An Analysis of Asynchronous Stochastic Accelerated Coordinate Descent *

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Abstract

Gradient descent, and coordinate descent in particular, are core tools in machine learning and elsewhere. Large problem instances are common. To help solve them, two orthogonal approaches are known: acceleration and parallelism. In this work, we ask whether they can be used simultaneously. The answer is “yes”.

More specifically, we consider an asynchronous parallel version of the accelerated coordinate descent algorithm proposed and analyzed by Lin, Liu and Xiao [13]. We give an analysis based on the efficient implementation of this algorithm. The only constraint is a standard bounded asynchrony assumption, namely that each update can overlap with at most $q$ others. ($q$ is at most the number of processors times the ratio in the lengths of the longest and shortest updates.) We obtain the following three results:

- A linear speedup for strongly convex functions so long as $q$ is not too large.
- A substantial, albeit sublinear, speedup for strongly convex functions for larger $q$.
- A substantial, albeit sublinear, speedup for convex functions.

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1 Introduction

We consider the problem of finding an (approximate) minimum point of a convex function \( f : \mathbb{R}^n \to \mathbb{R} \) having a Lipschitz bound on its gradient.

Gradient descent is the standard solution approach for huge-scale problems of this type. Broadly speaking, gradient descent proceeds by moving iteratively in the direction of the negative gradient of a convex function. Coordinate descent is a commonly studied version of gradient descent. It repeatedly selects and updates a single coordinate of the argument to the convex function. Stochastic versions are standard: at each iteration the next coordinate to update is chosen uniformly at random.

**Speed up by acceleration** Acceleration is a well-known technique for improving the rate of convergence. It improves the rate from \( \Theta \left( [1 - \Theta(\mu)]^T \right) \to \Theta \left( [1 - \Theta(\sqrt{\mu})]^T \right) \) on strongly convex functions with strong convexity parameter \( \mu \) (defined in Section 2), and from \( \Theta(1/T) \) to \( \Theta(1/T^2) \) on convex functions (here \( \mu = 0 \)). So the gains are most significant when \( \mu \) is small.

**Speed up by parallelism** Another way to achieve speedup and thereby solve larger problems is parallelism. There have been multiple analyses of various parallel implementations of coordinate descent [15, 14, 16, 20, 6, 7, 19, 4].

One important issue in parallel implementations is whether the different processors are all using up-to-date information for their computations. To ensure this requires considerable synchronization, locking, and consequent waiting. Avoiding the need for the up-to-date requirement, i.e. enabling asynchronous updating, was a significant advance. The advantage of asynchronous updating is it reduces and potentially eliminates the need for waiting. At the same time, as some of the data being used in calculating updates will be out of date, one has to ensure that the out-of-datedness is bounded in some fashion. In this paper, we ask the following question:

Can acceleration and asynchronous parallel updating be applied simultaneously and effectively to coordinate descent?

It was an open question whether the errors introduced by parallelism and asynchrony would preclude the speedups due to acceleration. For Devolder et al. [8] have shown that with arbitrary errors in the computed gradients \( g \) that are of size \( \Theta(\epsilon g) \) for some constant \( \epsilon > 0 \), in general, speedup due to acceleration cannot be maintained for more than a bounded number of steps. More specifically, they observe that the superiority of fast gradient methods over classical ones is no longer absolute when an inexact gradient is used. They show that, contrary to simple gradient schemes, fast gradient methods must necessarily suffer from error accumulation. In contrast, although the “errors” in the gradient values in our algorithm may be of size \( \Theta(\epsilon g) \), or even larger, it turns out there is sufficient structure to enable both the speedup due to acceleration and a further speedup due to parallelism.

**Modeling asynchrony** The study of asynchrony in parallel and distributed computing goes back to Chazan and Miranker [5] for linear systems and to Bertkasik and Tsitsiklis for a wider range of computations [3]. They obtained convergence results for both deterministic and stochastic algorithms along with rate of convergence results for deterministic algorithms. The first analyses to prove rate of convergence bounds for stochastic asynchronous computations were those by Avron, Druinsky and Gupta [2] (for the Gauss-Seidel algorithm), and Liu and Wright [14] (for coordinate descent); they called this the “inconsistent read” model. We follow this approach; and also,

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1 There are also versions in which different coordinates can be selected with different probabilities.

2 “Consistent reads” mean that all the coordinates a core read may have some delay, but they must appear
following Liu and Wright, we assume there is a bounded amount of overlap between the various
updates, but this is the only assumption we use in our analysis.

Analysis and results Our analysis has two starting points: the analysis of the sequential
accelerated stochastic coordinate descent by Lin et al. [13] and the analysis of the stochastic asyn-
chronous coordinate descent by Cheung et al. [7]. We now state our results informally.

Theorem 1 (Informal). Let \( q \) be an upper bound on how many other updates a single update can
overlap. \( L_{\text{res}} \) is a Lipshitz parameter defined in Section 2.

(i) Let \( f \) be a strongly convex function with strongly convex parameter \( \mu \). If \( q = O \left( \frac{\sqrt{\mu} \sqrt{n}}{\sqrt{T}} \right) \), then
\[
 f(x^T) - f^* = \Theta \left( \left( 1 - \frac{1}{5} \sqrt{\frac{n}{T}} \right)^T \right) \quad \text{(linear speedup)}.
\]
While if \( q = O \left( \frac{\sqrt{n}}{\sqrt{T}} \right) \), then
\[
 f(x^T) - f^* = \Theta \left( \left( 1 - \frac{\frac{\mu}{8}}{\frac{n}{T}} \right)^T \right) \quad \text{(sublinear speedup)}.
\]

(ii) Let \( f \) be a (non-strongly) convex function. If \( q = O \left( \frac{\sqrt{\epsilon n}}{\sqrt{T}} \right) \), then
\[
 f(x^{T+1}) - f^* = \Theta \left( \left( 1 - \frac{\epsilon}{T} \right)^3 \right) \quad \text{(sublinear speedup)}.
\]

Comparison to prior works Fang et al. [9] and Hannah et al. [11] have also recently analyzed
asynchronous accelerated coordinate descent. However, there are substantial differences. First,
their analyses at best only partially account for the efficient implementation of this algorithm.
Second, their analyses only consider the strongly convex case. Finally, they use the Common Value
assumption which significantly limits the possible asynchrony (a discussion of this issue phrased in
terms of “delay sequences” can be found in [20]).

The Common Value assumption strikes us as somewhat unnatural. It states that the random
choice of coordinate by one core does not affect the values it reads, and also does not affect the
overlapping computations performed by other cores. For a more detailed discussion, please see [7].
However, it simplifies the analysis of asynchronous coordinate descent, which may explain why it
has been used in multiple papers.

Related work Coordinate Descent is a method that has been widely studied; see Wright for
a recent survey [21].

Acceleration, in the spirit of accelerated gradient descent [18], has been used to achieve a faster
rate of convergence for coordinate descent. In particular, Nesterov [17] proposed an accelerated
version of coordinate descent. Lee and Sidford [12] also developed an accelerated coordinate descent
and focused on its application to solving linear systems. Xiao et al. [13] developed an accelerated
proximal coordinate descent method for minimizing convex composite functions. Fercoq et al. [10]
gave a generalized version of accelerated coordinate descent. Zhu et al. [11] developed a faster
accelerated coordinate descent by updating coordinates with different probabilities.

Considerable attention has also been given to applying asynchronous updates to coordinate
descent in order to achieve quicker convergence. There have been multiple analyses of various
asynchronous parallel implementations of coordinate descent [15, 14, 16, 20, 6, 7], several demon-
strating linear speedup, with the best bound, in [15, 7], showing linear speedup for up to \( \Theta(\sqrt{n}) \)
 simultanuously at some moment. Precisely, the vector of \( \tilde{x} \) values used by the update at time \( t \) must be \( x^{t-c} \) for some
c \( \geq 1 \). “Inconsistent reads” mean that the \( \tilde{x} \) values used by the update at time \( t \) can be any of the \((x^{t-c_1}, \ldots, x^{t-c_n})\),
where each \( c_j \geq 1 \) and the \( c_j \)'s can be distinct.
processors\cite{Zhu13} along with a matching lower bound \cite{Zhu13} showing this is essentially the maximum available speedup in general.

2 Preliminaries

We consider the problem of finding an (approximately) minimum point of a convex function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \). Let \( X^\ast \) denote the set of minimum points of \( f \); we use \( x^\ast \) to denote a minimum point of \( f \). Without loss of generality, we assume that \( f^\ast \), the minimum value of \( f \), is 0.

We recap a few standard terminologies. Let \( \bar{e}_j \) denote the unit vector along coordinate \( j \).

Definition 1. The function \( f \) is \( L \)-Lipschitz-smooth if for any \( x, \Delta x \in \mathbb{R}^n \), \( \|\nabla f(x + \Delta x) - \nabla f(x)\| \leq L \cdot \|\Delta x\| \). For any coordinates \( j, k \), the function \( f \) is \( L_{jk} \)-Lipschitz-smooth if for any \( x \in \mathbb{R}^n \) and \( r \in \mathbb{R} \), \( \nabla_k f(x + r \cdot \bar{e}_j) - \nabla_k f(x) \leq L_{jk} \cdot |r| \); it is \( L_{\text{res}} \)-Lipschitz-smooth if \( \|\nabla f(x + r \cdot \bar{e}_j) - \nabla f(x)\| \leq L_{\text{res}} \cdot |r| \). Finally, \( L_{\text{max}} := \max_{j,k} L_{jk} \) and \( L_{\text{res}} := \max_k \left( \sum_{j=1}^n (L_{kj})^2 \right)^{1/2} \).

\( L_{\text{res}} \) was introduced in \cite{Zhu13} to account for the effect of No Common Value on the averaging in our analysis. If Common Value was assumed, the parameter \( L_{\text{res}} \leq L_{\text{res}} \) would suffice. We recap the discussion of the difference between \( L_{\text{res}} \) and \( L_{\text{res}} \) from \cite{Zhu13} in Appendix F. We note that if the convex function is \( s \)-sparse, meaning that each term \( \nabla_k f(x) \) depends on at most \( s \) variables, then \( L_{\text{res}} \leq \sqrt{s} L_{\text{max}} \). When \( n \) is huge, this would appear to be the only feasible case.

Next, we define strong convexity.

Definition 2. Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a convex function. \( f \) is strongly convex with parameter \( 0 < \mu_f \leq 1 \), if for all \( x, y \), \( f(y) - f(x) - \langle \nabla f(x), y - x \rangle \geq \frac{1}{2} \mu_f \|y - x\|^2_L \), where \( \|y - x\|^2_L = \sum_j L_{jj} (y_j - x_j)^2 \).

By a suitable rescaling of variables, we may assume that \( L_{jj} \) is the same for all \( j \) and equals 1. This is equivalent to using step sizes proportional to \( L_{jj} \) without rescaling, a common practice. Note that rescaling leaves the strong convexity parameter \( \mu \) unchanged. Since we measure distances by \( \| \cdot \|_L \) (this is the same as rescaling and measuring distances by \( \| \cdot \|_2 \)) and as our strongly convex parameter is defined with respect to \( \| \cdot \|_L \), choosing the coordinate to update uniformly is the best choice. This is also the case for the accelerated algorithm analyzed in the work of Zhu et al. \cite{Zhu13}, where uniform sampling is the best choice with the measure \( \| \cdot \|_L \). \cite{Zhu13} also consider other measures, including the measure \( \| \cdot \|_2 \) without rescaling, where non-uniform sampling is a better choice. We note that the accelerated asynchronous algorithm in \cite{Zhu13} analyzes the non-uniform sampling case (which includes uniform sampling as a special case).

The update rule The basic time \( t \) iteration for our accelerated coordinate descent is shown in Algorithm \cite{Zhu13} (we will explain the meaning of the term \( \pi \) in the superscripts shortly — for now simply ignore this term and view the superscript as indicating a time). The values \( \psi_t, \varphi_t, \phi_t \) are suitable parameters satisfying \( 0 \leq \psi_t, \varphi_t, \phi_t \leq 1 \).

In a sequential implementation \( \hat{g}^{t,\pi}_{k_t} \) is the gradient \( \nabla_t f(y^{t,\pi}) \). In parallel implementations, in general, as we will explain, we will not have the coordinates \( y^{t,\pi} \) at hand, and instead we will compute a possibly different gradient \( \hat{g}^{t,\pi}_{k_t} = \nabla_t f(\hat{g}^{t,\pi}) \). The challenge for the analysis is to bound the effect of using \( \hat{g}^{t,\pi} \) rather than \( y^{t,\pi} \).
Algorithm 1: The basic iteration

1. Choose $k_t \in \{1, 2, \ldots, n\}$ uniformly at random;
2. $y^{(t, \pi)} = \psi_t z^{(t, \pi)} + (1 - \psi_t) z^{(t, \pi)}$;
3. $w^{(t, \pi)} = \varphi_t z^{(t, \pi)} + (1 - \varphi_t) y^{(t, \pi)}$;
4. $z^{(t+1, \pi)} = \arg \min_x \left\{ \frac{1}{2} \| x - w^{(t, \pi)} \|^2 + \langle g_{k_t}, x \rangle \right\}$;
5. $x^{(t+1, \pi)} = y^{(t, \pi)} + n\phi_t (z^{(t+1, \pi)} - w^{(t, \pi)})$;

We note that in general Algorithm 1 is not efficiently implementable, and we will be using a more efficient implementation which we will describe in Section 3.

**Bounded asynchrony** Following [14], we assume that each update overlaps the computation of at most $q$ other updates. $q$ is a function of the number of processors and the variation in the runtime of the different updates. This variation can be due to both variance in the inherent length of the updates, and variations coming from the computing environment such as communication delays, interrupts, processor loads, etc.

**Time in the analysis** Our analysis will be comparing the performance of our parallel implementation to that of the sequential algorithm. In order to do this, we need to impose an order on the updates in the parallel algorithm. As the algorithms are stochastic, we want an order for which at each step each coordinate is equally likely to be updated. While using commit times for the orderings seems natural, it does not ensure this property, so as in [16] [7], we instead use the ordering based on start times.

Suppose there are a total of $T$ updates. We view the whole stochastic process as a branching tree of height $T$. Each node in the tree corresponds to the moment when some core randomly picks a coordinate to update, and each edge corresponds to a possible choice of coordinate. We use $\pi$ to denote a path from the root down to some leaf of this tree. A superscript of $\pi$ on a variable will denote the instance of the variable on path $\pi$. A double superscript of $(t, \pi)$ will denote the instance of the variable at time $t$ on path $\pi$, i.e. following the $t$-th update.

**Notation** We let $k_t$ denote the coordinate selected at time $t$, which we call the coordinate being updated at time $t$, as in our efficient implementation, it will be the only coordinate being updated at time $t$. We let $\Delta z_{k_t}^{t, \pi} = z_{k_t}^{t+1, \pi} - w_{k_t}^{t, \pi}$ (note this is not the increment to $z_{k_t}^{t, \pi}$). Also, we let $\Delta x_{k_t}^{t, \pi} = n\phi_t \Delta z_{k_t}^{t, \pi}$.

Note that the computation starts at time $t = 0$, which is the “time” of the first update.

### 3 The Algorithm and its Performance

Algorithm 1 updates every coordinate in each iteration. This is unnecessary and could be very inefficient. Instead, we follow the approach taken in [13], which we now explain. Observe that the update rule could be written as follows.

\[
\begin{pmatrix}
  y_k^{(t+1, \pi)} \\
  z_k^{(t+1, \pi)}
\end{pmatrix}
= \begin{pmatrix}
  1 - \varphi_t (1 - \psi_{t+1}) & \varphi_t (1 - \psi_{t+1}) \\
  1 & \phi_t
\end{pmatrix} \begin{pmatrix}
  y_k^{(t, \pi)} \\
  z_k^{(t, \pi)}
\end{pmatrix}
+ \begin{cases}
  \Delta z_{k_t}^{t, \pi} \cdot \begin{pmatrix}
    0 \\
    1 - \psi_{t+1} (1 - n\phi_t)
  \end{pmatrix} & \text{if } k \neq k_t \\
  \Delta x_{k_t}^{t, \pi} \cdot \begin{pmatrix}
    0 \\
    1
  \end{pmatrix} & \text{if } k = k_t
\end{cases}
\]

\[\]
For short, we write \[
\begin{bmatrix}
y_k^{(t+1,\pi)} \\
y_k^{(t,\pi)} \\
z_k^{(t+1,\pi)} \\
z_k^{(t,\pi)}
\end{bmatrix} = A^t \begin{bmatrix}
y_k^{(t,\pi)} \\
y_k^{(t,\pi)} \\
z_k^{(t,\pi)} \\
z_k^{(t,\pi)}
\end{bmatrix} \text{ if } k \neq k_t \text{ and } \begin{bmatrix}
y_k^{(t+1,\pi)} \\
y_k^{(t,\pi)} \\
z_k^{(t+1,\pi)} \\
z_k^{(t,\pi)}
\end{bmatrix} = A^t \begin{bmatrix}
y_k^{(t,\pi)} \\
y_k^{(t,\pi)} \\
z_k^{(t,\pi)} \\
z_k^{(t,\pi)}
\end{bmatrix} + \Delta z_k^{t,\pi} D^{t_k} \text{ if } k = k_t.
\]

We let \( B^{t'} = A^t A^{t-1} \cdots A^1 \). Instead of storing the values \( y_k^{(t,\pi)} \) and \( z_k^{(t,\pi)} \), we store the values \( \begin{bmatrix} u_k^{(t,\pi)} \\ v_k^{(t,\pi)} \\ z_k^{(t,\pi)} \end{bmatrix} = (B^{(t)})^{-1} \begin{bmatrix} y_k^{(t,\pi)} \\ u_k^{(t+1,\pi)} \\ z_k^{(t+1,\pi)} \end{bmatrix} \), since for \( k \neq k_t \), \( \begin{bmatrix} u_k^{(t,\pi)} \\ v_k^{(t+1,\pi)} \end{bmatrix} = \begin{bmatrix} u_k^{(t,\pi)} \\ v_k^{(t,\pi)} \end{bmatrix} \), and so we need to update only \( u_k^{(t,\pi)} \) and \( v_k^{(t,\pi)} \) at time \( t \). This leads to Algorithm 2, an efficient version of Algorithm 1.

**Algorithm 2:** An efficient implementation of APCG

1. Let \( B^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \);  
2. Let \( u^{(0)} = v^{(0)} = x^{(0)} = y^{(0)} = z^{(0)} \);  
3. for \( t = 0 \) to \( T - 1 \) do  
   4. choose \( k_t \in 1, \ldots, n \) uniformly at random;  
   5. \( (y_k^{(t,\pi)}, z_k^{(t,\pi)}) \leftarrow B^{(t)} (u_k^{(t,\pi)}, v_k^{(t,\pi)}) \);  
   6. \( \Delta z_k^{(t)} \leftarrow \text{arg min}_h \{ \frac{1}{2} ||h||^2 + \langle \nabla f(y^{(k)}), h \rangle \} \);  
   7. \( A^{(t)} \leftarrow \begin{pmatrix} (1 - \varphi_t(1 - \psi_t)) & \varphi_t(1 - \psi_t) \\ \varphi_t & \varphi_t \end{pmatrix} \);  
   8. \( D^{(t)} \leftarrow \begin{pmatrix} 1 - \varphi_{t+1}(1 - n \phi_t) \\ 1 \end{pmatrix} \);  
   9. \( B^{(t+1)} \leftarrow A^{(t)} B^{(t)} \);  
10. \( (u_k^{(t+1)}, v_k^{(t+1)}) \leftarrow (u_k^{(t)}, v_k^{(t)}) + B^{(t+1)^{-1}} D^{(t)} \Delta z_k^{(t,\pi)} \);  

Now, we describe an asynchronous version of Algorithm 2 precisely. In this version there is a global counter which starts from 0. At each time, one core makes a request to the counter, the counter returns its current value and immediately increments its value by 1.

The initial values of \( u^{(0)} \) and \( v^{(0)} \) are in the shared memory. Each core iteratively performs the following tasks.

**Asynchronous Implementation of the loop in Algorithm 2**

1. Makes a request to the counter and receives an integer \( t \) as its rank order, or rank for short.  
   The assigned values are successive integers.
2. Chooses a random coordinate \( k_t \) uniformly.  
3. Retrieves values \( \tilde{u}^{(t)} \) and \( \tilde{v}^{(t)} \) from the shared memory.  
4. Calculates \( B^{(t)} \) and \( B^{(t)}^{-1} \).  
5. Sets \( (\tilde{y}^{(t)}, \tilde{z}^{(t)}) \leftarrow B^{(t)} (\tilde{u}^{(t)}, \tilde{v}^{(t)}) \).  
6. Computes \( \tilde{g}^{(t)}_{k_t} = \nabla f(\tilde{y}^{(t)}) \).  
7. Computes \( \Delta z_k^{(t)} = \text{arg min}_h \{ \frac{1}{2} ||h||^2 + \langle \tilde{g}^{(t)}_{k_t}, h \rangle \} \).
Theorem 3. Suppose that \( f \) is a strongly convex function with convex parameter \( \mu \) and dimension \( n \geq 19 \). Suppose we set \( \phi_t = \phi = \frac{\sqrt{3n} \mu}{\sqrt{20n}}, \varphi_t = \sqrt{\frac{20}{3}} \mu, \psi_t = 1 - \frac{\sqrt{3n} \mu}{\sqrt{20n}} \), and \( \psi_t = \frac{1}{1 + \frac{\sqrt{3n} \mu}{\sqrt{20n}}} \). Then,

\[
\mathbb{E} \left[ f(x(T)) - f^* \right] \leq \left( 1 - \frac{3 \sqrt{\mu}}{80 n} \right)^T \left[ f(x(0)) - f^* + \frac{1}{2} \mu \|x^* - x(0)\|^2 \right].
\]

Note that here \( A(t) = \left( \frac{1 + \phi^2}{1 + \phi} \right) \left( \frac{1 - 1 + \phi^2}{1 + \phi} \right) \) and \( B(t) = \left( \frac{1 + \phi^2}{1 + \phi} \right) \left( \frac{1}{1 - \phi} \right) \).

Theorem 3. Suppose that \( \epsilon < \frac{1}{3}, n \geq 19, \Gamma_t = \frac{20}{3} \sqrt{n \phi_t}, \) and \( q \leq \min \left\{ \frac{\sqrt{\epsilon}}{1 + \phi}, \frac{\sqrt{n}}{\sqrt{200}}, \frac{\sqrt{n}}{\sqrt{17}}, \frac{\sqrt{n}}{30} \right\}. \)

1. Suppose that \( f \) is a strongly convex function with strongly convex parameter \( \mu \), and we set \( \phi_t = \phi = \frac{\sqrt{3n} \mu}{n}, \varphi_t = 1 - \frac{\sqrt{3n} \mu}{n}, \) and \( \psi_t = \frac{1}{1 + \frac{\sqrt{3n} \mu}{n}} \). Then

\[
\mathbb{E} \left[ f(x(T)) - f^* \right] \leq \left( 1 - (1 - \epsilon) \frac{\sqrt{3n} \mu}{n} \right)^T \left[ f(x(0)) - f^* + \frac{10}{3} \|x^* - x(0)\|^2 \right].
\]

2. While if \( f \) is a convex function and we set \( \phi_t = \frac{2}{2n+1}, \varphi_t = 1, \) and \( \psi_t = \frac{2n+1}{2n+2}, \) then

\[
\mathbb{E} \left[ f(x(T)) - f^* \right] \leq \left( \frac{(2n)(2n+1)}{(2n+T)(2n+T+1)} \right)^{\frac{n+1}{n+1}} \left[ f(x(0)) - f^* + \frac{10}{3} \|x^* - x(0)\|^2 \right].
\]

Here, in the strongly convex case, \( A(t) = \left( \frac{1 + \phi^2}{1 + \phi} \right) \left( \frac{1 - 1 + \phi^2}{1 + \phi} \right) \) and \( B(t) = \left( \frac{1 + \phi^2}{1 + \phi} \right) \left( \frac{1}{1 - \phi} \right) \); in the non-strongly convex case, \( A(t) = \left( \frac{1 + \phi^2}{1 + \phi} \right) \left( \frac{2}{1 + \phi^2} \right) \) and \( B(t) = \left( \frac{1 + \phi^2}{1 + \phi} \right) \left( \frac{1}{1 - \phi} \right) \).
Remark 4.

(i) Computing $B^{(t)}$: In the strongly convex case, one simple observation is that it can be computed in $O(\log t)$ time. However, by assumption, each update can overlap at most $q$ other updates. If a process remembers its current $B^{(t)}$, then calculating the $B^{(t)}$ for its next update is just an $O(\log q)$ time calculation.

(ii) In the non-strongly convex case, $B^{(t)}$ can be calculated in $O(1)$ time.

(iii) Our analysis is for an efficient asynchronous implementation, in contrast to prior work.

(iv) The result in Hannah et al. [11] is analogous to Theorem 2 except that the constraints on $q$ replace $L_{\text{res}}$ by $L$. But, by allowing for non-uniform sampling of the coordinates, their bound also optimizes for non-scale-free measures $\mu$ of the strong convexity. However, as already noted, their analysis uses the Common Value assumption. In addition, their amortization does not account for the error magnification created by multiplying by $B^{(t)}$ to go from an out-of-date $(\hat{y}^{(t)}, \hat{z}^{(t)})$ to the corresponding $(\hat{y}^{(t)}, \hat{z}^{(t)})$. Rather, it appears to assume out-of-date $(\hat{y}^{(t)}, \hat{z}^{(t)})$ values can be read directly.

(v) The result in Fang et al. [9] does not consider the efficient implementation of the accelerated algorithm, and as noted in [11] it does not appear to demonstrate a parallel speedup.

4 The Analysis

4.1 The Series of Points Analyzed

We note that the commit time ordering of the updates need not be the same as their start time ordering. Nonetheless, as in [10, 7], we focus on the start time ordering as this guarantees a uniform distribution over the coordinates at each time step. For the purposes of our analysis, we suppose the updates are applied sequentially according to their start time ordering; so the time $t$ update is treated as if it updates the time $t$ variables. These need not be the same as the values the asynchronous algorithm encounters, because the algorithm encounters new values only when they commit. Recall that the updates are the values $B^{(t)}_{\text{res}}D^{(t)}\Delta \hat{z}^{(t,\pi)}_{k_t}$.

The precise definition follows. We first define $\hat{y}^{(t,\pi)}$ and $\hat{z}^{(t,\pi)}$ to be:

$$
(\hat{y}^{(t,\pi)}, \hat{z}^{(t,\pi)})^T = B^{(t)} \left[ (u^{(0)}, v^{(0)}) + \sum_{l=0}^{t-1} B^{(l+1)}_{\text{res}}D^{(l)}\Delta z^{(t,\pi)}_{k_t}1_{k_t} \right]^T,
$$

where $D^{(t)} = \begin{bmatrix} [n\psi_{l+1}\phi_l + (1-\psi_{l+1})] \end{bmatrix}^T$, and $1_{k_t}$ is the vector with one non-zero unit entry at coordinate $t$. Note that $\left[ (u^{(0)}, v^{(0)}) + \sum_{l=0}^{t-1} B^{(l+1)}_{\text{res}}D^{(l)}\Delta z^{(t,\pi)}_{k_t}1_{k_t} \right]^T$ may not appear in the memory at any time in the asynchronous computation, which is why we use the notation $\hat{y}$, $\hat{z}$. The key exception is that after the final update, at time $T$, the term $\left[ (u^{(0)}, v^{(0)}) + \sum_{l=0}^{T-1} B^{(l+1)}_{\text{res}}D^{(l)}\Delta z^{(t,\pi)}_{k_t}1_{k_t} \right]^T$ will be equal to the final $(u, v)^T$ in the shared memory.

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9 [11] state that they can remove the Common Value assumption as in their earlier work on non-accelerated coordinate descent [20]. However, in this earlier work, as noted in [7], this comes at the cost of having no parallel speedup.
In addition, we define $\hat{x}^{(t,\pi)}$ to satisfy
\[ y^{(t,\pi)} = (1 - \psi_t)\hat{z}^{(t,\pi)} + \psi_t\hat{x}^{(t,\pi)}, \]
and $\hat{w}^{(t,\pi)}$ as
\[ \hat{w}^{(t,\pi)} = \varphi_t\hat{z}^{(t,\pi)} + (1 - \varphi_t)\hat{y}^{(t,\pi)}. \]

Conveniently, as we shall see, for any fixed path $\pi$, the asynchronous version is equivalent to Algorithm 1. This will allow us to carry out much of the analysis w.r.t. the simpler Algorithm 1 rather than the asynchronous version of Algorithm 2. The only place we need to work with the latter algorithm is in bounding the differences $\tilde{g}_k - \nabla f(y^t_k)$, the differences between the computed gradients and the “correct” values.

The following theorem says that the $x^{(t,\pi)}$, $y^{(t,\pi)}$, $z^{(t,\pi)}$ and $w^{(t,\pi)}$ in Algorithm 1 are equal to $\hat{x}^{(t,\pi)}$, $\hat{y}^{(t,\pi)}$, $\hat{z}^{(t,\pi)}$ and $\hat{w}^{(t,\pi)}$.

**Theorem 5.** For a given path $\pi$, the values $\{x^{(t,\pi)}, y^{(t,\pi)}, z^{(t,\pi)}, w^{(t,\pi)}\}_{t=0,...,T}$ in Algorithm 1 are equal to the values $\{\hat{x}^{(t,\pi)}, \hat{y}^{(t,\pi)}, \hat{z}^{(t,\pi)}, \hat{w}^{(t,\pi)}\}_{t=0,...,T}$ in the asynchronous version of Algorithm 2 if each value $\{\hat{y}^{t_k}\}_{t=0,...,T}$, and the starting points are the same in both algorithms.

In the following analysis, with a slight abuse of notation, we let $\{x^{(t,\pi)}, y^{(t,\pi)}, z^{(t,\pi)}, w^{(t,\pi)}\}_{t=0,...,T}$ denote both $\{x^{(t,\pi)}, y^{(t,\pi)}, z^{(t,\pi)}, w^{(t,\pi)}\}_{t=0,...,T}$ in Algorithm 1 and $\{\hat{x}^{(t,\pi)}, \hat{y}^{(t,\pi)}, \hat{z}^{(t,\pi)}, \hat{w}^{(t,\pi)}\}_{t=0,...,T}$.

Given a path $\pi$, we will be considering alternate paths in which edge $k_t$ is changed to $k$ but every other choice of variable is unchanged; $\pi(k, t)$ denotes this alternate path. Then, $\{x^{(t,\pi(k,t))}, y^{(t,\pi(k,t))}, z^{(t,\pi(k,t))}, w^{(t,\pi(k,t))}\}$ are the instances of variables $\{x^{(t)}, y^{(t)}, z^{(t)}, w^{(t)}\}$ on path $\pi(k, t)$ at time $t$. Remember that $k_t$ denotes the coordinate selected at time $t$, which implies $\pi(\pi(k, t), t) = \pi$. The relationship between $\pi(k_t, t) = \pi$ and $\pi(k, t)$ is shown in the figure on the left.

### 4.2 Starting Point: The Progress Lemma (see Appendix C for the full version)

The starting point for the bounds in Theorem 2 and 3 is the following lemma.

Let $0 < \tau < 1$, $\zeta_t$, $\mathcal{P}_t$, $\mathcal{Q}_t$, $\mathcal{R}_t$, $\mathcal{S}_t$, and $\mathcal{T}_t$ denote some parameters. Then we show the following lemma.

**Lemma 6 (Informal Progress Lemma).** Define the potential function $\mathcal{F}^{(t)} = f(x^{(t)}) - f^* + \zeta_t \|x^* - z^{(t)}\|^2$. Then,
\[ \mathbb{E}_\pi [\mathcal{F}^{(t+1)}] \leq (1 - \tau \phi_t) \mathbb{E}_\pi [\mathcal{F}^{(t)}] - \text{Adj}_t, \]
where \( \text{Adj}_t = \mathbb{E}_\pi \left[ \mathcal{P}_t \left( \Delta z_{k,t}^{t,\pi} \right)^2 \right] \)

\[
- \mathbb{E}_\pi \left[ \sum_{k'} \left( Q_t \left( \nabla_{k'} f(y^{t,\pi}) - g_{k'}^{t,\pi(k',t)} \right)^2 \right. \\
\left. + R_t \left( \nabla_{k'} f(y^{t,\pi}) - \nabla_{k'} f(y^{t,\pi(k',t)}) \right)^2 \right) \right] \]

\[
- \mathbb{E}_\pi \left[ S_t \sum_{k'} \| w_{k'}^{t,\pi} - w_{k'}^{t,\pi(k',t)} \|^2 \right] \\
- \mathbb{E}_\pi \left[ T_t \sum_{k'} \| z_{k'}^{t,\pi} - z_{k'}^{t,\pi(k',t)} \|^2 \right] .
\]

\[\text{progress term}\]

\[\text{error terms}\]

**Remark 7.**

(i) The sequential analysis in [13][1] considers a single randomized update based on the same \( x', y', \) and \( z' \) and obtains the bound \( \mathbb{E}_\pi \left[ \mathcal{F}(t+1) \right] \leq (1 - \phi_t) \mathbb{E}_\pi \left[ \mathcal{F}(t) \right] \). In our asynchronous version, since we drop the common value assumption, \( x', y', \) and \( z' \) may be different when different choices of coordinates to update are made. This makes the analysis quite challenging.

(ii) Our analysis extracts progress and error terms. Our goal is to use the progress term to consume the error terms. We obtain the progress term by damping the reduction in the potential function obtained in the sequential analysis by a factor \( \tau \).

### 4.3 Amortization between Progress terms and Error terms in Progress Lemma

In order to prove the convergence result \( \mathbb{E}_\pi \left[ \mathcal{F}(T) \right] \leq \left( \prod_{t=0}^{T-1} (1 - \tau \phi_t) \right) \mathcal{F}(0) \), it suffices to show that \( \sum_t \prod_{l=t+1}^{T-1} (1 - \tau \phi_l) \text{Adj}_t \geq 0 \).

\( \text{Adj}_t \) consists of the progress term, \( \left( \Delta z_{k,t}^{t,\pi} \right)^2 \), and the error terms, \( \left( \nabla_{k'} f(y^{t,\pi}) - g_{k'}^{t,\pi(k',t)} \right)^2 \), \( \left( \nabla_{k'} f(y^{t,\pi}) - \nabla_{k'} f(y^{t,\pi(k',t)}) \right)^2 \), \( \| w_{k'}^{t,\pi} - w_{k'}^{t,\pi(k',t)} \|^2 \) and \( \| z_{k'}^{t,\pi} - z_{k'}^{t,\pi(k',t)} \|^2 \). It's hard to suitably bound the error terms by the progress term. To do this, in the spirit of [7], we introduce the new terms \( \left( \Delta_{\text{FE}}^{t,\pi} \right)^2 \) and \( \mathbb{E}_\pi \left[ \left( g_{\max,k,t}^{t,\pi} - g_{\min,k,t}^{t,\pi} \right)^2 \right] \) as a bridge, to connect the progress and error terms.

Roughly speaking, \( \left( \Delta_{\text{FE}}^{t,\pi} \right)^2 \) is the expectation, over all paths \( \pi \), of the difference between the maximal and minimal possible updates at time \( t \) on path \( \pi \), and \( \mathbb{E}_\pi \left[ \left( g_{\max,k,t}^{t,\pi} - g_{\min,k,t}^{t,\pi} \right)^2 \right] \) is the expectation of the difference between the maximal and minimal possible gradients at time \( t \). For more precise definitions, please see Appendix D. For simplicity, let \( E^\Delta_t \) denote the expected value of \( \left( \Delta z_{k,t}^{t,\pi} \right)^2 \) at time \( t \). We also suppose the \( B^{(t)} \) are good, which roughly speaking means that \( A^{(t)} \) is close to the identity matrix (the precise definition can be found in Definition 3 in Appendix E). We show in Lemma 23 that the \( B^{(t)} \) are good for the choices of parameters in Theorems 2 and 3. We show the following bounds on \( \left( \Delta_{\text{FE}}^{t,\pi} \right)^2 \), \( \mathbb{E}_\pi \left[ \left( g_{\max,k,t}^{t,\pi} - g_{\min,k,t}^{t,\pi} \right)^2 \right] \), and the error terms.

**Lemma 8** (Informal Amortization Lemma; full version in Appendix D). Let \( I = [0, T-1] \). If the
If the $B(t)$ are good, then:

$$(\Delta_t^{FE})^2 \leq \Theta\left(\frac{1}{T_t}\right) \mathbb{E}_\pi \left[ (g_{\max,k_t}^{\pi,t} - g_{\min,k_t}^{\pi,t})^2 \right]; \quad (2)$$

$$\mathbb{E}_\pi \left[ (g_{\max,k_t}^{\pi,t} - g_{\min,k_t}^{\pi,t})^2 \right] \leq \Theta(qL_{\text{res}}^2n\phi_t^2) \sum_{s \in I \cap [t-2q_t,t+2q_t]\setminus\{t\}} (\Delta_t^{FE})^2 + E_t^\Delta; \quad (3)$$

$$\mathbb{E}_\pi \left[ \sum_{k'} \left( w_{k'}^{(t,\pi)} - w_{k'}^{(t,\pi(k',t))} \right)^2 \right] \leq \Theta(q) \sum_{s \in I \cap [t-q-1,t-1]} (\Delta_t^{FE})^2; \quad (4)$$

$$\mathbb{E}_\pi \left[ \sum_{k'} \left( z_{k'}^{(t,\pi)} - z_{k'}^{(t,\pi(k',t))} \right)^2 \right] \leq \Theta(q) \sum_{s \in I \cap [t-q-1,t-1]} (\Delta_t^{FE})^2; \quad (5)$$

$$\mathbb{E}_\pi \left[ \sum_{k'} \left( \nabla_{k'} f(y^{(t,\pi)}) - g_{k'}^{t,\pi(k',t)} \right)^2 \right] \leq \Theta(n \cdot n^2 \phi_t^2 qL_{\text{res}}^2) \sum_{s \in I \cap [t-3q_t,t+q]} ((\Delta_t^{FE})^2 + E_t^\Delta)$$

$$+ \Theta(n \cdot n^2 \phi_t^2) ((\Delta_t^{FE})^2 + E_t^\Delta)$$

$$+ \Theta(n) \mathbb{E}_\pi \left[ (g_{\max,k_t}^{\pi,t} - g_{\min,k_t}^{\pi,t})^2 \right]; \quad (6)$$

$$\mathbb{E}_\pi \left[ \sum_{k'} \left( \nabla_{k'} f(y^{(t,\pi)}) - \nabla_{k'} f(y^{(t,\pi(k',t))}) \right)^2 \right] \leq \Theta(n \cdot n^2 \phi_t^2 qL_{\text{res}}^2) \sum_{s \in I \cap [t-3q_t,t+q]} ((\Delta_t^{FE})^2 + E_t^\Delta)$$

$$+ \Theta(n \cdot n^2 \phi_t^2) ((\Delta_t^{FE})^2 + E_t^\Delta). \quad (7)$$

Using (2) and (3), we obtain the following lemma, which bounds the sum of the series of $(\Delta_t^{FE})^2$ by $E_t^\Delta$.

**Lemma 9.** Let $\{a_t\}$ be a series of non-negative numbers, let $\Xi = \max_t \frac{216q_t^2 T_t^2 \phi_t^2}{n(\Gamma_t)^2}$, and let

$$\Phi_a = \min_{t \in [0,\ldots,T-1]} \min_{s \in [t-2q_t,t+2q_t] \cap [0,T-1]} \left\{ \frac{a_t \left( \prod_{l=t+1}^{T-1} (1 - \tau \phi_l) \right)}{a_t \left( \prod_{l=t+1}^{T-1} (1 - \tau \phi_l) \right)} \right\}.$$

If the $B(t)$ are good, then

$$\sum_t a_t \left( \prod_{l=t+1}^{T-1} (1 - \tau \phi_l) \right) (\Delta_t^{FE})^2 \leq \frac{\Xi}{\Phi_a} \sum_t a_t \left( \prod_{l=t+1}^{T-1} (1 - \tau \phi_l) \right) E_t^\Delta.$$

Using Lemma 8 and 9, we can bound the error terms by the progress term by using the bridges $(\Delta_t^a)\Delta^2$ and $\mathbb{E}_\pi \left[ (g_{\max,k_t}^{\pi,t} - g_{\min,k_t}^{\pi,t})^2 \right]$. By choosing the parameters carefully, we can deduce Theorems 2 and 3.

### 4.4 Note Regarding the Appendix

In Appendix A, we give a more general theorem, Theorem 10, that subsumes Theorems 2 and 3. In Appendix B, we show that Theorems 2 and 3 follow by carefully choosing the parameters in Theorem 10. In order to obtain this more general theorem, as in the main part, we demonstrate the full version of the Progress Lemma in Appendix C and then show the full version of the Amortization Lemma in Appendix D. Finally, in Appendix A, we give the proof of this general theorem.
A Proof of the general theorem: Theorem 10

We state and prove the general theorem: Theorem 10.

**Theorem 10.** Suppose that $0 < \tau \leq 1$, $0 < \tilde{\tau} \leq 1$, $\phi_t$, $\varphi_t$, $\psi_t$, $\Gamma_t$, $q_t$, $r = \max_t \left\{ \frac{36(3q)^2 I^2}{r \Gamma_t^2} \right\} \leq \frac{1}{32}$, $n$ and $\zeta_{t+1}$ satisfy the following conditions.

- **Strongly convex case:** $\phi_t = \phi$, $\varphi_t = 1 - \phi$, $\psi = \frac{1}{1+\phi}$, $\Gamma_t = \Gamma$, and $\zeta_{t+1} = \frac{n\phi \Gamma}{2} \left( 1 - \frac{n\phi(1-\tau)}{3} \right)$.
- **Non-strongly convex case:** $\phi_t = \frac{2}{t+t_0}$ for some $t_0 \geq 2(n+1)$, $\varphi_t = 1$, $\psi = 1 - \phi_t$, and $\zeta_{t+1} = \frac{n\phi \Gamma}{2} \left( 1 - \frac{n\phi(1-\frac{\tilde{\tau}}{4t_0})}{3} \right)$.

Let

$$\xi = \max_t \frac{\phi_t^2 \Gamma_t^2}{\phi_t^2 \Gamma_t^2 s} \leq \frac{6}{5},$$

$$\Xi = \max_t \frac{216n^2 \phi_t^2 q^2 I_t^2 \Xi}{n(G_t^2)},$$

$$\Phi_b = \min_{t \in [1, \ldots, T]} \min_{s \in [t-4q_t, t+2q]} \left\{ \frac{1 + 2n \phi s \xi \Gamma_t^2}{2 \Gamma_t} \frac{540q^2 I_t^2 \phi_t^2}{n} \left( \prod_{l=s+1}^{T} (1 - \tau \phi_l) \right) \right\};$$

$$\Phi_c = \min_{t \in [1, \ldots, T]} \min_{s \in [t-2q, t+2q]} \left\{ \frac{12 \Gamma_s \phi s \xi \Gamma_t^2}{12 \Gamma_t \phi t} \left( \prod_{l=t+1}^{T} (1 - \tau \phi_l) \right) \right\}.$$

in both cases. Suppose $\frac{(q)_t^2 I_t^2}{n} \leq 1$, the $B^{(t)}$ are good, and the following constraints hold:

**Strongly convex case:**

1. $\phi_t \leq \frac{1}{n+1}$;
2. $n\phi_t \Gamma_t \leq \mu$;
3. $\frac{4}{5} \Gamma_t \geq n\phi_t$; (subsumed by (vii))
4. $\frac{1}{(\Phi_b - \Xi)} 3 \frac{(896 + 897) \xi q^2 I_t^2 \phi_t^2}{n} \leq \frac{3}{5} n\phi_t \Gamma_t$;
5. $\frac{1}{(\Phi_c - \Xi)} 3 \frac{(896 + 897) \xi q^2 I_t^2 \phi_t^2}{n} \leq \frac{3}{5} n\phi_t \Gamma_t$;
6. $\phi_t \leq \frac{1}{1+\phi}$;
7. $\frac{1}{(\Phi_b - \Xi)} 3 \frac{(896 + 897) \xi q^2 I_t^2 \phi_t^2}{n} \leq \frac{3}{5} n\phi_t \Gamma_t$;
8. $\frac{1}{(\Phi_c - \Xi)} 3 \frac{(896 + 897) \xi q^2 I_t^2 \phi_t^2}{n} \leq \frac{3}{5} n\phi_t \Gamma_t$;

**Non-strongly convex case:**

1. $\tilde{\tau} \geq \frac{1}{2}$;
2. $(1 - \tilde{\tau} \phi_t) \frac{\Gamma_t}{n\phi_t} \leq (1 - \tau \phi_t) \frac{\Gamma_{t-1}}{n\phi_{t-1}}$;
3. $\frac{4}{5} \Gamma_t \geq n\phi_t$; (subsumed by (vii))
4. $\frac{1}{(\Phi_c - \Xi)} 3 \frac{(896 + 897) \xi q^2 I_t^2 \phi_t^2}{n} \leq \frac{3}{5} n\phi_t \Gamma_t$;
5. $\frac{1}{(\Phi_c - \Xi)} 3 \frac{(896 + 897) \xi q^2 I_t^2 \phi_t^2}{n} \leq \frac{3}{5} n\phi_t \Gamma_t$;
6. $\frac{1}{(\Phi_c - \Xi)} 3 \frac{(896 + 897) \xi q^2 I_t^2 \phi_t^2}{n} \leq \frac{3}{5} n\phi_t \Gamma_t$.
(vi) $\frac{3}{20} \Gamma_t \geq n \phi_t$; 

(ix) $q \leq \frac{\mathcal{R}}{25}$; 

(viii) $q \leq \min \left\{ \frac{n - 8}{12}, \frac{2n - 4}{10}, \frac{n}{25} \right\}$; 

(x) $n \geq 19$. 

Then,

$$
\mathbb{E} \left[ f(x(T)) - f^* + \zeta_{T+1} \left\| x^* - z(T) \right\|^2 \right] \leq \left( \prod_{t=0}^{T-1} (1 - \tau \phi_t) \right) \left[ f(x(0)) - f^* + \zeta_0 \left\| x^* - z(0) \right\|^2 \right].
$$

**Proof.** Let $\Upsilon = 1 - \tau$ in the strongly convex case and $\Upsilon = 1 - \tau - \frac{1}{400}$ in the non-strongly convex case. By Lemma [12], which requires constraints (i)–(iii), it suffices to show $\sum_t \prod_{l=t+1}^{T} (1 - \tau \phi_l) \text{Adj}_t \geq 0$. Our proof will also apply Lemmas [14] and [23], which assume constraint (viii).

Note that $r \leq \frac{73q^2 l_2^2 \xi}{100n}$ as $\frac{3}{20} \Gamma_t \geq n \phi_t$. By (17), (20), and (21) from Lemma [14] for the first inequality, and by Lemma [9] for the third inequality,

$$
\sum_t \left( \prod_{l=t+1}^{T} (1 - \tau \phi_l) \right) \cdot \left[ \frac{1}{n} \sum_{k} \left( \frac{3}{2} + 2n \phi_t \right) \left( \nabla_{k'} f(y(t, \pi)) - \gamma_{k', t} \right)^2 + \frac{10n \phi_t}{2t} \left( \nabla_{k'} f(y(t, \pi)) - \nabla_{k'} f(y(t, \pi(k', t))) \right)^2 \right] 
$$

$$
\leq \sum_t \left( \prod_{l=t+1}^{T} (1 - \tau \phi_l) \right) \cdot \left[ \frac{1}{n} \sum_{k} \left( \frac{3}{2} + 2n \phi_t \right) \left( \nabla_{k'} f(y(t, \pi)) - \gamma_{k', t} \right)^2 + \frac{10n \phi_t}{2t} \left( \nabla_{k'} f(y(t, \pi)) - \nabla_{k'} f(y(t, \pi(k', t))) \right)^2 \right] 
$$

$$
\leq \sum_t \left( \prod_{l=t+1}^{T} (1 - \tau \phi_l) \right) \cdot \left[ \frac{1}{n} \sum_{k} \left( \frac{3}{2} + 2n \phi_t \right) \left( \nabla_{k'} f(y(t, \pi)) - \gamma_{k', t} \right)^2 + \frac{10n \phi_t}{2t} \left( \nabla_{k'} f(y(t, \pi)) - \nabla_{k'} f(y(t, \pi(k', t))) \right)^2 \right] 
$$

(Using the definition of $\Phi_b$ and noting that there are at most $q$ occurrences of each $(\Delta_s^F)^2$)

$$
\leq \sum_t \frac{1}{\Phi_b} \left[ \frac{3}{(\Phi_b - 1)} + 1 \right] \left( \prod_{l=t+1}^{T} (1 - \tau \phi_l) \right) \frac{3}{2} + 2n \phi_t \left( \frac{896 + \frac{397}{1 - \tau} \xi q^2 L_{\text{res}}^2 n^2 \phi_t^2}{n} \right) (\Delta_t^F)^2 + E_t^\Delta 
$$

by Lemma [9] applied to the series $b_t = \frac{3}{2} + 2n \phi_t \left( \frac{896 + \frac{397}{1 - \tau} \xi q^2 L_{\text{res}}^2 n^2 \phi_t^2}{n} \right) (\Delta_t^F)^2 + E_t^\Delta$. 

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Similarly, by (18), (19), and Lemma 9
\[
\sum_t \left( \prod_{t=t+1}^{T-1} (1 - \tau \phi_t) \right) \cdot \left[ \mathbb{E}_\pi \left[ 6 \cdot \Gamma_t \phi_t \sum_{k'} \|w_{k'}^{(t, \pi)} - w_{k'}^{(t, \pi(k', t))}\|_2^2 \right] \right.
+ \mathbb{E}_\pi \left[ \frac{\phi_t \Gamma_t \phi_t n_t \Omega}{3} \sum_{k'} \|z_{k'}^{(t, \pi)} - z_{k'}^{(t, \pi(k', t))}\|_2^2 \right] \]
\[
\leq \sum_t \left( \prod_{t=t+1}^{T-1} (1 - \tau \phi_t) \right) \cdot \left[ 7T \phi_t \cdot 16q \sum_{s=t-q-1, t-1} (\Delta_s^{FE})^2 \right] \quad \text{(as } n \phi_t \varphi \leq 1)\]
\[
\leq \sum_t \left( \prod_{t=t+1}^{T-1} (1 - \tau \phi_t) \right) \cdot \left[ \frac{112q^2 \Gamma_t \phi_t (\Delta_s^{FE})^2}{\Phi_c} \right]
\]
(\text{using the definition of } \Phi_c \text{ and noting that there are at most } q \text{ occurrences}
\text{ of each } (\Delta_s^{FE})^2 \text{ term})
\[
\leq \sum_t \frac{\Xi}{\Phi_c(\Phi_c - \Xi)} \left( \prod_{t=t+1}^{T-1} (1 - \tau \phi_t) \right) \cdot \left[ \frac{112q^2 \Gamma_t \phi_t \varepsilon_t^{\lambda}}{\Phi_c(\Phi_c - \Xi)} \right].
\]
Therefore, to ensure \( \sum_t \prod_{t=t+1}^{T-1} (1 - \tau \phi_t) \text{ Adj}_t \geq 0 \), it suffices to have
\[
\frac{1}{(\Phi_b - \Xi)} \cdot \frac{3}{2T} + 2n \phi_t \cdot (896 + \frac{397}{n} \xi q^2 L^2 \tau n^2 \phi_t^2) + \Xi \cdot \frac{112q^2 \Gamma_t \phi_t}{\Phi_c(\Phi_c - \Xi)} \lesssim \frac{n \phi_t(4 \Gamma_t - n \phi_t)}{2},
\]
which is a consequence of constraints (iv)--(vii).

\[\square\]

**B Proofs of the Remaining Theorems**

**B.1 Proofs of Theorems 2 and 3**

Next, we obtain lower bounds on \( \Phi_b \) and \( \Phi_c \) under a condition that holds with our parameter choices in Theorems 2 and 3.

**Lemma 11.** Suppose that \( \frac{\Xi}{\phi_t} \) is a constant. Then if \( n \geq 50q \), \( \Phi_b, \Phi_c \geq \frac{4}{5}, \xi \leq \frac{6}{5} \), and if \( q \leq \frac{\sqrt{n}}{10} \) the condition on \( r \) holds.

**Proof.** We begin with the bound on \( \Phi_b \). By inspection,
\[
\Phi_b \geq \min_{1 \leq k \leq 4q} \left\{ \left[ \frac{\Gamma_t}{\Gamma_{t-k}} \right] \left[ \frac{\phi_t \cdot k}{\phi_t} \right]^{5/2} \prod_{t=k+1}^{t} (1 - \tau \phi_t) \right\}
\geq \min_{1 \leq k \leq 4q} \left\{ \left[ \frac{\phi_t \cdot k}{\phi_t} \right]^{5/2} \prod_{t=k+1}^{t} (1 - \tau \phi_t) \right\}
\geq \min \left\{ \left( 1 - \frac{1}{n} \right)^{5q}, \left( 1 - \frac{1}{2n} \right)^{10q} \right\}
\quad \text{(as by Lemma 22(ii), } \phi_t \leq \frac{1}{n} \text{ for all } l)\]
\[
\geq 1 - \frac{10q}{n} \quad \text{if } \frac{5q}{n} \leq 1
\geq \frac{4}{5} \quad \text{if } n \geq 50q.
\]

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Essentially the same argument yields the same lower bound on $\Phi_c$ and an upper bound on $\xi$.

Finally, a direct calculation shows the final claim.

**Proof of Theorem 2**: We first observe that the assumptions of Theorem 2 imply the conditions for Lemma 11 and the $B^{(t)}$ are good by Lemma 22 and 23. In addition, by assumption, $\frac{n\phi_t}{\Gamma_t} = \frac{3}{20}$, and $q \leq \frac{1}{2\sqrt{\frac{20}{3}}}$; thus it follows from the definition of $\Xi$ that $\Xi \leq \frac{1}{125}$. Furthermore, the choice of $\tau = \frac{1}{2}$ and $\frac{n\phi_t}{\Gamma_t} = \frac{3}{20}$ establishes constraints (i), (iii), and (vii). Next, the choice of $\phi_t = \frac{\sqrt{3\mu}}{\sqrt{20n}}$ and $\Gamma_t = \frac{\sqrt{2n\phi_t}}{\sqrt{3}}$ establishes constraint (ii). $n \geq 19$ implies $n\phi_t \Gamma_t = \frac{3}{20}$, and $q \leq \frac{1}{25} \sqrt{\frac{n}{L_{res}}}$; thus it follows from the definition of $\Xi$ that $\Xi \leq \frac{1}{125}$. Furthermore, the choice of $\tau = \frac{1}{2}$ and $\frac{n\phi_t}{\Gamma_t} = \frac{3}{20}$ establishes constraints (i), (iii), and (vii). Next, the choice of $\phi_t = \frac{\sqrt{3\mu}}{\sqrt{20n}}$ and $\Gamma_t = \frac{\sqrt{2n\phi_t}}{\sqrt{3}}$ establishes constraint (ii).

Constraint (iv).

\[
\frac{1}{(\Phi_b - \Xi)} \cdot \frac{3}{2\Gamma_t(1 - \tau)} \cdot (1388)q^2 \frac{L_{res}^2}{n} \frac{n^2 \phi_t^2}{\Gamma_t} \leq \frac{43}{200} n_\phi \Gamma_t.
\]

Using the assumption that $1 - \tau = \epsilon$, this reduces to

\[
\frac{125}{99} \cdot \frac{3}{2} \cdot 1388 \cdot \frac{200}{43} \cdot \frac{1}{(\frac{20\phi_t}{\Gamma_t})^2} \left( \frac{q^2 \frac{L_{res}^2}{n} \epsilon^2}{n} \right) \leq 1,
\]

and so it suffices that $q \leq \frac{1}{14} \sqrt{\frac{2\epsilon}{L_{res}}}$. Constraint (v).

\[
\frac{1}{(\Phi_b - \Xi)} \cdot \frac{2n\phi_t}{\Gamma_t} \cdot \frac{1388 \cdot q^2 \frac{L_{res}^2}{n} \phi_t^2}{n} \leq \frac{3}{50} n_\phi \Gamma_t.
\]

Proof of Theorem 3: Strongly convex case

In the strongly convex case, we choose $\tau = 1 - \epsilon$, $\Gamma_t = \frac{20}{3} \sqrt{\frac{n\phi_t}{\Gamma_t}}$, which establishes constraints (iii) and (vii) and choose $\phi_t = \frac{(\frac{\sqrt{3\mu}}{\sqrt{20n}})^2}{n}$ which establishes constraints (i) and (ii). Note that the assumptions of Theorem 3 imply the $B^{(t)}$ are good by Lemma 22 and 23.

**Constraint (iv):**

\[
\frac{1}{(\Phi_b - \Xi)} \cdot \frac{3}{2\Gamma_t(1 - \tau)} \cdot (1388)q^2 \frac{L_{res}^2}{n} \phi_t^2 \leq \frac{43}{200} n_\phi \Gamma_t.
\]

Using the assumption that $1 - \tau = \epsilon$, this reduces to

\[
\frac{125}{99} \cdot \frac{3}{2} \cdot 1388 \cdot \frac{200}{43} \cdot \frac{1}{(\frac{20\phi_t}{\Gamma_t})^2} \left( \frac{q^2 \frac{L_{res}^2}{n} \epsilon^2}{n} \right) \leq 1,
\]

and so it suffices that $q \leq \frac{1}{14} \sqrt{\frac{2\epsilon}{L_{res}}}$. Constraint (v).

\[
\frac{1}{(\Phi_b - \Xi)} \cdot \frac{2n\phi_t}{\Gamma_t} \cdot \frac{1388 \cdot q^2 \frac{L_{res}^2}{n} \phi_t^2}{n} \leq \frac{3}{50} n_\phi \Gamma_t.
\]
This reduces to
\[
\frac{125}{99} \cdot 2 \cdot 1388 \cdot \frac{n \phi_t}{(4n)^2} \cdot \left( \frac{q^2 L_{\text{res}}^2}{n} \right) \leq \frac{3}{50},
\]
and as \( n \alpha_t \leq 1 \), it suffices that \( q \leq \frac{\sqrt{n}}{37} \).

**Constraint (vi).**
\[
\frac{\Xi}{\Phi_c(\Phi_c - \Xi)} \cdot 112 \cdot q^2 \Gamma_t \phi_t \leq \frac{1}{50} n \phi_t \Gamma_t.
\]

As in the proof of Theorem 2, \( q \leq \frac{\sqrt{n}}{37} \) suffices.

**Non-strongly convex case** Remember that we chose \( \Gamma_t = \frac{20}{3} \sqrt{n \phi_t} \) and this establishes (iii) and (vii) as \( n \phi_t \leq 1 \). We set \( 1 - \frac{1}{4t_0} - \epsilon \geq \frac{1}{2} \) as \( \epsilon < \frac{1}{3} \) and \( n \geq 19 \).

**Constraint (i).**
\[
(1 - \Phi_b - \Xi) \cdot 2 \Gamma_t (1 - \frac{1}{4t_0} - \tau) \cdot 1388 q^2 L_{\text{res}}^2 n^2 \phi_t^2 \frac{n}{n} \leq \frac{1}{5} n \phi_t \Gamma_t.
\]

We know \( \ln \left( \frac{1 - \Phi_b}{1 - \tau \phi_t} \right)^2 \leq 2 \ln [1 - (\Phi_b - \tau) \phi_t] \leq -2(\Phi_b - \tau) \phi_t \), and by Lemma 22(iii), \( \ln \frac{\phi_{t+1}}{\phi_t} \geq \ln \left(1 - \frac{\phi_t}{2}\right) \geq -\frac{\phi_t}{2} \geq \left(\frac{\phi_t}{2}\right)^2 \) as \( \phi_t < 1 \). Thus it suffices to have \( 2(\Phi_b - \tau) \phi_t \geq \phi_t \left[\frac{1}{2} + \frac{\phi_t}{4}\right] \). Using the fact that \( \phi_t \leq \frac{1}{n} \), it suffices to let
\[
\Phi_b - \tau = \frac{1}{4} + \frac{1}{8n}. \tag{8}
\]

**Constraint (iv).**
\[
\frac{1}{(\Phi_b - \Xi)} \cdot 2 \Gamma_t (1 - \frac{1}{4t_0} - \tau) \cdot 1388 q^2 L_{\text{res}}^2 n^2 \phi_t^2 \frac{n}{n} \leq \frac{1}{5} n \phi_t \Gamma_t.
\]

Using the assumption that \( \Phi_b - \tau = 1 - \frac{1}{4t_0} - \epsilon > \frac{1}{2} \), this reduces to
\[
\frac{125}{99} \cdot \frac{3}{2} \cdot 1388 \cdot \frac{200}{43} \cdot \frac{1}{(4n)^2} \left( \frac{q^2 L_{\text{res}}^2}{n \epsilon} \right) \leq 1,
\]
and so it suffices that \( q \leq \frac{\sqrt{n}}{37} \).

**Constraint (v).**
\[
\frac{1}{(\Phi_b - \Xi)} \cdot 2n \phi_t \Gamma_t \cdot \frac{1388 \cdot q^2 L_{\text{res}}^2 n^2 \phi_t^2}{n} \leq \frac{3}{50} n \phi_t \Gamma_t.
\]
This reduces to
\[
\frac{125}{99} \cdot 2 \cdot 1388 \cdot \frac{n\phi t}{(2n^3)^2} \left( \frac{q^2 L_p^2}{n} \right) \leq \frac{3}{50},
\]
and as \( n\phi t \leq 1 \), it suffices that \( q \leq \frac{1}{3t} \sqrt{\frac{n}{T}} \).

**Constraint (vi)**
\[
\frac{2}{\Phi_c(\Phi_c - \Xi)} \cdot \cdot \cdot q^2 \Gamma_t \phi_t \leq \frac{1}{200} n\phi t \Gamma_t.
\]

As in the proof of Theorem 2, \( q \leq \frac{\sqrt{n}}{T} \) suffices.

Now, let’s look at the convergence rate.

1. In the strongly convex case, the convergence rate becomes
\[
\prod_{t}(1 - \tau \alpha_t) = \left( 1 - (1 - \epsilon) \left( \frac{3\mu}{n} \right)^T \right).
\]

2. In the non-strongly convex case, recall that the convergence rate of \( \prod_{k=0,\ldots,T-1}(1 - \phi_t) \) is
\[
\left( \frac{(t_0 - 2)(t_0 - 1)}{(t_0 + T - 1)(t_0 + T - 2)} \right)^{n\phi t} = \left( \frac{(t_0 - 2)(t_0 - 1)}{(t_0 + T - 1)(t_0 + T - 2)} \right)^{n\phi t}.
\]

**Proof of Theorem 5.** We know that when \( t = 0 \), \( \hat{y}^{(t)}, \hat{z}^{(t)} = x^{(t)} \). We will prove by induction that for any \( t > 0 \), \( \hat{y}^{(t)}, \hat{z}^{(t)} = x^{(t)} \).

At \( t = 0 \), Algorithm 1 sets \( z^{(t)} = x^{(t)} \), and for all \( t \) it sets \( y^{(t)} = (1 - \psi_t)z^{(t)} + \psi_t x^{(t)} \). Thus, \( \hat{y}^{(0)} = \hat{z}^{(0)} = z^{(0)} = y^{(0)} = \hat{z}^{(0)} \).

Now, suppose the hypothesis is true for all \( t \leq l \). We analyze the case \( t = l + 1 \). By (1), and
\[
B^{(l+1)} = A^{(l)} B^{(l)},
\]
we have
\[
(\hat{y}^{(l+1)}, \hat{z}^{(l+1)})^T = A^{(l)} ((\hat{y}^{(l)}, \hat{z}^{(l)})^T + D^{(l)}).
\]

By the definition of \( A^{(l)} \) and \( D^{(l)} \),
\[
\hat{y}^{(l+1)} = ((1 - \psi_{l+1})(1 - \varphi_t) + \psi_{l+1}) \hat{y}^{(l)} + (1 - \psi_{l+1}) \varphi_t \hat{z}^{(l)} + (n\psi_{l+1}\phi_t + (1 - \psi_{l+1})) \Delta z_{k_t}^{l+1} 1_{k_t},
\]
and
\[
\hat{z}^{(l+1)} = (1 - \varphi_t) \hat{y}^{(l)} + \varphi_t \hat{z}^{(l)} + (n\psi_{l+1}\phi_t + (1 - \psi_{l+1})) \Delta z_{k_t}^{l+1} 1_{k_t}.
\]

We treat \( y \) and \( z \) separately. First, we consider \( \hat{z}^{(l+1)} \). It’s easy to see that if \( \hat{z}^{(l)} = z^{(l)} \) and \( \hat{y}^{(l)} = y^{(l)} \), then \( \hat{z}^{(l+1)} = z^{(l+1)} \).

Next, we look at \( y^{(l+1)} \).
\[
y^{(l+1)} = \psi_{l+1} x^{(l+1)} + (1 - \psi_{l+1}) z^{(l+1)} = \psi_{l+1} (y^{(l)} + n\phi_t \Delta z_{k_t}^{l+1} 1_{k_t}) + (1 - \psi_{l+1}) \varphi_t z^{(l)} + (1 - \psi_{l+1})(1 - \varphi_t)y^{(l)} + (1 - \psi_{l+1}) \Delta z_{k_t}^{l+1} 1_{k_t}.
\]
Comparing this with (9), by the induction hypothesis, we get $y'(l+1,\pi') = \hat{y}'(l+1,\pi')$. Since $y'(l+1,\pi') = \hat{y}'(l+1,\pi')$ and $z'(l+1,\pi') = \hat{z}'(l+1,\pi')$, and as $\hat{x}'(l+1,\pi')$ and $\hat{x}'(l+1,\pi')$ satisfy, respectively:

$$y'(l+1,\pi') = (1 - \psi_{t+1})z'(l+1,\pi') + \psi_{t+1}x'(l+1,\pi')$$

and

$$\hat{y}'(l+1,\pi') = (1 - \psi_{t+1})\hat{z}'(l+1,\pi') + \psi_{t+1}\hat{x}'(l+1,\pi'),$$

also, as $w'(l+1,\pi')$ and $\hat{w}'(l+1,\pi')$ satisfy, respectively:

$$w'(l+1,\pi') = \phi_t z'(l+1,\pi') + (1 - \phi_t)y'(l+1,\pi')$$

and

$$\hat{w}'(l+1,\pi') = \phi_t \hat{z}'(l+1,\pi') + (1 - \phi_t)\hat{y}'(l+1,\pi'),$$

the theorem follows. □
C Proof of Lemma 12, the Progress Lemma

In this section, we prove Lemma 12. We begin by stating the full version of Lemma 12.

**Lemma 12.**

- **Strongly convex case**

  Suppose that \(0 < \tau \leq 1\), \(\phi_t = \phi\), \(\varphi_t = (1 - \phi)\), \(\psi_t = \frac{1}{1 + \phi}\), and that \(\Gamma_t = \Gamma\) and \(\tilde{\mu}\) satisfy the following constraints, for all \(t \geq 0\):

  \[
  \begin{align*}
  \text{i. } \phi &\leq \frac{1}{n}; \\
  \text{ii. } n\phi \Gamma &\leq \mu; \\
  \text{iii. } \frac{4}{5} \Gamma &\geq n\phi,
  \end{align*}
  \]

  and let \(\zeta_{t+1} = \frac{n\phi_t \Gamma_t}{2} \left(1 - \frac{n\phi_t (1 - \tau)}{3}\right)\).

  Then

  \[
  \mathbb{E}_\pi \left[ F^{(t+1)} \right] \leq (1 - \tau \phi_t) \mathbb{E}_\pi \left[ F^{(t)} \right] - \text{Adj}_t,
  \]

  where

  \[
  \text{Adj}_t = \mathbb{E}_\pi \left[ \frac{n\phi_t (\frac{4}{5} \Gamma_t - n\phi_t)}{2} \left( \Delta z_{k_t}^{t, \pi} \right)^2 \right]
  \]

  \[
  \begin{align*}
  &- \mathbb{E}_\pi \left[ \frac{1}{n} \sum_{k'} \left( \frac{3}{1 - \tau} + 2n\phi_t \right) \left( \nabla_k f(y^{(t, \pi)}) - g_{k'}^{t, \pi(k', t)} \right)^2 \\
  &\quad + \frac{10n\phi_t}{2\Gamma_t} \left( \nabla_k f(y^{(t, \pi)}) - \nabla_k f(y^{(t, \pi(k', t))}) \right)^2 \right]
  + 10n\phi_t \sum_{k'} \left( \nabla_k f(y^{(t, \pi)}) - \nabla_k f(y^{(t, \pi(k', t))}) \right)^2 \right]
  \]

  \[
  - \mathbb{E}_\pi \left[ 6\Gamma_t \phi_t \sum_{k'} \left( \nabla_k f(y^{(t, \pi)}) - \nabla_k f(y^{(t, \pi(k', t))}) \right)^2 \right]
  \]

  \[
  - \mathbb{E}_\pi \left[ \phi_t \Gamma_t \phi_t n\phi_t (1 - \tau) \sum_{k'} \left( \nabla_k f(y^{(t, \pi)}) - \nabla_k f(y^{(t, \pi(k', t))}) \right)^2 \right] .
  \]

  In this case, \(A^{(t)} = \left( \begin{array}{cc} \frac{1 + \phi^2}{1 + \phi} & 1 - \frac{1 + \phi^2}{1 + \phi} \\ \phi & 1 - \phi \end{array} \right)^t\) and \(B^{(t)} = \left( \begin{array}{cc} \frac{1 + \phi^2}{1 + \phi} & 1 - \frac{1 + \phi^2}{1 + \phi} \\ \phi & 1 - \phi \end{array} \right)^t\).

- **Non-strongly convex case**

  Suppose that \(0 < \tau \leq 1\), \(0 < \tilde{\tau} \leq 1\), \(\phi_t = \frac{2}{t_0 + 1}\) for \(t_0 \geq 2(n + 1)\), \(\varphi_t = 1\), \(\psi_t = 1 - \phi_t\), and \(\Gamma_t\) satisfy the following constraints, for all \(t \geq 0\):

  \[
  \begin{align*}
  \text{(i) } \tilde{\tau} &\geq \frac{1}{2}; \\
  \text{(ii) } (1 - \tilde{\tau} \phi_t) \frac{\Gamma_t}{n\phi_t} &\leq (1 - \tau \phi_t) \frac{\Gamma_t}{n\phi_t - 1}; \\
  \text{and let } \zeta_{t+1} = \frac{n\phi_t \Gamma_t}{2} \left(1 - \frac{n\phi_t (1 - \tilde{\tau} - \frac{1}{4n})}{3}\right).
  \end{align*}
  \]

  Then

  \[
  \mathbb{E}_\pi \left[ F^{(t+1)} \right] \leq (1 - \tau \phi_t) \mathbb{E}_\pi \left[ F^{(t)} \right] - \text{Adj}_t,
  \]
where $\text{Adj}_t = \mathbb{E}_\pi \left[ \frac{n\phi_t (\frac{4}{2} - n\phi_t) \left( \Delta z_{k_t}^{t,\pi} \right)^2}{2} \right]$

- $\mathbb{E}_\pi \left[ \frac{1}{n} \sum_{k'} \left( \frac{3}{2} \frac{1 - \frac{4}{2} \phi_t}{2\Gamma_t} \left( \nabla_{k'} f(y^{(t,\pi)}) - g_{k'}^{t,\pi} \right)^2 + \frac{10n\phi_t}{2\Gamma_t} \left( \nabla_{k'} f(y^{(t,\pi)}) - \nabla_{k'} f(y^{(t,\pi)}(k',t)) \right)^2 \right) \right]$

- $\mathbb{E}_\pi \left[ 6\Gamma_t \phi_t \sum_{k'} \left( w_{k'}^{(t,\pi)} - w_{k'}^{(t,\pi)}(k',t) \right)^2 \right]$

- $\mathbb{E}_\pi \left[ \frac{\phi_t \Gamma_t \phi_t}{3} \sum_{k'} \left( \int_{t,\pi}^{w_{k'}^{(t,\pi)}} \left( z_{k'}^{(t,\pi)} - z_{k'}^{(t,\pi)}(k',t) \right)^2 \right) \right]$

In this case, $A(t) = \begin{pmatrix} \frac{t+t_0-1}{t+t_0+1} & 2 \\ 0 & 1 \end{pmatrix}$ and $B(t) = \begin{pmatrix} \frac{t_0(t_0-1)}{(t_0+1)(t_0+t)} & 1 - \frac{t_0(t_0-1)}{(t_0+t)(t_0+t+1)} \end{pmatrix}$.

We restate Algorithm 1 for the reader’s convenience. We begin by determining constraints on these parameters for which Lemma 12 holds. We then show the parameter choice in Algorithm 1 satisfies these constraints.

**Algorithm 3:** The basic iteration

1. Choose $k_t \in \{1, 2, \ldots, n\}$ uniformly at random;
2. $y^{(t,\pi)} = \psi_t x^{(t,\pi)} + (1 - \psi_t) z^{(t,\pi)}$;
3. $w^{(t,\pi)} = \varphi_t z^{(t,\pi)} + (1 - \varphi_t) y^{(t,\pi)}$;
4. $z^{(t+1,\pi)} = \arg\min_x \left\{ \frac{1}{2} \| x - f^{(t,\pi)} \|^2 + \langle g_{k_t}^{t,\pi}, x_k \rangle \right\}$;
5. $x^{(t+1,\pi)} = y^{(t,\pi)} + n\phi_t (z^{(t+1,\pi)} - w^{(t,\pi)})$.

**Lemma 13.** Let $\zeta_{t+1} = \frac{n\phi_t \eta_t \Gamma_t}{2(\eta_t + 1)}$, and suppose that the parameters satisfy the following constraints:

i. $\eta_t > 1$;

ii. $\frac{n\phi_t \psi_t \varphi_t}{1 - \psi_t} = n(1 - \phi_t)$;

iii. $(1 - \phi_t) \left( \frac{n\phi_t \Gamma_t \psi_t \varphi_t}{2(\eta_t + 1)(1 - \phi_t)} \right) \leq (1 - \tau \phi_t) \left( \frac{n\phi_{t-1} \eta_{t-1} \Gamma_{t-1}}{2(\eta_{t-1} + 1)} \right)$;

iv. $\frac{n\Gamma_t (1 - \phi_t)}{2} \leq \frac{\mu}{2}$.
Then,
\[
\frac{1}{n} \sum_k \left[ f(x^{(t+1, \pi(k,t))}) - f(x^*) + \zeta_{t+1} \|x^* - z^{(t+1, \pi(k,t))}\|_2 \right]
\leq \frac{1}{n} \left( 1 - \tau \phi_t \right) \sum_k \left[ f(x^{(t, \pi(k,t))}) - f(x^*) + \zeta_t \|x^* - z^{(t, \pi(k,t))}\|_2 \right]
- \sum_k \phi_t \left( \frac{\Gamma_t}{2} - \frac{n \phi_t}{2} \right) \left( \Delta x_k^{t, \pi(k,t)} \right)^2
+ \frac{1}{n} \phi_t \sum_{k,k'} \left[ \frac{\eta_t + 2}{2 \Gamma_t} \left( \nabla_k f(y^{(t, \pi(k,t))}) - g_k^{t, \pi(k',t)} \right)^2 + \frac{10}{2 \Gamma_t} \left( \nabla_k f(y^{(t, \pi(k,t))}) - \nabla_k f(y^{(t, \pi(k',t))}) \right)^2 \right]
+ \frac{6}{n} \Gamma_t \phi_t \sum_{k,k'} \|w_k^{(t, \pi(k,t))} - w_{k'}^{(t, \pi(k',t))}\|^2
+ \frac{2}{n} \phi_t \Gamma_t \varphi_t \sum_{k,k'} \|z_k^{(t, \pi(k,t))} - z_k^{(t, \pi(k',t))}\|^2.
\]

Proof. We define \( \Delta x_k^{t, \pi, \pi(k,t)} \triangleq x^{(t+1, \pi)} - y^{(t, \pi)} \). Recall that \( \Delta z_k^{t+1, \pi(k,t)} = z_k^{t, \pi(k,t)} - w_k^{t, \pi(k,t)} \). Therefore, \( \Delta x_k^{t, \pi, \pi(k,t)} = n \phi_t \Delta z_k^{t+1, \pi(k,t)} \).

Since for any \( \pi' \), \( y^{(t, \pi')} = \psi_t x^{(t, \pi')} + (1 - \psi_t) z^{(t, \pi')} \),
\[ z^{(t, \pi')} - y^{(t, \pi')} = \frac{\psi_t}{1 - \psi_t} \left( y^{(t, \pi')} - x^{(t, \pi')} \right). \]

So,
\[
0 = n \phi_t \left( w^{(t, \pi')} - w^{(t, \pi')} \right)
= n \phi_t w^{(t, \pi')} - n \phi_t \left[ \psi_t z^{(t, \pi')} + (1 - \psi_t) y^{(t, \pi')} \right] \quad \text{(using Line 3, Algorithm 3)}
= n \phi_t w^{(t, \pi')} - n \phi_t \left[ y^{t, \pi'} + \frac{\psi_t \varphi_t}{1 - \psi_t} (y^{(t, \pi')} - x^{(t, \pi')}) \right] \quad \text{(using Line 2, Algorithm 3)}
= n \phi_t (w^{(t, \pi')} - y^{(t, \pi')}) + \frac{n \phi_t \psi_t \varphi_t}{1 - \psi_t} (x^{(t, \pi')} - y^{(t, \pi')})
= n \phi_t (w^{(t, \pi')} - y^{(t, \pi')}) + n (1 - \phi_t) (x^{(t, \pi')} - y^{(t, \pi')}) \quad \text{(using Constraint (ii)).} \quad (11)
\]

Note that \( x^{(t+1, \pi(k,t))} \) is the same as \( y^{(t, \pi(k,t))} \) on all the coordinates other than \( k \), and \( f \) is a\footnote{This is because \( z^{(t+1)} \) is the same as \( w^{(t)} \) on all the coordinates other than \( i_t \), and \( x^{(t+1)} - y^{(t)} = n \phi_t (z^{(t+1)} - w^{(t)}). \)
convex function with $L_{k_t,k_t} = 1$. Therefore,

$$f(x^{(t+1),\pi(k,t)}) \leq f(y^{(t,\pi(k,t))}) + \langle \nabla_k f(y^{(t,\pi(k,t))}), x_k^{(t+1,\pi(k,t))} - y_k^{(t,\pi(k,t))} \rangle + \frac{1}{2} \left( x_k^{(t+1,\pi(k,t))} - y_k^{(t,\pi(k,t))} \right)^2$$

$$= f(y^{(t,\pi(k,t))}) + \frac{1}{n} \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k',t))}), \Delta x_k^{t,\pi(k,t)} \rangle + \frac{1}{2} \left( \Delta x_k^{t,\pi(k,t)} \right)^2$$

$$+ \frac{1}{n} \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k',t))}) - \nabla_k f(y^{(t,\pi(k,t))}), \Delta x_k^{t,\pi(k,t)} \rangle$$

$$= f(y^{(t,\pi(k,t))}) + \frac{1}{n} \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k',t))}),$$

$$n\phi_t(w_k^{(t,\pi(k',t))} + \Delta z_k^{t,\pi(k,t)} - y_k^{(t,\pi(k',t))}) + n(1 - \phi_t)(x_k^{(t,\pi(k',t))} - y_k^{(t,\pi(k',t))})$$

$$+ \frac{n^2 \phi_t^2}{2} \left( \Delta z_k^{t,\pi(k,t)} \right)^2 + \frac{1}{n} \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k',t))}) - \nabla_k f(y^{(t,\pi(k,t))}), \Delta x_k^{t,\pi(k,t)} \rangle$$

(using (11) and the fact that $\Delta x_k^{t,\pi(k,t)} = n\phi_t \Delta z_k^{t,\pi(k,t)}$).

Summing over all $k$ gives

$$\frac{1}{n} \sum_k f(x^{(t+1,\pi(k,t))})$$

$$= \frac{1}{n} \sum_k f(y^{(t,\pi(k,t))}) + \frac{1}{n} \sum_k \frac{1}{n} \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k',t))}),$$

$$n\phi_t(w_k^{(t,\pi(k',t))} + \Delta z_k^{t,\pi(k,t)} - y_k^{(t,\pi(k',t))}) + n(1 - \phi_t)(x_k^{(t,\pi(k',t))} - y_k^{(t,\pi(k',t))})$$

$$+ \frac{n^2 \phi_t^2}{2} \left( \Delta z_k^{t,\pi(k,t)} \right)^2 + \frac{1}{n} \sum_k \frac{1}{n} \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k',t))}), x_k^{(t,\pi(k',t))} - y_k^{(t,\pi(k',t))} \rangle$$

$$+ \frac{n \phi_t}{n} \sum_k \frac{1}{n} \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k',t))}), w_k^{(t,\pi(k,t))} + \Delta z_k^{t,\pi(k,t)} - y_k^{(t,\pi(k',t))} \rangle$$

$$+ \frac{1}{n} \sum_k \frac{n \phi_t^2}{2} \left( \Delta z_k^{t,\pi(k,t)} \right)^2 + \frac{1}{n} \sum_k \frac{1}{n} \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k',t))}) - \nabla_k f(y^{(t,\pi(k,t))}), \Delta x_k^{t,\pi(k,t)} \rangle$$

$$\leq \frac{1}{n} (1 - \phi_t) \sum_k f(x^{(t,\pi(k,t))})$$

$$+ \frac{1}{n} \phi_t \sum_k \left[ f(y^{(t,\pi(k,t))}) + \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k,t))}), w_k^{(t,\pi(k,t))} + \Delta z_k^{t,\pi(k,t)} - y_k^{(t,\pi(k,t))} \rangle \right]$$

$$+ \sum_{k'} \frac{\Gamma_t}{2} \left( \Delta z_k^{t,\pi(k,t)} \right)^2$$

$$- \sum_k \frac{\phi_t (\Gamma_t - n \phi_t)}{2} \left( \Delta z_k^{t,\pi(k,t)} \right)^2 + \frac{1}{n} \sum_k \frac{1}{n} \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k,t))}) - \nabla_k f(y^{(t,\pi(k',t))}), \Delta x_k^{t,\pi(k,t)} \rangle.$$
Since $\Delta z^{t,\pi(k,t)}_k = -\frac{1}{\Gamma_t} \bar{g}^{t,\pi(k,t)}_k$, by a simple calculation,

$$
\langle \nabla_{k'} f(y(t,\pi(k,t))), \Delta z^{t,\pi(k',t)}_k \rangle + \frac{\Gamma_t}{2} \left( \Delta z^{t,\pi(k',t)}_k \right)^2
= \langle \nabla_{k'} f(y(t,\pi(k,t))), -\frac{1}{\Gamma_t} \bar{g}^{t,\pi(k',t)}_k \rangle + \frac{1}{2\Gamma_t} \left( -\bar{g}^{t,\pi(k',t)}_k \right)^2
= \langle \nabla_{k'} f(y(t,\pi(k,t))), -\frac{1}{\Gamma_t} \bar{g}^{t,\pi(k',t)}_k \rangle + \frac{1}{\Gamma_t} \left( -\bar{g}^{t,\pi(k',t)}_k \right)^2 + \frac{1}{2\Gamma_t} \left( -\bar{g}^{t,\pi(k',t)}_k \right)^2
\leq \langle \nabla_{k'} f(y(t,\pi(k,t))), -\frac{1}{\Gamma_t} \bar{g}^{t,\pi(k',t)}_k \rangle + \frac{1}{\Gamma_t} \left( -\bar{g}^{t,\pi(k',t)}_k \right)^2 + \frac{1}{2\Gamma_t} \left( -\bar{g}^{t,\pi(k',t)}_k \right)^2
$$

By Lemma 22, for any $\eta_t > 1$ and any $\bar{a}$ and $\bar{b}$, $\eta_t ||\bar{a} - \bar{b}||^2 + ||\bar{a}||^2 \geq \frac{\eta_t}{\eta_t + 1} ||\bar{b}||^2$. Therefore, putting $\bar{a} = x^*_k - w^{t,\pi(k',t)}_k + \frac{1}{\Gamma_t} \nabla_{k'} f(y(t,\pi(k,t)))$ and $\bar{b} = \bar{x}^*_k - w^{t,\pi(k',t)}_k + \frac{1}{\Gamma_t} \bar{g}^{t,\pi(k',t)}_k$, yields

$$
\langle \nabla_{k'} f(y(t,\pi(k,t))), \Delta z^{t,\pi(k',t)}_k \rangle + \frac{\Gamma_t}{2} \left( \Delta z^{t,\pi(k',t)}_k \right)^2 \leq -\frac{1}{\Gamma_t} \langle \nabla_{k'} f(y(t,\pi(k,t))), \bar{g}^{t,\pi(k',t)}_k \rangle + \frac{1}{2\Gamma_t} \left( \bar{g}^{t,\pi(k',t)}_k \right)^2
$$

\begin{align*}
&\leq \frac{1}{2\Gamma_t} \left( \bar{g}^{t,\pi(k',t)}_k - \nabla_{k'} f(y(t,\pi(k,t))) \right)^2 - \frac{\eta_t \Gamma_t}{2(\eta_t + 1)} ||\bar{a} - \bar{b}||^2
&\leq \frac{1}{2\Gamma_t} \left( \bar{g}^{t,\pi(k',t)}_k - \nabla_{k'} f(y(t,\pi(k,t))) \right)^2 - \frac{\eta_t \Gamma_t}{2(\eta_t + 1)} ||\bar{a} - \bar{b}||^2
&\leq \frac{1}{2\Gamma_t} \left( \bar{g}^{t,\pi(k',t)}_k - \nabla_{k'} f(y(t,\pi(k,t))) \right)^2
&\quad + \langle \nabla_{k'} f(y(t,\pi(k,t))), x^*_k - w^{t,\pi(k',t)}_k \rangle
&\quad + \frac{\Gamma_t}{2} \left( x^*_k - w^{t,\pi(k',t)}_k \right)^2 + \left( \eta_t \Gamma_t \right) \frac{1}{2(\eta_t + 1)} \left( x^*_k - w^{t,\pi(k',t)}_k \right)^2
&\quad + \eta_t \Gamma_t \frac{1}{2(\eta_t + 1)} \left( x^*_k - w^{t,\pi(k',t)}_k \right)^2
\end{align*}
Plugging into (12) gives
\[
\frac{1}{n} \sum_k f(x(t+1, \pi(k,t))) \\
\leq \frac{1}{n} (1 - \phi_t) \sum_k f(x^{(t,\pi(k,t))}) \\
+ \frac{1}{n} \phi_t \sum_k \left[ f(y^{(t,\pi(k,t))}) + \sum_{k'} \langle \nabla_{k'} f(y^{(t,\pi(k,t))}), x_{k'}^* - y_{k'}^{(t,\pi(k,t))} \rangle \\
+ \sum_{k'} \frac{\Gamma_t}{2} (x_{k'}^* - w_{k'}^{(t,\pi(k',t))})^2 \right] \\
- \frac{\phi_t (\Gamma_t - n \phi_t)}{2} \left( \Delta z_{k}^{(t,\pi(k,t))} \right)^2 \\
+ \frac{1}{n} \sum_k \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k,t))}) - \nabla_k f(y^{(t,\pi(k',t))}), \Delta x_{k}^{t,\pi(k,t)} \rangle \\
+ \frac{1}{n} \phi_t \sum_{k,k'} \left[ \frac{\eta_t + 1}{2 \Gamma_t} \left( \tilde{g}_{k'}^{(t,\pi(k,t))} - \nabla_{k'} f(y^{(t,\pi(k,t))}) \right)^2 \right. \\
\left. - \frac{\eta_t \Gamma_t}{2(\eta_t + 1)} \left( x_{k'}^* - w_{k'}^{(t,\pi(k',t))} + \frac{1}{\Gamma_t} \tilde{g}_{k'}^{(t,\pi(k',t))} \right)^2 \right] \\
+ \frac{1}{n} \phi_t \sum_{k,k'} \langle \nabla_{k'} f(y^{(t,\pi(k,t))}), w_{k'}^{(t,\pi(k,t))} - w_{k'}^{(t,\pi(k',t))} \rangle \\
\leq \frac{1}{n} (1 - \phi_t) \sum_k f(x^{(t,\pi(k,t))}) \\
+ \frac{1}{n} \phi_t \sum_k \left[ f(x^*) - \frac{\mu}{2} \| x^* - y^{(t,\pi(k,t))} \|^2 \right] \\
+ \sum_{k'} \frac{\Gamma_t}{2} (x_{k'}^* - w_{k'}^{(t,\pi(k',t))})^2 \right] \\
- \frac{\phi_t (\Gamma_t - n \phi_t)}{2} \left( \Delta z_{k}^{(t,\pi(k,t))} \right)^2 \\
+ \frac{1}{n} \sum_k \sum_{k'} \langle \nabla_k f(y^{(t,\pi(k,t))}) - \nabla_k f(y^{(t,\pi(k',t))}), \Delta x_{k}^{t,\pi(k,t)} \rangle \\
+ \frac{1}{n} \phi_t \sum_{k,k'} \left[ \frac{\eta_t + 1}{2 \Gamma_t} \left( \tilde{g}_{k'}^{(t,\pi(k,t))} - \nabla_{k'} f(y^{(t,\pi(k,t))}) \right)^2 \right. \\
\left. - \frac{\eta_t \Gamma_t}{2(\eta_t + 1)} \left( x_{k'}^* - w_{k'}^{(t,\pi(k',t))} + \frac{1}{\Gamma_t} \tilde{g}_{k'}^{(t,\pi(k',t))} \right)^2 \right] \\
+ \frac{1}{n} \phi_t \sum_{k,k'} \langle \nabla_{k'} f(y^{(t,\pi(k,t))}), w_{k'}^{(t,\pi(k,t))} - w_{k'}^{(t,\pi(k',t))} \rangle ,
\]
The labeling of terms is to facilitate matching terms in the next series of inequalities. We note the following inequalities:

1. If \( k \neq k' \), by line 3 of Algorithm 3,

\[
(x^*_{k'} - z_{k'}^{(t+1, \pi(k,t))})^2 = (x^*_{k'} - w_{k'}^{(t, \pi(k,t))})^2 \leq \varphi_t (x^*_{k'} - z_{k'}^{(t, \pi(k,t))})^2 + (1 - \varphi_t) (x^*_{k'} - y_{k'}^{(t, \pi(k,t))})^2.
\]

(13)

2. Otherwise, again by line 3 of Algorithm 3,

\[
\frac{\varphi_t \eta \Gamma_t}{2(\eta_t + 1)} \sum_{k, k' \neq k} (x^*_{k'} - z_{k'}^{(t+1, \pi(k,t))})^2 \leq \sum_{k' \neq k} \frac{n \varphi_t \eta \Gamma_t \varphi_t}{2(\eta_t + 1)} (x^*_{k'} - z_{k'}^{(t, \pi(k,t))})^2 + \varphi_t \sum_{k' \neq k} \frac{n \eta \Gamma_t (1 - \varphi_t)}{2(\eta_t + 1)} (x^*_{k'} - y_{k'}^{(t, \pi(k,t))})^2.
\]

(by 13).

This yields:

\[
\frac{1}{n} \sum_k \left[ f(x^{(t+1, \pi(k,t))}) - f(x^*) + \frac{n \varphi_t \eta \Gamma_t}{2(\eta_t + 1)} \|x^* - z^{(t+1, \pi(k,t))}\|^2 \right]
\]

\[
\leq \frac{1}{n} (1 - \phi_t) \sum_k \left[ f(x^{(t, \pi(k,t))}) - f(x^*) + \frac{n \varphi_t \eta \Gamma_t \varphi_t}{2(1 - \phi_t)} (x^*_{k'} - z_{k'}^{(t, \pi(k,t))})^2 + \sum_{k' \neq k} \frac{n \eta \Gamma_t (1 - \varphi_t)}{2(\eta_t + 1)(1 - \phi_t)} (x^*_{k'} - y_{k'}^{(t, \pi(k,t))})^2 \right]
\]

\[
- \frac{\phi_t}{n} \sum_k \left[ \frac{\mu}{2} \|x^* - y^{(t, \pi(k,t))}\|^2 - \frac{n \Gamma_t (1 - \varphi_t)}{2} (x^*_{k'} - y_{k'}^{(t, \pi(k,t))})^2 - \sum_{k' \neq k} \frac{n \eta \Gamma_t (1 - \varphi_t)}{2(\eta_t + 1)} (x^*_{k'} - y_{k'}^{(t, \pi(k,t))})^2 \right]
\]

\[
- \sum_k \frac{\phi_t (\Gamma_t - n \phi_t)}{2} \left( \Delta z_{k'}^{t, \pi(k,t)} \right)^2 + \frac{1}{n} \sum_k \sum_{k'} \frac{1}{n \Gamma_t} \langle \nabla_k f(y^{(t, \pi(k,t))}) - \nabla_k f(y^{(t, \pi(k',t))}), \Delta x_{k'}^{t, \pi(k,t)} \rangle
\]

\[
- \sum_k \phi_t \sum_{k, k'} \left[ \frac{\eta_t + 1}{2\Gamma_t} \left( g_{k'}^{t, \pi(k',t)} - \nabla_{k'} f(y^{(t, \pi(k,t))}) \right)^2 \right]
\]

\[
+ \frac{1}{n} \phi_t \sum_{k, k'} \langle \nabla_{k'} f(y^{(t, \pi(k,t))}), w_{k'}^{(t, \pi(k,t))} - w_{k'}^{(t, \pi(k',t))} \rangle
\]

\[
+ \frac{1}{n} \phi_t \sum_{k, k'} \langle \nabla_{k'} f(y^{(t, \pi(k,t))}), w_{k'}^{(t, \pi(k,t))} - w_{k'}^{(t, \pi(k',t))} \rangle.
\]
Note that the coefficient of $H$ is bigger than the coefficient of $J$. We intend to move some of $H$ to term $J$. As
\[
\frac{1}{n} \frac{\phi_t \Gamma_t \varphi_t}{2(\eta t + 1)(1 - \phi_t)} \left( x_k^* - z_k^{(t, \pi(k(t))} \right)^2 \leq 2 \frac{1}{n} \frac{\phi_t \Gamma_t \varphi_t}{2(\eta t + 1)(1 - \phi_t)} \left( x_k^* - z_k^{(t, \pi(k(t))} \right)^2 + \frac{2}{n} \frac{\phi_t \Gamma_t \varphi_t}{2(\eta t + 1)(1 - \phi_t)} \left( z_k^{(t, \pi(k(t)))} - z_k^{(t, \pi(k(t')))} \right)^2,
\]
we get
\[
\frac{1}{n} \sum_k (1 - \phi_t) \left[ \frac{n \phi_t \Gamma_t \varphi_t}{2(1 - \phi_t)} (x_k^* - z_k^{(t, \pi(k(t))} \right)^2 + \sum_{k' \neq k} \frac{n \phi_t \eta_t \Gamma_t \varphi_t}{2(\eta t + 1)(1 - \phi_t)} (x_k^* - z_k^{(t, \pi(k(t))} \right)^2 \leq \frac{1}{n} \sum_k (1 - \phi_t) \left[ \frac{n \phi_t \Gamma_t \varphi_t}{2(\eta t + 1)(1 - \phi_t)} \| x_k^* - z_k^{(t, \pi(k(t))} \| + \frac{2}{n} \frac{\phi_t \Gamma_t \varphi_t}{2(\eta t + 1)} \sum_{k, k'} (z_k^{(t, \pi(k(t)))} - z_k^{(t, \pi(k(t')))} \right]^2.
\]

By assumption $\frac{n \Gamma_t (1 - \varphi_t)}{2} \leq \mu^2$ and also $\frac{n \eta_t \Gamma_t (1 - \varphi_t)}{2(\eta t + 1)} \leq \frac{n \Gamma_t (1 - \varphi_t)}{2}$, so term $L$ is non-positive and hence can be dropped.

Term $D$ is bounded as follows.
\[
\frac{1}{n} \sum_k \frac{1}{n} \sum_{k'} \langle \nabla_k f(y^{(t, \pi(k(t))}) - \nabla_k f(y^{(t, \pi(k(t'))}), \Delta x_k^{(t, \pi(k(t))} \rangle = \frac{1}{n^2} \sum_{k, k'} \phi_t \langle \nabla_k f(y^{(t, \pi(k(t))}) - \nabla_k f(y^{(t, \pi(k(t'))}), \Delta z_k^{(t, \pi(k(t))} \rangle \leq \frac{1}{2n} \sum_{k, k'} \phi_t \left[ \frac{10}{\Gamma_t} \left( \nabla_k f(y^{(t, \pi(k(t))}) - \nabla_k f(y^{(t, \pi(k(t'))}) \right)^2 \right].
\]

Term $G$ is bounded as follows.
\[
\frac{1}{n} \sum_{k, k'} \langle \nabla_{k'} f(y^{(t, \pi(k(t))}, w_{k'}^{(t, \pi(k(t))} - w_{k'}^{(t, \pi(k(t'))}) \rangle = \frac{1}{n} \sum_{k, k'} \phi_t \langle \nabla_{k'} f(y^{(t, \pi(k(t))}) - z_{k'}^{(t, \pi(k(t))}, w_{k'}^{(t, \pi(k(t))} - w_{k'}^{(t, \pi(k(t'))}) \rangle + \frac{1}{n} \phi_t \langle -\Gamma_t \Delta z_{k'}^{(t, \pi(k(t))}, w_{k'}^{(t, \pi(k(t))} - w_{k'}^{(t, \pi(k(t'))}) \rangle \leq \frac{1}{2n} \phi_t \left[ \frac{10}{\Gamma_t} \left( \nabla_{k'} f(y^{(t, \pi(k(t))}) - z_{k'}^{(t, \pi(k(t))} \right)^2 + \frac{\Gamma_t}{10} \left( w_{k'}^{(t, \pi(k(t))} - w_{k'}^{(t, \pi(k(t'))} \right)^2 \right] + \frac{1}{2n} \phi_t \left[ \frac{\Gamma_t}{10} \left( \Delta z_{k'}^{(t, \pi(k(t))} \right)^2 + 10 \Gamma_t \left( w_{k'}^{(t, \pi(k(t))} - w_{k'}^{(t, \pi(k(t'))} \right)^2 \right].
\]
The coefficient of term $M$ is bounded as follows.
Recall that by assumption its coefficient $(1 - \phi_t) \left( \frac{n\phi_t \Gamma_t \phi_t (\eta_t + \frac{2}{\eta_t - 1})}{2(\eta_t + 1)(1-\phi_t)} \right) \leq (1 - \tau \phi_t) \left( \frac{n\phi_{t-1} \eta_{t-1} \Gamma_{t-1}}{2(\eta_{t-1} + 1)} \right) = (1 - \tau \phi_t) \zeta_k$, by the definition of $\zeta_t$.

Then, with the underbraces indicating matching terms, we obtain

\[
\frac{1}{n} \sum_k \left[ f(x^{(t+1,\pi(k,t))}) - f(x^*) + \zeta_{t+1} \|x^* - z^{(t+1,\pi(k,t))}\|^2 \right]
\leq \frac{1}{n} (1 - \tau \phi_t) \sum_k \left[ f(x^{(t,\pi(k,t))}) - f(x^*) + \zeta_{t} \|x^* - z^{(t,\pi(k,t))}\|^2 \right]
- \sum_k \frac{\phi_t (\frac{4\Gamma_t - n\phi_t}{2})}{2} \left( \Delta z_{t,\pi(k,t)} \right)^2\]

\[
+ \frac{1}{n} \phi_t \sum_{k,k'} \left[ \eta_t + 2 \left( \nabla_{k'} f(y^{(t,\pi(k,t))}) - B_{k'} f(y^{(t,\pi(k',t))}) \right)^2 \right] + \frac{10}{2\Gamma_t} \left( \nabla_k f(y^{(t,\pi(k,t))}) - \nabla_k f(y^{(t,\pi(k',t))}) \right)^2 \]

\[
+ \frac{6}{2n} \Gamma_t \phi_t \sum_{k,k'} \left[ \|w_{k'}^{(t,\pi(k,t))} - w_{k'}^{(t,\pi(k',t))}\|^2 \right] \]

\[
+ \frac{2}{n} \phi_t \Gamma_t \phi_t \sum_{k,k'} \left[ \|z_{k}^{(t,\pi(k,t))} - z_{k}^{(t,\pi(k',t))}\|^2 \right].\]

\[\square\]

**Proof of Lemma 12.** We will apply Lemma 13.

**Strongly convex case:**

In this case, we set $\phi_t = \phi$, $\varphi_t = (1 - \phi)$, $\psi_t = \frac{1}{1 + \phi}$, $\eta_t + 1 = \frac{3}{n\phi(1-\tau)}$, and let $\Gamma_t$ be a constant series. Let’s check the constraints of Lemma 13 one by one.

**Constraint 1.** $\eta_t > 1$.
This constraint holds since $\phi \leq \frac{1}{n}$ by assumption.

**Constraint 2.** $\frac{n\phi_t \psi_t \varphi_t}{1 - \psi_t} = n(1 - \phi_t)$.

Substituting yields

\[
\frac{\phi_t (\frac{4\Gamma_t - n\phi_t}{2})}{2(\eta_t + 1)(1-\phi_t)} = 1 - \phi.
\]

**Constraint 3.** $(1 - \phi_t) \left( \frac{n\phi_t \Gamma_t \phi_t (\eta_t + \frac{2}{\eta_t - 1})}{2(\eta_t + 1)(1-\phi_t)} \right) \leq (1 - \tau \phi_t) \left( \frac{n\phi_{t-1} \eta_{t-1} \Gamma_{t-1}}{2(\eta_{t-1} + 1)} \right).$
Substituting yields
\[(1 - \phi) \left(1 - \frac{n - 2}{3} \phi(1 - \tau)\right) \leq (1 - \tau \phi) \left(1 - \frac{n}{3} \phi(1 - \tau)\right).\]
This is true since \(\phi \leq \frac{1}{n}\).

**Constraint 4.** \(\frac{n \Gamma t (1 - \varphi t)}{2} \leq \mu\).

Substituting gives
\[n \phi \Gamma \leq \mu.\]
which is constraint (ii) of Lemma C

**Non-strongly convex case:**
In this case, we set \(\phi t = \frac{2}{t_0 + t}\) for \(t_0 \geq 2n\), \(\varphi t = 1\), \(\psi t = \frac{t_0 + t - 2}{t_0 + t}\), \(\eta t + 1 = \frac{3}{n \phi t (1 - \tau - \frac{1}{t_0})}\), and let \(\Gamma t\)
be a series such that \((1 - \tau \phi t) \frac{\Gamma t}{n \phi t} \leq (1 - \tau \phi t) \frac{\Gamma t - 1}{n \phi t - 1}\).

**Constraint 1.** \(\eta t > 1\).
This is easy to verify since \(\eta t + 1 = \frac{3}{n \phi t (1 - \tau - \frac{1}{t_0})}\).

**Constraint 2.** \(\frac{n \phi t \psi t \varphi t}{1 - \psi t} = n (1 - \phi t)\).
A simple calculation shows that this holds.

**Constraint 3.** \((1 - \phi t) \left(\frac{n \phi t \Gamma t \varphi t (\eta t + \frac{2}{n})}{2 (\eta t + 1) (1 - \phi t)}\right) \leq (1 - \tau \phi t) \left(\frac{n \phi t - 1 \eta t - 1 \Gamma t - 1}{2 (\eta t - 1 + 1)}\right)\).

The inequality is equivalent to
\[\frac{\phi t^2}{\phi t^2 - 1} \frac{\Gamma t}{n \phi t} \left(1 - \frac{n - 2}{n (\eta t + 1)}\right) \leq (1 - \tau \phi t) \left(1 - \frac{1}{\eta t - 1 + 1}\right) \frac{\Gamma t - 1}{n \phi t - 1}.\] (14)

Since \(\phi t = \frac{2}{t_0 + t}\), \(\frac{\phi t^2}{\phi t^2 - 1} = \left(1 - \phi t + \frac{\phi t^2}{4}\right)\) With some further calculation, given in the footnote 11 we obtain
\[\left(1 - \frac{n - 2}{n (\eta t + 1)}\right) (1 - \phi t + \frac{\phi t^2}{4}) \leq (1 - \tau \phi t) \left(1 - \frac{1}{\eta t - 1 + 1}\right).\] (15)

11 In order to prove this, we only need
\[\frac{1}{\eta t - 1 + 1} - \frac{n - 2}{n} \cdot \frac{1}{\eta t + 1} + \frac{n - 2}{n} \phi t (1 - \phi t) + \frac{n - 2}{n} \phi t^2 \leq 1 - \tau \phi t - \frac{1}{\eta t + 1} + \frac{\phi t^2}{n \phi t + 1},\] and this is equivalent to
\[\frac{1}{\eta t - 1 + 1} - \frac{n - 2}{n} \cdot \frac{1}{\eta t + 1} + \frac{n - 2}{n} \phi t (1 - \phi t) + \frac{n - 2}{n} \phi t^2 \leq (1 - \tau - \frac{\phi t^2}{3}) \phi t.\] Using the equality \(\eta t + 1 = \frac{3}{n \phi t (1 - \tau - \frac{1}{t_0})}\),
if suffices to have
\[\frac{n \phi t - 1 (1 - \tau - \frac{1}{t_0})}{3} \leq \frac{n - 2}{n} \phi t (1 - \tau - \frac{1}{t_0}) + \frac{n - 2}{n} \phi t^2 (1 - \tau - \frac{1}{t_0}) - \tau \phi t \frac{n \phi t - 1 (1 - \tau - \frac{1}{t_0})}{3} \leq (1 - \tau - \frac{1}{t_0}) \phi t.\]

Or equivalently,
\[n \phi t - 1 - (n - 2) \phi t + (n - 2) \phi t^2 - n \tau \phi t \phi t - 1 \leq 3 \phi t.\]
Rearranging terms yields
\[n \phi t - 1 \leq (n + 1) \phi t + n \tau \phi t \phi t - 1 - (n - 2) \phi t^2.\]
By constraint (v) of Lemma 12, \( (1 - \tilde{\tau}_t) \frac{\Gamma_t}{n\phi_t} \leq (1 - \tau_t) \frac{\Gamma_{t-1}}{n\phi_{t-1}} \); multiplying (15) by \( \frac{\Gamma_t}{n\phi_t} \), implies (14).

Constraint 4. \[ \frac{n\Gamma_t(1 - \varphi_t)}{2} \leq \frac{\mu}{2}. \]

This holds as \( \varphi_t = 1 \) in this case. \( \square \)

---

Since \( \frac{\phi_t}{\phi_{t-1}} \geq 1 - \frac{1}{2n} \) if \( t_0 \geq 2n \),

\[ n\phi_{t-1} \leq \left( n + \frac{n}{2n - 1} \right) \phi_t. \]

So the last thing to do is to prove \( \left( n + \frac{n}{2n - 1} \right) \phi_t \leq (n + 1)\phi_t + n\tilde{\tau}\phi_{t-1} - (n - 2)\phi_t^2 \), and this is equivalent to

\[ n + \frac{n}{2n - 1} \leq n + 1 - (n - 2)\phi_t + n\tilde{\tau}\phi_{t-1}. \]

This is equivalent to

\[ (n - 2)\phi_t - n\tilde{\tau}\phi_{t-1} \leq \frac{n - 1}{2n - 1}. \]

This inequality is easy to verify as \( \frac{1}{2} \geq \phi_{t-1} \geq \phi_t \) and \( \tilde{\tau} \geq \frac{1}{2} \).
D Proof of Lemma 14, the Amortization Bounds

Lemma 14. Let \( I = [0, T-1] \). Suppose \( r = \max_t \left\{ \frac{36(3q)^2 T^2 n^2 \xi \phi_t^2}{r^2} \right\} < 1 \), \( \xi = \max_{k \in [t-3q,t+q]} \frac{\phi_{t+1}^2}{\phi_t^2} \), 
\( \frac{224q^2 T^2 n^2}{r} \leq 1 \), and \( B^{(t)} \) are good, then:

\[
\begin{align*}
(\Delta_t^F)^2 &\leq \frac{1}{T} \mathbb{E}_\pi \left[ (g_{\max,k_t}^t - g_{\min,k_t}^t)^2 \right]; \\
\mathbb{E}_\pi \left[ (g_{\max,k_t}^t - g_{\min,k_t}^t)^2 \right] &\leq \frac{54q L^2}{n} \sum_{s \in I \cap [t-2q,t+2q] \setminus \{t\}} (\Delta_s^F)^2 + E_s^\Delta; \\
\mathbb{E}_\pi \left[ \sum_{k'} (w_{k'}^{(t,\pi)} - w_{k'}^{(t,\pi(k',t))})^2 \right] &\leq 16q \sum_{s \in I \cap [t-q-1,t-1]} (\Delta_s^F)^2; \\
\mathbb{E}_\pi \left[ \sum_{k'} (z_{k'}^{(t,\pi)} - z_{k'}^{(t,\pi(k',t))})^2 \right] &\leq 16q \sum_{s \in I \cap [t-q-1,t-1]} (\Delta_s^F)^2; \\
\mathbb{E}_\pi \left[ \sum_{k'} \left( \nabla_{k'} f(y_t^{(t,\pi)}) - \nabla_{k'} f(y_t^{(t,\pi(k',t))}) \right)^2 \right] &\leq \frac{9}{2} q L^2 r n \sum_{s \in I \cap [t-q-1,t-1]} (\Delta_s^F)^2 \\
&\quad + \frac{6r}{1-r} \sum_{s \in I \cap [t-3q,t+q] \setminus \{t\}} (\Delta_s^F)^2 + E_s^\Delta \\
&\quad + \frac{r}{108(1-r)} \sum_{s \in I \cap [t-q-1,t-1]} (\Delta_s^F)^2 + E_s^\Delta.
\end{align*}
\]

We introduce the following notation.

\( \Delta_{u, R}^{\pi, k} \) will denote the maximum value that \( \Delta_{z_{\pi, k}}^{\pi, t} \) can attain when the first \( u - q - 1 \) updates on path \( \pi \) have been fixed, assuming the update happens at coordinate \( k_t \), and it does not read any of the updates at times in \( R \), nor any of the variables updated at time \( v > u + q \). Here, \( R \) is either \( \emptyset \) or \{t\}. Let \( \Delta_{\min}^{u, R} \) denote the analogous minimum value.

Let \( \sum_{k_t}^{u, \pi} \) denote the maximum (and minimum) gradient with the same constraints as \( \sum_{k_t}^{\pi} \) (and \( \sum_{k_t}^{\pi} \)).

\[
|e^{u, \pi} - e^{u, R}| \leq 16q \mathbb{E}_\pi \left[ (g_{\max,k_t}^t - g_{\min,k_t}^t)^2 \right] + E_s^\Delta.
\]
Note that $\Delta z_{k_t}^{t,\pi} = \arg\min_h \left\{ \frac{f_t}{2} ||h||^2 + \langle \bar{g}_{k_t}^{t,\pi}, h \rangle \right\}$ (see Step 7 of the asynchronous version of Algorithm 2). So,

$$
(\Delta_{\text{max}} z_{k_t}^{t,\pi} - \Delta_{\text{min}} z_{k_t}^{t,\pi})^2 \leq \frac{1}{\Gamma_t^2} (g_{\text{max},k_t}^{t,\pi} - g_{\text{min},k_t}^{t,\pi})^2.
$$

Let $(\Delta_{\text{FE}}^t)^2$ denote the resulting expectation at time $t$:

$$(\Delta_{\text{FE}}^t)^2 \triangleq \mathbb{E}_\pi \left[ (\Delta_{\text{max}} z_{k_t}^{t,\pi} - \Delta_{\text{min}} z_{k_t}^{t,\pi})^2 \right].$$

Also, let $(E_{\Delta}^t) \triangleq \mathbb{E} \left[ (\Delta z_{k_t}^{t,\pi})^2 \right]$.

D.1 Proof of Lemma 14, Equation (17)

In order to bound the difference of the gradient on the RHS of (22), we first bound the possible difference on $\tilde{y}^{(t)}$, on which the algorithm calculates the gradient. Suppose that $t - q \leq t_1 \leq t$ (and $t - q \leq t_2 \leq t$), then we define $[\tilde{y}^{(t)}]_{\pi,R,t_1,k_t}$ and $[\tilde{y}^{(t)}]_{\pi,R,t_2,k_t}$ to be some $\tilde{y}^{(t)}$ when the first $t_1 - q - 1$ (resp. $t_2 - q - 1$) updates on path $\pi$ have been fixed, assuming the update happens at coordinate $k_t$, and it does not read any of the updates at times in $R$, nor any of the variables updated at time $v > t_1 + q$ (resp. $v > t_2 + q$).

**Lemma 15.** If the $B^{(t)}$ are good then

$$
\left| \left[ [\tilde{y}^{(t)}]_{\pi,R,t_1,k_t} - [\tilde{y}^{(t)}]_{\pi,R,t_2,k_t} \right] \right|_k \leq 3n\phi_t \sum_{k_s = k_t} \max_{k_s \in \{[t_2 - q + 1] \backslash (R \cup \{t\})\}} \left\{ \max_{t \in [\min\{s - q,t_1 - q\},\min\{s,t\}] \cup \{s\}} \{\Delta_{\text{max}}^{t,R \cup \{t\},s,\pi} z_{k_s}\} - \min_{t \in [\min\{s - q,t_1 - q\},\min\{s,t\}] \cup \{s\}} \{\Delta_{\text{min}}^{t,R \cup \{t\},s,\pi} z_{k_s}\} \right\},
$$

$$
\left| \left[ [\tilde{y}^{(t)}]_{\pi,R,t_1,k_t} - [\tilde{y}^{(t)}]_{\pi,R,t_2,k_t} \right] \right|_k \leq 3n\phi_t \sum_{k_s = k_t} \max_{t \in [\min\{s - q,t_1 - q\},\min\{s,t\}] \cup \{s\}} \left\{ \max_{t \in [\min\{s - q,t_1 - q\},\min\{s,t\}] \cup \{s\}} \{\Delta_{\text{max}}^{t,R \cup \{t\},s,\pi} z_{k_s}\} \right\},
$$

$$
\left| \left[ [\tilde{y}^{(t)}]_{\pi,R,t_1,k_t} - [\tilde{y}^{(t)}]_{\pi,R,t_2,k_t} \right] \right|_k \leq 3n\phi_t \sum_{k_s = k_t} \min_{t \in [\min\{s - q,t_1 - q\},\min\{s,t\}] \cup \{s\}} \left\{ \min_{t \in [\min\{s - q,t_1 - q\},\min\{s,t\}] \cup \{s\}} \{\Delta_{\text{min}}^{t,R \cup \{t\},s,\pi} z_{k_s}\} \right\}.
$$

Recall the definition of matrix $L$:

$$
(\nabla_{k_t} f(x) - \nabla_{k_t} f(x'))^2 \leq \left( \sum_k L_{k,k_t} |x_k - x'_k| \right)^2.
$$
Using Lemma 13 and the Cauchy-Schwarz inequality yields

\[
(g_{\max,k,t}^{\pi,t} - g_{\min,k,t}^{\pi,t})^2 
\leq 3q \cdot 9n^2 \phi_t^2 \sum_{s \in [t-2q,t+q] \setminus \{k\}} \frac{L_{k,s,k_t}^2}{n} \max_{\max \{s-q,t-q\}} \min_{\min \{s,t\} \cup \{s\}} \{\Delta_{\max,z_{s,k_t}}^{l(t)}\}^2 
- \min_{\max \{s-q,t-q\}} \min_{\min \{s,t\} \cup \{s\}} \{\Delta_{\min,z_{s,k_t}}^{l(t)}\}^2, 
\]

as there are at most \(3q\) terms on the RHS of (23).

Taking the average over \(\pi(k,t)\) yields

\[
E_k \left[ (g_{\max,k}^{\pi(k,t),t} - g_{\min,k}^{\pi(k,t),t})^2 \right] 
\leq 3q \cdot 9n^2 \phi_t^2 \sum_{s \in [t-2q,t+q] \setminus \{k\}} \frac{L_{\res}^2}{n} \max_{\max \{s-q,t-q\}} \min_{\min \{s,t\} \cup \{s\}} \{\Delta_{\max,z_{s,k_t}}^{l(t)}\}^2 
- \min_{\max \{s-q,t-q\}} \min_{\min \{s,t\} \cup \{s\}} \{\Delta_{\min,z_{s,k_t}}^{l(t)}\}^2, 
\]

This is legitimate because we exclude the update \(t\) on the right hand side, and also \(\pi\) will be equal to \(\pi(k,t)\) for times other than \(t\). Therefore,

\[
E_k \left[ (g_{\max,k}^{\pi(k,t),t} - g_{\min,k}^{\pi(k,t),t})^2 \right] \leq 3q \cdot 9n^2 \phi_t^2 \sum_{s \in [t-2q,t+q] \setminus \{k\}} \frac{L_{\res}^2}{n} \left[ 2(\Delta_{s,k_t}^{z_{s,k_t}})^2 + 2(\Delta_{\max,z_{s,k_t}} - \Delta_{\min,z_{s,k_t}})^2 \right] , 
\]

(24)

as \(\Delta_{s,k_t}^{z_{s,k_t}} \in [\Delta_{\min,z_{s,k_t}}, \Delta_{\max,z_{s,k_t}}]\).

By the definition of \((\Delta_{FE})^2\) and \(E_l^A\), the result follows.

D.2 Proof of Lemma 14, Equations (18) and (19)

Next, we show that \((w_{k'}^{t,\pi(t,\pi(k,t))} - w_{k'}^{t,\pi(k',t)})^2\) and \((z_{k'}^{t,\pi(t,\pi(k,t))} - z_{k'}^{t,\pi(k',t)})^2\) can be upper bounded by terms of the form \((\Delta_{\min,z_{s,k_t}} - \Delta_{\max,z_{s,k_t}})^2\).
Lemma 16. If the $B(t)$ are good then
\[
\sum_{k'} \left( w_{k'}^{t,(\pi)} - w_{k'}^{t,(\pi(k'),t)} \right)^2, \sum_{k'} \left( z_{k'}^{t,(\pi)} - z_{k'}^{t,(\pi(k'),t)} \right)^2 
\leq 8q \sum_{l \in [t-q,t-1]} \left[ \left( \Delta_{\min} z_{k_{l}}^{(l,\pi)} - \Delta_{\max} z_{k_{l}}^{(l,\pi)} \right)^2 + \left( \Delta_{\min} z_{k_{l}}^{(l,\pi(k'),t)} - \Delta_{\max} z_{k_{l}}^{(l,\pi(k'),t)} \right)^2 \right].
\]
(25)

D.3 Proof of Lemma 14, Equations (20) and (21)

Finally, we want to bound $\left( \nabla_{k'} f(y(t,\pi)) - z_{k'}^{t,(\pi(k'),t)} \right)^2$. We observe:

Observation 17.
\[
\sum_{k'} \left( \nabla_{k'} f(y(t,\pi)) - z_{k'}^{t,(\pi(k'),t)} \right)^2 
\leq 2 \sum_{k'} \left( \nabla_{k'} f(y(t,\pi)) - \nabla_{k'} f(y(t,\pi(k'),t)) \right)^2 + 2 \sum_{k'} \left( \nabla_{k'} f(y(t,\pi(k'),t)) - z_{k'}^{t,(\pi(k'),t)} \right)^2.
\]

The following lemma gives a bound on the first term.

Lemma 18. Suppose $r = \max_t \frac{36(3q)^2 L^2}{n^2 \phi_1^2}$, $\xi = \max_{t \in [t-3q,t+q]} \frac{\phi_1^2 r^2}{\phi_1^2 q}$, $\frac{36(3q)^2 L^2}{n} \leq 1$, and the $B(t)$ are good. Then
\[
\mathbb{E} \left[ \sum_{k'} \left( \nabla_{k'} f(y(t,\pi)) - \nabla_{k'} f(y(t,\pi(k'),t)) \right)^2 \right] 
\leq \frac{9}{2} n^2 \phi_1^2 q \sum_{l \in [t-q,t-1]} \mathbb{E} \left[ \left( \Delta_{\min} z_{k_{l}}^{(l,\pi)} - \Delta_{\max} z_{k_{l}}^{(l,\pi)} \right)^2 \right]
+ \frac{9}{2} n^2 \phi_1^2 q \mathbb{E} \left[ \frac{4r}{3(1-r)} \left( \Delta_{\min} z_{k_{t}}^{t,\pi} - \Delta_{\min} z_{k_{t}}^{t,\pi} \right)^2 + \frac{8r}{9(1-r)} \left( \Delta_{\max} z_{k_{t}}^{s,\pi} - \Delta_{\min} z_{k_{t}}^{s,\pi} \right)^2 
+ \frac{r}{486q(1-r)} \left( \Delta_{\max} z_{k_{s}}^{s,\pi} - \Delta_{\min} z_{k_{s}}^{s,\pi} \right)^2 
+ \frac{r}{486q(1-r)} \left( \Delta z_{k_{s}}^{s,\pi} \right)^2 \right].
\]

For the second term, note that $\nabla_{k'} f(y(t,\pi(k'),t))$ may not be in $[g_{\min,k'}^{t,\pi(k'),t}, g_{\max,k'}^{t,\pi(k'),t}]$ (because $y^{t,\pi(k'),t}$ are the actual values of the coordinates immediately prior to the time $t$ update, and some of the earlier updates that produced $y^{t,\pi(k'),t}$ may read the updated value at time $t$, while the terms $g_{\min,k'}^{t,\pi(k'),t}$ and $g_{\max,k'}^{t,\pi(k'),t}$ depend only on the values of earlier updates that do not read the time $t$ update). To obtain a bound, we consider the gradient value $g_{k'}^{t,\pi(k'),t}$ that would occur if there were synchronously updates from time $t - q$ to $t$. We have
\[
\left( \nabla_{k'} f(y(t,\pi(k'),t)) - z_{k'}^{t,\pi(k'),t} \right)^2 
\leq 2 \left( g_{k'}^{t,\pi(k'),t} - z_{k'}^{t,\pi(k'),t} \right)^2 + 2 \left( \nabla_{k'} f(y(t,\pi(k'),t)) - g_{k'}^{t,\pi(k'),t} \right)^2.
\]
(26)
Note that $g_{k'}^{t,\pi(k',t)} \in [g_{\min,k'}, g_{\max,k'}]$. Therefore,

$$
\left( g_{k'}^{t,\pi(k',t)} - g_{k'}^{t,\pi(k',t)} \right)^2 \leq \left( g_{\min,k'}^{t,\pi(k',t)} - g_{\max,k'}^{t,\pi(k',t)} \right)^2.
$$

Similarly to the proof in Appendix D.1, we obtain the bound

$$
\left( \nabla_{k'} f(y^{t,\pi(k',t)}) - g_{k'}^{t,\pi(k',t)} \right)^2 \leq \frac{9}{4} \frac{n^2 \varphi^2_t}{3(1-r)} \left( \Delta_{\min T_{k,t}^{s,\pi} - \Delta_{\min T_{k,t}^{s,\pi}} \right)^2 + \frac{8r}{9(1-r)} \left( \Delta z_{k,t}^{t,\pi} \right)^2
$$

By Lemma 25 in Appendix H, in expectation, we can bound the second term in (26) by

$$
\frac{9}{4} \frac{n^2 \varphi^2_t}{3(1-r)} \left( \Delta_{\min T_{k,t}^{s,\pi} - \Delta_{\min T_{k,t}^{s,\pi}} \right)^2 + \frac{8r}{9(1-r)} \left( \Delta z_{k,t}^{t,\pi} \right)^2 + \frac{r}{486q(1-r)} \left( \Delta z_{k,t}^{t,\pi} \right)^2.
$$

D.4 Proofs of the Subsidiary Lemmas

**Proof of Lemma 15.** WLOG, we assume $t - q \leq t_1 \leq t_2 \leq t$. Let $[\tilde{u}(t)]_{s,R,t,k_t}^{\pi,R,t_1,k_t}$ (resp. $[\tilde{v}(t)]_{s,R,t,k_t}^{\pi,R,t_2,k_t}$) denote the $\tilde{u}(t)$ (resp. $\tilde{v}(t)$) used to evaluate $[\tilde{y}(t)]_{s,R,t_1,k_t}^{\pi,R,t_1,k_t}$, and let $[\tilde{u}(t)]_{s,R,t,k_t}^{\pi,R,t_2,k_t}$ (resp. $[\tilde{v}(t)]_{s,R,t,k_t}^{\pi,R,t_2,k_t}$) denote the $\tilde{u}(t)$ (resp. $\tilde{v}(t)$) used to evaluate $[\tilde{y}(t)]_{s,R,t_2,k_t}^{\pi,R,t_2,k_t}$. Then,

$$
\begin{bmatrix}
[\tilde{y}(t)]_{s,R,t_1,k_t}^{\pi,R,t_1,k_t} - [\tilde{y}(t)]_{s,R,t_2,k_t}^{\pi,R,t_2,k_t}
\end{bmatrix}_k = \begin{bmatrix}
B(t)^T & \begin{bmatrix}
[\tilde{u}(t)]_{s,R,t_1,k_t}^{\pi,R,t_1,k_t} \\
[\tilde{v}(t)]_{s,R,t_1,k_t}^{\pi,R,t_1,k_t}
\end{bmatrix}_k - \begin{bmatrix}
[\tilde{u}(t)]_{s,R,t_2,k_t}^{\pi,R,t_2,k_t} \\
[\tilde{v}(t)]_{s,R,t_2,k_t}^{\pi,R,t_2,k_t}
\end{bmatrix}_k
\end{bmatrix}^{-1} \cdot \text{Update}^{s,t_1} - \begin{bmatrix}
B(t)^T & \begin{bmatrix}
[\tilde{u}(t)]_{s,R,t_1,k_t}^{\pi,R,t_1,k_t} \\
[\tilde{v}(t)]_{s,R,t_1,k_t}^{\pi,R,t_1,k_t}
\end{bmatrix}_k - \begin{bmatrix}
[\tilde{u}(t)]_{s,R,t_2,k_t}^{\pi,R,t_2,k_t} \\
[\tilde{v}(t)]_{s,R,t_2,k_t}^{\pi,R,t_2,k_t}
\end{bmatrix}_k
\end{bmatrix}^{-1} \cdot \text{Update}^{s,t_2},
$$

where Update$^{s,t_1}$ and Update$^{s,t_2}$ can be one of following, where $1_{k_s}$ denotes a vector which is 1 on coordinate $k_s$ and 0 on others:
\[
\begin{bmatrix}
[n\psi_{s+1}\phi_s + (1 - \psi_{s+1})] \Delta_{t_1,R_1\{t\}} z_{k_s}^{s,\pi} 1_{k_s} \\
\Delta_{t_1,R_1\{t\}} z_{k_s}^{s,\pi} 1_{k_s} \\
[n\psi_{s+1}\phi_s + (1 - \psi_{s+1})] \Delta_{t_2,R_2\{t\}} z_{k_s}^{s,\pi} 1_{k_s} \\
\Delta_{t_2,R_2\{t\}} z_{k_s}^{s,\pi} 1_{k_s} \\
[n\psi_{s+1}\phi_s + (1 - \psi_{s+1})] \Delta_{t_1,R_1\{t\}} z_{k_s}^{s,\pi} 1_{k_s} \\
\Delta_{t_1,R_1\{t\}} z_{k_s}^{s,\pi} 1_{k_s} \\
[n\psi_{s+1}\phi_s + (1 - \psi_{s+1})] \Delta_{t_2,R_2\{t\}} z_{k_s}^{s,\pi} 1_{k_s} \\
\Delta_{t_2,R_2\{t\}} z_{k_s}^{s,\pi} 1_{k_s}
\end{bmatrix}
\]

and Update\(^s,t_2\) can be one of following:

\[
\begin{bmatrix}
[n\psi_{s+1}\phi_s + (1 - \psi_{s+1})] \Delta_{t_2,R_2\{t\}} z_{k_s}^{s,\pi} 1_{k_s} \\
\Delta_{t_2,R_2\{t\}} z_{k_s}^{s,\pi} 1_{k_s} \\
[n\psi_{s+1}\phi_s + (1 - \psi_{s+1})] \Delta_{t_2,R_2\{t\}} z_{k_s}^{s,\pi} 1_{k_s} \\
\Delta_{t_2,R_2\{t\}} z_{k_s}^{s,\pi} 1_{k_s}
\end{bmatrix}
\]

Therefore,

\[
\begin{align*}
B(t) \left[ \begin{bmatrix} \tilde{u}(t) \end{bmatrix}^{\pi,R_1\{k_t\}^T} \right] - B(t) \left[ \begin{bmatrix} \tilde{v}(t) \end{bmatrix}^{\pi,R_1\{k_t\}^T} \right] &= \sum_{s \in [t-q,t+q] \setminus (R\cup\{t\})} B(t) \cdot B(t) \cdot B(t-1) \cdot \text{Update}\(^s,t_1\) - \sum_{s \in [t-2q,t+q] \setminus (R\cup\{t\})} B(t) \cdot B(t) \cdot B(t) \cdot B(t-1) \cdot \text{Update}\(^s,t_2\).
\end{align*}
\]

We know that \(B(t) \cdot B(t-1)\) is a 2 \times 2 matrix. Now let \(\delta_{t,s}^{t_1}\) (resp. \(\delta_{t,s}^{t_2}\)) denote the first entry of the vector

\[
\begin{bmatrix}
[n\psi_{s+1}\phi_s + (1 - \psi_{s+1})] \\
1
\end{bmatrix}
\]

or \(B(t) \cdot B(t-1) \cdot \begin{bmatrix} 0 \\
1
\end{bmatrix}\) or \(B(t) \cdot B(t-1) \cdot \begin{bmatrix} 0 \\
0
\end{bmatrix}\),

\[
\begin{bmatrix}
[n\psi_{s+1}\phi_s + (1 - \psi_{s+1})] \\
1
\end{bmatrix}
\]

corresponding to the choice of Update\(^s,t_1\) (resp. Update\(^s,t_2\)). Since \(t_1, t_2 \in [t-q, t], s \in [t-2q, t+2q]\) and \(B(t)\) are good, \(|\delta_{t,s}^{t_1}|, |\delta_{t,s}^{t_2}| \leq \frac{\phi_t}{2}\).

Since \(1_{k_s}\) is 1 on coordinate \(k_s\) and 0 on all other coordinates,

\[
\begin{bmatrix}
\tilde{y}(t) \end{bmatrix}^{\pi,R_1\{k_t\}} - \begin{bmatrix}
\tilde{y}(t) \end{bmatrix}^{\pi,R_1\{k_t\}} = \sum_{s \in [t-2q,t+q] \setminus (R\cup\{k\}) \text{ and } k_s=k} \left( \delta_{t,s}^{t_1} \Delta_{t_1,R_1\{t\}} z_{k_s}^{s,\pi} - \delta_{t,s}^{t_2} \Delta_{t_2,R_2\{t\}} z_{k_s}^{s,\pi} \right).
\]

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Then, by Lemma 19

\[
\begin{align*}
&\left| \left[ y(t) \right]_{\pi,R,t_1,k_1} - \left[ y(t) \right]_{\pi,R,t_2,k_2} \right|_k \\
\leq & \sum_{s \in [t-2q,t_1+q]\setminus (R_\cup(t)) \text{ and } k_s = k} 3n_\phi t \max \left\{ \left| \Delta_{t_1,R_\cup(t)} z_{s,\pi}^{s,\pi} - \Delta_{t_2,R_\cup(t)} z_{s,\pi}^{s,\pi} \right|, \right. \\
&\left. \left| \Delta_{t_1,R_\cup(t)} z_{s,\pi}^{s,\pi} - \Delta_{t_2,R_\cup(t)} z_{s,\pi}^{s,\pi} \right| \right. \\
&\left. + \sum_{s \in (t_1+q,t_2+q)\setminus (R_\cup(t)) \text{ and } k_s = k} 3n_\phi t \left| \Delta_{t_1,R_\cup(t)} z_{s,\pi}^{s,\pi} \right| \right.
\end{align*}
\]

Next, we make the following assertions:

- If \( s \in [t - 2q, t_1 + q] \) and \( t_1 \in [t - q, t] \), then

\[
\Delta_{t_1,R_\cup(t)} z_{s,\pi}^{s,\pi} \leq \max_{l \in [\max\{s - q, t - q\}, \min\{s, t\}] \cup \{s\}} \left\{ \Delta_{l,R_\cup(t)} z_{s,\pi}^{s,\pi} \right\}.
\]

- If \( s \in [t - 2q, t_1 + q] \) and \( t_1 \in [t - q, t] \), then

\[
\Delta_{t_1,R_\cup(t)} z_{s,\pi}^{s,\pi} \geq \min_{l \in [\max\{s - q, t - q\}, \min\{s, t\}] \cup \{s\}} \left\{ \Delta_{l,R_\cup(t)} z_{s,\pi}^{s,\pi} \right\}.
\]

- If \( s \in [t - 2q, t_2 + q] \) and \( t_2 \in [t - q, t] \), then

\[
\Delta_{t_2,R_\cup(t)} z_{s,\pi}^{s,\pi} \leq \max_{l \in [\max\{s - q, t - q\}, \min\{s, t\}] \cup \{s\}} \left\{ \Delta_{l,R_\cup(t)} z_{s,\pi}^{s,\pi} \right\}.
\]

- If \( s \in [t - 2q, t_2 + q] \) and \( t_2 \in [t - q, t] \), then

\[
\Delta_{t_2,R_\cup(t)} z_{s,\pi}^{s,\pi} \geq \min_{l \in [\max\{s - q, t - q\}, \min\{s, t\}] \cup \{s\}} \left\{ \Delta_{l,R_\cup(t)} z_{s,\pi}^{s,\pi} \right\}.
\]

We justify the first assertion. The arguments for the others are very similar.

We consider two cases.

**Case 1.** \( s \in [t_1, t_1 + q] \).

Then, \( t_1 \in [\max\{s - q, t - q\}, \min\{s, t\}] \). So the assertion is true.

**Case 2.** \( s \in [t - 2q, t_1 - 1] \).

We use the fact that \( \Delta_{max} z_{s,\pi}^{s,\pi} \leq \Delta_{max} z_{s,\pi}^{s,\pi} \) if \( l > s \) from [6] and [7].

\[
\Delta_{t_1,R_\cup(t)} z_{s,\pi}^{s,\pi} \leq \Delta_{max} z_{s,\pi}^{s,\pi} \quad (as \ t_1 > s)
\]

\[
\leq \max_{l \in [\max\{s - q, t - q\}, \min\{s, t\}] \cup \{s\}} \left\{ \Delta_{max} z_{s,\pi}^{s,\pi} \right\}.
\]
Now we can conclude that
\[
\left| \left[ \left[ \hat{y}(t)^\pi \right]_{\tau,1,k_1}^{R,\pi} - \left[ \hat{y}(t)^\pi \right]_{\tau,1,k_1}^{R,\pi} \right] \right|_k 
\leq 3n\phi_t \sum_{k_s = k} \max \left\{ \max_{t \in \{s, s - t - q\} \cup \{s\}} \left\{ \Delta^l_{\min, k_s} \right\}, \min_{t \in \{s, s - t - q\} \cup \{s\}} \left\{ \Delta^l_{\min, k_s} \right\} \right\} 
- \max_{t \in \{s, s - t - q\} \cup \{s\}} \left\{ \Delta^l_{\min, k_s} \right\}.
\]

**Proof of Lemma 16.** The proof of the bounds on \(\sum_{k'} \left( w_{k'}^{(t,\pi)} - w_{k'}^{(t,\pi(k',t))} \right)^2\) and \(\sum_{k'} \left( z_{k'}^{(t,\pi)} - z_{k'}^{(t,\pi(k',t))} \right)^2\) are similar. Here, we only give the proof for \(\sum_{k'} \left( w_{k'}^{(t,\pi)} - w_{k'}^{(t,\pi(k',t))} \right)^2\). Remember that \(w_{k'}^{(t,\pi)} = (\varphi_t, 1 - \varphi_t)(z_{k'}^{(t,\pi)}, y_{k'}^{(t,\pi)})^T\)
\[
= (\varphi_t, 1 - \varphi_t)B(t) \left[ (u_{k'}^{(t,\pi)}, v_{k'}^{(t,\pi)})^T + \sum_{l \in \{t, t-1\}} B(l+1)^{-1}D(l,\pi)\Delta z_{k'}^{(l,\pi)} \right].
\]
Remember that \(\Delta z_{k'}^{(l,\pi)} = 0\) for \(k' \neq k_l\).

Applying the Cauchy-Schwarz inequality gives:
\[
\left( w_{k'}^{(t,\pi)} - w_{k'}^{(t,\pi(k',t))} \right)^2 \leq \sum_{l \in \{t, t-1\}} (\varphi_t, 1 - \varphi_t)B(t) B(l+1)^{-1}D(l,\pi) \left( \Delta z_{k'}^{(l,\pi)} - \Delta z_{k'}^{(l,\pi(k',t))} \right)^2.
\]

Since \(B(t)\) are good, we know that \(\left| (\varphi_t, 1 - \varphi_t)B(t) B(l+1)^{-1} \left[ q\psi_{l+1}\phi_t + (1 - \psi_{l+1}) \right] \right| \leq 2\). So,
\[
\left( w_{k'}^{(t,\pi)} - w_{k'}^{(t,\pi(k',t))} \right)^2 \leq 4q \sum_{l \in \{t, t-1\}} \left( \Delta z_{k'}^{(l,\pi)} - \Delta z_{k'}^{(l,\pi(k',t))} \right)^2.
\]

We know that \(\Delta z_{k_l}^{(l,\pi)} = \left[ \Delta_{\min, k_l}^{(l,\pi)}, \Delta_{\max, k_l}^{(l,\pi)} \right]\), \(\Delta z_{k_l}^{(l,\pi(k',t))} = \left[ \Delta_{\min, k_l}^{(l,\pi(k',t))}, \Delta_{\max, k_l}^{(l,\pi(k',t))} \right]\); also the intervals \(\left[ \Delta_{\min, k_l}^{(l,\pi)}, \Delta_{\max, k_l}^{(l,\pi)} \right]\) and \(\left[ \Delta_{\min, k_l}^{(l,\pi(k',t))}, \Delta_{\max, k_l}^{(l,\pi(k',t))} \right]\) overlap (as \(\left[ \Delta_{\min, k_l}^{(l,\pi(k',t))}, \Delta_{\max, k_l}^{(l,\pi(k',t))} \right]\) is \(\left[ \Delta_{\min, k_l}^{(l,\pi)}, \Delta_{\max, k_l}^{(l,\pi)} \right]\)). Therefore
\[
\left( w_{k'}^{(t,\pi)} - w_{k'}^{(t,\pi(k',t))} \right)^2 \leq 8q \sum_{l \in \{t, t-1\}} \left[ \left( \Delta_{\min, k_l}^{(l,\pi)} - \Delta_{\max, k_l}^{(l,\pi)} \right)^2 + \left( \Delta_{\min, k_l}^{(l,\pi(k',t))} - \Delta_{\max, k_l}^{(l,\pi(k',t))} \right)^2 \right].
\]
Summing over $k'$ yields
\[
\sum_{k'} \left( w_{k'}^{(t,\pi)} - w_{k'}^{t,\pi(k',t)} \right)^2 \leq 8q \sum_{l \in [t-q,t-1]} \left[ \left( \Delta_{\min} z_{kl}^{(l,\pi)} - \Delta_{\max} z_{kl}^{(l,\pi)} \right)^2 + \left( \Delta_{\min} z_{kl}^{t,\pi(k',t)} - \Delta_{\max} z_{kl}^{t,\pi(k',t)} \right)^2 \right] .
\]

Proof of Lemma 18: Remember

\[
y_k^{(t,\pi)} = (1,0)(y_k^{(t,\pi)}, z_k^{t,\pi})^T = (1,0)B^{(t)} \left[ (u_k^{(t-q,\pi)}, v_k^{(t-q,\pi)})^T + \sum_{l \in [t-q,t-1] \text{ and } k_l = k} B^{(l+1)-1} D^{(l)} \Delta z_{kl}^{(l,\pi)} \right].
\]

Applying the Cauchy-Schwarz inequality gives:
\[
\left( y_k^{(t,\pi)} - y_k^{(t,\pi(k',t))} \right)^2 = \left( \sum_{l \in [t-q,t-1] \text{ and } k_l = k} (1,0)B^{(t)} B^{(l+1)-1} D^{(l)} \left( \Delta z_{kl}^{(l,\pi)} - \Delta z_{kl}^{(l,\pi(k',t))} \right) \right)^2 \leq q \sum_{l \in [t-q,t-1] \text{ and } k_l = k} \left( (1,0)B^{(t)} B^{(l+1)-1} D^{(l)} \left( \Delta z_{kl}^{(l,\pi)} - \Delta z_{kl}^{(l,\pi(k',t))} \right) \right)^2 ,
\]

where $D^{(l)} = \left[ n \psi_{l+1} \phi_l + (1 - \psi_{l+1}) \right]$. Since the $B^{(t)}$ are good, we know that the absolute value of the first element of $B^{(t)} B^{(l+1)-1} \left[ n \psi_{l+1} \phi_l + (1 - \psi_{l+1}) \right]$ is less than $\frac{2}{2} n \phi_l$. And so
\[
\left( y_k^{(t,\pi)} - y_k^{(t,\pi(k',t))} \right)^2 \leq \frac{9}{4} n^2 \phi_l^2 q \sum_{l \in [t-q,t-1] \text{ and } k_l = k} \left( \Delta z_{kl}^{(l,\pi)} - \Delta z_{kl}^{(l,\pi(k',t))} \right)^2 .
\]

We know that \( \Delta z_{kl}^{(l,\pi)} \in [\Delta_{\min} z_{kl}^{(l,\pi)}, \Delta_{\max} z_{kl}^{(l,\pi)}] \), also, that \( [\Delta_{\min} z_{kl}^{(l,\pi)}, \Delta_{\max} z_{kl}^{(l,\pi)}] \) overlap (as \([\Delta_{\min} z_{kl}^{(l,\pi)}, \Delta_{\max} z_{kl}^{(l,\pi)}] \subseteq [\Delta_{\min} z_{kl}^{(l,\pi)}, \Delta_{\max} z_{kl}^{(l,\pi)}]\)). Therefore,
\[
\left( y_k^{(t,\pi)} - y_k^{(t,\pi(k',t))} \right)^2 \leq \frac{9}{2} n^2 \phi_l^2 q \sum_{l \in [t-q,t-1] \text{ and } k_l = k} \left[ \left( \Delta_{\min} z_{kl}^{(l,\pi)} - \Delta_{\max} z_{kl}^{(l,\pi)} \right)^2 + \left( \Delta_{\min} z_{kl}^{t,\pi(k',t)} - \Delta_{\max} z_{kl}^{t,\pi(k',t)} \right)^2 \right] .
\]

Also,
\[
\left( \nabla k' f(y^{(t,\pi)}) - \nabla k' f(y^{(t,\pi(k',t))}) \right)^2 \leq \frac{9}{2} n^2 \phi_l^2 q \sum_{l \in [t-q,t-1]} L_{k_l,k_l'}^2 \left[ \left( \Delta_{\min} z_{kl}^{(l,\pi)} - \Delta_{\max} z_{kl}^{(l,\pi)} \right)^2 + \left( \Delta_{\min} z_{kl}^{t,\pi(k',t)} - \Delta_{\max} z_{kl}^{t,\pi(k',t)} \right)^2 \right] .
\]
Summing over $k'$ yields
\[
\sum_{k'} \left( \nabla_{k'} f(y(t,\pi)) - \nabla_{k'} f(y(t,\pi(k',t))) \right)^2
\leq \frac{9}{2} n^2 \phi_t^2 q \sum_{l \in [t-q,t-1]} \left[ L_{\text{res}}^2 \left( \Delta_{\min} z_{k_l}^{l,\pi} - \Delta_{\max} z_{k_l}^{l,\pi} \right)^2 \right]
+ \sum_{k'} L_{k_l,k'} \left[ \left( \Delta_{\min} z_{k_l}^{(l,\pi(k',t))} - \Delta_{\max} z_{k_l}^{(l,\pi(k',t))} \right)^2 \right].
\]

In Lemma 25 in Appendix H, we show that
\[
\mathbb{E} \left[ q \sum_{l \in [t-q,t-1]} L_{k_l,k_l}^2 \left( \Delta_{\max} z_{k_l}^{l,\pi} - \Delta_{\min} z_{k_l}^{l,\pi} \right)^2 \right]
\leq \frac{4r}{3(1-r)} \left( \Delta_{\min} z_{k_t}^{t,\pi} - \Delta_{\min} z_{k_t}^{t,\pi} \right)^2 + \frac{8r}{9(1-r)} \left( \Delta z_{k_t}^{t,\pi} \right)^2
+ \sum_{s \in [t-3q,t+q] \setminus \{t\}} \frac{r}{486q(1-r)} \left( \Delta z_{k_s}^{s,\pi} \right)^2.
\]

Therefore,
\[
\mathbb{E} \left[ \sum_{k'} \left( \nabla_{k'} f(y(t,\pi)) - \nabla_{k'} f(y(t,\pi(k',t))) \right)^2 \right]
\leq \frac{9}{2} n^2 \phi_t^2 q \sum_{l \in [t-q,t-1]} \left[ L_{\text{res}}^2 \left( \Delta_{\min} z_{k_l}^{l,\pi} - \Delta_{\max} z_{k_l}^{l,\pi} \right)^2 \right]
+ \frac{9}{2} n^2 \phi_t^2 \mathbb{E} \left[ \left( \Delta_{\min} z_{k_l}^{l,\pi} - \Delta_{\max} z_{k_l}^{l,\pi} \right)^2 \right]
+ \frac{4r}{3(1-r)} \left( \Delta_{\min} z_{k_t}^{t,\pi} - \Delta_{\min} z_{k_t}^{t,\pi} \right)^2 + \frac{8r}{9(1-r)} \left( \Delta z_{k_t}^{t,\pi} \right)^2
+ \sum_{s \in [t-3q,t+q] \setminus \{t\}} \frac{r}{486q(1-r)} \left( \Delta z_{k_s}^{s,\pi} \right)^2.
\]
\[\square\]
E  Some Technical Lemmas

Definition 3. We say $B^{(t)}$ are good if the absolute value of the first elements of these four matrices:
\[
B^{(t)} B^{(s+1)^{-1}} \left[ \begin{array}{c} n\psi_{s+1} \phi_s + (1 - \psi_{s+1}) \\ 1 \end{array} \right], \quad B^{(t)} B^{(s+1)^{-1}} \left[ \begin{array}{c} n\psi_{s+1} \phi_s + (1 - \psi_{s+1}) \\ 0 \end{array} \right],
\]
\[
B^{(t)} B^{(s+1)^{-1}} \left[ \begin{array}{c} 0 \\ 1 \end{array} \right], \quad and \quad B^{(t)} B^{(s+1)^{-1}} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right].
\]
are smaller than $\frac{2}{T} n\phi_t$, and the second element of
\[
B^{(t)} B^{(s+1)^{-1}} \left[ \begin{array}{c} n\psi_{s+1} \phi_s + (1 - \psi_{s+1}) \\ 1 \end{array} \right]
\]
is smaller than 2, for any $s$ and $t$ such that $|s - t| \leq 2q$.

Proof of Lemma 9 For any $t$ and any $s \in [t - 2q, t + 2q]$,
\[
a_t \left( \prod_{l=t+1}^{T-1} (1 - \tau \phi_l) \right) \leq \frac{1}{\Phi_a} a_s \left( \prod_{l=s+1}^{T-1} (1 - \tau \phi_l) \right).
\]
Therefore, by (16) and (17),
\[
\sum_t a_t \left( \prod_{l=t+1}^{T-1} (1 - \tau \phi_l) \right) (\Delta^F_s)^2 \leq \sum_t \sum_{s \in [t-2q, t+2q] \setminus \{t\}} \left[ a_t \left( \prod_{l=t+1}^{T-1} (1 - \tau \phi_l) \right) (\Delta^F_s)^2 + E_s^\Delta \right]
\]
\[
\leq \frac{54 n^2 \phi_t^2 q L^2 \max s}{n(\Gamma_k)^2} \sum_t a_t \left( \prod_{l=t+1}^{T-1} (1 - \tau \phi_l) \right) (\Delta^F_s)^2 + E_s^\Delta \]
\[
\leq \frac{3}{\Phi_a} \sum_t a_t \left( \prod_{l=t+1}^{T-1} (1 - \tau \phi_l) \right) (\Delta^F_s)^2 + E_s^\Delta \]

On rearranging, the result follows. \qed

Lemma 19. If $-\delta \leq \delta_1, \delta_2 \leq \delta$ then $|\delta_1 a - \delta_2 b| \leq \max\{2\delta|a - b|, 2\delta|a|, 2\delta|b|\}$.

Lemma 20. $\eta(a - b)^2 + a^2 \geq (1 - \frac{1}{\eta + 1})b^2$ for any $\eta > 0$, $a$ and $b$.

Lemma 21. Suppose $\prod_{t=0}^{T-1} (1 - \phi_t)$ has a convergence rate of $f(T)$ for any $T$, which means that $\frac{\prod_{t=0}^{T-1} (1 - \phi_t)}{f(T)} \leq 1$; then, for $\tau \geq 1$, the convergence rate of $\prod_{t=0}^{T-1} (1 - \tau \phi_t)$ is $f(T) \frac{\tau+1}{\eta + 1}$.

Lemma 22. $\{\phi_t\}_{t=1,2,\ldots}$ in Theorem 2 and 3 have the following properties:

(i) $\phi_t$ is a non-increasing series;

(ii) $\phi_t \leq \frac{1}{n+1}$;

(iii) $\frac{\phi_{t+1}}{\phi_t} \geq 1 - \frac{\phi_t}{2} \geq 1 - \frac{1}{2n}$.  

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Proof of Lemma 19: This is a straightforward calculation. □

Proof of Lemma 20: Expanding this inequality, we get \((\eta + 1)a^2 - 2\eta ab + (\eta - 1 + \frac{1}{\eta+1})b^2 \geq 0\). The LHS is equivalent to \((\eta + 1)(a - \frac{\eta}{\eta+1} b)^2 - \frac{\eta^2}{\eta+1} b^2 + (\eta - 1 + \frac{1}{\eta+1})b^2 = (\eta + 1)(a - \frac{\eta}{\eta+1} b)^2 = (\eta + 1)\).

□

Proof of Lemma 21: \[
\sum_{t=0}^{T-1} \ln(1 - \tau \phi_t) \leq \sum_{t=0}^{T-1} -\tau \phi_t \leq \sum_{t=0}^{T-1} -\frac{\tau}{1 + \frac{1}{n}} (\phi_t + \phi_t^2) \quad \text{(as } \phi_t \leq \frac{1}{n} \text{ by Lemma 22)}
\]
\[
\leq \sum_{t=0}^{T-1} \frac{\tau}{1 + \frac{1}{n}} \ln(1 - \phi_t) \leq \frac{\tau}{1 + \frac{1}{n}} \ln f(T).
\]

□

Proof of Lemma 22: It’s easy to check in the strongly convex case and in the non strongly convex case, it holds if \(t_0 \geq 2n\). □

Lemma 23. \(B^{(t)}\) is good in both the strongly convex and non-strongly convex cases if \(q \leq \min \left\{ \frac{9}{12}, \frac{2n-4}{10}, \frac{n}{20} \right\}\) and \(n \geq 10\).

Proof of Lemma 23: We first show that \(\left[ n \psi_{s+1} \phi_s + (1 - \psi_{s+1}) \right] \in \left[ n \phi_s, (n + 1) \phi_s \right] \).

• Strongly convex case: \(n \psi_{s+1} \phi_s + (1 - \psi_{s+1}) = (n + 1) \frac{\phi}{1 + \sigma} \in \left[ n \phi_s, (n + 1) \phi_s \right]\) as \(\phi \leq \frac{1}{n}\).

• Non-strongly convex case: \(n \psi_{s+1} \phi_s + (1 - \psi_{s+1}) = \frac{n \phi_{t+1} + 2}{t + \phi_{t+1}} \in \left[ n \phi_s, (n + 1) \phi_s \right]\) as \(t_0 \geq 2n\).

Next, we show that \(A^{(t)} \in \left[ 1 - e, e \right]\) for \(0 \leq e, f \leq \phi_t\).

• Strongly convex case: \(e = 1 - \frac{\phi^2}{1 + \sigma} \leq \phi = \phi_t\) and \(f = \phi = \phi_t\).

• Non-strongly convex case: \(e = \frac{2}{t + \phi_{t+1}} \leq \phi_t + 1 \leq \phi_t\) and \(f = 0 \leq \phi_t\).

Since \(\phi_t \leq \frac{1}{n}\) in both cases, we have the following observation:

Observation 24. If \(A^{(t)} \left( \begin{array}{c} p_1 \\ q_1 \end{array} \right) = \left( \begin{array}{c} p_2 \\ q_2 \end{array} \right)\) then \(p_1\) and \(q_1\) will be in the same order as \(p_2\) and \(q_2\) \((p_1 \leq q_1 \text{ if } p_2 \leq q_2; p_1 \geq q_1 \text{ if } p_2 \geq q_2)\); \(p_1\) and \(q_1\) will be in the interval \([p_2, q_2]\) if \(p_2 \leq q_2\), and in \([q_2, p_2]\) otherwise. Moreover,

\[|p_1 - q_1| \leq (1 - 2\phi_t)|p_2 - q_2|\]

This observation follows from the fact that \(A^{(t)} \in \left[ 1 - e, e \right]\) for \(0 \leq e, f \leq \phi_t\).

For simplicity, let \(c = n \psi_{s+1} \phi_s + (1 - \psi_{s+1}) \in \left[ n \phi_s, (n + 1) \phi_s \right]\).
• **Case 1:** \( t \leq s \). In this case,

\[
B^{(t)} \cdot B^{(s+1)-1} = \left( A^{(s)} \ldots A^{(t)} \right)^{-1}.
\]

- **Case 1a:** \( A^{(s)} \ldots A^{(t)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ 1 \end{pmatrix} \); we want to show \(|a| \leq (n + 1)\phi_t\) and \(|b| \leq 2\).

By Observation 24, we have \(a \leq c \leq b \leq 1\), and \(|1 - c| \geq \prod_{l=t}^{s} (1 - 2\phi_l) |b - a|\).

Since \(\phi_t\) is a decreasing series in both the strongly convex and non-strongly convex cases, \(|1 - c| \geq (1 - 2\phi_t)^{s-t+1} |b - a|\). As \(\frac{\phi_t}{a} \geq (1 - \frac{1}{2n})^{s-t} \geq 1 - \frac{s-t}{2n} \geq \frac{2(s-t+1)}{n}\) as 
\(s - t \leq 2q \leq \frac{2n - 4}{b}\), \(n\phi_s \geq 2(s - t + 1)\phi_t\), which implies \((1 - 2\phi_t)^{s-t+1} \geq 1 - n\phi_s\). Since \(c \geq n\phi_s\), \(|b - a| \leq 1\). We know that \(b \geq 1\), so \(0 \leq a \leq c \leq (n + 1)\phi_s \leq (n + 1)\phi_t\) and \(b \leq 1 + (n + 1)\phi_t \leq 2\).

- **Case 1b:** \( A^{(s)} \ldots A^{(t)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \); we want to show \(|a| \leq \frac{1}{2} n\phi_t\).

By Observation 24, \(a \leq 0 \leq b \leq 1\), and \(|1 - 0| \geq \prod_{l=t}^{s} (1 - 2\phi_l) |b - a| \geq \prod_{l=t}^{s} (1 - 2\phi_l) |b - a|\).

As \(\phi_t \leq \frac{1}{n}\),

\[
|b - a| \leq \left( \frac{1}{1 - 2\phi_t} \right)^{s-t+1} \leq \frac{1}{1 - 2(s - t + 1)\phi_t} \leq 1 + \frac{n\phi_t}{2}.
\]

The last inequality holds if \(s - t + 1 \leq 2q + 1 \leq \frac{2n}{5}\) and as \(\frac{1}{1 - ax} \leq 1 + \frac{ax}{1 - a}\) if \(x \leq 1\). Therefore, as \(b > 1\) and \(a < 0\), \(|a| \leq \frac{1}{2} n\phi_t\).

- **Case 1c:** \( A^{(s)} \ldots A^{(t)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} \); we want to show \(|a| \leq \frac{3}{2} n\phi_t\).

By Observation 24, \(b \leq 0 \leq c \leq a\), and \(|c - 0| \geq \prod_{l=t}^{s} (1 - 2\phi_l) |b - a|\).

Then, as \(c \leq (n + 1)\phi_t\),

\[
|b - a| \leq c \left( \frac{1}{1 - \phi_t} \right)^{s-t+1} \leq \frac{(n + 1)\phi_t}{1 - 2(s - t + 1)\phi_t} \leq \frac{3}{2} n\phi_t.
\]

The last inequality holds as \(1 - 2(s - t + 1)\phi_t \geq \frac{2n + 2}{3n}\) since \(s - t + 1 \leq 2q + 1 \leq \frac{n-2}{6}\) and \(\phi_t \leq \frac{1}{n}\). As \(b \leq 0 \leq a\), \(|a| \leq \frac{3}{2} n\phi_t\).

- **Case 2:** \( t > s \). When \(t = s + 1\), then \(B^{(t)} B^{(s+1)-1}\) is an identity matrix. It’s easy to check that Lemma 23 holds. So, here we assume \(t > s + 1\). Then,

\[
B^{(t)} \cdot B^{(s+1)-1} = A^{(t-1)} \ldots A^{(s+1)}.
\]

By Lemma 22(iii), for any \(l\) such that \(s + 1 \leq l \leq t\),

\[
1 - 2\phi_t \geq 1 - 2 \left( \frac{1}{1 - \frac{1}{2n}} \right)^{(t-l)} \phi_t.
\]
Therefore,
\[
\prod_{t=s+1}^{t-1} (1 - 2\phi_t) \geq \left(1 - 2 \left(\frac{2n}{2n - 1}\right)^{t-s} \phi_t \right)^{t-s} \\
\geq 1 - 2(t-s) \left(\frac{2n}{2n - 1}\right)^{t-s} \phi_t \\
\geq 1 - \frac{1}{4} n\phi_t.
\] (27)

The second inequality holds because \(2 \left(\frac{2n}{2n - 1}\right)^{t-s} \phi_t \leq 1\), and the last inequality holds because of the observation that \(2(t-s) \left(\frac{2n}{2n - 1}\right)^{t-s} \phi_t \leq \frac{1}{4} n\phi_t\), if \(t-s \leq 2q \leq \frac{n}{10}\) and \(n \geq 10\).

Also \(0 \leq c \leq (n + 1)\phi_s \leq \frac{5}{4} n\phi_t\) as \(t-s \leq 2q \leq \frac{n}{10}\) and \(n \geq 10\).

- **Case 2a:** \(A^{(t-1)} \cdots A^{(s+1)} \binom{c}{1} = \binom{a}{b} \); we want to show \(|a| \leq \frac{3}{4} n\phi_t\) and \(|b| \leq 2\).

  By Observation 24, \(c \leq a \leq b \leq 1\), and \(|b-a| \geq \prod_{t=s+1}^{t-1} (1 - 2\phi_t) \geq (1 - \frac{1}{4} n\phi_t)\), using (27). Then, \(0 \leq b \leq 1\), and \(0 \leq c \leq a \leq b - (1 - \frac{1}{4} n\phi_t - c) \geq \frac{3}{4} n\phi_t\), as \(c \leq \frac{5}{4} n\phi_t\).

- **Case 2b:** \(A^{(t-1)} \cdots A^{(s+1)} \binom{0}{1} = \binom{a}{b} \); we want to show \(|a| \leq \frac{1}{4} n\phi_t\).

  By Observation 24, \(0 \leq a \leq b \leq 1\), and \(|b-a| \geq (1 - \frac{1}{4} n\phi_t) \). Therefore, \(0 \leq a \leq \frac{1}{4} n\phi_t\).

- **Case 2c:** \(A^{(t-1)} \cdots A^{(s+1)} \binom{c}{0} = \binom{a}{b} \); we want to show \(|a| \leq \frac{5}{4} n\phi_t\).

  By Observation 24, \(0 \leq b \leq a \leq c\). So, \(0 \leq a \leq c \leq \frac{5}{4} n\phi_t\) as \(t-s \leq 2q \leq \frac{n}{10}\) and \(n \geq 10\).

\(\square\)

**F The difference between \(L_{\text{res}}\) and \(L_{\text{res}}^{\text{opt}}\)**

We review the discussion of this difference given in [7].

In general, \(L_{\text{res}}^{\text{opt}} \geq L_{\text{res}}\). \(L_{\text{res}} = L_{\text{res}}^{\text{opt}}\) when the rates of change of the gradient are constant, as for example in quadratic functions such as \(x^T Ax + bx + c\). All convex functions with Lipschitz bounds of which we are aware are of this type. We need \(L_{\text{res}}^{\text{opt}}\) because we do not make the Common Valuet assumption. We use \(L_{\text{res}}^{\text{opt}}\) to bound terms of the form \(\sum_j |\nabla_j f(y^j) - \nabla_j f(x^j)|^2\), where \(|y^j_k - x^j_k| \leq |\Delta_k|\), and for all \(h, i\), \(|y^h_k - y^i_k|, |x^h_k - x^i_k| \leq |\Delta_k|\), whereas in the Liu and Wright analysis, the term being bounded is \(\sum_j |\nabla_j f(y) - \nabla_j f(x)|^2\), where \(|y_k - x_k| \leq |\Delta_k|\); i.e., our bound is over a sum of gradient differences along the coordinate axes for pairs of points which are all nearby, whereas their sum is over gradient differences along the coordinate axes for the same pair of nearby points. Finally, if the convex function is \(s\)-sparse, meaning that each term \(\nabla_k f(x)\) depends on at most \(s\) variables, then \(L_{\text{res}}^{\text{opt}} \leq \sqrt{s} L_{\text{max}}\). When \(n\) is huge, this would appear to be the only feasible case.
G Managing the Counter in the Asynchronous Implementation

Here we discuss the effect of the counter on the computation.

If each update requires $\Omega(q)$ time, then the $O(q)$ time to update the counter will not matter.

If the updates are faster, each processor can update its counter every $r$ updates, for a suitable $r = O(q)$. Again, the cost of the counter updates will be modest. The effect on the analysis will be to increase $q$ to $qr$, reducing the possible parallelism by a factor of $r$.

If the required $r$ is too large, one can instead update the counter using a tree-based depth $O(\log q)$ computation. Then, even if the updates take just $O(1)$ time, choosing $r = O(\log n)$ will suffice.

H Amortization

In this section, we show the following lemma.

**Lemma 25.** Suppose $r = \max \left\{ \frac{36(3q)^2 t^2 n^2 \xi \phi^2}{\Gamma^2}, \frac{36(3q)^2 t^2}{n} \right\} < 1$, $\frac{36(3q)^2 t^2}{n} \leq 1$,

$$\xi = \max_t \max_{s \in [t-3q, t+q]} \frac{\phi^2 \Gamma^2}{t},$$

and the $B^{(t)}$ are good. Then,

$$\mathbb{E} \left[ q \sum_{t = [t-q, t-1]} l_{k_t, k_t}^l \left( \Delta_{\max} z_{k_t}^{l, \pi} - \Delta_{\min} z_{k_t}^{l, \pi} \right)^2 \right] \leq \mathbb{E} \left[ \frac{4}{3} \cdot \frac{r}{1 - r} \left( \Delta_{\min} z_{k_t}^{l, \pi} - \Delta_{\min} z_{k_t}^{l, \pi} \right)^2 + \frac{8}{9} \cdot \frac{r}{1 - r} (\Delta z_{k_t}^{l, \pi})^2 \right]$$

$$+ \sum_{s \in [t-3q, t+q] \setminus \{t\}} \frac{r}{486q(1 - r)} (\Delta z_{k_s}^{s, \pi})^2,$$

$$+ \sum_{s \in [t-3q, t+q] \setminus \{t\}} \frac{r}{486q(1 - r)} (\Delta z_{k_s}^{s, \pi})^2.$$

We first state a lemma similar to Lemma 25.

**Lemma 26.** If the $B^{(s)}$ are good, then for any $l_1, l_2 \in [a, b] \subseteq [l-q, l]$,

$$\left\| \left[ x^{l_1, l_2, k_t} \right] - \left[ x^{l_1, l_2, k_t} \right] \right\| \leq 3n \phi_t \left( \sum_{s \in [t-2q, t+q] \setminus \{t\}} \max_{l_t, l_t, k_t} \left\{ \Delta_{\max} z_{k_t}^{l, \pi} \right\} \right)$$

$$= \frac{4}{3} \cdot \frac{r}{1 - r} \left( \Delta_{\min} z_{k_t}^{l, \pi} - \Delta_{\min} z_{k_t}^{l, \pi} \right)^2 + \frac{8}{9} \cdot \frac{r}{1 - r} (\Delta z_{k_t}^{l, \pi})^2 \right]$$

$$+ \sum_{s \in [t-3q, t+q] \setminus \{t\}} \frac{r}{486q(1 - r)} (\Delta z_{k_s}^{s, \pi})^2,$$

$$+ \sum_{s \in [t-3q, t+q] \setminus \{t\}} \frac{r}{486q(1 - r)} (\Delta z_{k_s}^{s, \pi})^2.$$
Proof. The proof is very similar to that of Lemma 15. The only change beyond notation differences in replacing \( t_1 \) by \( l_1 \), \( t_2 \) by \( l_2 \) and \( t \) by \( L \) occur in the assertions at the end of proof. To illustrate the simple changes that are needed, we look at how the first assertion changes. It becomes: if \( s \in [l - 2q, l_1 + q] \) and \( l_1 \in [a, b] \) then
\[
\Delta_{\max}^{l_1; R \cup \{l\}} z_{k_s}^{s, \pi} \leq \max_{l' \in [\max\{a, s - q\}, b]} \left\{ \Delta_{\max}^{l' \cup \{l\}} z_{k_s}^{s, \pi} \right\}.
\]

We justify it as follows. If \( s \in [l_1, l_1 + q] \), then \( l_1 \in [\max\{a, s - q\}, b] \); otherwise, \( s \leq l_1 \), then again \( l_1 \in [\max\{a, s - q\}, b] \). So the assertion is true.

Proof of Lemma 25. Since we know that \( (\nabla_{k_l} f(x) - \nabla_{k'_l} f(x'))^2 \leq \left( \sum_k L_{k_l, k'_l} |x_k - x'_k| \right)^2 \), by Lemma 26 with \( R = \emptyset \),
\[
\left( g_{\max, k_l} - g_{\min, k_l} \right)^2 \leq 3n \phi_l \sum_{s \in [l - 2q, l + q] \setminus \{l\}} L_{k_s, k_l} \max \left\{ \max_{l' \in [\max\{a, s - q\}, b]} \left\{ \Delta_{\max}^{l' \cup \{l\}} z_{k_s}^{s, \pi} \right\}, \right. \\
\left. - \min_{l' \in [\max\{a, s - q\}, b]} \left\{ \Delta_{\min}^{l' \cup \{l\}} z_{k_s}^{s, \pi} \right\} \right. \\
\left. + \min_{l' \in [\max\{a, s - q\}, b]} \left\{ \Delta_{\max}^{l' \cup \{l\}} z_{k_s}^{s, \pi} \right\}, \right. \\
\left. \min_{l' \in [\max\{a, s - q\}, b]} \left\{ \Delta_{\min}^{l' \cup \{l\}} z_{k_s}^{s, \pi} \right\} \right\}^2.
\]
Therefore,

\[
\mathbb{E} \left[ q \sum_{t=[t-q,t-1]} L_{k_t,k_t}^2 \left( \sum_{l \in \mathbb{Z}} \sum_{q,t} L_{k_t,k_t}^2 \left( 3n \phi_l \sum_{s \in [l-2q,l+q]} \right) \right) \right] \leq \mathbb{E} \left[ \sum_{t=[t-q,t-1]} q \frac{L_{k_t,k_t}^2}{\Gamma_l^2} (g_{\max,k_t}^\pi - g_{\min,k_t}^\pi)^2 \right]
\]

\[
\leq \mathbb{E} \left[ \sum_{t=[t-q,t-1]} q \frac{L_{k_t,k_t}^2}{\Gamma_l^2} \left( 3n \phi_l \sum_{s \in [l-2q,l+q]} \right) \right] \leq \mathbb{E} \left[ \sum_{t=[t-q,t-1]} q \frac{L_{k_t,k_t}^2}{\Gamma_l^2} \right] \left( 3n \phi_l \sum_{s \in [l-2q,l+q]} \right) \left( 2 \left( \sum_{s \in [l-2q,l+q]} L_{k_s,k_s} \text{max} \left( \text{max}_{t' \in [s-q,s-q]} \left( \text{max}_{q,l} \Delta_{\max}^{t'}(l) \right) \right) \right)^2 \right)
\]

\[
\leq 2 \left( \sum_{s \in [l-2q,l+q]} L_{k_s,k_s} \text{max} \left( \text{max}_{t' \in [s-q,s-q]} \left( \text{max}_{q,l} \Delta_{\max}^{t'}(l) \right) \right) \right)^2.
\]
We note that by the Cauchy-Schwarz inequality,

\[
\left( \sum_{s \in [l-2q,l+q] \setminus \{l,t\}} L_{k_s,k_t} \max \left\{ \left( \max_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\max}^{t',(l)} z_{s,\pi}^{k_s} \} - \min_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\min}^{t',(l)} z_{s,\pi}^{k_s} \} \right) \right\}^2 \right) \leq \left( 3q \right) \left( \sum_{s \in [l-2q,l+q] \setminus \{l,t\}} L_{k_s,k_t}^2 \max \left\{ \left( \max_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\max}^{t',(l)} z_{s,\pi}^{k_s} \} \right) \right\}^2 \right)
\]

Also we note that, for any \( s \) in \([l - 2q, l + q] \setminus \{l\},\)

\[
\left( \min_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\min}^{t',(l)} z_{s,\pi}^{k_s} \} \right)^2 \leq \left( \max_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\max}^{t',(l)} z_{s,\pi}^{k_s} \} \right)^2 - \left( \min_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\min}^{t',(l)} z_{s,\pi}^{k_s} \} \right)^2 \leq 2 \left( \max_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\max}^{t',(l)} z_{s,\pi}^{k_s} \} \right)^2 + 2 \left( \min_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\min}^{t',(l)} z_{s,\pi}^{k_s} \} \right)^2
\]

Therefore,

\[
\mathbb{E} \left[ q \sum_{l=[t-q,t-1]} L_{l,k_l}^2 (\Sigma_{\max} z_{k_l}^{l,\pi} - \Sigma_{\min} z_{k_l}^{l,\pi})^2 \right] 
\]

\[
\leq \mathbb{E} \left[ \sum_{l=[t-q,t-1]} \left( q \frac{36 L_{k_l,k_t}^2 L_{k_l,k_t}^2 n^2 \phi^2 I_l^2}{I_l^2} \right) \right] 
\]

\[
+ \sum_{s \in [l-2q,l+q] \setminus \{l,t\}} \left( q \frac{36 L_{k_s,k_t}^2 L_{k_s,k_t}^2 q(3q)n^2 \phi^2 I_l^2}{I_l^2} \right)
\]

\[
\left[ \left( \max_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\max}^{t',(l)} z_{s,\pi}^{k_s} \} - \min_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\min}^{t',(l)} z_{s,\pi}^{k_s} \} \right)^2 + \left( \max_{t' \in [\max\{s-q,l-q\},l]} \{ \Delta_{\max}^{t',(l)} z_{s,\pi}^{k_s} \} \right)^2 \right]
\]
As $L^2_{k_s,k_t} \leq 4$, \[\begin{align*}
\mathbb{E} \left[ q \sum_{t=\max\{t-q,l-q\},l} L^2_{k_s,k_t} (\Delta_{\max}^{t,\pi} - \Delta_{\min}^{t,\pi})^2 \right] \\
\leq \mathbb{E} \left[ \sum_{t=\max\{t-q,l-q\},l} \left( q \frac{144 L^2_{k_s,k_t} n^2 \phi^2_I}{\Gamma^2_l} \left( \max_{t' \in \{t-q,l-q\},l} \{ \Delta_{\max}^{t',t,\pi} \} - \min_{t' \in \{t-q,l-q\},l} \{ \Delta_{\min}^{t',t,\pi} \} \right)^2 \right) \\
+ q \frac{144 L^2_{k_s,k_t} n^2 \phi^2_I}{\Gamma^2_l} \left( \max_{t' \in \{t-q,l-q\},l} \{ \Delta_{\max}^{t',t,\pi} \} - \min_{t' \in \{t-q,l-q\},l} \{ \Delta_{\min}^{t',t,\pi} \} \right)^2 \\
+ \sum_{s=\max\{t-q,l+q\},l} \left( q(3q) n^2 \phi^2_I \right) \frac{36 L^2_{k_s,k_t} L^2_{k_s,k_t} n^2 \phi^2_I}{\Gamma^2_l} \right].
\end{align*}\]

We let $\xi$ denote $\max_{t=\max\{t-q,l-q\},l} \frac{\phi^2_I}{\Gamma^2_l}$. Also, we use the bounds

$$\max_{t' \in \{t-q,l-q\},l} \{ \Delta_{\max}^{t',t,\pi} \} \leq \max_{t' \in \{t-q,l-q\},l} \{ \Delta_{\max}^{t',\pi} \} = \Delta_{\max}^{t,\pi},$$

which holds as $\Delta_{\max}^{t',\pi} \leq \Delta_{\max}^{t,\pi}$ if $t' \geq t$, and similarly,

$$\min_{t' \in \{t-q,l-q\},l} \{ \Delta_{\min}^{t',t,\pi} \} \geq \min_{t' \in \{t-q,l-q\},l} \{ \Delta_{\min}^{t',\pi} \} = \Delta_{\min}^{t,\pi}.$$

In addition, for $s \neq t$, $\max_{t' \in \{s-q,l-q\},l} \Delta_{\max}^{t',t,\pi} \Delta_{\min}^{t',t,\pi}$ $\Delta_{\max}^{t',t,\pi} \Delta_{\min}^{t',t,\pi}$ does not depend on either the time $l$ or the time $t$ updates; it is fixed over all paths $\pi$ obtained by varying $k_l$ and/or $k_t$. Hence, in the first and second terms, on averaging over $k_t$, we can replace $L^2_{k_s,k_t}$ by $\frac{L^2_{k_s}}{n}$. While in the final term, we can first average over $k_t$ which replaces $L^2_{k_s,k_t}$ by $\frac{L^2_{k_s}}{n}$. Next, we can average over $k_l$, causing $L^2_{k_s,k_t}$ to be replaced by $\frac{L^2_{k_s}}{n}$. Note that one cannot do the averaging in the opposite order because $L^2_{k_s,k_t} \max_{t' \in \{s-q,l-q\},l} \Delta_{\max}^{t',t,\pi} \Delta_{\max}^{t',t,\pi} \Delta_{\max}^{t',t,\pi} \Delta_{\max}^{t',t,\pi} \Delta_{\max}^{t',t,\pi} \Delta_{\max}^{t',t,\pi}$ need not be fixed as $k_s$ is averaged.

\textsuperscript{12}Since $f$ is a convex function, $x^T L x \geq 0$ for any $x$. Therefore, $L_{i,j} + L_{i,j} \leq L_{i,i} + L_{j,j} = 2$. Also, the $L_{i,j}$ are non-negative for all $i$ and $j$, so $L_{i,j} \leq 2$ for all $i$ and $j$.  

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This yields
\[
\mathbb{E} \left[ q \sum_{t=[t-q,t-1]} L^2_{k_1,k_1} \left( \sum_{t_q} e_{k_1}^{t_q} - \sum_{l_q} e_{k_1}^{l_q} \right)^2 \right] 
\leq \mathbb{E} \left[ \frac{144L^2_{res} \theta^2 n^2 \phi^2}{n \Gamma t} \cdot \left( \sum_{t_{max}} e_{k_1}^{t_{max}} - \sum_{l_{min}} e_{k_1}^{l_{min}} \right)^2 \right] 
+ \frac{144L^2_{res} \theta^2 n^2 \phi^2}{n \Gamma t} \cdot \Delta^{\{l,t\}}_{\max} \sum_{k_t} \left( \sum_{s,l} \frac{36L^2_{res} \theta^2 n^2 \phi^2}{n^2 \Gamma_t} \right) 
+ \frac{144L^2_{res} \theta^2 n^2 \phi^2}{n \Gamma t} \cdot \Delta^{\{l,t\}}_{\max} \sum_{k_t} \left( \sum_{s,l} \frac{36L^2_{res} \theta^2 n^2 \phi^2}{n^2 \Gamma_t} \right).
\]

Note that \( \max_{t\in[\max\{s_q,t_q\},l]} \Delta_{max_{s_k}}^{\{l,t\}} \leq \max_{t\in[\max\{s_q,t_q\},l]} \Delta_{max_{s_k}}^{\{l,t\}} \leq \Delta_{max_{s_k}}^{\{l,t\}} \) if \( l \geq s \). Similarly, \( \max_{t\in[\max\{s_q,t_q\},l]} \Delta_{max_{s_k}}^{\{l,t\}} \geq \min_{t\in[\max\{s_q,t_q\},l]} \Delta_{min_{s_k}}^{\{l,t\}} \). Therefore, \( \Delta_{max_{s_k}}^{\{l,t\}} \leq 2 \left( \Delta_{max_{s_k}}^{\{s,t\}} \right)^2 + 2 \left( \Delta_{min_{s_k}}^{\{s,t\}} \right)^2 \). And let \( \chi_{l,R'} = \max\{l_q, \max_{t'\in[R']\{l'-q\}}\}, l \}. In the next equation, \( R' = \{l_1\} \). Then

\[
\mathbb{E} \left[ q \sum_{t=[t-q,t-1]} L^2_{k_1,k_1} \left( \sum_{t_{max}} e_{k_1}^{t_{max}} - \sum_{l_{min}} e_{k_1}^{l_{min}} \right)^2 \right] 
\leq \mathbb{E} \left[ \frac{432L^2_{res} \theta^2 n^2 \phi^2}{n \Gamma t} \cdot \left( \sum_{t_{max}} e_{k_1}^{t_{max}} - \sum_{l_{min}} e_{k_1}^{l_{min}} \right)^2 + \frac{288L^2_{res} \theta^2 n^2 \phi^2}{n \Gamma t} \cdot \Delta^{\{l,t\}}_{\max} \sum_{k_t} \left( \sum_{s,l} \frac{36L^2_{res} \theta^2 n^2 \phi^2}{n^2 \Gamma_t} \right) \right] 
+ \sum_{s\in[t-3q,t+q]\{t\}} \frac{72L^2_{res} \theta^2 n^2 \phi^2}{n^2 \Gamma_t} \cdot \Delta^{\{s,t\}}_{\max} \sum_{k_t} \left( \sum_{s,l} \frac{36L^2_{res} \theta^2 n^2 \phi^2}{n^2 \Gamma_t} \right) 
+ \sum_{s\in[t-3q,t+q]\{t\}} \frac{72L^2_{res} \theta^2 n^2 \phi^2}{n^2 \Gamma_t} \cdot \Delta^{\{s,t\}}_{\max} \sum_{k_t} \left( \sum_{s,l} \frac{36L^2_{res} \theta^2 n^2 \phi^2}{n^2 \Gamma_t} \right) 
+ \sum_{t\in[t-q,t-1]} \left( \sum_{l_t\in[t-2q,t+q]\{t\}} \frac{36L^2_{res} \theta^2 n^2 \phi^2}{n^2 \Gamma_t} \right) \left( \sum_{s,l} \frac{36L^2_{res} \theta^2 n^2 \phi^2}{n^2 \Gamma_t} \right).
\]

Essentially the same argument yields the following bound. The main change is that the averaging will be over a sequence of \( m - 1 \) or \( m \) \( L^2_{res} \theta^2 n^2 \phi^2 \) terms, each of which yields a multiplier of \( \frac{L^2_{res} \theta^2 n^2 \phi^2}{n^2 \Gamma_t} \), and there
are at most $(3q)^{m-1}$ different choices of $l_1, \cdots, l_{m-1}$, which yields an additional factor of $(3q)^{m-1}$.

\[
\mathbb{E}\left[ \sum_{l \in [t-q,t-1]} \left( \sum_{l_1, l_2, \cdots, l_{m-1} \in [t-2q, l+q]} \left( \prod_{i=m-1}^{2} L_{k_{i}, k_{i-1}}^2 \right) L_{k_{i}, k_{i-1}}^2 \right) \right] 
\leq \mathbb{E}\left[ \frac{432(L_{\text{res}}^2)^m q(3q)^{m-1} n^2 \xi \phi_t^2}{n^m \Gamma_t^2} \left( \Delta_{\max} z_{k_{t}} \pi - \Delta_{\min} z_{k_{t}} \pi \right)^2 \right] 
+ \frac{288(L_{\text{res}}^2)^m q(3q)^{m-1} n^2 \xi \phi_t^2}{n^m \Gamma_t^2} (\Delta z_{k_{t}} \pi)^2 
+ \sum_{s \in [t-3q, t+q] \setminus \{t\}} \frac{72(L_{\text{res}}^2)^m q(3q)^{m-1} n^2 \xi \phi_t^2}{n^m \Gamma_t^2} (\Delta z_{k_{s}} \pi)^2 
+ \sum_{s \in [t-3q, t+q] \setminus \{t\}} \frac{36(3q) \left( \prod_{i=m}^{2} L_{k_{i}, k_{i-1}}^2 \right) L_{k_{i}, k_{i-1}}^2}{\Gamma_t^2} \left( \Delta_{\max} z_{k_{t}} \pi - \Delta_{\min} z_{k_{t}} \pi \right)^2 \right].
\]

For $m \geq 1$, let

\[
\mathcal{S}_m = \sum_{l \in [t-q, t-1]} \left( \sum_{l_1, l_2, \cdots, l_{m-1} \in [t-2q, l+q]} \left( \prod_{i=m-1}^{2} L_{k_{i}, k_{i-1}}^2 \right) L_{k_{i}, k_{i-1}}^2 \right) \right) \mathbb{E}\left[ \sum_{l \in [t-q, t-1]} \left( \sum_{l_1, l_2, \cdots, l_{m-1} \in [t-2q, l+q]} \left( \prod_{i=m-1}^{2} L_{k_{i}, k_{i-1}}^2 \right) L_{k_{i}, k_{i-1}}^2 \right) \right] 
\leq \mathbb{E}\left[ \frac{432(L_{\text{res}}^2)^m q(3q)^{m-1} n^2 \xi \phi_t^2}{n^m \Gamma_t^2} \left( \Delta_{\max} z_{k_{t}} \pi - \Delta_{\min} z_{k_{t}} \pi \right)^2 \right] 
+ \frac{288(L_{\text{res}}^2)^m q(3q)^{m-1} n^2 \xi \phi_t^2}{n^m \Gamma_t^2} (\Delta z_{k_{t}} \pi)^2 
+ \sum_{s \in [t-3q, t+q] \setminus \{t\}} \frac{72(L_{\text{res}}^2)^m q(3q)^{m-1} n^2 \xi \phi_t^2}{n^m \Gamma_t^2} (\Delta z_{k_{s}} \pi)^2 
+ \sum_{s \in [t-3q, t+q] \setminus \{t\}} \frac{36(3q) \left( \prod_{i=m}^{2} L_{k_{i}, k_{i-1}}^2 \right) L_{k_{i}, k_{i-1}}^2}{\Gamma_t^2} \left( \Delta_{\max} z_{k_{t}} \pi - \Delta_{\min} z_{k_{t}} \pi \right)^2 \right].
\]

Let $\mathcal{S}_0 = \mathbb{E}\left[ \sum_{l \in [t-q, t-1]} L_{k_{l}, k_{l-1}}^2 (\Delta_{\max} z_{k_{l}} \pi - \Delta_{\min} z_{k_{l}} \pi)^2 \right]$. Then, (28) and (29) can be restated as
follows:

\[ S_{m-1} \leq \frac{36(3q)n^2 \xi \phi^2}{\Gamma_t^2} S_m + \mathbb{E} \left[ \frac{432(L_{\text{res}}^2)^m q(3q)^m - n^2 \xi \phi^2}{n^{m+1} \Gamma_t^2} \cdot \left( \Delta_{\max}^{l,\pi} z_{k_{t}} - \Delta_{\min}^{l,\pi} z_{k_{t}} \right)^2 \right. \]

\[ + \frac{288(L_{\text{res}}^2)^m q(3q)^m - n^2 \xi \phi^2}{n^{m+1} \Gamma_t^2} (\Delta z_{k_{t}})^2 \]

\[ + \sum_{s \in [t-3q,t+q] \setminus \{t\}} \frac{72(L_{\text{res}}^2)^m q(3q)^m - n^2 \xi \phi^2}{n^{m+1} \Gamma_t^2} \left( \Delta_{\max}^{s,\pi} z_{k_{s}} - \Delta_{\min}^{s,\pi} z_{k_{s}} \right)^2 \]

\[ + \sum_{s \in [t-3q,t+q] \setminus \{t\}} \frac{72(L_{\text{res}}^2)^m q(3q)^m - n^2 \xi \phi^2}{n^{m+1} \Gamma_t^2} \left( \Delta z_{k_{s}} \right)^2 \].

Note that by definition, \( S_{3q} = 0 \). Let \( r = \frac{36(3q)^2 l_{\text{res}}^2 n^2 \xi \phi^2}{n \Gamma_t^2} \). Then,

\[ q \cdot S_0 \leq \mathbb{E} \left[ (1 + r + r^2 + r^3 + \cdots) \cdot \left( \frac{432L_{\text{res}}^2 q n^2 \xi \phi^2}{n^{m+1} \Gamma_t^2} \cdot \left( \Delta_{\min}^{l,\pi} z_{k_{t}} - \Delta_{\min}^{l,\pi} z_{k_{t}} \right)^2 \right. \right. \]

\[ + \sum_{s \in [t-3q,t+q] \setminus \{t\}} \frac{72L_{\text{res}}^2 q n^2 \xi \phi^2}{n^{m+1} \Gamma_t^2} \left( \Delta_{\max}^{s,\pi} z_{k_{s}} - \Delta_{\min}^{s,\pi} z_{k_{s}} \right)^2 \]

\[ + \sum_{s \in [t-3q,t+q] \setminus \{t\}} \frac{72L_{\text{res}}^2 q n^2 \xi \phi^2}{n^{m+1} \Gamma_t^2} \left( \Delta z_{k_{s}} \right)^2 \].

So long as \( r < 1, 1 + r + r^2 + r^3 + \cdots \leq \frac{1}{1-r} \), and as \( \frac{324q^2 L_{\text{res}}^2}{n} \leq 1 \), replacing \( S_0 \) by

\[ \mathbb{E} \left[ \sum_{l=[t-q,t-1]} L_{k_t,k_t}^2 (\Delta_{\max}^{l,\pi} z_{k_{l}} - \Delta_{\min}^{l,\pi} z_{k_{l}})^2 \right] \]

yields

\[ \mathbb{E} \left[ q \sum_{l=[t-q,t-1]} L_{k_t,k_t}^2 (\Delta_{\max}^{l,\pi} z_{k_{l}} - \Delta_{\min}^{l,\pi} z_{k_{l}})^2 \right] \]

\[ \leq \mathbb{E} \left[ \frac{4}{3} \cdot \frac{r}{1-r} (\Delta_{\min}^{l,\pi} z_{k_{t}} - \Delta_{\min}^{l,\pi} z_{k_{t}})^2 + \frac{8}{9} \cdot \frac{r}{1-r} (\Delta^{l,\pi} z_{k_{t}})^2 \right. \]

\[ + \sum_{s \in [t-3q,t+q] \setminus \{t\}} \frac{r}{486q(1-r)} (\Delta_{\max}^{s,\pi} z_{k_{s}} - \Delta_{\min}^{s,\pi} z_{k_{s}})^2 \]

\[ + \sum_{s \in [t-3q,t+q] \setminus \{t\}} \frac{r}{486q(1-r)} (\Delta z_{k_{s}})^2 \].

\( \square \)
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