Parameter Estimation of Heavy-Tailed AR Model with Missing Data via Stochastic EM

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Abstract—The autoregressive (AR) model is a widely used model to understand time series data. Traditionally, the innovation noise of the AR is modeled as Gaussian. However, many time series applications, for example, financial time series data are non-Gaussian, therefore, the AR model with more general heavy-tailed innovations are preferred. Another issue that frequently occurs in time series is missing values, due to the system data record failure or unexpected data loss. Although there are numerous works about Gaussian AR time series with missing values, as far as we know, there does not exist any work addressing the issue of missing data for the heavy-tailed AR model. In this paper, we consider this issue for the first time, and propose an efficient framework for the parameter estimation from incomplete heavy-tailed time series based on the stochastic approximation expectation maximization (SAEM) coupled with a Markov Chain Monte Carlo (MCMC) procedure. The proposed algorithm is computationally cheap and easy to implement. The convergence of the proposed algorithm to a stationary point of the observed data likelihood is rigorously proved. Extensive simulations on synthetic and real datasets demonstrate the efficacy of the proposed framework.

Index Terms—AR model, heavy-tail, missing values, SAEM, Markov chain Monte Carlo, convergence analysis

I. INTRODUCTION

In the recent era of data deluge, many applications collect and process time series data for inference, learning, parameter estimation and decision making. The autoregressive (AR) model is a commonly used model to analyze time series data, where observations taken closely in time are statistically dependent on others. In an AR time series, each sample is a linear combination of some previous observations with a stochastic innovation. An AR model of order $p$, AR($p$), is defined as

$$y_t = \varphi_0 + \sum_{i=1}^{p} \varphi_i y_{t-i} + \varepsilon_t,$$

(1)

where $y_t$ is the $t$-th observation, $\varphi_0$ is a constant, $\varphi_i$’s are the autoregressive coefficients, and $\varepsilon_t$ is the innovation associated with the $t$-th observation. The AR model has been successfully used in many real-world applications such as DNA microarray data analysis [1], EEG signal modeling [2], financial time series analysis [3], and animal population study [4], to name but a few.

Traditionally, the innovation $\varepsilon_t$ of the AR model is assumed to be Gaussian distributed, which, as a result of the linearity of the AR model, means that the observations are also Gaussian distributed. However, there are situations arising in applications of signal processing and financial markets where the time series are non-Gaussian and heavy-tailed, either due to the intrinsic data generation mechanism or the existence of outliers. Some examples are, the stock returns [3], [5], the brain fMRI [6], [7], and the black-swan events in animal population [4]. For these cases, one may seek an AR model with the innovations following a heavy-tailed distribution such as the Student’s $t$-distribution. The Student’s $t$-distribution is one of the most commonly used heavy-tailed distribution [8]. In this direction, the authors of [9] and [10] considered the AR model with the innovations following a Student’s $t$-distribution with a known degree of freedom, whereas [11] and [12] investigated the case with an unknown degree of freedom. The Student’s $t$ AR model performs well for the heavy-tailed AR time series, and can provide robust reliable estimates of the regressive coefficients when outliers occur.

Another issue that frequently occurs in practice is missing values during the data observation or recording process. There are various reasons that can lead to missing values: values may not be measured, values may be measured but get lost, or values may be measured but are considered unusable [13]. Some real-world cases are: some stocks may suffer a lack of liquidity resulting in no transaction and hence no price recorded, observation devices like sensors may break down during the measurement, and weather or other conditions disturb sample taking schemes. Therefore, the investigation of the AR time series with missing values is significant. Although there are numerous works about Gaussian AR time series with missing values [14]–[17], less attention has been paid to heavy-tailed AR time series with missing values, since the parameter estimation in such a case is much more complicated due to the intractable problem formulation. The frameworks for the parameter estimation for heavy-tailed AR time series in [9]–[12] require complete data, and thereby, are not suited for scenarios with missing data. The objective of the current paper is to deal with this challenge and develop an efficient framework for parameter estimation from incomplete data under the heavy-tailed time series model via the expectation-maximization (EM) type algorithm.

The EM algorithm is a widely used iterative method to obtain the maximum likelihood (ML) estimates of parameters when there are missing values or unobserved latent variables. In each iteration, the EM algorithm maximizes the conditional expectation of the complete data likelihood to update the estimates. Many variants of the EM algorithm have been proposed to deal with the specific challenges in different missing value problems. For example, to tackle the problem posed by the intractability of the conditional expectation of the complete data log-likelihood, the stochastic variant of the EM algorithm, which approximates the expectation by drawing samples of the latent variables from the conditional distribution, has been
proposed in [18], [19]. The stochastic EM has also been quite popular to curb the curse of dimensionality [14], [20], since its expectation conditional maximization (ECM) algorithm has been suggested to deal with the unavailability of the closed-form maximizer of the expected complete data log-likelihood [21]. The regularized EM algorithm has been used to enforce certain structures in parameter estimates like sparsity, low-rank, and network structure [22].

In this paper, we develop a provably convergent low cost algorithmic framework for the parameter estimation of the AR time series model with heavy-tailed innovations from incomplete time series. As far as we know, there does not exist any convergent algorithmic framework for such problem. Following [9]–[11], here we consider the AR model with the Student's t-distributed innovations. We formulate the ML estimation problem and develop an efficient algorithm to obtain the ML estimates of the parameters based on the stochastic EM framework. To tackle the complexity of the conditional distribution of the latent variables, we propose a Gibbs sampling scheme to generate the samples. Instead of directly sampling from the complicated conditional distribution, the proposed algorithm just need to sample from Gaussian distributions and gamma distributions alternatively. The convergence of the proposed algorithm to a stationary point is established. Simulations on real data and synthetic data show that the proposed framework can provide accurate estimation of parameters for incomplete time series, and is also robust against possible outliers. Although here we only focus on the Student's t-distributed innovation, the idea of the proposed approach and the algorithm can also be extended to the AR model with other heavy-tailed distributions.

This paper is organized as follows. The problem formulation is provided in Section II. The review of the EM and its stochastic variants is presented in Section III. The proposed algorithm is derived in Section IV. The convergence analysis is carried out in Section V. Finally, the simulation results for the proposed algorithm applied to both real and synthetic data are provided in Section VI and Section VII concludes the paper.

II. PROBLEM FORMULATION

For the simplicity of notations, we first introduce the AR(1) model. Suppose the univariate time series \( y_1, y_2, \ldots, y_T \) follows an AR(1) model

\[
y_t = \varphi_0 + \varphi_1 y_{t-1} + \epsilon_t,
\]

where the innovations \( \epsilon_t \)'s follow a zero-mean heavy-tailed Student's t-distribution \( \epsilon_t \overset{i.i.d.}\sim t(0, \sigma^2, \nu) \). The Student's t-distribution is more heavy-tailed as the degree of freedom \( \nu \) decreases. Note that the Gaussian distribution is a special case of the Student's t-distribution with \( \nu = +\infty \).

Given all the parameters \( \varphi_0, \varphi_1, \sigma^2 \) and \( \nu \), the distribution of \( y_t \) conditional on all the preceding data \( F_{t-1} \), which consists of \( y_1, y_2, \ldots, y_{t-1} \), only depends on the previous sample \( y_{t-1} \):

\[
p(y_t|\varphi_0, \varphi_1, \sigma^2, \nu, F_{t-1}) = p(y_t|\varphi_0, \varphi_1, \sigma^2, \nu, y_{t-1})
\]

\[= f_t(y_t; \varphi_0 + \varphi_1 y_{t-1}, \sigma^2, \nu)
\]

\[
= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi \nu \sigma^2} \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{(y_t - \varphi_0 - \varphi_1 y_{t-1})^2}{\nu \sigma^2}\right)^{-\frac{\nu+1}{2}}
\]

where \( f_t(\cdot) \) denotes the probability density function (pdf) of a Student's t-distribution.

In practice, a certain sample \( y_t \) may be missing due to various reasons, and it is denoted by \( y_t = \text{NA} \) (not available). Here we assume that the missing-data mechanism is ignorable, i.e., the missing does not depend on the value [13]. Suppose we have an observation of this time series with \( D \) missing blocks as follows:

\[
y_0, \ldots, y_{t_1}, \text{NA}, \ldots, \text{NA}, y_{t_1+n_1+1}, \ldots, y_{t_d}, \text{NA}, \ldots, \text{NA},
\]

\[
y_{t_d+n_d+1}, \ldots, y_{t_D}, \text{NA}, \ldots, \text{NA}, y_{t_D+n_D+1}, \ldots, y_T,
\]

where, in the \( d \)-th missing block, there are \( n_d \) missing samples \( y_{t_d+1}, \ldots, y_{t_d+n_d} \), which are surrounded from the left and the right by the two observed data \( y_{t_d} \) and \( y_{t_d+n_d+1} \). We set for convenience \( t_0 = 0 \) and \( n_0 = 0 \). Let us denote the set of the indexes of the observed \( y_t \)'s by \( C_o \), and the set of the indexes of the missing \( y_t \)'s by \( C_m \). Also denote \( y = (y_t, 1 \leq t \leq T) \), \( y_0 = (y_t, t \in C_o) \), and \( y_m = (y_t, t \in C_m) \).

Let us assume \( \Theta = (\varphi_0, \varphi_1, \sigma^2, \nu) \in \Theta \) with \( \Theta = \{\theta|\sigma^2 > 0, \nu > 0\} \). Ignoring the marginal distribution of \( y_1 \), the log-likelihood of the observed data is

\[
l(\theta; y_0) = \log \left( \int p(y; \theta) \, dy_m \right)
\]

\[
= \log \left( \int \prod_{t=2}^{T} p(y_t|\theta, F_{t-1}) \, dy_m \right)
\]

\[
= \log \left( \int \prod_{t=2}^{T} f_t(y_t; \varphi_0 + \varphi_1 y_{t-1}, \sigma^2, \nu) \, dy_m \right).
\]

Then the maximum likelihood (ML) estimation problem for \( \theta \) can be formulated as

\[
\max_{\theta \in \Theta} l(\theta; y_0).
\]

The integral in (4) has no closed-form expression; thus, the objective function is very complicated and we cannot solve the optimization problem directly. In order to deal with this, we resort to the EM framework, which circumvents such difficulty by optimizing a sequence of simpler approximations of the original objective function instead.

III. EM AND ITS STOCHASTIC VARIANTS

The EM algorithm is a general iterative algorithm to solve ML estimation problems with missing data or latent data. More specifically, given the observed data \( X \) generated from a statistical model with unknown parameter \( \theta \), the ML estimator of the parameter \( \theta \) is defined as the maximizer of the likelihood of the observed data

\[
l(X; \theta) = \log p(X|\theta).
\]

In practice, it often occurs that \( l(X; \theta) \) does not have manageable expression due to the missing data or latent data \( Z \), while the likelihood of complete data \( p(X, Z|\theta) \) has a manageable
expression. This is when the EM algorithm can help. The EM algorithm seeks to find the ML estimates by iteratively applying these two steps [23]:

(E) Expectation: calculate the expected log-likelihood of the complete data set \((X, Z)\) with respect to the current conditional distribution of \(Z\) given \(X\) and the current estimate of the parameter \(\theta^{(k)}\):

\[
Q \left( \theta \mid \theta^{(k)} \right) = \int \log p \left( X, Z \mid \theta \right) p \left( Z \mid X, \theta^{(k)} \right) dZ,
\]

where \(k\) is the iteration number.

(M) Maximization: find the new estimate

\[
\theta^{(k+1)} = \arg \max_\theta Q \left( \theta \mid \theta^{(k)} \right).
\]

The sequence \(\left\{ l \left( X; \theta^{(k)} \right) \right\}\) generated by the EM algorithm is non-decreasing, and the limit points of the sequence \(\{\theta^{(k)}\}\) are proven to be the stationary points of the observed data log-likelihood under mild regularity conditions [24]. In fact, the EM algorithm is a particular choice of the more general majorization-minimization algorithm [25].

However, in some applications of the EM algorithm, the expectation in the E-step cannot be obtained in closed-form. To deal with this, Wei and Tanner proposed the Monte Carlo EM (MCEM) algorithm, in which the expectation is computed by a Monte Carlo approximation based on a large number of independent simulations of the missing data [26]. The MCEM algorithm is computationally very intensive.

In order to reduce the amount of simulations required by the MCEM algorithm, the stochastic approximation EM (SAEM) algorithm replaces the E step of the EM algorithm by a stochastic approximation procedure, which approximates the expectation by combining new simulations with the previous ones [18]. At iteration \(k\), the SAEM is as follows:

(E-S1) Simulation: generate \(L\) realizations \(Z^{(k,l)}\) \((l = 1, 2, \ldots, L)\) from the conditional distribution \(p \left( Z \mid X, \theta^{(k)} \right)\)

(E-A) Stochastic approximation: update \(Q \left( \theta \mid \theta^{(k)} \right)\) according to

\[
\hat{Q} \left( \theta \mid \theta^{(k-1)} \right) = \hat{Q} \left( \theta \mid \theta^{(k-1)} \right) + \gamma^{(k)} \left( \frac{1}{L} \sum_{l=1}^{L} \log p \left( X, Z^{(k,l)} \mid \theta \right) - \hat{Q} \left( \theta \mid \theta^{(k-1)} \right) \right),
\]

where \(\{\gamma^{(k)}\}\) is a decreasing sequence of positive step sizes.

(M) Maximization: find the new estimate

\[
\theta^{(k+1)} = \arg \max_\theta \hat{Q} \left( \theta \mid \theta^{(k)} \right).
\]

The SAEM requires a smaller amount of samples per iteration due to the recycling of the previous simulations. A small value of \(L\) is enough to ensure satisfying results [27].

When the conditional distribution is very complicated and the simulation step (E-S1) of the SAEM cannot be directly performed, Kuhn and Lavielle proposed to combine the SAEM algorithm with a Markov Chain Monte Carlo (MCMC) procedure, which yields the SAEM-MCMC algorithm [19]. Assume the conditional distribution \(p \left( Z \mid X, \theta \right)\) is the unique stationary distribution of the transition probability density function \(\Pi_\theta\), the simulation step of the SAEM is replaced with

(E-S2) Simulation: draw realizations \(Z^{(k,l)}\) \((l = 1, 2, \ldots, L)\) based on the transition probability density function \(\Pi_{\theta^{(k)}}(Z^{(k,l)} \mid \cdot)\).

For each \(l\), the sequence \(\{Z^{(k,l)}\}_{k \geq 0}\) is a Markov chain with the transition probability density function \(\Pi_{\theta^{(k)}}\). The Markov Chain generation mechanism needs to be well designed so that the sampling is efficient and the computational cost is not too high.

IV. SAEM-MCMC FOR STUDENT’S \(t\) AR MODEL

For the \(t\) AR problem (5), if we only regard \(y_m\) as missing data and apply the EM type algorithm, the resulting conditional distribution of the missing data is too complicated, and it is difficult to maximize the expectation or the approximated expectation of the complete data log-likelihood. Interestingly, the Student’s \(t\)-distribution can be regarded as a Gaussian mixture [28]. Since \(\varepsilon_t \sim t(0, \sigma^2, \nu)\), we can present it as the Gaussian mixture

\[
\varepsilon_t | \sigma^2, \tau_t \sim N \left( 0, \frac{\sigma^2}{\tau_t} \right),
\]

\[
\tau_t \sim \text{Gamma} \left( \nu/2, \nu/2 \right),
\]

where \(\tau_t\) is the mixture weight. Denote \(\tau = \{\tau_t, 1 < t \leq T\}\).

We can use the EM type algorithm to solve the above optimization problem by regarding both \(y_m\) and \(\tau\) as latent data, and \(y_s\) as observed data.

The resulting complete data likelihood is

\[
L \left( \theta; y, \tau \right) = p \left( y, \tau; \theta \right)
\]

\[
= \prod_{t=2}^{T} \left\{ f_N \left( y_t; \varphi_0 + \varphi_1 y_{t-1}, \frac{\sigma^2}{\tau_t} \right) f_g \left( \tau_t, \frac{\nu}{2} \right) \right\}
\]

\[
= \prod_{t=2}^{T} \left\{ \frac{1}{\sqrt{2\pi \sigma^2/\tau_t}} \exp \left( -\frac{1}{2\sigma^2/\tau_t} \left( y_t - \varphi_0 - \varphi_1 y_{t-1} \right)^2 \right) \right\}
\]

\[
= \prod_{t=2}^{T} \left( \frac{1}{\sqrt{2\pi \sigma^2/\tau_t}} \exp \left( -\frac{\nu}{2} \right) \right)^{\frac{\nu}{2} - 1} \exp \left( -\frac{\nu}{2} \right)
\]

\[
= \prod_{t=2}^{T} \left( \frac{1}{\sqrt{2\pi \sigma^2/\tau_t}} \right)^{\frac{\nu}{2}} \exp \left( -\frac{\tau_t}{2\sigma^2} \left( y_t - \varphi_0 - \varphi_1 y_{t-1} \right)^2 - \frac{\nu}{2} \right)
\]

where \(f_N(\cdot)\) and \(f_g(\cdot)\) denote the pdf’s of the Normal (Gaussian) and gamma distributions, respectively. Through some derivation, it is observed that the likelihood of complete data belongs to the curved exponential family [29], i.e., the pdf can be written as

\[
L \left( \theta; y, \tau \right) = h \left( y, \tau \right) \exp \left( -\psi \left( \theta \right) + \langle s \left( y_0, y_m, \tau, \phi \left( \theta \right) \right), \right)
\]

(14)
where $\langle \cdot , \cdot \rangle$ is the inner product,
\begin{equation}
  h(y, \tau) = \prod_{t=2}^{T} \tau_t^{-\frac{x}{2}},
\end{equation}
\begin{equation}
  \psi(\theta) = (T-1) \left\{ \frac{\nu}{2} \log \left( \frac{\nu}{2} \right) - \log \left( \Gamma \left( \frac{\nu}{2} \right) \right) \right. \\
  \left. - \frac{1}{2} \log \left( \sigma^2 \right) - \frac{1}{2} \log (2\pi) \right\},
\end{equation}
\begin{equation}
  \phi(\theta) = \left[ \frac{\nu}{2} - \frac{1}{2\sigma^2} - \frac{\varphi_0^2}{2\sigma^2}, - \frac{\varphi_1^2}{2\sigma^2}, \frac{\varphi_0}{\sigma^2}, - \frac{\varphi_0\varphi_1}{\sigma^2} \right],
\end{equation}
and the minimal sufficient statistics
\begin{equation}
  s(y_o, y_m, \tau) = \left[ \sum_{t=2}^{T} (\log(\tau_t) - \tau_t), \sum_{t=2}^{T} \tau_t y_t^2, \sum_{t=2}^{T} \tau_t, \sum_{t=2}^{T} \tau_t y_{t-1}, \right. \\
  \left. \sum_{t=2}^{T} \tau_t y_t, \sum_{t=2}^{T} \tau_t y_{t-1}, \sum_{t=2}^{T} \tau_t y_{t-1} \right].
\end{equation}

Then the expectation of the complete data log-likelihood can be expressed as
\begin{equation}
  Q(\theta|\theta^{(k)}) = \iint \log(L(\theta; y, \tau)) p(y_m, \tau|y_o; \theta^{(k)}) dy_m d\tau
\end{equation}
\begin{equation}
  = \iint \log(h(y, \tau) \exp(\psi(\theta) + s(y_o, y_m, \tau))) dy_m d\tau
\end{equation}
\begin{equation}
  = \iint \log(h(y, \tau)) p(y_m, \tau|y_o; \theta^{(k)}) dy_m d\tau
\end{equation}
\begin{equation}
  - \psi(\theta) + \int s(y_o, y_m, \tau) p(y_m, \tau|y_o; \theta^{(k)}) dy_m d\tau
\end{equation}
\begin{equation}
  = -\psi(\theta) + \left\langle s(\theta^{(k)}), \phi(\theta) \right\rangle + \text{const.},
\end{equation}
where
\begin{equation}
  s(\theta^{(k)}) = \int s(y_o, y_m, \tau) p(y_m, \tau|y_o; \theta^{(k)}) dy_m d\tau.
\end{equation}

The EM algorithm is conveniently simplified by utilizing the properties of the exponential family. The E step of the EM algorithm is reduced to the calculation of the expected minimal sufficient statistics $\hat{s}(\theta^{(k)})$, and the M step is reduced to the maximization of the function (19).

A. E step

The conditional distribution of $y_m$ and $\tau$ given $y_o$ and $\theta$ is:
\begin{equation}
  p(y_m, \tau|y_o; \theta) = \frac{p(y, \tau; \theta)}{p(y|y_o; \theta)} = \frac{p(y, \tau; \theta)}{\int p(y, \tau; \theta) dy_m d\tau}
\end{equation}
\begin{equation}
  \propto p(y, \tau; \theta)
\end{equation}
\begin{equation}
  \propto \prod_{t=2}^{T} \left( \psi \frac{\tau_t^{\frac{\nu-1}{2}}}{\Gamma \left( \frac{\nu}{2} \right) 2\sigma^2} \exp \left( -\frac{\tau_t}{2\sigma^2} (y_t - \varphi_0 - \varphi_1 y_{t-1})^2 - \frac{\nu}{2} \tau_t \right) \right).
\end{equation}

Since the integral $\iint p(y, \tau; \theta) dy_m d\tau$ does not have a closed-from expression, we only know $p(y_m, \tau|y_o; \theta)$ up to a scalar. In addition, the proportional term is complicated, and we cannot get closed-form expression for the conditional expectations $\hat{s}(\theta^{(k)})$ or $Q(\theta|\theta^{(k)})$. Therefore, we resort to the SAEM-MCMC algorithm, which generates samples from the conditional distribution using a Markov chain process, and approximates the expectation $\hat{s}(\theta^{(k)})$ and $Q(\theta|\theta^{(k)})$ by a stochastic approximation.

We propose to use the Gibbs sampling method to generate the Markov chains. The Gibbs sampler divides the latent variables $(y_m, \tau)$ into two blocks $\tau$ and $y_m$, and then generates a Markov chain of samples from the distribution $p(y_m, \tau|y_o; \theta)$ by drawing realizations from its conditional distributions $p(\tau|y_m, \tau; \theta)$ and $p(y_m|\tau, y_o; \theta)$ alternatively. More specifically, at iteration $k$, given the current estimate $\theta^{(k)}$, the Gibbs sampler starts with $\tau^{(k-1)}, y_m^{(k-1)}$ $(l=1, 2, \ldots, L)$ and generate the next sample $(\tau^{(k,l)}, y_m^{(k,l)})$ via the following scheme:

- sample $\tau^{(k,l)}$ from $p(\tau|y_m^{(k-1)}, y_o; \theta^{(k)})$,
- sample $y_m^{(k,l)}$ from $p(y_m|\tau^{(k,l)}, y_o; \theta^{(k)})$.

Then the expected minimal sufficient statistics $\hat{s}(\theta^{(k)})$ and the expected complete data likelihood $Q(\theta|\theta^{(k)})$ are approximated by
\begin{equation}
  \hat{s}^{(k)} = \hat{s}^{(k-1)} + \gamma(k) \left( \frac{1}{L} \sum_{l=1}^{L} s(y_o, y_m^{(k,l)}, \tau^{(k,l)}) - \hat{s}^{(k-1)} \right),
\end{equation}
\begin{equation}
  \hat{Q}(\theta, \hat{s}^{(k)}) = -\psi(\theta) + \left\langle \hat{s}^{(k)}, \phi(\theta) \right\rangle + \text{const.}
\end{equation}

Lemmas 1 and 2 give the two conditional distributions $p(\tau|y_m, y_o; \theta)$ and $p(y_m|\tau, y_o; \theta)$. Basically, to sample from them, we just need to draw realizations from certain Gaussian distributions and gamma distributions, which is simple. Based on the above sampling scheme, we can get the transition probability density function of the Markov chain as follows:
\begin{equation}
  \Pi_\theta(y_m, \tau, y_m', \tau') = p(\tau'|y_m, y_o; \theta) p(y_m'|\tau', y_o; \theta).
\end{equation}

**Lemma 1.** Given $y_m, y_o, \theta$, the mixture weights $\{\tau_t\}$ are independent from each other, i.e.,
\begin{equation}
  p(\tau|y_m, y_o; \theta) = \prod_{t=2}^{T} p(\tau_t|y_m, y_o; \theta).
\end{equation}
In addition, \( \tau_t \) follows a gamma distribution:
\[
\tau_t | y_m, y_o; \theta \\
\sim \text{Gamma} \left( \frac{\nu + 1}{2}, \frac{\nu \varphi_0 - \varphi_1^2 y_{l-1}}{2\sigma^2 + \nu} \right). 
\]

(26)

**Proof:** See Appendix A.

**Lemma 2.** Given \( \tau, y_o \), and \( \theta \), the missing blocks \( y_d = [y_{d+1}, y_{d+2}, \ldots, y_{d+n}]^T \), where \( d = 1, 2, \ldots, D \), are independent from each other, i.e.,
\[
p (y_m | \tau, y_o; \theta) = \prod_{d=1}^D p (y_d | \tau, y_o; \theta).
\]

(27)

In addition, the conditional distribution of \( y_d \) only depends on the two nearest observed samples \( y_{td} \) and \( y_{td+n+d+1} \) with
\[
y_d | \tau, y_o; \theta \sim \mathcal{N} (\mu_d, \Sigma_d),
\]
where the \( i \)-th component of \( \mu_d \)
\[
(\mu_d)_i = \sum_{q=0}^{i-1} \varphi_1^q \varphi_0 + \varphi_1 y_{td} + \frac{\varphi_1^{2(q+1)+1-i} \sum_{j=1}^{i-1} \varphi_1^{2j} \tau_{td+q}}{\sum_{j=1}^{i-1} \varphi_1^{2j} \tau_{td+q}} \times \left( y_{td+n+d+1} - \sum_{q=0}^{n_d} \varphi_1^q \varphi_0 - \varphi_1^{n_d+1} y_{td} \right),
\]
and the component in the \( i \)-th column and the \( j \)-th row of \( \Sigma_d \)
\[
(\Sigma_d)_{ij} = \left( \varphi_1^{2i} \right) \sum_{q=1}^{\min(i,j)} \varphi_1^{2q-2(j-q)} \tau_{td+q} - \frac{\varphi_1^{2i} \tau_{td+q}}{\sum_{q=1}^{\min(i,j)} \varphi_1^{2q-2(j-q)} \tau_{td+q}} \sum_{q=1}^{n_d+1} \varphi_1^{2(n_d+q)} \tau_{td+q}.
\]

(29)

(30)

**Proof:** See Appendix A.

**B. M step**

After obtaining the approximation \( \hat{Q} (\theta, \hat{s}^{(k)}) \) in 23, we need to maximize it to update the estimates. The function \( \hat{Q} (\theta, \hat{s}^{(k)}) \) can be rewritten as
\[
\hat{Q} (\theta, \hat{s}^{(k)}) \\
= - \psi (\theta) + \left\{ \hat{s}^{(k)} \varphi (\theta) \right\} + \text{const}.
\]}
\[
= (T - 1) \left\{ \frac{\text{log} (\nu \varphi_0)}{2} - \text{log} \left( \Gamma \left( \frac{\nu}{2} \right) \right) - \frac{1}{2} \text{log} (\sigma^2) \right\} \\
+ \frac{\nu}{2} \hat{s}_1^{(k)} - \frac{\hat{s}_5^{(k)}}{2\sigma^2} - \frac{\hat{s}_6^{(k)}}{2\sigma} - \frac{\varphi_0 \hat{s}_5^{(k)}}{2\sigma^2} + \frac{\varphi_1 \hat{s}_6^{(k)}}{\sigma^2} - \frac{\varphi_0 \varphi_1 \hat{s}_7^{(k)}}{\sigma^2} + \text{const},
\]

(31)

where \( \hat{s}_i^{(k)} \) (\( i = 1, 2, \ldots, 7 \)) is the \( i \)-th component of \( \hat{s}^{(k)} \).

The optimization of \( \varphi_0, \varphi_1 \) and \( \sigma^2 \) is decoupled from the optimization of \( \nu \). Setting the derivatives of \( \hat{Q} (\theta, \hat{s}^{(k)}) \) with respect to to \( \varphi_0, \varphi_1 \) and \( \sigma^2 \) to 0 gives
\[
\varphi_0^{(k+1)} = \frac{s_5^{(k)}}{s_3^{(k)}} - \frac{s_6^{(k)}}{s_4^{(k)}} \varphi_1^{(k+1)},
\]

(32)

\[
\varphi_1^{(k+1)} = \frac{s_6^{(k)}}{s_4^{(k)}} - \frac{s_7^{(k)}}{s_5^{(k)}} \varphi_0^{(k+1)}.
\]

(33)

and
\[
(\sigma^{(k+1)})^2 = \frac{1}{T - 1} \left( s_2^{(k)} + \varphi_0^{(k+1)} s_3^{(k)} + \varphi_1^{(k+1)} s_4^{(k)} \right)^2 \\
- 2 \varphi_0^{(k+1)} s_5^{(k)} - 2 \varphi_1^{(k+1)} s_6^{(k)} \\
+ 2 \varphi_0^{(k+1)} \varphi_1^{(k+1)} s_7^{(k)}.
\]

(34)

The \( \nu^{(k+1)} \) can be found by one-dimensional search:
\[

\nu^{(k+1)} = \arg \max_{\nu > 0} f (\nu, \hat{s}^{(k)})
\]

(35)

with \( f (\nu, \hat{s}^{(k)}) = \{ \frac{\text{log} (\nu \varphi_0)}{2} - \text{log} \left( \Gamma \left( \frac{\nu}{2} \right) \right) \} + \frac{\nu \hat{s}_1^{(k)}}{2\sigma^2} \). According to Proposition 1 in [30], \( \nu^{(k+1)} \) always exists and is unique.

The resulting SAEM-MCMC algorithm is summarized in Algorithm 1.

**Algorithm 1** SAEM-MCMC Algorithm for Student’s t AR(1)

1: Initialize \( \theta^{(0)} \in \Theta, \hat{s}^{(0)} = 0, k = 0, \) and \( y_m^{(l)} \) for \( l = 1, 2, \ldots, L \).
2: for \( k = 1, 2, \ldots \) do
3: \hspace{0.5cm} Simulation:
4: \hspace{1cm} for \( l = 1, 2, \ldots, L \) do
5: \hspace{1.5cm} sample \( \tau^{(k,l)} \) from \( p (\tau | y_m^{(k-1,l)}, y_o; \theta^{(k)}) \) using Lemma 1,
6: \hspace{1.5cm} sample \( y_m^{(l,k)} \) for \( p (y_m | \tau^{(k,l)}, y_o; \theta^{(k)}) \) using Lemma 2.
7: \hspace{0.5cm} end for
8: \hspace{0.5cm} Stochastic approximation: evaluate \( \hat{s}^{(k)} \) and \( \hat{Q} (\theta, \hat{s}^{(k)}) \) as in (22) and (23) respectively.
9: \hspace{0.5cm} Maximization: update \( \theta^{(k+1)} \) as in (32), (33), (34) and (35).
10: \hspace{0.5cm} if stopping criteria is met then
11: \hspace{1cm} terminate loop
12: \hspace{0.5cm} end if
13: \hspace{0.5cm} end for

**C. Particular Cases**

In cases where some parameters in \( \theta \) are known, we just need to change the updates in M-step accordingly, and the simulation and approximation steps remain the same. For example, if we know that the time series is zero mean [1], [12], i.e., \( \varphi_0 = 0 \), then the update for \( \varphi_0^{(k+1)} \) and \( \varphi_1^{(k+1)} \) should be replaced with
\[
\varphi_0^{(k+1)} = 0,
\]

(36)
and
\[ \varphi_1^{(k+1)} = \frac{\varphi_0^{(k)}}{s_4^{(k)}}. \]  

(37)

If the time series is known to follow the random walk model [14], which is a special case of AR(1) model with \( \varphi_1 = 1 \), then the update for \( \varphi_0^{(k+1)} \) and \( \varphi_1^{(k+1)} \) should be replaced with
\[ \varphi_0^{(k+1)} = \frac{\varphi_0^{(k)}}{s_3^{(k)}} - \frac{\varphi_1^{(k)}}{s_3^{(k)}}, \]
\[ \text{and} \]
\[ \varphi_1^{(k+1)} = 1. \]

(38)

(39)

D. Generalization to AR(p)

The above ML estimation method can be immediately generalized to the Student’s t AR(p) model:
\[ y_t = \varphi_0 + \sum_{i=1}^{p} \varphi_i y_{t-i} + \varepsilon_t, \]
\[ \text{where } \varepsilon_t \overset{i.i.d.}{\sim} t(0, \sigma^2, \nu). \]

Similarly, we can apply the SAEM-MCMC algorithm to obtain the estimates by considering \( \tau \) and \( y_m \) as the latent data, and \( y_o \) as observed data. In each iteration, we draw some realizations of \( \tau \) and \( y_m \) from the conditional distribution \( p(y_m, \tau | y_o; \theta^{(k)}) \) to approximate the expectation function \( Q(\theta; \theta^{(k)}) \), and maximize the approximation \( \hat{Q}(\theta; \theta^{(k)}) \) to update the estimates. The main difference is that the conditional distribution of the AR(p) will become more complicated than that of the AR(1), since each sample of the AR(p) has more dependence on the previous samples. To deal with this challenge, when applying the Gibbs sampling, we can divide the latent data \( y_m, \tau \) into more blocks, \( \tau \) as a block and each \( y_{i,E} \in C_m \) as a block, so that the distribution of each block of latent variables conditional on other latent variables will be easy to obtain and sample from. For the sake of space, we do not go into details here, and we will consider this in our future work.

V. CONVERGENCE

In this section, we provide theoretical guarantee for the convergence of the proposed algorithm. The convergence of the simple deterministic EM algorithm has been addressed by many different authors, starting from the seminal work in [23], to a more general consideration in [24]. However, the convergence analysis of the stochastic variants of the EM algorithm, like the MCEM, SAEM and SAEM-MCMC algorithms, is much more difficult due to the randomness of sampling. See [18], [19], [31]–[34] for a more general overview of these stochastic EM algorithms and their convergence analysis. Of specific interest, the authors in [18] introduced the SAEM algorithm, and established the almost sure convergence to the stationary points of the observed data likelihood under mild additional conditions. The convergence analysis of such framework hinges upon the assumption that the complete data likelihood belongs to the curved exponential family. The authors in [19] coupled the SAEM framework with the MCMC procedure, and they have given the convergence conditions for the SAEM-MCMC algorithm when the complete data likelihood belongs to the curved exponential family. The given set of conditions is as follows.

(M1) For any \( \theta \in \Theta \),
\[ \int \int ||s(y_o, y_m, \tau)|| p(y_m, \tau | y_o; \theta) \, dy_m \, d\tau < \infty. \]

(41)

(M2) \( \psi(\theta) \) and \( \phi(\theta) \) are twice continuously differentiable on \( \Theta \).

(M3) The function
\[ \bar{s}(\theta) = \int \int s(y_o, y_m, \tau) p(y_m, \tau | y_o; \theta) \, dy_m \, d\tau \]

is continuously differentiable on \( \Theta \).

(M4) The objective function
\[ l(\theta; y_o) = \log \left( \int \int p(y, \tau; \theta) \, dy_m \, d\tau \right) \]

is continuously differentiable on \( \Theta \), and
\[ \partial_{\theta} \int \int p(y, \tau; \theta) \, dy_m \, d\tau = \int \int \partial_{\theta} p(y, \tau; \theta) \, dy_m \, d\tau. \]

(44)

(M5) For \( Q(\theta, \bar{s}) = -\psi(\theta) + \langle \bar{s}, \phi(\theta) \rangle + \text{const.} \), there exists a function \( \bar{\theta}(\bar{s}) \) such that \( \forall \bar{s} \in \Theta \), \( Q(\bar{\theta}(\bar{s}), \bar{s}) \geq Q(\theta, \bar{s}) \). In addition, the function \( \bar{\theta}(\bar{s}) \) is continuously differentiable.

(SAEM1) For all \( k, \gamma(k) \in [0, 1], \sum_{k=1}^{\infty} \gamma(k) = \infty \) and there exists \( \frac{1}{2} < \lambda \leq 1 \) such that \( \sum_{k=1}^{\infty} (\gamma(k))^{1+\lambda} < \infty \).

(SAEM2) \( l(\theta; y_o) \) is \( d \) times differentiable on \( \Theta \), where \( d = 7 \) is the dimension of \( s(y_o, y_m, \tau) \), and \( \theta(\bar{s}) \) is \( d \) times differentiable.

(SAEM3)
1) The chain takes its values in a compact set \( \Omega \).
2) The \( s(y_o, y_m, \tau) \) is bounded on \( \Omega \), and the sequence \( \{\bar{s}(k)\} \) takes its values in a compact subset.
3) For any compact subset \( V \) of \( \Theta \), there exists a real constant \( L \) such that for any \( (\theta, \theta') \) in \( V^2 \)
\[ \sup_{(y_m, \tau, y_m', \tau')} \left| \Pi_\theta(y_m, \tau, y_m', \tau') - \Pi_{\theta'}(y_m, \tau, y_m', \tau') \right| \leq L|\theta - \theta'|. \]

(45)

4) The transition probability \( \Pi_\theta \) generates a uniformly ergodic chain whose invariant probability is the conditional distribution \( p(y_m, \tau | y_o; \theta) \): there exist \( M_\theta > 0 \) and \( 0 < \rho_\theta < 1 \) such that
\[ ||\Pi_\theta^n(y_m, \tau, \cdot) - p(\cdot | y_o; \theta) ||_{TV} \leq M_\theta \rho_\theta^n \]
for any \( (y_m, \tau) \in \Omega \) and \( n \in N^+ \), where \( || \cdot ||_{TV} \) denotes the total variation norm.
In summary, the conditions (M1)-(M5) are all about the model, and are conditions for the convergence of the deterministic EM algorithm. The conditions (M1) and (M3) require the boundedness and continuous differentiability of the expectation of the sufficient statistics. The conditions (M2) and (M4) guarantee the continuous differentiability of the complete data log-likelihood \( l(\theta; y, \tau) \), the expectation of the complete data likelihood \( Q(\theta | \theta^{(k)}) \), and the observed data log-likelihood \( l(\theta; y_o) \). The condition (M5) indicates the existence of a global maximizer for \( Q(\theta, \theta) \).

The conditions (SAEM1)-(SAEM3) are additional requirement for the SAEM-MCMC to converge. The condition (SAEM1) is about the step sizes \( \gamma^{(k)} \). This condition can be easily satisfied by choosing the step sizes properly. It is recommended to set \( \gamma^{(k)} = 1 \) for \( 1 \leq k \leq K \) and \( \gamma^{(k)} = \frac{1}{K-k} \) for \( k \geq K+1 \), since the initial guess \( \theta^{(0)} \) may be far from the ML estimates we are looking for, and choosing the first step sizes equal to 1 allow the sequence \( \{ \theta^{(k)} \} \) to have a large variation and then converge to a neighborhood of the maximum likelihood. The condition (SAEM2) requires \( d = 7 \) times differentiability of \( l(\theta; y_o) \) and \( \theta (\hat{g}^{(k)}) \). The condition (SAEM3) imposes some constraints on the generated Markov chains.

In [19], the authors have established the convergence to the stationary points. However, their analysis assumes that complete data likelihood belongs to the curved exponential family, and all these conditions (M1)-(M5) and (SAEM1)-(SAEM3) are satisfied. These assumptions are very problem specific, and do not hold trivially for our case, since our conditional distribution of the latent variable is extremely complicated. To comment on the convergence of our proposed algorithm, we need to establish the conditions (M1)-(M5) and (SAEM1)-(SAEM3) one by one. The detailed proof for all the conditions is shown in the Appendix B. Consequently, we have the following conclusion about the convergence of the proposed Algorithm 1.

**Theorem 3.** Under the conditions (M1)-(M5) and (SAEM1)-(SAEM3), the iterates \( \{ \theta^{(k)} \} \) generated by Algorithm 1 has the following asymptotic property: with probability 1, \( \lim_{k \to +\infty} d(\theta^{(k)}, \mathcal{L}) = 0 \), where \( d(\theta^{(k)}, \mathcal{L}) \) denotes the distance from \( \theta^{(k)} \) to the set of stationary points of observed data log-likelihood \( \mathcal{L} = \{ \theta \in \Theta, \frac{\partial}{\partial \theta} Q(\theta | y_o) = 0 \} \).

**Proof:** Upon establishing the conditions (M1)-(M5) and (SAEM1)-(SAEM3), the proof of this theorem follows straightforward from the analysis of the work in [19].

VI. SIMULATIONS

In this section, we conduct a simulation study of the performance of the proposed ML estimator for the Student’s \( t \) AR time series with missing values and the convergence of the proposed algorithms. First, we show that the proposed estimator is able to make good estimates of the parameters from the incomplete time series which have been synthesized to fit the model. Second, we show its robustness to the innovation outliers. Finally, we test it on a real financial time series, the Hang Seng index.

### A. Parameter Estimation

In this subsection, we show the convergence of the proposed SAEM-MCMC algorithm and the performance of the proposed estimator on incomplete Student’s \( t \) AR(1) time series with different number of samples and missing percentages. The estimation error is measured by the mean square error (MSE):

\[
\text{MSE}(\hat{\theta}) := E \left[ (\hat{\theta} - \theta^{\text{true}})^2 \right],
\]

where \( \hat{\theta} \) is the estimate for the parameter \( \theta \), and \( \theta^{\text{true}} \) is its true value. The parameter \( \theta \) can be \( \phi_0 \), \( \phi_1 \), \( \sigma^2 \), and \( \nu \). The expectation is approximated via Monte Carlo simulations using 100 independent incomplete time series.

We set \( \nu^{\text{true}} = 1 \), \( \phi_1^{\text{true}} = 0.5 \), \( (\sigma^{\text{true}})^2 = 0.01 \), and \( \nu^{\text{true}} = 2.5 \). For each incomplete data set \( y_o \), we first randomly generate a complete time series \( \{ y_t \} \) with \( T \) samples based on the Student’s \( t \) AR(1) model. Then \( n_{\text{mis}} \) number of samples are randomly deleted to obtain an incomplete time series. The missing percentage of the incomplete time series is \( \rho := \frac{n_{\text{mis}}}{n} \times 100\% \).

In Section 5, we have established that the convergence of the proposed SAEM-MCMC algorithm to the stationary points of the observed data likelihood. However, it is observed...
TABLE I

|                | φ₀  | φ₁  | (σ²) | ν  |
|----------------|-----|-----|-------|----|
| True value     | 1.000 | 0.500 | 0.010 | 2.5 |
| Gaussian AR(1) | 1.119 | 0.442 | 0.033 | +∞  |
| Student’s t AR(1) | 0.989 | 0.501 | 0.009 | 2.234 |

that the estimation result can be sensitive to the initialization due to the existence of multiple stationary points. This is an inevitable problem since it is a non-convex optimization problem. Interestingly, it is also observed that when we initialize our algorithm using the ML estimates assuming Gaussian AR(1) model \((φ_0)_{\text{g}}, (φ_1)_{\text{g}}, (σ²)_{\text{g}}\), and initialize \(y^{(0)}_{m, l}\) using the mean of the conditional distribution \(p\left(\mathbf{y}_m; \mathbf{y}_o, (φ_0)_{\text{g}}, (φ_1)_{\text{g}}, (σ²)_{\text{g}}\right)\), which is a Gaussian distribution. The parameter \(ν(0)\) is initialized as a random positive number. In each iteration, we draw \(L = 10\) samples. For the step sizes, we set \(γ(k) = 1\) for \(1 ≤ k ≤ 30\) and \(γ(k) = \frac{1}{k - 30}\) for \(k ≥ 31\). Figure 1 gives an example of applying the proposed SAEM-MCMC algorithm to estimate the parameter on a synthetic AR(1) data set with \(T = 300\) and the missing percentage \(ρ = 10\%\). We can see the algorithm converges in less than 100 iterations, where each iteration just needs \(L = 10\) runs of Gibbs sampling, and also the final estimation error is small. Table I compares the estimation results of the Student’s t AR model and the Gaussian AR model. This testifies our argument that, for incomplete heavy-tailed data, the traditional method for incomplete Gaussian AR time series is too inefficient, and significant performance gain can be achieved by designing the algorithm under heavy-tailed model.

Figure 2 shows the estimation results with the number of samples \(T = 100, 200, 300, 400, 500\) and the missing percentages \(ρ = 10\%, 20\%, 30\%, 40\%\). For reference, we have also given the ML estimation result from the complete data sets \(ρ = 0\), which is obtained using the algorithm in [11]. We can observe that our method performs satisfactorily well even for high percentage of missing data, and, with increasing sample sizes, the estimates with missing values matches with the estimates of the complete data.

B. Robustness to Outliers

A useful characteristic of the Student’s t is its resilience to the outliers, which is not shared by the Gaussian distribution. Here we illustrate that the Student’s t AR model can provide robust estimation of autoregressive coefficients under innovation outliers.

Fig. 3. Incomplete AR(1) time series with four innovation outliers.
An innovation outlier is an outlier in the $\varepsilon_t$ process, and is a typical kind of outlier in the AR time series [35], [36]. Due to the temporal dependence of the AR time series data, an innovation outlier will affect not only the current observation $y_t$, but also subsequent observations. Figure 3 gives an example of a Gaussian AR(1) time series contaminated by four innovation outliers.

When an AR time series is contaminated by the outliers, the traditional ML estimation of the autoregressive coefficients based on the Gaussian AR model, which is equivalent to least squares fitting, will provide unreliable estimates for the autoregressive coefficients. Although, for complete time series, there are numerous works about the robust estimation of autoregressive coefficients under outliers, unfortunately, less attention was paid to robust estimation from incomplete time series. As far as we know, only Kharin and Voloshko have considered robust estimation with missing values [16]. In their paper, they assume that $\phi_0$ is known and equal to 0. To be consistent with Kharin’s method, in this simulation, we also assume $\phi_0^{true}$ is known and $\phi_0^{true} = 0$, although our method can also be applied to the case where $\phi_0^{true}$ is unknown.

We let $\phi_1^{true} = 0.5$ and $\varepsilon_t \sim N(0, 0.01)$. Note here the innovations follow Gaussian distribution. First, we randomly generate an incomplete Gaussian AR(1) time series with $T = 100$ samples and the missing percentage $\rho = 0.1$. Then we randomly generate another incomplete Gaussian AR(1) time series with four innovation outliers. The values of the innovation outliers are set to be $5$, $-5$, $5$, $-5$, and the positions are selected randomly. See the Figure 3 for this incomplete contaminated time series. The Gaussian AR(1) model, Student’s $t$ AR(1) model, and Kharin’s method are applied to estimate autoregressive coefficient $\phi_1$. After obtaining the estimate $\hat{\phi}_1$, we compute the one-step-ahead predictions $\hat{y}_t = \hat{\phi}_1 y_{t-1}$ and the prediction error $(\hat{y}_t - y_t)^2$ for $t \in C_o$ and $t - 1 \in C_o$. It is not surprising that the outliers are poorly predicted, so we omit it when compute the averaged prediction error. Table II show the estimation results and the one-step-ahead prediction errors. It is clear that the ML estimator based on Gaussian AR(1) has been significantly affected by the presence of the outliers, while the Student’s $t$ AR(1) model is robust to them, since the outliers cause the innovations to have a heavy-tailed distribution, which can be modeled by the Student’s $t$ distribution. Kharin’s method does not perform well either, as this method is designed for the additive outliers and replacement outliers, rather than the innovation outlier.

### C. Real Data

Here we consider the returns of the Hang Seng index over 260 working days from Jan. 2017 to Nov. 2017 (excluding weekends and public holidays). Figure 4 shows the quantile-quantile (QQ) plot of these returns. The derivation from the straight red line indicates that the returns are significantly non-Gaussian and heavy-tailed.

We divide the 260 returns into two parts: the estimation data, which involves the first 250 samples, and the test data, which involves the remaining 10 samples. First, we fit the estimation data to the Gaussian AR(1) model and Student’s $t$ AR(1) model, and estimate the parameters. Then we predict the test data using the one-step-ahead predication method based on the estimates, and compute the averaged prediction errors. Next, we randomly delete 10 of the estimation data, and estimate the parameters of the Gaussian AR(1) model and Student’s $t$ AR(1) model from this incomplete data set. Finally, we also make the prediction and compute the averaged prediction errors based on these estimates. The result is summarized in Table IV. We can get the following conclusions: i) the Student’s $t$ AR(1) model performs better than the Gaussian AR(1) model for this heavy-tailed time series, ii) the proposed parameter estimation method for incomplete Student’s $t$ AR(1) time series can provide similar estimates to the result of complete data.

### VII. Conclusions

In this paper, we have considered the parameter estimation of the heavy-tailed AR model with missing values. We have formulated an ML estimation problem and developed an efficient approach to obtain the estimates based on the stochastic EM. Since the conditional distribution of the latent data in our case is very complicated, we have proposed a Gibbs sampling scheme to draw realizations from it. The convergence of the proposed algorithm to the stationary points has been established. Simulations show that the proposed approach can provide reliable estimates from incomplete time series with different percentage of missing values, and is robust to the outliers. Although in this paper we only focus on the univariate AR model with the Student’s $t$ distributed innovations due

|                | $\hat{\phi}_1$ ($\phi_1^{true} = 0.5$) | Averaged prediction error |
|----------------|----------------------------------------|--------------------------|
| Gaussian AR(1) | 0.5337                                 | 0.0121                   |
| Student’s $t$ AR(1) | 0.4947                              | 0.0110                   |
| Kharin’s method | 0.4210                                 | 0.0212                   |

Fig. 4. Quantile-quantile plot of the the Hang Seng index returns showing that they are heavy-tailed.

TABLE II

**Estimation and prediction results for incomplete Gaussian AR(1) time series with outliers.**
to the limit of the space, our method can be extended to multivariate AR model and also other heavy-tailed distributed innovations.

**APPENDIX A**

**PROOF FOR LEMMAS 1 AND 2**

**A. Proof for Lemma 1**

The conditional distribution of \( \tau | y_m, y_o; \theta \) is

\[
p(\tau | y_m, y_o; \theta) \propto p(y; \tau, \theta)
\]

\[
= \prod_{t=2}^{T} \left( \frac{1}{\Gamma(\frac{\nu}{2})} \frac{\nu}{2} \tau_t^\frac{\nu}{2} \exp \left( -\frac{\tau_t}{2\sigma^2} (y_t - \varphi_0 - \varphi_1 y_{t-1})^2 \right) \right)
\]

which implies that the \( \{\tau_t\} \) are independent from each other with

\[
p(\tau_t | y_m, y_o; \theta) \propto \tau_t^{\frac{\nu}{2}-1} \exp \left( -\frac{\nu}{2} \frac{1}{\sigma^2} (y_t - \varphi_0 - \varphi_1 y_{t-1})^2 \right)
\]

Comparing this expression with the pdf of gamma distribution, we get that \( \tau_t | y_m, y_o; \theta \) follow a gamma distribution:

\[
\tau_t | y_m, y_o; \theta \sim \text{Gamma} \left( \frac{\nu}{2} \right), \left( y_t - \varphi_0 - \varphi_1 y_{t-1} \right)^2 / \sigma^2 + \nu
\]

**B. Proof for Lemma 2**

According to the Gaussian mixture representation (11) and (12), given \( \tau \) and \( \theta \), we have

\[
y_t = \varphi_0 + \varphi_1 y_{t-1} + \varepsilon_t,
\]

with \( \varepsilon_t \) following a Gaussian distribution: \( \varepsilon_t \sim i.i.d. N \left( 0, \sigma^2 \right) \).

We can see that the distribution of \( y_t \) conditional on all the preceding data \( F_{t-1} \), only depends on the previous sample \( y_{t-1} \):

\[
p(y_t | \tau, F_{t-1}; \theta) = p(y_t | \tau, y_{t-1}; \theta).
\]

In addition, the distribution of \( y_t \) conditional on all the preceding observed data \( F_{t-1} \), only depends on the nearest observed sample:

\[
p(y_t | \tau, F_{t-1}; \theta) = \left\{ \begin{array}{ll} p(y_t | \tau, y_{t-1}; \theta) & t = t_d + 1, \\
p(y_t | \tau, y_{t-n_d}; \theta) & t = t_d + n_d + 1, for \ d = 0, 1, \ldots, D. \end{array} \right.
\]

Table III

| \( \varphi^0 \) | \( \varphi^1 \) | \( \sigma^2 \) | \( \nu \) | Averaged prediction error |
|------------------|------------------|------------------|------------------|------------------|
| Complete data assuming Gaussian innovations | 7.548 x 10^{-4} | -1.058 x 10^{-1} | 1.702 x 10^{-3} | +\infty | 9.141 x 10^{-6} |
| Incomplete data assuming Gaussian innovations | 8.618 x 10^{-4} | -1.253 x 10^{-1} | 1.665 x 10^{-3} | +\infty | 9.455 x 10^{-6} |
| Complete data assuming Student’s t innovations | 5.440 x 10^{-4} | -9.580 x 10^{-2} | 6.524 x 10^{-6} | 2.622 | 8.836 x 10^{-6} |
| Incomplete data assuming Student’s t innovations | 5.538 x 10^{-4} | -9.459 x 10^{-2} | 6.331 x 10^{-6} | 2.671 | 8.831 x 10^{-6} |
for \( i = 1, 2, \ldots, n_d + 1 \), which means that \( y_{td+i} \) can be expressed as the sum of a constant (involves \( \tau \), \( y_{td} \) and \( \Theta \)) and a linear combination of the independent Gaussian random variables \( w_{td+1}, w_{td+2}, \ldots, w_{td+n_d} \). Therefore, we can obtain that \( y_{cd} \) follows a Gaussian distribution as follows:

\[
y_{cd|\tau, y_{td}; \Theta} \sim \mathcal{N}(\mu_{cd}, \Sigma_{cd}),
\]

(55)

where the \( i \)-th component of \( \mu_{cd} \)

\[
\mu_{cd}(i) = \sum_{q=0}^{i-1} \varphi_1^q \varphi_0 + \varphi_1^i y_{td},
\]

(56)

and the component in the \( i \)-th column and the \( j \)-th row of \( \Sigma_{cd} \)

\[
\Sigma_{cd(i,j)} = \sigma^2 \varphi_1^{\min(i,j)} \sum_{q=1}^\infty \varphi_1^{2\min(i,j)-q} \tau_d^{q+1},
\]

(57)

Then we can get that the conditional distribution of \( y_{td}|\tau, y_{td}, y_{td+n_d+1}; \Theta \) is also a Gaussian distribution with

\[
\mu_d = \mu_{cd(1:n_d)} + \frac{\sum_{cd(1:n_d, n_d+1)} y_{td+n_d+1} - \mu_{cd(n_d+1)}}{\sum_{cd(n_d+1, n_d+1)}},
\]

(58)

and

\[
\Sigma_d = \Sigma_{cd(1:n_d, n_d+1)} - \frac{\sum_{cd(1:n_d, n_d+1)} \Sigma_{cd(n_d+1, n_d+1)}}{\sum_{cd(n_d+1, n_d+1)}},
\]

(59)

where \( \mu_{cd(a_1, a_2)} \) denotes the subvector consisting of the \( a_1 \)-th to \( a_2 \)-th component of \( \mu_{cd} \), and the \( \Sigma_{cd(a_1, a_2)} \) means the submatrix consisting of the components in the \( a_1 \)-th to \( a_2 \)-th rows and the \( b_1 \)-th to \( b_2 \)-th columns of \( \Sigma_{cd} \).

**APPENDIX B**

**PROOF FOR CONDITIONS (M1)-(M5) AND (SAEM1)-(SAEM3)**

In this section we will establish the listed conditions one by one. We assume \( \Theta = \{ |\varphi_0| < \varphi_0^+, |\varphi_1| < \varphi_1^+, \sigma > \sigma^-, \nu^+ > \nu > \nu^- \} \) is a very large open set. The parameters \( \varphi_0^+, \varphi_1^+ \) and \( \nu^+ \) are very large positive number. The parameter \( \sigma^- \) is very small positive number. The parameter \( \nu^- \) is a small positive number that satisfies \( \nu^- > 2 \). In addition, we assume \( y_o \) is known and finite. We first prove the conditions (M1)-(M5), then prove the condition (SAEM2) and (SAEM3). The condition (SAEM1) can be easily satisfied by choosing the step sizes appropriately.

**A. Proof of (M1)-(M5)**

The proof begins by establishing the following two intermediary Lemmas.

**Lemma 4.** For any \( y_o \) and \( \Theta \) \( \in \Theta \), \( p(y_o; \Theta) = \int p(y, \tau; \Theta) \, dy \, d\tau = \int p(y; \Theta) \, dy < \infty \).

**Lemma 5.** For any \( y_o \), \( \Theta \in \Theta \) and \( 1 < t \leq T \)

\[
\int g(y, \tau) \, p(y, \tau; \Theta) \, dy \, d\tau < \infty,
\]

(60)

where \( g(y, \tau) \) can be \( \tau_1, \tau_1^2, y_1^2, \tau_1 y_1^2, \tau_2 y_2^2, \) or \( -\log(\tau) \).

Lemma 4 indicates that the observed data likelihood \( p(y_o; \Theta) \) is bounded, and Lemma 5 shows that the expectation of \( g(y, \tau) \) is bounded. These Lemmas provide the key ingredients required for establishing (M1)-(M5), and their usage for subsequent analysis is self-explanatory. Due to the space limitations, we do not include their proofs here. Interested readers may refer to the supplementary material.

**M1** For condition (M1), based on (18), we can get

\[
\int \int |s(y_o, y_m, \tau)| \, p(y_m, \tau|y_o; \Theta) \, dy_m \, d\tau = \int \int |s(y_o, y_m, \tau)| \, p(y_m, \tau; \Theta) \, dy_m \, d\tau
\]

\[
< \frac{1}{p(y_o; \Theta)} \sum_{t=2}^T \left( \int \left| \log(\tau_t) - \tau_{t-i} + |\tau_t y_{t-i}^2 + |\tau_t| + |\tau_t y_{t-i}^2| + |\tau_t y_{t-i} y_{t-i-1}| + |\tau_t y_{t-i-1}| \right) p(y_o, y_m, \tau; \Theta) \, dy_m \, d\tau
\]

\[
< \frac{1}{p(y_o; \Theta)} \sum_{t=2}^T \left( \tau_t - \log(\tau_t) + \tau_t y_{t-i}^2 + \tau_t + \tau_t y_{t-i-1} + \tau_t y_{t-i}^2 + \tau_t y_{t-i} y_{t-i-1} \right) \right) p(y_o, y_m, \tau; \Theta) \, dy_m \, d\tau
\]

\[
< \infty.
\]

(61)

where the three inequalities follow from the triangular inequality, the property of squares \( x_1 x_2 \leq \frac{x_1^2 + x_2^2}{2} \), and Lemma 5, respectively.

**M2** From the definition of \( \psi(\Theta) \) and \( \phi(\Theta) \) in (16) and (17), their continuous differentiability can be easily verified.

**M3** For condition (M3),

\[
s(\Theta) = \int \int s(y_o, y_m, \tau) \, p(y_m, \tau|y_o; \Theta) \, dy_m \, d\tau
\]

\[
= \int \int s(y_o, y_m, \tau) \, p(y_m, \tau; \Theta) \, dy_m \, d\tau
\]

\[
= \int \int s(y_o, y_m, \tau) \, p(y_m, \tau; \Theta) \, dy_m \, d\tau
\]

\[
= \int \int s(y_o, y_m, \tau) \, p(y_m, \tau; \Theta) \, dy_m \, d\tau.
\]

Since \( \int \int p(y_m, \tau; \Theta) \, dy_m \, d\tau = p(y_o; \Theta) > 0 \) and \( p(y, \tau; \Theta) \) is continuously differentiable, which can be easily checked from its definition (19), we can get that \( s(\Theta) \) is continuously differentiable.

**M4** Since \( \int \int p(y, \tau; \Theta) \, dy_m \, d\tau > 0 \), and \( p(y, \tau; \Theta) \) is 7 times differentiable, \( f(\Theta; y_o) = \log\left( \int \int p(y, \tau; \Theta) \, dy_m \, d\tau \right) \) is 7 times differentiable. For the verification of the equation (44), according to Leibniz integral rule, the equation (44) holds under the following three conditions:

1) \( \int \frac{\partial}{\partial \Theta} p(y, \tau; \Theta) \, dy \, d\tau < \infty \),
2) \( \frac{\partial}{\partial \Theta} \) exists for all \( \Theta \in \Theta \),
3) there is an integrable function \( g(y, \tau) \) such that \( \frac{\partial}{\partial \Theta} p(y, \tau; \Theta) \cdot \frac{\partial}{\partial \Theta} p(y, \tau; \Theta) \leq g(y, \tau) \) for all \( \Theta \in \Theta \) and almost every \( y \) and \( \tau \).

Since the first condition has been proved in Lemma 4, and the second condition can be easily verified from its definition, here we focus on the third condition.
The derivative of \( p(y, \tau; \theta) \) with respect to \( \varphi_0 \) is
\[
\frac{\partial p(y, \tau; \theta)}{\partial \varphi_0} = p(y, \tau; \theta) \sum_{j=2}^T \left( \tau_j (y_j - \varphi_0 - \varphi_1 y_{j-1}) \right)
\]
where \( \theta^* = \arg \max_{\theta \in \Theta} p(y, \tau; \theta) \). The first inequality follows from the triangle inequality, and the second inequality follows from \( p(y, \tau; \theta^*) \geq p(y, \tau; \theta) \), \( |\varphi_0| < \varphi_0^+ \), \( |\varphi_1| < \varphi_1^+ \), and \( \sigma > \sigma^- \).

The derivative with respect to \( \varphi_1 \) is
\[
\frac{\partial p(y, \tau; \theta)}{\partial \varphi_1} = p(y, \tau; \theta) \sum_{j=2}^T \left( \tau_j (y_j - \varphi_0 - \varphi_1 y_{j-1}) \right)
\]
where the first inequality follows from the triangle inequality, and the second inequality follows from \( p(y, \tau; \theta^*) \geq p(y, \tau; \theta) \), \( |\varphi_0| < \varphi_0^+ \), \( |\varphi_1| < \varphi_1^+ \), \( \sigma > \sigma^- \) and the property of squares.

The derivative with respect to \( \sigma^2 \) is
\[
\frac{\partial p(y, \tau; \theta)}{\partial \sigma^2} = p(y, \tau; \theta) \sum_{j=2}^T \left( \frac{\tau_j^2 (y_j - \varphi_0 - \varphi_1 y_{j-1})^2}{2 \sigma^2} \right)
\]
where \( \theta^* = \arg \max_{\theta \in \Theta} p(y, \tau; \theta) \). The first inequality follows from the triangle inequality, and the second inequality follows from \( p(y, \tau; \theta^*) \geq p(y, \tau; \theta) \), \( |\varphi_0| < \varphi_0^+ \), \( |\varphi_1| < \varphi_1^+ \), \( \sigma > \sigma^- \) and the property of squares.

The derivative with respect to \( \nu \) is
\[
\frac{\partial p(y, \tau; \theta)}{\partial \nu} = p(y, \tau; \theta) \sum_{j=2}^T \left( \frac{1}{2} + \frac{1}{2} \log \left( \frac{\nu^-}{2} \right) - \frac{1}{2} \Psi \left( \frac{\nu^-}{2} \right) \right)
\]
where \( \Psi(\cdot) \) is the digamma function. The first inequality follows from the triangle inequality, and the second inequality is due to that \( \log \left( \frac{\nu^-}{2} \right) - \Psi \left( \frac{\nu^-}{2} \right) \) is positive and strictly decreasing for \( \nu \geq \nu^- \).

Based on Lemmas 4 and 5, we can obtain that
\[
\int g_{\varphi_0}(y, \tau) \, dy \, d\tau < \infty, \quad \int g_{\varphi_1}(y, \tau) \, dy \, d\tau < \infty, \quad \int g_{\sigma^2}(y, \tau) \, dy \, d\tau < \infty, \quad \text{and} \quad \int g_{\nu}(y, \tau) \, dy \, d\tau < \infty.
\]

The condition M4 is verified.

(M5) This condition requires the existence of the global maximizer \( \theta(\bar{s}) \) for \( Q(\theta, \bar{s}) \) and its continuous differentiability. Since \( Q(\theta, \bar{s}) \) takes the same form with \( Q(\theta, s^{(k)}) \), the maximizer will also take the same form. From (32)-(35), we have
\[
\hat{\varphi}_0(\bar{s}) = \frac{s_5 - \varphi_1(\bar{s}) \bar{s}_7}{s_3},
\]
\[
\hat{\varphi}_1(\bar{s}) = \frac{s_3 s_8 - s_5 s_7}{s_3 s_4 - s_7^2},
\]
\[
(\hat{\sigma}(\bar{s}))^2 = \frac{1}{T^{-1}} \left( \bar{s}_2 + (\varphi_0(\bar{s}))^2 \bar{s}_3 + (\varphi_1(\bar{s}))^2 \bar{s}_4 - 2 \varphi_0(\bar{s}) \bar{s}_5 \right.
\]
and
\[
\hat{\nu}(\bar{s}) = \arg \max_{\nu^- < \nu < \nu^+} f(\nu, \bar{s}_1),
\]
where \( \bar{s}_i \ (i = 1, \ldots, 7) \) is the \( i \)-th component of \( \bar{s} \). It can be easily verified that \( \hat{\varphi}_0(\bar{s}), \hat{\varphi}_1(\bar{s}) \) and \( (\hat{\sigma}(\bar{s}))^2 \) are continuous
functions of \( \tilde{s} \), and are 7 times differentiable with respect to \( \tilde{s} \). For \( \nabla (\tilde{s}) \), the gradient of \( f (\nu, \tilde{s}_1) \) at \( \nabla \)
\[
g (\nabla, \tilde{s}_1) = \frac{\partial f (\nu, \tilde{s}_1)}{\partial \nu} \bigg|_{\nu=\nabla} = \frac{1}{2} \left( \log \left( \frac{\nu}{2} \right) - \Psi \left( \frac{\nu}{2} \right) + 1 + \frac{\tilde{s}_1}{T - 1} \right)
\]
\[= 0.
\]
According to the implicit function theorem, since \( g (\nabla, \tilde{s}_1) \) is 7 times continuously differentiable and \( \frac{\partial g (\nabla, \tilde{s}_1)}{\partial \nabla} = \frac{1}{T} \left( \frac{1}{2} - \frac{1}{T} \Psi \left( \frac{\nu}{2} \right) \right) \neq 0 \) for any \( \nabla \) and \( \tilde{s}_1 \), \( \nabla (s) \) is 7 times continuously differentiable with respect to \( \tilde{s} \).

**B. Proof of (SAEM2) and (SAEM3)**

The condition (SAEM2) has been verified in the proof of conditions (M4) and (M5). The condition (SAEM3.1) and (SAEM3.2) requires the compactness of the support of the conditional distribution \( p (y, m | \tau, \theta) \). In our case, the missing variables \( y_m \) and \( \tau \) can theoretically take their values over an infinite set. But from a practical point of view, this assumption is not a restriction, since in the practice, any variable takes its values in a (very large) compact set \( \Omega \) \[27\]. Here we focus on the proof of the conditions (SAEM3.3) and (SAEM3.4).

Let \( V \) be a compact subset of \( \Theta \). For the definition of the transition probability \( \Pi_\theta (y_m, \tau, y_m', \tau') \) in \( 24 \), we can easily verify that the transition probability \( \Pi_\theta (y_m, \tau, y_m', \tau') \) is continuously differentiable with respect to \( \theta \) for any \( \theta \in V \) and \( (y_m, \tau, y_m', \tau') \in \Omega^2 \), and its derivative is bounded. Therefore, \( \Pi_\theta (y_m, \tau, y_m', \tau') \) is Lipschitz continuous, i.e., for any \( (y_m, \tau, y_m', \tau') \in \Omega^2 \), there exists a constant \( K (y_m, \tau, y_m', \tau') \) such that for any \( (\theta, \theta') \in V^2 \),
\[
\Pi_\theta (y_m, \tau, y_m', \tau') - \Pi_{\theta'} (y_m, \tau, y_m', \tau') \leq K (y_m, \tau, y_m', \tau') |\theta - \theta'|.
\]

It follows that
\[
\sup_{(y_m, \tau, y_m', \tau') \in \Omega^2} \left| \Pi_\theta (y_m, \tau, y_m', \tau') - \Pi_{\theta'} (y_m, \tau, y_m', \tau') \right| \leq L |\theta - \theta'|.
\]

with \( L = \max_{(y_m, \tau, y_m', \tau') \in \Omega^2} K (y_m, \tau, y_m', \tau') \), which implies that the condition (SAEM3.3) is verified.

According to Theorem 8 in \[37\], a Markov chain is uniformly ergodic, if the transition probability satisfies some minorization condition, i.e., there exists \( \alpha \in N^+ \) and some probability measure \( \delta (\cdot) \) such that \( \Pi_\theta (y_m, \tau, y_m', \tau') \geq \epsilon \delta (y_m, \tau') \) for any \( (y_m, \tau, y_m', \tau') \in \Omega^2 \). Recall our transition probability \( \Pi_\theta (y_m, \tau, y_m, \tau') \) is a continuous function for \( (y_m, \tau) \in \Omega \), according to the extreme value theorem, there must exist an infimum \( g (y_m, \tau, \theta) = \inf_{(y_m, \tau) \in \Omega} \Pi_\theta (y_m, \tau, y_m', \tau') \). It follows that
\[
\Pi_\theta (y_m, \tau, y_m', \tau') \geq \epsilon \delta (y_m, \tau') \geq \epsilon g (y_m, \tau, \theta)
\]

with \( \epsilon = \int g (y_m, \tau, \theta) \, d\tau \) and \( \delta (y_m, \tau') = \epsilon^{-1} g (y_m, \tau', \theta) \). Therefore, the minorization condition holds in our case, and thus, condition (SAEM3.4) is verified.
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Supplementary Material

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In this supplementary material, we give the detailed proof for the Lemmas 4 and 5 in the submitted paper [1]:

**Lemma 4** For any \( y_0 \) and \( \theta \in \Theta \),
\[
p(y_0; \theta) = \int \int p(y; \tau; \theta) \, dy_m \, d\tau = \int p(y; \theta) \, dy_m < \infty.
\]

**Lemma 5** For any \( y_0 \), \( \theta \in \Theta \) and \( 1 < t \leq T \)
\[
\int \int g(y, \tau) \, p(y, \tau; \theta) \, dy_m \, d\tau < \infty,
\]
where \( g(y, \tau) \) can be \( \tau_1 \), \( \tau_2 \), \( y_1^2 \tau_1 \), \( \tau_1 y_1^2 \), or \( -\log(\tau_1) \).

To establish these lemmas, we first introduce some equations and inequalities in the first section. They are the key ingredients for the proof. Then we establish Lemma 4 and Lemma 5 in the second and third sections, respectively. For the simplicity of notations, we use \( f_t(y_j; y_{j-1}) \) to denote \( f_t(y_j; \varphi_0 + \varphi_1 y_{j-1}, \sigma^2, \nu) \), \( f_N(y_j; y_{j-1}, \tau_j) \) to denote \( f_N(y_j; \varphi_0 + \varphi_1 y_{j-1}, \sigma^2, \nu) \), and \( f_g(\tau_j) \) to denote \( f_g(\tau_j; \nu_2, \nu_2) \).

1. **Ingredients**

Recall that, given \( \varphi_0 \), \( \varphi_1 \), \( \sigma^2 \), \( \nu \) and \( y_{j-1} \), the variable \( y_j \) follows a Student’s \( t \)-distribution: \( y_j \sim t(\varphi_0 + \varphi_1 y_{j-1}, \sigma^2, \nu) \). Based on the property of the Student’s \( t \)-distribution, we can get the following equations and inequality about the variable \( y_j \) [2]:

1. The integral of the pdf should be 1:
\[
\int f_t(y_j; y_{j-1}) \, dy_j = 1.
\]

2. The first raw moment can be expressed as
\[
\int y_j f_t(y_j; y_{j-1}) \, dy_j = \varphi_0 + \varphi_1 y_{j-1}.
\]

3. The second raw moment can be expressed as
\[
\int y_j^2 f_t(y_j; y_{j-1}) \, dy_j = \frac{\nu \sigma^2}{\nu - 2} + (\varphi_0 + \varphi_1 y_{j-1})^2.
\]

4. Since the Student’s \( t \)-distribution can be represented as a Gaussian mixture [3], the pdf can be rewritten as
\[
f_t(y_j; y_{j-1}) = \int f_g(\tau_j) f_N(y_j; y_{j-1}, \tau_j) \, d\tau_j.
\]
5. For $\nu > \nu^- \geq 2$, the pdf of $y_j$ can be bounded as

$$f_i(y_j; y_{j-1}) = \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma^2} \Gamma \left( \frac{\nu}{2} \right)} \left( 1 + \frac{(y_j - \varphi_i - \varphi_1 y_{j-1})^2}{\nu \sigma^2} \right)^{-\frac{\nu+1}{2}} \leq \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma^2} \Gamma \left( \frac{\nu}{2} \right)} < \infty. \quad (6)$$

Then we introduce two important inequalities about $\tau_j$, which we will use later.

1. The first is about the expectation of $\tau_j^b$ with $b = 1, 2$:

$$\int \tau_j^b f_g (\tau_j) f_N (y_j; y_{j-1}, \tau_j) \, d\tau_j = \int \frac{(\nu)^{\frac{b}{2}}}{\Gamma \left( \frac{\nu}{2} \right) \sqrt{2\pi\sigma^2}} \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\Gamma \left( \frac{\nu}{2} \right)} \exp \left( - \frac{(y_j - \varphi_0 - \varphi_1 y_{j-1})^2}{2\sigma^2} + \frac{\nu}{2} \right) \tau_j \, d\tau_j \quad (7a)$$

$$= \int \frac{(\nu)^{\frac{b+1}{2}}}{\Gamma \left( \frac{\nu+1}{2} \right) \sqrt{2\pi\sigma^2}} \frac{\Gamma \left( \frac{\nu+2b}{2} \right)}{\Gamma \left( \frac{\nu}{2} \right)} \left( \frac{y_j - \varphi_0 - \varphi_1 y_{j-1}}{2\sigma^2} + \frac{\nu}{2} \right)^{\frac{\nu+2b}{2}} \, d\tau_j \quad (7b)$$

$$= f_i (y_j; y_{j-1}) \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\Gamma \left( \frac{\nu}{2} \right)} \frac{\Gamma \left( \frac{\nu+2b}{2} \right)}{\Gamma \left( \frac{\nu}{2} \right)} \left( \frac{(y_j - \varphi_0 - \varphi_1 y_{j-1})^2 + \nu}{2} \right)^b \quad (7c)$$

$$\leq f_i (y_j; y_{j-1}) \frac{\Gamma \left( \frac{\nu+2b+1}{2} \right)}{\nu^b \Gamma \left( \frac{\nu+1}{2} \right)} \frac{\Gamma \left( \frac{\nu+2b+1}{2} \right)}{\Gamma \left( \frac{\nu}{2} \right)} \left( \frac{(y_j - \varphi_0 - \varphi_1 y_{j-1})^2 + \nu}{2} \right)^b \quad (7d)$$

where the equations (7a) and (7d) follow from the definition of these pdf’s, the equation (7b) follows from $\int x^{\alpha-1} \exp(-\beta x) \, dx = 1$ (the integral of the pdf of the gamma distribution is 1), the last inequality (7d) follows from $\left( \frac{(y_j - \varphi_0 - \varphi_1 y_{j-1})^2 + \nu}{2} \right)^b \geq \left( \frac{\nu}{2} \right)^b$.

2. The second inequality is about the expectation of $\log (\tau_j)$,

$$\int \log (\tau_j) f_g (\tau_j) f_N (y_j; y_{j-1}, \tau_j) \, d\tau_j = \int \frac{(\nu)^{\frac{1}{2}}}{\Gamma \left( \frac{\nu}{2} \right) \sqrt{2\pi\sigma^2}} \log (\tau_j) \tau_j^{\frac{\nu-1}{2}} \exp \left( - \frac{(y_j - \varphi_0 - \varphi_1 y_{j-1})^2}{2\sigma^2} + \frac{\nu}{2} \right) \, d\tau_j \quad (8a)$$

$$= \int \frac{(\nu)^{\frac{1}{2}}}{\Gamma \left( \frac{\nu}{2} \right) \sqrt{2\pi\sigma^2}} \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\Gamma \left( \frac{\nu}{2} \right)} \left( \log \left( \frac{(y_j - \varphi_0 - \varphi_1 y_{j-1})^2}{2\sigma^2} + \frac{\nu}{2} \right) \right) f_i (y_j; y_{j-1}) \quad (8b)$$

$$\geq \left( \log \left( \frac{(y_j - \varphi_0 - \varphi_1 y_{j-1})^2}{2\sigma^2} + \frac{\nu}{2} \right) \right) f_i (y_j; y_{j-1}) \quad (8c)$$

$$\geq \left( \log \left( \frac{(y_j - \varphi_0 - \varphi_1 y_{j-1})^2}{2\sigma^2} + \frac{\nu}{2} \right) \right) f_i (y_j; y_{j-1}) \quad (8d)$$

$$\geq \left( \frac{\nu}{2} \right) - \frac{(y_j - \varphi_0 - \varphi_1 y_{j-1})^2}{2\sigma^2} \quad (8e)$$

$$\geq \left( \frac{\nu}{2} \right) - \frac{(y_j - \varphi_0 - \varphi_1 y_{j-1})^2}{2\sigma^2} \quad (8f)$$

where the equations (8a) and (8e) follow from the definition of these pdf’s, the equation (8b) follows from $\int \log (x) x^{\alpha-1} \exp(-\beta x) \, dx = \frac{\Gamma(\alpha)}{\beta^\alpha} (\Psi(\alpha) - \log (\beta)) \, [4]$, the inequality (8d) follows from $-\log (x) \geq -x$, the inequalities (8c) and (8e) follow from $(x_1 + x_2)^2 \leq 2x_1^2 + 2x_2^2$. 

2
2 Proof for Lemma 4

Lemma 4 is about the boundedness of the marginal pdf \( p (y_o; \theta) \). The pdf can be written as

\[
p (y_o; \theta) = \int p (y; \theta) \, dy_m
\]

\[
= \int \prod_{j=2}^{T} f_t (y_j; y_{j-1}) \, dy_m
\]

\[
= \left( \prod_{d=0}^{D} \prod_{j=t_d+n_d+2}^{t_d+n_d+1} f_t (y_j; y_{j-1}) \right) \left( \prod_{d=1}^{D} \int \prod_{j=t_d+1}^{t_d+n_d+1} f_t (y_j; y_{j-1}) \, dy_d \right),
\]

where we move the term that does not involve \( y_m \) outside the integral.

Since the pdf of the Student’s t-distribution \( f_t (y_j; y_{j-1}) \) is positive and bounded, we can get the first term \( c_1 (y_o, \theta) = \prod_{d=0}^{D} \prod_{j=t_d+n_d+2}^{t_d+n_d+1} f_t (y_j; y_{j-1}) < \infty \). Thus, in order to establish the Lemma 4, it is sufficient to establish the boundedness of the second term, i.e,

\[
\int \prod_{j=t_d+1}^{t_d+n_d+1} f_t (y_j; y_{j-1}, \sigma^2, \nu) \, dy_d < \infty.
\]

Before carrying out the general proof for the above inequality (10), we demonstrate the schematic and intuition with a simple example. We consider a time series as follows: \( y_1, y_2, NA, NA, NA, y_6, y_7 \). The corresponding second term (10) in this example can be expressed as

\[
\int \prod_{j=3}^{6} f_t (y_j; y_{j-1}) \, dy_1
\]

\[
= \left\{ \int \left( \int f_t (y_6; y_5)f_t (y_5; y_4) \, dy_5 \right) f_t (y_4; y_3) \, dy_4 \right\} f_t (y_3; y_2) \, dy_3
\]

\[
\leq \left\{ \int \left( \int \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma_1 \Gamma \left( \frac{\nu}{2} \right)}} f_t (y_5; y_4) \, dy_5 \right) f_t (y_4; y_3) \, dy_4 \right\} f_t (y_3; y_2) \, dy_3
\]

\[
= \left\{ \int \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma_1 \Gamma \left( \frac{\nu}{2} \right)}} f_t (y_4; y_3) \, dy_4 \right\} f_t (y_3; y_2) \, dy_3
\]

\[
= \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma_1 \Gamma \left( \frac{\nu}{2} \right)}} f_t (y_3; y_2) \, dy_3
\]

\[
< \infty,
\]

where the inequality (11b) follows from (6), the equations (11c)-(11e) hold from (2), and the inequality (11f) follows from the extreme value theorem. By applying (6), we find a upper bound \( \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma_1 \Gamma \left( \frac{\nu}{2} \right)}} \) for \( f_t (y_6; y_5) \), which does not involve \( y_5 \). Then the integral about \( y_5 \), \( \int \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma_1 \Gamma \left( \frac{\nu}{2} \right)}} f_t (y_5; y_4) \, dy_5 \), is easy to compute since \( \int f_t (y_5; y_4) \, dy_5 = 1 \) from (2), and the result is a function \( \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma_1 \Gamma \left( \frac{\nu}{2} \right)}} \), which does not involve \( y_4 \). Next, the integral about \( y_4 \), \( \int \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma_1 \Gamma \left( \frac{\nu}{2} \right)}} f_t (y_4; y_3) \, dy_4 \), also becomes simple, and the result does not involve \( y_3 \). Finally, we can get that the integral about \( y_3 \) equals to a continuous function \( \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma_1 \Gamma \left( \frac{\nu}{2} \right)}} \). According to the extreme value theorem, a continuous function on a closed bounded set is bounded \([5]\), therefore, \( \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu \pi \sigma_1 \Gamma \left( \frac{\nu}{2} \right)}} \) is bounded.
Lemma 5 is about the boundedness of the expectation of $g(y, \tau)$. For the convenience of proof, we divide the different cases of $g(y, \tau)$ into four groups: (1) $g(y, \tau) = \tau_1$ or $\tau_2^2$, (2) $g(y, \tau) = y_1^2$, (3) $g(y, \tau) = \tau_1 y_1^2 - 1$ or $\tau_1 y_2^2$, and (4) $-\log(\tau_1)$. We will prove that the inequality (11) is satisfied for these groups one by one.

### 3.1 $g(y, \tau) = \tau_1$ or $\tau_2^2$

For $g(y, \tau) = \tau_b$ with $b = 1, 2$, we have

\[
\begin{align*}
\int \int g(y, \tau) p(y, \tau; \theta) \, dy_m d\tau &= \int \int \tau_b^b p(\tau; \theta) p(y|\tau; \theta) \, dy_m d\tau \\
&= \int \int \tau_b^b \prod_{j=2}^{\tau} \left\{ f_{\theta}(\tau_j) \right\} \left\{ f_N(y_j; y_{j-1}, \tau_j) \right\} \, dy_m d\tau \\
&= \int \int \tau_b^b f_{\theta}(\tau_1) \left\{ f_N(y_1; y_{1-1}, \tau_1) \right\} \prod_{j=2}^{\tau} \left\{ \int f_{\theta}(\tau_j) \left\{ f_N(y_j; y_{j-1}, \tau_j) \right\} \, d\tau_j \right\} \, dy_m d\tau \\
&\leq \frac{2\Gamma\left(\frac{\nu+2b+1}{2}\right)}{\nu\Gamma\left(\frac{\nu+1}{2}\right)} \int f_{\theta}(y_1; y_{1-1}) \prod_{j=2}^{\tau} \left\{ f_{\theta}(y_{j-1}) \right\} \, dy_m \\
&= \frac{2\Gamma\left(\frac{\nu+2b+1}{2}\right)}{\nu\Gamma\left(\frac{\nu+1}{2}\right)} p(y_0; \theta) \\
&< \infty.
\end{align*}
\]

In (13a), we split the integral of $\{\tau_j\}$ into two parts: the first part involves $\tau_1$, while the second does not. In (13b), we apply the inequality (2d) to the first part of (13c). The inequality (13d) follows from the Lemma 4 and the extreme value theorem.
3.2 \( g(y, \tau) = y_t^2 \)

For \( g(y, \tau) = y_t^2 \), we need to consider two different cases: \( y_t \) is observed, and \( y_t \) is missing.

If \( y_t \) is observed, then we can easily get

\[
\int \int g(y, \tau) p(y, \tau; \theta) \, dy \, d\tau = \int \int y_t^2 p(y, \tau; \theta) \, dy \, d\tau
\]

\[
= \int y_t^2 \{ \int p(y, \tau; \theta) \, d\tau \} \, dy
\]

\[
= y_t^2 \int p(y; \theta) \, dy
\]

\[
< \infty.
\tag{14}
\]

where the last inequality holds from Lemma 4 and the fact that \( y_m \) is finite.

If \( y_t \) is missing, assume that \( y_t \) is in the \( d_1 \)-th missing block with \( t = t_{d_1} + i \), we have

\[
\int \int g(y, \tau) p(y, \tau; \theta) \, dy \, d\tau = \int \int y_t^2 p(y, \tau; \theta) \, dy \, d\tau
\]

\[
= \int y_t^2 \{ \int p(y, \tau; \theta) \, d\tau \} \, dy
\]

\[
= \int y_t^2 p(y; \theta) \, dy
\]

\[
\tag{15a}
\]

\[
= \prod_{d=0}^{D} \prod_{j=t_{d}+n_d+2}^{t_{d}+n_d+2} f_t(j; y_{j-1}) \int y_t^2 \prod_{d=1}^{t_{d}+n_d+1} \prod_{j=t_{d}+1}^{t_{d}+n_d+1} f_t(j; y_{j-1}) \, dy_d
\]

\[
= \prod_{d=0}^{D} \prod_{j=t_{d}+n_d+2}^{t_{d}+n_d+2} f_t(j; y_{j-1}) \left\{ \prod_{d \neq d_1} \int \prod_{j=t_{d}+1}^{t_{d}+n_d+1} f_t(j; y_{j-1}) \, dy_d \right\} \left\{ \int y_t^2 \prod_{j=t_{d_1}+1}^{t_{d_1}+n_d+1} f_t(j; y_{j-1}) \, dy_d \right\}.
\tag{15b}
\]

In (15d), we move the term that does not involve \( y_m = \{y_d\} \) outside the integral. In (15c), we split the integral of \( \{y_d\} \) into two parts: the first part does not involve \( y_{t_{d_1}+i} \), while the second part does.

Since the pdf \( f_t(j; y_{j-1}) \) is bounded from (6), the first term of (15a) is bounded:

\[
\prod_{d=0}^{D} \prod_{j=t_{d}+n_d+2}^{t_{d}+n_d+2} f_t(j; y_{j-1}) < \infty.
\tag{16}
\]

In addition, from (12d), the second term is also bounded:

\[
\prod_{d \neq d_1} \int \prod_{j=t_{d}+1}^{t_{d}+n_d+1} f_t(j; y_{j-1}) \, dy_d < \infty.
\tag{17}
\]

Thus, in order to prove that (15c) is bounded, it is sufficient to establish the boundedness of the third term, i.e.,

\[
\int y_t^2 \prod_{j=t_{d_1}+1}^{t_{d_1}+n_d+1} f_t(j; y_{j-1}) \, dy_d < \infty.
\tag{18}
\]

Before carrying out the general proof for the inequality (18), we demonstrate the schematic and intuition with a simple example. Again, we consider the time series as follows: \( y_1, y_2, NA, NA, NA, y_6, y_7 \).
Then we have
\[
\int y_1^2 \prod_{j=3}^{6} f_t(y_j; y_{j-1}) \, dy_1 = \left\{ \left\{ \int f_t(y_6; y_5) \, dy_5 \right\} \int f_t(y_5; y_4) \, dy_4 \right\} \int f_t(y_4; y_3) \, dy_3
\]
\[
\leq \left\{ \left\{ \int \left( \frac{\Gamma(\nu+1)}{\sqrt{\nu\pi\sigma} \Gamma(\frac{\nu}{2})} f_t(y_5; y_4) \right) \, dy_4 \right\} \int f_t(y_4; y_3) \, dy_3 \right\} \int f_t(y_3; y_2) \, dy_3
\]
\[
= \left\{ \left\{ \int \left( \frac{\Gamma(\nu+1)}{\sqrt{\nu\pi\sigma} \Gamma(\frac{\nu}{2})} \right) \, dy_4 \right\} \int f_t(y_4; y_3) \, dy_3 \right\} \int f_t(y_3; y_2) \, dy_3
\]
\[
= \left\{ \left\{ \int \left( \frac{\Gamma(\nu+1)}{\sqrt{\nu\pi\sigma} \Gamma(\frac{\nu}{2})} \right) \, dy_4 \right\} \int f_t(y_4; y_3) \, dy_3 \right\} \int f_t(y_3; y_2) \, dy_3
\]
\[
= \frac{\Gamma(\nu+1)}{\sqrt{\nu\pi\sigma} \Gamma(\frac{\nu}{2})} \left( \int f_t(y_3; y_2) \, dy_3 \right)
\]
\[
< \infty,
\]

where the inequality (19b) follows from (6), the equations (19c)-(19g) follow from (8)-(11), and the inequality (19h) holds due to the extreme value theorem. By applying (13), we find an upper bound \(\frac{\Gamma(\nu+1)}{\sqrt{\nu\pi\sigma} \Gamma(\frac{\nu}{2})}\) for \(f_t(y_6; y_5)\), which does not involve \(y_5\), so that the integral about \(y_5\), \(\int \frac{\Gamma(\nu+1)}{\sqrt{\nu\pi\sigma} \Gamma(\frac{\nu}{2})} f_t(y_5; y_4) \, dy_5\), is simple and equals to \(\frac{\Gamma(\nu+1)}{\sqrt{\nu\pi\sigma} \Gamma(\frac{\nu}{2})}\), which does not involve \(y_4\). Then the following integrals about \(y_4\) and \(y_3\) are also simple and the final result is bounded.

Now we come to the general proof for the inequality (18). The idea is the same as in the above example:
the case of $g$ follows from (4). Similar to the example, this integral of (20f) will finally reduce to a quadratic function where the inequality (20b) follows from (6), the equations (20d) and (20e) hold from (2), and the equation follows from (22d). The inequality (22f) follows from the result of last subsection. Similar to the example, this integral of (20f) will finally reduce to a quadratic function of $y_{d_1}$. Then, according to the extreme value theorem, we can obtain

$$
\int \cdots \int \Gamma \left( \frac{\nu+1}{2} \right) \frac{\nu \sigma^2}{\nu-2} \left( \frac{\varphi_0 + \varphi_1 y_{d_1} + i^{-1}}{2} \right) dy_{d_1} + \cdots f_t \left( y_{d_1} + i; y_{d_1} \right) dy_{d_1} + 1 (20e)
$$

where the inequality (20b) follows from (6), the equations (20d) and (20e) hold from (2), and the equation follows from (22d). The inequality (22f) follows from the result of last subsection. Similar to the example, this integral of (20f) will finally reduce to a quadratic function of $y_{d_1}$. Then, according to the extreme value theorem, we can obtain

$$
\int y_{d_1}^{\nu} \prod_{j=t_{d_1}+1}^{t_{d_1} + n_{d_1} + 1} f_t \left( y_j; y_j \right) dy_{d_1} < \infty, \quad (21)
$$

and thus, $\int y_{d_1}^{\nu} p(y, \tau, \theta) dy_{d_1} < \infty.$

### 3.3 $g(y, \tau) = \tau_t y_{t-1}^2$ or $\tau_t y_t^2$

In this subsection, we consider the cases of $g(y, \tau) = \tau_t y_{t-1}^2$ or $\tau_t y_{t-1}^2$. Here we only present the proof for the case of $g(y, \tau) = \tau_t y_t^2$, the case $\tau_t y_{t-1}^2$ can be verified similarly. For $g(y, \tau) = \tau_t y_t^2$, we have

$$
\int \cdots \int \tau_t y_t^2 p(y, \tau, \theta) dy_{m} d\tau
$$

$$
\int \cdots \int \tau_t y_t^2 \prod_{j=2}^{T} \left\{ f_N \left( y_j; y_{j-1}, \tau_t \right) f_g \left( \tau_t \right) \right\} dy_{m} d\tau
$$

$$
\int y_t^2 \left\{ \int f_N \left( y_t; y_{t-1}, \tau_t \right) f_g \left( \tau_t \right) d\tau_t \prod_{j \neq t} \left( \int f_N \left( y_j; y_{j-1}, \tau_j \right) f_g \left( \tau_j \right) d\tau_j \right) \right\} dy_{m}
$$

$$
\leq \int y_t^2 \frac{2\Gamma \left( \frac{\nu+1}{2} \right)}{\nu \Gamma \left( \frac{\nu+1}{2} \right)} f_t \left( y_t; y_{t-1} \right) \prod_{j \neq t} f_t \left( y_j; y_{j-1} \right) dy_{m}
$$

$$
= \frac{2\Gamma \left( \frac{\nu+1}{2} \right)}{\nu \Gamma \left( \frac{\nu+1}{2} \right)} \int y_t^2 p(y, \tau, \theta) dy_{m}
$$

$$
< \infty.
$$

In (22c), we split the integral of $\tau = \{ \tau_j \}$ into two parts: the first part involves $\tau_t$, while the second does not. The inequality (22a) holds from (7d). The inequality (22b) follows from the result of last subsection and the extreme value theorem.

### 3.4 $- \log(\tau_t)$

Finally, we consider the case of $g(y, \tau) = - \log(\tau_t)$:

$$
\int \cdots \int g(y, \tau) p(y, \tau, \theta) dy_{m} d\tau
$$

$$
= - \int \cdots \int \log(\tau_t) p(y, \tau, \theta) dy_{m} d\tau
$$

$$
= - \int \log(\tau_t) \prod_{j=2}^{T} \left\{ f_N \left( y_j; y_{j-1}, \tau_t \right) f_g \left( \tau_t \right) \right\} dy_{m} d\tau
$$

$$
= - \int \left\{ \int \log(\tau_t) f_N \left( y_t; y_{t-1}, \tau_t \right) f_g \left( \tau_t \right) d\tau_t \prod_{j \neq t} \left( \int f_N \left( y_j; y_{j-1}, \tau_j \right) f_g \left( \tau_j \right) d\tau_j \right) \right\} dy_{m}
$$

$$
= - \int \left\{ \int \log(\tau_t) f_N \left( y_t; y_{t-1}, \tau_t \right) f_g \left( \tau_t \right) d\tau_t \prod_{j \neq t} \left( \int f_N \left( y_j; y_{j-1}, \tau_j \right) f_g \left( \tau_j \right) d\tau_j \right) \right\} dy_{m}
$$

$$
= - \int \left\{ \int \log(\tau_t) f_N \left( y_t; y_{t-1}, \tau_t \right) f_g \left( \tau_t \right) d\tau_t \prod_{j \neq t} \left( \int f_N \left( y_j; y_{j-1}, \tau_j \right) f_g \left( \tau_j \right) d\tau_j \right) \right\} dy_{m}
$$
\[
\begin{align*}
&\leq - \int \left( \frac{\nu + 1}{2} - \frac{2\varphi_0^2 + \varphi_1^2 y_t^2}{\sigma^2} - \frac{\nu}{2} \right) f_t(y_t; y_{t-1}) \prod_{j \neq t} f_t(y_j; y_{j-1}) \, dy_m \\
&= \int \left( \frac{2\varphi_0^2 + \varphi_1^2 y_t^2}{\sigma^2} + \frac{\nu}{2} - \frac{\nu + 1}{2} \right) p(y_o, y_m; \theta) \, dy_m \\
&= \frac{2}{\sigma^2} \int y_t^2 p(y_o, y_m; \theta) \, dy_m + \frac{\varphi_1^2}{\sigma^2} \int y_{t-1}^2 p(y_o, y_m; \theta) \, dy_m + \left( \frac{2\varphi_0^2}{\sigma^2} + \frac{\nu}{2} - \frac{\nu + 1}{2} \right) p(y_o; \theta)
\end{align*}
\]

In (23c), we split the integral of \( \tau = \{\tau_j\} \) into two parts: the first part involves \( \tau_t \), while the second does not. The inequality (23d) holds from (7d). The inequality (23f) is from the result of last subsection, Lemma 4 and the extreme value theorem.

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