LOCAL ORBIT TYPES OF THE ISOTROPY REPRESENTATIONS FOR SEMISIMPLE PSEUDO-RIEMANNIAN SYMMETRIC SPACES

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Abstract. We list up all the possible local orbit types of hyperbolic or elliptic orbits for the isotropy representations of semisimple pseudo-Riemannian symmetric spaces. It is key to give a recipe to determine the local orbit types of hyperbolic principal orbits by using three kind of restricted root systems and Satake diagrams associated with semisimple pseudo-Riemannian symmetric spaces.

Introduction

Let $G/H$ be a semisimple pseudo-Riemannian symmetric space. The isotropy representation of $G/H$ is called an $s$-representation. In this paper, we investigate all the possible local orbit types (i.e., the conjugate classes of isotropy subalgebras) of hyperbolic or elliptic orbits for the $s$-representation of $G/H$ by using three kind of restricted root system associated with $G/H$. An orbit is said to be hyperbolic principal (resp. elliptic principal) if it is a hyperbolic orbit (resp. an elliptic orbit) whose local orbit type is the smallest one among the hyperbolic orbits (resp. the elliptic orbits). We also investigate the local orbit type of a hyperbolic principal orbit and an elliptic principal orbit by using the three kind of Satake diagrams associated with $G/H$. The present work is based on the paper [1]. In 1992, Heintze and Olmos ([11]) determined the isotropy subalgebras of the orbits in the case where $G/H$ is Riemannian. Moreover, in 2007, Boumuki ([1]) gave the isotropy subalgebras of elliptic orbits in the case where $G/H$ is a semisimple Lie group.

We state the main result of this paper. Let $G/H$ be a semisimple pseudo-Riemannian symmetric space. Denote by $\mathfrak{g}$ (resp. $\mathfrak{h}$) the Lie algebra of $G$ (resp. $H$). Let $\sigma$ be an involution of $\mathfrak{g}$ whose fixed point set coincides with...
\[ h. \] Set \( q := \{ X \in g \mid \sigma(X) = -X \} \), which is identified with the tangent space of \( G/H \) at \( eH \). Here \( e \) is the identity element of \( G \). Let \( a \) be a vector-type maximal split abelian subspace of \( q \). Denote by \( \Delta \) the restricted root system of \( G/H \) with respect to \( a \). Let \( \Psi \) be a simple root system of \( \Delta \), and \( \Delta_+ \) be the positive root system of \( \Delta \) with respect to \( \Psi \). Set, for any \( \Theta \subset \Psi \), \( h_\Theta := h_0 + \sum_{\lambda \in \Delta_0 \cap \Delta_+} h_\lambda \), where \( h_0 \) denotes the centralizer of \( a \) in \( h \), \( h_\lambda \) denotes the root subspace of \( h \) for \( \lambda \), and \( \Delta_0 := \Delta \cap \sum_{\lambda \in \Theta} R\lambda \). Denote by \( [h_\Theta] \) the conjugate class of \( h_\Theta \), and by \( (m^+(\lambda), m^-(\lambda)) \) the signature of \( \lambda \in \Delta \). Let \( \mathcal{W}(\Delta) \) be the Weyl group of \( \Delta \) and \( \mathcal{W}(\Delta^a) \) be the Weyl group of \( \Delta^a \), where \( \Delta^a := \{ \lambda \in \Delta \mid m^+(\lambda) > 0 \} \). Then we have the following result.

**Theorem A.** Let \( w_1, w_2, \ldots, w_l \) be a complete system of representatives for \( \mathcal{W}(\Delta)/\mathcal{W}(\Delta^a) \). The set of all local orbit types of the hyperbolic orbits for the \( s \)-representation of \( G/H \) coincides with

\[
\bigcup_{i=1}^{l} \{ [h_\Theta] \mid \Theta \subset w_i \cdot \Psi \},
\]

where \( w_i \cdot \Psi \) denotes the set of all the functions \( w_i \cdot \lambda(\lambda \in \Psi) \) defined by \( w_i \cdot \lambda(A) = \lambda(w_i^{-1}A) \) for all \( A \in a \).

A major difficulty in indefinite case arises from the fact that \( \mathcal{W}(\Delta) \) is not necessarily isomorphic to \( N_H(a)/Z_H(a) \), where \( N_H(a) \) (resp. \( Z_H(a) \)) denotes the normalizer (resp. the centralizer) of \( a \) in \( H \). Note that \( w_i \)'s \( (w_i \in \mathcal{W}(\Delta)/\mathcal{W}(\Delta^a)) \) are not necessarily preserve the signatures of the roots in \( \Psi \) invariantly, so that, for \( 1 \leq i \neq j \leq l \), \( \{ [h_\Theta] \mid \Theta \subset w_i \cdot \Psi \} \) and \( \{ [h_\Theta] \mid \Theta \subset w_j \cdot \Psi \} \) are not necessarily equal to each other (cf. Example 5.1). In positive definite case, \( \mathcal{W}(\Delta) \) is equal to \( \text{Ad}(H)_a \) and therefore \( l = 1 \) holds.

For classical-type semisimple pseudo-Riemannian symmetric spaces, we give the lists of the indices of \( \mathcal{W}(\Delta)/\mathcal{W}(\Delta^a) \) (cf. Table 2). From Theorem A in order to determine the set of local orbit types, it is sufficient to determine \( \{ [h_\Theta] \mid \Theta \subset w_i \cdot \Psi \} \) for each \( i \in \{ 1, 2, \ldots, l \} \). In [11], we gave a recipe to determine \( \{ [h_\Theta] \mid \Theta \subset \Pi \} \) for a simple root system \( \Pi \) of \( \Delta \). It follows from our recipe that for each \( \Theta \subset \Pi \), \( h_\Theta \) corresponds to the subdiagram of the Dynkin diagram associated with \( \Pi \), and is determined by the hyperbolic principal isotropy subalgebra and a semisimple subsymmetric pair of \( (g, h) \) associated with \( \Delta_\Theta \) (see page 315 in [11]). By applying our recipe to \( w_i \cdot \Psi \) for each \( i \in \{ 1, \ldots, l \} \), we can determine [11]. Our recipe is analogous to determine the local orbit types of the orbits for the isotropy action on Riemannian symmetric spaces of compact type by Tamaru ([13]).
Remark 1. Similarly, the set of all local orbit types of the elliptic orbits for the s-representation is also determined by using the restricted root system with respect to a toroidal-type maximal split abelian subspace.

The isotropy subalgebra of a hyperbolic principal orbit is isomorphic to $\mathfrak{z}_h$, which is called a hyperbolic principal isotropy subalgebra (abbreviated to HPIS). Suppose that $\theta$ is a Cartan involution of $\mathfrak{g}$ commuting with $\sigma$ and $\mathfrak{a}$ is a subspace of $\mathfrak{p}$, where $\mathfrak{p} := \text{Ker}(\theta + \text{id})$. Let $\mathfrak{a}_q$ (resp. $\mathfrak{a}_p$) be a maximal abelian subspace of $\mathfrak{q}$ (resp. $\mathfrak{p}$) containing $\mathfrak{a}$. By using the Satake diagrams associated with $G/H$ with respect to $\mathfrak{a}, \mathfrak{a}_q$ and $\mathfrak{a}_p$, we investigate the (Lie algebra) structure of $\mathfrak{z}_h$. Moreover, we give a recipe to determine $\mathfrak{z}_h$ (cf. Recipe 4.6 in Section 4). We obtained the these Satake diagrams for classical-type semisimple pseudo-Riemannian symmetric spaces in [2]. Then we have the following result in terms of Table 1 in [2].

**Theorem B.** The hyperbolic principal isotropy subalgebras of the s-representations associated with all classical-type semisimple pseudo-Riemannian symmetric spaces are as in Table 1.

| $\mathfrak{(g,h)}$ | HPIS | Remarks |
|--------------------|----------|---------|
| $(\mathfrak{sl}(n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{R}))$ | $R^{(n-1)/2} + \mathfrak{so}(2)^{n/2}$ | |
| $(\mathfrak{sl}(n,\mathbb{R})^\ast, \mathfrak{sl}(n,\mathbb{R}))$ | $R^{n-1}$ | |
| $(\mathfrak{sl}(n,\mathbb{C}), \mathfrak{so}(n,\mathbb{C}))$ | $\{0\}$ | |
| $(\mathfrak{sl}(2n,\mathbb{C}), \mathfrak{su}^\ast(2n))$ | $R^{n-1} + \mathfrak{so}(2)^n$ | |
| $(\mathfrak{su}(2n)^2, \mathfrak{su}(2n))$ | $R^{n-1} + \mathfrak{sp}(1)^n$ | |
| $(\mathfrak{sl}(2n,\mathbb{C}), \mathfrak{sp}(n,\mathbb{C}))$ | $\mathfrak{sp}(1,\mathbb{C})^n$ | |
| $(\mathfrak{sl}(n,\mathbb{C}), \mathfrak{su}(p,n-p))$ | $\mathfrak{so}(2)^{n-1}$ | |
| $(\mathfrak{su}(p,n-p)^2, \mathfrak{su}(p,n-p))$ | $R^p + \mathfrak{so}(2)^p + \mathfrak{su}(n-2p)$ | $n > 2p$ |
| | $R^p + \mathfrak{so}(2)^{p-1}$ | $n = 2p$ |
| $(\mathfrak{sl}(n,\mathbb{C}), \mathfrak{sl}(p,C) + \mathfrak{sl}(n-p,C) + C)$ | $\mathfrak{C}^p + \mathfrak{sl}(n-2p,C)$ | $n > 2p$ |
| | $\mathfrak{C}^{p-1}$ | $n = 2p$ |
| $(\mathfrak{so}(2n,\mathbb{C}), \mathfrak{so}^\ast(2n))$ | $\mathfrak{so}(2)^n$ | |
| $(\mathfrak{so}^\ast(2n)^2, \mathfrak{so}^\ast(2n))$ | $R^m + \mathfrak{su}(2)^m + \mathfrak{so}(2)$ | $n = 2m + 1$ |
| | $R^m + \mathfrak{su}(2)^m$ | $n = 2m$ |
| $(\mathfrak{so}(2n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C}) + C)$ | $\mathfrak{sl}(2,C)^m + C$ | $n = 2m + 1$ |
| | $\mathfrak{sl}(2,C)^m$ | $n = 2m$ |
Table 1: (continued)

| \((g, h)\) | HPIS | Remarks |
|---------|------|---------|
| \((\mathfrak{so}(n, C), \mathfrak{so}(p, n - p))\) | \(\mathfrak{so}(2)^m\) | \(n = 2m + 1\) |
| | | \(p = 2q\) |
| | \(\mathfrak{so}(2)^m + \mathfrak{R}\) | \(n = 2m\) |
| | | \(p = 2q + 1\) |
| | \(\mathfrak{so}(2)^m\) | \(n = 2m\) |
| | | \(p = 2q\) |
| \((\mathfrak{so}(p, n - p)^2, \mathfrak{so}(p, n - p))\) | \(\mathfrak{R}^p + \mathfrak{so}(n - 2p)\) | |
| \((\mathfrak{so}(n, C), \mathfrak{so}(p, C) + \mathfrak{so}(n - p, C))\) | \(\mathfrak{so}(n - 2p, C)\) | |
| \((\mathfrak{sp}(n, C), \mathfrak{sp}(n, R))\) | \(\mathfrak{so}(2)^n\) | |
| \((\mathfrak{sp}(n, R)^2, \mathfrak{sp}(n, R))\) | \(\mathfrak{R}^m\) | |
| \((\mathfrak{sp}(n, C), \mathfrak{sl}(n, n) + C)\) | \(\{0\}\) | |
| \((\mathfrak{sp}(n, C), \mathfrak{sp}(p, n - p))\) | \(\mathfrak{so}(2)^n\) | |
| \((\mathfrak{sp}(p, n - p)^2, \mathfrak{sp}(p, n - p))\) | \(\mathfrak{R}^p + \mathfrak{sp}(1)^p + \mathfrak{sp}(n - 2p)\) | |
| \((\mathfrak{sp}(n, C), \mathfrak{sp}(p, C) + \mathfrak{sp}(n - p, C))\) | \(\mathfrak{sp}(1, C)^p + \mathfrak{sp}(n - 2p, C)\) | |
| \((s\ell(n, R), \mathfrak{so}(p, n - p))\) | \(\{0\}\) | |
| \((s\ell(p, n - p), \mathfrak{so}(p, n - p))\) | \(\mathfrak{so}(n - 2p)\) | |
| \((s\ell(n, R), s\ell(p, R) + s\ell(n - p, R) + \mathfrak{R})\) | \(\mathfrak{R}^p + s\ell((n - 2p, R)\) | \(n > 2p\) |
| | | \(\mathfrak{R}^{p-1}\) | \(n = 2p\) |
| \((s\ell(2n), s\ell(p, n - p))\) | \(\mathfrak{sp}(1)^n\) | |
| \((s\ell(2n), s\ell^*(2n))\) | \(\mathfrak{u}(1)^n\) | |
| \((s\ell(n, n), s\ell^*(2n))\) | \(\{0\}\) | |
| \((s\ell(2n, R), s\ell(n, C) + \mathfrak{so}(2))\) | \(\mathfrak{R}^{n-1}\) | |
| \((s\ell^*(2n), s\ell(n, C) + \mathfrak{so}(2))\) | \(\mathfrak{R}^{n-1} + \mathfrak{so}(2)^m\) + \(\mathfrak{su}(2)^m\) | \(n = 2m + 1\) |
| | | \(\mathfrak{su}(2)^m + \mathfrak{so}(2)^m\) | \(n = 2m\) |
| \((s\ell(n, n), s\ell(p, R))\) | \(\mathfrak{sp}(1, C)^m + \mathfrak{sp}(1, R)\) | \(n = 2m + 1\) |
| | | \(\mathfrak{sp}(1, C)^m\) | \(n = 2m\) |
| \((s\ell(n, n), s\ell(n, C) + \mathfrak{R})\) | \(\mathfrak{so}(2)^{n-1}\) | |
| \((s\ell^*(2n), s\ell(p, n - p) + \mathfrak{su}(2))\) | \(\mathfrak{su}(2)^m + \mathfrak{so}(2)\) | \(n = 2m + 1\) |
| | | \(p = 2q\) | |
| \((s\ell(2n, 2(n - p)), s\ell(p, n - p) + \mathfrak{su}(2))\) | \(\mathfrak{su}(1)^p + \mathfrak{so}(n - 2p)\) | |
| | | \(\mathfrak{su}(1, 1)^p\) | \(n = 2m\) |
| | | \(p = 2q\) | |
**LOCAL ORBIT TYPE**

**Table 1:** (continued)

| (g, h) | HPIS | Remarks |
|--------|------|---------|
| (so*(2n), so*(2p) + so*(2(n − p))) | so(2)p + so*(2(n − 2p)) | |
| (so(n, n), so(n, C)) | {0} | |
| (so*(2n), so(n, C)) | so(2)\[n/2⟩ | |
| (so(n, n), sl(n, R) + R) | R + sl(2, R)\[m⟩ n = 2m + 1 sl(2, R)\[m⟩ n = 2m | |
| (so*(4n), su*(2n) + R) | sp(1)n | |
| (sp(n, R), su(p, n − p) + so(2)) | {0} | |
| (sp(p, n − p), su(p, n − p) + so(2)) | u(1)p + u(n − 2p) | |
| (sp(n, R), sp(p, R) + sp(n − p, R)) | sp(1, R)p + sp(n − 2p, R) | |
| (sp(n, R), sl(n, R) + R) | {0} | |
| (sp(n, n), sp(n, C)) | sp(1)n | |
| (sp(2n, R), sp(n, C)) | sp(1, R)n | |
| (sp(n, n), su*(2n) + R) | u(1)n | |

| (g, h) = (su(n, m), su(i, j) + su(n − i, m − j) + so(2)) | |
|---|---|---|
| HPIS | Remarks |
| so(2)^n | i + j = n = m |
| so(2)^n + su(m − n) | n < i + j = m |
| so(2)^m+n−(i+j) + su(i + j − n, i + j − m) | n ≤ m < i + j |
| so(2)^m+n | n = i + j < m |
| so(2)^m+1 + su(i + j − n) + su(m − (i + j)) | n < i + j < m |
| so(2)^m+j + su(n − (i + j), m − (i + j)) | i + j < n ≤ m |

| (g, h) = (so(n, m), so(i, j) + so(n − i, m − j)) | |
|---|---|---|
| HPIS | Remarks |
| {0} | i + j = n = m |
| so(m − n) | n < i + j = m |
| so(i + j − n, i + j − m) | n ≤ m < i + j |
| so(m − n) | n = i + j < m |
| so(i + j − n) + so(m − (i + j)) | n < i + j < m |
| so(n − (i + j), m − (i + j)) | i + j < n ≤ m |

| (g, h) = (sp(n, m), sp(i, j) + sp(n − i, m − j)) | |
|---|---|---|
| HPIS | Remarks |
| sp(1)^n | i + j = n = m |
| sp(1)^n + sp(m − n) | n < i + j = m |
| sp(1)^m+n−(i+j) + sp(i + j − n, i + j − m) | n ≤ m < i + j |
| sp(1)^n + sp(m − n) | n = i + j < m |
| sp(1)^n + sp(i + j − n) + sp(m − (i + j)) | n < i + j < m |
| sp(1)^m+j + sp(n − (i + j), m − (i + j)) | i + j < n ≤ m |
It is a difficult problem to determine the structures of the HPISs, since these are not necessarily compact. By using the Satake diagram associated with $G/H$ with respect to $a$, we determine the structure of $z^C_g$, where $z_g$ denotes the centralizer of $a$ in $g$. Moreover, we investigate the decomposition $z_g = z_g \cap h + z_g \cap q$ and $z_g = z_g \cap \mathfrak{k} + z_g \cap \mathfrak{p}$ by using the Satake diagrams associated with $G/H$ with respect to $a$, $a_q$ and $a_p$. In positive definite case, the isotropy subalgebras are compact, so that, in order to investigate the structures of the isotropy subalgebras, it is sufficient to investigate the dimension of their centers and the Dynkin diagrams of their semisimple parts by using the Satake diagram with respect to $a$.

The organization of this paper is as follows. In Section 1 we give preliminaries for the restricted root systems with respect to maximal split abelian subspaces for semisimple pseudo-Riemannian symmetric spaces. Moreover, we recall the notion of the Satake diagrams. In Section 2 we prove Theorem A. In Section 3, we shall give a complete representatives for $W(\Delta)/W(\Delta^a)$ in the case where $G/H$ is of classical-type except for $(\Delta, \Delta^a) = (A_{n-1}, A_{n-p-1})$. In the case of $(\Delta, \Delta^a) = (A_{n-1}, A_{p-1} \times A_{n-p-1})$, we give a recursive formula for $W(\Delta)/W(\Delta^a)$ (Proposition 3.1). By using the formula (2), we give a complete representative for $W(\Delta)/W(\Delta^a)$ in the case of $(\Delta, \Delta^a) = (A_4, A_1 \times A_2)$ (Example 3.2). In Section 4, we shall give a recipe to determine the (Lie algebra) structures of HPISs (Recipe 4.6). Moreover, we give examples for $G/H$ as $(\mathfrak{su}(2p, 2(n-p)), \mathfrak{sp}(p, n-p))$, $(\mathfrak{sl}(n, C), \mathfrak{sl}(n, R))$ (Example 4.7 and 4.8). In Section 5, we investigate the isotropy subalgebras of the hyperbolic orbits for the $s$-representation of a semisimple pseudo-Riemannian symmetric space $G/H$. We give all the possible local orbit types of hyperbolic orbits for the $s$-representation associated with $(\mathfrak{sl}(4, R), \mathfrak{so}(2, 2))$ (Example 5.1). We also give all the possible local orbit types of hyperbolic orbits for the $s$-representations associated with all classical semisimple pseudo-Riemannian symmetric spaces in the case where $\Delta = \Delta^a$, $(\Delta, \Delta^a) = ((BC)_r, B_r)$ or $(\Delta, \Delta^a) = (C_r, D_r)$ (Example 5.2 and 5.3). In Section 6, we discuss the relation between the local orbit types of the elliptic orbits and those of the hyperbolic orbits.

**Research plan in the future.** We will explicitly give a standard complete system of representatives for $W(\Delta)/W(\Delta^a)$ for exceptional-type semisimple pseudo-Riemannian symmetric spaces. Moreover, we will give HPISs for these spaces. We will develop the submanifold geometry of orbits for $s$-representations by using their isotropy subalgebras.
1. Preliminaries

Let $G$ be a connected semisimple noncompact Lie group, and $\sigma$ be an involution of $G$. Let $H$ be a closed subgroup of $G$ with $(G_\sigma)_0 \subset H \subset G_\sigma$, where $G_\sigma$ denotes the fixed point group of $\sigma$ and $(G_\sigma)_0$ denotes the identity component of $G_\sigma$. The pair $(G, H)$ is called a semisimple symmetric pair. Then the coset space $G/H$ equipped with the metric induced from the Killing form of the Lie algebra $\mathfrak{g}$ of $G$ is a semisimple pseudo-Riemannian symmetric space. The holonomy representation of $G/H$ is equivalent to the $s$-representation. Denote by $\text{Ad}_G$ (resp. $\text{ad}_\mathfrak{g}$) the adjoint representation of $G$ (resp. $\mathfrak{g}$). The involution $\sigma$ of $G$ induces an involution of $\mathfrak{g}$, which is also denoted by the same symbol $\sigma$. Then the Lie algebra $\mathfrak{h}$ of $H$ coincides with \{ $X \in \mathfrak{g} \mid \sigma(X) = X$ \}. The pair $(\mathfrak{g}, \mathfrak{h})$ is called a semisimple symmetric pair. Set $\mathfrak{q} := \{ X \in \mathfrak{g} \mid \sigma(X) = -X \}$. The $s$-representation is equivalent to the representation $\text{Ad}$ of $H$ on $\mathfrak{q}$ defined by $\text{Ad}(h) = \text{Ad}_G(h)|_\mathfrak{q}$ for all $h \in H$. An element $X \in \mathfrak{q}$ is said to be semisimple if the endomorphism $\text{ad}_\mathfrak{q}(X)^C$ is diagonalizable, where $\text{ad}_\mathfrak{q}(X)^C$ is the complexification of $\text{ad}_\mathfrak{q}(X)$. A semisimple element $X \in \mathfrak{q}$ is said to be hyperbolic (resp. elliptic) if any eigenvalue of $\text{ad}_\mathfrak{q}(X)^C$ is real (resp. pure imaginary). An orbit through a hyperbolic (resp. an elliptic) element is called a hyperbolic orbit (resp. an elliptic orbit). Denote by $H_X$ the isotropy subgroup of $H$ at $X \in \mathfrak{q}$, by $\mathfrak{h}_X$ the Lie algebra of $H_X$. It is clear that $H_{\text{Ad}(h)X} = hH_Xh^{-1}$ and $\mathfrak{h}_{\text{Ad}(h)X} = \text{Ad}_G(h)|_\mathfrak{h}_X$ hold for all $h \in H$. The conjugate classes $\{ hH_Xh^{-1} \mid h \in H \}$ and $\{ \text{Ad}_G(h)|_\mathfrak{h}_X \mid h \in H \}$ are called an orbit type and a local orbit type, respectively.

We recall the notion of the restricted root systems with respect to maximal split abelian subspaces for semisimple pseudo-Riemannian symmetric spaces (cf. [15, 14]). Let $\mathfrak{a}$ be a maximal split abelian subspace of $\mathfrak{q}$ (i.e., a maximal abelian subspace of $\mathfrak{q}$ which consists of only hyperbolic elements or only elliptic elements). We say that $\mathfrak{a}$ is vector-type (resp. troidal-type) if all elements of $\mathfrak{a}$ are hyperbolic (resp. elliptic). Set, for any $\lambda \in \mathfrak{a}^*$,

$$
\mathfrak{g}_\lambda := \{ X \in \mathfrak{g} \mid \text{ad}(A)X = (\sqrt{-1})^\epsilon\lambda(A)X, \forall A \in \mathfrak{a} \},
$$

$$
\mathfrak{h}_\lambda := \{ X \in \mathfrak{h} \mid \text{ad}(A)^2X = (-1)^\epsilon\lambda(A)^2X, \forall A \in \mathfrak{a} \},
$$

$$
\mathfrak{q}_\lambda := \{ X \in \mathfrak{q} \mid \text{ad}(A)^2X = (-1)^\epsilon\lambda(A)^2X, \forall A \in \mathfrak{a} \},
$$

where $\epsilon = 0$ (resp. 1) when $\mathfrak{a}$ is vector-type (resp. troidal-type). Denote by $\Delta := \{ \lambda \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}_\lambda \neq \{0\} \} = \{ \lambda \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{q}_\lambda \neq \{0\} \}$, which is called the restricted root system of $G/H$ (or $(\mathfrak{g}, \mathfrak{h})$) with respect to $\mathfrak{a}$. The
dimension of \( \mathfrak{a} \) is called the split rank of \( G/H \) (or \( (\mathfrak{g}, \mathfrak{h}) \)), which is denoted by \( \text{s-rank}(G/H) \) (or \( \text{s-rank}(\mathfrak{g}, \mathfrak{h}) \)). Note that, for each \( \lambda \in \Delta \), the restriction of the Killing form of \( \mathfrak{g} \) to \( q_\lambda \times q_\lambda \) is a nondegenerate inner product of \( q_\lambda \), whose index denotes \( m^- (\lambda) \). Set, for each \( \lambda \in \Delta \), \( m(\lambda) := \dim q_\lambda \) and \( m^+(\lambda) := m(\lambda) - m^- (\lambda) \). We call \( m(\lambda) \) and \((m^+(\lambda), m^-(\lambda))\) the multiplicity of \( \lambda \) and the signature of \( \lambda \), respectively. A symmetric pair \((\mathfrak{g}, \mathfrak{h})\) is called basic if \( m^+(\lambda) \geq m^- (\lambda) \) for any \( \lambda \in \Delta \) such that \( \frac{1}{2} \lambda \not\in \Delta \). Let \( \varepsilon \) be a signature of \( \Delta \), i.e., \( \varepsilon \) is a mapping from \( \Delta \) to \( \{ \pm 1 \} \) satisfying the two conditions: (i) \( \varepsilon(\lambda + \mu) = \varepsilon(\lambda) \varepsilon(\mu) \) (\( \lambda, \mu, \lambda + \mu \in \Delta \)), and (ii) \( \varepsilon(-\lambda) = \varepsilon(\lambda) \) (\( \lambda \in \Delta \)).

Denote by \( \sigma_\varepsilon \) the involution of \( \mathfrak{g} \) defined by

\[
\sigma_\varepsilon(X) = \begin{cases} 
\sigma(X) & (X \in \mathfrak{z}_\varepsilon), \\
\varepsilon(\lambda)\sigma(X) & (X \in \mathfrak{g}_\lambda, \lambda \in \Delta).
\end{cases}
\]

Then \((\mathfrak{g}, \mathfrak{h}_\varepsilon)\) is a semisimple symmetric pair, where \( \mathfrak{h}_\varepsilon = \ker(\sigma_\varepsilon - \text{id}) \). Denote by \( \mathfrak{z}_\varepsilon, \mathfrak{h}_\varepsilon \) and \( \mathfrak{q}_\varepsilon \) the centralizer of \( \mathfrak{a} \) in \( \mathfrak{g}, \mathfrak{h} \) and \( \mathfrak{q} \), respectively. Then we have the following decompositions.

**Lemma 1.1.** Let \( \Delta_+ \) be a positive root system of \( \Delta \).

\[
\mathfrak{g} = \mathfrak{z}_\varepsilon + \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda, \quad \mathfrak{h} = \mathfrak{z}_\varepsilon + \sum_{\lambda \in \Delta_+} \mathfrak{h}_\lambda, \quad \mathfrak{q} = \mathfrak{z}_\varepsilon + \sum_{\lambda \in \Delta_+} \mathfrak{q}_\lambda.
\]

Denote by \( s_\lambda \) the reflection of \( \mathfrak{a} \) along the hyperplane \( \lambda^{-1}(0) \), and \( \mathcal{W}(\Delta) \) by the Weyl group of \( \Delta \) (i.e., the group generated by \( s_\lambda \)'s (\( \lambda \in \Delta \))). Let \( \Psi \) be a simple root system of \( \Delta \). Then \( \{ \lambda \in \Psi \mid \lambda(A) > 0, \forall \lambda \in \Psi \} \) is a Weyl chamber associated with \( \Delta \), and \( \mathcal{W}(\Delta) \) acts simply transitively on the set of the Weyl chambers associated with \( \Delta \).

Next, we recall three Satake diagrams associated with three kinds of restricted root systems associated with semisimple pseudo-Riemannian symmetric spaces (cf. [2]). For simplicity, in the sequel, suppose that \( \mathfrak{a} \) is a vector-type maximal split abelian subspace of \( \mathfrak{q} \). It is known that \( \mathfrak{a} \) is vector-type if and only if \( \mathfrak{a} \) is a maximal abelian subspace of \( \mathfrak{p} \cap \mathfrak{q} \), where \( \mathfrak{p} \) is the \((-1)\)-eigenspace of certain Cartan involution commuting with \( \sigma \). Fix a such Cartan involution \( \theta \). Set \( \mathfrak{t} := \{ X \in \mathfrak{g} \mid \theta(X) = X \} \). Let \( \mathfrak{a}_\mathfrak{q} \) (resp. \( \mathfrak{a}_\mathfrak{p} \)) be a maximal abelian subspace of \( \mathfrak{q} \) (resp. \( \mathfrak{p} \)) containing \( \mathfrak{a} \). Let \( \tilde{\mathfrak{a}} \) be a maximal abelian subalgebra of \( \mathfrak{g} \) containing \( \mathfrak{a}_\mathfrak{q} \) and \( \mathfrak{a}_\mathfrak{p} \). Then \( \tilde{\mathfrak{a}}^C \) is a Cartan subalgebra of \( \mathfrak{g}^C \). Denote by \( R \) the root system of \( \mathfrak{g}^C \) with respect to \( \tilde{\mathfrak{a}}^C \). We define the vector \( A_\alpha (\alpha \in R) \) of \( \tilde{\mathfrak{a}}^C \) by \( B(A, A_\alpha) = \alpha(A) \) for all \( A \in \tilde{\mathfrak{a}}^C \), where \( B \) denotes the Killing form of \( \mathfrak{g}^C \). Set \( \tilde{\mathfrak{a}}_R := \text{Span}_R \{ A_\alpha \mid \alpha \in R \} \). Then we have \( \tilde{\mathfrak{a}}_R = \sqrt{-1} \tilde{\mathfrak{a}} \cap (\mathfrak{t} \cap \mathfrak{h}) + \mathfrak{a}_p \cap \mathfrak{h} + \sqrt{-1} \mathfrak{a}_q \cap \mathfrak{t} + \mathfrak{a} \). Fix a
basis $\mathcal{A} := (A_1, A_2, \ldots, A_r)$ of $\tilde{\mathfrak{a}}_R$ such that $(A_1, A_2, \ldots, A_l)$ is a basis of $\mathfrak{a}$, $(A_{l+1}, A_{l+2}, \ldots, A_m)$ is a basis of $\sqrt{-1}(\mathfrak{a}_q \cap \mathfrak{e})$, $(A_{m+1}, A_{m+2}, \ldots, A_{m+n-1})$ is a basis of $\mathfrak{a}_p \cap \mathfrak{h}$, and $(A_{m+n-l+1}, \ldots, A_r)$ is a basis of $\sqrt{-1}\mathfrak{a} \cap (\mathfrak{g} \cap \mathfrak{h})$, where $l := s\text{rank}(\mathfrak{g}, \mathfrak{h})$, $m := \text{rank}(\mathfrak{g}, \mathfrak{h})$, $n := \text{rank}(\mathfrak{g}, \mathfrak{e})$ and $r := \text{rank}(\mathfrak{g}^C)$. Take the lexicographic ordering $>$ of the dual space of $\tilde{\mathfrak{a}}_R$ with respect to $\mathcal{A}$. Denote by $\Phi$ the simple root system of $R$ with respect to $>$. From the Dynkin diagram of $R$ associated with $\Phi$ we construct the Satake diagram $S(G/H, \mathfrak{a})$ (or $S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$) associated with $G/H$ with respect to $\mathfrak{a}$ as follows. First, replace white circles in the Dynkin diagram, which imply elements in $\Phi_0 := \{\alpha \in \Phi \mid \alpha|_{\mathfrak{a}} = 0\}$, to a black circle. Second, if simple roots $\alpha, \beta \in \Phi \setminus \Phi_0$ and $\alpha|_{\mathfrak{a}} = \beta|_{\mathfrak{a}}$, then join $\alpha$ to $\beta$ with a curved arrow. Similarly, we construct the Satake diagrams $S(G/H, \mathfrak{a}_q)$ (or $S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_q)$) associated with $G/H$ with respect to $\mathfrak{a}_q$. Note that the Satake diagram with respect to $\mathfrak{a}_p$ coincides with the Satake diagram $S(G/K, \mathfrak{a}_p)$ of the Riemannian symmetric space $G/K$ with respect to $\mathfrak{a}_p$, where $K$ denotes the Lie subgroup of $G$ with Lie algebra $\mathfrak{k}$. By the definition of $>$, $S(G/H, \mathfrak{a})$, $S(G/H, \mathfrak{a}_q)$ and $S(G/K, \mathfrak{a}_p)$ are constructed from the Dynkin diagram of the same simple root system of $\Phi$. Then $S(G/H, \mathfrak{a})$, $S(G/H, \mathfrak{a}_q)$ and $S(G/K, \mathfrak{a}_p)$ are said to be compatible with one another. Set $\Phi_{0,p} := \{\alpha \in R \mid \alpha|_{\mathfrak{a}_p} = 0\}$. Denote by $p_\theta$ the Satake involution of $S(G/K, \mathfrak{a}_p)$. By the classification of the Satake diagrams of irreducible Riemannian symmetric spaces, we have the following fact.

**Lemma 1.2.** Let $G/K$ be an irreducible Riemannian symmetric space. Then the Satake diagram of $G/K$ has the following two properties.

1. If $\beta \in \Phi \setminus \Phi_{0,p}$ is disconnected with the roots of $\Phi_{0,p}$, then $(-\theta)\beta = p_\theta \beta$ holds.
2. If $\beta \in \Phi \setminus \Phi_{0,p}$ is connected with a root of $\Phi_{0,p}$, then $(-\theta)\beta \equiv p_\theta \beta \mod \{\langle \beta \rangle\}_Z$, where $\langle \beta \rangle \subset \Phi_{0,p}$ denotes the union of the connected components of $\Phi_{0,p}$ connected with $\beta$.

2. Proof of Theorem A

Let $(\mathfrak{g}, \mathfrak{h})$ be a semisimple symmetric pair, and $\sigma$ be an involution of $\mathfrak{g}$ with $\mathfrak{h} = \text{Ker}(\sigma - \text{id})$. Suppose that $\mathfrak{a}$ is a vector-type maximal split abelian subspace of $\mathfrak{q} := \text{Ker}(\sigma + \text{id})$. Denote by $\Delta$ the restricted root system of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}$. 
Proof of Theorem A. Let $\theta$ be a Cartan involution of $\mathfrak{g}$ commuting with $\sigma$, and $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ be the Cartan decomposition corresponding to $\theta$. Any vector-type maximal split abelian subspace is $\text{Ad}(H)$-conjugate to a $\theta$-invariant one. Without loss of generality, we may assume that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. By Lemma 2 of [15], if $A \in \mathfrak{a}$ and $\text{Ad}(h)A \in \mathfrak{a}$ ($h \in H$), then there exists a $k \in N_{H \cap K}(\mathfrak{a})$ such that $\text{Ad}(h)A = \text{Ad}(k)A$, where $N_{H \cap K}(\mathfrak{a}) := \{k \in H \cap K \mid \text{Ad}(k)\mathfrak{a} = \mathfrak{a}\}$. Therefore $\text{Ad}(H)X$ and $\text{Ad}(H)Y$ ($X, Y \in \mathfrak{a}$) are same orbit if and only if $X$ is conjugate in $Y$ in $N_{H \cap K}(\mathfrak{a})$. Set $Z_{H \cap K}(\mathfrak{a}) := \{h \in H \cap K \mid \text{Ad}(h)A = A, \forall A \in \mathfrak{a}\}$. Then the quotient group $N_{H \cap K}/Z_{H \cap K}$ is isomorphic to the Weyl group $W(\Delta^a)$ associated with $\Delta^a$.

Let $w_1, w_2, \ldots, w_l$ be a complete system of representatives for $W(\Delta)/W(\Delta^a)$. Then we can show that $\mathfrak{a} = \bigcup_{i=1}^l \overline{C(w_i \cdot \Psi)}$ and that $\bigcup_{i=1}^l \text{Ad}(H)(\overline{C(w_i \cdot \Psi)})$ coincides with the set of all the hyperbolic elements in $\mathfrak{q}$, where $C(w_i \cdot \Psi) := \{A \in \mathfrak{a} \mid \lambda(A) > 0, \forall \lambda \in w_i \cdot \Psi\}$ and $\overline{C(w_i \cdot \Psi)}$ denotes the closure of $C(w_i \cdot \Psi)$.

Hence we have

$$L_h(G/H) := \{[h_A] \mid A \in \mathfrak{q} \text{ is hyperbolic}\}$$

$$= \bigcup_{i=1}^l \{[h_A] \mid A \in \overline{C(w_i \cdot \Psi)}\}.$$ 

For each $\Theta \subset w_i \cdot \Psi$, set $C(w_i \cdot \Psi, \Theta) := \{A \in \mathfrak{a} \mid \lambda(A) > 0(\forall \lambda \in (w_i \cdot \Psi) \setminus \Theta), \mu(A) = 0(\forall \mu \in \Theta)\}(\subset \overline{C(w_i \cdot \Psi)})$. Then, for any two subsets $\Theta, \Theta'$ of $w_i \cdot \Psi$, we have $C(w_i \cdot \Psi, \Theta) \cap C(w_i \cdot \Psi, \Theta') = \emptyset$ if $\Theta \neq \Theta'$. Therefore $\overline{C(w_i \cdot \Psi)}$ is decomposed as

$$\overline{C(w_i \cdot \Psi)} = \bigcup_{\Theta \subset w_i \cdot \Psi} C(w_i \cdot \Psi, \Theta) \quad (\text{disjoint}).$$

Moreover, for any $A \in C(w_i \cdot \Psi, \Theta)$, $\Theta$ is a simple root system of $\Delta_A(\subset \Delta :\lambda \in \Delta \mid \lambda(A) = 0\}$. This implies that, for all $A \in C(w_i \cdot \Psi, \Theta)$,

$$h_A = h_b + \sum_{\lambda \in \Delta_A \cap \Delta_+} h_\lambda = h_b + \sum_{\lambda \in \Delta_\Theta \cap \Delta_+} h_\lambda = h_\Theta,$$

where $\Delta_\Theta = \Delta \cap \sum_{\lambda \in \Theta} R\lambda$. Hence we have

$$L_h(G/H) = \bigcup_{i=1}^l \bigcup_{\Theta \subset w_i \cdot \Psi} \{[h_A] \mid A \in C(w_i \cdot \Psi, \Theta)\}$$

$$= \bigcup_{i=1}^l \{[h_\Theta] \mid \Theta \subset w_i \cdot \Psi\}. \square$$
3. Complete systems of representatives for $\mathcal{W}(\Delta)/\mathcal{W}(\Delta^a)$

Let $(\mathfrak{g}, \mathfrak{h})$ be a semisimple symmetric pair and $\sigma$ be an involution of $\mathfrak{g}$ with $\mathfrak{h} = \text{Ker}(\sigma - \text{id})$. Suppose that $\theta$ is a Cartan involution of $\mathfrak{g}$ commuting with $\sigma$, and $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Denote by $\Delta$ the restricted root system of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}$. Set $\mathfrak{h}^a = \text{Ker}(\sigma \circ \theta - \text{id})$. It is clear that $\mathfrak{h}^a = \mathfrak{t} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}$ holds. Then $\mathfrak{h}^a$ is reductive, and $(\mathfrak{h}^a, \mathfrak{t} \cap \mathfrak{h})$ is a Riemannian symmetric pair whose restricted root system is $\Delta^a := \{\lambda \in \Delta \mid \mathfrak{h}^a \cap \mathfrak{h}^a \neq \{0\}\}$. Denote by $\mathcal{W}(\Delta)$ (resp. $\mathcal{W}(\Delta^a)$) the Weyl group of $\Delta$ (resp. $\Delta^a$). In this section, for classical-type semisimple symmetric pairs $(\mathfrak{g}, \mathfrak{h})$, we shall give a complete system of representatives for $\mathcal{W}(\Delta)/\mathcal{W}(\Delta^a)$.

First, we shall list up the types of $\Delta$ and $\Delta^a$, and the index $|\mathcal{W}(\Delta)/\mathcal{W}(\Delta^a)|$ of $\mathcal{W}(\Delta)/\mathcal{W}(\Delta^a)$ (i.e. the cardinality of $\mathcal{W}(\Delta)/\mathcal{W}(\Delta^a)$) (see, Table 2). In Table 2, $nC_k$ is a binomial coefficient and, for a Lie algebra $\mathfrak{g}$, we denote by $\mathfrak{f}^\perp$ the direct sum $\mathfrak{f} + \mathfrak{l}$.

Table 2: The index of $\mathcal{W}(\Delta)/\mathcal{W}(\Delta^a)$

| $(\mathfrak{g}, \mathfrak{h})$ | Type of $\Delta$ | Type of $\Delta^a$ | Index | Remarks |
|-----------------------------|------------------|-------------------|-------|---------|
| $(\mathfrak{sl}(n, \mathfrak{C}), \mathfrak{sl}(n, \mathfrak{R}))$ | $(\mathfrak{BC})_m$ | $(\mathfrak{BC})_n$ | $n = 2m + 1$ | |
| $(\mathfrak{sl}(n, \mathfrak{C}), \mathfrak{so}(n, \mathfrak{C}))$ | $(\mathfrak{A})_{n-1}$ | $(\mathfrak{A})_{n-1}$ | $n = 2m$ | |
| $(\mathfrak{sl}(n, \mathfrak{C}), \mathfrak{su}^*(2n))$ | $(\mathfrak{C})_n$ | $(\mathfrak{C})_n$ | $n = 2m$ | |
| $(\mathfrak{su}(2n, \mathfrak{C}), \mathfrak{sp}(2n, \mathfrak{C}))$ | $(\mathfrak{A})_{n-1}$ | $(\mathfrak{A})_{n-1}$ | $n = 2m$ | |
| $(\mathfrak{sl}(n, \mathfrak{C}), \mathfrak{su}(p, n - p))$ | $(\mathfrak{A})_{n-1}$ | $(\mathfrak{A})_{n-1} \times (\mathfrak{A})_{n-1}$ | $nC_k$ | |
| $(\mathfrak{su}(p, n - p), \mathfrak{su}(p, n - p))$ | $(\mathfrak{B})_p$ | $(\mathfrak{B})_p$ | $n > 2p$ | |
| $(\mathfrak{so}(2n, \mathfrak{C}), \mathfrak{so}^*(2n))$ | $(\mathfrak{D})_m$ | $(\mathfrak{D})_m$ | $n = 2m + 1$ | |
| $(\mathfrak{so}^*(2n), \mathfrak{so}^*(2n))$ | $(\mathfrak{C})_m$ | $(\mathfrak{C})_m$ | $n = 2m$ | |
| $(\mathfrak{so}(2n, \mathfrak{C}), \mathfrak{sl}(n, \mathfrak{C}) + \mathfrak{sl}(n - p, \mathfrak{C}) + \mathfrak{C})$ | $(\mathfrak{B})_m$ | $(\mathfrak{B})_m$ | $n = 2m + 1$ | |
| $(\mathfrak{so}(2n, \mathfrak{C}), \mathfrak{so}(p, n - p))$ | $(\mathfrak{B})_m$ | $(\mathfrak{B})_m$ | $n = 2m + 1$ | |
| $(\mathfrak{so}(n, \mathfrak{C}), \mathfrak{so}(p, n - p))$ | $(\mathfrak{B})_m$ | $(\mathfrak{B})_m$ | $n = 2m + 1$ | |
| $(\mathfrak{B})_m$ | $(\mathfrak{D})_m \times (\mathfrak{D})_m$ | $2mC_q$ | $n = 2m + 1$ | |
| $(\mathfrak{B})_m$ | $(\mathfrak{B})_m \times (\mathfrak{B})_m$ | $mC_q$ | $n = 2m + 1$ | |
| $(\mathfrak{D})_m$ | $(\mathfrak{D})_m \times (\mathfrak{D})_m$ | $2mC_q$ | $n = 2m + 1$ | |
| (g, h) | Type of $\Delta$ | Type of $\Delta^*$ | Index | Remarks |
|-------|------------------|-------------------|-------|---------|
| $\mathfrak{so}(p, n - p)^2, \mathfrak{so}(p, n - p)$ | $B_p$ | $B_p$ | 1 | $n > 2p$ |
| $\mathfrak{so}(n, C), \mathfrak{so}(p, C) + \mathfrak{so}(n - p, C)$ | $B_p$ | $B_p$ | 1 | $n > 2p$ |
| $\mathfrak{sp}(n, C), \mathfrak{sp}(n, C)$ | $C_n$ | $A_{n-1}$ | $2^n$ | |
| $(\mathfrak{sp}(n, R), \mathfrak{sp}(n, R))$ | $C_n$ | $C_n$ | 1 | |
| $(\mathfrak{sp}(n, C), \mathfrak{sl}(n, C) + C)$ | $C_n$ | $C_n$ | 1 | |
| $(\mathfrak{sp}(n, R)^2, \mathfrak{sp}(n, R))$ | $C_n$ | $C_n$ | 1 | |
| $(\mathfrak{sp}(n, C), \mathfrak{sp}(n, p - n))$ | $(BC)_p$ | $(BC)_p$ | 1 | $n > 2p$ |
| $(\mathfrak{sp}(n, C), \mathfrak{sp}(n, p - n))$ | $C_p$ | $C_p$ | 1 | $n = 2p$ |
| $(\mathfrak{sp}(n, R), \mathfrak{so}(n, p - n))$ | $A_{n-1}$ | $A_{n-1} \times A_{p-n-1}$ | $nC_p$ | |
| $(\mathfrak{su}(p, n - p), \mathfrak{so}(p, n - p))$ | $(BC)_p$ | $B_p$ | 1 | $n > 2p$ |
| $(\mathfrak{su}(p, n - p), \mathfrak{so}(p, n - p))$ | $C_p$ | $D_p$ | 2 | $n = 2p$ |
| $(\mathfrak{so}(n, R), \mathfrak{sl}(p, R) + \mathfrak{sl}(n - p, R) + R)$ | $(BC)_p$ | $B_p$ | 1 | $n > 2p$ |
| $(\mathfrak{su}^*(2n), \mathfrak{sp}(n, p - n))$ | $A_{n-1}$ | $A_{n-1} \times A_{p-n-1}$ | $nC_p$ | |
| $(\mathfrak{su}(2p, 2(n - p)), \mathfrak{sp}(n, p - n))$ | $(BC)_p$ | $(BC)_p$ | 1 | $n > 2p$ |
| $(\mathfrak{su}^*(2n), \mathfrak{su}^*(2p) + \mathfrak{su}^*(2(n - p)) + R)$ | $(BC)_p$ | $(BC)_p$ | 1 | $n > 2p$ |
| $(\mathfrak{sl}(2n, R), \mathfrak{sp}(n, R))$ | $A_{n-1}$ | $A_{n-1}$ | 1 | |
| $(\mathfrak{su}(2n), \mathfrak{so}^*(2n))$ | $A_{n-1}$ | $A_{n-1}$ | 1 | |
| $(\mathfrak{so}(n, n), \mathfrak{so}^*(2n))$ | $C_n$ | $C_n$ | 1 | |
| $(\mathfrak{so}(2n, R), \mathfrak{sl}(n, C) + \mathfrak{so}(2))$ | $(BC)_m$ | $(BC)_m$ | 1 | $n = 2m + 1$ |
| $(\mathfrak{so}(2n, \mathfrak{su}(n, n) + \mathfrak{so}(2))$ | $C_m$ | $C_m$ | 1 | $n = 2m$ |
| $(\mathfrak{su}(n, n), \mathfrak{sp}(n, R))$ | $(BC)_m$ | $(BC)_m$ | 1 | $n = 2m + 1$ |
| $(\mathfrak{su}(n, n), \mathfrak{sl}(n, C) + \mathfrak{R})$ | $C_n$ | $A_{n-1}$ | $2^n$ | |
| $(\mathfrak{so}^*(2n), \mathfrak{su}(p, n - p) + \mathfrak{so}(2))$ | $(BC)_m$ | $C_q \times (BC)_{m-q}$ | $nC_q$ | $n = 2m + 1$ |
| $(\mathfrak{so}(2p, 2(n - p)), \mathfrak{su}(p, n - p) + \mathfrak{so}(2))$ | $(BC)_p$ | $(BC)_p$ | 1 | $n > 2p$ |
| $(\mathfrak{so}^*(2n), \mathfrak{so}^*(2p) + \mathfrak{so}^*(2(n - p)))$ | $C_p$ | $C_p$ | 1 | $n = 2p$ |
Table 2: (continued)

| (g, h) | Type of $\Delta$ | Type of $\Delta^\ast$ | Index | Remarks |
|--------|------------------|------------------------|-------|---------|
| $(\mathfrak{so}(n, n), \mathfrak{so}(n, C))$ | $D_n$ | $A_{n-1}$ | $2^{n-1}$ | |
| $(\mathfrak{so}^\ast(2n), \mathfrak{so}(n, C))$ | $(BC)_m$ | $B_m$ | 1 | $n = 2m + 1$ |
| $(\mathfrak{so}(n, n), \mathfrak{sl}(n, R) + R)$ | $(BC)_m$ | $B_m$ | 1 | $n = 2m + 1$ |
| $(\mathfrak{so}^\ast(4n), \mathfrak{su}^\ast(2n) + R)$ | $C_n$ | $A_{n-1}$ | $2^n$ | |
| $(\mathfrak{sp}(n, R), \mathfrak{su}(p, n - p) + \mathfrak{so}(2))$ | $(BC)_p$ | $B_p$ | 1 | $n > 2p$ |
| $(\mathfrak{sp}(p, n - p), \mathfrak{su}(p, n - p) + \mathfrak{so}(2))$ | $C_p$ | $C_p$ | 1 | $n = 2p$ |
| $(\mathfrak{sp}(n, R), \mathfrak{sp}(p, R) + \mathfrak{sp}(n - p, R))$ | $(BC)_p$ | $B_p$ | 1 | $n > 2p$ |
| $(\mathfrak{sp}(n, n), \mathfrak{su}(n, R) + R)$ | $C_n$ | $A_{n-1}$ | $2^n$ | |
| $(\mathfrak{sp}(n, n), \mathfrak{sp}(n, C))$ | $C_n$ | $C_n$ | 1 | |
| $(\mathfrak{sp}(n, n), \mathfrak{su}^\ast(2n) + R)$ | $C_n$ | $C_n$ | 1 | |

$(g, h) = (\mathfrak{su}(n, m), \mathfrak{su}(i, j) + \mathfrak{su}(n - i, m - j) + \mathfrak{so}(2))$

| Type of $\Delta$ | Type of $\Delta^\ast$ | Index | Remarks |
|------------------|------------------------|-------|---------|
| $\mathfrak{C}_n$ | $\mathfrak{C}_i \times \mathfrak{C}_{n-i}$ | $n\mathfrak{C}_i$ | $i + j = n = m$ |
| $(BC)_n$ | $(BC)_{n-i}$ | $n\mathfrak{C}_i$ | $n < i + j = m$ |
| $(BC)_{m+n-(i+j)}$ | $(BC)_{m-j} \times (BC)_{n-i}$ | $m+n-(i+j)\mathfrak{C}_{n-i}$ | $n \leq m < i + j$ |
| $(BC)_n$ | $(BC)_i \times \mathfrak{C}_{n-i}$ | $n\mathfrak{C}_i$ | $n = i + j < m$ |
| $(BC)_{i+j}$ | $(BC)_i \times (BC)_j$ | $i+j\mathfrak{C}_i$ | $i + j < n \leq m$ |

$(g, h) = (\mathfrak{so}(n, m), \mathfrak{so}(i, j) + \mathfrak{so}(n - i, m - j))$

| Type of $\Delta$ | Type of $\Delta^\ast$ | Index | Remarks |
|------------------|------------------------|-------|---------|
| $D_n$ | $D_i \times D_{n-i}$ | $2n\mathfrak{C}_i$ | $i + j = n = m$ |
| $B_n$ | $B_i \times B_{n-i}$ | $2n\mathfrak{C}_i$ | $n < i + j = m$ |
| $B_{m+n-(i+j)}$ | $B_{m-j} \times B_{n-i}$ | $m+n-(i+j)\mathfrak{C}_{n-i}$ | $n \leq m < i + j$ |
| $B_n$ | $B_i \times B_{n-i}$ | $2n\mathfrak{C}_i$ | $n = i + j < m$ |
| $B_{i+j}$ | $B_i \times B_j$ | $i+j\mathfrak{C}_i$ | $i + j < n \leq m$ |

$(g, h) = (\mathfrak{sp}(n, m), \mathfrak{sp}(i, j) + \mathfrak{sp}(n - i, m - j))$

| Type of $\Delta$ | Type of $\Delta^\ast$ | Index | Remarks |
|------------------|------------------------|-------|---------|
| $\mathfrak{C}_n$ | $\mathfrak{C}_i \times \mathfrak{C}_{n-i}$ | $n\mathfrak{C}_i$ | $i + j = n = m$ |
| $(BC)_n$ | $(BC)_{n-i}$ | $n\mathfrak{C}_i$ | $n < i + j = m$ |
| $(BC)_{m+n-(i+j)}$ | $(BC)_{m-j} \times (BC)_{n-i}$ | $m+n-(i+j)\mathfrak{C}_{n-i}$ | $n \leq m < i + j$ |
| $(BC)_n$ | $(BC)_i \times \mathfrak{C}_{n-i}$ | $n\mathfrak{C}_i$ | $n = i + j < m$ |
| $(BC)_{i+j}$ | $(BC)_i \times (BC)_j$ | $i+j\mathfrak{C}_i$ | $i + j < n \leq m$ |
If $\Delta = \Delta^a$ holds, then $\{\text{id}\}$ (id: the identity transformation of $a$) gives a complete system of representative for $\mathcal{W}(\Delta)/\mathcal{W}(\Delta^a)$. Assume that $(g, h)$ is an irreducible symmetric pair with $\Delta^a \subseteq \Delta$. Without loss of generality, we assume that $(g, h) = (g', h'_{\varepsilon(\lambda)})$ for suitable basic symmetric pair $(g', h')$ and $\lambda \in \Psi'$, where $\Psi'$ is a simple root system of the restricted root system of $(g', h')$ and $\varepsilon(\lambda)$ denotes the signature of $\Delta$ defined by $\varepsilon(\mu) = 1$ if $\mu \in \Psi' \setminus \{\lambda\}$, $\varepsilon(\lambda) = -1$ (cf. Section 6 in [14]). Note that by using this assumption, we can express the inclusion $\Delta^a \subset \Delta$ explicitly (see, Table V in [14] and Table 1, 2 in [13]).

In the sequel, we shall follow notations of irreducible root systems in [7]:

$$A_n = \{e_i - e_j \mid 1 \leq i \neq j \leq n + 1\},$$
$$B_n = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\},$$
$$C_n = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{2e_i \mid 1 \leq i \leq n\},$$
$$D_n = \{e_i \pm e_j \mid 1 \leq i < j \leq n\},$$
$$(BC)_n = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i, 2e_i \mid 1 \leq i \leq n\}.$$

Denote by $s_\lambda (\lambda \in \Delta)$ the reflection on $a$ along $\lambda^{-1}(0)$.

3.1. Type $(\Delta, \Delta^a) = (C_n, A_{n-1})$. Assume that

$$A_{n-1} = \{e_i - e_j \mid 1 \leq i \neq j \leq n\} \subset C_n.$$

$\Psi := \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{2e_n\}$ is a simple root system of $\Delta$. For $\Delta = C_n$, $\mathcal{W}(\Delta)$ is generated by all permutations of $e_1, \ldots, e_n$, and all sign changes of the coefficients for $e_1, \ldots, e_n$. Set

$$t_i = \begin{cases} s_{e_i - e_{i+1}} \cdots s_{e_{n-1} - e_n} s_{2e_n} s_{e_n - e_{n-1}} \cdots s_{e_1 - e_{i+1}} (1 \leq i \leq n-1), \\ s_{2e_n} (i = n). \end{cases}$$

Since $\mathcal{W}(\Delta^a)$ is generated all permutations of $e_1, \ldots, e_n$, we have $t_i \notin \mathcal{W}(\Delta^a)$. Therefore all sign changes, that is, $\{\prod_{i=1}^{n} t_i^{l_i} \mid l_i = 0, 1\}$ gives a complete system of representatives for $\mathcal{W}(\Delta)/\mathcal{W}(\Delta^a)$. Indeed,

$$\left(\prod_{i=1}^{n} t_i^{l_i}\right)^{-1} \prod_{i=1}^{n} t_i^{k_i} = \prod_{i=1}^{n} t_i^{l_i+k_i} \notin \mathcal{W}(\Delta^a)$$

is equivalent to $l_{i_0} \neq k_{i_0}$ for some $1 \leq i_0 \leq n$. 
3.2. **Type** $(\Delta, \Delta^a) = (C_n, D_n)$. Assume that

$$D_n = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \subset C_n.$$  

$\Psi := \{e_i - e_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{2e_n\}$ is a simple root system of $\Delta$. Since $W(\Delta^a)$ is generated by all permutations of $e_1, \ldots, e_n$ and all even sign changes of $e_1, \ldots, e_n$, we have $s_{2e_n} \notin W(\Delta^a)$. Hence $\{id, s_{2e_n}\}$ gives a complete system of representatives for $W(\Delta)/W(\Delta^a)$.

3.3. **Type** $(\Delta, \Delta^a) = (D_n, A_{n-1})$. Assume that

$$A_{n-1} = \{e_i - e_j \mid 1 \leq i \neq j \leq n\} \subset D_n.$$  

$\Psi := \{e_i - e_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{e_{n-1} + e_n\}$ is a simple root system of $\Delta$. By the assumption, $W(\Delta^a)$ is generated by all permutations of $e_1, \ldots, e_n$. By a similar argument for the case of type $(C_n, A_{n-1})$, all even sign changes,

that is, $\{\prod_{i=1}^{n-1} (s_{e_i - e_{i+1}}s_{e_i + e_{i+1}})^{l_i} \mid l_i = 0, 1\}$ give a complete system of representatives for $W(\Delta)/W(\Delta^a)$.

**Remark 2.** Assume that $\Delta = D_n$ and

$$\Delta^a = \{e_i - e_j \mid 1 \leq i \neq j \leq n - 1\} \cup \{\pm(e_i + e_n) \mid 1 \leq i \leq n - 1\}(\cong A_{n-1}).$$

Then we can prove that $\{\prod_{i=1}^{n-1} (s_{e_i - e_{i+1}}s_{e_i + e_{i+1}})^{l_i} \mid l_i = 0, 1\}$ gives a complete system of representatives for $W(\Delta)/W(\Delta^a)$.

3.4. **Type** $(\Delta, \Delta^a) = (A_{n-1}, A_{p-1} \times A_{n-p-1})$. Assume that

$$A_{p-1} \times A_{n-p-1}$$

$$= \{e_i - e_j \mid 1 \leq i \neq j \leq p\} \cup \{e_i - e_j \mid p + 1 \leq i \neq j \leq n\} \subset A_{n-1}.$$  

$\Psi := \{e_i - e_{i+1} \mid 1 \leq i \leq n - 1\}$ is a simple root system of $\Delta$ and $\Psi \cap \Delta^a = \{e_i - e_{i+1} \mid i \neq p\}$ is a simple root system of $\Delta^a$. For $p = 0, n$, we have $\Delta = \Delta^a$, so that id gives a complete system of representative. For $p = 1, \ldots, n - 1$, we give the following formula for $W(\Delta)/W(\Delta^a)$.

**Proposition 3.1.**

$$W(\Delta)/W(\Delta^a) = W(\Gamma)/W(\Gamma^a) \cup \left\{w_{s_{e_n - e_n}} \cdots s_{e_{p} - e_{p+1}}w \in W(\Gamma)/W(\tilde{\Gamma}) \right\},$$

where $\Gamma := \{e_i - e_j \mid 1 \leq i \neq j \leq n - 1\}(\subset \Delta)$, $\Gamma^a := \Gamma \cap \Delta^a$ and $\tilde{\Gamma} := \{e_i - e_j \mid 1 \leq i \neq j \leq p - 1\} \cup \{e_i - e_j \mid p \leq i \neq j \leq n - 1\}$. 
Proof. By using the formula $nC_p = n-1C_{p-1} + n-1C_p$, the cardinalities of the both sides in (2) are the same. Let $w_i (1 \leq i \leq n-1C_{p-1})$ (resp. $z_j (1 \leq j \leq n-1C_p)$) be a complete system of representatives for $W(\Gamma)/W(\Gamma^a)$ (resp. $W(\Gamma)/W(\Gamma^a)$). By the definition of $\Gamma^a$, $W(\Gamma^a)$ is generated by all permutations of $e_1, \ldots, e_p$ and all permutations of $e_{p+1}, \ldots, e_{n-1}$. For $w_i, w_j (i \neq j)$, we have $w_i^{-1}w_j(e_p) = e_k$ for some $p+1 \leq k \leq n-1$. This implies that $w_i^{-1}w_j \notin W(\Delta^a)$ holds. Indeed, any element in $W(\Delta^a)$ must preserve $\{e_1, \ldots, e_p\}$ invariantly. For $w_i, z_j$, we have

$$w_i^{-1}(z_js_{e_{n-1}-e_n}\cdots s_{e_p-e_{p+1}})(e_p) = w_i^{-1}z_j(e_n) = e_n.$$ 

This implies that $w_i^{-1}(z_js_{e_{n-1}-e_n}\cdots s_{e_p-e_{p+1}}) \notin W(\Delta^a)$ holds. For $z_i, z_j (i \neq j)$, we have $z_i^{-1}z_j(e_k) = e_{k+1}$ for some $1 \leq k_1 \leq p - 1$ and $p \leq k_2 \leq n-1$. Then we have

$$(z_is_{e_{n-1}-e_n}\cdots s_{e_p-e_{p+1}})^{-1}(z_js_{e_{n-1}-e_n}\cdots s_{e_p-e_{p+1}})(e_{k_1}) = (s_{e_p-e_{p+1}}\cdots s_{e_{n-1}-e_n})(z_i^{-1}z_j)(e_{k_1}) = (s_{e_p-e_{p+1}}\cdots s_{e_{n-1}-e_n})(e_{k_2})$$

$$\in \{e_{p+1}, \ldots, e_n\}.$$ 

Therefore $(z_is_{e_{n-1}-e_n}\cdots s_{e_p-e_{p+1}})^{-1}(z_js_{e_{n-1}-e_n}\cdots s_{e_p-e_{p+1}}) \notin W(\Delta^a)$. Hence all $w_i$'s and $z_j$'s are not equivalent to each other in $W(\Delta^a)$. This proves Proposition 3.1.

Remark 3. By using Proposition 3.1 we can give a complete system of representatives for $W(\Delta)/W(\Delta^a)$, recursively.

Example 3.2 (Type $(\Delta, \Delta^a) = (A_4, A_1 \times A_2)$). We shall give a complete system of representatives for $W(\Delta)/W(\Delta^a)$ by using Proposition 3.1. Set $\Gamma^k := \{e_i - e_j \mid 1 \leq i \neq j \leq k\}(\subset \Delta)$ for $1 \leq k \leq 5$, and

$$\Gamma^l := \{e_i - e_j \mid 1 \leq i \neq j \leq p\} \cup \{e_i - e_j \mid p+1 \leq i \neq j \leq l\}(\subset \Gamma^l)$$

for $0 \leq p \leq l \leq 5$. Set $s_i = s_{e_{i-1}-e_{i+1}}$ for $1 \leq i \leq 4$. Then we have

$$W(\Delta)/W(\Delta^a)$$

$$= W(\Gamma^5)/W(\Gamma^{5,2})$$

$$= W(\Gamma^4)/W(\Gamma^{4,2}) \cup \{ws_4s_3s_2 \mid w \in W(\Gamma^4)/W(\Gamma^{4,1})\}$$

$$= W(\Gamma^3)/W(\Gamma^{3,2}) \cup \{ws_3s_2 \mid w \in W(\Gamma^3)/W(\Gamma^{3,1})\}$$

$$\cup \{ws_4s_3s_2 \mid w \in W(\Gamma^4)/W(\Gamma^{4,1})\}$$

$$= W(\Gamma^2)/W(\Gamma^{2,2}) \cup \{ws_2 \mid w \in W(\Gamma^2)/W(\Gamma^{2,1})\}$$

$$\cup \{ws_3s_2 \mid w \in W(\Gamma^3)/W(\Gamma^{3,1})\} \cup \{ws_4s_3s_2 \mid w \in W(\Gamma^4)/W(\Gamma^{4,1})\}$$
Moreover, we have $W(\Gamma^2)/W(\Gamma^{2,2}) = \{\text{id}\}$,
$$W(\Gamma^2)/W(\Gamma^{1,1}) = W(\Gamma^1)/W(\Gamma^{1,0}) \cup \{ws_1 \mid w \in W(\Gamma^1)/W(\Gamma^{1,0})\}$$
$$= \{\text{id}, s_1\},$$
$$W(\Gamma^3)/W(\Gamma^{2,1}) = W(\Gamma^2)/W(\Gamma^{2,0}) \cup \{ws_2s_1 \mid w \in W(\Gamma^2)/W(\Gamma^{2,0})\}$$
$$= \{\text{id}, s_1, s_2s_1\},$$
$$W(\Gamma^4)/W(\Gamma^{3,1}) = W(\Gamma^3)/W(\Gamma^{3,0}) \cup \{ws_3s_2s_1 \mid w \in W(\Gamma^3)/W(\Gamma^{3,0})\}$$
$$= \{\text{id}, s_1, s_2s_1, s_3s_2s_1\}.$$

Therefore we conclude that
$$W(\Delta)/W(\Delta^a) = \left\{\text{id}, s_1, s_2, s_3s_2, s_1s_3s_2, s_2s_1s_3s_2, s_4s_3s_2, \right\}.$$

3.5. **Type** $(\Delta, \Delta^a) = (B_n, B_p \times B_{n-p}), (C_n, C_p \times C_{n-p}), ((BC)_n, C_p \times (BC)_{n-p})$ or $((BC)_n, (BC)_p \times (BC)_{n-p})$. We consider the case where $(\Delta, \Delta^a)$ is of type $(B_n, B_p \times B_{n-p})$. Assume that
$$B_p \times B_{n-p} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq p\}$$
$$\cup \{\pm e_i \pm e_j \mid p + 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\} \subset B_n.$$

Since $W(\Delta)$ is equal to the Weyl group for $C_n$ (cf. Subsection 3.1), and $W(\Lambda^a)$ is generated by all permutations of $e_1, \ldots, e_p$, all permutations of $e_{p+1}, \ldots, e_n$ and all sign changes of the coefficients of $e_1, \ldots, e_n$, we have $W(\Delta)/W(\Delta^a) = W(\Lambda)/W(\Lambda^a)$, where $\Lambda := \{e_i - e_j \mid 1 \leq i \neq j \leq n\}(\subset \Delta)$ and $\Lambda^a = \Lambda \cap \Delta^a$. Moreover, by applying Proposition 3.1 to $W(\Lambda)/W(\Lambda^a)$, we can give a complete system of representatives for $W(\Delta)/W(\Delta^a)$. The arguments in the other cases are similar.

3.6. **Type** $(\Delta, \Delta^a) = (D_n, D_p \times D_{n-p})$ or $(B_n, D_p \times B_{n-p})$. We consider the case where $(\Delta, \Delta^a)$ is of type $(D_n, D_p \times D_{n-p})$. Assume that
$$D_p \times D_{n-p} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq p\} \cup \{\pm e_i \pm e_j \mid p + 1 \leq i < j \leq n\} \subset D_n.$$

Then $W(\Delta^a)$ is generated by all permutations of $e_1, \ldots, e_p$, all permutations of $e_{p+1}, \ldots, e_n$, all even sign changes of $e_1, \ldots, e_p$ and all even sign changes of $e_{p+1}, \ldots, e_n$. In particular, $s_{e_{p} - e_{p+1}}s_{e_{p} + e_{p+1}} \not\in W(\Delta^a)$ holds. Hence
$$W(\Delta)/W(\Delta^a) = \{(s_{e_{p} - e_{p+1}}s_{e_{p} + e_{p+1}})^l w \mid l = 0, 1, w \in W(\Lambda)/W(\Lambda^a)\},$$
where $\Lambda := \{e_i - e_j \mid 1 \leq i \neq j \leq n\}(\subset \Delta)$ and $\Lambda^a = \Lambda \cap \Delta^a$. By applying Proposition 3.1 to $W(\Lambda)/\mathcal{W}(\Lambda^a)$, we can give a complete system of representatives for $\mathcal{W}(\Lambda)/\mathcal{W}(\Lambda^a)$. By a similar argument, we can give a complete system of representatives in the case of type $(B_n, D_p \times B_{n-p})$.

4. Determination of hyperbolic principal isotropy subalgebras

In this section, we shall determine the (Lie algebra) structure of the HPIS for $s$-representation of an irreducible semisimple pseudo-Riemannian symmetric space $G/H$ by using the Satake diagrams associated with $G/H$. Denote by $\mathfrak{g}$ (resp. $\mathfrak{h}$) the Lie algebra of $G$ (resp. $H$). Let $\sigma$ be an involution of $\mathfrak{g}$ with $\mathfrak{h} = \text{Ker}(\sigma - \text{id})$. Suppose that $\theta$ is a Cartan involution of $\mathfrak{g}$ commuting with $\sigma$. Set $\mathfrak{k} := \text{Ker}(\theta - \text{id})$ and $\mathfrak{p} := \text{Ker}(\theta + \text{id})$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{a}_q$ (resp. $\mathfrak{a}_p$) be a maximal abelian subspace of $\mathfrak{q}$ (resp. $\mathfrak{p}$) containing $\mathfrak{a}$. Let $\tilde{\mathfrak{a}}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_q + \mathfrak{a}_p$. Note that any hyperbolic principal isotropy subalgebra is equal to the centralizer $\mathfrak{z}_\theta$ of $\mathfrak{a}$ in $\mathfrak{h}$. Denote by $\mathfrak{z}_\theta$ the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Since $\mathfrak{z}_\theta$ is invariant under $\sigma$ and $\theta$, we have the decomposition

$$\mathfrak{z}_\theta = \mathfrak{z}_\theta \cap (\mathfrak{k} \cap \mathfrak{h}) + \mathfrak{z}_\theta \cap (\mathfrak{p} \cap \mathfrak{h}) + \mathfrak{z}_\theta \cap (\mathfrak{k} \cap \mathfrak{q}) + \mathfrak{z}_\theta \cap (\mathfrak{p} \cap \mathfrak{q}).$$

It is clear that $\mathfrak{z}_\theta \cap (\mathfrak{p} \cap \mathfrak{q}) = \mathfrak{a}$. We also have the decomposition $\mathfrak{z}_\theta = \mathfrak{z}_\theta^c + \mathfrak{z}_\theta^s$, where $\mathfrak{z}_\theta^c$ (resp. $\mathfrak{z}_\theta^s$) denotes the center (resp. the semisimple part) of $\mathfrak{z}_\theta$. It is clear that $\mathfrak{z}_\theta^c$ is contained in $\tilde{\mathfrak{a}}$. Then their decomposition are compatible, i.e.,

$$\mathfrak{z}_\theta^c = \mathfrak{z}_\theta^c \cap (\mathfrak{k} \cap \mathfrak{h}) + \mathfrak{z}_\theta^c \cap (\mathfrak{p} \cap \mathfrak{h}) + \mathfrak{z}_\theta^c \cap (\mathfrak{k} \cap \mathfrak{q}) + \mathfrak{z}_\theta^c \cap (\mathfrak{p} \cap \mathfrak{q}),$$

$$\mathfrak{z}_\theta^s = \mathfrak{z}_\theta^s \cap (\mathfrak{k} \cap \mathfrak{h}) + \mathfrak{z}_\theta^s \cap (\mathfrak{p} \cap \mathfrak{h}) + \mathfrak{z}_\theta^s \cap (\mathfrak{k} \cap \mathfrak{q}) + \mathfrak{z}_\theta^s \cap (\mathfrak{p} \cap \mathfrak{q}).$$

Set $\tilde{\mathfrak{a}}^c = \tilde{\mathfrak{a}} \cap \mathfrak{z}_\theta^c$ and $\tilde{\mathfrak{a}}^s = \tilde{\mathfrak{a}} \cap \mathfrak{z}_\theta^s$. We can obtain the dimension of $\tilde{\mathfrak{a}}^c \cap (\mathfrak{k} \cap \mathfrak{h})$ (resp. $\tilde{\mathfrak{a}}^c \cap (\mathfrak{p} \cap \mathfrak{h})$, $\tilde{\mathfrak{a}}^c \cap (\mathfrak{k} \cap \mathfrak{q})$) by calculating $\dim \tilde{\mathfrak{a}} \cap (\mathfrak{k} \cap \mathfrak{h}) - \dim \tilde{\mathfrak{a}}^c \cap (\mathfrak{k} \cap \mathfrak{h})$ (resp. $\dim \tilde{\mathfrak{a}} \cap (\mathfrak{p} \cap \mathfrak{h}) - \dim \tilde{\mathfrak{a}}^c \cap (\mathfrak{p} \cap \mathfrak{h})$, $\dim \tilde{\mathfrak{a}} \cap (\mathfrak{k} \cap \mathfrak{q}) - \dim \tilde{\mathfrak{a}}^c \cap (\mathfrak{k} \cap \mathfrak{q})$).

(4.1) The structure of $\mathfrak{z}_\theta^c$. Denote by $R$ the root system of $\mathfrak{g}^C$ with respect to $\tilde{\mathfrak{a}}^C$, and by $\mathfrak{g}_\alpha^C$ the root space of $\mathfrak{g}^C$ associated with $\alpha \in R$. Then we have the decomposition

$$(\mathfrak{z}_\theta^c)^C = \text{Span}_C\{A_\alpha \mid \alpha \in R_0\} + \sum_{\alpha \in R_0} \mathfrak{g}_\alpha^C,$$
and $\dim C(z^i_a)^C = \dim C \overline{a}^C - \text{rank } R_0$, where $A_\alpha \in \overline{a}^C$ ($\alpha \in R$) defined by $\alpha(A) = B(A, A_\alpha)$ for all $A \in \overline{a}^C$ ($B$ is the Killing form of $g^C$), and $R_0 = \{\alpha \in R \mid \alpha|_a = 0\}$. Let $\Phi$ be a simple root system of $R$. Since $\Phi_0 := \Phi \cap R_0$ is a simple root system of $R_0$, the black circles in the Satake diagram associated with $G/H$ with respect to $a$ determines the Dynkin diagram of $(z^i_a)^C$. In particular, the rank of $R_0$ is equal to the number of the black circles.

4.2. The structures of the semisimple part of $\mathfrak{z}_0$ and $\mathfrak{z}_b$. In this subsection, we shall determine the structures $\mathfrak{z}_0^i$ and $\mathfrak{z}_b$. Suppose that the Satake diagrams associated with $G/H$ with respect to $a$, $a_q$ and $a_p$ are compatible with one another. Denote by $p_\sigma$ (resp. $p_\theta$) the Satake involution of the Satake diagram associated with $G/H$ with respect to $a_q$ (resp. $a_p$).

Let $R_0 = R_0^1 \cup R_0^2 \cdots \cup R_0^k$ be the irreducible decomposition of $R_0$. Denote by $\mathfrak{z}(G)$ ($G$ is a closed subsystem of $R_0$) the subalgebra of $(\mathfrak{z}_0^i)^C$ generated by $\{g^C \mid \alpha \in \Gamma\}$. Set, for each $i \in \{1, \ldots, k\}$, $\mathfrak{z}_0^i := \Phi_0 \cap R_0^i$, $\Phi_{0,q}^i := \{\alpha \in \Phi_0^i \mid \alpha|_{a_q} = 0\}$ and $\Phi_{0,p}^i := \{\alpha \in \Phi_0^i \mid \alpha|_{a_p} = 0\}$. Then, for each $i \in \{1, \ldots, k\}$, $\Phi_0^i$ satisfies one of the following.

Case 1: $\Phi_0^i = \Phi_{0,q}^i = \Phi_{0,p}^i$.

Case 2: $\Phi_0^i \setminus \Phi_{0,p}^i(\neq \emptyset)$ is $p_\theta$-invariant and $\Phi_0^i = \Phi_{0,q}^i$.

Case 3: $\Phi_0^i = \Phi_{0,p}^i$ and $\Phi_0^i \setminus \Phi_{0,q}^i(\neq \emptyset)$ is $p_\sigma$-invariant.

Case 4: $\Phi_0^i \setminus \Phi_{0,p}^i(\neq \emptyset)$ is $p_\theta$-invariant and $\Phi_0^i \setminus \Phi_{0,q}^i(\neq \emptyset)$ is $p_\sigma$-invariant.

Case 5: $\Phi_0^i \setminus \Phi_{0,p}^i(\neq \emptyset)$ is not $p_\theta$-invariant.

Case 6: $\Phi_0^i \setminus \Phi_{0,p}^i(\neq \emptyset)$ is not $p_\sigma$-invariant.

Set $g^d := \mathfrak{t} \cap \mathfrak{h} + \sqrt{-1}\mathfrak{p} \cap \mathfrak{h} + \sqrt{-1}\mathfrak{k} \cap \mathfrak{q} + \mathfrak{p} \cap \mathfrak{q}(\subset g^C)$. Denote by $\tau$ (resp. $\tau^d$) the conjugation of $g^C$ with respect to $g$ (resp. $g^d$). For each $\alpha \in R$, the linear functions $\tau \cdot \alpha$, $\tau^d \cdot \alpha$, $\theta \cdot \alpha$ and $\sigma \cdot \alpha$ defined by, for all $A \in \overline{a}^C$,

$$(\tau \cdot \alpha)(A) = \overline{\alpha(\tau A)}, \quad (\tau^d \cdot \alpha)(A) = \overline{\alpha(\tau^d A)},$$

$$(\theta \cdot \alpha)(A) = \alpha(\theta A), \quad (\sigma \cdot \alpha)(A) = \alpha(\sigma A),$$

are again elements of $R$.

**Lemma 4.1.** If $\Phi_0^i \setminus \Phi_{0,p}^i(\neq \emptyset)$ is $p_\theta$-invariant, then $R_0^i$ is $\theta$-invariant.

**Proof.** Since $\theta \cdot \alpha = \alpha \in \Phi_{0,p}^i$ for all $\alpha \in \Phi_{0,p}^i$, $\Phi_{0,p}^i$ is $\theta$-invariant. Let $\alpha$ be a root in $\Phi_0^i$ such that $\alpha|_{a_p} \neq 0$. It follows from Lemma 4.2 that if $\alpha \in \Phi_0^i \setminus \Phi_{0,p}^i$ is disconnected with all roots of $\Phi_{0,p}^i$, then we have $\theta \cdot \alpha = -p_\theta \alpha \in R_0^i$. In the case where $\alpha \in \Phi_0^i \setminus \Phi_{0,p}^i$ is connected with a root of $\Phi_{0,p}$, $\theta \cdot \alpha$ has the form $-p_\theta \alpha + \sum_{\beta \in \langle \alpha \rangle} \mathbb{Z} \beta$, and $\langle \alpha \rangle$ is contained in $\Phi_0^i$, where $\langle \alpha \rangle$ is the union of the
connected components of $\Phi_{0,\theta}$ connected with $\alpha$. Hence we have $\theta \cdot \alpha \in R_0$. Since $\Phi_0$ is a simple root system of $R_0$, $R_0$ is $\theta$-invariant. □

By a similar argument as Lemma 4.1 we have the following result.

**Lemma 4.2.** If $\Phi_{0,\theta}(\neq \emptyset)$ is $p_\sigma$-invariant, then $R_0$ is $\sigma$-invariant.

**Lemma 4.3.** For any $\alpha \in R_0$ with $\alpha|_{a_p} = 0$, $g_{\alpha}^C \subset \mathfrak{k}^C$ holds.

**Proof.** For any $\alpha \in R_0$ with $\alpha|_{a_p} = 0$, we have $\theta \cdot \alpha = \alpha$, so that $g_{\alpha}^C$ is $\theta$-invariant. Then, for any $X \in g_{\alpha}^C$, we have $[A, X - \theta X] = 0$ for all $A \in a_p$. By the maximality of $a_p$ in $p$, $X - \theta X \in a_p^C$ holds. Since $X - \theta X \in a_p^C \cap g_{\alpha}^C = \{0\}$, we have $\theta X = X$. Hence $g_{\alpha}^C \subset \mathfrak{k}^C$ holds. □

By a similar argument as Lemma 4.3 we have the following fact.

**Lemma 4.4.** For any $\alpha \in R_0$ with $\alpha|_{a_q} = 0$, $g_{\alpha}^C \subset \mathfrak{h}^C$ holds.

Note that, for each $\alpha \in R$, $\tau \cdot \alpha = -\theta \cdot \alpha$ and $\tau^d \cdot \alpha = -\sigma \cdot \alpha$ hold. Hence it is shown that, for any closed subsystem $\Gamma$, $\Gamma$ is $\tau$-invariant (resp. $\tau^d$-invariant) if and only if $\Gamma$ is $\theta$-invariant (resp. $\sigma$-invariant). We also have the following fact.

**Lemma 4.5.** Let $\alpha$ be a root in $R_0$. If $\alpha$ satisfies $\alpha|_{a_p} = 0$ (resp. $\alpha|_{a_q} = 0$), then $g_{\alpha}^C + g_{\alpha}^C$ is $\tau$-invariant (resp. $\tau^d$-invariant).

In the sequel, we shall determine the structures of the irreducible factors of $s_\theta^*$ and associated with $\Phi^i_0$ and these $\mathfrak{h}$-parts.

Case 1. It is clear that $R_0^i$ is invariant under $\sigma$ and $\theta$. It follows from Lemma 4.3 and Lemma 4.4 that $\tilde{\mathfrak{z}}(R_0^i)$ is a subalgebra of $\mathfrak{k}^C \cap \mathfrak{h}^C$. Set $\tilde{\mathfrak{z}}_\theta^i := \tilde{\mathfrak{z}}(R_0^i) \cap \mathfrak{g}$, which is a real form of $\tilde{\mathfrak{z}}(R_0)$ by using Lemma 4.5. Moreover, we have $\tilde{\mathfrak{z}}_\theta^i = \tilde{\mathfrak{z}}_\theta^i \cap \mathfrak{h} \subset \mathfrak{k} \cap \mathfrak{h}$. In particular, $\tilde{\mathfrak{z}}_\theta^i$ is a compact real form of $\tilde{\mathfrak{z}}(R_0)$, which is uniquely determined (up to isomorphism) by the Dynkin diagram of $\Phi^i_0$.

Case 2. It follows from Lemma 4.1 that $R_0^i$ is $\theta$-invariant. This implies that $\tilde{\mathfrak{z}}(R_0)$ is $\tau$-invariant. By using Lemma 4.4, we have $\tilde{\mathfrak{z}}_\theta^i := \tilde{\mathfrak{z}}(R_0) \cap \mathfrak{g} \subset \mathfrak{h}$. Moreover, $\theta|_{\tilde{\mathfrak{z}}_\theta^i}$ is a Cartan involution of $\tilde{\mathfrak{z}}_\theta^i$. Then $(\tilde{\mathfrak{z}}_\theta^i, \tilde{\mathfrak{z}}_\theta^i \cap \mathfrak{k})$ is an irreducible Riemannian symmetric pair (of noncompact-type). Moreover, its Satake diagram is given by the Dynkin diagram of $\Phi^i_0$ and $p_\theta|_{\Phi^i_0 \setminus \Phi^i_0}$.

Case 3. By using Lemma 4.3 we have $\tilde{\mathfrak{z}}(R_0^i) \subset \mathfrak{k}^C$. Then $\tilde{\mathfrak{z}}_\theta^i := \tilde{\mathfrak{z}}(R_0) \cap \mathfrak{g} \subset \mathfrak{k}$ and $\tilde{\mathfrak{z}}_\theta^i \cap \mathfrak{h} \subset \mathfrak{k} \cap \mathfrak{h}$ hold. Note that $\sigma|_{\tilde{\mathfrak{z}}_\theta^i}$ is not trivial. Then $(\tilde{\mathfrak{z}}_\theta^i, \tilde{\mathfrak{z}}_\theta^i \cap \mathfrak{h})$ is an irreducible Riemannian symmetric pair (of compact-type). Moreover, its Satake diagram is given by the Dynkin diagram of $\Phi^i_0$ and $p_\sigma|_{\Phi^i_0 \setminus \Phi^i_0}$.
Case 4. In this case, \( \mathfrak{g}_0^i := \mathfrak{z}(R_0^i) \cap \mathfrak{g} \) is a noncompact subalgebra of \( \mathfrak{g} \), and \((\mathfrak{z}_0^i, \mathfrak{z}_0^i \cap \mathfrak{h})\) is an irreducible semisimple symmetric pair. Since \( \mathfrak{z}_0 \cap (\mathfrak{p} \cap \mathfrak{q}) = a \subset \mathfrak{z}_0^i \) holds, we have \( \mathfrak{z}_0^i \cap (\mathfrak{p} \cap \mathfrak{q}) \subset \mathfrak{z}_0^i \cap (\mathfrak{p} \cap \mathfrak{q}) = \{0\} \). This contradicts the fact that any noncompact irreducible semisimple symmetric pair has the split rank greater than or equal to one. Hence Case 4 cannot occur.

Case 5. Denote by \( \tilde{R}_0^i \) the smallest \( \theta \)-invariant closed subsystem of \( R_0^i \) containing \( R_0^i \). Then \( \tilde{R}_0^i \) is not connected, and \( \tilde{z}(\tilde{R}_0^i) \) is \( \tau \)-invariant. Set \( \tilde{z}_0^i := \tilde{z}(\tilde{R}_0^i) \cap \mathfrak{g} \). Since \( \theta|_{\mathfrak{z}_0^i} \) is not trivial, \((\tilde{z}_0^i, \tilde{z}_0^i \cap \mathfrak{t})\) is an irreducible Riemannian symmetric pair. By the classification of irreducible Riemannian symmetric pairs, \((\tilde{z}_0^i, \tilde{z}_0^i \cap \mathfrak{t})\) is isomorphic to a Riemannian symmetric pair \((\mathfrak{m}^C, \mathfrak{m})\) (of compact real form type), where \( \mathfrak{m} \) is a simple compact Lie algebra. Therefore we have \( \Phi_0^i = \emptyset \). Moreover, the Dynkin diagram of \( \mathfrak{l} \) is equal to \( \Phi_0^i \). It follows from Lemma 2.8 in [14] that \( \Phi_0^i = \Phi_0^i,0 \) holds. Hence \( \tilde{z}_0^i \) is contained in \( \mathfrak{h} \).

Case 6. Denote by \( \tilde{R}_0^i \) the smallest \( \sigma \)-invariant closed subsystem of \( R_0^i \) containing \( R_0^i \), which is not connected. Then \( \tilde{z}(\tilde{R}_0^i) \) is \( \tau \)-invariant. Set \( \tilde{z}_0^i := \tilde{z}(\tilde{R}_0^i) \). Since \( \sigma|_{\tilde{z}_0^i} \) is not trivial, \((\tilde{z}_0^i, \tilde{z}_0^i \cap \mathfrak{t}^e)\) is isomorphic to a Riemannian symmetric pair \((\mathfrak{m}^C, \mathfrak{m})\) (of compact real form type), where \( \mathfrak{m} \) is a simple compact Lie algebra. Note that the Dynkin diagram of \( \mathfrak{m} \) is equal to \( \Phi_0^i \). Then we have \( \Phi_0^i,0 = \emptyset \). It follows from Lemma 2.8 in [14] that \( \Phi_0^i = \Phi_0^i,0 \) holds. Hence \( \tilde{z}(\tilde{R}_0^i) \) is \( \tau \)-invariant, and \( \tilde{z}_0^i := \tilde{z}(\tilde{R}_0^i) \cap \mathfrak{g} \) is subalgebra of \( \mathfrak{t} \). Then \((\tilde{z}_0^i, \tilde{z}_0^i \cap \mathfrak{h})\) is isomorphic to \((\mathfrak{m} + \mathfrak{m}, \mathfrak{m})\).

Here, we give a recipe to determine the hyperbolic principal isotropy subalgebra as follows.

**Recipe 4.6 (hyperbolic principal isotropy subalgebras).** Let \((\mathfrak{g}, \mathfrak{h})\) be an irreducible semisimple symmetric pair.

- **Step 1.** We calculate all the irreducible components \( \Phi_0^i \) of \( \Phi_0 \) by using the Satake diagram \( S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}) \).
- **Step 2.** For each \( i \), we investigate whether \( \Phi_0^i \) corresponds to either Case 1–3, 5 or 6 by using the Satake diagrams \( S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}) \) and \( S(\mathfrak{g}, \mathfrak{t}, \mathfrak{a}_p) \).
- **Step 3.** For each \( i \), we determine the following subalgebra of \( \mathfrak{z}_0^i \) associated with \( \Phi_0^i \).
  - **Case 1:** We determine \( \mathfrak{z}_0^i(\subset \mathfrak{h}) \) by investigating the Dynkin diagram of \( \Phi_0^i \).
Case 2: We determine \( (\tilde{\mathfrak{f}}^i, \tilde{\mathfrak{f}}^i \cap \mathfrak{k}) \) by investigating the Satake diagram obtained from the Dynkin diagram of \( \Phi_0^i \) and \( p_\theta|_{\tilde{\mathfrak{f}}^i \cap \mathfrak{k}_{0, p}} \). Note that \( \tilde{\mathfrak{f}}^i \) is contained in \( \mathfrak{h} \).

Case 3: We determine \( (\tilde{\mathfrak{f}}^i, \tilde{\mathfrak{f}}^i \cap \mathfrak{h}) \) by investigating the Satake diagram obtained from the Dynkin diagram of \( \Phi_0^i \) and \( p_\sigma|_{\tilde{\mathfrak{f}}^i \cap \mathfrak{h}_{0, q}} \).

Case 5: We calculate \( \Phi_0^i \cup p_\theta \Phi_0^i (=: \hat{\Phi}_0^i) \) by using the Satake diagrams \( S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}) \) and \( S(\mathfrak{g}, \mathfrak{t}, \mathfrak{a}_p) \). We determine \( (\tilde{\mathfrak{f}}^i, \tilde{\mathfrak{f}}^i \cap \mathfrak{t}) \) by investigating the Satake diagram obtained from the Dynkin diagram of \( \Phi_0^i \) and \( p_\theta|_{\tilde{\mathfrak{f}}^i \cap \mathfrak{t}} \). Note that \( \tilde{\mathfrak{f}}^i \) is contained in \( \mathfrak{h} \).

Case 6: We calculate \( \Phi_0^i \cup p_\sigma \Phi_0^i (=: \hat{\Phi}_0^i) \) by using the Satake diagrams \( S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}) \) and \( S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_q) \). We determine \( (\tilde{\mathfrak{f}}^i, \tilde{\mathfrak{f}}^i \cap \mathfrak{h}) \) by investigating the Satake diagram obtained from the Dynkin diagram of \( \Phi_0^i \) and \( p_\sigma|_{\tilde{\mathfrak{f}}^i \cap \mathfrak{h}} \).

Step 4. We calculate the following dimensions.

\[
\dim \tilde{\mathfrak{a}} \cap (\mathfrak{t} \cap \mathfrak{h}) = \text{rank } \mathfrak{g}^C - \text{rank } (\mathfrak{g}, \mathfrak{h}) - \text{rank } (\mathfrak{g}, \mathfrak{t}) + \text{s-rank } (\mathfrak{g}, \mathfrak{h}),
\]

\[
\dim \tilde{\mathfrak{a}}^c \cap (\mathfrak{t} \cap \mathfrak{h}) = \dim \tilde{\mathfrak{a}} \cap (\mathfrak{t} \cap \mathfrak{h}) - \dim \tilde{\mathfrak{a}} \cap (\mathfrak{t} \cap \mathfrak{h}),
\]

\[
\dim \tilde{\mathfrak{a}}^c \cap (\mathfrak{p} \cap \mathfrak{h}) = \dim \tilde{\mathfrak{a}} \cap (\mathfrak{p} \cap \mathfrak{h}) - \dim \tilde{\mathfrak{a}} \cap (\mathfrak{p} \cap \mathfrak{h}).
\]

Step 5. From the data in Steps 1–4, we determine \( \tilde{\mathfrak{f}}^i \).

**Example 4.7** \( ((\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(2p, 2(n - p)), \mathfrak{sp}(p, n - p))) \). First, we calculate \( \tilde{\mathfrak{f}}^i \) in the case of \( n > 2p \). We give the Satake diagrams associated with \( (\mathfrak{g}, \mathfrak{h}) \) with respect to \( \mathfrak{a}, \mathfrak{a}_p \) and \( \mathfrak{a}_q \), respectively (see Table 3).

### Table 3: \( (\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(2p, 2(n - p)), \mathfrak{sp}(p, n - p))(n > 2p) \)

| the Satake diagram \( S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}) \) | the Satake diagram \( S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_p) \) | the Satake diagram \( S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_q) \) |
| --- | --- | --- |
| ![Diagram 1](attachment:diagram1.png) | ![Diagram 2](attachment:diagram2.png) | ![Diagram 3](attachment:diagram3.png) |
Step 1. Set $\Phi_0^i := \{\alpha_{2i-1}\}$, $\Phi_0^{p+i} := \{\alpha_{2n-(2i-1)}\}$ for $1 \leq i \leq p$, and $\Phi_0^{2p+1} := \{\alpha_{2p+1}, \ldots, \alpha_{2n-(2p+1)}\}$. From $S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$, $\Phi_0^i$ is an irreducible component of $\Phi_0$ for $1 \leq j \leq 2p + 1$.

Step 2. It follows from $S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ and $S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_q)$ that $\Phi_0^i$ and $\Phi_0^{p+i}$ correspond to Case 5, and $\Phi_0^{2p+1}$ corresponds to Case 3. We obtain $\Phi_0^i = \Phi_0^i \cup p_0^i \Phi_0^i = \Phi_0^i \cup \Phi_0^{p+i}$ for $1 \leq i \leq p$.

Step 3. For $1 \leq i \leq p$, $(\mathfrak{z}_h^i, \mathfrak{z}_h^i \cap \mathfrak{h})$ is isomorphic to $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2))$ because their Satake diagram are “◦ ◦”. We also have $(\mathfrak{z}_h^{2p+1}, \mathfrak{z}_h^{2p+1} \cap \mathfrak{h}) \cong (\mathfrak{su}(2(n-2p)), \mathfrak{sp}(n-2p))$.

Step 4. We obtain $\dim \mathfrak{a} \cap (\mathfrak{f} \cap \mathfrak{h}) = 0$, $\dim \mathfrak{a} \cap (\mathfrak{p} \cap \mathfrak{h}) = 0$. Here, we have $\text{rank} \mathfrak{g}^C = 2n - 1$, $\text{s-rank}(\mathfrak{g}, \mathfrak{h}) = p$, $\text{rank}(\mathfrak{g}, \mathfrak{h}) = n - 1$, $\text{rank}(\mathfrak{g}, \mathfrak{f}) = 2p$, $\dim \mathfrak{a}^c \cap (\mathfrak{f} \cap \mathfrak{h}) = n - p$, and $\dim \mathfrak{a} \cap (\mathfrak{p} \cap \mathfrak{h}) = p$.

Step 5. It follows from Step 1–4 that $\mathfrak{z}_h$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})^p + \mathfrak{sp}(-2p)(n > 2p)$. In the case of $n = 2p$, we have $\mathfrak{z}_h \cong \mathfrak{sl}(2, \mathbb{C})^p$, by the similar calculation. Hence we have $\mathfrak{z}_h \cong \mathfrak{sl}(2, \mathbb{C})^p + \mathfrak{sp}(-2p)(n \geq 2p)$. Note that we have $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3, 1) \subset \mathfrak{so}(4, 1) \cong \mathfrak{sp}(1, 1)$ by using the list of special isomorphisms (see, Section 4 of Chapter X in [6]).

Example 4.8 \(((\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}(n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{R})))\). First, we determine $\mathfrak{z}_h$ in the case of $n = 2m$. We give the Satake diagrams associated with $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}, \mathfrak{a}_p$ and $\mathfrak{a}_q$, respectively (see Table 4).

| Table 4: $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}(2m, \mathbb{C}), \mathfrak{sl}(2m, \mathbb{R}))$ |
|---|---|
| the Satake diagram $S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ | the Satake diagrams $S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_p)$ |
| the Satake diagram $S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_q)$ |
| ![Satake Diagram](image) | ![Satake Diagram](image) |

Step 1. From $S(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ we have $\Phi_0 = \emptyset$.

Step 2. It is clear that $\Phi_0$ corresponds to Case 1.

Step 3. It is clear that $\mathfrak{z}_h^i \cap \mathfrak{h} = \{0\}$.

Step 4. We have $\dim \mathfrak{a}^c \cap (\mathfrak{f} \cap \mathfrak{h}) = m$ and $\dim \mathfrak{a} \cap (\mathfrak{p} \cap \mathfrak{h}) = m - 1$ by using $\text{rank} \mathfrak{g}^C = 2(2m - 1)$, $\text{s-rank}(\mathfrak{g}, \mathfrak{h}) = m$, $\text{rank}(\mathfrak{g}, \mathfrak{h}) = 2m - 1$, and $\text{rank}(\mathfrak{g}, \mathfrak{f}) = 2m - 1$. 


Step 5. It follows from Step 1–4 that \( \mathfrak{h} \) is isomorphic to \( R^{m-1} + \mathfrak{so}(2)^m \) \((n = 2m)\). In the case of \( n = 2m + 1 \), we have \( \mathfrak{h} \cong R^n + \mathfrak{so}(2)^m \), by a similar calculation as above. Hence we have \( \mathfrak{h} \cong R^{(n-1)/2} + \mathfrak{so}(2)^{[n/2]} \), where \([n-1)/2] \) (resp. \([n/2] \)) is the greatest integer less than or equal to \((n-1)/2 \) (resp. \(n/2\)).

5. Determination of Local Orbit Types

Let \((\mathfrak{g}, \mathfrak{h})\) be a semisimple symmetric pair and \(\sigma\) be an involution of \(\mathfrak{g}\) with \(\mathfrak{h} = \text{Ker}(\sigma - \text{id})\). Suppose that \(\theta\) is a Cartan involution of \(\mathfrak{g}\) commuting with \(\sigma\), and \(\mathfrak{a}\) is maximal abelian subspace of \(\mathfrak{p} \cap \mathfrak{q}\). Denote by \(\Delta\) the restricted root system of \((\mathfrak{g}, \mathfrak{h})\) with respect to \(\mathfrak{a}\). Set \(\Delta^\sigma := \{\lambda \in \Delta \mid \mathfrak{g}_\lambda \cap \mathfrak{h}^\sigma \neq \{0\}\}\). Denote by \(\mathcal{W}(\Delta) \) (resp. \(\mathcal{W}(\Delta^\sigma)\)) the Weyl group of \(\Delta\) (resp. \(\Delta^\sigma := \{\lambda \in \Delta \mid m^+(\lambda) > 0\}\)). Let \(w_1, \ldots, w_l\) be a complete system of representatives for \(\mathcal{W}(\Delta)/\mathcal{W}(\Delta^\sigma)\). From Theorem [A] we can determine the set of all local orbit types of the hyperbolic orbits by investigating \(\{[\mathfrak{h}_\Theta] \mid \Theta \subset w_i \cdot \Psi\}\) for all \(i \in \{1, \ldots, l\}\). By using the recipe in [I], we can determine \(\{[\mathfrak{h}_\Theta] \mid \Theta \subset w_i \cdot \Psi\}\) for each \(i \in \{1, \ldots, l\}\). In fact, we can determine \(\mathfrak{h}_\Theta\) for each \(\Theta \subset w_i \cdot \Psi\), by using the hyperbolic principal isotropy subalgebra and a subsymmetric pair of \((\mathfrak{g}, \mathfrak{h})\) associated with \(\Delta_\Theta := \Delta \cap \sum_{\lambda \in \Theta} \mathbb{R}\lambda\) (see, page 315 in [I] for more detail).

**Example 5.1** ((\(\mathfrak{g}, \mathfrak{h}\) = (\(\mathfrak{sl}(4, R), \mathfrak{so}(2, 2)\))). The Dynkin diagram of the restricted root system of \((\mathfrak{g}, \mathfrak{h})\) is the following.

\[
\begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\hline
m^+(\lambda_i) & m^+(2\lambda_i) & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (i = 1, 3) \\
m^-(\lambda_i) & m^-(2\lambda_i) & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (i = 2)
\end{array}
\]

Set \(\Psi := \{\lambda_1, \lambda_2, \lambda_3\} = \{e_1 - e_2, e_2 - e_3, e_3 - e_4\}\). It follows from Table [I] that the HPIS is equal to \(\{0\}\). Moreover, from Example 3.2 we have \(\mathcal{W}(\Delta)/\mathcal{W}(\Delta^\sigma) = \{\text{id}, s_2, s_1s_2, s_3s_2, s_1s_3s_2, s_2s_1s_3s_2\}\).

By using Theorem [A] all the possible isotropy subalgebras of the hyperbolic orbits for the \(s\)-representation associated with \((\mathfrak{g}, \mathfrak{h})\) are equal to \(\mathfrak{h}_\Theta\)'s for all \(\Theta \subset w \cdot \Psi, w \in \mathcal{W}(\Delta)/\mathcal{W}(\Delta^\sigma)\). For each \(w \in \mathcal{W}(\Delta)/\mathcal{W}(\Delta^\sigma)\), we can determine \(\{[\mathfrak{h}_\Theta] \mid \Theta \subset w \cdot \Psi\}\) by using the recipe in [I]. In Table 5 we shall give the lists of all the possible isotropy subalgebras of the hyperbolic orbits for the \(s\)-representation associated with \((\mathfrak{g}, \mathfrak{h})\).
Example 5.2. Let $G/H$ be a semisimple pseudo-Riemannian symmetric space. Suppose that the restricted root system $\Delta$ with respect to a vector-type maximal split abelian subspace satisfies $\Delta = \Delta^a$ or $(\Delta, \Delta^a) = ((BC)_r, B_r)$. Then we have $\mathcal{W}(\Delta)/\mathcal{W}(\Delta^a) = \{\text{id}\}$. It follows from Theorem $\Delta$ that $\mathcal{L}_h(G/H) = \{[h_\Theta] | \Theta \subset \Psi\}$ holds, where $\Psi$ is a standard simple root system of $\Delta$. In Table $\Psi$ we list up the set of all the possible local orbit types of the hyperbolic orbits for the $s$-representations associated with all classical-type semisimple pseudo-Riemannian symmetric spaces satisfying $\Delta = \Delta^a$ or $(\Delta, \Delta^a) = ((BC)_r, B_r)$.
Table 6: Local orbit types

(I) $\Delta = \Delta^a = A_r, \Psi = \{e_i - e_{i+1} \mid 1 \leq i \leq r\}$

$\Gamma(g, h) = (\mathfrak{sl}(n, R) + \mathfrak{sl}(n, R), \mathfrak{sl}(n, R))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i_k} - e_{i_{k+1}}\}$ | $h_\Theta$ |
|----------------------|----------------------------------|------------------|
| $R^k + \sum_{l=1}^{k+1} \mathfrak{sl}(i_l - i_{l-1}, R)$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n$ |

$(g, h) = (\mathfrak{sl}(n, C), \mathfrak{so}(n, C))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i_k} - e_{i_{k+1}}\}$ | $h_\Theta$ |
|----------------------|----------------------------------|------------------|
| $\sum_{l=1}^{k+1} \mathfrak{so}(i_l - i_{l-1}, C)$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n$ |

$(g, h) = (\mathfrak{su}^*(2n) + \mathfrak{su}^*(2n), \mathfrak{su}^*(2n))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i_k} - e_{i_{k+1}}\}$ | $h_\Theta$ |
|----------------------|----------------------------------|------------------|
| $R^k + \sum_{l=1}^{k+1} \mathfrak{su}^*(2i_l - i_{l-1})$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n$ |

$(g, h) = (\mathfrak{sl}(2n, C), \mathfrak{sp}(n, C))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i_k} - e_{i_{k+1}}\}$ | $h_\Theta$ |
|----------------------|----------------------------------|------------------|
| $\sum_{l=1}^{k+1} \mathfrak{sp}(i_l - i_{l-1}, C)$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n$ |

$(g, h) = (\mathfrak{sl}(2n, R), \mathfrak{sp}(n, R))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i_k} - e_{i_{k+1}}\}$ | $h_\Theta$ |
|----------------------|----------------------------------|------------------|
| $\sum_{l=1}^{k+1} \mathfrak{sp}(i_l - i_{l-1}, R)$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n$ |

$(g, h) = (\mathfrak{su}^*(2n), \mathfrak{so}^*(2n))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i_k} - e_{i_{k+1}}\}$ | $h_\Theta$ |
|----------------------|----------------------------------|------------------|
| $\sum_{l=1}^{k+1} \mathfrak{so}^*(2i_l - i_{l-1})$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n$ |

(II) $\Delta = \Delta^a = B_r, \Psi = \{e_i - e_{i+1} \mid 1 \leq i \leq r - 1\} \cup \{e_r\}$

$\Gamma(g, h) = (\mathfrak{so}(p, n-p) + \mathfrak{so}(p, n-p), \mathfrak{so}(p, n-p)) \ (n > 2p)$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i_k} - e_{i_{k+1}}, e_p\}$ | $h_\Theta$ |
|----------------------|----------------------------------|------------------|
| $R^k + \sum_{l=1}^{k} \mathfrak{so}(n - 2p)$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k = p$ |

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i_k} - e_{i_{k+1}}\}$ | $h_\Theta$ |
|----------------------|----------------------------------|------------------|
| $R^k + \sum_{l=1}^{k} \mathfrak{so}(p - i_k, n - p - i_k)$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k < p$ |
(III) $\Delta = \Delta^0 = C_r$, $\Psi = \{e_i - e_{i+1} \mid 1 \leq i \leq r - 1\} \cup \{2e_r\}$

$(g, h) = (\mathfrak{so}(2n, C), \mathfrak{su}^*(2n))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k} - e_{i_k + 1}, 2e_n\}$ | $h_\Theta$ | $R^k + \mathfrak{so}(2)^k + \sum_{l=1}^{k} \mathfrak{s}(i_l - i_{l-1}, C) + \mathfrak{su}(2(n - i_k))$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k < n$ |
|-------------------------|-------------------------------------------------|------------|-------------------------------------------------|-----------------|--------------------------|
| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k} - e_{i_k + 1}\}$ | $h_\Theta$ | $R^k + \mathfrak{so}(2)^k + \sum_{l=1}^{k} \mathfrak{s}(i_l - i_{l-1}, C) + \mathfrak{su}(n - i_k, n - i_k)$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k < n$ |

$(g, h) = (\mathfrak{su}(n, n) + \mathfrak{su}(n, n), \mathfrak{su}(n, n))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k} - e_{i_k + 1}, 2e_n\}$ | $h_\Theta$ | $C^{k-1} + \sum_{i=1}^{k} \mathfrak{s}(i_l - i_{l-1}, C)$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k = n$ |
| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k} - e_{i_k + 1}\}$ | $h_\Theta$ | $C^k + \sum_{i=1}^{k} \mathfrak{s}(i_l - i_{l-1}, C) + \mathfrak{s}(n - i_k, C)^2 + C$ | Remarks | $0 = i_0 < i_1 < \cdots < i_k < n$ |
Table 6: (continued)

\((g, h) = (\mathfrak{so}^*(4n) + \mathfrak{so}^*(4n), \mathfrak{so}^*(4n))\)

| \(\Theta(\subset \Psi)\) | \(\psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n\}\) |
|-----------------|---------------------------------|
| \(h_{\Theta}\) | \(R^k + \sum_{l=1}^{k} \mathfrak{su}^*(2(i_l - i_{l-1})))\) |
| Remarks | 0 = \(i_0 < i_1 < \cdots < i_k = n\) |

\((g, h) = (\mathfrak{so}(4n, C), \mathfrak{sl}(2n, C) + C)\)

| \(\Theta(\subset \Psi)\) | \(\psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n\}\) |
|-----------------|---------------------------------|
| \(h_{\Theta}\) | \(\sum_{l=1}^{k} \mathfrak{sp}(i_l - i_{l-1}, C)\) |
| Remarks | 0 = \(i_0 < i_1 < \cdots < i_k = n\) |

\((g, h) = (\mathfrak{sp}(n, R) + \mathfrak{sp}(n, R), \mathfrak{sp}(n, R))\)

| \(\Theta(\subset \Psi)\) | \(\psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n\}\) |
|-----------------|---------------------------------|
| \(h_{\Theta}\) | \(R^k + \sum_{l=1}^{k} \mathfrak{sl}(i_l - i_{l-1}, R)\) |
| Remarks | 0 = \(i_0 < i_1 < \cdots < i_k = n\) |

\((g, h) = (\mathfrak{sp}(n, C), \mathfrak{sl}(n, C) + C)\)

| \(\Theta(\subset \Psi)\) | \(\psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n\}\) |
|-----------------|---------------------------------|
| \(h_{\Theta}\) | \(\sum_{l=1}^{k} \mathfrak{so}(i_l - i_{l-1}, C)\) |
| Remarks | 0 = \(i_0 < i_1 < \cdots < i_k = n\) |

\((g, h) = (\mathfrak{sp}(n, C), \mathfrak{sl}(n, C) + C)\)

| \(\Theta(\subset \Psi)\) | \(\psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n\}\) |
|-----------------|---------------------------------|
| \(h_{\Theta}\) | \(\sum_{l=1}^{k} \mathfrak{so}(i_l - i_{l-1}, C) + \mathfrak{sl}(n - i_k, C) + C\) |
| Remarks | 0 = \(i_0 < i_1 < \cdots < i_k < n\) |
Table 6: (continued)

\[(g, h) = (\text{sp}(n, n) + \text{sp}(n, n), \text{sp}(n, n))\]

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k - 1} - e_{i_k - 1 + 1}, 2e_n\}\) | \(h_\theta\) | \(R^k + \sum_{l=1}^k \text{su}^*(2(i_l - i_{l-1}))\) |
|------------------------|---------------------------------|-------------|---------------------------------|
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k = n\) | Remarks | \(0 = i_0 < i_1 < \cdots < i_k < n\) |

\[(g, h) = (\text{sp}(2n, C), \text{sp}(n, C) + \text{sp}(n, C))\]

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k - 1} - e_{i_k - 1 + 1}, 2e_n\}\) | \(h_\theta\) | \(\sum_{l=1}^k \text{sp}(i_l - i_{l-1}, C)\) |
|------------------------|---------------------------------|-------------|---------------------------------|
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k = n\) | Remarks | \(0 = i_0 < i_1 < \cdots < i_k < n\) |

\[(g, h) = (\text{su}(2n, 2n), \text{sp}(n, n))\]

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k - 1} - e_{i_k - 1 + 1}, 2e_n\}\) | \(h_\theta\) | \(\sum_{l=1}^k \text{sp}(i_l - i_{l-1}, C)\) |
|------------------------|---------------------------------|-------------|---------------------------------|
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k = n\) | Remarks | \(0 = i_0 < i_1 < \cdots < i_k < n\) |

\[(g, h) = (\text{su}^*(4n), \text{su}^*(2n) + \text{su}^*(2n) + R)\]

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k - 1} - e_{i_k - 1 + 1}, 2e_n\}\) | \(h_\theta\) | \(R^{k-1} + \sum_{l=1}^k \text{su}^*(2(i_l - i_{l-1}))\) |
|------------------------|---------------------------------|-------------|---------------------------------|
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k = n\) | Remarks | \(0 = i_0 < i_1 < \cdots < i_k < n\) |

\[(g, h) = (\text{su}^*(4n), \text{su}^*(2n) + \text{su}^*(2n) + R)\]

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k - 1} - e_{i_k - 1 + 1}, 2e_n\}\) | \(h_\theta\) | \(R^k + \sum_{l=1}^k \text{su}^*(2(i_l - i_{l-1})) + \text{su}^*(2(n - i_k))^2 + R\) |
|------------------------|---------------------------------|-------------|---------------------------------|
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k < n\) | Remarks | \(0 = i_0 < i_1 < \cdots < i_k < n\) |
Table 6: (continued)

\[(g, h) = (\mathfrak{su}(n, n), \mathfrak{so}^*(2n))\]

| \(\Theta(\subset \Psi)\) | \(\mathfrak{sl}(\{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}, 2e_n\})\) |
|-------------------------|------------------------------------------------------------------|
| \(\mathfrak{h}_\Theta\) | \(\sum_{l=1}^k \mathfrak{so}(i_l - i_{l-1}, C)\) |
| Remarks                 | \(0 = i_0 < i_1 < \cdots < i_k = n\) |

| \(\Theta(\subset \Psi)\) | \(\mathfrak{sl}(\{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}\})\) |
|-------------------------|------------------------------------------------------------------|
| \(\mathfrak{h}_\Theta(\subset \Psi)\) | \(\sum_{l=1}^k \mathfrak{so}(i_l - i_{l-1}, C) + \mathfrak{so}^*(2(n - i_k))\) |
| Remarks                 | \(0 = i_0 < i_1 < \cdots < i_k < n\) |

| \(\Theta(\subset \Psi)\) | \(\mathfrak{sl}(\{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}\})\) |
|-------------------------|------------------------------------------------------------------|
| \(\mathfrak{h}_\Theta\) | \(R^{k-1} + \sum_{l=1}^k \mathfrak{sl}(i_l - i_l, R)\) |
| Remarks                 | \(0 = i_0 < i_1 < \cdots < i_k = n\) |

| \(\Theta(\subset \Psi)\) | \(\mathfrak{sl}(\{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}\})\) |
|-------------------------|------------------------------------------------------------------|
| \(\mathfrak{h}_\Theta\) | \(R^k + \sum_{l=1}^k \mathfrak{sl}(i_l - i_{l-1}, R) + \mathfrak{sl}(n - i_k, C) + \mathfrak{so}(2)\) |
| Remarks                 | \(0 = i_0 < i_1 < \cdots < i_k < n\) |

| \(\Theta(\subset \Psi)\) | \(\mathfrak{sl}(\{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}\})\) |
|-------------------------|------------------------------------------------------------------|
| \(\mathfrak{h}_\Theta\) | \(R^{k-1} + \sum_{l=1}^k \mathfrak{su}^*(2(i_l - i_{l-1}))\) |
| Remarks                 | \(0 = i_0 < i_1 < \cdots < i_k = n\) |

| \(\Theta(\subset \Psi)\) | \(\mathfrak{sl}(\{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}\})\) |
|-------------------------|------------------------------------------------------------------|
| \(\mathfrak{h}_\Theta\) | \(R^k + \sum_{l=1}^k \mathfrak{su}^*(2(i_l - i_{l-1})) + \mathfrak{sl}(2(n - i_k), C) + \mathfrak{so}(2)\) |
| Remarks                 | \(0 = i_0 < i_1 < \cdots < i_k < n\) |

| \(\Theta(\subset \Psi)\) | \(\mathfrak{sl}(\{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}\})\) |
|-------------------------|------------------------------------------------------------------|
| \(\mathfrak{h}_\Theta\) | \(\sum_{l=1}^k \mathfrak{sp}(i_l - i_{l-1}, C)\) |
| Remarks                 | \(0 = i_0 < i_1 < \cdots < i_k = n\) |

| \(\Theta(\subset \Psi)\) | \(\mathfrak{sl}(\{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}\})\) |
|-------------------------|------------------------------------------------------------------|
| \(\mathfrak{h}_\Theta\) | \(\sum_{l=1}^k \mathfrak{sp}(i_l - i_{l-1}, C) + \mathfrak{sp}(2(n - i_k), R)\) |
| Remarks                 | \(0 = i_0 < i_1 < \cdots < i_k < n\) |
Table 6: (continued)

| (g, h) = (so(2n, 2n), su(n, n) + so(2)) | \( \Theta(\subset \Psi) \) | \( \psi \setminus \{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n \} \) | \( h_\Theta \) | \( \sum_{l=1}^k \text{sp}(i_l - i_{l-1}, R) \) | Remarks |
|------------------------------------------|---------------------------------|-----------------|------|-----------------|----------|
| Remarks                                  | 0 = i_0 < i_1 < \cdots < i_k = n |

| (g, h) = (so^*(4n), so^*(2n) + so^*(2n)) | \( \Theta(\subset \Psi) \) | \( \psi \setminus \{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n \} \) | \( h_\Theta \) | \( \sum_{l=1}^k so^*(2(i_l - i_{l-1})) \) | Remarks |
|------------------------------------------|---------------------------------|-----------------|------|-----------------|----------|
| Remarks                                  | 0 = i_0 < i_1 < \cdots < i_k = n |

| (g, h) = (sp(n, n), su(n, n) + so(2)) | \( \Theta(\subset \Psi) \) | \( \psi \setminus \{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n \} \) | \( h_\Theta \) | \( \sum_{l=1}^k so^*(2(i_l - i_{l-1})) \) | Remarks |
|------------------------------------------|---------------------------------|-----------------|------|-----------------|----------|
| Remarks                                  | 0 = i_0 < i_1 < \cdots < i_k = n |

| (g, h) = (sp(2n, R), sp(n, R) + sp(n, R)) | \( \Theta(\subset \Psi) \) | \( \psi \setminus \{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n \} \) | \( h_\Theta \) | \( \sum_{l=1}^k \text{sp}(i_l - i_{l-1}, R) \) | Remarks |
|------------------------------------------|---------------------------------|-----------------|------|-----------------|----------|
| Remarks                                  | 0 = i_0 < i_1 < \cdots < i_k = n |

| (g, h) = (sp(2n, R), sp(n, R) + sp(n, R)) | \( \Theta(\subset \Psi) \) | \( \psi \setminus \{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n \} \) | \( h_\Theta \) | \( \sum_{l=1}^k \text{sp}(i_l - i_{l-1}, R) + \text{sp}(n - i_k, R)^2 \) | Remarks |
|------------------------------------------|---------------------------------|-----------------|------|-----------------|----------|
| Remarks                                  | 0 = i_0 < i_1 < \cdots < i_k = n |
Table 6: (continued)

$$(g, h) = (sp(2n, R), sp(n, C))$$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \left\{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n \right\}$ |
|------------------------|---------------------------------------------------------------------|
| $h_\Theta$             | $\sum_{l=1}^{k} sp(i_l - i_{l-1}, R)$                                 |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                                     |

$$(g, h) = (sp(n, n), su^*(2n) + R)$$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \left\{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, 2e_n \right\}$ |
|------------------------|---------------------------------------------------------------------|
| $h_\Theta$             | $\sum_{l=1}^{k} so^*(2(i_l - i_{l-1}))$                               |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                                     |

$$(g, h) = (so(n, n) + so(n, n), so(n, n))$$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \left\{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_{n-1} + e_n \right\}$ |
|------------------------|---------------------------------------------------------------------|
| $h_\Theta$             | $R^k + \sum_{l=1}^{k} sl(i_l - i_{l-1}, R)$                          |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                                     |

$$(g, h) = (so(2n, C), so(n, C) + so(n, C))$$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \left\{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_{n-1} + e_n \right\}$ |
|------------------------|---------------------------------------------------------------------|
| $h_\Theta$             | $\sum_{l=1}^{k} so(i_l - i_{l-1}, C)$                                |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                                     |

(IV) $\Delta = D_r$, $\Delta^a = \Delta$, $\Psi = \{ e_i - e_{i+1} | 1 \leq i \leq r - 1 \} \cup \{ e_{r-1} + e_r \}$

$$(g, h) = (so(n, n) + so(n, n), so(n, n))$$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \left\{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_{n-1} + e_n \right\}$ |
|------------------------|---------------------------------------------------------------------|
| $h_\Theta$             | $R^k + \sum_{l=1}^{k} sl(i_l - i_{l-1}, R) + so(n - i_k, n - i_k)$ |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                                     |

$$(g, h) = (so(2n, C), so(n, C) + so(n, C))$$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \left\{ e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_{n-1} + e_n \right\}$ |
|------------------------|---------------------------------------------------------------------|
| $h_\Theta$             | $\sum_{l=1}^{k} so(i_l - i_{l-1}, C) + so(n - i_k, C)^2$ |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                                     |
(V) $\Delta = \Delta^v = (BC)_r$, $\Psi = \{e_i - e_{i+1} \mid 1 \leq i \leq r - 1\} \cup \{e_r\}$

$(g, h) = (\text{su}(p, n - p) + \text{su}(p, n - p), \text{su}(p, n - p)) (n > 2p)$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}, e_p\}$ |
|------------------------|------------------------------------------------------------------|
| $h_{\Theta}$           | $R^k + \text{so}(2)^k + \sum_{l=1}^{k} \text{sl}(i_l - i_{l-1}, C)$ |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = p$                              |

$(g, h) = (\text{sl}(n, C), \text{sl}(p, C) + \text{sl}(n - p, C) + C) (n > 2p)$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}, e_p\}$ |
|------------------------|------------------------------------------------------------------|
| $h_{\Theta}$           | $C^k + \sum_{l=1}^{k} \text{sl}(i_l - i_{l-1}, C) + \text{sl}(n - 2p, C)$ |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = p$                              |

$(g, h) = (\text{so}^*(2(2n+1)) + \text{so}^*(2(2n+1)), \text{so}^*(2(2n+1)))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}, e_n\}$ |
|------------------------|------------------------------------------------------------------|
| $h_{\Theta}$           | $R^k + \text{so}(2) + \sum_{l=1}^{k} \text{su}^*(2(i_l - i_{l-1}))$ |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                              |

$(g, h) = (\text{so}(2(2n+1), C), \text{sl}(2n + 1, C) + C)$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}, e_n\}$ |
|------------------------|------------------------------------------------------------------|
| $h_{\Theta}$           | $C + \sum_{l=1}^{k} \text{sp}(i_l - i_{l-1}, C)$ |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                              |

$(g, h) = (\text{so}(2(2n+1), C), \text{sl}(2n + 1, C) + C)$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k} - e_{i_k+1}, e_n\}$ |
|------------------------|------------------------------------------------------------------|
| $h_{\Theta}$           | $\sum_{l=1}^{k} \text{sp}(i_l - i_{l-1}, C) + \text{sl}(2(n - i_k) + 1, C) + C$ |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k < n$                              |
Table 6: (continued)

\((g, h) = (sp(p, n-p) + sp(p, n-p), sp(p, n-p))\) \((n > 2p)\)

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_{1+1}}, \ldots, e_{i_{k-1}} - e_{i_{k-1+1}}, e_p\}\) | \(b_\Theta\) | \(R^k + \sum_{l=1}^{k} su^* (2(i_l - i_{l-1})) + sp(n-2p)\) |
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k = p\) |

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_{1+1}}, \ldots, e_{i_{k-1}} - e_{i_{k-1+1}}, e_p\}\) | \(b_\Theta\) | \(R^k + \sum_{l=1}^{k} su^* (2(i_l - i_{l-1})) + sp(p - i_k, n-p - i_k)\) |
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k < p\) |

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_{1+1}}, \ldots, e_{i_{k-1}} - e_{i_{k-1+1}}, e_p\}\) | \(b_\Theta\) | \(\sum_{i=1}^{k} sp(i_l - i_{l-1}, C) + sp(p - i_k, C) + sp(n-2p, C)\) |
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k = p\) |

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_{1+1}}, \ldots, e_{i_{k-1}} - e_{i_{k-1+1}}\}\) | \(b_\Theta\) | \(\sum_{i=1}^{k} sp(i_l - i_{l-1}, C) + sp(p - i_k, C) + sp(n-p - i_k, C)\) |
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k < p\) |

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_{1+1}}, \ldots, e_{i_{k-1}} - e_{i_{k-1+1}}, e_p\}\) | \(b_\Theta\) | \(\sum_{i=1}^{k} sp(i_l - i_{l-1}, C) + sp(p - i_k, C) + sp(n-2p, C)\) |
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k = p\) |

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_{1+1}}, \ldots, e_{i_{k-1}} - e_{i_{k-1+1}}, e_p\}\) | \(b_\Theta\) | \(R^k + \sum_{l=1}^{k} su^* (2(i_l - i_{l-1})) + su^* (2(n-2p))\) |
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k = p\) |

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_{1+1}}, \ldots, e_{i_{k-1}} - e_{i_{k-1+1}}\}\) | \(b_\Theta\) | \(R^k + \sum_{l=1}^{k} su^* (2(i_l - i_{l-1})) + su^* (2(n-p - i_k)) + su^* (2(n - p - i_k)) + R\) |
| Remarks | \(0 = i_0 < i_1 < \cdots < i_k < p\) |
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Table 6: (continued)

\[(g, h) = (\text{su}^*(2(2n + 1)), \text{sl}(2n + 1, C) + \text{so}(2))\]

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k - 1} - e_{i_k - i_1}, e_n\}\) |
|--------------------------|---------------------------------------------------------------------|
| \(h_\theta\)            | \(R^k + \text{so}(2) + \sum_{l=1}^k \text{su}^*(2(i_l - i_{l-1}))\) |
| Remarks                  | 0 = i_0 < i_1 < \cdots < i_k = n                                   |

\[(g, h) = (\text{su}(2(2n + 1, 2n + 1, \text{sp}(2n + 1, R)))\]

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k - 1} - e_{i_k - i_1}, e_n\}\) |
|--------------------------|---------------------------------------------------------------------|
| \(h_\theta\)            | \(\sum_{l=1}^k \text{sp}(i_l - i_{l-1}, C)\)                       |
| Remarks                  | 0 = i_0 < i_1 < \cdots < i_k = n                                   |

\[(g, h) = (\text{so}(2p, 2(n-p)), \text{su}(p, n-p) + \text{so}(2)) (n > 2p)\]

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k - 1} - e_{i_k - i_1}, e_p\}\) |
|--------------------------|---------------------------------------------------------------------|
| \(h_\theta\)            | \(\sum_{l=1}^k \text{sp}(i_l - i_{l-1}, R)\)                       |
| Remarks                  | 0 = i_0 < i_1 < \cdots < i_k = p                                    |

\[(g, h) = (\text{so}^*(2n), \text{so}^*(2p) + \text{so}^*(2(p-n))) (n > 2p)\]

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k - 1} - e_{i_k - i_1}, e_p\}\) |
|--------------------------|---------------------------------------------------------------------|
| \(h_\theta\)            | \(\sum_{l=1}^k \text{so}^*(2(i_l - i_{l-1}))\)                     |
| Remarks                  | 0 = i_0 < i_1 < \cdots < i_k = p                                    |

\[(g, h) = (\text{so}^*(2n), \text{so}^*(2p) + \text{so}^*(2(n-p))) (n > 2p)\]

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1 + 1}, \ldots, e_{i_k - 1} - e_{i_k - i_1}, e_p\}\) |
|--------------------------|---------------------------------------------------------------------|
| \(h_\theta\)            | \(\sum_{l=1}^k \text{so}^*(2(i_l - i_{l-1})) + \text{so}^*(2(p-i_k)) + \text{so}^*(2(n-p-i_k))\) |
| Remarks                  | 0 = i_0 < i_1 < \cdots < i_k < p                                   |
Table 6: (continued)

(g, h) = (sp(p, n - p), su(p, n - p) + so(2)) (n > 2p)

| θ(⊂ ψ) | ψ \ \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_p\} |
|--------|---------------------------------------------------------------|
| h_{θ}  | u(n - 2p) + \sum_{l=1}^{k} so^2(2(i_l - i_{l-1})) |
| Remarks| 0 = i_0 < i_1 < \ldots < i_k = p |

(g, h) = (sp( n, R), sp(p, R) + sp(n - p, R)) (n > 2p)

| θ(⊂ ψ) | ψ \ \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_p\} |
|--------|---------------------------------------------------------------|
| h_{θ}  | \sum_{l=1}^{k} sp(i_l - i_{l-1}, R) + sp(n - 2p, R) |
| Remarks| 0 = i_0 < i_1 < \ldots < i_k = p |

(g, h) = (sl(2n + 1, C), sl(2n + 1, R))

| θ(⊂ ψ) | ψ \ \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_n\} |
|--------|---------------------------------------------------------------|
| h_{θ}  | R^{k} + so(2)^{k} + \sum_{l=1}^{k} sl(i_l - i_{l-1}, C) |
| Remarks| 0 = i_0 < i_1 < \ldots < i_k = n |

(g, h) = (su(p, n - p), so(p, n - p))

| θ(⊂ ψ) | ψ \ \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_p\} |
|--------|---------------------------------------------------------------|
| h_{θ}  | \sum_{l=1}^{k} so(i_l - i_{l-1}, C) + so(n - 2p) |
| Remarks| 0 = i_0 < i_1 < \ldots < i_k = p |

(VI) \(Δ = (BC)_r, Δ^a = B_r, ψ = \{e_i - e_{i+1} | 1 \leq i \leq r - 1\} \cup \{e_r\}\)

| θ(⊂ ψ) | ψ \ \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_n\} |
|--------|---------------------------------------------------------------|
| h_{θ}  | R^{k} + so(2)^{k} + \sum_{l=1}^{k} sl(i_l - i_{l-1}, C) + sl(2(n - i_k) + 1, R) |
| Remarks| 0 = i_0 < i_1 < \ldots < i_k < n |

(g, h) = (su(p, n - p), so(p, n - p))

| θ(⊂ ψ) | ψ \ \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_p\} |
|--------|---------------------------------------------------------------|
| h_{θ}  | \sum_{l=1}^{k} so(i_l - i_{l-1}, C) + so(n - 2p) |
| Remarks| 0 = i_0 < i_1 < \ldots < i_k = p |

(g, h) = (su(p, n - p), so(p, n - p))

| θ(⊂ ψ) | ψ \ \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_p\} |
|--------|---------------------------------------------------------------|
| h_{θ}  | \sum_{l=1}^{k} so(i_l - i_{l-1}, C) + so(p - i_k, n - p - i_k) |
| Remarks| 0 = i_0 < i_1 < \ldots < i_k < p |
Table 6: (continued)

$$(g, h) = (\mathfrak{sl}(n, R), \mathfrak{sl}(p, R) + \mathfrak{sl}(n-p, R) + R) \ (n > 2p)$$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_n\}$ |
|------------------------|---------------------------------------------------------------------|
| $h_\Theta$             | $R_k + \sum_{l=1}^k \mathfrak{so}(2(i_l - i_{l-1}))$               |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                                 |

$$(g, h) = (\mathfrak{so}^*(2(2n+1)), \mathfrak{so}(2n+1, C))$$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_n\}$ |
|------------------------|---------------------------------------------------------------------|
| $h_\Theta$             | $\sum_{l=1}^k \mathfrak{so}^*(2(i_l - i_{l-1}))$ + $\mathfrak{so}(2(n-i_k) + 1, C)$ |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k < n$                                 |

$$(g, h) = (\mathfrak{so}(2n+1, 2n+1), \mathfrak{sl}(2n+1, R) + R)$$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k-1+1}, e_n\}$ |
|------------------------|---------------------------------------------------------------------|
| $h_\Theta$             | $R + \sum_{l=1}^k \mathfrak{sp}(i_l - i_{l-1}, R)$                 |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                                 |

Example 5.3. Let $G/H$ be a semisimple pseudo-Riemannian symmetric space. Suppose that the restricted root system $\Delta$ with respect to a vector-type maximal split abelian subspace satisfies $(\Delta, \Delta^a) = (C_r, D_r)$. By using the argument as in Subsection 3.2 and Theorem A, we have $\mathcal{L}_h(G/H) = \{[h_\Theta] \mid \Theta \subset \Psi\} \cup \{[h_\Theta] \mid \Theta \subset s_{2r} \cdot \Psi\}$, where $\Psi = \{e_i - e_{i+1} \mid 1 \leq i \leq r - 1\} \cup \{2e_r\}$. In Table 7, we list up the set of all the possible local orbit types of the hyperbolic orbits for the s-representations associated with all classical-type semisimple pseudo-Riemannian symmetric spaces satisfying $(\Delta, \Delta^a) = (C_r, D_r)$. 
Table 7: Local orbit types

$\Delta = C_r$, $\Delta^a = D_r$, $\Psi = \{e_i - e_{i+1} \mid 1 \leq i \leq r - 1\} \cup \{2e_r\}$

$(g, h) = (\mathfrak{sl}(2n, C), \mathfrak{sl}(2n, R))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i-k+1} - e_{i-k+1} + 2\} \{2n\}$ | $R^{k-1} + \mathfrak{so}(2)^k + \sum_{l=1}^{k} \mathfrak{sl}(i_l - i_{l-1}, C)$ |
|----------------------|-------------------------------------------------|--------------------------------------------------|
| Remarks              | $0 = i_0 < i_1 < \cdots < i_k = n$               |                                                  |

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i-k+1} - e_{i-k+1} + 2\} \{2n\}$ | $R^{k-1} + \mathfrak{so}(2)^k + \sum_{l=1}^{k} \mathfrak{sl}(i_l - i_{l-1}, C)$ |
|----------------------|-------------------------------------------------|--------------------------------------------------|
| Remarks              | $0 = i_0 < i_1 < \cdots < i_k = n$               |                                                  |

| $\Theta(\subset s_{2e_n}, \Psi)$ | $s_{2e_n}(\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i-k+1} - e_{i-k+1} + 2\} \{2n\}$ | $R^{k-1} + \mathfrak{so}(2)^k + \sum_{l=1}^{k} \mathfrak{sl}(i_l - i_{l-1}, C)$ |
|----------------------|-------------------------------------------------|--------------------------------------------------|
| Remarks              | $0 = i_0 < i_1 < \cdots < i_k = n$               |                                                  |

$\Theta(\subset \Psi)$

| Remarks              | $0 = i_0 < i_1 < \cdots < i_k = n$               |
|----------------------|-------------------------------------------------|

$(g, h) = (\mathfrak{su}(n, n), \mathfrak{so}(n, n))$

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i-k+1} - e_{i-k+1} + 2\} \{2n\}$ | $\sum_{l=1}^{k} \mathfrak{so}(i_l - i_{l-1}, C)$ |
|----------------------|-------------------------------------------------|--------------------------------------------------|
| Remarks              | $0 = i_0 < i_1 < \cdots < i_k = n$               |                                                  |

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_i - e_{i+1}, \ldots, e_{i-k+1} - e_{i-k+1} + 2\} \{2n\}$ | $\sum_{l=1}^{k} \mathfrak{so}(i_l - i_{l-1}, C)$ |
|----------------------|-------------------------------------------------|--------------------------------------------------|
| Remarks              | $0 = i_0 < i_1 < \cdots < i_k = n$               |                                                  |
Table 7: (continued)

\((g, h) = (\mathfrak{sl}(2n, R), \mathfrak{sl}(n, R) + \mathfrak{sl}(n, R) + R)\)

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_{k-1}} - e_{i_{k-1}+1}, 2e_n\}\) |
|-------------------------|---------------------------------------------------------------------------------|
| \(h_\Theta\)          | \(R^{k-1} + \sum_{l=1}^{k} \mathfrak{sl}(i_l - i_{l-1}, R)\)                   |
| Remarks                | \(0 = i_0 < i_1 < \cdots < i_k = n\)                                           |

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_{k-1}} - e_{i_{k-1}+1}\}\) |
|-------------------------|---------------------------------------------------------------------------------|
| \(h_\Theta\)          | \(R^k + \sum_{l=1}^{k} \mathfrak{sl}(i_l - i_{l-1}, R) + \mathfrak{sl}(n - i_k, R)^2 + R\) |
| Remarks                | \(0 = i_0 < i_1 < \cdots < i_k < n\)                                           |

| \(\Theta(\subset s_{2c_n} \Psi)\) | \(s_{2c_n} (\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_{k-1}} - e_{i_{k-1}+1}\})\) |
|-----------------------------|---------------------------------------------------------------------------------|
| \(h_\Theta\)              | \(R^{k-1} + \sum_{l=1}^{k} \mathfrak{sl}(i_l - i_{l-1}, R)\)                   |
| Remarks                    | \(0 = i_0 < i_1 < \cdots < i_k = n\)                                           |

\(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_{k-1}} - e_{i_{k-1}+1}, 2e_n\}\) |
|-------------------------|---------------------------------------------------------------------------------|
| \(h_\Theta\)          | \(\sum_{l=1}^{k} \mathfrak{so}^*(2(i_l - i_{l-1}))\)                           |
| Remarks                | \(0 = i_0 < i_1 < \cdots < i_k = n\)                                           |

\(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_{k-1}} - e_{i_{k-1}+1}\}\) |
|-------------------------|---------------------------------------------------------------------------------|
| \(h_\Theta\)          | \(\sum_{i=1}^{k} \mathfrak{so}^*(2(i_l - i_{l-1})) + \mathfrak{so}(2(n - i_k), C)\) |
| Remarks                | \(0 = i_0 < i_1 < \cdots < i_k = n\)                                           |

\((g, h) = (\mathfrak{so}^*(4n), \mathfrak{so}(2n, C))\)

| \(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_{k-1}} - e_{i_{k-1}+1}, 2e_n\}\) |
|-------------------------|---------------------------------------------------------------------------------|
| \(h_\Theta\)          | \(\sum_{l=1}^{k} \mathfrak{so}^*(2(i_l - i_{l-1}))\)                           |
| Remarks                | \(0 = i_0 < i_1 < \cdots < i_k = n\)                                           |

\(\Theta(\subset \Psi)\) | \(\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_{k-1}} - e_{i_{k-1}+1}\}\) |
|-------------------------|---------------------------------------------------------------------------------|
| \(h_\Theta\)          | \(\sum_{i=1}^{k} \mathfrak{so}^*(2(i_l - i_{l-1})) + \mathfrak{so}(2(n - i_k), C)\) |
| Remarks                | \(0 = i_0 < i_1 < \cdots < i_k = n\)                                           |
Table 7: (continued)

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k+1}\}$ |
|------------------------|-------------------------------------------------------------|
| $h_\theta$             | $\sum_{i=1}^k \text{ap}(i_l - i_{l-1}, R)$                  |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                         |

| $\Theta(\subset \Psi)$ | $\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k+1}\}$ |
|------------------------|-------------------------------------------------------------|
| $h_\theta$             | $\sum_{i=1}^k \text{ap}(i_l - i_{l-1}, R)$                  |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                         |

| $\Theta(\subset s_{2c_n} \Psi)$ | $s_{2c_n}(\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k+1}\})$ |
|------------------------|--------------------------------------------------------------------------------|
| $h_\theta$             | $\sum_{i=1}^k \text{ap}(i_l - i_{l-1}, R)$                                    |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k = n$                                          |

| $\Theta(\subset s_{2c_n} \Psi)$ | $s_{2c_n}(\Psi \setminus \{e_{i_1} - e_{i_1+1}, \ldots, e_{i_k-1} - e_{i_k+1}\})$ |
|------------------------|--------------------------------------------------------------------------------|
| $h_\theta$             | $\sum_{i=1}^k \text{ap}(i_l - i_{l-1}, R) + \text{sl}(2(n - i_k), R) + R$ |
| Remarks                | $0 = i_0 < i_1 < \cdots < i_k < n$                                          |

6. Local orbit types of the elliptic orbits

Let $(g, h)$ be a semisimple symmetric pair and $\sigma$ be an involution of $g$ with $h = \text{Ker}(\sigma - \text{id})$. Set $g^c := h + \sqrt{-1}q(\subset g^C)$. The symmetric pair $(g^c, h)$ is called the $c$-dual pair of $(g, h)$. In this section, we discuss the relation between the isotropy subalgebras of elliptic orbits for the $s$-representation associated with $(g, h)$ and those of hyperbolic orbits for the $s$-representation associated with $(g^c, h)$. Then we have the following fact.

**Lemma 6.1.** For any elliptic element $X \in q$, the centralizer of $X$ in $h(\subset g)$ coincides with that of $\sqrt{-1}X$ in $h(\subset g^c)$.

From Lemma 6.1 we can determine the local orbit types of elliptic orbits for the $s$-representation of $(g, h)$ by investigating those of hyperbolic orbits for the $s$-representation of $(g^c, h)$. For example, in the case of $(g, h) = (\text{sl}(n, R) + \text{sl}(n, R), \text{sl}(n, R))$, we have $(g^c, h) = (\text{sl}(n, C), \text{sl}(n, R))$. Then it follows from the above argument that the elliptic principal isotropy subalgebra for the $s$-representation associated with $(\text{sl}(n, R) + \text{sl}(n, R), \text{sl}(n, R))$ coincides with $R^{(n-1)/2} + \text{so}(2)^{[n/2]}$ for any $n \in N$ (see, Table 1). In the case of $n = 4$, this result was shown by Boumuki (PROPOSITION 5.1 in [3]). He
actually determined all the isotropy subalgebras of elliptic orbits for the s-representation associated with $(\mathfrak{sl}(4, \mathbb{R}) + \mathfrak{sl}(4, \mathbb{R}), \mathfrak{sl}(4, \mathbb{R}))$. Our method of determining the isotropy subalgebras depends on restricted root system theory for semisimple symmetric pairs, and is different from Boumuki’s method, which depends on root system theory for semisimple complex Lie algebras and that for compact Lie algebras.

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References

[1] K. Baba, Local orbit types of s-representations for exceptional semisimple symmetric spaces, SUT J. Math. 44 (2008), 307–328.
[2] K. Baba, Satake Diagrams and Restricted Root Systems of Semisimple Pseudo-Riemannian Symmetric Spaces, Tokyo J. Math. 32 (2009), 127–158.
[3] M. Berger, Les espaces symétriques noncompacts, Ann. Sci. École Norm. Sup. 74 (1957), 85–177.
[4] N. Boumuki, Isotropy subalgebras of elliptic orbits in semisimple Lie algebras, and the canonical representatives of pseudo-Hermitian symmetric elliptic orbits, J. Math. Soc. Japan, 59, (2007), 1135–1177.
[5] E. Heintze and C. Olmos, Normal holonomy groups and s-representations, Indiana Univ. Math. J., 41, (1992), 869–874.
[6] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Graduate Studies in Mathematics, American Mathematical Society, Providence, Rhode Island, 2001.
[7] A. Knapp, Lie groups beyond an introduction, Progress in Mathematics, Birkhäuser Boston Inc. 2002.
[8] K. Kondo, Local orbit types of S-representations of symmetric R-spaces, Tokyo J. Math. 26 (2003), 67–81.
[9] O. Loos, Symmetric spaces. I: General theory, W. A. Benjamin, Inc., New York-Amsterdam, 1969.
[10] O. Loos, Symmetric spaces. II: Compact spaces and classification, W. A. Benjamin, Inc., New York-Amsterdam, 1969.
[11] C. Olmos, The normal holonomy group, Proc. Amer. Math. Soc., 110, (1990), 813–818.
[12] T. Oshima and T. Matsuki, Orbits on affine symmetric spaces under the action of the isotropy subgroups, J. Math. Soc. Japan, 32 (1980), 399–414.
[13] T. Oshima and J. Sekiguchi, Eigenspaces of Invariant Differential Operators on an Affine Symmetric Space, Inventiones math. 57 (1980), 1–81.
[14] T. Oshima and J. Sekiguchi, The restricted root system of a semisimple symmetric pair, Adv. Stud. Math. 4 (1984), 433–497.
[15] W. Rossmann, *The structure of semisimple symmetric spaces*, Canad. J. Math. **31** (1979), 157–180.

[16] H. Tamaru, *The local orbit types of symmetric spaces under the actions of the isotropy subgroups*, Differential Geom. Appl. **11** (1999), 29–38.

[17] G. Warner, *Harmonic analysis on semi-simple Lie groups. I*, Springer-Verlag, New York, 1972.

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