Canonical formulation treatment of a free relativistic spinning particle

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Abstract

The Hamilton - Jacobi method of constrained systems is discussed. The equations of motion for a free relativistic spinning particle are obtained without using any gauge fixing conditions. The quantization of this model is discussed.

1 Introduction

The theory of constrained systems was developed by Dirac [1,2] and is becoming the fundamental tool for the study of classical systems of particles and fields [3,4]. In his method Dirac distinguish between two types of constraints, first and second class constraints. As there is an even number of second class constraints, these are used to eliminate a certain number of canonically conjugate pairs of variables. The Dirac Hamiltonian is then the canonical Hamiltonian plus first class constraints, which are considered as generators of gauge transformations. To eliminate the gauge freedom, one has to impose an external gauge fixing conditions for each first class constraint and evaluate the new Dirac bracket.

Recently, the canonical method [5-7] has been developed to investigate constrained systems. The equations of motion are obtained as total differential equations in many variables which require the investigation of integrability conditions. If the system is integrable, one can solve the equations of motion without using any gauge fixing conditions. In this paper, we shall treat the free relativistic spinning particle as a constrained system. In fact, this work is a continuation of previous paper [5] in which we have obtained the equations of motion for a free relativistic spineless particle and the canonical phase space coordinates \( q_i, p_i \) are obtained in terms of parameter \( x_0 \).
Now we would like to give a brief discussion of the canonical method [5-7]. This method gives the set of Hamilton - Jacobi partial differential equations [HJPDE] as

$$H'_{\alpha}(t, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_a}) = 0,$$

$$\alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r,$$

(1)

where

$$H'_{\alpha} = H_\alpha(t, q_a, p_a) + p_\alpha,$$

(2)

and $H_0$ is defined as

$$H_0 = p_a w_a + p_\mu q_\mu |_{p_\nu = -H_\nu - L(t, q_\mu, q_\nu, q_a = w_a)},$$

$$\mu, \nu = n - r + 1, ..., n.$$

(3)

The equations of motion are obtained as total differential equations in many variables as follows:

$$dq_a = \frac{\partial H'_{\alpha}}{\partial p_a} dt_a;$$

(4)

$$dp_a = -\frac{\partial H'_{\alpha}}{\partial q_a} dt_a;$$

(5)

$$dp_\beta = -\frac{\partial H'_{\alpha}}{\partial t_\beta} dt_a;$$

(6)

$$dZ = (-H_a + p_a \frac{\partial H'_{\alpha}}{\partial p_a}) dt_a;$$

(7)

$$\alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r$$

where $Z = S(t_a; q_a)$. The set of equations (4-6) is integrable [5] if

$$dH'_{0} = 0,$$

(8)

$$dH'_{\mu} = 0, \mu = n - p + 1, ..., n.$$

(9)

If conditions (8) and (9) are not satisfied identically, one considers them as new constraints and again tests the consistency conditions. Hence, the canonical formulation leads to obtain the set of canonical phase space coordinates $q_a$ and $p_a$ as functions of $t_a$, besides the canonical action integral is obtained in terms of the canonical coordinates. The Hamiltonians $H'_{\alpha}$ are considered as the infinitesimal generators of canonical transformations given by parameters $t_\alpha$ respectively.

For the quantization of constrained systems we can use the Dirac’s method of quantization [1]. In this case we have

$$H'_\alpha \Psi = 0, \quad \alpha = 0, n - r + 1, ..., n,$$

(10)
where $\Psi$ is the wave function. The consistency conditions are

$$[H'_\mu, H'_\nu] \Psi = 0, \quad \mu, \nu = 1, ..., r,$$

where $[,]$ is the commutator. The constraints $H'_\alpha$ are called first-class constraints if they satisfy

$$[H'_\mu, H'_\nu] = C^\gamma_{\mu\nu} H'_\gamma,$$  \hspace{1cm} (12)

In the case when the Hamiltonians $H'_\mu$ satisfy

$$[H'_\mu, H'_\nu] = C_{\mu\nu},$$  \hspace{1cm} (13)

with $C_{\mu\nu}$ do not depend on $q_i$ and $p_i$, then from (11) there arise naturally Dirac’s brackets and the canonical quantization will be performed taking Dirac’s brackets into commutators.

On the other hand, the path integral quantization is an alternative method to perform the quantization of constrained systems. If the system is integrable then one can solve equations (4-6) to obtain the canonical phase-space coordinates as

$$q_a \equiv q_a(t, t_\mu), \quad p_a \equiv p_a(t, t_\mu), \quad \mu = 1, ..., r,$$  \hspace{1cm} (14)

then we can perform the path integral quantization using Muslih method [8-12] with the action given by (7).

After this introduction we shall treat the relativistic spinning particle as a constrained system and demonstrate the fact that gauge fixing is solved naturally if the canonical method is used.

2 The free relativistic spin particle as a constrained system

Let us consider the action of the free relativistic spinning particle as [4]

$$S = \int L d\tau,$$  \hspace{1cm} (15)

$$L = -\frac{\dot{x}^2}{2e} + \frac{i\dot{x}\chi\psi}{e} - \frac{em^2}{2} - i\psi\dot{\psi} + i\psi_5\dot{\psi}_5 + im\chi\psi_5,$$  \hspace{1cm} (16)

where $x^\mu$, $e$ are even variables and $\psi^\mu$, $\chi$, $\psi_5$ are odd variables. With the action (15) there exist two types of gauge transformations:

$$\delta x = \dot{x} \zeta, \quad \delta e = \frac{d}{d\tau}(e\zeta), \quad \delta \psi_5 = \dot{\psi}_5 \zeta, \quad \delta \chi = \frac{d}{d\tau}(\chi \zeta), \quad \delta \psi_5 = \dot{\psi}_5 \zeta,$$  \hspace{1cm} (18)
where $\zeta(\tau)$ are even parameters, while $\epsilon(\tau)$ are odd parameters. The canonical momenta are defined as

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu} = -\frac{1}{e}(\dot{x}_\mu - i\chi\psi_\mu),$$  \hspace{1cm} (19)

$$\pi_e = \frac{\partial L}{\partial \dot{e}} = 0 = -H_1,$$  \hspace{1cm} (20)

$$\pi_\chi = \frac{\partial_r L}{\partial \dot{\chi}} = 0 = -H_2,$$  \hspace{1cm} (21)

$$\pi_\mu = \frac{\partial_r L}{\partial \dot{\psi}_\mu} = -i\dot{\psi}_\mu = -H_3,$$  \hspace{1cm} (22)

$$\pi_5 = \frac{\partial_r L}{\partial \dot{\psi}_5} = i\dot{\psi}_5 = -H_4,$$  \hspace{1cm} (23)

where $\partial_r$ means right derivatives and the metric convention $g = (+1, -1, -1, -1)$. Now equation (19) leads us to obtain the velocities $\dot{x}_\mu$ in terms of momenta and coordinates as

$$\dot{x}_\mu = (-ep_\mu + i\chi\psi_\mu) = w_\mu.$$  \hspace{1cm} (24)

The canonical Hamiltonian $H_0$ can be obtained as

$$H_0 = p_\mu w_\mu - \dot{e}H_1 - \dot{\chi}H_2 - \dot{\psi}_\mu H_3 - \dot{\psi}_5 H_4 - L,$$  \hspace{1cm} (25)

$$H_0 = -\frac{e}{2}(p^2 - m^2) + i\chi(\psi \cdot p - m\psi_5).$$  \hspace{1cm} (26)

Making use of equations (1,2) and (26), the set of Hamilton- Jacobi partial differential equations reads

$$H'_0 = p^{(\tau)} + H_0 = 0; \quad p^{(\tau)} = \frac{\partial S}{\partial \tau},$$  \hspace{1cm} (27)

$$H'_e = \pi_e = 0; \quad \pi_e = \frac{\partial S}{\partial e},$$  \hspace{1cm} (28)

$$H'_\chi = \pi_\chi = 0; \quad \pi_\chi = \frac{\partial S}{\partial \chi},$$  \hspace{1cm} (29)

$$H'_\mu = \pi_\mu + i\dot{\psi}_\mu = 0; \quad \pi_\mu = \frac{\partial S}{\partial \psi_\mu},$$  \hspace{1cm} (30)

$$H'_5 = \pi_5 - i\dot{\psi}_5 = 0; \quad \pi_5 = \frac{\partial S}{\partial \psi_5}.$$  \hspace{1cm} (31)

This set leads to the total differential equations as

$$dx_\mu = (-ep_\mu + i\chi\psi_\mu) d\tau,$$  \hspace{1cm} (32)

$$dp_\mu = 0.$$  \hspace{1cm} (33)
\[ d\pi_e = -\frac{1}{2}(p^2 - m^2)d\tau, \quad (34) \]
\[ d\pi\chi = i(\psi \cdot p - m\psi_5)d\tau, \quad (35) \]
\[ d\pi_\mu = -i\chi p_\mu d\tau + i\psi_\mu, \quad (36) \]
\[ d\pi_5 = i\chi d\tau - i\psi_5, \quad (37) \]
\[ dp(\tau) = 0. \quad (38) \]

To check whether the set of equations (32-38) is integrable or not let us consider the total variations of \( H'_0, H'_1, H'_2, H'_3 \) and \( H'_4 \). In fact, the total variations of \( H'_3 \) and \( H'_4 \) determine \( d\psi_\mu \) and \( d\psi_5 \) in terms of \( d\tau \) as

\[ d\psi_\mu = \frac{1}{2}\chi p_\mu d\tau, \quad (39) \]
\[ d\psi_5 = \frac{m}{2}\chi d\tau. \quad (40) \]

The total variation of \( H'_1 \) leads to the constraint \( H'_5 \) as

\[ H'_5 = (p^2 - m^2), \quad (41) \]

the total variation of \( H'_4 \) is identically zero and the total variation of \( H'_2 \) leads to the constraint \( H'_6 \) as

\[ H'_6 = (\psi \cdot p - m\psi_5). \quad (42) \]

Taking the total variations of \( H'_6 \) one may obtain

\[ dH'_6 = \psi_\mu dp^\mu + p^\mu d\psi_\mu - m\psi_5, \quad (43) \]
\[ dH'_6 = \frac{1}{2}\chi(p^2 - m^2)d\tau. \quad (44) \]

Calculations shows that the total variations of \( H'_0 \) is zero. The set of equations (32-38) is integrable and the phase space coordinates \( (x_\mu, \psi_\mu, \psi_5) \) and \( (p_\mu, \pi_\mu, \pi_5) \) are obtained in terms of independent parameters \( (\tau, e, \chi) \). \( H'_0, H'_1, H'_2 \) and \( H'_3 \) can be interpreted as infinitesimal generators of canonical transformations given by parameters \( \tau, e \) and \( \chi \) respectively.

Now we would like to discuss the operator quantization of the free relativistic spinning particle. The anti commutator relations corresponding to the pseudoclassical brackets of \( \psi_\mu \) and \( \psi_5 \) are

\[ [\psi_\mu, \psi_\nu] = g_{\mu\nu}, \quad [\psi_5, \psi_5] = -1, \quad [\psi_\mu, \psi_5] = 0, \quad (45) \]

which may realized as

\[ \psi_\mu = \frac{1}{\sqrt{2}}\gamma_5\gamma_\mu, \quad \psi_5 = \frac{1}{\sqrt{2}}\gamma_5. \quad (46) \]

Here \( \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \), such that \( \gamma_5^2 = -1, \quad [\gamma_5, \psi_\mu]_+ = 0. \)
Consider the physical Hilbert space as a subspace of a Hilbert space \( \mathbb{R} \) in which the fields \( x^\mu, \psi_\mu, \psi_5 \) and their momenta are not submitted to the constraint. Then only those states of \( \mathbb{R} \) which fulfill

\[
(p^2 - m^2) | \psi_{\text{phys}} \rangle = 0, \quad (47)
\]
\[
\gamma_5 (p_\mu \gamma^\mu - m) | \psi_{\text{phys}} \rangle = 0, \quad (48)
\]

belong to the physical Hilbert space. One should notice that equation (48) is nothing but the Dirac equation.

Now to obtain the path integral quantization of this system, we can use equation (7) to obtain the canonical action as

\[
S = \int \left\{ \left[ -\frac{e}{2} (p^2 - m^2) - i \chi (\psi \cdot p - m \psi_5) + p_\mu (-ep^\mu + i \chi \psi^\mu) \right] d\tau + \left[ -i \psi_\mu + \pi_\mu \right] d\psi_\mu + \left[ i \psi_5 + \pi_5 \right] d\psi_5 \right\}. \quad (49)
\]

Making use of the equations of motion (39,40) and using the definition of the canonical momenta, one can recover the original action.

3 Conclusion

A free relativistic spinning particle is treated as a constrained system. The canonical method [5-7] treatment of this system leads us to obtain the equations of motion as total differential equations in many variables. Since the integrability conditions are satisfied, this system is integrable. Hence, one can solve the classical dynamics of this system in terms of parameters \( \tau, e, \chi \) without using any gauge fixing conditions. Although \( e \) and \( \chi \) are treated as coordinates in the Lagrangian, the presence of constraints and the integrability conditions force us to treat them as parameters like \( \tau \).

The operator quantization of the relativistic spinning particle leads to obtain the Dirac equation, which describe simultaneously the particle and the anti-particle.

Unlike conventional methods one can perform the path integral quantization of this system using Muslih method to obtain the action directly without considering any Lagrange multipliers.

In fact, Dirac’s method treatment needs gauge fixing conditions to determine the classical dynamics of the constrained system. In the the free relativistic spinning particle system, since the number of the first class constraints are four, one has to impose four supplementary gauge fixing conditions.

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