Transverse instability and disintegration of domain wall of relative phase in coherently coupled two-component Bose-Einstein condensates

Kousuke Ihara and Kenichi Kasamatsu

Department of Physics, Kindai University, Higashi-Osaka, Osaka 577-8502, Japan

(Dated: April 5, 2019)

We study transverse instability and disintegration dynamics of a domain wall of a relative phase in two-component Bose-Einstein condensates with coherent Rabi coupling. We obtain analytically the phase diagram of the stationary solution of the domain wall in the plane of the Rabi-coupling and the intercomponent coupling constant, and study numerically the energetic and dynamical stability. Outside the stable region, the domain wall is always dynamically unstable for the transverse modulation. The nonlinear evolution associated with the instability is demonstrated through numerical simulations for both the domain wall without edges and that with edges formed by the quantized vortices.

PACS numbers: 03.75.Kk 03.75.Lm

I. INTRODUCTION

Solitons in multidimensional systems are generally unstable, known as the transverse instability [1, 2], where the solitonic structure is dynamically unstable against the symmetry-breaking modulation along the unbounded spatial dimension. Bose-Einstein condensates (BECs) are well-suited system to study such an unstable properties of solitons. This is because: (i) The properties of BECs are described by the Gross-Pitaevskii (GP) (non-linear Schrödinger) equation, which allows various solitary wave solutions. (ii) The system consists of dilute gases and is almost far from dissipation so that nonequilibrium processes of unstable dynamics can be directly observed from the well-defined initial states. (iii) The topological solitons can be created by well-developed phase-engineering techniques. For example, the dark solitons in two-dimensional (2D) BECs are broken up to an array of vortex pairs due to the transverse instability, called “snake instability” [3, 4]. The decay of a planar dark soliton into a vortex ring was observed experimentally in a 3D BEC [5, 6].

For two-component BECs described by the two-component order parameters, the structures of solitons are richer than those in single-component BECs, e.g., dark solitons [7], dark-bright solitons [8, 9], domain walls [10], and magnetic solitons [11]. For a typical two-component mixture of BECs, the U(1)-phases in each component are independent variables, because the two components are coupled only through the density-density coupling. However, when there are the Rabi coupling between two components, the symmetry associated with one of the two U(1)-phase is broken, and the relative phase between the two components makes sense. Then, the new type of soliton can exist as “a domain wall of the relative phase”, which is obtained as a solution of the sine-Gordon equation for the two-component BECs. This domain wall, firstly predicted by Son and Stephanov [12], can exist as a bound string between two vortices in two-component BECs, which results in a molecule of the vortices [13].

Recently, real time dynamics of the aforementioned vortex molecule in the Rabi-coupled condensates has been studied by some authors [14,16]. The authors in Refs. [14,16] have especially stressed that this system can be used as the analogous simulations of the “confinement” in quark–anti-quark phenomena. Tylutki et al. studied the precession dynamics of the vortex molecule as a function of the Rabi coupling and the molecular distance [14]. In a certain regime, the domain wall was found to disintegrate into some parts, which is analogous to the string breaking phenomena in quantum chromodynamics. This observation was further confirmed by the simulations by Eto and Nitta [10].

Here, we reveal the physical origin of this disintegration as the instability associated with the transverse displacement of the domain wall, i.e, the snake instability. We obtain an exact solution of the domain wall for the 1D coupled GP equations for two-component BECs, making a phase diagram of the stationary solutions in the plane of the intercomponent coupling strength and the Rabi frequency. The energetic stability of the solution reproduces the previous work by Usui and Takeuchi [17]. In the unstable regime, the dynamical instability takes place for the modulation along the transverse direction parallel to the wall, which is analyzed by the Bogoliubov-de Gennes (BdG) equation. The nonlinear dynamics associated with this instability is shown by direct numerical simulations of the 2D GP equation for the domain wall without edges and with edges formed by quantized vortices. The disintegration observed in previous works can be explained as the manifestation of the snake instability.

The paper is organized as follows. After introducing the formulation of the problem in Sec. I, we first study the stability of the domain wall of the relative phase based on the exact solution and the BdG analysis around the solution in Sec. II. In Sec. III, we demonstrate the transverse instability through the direct numerical simulations of the 2D GP equations. Section IV is devoted to the conclusion.
II. FORMULATION

We consider two-component BECs of ultracold atoms having the same mass \( m \) and residing in two different hyperfine states. The BECs are described by the condensate wave functions \( \Psi_j \) \((j = 1, 2)\). The equilibrium state of the system is obtained by minimizing the Gross-Pitaevskii (GP) energy functional

\[
E[\Psi_1, \Psi_2] = \int dr \left[ \sum_{j=1,2} \left( \frac{\hbar}{2} \nabla \Psi_j + \frac{g_j}{2} |\Psi_j|^4 \right) + g_{12} |\Psi_1|^2 |\Psi_2|^2 - \frac{\hbar \Omega_{\text{R}}}{2} (\Psi_1^* \Psi_2 + \Psi_2^* \Psi_1) \right]. \tag{1}
\]

Here, \( h = \hbar^2 \nabla^2 / (2m) + V_{\text{ext}} \) is the single-particle Hamiltonian with the trapping potential \( V_{\text{ext}} \). The coupling constants \( g_j \) \((j = 1, 2)\) and \( g_{12} \) represent the strength of intra- and the intercomponent interactions, respectively, described as \( g_j = 4\pi \hbar^2 a_j / m \) and \( g_{12} = 4\pi \hbar^2 a_{12} / m \) with the s-wave scattering lengths \( a_j \) and \( a_{12} \) between the corresponding atoms. We assume that the intercomponent coupling constants satisfy \( g_1 = g_2 = g \) for simplicity. The last term in Eq. (1) describes a coherent Rabi coupling induced by an external electromagnetic field, which allows atoms to transfer their internal states coherently \cite{18, 19}; \( \Omega_{\text{R}} \) stands for the Rabi frequency. Then, the total number \( N = N_1 + N_2 = \int \text{d}r |\Psi_1|^2 + |\Psi_2|^2 \) is a constant of motion. In the following analysis except the numerical simulations in Sec. IV, we assume the homogeneous system by setting \( V_{\text{ext}} = 0 \). For the coupling constants, we confine ourselves to the range \(-1 < g_{12} / g < 1\) corresponding to a miscible regime without the Rabi coupling, otherwise the components phase separate for \( g_{12} > g \) \cite{20, 21} or undergo mean-field collapse for \( g_{12} < g \) \cite{22}.

The time-dependent GP equations are given by the variational procedure \( i\hbar \partial \Psi_j / \partial t = \delta E / \delta \Psi_j^* \) as

\[
\begin{align*}
\hbar \frac{\partial \Psi_1}{\partial t} &= -\frac{\hbar^2 \nabla^2}{2m} \Psi_1 + g |\Psi_1|^2 \Psi_1 + g_{12} |\Psi_2|^2 \Psi_1 - \frac{\hbar \Omega_{\text{R}}}{2} \Psi_2, \tag{2} \\
\hbar \frac{\partial \Psi_2}{\partial t} &= -\frac{\hbar^2 \nabla^2}{2m} \Psi_2 + g |\Psi_2|^2 \Psi_2 + g_{12} |\Psi_1|^2 \Psi_2 - \frac{\hbar \Omega_{\text{R}}}{2} \Psi_1. \tag{3}
\end{align*}
\]

First, we consider the ground state in the homogeneous system by ignoring the time and spatial derivative terms in Eqs. (2) and (3). Because of the conservation of the total particle number, the chemical potential as a Lagrange multiplier for both components should be common as \( \mu_1 = \mu_2 = \mu \), the stationary wave function being written as \( \Psi_j = e^{i \mu t / \hbar} \sqrt{n_j} \). Then, the relative phase \( \theta = \theta_1 - \theta_2 \) should be vanished, because the energy of the Rabi-coupling term \(-\hbar \Omega_{\text{R}} \sqrt{n_1 n_2} \cos \theta \) is minimized at \( \theta = 0 \). As a result, the densities in a miscible regime satisfy \( n_1 = n_2 \equiv n_0 \) with

\[
n_0 = \frac{\mu + \hbar \Omega_{\text{R}} / (g + g_{12})}{g + g_{12}}. \tag{4}
\]

This miscible phase holds when the intercomponent coupling constant satisfies \( g_{12} < g + \hbar \Omega_{\text{R}} / (n_1 + n_2) \) \cite{23}, otherwise the equilibrium state involves spontaneous density imbalance.

We scale the wave function as \( \Psi_j = \sqrt{n_0} \psi_j \), and introduce the length, time, and energy scale as \( \xi = \hbar / \sqrt{2m n_0} \), \( \Omega_{\text{R}} / (g n_0 \gamma) \), and \( gn_0 \), respectively. Then, we get the dimensionless GP equation,

\[
\begin{align*}
i \frac{\partial \psi_1}{\partial t} &= -\nabla^2 \psi_1 - \mu \psi_1 + |\psi_1|^2 \psi_1 + \gamma |\psi_2|^2 \psi_1 - \omega_R \psi_2, \\
i \frac{\partial \psi_2}{\partial t} &= -\nabla^2 \psi_2 - \mu \psi_2 + |\psi_2|^2 \psi_2 + \gamma |\psi_1|^2 \psi_2 - \omega_R \psi_1.
\end{align*} \tag{5}
\]

Here, the all variables are dimensionless and the coefficients are given as

\[
\omega_R = \frac{\Omega_{\text{R}}}{2gn_0}, \quad \gamma = \frac{g_{12}}{g}, \quad \mu = 1 - \gamma - \omega_R. \tag{7}
\]

III. TRANSVERSE INSTABILITY OF A DOMAIN WALL OF THE RELATIVE PHASE

In the seminal paper \cite{12}, Son and Stephanov showed that the GP equation for the Rabi-coupled two-component BECs can be reduced to the sine-Gordon equation when the gradient of the density is neglected. Considering the one-dimensional system along the \( x \)-axis and substituting the expression \( \psi_j(x) = e^{i \theta_j(x)} \) into Eqs. (5) and (6), we get the sine-Gordon equation

\[
\frac{\partial^2 \theta}{\partial x^2} = 2 \omega_R \sin \theta \tag{8}
\]

with the relative phase \( \theta \equiv \theta_1 - \theta_2 \) and the constant \( C \) of integral. The stationary solution for the boundary conditions \( \theta = \pi \pm \pi \) and \( \partial_x \theta = 0 \) for \( x \rightarrow \pm \infty \) is given by

\[
\theta(x) = 4 \arctan e^{\sqrt{2\omega_R}x}. \tag{9}
\]

This solution is known as the sine-Gordon soliton of the relative phase \cite{12}. They proposed that, when vortices are present in two-component BECs, the Rabi coupling can bind a pair of vortices in the different components via the sine-Gordon domain wall. After that the binding of vortices and composite structures of the resulting vortex molecules have been studied in various situations \cite{10, 23, 50}.

In this section, we consider the stability of the domain wall Eq. (9) based on the GP equations (5) and (6), where we take account of the contribution of the density gradient. We first consider the energetic stability of the domain wall. Although this problem has been considered by several papers \cite{12, 17, 31}, but we employ the exact solution of the domain wall and extend the phase diagram to the region of the negative intercomponent
coupling $\gamma < 0$. Next, we consider the transverse in-stability by extending the domain wall to the additional spatial dimension. The stability can be studied by the BdG analysis, where the signal of the instability is the appearance of the imaginary excitation frequency of the Bogoliubov modes.

A. Exact solution and energetic instability

We consider the energetic stability of the domain wall of the relative phase by solving the one-dimensional version of the GP equations (5) and (6). We seek the solutions which satisfy $|\psi_j| = 1$ at infinity and the phases change continuously as $0 \to +(-\pi)$ for $\psi_1$ ($\psi_2$) from $x = -\infty$ to $x = +\infty$. For the sine-Gordon kink and our symmetric parameters, the solution should satisfy the relation $\psi_1 = \psi_2^*$. This restriction reduces the equation as

$$-\frac{\partial^2 \psi_1}{\partial x^2} - \mu \psi_1 + (1 + \gamma)|\psi_1|^2\psi_1 - \omega_R \psi_1^* = 0,$$

which allows us to get an expression of the exact solution of Eqs. (5) and (6) as

$$\psi_1 = \psi_2^* = -\tanh(\sqrt{2\omega_R}x) + iA \text{sech}(\sqrt{2\omega_R}x),$$

$$A = \sqrt{1 + \gamma - 4\omega_R \gamma \omega_R}.$$

The typical profile of the solution is shown in Fig. 1(a) and (b). The relative phase changes $2\pi$ around the origin with the length scale $\sim (2\omega_R)^{-1/2}$. The density profile is written as $n_1 = n_2 = \tanh^2(\sqrt{2\omega_R}x) + A^2 \text{sech}^2(\sqrt{2\omega_R}x)$, being uniform only for $\omega_R = 0$. With increasing $\omega_R$, the spatial profile of $\theta$ approaches to a step function and the density depression becomes deeper. The solution of Eq. (11) is effective below the upper bound of the Rabi frequency $\omega_R^c = (1 + \gamma)/4$, at which the solution coincides with the form of the dark soliton with exactly zero density at the center. Physically, this boundary is interpreted as that the length scale $(2\omega_R)^{-1/2}$ is equal to the healing length given by the effective coupling constant $1 + \gamma$.

Figure 1(c) shows the phase diagram representing the energetic stability of the single domain wall in the $\gamma$-$\Omega_R$ plane. Below the line $\omega_R = \omega_R^c$, we have the solution of the sine-Gordon soliton. Here, we check numerically the energetic stability of the solution Eq. (11) through the imaginary time propagation of the GP equations (5) and (6), which is plotted in Fig. 1(c). When the solution is unstable, the density difference grows and unwinds the $2\pi$ difference of the relative phase to zero, leading to the uniform solution [17]. The stable range of $\omega_R$ becomes narrower with increasing $|\gamma|$ and vanishes at $|\gamma| = \pm 1$.

This result agrees with the previous works [12] [17]. In Ref. [12], the stability criterion was obtained as $\omega_R^c = (1 - \gamma)/3$ by neglecting the density gradient, which is valid for $\gamma \sim 1$. Usui and Takeuchi extended the analysis by considering the density depression through the numerical and variational analyses [17]. They proposed that the instability is associated with the Landau instability of the local counterflow across the domain wall [12] and the critical velocity was estimated by using the local density at the depression. Since the instability occurs when the wavelength of the unstable excitation is smaller than the length scale of the domain wall, they obtained the expression for the stability as

$$\omega_R^c = \frac{1}{3} n_{\text{min}}(\omega_R^c)(1 - \gamma).$$

Here, $n_{\text{min}}(\omega_R^c)$ represents the density at the density minimum for $\omega_R = \omega_R^c$, being written as $n_{\text{min}} = (1 + \gamma - 4\omega_R^c)/(1 + \gamma)$ by using Eq. (12). This stability criterion is also shown in the dashed curve in Fig. 1, which is good agreement with the numerical result.
B. Transverse instability of the domain wall

Next, we further study the stability of the sine-Gordon domain wall by the BdG analysis. Especially, we include the fluctuation along the direction parallel to the domain wall to study the transverse instability.

In the standard BdG analysis, the wave function is expanded around the stationary solution $\psi^0_j$ as

$$
\psi_j = \psi^0_j + \left[ u_j(x) e^{i k_y - i \omega t} - v^*_j(x) e^{-i k_y + i \omega t} \right].
$$

Here, the fluctuation along the domain wall is included by the plane wave $\propto e^{i k_y}$. Substituting this expression into Eqs. (5) and (6), we get the eigenvalue equation. The eigenfrequency $\omega$ is calculated by solving the BdG equation

$$
\mathbf{\tilde{H}} \mathbf{u} = i \hbar \omega \mathbf{u}, \quad \mathbf{\tilde{H}} = \begin{pmatrix}
\tilde{h}_1 & -\left( \psi^0_j \right)^2 & -h_1 \\
\gamma \psi^0_1 \psi^0_2 - \omega_R & \gamma \psi^0_1 \psi^0_2 - \omega_R & -\gamma \psi^0_1 \psi^0_2 \\
\gamma \psi^0_2 \psi^0_2 - \omega_R & h_2 & (\psi^0_2)^2 - h_2 \\
\end{pmatrix},
$$

where $\mathbf{u} = (u_1, u_2, v_1, v_2)^T$ and

$$
\tilde{h}_j = -\partial^2 + k^2 - \mu + 2|\psi_j^0|^2 + \gamma |\psi_j^0|^2, \quad \text{where } j = 1(2) \text{ for } j = 2(1).
$$

Employing the domain wall solution for $\psi^0_j$, we numerically solve Eq. (15) with the finite system size $-60 \leq x \leq 60$ and the 600 grid points. To this end, starting from Eq. (11) which is a solution in an infinite system, we make imaginary time propagation of the GP equation to get the proper solution $\psi_j^0$ for the finite size system.

The eigenvalues of the BdG equation can be used to clarify the stability properties of the stationary solutions. Let us first consider the situation $k = 0$, where the only one-dimensional perturbation is present. In the stable region in Fig. 1, there are positive eigenfrequencies and one zero-energy mode when we take only eigenmodes with positive norm $\sum_j \int dx |u_j|^2 - |v_j|^2 > 0$. When the solution enters the unstable region in Fig. 1, there appears the negative eigenvalue, the signature of the energetic Landau instability. The magnitude of the negative eigenvalues is very small in the most of unstable region, which implies that the Landau instability is very weak. The imaginary time propagation confirms this feature, because the decay of the initial domain wall solution to the uniform one needs very long time. Thus, we expect that this instability is not significant in cold gas experiments at ultralow temperatures.

More significant instability is the dynamical one. When we include the eigenmodes with finite $k$, there appears imaginary component in the excitation frequency. Figure 2(a) shows the imaginary part of the excitation frequency as a function of $k$ for $\gamma = 0$ and several values of $\omega_R$. The imaginary part appears for finite range $0 < k < k_{\text{max}}$ and takes a maximum value at a certain wave number $k_0$. The unstable range of $k$ is extended and the maximum of $\text{Im}[\omega]$ is enhanced with increasing $\omega_R$. The instability becomes stronger with increasing $\omega_R$. In Fig. 2(b), we also plot $k_0$ and $k_{\text{max}}$ as a function of $\omega_R$ for $\gamma = 0, \pm 0.5$. The plots can be fitted well by the linear functions; the inclination of the linear function becomes large as $\gamma$ increases. Interpolating the lines to $k_0 = k_{\text{max}} = 0$ gives the critical values $\omega^c_R$ of the dynamical stability for each $\gamma$, which agrees with the numerically obtained $\omega^c_R$ in Fig. 1. Thus, the parameter region exhibiting the transverse instability coincides with the unstable region of Fig. 1. For $\omega_R > \omega^c_R$ the solution reduces to the dark soliton, where the transverse instability is expected according to the previous literature [3, 4].

IV. DISINTEGRATION DYNAMICS OF A DOMAIN WALL OF RELATIVE PHASE

In this section, we perform numerical simulations of the GP equations to demonstrate the disintegration of the domain wall of the relative phase through the trans-
verse instability. We consider two cases: (i) an extended domain wall without the edges in a uniform system, and (ii) a domain wall with the finite length in a cylindrical trap. In the latter case, the edges of the wall correspond to the vortices with the same circulation in each component. The disintegration of the domain wall has been found in the numerical simulations in Refs. [14][16], when the separation of the vortices and the Rabi frequency are large. The transverse instability of a moving domain wall without edges in trapped BECs has been reported by C. Qu et al. [31].

![Diagram](image)

FIG. 3. The structure of the domain wall of the relative phase in a 2D space. The panels (a) and (b) associate with the situation (i) for $\gamma = 0$ and $\omega_R = 0.15$, while (c) and (d) with the situation (ii) for $\gamma = 0$, $\omega_R = 0.1$, and the length of the wall $L = 45$ in a cylindrical trap Eq. (16). In (c) and (d), the vortex in $\psi_1$- ($\psi_2$-) component is located at $(x, y) = (0, (+)22.5)$. This state is not a stationary solution and corresponds to the initial state of the simulations. (a) and (c): Two-dimensional profile of the relative phase $\theta = \theta_1 - \theta_2$. The range of the contour plot is $-\pi \leq \theta \leq \pi$. (b): The cross section of $\theta$ along the $y = 0$ line. Here, the solid curves represent $\theta$ within the range $-\pi \leq \theta \leq \pi$. The shift of $\theta$ by $2\pi$ for $x > 0$ highlights the structure of the sine-Gordon domain wall as shown by the dashed curve, agreement with the form of Eq. (9). (d): The density profile of $\psi_1$ component.

Typical structures of the domain wall corresponding to the situations (i) and (ii) are shown in Fig. 3. In the contour plots of $\theta$, we show it within the range $-\pi \leq \theta \leq \pi$ for clarity instead of $0 \leq \theta \leq 2\pi$, where we have $2\pi$ phase jump at the center of the wall. Then, the wall can be visualized as the localized pattern as shown in Figs. 3(a) and (c), as done in the previous papers [13][14][16]. Figure 3(a) shows the initial state of the time development in Fig. 4(b). Figures 3(c) and (d) show the typical structure of the vortex molecule. The $\psi_1$-component has a vortex at $(x, y) = (0, 22.5)$ with the positive unit winding and $\psi_2$-component has a vortex at $(x, y) = (0, -22.5)$ with the same positive unit winding. When the Rabi coupling is applied, the relative phase between two components tends to becomes zero over the entire space. However, the phase kinks due to the vortices leave a localized structure in the relative phase, the domain wall being naturally formed.

### A. Domain wall in a uniform system

We first demonstrate the nonlinear dynamics associated with the transverse instability through the simulation of the 2D GP equations [5] and [6] in a uniform system. We first prepare the initial state as $\psi_{\text{ini}}(x, y) = \psi_j(x)$ with the solution of Eq. (11), and calculate the time development using the Crank-Nicholson method. The system size is $[-50, 50]$ in the $x$-$y$ plane with numerical grids $1000 \times 1000$. We take the Neumann and periodic boundary condition for $x$- and $y$- direction, respectively. To initiate the dynamical instability, we add a small random noise $\sim 10^{-6}$ to the initial wave functions. We confirm that the domain wall in the stable region in Fig. 1 is certainly stable in the real time development.

Figure 4 shows the typical snapshots of the disintegration dynamics of the domain wall of the relative phase in the unstable regime for $\gamma = 0, \pm 0.5$. The snapshots show that the small transverse modulation of the wall is amplified after some time and leads to the disintegration of the domain wall. According to the BdG analysis, the growing time scale and the wave length of the unstable excitations are estimated as $\tau \sim 2\pi/\text{Im}[\omega(k_0)]$ and $\lambda \sim 2\pi/k_0$, respectively; for the parameters in Fig. 4 we have $(\tau, \lambda) = (114.9, 32.2)$ for (a), $(\tau, \lambda) = (247.2, 64.1)$ for (b), and $(\tau, \lambda) = (149.5, 46.5)$ for (c). The simulations results are fairly agreement with these estimations; the deviations may be due to the finite size effect in the simulations.

After the disintegration, the dynamics of the walls exhibits different behaviors depending on $\gamma$. There, the domain wall with the finite size involves the vortices at the edges. For $\gamma = -0.5$ in Fig. 4(a), the walls rapidly shrink to zero size due to the combined attractive force by the intercomponent coupling and the Rabi coupling between the two components. In other words, the vortices at the wall edge attract each other due the attractive vortex-vortex interaction [34] as well as the string tension caused by the Rabi coupling. For $\gamma = 0$ [Fig. 4(b)], some fragmented walls also tend to shrink due to the attraction by the Rabi coupling, but the others keep their separation. Such vortex molecules undergo center-of-mass motions, going to outside. Eventually, the all walls shrink to zero size at the later stage. Contrary to these, the subsequent dynamics for $\gamma = 0.5$ is different. Although the initial wall disintegrates into small pieces, they keep the disintegration-merge cycle for a while, and after that some walls go to outsides. In this case, the repulsive intercomponent interaction $\gamma > 0$ prevents the shrink
of the wall. In equilibrium, the balance of the repulsive vortex-vortex interaction [34] and the attractive force by the Rabi coupling realizes the stable vortex molecule [13, 25, 26, 28].

B. domain wall connecting vortices in a cylindrical trap

We next consider the dynamics of the domain wall having initially the finite length. This situation has been demonstrated in Refs. [14, 16] and is deeply connected with the experimental observation. A natural way to prepare the domain wall of the relative phase in two-component BECs is that vortices are prepared in each component and then the internal states are coupled by rf fields to induce the Rabi coupling. Then, the vortices are connected with the domain wall of the relative phase. This structure has been studied very well in the previous literature [12, 13, 25]. We conclude that the numerical observations in Ref. [14, 16] are actually due to the transverse instability.

In simulations, we numerically solve the 2D GP equation with the cylindrical trap

\[ V_{\text{ext}} = V_0 \left[ 2 + \tanh(ar + R) - \tanh(ar + R) \right] \]  (16)

with the radial coordinate \( r \), the potential depth \( V_0 = 10 \) and the radius \( R = 50 \) of the cylinder. The sharpness of the wall boundary is represented by the parameter \( a = 1 \).

We use the trap of Eq. (16) because there is additional contribution to the vortex dynamics from the density inhomogeneity when we employ the harmonic trap [13, 15].

We simulate the time development with the following procedure. First, the initial state without vorticity is prepared for given \( \gamma \) and \( \omega_R \) through the imaginary time evolution of the GP equation. Next, we imprint a vortex in both components by multiplying the phase factor \( e^{i\theta_v(r)} \) with the profile \( \theta_v(r) = \arctan \left( \frac{y \pm y_0}{x} \right) \) (− for \( \psi_1 \) and + for \( \psi_2 \)) and make additional imaginary time evolution. Then, the vortex separation gradually decreases from \( 2y_0 \), but after some time, the rate of the decrease reach a quasi-stationary behavior characterized by the slow linear decrease of the energy, which has been also reported in Ref. [16]. We stop the imaginary time evolution at a certain time in this quasi-stationary regime and use this configuration with the vortex separation \( L < 2y_0 \) as the initial state of the simulation.

In this situation, we have to consider the finite size effect for the transverse instability. It is necessary for the unstable modes to grow, a half of their wave length should be smaller than the length \( L \) of the domain wall. The condition is given by \( \lambda/2 = \pi/k_{\text{max}} < L \), where \( k_{\text{max}} \) can be written as a linear function of \( \omega_R \) [see Fig. 2(b)]. Using this condition, we get the boundary for the dis-
integration of the finite-size domain wall, as depicted in Fig. 5. With increasing the Rabi frequency $\omega_R$, the critical length $L$ of the wall decreases, which is consistent with the observation in Ref. 14 [16]. The stability diagram is weakly dependent on the intercomponent coupling $\gamma$.

We see that the behavior is qualitatively consistent with the BdG prediction, but quantitatively the disintegration takes place even in the region of the Rabi frequencies smaller than the critical values.

This quantitative difference may attribute to the dynamical precession of the domain wall with the finite size [14]; we supposed that the wall is static in the BdG analysis. To see the stability of moving domain wall in a simple situation, we consider the energetic stability of a moving domain wall with the velocity $V$ in the 1D system. In a frame of the co-moving frame of the velocity $V$, the GP equation reads

$$i\frac{\partial \psi_j}{\partial t} = -\frac{\partial^2 \psi_j}{\partial x^2} - \mu \psi_j + |\psi_j|^2 \psi_j + \gamma |\psi_j|^2 \psi_j - \omega_R \psi_j + iV \frac{\partial \psi_j}{\partial x}. \quad (17)$$

Here, the velocity $V$ is scaled by the sound velocity $\xi/\tau$. Using Eq. (17), we analyze the energetic stability of the domain wall solution satisfying the boundary condition similar to Sec. IIIA through the imaginary time evolution. We show the stability boundary for $V = 0.2$ and $V = 0.4$ in Fig. 6 for given sets of $\gamma$ and $\omega_R$.

Starting from the initial state obtained by the above procedure, we monitor the real time dynamics of the domain wall with the length $L$ for given sets of $\gamma$ and $\omega_R$. The typical integration process is shown in Fig. 6 for $L = 45$, $\gamma = 0$, and $\omega_R = 0.1$. The wall initially deforms into S-shape and breaks into small pieces at the curved points. After that, the remaining wall again deforms into inverted S-shape, repeating the similar breaking process. Eventually, the wall breaks into five pieces. In this simulation, we take $\gamma = 0$ so that there is no repulsive vortex-vortex interaction. Thus, the fragmented walls eventually shrink as a result of the tension of the sine-Gordon wall.

When the domain wall disintegrates, the additional vortices are nucleated in each component through the snake instability. Thus, monitoring the number of vortices (phase defects) during the time developments provides a clear criterion for the disintegration; the number of vortices in each component is kept to be unity when the domain wall is stable, otherwise it is unstable. Using this criterion, we calculate the critical length of the wall as a function of $\omega_R$ for $\gamma = 0, \pm 0.5$ and plot in Fig. 5.

We study the transverse instability and disintegration dynamics of the domain wall of the relative phase in Rabi-coupled two-component BECs, motivated by the numerical observation in Refs. 14 [16]. Using the exact solutions, we construct the phase diagram representing the stability of the domain wall. In the unstable regime, the domain wall exhibits dynamical instability associated with the transverse displacement, known as the snake instability. The instability causes the disintegration of the domain wall into the small pieces of wall, namely the vortex molecules. The growth time and the size of the resulting vortex molecules are fairly agreement with the prediction by the BdG analysis.

V. CONCLUSION

We study the transverse instability and disintegration dynamics of the domain wall of the relative phase in Rabi-coupled two-component BECs, motivated by the numerical observation in Refs. 14 [16]. Using the exact solutions, we construct the phase diagram representing the stability of the domain wall. In the unstable regime, the domain wall exhibits dynamical instability associated with the transverse displacement, known as the snake instability. The instability causes the disintegration of the domain wall into the small pieces of wall, namely the vortex molecules. The growth time and the size of the resulting vortex molecules are fairly agreement with the prediction by the BdG analysis.

ACKNOWLEDGMENTS

The work of K.K. is supported by KAKENHI from the Japan Society for the Promotion of Science (JSPS) Grant-in- Aid for Scientific Research (KAKENHI Grant No. 18K03472).
FIG. 6. The snapshot of the unstable dynamics of the domain wall of the relative phase for $L = 45$. The panels show the profile of the relative phase defined with the range $-\pi \leq \theta \leq \pi$. The parameters are $\gamma = 0$ and $\omega_R = 0.1$

[1] Y. S. Kivshar and D. E. Pelinovsky, Self-focusing and transverse instabilities of solitary waves, Physics Reports 331, 117 (2000).
[2] L. Carr and J. Brand, Multidimensional solitons: Theory, in Emergent Nonlinear Phenomena in Bose-Einstein Condensates (Springer, 2008) pp. 133–156.
[3] A. E. Muryshev, H. B. van Linden van den Heuvel, and G. V. Shlyapnikov, Stability of standing matter waves in a trap, Phys. Rev. A 60, R2665 (1999).
[4] J. Brand and W. P. Reinhardt, Solitonic vortices and the fundamental modes of the “snake instability”: Possibility of observation in the gaseous bose-einstein condensate, Phys. Rev. A 65, 043612 (2002).
[5] D. L. Feder, M. S. Pindzola, L. A. Collins, B. I. Schneider, and C. W. Clark, Dark-soliton states of bose-einstein condensates in anisotropic traps, Phys. Rev. A 62, 053606 (2000).
[6] B. P. Anderson, P. C. Haljan, C. A. Regal, D. L. Feder, L. A. Collins, C. W. Clark, and E. A. Cornell, Watching dark solitons decay into vortex rings in a bose-einstein condensate, Phys. Rev. Lett. 86, 2926 (2001).
[7] P. Öhberg and L. Santos, Dark solitons in a two-component bose-einstein condensate, Phys. Rev. Lett. 86, 2918 (2001).
[8] P. Busch and J. R. Anglin, Dark-bright solitons in inhomogeneous bose-einstein condensates, Phys. Rev. Lett. 87, 010401 (2001).
[9] C. Becker, S. Stellmer, P. Soltan-Panahi, S. Dörscher, M. Baumert, E.-M. Richter, J. Kronjäger, K. Bongs, and K. Sengstock, Oscillations and interactions of dark and dark–bright solitons in a “snake”-bose-einstein condensates, Nature Physics 4, 496 (2008).
[10] S. Coen and M. Haelterman, Domain wall solitons in binary mixtures of bose-einstein condensates, Phys. Rev. Lett. 87, 140401 (2001).
[11] C. Qu, L. P. Pitaevskii, and S. Stringari, Magnetic solitons in a binary bose-einstein condensate, Phys. Rev. Lett. 116, 160402 (2016).
[12] D. Son and M. A. Stephanov, Domain walls of relative phase in two-component bose-einstein condensates, Physical Review A 65, 063621 (2002).
[13] K. Kasamatsu, M. Tsubota, and M. Ueda, Vortex molecules in coherently coupled two-component bose-einstein condensates, Physical review letters 93, 250406 (2004).
[14] M. Tylutki, L. P. Pitaevskii, A. Recati, and S. Stringari, Confinement and precession of vortex pairs in coherently coupled bose-einstein condensates, Physical Review A 93, 043623 (2016).
[15] L. Calderaro, A. L. Fetter, P. Massignan, and P. Wittke, Vortex dynamics in coherently coupled bose-einstein condensates, Physical Review A 95, 023605 (2017).
[16] M. Eto and M. Nitta, Confinement of half-quantized vortices in coherently coupled bose-einstein condensates: Simulating quark confinement in a qcd-like theory, Physical Review A 97, 023613 (2018).
[17] A. Usui and H. Takeuchi, Rabi-coupled countersuperflow in binary bose-einstein condensates, Physical Review A 91, 063635 (2015).
[18] D. Hall, M. Matthews, C. Wieman, and E. A. Cornell, Measurements of relative phase in two-component bose-einstein condensates, Physical Review Letters 81, 1543 (1998).
[19] M. Matthews, B. Anderson, P. Haljan, D. Hall, M. Holland, J. Williams, C. Wieman, and E. Cornell, Watching a superfluid untwist itself: Recurrence of rabi oscillations in a bose-einstein condensate, Physical review letters 83, 3358 (1999).
[20] E. Timmermans, Phase separation of bose-einstein condensates, Physical review letters 81, 5718 (1998).
[21] P. Ao and S. Chui, Binary bose-einstein condensate mixtures in weakly and strongly segregated phases, Physical Review A 58, 4836 (1998).
[22] S. K. Adhikari, Stability and collapse of a coupled bose-einstein condensate, Physics Letters A 281, 265 (2001).
[23] M. Abad and A. Recati, A study of coherently coupled two-component bose-einstein condensates, The European Physical Journal D 67, 148 (2013).
[24] K. Kasamatsu, M. Tsubota, and M. Ueda, Spin textures in rotating two-component bose-einstein condensates, Physical Review A 71, 043611 (2005).
[25] K. Kasamatsu, M. Tsubota, and M. Ueda, Vortices in multicomponent bose-einstein condensates, International Journal of Modern Physics B 19, 1835 (2005).
[26] M. Eto and M. Nitta, Vortex trimer in three-component bose-einstein condensates, Physical Review A 85, 053645 (2012).
[27] M. Cipriani and M. Nitta, Crossover between integer and fractional vortex lattices in coherently coupled two-component bose-einstein condensates, Physical review letters 111, 170401 (2013).
[28] D. S. Dantas, A. R. Lima, A. Chaves, C. Almeida, G. Farias, and M. Milošević, Bound vortex states and exotic lattices in multicomponent bose-einstein condensates: The role of vortex-vortex interaction, Physical Review A 91, 023630 (2015).

[29] A. Aftalion and P. Mason, Rabi-coupled two-component bose-einstein condensates: Classification of the ground states, defects, and energy estimates, Physical Review A 94, 023616 (2016).

[30] B. M. Uranga and A. Lamacraft, Infinite lattices of vortex molecules in rabi-coupled condensates, Physical Review A 97, 043609 (2018).

[31] C. Qu, M. Tylutki, S. Stringari, and L. P. Pitaevskii, Magnetic solitons in rabi-coupled bose-einstein condensates, Physical Review A 95, 033614 (2017).

[32] H. Takeuchi, N. Suzuki, K. Kasamatsu, H. Saito, and M. Tsubota, Quantum kelvin-helmholtz instability in phase-separated two-component bose-einstein condensates, Physical Review B 81, 094517 (2010).

[33] N. Suzuki, H. Takeuchi, K. Kasamatsu, M. Tsubota, and H. Saito, Crossover between kelvin-helmholtz and counter-superflow instabilities in two-component bose-einstein condensates, Physical Review A 82, 063604 (2010).

[34] M. Eto, K. Kasamatsu, M. Nitta, H. Takeuchi, and M. Tsubota, Interaction of half-quantized vortices in two-component bose-einstein condensates, Physical Review A 83, 063603 (2011).