Soliton in Gravitating Gas. Hoag’s Object.

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Abstract

We explore the possibility of creating of solitons in gravitating gas. It is shown that the virial arguments does not put an obstacle for the existence of localized static solutions. The simplest toroidal soliton of gravitating gas could be the explanation of the peculiar galaxy named Hoag’s object.

1 Introduction

In the recent paper [1] we have considered the Hamiltonian formalism for fluid and gas based on the Lagrangian description. It was pointed out in this paper that apart from other advantages, the Lagrangian description, which uses the trajectories of the particles of fluid (gas) as the dynamical variables, is the most convenient for the introduction of interaction. In particular it was demonstrated in [1] how the introduction of the electromagnetic interaction of particles which constitute the fluid, provide us with the theory of plasma. In the present paper we are going to consider in analogous way the theory of gas of particles which interact with each other through gravitational Newton potential. The system of such particles could be considered as a model for the motion of stars in a galaxy when the gravitation interaction prevails all other interaction. The total number of stars in typical galaxy is of the order of $10^{13} - 10^{14}$, so it may be reasonable to consider this collection of ”particles” as a gas.

The simplest model, which is usually used for numerical simulation of N-body model of galaxy is described by the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m_i} - \gamma \sum_{i \neq j}^{N} \frac{1}{|\vec{x}_i - \vec{x}_j|},$$

where $\vec{x}_i, \vec{p}_i$ are canonical coordinates of particles (stars) with masses $m_i$. The model we are going to consider is based on the dynamics described by $H$ with the assumption $m_i = m$, when $N \rightarrow \infty$. For this limit there appears
a natural desire to consider a continuous distribution of the particles as it is
done in the theory of fluid or gas [2].

Apparently, the attractive potential provided by gravity will produce es-
sential difference of the properties of such a media. The potential part of
the energy will produce a collapse, which will be prevented by kinetic term.
Such a competition sometimes results in a creation of the steady solution —
a soliton.

It is important to mention here that the solitons in gas and fluid is in-
trinsically different with e.g. single solitons considered in particle physic.
In the case of fluid or gas only the field of density and velocities are time
independent, while the constituents of the media are moving. The simplest
example of such a soliton is a tornado.

2 Description of the Model

In fluid (gas) mechanics there are two different pictures of description. The
first, usually refereed as Eulerian, uses as the coordinates the space dependent
fields of velocity and density. The second, Lagrangian description, uses the
coordinates of the particles \( \vec{x}(\xi_i, t) \) labeled by the set of the parameters \( \xi_i \),
which could be considered as the initial positions \( \vec{\xi} = \vec{x}(\xi_i, t = 0) \) and time
\( t \). These initial positions \( \vec{\xi} \) as well, as the coordinates \( \vec{x}(\xi_i, t) \) belong to some
domain \( D \subseteq R^3 \). In sequel we shall consider only conservative systems,
where the paths of different particles do not cross, therefore it is clear that
the functions \( \vec{x}(\xi_i, t) \) define a diffeomorphism of \( D \subseteq R^3 \) and the inverse
functions \( \vec{\xi}(x_i, t) \) should also exist.

\[
\begin{align*}
x_j(\xi_i, t) \bigg|_{\vec{\xi}=\vec{\xi}(x_i, t)} &= x_j, \\
\xi_j(x_i, t) \bigg|_{\vec{x}=\vec{x}(\xi_i, t)} &= \xi_j.
\end{align*}
\]

The density of the particles in space at time \( t \) is

\[
\rho(x, t) = \int d^3 \xi \rho_0(\xi_i) \delta(\vec{x} - \vec{x}(\xi_i, t)),
\]

where \( \rho_0(\xi) \) is the initial density at time \( t = 0 \). The velocity field \( \vec{v} \) as a
function of coordinates \( \vec{x} \) and \( t \) is:

\[
\vec{v}(x_i, t) = \vec{x}(\vec{\xi}(x_i, t), t),
\]
where $\vec{\xi}(x, t)$ is the inverse function (1). The velocity also could be written in the following form:

$$
\vec{v}(x_i, t) = \int d^3 \xi \rho_0(\xi_i) \dot{x}(\xi_i, t) \delta(\vec{x} - \vec{x}(\xi_i, t))
$$

or

$$
\rho(x_i, t)\vec{v}(x_i, t) = \int d^3 \xi \rho_0(\xi_i) \dot{x}(\xi_i, t) \delta(\vec{x} - \vec{x}(\xi_i, t)).
$$

Let us calculate the time derivative of the density using its definition (2):

$$
\dot{\rho}(x_i, t) = \int d^3 \xi \rho_0(\xi_i) \frac{\partial}{\partial t} \delta(\vec{x} - \vec{x}(\xi_i, t))
$$

$$
= \int d^3 \xi \rho_0(\xi_i) \left( -\ddot{x}(\xi_i, t) \right) \frac{\partial}{\partial \vec{x}} \delta(\vec{x} - \vec{x}(\xi_i, t))
$$

$$
= -\frac{\partial}{\partial \vec{x}} \rho(x_i, t) \vec{v}(x_i, t)
$$

In such a way we verify the continuity equation of fluid dynamics:

$$
\dot{\rho}(x_i, t) + \frac{\partial}{\partial \vec{x}} \left( \rho(x_i, t) \vec{v}(x_i, t) \right) = 0.
$$

Using the coordinates $\vec{x}(\xi_i, t)$ as a configurational variables we can consider the simplest motion of the fluid described by the Lagrangian

$$
L = \int d^3 \xi \rho_0(\xi_i) \frac{m\dot{\vec{x}}^2(\xi_i, t)}{2}.
$$

The equations of motion which follow from (8) apparently are

$$
m\ddot{x}(\xi_i, t) = 0
$$

Now let us find what does this equation mean for the density and velocity of the fluid. For that we shall differentiate both sides of (5) with respect to time

$$
\frac{\partial}{\partial t} \rho(x_i, t)\vec{v}(x_i, t) = \int d^3 \xi \rho_0(\xi_i) \ddot{x}(\xi_i, t) \delta(\vec{x} - \vec{x}(\xi_i, t))
$$

$$
+ \int d^3 \xi \rho_0(\xi_i) \dot{x}(\xi_i, t) \frac{\partial}{\partial t} \delta(\vec{x} - \vec{x}(\xi_i, t))
$$
The first term in the r.h.s. of (10) vanishes due to the equations of motion (9) and transforming the second in the same way, as we did in (6) we arrive at

\[ \frac{\partial}{\partial t} \rho(x_i, t) \mathbf{v}(x_i, t) + \frac{\partial}{\partial x_k} \left( \rho(x_i, t) \mathbf{v}(x_i, t) v_k(x_i, t) \right) = 0 \]  

(11)

Let us rewrite (11) in the following form:

\[ \mathbf{v}(x_i, t) \left[ \dot{\rho}(x_i, t) + \frac{\partial}{\partial x_k} \left( \rho(x_i, t) v_k(x_i, t) \right) \right] + \rho(x_i, t) \left[ \dot{v}(x_i, t) + v_k(x_i, t) \frac{\partial}{\partial x_k} \mathbf{v}(x_i, t) \right] = 0. \]  

(12)

The first term in (12) vanishes due to the continuity equation, while the second gives Euler’s equation in the case of the free flow:

\[ \dot{v}(x_i, t) + v_k(x_i, t) \frac{\partial}{\partial x_k} \mathbf{v}(x_i, t) = 0 \]  

(13)

In order to get the usual Euler equations for fluid or gas with the internal pressure we need to add to the Lagrangian (8) the ”potential” part, as it has been shown in [1]. This ”potential” part describes the repulsive interaction between particles which constitute the media (gas or fluid). As was announced in Introduction, we are going to introduce another interaction — gravitational attraction between particles. For that we have to modify the Lagrangian (8) in the following way:

\[ L = \int d^3 \xi \rho_0(\xi_i) \frac{m \ddot{x}(\xi_i, t)}{2} + \frac{\gamma}{2} \int d^3 \xi d^3 \xi' \rho_0(\xi_i)\rho_0(\xi'_i) \frac{\dot{\rho}(\xi_i, t)\dot{\rho}(\xi'_i, t)}{|\ddot{x}(\xi_i, t) - \ddot{x}(\xi'_i, t)|^3}, \]  

(14)

where \( \gamma \) denotes the gravitational constant. The equations of motion, which follow from the Lagrangian (14) have the form:

\[ m \ddot{x}(\xi_i, t) + \gamma \int d^3 \xi' \rho_0(\xi'_i) \frac{\ddot{x}(\xi_i, t) - \ddot{x}(\xi'_i, t)}{|\ddot{x}(\xi_i, t) - \ddot{x}(\xi'_i, t)|^3} = 0 \]  

(15)

Translating equation (15) on to the language of Euler variables, as has been done above we arrive at the following set, including the continuity equation:

\[ \dot{v}(x_i, t) + v_k(x_i, t) \frac{\partial}{\partial x_k} \mathbf{v}(x_i, t) = \frac{\gamma}{m} \frac{\partial}{\partial \ddot{x}} \int d^3 y \rho(y_i, t) \frac{\ddot{x}(y_i, t)}{|\dot{x} - \dot{y}|}, \]  

(16)

\[ \dot{\rho}(x_i, t) + \frac{\partial}{\partial \ddot{x}} \left( \rho(x_i, t) \mathbf{v}(x_i, t) \right) = 0. \]  

(17)
Note that the r.h.s of the equation (16) in the case of ordinary gas or fluid is expressed through the internal pressure \( p(x_i) \):

\[
\dot{v}(x_i, t) + v_k(x_i, t) \frac{\partial}{\partial x_k} \vec{v}(x_i, t) = -\frac{1}{\rho(x_i)} \frac{\partial}{\partial x} p(x_i)
\]  

(18)

The set (16),(17) defines the evolution of initial distribution of \( \rho(x_i, t_0) \), \( \vec{v}(x_i, t_0) \) of gravitating gas and, besides it could be used to find the static configuration of this gas for different boundary conditions. In particular we can explore the possibility of the existence of the static isolated configurations of gravitating gas. Isolation here means that density \( \rho(x_i) \) vanishes at infinity. Note, that for usual fluid, gas or plasma these kind of solutions are forbidden due to virial arguments, known in the case of plasma as Shafranov’s theorem [3],[4].

For the gas, describing by equation (18) this theorem could be proven as follows. First, using the continuity equation let us rewrite equation (18) for static case in the following form:

\[
\frac{\partial}{\partial x_k} \left[ \rho(x_i)v_k(x_i)v_j(x_j) + \delta_{jk}p(x_i) \right] = 0.
\]  

(19)

Integrating (19) with \( x_j \) over \( R^3 \) we obtain:

\[
0 = \int d^3x \frac{\partial}{\partial x_k} \left[ \rho(x_i)v_k(x_i)v_j(x_j) + \delta_{jk}p(x_i) \right] = \int d^3x \frac{\partial}{\partial x_k} \left( x_j \left[ \rho(x_i)v_k(x_i)v_j(x_j) + \delta_{jk}p(x_i) \right] \right) - \int d^3x \delta_{jk} \left[ \rho(x_i)v_k(x_i)v_j(x_j) + \delta_{jk}p(x_i) \right]
\]  

(20)

For usual gases \( p(x_i) \sim \rho^\gamma(x_i), \gamma > 0 \), therefore the integral over divergence will vanish for isolated solutions for which \( \rho(x_i) \to 0 \) when \( |\vec{x}| \to \infty \) and we the obtain the following equation:

\[
\int d^3x \left[ \rho(x_i)\vec{v}^2(x_j) + 3p(x_i) \right] = 0.
\]  

(21)

Apparently, this equation could be satisfied only for the case \( \rho(x_i) = 0 \), i.e. there is no isolated in the above formulated sense, static solutions of the equation (18). In order to have a static solution of (18) we need to change the boundary condition \( \rho(x_i) \to 0 \) to the condition \( \rho(x_i) \to \rho_{as} \), where \( \rho_{as} \) is asymptotic uniform density.
Now we shall show that the arguments of this theorem bring no obstacles for gravitating gas. For this we again will rewrite the equations (16) for the static case, using continuity equation in the following form:

\[
\frac{\partial}{\partial x_j} \left( \rho(x_i) v_k(x_i) v_j(x_i) \right) - \frac{\gamma}{m} \rho(x_i) \frac{\partial}{\partial x_k} \int d^3y \frac{\rho(y_i, t)}{|\vec{x} - \vec{y}|} = 0, \quad (22)
\]

Integrating (22) with respect to \(x_k\) over \(R^3\) we obtain:

\[
\int d^3x_k \left[ \frac{\partial}{\partial x_j} \left( \rho(x_i) v_k(x_i) v_j(x_i) \right) - \frac{\gamma}{m} \rho(x_i) \frac{\partial}{\partial x_k} \int d^3y \frac{\rho(y_i, t)}{|\vec{x} - \vec{y}|} \right] = 0. \quad (23)
\]

Consider the first term of the integrand in (23). Integrating by parts as above and taking into account the asymptotic conditions for \(\rho(x_i)\) we obtain:

\[
\int d^3x_k \frac{\partial}{\partial x_j} \left( \rho(x_i) v_k(x_i) v_j(x_i) \right) = - \int d^3x \rho(x_i) \bar{v}^2(x_i). \quad (24)
\]

Integration of the second term of the integrand in (23) is straightforward, yielding

\[
- \frac{\gamma}{m} \int d^3x_k \rho(x_i) \frac{\partial}{\partial x_k} \int d^3y \frac{\rho(y_i, t)}{|\vec{x} - \vec{y}|} = \frac{\gamma}{2m} \int d^3x d^3y \frac{\rho(x_i) \rho(x_i)}{|\vec{x} - \vec{y}|}. \quad (25)
\]

In such a way from equation (23) we obtain:

\[
\int d^3x \rho(x_i) \bar{v}^2(x_i) - \frac{\gamma}{2m} \int d^3x d^3y \frac{\rho(x_i) \rho(x_i)}{|\vec{x} - \vec{y}|} = 0. \quad (26)
\]

This relation apparently could be satisfied for non-trivial configurations of \(\rho(x_i), \bar{v}(x_i)\). The energy functional, corresponding to the Lagrangian (14) has the following form:

\[
E = \frac{m}{2} \int d^3x \rho(x_i, t) \bar{v}^2(x_i, t) - \frac{\gamma}{2} \int d^3x d^3y \frac{\rho(x_i, t) \rho(y_i, t)}{|\vec{x} - \vec{y}|}. \quad (27)
\]

The two terms of the energy functional have clear interpretation as kinetic \(T\) and potential \(U\) parts of energy and equation (23) expresses famous "virial theorem" [5]:

\[
2T = -U. \quad (28)
\]
Note here, that in the "virial theorem" equation (28) holds true for mean values of kinetic and potential energies, while in our case of static solutions there is no need to average over time.

Using the relation (28) we can easily find the total energy of static configuration of the gravitating gas:

\[
E_{\text{static}} = -\frac{m}{2} \int d^3x \rho(x) \vec{v}^2(x) = \]

\[
= -\frac{\gamma}{4} \int d^3x d^3y \frac{\rho(x,t)\rho(y,t)}{|\vec{x} - \vec{y}|}.
\]

So, the total energy of the static solution is negative, as for the bound state of Kepler problem.

### 3 Properties of Static Solutions

The equations which define our static configurations of gravitating gas have the following form:

\[
v_k(x_i) \frac{\partial}{\partial x_k} \vec{v}(x_i) = \frac{\gamma}{m} \frac{\partial}{\partial \vec{x}} \int d^3y \frac{\rho(y)}{|\vec{x} - \vec{y}|},
\]

\[
\frac{\partial}{\partial \vec{x}} \left( \rho(x_i) \vec{v}(x_i) \right) = 0
\]

(30)

Taking divergence of the first equation:

\[
\vec{\partial} \left( v_k(x_i) \frac{\partial}{\partial x_k} \vec{v}(x_i) \right) = \frac{\gamma}{m} \Delta \int d^3y \frac{\rho(y)}{|\vec{x} - \vec{y}|} =
\]

\[
= \frac{\gamma}{m} (-4\pi) \rho(x_i),
\]

(31)

we can write the whole set of the equations for static configuration in a pure local form:

\[
\vec{\partial} \left( v_k(x_i) \frac{\partial}{\partial x_k} \vec{v}(x_i) \right) = -4\pi \frac{\gamma}{m} \rho(x_i),
\]

\[
\frac{\partial}{\partial \vec{x}} \left( \rho(x_i) \vec{v}(x_i) \right) = 0,
\]

\[
\vec{\partial} \times \left( v_k(x_i) \frac{\partial}{\partial x_k} \vec{v}(x_i) \right) = 0,
\]

(32)
where the last equation requires the expression \( v_k(x_i) \frac{\partial}{\partial x_k} \vec{v}(x_i) \) to be a gradient.

Now we are going to derive an important inequality, which bounds the \((-E_{\text{static}})\) of any solution of (32) from below. For this let us introduce the notation for the potential \( U(x_i) \):

\[
U(x_i) = - \int d^3y \frac{\rho(y_i)}{|\vec{x} - \vec{y}|}
\]  

(33)

Integrating by parts we obtain the following relation:

\[
\int d^3x \left( \partial U(x_i) \right)^2 = 4\pi \int d^3x d^3y \frac{\rho(x_i,t)\rho(y_i,t)}{|\vec{x} - \vec{y}|}
\]  

(34)

Using (34) we can write the expression for the energy given by (29) of any static configuration, which is the solution of (32) in the following form:

\[
E_{\text{static}} = -\frac{\gamma}{16\pi} \int d^3x \left( \partial U(x_i) \right)^2
\]  

(35)

Substituting into (35) the expression for the gradient of the potential \( U(x_i) \) from the first equation (30), we obtain the expression for the \( E_{\text{static}} \) only through the field of velocity:

\[
-E_{\text{static}} = \frac{m^2}{16\pi\gamma} \int d^3x \left( v_k(x_i) \frac{\partial}{\partial x_k} \vec{v}(x_i) \right)^2
\]  

(36)

This form is most convenient for the derivation of desired inequality. Now let us consider a function \( f(x_i) \) from the Hilbert space \( W^1_2 \), which consists of all measurable functions on \( R^3 \), which have at least one derivative and square integrable on \( R^3 \) together with its derivatives. In particular we assume that the density \( \rho(x_i) \) belongs to \( W^1_2 \). Taking into account the equation (31) we have

\[
\left| \int d^3x \rho(x_i) f(x_i) \right| = \left| \frac{m}{4\pi\gamma} \int d^3x \rho(x_i) v_j(x_i) \partial_j v_k(x_i) \right|
\]

\[
= \left| \frac{m}{4\pi\gamma} \int d^3x \partial_k f(x_i) v_j(x_i) \partial_j v_k(x_i) \right|
\]

\[
\leq \frac{m}{4\pi\gamma} \left( \int d^3x (\partial_k f(x_i))^2 \right)^{1/2} \left( \int d^3x (v_j(x_i) \partial_j v_k(x_i))^2 \right)^{1/2},
\]  

(37)

where on the last step we have used Cauchy inequality. From (37) we immediately obtain the inequality:

\[
-E_{\text{static}} \geq \pi\gamma \left( \frac{\int d^3x \rho(x_i) f(x_i)}{\int d^3x (\partial_k f(x_i))^2} \right)^2,
\]  

(38)
which is valid for any \( f(x_i) \) from \( W^1_2 \) and is saturated for \( f(x_i) = U(x_i) \). Indeed, let us calculate the derivative of the functional in the r.h.s of (38) with respect to \( f(x_i) \):

\[
\frac{\delta}{\delta f(x_i)} \left( \int d^3x \rho(x_i) f(x_i) \right)^2 \frac{1}{\int d^3x (\partial_k f(x_i))^2} \]

\[
= 2 \left( \int d^3x \rho(x_i) f(x_i) \right)^2 \frac{\rho(x_i)}{\int d^3x (\partial_k f(x_i))^2} \left[ \frac{\Delta f(x_i)}{\int d^3x (\partial_k f(x_i))^2} \right].
\]

This derivative vanishes for \( f(x_i) = f_{\text{max}}(x_i) \) given by

\[
f_{\text{max}}(x_i) = C\Delta^{-1} \rho(x_i) = C' U(x_i),
\]
where \( C \) and \( C' \) are inessential constants, which do not enter into the functional. It is easy to prove that the second variation of this functional is negative on the \( f_{\text{max}}(x_i) \), so this function provides the absolute maximum for the functional and due to the equations (34-36) its value coincides with the l.h.s. of (38). However, the function \( f_{\text{max}}(x_i) \) does not belong to the Hilbert space \( W^1_2 \), because the potential \( U(x_i) \) has the asymptotic behaviour \( \frac{1}{|x|} \) at infinity and therefore is not square integrable.

The integral \( J \) which enters into inequality (38) could be written in the following form

\[
J = \int d^3x \rho(x_i) f_N(x_i),
\]
where we denoted as \( f_N(x_i) \) the normalized function \( f(x_i) \):

\[
f_N(x_i) = \frac{f(x_i)}{\sqrt{\int d^3x (\partial_k f(x_i))^2}} = \frac{f(x_i)}{\|\partial_k f(x_i)\|_2}.
\]

In the Hilbert space \( W^1_2 \) the Hoelder’s inequality holds true:

\[
| \int d^3x f(x_i) g(x_i) | \leq \|f\|_p \|g\|_{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1
\]

as well, as the remarkable Ladyjenskaya’s [6] inequality:

\[
\|f\|_6 \leq (48)^{1/6} \|\partial_k f\|_2
\]
Here we use the standard notations:

$$\|f\|_p = \left( \int d^3x f(x_i)^p \right)^{1/p},$$

$$\|\partial_k f\|_p = \left( \int d^3x |\partial_k f(x_i)|^p \right)^{1/p}.$$

From these two inequalities we obtain the following bound for the integral $J$:

$$|J| \leq (48)^{1/6}\|\rho\|_{6/5}$$

(45)

The facts mentioned above could possibly support the statement that the $(-E_{\text{static}})$ is bounded from below by appropriate norm of the density. Indeed, the absolute maximum of the functional $|J|$ should be bigger when its maximum in a restricted space like $W^1_2$, given by (45). However we can not present the rigorous proof of this statement.

One of the other general property of a static configurations of gas or fluid is the existence of the topological charge — ”helicity” (or Hopf invariant), which explicit form is

$$q = \int d^3x \vec{v}(x_i) \cdot \text{rot} \vec{v}(x_i).$$

(46)

This object is not only the integral of motion of the equations (16) but it also is the central element of the algebra of Poisson brackets of $(\vec{v}(x_i), \rho(x_i))$ [1]. The existence of such an object brings additional argument for the stability of the solitons. The role of ”helicity” in the case of the solitons in plasma was pointed out in [7],[8]. Moreover, in [9] it was shown that there exists a remarkable inequality which bounds the energy of plasma solitons from below by $q^{3/4}$. The derivation of such inequality for our case (and in general for fluid solitons ) is highly desirable and we going to consider this question in the future publications.

4 On the Possible Structure of the Static Solutions

The equations (32) which define the static configurations of gravitating gas are 3-dimensional nonlinear partial differential equations and the probability to find an analytic solution is very low. The only case where a class of solutions was found in a similar situation is the t’Hooft-Polyakov monopole, but there the requirement of the spherical symmetry simplified essentially the
problem. In our case we can not expect the spherically symmetric solution because the continuity equation requires the trajectories of the particles be closed in order to provide the static configuration for $\vec{v}(x_i), \rho(x_i)$. The simplest and most symmetric configuration we could expect for our case is the toroidal structure, where the density is concentrated in the vicinity of the axis of the toroid, while the field of velocity is tangential to the embedded one into the other toroidal surfaces. It is not the first time the toroidal-shape soliton appeared in the context of the theory of continuous media. Since the pioneer works of Lord Kelven in XIX century to the present time it was studied by many scientists both mathematician and physicists and recently the interest to the subject was again attracted by the works of Faddeev and Niemi [7],[8]

It is clear that in the case of attractive gravitational interaction the particles, which constitute the gas move around the region with bigger density (like planets around the Sun) so there should exists a collective motion in the approximation when the radius of torus tends to infinity and we can speak about cylindrical rather when toroidal configuration. Indeed, let us consider this axially symmetric tornado-like solution of (32). For that we shall write the Ansatz:

$$\vec{v}(x) = v(r)(-\frac{y}{r}, \frac{x}{r}, 0), \quad \rho(x) = \rho(r),$$

where $r = \sqrt{x^2 + y^2}$. The second and the third equations (32) are satisfied by (47), while the first gives

$$\frac{1}{r} \partial_r v(r)^2 = 4\pi \frac{\gamma}{m} \rho(r).$$

This solution shows that the density stays an arbitrary function (what is expected for the partial differential equations) and the velocity grows with radius up to its asymptotic value. The later is the consequence of the approximation— here we actually have 2-dimensional potential $log r$ instead of Coulomb $\frac{1}{r}$. This example is to demonstrate that the particles which constitute the gravitating gas in their collective motion form the localized object.

The traditional way to tackle 3-dimensional gas or fluid is to introduce for the velocity Clebsh parametrization suggested in [10] and recently discussed
in[11]. In general case this parametrization has the following form:

\[ \vec{v}(x_i) = f(x_i)\vec{g}(x_i) + \vec{h}(x_i), \]  

(49)

where \( f(x_i), g(x_i), h(x_i) \) are scalar functions. For the toroidal solution with \( z \) – as the axis of symmetry we shall assume that the density \( \rho(x_i) \) does not depend upon the azimuth angle \( \phi \) and Clebsh parametrization takes the form:

\[ \vec{v}(x_i) = \cos\alpha(r, z)\vec{\beta}(r, z) + k\vec{\phi}(x_i), \]

(50)

where \( r, z, \phi \) are cylindrical coordinates,

\[
\begin{align*}
  r &= \sqrt{x^2 + y^2} \\
  \phi(x_i) &= \arctan \frac{y}{x}
\end{align*}
\]

In fluid dynamics the parametrization (50) has an interesting mechanical interpretation which we shall discuss elsewhere. The ”helicity” functional for this case has the following form

\[ q = \int \cos\alpha(r, z)\vec{\beta}(r, z) \wedge d\cos\alpha(r, z) \wedge d\beta(r, z) \]

\[ = 2\pi k \int \cos\alpha(r, z) \wedge d\beta(r, z) \]  

(51)

The function \( \beta(r, z) \) in (50) should has a singularity at the point \((r_c, 0)\) in the semiplane \((r, z)\) where \( r_c \) is the radius of axis line of the torus. When the point \((r, z)\) goes around \((r_c, 0)\), the function \( \beta(r, z) \) increases on \( 2\pi \). The gradient of such function will be the vector tangential to the toroidal surfaces. The addition of the gradient of azimuthal angle \( \phi \) makes the velocity (50) winding also in the azimuth direction. Such a behaviour of the field of velocity leads to the nontrivial Hopf invariant, which characterizes the homotopy classes \( \pi_3(S^2) \). Described this way the Clebsh parameters, together with the density \( \rho(x_i) \) could be found by numerical integration of the equations (32).

The construction we presented and discussed above would be incomplete and too academic without an example of the galaxy which indeed may demonstrate the toroidal structure. Fortunately such galaxy does exists. It was discovered back in 1950 by Art Hoag and recently a very good picture was obtained by Hubble telescope (see figure 1). On this picture it is clearly seen

\footnote{Compare this with the similar parametrization of the dynamical variables in [8] for the case of plasma.}
the toroidal structure and moreover the motion of the stars seem to be very close to what we should expect for the configuration with nontrivial ”helicity” functional. In the gallery of the galaxies which is available on the sites http://www.astronomy.com or http://hubblesite.ogr we can find some other examples with the form more or less close to the torus, but the Hoag’s object is the best of all. these examples show that the model we considered might be not so trivial and could explain more sophisticated structures like spiral galaxy.

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Figure 1: The above photo taken by the Hubble Space Telescope in July 2001 reveals unprecedented details of Hoag’s Object and may yield a better understanding. Hoag’s Object spans about 100,000 light years and lies about 600 million light years away toward the constellation of Serpens.
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This figure "HoagsObject.jpg" is available in "jpg" format from:

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