Solution of the proton radius puzzle?
Low momentum transfer electron scattering data are not enough.

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In two recent papers it is argued that the “proton radius puzzle” can be explained by truncating the electron scattering data to low momentum transfer and fit the rms radius in the low momentum expansion of the form factor. It is shown that this procedure is inconsistent and violates the Fourier theorem. The puzzle cannot be explained in this way.

The “proton radius puzzle” is the difference of the rms radius \( R_p = \langle r^2 \rangle^{1/2} \) as determined from elastic electron scattering and as derived from a very precise Lamb shift measurement of muonic hydrogen. The electron scattering result \( R_p^e = 0.877(5) \) fm \([1, 3]\) deviates from the muonic result \( R_p^μ = 0.8409(4) \) fm by 0.036(5) fm \([1, 3]\) or 7 standard deviations.

In two recent papers Keith Griffioen, Carl Carlson, and Sarah Maddox \([6]\) and Douglas W. Higinbotham, et al. \([7]\) conjecture that this difference, sometimes called the “proton radius puzzle”, could be resolved by just restricting the analysis of the electron scattering data to low momentum transfer and fit the rms radius in the low momentum expansion of the form factor. It is shown that this procedure is inconsistent and violates the Fourier theorem. The puzzle cannot be explained in this way.

The method used in refs. \([6, 7]\) is well known since the early days of electron scattering and uses the expansion of eq. (1):

\[
G(Q^2) = 1 - \frac{1}{6} \langle r^2 \rangle Q^2 + \frac{1}{120} \langle r^4 \rangle Q^4 - \frac{1}{5040} \langle r^6 \rangle Q^6 + \ldots \tag{3}
\]

where

\[
\langle r^n \rangle = 4\pi \int r^2 dr \rho(r) r^n. \tag{4}
\]

In the two papers the rms radius in eq. (4) is determined by fitting the truncated data basis for low \( Q^2 \) by eq. (3) to order \( Q^2 \) (linear), \( Q^4 \) (quadratic), and \( Q^6 \) (cubic) \([6]\) and order \( Q^2 \) \([7]\). It is a well known fact though, that a sharp truncation in the coordinate space (e.g. a uniformly charged sphere) produces an oscillating behaviour in momentum space. The same is true for a sharp truncation in momentum space. A “saw tooth” like form factor as given by eq. (3) corresponds to the charge distribution as given by eq. (1), i.e. produces an oscillating behaviour in coordinate space.

\[
G(q) = \left( 1 - \frac{q^2 R^2}{6(hc)^2} \right) \Theta \left( \frac{6(hc)^2}{R^2} - q^2 \right) \tag{5}
\]

FIG. 1. The form factor \( G(Q^2) \) for the “saw tooth” model.
Table I. Moments of the indicated charge distributions also including the first and third Zemach moments. The moments are given for an rms radius of 0.840 fm. Experimental values are taken from [10]. All charge distributions are equivalent to simple form factor models which are evidently at variance with the experimentally observed form factor of the proton. The saw tooth model demonstrates clearly the impossibility to neglect the high $Q^2$ range.

| $\rho(r)$ label | $\langle r^2 \rangle$ | $\langle r^4 \rangle$ | $\langle r^6 \rangle$ | $\langle r^4 \rangle_{(2)}$ | $\langle r^8 \rangle_{(2)}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Exponential     | 0.7056          | 1.2447          | 4.0985          | 1.0609          | 2.2457          |
| Gaussian        | 0.7056          | 0.8298          | 1.3662          | 1.0945          | 2.0594          |
| Uniform         | 0.7056          | 0.5927          | 0.5421          | 1.1154          | 1.9433          |
| Saw tooth       | 0.7056          | 0.0000          | 0.7277          | 0.6162          |                  |
| Experiment      | 0.774           | 2.59            | 29.8            | 1.085           | 2.85            |

\[ \rho(r) = -\frac{(2\pi^2 - R^2)}{2\pi^2 r^3} \left( \frac{\sqrt{6} r}{R} \right) \sin \left( \frac{\sqrt{6} r}{R} \right) + \sqrt{6} r R \cos \left( \frac{\sqrt{6} r}{R} \right) \]

**FIG. 2.** The charge distribution $\rho(r)$ for the “saw tooth” model.

For this form factor all values $\langle r^n \rangle$ for $n \geq 4$ obtained with the derivatives of eq. (6) are zero. It produces also unreasonably small values for the Zemach moments (see Tab. 1) inconsistent with both electron scattering and spectroscopy [5-11]. This is of course not observed and unphysical as there is no known mechanism that could produce oscillating charges at large distances in the proton. However, this pathological situation is within the statistical errors of the fits of Carl Carlson [11] to the truncated data range taking $\langle r^2 \rangle$ and $\langle r^4 \rangle$ as fit parameters and with Higinbotham et al. [7] putting unjustifiably $\langle r^4 \rangle \equiv 0$.

In ref. [6] the problem is realized and its influence is determined for three of the charge distribution in the Tab. 1: exponential, Gaussian and uniformly charged sphere. But these are just very crude models. We know more about the proton. Two independent studies of the world data have been published by John Arrington et al. [3-12] and Jan Bernauer et al. [1-2] based on the measured form factors up to large $Q^2 \lesssim 10$ (GeV/c)$^2$. It should be unnecessary to state that models like the Gaussian or homogeneously charged sphere are excluded for the proton since the work of Robert Hofstadter [13].

The extracted radius in ref. [6] depends strongly on the $Q^4$ term, the curvature of $G(Q^2)$ at small $Q^2$: higher values $\langle r^4 \rangle$ correlate with larger radii $\langle r^2 \rangle$. The fits over the whole $Q^2$ range by Bernauer et al. [1-2] indeed find a strong curvature. In Kraus et al. [14] it has been shown that a low-order fit including a fit of the curvature to a truncated data set is not reliable.

It is noted that the form factor and charge distributions going into the rms radius in the two independent studies [1-3, 12] are automatically fulfilling the Fourier relation, since they are determined from one consistent fit over the full $Q^2$ range. The two papers [6-7] using the truncated $G(Q^2)$ disregard this consistency. Griffioen et al. [6] insert the rms radius fitted for small $Q^2$ into a model assumed to be valid for large $Q^2 \geq 0.02$ (GeV/c)$^2$. However, this form factor is not in agreement with the cited measurements [1-3, 12].

Higinbotham et al. [7] neglect the $Q^4$ term completely though they fit the data of the larger region $Q^2 \lesssim 0.03$ (GeV/c)$^2$. Yet, their reasoning is erroneous on several aspects. First, we know that the true form factor has a finite curvature, so using an F-test to decide about the significance of the $Q^4$-term in the expansion of eq. (3) is not justified. Any hypothesis test in classical statistics is based on a very important assumption: one has to know the true function. It may be the case that we do not know the precise true functions for the form factors of the proton, but the truncated polynomial of eq. (3) can definitely be excluded. A p-value or a significance level calculated with a wrong model assumption is not valid.

Second, large off-diagonal elements in the covariance matrix are not a “problem”. On the contrary, if one fits a polynomial expansion like the one in eq. (3) one will always end up with highly correlated parameters. It is also always possible to construct an orthogonal functional basis (ref. [15] recommends the Forsythe method for polynomials) where the resulting covariance matrix of the parameters is indeed diagonal. For a quantitative discussion of the correlation it is useful to calculate the correlation matrix:

\[ \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \quad \text{with} \quad -1 \leq \rho_{ij} \leq 1 \]

where the $\sigma_{ij}$ are the elements of the covariance matrix and $\sigma_i = \sqrt{\sigma_{ii}}$. For the covariance matrix in [7], eq. (5) therein, one gets $\rho_{23} = -0.95$ and therefore a strong negative correlation between the linear and the quadratic parameter. Taking into account the factor $-1/6$ of eq. (3) one gets a large positive correlation between the quadratic parameter and the extracted radius.
The authors of [7] have shown themselves that if they artificially reduce the quadratic term to zero, they reduce the radius from $R_p = 0.875$ fm to 0.840 fm therefore giving a very strong reason not to neglect the quadratic term.

But the reasoning in ref. [7] has a more serious problem still. In their F-test two regression models are compared, where one model, the order 2 polynomial, includes the second, linear, model. It is clear that the order 2 polynomial will always fit the data better than the linear model, unless the quadratic term becomes zero and both models are identical. So, Higinbotham al. are rejecting a model not because it gives a worse fit but because the fit is not “significantly” better. Moreover, fitting the data well is only a precursor to the more important goal: getting a robust estimate of the rms radius. As we have argued in the previous paragraph, the extracted radius changes dramatically when the order of the polynomial model is reduced from quadratic to linear. This makes the quadratic term very significant for the extraction of the radius.

In addition, their two parameter fits are not influenced by the form factor at large $Q^2$. Therefore, their considerations of this form factor in the second part of their paper are in view of refs. [12] not only misplaced, but wrong.

A numerically precise calculation with the charge distribution derived from eq. [1] and based on the data and fits of refs. [12] of moments with $n \leq 6$ and Zemach moments is published in ref. [10].

One may ask why the method of small $Q^2$ expansion was relatively successful for nuclei. This is due to the fact that the short range nuclear force produces to a good approximation a uniformly charged sphere and one can derive the $R_A \propto A^{1/3}$ dependence of the rms radius. This is a good model for nuclei. However, for the charge distribution of the proton we do not yet have a good model and consequently the form factor and the rms radius are only derivable from a fit over a sufficiently wide $Q^2$ range as performed in refs. [13] [12]. Ingo Sick has investigated the minimal $Q^2$ required in a recent paper [16].

The approach of the two papers is, nonetheless, not only wrong in analytical terms it also misunderstands the statistical evaluation of physics data. The truncated data set represents just one statistical sample. The $\chi^2$ evaluation has originally nothing to do with finding a optimal “model/theory function” by fitting. The minimal sum of the weighted squares of deviations of the data from a model function should be distinguished from $\chi^2$ and we call it $M^2$. If one knows the model function with certainty - this includes the knowledge of the parameters - there is no fitting and $\chi^2 = M^2$ representing a test of statistical pureness of the data (Pearson test). In Physics, however, one has neither a certain model/theory function nor the certainty that the sample is statistically pure. One has therefore to deal with a mixed and dirty situation. It may very well be that the fits of eq. [3] with $(r^n)$ as free fit parameters are giving a small $M^2$, but this cannot be interpreted as a value of the $\chi^2$ distribution and consequently as measure of the significance of the fit. The sample is not “true” but just one of the possible statistical fluctuations. Since $M^2$ is not a $\chi^2$ an estimate of errors from such equating is an approximation. The physics constraints discussed above have to be realized even if the $M^2$ gets worse. It just means that the sample is not following so closely the model expectations that the $M^2$ is absolutely minimal. Therefore it is really light hearted to neglect the information contained in 80% of the data and believe that this is a valid approach. A similar remark holds for the Fisher-Snedecor test variable in the F-test. It is recommended to study the landmark book of Frederick James [15] which was exactly meant as an educational means for CERN users in 1970 to get out off the over simplified application of statistics in physics. It served as the basis for the chapters about statistics in the Review of Particle Physics of the Particle Data Group [17] which serve as the standard in particle physics.

In the second part of the papers of Griffioen et al. [6] and Higinbotham et al. [7] the “continued fraction expansion” for the form factor $G(Q^2)$ is tried as an alternative to the many ansätze in refs. [1–3, 12] yielding a radius in accord with the muonic hydrogen measurement albeit with relatively large error. The statistical evaluation is based on a markedly worse normalized $\chi^2/dof$. (We continue to call it $\chi^2$ since people are so used to it.) It has to be noted that the “continued fraction expansion” of $G(Q^2)$ is the only one giving a small rms radius and was excluded from the analysis of refs. [12] since it was to stiff to fit the data. It is not better justified by any theoretical argument than the others used, and, therefore, the “model error” assigned to the rms radius had to include all models with a sufficiently good $\chi^2$ disfavoring the small radius. A detailed discussion of the correct statistical analysis of fits to the new Mainz cross section data (including the constant fraction expansion) will be part of a forthcoming paper [18].

In summary, a low order expansion of the form factors has to be consistent with our knowledge of the shape at large $Q^2$ derived from experiments over the 50 years since the work of Robert Hofstadter. A fit to a truncated $Q^2$-range data set cannot be used to extract a robust value for the radius since it neglects this knowledge. The full $Q^2$ range of $G(Q^2)$ has to be used to be able to determine the mandatory knowledge of $\rho(r)$. Since the two papers are neglecting this requirement they do not explain the “proton radius puzzle”.

It is worth noting that the realization of the importance of the full form factor also limits the conjectures of spikes or bumps at very low $Q^2$ where no electron scattering measurements are possible. Any structure there must introduce significant long range contribution to the charge distribution.
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