SMOOTH CONJUGACY OF ANOSOV DIFFEOMORPHISMS ON HIGHER DIMENSIONAL TORI

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ABSTRACT. Let \( L \) be a hyperbolic automorphism of \( T^d, d \geq 3 \). We study the smooth conjugacy problem in a small \( C^1 \)-neighborhood \( \mathcal{U} \) of \( L \).

The main result establishes \( C^{1+\nu} \) regularity of the conjugacy between two Anosov systems with the same periodic eigenvalue data. We assume that these systems are \( C^1 \)-close to an irreducible linear hyperbolic automorphism \( L \) with simple real spectrum and that they satisfy a natural transitivity assumption on certain intermediate foliations.

We elaborate on the example of de la Llave of two Anosov systems on \( T^4 \) with the same constant periodic eigenvalue data that are only Hölder conjugate. We show that these examples exhaust all possible ways to perturb \( C^{1+\nu} \) conjugacy class without changing periodic eigenvalue data. Also we generalize these examples to majority of reducible toral automorphisms as well as to certain product diffeomorphisms of \( T^d \) \( C^1 \)-close to the original example.

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1. INTRODUCTION AND STATEMENTS

Consider an Anosov diffeomorphism $f$ of a compact smooth manifold. Structural stability asserts that if a diffeomorphism $g$ is $C^1$ close to $f$, then $f$ and $g$ are topologically conjugate, i.e.,

$$h \circ f = g \circ h.$$ 

The conjugacy $h$ is unique in the neighborhood of identity. It is known that $h$ is Hölder-continuous.

There are simple obstructions to the smoothness of $h$. Namely, if $x$ is a periodic point of $f$ with period $p$, that is, $f^p(x) = x$, then $g^p(h(x)) = h(x)$. If $h$ were differentiable, then

$$Df^p(x) = (Dh(x))^{-1} Dg^p(h(x)) Dh(x),$$

i.e., $Df^p(x)$ and $Dg^p(h(x))$ are conjugate. We see that every periodic point carries a modulus of smooth conjugacy.

Suppose that for every periodic point $x$ of period $p$, the differentials of the return maps $Df^p(x)$ and $Dg^p(h(x))$ are conjugate. Then we say that the periodic data (p. d.) of $f$ and $g$ coincide.

**Question 1.** Suppose that the p. d. coincide. Is then $h$ differentiable? If it is, how smooth is it?

1.1. **Positive answers.** We describe situations when the p. d. form a full set of moduli of $C^1$ conjugacy.

The only surface that supports Anosov diffeomorphisms is the two-dimensional torus. For Anosov diffeomorphisms of $\mathbb{T}^2$, the complete answer to Question 1 was given by de la Llave, Marco and Moriyón.

**Theorem (LMM88, L92).** Let $f$ and $g$ be topologically conjugate $C^r$, $r > 1$, Anosov diffeomorphisms of $\mathbb{T}^2$ with coinciding p. d. Then the conjugacy $h$ is $C^{r-\varepsilon}$, where $\varepsilon > 0$ is arbitrarily small.

De la Llave [L92] also observed that the answer is negative for Anosov diffeomorphisms of $\mathbb{T}^d$, $d \geq 4$. He constructed two diffeomorphisms with the same p. d. which are only Hölder conjugate. We describe this example in Section 2.

In dimension three, the only manifold that supports Anosov diffeomorphisms is the three-dimensional torus. Moreover, all Anosov diffeomorphisms of $\mathbb{T}^3$ are topologically conjugate to linear automorphisms of $\mathbb{T}^3$. Nevertheless, the answer to Question 1 is not known.

**Conjecture 1.** Let $f$ and $g$ be topologically conjugate $C^r$, $r > 1$, Anosov diffeomorphisms of $\mathbb{T}^3$ with coinciding p. d. Then the conjugacy $h$ is at least $C^1$.

There are partial results that support this conjecture.
Theorem (GG08). Let $L$ be a hyperbolic automorphism of $\mathbb{T}^3$ with real eigenvalues. Then there exists a $C^1$-neighborhood $\mathcal{U}$ of $L$ such that any $f$ and $g$ in $\mathcal{U}$ having the same p. d. are $C^{1+\nu}$ conjugate.

Theorem (KS07). Let $L$ be a hyperbolic automorphism of $\mathbb{T}^3$ that has one real and two complex eigenvalues. Then any $f$ sufficiently $C^1$ close to $L$ that has the same p. d. as $L$ is $C^\infty$ conjugate to $L$.

In higher dimensions, not much is known. In recent years, much progress has been made (see L02, KS03, L04, F04, S05, KS06, KS07) in the case when the stable and unstable foliations carry invariant conformal structures. To ensure existence of these conformal structures one has to at least assume that every periodic orbit has only one positive and one negative Lyapunov exponent. This is a very restrictive assumption on the p. d.

In contrast to the above, we will study the smooth-conjugacy problem in the proximity of a hyperbolic automorphism $L: \mathbb{T}^d \to \mathbb{T}^d$:

Let $l$ be the dimension of the stable subspace of $L$ and $k$ be the dimension of the unstable subspace of $L$, so $k + l = d$. Consider the $L$-invariant splitting $T \mathbb{T}^d = F_l \oplus F_{l-1} \oplus \ldots \oplus F_1 \oplus E_1 \oplus E_2 \oplus \ldots \oplus E_k$ along the eigendirections with corresponding eigenvalues $\mu_1 < \mu_1 < \ldots < \mu_l < 1 < \lambda_1 < \lambda_2 < \ldots < \lambda_k$. Let $\mathcal{U}$ be a $C^1$-neighborhood of $L$. The precise choice of $\mathcal{U}$ is described in Section 6.1. The theory of partially hyperbolic dynamical systems guarantees that for any $f$ in $\mathcal{U}$ the invariant splitting survives (e. g. see Pes04); that is,

$$TT \mathbb{T}^d = F'_l \oplus F'_{l-1} \oplus \ldots \oplus F'_1 \oplus E'_1 \oplus E'_2 \oplus \ldots \oplus E'_k.$$ 

We will see in Section 6.1 that these one-dimensional invariant distributions integrate uniquely to foliations $U'_l, U'_{l-1}, \ldots, U'_1, V'_1, V'_2, \ldots, V'_k$.

Given a foliation $\mathcal{F}$ on $\mathbb{T}^d$ and an open set $B$, define $\mathcal{F}(B) = \bigcup_{y \in B} \mathcal{F}(y)$.

We will assume that $f$ has the following property:

Property A. For every $x \in \mathbb{T}^d$ and every open ball $B \ni x$,

$U'_{l-1}(B) = U'_{l-2}(B) = \ldots = U'_1(B) = V'_1(B) = V'_2(B) = \ldots = V'_{k-1}(B) = \mathbb{T}^d$.

We discuss this property in Section 4.1

Theorem A. Let $L$ be a hyperbolic automorphism of $\mathbb{T}^d$, $d \geq 3$, with a simple real spectrum. Assume that the characteristic polynomial of $L$ is irreducible over $\mathbb{Z}$. There exists a $C^1$-neighborhood $\mathcal{U} \subset \text{Diff}^r(\mathbb{T}^d)$, $r \geq 2$, of $L$ such that any $f \in \mathcal{U}$ satisfying Property A and any $g \in \mathcal{U}$ with the same p. d. are $C^{1+\nu}$ conjugate.
Remark.

1. We will see in Section 4.1 that irreducibility of the characteristic polynomial of $L$ is necessary for $f$ to satisfy $A$. Formally, we could have omitted the irreducibility assumption above. Theorem B below shows that the irreducibility of $L$ is a necessary assumption for the conjugacy to be $C^1$. We believe that Theorem A holds when $L$ is irreducible without assuming that $f$ satisfies $A$.

2. $\nu$ is a small positive number. It is possible to estimate $\nu$ from below in terms of the eigenvalues of $L$ and the size of $U$.

3. Obviously an analogous result holds on finite factors of tori. But we do not know how to prove it on nilmanifolds. The problem is that for an algebraic Anosov automorphism of a nilmanifold, various intermediate distributions may happen to be nonintegrable.

Theorem A is a generalization of the theorem from [GG08] quoted above. Our method does not lead to higher regularity of the conjugacy (see the last section of [GG08] for an explanation). Nevertheless we conjecture that the situation is the same as in dimension two.

**Conjecture 2.** In the context of Theorem A one can actually conclude that $f$ and $g$ are $C^{r-\varepsilon}$ conjugate, where $\varepsilon$ is an arbitrarily small positive number.

Simple examples of diffeomorphisms that possess Property $A$ include $f = L$ and any $f \in U$ when $\max(k,l) \leq 2$ (see Section 4). In addition, we construct a $C^1$-open set of Anosov diffeomorphisms of $T^5$ and $T^6$ close to $L$ that have Property $A$. It seems that this construction can be extended to any dimension.

We describe this open set when $l = 2$ and $k = 3$. Given $f \in U$, denote by $D^w_f$ the derivative of $f$ along $V^f_1$. Choose $f \in U$ in such a way that

\[ \forall x \neq x_0, \quad D^w_f(x) > D^w_f(x_0), \]

where $x_0$ is a fixed point of $f$. Then any diffeomorphism sufficiently $C^1$ close to $f$ satisfies Property $A$.

1.2. When the coincidence of periodic data is not sufficient. First let us briefly describe the counterexample of de la Llave.

Let $L: T^d \to T^d$ be a hyperbolic automorphism of product type,

\[ L(x,y) = (Ax, By), \quad (x,y) \in T^2 \times T^2, \quad (1) \]

where $A$ and $B$ are Anosov automorphisms. Let $\lambda, \lambda^{-1}$ be the eigenvalues of $A$ and $\mu, \mu^{-1}$ the eigenvalues of $B$. We assume that $\mu > \lambda > 1$. Consider perturbations of the form

\[ \tilde{L} = (Ax + \varphi(y), By), \quad (2) \]

where $\varphi: T^2 \to \mathbb{R}^2$ is a $C^1$-small $C^r$-function, $r > 1$. Obviously the p. d. of $L$ and $\tilde{L}$ coincide. We will see in Section 2 that the majority of perturbations (2) are only Hölder conjugate to $L$. The following theorem is a simple generalization of this counterexample.

**Theorem B.** Let $L: T^d \to T^d$ be a hyperbolic automorphism such that the characteristic polynomial of $L$ factors over $\mathbb{Q}$. Then there exist $C^\infty$-diffeomorphisms $\tilde{L}: T^d \to T^d$ and $\hat{L}: T^d \to T^d$ arbitrarily $C^1$-close to $L$ with the same p. d. such that the conjugacy between $\tilde{L}$ and $\hat{L}$ is not Lipschitz.
Remark. In the majority of cases, one can take $\hat{L} = L$. The need to take $\hat{L}$ and $L$ both different from $L$ appears, for instance, when $L(x, y) = (Ax, Ay)$. It was shown in [L02] that the p. d. form a complete set of moduli for the smooth-conjugacy problem to $L$. This is a remarkable phenomenon due to the invariance of conformal structures on the stable and unstable foliations. Nevertheless we still have a counterexample if we move a little bit away from $L$.

Next we study the smooth conjugacy problem in the neighborhood of (1) assuming that $\mu > \lambda > 1$. We show that the perturbations (2) exhaust all possibilities. Before formulating the result precisely let us move to a slightly more general setting. Let $A$ and $B$ be as in (1) with $\mu > \lambda > 1$. Consider the Anosov diffeomorphism

$$L(x, y) = (Ax, g(y)), \quad (x, y) \in T^2 \times T^2,$$

where $g$ is an Anosov diffeomorphism sufficiently $C^1$-close to $B$, so $L$ can be treated as a partially hyperbolic diffeomorphism with the automorphism $A$ acting in the central direction. Consider perturbations of the form

$$\tilde{L} = (Ax + \varphi(y), g(y)).$$

As before, it is obvious that the p. d. of $L$ and $\tilde{L}$ coincide. In Section 8 we will see that $L$ and $\tilde{L}$ with nonlinear $g$ also provide a counterexample to Question 1.

**Theorem C.** Given $L$ as in (1) with $\mu > \lambda > 1$, there exists a $C^1$-neighborhood $\mathcal{U} \subset \text{Diff}^r(T^4)$, $r \geq 2$, of $L$ such that any $f \in \mathcal{U}$ that has the same p. d. as $L$ is $C^{1+\nu}$-conjugate, $\nu > 0$, to a diffeomorphism $\hat{L}$ of type (4).

1.3. **Additional moduli of $C^1$ conjugacy in the neighborhood of the counterexample of de la Llave.** Let $L$ be given by (1) with $\mu > \lambda > 1$ and let $\mathcal{U}$ be a small $C^1$-neighborhood of $L$. It is useful to think of diffeomorphisms from $\mathcal{U}$ as partially hyperbolic diffeomorphisms with two-dimensional central foliations. Consider $f, g \in \mathcal{U}$, $h \circ f = g \circ h$. According to the celebrated theorem of Hirsch, Pugh and Shub [HPS77], the conjugacy $h$ maps the central foliation of $f$ into the central foliation of $g$.

Assume that the p. d. of $f$ and $g$ are the same. We will show that $h$ is $C^{1+\nu}$ along the central foliation. As described above, it can still happen that $h$ is not a $C^1$-diffeomorphism. This means that the conjugacy is not differentiable in the direction transverse to the central foliation. The geometric reason for this is a mismatch between the strong stable (unstable) foliations of $f$ and $g$ — the conjugacy $h$ does not map the strong stable (unstable) foliation of $f$ into the strong stable (unstable) foliation of $g$.

Motivated by this observation, we now introduce additional moduli of $C^1$-differentiable conjugacy. Roughly speaking, these moduli measure the tilt of the strong stable (unstable) leaves when compared to the model (1).

We define these moduli precisely. Let $W^s_L$, $W^u_L$, $W^u_w$ and $W^u_w$ be the foliations by straight lines along the eigendirections with eigenvalues $\mu^{-1}, \lambda^{-1}, \lambda$ and $\mu$ respectively. For any $f \in \mathcal{U}$ these invariant foliations survive. We denote them by $W^s_f$, $W^u_f$, $W^u_w$ and $W^u_w$. We will also write $W^s_f$ and $W^u_f$ for two-dimensional stable and unstable foliations.

Let $h_f$ be the conjugacy to the linear model, $h_f \circ f = L \circ h_f$. Then

$$h_f(W^\sigma_f) = W^\sigma_L, \quad \sigma = s, u, ws, wu.$$  

(5)
Fix orientations of $W^\sigma_L$, $\sigma = ss, ws, wu, su$. Then for every $x \in T^4$ there exists a unique orientation-preserving isometry $\mathcal{J}^\sigma(x): W^\sigma_L(x) \to \mathbb{R}$, $\mathcal{J}^\sigma(x)(x) = 0$, $\sigma = ss, ws, wu, su$.

Define $\Phi^u_f: T^4 \times \mathbb{R} \to \mathbb{R}$ by the formula

$$\Phi^u_f(x, t) = I^w_L(\mathcal{J}^su_f(h^{-1}(x)) \cap W^wu_L(h_f^{-1}(x), \mathcal{J}^su_f(x)^{-1}(t)))$$

The geometric meaning is transparent and illustrated on Figure 1. The image of the strong unstable manifold $h_f(W^su_f(h_{-1}(x)))$ can be viewed as a graph of the function $\Phi^u_f(x, \cdot)$ over $W^su_L(x)$. Analogously we define $\Phi^s_g: T^4 \times \mathbb{R} \to \mathbb{R}$.

Clearly, $\Phi^{s/u}_f$ are moduli of $C^1$-conjugacy. Indeed, assume that $f$ and $g$ are $C^1$-conjugate by $h$. Then $h(W^su_f) = h(W^su_g)$ and $h(W^ss_f) = h(W^ss_g)$ since strong stable and unstable foliations are characterized by the speed of convergence which is preserved by $C^1$-conjugacy. Hence $\Phi^{s/u}_f = \Phi^{s/u}_g$.

It is possible to choose a subfamily of these moduli in an efficient way. We say that $f$ and $g$ from $\mathcal{U}$ have the same strong unstable foliation moduli if

$$\exists t \neq 0 \text{ such that } \forall x \in T^4, \quad \Phi^u_f(x, t) = \Phi^u_g(x, t)$$

or

$$\exists x \in T^4 \text{ and } \exists I = (a, b) \subset \mathbb{R} \text{ such that } \forall t \in I, \quad \Phi^u_f(x, t) = \Phi^u_g(x, t).$$

The definition of the strong stable foliation moduli is analogous.

**Theorem D.** Given $L$ as in (1) with $\mu > \lambda > 1$, there exists a $C^1$-neighborhood $\mathcal{U} \subset \text{Diff}^r(T^4)$, $r \geq 2$, of $L$ such that if $f, g \in \mathcal{U}$ have the same p. d. and the same strong unstable and strong stable foliation moduli, then $f$ and $g$ are $C^{1+\nu}$-conjugate.

**Remark.** In this case $C^{1+\nu}$-differentiability is in fact the optimal regularity.

### 1.4. Organization of the paper and a remark on terminology.

In Section 2 we describe the counterexample of de la Llave in a way that allows us to generalize it to Theorem B in Section 3. Sections 2 and 3 are independent of the rest of the paper.

In Sections 4 and 5 we discuss Property $A$ and construct examples of diffeomorphisms that satisfy Property $A$. These sections are self-contained.
Section 6 is devoted to the proof of our main result, Theorem A. It is self-contained but in number of places we refer to [GG08], where the three-dimensional version of Theorem A was established.

Theorem C is proved in Section 7. It is independent of the rest of the paper with the exception of a reference to Proposition 10.

The proof of Theorem D appears in Section 8 and relies on some technical results from [GG08].

Throughout the paper we will prove that various maps are $C^{1+\nu}$-differentiable. This should be understood in the usual way: the map is $C^1$-differentiable and the derivative is Hölder-continuous with some positive exponent $\nu$. The number $\nu$ is not the same in different statements.

When we say that a map is $C^{1+\nu}$-differentiable along a foliation $\mathcal{F}$, we mean that restrictions of the map to the leaves of $\mathcal{F}$ are $C^{1+\nu}$-differentiable and the derivative is a Hölder-continuous function on the manifold, not only on the leaf.

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2. The counterexample on $\mathbb{T}^4$

Here we describe the example of de la Llave of two Anosov diffeomorphisms of $\mathbb{T}^4$ with the same p. d. that are only Hölder conjugate. Understanding of the example is important for the proof of Theorem B.

Recall that we start with an automorphism $L : \mathbb{T}^4 \to \mathbb{T}^4$ such that

$$L(x, y) = (Ax, By), \quad (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2,$$

where $A$ and $B$ are Anosov automorphisms, $Av = \lambda v$, $A\tilde{v} = \lambda^{-1}\tilde{v}$, $Bu = \mu u$, $B\tilde{u} = \mu^{-1}\tilde{u}$. We assume that $\mu \geq \lambda > 1$.

To simplify computations we consider a special perturbation of the form

$$\tilde{L} = (Ax + \varphi(y)v, By).$$

We look for the conjugacy $h$ of the form

$$h(x, y) = (x + \psi(y)v, y). \quad (8)$$

The conjugacy equation $h \circ \tilde{L} = L \circ h$ transforms into a cohomological equation on $\psi$

$$\varphi(y) + \psi(By) = \lambda \psi(y). \quad (9)$$

Let us solve for $\psi$ using the recurrent formula

$$\psi(y) = \lambda^{-1}\varphi(y) + \lambda^{-1}\psi(By).$$

We get a continuous solution to (9),

$$\psi(y) = \lambda^{-1}\sum_{k \geq 0} \lambda^{-k} \varphi(B^k y). \quad (10)$$

Hence the conjugacy is indeed given by the formula (8).

In the following proposition we denote by the subscript $u$ the partial derivative in the direction of $u$.
**Proposition 1.** Assume that $\mu > \lambda > 1$. Then function $\psi$ is Lipschitz in the direction of $u$ if and only if

$$\sum_{k \in \mathbb{Z}} \left( \frac{\mu}{\lambda} \right)^k \varphi_u(B^k y) = 0,$$

i.e., the series on the left converges in the sense of distribution convergence and the limit is equal to zero.

**Proof.** First assume (11). Let us consider the series (10) as a series of distributions that converge to $\psi$. Then, as a distribution, $\psi_u$ is obtained by differentiating (10) termwise:

$$\psi_u = \lambda^{-1} \sum_{k \geq 0} \lambda^{-k} \mu^k \varphi_u(B^k).$$

Applying (11), we get

$$\psi_u = \lambda^{-1} \sum_{k < 0} \lambda^{-k} \mu^k \varphi_u(B^k).$$

Since $\mu > \lambda$ the above series converges and the distribution is regular. Hence $\psi$ is differentiable in the direction of $u$.

Now assume that $\psi$ is $u$-Lipschitz. By differentiating (9), we get a cohomological equation on $\psi_u$,

$$\varphi_u(x) + \mu \varphi_u(B y) = \lambda \psi_u(y),$$

that is satisfied on a $B$-invariant set of full measure. We solve it using the recurrent formula

$$\psi_u(y) = -\frac{1}{\mu} \varphi_u(B^{-1} y) + \frac{\lambda}{\mu} \psi_u(B^{-1} y).$$

Hence

$$\psi_u = \lambda^{-1} \sum_{k < 0} \lambda^{-k} \mu^k \varphi_u(B^k).$$

(13)

On the other hand we know that as a distribution $\psi_u$ is given by (12). Combining (12) and (13) we get the desired equality (11).

If $\mu = \lambda$ then the argument above works only in one direction. We will see that in this case $L$ and $\tilde{L}$ do not provide a counterexample since the p. d. are different.

**Proposition 2.** Assume that $\mu = \lambda$. Then (11) is a necessary assumption for $\psi$ to be Lipschitz in the direction of $u$.

**Proof.** As in the proof of Proposition 1, viewed as distribution, $\psi_u$ is given by

$$\psi_u = \lambda^{-1} \sum_{k \geq 0} \varphi_u(B^k).$$

(14)

Assume that $\psi$ is $u$-Lipschitz. Then, analogously to (13), we get

$$\psi_u = \lambda^{-1} \sum_{-N \leq k < 0} \varphi_u(B^k) + \psi(B^N).$$

(15)

Note that in the sense of distributions, $\psi(B^N) \to 0$ as $N \to \infty$ since $B$ is mixing. Hence, as a distribution, $\psi_u$ is given by

$$\psi_u = \lambda^{-1} \sum_{k < 0} \varphi_u(B^k).$$

(16)

Combining (14) and (16), we get (11).
By rewriting condition (11) in terms of Fourier coefficients of \( \varphi \), one can see that it is an infinite codimension condition. Moreover, one can easily construct functions that do not satisfy (11): one only needs to make sure that some Fourier coefficients of the sum (11) are nonzero. For instance, for any \( \varepsilon > 0 \) and positive integer \( p \), the function

\[
\varphi(y) = \varphi(y_1, y_2) = \varepsilon \sin(p\pi y_1)
\]

works. Thus the corresponding \( \tilde{L} \) is not \( C^1 \)-conjugate to \( L \). Note that \( \tilde{L} \) may be chosen arbitrarily close to \( L \).

**Remark.**

1. Perturbations of the general type (2) can be treated analogously by decomposing \( \vec{\phi} = \phi_1 v + \phi_2 \tilde{v} \).

2. The assumption \( \mu \geq \lambda > 1 \) is crucial in this construction.

3. By choosing appropriate \( \lambda \) and \( \mu \), one can get any desired regularity of the conjugacy (see [L92] for details). For example, if \( \mu^2 > \lambda > \mu > 1 \), the conjugacy is \( C^1 \) but not \( C^2 \).

From now on let us assume that \( \mu = \lambda \). As we remarked in the introduction, \( L \) and \( \tilde{L} \) do not provide a counterexample. Indeed, the derivative of \( \tilde{L} \) in the basis \( \{v, u, \tilde{v}, \tilde{u}\} \) is

\[
\begin{pmatrix}
\lambda & \varphi_u & 0 & \varphi_{\tilde{u}} \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda^{-1} & 0 \\
0 & 0 & 0 & \lambda^{-1}
\end{pmatrix}.
\]

Let \( x \) be a periodic point, \( \tilde{L}^p(x) = x \). Then the derivative of the return map at \( x \) is

\[
\begin{pmatrix}
\lambda^p & \lambda^{p-1} \sum_{y \in O(x)} \varphi_u(y) & 0 & * \\
0 & \lambda^p & 0 & 0 \\
0 & 0 & \lambda^{-p} & 0 \\
0 & 0 & 0 & \lambda^{-p}
\end{pmatrix}.
\]  

(18)

We see that it is likely to have a Jordan block while \( L \) is diagonalizable. Hence \( L \) and \( \tilde{L} \) have different p. d.

It is still easy to construct a counterexample in a neighborhood of \( L \). Let

\[
\tilde{L} = (Ax + \xi(y)v, By)
\]

and let

\[
h(x, y) = (x + \psi(y)v, y)
\]

be the conjugacy between \( \tilde{L} \) and \( \tilde{L} \).

**Proposition 3.** The condition

\[
\sum_{k \in \mathbb{Z}} (\xi - \varphi)_u(B^k y) = 0,
\]

is necessary for \( \phi \) to be Lipschitz in the direction of \( u \).

The proof of Proposition 3 is exactly the same as the one of Proposition 2.

Now take \( \varphi \) that does not satisfy (11) as before and take \( \xi = 2\varphi \). Then obviously the condition of Proposition 3 is not satisfied. Hence \( h \) is not Lipschitz. By looking at (18) it is obvious that our choice of \( \xi \) guarantees that the Jordan normal forms of the derivatives of the return maps at periodic points of \( \tilde{L} \) and \( \tilde{L} \) are the same.
Remark. Due to the special choice of $\xi$ it was easy to ensure that the p. d. of $\tilde{L}$ and $\hat{L}$ are the same. We could have taken a different and somewhat more general approach. It is possible to show that for many choices of $\varphi$, the sum that appears over the diagonal in $[\mathbf{10}]$ is nonzero for every periodic point $x$. All of the corresponding diffeomorphisms will have the same p. d. with a Jordan block at every periodic point.

3. Proof of Theorem B

Here we consider $L: \mathbb{T}^d \to \mathbb{T}^d$ with a reducible characteristic polynomial. We show how to construct $\tilde{L}$ and $\hat{L}$ with the same p. d. which are not Lipschitz conjugate.

Assume that all real eigenvalues of $L$ are positive. Otherwise we may consider $L^2$. Let $M: \mathbb{R}^d \to \mathbb{R}^d$ be the lift of $L$ and let $\{e_1, e_2, \ldots e_d\}$ be the canonical basis, so $\mathbb{T}^d = \mathbb{R}^d / \text{span}_\mathbb{Z}\{e_1, e_2, \ldots e_d\}$.

It is well known that the characteristic polynomial of $M$ factors over $\mathbb{Z}$ into the product of polynomials irreducible over $\mathbb{Q}$:

$$P(x) = P_1(x)P_2(x)\ldots P_r(x), \ r \geq 2.$$  

Let $\lambda$ be the eigenvalue of $M$ with the smallest absolute value which is greater than one. Without loss of generality we assume that $P_1(\lambda) = 0$.

Let $V_i$ be the invariant subspace that corresponds to the roots of $P_i$. Then $\dim V_i = \deg P_i$ and it is easy to show that

$$V_i = \text{Ker}(P_i(M)).$$

Matrices of $P_i(M)$ have integer entries. Hence there is a basis $\{\tilde{e}_1, \tilde{e}_2, \ldots \tilde{e}_d\}$, $\tilde{e}_i \in \text{span}_\mathbb{Z}\{e_1, e_2, \ldots e_d\}$, $i = 1, \ldots d$, such that the matrix of $M$ in this basis has integer entries and is of block diagonal form with blocks corresponding to the invariant subspaces $V_i, i = 1, \ldots r$.

We consider projection of $M$ to $\tilde{\mathbb{T}}^d = \mathbb{R}^d / \text{span}_\mathbb{Z}\{\tilde{e}_1, \tilde{e}_2, \ldots \tilde{e}_d\}$. Denote by $N$ the induced map on $\tilde{\mathbb{T}}^d$. We have the following commutative diagram, where $\pi$ is a finite-to-one projection.

$$\begin{array}{ccc}
\mathbb{R}^d & \xrightarrow{M} & \mathbb{R}^d \\
\downarrow & & \downarrow \\
\mathbb{T}^d & \xrightarrow{N} & \tilde{\mathbb{T}}^d \\
\pi \downarrow & & \pi \downarrow \\
\mathbb{T}^d & \xrightarrow{L} & \mathbb{T}^d \\
\end{array}$$

Notice that $N$ has the form $N(x, y) = (Ax, By)$, $(x, y) \in \mathbb{T}^{\deg P_1} \times \mathbb{T}^{d-\deg P_1}$. Let $\mu$ be an eigenvalue of $B$. By construction, $|\lambda| \leq |\mu|$, is an eigenvalue of $A$.

With certain care, the construction of Section 2 can be applied to $N$. We have to distinguish the following cases:

1. $\lambda$ and $\mu$ are real.
2. $\lambda$ is real and $\mu$ is complex.
3. $\lambda$ is complex and $\mu$ is real.
4. $\lambda$ and $\mu$ are complex.
Assume that $|\lambda| < |\mu|$. Then we take $\hat{L} = L$.

In the first case the construction of Section 2 applies straightforwardly. We use a function of the type (17) to produce $\tilde{N}$. Now we only need to make sure that $\tilde{N}$ can be projected to a map $\hat{L} : \mathbb{T}^d \to \mathbb{T}^d$. Since $\pi$ is a finite-to-one covering map this can be achieved by choosing suitable $p$ in (17).

Other cases require heavier calculations but follow the same scheme as Proposition 1. We outline the construction in case 4, which can appear, for instance, if $A$ and $B$ are hyperbolic automorphisms of four-dimensional tori without real eigenvalues.

Let $V_A = \text{span}\{v_1, v_2\}$ be the two-dimensional $A$-invariant subspace corresponding to $\lambda$ and $V_B = \text{span}\{u_1, u_2\}$ be the two-dimensional $B$-invariant subspace corresponding to $\mu$. Then $A$ acts on $V_A$ by multiplication by $|\lambda|R_A$ and $B$ acts on $V_B$ by multiplication by $|\mu|R_B$, where $R_A$ and $R_B$ are rotation matrices expressed in the bases $\{v_1, v_2\}$ and $\{u_1, u_2\}$, respectively.

We are following the construction from the previous section. Let

$$\tilde{N}(x, y) = (Ax + \varphi(y)\tilde{v}, By) \overset{\text{def}}{=} (Ax + \varphi_1(y)v_1 + \varphi_2(y)v_2, By).$$

Then we look for a conjugacy of the form

$$h(x, y) = (x + \tilde{\psi}(y)\tilde{v}, y) \overset{\text{def}}{=} (x + \psi_1(y)v_1 + \psi_2(y)v_2, y).$$

The conjugacy equation $h \circ \tilde{N} = N \circ h$ transforms into

$$\varphi(y)\tilde{v} + \tilde{\psi}(By)\tilde{v} = |\lambda|R_A\tilde{\psi}(y).$$

Solving for $\tilde{\psi}$ gives

$$\tilde{\psi}(y) = \sum_{k \geq 0} |\lambda|^{-k-1}R_A^{-k-1}\varphi(B^k y),$$

which we would like to differentiate along the directions $u_1$ and $u_2$. We use the formula

$$\varphi(B y)u = \begin{pmatrix} \varphi_1(B y)u_1 & \varphi_2(B y)u_2 \end{pmatrix} = |\mu| \begin{pmatrix} (\varphi_1)_{u_1} & (\varphi_1)_{u_2} \\ (\varphi_2)_{u_1} & (\varphi_2)_{u_2} \end{pmatrix} (By)R_B = \varphi_B(B y)R_B$$

to get that, as a distribution,

$$\tilde{\psi}_u = \sum_{k \geq 0} |\lambda|^{-k-1}|\mu|^k R_A^{-k-1}\varphi_B(B^k)R_B^k.$$

Now we assume that $\tilde{\psi}$ is Lipschitz and we differentiate (19) along the directions $u_1$ and $u_2$:

$$\varphi_B(u) + |\mu|\tilde{\psi}_u(B y)R_B = |\lambda|R_A\tilde{\psi}(y).$$

Hence, by the recurrent formula,

$$\tilde{\psi}_u = \sum_{k < 0} |\lambda|^{-k-1}|\mu|^k R_A^{-k-1}\varphi_B(B^k)R_B^k.$$

Combining the expressions for $\tilde{\psi}_u$, we get

$$\sum_{k \in \mathbb{Z}} |\lambda|^{-k}|\mu|^k R_A^{-k-1}\varphi_B(B^k)R_B^k = 0.$$

Using Fourier decompositions, one can find functions $\varphi$ that do not satisfy the condition above. One also needs to make sure that the choice of $\varphi$ allows one to project $\tilde{N}$ down to $\hat{L}$. We omit this analysis since it is routine.
This is a contradiction and therefore $\tilde{\psi}$ (and hence $h$) is not Lipschitz.

If $|\lambda| = |\mu|$ but $\lambda \neq \mu$, then the scheme above still works. Obviously, extra Jordan blocks do not appear in the normal forms at periodic points of $\tilde{L}$.

Finally, the case $\lambda = \mu$ must be treated separately. We use the same trick as in Section 2 to find $\tilde{L}$ and $\hat{L}$ with the same p. d. that are only Hölder conjugate. This trick also works well in the case of complex eigenvalues; we omit the details.

4. On the Property $\mathcal{A}$

4.1. Transitivity versus minimality. Here we discuss Property $\mathcal{A}$. Let $\mathcal{F}$ be a foliation of a compact manifold $M$. As usually $\mathcal{F}(x)$ stands for the leaf of $\mathcal{F}$ that contains $x$ and $\mathcal{F}(x, R)$ stands for the ball of radius $R$ centered at $x$ inside of $\mathcal{F}(x)$.

Definition 1. The foliation $\mathcal{F}$ is called minimal if every leaf of $\mathcal{F}$ is dense in $M$.

Definition 2. The foliation $\mathcal{F}$ is called transitive if there exists a leaf of $\mathcal{F}$ that is dense in $M$.

Definition 3. The foliation $\mathcal{F}$ is called tubularly minimal if for every $x$ and every open ball $B \ni x$,

$$\bigcup_{y \in B} \mathcal{F}(y) = M.$$  

Property $\mathcal{A}$ simply requires the foliations $U^f_{l-1}, U^f_{l-2}, \ldots, U^f_1, V^f_1, V^f_2, \ldots, V^f_{k-1}$ to be tubularly minimal. We introduce the following related property:

Property $\mathcal{A}'$. The foliations $U^f_{l-1}, U^f_{l-2}, \ldots, U^f_1, V^f_1, V^f_2, \ldots, V^f_{k-1}$ are minimal.

Proposition 4. The foliation $\mathcal{F}$ is transitive if and only if it is tubularly minimal.$^1$

Proof. Transitivity obviously implies tubular minimality.

Assume that $\mathcal{F}$ is tubularly minimal. Let $\{B_n, n \geq 1\}$ be a countable basis for the topology of $M$. By the definition of tubular minimality, the sets $\mathcal{F}(B_n)$ are open and dense in $M$. Hence by the Baire Category Theorem, the set

$$B = \bigcap_{n \geq 1} \mathcal{F}(B_n)$$

is nonempty and for every $x \in B$ the leaf $\mathcal{F}(x)$ is dense in $M$.

Remark. We define Property $\mathcal{A}$ in terms of tubular minimality rather than transitivity since we need denseness of the tubes for the proof of Theorem A.

A priori, transitivity is weaker than minimality. Hence, a priori, Property $\mathcal{A}$ is weaker than Property $\mathcal{A}'$.

If, in Theorem A, we had required $f$ to satisfy Property $\mathcal{A}'$ instead of Property $\mathcal{A}$, then the induction procedure that we use (the first induction step) is much simpler. The proof of this step, assuming only Property $\mathcal{A}$, requires a much more lengthy and delicate argument. It is not clear to us what the relationship is between Properties $\mathcal{A}$ and $\mathcal{A}'$; they may be equivalent. Thus, we will first provide a proof of Theorem A assuming that $f$ has Property $\mathcal{A}'$, then we will present a separate proof of this first induction step (namely Lemma 6.6) that uses only Property $\mathcal{A}$.

Minimality of a foliation can be characterized similarly to tubular transitivity.

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$^1$We would like to thank the referee for pointing out this fact.
Proposition 5. The foliation $\mathcal{F}$ is minimal if and only if for every $x$ and every open ball $B \ni x$,

$$\bigcup_{y \in B} \mathcal{F}(y) = M.$$ 

The proof is simple, so we omit it. As a corollary the foliation $\mathcal{F}$ is minimal if and only if for every $x$ and every open ball $B \ni x$, there exists a number $R$ such that

$$\bigcup_{y \in B} \mathcal{F}(y, R) = M. \quad (20)$$

This is the property which we will actually use in the proof of the induction step 1.

4.2. Examples of diffeomorphisms that satisfy Property $A$.

Proposition 6. Assume that $L$ is irreducible. Then the foliations $U_j^L, V_i^L$, $j = 1 \ldots l$, $i = 1 \ldots k$, are minimal.

Proof. Denote by $\mathcal{F}$ one of the foliations under consideration. Since $\mathcal{F}$ is a foliation by straight lines, the closure of a leaf $\mathcal{F}(x)$ is a subtorus of $\mathbb{T}^d$. This subtorus lifts to a rational invariant subspace of $\mathbb{R}^d$. The invariant subspace corresponds to a rational factor of the characteristic polynomial of $L$, but we assumed that it is irreducible over $\mathbb{Q}$. Hence the invariant subspace is the whole of $\mathbb{R}^d$ and the subtorus is the whole $\mathbb{T}^d$. \hfill $\square$

So we can see that the conclusion of Theorem A holds at least for $f = L$.

We will see in Section 6.1 that for any $f \in \mathcal{U}$, the foliations $U_j^f$ and $V_i^f$ are minimal. Hence the conclusion of Theorem A holds for any $f \in \mathcal{U}$ if $\max(k, l) \leq 2$.

It is easy to construct $f \neq L$ that satisfies $A$ when $k = 3$ and $l = 2$ since we only have to worry about the foliation $V_2^L$. We let $f = s \circ L$ where $s$ is any small shift along $V_2^L$. Clearly $V_2^f = V_2^L$ and hence $f$ satisfies $A$.

Questions about robust minimality of the foliations $U_{i-1}^f, U_{i-2}^f, \ldots, U_1^f, V_1^f, V_2^f, \ldots, V_k^f$ arise naturally. Robust minimality of strong stable and strong unstable foliations of partially hyperbolic systems has received some attention in the literature due to its intimate connection with robust transitivity; see $[\text{Ma78}]$ and the more recent papers $[\text{BDU02}, \text{PS06}]$, where robust minimality of the full expanding foliation is established under some assumptions. We do not have this luxury in our setting: the expanding foliations that we are interested in subfoliate the full unstable foliation. A representative problem here is the following.

Question 2. Let $L : \mathbb{T}^3 \to \mathbb{T}^3$ be a hyperbolic linear automorphism with real spectrum $\lambda_1 < 1 < \lambda_2 < \lambda_3$. Consider the one-dimensional strong unstable foliation. Is it true that this foliation is robustly minimal? In other words, is it true that for any $f$ sufficiently $C^1$-close to $L$ the strong unstable foliation of $f$ is minimal?

In addition to the simple examples above, in the next section we construct a $C^1$-open set of diffeomorphisms that possess Property $A$. The following statement can be obtained by applying the construction and the arguments of the next section in the context of Question 2.

Proposition 7. Let $L$ be as in Question 2. Then there exists a $C^1$-open set $\mathcal{U}$ $C^1$-close to $L$ such that for every $f \in \mathcal{U}$ the strong unstable foliation of $f$ is transitive.
5. An example of an open set of diffeomorphisms with Property $A$

Let $L: \mathbb{T}^5 \to \mathbb{T}^5$ be a hyperbolic automorphism as in Theorem A, $l = 2, k = 3$, and let $U$ be a $C^1$-neighborhood of $L$ chosen as in Section 6.1.

Recall that $D_f^{wu}$ stands for the derivative of $f \in U$ along $V^f_1$. Choose $f \in U$ in such a way that
\[ \forall x \neq x_0 \quad D_f^{wu}(x) > D_f^{wu}(x_0), \]
where $x_0$ is a fixed point of $f$.

**Proposition 8.** There exists a $C^1$-neighborhood $U$ of $f$ such that any diffeomorphism $g \in U$ has Property $A$.

**Remark.** A similar example can be constructed on $\mathbb{T}^6$ with $l = 3, k = 3$. We only need to do the trick described below for both the stable and unstable manifolds of the fixed point $x_0$.

Before proving the proposition let us briefly explain the idea behind the proof. We know that $U_1^x$ and $V_1^y$ are minimal. Hence we only need to show that the foliation $V_2^y$ is tubularly minimal, *i.e.*, for every $x \in \mathbb{T}^5$ and every open ball $B \ni x$
\[ \bigcup_{y \in B} V_2^y(y) = \mathbb{T}^5. \]

To illustrate the idea we take $g = f$ and $x = x_0$. We work on the universal cover $\mathbb{R}^5$ with lifted foliations. Let
\[ \mathcal{T} \overset{\text{def}}{=} \bigcup_{y \in B} V_2^f(y) \subset \mathbb{R}^5, \]
which is an open tube.

We show that $\mathcal{T}$ contains arbitrarily long connected pieces of the leaves of $V_1^f$ as shown on Figure 2. It will then follow that $\mathcal{T}$ is dense in $\mathbb{T}^5$. Indeed, the foliation $V_1^f$ is not just minimal but uniformly minimal: for any $\varepsilon > 0$ there exists $R > 0$ such that $\forall z \in \mathbb{T}^5$ $V_1^f(z, R)$ is $\varepsilon$-dense in $\mathbb{T}^5$. This property follows from the fact that $V_1^f$ is conjugate to the linear foliation $V_1^L$.

Pick $y_0 \in B \cap V_1^f(x_0)$ close to $x_0$. Let $x \in V_2^f(x_0)$ be a point far away in the tube $\mathcal{T}$ and $y = V_1^f(x) \cap V_2^f(y_0)$. To show that $\mathcal{T}$ contains arbitrarily long pieces of leaves of $V_1^f$ we prove that $d_1^f(x, y)$ (recall that $d_1^f$ is the Riemannian distance along $V_1^f$) is an unbounded function of $x$.

We make use of the affine structure on $V_1^f$. We refer to [CG10] for the definition of affine distance-like function $d_1$. Recall the following crucial properties of $d_1$:

(D1) \[ d_1(x, y) = d_1^f(x, y) + o(d_1^f(x, y)), \]
(D2) \[ d_1(f(x), f(y)) = D_f^{wu}(x)d_1(x, y), \]
(D3) \[ \forall K > 0 \exists C > 0 \text{ such that } \frac{1}{C}d_1(x, y) \leq d_1^f(x, y) \leq Cd_1(x, y) \]
whenever $d_1(x, y) < K$.

Using Property (D3), we can see that it is enough to show that $d_1(x, y)$ is unbounded.

Given $x$ as above, pick $N$ large enough such that the ratio
\[ \frac{d_1(f^{-N}(x), f^{-N}(y))}{d_1(x_0, f^{-N}(y_0))} \]
is close to 1 as shown in Figure 3. This is possible since $V_2^f$ contracts exponentially faster than $V_1^f$ under the action of $f^{-1}$. It is not hard to see that, given a large number $n$, we can pick $x$ (and $N$ correspondingly) far enough from $x_0$ such that at least $n$ points from the orbit $\{x, f^{-1}(x), \ldots, f^{-N}(x)\}$ lie outside of $B$. For such a point $z = f^{-i}(x)$ that is not in $B$,

$$D_{wu}^f(z) \geq D_{wu}^f(x_0) + \delta,$$

where $\delta > 0$ depends only on the size of $B$.

Using (D2), we get

$$\tilde{d}_1(x, y) = \frac{\prod_{i=1}^{N} \frac{D_{wu}^f(f^{-i}(x))}{D_{wu}^f(x_0)}}{\frac{\tilde{d}_1(f^{-N}(x), f^{-N}(y))}{\tilde{d}_1(x_0, f^{-N}(y))}} \geq \left(\frac{D_{wu}^f(x_0) + \delta}{D_{wu}^f(x_0)}\right)^n \frac{\tilde{d}_1(f^{-N}(x), f^{-N}(y))}{\tilde{d}_1(x_0, f^{-N}(y))},$$

which is an arbitrarily large number. Hence $\tilde{d}_1(x, y)$ is arbitrarily large and we are done.

**Remark.** Although Proposition 8 deals with a pretty special situation we believe that the picture on Figure 2 is generic. To be more precise, we think that for any $g \in U$ the following alternative holds. Either $V_2^g$ is conjugate to the linear foliation $V_2^L$ or there exists a dense set $\Lambda$ such that for any $x \in \Lambda$ and any $B \ni x$ the tube

$$\bigcup_{y \in B} V_2^f(y) \subset \mathbb{R}^5$$

contains arbitrarily long connected pieces of the leaves of $V_1^g$.

**Proof of Proposition 8.** The argument is more delicate than the one presented above since we do not know that the minimum of the derivative is achieved at $x_0$. 
Let $B_0$ be a small ball around $x_0$ and $B_1 \supset B_0$ a bigger ball. Condition (21) guarantees that we can choose them in such a way that
\[ m_0 < D_{f}^{wu}(x_0) < \sup_{x \in B_0} D_{f}^{wu}(x) < m_1 < M < \min_{x \notin B_1} D_{f}^{wu}(x), \]
with $m_0$, $m_1$ and $M$ satisfying
\[ \frac{Mm_0^{q-1}}{m_1^q} > 1, \tag{25} \]
where $q$ is an integer that depends only on the size of $\mathcal{U}$ and the size of $B_1$. After that we choose $\hat{\mathcal{U}} \subset \mathcal{U}$ so the fixed point of $g$ (that corresponds to $x_0$) is inside of $B_0$ and the property above persists. Namely,
\[ \forall g \in \hat{\mathcal{U}} \quad m_0 < \inf_{x \in B_0} D_{g}^{wu}(x) < \sup_{x \in B_0} D_{g}^{wu}(x) < m_1 < M < \min_{x \notin B_1} D_{f}^{wu}(x). \tag{26} \]

Note that provided that $f$ is sufficiently $C^1$-close to $L$ and the ball $B_1$ is small enough, any piece of a leaf of $V_{g}^{2}$ outside of $B_1$ that starts and ends on the boundary of $B_1$ cannot be homotoped to a point keeping the endpoints on the boundary. This is a minor technical detail that makes sure that the picture shown on Figure 4 does not occur. Thus there is a lower bound $R$ on the lengths of pieces of leaves of $V_{g}^{2}$ outside of $B_1$ with endpoints on the boundary of $B_1$. Obviously, there is also an upper bound $r$ on the lengths of pieces of leaves of $V_{g}^{2}$ inside $B_1$. 

**Figure 3.** Illustration to the argument. Quadrilateral in the box is much smaller then the one outside.
Figure 4. (a) does not occur if $B$ is sufficiently small; (b) choice of $I_1$.

It is enough to check (22) for a dense set $\Lambda$ of points $x \in \mathbb{T}^5$. We take $\Lambda$ to be a subset of the set of periodic points $g\Lambda = \{p : D_{f^{n(p)}(p)}^w(p) \leq m_1^{n(p)}\}$, where $n(p)$ stands for the period of $p$. The set $\Lambda$ consists of periodic points that spend a large but fixed percentage of time inside of $B_0$. It is fairly easy to show that $\Lambda$ is dense in $\mathbb{T}^5$. The proof is a trivial corollary of the specification property (e.g. see [KH95]).

So we fix $\tilde{x}_0 \in \Lambda$, a small ball $B_1$ centered at $\tilde{x}_0$ and $y_0 \in B \cap V_1^g(\tilde{x}_0)$ close to $\tilde{x}_0$. Our goal now is to find $x \in V_2^g(\tilde{x}_0)$ far in the tube $\mathcal{I}$ defined by (23) for which we can carry out estimates similar to (24).

We will be working with pieces of leaves of $V_2^g$. Given a piece $I$ with endpoints $z_1$ and $z_2$ let $|I| = d_2^I(z_1, z_2)$. Let $q$ be a number such that for any piece $I$, $|I| = R$, we have

$$|g^q(I)| > 2R + r.$$  \hspace{1cm} (28)

Notice that $q$ can be chosen to be independent of $g$ and depends only on $\hat{\beta}_2$, $R$ and $r$.

Pick $I_1 \subset V_2^g(\tilde{x}_0)$, $|I_1| = R$, $I_1 \cap B_1 = \emptyset$, as close to $\tilde{x}_0$ as possible if $\tilde{x}_0 \in B_1$ (see Figure 4b) or passing through $\tilde{x}_0$ if $\tilde{x}_0 \notin B_1$. Given $I_i$, $i \geq 1$ we choose $I_{i+1} \subset f^q(I_i)$, $|I_{i+1}| = R$, $I_{i+1} \cap B_1 = \emptyset$. Condition (28) guarantees that such choice is possible.

We fix $N$ large and take $x \in I_{Nq} \subset V_2^g(\tilde{x}_0)$. Let $y = V_1^g(x) \cap V_2^g(y_0)$ as before. The construction of the sequence $\{I_i, i \geq 1\}$ ensures that the points $f^{-q^i}(x)$, $i = 0, \ldots N - 1$, are outside $B_1$. This fact together with (26) and (27) allows to carry
out the following estimate:
\[
\frac{\tilde{d}_1(x, y)}{d_1(\tilde{x}_0, y_0)} = \prod_{i=1}^{Nq} \frac{D_{g^{-i}}^u(\gamma^{-1}(x))}{D_{g^{-i}}^u(\gamma^{-1}(\tilde{x}_0))} \cdot \frac{\tilde{d}_1(f^{-Nq}(x), f^{-Nq}(y))}{d_1(\tilde{x}_0, f^{-Nq}(y_0))} \\
\geq \frac{M_{\alpha}^N m_{\alpha}^{N(q-1)}}{m_{\alpha}^N} \cdot \frac{\tilde{d}_1(f^{-Nq}(x), f^{-Nq}(y))}{d_1(\tilde{x}_0, f^{-Nq}(y_0))}.
\]

The affine-like distance ratio on the right is bounded away from 0 independently of \( N \) since \( f^{-Nq}(x) \in I_1 \), while the coefficient in front of it is arbitrarily large according to (25). Hence \( \tilde{d}_1(x, y) \) is arbitrarily large and the projection of the tube \( T \) is dense in \( T^5 \). 

6. Proof of Theorem A

For reasons explained in Section 4 we first prove Theorem A assuming that \( f \) has Property \( \mathcal{A}' \). The only place where we use Property \( \mathcal{A}' \) is in the proof of Lemma 6.6. In Section 6.6 we give another proof of Lemma 6.6 that uses Property \( \mathcal{A} \) only.

6.1. Scheme of the proof of Theorem A. Recall the notation from [1,14] for the \( L \)-invariant splitting
\[
TT^d = F_1 \oplus F_{i-1} \oplus \ldots \oplus F_1 \oplus E_1 \oplus E_2 \oplus \ldots \oplus E_k
\]
along the eigendirections with corresponding eigenvalues
\[
\mu_1 < \mu_{i-1} < \ldots < \mu_1 < 1 < \lambda_1 < \lambda_2 < \ldots < \lambda_k.
\]
We choose a neighborhood \( \mathcal{U} \) in such a way that, for any \( f \in \mathcal{U} \), the invariant splitting survives:
\[
TT^d = F_{i}^{f} \oplus F_{i-1}^{f} \oplus \ldots \oplus F_{1}^{f} \oplus E_{1}^{f} \oplus E_{2}^{f} \oplus \ldots \oplus E_{k}^{f},
\]
with
\[
\angle(F_i, F_i^f) < \frac{\pi}{2}, \quad \angle(E_j, E_j^f) < \frac{\pi}{2}, \quad i = 1, \ldots, l, \quad j = 1, \ldots, k \quad (29)
\]
and \( f \) is partially hyperbolic in the strongest sense; that is, there exist \( C > 0 \) and constants
\[
\alpha_l < \alpha_{i-1} < \ldots < \alpha_1 < 1 < \beta_1 < \beta_2 < \ldots < \beta_k
\]
independent of the choice of \( f \) in \( \mathcal{U} \) such that for \( n > 0 \)
\[
\|D(f^n)(x)(v)\| \leq C\alpha_l^n\|v\|, \quad v \in F_{i}^{f}(x),
\]
\[
\frac{1}{C}\tilde{\alpha}_l^{-1}\|v\| \leq \|D(f^n)(x)(v)\| \leq C\alpha_l^{-1}\|v\|, \quad v \in F_{i-1}^{f}(x),
\]
\[
\ldots
\]
\[
\frac{1}{C}\tilde{\alpha}_1^{-1}\|v\| \leq \|D(f^n)(x)(v)\| \leq C\alpha_1^{-1}\|v\|, \quad v \in F_{1}^{f}(x),
\]
\[
\frac{1}{C}\tilde{\beta}_1^{-1}\|v\| \leq \|D(f^n)(x)(v)\| \leq C\beta_1^{-1}\|v\|, \quad v \in E_{1}^{f}(x),
\]
\[
\ldots
\]
\[
\frac{1}{C}\tilde{\beta}_k^{-1}\|v\| \leq \|D(f^n)(x)(v)\|, \quad v \in E_{k}^{f}(x). \quad (30)
\]
Equivalently, the Mather spectrum of \( f \) does not contain 1 and has \( d \) connected components.
Such a choice is possible — see Theorem 1 in [JPL]. This theorem also guarantees that $C^1$-size of $\mathcal{U}$ is rather large.

We show that the choice of $\mathcal{U}$ guarantees unique integrability of intermediate distributions. From now on, for the sake of concreteness, we work with unstable distributions and foliations.

For a given $f \in \mathcal{U}$ let $E^f(i, j) = E^f_i \oplus E^f_{i+1} \oplus \ldots \oplus E^f_j$, $i \leq j$.

**Lemma 6.1.** For any $f$ in $\mathcal{U}$ distribution $E^f(1, 1), E^f(1, 2), \ldots E^f(1, k)$ are uniquely integrable.

Let $W^f_1 \subset W^f_2 \subset \ldots \subset W^f_k$ be the corresponding flag of weak unstable foliations. The last foliation in the flag is the unstable foliation $W^f = W^f_k$.

**Lemma 6.2.** For any $f$ in $\mathcal{U}$ and $i \leq j$, the distribution $E^f(i, j)$ is uniquely integrable.

Denote by $W^f(i, j)$, $i \leq j$, the integral foliation of $E^f(i, j)$. Also recall that we denote by $V^f_1, V^f_2, \ldots V^f_k$ the integral foliations of $E^f_1, E^f_2, \ldots E^f_k$ correspondingly. Notice that $V^f_i = W^f(i, i)$ and $W^f_i = W^f(1, i)$, $i = 1, \ldots k$.

Now we consider $f$ and $g$ as in Theorem A, $h \circ f = g \circ h$. The conjugacy $h$ maps the unstable (stable) foliation of $f$ into the unstable (stable) foliation of $g$. Moreover, $h$ preserves the whole flag of weak unstable (stable) foliations.

**Lemma 6.3.** Fix an $i = 1, \ldots k$. Then $h(W^f_i) = W^g_i$.

**Remark.** The proof of this lemma does not use the assumption on the p. d. We only need $f$ and $g$ to be in $\mathcal{U}$.

Lemmas 6.1, 6.2 and 6.3 can be proved under a milder assumption. Instead of requiring $f$ and $g$ to be in $\mathcal{U}$ we can require the following.

**Alternative Assumption:** $f$ and $g$ are partially hyperbolic in the sense of (30) with the rate constants satisfying

$$\mu_l < \alpha_l < \tilde{\alpha}_l - 1 < \mu_{l-1} < \alpha_{l-1} < \ldots < \tilde{\beta}_{k-1} < \lambda_{k-1} < \beta_{k-1} < \tilde{\beta}_k < \lambda_k.$$ (⋆)

We think that (⋆) is actually automatic from (30).

**Remark.** To carry out proofs of the Lemmas above under the Alternative Assumption one needs to transfer the picture to the linear model by the conjugacy and use the inequalities (⋆) for growth arguments. This way one uses quasi-isometric foliations by straight lines of the linear model instead of foliations of $f$ which are a priori not known to be quasi-isometric.

**Conjecture 3.** Suppose that $f$ is homotopic to $L$ and partially hyperbolic in the strongest sense (30). Then the rate constants satisfy (⋆).

**Remark.** The proofs of Lemmas 6.1, 6.2 and 6.3 are the only places where we really need $f$ and $g$ to be in $\mathcal{U}$. So, in Theorem A, the assumption that $f, g \in \mathcal{U}$ can be substituted by the alternative assumption.

**Lemma 6.4.** A leaf $W^f_1(x)$ is dense in $\mathbb{T}^d$.

**Proof.** By Lemma 6.3 the conjugacy between $L$ and $f$ takes the foliation $W^L_1$ into the foliation $W^f_1$. According to Proposition 6 leaves of $W^L_1$ are dense. Hence leaves of $W^f_1$ are dense. \qed
Next we describe the inductive procedure which leads to the smoothness of \( h \) along the unstable foliation.

**Induction base.** We know that \( h \) takes \( W^f_i \) into \( W^g_i \).

**Lemma 6.5.** The conjugacy \( h \) is \( C^{1+\nu} \)-differentiable along \( W^f_i \), i.e., the restrictions of \( h \) to the leaves of \( W^f_i \) are differentiable and the derivative is a \( C^\nu \)-function on \( \mathbb{T}^d \).

Provided that we have Lemma 6.4, the proof of Lemma 6.5 is the same as the proof of Lemma 5 from [GG08].

**Induction step.** The induction procedure is based on the following lemmas.

**Lemma 6.6.** Assume that \( h \) is \( C^{1+\nu} \)-differentiable along \( W^f_{m-1} \) and \( h(V^f_i) = h(V^g_i), \ i = 1, \ldots, m - 1, \ 1 < m \leq k \). Then \( h(V^f_m) = V^g_m \).

**Lemma 6.7.** Assume that \( h(V^f_m) = V^g_m \) for some \( m = 1, \ldots k \). Then \( h \) is \( C^{1+\nu} \)-differentiable along \( V^f_m \).

We also use a regularity result due to Journé.

**Regularity Lemma (JSS).** Let \( M_j \) be a manifold and \( W^s_j, W^u_j \) be continuous transverse foliations with uniformly smooth leaves, \( j = 1, 2 \). Suppose that \( h : M_1 \to M_2 \) is a homeomorphism that maps \( W^s_1 \) into \( W^s_2 \) and \( W^u_1 \) into \( W^u_2 \). Moreover, assume that the restrictions of \( h \) to the leaves of these foliations are uniformly \( C^{r+\nu}, r \in \mathbb{N}, 0 < \nu < 1 \). Then \( h \) is \( C^{r+\nu} \).

**Remark.** There are two more methods of proving analytical results of this flavor besides Journé’s. One is due to de la Llave, Marco, Moriyón and the other one is due to Hurder and Katok (see [KN03] for a detailed discussion and proofs). We remark that we really need Journé’s result since the alternative approaches require foliations to be absolutely continuous while we apply the Regularity Lemma to various foliations that do not have to be absolutely continuous.

Now the inductive scheme can be described as follows. Assume that \( h \) is \( C^{1+\nu} \) along \( W^f_{m-1} \) for some \( m \leq k \) and \( h(V^f_i) = h(V^g_i), \ i = 1, \ldots, m - 1 \). By Lemma 6.6 we have that \( h(V^f_m) = V^g_m \) and by Lemma 6.5, \( h \) is \( C^{1+\nu} \) along \( V^f_m \). Fix a leaf \( W^f_m(x) \). Leaves of \( W^f_{m-1} \) and \( V^f_m \) subfoliate \( W^f_m(x) \) and it is clear that the Regularity Lemma can be applied for \( h : W^f_m(x) \to W^g_m(h(x)) \). Hence \( h \) is \( C^{1+\nu} \) on every leaf of \( W^f_m \). Hölder-continuity of the derivative of \( h \) in the direction transverse to \( W^f_m \) is a direct consequence of Hölder continuity of the derivatives along \( W^f_{m-1} \) and \( V^f_m \). We conclude that \( h \) is \( C^{1+\nu} \)-differentiable along \( W^f_m \).

By induction \( h \) is \( C^{1+\nu} \)-differentiable along the unstable foliation and analogously along the stable foliation. We finish the proof of Theorem A by applying the Regularity Lemma to stable and unstable foliations.

**6.2. Proof of the integrability lemmas.** In the proofs of Lemmas 6.1 and 6.2 we work with lifts of maps, distributions and foliations to \( \mathbb{R}^d \). We use the same notation for lifts as for the objects themselves.
Proof of Lemma 6.7. Fix $i < k$. We assume that the distribution $E^f(i, i)$ is not integrable or it is integrable but not uniquely. In any case it follows that we can find distinct points $a_0, a_1, \ldots, a_m$ such that

1. $\{a_1, a_2, \ldots, a_m\} \subset W^f(a_0)$,
2. there are smooth curves $\tau_j: [0, 1] \to W^f(a_0), j = 1, \ldots, m$, such that $\tau_j(0) = a_{j-1}, \tau_j(1) = a_j$ and $\dot{\tau}_j \in E^f_{\tau(j)}$, where $\tau(j) \leq i$,
3. there are smooth curves $\omega_j: [0, 1] \to W^f(a_0), j = 1, \ldots, m$, such that $\omega_j(0) = a_0, \omega_j(1) = a_m$, $\dot{\omega}_j(0) = \omega_{j+1}(0)$ for $j = 1 \ldots m - 1$ and $\dot{\omega}_j \in E^f_{\omega(j)}$ with $q(j) > i$ if $q(j) \neq q(j_2)$ if $j \neq j_2$.

Assume that $\tilde{m} = 1$ and let $\omega = \omega_1, q = q(1)$. The general case can be established in the same way by working with $\omega = \omega_j$ where $j$ is chosen so that $q(j) > i$ for $j \neq j$.

Let $\tilde{\tau}$ be a piecewise smooth curve obtained by concatenating $\tau_1, \tau_2, \ldots, \tau_{m-1}$ and $\tau_m$. From the second property above and (30), we get the following rough estimate:

$$\forall n \geq 0 \quad \text{length}(f^n(\tilde{\tau})) \leq \beta^n \text{length}(\tilde{\tau}).$$

Similarly,

$$\forall n \geq 0 \quad \text{length}(f^n(\omega)) \geq \beta^n \text{length}(\omega).$$

Denote by $d(\cdot, \cdot)$ the usual distance in $\mathbb{R}^d$. It follows from the assumption (29) that any curve $\gamma: [0, 1] \to \mathbb{R}^d$ tangent to the distribution $E^f_{\gamma}$ is quasi-isometric:

$$\exists c > 0 \text{ such that } \text{length}(\gamma) \leq c d(\gamma(0), \gamma(1)).$$

In particular,

$$\forall n \geq 0 \quad d(f^n(a_0), f^n(a_m)) \geq \frac{1}{c} \text{length}(f^n(\omega)).$$

The inequalities (31), (32) and (33) sum up to a contradiction. □

Proof of Lemma 6.8. The theory of partial hyperbolicity guarantees that the distributions $E^f(i, k), i = 1, \ldots, k$, integrate uniquely to foliations $W^f(i, k)$. Let us fix $i$ and $j$, $i < j$, and define $W^f(i, j) = W^f(1, j) \cap W^f(i, k)$. Obviously $W^f(i, j)$ is an integral foliation for $E^f(i, j)$. Unique integrability of $E^f(i, j)$ is a direct consequence of the unique integrability of $E^f(1, j)$ and $E^f(i, k)$. □

6.3. Weak unstable flag is preserved: proof of Lemma 6.3.

Proof. We continue working on the universal cover. Pick two points $a$ and $b, a \in W^f_i(b)$. Since

$$h(x + \tilde{m}) = h(x) + \tilde{m}, \quad \tilde{m} \in \mathbb{Z}^d$$

we have that $d(h(x), h(y)) \leq c_1 d(x, y)$ for any $x$ and $y$ such that $d(x, y) \geq 1$. Hence, for any $n > 0$,

$$d(g^n(h(a)), g^n(h(b))) = d(h(f^n(a)), h(f^n(b))) \leq c_2 d(f^n(a), f^n(b)) \leq c_2 c_3 \beta^n,$$

where $c_2$ and $c_3$ depend on $d(a, b)$. This inequality guarantees $h(a) \in W^g_i(h(b))$. Since the choice of $a$ and $b$ was arbitrary we conclude that $h(W^f_i) = W^g_i$. □
6.4. Induction step 1: the conjugacy preserves the foliation $V_m$. We now prove Lemma 6.6, which is the key ingredient in the proof of Theorem A. The proof is based on our idea from [GG08] but we take a rather different approach in order to deal with the high dimension of $W^f$. We provide a complete proof almost without referring to [GG08]. Nevertheless, we strongly encourage the reader to read Section 4.4 of [GG08] first.

The goal is to prove that $h(V^f_m) = V^g_m$. So we will consider the foliation $U = h^{-1}(V^g_m)$. As in the case for usual foliation, $U(x)$ stands for the leaf of $U$ passing through $x$ and $U(x, R)$ stands for the local leaf of size $R$. A priori, the leaves of $U$ are just Hölder-continuous curves. Hence the local leaf needs to be defined with a certain care. One way is to consider the lift of $U$ and define the lift of the local leaf $U(x, R)$ as a connected component of $x$ of the intersection $U(x) \cap B(x, R)$.

We prove Lemma 6.6 by induction.

**Induction base.** We will be working on $m$-dimensional leaves of $W^f_m$. By Lemma 6.3, $U$ subfoliates $W^f_m$. In other words, for any $x \in \mathbb{T}^d$, $U(x) \subset W^f_m(x)$.

**Induction step.** Suppose that $U$ subfoliates $W^f(i, m)$ for some $i < m$. Then $U$ subfoliates $W^f(i + 1, m)$.

By induction $U$ subfoliate $W^f(m, m) = V^f_m$. Hence $U = V_m$.

First let us prove several auxiliary claims. Note that all of the foliations that we are dealing with are oriented and the orientation is preserved under the dynamics. Denote by $d_i^f$ and $d_i^g$ the induced distances on the leaves of $V^f_i$ and $V^g_i$ correspondingly, $j = 1, \ldots, k$.

**Lemma 6.8.** Consider a point $a \in \mathbb{T}^d$. Pick a point $b \in U(a)$ and let $\tilde{b} = V^f_i(b) \cap W^f(i + 1, m)(a)$. Assume that $\tilde{b} \neq b$. Pick a point $c \in V^f_i(a)$ and let $d = U(c) \cap W^f(i, m - 1)(b)$, $\tilde{d} = V^f_i(d) \cap W^f(i + 1, m)(c)$. Then $\tilde{d} \neq d$ and the orientations of the pairs $(b, \tilde{b})$ and $(d, \tilde{d})$ in $V^f_i$ are the same.

The statement of the lemma when $i = 1$ and $m = 3$ is illustrated on Figure 5.

**Remark.** Since by the induction hypothesis, $h(W^f(i, m - 1)) = W^g(i, m - 1)$, we see that the leaf $U(a)$ intersects each leaf $W^f(i, m - 1)(x)$, $x \in W^f(i, m - 1)(a)$ exactly once.

**Proof.** Let $e = V^f_i(b) \cap W^f(i + 1, m)(d)$ and $\bar{e} = V^f_i(b) \cap W^f(i + 1, m)(\bar{d})$. Obviously $(e, \bar{e})$ has the same orientation as $(d, \bar{d})$ and also has the advantage of lying on the leaf $V^f_i(b)$. Therefore, we forget about $(d, \bar{d})$ and work with $(e, \bar{e})$.

We use an affine structure on the expanding foliation $V^f_i$. Namely, we work with the affine distance-like function $\tilde{d}_i$. We refer to [GG08] for the definition. There we define the affine distance-like function on the weak unstable foliation. The definition for the foliation $V^f_i$ is the same with obvious modifications. Recall the crucial properties of $\tilde{d}_i$:

1. $\tilde{d}_i(x, y) = d_i^f(x, y) + o(d_i^f(x, y))$,
2. $\tilde{d}_i(f(x), f(y)) = D^f_j(x)\tilde{d}_i(x, y)$, where $D^f_j$ is the derivative of $f$ along $V^f_i$,
3. $\forall K > 0 \exists C > 0$ such that $\frac{1}{C}d_i(x, y) \leq \tilde{d}_i(x, y) \leq Cd_i(x, y)$ whenever $d_i(x, y) < K$. 

The crucial properties of $\tilde{d}_i$ ensure the existence of an affine distance-like function $\tilde{d}_i$ on the expanding foliation $V^f_i$. We refer to [GG08] for the definitions.

**Remark.** Since by the induction hypothesis, $h(W^f(i, m - 1)) = W^g(i, m - 1)$, we see that the leaf $U(a)$ intersects each leaf $W^f(i, m - 1)(x)$, $x \in W^f(i, m - 1)(a)$ exactly once.

**Proof.** Let $e = V^f_i(b) \cap W^f(i + 1, m)(d)$ and $\bar{e} = V^f_i(b) \cap W^f(i + 1, m)(\bar{d})$. Obviously $(e, \bar{e})$ has the same orientation as $(d, \bar{d})$ and also has the advantage of lying on the leaf $V^f_i(b)$. Therefore, we forget about $(d, \bar{d})$ and work with $(e, \bar{e})$.

We use an affine structure on the expanding foliation $V^f_i$. Namely, we work with the affine distance-like function $\tilde{d}_i$. We refer to [GG08] for the definition. There we define the affine distance-like function on the weak unstable foliation. The definition for the foliation $V^f_i$ is the same with obvious modifications. Recall the crucial properties of $\tilde{d}_i$:

1. $\tilde{d}_i(x, y) = d_i^f(x, y) + o(d_i^f(x, y))$,
2. $\tilde{d}_i(f(x), f(y)) = D^f_j(x)\tilde{d}_i(x, y)$, where $D^f_j$ is the derivative of $f$ along $V^f_i$,
3. $\forall K > 0 \exists C > 0$ such that $\frac{1}{C}d_i(x, y) \leq \tilde{d}_i(x, y) \leq Cd_i(x, y)$ whenever $d_i(x, y) < K$. 

The crucial properties of $\tilde{d}_i$ ensure the existence of an affine distance-like function $\tilde{d}_i$ on the expanding foliation $V^f_i$. We refer to [GG08] for the definitions.
Assume that \((e, \tilde{e})\) has orientation opposite to \((b, \tilde{b})\) or \(e = \tilde{e}\). For the sake of concreteness we assume that these points lie on \(V^f_i(b)\) in the order \(b, \tilde{b}, \tilde{e}, e\). All other cases can be treated similarly. Then
\[
\tilde{d}_i(b, e) \geq \tilde{d}_i(b, \tilde{e}) > \tilde{d}_i(b, \tilde{b}).
\]

**Remark.** Notice that \(\tilde{d}_i(b, \tilde{e}) - \tilde{d}_i(b, \tilde{b}) \neq \tilde{d}_i(\tilde{b}, \tilde{e})\) since \(\tilde{d}_i\) is neither symmetric nor additive. The distance \(\tilde{d}_i\) is given by an integral of a certain density with normalization defined by the first argument. As long as the first argument (point \(b\) in the above inequality) is the same, all natural inequalities hold.

Applying (D2), we get that
\[
\forall n > 0 \quad \frac{\tilde{d}_i(f^{-n}(b), f^{-n}(e))}{\tilde{d}_i(f^{-n}(b), f^{-n}(\tilde{e}))} = c_1 > 1,
\]
where \(c_1\) does not depend on \(n\). By property (D1) we can switch to the usual distance:
\[
\exists N : \forall n > N \quad \frac{d_i^f(f^{-n}(b), f^{-n}(e))}{d_i^f(f^{-n}(b), f^{-n}(\tilde{e}))} > c_2 > 1,
\]
where \(c_2\) does not depend on \(n\).

Under the action of \(f^{-1}\), strong unstable leaves of \(W^f(i + 1, m)\) contract exponentially faster than weak unstable leaves of \(V^f_i\). Thus
\[
\forall \varepsilon > 0 \quad \exists N : \forall n > N \quad \left| \frac{d_i^f(f^{-n}(a), f^{-n}(c))}{d_i^f(f^{-n}(\tilde{b}), f^{-n}(\tilde{e}))} - 1 \right| < \varepsilon.
\]

We have that \(h(e) \in W^g(i + 1, m)(h(c))\). Indeed, notice that
\[
e = V^f_i(b) \cap W^f(i + 1, m)(d) = V^f_i(b) \cap W^f(i + 1, m - 1)(d).
\]
(if \(i = m - 1\) than we have \(e = d\)). Thus
\[
h(e) = h(V_i^f(b) \cap W_i^f(i + 1, m - 1(d)) = V_i^g(h(b)) \cap W_i^g(i + 1, m - 1(h(d)))
\]
\[
= V_i^g(h(b)) \cap W_i^g(i + 1, m)(h(d)) = V_i^g(h(b)) \cap W_i^g(i + 1, m(h(c)),
\]
where the last equality is justified by the fact that \(h(d) \in V_m^g(h(c))\). We know also
that \(h(b) \in W^g(i + 1, m)(b(a)).\) Hence, analogously to (38), we have
\[
\forall \varepsilon > 0 \exists N: \forall n > N \quad \left| \frac{d_i^f(g^{-n}(h(a)), g^{-n}(h(c)))}{d_i^f(f^{-n}(a), f^{-n}(c))} - D_i^h(f^{-n}(a)) \right| < \varepsilon. \tag{37}
\]
On the other hand, we also know that \(h\) is continuously differentiable along \(V_i^f\). Hence
\[
\forall \varepsilon > 0 \exists N: \forall n > N \quad \left| \frac{d_i^f(g^{-n}(h(b)), g^{-n}(h(e)))}{d_i^f(f^{-n}(b), f^{-n}(e))} - D_i^h(f^{-n}(a)) \right| < \varepsilon. \tag{38}
\]
and
\[
\forall \varepsilon > 0 \exists N: \forall n > N \quad \left| \frac{d_i^f(g^{-n}(h(b)), g^{-n}(h(e)))}{d_i^f(f^{-n}(b), f^{-n}(e))} - D_i^h(f^{-n}(a)) \right| < \varepsilon.
\]
Therefore from (37) and (38) we have
\[
\forall \varepsilon > 0 \exists N: \forall n > N \quad \left| \frac{d_i^f(f^{-n}(a), f^{-n}(c))}{d_i^f(f^{-n}(b), f^{-n}(e))} - 1 \right| < \varepsilon,
\]
which we combine with (36) to get
\[
\forall \varepsilon > 0 \exists N: \forall n > N \quad \left| \frac{d_i^f(f^{-n}(b), f^{-n}(e))}{d_i^f(f^{-n}(a), f^{-n}(c))} - 1 \right| < \varepsilon.
\]
We have reached a contradiction with (34). \(\square\)

**Remark.** By the same argument one can prove that if \(b = \tilde{b}\) then \(d = \tilde{d}\).

**Lemma 6.9.** Consider a weak unstable leaf \(W^f_{m-1}(a)\) and \(b \in V^f_{m-1}(a),\ b \neq a\). For
any \(y \in W_{m-1}^f(a),\ let \ y' = W_{m-1}^f(b) \cap V^f_{m-1}(y).\ Then there exist \(c_1, c_2 > 0\) such that
\(\forall y \in W_{m-1}^f(a),\ c_1 > d_i^f(y, y') > c_2.\)

**Proof.** We will be working on the universal cover \(\mathbb{R}^d.\) We abuse notation slightly
by using the same notation for the lifted objects. Note that the leaves on \(\mathbb{R}^d\) are
connected components of preimages by the projection map of the leaves on \(\mathbb{T}^d.\)

Let \(h_f\) be the conjugacy with the linear model, \(h_f \circ f = L \circ h_f.\) Lemma 6.8
holds for \(h_f; h_f(W_{m-1}^f) = W_{m-1}^L.\) The leaves \(W_{m-1}^L(h_f(a))\) and \(W_{m-1}^L(h_f(b))\) are
parallel hyperplanes. Thus the lower bound follows from the uniform continuity
of \(h_f.\)

It follows from (34) that \(h_f^{-1} - Id\) is bounded. Hence we can find positive \(R\) that
depends only on size of \(\mathcal{U}\) such that
\[
W_{m-1}^f(a) \subset \text{Tube}_a \overset{\text{def}}{=} \bigcup_{x \in B(a, R)} W_{m-1}^L(x)
\]
and
\[ W^f_{m-1}(b) \subset \text{Tube}_b \overset{\text{def}}{=} \bigcup_{x \in B(b, R)} W^L_{m-1}(x). \]

Then, obviously,
\[ d^f_m(y, y') \leq \sup \{ d^f_m(x, x') \mid x \in \text{Tube}_a, x' \in \text{Tube}_b \cap V^f_m(x) \}. \]

Assumption (29) guarantees that \( E^f_m \) is uniformly transverse to \( TW^L_{m-1} = E^L_1 \oplus E^L_2 \oplus \ldots \oplus E^L_{m-1} \). Thus the supremum above is finite. □

**Remark.**
1. Given two points \( a, b \in \mathbb{R}^d \) let
\[ \hat{d}(a, b) = \text{distance}(W^L_{m-1}(h_f(a)), W^L_{m-1}(h_f(b))). \]
It is clear from the proof that constants \( c_1 \) and \( c_2 \) can be chosen in such a way that they depend only on \( \hat{d}(a, b) \).
2. In the proof above we do not use the fact that both \( W^f_{m-1} \) and \( V^f_m \) are expanding. We only need them to be transverse. Thus, if we substitute for the weak unstable foliation \( W^f_{m-1} \) some weak stable foliation \( F^i \), the statement still holds.
3. As mentioned earlier the assumption (29) is crucial only for Lemmas 6.1, 6.2 and 6.3. We used this assumption in the proof above only for convenience. A slightly more delicate argument goes through without using assumption (29).

**Proof of the induction step.** We will be working inside of the leaves of \( W^f(i, m) \). Assume that \( U \) does not subfoliate \( W^f(i+1, m) \). Then there exists a point \( x_0 \) and \( x_1 \in U(x_0) \) close to \( x_0 \) such that \( x_1 \not\in W^f(i+1, m)(x_0) \).

We fix an orientation \( O \) of \( U \) and \( V^f_i \) that is defined on pairs of points \((x, y), y \in U(x) \) and \((x, y), y \in V^f_i(x) \). Although we denote these orientations by the same symbol it will not cause any confusion since \( U \) and \( V^f_i \) are topologically transverse.

For every \((x, y), y \in U(x) \) with \( O(x, y) = O(x_0, x_1) \), define \[ [x, y] = W^f(i + 1, m)(x) \cap V^f_i(y). \]
For instance, in Lemma 6.8, \( \tilde{b} = [a, b], \tilde{d} = [c, d]. \)

![Figure 6. Definition of [x, y].](image-url)
Lemma 6.10. For every \((x, y)\) as above, either \([x, y] = y\) or \(O([x, y], y) = O^+ \eqdef O([x_0, x_1], x_1)\).

Proof. Let \(a_0 = \hat{d}(x_0, x_1)\) (for definition of \(\hat{d}\) see the remark after the proof of Lemma 6.9). The number \(a_0\) is positive since \(U(x)\) is transverse to \(W^{f}_{m-1}\).

For any \(y \in T^d\), there is a unique point \(sh(y) \in U(y)\) such that \(\hat{d}(y, sh(y)) = a_0\) and \(O(y, sh(y)) = O(x_0, x_1)\).

The leaves of all the foliations that we consider depend continuously on the point. Therefore we can find a small ball \(B\) centered at \(x_0\) such that \(\forall y \in B\), \([y, sh(y)] \neq sh(y)\) and \(O([y, sh(y)], sh(y)) = O^+\) as shown on the Figure 7.

By Property \(\mathcal{A}\),

\[
\bigcup_{y \in B} V^f_i = T^d.
\]

Thus

\[
\forall z \in T^d \quad [z, sh(z)] \neq sh(z) \quad \text{and} \quad O([z, sh(z)], sh(z)) = O^+. \tag{39}
\]

FIGURE 7. Orientation of \(([z, sh(z)], sh(z))\) is positive for any \(z\) in the \(V^f_i\)-tube through the ball \(B\). Foliation \(W^f_{i+1}(m)\) is two-dimensional on the picture.

Now let us assume the contrary of the statement of the lemma. Namely, assume that there exists \(\tilde{x}_0\) and \(\tilde{x}_1, \tilde{x}_1 \in U(\tilde{x}_0), O(\tilde{x}_0, \tilde{x}_1) = O(x_0, x_1)\), such that \([\tilde{x}_0, \tilde{x}_1] \neq \tilde{x}_1\) and \(O([\tilde{x}_0, \tilde{x}_1], \tilde{x}_1) \eqdef O^- \neq O^+\). By perturbing \(\tilde{x}_1\) infinitesimally along \(U(\tilde{x}_0)\) we can ensure that \(N_1a_0 = N_2\hat{d}(\tilde{x}_0, \tilde{x}_1)\), where \(N_1\) and \(N_2\) are some large integer numbers.
For any \( y \in \mathbb{T}^d \) there is a unique point \( \tilde{sh}(y) \in U(y) \) such that \( \hat{d}(y, \tilde{sh}(y)) = \hat{d}(\tilde{x}_0, \tilde{x}_1) \) and \( O(y, \tilde{sh}(y)) = O(\tilde{x}_0, \tilde{x}_1) \). Then by the same argument we show an analog of (39):

\[
\forall z \in \mathbb{T}^d [z, \tilde{sh}(z)] \neq \tilde{sh}(z) \quad \text{and} \quad O([z, \tilde{sh}(z)], \tilde{sh}(z)) = O^-. \quad (40)
\]

Pick a point \( x \in \mathbb{T}^d \) and \( y, z \in U(x) \), \( O(x, y) = O(y, z) \). Assume that \( O([x, y], y) = O([y, z], z) \). Then \( O([x, z], z) = O([x, y], y) \). This obvious property allows us to “iterate” \( sh \) and \( \tilde{sh} \).

Choose any \( z \) and “iterate” (39) and (40) \( N_1 \) and \( N_2 \) times correspondingly as shown on the Figure 8.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure8.pdf}
\caption{Illustration to the argument with shifts along \( U(z) \). Foliation \( W^f(i + 1, m) \) is one-dimensional here, \( N_1 = 3 \), \( N_2 = 2 \). The black segments of \( V^f_i \) carry known information about the orientations of \( ([\cdot, sh(\cdot)], sh(\cdot)) \) and \( ([\cdot, \tilde{sh}(\cdot)], \tilde{sh}(\cdot)) \). This picture is clearly impossible if \( sh^{N_1} = \tilde{sh}^{N_2} \).}
\end{figure}

We get that

\[
O([z, sh^{N_1}(z)], sh^{N_1}(z)) = O^+ \quad \text{and} \quad O([z, \tilde{sh}^{N_2}(z)], \tilde{sh}^{N_2}(z)) = O^-.
\]

To get a contradiction it remains to notice that \( sh^{N_1} = \tilde{sh}^{N_2} \). Hence the lemma is proved. \( \square \)

From (39) we see that for any \( z \in \mathbb{T}^d \), \( d^f_i ([z, sh(z)], sh(z)) > 0 \). Hence, due to the compactness and continuity of the function \( d^f_i ([\cdot, sh(\cdot)], sh(\cdot)) \), we have \( \delta < d^f_i ([z, sh(z)], sh(z)) < \Delta \) for some positive \( \delta \) and \( \Delta \). Lemma 6.10 guarantees even more:

\[
\forall x \in \mathbb{T}^d \quad \text{and} \quad y \in U(x), O(x, y) = O(x_0, x_1) \quad \text{such that} \quad \hat{d}(x, y) \leq a_0,
\]

we have \( d^f_i ([x, y], y) < \Delta \). (41)
From now on it is more convenient to work on the universal cover, although formally we do not have to do it since we are working inside of the leaves of $W^f(i, m)$, which are isometric to their lifts.

Let $x_n = sh^n(x_0)$, $n > 0$. For every $n \geq 0$, we have $O([x_n, x_{n+1}], x_{n+1}) = O^+$ and $d_f^i([x_n, x_{n+1}], x_{n+1}) > \delta$. Lemma 6.10 also tells us that the leaves of $U$ are monotone with respect to foliation $W^f(i + 1, m)$. Namely, for any $x \in T^d$, the intersection $U(x) \cap W^f(i + 1, m)(x)$ is a connected piece of $U(x)$.

Denote by $x_n, x_{n+1}$ the piece of $U(x_0)$ that lies between $x_n$ and $x_{n+1}$. We know that for any $n \geq 0$, $x_n, x_{n+1}$ is confined between the leaves $W^f(i, m - 1)(x_n)$ and $W^f(i, m - 1)(x_{n+1})$. Lemma 6.10 guarantees that $x_n, x_{n+1}$ is also confined between $W^f(i + 1, m)(x_n)$ and $W^f(i + 1, m)(x_{n+1})$, as shown on Figure 10. Thus, it makes sense to measure two different “dimensions” of $x_n, x_{n+1}$. Namely, let $a_n = d(x_n, x_{n+1})$ and $b_n = d_f^i([x_n, x_{n+1}], x_{n+1})$. As we have remarked earlier $b_n > \delta > 0$ and $a_n = a_0$ by the definition of $d$ and $sh$.

These “dimensions” behave nicely under the dynamics. Namely,

$$\forall N > 0 \ (f_*)^{-N}(b_n) \overset{\text{def}}{=} d_f^i([f^{-N}(x_n), f^{-N}(x_{n+1})), f^{-N}(x_{n+1})) \geq \delta \beta^{-N},$$

and

$$\forall N > 0 \ (f_*)^{-N}(a_n) \overset{\text{def}}{=} d(f^{-N}(x_n), f^{-N}(x_{n+1})) = a_0 \lambda^{-N}.$$

Recall that $\lambda > \beta_i$.

The idea now is to show that the leaf $U(f^{-N}(x_0))$ is lying “too close” to $W^f(i, m - 1)(x_0)$ for $N$ large, which would lead to a contradiction.

Take $N$ large and let $M = \lfloor \lambda^{-N} \rfloor$. Then

$$d(f^{-N}(x_0), f^{-N}(x_M)) = \sum_{j=0}^{M-1} d(f^{-N}(x_j), f^{-N}(x_{j+1})) = \sum_{j=0}^{M-1} (f_*)^{-N}(a_j) = M a_0 \lambda^{-N} \leq a_0.$$

The first equality holds since the holonomy along $W^f(i, m - 1)$ is isometric with respect to $d$.

To estimate $d_f^i([f^{-N}(x_0), f^{-N}(x_M)], f^{-N}(x_M))$ in a similar way we need to have control over holonomies along $W^f(i, m - 1)$.

Fix two small one-dimensional transversals $T(x) \subset V_i^f(x)$ and $T(y) \subset V_i^f(y)$, $y \in U(x)$ with $d(x, y) \leq a_0$. This condition ensures that the distance between $x$ and $y$ along $W^f(i, m)(x)$ is uniformly bounded from above. To see this we only need to bound the distance between $h(x)$ and $h(y)$ along $W^g(i, m)(h(x))$. This, in turn, is a direct consequence of Lemma 6.10 applied to $g$ since $h(y) \in V_n^g(h(x))$.

Consider the holonomy map along $W^f(i + 1, m)$ $H : T(x) \to T(y)$. This holonomy can be viewed as the holonomy along $W^f(i + 1, k)$. Recall that $W^f(i + 1, k)$ is the fast unstable foliation. Since $f$ is at least $C^2$-differentiable, $W^f(i + 1, k)$ is Lipschitz inside of $W^f(i, k)$. Moreover, since the distance between $x$ and $y$ is bounded from above, the Lipschitz constant $C_{3dol}$ of $H$ is uniform in $x$ and $y$. For a proof, see [LYS5], Section 4.2. They prove that the unstable foliation is Lipschitz within center-unstable leaves but the proof goes through for $W^f(i + 1, k)$ within the leaves of $W^f(i, k)$. 
Figure 9. Piece $x_n x_{n+1}$ is “monotone” with respect to foliation $W^f(i, m - 1)$. By Lemma 6.10 $x_n x_{n+1}$ is also “monotone” with respect to $W^f(i + 1, m)$: the intersections of $x_n x_{n+1}$ with local leaves of $W^f(i + 1, m)$ are points or connected components of $x_n x_{n+1}$. On this picture foliations $W^f(i, m - 1)$ and $W^f(i + 1, m)$ are two-dimensional.

Let $\tilde{x}_j = W^f(i + 1, m)(f^{-N}(x_j)) \cap V^f_i(f^{-N}(x_M))$, $j = 1, \ldots, M$. Then

$$d_i^f([f^{-N}(x_0), f^{-N}(x_M), f^{-N}(x_M)])$$

$$= \sum_{j=0}^{M-1} d_i^f(\tilde{x}_j, \tilde{x}_{j+1})$$

$$\geq C_{Hol} \sum_{j=0}^{M-1} d_i^f([f^{-N}(x_j), f^{-N}(x_{j+1})], f^{-N}(x_{j+1}))$$

$$= C_{Hol} \sum_{j=0}^{M-1} (f^*)^{-N}(b_j)$$

$$\geq C_{Hol} M \delta \beta_i^{-N}. $$
The holonomy constant $C_{Hol}$ is uniform since
\[
\hat{d}(f^{-N}(x_j), \tilde{x}_j) \leq \hat{d}(f^{-N}(x_0), \tilde{x}_j) = \hat{d}(f^{-N}(x_0), f^{-N}(x_M)) \leq a_0
\]
by \[42\]. Notice that $C_{Hol} M \beta_i^{-N}$ can be arbitrarily big when $N \to \infty$, while $d(f^{-N}(x_0), f^{-N}(x_M)) \leq a_0$ which contradicts to \[41\]. Hence the induction step is established.

6.5. **Induction step 2: proof of Lemma 6.7 by transitive point argument.** The proof of Lemma 6.7 is carried out in a way similar to the proofs of Lemmas 4 and 5 from \textbf{GG08}. Here we overview the scheme and deal with the complications that arise due to higher dimension.

First, using the assumption on the p. d., we argue that $h$ is uniformly Lipschitz along $V^f_m$, i.e., for any point $x$ the restriction $h|_{V^f_m(x)}: V^f_m(x) \to V^g_m(x)$ is a Lipschitz map with a Lipschitz constant that does not depend on $x$. At this step, the assumption on the p. d. along $V^f_m$ is used.

The Lipschitz property implies differentiability at almost every point with respect to the Lebesgue measure on the leaves of $V^f_m$. The next step is to show that the differentiability of $h$ along $V^f_m$ at a transitive point $x$ implies that $h$ is $C^{1+\nu}$-differentiable along $V^f_m$. This is done by a direct approximation argument (see Step 1 in Section 4.3 in \textbf{GG08}). The transitive point $x$ “spreads differentiability” all over the torus.

Last but not least, we need to find such a transitive point $x$. Ideally, for that we would find an ergodic measure $\mu$ with full support such that the foliation $V^f_m$ is absolutely continuous with respect to $\mu$. Then, by the Birkhoff Ergodic Theorem, almost every point would be transitive. Since $V^f_m$ is absolutely continuous, we would then have that almost every point, with respect to the Lebesgue measure on the leaves, is transitive. Hence we would have a full measure set of the points that we are looking for.
Unfortunately, we cannot carry out the scheme described above. The problem is that the foliation $V^f_m$ is not absolutely continuous with respect to natural ergodic measures (see \cite{GG08} for detailed discussion and \cite{SX08} for in-depth analysis of this phenomenon). Instead, we construct a measure $\mu$ such that almost every point is transitive and $V^f_m$ is absolutely continuous with respect to $\mu$. This is clearly sufficient.

The construction follows the lines of Pesin–Sinai’s \cite{PS83} construction of $u$-Gibbs measures. Given a partially hyperbolic diffeomorphism, they construct a measure such that the unstable foliation is absolutely continuous with respect to the measure. In fact, this construction works well for any expanding foliation. We apply this construction to $m$-dimensional foliation $W^f_m$.

The construction is described as follows. Let $x_0$ be a fixed point of $f$. For any $y \in W^f_m(x_0)$, define

$$\rho(y) = \prod_{n \geq 0} \frac{J^f_m(f^{-n}(y))}{J^f_m(x_0)},$$

where $J^f_m$ is the Jacobian of $f|_{W^f_m}$.

Let $V_0$ be an open bounded neighborhood of $x_0$ in $W^f_m(x_0)$. Consider a probability measure $\eta_0$ supported on $V_0$ with density proportional to $\rho(\cdot)$. For $n > 0$, define

$$V_n = f^n(V_0), \quad \eta_n = (f^n)_* \eta_0.$$

Let

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \eta_i.$$

By the Krylov–Bogolyubov Theorem, $\{\mu_n; n \geq 0\}$ is weakly compact and any of its limits is $f$-invariant. Let $\mu$ be an accumulation point of $\{\mu_n; n \geq 0\}$. This is the measure that we are looking for.

Foliation $W^f_m$ is absolutely continuous with respect to $\mu$. We refer to \cite{PS83} or \cite{GG08} for the proof. The proof of \cite{GG08} requires some minimal modifications that are due to the higher dimension of $W^f_m$.

Since the foliation $W^f_m$ is conjugate to the linear foliation $W^L_m$, we have that for any open ball $B$,

$$\exists R > 0 \quad \bigcup_{y \in B} W^f_m(y, R) = \mathbb{T}^d,$$

where $W^f_m(y, R)$ is a ball of radius $R$ inside of the leaf $W^f_m(y)$. Together with absolute continuity, this guarantees that $\mu$ almost every point is transitive. See \cite{GG08}, Section 4.3, Step 3 for the proof. We stress that we do not need to know that $\mu$ has full support in that argument.

It is left to show that the conjugacy $h$ is $C^{1+\nu}$-differentiable in the direction of $V^f_m$ at $\mu$ almost every point. For this we need to argue that $V^f_m$ is absolutely continuous with respect to $\mu$.

The foliation $W^f(m, k)$ is Lipschitz inside of a leaf of $W^f$ (again we refer to \cite{LY85}, Section 4.2). Hence $V^f_m = W^f(m, k) \cap W^f_m$ is Lipschitz inside of a leaf of $W^f_m$. So $V^f_m$ is absolutely continuous with respect to the Lebesgue measure on a leaf of $W^f_m$, while $W^f_m$ is absolutely continuous with respect to $\mu$. Therefore $V^f_m$ is absolutely continuous with respect to $\mu$. 
6.6. Induction step 1 revisited. To carry out the proof of Lemma 6.1 assuming Property \( A \) only, we shrink the neighborhood \( U \) even more. In addition to (29) and (30), we require \( \rho > \delta > 0 \) and (31), we require \( x \) with base point \( \bar{\varepsilon} \) and \( \bar{\delta} \). The following condition that we will actually use is obviously a consequence of the above one.

\[
\forall i < k \quad \text{and} \quad \forall m, i \leq m \leq k, \quad \rho \overset{\text{def}}{=} \frac{\log \beta_m}{\log \beta_m} > \frac{\log \beta_i}{\log \beta_m}.
\]  
(43)

This inequality can be achieved by shrinking the size of \( U \) since \( \beta_j \) and \( \tilde{\beta}_j \) get arbitrarily close to \( \lambda_j \), \( j = 1, \ldots, k \).

Remark. Condition (43) greatly simplifies the proof of Lemma 6.6. We have yet another, longer, proof (but based on the same idea) of Lemma 6.6 that works for any \( f \) with Property \( A \) in \( U \) as defined in Section 6.1. It will not appear here.

We start the proof as in Section 6.4.1. The first place where we use Property \( A' \) is the proof of Lemma 6.10. So we reprove induction step 1 with Property \( A \) only assuming that we have got everything that preceded Lemma 6.10. With Property \( A' \), the proof of Lemma 6.10 still goes through, although instead of (39), we get

\[
\forall z \in \mathbb{T}^d \quad \text{either} \quad \{z, sh(z)\} = sh(z) \quad \text{or} \quad O([z, sh(z)], sh(z)) = O^+.
\]

Thus we still have Lemma 6.10 and the upper bound (41) but not the lower bound \( d^f_i([z, sh(z)], sh(z)) > \delta \). This is the reason why we cannot proceed with the proof of the induction step as at the end of Section 6.4.1.

Proof of the induction step. As before, we need to show that \( U \) subfoliates foliation \( W^f(i + 1, m) \).

Fix an orientation \( O \) on \( V_m^f \) and \( V_i^f \). Given \( x \in \mathbb{T}^d \) and \( \varepsilon > 0 \) choose \( \bar{x} \in V_m^f(x) \) such that \( d^f_i(x, \bar{x}) = \varepsilon \) and \( O(x, \bar{x}) = O^+ \). Let \( \bar{y} = U(x) \cap W^f(i, m - 1)(\bar{x}) \) and \( y = V_i^f(x) \cap W^f(i + 1, m)(\bar{y}) \). This way we define an \( \varepsilon \)-“rectangle” \( R = R(x, \bar{x}, y, \bar{y}) \) with base point \( x \), vertical size \( d^f_i(x, \bar{x}) = \varepsilon \), and horizontal size \( d^f_i(x, y) = \bar{\varepsilon} \).

Remark. Notice that we measure vertical size in a way different from the one in Section 6.4.

It is clear that this “rectangle” is uniquely defined by its “diagonal” \( (x, \bar{y}) \) (Figure 9 is the picture of “rectangle” with diagonal \( (x_n, x_{n+1}) \)). Sometimes we will use the notation \( R(x, \bar{y}) \). Note that by Lemma 6.11, \( O(x, y) \) does not depend on \( x \) and \( \varepsilon \). It also guarantees that the piece of \( U(x) \) between \( x \) and \( \bar{y} \) lies “inside” of \( R(x, y) \). The horizontal size \( \bar{\varepsilon} \) might happen to be equal to zero.

Next we define a set of base points \( X_z \) such that \( U(x), x \in X_z \), has big Hölder slope inside of corresponding \( \varepsilon \)-rectangle,

\[
X_z = \{ x \in \mathbb{T}^d: \quad \bar{\varepsilon} \leq \varepsilon^z \},
\]

with some \( \delta \) satisfying inequality \( \rho > \delta > \log \beta_i/\log \beta_m \).

Let \( \mu \) be the measure constructed in Section 6.5. Recall that \( \mu \)-almost every point is transitive. The foliation \( W^f(i, m) \) is absolutely continuous with respect to
Hence by Lemma 6.10 we conclude that the piece of arbitrarily large measure, the horizontal size of rectangles is equal to zero. Hence the leaves of \( W_C \) lie inside of the leaves of \( W^f(i + 1, m) \).

Given \( n \), let \( N = N(n) \) be the largest number such that \( \frac{1}{N} \beta_n \varepsilon_n < 1 \) (constant \( C \) here is from Definition 3.10). Take \( x \in X_{\varepsilon_n} \) and the corresponding \( \varepsilon_n \)-rectangle \( \mathcal{R}(x, y, \bar{x}, \bar{y}) \) and consider its image \( \mathcal{R}(f^N(x), f^N(y), f^N(\bar{x}), f^N(\bar{y})) \). The choice of \( N \) provides a lower bound on the vertical size,

\[
\text{VS}(\mathcal{R}(f^N(x), f^N(y), f^N(\bar{x}), f^N(\bar{y}))) = d_m(f^N(x), f^N(\bar{x})) \geq \frac{1}{\beta_m},
\]

while the horizontal size can be estimated as follows:

\[
\text{HS}(\mathcal{R}(f^N(x), f^N(y), f^N(\bar{x}), f^N(\bar{y}))) = d_{\beta_m}(f^N(x), f^N(\bar{x})) \leq C\beta_{\delta}(C\beta_{\delta})^{N} \leq C\beta_{\delta}(C\beta_{\delta})^{N} \leq C\beta_{\delta}(C\beta_{\delta})^{N}.
\]

Rather than continuing to look at the rectangle \( \mathcal{R}(f^N(x), f^N(y), f^N(\bar{x}), f^N(\bar{y})) \), we will now consider the rectangle \( \tilde{\mathcal{R}}(f^N(x)) \) with base point \( f^N(x) \) and fixed vertical size \( 1/\beta_m \). Lemma 6.10 together with the estimate on the vertical size of \( \mathcal{R}(f^N(x), f^N(y), f^N(\bar{x}), f^N(\bar{y})) \), guarantees that horizontal size of \( \tilde{\mathcal{R}}(f^N(x)) \) is less than \( C^{1+\delta}(\beta_{\delta}/\beta_m)^N \) as well.

Thus, for every \( x \in f^N(X_{\varepsilon_n}) \), the horizontal size of \( \tilde{\mathcal{R}}(x) = \tilde{\mathcal{R}}(x, z, \bar{x}, \bar{z}) \) is less than \( C^{1+\delta}(\beta_{\delta}/\beta_m)^N \). Note that \( (\beta_{\delta}/\beta_m)^N \to 0 \) as \( n \to \infty \) since \( \beta_{\delta}/\beta_m < 1 \) and \( N \to \infty \) as \( n \to \infty \).

Let \( X = \lim_{n \to \infty} f^N(X_{\varepsilon_n}) \). Since any \( x \in X \) also belongs to \( f^N(X_{\varepsilon_n}) \) with arbitrarily large \( N \) we conclude that \( \tilde{\mathcal{R}}(x) \) has zero horizontal size, i.e., \( x = z \). Hence by Lemma 6.10 we conclude that the piece of \( U(x) \) from \( x \) to \( \bar{z} \) lies inside of \( W^f(i + 1, m)(x) \).

It is a simple exercise in measure theory to show that

\[
\mu(X) \geq \lim_{n \to \infty} \mu(f^N(X_{\varepsilon_n})) = \lim_{n \to \infty} \mu(X_{\varepsilon_n}) > 0.
\]

Finally recall that \( \mu \)-almost every point is transitive, \( \{f^j(x), j \geq 1\} = \mathbb{T}^d \). Hence by taking a transitive point \( x \in X \) and applying a straightforward approximation argument, we get that \( \forall y \ U(y) \subset W^f(i + 1, m)(y) \).

**Case 2.** \( \lim_{\varepsilon \to 0} \mu(\mathcal{E}) = 0 \).

In this case, the idea is to use the assumption above to find a leaf \( U(x) \) which is “flat”, i.e., arbitrarily close to \( W^f(i, m - 1)(x) \). Since the leaf \( U(x) \) has to “feel”
the measure \( \mu \), we need to take it together with a small neighborhood. The choice of this neighborhood is done by multiple applications of the pigeonhole principle.

Given a point \( \tilde{y} \in U(x) \), denote by \( U_{\tilde{x} \tilde{y}} \) the piece of \( U(x) \) between \( x \) and \( \tilde{y} \). As before, by \( R(x, \tilde{y}) \) we denote the rectangle spanned by \( x \) and \( \tilde{y} \). Recall that \( HS(R(x, \tilde{y})) \) and \( VS(R(x, \tilde{y})) \) stand for the horizontal and vertical sizes of \( R(x, \tilde{y}) \). We will also need to measure the sizes of \( U_{\tilde{x} \tilde{y}} \). Let \( HS(U_{\tilde{x} \tilde{y}}) = HS(R(x, \tilde{y})) \) and \( VS(U_{\tilde{x} \tilde{y}}) = VS(R(x, \tilde{y})) \).

**Iterating Pigeonhole Principle.** Divide \( \mathbb{T}^d \) into finite number of tubes \( T_1, T_2, \ldots, T_q \) foliated by \( U \) such that any connected component of \( U(x) \cap T_j \), \( j = 1, \ldots, q \), has horizontal size between \( S_0 \) and \( S_1 \). The numbers \( S_0 \) and \( S_1 \) are fixed, \( 0 < S_0 < S_1 \). We also require every tube \( T_j \) to be \( W^f(i, m - 1) \)-foliated so it can be represented as

\[ T_j = \bigcup_{y \in \text{Transv}} \text{Plaque}(y), \]

where \( \text{Transv} \) is a plaque of \( U \) and \( \text{Plaque}(y) \) are plaques of \( W^f(i, m - 1) \).

Given a small number \( \tau > 0 \), we can find an \( \varepsilon > 0 \) such that \( \mu(\cdot) < \tau \). Then by the pigeonhole principle we can choose a tube \( T_j \) such that \( \mu(T_j) \neq 0 \) and

\[ \frac{\mu(T_j \cap \cdot)}{\mu(T_j)} < \tau. \]

The tube \( T_j \) can be represented as \( T_j = \bigcup_{z \in \tilde{T}_j} W(z) \), where \( \tilde{T}_j \) is a transversal to \( W^f(i, m) \) and \( W(z), z \in \tilde{T}_j \), are connected plaques of \( W^f(i, m) \). By absolute continuity,

\[ \mu(T_j) = \int_{\tilde{T}_j} d\hat{\mu}(z) \int_{W(z)} d\mu_W(z), \]

where \( \hat{\mu} \) is the factor measure on \( \tilde{T}_j \) and \( \mu_W(z) \) is the conditional measure on \( W(z) \).

Applying the pigeonhole principle again, we choose \( W = W(z) \) such that

\[ \mu_W(W \cap \cdot) < \tau. \]

Recall that \( \mu_W(W) = 1 \) by definition of the conditional measure and \( \mu_W \) is equivalent to the induced Riemannian volume on \( W \) by the absolute continuity of \( W^f(i, m) \).

The plaque \( W \) is subfoliated by plaques of \( U \) of sizes between \( S_0 \) and \( S_1 \). Unfortunately, we do not know if \( U \) is absolutely continuous with respect to \( \mu_W \). So we construct a finite partition of \( W \) into smaller plaques of \( W^f(i, m) \) which are thin \( U \)-foliated tubes.

To construct this partition, we switch to \( h(W) \), which is a plaque of \( W^g(i, m) \) subfoliated by the plaques of \( h(U) = V_m^g \). The partition \( \{ \tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_p \} \) will consist of \( V_m^g \)-tubes inside of \( h(W) \) that can be represented as

\[ \tilde{T}_j = \bigcup_{z \in \tilde{T}_j} V(z), \quad j = 1, \ldots, p, \]

where \( \tilde{T}_j \) is a transversal to \( V_m^g \) inside of \( h(W) \) and \( V(z) \) are plaques of \( V_m^g \). For every \( j = 1, \ldots, p \), choose \( z_j \in \tilde{T}_j \). Then the tube \( \tilde{T}_j \) can also be represented as

\[ \tilde{T}_j = \bigcup_{y \in V(z_j)} \tilde{P}_j(y), \]
where $\tilde{P}_j(y) \subset W^g(i, m - 1)(y)$ are connected plaques.

Recall that $V^g_m$ is Lipschitz inside of $W^g(i, m)$. Hence for any $\xi > 0$ it is possible to find a partition $\{\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_p\}$, $p = p(\xi)$, such that

$$\forall j = 1, \ldots, p \ \forall y \in V(z_j) \ \exists B_j(\tilde{C}_1\xi), B_j(\tilde{C}_2\xi) \subset W^g(i, m - 1)(y)$$

such that $B_j(\tilde{C}_1\xi) \subset \tilde{P}_j(y) \subset B_j(\tilde{C}_2\xi)$.  \(44\)

where $B_j(\tilde{C}_1\xi)$ and $B_j(\tilde{C}_2\xi)$ are balls inside of $(W^g(i, m - 1)(y))$, induced Riemannian distance) of radii $\tilde{C}_1\xi$ and $\tilde{C}_2\xi$ respectively. The constants $\tilde{C}_1$ and $\tilde{C}_2$ are independent of $\xi$. Since we are working in a bounded plaque $h(W)$ they also do not depend on any other choices but $S_1$.

In the sequel we will need to take $\xi$ to be much smaller than $\varepsilon$.

Now we pool this partition back into a partition of $W$.

$$\{T_1, T_2, \ldots, T_p\} = \{h^{-1}(\tilde{T}_1), h^{-1}(\tilde{T}_2), \ldots h^{-1}(\tilde{T}_p)\}.$$  

Although we use the same notation for this partition, it is clearly different from the initial partition of $\mathbb{T}^d$.

Each tube $T_j$ can be represented as

$$T_j = \bigcup_{y \in U(h^{-1}(z_j))} P_j(y),$$  \(45\)

where $P_j(y) = h^{-1}(\tilde{P}_j(y)) \subset W^f(i, m - 1)(y)$.

By Lemma 6.7, $h$ is continuously differentiable along $W^f(i, m - 1)$. Moreover, the derivative depends continuously on the points in $W$. Hence property (44) persists:

$$\forall j = 1, \ldots, p \ \forall y \in U(h^{-1}(z_j)) \ \exists B_j(C_1\xi), B_j(C_2\xi) \subset W^f(i, m - 1)(y)$$

such that $B_j(C_1\xi) \subset P_j(y) \subset B_j(C_2\xi)$.  \(46\)

The constants $C_1$ and $C_2$ differ from $\tilde{C}_1$ and $\tilde{C}_2$ by a finite factor due to the bounded distortion along $W^f(i, m - 1)$ by the differential of $h$. 

**Figure 11.** We construct the partition $\{T_1, T_2, \ldots, T_p\}$ as a pull-back of the partition of $h(W)$ by $V^g_m$-tubes. The foliation $V^g_m$ is Lipschitz and $h$ is continuously differentiable along $W^f(i, m - 1)$. This guarantees that the “width” of a tube $T_j$ is of the same order as we move along $T_j$  \(46\). Hence $\mu_W$ is “uniformly distributed” along $T_j$. 

The constants $C_1$ and $C_2$ differ from $\tilde{C}_1$ and $\tilde{C}_2$ by a finite factor due to the bounded distortion along $W^f(i, m - 1)$ by the differential of $h$. 


Applying the pigeonhole principle for the last time, we find \( \mathcal{T} \in \{ \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_p \} \) such that
\[
\frac{\mu_W(\mathcal{T} \cap X_x)}{\mu_W(\mathcal{T})} < \tau. \tag{47}
\]
Take a plaque \( U_{x\bar{y}} \) inside of \( \mathcal{T} \). By construction,
\[
S_0 < V_S(U_{x\bar{y}}) < S_1.
\]

**Estimating horizontal size of \( U_{x\bar{y}} \) from below.** We have constructed \( U_{x\bar{y}} \) such that a lot of points in the neighborhood of \( U_{x\bar{y}} \) \( \mathcal{T} \) lie outside of \( X_x \). The corresponding \( \varepsilon \)-rectangles \( \mathcal{R}(x) \) have vertical size greater than \( \varepsilon^\beta \). It is clear that we can use this fact to show that \( V_S(U_{x\bar{y}}) \) is large.

Choose a sequence \( \{x_0 = x, x_1, \ldots, x_N\} \subset U_{x\bar{y}} \) such that
\[
V_S(\mathcal{R}(x_0, x_N)) \geq S_0 \quad \text{and} \quad V_S(\mathcal{R}(x_j, x_{j+1})) = \varepsilon, \ j = 0, \ldots, N - 1.
\]

First we estimate the number of rectangles \( N \).

**Lemma 6.11.** The holonomy map \( \text{Hol} : T(a) \to T(b) \), \( b \in W^f(i, m)(a) \), \( T(a) \subset V^f_i(a) \), \( T(b) \subset V^f_{i,m}(b) \), along \( W^f(i, m - 1) \) is Hölder-continuous with exponent
\[
\rho \defeq \frac{\log \beta_m}{\log \beta_m}.
\]

We postpone the proof until the end of the current section.

Let \( \tilde{x}_j = W^f(i, m - 1)(x_j) \cap V^f_{i,m}(x_0) \), \( j = 0, \ldots, N \). Then according to the lemma above,
\[
d^f_{i,m}(\tilde{x}_{j-1}, \tilde{x}_j) \leq C_{\text{Hol}} V_S(\mathcal{R}(x_{j-1}, x_j)) \rho = C_{\text{Hol}} \varepsilon^\rho, \quad j = 1, \ldots, N,
\]
which allows us to estimate \( N \)
\[
S_0 \leq V_S(\mathcal{R}(x_0, x_N)) = \sum_{j=1}^{N} d^f_{i,m}(\tilde{x}_{j-1}, \tilde{x}_j) \leq NC_{\text{Hol}} \varepsilon^\rho.
\]

Hence
\[
N \geq \frac{S_0}{C_{\text{Hol}} \varepsilon^\rho}. \tag{48}
\]

Along with the rectangles \( \mathcal{R}(x_j, x_{j+1}) \), let us consider sets \( A(x_j, x_{j+1}) \subset \mathcal{T}, \ j = 0, \ldots, N - 1 \), given by the formula
\[
A(x_j, x_{j+1}) = \bigcup_{y \in U_{x_j, x_{j+1}}} P(y),
\]
where \( P(y) \) are the plaques of \( W^f(i, m - 1) \) from the representation \( \mathcal{R} \) for \( \mathcal{T} \). The sets \( A(x_j, x_{j+1}) \) have the same vertical size. The following property of these sets is a direct consequence of \( \mathcal{R} \) and the fact that \( \mu_W \) is equivalent to the Riemannian volume on \( W \).

\[
\exists C_{\text{univ}} \text{ such that } \forall j, \tilde{j} = 1, \ldots, N - 1 \quad \frac{1}{C_{\text{univ}}} \leq \frac{\mu_W(A(x_j, x_{j+1}))}{\mu_W(A(\tilde{x}_j, x_{j+1}))} \leq C_{\text{univ}}. \tag{49}
\]

The constant \( C_{\text{univ}} \) depends on \( C_1, C_2 \) and size of \( W \), but is independent of \( \varepsilon \) and \( \xi \). Let
\[
A_1 = \bigcup_{j=1}^{N-1} A(x_j, x_{j+1}) \quad \text{and} \quad A_2 = \bigcup_{j=1}^{N-1} A(x_j, x_{j+1})
\]
where
It follows from (17) that either
\[ \frac{\mu_W(A_1 \cap X_\varepsilon)}{\mu_W(A_1)} < \tau \quad \text{or} \quad \frac{\mu_W(A_2 \cap X_\varepsilon)}{\mu_W(A_2)} < \tau. \]
For concreteness, assume that the first possibility holds.

The bounds (19) allow us to estimate the number \( N_1 \) of sets \( A(x_j, x_{j+1}) \subset A_1 \) that have a point \( q_j \in A(x_j, x_{j+1}) \) such that \( q_j \notin X_\varepsilon \).

\[ N_1 \geq \left\lfloor \frac{N}{2} \right\rfloor - [\text{univ} \tau N]. \]

Here \( \lfloor N/2 \rfloor \) is the total number of sets \( A(x_j, x_{j+1}) \in A_1 \) and \([\text{univ} \tau N]\) is the maximal possible number of sets \( A(x_j, x_{j+1}) \in A_1 \cap X_\varepsilon \). Clearly we can choose \( \tau \) and \( \varepsilon \) accordingly so \( N_1 \geq N/3 \).

For every \( A(x_j, x_{j+1}) \) as above, fix \( q_j \in A(x_j, x_{j+1}) \), \( q_j \notin X_\varepsilon \), and consider rectangle \( R(q_j) \) of vertical size \( \varepsilon \). Then
\[ HS(R(q_j)) \geq \varepsilon^\delta. \]

Consider two rectangles \( R(q_j) \) and \( R(q_j') \) as above. Since \( |j - j'| \geq 2 \), they do not “overlap” vertically if \( \xi \) is sufficiently small (although this is not important to us). They might happen to “overlap” horizontally as shown on the Figure 12 but the size of the overlap cannot exceed the diameter of the tube \( T \), which, according to (12), is bounded by \( C_2 \xi \).

The above considerations result in the following estimate:
\[ HS(U_{xy}) \geq HS(U_{x_0x_N}) \geq \frac{1}{C_H} \sum_{j=1}^{N_1} HS(R(q_j)) - C_H N_1 C_2 \xi \]
\[ \geq \frac{1}{C_H} N_1 \varepsilon^\delta - C_H N C_2 \xi \geq \frac{N}{3C_H} \varepsilon^\delta - NC_H C_2 \xi \]
\[ \geq \frac{S_0}{3C_HC_{\text{hol}}} \delta - \rho - NC_H C_2 \xi, \]

where \( C_H \) is the Lipschitz constant of the holonomy map along \( W_f(i+1, m) \). We used estimate on \( N_1 \) and estimate (15) on \( N \).

Finally, recall that \( \delta - \rho < 0 \), while \( \xi \) can be chosen arbitrarily small independently of \( \varepsilon \) (and hence \( N \)). Hence by choosing \( \varepsilon \) small, we can find \( U_{xy} \) with arbitrarily big horizontal size, which contradicts to the uniform upper bound (11) that follows from compactness. Hence Case 2 is impossible. \( \Box \)

Remark. Note that we do not need to take \( \tau \) arbitrarily small. The constant \( \tau \) just needs to be small enough to provide the estimate on \( N_1 \).

Proof of Lemma 6.11. Take points \( x \) and \( y \in V_m(x) \) such that
\[ 1 \leq d_m^f(x, y) \leq C \beta_m \]
By Lemma 6.9, there exist constants \( c_1 \) and \( c_2 \) such that
\[ \forall \tilde{x}, \tilde{y} \in V_m(\tilde{x}), \quad \tilde{x} \in W_f(i, m-1)(x), \quad \tilde{y} \in W_f(i, m-1)(y) \]
\[ c_1 < d_m^f(\tilde{x}, \tilde{y}) < c_2. \]
Figure 12. This picture illustrates the key estimate (50). Since the holonomy along $W_f(i + 1, m)$ is Lipschitz, the horizontal size of $U_{x_0 x}$ can be estimated from below by the sum of horizontal sizes of “flat” rectangles with base points $q_j \in A_1 \subset \mathcal{T}$, $j = 1, \ldots, N_1$. They might overlap horizontally as shown, but the overlap is of order $\xi \ll \varepsilon$.

Moreover, since $c_1$ and $c_2$ depend only on $\hat{d}(x, y)$ (see the remark after the proof of Lemma 6.9), they can be chosen independently of $x$ and $y$ as long as $x$ and $y$ satisfy (51).

Take $x, y \in T(a)$ close to each other. Let $N$ be the smallest integer such that $d^f_m(f^N(x), f^N(y)) \geq 1$. Then

$$d^f_m(f^N(x), f^N(y)) \geq \frac{1}{C\beta_m} d^f_m(x, y),$$

(53)

and, obviously,

$$d^f_m(f^N(x), f^N(y)) \leq C\beta_m.$$  

(54)

Hence by taking in (52) $\bar{x} = f^N(\text{Hol}(x))$ and $\bar{y} = f^N(\text{Hol}(y))$, we get

$$d^f_m\left(f^N(\text{Hol}(x)), f^N(\text{Hol}(y))\right) > c_1.$$  

(55)

On the other hand,

$$d^f_m\left(f^N(\text{Hol}(x)), f^N(\text{Hol}(y))\right) \leq C\beta_m d^f_m(\text{Hol}(x), \text{Hol}(y)).$$  

(56)
Combining (53), (54), (55) and (56), we finish the proof
\[
d^f_m(x, y) \leq \frac{C}{\beta_m^N} d^f_m(f^N(x), f^N(y)) \leq \frac{C^2 \beta_m}{c_1^\rho \beta_m^N} \cdot c_1^\rho
\]
\[
< \frac{C^2 \beta_m}{c_1^\rho} \cdot \frac{1}{\beta_m^N} = \frac{1}{\beta_m^N} \cdot \frac{d^f}{\beta_m^N} \cdot (f^N(\operatorname{Hol}(x)), f^N(\operatorname{Hol}(y)))^\rho
\]
\[
\leq C_{\operatorname{Hol}} \frac{\beta_m^N}{\beta_m} d^f_m(\operatorname{Hol}(x), \operatorname{Hol}(y))^\rho
\]
\[
= C_{\operatorname{Hol}} d^f_m(\operatorname{Hol}(x), \operatorname{Hol}(y))^\rho.
\]

We used (43) for the last equality. \(\square\)

7. Proof of Theorem C

7.1. Scheme of the proof of Theorem C. The way we choose the neighborhood \(\mathcal{U}\) is the same as in Theorem A. We look at the \(L\)-invariant splitting
\[
TT^4 = E^s_E^u \oplus E^s_L^u \oplus E^w_L^w \oplus E^w_L^w,
\]
where \(E^s_L^w, E^w_L^w\) are eigendirections with eigenvalues \(\lambda^{-1} < \lambda\) and \(E^s_L^u \oplus E^w_L^w\) is the Anosov splitting of \(g\). We choose \(\mathcal{U}\) in such a way that for any \(f \in \mathcal{U}\) the invariant splitting survives,
\[
TT^4 = E^s_E^u \oplus E^s_L^u \oplus E^w_L^w \oplus E^w_L^w,\tag{57}
\]
with
\[
\max_{x \in T^4, \sigma=s,s,sw,ws} (\angle(E^f_x, E^f_x)) < \frac{\pi}{2}\tag{58}
\]
and \(f\) is partially hyperbolic in the strongest sense (30) with respect to the splitting (57).

Lemma 6.1 works for \(f \in \mathcal{U}\). Hence the distributions \(E^s, E^u, E^u, E^u\) integrate uniquely to foliations \(W^s, W^u, W^u, W^u\). Also, as usual, \(W^s\) and \(W^u\) stand for two-dimensional stable and unstable foliations.

Fix \(f \in \mathcal{U}\) and let \(H\) be the conjugacy with the model, \(H \circ f = L \circ H\). The distribution \(E^s_L^w \oplus E^u_L^w\) obviously integrates to the foliation \(W^s_L\), which is subfoliated by \(W^s_L\) and \(W^u_L\). Applying Lemma 6.3 to the weak foliations, we get \(H(W^w_L) = W^w_L\) and \(H(W^u_L) = W^u_L\). Hence the distribution \(E^w_L^w \oplus E^w_L^w\) integrates to the foliation \(W^u_L\), which is subfoliated by \(W^w_L\) and \(W^u_L\).

Note that the leaves of \(W^u_L\) are embedded two-dimensional tori.

Lemma 7.1. The conjugacy \(H\) is \(C^{1+\nu}\) along \(W^w_L\) and \(W^u_L\). Hence, by the Regularity Lemma, \(H\) is \(C^{1+\nu}\) along \(W^u_L\).

Proposition 10 is a more general statement which we prove in Section 8. So we omit the proof of Lemma 7.1 here.

We establish smoothness of central holonomies.

Lemma 7.2. Let \(T_1\) and \(T_2\) be open \(C^{1+\nu}\)-disks transverse to \(W^u_L\). Then the holonomy map along \(W^u_L\), \(H^u\): \(T_1 \to T_2\), is \(C^{1+\nu}\)-differentiable.

Next we introduce a distance on the leaves of \(W^w_L\) and \(W^u_L\) by simply letting \(d^x(x, y) = d^y(H(x), H(y)), y \in W^u_L(x), \sigma = ws, wu\). Notice that by Lemma 7.1, \(d^{ws}\) and \(d^{wu}\) are induced by a Hölder-continuous Riemannian metric — the pullback by \(DH^{-1}\) of the Riemannian metric on \(W^u_L\).
Let \( x_0 \) be the fixed point of \( f \) and let \( S_0 \) be the two-dimensional torus passing through \( x_0 \) and tangent to \( E^s_L \oplus E^u_L \). Assumption (58) guarantees that \( S_0 \) is transverse to \( W^c_f \).

Now we construct a foliation \( S \) that is transverse to \( W^c_f \). For any point \( x \in T^4 \) let \( x_1 = W^c_f(x) \cap S_0 \) and \( x_2 \) be some point of intersection of \( W^{us}_f(x_1) \) and \( W^{wu}_f(x) \). Fix \( \tilde{x} \in T^4 \) and define

\[
S(\tilde{x}) = \{ x : \text{such that } (x_1, x_2) \text{ and } (\tilde{x}_1, \tilde{x}_2) \text{ have the same orientation in } W^{us}_f; (x_2, x) \text{ and } (\tilde{x}_2, x) \text{ have the same orientation in } W^{wu}_f; \\
d^{ws}(x_1, x_2) = d^{ws}(\tilde{x}_1, \tilde{x}_2); \quad d^{wu}(x_2, x) = d^{wu}(\tilde{x}_2, \tilde{x}) \}. \]

Figure 13. Definition of \( S \). Point \( x \in S(\tilde{x}) \).

According to this definition, \( S(\tilde{x}) \) intersects each leaf of \( W^c_f \) exactly once. Also note that, since the distances came from the model \( L \), the definition above does not depend on the choice of \( \tilde{x}_2 \). It is clear that \( S \) is a topological foliation into topological two-dimensional tori. We show that these tori are in fact regular.

**Lemma 7.3.** Leaves of \( S \) are \( C^{1+\nu} \)-embedded two-dimensional tori.

Let \( f_0 : S_0 \to S_0 \) be the factor map of \( f \), \( f_0(x) = W^c_f(f(x)) \cap S_0 \). Lemma 7.2 guarantees that \( f_0 \) is a \( C^{1+\nu} \)-diffeomorphism. Every periodic point of \( f_0 \) lifts to a periodic point of \( f \). Applying Lemma 7.2 again, we see that the p. d. of \( f_0 \) are the same as the strong stable and unstable p. d. of \( f \) which is the same as the p. d. of \( g \). Hence there is a \( C^{1+\nu} \)-diffeomorphism \( h_0 \) homotopic to identity such that \( h_0 \circ f_0 = g \circ h_0 \).
Let \( f_c : W^s_f(x_0) \to W^s_f(x_0) \) be the restriction of \( f \) to \( W^s_f(x_0) \). Obviously the p. d. of \( f_c \) and \( A \) are the same. Hence there is a \( C^{1+\nu} \)-diffeomorphism \( h_c \) homotopic to identity such that \( h_c \circ f_c = A \circ h_c \).

We are ready to construct the conjugacy \( h : \mathbb{T}^4 \to \mathbb{T}^2 \times \mathbb{T}^2 \).

\[
h(x) = (h_c(S(x) \cap W^s_f(x_0)), h_0(W^s_f(x) \cap S_0))
\]

The homeomorphism \( h \) maps the central foliation into the vertical foliation and the foliation \( S \) into the horizontal foliation.

**Remark.** Notice that at this point we do not know if \( h \) is a \( C^{1+\nu} \)-diffeomorphism, although \( h_c \) and \( h_0 \) are \( C^{1+\nu} \)-differentiable.

**Lemma 7.4.** Homeomorphism \( h \) is \( C^{1+\nu} \)-differentiable along \( W^s_f \).

**Proof.** The projection \( x \mapsto S(x) \cap W^s_f(x_0) \) projects a weak stable leaf \( W^s_f(x) \) into \( W^s_f(pr(x)) \). Moreover, it is clear from the definition of \( S \) that the restriction of this projection to \( W^s_f(x) \) is an isometry with respect to the distance \( d^w \). According to the formula for the first component of \( h \), we compose \( h \) with \( h_c \), which is an isometry when restricted to the leaf \( W^s_f(pr(x)) \) by the definition of \( d^w \). The diffeomorphism \( h_c \) straightens the weak stable foliation into a foliation by straight lines \( W^w_L \). Hence \( h(W^w_L) = W^w_L \) and \( h \) is an isometry as a map \((W^w_L(x), d^w) \mapsto (W^w_L(h(x)), \text{Riemannian metric}) \). Thus \( h \) is \( C^{1+\nu} \) along \( W^w_L \).

Everything above can be repeated for the weak unstable foliation. Applying the Regularity Lemma, we get the desired statement. \( \square \)

**Lemma 7.5.** The homeomorphism \( h \) is \( C^{1+\nu} \)-differentiable along \( S \).

**Proof.** The restriction of \( h \) to \( S_0 \) is just \( h_0 \). The restriction of \( h \) to some other leaf \( S(x) \) can be viewed as composition of the holonomy \( H^s_f \), \( h_0 \) and the holonomy \( H^s_L \). Hence this restriction is \( C^{1+\nu} \)-differentiable as well. We need to make sure that the derivative of \( h \) along \( S \) is Hölder-continuous on \( \mathbb{T}^4 \). For this we need only show that the derivative of \( H^s_L : S(x) \to S_0 \) depends Hölder-continuously on \( x \). This assertion will become clear in the proof of Lemma 7.3. \( \square \)

Now by the Regularity Lemma, we conclude that \( h \) is a \( C^{1+\nu} \)-diffeomorphism.

Let \( \tilde{L} = h \circ f \circ h^{-1} \). Clearly the foliations \( W^w_L \) and \( W^u_L \) are \( \tilde{L} \)-invariant. By construction, \( h \) and \( h^{-1} \) are isometries when restricted to the leaves of the weak foliations. Recall that \( f \) stretches by a factor \( \lambda \) the distance \( d^u \) on \( W^u_f \) and contracts by a factor \( \lambda^{-1} \) the distance \( d^w \) on \( W^w_f \). Hence, if we consider the restriction of \( \tilde{L} \) on a fixed vertical two-torus \( W^s_f(L(x)) \), then it acts by a hyperbolic automorphism \( A \).

Also, it is obvious from the construction of \( h \) that the factor map of \( \tilde{L} \) on a horizontal two-torus is \( g \). These observations show that \( \tilde{L} \) is of the form

\[
\tilde{L} = (Ax + \varphi(y), g(y)).
\]  \hspace{1cm} (4)

Note that we do not have to additionally argue that \( \varphi \) is smooth since we know that \( \tilde{L} \) is a \( C^{1+\nu} \)-diffeomorphism.

**Remark.** An observant reader would notice that our choice of \( h \) and hence \( \tilde{L} \) is far from being unique. The starting point of the construction of \( h \) is the torus \( S_0 \).
Although we have chosen a concrete $S_0$, in fact, the only thing we need from $S_0$ is transversality to $W_f^s$. This is not surprising. Many diffeomorphisms of type [4] are $C^1$-conjugate to each other. In the linear case this is controlled by the invariants [11].

In the rest of this section we prove Lemmas 7.2 and 7.3.

7.2. A technical Lemma. Before we proceed with proofs of Lemmas 7.2 and 7.3 we establish a crucial technical lemma which is a corollary of Lemma 7.1.

Let $U^\sigma = H(W^\sigma_f)$, $\sigma = ss, su$. These are foliations by Hölder-continuous curves.

Lemma 7.6. Fix $x \in \mathbb{T}^d$ and $y \in W^s_f(x)$. Let $\bar{v}$ be a vector connecting $x$ and $y$ inside of $W^s_f(x)$. Then

$$U^\sigma(y) = U^\sigma(x) + \bar{v}.$$ 

In other words, the foliation $U^\sigma$ is invariant under translations along $W^s_f$, $\sigma = ss, su$.

Proof. For concreteness, we take $\sigma = ss$. The proof in the case where $\sigma = su$ is the same.

First let us assume that $y \in W^{ss}_f(x)$. This allows us to restrict our attention to the stable leaf $W^s_f(x)$, since $U^{ss}(x)$ and $U^{ss}(y)$ lie inside of $W^s_f(x)$. Pick a point $z \in U^{ss}(x)$ and let $\bar{z} = W^{ss}_f(z) \cap U^{ss}(y)$. We only need to show that $d(x,y) = d(z,\bar{z})$, where $d$ is the Riemannian distance along weak stable leaves. The simple idea of the proof of Claim 1 from [GG08] works here. We briefly outline the argument.

Let $c = d(z,\bar{z})/d(x,y)$. Obviously

$$\forall n \quad \frac{d(L^n(z), L^n(\bar{z}))}{d(L^n(x), L^n(y))} = c. \quad (59)$$

Since $H^{-1}(z) \in W^{ss}_f(x)$, $H^{-1}(\bar{z}) \in W^{ss}_f(y)$, and strong stable leaves contract exponentially faster than weak stable leaves, we have

$$\forall \varepsilon > 0 \quad \exists N : \forall n > N : \left| \frac{d(H^{-1}(L^n(z)), H^{-1}(L^n(\bar{z})))}{d(H^{-1}(L^n(x)), H^{-1}(L^n(y)))} - 1 \right| = \frac{d(f^n(H^{-1}(z)), f^n(H^{-1}(\bar{z})))}{d(f^n(H^{-1}(x)), f^n(H^{-1}(y)))} - 1 < \varepsilon. \quad (60)$$

On the other hand, since the derivative of $H$ along $W^{ss}_f$ is continuous, the ratios

$$\frac{d(L^n(z), L^n(\bar{z}))}{d(H^{-1}(L^n(z)), H^{-1}(L^n(\bar{z})))} \quad \text{and} \quad \frac{d(L^n(x), L^n(y))}{d(H^{-1}(L^n(x)), H^{-1}(L^n(y)))}$$

are arbitrarily close when $n \to +\infty$. Together with (60), this shows that the constant $c$ from (59) is arbitrarily close to 1. Hence $c = 1$.

Finally, recall that for any $x$ the leaf $W^{ss}_L(x)$ is dense in $W^s_f(x)$. Hence by continuity, we get the statement of the lemma for any $y \in W^s_f(x)$.

Lemma 7.6 leads to some nontrivial structural information about $f$ which is of interest on its own.

Proposition 9. The distributions $E^{wu}_f \oplus E^{ss}_f$ and $E^{ws}_f \oplus E^{su}_f$ are integrable.
Proof. It follows from Lemma [7.4] that the foliations $W^u_w$ and $U^{ss}$ integrate together. Thus the foliations $W^u_f$ and $W^{ss}$ integrate to a foliation with tangent distribution $E^u_f \oplus E^{ss}_f$.

7.3. Smoothness of central holonomies. We assume that the holonomy map $H^u_f: T_1 \to T_2$ is a bijection. It can be represented as a composition of holonomies along $W^u_w$ and $W^u_f$. Indeed, let us work on the universal cover and consider two open three-dimensional submanifolds of $\mathbb{R}^4$: $M_1 = \bigcup_{x \in T_1} W^u_f(x)$ and $M_2 = \bigcup_{x \in T_2} W^{ss}_f(x)$. Let $T_3 = M_1 \cap M_2$. Obviously, $T_3$ is a smooth two-dimensional open submanifold. Also, it is easy to see that $T_3$ is connected since we are working on the universal cover. Then $H^u_f: T_1 \to T_2$ is the composition of $H^u_f: T_1 \to T_3$ and $H^u_f: T_3 \to T_2$.

So, it is sufficient to study the holonomy map along $W^u_f \rightarrow H^u_f: T_1 \to T_2$. The study of holonomies along $W^u_f$ is the same.

First we make a reduction that will allow us to work with one-dimensional transversals instead of two-dimensional transversals. Let $\tilde{W}_f$ and $\tilde{W}_L$ be the integral foliations of $E^u_f \oplus E^{uu}_f \oplus E^{uu}_f$ and $E^{ss}_f \oplus E^{uu}_f \oplus E^{uu}_f$, respectively. Also, let $\tilde{W}_f$ and $\tilde{W}_L$ be the integral foliations of $E^{ss}_f \oplus E^u_f \oplus E^{uu}_f$ and $E^{ss}_f \oplus E^u_f \oplus E^{uu}_f$, respectively.

Any transversal $T$ to $W^u_f$ can be foliated by connected components of intersections with leaves of $\tilde{W}_f$. Call this foliation $\tilde{T}$. This is a well-defined one-dimensional foliation since $T$ is two-dimensional while the leaves of $\tilde{W}_f$ are three-dimensional and both $T$ and $W_f$ are transverse to $W^u_f$. The holonomy map $H^u_f: T_1 \to T_2$ maps $\tilde{T}_1$ into $\tilde{T}_2$ since $W^u_f$ subfoliates $\tilde{W}_f$.

Analogously, any transversal $T$ can be foliated by connected components of intersections with leaves of $\tilde{W}_f$. Call this foliation $\tilde{T}$. Then $H^u_f(\tilde{T}_1) = \tilde{T}_2$ since $W^u_f$ subfoliates $\tilde{W}_f$.

Hence we can consider restrictions of $H^u_f$ to the leaves of $\tilde{T}_1$ and $\tilde{T}_2$.

Lemma 7.7. The restriction of the holonomy $H^u_f$ to a leaf of $\tilde{T}_1$, $H^u_f: \tilde{T}_1(x) \to \tilde{T}_2(H^u_f(x))$, is $C^{1+\nu}$-differentiable.

Lemma 7.8. The restriction of the holonomy $H^u_f$ to a leaf of $\tilde{T}_1$, $H^u_f: \tilde{T}_1(x) \to \tilde{T}_2(H^u_f(x))$, is $C^{1+\nu}$-differentiable.

Note that $\tilde{T}_1$ and $\tilde{T}_2$ are transverse since $T_i$ is transverse to $W^u_f$, $i = 1, 2$. Hence, by the Regularity Lemma, the holonomy $H^u_f: T_1 \to T_2$ is $C^{1+\nu}$-differentiable.

To prove Lemmas 7.7 and 7.8 we need to establish regularity of $H$ in the strong unstable direction.

Given $x \in \mathbb{T}^4$, define $H_x: W^u_f(x) \to W^u(L)(H(x))$ by the following composition:

$$W^u_f(x) \xrightarrow{H} U^u(L)(H(x)) \xrightarrow{H^u_L} W^u(L)(H(x)).$$

First, we map $W^u_f(x)$ into a Hölder-continuous curve $U^u(L)(H(x)) \subset W^u_L(H(x))$ and then we project it on $W^u(L)(H(x))$ along the linear foliation $W^u_L$, as shown on the Figure 14.

Lemma 7.9. For any $x \in \mathbb{T}^4$, the map $H_x$ is $C^{1+\nu}$-differentiable.
Proof. Let us first show that $H_x$ is uniformly Lipschitz with a constant that does not depend on $x$. Denote by $d$, $d^u_L$, $d^u_W$ and $d^wu_L$ the Riemannian distances on the universal cover $\mathbb{R}^4$ along the leaves of $W^u_f$, along the leaves of $W^s_f$, and along the leaves of $W^u_L$, respectively. First, we show that $H_x$ is Lipschitz if the points are far enough apart. Assume that $y, z \in W^u_f(x)$ and $d^u_f(y, z) \geq 1$. Then on the universal cover,

$$d^u_L(H_x(y), H_x(z)) \leq c_1 d^u_L(H_x(y), H_x(z))$$

$$\leq c_1 c_2 \inf \{d^u_L(\tilde{y}, \tilde{z}) : \tilde{y} \in W^u_L(H_x(y)), \tilde{z} \in W^u_L(H_x(z))\}$$

$$\leq c_1 c_2 d^u_L(H(x), H(y))$$

$$\leq c_1 c_2 c_3 d(y, z)$$

$$\leq c_1 c_2 c_3 c_4 d^u_f(y, z).$$

The first and fourth inequalities hold since $W^u_L$ and $W^s_L$ are quasi-isometric. The second inequality holds with a universal constant $c_2$ due to the uniform transversality of $W^u_L$ and $W^s_L$. Inequalities 3 and 6 are obvious. The fifth inequality holds since $d^u_f(y, z) \geq 1$ and the lift of the conjugacy satisfies

$$H(x + \tilde{m}) = H(x) + \tilde{m}, \quad x \in \mathbb{R}^4, \quad \tilde{m} \in \mathbb{Z}^4.$$

Here we slightly abuse notation by denoting the lift and the map itself by the same letter.

Now we need to show that $H_x$ is Lipschitz if $y$ and $z$ are close on the leaf. Notice that $H_x$ is the composition of $H_y$ and the holonomy $H^u_L : W^u_L(H(y)) \to W^u_L(H(x))$, which is just a translation. Hence, to show that $H_x$ is Lipschitz at $y$ we only need to show that $H_y$ is Lipschitz at $y$.

So we fix $x$ and $y$ on $W^u_f(x)$ close to $x$ and show that $d^u_L(H_x(x), H_x(y)) \leq c d^u_L(x, y)$. The argument here is an adapted argument from the proof of Lemma 4 from [CG08]. The two major tools here are the Livshitz Theorem and the affine distance-like functions $d^u_L$ and $\tilde{d}^u_L$ on $W^u_f$ and $W^u_L$, respectively. We used the same distance-like function on the foliation $V^f_l$ in the proof of Lemma 6.8. Recall the properties of $d^u_f$.
(D1) \( \hat{d}^u_j(x,y) = d^u_j(x,y) + o(d^u_j(x,y)) \),
(D2) \( \hat{d}^u_j(f(x), f(y)) = D^u_j(x)\hat{d}^u_j(x,y) \),
(D3) \( \forall K > 0 \exists C > 0 \) such that

\[
\frac{1}{C} d^u_j(x,y) \leq \hat{d}^u_j(x,y) \leq Cd^u_j(x,y)
\]

whenever \( d^u_j(x,y) < K \).

Consider the Hölder-continuous functions \( D^u_j(\cdot) \) and \( D^u_L(\cdot) \). The assumption on the p. d. of \( f \) and \( L \) guarantee that the products of these derivatives along periodic orbits coincide. Thus we can apply the Livshitz Theorem and get the Hölder-continuous positive transfer function \( P \) such that

\[
\forall n > 0 \quad \prod_{i=0}^{n-1} \frac{D^u_j(H(f^i(x)))}{D^u_j(f^i(x))} = \frac{P(x)}{P(f^n(x))}.
\]

Choose the smallest \( N \) such that \( d^u_j f^N(x) f^N(y) \geq 1 \). Then

\[
\frac{\hat{d}^u_j(H_x(x), H_x(y))}{d^u_j(x,y)} = \prod_{i=0}^{N-1} \frac{D^u_j(L(H_x(x)))}{D^u_j(f^i(x))} \cdot \frac{\hat{d}^u_j(L^N(H_x(x)), L^N(H_x(y)))}{\hat{d}^u_j(f^N(x), f^N(y))}
\]

\[
= \frac{P(x)}{P(f^n(x))} \cdot \frac{\hat{d}^u_j(H_{f^N(x)}(f^N(x)), H_{f^N(x)}(f^N(y)))}{\hat{d}^u_j(f^N(x), f^N(y))}
\]

\[
\leq \frac{P(x)}{P(f^n(x))} \cdot c_1 c_2 c_3 c_4.
\]

The function \( P \) is uniformly bounded away from zero and infinity. Hence, together with (D3), this shows that \( H_x \) is Lipschitz at \( x \) uniformly in \( x \) and hence is uniformly Lipschitz.

Next we apply the transitive point argument. Consider the SRB measure \( \mu^u \) which is the equilibrium state for the potential minus the logarithm of the unstable jacobian of \( f \). It is well known that \( W^u_y \) is absolutely continuous with respect to \( \mu^u \). On a fixed leaf of \( W^u_y \), the foliation \( W^u_j \) is absolutely continuous with respect to the Lebesgue measure on the leaf (for proof see [LY85], Section 4.2; they prove that the unstable foliation is Lipschitz with center-unstable leaves, but the proof goes through for strong unstable foliation within unstable leaves). Hence \( W^u_j \) is absolutely continuous with respect to \( \mu^u \).

We know that \( H_x \) is Lipschitz and hence almost everywhere differentiable on \( W^u_j(x) \). It is clear from the definition that \( H_x \) is differentiable at \( y \) if and only if \( H_y \) is differentiable at \( y \). Thus it makes sense to speak about differentiability at a point on a strong unstable leaf without referring to a particular map \( H_x \). The absolute continuity of \( W^u_j \) allows to conclude that \( H_x \) is differentiable at \( x \) for \( \mu^u \)-almost every \( x \).

Since \( \mu^u \) is ergodic and has full support we can consider a transitive point \( \bar{x} \) such that \( H_{\bar{x}} \) is differentiable at \( \bar{x} \). Now \( C^1 \)-differentiability of \( H_{\bar{x}} \) for any \( x \in \mathbb{T}^4 \) can be shown by an approximation argument: we approximate the target point by iterates of \( \bar{x} \). The argument is the same as the proof of Step 1, Lemma 5 from [GGDS] with minimal modifications, so we omit it. This argument shows even more, namely,

\[
D(H_{\bar{x}})(x) = \frac{P(x)}{P(\bar{x})} D(H_{\bar{x}})(\bar{x}).
\]
Note that \( D(H_x)(y) = D(H_y)(y) \). Hence \( H_x \) maps the Lebesgue measure on the leaf \( W^s_f(x) \) into an absolutely continuous measure, \( dy \mapsto \frac{P(y)^2}{P(x)} d\text{Leb} \). Recall that \( P \) is Hölder-continuous. Hence \( H_x \) is \( C^{1+\nu} \)-differentiable.

**Proof of Lemma 7.7** We work in a ball \( B \) inside of the leaf \( \tilde{W}_f(x) \) that contains \( \tilde{T}_1(x) \) and \( \tilde{T}_2(H^s_f(x)) \). Recall that \( B \) is subfoliated by \( W^c_f \) and \( W^u_f \). We apply the conjugacy map \( H \) to the ball \( B \). It maps \( W^s_f \) and \( W^c_f \) into \( U^s \) and \( W^c_L \), respectively. We construct a shift map \( sh: H(B) \to \tilde{W}_L(H(x)) \) in such a way that, for any \( z \), the leaf \( W^c_L(z) \) is \( sh \)-invariant and the action of \( sh \) on the leaf is a rigid translation.

Given a point \( z \in H(B) \), let \( y(z) = W^c_L(H(x)) \cap U^s(z) \). Define

\[
sh(z) = W^s_L(y(z)) \cap W^u_L(z).
\]

Clearly \( sh(U^s(H(x))) = W^s_L(H(x)) \). Moreover, by Lemma 7.8, \( sh(U^s) = W^s_L \).

The shift \( sh \) is designed such that the composition \( sh \circ H \) maps the foliation \( W^c_f \) into \( W^c_L \) and the foliation \( W^u_f \) into \( W^u_L \). According to Lemma 7.1, \( sh \circ H \) is \( C^{1+\nu} \)-differentiable along \( W^c_f \). Also notice that the restriction of \( sh \circ H \) to a strong unstable leaf \( W^u_f \) is nothing but \( H_y \) composed with constant parallel transport along \( W^u_L \). Recall that \( H_y \) is \( C^{1+\nu} \)-differentiable by Lemma 7.9. Hence, by the Regularity Lemma, we conclude that \( sh \circ H \) is \( C^{1+\nu} \)-diffeomorphism.

Therefore \( \tilde{T}_1 = sh \circ H(\tilde{T}_1(x)) \) and \( \tilde{T}_2 = sh \circ H(\tilde{T}_2(H^s_f(x))) \) are smooth curves inside of \( H(B) \) and the holonomy map \( H^s_f \) can be represented as a composition...
as shown on the commutative diagram

$$
\begin{array}{ccc}
\tilde{T}_1(x) & \xrightarrow{H^w_{\text{wu}}} & \tilde{T}_2(H^w_{\text{wu}}(x)) \\
\downarrow \text{sh} \circ H & & \downarrow \text{sh} \circ H \\
\hat{T}_1 & \xrightarrow{H^w_{\text{wu}}} & \hat{T}_2
\end{array}
$$

The holonomy $H^w_{\text{wu}}$ is smooth since $W^w_{\text{wu}}$ is a foliation by straight lines. Hence $H^w_{\text{wu}}$ is $C^{1+\nu}$-differentiable. \hfill \square

Remark. Notice that this argument completely avoids dealing with the geometry of transversals, i.e., their relative position to the foliations.

Proof of Lemma 7.8. We use exactly the same argument as in the previous proof. Notice that the picture is not completely symmetric compared to the picture in Lemma 7.7 since we are dealing with the weak unstable holonomy. Nevertheless the argument goes through by looking at transversals $\tilde{T}_1(x)$ and $\tilde{T}_2(H^w_{\text{wu}}(x))$ on the leaf of $\tilde{W}_f$. The shift map must be constructed in such a way that it maps $U^{ss}$ into $W_L^{ss}$. \hfill \square

Proof of Lemma 7.8. In this proof we exploit the same idea of composing $H$ with some shift map. We fix $S_1 = S(x_1) \in S$ which is, a priori, just an embedded topological torus. We assume that $x_1 \in W^f_{\text{wu}}(x_0)$. It is easy to see that this is not restrictive.

Foliate $S_0$ and $S_1$ by $\tilde{T}_0$, $\tilde{T}_0$ and $\tilde{T}_1$, respectively, by taking intersections with leaves of $W_f$ and $W_f$. To prove the lemma we only have to show that the leaves of $\tilde{T}_0$ and $\tilde{T}_1$ are $C^{1+\nu}$-differentiable curves.

We restrict our attention to a leaf of $\tilde{W}_f$. Construct the shift map $\text{sh}$ in the same way as in Lemma 7.7. Fix an $x \in S_0$ and let $\tilde{T}_0 = \text{sh} \circ H(\tilde{T}_0(x))$, $\tilde{T}_1 = \text{sh} \circ H(\tilde{T}_1(H^w_{\text{wu}}(x)))$.

$\tilde{T}_0$ is a $C^{1+\nu}$-curve since $\text{sh} \circ H$ is $C^{1+\nu}$-diffeomorphism. By the definition of $S_1$,

$$\forall y \in \tilde{T}_0 \quad d^w(y, H^w_{\text{wu}}(y)) = d^w(x, H^w_{\text{wu}}(x)).$$

Recalling the definition of $d^w$, we see that the conjugacy $H$ acts as an isometry on a weak unstable leaf. Obviously $\text{sh}$ is an isometry when restricted to a weak unstable leaf as well. Therefore

$$\forall y \in \tilde{T}_0 \quad d(y, H^w_{\text{wu}}(y)) = d(\text{sh} \circ H(x), H^w_{\text{wu}}(\text{sh} \circ H(x))),$$

where $d$ is the Riemannian distance along $W^w_{\text{wu}}$. Hence $\tilde{T}_1$ is smooth as a parallel translation of $\tilde{T}_0$. We conclude that $\tilde{T}_1(H^w_{\text{wu}}(x)) = (\text{sh} \circ H)^{-1}(\tilde{T}_1)$ is $C^{1+\nu}$-curve.

Repeating the same argument for $\tilde{T}_0(x)$ and $\tilde{T}_1(H^w_{\text{wu}}(x))$, we can show that $\tilde{T}_1(H^w_{\text{wu}}(x))$ is $C^{1+\nu}$-curve. Hence the lemma is proved. \hfill \square

8. Proof of Theorem D

8.1. Scheme of the proof of Theorem D. We choose $\mathcal{U}$ in the same way as in 7.1. The only difference is that $L$ is given by 1 not by 3.

Given $f \in \mathcal{U}$ we denote by $W^\tau_f$ the two-dimensional central foliation. Take $f$ and $g$ in $\mathcal{U}$. Then they are conjugate, $h \circ f = g \circ h$. 

Proposition 10. Assume that \( f \) and \( g \) have the same p. d. Then \( h(W_f^c) = W_g^c \) and the conjugacy \( h \) is \( C^{1+\nu} \)-differentiable along \( W_f^c \).

Remark. In the proof we only need coincidence of the p. d. in the central direction. After we have differentiability along the central foliation, the strong stable and unstable foliation moduli come into the picture.

Lemma 8.1. Assume that \( f \) and \( g \) have the same p. d. and the same strong unstable foliation moduli. Then \( h(W_f^{su}) = W_g^{su} \).

Now the proof of Theorem D follows immediately. Coincidence of the p. d. in the strong unstable direction guarantees \( C^{1+\nu} \)-differentiability of \( h \) along \( W_f^{su} \). This can be done by a transitive-point argument with the SRB-measure in the same way as the proof of Lemma 6.7. Then we repeat everything for the strong stable foliation. After this, we apply the Journé Regularity Lemma twice to conclude that \( h \) is \( C^{1+\nu} \)-differentiable.

In particular, this argument shows that in the counterexample of de la Llave, the strong stable and unstable foliations are not preserved by the conjugacy. We can make use of this fact by extending the counterexample for diffeomorphisms of the form \((x,y) \mapsto (Ax + \varphi(y), g(y))\). Namely, take \( L = (Ax, By) \) and \( \tilde{L} = (Ax + \varphi(y), By) \) as in \( \text{[I]} \) and \( \text{[II]} \), respectively. We know that the strong foliations of \( L \) and \( \tilde{L} \) do not match. The strong foliations depend continuously on the diffeomorphism in the \( C^1 \)-topology. If we consider diffeomorphisms \( L'(x,y) = (Ax, g(y)) \) and \( \tilde{L}'(x,y) = (Ax + \varphi(y), g(y)) \) with \( g \) sufficiently \( C^1 \)-close to \( B \), then the conjugacy between \( L' \) and \( \tilde{L}' \) is \( C^0 \) close to the conjugacy between \( L \) and \( \tilde{L} \). Therefore the strong foliations of \( L' \) and \( \tilde{L}' \) do not match as well. We conclude that \( L' \) and \( \tilde{L}' \) are not \( C^1 \)-conjugate as well.

We do not know how to show that the counterexample extends to the whole neighborhood \( U \).

Conjecture 4. For any \( f \in \mathcal{U} \) there exists \( g \in \mathcal{U} \) with the same p. d. which is not \( C^1 \)-conjugate to \( f \).

Proof of Lemma 8.1. Let \( U = h^{-1}(W_g^{su}) \). We need to show that \( U = W_f^{su} \). The main tool is the following statement.

Lemma 8.2. Consider a point \( a \in \mathbb{T}^1 \). Suppose that there is a point \( b \neq a, b \in W_f^{su}(a) \cap U(a) \). Let \( c \in W_f^{wu}(a) \) and \( d = W_f^{wu}(b) \cap W_f^{su}(c) \), \( e = W_f^{wu}(b) \cap U(c) \). Then \( d = e \).

This means that the “intersection structure” of \( U \) and \( W_f^{su} \) is invariant under the shifts along \( W_f^{wu} \). We refer to [GG08] for the proof. Claim 1 in [GG08] is exactly the same statement in the context of \( \mathbb{T}^1 \). The proof uses Proposition 11.

According to the definition of strong unstable foliation moduli, we have to distinguish two cases.

First assume \([\text{I}] \). It follows that there is a curve \( \mathcal{C} \subset W_f^{su}(x) \) that corresponds to the interval \( I \) such that \( \mathcal{C} \subset U \) as well. Let

\[ S = \bigcup_{a \in \mathcal{C}} W_f^{wu}(a). \]

Obviously \( S \subset W_f^{wu}(x) \). It follows from Lemma 8.1 that \( W_f^{su} = U \) when restricted to \( S \). Then \( W_f^{su} = \hat{U} \) when restricted to \( f^n(S), n > 0 \) as well. It remains to notice that \( \bigcup_{n>0} f^n(S) \) is dense in \( \mathbb{T}^1 \) since \( \text{length}(f^n(\mathcal{C})) \to \infty \) as \( n \to \infty \). Hence \( W_f^{su} = U \).
Now let us consider the second case. Namely, assume $\emptyset$. Let $x_0$ be a fixed point. Define $x_1 = \mathcal{F}^u(x_0)^{-1}(t)$. Then by (6), $x_1 \in W^u_f(x_0) \cap U(x_0)$. We continue to define a sequence $\{x_k; k \geq 0\}$ inductively. Given $x_k$, define $x_{k+1} = \mathcal{F}^u(x_k)^{-1}(t)$. Then for any $k$, $x_{k+1} \in W^u_f(x_k) \cap U(x_k)$. Obviously $f^{-n}(x_k) \in W^u_f(x_0) \cap U(x_0)$ as well.

The map $\mathcal{F}^u(x_0)$ is an isometry, hence $d^u_f(x_k, x_{k+1})$ does not depend on $k$. Therefore the set $\{f^{-n}(x_k); n \geq 0, k \geq 0\}$ is dense in $W^u_f(x_0)$, which guarantees that $W^u_f(x_0) = U(x_0)$. We can proceed as in the first case now to conclude that $W^u_f = U$.

8.2. Smoothness along the central foliation: proof of Proposition 10. We apply the transitive point argument as in the proof of Lemma 6.7. The technical difficulty that we have to deal with is that the leaves of $W^c$ are not dense in $T^4$.

The conjugacy $h$ preserves the weak stable and unstable foliations. By the Regularity Lemma we only need to show $C^{1+\epsilon}$-differentiability of $h$ along these one-dimensional foliations. For concreteness, we work with the weak unstable foliation $W^u_f$.

For the transitive point argument to work, we have to find an invariant measure $\mu$ such that $\mu$-a. e. point is transitive ($\{f^n(x); n \geq 0 \} = T^4$) and $W^u_f$ is absolutely continuous with respect to $\mu$. Provided that we have such a measure $\mu$, $C^{1+\epsilon}$-differentiability of $h$ along $W^u_f$ is proved in the same way as Lemma 5 from [GG08].

We modify the construction from the proof of Lemma 6.7. Consider the space $T$ of the leaves of $W^u_f$. Clearly this is a topological space homeomorphic to a two-torus. Let $\tilde{f} : T \to T$ be the factor dynamics of $f$. Since the conjugacy to the linear model $L$ maps the central leaves to the central leaves, $\tilde{f}$ is conjugate to the automorphism $B : T^2 \to T^2$, $\tilde{h} \circ B = \tilde{f} \circ \tilde{h}$. Then the measure $\tilde{\mu} = \tilde{h}_*(\text{Lebesgue})$ is $\tilde{f}$-invariant and ergodic.

Pick a point $x_0$ on a $\tilde{\mu}$-typical central leaf. Let $V_0$ be an open bounded neighborhood of $x_0$ in $W^u_f(x_0)$. Given $x$ and $y \in W^u_f(x)$, let

$$\rho(x, y) = \prod_{n \geq 0} \frac{D^u_f(f^{-n}(y))}{D^u_f(f^{-n}(x))}.$$ 

Consider a probability measure $\eta_0$ supported on $V_0$ with density proportional to $\rho(x_0, \cdot)$. For $n > 0$ define

$$V_n = f^n(V_0), \quad \eta_n = (f^n)_* \eta_0.$$ 

Let

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \eta_i.$$ 

An accumulation point of $\{\mu_n; n \geq 0\}$ is the measure $\mu$ that we are looking for.

By the choice of $x_0$ the projection of $\mu$ to $T$ is $\tilde{\mu}$.

The foliation $W^u_f$ is absolutely continuous with respect to $\mu$. We refer to [PS83] or [GG08] for the proof. In [GG08], $x_0$ is a fixed point but we do not use it in the proof of absolute continuity.

Now we have to argue that $\mu$-a. e. point is transitive. We fix a ball in $T^4$ and we show that $\mu$-a. e. point visits the ball infinitely many times. Then to prove
transitivity, we only need to cover $T^4$ by a countable collection of balls such that every point is contained in an arbitrarily small ball.

So let us fix a ball $B'$ and a slightly smaller ball $B$, $B \subset B'$. Let $\psi$ be a nonnegative continuous function supported on $B'$ and equal to 1 on $B$. By the Birkhoff Ergodic Theorem,

$$E(\psi|\mathcal{I}) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi \circ f^i$$

(61)

where $\mathcal{I}$ is the $\sigma$-algebra of $f$-invariant sets.

Let $A = \{ x : E(\psi|\mathcal{I})(x) = 0 \}$. Then $\mu(A \cap B) = 0$ since $\int_A \psi \, d\mu = \int_A E(\psi|\mathcal{I}) \, d\mu = 0$. Hence

$$E(\psi|\mathcal{I})(x) > 0 \text{ for } \mu - \text{ a. e. } x \in B.$$

Let $\bar{B} \subset B$ be a slightly smaller ball and let $W^c(\bar{B}) = \bigcup_{x \in \bar{B}} W^c_f(x)$. Since weak unstable leaves are dense in the corresponding central leaves it is possible to find $R > 0$ such that

$$W^c(\bar{B}) \subset \bigcup_{x \in B} W^u_f(x, R).$$

Applying the standard Hopf argument, for $\mu$-a. e. $x$, the function $E(\psi|\mathcal{I})$ is constant on $W(x, R)$. Now the absolute continuity of $W^u_f$ together with the above observations show that

$$E(\psi|\mathcal{I})(x) > 0 \text{ for } \mu - \text{ a. e. } x \in W^c(\bar{B}).$$

Obviously

$$\forall n \ E(\psi|\mathcal{I})(x) > 0 \text{ for } \mu - \text{ a. e. } x \in f^n(B).$$

Repeat the same argument to get

$$\forall n \ E(\psi|\mathcal{I})(x) > 0 \text{ for } \mu - \text{ a. e. } x \in W^c(f^n(\bar{B})).$$

Let $\mathcal{O}(\bar{B}) = \bigcup_{n \in \mathbb{Z}} f^n(\bar{B})$ and $W^c(\mathcal{O}(\bar{B})) = \bigcap_{x \in \mathcal{O}(\bar{B})} W^c_f(x)$. Then

$$E(\psi|\mathcal{I})(x) > 0 \text{ for } \mu - \text{ a. e. } x \in W^c(\mathcal{O}(\bar{B})).$$

Set $W^c(\mathcal{O}(\bar{B}))$ is $f$-saturated. Hence $\mu(W^c(\mathcal{O}(\bar{B})))$ is equal to the $\bar{\mu}$-measure of its projection $\text{proj}(W^c(\mathcal{O}(\bar{B}))) = \text{proj}(\mathcal{O}(\bar{B}))$ on $T$. Set $\text{proj}(\mathcal{O}(\bar{B}))$ is an open $f$-invariant set. By ergodicity of $f$, it has full measure. Hence $\mu(W^c(\mathcal{O}(\bar{B}))) = 1$ and

$$E(\psi|\mathcal{I})(x) > 0 \text{ for } \mu - \text{ a. e. } x \in T^4.$$  

According to (61) this means that $\mu$-a. e. $x$ visits $B'$ infinitely many times.

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