 4.** Counting rooted trees on $n + 1$ vertices with properly 2-colored non-root vertices is not the same as counting properly 2-colored forests on $n$ vertices.
Table 1. Number of RB-operators of nonzero weight on a sum of $n$ fields

| Class of RB-operators | Description                                                                 | formula and OEIS [18]                                                                 |
|-----------------------|------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|
| all                   | labeled rooted trees on $n + 1$ vertices with 2-colored non-root vertices   | $2^n(n + 1)^{n-1}$, A097629, 2.12.128, 2000.41472                                   |
| splitting             | labeled rooted trees on $n + 1$ vertices with properly 2-colored non-root vertices | $2(n + 2)^{n-1}$, A007830, 2.8.50, 432.4802                                         |
| inner-splitting       | labeled rooted trees on $n + 1$ vertices (twice)                             | $2(n + 1)^{n-1}$, 2·A000272, 2.6.32, 250.2592                                       |
| non-splitting         | labeled rooted trees on $n + 1$ vertices with improperly 2-colored non-root vertices | —, 0.4.78, 1568.36670                                                               |

Table 2. Number of RB-operators of nonzero weight on a sum of $n$ fields (up to conjugation with an automorphism)

| Class of RB-operators | Description                                                                 | OEIS [18]                                                                 |
|-----------------------|------------------------------------------------------------------------------|--------------------------------------------------------------------------------|
| all                   | rooted trees on $n + 1$ vertices with 2-colored non-root vertices           | A000151, 2.7.26.107.458                                                        |
| splitting             | rooted trees on $n + 1$ vertices with properly 2-colored non-root vertices  | A000106, 2.5.12.30.74                                                          |
| inner-splitting       | rooted trees on $n + 1$ vertices (twice)                                     | 2·A000081, 2.4.8.18.40                                                         |
| non-splitting         | rooted trees on $n + 1$ vertices with improperly 2-colored non-root vertices | —, 0.2.14.77.384                                                                |

Let us write down all non-splitting pairwise nonisomorphic RB-operators for $n = 2, 3$.

**Statement 5.** Up to $\phi$, we have the following non-splitting pairwise nonisomorphic RB-operators

a) for $n = 2$: $R(e_1) = e_2$, $R(e_2) = 0$;

b) for $n = 3$:

(RB1) $R(e_1) = e_2 + e_3$, $R(e_2) = e_3$, $R(e_3) = 0$,

(RB2) $R(e_1) = e_2 + e_3$, $R(e_2) = e_3$, $R(e_3) = -e_3$,

(RB3) $R(e_1) = e_2 + e_3$, $R(e_2) = -e_2 - e_3$, $R(e_3) = -e_3$,

(RB4) $R(e_1) = e_2 + e_3$, $R(e_2) = R(e_3) = 0$,

(RB5) $R(e_1) = e_2 + e_3$, $R(e_2) = -e_2$, $R(e_3) = 0$,

(RB6) $R(e_1) = e_2$, $R(e_2) = R(e_3) = 0$,

(RB7) $R(e_1) = e_2$, $R(e_2) = 0$, $R(e_3) = -e_3$.

**Proof.** a) Non-splitting case appears only when the graph $T'$ is non-empty and improperly 2-colored. Up to $\phi$, we may assume that two vertices are colored in white.
b) Cases (RB1)–(RB3) correspond to improperly 2-colorings of the graph $T'$ with $V(T') = \{1, 2, 3\}$ and $E(T') = \{(1, 2), (2, 3)\}$. Cases (RB4), (RB5) correspond to improperly 2-colorings of the graph $T'$ with $E(T') = \{(1, 2), (1, 3)\}$. Finally, cases (RB6), (RB7) correspond to improperly 2-colorings of the graph $T'$ with $E(T') = \{(1, 2)\}$.

**Statement 6.** Up to $\phi$, we have the following splitting but not inner-splitting pairwise nonisomorphic RB-operators:

a) for $n = 2$: $R(e_1) = -e_1, R(e_2) = 0$;
b) for $n = 3$:
(RB1') $R(e_1) = e_2, R(e_2) = 0, R(e_3) = -e_3$,
(RB2') $R(e_1) = -e_1, R(e_2) = R(e_3) = 0$.

4 RB-induced algebra structures on a sum of fields

Let $C$ be an associative algebra and $R$ be an RB-operator on $C$ of weight $\lambda$. Then the space $C$ under the product

$$x \circ_R y = R(x)y + xR(y) + \lambda xy \tag{5}$$

is an associative algebra [14, 13]. Let us denote the obtained algebra as $C^R$. It is easy to see that $C^{\phi(R)} \cong C^R$.

Let us denote by $A_n$ the $n$-dimensional algebra with zero (trivial) product.

**Theorem 3.** Given an algebra $A = Fe_1 \oplus \ldots \oplus Fe_n$ and an RB-operator $R$ of weight $\lambda$ on $A$, we have $A^R \cong \begin{cases} A_n, & \lambda = 0, \\ A, & \lambda \neq 0. \end{cases}$

**Proof.** If $\lambda = 0$, then $R = 0$ [12] and $x \circ_R y = 0$. For $\lambda \neq 0$, we may assume that $\lambda = 1$, since recalling of the product does not exchange the algebraic structure.

Let us prove the statement by induction on $n$. For $n = 1$, we have either $R = 0$ or $R = -\text{id}$. Due to (5) we get either $x \circ y = xy$ or $x \circ y = -xy$, in both cases $A^R \cong A$.

Suppose that we have proved Theorem 3 for all numbers less $n$. Let a graph $T' = T'(R)$ with $n$ vertices be disconnected, denote by $T_1, \ldots, T_k$ the connected components of $T'$. As earlier, we define $A = A_1 \oplus \ldots \oplus A_k$ and $A_s = \text{Span}\{e_j \mid j \in V(T_s)\}$ and define $R_s$ as the induced RB-operator $R_s|_{A_s}$. By the induction hypothesis, $A_s^R \cong A_s$ for every $s$ and so $A = A_1 \oplus \ldots \oplus A_k \cong A_1^R \oplus \ldots \oplus A_k^R = A^R$.

Now consider the case when $T'$ is connected. We may assume that $e_1$ corresponds to the vertex 1, the only source in $G$. Note the space $I_1 = \text{Span}\{e_j \mid j \geq 2\}$ is an ideal in $A^R$ which is isomorphic to $Fe_2 \oplus \ldots \oplus Fe_n$ by the induction hypothesis. Up to $\phi$, we may assume that the vertex 1 in $T'$ is colored in white and $2, \ldots, t$ is a list of all neighbours of 1 in $T'$. Let us consider the one-dimensional space $I_2$ in $A^R$ generated by the vector $a = e_1 - c(2)e_2 - \ldots - c(t)e_t$, where

$$c(i) = \begin{cases} 1, & \text{i is colored in white,} \\ -1, & \text{i is colored in black.} \end{cases}$$
In terms of the matrix entries, $c(i) = 1 + 2r_{ii}$. We may assume that $c(2) = c(3) = \ldots = c(s) = 1$ and $c(s + 1) = \ldots = c(t) = -1$ for some $s \in \{2, \ldots , t\}$.

By (5) we compute the product of $a$ with $e_k$ for $k > t$:

$$a \circ e_k = (e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t) \circ e_k$$

$$= R(e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t)e_k.$$

Since $k$ is connected with only one vertex from $2, \ldots , t$ (due to (SF2)), say $j$, we have

$$a \circ e_k = R(e_1 - c(j)e_j)e_k = e_k - c(j)(1 + 2r_{jj})e_k = (1 - (c(j))^2)e_k = 0.$$

Analogously we can check that $a \circ e_k = 0$ for all $k > 1$. Thus, $I_2$ is an ideal in $A^R$.

Now, we calculate

$$a \circ a = e_1 \circ (e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t)$$

$$= R(e_1)(e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t) + e_1$$

$$= (e_2 + \ldots + e_s + e_{s+1} + \ldots + e_t)(e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t) + e_1$$

$$= e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t = a$$

and so $I_2$ is isomorphic to $F$.

Summarising, we have $A^R = I_1 \oplus I_2 \cong (Fe_2 \oplus \ldots \oplus Fe_n) \oplus F \cong A$. Theorem is proved.

Acknowledgements

The main part of the paper was done while working in Sobolev Institute of Mathematics in 2017. The research is supported by RSF (project N 14-21-00065).

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