VECTOR VALUED THETA FUNCTIONS ASSOCIATED WITH
BINARY QUADRATIC FORMS

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Abstract. We study the space of vector valued theta functions for the Weil representation of a positive definite even lattice of rank two with fundamental discriminant. We work out the relation of this space to the corresponding scalar valued theta functions of weight one and determine an orthogonal basis with respect to the Petersson inner product. Moreover, we give an explicit formula for the Petersson norms of the elements of this basis.

1. Introduction and statement of results

Integral binary quadratic forms and the automorphic properties of their theta functions are well known. It is the purpose of the present note to describe the related space of vector valued theta functions transforming with the Weil representation.

Let $P$ be an even positive-definite lattice of rank 2 with quadratic form $Q$. For simplicity, we assume that the discriminant $D < 0$ of $Q$ is a fundamental discriminant. The theta function attached to $P$ is a holomorphic modular form of weight 1 and transforms with the Weil representation $\rho_P$ of $\text{SL}_2(\mathbb{Z})$ (see Section 3.1). In fact, there is a family of theta functions attached to $P$ that have the same weight and transformation behaviour. These theta functions essentially correspond to the lattices in the genus of $P$.

Let $U = P \otimes \mathbb{Z} \mathbb{Q}$ be the corresponding rational quadratic space containing these lattices. The general spin group $T(\mathbb{A}_f) = G\text{Spin}_U(\mathbb{A}_f)$, a central extension of the special orthogonal group, acts transitively on the lattices in the genus of $P$.

We describe this action in detail in Section 2.2. We let $K \subset T(\mathbb{A}_f)$ be an open compact subgroup that preserves $P$ and acts trivially on $P'/P$, where $P'$ is the dual lattice of $P$. Consider the class group $\text{Cl}(K) = H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K$ and define a theta function on the product of the complex upper half-plane $\mathbb{H}$ and $\text{Cl}(K)$ as

$$\Theta_P(\tau, h) = \sum_{\beta \in P'/P} \sum_{\lambda \in h(P + \beta)} e(Q(\lambda)\tau) e_\beta,$$

where $e(x) = e^{2\pi i x}$ and $e_\beta$ denotes the standard basis element of the group ring $\mathbb{C}[P'/P]$ corresponding to $\beta \in L'/L$. We will frequently write $e_0$ for $e_{0+P}$. We define the space $\Theta(P)$ of theta functions associated with $P$ to be the complex vector space generated by the forms $\Theta_P(\tau, h)$ for $h \in \text{Cl}(K)$. It is a subspace of $M_{1,P}$, the space of modular forms of weight 1 and representation $\rho_P$. Recall the definition of the Petersson inner product $(f, g)$, where $f, g$ are

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both modular forms of the same weight \( k \) and representation \( \rho_P \) as
\[
(f, g) = \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}} \langle f(\tau), g(\tau) \rangle e^{ku^2} \frac{dudv}{v^2},
\]
where \( \tau = u + iv \) with \( u, v \in \mathbb{R} \) and \( \langle \cdot, \cdot \rangle \) denotes the bilinear pairing on \( \mathbb{C}[P'/P] \), such that \( \langle \epsilon_\mu, \epsilon_\nu \rangle = \delta_{\mu, \nu} \). The integral converges if at least one of \( f \) and \( g \) is a cusp form.

It is useful to consider the following linear combinations of theta functions in \( \Theta(P) \). Let \( \psi \) be a character of \( \text{Cl}(K) \). We let
\[
\Theta_P(\tau, \psi) = \sum_{h \in \text{Cl}(K)} \psi(h) \Theta_P(\tau, h).
\]

Our focus lies on lattices of the following form. Let \( P = a \) be a fractional ideal in the imaginary quadratic field \( k \) of discriminant \( D < 0 \) with quadratic form \( \frac{N(x)}{N(a)} \). Moreover, let \( K = \hat{O}_D = \left( \mathcal{O}_D \otimes \mathbb{Z} \hat{\mathbb{Z}} \right)^\times \), where \( \mathcal{O}_D \) is the ring of integers of \( k \) and \( \mathbb{Z} \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \). In this case, the group \( \text{Cl}(K) \) is isomorphic to the class group \( \text{Cl}_k \) of \( k \).

**Theorem 1.1.** For \( P \) and \( K \) as above we have:

(i) If \( \psi = 1 \), then \( \Theta_P(\tau, \psi) = E_P(\tau) \) is an Eisenstein series, spanning the space of Eisenstein series of weight one and representation \( \rho_P \).

(ii) If \( \psi \neq 1 \), then \( \Theta_P(\tau, \psi) \) is a cusp form.

(iii) Choose a system \( C \) of representatives of characters on \( \text{Cl}_k \) modulo complex conjugation. Then the set
\[
\mathcal{B}(P) = \{ \Theta_P(\tau, \psi) \mid \psi \in C \}
\]
is an orthogonal basis for \( \Theta(P) \).

(iv) In particular, the dimension of the space \( \Theta(P) \) is equal to
\[
\dim \Theta(P) = \frac{h_k + 2^{t-1}}{2},
\]
where \( h_k \) is the class number of \( k \) and \( t \) is the number of prime divisors of \( D \).

**Remark 1.2.** Note that the set \( \mathcal{B}(P) \) does depend on the choice of representatives, but only up to scalar factors.

We also give an explicit formula for the Petersson inner products of these basis elements in terms of special values of Dedekinds \( \eta \)-function. Recall that given an ideal \( b \) of \( k \) which corresponds to the binary quadratic form \([a, b, c]\), there is a CM point given by the unique root of the polynomial \( a\tau^2 + b\tau + c \) that lies in \( \mathbb{H} \). We write \( \tau(b) = u(a) + iv(a) \in \mathbb{H} \) for this point.

**Proposition 1.3.** Let \( \chi \) and \( \psi \) be characters of \( \text{Cl}(K) \), not both trivial. With the assumptions of Theorem 1.1, the following holds.

(i) We have
\[
(\Theta_P(\tau, \psi), \Theta_P(\tau, \chi)) = 0
\]
unless \( \psi = \bar{\chi} \) or \( \psi = \chi \).

(ii) If \( \psi^2 \neq 1 \) and \( \psi = \bar{\chi} \), we obtain
\[
(\Theta_P(\tau, \psi), \Theta_P(\tau, \bar{\psi})) = -\psi(a)h_k \sum_{b \in \text{Cl}_k} \psi(b) \log |v(b)\eta^4(\tau(b))|,
\]
(iii) and if $\psi^2 \neq 1$ but $\psi = \chi$, we have

$$ (\Theta_P(\tau, \psi), \Theta_P(\tau, \psi)) = -h_k \sum_{b \in \text{Cl}_k} \psi(b) \log |v(b)\eta^4(\tau(b))|.$$ 

(iv) If $\psi = \chi$ and $\psi^2 = \chi^2 = 1$, the result is the sum of these two expressions.

The analogous formula is well known in the scalar valued case (see Corollary 6.10). However, our proof is very different from the classical one that uses Kronecker’s limit formula (see Proposition 3.1 in [DL15] for a proof). We prove the formulas by using a certain seesaw identity and expressing the Petersson inner products as CM values of a regularized theta lift. This principle in fact generalizes to arbitrary dimensions, which will be the subject of a sequel to this article.

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2. Shimura varieties for quadratic spaces of type $(2,0)$

We let $k = k_D = \mathbb{Q}(\sqrt{D})$ be the imaginary quadratic field of discriminant $D$ and we write $\text{Cl}_k$ for the ideal class group of $k$. We let $\mathcal{O}_D \subset k$ be the ring of integers in $k$. We write $\mathbb{A}_k$ for the ring of adeles over $k$. Recall that idele class group of $k$ is defined as the quotient

$$I_k = k^\times \backslash \mathbb{A}_k^\times.$$ 

Here, $k^\times$ is embedded diagonally into $\mathbb{A}_k^\times$ and the elements of the subgroup $k^\times$ are called principal ideles. We also write

$$I_k = k^\times \backslash k_{k,f}^\times$$

for the finite idele class group.

Theorem 2.1 (VI. Satz 1.3, [Neu07]). We have a surjective homomorphism

$$I_k \rightarrow \text{Cl}_k, \quad (\alpha_p)_p \mapsto \prod_{p|\infty} \mathfrak{p}^{p_\mathfrak{p}((\alpha))},$$

inducing an isomorphism

$$I_k/\hat{\mathcal{O}}_D^\times = k^\times \backslash \text{Cl}_k = \hat{\mathcal{O}}_D^\times \cong \text{Cl}_k,$$

where $\hat{\mathcal{O}}_D^\times$ is the subgroup

$$\hat{\mathcal{O}}_D^\times = \prod_{p|\infty} \mathcal{O}_p^\times.$$

2.1. Binary quadratic forms and ideals. Recall that a fractional ideal $\mathfrak{a}$ of $k$ defines an integral binary quadratic form in the following way. If $\mathfrak{a}$ is generated as a $\mathbb{Z}$-module by two elements

$$\mathfrak{a} = (\alpha, \beta) = \mathbb{Z}\alpha + \mathbb{Z}\beta,$$

then

$$Q_\mathfrak{a}(x, y) = \frac{N(\alpha)}{N(\mathfrak{a})}x^2 + \frac{\text{tr}(\alpha\beta)}{N(\mathfrak{a})}xy + \frac{N(\beta)}{N(\mathfrak{a})}y^2 = \frac{N(x\alpha + y\beta)}{N(\mathfrak{a})}$$

is an integral binary quadratic form of discriminant $D$. This induces a bijective correspondence between equivalence classes of positive definite integral binary quadratic forms of discriminant
$D$ and the class group $\text{Cl}_k$ of $k$ (if we also restrict to oriented bases). A good reference for this correspondence is [Zag81].

2.2. The action of $\text{GSpin}_U(\mathbb{A}_f)$. In this section, we let $U = a \otimes \mathbb{Z} \mathbb{Q}$ for a fractional ideal $a$ of $k$ and we view $U$ simply as a 2-dimensional rational quadratic space with quadratic form $Q(x) = N(x)/N(a)$. We write $T = \text{GSpin}_U$. Over $\mathbb{Q}$ we have that $C_U^0 \cong \mathbb{Q}(\sqrt{-|\text{det}(U)|})$, the even part of the Clifford algebra, is isomorphic to $k$. The Clifford norm corresponds to the norm $N(x) = x\bar{x}$ in $k$. Here, $\bar{x}$ denotes complex conjugation. Moreover, the group $SO_U(\mathbb{Q})$ is isomorphic to

$$k^1 = \{x \in k \mid N(x) = 1\}$$

and $T(\mathbb{Q}) \cong k^\times$ is the multiplicative group of $k$.

Under this identification the map $T(\mathbb{Q}) \mapsto SO_U(\mathbb{Q})$ is given by $x \mapsto x/\bar{x}$. This is essentially Hilbert’s theorem 90 but can also be seen directly by a short calculation using the definition of $k$. To see this, we consider the orthogonal basis $\{v_1 = N(a), v_2 = -\sqrt{D}\}$ of $k$ as a vector space over $\mathbb{Q}$, where $D$ is the discriminant of $k$. We have $Q(v_1) = N(a)$ and $Q(v_2) = N(\sqrt{D})/N(a) = -D/N(a)$. The even Clifford algebra $C_U^0$ is generated (as a $\mathbb{Q}$-algebra) by 1 and $\delta = v_1v_2$. Note that $\delta^2 = D$.

The group $\text{GSpin}_U$ is given by all non-zero elements in $C_U^0$ in our case. By definition, an element $a + b\delta \in \text{GSpin}_U$ acts on $x \in k = U$ via

$$(a + b\delta) \cdot x \cdot (a + b\delta)^{-1}$$

where the multiplication is in the Clifford algebra $C_U$. It is enough to compute this on the basis vectors $v_1, v_2$ of $k$. It is easy to see that $\delta v_j^{-1} = -v_j^{-1}\delta$ and we obtain

$$(a + b\delta) \cdot v_j \cdot (a + b\delta)^{-1} = (a + b\delta) \cdot ((a + b\delta)v_j^{-1})^{-1} = (a + b\delta) \cdot (a - b\delta)^{-1} \cdot v_j.$$  

The element $x = (a + b\delta) \cdot (a - b\delta)^{-1}$ is contained in $k^\times$. The isomorphism $k^\times \cong T(\mathbb{Q})$ is explicitly given via $a + b\sqrt{D} \mapsto a + b\delta$. Under this identification, the action of $x \in k^\times \cong \text{GSpin}_U$ on $k$ is given by multiplication with $x/\bar{x}$.

Using this, we see that $T(\mathbb{A}_f) \cong \mathbb{A}_{k,f}^\times$ is isomorphic to the multiplicative group of ideles over $k$. To avoid confusion, in this section we write $h.x$ for the action of $h \in T(\mathbb{A}_f)$ on $x$ and simply $hx$ for multiplication of adeles. Recall that the group $T(\mathbb{A}_f) = \text{GSpin}_U(\mathbb{A}_f)$ acts on lattices in $U$. If $h = (h_p)_p \in \text{GSpin}_U(\mathbb{A}_f)$ and $L = \hat{L} \cap V(\mathbb{Q})$ is a lattice in $V$, then $h.L = (h.\hat{L}) \cap V(\mathbb{Q}) = \prod_p (h_p.\hat{L}_p)_p \cap U(\mathbb{Q})$.

In the following, we will examine the action of $T(\mathbb{A}_f)$ on lattices in $U$ more closely. It is important to note that this action is different form the “natural” action on ideals (or lattices) in $k$. Recall that this natural action is simply given by the linear action of $\mathbb{Q}_p^\times$ on $k \otimes \mathbb{Q}_p$.

The $\mathbb{Q}_p$ vector space $k \otimes \mathbb{Q}_p$ is an algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$, isomorphic to $C_U^0(\mathbb{Q}_p)$. It is also isomorphic [Neu07, II, Theorem 8.3] to the product

$$(2.1) \quad \prod_{p \mid p} k_p,$$

where the product is over all prime ideals $p$ of $k$ that lie above $p$.

For our purposes, it is enough to consider a lattice given by a fractional ideal $a \subset k$. Then the action of $x \in T(\mathbb{A}_f)$ is given as follows.
We write $\pi_p \in \mathcal{O}_p$ for a uniformizer in $\mathcal{O}_p$. This means that the only prime ideal in $\mathcal{O}_p$ is generated by $\pi_p$ and every element in $k_p$ can be written as $\pi_p^m u$, where $m \in \mathbb{Z}$ and $u \in \mathcal{O}_p^\times$. We can write
\[
a = \prod_p p^{\nu_p(a)} = (\pi_p^{\nu_p(a)})_p \cap k,
\]
where we view $k$ as diagonally embedded into $\mathbb{A}_{k,f}$.

**Lemma 2.2.** Let $h = (h_p)_p \in T(\mathbb{A}_f)$. Then we have
\[
h \cdot a = \prod_p p^{\nu_p(a) + \mu_p(h)},
\]
where
\[
\mu_p(h) = \begin{cases} 
0, & \text{if } p = \mathfrak{p} \\
v_p(h_p) - v_p(h_p), & \text{otherwise}.
\end{cases}
\]

**Proof.** For primes $p$ with $p = \mathfrak{p}$, that is for inert and ramified primes, the action of $T(\mathbb{Q}_p)$ does not change the valuation $v_p$. In those cases $h_p \in T(\mathbb{Q}_p) \cong k_p^\times$ acts by multiplication with $h_p/\bar{h}_p$, where $\bar{h}_p$ denotes the image of $h_p$ under the non-trivial Galois automorphism of the extension $k_p/\mathbb{Q}_p$.

If the rational prime $p$ however splits in $k$ as $p\mathcal{O}_D = \mathfrak{p}\mathfrak{p}$, the action is necessarily slightly different. We have
\[
k \otimes \mathbb{Q}_p \cong k_p \times k_p \cong \mathbb{Q}_p^2,
\]
as in Eq. (2.1). The isomorphism is given explicitly as follows. Let $\mathfrak{d} \in \mathcal{O}_p$ with $\mathfrak{d}^2 = D$. Such a square-root exists because $p$ is split in $k$ and therefore $D$ is a square modulo $p$. Then the isomorphism $k \otimes \mathbb{Q}_p \cong \mathbb{Q}_p^2$ is realized by
\[
(a + b\sqrt{D}) \otimes c \mapsto ((a + b\mathfrak{d})c, (a - b\mathfrak{d})c) \in \mathbb{Q}_p^2.
\]
Therefore,
\[
T(\mathbb{Q}_p) \cong (k \otimes \mathbb{Q}_p)^\times \cong k_p^\times \times k_p^\times \cong \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times.
\]
Using the same arguments as over $\mathbb{Q}$, we see that an element $x \otimes c \in T(\mathbb{Q}_p)$ acts by multiplication with $x/\bar{x} \otimes 1$. Therefore, if $x = a + b\sqrt{D}$, then this corresponds to multiplication with
\[
\begin{pmatrix} a + b\mathfrak{d} & a - b\mathfrak{d} \\ a - b\mathfrak{d} & a + b\mathfrak{d} \end{pmatrix},
\]
giving the formula in the lemma. \hfill \square

In particular, we see that the action of $T(\mathbb{A}_f)$ on lattices in $U$ is really fundamentally different from multiplication in the class group. We denote the class of $h$ under the surjective map $\mathbb{A}_{k,f} \to \text{Cl}_k$ in Theorem 2.1 by $[h]$. From the formulas above, we see that the action of $T(\mathbb{A}_f)$ on the class $[a]$ of $a$ corresponds to multiplication by the class $[h]/[\bar{h}]$. Here, $[\bar{h}]$ denotes the complex conjugate class of $[h]$. Note that $[h]/[\bar{h}] = [h]^2$ since in an imaginary quadratic field the ideal $\mathfrak{pp}$ is a principal ideal for all prime ideals $p \subset \mathcal{O}_D$. (It is either generated by $p = N(p)$ or by $p^2$.)

Therefore, the class $[h,a] \in \text{Cl}_k$ is given by $[h,a] = [h]^2[a]$, in accordance with the fact that $\text{GSpin}_U(\mathbb{A}_f)$ acts on lattices in the same genus.
3. Petersson inner products of theta functions

3.1. Regularized theta lifts. We briefly recall Borcherds’ regularized theta lift [Bor98]. We refer to the literature for details [Kud03, Bor98, Bru02, Ehl13].

Let \( V \) be a rational quadratic space with quadratic form \( Q \) of signature \((b^+,b^-)\) and let \( H = \text{GSpin}_V \). We write \((x,y) = Q(x+y) - Q(x) - Q(y)\) for the associated bilinear form. Let \( L \subset V \) be an even lattice and denote by \( \Theta_L(\tau,z,h) \) the Siegel theta function associated with \( L \). For an appropriate choice of an open compact subgroup \( K \subset H(\mathbb{A}_f) \), it is a function in \((z,h)\) on the Shimura variety with complex points

\[
X_K(\mathbb{C}) = H(\mathbb{Q}) \backslash (\mathbb{D} \times H(\mathbb{A}_f)/K),
\]

where \( \mathbb{D} \) is the symmetric space attached to \( V \). Assume that the signature \( b^+ - b^- \) is even. Recall that \( \text{SL}_2(\mathbb{Z}) \) is generated by

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

There is a unitary representation \( \rho_L \) of \( \text{SL}_2(\mathbb{Z}) \) on the group ring \( \mathbb{C}[L'/L] \), called the Weil representation. The action of \( \rho_L \) is defined as follows:

\[
\rho_L(T)\epsilon_\mu = e(Q(\mu))\epsilon_\mu,
\]

\[
\rho_A(S)\epsilon_\mu = \frac{e((b^+ - b^-)/8)}{\sqrt{|L'/L|}} \sum_{\nu \in L'/L} e(- (\mu,\nu))\epsilon_\nu.
\]

We write \( M_{k,L} \) for the complex vector space of modular forms of weight \( k \) and representation \( \rho_L \). Moreover, cusp forms are denoted \( S_{k,L} \) and weakly holomorphic modular forms (which are allowed to have a pole at the cusp at \( \infty \)) by \( M_{k,L}^! \). Put \( k = b^+ - b^- \). As a function of \( \tau \), the theta function \( \Theta_L(\tau,z,h) \) is a vector valued (non-holomorphic unless \( L \) is positive definite) modular form of weight \( k \) and representation \( \rho_L \).

Denote by \( \mathcal{F} := \{ \tau \in \mathbb{H}; |\tau| \geq 1, -1/2 \leq \Re(\tau) \leq 1/2 \} \) the standard fundamental domain for the action of \( \text{SL}_2(\mathbb{Z}) \) and let \( \mathcal{F}_T := \{ \tau \in \mathcal{F}; \Im(\tau) \leq T \} \). Here and throughout, we write \( d\mu(\tau) = dudv/uv^2 \) for \( \tau = u + iv \in \text{uhp} \). For a vector valued modular form \( f \in M_{k,L}^! \), let

\[
\Phi_L(z,h,f) = \int_{\mathcal{F}} \sum_{i=0}^{\text{reg}} \langle f(\tau), \Theta_L(\tau,z,h) \rangle v^k d\mu(\tau) := \text{CT} \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle f(\tau), \Theta_L(\tau,z,h) \rangle v^{k-s} d\mu(\tau).
\]

Here, \( \text{CT} \) denotes the constant term in the Laurent expansion at \( s = 0 \) of the meromorphic continuation of the function in brackets defined by the limit.

3.2. Special values of a theta lift and inner products. We will now obtain an explicit expression for the Petersson inner products of the cusp forms contained in \( \Theta(P) \). We will utilize a seesaw identity that relates these inner products to special values of the Borcherds lift for \( \text{O}(2,2) \).

Suppose that we are given a lattice \( P \) of signature \((2,0)\) that corresponds to the integral binary quadratic form \([A,B,C]\) of negative fundamental discriminant \( D \equiv 1 \text{ mod } 4 \). Equivalently, \( P \) corresponds to an integral ideal \( a \subset \mathcal{O}_D \) generated by \( A \) and \((B + \sqrt{D})/2 \). Here, \( \mathcal{O}_D \subset k \) is the ring of integers in \( k = \mathbb{Q}(\sqrt{D}) \).

The lattice \( P \oplus P^- \) has type \((2,2)\) and level \(|D|\). We write \( P^- \) for the lattice given by \( P \) together with the negative of the quadratic form. The discriminant group has order \( D^2 \). We
take a \(\mathbb{Z}\)-basis \(\{p_1, p_2\}\) of \(P\) with \(Q(p_1) = A, Q(p_2) = C\) and bilinear form \((p_1, p_2) = B\). We use the same basis for \(P^-\). The starting point is the following embedding.

Consider the even unimodular lattice \(L = M_2(\mathbb{Z})\) with the quadratic form given by \(Q(X) = -\det(X)\). The bilinear form is

\[
(X, Y) = -\text{tr}(XY^*), \quad \text{where} \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)^* = \left(\begin{array}{cc} d & -b \\ -c & a \end{array}\right)
\]

and the type of \(L\) is \((2, 2)\).

The symmetric domain \(\mathbb{D}\) attached to \(H = \text{GSpin}_V\) can be identified with \(\mathbb{H}^2 \cup \overline{\mathbb{H}}^2\) in this case via

\[
(3.1) \quad (z_1, z_2) \mapsto \mathbb{R}\Re\left(\begin{array}{cc} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{array}\right) \oplus \mathbb{R}\Im\left(\begin{array}{cc} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{array}\right).
\]

**Lemma 3.1.** Under this identification, the group \(H = \text{GSpin}_V\) for \(V = L \otimes \mathbb{Q}\) can be identified with the subgroup \(G\) of \(\text{GL}_2 \times \text{GL}_2\) defined by

\[
G = \{(g_1, g_2) \in \text{GL}_2 \times \text{GL}_2 \mid \det g_1 = \det g_2\},
\]

which acts on \(\mathbb{D}\) via fractional linear transformations in both components. The corresponding action of \((g_1, g_2) \in H\) on \(x \in M_2(\mathbb{Q})\) is given by

\[
(g_1, g_2).x = g_1 x g_2^{-1}.
\]

**Proof.** Consider the orthogonal basis

\[
v_0 = \left(\begin{array}{c} 1 \\ 0 \end{array}\right) = I_2, \quad v_1 = \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \quad v_2 = \left(\begin{array}{c} 0 \\ 1 \end{array}\right), \quad v_3 = \left(\begin{array}{c} 0 \\ -1 \end{array}\right).
\]

According to Example 2.10 in the second contribution to [BvdGHZ08], we have that the center \(Z(C^0_V)\) of the even Clifford algebra is given by \(\mathbb{Q} \oplus \mathbb{Q}\) and

\[
C^0_V = Z + Z v_1 v_2 + Z v_2 v_3 + Z v_1 v_3.
\]

We obtain an isomorphism

\[
C^0_V \cong M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q})
\]

via

\[
1 \mapsto (I_2, I_2), \quad v_i v_j \mapsto (v_i v_j^*, v_i^* v_j).
\]

Under this isomorphism, the canonical involution of \(C_V\) corresponds to

\[
(A, B) \mapsto (A^*, B^*)
\]

and the Clifford norm is given by

\[
N(A, B) = (\det(A) I_2, \det(B) I_2).
\]

Therefore, \(N(A, B) \in \mathbb{Q}^\times\) is equivalent to \(A, B \in \text{GL}_2(\mathbb{Q})\) with \(\det(A) = \det(B)\).

It is straightforward to check that under the identification Eq. (3.1), the action of \(H\) corresponds to fractional linear transformations.

We let \(K = H(\hat{\mathbb{Z}})\), that is

\[
K = H(\hat{\mathbb{Z}}) = \{(g_1, g_2) \in \text{GL}_2(\hat{\mathbb{Z}}) \times \text{GL}_2(\hat{\mathbb{Z}}) \mid \det g_1 = \det g_2 \in \hat{\mathbb{Z}}\}.
\]

It is clear that \(K\) preserves \(L\). By strong approximation and the theory of Shimura varieties, the associated Shimura variety \(X_K\) is a product of two modular curves

\[
X_K = H(\mathbb{Q}) \backslash \mathbb{D} \times H(\mathbb{A}_f)/K \cong \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \times \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}.
\]
It turns out that the additive Borcherds lift of the constant function is equal to

\[(3.2) \quad \Phi_L(z_1, z_2, 1) = -4 \log \left| (y_1 y_2)^{1/4} \eta(z_1) \eta(z_2) \right| - \log(2\pi) - \Gamma'(1) \]

as a function on $\mathbb{H}^2$. We refer to Section 5.1 of the thesis of Hofmann [Hof11] for details.

Consider the point $z_0 = \left( \frac{-B + \sqrt{D}}{2A}, \frac{-B + \sqrt{D}}{2} \right) \in \mathbb{H}^2$. It corresponds to the two rational points $z_{p \pm}^\pm \in D$, as we shall see below. For simplicity, we drop the sign ± indicating the orientation from our notation.

A basis of $z_0 \cap V(\mathbb{Q})$ is given by $Q f_1 \oplus Q f_2$, with $f_1 = \begin{pmatrix} -1 & -B \\ 0 & A \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 & -B^2 - D \\ -1 & 4A \end{pmatrix}$.

Indeed, we have $z_0 = \mathbb{R} \left( \frac{-B}{2A} \begin{pmatrix} -B^2 + D \\ 1 \end{pmatrix} \right) \oplus \mathbb{R} \sqrt{|D|} \left( \begin{pmatrix} B \\ 2A \\ \frac{1}{4} \end{pmatrix} \right) \oplus \mathbb{R} \sqrt{|D|} \left( \begin{pmatrix} B \\ 2A \\ \frac{1}{4} \end{pmatrix} \right)$.

We obtain $f_1$ as

$$f_1 = -2A \begin{pmatrix} 1 & B \\ 0 & \frac{2A}{1} \end{pmatrix}.$$

and

$$f_2 = - \left( \begin{pmatrix} -B \\ 1 \\ \frac{B^2 + D}{4A} \end{pmatrix} \right) \oplus B \begin{pmatrix} 1 \\ B \\ \frac{1}{2} \end{pmatrix}.$$

In fact, with this choice of basis, we get an isometry of even lattices.

**Lemma 3.2.** We have an isometry of lattices $(P, Q) \cong \mathbb{Z} f_1 \oplus \mathbb{Z} f_2 \subset L$ given by $p_1 \mapsto f_1, \ p_2 \mapsto f_2$.

or, equivalently of $(a, N(x)/N(a)) \cong (P, Q)$ given by $A \mapsto f_1, \ \frac{B + \sqrt{D}}{2} \mapsto f_2$.

Moreover, we have for $U = \mathbb{Q} f_1 \oplus \mathbb{Q} f_2$ that $L \cap U = \mathbb{Z} f_1 \oplus \mathbb{Z} f_2 = P$ and

$L \cap U^\perp = \mathbb{Z} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \oplus \mathbb{Z} \begin{pmatrix} 0 & -B^2 - D \\ 1 & 4A \end{pmatrix}$

is isometric to $P^\perp$.

**Proof.** It is trivial to check that $Q(f_1) = A, \ Q(f_2) = (B^2 - D)/4A$ and $(f_1, f_2) = B$. Similarly, the matrices

$$\tilde{f}_1 = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \quad \tilde{f}_2 = \begin{pmatrix} 0 & -B^2 - D \\ 1 & 4A \end{pmatrix}.$$

satisfy $Q(\tilde{f}_1) = -A, \ Q(\tilde{f}_2) = -(B^2 - D)/4A$ and $(\tilde{f}_1, \tilde{f}_2) = -B$. Moreover, $f_1$ and $f_2$ are both orthogonal to $\tilde{f}_1$ and $\tilde{f}_2$. 
As for the equalities \( L \cap U = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \) and \( L \cap U^\perp = \mathbb{Z}\tilde{f}_1 \oplus \mathbb{Z}\tilde{f}_2 \), the inclusions “\( \subset \)” are clear and the other direction is easy to see because any non-integral linear combination of these vectors has a non-integral entry.

The lemma provides an embedding of \( P \oplus P^- \) into \( L \) as an orthogonal sum. Under this embedding, \( z_P = z_U = z_0 \). We write \( T = \text{GSpin}_U \) and identify it with \( k^\times \) as an algebraic group over \( \mathbb{Q} \), as before. We now come to the corresponding embedding on the level of orthogonal groups. Note that we have \( K_T := K \cap T(\mathbb{A}_f) \cong \mathcal{O}_D^\times \). Recall that given an ideal \( \mathfrak{b} \) of \( k \) which corresponds to the binary quadratic form \([a, b, c]\), there is a CM point given by the unique root of the polynomial \( a\tau^2 + b\tau + c \) that lies in \( \mathbb{H} \). We write \( \tau(\mathfrak{b}) = u(\mathfrak{a}) + iv(\mathfrak{a}) \in \mathbb{H} \) with \( u(\mathfrak{a}), v(\mathfrak{a}) \in \mathbb{R} \) for this point.

**Lemma 3.3.** The group \( T = \text{GSpin}_U \) embeds into \( G \) via

\[
1 \mapsto (I_2, I_2),
\]

where \( I_2 \in \text{GL}_2 \) is the identity matrix and

\[
\sqrt{D} \mapsto (X, Y), \text{ where } X = \begin{pmatrix} -B & \frac{D-B^2}{2A} \\ 2A & B \end{pmatrix} \text{ and } Y = \begin{pmatrix} -B & \frac{D-B^2}{2} \\ 2 & B \end{pmatrix}.
\]

Similarly, the image of \( T^- = \text{GSpin}_{U^\perp} \) is given by

\[
\sqrt{D} \mapsto (X, Y^*)\]

**Proof.** This can easily be seen by determining the stabilizer of the point \( z_P \) as given above on \( \mathbb{H}^2 \). We also refer to Section 4.4 in [Shi94]. Proposition 4.6, ibid., tells us that if \( C/\Lambda \) is an elliptic curve with complex multiplication, \( \Lambda = \mathbb{Z} + \mathbb{Z}\tau \), then there is an embedding \( q_\tau \) of \( k \) into \( M_2(\mathbb{Q}) \), such that

\[
q_\tau(k^\times) = \{ A \in \text{GL}_2^+(\mathbb{Q}) \mid A\tau = \tau \}.
\]

There are exactly two embeddings with this property for a given point \( \tau \). One of them has the property

\[
q_\tau(\mu) \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \mu \begin{pmatrix} \tau \\ 1 \end{pmatrix}.
\]

The other one, denoted \( \bar{q}_\tau \), satisfies the same property with \( \tau \) replaced by \( \bar{\tau} \), that is,

\[
\bar{q}_\tau(\mu) \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \bar{\mu} \begin{pmatrix} \tau \\ 1 \end{pmatrix}.
\]

It is easy to check that \( q_{\tau(\mathfrak{a})}(\sqrt{D}) = X \) and \( q_{\tau(\mathcal{O}_D)}(\sqrt{D}) = Y \) as well as \( \bar{q}_{\tau(\mathcal{O}_D)}(\sqrt{D}) = Y^* \). Using these formulas, we see that the correct embedding of \( k^\times \times k^\times \) in our case is given by

\[
(\lambda, \mu) \mapsto (q_{\tau(\mathfrak{a})}(\lambda)q_{\tau(\mathfrak{a})}(\mu), q_{\tau(\mathcal{O}_D)}(\lambda)\bar{q}_{\tau(\mathcal{O}_D)}(\mu)) \text{ for } \lambda, \mu \in k.
\]

Similar to the proof of Theorem 6.31 in [Shi94], we have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{Q}^2 & \xrightarrow{\iota_\tau} & k^\times \\
\downarrow{q_{\tau(\mathfrak{a})}} & & \downarrow{\mu} \\
\mathbb{Q}^2 & \xrightarrow{\iota_\tau} & k^\times,
\end{array}
\]

where

\[
\iota_\tau(x_1, x_2) = (x_1, x_2) \begin{pmatrix} \tau \\ 1 \end{pmatrix}
\]
Therefore, we have \( \mu \in \mathbb{A}_f \) by an idele on the right and these actions commute with the map \( \tau \) in the same way. We let \( a_r = \mathbb{Z} \tau + \mathbb{Z} \) and \( q_r(\tau) = \gamma k \) for \( h \in \mathbb{A}_k \) and with \( \gamma \in H(\mathbb{Q}) \) and \( k \in K \). There is an element \( \mu \in k^\times \), such that
\[
\gamma^{-1}(\begin{pmatrix} \tau \\ 1 \end{pmatrix}) = \mu \left( \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right).
\]

Therefore, we have
\[
h^{-1}a_r = \tau(\mathbb{Z}^2q(\tau)^{-1}) = \tau(\mathbb{Z}^2\gamma^{-1}) = \mu a_w,
\]
where \( w = \gamma^{-1} \tau \).

This shows that for \( g \in T(\mathbb{A}_f) \) and \( h \in T^-(\mathbb{A}_f) \), we have
\[
H(\mathbb{Q})(\tau(\mathbb{A}), \tau(\mathbb{O}_D)), (g, h)) K = H(\mathbb{Q})(\tau((gh)^{-1} a), \tau((g^{-1} h))) (1, 1) K.
\]

Here, we used the notation \( (h) \) for the ideal corresponding to \( h \) and \( (h)a \) means multiplication of fractional ideals (and not the action of \( \text{GSpin}_U \) on lattices in \( U \)).

**Proposition 3.4.** Let \( g, h \in T(\mathbb{A}_f) \cong \mathbb{A}_k \). We write \( \tau_1 = \tau((hg)^{-1} a) = u_1 + iv_1 \) and \( \tau_2 = \tau((gh^{-1} )) = u_2 + iv_2 \) and obtain
\[
\Phi_{p}(\Theta_p(\tau, g), h) = -4 \log \left| (v_1 v_2)^{1/4} e(\tau_1) e(\tau_2) \right| \log(2\pi) - \Gamma'(1).
\]

*Note that the value depends only on the ideal classes of \( (h), (g) \) and \( a \).*

**Proof.** The proposition essentially follows from the identity
\[
-4 \log \left| (v_1 v_2)^{1/4} e(\tau_1) e(\tau_2) \right| \log(2\pi) - \Gamma'(1) = \Phi_{L}(\tau(z_p, (h, g)), 1),
\]

which is a consequence of Eq. (3.2) and our considerations above as follows: We use the maps \( \text{res}_{L/(P \oplus P^-)} \) and \( \text{tr}_{L/(P \oplus P^-)} \) defined in Lemma 3.1 in [BY09]. Note that the Siegel theta function satisfies
\[
\Theta_{P \oplus P^-}(\tau, (h, g)) = \Theta_p(\tau, h) \otimes \Theta_{p^-}(\tau, g)
\]
and \( \Theta^L_{P \oplus P^-} = \Theta_L \). Moreover, we have that
\[
(f(\tau), \Theta_L(\tau, (h, g))) = (f_{p \oplus p^-}(\tau), \Theta_p(\tau, h) \otimes \Theta_{p^-}(\tau, g))
\]
\[
= (f_{P \oplus p^-}(\tau), \Theta_{p^-}(\tau, h) \otimes \Theta_p(\tau, g)).
\]

With the embeddings defined above, we consider \( P \oplus P^- \) as a sublattice of \( L \). Then we have \( P \oplus P^- \subset L \subset L' \subset P' \oplus (P^-)' \) and
\[
L/(P \oplus P^-) \subset P'/P \oplus (P^-)' \subset P'/P \oplus P'/P.
\]

Using our embeddings defined above, it is not hard to see that for the constant function 1, we have
\[
1_{P \oplus P^-} = \text{res}_{L/(P \oplus P^-)}(1) = \sum_{\beta \in P'/P} \epsilon_{\beta + P} \otimes \epsilon_{\beta + P^-}.
\]
Thus, we obtain
\[ \Phi_L((z_P, (h, g)), 1) = \int_{\text{reg} SL_2(\mathbb{Z}) \backslash \mathbb{H}} \langle 1_{p \otimes p}, \Theta_{\tau, h} \otimes \Theta_{\tau, g} \rangle d\mu(\tau) \]
\[ = \int_{\text{reg} SL_2(\mathbb{Z}) \backslash \mathbb{H}} \langle \Theta_{\tau, g}, \overline{\Theta_{\tau, h}} \rangle v d\mu(\tau) \]
\[ = \Phi_P(\Theta_{\tau, g}, h). \]

4. PROOFS OF PROPOSITION 1.4 AND THEOREM 1.2

As in the introduction, let \((P, Q)\) be a two-dimensional positive definite even lattice. We let \(U = P \otimes \mathbb{Z} Q\) be the associated rational quadratic space. We will assume that \((P, Q)\) is given by a fractional ideal \(a \in \text{Im} \mathbb{N} \) as in the last sections and only use the letter \(P\) to distinguish between the scalar valued and vector valued case. We have that the dual lattice of \(P\) is given by \(P' \sim d_k^{-1} a\), where \(d_k\) denotes the different ideal of \(k\).

Recall the definition of the theta function \(\Theta_{\tau, h}\) attached to \(P\) from the introduction.

Remark 4.1. We should warn the reader that if \(P = a \subset k\) is a fractional ideal, the theta function \(\Theta_{\tau, h}\) is in general not verbatim equal to the vector-valued theta function corresponding to \((h)^2 a\), if \((h)\) denotes the ideal corresponding to \(h\) (defined as in Theorem 2.1). This is due to the fact that \(T(\mathbb{A}_f)\) also acts on the components via automorphisms.

We can prove the explicit expression in Proposition 1.3 for the Petersson inner products of vector valued theta functions in \(\Theta(P)\) using Proposition 3.4.

Proof of Proposition 1.3. Let us abbreviate
\[ f(b) = v(b)^{1/4} \eta(\tau(b)) \]
for any fractional ideal (class) \(b \subset k\). We have by definition and Proposition 3.4 that
\[ \langle \Theta_{\tau, \psi}, \Theta_{\tau, \chi} \rangle = \sum_{h, g \in T(\mathbb{A}_f)/K_T} \psi(g) \overline{\chi}(h) \Phi_P(\Theta_{\tau, g}, h) \]
\[ = -4 \sum_{h, g \in T(\mathbb{A}_f)/K_T} \psi(g) \overline{\chi}(h) \log |f((hg)^{-1} a) f(\tau((h^{-1} g)))| \]
because for non-trivial characters the constant does not contribute to the sum by orthogonality of characters. We split the sum above into
\[ \sum_{g, h} \psi(g) \overline{\chi}(h) \log |f((hg)^{-1} a)| + \sum_{g, h} \psi(g) \overline{\chi}(h) \log |f((h^{-1} g))| \]
\[ = \sum_g \psi(g) \chi(g) \sum_h \chi(h) \log |f(h a)| + \sum_g \psi(g) \overline{\chi}(g) \sum_h \chi(h) \log |f(h)| \]
\[ = \begin{cases} h_k \sum_h \chi(h) \log |f(h a)|, & \text{if } \psi = \overline{\chi}, \\ h_k \sum_h \chi(h) \log |f(h)|, & \text{if } \psi = \chi, \\ 0, & \text{otherwise}, \end{cases} \]
as long as we do not have $\chi = \psi = \bar{\psi}$, in which case we get the sum of the two terms. For the first sum, we obtain
\[
\sum_{h} \chi(h) \log |f((h)\mathbf{a})| = \sum_{h} \chi(h) \log |v((h)\mathbf{a})^{1/4} \eta(\tau(h))| = \tilde{\chi}(\mathbf{a}) \sum_{h} \chi(h) \log |v((h))^{1/4} \eta(\tau(h)))| \]
\[\square\]

We can now give the proof of Theorem 1.1.

Proof of Theorem 1.1. That $E_P(\tau)$ as defined above is really an Eisenstein series follows from the Siegel-Weil formula (Theorem 2.1 of [BY09]). The Eisenstein series correspond to isotropic vectors in the discriminant group $P'/P$ (see [Bru02]) and we assumed that $|P'/P| = |D|$ is square-free, which implies (i).

That $\Theta_P(\tau, \bar{\psi})$ is a cusp form for non-trivial $\psi$ is clear.

To see that $\mathcal{B}(P)$ is a basis of $\Theta(P)$, first note that Proposition 1.3 implies that the set $\mathcal{B}(P)$ is linear independent. Moreover, if $\psi^2 \neq 1$ the Proposition also implies
\[(\bar{\psi}(\psi)\Theta_P(\tau, \bar{\psi}) - \Theta_P(\tau, \bar{\psi}), f(\tau)) = 0\]
for all $f \in \Theta(P)$. Therefore, $\bar{\psi}(\psi)\Theta_P(\tau, \bar{\psi}) - \Theta_P(\tau, \bar{\psi}) \in \Theta(P) \cap \Theta(P)^\perp$, where $\Theta(P)^\perp$ is the orthogonal complement of $\Theta(P)$ with respect to the Petersson inner product. Consequently, $\Theta_P(\tau, \bar{\psi}) = \bar{\psi}(\psi)\Theta_P(\tau, \bar{\psi})$.

Finally, let $A$ be the set of elements $x \in \text{Cl}_k$, such that $\bar{x} = x$ and let $B = \text{Cl}_k \setminus A$. Then $|\mathcal{B}(P)| = |A| + |B|/2$. Moreover, it is well known that $|A| = 2^t - 1$ and $|B| = h_k - 2^t - 1$ which implies the assertion. $\square$

5. Liftings of newforms in the case of square-free level

In this section we will show some general properties of liftings of scalar valued modular forms to vector valued modular forms in the case of square-free level. We will apply these results to relate scalar valued theta series to vector valued ones. This lifting has been used by Bundschuh in his thesis [Bun01], by Bruinier and Bundschuh [BB03] and Scheithauer [Sch11].

Let $L$ be an even lattice with quadratic form $Q$ of type $(2, n)$, level $N$ and determinant $D = |L'/L|$. The group $\Gamma_0(N)$ acts on $\mathfrak{e}_0$ via the Weil representation $\rho_L$ by a character. It is given by
\[
\chi_L \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} (-1)^{\frac{n+2}{2}} \frac{D}{d} & \text{if } d > 0, \\ (-1)^{\frac{n+2}{2}} \left( \frac{(-1)^{\frac{n+2}{2}} D}{-d} \right) & \text{if } d < 0. \end{cases}
\]
We will throughout assume that $N$ is square-free. Then $2 + n$ is even and the character is quadratic. Moreover, this implies that for any $f \in M_{k,L}$, the component function $f_0$ is a modular form in $M_k(N, \chi_L)$. Conversely, we can “lift” any $f \in M_k(N, \chi_L)$ to a vector-valued modular form by defining
\[(5.1) S_L(f) = \sum_{\gamma \in \Gamma_0(N) \setminus \text{SL}_2(\mathbb{Z})} (f | k \gamma) \rho_L(\gamma^{-1}) \mathfrak{e}_0 \in M_{k,L}.
\]
There is also a map that is adjoint to the lift with respect to the Petersson inner product. It is simply given by the map $F \mapsto F_0$ for $F \in M_{k,L}$. We also write $(f, g)$ for the Petersson
inner product on the space of cusp forms for $\Gamma_0(N)$ (possible with character), i.e.

$$(f, g) = \int_{\Gamma_0(N) \backslash \mathcal{H}} f(\tau) \overline{g(\tau)} \Im(\tau)^k d\mu(\tau)$$

for $f, g \in S_k(N, \chi)$.

**Proposition 5.1.** Let $f \in S_k(N, \chi_L)$ and let $F \in M_{k, L}$. Then, we have for the Petersson inner product

$$(S_L(f), F) = (f, F_0).$$

**Proof.** Using the definitions, we obtain

$$(S_L(f), F) = \int_{\mathcal{F}} \left( \sum_{\gamma \in \Gamma_0(N) \backslash \text{SL}(2, \mathbb{Z})} (f | k \gamma) \rho_L(\gamma^{-1}) e_0, F(\gamma) \right) v^k d\mu(\tau)$$

$$= \int_{\mathcal{F}} \left( \sum_{\gamma \in \Gamma_0(N) \backslash \text{SL}(2, \mathbb{Z})} (f | k \gamma) \langle \epsilon_0, \rho_L(\gamma) \rangle \right) v^k d\mu(\tau)$$

$$= \int_{\mathcal{F}} \left( \sum_{\gamma \in \Gamma_0(N) \backslash \text{SL}(2, \mathbb{Z})} \Im(\gamma)^k f(\gamma) \langle \epsilon_0, F(\gamma) \rangle \right) d\mu(\tau)$$

$$= \sum_{\gamma \in \Gamma_0(N) \backslash \text{SL}(2, \mathbb{Z})} \int_{\mathcal{F}} \Im(\gamma)^k f(\gamma) \overline{F(\gamma)} d\mu(\tau).$$

The last line is equal to the Petersson inner product of $f$ and $F_0$ defined as in the statement of the Proposition. \qed

Following Bundschuh [Bun01], we define a subspace of the newforms in $S_k(N, \chi_L)$. Let $A = L'/L$ and for a prime $p$ denote by $A_p$ the $p$-component of $A$. Moreover, write $\chi_L = \prod_{p\mid N} \chi_{L, p}$ as a product of characters modulo $p$ for $p \mid N$. For each prime $p_i$ dividing $N = p_1 \ldots p_r$, we define an element $\varepsilon_i \in \{0, 1, -1\}$.

**Definition 5.2.** If $\dim_{\mathbb{F}_{p_i}} A_{p_i} \geq 2$ or $p_i = 2$, we define $\varepsilon_i = 0$. If $\dim_{\mathbb{F}_{p_i}} A_{p_i} = 1$, $p_i \neq 2$ and $NQ | A_{p_i}$ represents the squares modulo $p_i$, we define $\varepsilon_i = 1$. Otherwise, we define $\varepsilon_i = -1$. Using these signs, we let

$$S^\varepsilon_{k, \ldots, \varepsilon_r}(N, \chi_L) = \{ f \in S^\text{new}_k(N, \chi_L) \mid \exists i \text{ with } \varepsilon_i \neq 0 \text{ and } \chi_{L, p_i}(n) = -\varepsilon_i \Rightarrow c_f(n) = 0 \}.$$

**Remark 5.3.** Note that we have

$$S^\text{new}_k(N, \chi_L) = \bigoplus_{(\varepsilon_1, \ldots, \varepsilon_r) \in \{\pm 1\}^r} S^\varepsilon_{k, \ldots, \varepsilon_r}.$$

We refer to the thesis of Bundschuh [Bun01, Satz 4.3.4] for details.

**Theorem 5.4.** Let $L$ be an even lattice of square-free level $N$ and $f \in S^\varepsilon_{k, \ldots, \varepsilon_r}(N, \chi_L)$. Assume that $\dim_{\mathbb{F}_{p_i}} A_{p_i} = 1$ or $\dim_{\mathbb{F}_{p_i}} A_{p_i} \geq 2$ even for all odd $p_i$. We have

$$\langle S_L(f), \epsilon_{0+L} \rangle = \nu \frac{N}{|L'/L|} f.$$

Here, we let

$$\nu = \nu(m) = \#\{ \mu \in L'/L \mid NQ(\mu) \equiv m \mod N \}$$

for any $m \in \mathbb{Z}$ with $(m, N) = 1$ and $\nu(m) \neq 0$, which is independent of the choice of $m$.\/
Proof. We follow the proof of Satz 4.3.9 in [Bun01]. Let
\[ f(\tau) = \sum_{n=1}^{\infty} a(n)e(n\tau) \]
be the Fourier expansion of \( f \) (at the cusp \( \infty \)) and let
\[ f|_k W_N = \sum_{n=1}^{\infty} a_N(n)e(n\tau). \]
Let \( \mu \in L'/L \) with \((NQ(\mu), N) = 1\). In this case it is not hard to see that
\begin{equation}
F_{\mu}(\tau) = \frac{N^{1-k/2}e(\text{sgn}(L)/8)}{\sqrt{|L'/L|}} \sum_{n \equiv NQ(\mu) \mod N} a_N(n)e\left(\frac{n}{N}\tau\right),
\end{equation}
where \( S_L(f) = F(\tau) = \sum_{\mu \in L'/L} F_{\mu}(\tau)e_{\mu} \). This follows from Theorem 4.2.8 in [Bun01] and can also be deduced from explicit formulas for the Weil representation [Sch09, Str13]. We obtain
\begin{equation}
F_{0|_k W_N} = N^{k/2}(F_{0|_k S})(N\tau) = N^{k/2}e\left(-\text{sgn}(L)/8\right) \frac{1}{\sqrt{|L'/L|}} \sum_{\mu \in L'/L} F_{\mu}(N\tau)
\end{equation}
with certain coefficients \( b(n) \).

By the assumptions of the theorem on the dimension of \( A_p \) over \( \mathbb{F}_p \) for \( p \mid N \), we have that the representation number
\[ \nu(m) = \left| \{ \mu \in L'/L \mid NQ(\mu) \equiv m \mod N \} \right| \]
is in fact equal for all \( m \neq 0 \) with \( \nu(m) \neq 0 \) (cf. [Kne02, Section 13]). Therefore, if we put \( \nu = \nu(m) \) for any \( m \in \mathbb{Z} \) with \( (m,N) = 1 \) and \( \nu(m) \neq 0 \), the last expression simplifies to
\begin{equation}
F_{0|_k W_N} = \frac{N}{|L'/L|} \nu \sum_{(n,N) = 1} a_N(n)e(n\tau) + \sum_{n > 1 \atop (n,N) > 1} b(n)e(n\tau),
\end{equation}
Here, we used the assumption that \( f \in S^{\varepsilon_1,\ldots,\varepsilon_r}_k(N,\chi_L) \). Therefore, we can express the difference to \( f|_k W_N \) as
\[ F_{0|_k W_N} - \frac{N}{|L'/L|} \nu f|_k W_N = \sum_{n \geq 1 \atop (n,N) > 1} c(n)e(n\tau) \]
for some complex numbers \( c(n) \). However, we also have that \( F_0 \) is a newform (see for instance Proposition 7.3 in [SVar]). Thus, \( F_{0|_k W_N} \) is also a newform, and hence the difference vanishes. \( \square \)

The group \( O(L'/L) \) acts on vector-valued modular forms by permuting the basis vectors \( e_{\mu} \). That is, \( \sigma \in O(L'/L) \) acts via \( e_{\mu} \mapsto e_{\sigma(\mu)} \). Using this action, we define the symmetrization
of a modular form $f \in M_{k,L}$ as

$$f^{\text{sym}}(\tau) = \sum_{\sigma \in \text{O}(L'/L)} f^\sigma(\tau) = \sum_{\mu \in L'/L} \sum_{\sigma \in \text{O}(L'/L)} f_\mu(\tau) \epsilon(\mu).$$

This function is clearly invariant under the action of $\text{O}(L'/L)$. We write $M_{k,L}^{\text{sym}}$ for the subspace of $M_{k,L}$ that is invariant under $\text{O}(L'/L)$. The map

$$(5.5) \quad M_{k,L} \longrightarrow M_{k,L}^{\text{sym}}, \quad f \mapsto f^{\text{sym}}$$

is obviously surjective.

The following proposition can be found in Propositions 5.1 and 5.3 of [Sch11].

**Proposition 5.5.** Let $L$ be an even lattice of square-free level $N$. Then the orthogonal group $\text{O}(L'/L)$ acts transitively on all elements of the same norm and order in $L'/L$. Moreover, if $F \in M_{k,L}^{\text{sym}}$ and $F_0 = 0$, then $F = 0$.

6. **Lifting scalar valued theta functions**

As before, let $D < 0$ be an odd fundamental discriminant and let $k = \mathbb{Q}(\sqrt{|D|})$ be the imaginary quadratic field of discriminant $D$. We write $\mathcal{O}_D$ for the ring of integers in $k$ and $\text{Cl}_k \cong \text{Cl}(K)$ for the ideal class group of $k$ as in Section 4. We assume that $D$ is odd.

6.1. **Scalar valued theta functions.** For an integral ideal $a \subset \mathcal{O}_D$, we can consider the associated theta function

$$(6.1) \quad \theta_a(\tau) = \sum_{a \in \mathcal{O}_D} e\left(\frac{N(a)}{N(a)} \tau\right) = 1 + \sum_{n \geq 1} \rho(n,a) e(n\tau).$$

Since $Q_a$ is positive definite, the series converges normally and defines a holomorphic modular form of weight one. It is contained in $M_1(|D|, \chi_D)$, where $\chi_D$ is the primitive Dirichlet character of conductor $|D|$.

It is easy to see that the representation number $\rho(n,a)$, and therefore also the theta function $\theta_a$, only depends on the class $[a] \in \text{Cl}_k$ of $a$.

We denote by $\Theta(k) \subset M_1(|D|, \chi_D)$ the space generated by all theta functions $\theta_a$ for $[a] \in \text{Cl}_k$. Note that since $D$ is a fundamental discriminant, the theta functions $\theta_a$ are all newforms.

Let $\psi \in \text{Cl}_k^*$ be a class group character and define

$$(6.2) \quad \theta_\psi(\tau) = \frac{1}{w_k} \sum_{[a] \in \text{Cl}_k} \psi([a]) \theta_a(\tau).$$

Here, $w_k$ is the number of roots of unity contained in $k$.

The following well known theorem (see [Kan12]) describes the space of scalar valued theta functions. In the case of a prime discriminant it straightforward to derive the theorem from our results in Section 4.

**Theorem 6.1.**

(i) If $\psi^2 = 1$, then $\theta_\psi$ is an Eisenstein series.

(ii) If $\psi^2 \neq 1$, then $\theta_\psi$ is a primitive cuspidal newform.

(iii) Choose a system $\mathcal{C}$ of representatives of characters on $\text{Cl}_k$ modulo complex conjugation. Then the set $\mathcal{B}(k) = \{\theta_\psi \mid \psi \in \mathcal{C}\}$ is an orthogonal basis for $\Theta(k)$ with respect to the Petersson inner product.
It is in fact easy to see that
\begin{equation}
\theta_a(\tau) = \frac{w_k}{h_k} \sum_{\chi \in \text{Cl}_k^*} \bar{\chi}(\{a\}) \theta_\chi(\tau).
\end{equation}

**Definition 6.2.** Let \( \mathfrak{A} \in \text{Cl}_k / \text{Cl}_k^2 \) be a genus and let \( a \in \mathfrak{A} \). The Eisenstein series
\[ E_\mathfrak{A}(\tau) = \frac{1}{h_k} \sum_{[b] \in \text{Cl}_k} \theta_{ab^2}(\tau) \]
is called the (normalized) genus Eisenstein series of \( \mathfrak{A} \).

**Remark 6.3.** The fact that \( E_\mathfrak{A}(\tau) \) is an Eisenstein series is again a special case of the Siegel-Weil formula (see Theorem 2.1 of [BY09]).

### 6.2. Liftings.

**Definition 6.4.** We define the subspace of symmetric theta functions as
\[ \Theta^{\text{sym}}(P) = \langle \Theta^{\text{sym}}_P(\tau, h) \mid h \in \text{Cl}(K) \rangle \subset \Theta(P), \]
where \( \Theta^{\text{sym}}_P(\tau, h) \) is defined in Eq. (5.5).

**Proposition 6.5.** Let \( a \subset \mathcal{O}_D \) be an ideal and let \((P, Q) = (a, N(x)) = (b, N(a))\) be the corresponding even quadratic lattice. For \( h \in I_k / \hat{\mathcal{O}}_D^\times \) corresponding to the ideal class of \( b \subset \mathcal{O}_D \), we have
\[ S_P(\theta_{ab^2})(\tau) = \Theta^{\text{sym}}_P(\tau, h). \]

**Proof.** Note that in our case the level \( N \) is equal to \( |D| \), the order of the discriminant group. Moreover, for \( p \mid D \), the \( \mathbb{F}_p \)-rank of \( A_p \) is equal to one. That means the “signs” \( \epsilon_1, \ldots, \epsilon_t \), where \( t \) is the number of prime divisors of \( D \) in Definition 5.2 are all nonzero.

We first show that the 0-th components of \( S_P(\theta_{ab^2})(\tau) \) and \( \Theta^{\text{sym}}_P(\tau, h) \) agree and then the claim follows from Proposition 5.5. It is clear that the 0-th component of the function \( \Theta_P(\tau, h) \) is equal to \( \theta_{ab^2} \). We write
\[ \theta_{ab^2} = E_\mathfrak{A}(\tau) + g_{ab^2}(\tau) \]
for a cusp form \( g_{ab^2}(\tau) \in S_1(|D|, \chi_D) \). Then, it is not hard to see that in fact
\[ g_{ab^2}(\tau) \in S(|D|, \chi_D)^{\epsilon_1, \ldots, \epsilon_t} \]
for \( \epsilon_1, \ldots, \epsilon_t \) as in Definition 5.2 for the lattice \( P \). Indeed, we write \( \chi_P = \chi_D = \prod_{i=1}^t \chi_{p_i^*} \), where
\[ \chi_{p_i^*}(n) = \left( \frac{p_i^*}{n} \right) \text{ with } p_i^* = \left( \frac{-1}{p} \right) p \]
for a prime divisor \( p \) of \( D \). Then \( \chi_{p_i^*}(n) = -\epsilon_i \) implies that the coefficient of index \( n \) of \( \theta_{ab^2} \) and of \( E_\mathfrak{A} \) vanish because the characters \( \chi_{p_i^*}(n) \) are the basis of the genus characters.

Moreover, the normalized Eisenstein series \( E_\mathfrak{A} \in M_1(|D|, \chi_D) \), where \( \mathfrak{A} \) is the genus of \( a \), lifts to
\[ S_L(E_\mathfrak{A}) = \nu E_P. \]

Proposition 5.1 shows that the lift of an Eisenstein series is again an Eisenstein series. Since the Eisenstein subspace of \( M_{1, P} \) is one-dimensional, the lift of it has to be a multiple of \( E_P \). The correct multiple can be read off from Eq. (5.2) and Eq. (5.3) in the proof of Theorem 5.4.
Note that under the assumptions of the Proposition, we have $\nu = \mid O(L'/L) \mid = 2^{t-1}$ by Proposition 5.5. Thus, by Theorem 5.4, the 0-th component of
\[
\Theta^\text{sym}_P(\tau, h) - S_P(\theta_{ab^2})(\tau)
\]
vanishes. Then the lemma follows from Proposition 5.5 since $\Theta^\text{sym}_P(\tau)$ and $S_P(\theta_{ab^2})(\tau)$ are invariant under $O(P'/P)$.

\[\square\]

Remark 6.6. It follows from Proposition 6.5 that the space $\Theta^\text{sym}(P)$ is the space spanned by the lifts $S_P(\theta_{ab^2})$ of the scalar valued theta functions $\theta_{ab^2}$ in the genus of $a$. This establishes an isomorphism between $\Theta^\text{sym}_P$ and the space of scalar valued theta functions in the genus of $a$.

Proposition 6.7. Let $C = Cl^*_k$ be the group of class group characters. Then the set
\[
\mathcal{B}^\text{sym}(P) = \{ \Theta^\text{sym}_P(\tau, \psi) \mid \psi \in C^2 \}
\]
spans $\Theta^\text{sym}(P) \subset M_{1.P}$. The elements of $\mathcal{B}^\text{sym}(P)$ are permuted by the action of $\text{Aut}(C)$. Moreover, $(\Theta^\text{sym}_P(\tau, \psi), \Theta^\text{sym}_P(\tau, \chi)) = 0$ unless $\psi = \chi$ or $\psi = \overline{\chi}$.

Proof. Using Proposition 6.5, we see that $\Theta^\text{sym}_P(\tau, \psi)$ is in fact equal to the lift of
\[
w_k \sum_{\chi^2 = \psi} \overline{\chi}(a) \theta_\lambda.
\]
Moreover, we have the relation $\psi(a) \Theta_P(\tau, \psi) = \overline{\Theta_P(\tau, \psi)} = \Theta_P(\tau, \overline{\psi})$. The result follows from Theorem 1.1.

\[\square\]

Corollary 6.8. Let $C^2$ be a set of representatives of $C^2$ modulo the relation $\chi \mapsto \overline{\chi}$. Then the set
\[
\{ \Theta^\text{sym}_P(\tau, \psi) \mid \psi \in C^2 \}
\]
is an orthogonal basis of $\Theta^\text{sym}(P)$.

Corollary 6.9. Let $\psi \in Cl^*_k, \psi \neq 1$. We have
\[
w_k^2 \sum_{\chi^2 = \psi} (1 + \chi^2(a))(\theta_\lambda(\tau), \theta_\lambda(\tau)) = (\Theta^\text{sym}_P(\tau, \psi), \Theta^\text{sym}_P(\tau, \psi)).
\]

Proof. We expand the right hand side and obtain
\[
(\Theta^\text{sym}_P(\tau, \psi), \Theta^\text{sym}_P(\tau, \psi)) = w_k^2 \left( \sum_{\chi^2 = \psi} \overline{\chi}(a) \theta_\lambda(\tau), \sum_{\lambda^2 = \psi} \overline{\lambda}(a) \theta_\lambda(\tau) \right)
= w_k^2 \sum_{\chi^2 = \psi} (1 + \chi^2(a))(\theta_\lambda(\tau), \theta_\lambda(\tau)).
\]

We obtain the following well-known formula for the Petersson norm of the scalar valued cusp forms associated with theta functions of positive definite binary quadratic forms.

Corollary 6.10. Suppose that $D = -p$ is a prime discriminant. Let $\chi \in Cl^*_k$ with $\chi \neq 1$ and write again $\tau(a) = u(a) + iv(a)$. Then we have
\[
(\theta_\lambda(\tau), \theta_\lambda(\tau)) = -\frac{4h_k}{w_k^2} \sum_{a \in Cl_k} \chi^2(a) \log \mid v(a)^{1/2} \eta^2(\tau(a)) \mid.
\]
Proof. We use Corollary 6.9 with $P \cong \mathcal{O}_D$ for $D = -p$ together with Theorem 1.1. Moreover, we have to use the fact that for prime discriminants, the class number is odd and therefore, the sum in Corollary 6.9 reduces to a single term. □

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