MINISUPERSPACE EXAMPLES OF QUANTIZATION USING CANONICAL VARIABLES OF THE ASHTEKAR TYPE: STRUCTURE AND SOLUTIONS

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ABSTRACT: The Ashtekar variables have been use to find a number of exact solutions in quantum gravity and quantum cosmology. We investigate the origin of these solutions in the context of a number of canonical transformations (both complex and real) of the basic Hamiltonian variables of general relativity. We are able to present several new solutions in the minisuperspace (quantum cosmology) sector. The meaning of these solutions is then discussed.

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I Introduction

a) General considerations:

The formulation of the Ashtekar variables\cite{1} has led to a considerable body of work applying them to problems of quantum gravity. These variables are the result of a complex canonical transformation on a set of 3+1 Hamiltonian variables of general relativity related to the ADM variables. One point about canonical transformations that is perhaps slighted is that they sometimes allow one to find particular quantum solutions. The usual mapping of quantum solutions from one set of canonical variables to another is generated by multiplying a solution in one set of variables, $\Psi$, by $e^{iG}$, where $G$ is the generator of the particular canonical transformation to give $\psi = e^{iG}\Psi$. Notice that if one manages by some technique to find a particular solution $\Psi$ in one set of variables and the generating function is known, it is possible to find a more complicated solution $\psi$ in terms of the old variables. This technique has been used with the Ashtekar variables to find a few exact solutions in quantum cosmology\cite{2}\cite{3}\cite{4}.

There are a number of ways in which this concept can be extended beyond these results. One is to attempt to promote quantum cosmology solutions to full quantum gravity solutions as was done with the Chern-Simons solution in terms of the Ashtekar variables\cite{2}. Another route is to study the concept of canonical transformations on the usual 3+1 variables and attempt to use the new variables to generate new exact solutions to quantum gravity.

The plan of this article is to investigate several canonical transformations related to the transformation that gives the Ashtekar variables and use them in the sense mentioned above to generate new exact solutions. At this point all the solutions we have been able to find are minisuperspace (quantum cosmology) solutions, specifically solutions for diagonal Bianchi type IX cosmological models (Mixmaster models). We will begin with a brief discussion of canonical transformations in Hamiltonian formulations of general relativity and their relation to quantum gravity. We will then use these concepts to write down a series of equations that can be solved to give families of exact solutions. Next we will discuss the form and meaning of these solutions. The last section of the article will be devoted to the significance of our solutions in the context of quantum gravity and suggestions for using canonical transformations to find new solutions in quantum gravity.

b) Canonical transformations and Ashtekar Variables:

Both the ADM Hamiltonian variables for the gravitational field and the Ashtekar variables are based on a 3+1 decomposition of the four-metric of space time. The ADM variables consist of the three-metric components $g_{ab}$ on $t = \text{constant}$ slices, and their conjugate momenta $\pi^{ab}$ constructed from the extrinsic curvature of these surfaces, $K^{ab}$. The Ashtekar variables result from a “Bargmannization”\cite{5} of these variables that is based on a complex canonical transformation similar to $p \rightarrow f(q)+ip$, $q \rightarrow q$ in an ordinary one-dimensional classical system. It is useful to look at a similar
transformation in terms of the ADM variables. If we write the ADM action as

\[
I = \int \left\{ \pi^{ab} \dot{g}_{ab} - N \left( \frac{1}{\sqrt{g}} G_{abcd} \pi^{ab} \pi^{cd} - \sqrt{g} (\zeta R - 2 \Lambda) \right) + N_a \pi^{ab} \right\} d^4 x,
\]

where, as usual, \(3R\) is the Ricci scalar on \(t = \text{const.}\) surfaces, \(\mid\) is a covariant derivative on these surfaces, \(\Lambda\) is the cosmological constant, and \(G_{abcd}\) is the DeWitt metric,

\[ G_{abcd} = g_{ac} g_{bd} - \frac{1}{2} g_{ab} g_{cd}. \]

The parameter \(\zeta\) is used to control the signature of space-time; it is chosen to be +1 for Lorentzian signatures and -1 for Euclidean signatures. It is possible to attempt the complex canonical transformation

\[
g_{ab} \rightarrow g_{ab}, \quad \pi^{ab} \rightarrow \tau^{ab}, \quad \tau^{ab} = f^{ab}(g_{cd}) + i\pi^{ab}. \]

Here we assume that \(f^{ab}\) could depend explicitly on \(g_{ab,c}\) as well as \(g_{ab}\). The transformation is canonical (ignoring topological complications in the space of metrics) if and only if

\[
\frac{\delta f^{ab}}{\delta g_{cd}} - \frac{\delta f^{cd}}{\delta g_{ab}} = 0.
\]

The action now becomes

\[
I = \int \left\{ \left( \frac{1}{i} \right) \tau^{ab} \dot{g}_{ab} + i(f^{ab} \dot{g}_{ab}) - N \left( \frac{1}{\sqrt{g}} [-G_{abcd} \tau^{ab} \tau^{cd} + 2G_{abcd} f^{ab} \tau^{cd} - \\
- G_{abcd} f^{ab} f^{cd}] - \sqrt{g} (\zeta R - 2 \Lambda) \right) - 2iN_a (\tau^{ab} \mid_b - f^{ab} \mid_b) \right\} d^4 x.
\]

Notice that the term \(if^{ab} \dot{g}_{ab}\) is a total time derivative because (1.3) implies that \(f^{ab}\) is the functional derivative of a functional \(S\), i.e. \(f^{ab} = \delta S / \delta g_{ab}\). It is also possible to remove the three-curvature term from the Hamiltonian constraint if we take

\[
G_{abcd} \frac{\delta S}{\delta g_{ab}} \frac{\delta S}{\delta g_{cd}} + g \zeta R = 0,
\]

which is the Einstein-Hamilton-Jacobi equation [6] for \(S\) with the sign of \(g \, 3R\) reversed.

There are a number of observations that we can make about the action (1.4). First of all, if \(S\) is taken to satisfy (1.5), the Hamiltonian constraint,

\[
\mathcal{H}_\perp = 0 = -G_{abcd} \tau^{ab} \tau^{cd} + 2G_{abcd} \frac{\delta S}{\delta g_{ab}} \tau^{cd} + 2g \Lambda,
\]

is an algebraic function of \(\tau^{ab}\) with at most second-order terms. An obvious solution to \(\mathcal{H}_\perp = 0\) (when \(\Lambda = 0\)) is \(\tau^{ab} = 0\), which is the complex equivalent of the usual Einstein-Hamilton-Jacobi formulation where from (1.2b) \(\pi^{ab} = i\delta S / \delta g_{ab}\), \(S\) obeys (1.5), and the space constraint reduces to \((\delta S / \delta g_{ab}) \mid_b = 0\).

Notice that the complex version of the canonical transformation is not necessary, one can define \(\pi^{ab} \rightarrow \tau^{ab} = \delta S / \delta g_{ab} + \pi^{ab}\), and the action becomes

\[
I = \int \left\{ \tau^{ab} \dot{g}_{ab} - N \left( \frac{1}{\sqrt{g}} G_{abcd} \tau^{ab} \tau^{cd} - 2G_{abcd} \frac{\delta S}{\delta g_{ab}} \tau^{cd} + \\
+ 2G_{abcd} f^{ab} \tau^{cd} - \sqrt{g} (\zeta R - 2 \Lambda) \right) - 2iN_a (\tau^{ab} \mid_b - f^{ab} \mid_b) \right\} d^4 x.
\]
\[ G_{abcd} \frac{\delta S}{\delta g_{ab}} \frac{\delta S}{\delta g_{cd}} - \sqrt{g}(3R - 2\Lambda) + 2N_a [\tau^{ab}_{\mid b} + (\frac{\delta S}{\delta g_{ab}})_{\mid b}] d^4x, \]  
and if \( S \) obeys the ordinary Einstein-Hamilton-Jacobi equation

\[ G_{abcd} \frac{\delta S}{\delta g_{ab}} \frac{\delta S}{\delta g_{cd}} - g \zeta (3R - 2\Lambda) + 2N_a [\tau^{ab}_{\mid b} + (\frac{\delta S}{\delta g_{ab}})_{\mid b}] d^4x, \]  

(1.7)

\[ \tau^{ab} = 0 \] is a solution to the Hamiltonian constraint (for \( \Lambda = 0 \)) and \( (\frac{\delta S}{\delta g_{ab}})_{\mid b} = 0 \) is again the content of the space constraint. Of course it is not necessary to assume that \( \tau^{ab} \) is equal to zero, or that \( S \) be a solution to the Einstein-Hamilton-Jacobi equation.

If \( S \) is assumed to be any function of \( g_{ab} \) (and its derivatives), then, for example, in (1.7) the Hamiltonian constraint becomes

\[ \mathcal{H}_{\perp} = \frac{1}{\sqrt{g}} [G_{abcd} \tau^{ab} \tau^{cd} - 2G_{abcd} \frac{\delta S}{\delta g_{ab}} \tau^{cd}] - \sqrt{g} \zeta (3R' - 2\Lambda), \]  

(1.9)

where \( 3R' \) is a new “scalar curvature” defined by

\[ 3R' = 3R - \frac{1}{\zeta} G_{abcd} \frac{\delta S}{\delta g_{ab}} \frac{\delta S}{\delta g_{cd}}. \]  

(1.10)

In view of the fact that there are a number of possible linear combinations of \( \pi^{ab} \) and \( \frac{\delta S}{\delta g_{ab}} \) that are, in principle, acceptable, we would like to study general transformations of the form

\[ \pi^{ab} \rightarrow \tau^{ab} = \delta S/\delta g_{ab} + \beta \pi^{ab}, \]  

(1.10a)

\[ g^{ab} \rightarrow \tilde{g}^{ab}, \]  

(1.10b)

where \( S \) will not be assumed to be a solution of the resulting Hamilton-Jacobi equation and \( \tau^{ab} \) will not necessarily be taken to be zero. Which of the approaches outlined above one chooses depends on the system one is studying and the goal one is trying to achieve. For the classical theory it might seem to be less desirable to use an \( S \) that does not remove the curvature term from the Hamiltonian constraint, but if one is using a complex canonical transformation, the quantum theory can be made more difficult by the necessity of imposing a reality condition on quantum states, and one can trade the existence of \( 3R' \neq 0 \) for explicitly real quantum variables[7].

Notice that even for complex canonical transformation we mentioned above there are, in principle, as many such transformations as there are solutions \( S \) to (1.5). The main difficulty in finding functions, \( S \), of \( g_{ab} \) and \( g_{ab,c} \) is that \( f^{ab} = \delta S/\delta g_{ab} \) is a two-index object, and it is difficult to construct such an object that satisfies (1.5) solely from \( g_{ab} \) and \( g_{ab,c} \) (or the Christoffel symbols \( \Gamma^a_{bc} \)). However, if one considers an orthonormal basis of one-forms on \( t = \text{const.} \) surfaces, \( \sigma^i = e^i_a(x^c)dx^a \), \( (ds^2 = \sigma^i \sigma^i) \), then the connection coefficients \( \Gamma^i_{jk} \) have the natural symmetry \( \Gamma^i_{jk} = \Gamma^i_{kj} \) and it is possible to construct the spin coefficients \( \Gamma^i_j = (1/2)\varepsilon^{ikj} \Gamma_{kj} \). Using these, Ashtekar was able to find an elegant solution for the equivalent of \( S \).

Before we continue, we introduce some notation that will be used throughout the paper. \( SO(3) \) indices will be denoted by lower case latin letters from the middle of the
alphabet, $i, k, \ell \cdots = 1, 2, 3$ (we reserve letters from the beginning of the alphabet for tangent space indices). We will use additional indices $I, J, K, \cdots = 1, 2, 3$ as labels for certain geometrical objects. The 3-dimensional Levi-Civita tensor density and its inverse are denoted as $\tilde{\eta}^{abc}$ and $\eta_{abc}$ and the internal Levi-Civita tensors will be denoted as $\varepsilon_{ijk}$, $\varepsilon_{ij}$, $\varepsilon_{ijk}$, $\varepsilon_{ijk}$. The basic fields in the ADM formalism with an $SO(3)$ internal symmetry are a densitized triad $\tilde{E}^a_i$ and an object $K^i_a$ closely related to the extrinsic curvature. We introduce $e^a_i$ as the inverse of $e^i_a$ (the coefficients of $\sigma^i$) satisfying $e^a_i e^i_a = \delta^a_b$, $e^a_i e^i_a = \delta^i_l$. The determinant of $e^i_a$ is defined as,

$$\det e^i_a = \tilde{e} = \frac{1}{6} \tilde{\eta}^{abc} \varepsilon_{ijk} e^i_a e^j_b e^k_c.$$  (1.11)

Finally, the $SO(3)$ connection $\Gamma^i_a$ (where $\Gamma^i_a = \Gamma^j_a e^i_j$, $\Gamma^i_j$ defined above) compatible with $e^i_a$ is

$$\Gamma^i_a = -\frac{1}{2\tilde{e}} (e^i_a e^j_b - 2e^j_a e^i_b) \tilde{\eta}^{bcd} \partial_c \epsilon_{bdj}.$$  (1.12)

The Ashtekar variables are a densitized basis $\tilde{E}^a_i \equiv 2\tilde{e} e^a_i$ and the equivalent of $\tau^{ab}$, $A^i_a = \Gamma^i_a + iK^i_a$. We will not go into the details of the ADM action written in terms of these variables (although we will give the constraints later) but just point out that $\tilde{E}^a_i$ and $A^i_a$ are new canonical variables derived from $\tilde{E}^a_i$ and $K^i_a$ by means of a complex canonical transformation generated by the equivalent of $\mathcal{S}$, $\tilde{\mathcal{S}} = 2i \int \tilde{E}^a_i \Gamma^i_a d^3 x$. Notice that we have an explicit form for $\tilde{S}$, whereas in the previous formulation embodied in (1.2) we would have had to find a solution to the Einstein-Hamilton-Jacobi equation in order to give an explicit expression for $\mathcal{S}$. Introducing $\tilde{E}^a_i$ as a basic variable introduces a new symmetry, the freedom to perform $SO(3)$ rotations in the $ijk$-indices without changing $g_{ab}$. This symmetry is mirrored in a new constraint which needs to be added to the usual diffeomorphism and Hamiltonian constraints. The constraint structure in the new variables is

$$\nabla_a \tilde{E}^a_i = 0,$$  (1.13a)

$$F^i_a \tilde{E}^a_i = 0,$$  (1.13b)

$$\varepsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F_{abk} + 2(\det \tilde{E}^a_i)\Lambda = 0,$$  (1.13c)

where (1.13a) is the basis-rotation or Gauss-law constraint, (1.13b) is the diffeomorphism constraint, and (1.13c) is the Hamiltonian constraint (modulo a term proportional to the Gauss law constraint). The quantity

$$F^i_{ab} = 2\partial_{[a} a^i_{b]} + \varepsilon^{ijk} A^i_a A^k_b,$$  (1.14)

is the curvature of the connection $A^i_a$, and $\nabla_a \lambda_i = \partial_a \lambda_i + \varepsilon^{ijk} A^j_{ab} \lambda_k$ is the $SO(3)$ covariant derivative acting on internal indices.

Of course, the definition $A^i_a = \Gamma^i_a + iK^i_a$ is equivalent to $\tau^{ab} = \delta \mathcal{S}/\delta g_{ab} - i\pi^{ab}$, and as we mentioned above, the possibility of other linear combinations of momenta and functions of the metric (here triad) still hold. It should be possible to find a generator for a real canonical transformation of the form $A^i_a = -\mathcal{F}^i_a + K^i_a$ that removes the potential term from the Hamiltonian constraint just as the transformation
\[ \tau^{ab} = -\delta S/\delta g_{ab} + \pi^{ab} \] does if the generator \( S \) satisfies the equivalent of the normal Einstein-Hamilton-Jacobi equation. Unfortunately, no such generator is known at this time. It is also possible to construct general transformations of the type given by (1.10) which do not remove the curvature terms from (1.3). In fact, one of us (F. B.) has studied canonical transformations of the type \( \tilde{E}_i^a = \tilde{E}_i^a, \ A_i^a = \Gamma_i^a + \beta K_i^a \) where \( \beta \) was taken to be -1 and the Hamiltonian constraint became\[ \varepsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b (F_{abk} - 2R_{abk}) - (\det \tilde{E}_i^a)\Lambda = 0, \tag{1.15} \]
where \( R_{ab} \) is the curvature of the connection \( \Gamma_i^a, \)
\[ R_{ab}^i = 2\partial[a\Gamma_i^b] + \varepsilon_{ijk}^a \Gamma_j^a \Gamma_k^b. \tag{1.16} \]
As mentioned above, real transformations have the advantage (at the possible cost of more complicated equations) of not requiring reality conditions in the quantum formulation.

c) Minisuperspace Models

In order to give concrete examples of possible Hamiltonian formulations that we discussed above, we would like to apply them to minisuperspace models where, as we will show, they lead to a number of exact solutions in the minisuperspace sector, some of which were known and some of which are new.

The minisuperspace examples we will use are the diagonal Class A Bianchi cosmological models where the metric has the form
\[ ds^2 = -dt^2 + g_{IJ}(t)\sigma^I\sigma^J, \tag{1.17} \]
where \( g_{IJ} \) is a diagonal matrix and the \( \sigma^I \) are invariant one-forms that satisfy \( d\sigma^I = (1/2)C^I_{JK}\sigma^J \wedge \sigma^K \) where the \( C^I_{JK} \) are structure constants of the form \( C^I_{JK} = m^{IJ\ell} \varepsilon_{\ell\ell\ell}, \) where \( m^{IJ} \) is a matrix of constants \[8\]. We will be most interested in the Bianchi IX case where \( m^{IJ} = \delta^{IJ} \).

For Class A Bianchi we can write the ADM action in terms of \( g_{IJ}(t) \) and the basis components of the ADM momentum as
\[ I = \int \left\{ \pi^{IJ}g_{IJ} - \frac{1}{\sqrt{g}}\left| G_{IJKL}\pi^{IJ}\pi^{KL} \right| - \sqrt{g}(\zeta^3R - 2\Lambda) \right\} dt\sigma^1 \wedge \sigma^2 \wedge \sigma^3, \tag{1.18} \]
since for all vacuum diagonal Class A models the diffeomorphism constraint is identically zero. Here \( \zeta^3R \) is an algebraic function of \( g_{IJ}(t) \) and the structure constants \( C^I_{JK} \) (or \( m^{IJ} \)), and \( g = \det(g_{IJ}) \).

It is now possible to make the same canonical transformation \( \tau^{IJ} = f^{IJ} + i\pi^{IJ} \) as in (1.2), but the advantage in the minisuperspace is that we can realize \( f^{IJ} \) as \( \partial S/\partial g_{IJ} \), replacing the functional derivative by a partial derivative. The Einstein-Hamilton-Jacobi equation (1.5) becomes a partial differential equation,
\[ G_{IJKL} \frac{\partial S}{\partial g_{IJ}} \frac{\partial S}{\partial g_{KL}} + g\zeta^3R = 0. \tag{1.19} \]
This equation has a number of particular solutions for Bianchi IX models, some of which have been given elsewhere \[3\] and several more which will be discussed below. Notice again that this equation contains the entire classical problem (both Lorentzian and Euclidean), so, in principle, it has a rich solution space and consequently a large family of canonical transformations of the form \(\tau^{IJ} = (\partial S/\partial g_{IJ}) + i\pi^{IJ}\).

As before, \(\tau^{IJ} = 0\) is a solution to the Hamiltonian constraint (\(\Lambda = 0\))

\[
\mathcal{H}_\perp = 0 = -G_{IJKL}\tau^{IJ}\tau^{KL} + 2G_{IJKL}\partial S/\partial g_{IJ}\tau^{KL} + 2g\Lambda, \tag{1.20}
\]

but this is not the only possible solution, and as before, linear canonical transformations of the form \(\tau^{IJ} = (\partial S/\partial g_{IJ}) + \beta\pi^{IJ}\) are possible, where \(S\) may or may not be chosen to annihilate the \(\mathcal{3}\)R term. We will discuss these possibilities in the minisuperspace context below.

While we will refer to the connection between the formulation in terms of the \(S\) solutions given above and the Ashtekar variables and similar variables, we will generally work in terms of the variables themselves in order to write the minisuperspace quantum equations. Writing the metric in the form of (1.7), the one-forms \(\sigma^I\) are

\[
\sigma^I = \sigma^I_a(x^c)\,dx^a,
\]

and the orthonormal one-forms \(e^i_I\) are \(e^i_I(t)\sigma^I_a(x^c).\) The variables given above become:

\[
\begin{align*}
e^i_a &= e^i_I(t)\sigma^I_a(x^c), \tag{1.21a} \\
A^i_a &= a^i_I(t)\sigma^I_a(x^c), \tag{1.21b} \\
K^i_a &= k^i_I(t)\sigma^I_a(x^c), \tag{1.21c} \\
\tilde{E}^a_i &= E^a_I(t)\det(\sigma)\sigma^I_a(x^c), \tag{1.21d}
\end{align*}
\]

where all the \(x\)-dependence is contained in \(\sigma^I_a(x^c).\) The basic variables for our presentation will be functions of \(t\) (and \(m^{IJ}\)) only that can be written in the form

\[
F^i_{jk} = (a^i_I m^{LK}\varepsilon_{LJK} + \varepsilon^i_a a^a_j a^a_K)e^j_J e^K_k, \tag{1.22}
\]

\[
\Gamma^i = -\frac{1}{2(\det e^I_I)}[\delta^i_j e^j_{ik} m^{KJ} - 2e^j_i e_{ik} m^{JK}], \tag{1.23}
\]

and the curvature

\[
\varepsilon^{ijk}\tilde{E}^a_i \tilde{E}^b_j R_{abk} = -2(\det\sigma^I_a)^2 e^i_I e^j_J e^k_L (m^{LK} m^{IJ} - 2m^{LJ} m^{IK})], \tag{1.24}
\]

which, since it is a density, retains the \(x\)-dependent determinant \((\det\sigma^I_a).\)

Diagonal Class A Bianchi models, as we will see below, satisfy identically the Gauss-law and diffeomorphism constraints, so the only constraint that survives and provides the quantum operator we will need to determine the minisuperspace wave function of our models is the Hamiltonian constraint.

**d) The Quantum Problem**

As we mentioned above, we would like to use quantum minisuperspaces as models of quantum gravity in which it is possible to find exact particular solutions that can
be used to compare quantization in the different sets of canonical variables discussed above. We will use the Dirac scheme of quantization where we will apply the Hamiltonian constraint to a state function \( \Psi \) and obtain a form of the Wheeler-DeWitt equation for the function \( \Psi \). In order to do this we will realize some of the operators in the quantum system as derivative operators on functions of the others. In the ADM formulation, for example, the metric \( g_{IJ} \) and its conjugate momentum \( \pi^{IJ} \) become operators and one can make \( \Psi \) a function of \( g_{IJ} \) and realize \( \pi^{IJ} \) as \(-i\partial/\partial g_{IJ}\). It is also possible to choose the “momentum representation” in which \( \Psi \) is a function of \( \pi^{IJ} \) and \( g_{IJ} \) becomes \(-i\partial/\partial \pi^{IJ}\). It is also possible to have \( \Psi(g_{IJ}) \) and realize \( \tau^{IJ} \) as \( \partial/\partial g_{IJ} \) as in the Bargmann-Segal formulation\(^5\) or choose the “connection representation” in which \( \Psi = \Psi(\tau^{IJ}) \) and \( g_{IJ} \) becomes \(-\partial/\partial \tau^{IJ}\). Since \( a^I_I \) in the Ashtekar representation is the conjugate of \( E^I_i \), we have the same possibilities, that of \( E^I_i \rightarrow E^I_i , \ a^I_i \rightarrow \partial/\partial E^I_i \) or \( E^I_i \rightarrow \partial/\partial a^I_i , \ a^I_i \rightarrow a^I_i \). We will investigate a number of them.

Notice that for any solution to (1.19) the operator version of (1.20) in the \( \tau_{IJ} \rightarrow \partial/\partial g_{IJ} \) representation (with \( \Lambda = 0 \) and all derivatives standing to the right) has the form

\[
\hat{H}_\perp = -G_{IJKL} \frac{\partial^2}{\partial g_{IJ} \partial g_{KL}} + 2G_{IJKL} \frac{\partial S}{\partial g_{IJ}} \frac{\partial}{\partial g_{KL}},
\]

which has as a solution to \( \hat{H}_\perp \Psi = 0, \Psi = \Psi_0 = \text{const} \). By the usual transformation of variables we can construct a solution of the usual Wheeler-DeWitt equation of the form \( \psi = e^{\pm S} \Psi_0 \). This formal solution becomes a true solution if we have an explicit solution for \( S \). Notice that for the usual Hamilton-Jacobi formulation a solution \( \psi = e^{\pm is} \Psi_0 \) is possible if a solution \( S \) can be found for the usual Einstein-Hamilton-Jacobi equation. It is also possible to study the quantum problem for any of the linear canonical transformations of the form \( g_{IJ} \rightarrow g_{IJ}, \pi^{IJ} \rightarrow f^{IJ} + \beta \pi^{IJ} \). Of course, the equivalent Ashtekar-like transformations also lead to quantum equations and there exist similar maps among the quantum solutions. Our plan is to investigate a number of these possibilities, present several exact solutions, and use them as model examples of possible quantum solutions in the full theory of gravity and discuss the relation between them and such solutions.

II Solutions to the Hamilton–Jacobi Equation

The purpose of this section is to discuss new solutions to the Hamilton–Jacobi equations as an intermediate step to finding solutions to the Wheeler–DeWitt equation for Bianchi IX models. We do this both in the real ADM and the real Ashtekar formulations. Whereas in the ADM case the theory is explicitly real for both Euclidean and Lorentzian signatures (that we describe collectively by using the parameter \( \zeta \) introduced above) the usual way to treat Lorentzian signatures with Ashtekar variables is by working with complex fields and imposing some “reality” conditions that can be used, for example, as a tool for fixing the scalar product. A less conventional attitude is to use explicitly real “Ashtekar-like” variables (in the sense that they keep their geometrical meaning)\(^7\). The Lorentzian theory is recovered by modifying the
Hamiltonian constraint through the introduction of a potential term. In this paper we choose to concentrate on this second, and more novel, approach, motivated by the desire to know whether this new formulation provides us with a useful alternative to the use of reality conditions. To this end it proves to be convenient to compare the results obtained with ones corresponding to the ADM case, so we begin by studying the geometrodynamical formulation. In order to facilitate the comparison between the ADM and Ashtekar formalisms we slightly modify the ADM constraints by introducing an internal $SO(3)$ symmetry. For the specific example of the Bianchi IX model we get the constraints

\begin{align}
\varepsilon_{ijk}k_i^j\tilde{E}^{kl} &= 0 \\
\varepsilon^I{}_Jk^J_i E^I_i - 2\varepsilon_{ijk}e^I_j e^N_k \delta^{TN}k_i^k &= 0 \\
(dets)^2 \left\{ 2\zeta [(T\text{re}^2)^2 - 2(T\text{re}^4)] - 2\Lambda (det E^I_i) + 2k_i^i k_J^J E^I_i E^I_J \right\} &= 0,
\end{align}

where $T\text{re}^2 \equiv e^I_i e^J_j \delta^{IJ} \delta_{ij}$ and $T\text{re}^4 \equiv e^I_i e^J_j e^K_k e^L_l \delta^{IJ} \delta^{KL} \delta_{ij} \delta_{kl}$ and $\zeta$ controls the space–time signature ($\zeta = \pm 1$ for Lorentzian and Euclidean signatures respectively). In this paper we concentrate on the study of Mixmaster models for which $k_i^i$ and $\tilde{E}^i_i$ are taken to be diagonal,

\begin{align}
k_i^i &\equiv \begin{bmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \\
E^I_i &\equiv \begin{bmatrix} M & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & L \end{bmatrix},
\end{align}

the variables $\mu, \nu, \lambda, M, N, L$ introduced above are canonically conjugate pairs, i.e.

\begin{align}
\{ M, \mu \} &= 1, \\
\{ N, \nu \} &= 1, \\
\{ L, \lambda \} &= 1,
\end{align}

and the remaining Poisson brackets are zero. The previous expressions, together with (1.21d) allow us to write

\begin{align}
e^I_i &= \frac{1}{\sqrt{2}(LMN)^{1/2}} \begin{bmatrix} 1/M & 0 & 0 \\ 0 & 1/N & 0 \\ 0 & 0 & 1/L \end{bmatrix}.
\end{align}

Introducing (2.2) and (2.4) in (2.1) we find that the first two constraint equations are identically satisfied and the scalar constraint is given by

\begin{align}
2(\mu \nu MN + \mu \lambda ML + \nu \lambda NL) + \zeta (M^2 + N^2 + L^2) - \\
- \zeta \left( \frac{M^2 L^2}{N^2} + \frac{N^2 L^2}{M^2} + \frac{M^2 N^2}{L^2} \right) - 2\Lambda MNL = 0.
\end{align}

The usual Ashtekar constraints for type A Bianchi models are ($m^{IJ} = \delta^{IJ}$ gives Bianchi IX)

\begin{align}
\varepsilon_{ijk}a^j_i E^{ij} &= 0, \\
(m^{IL} \varepsilon_{LJK}a^I_l + \varepsilon^{ijk} a_{j;jk} a_{kk})E^K_i &= 0,
\end{align}
\[ \varepsilon_{ijk} \left[ a^i_m m^{IL} \varepsilon_{LJK} + \varepsilon^i \ell_m a^j \ell a^K \right] E_j^I E^K_m = 0. \]

With this form of the constraints real variables describe Euclidean signatures and complex variables, with the addition of reality conditions, Lorentzian signatures. For Bianchi IX diagonal models we write

\[ a^i_j \equiv \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad E^i_j \equiv \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & G \end{bmatrix}, \]

where the pairs \((\alpha, A), (\beta, B), (\gamma, G)\) are canonically conjugate. If we use the real formulation in terms of Ashtekar-like variables introduced in Ref. and follow the same steps as before we find that the Hamiltonian constraint has a potential term. For the Bianchi IX model the constraint is given by

\[ \varepsilon_{ijk} \left[ a^i_m m^{IL} \varepsilon_{LJK} + \varepsilon^i \ell_m a^j \ell a^K \right] E_j^I E^K_m - 2\Lambda (\det E^i_j) + 4(e^i_j e^j_i) (\delta^K_L m^{LJ} - 2m^{LJ} m^{IK}) = 0, \]

with \(m^{IJ} = \delta^{IJ}\). If we introduce (2.7), the Gauss law and vector constraints are identically satisfied (as before) and the scalar constraint (2.6c) becomes

\[ 2(\alpha \beta + \gamma) AB + 2(\alpha \gamma + \beta) AG + 2(\beta \gamma + \alpha) BG + 2\Lambda ABG = 0, \]

while the constraint (2.8) for the real Ashtekar-like variables becomes

\[ 2(\alpha \beta + \gamma) AB + 2(\alpha \gamma + \beta) AG + 2(\beta \gamma + \alpha) BG +
+ 2(A^2 + B^2 + G^2) - \left( \frac{A^2 B^2}{G^2} + \frac{A^2 G^2}{B^2} + \frac{B^2 G^2}{A^2} \right) - 2\Lambda ABG = 0. \]

In conclusion, we see that to quantize these models we only need to consider the Hamiltonian constraints (2.5), (2.9), and (2.10). In all the cases we can use either a “position” representation or a momentum representation. Since both (2.5) and (2.10) are non-polynomial in some of the variables, we multiply them by appropriate factors in order to avoid the appearance of derivatives in the denominators in some of the quantizations of the model. The unpleasant consequence of this is that the differential equations that the wave functions will have to satisfy are of very high order.

Starting from (2.5), if we quantize by realizing the operators \( \hat{\mu}, \hat{\nu}, \hat{\lambda}, \hat{M}, \hat{N}, \hat{L} \) as

\[ \mu, \nu, \lambda, \hat{M}, \hat{N}, \hat{L} \rightarrow i\partial_\mu, i\partial_\nu, i\partial_\lambda, \]

\((\hbar = 1)\) we have \([\hat{\mu}, \hat{M}] = -i, [\hat{\nu}, \hat{N}] = -i, \) and \([\hat{\lambda}, \hat{L}] = -i\). The Wheeler-DeWitt equation now becomes (choosing the operator ordering corresponding to writing all the derivatives to the right and multiplying the constraint by \(N^2 M^2 L^2\))

\[ \left\{ 2\left[ \mu \nu \partial_\mu \partial_\nu + \mu \lambda \partial_\mu \partial_\lambda + \nu \lambda \partial_\nu \partial_\lambda \right] \partial^2_\mu \partial^2_\nu \partial^2_\lambda + \zeta \partial^2_\mu \partial^2_\nu \partial^2_\lambda \left( \partial^2_\mu + \partial^2_\nu + \partial^2_\lambda \right) - \frac{1}{2} \zeta \left( \partial^2_\mu \partial^4_\mu + \partial^4_\mu \partial^2_\mu + \partial^2_\lambda \partial^4_\lambda \right) - 2i\Lambda \partial^2_\mu \partial^2_\nu \partial^2_\lambda \right\} \Psi = 0. \]

\(1\)For the Euclidean theory this means just that \(\{\alpha, A\} = \{\beta, B\} = \{\gamma, G\} = 1\) with the remaining Poisson brackets equal to zero. If we consider complex variables the previous Poisson brackets pick a purely imaginary factor and become \(\{\alpha, A\} = \{\beta, B\} = \{\gamma, G\} = i\).
If instead, we quantize using
\[
\hat{\mu}, \hat{\nu}, \hat{\lambda} \rightarrow -i\partial_M, -i\partial_N, -i\partial_L,
\]
\[
\hat{M}, \hat{N}, \hat{L} \rightarrow M, N, L,
\]
we get
\[
\left\{ -2(MN\partial_M\partial_N + ML\partial_M\partial_L + NL\partial_N\partial_L) + \zeta(M^2 + N^2 + L^2) - \frac{1}{2} \zeta \left( \frac{M^2L^2}{N^2} + \frac{L^2N^2}{M^2} + \frac{M^2N^2}{L^2} \right) - 2\Lambda MNL \right\} \Psi = 0.
\]
We consider now the Hamiltonian constraint (2.9). If we quantize according to
\[
\hat{\alpha}, \hat{\beta}, \hat{\gamma} \rightarrow \alpha, \beta, \gamma,
\]
\[
\hat{A}, \hat{B}, \hat{G} \rightarrow -i\partial_\alpha, -i\partial_\beta - i\partial_\gamma,
\]
and write all the derivatives to the left we get the Wheeler-DeWitt equation (see Refs. [2,3])
\[
[\partial_\alpha \partial_\beta (\alpha\beta + \gamma) - \partial_\alpha \partial_\gamma (\alpha\gamma + \beta) - \partial_\beta \partial_\gamma (\beta\gamma + \alpha) + i\Lambda \partial_\alpha \partial_\beta \partial_\gamma] \Psi = 0.
\]
If we quantize using
\[
\hat{\alpha}, \hat{\beta}, \hat{\gamma} \rightarrow -i\partial_A, -i\partial_B - i\partial_G,
\]
\[
\hat{A}, \hat{B}, \hat{G} \rightarrow A, B, G,
\]
we find (See Ref. [4])
\[
[AB(-\partial_A \partial_B + i\partial_G) + AG(-\partial_A \partial_G + i\partial_B) + BG(-\partial_B \partial_G + i\partial_A) + \Lambda AGB] \Psi = 0.
\]
The equivalent of this equation for supergravity has been considered in Ref. [1].
Finally we consider the new Hamiltonian constraint (2.10). Using the quantizations introduced above we find, respectively,
\[
\left\{ -2[(\alpha\beta + \gamma)\partial_\alpha \partial_\beta + (\alpha\gamma + \beta)\partial_\alpha \partial_\gamma + (\beta\gamma + \alpha)\partial_\beta \partial_\gamma] \partial_\alpha^2 \partial_\beta^2 \partial_\gamma^2 - 2i\Lambda \partial_\alpha^3 \partial_\beta^3 \partial_\gamma^3 + 2\partial_\alpha^2 \partial_\beta^2 \partial_\gamma^2 \partial_\alpha + \partial_\gamma^2 \partial_\beta^2 (\partial_\alpha^2 + \partial_\beta^2 + \partial_\gamma^2) - (\partial_\alpha^2 \partial_\beta + \partial_\alpha^2 \partial_\gamma + \partial_\beta^2 \partial_\gamma) \right\} \Psi = 0,
\]
and
\[
\left\{ 2[AB(-\partial_A \partial_B + i\partial_G) + AG(-\partial_A \partial_G + i\partial_B) + BG(-\partial_B \partial_G + i\partial_A)] + +2(A^2 + B^2 + G^2) - \left( \frac{A^2B^2}{G^2} + \frac{A^2G^2}{B^2} + \frac{B^2G^2}{A^2} \right) \right\} \Psi = 0,
\]
where in (2.19) we have multiplied by $A^2B^2G^2$ to avoid derivatives in the denominators. Equations (2.17) and (2.19) are ninth order partial differential equations (PDE’s)(eighth order if we do not include the cosmological constant term); they are quite complicated and will have a number of “spurious” solutions introduced by multiplying by sixth order polynomials. Their solutions must be related to those of (2.14) and (2.20) by Fourier transform. Equations (2.17) and (2.18) were studied by Kodama[2]. In this paper we will concentrate on the discussion of the solutions to Eq. (2.20) and their relationship with solutions to the other equations.
In order to study equations (2.14), (2.16), (2.18), and (2.20) we will write $\Psi = W e^{-S}$ where $W$ and $S$ are functions of $(M, N, L, (\alpha, \beta, \gamma), (A, B, G),$ and $(A, B, G)$ respectively. In this way we get the equations shown in Appendix A. We will look for particular solutions having the property that $S$ satisfies a Hamilton–Jacobi equation (much in the spirit of the WBK approximation scheme) In this way each of the equations in Appendix A divides into two; an equation for $S$ and another for $W$. The Hamilton–Jacobi equations obtained (corresponding to (2.14), (2.16), (2.18), and (2.20) respectively) are

\[
-2 [ MN \partial_M S \partial_N S + LM \partial_L S \partial_M S + LN \partial_L S \partial_N S ] + \\
\left[ \zeta (M^2 + N^2 + L^2) - \frac{1}{2} \zeta \left( \frac{M^2 L^2}{N^2} + \frac{N^2 L^2}{M^2} + \frac{M^2 N^2}{L^2} \right) - 2 \Lambda MN L \right] = 0, \tag{2.21}
\]

\[
(\alpha \beta + \gamma) \partial_\alpha S \partial_\beta S + (\alpha \gamma + \beta) \partial_\alpha S \partial_\gamma S + (\beta \gamma + \alpha) \partial_\beta S \partial_\gamma S + i \Lambda \partial_\alpha S \partial_\beta S \partial_\gamma S = 0, \tag{2.22}
\]

\[
AB \partial_A S \partial_B S + AG \partial_A S \partial_G S + BG \partial_B S \partial_G S + \\
i(AB \partial_A S + AG \partial_B S + BG \partial_A S) - \Lambda ABG = 0, \tag{2.23}
\]

\[
AB \partial_A S \partial_B S + AG \partial_A S \partial_G S + BG \partial_B S \partial_G S + \\
i(AB \partial_A S + AG \partial_B S + BG \partial_A S) - (A^2 + B^2 + G^2) + \\
\frac{1}{2} \left( \frac{A^4 B^2}{G^2} + \frac{A^2 G^2}{B^2} + \frac{B^2 G^2}{A^2} \right) + \Lambda ABG = 0. \tag{2.24}
\]

We will now discuss some solutions (among them several new ones) to the Hamilton-Jacobi equations (2.21–2.24). Starting from (2.21), and putting $\Lambda = 0$, we will take an ansatz of the form

\[
S = a \left( \frac{LM}{N} + \frac{LN}{M} + \frac{MN}{L} \right) + bL + cM + dN + e, \tag{2.25}
\]

where $a, b, c, d, e$ are constants that we have to fix. The Moncrief-Ryan solution to (2.21) is contained in this family and corresponds to $b = c = d = e = 0$. Notice that the addition of linear terms is not trivial due to the non-linear character of the Hamilton-Jacobi equation. Substituting (2.25) in (2.21) we find that the equation is satisfied if the constants in (2.25) are solutions of the following equations

\[
-4a^2 + \zeta = 0, \\
b + 2ad = 0, \\
bd + 2ac = 0, \\
\tag{2.26}
\]

\[
cd + 2ab = 0,
\]

and $e$ is arbitrary. For Lorentzian signatures ($\zeta = +1$) the possible solutions to (2.26) are

\[
(a, b, c, d) = (\frac{1}{2}, 0, 0, 0), (\frac{1}{2}, -1, -1, -1), (\frac{1}{2}, 1, 1, -1), (\frac{1}{2}, 1, -1, 1), (\frac{1}{2}, -1, 1, 1), \\
(\frac{1}{2}, 0, 0, 0), (\frac{1}{2}, 1, 1, 1), (\frac{1}{2}, -1, -1, 1), (\frac{1}{2}, -1, 1, -1), (\frac{1}{2}, 1, -1, -1). \tag{2.27}
\]

If $\zeta = -1$ (Euclidean signatures) the solutions to (2.26) are given by

\[
(a, b, c, d) = (\frac{1}{2}, 0, 0, 0), (\frac{1}{2}, -i, -i, -i), (\frac{1}{2}, i, i, -i), (\frac{1}{2}, i, -i, i), (\frac{1}{2}, -i, i, i), \\
(\frac{1}{2}, 0, 0, 0), (\frac{1}{2}, i, i, i), (\frac{1}{2}, -i, -i, i), (\frac{1}{2}, -i, i, -i), (\frac{1}{2}, i, -i, -i). \tag{2.28}
\]
The solutions shown above seem to be in correspondence with the analytical solutions known in closed form for the Bianchi IX model[11]. It is possible that there are no more solutions for $S$ that can be written in analytic form[12]. The difference between the solutions for Euclidean and Lorentzian formulations is the appearance of a global, purely imaginary, factor. We have not been able to find solutions for non-zero cosmological constant. Equation (2.22) has been studied in detail by Kodama[2]. Here we give the known solutions for completeness (in this case the cosmological constant $\Lambda$ must be different from zero)

$$S = \frac{3i}{2\Lambda} \left[ \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta\gamma \right]. \quad (2.29)$$

We are not aware of any solution for $\Lambda = 0$.

Equations (2.23) and (2.24) differ only in the potential term and the sign of the term with the cosmological constant. For $\Lambda = 0$ we try solutions of the form

$$S = a \left( \frac{AB}{G} + \frac{AG}{B} + \frac{BG}{A} \right) + bA + cB + dG + e. \quad (2.30)$$

It is easy to verify that (2.30) are solutions to the Hamilton-Jacobi equation (2.23) provided that the constants satisfy the following conditions (e arbitrary)

$$a^2 + ia = 0, \quad bc + id + 2ad = 0, \quad bd + ic + 2ac = 0, \quad cd + ib + 2ab = 0, \quad (2.31)$$

the solutions to the previous equations are

$$(a, b, c, d) = (0, 0, 0, 0), \, (0, -i, -i, -i), \, (0, i, -i, i), \, (0, -i, i, i), \, (0, i, i, -i), \, (0, -i, i, i), \, (-i, 0, 0, 0), \, (-i, i, i, i), \, (-i, -i, -i, i), \, (-i, i, -i, -i). \quad (2.32)$$

If we consider instead equation (2.24) with its potential term, (2.30) is a solution (for $\Lambda = 0$) when the constants $a, b, c, d, e$ are solutions to the equations

$$2a^2 + 2ia - 1 = 0, \quad bc + id + 2ad = 0, \quad bd + ic + 2ac = 0, \quad cd + ib + 2ab = 0, \quad (2.33)$$

where, as before, $e$ is arbitrary. The solutions to (2.33) are

$$(a, b, c, d) = \left( \frac{i+1}{2}, 0, 0, 0 \right), \left( \frac{i+1}{2}, -1, 1, 1 \right), \left( \frac{i-1}{2}, 1, 1, -1 \right), \left( -1, -1, -1, -1 \right), \left( \frac{i+1}{2}, 0, 0, 0 \right), \left( \frac{i+1}{2}, 1, 1, 1 \right), \left( \frac{i-1}{2}, 1, -1, 1 \right), \left( \frac{i+1}{2}, 1, -1, 1 \right). \quad (2.34)$$

As before we know of no solution for the $\Lambda \neq 0$ case.
III Discussion of Solutions

In order to display the solutions given in the previous section we will give them in terms of the Misner parametrization of the Mixmaster model. For the Lorentzian ADM equations (2.1) the metric variables given in Eq. (2.2), $M$, $N$, $L$, are, in terms of the Misner variables,

$$g_{11} = e^{2\alpha} e^{2\beta_+ + 3\sqrt{3} \beta_-}, \quad g_{22} = e^{2\alpha} e^{2\beta_+ - 2\sqrt{3} \beta_-}, \quad g_{33} = e^{2\alpha} e^{-4\beta_+},$$

$$M = e^{2\alpha} e^{2\beta_+}, \quad N = e^{2\alpha} e^{-\beta_+ + \sqrt{3} \beta_-}, \quad L = e^{2\alpha} e^{-\beta_+ - \sqrt{3} \beta_-}.$$  

(3.1)

This means that the Lorentzian solutions for $S$ given in Eq. (2.25) have the form (with the trivial constant $e = 0$)

$$S = e^{2\alpha} \left( a e^{-4\beta_+} + 2 e^{2\beta_+} \cosh 2\sqrt{3} \beta_- \right) + e^{2\beta_+} + e^{-\beta_+} \left( (b + d) \cosh \sqrt{3} \beta_- + (d - b) \sinh \sqrt{3} \beta_- \right).$$  

(3.2)

Notice that the solutions (2.30) for the real Ashtekar equations have exactly the same form as (3.2) since $A = M$, $B = N$, $G = L$, but with different constants $a, \ldots, d$.

We would like to display the solutions for $\alpha = \text{const.}$ (average radius of the universe constant) in the $\beta_+ \beta_-$-plane. The solution for $S$ (3.2) with $b = c = d = 0$ is, as was mentioned above, the solution given by Moncrief and Ryan (and by Graham as the bosonic sector of supergravity) and Figure 1 is a three-dimensional plot of $|\Psi|^2 = e^{-2S}$ for $\alpha = 0$ in the $\beta_+ \beta_-$-plane. The line shown on the plot represents a contour of the potential $V(\beta_\pm)$ given in Ref. 3 which drives the Mixmaster model. The potential has a triangular symmetry, where rotation by $\pi/3$ in the $\beta_+ \beta_-$-plane leaves $V$ invariant. There are also soft “channels” where $V$ goes to zero for large values of $\beta_\pm$ that begin at the corners of the triangle and run directly to infinity. These channels become exponentially narrower at large values of $\beta_+$ and $\beta_-$. The straight lines that define the center of these channels each represent the Taub model, a special case of the Mixmaster model[13]. The point $\beta_+ = \beta_- = 0$ is the $k = +1$ Robertson-Walker universe. The solution shown in Fig. 1 is peaked over $\beta_+ = \beta_- = 0$ and has a roughly triangular form with the points of the triangle lying in the directions of the three “Taub” channels of the potential.

At first glance there seems to be a large number of new solutions with $b, \ldots, d$ nonzero, but these solutions share the triangular symmetry of the Misner potential, so some of them are just copies of the others related by a rotation by $\pi/3$ in the $\beta_+ \beta_-$-plane. Of the Lorentzian “wormhole” solutions of Eqs. (2.26), with $a = 1/2$, there is one isolated new solution with $b = c = d = -1$, which has the triangular symmetry of the potential. This solution, for $\alpha = 0$ is shown as a graph of $e^{-2S}$ in Fig. 2. It has a peak over $\beta_+ = \beta_- = 0$ as before, but it seems to single out the Taub model channels with “arms” where the solution goes asymptotically to one as the distance out along the three channels becomes infinite. The “arms” become narrower rapidly as the distance from $\beta_+ = \beta_- = 0$ becomes large. For the arm along the $\beta_+$-axis at $\alpha = 0$ and for small $\beta_-$ and large $\beta_+$, $e^{-2S}$ becomes

$$\exp\{-12e^{2\beta_+} \beta_-^2\},$$  

(3.3)
a Gaussian in $\beta_-$ with width $\sqrt{1/12}e^{-\beta_+}$. We will discuss the problem of “normalization” of these solutions below.

Of the last three Lorentzian $a = 1/2$ solutions given in (2.17), only one is relevant, with the others given by $\pi/3$ rotations in the $\beta_+\beta_-$-plane. We choose $b = d = 1$, $c = -1$, and for these constants $e^{-2S}$ is shown in Figure 3. This solution is very strongly peaked over almost the entire Taub model line that is represented by $\beta_- = 0$. As before, for large $\beta_+$ and small $\beta_-$ we have have exactly the same form for $e^{-S}$ as given in (3.3) and $e^{-2S} \to 1$ for $\beta_- = 0$ and $\beta_+ \to \infty$. This solution falls rapidly to zero for $\beta_+ < 0$, so the peak near $\beta_+ = \beta_- = 0$ evident in the solutions given above has disappeared.

For the solution given by (2.30), as we have mentioned, Eq. (3.2) still describes the “real Ashtekar” solutions with $a, \cdots, d$ given by (2.34). Since the only difference between this solution and the Lorentzian solutions is that $a$ becomes complex, but with real part the same as for the Lorentzian solutions, so $|\Psi|^2$ is the same as in the Lorentzian case, and Figures 1-3 give this function also.

For the ADM Euclidean solution and the true Ashtekar solutions $S$ becomes pure imaginary, and $|\Psi|^2$ is one, so we give no graphs of these functions. Of course, there is no reason to suppose that $|\Psi|^2$ in the Ashtekar case has any intrinsic meaning (such as the “probability” of finding the universe at some $\beta_+\beta_-$ point at $\alpha = 0$), since there is no agreement on probability measures for these complexified theories.

It is obvious, however, that our graphing of $|\Psi|^2$ implies that we are thinking of the Hartle–Hawking\[15]\ definition of the probability associated with the wave function of the universe. The definition of probability measures on solutions to various formulations of quantum gravity has been a difficult problem. The Hartle–Hawking definition for ordinary ADM variables is one possibility, but other definitions in terms of superspace currents have also been proposed\[16]. For complexified variables of the Bargmann-Segal or Ashtekar type, the construction of probability measures is even more complicated, so we will not attempt to give any such measures for the solutions given in (2.22) and (2.23).

The situation is slightly different for the ADM solutions (2.25) and the real Ashtekar-like solutions (2.30). Notice that the derivative terms in (2.14) form a Laplace-Beltrami operator for the superspace metric

$$g_{ij} = \begin{pmatrix} \frac{1}{M^2} & -\frac{1}{M} & -\frac{1}{M} & -\frac{1}{M} \\ -\frac{1}{M} & \frac{1}{N^2} & -\frac{1}{N^2} & -\frac{1}{N^2} \\ -\frac{1}{M} & -\frac{1}{N^2} & \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{M} & -\frac{1}{N^2} & -\frac{1}{L^2} & \frac{1}{L^2} \end{pmatrix},$$

(3.4)

so we can define a conserved superspace current

$$j_0^k = -i \left( MN\Psi^*\partial_N\Psi + ML\Psi^*\partial_L\Psi - MN\Psi\partial_N\Psi^* - ML\Psi\partial_L\Psi^* \\
MN\Psi^*\partial_M\Psi + NL\Psi^*\partial_L\Psi - MN\Psi\partial_M\Psi^* - NL\Psi\partial_L\Psi^* \\
ML\Psi^*\partial_M\Psi + NL\Psi^*\partial_N\Psi - ML\Psi\partial_M\Psi^* - NL\Psi\partial_N\Psi^* \right)$$

(3.5)

valid for any solution to (2.14). Unfortunately $\Psi = We^{-S}$ for $W = \text{const.}$ and $S$ given by (2.25) is not a solution to (2.14). For $b = c = d = 0$ Moncrief and Ryan\[3] showed that the factor ordering for the derivative terms of (2.14) that allows $W =
const. is
\[ \frac{1}{L} \partial_M MNL \partial_N + \frac{1}{N} \partial_L MNL \partial_M + \frac{1}{M} \partial_N MNL \partial_L, \]
and a conserved superspace current for this equation is
\[ j^k = (MNL) j^k_0, \]
where \( j^k_0 \) is defined above. Unfortunately both these currents are zero for \( \Psi \) given by (2.25) with \( b = c = d = 0 \), since \( \Psi \) is real. It might appear that the solutions (2.30) would have nonzero currents since they are complex, but the equation of motion (2.20) has pure imaginary terms, and we have been unable to find a conserved current that is compatible with it, so we cannot say that such a current would be nonzero. The only interesting “probability” measure is then the Hartle–Hawking \( |\Psi|^2 \) which we have given in Figs. 1-3.

We might worry about the “normalization” of the new functions given by (2.25) with \( b, c, d \neq 0 \) since they do not fall off rapidly for large \( \beta_\pm \) (we are thinking of “normalization” in the sense of an ADM equation solution with \( \alpha \) taken as an internal time, which means that \( \int d\beta_+ d\beta_- |\Psi|^2 \) over the \( \beta_+ \beta_- \)-plane should be finite). However, the “arms” of these solutions that remain finite as we move out the Taub channels begin near \( \beta_+ = \beta_- = 0 \) and become very narrow rapidly, and the fact that
\[ \int_0^\infty d\beta_+ \int_{-\infty}^\infty d\beta_- \exp\{-12e^{2\beta_+} \beta_-^2\} = \frac{1}{2} \sqrt{\pi/3} \]
implies that \( \int d\beta_+ d\beta_- |\Psi|^2 \) will remain finite.

IV Conclusions and Suggestions for Further Research

In this paper we have studied the use of canonical transformations to find particular solutions to quantum gravity, at least in the minsuperspace sector. What we showed was that any solution, \( S \), to the Einstein-Hamilton-Jacobi equation can be used to generate a canonical transformation that leads to a solution of the form \( \Psi = We^{-S} \). The prefactor \( W \) serves to allow us to adjust the factor ordering of the Wheeler-DeWitt equation, since any choice of \( W \) is valid for some factor ordering. For some solutions \( S \) the factor ordering that allows \( W = \text{const.} \) is relatively simple, as in (3.6), but for others, the only such factor ordering we have found is somewhat clumsy (see Appendix B), but simpler expressions for \( W \) probably exist.

We also showed that there are, in principle, a large number of functions \( S \) that are solutions to modified Einstein-Hamilton-Jacobi equations that represent other canonical transformations that do not annihilate the curvature term in the Hamiltonian constraint. Each of these canonical transformations allows us to find exact solutions of the form \( We^{-S} \). The problem is that one would like to have an analytic solution for \( S \), and, in principle, one must solve a nonlinear functional differential equation such as (1.5) in order to find \( S \). If one does not insist that \( S \) be a true solution to the Einstein-Hamilton-Jacobi equation, but rather a solution to the modified
equation that would come from (1.9), it may still be possible to simplify the curvature term in the equation to the point where $S$ may be found easily.

Seen from this viewpoint, the Ashtekar variables, based on a complex canonical transformation, have the advantage that $S$ is known exactly for the transformation, and one can construct solutions of the form $We^{-S}$ with $W = \text{const.}$ easily. Unfortunately there are two drawbacks to this procedure. One is that often the $S$ one calculates is zero (for example, $S = 0$ is a possible solution to Eqs. [2.22] and [2.23] with $\Lambda = 0$). The other is that since the transformation is complex, there is a need for a “reality condition” on the functions $\Psi$. Because of this we investigated a real transformation similar to the one that generates the Ashtekar variables that requires no reality condition, but makes it necessary to solve a more complicated equation for $S$.

In order to give concrete examples of the idea of finding solutions given by generators $S$ obtained by solving the equivalent of the Einstein-Hamilton-Jacobi equation we investigated Bianchi-type minisuperspaces, and by writing the equations in terms of variables of the Ashtekar type, we were able to find several solutions for $S$, some of them new. Unfortunately, the only explicit factor ordering which allowed $W = \text{const.}$ we were able to find is not very appealing.

Of course, minisuperspace solutions, while they may be of interest as “wave functions of the universe”, are perhaps better thought of as clues to finding solutions to full quantum gravity. The Chern-Simons solution in quantum gravity was found in just this way. In the future we might hope to find solutions to full quantum gravity that are suggested by the minisuperspace exact solutions given above. This would require solutions $S$ to the functional Einstein-Hamilton-Jacobi equation or its analogues where the curvature term is not annihilated.

This idea is perhaps the strongest suggestion for further research. It has proved fruitful to look for solutions to the Einstein-Hamilton-Jacobi equation and its analogues, since it has been possible to find a number of exact solutions to a problem such as the Mixmaster model which has a reasonably complicated structure, and one could easily have assumed that it would be impossible to find any analytic particular solutions to the problem. That it is fairly easy to find such solutions in this minisuperspace case leads one to believe that it would not be impossible to find such solutions in full quantum gravity. The existence of the Chern-Simons solution seems to point in this direction.

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A Equations for $S$ and $W$

The equations obtained by writing the wave function as $\Psi = W e^{-S}$ and substituting it in (2.14, 2.16, 2.18, 2.20) are

\[
-2 [MN \partial_M \partial_N W + LM \partial_L \partial_M W + LN \partial_L \partial_N W - \\
MN (\partial_M \partial_N S)W - LM (\partial_L \partial_M S)W - LN (\partial_L \partial_N S)W - \\
MN \partial_M S \partial_N W - M L \partial_M S \partial_L W - LN \partial_L S \partial_N W - \\
MN \partial_M W \partial_N S - M L \partial_M W \partial_L S - LN \partial_L W \partial_N S + \\
MN (\partial_M S \partial_N S)W + ML (\partial_M S \partial_L S)W + LN (\partial_L S \partial_N S)W] + \\
\left[ \zeta(M^2 + N^2 + L^2) - \frac{1}{2} \zeta \left( \frac{M^2 N^2}{N^2} + \frac{N^2 L^2}{M^2} + \frac{M^2 N^2}{L^2} \right) - 2 \Lambda MN L \right] W = 0 \tag{A.1}
\]

\[
-(\alpha \beta + \gamma) [\partial_a \partial_B W - (\partial_a \partial_B S)W - \partial_a S \partial_B W - \partial_a W \partial_B S + (\partial_a S \partial_B S)W] - \\
-(\alpha \gamma + \beta) [\partial_a \partial_L W - (\partial_a \partial_L S)W - \partial_a S \partial_L W - \partial_a W \partial_L S + (\partial_a S \partial_L S)W] - \\
-(\beta \gamma + \alpha) [\partial_a \partial_M W - (\partial_a \partial_M S)W - \partial_a S \partial_M W - \partial_a W \partial_M S + (\partial_a S \partial_M S)W] + \\
+i \Lambda [\partial_a \partial_B \partial_L W - (\partial_a \partial_B \partial_L S)W - (\partial_a \partial_B S)\partial_L W - (\partial_a \partial_L S)\partial_B W] - \\
-(\partial_B \partial_L S)\partial_a W - (\partial_a \partial_B \partial_L S)W - (\partial_a \partial_B S)\partial_L W - (\partial_a \partial_L S)\partial_B W + \\
+(\partial_a \partial_L \partial_B \partial_S)W + (\partial_a \partial_L S)\partial_B W + (\partial_a \partial_B S)\partial_L W + (\partial_a \partial_B \partial_L S)W] = 0 \tag{A.2}
\]

\[
-AB \partial_A \partial_B W - AG \partial_A \partial_G W - BG \partial_B \partial_G W + \\
+AB \partial_A W \partial_B S + AG \partial_A W \partial_G S + BG \partial_B W \partial_G S + \\
+AB \partial_A S \partial_B W + AG \partial_A S \partial_G W + BG \partial_B S \partial_G W + \\
+AB (\partial_A \partial_B S)W + AG (\partial_A \partial_G S)W + BG (\partial_B \partial_G S)W - \\
-AB (\partial_A S \partial_B S)W - AG (\partial_A S \partial_G S)W - BG (\partial_B S \partial_G S)W + \\
+i (AB \partial_G W + AG \partial_B W + BG \partial_A W) - \\
-i (AB (\partial_G S)W + AG (\partial_B S)W + BG (\partial_A S)W + \Lambda AB G W = 0 \tag{A.3}
\]

\[
-AB \partial_A \partial_B W - AG \partial_A \partial_G W - BG \partial_B \partial_G W + \\
+AB \partial_A W \partial_B S + AG \partial_A W \partial_G S + BG \partial_B W \partial_G S + \\
+AB \partial_A S \partial_B W + AG \partial_A S \partial_G W + BG \partial_B S \partial_G W + \\
+AB (\partial_A \partial_B S)W + AG (\partial_A \partial_G S)W + BG (\partial_B \partial_G S)W - \\
-AB (\partial_A S \partial_B S)W - AG (\partial_A S \partial_G S)W - BG (\partial_B S \partial_G S)W + \\
+i (AB \partial_G W + AG \partial_B W + BG \partial_A W) - \\
-i (AB (\partial_G S)W + AG (\partial_B S)W + BG (\partial_A S)W) - \Lambda AB G W + \\
+(A^2 + B^2 + G^2)W - \frac{1}{2} \left( \frac{A^2 B^2}{G^2} + \frac{A^2 G^2}{B^2} + \frac{B^2 G^2}{A^2} \right) W = 0 \tag{A.4}
\]

The equations for $W$ are then

\[
MN \partial_M \partial_N W + LM \partial_L \partial_M W + LN \partial_L \partial_N W - \\
-MN (\partial_M \partial_N S)W - LM (\partial_L \partial_M S)W - LN (\partial_L \partial_N S)W - \\
-MN \partial_M S \partial_N W - M L \partial_M S \partial_L W - LN \partial_L S \partial_N W - \\
-MN \partial_M W \partial_N S - M L \partial_M W \partial_L S - LN \partial_L W \partial_N S = 0 \tag{A.5}
\]
\[-(\alpha \beta + \gamma) [\partial_\alpha \partial_\beta W - (\partial_\alpha \partial_\beta S)W - \partial_\alpha S \partial_\beta W - \partial_\alpha W \partial_\beta S] -
\[-(\alpha \gamma + \beta) [\partial_\alpha \partial_\gamma W - (\partial_\alpha \partial_\gamma S)W - \partial_\alpha S \partial_\gamma W - \partial_\alpha W \partial_\gamma S] -
\[-(\beta \gamma + \alpha) [\partial_\beta \partial_\gamma W - (\partial_\beta \partial_\gamma S)W - \partial_\beta S \partial_\gamma W - \partial_\beta W \partial_\gamma S] +
+i \Lambda [\partial_\alpha \partial_\beta \partial_\gamma W - (\partial_\alpha \partial_\beta \partial_\gamma S)W - (\partial_\alpha \partial_\beta S)\partial_\gamma W - (\partial_\alpha \partial_\gamma S)\partial_\beta W -
+(\partial_\alpha \partial_\beta S)(\partial_\gamma W) + (\partial_\alpha \partial_\gamma S)(\partial_\beta W) + (\partial_\beta \partial_\gamma S)(\partial_\alpha W) +
+\partial_\alpha S \partial_\beta S \partial_\gamma W + \partial_\alpha S \partial_\gamma S \partial_\beta W + \partial_\beta S \partial_\gamma S \partial_\alpha W] = 0
\]

\[\begin{align*}
-AB \partial_A \partial_B W - AG \partial_A \partial_G W - BG \partial_B \partial_G W + \\
+AB \partial_A W \partial_B S + AG \partial_A W \partial_G S + BG \partial_B W \partial_G S + \\
+AB \partial_A S \partial_B W + AG \partial_A S \partial_G W + BG \partial_B S \partial_G W + \\
+AB(\partial_A \partial_B S)W + AG(\partial_A \partial_G S)W + BG(\partial_B \partial_G S)W + \\
i(AB \partial_G W + AG \partial_B W + BG \partial_A W) = 0
\end{align*}\]

where in each case \(S\) must be a solution for the corresponding Hamilton-Jacobi equation. Notice that the equations for \(W\) that come from (A.3) and (A.4) appear to be the same, but the functions \(S\) that appear in them are different because they are solutions to different Hamilton-Jacobi equations.

**B  Factor ordering for the ADM solutions**

If we want \(W = \text{const.}\), one possible factor ordering for the derivative terms in (2.14) is

\[-2 \left( MN \frac{1}{\rho_1} \frac{\partial M \rho_1 \partial N}{\partial L} + ML \frac{1}{\rho_2} \frac{\partial M \rho_2 \partial L}{\partial N} + NL \frac{1}{\rho_3} \frac{\partial N \rho_3 \partial L}{\partial M} \right), \quad (B.1)\]

where \(\rho_1, \rho_2, \rho_3\) are functions of \(M, N, L\). Inserting \(\Psi = W_0 e^{-S}\) into (2.14) with the factor ordering (B.1) and \(W_0 = \text{const.}\) gives (for \(S\) given by [2.25])

\[a \left( \frac{MN}{L} + \frac{LN}{M} + \frac{LM}{N} \right) - aM \frac{\partial \rho_1}{\partial M} \left( - \frac{LM}{N} + \frac{LN}{M} + \frac{MN}{L} \right) -
\[\frac{aL}{\rho_2} \frac{\partial \rho_2}{\partial L} \left( \frac{LM}{N} - \frac{LN}{M} + \frac{MN}{L} \right) - aN \frac{\partial \rho_3}{\partial N} \left( \frac{ML}{N} + \frac{NL}{M} - \frac{MN}{L} \right) -
-dN \frac{M \partial \rho_1}{\rho_1 \partial M} - cM \frac{L \partial \rho_2}{\rho_2 \partial L} - bL \frac{N \partial \rho_3}{\rho_3 \partial N} = 0. \quad (B.2)\]

This is one equation for the three unknowns \(\rho_1, \rho_2, \rho_3\), so a solution is always possible. If, as an example, we take \(\rho_1 = NL, \rho_2 = MN, \) we find that \(\rho_3\) becomes

\[\rho_3 = MNL \exp \left\{ \frac{2 \left( \frac{MN}{L} + bL - \frac{M}{L} \right)}{\sqrt{M^2 + (1-b^2)L^2}} \tan^{-1} \left( \frac{-bL/2a + MNL}{\sqrt{M^2 + (1-b^2)L^2}} \right) \right\}. \quad (B.3)\]

This solution is not elegant, and there are probably more symmetric solutions that would be simpler.
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FIGURE CAPTIONS

Figure 1. The square of the wave function $\Psi = e^{-S}$ with $S$ given by (2.25) with $a = 1/2$, $b = c = d = e = 0$. The solid line is an equipotential of the Mixmaster potential. This solution is peaked around the Robertson-Walker universe which is represented by $\beta_+ = \beta_- = 0$.

Figure 2. $|\Psi|^2 = e^{-2S}$ for $S$ given by (2.25) with $a = 1/2$, $b = c = d = -1$, $e = 0$. The solid line is an equipotential of the Mixmaster potential. While the wave function is still peaked around the Robertson-Walker universe, it has “arms” that single out the Taub models represented by the “channels” in the potential.

Figure 3. $|\Psi|^2 = e^{-2S}$ for $S$ given by (2.25) with $a = 1/2$, $b = d = 1$, $c = -1$, $e = 0$. The solid line is again an equipotential of the Mixmaster potential. This solution is no longer peaked about the Robertson-Walker model, and lies almost entirely over the $\beta_+ > 0$ portion of the Taub model represented by $\beta_- = 0$. 
Figure 1.
Figure 3.