SL$_2(\mathbb{C})$-CHARACTER VARIETY OF A HYPERBOLIC LINK AND REGULATOR

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Abstract. In this paper, we study the SL$_2(\mathbb{C})$ character variety of a hyperbolic link in S$^3$. We analyze a special smooth projective variety $Y^h$ arising from some 1-dimensional irreducible slices on the character variety. We prove that a natural symbol obtained from these 1-dimensional slices is a torsion in $K_2(\mathbb{C}(Y^h))$. By using the regulator map from $K_2$ to the corresponding Deligne cohomology, we get some variation formulae on some Zariski open subset of $Y^h$. From this we give some discussions on a possible parametrized volume conjecture for both hyperbolic links and knots.

1. Introduction

This is the sequel of our previous work [LW2] on the generalized volume conjecture for a hyperbolic knot in $S^3$. In this paper we shall study a hyperbolic link in $S^3$, and extend several results from the knot case. The main idea is to apply the regulator map in the K-theory to the SL$_2(\mathbb{C})$ character varieties of hyperbolic links.

For a link $L$ in $S^3$, Kashaev ([Ka1]) introduced a sequence of complex numbers $\{K_N|N > 1, \text{odd integer}\}$, which were derived from a matrix version of the quantum dilogarithms. Kashaev’s Volume Conjecture ([Ka2]) predicts that for any hyperbolic link $L$ in $S^3$ the asymptotic behavior of his invariants $\{K_N\}$ regains the hyperbolic volume of $S^3 - L$. This was verified for the figure eight knot ([Ka2]). The Volume Conjecture provides an intriguing relationship between the quantum invariants and the hyperbolic volume, but we still do not fully understand it.

For the knot case, Murakami-Murakami ([MM]) showed the Kashaev invariants $\{K_N\}$ can be identified with the values of normalized colored Jones polynomial at the primitive $N$-th root of unity. From this, a new formulation of the Volume Conjecture is stated as that the asymptotic behavior of the colored Jones invariants of any knot equals the Gromov simplicial volume of its complement in $S^3$. This new version of the volume conjecture bridges the quantum invariants of the knot with its classical geometry and topology. However, this formulation does not fit well for links, since it does not hold for many split links (see [MMOTY]). Hence it is a very interesting question for us to see what is really behind the volume conjecture for links.

Following Witten’s SU(2) topological quantum field theory, Gukov ([Gu]) proposed a complex version of Chern-Simons theory and generalized the volume conjecture to a $\mathbb{C}^*$-parametrized version with parameter lying on the zero locus of the $A$-polynomial of the knot. In [LW2], we constructed a natural torsion element in the $K_2$ of the function field of the curve defined by the $A$-polynomial. We then showed that the part from the $A$-polynomial
in Gukov’s generalized volume conjecture can be interpreted by the regulator map on this torsion element. In particular, this implied the quantization condition posed by Gukov [Guk. Page 597].

It is natural to ask if there exists a parametrized volume conjecture for links in \( S^3 \) as Gukov did for the knot case. This is the motivation of this paper. Now we have to deal with two problems for links with more than one component. First, its \( SL_2(\mathbb{C}) \) character variety has dimension \( >1 \). Hence it is not clear how to define an \( A \)-polynomial for such a link, which will contain the geometric information like volume and Chern-Simons as the knot case. Secondly, it is not clear how to relate the colored Jones polynomials to \( SL_2(\mathbb{C}) \) character variety of dimension \( >1 \). In this paper, we shall focus on the first problem. We introduce \( n \) curves on the geometric component of the character variety. From these curves, we obtain an \( n \)-dimensional smooth projective variety \( Y^h \), where \( n \) is the number of the components of the link. We construct a natural torsion element in \( K_2 \) of the function field of \( Y^h \). By applying the regulator map on this torsion element, we get the variation formulae (Theorem 3.12) on some Zariski open subset of \( Y^h \). When the link has one component, it recovers the results for hyperbolic knots. This suggests that there should exist a parametrized volume conjecture for hyperbolic links and the \( Y^h \) may provide a replacement of the locus of the \( A \)-polynomial of a knot. For the second problem, we give some speculations in the end of Section 4.

On the other hand, using dilogarithm, Dupont (Duf) constructed explicitly the Cheeger-Chern-Simons class associated to the second Chern polynomial. Apply it to a closed hyperbolic 3-manifold \( M \), we get a number in \( \mathbb{C}/\mathbb{Z} \). He (loc.cit.) showed that its imaginary part equals the hyperbolic volume of \( M \) and the real part is the Chern-Simons invariant of \( M \). In general, for an odd dimension hyperbolic manifold of finite volume, Goncharov (Gon) constructed an element in the Quillen’s algebraic K-group of \( \mathbb{C} \) and proved that after applying it to the Borel regulator, we get the volume of the manifold. Our approach can be regarded as a family version of theirs for the \( SL_2(\mathbb{C}) \) character variety of a hyperbolic link.

Our paper is organized as follows. In section 2, we review the basics of the \( SL_2(\mathbb{C}) \) character variety of a hyperbolic link. We then study the properties of a smooth projective variety \( Y^h \) coming from the 1 dimensional slices of the character variety. In section 3, we recall the definitions and basic properties of \( K_2 \) of a commutative ring. Then we state and prove our main results in this section. In Section 4, we give some discussions related to a possible parametrized volume conjecture for hyperbolic links.

2. Character Variety of a Hyperbolic Link

2.1. Let \( L \) be a hyperbolic link in \( S^3 \) with \( n \) components \( K_1, \ldots, K_n \). This means that the complement \( S^3 - L \) carries a complete hyperbolic structure of finite volume. Let \( N(L) \) be an open tubular neighborhood of \( L \) in \( S^3 \). Put \( M_L = S^3 - N(L) \), then it is a compact 3-manifold with boundary \( \partial M_L \) a disjoint union of \( n \) tori \( T_1, \ldots, T_n \). Note that \( \pi_1(S^3 - L) \) and \( \pi_1(M_L) \) are isomorphic. In the following, we shall identify them.

Let \( R(M_L) = \text{Hom}(\pi_1(M_L), SL_2(\mathbb{C})) \) and \( R(T_i) = \text{Hom}(\pi_1(T_i), SL_2(\mathbb{C})) \), \( i = 1, \ldots, n \). \( SL_2(\mathbb{C}) \) acts on them by conjugation. According to [CS1], they are affine algebraic sets and so are the corresponding character varieties \( X(M_L) \) and \( X(T_i) \), which are the algebra-geometric quotients of \( R(M_L) \) and \( R(T_i) \) by \( SL_2(\mathbb{C}) \). We then have the canonical surjective morphisms \( t : R(M_L) \to X(M_L) \) and \( t_i : R(T_i) \to X(T_i) \) which map a representation to its character. Induced by the inclusions of \( \pi_1(T_i) \) into \( \pi_1(M_L) \), we have the restriction map:

\[
r : X(M_L) \to X(T_1) \times \cdots \times X(T_n).
\]
For the details on the character varieties, we refer to [CSI, Sha].

2.2. Let \( \rho_0 : \pi_1(M_L) \rightarrow SL_2(\mathbb{C}) \) be a representation associated to the complete hyperbolic structure on \( S^3 - L \). Then it is irreducible. Denote \( \chi_0 \) its character. Fix an irreducible component \( R_0 \) of \( R(M_L) \) containing \( \rho_0 \). Let \( X_0 = t(R_0) \), then \( X_0 \) is an affine variety of dimension \( n \) ([CSI, Sha]). We call it the geometric component of the character variety. Define \( Y_0 \) to be the Zariski closure of the image \( r(X_0) \) in \( X(T_1) \times \cdots \times X(T_n) \).

For \( g \in \pi_1(M_L) \), there is a regular function \( I_g : X_0 \rightarrow \mathbb{C} \) defined by \( I_g(\chi) = \chi(g) \), for \( \forall \chi \in X_0 \). The following proposition was proved in [CS2].

**Proposition 2.1.** Let \( \gamma_i \) be a non-contractible simple closed curve in the boundary torus \( T_i \), \( 1 \leq i \leq n \). Let \( g_i \in \pi_1(M_L) \) be an element whose conjugacy class corresponds to the free homotopy class of \( \gamma_i \). Let \( k \) be an integer with \( 0 \leq k \leq n \), and let \( V \) be the algebraic subset of \( X_0 \) defined by the equations \( I^2_{g_i}(\chi) = 4 \), \( k < i \leq n \). Let \( V_0 \) denote an irreducible component of \( V \) containing \( \chi_{\rho_0} \). Then if \( \chi \) is a point of \( V_0 \), \( i \) is an integer with \( k < i \leq n \), and \( g \) is an element of the subgroup (defined up to conjugacy) \( \text{Im}(\pi_1(T_i) \rightarrow \pi_1(M_L)) \), we have \( I_g(\chi) = \pm 2 \). Furthermore, if \( k = 0 \), then \( V_0 = \{ \chi_{\rho_0} \} \).

**Proof.** See [CS2] Proposition 2, Page 539].

The following is a generalization of the knot case (c.f. [CSI, CS2]).

**Proposition 2.2.** \( Y_0 \) is an \( n \)-dimensional affine variety.

**Proof.** It is clear that \( Y_0 \) is an affine variety. We need to show that \( \dim Y_0 = n \). Since \( \dim X_0 = n \), \( \dim Y_0 \leq n \). Assume that \( \dim Y_0 = m < n \). Then for \( y \in r(X_0) \), every component of the fibre \( r^{-1}(y) \) has dimension \( \geq n - m \geq 1 \). Take \( y = r(\chi_0) \), then there is an irreducible component \( C \) of the fibre \( r^{-1}(y) \) containing \( \chi_0 \) and \( \dim C \geq 1 \). For each boundary torus \( T_i \) and a non-trivial \( g_i \in \text{Im}(\pi_1(T_i) \rightarrow \pi_1(M_L)) \), consider the regular function \( I_{g_i} : X_0 \rightarrow \mathbb{C} \). For all \( \chi \in C \), \( I_{g_i}(\chi) = I_{g_i}(\chi_0) \). Since \( \chi_0 \) is the character of the complete hyperbolic structure on \( M_L \), \( I^2_{g_i}(\chi) - 4 = I^2_{g_i}(\chi_0) - 4 = 0 \) for all \( \chi \in C \), \( g_i \in \text{Im}(\pi_1(T_i) \rightarrow \pi_1(M_L)) \), \( 1 \leq i \leq n \). Now we fix \( n \) non-trivial \( g_i \in \text{Im}(\pi_1(T_i) \rightarrow \pi_1(M_L)) \), \( 1 \leq i \leq n \). Consider the algebraic subset \( V \) of \( X_0 \) defined by the equations \( I^2_{g_i} - 4 = 0 \), \( 1 \leq i \leq n \). By its construction, \( C \) is contained in an irreducible component say \( V_0 \) of \( V \) containing \( \chi_0 \). Hence \( \dim V_0 \geq 1 \). On the other hand, by Proposition 2.1, \( V_0 = \{ \chi_0 \} \), a contradiction. Therefore, \( \dim Y_0 = n \).

For every boundary torus \( T_i \), fix a meridian-longitude basis \( \{ \mu_i, \lambda_i \} \) for \( \pi_1(T_i) = H_1(T_i, \mathbb{Z}) \). Given \( 1 \leq i \leq n \), we define \( X_0^i \) as the subvariety of \( X_0 \) defined by the equations \( I^2_{\mu_j} - 4 = 0 \), \( j \neq i, 1 \leq j \leq n \). Let \( V_i \) be an irreducible component of \( X_0^i \) containing \( \chi_0 \).

**Proposition 2.3.** For each \( i = 1, \cdots, n \), \( V_i \) has dimension \( 1 \).

**Proof.** Since \( X_0^i \) is defined by \( n - 1 \) equations and \( \dim X_0 = n \), every component of \( X_0^i \) has dimension at least \( 1 \). Now assume that \( \dim V_i \geq 2 \). Let \( U \) be the subvariety of \( V_i \) defined by the equation \( I^2_{\mu_i} - 4 = 0 \) and let \( U_0 \) be the irreducible component of \( U \) containing \( \chi_0 \). Then \( \dim U_0 \geq 2 \) implies that \( \dim U_0 \geq 1 \). But this contradicts the last assertion in Proposition 2.1. Hence, \( \dim V_i = 1 \).

**Lemma 2.4.** Given a non-trivial \( g_i \in \text{Im}(\pi_1(T_i) \rightarrow \pi_1(M_L)) \), \( 1 \leq i \leq n \), then

1. On every \( V_j \) with \( j \neq i \), we have \( I_{g_i} = \pm 2 \) is a constant.
2. On \( V_i \), \( I_{g_i} \) is not a constant, hence it is not a constant on \( X_0 \) either.
Proposition 2.1. \( \text{in some irreducible component } V \)

Definition 2.6. \( V^1 \) has dimension \( f \)

For (2), suppose \( I_{g_i} \) were a constant on \( V_i \), then \( I_{g_i} = I_{g_i}(\chi_0) = \pm 2 \). Consider algebraic subset \( V \) of \( X_0 \) defined by the \( n \) equations \( I_{g_j}^2 = 4 \), \( (j \neq i) \), and \( I_{g_i}^2 = 4 \). Then \( V_i \) is contained in some irreducible component \( V_0 \) of \( V \) containing \( \chi_{p_0} \). Hence \( \dim V_0 \geq 1 \). Contradiction to Proposition 2.1.

For each \( i = 1, \ldots, n \), let \( p_i \) be the projection map from \( X(T_1) \times \cdots \times X(T_n) \) to the \( i \)-th factor \( X(T_i) \). Denote by \( r_i : X_0 \rightarrow X(T_i) \) the composite of \( r \) and \( p_i \). Then we have

Proposition 2.5. For every \( i = 1, \ldots, n \), the Zariski closure \( W_i \) of the image \( r_i(V_i) \) in \( X(T_i) \) has dimension 1.

Proof. It is sufficient to consider the case \( i = 1 \). Since \( \dim V_1 = 1 \) and \( r_1 \) is regular, \( \dim W_1 \leq 1 \). Assume that \( \dim W_1 = 0 \). This means that \( r_1(V_i) \) consists of a single point. Therefore, for any \( g_1 \in \text{Im}(\pi_1(T_1) \rightarrow \pi_1(M_L)) \), \( I_{g_1} \) is a constant on \( V_1 \). This contradicts Lemma 2.4 part 2.

2.3. For \( 1 \leq i \leq n \), denote by \( R_D(T_i) \) the subvariety of \( R(T_i) \) which consists of the diagonal representations. For such a representation \( \rho \), by taking the eigenvalues of \( \rho(\mu_i) \) and \( \rho(\lambda_i) \), it is clear that \( R_D(T_i) \) is isomorphic to \( \mathbb{C}^* \times \mathbb{C}^* \). We shall denote the coordinates by \( (l_i, m_i) \). Let \( t_{i\mid D} \) be the restriction of \( t_i \) on \( R_D(T_i) \). \( D_i = t_{i\mid D}^{-1}(W_i) \). By the proof of \cite{LW1}, Proposition 3.3], \( D_i \) is either irreducible or has two isomorphic irreducible components. Let \( y^i \in D_i \) be the point corresponding to the character of the representation of the hyperbolic structure on \( S^3 - L \). Let \( Y_i \) be an irreducible component of \( D_i \) containing \( y^i \). Then \( Y_i \) is an algebraic curve. Denote by \( Y_i \) the smooth projective model of \( Y_i \). Denote \( \mathbb{C}(Y_i) \) the function field of \( Y_i \) which is isomorphic to the function field \( \mathbb{C}(Y_i) \) of \( Y_i \).

Definition 2.6. \( Y^h = \bigoplus_{i=1}^{n} K_2(\mathbb{C}(Y_i)) \rightarrow K_2(\mathbb{C}(Y^h)) \).

For \( f_i, g_i \in \mathbb{C}(Y_i), i = 1, \ldots, n, j(\sum_{i=1}^{n} \{f_i, g_i\}) = \prod_{i=1}^{n} \{f_i, g_i\}, \) where we identify \( f_i, g_i \) as rational functions on \( Y^h \) via the injection \( j_i \). Note in this paper we shall use the multiplication in \( K_2 \) instead of addition.

Proposition 2.7. There exists a finite field extension \( F \) of \( \mathbb{C}(Y^h) \) with the property that for every \( i = 1, \ldots, n \), there is a representation

\[ P_i : \pi_1(M_L) \rightarrow SL_2(F) \]

such that for \( 1 \leq j \leq n \), if \( j \neq i \), then traces of \( P_i(\lambda_j) \) and \( P_i(\mu_j) \) are either 2 or \(-2\). If \( j = i \), then

\[ P_i(\lambda_i) = \begin{bmatrix} l_i & 0 \\ 0 & l_i^{-1} \end{bmatrix} \text{ and } P_i(\mu_i) = \begin{bmatrix} m_i & 0 \\ 0 & m_i^{-1} \end{bmatrix}. \]
Proof. By definition, for each \( i \), \( W_i \) is the Zariski closure of \( r_i(V_i) \) in \( X(T_i) \) and \( Y_i \) is mapped dominatingly to \( W_i \). The canonical morphism \( t : R_0 \to X_0 \) is surjective, so we can choose a curve \( D_i \subset R_0 \) such that \( t(D_i) \) is dense in \( V_i \). Hence \( r_i \circ t : D_i \to W_i \) is dominating. Then the function fields \( \mathbb{C}(D_i) \) and \( \mathbb{C}(Y_i) \) are finite extensions of \( \mathbb{C}(W_i) \). By [CSII, Page 115], there is a tautological representation \( \pi_i : \pi_1(M_i) \to SL_2(\mathbb{C}(D_i)) \), and for any \( g \in \pi_1(M_i) \) the trace of \( \pi_i(g) = I_g \). Let \( F_i \) be the composite field of \( \mathbb{C}(D_i) \) and \( \mathbb{C}(Y_i) \). It is finite over both \( \mathbb{C}(D_i) \) and \( \mathbb{C}(Y_i) \). We shall view \( \pi_i \) as a representation in \( SL_2(F_i) \). Since \( t(D_i) \) is dense in \( V_i \), by Lemma 2.4, if \( j \neq i \), traces of \( \pi_i(\lambda_j) \) and \( \pi_i(\mu_j) \) are \( \pm 2 \); if \( j = i \), traces of \( \pi_i(\lambda_i) \) and \( \pi_i(\mu_i) \) are non-constant functions on \( D_i \). Since \( \pi_i(\lambda_i) \) and \( \pi_i(\mu_i) \) are commuting and their eigenvalues \( l_i, m_i \) are in \( F_i \), \( \pi_i \) is conjugate in \( GL_2(F_i) \) to a representation

\[
P_i : \pi_1(M_i) \to SL_2(F_i)
\]

such that if \( j \neq i \), then traces of \( P_i(\lambda_j) \) and \( P_i(\mu_j) \) are either \( 2 \) or \( -2 \). If \( j = i \), then

\[
P_i(\lambda_i) = \begin{bmatrix} l_i & 0 \\ 0 & l_i^{-1} \end{bmatrix} \quad \text{and} \quad P_i(\mu_i) = \begin{bmatrix} m_i & 0 \\ 0 & m_i^{-1} \end{bmatrix}.
\]

Fix an algebraic closure \( \overline{\mathbb{C}(Y^h)} \) of \( \mathbb{C}(Y^h) \). As above, by viewing \( \mathbb{C}(Y_i) \) as a subfield of \( \overline{\mathbb{C}(Y^h)} \), we can identify the finite field extension \( F_i \) as a subfield of \( \overline{\mathbb{C}(Y^h)} \). In \( \overline{\mathbb{C}(Y^h)} \), take the composite of \( F_i \) and \( \mathbb{C}(Y^h) \) over \( \mathbb{C}(Y_i) \), denoted it by \( K_i \). Then \( F_i \subset K_i \) and \( K_i \) is a finite extension of \( \mathbb{C}(Y^h) \) because the extension \( F_i/\mathbb{C}(Y_i) \) is finite. Now let \( F \) be the composite of the fields \( K_1, \cdots , K_n \) in \( \overline{\mathbb{C}(Y^h)} \). Then \( F \) is a finite extension of \( \mathbb{C}(Y^h) \) since each \( K_i \) is. Now compose each \( P_i \) with the embedding \( SL_2(F_i) \hookrightarrow SL_2(F) \) and the proposition follows. \( \square \)

3. K-theory and Deligne Cohomology

First we shall recall the basic definitions of \( K_2 \) of a commutative ring \( A \). The reference is [Mil]. Let \( GL(A) \) be the direct limit of the groups \( GL_n(A) \), and let \( E(A) \) be the direct limit of the groups \( E_n(A) \) generated by all \( n \times n \) elementary matrices.

**Definition 3.1.** For \( n \geq 3 \), the Steinberg group \( St(n, A) \) is the group defined by generators \( x_{ij}^\lambda, 1 \leq i \neq j \leq n, \lambda \in A \), subject to the following three relations:

\[
(i) \quad x_{ij}^\lambda \cdot x_{ij}^\mu = x_{ij}^{\lambda + \mu}; \\
(ii) \quad [x_{ij}^\lambda, x_{jl}^\mu] = x_{il}^{\lambda \mu}, \text{ for } i \neq l; \\
(iii) \quad [x_{ij}^\lambda, x_{kl}^\mu] = 1, \text{ for } j \neq k, l \neq i.
\]

We have the canonical homomorphism \( \phi_n : St(n, A) \to GL_n(A) \) by \( \phi(x_{ij}^\lambda) = e_{ij}^\lambda \), where \( e_{ij}^\lambda \in GL_n(A) \) is the elementary matrix with entry \( \lambda \) in the \((i, j)\) place. Take the direct limit as \( n \to \infty \), we get

\[
\phi : St(A) \to GL(A).
\]

Its image \( \phi(St(A)) \) is equal to \( E(A) \), the commutator subgroup of \( GL(A) \).

**Definition 3.2.** \( K_2(A) = \text{Ker} \phi \).

It is well-known that \( K_2(A) \) is the center of the Steinberg group \( St(A) \) (See [Mil, Theorem 5.1]) and there is a canonical isomorphism \( \alpha : H_2(E(A); \mathbb{Z}) \to K_2(A) \) (See [Mil, Theorem 5.10]).
3.1. The Symbol. Let $U, V$ be two commutative elements of $E(A)$. Choose $u, v \in St(A)$ such that $U = \phi(u)$ and $V = \phi(v)$. Then the commutator $[u, v] = uvu^{-1}v^{-1}$ is in the kernel of $\phi$. Hence $[u, v] \in K_2(A)$. We can check it is independent of the choices of $u$ and $v$, and denote it by $U \star V$.

**Lemma 3.3.** (1). The construction is skew-symmetric: $U \star V = (V \star U)^{-1}$.

(2). It is bi-multiplicative: $(U_1 \cdot U_2) \star V = (U_1 \star V) \cdot (U_2 \star V)$.

(3). It is invariant under conjugation: if $P \in GL(A)$, then $(PUP^{-1}) \star (PVP^{-1}) = U \star V$.

**Proof.** This is [Mil, Lemma 8.1]. For (3), we remark that since $E(A)$ is a normal subgroup of $GL(A)$, the left-hand side of the formula makes sense. If $P, U, V$ are in $GL(n, A)$, then choose $p \in St(A)$ such that

$$\phi(p) = \begin{bmatrix} P & 0 \\ 0 & P^{-1} \end{bmatrix} \in E(A).$$

Now we have $\phi(pup^{-1}) = PUP^{-1}$ and $\phi(pvp^{-1}) = PVP^{-1}$. Hence,

$$[pup^{-1}, pvp^{-1}] = p[u, v]p^{-1} = [u, v].$$

$\square$

Given two units $f, g$ of $A$, consider the matrices:

$$D_f = \begin{bmatrix} f & 0 & 0 \\ 0 & f^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad D_g' = \begin{bmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g^{-1} \end{bmatrix}.$$

They are in $E(A)$ and commutative. Define the symbol $\{f, g\} := D_f \star D_g'$.

**Lemma 3.4.** (1). The symbol $\{f, g\}$ is skew-symmetric: $\{f, g\} = \{g, f\}^{-1}$.

(2). It is bi-multiplicative: $\{f_1, f_2, g\} = \{f_1, g\}\{f_2, g\}$.

(3). Denote $\text{diag}(f_1, \ldots, f_n)$ the diagonal matrix with diagonal entries $f_1, \ldots, f_n$. If $f_1 \cdots f_n = g_1 \cdots g_n = 1$, then

$$\text{diag}(f_1, \ldots, f_n) \star \text{diag}(g_1, \ldots, g_n) = \{f_1, g_1\}\{f_2, g_2\} \cdots \{f_n, g_n\}.$$ 

where the right-hand side means the product of the symbols $\{f_i, g_i\}$, $1 \leq i \leq n$.

**Proof.** [Mil] Lemma 8.2 Lemma 8.3].

Let $F$ be a field. Let $SL(F)$ be the direct limit of the groups $SL_n(F)$. We know that $SL(F) = E(F)$ and any element of $SL_n(F)$ is also naturally an element of $E(F)$.

**Lemma 3.5.** Let $u, t \in F$, then

(1).

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \star \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} = 1.$$

(2).

$$\begin{bmatrix} -1 & t \\ 0 & -1 \end{bmatrix} \star \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \star \begin{bmatrix} -1 & u \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & t \\ 0 & -1 \end{bmatrix} \star \begin{bmatrix} -1 & u \\ 0 & -1 \end{bmatrix}$$

are 2-torsions in $K_2(F)$.

(3). If $U$ and $V$ are two commuting matrices in $SL_2(F)$ and their traces are 2 or $-2$, then $U \star V$ is a 2-torsion in $K_2(F)$. In particular if both have trace 2, then $U \star V = 1$. 

Proof. We shall use the following notations. For \( s \in F \),
\[
M(1, s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M(-1, s) = \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}.
\]
In particular, \( M(1, 0) \) is the 2 \( \times \) 2 identity matrix and \( M(-1, 0) \) is the 2 \( \times \) 2 diagonal matrix with diagonal entries \(-1\).

For (1), \( M(1, t) \ast M(1, u) = [x_{12}^t, x_{12}^u] = 1 \) by the definition of \( St(A) \).

For (2), notice that by the definition, \( M(1, 0) \ast A = 1 \) and \( A \ast A = 1 \) for any \( A \in E(F) \).

By Lemma 3.3
\[
1 = (M(-1, 0) \ast M(-1, 0)) \ast M(1, s) = (M(-1, 0) \ast M(1, s))^2,
\]
so \( M(-1, 0) \ast M(1, s) \) is a 2-torsion in \( K_2(F) \). Since
\[
M(-1, t) = M(-1, 0) \ast M(1, -t), \quad M(-1, u) = M(-1, 0) \ast M(1, -u),
\]
by Lemma 3.3 and the first part, we have
\[
M(-1, t) \ast M(1, u) = (M(-1, 0) \ast M(1, u))(M(1, -t) \ast M(1, u)) = M(-1, 0) \ast M(1, u)
\]
and
\[
M(-1, t) \ast M(-1, u) = (M(-1, 0) \ast M(1, -u))(M(1, -t) \ast M(-1, 0))
\]

hence they are 2-torsion.

For (3), we can find \( P \in GL_2(F) \) such that
\[
PUP^{-1} = \begin{bmatrix} \pm 1 & t \\ 0 & \pm 1 \end{bmatrix} \quad \text{and} \quad PV^{-1} = \begin{bmatrix} \pm 1 & u \\ 0 & \pm 1 \end{bmatrix}.
\]

Then it follows from the first two parts and Lemma 3.3 (3). \( \square \)

The following proposition slightly generalizes [CCGLS, Lemma 4.1]. The proof is the same.

**Proposition 3.6.** Let \( \pi \) be a free abelian group of rank two with \( \{e_1, e_2\} \) its basis. Let \( f : \pi \to E(A) \) be a group homomorphism defined by \( f(e_1) = U, \ f(e_2) = V \). Then there is a generator \( t \) of \( H_2(\pi; \mathbb{Z}) \) such that \( \alpha(f_*(t)) = U \ast V \), where \( \alpha : H_2(E(A); \mathbb{Z}) \to K_2(A) \) is the canonical isomorphism and \( f_* : H_2(\pi; \mathbb{Z}) \to H_2(E(A); \mathbb{Z}) \) is the homomorphism induced by \( f \).

**Proof.** Since \( \pi \) is abelian, \( U \) and \( V \) are commutative. \( U \ast V \) is well-defined. Let \( F \) be a free group on \( \{e_1, e_2\} \). The homomorphism \( f \) gives rise to the following commutative diagram of short exact sequences of groups:
\[
\begin{array}{cccccc}
0 & \longrightarrow & [F, F] & \longrightarrow & F & \longrightarrow & \pi & \longrightarrow & 0 \\
\downarrow & & f_2 & \downarrow f_1 & \downarrow f & & \downarrow \\
0 & \longrightarrow & K_2(A) & \longrightarrow & St(A) & \phi & \longrightarrow & E(A) & \longrightarrow & 0
\end{array}
\]
where \( f_2([e_1, e_2]) = U \ast V \). Apply the homology spectral sequence to the above diagram, we obtain the following diagram:
\[
\begin{array}{c}
H_2(\pi; \mathbb{Z}) \longrightarrow H_0(\pi; H_1([F, F]; \mathbb{Z})) \\
\downarrow f_* \downarrow g \\
H_2(E(A); \mathbb{Z}) \longrightarrow K_2(A)
\end{array}
\]
The top horizontal arrow is an isomorphism. The class of \([e_1, e_2]\) is the generator of \(H_0(\pi; H_1([F, F]; \mathbb{Z}))\). It is mapped to \(U \star V\) by \(g\) which is induced by \(f_2\). Let \(t\) be the generator of \(H_2(\pi; \mathbb{Z})\) mapped to the class of \([e_1, e_2]\). Then we have \(\alpha(f_*(t)) = U \star V\) by the commutative diagram. 

**Corollary 3.7.** (1) If \(U = \text{diag}(u, u^{-1})\) and \(V = \text{diag}(v, v^{-1})\), where \(u, v\) are units of \(A\), then there is a generator \(t\) of \(H_2(\pi; \mathbb{Z})\) such that \(\alpha(f_*(t)) = \{u, v\}^2\).

(2) Suppose \(A\) is a field. If \(U\) and \(V\) are two commuting matrices in \(SL_2(A)\) and their traces are 2 or \(-2\), then the image of any generator of \(H_2(\pi; \mathbb{Z})\) is a 2-torsion in \(K_2(A)\).

**Proof.** For (1), by Lemma 3.4, we have \(U \star V = \{u, v\}\{u^{-1}, v^{-1}\} = \{u, v\}^2\).

For (2), by Lemma 3.4 (3), \(U \star V\) is a 2-torsion in \(K_2(F)\).

Now we can prove the following:

**Theorem 3.8.** For each \(i = 1, \ldots, n\), there is an integer \(\epsilon(i) = 1\) or \(-1\), such that the symbol \(\prod_{i=1}^n \{i_i, m_i\}^{\epsilon(i)}\) is a torsion element in \(K_2(\mathbb{C}Y^h)\).

**Proof.** First, by Proposition 2.7 for each \(i = 1, \ldots, n\), there exists a finite extension \(F\) of \(\mathbb{C}Y^h\) and a representation \(P_i : \pi_1(M_L) \to SL_2(F)\) such that for \(1 \leq j \leq n\), if \(j \neq i\), then traces of \(P_i(\lambda_j)\) and \(P_i(\mu_j)\) are either 2 or \(-2\); if \(j = i\), then

\[
P_i(\lambda_i) = \begin{bmatrix} i_i & 0 \\ 0 & i_i^{-1} \end{bmatrix} \quad \text{and} \quad P_i(\mu_i) = \begin{bmatrix} m_i & 0 \\ 0 & m_i^{-1} \end{bmatrix}.\]

The inclusions of \(\pi_1(T_i)\) into \(\pi_1(M_L)\) induce the homomorphisms \(\pi_1(T_i) \to E(F)\) by composing with \(P_i\). This gives rise to the following homomorphisms on the group homology:

\[
\bigoplus_{i=1}^n H_2(\pi_1(T_i), \mathbb{Z}) \xrightarrow{\alpha} H_2(\pi_1(M_L), \mathbb{Z}) \xrightarrow{\beta} H_2(E(F), \mathbb{Z}) = K_2(F),
\]

where \(\alpha = j_{1*} + \cdots + j_{n*}\), \(\beta = P_1 + \cdots + P_n\); \(j_{i*}\) are the morphisms on the group homology induced by the inclusions \(j_i : \pi_1(T_i) \hookrightarrow \pi_1(M_L)\), and \(P_i\) are those induced by \(P_i\).

The orientation of \(M_L\) induces the orientation on each boundary torus \(T_i\). Let \([T_i]\) be the oriented class of \(H_2(T_i, \mathbb{Z}) = \mathbb{Z}\). By Corollary 3.7 (1), for each \(i\), there is a generator \(\xi_i\) of \(H_2(\pi_1(T_i))\) such that \(P_i(j_{i*}(\xi_i)) = \{i_i, m_i\}^2\). Since \(T_i\) is the \((\pi_1(T_i), 1)\) space, \(H_2(\pi_1(T_i), \mathbb{Z}) = H_2(T_i, \mathbb{Z})\). If \(\xi_i = [T_i]\), define \(\epsilon(i) = 1\); if \(\xi_i = -[T_i]\), then define \(\epsilon(i) = -1\).

Since \(L\) is a hyperbolic link, \(M_L\) is a \((\pi_1(M_L), 1)\) space. Hence we have \(H_2(\pi_1(M_L), \mathbb{Z}) = H_2(M_L, \mathbb{Z})\). Under this identification,

\[
\alpha(\epsilon(1)\xi_1, \ldots, \epsilon(n)\xi_n) = \sum_{i=1}^n [T_i] = [\partial M_L] = 0 \text{ in } H_2(M_L, \mathbb{Z}).
\]

Therefore,

\[
\beta(\alpha(\epsilon(1)\xi_1, \ldots, \epsilon(n)\xi_n)) = 1 \text{ in } K_2(F).
\]
On the other hand, we have

$$
\beta(\alpha(\epsilon(1)\xi_1, \cdots, \epsilon(n)\xi_n)) = \beta\left(\sum_{i=1}^{n} j_{is}(\epsilon(i)\xi_i)\right)
= \sum_{k=1}^{n} P_{ks}\left(\sum_{i=1}^{n} j_{is}(\epsilon(i)\xi_i)\right)
= \sum_{i=1}^{n} P_{is}(j_{is}(\epsilon(i)\xi_i)) + \sum_{1 \leq k \leq n} P_{ks}(j_{is}(\epsilon(i)\xi_i))
= \prod_{i=1}^{n} \{l_i, m_i\}^{2(\epsilon(i))} \cdot \prod_{1 \leq i \neq k \leq n} P_k(\mu_i) \star P_k(\lambda_i),
$$

where the last step follows from Proposition 3.6 and Corollary 3.7. Note also we use multiplication in $K_2(F)$.

By Corollary 3.7 (2), $\prod_{1 \leq i \neq k \leq n} P_k(\mu_i) \star P_k(\lambda_i)$ is a 2-torsion. Compare with (3.2), we see that $\prod_{i=1}^{n} \{l_i, m_i\}^{2(\epsilon(i))}$ is a 2-torsion in $K_2(F)$. By the same argument in [LW2 Proposition 3.2], $\prod_{i=1}^{n} \{l_i, m_i\}^{(\epsilon(i))}$ is a torsion in $K_2(C(Y^h))$. 

**Remark 3.1.** This theorem is a natural generalization of our previous result [LW2 Proposition 3.2] about the hyperbolic knot case.

**3.2. Deligne cohomology.** Let $X$ be a nonsingular variety over $\mathbb{C}$. First let us recall the definition of the (holomorphic) Deligne cohomology groups of $X$. For more details, see [EV].

We define the complex $\mathbb{Z}(p)_\varnothing$ of sheaves on $X$ as follows:

$$
\mathbb{Z}(p)_\varnothing : \mathbb{Z}(p) \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_X,
$$

where $\mathbb{Z}(p)$ is the constant sheaf $(2\pi i)^p\mathbb{Z}$ and sits in degree zero, $\mathcal{O}_X$ is the sheaf of holomorphic functions on $X$, and $\Omega^i_X$ is the sheaf of holomorphic $i$-forms on $X$. The first map in (3.3) is the inclusion and $d$ is the exterior differential. The Deligne cohomology groups of $X$ are defined as the hypercohomology of the complex $\mathbb{Z}(p)_\varnothing$:

$$
H^q_{\varnothing}(X; \mathbb{Z}(p)) := \mathbb{H}^q(X; \mathbb{Z}(p)_\varnothing).
$$

For example, the exponential exact sequence of sheaves on $X$

$$
0 \to \mathbb{Z}(1) \to \mathcal{O}_X \to \mathcal{O}^*_X \to 0
$$

gives rise to a quasi-isomorphism between $\mathbb{Z}(1)_\varnothing$ and $\mathcal{O}^*_X[-1]$, where $\mathcal{O}^*_X$ is the sheaf of non-vanishing holomorphic functions on $X$. Moreover there is a quasi-isomorphism between $\mathbb{Z}(2)_\varnothing$ and the complex ([EV page 46])

$$
\mathcal{O}^*_X \xrightarrow{d\log} \Omega^1_X[-1].
$$

Therefore, we have for any integer $q$, 

$$
H^q_\varnothing(X; \mathbb{Z}(1)) = H^{q-1}(X; \mathcal{O}^*_X); \quad H^q_\varnothing(X; \mathbb{Z}(2)) = \mathbb{H}^{q-1}(X; \mathcal{O}^*_X \to \Omega^1_X).
$$

On the other hand, Degline ([De]) interprets $\mathbb{H}^q(X; \mathcal{O}^*_X \to \Omega^1_X) = H^q_\varnothing(X; \mathbb{Z}(2))$ as the group of holomorphic line bundles with (holomorphic) connections over $X$. 

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**Remark 3.1.** This theorem is a natural generalization of our previous result [LW2 Proposition 3.2] about the hyperbolic knot case.
Let \( \mathbb{C}(X) \) be the function field of \( X \). Given two functions \( f, g \in \mathbb{C}(X) \), let \( D(f, g) \) be the divisors of the zeros and poles of \( f \) and \( g \). \( |D(f, g)| \) denotes its support. Then we have the morphism:

\[
(f, g) : X - |D(f, g)| \longrightarrow \mathbb{C}^* \times \mathbb{C}^*
\]
given by \( (f, g)(x) = (f(x), g(x)) \).

Let \( \mathcal{H} \) be the Heisenberg line bundle with connection on \( \mathbb{C}^* \times \mathbb{C}^* \). For its construction, see \[Bl\, Ram\]. Pull back \( \mathcal{H} \) along \( (f, g) \), we obtain a line bundle with connection on \( X - |D(f, g)| \), denoted by \( r(f, g) \). Hence \( r(f, g) \in \mathbb{H}^1(V; \mathcal{O}_V^* \to \Omega_V^1) = H^2_\mathbb{Z}(V; \mathbb{Z}(2)) \), where \( V = X - |D(f, g)| \). Moreover we can represent \( r(f, g) \) in terms of \( \check{\text{C}} \)-ech cocycles for \( \mathbb{H}^1(V; \mathcal{O}_V^* \to \Omega_V^1) \). Indeed, choose an open covering \( (U_i)_{i \in I} \) of \( V \) such that the logarithm of \( f \) is well-defined on every \( U_i \), denoted by \( \log_i f \). Then \( r(f, g) \) is represented by the cocyle \( (c_{ij}, \omega_i) \), with

\[
c_{ij} = g^{\frac{1}{2\pi i}(\log f - \log g)}, \quad \text{on } U_i \cap U_j;
\]

\[
\omega_i = \frac{1}{2\pi i} \log_i f \frac{dg}{g}, \quad \text{on } U_i.
\]

Its curvature is

\[
R = \frac{1}{2\pi i} \frac{df}{f} \wedge \frac{dg}{g}.
\]

**Remark 3.2.** There is a cup product \( \cup \) on the Deligne cohomology groups (see \[Be\, EV\]). For \( f, g \in H^0(X; \mathcal{O}_X^*) = H^1_\mathbb{Z}(X; \mathbb{Z}(1)) \) as above, the cup product \( f \cup g \) is exactly the line bundle \( r(f, g) \in H^2_\mathbb{Z}(X; \mathbb{Z}(2)) \).

Furthermore, we have the following properties about \( r(f, g) \):

**Lemma 3.9.** \( r(f_1f_2, g) = r(f_1, g) \otimes r(f_2, g), r(f, g) = r(g, f)^{-1} \), and the Steinberg relation \( r(f, 1 - f) = 1 \) holds if \( f \neq 0, f \neq 1 \).

**Proof.** See \[Bl\, EV\] and \[Ram\, Section 4\]. The proofs there assume that \( X \) is a curve. But they are valid for arbitrary \( X \) without change. Note that in order to prove the Steinberg relation, we need the ubiquitous dilogarithm function. \( \square \)

Now proceed as in \[Bl\, Ram\] for curves, we have the regulator map:

\[
r : K_2(\mathbb{C}(X)) \longrightarrow \varinjlim_{U \subset X: \text{Zariski open}} H^2_\mathbb{Z}(U; \mathbb{Z}(2))
\]

Notice that when \( \dim X > 1 \), the only difference is that the line bundle \( r(f, g) \) is not necessarily flat. However, we have the following

**Proposition 3.10.** If \( x \in K_2(\mathbb{C}(X)) \) is a torsion, then the corresponding line bundle \( r(x) \) is flat.

**Proof.** Let \( U \) be the Zariski open subset over which the line bundle \( r(x) \) is defined. Since \( x \) is a torsion in \( K_2(\mathbb{C}(X)) \), \( r(x) \) is a torsion in \( \mathbb{H}^1(U; \mathcal{O}_U^* \to \Omega_U^1) \). Choose a suitable open covering \( (U_i)_{i \in I} \) of \( U \) such that \( r(x) \) is represented by \( \check{\text{C}} \)-ech cocyle \( (c_{ij}, \omega_i) \) with \( c_{ij} \in \mathcal{O}_U^*(U_i \cap U_j) \) and \( \omega_i \in \Omega^1(U_i) \). Then there exists an integer \( n > 0 \), such that the class represented by the cocycle \( ((c_{ij})^n, n\omega_i) \) is zero. Hence, there exists \( t_i \in \mathcal{O}_X^*(U_i) \) (or by a refinement covering of \( \{U_i\} \) ), such that

\[
c_{ij}^n = \frac{t_j}{t_i}, \quad \omega_i = \frac{1}{n} \frac{dt_i}{t_i}.
\]
Therefore, $d\omega = 0$ for all $i$ and the curvature is 0. 

Let $|D|$ be the support of the divisors of zeros and poles of the rational functions $m_i$, $l_i$ on $Y^h$, $1 \leq i \leq n$. Define $Y^h_0 = Y^h - |D|$. The line bundle $r(\prod_{i=1}^{n}\{l_i, m_i\})^{(i)}$ is well-defined over $Y^h_0$.

**Corollary 3.11.** The line bundle $r(\prod_{i=1}^{n}\{l_i, m_i\})^{(i)}$ over $Y^h_0$ is flat, therefore it is an element of $H^1(Y^h_0; \mathbb{C}^*)$.

**Proof.** This follows from Theorem 3.8 and Proposition 3.10.

By the Čech cocycle for $r(f, g)$ given in (3.4) and (3.5), we can represent $r(\prod_{i=1}^{n}\{l_i, m_i\})^{(i)}$ as follows. Choose an open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of $Y^h_0$ such that on every $U_\alpha$, the logarithms of $l_i$ are well-defined and denoted by $\log \alpha l_i$. Then $r(\prod_{i=1}^{n}\{l_i, m_i\})^{(i)}$ is represented by the cocycle $(c_\alpha, \omega_\alpha)$:

$$c_\alpha = \prod_{i=1}^{n} l_i^{\epsilon(i)\left(\frac{1}{2\pi i} (\log \beta l_i - \log \alpha l_i)\right)}$$

on $U_\alpha \cap U_\beta$,

$$\omega_\alpha = \sum_{i=1}^{n} \epsilon(i) \left(\frac{1}{2\pi i} \log m_i \right) \frac{dm_i}{m_i}$$

on $U_\alpha$.

Let $t_0 = (l_0^1, m_1^0, \cdots, l_0^n, m_n^0) \in Y^h_0$ be a point corresponding to the hyperbolic structure of the link complement $S^3 - L$. Then the monodromy of the flat line bundle $r(\prod_{i=1}^{n}\{l_i, m_i\})^{(i)}$ give rises to the representation $M : \pi_1(Y^h_0, t_0) \to \mathbb{C}^*$. With its explicit descriptions (3.7), (3.8), we have the following formula for $M$. Let $\gamma$ be a loop based at $t_0$. Let $\log l_i$ be a branch of logarithm of $l_i$ over $\gamma - \{t_0\}$, then we have (c.f. Def. (2.7.2))

$$M(\gamma) = \exp \left\{ \sum_{i=1}^{n} \left( - \frac{\epsilon(i)}{2\pi i} \left( \int_{\gamma} \log l_i \frac{dm_i}{m_i} - \log m_i(t_0) \int_{\gamma} \frac{dl_i}{l_i} \right) \right) \right\}.$$

**Theorem 3.12.** (i) The real 1-form $\eta = \sum_{i=1}^{n} \epsilon(i)(\log |l_i| d \arg m_i - \log |m_i| d \arg l_i)$ is exact on $Y^h_0$. Hence there exists a smooth function $V : Y^h_0 \to \mathbb{R}$ such that

$$dV = \sum_{i=1}^{n} \epsilon(i)(\log |l_i| d \arg m_i - \log |m_i| d \arg l_i).$$

(ii) Suppose $m_i^0 = 2$, $1 \leq i \leq n$. For a loop $\gamma$ with initial point $t_0$ in $Y^h_0$

$$\frac{1}{4\pi^2} \sum_{i=1}^{n} \epsilon(i) \left( \int_{\gamma} \log |m_i| d \log |l_i| + \arg l_i d \arg m_i \right) = \frac{p}{q},$$

where $p$ is some integer and $q$ is the order of the symbol $\prod_{i=1}^{n}\{l_i, m_i\}^{(i)}$ in $K_2(\mathbb{C}(Y^h))$.

**Proof.** First, by (3.8), the curvature of the flat line bundle is

$$R = \sum_{i=1}^{n} \frac{\epsilon(i)}{2\pi i} \left( \frac{dl_i}{l_i} \wedge \frac{dm_i}{m_i} \right) = 0.$$

On the other hand, we have $d\eta = \text{Im}(\sum_{i=1}^{n} \epsilon(i)(\frac{dl_i}{l_i} \wedge \frac{dm_i}{m_i}))$, hence $\eta$ is a real closed 1-form.
Since the symbol $\prod_{i=1}^{n} \{l_i, m_i\}^{(i)}$ has order $q$ in $K_2(\mathbb{C}(Y^h))$, for a loop $\gamma$ in $Y^h$, by (3.9) we have
\[
1 = M(\gamma)^q = (\exp \left\{ \sum_{i=1}^{n} \left( \frac{\epsilon(i)}{2\pi i} \right) \int_{\gamma} \log l_i \, \frac{dm_i}{m_i} - \log m_i(t_0) \int_{\gamma} \frac{dl_i}{l_i} \right\})^q.
\]
Write $\sum_{i=1}^{n} \epsilon(i) \int_{\gamma} \log l_i \, \frac{dm_i}{m_i} - \log m_i(t_0) \int_{\gamma} \frac{dl_i}{l_i} = Re + i Im$, where $Re$ and $Im$ are the real and imaginary parts respectively. Then we have $\exp \left( \frac{q \cdot Im}{2\pi} + \frac{q \cdot Re}{2\pi i} \right) = 1$. Therefore, $Im = 0$ and $\frac{q \cdot Re}{2\pi i} = 2\pi ip$, for some integer $p$.

A straightforward calculation or [LW2, Lemma 3.4] shows that
\[
(3.10) \quad Im = \int_{\gamma} \eta, \quad Re = -\sum_{i=1}^{n} \epsilon(i) \int_{\gamma} (\log |m_i| d\log |l_i| + \arg l_i d\arg m_i) = \int_{\gamma} \xi.
\]
These immediately imply both parts of the theorem. \hfill \Box

**Remark 3.3.** When $n = 1$, our $V$ is (up to sign) the volume function of the representation of the knot complement ([Dun]). For $n \geq 2$, up to some constant and signs related to the orientations on each boundary component of the hyperbolic link complement, the function $V$ should be closely related to the volume function given in [Ho, Theorem 5.5].

**Remark 3.4.** If there exists any representation $\rho: \pi_1(Y^h) \to GL_n(\mathbb{C}), \ n \geq 2$, then Reznikov [Re, Theorem 1.1] proved that the Chern classes $c_i \in H^i(\mathbb{C}(Y^h, \mathbb{Z}(i))$ in the Deligne cohomology groups are torsion for all $i \geq 2$.

### 3.3. On the Bohr-Sommerfeld quantization condition for hyperbolic links

In this section, we shall discuss the above Theorem [3.12][ii] from symplectic point of view. When $n = 1$, this is the Bohr-Sommerfeld quantization condition proposed by Gukov for knots [Guk, Page 597], and is proved in [LW2, Theorem 3.3 (2)].

Let $\Sigma$ be a closed surface with fundamental group $\pi$. Its $SL_2(\mathbb{C})$-character variety is the space of equivalence classes of representations from $\pi$ into $SL_2(\mathbb{C})$. This variety carries a natural complex-symplectic structure, where a complex-symplectic structure is a nondegenerate closed holomorphic exterior 2-form (see [Go1, Go2]).

A homomorphism $\rho: \pi \to SL_2(\mathbb{C})$ is irreducible if it has no proper linear invariant subspace of $\mathbb{C}^2$, and irreducible representations are stable points, denoted by $\text{Hom}(\pi, SL_2(\mathbb{C}))^s$. Now $SL_2(\mathbb{C})$ acts freely and properly on $\text{Hom}(\pi, SL_2(\mathbb{C}))^s$, and the quotient $X^s(\Sigma) = \text{Hom}(\pi, SL_2(\mathbb{C}))^s / SL_2(\mathbb{C})$ is an embedding onto an open subset in the geometric quotient $\text{Hom}(\pi, SL_2(\mathbb{C}))/\text{SL}_2(\mathbb{C})$. Thus $X^s(\Sigma)$ is a smooth irreducible complex quasi-affine variety which is dense in the geometric quotient (see [Go2, Section 1]). Note that $\rho$ is a nonsingular point if and only if $\dim Z(\rho)/Z(SL_2(\mathbb{C})) = 0$, and this corresponds to the top stratum $X^s(\Sigma)$, where $Z(\rho)$ is the centralizer of $u$ in $SL_2(\mathbb{C})$. If $\rho \in \text{Hom}(\pi, SL_2(\mathbb{C}))$ is a singular point (i.e., $\dim Z(\rho)/Z(SL_2(\mathbb{C})) > 0$), then all points of $\sigma \in \text{Hom}(\pi, Z(\rho))$ with $\text{stab}(\sigma) = Z(\sigma) = Z(\rho)$ have the same orbit type and form a stratification of the $SL_2(\mathbb{C})$-character variety (see [Go1, Section 1]).

We have the $SL_2(\mathbb{C})$-character variety $X(T^2)$ of the torus $T^2$ as a surface in $\mathbb{C}^3$ given by
\[
x^2 + y^2 + z^2 - xyz - 4 = 0.
\]
See [LW1, Proposition 3.2]. There exists a natural symplectic structure on the smooth top stratum \( X^s(T^2) \) of \( X(T^2) \), and there exists a symplectic structure \( \omega \) on the character variety \( X^s(\partial M_L) = \prod_{i=1}^n X^s(T^2_i) \) such that \( X(M_L) \cap X^s(\partial M_L)(\subset X(M_L)) \) is a Lagrangian subvariety of \( X^s(\partial M_L) \), where \( X^s(\partial M_L) \) is a smooth irreducible variety which is open and dense in \( X(\partial M_L) \).

The inclusion \( \partial M_L \to M_L \) indeed induces a degree-one map on the irreducible components. Thus \( r(X_0)^s \) (the smooth part of the image \( r(X_0) \)) is a Lagrangian submanifold of the symplectic manifold \( X^s(\partial M_L) \). Note that the pullback of the symplectic 2-form on the double covering of \( X^s(T^2) \) is again a skew-symmetric and nondegenerate. The symplectic form \( \tilde{\omega} \) through the map \( t_i \) on the irreducible component gives the Lagrangian property for the corresponding pullback of the Lagrangian part \( r(X_0)^s \). Hence we have the product Lagrangian smooth part of the pullback of \( \prod_{i=1}^n r(X_0)^s \). Then we need to see that the smooth projective model preserves the Lagrangian and symplectic property.

Let \( \tilde{X}(T^2) \) be the symplectic blowup of the double covering of \( X(T^2) \) as in [MS]. The blowup in the complex category carries a natural symplectic structure on \( \tilde{X}(T^2) \) ([MS, Section 7.1]). On the other hand, the corresponding part \( \tilde{Y}_i \) of \( Y_i \) (the irreducible component of \( D_i \) containing \( y_i \)) lies in the symplectic manifold \( \tilde{X}(T^2) \).

Define a compatible Lagrangian blowup with respect to the complex blowup as following. Define a real submanifold \( \mathbb{R}^n \) of \( \mathbb{R}^n \times \mathbb{R}P^{n-1}(\subset \mathbb{C}^n \times \mathbb{C}P^{n-1}) \) as a subspace of pairs \((x,l)\) with \( x = Re(z) \in l \), where \( l \in \mathbb{R}P^{n-1} \) is a real line in \( \mathbb{R}^n \). If \( I_C \) is an complex conjugation on \( \mathbb{C}^n \) and \( J_{CP^{n-1}} \) be the complex involution on \( CP^{n-1} \) as complex conjugation on each components, then

\[
\mathbb{R}^n = \text{Fix}(I_C \times J_{CP^{n-1}}|_{\mathbb{C}^n}) \subset \mathbb{C}^n = \{(z_1, \cdots, z_n; [w_1 : \cdots : w_n]) | w_j z_k = w_k z_j, 1 \leq j, k \leq n\}.
\]

It is clear that \( \mathbb{R}^n \) is Lagrangian in \( \mathbb{C}^n \). Hence the real Lagrangian blowup \( \tilde{Y}_i \) is Lagrangian in \( \tilde{X}(T^2) \), and the Lagrangian submanifold \( \tilde{Y}_h \) is Lagrangian in the symplectic manifold \( \prod_{i=1}^n \tilde{X}(T^2) \). This only gives a way to have the symplectic and Lagrangian properties being preserved under the blowup, and treat the Lagrangian blowup in a real blowup with respect to the complex one.

Now we have a Lagrangian submanifold \( Y_i^h \) in a symplectic manifold. Suppose \( m_i^0 = 2 \), \( 1 \leq i \leq n \). For a loop \( \gamma \) with initial point \( t_0 \) in \( Y_i^h \), by Theorem [3.12(ii)]

\[
\frac{1}{4\pi^2} \sum_{i=1}^n \epsilon(i) \int_{\gamma} (\log |m_i| \ d \log |l_i| + \arg l_i \ d \arg m_i) = \frac{p}{q},
\]

where \( p \) is some integer and \( q \) is the order of the symbol \( \prod_{i=1}^n \{l_i, m_i \}^{\epsilon(i)} \) in \( K_2(\mathbb{C}(Y_i^h)) \). We will call this result the Bohr-Sommerfield quantization condition for hyperbolic links. It would be interesting to give an interpretation from mathematical physics, as what Gukov did for hyperbolic knots.

4. On a possible unified Volume Conjecture for both knots and links

In this section, we shall give some descriptions and speculations of a possible parametrized volume conjecture which includes both hyperbolic knots and links.

By Corollary 3.11, the class \( r(\prod_{i=1}^n \{m_i, l_i \}^{\epsilon(i)}) \) corresponds to a flat line bundle over \( Y_i^h \), therefore the curvature of the holomorphic connection is zero. Formally this can be expressed
as \(d(\xi + i\eta) = 0\), where \(\xi\) and \(\eta\) are defined in (3.10). Hence, \(\frac{1}{2\pi i}(\xi + i\eta)\) can be viewed as the 1-form Chern-Simons of the line bundle \(r(\prod_{i=1}^{n}(m_i, l_i)^{\epsilon_i})\).

Let \(\gamma : [0, 1] \to Y^h_0\) be a path with initial point \(\gamma(0) = t_0\) the point corresponding to the complete hyperbolic structure. Write \(\gamma(t) = (m(t), l(t)) = (m_1(t), l_1(t), \ldots, m_n(t), l_n(t))\). Recall that \(q\) is the order of the symbol \(\prod_{i=1}^{n}(m_i, l_i)^{\epsilon_i}\) in \(K_2(\mathbb{C}(Y^h))\). Let \(Vol(L)\) and \(CS(L)\) be the volume and usual Chern-Simons invariant of the complete hyperbolic structure on \(S^3 - L\) respectively. Now according to Theorem 3.12, we define

\[
V(\gamma(1)) = Vol(L) + 2 \cdot \sum_{i=1}^{n} \epsilon(i) \int_{\gamma} (\log |l_i| d\arg m_i - \log |m_i| d\arg l_i).
\]

and call (4.2) the special Chern-Simons invariant of the hyperbolic link \(L\) at \(\gamma(1)\). By Theorem 3.12, the quantity \(\sqrt[2\pi]{V(\gamma(1))} + \sqrt[2\pi]{U(\gamma(1))}\) lies in \(\mathbb{C}/\mathbb{Z}\).

**Remark 4.1.** The above \(U(\gamma(1))\) in (4.2) is different from the usual Chern-Simons invariant for a 3-dimensional manifold. The latter comes from the transgressive 3-form of the second Chern class of the 3-dimensional manifold.

When \(\gamma(1)\) varies in a neighborhood of \(t_0\), the \((\mathbb{C}^*)^n\)-parametrized invariant \(\sqrt[2\pi]{V(\gamma(1))} + \sqrt[2\pi]{U(\gamma(1))}\) is the generalization of the right-hand side of [LW2, Conjecture 3.9].

In order to formulate a conjecture parallel to the knot case as in [LW2, Conjecture 3.9], we have to find a way of relating the quantum invariants to the \(n\)-dimensional variety \(Y^h_0\) which comes from the \(SL_2(\mathbb{C})\) character variety. By the work of Kashaev and Baseilhac-Benedetti ([BB, Ka1]), there exists an \(SL_2(\mathbb{C})\) quantum hyperbolic invariant for a hyperbolic link in \(S^3\), which is conjectured to give the information of the volume and Chern-Simons at the point for the complete hyperbolic structure. We speculate that there should exist a “family” version of their quantum hyperbolic invariants parametrized by \(Y^h_0\), then we can replace the left-hand side of [LW2, Conjecture 3.9] by its logarithm limit to formulate the generalized volume conjecture for a hyperbolic link in \(S^3\).

Here is a conjectural description. First we assume that for a point \(p \in Y^h_0\) near \(t_0\) of the complete hyperbolic structure, we can define certain quantum invariants \(K_N(L, p)\). Then for fixed number \(a_j, 1 \leq j \leq n\), we take \(m_j = -\exp(i\pi a_j), 1 \leq j \leq n\), in (4.1) and (4.2). We formulate the following:

**Conjecture:** (A Possibly Unified Parametrized Volume Conjecture)

\[
\lim_{N \to \infty} \frac{\log K_N(L, \gamma(1))}{N} = \frac{1}{2\pi} (Vol(\gamma(1)) + i\frac{1}{2\pi} U(\gamma(1))).
\]

**Remark 4.2.** When \(L\) is a hyperbolic knot (i.e., \(n = 1\), we can take \(K_N(L, \gamma(1))\) as the colored Jones polynomial \(J_N(K, e^{2\pi i a/N})\). In this case our unified Conjecture 4.3 is reduced to [LW2, The Reformulated Generalized Volume Conjecture (3.9)] for hyperbolic knots.
Remark 4.3. When $n \geq 2$, we can take $K_N(L, \gamma(0)) = K_N(L, t_0)$ as the Kashaev and Baseilhac-Benedetti invariant which is well-defined and conjectured to give the information of the volume and Chern-Simons at the complete hyperbolic structure $t_0$. For general $p$, although we expect that there is a way of deforming $K_N(L, t_0)$ to get $K_N(L, p)$, we do not have a rigorous definition.

Remark 4.4. Since the usual colored Jones polynomial formulation of volume conjecture does not hold for all links ([MMOTY]), it is a very interesting question to see what is really behind the volume conjecture for links. From the regulator point of view developed here, we expect that our unified volume conjecture (4.3) is a good candidate which gives a $(\mathbb{C}^*)^n$-parametrized version of the volume conjecture for both links and knots.

Acknowledgements. Q. Wang is grateful for the support and hospitality of the Marie Curie Research Programme at DPMMS, University of Cambridge and the program ANR “Galois” at the Université Pierre et Marie Curie (Paris 6) and École Normale Supérieure, Paris. He wants to thank professors Y. André and A. Scholl for their helpful discussions.

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