GAUGE-INDEPENDENT EXTRACTION OF
THE NEXT-TO-LEADING-ORDER DEBYE MASS
FROM THE GLUON PROPAGATOR

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of the static gluon propagator rather than the zero-momentum limit of
the time-time component of the gluon self-energy, a gauge-independent
result for the next-to-leading order correction can be derived upon resum-
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logarithmically sensitive to the magnetic screening mass.

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1. The Hot QCD Debye Mass Puzzle

In QED, the electric permittivity \( \epsilon(\omega, k) \) is given by the time-time component of the photon self-energy,

\[
\epsilon(\omega, k) = 1 + \frac{\Pi_{00}(\omega, k)}{k^2}.
\]  

In the static limit \( \omega = 0 \), this is the screening factor of longitudinal electric fields,

\[
\langle E_i^i(k) \rangle = -i \frac{k^i J^0}{k^2 + \Pi_{00}(0, k)}
\]  

where \( J^\mu \) is a weak, conserved external current. The long-wavelength limit of \( \Pi_{00}(0, k) \) is usually identified with the Debye screening mass, \( m_{\text{el}}^2 \overset{\text{def}}{=} \Pi_{00}(0, k \to 0) \).

In fact, in the high-temperature limit \( T \gg k \) the leading contribution

\[
\Pi_{00}(0, k) = \frac{e^2 T^2}{3}
\]

provides a momentum-independent mass term, which modifies the classical Coulomb potential by a factor \( e^{-m_{\text{el}} r} \). The leading high-temperature result in QCD is very similar; one just has to replace \( e^2 \to g^2 (N + N_f/2) \) for color group SU(\( N \)) and \( N_f \) fermions.

In QED, higher-order corrections to the long-wavelength limit of \( \Pi_{00}(0, k) \) are known through an exact relation to the equation of state

\[
\Pi_{00}(0, k \to 0) = e^2 \frac{\partial^2 P(\mu, T)}{\delta \mu^2},
\]
where \( P(\mu, T) \) is the pressure and \( \mu \) the chemical potential. Since the first three terms in the perturbation expansion of \( P \) are known, one also knows the first three terms in (4). With \( \mu = 0 \),

\[
\Pi_{00}(0, k \to 0) = \frac{e^2 T^2}{3} \left( 1 - \frac{3e^2}{8\pi^2} + \frac{\sqrt{3}e^3}{4\pi^3} + \ldots \right). \tag{5}
\]

In QCD, one would expect the first correction term to be of relative order \( g \) rather than \( g^2 \), because of the “plasmon effect”. However, after ring resummation a gauge-fixing dependent result is found for \( \Pi_{00}(0, k \to 0) \) in covariant gauges

\[
\Pi_{00}(0, k \to 0) = m_{0el}^2 \left( 1 + \frac{\alpha N}{4\pi} \frac{\sqrt{3}}{2N + N_f}g \right) \tag{6}
\]

with gauge fixing parameter \( \alpha \), which shows that it cannot be identified with a directly measurable quantity that the Debye mass presumably should be.

The reason for the gauge dependence of (6) of course is that the self-energy of non-Abelian gauge fields is a gauge variant quantity. In QED, on the other hand, the self-energy of photons is directly related to the correlator of the gauge-invariant electromagnetic field strengths, so this problem does not arise there.

It was therefore argued that the non-Abelian Debye mass cannot in general be derived from the gluon propagator except in the temporal axial gauge, where the time-time component of the propagator is again directly related to the correlator of two chromoelectric field operators.

The temporal axial gauge, however, is notoriously difficult at finite temperatures, and actually the principal-value prescription commonly used to deal with its singularities at zero frequencies has proved to be flawed. These singularities moreover prevent a straightforward implementation of ring resummation, because the static mode cannot be isolated. A first attempt of a ring-resummed calculation of the non-Abelian Debye mass yielded a negative correction term to the Debye mass of relative order \( g \), but this could not be reproduced by taking the temporal limit of a corresponding calculation in general axial gauge, which by the way gave a positive result. The former was consequently withdrawn and replaced by a calculation which resums the asymptotic gluon mass rather than the leading-order Debye mass (not taking into account vertex corrections, however). This yielded again a negative correction to the Debye mass as defined through the infrared limit of the self-energy in temporal axial gauge, to wit,

\[
\Pi_{00}^{TAG}(0, k \to 0) \approx m_{0el}^2 \left( 1 - \frac{g}{2\pi} \frac{3}{2} (N + N_f/2) \right). \tag{7}
\]

Leaving aside for the moment the open questions about pole prescriptions in temporal axial gauge and even which resummation scheme should be employed, there still remains the question whether the analysis of correlators of chromoelectric
field operators really guarantees to give gauge independent answers where the gluon propagator evidently failed. The temporal axial gauge is singled out as the one where no higher vertex functions are needed to obtain this correlator, but one could of course use any gauge. Because the chromoelectric field is not a gauge-invariant operator, there is a priori no reason to expect gauge fixing in dependence of its correlation functions. Indeed, one finds that under a change of gauge condition \( f_\mu A^\mu \rightarrow (f_\mu + \delta f_\mu) A^\mu \) the correlator of two chromoelectric field operators varies according to

\[
\delta \langle E^a_j(x)E^e_k(y) \rangle = -gf^{abc} \int d^4z \langle E^b_j(x)\bar{c}^c(x)c^d(z)\delta f_\mu A^{d\mu}(z)E^e_k(y) \rangle + (a, j, x \leftrightarrow e, k, y),
\]

where \( \bar{c} \) and \( c \) are Faddeev-Popov ghost fields, which make their appearance even in gauges which are otherwise ghost-free.

Thus one is not necessarily on safe grounds by studying the correlation of field operators rather than of the gauge fields. On the other hand, if one succeeded in extracting gauge independent information from the gauge variant quantity \( \Pi_{00} \), then this information should equally be found in the correlator of electric field operators, since in a particular gauge (the temporal one) the two are directly related.

2. Changing the Definition of the Debye Mass

A strong hint that the very definition of the Debye mass as \( \Pi_{00}(0, k \rightarrow 0) \) might not be sufficient beyond leading order is seen by the consequences of keeping the term proportional to \( k^2 \) in the resummed result (6). Then

\[
\Pi_{00}(0, k \rightarrow 0) = m^2_{0el} \left( 1 + \alpha \frac{N}{4\pi} \sqrt{\frac{6}{2N + N_f}} g \right) - \frac{2N}{3\pi} \sqrt{\frac{6}{2N + N_f}} g k^2 + \ldots,
\]

which, when inserted into (2) leads to a different mass term besides an over-all factor that is constant. This does not remove the gauge dependence of the putative correction to the Debye mass (it just replaces \( \alpha \) by \( \alpha + \frac{8}{3} \)), but it does tell that the \( k \)-dependence of \( \Pi_{00}(0, k) \) still can change things!

Let us therefore go back to the linear response formula for longitudinal electric fields, eq. (2). Actually, this formula is valid also in the non-Abelian case, since with a single source \( J \) there is also only a single direction in color space to which the gauge potentials can point, and the nonlinear terms in the chromoelectric field strength vanish trivially. If \( J_0 \) is a static point charge \( Q \) located at the origin, then the Fourier transform of eq. (2) equals minus the gradient of a potential \( \Phi \) given by

\[
\Phi(r) = Q \int \frac{d^3k}{(2\pi)^3} \frac{e^{ikr}}{k^2 + \Pi_{00}(k_0 = 0, k)}
\]

\[
= \frac{Q}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{e^{ikr} - e^{-ikr}}{2ir} \frac{k dk}{k^2 + \Pi_{00}(0, k)}.
\]
The usual definition of the Debye mass as \( m_{\text{el}}^2 \stackrel{\text{def}}{=} \Pi_{00}(0, k \to 0) \) is commonly motivated by saying that when \( r \) is very large, the dominant contribution to the integral comes from \( k = 0 \). This is not quite correct, however. Inserting for instance the leading high-temperature result (3), which is a simple constant mass squared, one can readily evaluate by appropriately closing the contour in the complex \( k \)-plane wherein the integrand has simple poles at \( k = \pm i m_{\text{el}} \), where only \( \Re k = 0 \). If \( \Pi_{00}(0, k) \) depends on \( k \), as it will be the case in general (i.e., beyond leading order), then this dependence will be important to determine the location of the pole and thus the magnitude of the mass that will appear in the exponent of \( e^{-mr} \). It is therefore rather obvious that one should define the Debye mass, which certainly should account for the actual exponential fall-off, self-consistently by the zeros of the denominator in (10). I therefore propose the implicit definition

\[
m_{\text{el}}^2 = \Pi_{00}(0, k) \bigg|_{k^2 = -m_{\text{el}}^2} \tag{11}
\]

for the (chromo)electric Debye mass.

The Debye mass is thus defined through the singularity of the propagator appearing in (10), and this propagator is in fact one of the components of the (static) gauge-field propagator for which formal arguments showing gauge independence of its poles can be derived from gauge fixing identities. In non-Abelian gauge theories, the gauge dependence of the usual definition of the Debye mass through the zero-momentum limit of the self-energy is only to be expected because the leading high-temperature corrections move the pole away from strictly \( k = 0 \).

One might now wonder why there appeared to be no problem with the old definition in the case of QED. In QED there is no problem of gauge dependence for the photon self-energy, for the latter is gauge independent through all orders of perturbation theory. Nevertheless, if one wants the Debye mass to truly describe the exponential fall-off of the electrostatic potential, then it is clear that one has to adopt the self-consistent definition (11). But then there is a correction term to be added to (5) according to

\[
m_{\text{el}}^2 = \Pi_{00}(0, k \to 0) + \left[ \Pi_{00}(0, k)|_{k^2 = -m_{\text{el}}^2} - \Pi_{00}(0, k \to 0) \right], \tag{12}
\]

and this correction is

\[
m_{\text{el}}^2 - \Pi_{00}(0, k \to 0) = m_{\text{el}}^2 \left( \frac{2e^2}{9\pi^2} - \frac{e^2}{6\pi^2} \left[ \ln \frac{\bar{\mu}}{\pi T} + \gamma_E \right] + O(e^4) \right), \tag{13}
\]

where \( \bar{\mu} \) is the mass scale introduced by dimensional regularization in which minimal subtraction has been performed. This correction term shows what has been missing in the first place when trying to identify with a physical quantity. As a physical quantity, it has to be a renormalization-group invariant, which is indeed what (13) brings about: the coefficient of the logarithmic term in (13) is exactly such that \( \partial m_{\text{el}}^2 / \partial \bar{\mu} = 0 \) because \( de/d(\ln \bar{\mu}) = \beta(e) = e^3/(12\pi^2) + O(e^4) \).
In fact, the importance of the $k$-dependence of $\Pi_{00}(0, k)$ for Debye screening has been recognized previously in the case of a degenerate electron gas, giving rise to the so-called Friedel oscillations. Yet the self-consistent definition (11) of the Debye mass has not been adopted before as far as I know.

3. Next-to-Leading Order Calculation of the Non-Abelian Debye Mass

A complete calculation of perturbative corrections to the high-temperature dispersion laws beyond those determined by the gauge-independent hard thermal loops (HTL’s) has been shown by Braaten and Pisarski\cite{14} to require the resummation of all of the HTL contributions to self-energy and vertices. This rather involved resummation scheme has been applied in particular to determine the damping of collective excitations\cite{13}, and most recently also to compute the next-to-leading order contribution to the chromodynamical plasma frequency\cite{15}.

The problem of calculating the next-to-leading order contribution to the non-Abelian Debye mass is in fact a problem of the same kind. With the definition (11), the Debye mass can be regarded as being given by the position of the plasmon pole when the frequency is lowered below the plasma frequency and eventually put to zero, whilst the wavevector becomes imaginary.

Fortunately, with zero external frequencies the Braaten-Pisarski scheme can be simplified tremendously\cite{17}. In the imaginary time formalism, one may separate the modes of a resummed bosonic propagator $1/[(2\pi n T)^2 + k^2 + \Pi]$ into the static mode $n = 0$ and the non-static ones, where the latter are automatically hard in the terminology of Braaten and Pisarski. Hence, only the static mode needs resummation of the HTL, which in the propagator is just the leading-order Debye mass $m_{\text{rel}}$. This separation of course makes analytic continuation to non-zero external frequencies rather impossible, but this is no shortcoming in the static case. With all the external frequencies being zero, all potentially soft lines of a diagram are static, and so are the vertices that would need resummation of HTL contributions. However, the latter vanish entirely in the purely static limit.

Hence, as far as the relative order $g$ correction to the static gluon self-energy is concerned, Braaten-Pisarski resummation boils down to a conventionally ring-resummed\cite{18} one-loop calculation. The static ring-resummed propagator in general covariant gauge reads

$$\Delta_{\mu\nu}\big|_{p_0=0} = \left[ \frac{1}{p^2 + m_0^2} \delta^0_{\mu} \delta^0_{\nu} + \frac{1}{p^2} \left( \eta_{\mu\nu} - \delta^0_{\mu} \delta^0_{\nu} + \frac{P_\mu P_\nu}{p^2} \right) + \alpha \frac{P_\mu P_\nu}{(p^2)^2} \right]_{p_0=0}, \quad (14)$$

and the complete next-to-leading order contribution to $\Pi_{00}(0, k)$ is found as\cite{18}

$$\delta\Pi_{00}(k_0 = 0, k) = gmN \sqrt{\frac{6}{2N + N_f}} \int \frac{d^3-2\epsilon p}{(2\pi)^{3-2\epsilon}} \left( \frac{1}{p^2 + m^2} + \frac{1}{p^2} \right).$$


\[
\frac{4m^2 - (k^2 + m^2)[3 + 2pk/p^2]}{p^2(q^2 + m^2)} + \alpha(k^2 + m^2) \frac{p^2 + 2pk}{p^4(q^2 + m^2)}, \quad (15)
\]

where \( m \equiv m_{\text{qel}} \) and \( q = p + k \). (Here dimensional regularization has been used when separating the static modes from the sum over Matsubara frequencies; the limit \( \epsilon \to 0 \) gives a regular expression because of the odd integration dimension.)

The new definition (11) requires to evaluate at \( k^2 = -m^2 \). There the gauge parameter \( \alpha \) dependent part vanishes algebraically, in accordance with the gauge fixing identities. However, closer inspection reveals that the integrals in (15) develop “mass-shell” singularities, caused by massless transverse and massless unphysical modes in the gluon propagator.

The third term of the integrand in (15) is logarithmically singular as \( k^2 \to -m^2 \), and the singularity is caused exclusively by the massless denominator in the spatially transverse part of the gluon propagator (14). A magnetic screening mass would remove this singularity, and because the latter is only logarithmic, the coefficient of the corresponding logarithm is determined by (15),

\[
\delta \Pi_{00}(0, k) \mid_{k^2 \to -m^2_{\text{qel}}} \rightarrow g^2 N m_{\text{qel}} T \ln \frac{2m_{\text{qel}}}{m_{\text{magn.}}}, \quad (16)
\]

up to terms that are regular as \( m_{\text{magn.}} \to 0 \). Assuming that \( m_{\text{magn.}} \sim g m_{\text{qel}} \), the next-to-leading order contribution to \( m_{\text{qel}}^2 \) is found to be of order \( g \ln(1/g) \) rather than \( g \),

\[
\frac{\delta m_{\text{qel}}^2}{m_{\text{qel}}^2} = \frac{N}{2\pi} \sqrt{\frac{6}{2N + N_f}} g \ln \frac{1}{g} + O(g), \quad (17)
\]

and it is positive, at least at weak coupling \( g \ll 1 \). Taken seriously for larger coupling \( g \sim 1 \), the logarithm would eventually switch sign, but there the sublogarithmic terms would be of equal importance.

Unfortunately, the sublogarithmic terms cannot be calculated completely, because the presumed phenomenon of magnetic screening is nonperturbative. However, in order to obtain an estimate of those, let us assume that a simple replacement of \( 1/k^2 \to 1/(k^2 + m_{\text{magn.}}^2) \) in the transverse part of the static propagator (14) correctly summarizes the effects at \( k \sim g^2 T \). Then we may go on to evaluate the remaining contributions in (13). Here one encounters a difficulty with the \( \alpha \) dependent term in Eq. (15), because by approaching the imaginary pole \( k^2 \to -m^2 \), the explicit factor that apparently ensures gauge independence gets cancelled by a linear singularity in the momentum integral. Exactly the same phenomenon was encountered in the recalculation of plasmon damping rates in general covariant gauges. I have argued previously that this behaviour just reflects a singular, gauge dependent behaviour of the residue of the propagator rather than an actual gauge dependence of the pole determining the dispersion laws. Indeed, introducing an (unphysical) cut-off again moves the gauge dependence seemingly affecting the pole position into the residue, while the correction to the pole position becomes independent of this infrared regularization. Alternatively, the gauge dependent contributions can be avoided altogether by a quantization procedure that keeps unphysical
modes unthermalized. The gauge independent correction term then reads

\[ \delta \equiv \frac{\delta m^2_{\text{el}}}{m^2_{\text{el}}} = gN \sqrt{\frac{6}{2N + N_f}} \frac{1}{2\pi} \left( \ln \frac{m_{\text{el}}}{m_{\text{magn.}}} + \ln 2 - \frac{1}{2} \right) + O(g^2). \] (18)

In pure SU(2) lattice gauge theory high-statistics results on Debye screening have rather recently been obtained at temperatures well above the critical temperature, finding a positive excess in the screening mass squared of \( \delta = +0.30(9) \). Inserting the parameters of this lattice calculation as well as an older lattice result for the magnetic screening mass, (18) yields \( \delta \approx +0.5 \), which comes remarkably close in view of \( g \approx 1 \). (The renormalized value of \( g \) used in this calculation appears to be in good agreement with independent lattice calculations of the SU(2) pressure when the latter is equated to the perturbative result.)

In conclusion, by defining the Debye mass through the relevant pole of the static gluon propagator rather than the (gauge-dependent) zero-momentum limit of the gluon self-energy, a gauge-independent result at next-to-leading order has been derived after identifying and resumming the relevant hard-thermal-loop contributions. The location of the pole turns out to be logarithmically sensitive to the nonperturbative magnetic screening mass, but the coefficient of the corresponding logarithm can be calculated perturbatively. The latter is in fact related to a logarithmic divergence \( \sim gm_{\text{el}} \ln(m_{\text{el}}r) \) encountered by Nadkarni when trying to extract corrections to the non-Abelian Debye mass from the correlation function of two Polyakov loops.

A similar logarithmic sensitivity to the scale \( g^2T \) has been encountered in the calculations of plasmon and fermion damping, which even appears in the Abelian case where there is no magnetic screening mass. In the case of the Debye mass, however, the origin of the logarithmic term is genuinely non-Abelian. Another difference which is important in view of the discussions surrounding the calculations on damping is that the position of the pole defining the Debye mass is on the imaginary axis of \( k \), and it stays there when the corrections are included. This makes it rather unnatural to attempt anything else than a self-consistent procedure. As concerns the gauge dependences which in both calculations require regularization of mass-shell singularities (unless unphysical modes are frozen), it should be kept in mind that they occur in the sublogarithmic terms which are strictly speaking beyond the reach of perturbation theory, at least in the non-Abelian case. It is therefore left open whether the ticklish “mass-shell singularities” caused by the massless unphysical modes in covariant gauges might not disappear in a calculation which is complete down to and including the order \( k \sim g^2T \).

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