SIMPSON TYPE INEQUALITIES VIA ϕ–CONVEXITY

M.EMİN ÖZDEMİR♦, MERVE AVCI♦, AND A. OCAK AKDEMIR♣

Abstract. In this paper, we obtain some Simpson type inequalities for functions whose derivatives in absolute value are ϕ–convex.

1. INTRODUCTION AND PRELIMINARIES

Suppose \( f : [a, b] \to \mathbb{R} \) is a four times continuously differentiable mapping on \((a, b)\) and \( \|f^{(4)}\|_{\infty} = \sup |f^{(4)}(x)| < \infty \). The following inequality

\[
\left| \frac{1}{3} \left[ f(a) + f(b) \right] + 2f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4
\]

is well known in the literature as Simpson’s inequality.

For some results about Simpson inequality see [3]-[7].

In [3], Alomari et al. proved some inequalities of Simpson type for \( s \)-convex functions by using the following Lemma.

Lemma 1. Let \( f, \varphi : K \to \mathbb{R} \) be two continuous functions. We recall the following results, which are due to Noor and Noor [1], Noor [2] as follows:

\[ p(t) = \begin{cases} 
  t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right) \\
  t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right]. 
\end{cases} \]

Let \( f, \varphi : K \to \mathbb{R} \), where \( K \) is a nonempty closed set in \( \mathbb{R}^n \), be continuous functions. We recall the following results, which are due to Noor and Noor [1], Noor [2] as follows:

Definition 1. Let \( u \in K \). Then the set \( K \) is said to be \( \varphi \)–convex at \( u \) with respect to \( \varphi \), if

\[ u + t\varphi(v-u) \in K, \quad \forall u, v \in K, \quad t \in [0,1]. \]

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Corresponding Author.
Remark 1. We would like to mention that the Definition[7] of a \( \varphi \)-convex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point \( u \) which is contained in \( K \). We don’t require that the point \( v \) should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that \( v \) should be an end point of the path for every pair of points, \( u, v \in K \), then \( e^{i\varphi}(v-u) = v-u \) if and only if, \( \varphi = 0 \), and consequently \( \varphi \)-convexity reduces to convexity. Thus, it is true that every convex set is also an \( \varphi \)-convex set, but the converse is not necessarily true.

Definition 2. The function \( f \) on the \( \varphi \)-convex set \( K \) is said to be \( \varphi \)-convex with respect to \( \varphi \), if
\[
f(u + te^{i\varphi}(v-u)) \leq (1-t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0, 1].
\]
The function \( f \) is said to be \( \varphi \)-concave if and only if \(-f\) is \( \varphi \)-convex. Note that every convex function is a \( \varphi \)-convex function, but the converse is not true.

The following inequality is known as the Hölder inequality[8]:

**Theorem 1.** Let \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( f \) and \( g \) are real functions defined on \([a, b]\) and if \( |f|^p \) and \( |g|^q \) are integrable functions on \([a, b]\) then
\[
\int_a^b |f(x)g(x)| \, dx \leq \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_a^b |g(x)|^q \, dx \right)^{\frac{1}{q}},
\]
with equality holding if and only if \( A |f(x)|^p = B |g(x)|^q \) almost everywhere, where \( A \) and \( B \) are constants.

2. Simpson type inequalities for \( \varphi \)-convex functions

Throughout this section, let \( K = [a, a + e^{i\varphi}(b-a)] \) and \( 0 \leq \varphi \leq \frac{\pi}{2} \).

We used the following Lemma to obtain our main results.

**Lemma 2.** Let \( K \subset \mathbb{R} \) be a \( \varphi \)-convex subset and \( f : K \to (0, \infty) \) be a differentiable function on \( K^\circ \) (the interior of \( K \)), \( a, b \in K \) with \( a < a + e^{i\varphi}(b-a) \). If \( f' \) is integrable on \([a, a + e^{i\varphi}(b-a)]\), following equality holds:
\[
\left| \int_0^1 f(a + 4f \left( \frac{2a + e^{i\varphi}(b-a)}{2} \right) + f(a + e^{i\varphi}(b-a)) \right| - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) \, dx
\]
\[
= e^{i\varphi}(b-a) \int_0^1 p(t)f'(a + te^{i\varphi}(b-a)) \, dt,
\]
where
\[
p(t) = \begin{cases} 
  t - \frac{1}{6}, & t \in [0, \frac{1}{2}) \\
  t - \frac{5}{6}, & t \in [\frac{1}{2}, 1].
\end{cases}
\]
Proof. Since \( K \) is a \( \varphi \)-convex set, for \( a, b \in K \) and \( t \in [0, 1] \) we have \( a + e^{i\varphi}(b-a) \in K \). Integrating by parts implies that
\[
\int_0^1 \left( t - \frac{1}{6} \right) \frac{f'(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} \, dt + \int_0^1 \left( t - \frac{5}{6} \right) \frac{f'(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} \, dt
\]

\[
= \left( t - \frac{1}{6} \right) \frac{f(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} \bigg|_0^1 - \int_0^1 \frac{f(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} \, dt
\]

\[
+ \left( t - \frac{5}{6} \right) \frac{f(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} \bigg|_0^1 - \int_0^1 \frac{f(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} \, dt
\]

\[
= \frac{1}{6e^{i\varphi}(b-a)} \left[ f(a) + 4f \left( \frac{2a + e^{i\varphi}(b-a)}{2} \right) + f(a + e^{i\varphi}(b-a)) \right]
\]

\[
- \frac{1}{e^{i\varphi}(b-a)} \left[ \int_0^1 f(a + te^{i\varphi}(b-a)) \, dt + \int_0^1 f(a + te^{i\varphi}(b-a)) \, dt \right].
\]

If we change the variable \( x = a + te^{i\varphi}(b-a) \) and multiply the resulting equality
with \( e^{i\varphi}(b-a) \) we get the desired result. \( \square \)

**Theorem 2.** Let \( f : K \rightarrow (0, \infty) \) be a differentiable function on \( K^o \). If \( |f'| \) is \( \varphi \)-convex function on \( K^o \) and \( a, b \in K \) with \( a < a + e^{i\varphi}(b-a) \). Then, the following inequality holds:

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + e^{i\varphi}(b-a)}{2} \right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) \, dx \right|
\]

\[
\leq \frac{5}{72} e^{i\varphi}(b-a) \left[ |f'(a)| + |f'(b)| \right].
\]

**Proof.** From Lemma 2 and using the \( \varphi \)-convexity of \( |f'| \) we have

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + e^{i\varphi}(b-a)}{2} \right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) \, dx \right|
\]

\[
\leq e^{i\varphi}(b-a) \left\{ \int_0^1 \left( t - \frac{1}{6} \right) \left| f'(a + te^{i\varphi}(b-a)) \right| \, dt + \int_0^1 \left| t - \frac{5}{6} \right| \left| f'(a + te^{i\varphi}(b-a)) \right| \, dt \right\}
\]

\[
\leq e^{i\varphi}(b-a) \left\{ \int_0^1 \left( 1 - t \right) \left| f'(a) \right| + t \left| f'(b) \right| \, dt + \int_0^1 \left( t - \frac{1}{6} \right) \left| f'(a) \right| + t \left| f'(b) \right| \, dt + \int_0^1 \left( t - \frac{5}{6} \right) \left| f'(a) \right| + t \left| f'(b) \right| \, dt \right\}
\]

\[
= \frac{5}{72} e^{i\varphi}(b-a) \left[ |f'(a)| + |f'(b)| \right]
\]

which completes the proof. \( \square \)
Theorem 3. Let $f : K \to (0, \infty)$ be a differentiable function on $K$, $a, b \in K$ with $a < a + e^{i\varphi}(b - a)$. If $|f'|^q$ is $\varphi$–convex function on $K$ for some fixed $q > 1$ then the following inequality holds

$$\left| \frac{1}{6} \left[ f(a) + 4f\left( \frac{2a + e^{i\varphi}(b - a)}{2} \right) + f(a + e^{i\varphi}(b - a)) \right] - \frac{1}{e^{i\varphi}(b - a)} \int_a^{a + e^{i\varphi}(b - a)} f(x)dx \right| \leq e^{i\varphi}(b - a) \left( 1 + \frac{2p+1}{6p+1(p+1)} \right)^{\frac{q}{p}}$$

$$\times \left\{ \left( \frac{3}{8} |f'(a)|^q + \frac{1}{8} |f'(b)|^q \right)^{\frac{1}{q}} + \left( \frac{1}{8} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right)^{\frac{1}{q}} \right\}.$$ 

where $p = \frac{q}{q-1}$.

Proof. From Lemma 2 and using the Hölder inequality, we have

$$\left| \frac{1}{6} \left[ f(a) + 4f\left( \frac{2a + e^{i\varphi}(b - a)}{2} \right) + f(a + e^{i\varphi}(b - a)) \right] - \frac{1}{e^{i\varphi}(b - a)} \int_a^{a + e^{i\varphi}(b - a)} f(x)dx \right| \leq e^{i\varphi}(b - a) \left( \int_0^{\frac{1}{2}} \left| \int_0^t \left( \frac{1}{6} - t \right)^p dt \right|^q dt \right)^{\frac{1}{q}}$$

$$\times \left( \int_0^{\frac{1}{2}} [(1 - t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{\frac{1}{q}}$$

$$+ \left( \int_{\frac{1}{2}}^1 \left( t - \frac{5}{6} \right)^p dt + \int_{\frac{1}{2}}^1 \left( t - \frac{5}{6} \right)^p dt \right)^{\frac{1}{q}}$$

$$\times \left( \int_{\frac{1}{2}}^1 [(1 - t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{\frac{1}{q}} \right\} \right.$$ 

$$= e^{i\varphi}(b - a) \left( 1 + \frac{2p+1}{6p+1(p+1)} \right)^{\frac{q}{p}}$$

$$\times \left\{ \left( \frac{3}{8} |f'(a)|^q + \frac{1}{8} |f'(b)|^q \right)^{\frac{1}{q}} + \left( \frac{1}{8} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right)^{\frac{1}{q}} \right\}.$$ 

which is the desired. \qed
Theorem 4. Under the assumptions of Theorem 3 we have the following inequality

\[
\left| \frac{1}{6} \left[ f(a) + 4f\left( \frac{2a + e^{i\varphi}(b-a)}{2} \right) \right] + f(a + e^{i\varphi}(b-a)) \right| - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \leq e^{i\varphi}(b-a) \left( \frac{2(1 + 2^{p+1})}{6^{p+1}(p+1)} \right)^{\frac{1}{q}} \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right].
\]

Proof. From Lemma 2 \( \varphi \)–convexity of \(|f'|^q\) and using the Hölder inequality, we have

\[
\left| \frac{1}{6} \left[ f(a) + 4f\left( \frac{2a + e^{i\varphi}(b-a)}{2} \right) \right] + f(a + e^{i\varphi}(b-a)) \right| - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \leq e^{i\varphi}(b-a) \left( \int_0^1 |p(t)| \left| f'(a + te^{i\varphi}(b-a)) \right| dt \right)
\]

\[
\leq e^{i\varphi}(b-a) \left( \int_0^1 |p(t)|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'(a + te^{i\varphi}(b-a)) \right|^q dt \right)^{\frac{1}{q}}
\]

\[
\leq e^{i\varphi}(b-a) \left( \int_0^1 t - \frac{1}{6} |t - \frac{5}{6} |^p dt + \int_0^1 |t - \frac{5}{6} |^p dt \right) \left( \int_0^1 \left| t - \frac{5}{6} \right| \left| f'(a) \right|^q + t \left| f'(b) \right|^q dt \right)^{\frac{1}{q}}
\]

\[
= e^{i\varphi}(b-a) \left( \frac{2(1 + 2^{p+1})}{6^{p+1}(p+1)} \right)^{\frac{1}{q}} \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right]
\]

where we used the fact that

\[
\int_0^1 \left| t - \frac{1}{6} \right|^p dt = \int_\frac{1}{2}^1 \left| t - \frac{5}{6} \right|^p dt = \frac{(1 + 2^{p+1})}{6^{p+1}(p+1)}.
\]

The proof is completed. \( \square \)

Theorem 5. Let \( f : K \to (0, \infty) \) be a differentiable function on \( K^o \), \( a, b \in K \) with \( a < a + e^{i\varphi}(b-a) \). If \(|f'|^q\) is \( \varphi \)–convex function on \( K^o \) for some fixed \( q \geq 1 \) then the following inequality holds

\[
\left| \frac{1}{6} \left[ f(a) + 4f\left( \frac{2a + e^{i\varphi}(b-a)}{2} \right) \right] + f(a + e^{i\varphi}(b-a)) \right| - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \leq e^{i\varphi}(b-a) \left( \frac{5}{72} \right)^{1-\frac{1}{q}}
\]

\[
\times \left\{ \left( \frac{61 |f'(a)|^q + 29 |f'(b)|^q}{1296} \right)^{\frac{1}{q}} + \left( \frac{29 |f'(a)|^q + 61 |f'(b)|^q}{1296} \right)^{\frac{1}{q}} \right\}.
\]
Proof. From Lemma 2 and using the power-mean inequality, we have
\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + e^{i\varphi} (b - a)}{2} \right) + f \left( a + e^{i\varphi} (b - a) \right) \right] - \frac{1}{e^{i\varphi} (b - a)} \int_{a}^{a + e^{i\varphi} (b - a)} f(x) dx \right| 
\]
\[
\leq e^{i\varphi} (b - a) \times \left\{ \left( \int_{0}^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^{1 - \frac{q}{2}} dt \right)^{\frac{1}{q}} \left( \int_{0}^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(a + te^{i\varphi} (b - a))|^{q} dt \right)^{\frac{1}{q}} \right. 
+ \left. \left( \int_{\frac{1}{2}}^{1} \left| t - \frac{5}{6} \right|^{1 - \frac{q}{2}} dt \right)^{\frac{1}{q}} \left( \int_{\frac{1}{2}}^{1} \left| t - \frac{5}{6} \right| |f'(a + te^{i\varphi} (b - a))|^{q} dt \right)^{\frac{1}{q}} \right\}.
\]

Since $|f'|^{q}$ is $\varphi-$convex function we have
\[
\int_{0}^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(a + te^{i\varphi} (b - a))|^{q} dt 
\leq \int_{0}^{\frac{1}{2}} \left( \frac{1}{6} - t \right) [(1 - t) |f'(a)|^{q} + t |f'(b)|^{q}] dt 
+ \int_{\frac{1}{2}}^{1} \left( t - \frac{1}{6} \right) [(1 - t) |f'(a)|^{q} + t |f'(b)|^{q}] dt 
= \frac{61 |f'(a)|^{q} + 29 |f'(b)|^{q}}{1296}.
\]

and
\[
\int_{\frac{1}{2}}^{1} \left| t - \frac{5}{6} \right| |f'(a + te^{i\varphi} (b - a))|^{q} dt 
\leq \int_{\frac{1}{2}}^{1} \left( \frac{5}{6} - t \right) [(1 - t) |f'(a)|^{q} + t |f'(b)|^{q}] dt 
+ \int_{\frac{1}{2}}^{1} \left( t - \frac{5}{6} \right) [(1 - t) |f'(a)|^{q} + t |f'(b)|^{q}] dt 
= \frac{29 |f'(a)|^{q} + 61 |f'(b)|^{q}}{1296}.
\]

Combining all the above inequalities gives us the desired result.

\[\square\]

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♦ Atatürk University, K.K. Education Faculty, Department of Mathematics, Erzurum 25240, Turkey
$E$-mail address: emos@atauni.edu.tr

★ Adiyaman University, Faculty of Science and Arts, Department of Mathematics, Adiyaman 02040, Turkey
$E$-mail address: mavci@posta.adiyaman.edu.tr

♣ Ağrı İbrahim Çeçen University, Faculty of Science and Arts, Department of Mathematics, Ağrı 04100, Turkey