ON THE NUMBER OF INVARIANT MEASURES FOR RANDOM EXPANDING MAPS IN HIGHER DIMENSIONS

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Abstract. In [26], Jabłoński proved that a piecewise expanding \( C^2 \) multi-dimensional Jabłoński map admits an absolutely continuous invariant probability measure (ACIP). In [6], Boyarsky and Lou extended this result to the case of i.i.d. compositions of the above maps, with an on average expanding condition. We generalize these results to the (quenched) setting of random Jabłoński maps, where the randomness is governed by an ergodic, invertible and measure preserving transformation. We prove that the skew product associated to this random dynamical system admits a finite number of ergodic ACIPs. Furthermore, we provide two different upper bounds on the number of mutually singular ergodic ACIP’s, motivated by the works of Buzzi [9] in one dimension and Gora, Boyarsky and Proppe [23] in higher dimensions.

1. Introduction

A fundamental problem in ergodic theory is to describe the asymptotic statistical behavior of orbits defined by a dynamical system. In this approach, one attempts to understand and quantify the different invariant measures of the system, in particular those which have physical relevance. This problem has been studied intensively for several classes of piecewise smooth systems, starting with one dimensional deterministic systems in the key paper [30] by Lasota and Yorke in 1973. In 2000, Buzzi [9] identified bounds on the number of physical measures for random compositions of Lasota–Yorke maps. In higher-dimensional frameworks, including random versions of [20] [38] [13] [40], understanding and, specifically, bounding the number of physical measures is still an unsolved problem. This challenge is related to open questions in multiplicative ergodic theory, regarding multiplicity of Lyapunov exponents. The focus of this work is on investigating and bounding the number of physical measures for a class of higher dimensional expanding-on-average random dynamical systems, where the randomness is driven by a rather general type of ergodic process, including but not limited to the i.i.d. case.

In this paper we study a class of discrete time dynamical systems in which, at each iteration of the process, one of a collection of maps is selected and applied. Ulam and von Neumann [41], Morita [32], Pelikan [35] and Buzzi [9] were among those who started working on such systems, which have been named time dependent, random or non autonomous dynamical systems. In general, there is no measure which is invariant under all these maps simultaneously. Therefore, we instead consider random invariant measures which are absolutely continuous with respect to Lebesgue (ACIPs), and their associated marginals, which give rise to physical measures.
This work focuses on dynamical systems modeled by random compositions of so-called Jabłoński maps. These maps have been studied by several researchers after the first paper [26] by Jabłoński in 1983. In [21], Góra and Boyarsky used Jabłoński transformations as a model for interacting cellular systems. In [7], Boyarsky and Lou presented a method for approximating the ACIPs in [26] by means of approximating the transfer operator by finite dimensional operators, which is a version of Ulam’s conjecture in a multidimensional setting. In [8], Boyarsky, Góra and Lou considered a larger class of $C^2$ transformations defined on a rectangular partition of the $n$ dimensional cube. The authors approximated any such map by a sequence of Jabłoński transformations and proved that the sequence of invariant densities associated with these Jabłoński maps converges weakly in $L^1$ to the invariant density associated with that map. In [5], Bose replaced the weak approximation of the invariant density in [7] by strong approximation using a compactness argument. The special case of random i.i.d. Jabłoński maps was studied in [6, 24].

Random Jabłoński maps, introduced in Definition 2.16, are defined by a collection of piecewise smooth maps $(f_\omega)_{\omega \in \Omega}$ defined on the state or phase space $I^n$, where $I = [0, 1]$ and $n \in \mathbb{N}$ is the dimension, equipped with the Borel sigma algebra of measurable sets and the $n$ dimensional Lebesgue measure $m$. The family of maps is assumed to satisfy an expanding-on-average condition.

Our approach relies on so-called transfer operators, acting on the space of higher dimensional functions of bounded variation. Given a nonsingular map $f$, its transfer operator $L_f$ encodes information about the application of $f$ and describes how densities, i.e. nonnegative integrable functions with integral one, evolve in time. If a collection of points in phase space is distributed according to a probability density function $h$, and pushed forward by $f$, then the resulting collection of points will be distributed according to a new density denoted by $L_f(h)$ or $L_f h$.

The first appearance of one dimensional functions of bounded variation is due to C. Jordan in 1881 in connection with Dirichlet’s test for the convergence of Fourier series. In 1905, G. Vitali gave the first definition of bounded variation function in two dimensions. Later on, L. Tonelli observed that Vitali’s generalization was not the right generalization of the one dimensional variation because it contains second order elements related to the curvature of the graph rather than its area. In 1936, in a closer analogy to the one dimensional variation, Tonelli introduced his generalization which measures the length of the projection of the graph onto the vertical axis counting multiplicities at least for continuous functions. Tonelli’s definition is more convenient for continuous functions since the definition depends on the choice of the coordinate axes if the function is not continuous. To solve this issue, in the same year, L. Cesari modified Tonelli’s definition by requiring the integrals in Tonelli’s definition to be finite for functions equal almost everywhere. This definition does not depend on the coordinates even for discontinuous functions. Functions of bounded variation in this sense were called bounded variation functions in the sense of Tonelli–Cesari. However, the point of view which is popular these days and adapted in most of the literature [20] as the most suitable generalization of the one dimensional theory is due to De Giorgi and Fichera. Krickeberg and Fleming independently showed that a bounded variation function in the sense of Tonelli–Cesari has a vector measure as its distributional gradient, thus obtaining the equivalence with De Giorgi’s definition. For more information on historical details about higher dimensional functions of bounded variation, we refer the reader to [2].
In the deterministic case, an early use of transfer operators in the one dimensional bounded variation setting is due to Lasota and Yorke, who in [30] proved the existence of ACIPs for piecewise $C^2$ transformations $f$ on $I$, with the assumption of a uniform expanding condition $\inf |f'| > 1$. The authors exploited the fact that the transfer operator corresponding to the point transformation under consideration has the property of keeping the variation of the functions $h, L_fh, \ldots, L^n_fh, \ldots$ under control. This result was later on referred to as Lasota-Yorke inequality. In [26], Jabłoński generalized the one dimensional work of Lasota and Yorke [30] to piecewise continuous maps on the multidimensional cube $I^n$ with similar type of uniform expanding condition on the rectangles of a rectangular partition. The proof of this result was similar to the proof of Theorem 1 in [30], but it uses the notion of variation of functions of several variables due to Tonelli–Cesari, which we also use in this paper.

In higher dimensions, the situation is more challenging than in one dimension. For example, in the general case, crucial difficulties come from the much richer geometry which can arise from the phase space partitions, and from the growing complexity of the partitions arising from the iterated dynamics. To overcome these issues, one may impose conditions on the geometry of the partitions and, roughly speaking, to ensure the amount of expansion is enough to overcome the dynamical complexity. See, for example, the conditions given in Theorem 4 in [23], Theorem 3.1 in [13] and equation (1.8) in [40].

A number of authors have studied the existence of ACIPs for piecewise expanding maps in higher dimensions. In [20], Góra and Boyarsky proved the existence of ACIPs with densities of bounded variation for piecewise $C^2$ transformations in $\mathbb{R}^n$ for domains with piecewise $C^2$ boundaries with the assumption that where the $C^2$ segments of the boundaries meet, the angle subtended by the tangents to these segments at the point of contact is bounded away from zero. The case when the boundaries for which the angle mentioned may become zero (i.e. the boundaries of partitions may contain 'cusps') is studied in [28, 1] by Keller and Adl-Zarabi. In [38], Saussol developed a Lasota-Yorke inequality for a class of piecewise expanding maps defined on a compact subset of $\mathbb{R}^n$ and used it to prove the existence of a finite number of ACIPs with densities in the Quasi-Hölder space. The author also provided an upper bound on the number of these ACIPs. In [13], Cowieson extended the work of Góra and Boyarsky by establishing a simpler condition which guarantees the existence of an ACIPs. The condition is that, the expansion must be greater than the cut index defined in [13, Section 2.2]. The author made some statements about random perturbations of such maps in [13, Theorem 3.2]. In [40], Thomine gave a sufficient condition shown in [10, Equation (1.7)] under which a piecewise $C^{1+\alpha}$ uniformly expanding map admits a finite number of ACIPs. The author also compares his results with the work of Saussol [38] and Cowieson [13]. Although no upper bounds on the number of these ACIPs are explicitly given in [40], the author mentions that the results of [38] could perhaps be adapted to his setting.

In the random one-dimensional case, in [9], Buzzi considers random expanding-on-average Lasota–Yorke maps that have neither too many branches nor too large distortion, and proves that the associated skew product transformation possesses a finite number of mutually singular ergodic ACIPs, each giving a family of random invariant measures with densities of bounded variation. In [3], Araujo and Solano
proved existence of ACIPs for random one dimensional dynamical systems with asymptotic expansion. Their work can be seen as a generalization of the work of Keller [27] which proves that for maps on the interval with finite number of critical points and non-positive Schwarzian derivative, existence of absolutely continuous invariant probability is earned by positive Lyapunov exponents. They also prove similar results for higher dimensional random systems under the assumption of slow recurrence to the set of discontinuities and/or criticalities, which are of a certain non-degenerate type, shown in [3 Equation (1.5)].

In [6], Boyarsky and Lou studied the case of i.i.d. compositions of Jabłoński maps as defined in [26]. The authors considered the setting of a finite number of piecewise $C^2$ and monotonic Jabłoński maps $f_1, \ldots, f_l$ where $l$ is a finite positive integer and

$$f_{k,i}(x_1, \ldots, x_n) = \varphi_{k,i,j}(x_i),$$

for $(x_1, \ldots, x_n) \in D_{k,j}$ where $D_{k,j}$ is the $j^{th}$ rectangle in the partition of $f_k$. The random map $f$ is defined by choosing $f_i$ with probability $p_i$, where the $p_i$’s are positive and add up to one. The authors assumed an expanding-on-average condition that is, there exists a positive constant $0 < \gamma < 1$ such that

$$(1.1) \quad \sum_{i=1}^{l} \sup_j p_i |\varphi_{k,i,j}'(x_i)| \leq \gamma,$$

for all $i = 1, \ldots, l$ and $(x_1, \ldots, x_n) \in cl(D_{k,j})$ (the closure of $D_{k,j}$) and proved that $f$ admits an ACIP with respect to the Lebesgue measure. This measure has density $h$ which is a fixed point of the averaged transfer operator $L_f$ of $f$, given by

$$(1.2) \quad L_f = \sum_{i=1}^{l} p_i L_{f_i},$$

where $L_{f_i}$ is the transfer operator of the corresponding map $f_i$.

Our conditions on the maps are much more general than the ones in [6, 35]. In both articles, the maps driving the dynamics or defining the random orbits are given by an i.i.d. process. Moreover, the maps must be chosen from a finite set. However, in our situation the way of selecting the maps comes from the base map $\sigma$ defined on a probability measure space $(\Omega, \mathcal{F}, \mathbb{P})$. A difficulty of this setting is that there is no known formula for an averaged transfer operator that corresponds to the one described in (1.2) in [35, 6]. The way we overcome this obstacle, as it has been done in [9, 16, 18], is by developing a random Lasota-Yorke inequality, Equation (3.1), which we use to prove several results in this paper.

Quasi-compactness is one of the concepts which has played a key role in the modern approach to investigate transport properties of random dynamical systems through transfer operators [16, 18]. In the case of autonomous systems, this property was introduced in the work of Ionescu Tulcea and Marinescu [22]. A bounded linear operator defined on a Banach space is called quasi-compact if its spectral radius is strictly larger than its essential spectral radius. The formulation of a non-autonomous analogue of the quasi-compactness property goes back to Thieullen [39]. It is now widely known that quasi-compactness can be usually derived from Lasota-Yorke type inequalities, and this is the route we pursue.

The quasi-compactness theorem of Ionescu Tulcea and Marinescu [22] is used to provide spectral decompositions and properties in the case of deterministic dynamical systems similar to the ones given in Section 3 in [20]. However, in the random
case one instead uses Oseledets type multiplicative ergodic theorems. In 1968, Oseledets multiplicative ergodic theorem was first introduced by Oseledets [34] in the context of random multiplication of matrices. In its basic form, it describes the asymptotic behavior of a product of matrices sampled from a dynamical system. After that, different proofs were provided and different generalizations have been developed and applied to transfer operator cocycles, see [16, 18]. In this paper we adapt Theorem 17 from [16] to provide an Oseledets splitting for random Jabłoński maps.

When applicable, multiplicative ergodic theorems provide existence and finiteness of random ACIPs. However, explicit bounds do not come directly from this machinery. Despite some progress by Buzzi [9] and Araujo–Solano [3], the question of how to find bounds on the number of ACIPs in random dynamical systems is largely open. In [23 Theorem 2], Góra, Boyarsky and Proppe proved that, in their setting, the support of absolutely continuous invariant measures is open Lebesgue almost everywhere. They used this key fact to obtain their result [23 Theorem 3] that the number of ergodic ACIP’s for deterministic dynamical systems modeled by Jabłoński transformations is at most equal to the number of crossing points. We combine elements of their arguments with ideas from the one dimensional work of Buzzi on random Lasota-Yorke maps (see Section 3 in [9]) to develop a bound on the number of mutually singular ergodic ACIP’s for a class of admissible random Jabłoński maps. Another bound is also developed, and these bounds are compared in Section 5.3.

This paper is structured as follows: in Section 2, we state the definition of admissible random Jabłoński maps, which involves the formulation of an expanding-on-average condition motivated from the expanding condition given in (1.1). In Section 3, in Theorem 3.1 we prove that this random map is quasi-compact and the maximal Lyapunov exponent is indeed zero. In Section 4, in Theorem 4.3, we prove that the random invariant densities of admissible random Jabłoński maps are of bounded variation and equivariant. In Corollary 4.6, we prove these densities belong to the leading Oseledets subspace and the number of ergodic ACIPs with respect to the associated skew product is finite. Theorem 4.8 is a probabilistic conclusion that shows that the marginals of the measures in Theorem 4.3 are physical, which means that for Lebesgue almost initial condition, the asymptotic long term behaviour of the corresponding random orbit will be described by one of these physical measures. In Section 5, we establish upper bounds on the number of mutually singular ergodic ACIPs for a class of admissible random Jabłoński maps, and present an example in Section 5.3.

2. Terminology and background

2.1. Preliminaries. In this subsection we state the basic definitions and tools that will be used throughout the paper.

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A measurable transformation $\sigma : \Omega \rightarrow \Omega$ is said to be nonsingular if

$$\mathbb{P}(\sigma^{-1}(A)) = 0,$$

for all $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$.

Definition 2.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A transformation $\sigma : \Omega \rightarrow \Omega$ is said to be a measure-preserving transformation or, equivalently, $\mathbb{P}$ is said to be a
\(\sigma\)-invariant measure, if

\[
P(\sigma^{-1}(A)) = P(A),
\]

for all \(A \in \mathcal{F}\).

**Definition 2.3.** Let \((\Omega, \mathcal{F}, P)\) be a probability space. A nonsingular transformation \(\sigma: \Omega \to \Omega\) is said to be **ergodic** if for all \(A \in \mathcal{F}\), with \(\sigma^{-1}(A) = A\), we have \(P(A) = 0\) or \(P(\Omega \setminus A) = 0\).

**Definition 2.4.** Let \((X, \mathcal{B}, \mu)\) be a measure space and \(f: X \to X\) a nonsingular transformation. The unique operator \(L_f: L^1(X) \to L^1(X)\) satisfying the dual relation

\[
\int_A L_f h(x) \mu(dx) = \int_{f^{-1}(A)} h(x) \mu(dx),
\]

for every \(A \in \mathcal{B}\) and \(h \in L^1(X)\) is called the transfer or Perron-Frobenius operator corresponding to \(f\).

For \(x = (x_1, \ldots, x_n)\), if \(A = \prod_{i=1}^n [0, x_i]\) in the above definition, then differentiating both sides, we obtain

\[
L_f h(x) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{f^{-1}(\prod_{i=1}^n [0, x_i])} h(y) \mu(dy),
\]

where \(\partial x_i\) is the derivative with respect to \(x_i\), \(i = 1, \ldots, n\). This formula can be seen in Section 2 in [8]. It is well known that the transfer operator is linear, positive, contractive and \(L_f h = h\) if and only if the measure \(\nu\) where \(d\nu = h \cdot d\mu\) is invariant under \(f\), see [4].

Given sets \(A_i, i = 1, \ldots, n\), denote the Cartesian product of the sets \(A_i\) by

\[
\prod_{i=1}^n A_i = \{(a_1, \ldots, a_n) : a_i \in A_i, i = 1, \ldots, n\}.
\]

For \(i = 1, \ldots, n\), let \(P_i\) be the projection of \(\mathbb{R}^n\) onto \(\mathbb{R}^{n-1}\) given by

\[
P_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).
\]

We next describe the definition of the total variation of an integrable function of several variables, due to Tonelli and Cesari, which was first used in the context of transfer operators in [6, 26].

**Definition 2.5.** Consider the \(n\) dimensional rectangle \(A = \prod_{i=1}^n [a_i, b_i]\) where \(a_i, b_i \in \mathbb{R}\) and \(a_i < b_i\) and a function \(g: A \to \mathbb{R}\). For \(i = 1, \ldots, n\), consider a real valued function \(\mathcal{V}_i g\) of \((n-1)\) variables \((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\) given by

\[
\sup_{a_i = x_1 \leq \cdots \leq x_i = b_i} \sum_{k=1}^r \left( g(x_1, \ldots, x_i^k, \ldots, x_n) - g(x_1, \ldots, x_i^{k-1}, \ldots, x_n) \right).
\]

For a measurable function \(f \in L^1(A)\) and \(i = 1, \ldots, n\), define

\[
\mathcal{V}_i f = \inf_{g = f \text{ a.e.}} \int_{P_i(A)} \mathcal{V}_i g dm,
\]

where \(\mathcal{V}_i g\) is measurable.
Definition 2.6. where $P_i$ is defined in (2.1) and let the total variation of $f$ be

$$\mathcal{A}^{\Lambda} f = \max_{i=1,\ldots,n} \mathcal{A}^{\Lambda} f_i.$$ 

If the total variation $\mathcal{A}^{\Lambda} f$ of $f$ on $\Lambda$ is a finite, then $f$ is said to be of bounded variation on $\Lambda$, and the set of all such maps is denoted by $BV(\Lambda)$. For $f \in BV(\Lambda)$, the norm of $f$ is defined by

$$\|f\|_{BV} = \|f\|_1 + \mathcal{A}^{\Lambda} f.$$ 

The space $BV(\Lambda)$ is a Banach space by Remark 1 in [17] and compactly embedded in $L^1(\Lambda)$ by Corollary 3.49 in [2].

2.2. Random dynamical systems.

Definition 2.7. A random dynamical system is a tuple $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{X}, \mathcal{L})$, where the base $\sigma$ is an invertible measure-preserving transformation of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{X}, \|\cdot\|)$ is a Banach space and $\mathcal{L} : \Omega \to L(\mathcal{X}, \mathcal{X})$ is a family of bounded linear maps of $\mathcal{X}$, called the generator.

For convenience, we let $L_\omega := \mathcal{L}(\omega)$. A random dynamical system defines a cocycle, given by

$$(k, \omega) \mapsto L^{(k)}_\omega := \mathcal{L}_{\sigma^{k-1} \omega} \circ \cdots \circ \mathcal{L}_{\sigma \omega} \circ L_\omega.$$ 

Different regularity conditions may be imposed on the generator $\mathcal{L}$. The following concept of $\mathbb{P}$-continuity, which was first introduced by Thieullen in [39], will be used in the sequel.

Definition 2.8. Let $\Omega$ be a topological space, equipped with a Borel probability measure $\mathbb{P}$ and let $Y$ be a topological space. A mapping $L : \Omega \to Y$ is said to be $\mathbb{P}$-continuous if $\Omega$ can be expressed as a countable union of Borel sets such that the restriction of $L$ to each of them is continuous.

In the rest of this work, we consider random dynamical systems whose generators $\mathcal{L} : \Omega \to L(\mathcal{X}, \mathcal{X})$, given by $\omega \mapsto L_\omega$, are $\mathbb{P}$-continuous and that $\Omega$ is a Polish space. That is, a complete separable metric space.

Definition 2.9. The index of compactness (or Kuratowski measure of noncompactness) of a bounded linear map $A : \mathcal{X} \to \mathcal{X}$ is

$$\|A\|_{ic(\mathcal{X})} = \inf \{r > 0 : A(B_X) \text{ can be covered by finitely many balls of radius } r\},$$

where $B_X$ denotes the unit ball in $\mathcal{X}$.

Definition 2.10. Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{X}, \mathcal{L})$ be a random dynamical system. Assume that $\int_{\Omega} \log^+ \|L_\omega\|d\mathbb{P}(\omega) < \infty$. For each $\omega \in \Omega$, the maximal Lyapunov exponent $\lambda(\omega)$ for $\omega$ is defined as

$$\lambda(\omega) = \lim_{k \to \infty} \frac{1}{k} \log \|L^{(k)}_\omega\|,$$

whenever the limit exists. The index of compactness $K(\omega)$ for $\omega$ is defined as

$$K(\omega) = \lim_{k \to \infty} \frac{1}{k} \log \|L^{(k)}_\omega\|_{ic(\mathcal{X})},$$

whenever the limit exists.

The following is established in [18].
Remark 2.11. If a random dynamical system $\mathcal{R}$ has an ergodic base $\sigma$, then $\lambda$ and $K$ in the previous definition are $\mathbb{P}$–almost everywhere constant. We call these constants $\lambda^*(\mathcal{R})$ and $K^*(\mathcal{R})$, or simply $\lambda^*$ and $K^*$, if $\mathcal{R}$ is clear from the context. It follows from the definition that $K^* \leq \lambda^*$. The assumption $\int_{\Omega} \log^+ \|L_\omega\|d\mathbb{P}(\omega) < \infty$ implies that $\lambda^* < \infty$.

**Definition 2.12.** A random dynamical system $\mathcal{R}$ with an ergodic base $\sigma$ is called **quasi-compact** if $K^* < \lambda^*$.

The next proposition relates the maximal Lyapunov exponent $\lambda^*(\mathcal{R})$ and the index of compactness $K^*(\mathcal{R})$ of a random dynamical system $\mathcal{R}$ with the corresponding quantities for $\mathcal{R}^{(n)} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma^n, X, \mathcal{L}^{(n)})$, $n \in \mathbb{N}$.

**Proposition 2.13.** Consider a random dynamical system $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ with an ergodic base $\sigma$. Then, for each $n \in \mathbb{N}$, $\mathcal{R}^{(n)} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma^n, X, \mathcal{L}^{(n)})$ is a random dynamical system, with a possibly non-ergodic base $\sigma^n$. For each $\omega \in \Omega$, let

$$\lambda_n(\omega) = \lim_{k \to \infty} \frac{1}{k} \log \|L^{(nk)}(\omega)\|,$$

$$K_n(\omega) = \lim_{k \to \infty} \frac{1}{k} \log \|L^{(nk)}(\omega)\|_{\text{ic}(X)}.$$

Then

$$\lambda_n(\omega) = n\lambda^*(\mathcal{R}),$$

$$K_n(\omega) = nK^*(\mathcal{R}),$$

$\mathbb{P}$–almost everywhere.

**Proof.** For each $n \in \mathbb{N}$, $\{L^{(nk)}(\omega)\}_{k=1}^{\infty}$ is a subsequence of $\{L^{(k)}(\omega)\}_{k=1}^{\infty}$. Thus, the proof follows from Remark 2.11. \qed

While the transformation $\sigma^n$ may be non-ergodic when $\sigma$ is ergodic, the following result ensures that $\sigma^n$ is ergodic on some subset $Z \subset \Omega$. This result will be used in the proof of Proposition 2.12.

**Lemma 2.14** (González-Tokman & Quas [19] Lemma 35). Let $\sigma$ be an ergodic $\mathbb{P}$-preserving transformation of $(\Omega, \mathcal{F}, \mathbb{P})$ and let $n \in \mathbb{N}$. Then there exists $k$, a factor of $n$, and a $\sigma^n$-invariant subset $Z$ of $\Omega$ of measure $1/k$ such that $\Omega = \bigcup_{s=0}^{k-1} \sigma^{-s} Z$ and $\sigma^n|Z$ is ergodic. When $\sigma$ is invertible, this argument also applies to $n < 0$.

2.3. **Admissible random Jabłoński maps and quasi-compactness.**

**Definition 2.15.** A partition $\mathcal{B} = \{B_1, \ldots, B_q\}$ of $I^n$ is called **rectangular** if for each $j = 1, \ldots, q$,

$$B_j = \prod_{i=1}^{n} B_{ij},$$

where $B_{ij} = [a_{ij}, b_{ij}]$ if $b_{ij} < 1$ and $B_{ij} = [a_{ij}, b_{ij}]$ if $b_{ij} = 1$.

A piecewise map $f : I^n \cap \Omega$ defined on the rectangular partition given in Definition 2.15 is generally written as

$$f(x_1, \ldots, x_n) = (\varphi_{1,j}(x_1, \ldots, x_n), \ldots, \varphi_{n,j}(x_1, \ldots, x_n)),$$
where \((x_1, \ldots, x_n) \in B_j, j = 1, \ldots, q\). Following [26], we next introduce Jabłoński maps as a special case of such maps. We then define random Jabłoński maps and admissible random Jabłoński maps.

**Definition 2.16 (Jabłoński [26]).** A map \(f : I^n \in B = \{B_1, \ldots, B_q\}\) of \(I^n\) and is given by the formula

\[
f(x_1, \ldots, x_n) = (\varphi_{1,j}(x_1), \ldots, \varphi_{n,j}(x_n)),
\]

where \((x_1, \ldots, x_n) \in B_j, j = 1, 2, \ldots, q\). The vertices of the rectangles in \(B\) which lie in the interior of \(I^n\) are called the crossing points of \(f\). The real valued maps \(\varphi_{i,j} : B_{ij} \to [0,1]\) are called the components of \(f\). We use \(J\) to denote for the class of Jabłoński maps on \(I^n\).

While the above family of Jabłoński maps may seem restrictive, in [8], Boyarsky, Góra and Lou proved that for any piecewise \(C^2\) map \(f\) defined on a rectangular partition of \(I^n\), \(f\) can be approximated by a sequence of piecewise \(C^2\) Jabłoński transformations. In other words, there exists a sequence of Jabłoński maps \(f_n\) that converges pointwise to \(f\). Moreover, the corresponding sequence of invariant densities of \(f_n\) (which exists by [26]) converges weakly to an invariant density of \(f\). Generally speaking, Jabłoński maps can be seen as the basis maps for a much larger class of piecewise defined maps on the \(n\) dimensional rectangle.

**Definition 2.17.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\sigma : \Omega \in \mathcal{F}\) over \(\sigma\) is a random Jabłoński map over \(\sigma\) is a map \(F : \Omega \to J\), where \(f_\omega := F(\omega) : I^n \in B\), for each \(\omega \in \Omega\), there exists a rectangular partition \(B^\omega\) of \(I^n\), say,

\[
B^\omega = \{B^{\omega}_1, \ldots, B^{\omega}_{q_\omega}\},
\]

where \(q_\omega\) is a positive integer. If \(x = (x_1, \ldots, x_n) \in B^\omega_j\), then we have

\[
f_\omega(x) = (\varphi_{\omega,1,j}(x_1), \ldots, \varphi_{\omega,n,j}(x_n)),
\]

where \(B^\omega_j = \prod_{i=1}^n [a^{\omega,j}_i, b^{\omega,j}_i]\) and \(\varphi_{\omega,i,j}\) is a map from \([a^{\omega,j}_i, b^{\omega,j}_i]\) into \([0,1]\). For \(k \in \mathbb{N}\), the \(k\) fold composition \(f^{(k)}_\omega\) is defined as

\[
f^{(k)}_\omega := f_{\sigma^{k-1}\omega} \circ \cdots \circ f_{\sigma\omega} \circ f_\omega.
\]

For simplicity, we sometimes refer to the range of \(F\), that is \(\{f_\omega\}_{\omega \in \Omega}\), as the random Jabłoński map. A random Jabłoński map gives rise to a random dynamical system, where \(\mathcal{X} = BV(I^n)\) and \(\mathcal{L}_\omega = \mathcal{L}_{f_\omega}\). The next proposition proves that for all \(\omega \in \Omega\), \(\mathcal{L}_\omega\) is a bounded operator on \(BV(I^n)\).

**Proposition 2.18.** For all \(\omega \in \Omega\), \(\mathcal{L}_\omega\) is a bounded operator on \(BV(I^n)\).
Proof. Let $\omega \in \Omega$,
\[
\|L_\omega h\|_{BV} = \int_{I^n} |L_\omega h| dm + \mathcal{L}_\omega h
\leq \int_{I^n} L_\omega |h| dm + \mathcal{L}_\omega h
= \int_{I^n} |h| dm + \mathcal{L}_\omega h
= \|h\|_1 + \mathcal{L}_\omega h
\]
By the Lasota-Yorke inequality provided in the proof of Theorem 1 in [26], the last term is less than or equal
\[
\|h\|_1 + \alpha\|h\|_1 + \beta\mathcal{L}_\omega h \leq \max(1 + \alpha, \beta)\|h\|_{BV}.
\]
for some $\alpha, \beta > 0$. □

We note that
\[
(2.3) \quad L^{(k)}_\omega = L_{f_{\sigma^{-1}} \circ \cdots \circ f_{\omega} \circ f_\omega} = L_{\sigma^{-k} \circ \cdots \circ \sigma \circ \omega} \circ L_\omega.
\]
In what follows, we will use the notation $\vec{r} = (r_1, \ldots, r_n) \in \mathbb{N}^n$ and denote the set of vector indices by
\[
Z_{\vec{r}} := \{ \vec{s} = (s_1, s_2, \ldots, s_n) : 1 \leq s_i \leq r_i \}.
\]

Remark 2.19. If the random Jabłoński map $F$ has a finite range, then for each $k_0 \in \mathbb{N}$, there exists a common partition $\mathcal{B} = \mathcal{B}(k_0)$ of $I^n$ into maximal rectangles such that the components of the maps $\{f^{(k_0)}_\omega\}_{\omega \in \Omega}$ are $C^2$ and monotonic on their interval domains.

Remark 2.20. The generator $\mathcal{L} : \Omega \rightarrow L(X, \mathcal{X})$ of the random dynamical system generated by a random Jabłoński map $\mathcal{F}$ is $\mathbb{P}$-continuous if its range is at most countably infinite (consisting of, say, $f_1, f_2, \ldots$) and the preimage of each $f_j$ is a measurable set. Our results will be valid when there exists a common partition $\mathcal{B}$ of $I^n$ into rectangles such that the components of the maps $\{f^{(N)}_\omega\}_{\omega \in \Omega}$ are $C^2$ and monotonic on their interval domains, where $N \in \mathbb{N}$ satisfies the condition given in (3.2). In this case, for each $i = 1, \ldots, n$, there exists a partition
\[
0 = a_{i,0} < a_{i,1} < \cdots < a_{i,r_i} = 1,
\]
for some $r_i \in \mathbb{N}$. Let $B_{s_i} = [a_{i,s_i-1}, a_{i,s_i})$ when $s_i = 1, 2, \ldots, r_i - 1$ and $B_{r_i} = [a_{i,r_i-1}, a_{i,r_i})$. For each vector index $\vec{s} \in Z_{\vec{r}}$, we denote the $n$ dimensional rectangle by $B_\vec{s} = \prod_{i=1}^n B_{s_i}$. The common rectangular partition is given by
\[
\mathcal{B} = \{ B_\vec{s} : \vec{s} \in Z_{\vec{r}} \}.
\]
For each $\omega \in \Omega$ and $\vec{s} \in Z_{\vec{r}}$, we write the map $f_\omega$ with respect to $\mathcal{B}$ as
\[
f_\omega(x) = (\varphi_{\omega,1}, \vec{s}(x_1), \ldots, \varphi_{\omega,n, \vec{s}}(x_n)), x = (x_1, \ldots, x_n) \in B_\vec{s},
\]
and, for each \( k \in \mathbb{N} \), the map \( f^{(k)}_\omega \) as
\[
f^{(k)}_\omega(x) = (\varphi_{\omega,1,k,s}(x_1), \ldots, \varphi_{\omega,n,k,s}(x_n)), \quad x = (x_1, \ldots, x_n) \in B_{s}.
\]

**Remark 2.21.** One can associate to the random Jabłoński map \( \mathcal{F} = \{f_\omega\}_{\omega \in \Omega} \), the skew product map \( F \) on \( \Omega \times I^n \) which encodes the dynamics of the whole system (2.4)
\[
F(\omega, x) = (\sigma_\omega, f_\omega(x)).
\]
Expanding properties for dynamical systems lead to chaotic behavior of the orbits. However, they usually give rise to good ergodic properties like the existence of absolutely continuous invariant measures. Next we introduce the admissible random Jabłoński maps. This definition involves a formulation of an expanding-on-average condition.

**Definition 2.22.** Using the notation in Remark[2.20] a random Jabłoński map \( \mathcal{F} \) is called admissible if all the components \( \varphi_{\omega,i,s} \) are \( C^2 \) and monotonic on \([a_{i,s_i}-1, a_{i,s_i}]\) and there exists a constant \( \gamma > 0 \) such that
\[
\Gamma := \int_{\Omega} \min_{i=1,\ldots,n} \log(\gamma_i(\omega))dP(\omega) > \gamma,
\]
where
\[
\gamma_i(\omega) := \inf_{s \in \mathbb{Z}_{r_i}} (|\varphi'_{\omega,i,s}(x_i)|).
\]
In addition, we assume the mapping \( \omega \mapsto \mathcal{L}_\omega \) is \( P \)-continuous.

### 3. Random Lasota-Yorke Inequality and Quasi-Compactness

In the next theorem, we establish a suitable Lasota-Yorke inequality on the space of bounded variation \( BV(I^n) \) and we use it to prove the quasi-compactness property for admissible random Jabłoński maps.

**Theorem 3.1.** Let \( \mathcal{F} = \{f_\omega\}_{\omega \in \Omega} \) be an admissible random Jabłoński map. Then:

(i) the random dynamical system generated by \( \mathcal{F} \) is quasi-compact; and

(ii) its maximal Lyapunov exponent \( \lambda^* \) is zero.

**Proof of Theorem 3.1 (i).** The first step is to show that there are \( N \in \mathbb{N} \) and positive measurable functions \( \alpha_1, \alpha_2 : \Omega \to \mathbb{R}^+ \) such that
\[
\int_{\Omega} \log \alpha_1(\omega)dP(\omega) < 0
\]
and
\[
\int_{\Omega} \mathcal{L}^{(N)}_\omega h \leq \alpha_1(\omega)\|h\|_1 + \alpha_2(\omega)\|h\|_1,
\]
for all \( h \in BV(I^n) \), where \( \mathcal{L}^{(N)}_\omega \) is defined in (2.3). Let \( x = (x_1, \ldots, x_n) \in I^n \) and \( \omega \in \Omega \), choose \( N \in \mathbb{N} \) such that
\[
N\gamma > \log(3),
\]
where \( \gamma \) satisfies the condition (2.5). Let \( s_0 = \bar{s} \) be the label of the unique rectangle in \( B \) for which \( x \in B_{s_0} \) and for \( k = 1, 2, \ldots \), let \( s_k \in \mathbb{Z}_{r} \) be such that
\[
f^{(k)}_\omega(x) = (\varphi_{\omega,1,k,s}(x_1), \ldots, \varphi_{\omega,n,k,s}(x_n))
\]
and
\[
= (\varphi_{\sigma^{k-1}\omega,1,s_{k-1}}, \varphi_{\omega,1,s_1}(x_1), \ldots, \varphi_{\omega,n,s_n}(x_n))
\]
in \( B_{s_k} \).
Note that for any $i = 1, 2, \ldots, n$,  

$$
\int_{\Omega} \inf_{\bar{x} \in \mathbb{Z}_r} \int_{x_i \in [a_i, s_i - 1, a_i, s_i]} \log(|\varphi_{\omega,i,N,S}(x_i)|) d\mathbb{P}(\omega)
$$

(3.3)

$$
\geq \sum_{k=0}^{N-1} \int_{\Omega} \inf_{\bar{x} \in \mathbb{Z}_r} \int_{x_i \in [a_i, s_i - 1, a_i, s_i]} \log(|\varphi_{\sigma^k(\omega),i,k-1}(\varphi_{\omega,i,k-1}(x_i))|) d\mathbb{P}(\omega)
$$

$$
\geq \sum_{k=0}^{N-1} \int_{\Omega} \log(\gamma_i(\omega)) d\mathbb{P}(\omega) \geq N\gamma > \log(3).
$$

Let $\mathcal{E}$ be the set of functions of the form $g = \sum_{j=1}^{M} g_j \chi_{A_j}$, where $A_j = \prod_{i=1}^{n} [\alpha_{ij}, \beta_{ij}] \subseteq \mathbb{I}^n$ and $g_j: \mathbb{I}^n \to \mathbb{R}$ is a $C^1$ function on $A_j$. By [26, Remark 1], $\mathcal{E}$ forms a dense subset of the space $L^1(\mathbb{I}^n)$. By [26, Remark 5], $\mathcal{E} \subset BV(\mathbb{I}^n)$.

We argue in a similar way to the proof of Theorem 1 in [26]. We provide the Lasota-Yorke inequality on elements of $\mathcal{E}$. Since the BV norm is a continuous function and by Proposition 2.18, the transfer operator is bounded, using a density argument, the inequality can be extended to elements of $BV(\mathbb{I}^n)$.

Let $h \in \mathcal{E}$ be such that $h \geq 0$ and for any $i = 1, \ldots, n$, let $h_i \in \mathcal{E}$ be such that $h_i = h$ Lebesgue almost everywhere with the property

$$
\int_{P_i(\mathbb{I}^n)} \mathbb{I}^n \mathbb{V}_i dm = \mathbb{I}^n h_i.
$$

Let

$$
\Psi_{\omega,i,N,S} := \varphi_{\omega,i,N,S}^{-1},
$$

$$
\delta_{\omega,i,N,S} := |\varphi_{\omega,i,N,S}'|,
$$

$$
I_{\omega,N,S} := \prod_{i=1}^{n} \varphi_{\omega,i,N,S}([a_i, s_i - 1, a_i, s_i]).
$$

The transfer operator $\mathcal{L}^{(N)}(\omega)$ applied to $h$ evaluated at $x = (x_1, \ldots, x_n) \in \mathbb{I}^n$ is given by

$$
\mathcal{L}^{(N)}(\omega)h(x) = \sum_{\bar{x} \in \mathbb{Z}_r} h\left(\Psi_{\omega,1,N,S}(x_1), \ldots, \Psi_{\omega,n,N,S}(x_n)\right) \prod_{j=1}^{n} \delta_{\omega,j,N,S}(x_j) I_{\omega,N,S} (x).
$$
If we apply $V_i$ for the the $L^1(I^n)$-function $L_{\omega}^{(N)} h_i$ and take the integral over $P_i(I^n)$, we get

$$\int_{P_i(I^n)} V_i L_{\omega}^{(N)} h_i dm \leq I_1 + I_2,$$

where

$$I_1 = \sum_{s \in \mathbb{Z}} \int_{P_i(I_{\omega,N,s}^{(N)})} h_i \left( \Psi_{\omega,1,N,s}^{-1}(x_1), \ldots, \Psi_{\omega,n,N,s}^{-1}(x_n) \right) \prod_{j=1}^{n} \delta_{\omega,j,N,s}^{-1}(x_j) dm,$$

$$I_2 = \sum_{s \in \mathbb{Z}} \int_{P_i(I_{\omega,N,s}^{(N)})} \left( |h_i(\Psi_{\omega,1,N,s}^{-1}(x_1), \ldots, \Psi_{\omega,n,N,s}^{-1}(x_n))| \delta_{\omega,i,N,s}^{-1}(\varphi_{\omega,i,N,s}^{-1}(a_i,s)) \right) \prod_{j=1}^{n} \delta_{\omega,j,N,s}^{-1}(x_j) dm.$$

Let

$$\rho_{\omega,i,N} = \sup_{s \in \mathbb{Z}} \delta_{\omega,i,N,s},$$

and

$$K_{\omega,i,N} = \sup_{s \in \mathbb{Z}} \int_{P_i(I_{\omega,N,s}^{(N)})} \left( |h_i(\Psi_{\omega,1,N,s}^{-1}(x_1), \ldots, \Psi_{\omega,n,N,s}^{-1}(x_n))| \delta_{\omega,i,N,s}^{-1}(\varphi_{\omega,i,N,s}^{-1}(a_i,s)) \right) \prod_{j=1}^{n} \delta_{\omega,j,N,s}^{-1}(x_j) dm.$$

These constants are motivated from the ones given in Theorem 1 in [35] also Theorem 2 in [6]. By Inequality 7 in [35] adapted to our notation, we have

$$I_1 \leq 2 \rho_{\omega,i,N} \sum_{s \in \mathbb{Z}} \int_{P_i(I_{\omega,N,s}^{(N)})} P_i(I_{\omega,N,s}^{(N)}) \times I \left( \Psi_{\omega,1,N,s}^{-1}(x_1), \ldots, x_i, \ldots, \Psi_{\omega,n,N,s}^{-1}(x_n) \right) \prod_{j=1}^{n} \delta_{\omega,j,N,s}^{-1}(x_j) dm$$

$$+ K_{\omega,i,N} \sum_{s \in \mathbb{Z}} \int_{P_i(I_{\omega,N,s}^{(N)})} \int_{0}^{1} h_i \left( \Psi_{\omega,1,N,s}^{-1}(x_1), \ldots, x_i, \ldots, \Psi_{\omega,n,N,s}^{-1}(x_n) \right) dx_i \prod_{j=1}^{n} \delta_{\omega,j,N,s}^{-1}(x_j) dm.$$
Since \( \{B_s : s \in \mathbb{Z}_T \} \) forms a partition for \( I^n \), the last sum is equal to

\[
2 \rho_{\omega,i,N} \sum_{s \in Z_{P_i(\mathbb{T})}} \int_{P_i(B_s)} \frac{P_i(B_s) \times I}{V} h_i(x_1, \ldots, x_n) dm
\]

\[
+ K_{\omega,i,N} \sum_{s \in Z_{P_i(\mathbb{T})}} \int_{P_i(B_s)} \int_0^1 h_i(x_1, \ldots, x_n) dx_i dm.
\]

Since \( \{B_s : s \in Z_{\mathbb{T}} \} \) forms a partition for \( I^n \), the last sum is equal to

\[
2 \rho_{\omega,i,N} \int_{P_i(I^n)} \frac{I^n}{V} h_i dm + K_{\omega,i,N} \int_{P_i(I^n)} \left( \int_0^1 h_i dx_i \right) dm
\]

(3.6)

\[
\leq 2 \rho_{\omega,i,N} \int_{P_i(I^n)} \frac{I^n}{V} h_i dm + K_{\omega,i,N} \| h_i \|_1.
\]

The expression in \( I_2 \) is less than or equal to

\[
sup_{s \in Z_{\mathbb{T}}} | \sum_{s \in Z_{\mathbb{T}}} \int_{P_i(I^n)} \left( \left| h_i(\Psi_{\omega,1,N,s}(x_1), \ldots, a_{i,s}, \ldots, \Psi_{\omega,n,N,s}(x_n)) \right| + | h_i(\Psi_{\omega,1,N,s}(x_1), \ldots, a_{i,s}, \ldots, \Psi_{\omega,n,N,s}(x_n)) | \right) \prod_{j \neq i} \delta_{\omega,j,N,s}(x_j) dm.
\]

Since \( h \geq 0 \), the argument after Equation (5) in [35], implies

\[
\left( \left| h_i(\Psi_{\omega,1,N,s}(x_1), \ldots, a_{i,s}, \ldots, \Psi_{\omega,n,N,s}(x_n)) \right| + | h_i(\Psi_{\omega,1,N,s}(x_1), \ldots, a_{i,s}, \ldots, \Psi_{\omega,n,N,s}(x_n)) | \right) \prod_{j \neq i} \delta_{\omega,j,N,s}(x_j) dm.
\]

\[
\leq \int_{P_i(I^n)} \frac{I^n}{V} h_i(\Psi_{\omega,1,N,s}(x_1), \ldots, x_i, \ldots, \Psi_{\omega,n,N,s}(x_n))
\]

\[
+ 2 \int_0^1 h_i(\Psi_{\omega,1,N,s}(x_1), \ldots, x_i, \ldots, \Psi_{\omega,n,N,s}(x_n)) dx_i,
\]

then we have the expression in \( I_2 \) is less than or equal to

\[
\rho_{\omega,i,N} \sum_{s \in Z_{P_i(\mathbb{T})}} \int_{P_i(I^n)} \left( \frac{P_i(I^n) \times I}{V} h_i(\Psi_{\omega,1,N,s}(x_1), \ldots, x_i, \ldots, \Psi_{\omega,n,N,s}(x_n))
\]

\[
+ 2 \int_0^1 h_i(\Psi_{\omega,1,N,s}(x_1), \ldots, x_i, \ldots, \Psi_{\omega,n,N,s}(x_n)) dx_i \right) \prod_{j \neq i} \delta_{\omega,j,N,s}(x_j) dm.
\]

Using again the change of variables \( U = \Psi^{-1} \), we get the last sum is equal to
\[ \rho_{\omega,i,N} \sum_{x \in Z_{P_i(\mathcal{F})}} \int_{P_i(\mathcal{F})} \left( \nabla \log h \right)_i (x_1, \ldots, x_n) + 2 \int_0^1 h_i(x_1, \ldots, x_n) dx_i d\mu \]

\[ = \rho_{\omega,i,N} \int_{P_i(\mathcal{F})} t h_i dx + 2 \rho_{\omega,i,N} \int_{P_i(\mathcal{F})} 0 h_i dx + \rho_{\omega,i,N} ||h||_1. \]

(3.7) \leq \rho_{\omega,i,N} \int_{P_i(\mathcal{F})} t h_i dx + 2 \rho_{\omega,i,N} ||h||_1.

Now, combining the results from (3.6), (3.7) and (3.4), we get

\[ \int_{P_i(\mathcal{F})} t L^{(N)} h dx \leq 3 \rho_{\omega,i,N} \int_{P_i(\mathcal{F})} t h_i dx + (K_{\omega,i,N} + 2 \rho_{\omega,i,N}) ||h||_1. \]

Thus, letting

(3.8) \quad \alpha_1(\omega) = \max_{i=1,\ldots,n} 3 \rho_{\omega,i,N},

(3.9) \quad \alpha_2(\omega) = \max_{i=1,\ldots,n} (K_{\omega,i,N} + 2 \rho_{\omega,i,N}),

we have, for each \( i = 1, \ldots, n, \)

\[ \int_{P_i(\mathcal{F})} t L^{(N)} h dx \leq \alpha_1(\omega) \int_{P_i(\mathcal{F})} t h dx + \alpha_2(\omega) ||h||_1. \]

By [18, Lemma C.5] and Lemma 2.14, the index of compactness \( K_N(\omega) \) is less than

\[ \int_{\sigma^{-\ell}Z} \log \alpha_1(\omega) dP(\omega), \]

where \( \ell \) is such that \( \sigma^{-\ell}Z \) is the ergodic component of \( \sigma^N \) containing \( \omega \). Since

\[ \Omega = \bigcup_{s=0}^{k-1} \sigma^{-s}Z \quad \text{and} \quad \int_{\sigma^{-k}Z} \log \alpha_1(\omega) dP(\omega) < 0, \]

we have \( \int_{\sigma^{-\ell_0}Z} \log \alpha_1(\omega) dP(\omega) < 0 \) for some \( \ell_0 = 0, 1, \ldots, k-1 \). By Proposition 2.13, we have \( K^* = \frac{K_N(\omega)}{N} < 0 \).

Since the transfer operator \( L^{(n)}(\omega) \) is a Markov operator for each \( \omega \in \Omega \), for any density function \( h \in BV(I^n) \), we have that \( ||L^{(n)}(\omega) h||_{BV} \geq ||L^{(n)}(\omega) h||_{1} = ||h||_{1} = 1 \).

This shows that

(3.9) \quad \lambda^* \geq 0,

and therefore \( K^* < \lambda^* \). This finishes the proof of Theorem 3.1 (i).

Proof of Theorem 3.1 (ii). In the proof of Theorem 3.1 (i), we proved that there are \( N \in \mathbb{N} \) where \( N \) satisfies the condition in (3.2) and \( \alpha_1, \alpha_2 : \Omega \to \mathbb{R}^+ \) such that

\[ \int_{\Omega} \log \alpha_1(\omega) dP(\omega) < 0 \]

with the property that

(3.10) \quad \int_{I^n} L^{(N)} h \leq \alpha_1(\omega) \int_{I^n} h + \alpha_2(\omega) ||h||_1,

for all \( h \in BV(I^n) \) and \( \omega \in \Omega \). We also proved that \( \lambda^* \geq 0 \) in (3.9). It remains to prove \( \lambda^* \leq 0 \). Since \( ||L^{(n)}(\omega)||_{1} \leq 1 \), it is enough to consider the growth of the variation of the term \( L^{(n)}(\omega) h \). Using the argument in [18, Lemma C.5] and [9, Proposition
1.4, $\alpha_1(\omega)$ and $\alpha_2(\omega)$ can be redefined so that (3.10) holds and $\alpha_2(\omega)$ is uniformly bounded by positive constant $\tilde{\alpha}_2$, which gives a hybrid Lasota-Yorke inequality

\[(3.11)\]

$$I^n V \mathcal{L}_n^{(N)} h \leq \alpha_1(\omega) I^n h + \tilde{\alpha}_2 \|h\|_1.$$ 

By iterating the hybrid Lasota-Yorke inequality (3.11), we get a bound on the sequence $(\mathcal{V} L^{(Nk)} h)_{k=1}^\infty$. Therefore,

$$\lim_{k \to \infty} \frac{1}{Nk} \log \|L^{(Nk)} h\|_{BV} \leq 0.$$ 

and since this is true for almost every $\omega \in \Omega$, Proposition 2.13 implies that $\lambda^* \leq 0$. □

4. Random invariant densities and ACIPs, skew product ACIPs and Physical measures

The concept of random invariant measures (for random dynamical systems) is a natural generalization of the notion of invariant measures (for deterministic dynamical systems). In this section we introduce our main results regarding the existence of random invariant densities and measures as well as skew product ACIPs. After that, we deduce the existence of physical measures. We shall assume throughout the rest of the paper that

$$\int_{\Omega} \log^+ \|L^{(N)} h\|_{BV} d\mathbb{P}(\omega) < \infty.$$ 

Definition 4.1. Let $\mathcal{F} = \{f_\omega\}_{\omega \in \Omega}$ be an admissible random Jabłoński map. A family $\{\mu_\omega\}_{\omega \in \Omega}$ of random invariant measures for $\mathcal{F}$ is a family of probability measures $\mu_\omega$ on $I^n$ where the map $\omega \mapsto \mu_\omega$ is measurable and

$$f_\omega \mu_\omega = \mu_{\sigma \omega},$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. A family $\{h_\omega\}_{\omega \in \Omega}$ of random invariant densities for $\mathcal{F}$ is a family such that $h_\omega \geq 0$, $h_\omega \in L^1(I^n)$, $\|h_\omega\|_1 = 1$, the map $\omega \mapsto h_\omega$ is measurable and

\[(4.1)\]

$$L^{(N)} h_\omega = h_{\sigma \omega},$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Proposition 4.2. Let $N$ be as in (3.2). Then, for $\mathbb{P}$-almost all $\omega \in \Omega$, we have

$$\lim_{j \to \infty} \frac{1}{j} \sum_{t=1}^j \log(\alpha_1(\sigma^{-tN} \omega)) < 0.$$ 

Proof. By Lemma 2.14 there exists $k$, a factor of $N$, and a $\sigma^{-N}$-invariant subset $Z$ of $\Omega$ of measure $1/k$ such that $\Omega = \bigcup_{s=0}^{k-1} \sigma^s Z$ and $\sigma^{-N}|_Z$ is ergodic. In fact, since $\sigma$ is invertible, ergodic and $\mathbb{P}$-preserving, $\sigma^{-N}|_{\sigma^\ell Z}$ is ergodic and $\mathbb{P}(\sigma^\ell Z) = \frac{1}{k}$, for all $\ell = 0, 1, \ldots, k-1$. By Birkhoff ergodic theorem, we have

$$\lim_{j \to \infty} \frac{1}{j} \sum_{t=1}^j \log(\alpha_1(\sigma^{-tN} \omega)) = k \int_{\sigma^\ell Z} \log \alpha_1(\omega) d\mathbb{P}(\omega),$$

for $\mathbb{P}$-almost all $\omega \in \sigma^\ell Z$, and $\ell = 0, 1, \ldots, k-1$. Note that for any $\ell = 0, 1, \ldots, k-1$ and $\mathbb{P}$-almost all $\omega \in \sigma^\ell Z$, the definition of $\alpha_1(\omega)$ in (3.8) and the argument in (3.3)
imply
\[
\int_{\sigma^t Z} \log \alpha_1(\omega) d\mathbb{P}(\omega) = \int_{\sigma^t Z} \log \left( \max_{i=1,\ldots,n} 3 \left( \sup_{s \in \mathbb{Z}} \left| (\varphi_{\omega,i,N,s}^{-1})' \right| \right) \right) d\mathbb{P}(\omega)
\]
\[
= \int_{\sigma^t Z} \log \left( \max_{i=1,\ldots,n} 3 \left( \sup_{s \in \mathbb{Z}} \left| (\varphi_{\sigma^N,\omega,i,s}^{-1} \circ \cdots \circ \varphi_{\omega,i,s_1} \circ \varphi_{\omega,i,s})^{-1} \right| \right) \right) d\mathbb{P}(\omega)
\]
\[
\leq \int_{\sigma^t Z} \log \left( \max_{i=1,\ldots,n} 3 \left( \prod_{t=0}^{N-1} \sup_{s \in \mathbb{Z}} \frac{1}{(\varphi_{\sigma^t,\omega,i,s_1}^{-1} \circ \cdots \circ \varphi_{\sigma^0,\omega,i,s_1})'(x_i)} \right) \right) d\mathbb{P}(\omega).
\]

By definition of \( \gamma_i \) in Equation (2.6), we have
\[
\int_{\sigma^t Z} \log \alpha_1(\omega) d\mathbb{P}(\omega) \leq \int_{\sigma^t Z} \log 3 - \min_{i=1,\ldots,n} \log \left( \prod_{t=0}^{N-1} \gamma_i(\sigma^t \omega) \right) d\mathbb{P}(\omega)
\]
\[
= \frac{\log(3)}{k} - \sum_{t=0}^{N-1} \int_{\sigma^t Z} \min_{i=1,\ldots,n} \log(\gamma_i(\sigma^t \omega)) d\mathbb{P}(\omega)
\]
\[
= \frac{1}{k} (\log(3) - N \gamma) < \frac{1}{k} (\log(3) - N \gamma) < 0,
\]
by Definition 2.22 and (3.2).

\[\square\]

**Theorem 4.3.** Consider an admissible random Jabloński map \( \mathcal{F} \). For each \( \omega \in \Omega \) and \( k = 1, 2, \ldots, \), we define
\[
h^k_\omega = (\mathcal{L}_{\sigma^{-1}} \circ \cdots \circ \mathcal{L}_{\sigma^{-(k-1)}} \circ \mathcal{L}_{\sigma^{-k}}) 1,
\]
where \( 1 \in BV(I^n) \) is the constant function and for each \( s = 1, 2, \ldots, \), we define
\[
H^s_\omega = \frac{1}{s} \sum_{k=1}^{s} h^k_\omega.
\]

Then, for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \):
(i) the sequence \( \{H^s_\omega\}_{s \in \mathbb{N}} \) is relatively compact in \( L^1 \); and
(ii) the following limit exists,
\[
\lim_{s \to \infty} H^s_\omega =: h_\omega \in BV(I^n) \text{ in } L^1.
\]

Moreover, \( \{h_\omega\}_{\omega \in \Omega} \) is a family of random invariant densities for \( \mathcal{F} \).

**Proof.** Recall from the proof of Theorem 3.1 (ii), there are \( N \in \mathbb{N} \) where \( N \) satisfies the condition in (3.2), a constant \( \tilde{\alpha}_2 \) and a positive measurable function \( \alpha_1 : \Omega \to \mathbb{R}^+ \) such that
\[
\int_{\Omega} \log \alpha_1(\omega) d\mathbb{P}(\omega) < 0
\]
and the hybrid Lasota-Yorke inequality (3.11) is satisfied. That is,
\[
\mathbb{V} \mathcal{L}^{(N)}_\omega h \leq \alpha_1(\omega) \mathbb{V} h + \tilde{\alpha}_2 ||h||_1,
\]
for all \( h \in BV(I^n) \) and \( \omega \in \Omega \). For \( k = 1, 2, \ldots, \), and \( \mathbb{P} \)-almost all \( \omega \in \Omega \), the following holds,
\[
h^{Nk}_\omega = \mathcal{L}^{(Nk)}_{\sigma^{-Nk}\omega} 1.
Applying (3.11) to upper bound the variation of \( h_{\omega N}^{Nk} \) on \( I^n \) yields
\[
\mathbb{V} h_{\omega N}^{Nk} = \mathbb{V} L_{\sigma Nk}^{(Nk)} I^n \\
\quad \leq \alpha_1(\sigma^{-N}\omega) \mathbb{V} (L_{\sigma^{-(N+1)}(\omega)} \circ \cdots \circ L_{\sigma^{-2N}(\omega)} \circ L_{\sigma^{-Nk}(\omega)}) 1 \\
\quad + \tilde{\alpha}_2 \| (L_{\sigma^{-(N+1)}(\omega)} \circ \cdots \circ L_{\sigma^{-2N}(\omega)} \circ L_{\sigma^{-Nk}(\omega)}) 1 \|_1 \\
\quad \leq \alpha_1(\sigma^{-N}\omega) \alpha_1(\sigma^{-2N}\omega) \mathbb{V} (L_{\sigma^{-(3N+1)}(\omega)} \circ \cdots \circ L_{\sigma^{-3N}(\omega)} \circ L_{\sigma^{-Nk}(\omega)}) 1 \\
\quad + \tilde{\alpha}_2 \| (L_{\sigma^{-(N+1)}(\omega)} \circ \cdots \circ L_{\sigma^{-2N}(\omega)} \circ L_{\sigma^{-Nk}(\omega)}) 1 \|_1 \\
\quad + \alpha_1(\sigma^{-N}\omega) \tilde{\alpha}_2 \| (L_{\sigma^{-(3N+1)}(\omega)} \circ \cdots \circ L_{\sigma^{-3N}(\omega)} \circ L_{\sigma^{-Nk}(\omega)}) 1 \|_1 \\
\quad \leq \cdots \leq \alpha_1(\sigma^{-N}\omega) \alpha_1(\sigma^{-2N}\omega) \cdots \alpha_1(\sigma^{-kN}\omega) \mathbb{V} 1 \\
\quad + \tilde{\alpha}_2 \| (L_{\sigma^{-(N+1)}(\omega)} \circ \cdots \circ L_{\sigma^{-2N}(\omega)} \circ L_{\sigma^{-Nk}(\omega)}) 1 \|_1 \\
\quad + \alpha_1(\sigma^{-N}\omega) \tilde{\alpha}_2 \| (L_{\sigma^{-(3N+1)}(\omega)} \circ \cdots \circ L_{\sigma^{-3N}(\omega)} \circ L_{\sigma^{-Nk}(\omega)}) 1 \|_1 \\
\quad + \cdots + \alpha_1(\sigma^{-N}\omega) \alpha_1(\sigma^{-2N}\omega) \cdots \alpha_1(\sigma^{-kN}\omega) \tilde{\alpha}_2 1 \|_1,
\]
and since \( \mathbb{V} 1 = 0 \), \( \| 1 \|_1 = 1 \) and the transfer operator is contractive, we have
\[
\mathbb{V} h_{\omega N}^{Nk} \leq \tilde{\alpha}_2 \left( 1 + \alpha_1(\sigma^{-N}\omega) + \alpha_1(\sigma^{-N}\omega) \alpha_1(\sigma^{-2N}\omega) + \cdots \\
\quad + \alpha_1(\sigma^{-N}\omega) \alpha_1(\sigma^{-2N}\omega) \cdots \alpha_1(\sigma^{-kN}\omega) \right) \\
\quad = \tilde{\alpha}_2 \left( 1 + \sum_{j=1}^{k} \alpha_1^{(j)}(\sigma^{-jN}\omega) \right),
\]
where for \( j = 1, 2, \ldots \), we let \( \alpha_1^{(j)}(\sigma^{-jN}\omega) = \alpha_1(\sigma^{-N}\omega) \alpha_1(\sigma^{-2N}\omega) \cdots \alpha_1(\sigma^{-jN}\omega) \).

By Proposition 4.2, there exists \( 0 < \hat{\alpha}(\omega) < 1 \) such that the time averages \( \frac{1}{N} \log \alpha_1^{(j)}(\sigma^{-jN}\omega) \)
converge to \( \log(\hat{\alpha}(\omega)) < 0 \). Choose \( \alpha(\omega) \) such that \( 0 < \hat{\alpha}(\omega) < \alpha(\omega) < 1 \). For sufficiently large \( j_0(\omega) \), we have that \( \alpha_1^{(j)}(\sigma^{-jN}\omega) < \alpha(\omega)^j \), for all \( j \geq j_0(\omega) \).

Let \( c(\omega) \) be defined as
\[
c(\omega) = \max_{1 \leq j \leq j_0(\omega)} \left( \frac{\alpha_1^{(j)}(\sigma^{-jN}\omega)}{\alpha(\omega)^j} \right),
\]
and hence for all \( j \), we have that
\[
\alpha_1^{(j)}(\sigma^{-jN}\omega) < c(\omega) \alpha(\omega)^j.
\]
Taking the sum over \( j \), we get that
\[
\tilde{\alpha}_2 (1 + \sum_{j=1}^{k} \alpha_1^{(j)}(\sigma^{-jN}\omega)) \leq \tilde{\alpha}_2 (1 + c(\omega) \sum_{j=0}^{\infty} \alpha(\omega)^j) \\
\quad = \tilde{\alpha}_2 (1 + c(\omega) \hat{\alpha}(\omega)),
\]
where \( \hat{\alpha}(\omega) = \frac{1}{1 - \alpha(\omega)} \). Let
\[
c_1(\omega) = \tilde{\alpha}_2 (1 + c(\omega) \hat{\alpha}(\omega)),
\]
then we have proven that for every \( k \in \mathbb{N} \)
\[
\frac{1}{k} \mathbb{V} h^{\mathbb{N}k}_\omega \leq c_1(\omega).
\]
From this inequality, it follows \( \{ \frac{1}{k} \mathbb{V} h^{\mathbb{N}k}_\omega \}_{k \in \mathbb{N}} \) is bounded. The same holds for the whole sequence \( \{ \mathbb{V} h^{\mathbb{N}k}_\omega \}_{k \in \mathbb{N}} \) and indeed for the averages \( \{ H_s^\omega \}_{s \in \mathbb{N}} \) is relatively compact in \( L^1 \) by [31, Lemma A.1]. This establishes (i).

Then, the random mean ergodic theorem [33, Theorem B] shows that \( \{ H_s^\omega \}_{s \in \mathbb{N}} \) converges in the strong sense to a random invariant density \( h^\omega \), as in (4.2). The fact that \( h^\omega \in BV(I^n) \) follows once again from the relative compactness of \( BV(I^n) \) in \( L^1 \). This establishes (ii).

Remark 4.4. For \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), define \( \mu_\omega \) on the fiber \( \{ \omega \} \times I^n \subset \Omega \times I^n \), as
\[
\frac{d\mu_\omega}{dm} = h^\omega,
\]
where \( h^\omega \) is given by (4.2). Then \( \mu_\omega \) is a random invariant ACIP and the measure \( \mu \) defined on \( \mathbb{P} \times m \)-measurable sets \( A \subseteq \Omega \times I^n \) by
\[
\mu(A) = \int_\Omega \mu_\omega(A) d\mathbb{P}(\omega),
\]
is an ACIP for the associated skew product \( F \) defined in (2.4).

Multiplicative ergodic theorems are concerned with random dynamical systems \( R = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{X}, \mathcal{L}) \). They give rise to an \( \omega \)-dependent hierarchical decomposition of \( \mathcal{X} \) into equivariant subspaces, called Oseledets spaces. In the literature, multiplicative ergodic theorems are divided into two types, according to the invertibility of the base map \( \sigma \) and the operators \( \mathcal{L}_\omega \). In [10], Froyland, Lloyd and Quas show a semi-invertible multiplicative ergodic theorem, where the base is assumed to be invertible, but there is no assumption about invertibility of the operators \( \mathcal{L}_\omega \). We will apply this theorem to show that the random invariant densities \( h^\omega \) found in Theorem 4.3 belong to the leading Oseledets subspace. Moreover, we will deduce the finiteness of the number of ergodic ACIPs in Corollary 4.6.

An Oseledets splitting for a random dynamical system \( R = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{X}, \mathcal{L}) \) consists of

- A sequence of isolated (exceptional) Lyapunov exponents
  \[
  \infty > \lambda^* = \lambda_1 > \lambda_2 > \cdots > \lambda_l > \lambda^{**} \geq -\infty,
  \]
  where the index \( l \geq 1 \) is allowed to be finite or countably infinite, and
- A family of \( \omega \)-dependent splittings,
  \[
  \mathcal{X} = Y_1(\omega) \oplus \cdots \oplus Y_l(\omega) \oplus V(\omega),
  \]
  where for \( j = 1, \ldots, l \), \( d_j := \text{dim}(Y_j(\omega)) < \infty \) and \( V(\omega) \in \mathcal{G}(\mathcal{X}) \) where \( \mathcal{G}(\mathcal{X}) \) is the Grassmannian of \( \mathcal{X} \).
For all \( j = 1, \ldots, l \) and \( \mathbb{P}-\text{a.e.} \ \omega \in \Omega \), we have
\begin{align*}
\mathcal{L}_\omega Y_j(\omega) &= Y_j(\sigma\omega), \quad (4.4) \\
\mathcal{L}_\omega V(\omega) &\subseteq V(\sigma\omega), \quad (4.5)
\end{align*}

and
\begin{align*}
\lim_{s \to \infty} \frac{1}{s} \log \| \mathcal{L}_\omega^{(s)} y \| &= \lambda_j, \quad \forall y \in Y_j(\omega) \setminus \{0\}, \quad (4.6) \\
\lim_{s \to \infty} \frac{1}{s} \log \| \mathcal{L}_\omega^{(s)} v \| &\leq \mathcal{K}^*, \quad \forall v \in V(\omega). \quad (4.7)
\end{align*}

**Theorem 4.5** (Froyland, Lloyd and Quas [16, Theorem 17]). Let \( \Omega \) be a Borel subset of a separable complete metric space, \( \mathcal{F} \) the Borel sigma-algebra and \( \mathbb{P} \) a Borel probability measure. Let \( X \) be a Banach space. Consider a random dynamical system \( R = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L}) \) with base transformation \( \sigma : \Omega \to \Omega \) an ergodic homeomorphism, and suppose that the generator \( \mathcal{L} : \Omega \to L(X, X) \) is \( \mathbb{P} \)-continuous and satisfies
\[
\int_{\Omega} \log^+ \| \mathcal{L}_\omega \| d\mathbb{P}(\omega) < \infty.
\]
If \( R \) is quasi-compact, that is, if \( \mathcal{K}^* < \lambda^* \), then \( R \) admits a unique \( \mathbb{P} \)-continuous Oseledets splitting.

By Theorem 3.1 admissible random Jabloiski maps give rise to quasi-compact random dynamical systems with \( \lambda_1 = 0 \). Therefore, Theorem 4.5 implies the following.

**Corollary 4.6.** For \( \mathbb{P}-\text{a.e.} \ \omega \in \Omega \), the random invariant density \( h_\omega \) given in (4.2) belongs to the Oseledets space \( Y_1(\omega) \) given in (4.3). Moreover, the number \( r \) of ergodic ACIPS \( \mu_1, \ldots, \mu_r \) with respect to the associated skew product is finite; indeed, we have
\[
r \leq d_1 = \dim(Y_1(\omega)). \quad (4.8)
\]

**Proof.** Let \( \omega \in \Omega \), by the equivariance property given in (4.1), we have \( \mathcal{L}_\omega^{(m)} h_\omega = h_{\sigma^m \omega} \), for \( m \in \mathbb{N} \). To show that \( h_\omega \in Y_1(\omega) \), we verify the limit condition given in (4.6) for \( j = 1 \). Note that
\[
\lim_{m \to \infty} \frac{1}{m} \log \| \mathcal{L}_\omega^{(m)} h_\omega \|_{BV} \\
= \lim_{m \to \infty} \frac{1}{m} \log \| h_{\sigma^m \omega} \|_{BV} \\
\geq \lim_{m \to \infty} \frac{1}{m} \log \| h_{\sigma^m \omega} \|_1 = 0 = \lambda^*,
\]
on the other hand
\[
\lim_{m \to \infty} \frac{1}{m} \log \| \mathcal{L}_\omega^{(m)} h_\omega \|_{BV} \\
\leq \lim_{m \to \infty} \frac{1}{m} \log \| \mathcal{L}_\omega^{(m)} \|_{BV} = 0 = \lambda^*,
\]
by Theorem 3.1. Since the splitting in Theorem 4.5 is unique, this gives that \( h_\omega \in Y_1(\omega) \). By the finite dimensionality of the leading Oseledets subspace \( Y_1(\omega) \), we get the bound given in (4.8). \( \square \)
Next, we define physical measures and show how the measures given in Corollary 4.6 are physical measures.

**Definition 4.7.** Consider the tuple $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, f)$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\sigma : \Omega \rightarrow \Omega$ an invertible, ergodic and $\mathbb{P}$-preserving transformation and $f = \{f_\omega : M \rightarrow M\}_{\omega \in \Omega}$ where $M \subseteq \mathbb{R}^n$. A probability measure $\nu$ on $M$ is called physical if for $\mathbb{P}$-a.e. $\omega \in \Omega$, the Lebesgue measure of the random basin $RB_\omega(\nu)$ of $\nu$ at $\omega$ is positive where

$$RB_\omega(\nu) = \{x \in M : \frac{1}{s} \sum_{k=0}^{s-1} \delta_{f_\omega^{(k)}}(x) \rightarrow \nu\},$$

where $\delta_x$ is the Dirac measure at a point $x$.

The convergence in Definition 4.7 is in the weak convergence sense. In the case where $f_\omega$ is independent of $\omega$, this reduces to the definition of physical measure for a deterministic dynamical system. The next probabilistic result due to Buzzi applies in our setting.

**Theorem 4.8** (Buzzi [9, Proposition 4.1]). Let $\mu_i$ be one of the measures $\mu_i : i = 1, \ldots, r$ given in Corollary 4.6. Then, the marginal measure of $\mu_i$ on $I^n$, denoted by $\nu_i$, is a physical measure on $I^n$.

The union of all basins of the of the physical measures $\nu_i$ coming from the marginals of $\mu_i$ on $I^n$, $i = 1, \ldots, r$ has full Lebesgue measure, which means Lebesgue almost everywhere, the asymptotic long term behaviour of the random orbits will be described by one of these physical measures. Another immediate consequence of the proof of Theorem 4.8 is the following.

**Corollary 4.9.** There exists a constant $b > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $i = 1, \ldots, r$, $m(RB_\omega(\nu_i)) > b$.

5. **Bounds on the number of ergodic skew product ACIPs**

A difficulty in the general study of ACIPs of piecewise expanding maps in higher dimensions is that the geometric complexity around discontinuities or interior crossing points might grow rapidly as the dynamical partitions are refined [10]. This is in contrast to one-dimensional maps, where the geometry is much simpler and such a complexity growth can not happen. However, this complication does not occur in the context of random Jabłoński maps. In [23], Góra, Boyarsky and Proppe proved that for a class of deterministic dynamical systems modeled by Jabłoński transformations, the number of crossing points gives an upper bound for the number of ergodic ACIPs.

In this section, we establish bounds on the number of ergodic ACIPs for random Jabłoński maps. The first bound, presented in Section 5.1 is motivated by the work of Buzzi [9] in the one dimensional case of random Lasota-Yorke maps. The second bound, presented in Section 5.2 is inspired by the work of Góra, Boyarsky and Proppe on absolutely continuous invariant measures for deterministic dynamical systems given by multidimensional expanding maps [23]. An example is presented in Section 5.3.

Let $\mathcal{F} = \{f_\omega\}_{\omega \in \Omega}$ be an admissible random Jabłoński map. Suppose that there exist $r$ mutually singular ergodic ACIPs $\mu_1, \ldots, \mu_r$ for the associated skew product map $F$. Fix $i \in \{1, \ldots, r\}$ and $\omega \in \Omega$, then the fiber measure $\mu_\omega^i$ is a measure on
By Theorem 2 in [23], the support $\text{Supp}(\mu^i_{\omega})$ of $\mu^i_{\omega}$ is open Lebesgue almost everywhere. This fact was before introduced in Keller’s thesis [28]. Let $I_{i,\omega}(0) \subseteq \text{Supp}(\mu^i_{\omega})$ be a nontrivial rectangle lying inside one of the rectangles of $B^\omega$. Define the sequence

\begin{equation}
I_{i,\omega}(s + 1) = f_{\sigma^s \omega}(I_{i,\omega}(s)) \cap J, \quad s \in \mathbb{N} \cup \{0\},
\end{equation}

where $J$ is the open rectangle in the partition $B^{\sigma^s + 1}\omega$ of the Jabłoński map $f_{\sigma^s + 1}\omega$ which maximizes the Lebesgue measure of $I_{i,\omega}(s + 1)$. For $s \in \mathbb{N} \cup \{0\}$, define $c_{i,\omega}(s)$ to be the number of crossing points in the partition $B^{\sigma^s + 1}\omega$ lying inside the image $f_{\sigma^s \omega}(I_{i,\omega}(s))$. Let

\begin{equation}
M(\omega) = \max_{z \in \mathbb{R}} \max_{d=1,...,n} \{ \text{number of rectangles } B \in B^{\sigma^s \omega} \text{ s.t. } H^{(d)}_{n-1}(z) \cap \text{Int}(B) \neq \emptyset \},
\end{equation}

where $H^{(d)}_{n-1}(z)$ is the $(n-1)$ dimensional hyperplane given by the equation $x_d = z$. This definition of $M$ is motivated by a deterministic analogue, Definition 3 in [23].

For $i = 1, \ldots, r$, denote by

\begin{equation}
\mathcal{D}_i = \{ \omega \in \Omega : \text{Supp}(\mu^i_{\omega}) \text{ has a crossing point in its interior} \}.
\end{equation}

Also, let

\begin{equation}
\gamma(\omega) = \prod_{i=1}^{n} \gamma_i(\omega),
\end{equation}

and $\gamma_i(\omega)$ is defined in equation (2.6).

5.1. Multidimensional bound à la Buzzi. In this section, we assume the following.

\begin{equation}
\delta := \int_{\Omega} \log(\frac{\gamma(\omega)}{M(\omega)}) d\mathbb{P}(\omega) > 0,
\end{equation}

This condition means that, on average, the fiber expansion constants dominate the partition complexities.

**Lemma 5.1.** Let $\mathcal{F} = \{ f_{\omega} \}_{\omega \in \Omega}$ be an admissible random Jabłoński map and assume that [5.4] is satisfied. Then, the number $r$ of mutually singular ergodic ACIPs for the associated skew product map $F$ satisfies

\begin{equation}
\int_{\Omega} \log \left( 2^{n-1} \left( \frac{c_{i}(\omega)}{r} + 1 \right) \right) d\mathbb{P}(\omega) \geq \delta.
\end{equation}

**Proof.** First we show that at least one of the sets in

\begin{equation}
f_{\sigma^s \omega}(I_{i,\omega}(s)), \quad s \in \mathbb{N} \cup \{0\}
\end{equation}

has a crossing point in its interior. The argument proceeds by contradiction. Suppose that for none of the sets in [5.6] has a crossing point in the interior. Then,

\[
m(I_{i,\omega}(s + 1)) \geq \frac{\gamma(\sigma^s \omega)}{M(\sigma^s \omega)} m(I_{i,\omega}(s)) \geq \frac{\gamma(\sigma^s \omega)}{M(\sigma^s \omega)} \cdots \frac{\gamma(\omega)}{M(\omega)} m(I_{i,\omega}(0)).
\]
By \([5.4]\), we have \(\delta = \int_0^1 \log \frac{\gamma(\omega)}{M(\omega)} d\overline{\mu}(\omega) > 0\). Hence, Birkhoff ergodic theorem implies that \(m(I_{i,\omega}(s+1)) \to \infty\) as \(s \to \infty\), and this is a contradiction. Hence, at least one of the sets in \([5.6]\) has a crossing point in its interior.

For \(k = 0, 1, 2, 3, \ldots\) and \(\omega \in \Omega\), define

\[
g_{i,k}(\omega) = \begin{cases} \frac{\gamma(\sigma^k \omega)}{M(\sigma^k \omega)} & : \sigma^k \omega \in D_i \\ \frac{\gamma(\sigma^k \omega)}{M(\sigma^k \omega)} & : \sigma^k \omega \in \Omega \setminus D_i \end{cases}
\]

By equation \([5.1]\), for \(s \in \mathbb{N}\), \(I_{i,\omega}(s)\) comes from evolving \(I_{i,\omega}(s-1)\) by the map \(f_{\sigma^{s-1} \omega}\), and then taking the largest intersection of its image with one of the partition rectangles of \(\mathcal{B}^{\sigma^s \omega}\). Therefore, the volume of \(I_{i,\omega}(s)\) depends on whether the set \(f_{\sigma^{s-1} \omega}(I_{i,\omega}(s-1))\) has a crossing point in its interior or not. In case the interior of this set has a crossing point, the volume of \(I_{i,\omega}(s)\) is bounded below by the volume of \(I_{i,\omega}(s-1)\) expanded by \(\gamma(\sigma^{s-1} \omega)\) and scaled by \(2\pi^{s-1}(c_{i,\omega}(s-1) + 1)\). This last scaling term is an upper bound on the number of rectangles of \(\mathcal{B}^{\sigma^s \omega}\) meeting \(f_{\sigma^{s-1} \omega}(I_{i,\omega}(s-1))\). On the other hand, if the interior of \(f_{\sigma^{s-1} \omega}(I_{i,\omega}(s-1))\) has no crossing points, the volume of \(I_{i,\omega}(s)\) is bounded below by the volume of \(I_{i,\omega}(s-1)\) expanded by \(\gamma(\sigma^{s-1} \omega)\) and scaled by \(M(\sigma^{s-1} \omega)\). Thus, in general,

\[
m(I_{i,\omega}(s)) \geq g_{i,s-1}(\omega)m(I_{i,\omega}(s-1)).
\]

Therefore, inductively, we have

\[
m(I_{i,\omega}(s)) \geq g_{i,s-1}(\omega) \ldots g_{i,0}(\omega)m(I_{i,\omega}(0)).
\]

Since \(m(I_{i,\omega}(s)) \leq 1\), for all \(s = 1, 2, 3, \ldots\), we have

\[
\sum_{k=0}^{s-1} \log \left( \frac{1}{g_{i,k}(\omega)} \right) \geq \log(m(I_{i,\omega}(0)).
\]

By summing over \(i = 1, \ldots, r\) and dividing by \(r\), we get

\[
\frac{1}{r} \sum_{k=0}^{s-1} \log \left( \frac{1}{g_{i,k}(\omega)} \right) \geq \xi,
\]

where \(\xi := \frac{1}{r} \sum_{i=1}^{r} \log(m(I_{i,\omega}(0))).\) This gives that

\[
\sum_{k=0}^{s-1} \log \left( \frac{1}{(g_{1,k}(\omega) \ldots g_{r,k}(\omega))^{\frac{1}{r}}} \right) \geq \xi.
\]

Since the measures \(\mu_i\) are mutually singular, for all \(\omega \in \Omega\), we have

\[
c_{1,\omega}(k) + \cdots + c_{r,\omega}(k) \leq c_{\epsilon}(\sigma^k \omega),
\]

where we recall that \(c_{\epsilon}(\omega)\) is the total number of interior crossing points in the partition \(\mathcal{B}^{\sigma^s \omega}\) of \(f_{\sigma^s \omega}\). By adding \(r\) to both sides, dividing by \(r\) and using the arithmetic-geometric mean inequality, we get

\[
\left( (c_{1,\omega}(k) + 1) \ldots (c_{r,\omega}(k) + 1) \right)^{\frac{1}{r}} \leq \frac{c_{\epsilon}(\sigma^k \omega) + r}{r}.
\]
Therefore, (5.9) and the definition of \( g_{i,k}(\omega) \) yield
\[
\frac{1}{s} \sum_{k=0}^{s-1} \log \left( \frac{2^{n-1}(c_t(\sigma^k \omega) + r)}{\gamma(\sigma^k \omega)} \right) M(\sigma^k \omega) \geq \frac{1}{s} \sum_{k=0}^{s-1} \log \left( \frac{1}{(g_{1,k}(\omega), \ldots, g_{r,k}(\omega))^T} \right) \geq \frac{\xi}{s}.
\]

(5.10)

Applying Birkhoff ergodic theorem, we get
\[
\int_{\Omega} \log \left( \frac{2^{n-1}(c_t(\omega) + 1) M(\omega)}{\gamma(\omega)} \right) d\mathbb{P}(\omega) \geq 0.
\]

This gives that
\[
\int_{\Omega} \log \left( 2^{n-1}(c_t(\omega) + 1) \right) d\mathbb{P}(\omega) + \int_{\Omega} \log \left( \frac{M(\omega)}{\gamma(\omega)} \right) d\mathbb{P}(\omega) \geq 0,
\]

and therefore we have
\[
(5.11) \quad \int_{\Omega} \log \left( 2^{n-1}(\frac{c_t(\omega)}{r} + 1) \right) d\mathbb{P}(\omega) \geq \delta.
\]

Lemma 5.1 may be used to obtain explicit bounds on \( r \).

Lemma 5.2. Suppose (5.4) holds, for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), \( c_t(\omega) \leq c \) and \( \log(2^{n-1}) < \delta \). Then (5.5) gives an explicit bound on \( r \), that is
\[
(5.12) \quad r \leq \frac{c}{\exp(\delta) - 1}.
\]

Proof. Since for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), \( c_t(\omega) \leq c \), we get
\[
\int_{\Omega} \log \left( 2^{n-1}(\frac{c_t(\omega)}{r} + 1) \right) d\mathbb{P}(\omega) \leq \log(2^{n-1}(\frac{c}{r} + 1)).
\]

By (5.5), we have \( \log(2^{n-1}(\frac{c}{r} + 1)) \geq \delta \) which implies
\[
(5.13) \quad \frac{c}{r} + 1 \geq \exp(\delta) \frac{1}{2^{n-1}}.
\]

Since \( \log(2^{n-1}) < \delta \), we have \( \frac{\exp(\delta)}{2^{n-1}} > 1 \) and thus (5.13) gives a nontrivial bound on \( r \). By solving (5.13) for \( r \), we get the upper bound given in (5.12).

The next corollary shows another way of getting finiteness of the number of measures \( r \), previously obtained in Corollary 4.6 using multiplicative ergodic theory.

Corollary 5.3. Consider the assumptions in Lemma 5.2. Then the number of measures \( r \) in Corollary 4.6 is finite.

Proof. The integrand in (5.5) is a non-increasing function of \( r \). Hence, as \( r \to \infty \), we get \( \int_{\Omega} \log(2^{n-1}) d\mathbb{P}(\omega) \geq \delta \), which contradicts the assumption.

Another immediate consequence of Lemma 5.2 is the following.

Corollary 5.4. If \( \frac{c}{\exp(\delta) - 1} < 2 \), then there exists a unique ergodic ACIP for the skew product.
5.2. Another bound on $r$. We recall that $\gamma(\omega)$, introduced in (5.3), quantifies the expansion in the random system. The geometry of the partitions $\{B^\omega\}_{\omega \in \Omega}$ is related to the quantities $q_\omega$, the number of rectangles in the partition $B^\omega$; $M(\omega)$, defined in (5.2); and $c_t(\omega)$, the total number of interior crossing points in the partition $B^{\sigma^t}\omega$.

**Lemma 5.5.** Assume for $\mathbb{P}$-a.e. $\omega \in \Omega$, $M(\omega) \leq M$, $c_t(\omega) \leq c$ and $q_\omega \leq q$. Then,

\begin{equation}
(5.14) \quad r \leq c \left( \log(q) - \log(M) \right) \frac{1}{\int_{\Omega} \log(\gamma(\omega))d\mathbb{P}(\omega) - \log(M)}.
\end{equation}

**Proof.** Recall that $M < q$, by the definition of $M(\omega)$ in (5.2). For $i = 1, 2, \ldots, r$,

\[ g_i(\omega) = \begin{cases} \frac{\gamma(\omega)}{q_\omega} & : \omega \in D_i \\ \frac{\gamma(\omega)}{M(\omega)} & : \omega \in \Omega \setminus D_i \end{cases}. \]

In a similar argument to (5.7), note that for all $\omega \in \Omega$ and $s = 1, 2, 3, \ldots$, we have

\[ m(I_{i,\omega}(s)) \geq g_i(\sigma^{s-1}\omega) \ldots g_i(\omega) m(I_{i,\omega}(0)). \]

Then, for all $s = 1, 2, 3, \ldots$, we have

\begin{equation}
(5.15) \quad \frac{1}{s} \sum_{k=0}^{s-1} \log(g_i(\sigma^k\omega)) \leq \frac{1}{s} \log \left( \frac{1}{m(I_{i,\omega}(0))} \right).
\end{equation}

It is also clear that

\begin{equation}
(5.16) \quad g_i(\omega) = \begin{cases} \frac{\gamma(\omega)}{q} & : \omega \in D_i \\ \frac{\gamma(\omega)}{M(\omega)} & : \omega \in \Omega \setminus D_i \end{cases}.
\end{equation}

Using (5.16) and Birkhoff ergodic theorem, from (5.15), we get

\[ \int_{D_i} \log \left( \frac{\gamma(\omega)}{q} \right) d\mathbb{P}(\omega) + \int_{\Omega \setminus D_i} \log \left( \frac{\gamma(\omega)}{M(\omega)} \right) d\mathbb{P}(\omega) \leq 0, \]

which simplifies to

\[ \int_{\Omega} \log(\gamma(\omega))d\mathbb{P}(\omega) - \log(M) + m_i \log \left( \frac{M}{q} \right) \leq 0, \]

where $m_i = m(D_i)$. Therefore, for all $i = 1, 2, \ldots, r$, we have

\begin{equation}
(5.17) \quad m_i \geq \frac{\int_{\Omega} \log(\gamma(\omega))d\mathbb{P}(\omega) - \log(M)}{\log(q) - \log(M)}.
\end{equation}

For $i = 1, 2, \ldots, r$, define

\[ a_i(\omega) = \begin{cases} 1 & : \omega \in D_i \\ 0 & : \omega \in \Omega \setminus D_i \end{cases}, \]

then for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\sum_{i=1}^{r} a_i(\omega) \leq c$. Note that $m_i = \int_{\Omega} a_i(\omega) d\mathbb{P}(\omega)$. Taking the sum over all $i = 1, 2, \ldots, r$, we get $r \leq c \frac{\sum_{i=1}^{r} m_i}{\min(m_1, \ldots, m_r)}$. By (5.17), we get the bound given in (5.14).

Since $m_i \leq 1$ for all $i = 1, 2, \ldots, r$, an immediate consequence of (5.17) is the following.
Corollary 5.6. We have $\int_{\Omega} \log(\gamma(\omega)) dP(\omega) \leq \log(q)$, where $q$ is defined in Lemma 5.5.

Corollary 5.7. If $\frac{c(\log(q) - \log(M))}{\int_{\Omega} \log(\gamma(\omega)) dP(\omega) - \log(M)} < 2$, then there exists a unique ergodic ACIP for the skew product.

5.3. Example. Consider an admissible random Jabłoński map where the common partition is taken to be the equally sized 25 squares partition shown in Figure (1). For this partition, we have $M = 5$, $c = 16$ and $q = 25$.

Let $\gamma_1, \gamma_2 > 0$ be such that for all $\omega \in \Omega$, $\gamma_1(\omega) \geq \gamma_1$ and $\gamma_2(\omega) \geq \gamma_2$ where $\gamma_1(\omega)$ is defined in (2.6). By (5.3), we have

$$\gamma(\omega) = \gamma_1(\omega) \gamma_2(\omega) \geq \gamma_1 \gamma_2,$$

for all $\omega \in \Omega$. Note that $\gamma_1$ and $\gamma_2$ can not take values such that $\gamma_1 \gamma_2 > 25$, because the rectangles of the partition would be mapped outside $I^2$. The constant $\delta$ defined in (5.4) is

$$\delta = \int_{\Omega} \log\left(\frac{\gamma(\omega)}{M(\omega)}\right) dP(\omega) \geq \log\left(\frac{\gamma_1 \gamma_2}{5}\right).$$

For the bound in Section 5.1, we must have, in addition, that $\gamma_1$ and $\gamma_2$ can not be such that $\gamma_1 \gamma_2 \leq 10$. Since this contradicts the condition in Lemma 5.2 that $\log(2^{n-1}) < \delta$, we can make the restriction that

$$10 < \gamma_1 \gamma_2 \leq 25.$$

Then, (5.12) implies that

(5.18) $$r \leq \frac{160}{\gamma_1 \gamma_2 - 10},$$

The bound in (5.14), implies that

(5.19) $$r \leq \frac{16 \log(5)}{\log\left(\frac{\gamma_1 \gamma_2}{5}\right)}.$$

Figure (2) shows the dependence of the two bounds on $\gamma_1 \gamma_2$ and the regions on which each of the bounds is sharper. The bounds from Sections 5.1 and 5.2 are shown in black/solid and orange/dashed, respectively.
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