TROPICAL PROBABILITY THEORY AND AN APPLICATION TO THE ENTROPIC CONE

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ABSTRACT. In a series of articles, we have been developing a theory of tropical diagrams of probability spaces, expecting it to be useful for information optimization problems in information theory and artificial intelligence. In this article, we give a summary of our work so far and apply the theory to derive a dimension-reduction statement about the shape of the entropic cone.

1. Introduction

With the aim of developing a systematic approach to an important class of problems in information theory and artificial intelligence, we started in [MP18] the development of a theory of tropical diagrams of probability spaces. One of our intended applications is to characterize, or at least derive important properties of, the entropic cone: an important open problem in information theory, to which Fero Matuš made invaluable contributions.

In this article, we give a summary of our work on tropical diagrams so far and apply the technology to derive a statement about the entropic cone.

We briefly recall the definition of the entropic cone. Given a collection of \( k \) random variables \( X_1, \ldots, X_k \) and a subset \( I \subset \{1, \ldots, k\} \), we can record the entropy of the joint random variable \( X_I \). This way, we get a function from \( \Lambda_k \) to \( \mathbb{R} \), where \( \Lambda_k \) denotes the set of nonempty subsets of \( \{1, \ldots, k\} \). We interpret the function as an element of the vector space \( \mathbb{R}^{\Lambda_k} \) and call it the entropy vector of the random variables \( X_1, \ldots, X_k \). In general, we say that a vector in \( \mathbb{R}^{\Lambda_k} \) is entropically representable if it is the entropy vector of some collection of random variables \( X_1, \ldots, X_k \).

The entropic cone is the closure of the set of entropically representable vectors. Entropies of random variables, conditional entropies, mutual information and conditional mutual information are all nonnegative. These conditions are called the Shannon inequalities. For \( k \leq 3 \), the Shannon inequalities completely describe the entropic cone, but for \( k \geq 4 \) the situation is much more complicated. Zhang and Yeung showed that the entropic cone and the submodular cone (i.e., the cone cut out by Shannon inequalities) are different [ZY98], by identifying a non-Shannon inequality satisfied by all entropically representable vectors. Subsequently, more non-Shannon inequalities were discovered, e.g. [MMRV02, DFZ06].

In [Mat07] Matuš discovered several infinite families of linear inequalities satisfied by the entropic cone and used it in a clever way to show that the cone
is not polyhedral. Other infinite families of information inequalities were found in [DFZ11] as well as many (more than 200) sporadic inequalities.

In the case of four random variables, the entropic cone is a closed convex cone in $\mathbb{R}^{15}$. Using techniques developed in [MP18, MP19c, MP19b, MP19a], we show how the dimension of the problem of determining the entropic cone could be reduced from 15 to 11.

During our work on the development of tropical probability we were greatly influenced by the article of Gromov [Gro12] and by numerous discussions with Fero Matúš as well as by his published work, such as [Mat93, MS95, Mat07].

2. Tropical Diagrams

The language of random variables was introduced by Fréchet, Kolmogorov and others, so that joint distributions are automatically defined. For our purposes, this is not a convenient setup, as we often need to vary the joint distributions. That’s why we use a different language of diagrams of probability spaces, which we introduce below. A more detailed discussion and proofs of the statements below can be found in [MP18], [MP19c] and [MP19a].

2.1. Probability spaces. For the purposes of this article, a probability space is a set with a probability measure on it which is supported on a finite subset. A reduction from one probability space to another is an equivalence class of measure-preserving maps, where two maps are considered equivalent if they coincide on a set of full measure. Note that the target space of a random variable taking values in a finite set is a probability space according to this definition.

The tensor product $X \otimes Y$ of two probability spaces $X$ and $Y$ is the independent product.

2.2. Diagrams of probability spaces. We will consider commutative diagrams of probability spaces and reductions, such as a two-fan and a diamond, pictured below

\begin{equation}
\begin{array}{c}
X & \leftarrow & Z & \rightarrow & Y \\
 & & & & \\
X & \leftrightarrow & W & \leftrightarrow & Y
\end{array}
\end{equation}

In these diagrams, $X$, $Y$, $Z$ and $W$ are probability spaces, and the arrows are reductions. To speak about general diagrams, we will need to specify the arrangement of probability spaces and reductions, i.e. we need to record the underlying combinatorial structure. There are several, equivalent, ways to do so: using a poset category, a partially ordered set (poset), or a directed acyclic graph (DAG) with some additional properties as described below. From our perspective, the language of categories is most convenient for this purpose, but it may not be as familiar as the other two concepts. That is why we will provide a dictionary to convert from one setup to the other.
2.2.1. **Categories, posets and DAGs.** A **poset category** is a finite category $G$ such that for any pair of objects $i,j \in G$ there is at most one morphism either way. We will require the poset categories used for indexing diagrams to have an additional property, that we describe below after introducing some convenient terminology.

For a morphism $i \to j$ in $G$, the object $i$ will be called an **ancestor** of $j$ and object $j$ will be called a **descendant** of $i$.

An **indexing category** $G$ is a finite poset category such that for any two objects $i,j \in G$ there exists a **minimal common ancestor** $\hat{i}$, that is an object $\hat{i}$ which is an ancestor to both $i$ and $j$ and such that any other common ancestor of $i$ and $j$ is also an ancestor of $\hat{i}$. For an interested reader an example of a poset category that fails this property is shown below.

![Diagram](image)

Given a poset $(P, \geq)$ such that any subset in $P$ has a supremum (a least common upper bound), one can construct an indexing category $G$, having as objects the points in the poset, and a unique morphism $i \to j$ for any pair $i \geq j$.

Starting with a DAG, one can construct a poset category by taking the transitive closure of the DAG and considering vertices as objects and arrows as morphisms. The translation of the defining property of indexing categories is straightforward in the DAG language.

A **fan** in a category is a pair of morphisms with the same domain $(i \leftarrow k \to j)$. Such a fan is called **minimal** if whenever it is included in a commutative diagram

![Diagram](image)

the vertical arrow $k \to l$ must be an isomorphism.

Indexing categories have the following useful properties, which are elementary to establish. First, for any pair of objects $i,j$ in an indexing category $G$, there exists a **unique minimal fan** in $G$ with target objects $i$ and $j$. Secondly, any indexing category is **initial**, i.e. it has an **initial object** that is an ancestor to any other object in $G$.

2.2.2. **Diagrams.** A **diagram of probability spaces** is a functor $\mathcal{X}$ from an indexing category $G = \{i; \gamma_{ij}\}$ to the category of probability spaces. Essentially, this means that given an indexing category, poset or DAG, we get a $G$-diagram of probability spaces $\mathcal{X} = \{X_i; \chi_{ij}\}$ by assigning to each object/vertex $i$ a probability space $X_i$ and to each morphism/arrow $\gamma_{ij}$ a reduction $\chi_{ij}$, requiring that the resulting diagram commutes. We denote the set of all $G$-diagrams of probability spaces by $\text{Prob}(G)$.
2.2.3. Full diagrams and random variables. Important examples of diagrams are $\Lambda_n$-diagrams, which we call full diagrams, where $\Lambda_n$ is the poset of non-empty subsets of the set $\{1, \ldots, n\}$ ordered by inclusion. Given an $n$-tuple of random variables $(X_1, \ldots, X_n)$ we can construct a $\Lambda_n$-diagram

$$(X_1, \ldots, X_n) := \{X_I; \chi_{IJ}\}$$

by setting $X_I$ equal to the target space of $(X_i : i \in I)$ with the induced distribution and $\chi_{IJ}$ equal to the natural projections. On the other hand, starting with a $\Lambda_n$-diagram we can construct an $n$-tuple of random variables as reductions from the initial space to $n$ terminal spaces. Diagrams of combinatorial type $\Lambda_2$ are two-fans, pictured above in (2.1), and $\Lambda_1$-diagrams are single probability spaces.

2.2.4. Diagrams of diagrams. A reduction $\rho : \mathcal{X} \to \mathcal{Y}$ from a $\mathcal{G}$-diagram $\mathcal{X}$ to a $\mathcal{G}$-diagram $\mathcal{Y}$ is a natural transformation from (the functor) $\mathcal{X}$ to $\mathcal{Y}$. It amounts to specifying a reduction $\rho_i : X_i \to Y_i$ for every $i$, such that the diagram obtained from $\mathcal{X}$, $\mathcal{Y}$ and the $\rho_i$'s is commutative. Thus, $\text{Prob}(\mathcal{G})$ is itself a category.

Hence, we can also construct diagrams of diagrams. Most important for us are two-fans of $\mathcal{G}$ diagrams,

$$\begin{tikzcd}
\mathcal{X} & \mathcal{Z} & \mathcal{Y} \\
\mathcal{A} & \mathcal{C} & \mathcal{B}
\end{tikzcd}$$

where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are $\mathcal{G}$-diagrams, and the arrows are reductions of diagrams. For the space of $\mathcal{H}$-diagrams of $\mathcal{G}$-diagrams we will use the notation $\text{Prob}(\mathcal{G})(\mathcal{H}) = \text{Prob}(\mathcal{G}, \mathcal{H})$. Note that an $\mathcal{H}$-diagram of $\mathcal{G}$-diagrams can equivalently be interpreted as a $\mathcal{G}$-diagram of $\mathcal{H}$-diagrams.

2.2.5. Minimal diagrams. A $\mathcal{G}$-diagram $\mathcal{X}$ is called minimal if it maps minimal fans in $\mathcal{G}$ to minimal fans in the target category.

A minimization of a two-fan $\hat{\mathcal{Z}} := (\mathcal{X} \leftarrow \mathcal{Z} \to \mathcal{Y})$ of either of probability spaces or of diagrams is the minimal fan $\hat{\mathcal{C}}$ and a reduction

$$\begin{tikzcd}
\mathcal{X} & \mathcal{Z} & \mathcal{Y} \\
\mathcal{A} & \mathcal{C} & \mathcal{B}
\end{tikzcd}$$

such that $f$ and $g$ are isomorphisms.

It is shown in [MP18, Proposition 2.1] that a minimization always exists and is unique up to isomorphism.

We will also refer to a minimal two-fan with $\mathcal{X}$ and $\mathcal{Y}$ as targets, as a coupling between $\mathcal{X}$ and $\mathcal{Y}$. 
2.2.6. Tensor product and conditioning. The tensor product of two $G$-diagrams $X = \{ X_i; \chi_{ij} \}$ and $Y = \{ Y_i; \upsilon_{ij} \}$ is $X \otimes Y := \{ X_i \otimes Y_i; \chi_{ij} \times \upsilon_{ij} \}$.

If $X$ is a $G$-diagram, and $U$ is a probability space in $X$, then the whole diagram $X$ can be conditioned on an outcome $u \in U$ with positive weight. We denote the conditioned diagram by $X,u$. A precise definition of this construction is given in [MP18, Section 2.8].

2.3. The intrinsic and asymptotic entropy distances. For a given a $G$-diagram $X$ we may evaluate the entropies of the individual probability spaces, which gives a map 

$$\text{Ent}_*: \text{Prob}(G) \rightarrow \mathbb{R}^G$$

where the target space $\mathbb{R}^G$ is the vector space of all real-valued functions on the set of objects in $G$, equipped with the $\ell^1$-norm. The entropy is a homomorphism in the sense that $\text{Ent}_*(X \otimes Y) = \text{Ent}_*(X) + \text{Ent}_*(Y)$.

Given a two-fan of $G$-diagrams $K = (X \leftarrow Z \rightarrow Y)$, the entropy distance between $X$ and $Y$ is defined by 

$$kd(K) := \| \text{Ent}_*(Z) - \text{Ent}_*(X) \|_1 + \| \text{Ent}_*(Z) - \text{Ent}_*(Y) \|_1$$

We use the entropy distance as a measure of deviation of a fan $K$ from being an isomorphism between $X$ and $Y$. Indeed, the entropy distance $kd(K)$ vanishes if and only if both arrows in $K$ are isomorphims.

We obtain the intrinsic entropy distance $k(X,Y)$ between two $G$-diagrams $X$ and $Y$ by taking an infimum over all couplings between $X$ and $Y$

$$k(X,Y) := \inf \{ kd(K) : K \text{ coupling between } X \text{ and } Y \}$$

For probability spaces, the intrinsic entropy distance was introduced in [KSS12, Vid12].

We also define the asymptotic entropy distance $\kappa(X,Y)$ by 

$$\kappa(X,Y) := \lim_{n \to \infty} \frac{1}{n} k(X^n, Y^n)$$

where $X^n$ denotes the $n$-fold independent product of $X$.

Both $k$ and $\kappa$ are pseudo-distance functions in that they satisfy all axioms of a distance function, except that they may vanish on pairs of non-identical points, see [MP18] and [MP19c] for the proofs.

2.4. Tropical diagrams. Tropical objects, as for instance encountered in algebraic geometry, are, roughly speaking, divergent sequences of classical objects (e.g. algebraic varieties), renormalized by viewing them on a log scale with increasing base.

The space of tropical diagrams of probability spaces is defined along similar lines: it consists of certain divergent sequences of diagrams and is endowed with an asymptotic entropy distance, thus achieving a similar renormalization.

Our description below is extremely brief. For details and proofs, we refer the reader to [MP19c].
2.4.1. **Quasi-linear sequences.** We define the linear sequence generated by a $G$-diagram $\mathcal{X}$ as the sequence $\overrightarrow{\mathcal{X}} := (\mathcal{X}^n : n \in \mathbb{N}_0)$ and we define the distance between two such sequences by

$$\kappa \left( \overrightarrow{\mathcal{X}}, \overrightarrow{\mathcal{Y}} \right) := \lim_{n \to \infty} \frac{1}{n} k \left( \mathcal{X}^n, \mathcal{Y}^n \right)$$

Tropical diagrams of probability spaces will be sequences that are almost linear, so that it allows us to define algebraic operations on them, and establish completeness of the space of all tropical diagrams.

We call a sequence $[\mathcal{X}] := (\mathcal{X}(n) : n \in \mathbb{N})$ quasi-linear if for every $m, n \in \mathbb{N}$,

$$\kappa \left( \mathcal{X}(m + n), \mathcal{X}(m) \otimes \mathcal{X}(n) \right) \leq C(m + n)^{3/4}$$

and define the distance between two quasi-linear sequences by

$$\kappa \left( [\mathcal{X}], [\mathcal{Y}] \right) := \lim_{n \to \infty} \frac{1}{n} \kappa \left( \mathcal{X}(n), \mathcal{Y}(n) \right)$$

and denote by $\text{Prob}[G]$ the (pseudo-)metric space of all quasi-linear sequences endowed with $\kappa$. Two quasi-linear sequences will be called asymptotically equivalent if they are zero distance apart. Equivalence classes of quasi-linear sequences will be called tropical diagrams of probability spaces. In our discussions we will sometimes be sloppy, and make no distinction between equivalence classes and their representatives (quasi-linear sequences). This is harmless, as operations we consider are all $\kappa$-continuous and preserve asymptotic equivalence.

The sum of two sequences is defined as element-wise tensor product, and multiplication by a scalar $\lambda \geq 0$ is defined by

$$\lambda \cdot [\mathcal{X}] := (\mathcal{X}(\lambda \cdot n) : n \in \mathbb{N}_0)$$

The addition and scalar multiplication satisfy the usual associative, commutative and distributive laws up to asymptotic equivalence. Therefore, the space $\text{Prob}[G]$ has the structure of a convex cone.

The asymptotic distance $\kappa$ is 1-homogeneous

$$\kappa \left( \lambda \cdot [\mathcal{X}], \lambda \cdot [\mathcal{Y}] \right) = \lambda \kappa \left( [\mathcal{X}], [\mathcal{Y}] \right)$$

and translation-invariant,

$$\kappa \left( [\mathcal{X} + \mathcal{Z}], [\mathcal{Y} + \mathcal{Z}] \right) = \kappa \left( [\mathcal{X}], [\mathcal{Y}] \right)$$

We show in [MP19c] that the space $\text{Prob}[G]$ is complete. Together with the algebraic structure, it implies that $\text{Prob}[G]$ is a closed convex cone in some (generally infinite-dimensional) Banach space $B$. We call elements in the dual space $B^*$ entropic quantities. The entropy functional defined by

$$\text{Ent}_*([\mathcal{X}]) := \lim_{n \to \infty} \frac{1}{n} \text{Ent}_* (\mathcal{X}(n))$$

yields an example of such dual elements.
2.5. **Homogeneous diagrams and asymptotic equipartition property.**

We call a diagram of probability spaces $\mathcal{X}$ *homogeneous* if its automorphism group $\text{Aut}(\mathcal{X})$ acts transitively on every space in $\mathcal{X}$.

Examples of homogeneous diagrams can be constructed in the following way. A **G-diagram of groups** is a pair consisting of an ambient finite group $G$ and a $G$-diagram of subgroups of $G$, $(H_i : i \in G)$, where the arrows are inclusions. Starting with a $G$-diagram of groups, we construct a $G$-diagram of probability spaces $\mathcal{X} := \{X_i; \chi_{ij}\}$ by setting $X_i := G/H_i$ with the uniform measure and defining the reduction $\chi_{ij}$ to be the natural projection $\chi_{ij} : G/H_i \to G/H_j$ whenever $H_i \subseteq H_j$. In fact, every homogeneous diagram arises in this way [MP18, Section 2.7.1]. We call a homogeneous diagram *Abelian* if it can be constructed in this way with Abelian $G$.

Starting with a diagram of groups $(G; H_i : i \in G)$ the resulting homogeneous diagram will be minimal if and only if for any $i, j \in G$ there exists $k \in G$, such that $H_k = H_i \cap H_j$. If a diagram of groups satisfies this property, we will call it **minimal** as well. When $G$ is a full indexing category $G = \Lambda_n$, the description of minimal diagrams of groups is especially simple: One needs to specify only the terminal groups, others being obtained by appropriate intersections. We will write $(G; H_1, \ldots, H_n)$ for such minimal $\Lambda_n$-diagram of groups.

We denote the space of quasi-linear sequences of homogeneous $G$-diagrams by $\text{Prob}[G]_h$, the subspace of sequences of Abelian $G$-diagrams by $\text{Prob}[G]_{\text{Ab}}$, and the space quasi-linear sequences of minimal $G$-diagrams by $\text{Prob}[G]_m$.

The Asymptotic Equipartition Property for diagrams of probability spaces that we have shown in [MP18, Theorem 6.1] essentially says that any linear sequences of diagrams of probability spaces is asymptotically equivalent to a quasi-linear sequence of homogeneous diagrams. Together with the density of linear sequences in $\text{Prob}[G]$, [MP19c, Theorem 5.2], it implies the following theorem.

**Theorem 2.1.** For any indexing category $G$ the spaces $\text{Prob}[G]_h$ and $\text{Prob}[G]_{h,m}$ are dense in $\text{Prob}[G]$ and $\text{Prob}[G]_m$, respectively.

Intuitively, according to the theorem above, one may think of a tropical diagram as a homogeneous diagram of very large probability spaces. Thus, whenever one wants to evaluate a continuous linear functional on a diagram $\mathcal{X}$, one may assume that it is homogeneous and consists of arbitrarily large spaces. We take advantage of this point of view in the next section.

As a trivial, but enlightening, example, consider a two-fan $(X \leftarrow Z \rightarrow Y)$. By Theorem 2.1, a high power $(X^n \leftarrow Z^n \rightarrow Y^n)$ can be approximated by a homogeneous fan $H_X \leftarrow H_Z \rightarrow H_Y$. The entropies of $X$, $Z$ and $Y$ (and therefore also the mutual information $\text{Ent}(X) + \text{Ent}(Y) - \text{Ent}(Z)$ between $X$ and $Y$) can be established by just counting in the homogeneous fan: $\text{Ent}(X) \approx \frac{1}{n} \log |H_X|$ where $|H_X|$ denotes the cardinality of $H_X$. However, the three entropies do not determine the asymptotic equivalence class of the fan, there are many more (in fact, infinitely many) independent entropic quantities. But all can be determined from the homogeneous approximation.
2.6. Tropical conditioning.

2.6.1. Conditioning. One of the advantages of homogeneous diagrams is that if a homogeneous diagram $\mathcal{X}$ contains a space $U$, then the isomorphism class of the conditioned diagram $\mathcal{X}|u$ does not depend on the choice of an atom $u \in U$.

Since any tropical diagram is asymptotically equivalent to a homogeneous tropical diagram, we can use this independence of $u$ to define an operation of conditioning of a tropical $G$-diagram $[\mathcal{X}]$ on a space $[U]$ in it, obtaining another tropical $G$-diagram denoted by $[\mathcal{X}|U]$. In the tropical setting the diagram $[\mathcal{X}|U]$ depends (Lipschitz-)continuously on $[\mathcal{X}]$ and $[U]$. This subject is discussed in more details in [MP19a].

2.6.2. Entropy and mutual information. Now that we defined $[\mathcal{X}|U]$ as a tropical diagram, its entropy $\text{Ent}_*([\mathcal{X}|U])$ is defined by the limit in (2.3). At the same time, it equals the limit

$$\lim_{n \to \infty} \frac{1}{n} \text{Ent}_*(\mathcal{X}(n)|U(n)) := \lim_{n \to \infty} \frac{1}{n} \int_{u \in U(n)} \text{Ent}_*(\mathcal{X}(n)|u) \, dp(u)$$

In [MP19c] it is shown that the space of single tropical probability spaces, $\text{Prob}[\Lambda_1]$, is isomorphic to $\mathbb{R}_{\geq 0}$, with the isomorphism given by the entropy. Thus, a tropical probability space is completely determined by its entropy, and we will simply write

- $[X]$ for $\text{Ent}([X])$.
- $[X : Y] := [X] + [Y] - [Z]$ for the mutual information between $X$ and $Y$ in the minimal two-fan $[X] \leftarrow [Z] \rightarrow [Y]$.
- $[X : Y|U] := [\hat{X}] + [\hat{Y}] - [\hat{Z}] - [U]$ for the conditional mutual information between $X$ and $Y$, where $[\hat{X}]$, $[\hat{Y}]$, $[\hat{Z}]$ and $[U]$ are the spaces in the minimal diagram

$$[X] \leftarrow [\hat{X}] \quad [U] \quad [\hat{Y}] \quad [Y]$$

3. Arrow Contraction and Expansion

In this section we describe two operations on tropical diagrams, arrow contraction and expansion. Given a tropical diagram $[Z]$, arrow contraction is a modification of the diagram in such a way that a certain arrow becomes an isomorphism, while keeping control of what happens to some other parts of $[Z]$. Arrow expansion is an inverse operation. We will apply these techniques in the next section to derive a dimension reduction result for the entropic cone.

The full construction is quite involved. Here we will only describe a corollary necessary for our purposes, and refer the reader to [MP19a] for the full results and details.
3.1. **Admissible and reduced sub-fans.** Suppose \( Z \) is a \( G \)-diagram and \( X \) is an element in it. By the *ideal* generated by \( X \) we mean the sub-diagram \([X]\) of \( Z\), that consists of the target spaces of all morphisms starting in \( X \) and all (available in \( Z \)) arrows between them. We will sometimes refer to spaces in \([X] \) as the *descendants* of \( X \). The ideal generated by a space \( X \) included in some diagram will be denoted \([X]\).

If \([Z]\) is a diagram of tropical probability spaces with the tropical space \([X]\) in it, in order to unclutter notations we will write

\[
[X] := [X]\]

An *admissible sub-fan* \((X \leftarrow Z \rightarrow U)\) in a diagram \( Z \) is a *minimal* sub-fan such that the space \( U \) is terminal, i.e. it is not the domain of definition of any (non-identity) morphism in \( Z \). An admissible fan will be called reduced if \( Z \rightarrow X \) is an isomorphism.

A diagram with an admissible fan is illustrated schematically in Figure 1. Two more concrete examples are shown in Figures 2 and 3.

![Diagram](image)

**Figure 1.** Arrow contraction/expansion and arrow collapse. Here \([X] := [X]\). In the left diagram the fan \([X] \leftarrow [Z] \rightarrow [U]\) is admissible. In the middle diagram \([X] \leftarrow [Z'] \rightarrow [U']\) is admissible and reduced. The diagrams may contain some other spaces beyond those shown. During contraction/expansion we don’t have control over the other parts of the diagram.

If an arrow \( Y \rightarrow X \) in a diagram \( Z \) is an isomorphism, then we can identify the spaces \( X \) and \( Y \), thus changing the combinatorial structure of \( Z \). We call such change *arrow collapse*. Examples of the process of collapsing an arrow can be seen in Figures 1, 2 and 3.

3.2. **Arrow contraction and expansion.** Suppose \([Z]\) and \([Z']\) are two tropical \( G \)-diagrams, containing admissible fans \(( [X] \leftarrow [Z] \rightarrow [U] ) \) and \(( [X'] \leftarrow [Z'] \rightarrow [U'] ) \), respectively, which correspond to each other under the combinatorial isomorphism between \([Z]\) and \([Z']\). Suppose the fan \(( [X'] \leftarrow [Z'] \rightarrow [U'] ) \) is reduced. Denote \([X'] := [X]\) and \([X'] := [X']\).
We say that \( [Z'] \) is obtained from \([Z]\) by arrow contraction or, alternatively, \([Z]\) is obtained from \([Z']\) by arrow expansion, if

\[
\begin{align*}
[X] &= [X'] \\
[X\mid U] &= [X'\mid U']
\end{align*}
\] (3.1)

The other spaces in \([Z]\) outside of \([X]\) and \([U]\) may change in an uncontrolled manner. In view of equality (3.1) we will identify diagrams \([X]\) and \([X']\).

Arrow contraction, expansion and collapse are illustrated in Figures 1, 2 and 3.

Note that equation (3.2) is in general an equality in an infinite-dimensional space. But as a simple consequence we have that for any two spaces \([X_1]\) and \([X_2]\) in \([X]\) and the corresponding spaces \([X'_1]\) and \([X'_2]\) in \([X']\) the following equalities hold:

\[
\begin{align*}
[X_1\mid U] &= [X'_1\mid U'] \\
[X_1 : X_2\mid U] &= [X'_1 : X'_2\mid U'] \\
[X_1 : U] &= [X'_1 : U'] \\
[X_1 : U\mid X_2] &= [X'_1 : U'\mid X'_2] \\
[U'] &= [X : U]
\end{align*}
\] (3.3)

Indeed, the first equality follows directly from equality (3.2). The next three can be proven by expanding the right- and left-hand sides into summands of the form \([A\mid U]\) and \([B]\) for some \([A]\) and \([B]\) in \([X]\). The last one follows from the fact that \([U']\) is a descendant of \([X]\) in \([Z']\) and therefore \([X\mid U'] = [X] - [U']\).

We expect that arrow contraction is possible for any diagram with an admissible fan, but for the purposes of this article the following approximate contraction result from [MP19a] suffices.

**Theorem 3.1.** Let \( ([X] \leftarrow [Z] \rightarrow [U]) \) be an admissible fan in some tropical \(G\)-diagram \([Z]\). Then for every \(\varepsilon > 0\) there exists a \(G\)-diagram \([Z']\) containing an admissible fan \( ([X'] \leftarrow [Z'] \rightarrow [U']) \), corresponding to the original admissible fan through the combinatorial isomorphism, such that, with the notations \(X = [X]\) and \(X' = [X']\), the diagram \([Z']\) satisfies

\[
\begin{align*}
\text{(i)} \quad & \kappa([X'\mid U'], [X\mid U]) \leq \varepsilon \\
\text{(ii)} \quad & \kappa(X', X) \leq \varepsilon \\
\text{(iii)} \quad & [Z\mid X'] \leq \varepsilon
\end{align*}
\]

The evaluation of entropy of an individual space in a tropical diagram is a 1-Lipschitz linear functional, while the operation of conditioning is also a Lipschitz map, see [MP19b]. Thus in the settings of Theorem 3.1 the following inequalities hold: for any two spaces \([X_1]\) and \([X_2]\) in \([X]\) and corresponding
spaces \([X'_1]\) and \([X'_2]\) in \([X']\) the following inequalities hold:
\[
\left| [X_1|U] - [X'_1|U'] \right| \leq \varepsilon \\
\left| [X_1 : X_2|U] - [X'_1 : X'_2|U'] \right| \leq 2\varepsilon \\
(3.4) \left| [X_1 : U] - [X'_1 : U'] \right| \leq 2\varepsilon \\
\left| [X_1 : U_2] - [X'_1 : U'_2|X'_2] \right| \leq 3\varepsilon \\
\left| [U'] - [X : U] \right| \leq \varepsilon
\]

The following much simpler theorem from [MP19a] is the reverse of Theorem 3.1.

**Theorem 3.2.** (Expansion) Given a reduced admissible sub-fan \(([X] \leftarrow [Z'] \rightarrow [U'])\) in a tropical \(G\)-diagram \([Z']\) and a non-negative number \(\lambda \geq 0\), there is another \(G\)-diagram \([Z]\) obtained from \([Z]\) by arrow expansion such that \([Z|X] = \lambda\).

3.3. **Examples.** To illustrate the discussion above, we consider two examples of admissible sub-fans and arrow contraction and expansion.

3.3.1. As a first example, suppose we are given a tropical two-fan \([Z] = ([X] \leftarrow [Z] \rightarrow [U])\) as in Figure 2. We may ask the following question:

*Can the mutual information between \([X]\) and \([U]\) be captured by a tropical space \([V]\)? More precisely, is there a diamond extension

\[
[X] \leftarrow [Z] \rightarrow [V] \leftarrow [U]
\]

such that \([V] = [X : U]\) (or equivalently \([X : U|V] = 0\))?*

The answer is, in general, no. However, by contracting and collapsing the arrow \([Z] \rightarrow [X]\) we can still obtain a reduction \(([X] \rightarrow [V])\), where \([V]\) has the required “size”, i.e. its entropy equals the mutual information between \([X]\) and \([U]\). If we want to, we can still keep the spaces \([Z]\) and \([U]\) in the diagram after contraction/collapse. Note, however, that in general there will be no reduction \([U] \rightarrow [V]\) commuting with the other reductions.

![Figure 2](image-url)

**Figure 2.** Contraction/expansion and arrow collapse in a two-fan
3.3.2. As a second example, consider the tropical $\Lambda_3$-diagram

$$[Z] = ([X_1], [X_2], [U])$$

shown in Figure 3. Such examples can be particularly useful when the space $[U]$ is chosen to satisfy additional properties. For instance, it could be chosen such that the diagrams $[X_1]$ and $[X_2]$ are independent conditioned on $[U]$. We will discuss such extensions elsewhere. The fan $([X] \leftarrow [Z] \rightarrow [U])$ is admissible and the ideal $[X]$ is the fan $([X_1] \leftarrow [X] \rightarrow [X_2])$.

If we contract $[Z] \rightarrow [U]$, we obtain a diagram with a new space $[V]$, that has the same properties relative to $[X] = ([X_1], [X_2])$. The arrow expansion can be seen by reading the picture backwards.

### 4. Entropic Cone

In this section we define the submodular, entropic and Abelian cones associated to an indexing category $G$.

#### 4.1. Vector-spaces $\mathbb{R}^G$ and $\mathbb{R} \otimes G$

Given an indexing category $G$ we consider two linear spaces associated to it. Recall that the vector space $\mathbb{R}^G$ is the space of functions from $G$ to $\mathbb{R}$. The second space, dual to the first one, is $\mathbb{R} \otimes G$ – the vector-space of formal finite linear combinations of objects in $G$ with real coefficients. These two vector spaces are in natural duality defined by

$$\text{For } f \in \mathbb{R}^G \text{ and } \sum_{i \in G} \lambda_i \otimes i \in \mathbb{R} \otimes G, \quad \left( f, \sum_{i \in G} \lambda_i \otimes i \right) := \sum_{i \in G} \lambda_i f(i)$$

The collection of vectors $\{1 \otimes i\}_{i \in G} = \{[i]\}_{i \in G}$ forms a basis of the space $\mathbb{R} \otimes G$, and we denote the dual basis in $\mathbb{R}^G$ by $\{f_i\}_{i \in G}$. We also consider the following special vectors in $\mathbb{R} \otimes G$:

- The basis vectors $[i] := 1 \otimes i$.
- $[i : j] := [i] - [j]$, where $i$ is the top object in a minimal fan $i \leftarrow \hat{i} \rightarrow j$ in $G$.
- $[\hat{i} : j] := [\hat{i}] + [j] - [i]$, where $\hat{i}$ is top object in a minimal fan $i \leftarrow \hat{i} \rightarrow j$ in $G$. 

---

**Figure 3.** Arrow contraction and expansion in a $\Lambda_3$-diagram
4. Submodular, entropic and Abelian cones. Let $\mathcal{G} = \{i; \gamma_{ij}\}$ be an indexing category. We define three closed, convex cones in $\mathbb{R}^\mathcal{G}$: the submodular cone $\Gamma_{sm}(\mathcal{G})$, the entropic cone $\Gamma(\mathcal{G})$ and the Abelian cone $\Gamma_{Ab}(\mathcal{G})$.

4.2.1. The submodular cone. The submodular cone $\Gamma_{sm}(\mathcal{G}) \subset \mathbb{R}^\mathcal{G}$ consists of nonnegative, non-decreasing, submodular functions on the set of objects in the category (points in the poset or vertices in the DAG) $\mathcal{G}$. In essence, these are the functions on $\mathcal{G}$ that satisfy Shannon-like inequalities. More formally it is defined as follows.

The properties nonnegativity, monotonicity and submodularity are defined through linear inequalities. Every linear inequality for $f \in \mathbb{R}^\mathcal{G}$ can be written in the form $\langle f, v \rangle \geq 0$ for some $v \in \mathbb{R}^\otimes \mathcal{G}$. A function $f \in \mathbb{R}^\mathcal{G}$ is called

- **positive**, if $\langle f, [i] \rangle \geq 0$ for every object $i \in \mathcal{G}$
- **monotone**, if $\langle f, [i:j] \rangle \geq 0$, for every $i, j \in \mathcal{G}$.
- **submodular**, if $\langle f, [i:j] \rangle \geq 0$ and $\langle f, [i:j]\tilde{k} \rangle \geq 0$ for every $i, j, k \in \mathcal{G}$

The submodular cone is dual to the cone spanned by Shannon-like inequalities

$$\Gamma_{sm}(\mathcal{G}) := \{f \in \mathbb{R}^\mathcal{G} : \langle f, v \rangle \geq 0 \text{ for all } v \in \mathbb{SH}\}$$

where $\mathbb{SH} := \{[i], [i:j], [i:j], [i:j]\tilde{k} : i, j, k \in \mathcal{G}\}$.

4.2.2. The entropic cone. The entropic cone consists of functions on $\mathcal{G}$ that are realizable as entropies of tropical $\mathcal{G}$-diagrams of probability spaces, i.e. it is the image under the entropy map $\text{Ent}_*$ of the tropical cone of minimal diagrams indexed by $\mathcal{G}$:

$$\Gamma(\mathcal{G}) := \text{Ent}_*(\text{Prob}[\mathcal{G}_m])$$

In view of the tropical AEP Theorem 2.1 one can equivalently define

$$\Gamma(\mathcal{G}) := \text{Closure} \left( \text{Ent}_*(\text{Prob}[\mathcal{G}_{m,n}]) \right)$$
where by $\text{Prob}[G]_{m,h}$ we mean the space of minimal, homogeneous, tropical $G$-diagrams. As we explained in Section 2.2.3, when $G = \Lambda_n$, diagrams correspond to $n$-tuples of random variables. In this case, the entropic cone is equal to the closure of the set of entropically representable vectors, i.e. vectors whose coordinates are entropies of the $n$ random variables and their joints, see [Yeu08].

4.2.3. The Abelian cone. The Abelian cone consists of entropy vectors of Abelian tropical diagrams

$$\Gamma_{\text{Ab}}(G) := \text{Ent}_\ast(\text{Prob}[G]_{\text{Ab},m})$$

The following two inclusions follow from the definitions and the fact that entropy satisfies Shannon inequalities.

$$\Gamma_{\text{sm}}(G) \supset \Gamma(G) \supset \Gamma_{\text{Ab}}(G) \tag{4.1}$$

4.2.4. The cases of $G = \Lambda_1$, $\Lambda_2$, and $\Lambda_3$. In this cases all three cones coincide. Essentially it means that any tuple of numbers, that satisfy Shannon inequalities can be realized as entropies of Abelian diagrams, see [ZY98].

4.3. The case $G = \Lambda_4$. The Zhang-Yeung non-Shannon information inequality ([ZY98]) shows that the submodular cone $\Gamma_{\text{sm}}(\Lambda_4)$ is strictly larger than the entropic cone $\Gamma(\Lambda_4)$. It is also known that $\Gamma(\Lambda_4)$ is strictly larger than $\Gamma_{\text{Ab}}(\Lambda_4)$, see for example [Mat07]. Hence, both inclusions in (4.1) are proper.

The cone $\Gamma_{\text{sm}}(\Lambda_4)$ is polyhedral by definition, and it is known that the cone $\Gamma_{\text{Ab}}(\Lambda_4)$ is polyhedral as well, see, for example, [DFZ11]. In contrast, the entropic cone $\Gamma(\Lambda_4)$ is not polyhedral, as has been shown by Matúš in [Mat07].

There are many upper and lower bounds for $\Gamma(\Lambda_4)$. The upper bounds are in the form of linear inequalities, some of them organized in infinite families. A large list can be found in [DFZ11]. Lower bounds are in the form of points in the complement $\Gamma(\Lambda_4) \setminus \Gamma_{\text{Ab}}(\Lambda_4)$.

Note that there is an action of symmetric group $S_4$ on $\Lambda_4$, $\text{Prob}[\Lambda_4]$, $\Gamma_{\text{sm}}(\Lambda_4)$, $\Gamma(\Lambda_4)$ and $\Gamma_{\text{Ab}}(\Lambda_4)$.

We will adopt Matúš’ notations, where an integer (in small bold face) represents the set of its decimal digits (eg $24 \leftrightarrow \{2, 4\} \in \Lambda_4$).

4.3.1. Ingleton inequalities and the Abelian cone $\Gamma_{\text{Ab}}(\Lambda_4)$. In addition to the Shannon inequalities, Abelian diagrams also satisfy six Ingleton inequalities, corresponding to the Ingleton vector

$$\text{ing}(12; 34) := -[1:2] + [1:2|3] + [1:2|4] + [3:4] \in \mathbb{R} \otimes \Lambda_4$$

and five other vectors obtained by permuting the coordinates.

The cone $\Gamma_{\text{Ab}}(\Lambda_4)$ is a polyhedral cone dual to the cone spanned by $\text{SH}$ and six Ingleton vectors. Its structure is well-known: it coincides with the cone called $\text{H}^\oplus$ in [MS95]. It has 35 extremal rays, grouped into ten $S_4$-orbits.
4.3.2. *The submodular cone* \( \Gamma_{\text{sm}}(\Lambda_4) \). We will represent vectors in \( \mathbb{R}^{\Lambda_4} \) by writing their coordinates in the following order

\[
\begin{pmatrix}
1234 \\
123, 124, 134, 234 \\
12, 13, 14, 23, 24, 34 \\
1, 2, 3, 4
\end{pmatrix}
\]

The cone \( \Gamma_{\text{sm}}(\Lambda_4) \) has 41 extremal rays, grouped into eleven \( S_4 \)-orbits: the 35 rays that are extremal for \( \Gamma_{\text{Ab}}(\Lambda_4) \) and six special rays in the \( S_4 \)-orbit of a ray generated by the vector

\[
\text{spc}(12; 34) := \begin{pmatrix}
4, 4, 4, 4 \\
3, 3, 3, 3, 3, 4 \\
2, 2, 2, 2
\end{pmatrix}
\]

Note that \( \text{spc}(12; 34), \text{ing}(12; 34) = -1 \). It is known that \( \text{spc}(12; 34) \) and the other special vectors are not in \( \Gamma(\Lambda_4) \); they are neither representable as entropy vectors of some diagram of probability spaces nor can they be approximated by representable vectors.

4.3.3. *The non-Ingleton cone.* The closure of the complement

\[
\Gamma_{\text{sm}}(\Lambda_4) \setminus \Gamma_{\text{Ab}}(\Lambda_4)
\]

is the union of six cones with disjoint interiors, permuted by the action of \( S_4 \). The stabilizer \( D_2 \) of this action is the dihedral subgroup of \( S_4 \) preserving the partition \( 1234 = 12 \cup 34 \). It has order four and is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Consider one of these cones, containing \( \text{spc}(12; 34) \) and denote it by \( N \). We will call it the non-Ingleton cone. The cone \( N \) has a 14-dimensional simplex as a base. The vertices \( a_1, \ldots, a_{14} \) and the dual faces \( \alpha_1, \ldots, \alpha_{15} \) of the simplex are listed in Table 1.

The covectors \( \alpha_1, \ldots, \alpha_{15} \) give convex coordinates in the simplex.

4.3.4. *The cone* \( \Gamma(\Lambda_4) \). The cone \( \Gamma(\Lambda_4) \) is squeezed between \( \Gamma_{\text{Ab}} \) and \( \Gamma_{\text{sm}} \) and the whole picture is \( S_4 \)-symmetric. Thus the “unknown” part of the \( \Gamma(\Lambda_4) \) is the intersection \( \Gamma' := \Gamma(\Lambda_4) \cap N \). It contains the rays spanned by vectors \( a_1, \ldots, a_{14} \) and therefore the whole face \( \{ \alpha_{15} = 0 \} \). The remaining part of the boundary \( \partial_+ \Gamma' \) is what we are after. From convexity of \( \Gamma' \) it follows that this part of the boundary is the graph of a certain function defined on the cone spanned by \( a_1, \ldots, a_{14} \)

\[
\partial_+ \Gamma' = \{ \alpha_{15} = \Phi(\alpha_1, \ldots, \alpha_{14}) \}
\]

where \( \Phi \) is defined by

\[
\Phi(x_1, \ldots, x_{14}) := \sup \{ \alpha_{15}(x) : (\alpha_1(x), \ldots, \alpha_{14}(x)) = (x_1, \ldots, x_{14}), \ x \in \Gamma(\Lambda_4) \}
\]

Obviously, the function \( \Phi \) is 1-homogeneous.

**Theorem 4.1.** The function \( \Phi \) does not depend on the first four arguments.
| Vertex          | Dual face | Representative                          | $D_2$-orbit |
|-----------------|-----------|----------------------------------------|-------------|
| $a_1 = \begin{pmatrix} 1 \\ 1,1,1,0 \\ 1,1,0,0 \\ 1,0,0,0 \end{pmatrix}$ | $\alpha_1 = [1|234]$ | $l_2 \cdot (\mathbb{Z}_2; \{0\}, \mathbb{Z}_2, \mathbb{Z}_2)$ | $a_1, a_2$ |
| $a_2 = \begin{pmatrix} 0,1,1,1 \\ 0,1,0,1,0 \\ 0,0,1,0 \\ 1 \end{pmatrix}$ | $\alpha_3 = [3|124]$ | $l_2 \cdot (\mathbb{Z}_2; \{0\}, \{0\}, \mathbb{Z}_2)$ | $a_3, a_4$ |
| $a_3 = \begin{pmatrix} 1,1,1,1 \\ 1,1,1,1,0 \\ 1,1,0,0 \\ 1 \end{pmatrix}$ | $\alpha_5 = [1:3|2]$ | $l_2 \cdot (\mathbb{Z}_2; \{0\}, \{0\}, \{0\})$ | $a_5, a_6, a_7, a_8$ |
| $a_4 = \begin{pmatrix} 1,1,1,1 \\ 1,1,1,1,1 \\ 1,1,1,0 \\ 1 \end{pmatrix}$ | $\alpha_9 = [1:2|4]$ | $l_2 \cdot (\mathbb{Z}_2; \{0\}, \{0\}, \{0\})$ | $a_9, a_{10}$ |
| $a_5 = \begin{pmatrix} 1,1,1,1 \\ 1,1,1,1,1 \\ 1,1,1,1 \\ 2 \end{pmatrix}$ | $\alpha_{11} = [3:4]$ | $l_2 \cdot (\mathbb{Z}_2; \{0\}, \{0\}, \{0\})$ | $a_{11}$ |
| $a_6 = \begin{pmatrix} 2,2,2,2 \\ 1,2,2,1,1,2 \\ 1,0,1,1 \\ 3 \end{pmatrix}$ | $\alpha_{12} = [3:4|1]$ | $l_2 \cdot ((\mathbb{Z}_2)^2; \langle \chi_1 \rangle, \langle \chi_1 \chi_2 \rangle)$ | $a_{12}, a_{13}$ |
| $a_7 = \begin{pmatrix} 3,3,3,3 \\ 2,2,2,2,2,2 \\ 1,1,1,1 \\ 4 \end{pmatrix}$ | $\alpha_{14} = [1:2|34]$ | $l_3 \cdot ((\mathbb{Z}_3)^3; \langle \chi_1, \chi_2 \rangle, \langle \chi_2, \chi_3 \rangle, \langle \chi_3, \chi_1 \rangle, \langle \chi_1 + \chi_2 + \chi_3 \rangle)$ | $a_{14}$ |
| $a_8 = \begin{pmatrix} 4,4,4,4 \\ 3,3,3,3,3,4 \\ 2,2,2,2 \\ 4 \end{pmatrix}$ | $\alpha_{15} = -\text{ing}(12;34)$ | Not representable | $a_{15}$ |

Table 1. The vertices and faces of the base simplex of non-Ingleton cone. The dihedral group $D_2$ acts on the simplex by transposing 1 and 2 and, independently, 3 and 4, so we list only one representative in each orbit. To shorten notations we set $l_2 = (\ln 2)^{-1}$ and $l_3 := (\ln 3)^{-1}$. By $(\mathbb{Z}_n)^k$ we mean the direct product of $k$ copies of the cyclic group of order $n$ and $\chi_1, \ldots, \chi_k$ stand for the standard generators in $(\mathbb{Z}_n)^k$. 
Proof: For convenience, for a tropical $\Lambda_4$-diagram we write

$$A_i[\mathcal{X}] := \langle \text{Ent}_i[\mathcal{X}], \alpha_i \rangle$$

e.g. $A_1[\mathcal{X}] = [X_1|X_{234}]$, $A_5[\mathcal{X}] = [X_1 : X_3|X_2]$, etc. Note that all $A_i$'s are Lipschitz-continuous with respect to the input diagram with Lipschitz constant at most 14.

In terms of functionals $A_i$ the definition of the function $\Phi$ can be rewritten as

$$\Phi(x_1, \ldots, x_{14}) := \sup\{A_{15}[\mathcal{X}] : A_i[\mathcal{X}] = x_i \text{ for } 1 \leq i \leq 14; [\mathcal{X}] \in \text{Prob}[\Lambda_4]m\}$$

Consider a minimal tropical $\Lambda_4$-diagram $[\mathcal{X}] = ([X_1],[X_2],[X_3],[X_4])$. It contains an admissible sub-fan $([X_{234}] \leftarrow [X_{1234}] \rightarrow [X_1])$.

Applying Theorem 3.1 to $[\mathcal{X}]$ and parameter $\varepsilon > 0$ we obtain another diagram $[\mathcal{X}']$ such that

$$A_1[\mathcal{X}'] \leq \varepsilon$$
$$|A_i[\mathcal{X}'] - A_i[\mathcal{X}]| \leq 14\varepsilon \quad \text{for } i = 2, \ldots, 15$$

Repeatedly applying Theorem 3.1 to the resulting diagram after circular permutation of terminal spaces we obtain a $\Lambda_4$-diagram

$$[\mathcal{X}'''] = ([X_1'''],[X_2'''],[X_3'''],[X_4'''])$$

such that

$$A_i[\mathcal{X}'''] \leq (3 \cdot 14 + 1)\varepsilon \quad \text{for } i = 1, 2, 3, 4$$
$$|A_i[\mathcal{X}'''] - A_i[\mathcal{X}]| \leq (4 \cdot 14)\varepsilon \quad \text{for } i = 5, \ldots, 15$$

Therefore, for any tuple $(x_1, \ldots, x_{15})$ of non-negative numbers there exists a tuple $(x_1'', \ldots, x_{15}'')$ such that

$$\Phi(x_1, \ldots, x_{14}) \leq \Phi(x_1'', \ldots, x_{14}'') + 56\varepsilon$$
$$x_i'' \leq 43\varepsilon \quad \text{for } i = 1, 2, 3, 4$$
$$|x_i'' - x_i| \leq 56\varepsilon \quad \text{for } i = 5, \ldots, 14$$

Since the function $\Phi$ is convex and therefore continuous, we can pass to the limit with $\varepsilon \to 0$, obtaining the following result. For any tuple $(x_1, \ldots, x_{15})$ of non-negative numbers holds

$$\Phi(x_1, \ldots, x_{14}) \leq \Phi(0, 0, 0, 0, x_5, \ldots, x_{14})$$

On the other hand, given a diagram $[\mathcal{Y}]$ with $A_i[\mathcal{Y}] = 0$, $i = 1, 2, 3, 4$, and a tuple of non-negative numbers $(x_1, x_2, x_3, x_4)$, we can expand the arrows in the four admissible fans, that we described above, to lengths $(x_1, x_2, x_3, x_4)$. The resulting diagram $[\mathcal{Y}''']$ satisfies

$$A_i[\mathcal{Y}'''] = \begin{cases} x_i & i = 1, 2, 3, 4 \\ A_i[\mathcal{Y}] & i = 5, \ldots, 15 \end{cases}$$

This implies

$$\Phi(x_1, \ldots, x_{14}) \geq \Phi(0, 0, 0, 0, x_5, \ldots, x_{14})$$
for any non-negative \((x_1, \ldots, x_{14})\). 

\[\Xi\]

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