ASYMPTOTICS OF PARTIAL THETA FUNCTIONS WITH A DIRICHLET CHARACTER

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Abstract. In this paper, we prove asymptotic expansions of partial theta functions with a nonprincipal Dirichlet character, and these asymptotic expansions have connections with certain $L$-series.

1. Introduction

In this paper, we use the character analogue of Euler-Maclaurin summation formula to obtain asymptotic expansions, as $t \to 0^+$, for the sums

\begin{equation}
\sum_{n=0}^{\infty} \chi(n)e^{-(n+b/r)r^\theta} \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n \chi(n)e^{-(n+b/r)r^\theta},
\end{equation}

where $r$ is a positive integer, $b$ is a real number, and $\chi$ is a nonprincipal Dirichlet character. When $r = 2$, these sums are partial theta functions with a Dirichlet character. Our asymptotic expansions are in terms of the generalized Bernoulli and Euler polynomials. (See Theorems 1.1 and 1.2). Inspired by the works of Lawrence and Zagier [7], we also show a connection between certain $L$-series and our asymptotic expansions. (See Theorem 3.1).

1.1. Background. Let $\theta > 0$, $b$ real, and $r \in \mathbb{N}$. Write

\begin{equation}
G_{1,\chi}(\theta, b, r) = r \sum_{n=0}^{\infty} (-1)^n \chi(n)e^{-(n+b/r)r^\theta}
\end{equation}

and

\begin{equation}
G_{2,\chi}(\theta, b, r) = r \sum_{n=0}^{\infty} \chi(n)e^{-(n+b/r)r^\theta}
\end{equation}

for any periodic function $\chi(n)$, such as a nonprincipal Dirichlet character. For $r = 2$ and $\chi = \chi^0$ (the principal character, i.e., its conductor $f_\chi = 1$ and $\chi(m) = 1$ for all integers $m$, see [6, p. 4]), the function $G_{1,\chi^0}(\theta, b, 2)$ and $G_{2,\chi^0}(\theta, b, 2)$ were introduced by Berndt and Kim in 2011 [3] as $G_1(\theta) = G_{1,\chi^0}(\theta, b, 2)$ and $G_2(\theta) = G_{2,\chi^0}(\theta, b, 2)$. They also [3, Theorem 3.4] obtained asymptotic expansions of $G_1(\theta)$ and $G_2(\theta)$ involving Bernoulli and Hermite polynomials. Their works were inspired by the following asymptotic expansion for the partial theta function in S. Ramanujan’s 2000 Mathematics Subject Classification. 34E05, 41A60, 11F27, 11B68.

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second notebook [2, p. 324]

\[
2 \sum_{n=0}^{\infty} (-1)^n q^{n^2+n} = 2 \sum_{n=0}^{\infty} (-1)^n \left( \frac{1-t}{1+t} \right)^{n^2+n} \sim 1 + t + t^2 + 2t^3 + 5t^4 + \cdots ,
\]

where \( q = \frac{1-t}{1+t} \to 1^- \), or \( t \to 0^+ \). As remarked by Berndt and Kim [3], the above asymptotic expansion is very interesting since there is no a priori reason to believe that the coefficients (in the variable \( t \)) are positive integers.

Recently, McIntosh [9] proved asymptotic expansions for general sums

\[
\sum_{n=0}^{\infty} e^{-(n+c)t} \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n e^{-(n+c)t},
\]

where \( r \) is a positive integer and \( c \) is a real number.

In this paper, we will consider character analogues of the above sums. (See (1.2) and (1.3) above).

### 1.2. The generalized Bernoulli and Euler polynomials.

The Bernoulli polynomials \( B_n(x) \) are defined by the generating function

\[
\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi)
\]

and \( B_n = B_n(0) \) are the Bernoulli numbers with \( B_0 = 1, B_1 = -1/2 \) and \( B_{2n+1} = B_{2n-1}(1/2) = 0 \) for \( n \ge 1 \).

Suppose that \( \chi \) is a primitive character modulo \( f_\chi \). The generalized Bernoulli polynomials \( B_{n,\chi}(x) \) are defined by the following generating function ([1, Proposition 6.2])

\[
\frac{f_\chi^{-1} \chi(m)t e^{(m+x)t}}{e^{f_\chi t} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!} \quad (|t| < 2\pi/f_\chi)
\]

and \( B_{n,\chi} = B_{n,\chi}(0) \) are the generalized Bernoulli numbers. In particular, if \( \chi^0 \) is the principal character, i.e., if \( f_\chi = 1 \) and \( \chi(m) = 1 \) for all integers \( m \) then \( B_{n,\chi^0}(x) = B_n(x) \) for \( n \ge 0 \) and \( B_{0,\chi}(x) = 0 \) for \( \chi \neq \chi^0 \).

The generalized Bernoulli functions \( \overline{B}_{n,\chi}(x) \) are functions with period \( f_\chi \). They are defined by ([1, Theorem 3.1])

\[
\overline{B}_{n,\chi}(x) = f_\chi^{-1} \sum_{m=0}^{f_\chi-1} \chi(m) \overline{B}_n \left( \frac{m+x}{f_\chi} \right), \quad n \ge 1
\]

for all real \( x \). Here we recall some properties for the generalized Bernoulli functions which will be needed in the sequel (see [5, (9), (10) and (11)])

\[
\frac{d}{dx} B_n(x) = nB_{n-1}(x) \quad \text{and} \quad \frac{d}{dx} B_{n,\chi}(x) = nB_{n-1,\chi}(x), \quad n \ge 1,
\]

\[
\frac{d}{dx} \overline{B}_{n,\chi}(x) = n\overline{B}_{n-1,\chi}(x), \quad n \ge 2 \quad \text{and} \quad \overline{B}_{n,\chi}(f_\chi) = \overline{B}_{n,\chi}(0) = B_{n,\chi},
\]

\( B_{n,\chi}(-x) = (-1)^n \chi(-1) B_{n,\chi}(x), \quad n \ge 0 \).
The Euler polynomials $E_n(x)$ are defined by the generating function

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (|t| < \pi) \quad (1.9)$$

(see [11]). Suppose that $\chi$ is a primitive character modulo $f_\chi$ and $f_\chi > 1$ is odd. The generalized Euler polynomials $E_{n,\chi}(x)$ are defined by

$$\sum_{m=0}^{f_\chi - 1} \frac{2(-1)^m \bar{\chi}(m) e^{(m+x)t}}{e^{fx^t} + 1} = \sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!} \quad (|t| < \pi/f_\chi) \quad (1.10)$$

In particular, $E_{n,\chi} = E_{n,\chi}(0)$ are the generalized Euler numbers. If $\chi^0$ is the principal character, i.e., its conductor $f_{\chi^0} = 1$ and $\chi(m) = 1$ for all integers $m$, then $E_{n,\chi^0}(x) = E_n(x)$ for $n \geq 0$. For $0 \leq x < 1$, $E_{n,\chi}(x)$ denotes the character Euler function, with odd period $f_\chi$, defined by Can and Dağlı [4].

Recall that (see [4, (2.4) and (3.5)])

$$E_{n,\chi}(x) = f_\chi^n \sum_{m=0}^{f_\chi - 1} (-1)^m \bar{\chi}(m) E_n \left( \frac{m + x}{f_\chi} \right), \quad n \geq 0 \quad (1.11)$$

and

$$2^{n+1} \bar{\chi}(2) \mathcal{B}_{n+1,\chi} \left( \frac{x}{2} \right) - \mathcal{B}_{n+1,\chi}(x) = -\frac{n + 1}{2} E_{n,\chi}(x) \quad (1.12)$$

1.3. **Main results.** We now state a theorem which generalises [8, Theorem 1.1] and [9, Theorem 1].

**Theorem 1.1.** Let $G_{2,\chi}(\theta, b, r)$ be defined in (1.3) and let $\chi$ be a nonprincipal character. Then we have the following asymptotic expansions.

(i) For any nonnegative integer $N$, we have

$$G_{2,\chi}(\theta, 0, r) = -r \sum_{n=0}^{N} (-1)^n \frac{B_{rn+1,\chi}}{(rn + 1)n!} \theta^n + O(\theta^{N+1})$$

as $\theta \to 0^+$. 

(ii) When $b > 0$, for any nonnegative integer $N$, we have

$$G_{2,\chi}(\theta, b, r) = -r \sum_{n=0}^{N} (-1)^n \frac{B_{rn+1,\chi}(b)}{(rn + 1)n!} \theta^n + O(\theta^{N+1})$$

as $\theta \to 0^+$. 

We also have the following result.

**Theorem 1.2.** Let $G_{1,\chi}(\theta, b, r)$ be defined in (1.2) and let $\chi$ be a nonprincipal character. Then we have the following asymptotic expansions.

(i) Suppose that the conductor $f_\chi$ of $\chi$ is odd. For any nonnegative integer $N$, we have

$$G_{1,\chi}(\theta, 0, r) = \frac{r}{2} \sum_{n=0}^{N} (-1)^n \frac{E_{rn,\chi}}{n!} \theta^n + O(\theta^{N+1})$$

as $\theta \to 0^+$. 


(ii) Suppose that the conductor $f_\chi$ of $\chi$ is odd. When $0 < b < r$, for any nonnegative integer $N$, we have

$$G_1,\chi(\theta, b, r) = \frac{r}{2} \sum_{n=0}^{N} (-1)^n \frac{E_{rn,\bar{\chi}}(\frac{b}{r})}{n!} \theta^n + O(\theta^{N+1})$$

as $\theta \to 0^+$.

1.4. Some consequences.

Before proving our main results, we show some of their consequences.

Let $\chi$ be a nonprincipal character. When $r = 1$ in (1.3), the sum $G_{2,\chi}$ is a geometric series and for $|\theta| < 2\pi/f_\chi$, we have

$$G_{2,\chi}(\theta, b, 1) = \sum_{a=0}^{f_\chi-1} \chi(a) \sum_{m=0}^{\infty} e^{-(m f_\chi + a + b) \theta} \tag{1.13}$$

Using (1.6) and (1.13), the asymptotic expansion of $G_{2,\chi}(\theta, b, 1)$ at $\theta = 0$ has the form

$$G_{2,\chi}(\theta, b, 1) = - \sum_{n=0}^{\infty} (-1)^n \frac{B_{n+1,\bar{\chi}}(b)}{(n+1)!} \theta^n. \tag{1.14}$$

In addition, for $r \in \mathbb{N}$, we will show that in Theorem 1.1(ii), the statement with $b$ replaced by $b + r f_\chi$ follows from the statement for $b$.

From Theorem 1.1, we have

$$G_{2,\chi}(\theta, b + r f_\chi, r) = r \sum_{n=1}^{\infty} \chi(n + f_\chi) e^{-(n + b/r + f_\chi) \theta} \tag{1.15}$$

$$= r \sum_{n=1}^{\infty} \chi(n) e^{-(n+b/r) \theta} - r \sum_{m=1}^{f_\chi} \chi(m) e^{-(m+b/r) \theta}$$

$$= -r \sum_{n=0}^{N} (-1)^n \frac{B_{rn+1,\bar{\chi}}(\frac{b}{r})}{rn+1} \frac{\theta^n}{n!} + O(\theta^{N+1})$$

$$- r \sum_{n=0}^{\infty} (-1)^n \sum_{m=1}^{f_\chi} \chi(m) \left( \frac{m+b}{r} \right)^{rn} \frac{\theta^n}{n!}$$

$$= -r \sum_{n=0}^{N} (-1)^n \left[ \frac{1}{rn+1} B_{rn+1,\bar{\chi}}(\frac{b}{r}) \right] + \sum_{m=1}^{f_\chi} \chi(m) \left( \frac{m+b}{r} \right)^{rn} \frac{\theta^n}{n!} + O(\theta^{N+1}).$$
By the definition of the generalized Bernoulli polynomials (1.6), we easily find that

\[(1.16) \quad B_{n, \chi}(x + lf_\chi) - B_{n, \chi}(x) = n \sum_{m=1}^{l f_\chi} \bar{\chi}(m)(m + x)^{n-1},\]

where \(n \geq 0\) and \(l \in \mathbb{N}\). Letting \(l = 1\) and \(x = b/r\), replacing \(n\) by \(rn + 1\), \(\chi\) by \(\bar{\chi}\) in (1.16), then substituting into (1.15), we obtain the following identity

\[G_{2, \chi}(\theta, b + r f_\chi, r) = -r \sum_{n=0}^{N} (-1)^n \frac{B_{rn+1, \bar{\chi}} \left( \frac{b}{r} + f_\chi \right) \theta^n}{rn + 1} + O(\theta^{N+1}).\]

2. Proofs

Our main technical tool comes from Zagier’s treatment of asymptotic expansions for infinite series, which can be found in Section 4 of [12].

First we need a character analogue of Euler-Maclaurin summation formula due to Berndt [1].

**Theorem 2.1** ([1] Theorem 4.1). Let \(f \in C^{(N+1)}[\alpha, \beta], -\infty < \alpha < \beta < \infty\). Then

\[\sum_{\alpha \leq m \leq \beta} \chi(m)f(m) = \chi(-1) \sum_{n=0}^{N} \frac{(1)^{n+1}}{n+1!} \left( B_{n+1, \bar{\chi}} (\beta) f^{(n)}(\beta) - B_{n+1, \bar{\chi}} (\alpha) f^{(n)}(\alpha) \right) + \chi(-1) \frac{(1)^{N}}{N+1!} \int_{\alpha}^{\beta} B_{N, \bar{\chi}}(x) f^{(N+1)}(x) dx,\]

where the dash indicates that if \(m = \alpha\) or \(m = \beta\), then only \(\frac{1}{2} \chi(\alpha)f(\alpha)\) or \(\frac{1}{2} \chi(\beta)f(\beta)\) is counted, respectively.

Let \(f : (0, \infty) \to \mathbb{C}\) be a smooth function which has an asymptotic power series expansion around 0. This means that

\[(2.1) \quad f(t) = \sum_{n=0}^{\infty} b_n t^n\]

as \(t \to 0^+\). Also, we assume that \(f(t)\) and all of its derivatives rapidly decay at infinity, i.e., the function \(t^A f^{(n)}(t)\) is bounded on \(\mathbb{R}_+\) for any \(A \in \mathbb{R}\) and \(n \in \mathbb{Z}^+\) (see [3]).

For any \(a \geq 0\), we consider the summation

\[(2.2) \quad g_{a, \chi}(t) = \begin{cases} \sum_{m=0}^{\infty} \chi(m)f(mt) & \text{if } a = 0, \\ \sum_{m=0}^{\infty} \chi(m)f((m + a)t) & \text{if } a > 0, \end{cases}\]

and in the next lemma we prove that its asymptotic behaviour can be simply described in terms of the coefficients of the expansion (2.1).

**Lemma 2.2.** Suppose that \(f\) has the asymptotic expansions (2.1) and \(f\) together with all of its derivatives are of rapid decay at infinity. Then the function \(g_{a, \chi}(t)\) defined in (2.2) has the following asymptotic expansions.
(i) Suppose that $\chi$ is a nonprincipal character. For any nonnegative integer $N$, we have
\[
g_0,\chi(t) = -\sum_{n=0}^{N} b_n \frac{B_{n+1,\chi}}{n+1} t^n + O(t^{N+1})
\]
as $t \to 0^+$.

(ii) Suppose that $\chi$ is a nonprincipal character. When $a > 0$, for any nonnegative integer $N$, we have
\[
g_{a,\chi}(t) = -\sum_{n=0}^{N} b_n \frac{B_{n+1,\chi}(a)}{n+1} t^n + O(t^{N+1}),
\]
as $t \to 0^+$.

Proof. Let $\alpha = 0$ and $\beta = f_\chi M$ in Theorem 2.1. Then
\[
\sum_{m=0}^{f_\chi M} \chi(m) f(m) = \chi(-1) \sum_{n=0}^{N} \frac{(-1)^{n+1} B_{n+1,\chi}}{(n+1)!} (f^{(n)}(f_\chi M) - f^{(n)}(0)) + \chi(-1) \frac{(-1)^N}{(N+1)!} \int_{0}^{f_\chi M} \overline{B}_{N+1,\chi}(x) f^{(N+1)}(x) dx,
\]
where we have used $\overline{B}_{n,\chi}(x) = \overline{B}_{n,\chi}(0) = B_n, \chi$ (see (1.8) above). From the assumption, the function $f$ and each of its derivatives are of rapid decay at infinity; we have $\int_{0}^{\infty} |f^{(N)}(x)| \, dx$ converges. Since $\overline{B}_{N,\chi}(x)$ is periodic and hence bounded, letting $M \to \infty$ in (2.3), we get
\[
\sum_{m=0}^{\infty} \chi(m) f(m) = \chi(-1) \sum_{n=0}^{N} \frac{(-1)^n B_{n+1,\chi}}{(n+1)!} f^{(n)}(0)
\]
\[
+ \chi(-1) \frac{(-1)^N}{(N+1)!} \int_{0}^{\infty} \overline{B}_{N+1,\chi}(x) f^{(N+1)}(x) dx.
\]
Replacing $f(x)$ by $f(tx)$ and then $x$ by $x/t$ with $t > 0$ in the above equation, we have
\[
\sum_{m=0}^{\infty} \chi(m) f(mt) = \chi(-1) \sum_{n=0}^{N} \frac{(-1)^n B_{n+1,\chi}}{(n+1)!} f^{(n)}(0)t^n
\]
\[
+ (-t)^N \chi(-1) \int_{0}^{\infty} \frac{\overline{B}_{N+1,\chi}(x/t)}{(N+1)!} f^{(N+1)}(x) dx.
\]
From the same reason as above, the last integral $\int_{0}^{\infty} \frac{\overline{B}_{N+1,\chi}(x/t)}{(N+1)!} f^{(N+1)}(x) dx$ is bounded as $t \to 0^+$ if $N$ is fixed, so the last term in the above equation can be denoted by $O(t^N)$. From (2.1), we have $f^{(n)}(0) = n! b_n$, substituting into (2.5) we get the desired asymptotic formula (i) by using (1.8) and (2.2).

To see (ii), if we write $g(x) = f((x + a)t)$ with $a > 0$ and use (2.1), then we get
\[
g(x) = f((x + a)t) = \sum_{n=0}^{\infty} b_n (x + a)^n t^n.
\]
Therefore,

\[(2.7) \quad g^{(j)}(x) = \sum_{n=j}^{\infty} b_n \binom{n}{j} j! (x + a)^{n-j} t^n.\]

Put \(x = 0\) in (2.7) and multiply both sides of the result equality by \(\frac{B_{j+1, \bar{\chi}}}{(j+1)!}\), we have

\[(2.8) \quad \frac{B_{j+1, \bar{\chi}} g^{(j)}(0)}{(j+1)!} = \sum_{n=j}^{N} b_n \binom{n}{j} \frac{B_{j+1, \bar{\chi}} a^{n-j} t^n}{j+1} + O(t^{N+1}).\]

Then we get

\[(2.9) \quad \sum_{j=0}^{N} (-1)^j \frac{B_{j+1, \bar{\chi}} g^{(j)}(0)}{(j+1)!} = \sum_{j=0}^{N} (-1)^j \sum_{n=j}^{N} b_n \binom{n}{j} \frac{B_{j+1, \bar{\chi}} a^{n-j} t^n}{j+1} + O(t^{N+1})
\]

\[= \sum_{n=0}^{N} b_n \left[ \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{B_{j+1, \bar{\chi}} a^{n-j}}{j+1} \right] t^n + O(t^{N+1}).\]

Applying (2.4) with \(f(x)\) replaced by \(g(x)\), we have

\[(2.10) \quad \sum_{m=0}^{\infty} \chi(m) g(m) = \chi(-1) \sum_{j=0}^{N} \frac{(-1)^j B_{j+1, \bar{\chi}}}{(j+1)!} g^{(j)}(0)
\]

\[+ \chi(-1) \frac{(-1)^N}{(N+1)!} \int_{0}^{\infty} B_{N+1, \bar{\chi}}(x) g^{(N+1)}(x) \, dx.\]

Now combining the results of (2.6), (2.9) and (2.10), we find that

\[(2.11) \quad \sum_{m=0}^{\infty} \chi(m) f((m + a)t) = \chi(-1) \sum_{n=0}^{N} b_n \left[ \sum_{j=0}^{n} \binom{n}{j} \frac{B_{j+1, \bar{\chi}} a^{n-j}}{j+1} \right] t^n
\]

\[+ (-t)^N \chi(-1) \int_{at}^{\infty} \frac{B_{N+1, \bar{\chi}}(x/t - a)}{(N+1)!} f^{(N+1)}(x) \, dx
\]

\[+ O(t^{N+1}),\]

where we have used the fact that \(g^{(N+1)}(x) = t^{(N+1)} f^{(N+1)}((x + a)t)\). From the same reason as before, the integral \(\int_{at}^{\infty} \frac{B_{N+1, \bar{\chi}}(x/t - a)}{(N+1)!} f^{(N+1)}(x) \, dx\) is bounded as \(t \to 0^+\) with \(N\) fixed, so the remainder term in the above expansion becomes \(O(t^N)\). Using the identity

\[B_{n, \bar{\chi}}(x) = \sum_{j=0}^{n} \binom{n}{j} B_{j, \bar{\chi}} x^{n-j},\]
and \( B_{0, \chi} = 0 \) if \( \chi \neq \chi^0 \) (see [6, p. 9]), we get the expression

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{B_{j+1, \chi} a^{n-j}}{j+1} = \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} B_{j+1, \chi} a^{n-j}
\]

\[
= \frac{1}{n+1} \sum_{j=-1}^{n+1} (-1)^{j-1} \binom{n+1}{j} B_{j, \chi} a^{n-j+1}
\]

\[
= \frac{(-1)^n}{n+1} B_{n+1, \chi} (-a).
\]

(2.12)

So, combining (2.11) and (2.12), and noticing that

\[
B_{n, \chi} (-x) = (-1)^n \bar{\chi} (-1) B_{n, \chi} (x), \quad n \geq 0
\]

(see (1.8) above), we obtain the desired asymptotic formula (ii).

\[\square\]

**Proof of Theorem 1.1.** Let \( r \) be any positive integer and let \( \chi \) be a non-principal character. Define \( f(t) = e^{-t^r} \) for \( t > 0 \). It is easy to see that this function is a smooth function and it is of rapid decay at infinity. From the Taylor expansion

\[
e^{-t^r} = \sum_{n=0}^{\infty} \frac{(-1)^n r^n t^{rn}}{n!},
\]

we see that the function \( f(t) = e^{-t^r} \) has the expansion form (2.1) at \( t = 0 \)

\[
f(t) = \sum_{m=0}^{\infty} b_m t^m
\]

with

\[
b_m = b_{rn} = \frac{(-1)^n}{n!} \text{ if } r \mid m, \text{ and } b_m = 0 \text{ if } r \nmid m,
\]

for \( m \geq 0 \).

First we prove (i). Set \( \theta = t^r \) with \( r \in \mathbb{N} \). By (1.3), we have

\[
G_{2, \chi}(\theta, 0, r) = r \sum_{m=0}^{\infty} \chi(m) e^{-m^r \theta} = r \sum_{m=0}^{\infty} \chi(m) e^{-(mt)^r},
\]

(2.15)
then by (1.2), (2.2), (2.13), (2.14) and Lemma 2.2(i), we get

\[ G_{2,\chi}(\theta, 0, r) = \sum_{m=0}^{\infty} \chi(m)e^{-(mt)r} \]

\[ = - \sum_{n=0}^{N} b_{rn} B_{rn+1,\bar{\chi}}t^{rn} + O(t^{r(N+1)}) \]

(the remainder term is \( O(t^{r(N+1)}) \) in the above, since by (2.14) the coefficients of \( t^m \) equals to 0 if \( r \nmid m \))

\[ = - \sum_{n=0}^{N} (-1)^n \frac{B_{rn+1,\bar{\chi}}}{(rn + 1)n!}t^{rn} + O(t^{r(N+1)}) \]

(since by (2.14) we have \( b_{rn} = (-1)^n/n! \))

\[ = - \sum_{n=0}^{N} (-1)^n \frac{B_{rn+1,\bar{\chi}}}{(rn + 1)n!} \theta^n + O(\theta^{N+1}). \]

Thus the first part follows.

Next, we prove (ii). Again, let \( \theta = r^t \) with \( r \in \mathbb{N} \). From (1.3), (2.2), (2.13), (2.14) and Lemma 2.2(ii), for any positive integer \( N \), we have (2.16)

\[ G_{2,\chi}(\theta, b, r) = r \sum_{m=0}^{\infty} \chi(m)e^{-(m+b/r)t}r \]

\[ = r \sum_{m=0}^{\infty} \chi(m)f((m + b/r)t) \]

\[ = -r \sum_{n=0}^{N} b_{rn} \frac{B_{rn+1,\bar{\chi}}}{rn + 1} \theta^n + O(t^{r(N+1)}) \]

(the remainder term is \( O(t^{r(N+1)}) \) in the above, since by (2.14) the coefficients of \( t^m \) equals to 0 if \( r \nmid m \))

\[ = -r \sum_{n=0}^{N} (-1)^n \frac{B_{rn+1,\bar{\chi}}}{(rn + 1)n!} \theta^n + O(\theta^{N+1}) \]

as \( \theta \to 0^+ \). This completes the proof.

**Proof of Theorem 1.2.** We prove an asymptotic expansion for \( G_{1,\bar{\chi}}(\theta, b, r) \) with \( 0 < b < r \). By separating even and odd terms, we find that

\[ G_{1,\bar{\chi}}(\theta, b, r) = \chi(2)G_{2,\chi}(2^r\theta, b/2, r) - r \sum_{n=0}^{\infty} \chi(2n + 1)e^{-(2n+1+b/r)^r} \theta. \]
We can rewrite the sum $G_{2,\chi}(\theta, b, r)$ as

$$G_{2,\chi}(\theta, b, r) = r \sum_{n=0}^{\infty} \chi(2n+1)e^{-(2n+1+b/r)\theta}. \tag{2.18}$$

Substituting the above equality into (2.17), we have

$$G_{1,\chi}(\theta, b, r) = 2\chi(2)G_{2,\chi}(2\theta, b/2, r) - G_{2,\chi}(\theta, b, r). \tag{2.19}$$

Letting $x = b/r$ with $0 < b < r$ in (1.12), we find easily that

$$2^{n+1}\bar{\chi}(2)B_{n+1,\chi}(b/r) - B_{n+1,\chi}(b/r) = -\frac{n+1}{2}E_{n,\chi}(b/r), \tag{2.20}$$

where we have used the fact that $B_{n,\chi}(x) = B_{n,\chi}(x)$ and $E_{n,\chi}(x) = E_{n,\chi}(x)$ for $0 \leq x < 1$. Let $\chi$ be a Dirichlet character with odd conductor $f$. Combining the results of Theorem 1.1(ii), (2.19), and (2.20), we can deduce that

$$G_{1,\chi}(\theta, b, r) = r \sum_{n=0}^{N} \frac{(-1)^n}{rn+1} \left[ B_{rn+1,\chi}(b/r) - 2^{rn+1}\chi(2)B_{rn+1,\chi}(b/r) \right] \theta^n \frac{1}{n!} + O(\theta^{N+1})$$

as $\theta \to 0^+$. This completes the proof of (ii).

The proof of (i) is similar to that of (ii) and we may omit it. □

### 3. Connection with certain $L$-series

Let $C : \mathbb{Z} \to \mathbb{C}$ be a periodic function with mean value 0 and $L(s, C) = \sum_{n=1}^{\infty} C(n)n^{-s}$ (Re($s$) $> 1$) be the associated $L$-series. Lawrence and Zagier [7] proved that $L(s, C)$ can be extended to the whole complex plane $\mathbb{C}$, and by using Mellin transformation they also showed that the two functions $\sum_{n=1}^{\infty} C(n)e^{-nt}$ and $\sum_{n=1}^{\infty} C(n)e^{-n^2t}$ ($t > 0$) have the asymptotic expansions

$$\sum_{n=1}^{\infty} C(n)e^{-nt} \sim \sum_{r=0}^{\infty} L(-r, C) \frac{(-t)^r}{r!}$$

and

$$\sum_{n=1}^{\infty} C(n)e^{-n^2t} \sim \sum_{r=0}^{\infty} L(-2r, C) \frac{(-t)^r}{r!}$$

as $t \to 0^+$. Furthermore, the number $L(-r, c)$ are given explicitly by

$$L(-r, C) = \frac{-M^r}{r+1} \sum_{n=1}^{M} C(n)B_{r+1} \left( \frac{n}{M} \right) \quad (r = 0, 1, \ldots)$$

where $B_k(x)$ denotes the $k$th Bernoulli polynomial and $M$ is any period of the function $C(n)$.
Let
\[ A(q) = 1 + q + q^3 + q^7 - q^8 - q^{14} - q^{20} - \cdots \quad (|q| < 1) \]
be a holomorphic function in the unit disk and \( \zeta \) be a root of unity. By using
the above expansions, they showed that the radical limit of \( 1 - \frac{1}{2}A(q) \) as \( q \)
tends to \( \zeta \) equals \( W(\zeta) \), the (rescaled) Witten-Reshetikhin-Turaev (WRT)
invariant of the Poincaré homology sphere. As pointed out by Lawrence and
Zagier [7, p. 95], although for a general 3-manifold it is hopeless to give nice
formulae for WRT invariants, the WRT invariant of the Poincaré homology
sphere is accessible to computations.

In this section, inspired by the works of Lawrence and Zagier [7], we
show a connection between certain \( L \)-series and our asymptotic expansions.

Suppose \( \epsilon \in \{1, 2\} \). Define the series
\[ G_\epsilon(\theta, b, r) = r \sum_{n=0}^{\infty} (-1)^{\epsilon n} e^{-(n+b/r)^{\epsilon} \theta}. \]
If \( \chi \) is a nonprincipal character with conductor \( f_\chi \), then define
\[ G_{\epsilon, \chi}(\theta, b, r) \]
by
\[ G_{\epsilon, \chi}(\theta, b, r) = r \sum_{n=0}^{\infty} (-1)^{\epsilon n} \chi(n) e^{-(n+b/r)^{\epsilon} \theta}. \]
We can express \( G_{\epsilon, \chi}(\theta, b, r) \) in terms of series \( G_\epsilon(\theta, b, r) \).
If \( \chi \) is a character \( \text{mod } f_\chi \), then we rearrange the terms in the series for
\( G_{\epsilon, \chi}(\theta, b, r) \) according to the residue classes \( \text{mod } f_\chi \).
That is, we write
\[ n = mf_\chi + a, \quad \text{where } 0 \leq a \leq f_\chi - 1 \text{ and } m = 0, 1, 2, \ldots, \]
and obtain
\[ G_{\epsilon, \chi}(\theta, b, r) = r \sum_{a=0}^{f_\chi-1} \sum_{m=0}^{\infty} (-1)^{\epsilon m(mf_\chi + a)} \chi(mf_\chi + a) e^{-(mf_\chi + a + b/r)^{\epsilon} \theta} \]
\[ = \sum_{a=0}^{f_\chi-1} (-1)^{\epsilon a} \chi(a) G_\epsilon (mf_\chi, ar + b, \theta). \]

On the other hand, we define the coefficients \( \{\gamma_n\} \) from the asymptotic
expansion
\[ G_{\epsilon, \chi}(\theta, b, r) = r \sum_{n=0}^{\infty} (-1)^{\epsilon n} \chi(n) e^{-(n+b/r)^{\epsilon} \theta} \sim \sum_{n=0}^{\infty} \gamma_n \theta^n \quad \text{as } \theta \to 0^+ \]
and consider the Mellin transform of \( G_{\epsilon, \chi}(\theta, b, r) \):
\[ \int_0^\infty G_{\epsilon, \chi}(\theta, b, r) \theta^{s-1} d\theta = \sum_{n=0}^{\infty} (-1)^{\epsilon n} \chi(n) \int_0^\infty e^{-(n+b/r)^{\epsilon} \theta} \theta^{s-1} d\theta \]
\[ = r \sum_{n=0}^{\infty} \frac{(-1)^{\epsilon n} \chi(n)}{(n+b/r)^{rs}} \int_0^\infty e^{-t^{s-1}} dt \]
\[ = \Gamma(s) L_{r, \epsilon}(rs, b; \chi). \]
Here $L_{r,\epsilon}(s, b; \chi)$ is the modified Dirichlet $L$-series defined by

$$
L_{r,\epsilon}(s, b; \chi) = r \sum_{m=0}^{\infty} \frac{(-1)^m \chi(m)}{(m + b/r)^s},
$$

where $\chi$ is a primitive character modulo $f_\chi$, $b$ is a positive real number, and $r \in \mathbb{N}$. On the other hand, we find that

$$
\int_0^\infty G_{\epsilon,\chi}(\theta, b, r)\theta^{s-1}d\theta = \int_0^\infty \left( \sum_{n=0}^{N-1} \gamma_n \theta^n + O(\theta^N) \right) \theta^{s-1}d\theta
$$

$$
= \sum_{n=0}^{N-1} \frac{\gamma_n}{s+n} + R_N(s),
$$

where $R_N(s)$ is an analytic function for $\text{Re}(s) > -N$. And it is clear that the residue of $R_N(s)$ at $s = -n$ is $\gamma_n$. By (3.5), this implies that

$$
\gamma_n = \text{res}_{s=-n} \{ L_{r,\epsilon}(rs, b; \chi) \Gamma(s) \} = \frac{(-1)^n}{n!} L_{r,\epsilon}(-rn, b; \chi).
$$

Therefore, from (3.4) and (3.8), we see that

$$
G_{\epsilon,\chi}(\theta, b, r) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L_{r,\epsilon}(-rn, b; \chi) \theta^n \quad \text{as} \quad \theta \to 0^+,
$$

where $\epsilon \in \{1, 2\}$. From Theorem 1.1 and 1.2, the numbers $L_{r,\epsilon}(-rn, b; \chi)$ are given explicitly by

$$
L_{r,\epsilon}(-rn, b; \chi) = \begin{cases} 
\frac{1}{2} r E_{rn, \bar{\chi}} \left( \frac{b}{r} \right) & \text{if } \epsilon = 1 \text{ and odd } f_\chi, \\
-r \frac{B_{rn+1, \bar{\chi}}(\bar{\chi})}{rn+1} & \text{if } \epsilon = 2,
\end{cases}
$$

where $\chi \neq \chi^0$ is a primitive character modulo $f_\chi$, $b$ is a real number with $0 < b < r$, $n$ is any nonnegative integer, and $r \in \mathbb{N}$. By (3.4), (3.8) and (3.10), we have established the following theorem.

**Theorem 3.1.** Let the notations such as $\epsilon$ be defined as above. Then the series $G_{\epsilon,\chi}(\theta, b, r)$ have the asymptotic expansions

$$
G_{\epsilon,\chi}(\theta, b, r) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L_{r,\epsilon}(-rn, b; \chi) \theta^n \quad \text{as} \quad \theta \to 0^+.
$$

Furthermore, the numbers $L_{r,\epsilon}(-rn, b; \chi)$ are given explicitly by

$$
L_{r,\epsilon}(-rn, b; \chi) = \begin{cases} 
\frac{1}{2} r E_{rn, \bar{\chi}} \left( \frac{b}{r} \right) & \text{if } \epsilon = 1 \text{ and odd } f_\chi, \\
-r \frac{B_{rn+1, \bar{\chi}}(\bar{\chi})}{rn+1} & \text{if } \epsilon = 2,
\end{cases}
$$

where $\chi \neq \chi^0$ is a primitive character modulo $f_\chi$, $b$ is a real number with $0 < b < r$, $n$ is any nonnegative integer, and $r \in \mathbb{N}$.

In particular, it is easy to see that

$$
r \sum_{m=0}^{\infty} \frac{\chi(2m+1)}{(2m+1+b/r)^s} = r \sum_{m=0}^{\infty} \frac{\chi(m)}{(m+b/r)^s} - r \sum_{m=0}^{\infty} \frac{\chi(2m)}{(2m+b/r)^s}.
$$
This yields, after simplification, (cf. (10.24) on p. 176 of \([10]\))

\[
L_{r,1}(s, b; \chi) = r \sum_{m=0}^{\infty} \frac{(-1)^m \chi(m)}{(m + b/r)^s}
\]

\[= r \sum_{m=0}^{\infty} \frac{\chi(2m)}{(2m + b/r)^s} - r \sum_{m=0}^{\infty} \frac{\chi(2m + 1)}{(2m + 1 + b/r)^s}
\]

\[= 2^{-s+1} \chi(2) r \sum_{m=0}^{\infty} \frac{\chi(m)}{(m + b/2r)^s} - r \sum_{m=0}^{\infty} \frac{\chi(m)}{(m + b/r)^s}.
\]

We use (3.11) in the last step above. Using (3.6) and (3.12) we see that

\[
L_{r,1}(s, b; \chi) = 2^{-s+1} \chi(2) L_{r,2}(s, b/2; \chi) - L_{r,2}(s, b; \chi).
\]

We consider \(s = -rn\) in (3.13), using Theorem 3.1 we have the following result.

**Corollary 3.2.** Let \(\chi, \text{ etc.}, \) be as above. Then

\[
E_{rn, \bar{\chi}} \left(\frac{b}{r}\right) = \frac{1}{rn + 1} \left(B_{rn+1, \bar{\chi}} \left(\frac{b}{r}\right) - 2^{rn+1} \chi(2) B_{rn+1, \bar{\chi}} \left(\frac{b}{2r}\right)\right),
\]

where \(\chi \neq \chi^0\) is a primitive character modulo \(f_{\chi}\), \(b\) is a real number with \(0 < b < r\), \(n\) is any nonnegative integer, and \(r \in \mathbb{N}\).

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