Global Well-posedness of the Parabolic-parabolic Keller-Segel Model in $L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ and $H^1_b(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$

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Abstract

In this paper, we study global well-posedness of the two-dimensional Keller-Segel model in Lebesgue space and Sobolev space. Recall that in the paper “Existence and uniqueness theorem on mild solutions to the Keller-Segel system in the scaling invariant space. J. Differential Equations, 252 (2012), 1213–1228”, Kozono, Sugiyama & Wachi studied global well-posedness of $n(\geq 3)$ dimensional Keller-Segel system and posted a question about the even local in time existence for the Keller-Segel system with $L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ initial data. Here we give an affirmative answer to this question: in fact, we show the global in time existence and uniqueness for $L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ initial data. Furthermore, we prove that for any $H^1_b(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ initial data with $H^1_b(\mathbb{R}^2) := H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, there also exists a unique global mild solution to the parabolic-parabolic Keller-Segel model. The estimates of $\sup_{t>0} t^{-\frac{n}{p}} \|u\|_{L^p}$ for $(n, p) = (2, \infty)$ and the introduced special half norm, i.e. $\sup_{t>0} (1+t)^{-\frac{1}{2}} \|\nabla v\|_{L^\infty}$, are crucial in our proof.

Keywords: Keller-Segel model; Fourier transformation; well-posedness; decay property; parabolic-parabolic system.

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1 Introduction

In this article, we study the following two-dimensional (2D) Keller-Segel model:

\begin{align*}
  u_t - \Delta u + \nabla \cdot (u \nabla v) &= 0 \quad \text{in } (0, \infty) \times \mathbb{R}^2, \\
  v_t - \Delta v + v - u &= 0 \quad \text{in } (0, \infty) \times \mathbb{R}^2, \\
  (u, v)|_{t=0} &= (u_0, v_0) \quad \text{in } \mathbb{R}^2,
\end{align*}

where $(t, x) \in (0, \infty) \times \mathbb{R}^2$, $u = u(t, x)$ and $v = v(t, x)$ are the scalar valued density of amoebae and the scalar valued concentration of chemical attractant, respectively, while $(u_0, v_0)$ is the given initial data. For the derivation of the equation, we refer to Childress and Percus [3] and Keller and Segel [14].
Noticing that \( (1.1)-(1.2) \) is “almost” scale invariant since \( u_t - \Delta u + \nabla \cdot (u \nabla v) = 0 \) and \( v_t - \Delta v - u = 0 \) are invariant under the following transformations
\[
(u(t,x), v(t,x)) \rightarrow (\lambda^2 u(\lambda^2 t, \lambda x), \lambda(\lambda^2 t, \lambda x)) \quad \text{for } \lambda > 0.
\]

The idea of using a functional setting invariant by scaling is now classical and originates several works, see for instance, global existence of mild solutions to system \((1.1)-(1.3)\) for initial \((u_0, v_0) \in H_t^{\frac{\nu}{2},r}(\mathbb{R}^n) \times H^{\frac{\nu}{2},r}(\mathbb{R}^n)\) with \( \max \{1, \frac{n}{2}\} < r < \frac{n}{2} \) in \(17\), for initial \((u_0, v_0) \in L^{n/2} \mathbb{R}^n \times BMO(\mathbb{R}^n)\) with \( n \geq 3 \) in \(18\), and for initial \((u_0, v_0) \in L^2(\mathbb{R}^n) \times H^{2} \mathbb{R}^n \times \dot{H}^{2n} \mathbb{R}^n\) with \( n \geq 3 \) and \( \frac{n}{2} < \alpha < \frac{1}{2} \) in \(19\). It is also known that apart from existence and uniqueness of mild solutions in scale invariant spaces, there are papers on asymptotic behaviors (see e.g. \[12\], \[32\]) and stationary solutions (see e.g. \[9\], \[24\]). We also refer readers to, for instance \[11\] and references cited therein, to see results on the quasilinear degenerate Keller-Segel system.

The first goal of this paper is to answer Kozono, Sugiyama and Wachi’s question in \(19\) of figuring out whether there exists a solution to system \((1.1)-(1.3)\) even locally in time for \((u_0, v_0) \in L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)\). In fact, we prove that there does exist a unique global mild solution to system \((1.1)-(1.3)\) with \((u_0, v_0) \in L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)\) by estimating \(t \|u(t,\cdot)\|_{L^\infty}, \|u(t,\cdot)\|_L, \text{ and } t^{\frac{1}{2}} \|\nabla v(t,\cdot)\|_{L^\infty}\) in the \(L^p\)-framework, see for instance \[15\]. Moreover, by exploring the special structure of system \((1.1)-(1.2)\), Deng and Li \[2\] established global existence of mild solution for initial data \((u_0, v_0) \in L^q(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)\) with \( 1 < q < \infty \), where global existence of mild solution for initial data \((u_0, v_0) \in L^2(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)\) was left as an open question.

The second goal of this paper is to study global well-posedness of system \((1.1)-(1.3)\) with \( H_t^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\) initial data. Up to now, there are several results on local and global existence of system \((1.1)-(1.3)\) for \((u_0, v_0) \in H^\nu(\mathbb{R}^2) \times H^\nu(\mathbb{R}^2)\) with \( \nu > 1 \) (cf. Nagai, Senba and Yoshida \[26\], and Yagi \[33\]) and result on global existence of system \((1.1)-(1.3)\) with initial data \((u_0, v_0) \in H^{-1}(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\) (cf. \[2\]). Recalling that \( H^\nu(\mathbb{R}^2) \hookrightarrow H_t^1(\mathbb{R}^2) \) and \( H^1(\mathbb{R}^2)\) can not be embedded into \( L^\infty(\mathbb{R}^2)\), hence global existence of mild solution to system \((1.1)-(1.3)\) with \((u_0, v_0) \in H^\nu(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\) improves the previous results. The proof is based on a combination of the \(L^2\)-Fourier multiplier theory, the smoothing properties of heat kernel and the new half norm of \( v \), i.e. \( \sup_{t > 0} t^{\frac{1}{4}}(1 + t)^{-\frac{1}{2}}\|\nabla v\|_{L^\infty} \) which balances the need for \( t \) near zero and \( t \) near infinity. With this unusual half norm, different form the usual scaling invariant ones, enables us to overcome the main difficulty and to close the iteration scheme. At last, global well-posedness of system \((1.1)-(1.3)\) with initial data \((u_0, v_0) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\) is left as another open question.

Next we recall some results concerning the parabolic-elliptic/parabolic-hyperbolic Keller-Segel systems. Concerning the parabolic-elliptic Keller-Segel model
\[
u_t = \Delta u - \nabla \cdot (u \nabla v), \quad \Delta v = v - u.
\]
It was conjectured by Childress and Percus \[4\] that in a two-dimensional domain \( \Omega \) there exists a critical number \( c* \) such that if \( \int_\Omega u_0(x)dx < c* \) then the solution exists
globally in time, and if \( \int_\Omega u_0(x)dx > c^* \) then blowup happens. For different versions of the Keller-Segel model, the conjecture has been essentially proved; for a complete review of this topic, we refer the reader to the paper [10] and the references therein, also see e.g. Diaz, Nagai, and Rakotoson [7], Blanchet, Dolbeault and Perthame [1]. As for the hyperbolic-hyperbolic Keller-Segel model

\[
\partial_t u = \Delta u + \nabla \cdot (u\nabla w), \quad \partial_t w = u,
\]

it was used in [31] for one dimensional case and was extended to multidimensional cases in [22], and has been studied in [21, 27] and a comprehensive qualitative and numerical analysis was provided there. We refer readers to references [5, 6, 8, 15, 16, 23, 25, 28, 29, 34] for more discussions in this direction.

Throughout this paper, both \( \mathcal{F}f \) and \( \hat{f} \) stand for Fourier transform of \( f \) with respect to space variable and \( F^{-1} \) stands the inverse Fourier transform. Let \( C \) and \( e \) be positive constants that may vary from line to line. \( A \leq B \) stands for \( A = CB \) and \( A \sim B \) stands for \( A \leq B \leq A \). For any \( (p, q) \in [1, \infty]^2 \), we denote \( L^p(0, \infty) \), \( L^q(\mathbb{R}^2), H^s(\mathbb{R}^2) \) and \( L^p(0, \infty; L^q(\mathbb{R}^2)) \) by \( L^p_t, L^q_t, H^s_t \) and \( L^p_t L^q_t \), respectively.

**Theorem 1.1.** For any initial data \( (u_0, v_0) \in L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2) \) with \( \sup_{t>0} \| e^{t \Delta} u_0 \|_{L^1} \) and \( \sup_{t>0} \| \nabla e^{t \Delta} v_0 \|_{L^\infty} \) being small, there exist a unique global mild solution \( (u, v) \) to system (1.1) and (1.2) and positive constant \( c \) such that

\[
(u, v) \in C([0, \infty); L^1(\mathbb{R}^2)) \times C_w([0, \infty); L^\infty(\mathbb{R}^2))
\]

with \( C_w([0, \infty); X) \) being the set of weakly-star continuous functions on \( [0, \infty) \) valued in Banach space \( X \), and

\[
\sup_{t>0} \left( \| u \|_{L^1} + t \| u \|_{L^\infty} + \frac{1}{4c} t^{\frac{1}{2}} \| \nabla v \|_{L^\infty} \right) \\
\leq 2 \sup_{t>0} \left( \| e^{t \Delta} u_0 \|_{L^1} + t \| e^{t \Delta} u_0 \|_{L^\infty} + \frac{1}{4c} t^{\frac{1}{2}} \| e^{t \Delta} \nabla v_0 \|_{L^\infty} \right), \tag{1.4}
\]

which yields that \( \| u \|_{L^\infty} \leq o(t^{-1}) \) and \( \| \nabla v \|_{L^\infty} \leq o(t^{-\frac{1}{2}}) \) as \( t \to \infty \).

**Remark:** (i) Applying Lemma 2.5 to Proposition 3.1 we observe that (1.4) holds if

\[
\sup_{t>0} \left( \| e^{t \Delta} u_0 \|_{L^1} + t \| e^{t \Delta} u_0 \|_{L^\infty} + \frac{1}{4c} t^{\frac{1}{2}} \| e^{t \Delta} \nabla v_0 \|_{L^\infty} \right) \leq \frac{3}{32c^2}. \tag{1.5}
\]

Applying \( \frac{1}{16t} e^{-\frac{1}{16}t} \) to the left hand side of (1.5), it suffices to assume that \( 2 \| u_0 \|_{L^1} + \frac{1}{16} \| \nabla v_0 \|_{\dot{B}^{-1}_{\infty, \infty}} \leq \frac{3}{32c^2} \), where

\[
\| v_0 \|_{\dot{B}^{-1}_{\infty, \infty}} \sim \| \nabla v_0 \|_{\dot{B}^{-1}_{\infty, \infty}} = \sup_{t>0} t^{\frac{1}{2}} \| e^{t \Delta} \nabla v_0 \|_{L^\infty} \leq \| v_0 \|_{L^\infty}
\]

since Riesz transforms \( \sqrt{-\Delta} \) are bounded in homogeneous Besov spaces, \( \sqrt{-\Delta} \) maps \( \dot{B}^0_{\infty, \infty} \) isomorphically onto \( \dot{B}^{-1}_{\infty, \infty} \) and \( \frac{1}{\sqrt{-\Delta}} \) maps \( \dot{B}^{-1}_{\infty, \infty} \) isomorphically onto \( \dot{B}^0_{\infty, \infty} \) (see [30], Theorem 1, p.242). Therefore, only \( \dot{B}^0_{\infty, \infty} \) smallness of \( v_0 \) and \( L^1 \) smallness
of $u_0$ are needed. Local existence of mild solution follows directly by changing time interval $[0, \infty)$ into $[0, T]$. However, if $v_0(x_1, x_2) = 1_{[0, 1]}(x_1)$ and $(t, x_1, x_2) \in (0, \frac{1}{64}) \times (t^4, 2t^2) \times (-\infty, \infty)$, then there holds
\[ t^1 \int_{-\infty}^\infty \frac{1}{\sqrt{4t}} \left| x_1 - y_1 \right| e^{-\frac{(x_1-y_1)^2}{4t}} dy_1 dy_2 = \frac{1}{\sqrt{\pi}} \int_{y_1 \in [0, 1]} \frac{1}{\sqrt{4t}} \left| x_1 - y_1 \right| e^{-\frac{(x_1-y_1)^2}{4t}} dy_1 \]
\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{1}{\sqrt{4t}} \left| x_1 - y_1 \right| e^{-\frac{(x_1-y_1)^2}{4t}} dy_1 - \frac{1}{\sqrt{\pi}} \int_{\infty}^\infty \frac{1}{\sqrt{4t}} \left| x_1 - y_1 \right| e^{-\frac{(x_1-y_1)^2}{4t}} dy_1 \]
\[ = \frac{1}{\sqrt{\pi}} \int_{1}^3 \left| x_1 - y_1 \right| e^{-\frac{(x_1-y_1)^2}{4t}} dy_1 = c_0. \]

For such $v_0$, if we set $\tilde{v}_0 = v_0/c_0$, then we have
\[ \lim_{T \to 0^+} \sup_{0 < t < T} t^\frac{1}{2} \| \nabla e^{t\Delta} \tilde{v}_0 \|_{L^\infty} \geq 1. \]

Hence it seems difficult to prove local (global) existence of mild solution for arbitrary large $L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ initial data.

(ii) Proof of Theorem 1.1 also applies for $u_t - \Delta u + \nabla \cdot (u \nabla v) = 0$ and $v_t - \Delta v + u = 0$ with initial data $(u_0, v_0) \in L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$.

Here and hereafter, we set $\sigma(t) = \frac{1}{4}(1 + t)^{-\frac{1}{2}}$. Then we state the following result.

**Theorem 1.2.** For any initial data $(u_0, v_0) \in H^1_0(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, there exist positive constants $\varepsilon_0$ and $c$ so that if $\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} \geq \sqrt{c}$, then system (1.1)–(1.3) has a unique global solution $(u, v)$ satisfying
\[ (u \pm v, u \pm \sigma \nabla v, \nabla u \pm \nabla v) \in C([0, \infty); H^1(\mathbb{R}^2)) \times L^\infty_t L^\infty \times L^1_t H^1. \]
Moreover, $\|u \pm \sigma \nabla v\|_{L^\infty_t H^1} + \|u \pm \sigma \nabla v\|_{L^\infty_t L^\infty} + \|
abla u\|_{L^1_t L^2} + \frac{1}{2\sigma} \|\nabla v\|_{L^1_t H^1} \leq 2\varepsilon_0$.

**Plan of the paper:** In Sect. 2 we introduce several preliminary lemmas, while in Sect. 3 we prove Theorems 1.1 and 1.2.

## 2 Preliminaries

In this section, we list several known lemma and prove some key lemmas which will be used in proving the well-posedness of the parabolic-parabolic chemotaxis. The first lemma given below is concerned with initial data belonging to $H^M(\mathbb{R}^2)$. For simplicity, here and hereafter, we omit the space domain in various function spaces, for instance $H^1(\mathbb{R}^2)$ is denoted by $H^1$, if there is no confusion.
Lemma 2.1. Let \( n = 2 \), \( \Lambda = \sqrt{-\Delta} \), \((s, \delta, r, \rho) \in (-\infty, \infty) \times [0, \infty) \times [1, \infty] \times [2, \infty) \) and \( v \in H^s \). If \( m(t, \xi) \in L_t^r L_\xi^\infty \) and \( m(t, D)v = \mathcal{F}^{-1}m(t, \xi)\hat{v}(\xi) \), then we get
\[
\|m(t, D)v\|_{L_t^r H^s} \lesssim \|m\|_{L_t^r L_\xi^\infty} \|v\|_{H^s}; \tag{2.1}
\]
Else if \( m_\delta(t, \xi) := m(t, \xi)|\xi|^\delta \in L_\xi^\infty L_t^p \) and \( m_\delta(t, D)v = \mathcal{F}^{-1}m_\delta(t, \xi)\hat{v}(\xi) \), then we get
\[
\|m_\delta(t, D)v\|_{L_t^p H^s} \lesssim \|m_\delta\|_{L_\xi^\infty L_t^p} \|v\|_{H^s}. \tag{2.2}
\]

Proof. Proof of (2.1) follows from classical Fourier multiplier theory and, for readers convenience, we give the detail proof as follows:
\[
\|m(t, D)v\|_{L_t^r H^s} \lesssim \|m(t, \xi)(1 + |\cdot|^2)^{\frac{s}{2}}\hat{v}(\cdot)\|_{L_t^r L_\xi^2}
\lesssim \|m\|_{L_t^r L_\xi^\infty}(1 + |\cdot|^2)^{\frac{s}{2}}\|\hat{v}(\cdot)\|_{L_\xi^2}
\lesssim \|m\|_{L_t^r L_\xi^\infty} \|v\|_{H^s}, \tag{2.3}
\]
where we have used Plancherel equality twice.

In order to prove (2.2), by making use of Plancherel equality, Minkowski’s inequality, Hölder’s inequality and Plancherel equality again, we get
\[
\|m_\delta(t, D)v\|_{L_t^p H^s} \lesssim \|m_\delta\|_{L_\xi^\infty L_t^p} \|v\|_{H^s} \tag{2.4}
\]
where
\[
\|m_\delta\|_{L_\xi^\infty L_t^p} \lesssim \|m_\delta\|_{L_\xi^\infty L_t^p} \lesssim \|m_\delta\|_{L_\xi^\infty L_t^p} \|v\|_{H^s}.
\]

Hence, we finish the proof. \( \square \)

The skill used in the above Lemma will be used repeatedly in the following parts. The next Lemma is devoted to estimate the bilinear term which is known as the maximal \( L_t^p L_\xi^q \) regularity result for the heat kernel (cf. \cite{20}, Theorem 7.3, p. 64).

Lemma 2.2. (Maximal \( L_t^p L_\xi^q \) regularity for heat kernel) The operator \( T \) defined by
\[
g(t, x) \mapsto Tg(t, x) = \int_0^t e^{(t-\tau)\Delta} \Delta g(\tau, x) d\tau \tag{2.5}
\]
is bounded from \( L_t^p L_\xi^q \) to \( L_t^p L_\xi^q \) with \( 1 < p < \infty \) and \( 1 < q < \infty \).

The next Lemma is also dedicated to estimating the bilinear term.
Lemma 2.3. For any \((s, c, c_1, p, r, p_1) \in \mathbb{R} \times (0, \infty)^2 \times [2, \infty] \times [1, 2] \times [1, \infty], p_1 \geq r\) and \(0 \leq \theta < 2(1 + \frac{1}{p_1} - \frac{1}{r})\), if \(m(t, \xi) = \frac{c_1}{e^{ct} |\xi|^2}\) and \(\mu(t, \xi) = \frac{c_1}{e^{ct} + \xi^2}\), then there exists constant \(C_{\theta, p_1, r}\) depending on \(\theta, p_1\) and \(r\) such that have

\[
\| \int_0^t m(t-\tau, D) \Lambda^{2+\frac{\theta}{p} - \frac{\theta}{r}} F(\tau, x) d\tau \|_{L^p_t H^s} \lesssim \| F \|_{L^p_t H^s}, \tag{2.6}
\]

\[
\| \int_0^t \mu(t-\tau, D) \Lambda^\theta F(\tau, x) d\tau \|_{L^p_t H^s} \lesssim C_{\theta, p_1, r} \| F \|_{L^p_t H^s}. \tag{2.7}
\]

Proof. In order to prove (2.6), setting \(\langle \xi \rangle = \sqrt{1 + |\xi|^2}\) and by using Plancherel equality, the Minkowski inequality, the Young inequality and the Minkowski inequality as well as Plancherel equality, we have

\[
\| \int_0^t m(t-\tau, D) \Lambda^{2+\frac{\theta}{p} - \frac{\theta}{r}} F(\tau, x) d\tau \|_{L^p_t L^2_x} = \| \int_0^t m(t-\tau, \xi) |\xi|^{2+\frac{\theta}{p} - \frac{\theta}{r}} \langle \xi \rangle^s \hat{F}(\tau, \xi) d\tau \|_{L^p_t L^2_x}
\]

\[
\lesssim \| \int_0^t m(t-\tau, \xi) |\xi|^{2+\frac{\theta}{p} - \frac{\theta}{r}} \|_{L^p_t L^2_x} \| \langle \xi \rangle^s \hat{F}(\tau, \xi) \|_{L^2_x}
\]

\[
\lesssim \sup_{\xi \in \mathbb{R}^2} \| m(\cdot, \xi) |\xi|^{2+\frac{\theta}{p} - \frac{\theta}{r}} \|_{L^p_t L^2_x} \| \langle \xi \rangle^s \hat{F}(\cdot, \xi) \|_{L^2_x}
\]

\[
\lesssim \| \langle \xi \rangle^s \hat{F}(\cdot, \xi) \|_{L^p_t L^2_x}
\]

\[
\lesssim \| F \|_{L^p_t H^s}.
\]

It remains to prove (2.7). Using the \(L^1\) integrability of \(e^{-ct} t^{-\frac{\theta}{p} + \frac{\theta}{r}}\), we get

\[
\| \int_0^t \mu(t-\tau, D) \Lambda^\theta F(\tau, x) d\tau \|_{L^p_t H^s} = \| \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{\theta}{p}} \| F \|_{H^s} d\tau \|_{L^p_t}
\]

\[
\lesssim \left( \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{\theta}{p}} \right)^{\frac{1}{p_1}} \left( \int_0^t (t-\tau)^{-\frac{\theta}{r}} \right)^{\frac{1}{r_1}} \| F \|_{L^p_t H^s}
\]

\[
\lesssim C_{\theta, p_1, r} \| F \|_{L^p_t H^s}.
\]

Therefore, we finish the whole proof. \(\square\)

Let us state the equivalent definition of Besov space \(\dot{B}^s_{p,q} := \dot{B}^s_{p,q}(\mathbb{R}^2)\) using heat semigroup method (for a proof see, for instance \([30]\) p.192 or \([20]\) Theorem 5.4, p.45).

**Proposition 2.4.** Let \((s, p, q) \in (-\infty, 0) \times [1, \infty]^2\). The homogeneous Besov space \(\dot{B}^s_{p,q}\) is defined as the set of tempered distribution \(f\) such that

\[
\| f \|_{\dot{B}^s_{p,q}} = \left( \int_0^\infty (t^{-\frac{\alpha}{2}} \| e^{t\Delta} f \|_{L^p})^q \frac{dt}{t} \right)^{\frac{1}{q}} \quad \text{if} \quad 1 \leq q < \infty,
\]

\[
\| f \|_{\dot{B}^s_{p,\infty}} = \sup_{t > 0} t^{-\frac{\alpha}{2}} \| e^{t\Delta} f \|_{L^p} \quad \text{if} \quad q = \infty.
\]
The last lemma of this section is a slightly generalized version about the well-known Picard contraction principle (see for instance [20], Theorem 13.2, p.124) which is used to prove the main results concerning well-posedness of (1.1)-(1.3) with \((u_0, v_0)\) either belonging to \(L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)\) or belonging to \(H_b^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\).

**Lemma 2.5.** (The Picard contraction principle) Let \((X \times Y, \| \cdot \|_X + \| \cdot \|_Y)\) be an abstract Banach product space, \(L : X \rightarrow Y\) and \(B : X \times Y \rightarrow X\) are a linear operator and a bilinear operator, respectively, such that for any \((u,v) \in X \times Y\), there exist positive constant \(c\) and if

\[
\|L(u)\|_Y \leq c\|u\|_X, \quad \|B(u,v)\|_X \leq c\|u\|_X\|v\|_Y,
\]

then for any \((e^{t\Delta}u_0, e^{(t-1)\Delta}v_0) \in X \times Y\) with \(\|(e^{t\Delta}u_0, \frac{1}{4c}e^{(t-1)\Delta}v_0)\|_{X \times Y} < \frac{3}{4c}\), the following system

\[
(u, v) = (e^{t\Delta}u_0, e^{(t-1)\Delta}v_0) + (B(u,v), L(u))
\]

has a solution \((u,v)\) in \(X \times Y\). In particular, the solution is such that \(\|(u, \frac{4c}{4c})\|_{X \times Y} \leq 2\|(e^{t\Delta}u_0, \frac{1}{4c}e^{(t-1)\Delta}v_0)\|_{X \times Y}\) and it is the only one such that \(\|(u, \frac{4c}{4c})\|_{X \times Y} < \frac{3}{16c}\).

**Proof.** The proof is standard now. However, for reader’s convenience, we give a brief proof. We first define a mapping \(\Phi : X \times Y \rightarrow X \times Y\) such that

\[
\Phi(u, v) = (e^{t\Delta}u_0, e^{(t-1)\Delta}v_0) + (B(u,v), L(u)).
\]

Applying simple transformations, i.e. \(w = \frac{4c}{4c}u_0\) and \(w_0 = \frac{1}{4c}v_0\) to (2.10), we get

\[
\Phi(u, w) = (e^{t\Delta}u_0, e^{(t-1)\Delta}w_0) + (4cB(u, w), \frac{1}{4c}L(u)).
\]

By applying (2.8) to (2.11), we have

\[
\|\Phi(u, w)\|_{X \times Y} \leq \|(e^{t\Delta}u_0, e^{(t-1)\Delta}w_0)\|_{X \times Y} + 4e^2\|u\|_X\|w\|_Y + \frac{1}{4}\|u\|_Y
\]

\[
\leq A_0 + e^2\|u, w\|_{X \times Y} + \frac{1}{4}\|u, w\|_{X \times Y},
\]

where \(A_0 := \|(e^{t\Delta}u_0, e^{(t-1)\Delta}w_0)\|_{X \times Y}\). Let \(\overline{B(0, 2A_0)} \subset X \times Y\) be a closed ball centered at origin with radius \(2A_0\). From (2.12), we observe that \(\Phi\) is well defined in \(B(0, 2A_0)\) and maps \(B(0, 2A_0)\) into itself. Moreover, for any \((u_1, w_1), (u_2, w_2) \in B(0, 2A_0)\), by making use of (2.8), we get

\[
\|\Phi(u_1, w_1) - \Phi(u_2, w_2)\|_{X \times Y} = \|(4cB(u_1, w_1) - 4cB(u_2, w_2), \frac{L(u_1) - L(u_2)}{4c})\|_{X \times Y}
\]

\[
\leq 4e^2\max\{\|u_2\|_X, \|w_1\|_Y\}\|(u_1 - u_2, w_1 - w_2)\|_{X \times Y} + \frac{1}{4}\|(u_1 - u_2, w_1 - w_2)\|_{X \times Y}
\]

\[
\leq 8e^2A_0\|(u_1 - u_2, w_1 - w_2)\|_{X \times Y} + \frac{1}{4}\|(u_1 - u_2, w_1 - w_2)\|_{X \times Y}
\]
\[ \leq (8c^2A_0 + \frac{1}{4})\|(u_1 - u_2, w_1 - w_2)\|_{X \times Y}, \quad (2.13) \]

where \(8c^2A_0 + \frac{1}{4} < 1\) since \(A_0 < \frac{3}{32c^2}\). From (2.13), we observe that \(\Phi : (u, w) \mapsto \Phi(u, w)\) in (2.11) is contractive. Thus there exists a unique solution \((u, w)\) to (2.11), which shows that (2.10) also has a unique solution \((u, v)\) to (2.10) provides that \(\|(e^{t\Delta}u_0, \frac{1}{4c}e^{t(\Delta-1)}v_0)\|_{X \times Y} < 3/32c^2\).

### 3 Proof of Theorem 1.1

As usual, we apply the heat semigroup \(e^{t\Delta}\) with heat kernel \(\frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}\) to invert system (1.1)–(1.3) into the following integral equations via the Duhamel principle:

\[
\begin{align*}
\begin{cases}
u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (u \nabla v) d\tau : = e^{t\Delta}u_0 - B(u, v), \\
v = e^{t(\Delta-1)}w_0 + \int_0^t e^{(t-\tau)(\Delta-1)} ud\tau := e^{t(\Delta-1)}w_0 + L(u).
\end{cases}
\end{align*}
\]

(3.1)

Let \(c\) be the largest positive constant that appears in the linear and bilinear estimates and depends only on dimension. By denote \(\frac{u}{4c}\) by \(w\), we get the following system

\[
\begin{align*}
\begin{cases}
u = e^{t\Delta}u_0 - 4c \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (u \nabla w) d\tau : = e^{t\Delta}u_0 - 4cB(u, w), \\
w = e^{t(\Delta-1)}w_0 + \frac{1}{4c} \int_0^t e^{(t-\tau)(\Delta-1)} ud\tau := e^{t(\Delta-1)}w_0 + \frac{1}{4c}L(u),
\end{cases}
\end{align*}
\]

(3.2)

where we regard equations (3.2) as a fixed point system and let mapping \(\Phi\) be

\[
\Phi : (u, w) \mapsto \left( e^{t\Delta}u_0, e^{t(\Delta-1)}w_0 \right) + \left( -4cB(u, w), \frac{1}{4c}L(u) \right).
\]

(3.3)

We call solution \((u, 4cw)\) to (3.1) mild solution of (1.1)–(1.3) if \((u, w)\) solves (3.2).

#### 3.1 Proof of Theorem 1.1

In this subsection, we prove global well-posedness of system (3.2) with initial data \((u_0, w_0) \in L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)\) by making use of the Kato’s \(L^p\)-framework. At first, we set

\[
\begin{align*}
X &= \{ u \in \mathcal{S}'(\mathbb{R}^2 \times (0, \infty)) ; \sup_{t > 0} \| u(\cdot, t) \|_{L^1} + \sup_{t > 0} t \| u(\cdot, t) \|_{L^\infty} < \infty \}, \\
Y &= \{ w \in \mathcal{S}'(\mathbb{R}^2 \times (0, \infty)) ; \sup_{t > 0} t^{\frac{1}{2}} \| \nabla w(\cdot, t) \|_{L^\infty} < \infty \}.
\end{align*}
\]

(3.4)

Then we prove that for suitably small initial data \((u_0, w_0)\) the mapping \(\Phi\) is contractive and maps a closed ball of \(X \times Y\) into itself.
Proposition 3.1. For any initial data $(u_0, w_0) \in L^1 \times L^\infty$, there exists positive constant $c$ such that
\[
\begin{cases}
\|e^{t\Delta} u_0, e^{t(\Delta - 1)}w_0\|_{X \times Y} \leq c\|(u_0, w_0)\|_{L^1 \times \dot{B}^{0}_{\infty, \infty}} \leq c\|(u_0, w_0)\|_{L^1 \times L^\infty}, \\
\|4cB(u, w)\|_X \leq 4c^2\|u\|_X\|w\|_Y \quad \text{and} \quad \|\frac{1}{4c}L(u)\|_Y \leq \frac{1}{4}\|u\|_X.
\end{cases}
\] (3.5)

Proof. We divide the whole proof into two parts concerning with $e^{t\Delta} u_0$, $e^{-t}e^{t\Delta} w_0$ and $B(u, w)$, $L(u)$, respectively.

PART I. Estimates for $\|e^{t\Delta} u_0\|_X$ and $\|e^{-t}e^{t\Delta} w_0\|_Y$. Recall that the heat kernel is $\frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}$. Then for any $t > 0$ and $1 \leq p \leq \infty$, there hold
\[
\left\|\frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}\right\|_{L^p} \leq t^{-1 + \frac{1}{p}} \quad \text{and} \quad \left\|\frac{1}{4\pi t}\nabla e^{-\frac{|x|^2}{4t}}\right\|_{L^p} \leq t^{-\frac{1}{2} + \frac{1}{p}}.
\] (3.6)

Applying Young’s inequality and (3.6) to $e^{t\Delta} u_0(x) = \int_{\mathbb{R}^2} \frac{1}{4\pi t}e^{-\frac{|x-y|^2}{4t}} u_0(x-y)dy$, we get
\[
\begin{align*}
\|e^{t\Delta} u_0\|_X &= \sup_{t>0} \|e^{t\Delta} u_0\|_{L^1} + \sup_{t>0} t\|e^{t\Delta} u_0\|_{L^\infty} \\
&\leq \sup_{t>0} \left\|\frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}\right\|_{L^1}\|u_0\|_{L^1} + \sup_{t>0} t\left\|\frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}\right\|_{L^\infty}\|u_0\|_{L^1} \\
&\leq 2\|u_0\|_{L^1}
\end{align*}
\] (3.7)
or from Proposition 2.4 and embedding theorem $L^1 \hookrightarrow \dot{B}^{-2}_{\infty, \infty}$, we have
\[
\sup_{t>0} t\|e^{t\Delta} u_0\|_{L^\infty} = \|u_0\|_{\dot{B}^{-2}_{\infty, \infty}} \leq c_1\|u_0\|_{L^1}.
\]

We emphasize here that for any $(s, \alpha, p, q) \in \mathbb{R}^2 \times [1, \infty)^2$, $(-\Delta)^{\alpha}$ maps $\dot{B}^s_{p,q}$ isomorphically onto $\dot{B}^{s-\alpha}_{p,q}$ (cf. [30], Theorem 1, p.242), which is a direct consequence of the well known Bernstein’s inequalities. Thus following similar arguments of $e^{t\Delta}$ and using (3.6) as well as Proposition 2.4, we get
\[
\begin{align*}
\|e^{-t}e^{t\Delta} w_0\|_Y &= \sup_{t>0} t^{\frac{1}{2}}\|e^{-t}e^{t\Delta}\nabla w_0\|_{L^\infty} \leq \sup_{t>0} t^{\frac{1}{2}}\|e^{t\Delta}\nabla w_0\|_{L^\infty} \\
&= \|\nabla w_0\|_{\dot{B}^{-1-\alpha}_{\infty, \infty}} \sim \|w_0\|_{\dot{B}^{0}_{\infty, \infty}} \\
&\leq c_1\|w_0\|_{L^\infty},
\end{align*}
\] (3.8)

where the fourth and the fifth inequalities follow from Theorem 1 of [30] p.242 and boundedness of Riesz transforms $\frac{\nabla}{\sqrt{-\Delta}}$ as well as $L^\infty \hookrightarrow \dot{B}^0_{\infty, \infty}$.

PART II. Estimates for $\|B(u, w)\|_X$ and $L(u)$ $Y$. As for $\|B(u, w)\|_X$, we have
\[
\|B(u, w)\|_X = \sup_{t>0} \left\|\int_0^t e^{(t-\tau)\Delta}\nabla \cdot (u\nabla w) d\tau\right\|_{L^1} + \sup_{t>0} t\left\|\int_0^t e^{(t-\tau)\Delta}\nabla \cdot (u\nabla w) d\tau\right\|_{L^\infty}
\]
As for \( L(u) \), from definition of \( \| \cdot \|_Y \), we need to estimate

\[
\sup_{t>0} t^{\frac{1}{2}} \| \nabla L(u) \|_{L^\infty} = \sup_{t>0} \frac{1}{t^{\frac{1}{2}}} \| \int_0^t e^{-(t-\tau)} e(t-\tau) \Delta u \, d\tau \|_{L^\infty} \\
\leq \sup_{t>0} \left( t^{\frac{1}{2}} \int_0^t (t-\tau)^{-\frac{3}{2}} \| u(\tau) \|_{L_1} \, d\tau + \frac{1}{t^{\frac{1}{2}}} \int_0^t (t-\tau)^{-\frac{3}{2}} \tau^{-1} \| u(\tau) \|_{L^\infty} \, d\tau \right) \\
\leq c_3 \sup_{\tau>0} (\| u(\tau) \|_{L_1} + \tau \| u(\tau) \|_{L^\infty}) \\
\leq c_3 \| u \|_X. \tag{3.10}
\]

Setting \( c = \max\{c_1, c_2, c_3\} \), combining (3.1)–(3.10), multiplying \( B(u, w) \) by 4c and multiplying \( L(u) \) by \( \frac{1}{4c} \), we prove (3.9). \( \square \)

**Proof of Theorem 1.1** At first, applying Proposition 3.1 following similar arguments as in the proof of Lemma 2.3, we prove that there exists a unique solution \((u, w) \in B(0, 2A_{10}) \subset X \times Y\) to system (3.2) if \( A_{10} := \| (e^{t \Delta} u_0, e^{t(\Delta-1)} w_0) \|_{X \times Y} < 3/32c^2 \). Moreover, this solution also satisfies \( \Phi(u, w) = (u, w) \). From (3.7)–(3.8), it suffices to assume that \( \|(u_0, w_0)\|_{L^1 \times B_{0, \infty}} < 3/32c^3 \) since \( A_{10} \leq c \|(u_0, w_0)\|_{L^1 \times B_{0, \infty}} < 3/32c^2 \).

Next we show that \( w \in C_w([0, \infty); L^\infty(\mathbb{R}^2)) \). From (3.2) and (3.6), we have

\[
\sup_{t>0} \| w(t) \|_{L^\infty} = \sup_{t>0} \| e^{t(\Delta-1)} w_0 + \frac{1}{4c} L(u) \|_{L^\infty} \leq \| w_0 \|_{L^\infty} + \frac{1}{4} \| u \|_X,
\]

where in (3.10), \( c_3 = \sup_{t>0} (t^{\frac{1}{2}} \int_0^t (t-\tau)^{-\frac{3}{2}} \| u(\tau) \|_{L_1} \, d\tau + t^{\frac{1}{2}} \int_0^t (t-\tau)^{-\frac{3}{2}} \tau^{-1} \| u(\tau) \|_{L^\infty} \, d\tau) \) and similarly

\[
\| \frac{1}{4c} L(u) \|_{L^\infty} = \frac{1}{4c} \| \int_0^t e^{t(\Delta-1)} \Delta u \, d\tau \|_{L^\infty} \leq \frac{1}{4c} \| \int_0^t e^{t(\Delta-1)} \, d\tau \|_{L^\infty} \\
\leq \frac{1}{4c} \int_0^t (t-\tau)^{-\frac{3}{2}} \| u(\tau) \|_{L^\infty} \, d\tau + \frac{1}{4c} \int_0^t \tau^{-1} \| u(\tau) \|_{L^\infty} \, d\tau
\]
\[ \leq \frac{c_3}{4c} \| u \|_X \leq \frac{1}{4} \| u \|_X \]

since \( \int_0^2 (t-\tau)^{-1} d\tau < t^\frac{2}{3} \int_0^2 (t-\tau)^{-\frac{3}{2}} d\tau, \int_0^\frac{1}{2} \tau^{-1} d\tau < t^\frac{2}{3} \int_0^\frac{1}{2} (t-\tau)^{-\frac{3}{2}} \tau^{-1} d\tau \) and \( c \geq c_3 \).

Moreover, following a dense argument in \( L^1(\mathbb{R}^2) \) we can prove the time continuity of \( u \). Since Schwartz function space is not dense in \( L^\infty(\mathbb{R}^2) \), we can only obtain the weakly star time continuity of solution \( w \).

Finally, performing transformation: \( (u, v) = (u, 4cw) \), we get the unique solution \((u, v)\) of (1.1) - (1.3).

### 3.2 Proof of Theorem 1.2

In this subsection, we prove global well-posedness of system (3.2) with initial data \((u_0, w_0) \in H^1_b(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\) by making use of the Kato’s framework, see [13] for instance. At first, we recall that \( \sigma(t) = t^\frac{3}{2}(1+t)^{-\frac{1}{2}} \) and then we set

\[
X = \{ u \in \mathcal{S}'(\mathbb{R}^2 \times (0, \infty)) ; \sup_{t>0} \| u(\cdot, t) \|_{H^1} + \| \nabla u \|_{L^2_t H^1} + \| u \|_{L^\infty_t L^\infty} < \infty \},
\]

\[
Y = \{ w \in \mathcal{S}'(\mathbb{R}^2 \times (0, \infty)) ; \sup_{t>0} \| w(\cdot, t) \|_{H^1} + \| \nabla w \|_{L^2_t H^1} + \| \sigma \nabla w \|_{L^\infty_t L^\infty} < \infty \}. \tag{3.11}
\]

The following Proposition will play a central role in proving Theorem 1.2.

**Proposition 3.2.** For any initial data \((u_0, w_0) \in H^1_b(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\), there exists positive constant \( c \) such that

\[
\| 4cB(u, w) \|_X \leq 4c^2 \| u \|_X \| w \|_Y \leq c^2 \| (u, w) \|_{X \times Y}^2, \quad \| \frac{1}{4c} L(u) \|_Y \leq \frac{1}{4} \| u \|_X \tag{3.12}
\]

and \( \| (e^{t\Delta} u_0, e^{t(\Delta-1)} w_0) \|_{X \times Y} \leq c \| (u_0, v_0) \|_{H^1_b \times H^1} \).

**Proof.** We divide the whole proof into two parts concerning with \((e^{t\Delta} u_0, e^{-t} e^{t\Delta} w_0)\) and \((-4cB(u, w), \frac{1}{4c} L(u))\), respectively.

**PART I.** *Estimates for \( \| e^{t\Delta} u_0 \|_X \) and \( \| e^{-t} e^{t\Delta} w_0 \|_Y \).* As for \( \| e^{t\Delta} u_0 \|_X \), noticing that \( e^{-c|\xi|^2} \in L^\infty_t L^\infty_\xi \) and \( e^{-c|\xi|^2} \xi \in L^\infty_t L^2_\xi \), then by applying Lemma 2.1 and Young’s inequality, we have

\[
\| e^{t\Delta} u_0 \|_X = \| e^{t\Delta} u_0 \|_{L^\infty_t H^1} + \| e^{t\Delta} \nabla u_0 \|_{L^2_t H^1} + \| e^{t\Delta} u_0 \|_{L^\infty_t L^\infty} \\
\leq \| u_0 \|_{H^1} + \| u_0 \|_{H^1} + \| u_0 \|_{L^\infty} \\
\leq c \| u_0 \|_{H^1}. \tag{3.13}
\]

Similarly, we have

\[
\| e^{-t} e^{t\Delta} w_0 \|_{L^\infty_t H^1} + \| e^{-t} e^{t\Delta} \nabla w_0 \|_{L^2_t H^1} \leq c \| w_0 \|_{H^1}. \tag{3.14}
\]
Recall that $\sigma(t) = t^{\frac{1}{2}}(1 + t)^{-\frac{1}{2}}$. Then we get
\[
\|\sigma \nabla e^{-t}\nabla X_0\|_{L^\infty_t L^\infty_x} = \sup_{t > 0} t^{\frac{1}{2}}(1 + t)^{-\frac{1}{2}} e^{-t} \|e^t \nabla X_0\|_{L^\infty_x} \\
\leq \sup_{t > 0} t^{\frac{1}{2}} \|e^t \nabla X_0\|_{L^\infty_x} = \|\nabla X_0\|_{\dot{B}^{-\frac{1}{2}}_\infty^1} \\
\leq c \|\nabla X_0\|_{L^2} \leq c \|X_0\|_{H^1},
\]
(3.15)
where we have used Proposition 2.3 embedding theorems of Besov spaces (cf. [30]).

**PART II. Estimates for $\|B(u,w)\|_X$ and $\|L(u)\|_Y$.** As for $\|B(u,v)\|_X$, we get
\[
\|B(u,w)\|_X = \|B(u,w)\|_{L^\infty_t H^1} + \|\nabla B(u,w)\|_{L^2_t H^1} + \|B(u,w)\|_{L^\infty_t L^\infty_x} \\
\leq \|\int_0^t e^{(t-\tau)\nabla \cdot (u \nabla w) d\tau}\|_{L^\infty_t L^2} + \|\int_0^t e^{(t-\tau)\nabla \nabla \cdot (u \nabla w) d\tau}\|_{L^\infty_t L^2} \\
+ \|\int_0^t e^{(t-\tau)\nabla \nabla \cdot (u \nabla w) d\tau}\|_{L^2_t L^2} + \|\int_0^t e^{(t-\tau)\nabla \nabla \cdot (u \nabla w) d\tau}\|_{L^2_t L^2} \\
+ \|\int_0^t e^{(t-\tau)\nabla \nabla \cdot (u \nabla w) d\tau}\|_{L^\infty_t L^\infty_x} \\
:= I_1 + I_2 + I_3 + I_4 + I_5,
\]
where by applying Lemma 2.3 we have
\[
I_1 \leq c \|u \nabla w\|_{L^2_t L^2} \leq c \|u\|_{L^\infty_t L^\infty_x} \|\nabla w\|_{L^2_t L^2},
\]
(3.16)
\[
I_2 \leq c \|u \nabla w\|_{L^\infty_t L^2} \leq c \|u\|_{L^\infty_t L^\infty_x} \|\nabla w\|_{L^2_t L^2}
\]
(3.17)
and by applying Lemma 2.2 Hölder’s inequality and interpolation theorem, we have
\[
I_3 \leq c \|u \nabla w\|_{L^2_t L^2} \leq c \|u\|_{L^\infty_t L^\infty_x} \|\nabla w\|_{L^2_t L^2},
\]
(3.18)
Similarly, we have
\[
I_4 \leq c \|\nabla \cdot (u \nabla w)\|_{L^2_t L^2} \leq c (\|\nabla u \cdot \nabla w\|_{L^2_t L^2} + \|\nabla w\|_{L^2_t L^2}) \\
\leq c (\|\nabla u\|_{L^4_t L^4} \|\nabla w\|_{L^4_t L^4} + \|\nabla w\|_{L^\infty_t L^\infty} \|\nabla w\|_{L^2_t H^1}) \\
\leq c (\|u\|_{L^\infty_t L^2} \|\nabla u\|_{L^\infty_t L^2} \|\nabla w\|_{L^\infty_t H^1} \|\nabla w\|_{L^2_t H^1} + \|\nabla w\|_{L^\infty_t L^\infty} \|\nabla w\|_{L^2_t H^1}),
\]
(3.19)
where $\|f\|_{L^4_t L^4} \leq c \|f\|_{L^2_t H^1} \leq c \|f\|_{L^\infty_t L^2} \|\nabla f\|_{L^2_t H^1}$. As for $I_5$, by splitting the time interval, we obtain that if $t > 2$, then $1 < t - 1 < \tau < t, 1 < \frac{1}{\sigma(\tau)} = \frac{\sqrt{1 + \tau}}{\sqrt{\tau}} < 2$ and
\[
I_5 = \|\int_0^t e^{(t-\tau)\nabla \cdot (u \nabla w) d\tau}\|_{L^\infty_t L^\infty_x} \\
\leq \|\int_0^{t-1} e^{(t-\tau)\nabla \cdot (u \nabla w) d\tau}\|_{L^\infty_t L^\infty_x} + \|\int_{t-1}^t e^{(t-\tau)\nabla \cdot (u \nabla w) d\tau}\|_{L^\infty_t L^\infty_x}
\]
It remains to estimate $I \theta = 1$, we have
\[
\leq c \int_{t-1}^{t} (t - \tau)^{-\frac{3}{2}} \|u \nabla w\|_{L^1} d\tau + c \int_{t-1}^{t} (t - \tau)^{-\frac{3}{2}} \|u \nabla w\|_{L^\infty} d\tau
\leq c \left( \|u \nabla w\|_{L^\infty L^1} + \|u \nabla w\|_{L^\infty L^\infty} \right)
\leq c \left( \|u\|_{L^\infty T^2} \|\nabla w\|_{L^\infty L^2} + \|u\|_{L^\infty L^\infty} \|\sigma \nabla w\|_{L^\infty L^\infty} \right);
\] (3.20)

Else if $0 < t \leq 2$ and $0 < \tau < t$, then we get $\tau^2/2 < \sigma(\tau) < \tau^2$ and

\[
I_5 = \left\| \int_{0}^{t} e^{(t-\tau)\Delta} \cdot (u \nabla w) d\tau \right\|_{L^\infty L^\infty}
\leq \left\| \int_{0}^{t} (t - \tau)^{-\frac{1}{2}} \frac{1}{\sigma(\tau)} \|\sigma(\tau)\|_{L^\infty} \|u \nabla w\|_{L^\infty} d\tau \right\|
\leq c \int_{0}^{t} (t - \tau)^{-\frac{1}{2}\tau^{-\frac{3}{2}}} \|u\|_{L^\infty L^\infty} \|\sigma \nabla w\|_{L^\infty L^\infty}
\leq c \left\| \|u\|_{L^\infty L^\infty} \|\sigma \nabla w\|_{L^\infty L^\infty} \right\|.
\] (3.21)

In order to estimate $\|L(u)\|_{Y}$, we have

\[
\|L(u)\|_{Y} = \|L(u)\|_{L^\infty H^1} + \|\nabla L(u)\|_{L^2 H^1} + \|\sigma \nabla w\|_{L^\infty L^\infty}
\] : $= I_6 + I_7 + I_8$,
\] (3.22)

where by applying Lemma 2.3 (2.7) to $I_5$, we have

\[
I_6 = \|L(u)\|_{L^\infty H^1} = \left\| \int_{0}^{t} e^{-t+\tau} e^{(t-\tau)\Delta} u \right\|_{L^\infty H^1}
\leq \sup_{t>0} \int_{0}^{t} e^{-t+\tau} \|u\|_{H^1} d\tau
\leq c \left\| \|u\|_{L^\infty H^1} \right\|
\] (3.23)

and by applying (2.7) to $\nabla L(u) = L(\nabla u)$ and $\Delta L(u) = \nabla L(\nabla u)$ with $\theta = 0$ and $\theta = 1$, we have

\[
I_7 = \|\nabla L(u)\|_{L^2 H^1} \leq \|\nabla L(u)\|_{L^2 L^2} + \|\nabla \nabla L(u)\|_{L^2 L^2}
\leq \|L(\nabla u)\|_{L^2 L^2} + \|\nabla L(\nabla u)\|_{L^2 L^2}
\leq c \left\| \|\nabla u\|_{L^2 L^2} \right\|.
\] (3.24)

It remains to estimate $I_8 = \|\sigma \nabla w\|_{L^\infty L^\infty}$. Recalling the definition of $w$, we get

\[
I_8 = \sup_{t>0} t^2 (1 + t)^{-\frac{3}{2}} \left\| \int_{0}^{t} e^{-t+\tau} e^{(t-\tau)\Delta} u d\tau \right\|_{L^\infty}
\leq c \sup_{t>0} \int_{0}^{t} e^{-t+\tau} (t - \tau)^{-\frac{1}{2}} \|u\|_{L^\infty L^\infty} d\tau
\leq c \left\| \|u\|_{L^\infty L^\infty} \right\|.
\] (3.25)

Combining (3.13)–(3.25), we prove (3.12) and hence finish the proof. □
Proof of Theorem 1.1: Applying Proposition 3.2, following similar arguments as in Lemma 2.5 we can prove Theorem 1.2 and hence we omit the details.

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