Nonparametric Estimation of Uncertainty Sets for Robust Optimization

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Abstract—We investigate a data-driven approach to constructing uncertainty sets for robust optimization problems, where the uncertain problem parameters are modeled as random variables whose joint probability distribution is not known. Relying only on independent samples drawn from this distribution, we provide a nonparametric method to estimate uncertainty sets whose probability mass is guaranteed to approximate a given target mass within a given tolerance with high confidence. The nonparametric estimators that we consider are also shown to obey distribution-free finite-sample performance bounds that imply their convergence in probability to the given target mass. In addition to being efficient to compute, the proposed estimators result in uncertainty sets that yield computationally tractable robust optimization problems for a large family of constraint functions.

Index Terms—Chance-constrained optimization, robust optimization, data-driven optimization, nonparametric estimation.

I. INTRODUCTION

In this paper, we consider a class of optimization problems introduced by [1] whose feasible regions are defined in terms of chance constraints of the form

\[ \Pr\left( f(x, u) \leq 0 \right) \geq \alpha. \quad (1) \]

Here, \( f : \mathbb{R}^\ell \times \mathbb{R}^d \to \mathbb{R} \) denotes the constraint function, \( x \in \mathbb{R}^\ell \) denotes the decision variable, and \( u \) is an \( \mathbb{R}^d \)-valued random vector that reflects uncertainty in the constraint parameters. The chance constraint (1) requires that the decision variable satisfy the constraint \( f(x, u) \leq 0 \) with probability no smaller than \( \alpha \in [0, 1] \).

Chance constrained optimization problems are challenging to solve for a variety of reasons. First, their feasible regions are generally nonconvex [2]. For example, chance constrained problems have been shown to be NP-hard even in the most basic setting where the constraint function is affine in both the decision variable and uncertain parameters [3]. Furthermore, verifying the feasibility of a candidate solution to a chance constrained problem is difficult, because it involves the evaluation of a multivariate integral which can be computationally expensive in high dimensions [2].

To complicate matters further, the underlying distribution according to which the random vector is distributed may be unknown.

A variety of approaches to the solution of chance constrained problems have been explored in the literature. There are several works that develop exact convex reformulations of chance constrained problems [4]–[8] and their distributionally robust generalizations [9]–[11]. These reformulations rely on a variety of assumptions, including restrictions on the family of probability distributions according to which the random vector is distributed and the functional form of the chance constraints. Ultimately, many problems of practical interest may fail to satisfy these narrow structural and distributional assumptions.

Because of the rarity of problem instances amenable to exact convex reformulation, there is another line of research focused on the design of approximation methods for chance constrained problems. One approach that has been explored extensively involves the construction of explicit convex inner (conservative) approximations to chance constrained optimization problems [12]–[17]. Again, a potential drawback of these methods stems from their reliance (to varying degrees) on knowledge of certain features of the underlying distribution, e.g., support or moment information. Data-driven approximation methods seek to alleviate the reliance on distributional assumptions that may be overly stringent or difficult to verify in practice. Instead, they utilize data sampled from the underlying distribution. For instance, the sample average approximation method [2] involves selecting an optimal subset of the sampled data that has empirical probability mass no smaller than the target reliability level \( \alpha \).

While this approach may be less conservative than other data-driven methods, it gives rise to approximations in the form of mixed-integer optimization problems—which become computationally intractable to solve in the large sample regime.

There is another stream of literature focusing on data-driven approximations that give rise to tractable convex optimization problems. A particular category of methods uses the given data to construct estimates of the unknown distribution or its lower-order moments [9], [18]–[21]. These estimates, together with suitably defined confidence regions, give rise to distributionally robust approximations to the original chance constrained program that, in some instances, admit tractable convex reformulations.

Another family of data-driven methods known as scenario approximations have also been widely studied as another tractable alternative to the approximation of chance constrained problems [22], [23] and their distributionally robust counterparts [24], [25]. Specifically, scenario-based methods approximate the chance constraint (1) with \( n \) sampled constraints of the form

\[ f(x, u_i) \leq 0, \quad i = 1, \ldots, n, \]

where \( u_1, \ldots, u_n \) is an i.i.d. sample drawn from the unknown distribution. In addition to being distribution-free, scenario-based methods do not impose restrictions on the
functional form of the constraint function beyond requiring its convexity in the decision variable. However, these methods require a sample size that is at least $O \left( \frac{\log(1/\delta)}{\alpha^2} \right)$ in order to yield a solution that is feasible for the original chance constrained problem with probability at least $1 - \delta$. Therefore, when the dimension of the decision variable is large or when the reliability needed is high, scenario-based approximations can become numerically challenging to solve due to the large number of constraints that must be enforced.

A. Contribution and Related Work

Our approach is predicated on the conservative approximation of (1) in the form of a robust constraint:

$$f(x,u) \leq 0 \quad \forall u \in U,$$

(2)

where the uncertainty set $U \subset \mathbb{R}^d$ is constructed from data in a manner that ensures its satisfaction of the mass requirement $\Pr (u \in U) \geq \alpha$ with high probability. In this paper, we provide a nonparametric method to construct such uncertainty sets from data in a manner that guarantees that their probability mass is within a given tolerance of the target mass $\alpha$ with high confidence. The uncertainty sets that we propose are trivial to compute, satisfy distribution-free finite-sample statistical guarantees, and give rise to robust constraints (2) that are computationally tractable for a large family of constraint functions.

Additionally, the proposed methodology allows for the explicit representation of nonconvex uncertainty sets as finite unions of convex sets—e.g., as the union of $m \in \mathbb{N}$ balls centered at the sampled data points. In this manner, the resulting robust constraint (2) can be equivalently reformulated as a finite intersection of $m$ simpler robust constraints, where each robust constraint is defined in terms of a convex uncertainty set. Importantly, the geometry of the underlying convex sets can be tailored to accommodate the structure of the constraint function to ensure the tractability of the resulting robust constraints. As another degree of freedom in the parameterization of these sets, the user is free to specify the number of constitutive sets used in the representation of the uncertainty set, and, hence, the number of robust constraints that must be enforced in the approximation. This provides the user with some degree of control over the size of the resulting robust optimization problem, unlike scenario approximation methods.

Our approach is closely related to a class of existing methods that utilize data to construct uncertainty sets that yield tractable robust approximations to chance-constrained optimization problems. For example, Margellos et al. [26] develop a data-driven approach to estimate intervals that cover each component of the random vector with high probability, resulting in uncertainty sets that take the form of hyperrectangles. Bertsimas et al. [27] utilize statistical hypothesis tests to construct convex uncertainty sets from data. However, due to the convex geometry of the uncertainty sets produced by these methods, they may be limited in terms of their ability to accurately describe nonconvex high probability regions that reflect multimodality in the underlying distribution. Because of the limited expressiveness of these uncertainty sets, they may result in overly conservative approximations to the original chance constraint. Closer to the approach adopted in this paper, Campbell and How [28] consider representations in the form of unions of ellipsoids. A potential drawback of their approach, however, stems from the reliance of their results on the assumption that the unknown distribution belong to a family of Dirichlet process Gaussian mixtures.

The problem of estimating high probability sets from data also has similarities to minimum-volume (MV) set estimation problems. MV-set estimators based on the estimation of density level sets [29]–[32] typically yield nonconvex sets that are not readily expressible as finite unions of convex sets—limiting their tractability from a robust optimization perspective. More broadly, MV-set estimation methods include those based on nonparametric set estimation [33], [34], empirical quantile-based estimation [35], [36], classification via support vector machines [37], [38], and network flow-based methods [39]. With the exception of [36], the methods presented in these papers lack explicit finite-sample guarantees on their performance. A practical limitation of the algorithms proposed by Scott et al. [36], however, is that the computation of their MV-set estimates becomes intractable in high dimensions.

Organization: The remainder of the paper is organized as follows. In Section II, we introduce a data-driven method to construct uncertainty sets with a given probability mass. In Section III, we provide finite-sample statistical guarantees on the performance of the proposed class of estimators. Section IV illustrates the behavior of the estimators with synthetic numerical experiments, and Section V concludes the paper.

Notation: We employ the following notational conventions throughout the paper. Let $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real numbers and integers, respectively. Given a positive integer $n \in \mathbb{Z}$, we let $[n] := \{1, \ldots, n\}$ denote the set of the first $n$ positive integers. Given a real number $x \in \mathbb{R}$, we denote its ceiling by $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geq x\}$. Throughout, we use boldface symbols to denote random variables, and non-boldface symbols to denote particular values in the range of a random variable and other deterministic quantities.

We use $\Pr (A)$ to denote the probability of an event $A$. The conditional probability of an event $A$ conditioned on a random variable $x$ is denoted by $\Pr (A \mid x)$. It is a random variable since it is a function of the random variable $x$. It takes the value $\Pr (A \mid x) = x$ whenever the realization of the conditioning random variable $x$ equals $x$.

II. Uncertainty Set Estimation

In this section, we provide a data-driven method to construct uncertainty sets whose probability mass is both close to and no smaller than a given target mass $\alpha \in (0, 1)$ with high probability, based only on a finite random sample drawn from the unknown distribution of interest. In addition to the mass requirement, we seek representations that result in computationally tractable robust constraints.
The remainder of this section is organized as follows. In Sec. [II-A], we provide a general approach to learning \( \alpha \)-probability sets via the estimation of sublevel sets for a general class of continuous functions. In Sec. [II-B], we introduce a particular class of functions whose level sets (expressible as finite unions of convex sets) yield tractable robust constraints for several important families of constraint functions. We close with a discussion on the connections between the proposed family of approximations and scenario-based approximation methods.

A. Estimation via Level Sets

With the aforementioned objectives in mind, we consider a family of uncertainty sets defined as \( r \)-sublevel sets \( \{\} \) of a given continuous function \( \phi : \mathbb{R}^d \rightarrow \mathbb{R} \), i.e.,

\[
\mathcal{U}_\phi(r) := \{ u \in \mathbb{R}^d \mid \phi(u) \leq r \}. \tag{3}
\]

The proposed family of uncertainty sets \( \{\} \) is parameterized by the user-specified inputs: the shape function \( \phi \) and level \( r \). Intuitively, the shape function \( \phi \) should be chosen so that it is smallest over those regions with the greatest concentration of probability mass, in order to limit the volume (Lebesgue measure) of the resulting sublevel set. While this might suggest an approach to specifying \( \phi \) in terms of a density estimate based on an i.i.d. sample drawn from the unknown distribution, the particular functional form of \( \phi \) must also yield uncertainty (level) sets that are computationally tractable from a robust optimization perspective—a condition that most nonparametric (kernel) density estimators fail to satisfy. In Sec. [II-B], we suggest a particular functional form for the shape function \( \phi \) that is expressive enough to represent compact sets of arbitrary form, while ensuring the tractability of the robust constraints that it yields for a large family of constraint functions. It is important to note that, in the interest of lightening notation, we have omitted the potential dependence that the function \( \phi \) may have on data drawn from the unknown distribution of interest.

For the remainder of this section, we treat the shape function \( \phi \) as being fixed (implicitly conditioning all probabilistic statements on the data on which it is based), and focus our discussion on the role of the level \( r \) in controlling the behavior of the sublevel sets of \( \phi \). Clearly, the level \( r \in \mathbb{R} \) directly influences the volume and probability mass of the sublevel sets of \( \phi \), with larger levels resulting in sublevel sets of greater volume. Thus, with the aim of limiting the conservatism of the robust approximations \( \{\} \) induced by the proposed family of uncertainty sets, we are interested in characterizing the smallest level \( r \) such that the target mass requirement is satisfied. We refer to this as the \( \alpha \)-covering level, which we formally define as follows.

**Definition 1 (\( \alpha \)-covering level):** Let \( \alpha \in (0, 1) \). The \( \alpha \)-covering level is defined as

\[
\rho(\alpha) := \inf \{ r \in \mathbb{R} \mid \Pr (u \in \mathcal{U}_\phi(r)) \geq \alpha \}. \tag{4}
\]

As one of the objectives of this paper, we are interested in constructing an estimator for the \( \alpha \)-covering level \( \rho(\alpha) \) based only on an i.i.d. training sample \( u_1, \ldots, u_n \) drawn from the unknown distribution. Before introducing the particular family of estimators that we consider in this paper, it will be informative to express the \( \alpha \)-covering level in terms of the quantile

\[
\rho(\alpha) = F^{-1}(\alpha),
\]

where \( F^{-1}(\alpha) := \inf \{ z \in \mathbb{R} \mid \Pr (\zeta \leq z) \geq \alpha \} \) denotes the \( \alpha \)-quantile of the transformed random variable \( \zeta := \phi(u) \). This reformulation suggests a natural estimator for \( \rho(\alpha) \) in the form of an empirical quantile

\[
F_n^{-1}(\alpha_n) := \inf \{ z \in \mathbb{R} \mid F_n(z) \geq \alpha \}, \tag{4}
\]

where \( \alpha_n \in (0, 1) \) is a sequence of probability levels, and \( F_n(z) := (1/n) \sum_{i=1}^n I(\zeta_i \leq z) \) denotes the empirical distribution function associated with the transformed training sample \( \zeta_i := \phi(u_i) \) for \( i = 1, \ldots, n \).

In Section III, we investigate the role of \( \alpha_n \) in controlling the limiting behavior and rate of convergence of the proposed empirical quantile to the \( \alpha \)-covering level. In particular, we establish minimum training sample size requirements on \( n \) ensuring that the probability mass covered by the resulting uncertainty set \( \mathcal{U}_\phi(F_n^{-1}(\alpha_n)) \) is within a given tolerance of (and no smaller than) the given target mass \( \alpha \) with high probability.

B. Shape Functions \( \phi \) Compatible with Robust Optimization

The functional form of the shape function \( \phi \) (and the geometry of its sublevel sets) will play a critical role in governing the tractability of the family of robust constraints that it gives rise to. With this in mind, we suggest a class of shape functions whose sublevel sets are rich enough to represent compact sets of arbitrary form, while being simple enough to ensure the tractability of the resulting robust constraints. Specifically, we propose a particular class of shape functions whose sublevel sets admit explicit representations as finite unions of regular convex sets, e.g., \( p \)-norm balls. In this manner, the resulting robust constraint \( \{\} \) can be equivalently reformulated as a finite intersection of simpler robust constraints, where each robust constraint is defined in terms of a convex uncertainty set. Being expressed as \( p \)-norm balls, the particular geometry of the underlying convex sets (e.g., \( \ell_1 \) versus \( \ell_2 \)) can be tailored to accommodate the structure of the given constraint function to facilitate the tractability of the corresponding robust constraint. In addition to determining the number of constraints, the number of sets in the union will also influence the volume of the uncertainty sets that we learn, and hence the conservatism of the resulting robust constraint approximations. Therefore, in treating the number of sets in the approximation as a user-specified parameter, one can tradeoff the computational complexity of the resulting optimization problem against the quality (conservatism) of its solutions.

With these design criteria in mind, we now introduce the specific class of estimators we consider. Given an i.i.d.
sample $\bar{u}_1, \ldots, \bar{u}_m$ drawn from the unknown distribution of interest\(^2\), define
\[ \phi(u) := \min_{i \in [m]} \| u - \bar{u}_i \| , \]  \hspace{1cm} (5)
where $\| \cdot \|$ denotes the $p$-norm ($p \geq 1$) on $\mathbb{R}^d$. The specification of the shape function according to (5) gives rise to a class of uncertainty sets that are defined as a union of balls with a common radius, where each ball is centered at a randomly sampled point
\[ \mathcal{U}_\phi(r) = \bigcup_{i=1}^{m} B(\bar{u}_i, r) . \] \hspace{1cm} (6)
Here $B(u, r) := \{ v \in \mathbb{R}^d \mid \| u - v \| \leq r \}$ denotes the closed ball of radius $r \geq 0$ centered at $u \in \mathbb{R}^d$. The parameter $r$ plays an analogous role to that of bandwidth parameters in the context of nonparametric kernel density estimation.

It is worth noting that set estimators in the form of unions of balls have been studied in the context of support estimation by [33], [34]. In particular, Bafillo et al. [34] establish conditions on the decay rate of the ball radius $r$ in $m$ ensuring that the resulting sequence of set estimates converges in probability to the support of the underlying distribution as the number of balls $m$ goes to infinity. By contrast, in our approach we treat the number of balls as fixed and calibrate the radius $r$ from data in a manner guaranteeing that the probability mass covered by the resulting uncertainty set is within a given tolerance of the target probability mass with high confidence\(^3\).

Given an uncertainty set $\mathcal{U}_\phi(r)$ defined according to (6), one can equivalently reformulate the robust constraint\(^2\) as the intersection of $m$ robust constraints of the form
\[ f(x, u) \leq 0 \quad \forall u \in B(\bar{u}_i, r), \quad i = 1, \ldots, m. \]

Robust constraints defined in terms of uncertainty sets of this form result in computationally tractable optimization problems for a large family of constraint functions. For example, given constraint functions that are affine in the decision variable, robust constraints defined over polyhedral regions such as those specified by the $\ell_1$ or $\ell_\infty$ norm result in linear programs, while those specified in terms of the $\ell_2$ norm result in second-order cone programs. Furthermore, robust second-order cone constraints defined over ellipsoidal uncertainty sets result in semidefinite programs. We refer the reader to [40] and [41] for a more extensive discussion surrounding the tractable reformulation of robust constraints.

\(^2\)It is important to note that we have modified the notation used to denote the sample $(\bar{u}_1, \ldots, \bar{u}_m)$ in order to distinguish its specification from the training sample $(\bar{u}_1, \ldots, \bar{u}_m)$ that is subsequently used to estimate the $\alpha$-covering level.

\(^3\)As matter of notational convenience, that we have suppressed the dependency of the uncertainty set $\mathcal{U}_\phi(r)$ on $m$, as it will be considered fixed for the majority of the discussion that follows.

C. Comparison to Scenario Approximation Methods

The approach considered here gives rise to optimization problems that are similar in structure to those based on scenario approximations of chance constrained problems and their distributionally robust generalizations. Specifically, when the radius $r$ of the balls composing the uncertainty sets is equal to zero we recover the standard scenario approximations to chance constrained problems such as those presented in [22], [23]. When $r > 0$, we recover the robust scenario approximation to distributionally robust chance constrained problems studied in [24], [25]. Interestingly, the latter connection suggests that the class of approximations considered in this paper also possess an intrinsic distributional robustness—a point that merits further examination as part of future research.

Although the class of approximations that we propose are similar in structure to those based on scenario approximation methods, our method offers an important computational advantage in terms of the number of constraints that must be enforced in the resulting approximation to the original chance constraint. Specifically, while scenario approximation methods result in approximations based on a number of sample-based constraints that is at least $O\left(\frac{1}{r^2} \ln \left( \frac{1}{\delta} \right) \right)$, the method proposed in this paper treats the number of constraints $m$ in the resulting robust approximation as a parameter that can be set by the user. Importantly, our theoretical guarantees hold for any value of $m$.

It is also important to note that the computational advantages of our approach as compared to scenario-based methods are accompanied by several drawbacks. First, while scenario approximations can be applied to arbitrary functions that are convex in the decision variable $x$, we impose additional restrictions on the class of constraint functions that can be handled by our method. Secondly, our results rely on the assumption of continuity of the distribution according to which the random vector is distributed, while the scenario approximations place no restrictions on this distribution.

III. Finite-Sample Statistical Guarantees

We now establish a bound on the rate at which the probability mass of the proposed class of uncertainty sets converges to the target mass as a function of the training sample size $n$. To lighten the notation in the following discussion, we denote the probability mass of the uncertainty set $\mathcal{U}_\phi(r)$ conditioned on a level $r$ by
\[ \pi_\phi(r) := \Pr (u \in \mathcal{U}_\phi(r)) . \] \hspace{1cm} (7)
It is also important to note that, throughout this section, we treat the shape function $\phi$ as being fixed. Thus, all subsequent statements should be interpreted as being conditional on $\phi$.

Before stating the main result of this section, we require a technical assumption that is assumed to hold throughout the remainder of the paper.

Assumption 1: The transformed random variable $\zeta = \phi(u)$ is assumed to have a continuous distribution.

In particular, Assumption 1 is satisfied under the shape function $\phi$ given by (5) if the underlying random vector...
Let \( u \) be continuous. With Assumption 1 in hand, we have the following result.

**Theorem 1:** Let \( \varepsilon \in (0, 1 - \alpha) \) and \( \alpha_n \in (\alpha, \alpha + \varepsilon) \) for all \( n \geq 1 \). It follows that
\[
\Pr \left( \pi \left( F^{-1} (\alpha) \right) < \alpha \right) \leq \exp \left( -\frac{n(\alpha - \alpha_n)^2}{2(1 - \varepsilon)} \right)
\] (8)
and
\[
\Pr \left( \pi \left( F^{-1} (\alpha_n) \right) > \alpha + \varepsilon \right) \leq \exp \left( -\frac{n(\alpha + \varepsilon - \alpha_n)^2}{2(\alpha + \varepsilon)} \right).
\] (9)

It follows that, for any fixed tolerance \( \varepsilon \in (0, 1 - \alpha) \), the probability that the mass covered by the uncertainty set \( \mathcal{U}_\varepsilon (F^{-1} (\alpha_n)) \) is within \( \varepsilon \) of (and no less than) the target mass \( \alpha \) approaches one at an exponential rate in the sample size. One may also use this result to establish conditions on the sequence \( \alpha_n \) guaranteeing that \( \pi \left( F^{-1} (\alpha_n) \right) \rightarrow \alpha \) in probability as the sample size \( n \) tends to infinity. We note that the proof of Theorem 1 is largely standard. It involves the reformation of the error probabilities (9) and (8) as binomial tail probabilities, which are then bounded from above using Chernoff's inequality.

**Proof:** We first prove inequality (8). First notice that, conditioned on a level \( r \), it holds that \( \pi(u) = \Pr(\phi(u) \leq r) = F(r) \), where \( F \) denotes the cumulative distribution function of the transformed random variable \( \zeta := \phi(u) \). This identity, combined with the assumed continuity of the distribution \( F \), implies that
\[
\Pr \left( \pi \left( F^{-1} (\alpha_n) \right) < \alpha \right) = \Pr \left( F^{-1} (\alpha_n) < F^{-1} (\alpha) \right).
\]


The empirical quantile function can be expressed in terms of the order statistics of the transformed training sample, which we denote by \( \zeta(1) \leq \zeta(2) \leq \cdots \leq \zeta(n) \). Specifically, it holds that
\[
F^{-1} (\gamma) = \zeta(\lceil n\gamma \rceil)
\] (10)
for all \( \gamma \in (0, 1) \). Importantly, the cumulative distribution function of the \( k \)-th order statistic can be expressed according to the upper tail of a binomial:
\[
\Pr \left( \zeta(k) \leq z \right) = \sum_{i=k}^{n} \binom{n}{i} F(z)^i (1 - F(z))^{n-i}
\] (11)
for each \( k = 1, \ldots, n \). It follows that
\[
\Pr \left( \pi \left( F^{-1} (\alpha_n) \right) < F^{-1} (\alpha) \right) = \Pr \left( \zeta(\lceil n\alpha_n \rceil) < F^{-1} (\alpha) \right) = \sum_{i=\lceil n\alpha_n \rceil}^{n} \binom{n}{i} \alpha^i (1 - \alpha)^{n-i}
\] (12)
and
\[
= \sum_{i=0}^{\lceil n\alpha_n \rceil} \binom{n}{i} (1 - \alpha)^i \alpha^{n-i}.
\] (13)

The first equality follows from (10). The second equality follows from (11) and Assumption 1 which implies that \( F(F^{-1} (\alpha)) = \alpha \). The third equality stems from an equivalent reformulation of the upper binomial tail [12] as a lower binomial tail. One can bound (13) from above using the following well-known upper bound on the lower tail of a binomial [43][Theorem 4.5], which is a direct consequence of the Chernoff bound.

**Theorem 2:** Let \( \xi \) be a binomial random variable with parameters \( n \in \mathbb{N} \) and \( p \in [0, 1] \). For \( k \leq np \), it holds that
\[
\Pr (\xi \leq k) \leq \exp \left( -\frac{(np - k)^2}{2np} \right).
\]

Inequality (8) follows from an application of Theorem 2 to Corollary 1, which is a direct consequence of [43][Theorem 4.5], where we use the fact that \( n - \lfloor n\alpha \rfloor \leq n(1 - \alpha_n) \).

The proof of inequality (9) is analogous in nature to the proof of (8). Using similar arguments, it is possible to show that
\[
\Pr \left( \pi \left( F^{-1} (\alpha_n) \right) > \alpha + \varepsilon \right) = \sum_{i=0}^{\lfloor n\alpha_n \rfloor - 1} \binom{n}{i} (\alpha + \varepsilon)^i (1 - \alpha - \varepsilon)^{n-i}.
\] (14)

Inequality (9) follows from an application of Theorem 2 to (14), where we use the fact that \( \lfloor n\alpha_n \rfloor - 1 \leq n\alpha_n \).

It is also possible to use Theorem 1 to characterize a distribution-free bound on the sample size requirement ensuring that the probability mass covered by the uncertainty set satisfies the given tolerance with a given confidence \( 1 - \delta \). We state the following corollary without proof, as it is an immediate consequence of Theorem 1.

**Corollary 1:** Let \( \delta \in (0, 1) \), \( \varepsilon \in (0, 1 - \alpha) \), and \( \lambda \in (0, 1) \). Set \( \alpha_n = \alpha + \lambda \varepsilon \) for all \( n \geq 1 \). If
\[
\frac{n}{\lambda} \geq c(\lambda, \alpha, \varepsilon) \left( \frac{2}{\varepsilon^2} \right) \ln \left( \frac{2}{\delta} \right),
\] (15)
where \( c(\lambda, \alpha, \varepsilon) := \max \left\{ (1 - \alpha) / \lambda^2, (\alpha + \varepsilon) / (1 - \lambda)^2 \right\} \), then
\[
\Pr \left( \alpha \leq \pi \left( F^{-1} (\alpha_n) \right) \leq \alpha + \varepsilon \right) \geq 1 - \delta.
\]

It is important to note that the sample size requirement (15) is dimension-free in that it does not depend on the dimension of the random vector or that of the decision variable. Additionally, for \( \alpha \in (1/2, 1) \), it is straightforward to show that the value of \( \lambda \) which minimizes the sample size requirement (15) is given by
\[
\lambda^* = \frac{1 - \alpha - \sqrt{(1 - \alpha) (\alpha + \varepsilon)}}{1 - 2\alpha - \varepsilon}.
\] (16)

We also remark that it is possible to improve upon the sample size requirement (15) through a refinement of the upper bounds on the lower binomial tails [13]-[14], using arguments analogous to those in [44]. In particular, the dependence on the tolerance parameter \( \varepsilon \) can be improved to \( O(1/\varepsilon) \).

**IV. EXPERIMENTS**

In this section, we present numerical experiments illustrating certain features and properties of the proposed uncertainty set estimator. Throughout these experiments, we consider an \( \mathbb{R}^2 \)-valued random vector distributed according to a Gaussian mixture with two components. Its probability density function is depicted in Figure 14.
A. Examining the Role of \( m \)

Figure 1 illustrates the impact of the number of balls \( m \) on the shape and volume of the uncertainty sets produced by our method. In this experiment, we consider a fixed probability mass \( \alpha = 0.9 \), tolerance parameter \( \varepsilon = 0.05 \), and confidence parameter \( \delta = 0.05 \). In specifying the uncertainty set estimator, we set \( \alpha_n = \alpha + \lambda^* \varepsilon \) for all \( n \), where \( \lambda^* \) is specified according to (10). For each value of \( m \in \{1, 10, 10^2, 10^3\} \), we construct an uncertainty set \( \mathcal{U}_\phi(F_n^{-1}(\alpha_n)) \) according to (6) using a training sample with size satisfying the bound (15). As expected, Figures 1b-1e show that the proposed quantile estimate for the empirical quantile \( \frac{t}{n} \) is robust to the number of balls used in the approximation. And, on balance, the volume of the uncertainty sets appears to shrink with \( m \), resulting in a potential decrease in the conservativeness of the approximation to the original chance constrained problem. It is important to note, however, that as \( m \) increases, the volume of outliers also appears to increase, potentially resulting in increased conservativeness of the approximation. Thus, from a practical perspective, the number of balls used in the approximation should be tuned to balance this tradeoff.

B. Illustrating Mass Consistency

In Figure 2, we conduct a Monte-Carlo analysis to illustrate the mass consistency of the proposed uncertainty set estimator by varying the tolerance parameter value. Throughout these experiments, we fix the number of balls to \( m = 10 \) and consider a target probability mass of \( \alpha = 0.9 \). We vary the tolerance parameter on a logarithmic scale between 0.005 and 0.05, and set \( \alpha_n = \alpha + \lambda^* \varepsilon \), where \( \lambda^* \) is defined according to (16). For each value of the tolerance parameter, we chose the number of training samples to be the smallest integer satisfying the sample size requirement (15).

The numerical experiments are conducted as follows for each value of the tolerance parameter \( \varepsilon \). First, we generate a random sample \( \mathbf{u}_1, \ldots, \mathbf{u}_m \) that determines the centers of the balls composing the uncertainty set that we construct. For each experiment, we draw an i.i.d. training sample to evaluate \( F_n^{-1}(\alpha_n) \) according to (4) and \( \mathcal{U}_\phi(F_n^{-1}(\alpha_n)) \) according to (6). We then estimate the probability mass covered by each uncertainty set \( \mathcal{U}_\phi(F_n^{-1}(\alpha_n)) \) using an empirical average based on one million independent samples of the random vector \( \mathbf{u} \). We estimate empirical confidence intervals associated with \( \pi_\phi(F_n^{-1}(\alpha_n)) \) using one thousand independent experiments. Figure 2 depicts the middle 90% empirical confidence interval associated with the probability mass covered by the uncertainty sets for each value of \( \varepsilon \). Notice that, consistent with the result in Theorem 1, the fifth percentile of the probability mass covered by the uncertainty set remains above the target probability mass. Furthermore, the mass captured by the uncertainty set approaches the target probability mass with high confidence as the tolerance parameter value decreases to zero.

V. Conclusion

We provide a data-driven method to construct uncertainty sets for robust optimization problems. The estimators we consider are consistent, satisfy finite-sample performance guarantees, are efficient to compute, and give rise to tractable robust constraints for a large family of constraint functions. Furthermore, the the proposed method provides the user with two mechanisms by which to control the complexity of the resulting robust optimization problem. First, the user can directly control the number of constraints in the resulting approximation while preserving the conservativeness of the approximation with respect to the original chance constraint. Second, the geometry of the uncertainty sets can be tailored to accommodate the structure of the given constraint function to ensure the computational tractability of the robust approximation. As a direction for future research, it would be interesting to refine our sample complexity results to explicitly reflect the role of the number of balls used in the approximation in determining the volume of the sets learned.
