Squares that look round:
Transforming Spherical Images

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But first... Himmeli

Photo credit: http://kaylovesvintage.blogspot.de
But first… Himmeli

Joint work with Marco Mahler.
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Equirectangular projection
Equirectangular projection
Stereographic projection  \( \rho : S^2 \to \mathbb{C} \)

\[
\rho(u, v, w) = \frac{u + iv}{1 - w}
\]
Stereographic projection
Transform by $z \mapsto 2z$ (or pull back by $z \mapsto z/2$)
Pull back by $z \mapsto z^2$
Pull back by $z \mapsto z^2$
The Droste effect
The Droste effect
Droste annulus
Apply log, then tile horizontally, apply exp.
Twisted Droste effect (Escher, De Smit-Lenstra)
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Other kinds of “twist”, in analogy with the Droste effect

The Weierstrass $\wp$–function (for the square lattice) can be given as

$$\wp_i(z) = \frac{1}{z^2} + \sum' \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right),$$

where the sum ranges over the non-zero Gaussian integers $w \in \mathbb{Z}[i]$. 
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where the sum ranges over the non-zero Gaussian integers \( w \in \mathbb{Z}[i] \).

The function is doubly periodic:

\[
\wp_i(z + 1) = \wp_i(z + i) = \wp_i(z),
\]

so we can view it as a map from the torus to \( \widehat{\mathbb{C}} \).
Charles Sanders Pierce used the Weierstrass $\wp$–function on spherical images in 1879.
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https://skfb.ly/NJRx
Our version of a torus Earth

https://skfb.ly/MYpC
Our version of a torus Earth

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Tile, take a different square

Scale by $1 + i$
Tile, take a different square, then map back to the sphere using a Schwarz-Christoffel map.

Scale by $1 + i$, composition is $z \mapsto \frac{i}{2}(-z + 1/z)$. 
Tile, take a different square, then map back to the sphere using a Schwarz-Christoffel map.

Scale by 2, composition is $z \mapsto \frac{(z^2+1)^2}{4z(z^2-1)}.$
Tile, take a different square, then map back to the sphere using a Schwarz-Christoffel map.

Scale by $2 + i$, composition is $z \mapsto z \frac{((-1+2i)+z^2)^2}{(-i+(2+i)z^2)^2}$. 

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Hexagonal variation

Instead, we can pull back by the Weierstrass function $\wp_\omega$, where $\omega = e^{\pi i/3}$, giving a hexagonal torus.
Tile, take a different hexagon

Scale by $1 + \omega$
Tile, take a different hexagon, then map back to the sphere using a Schwarz-Christoffel map.

Scale by $1 + \omega$, composition is $z \mapsto \frac{z^3 + \sqrt{2}}{3\omega \cdot z^2}$.
Book: Visualizing Mathematics with 3D Printing

http://3dprintmath.com
Thanks!

- Videos at youtube.com/user/henryseg
- Paper at http://archive.bridgesmathart.org/2016/bridges2016-15.html
- (Some) source code at github.com/henryseg/spherical_image_editing