Signless Laplacian spectral radius and fractional matchings in graphs*

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Abstract

A fractional matching of a graph $G$ is a function $f$ giving each edge a number in $[0, 1]$ such that $\sum_{e \in \Gamma(v)} f(e) \leq 1$ for each vertex $v \in V(G)$, where $\Gamma(v)$ is the set of edges incident to $v$. The fractional matching number of $G$, written $\alpha'_f(G)$, is the maximum value of $\sum_{e \in E(G)} f(e)$ over all fractional matchings. In this paper, we investigate the relations between the fractional matching number and the signless Laplacian spectral radius of a graph. Moreover, we give some sufficient spectral conditions for the existence of a fractional perfect matching.

1 Introduction

Graphs considered in this paper are simple and undirected. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. As usual, $d(u)$ stands for the degree of a vertex $u$ in $G$. The adjacent matrix of $G$ is $A(G) = (a_{ij})_{n \times n}$, where $a_{ij} = 1$ if $i$ and $j$ are adjacent, and $a_{ij} = 0$ otherwise. The diagonal matrix of $G$ is $D(G) = (d(i))_{n \times n}$, where $d(i)$ is the degree of vertex $i$. Let $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$, $\mu_1(G) \geq \mu_2(G) \geq \ldots \geq \mu_n(G)$ and $q_1(G) \geq q_2(G) \geq \ldots \geq q_n(G)$ be the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively, where $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$. Particularly, the eigenvalues $\lambda_1(G)$, $\mu_1(G)$ and $q_1(G)$ are called the spectral radius, Laplacian spectral radius and signless Laplacian spectral radius of $G$, respectively. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ induced by $S$, and let $G - S$ be the graph obtained from $G$ by deleting the vertices in $S$ together with their incident edges. The complement graph $G^c$ of $G$ is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of $G$. For any two vertex-disjoint graphs $G_1$ and $G_2$, we use $G_1 \vee G_2$ to denote the join of graphs $G_1$ and $G_2$ and $G_1 \cup G_2$ to denote the disjoint union of graphs $G_1$ and $G_2$.

An edge set $M$ of $G$ is called a matching if any two edges in $M$ have no common vertices. If each vertex of $G$ is incident with exactly one edge of $M$, then $M$ is called a perfect matching of $G$. The matching number of a graph $G$, denoted by $\alpha'(G)$, is the number of edges in a maximum matching. A fractional matching of a graph $G$ is a function $f$ giving each edge a number in $[0, 1]$

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such that $\sum_{e \in \Gamma(v)} f(e) \leq 1$ for each vertex $v \in V(G)$, where $\Gamma(v)$ is the set of edges incident to $v$.

The fractional matching number of $G$, written $\alpha'_*(G)$, is the maximum value of $\sum_{e \in E(G)} f(e)$ over all fractional matchings $f$. A fractional perfect matching of a graph $G$ is a fractional matching $f$ with $\alpha'_*(G) = \frac{n}{2}$, and a fractional perfect matching $f$ of a graph $G$ is a perfect matching if it takes only the values 0 or 1.

Fractional matching has attracted many researchers’ attention. Behrend et al. [2] established a lower bound on the fractional matching number of a graph with given some graph parameters and characterized the graphs whose fractional matching number attains the lower bound. Choi et al. [5] gave the tight upper bounds on the difference and ratio of the fractional matching number and matching number among all $n$-vertex graphs, and characterized the infinite family of graphs where equalities hold. O [6] investigated the relations between the spectral radius of a connected graph with minimum degree $\delta$ and its fractional matching number, and gave a lower bound on the fractional matching number in terms of the spectral radius and minimum degree. Xue [9] studied the connections between the fractional matching number and the Laplacian spectral radius of a graph, and obtained some lower bounds on the fractional matching number of a graph. Moreover, they presented some sufficient spectral conditions for the existence of a fractional perfect matching.

Motivated by [6, 9], we investigate the relations between the signless Laplacian spectral radius of a graph and its fractional matching number. In Section 2, we list some useful lemmas. In Section 3, we establish a lower bound on the fraction number of a graph in terms of its signless Laplacian spectral radius and minimum degree. In Section 4, we obtain some sufficient spectral conditions for the existence of a fractional perfect matching.

## 2 Preliminaries

In this section, we list some lemmas which will be used in our paper later. Some fundamental properties of fractional matching were obtained in [7].

**Lemma 2.1.** [7] For any graph $G$, let $\alpha'_*(G)$ be the fractional matching number of $G$. Then

(i) $2\alpha'_*(G)$ is an integer.
(ii) $\alpha'_*(G) = \frac{1}{2}(n - \max\{i(G - S) - |S|\})$, where the maximum is taken over all $S \subseteq V(G)$.

**Lemma 2.2.** [8] Let $G$ be a connected graph. If $H$ is a subgraph of $G$, then $q_1(H) \leq q_1(G)$.

**Lemma 2.3.** [4] Let $K_n$ be a complete graph of order $n$, where $n \geq 2$. Then $q_1(K_n) = 2n - 2$.

We now explain the concepts of the equitable matrix and equitable partition.

**Definition 2.4.** [1] Let $M$ be a real matrix of order $n$ described in the following block form

$$M = \begin{pmatrix}
M_{11} & \cdots & M_{1t} \\
\vdots & \ddots & \vdots \\
M_{t1} & \cdots & M_{tt}
\end{pmatrix},$$
where the blocks $M_{ij}$ are $n_i \times n_j$ matrices for any $1 \leq i, j \leq t$ and $n = n_1 + \ldots + n_t$. For $1 \leq i, j \leq t$, let $b_{ij}$ denote the average row sum of $M_{ij}$, i.e. $b_{ij}$ is the sum of all entries in $M_{ij}$ divided by the number of rows. Then $B(M) = (b_{ij})$ (simply by $B$) is called the quotient matrix of $M$. If for each pair $i, j$, $M_{ij}$ has constant row sum, then $B$ is called the equitable quotient matrix of $M$ and the partition is called equitable.

Lemma 2.5. [1] Let $M$ be a symmetric real matrix. If $M$ has an equitable partition and $B$ is the corresponding matrix, then each eigenvalue of $B$ is also an eigenvalue of $M$.

The relation between $\lambda_1(B)$ and $\lambda_1(M)$ is obtained as below.

Lemma 2.6. [10] Let $B$ be the equitable matrix of $M$ as defined in Definition 2.4, and $M$ be a nonnegative matrix. Then $\lambda_1(B) = \lambda_1(M)$.

O [6] constructed a family of connected bipartite graphs $H(\delta, k)$, where $\delta$ and $k$ are two positive integers. For each graph $G \in H(\delta, k)$ with the bipartition $V(G) = V_1 \cup V_2$, $G$ satisfies the following conditions:

(i) every vertex in $V_1$ has degree $\delta$,
(ii) $|V_1| = |V_2| + k$,
(iii) the degrees of vertices in $V_2$ are equal.

The exact values of the fractional matching number and the spectral radius for graphs in $H(\delta, k)$ are obtained as below.

Lemma 2.7. [6] If $H \in H(\delta, k)$, then $\alpha'(H) = |V(H)| - k$ and $\lambda_1(H) = \delta \sqrt{1 + \frac{2k}{|V(H)| - k}}$.

We now determine the signless Laplacian spectral radius of graphs in $H(\delta, k)$.

Lemma 2.8. If $H \in H(\delta, k)$, then $q_1(H) = \frac{2\delta|V(H)|}{|V(H)| - k}$.

Proof. If $H \in H(\delta, k)$, by the partition $V(H) = V_1 \cup V_2$, we can obtain the quotient matrix of $Q(H)$:

$$B = \left( \begin{array}{cc} \delta & \delta \\
\frac{|V_1|}{|V_2|} & \frac{|V_1|}{|V_2|} \end{array} \right).$$

It is easy to calculate that $\lambda_1(B) = \delta (1 + \frac{|V_1|}{|V_2|}) = \delta (2 + \frac{k}{|V_2|})$. By the construction of $H(\delta, k)$, the partition $V(H) = V_1 \cup V_2$ is equitable and $|V_2| = \frac{|V(H)| - k}{2}$. By Lemma 2.6 we have

$$q_1(H) = \lambda_1(Q(H)) = \lambda_1(B) = \delta \left( 2 + \frac{k}{|V_2|} \right) = \delta \left( 2 + \frac{2k}{|V(H)| - k} \right) = \frac{2\delta|V(H)|}{|V(H)| - k}$$

as desired. 

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3 A relationship between $q_1(G)$ and $\alpha'_*(G)$

In this section, we investigate the relationship between the signless Laplacian spectral radius of a graph with minimum degree $\delta$ and its fractional matching number. Similar to the proof of Lemma 3.2 in [6], we can obtain the following lemma.

**Lemma 3.1.** Let $G$ be an $n$-vertex connected graph with minimum degree $\delta$, and let $k$ be a real number between 0 and $n$. If $q_1(G) < \frac{2n\delta}{n-k}$, then $\alpha'_*(G) > \frac{n-k}{2}$.

**Proof.** If $\alpha'_*(G) \leq \frac{n-k}{2}$, by Lemma 2.7 there exists a vertex set $S \subseteq V(G)$ such that $i(G-S) - |S| \geq k$. Since $i(G-S)$ is an integer, then $i(G-S) - |S| \geq \lceil k \rceil$. Let $A$ be the set of all isolated vertices in $G - S$. Then,

$$|A| = i(G-S) \geq |S| + \lceil k \rceil.$$

Consider the bipartite subgraph $H$ with the partitions $V(H) = A \cup S$ such that $E(H)$ is the set of edges of $G$ having one endpoint in $A$ and the other in $S$. Let $r$ be the number of edges in $H$. Then $r \geq \delta |A|$. For the partition $V(H) = A \cup S$, we can obtain a quotient matrix of $Q(H)$ as below:

$$B = \left( \begin{array}{cc} \frac{r}{|A|} & \frac{r}{|S|} \\ \frac{r}{|A|} & \frac{r}{|S|} \end{array} \right).$$

It is easy to calculate that $\lambda_1(B) = \frac{r(|A| + |S|)}{|A||S|}$. Since the partition is equitable, by Lemma 2.5, we have

$$q_1(G) = \lambda_1(Q(G)) \geq \lambda_1(B) = \frac{r(|A| + |S|)}{|A||S|} \geq \delta \frac{|A| + |S|}{|S|} \geq \delta 2|S| + \lceil k \rceil \geq \delta \left( 2 + \frac{2|k|}{n-k} \right) \geq \frac{2n\delta}{n-k}$$

since $r \geq \delta |A|, |A| \geq |S| + \lceil k \rceil, n \geq |A| + |S| \geq 2|S| + k$ and $|S| \geq \delta$.

**Theorem 3.2.** If $G$ is an $n$-vertex graph with minimum degree $\delta$, then we have

$$\alpha'_*(G) \geq \frac{n\delta}{q_1(G)},$$

with equality if and only if $k = \frac{n(q_1(G)-2\delta)}{q_1(G)}$ is an integer and $G$ is an element of $\mathcal{H}(\delta, k)$.

**Proof.** By Lemma 3.1 $\alpha'_*(G) > \frac{n-k}{2}$ if $q_1(G) < \frac{2n\delta}{n-k}$. Note that $\frac{2n\delta}{n-k}$ is an increasing function of $k$ on $[0, n)$, thus $\frac{2n\delta}{n-k}$ decreases towards $q_1(G)$ as $k$ decreases towards $z$, where $z = \frac{n(q_1(G)-2\delta)}{q_1(G)}$. Then for each value $k \in (z, n)$, we have $\alpha'_*(G) > \frac{n-k}{2}$ by Lemma 3.1. Let $k$ tend to $z$ and finally to $z$, we obtain $\alpha'_*(G) \geq \frac{n\delta}{q_1(G)}$ as desired.

If $k = \frac{n(q_1(G)-2\delta)}{q_1(G)}$ is an integer and $G \in \mathcal{H}(\delta, k)$, then by Lemma 2.7 we have $\alpha'_*(G) = \frac{n-k}{2} = \frac{n\delta}{q_1(G)}$. For the ‘only if’ part, assume that $\alpha'_*(G) = \frac{n\delta}{q_1(G)}$. Then $k = z$ and the inequalities in Lemma 3.1 become equality. Since $\lceil k \rceil = k$, $k$ must be an integer. In addition, note that $r = \delta |A|, |A| = |S| + k, n = 2|S| + k$ and $|S| = \delta, G$ must be included in $\mathcal{H}(\delta, k)$.

Let $G$ be a bipartite graph with partition $V(G) = V_1 \cup V_2$. Then $G$ is said to be semi-regular if all vertices in $V_i$ have the same degree $d_i$ for $i = 1, 2$. 


Lemma 3.3. Let $G$ be a connected graph graph. Then

$$q_1(G) \leq \max\{d(u) + d(v) : uv \in E(G)\},$$

with equality if and only if $G$ is a regular bipartite graph or a semi-regular bipartite graph.

Proof. Without loss of generality, assume that $G$ is connected and $d(u_1) + d(v_1) = \max\{d(u) + d(v) : uv \in E(G)\}$. Let $A = N(u_1) \setminus \{v_1\}$ and $B = N(v_1) \setminus \{u_1\}$. Since $g(G) \geq 5$, then $|A| + |B| \leq \alpha(G)$ and thus $d(u_1) + d(v_1) = 2 + |A| + |B| \leq 2 + \alpha(G)$. By Lemma 3.3, $q_1(G) \leq 2 + \alpha(G)$. If $q_1(G) = 2 + \alpha(G)$, then $\alpha(G) = |A| + |B|$ and thus $G$ is bipartite regular or semi-regular. Suppose that $|A| \geq |B|$ for convenience. Let $w_1$ be a vertex of $B$. Then $u_1, w_1 \in B$ since both $u_1$ and $w_1$ are adjacent to $v_1$. Since $G$ is regular or semi-regular, then $d(u_1) = d(w_1)$ and thus $|N(w_1) \setminus \{v_1\}| = |A|$. Note that $N(w_1) \cup A$ is an independent set of $G$, then $\alpha(G) \geq |N(w_1) \cup A| = 2|A| + 1$, a contradiction to the fact $\alpha(G) = |A| + |B|$. $\square$

Together with Theorems 3.2 and 3.4, we obtain a lower bound on the fractional matching number in terms of the independence number and minimum degree, which improves the lower bound obtained in [9].

Corollary 3.5. Let $G$ be a connected graph with independence number $\alpha(G)$ and minimum degree $\delta$. If $g(G) \geq 5$, then

$$\alpha'_s(G) > \frac{n \delta}{\alpha(G) + 2}.$$ 

4 Signless Laplacian spectral radius and fractional perfect matching

Some sufficient condition for the existence of a fractional perfect matching in a graph in terms of the spectral radius were obtain in [9]. In this section, we are devoted to give some sufficient conditions for a graph to have a fractional perfect matching from the viewpoint of signless Laplacian spectral radius.

Theorem 4.1. Let $G$ be an $n$-vertex connected graph with minimum degree $\delta$. If $q_1(G) < \frac{2n \delta}{n-1}$, then $G$ has a fractional perfect matching.

Proof. If $q_1(G) < \frac{2n \delta}{n-1}$, then it follows from Lemma 3.1 that $\alpha'_s(G) > \frac{n \delta - 1}{2}$. By Lemma 2.1, $2\alpha'_s(G)$ is an integer, then $\alpha'_s(G) = \frac{n \delta}{2}$, which means that $G$ has a fractional perfect matching. $\square$
We now give a sufficient condition for the existence of a fractional perfect matching in a graph in terms of the signless Laplacian spectral radius of its complement.

**Theorem 4.2.** Let $G$ be an $n$-vertex connected graph with minimum degree $\delta$ and $G^c$ be the complement of $G$. If $q_1(G^c) < \frac{\delta}{2}$, then $G$ has a fractional perfect matching.

**Proof.** Assume to the contrary that $\alpha'_*(G) < \frac{n}{2}$. By Lemma 2.1 there exists a vertex set $S \subseteq V(G)$ such that $i(G - S) - |S| > 0$. Denote by $A$ the set of isolated vertices in $G - S$. Note that the neighbours of each isolated vertex belong to $S$, then $|S| \geq \delta$, which implies that $|A| \geq |S| + 1 \geq \delta + 1$. Since $G^c[A]$ is a clique, by Lemmas 2.2 and 2.3 we have

$$q_1(G^c) \geq q_1(G^c[A]) = 2(|A| - 1) = 2\delta,$$

a contradiction. This completes the proof.

**Theorem 4.3.** Let $G$ be an $n$-vertex connected graph with minimum degree $\delta$ and $G^c$ be the complement of $G$. If $q_1(G^c) < 2\delta + 1$, then $G$ has a fractional perfect matching unless $G \cong H_1 \lor H_2$, where $H_1$ is a $(\delta + 1)$-independent set and $H_2$ is any graph of order $\delta$.

**Proof.** Suppose that $\alpha'_*(G) < \frac{n}{2}$. By Lemma 2.1 there exists a vertex set $S \subseteq V(G)$ such that $i(G - S) - |S| > 0$. Let $A$ be the set of isolated vertices in $G - S$. Then $|A| \geq |S| + 1 \geq \delta + 1$. If $|A| \geq \delta + 2$, then there is a clique of order $\delta + 2$ in $G^c$ and thus $q_1(G^c) \geq 2(\delta + 1)$, a contradiction. Furthermore, we have $|A| = |S| + 1 = \delta + 1$. If $V(G) \neq A \cup S$, then there is a clique of order $\delta + 2$ in $G^c$ and thus $q_1(G^c) \geq 2(\delta + 1)$, a contradiction. Hence, we have $V(G) = A \cup S$. Therefore, we have $G \cong H_1 \lor H_2$. This completes the proof.

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