Violation of Locality Without Inequalities for Multiparticle Perfect Correlations

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We prove that for a three-qubit system in the Greenberger-Horne-Zeilinger (GHZ) state, locality per se is in conflict with the perfect GHZ correlations. The proof does not in any way use the realism assumption and can lead to a refutation of locality. We also provide inequalities that are imposed by locality. The experimental confirmation of the present reasoning may imply a genuine quantum nonlocality and will deepen our understanding of nonlocality of nature.

Bell’s discovery of his celebrated inequalities (or more generally, Bell’s theorem) [1, 2, 3], which are derived from Einstein, Podolsky, and Rosen’s notion of local realism [4], is recognized as “one of the profound scientific discoveries of the century” [5]. Bell’s inequalities have already been generalized to the multiparticle cases [6, 7]. Bell’s theorem with inequalities demonstrates the conflict between local realism and quantum mechanics (QM) for certain statistical predictions of quantum theory. Bell’s theorem without inequalities has also been demonstrated for multiparticle Greenberger-Horne-Zeilinger (GHZ) states [8, 9, 10] and for nonmaximally entangled biparticle states [11]. Recently, a GHZ-type theorem was proved for the case of two pairs of maximally entangled qubits [12] and for the case of two doubly-entangled photons [13]. Strikingly, for Bell’s theorem without inequalities the contradiction between QM and local realism is of a non-statistical nature. This is known as the “all versus nothing” proof of Bell’s theorem.

Locality and realism are two indispensable assumptions [14, 15] that are used in Bell’s theorem demonstrating the contradictions between QM and local realism. So far, many experiments [14, 16] completely confirmed the statistical predictions of QM against local realistic theories though some loopholes still remain to be closed [10]. Experimental tests of the GHZ theorem [17] and Hardy’s theorem [18] have also been performed, confirming again the correctness of QM. These experiments to test Bell’s theorem rule out local realism and necessarily imply that at least one of the two assumptions (realism or locality) should be incompatible with QM [14, 15].

However, since both realism and locality are necessary ingredients in Bell’s theorem, all experiments designed to test Bell’s theorem (with and without inequalities) rule out neither locality nor realism separately. Logically, a new falsifiable formulation beyond Bell’s theorem is thus desirable to test QM versus realism and versus locality separately. Very recently, this kind of formulation has been suggested such that the separate role of the locality or realism assumption can be tested for statistical predictions of QM [19]. In particular, we derived for the Bell experiments two testable inequalities by considering realism or locality separately. This result can lead to experimental tests of locality of nature beyond Bell’s theorem. In Ref. [19], however, only statistical predictions of QM were considered. In QM, there are also non-statistical (i.e., definite) predictions, or perfect correlations, such as those considered in Bell’s theorem without inequalities [8, 9, 10, 11, 12, 13]. It is thus important to see if one can test locality of nature for these definite predictions of QM.

In this Letter we prove that for a three-qubit system in the GHZ state locality is indeed in conflict with the perfect correlations predicted by QM. To this end, one needs to adopt a locality condition without resorting to any specific theory (quantum or realistic). Then it is shown that the definite predictions as used in the usual GHZ reasoning [8, 9, 10] are incompatible with the locality condition. The proof thus does not in any way use the realism assumption or other counterfactual reasons [20].

To warm up and for comparison, let us first recall the standard reasoning of the GHZ theorem [8, 9, 10, 17]. Consider the three qubits, each of which is spacelike separated from the remaining qubits, in the GHZ state

$$|\Delta\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3 + |\downarrow\rangle_1 |\downarrow\rangle_2 |\downarrow\rangle_3),$$

where $|\uparrow\rangle \equiv |+1\rangle$ and $|\downarrow\rangle \equiv |-1\rangle$ are two orthonormal states (e.g., the spin-up and spin-down states) of a qubit. Following the GHZ argument [8, 9, 10, 17], it is directly verifiable that the GHZ state $|\Delta\rangle$ satisfies the eigenvalue equations as

$$\sigma_{x_1}\sigma_{x_2}\sigma_{x_3} |\Delta\rangle = |\Delta\rangle, \quad \sigma_{x_1}\sigma_{y_2}\sigma_{y_3} |\Delta\rangle = -|\Delta\rangle, \quad \sigma_{y_1}\sigma_{y_2}\sigma_{x_3} |\Delta\rangle = |\Delta\rangle.$$ (2)

Due to the perfect GHZ correlations, one can determine the value of any observable $(\sigma_{x_1}, \sigma_{y_1}, \sigma_{x_2}, \ldots)$ for a qubit by performing appropriate measurements on the other two qubits. This allows one to establish a local realistic interpretation of the quantum-mechanical results [2], i.e., one assumes that the individual value of any operator $(\sigma_{x_1}, \sigma_{y_1}, \sigma_{x_2}, \ldots)$ is predetermined, e.g., by local hidden variables, regardless of any set of measurements on the three qubits. However, the GHZ theorem states that
such a local realistic interpretation is always incompatible with the non-statistical quantum predictions \[2\].

In the GHZ reasoning, the realism and locality assumptions are explicitly used to establish the conflict with QM. By realism one means that for any physical quantity, there is a definite value predetermined possibly by certain classical hidden variables. Meanwhile, locality requires that for any physical theory (quantum or classical), the experimental results obtained from a physical system at one location should be independent of any observations or actions made at any other spacelike separated locations. Local realism in the context of the GHZ experiment has been ruled out by recent experiments \[17\]. We notice that Stapp \[20\] made an alternative argument of the GHZ experiment. However, the argument does not use realism explicitly, it resorts to some other counterfactual reasonings, which remain controversial \[21\].

The Pauli operators \(\sigma_x \) and \(\sigma_y \) for qubit-1 can be represented by: \(\sigma_x = \sum \delta_{i} |i\rangle \langle i| \) and \(\sigma_y = \sum \delta_{j} |j\rangle \langle j| \), where \(|i\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \) and \(|j\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle) \) are the eigenstates of \(\sigma_x \) (\(\sigma_y \)) with eigenvalues \(i \) (\(j \) respectively). The corresponding Pauli operators for the remaining qubits can be similarly represented, but with \(x \) being replaced by \(y \) for qubit-2 and \(z \) for qubit-3. In terms of these eigenstates, the GHZ state \[11\] reads

\[
|\Delta\rangle = \sum \Psi_{i,j,k} |i\rangle |j\rangle |k\rangle,
\]

where \(\Psi_{i,j,k} \) are the probability amplitudes in the eigenstate base. The first eigen-equation in \[2\] can then be rewritten as

\[
\sum \Psi_{i,j,k} |i\rangle |j\rangle |k\rangle = |\Delta\rangle,
\]

which, together with Eq. \[3\], gives immediately

\[
\sum \Psi_{i,j,k} \Psi_{|i\rangle |j\rangle |k\rangle}^* = 1.
\]

Standard QM states that \(|\Psi_{i,j,k}\rangle^2 \) are the probabilities of measuring \(\sigma_x \), \(\sigma_y \), and \(\sigma_z \) with the outcome \(i \), \(j \), and \(k \) respectively. One obtains

\[
\sum |i\rangle |j\rangle |k\rangle \Psi_{i,j,k}^2 = -1,
\]

\[
\sum |i\rangle |j\rangle |k\rangle \Psi_{i,j,k}^* = -1,
\]

\[
\sum |i\rangle |j\rangle |k\rangle \Psi_{i,j,k} = -1,
\]

where the probabilities \(|\Psi_{i,j,k}\rangle^2 \), \(|\Psi_{i,j,k}\rangle^* \), and \(|\Psi_{i,j,k}\rangle \) are to be understood similarly to \(|\Psi_{i,j,k}\rangle^2 \). Note that \[3\], \[4\], \[5\], \[6\], \[7\], \[8\] are derived using only standard QM.

Since \(|i\rangle = |j\rangle = |k\rangle = 1 \) (Note that \(\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \)), for nonzero \(|\Psi_{i,j,k}\rangle^2 \) \(( \leq 1 \)

\[
i_x i_y k_x = 1, \quad i_x j_y k_y = -1, \quad i_y j_y k_y = -1,
\]

which are actually the measured results of a single run of the GHZ experiment. However, the relations in \[9\] cannot lead to the usual GHZ conflict except that there is a logic link ensuring \(\gamma_x = \gamma_y \) and \(\gamma_y = \gamma_z \) (with \(x = i, j, k \)). The realism assumption underlying the GHZ reasoning \[8\], \[9\], \[10\], \[17\] or Stapp’s counterfactual reasoning \[20\] can provide such a logic link that gives an all-versus-nothing conflict for only a single run of the GHZ experiment.

By contrast to the GHZ/Stapp reasoning, here we assume only locality. Then how the above three-qubit GHZ correlations can be interpreted by a localist? According to local causality, events occurring in the backward light core of a qubit may affect the events occurring on the qubit. Thus, only events in the overlap of the backward light cores of the three qubits may be “common causes” \[3\] of the events occurring on the three qubits. The localist may then interpret the GHZ correlations being solely coming from the common causes that are certain physical events. Similarly to Ref. \[19\], the locality condition in the present three-qubit case reads

\[
|\Psi_{i,j,k}\rangle^2 = \sum p_{\mu} p_{\mu}^x p_{\mu}^y p_{\mu}^z,
\]

where \(p_{\mu}^x \) are local probabilities (e.g., \(p_{\mu}^x \) represents the probability of measuring \(\sigma_x \) with the outcome \(i \)) conditioned on a given common cause \(\mu \); \(p_{\mu} (\geq 0) \) are the probabilities for the given cause \(\mu \) to occur, and \(\sum_{\mu} p_{\mu} = 1 \). The locality condition \[10\] for the GHZ state \[11\] has been imposed in such a way that the given common cause can affect all local probabilities; conditioned on the same common cause, observable probabilities for the qubits must be mutually independent.

Using the locality condition \[10\] and defining

\[
\bar{v}_x^\mu \equiv \sum_{i_x=\pm 1} i_x p_{i_x}^\mu, \quad \bar{y}_y^\mu \equiv \sum_{i_y=\pm 1} i_y p_{i_y}^\mu,
\]

\[
\bar{j}_x^\mu \equiv \sum_{j_x=\pm 1} j_x p_{j_x}^\mu, \quad \bar{j}_y^\mu \equiv \sum_{j_y=\pm 1} j_y p_{j_y}^\mu, \quad \bar{k}_z^\mu \equiv \sum_{k_z=\pm 1} k_z p_{k_z}^\mu,
\]

Eqs. \[3\], \[4\], \[5\] then become

\[
\sum_{\mu} p_{\mu} \bar{v}_x^\mu \cdot \bar{j}_x^\mu \cdot \bar{k}_y^\mu = +1, \quad \sum_{\mu} p_{\mu} \bar{v}_x^\mu \cdot \bar{y}_y^\mu \cdot \bar{k}_y^\mu = -1,
\]

\[
\sum_{\mu} p_{\mu} \bar{v}_y^\mu \cdot \bar{j}_x^\mu \cdot \bar{k}_y^\mu = -1, \quad \sum_{\mu} p_{\mu} \bar{v}_y^\mu \cdot \bar{y}_y^\mu \cdot \bar{k}_y^\mu = -1,
\]

\[
\sum_{\mu} p_{\mu} \bar{v}_x^\mu \cdot \bar{y}_y^\mu \cdot \bar{k}_y^\mu = -1, \quad \sum_{\mu} p_{\mu} \bar{v}_x^\mu \cdot \bar{y}_y^\mu \cdot \bar{k}_y^\mu = -1,
\]
which, for nonzero $p_\mu$ ($\leq 1$) and any given $\mu$, lead to
\[
M\mu^{x,y}= 1, \quad M\mu^{x,y} = -1, \\
M\mu^{y,x} = -1, \quad M\mu^{y,x} = -1. \tag{13}
\]
Obviously, $|M_{x,y}| = 1, |M_{x,y}| = 1, \text{ and } |k_{x,y}| = 1$. Similarly to the GHZ argument, these relations are also mutually consistent: The latter three relations in Eq. (13) give $M\mu^{x,y} = (\mu_x^2 + \mu_y^2) = -1$, which conflicts with the first relation in Eq. (13), regardless of the explicit forms of the probabilities in (10) and the specific values of $\mu_x, \mu_y$, and $k_{x,y}$. Thus, quantum-mechanical predictions (AS) are incompatible with locality [Eq. (10)] for the GHZ states (11) and the incompatibility is manifested without inequalities. The present proof implies that the GHZ experiment performed by Pan et al. (11) actually ruled out locality, but not realism.

Some interesting new features emerge in the present proof. First, although the procedure for proving the conflict is somewhat similar to that used in the GHZ argument (12), here we do not in any way use the realism assumption: $\mu_x, \mu_y$, and $k_{x,y}$ in Eqs. (11) should not be confused with the corresponding local realistic predictions occurring in the GHZ argument, they are predictions under the locality assumption. The perfect correlations among the three qubits and the logic link leading to the conflict, as shown in (13), are recovered at the level of average values. This is in sharp contrast to the GHZ argument and Eq. (11). Thus, the conflict proved here is not of an all-versus-nothing type due to the probabilistic nature of the locality condition (10).

By assuming QM, the conflict will be even stronger than that shown in (13). To see this, recall that in QM, the quantities defined in (11) are quantum mechanical predictions of the three local parties and as such, they are constrained by the following Heisenberg-Robertson relations $\mu_{x}^2 + \mu_{y}^2 \leq 1$ and thus, $|\mu_{x}|, |\mu_{y}| \leq 1$, which imply that either $|\mu_{x}|, |\mu_{y}| < 1$ or one of $|\mu_{x}|$ and $|\mu_{y}|$ is 1 (and another will necessarily be zero). For the former case, each of the four relations in (13) cannot be valid, while for the latter case, at most one relation in (13) is valid. This stronger conflict between locality and the GHZ correlations is in sharp contrast to that shown in (13) (as well as in the GHZ (8, 9, 10, 17) or Stapp’s reasoning (21)), where a combination of three relations conflicts the remaining one. Since here we have used the quantum probabilities for envents in each location, this nonlocality is of a quantum nature, i.e., quantum nonlocality.

There are also bipartite perfect correlations as used in the usual EPR argument. Following the above discussion, one cannot obtain a similar conflict for the perfect EPR correlations if only locality is assumed. When quantum results (i.e., the Heisenberg-Robertson relations) are used as above, a conflict does occur. We anticipate this to be the essence of the EPR reasoning.

In the above gedanken GHZ experiment, one assumes perfect correlations and ideal measurement devices. In a real experiment, these ideal requirements are practically impossible. To face this difficulty, a “local inequality” for the GHZ state is desirable to take into account the imperfections of real experiments. For this purpose, one can introduce the “GHZ-Mermin operators” (4)

\[
M = (\sigma_1x\sigma_2y + \sigma_1y\sigma_2x)\sigma_3y + (\sigma_1y\sigma_2y - \sigma_1x\sigma_2x)\sigma_3x, \\
M' = (\sigma_1x\sigma_2y + \sigma_1y\sigma_2x)\sigma_3x - (\sigma_1y\sigma_2y - \sigma_1x\sigma_2x)\sigma_3y. \tag{14}
\]

Using the locality condition (10), one can prove the locality inequality
\[
\max \{|\langle M \rangle_{LT}|, |\langle M' \rangle_{LT}|\} \leq 2, \tag{15}
\]
which is imposed on any local theory. As a comparison, recall that local realistic theories satisfy (2, 7) the same inequality as (16), i.e., $\max \{|\langle M \rangle_{LT}|, |\langle M' \rangle_{LT}|\} \leq 2$. Meanwhile, realistic theories (without the locality assumption) predict
\[
\max \{|\langle M \rangle_{LT}|, |\langle M' \rangle_{LT}|\} \leq 4. \tag{16}
\]

Similarly to Ref. (13), the inequality (16) can be proved by noticing that the absolute values of each of the eight observable quantities in Eq. (14) are equal to or less than 1 in a realistic theory.

However, if the local probabilities in the locality condition (10) are given by QM, then a quantum locality inequality, stronger than the locality inequality (14), can be proved for three-qubit states $\rho$:
\[
\langle M \rangle_{\rho,L}^2 + \langle M' \rangle_{\rho,L}^2 \leq 1, \tag{17}
\]
where $\langle M \rangle_{\rho} = Tr(\rho M)$. For the three qubits in the GHZ state (11) one obtains, from Eqs. (2), $M = 4|\Delta|$, resulting in $\langle M \rangle_{\rho}^2 + \langle M' \rangle_{\rho}^2 = 16$, which conflicts the inequality (17). Violation of the quantum locality inequality (17) implies genuine quantum nonlocality. Without assuming locality, QM predicts an inequality (22, 23)
\[
\langle M \rangle_{\rho,QM}^2 + \langle M' \rangle_{\rho,QM}^2 \leq 16 \tag{18}
\]
for all three-qubit states.

As we proved in Ref. (24), the locality condition given by QM is a necessary and sufficient condition for $N$-qubit states to be totally separable so far as the $N$ qubits are mutually spacelike separated. Thus, the quantum locality inequality (17) is fulfilled by totally separable quantum states, which are quantum mechanically local by definition (10). Any three-qubit state has quantum nonlocality if it violates the quantum locality inequality (17). It is worthwhile to mention that three-qubit states satisfying $8 < \langle M \rangle_{\rho,3}^2 + \langle M' \rangle_{\rho,3}^2 \leq 16$ are totally entangled (i.e., 3-entangled), while partially entangled (i.e., 2-entangled)
three-qubit states fulfill $1 < \langle M \rangle^2_{\rho(2,1)} + \langle M' \rangle^2_{\rho(2,1)} \leq 8$. This result is useful in detecting three-qubit quantum nonlocality (or entanglement).

The four inequalities (15-18) are plotted in the $\langle M \rangle$-$\langle M' \rangle$ plane (Fig. 1). The diagram shows that for the GHZ experiment, all quantum predictions are a subset of the predictions of classical realism: locality alone can lead to contradictions with the GHZ correlations. If the probabilities in the locality condition (10) are quantum mechanical predictions, then stronger contradictions can be revealed than locality alone or local realism. This observation is similar to the results obtained in Ref. [19] for the Bell experiments. The experimental verification of violating the quantum locality inequality (17) implies a genuine quantum nonlocality that is not masked by the realism assumption.

To summarize, we have proved that for a three-qubit system in the GHZ state locality is in conflict with the perfect GHZ correlation. By contrast to the GHZ theorem refuting local realism, the present proof refutes only locality and does not in any way use the realism assumption, which cannot be ruled out by the standard GHZ experiments. Thus, the present result reveals a genuine (multiparticle) quantum nonlocality, i.e., quantum nonlocality beyond Bell’s nonlocality. We have also provided inequalities that are imposed by locality and can be maximally violated by the GHZ-entangled qubits. This is essential for a practical GHZ experiment. Similar considerations on Hardy’s theorem [11] and Cabello’s theorem [12] will be given elsewhere.

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