AN APPLICATION OF MOSER’S TWIST THEOREM TO
SUPERLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider a simple superlinear Duffing equation
\[ x'' + 2x^3 + p(t) = 0 \]  (0.1)
with impulses, where \( p(t + 1) = p(t) \) is an integrable function in \( \mathbb{R} \). In order
to apply Moser’s twist theorem, we need to ensure that the corresponding
Poincaré map of (0.1) is quite close to a standard twist map but it is not usually
achieved due to the existence of impulses. Two types of impulsive functions
which overcome this problem with different effects in the Poincaré map are
provided here. In both cases, there are large invariant curves diffeomorphism
to circles surrounding the origin and going to the infinity, which confine the
solutions in its interior and therefore lead to the boundedness of all solutions.
Furthermore, it turns out that the solutions starting at \( t = 0 \) on the invariant
curves are quasiperiodic.

1. Introduction and results. In this paper, we discuss the superlinear Duffing
impulsive equation
\[
\begin{cases}
  x'' + 2x^3 + p(t) = 0, & t \neq t_j; \\
  \Delta x|_{t=t_j} = I(x(t_j-), x'(t_j-)), \\
  \Delta x'|_{t=t_j} = J(x(t_j-), x'(t_j-)), & j \in \mathbb{Z},
\end{cases}
\]  (1.1)
where \( 0 < t_1 < 1 \), \( \Delta x|_{t=t_j} = x(t_j+) - x(t_j-) \) and \( \Delta x'|_{t=t_j} = x'(t_j+) - x'(t_j-) \).
In addition, assume that the impulsive time is 1-period, that is, \( t_{j+1} = t_j + 1 \)
for \( j \in \mathbb{Z} \), and \( p(t+1) = p(t) \) being integrable, is bounded.

It is well known that every solution of the Duffing equation
\[ x'' + x^3 = p(t), \]
\( p(t+1) = p(t) \) being continuous, is bounded. This result, prompted by questions of
J. Littlewood in [7], is due to G. Morris [8] and is developed by R. Dieckerhoff and
E. Zehnder in [4]. We aim to extend the result to the cases with impulsive effects.

Recently, as impulsive equations widely arise in applied mathematics, they
attracted a lot of attentions and many researchers studied the basic theories in

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and the applications in theory of control \cite{10, 13, 20} and so on. Among these, the existence of periodic solutions of impulsive differential equations has been discussed as a hot topic via fixed point theory in \cite{3, 10, 11}, topological degree theory in \cite{14, 18} and variational method in \cite{1, 12, 22, 23, 25}.

Generally speaking, the behaviors of solutions with impulsive effects may have great changes compared with solutions without. Here we take a simple linear autonomous Duffing equation for example. Let consider

\begin{equation}
\dddot{x} + (2\pi)^2 x = 0
\end{equation}

with the impulsive conditions

\begin{equation}
x(t_j^+) = 2x(t_j^-), \quad x'(t_j^+) = \frac{1}{2}x'(t_j^-),
\end{equation}

where \(t_j = \frac{j}{2}\) for \(j \in \mathbb{Z}\). It is obvious to see that without impulses all solutions of \eqref{linear_duffing} are 1-periodic and they satisfy

\begin{equation}
(2\pi x(t))^2 + (x'(t))^2 = C,
\end{equation}

where \(C\) is a constant related to the initial values. However, under the influence of impulses in \eqref{impulsive_conditions}, all solutions except for the trivial one are unbounded. In fact, with the initial point \((x(0), x'(0)) = (x_0, 0)\) \((x_0 \neq 0)\), the solution at each \(t_j^\pm\) is located on the x-axis and the radius of the trajectory \eqref{radius} at \(t_j^+\) becomes two times larger than the previous one at \(t_j^-\), that is, \((x(\frac{1}{2}^+), x'(\frac{1}{2}^+)) = (-2x_0, 0)\), \((x(1^+), x'(1^+)) = (4x_0, 0), \cdots\), which implies that the solution tends to infinity as \(t \to +\infty\).

From the example above, we find some behaviors, such as boundedness and periodic or quasiperiodic solutions, worth exploring for impulsive Duffing equations. Besides the commonly used variational method and some functional methods, KAM theory which includes Moser’s twist theorem and related fixed point theorems for instance Poincaré-Birkhoff twist theorem is well applied to solving the existence of periodic solutions (see \cite{5, 15, 17}).

In \cite{17}, D. Qian etc considered the superlinear impulsive differential equation

\begin{equation}
\dddot{x} + g(x) = p(t, x, x')
\end{equation}

with general impulsive functions

\[
\Delta x|_{t=t_j} = I_j(x(t_j^-), x'(t_j^-)), \quad \Delta x'|_{t=t_j} = J_j(x(t_j^-), x'(t_j^-)), j \in \mathbb{Z}.
\]

The authors proved via Poincaré-Birkhoff twist theorem the existence of infinitely many periodic solutions of \eqref{superlinear_duffing} with \(p = p(t)\), and also the existence of periodic solutions for non-conservative case with degenerate impulsive terms by developing a new twist fixed point theorem. Subsequently, Y. Niu and X. Li in \cite{15} discussed the periodic solutions of \eqref{superlinear_duffing} with sublinear \(g\). The impulsive functions are special ones \(\Delta x|_{t=t_j} = ax(t_j^-)\), \(\Delta x'|_{t=t_j} = ax'(t_j^-)\), \(j \in \mathbb{Z}\). Basing on the Poincaré-Bohlf fixed point theorem and the fixed point theorem established by D. Qian etc in \cite{17}, they obtained the existence of harmonic solutions and subharmonic solutions respectively. However, the study about the boundedness of solutions and the existence of quasiperiodic solutions for impulsive equations by the twist theorem is few up to now.
As we all know, one sufficient condition for applying the twist theorem on a twist map is that $f$, $g$, together with its derivatives, should be small enough with the form $O(r^{-\varepsilon})$ ($\varepsilon > 0$) as $r \to +\infty$. Due to the existence of impulses, some items will be added to the right hands of Eq. (1.6). If these items in $r_1$ and $\theta_1$ are small enough or if the added item in $\theta_1$ is a constant while the added item in $r_1$ is $0$, we can treat them equally with $f$ and $g$ or $\alpha(r)$. In both cases, the existence of impulses does not influence the application of the twist theorem. This provides us with two ideas for finding the impulsive functions.

One type of impulsive functions is that $I, J$ satisfy assumption (i):

$$
\lim_{h_0 \to +\infty} \left| \frac{\partial^{r+s} I(x,y)}{\partial x^r \partial y^s} \cdot (h_0(x,y)) \right| < +\infty,
\lim_{h_0 \to +\infty} \left| \frac{\partial^{r+s} J(x,y)}{\partial x^r \partial y^s} \cdot (h_0(x,y)) \right| < +\infty,
$$

(1.7)

where $r$, $s$ are any non-positive integers with $r + s \leq 5$, $h_0(x,y) = \frac{1}{2}(x^4 + y^2)$ and $\varepsilon > 0$ is a given number.

We suppose the item brought by impulses in the expression of $r_1$ has the form $O(r^{-\varepsilon_1})$ ($\varepsilon_1 > 0$). This hypothesis is reasonable. In fact, if the added item in the expression of $r_1$ is $O(r^{\varepsilon_2})$ ($\varepsilon_2 > 0$), $r_1$ dose not satisfy the twist theorem, let alone the boundedness. According to Eq. (1.1) and its Poincaré map, we can calculate the added item in the expression of $\theta_1$ is $O(r^{-\varepsilon_3})$, where $\varepsilon_3 > 0$ is uniquely determined by $\varepsilon_1$. Assumption (i) exactly satisfies the relation between $\varepsilon_1$ and $\varepsilon_3$ when we unify the variables, and it is the weakest condition which guarantees the application of the twist theorem from the discussions above.

The another type of impulsive functions has specific form

$$I(x(t^-), x'(t^-)) = -2x(t_j^-), \quad J(x(t^-), x'(t^-)) = -2x'(t_j^-).$$

At impulsive times, the point $(x(t_j^-), x'(t_j^-))$ jumps to the origin-symmetric point $(x(t_j^+), x'(t_j^+)) = (-x(t_j^-), -x'(t_j^-))$ on the $(x,x')$ plane. Under these impulses, the item caused by impulses in the expression of $r_1$ is $0$ and the change in the expression of $\theta_1$ is only a constant. We regard this constant as a part of $\alpha(r)$, which implies the twist theorem is still valid.

Consequently, we obtain the boundedness of solutions and the existence of quasiperiodic solutions of (1.1) with two types of impulsive functions mentioned above. The main results are as follows.

**Theorem 1.1.** Assume that $p(t)$ is 1-periodic, integrable, bounded and $I, J$ defined in (1.1) satisfy (i) mentioned in (1.7) and (ii) the jumping map

$$\Lambda : (x, y) \to (x + I(x,y), y + J(x,y))$$

(1.8)

is area-preserving. Then every solution $x(t)$ of (1.1) is bounded, i.e. it exists for all $t \in \mathbb{R}$ and

$$\sup_{t \in \mathbb{R}/\{t_i\}} (|x(t)| + |x'(t)|) + \sup_{j \in \mathbb{Z}} (|x(t_j)| + |x'(t_j)|) < \infty.$$
Theorem 1.2. Under the same assumptions as in Theorem 1.1, there is a large \( \omega^* > 0 \) such that for every irrational number \( \omega > \omega^* \) satisfying
\[
\left| \omega - \frac{p}{q} \right| \geq c|q|^{-2-\beta}
\]
for all integers \( p \) and \( q \neq 0 \) with two constants \( \beta > 0 \) and \( c > 0 \), there is a quasiperiodic solution of (1.1) having frequencies \((\omega, 1)\), i.e. there is a \( 1 \)-periodic function \( F(\theta_1, \theta_2) \) in \( \theta_1 \) and \( \theta_2 \), such that
\[
x(t) = F(\omega t, t)
\]
is a solution of (1.1).

Theorem 1.3. Assume that the impulsive functions in Eq. (1.1) are
\[
I(x(t_j-), x'(t_j-)) = -2x(t_j-), \quad J(x(t_j-), x'(t_j-)) = -2x'(t_j-).
\]
Then the results in Theorems 1.1 and 1.2 are also valid.

Remark 1. Our discussions start from the relatively simple Eq. (1.1) with only one impulsive time in \((0, 1)\). Actually, for the general Duffing type equations with polynomial potentials like
\[
x'' + x^{2n+1} + \sum_{i=0}^{2n} x^ip_i(t) = 0, \quad t \neq t_j
\]
and with finite number of impulsive times in \((0, 1)\), the discussion is similar and one can also obtain the results in all above Theorems.

Remark 2. Except the quasiperiodic solutions, one also obtain infinitely many periodic solutions of (1.1) with minimal period \( m \) \((m \geq 1, m \in \mathbb{N})\) by the Poincaré-Birkhoff fixed point theorem. The proof is similar with the proof in [4] then we omit here.

The structure of the paper is as follows. In section 2 we transform equation (1.1) into Hamiltonian system (3.1) under action and angle variables. Then some estimations of the impulsive functions and the Poincaré mapping of (3.1) under a scalar transformation are given in section 3. Finally, we prove the main results by the twist theorem in the last section.

2. Action and angle-variables. Dropping the time dependent term, we first consider equation (1.1) without impulse effects
\[
x'' + 2x^3 = 0,
\]
which can be written as an equivalent system of the form
\[
\begin{cases}
x' = y, \\
y' = -2x^3.
\end{cases}
\]
This is a time independent Hamiltonian system on \( \mathbb{R}^2 \)
\[
\begin{cases}
x' = \frac{\partial}{\partial y} h_0(x, y), \\
y' = -\frac{\partial}{\partial x} h_0(x, y),
\end{cases}
\]
where
\[
h_0(x, y) = \frac{1}{2}(x^4 + y^2).
\]
Clearly $h_0 > 0$ on $\mathbb{R}^2$ except at the only equilibrium point $(x, y) = (0, 0)$ where $h_0 = 0$. All solutions of (2.1) are periodic, the periods tending to zero as $h_0 = E$ tends to infinity. We take $(C(t), S(t))$ for a solution of system (2.1) with the initial conditions $(C(0), S(0)) = (1, 0)$. Let $T^* > 0$ be its minimal period. Then functions $C$ and $S$ satisfy

\[(1)\ C(t) = C(t + T^*),\ S(t) = S(t + T^*) \text{ and } C(0) = 1,\ S(0) = 0.
\]

\[(2)\ C'(t) = S(t),\ S'(t) = -2C^3(t).
\]

\[(3)\ S^2(t) + C^4(t) = 1.
\]

\[(4)\ C(-t) = C(t) \text{ and } S(-t) = -S(t).
\]

The action and angle variables are now defined by the map $\varphi : \mathbb{R} \times S^1 \to \mathbb{R}^2 \setminus 0$, where $S^1 = \mathbb{R}/\mathbb{Z}$ and $(x, y) = \varphi(\lambda, \theta)$ with $\lambda > 0$ and with $\theta \mod 1$ is given by the formulae:

\[\varphi : \begin{cases} x = c^\frac{1}{2} \lambda^\frac{1}{2} C(\theta T^*), \\ y = c^\frac{3}{2} \lambda^\frac{5}{2} S(\theta T^*), \end{cases}\]

where $c = 3/(2T^*)$.

We claim that $\varphi$ is a symplectic diffeomorphism from $\mathbb{R}^+ \times S^1$ onto $\mathbb{R}^2 \setminus 0$. Indeed, from the Jacobian $\Delta$ of $\varphi$ one finds by (2) and (3) $|\Delta| = 1$, so that $\varphi$ is measure preserving. Moreover since $(C, S)$ is a solution of a differential equation having $T^*$ as minimal period, one concludes that $\varphi$ is one to one and onto, which proves the claim.

As for (1.1) without the time dependent term, it can be written in the form of Hamiltonian system

\[X_{h_0} : \begin{cases} x' = \frac{\partial}{\partial y} h_0(x, y), & y' = -\frac{\partial}{\partial x} h_0(x, y), \quad t \neq t_j, \\ x(t_j+) = x(t_j) + I(x(t_j), y(t_j)), \\ y(t_j+) = y(t_j) + J(x(t_j), y(t_j)), \quad j \in \mathbb{Z}, \end{cases}\]

where $h_0$ is defined in (2.2). Here and hereafter we denote $(x(t_j), y(t_j))$ by $(x(t_j), y(t_j))$ for simplicity.

By the transformation (2.3) and property (3) of $C(t)$ and $S(t)$, the Hamiltonian function of (2.4) becomes

\[h_0 \circ \psi(\lambda, \theta) = \frac{1}{2} c^2 \lambda^\frac{2}{3} \hat{h}_0^* (\lambda).\]

The trajectories $h_0 = c$ ($c$ is a parameter) of (2.1) in $(x, y)$ plane are closed curves which are symmetric with respect to the $x$-axis and $y$-axis. In the new coordinates $(\lambda, \theta)$, action variable $\lambda$ denotes the area encircled by the solution orbit $h_0 = c$, while angle variable $\theta$ is the angle with the $x$-axis along the orientation of solutions. By (2.3), we have

\[\lambda = \frac{1}{c} (x^4 + y^2)^\frac{3}{4},\]

which together with (1.8) implies that

\[\lambda(t_j-) = \frac{1}{c} (x(t_j^-)^4 + y(t_j^-)^2)^\frac{3}{2},\]

and

\[\lambda(t_j+) = \frac{1}{c} ((x(t_j))^4 + (y(t_j) + J)^2)^\frac{3}{2}.\]
for each \( t_j, \ j \in \mathbb{Z} \).

According to the initial values and the motions of solutions of Eq. (2.4), we can define \( \theta(t_j-) \) well. One difficulty we encountered is the definition and estimation of \( \theta(t_j+) \), since the arguments which are up to any integer multiple of 1 are generally regarded as the same ones. Therefore we need an exact expression of \( \theta(t_j+) \) to make sure the jumping map \( \Lambda \) to be a homeomorphism.

By letting \( z = (x, y) \) and then

\[
z(t_j+) = (x(t_j+), y(t_j+)) = (x(t_j) + I, y(t_j) + J),
\]

we define

\[
\theta(t_j+) = \arg z(t_j+) + k,
\]

where \( k \) is chosen to satisfy

\[
|\theta(t_j+) - \theta(t_j-)| \leq \frac{1}{2}
\]

and \( \arg z \) denotes the argument of \( z \) with \( \arg z \in [0, 1) \). Particularly, if for some \( j \in \mathbb{Z} \) there exists \( k \) such that \( |\theta(t_j+) - \theta(t_j-) - \frac{1}{2}| = \frac{1}{2} \) we only choose the suitable \( k \) satisfying \( \theta(t_j+) - \theta(t_j-) = \frac{1}{2} \) (or \( \theta(t_j+) - \theta(t_j-) = -\frac{1}{2} \)) for all those \( j \).

By this definition, \( \theta(t_j+) \) can be decided by \( \theta(t_j-) \) with the unique \( k \). Moreover, if \( \theta(t_i-) = \theta(t_j-) + 1 \), then \( \theta(t_i+) = \theta(t_j+) + 1 \). This implies that in the new coordinates the jumping map is a homeomorphism.

Thus by (2.5), in new action and angle variables system \( X_{ho} \) becomes

\[
X_{ho} : \begin{cases} 
\theta' = \frac{\partial}{\partial \lambda} h_0^* = \frac{2}{3} c^{\frac{4}{3}} \lambda^\frac{1}{3}, \\
\lambda' = -\frac{\partial}{\partial \theta} h_0^* = 0, \quad t \neq t_j; \\
\Delta \lambda(t_j) = I^*(\lambda(t_j-), \theta(t_j-)), \\
\Delta \theta(t_j) = J^*(\lambda(t_j-), \theta(t_j-)), \quad j \in \mathbb{Z},
\end{cases}
\]

where

\[
\begin{align*}
I^*(\lambda(t_j-), \theta(t_j-)) &= \lambda(t_j+) - \lambda(t_j-), \\
J^*(\lambda(t_j-), \theta(t_j-)) &= \theta(t_j+) - \theta(t_j-)
\end{align*}
\]

and \( \lambda(t_j\pm), \theta(t_j\pm) \) are defined in (2.7)-(2.9). Hereafter we always denote \((\lambda(t_j-), \theta(t_j-))\) by \((\lambda(t_j), \theta(t_j))\).

We turn back to rewrite full equation (1.1) into the Hamilton system in \((x, y)\) variables with the Hamiltonian function

\[
h(x, y, t) = \frac{1}{2} (x^4 + y^2) + p(t)x.
\]

By the symplectic transformation \( \varphi \), \( h \) is transformed into

\[
h_1(\lambda, \theta, t) = \frac{1}{2} c^{\frac{4}{3}} \lambda^\frac{1}{3} + g(\theta, t)\lambda^\frac{4}{3},
\]

where

\[
g(\theta, t) = c^{\frac{1}{3}} p(t) C(\theta T^*)
\]
and \( g \in C^\infty(S^2) \) with \( S^2 = \mathbb{R}^2/\mathbb{Z}^2 \). Finally, the Hamiltonian system of (1.1) becomes

\[
\begin{align*}
\theta' &= \frac{\partial}{\partial h_1} = \frac{2}{7} c \frac{d}{dt} \lambda + \frac{1}{7} g(t, \theta) \lambda^{-\frac{3}{7}}, \\
\lambda' &= -\frac{\partial}{\partial h_1} = -\frac{2}{7} g(t, \theta) \cdot \lambda^{\frac{4}{7}}, \quad t \neq t_j; \\
\Delta \lambda(t_j) &= I^*(\lambda(t_j), \theta(t_j)), \\
\Delta \theta(t_j) &= J^*(\lambda(t_j), \theta(t_j)), \quad j \in \mathbb{Z}.
\end{align*}
\]

3. The estimation of the time 1 map. We now consider system \( X_{h_1} \) for \( t \in [0, 1] \), where

\[
\begin{align*}
\theta' &= a \lambda^{\frac{2}{7}} + \frac{1}{7} g(t, \theta) \lambda^{-\frac{3}{7}}, \\
\lambda' &= -\frac{\partial}{\partial \theta} g(t, \theta) \cdot \lambda^{\frac{4}{7}}, \quad t \neq t_j; \\
\Delta \lambda(t_j) &= I^*(\lambda(t_j), \theta(t_j)), \\
\Delta \theta(t_j) &= J^*(\lambda(t_j), \theta(t_j)), \quad j \in \mathbb{Z}
\end{align*}
\]

with \( a = \frac{2}{7} c \frac{d}{dt} \). Denote by \( A_{\lambda_0} \subset \mathbb{R}^+ \times S^2 \) the annulus

\[
A_{\lambda_0} := \{(\lambda, \theta, t) \mid \lambda \geq \lambda_0 \text{ and } (\theta, t) \in S^2\}.
\]

Let \((\lambda(t), \theta(t))\) be the solution of (3.1) with the initial value \((\lambda(0), \theta(0)) = (\lambda, \theta)\), and denote

\[
\begin{align*}
P_0 : (\lambda, \theta) &\mapsto (\lambda(t_1-), \theta(t_1-)), \\
\Lambda^* : (\lambda(t_1-), \theta(t_1-)) &\mapsto (\lambda(t_1+), \theta(t_1+)), \\
P_1 : (\lambda(t_1+), \theta(t_1+)) &\mapsto (\lambda(1), \theta(1)),
\end{align*}
\]

where \( \Lambda^* = \Lambda \circ \varphi \). Then the Poincaré map of (3.1) denoted by \( P : (\lambda, \theta) \mapsto (\lambda(1), \theta(1)) \) can be expressed by

\[
P = P_1 \circ \Lambda^* \circ P_0.
\]

Since that \( P_j, \ j = 0, 1 \) are symplectic by the equation \( x'' + 2x + p(t) = 0 \) being conservative and the transformation \( \varphi \) and the jumping map \( \Lambda \) are symplectic, the Poincaré map \( P \) is area-preserving. Moreover, because the jumping map \( \Lambda^* \) is a homeomorphism by our definition of \( \theta(t_j) \), the Poincaré map \( P \) is continuous about the initial values \((\lambda, \theta)\) and is 1-periodic in \( \theta \).

Before the estimation of \( P \), we first give some estimations about \( I^* \) and \( J^* \).

**Lemma 3.1.** Let

\[
\begin{align*}
\hat{I}(\lambda, \theta) &= I \left( c^\frac{d}{dt} \lambda^\frac{2}{7} C(\theta T^*), c^\frac{d}{dt} \lambda^\frac{2}{7} S(\theta T^*) \right), \\
\hat{J}(\lambda, \theta) &= J \left( c^\frac{d}{dt} \lambda^\frac{2}{7} C(\theta T^*), c^\frac{d}{dt} \lambda^\frac{2}{7} S(\theta T^*) \right).
\end{align*}
\]

Under the assumption (i) mentioned in (1.7), it holds that for any non-negative integers \( k_1, k_2 \) \((k_1 + k_2 \leq 5)\),

\[
\begin{align*}
\frac{\partial^{k_1+k_2}}{\partial \lambda^{k_1} \partial \theta^{k_2}} \hat{I}(\lambda, \theta) &= O(\lambda^{-\frac{d}{2} - k_1}), \quad \lambda \to +\infty, \\
\frac{\partial^{k_1+k_2}}{\partial \lambda^{k_1} \partial \theta^{k_2}} \hat{J}(\lambda, \theta) &= O(\lambda^{-\frac{d}{2} - k_1}), \quad \lambda \to +\infty.
\end{align*}
\]
Proof. By (2.6) and the assumption (i), we have
\[ \frac{\partial^{++} I(x, y)}{\partial x \partial y} = O(\lambda^{-\frac{1}{2} - \frac{1}{2} - \frac{3}{2} - k}), \quad \lambda \to +\infty, \quad (3.5) \]
\[ \frac{\partial^{++} J(x, y)}{\partial x \partial y} = O(\lambda^{\frac{1}{2} - \frac{1}{2} - \frac{3}{2} - k}), \quad \lambda \to +\infty. \quad (3.6) \]

For \( k_1 = k_2 = 0 \), it is obvious from (3.5) and (3.6) that
\[ \hat{I}(\lambda, \theta) = I \left( c^{\frac{1}{2}} \lambda^{\frac{1}{2}} C(\theta T^{*}), c^{\frac{5}{2}} \lambda^{\frac{3}{2}} S(\theta T^{*}) \right) = O(\lambda^{-\frac{1}{2}}), \quad \lambda \to +\infty, \]
\[ \hat{J}(\lambda, \theta) = J \left( c^{\frac{1}{2}} \lambda^{\frac{1}{2}} C(\theta T^{*}), c^{\frac{5}{2}} \lambda^{\frac{3}{2}} S(\theta T^{*}) \right) = O(\lambda^{\frac{1}{2} - \frac{1}{2}}), \quad \lambda \to +\infty. \]

For \( k_1 = 1, \ k_2 = 0 \), since
\[ \frac{\partial \hat{I}(\lambda, \theta)}{\partial \lambda} = \frac{\partial I(x, y)}{\partial x} + \frac{\partial I(x, y)}{\partial y} \cdot \frac{\partial y}{\partial \lambda}, \]
\[ \frac{\partial \hat{J}(\lambda, \theta)}{\partial \lambda} = \frac{\partial J(x, y)}{\partial x} + \frac{\partial J(x, y)}{\partial y} \cdot \frac{\partial y}{\partial \lambda}, \]
and again by (2.3), (3.5) and (3.6), we conclude
\[ \frac{\partial \hat{I}(\lambda, \theta)}{\partial \lambda} = O(\lambda^{-\frac{1}{2} - 1}), \quad \lambda \to +\infty, \]
\[ \frac{\partial \hat{J}(\lambda, \theta)}{\partial \lambda} = O(\lambda^{\frac{1}{2} - \frac{1}{2} - 1}), \quad \lambda \to +\infty. \]

For \( k_1 = 0, \ k_2 = 1 \),
\[ \frac{\partial \hat{I}(\lambda, \theta)}{\partial \theta} = \frac{\partial I(x, y)}{\partial x} + \frac{\partial I(x, y)}{\partial y} \cdot \frac{\partial y}{\partial \theta}, \]
\[ \frac{\partial \hat{J}(\lambda, \theta)}{\partial \theta} = \frac{\partial J(x, y)}{\partial x} + \frac{\partial J(x, y)}{\partial y} \cdot \frac{\partial y}{\partial \theta}. \]

Combining (2.3) with (3.5) and (3.6), we have
\[ \frac{\partial \hat{I}(\lambda, \theta)}{\partial \theta} = O(\lambda^{-\frac{1}{2}}), \quad \lambda \to +\infty, \]
\[ \frac{\partial \hat{J}(\lambda, \theta)}{\partial \theta} = O(\lambda^{\frac{1}{2} - \frac{1}{2}}), \quad \lambda \to +\infty. \]

The other cases where \( k_1, k_2 \) are any non-negative integers with \( k_1 + k_2 \leq 5 \) can be discussed by the similar way and we obtain
\[ \frac{\partial^{k_1+k_2} \hat{I}(\lambda, \theta)}{\partial \lambda^{k_1} \partial \theta^{k_2}} = O(\lambda^{-\frac{1}{2} - k_1}), \quad \lambda \to +\infty, \]
\[ \frac{\partial^{k_1+k_2} \hat{J}(\lambda, \theta)}{\partial \lambda^{k_1} \partial \theta^{k_2}} = O(\lambda^{\frac{1}{2} - \frac{1}{2} - k_1}), \quad \lambda \to +\infty. \]

The proof then is finished. \[ \square \]

Next we estimate the derivatives of \( I^* \) and \( J^* \) about \( \lambda(t_1) \) and \( \theta(t_1) \), where for simplicity these two variables are denoted by \( \lambda \) and \( \theta \) in the proof. Correspondingly, we also denote by \( I^*(\lambda, \theta) = I^* \), \( J^*(\lambda, \theta) = J^* \), \( \hat{I}(\lambda, \theta) = \hat{I} \), \( \hat{J}(\lambda, \theta) = \hat{J} \), where \( I, J \) are defined in (3.3) and (3.4).
Lemma 3.2. Under the assumption (i) mentioned in (1.7), it holds that for any non-negative integers $k_1, k_2$ ($k_1 + k_2 \leq 5$),

\[
\frac{\partial^{k_1+k_2} I^* (\lambda, \theta)}{\partial \lambda^{k_1} \partial \theta^{k_2}} = O(\lambda^{\frac{2}{3} - \frac{1}{3} \varepsilon - k_1}), \quad \lambda \to +\infty,
\]

\[
\frac{\partial^{k_1+k_2} J^* (\lambda, \theta)}{\partial \lambda^{k_1} \partial \theta^{k_2}} = O(\lambda^{-\frac{1}{3} - \frac{1}{3} \varepsilon - k_1}), \quad \lambda \to +\infty.
\]

Proof. By (2.6) and (2.10), we have

\[
I^* = \lambda (t_1) - \lambda (t_1)
\]

\[
= \frac{1}{\varepsilon} \left[ (y(t_1) + J)^2 + (x(t_1) + J) \right]^{\frac{3}{2}} - \lambda
\]

\[
= \frac{1}{\varepsilon} \left[ (c^2 \lambda^2 S(\theta T^*) + \dot{J})^2 + (c^2 \lambda^2 C(\theta T^*) + \dot{I}) \right]^{\frac{3}{2}} - \lambda
\]

\[
= \lambda \left[ 1 + 2c^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} S(\theta T^*) \dot{J} + c^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \dot{J}^2
\]

\[
+ \sum_{i=1}^{4} C^3 \dot{I} e^{-\frac{1}{2} \lambda^{-\frac{1}{2}} C(\theta T^*)^{4-1}} \right]^{\frac{3}{2}} - \lambda
\]

\[
= \lambda \sum_{j=1}^{n} \left[ (\dot{I} e^{-\frac{1}{2} \lambda^{-\frac{1}{2}} C(\theta T^*)^{4-1}} \right]^{\frac{3}{2}} - \lambda
\]

\[
= \lambda \sum_{i=1}^{4} C^3 \dot{I} e^{-\frac{1}{2} \lambda^{-\frac{1}{2}} C(\theta T^*)^{4-1}} \right]^{\frac{3}{2}} - \lambda
\]

the last line of which is due to Taylor’s formula. The estimation above and Lemma 3.1 imply that for any non-negative integers $k_1, k_2$ ($k_1 + k_2 \leq 5$),

\[
\frac{\partial^{k_1+k_2} I^*}{\partial \lambda^{k_1} \partial \theta^{k_2}} = O(\lambda^{\frac{2}{3} - \frac{1}{3} \varepsilon - k_1}), \quad \lambda \to +\infty.
\]

Now we are in the position to estimate $J^*$. By (2.3), (3.3) and (3.4), it holds that

\[
\dot{I} = c^2 (\lambda + I^*)^2 C(\theta + J^*) T^* - c^2 \lambda^2 C(\theta T^*)
\]

\[
\dot{J} = c^2 \lambda^2 S((\theta + J^*) T^*). \quad (3.8)
\]

When $|C(\theta T^*)| \leq (\frac{1}{4})^4 \frac{3}{2} \leq |S(\theta T^*)| \leq 1$, by the mean value theorem, we have

\[
\dot{I} = \frac{1}{3} c^2 (\lambda + \xi_1 I^*)^2 C \left( \left( \theta + J^* \right) T^* \right) I^* + c^2 \lambda^2 T^* S \left( \left( \theta + \xi_2 J^* \right) T^* \right) J^*, \quad (3.9)
\]

where $0 < \xi_1, \xi_2 < 1$. Since that from Lemma 3.1 and (3.7),

\[
\frac{\partial^{k_1+k_2} \dot{I}(\lambda, \theta)}{\partial \lambda^{k_1} \partial \theta^{k_2}} = O(\lambda^{-\frac{1}{3} \varepsilon - k_1}), \quad \lambda \to +\infty,
\]

and

\[
c^2 (\lambda + \xi_1 I^*)^2 C \left( \left( \theta + J^* \right) T^* \right) I^* = O(\lambda^{-\frac{1}{3} \varepsilon}), \quad \lambda \to +\infty
\]

hold, it can be proved easily that $|S \left( \left( \theta + \xi_2 J^* \right) T^* \right)| \geq c$ ($c$ is a positive parameter) and therefore

\[
J^* = O(\lambda^{-\frac{1}{3} - \frac{1}{3} \varepsilon}), \quad \lambda \to +\infty. \quad (3.10)
\]
With the property (2) of $C$ and $S$, we take the partial derivative of $\theta$ on both sides of Eq. (3.9) and therefore we figure out

$$
\frac{\partial J^*}{\partial \theta} = \left[ \frac{\partial I}{\partial \theta} + \frac{2}{3} \xi_1 c^{\frac{1}{2}} (\lambda + \xi_1 I^*)^{-\frac{2}{3}} C ((\theta + J^*)^T) \frac{\partial I}{\partial \theta} I^* 
\right.
- \frac{1}{3} c^{\frac{1}{2}} T^* \lambda^{\frac{2}{3}} S ((\theta + \xi_2 J^*)^T) \frac{\partial I}{\partial \theta} I^*
\left. + 2c^{\frac{1}{2}} \lambda^{\frac{1}{3}} (T^*)^2 C^3 ((\theta + \xi_2 J^*)^T) J^* \right]
\times \left[ \frac{1}{3} c^{\frac{1}{2}} T^* (\lambda + \xi_1 I^*)^{-\frac{2}{3}} S ((\theta + J^*)^T) I^*
\right.
\left. + \frac{1}{3} c^{\frac{1}{2}} (\lambda + \xi_1 I^*)^{-\frac{2}{3}} C ((\theta + J^*)^T)
- 2\xi_2 c^{\frac{1}{2}} \lambda^{\frac{1}{3}} (T^*)^2 C^3 ((\theta + \xi_2 J^*)^T) J^* 
\right.
\left. + \frac{1}{3} \lambda^{\frac{1}{3}} T^* S ((\theta + \xi_2 J^*)^T) \right]^{-1}.
$$

By (3.7), (3.10) and Lemma 3.1,

$$
\frac{\partial J^*}{\partial \theta} = O(\lambda^{-\frac{1}{3} - \frac{1}{2} \varepsilon}), \; \lambda \to +\infty.
$$

Similarly, taking the partial derivative of $\lambda$ on both sides of the Eq. (3.9), we have

$$
\frac{\partial J^*}{\partial \lambda} = \left[ \frac{\partial I}{\partial \lambda} + \frac{2}{3} c^{\frac{1}{2}} (\lambda + \xi_1 I^*)^{-\frac{2}{3}} C ((\theta + J^*)^T) \left( 1 + \xi_1 \frac{\partial I}{\partial \lambda} \right) I^*
\right.
\left. - \frac{1}{3} c^{\frac{1}{2}} T^* \lambda^{\frac{2}{3}} S ((\theta + \xi_2 J^*)^T) \frac{\partial I}{\partial \lambda} I^*
\right.
\left. + \frac{1}{3} c^{\frac{1}{2}} (\lambda + \xi_1 I^*)^{-\frac{2}{3}} C ((\theta + J^*)^T)
- 2\xi_2 c^{\frac{1}{2}} \lambda^{\frac{1}{3}} (T^*)^2 C^3 ((\theta + \xi_2 J^*)^T) J^* 
\right.
\left. + \frac{1}{3} \lambda^{\frac{1}{3}} T^* S ((\theta + \xi_2 J^*)^T) \right]^{-1},
$$

which along with (3.7), (3.10) and Lemma 3.1 implies

$$
\frac{\partial J^*}{\partial \lambda} = O(\lambda^{-\frac{1}{3} - \frac{1}{2} \varepsilon - 1}), \; \lambda \to +\infty.
$$

When $(\frac{3}{2})^\frac{1}{3} < |C(\theta T^*)| \leq 1, |S(\theta T^*)| \leq \frac{3}{2}$, we discuss the second equation of (3.8) and obtain (3.11) in the similar way with the discussions above. Thus we finish the proof. □

Let

$$
\rho = a \lambda^{\frac{1}{2}}, \; \theta = \theta.
$$

Under this scalar transformation, Eq. (3.1) has a form of

$$
\mathbf{X} : \left\{ \begin{array}{l}
\theta' = \rho + g_1(\theta, t) \rho^{-2}, \\
\rho' = -\frac{\partial I}{\partial \rho} g_1(\theta, t) \cdot \rho^{-1}, \; t \neq t_1,
\end{array} \right.
$$

where $g_1(\theta, t) = \frac{a^2}{3} g(\theta, t)$, with impulsive effects

$$
\Delta \rho(t_1) \equiv I^{**}(\rho(t_1), \theta(t_1)), \; \Delta \theta(t_1) \equiv J^{**}(\rho(t_1), \theta(t_1)).
$$

Indeed,

$$
J^{**}(\rho(t_1), \theta(t_1)) = J^* \left( (\rho(t_1)/a)^{\frac{3}{2}}, \theta(t_1) \right).
$$
and

\[ I^{**}(\rho(t_1), \theta(t_1)) = \rho(t_1^+) - \rho(t_1) \]

\[ = a\lambda \frac{1}{t_1^+} - a\lambda \frac{1}{t_1} \]

\[ = \frac{\lambda}{3}(\lambda(t_1) + \eta I^*(\lambda(t_1), \theta(t_1)))^{-\frac{2}{3}} \cdot I^*(\lambda(t_1), \theta(t_1)), \]

where \(0 < \eta < 1\), \(I^*, J^*\) are given in (3.1).

It follows Lemma 3.2 that for any non-negative integers \(k_1, k_2 (k_1 + k_2 \leq 5)\),

\[ \frac{\partial^{k_1+k_2} I^{**}}{\partial \rho^{k_1} \partial \theta^{k_2}} = O(\rho^{-\varepsilon-k_1}), \quad \lambda \to +\infty, \quad (3.14) \]

\[ \frac{\partial^{k_1+k_2} J^{**}}{\partial \rho^{k_1} \partial \theta^{k_2}} = O(\rho^{-1-\varepsilon-k_1}), \quad \lambda \to +\infty. \quad (3.15) \]

Now we give the Poincaré map of (3.13) in the following lemma.

**Lemma 3.3.** The time 1 map \(\mathcal{P}\) of the flow \(\psi^t\) of the vectorfield \(X\) given by (3.13) is of the form

\[ \mathcal{P} : \begin{cases} 
\theta_1 = \theta + \rho + f_1(\rho, \theta), \\
\rho_1 = \rho + f_2(\rho, \theta), 
\end{cases} \quad (3.16) \]

where \((\theta, \rho) = (\theta(0), \rho(0))\). Moreover for every non-negative pair \((r, s) (r + s \leq 5)\),

\[ \left| \frac{\partial^{r+s} f_1(\rho, \theta)}{\partial \rho^r \partial \theta^s} \right|, \left| \frac{\partial^{r+s} f_2(\rho, \theta)}{\partial \rho^r \partial \theta^s} \right| \leq \rho^{-\varepsilon} \]

for \(\rho \geq \rho^*(r, s)\), where \(\varepsilon\) is given in the assumption (i) of Theorem 1.1.

**Proof.** We estimate \((\theta(t), \rho(t))\) on \(\left[0, t_1\right)\) and \([t_1, 1]\) respectively. First of all, let set for the flow of (3.13)

\[ (\theta(t), \rho(t)) = \psi^t(\theta, \rho) \]

with \(\psi^0 = \text{id}\) and assume

\[ \theta(t) = \theta + tp + A_1(\rho, \theta, t), \]

\[ \rho(t) = \rho + B_1(\rho, \theta, t) \]

for \(t \in [0, t_1]\). Then the integral equation of (3.13)

\[ \psi^t(\theta, \rho) = \psi^0(\theta, \rho) + \int_0^t X \circ \psi^s ds \]

for the flow is equivalent to the following equations for \(A_1\) and \(B_1\):

\[ A_1(\rho, \theta, t) = \int_0^t B_1 ds + \int_0^t g_1(\theta + sp + A_1, s)(\rho + B_1)^{-2} ds, \]

\[ B_1(\rho, \theta, t) = -\int_0^t \left( \frac{\partial g_1}{\partial \theta} \right)(\theta + sp + A_1, s)(\rho + B_1)^{-1} ds. \quad (3.18) \]

One verifies easily that there is some large \(\rho_1 > \rho_0 \triangleq a\lambda(0)^{\frac{1}{2}}\) and for \(\rho \geq \rho_1\)

Eq. (3.18) has a unique solution in the space \(|A_1|, |B_1| \leq 1\) using the contraction principle. Moreover \(|A_1|, |B_1|\) are smooth and \(|A_1| \leq c_1\rho^{-1}, |B_1| \leq c_2\rho^{-1}\) in view of (3.18). From (3.17),

\[ \theta(t_1) = \theta + t_1^\rho + A_1(\rho, \theta, t_1), \]

\[ \rho(t_1) = \rho + B_1(\rho, \theta, t_1), \]
which together with impulsive conditions results
\[
\theta(t_1) = \theta(t_1-1) + J^{**}(\rho(t_1), \theta(t_1))
= \theta + t_1 \rho + J^{**}(\rho(t_1), \theta(t_1)) + A_1(\rho, \theta, t_1),
\]
\[
\rho(t_1) = \rho(t_1-1) + J^{**}(\rho(t_1), \theta(t_1))
= \rho + I^{**}(\rho(t_1), \theta(t_1)) + B_1(\rho, \theta, t_1).
\tag{3.19}
\]

Secondly, for \( t \in [t_1, 1] \), the integral equation becomes
\[
\psi'(\theta, \rho) = \psi^{t_1+}(\theta, \rho) + \int_{t_1}^{t} \overline{X} \circ \psi^s ds,
\]
and we assume
\[
\theta(t) = \theta(t_1) + (t - t_1) \rho(t_1) + A_2(\rho, \theta, t),
\rho(t) = \rho(t_1) + B_2(\rho, \theta, t).
\tag{3.20}
\]

Similarly, \( A_2, B_2 \) can be obtained from equations
\[
A_2(\rho, \theta, t) = \int_{t_1}^{t} B_{22} ds + \int_{t_1}^{t} g_1(\theta(t_1) + (s - t_1) \rho(0t_1) + A_2, s)
\times (\rho(t_1) + B_2)^{-2} ds,
\]
\[
= \int_{t_1}^{t} B_{22} ds + \int_{t_1}^{t} g_1(\theta(t_1) + (s - t_1) \rho(t_1) + A_2, s)
\times (\rho + I^{**} + B_1 + B_2)^{-2} ds,
\]
\[
B_2(\rho, \theta, t) = -\int_{t_1}^{t} \left( \frac{\partial}{\partial \rho} g_1 \right)(\theta(t_1) + (s - t_1) \rho(t_1) + A_2, s)
\times (\rho(t_1) + B_2)^{-1} ds,
\]
\[
= -\int_{t_1}^{t} \left( \frac{\partial}{\partial \rho} g_1 \right)(\theta(t_1) + (s - t_1) \rho(t_1) + A_2, s)
\times (\rho + I^{**} + B_1 + B_2)^{-1} ds,
\]
by the contraction principle in the space \( |A_2|, |B_2| \leq 1 \) for \( \rho \geq \rho_2 \), where \( \rho_2 > \rho_0 \) is a large number. In view of (3.14) and the estimations of \( |A_1|, |B_1| \), one concludes
\[
|A_2| \leq c_3 \rho^{-1}, |B_2| \leq c_4 \rho^{-1}.
\]

Then from (3.20), we have
\[
\theta(1) = \theta(t_1) + (1 - t_1) \rho(t_1) + A_2(\rho, \theta, 1),
\rho(1) = \rho(t_1) + B_2(\rho, \theta, 1),
\]
which together with (3.19) implies
\[
\theta(1) = \theta + \rho + (1 - t_1) I^{**} + J^{**} + A_1(\rho, \theta, t_1) + A_2(\rho, \theta, 1) + (1 - t_1) B_1(\rho, \theta, t_1),
\rho(1) = \rho + I^{**} + B_1(\rho, \theta, t_1) + B_2(\rho, \theta, 1).
\]

By letting
\[
f_1(\rho, \theta) \triangleq (1 - t_1) I^{**} + J^{**} + A_1(\rho, \theta, t_1) + A_2(\rho, \theta, 1) + (1 - t_1) B_1(\rho, \theta, t_1)
\]
and
\[
f_2(\rho, \theta) \triangleq I^{**} + B_1(\rho, \theta, t_1) + B_2(\rho, \theta, 1),
\]
we rewrite \( \rho(1) \) and \( \theta(1) \) in the form of
\[
\theta(1) = \theta + \rho + f_1(\rho, \theta)
\rho(1) = \rho + f_2(\rho, \theta).
\tag{3.21}
For $\rho \geq \rho^*$ large enough where $\rho^* = \max_{i=1,2} |\rho_i|$, since $|A_i|, |B_i|(i = 1, 2) \leq C\rho^{-1}$ and $I^{**} = O(\rho^{-\epsilon}), J^{**} = O(\rho^{-1-\epsilon})$ by (3.14) and (3.15), the required estimations then can be obtained. We complete the proof. □

4. Proof of Theorems 1.1-1.3. In this section, we will conclude that the time 1 map $\mathcal{P}$ is close to a twist map. To apply the twist theorem, we need another essential condition—the intersection property. Although there exist impulses for system $X_{h_1}$, the Poincaré map $P$ defined in (3.2) is always area-preserving or symplectic. Thus the following lemma appeared in [4] is valid for $P$ and we omit the proof.

**Lemma 4.1.** (Lemma 5 in [4]) The time 1 map $P$ of $X_{h_1}$ has the intersection property on $A_{\lambda_0}$, i.e., if $C$ is an embedded circle in $A_{\lambda_0}$ homotopic to a circle $\lambda = \text{const}$ in $A_{\lambda_0}$, then $P(C) \cap C \neq \emptyset$.

From Lemma 3.3, if $\rho$ is sufficiently large, the map $\mathcal{P}$ is, with its derivatives, close to a standard twist map. Moreover, it has the intersection property by the transformation (3.12) and Lemma 4.1, so that the assumptions of Moser’s twist theorem are met.

It follows that for $\omega \geq \omega^* (\omega^* \text{ sufficiently large})$ with

$$|\omega - \frac{p}{q}| \geq c|q|^{-2-\beta} \quad (4.1)$$

for two constants $\beta > 0$ and $c > 0$ and for all integers $p$ and $q \neq 0$, there is an embedding $\psi : S^1 \to A_{\rho}\eta\rho$ of a circle, which is invariant under the map $\mathcal{P}$. Here $\rho^*$ is a large number defined in Lemma 3.3. More specifically, on this invariant curve the map $\mathcal{P}$ is conjugated to a rotation with number $\omega$

$$\mathcal{P} \circ \psi(s) = \psi(s + \omega) \quad \text{with } s \pmod{1}. \quad (4.2)$$

Under the scalar transformation (3.12), (4.2) also holds for $P$ on $A_{\lambda_0}$ in $(\lambda, \theta)$ coordinates, where $\lambda_0 = (\rho^*/a)^3$. May as well denote the invariant curve by $\psi$ with rotation number $\omega$. The solution of the Hamiltonian system (3.1) starting at time $t = 0$ on this invariant curve determines a 1-periodic cylinder in the space $(\lambda, \theta, t) \in A_{\lambda_0} \times \mathbb{R}$. Since the Hamiltonian vector field $X_{h_1}$ is time-periodic, the phase space is $A_{\lambda_0} \times S^1$.

Let $\Phi^\tau$ with $\Phi^0 = id$ be the flow of the time independent vector field $(X_{h_1}, 1)$ on $A_{\lambda_0} \times S^1$ and define $\Psi(s, \tau) : S^2 \to A_{\lambda_0} \times S^1$ by setting

$$\Psi(s, \tau) = \Phi^\tau(\psi(s - \tau \omega), 0) = (\phi^\tau \circ \psi(s - \tau \omega), \tau).$$

By (4.2), we have $P = \phi^1$. Then

$$\Psi(s, \tau + 1) = (\phi^{\tau+1} \circ \psi(s - (\tau + 1)\omega), \tau + 1)$$

$$= (\phi^\tau \circ \phi^1 \circ \psi(s - \tau \omega - \omega), \tau + 1)$$

$$= (\phi^\tau \circ \psi(s - \tau \omega), \tau)$$

$$= \Psi(s, \tau), \quad (4.3)$$

and

$$\Psi(s + 1, \tau) = \Phi^\tau(\psi(s + 1 - \tau \omega), 0) = \Psi(s, \tau). \quad (4.4)$$

Moreover, since $(X_{h_1}, 1)$ is an autonomous system, we have

$$\Phi^t \circ \Psi(s, \tau) = \Psi(s + \omega t, \tau + t). \quad (4.5)$$

Assume that

$$X(t; s, \tau) \triangleq \Psi(s + \omega t, \tau + t) \quad (4.6)$$
with $X(0; s, \tau) = \Psi(s, \tau)$. By (4.5),
\[ X(t; s, \tau) = \Phi^t \circ X(0; s, \tau), \]
which implies $X(t; s, \tau)$ is a solution of $X_{h_1}(1)$. Combining with (4.3), (4.4) and choosing $(s, \tau) = (0, 0)$ in (4.6), we conclude that solution $X(t; s, \tau)$ of $X_{h_1}$ is quasiperiodic with frequencies $(\omega, 1)$ and $\Psi$ is its shell function, i.e.
\[ X(t; 0, 0) = \Psi(\omega t, t). \]

Here we give some explanations about the quasiperiodic solutions. The discontinuous of $\Phi$ results that $X(t)$ is piecewise continuous. On each interval without impulses, $X(t)$ has the form of quasiperiodic solutions while at impulsive times $X(t)$ satisfies impulsive conditions. Strictly speaking, $\Psi(\omega t, t)$ is not a continuous shell function for all $t \in \mathbb{R}$. But according to the form of $X(t)$ at $t \neq t_j, j \in \mathbb{Z}$, we also regard $X(t)$ is quasiperiodic with piecewise continuous shell function $\Psi$.

Through the symplectic transformation (2.3), solutions of impulsive system with Hamiltonian (2.11) are also quasiperiodic in $(x, y)$ plane. This prove the statement of Theorem 1.2.

In order to verify the statement of Theorem 1.1, we transform the invariant curves obtained for the Poincaré map $P$ of $X_{h_1}$ into the $(x, y)$ coordinates. Since the transformation $\varphi$ (2.3) is symplectic, there are invariant curves of the time 1 map of (1.1) in $(x, y)$ plane. Meanwhile, the solutions starting at $t = 0$ on the invariant curves are quasiperiodic. These families of quasiperiodic solutions can be visualized as invariant cylinders in the space $(t, x, y)$. By uniqueness, the solution starting inside a cylinder will never escape. The solution is therefore confined in the interior of the time periodic cylinder above the invariant curve and hence is bounded. This ends the proof of Theorem 1.1.

At last, we give the proof of Theorem 1.3. Under the assumption of this Theorem, Eq. (1.1) is equal to
\[
\begin{align*}
x' &= y, \quad y' = -2x^3 - p(t), \quad t \neq t_j; \\
x(t_j^+) &= -x(t_j^-), \quad j \in \mathbb{Z}; \\
y(t_j^+) &= -y(t_j^-).
\end{align*}
\]
(4.7)

By transformation (2.3) and the definition of $\theta(t_1^+)$ in (2.9), impulsive functions of (4.7) become
\[ \lambda(t_1^+) = \lambda(t_1^-), \quad \theta(t_1^+) = \theta(t_1^-) + 1/2. \]

Therefore, $I^* = 0$ and $J^* = 1/2$. Correspondingly, through the transformation (3.12), $I^{**} = 0$ and $J^{**} = 1/2$. By the similar way with the proof of Lemma 3.3, we calculate that the Poincaré map of (4.7) in $(\rho, \theta)$ coordinates has the form
\[
\begin{align*}
\theta(1) &= \theta + \rho + 1/2 + A_1(\rho, \theta, t_1) + A_2(\rho, \theta, 1) + (1 - t_1)B_1(\rho, \theta, t_1), \\
\rho(1) &= \rho + B_1(\rho, \theta, t_1) + B_2(\rho, \theta, 1),
\end{align*}
\]
where $|A_i|, |B_i| \leq cp^{-1}$ for $p$ large enough. It satisfies the twist theorem, hence the results in Theorems 1.1 and 1.2 are also true by the similar proofs.

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