ON RICKART MODULES

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Abstract

Gangyong Lee, S.Tariq Rizvi, and Cosmin S.Roman studied Rickart modules. The main purpose of this paper is to develop the properties of Rickart modules. We prove that each injective and prime module is a Rickart module. And we give characterizations of some kind of rings in term of Rickart modules.

Keywords: Endomorphism ring, Direct summand, Kernel of endomorphism, Rickart modules and modules with the summand intersection property.

1 INTRODUCTION

A module $M$ is called a Rickart module if for every $\varphi \in S = \text{End}(M)$, then $\ker \varphi = eM$ for some $e^2 = e \in S$. Equivalently a module is a Rickart module if and only if for every $\varphi \in S = \text{End}(M)$, then $\ker \varphi$ is a direct summand of $M$, See [1, 2].

In this paper, we give some results on the Rickart modules.

In §2, we give characterization of the Rickart modules. Also we study the direct sum of Rickart modules. For example we prove that an $R$-module $M$ is Rickart if and only if $A_M \cap T_f \leq M \oplus M$, for every endomorphism $f : M \rightarrow M$, see Theorem (2.2).

In section 3, we give characterizations of certain classes of rings in term of the Rickart modules. For example we prove that a ring $R$ is semisimple if and only if all injective $R$-module is Rickart, see Theorem (3.12).

Throughout this article, $R$ is a ring with identity and $M$ is a unital left $R$-module. For a left module $M$, $S = \text{End}_R(M)$ will denote the endomorphism ring of $M$. The notations $N \leq M, N \leq M$ mean that $N$ is a submodule, a direct summand of $M$.

2 CHARACTERIZATIONS OF RICKART MODULES

In this section, we give a characterizations for the Rickart modules. Following [1], A module $M$ is called a Rickart module if for every $\varphi \in S = \text{End}(M)$, $\ker \varphi = eM$, for some $e^2 = e \in S$. It’s known that every direct summand of a Rickart module is a Rickart module.

Remark 2.1: Let $M$ be an $R$-module and $f : M \rightarrow M$ be an $R$-homomorphism.
Let $A_M = M \oplus 0$, $B_M = 0 \oplus M$ and $\bar{f}: A_M \to B_M$ be a map defined by
$\bar{f}(m, 0) = (0, f(m))$, for every $m \in M$. It is clear that $M \oplus M = A_M \oplus B_M$. $\bar{f}$ is an $R$-homomorphism and $\text{ker}\bar{f} = \text{ker}f \oplus 0$. Let $T_f = \{x + \bar{f}(x), x \in A_M\}$. Clearly that $T_f$ is a submodule of $M \oplus M$.

In this paper by $A_M, B_M, \bar{f}, T_f$ we mean the same concepts in the previous above Remark.

**Theorem 2.2:** An $R$-module $M$ is Rickart module if and only if for every $R$-homomorphism $f: M \to A_M \cap T_f$ is a direct summand of $M \oplus M$.

**Proof:** Let $M$ is Rickart module and $f: M \to M$ be an $R$-homomorphism. Then $\text{ker}f$ is a direct summand of $M$ and hence $\text{ker}\bar{f} = \text{ker}f \oplus 0$ is a direct summand of $M \oplus M$. Claim that $\text{ker}\bar{f} = A_M \cap T_f$. To show that, let $(m, 0) \in \text{ker}\bar{f}$. Then $(m, 0) = (m, 0) + \bar{f}(m, 0) \in A_M \cap T_f$. Now let $(m, 0) \in A_M \cap T_f$. So there exists $m_1 \in M$ such that $(m, 0) = (m_1, 0) + \bar{f}(m_1, 0) = (m_1, 0) + (0, f(m_1)) = (m_1, f(m_1))$. Hence $m = m_1$ and $f(m_1) = 0$. Thus $(m, 0) \in \text{ker}\bar{f}$.

For the converse, since $A_M \cap T_f = \text{ker}\bar{f} \leq M \oplus M$ and $\text{ker}\bar{f} \leq A_M$. Then $\text{ker}\bar{f} = A_M \cap T_f \leq M \oplus M$. Thus $M$ is a Rickart module.

Recall that An $R$-module $M$ is called a prime $R$-module if $\text{ann}(x) = \text{ann}(y)$, for every non zero elements $x$ and $y$ in $M$ [3].

In the following proposition we give conditions under which an $R$-module $M$ can be Rickart.

**Propositions 2.3:** Let $M$ be an injective and prime $R$-module, then $M$ is a Rickart module.

**Proof:** Let $f: M \to M$ be an $R$-homomorphism. Since $M$ is injective and prime. Then $M \oplus M$ is injective and prime. Since $A_M \leq M \oplus M$, then $A_M$ is injective. First claim that $M \oplus M = T_f \oplus B_M$. To show that $(x, y) \in M \oplus M$. Hence $(x, y) = (x, 0) + (0, f(x)) - (0, f(x)) + (0, y)$. It is clear that $(x, 0) + (0, f(x)) \in T_f$ and $-(0, f(x)) + (0, y) \in B_M$. So $M \oplus M = T_f + B_M$.

Now let $(m, 0) + \bar{f}(m, 0) \in T_f \cap B_M$. $(m, f(m)) \in B_M = 0 \oplus M$ and hence $m = 0$. Thus $M \oplus M = T_f + B_M$. Thus $T_f$ is injective.

Let $I$ be an ideal of $R$ and $g: I \to A_M \cap T_f$ be a non zero homomorphism. Let $i_1: A_M \cap T_f \to A_M$ and $i_2: A_M \cap T_f \to T_f$ be the inclusion homomorphisms. Thus $i_1 \circ g: I \to A_M$ and $i_2 \circ g: I \to T_f$. By Baer’s Criterion [4, Th(7.1.7)P.13] there exists $a \in A_M$ and $b \in T_f$, such that $g(w) = wa$ and $g(w) = wb$ for each $w \in I$. Thus $w(a - b) = 0$. Assume that $a \neq b$. Since $w \in \text{ann}(a - b)$ and $M$ is prime, then $w \in \text{ann}(a)$. Thus $g = 0$, which is a contradiction, therefore $a = b \in A_M \cap T_f$ and hence $A_M \cap T_f$ is injective.

Now consider the short exact sequence
$$0 \to A_M \cap T_f \to A_M \to \frac{A_M \oplus M}{A_M \cap T_f} \to 0$$

Where $i$ is the inclusion map and $\pi$ be the natural epimorphism. The sequence splits, as shown by [4]. Hence $A_M \cap T_f \leq M \oplus M$. By theorem (2.2), $M$ is a Rickart module.

The converse of the above proposition is not always true. For example. Consider $Z_6$ as $Z$-module. $Z_6$ is semisimple and hence $Z_6$ is Rickart. But $Z_6$ is neither injective nor prime.

**Propositions 2.4:** Let $M$ be an $R$-module such that for every homomorphism $f: M \to M$, $A_M + T_f$ is projective, then $M$ is Rickart module.

**Proof:** Let $f: M \to M$ be an $R$-homomorphism, consider the following short exact sequences
$$0 \to A_M \cap T_f \to A_M \xrightarrow{i_1} A_M \to \frac{A_M}{A_M \cap T_f} \to 0$$
$$0 \to T_f \to A_M + T_f \xrightarrow{i_2} \frac{A_M + T_f}{T_f} \to 0$$

Where $i_1, i_2$ are the inclusion homomorphisms and $f_1, f_2$ are the natural epimorphisms.

By the second isomorphism theorem, $\frac{A_M}{A_M \cap T_f} = \frac{A_M + T_f}{T_f}$. Since $T_f$ is a direct summand of $M \oplus M$ and $T_f \leq A_M + T_f$, then $T_f$ is a summand of $A_M + T_f$. Thus the second sequence splits. But $A_M + T_f$ is...
projective, thus \( \frac{A_M}{A_M \cap T_f} \cong \frac{A_M + \kappa f}{T_f} \) is projective. Hence the first sequence splits. Thus \( A_M \cap T_f \) is a summand of \( A_M \). Since \( A_M \) is a summand of \( M \oplus M \), then \( A_M \cap T_f \) is a summand of \( M \oplus M \). By the same argument of the prove of theorem 2.2, \( \ker f \oplus 0 = \ker f = T_f \cap A_M \), therefor \( \ker f \oplus 0 \) is a summand of \( M \oplus M \). Since \( \ker f \oplus 0 \subseteq A_M \), Then \( \ker f \oplus 0 \) is a summand of \( A_M \) and hence \( \ker f \) is a summand of \( M \). Thus \( M \) is a Rickart module.

The converse is not true as the following example shows:

Consider \( Z_6 \) as \( Z \)-module, \( Z_6 \) is semisimple and hence is a Rickart. It’s is known that \( Z_6 \) is not projective. Let \( f: Z \to Z \) be an \( R \)-homomorphism. One can easily show that \( A_{Z_6} + T_f = A_{Z_6} \oplus \text{Im} f \).

Now assume that \( A_{Z_6} + T_f \) is projective, then \( A_{Z_6} \cong Z_6 \) is projective which is a contradiction.

However, we have the following

**Theorem 2.5:** Let \( M \) be a projective \( R \)-module, Then \( M \) is a Rickart module if and only if for every \( R \)-homomorphism \( f: M \to A_M + T_f \) is a projective \( R \)-module.

**Proof:** Suppose that \( M \) is a Rickart module and let \( f: M \to A_M + T_f \) be an \( R \)-homomorphism. Now consider the following short exact sequences

\[
0 \to A_M \cap T_f \xrightarrow{i_1} A_M \xrightarrow{\pi_1} A_M / A_M \cap T_f \to 0
\]

\[
0 \to T_f \xrightarrow{i_2} A_M + T_f \xrightarrow{\pi_2} A_M / T_f \to 0
\]

Where \( i_1, i_2 \) are the inclusion homomorphisms and \( \pi_1, \pi_2 \) are the natural epimorphisms. Since \( M \) is a Rickart module, then \( \ker f \) is a summand of \( M \). By the same argument of the prove of theorem 2.2 \( \ker f \oplus 0 = \ker f \cap T_f = T_f \cap A_M \), hence \( A_M / T_f \cap A_M \) is a summand of \( A_M \). Thus the first sequence splits. But \( A_M = M \oplus 0 = M \) and \( M \) is projective, there for \( A_M \) is projective. Then \( A_M / T_f \cap A_M \) is projective. By the second isomorphism theorem

\[
A_M / T_f \cap A_M \cong \frac{A_M + T_f}{T_f}
\]

Thus \( \frac{A_M + T_f}{T_f} \) is projective. Hence the second sequence splits. But \( M \oplus M = T_f \oplus B_M \), therefor \( T_f \) is projective. Thus \( A_M + T_f \) is projective, (where \( A_M + T_f = T_f \oplus \frac{A_M + T_f}{T_f} \)).

Let \( M \) and \( N \) be two \( R \)-modules. Recall that \( M \) is called \( N \)-Rickart (or relatively Rickart to \( N \)) if, for every \( R \)-homomorphism \( f: M \to N \), \( \ker f \) is a summand of \( M \) [1].

Before giving our next result, we need the following.

**Propositions 2.6.1:** The following are equivalent for a module \( M \)

1. \( M \) is a Rickart module;
2. For every submodule \( N \) of \( M \), every direct summand \( L \) of \( M \) is \( N \)-Rickart.

**Propositions 2.7:** Let \( M \) be an indecomposable \( R \)-module and let \( N \) be any \( R \)-module if \( M \) is \( N \)-Rickart, then either

1. \( \text{Hom}(M,N) = 0 \) or
2. Every nonzero \( R \)-homomorphism from \( M \) to \( N \) is a monomorphism.

**Proof:** Assume that \( \text{Hom}(M,N) \neq 0 \) and let \( f: M \to M \) be a nonzero \( R \)-homomorphism. Since \( M \) is \( N \)-Rickart, then \( \ker f \) is a summand of \( M \). But \( M \) is indecomposable. So \( \ker f = \{0\} \) and \( f \) is a monomorphism.

Recall that an \( R \)-module \( M \) is called a Quasi-Dedekind \( R \)-module if every nonzero endomorphism of \( M \) is a monomorphism [6, Th(1.5), CH2].

**Corollary 2.8:** Let \( M \) be an indecomposable \( R \)-module and let \( N \) be any \( R \)-module such that \( \text{Hom}(M,N) \neq 0 \). If \( M \) is \( N \)-Rickart, then \( M \) is Quasi-Dedekind. In particular if \( M \) is Rickart, then \( M \) is Quasi-Dedekind.

**Proof:** By Prop.2.7, there is a monomorphism \( f: M \to N \). Assume \( M \) is not Quasi-Dedekind \( R \)-module, therefore there exists a nonzero endomorphism \( g: M \to M \) such that \( \ker g \neq 0 \). Since \( f \) is a monomorphism, then \( \ker(f \circ g) = \ker g = 0 \). Since \( M \) is \( N \)-Rickart, then \( \ker g \subseteq \text{Hom}(M,N) \) and hence \( \ker g = \text{Hom}(M,N) \). Thus \( g = 0 \), which is a contradiction. Thus \( M \) is a Quasi-Dedekind \( R \)-module.
3 CHARACTERIZATIONS OF RINGS BY MEANS OF RICKART MODULES

It’s known the direct sum of the Rickart modules need not be a Rickart module, see [1], [2].

In this section, we give a conditions under which a direct sum of Rickart modules is a Rickart module.

**Proposition 3.1**: Let $M$ be an $R$-module, If $R$ is $M$-Rickart, then every cyclic submodule of $M$ is projective. In particular if $R$ is an $R$-Rickart module, then every Principale ideal is projective ideal, i.e. $R$ is a p.p. ring.

**Proof**: Let $m \in M$, consider the following short exact sequence

$$0 \to ker f \to R \to Rm \to 0$$

where $i$ is the inclusion homomorphism and $f$ is defined as follows $f(r) = rm, \forall r \in R$.

Let $i_2: Rm \to M$ be the inclusion homomorphism. Now consider $i_2 \circ f: R \to M$. Since $R$ is $M$-Rickart. Then $ker(i_2 \circ f)$ is a summand of $R$. But $i_2$ is a monomorphism, therefore $ker f = ker(i_2 \circ f) = 0$. Thus the sequence is split and hence $Rm$ is summand of $R$, since $R$ is a projective $R$-module. Then $Rm$ is projective.

Recall that an $R$-module $M$ is called dualizable if $Hom(M, R) \neq 0$, [5]

**Corollary 3.2**: Let $M$ be a dualizable indecomposable $R$-module and $M$ is $R$-Rickart, then $M$ is isomorphic to an ideal of $R$. Hence if $R$ has no nonzero nilpotent elements, then $E(M)$ is commutative, where $E(M)$ is the ring of $R$-endomorphism of $M$.

**Proof**: Since $Hom(M, R) \neq 0$, then by Prop (2.7) $M$ is isomorphic to an ideal $I$ of $R$ and hence $E(M) \cong E(I)$. For the second part, since $R$ has no nilpotent elements $I$ is an ideal in $R$, Then $E(I)$ is commutative[7,prop (2.1)CH1]. Thus $E(M)$ is commutative

**Corollary 3.3**: Let $M$ be a projective indecomposable $R$-module and $R$ has no nonzero nilpotent element. If $M$ is an $R$-Rickart module and $Hom(M, R) \neq 0$, then $M$ is a multiplication module.

**Proof**: By the same argument of the proof of Cor (3.2), $E(M)$ is a commutative and hence $M$ is multiplication $[8]$

Recall that an $R$-module $M$ is called an SIP module if the intersection of any two direct summands of $M$ is also a direct summand of $M$ [9]. It is known that every Rickart module is an SIP module [1].

Before we give our next result, we need the following

**Theorem 3.4, [9]**: Let $R$ be a Noetherian domain and let $M$ be an injective $R$-module. If $M$ has the SIP, then either

1. $M$ is torsion free or
2. $M$ is torsion and for any two distinct indecomposable summands $A$ and $B$ of $M$, $Hom(A, B) = 0$

Now, we prove that

**Theorem 3.5**: Let $R$ be a Noetherian domain and let $M$ be an injective $R$-module, then the following are equivalent

1. $M \oplus M$ is a Rickart module.
2. $M$ is torsion free.
3. $\bigoplus_\Lambda M$ is Rickart module, for every index set $\Lambda$.

**Proof**: (1)$\Rightarrow$(2) Since $M$ is a summand of $M \oplus M$, then $M$ is Rickart and hence $M$ has the SIP. By Th (3.4) $M$ is either torsion or torsion free. Suppose $M$ is torsion, so $M \oplus M$ is torsion. Since $R$ is noetherian domain, then by [4, Th(6.6.2), p.162], $M = \bigoplus_{\alpha \in \Lambda} M_\alpha$ where $M_\alpha$ is an indecomposable submodule of $M$, now let $\alpha, \beta \in \Lambda$ and hence

$$M \oplus M \cong M_\alpha \oplus M_\beta \oplus \bigoplus_{\alpha \neq \beta} M_\alpha \oplus \bigoplus_{\alpha \neq \beta} M_\beta$$

$mnmnmNm_\alpha \oplus M_\alpha$ is Rickart and injective, thus by Cor (2.8) $M_\alpha$ is Quasi-Dedekind and hence $M_\alpha$ is prime by [6, prop (1.7), p.26] which is a contradiction. To verify this suppose $M_\alpha$ is prime. Since $M$ is torsion, then $M_\alpha$ is torsion. But $M_\alpha$ is injective over integral domain, therefore $M_\alpha$ is divisible. Now let $0 \neq x \in M_\alpha$, and let $0 \neq y \in M_\alpha$. Since $M_\alpha$ is divisible, then $x = ry$ for some $y \in M_\alpha$. Thus $M_\alpha$ is not prime.

(2)$\Rightarrow$(1) since $M$ is torsion free, then $M \oplus M$ is torsion free. Hence $M \oplus M$ is prime and injective. Thus $M \oplus M$ is a Rickart module, by Prop (2.3).
(2)⇒ (3) Since $M$ is torsion free, then $M$ is prime and hence $\bigoplus \lambda M$ is prime for every index set $\lambda$. But $\bigoplus \lambda M$ is injective, then by Prop (2.3), $\bigoplus \lambda M$ is a Rickart module.
Recall that A ring $R$ is a left semihereditary if every finitely generated left ideal is projective[10].
Befor we give our next result, we need the following

**Theorem 3.6:** A ring $R$ is a left semihereditary if and only if every finitely generated projective (free) $R$-module is a Rickart module.

**Theorem 3.7:** The following statements are equivalent for a commutative ring $R$
(1) $R$ is a semihereditary ring.
(2) $\bigoplus I R$ is Rickart for every finite index set $I$.
(3) $R \bigoplus R \bigoplus R$ is a Rickart $R$-module.

**Proof:** (1)⇒(2) Clear by Th 3.6.
(2)⇒ (3) Clear.
(3)⇒(1) Let $I = Ra + Rb$ be two generated ideal in $R$.
Define $f: R \bigoplus R \rightarrow Ra + Rb$ by $f(r_1, r_2) = r_1a + r_2b$. It is clear that $f$ is an epimorphism. Let $i: Ra + Rb \rightarrow R$ be the inclusion map. Since $i \circ f: R \bigoplus R \rightarrow R$ and $R \bigoplus R \bigoplus R$ is Rickart, then $\ker(i \circ f)$ is a summand by Prop(2.6). It is clear that $i$ is a monomorphism, therefore $\ker(i \circ f) = \ker f$ is a summand of $R$. Thus $Ra + Rb$ is a projective $R$-module. One can show that $R$ is semihereditary[11].

We end this section by the following

**Theorem 3.8:** The following conditions are equivalent for a ring $R$
(1) $R$ is semisimple.
(2) All $R$-modules are Rickart.
(3) All injective $R$-modules are Rickart.

**Proof:** (1)⇒(2)⇒(3) It is clear.
(3)⇒(1) Let $M$ be any $R$-module, there is an injective $R$-module $E_1$ and a monomorphism $g_1: M \rightarrow E_1$, by [4] Likewise, there is an a monomorphism $g_2: \frac{E_1}{\text{Im} g_1} \rightarrow E_2$, for some injective $R$-module $E_2$. Let $f: E_1 \rightarrow \frac{E_1}{\text{Im} g_1}$ be the natural epimorphism. Now consider $g_2 \circ f: E_1 \rightarrow E_2$. Note that $E_1 \bigoplus E_2$ is injective and hence by assumption $E_1 \bigoplus E_2$ is Rickart, then by Prop(2.6) $E_1$ is $E_2$-Rickart. Thus $\ker g_2 \circ f$ is a summand of $E_1$. But $g_2$ is a monomorphism, then $\ker (g_2 \circ f) = \ker f = \text{Im} g_1$ is summand of $E_1$. Thus $\text{Im} g_1$ is injective. Since $M \cong \text{Im} g_1$. Then $M$ is injective. Then by [4, cor(8.2.2)] $R$ is semisimple.

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