POLYNOMIALS FOR TORUS LINKS FROM
CHERN-SIMONS GAUGE THEORIES

J. M. Isidro, J. M. F. Labastida and A. V. Ramallo

Departamento de Física de Partículas
Universidad de Santiago
E-15706 Santiago de Compostela, Spain

ABSTRACT

Invariant polynomials for torus links are obtained in the framework of the Chern-Simons topological gauge theory. The polynomials are computed as vacuum expectation values on the three-sphere of Wilson line operators representing the Verlinde algebra of the corresponding rational conformal field theory. In the case of the $SU(2)$ gauge theory our results provide explicit expressions for the Jones polynomial as well as for the polynomials associated to the $N$-state ($N > 2$) vertex models (Akutsu-Wadati polynomials). By means of the Chern-Simons coset construction, the minimal unitary models are analyzed, showing that the corresponding link invariants factorize into two $SU(2)$ polynomials. A method to obtain skein rules from the Chern-Simons knot operators is developed. This procedure yields the eigenvalues of the braiding matrix of the corresponding conformal field theory.
1. Introduction

Chern-Simons gauge theories have provided a framework to obtain topological invariants associated to knots, links and graphs on compact oriented three-manifolds [1,2]. Although there seems to exist a systematic procedure to carry out the computation of these topological invariants in $S^3$ within this framework [3], invariants have been calculated only in a few situations [1,2,3,4,5,6,7]. For a given knot, link or graph, the systematic procedure quoted above [3] consists of a finite number of steps which in general are rather involved. Though well defined, this procedure seems complicated to obtain general expressions for general sets of knots. For example, just take the simplest set of knots: torus knots. An insurmountable amount of effort would be needed to obtain an explicit expression for the topological invariants associated to a general torus knot labelled by two coprime integers $r$ and $s$. A similar disadvantage is shared by methods which use skein rules [8,9,6,7]. It seems preferable to build a framework of calculation where one possesses explicit general forms of matrix elements in some Hilbert space for the operators associated to different sets of knots, links and graphs. Furthermore, this procedure would allow to compute topological invariants on three-manifolds other than $S^3$. The knot operators for torus knots carrying arbitrary representations of $SU(N)$ constructed in [4,5] constitute an example of such a framework. For a given oriented torus knot carrying an arbitrary representation of $SU(N)$ the explicit form of the matrix elements of these operators is given in [5]. This allows to compute the topological invariant associated to an oriented torus knot carrying an arbitrary representation of $SU(N)$ for any three-manifold which can be constructed by gluing two solid tori together, namely, for any lens space. One would like to possess a similar construction for arbitrary Riemann surfaces. For genus $g \geq 2$ this is not yet available.

Most of the explicit calculations of topological invariants in Chern-Simons gauge theory have been carried out making use of knowledge on conformal field theories. For example, the identification of the kind of resulting invariant polynomials
has been done by obtaining skein rules using the half-monodromy matrix of the corresponding conformal field theory. This is the procedure originally used in [1] for the fundamental representation of $SU(N)$ which was later extended to arbitrary representations of $SU(2)$ [8,6,7], and to the fundamental representations of $SO(N)$, $Sp(2n)$, $SU(m|n)$, and $OSp(m|2n)$ [9,6]. These derivations showed that, while invariant polynomials related to the fundamental representation of $SU(2)$ lead to the Jones polynomials [10], the ones related to the fundamental representation of $SU(N)$ lead to their two variable generalization, the HOMFLY polynomials [11]. In addition, it turns out that the invariants related to the fundamental representations of $SO(N)$ and $Sp(2n)$ lead to Kauffman polynomials [12]. The invariants associated to $SU(2)$ for arbitrary representations are related to the Akutsu-Wadati polynomials [13]. These polynomials were derived from exactly solvable models in statistical mechanics, namely the $N$-state vertex models. Each of these vertex models provides a set of skein rules which turn out to be the same as the ones derived for arbitrary representations of $SU(2)$ in Chern-Simons gauge theory. For $N = 2$ the Akutsu-Wadati polynomials are the Jones polynomials. Skein rules are not enough in general to determine invariant polynomials. Only for knots and links in the fundamental representation do they turn out to be enough. This means that in general one has to develop more powerful methods to compute these invariants. The systematic procedure presented in [3] seems to be complete but difficult to apply even in simple situations. Furthermore, its construction relies entirely on knowledge on the corresponding conformal field theory. Other methods to compute invariants have been recently presented in [7]. These methods can be applied to the case of knots and links and also are based on knowledge on conformal field theory. They are simpler to implement than the ones in [3] and, indeed, invariant polynomials for knots with up to seven crossings carrying arbitrary representations of $SU(2)$ have been obtained [7]. These methods, however, do not seem to provide general expressions for general sets of links as, for example, torus links. The approach based on the construction of knot operators seems to be the most promising one in providing these kinds of general expressions. In addition, it does not rely
on knowledge on conformal field theory and therefore it could also be applied in situations where the corresponding data on conformal field theory is absent. Furthermore, such data could be obtained from knot operators. In this paper we will show how to obtain skein rules from torus knot operators for arbitrary representations of $SU(2)$. Though knot operators seem to be very promising in providing general expressions of invariant polynomials, they are known only for the case of torus knots. Thus, so far, the scope of this approach is limited and in order to compute invariants which cannot be obtained from knot operators one has to use methods as the ones in [1,2,3,7].

Rational conformal field theories seem to be suitable to provide invariants for compact oriented three-manifolds similar to the ones associated to Chern-Simons gauge theories [14,15]. In [16] we initiated a program to obtain explicit constructions in this respect. We carried out a Chern-Simons coset construction which allowed to obtain explicit expressions for knot operators associated to minimal unitary models for arbitrary torus knots. In this paper we will present the invariant polynomials for torus knots carrying arbitrary highest weights for any minimal unitary model. In addition, we will compute the corresponding skein rules. These skein rules are consistent with the structure of the half-monodromy matrix of the minimal unitary models.

This paper is organized as follows. In sect. 2 we recall the basic features of the operator formalism and introduce the knot operators. The computation of the link polynomials for the $SU(2)$ gauge theory is performed in sect. 3. In this section the properties of the polynomials are studied and the explicit expression for torus links is given. In the appendix we explicitly prove the polynomial character of our $SU(2)$ invariants for torus knots. The coset construction giving rise to the minimal unitary models is briefly reviewed in sect. 4 and the corresponding link polynomials are obtained. In sect. 5 the skein rules satisfied by our polynomials are obtained both in the $SU(2)$ and minimal model case. Finally, in sect. 6 we summarize our results, draw some conclusions and indicate some directions for future work.
2. Operator formalism

The Chern-Simons (CS) topological gauge theory is based on the action,

\[ S = \frac{k}{4\pi} \int_M \text{Tr} \left[ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right], \tag{2.1} \]

where \( A \) is a one-form connection taking values in a Lie algebra and the trace is taken in the fundamental representation. In this paper we shall restrict ourselves to the case in which the gauge group is \( SU(2) \). In (2.1) \( k \) is a positive integer and \( M \) is a three-manifold without boundary. The normalization chosen in (2.1) is such that the exponential \( e^{iS[A]} \), appearing in the functional integral, is invariant under gauge transformations \( A \to h^{-1}Ah + h^{-1}dh \). The partition function of the theory is obtained by integrating over all possible orbits of gauge fields living on \( M \),

\[ Z(M) = \int [DA]_M e^{iS[A]}. \tag{2.2} \]

Notice that the action, being the integral of a three-form, does not depend on the metric of \( M \). Therefore the partition function \( Z(M) \) is a topological invariant associated to the manifold \( M \). Furthermore, in CS theories there is a class of observables which are both gauge invariant and metric independent. These are the Wilson line operators, defined for each closed curve \( \gamma \) in \( M \) and for any irreducible representation \( R \) of the gauge group,

\[ W^\gamma_R = \text{Tr}_R(\text{P exp} \int_\gamma A), \tag{2.3} \]

where \( \text{P} \) denotes a path-ordered product along \( \gamma \). The vacuum expectation value of a product of Wilson line operators is given by,

\[ \langle W^\gamma_{R_1} \cdots W^\gamma_{R_n} \rangle_M = (Z(M))^{-1} \int [DA]_M \prod_{i=1}^n W^\gamma_{R_i} e^{iS[A]}. \tag{2.4} \]

The topological nature of our gauge theory ensures that the functional integral (2.4) only depends on the topological properties of the embedding in the three-
manifold $M$ of the link defined by the set of curves $\gamma_i$. It is thus natural to suppose that $\langle \prod_{i=1}^{n} W_{R_i}^{\gamma_i} \rangle$ will be related to a link polynomial. This is indeed the case as we shall check below. The representations $R_i$ determine the quantum numbers running along $\gamma_i$ and can be regarded as the colors associated to each link component.

In order to compute the expectation value given in eq. (2.4) we proceed as follows. First of all we shall decompose $M$ as the connected sum of two three-manifolds $M_1$ and $M_2$ sharing a common boundary (see Fig. 1). The joint of $M_1$ and $M_2$ to build $M$ is performed by identifying their boundaries via a homeomorphism. In this paper we will only consider the case in which $M_1$ and $M_2$ are solid tori. The homeomorphism needed to obtain $M$ is just a modular transformation of the torus. The class of manifolds we get with this construction are the so-called lens spaces.

Any functional integral over $M$ can be decomposed into two path integrals over $M_1$ and $M_2$, each of which defines a wave functional depending on the gauge field configurations on the common boundary $\partial M_1 = \partial M_2$ (see Fig. 2). The complete path integral over $M$ is just an inner product in the space of wave functionals,

$$
\int [DA]_M \prod_{i=1}^{n} W_{R_i}^{\gamma_i} e^{iS[A]} = \langle \Psi_2 | S | \Psi_1 \rangle, \quad (2.5)
$$

where $S$ is the operator representation in the space of wave functionals of the homeomorphism that identifies $\partial M_1$ and $\partial M_2$.

The operator formalism developed in [4,5] provides an explicit representation of the CS wave functionals and determines the form of the inner product appearing in (2.5). It will allow us to evaluate vacuum expectation values of some classes of Wilson lines. In this section we shall briefly review the main ingredients and results of this approach.

Let us perform a hamiltonian analysis of our theory by considering the boundary $\Sigma = \partial M_1$ as equal-time surface. Define local complex coordinates on $\Sigma$ as
\[ z = \sigma_1 + i\sigma_2, \ z = \sigma_1 - i\sigma_2. \] The corresponding components of the gauge field are \[ A_z = \frac{1}{2}(A_1 - iA_2) \] and \[ A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2). \] Choosing a gauge in which the component of \( A \) along the direction perpendicular to \( \Sigma \) (i.e., the “time” direction) vanishes, one finds the following commutation relations from the action (2.1),

\[ [A^a_z(\sigma), A^b_{\bar{z}}(\sigma')] = -\frac{\pi}{k} \delta^{ab} \delta^{(2)}(\sigma - \sigma'). \] (2.6)

Equation (2.6) implies that \( A_z \) and \( A_{\bar{z}} \) are canonically conjugate. We can conventionally choose the antiholomorphic component \( A_{\bar{z}} \) to play the role of a “coordinate”, whereas \( A_z \) will be considered as the “momentum”. The inner product in the Hilbert space takes the standard form in the holomorphic quantization formalism. In this scheme the wave functionals depend on \( A_{\bar{z}} \) and the holomorphic component \( A_z \) is represented as a derivative with respect to \( A_{\bar{z}} \), as dictated by equation (2.6).

It is thus natural to implement this operator representation in such a way that the inner product of the wave functionals corresponding to \( M_1 \) and \( M_2 \) reconstructs the original functional integral over \( M \). In so doing, one discovers that, due to a two-dimensional chiral anomaly, the coefficient \( k \) in (2.6) must be renormalized by a finite amount, \( k \to k + c_v \), where \( c_v \) is the quadratic Casimir in the adjoint representation (see references [4,5] for details). Actually, since we are dealing with a topological theory, the gauge field \( A \) has no true local degrees of freedom. The only relevant components of \( A \) are its zero modes, which parametrize the holonomy of the gauge field around non-trivial cycles. Within this path integral representation one can define an effective quantum-mechanical problem by integrating in our inner product over the non-zero modes of \( A \). Let us recall the results of [4,5] for the case in which \( M_1 \) and \( M_2 \) are solid tori and the gauge group is \( SU(2) \). In this case the boundary of \( M_1 \) and \( M_2 \) is a torus \( T^2 \). Choose a basis of the first homology of \( T^2 \) as depicted in Fig. 3, in which the \( A \) cycle is the one which is contractible in the solid torus. The holomorphic one-form \( \omega(z) \) is defined by giving its integrals along the \( A \) and \( B \) cycles of the canonical homology basis:

\[ \int_A \omega = 1, \int_B \omega = \tau, \] where \( \tau \) is the modular parameter of \( T^2 \). As the first homology group

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of the torus is abelian, the zero-mode part of the gauge connection will live in the Cartan subalgebra of $SU(2)$. Let us parametrize it as follows,

$$A = \frac{\pi a}{2 \text{Im} \tau} \bar{\omega} T_3 - \frac{\pi \bar{a}}{2 \text{Im} \tau} \omega T_3,$$

(2.7)

where $T_3$ is a diagonal $SU(2)$ matrix (in the fundamental representation $T_3$ is the Pauli matrix $\sigma_3$). In equation (2.7) the antiholomorphic component of the gauge field is determined by the variable $a$. Notice that the canonical commutation relations (2.6) (including the shift $k \to k + 2$ discussed above) become,

$$[\bar{a}, a] = \frac{2 \text{Im} \tau}{\pi (k + 2)},$$

(2.8)

and therefore we can represent $\bar{a}$ as,

$$\bar{a} = \frac{2 \text{Im} \tau}{\pi (k + 2)} \frac{\partial}{\partial a}.$$

(2.9)

The states appearing in the effective zero-mode problem will be functions of the variable $a$. Their explicit representation can be obtained by solving the Gauss Law of the theory. If the torus $T^2$ in which our wave functions are defined does not cut any Wilson line (i.e., if all the Wilson lines are contained in the interior of the solid tori), the Hilbert space is spanned by the finite set of functions,

$$\Phi_{j,k}(a) = \frac{\lambda_{j,k+2}(a)}{\Pi(a)},$$

(2.10)

the $\lambda$'s being

$$\lambda_{j,k+2}(a) = e^{\frac{\pi (k+2) a^2}{4 \text{Im} \tau}} \left[ \Theta_{j+1,k+2}(a, \tau) - \Theta_{j-1,k+2}(a, \tau) \right],$$

(2.11)

and $\Pi(a) = \lambda_{0,2}(a)$. In (2.11) $\Theta_{l,m}$ are classical theta-functions [17,18] of level $m$,

$$\Theta_{l,m}(a, \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi im \left( n + \frac{l}{2m} \right)^2 + 2\pi im \left( n + \frac{l}{2m} \right) a}.$$

(2.12)

The wave functionals $\Phi_{j,k}(a)$ represent the characters of an $SU(2)$ Kac-Moody algebra at level $k$ for an isospin $\frac{j}{2}$ [18]. Using the periodicity properties of the theta-functions, one can verify that $\lambda_{j,k+2} = -\lambda_{2k+2-j,k+2}$ and $\lambda_{-j,k+2} = -\lambda_{j-2,k+2}$ and
therefore there are \( k + 1 \) independent states labelled by \( j = 0, \ldots, k \). If \( \Psi_1 \) and \( \Psi_2 \) are arbitrary combinations of the states (2.10), then their inner product is,

\[
\langle \Psi_2 | \Psi_1 \rangle = \int \frac{d\bar{a}a}{2\sqrt{\text{Im} \tau}} \Pi(a) \Pi(\bar{a}) e^{-\frac{(k+1)\pi a\bar{a}}{2\text{Im} \tau}} \Psi_2(a) \Psi_1(a).
\] (2.13)

It can be easily checked that the wave functions (2.10) are orthonormal with respect to the inner product (2.13).

An important property of the functions (2.10) is that they provide a unitary representation of the modular group of the torus. In fact under the generating transformations of the modular group \( S : a \to \frac{a}{\tau}, \tau \to -\frac{1}{\tau} \) and \( T : a \to a, \tau \to \tau + 1 \), they behave as follows,

\[
\Phi_{j,k}(\frac{a}{\tau}, -\frac{1}{\tau}) = \sum_{l=0}^{k} S_{jl} \Phi_{l,k}(a, \tau),
\] (2.14)

\[
\Phi_{j,k}(a, \tau + 1) = e^{2\pi i (h_j - \frac{c}{k+2})} \Phi_{j,k}(a, \tau),
\] (2.15)

where the matrix \( S \) is given by,

\[
S_{jl} = \left( \frac{2}{k+2} \right)^{\frac{1}{2}} \sin \frac{\pi(j+1)(l+1)}{k+2}.
\] (2.16)

In (2.15) \( h_j = \frac{j(j+2)}{4(k+2)} \) and \( c = \frac{3k}{k+2} \) are the conformal weight for a primary field of isospin \( \frac{j}{2} \) of the \( SU(2)_k \) Wess-Zumino-Witten model [19,20], and the central charge of this model respectively. It is a straightforward exercise to check that the inner product (2.13) is modular invariant.

In order to compute in this formalism expectation values like the ones appearing in eq. (2.5), we must find a way of determining what linear combination of the states (2.10) corresponds to the insertion inside the solid torus of a given set of Wilson lines. This state encodes three-dimensional topological information on the knot or link. Notice that any link can be placed inside a genus one handlebody.
This is easy to understand if one represents the link as the closure of a braid. The strands of this braid can be put inside a cylinder, which gives rise to a solid torus when one identifies its upper and lower boundaries (see Fig. 4). Unfortunately, we will not be able to find the toroidal state created by an arbitrary knot. Only for torus knots we will be able to determine it. In order to achieve this purpose let us first remark that in the gauge we are using the hamiltonian of the system is zero, which means that the “time” evolution of our theory is trivial (remember that the time direction is the one perpendicular to $T^2$). This fact is a reflection of the topological nature of our theory and, in particular, it implies that any portion of a Wilson line can be continuously deformed and translated to the boundary of the solid torus with no change in the path integral. Only if the curve on which our Wilson line is defined is a torus knot will we be able to place it completely (without self-intersections) on the boundary $T^2$. Of course, this can only be done if there are no topological obstructions due to the other components of the link. Once on the boundary, our Wilson line may be regarded as an operator acting on the state created by the insertion of the other components of the link. If we were able to move all the components of a given link to the boundary (which is only possible if our link is composed of torus knots), we would have a representation of the torus state $\Psi_1$ associated to the link as the result of acting with a product of operators on a “vacuum” state. The natural candidate for this vacuum state is $\Phi_{0,k}$ (i.e., the $SU(2)_k$ character of the identity). Indeed, we will show below that all the states (2.10) can be obtained from $\Phi_{0,k}$ by acting with Wilson line operators on it. Therefore we can write,

$$\Psi_1 = \prod_{i=1}^n \hat{W}_{R_i}^{\gamma_i} \Phi_{0,k},$$

(2.17)

where we have put a hat over the $W'$s to stress the fact that they have to be considered as operators depending on $a$ and $\bar{a}$. Recall that a torus link is characterized by two integers $r$ and $s$. It can be represented as the closure of the braid with $s$ strands $(\sigma_1 \ldots \sigma_{s-1})^r$, where $\sigma_i$ is the operation that interchanges the strands
numbered \( i \) and \( i + 1 \). When \( r \) and \( s \) are coprime integers the link is a knot that can be drawn on the surface of a torus. An \((r,s)\) torus knot is a curve on \( T^2 \) belonging to the same homology class as \( rA + sB \).

It is now straightforward to obtain the general form of the operators \( \hat{W}_j^{(r,s)} \) for a torus knot. Notice first of all that the \( \Pi \) factors coming from the measure and the states cancel in the inner product (2.13). Therefore we can ignore them everywhere and take the numerator of (2.10) as wave functionals. In this basis of states the operator formalism simplifies greatly. Let us denote by \( \Lambda_j \) the set of weights for the isospin \( j \) irreducible representation of \( SU(2) \) (i.e., the eigenvalues of the \( T^3 \) generator in (2.7)). With our conventions \( \Lambda_j = \{-j, -j + 2, \ldots, j - 2, j\} \). Using the parametrization (2.7) together with (2.9), we can write down the explicit form of the knot operators for an \((r,s)\) torus knot carrying isospin \( j \):

\[
\hat{W}_j^{(r,s)} = \sum_{n \in \Lambda_j} \exp \left[ -\frac{n \pi}{2 \text{Im} \tau} (r + s \bar{\tau})a + \frac{n}{k + 2} (r + s \tau) \frac{\partial}{\partial a} \right]. \tag{2.18}
\]

Using the well-known behaviour of the theta functions (2.11) under shifts in their argument [17], we can easily get the general form of the matrix elements of the knot operators (2.18):

\[
\hat{W}_j^{(r,s)} \lambda_{l,k+2} = \sum_{n \in \Lambda_j} \exp \left[ \frac{i \pi n^2 r s}{2(k + 2)} + \frac{i \pi nr}{k + 2} (l + 1) \right] \lambda_{l + sn,k+2}. \tag{2.19}
\]

Notice that in (2.19) we have used the \( \lambda \) functions (2.11) as basis of our Hilbert space. Two particular cases of (2.19) are very interesting. Consider first the case \( r = 0, s = 1 \) (i.e., the Wilson lines along a \( B \) cycle). Acting on the vacuum, these operators create the state corresponding to their isospin,

\[
\hat{W}_j^{(0,1)} \lambda_{0,k+2} = \lambda_{j,k+2}. \tag{2.20}
\]

In order to prove (2.20) one has to use in (2.19) the periodicity properties of the \( \lambda \) functions (see above). This result confirms our previous conclusion that the
state having zero isospin is the one obtained by doing the path integral with no Wilson line insertions. On the other hand, for an A-cycle, the operators (2.18) act diagonally and, remarkably, their matrix elements are ratios of the S matrix entries,

\[ \hat{W}_j^{(1,0)} \lambda_{l,k+2} = \frac{S_{lj}}{S_{l0}} \lambda_{l,k+2} . \quad (2.21) \]

Notice that the non-diagonal (diagonal) form of eq. (2.20) (eq. (2.21)) is to be expected from the non-contractible (contractible) nature of the B(A)-cycle in the solid torus. This is a first example of how the operators (2.18) are able to capture three-dimensional information. From the point of view of rational conformal field theories eqs. (2.20) and (2.21) imply that \( \hat{W}_j^{(r,s)} \) are Verlinde operators [21] associated to arbitrary torus knots. In fact one can prove [5] that for a fixed \((r, s)\) torus knot the operators \( \hat{W}_j^{(r,s)} \) satisfy the fusion rules of the corresponding primary fields in the 2D conformal theory.

Let us apply now our formalism to the computation of vacuum expectation values on the three-sphere \( S^3 \). It is well known that one can get \( S^3 \) by joining together two solid tori whose boundaries are identified by means of an \( S \) modular transformation. It is obvious that the partition function is the vacuum expectation value of the \( S \) matrix (i.e., \( Z(S^3) = S_{00} \)). Let us insert a set of Wilson lines in one of the solid tori by putting them first on the boundary \( T^2 \) and moving them afterwards to the interior of the solid torus. If the \( i^{th} \) Wilson line \((i = 1, \ldots , n)\) has isospin \( \frac{h}{2} \) and winds \( r_i(s_i) \) times around the A(B) cycle, the corresponding vacuum expectation value is,

\[ \langle \hat{W}_j^{(r_1,s_1)} \cdots \hat{W}_j^{(r_n,s_n)} \rangle_{S^3} = \frac{(S_{j1}^{(r_1,s_1)} \cdots S_{jn}^{(r_n,s_n)})_{00}}{S_{00}} . \quad (2.22) \]

Notice that the order in which we insert the different operators is relevant. The same set of operators inserted in different order give rise in general to different links. To illustrate this point consider for example two Wilson lines, one defined for the A cycle and the other for the B cycle. If we insert the A cycle first, both curves
are not linked, whereas if we reverse the order of insertion we get the Hopf link. This fact is illustrated in Fig. 5. The knot operators in (2.22) in general do not commute and it is precisely this fact that reflects the different three-dimensional possibilities in constructing links by inserting from the boundary torus knots in a genus one handlebody.

Let us finish this section by writing down explicitly the particular case of a one component link in eq. (2.22). Using our result (2.19) for the matrix elements of the knot operators, we get,

$$
\langle W_{j}^{(r,s)} \rangle_{S^3} = \sum_{n \in \Lambda_j} \exp \left[ \frac{i\pi n^2 r s}{2(k + 2)} + \frac{i\pi n r}{k + 2} \right] \frac{S_{ns,0}}{S_{00}}. \quad (2.23)
$$

In the next section we will use (2.23) to obtain the general expression of knot polynomials for torus knots.
3. SU(2) polynomials

Let us apply now our formalism to the computation of invariant polynomials for knots and links. Our first step will consist in rewriting eq. (2.23) in terms of the variable \[ t = e^{\frac{2\pi i}{k+2}}. \] (3.1)

Using the explicit form of the elements of the modular \( S \) matrix we get,

\[
\langle W_{j}^{(r,s)} \rangle_{S^3} = \sum_{n \in \Lambda_j} t^{\frac{\nu n^2 + \nu}{2}} t^{\frac{1+n}{2}} - t^{-\frac{1+n}{2}} \frac{1}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}.
\] (3.2)

Let us study the properties of the sum appearing in (3.2). In the case \( s = 1 \) it can be performed explicitly. We shall see now how this can be done. First of all we parametrize the elements \( n \in \Lambda_j \) as \( n = j - 2\alpha \) with \( \alpha = 0, 1, \ldots, j \). If we split the sum into the term \( \alpha = 0 \) and the contribution coming from \( \alpha > 0 \) we get (forgetting for a moment the term \( (t^{\frac{j}{2}} - t^{-\frac{j}{2}})^{-1} \)),

\[
t^{j} j(j+2) (t^{\frac{j+1}{2}} - t^{-\frac{j+1}{2}}) + \sum_{\alpha=1}^{j} t^{j} (j-2\alpha)(j-2\alpha+2) (t^{\frac{j-2\alpha+1}{2}} - t^{-\frac{j-2\alpha+1}{2}}).
\] (3.3)

The sum from \( \alpha = 1 \) to \( j \) in (3.3) vanishes identically. To prove this fact it is enough to change the summation index as \( \alpha \rightarrow j + 1 - \alpha \). Under this transformation the sum (3.3) goes into minus itself, which proves the statement. Therefore we may write,

\[
\langle W_{j}^{(r,1)} \rangle_{S^3} = t^{j} j(j+2) t^{\frac{j+1}{2}} - t^{-\frac{j+1}{2}} \frac{1}{t^{\frac{j}{2}} - t^{-\frac{j}{2}}}.
\] (3.4)

The \((r, 1)\) torus knot is depicted in Fig. 6. Notice that under \( r \) moves of the type I represented in Fig. 7 it can be transformed into the unknot \((i.e., \text{the trivial knot})\). In knot theory [22] the moves in Fig. 7 are called Reidemeister moves. If two knots can be transformed into each other by a series of these moves they are topologically
equivalent or, to be more precise, ambient isotopy equivalent. We see from (3.4) that in the Chern-Simons theory the vacuum expectation value of a Wilson line depends on the number \( r \) of type I moves. This means that one should consider the knots appearing in Chern-Simons theory as bands. Actually it is well known [22] that move I of Fig. 7, when applied to a band, introduces a twist, which makes the topological equivalence between the \((r,1)\) knot and the trivial knot no longer true (see Fig. 8). One can understand the result (3.4) from the point of view of the gauge theory if one regards the Wilson lines as knots with a frame. In fact the knots we are working with are not “bare” curves. They carry some quantum numbers, which in the present case correspond to an SU\((2)\) Kac-Moody algebra. These quantum numbers can be considered as a frame which converts our Wilson lines into bands. Actually the framing dependence of (3.4) is a multiplicative factor which can be rewritten as,

\[
e^{2\pi irh_j},
\]

where \( h_j \) is the conformal weight defined above (see eq. (2.15)). It is also clear the effect due to framing for a general \((r,s)\) knot is a factor \( e^{2\pi irsh_j} = t^{\frac{r}{4} j(j+2)} \).

Therefore, in order to construct ambient isotopy invariant polynomials we have to eliminate this factor from our expectation values. On the other hand, we must be careful with the orientations. It turns out that our conventions for torus knots are opposite to those of refs. [10,23] and hence we must change the sign of \( r \) in (3.2) before making contact with the results of [10,23]. Furthermore, it is customary to normalize the knot polynomials in such a way that the polynomial of the trivial knot is 1. These considerations lead us to define,

\[
P_j^{(r,s)}(t) = \frac{t^{\frac{r}{4} j(j+2)}}{\langle W_j^{(0,1)} \rangle_{S^3}} \langle W_j^{(-r,s)} \rangle_{S^3}.
\]

Notice the change of sign in the exponent of the deframing factor due to the change
in our orientation conventions. Using (3.2) and (3.4) we can write,

\[
P_j^{(r,s)}(t) = \frac{t^j t^j j(j+2) + \frac{j(j+1)}{2}}{t^{j+1} - 1} \sum_{n \in \Lambda_j} t^{-\frac{n^2}{4} - \frac{rn}{2}} (t^{\frac{1+sn}{2}} - t^{-\frac{1+sn}{2}}). \tag{3.7}
\]

Let us rearrange (3.7) in a more convenient way. First of all we extract the minimal power of \(t\) (for \(r, s \geq 0\)) appearing in the sum over \(\Lambda_j\). This minimal power is generated by the second term in the sum when \(n = j\) and it is equal to \(t^{-\frac{1}{4}(r)j^2 + 2(r+s)j + 2}\). Therefore, we rewrite (3.7) as,

\[
P_j^{(r,s)}(t) = \frac{t^{\frac{1}{2}(r-1)(s-1) + \frac{1}{4}(r)j^2 + 2(r+s)j + 2}}{t^{j+1} - 1} \sum_{n \in \Lambda_j} t^{-\frac{rn}{4} - \frac{rn}{2}} (t^{\frac{1+sn}{2}} - t^{-\frac{1+sn}{2}}). \tag{3.8}
\]

Finally, if we parametrize \(n\) as \(n = -j + 2l\), with \(l = 0, 1, \cdots, j\) it is easy to obtain the expression:

\[
P_j^{(r,s)}(t) = \frac{t^{\frac{1}{2}(r-1)(s-1)}}{t^{j+1} - 1} \sum_{l=0}^{j} t^{r(1+sl)(j-l)} (t^{1+sl} - t^{s(j-l)}). \tag{3.9}
\]

Let us work out some particular cases of our general equation (3.9). Putting \(j = 1\) (i.e., for the fundamental representation) we obtain,

\[
P_1^{(r,s)}(t) = \frac{t^{\frac{1}{2}(r-1)(s-1)}}{1 - t^2} (1 - t^{r+1} - t^{s+1} + t^{r+s}). \tag{3.10}
\]

This result coincides with the one obtained by Jones in [23]. In the context of Chern-Simons theory it was obtained in [5]. On the other hand for the adjoint representation \((j = 2)\) eq. (3.9) yields,

\[
P_2^{(r,s)}(t) = \frac{t^{(r-1)(s-1)}}{1 - t^3} (1 - t^{2r+1} - t^{2s+1} + t^{2(r+s)} + t^{r+s+rs} - t^{(r+1)(s+1)}). \tag{3.11}
\]

For \(s = 2\) eq. (3.11) was first given by Akutsu and Wadati [13], who obtained it from the study of a three-state solvable vertex model in Statistical Mechanics.
The corresponding link polynomials are defined by means of a Markov trace, which is evaluated recursively by using the defining relations of the braid group. This makes their procedure rather cumbersome to obtain general expressions like (3.9). Another possible approach is trying to use the recursion relations (skein rules) generated by the representation of the braid group. In fact, the polynomial for \((r, 2)\) torus knots was obtained in [13] by solving the skein rules for the closure of a two-strand braid. The result coincides with eq. (3.11). To the best of our knowledge the general equation (3.9) was not known previously. As we pointed out in the introduction, the main advantage of our approach is that it allows a direct evaluation of some link polynomials. Once the general expression is obtained, we can search for recursion relations fulfilled by our results. This shall be done in sect. 5, where we check that the skein rules satisfied by our polynomials are indeed those obtained in ref. [13]. Hence our conclusion is that the isospin \(\frac{j}{2}\) Wilson lines in Chern-Simons theory give rise to knot polynomials identical to the Akutsu-Wadati polynomials for a \((j + 1)\)-state vertex model.

Two important properties of our general expression (3.9) are worth mentioning. First of all it can be shown by simple inspection that \(P_j^{(r,s)}\) is symmetric under the interchange of \(r\) and \(s\),

\[
P_j^{(r,s)}(t) = P_j^{(s,r)}(t).
\]  \[(3.12)\]

This result should be expected since the torus knots \((r, s)\) and \((s, r)\) are equivalent in \(S^3\). On the other hand it is easily proved that substituting the argument \(t\) of \(P_j^{(s,r)}\) by \(\frac{1}{t}\) one gets the polynomial for the mirror image knot, namely,

\[
P_j^{(r,s)}\left(\frac{1}{t}\right) = P_j^{(-r,s)}(t).
\]  \[(3.13)\]

This is a well known property of the Jones polynomial [10,23] that generalizes to the higher isospin case.

It is far from obvious from (3.9) that \(P_j^{(r,s)}(t)\) is always a polynomial in the variable \(t\). One should check that any root of the denominator of (3.9) \((i.e., \text{any})\)
such that \( t^{j+1} = 1 \) is also a root of the sum appearing in the numerator of this equation. In order to get the general pattern of how this occurs it is useful to study some particular cases first. For \( j = 1 \) one easily shows that the numerator of (3.10) vanishes for \( t = \pm 1 \). In order to prove it one must use the fact that \( r \) and \( s \) are coprime and therefore they cannot be both even. In the same way as \( r \) and \( s \) are not both multiple of three, one can verify that any third root of unity is also a root of the numerator of (3.11). The general proof of the polynomial character of \( P_j^{(r,s)}(t) \) goes along the same lines of these two examples and is presented in the Appendix.

Let us consider now the case in which several operators are inserted. In this case our general expression (2.22) will give rise to link invariants. The number of components of the link is precisely the number of knot operators appearing in (2.22). As was previously discussed, we must be careful about the order in which we take the product of the knot operators. Computing, for example, \( \langle W_i^{(0,1)} W_j^{(1,0)} \rangle_{S^3} \) we get, using (2.20) and (2.21),

\[
\langle W_i^{(0,1)} W_j^{(1,0)} \rangle_{S^3} = \frac{S_{i0} S_{j0}}{(S_{00})^2}.
\]

(3.14)

This result is easy to interpret. Remember that according to eqs. (2.16), (3.4) and (3.12) one has,

\[
\langle W_i^{(0,1)} \rangle_{S^3} = \langle W_i^{(1,0)} \rangle_{S^3} = \frac{S_{i0}}{S_{00}}.
\]

(3.15)

Therefore we can rewrite eq. (3.14) as,

\[
\langle W_i^{(0,1)} W_j^{(1,0)} \rangle_{S^3} = \langle W_i^{(0,1)} \rangle_{S^3} \langle W_j^{(1,0)} \rangle_{S^3}.
\]

(3.16)

This factorized result is quite natural since, as argued at the end of sec. 2, we are computing the vacuum expectation value for a link consisting of two unlinked trivial knots. Actually Witten has proved in [1] that, with our normalizations, the expectation value for a link having several unlinked components always factorizes
into the product of the average values of its unlinked parts. On the other hand reversing the order of the operators inside the vacuum expectation value we get,

$$\langle W^{(1,0)}_j W^{(0,1)}_i \rangle_{S^3} = \frac{S_{ji}}{S_{00}}. \quad (3.17)$$

In this case, as we insert the $B$ cycle into the solid torus first, the two components are linked (we are constructing the Hopf link, see Fig. 5) and the result does not factorize.

It is important to point out that with our approach we can construct the same link in many ways. Nevertheless, as we shall check in some examples, the result for the invariant polynomial is unique. For instance we could construct the link consisting of two unlinked unknots by inserting two $A$-cycles or two $B$-cycles as shown in Fig. 9. The corresponding expectation values would be $\langle W^{(1,0)}_i W^{(1,0)}_j \rangle_{S^3}$ and $\langle W^{(0,1)}_i W^{(0,1)}_j \rangle_{S^3}$ (the order of the operators in this case does not matter since they commute). In the case of two $A$-cycle insertions it is straightforward to verify using (2.21) that we get precisely the right-hand side of eq. (3.14). On the other hand by a direct calculation we can prove the following property of the modular $S$ matrix:

$$\sum_{n \in \Lambda_j} S_{i+n,0} = \frac{S_{i0} S_{j0}}{S_{00}}, \quad (3.18)$$

which is the basic equation needed to prove that the insertion of two $B$-cycles gives the same result as in (3.16).

Another important issue is the question of the framing of the different link components. Let us compare the vacuum expectation values $\langle W^{(r,s)}_k W^{(l,1)}_j \rangle_{S^3}$ and $\langle W^{(r,s)}_k W^{(0,1)}_j \rangle_{S^3}$. It is clear that in both cases we are constructing the same link because the $(l,1)$ knot, which is inserted first into the solid torus, can be converted into a knot parallel to the $B$-cycle by a series of type I Reidemeister moves. For example, the link of Fig. 10 is ambient isotopic to the one depicted in Fig. 11. Therefore both expectation values should only differ by a framing factor. Let us
check this fact explicitly. Using equations (2.16) and (2.19) for the matrix elements, we get,
\[
\langle W^{(r,s)}_k W^{(l,1)}_j \rangle_{S^3} = \sum_{m \in \Lambda_k, n \in \Lambda_j} t^{n(n+2) + m(m+2n+2)} \frac{t^{\frac{n+m+1}{2}} - t^{-\frac{n+m+1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}.
\] (3.19)

As happened with the case of a one-component link, the sum over \(\Lambda_j\) can be explicitly performed. In complete analogy with what we have done to obtain (3.4), let us parametrize \(n \in \Lambda_j\) as \(n = j - 2\alpha\) with \(\alpha = 1, \ldots, j\). The contribution of \(\alpha = 0\) (i.e., \(n = j\)) in (3.19) is:
\[
t^j(j+2) \sum_{m \in \Lambda_k} t^{jm} \frac{t^{\frac{ms+1}{2}} - t^{-\frac{ms+1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}},
\] (3.20)
which is equal to \(t^j(j+2) \langle W^{(r,s)}_k W^{(0,1)}_j \rangle_{S^3}\). The remaining terms in the sum over \(\Lambda_j\) cancel among themselves. To prove it we notice that redefining the summation indices in (3.19) as \(m \rightarrow -m\) and \(\alpha \rightarrow j + 1 - \alpha\) the sum changes its sign. Hence we have found that,
\[
\langle W^{(r,s)}_k W^{(l,1)}_j \rangle_{S^3} = t^j(j+2) \langle W^{(r,s)}_k W^{(0,1)}_j \rangle_{S^3},
\] (3.21)
which means that both vacuum expectation values will give rise to the same result once they are conveniently deframed. In general for any knot operator \(W^{(r,s)}_j\) appearing inside an expectation value, we shall include a factor \(t^{-j(j+2)} = e^{-2\pi irsh_j}\) in order to obtain link polynomials invariant under type I Reidemeister moves.

On the other hand, if we perform a two dimensional projection of our link, we shall have crossings corresponding to the same component, as well as others involving two different components of the link. This last type of crossings is characterized by the linking number, which is a topological invariant quantity. In the context of these two dimensional projections one can understand the framing dependence (3.4) and (3.21) as a non topological factor that our gauge theory introduces in
the expectation value each time a component knot crosses itself. The deframing procedure eliminates this factor, giving rise to ambient-isotopy invariant quantities. From this point of view it is natural to treat all the crossings of a given link on equal footing. Only in this case will we make contact with the formulations in which the crossing of two lines is obtained by the action of an operator (the so-called braiding matrix) which does not distinguish if the lines belong to the same component of the link or not. In these formulations the link polynomial satisfies a set of skein rules, which are obtained from the characteristic equation of the braiding matrix (see sect. 5). Therefore, we should correct our expectation values by including, for each crossing of two different components, a factor of the same type as the one we took into account for the self-crossings of a knot. We must do that without losing the topological invariance of the corrected result. As was mentioned above, the natural topological invariant quantity associated to two link components is their linking number. It is easy to convince oneself that the number of crossings between two link components is twice their linking number (see, for example, the closed braid of Fig. 10). Therefore, if all the linking components carry the same isospin \( \frac{j}{2} \), the correcting factor we are looking for is,

\[
e^{-2\pi ih_j(2L)},
\]

(3.22)

where \( L \) is the total linking number. Taking all these facts into account, we can write down the general expression for the invariant associated to the insertions of the Wilson lines \( W_j^{(r_1,s_1)} \cdots W_j^{(r_n,s_n)} \). If the order of insertion is the same as in eq. (2.22), the linking number will be,

\[
L = \sum_{i<j}^{i,j=1, \ldots, n} r_i s_j.
\]

(3.23)

Notice that (3.23) depends on the order of insertion. Therefore the corresponding invariant polynomial can be written as,

\[
t^{-\frac{1}{2}j(j+2)}(\sum_{i=1}^{n} r_i s_i + 2L) \langle W_j^{(r_1,s_1)} \cdots W_j^{(r_n,s_n)} \rangle_{S^3} \frac{\langle W_j^{(0,1)} \rangle_{S^3}}{\langle W_j^{(0,1)} \rangle_{S^3}}.
\]

(3.24)
The formalism we have developed so far can be used to compute the polynomials for general torus links. Remember that an \((r, s)\) torus link is the closure of the braid \((\sigma_1 \ldots \sigma_{s-1})^r\). In the following we shall denote by \(n\) the greatest common divisor of \(r\) and \(s\). It is easy to convince oneself that an \((r, s)\) torus link has \(n\) components, each of which is an \((r/n, s/n)\) torus knot. In Fig. 10 we have shown this fact for the \((4, 2)\) torus link. Furthermore, the full link can be obtained by a successive insertion of its components into the interior of \(T^2\). We shall restrict ourselves to the case in which all the lines carry the same isospin. With our conventions for the orientation of the homology cycles \(A\) and \(B\), it is clear that we must compute the average value \(\langle (W_j^{(-\frac{r}{n}, \frac{s}{n})})^n \rangle_{S^3}\). Using (3.23) we see that in this case the linking number is \(L = -\frac{rs}{n^2} \frac{n(n-1)}{2}\). We can use (3.24) to write down the generalization of (3.6) to the case in which \(r\) and \(s\) are not coprime, \(P_j^{(r,s)}(t) = \frac{t^{rj+2}j(n+2)}{t^{j+1}} \langle (W_j^{(-\frac{r}{n}, \frac{s}{n})})^n \rangle_{S^3}\). (3.25)

Notice that in (3.25) the same deframing factor as in (3.6) appears. In order to compute the expectation value appearing in (3.25) the easiest way is to notice that all the operators we are multiplying have the same pair of indices \((-\frac{r}{n}, \frac{s}{n})\) and hence they commute. Therefore using (2.18) we can write:

\[
(W_j^{(-\frac{r}{n}, \frac{s}{n})})^n = \sum_{p_1, \ldots, p_n \in \Lambda_j} \exp \left[ -\frac{\pi}{2\text{Im}\tau} \sum_{i=1}^n p_i \left( -\frac{r}{n} + \frac{s}{n} \tau \right) a \right] + \sum_{i=1}^n \frac{(-r+s\tau)}{n} \frac{a}{k+2} \frac{\partial}{\partial a}.
\]

(3.26)

Using (3.26) it is clear that the matrix element appearing in (3.25) can be obtained from the one corresponding to torus knots (eq. (3.2)) just substituting \(r \to \frac{r}{n}, s \to \frac{s}{n}\) and \(n \to \sum_{i=1}^n p_i\) and summing over all \(p_i \in \Lambda_j\). Using this result in (3.25) we get,

\[
P_j^{(r,s)}(t) = \frac{t^{rj+2}j(n+2)}{t^{j+1}} \sum_{p_1, \ldots, p_n \in \Lambda_j} t^{-\frac{r}{n}(\sum p_i)^2 - \frac{s}{n}(\sum p_i)} \left( t^{\frac{11}{2} \sum p_i} - t^{\frac{11}{2} \sum p_i} \right).
\]

(3.27)

We can perform in (3.27) the same sort of manipulations we have done to obtain
First we parametrize the indices $p_i$ as $p_i = -j + 2l_i$, with $l_i = 0, 1, \ldots, j$. From (3.27) it is obvious that only the combination $\sum_{i=0}^{n} l_i$ will appear in the sum. If we define the following combinatorial factor,

$$C_j(n, l) = \sum_{l_1, \ldots, l_n = 0}^{j} \delta_{l_1 + \cdots + l_n, l},$$

we can convert the multiple sum of (3.27) into a single sum,

$$P_j^{(r,s)}(t) = \frac{t^j t^{(r-1)(s-1)}}{t^j + 1 - 1} \sum_{l=0}^{nj} C_j(n, l) t^{\frac{r}{n}} (1 + \frac{s}{n}) (n - l) (t^{\frac{s}{n}} - t^{\frac{r}{n}}),$$

which is the expression generalizing (3.9) to the case in which $r$ and $s$ are not coprime. The combinatorial factors (3.28) have the following reflection symmetry:

$$C_j(n, l) = C_j(n, nj - l),$$

which can be easily obtained from their definition (eq. (3.28)). Using (3.30) one immediately proves that eq. (3.12) is also satisfied when $r$ and $s$ are not coprime. Moreover changing $t \to \frac{1}{t}$ in (3.29) one can prove that the equation (3.13) also holds for torus links.

In the case of the fundamental representation (i.e., for the Jones polynomial) the combinatorial factors (3.28) are easy to compute. They are simply,

$$C_1(n, l) = \binom{n}{l}.\tag{3.31}$$

Therefore we get the following equation for the Jones polynomial for torus links,

$$P_1^{(r,s)}(t) = \frac{t^j t^{(r-1)(s-1)}}{1 - t^2} \sum_{l=0}^{n} \binom{n}{l} t^{\frac{r}{n} (1 + \frac{s}{n}) (n - l) (t^{\frac{s}{n}} - t^{\frac{r}{n}})},$$

to be compared with eq. (3.10). The sum in (3.32) obviously vanishes for $t = 1$.\textsuperscript{22}
For \( t = -1 \) it is equal to,

\[
(-1)^{r+1}(1 + (-1)^s)(1 + (-1)^{r+1+ns/n^2})^n,
\]

and can be shown to be always zero due to the fact that \( \frac{r}{n} \) and \( \frac{s}{n} \) are coprime. In conclusion \( P_1^{(r,s)} \) is always a polynomial. Actually, due to the term \( t^{\frac{1}{4}(r-1)(s-1)} \) in (3.32), when the number \( n \) of link components is odd \( P_1^{(r,s)} \) is a Laurent polynomial in the variable \( t \), whereas for a link with an even number of components \( P_1^{(r,s)} \) is \( \sqrt{t} \) times a Laurent polynomial in \( t \), in agreement with Jones’ result in [10,23].

For general \( j \) we can represent the combinatorial factor \( C_j(n, l) \) as the coefficient of \( x^l \) in the power expansion of \( (1 + x + x^2 + \cdots + x^n)^n \). Therefore we can write,

\[
C_j(n, l) = \sum_{k_1, \ldots, k_j=0}^n \binom{n}{k_1} \binom{k_1}{k_2} \cdots \binom{k_{j-1}}{k_j} \delta_{k_1+\cdots+k_j,l}.
\]

(3.34)

This equation generalizes (3.31) and allows to perform explicit computations in eq. (3.29). The question remains whether (3.29) gives rise to a polynomial or not. We will not attempt to prove this here but let us mention that using the representation (3.34) we have checked it explicitly in many particular cases and we conjecture that it is true in general.

The values of the Jones polynomials and its derivatives at \( t = 1 \) have been studied in refs. [23] and [24]. It is clear that these values must be topological invariant quantities characterising the link. We can use our general expression (3.29) for torus links in order to get the generalization of the results of [23] and [24] for an arbitrary isospin. When one takes \( t = 1 \) in the right-hand side of (3.29) one encounters an indefinite expression that can be readily evaluated by using l’ Hôpital’s rule together with the following sums involving the combinatorial factors \( C_j(n, l) \):

\[
\sum_{l=0}^{n^j} C_j(n, l) = (j + 1)^n, \quad \sum_{l=0}^{n^j} lC_j(n, l) = n \frac{j(j + 1)^n}{2}.
\]

(3.35)
The final result only depends on the number \( n \) of components of the link:

\[
P^{(r,s)}_j(1) = (j + 1)^{n-1}.
\]  

(3.36)

In the same way one can compute the first derivative of \( P^{(r,s)}_j(t) \) at \( t = 1 \). In the course of the calculation one needs the sum:

\[
\sum_{l=0}^{n_j} l^2 C_j(n, l) = n \frac{j(j+1)^n}{2} \left[ \frac{2j+1}{3} + \frac{(n-1)j}{2} \right].
\]  

(3.37)

After some algebra we obtain,

\[
\frac{d}{dt} P^{(r,s)}_j(1) = -\frac{j(j+2)(j+1)^{n-1}}{2} L,
\]

(3.38)

where \( L = -\frac{rs}{n^2} \frac{n(n-1)}{2} \) is the total linking number. For \( j = 1 \) eqs. (3.36) and (3.38) coincide with the ones obtained in [23] and [24] (after taking into account the different conventions used in the definition of the polynomials for links and in the linking number, see sect. 5). Moreover although we have obtained (3.36) and (3.38) using our general expression (3.29), which is only valid for torus links, we have checked that they are also satisfied by all the link polynomials obtained in [13] and [7], which include more general links than those considered here. We therefore conjecture that our expressions for the value of the Akutsu-Wadati polynomials and their first derivatives at \( t = 1 \) are valid for any link or knot, generalizing in this way the results of refs. [23] and [24].
4. Minimal model polynomials

The Chern-Simons program can be extended to include the three-dimensional description of conformal field theories which do not possess an underlying Kac-Moody symmetry. In this case one must be able to formulate a quantum-mechanical problem whose toroidal states represent the characters of the corresponding conformal field theory. One should find in this Hilbert space an operator for any torus knot carrying the quantum numbers of the two-dimensional primary fields. These operators have to satisfy the Verlinde property, i.e., some class of them must create the states when acting on the vacuum. From the point of view of knot theory, the expectation values of these operators will define polynomial invariants for knots and links coloured with the quantum numbers appearing in the representation theory of the two-dimensional model. Exploring the space of conformal field theories and trying to find a three-dimensional formulation for some of these models, one could discover new hierarchies of link polynomials. It is natural to think that new classes of theories, having different chiral algebras, could provide invariants capable of detecting new three-dimensional topological properties.

In [16] we have taken a modest step in this direction. Using the coset construction [25], we were able to develop the Chern-Simons program for the discrete series of minimal unitary models [26] having central charge $c < 1$. Our starting point was the zero-mode problem of the Chern-Simons gauge theory $SU(2)_m \times SU(2)_1$, consisting of two gauge fields having a Chern-Simons action with levels $m$ and 1. In this zero-mode problem one finds that there exists a variable in terms of which the states in the Hilbert space decompose into $SU(2)_{m+1}$ characters. Integrating out this variable one is left with an effective problem that implements the coset $SU(2)_m \times SU(2)_1 / SU(2)_{m+1}$. A basis in this effective Hilbert space is given by the following set of wave functionals depending on the remaining variable $d$:

$$\chi_{p,q}(d, \tau) = \frac{e^{i \pi d}}{2\eta(\tau)} \left[ \Theta_{n_{-},k}(d, \tau) - \Theta_{n_{+},k}(d, \tau) + \Theta_{-n_{-},k}(d, \tau) - \Theta_{-n_{+},k}(d, \tau) \right], \quad (4.1)$$

where $p$ and $q$ are integers ($1 \leq p \leq m+1$, $1 \leq q \leq m+2$), $n_{\pm} = p(m+3) \pm q(m+2)$.
and \( k = (m + 2)(m + 3) \). The wave functions (4.1) evaluated at the origin \((d = 0)\) are the standard Rocha-Caridi characters [27] for the minimal unitary models with central charge \( c_m = 1 - \frac{6}{(m+2)(m+3)} < 1 \). The set of all possible values of \( p \) and \( q \) define the so-called conformal grid. Changing \( p \) and \( q \) as \( p \to m + 2 - p \) and \( q \to m + 3 - q \), we get the same wave functional: \( \chi_{p,q} = \chi_{m+2-p,m+3-q} \). This is the reflection symmetry of the conformal grid, which allows to determine the number of independent states of the form (4.1). Under the action of the modular group the wave functionals \( \chi_{p,q} \) behave as the characters of the minimal models, 

\[
\chi_{p,q}(d, \tau)|_T \equiv \chi_{p,q}(d, \tau + 1) = e^{2\pi i(h_{p,q} - \frac{c_m}{12})} \chi_{p,q}(d, \tau),
\]

\[
\chi_{p,q}(d, \tau)|_S \equiv \chi_{p,q}(\frac{d}{\tau}, -\frac{1}{\tau}) = \sum_{p',q'} S_{p,q}^{p',q'} \chi_{p',q'}(d, \tau),
\]

(4.2)

where \( h_{p,q} = \frac{[(m+3)p-(m+2)q]^2-1}{4(m+2)(m+3)} \) are the conformal weights of the primary fields and the matrix \( S \) is given by [26],

\[
S_{p,q}^{p',q'} = \left( \frac{8}{(m+2)(m+3)} \right)^{\frac{1}{2}} (-1)^{(p+q)(p'+q')} \sin \frac{\pi pp'}{m+2} \sin \frac{\pi qq'}{m+3}.
\]

(4.3)

The knot operators acting in the Hilbert space spanned by the states (4.1) are Wilson line operators constructed with the one-form:

\[
D = \frac{\pi d}{2\text{Im} \tau} \bar{\omega} - \frac{\pi \bar{d}}{2\text{Im} \tau} \omega.
\]

(4.4)

To a given closed curve on \( T^2 \) we shall associate observables obtained by combining abelian Wilson lines for the gauge field \( D \) with different charges. In order to find out the precise combination of charges that builds up the quantum numbers of the minimal models, let us rewrite the states (4.1) as a double sum over the \( SU(2) \) Weyl group (i.e., over \( Z_2 \)):

\[
\chi_{p,q}(d, \tau) = \frac{e^{\pi m^2 \text{Im} \tau}}{2\eta(\tau)} \sum_{\omega, \omega' \in Z_2} \epsilon(\omega) \epsilon(\omega') \Theta_{(m+3)\omega(p)-(m+2)\omega'(q),k}(d, \tau),
\]

(4.5)

where \( \omega(p) = \pm p \) and \( \epsilon(\omega) \) is the signature of the Weyl group element \( \omega \). Eq. (4.5) suggests that one should regard \( p \) and \( q \) as labels of \( SU(2) \) representations. In
fact, the $U(1)$ charges entering the knot operators for the $(p, q)$ state are obtained by combining the weights of two $SU(2)$ representations of isospin $\frac{p-1}{2}$ and $\frac{q-1}{2}$. Define the set of weights $\Gamma_{p,q} = (m+3)\Lambda_{p-1} + (m+2)\Lambda_{q-1}$. For any curve $\gamma$ on $T^2$ we define:

$$ W_{p,q}^\gamma = \sum_{n \in \Gamma_{p,q}} e^{-n \int_\gamma D}. $$

(4.6)

By looking at the inner product measure in the effective problem for the variable $d$, one gets the operator representation (analogous to eq. (2.9)):

$$ \bar{d} = \frac{2 \text{Im} \tau}{\pi(m+2)(m+3)} \frac{\partial}{\partial d}. $$

(4.7)

Therefore using (4.4) and (4.7) in (4.6), we can write the explicit representation of the knot operator for a torus knot $\gamma = rA + sB$:

$$ \hat{W}_{p,q}^{(r,s)} = \sum_{n \in \Gamma_{p,q}} \exp\left[ -\frac{n \pi}{2 \text{Im} \tau} (r + s\bar{\tau})d + \frac{n}{(m+2)(m+3)} (r + s\tau) \frac{\partial}{\partial d} \right]. $$

(4.8)

As a check we can calculate the action of the $B$-cycle operators on the vacuum $\chi_{1,1}$:

$$ \hat{W}_{p,q}^{(0,1)} \chi_{1,1} = \chi_{p,q}; $$

(4.9)

which is the equation equivalent to (2.20) and confirms that our operators (4.8) carry the quantum numbers of the degenerate representations of the Virasoro algebra. For the $A$-cycle one gets an equation similar to (2.21):

$$ \hat{W}_{p,q}^{(1,0)} \chi_{p',q'} = \frac{S_{p,q}^{p',q'}}{S_{1,1}^{p,q}} \chi_{p',q'}. $$

(4.10)

In fact one can check that for $r$ and $s$ relatively prime, the $\hat{W}_{p,q}^{(r,s)}$ operators satisfy the fusion rules of the minimal unitary models. The general matrix element was
given in [16]. Let us rewrite it as a function of two variables analogous to (3.1):

\[ t_1 = e^{2i\pi \frac{m+3}{m+2}}, \quad t_2 = e^{2i\pi \frac{m+2}{m+3}}. \]  

(4.11)

In terms of \( t_1 \) and \( t_2 \) one can write,

\[ \hat{W}^{(r,s)}_{p,q} \chi_{p'q'} = \sum_{n \in \Lambda_{p-1}} \sum_{l \in \Lambda_{q-1}} e^{i\pi r[s(p-1)(q-1)+(p-1)q'+(q-1)p'] \frac{m}{m+2} n^2 \frac{m}{m+3} l^2 + \frac{m}{m+3} q'} \chi_{p'+sn,q'+sl}. \]  

(4.12)

It is important to point out that the variables \( t_1 \) and \( t_2 \) defined in (4.11) have been also considered in ref. 28, where the braiding matrices of the minimal models were studied within a Feigin-Fuchs approach.

In order to compute the knot invariants associated to the operators (4.8) we proceed in complete analogy with the \( SU(2) \) case. The expectation values of our operators on the three-sphere will be the result of a certain functional integral over fields living in \( S^3 \). They can be obtained by the equivalent of eq. (2.22):

\[ \langle W^{(r_1,s_1)}_{p_1,q_1} \cdots W^{(r_n,s_n)}_{p_n,q_n} \rangle_{S^3} = \frac{(S\hat{W}^{(r_1,s_1)}_{p_1,q_1} \cdots \hat{W}^{(r_n,s_n)}_{p_n,q_n})^1_{1,1}}{S^1_{1,1}}. \]  

(4.13)

When one performs the explicit evaluation of the matrix element (4.13) it is useful to write the form of the modular matrix \( S \) in terms of \( t_1 \) and \( t_2 \). One has,

\[ S_{p'q'}^{p,q} = \frac{1}{\sqrt{2(m+2)(m+3)}} (-1)^{pq'+p'q} (t_1 \frac{m}{2} - t_1 t_2 \frac{m}{2} - t_2 \frac{m}{2}). \]  

(4.14)

From eqs. (4.12) and (4.14) one suspects that the minimal models will give rise to two-variable polynomials. The result of our calculations will show that these two-variable polynomials are in fact products, with a global sign, of one-variable \( SU(2) \) polynomials. Suppose, for example, that we want to obtain \( \langle W^{(r,1)}_{p,q} \rangle_{S^3} \). Plugging eqs. (4.12) and (4.14) into the general expression (4.13) one finds two sums (one
for each variable) identical to the one evaluated in eqs. (3.3) and (3.4). Therefore one can use the results of sect. 3 and write:

\[ \langle W_{r,1}^{(r,1)} \rangle_{S^3} = (-1)^{r(pq+1)} t_1^{\frac{r}{4}(p^2-1)} t_2^{\frac{r}{4}(q^2-1)} S_{1,1}^{pq} / S_{1,1}^{1,1}. \] (4.15)

The \( r \)-dependent factor in (4.15) is quite natural since,

\[ e^{2\pi i h_{p,q}} = (-1)^{pq+1} t_1^{\frac{p^2-1}{4}} t_2^{\frac{q^2-1}{4}}. \] (4.16)

Hence we can write,

\[ \langle W_{p,q}^{(r,1)} \rangle_{S^3} = e^{2\pi i r h_{p,q}} \langle W_{p,q}^{(0,1)} \rangle_{S^3} \] (4.17)

which confirms that the knots under consideration are framed with degenerate representations of the Virasoro algebra. Eq. (4.17) implies that to define ambient isotopy polynomials we have to multiply each operator \( \hat{W}_{r,s} \) by a factor \( e^{-2\pi i r h_{p,q}} \).

In the case in which we have several knot operators inside the expectation value, one can check that consistency conditions similar to eqs. (3.14)-(3.18) are also verified. The equivalent of relation (3.21) is also fulfilled. In conclusion we have formulated a well-defined scheme which associates an invariant link polynomial to any integers (\( p, q \)) labelling the primary fields of the minimal unitary series of the Virasoro algebra. Let us obtain the general expression of these invariants for torus links. If \( n \) denotes the number of components of a (\( r, s \)) torus link (i.e., \( n \) is the greatest common divisor of \( r \) and \( s \), see sect. 3) one can write in complete analogy with eq. (3.25):

\[ \Pi_{p,q}^{(r,s)}(t_1, t_2) = \frac{(-1)^{rs(pq+1)} t_1^{\frac{r}{4}(p^2-1)} t_2^{\frac{r}{4}(q^2-1)}}{\langle W_{p,q}^{(0,1)} \rangle_{S^3}} \langle W_{p,q}^{(-\frac{r}{2}, \frac{s}{2})} \rangle_{S^3}^{n}. \] (4.18)

The computation of eq. (4.18) proceeds along the same steps as the \( SU(2) \) case. First, one realizes that the operators appearing in (4.18) commute. The calculation
of the required matrix element is then straightforward and the result factorizes into two $SU(2)$ polynomials in the variables $t_1$ and $t_2$:

$$
\Pi_{p,q}^{(r,s)}(t_1, t_2) = (-1)^{(p+q)(n-1)} P_{p-1}^{(r,s)}(t_1) P_{q-1}^{(r,s)}(t_2). \tag{4.19}
$$

In (4.19) we are using the notation $P_{0}^{(r,s)}(t) = 1$. In particular the polynomials for the representations $(1, q)$ and $(p, 1)$ are, except for a global sign depending on the number of components $n$, equal to the Akutsu-Wadati polynomials for isospins $\frac{q-1}{2}$ and $\frac{p-1}{2}$ respectively. For $(1, 2)$ and $(2, 1)$ primary fields this same conclusion was reached in [29] from considerations of rational conformal field theories. Our result (4.19) generalizes the one in ref. [29] and shows that, from the point of view of knot theory, little is gained with respect to $SU(2)$ by placing minimal model quantum numbers on the curves defining a knot or link. However our aim was to show how to construct polynomials in a theory without a Kac-Moody chiral algebra. The space of rational conformal field theories is a vast one and can provide many surprises in three-dimensional topology.
5. Skein rules from knot operators

In this section we will derive the skein rules corresponding to links carrying an arbitrary integrable representation of $SU(2)_k$, and to links carrying arbitrary quantum numbers of a primary field of a minimal unitary model. In the Literature the skein rules have been computed using information from conformal field theory in several situations. The procedure originally proposed in [1] for the fundamental representation of $SU(N)$ has been extended to arbitrary representations of $SU(2)$ [8,6,7], and to the fundamental representation of $SO(N)$, $Sp(2n)$, $SU(m|n)$ and $OSp(m,2n)$ [9,6]. This procedure uses the half-monodromy matrix of the corresponding conformal field theory. The novelty of the derivation of skein rules we will carry out in this section is that it does not make use of such a matrix. In our derivation we make use of the knot operators obtained in [5,16] and, therefore, the approach does not need information from conformal field theory. Because of this feature this method could be used in situations where the data from conformal field theory is not available. Furthermore, as will be shown, one could then extract quantities like the eigenvalues of the half-monodromy matrix from this approach.

The skein rules are not useful to compute link invariants in general. Only for the case of skein rules involving three entries does a systematic procedure exist to compute arbitrary link invariants. However, one can write down in general many types of skein rules which might help to design a complete procedure for a given situation. On the other hand, the skein rules provide a simple way to compare and identify link invariants. For example, the invariant polynomials for arbitrary representations of $SU(2)$ have been identified with the Akutsu-Wadati polynomials [13] after comparing their corresponding skein rules [8,6,7]. It is worth therefore having a systematic procedure to obtain skein rules from knot operators. Since knot operators can be constructed for any compact group [5] and for some coset constructions [16] one could obtain skein rules for a large variety of situations.
5.1. Skein rules for $SU(2)$

Let us consider a three-ball with boundary $S^2$ and four marked points connected by Wilson lines as shown in Fig. 12. The structure inside the box $A$ in this figure is arbitrary. We will assume that the four lines attached to the marked points carry the same integrable representation of $SU(2)_k$. Let us choose this representation to be the one with isospin $j/2$. It is well known that for large enough $k$ the Hilbert space associated to the three-ball with four marked points pictured in Fig. 12 is $j + 1$-dimensional. This implies that there are at most $j + 1$ linearly independent states of this form. The procedure to obtain skein rules is to take $j + 2$ specific states of the form depicted in Fig. 12 and build a linear relation among them. In this way all link invariants obtained by gluing the same three-ball with four marked points to the $j + 2$ ones chosen are linearly related. Knowledge of the skein rules means knowledge of the coefficients entering these linear relations. The way one takes $j + 2$ states of the form shown in Fig. 12 is arbitrary as long as they are topologically unequivalent and this leads to different types of skein rules. The procedure presented in this paper could be extended to more general situations. For instance notice that we have started with all four points associated to the same representation. We could have chosen different representations and even increase the number of marked points on the boundary of the three-ball.

We have indicated above that the procedure to obtain skein rules we are about to describe does not make use of the half-monodromy matrix. However, it seems that we are using some information from conformal field theory when we state the dimensionality of the Hilbert space associated to the states represented in Fig. 12. As we will discuss below not even this information is needed. The only feature we need to use about this Hilbert space is that its dimension is finite.

Let us consider states of the type depicted in Fig. 12 which have the form shown in Fig. 13. These states involve $i$ half-monodromy twists of two strands going through the three-ball. Let us denote such state as $|j; i\rangle$, where $j$ labels the representation carried by the two strands, having isospin $j/2$, and $i$ labels the
number of half-monodromy twists. A negative value of \( i \) corresponds to a twist in the opposite sense to the one drawn in Fig. 13. It is known from general arguments in Chern-Simons theory [1] that for a fixed \( j \) the dimension of the Hilbert space to which \(|j; i\rangle\) belongs is finite. This implies that one can choose a finite number of states for a fixed \( j \) in such a way that any other is a linear combination of them. Clearly, the states \(|j; i\rangle\) for a fixed \( j \) can be chosen in a variety of ways and this leads to different types of skein rules. Here we will only consider the case in which all the states involved differ by one half-monodromy twist respect to some other state. The extension of our analysis to obtain other sets of skein rules by considering, for example, states differing by \( m \) half-monodromy twists, is straightforward. The resulting skein rules are in fact linear combinations of the ones for \( m = 1 \).

Using the fact that the dimension of the Hilbert space corresponding to configurations of the form pictured in Fig. 13 is finite one can be sure that there exist a finite number \( N > 0 \) and coefficients \( \alpha_i, \ i = 0, 1, ..., N \), not all zero, such that,

\[
\sum_{i=0}^{N} \alpha_i |j; M + i\rangle = 0, \tag{5.1}
\]

for some arbitrary value of \( M \). Certainly there exist many values of \( N \) for which (5.1) holds. We are interested in determining the smallest value of \( N \) for which (5.1) is satisfied. Of course, this value of \( N \) as well as the form of the coefficients \( \alpha_i \) depends on the representation chosen, i.e., it depends on \( j \) although it is not explicitly specified. Since all the states entering (5.1) are topologically inequivalent it is clear that \( N \) cannot depend on \( M \). On the other hand if \( \alpha_i \) had any dependence on \( M \), it should be a global one and, therefore, removable from (5.1) after extracting a multiplicative factor independent of \( i \).

Equation (5.1) represents the skein rules we are trying to find out once the values of \( N \) and \( \alpha_i, \ i = 0, 1, ..., N \) are computed. The knot operators provide a way to obtain these quantities. The procedure is as follows. Consider the state
$|j; 0\rangle$ and take the inner product of this state with eq. (5.1),

$$\sum_{i=0}^{N} \alpha_i \langle j; 0 | j; M + i \rangle = 0. \quad (5.2)$$

Geometrically what has been done is the gluing of a three-ball represented by $|j; M + i\rangle$ to another three-ball with surface $S^2$ with four marked points as the one depicted in Fig. 14. The resulting structure is a link in $S^3$ carrying a representation of isospin $j/2$. For example, for $M + i = 1$ one obtains the unknot, for $M + i = 2$ the Hopf link, and for $M + i = 3$ the trefoil. It is clear that if $M + i$ is even the resulting link has two components while if $M + i$ is odd it has only one component.

We have obtained in the previous section the vacuum expectation value for torus links in $S^3$ with an arbitrary number of components for a fixed representation. In (5.2) one deals with vacuum expectation values corresponding to torus links with one and two components. Let us work out their form explicitly from the general expression given in (3.29). The value of $\langle j; 0 | j; M + i \rangle$ is related, up to a constant depending only on $j$, to a polynomial invariant in (3.29):

$$\langle j; 0 | j; M + i \rangle \sim P_j(M+i,2). \quad (5.3)$$

Notice that in this equation all the inner products are associated to deframed knots. This setting, in which all crossings are treated on equal footing, is the right one to derive skein rules. We will consider the polynomial $P_j^{(r,2)}$ for the case in which $r$ is odd first. After using (3.29) one finds:

$$P_j^{(r,2)}(t) = \frac{t^{\frac{r}{2}(r-1)}}{t^j - 1} \sum_{l=0}^{j} t^{r(1+2l)(j-l)}(t^{1+2l} - t^{2(j-l)}), \quad r \text{ odd}, \quad (5.4)$$

where we have utilized the fact that $C_j(1, l) = 1$. Let us rearrange the sum in a more convenient way. We will consider first the case in which $j$ is even. Later we
will work out the case in which $j$ is odd. After performing the change \( l = \frac{1}{2}(j - m) \), \( m = -j, -j + 2, \ldots, j - 2, j \), in the sum we have:

\[
P^{(r,2)}_j(t) = \frac{t^{\frac{z}{2}j(j+2)-\frac{1}{2}}}{t^{\frac{1+z}{2}} - t^{-\frac{1+z}{2}}} \sum_{m=-j}^{j} t^{-\frac{z}{2}m(m+1)-m(t^{2m+1} - 1)}, \quad r \text{ odd, } j \text{ even.} \tag{5.5}
\]

Splitting the sum into two parts, one corresponding to \( m \geq 0 \), and another one to \( m < 0 \), and performing the shift \( m = -m' - 1 \), \( m' = 1, 3, \ldots, j - 1 \), in the latter, one finds that (5.5) can be written as,

\[
P^{(r,2)}_j(t) = \frac{t^{\frac{z}{2}j(j+2)-\frac{1}{2}}}{t^{\frac{1+z}{2}} - t^{-\frac{1+z}{2}}} \sum_{l=0}^{j} (-1)^l t^{-\frac{z}{2}(l+1)-l(t^{2l+1} - 1)}, \quad r \text{ odd, } j \text{ even.} \tag{5.6}
\]

It is straightforward to follow the same steps for the case of \( j \) odd. The final expression for any \( j \) turns out to be:

\[
P^{(r,2)}_j(t) = \frac{t^{\frac{z}{2}(r-1)}}{t^{r+1} - 1} \sum_{l=0}^{j} C_j(2,l) t^{\frac{z}{2}(1+l)(2j-l)} (t^{1+l} - t^{2j-l}), \quad r \text{ even,} \tag{5.7}
\]

where,

\[
C_j(2,l) = \sum_{m_1,m_2=0}^{j} \delta_{m_1+m_2,l}. \tag{5.9}
\]

It turns out that,

\[
C_j(2,l) = \begin{cases} 
  l + 1, & \text{if } 0 \leq l \leq j, \\
  2j - l + 1, & \text{if } j < l \leq 2j,
\end{cases} \tag{5.10}
\]
so we can write (5.8) as,

\[
P^{(r,2)}_j(t) = \frac{t^{j(r-1)}}{t^{j+1} - 1} \left\{ \sum_{l=0}^{j} (l+1)t^{(1+l)(2j-l)}(t^{1+l} - t^{2j-l}) \right. \\
+ \left. \sum_{l=j+1}^{2j} (2j-l+1)t^{(1+l)(2j-l)}(t^{1+l} - t^{2j-l}) \right\}, \quad r \text{ even.} 
\]

(5.11)

After performing the change \( l = 2j - m - 1, m = -1, 0, ..., j - 2 \), in the second sum, it turns out that (5.11) can be written as:

\[
P^{(r,2)}_j(t) = \frac{t^{j(r-1)}}{t^{j+1} - 1} \sum_{l=-1}^{j-1} t^{(1+l)(2j-l)}(t^{2j-l} - t^{1+l}), \quad r \text{ even.} 
\]

(5.12)

Finally, performing the shift \( l = j - m - 1, m = 0, 1, ..., j \), in the sum entering (5.12) one finds after some rearrangements:

\[
P^{(r,2)}_j(t) = \frac{t^{j(j+2)-\frac{1}{2}}}{t^{j+1} - t^{-\frac{1}{2}}} \sum_{l=0}^{j} (-1)^l t^{-\frac{1}{2}(l+1)-l}(t^{2l+1} - 1), \quad r \text{ even,} 
\]

which has a form very similar to the expression obtained for \( r \) odd in (5.7). In fact, comparing (5.7) and (5.13), it is simple to write down the general form. For any \( r \) one has,

\[
P^{(r,2)}_j(t) = \frac{t^{j(j+2)-\frac{1}{2}}}{t^{j+1} - t^{-\frac{1}{2}}} \sum_{l=0}^{j} (-1)^l (j+l)^r t^{-\frac{1}{2}(l+1)-l}(t^{2l+1} - 1). 
\]

(5.14)

Let us return to our considerations regarding the quantity \( \langle j; 0 | j; M + i \rangle \) in (5.2). This inner product is related, up to a constant which only depends on \( j \), to the polynomial invariant in (5.14) when all the links are deframed. The presence of this constant is due to the normalization used in getting (3.29). Since such a constant depends only on \( j \) it is a global one and (5.2) is equivalent to the equation:

\[
\sum_{i=0}^{N} \alpha_i P^{(M+i,2)}_j(t) = 0, 
\]

(5.15)

for any value of \( M \).
After using (5.14) and removing an irrelevant multiplicative factor, (5.15) takes the form,

\[ \sum_{i=0}^{N} \sum_{l=0}^{j} \alpha_i (-1)^{(M+i)(j+l)} t^{\frac{M+i}{2}} (j(j+2)-l(l+1))^{l} (t^{2l+1} - 1) = 0. \]  

(5.16)

As we argued above, \( N \) and \( \alpha_i, \ i = 0, 1, ..., N \), do not depend on \( M \). Since \( N \) is finite one can always consider a value of \( M \) such that \( M >> N \). Let us consider eq. (5.16) in such a situation. Since \( M \) multiplies \( l(l+1) \) in the exponent of \( t \) and one can choose \( M \) as large as one wish while \( N \) remains finite, the only possible way to realize (5.16) is to demand that it vanish for any value of \( l \):

\[ \sum_{i=0}^{N} \alpha_i (-1)^{(M+i)(j+l)} t^{\frac{M+i}{2}} (j(j+2)-l(l+1))^{l} (t^{2l+1} - 1) = 0, \quad l = 0, 1, ..., j. \]  

(5.17)

After factoring out all the irrelevant dependence from this equation one finds that the coefficients \( \alpha_i, \ i = 0, 1, ..., N \), must satisfy:

\[ \sum_{i=0}^{N} \alpha_i (-1)^{i(j+l)} t^{\frac{i}{2}} (j(j+2)-l(l+1))^{l} = 0, \quad l = 0, 1, ..., j. \]  

(5.18)

This is a linear homogeneous system of \( j + 1 \) equations for the unknown coefficients \( \alpha_i, \ i = 0, 1, ..., N \). Clearly, for \( N = j + 1 \) one has \( j + 2 \) unknowns and it is possible to find a solution where not all the \( \alpha_i \) are zero. To show that this is the minimum value of \( N \) such that (5.18) has a non-trivial solution we must prove that for \( N = j \) the linear system (5.18) has only the trivial solution, \( i.e. \), that the matrix \( A^{(j)} \) whose matrix elements are,

\[ A_{il}^{(j)} = (-1)^{i(j+l)} t^{\frac{i}{2}} (j(j+2)-l(l+1))^{l}, \quad i, l = 0, 1, ..., j. \]  

(5.19)

has a determinant different from zero. The determinant of the matrix \( A^{(j)} \) is a polynomial in \( t \). To prove that this polynomial is not identically zero it is enough
to prove that among the terms appearing in it there is one which cannot cancel with any other one. Clearly, after multiplying the rows of the matrix \( A^{(j)} \) by adequate coefficients we have \( \det A^{(j)} \neq 0 \) if and only if \( \det a \neq 0 \) where \( a \) is a matrix whose entries are:

\[
a_{il} = (-1)^{il} t^{-\frac{1}{2}l(l+1)}, \quad i, l = 0, 1, ..., j. \tag{5.20}
\]

Notice that we have got rid of all the \( j \)-dependence. The determinant of \( a \) can be explicitly written as:

\[
\det a = \sum_{i_0, ..., i_j=0}^{j} (-1)^{\sum_{n=0}^{j} n_i t^{-\frac{1}{2} \sum_{n=0}^{j} n_i (i_n + 1)}} \varepsilon_{i_0 ... i_j}, \tag{5.21}
\]

where \( \varepsilon_{i_0 ... i_j} \) is the totally antisymmetric epsilon symbol of \( j + 1 \) indices. The exponent of \( t \) becomes minimum when \( i_n = n \) for \( n = 0, 1, ..., j \). Clearly, all other configurations of \( i_0, i_1, ..., i_j \) contribute with strictly larger values. The term with a minimum exponent stands by itself and therefore \( \det a \neq 0 \). This implies that the matrix \( A^{(j)} \) in (5.19) has a non-vanishing determinant and therefore the minimum value of \( N \) such that (5.18) has a non-trivial solution is \( N = j + 1 \). Notice that this is the value we anticipated.

We are now in a position to solve (5.18) for \( N = j + 1 \). Since eq. (5.18) is homogeneous one can rescale it or, equivalently, one can fix one of the coefficients to 1. Once this is done the solution is unique. Let us define:

\[
\beta_{j+1-i} = \frac{\alpha_i}{\alpha_{j+1}}, \quad i = 0, 1, ..., j + 1. \tag{5.22}
\]

In terms of these new coefficients eq. (5.18) becomes:

\[
\sum_{i=0}^{j+1} \beta_{j+1-i} (-1)^{ij(j+1)} t^{i(j+2)-\frac{1}{2}l(l+1)} = 0, \quad l = 0, 1, ..., j. \tag{5.23}
\]

Notice that we have chosen a normalization such that \( \beta_0 = 1 \). The simplest way to solve for \( \beta_n, n = 1, ..., j + 1, \) is the following. First notice that eq. (5.23) can
be written as a polynomial equation for a diagonal \((j + 1) \times (j + 1)\) matrix \(B\),

\[
\sum_{i=0}^{j+1} \beta_{j+1-i} B^i = 0, \quad (5.24)
\]

where,

\[
B_{mn} = \delta_{mn}(-1)^{j+m} t^{\frac{1}{2}j(j+2) - \frac{1}{2}m(m+1)}, \quad m, n = 0, 1, \ldots, j. \quad (5.25)
\]

The matrix polynomial entering (5.24) must be a multiple of the minimal polynomial of the matrix \(B\). On the other hand, it is easy to verify that all the eigenvalues of \(B\) are different. This implies that actually (5.24) must contain its minimal polynomial and that this coincides with its characteristic polynomial. Since we have chosen \(\beta_0 = 1\) this identification allows to write the coefficients \(\beta_n\) in terms of the eigenvalues of the matrix \(B\) in the standard form. Let us introduce the following notation for these eigenvalues:

\[
\lambda_m = (-1)^{j+m} t^{\frac{1}{2}j(j+2) - \frac{1}{2}m(m+1)}, \quad m = 0, 1, \ldots, j, \quad (5.26)
\]

then, using the standard form of the coefficients of the characteristic polynomial one has,

\[
\beta_n = (-1)^n \sum_{\substack{i_1, \ldots, i_n = 0 \atop i_1 < i_2 < \ldots < i_n}}^{j} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_n}, \quad n = 1, \ldots, j + 1. \quad (5.27)
\]

The matrix \(B\) appearing in (5.24) is intrinsically related to the half-monodromy matrix of the corresponding conformal field theory [30]. In the standard approach to build skein rules using that matrix one encounters an equation like (5.24) once the half-monodromy matrix has been corrected by framing factors to have all links in the standard framing. This implies that the eigenvalues of the half-monodromy matrix \(\mu_n\), \(n = 0, 1, \ldots, j\), must be related to the eigenvalues \(\lambda_n\), \(n = 0, 1, \ldots, j\), in
(5.26) in the following form:

$$\mu_n = e^{-2\pi i h_j} \lambda_n, \quad n = 0, \ldots, j,$$

(5.28)

where $h_j$ is the conformal weight corresponding to the integrable representation of $SU(2)$ of isospin $j/2$. After using the fact that $t = \exp\left(\frac{2\pi i}{k+2}\right)$ and $h_j = \frac{j(j+2)}{4(k+2)}$, one finds,

$$\mu_n = (-1)^{(j+n)} e^{i\pi (2h_j - h_{2n})} \quad n = 0, \ldots, j,$$

(5.29)

which are indeed the eigenvalues of the half-monodromy matrix [30].

To make contact with the standard way of expressing skein rules we must rescale the coefficients of (5.1) in such a way that the exponents of $\alpha_0$ and $\alpha_{j+1}$ have opposite signs. Let us first compute $\beta_{j+1}$. We have from (5.27) and (5.26),

$$\beta_{j+1} = (-1)^{(j+1)} \lambda_0 \lambda_1 \ldots \lambda_j$$

$$= (-1)^{j^2 + 1 + \sum_{n=0}^{j} n t^{1/2} j(j+1)(j+2) - \frac{1}{2} \sum_{n=0}^{j} n(n+1)}.$$

(5.30)

After using the fact that

$$\sum_{n=0}^{j} n = \frac{1}{2} j(j+1), \quad \sum_{n=0}^{j} n^2 = \frac{1}{6} j(j+1)(2j+1),$$

(5.31)

one finally obtains,

$$\beta_{j+1} = (-1)^{\frac{1}{2} j(j+1)(j+2)} t^{\frac{1}{2} j(j+1)(j+2)}.$$

(5.32)

This implies that we must take,

$$\alpha_{j+1} = t^{-\frac{1}{6} j(j+1)(j+2)},$$

(5.33)

to agree with the standard form of expressing skein rules. From eq. (5.22) one obtains the rest of the coefficients:

$$\alpha_i = t^{-\frac{1}{6} j(j+1)(j+2)} \beta_{j+1-i}, \quad i = 0, 1, \ldots, j.$$

(5.34)

We are now in a position to write down the skein rules. Let us consider the states $|j; M + i\rangle$, $i = 0, 1, \ldots, j + 1$. These states have the form depicted in Fig. 13.
Let us take an arbitrary state $|L\rangle$ of the form shown in Fig. 12 and glue it to a state $|j; M + i\rangle$. We will denote the resulting link by,

$$L_{M+i} = \langle L|j; M + i\rangle, \quad i = 0, 1, ..., j + 1. \quad (5.35)$$

Eq. (5.1) guarantees that the invariant polynomials associated to the links $L_{M+i}$, $i = 0, 1, ..., j + 1$, satisfy the linear relation:

$$\sum_{i=0}^{j+1} \alpha_i L_{M+i} = 0, \quad (5.36)$$

where the coefficients $\alpha_i$, $i = 0, 1, ..., j + 1$, are given in (5.34), (5.33) and (5.27).

It is worth working out the resulting skein rules for the representations of lower dimensions. One finds,

- **isospin $\frac{1}{2}$, $j = 1$,**
  $$-tL_M + (t^\frac{1}{2} - t^{-\frac{1}{2}})L_{M+1} + t^{-1}L_{M+2} = 0$$

- **isospin 1, $j = 2$,**
  $$t^4 L_M - (t^3 - t + 1)L_{M+1} - (t^{-3} - t^{-1} + 1)L_{M+2} + t^{-4}L_{M+3} = 0$$

- **isospin $\frac{3}{2}$, $j = 3$,**
  $$t^{10} L_M - (t^{\frac{17}{2}} - t^{\frac{11}{2}} + t^{\frac{7}{2}} - t^{\frac{5}{2}})L_{M+1} - (t^4 - t^2 + t + t^{-1} - t^{-2} + t^{-4})L_{M+2}$$
  $$- (t^{-\frac{17}{2}} - t^{-\frac{11}{2}} + t^{-\frac{7}{2}} - t^{-\frac{5}{2}})L_{M+3} + t^{-10}L_{M+4} = 0$$

- **isospin 2, $j = 4$,**
  $$-t^{20} L_M + (t^{18} - t^{14} + t^{11} - t^9 + t^8)L_{M+1}$$
  $$+ (t^{12} - t^9 + t^7 - t^6 + t^5 - t^3 + t^2 + 1 - t^{-1} + t^{-3})L_{M+2}$$
  $$- (t^{-12} - t^{-9} + t^{-7} - t^{-6} + t^{-5} - t^{-3} + t^{-2} + 1 - t + t^3)L_{M+3}$$
  $$- (t^{-18} - t^{-14} + t^{-11} - t^{-9} + t^{-8})L_{M+4} + t^{-20}L_{M+5} = 0$$
Notice that for isospin 1/2 we obtain the skein rules for the Jones polynomial as given in [23] after making the replacement $t^{\frac{1}{2}} \rightarrow -t^{\frac{1}{2}}$. This feature is also shared by the skein rules obtained in [1] which are the same as ours. Taking $M = -1$ in (5.36) one finds the standard way of expressing the skein rules of the Jones polynomial. This is depicted in Fig. 15. For higher isospin representations the skein rules we have obtained are equivalent to the ones leading to the Akutsu-Wadati polynomials [13]. These polynomials are obtained from $N$-state vertex models. The relation between these models and the representations of $SU(2)$ is such that $N = j + 1$. The skein rules given by eq. (5.36), (5.34), (5.33) and (5.23) are the same as the ones obtained by using the half-monodromy matrix in [7].

The skein rules (5.36) we have derived constitute a sort of basis of skein rules in the sense that one can take different values of $M$. For example, it is clear that, taking adequate linear combinations of skein rules for different values of $M$, one could write down a skein rule where only diagrams of the type shown in Fig. 13 with an even number of half-monodromy twists would enter. Of course, this skein rule could also be obtained by repeating the analysis described in this subsection to that situation. It is clear that for such a case the matrix entering the analogous to eq. (5.24) would be $B^2$. However, there is a case in which the skein rules we have obtained do not apply. When the number of half-monodromy twists in diagrams as the one in Fig. 13 is even it is possible to consider an analogous diagram where the two strands have opposite orientations. This situation is pictured in Fig. 16. Notice that in Fig. 16, apart from changing the orientation of one of the two strands, we have also converted overcrossings into undercrossings and vice versa. We have done this in order to keep the same sign for the linking number as in Fig. 13 for the same number of half-monodromies in both figures. Again, there is a finite number of configurations of the type depicted in Fig. 16 and therefore, by considering sets of them, one can construct new types of skein rules. Let us work out their form.

We will denote by $|j; 2(M + i)\rangle$ the state corresponding to the three-ball drawn in Fig. 16. Since there are only a finite number of linearly independent states there
exist $N$ and $\hat{\alpha}_i$, $i = 0, 1, ..., N$, not all zero, such that,

$$
\sum_{i=0}^{N} \hat{\alpha}_i |j; 2(M + i)\rangle = 0. \quad (5.37)
$$

Our goal is to determine $N$ and $\hat{\alpha}_i$, $i = 0, 1, ..., N$. Let us consider the state with no twists pictured in Fig. 17, $|j; 0\rangle$, and let us glue it to the $N + 1$ states entering (5.37). One has:

$$
\sum_{i=0}^{N} \hat{\alpha}_i \langle j; 0|j; 2(M + i)\rangle = 0. \quad (5.38)
$$

The inner products appearing in this equation represent link invariants in $S^3$ of two components, both carrying the same representation. However, their relative orientation is the opposite to the one of the two component link invariant in (5.2). This means in particular that now we do not possess a relation like (5.3). Recall that $P_j^{(r,s)}$ in (5.3) represented a torus link whose components had the same orientation. We must then compute the link invariant corresponding to two components with opposite orientations, carrying the same representation, from the knot operators (2.18).

Recall that, for even $r$, $P_j^{(r,2)}$ is obtained from the vacuum expectation value $\langle W_j^{(-\frac{r}{2},1)} W_j^{(-\frac{r}{2},1)} \rangle_{S^3}$. Reversing the orientation of one of the two strands corresponds to the change $W_j^{(-\frac{r}{2},1)} \to W_j^{(\frac{r}{2},-1)}$ in the corresponding knot operator. On the other hand we interchange overcrossings and undercrossings by doing $W_j^{(-\frac{r}{2},1)} \to W_j^{(\frac{r}{2},1)}$ in one of the two operators. Performing one of these two changes on one operator and the other on the remaining operator, it is evident that we must consider the vacuum expectation value $\langle W_j^{(\frac{r}{2},1)} W_j^{(\frac{r}{2},-1)} \rangle_{S^3}$. The corresponding invariant polynomial can be obtained from our general prescription (3.24):

$$
Q_j^{(r,2)} = \frac{t_2^{e_{j+2}}}{\langle W_j^{(0,1)} \rangle_{S^3}} \langle W_j^{(\frac{r}{2},1)} W_j^{(\frac{r}{2},-1)} \rangle_{S^3}, \quad (5.39)
$$
where \( r \) is an even integer. Indeed, we will have then,

\[
\hat{\gamma}(j; 0|j; 2(M + i)) \sim Q_j^{(2(M+i),2)},
\]

and therefore we will be in a position to carry out the analysis leading to the new sets of skein rules. Using (2.19) one finds,

\[
Q_j^{(2(M+i),2)}(t) = \frac{t^2}{t^j+1 - 1} \sum_{l=0}^{j} t^{(M+i)l(l+1)-l}(t^{2l+1} - 1).
\]

Plugging this expression into (5.38) and removing irrelevant global factors one obtains,

\[
\sum_{i=0}^{N} \sum_{l=0}^{j} \alpha_i t^{(M+i)l(l+1)-l}(t^{2l+1} - 1) = 0.
\]

The arguments that led to (5.17) are also applicable here. It turns out that (5.42) implies:

\[
\sum_{i=0}^{N} \alpha_i t^{il(l+1)} = 0, \quad l = 0, 1, ..., j.
\]

The minimum value for which this system of \( j+1 \) linear equations possesses a solution where not all \( \alpha_i \) are zero is \( N = j + 1 \). In that case the solution to (5.43) is unique once a normalization is chosen. Let us define,

\[
\hat{\beta}_{j+1-i} = \frac{\hat{\alpha}_i}{\hat{\alpha}_{j+1}}, \quad i = 0, 1, ..., j + 1,
\]

so that \( \beta_0 = 1 \). Equation (5.43) becomes:

\[
\sum_{i=0}^{j+1} \hat{\beta}_{j+1-i} t^{il(l+1)} = 0, \quad l = 0, 1, ..., j.
\]
or, in matrix form,

$$\sum_{i=0}^{j+1} \hat{\beta}_{j+1-i} \hat{B}^i = 0, \quad (5.46)$$

where $\hat{B}$ is a $(j + 1) \times (j + 1)$ diagonal matrix,

$$\hat{B}_{mn} = \delta_{mn} t^{m(m+1)} \quad m, n = 0, 1, \ldots, j. \quad (5.47)$$

After introducing the eigenvalues of $\hat{B}$,

$$\hat{\lambda}_m = t^{m(m+1)} \quad m = 0, 1, \ldots, j, \quad (5.48)$$

it is straightforward to write down the solution to (5.46):

$$\hat{\beta}_n = (-1)^n \sum_{i_1, \ldots, i_n=0}^{j} \hat{\lambda}_{i_1} \hat{\lambda}_{i_2} \cdots \hat{\lambda}_{i_n}, \quad n = 1, \ldots, j + 1. \quad (5.49)$$

In order to obtain the symmetric standard normalization of skein rules we take a particular value of $\hat{\alpha}_{j+1}$. First, we must compute $\hat{\beta}_{j+1}$. Using (5.49) and (5.31),

$$\hat{\beta}_{j+1} = (-1)^{j+1} t^{\sum_{n=0}^{j} n(n+1)} = (-1)^{j+1} t^{\frac{1}{6}j(j+1)(j+2)}, \quad (5.50)$$

then $\hat{\alpha}_{j+1}$ is chosen as,

$$\hat{\alpha}_{j+1} = t^{-\frac{1}{6}j(j+1)(j+2)}, \quad (5.51)$$

so that,

$$\hat{\alpha}_i = t^{-\frac{1}{6}j(j+1)(j+2)} \hat{\beta}_{j+1-i}, \quad i = 0, 1, \ldots, j. \quad (5.52)$$

Let us now consider a state $|L\rangle$ of the form pictured in Fig. 18, where the interior of the box $\hat{A}$ is arbitrary. Gluing this state to the $j + 2$ states entering
(5.37) one builds the link invariants,
\[ \hat{L}_{2(M+i)} = \langle L|j; 2(M+i) \rangle; \quad i = 0, \ldots, j + 1. \] 
(5.53)

These link invariants must satisfy the linear relations,
\[ \sum_{i=1}^{j+1} \hat{\alpha}_i \hat{L}_{2(M+i)} = 0, \] 
(5.54)
where \( \hat{\alpha}_i \) are given in (5.52), (5.49) and (5.48). The skein rules we have obtained are equivalent to the ones obtained by a different method in [7]. We present some examples of these skein rules (5.54):

- **isospin \( \frac{1}{2}, j = 1, \)**
  \[ t \hat{L}_{2M} - (t + t^{-1}) \hat{L}_{2M+2} + t^{-1} \hat{L}_{2M+4} = 0 \]

- **isospin 1, \( j = 2, \)**
  \[-t^4 \hat{L}_{2M} + (t^4 + t^2 + t^{-2}) \hat{L}_{2M+2} - (t^{-4} + t^{-2} + t^2) \hat{L}_{2M+4} + t^{-4} \hat{L}_{2M+6} = 0 \]

- **isospin \( \frac{3}{2}, j = 3, \)**
  \[ t^{10} \hat{L}_{2M} - (t^{10} + t^8 + t^4 + t^{-2}) \hat{L}_{2M+2} \\
  + (t^8 + t^4 + t^2 + t^{-2} + t^4 + t^{-8}) \hat{L}_{2M+4} \\
  - (t^{-10} + t^{-8} + t^{-4} + t^2) \hat{L}_{2M+6} + t^{-10} \hat{L}_{2M+8} = 0 \]

- **isospin 2, \( j = 4, \)**
  \[-t^{20} \hat{L}_{2M} + (t^{20} + t^{18} + t^{14} + t^8 + 1) \hat{L}_{2M+2} \\
  -(t^{18} + t^{14} + t^{12} + t^8 + t^6 + t^2 + 1 + t^{-2} + t^{-6} + t^{-12}) \hat{L}_{2M+4} \\
  +(t^{-18} + t^{-14} + t^{-12} + t^{-8} + t^{-6} + t^{-2} + 1 + t^2 + t^6 + t^{12}) \hat{L}_{2M+6} \\
  - (t^{-20} + t^{-18} + t^{-14} + t^{-8} + 1) \hat{L}_{2M+8} + t^{-20} \hat{L}_{2M+10} = 0 \]

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5.2. Skein rules for minimal models

In this subsection we will derive the skein rules corresponding to minimal models. The procedure is completely analogous to the one carried out for $SU(2)$. Our starting point is eq. (5.1) which now takes the form:

$$\sum_{i=0}^{N} \alpha_i |p, q; M + i\rangle = 0,$$

where $p, q$ denote the quantum numbers carried by the strands in Fig. 13. Gluing these states to the one pictured in Fig. 14 we have,

$$\sum_{i=0}^{N} \alpha_i \langle p, q; 0 | p, q; M + i\rangle = 0.$$

The inner products in (5.56) are of the form computed in (4.18) up to a factor depending only on $p$ and $q$. Indeed, one has,

$$\langle p, q; 0 | p, q; M + i\rangle \sim \Pi_{p,q}^{(M+i,2)}.$$

The form of the invariant polynomial $\Pi_{p,q}^{(M+i,2)}$ can be easily obtained due to its structure in terms of the invariant polynomial (3.29) of $SU(2)$ (see eq. (4.19)):

$$\Pi_{p,q}^{(M+i,2)}(t_1, t_2) = (-1)^{(p+q)(n-1)} P_{p-1}^{(M+i,2)}(t_1)P_{q-1}^{(M+i,2)}(t_2),$$

where $n$ is the greatest common divisor of $M + i$ and 2. After using (5.14) we have,

$$\Pi_{p,q}^{(M+i,2)}(t_1, t_2) = (-1)^{(p+q)} \frac{t_1^{\frac{M+i}{2}(p^2-1)-1} t_2^{\frac{M+i}{2}(q^2-1)-1}}{(t_1^{\frac{p}{2}} - t_1^{-\frac{p}{2}})(t_2^{\frac{q}{2}} - t_2^{-\frac{q}{2}})} \times$$

$$\sum_{l=0}^{p-1} \sum_{m=0}^{q-1} (-1)^{(m+l)(M+i)} t_1^{\frac{M+i}{2}l(l+1) - l} t_2^{-\frac{M+i}{2}m(m+1) - m} (t_1^{2l+1} - 1)(t_2^{2m+1} - 1).$$

Following the same arguments that led to (5.18) one finds after utilizing (5.59),
(5.57) and (5.56),

\[ \sum_{i=0}^{N} \alpha_i (-1)^{i(m+l)} t^1_2 (p^2 - 1) - \frac{1}{2} (l+1) t^2_1 (q^2 - 1) - \frac{1}{2} m(m+1) = 0, \]  

\[ l = 0, ..., p - 1, \quad m = 0, ..., q - 1. \]  

(5.60)

This is a system of \( pq \) linear equations. An argument entirely similar to the one discussed for the case of \( SU(2) \) shows that the minimum value of \( N \) needed to have a solution of (5.60) with not all \( \alpha_i \) zero is \( pq + 1 \). We will choose the same type of normalization as before, \( i.e., \) we define

\[ \beta_{pq-i} = \frac{\alpha_i}{\alpha_{pq}}, \quad i = 0, 1, ..., pq, \]  

(5.61)

so that \( \beta_0 = 1 \). Eq. (5.60) can then be written as,

\[ \sum_{i=0}^{pq} \beta_{pq-i} (-1)^{i(m+l)} t^1_2 (p^2 - 1) - \frac{1}{2} (l+1) t^2_1 (q^2 - 1) - \frac{1}{2} m(m+1) = 0, \]  

\[ l = 0, ..., p - 1, \quad m = 0, ..., q - 1. \]  

(5.62)

Similarly to the case of \( SU(2) \), (5.62) can be expressed in terms of a \( pq \times pq \) matrix \( B \),

\[ \sum_{i=0}^{pq} \beta_{pq-i} B^i = 0, \]  

(5.63)

which is the direct product of two diagonal matrices, \( U \) and \( V \),

\[ B = U \otimes V, \]  

(5.64)

with

\[ U_{mn} = \delta_{mn} (-1)^{n(m+l)} t^1_2 (p^2 - 1) - \frac{1}{2} m(m+1), \quad m, n = 0, 1, ..., p - 1, \]  

\[ V_{mn} = \delta_{mn} (-1)^{n(m+l)} t^2_1 (q^2 - 1) - \frac{1}{2} m(m+1), \quad m, n = 0, 1, ..., q - 1. \]  

(5.65)

The eigenvalues of the matrix \( B \) in (5.64) are all the possible products of eigenvalues of the matrices \( U \) and \( V \). These eigenvalues can therefore be labeled
with two indices. They take the following form:

$$
\lambda_{lm} = (-1)^{l+m} t_1^{\frac{l}{2}(p^2-1) - \frac{1}{2}l(l+1)} t_2^{\frac{m}{2}(q^2-1) - \frac{1}{2}m(m+1)}, \quad l = 0, \ldots, p-1, \quad m = 0, \ldots, q-1.
$$

(5.66)

It is simple to verify that all $\lambda_{lm}$ are different and therefore (5.63) must correspond to the characteristic equation of the matrix $B$ in (5.64). This identification allows to write down immediately the coefficients appearing in (5.63):

$$
\beta_n = (-1)^n \sum_{\substack{l_1, \ldots, l_n = 0, \ldots, p-1 \\
m_1, \ldots, m_n = 0, \ldots, q-1 \\
all \text{pairs } (l_i, m_i) \text{ different}}} \lambda_{l_1m_1} \lambda_{l_2m_2} \cdots \lambda_{l_nm_n}, \quad n = 1, 2, \ldots, pq. \quad (5.67)
$$

As in the case of $SU(2)$ the matrix $B$ of (5.63) is related to the half-monodromy matrix of the corresponding minimal model. The eigenvalues of this matrix can be obtained from (5.66) after taking care of the framing. Indeed, these eigenvalues are,

$$
\mu_{lm} = e^{-2\pi i h_{p,q} \lambda_{lm}}, \quad l = 0, \ldots, p-1, \quad m = 0, \ldots, q-1, \quad (5.68)
$$

where $h_{p,q} = \left(\frac{(m+3)p-(m+2)q}{4(m+2)(m+3)}\right)^{2-1}$. After taking into account the fact that $t_1 = \exp(2\pi i \frac{m+3}{m+2})$ and $t_2 = \exp(2\pi i \frac{m+1}{m+3})$ one finds,

$$
\mu_{lm} = e^{i\pi(2h_{p,q}-h_{2l+1,2n+1})}, \quad l = 0, \ldots, p-1, \quad m = 0, \ldots, q-1. \quad (5.69)
$$

As in the case of $SU(2)$ skein rules are usually written in a symmetric fashion. Let us work out the form of $\alpha_{pq}$ giving rise to this type of expression for the corresponding relations. First we must compute $\beta_{pq}$. Using (5.66) and (5.31) one finds:

$$
\beta_{pq} = (-1)^{pq} \prod_{\substack{l=0, \ldots, p-1 \\
m=0, \ldots, q-1}} \lambda_{lm}
$$

$$
= (-1)^{\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)+\frac{1}{2}pq(p^2-1)+\frac{1}{2}qp(q^2-1).} \quad (5.70)
$$
This implies that we must define $\alpha_{pq}$ as,

$$\alpha_{pq} = t_1^{-\frac{1}{6}}p(q^2-1) t_2^{-\frac{1}{6}}q(p^2-1),$$  \hspace{1cm} (5.71)$$
and therefore one has after using (5.61),

$$\alpha_i = t_1^{-\frac{1}{6}}p(q^2-1) t_2^{-\frac{1}{6}}q(p^2-1) \beta_{pq-i}, \quad i = 0, 1, ..., pq - 1. \hspace{1cm} (5.72)$$

In order to write down the final form of the skein rules let us consider a state $|L\rangle$ of the type pictured in Fig. 12 and let us define the following links:

$$L_{M+i} = \langle L|p, q; M + i\rangle, \quad i = 0, 1, ..., pq. \hspace{1cm} (5.73)$$

Eq. (5.55) implies that the invariant polynomials associated to the links $L_{M+i}$ satisfy the relation,

$$\sum_{i=0}^{pq} \alpha_i L_{M+i} = 0, \hspace{1cm} (5.74)$$

where the coefficients $\alpha_i$ are given by equations (5.72), (5.71) and (5.67). It is worth working out some specific cases. One finds:

- $p = 1, q = 2,$
  $$-t_2 L_M - (t_2^{\frac{1}{2}} - t_2^{-\frac{1}{2}}) L_{M+1} + t_2^{-1} L_{M+2} = 0$$

- $p = 2, q = 2,$
  $$t_1^2 t_2^3 L_M - (t_1^2 - t_1^{-2})(t_2^{\frac{3}{2}} - t_2^{-\frac{3}{2}}) L_{M+1} - (t_1 + t_1^{-1} - 2 + t_2 + t_2^{-1}) L_{M+2}$$
  $$- (t_1^{\frac{3}{2}} - t_1^{-\frac{3}{2}})(t_2^{\frac{3}{2}} - t_2^{-\frac{3}{2}}) L_{M+3} + t_1^{-2} t_2^{-2} L_{M+4} = 0$$
\( p = 2, q = 3, \)

\[
-t_1^3 t_2^8 L_M - (t_1^2 - t_1^3)(t_2^7 - t_2^5 + t_2^4) L_{M+1} \\
+ ((t_1^2 - 2t_1 + 1)(t_2^4 - t_2^3 + t_2) + t_1(t_2^6 + t_2^2 + 1)) L_{M+2} \\
+ (t_1^2 - t_1^3 - (t_1^2 - t_1^3)(t_2^3 - t_2^2 - t_2 + 3 + t_2^{-1} - t_2^{-2} + t_2^{-3})) L_{M+3} \\
- ((t_1^2 - 2t_1^{-1} + 1)(t_2^{-4} - t_2^{-3} + t_2^{-1}) + t_1^{-1}(t_2^{-6} + t_2^{-2} + 1)) L_{M+4} \\
+ (t_1^{-2} - t_1^{-3})(t_2^{-7} - t_2^{-5} + t_2^{-4}) L_{M+5} + t_1^{-3} t_2^{-8} L_{M+6} = 0
\]

Notice that for \( p = 1, q = 2 \) the resulting skein rules are the ones of the Jones polynomial as presented in [23]. It is clear from the construction that the skein rules corresponding to \( q, p \) are the same as the ones for \( p, q \) after the replacement \( t_1 \leftrightarrow t_2 \).
6. Summary and conclusions

In this paper we have developed a formalism for Chern-Simons theories that allows the direct calculation of invariant polynomials for some classes of links. The main advantage of our approach relies on the fact that there is no need of using any kind of recursion relations such as skein rules or the ones used to evaluate Markov traces. The basic ingredient in our approach are the knot operators associated to any closed curve on the torus. In general with our method we can compute polynomials for links obtained by inserting, inside a solid torus, several curves defined on its boundary. It is interesting to notice that our approach is three-dimensional in nature. The topological character of the underlying quantum field theory ensures this fact, which is reflected in the way in which our knot operators act on the Hilbert space of toroidal states. Our formalism, applied to the $SU(2)$ Chern-Simons gauge theory, has allowed us to obtain the general form of the Akutsu-Wadati polynomials for torus links. To the best of our knowledge, this expression was not previously known. Moreover we have been able to extend our method to the minimal unitary models corresponding to the degenerate representations of the Virasoro algebra. The basic tool used in this extension is the coset construction implemented in the context of Chern-Simons theories. We have been able to obtain knot operators carrying quantum numbers of the Virasoro algebra and whose vacuum expectation values on the three-sphere fulfill the conditions required to define link polynomials. The minimal model invariants are two-variable polynomials which (up to a global sign depending on the number of link components) are equal to products of Akutsu-Wadati polynomials. Although this result is somehow disappointing, we think these kinds of coset constructions are potentially very relevant in order to obtain new classes of link invariants. Our hope is that some types of rational conformal field theories (having, for example a supersymmetric or, say, a parafermionic algebra) could provide new invariants capable of discriminating between links that the known polynomials are unable to distinguish. Of course this is, for the time being, just a mere possibility and much more work remains to be done in order to find out if the Chern-Simons
coset approach can shed new light on knot theory. On the other hand, once the
knot operators associated to a given rational conformal field theory are known,
we can obtain the skein rules satisfied by the corresponding polynomials. In this
way we may get information about the braiding properties of the corresponding
two-dimensional theory.

Our work can be extended in many directions. First of all, our formalism can
be used to study multicolored links, in which different representations are placed
on the link components. We can also consider the $SU(N)$ gauge theory, whose op-
erator formalism was analyzed in Ref. [5]. In this case one would expect that knot
operators in the fundamental representations will give rise to a particular special-
ization of the HOMFLY polynomial. Another possible generalization could consist
in working with three-manifolds other than the three-sphere $S^3$. Our formalism
can be easily extended to more general lens spaces. It is worthwhile noticing that
in this case the same knot operators as in $S^3$ should be used to compute the knot
invariants. Only the modular transformation of the Heegaard splitting is different.
This means that our knot operators somehow encode information intrinsic to the
knot.

The main drawback of our approach is that it is only valid for a restricted class
of links. It would be quite interesting to extend it to more general types of links.
There are several possible directions to accomplish this objective. First of all we
could follow the approach of [7] and introduce duality matrices, which would allow
us to obtain polynomials for knots and links not considered here. Notice that from
our knot operators we can compute the eigenvalues of the half-monodromy matrix,
which is a basic ingredient in order to obtain the duality matrix. Another possible
approach to tackle this problem consists in the quantization of the Chern-Simons
theory by considering a Heegaard splitting of genus $g > 1$. There exists however a
great difficulty in this quantization program: the parametrization of non-abelian
flat connections over $g > 1$ Riemann surfaces is not known. This fact makes
this approach very hard to pursue and, actually, we think that the knowledge of
the knot invariants could give some insight which could help to understand the
modular space of flat gauge connections over higher genus Riemann surfaces. A (maybe) more promising approach would consist in keeping the theory formulated in a genus one handlebody and build the knots and links by joining open Wilson lines that are subsequently inserted into the solid torus. Hopefully this method would allow to determine (as knot operators for closed curves did) the state created on the boundary by the knot contained in the solid torus. The computation of the invariant polynomial could be done as in this paper. We intend to study these topics in a near future.

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APPENDIX

In this appendix we shall demonstrate that the equation (3.9) for torus knots is always a Laurent polynomial in the variable \( t \). We have to prove that the numerator of \( P^r_s(t) \) always vanishes when \( t^{j+1} = 1 \). Therefore we must study the sums:

\[
\sum_{l_1=0}^j t^{1+r(j-l_1)+sl_1+rsl_1(j-l_1)} - \sum_{l_2=0}^j t^{(r+s)(j-l_2)+rs l_2(j-l_2)}.
\] (A.1)

We shall show that for each term with index \( l_1 \) in the first sum there is another term with index \( l_2 \) in the second sum whose contribution cancels the first one when \( t^{j+1} = 1 \). In fact we shall verify that to any \( 0 \leq l_1 \leq j \) one can assign a unique \( l_2 \) in the same range in such a way that the correspondence between \( l_1 \) and \( l_2 \) is one-to-one. Let us write the difference between two generic terms in (A.1) as:

\[
t^{1+r(j-l_1)+sl_1+rsl_1(j-l_1)}(1 - t^\Delta),
\] (A.2)

where \( \Delta \) is the difference of the exponents of \( t \) in the second and first terms in (A.1). As \( t^{j+1} = 1 \), we can compute \( \Delta \) modulo \( j+1 \). The result is,

\[
\Delta = [r(l_1 - l_2) - 1][s(l_1 + l_2 + 1) + 1] \mod (j+1).
\] (A.3)

We are going to study how \( l_1 \) and \( l_2 \) can be chosen in such a way that \( \Delta = 0 \mod (j+1) \). Suppose that \( p_i \) (\( i = 1, \ldots, n \)) are all the prime numbers that divide both \( s \) and \( j+1 \). Let be \( a_i \) the largest positive integer such that \( p_i^{a_i} \) divides \( j+1 \). We define the integer \( \alpha \) as,

\[
\alpha = \prod_{i=1}^n p_i^{a_i}.
\] (A.4)

Obviously \( \alpha \) divides \( j+1 \) and the definition of \( \alpha \) implies that \( \frac{j+1}{\alpha} \) and \( s \) are
relatively prime,
\( (s, \frac{j + 1}{\alpha}) = 1 \), \hspace{1cm} (A.5)
where by \((m, n)\) we have denoted the greatest common divisor of two integer numbers \(m\) and \(n\). By construction it is clear that \(\alpha\) and \(\frac{j + 1}{\alpha}\) cannot have any common prime in their prime decomposition and thus we have,
\( (\alpha, \frac{j + 1}{\alpha}) = 1 \). \hspace{1cm} (A.6)

Furthermore \(r\) and \(\alpha\) are also coprime since the prime numbers dividing \(\alpha\) also divide \(s\) and, as \((r, s) = 1\), they cannot appear in the prime decomposition of \(r\). Therefore,
\( (r, \alpha) = 1 \). \hspace{1cm} (A.7)

Let us now solve the equation \(\Delta = 0 \mod(j + 1)\) in the following way. Imagine that we equal the factor in \(\Delta\) depending on \(s\) to zero modulo \(j + 1\) and we put the part depending on \(r\) equal to zero modulo \(\alpha\):
\( s(l_1 + l_2 + 1) + 1 = 0 \mod\left(\frac{j + 1}{\alpha}\right), \hspace{1cm} (A.8) \)
\( r(l_1 - l_2) - 1 = 0 \mod(\alpha). \hspace{1cm} (A.9) \)

Obviously any \(l_1\) and \(l_2\) solving simultaneously (A.8) and (A.9) also satisfy the equation \(\Delta = 0 \mod(j + 1)\). Let us check that (A.8) can always be solved. Eq. (A.5) implies that there exist two integers \(n_1\) and \(n_2\) such that,
\( n_1 \frac{j + 1}{\alpha} - n_2 s = 1, \hspace{1cm} (A.10) \)
which means
\( s n_2 + 1 = 0 \mod\left(\frac{j + 1}{\alpha}\right). \hspace{1cm} (A.11) \)
Taking \(l_1 + l_2 = n_2 - 1\) we have a solution of (A.8). In the same way as \((r, \alpha) = 1\) we can solve (A.9) and determine \(l_1 - l_2 \mod \alpha\). Therefore for a given \(0 \leq l_1 \leq j\)
the solutions of (A.8) and (A.9) give \( l_2 \) modulo \( \frac{j+1}{\alpha} \) and modulo \( \alpha \) respectively. We are going to prove that this is enough to determine a unique \( l_2 \) in the interval \( 0 \leq l_2 \leq j \). Suppose we are given the following equations,

\[
l_2 = a \mod(\frac{j+1}{\alpha}),
\]

(A.12)

\[
l_2 = b \mod(\alpha).
\]

(A.13)

We want to show that for any \( a \) and \( b \) equations (A.12) and (A.13) are compatible (i.e., they can be simultaneously solved). In virtue of (A.6) there are two integers \( m \) and \( n \) satisfying,

\[
m \frac{j+1}{\alpha} - n\alpha = b - a.
\]

(A.14)

Then \( l_2 = a + m \frac{j+1}{\alpha} = b + n\alpha \) solves (A.12) and (A.13). Let us now check the uniqueness of the solution in the interval \( 0 \leq l_2 \leq j \). If \( \bar{l}_2 \) also satisfies (A.12) and (A.13) we have that \( l_2 - \bar{l}_2 = 0 \mod(\frac{j+1}{\alpha}) \) and \( l_2 - \bar{l}_2 = 0 \mod(\alpha) \). Thus both \( \frac{j+1}{\alpha} \) and \( \alpha \) divide \( l_2 - \bar{l}_2 \) and in view of (A.6) this implies that their product (i.e., \( j + 1 \)) also divides \( l_2 - \bar{l}_2 \). Hence \( l_2 = \bar{l}_2 \mod(j + 1) \) and the statement is proved.

In the same way one can show that for a given \( l_2 \) there exists a unique \( 0 \leq l_1 \leq j \) that solves (A.8) and (A.9). This completes the proof of the polynomial character of \( P_j^{(r,s)}(t) \) when \( (r, s) = 1 \).
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FIGURE CAPTIONS

1) The three-manifold $M$ is constructed as the connected sum of two three-manifolds $M_1$ and $M_2$ joined along their common boundary $\Sigma$.

2) The wave-functionals $\Psi_1$ and $\Psi_2$ are the result of the functional integration over gauge configurations on $M_1$ and $M_2$ respectively.

3) Canonical homology basis for the torus.

4) Any braid can be put inside a cylinder. By identifying the lower and upper boundaries of the cylinder one constructs a solid torus having the closure of the braid in its interior.

5) Inserting an $A$-cycle inside a solid torus followed by a $B$-cycle we construct two unlinked unknotted. If the order of insertion is reversed we end up with the Hopf link.

6) The $(r, 1)$ torus knot is ambient isotopy equivalent to the trivial knot.

7) The Reidemeister moves. The moves II and III (I, II and III) define an equivalence relation called regular isotopy (ambient isotopy respectively).

8) The type I Reidemeister move introduces a twist in a band.

9) Two $A$-cycles or two $B$-cycles on $T^2$ give rise to two unlinked trivial knots.

10) The $(4, 2)$ torus link is the closure of the braid $(\sigma_1)^4$. It has two components each of which is a $(2, 1)$ torus knot.

11) One of the components of the $(4, 2)$ torus link shown in Fig. 10 can be moved to the interior of the torus and, after applying two type I Reidemeister moves, can be converted into a $(0, 1)$ torus knot.

12) Three-ball with boundary $S^2$ with four marked points.

13) Diagram representing a state involving $i$ half-monodromy twists.

14) Diagram representing a two-strand state with no twists.

15) Standard diagrams corresponding to the skein rules of the Jones polynomial.
16) Diagram corresponding to an even number of half-monodromy twists with strands carrying opposite orientations.

17) Diagram corresponding to a state with no twists and two strands with opposite orientations.

18) Diagram for the general form of a state representing a three-ball whose boundary $S^2$ has four marked points carrying the same representation and opposite orientations. The interior of the box $\hat{A}$ is arbitrary.