Pattern formation in terms of semiclassically limited distribution on lower dimensional manifolds for the nonlocal Fisher–Kolmogorov–Petrovskii–Piskunov equation

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Abstract
We have investigated the pattern formation in systems described by the nonlocal Fisher–Kolmogorov–Petrovskii–Piskunov equation for the cases where the dimension of the pattern concentration domain is lower than that of the domain of independent variables. We have obtained a system of integro-differential equations which describe the dynamics of the concentration domain and the semiclassically limited density distribution for a pattern in the class of trajectory concentrated functions. Also, asymptotic large time solutions have been obtained that describe the semiclassically limited distribution for a quasi-steady-state pattern on the concentration manifold. The approach is illustrated by an example for which the analytical solution is in good agreement with the results of numerical calculations.

Keywords: pattern formation, nonlocal population dynamics, Fisher–Kolmogorov–Petrovskii–Piskunov equation, semiclassical approximation

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1. Introduction

Nonlocal reaction-diffusion (RD) models are generally used to describe structures ordered in space and time. Structures of this type, formed by self-organization mechanisms, are involved in many important phenomena in biology, medicine, epidemiology, and ecology, such as the pattern formation in population dynamics, cancer treatment, evolution of infectious diseases, etc (see, e.g., the review papers [1, 2], and references therein).

The evolution of one-species microbial populations with long-range interactions between individuals is modeled by a nonlocal generalization of the classical Fisher–Kolmogorov–Petrovskaï–Piskunov (FKPP) equation [3, 4] for population density $u(x, t)$:

$$u_t(x, t) = D\Delta u(x, t) + au(x, t) - bu^2(x, t).$$

Equation (1.1) contains terms that describe a diffusion process with coefficient $D$, population growth with rate $a$, and the local competition between individuals with rate $b$.

Nonlocal effects arise in competitive interactions of microbial populations due to the diffusion of nutrients, the release of toxic substances, chemotaxis, and molecular communications between individuals [1, 2, 5–7].

No space ordered structures (patterns) occur during the evolution of a system governed by equation (1.1). In nonlocal FKPP models, patterns appear due to nonlocal competitive losses and diffusion [5–7], convection [8], and nonlocal population growth [9] under a certain choice of parameters. Note that the pattern formation in nonlocal FKPP models is different from the well-known Turing morphogenesis where the mechanism of pattern formation relies on the competition between the activator and the inhibitor [10, 11].

In this work, we consider the following version of the nonlocal FKPP equation which is a generalization of the equations described elsewhere [5–9, 11–20]:

$$u_t = D\Delta u - \left(\nabla, u \right) V_\gamma(x, \vec{y}, t) u(x, \vec{y}, t) d\vec{y} + a(x, t)u - bu^2,$$

where $u(x, \vec{y}, t)$ is a smooth scalar function belonging to a Schwartz space $\mathcal{S}$ in the spatial variable $\vec{x} \in \mathbb{R}^n$ at each point in time $t$ and $\langle \vec{a}, \vec{b} \rangle$ is the Euclidian scalar product of $\vec{a}, \vec{b} \in \mathbb{R}^n$, $|\vec{a}|^2 = \langle \vec{a}, \vec{a} \rangle$. Here, the local competition term in (1.1) has been replaced by the nonlocal loss term controlled by the influence function $b_\gamma(x, \vec{y})$ with a range parameter $\gamma$.

Some external factors can cause convective processes, which will contribute to the population dynamics [8]. The gradient vectors $V_\gamma = \nabla, V(x, t)$ and $W_\gamma = \nabla, W(x, \vec{y}, t)$ in equation (1.2) describe the local and the nonlocal average convective velocity, respectively. In a bacteria population, the nonlocal convection term describes the flow of bacteria that move under the action produced by other bacteria [14]. Note that the FKPP equation with a local convection term was investigated by da Cunha et al [8] and that with a nonlocal convection term by Shin-Ichiro [12] and Clerc et al [14]. The (one-dimensional) 1D FKPP equation with both a local and a nonlocal convection term was also considered by the authors [13].

Note that equations (1.1) and (1.2) are of a mean-field character, wherein fluctuations and correlations are neglected and any stochastic noise is not taken into account.

Some nonlocal FKPP equations were treated analytically and numerically by several authors.

Equation (1.2), not containing a convection term ($V(x, \vec{y}, t) = W(x, \vec{y}, t) = 0$), was solved numerically for Gaussian and cutoff influence functions $b_\gamma(x, \vec{y})$ with periodic boundary conditions in a 2D case and with a null flow boundary condition in a 1D case [5]. Spatial
structures were obtained and analyzed for a given relationship between the width of the influence function and the dimension of the population domain.

The stability of homogeneous steady-state 1D solutions was examined by Fuentes et al [6] with the use of the dispersion relation between the wavenumber of any mode of the pattern and the rate of its growth.

The transition from a homogeneous steady-state to a spatially modulated stable state was considered by Maruvka and Shnerb [15]. They also studied the spatial invasion of a stable into an unstable phase for a branching coalescence process with nonlocal competition [16].

The nonlocal FKPP equation (1.2) with the cutoff influence function $b_\gamma(\vec{x}, \vec{y})$ and the term describing the convection caused by a constant or a spatially nonuniform velocity field was investigated for a case where diffusion contributed insignificantly to the pattern formation [8].

For the cases of patterns appearing in the presence of convection, limit values of the parameters were estimated using the dispersion relation obtained by the perturbation method for the 1D equation (1.2) similar to that used in [6]. The influence of convection on the pattern formation was investigated numerically.

To study the pattern formation as applied to ecological invasion, the nonlocal 1D FKPP equation was used with the spatial variable $x$ treated as a physiological trait [17].

Nonlocal interactions in one-species RD systems can also manifest themselves as population traveling waves [15, 16, 18, 19], swarm formation [20], etc.

From the above references we see that only some properties of the pattern formation, such as the necessary conditions of their emergence and sustainability issues, can be investigated analytically. The overall view of the structure can be determined by solving numerically the corresponding 1D FKPP equation. The solution construction and the analysis of the pattern properties depending on the model parameters become much more complicated in a multidimensional case.

Note that the special cases of equation (1.2) describe a wide range of RD phenomena, some of which are discussed in the above references for a special choice of the functions $V(\vec{x}, t), W(\vec{x}, \vec{y}, t), b(\vec{x}, \vec{y})$. This leads to the idea to develop methods for analysis of equation (1.2) in the general form which explore the combined effects of diffusion, local and nonlocal convection and nonlocal competitive losses.

In this paper, we investigate the patterns described by equation (1.2) and concentrated on manifolds whose dimension $k$ is less than the number of independent variables in the equation, $n, k < n$. Such patterns can be studied using a system of equations describing the evolution of the pattern concentration domain.

For easy consideration, we restrict ourselves to a simply connected manifold

$$\Lambda^k_t = \{\vec{x} \in \mathbb{R}^n | \vec{x} = \vec{X}(t, s), s \in G \subset \mathbb{R}^k\}. \quad (1.3)$$

Here, the real variables $s, s \in G \subset \mathbb{R}^k$, parametrize the manifold $\Lambda^k_t$, and the real vector $\vec{X}(t, s)$ smoothly depends on $t \in \mathbb{R}^1$ and on the parameters $s$.

The manifold $\Lambda^k_t$ carries information about the evolution of the pattern geometry. A similar approach is embodied in the Cartan’s method of moving frames in which a moving frame is adapted to the kinematic properties of the observer in motion.

Manifolds naturally arise as concentration domains for the solutions of multidimensional (integro-) differential equations in the WKB-Maslov formalism of semiclassical asymptotics [21–23]. Lower dimensional manifolds with a complex germ [22, 23] allow one to construct asymptotic solutions of the original equation (1.2) for $D \to 0$. The small diffusion approximation seems to be quite reasonable (see, e.g., [8] where diffusion is neglected at all).
The solution \( u(\vec{x}, t) \) of equation (1.2) generates a distribution \( \rho(t, s) \) on the manifold \( \Lambda^k \), which can be assumed to be a \textit{semiclassically limited distribution} (SLD), as \( D \to 0 \), on the space \( \mathbb{R}^k \). The SLD is determined by simpler equations compared to the original equation (1.2), and it carries the most significant information about the pattern.

In terms of the semiclassical formalism, we deal with a special case of the 2D pattern formation considering the 2D equation (1.2) in a class of functions concentrated in a neighborhood of a 1D curve in a 2D space (\( \mathbb{R}^2 \)). To illustrate the pattern formation explicitly in analytical form, we omit the convection term in equation (1.2).

In section 2, we describe the lower dimensional manifold \( \Lambda^k \) where solutions of the nonlocal FKPP equation (1.2) are concentrated. In section 3, a dynamical system is deduced to describe the evolution of \( \Lambda^k \) and \( \rho(t, s) \). In section 4, we propose a method for solving the dynamical system for \( V(\vec{x}, t) = W(\vec{x}, \vec{y}, t) = 0 \) in (1.2). In section 5, we construct an exact solution of the 2D dynamical system for the case where the influence function \( b_Y(\vec{x}, \vec{y}) \) is symmetric. In section 6, a class of asymptotic solutions is found. These solutions are perturbations of the exact solution obtained in section 5 and tend to this solution as \( T \to \infty \). Some of the asymptotic solutions are treated as describing the pattern formation. In section 7, we consider the evolution of the SLD \( \rho(t, s) \) in the presence of diffusion. In Conclusion, the basic results are discussed.

2. Concentration manifold

Geometric properties of bacterial patterns [11] are characterized by the bacterial density distribution on a geometric object (manifold). For instance, a bacterial colony having a density maximum at a point is concentrated in a neighborhood of the point (zero-dimensional manifold), a ring distribution (see, e.g., [24–26]) is concentrated in a neighborhood of a circumference (1D manifold), etc. Generally speaking, the concentration manifold of a pattern and the SLD on the manifold are important pattern characteristics carrying information about the entire density distribution.

From a mathematical point of view, a pattern is described by the solution of equation (1.2), and so it is necessary to describe the concentration domain of the solution and find the SLD.

Let us define a class \( J_D(\Lambda^k) \) of functions \( u(\vec{x}, t, D) \in J_D(\Lambda^k) \) depending on a parameter \( D \) and concentrated on a manifold \( \Lambda^k \). Suppose that functions \( u(\vec{x}, t, D) \) decrease, as \( |\vec{x}| \to \infty \), faster than any power of \( \vec{x} \), and so moments of any finite order exist for these functions.

For a smooth function \( A(\vec{x}, t) \) and for \( u(\vec{x}, t, D) \in J_D(\Lambda^k) \), we define

\[
A_u(t, D) = \frac{1}{m_u(t, D)} \int_{\mathbb{R}^n} A(\vec{x}, t) u(\vec{x}, t, D) \, d\vec{x},
\]

(2.1)

where \( m_u(t, D) \) is the zero moment of the function \( u(\vec{x}, t, D) \):

\[
m_u(t, D) = \int_{\mathbb{R}^n} u(\vec{x}, t, D) \, d\vec{x}.
\]

(2.2)

Assume that there exists a limit

\[
\lim_{D \to 0} m_u(t, D) = \lim_{D \to 0} \int_{\mathbb{R}^n} u(\vec{x}, t, D) \, d\vec{x} = \int_G \rho(t, s) \, ds
\]

(2.3)

and denote

\[
m_\rho(t) = \int_G \rho(t, s) \, ds.
\]

(2.4)

As \( u(\vec{x}, t, D) \) has the meaning of population density, the functions \( u(\vec{x}, t, D) \) and \( \rho(t, s) \) are non-negative.
We say that a function \( u(\vec{x}, t, D) \) belongs to the class \( J_0(\Lambda^k_1) \) if

\[
\lim_{D \to 0} A_\mu(t, D) = \frac{1}{m_\rho(t)} \int_G A(\vec{X}(t, s), t) \rho(t, s) \, ds
\]

and refer to \( J_0(\Lambda^k_1) \) as the class of functions semiclassically concentrated on the manifold \( \Lambda^k_1 \) [27]. The solutions of equation (1.2) found in this class describe the patterns above, i.e. the patterns concentrated in a neighborhood of the manifold \( \Lambda^k_1 \).

Equations (2.1) and (2.5) can be written in the equivalent form

\[
\lim_{D \to 0} \frac{u(\vec{x}, t, D)}{m_\rho(t, D)} = \int_G \delta(\vec{x} - \vec{X}(t, s)) \rho(t, s) \, ds.
\]

Note that the functions \( \rho(t, s) \) and \( u(\vec{x}, t, D) \) are explicitly connected to one another. Let \((s, \xi)\) be a coordinate system in a space \( \mathbb{R}^n \) where the variables \( \xi \in \mathbb{W} \subset \mathbb{R}^{n-1} \) complement the variables \( s \) to form a coordinate system in \( \mathbb{R}^n \), such that \( \vec{x} = \vec{x}(s, \xi), \ s = s(\vec{x}), \) and \( \xi = \xi(\vec{x}) \) with the Jacobian \( J(\vec{x}(s, \xi)) \neq 0 \). Define the variables \( \xi \) so that their coordinate lines are orthogonal to the manifold \( \Lambda^k_1 \) with respect to the Euclidean inner product in a tangent space. So we obtain

\[
\rho(t, s) = \lim_{D \to 0} \int_{\mathbb{W}} u(\vec{x}(s, \xi), t, D) \frac{d\xi}{\rho(s, \xi)}.
\]

3. Evolution of the manifold

Let us obtain a system of equations to describe the evolution of the function \( \rho(t, s) \) and vector \( \vec{X}(t, s) \) related to a solution \( u(\vec{x}, t, D) \) of equation (1.2) in the class \( J_0(\vec{X}(t, s)) \).

According to (2.1), we define the first normalized moment of the function \( u(\vec{x}, t, D) \) as

\[
\vec{x}_\mu(t, D) = \frac{1}{m_\rho(t, D)} \int_{\mathbb{R}^n} \vec{x} u(\vec{x}, t, D) \, d\vec{x}.
\]

Using (2.5) and (3.1), we obtain

\[
\vec{x}_\mu(t) = \lim_{D \to 0} \vec{x}_\mu(t, D) = \frac{1}{m_\rho(t)} \int_G \vec{X}(t, s) \rho(t, s) \, ds.
\]

From (2.4) and (3.2) it follows that

\[
\dot{m}_\rho = \int_G \dot{\rho}(t, s) \, ds,
\]

\[
\ddot{x}_\mu = -\frac{\dot{m}_\rho}{m_\rho} \dot{x}_\mu(t) + \frac{1}{m_\rho} \int_G [\dot{\vec{x}}(t, s) \rho(t, s) + \vec{X}(t, s) \dot{\rho}(t, s)] \, ds.
\]

On the other hand, differentiating equations (2.2) and (3.1) with respect to \( t \) and taking into account (1.2), we obtain

\[
\dot{m}_u = \int_{\mathbb{R}^n} u_t(\vec{x}, t, D) \, d\vec{x} = \int_{\mathbb{R}^n} \left[ a(\vec{x}, t) - \dot{\gamma} \right] \int_{\mathbb{R}^n} b(\vec{x}, \vec{y}, t) \, d\vec{y} \right] u(\vec{x}, t, D) \, d\vec{x},
\]

\[
\dot{x}_u = \frac{1}{m_u} \int_{\mathbb{R}^n} \left( V(\vec{x}, t) + \dot{\gamma} \right) u(\vec{x}, t, D) \, d\vec{x} + \frac{1}{m_u} \int_{\mathbb{R}^n} [a(\vec{x}, t) - \dot{\gamma}] \int_{\mathbb{R}^n} b(\vec{x}, \vec{y}, t) \, d\vec{y} \right] u(\vec{x}, t, D) \, d\vec{x}.
\]
The equations describing the evolution of the SLD $\rho(t, s)$ and of the vector $\vec{X}(t, s)$ are obtained from equations (3.5) and (3.6), respectively, by a limiting process (as $D \to 0$):

$$\dot{\rho}(t, s) = \rho(t, s) \left[a(\vec{X}(t, s), t) - \chi \int_{\Gamma} b^{\gamma} (\vec{X}(t, s), \vec{X}(t, s')) \rho(t, s') \, ds'\right] , \quad (3.7)$$

$$\dot{\vec{X}}(t, s) = V_{\gamma}(\vec{X}(t, s), t) + \chi \int_{\Gamma} W_{\gamma}(\vec{X}(t, s), \vec{X}(t, s'), t) \rho(t, s') \, ds' . \quad (3.8)$$

From (3.8) it follows that the dynamics of the manifold $\Lambda_t^\delta$ is determined by the convective terms $V_{\gamma}(\vec{x}, t)$ and $W_{\gamma}(\vec{x}, \vec{y}, t)$ in the FKPP equation (1.2).

The system of equations (3.7), (3.8) is closed and it describes the evolution of the vector $\vec{X}(t, s)$ determining the manifold $\Lambda_t^\delta$ and the SLD $\rho(t, s)$ on the manifold $\Lambda_t^\delta$. To each solution $u(\vec{x}, t, D)$ of equation (1.2) with an initial condition

$$u(\vec{x}, t, D)|_{t=0} = \phi(\vec{x}, D) \quad (3.9)$$

there corresponds a solution of system (3.7), (3.8) with initial conditions

$$\rho(t, s)|_{t=0} = \rho_0(s), \quad \vec{X}(t, s)|_{t=0} = \vec{X}_0(s), \quad \rho(\vec{x}, t)|_{t=0} = \phi(\vec{x}, D), \quad (3.10)$$

where $\rho_0(s)$ and $\vec{X}_0(s)$ are related to $\phi(\vec{x}, D)$ by (2.3) and (3.2), respectively:

$$\lim_{D \to 0} m_\rho(D) = m_\rho, \quad \lim_{D \to 0} \frac{1}{m_\rho} \int_{\mathbb{R}^n} \vec{X}_0(\vec{x}, D) \, d\vec{x} = \frac{1}{m_\rho} \int_{\mathbb{R}^n} \vec{X}_0(s) \rho_0(s) \, ds, \quad \rho_0(s), \quad (3.11)$$

where

$$m_\rho(D) = \int_{\mathbb{R}^n} \phi(\vec{x}, D) \, d\vec{x}. \quad (3.12)$$

We refer to equations (3.7), (3.8) as the Einstein–Ehrenfest (EE) dynamical system of $(k, M)$ type for $M=1$. Here, $k$ is the dimension of the manifold $\Lambda_t^\delta$, $M$ is the highest order of the moments in the system.

Therefore, the study of the patterns described by equation (1.2) in terms of the SLD $\rho(t, s)$ on the manifold $\Lambda_t^\delta$ is reduced to solving the EE system (3.7), (3.8) with initial conditions (3.10).

### 4. Solution of the Einstein–Ehrenfest system without a convection term

We next consider a method for solving the Cauchy problem for the EE system (3.7), (3.8) without a convection term, i.e. with

$$V(\vec{x}, t) = 0, \quad W(\vec{x}, \vec{y}, t) = 0, \quad \vec{x} \in \mathbb{R}^n \quad (4.1)$$

in equation (1.2).

For this case, equation (1.2) takes the form

$$u_t = D \Delta u + a(\vec{x}, t) u - \chi u \int_{\mathbb{R}^n} b^{\gamma}(\vec{x}, \vec{y}) u(\vec{y}, t) \, d\vec{y} \quad (4.2)$$

and the EE system (3.7), (3.8) is greatly simplified. From (3.8) it follows that

$$\vec{X}(t, s) = \vec{X}_\phi(s). \quad (4.3)$$

Substituting (4.3) in equation (3.7) we get

$$\dot{\rho}(t, s) = \rho(t, s) \left[\tilde{a}(t, s) - \chi \int_{\Gamma} \tilde{b}^{\gamma}(s, s') \rho(t, s') \, ds'\right] , \quad (4.4)$$
\[ \rho(t, s)|_{t=0} = \rho_0(s), \]  

(4.5)

where

\[ \tilde{b}_j(s, s') = b_j(\tilde{X}_j(s), \tilde{X}_j(s')), \quad \tilde{a}(t, s) = a(\tilde{X}_j(s), t). \]  

(4.6)

Consider an auxiliary linear problem of finding the eigenfunctions \( v_j(s) \) and eigenvalues \( \lambda_j \) of a Fredholm equation with kernel \( \tilde{b}_j(s, s') \) (see, e.g., [33]):

\[ \int_G \tilde{b}_j(s, s')v_j(s') \, ds' = \lambda_j v_j(s). \]  

(4.7)

Here \( j = (j_1, j_2, \ldots, j_k) \) is a multi-index, \( j \in \mathbb{Z}_+^k \).

To be definite, we assume that (4.7) is a Fredholm equation with symmetric kernel \( \tilde{b}_j(s, s') = \tilde{b}_j(s', s) \) and its solutions form an orthogonal system

\[ \int_G v^*_j(s)v_k(s) \, ds = \delta_{jk}, \]  

(4.8)

where \( v^*_j(s) \) is the complex conjugate of \( v_j(s) \). So, we can find solutions to equation (4.4) as an expansion of \( \rho(t, s) \) in terms of the eigenfunctions of the kernel \( \tilde{b}_j(s, s') \) given by (4.6):

\[ \rho(t, s) = \sum_{|j|=0}^\infty \tilde{\beta}_j(t)v_j(s). \]  

(4.9)

Similarly,

\[ \rho_0(s) = \sum_{|j|=0}^\infty \tilde{\beta}_0(j)v_j(s). \]  

(4.10)

In view of (4.8), the Fourier coefficients \( \tilde{\beta}_j(t) \) and \( \tilde{\beta}_0 \) are calculated as

\[ \tilde{\beta}_j(t) = \int_G v^*_j(s)\rho(t, s) \, ds, \]  

(4.11)

\[ \tilde{\beta}_j(t)|_{t=0} = \tilde{\beta}_0 = \int_G v^*_j(s)\rho_0(s) \, ds. \]  

(4.12)

The kernel \( \tilde{b}_j(s, s') \) of the form (4.6) can be represented as [33]

\[ \tilde{b}_j(s, s') = \sum_{|j|=0}^\infty \lambda_j v_j(s)v^*_j(s'). \]  

(4.13)

In view of (4.7) and (4.9), equation (4.4) takes the form

\[ \dot{\rho}(t, s) = \rho(t, s) \left[ \sum_{|j|=0}^\infty [a_j(t) - \kappa \lambda_j \beta_j(t)]v_j(s) \right]. \]  

(4.14)

Here we have used the notation

\[ a_j(t) = \int_G \tilde{a}(t, s)v^*_j(s) \, ds. \]

Differentiation of equation (4.11) with respect to \( t \) yields

\[ \dot{\beta}_j(t) = \int_G \dot{\rho}(t, s)v^*_j(s) \, ds \\
= \sum_{|j|=0}^\infty \int_G \rho(t, s)(a_j(t) - \kappa \lambda_j \beta_j(t))v_j(s)v^*_j(s) \, ds. \]
In view of the expansion

\[ v_j^*(s)v_j(s) = \sum_{|j'|=0}^{\infty} \Omega_{jj'}^{\infty} v_{jj'}(s), \]  

(4.15)

we obtain a system

\[ \hat{\beta}_j = \sum_{|j'|=0}^{\infty} [a_{j'}(t) - \kappa \lambda_j \beta_j] \sum_{|j'|=0}^{\infty} \Omega_{jj'}^{\infty} \hat{\beta}_{jj'} \]

(4.16)

with the initial condition \(4.12\).

System \(4.16\) is equivalent to equation \(4.4\), and its solution can be found independently. This property provides a way of solving equation \(4.4\). Namely, let the solution of the Cauchy problem \(4.16\), \(4.12\) be known. Then the solution of equation \(4.4\) with the initial condition \(4.5\) is given by \(4.9\). In addition, we can obtain another representation for the solution of the problem \(4.4\), \(4.5\) as

\[ \rho(t, s) = \rho_0(s) \exp \left[ \sum_{|j|=0}^{\infty} \int_0^t (a_j(\tau) - \kappa \lambda_j \beta_j(\tau))v_j(s) d\tau \right]. \]  

(4.17)

We next consider an example which illustrates the method described.

5. Exact solution of the Einstein–Ehrenfest system

Let us construct an exact solution of equation \(4.4\) with coefficients \(4.6\) in which

\[ a(\vec{x}, t) = a = \text{const}, \quad b_j(\vec{x}, \vec{y}) = b_0 \exp \left\{ -\frac{(\vec{x} - \vec{y})^2}{2\gamma^2} \right\}, \quad \vec{x} \in \mathbb{R}^2, \]  

(5.1)

such that it would be concentrated on the manifold \(\Lambda_1\) determined by \(4.3\), where

\[ \vec{X}(t, s) = \vec{X}_0(s) = (R\cos s, R\sin s), \quad s \in \mathbb{G} = [-\pi, \pi] \subset \mathbb{R}. \]  

(5.2)

In this case, the manifold \(\Lambda_1\) is compact, and equation \(4.4\) becomes

\[ \dot{\rho}(t, s) = a \rho(t, s) - \kappa \rho(t, s) \int_{-\pi}^{\pi} \tilde{b}_j(s, s') \rho(t, s') ds', \]  

(5.3)

where

\[ \tilde{b}_j(s, s') = b_0 \exp \left\{ -\frac{(X_0(s) - X_0(s'))^2}{2\gamma^2} \right\} = b_0 \exp \left( -\frac{R^2}{\gamma^2} [1 - \cos(s - s')] \right). \]  

(5.4)

The eigenfunctions \(v_j(s)\) and eigenvalues \(\lambda_j\) of the Fredholm operator \(4.7\) with the kernel \(b_j(s, s')\) \(5.4\) have the form \[33\]

\[ v_j(s) = \frac{1}{\sqrt{2\pi}} e^{js}, \quad \lambda_j = 2\pi b_0 e^{-\mu} I_j(\mu), \quad j = -\infty, \infty, \]  

(5.5)

where \(\mu = R^2/\gamma^2\) and \(I_j(\mu)\) is a modified Bessel function of the first kind \[34\]. The functions \(v_j(s)\) constitute an orthogonal system of the form \(4.8\). Then the kernel \(\tilde{b}_j(s, s')\) given by \(5.4\), in view of \(4.13\), can be written as

\[ \tilde{b}_j(s, s') = \sum_{j=-\infty}^{\infty} 2\pi b_0 e^{-\mu} I_j(\mu) v_j(s) v_{-j}(s'). \]

(5.6)
Here, we have taken into account that $v^*_j(s) = v_{-j}(s)$. In view of (5.5) and (5.6), equation (4.14) can be written as

$$
\dot{\rho}(t, s) = a\rho(t, s) - \kappa \rho(t, s) \left\{ \sum_{j=-\infty}^{\infty} \lambda_j \beta_j(t) v_j(s) \right\},
$$

(5.7)

where $\beta_j(t)$ is defined by (4.11). From (5.5) and (4.15) we get that $\Omega_0^{ij} = (\sqrt{2\pi})^{-1} \delta_{ij} \langle j-j' \rangle$, and equation (4.15) takes the form

$$
v^*_j(s) v_j^*(s) = \frac{1}{\sqrt{2\pi}} v_{-j+j'}(s).
$$

(5.8)

Equations (5.8) and (4.16) yield the system

$$
\dot{\beta}_j = a\beta_j - \frac{\kappa}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} \lambda_l \beta_{j-l} \beta_l, \quad j = -\infty, \infty,
$$

(5.9)

with the initial condition (4.12). In this case, relation (4.17) becomes

$$
\rho(t, s) = \rho_\varphi(s) \exp \left[ at - \kappa \sum_{j=-\infty}^{\infty} \lambda_j v_j(s) \int_0^t \beta_j(t') \, dt' \right].
$$

(5.10)

Direct calculations show that the function $\beta_0(t)$ is a solution of the logistic equation

$$
\dot{\beta}_0 = a\beta_0 - \frac{\kappa}{\sqrt{2\pi}} \lambda_0 \beta_0^2, \quad \beta_0|_{t=0} = \beta_{00}.
$$

(5.11)

where $\beta_{00}$ is a given constant. So, the functions

$$
\beta_j(t) = \beta_0(t) \delta_{j,0}
$$

(5.12)

are solutions of system (5.9) with initial conditions

$$
\beta_{0j} = \beta_{00} \delta_{j,0}.
$$

(5.13)

From (4.10) and (5.13) we have

$$
\rho_\varphi(s) = \sum_{j=-\infty}^{\infty} \beta_{00} \delta_{j,0} v_j(s) = \beta_{00} v_0,
$$

(5.14)

where $v_0 = (\sqrt{2\pi})^{-1}$ is determined by (5.5). The solution of the Cauchy problems (5.11) has the form

$$
\beta_0(t) = \frac{\beta_{00} e^{at}}{1 + \kappa \lambda_0 \beta_{00} (a \sqrt{2\pi})^{-1} (e^{at} - 1)}.
$$

(5.15)

Then from (4.9) it follows that

$$
\rho(t, s) = \rho_0(t, s) = v_0 \beta_0(t) = v_0 \frac{\beta_{00} e^{at}}{1 + \kappa \lambda_0 \beta_{00} (a \sqrt{2\pi})^{-1} (e^{at} - 1)}.
$$

(5.16)

Note that the solution $\rho_0(t, s)$ of the form (5.16) is spatially homogeneous.

On the other hand, substituting (5.15) in (5.10), we also obtain (5.16).
This equality is illustrated by figures 1 and 3. Taking into account the explicit form (5.16) and (6.2) of the functions $\rho_0(t, s)$ and $\rho_{lim}$, respectively, we get

$$T_c(\alpha) = \frac{1}{\alpha} \ln \left( \frac{\alpha \rho_{lim} \sqrt{2\pi} / \beta_{00}}{1 - \alpha} \right).$$

Figures 1 and 3 display graphs of the functions $\rho_0(t, s)$ (solid line), $\rho_{lim}$ (dashed line), and $\alpha \rho_{lim}$ (dash-dot line) for $\alpha = 0.95$ and 1.05, respectively, for $b_0 = 1$, $\kappa = 0.2$, $R = 1$, $\gamma = 1$, $\rho_c = 1/\sqrt{2\pi}$. Figures 2 and 4 display graphs of the derivative modules $|\rho_0(t, s)|$ for $\alpha = 0.95$ and 1.05, respectively, and for the same values of the equation parameters.
Figure 2. Graph of the function \((\rho_0)(t, s)\) for \(a = 1, a > \alpha \lambda_0 v_0 \beta_0\), \(\alpha = 0.95\).

Figure 3. Graph of the function \(\rho_0(t, s)\) for \(a = 0.1, a < \alpha \lambda_0 v_0 \beta_0\), \(\alpha = 1.05\).

Figure 4. Graph of the function \(|(\rho_0)(t, s)|\) for \(a = 0.1, a < \alpha \lambda_0 v_0 \beta_0\), \(\alpha = 1.05\).

We see that for \(t > T_c(\alpha)\) the function \(\rho_0(t, s)\) monotonically tends to \(\rho_{\text{lim}}\) and the function \(|(\rho_0)(t, s)|\) monotonically tends to zero. According to [5, 8], the parameters of equation (5.3) are considered to be non-dimensional.

The solutions of equation (5.3) tend to a steady-state solution at large times [28–32]. This property has led us to seek solutions to equation (5.3) in a class of functions closely similar to (5.16).

Let \(T (T > T_c(\alpha))\) be a large parameter that characterizes the evolution time for equation (5.3) and let \(t = T \tau\), where \(\tau \in [0, 1]\). Let \(K_T = K_T(\phi)\) denote a class of functions of the form

\[
K_T = \left\{ \beta(t) \Big| \beta(t) = \beta(\theta, \tau, T) = \beta^{(0)}(\theta, \tau) + \frac{1}{T} \beta^{(1)}(\theta, \tau) + \cdots, \quad \theta = T \phi(\tau) \right\}. \tag{6.8}
\]

Here, the function \(\phi(\tau)\) is a functional parameter of the class \(K_T\). We refer to the variables \(\theta\) and \(\tau\) as a fast and a slow variable, respectively.

Let us seek solutions to system (5.9) in the class of functions \(K_T\). In view of (4.9) and (6.8), we obtain

\[
\rho(t, s) = \rho(\theta, \tau, s) = \rho^{(0)}(\theta, \tau, s) + \frac{1}{T} \rho^{(1)}(\theta, \tau, s) + \cdots, \quad T \to \infty. \tag{6.9}
\]
Choose the initial condition for the function \( \rho (t, s) \) as

\[
\rho_\phi (s) = \bar{\rho}_0 v_0 + \frac{1}{T} \rho_\phi (s). \tag{6.10}
\]

We refer to a solution of the form (6.9) of equation (5.3) as a quasi-steady-state solution. It is important that the function \( \rho^{(0)} (\theta, \tau, s) \) determines the behavior of the quasi-steady-state solution as \( T \to \infty \).

The slow variable \( \tau \) and the fast variable \( \theta \) scale time \( t \), so that

\[
\frac{d}{dt} = \frac{\partial}{\partial \theta} + \frac{1}{T} \frac{\partial}{\partial \tau} = \phi_t \frac{\partial}{\partial \theta} + \frac{1}{T} \frac{\partial}{\partial \tau}. \tag{6.11}
\]

Then (5.9) reads

\[
\left[ \phi_t \frac{\partial}{\partial \theta} + \frac{1}{T} \frac{\partial}{\partial \tau} \right] \left( \beta_j^{(0)} + \frac{1}{T} \beta_j^{(1)} + \cdots \right) = a \left( \beta_j^{(0)} + \frac{1}{T} \beta_j^{(1)} + \cdots \right)
- \frac{\lambda}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} \lambda_l \left( \beta_{j-l}^{(0)} + \frac{1}{T} \beta_{j-l}^{(1)} + \cdots \right) \left( \beta_l^{(0)} + \frac{1}{T} \beta_l^{(1)} + \cdots \right). \tag{6.12}
\]

Equating terms with the same power of \( 1/T \), we obtain

\[
\frac{1}{T^j} \phi_t \frac{\partial}{\partial \theta} \beta_j^{(0)} = a \beta_j^{(0)} - \frac{\lambda}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} \lambda_l \beta_{j-l}^{(0)} \beta_l^{(0)}, \tag{6.13}
\]

\[
\frac{1}{T} \phi_t \frac{\partial}{\partial \theta} \beta_j^{(1)} = a \beta_j^{(1)} - \frac{\lambda}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} \lambda_l (\beta_{j-l}^{(1)} \beta_l^{(0)} + \beta_{j-l}^{(0)} \beta_l^{(1)}) - \frac{\partial}{\partial \tau} \beta_j^{(0)}, \ldots \tag{6.14}
\]

Equation (6.13) for \( j = 0 \) reads

\[
\phi_t \frac{\partial}{\partial \theta} \beta_0^{(0)} = a \beta_0^{(0)} - \frac{\lambda}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} \lambda_l \beta_{-l}^{(0)} \beta_l^{(0)}. \tag{6.15}
\]

With the initial conditions (6.10), we obtain the following solution of equations (6.13) and (6.15):

\[
\beta_0^{(0)} (\theta, \tau) = \frac{\bar{\rho}_0 e^{x_0 / \phi_t}}{1 + \lambda \bar{\rho}_0 (\phi_t / (\sqrt{2\pi})^{-1} (e^{x_0 / \phi_t} - 1) ,
\beta_j^{(0)} (\theta, \tau) = 0, \quad j \neq 0. \tag{6.16}
\]

Without loss of generality, we have (see, e.g., [22, 35])

\[
\phi (t \tau) = at \tau. \tag{6.17}
\]

Then (6.16) takes the form \( \beta_0^{(0)} (t) = \beta_0 (t) \), where \( \beta_0 (t) \) is given by (5.15). Similarly, from (4.9) we obtain \( \rho^{(0)} (t, s) = \rho_0 (t, s) \). Here, \( \rho_0 (t, s) \) is determined by (5.16). Then an asymptotic solution \( \rho (t, s) \) of equation (5.3) is

\[
\rho (t, s) = \rho^{(0)} (t, s) [1 + O(1/T)], \quad T \to \infty. \tag{6.18}
\]

By analogy with [8, 9], where a 1D pattern is considered as a perturbation of the exact steady-state solution of the FKPP equation (1.1), we describe patterns as perturbations of the exact non-steady-state solution (5.16) of equation (5.3).

We refer to patterns of this type as quasi-steady-state patterns. They evolve monotonically to the steady-state \( \rho_{\infty} \), given by (6.2) (as \( T \to \infty \)).
From (6.16) and (6.17) it follows that

$$\frac{\partial}{\partial \tau} \beta_j^{(0)}(\theta, \tau) = 0.$$  

We write equation (6.14) as

$$\frac{\partial}{\partial \theta} \beta_0^{(1)} = \frac{\partial}{\partial \theta} \beta_0^{(0)} - \frac{2x\lambda_0}{a\sqrt{2\pi}} \rho_0^{(0)} \beta_0^{(1)},$$

$$\frac{\partial}{\partial \theta} \beta_j^{(1)} = \frac{\partial}{\partial \theta} \beta_j^{(0)} - \frac{\kappa}{a\sqrt{2\pi}} \rho_0^{(0)} \left( \lambda_0 \beta_j^{(1)} + \lambda_j \beta_j^{(1)} \right), \quad (6.19)$$

$$\frac{\partial}{\partial \theta} \beta_{-j}^{(1)} = \beta_{-j}^{(1)} - \frac{\kappa}{a\sqrt{2\pi}} \rho_0^{(0)} \left( \lambda_0 \beta_{-j}^{(1)} + \lambda_{-j} \beta_{-j}^{(1)} \right).$$

If the initial distribution (6.10) is symmetric, $\lambda_{-j} = \lambda_j$, then the equation for $\beta_{j}^{(1)}$ is identical to that for $\beta_{-j}^{(1)}$:

$$\frac{\partial}{\partial \theta} \beta_j^{(1)} = \beta_j^{(1)} - \frac{\kappa(\lambda_j + \lambda_0)}{a\sqrt{2\pi}} \rho_0^{(0)} \beta_j^{(1)}. \quad (6.20)$$

The solution of system (6.19) is

$$\beta_0^{(1)}(\theta, \tau) = \beta_{10} e^{\theta} \left[ 1 + \frac{\kappa(\lambda_0 + \lambda_0)}{a\sqrt{2\pi}} (e^{\theta} - 1)^2 \right],$$

$$\beta_j^{(1)}(\theta, \tau) = \frac{\beta_{1j} e^{\theta}}{\beta_{1j} e^{\theta} + \left[ 1 + \frac{\kappa(\lambda_j + \lambda_0)}{a\sqrt{2\pi}} (e^{\theta} - 1)^2 \right]^{(\lambda_j + \lambda_0)/\lambda_0}}, \quad (6.21)$$

where

$$\beta_{1j} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^\pi \tilde{\rho}_j(s) e^{-js} \, ds. \quad (6.22)$$

Here, $\tilde{\rho}_j(s)$ is given by (6.10).

Then, for the case of a symmetrical initial density distribution, we get

$$\rho(t, s) = \rho^{(0)}(t, s) + \frac{1}{T} \rho^{(1)}(t, s) = v_0 \frac{\beta_{10} e^{\theta}}{1 + \frac{\kappa(\lambda_0 + \lambda_0)}{a\sqrt{2\pi}} (e^{\theta} - 1)^2} + \frac{1}{T} \sqrt{\frac{2\pi}{\lambda_0}} \sum_{j = -\infty}^{\infty} \frac{\beta_{1j} e^{\theta} e^{js}}{1 + \frac{\kappa(\lambda_j + \lambda_0)}{a\sqrt{2\pi}} (e^{\theta} - 1)^2}^{(\lambda_j + \lambda_0)/\lambda_0} + \cdots. \quad (6.23)$$

Note, that substitution of (6.8) directly in (5.10) and in view of (6.16) and (6.21) also yields (6.23) (see appendix A).

The proposed procedure for constructing asymptotic solutions can be applied immediately to equation (5.3). Representing the function $\rho(t, s)$ as (6.9), substituting (6.9) in (5.3), and taking into account (6.11), we obtain

$$\left[ \phi_t \frac{\partial}{\partial \theta} + \frac{1}{T} \frac{\partial}{\partial \tau} \right] \left( \rho^{(0)}(0) + \frac{1}{T} \rho^{(1)}(0) + \cdots \right) = a \left( \rho^{(0)}(0) + \frac{1}{T} \rho^{(1)}(0) + \cdots \right) - \kappa \left( \rho^{(0)} + \frac{1}{T} \rho^{(1)} + \cdots \right)$$

$$\times \int_{-\pi}^\pi \tilde{\rho}_j(s') \left( \rho^{(0)}(\theta, \tau, s') + \frac{1}{T} \rho^{(1)}(\theta, \tau, s') + \cdots \right) \, ds'. \quad (6.24)$$

Equating terms with the same powers of $1/T$ yields

$$\phi_t \rho^{(0)}_j = a \rho^{(0)} - \kappa \rho^{(0)} \int_{-\pi}^\pi \tilde{b}_j(s, s') \rho^{(0)}(s') \, ds'. \quad (6.25)$$
Figure 5. Graph of the function $\rho(t,s)$ for $t=200$ and $\gamma=0.05$ (a), 1 (b), 1.5 (c), and 50 (d). The solid curve shows $\rho(t,s)$ obtained by numerical calculations and the dashed curve shows the analytical solution $\rho(t,s)$ given by (6.23).

\[
\phi_1 \rho^{(1)}_\theta = a \rho^{(1)} - \chi \rho^{(1)} \int_{-\pi}^{\pi} \tilde{b}_y(s,s') \rho^{(0)}(s') \, ds' \\
- \chi \rho^{(0)} \int_{-\pi}^{\pi} \tilde{b}_y(s,s') \rho^{(1)}(s') \, ds' - \rho^{(0)}_t.
\]

For the initial condition (6.10) we obtain $\rho^{(0)}(\theta,\tau,s) = \rho_0(t,s)$, where $\rho_0(t,s)$ is given by (5.16), as in section 5. If we seek a solution to equation (6.26) as an expansion in terms of the eigenfunctions of the kernel (5.5)

\[
\rho^{(1)}(\theta,\tau,s) = \sum_{j=-\infty}^{\infty} C_j(\tau,\theta) v_j(s),
\]

we also get (6.23) (see appendix B).

To verify the asymptotic formula (6.23), we have solved equation (5.3) by direct numerical simulation. We have put $j=10$ in (6.23), i.e. we have taken into account 21 terms. We have used a well-known finite difference scheme for updating the function $\rho(t,s)$ [36]. Figure 5 displays the SLD $\rho(t,s)$ obtained by formula (6.23) (dashed line) and by numerical calculations (solid line) for $a=1$, $b_0=1$, $\chi=0.2$, $R=1$, $T=10$, $\rho_0(s) = (\sqrt{2\pi})^{-1} + T^{-1} \exp(-s^2/0.6)$.

If $\gamma \ll R$, the function $\tilde{b}_y(s,s')$ of the form (5.4) is a $\delta$-shaped sequence which tends to the $\delta$-function as $\gamma \to 0$. In this case, equation (5.3) becomes local and patterns do not form (see figure 5(a)). If $\gamma \gg R$, the function $\tilde{b}_y(s,s')$ can be represented asymptotically: $\tilde{b}_y(s,s') \approx b_0$ as $\gamma \to \infty$. Then equation (5.3) tends to the form

\[
\rho_t(t,s) = a \rho(t,s) - \sqrt{2\pi} \chi b_0 \rho(t,s) \beta_0(t)
\]

where $\beta_0(t)$ is defined in (4.11). In this case, patterns are also not observed (see figure 5(d)). Figures 5(b) and (c) display patterns for $t=200$. Note that the number of peaks formed decreases with increasing $\gamma$ for a given $R$.

At the early stage of the structure evolution, additional peaks may form. Then the structure evolves steadily without changing its quality, i.e. it becomes quasi-steady-state.
As can be seen from figure 5, the numerical and the analytical approaches are in good agreement. Note that all graphs in figure 5 are 1D sweeps of truncated 2D distributions. For clarity, we represent the pattern displayed in figure 5(b) (dashed line) in a 3D space (see figure 6).

7. The equation for SLD with a diffusion term

Equation (4.4) does not account for diffusion and the transport effects, whereas the numerical simulations using models which take into account either diffusion or convection [5, 8] predict the occurrence of patterns. Note that the convection term \( V_\gamma(x, t) \neq 0 \) involved in equation (1.2) affects the lower dimensional patterns, which are qualitatively different from the full dimensional patterns considered, e.g., by da Cunha et al [8, 9]. Addition of a linear convection term, \( V_\gamma = -k_0 \bar{x}, k_0 > 0 \), to equation (1.2) results, according to (3.8), in a compression of the original manifold. Therefore, density peaks form in smaller numbers at the same values of \( \gamma \) (see figure 7). It has been found [8] that convection is responsible for the motion of the population in space. For the lower dimensional patterns, convection affects mainly the evolution of the original manifold.

Let us next investigate how diffusion affects the pattern formation by adding a diffusion term to equation (5.3). Thus we have the equation

\[
\rho_\gamma(t, s) = D\rho_{ss}(t, s) + a\rho(t, s) - k\rho(t, s) \int_{-\pi}^{\pi} \hat{b}_\gamma(s, s')\rho(t, s') \, ds', \quad D = \text{const},
\]

with the following boundary and initial conditions, respectively:

\[
\rho(0, s) = \rho_\phi(s), \quad \rho(t, s + 2\pi) = \rho(t, s).
\]
The eigenfunctions and eigenvalues of the Fredholm operator with kernel \( \tilde{b}(s, s') \) given by (5.4) are defined by (5.5). We will seek a solution to equation (7.1) in the form (4.9), where \( v_j(s) \) are the eigenfunctions of the kernel \( \tilde{b}(s, s') \) and the Fourier coefficients \( \beta_j(t) \) are calculated by (4.11). In view of (4.9), equation (7.1) can be written as

\[
\rho(t, s) = D \rho_{ss}(t, s) + a \rho(t, s) - \frac{1}{\kappa_1} \rho(t, s) \left\{ \sum_{p=-\infty}^{\infty} \lambda_p \beta_p(t) v_p(s) \right\}. \tag{7.2}
\]

As a result, system (4.16) becomes

\[
\dot{\beta}_j = \tilde{a}_j \beta_j - \frac{\kappa}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} \lambda_p \beta_{j-p} \beta_p, \quad j = -\infty, \infty, \tag{7.3}
\]

where

\[
\tilde{a}_j = -D j^2 + a \tag{7.4}
\]

and the initial condition \( \beta_{0j} \) is given by (4.12).

Repeating the reasoning used in the previous sections, we obtain for the initial density distribution (6.10)

\[
\rho(t, s) = v_0 \frac{\beta_{0j} e^{\alpha s}}{1 + \alpha \lambda_0 \beta_{0j} (\alpha \sqrt{2\pi})^{-1} (e^{\alpha t} - 1)} + \frac{1}{T^{1/2} \pi} \sum_{j=-\infty}^{\infty} \frac{\beta_{1j} e^{\tilde{a}_j t} e^{js}}{1 + \alpha \lambda_0 \beta_{0j} (\alpha \sqrt{2\pi})^{-1} (e^{\alpha t} - 1)^{1+j(\mu)/(\ell_0(\mu))} + \cdots} \tag{7.5}
\]

where \( \beta_{1j} \) is given by (6.22).

From (7.5) we see that as \( D \) increases, the quantity \( \tilde{a}_j \) given by (7.4) decreases, and patterns do not form. This fact is in good agreement with the results obtained by Kenkre [7].

To show the part played by diffusion in pattern formation, we set an initial condition of another type by using a cutoff function. Figure 8 displays the solution of equation (7.1) obtained by numerical calculations (which is concentrated on a circumference) for \( a = 1 \),

**Figure 8.** Graphs of the function \( \rho(t, s) \) for \( t = 100 \) and \( D = 0 \) (a), 0.005 (b), and 0.5 (c).
If the initial distribution is a cutoff function and $D = 0$, patterns form in a closed region where the population has been initially located (see figure 8(a)). If $D$ is small enough, patterns are formed but they experience qualitative changes because of the expansion of the concentration domain (see figure 8(b)). For large values of $D$, pattern formation is not observed (see figure 8(c)).

8. Conclusion

The phenomenon of pattern formation in one-species populations was studied using a number of models based on generalized Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equations taking into account nonlocal interaction effects. We have focused on a special type of pattern formation with the patterns concentrated on an evolving lower dimensional manifold $\Lambda_{1}^{k}$ whose dimension $k$ is lower then the space dimension $n$.

This property has allowed us to describe this type of pattern formation in a simpler way. Using the ideas of the semiclassical approximation method developed for nonlocal FKPP equations, we characterize the dynamics of formation of concentrated patterns by the Einstein–Ehrenfest (EE) system that describes the evolution of the manifold $\Lambda_{1}^{k}$ and the semiclassically limited distribution (SLD) $\rho(t, s)$.

We have found an exact solution of the dynamical equation determining the SLD $\rho(t, s)$. This solution is spatially homogeneous and monotonically depending on time. By analogy with previous studies [28–32], we have assumed that the patterns above can be described as large time perturbations of this exact solution. The large time asymptotics for the SLD $\rho(t, s)$ are constructed explicitly, to within $O(1/T^{2})$, in the class of functions which tend to the above exact solution as $T \to \infty$. Thereby, the exact solution can be regarded as an attractor of the constructed class of asymptotic solutions and, hence, of the corresponding concentrated patterns. As the patterns evolve monotonically without qualitative changes to some steady-state, we conclude that these asymptotic solutions describe approximately the quasi-steady-state patterns. We have obtained the analytical solutions to equation (1.2) for $V(\vec{x}, t) = W(\vec{x}, \vec{y}, t) = 0$. Taking into account a linear convection term $V(\vec{x}, t) = -k_{0}\vec{x}$ arrives at deformation of the localization manifold $\Lambda_{1}^{k}$. A more detailed analysis of the convection role in the pattern formation needs a separate study. Note that the patterns under consideration were observed to form only under a special choice of the model parameters. The contribution of diffusion to the pattern formation has been investigated.

The approach used allows one, on the one hand, to gain information on the most essential characteristics of patterns and, on the other hand, to apply the methods developed for 1D problems to multidimensional problems. It should be noted that the natural extension of the work is the problem of finding solutions $u(\vec{x}, t)$ to FKPP equations using the functions $\vec{X}(t, s)$ and $\rho(t, s)$. The WKB-Maslov method [21–23] provides a conceptual way of finding an asymptotical solution to the problem. Also, a direct study of the EE system for $k > 1$ is of interest.

The formalism proposed can be generalized to concentration manifolds of more general topological structure, such as multiply connected manifolds [21], and to curved manifolds describing the growth of microbial populations on complex structure objects.

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Appendix A

Let us show that substitution of (6.8) in (5.10) in view of (6.16) and (6.21) yields (6.23). The coefficient of \( v_j(s) \) in (6.23) is \( \beta_j^{(1)}(t) \). To compare the solutions obtained by (4.9) and (4.17), we calculate the coefficient of \( v_j(s) \) using (5.10) for the initial distribution \( \rho_0(s) \). The function \( \rho(t, s) \) is

\[
\rho(t, s) = \rho_0(s) \exp \left[ at - x\lambda_0 v_0 \int_0^t \beta_0^{(0)}(t') \, dt' \right] \times \exp \left[ - \frac{1}{T} \sum_{l=-\infty}^{\infty} x\lambda_l v_l(s) \int_0^t \beta_l^{(1)}(t') \, dt' \right] = \left( \beta_0 v_0 + \frac{1}{T} \sum_{l=-\infty}^{\infty} \beta_l v_l(s) \right) \exp \left[ at - x\lambda_0 v_0 \int_0^t \beta_0^{(0)}(t') \, dt' \right] \times \exp \left[ - \frac{1}{T} \sum_{l=-\infty}^{\infty} x\lambda_l v_l(s) \int_0^t \beta_l^{(1)}(t') \, dt' \right] = \left( \beta_0 v_0 + \frac{1}{T} \sum_{l=-\infty}^{\infty} \beta_l v_l(s) \right) \times \frac{\beta_0^{(0)}(t)}{\beta_0} \left[ 1 - \frac{1}{T} \sum_{l=-\infty}^{\infty} x\lambda_l v_l(s) \int_0^t \beta_l^{(1)}(t') \, dt' \right] + O\left( \frac{1}{T^2} \right). \quad (A.1)
\]

The coefficient of \( v_j(s) \) is

\[
\frac{\beta_j^{(0)}(t)}{\beta_0} = \frac{\beta_j^{(0)}(t)}{\beta_0} \left[ 1 - \frac{1}{T} \sum_{l=-\infty}^{\infty} x\lambda_l v_l(s) \int_0^t \beta_l^{(1)}(t') \, dt' \right] + O\left( \frac{1}{T^2} \right)
\]

As the coefficient of \( v_j(s) \) is also equal to \( \beta_j^{(1)}(t) \), we can state that the approach based on relation (4.9) is equivalent, to within \( O(1/T^2) \), to that based on relation (4.17).

Appendix B

To show that the expansion of the function \( \rho^{(1)}(\theta, \tau, s) \) in the series (6.27) for equation (6.26) also yields (6.23), we substitute (6.27) in equation (6.26) and obtain

16.740,11.0469) and ‘Nauka’ (contract no. 1.604.2011) and by Tomsk State University project no. 2.3684.2011.
\[ a \sum_{l=-\infty}^{\infty} (C_l)v_l(s) - a \sum_{l=-\infty}^{\infty} C_l v_l(s) + \kappa \lambda_0 v_0 \beta_0^{(0)}(\theta) \sum_{l=-\infty}^{\infty} C_l v_l(s) \]
\[ + \kappa v_0 \beta_0^{(0)}(\theta) \sum_{l=-\infty}^{\infty} C_l \lambda_j v_l(s) = - (v_0 \beta_0^{(0)}(\theta, \tau), \tau) \]  
\[ \text{(B.1)} \]

where \( \beta_0^{(0)}(\theta, \tau) \) is given by (6.16).

Multiplying (B.1) by \( v_{-j}(s) \) and integrating it from \(-\pi\) to \(\pi\), we get

\[ j \neq 0, \quad a(C_j) - aC_j + \kappa \lambda_0 v_0 \beta_0^{(0)} C_j + \kappa \lambda_j v_0 \beta_0^{(0)} C_j = 0. \]  
\[ \text{(B.2)} \]

The solutions of equations (B.2) are

\[ j = 0, \quad C_0 = C_0(0) \exp \left[ \theta - 2 \ln \left( 1 + \frac{\chi \lambda_0 v_0 \beta_0}{a} (e^\theta - 1) \right) \right] \]
\[ = \frac{C_0(0) e^{\theta}}{[1 + \kappa \lambda_0 v_0 \beta_0 a^{-1} (e^\theta - 1)]^2} \]
\[ j \neq 0, \quad C_j = C_j(0) \exp \left[ \theta \frac{\lambda_0 + \lambda_j}{\lambda_0} \ln \left( 1 + \frac{\kappa \lambda_0 v_0 \beta_0}{a} (e^\theta - 1) \right) \right] \]
\[ = \frac{C_j(0) e^{\theta}}{[1 + \kappa \lambda_0 v_0 \beta_0 a^{-1} (e^\theta - 1)]^{1 + l(\mu)/h(\mu)}}. \]  
\[ \text{(B.3)} \]

Thus, the final solution of equation (5.3), to within \( O(1/T^2) \), can be written as

\[ \rho(t, s) = \rho(\theta, \tau, s) = \frac{1}{\sqrt{2\pi}} \frac{\beta_{00} e^{\theta}}{[1 + \kappa \lambda_0 v_0 \beta_0 a^{-1} (e^\theta - 1)]^2} \]
\[ + \frac{1}{T \sqrt{2\pi}} \sum_{l=-\infty}^{\infty} \frac{C_l(0) e^{\theta} e^{il\mu}}{\beta_{00} e^{\theta}} + \cdots \]
\[ = \frac{1}{\sqrt{2\pi}} \frac{\beta_{00} e^{\theta}}{[1 + \kappa \lambda_0 v_0 \beta_0 a^{-1} (e^\theta - 1)]^2} \]
\[ + \frac{1}{T \sqrt{2\pi}} \sum_{l=-\infty}^{\infty} \frac{C_l(0) e^{\theta} e^{il\mu}}{\beta_{00} e^{\theta}} + \cdots, \]  
\[ \text{(B.4)} \]

which is the same as (6.23) for \( C_l(0) = \beta_{ll} \).

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