$L^2$-harmonic $p$-forms on submanifolds with finite total curvature

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Abstract

Let $H^p(L^2(M))$ be the space of all $L^2$-harmonic $p$-forms ($2 \leq p \leq n - 2$) on complete submanifolds $M$ with flat normal bundle in spheres. In this paper, we first show that $H^p(L^2(M))$ is trivial if the total curvature of $M$ is less than a positive constant depending only on $n$. Second, we show that the dimension of $H^p(L^2(M))$ is finite if the total curvature of $M$ is finite. The vanishing theorem is a generalized version of Gan-Zhu-Fang theorem and the finiteness theorem is an extension of Zhu-Fang theorem.

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1 Introduction

$L^2$-harmonic forms on submanifolds in various ambient spaces had been studied extensively during past few years. Many results demonstrated the fact that there is a close relation between the topology of the submanifold and the total curvature by using theory of $L^2$-harmonic forms. In [5, 6], it was showed that a complete minimal hypersurface $M^m$ ($m \geq 3$) with sufficient small the total scalar curvature in $R^{n+1}$ has only one end. In 2008, Seo [7] improved the upper bound of the total scalar curvature which was given by Ni [6]. Later Seo [8] proved that if an $n$-dimensional complete minimal submanifold $M$ in hyperbolic space has sufficiently small total scalar curvature, then $M$ has only one end. It is well-known that Euclidean space and hyperbolic space are space forms all. Fu and Xu [9] studied $L^2$-harmonic 1-forms on complete submanifolds in space forms and proved that a complete submanifold $M^n$ ($n \geq 3$) with finite total curvature and some conditions on mean curvature must have finitely many ends. Furthermore, Cavalcante, Mirandola and Vitório [10] obtained that if $M^n$ ($n \geq 3$) is a complete noncompact submanifold in Cartan-Hadamard manifold with finite total curvature and the first eigenvalue of the Laplacian of $M^n$ is bounded from below by a suitable constant, then the space of the $L^2$-harmonic 1-forms on $M^n$ has finite dimension. Zhu and Fang [11] investigated complete noncompact submanifolds in a sphere and obtained a result which was an improvement of Fu-Xu theorem on submanifolds.

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in spheres. Meanwhile, Zhu-Fang result was a generalized version of Cavalcante, Mirandola and Vitório’s result on submanifolds in Hadamard manifolds. The following theorem A is Zhu-Fang result.

**Theorem A** ([11]) Let $M^n (n \geq 3)$ be an $n$-dimensional complete noncompact oriented manifold isometrically immersed in an $(n + p)$-dimensional sphere $S^{n+p}$. If the total curvature is finite, then the dimension of $H^1 (L^2 (M))$ is finite and there are finitely many non-parabolic ends on $M$.

In 2015, Lin [14] studied $L^2$-harmonic $p$-forms on complete submanifolds in Euclidean space and proved that if a complete submanifold $M^n (n \geq 3)$ with flat normal bundle in $\mathbb{R}^{n+p}$ has sufficient small the total curvature, then the space of the $L^2$-harmonic $p$-forms on $M^n$ is trivial. Recently, Gan, Zhu and Fang [20] studied $L^2$-harmonic 2-forms on complete noncompact minimal hypersurface in spheres and proved the following result.

**Theorem B** ([20]) Let $M^n (n \geq 3)$ be an $n$-dimensional complete noncompact minimal hypersurface isometrically immersed in an $(n + 1)$-dimensional sphere $S^{n+1}$. There exists a positive constant $\delta(n)$ depending only on $n$ such that if the total curvature is less than $\delta(n)$, then the second space of reduced $L^2$ cohomology of $M$ is trivial.

Inspired by Li-Wang work [4] and the above results, in this paper, we study the space of $L^2$-harmonic $p$-forms on submanifold in spheres and prove the following vanishing and finiteness theorems.

**Theorem 1.1** Let $M$ be an $n$-dimensional $(n \geq 4)$ complete noncompact submanifold with flat normal bundle in sphere $S^{n+l}$. There exists a positive constant $c(n)$ depending only on $n$ such that if the total curvature is less than $c(n)$, then $H^p (L^2 (M)) = \{0\}$, $2 \leq p \leq n - 2$, where constant $c(n)$ is given by (3.8).

**Theorem 1.2** Let $M$ be an $n$-dimensional $(n \geq 4)$ complete noncompact submanifold with flat normal bundle in sphere $S^{n+l}$. If the total curvature is finite and $2 \leq p \leq n - 2$, then the dimension of $H^p (L^2 (M))$ is finite.

**Remark 1.1.** Theorem 1.1 is a generalization of Theorem B. On the other hand, harmonic $p$-forms $2 \leq p \leq n - 2$ are studied in Theorem 1.2 which is an extension of Theorem A. It is interesting to ask whether there are finitely many non-parabolic ends on $M$ in Theorem 1.2.

## 2 Preliminaries

Suppose $M$ is an $n$-dimensional complete submanifold immersed in an $n + l$ dimensional sphere $S^{n+l}$, $A$ is the second fundamental form and $H$ is the mean curvature vector of $M$. The traceless second fundamental form $\Phi$ is defined by

$$\Phi(X, Y) = A(X, Y) - \langle X, Y \rangle H,$$

for all vector field $X$ and $Y$, where $\langle , \rangle$ is the metric of $M$. Obviously

$$|\Phi|^2 = |A|^2 - n|H|^2.$$

We say $M$ has finite total curvature if

$$\| \Phi \|_{L^n (M)} = \left( \int_M |\Phi|^n \right)^{\frac{1}{n}} < \infty.$$
Lemma 2.1. ([11, 20]) Let $M^n$ be an $n$-dimensional complete noncompact oriented submanifold in sphere, then
\[ \left( \int_M |f|_{n-2}^{2n} \right)^{n-2} \leq C_0 \left( \int_M |\nabla f|^2 + \int_M (|H^2| + 1)f^2 \right), \]
for all $f \in C^1_0(M)$, where $C_0$ depends only on $n$.

Lin, Han and Li proved the following estimate.

Lemma 2.2. ([11, 22, 23]) Let $M^n$ be a complete submanifold with flat normal bundle in $S^{n+l}$, $\omega$ be a $L^2$-harmonic $p$-form ($2 \leq p \leq n-2$) on $M^n$, then
\[ |\omega|\Delta |\omega| \geq K_p|\nabla |\omega| |^2 + p(n-p)|\omega|^2 + Q_p|\omega|^2, \]
where $Q_p = \inf_{i_1, \ldots, i_n}(h^\alpha_{i_1i_1} + \ldots + h^\alpha_{i_pi_p})(h^\alpha_{i_{p+1}i_{p+1}} + \ldots + h^\alpha_{i_ni_n})$.

3 Proof of our main Theorems

Proof of Theorem 1.1. $M$ has flat normal bundle implies that there exists an orthonormal frame diagonalizing $h^\alpha_{ij}$ simultaneously. Choose proper local orthonormal frames, $h^\alpha_{ij}$ are diagonalized simultaneously. Direct computation yields
\[
2 \sum_{\alpha=n+1}^{n+l} (h^\alpha_{i_1i_1} + \ldots + h^\alpha_{i_pi_p})(h^\alpha_{i_{p+1}i_{p+1}} + \ldots + h^\alpha_{i_ni_n}) \\
= \sum_{\alpha=n+1}^{n+l} (h^\alpha_{i_1i_1} + \ldots + h^\alpha_{i_ni_n})^2 - \sum_{\alpha=n+1}^{n+l} (h^\alpha_{i_{p+1}i_{p+1}} + \ldots + h^\alpha_{i_ni_n})^2 \\
\geq n^2 |H|^2 - \max\{p, n-p\} |A|^2 \\
= \{p, n-p\} n |H|^2 - \max\{p, n-p\} |\Phi|^2. \tag{3.1}
\]
Substituting (3.1) into Lemma 2.2, we have
\[
|\omega|\Delta |\omega| \geq K_p|\nabla |\omega| |^2 + p(n-p)|\omega|^2 + \min\{p, n-p\} \frac{n}{2} |H|^2 |\omega|^2 - \frac{1}{2} \max\{p, n-p\} |\Phi|^2 |\omega|^2. \tag{3.2}
\]
This together with the condition of $2 \leq p \leq n-2$ yields
\[
|\omega|\Delta |\omega| \geq \frac{1}{n-2} |\nabla |\omega| |^2 + 2(n-2)|\omega|^2 + n |H|^2 |\omega|^2 - \frac{n-2}{2} |\Phi|^2 |\omega|^2. \tag{3.3}
\]
Setting $\eta \in C_0^\infty(M)$, multiplying (3.3) by $\eta^2$ and integrating over $M$, we obtain

$$\frac{n-2}{2} \int_M |\Phi|^2 |\omega|^2 \eta^2 \geq \frac{n-1}{n-2} \int_M |\nabla |\omega||^2 \eta^2 + 2(n-2) \int_M |\omega|^2 \eta^2$$

$$+ n \int_M |H|^2 |\omega|^2 \eta^2 + 2 \int_M |\omega|(\nabla \eta, \nabla |\omega|). \quad (3.4)$$

Combining the Hölder inequality with Lemma 2.1, we get

$$\int_M |\Phi|^2 |\omega|^2 \eta^2 \leq (\int_M |\Phi|^n)^\frac{2}{n} (\int_M (|\omega|\eta)^{\frac{2n}{n-2}}) \frac{n-2}{n}$$

$$\leq C_0(\int_M |\Phi|^n)^\frac{2}{n} (\int_M (|\nabla (\eta|\omega|)|)^2 + \int_M (|H|^2 +1)|\omega|^2 \eta^2)$$

$$\leq C_0(\int_M |\Phi|^n)^\frac{2}{n} (\int_M (|\nabla |\omega||^2 + |\omega|^2 |\nabla \eta|^2 + 2|\omega|\eta(\nabla \eta, \nabla |\omega|))$$

$$+ \int_M (|H|^2 +1)|\omega|^2 \eta^2]. \quad (3.5)$$

Setting $E = \frac{n-2}{2} C_0(\int_M |\Phi|^n)^\frac{1}{n}$ and using (3.4) and (3.5) it follows that

$$E \int_M |\omega|^2 |\nabla \eta|^2 + 2(E-1) \int_M |\omega|\eta(\nabla \eta, \nabla |\omega|)$$

$$\geq (\frac{n-1}{n-2} - E) \int_M |\nabla |\omega||^2 \eta^2 + [2(n-2) - E] \int_M |\omega|^2 \eta^2$$

$$+ (n-E) \int_M |H|^2 |\omega|^2 \eta^2. \quad (3.6)$$

Using the Cauchy-Schwarz inequality in (3.6) , we get

$$(E + \frac{|E-1|}{\epsilon}) \int_M |\omega|^2 |\nabla \eta|^2$$

$$\geq (\frac{n-1}{n-2} - E - |E-1|\epsilon) \int_M |\nabla |\omega||^2 \eta^2 + [2(n-2) - E] \int_M |\omega|^2 \eta^2$$

$$+ (n-E) \int_M |H|^2 |\omega|^2 \eta^2. \quad (3.7)$$

If

$$(\int_M |\Phi|^n)^\frac{1}{n} < \frac{2}{n-2} \sqrt{\frac{n-1}{2C_0}} = c(n), \quad (3.8)$$

then

$$\frac{n-1}{n-2} - E > 0.$$

Choosing sufficient small $\epsilon$, we obtain

$$\frac{n-1}{n-2} - E - |E-1|\epsilon > 0, \quad n-E > 0, \quad 2(n-2) - E > 0.$$
Let $\rho(x)$ be the geodesic distance on $M$ from $x_0$ to $x$ and $B_r(x_0) = \{x \in M : \rho(x) \leq r\}$ for some fixed point $x_0 \in M$. Choose $\eta \in C_0^\infty(M)$ as

$$
\eta = \begin{cases} 
1 & \text{on } B_r(x_0), \\
0 & \text{on } M \setminus B_{2r}(x_0), \\
|\nabla \eta| \leq \frac{2}{r} & \text{on } B_{2r}(x_0) \setminus B_r(x_0), 
\end{cases}
$$

and $0 \leq \eta \leq 1$. Substituting the above $\eta$ into (3.7), we finally have

$$
\frac{4}{r^2}(E + \frac{|E - 1|}{\varepsilon}) \int_{B_{2r}(x_0)} |\omega|^2 \\
\geq \frac{n - 1}{n - 2} - E - |E - 1|\varepsilon \int_{M \setminus B_{r}(x_0)} |\nabla \omega||^2 + [2(n - 2) - E] \int_{B_r(x_0)} |\omega|^2 \\
+ (n - E) \int_{B_r(x_0)} |H|^2|\omega|^2.
$$

Since $\int_M |\omega|^2 < \infty$, by taking $r \to \infty$, we have $\nabla|\omega| = 0$ and $\omega = 0$. That is $H^p(L^2(M)) = \{0\}$. This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** Let $\omega \in H^p(L^2(M))$, $2 \leq p \leq n - 2$ and $\eta \in C_0^\infty(M \setminus B_r(x_0))$. Similar to (3.7), we get

$$
(F + \frac{|F - 1|}{\varepsilon}) \int_{M \setminus B_{r}(x_0)} |\omega|^2|\nabla \eta|^2 \\
\geq \frac{n - 1}{n - 2} - F - |F - 1|\varepsilon \int_{M \setminus B_{r}(x_0)} |\nabla \omega||^2 \eta^2 + [2(n - 2) - F] \int_{M \setminus B_{r}(x_0)} |\omega|^2 \eta^2 \\
+ (n - F) \int_{M \setminus B_{r}(x_0)} |H|^2|\omega|^2 \eta^2,
$$

where $F = \frac{n - 2}{2} C_0(\int_{M \setminus B_{r}(x_0)} |\Phi|^n)^{\frac{2}{n}}$. The condition $(\int_M |\Phi|^n)^{\frac{2}{n}} < \infty$ implies that there is a decreasing positive function $\varepsilon(r)$ satisfying

$$
\lim_{r \to \infty} \varepsilon(r) = 0, \quad (\int_{M \setminus B_{r}(x_0)} |\Phi|^n)^{\frac{2}{n}} < \varepsilon(r).
$$

Thus we can choose $r = r_0 > 0$ such that

$$
\frac{n - 1}{n - 2} - F = \frac{n - 1}{n - 2} - \frac{n - 2}{2} C_0(\int_{M \setminus B_{r_0}(x_0)} |\Phi|^n)^{\frac{2}{n}} > 0.
$$

Choosing sufficient small $\varepsilon$, we get

$$
\frac{n - 1}{n - 2} - F - |F - 1|\varepsilon > 0.
$$

This together with (3.9) yields that

$$
\int_{M \setminus B_{r_0}(x_0)} |\nabla \omega|^2 \eta^2 \leq \frac{F + |F - 1|}{\varepsilon} \int_{M \setminus B_{r_0}(x_0)} |\omega|^2|\nabla \eta|^2
$$
where the positive constant $C_1$ depends only on $n$. Applying lemma 2.1 to $\eta|\omega|$ and combining with (3.10), (3.11) and (3.12), we obtain

$$
\int_{M \setminus B_r(x_0)} (\eta|\omega|)^{\frac{2a}{n-2}} \frac{n-2}{n} 
\leq C_0 \int_{M \setminus B_r(x_0)} \left( \frac{\nabla|\omega|^2}{\nabla|\omega|^2 + 2|\omega||\nabla\eta|} + 2|\omega|(|H|^2 + 1)|\omega|^2 \eta^2 \right) 
\leq C_0 \int_{M \setminus B_r(x_0)} \left[ 2\nabla|\omega|^2 \eta^2 + 2|\omega|^2 \nabla\eta^2 + (|H|^2 + 1)|\omega|^2 \eta^2 \right] 
\leq C_2 \int_{M \setminus B_r(x_0)} |\omega|^2 \nabla\eta^2, 
$$

(3.13)

where positive constant $C_2$ depends only on $n$.

Choose $\eta \in C_0^\infty(M \setminus B_r(x_0))$ as

$$
\eta = \begin{cases} 
0 & \text{on } B_r(x_0), \\
\rho(x) - r_0 & \text{on } B_{r_0+1}(x_0) \setminus B_r(x_0), \\
1 & \text{on } B_r(x_0) \setminus B_{r_0+1}(x_0), \\
\frac{2r - \rho(x)}{r} & \text{on } B_{2r}(x_0) \setminus B_r(x_0), \\
0 & \text{on } M \setminus B_{2r}(x_0), 
\end{cases}
$$

where $\rho(x)$ is the geodesic distance on $M$ from $x_0$ to $x$ and $r > r_0 + 1$. By substituting $\eta$ into (3.13) it follows that

$$
\int_{B_r(x_0) \setminus B_{r_0+1}(x_0)} \left( |\omega|^{\frac{2a}{n-2}} \right)^{\frac{n-2}{n}} \leq C_2 \int_{B_{r_0+1}(x_0) \setminus B_r(x_0)} |\omega|^2 + \frac{C_2}{r^2} \int_{B_{2r}(x_0) \setminus B_r(x_0)} |\omega|^2. 
$$

(3.14)

Since $|\omega| \in L^2(M)$, Letting $r \to \infty$, we conclude that

$$
\int_{B_r(x_0) \setminus B_{r_0+1}(x_0)} \left( |\omega|^{\frac{2a}{n-2}} \right)^{\frac{n-2}{n}} \leq C_2 \int_{B_{r_0+1}(x_0) \setminus B_r(x_0)} |\omega|^2. 
$$

(3.15)
On the other hand, the Hölder inequality asserts that
\[
\int_{B_{r_0+2}(x_0) \setminus B_{r_0+1}(x_0)} |\omega|^2 \leq \left( \text{vol}(B_{r_0+2}(x_0)) \right)^{\frac{n}{n-2}} \int_{B_{r_0+2}(x_0) \setminus B_{r_0+1}(x_0)} \left( |\omega|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2}}. \tag{3.16}
\]

From (3.15) and (3.16), we conclude that there exists a constant \( C_3 > 0 \) depending on \( \text{vol}(B_{r_0+2}(x_0)) \) and \( n \) such that
\[
\int_{B_{r_0+2}(x_0)} |\omega|^2 \leq C_3 \int_{B_{r_0+1}(x_0)} |\omega|^2. \tag{3.17}
\]

Fix a point \( x \in M \) and take \( \tau \in C^1_0(B_1(x)) \). Multiplying (3.3) by \( |\omega|^{q-2} \tau^2 \) with \( q > 2 \) and integrating by parts on \( B_1(x) \), we obtain
\[
-2 \int_{B_1(x)} \tau |\omega|^{q-1} (\nabla \tau, \nabla |\omega|) + \frac{n-2}{2} \int_{B_1(x)} |\Phi|^2 |\omega|^{q} \tau^2 
\geq \left( \frac{1}{n-2} + q - 1 \right) \int_{B_1(x)} |\omega|^{q-2} |\nabla |\omega||^2 \tau^2 + 2(n-2) \int_{B_1(x)} |\omega|^{q} \tau^2 
+ n \int_{B_1(x)} |H|^2 |\omega|^{q} \tau^2. \tag{3.18}
\]

By using the Cauchy-Schwarz inequality it follows that
\[
-2 \int_{B_1(x)} \tau |\omega|^{q-1} (\nabla \tau, \nabla |\omega|) \leq \frac{1}{n-2} \int_{B_1(x)} |\omega|^{q-2} |\nabla |\omega||^2 \tau^2 + (n-2) \int_{B_1(x)} |\omega|^{q} |\nabla \tau|^2. \tag{3.19}
\]

It follows from (3.18) and (3.19) that
\[
(n-2) \int_{B_1(x)} |\omega|^{q} |\nabla \tau|^2 + \frac{n-2}{2} \int_{B_1(x)} |\Phi|^2 |\omega|^{q} \tau^2 
\geq (q-1) \int_{B_1(x)} |\omega|^{q-2} |\nabla |\omega||^2 \tau^2 + 2(n-2) \int_{B_1(x)} |\omega|^{q} \tau^2 
+ n \int_{B_1(x)} |H|^2 |\omega|^{q} \tau^2. \tag{3.20}
\]

On the other hand, setting \( f \in C^1_0(B_1(x)) \), similar to Lemma 2.1, we have
\[
\left( \int_{B_1(x)} |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C_0 \int_{B_1(x)} |\nabla f|^2 + \int_{B_1(x)} (|H|^2 + 1) f^2. \tag{3.21}
\]

Applying (3.21) to \( \tau |\omega|^{\frac{q}{2}} \), we obtain
\[
\left( \int_{B_1(x)} (\tau^2 |\omega|^q)^{\frac{n-2}{n}} \right)^{\frac{n-2}{n}} \leq C_0 \int_{B_1(x)} |\nabla (\tau |\omega|^\frac{q}{2})|^2 + C_0 \int_{B_1(x)} (|H|^2 + 1) \tau^2 |\omega|^q 
\leq 2C_0 \int_{B_1(x)} |\nabla \tau|^2 |\omega|^q + \frac{q^2}{2} C_0 \int_{B_1(x)} \tau^2 |\omega|^{q-2} |\nabla |\omega||^2 
+ C_0 \int_{B_1(x)} (|H|^2 + 1) \tau^2 |\omega|^q. \tag{3.22}
\]
Inequality (3.22) and (3.20) imply that
\[(\int_{B_1(x)} (\tau^2 |\omega|^q)^{\frac{n}{n-2}})^{\frac{n-2}{n}} \leq 2C_0 \int_{B_1(x)} |\nabla \tau|^2 |\omega|^q + \frac{q^2}{2(q-1)} C_0 \int_{B_1(x)} ((n-2)|\nabla \tau|^2 + \frac{n-2}{2} |\Phi|^2 \tau^2) |\omega|^q\]
\[\leq \frac{q^2}{2(q-1)} C_0 \int_{B_1(x)} [2(n-2) + n|H|^2] |\omega|^q \tau^2 + C_0 \int_{B_1(x)} (|H|^2 + 1) \tau^2 |\omega|^q\]
\[\leq qC_4 \int_{B_1(x)} (|\nabla \tau|^2 + |\Phi|^2 \tau^2) |\omega|^q, \quad (3.23)\]
where $C_4$ is a positive constant depending only on $n$. Let $q_k = \frac{2n^k}{(n-2)^k}$ and $r_k = \frac{1}{2} + \frac{k}{2(k+1)}$ for an integer $k \geq 0$. Choose $\tau_k \in C_0^\infty(B_{r_k}(x))$ such that $\tau_k = 1$ on $B_{r_{k+1}}(x)$ and $|\nabla \tau_k| \leq 2^{k+3}$. Replacing $q$ and $\tau$ in (3.23) by $q_k$ and $\tau_k$ respectively, we obtain
\[(\int_{B_{r_{k+1}}(x)} |\omega|^q_{k+1})^{\frac{1}{q_{k+1}}} \leq [q_k C_4 (4^{k+3} + \sup_{B_{r_k}(x)} |\Phi|^2)]^{\frac{1}{q_k}} \left(\int_{B_{r_k}(x)} |\omega|^q_k\right)^{\frac{1}{q_k}}. \quad (3.24)\]
Apply the Morse iteration to $|\omega|$ via (3.24), we conclude that
\[\|\omega\|^2_{L^\infty(B_{\frac{1}{2}}(x))} \leq C_5 \int_{B_1(x)} |\omega|^2,\]
where $C_5$ is a positive constant depending only on $n$. Obviously
\[|\omega(x)|^2 \leq C_5 \int_{B_1(x)} |\omega|^2. \quad (3.25)\]
Choose $x \in B_{r_0+1}(x_0)$ such that
\[|\omega(x)|^2 = \|\omega\|^2_{L^\infty(B_{r_0+1}(x_0))}.\]
This together with (3.25) yields that
\[\|\omega\|^2_{L^\infty(B_{r_0+1}(x_0))} = |\omega(x)|^2 \leq C_5 \int_{B_1(x)} |\omega|^2 \leq C_5 \int_{B_{r_0+2}(x_0)} |\omega|^2. \quad (3.26)\]
This together with (3.17) implies that there exists a positive constant $C_6$ depending on $n$ and $\text{vol}(B_{r_0+2}(x_0))$, such that
\[\sup_{B_{r_0+1}(x_0)} |\omega|^2 \leq C_6 \int_{B_{r_0+1}(x_0)} |\omega|^2. \quad (3.27)\]
Let $\varphi$ be a finite dimensional subspace of $H^p(L^2(M))$. Lemma 11 in [25] implies that there exits $\omega \in \varphi$ such that
\[\frac{\text{dim} \varphi}{\text{vol}(B_{r_0+1}(x_0))} \int_{B_{r_0+1}(x_0)} |\omega|^2 \leq \left\{\binom{n}{p}, \text{dim} \varphi \right\} \sup_{B_{r_0+1}(x_0)} |\omega|^2.\]
This together with (2.27) yields $\text{dim} \varphi \leq C_7$, where $C_7$ depending on $n$ and $\text{vol}(B_{r_0+1}(x_0))$. This implies that $\text{dim} H^p(L^2(M)) < \infty$, which completes the proof of Theorem 1.2.
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