2-NEIGHBORLY 0/1-POLYTOPES OF DIMENSION 7

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ABSTRACT. We give a complete enumeration of all 2-neighborly 0/1-polytopes of dimension 7. There are 13,959,358,918 different 0/1-equivalence classes of such polytopes. They form 5,850,402,014 combinatorial classes and 1,274,089 different \(f\)-vectors. It enables us to list some of their combinatorial properties. In particular, we have found a 2-neighborly polytope with 14 vertices and 16 facets.

1. Introduction

A 0/1-polytope is a convex polytope whose set of vertices is subset of \(\{0, 1\}^d\). For beautiful introduction to the world of 0/1-polytopes, we refer to Ziegler [17]. Some recent results can be found in [10]. The classification of all 1,226,525 different 0/1-equivalence classes of 0/1-polytopes of dimension 5 was done by Aichholzer [1]. Also he completed the classification of 6-dimensional 0/1-polytopes up to 12 vertices. Recently, Chen and Guo [4] computed the numbers of 0/1-equivalence classes of 6-dimensional polytopes for each number of vertices \(k \in [13, 64]\). But nowadays it is too hard to list all these \(\approx 4.0 \cdot 10^{14}\) classes and investigate their properties explicitly. (If every polytope will occupy 8 bytes, then all the database will take about 3 petabytes.) Thus, it makes sense to focus on some interesting families of 0/1-polytopes.

A convex polytope \(P\) is called 2-neighborly if any two vertices form a 1-face (i.e. edge) of \(P\). There are at least two reasons for investigation of 2-neighborly 0/1-polytopes:

1. Let \(P_{d,n}\) is a random \(d\)-dimensional 0/1-polytope with \(n\) vertices. In 2008, Bondarenko and Brodskiy [3] showed, that if \(n = O(2^{d/6})\), then the probability \(\Pr(P_{d,n}\text{ is 2-neighborly})\) tends to 1 as \(d \to \infty\). Similar results are obtained by Gillmann [10].

2. Special 0/1-polytopes, such as the cut polytopes, the traveling salesman polytopes, the knapsack polytopes, the \(k\)-SAT polytopes, the 3-assignment polytopes, the set covering polytopes and many others have 2-neighborly faces with superpolynomial (in the dimension) number of vertices [6, 13, 15].

We enumerated and classified all 13,959,358,918 0/1-equivalence classes of 7-dimensional 2-neighborly 0/1-polytopes. It enables us to investigate extremal properties of these polytopes. For example, we have found a 2-neighborly polytope with 14 vertices and 16 facets. This is the first known example of a 2-neighborly polytope (except a simplex) whose number of facets is not greater than the number of vertices plus 2.

In [1], Aichholzer stated the question about the maximal number \(N_2(d)\) of vertices of a 2-neighborly \(d\)-dimensional 0/1-polytope. He showed that \(N_2(6) = 13, 18 \leq N_2(7) \leq 24, N_2(8) \geq 25, N_2(9) \geq 33, N_2(10) \geq 44\). We improve these estimations: \(N_2(7) = 20, 28 \leq N_2(8) \leq 34, N_2(9) \geq 38, N_2(10) \geq 52\).

The entire database occupies about 1TB. The part of it (in particular, all 6-polytopes) and the list of all \(f\)-vectors are available at [https://github.com/maksimenko-a-n/2neighborly-01polytopes](https://github.com/maksimenko-a-n/2neighborly-01polytopes).

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Every 0/1-polytope is a convex hull of a set $X \subseteq \{0, 1\}^d$. Since the natural way for defining a 0/1-polytope is the defining its set of vertices $X$, in the following we will frequently call $X$ by a “polytope”, having in mind the convex hull $\text{conv}(X)$.

We will use the following trivial facts. Every 0/1-polytope $\text{conv}(X)$, $X \subseteq \{0, 1\}^d$, is the convex hull of a 0/1-polytope $\text{conv}(X \setminus \{x\})$ and a vector $x \in X$. The same is true for 2-neighbourly polytopes. Let $P$ be a 2-neighbourly polytope and $X = \text{ext}(P)$ be its set of vertices. Then for every $x \in X$ the polytope $\text{conv}(X \setminus x)$ is also 2-neighbourly. Thus, we can enumerate 2-neighbourly 0/1-polytopes iteratively, starting with a polytope consisting of a single point and adding one point each time.

Two polytopes are 0/1-equivalent if one can be transformed into the other by a symmetry of the 0/1-cube. More precisely, this transformation means the using of two operations: permuting of coordinates and replacing some coordinates.

A polytope $X$ can contain up to $2^d$ of 0/1-polytopes. Let $P$ be a 2-neighborly polytope consisting of a single point and adding one point each time.

In the first step, $T_1$ contains only one polytope $(0, \ldots, 0)$.

For testing 2-neighbourliness of a 0/1-polytope $X \subseteq \{0, 1\}^d$ we use the ideas described in [11, Sec. 2.2]. Let $v, w \in X$ and we want to check the adjacency of $v$ and $w$ in $\text{conv}(X)$. First of all, we switch $X$ to $Y = \{x \oplus v \mid x \in X\}$. (Here, $\oplus$ is a coordinatewise XOR operation.) Hence, $v$ will be switched to 0. It is easy to prove that the vertices $0, y \in Y$ form an edge of a polytope $Y$ if they form an edge of a polytope $Z = \{z \in Y \mid z \wedge y = z\}$. (Here, $\wedge$ is a coordinatewise AND operation.) Thus, we have to check whether $y$ is in the conical hull

$$\text{cone}(Z) = \left\{ \sum_{z \in Z} \lambda_z z \left| \lambda_z \geq 0 \right. \right\}.$$

Namely, vertices 0 and $y$ form an edge in $Z$ if $y \notin \text{cone}(Z)$. The checking of $y \notin \text{cone}(Z)$ can be done by solving the corresponding linear programming problem. We did it with COIN-OR Linear Programming Solver [5].

We have run this algorithm on the computer cluster of Discrete and computational geometry laboratory of Yaroslavl state university (https://dcgcluster.accelomp.org). The cluster has a hundred 2.9GHz-cores. After several weeks of computations we had got the results collected in Table 11. Every 0/1-vector $x \in \{0, 1\}^d$, $d \leq 8$, we store as a 1-byte integer. Thus, a polytope with $n$ vertices occupies $n$ bytes and all the database — about 173GB.

Our results for the dimension 6 coincide with Aichholzer database [11]. In addition, we enumerate all 2-neighborly 0/1-polytopes of dimension 6 with 13 vertices.

3. Evaluating of combinatorial types and f-vectors

It is well known (see e.g. [13]) that the combinatorial type (face lattice) of a polytope $P$ with vertices $\{v_1, \ldots, v_n\}$ and facets $\{f_1, \ldots, f_k\}$ is uniquely determined by its facet-vertex incidence matrix $M = (m_{ij}) \in \{0, 1\}^{k \times n}$, where $m_{ij} = 1$ if facet $f_i$ contains vertex $v_j$, and $m_{ij} = 0$ if facet $f_i$ contains another vertex $v_k$. The $i$-th row of $M$ represents vertex $v_i$.
Algorithm 1: The enumeration of 2-neighborly 0/1-polytopes

**Input:** the dimension $d$, the array $T_n$ of 2-neighborly 0/1-polytopes with $n$ vertices
  (every polytope is an array of $n$ 0/1-vectors)

**Output:** the array $T_{n+1}$ of 2-neighborly 0/1-polytopes with $n+1$ vertices

**Function** enumerate_2neighborly($d$, $T_n$)

```plaintext
for $X \in T_n$ do
  for $v \in \{0, 1\}^d \setminus X$ do
    if is_2neighborly($X$, $v$) then
      add representative($X \cup \{v\}$) to $T_{n+1}$;
    end
  end
end
sort $T_{n+1}$ and remove duplicates;
return $T_{n+1}$;
end
```

// Let conv($X$) be 2-neighborly. Is conv($X \cup \{v\}$) 2-neighborly?

**Function** is_2neighborly($X$, $v$)

```plaintext
Y := $\emptyset$;
// firstly, test edges for $v$ and $x \in X$
for $x \in X$ do add $x \oplus v$ to $Y$; // switch $v$ to 0
for $y \in Y$ do
  if no_edge_0y($Y$, $y$) then return false;
end
// test edges $\{x, y\} \subseteq X$
for $x \in X$ do
  $w := v \oplus x$;
  $Y := \emptyset$;
  for $y \in X \setminus x$ do add $y \oplus x$ to $Y$; // switch $x$ to 0
  for $y \in Y$ do
    if $w \land y = w$ then
      if no_edge_0y($Y \cup \{w\}$, $y$) then return false;
    end
  end
end
return true;
end
```

// Isn’t $\{0, y\}$ an edge of conv($Y \cup \{0\}$)?

**Function** no_edge_0y($Y$, $y$)

```plaintext
Z := $\emptyset$;
for $z \in Y \setminus y$ do
  if $z \land y = z$ then add $z$ to $Z$;
end
if $y \in \text{cone}(Z)$ then return true;
return false;
end
```
There is a table showing the number of 0/1-equivalence classes of 2-neighborly polytopes of dimensions 6 and 7. The table is divided into two parts, labeled (A) and (B), corresponding to dimensions 6 and 7, respectively. Each part lists the number of vertices followed by the number of 0/1-equivalence classes for each number of vertices. The table entries are as follows:

### (A) The dimension 6

| Vertices | 0/1-equivalence classes |
|----------|-------------------------|
| 1        | 1                       |
| 2        | 6                       |
| 3        | 16                      |
| 4        | 94                      |
| 5        | 445                     |
| 6        | 2,528                   |
| 7        | 12,359                  |
| 8        | 47,445                  |
| 9        | 108,220                 |
| 10       | 110,032                 |
| 11       | 38,221                  |
| 12       | 3,222                   |
| 13       | 36                      |
| Total    | 322,625                 |

### (B) The dimension 7

| Vertices | 0/1-equivalence classes |
|----------|-------------------------|
| 1        | 1                       |
| 2        | 7                       |
| 3        | 23                      |
| 4        | 191                     |
| 5        | 1,510                   |
| 6        | 16,373                  |
| 7        | 183,209                 |
| 8        | 1,985,525               |
| 9        | 136,197,421             |
| 10       | 707,274,277             |
| 11       | 2,345,160,234           |
| 12       | 4,456,209,397           |
| 13       | 4,284,931,624           |
| 14       | 1,757,834,961           |
| 15       | 244,831,279             |
| 16       | 8,967,617               |
| 17       | 73,512                  |
| 18       | 180                     |
| 19       | 3                       |
| 20       | 3                       |
| Total    | 13,962,232,498          |

Table 1. The number of 0/1-equivalence classes of 2-neighborly polytopes of dimensions 6 and 7

otherwise. Thus, polytopes are combinatorially equivalent iff their facet-vertex incidence matrices differ only by column and row permutations.

For every polytope in our database we computed its facet-vertex incidence matrix \( M \) by using \textit{lrs}\cite{2}. This evaluation takes about 10 days on the computer cluster with 32 cores. After that, for every matrix \( M \), we computed the canonical form of a vertex-facet digraph of \( M \) by using \textit{bliss}\cite{12} (as it was done in \cite{7}). This evaluation takes about 2 days on the computer cluster with 32 cores. Having sorted canonical forms, we have splitted all the polytopes into combinatorial equivalence classes.

For computing f-vector of a polytope from its facet-vertex incidence matrix, we used Kaibel&Pfetsch algorithm\cite{13} and modified it for the case, when the number of vertices is small (an incidence matrix row can be stored in a 32-bit integer). The computing of f-vectors of all polytopes took about two weeks on the cluster.

The results of these computations are collected in Tables 2\textsuperscript{4} We enumerate only full-dimensional 0/1-polytopes, since any nonfull-dimensional 0/1-polytope is affinely equivalent to some full-dimensional one\cite{17}.

To give an idea of the magnitude of the obtained numbers, we give a couple of examples: f-vector \((13, 78, 266, 531, 603, 355, 84)\) consists of 2,448,144 combinatorial classes; f-vector \((9, 36, 82, 114, 97, 48, 12)\) consists of one combinatorial class with 5,160,979 0/1-equivalence classes.
### Table 2. Full-dimensional 2-neighborly 0/1-polytopes of dimension 5

| vertices | 0/1-equivalence classes | combinatorial classes | f-vectors |
|----------|-------------------------|-----------------------|-----------|
| 6        | 237                     | 1                     | 1         |
| 7        | 334                     | 2                     | 2         |
| 8        | 102                     | 8                     | 5         |
| 9        | 10                      | 7                     | 4         |
| 10       | 1                       | 1                     | 1         |
| total    | 684                     | 19                    | 13        |

### Table 3. Full-dimensional 2-neighborly 0/1-polytopes of dimension 6

| vertices | 0/1-equivalence classes | combinatorial classes | f-vectors |
|----------|-------------------------|-----------------------|-----------|
| 7        | 9892                    | 1                     | 1         |
| 8        | 46813                   | 4                     | 4         |
| 9        | 108178                  | 81                    | 32        |
| 10       | 110029                  | 9651                  | 180       |
| 11       | 38221                   | 17782                 | 411       |
| 12       | 32222                   | 2730                  | 455       |
| 13       | 36                      | 35                    | 34        |
| total    | 316391                  | 30284                 | 1117      |

### Table 4. Full-dimensional 2-neighborly 0/1-polytopes of dimension 7

| vertices | 0/1-equivalence classes | combinatorial classes | f-vectors |
|----------|-------------------------|-----------------------|-----------|
| 8        | 1456318                 | 1                     | 1         |
| 9        | 17588780                | 6                     | 6         |
| 10       | 135330686               | 419                   | 108       |
| 11       | 706996729               | 4790131               | 2090      |
| 12       | 2345138023              | 271351237             | 17113     |
| 13       | 4456209206              | 1414858979            | 66929     |
| 14       | 4284931624              | 2487091476            | 171289    |
| 15       | 1757834961              | 1431813684            | 303063    |
| 16       | 244831279               | 231549854             | 382319    |
| 17       | 8967617                 | 8872600               | 282000    |
| 18       | 73512                   | 73444                 | 48988     |
| 19       | 180                     | 180                   | 180       |
| 20       | 3                       | 3                     | 3         |
| total    | 13959358918             | 5850402014            | 1274089   |

For every combinatorial type, we store its facet-vertex incidence matrix. If the number of vertices (columns of the matrix) is not greater than 16, one row of the matrix occupies
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2 bytes. The average number of facets (rows of the matrix) is 97. Thus, all combinatorial types occupy about 1TB. The part of the database (in particular, all 6-polytopes) is available at https://github.com/maksimenko-a-n/2neighborly-01polytopes The full information can be requested from the author (by e-mail).

4. d-Polytopes with $d+3$ vertices

Combinatorial types of $d$-polytopes with $d+3$ or less vertices can be enumerated by using Gale diagrams [11, Chap. 6]. For $d+1$ vertices there are only one $d$-polytope — a simplex. For $d+2$ vertices, the number of combinatorial types is equal to the number of tuples $(m_0, \{m_1, m_{-1}\})$, where $m_0, m_1, m_{-1} \in \mathbb{Z}$, $m_0 \geq 0$, $m_1 \geq 2$, $m_{-1} \geq 2$, and $m_0 + m_1 + m_{-1} = d + 2$ [11 Sec. 6.3]. The appropriate polytope is 2-neighborly iff $m_1 \geq 3$, $m_{-1} \geq 3$. For small $d$, these tuples can be easily enumerated by hands.

The combinatorial type of every $d$-polytope with $d+3$ vertices is defined by the appropriate reduced Gale diagram or wheel-sequence [9]. We don’t list here the properties of these interesting objects, since it was done in [11, 9, 16]. The results of enumerating wheel-sequences by a computer are collected in Table 5. They coincide with the first values of the sequence A114289: https://oeis.org/search?q=A114289 and with the Fukuda–Miyata–Moriyama collection of $d$-polytopes for $d \leq 6$ [8].

| $d+2$ vertices | $d+3$ vertices |
|----------------|----------------|
| $d$ | all 2-neighborly polytopes | 0/1-polytopes | all 2-neighborly polytopes | 0/1-polytopes |
| 4 | 4 | 1 | 1 | 31 | 1 | 0 |
| 5 | 6 | 2 | 2 | 116 | 11 | 8 |
| 6 | 9 | 4 | 4 | 379 | 85 | 81 |
| 7 | 12 | 6 | 6 | 1133 | 423 | 419 |

Table 5. Combinatorial types of $d$-polytopes with $d+2$ and $d+3$ vertices

For $d \leq 7$, every combinatorial type of a 2-neighborly $d$-polytope with $d+2$ vertices can be represented by a 0/1-polytope. Almost the same is true for polytopes with $d+3$ vertices. The exceptions are 4 polytopes and the pyramids over them. The first one is a cyclic 4-polytope with 7 vertices. The second can be represented by the wheel-sequence $(0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1)$. It has f-vector $(8, 28, 50, 44, 16)$ and its facet-vertex incidence matrix has two columns with 12 ones as opposed to other polytopes with the same f-vector. The third polytope can be represented by the wheel-sequence $(0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1)$. It has f-vector $(8, 28, 51, 47, 18)$ and its facet-vertex incidence matrix has a column with 14 ones as opposed to other polytopes with the same f-vector. The forth polytope is represented by the sequence $(0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1)$. It has f-vector $(9, 36, 80, 103, 72, 22)$ and its incidence matrix has no column with 12 ones as opposed to other polytopes with the same f-vector.

5. Polytopes with a small number of facets

The minimal and the maximal numbers of facets of 2-neighborly 0/1-polytopes listed in Table 6. As can be seen, there is a 2-neighborly 7-polytope $P_{14,16}$ with 14 vertices and 16 facets. In Figure 1, we list vertices of $P_{14,16}$. As far as we know, any other 2-neighborly polytope (except a simplex) has the property (facets – vertices) $\geq 3$. The polytope $P_{14,16}$ has several other special properties. It is the only 2-simple polytope in our database. (A d-polytope is 2-simple if every $(d-3)$-face is
incident to exactly three facets.) All its vertex figures are combinatorially equivalent 6-polytopes with 13 vertices and 11 facets. For any vertex figure of any other polytope in our database, the number of facets is not less than the number of vertices.

| dimension 5 | vertices | 6 | 7 | 8 | 9 | 10 |
|-------------|----------|---|---|---|---|----|
| facets min  | 6        | 10| 12| 16| 22|    |
| facets max  | 6        | 12| 20| 22| 22|    |

| dimension 6 | vertices | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-------------|----------|---|---|---|----|----|----|----|
| facets min  | 7        | 11| 13| 14| 17 | 21 | 26 |    |
| facets max  | 7        | 16| 30| 47| 55 | 65 | 76 |    |

| dimension 7 | vertices | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-------------|----------|---|---|----|----|----|----|----|----|----|----|----|----|----|
| facets min  | 8        | 12| 14| 15 | 18 | 20 | 39 | 55 | 67 | 100| 139| 219|    |    |
| facets max  | 8        | 20| 40| 70 | 104| 134| 163| 198| 239| 254| 281| 244| 228|    |

Table 6. The number of facets of a 2-neighborly 0/1-polytope

| coordinates |
|-------------|
| 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 1 |
| 0 0 0 0 0 1 0 0 |
| 0 0 0 1 0 0 0 0 |
| 0 0 1 0 0 0 0 0 |
| 0 1 0 0 0 0 0 0 |
| 1 0 0 0 0 1 1 1 |
| 1 0 0 1 1 1 0 0 |
| 1 0 1 0 1 0 1 0 |
| 1 0 1 1 0 1 0 0 |
| 1 1 0 0 1 1 0 0 |
| 1 1 0 1 0 0 1 0 |
| 1 1 1 0 0 0 0 0 |

Figure 1. The 2-neighborly 0/1-polytope with 14 vertices and 16 facets

6. POLYTOPES WITH A BIG NUMBER OF VERTICES

Let \( N_2(d) \) be the maximal number of vertices of a 2-neighborly \( d \)-dimensional 0/1-polytope. In [1], it was showed that \( N_2(d - 1) + 1 \leq N_2(d) \leq 2N_2(d - 1) \) and given some estimations for \( d \leq 10 \). By using Algorithm [1] we improve these estimations (see Table 7).

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| dimension | 7  | 8  | 9  | 10 |
|-----------|----|----|----|----|
| best example | 18 | 25 | 33 | 44 |
| upper bound  | 24 | 48 | 96 | 192 |
| new best example | 20 | 28 | 38 | 52 |
| upper bound  | 20 | 34 | 68 | 136 |

Table 7. The maximal number of vertices of a 2-neighborly 0/1-polytope

REFERENCES

[1] O. Aichholzer. Extremal properties of 0/1-polytopes of dimension 5. In Polytopes — Combinatorics and Computation. Birkhauser, 2000, pp. 111–130, http://www.ist.tugraz.at/staff/aichholzer/research/rp/rcs/info01poly

[2] D. Avis. lrs: A revised implementation of the reverse search vertex enumeration algorithm. In Polytopes — Combinatorics and Computation. Birkhauser, 2000, pp. 177–198, http://cgm.cs.mcgill.ca/~avis/C/lrs.html

[3] V.A. Bondarenko, A.G. Brodsky. On random 2-adjacent 0/1-polyhedra. Discrete Mathematics and Applications, 18(2): 181–186, 2008.

[4] W.Y.C. Chen, P.L. Guo. Equivalence classes of full-dimensional 0/1-polytopes with many vertices. Discrete & Computational Geometry, 52(4): 630–662, 2014.

[5] CLP: COIN-OR linear programming solver: https://github.com/coin-or/Clp

[6] M.M. Deza, M. Laurent. Geometry of cuts and metrics, Springer, 1997.

[7] M. Firsching. The complete enumeration of 4-polytopes and 3-spheres with nine vertices, preprint, 22 pages. https://arxiv.org/abs/1803.05205v2

[8] K. Fukuda, H. Miyata, S. Moriyama. Complete enumeration of small realizable oriented matroids. Discrete & Computational Geometry, 49(2), 2013, 359–381. The database is available at http://www-imai.is.s.u-tokyo.ac.jp/~hmiyata/oriented_matroids/

[9] E. Fusy. Counting d-polytopes with d + 3 vertices. The electronic journal of combinatorics, 13, #R23, 2006.

[10] R. Gillmann. 0/1-Polytopes: typical and extremal properties. PhD Thesis, TU Berlin, 2006.

[11] B. Grünbaum. Convex polytope, 2nd edition (V. Kaibel, V. Klee and G.M. Ziegler, eds.), Springer, 2003.

[12] T. Junttila and P. Kaski. Engineering an efficient canonical labeling tool for large and sparse graphs. In: Proceedings of the Ninth Workshop on Algorithm Engineering and Experiments (ALENEX07), pages 135-149, SIAM, 2007.

[13] bliss: A Tool for Computing Automorphism Groups and Canonical Labelings of Graphs: version 0.73. http://www.tcs.hut.fi/Software/bliss/

[14] V. Kaibel and M. E. Pfetsch. Computing the face lattice of a polytope from its vertex-facet incidences. Comput.Geom., 23(3): 281–290, 2002.

[15] A. Maksimenko. The common face of some 0/1-polytopes with NP-complete nonadjacency relation. Journal of Mathematical Sciences, 203(6): 823–832, 2014.

[16] A. Maksimenko. Boolean quadric polytopes are faces of linear ordering polytopes. Siberian Electronic Mathematical Reports, 14: 640–646, 2017.

[17] G.M. Ziegler. Lectures on 0/1-polytopes. In Polytopes — Combinatorics and Computation, 2000, pp. 1–41.

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