Physical Network Coding in Two-Way Wireless Relay Channels

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Abstract—It has recently been recognized that the wireless networks represent a fertile ground for devising communication modes based on network coding. A particularly suitable application of the network coding arises for the two-way relay channels, where two nodes communicate with each other assisted by using a third, relay node. Such a scenario enables application of physical network coding, where the network coding is either done (a) jointly with the channel coding or (b) through physical combining of the communication flows over the multiple access channel. In this paper we first group the existing schemes for physical network coding into two generic schemes, termed 3-step and 2-step scheme, respectively. We investigate the conditions for maximization of the two-way rate for each individual scheme: (1) the Decode–and–Forward (DF) 3-step schemes (2) three different schemes with two steps: Amplify–and–Forward (AF), JDF and Denoise–and–Forward (DNF). While the DNF scheme has a potential to offer the best two-way rate, the most interesting result of the paper is that, for some SNR configurations of the source–relay links, JDF yields identical maximal two-way rate as the upper bound on the rate for DNF.

I. INTRODUCTION

It has been recently noted [1] that broadcast and unreliable nature of the wireless medium sets a fertile ground for developing network–coding [2] solutions. The network coding can offer performance improvement in the wireless networks for two-way (or multi–way) communication flows [3] [4] [5] [6] [7] [8] [9]. In general, there are two generic schemes for two–way wireless relay (Fig. 1): (a) 3–step scheme (b) 2–step scheme. The node $A$ has packets for the node $C$ and vice versa. In Step 1 of the 3–step scheme, $A$ transmits the packet $D_{AC}$, in Step 2 $C$ transmits the packet $D_{CA}$. Here $B$ decodes both packets, such that the 3–step schemes are Decode–and–Forward (DF) schemes. In the simpler DF schemes [3] [4] [5], the direct link between $A$ and $C$ is ignored by the receivers in Steps 1 and 2, such that in Step 3 $B$ broadcasts the packet $D_{BC} \oplus D_{BA} = D_{AC} \oplus D_{CA}$, where $\oplus$ is XOR operation, after which the node $A$ is able to decode the packet $D_{CA}(D_{AC})$. While it is hard to characterize such a simple DF scheme as “physical” network coding, such an attribute can be attached to the 3–step DF scheme [7], where the direct link $A–C$ is not ignored in the Steps 1 and 2 and a joint network–channel coding is needed. In that case, the packet $D_{BA}(D_{BC})$ is a many–to–one function of the packet $D_{CA}(D_{AC})$, since $A$ already has some information from the Step 2 (1). In the 2–step schemes the communication flows are combined through a simultaneous transmission over a multiple access channel. In Step 1 $B$ receives a noisy signal that consists of interference between the signals of $A$ and $C$. Due to the half–duplex operation, the direct link is naturally ignored in the 2–step schemes. The signal $x_B$ that is broadcasted in Step 2 depends on the applied 2–step scheme. In Amplify–and–Forward (AF) [5], $x_B$ is simply an amplified version of the signal received by $B$ in step 1. After receiving $x_B$, the node $A$ subtracts its own signal and decodes the signal sent by $C$ in Step 1. The 2–step scheme termed Denoise–and–Forward (DNF) has been introduced in [6]. A related scheme appeared in [10]. In DNF, the node $B$ again does not decode the packets sent by $A$ and $C$ in Step 1, but it maps the received signal to a codeword from a discrete set. Hence, the signal $x_B$ carries now the information about the set of codeword pairs $\{(x_{AC}, x_{CA})\}$ which are considered by the node $B$ as likely to have been sent in the Step 1. In general, this set can consist of several codeword pairs, such that $B$ has an ambiguity which information has been sent. Nevertheless, since $A$ knows $x_{AC}(x_{CA})$, after receiving $x_B$, it will extract exactly one codeword as a likely one to have been sent by $C$ (A) in Step 1. The final considered 2–step scheme is Joint Decode–and Forward (JDF), recently considered in [9]. In JDF, the transmission rates in Step 1 of Fig. 1(b) are selected such that $B$ can jointly decode both $x_{AC}$ and $x_{CA}$, and then use XOR to obtain the signal for broadcast in Step 2.

Fig. 1. Generic schemes for physical network coding over the two–way relay channel. (a) Three–step scheme (b) 2–step scheme.
In this paper we investigate the strategies that can maximize the overall two-way rate for several 2- and 3-step schemes for physical network coding. We show that the key to maximizing the two-way rate in the system for the 3-step schemes is the relation between the durations of Step 1 and Step 2. On the other hand, we show that the key factor for maximizing the two-way rate in the 2-step schemes is the choice of the rates at which A and C transmit in Step 1. Note that we are not providing the absolute capacities of the two-way relay channel, since we are putting some operational restrictions to the applied schemes. Nevertheless, the results give an excellent overview of what can be achieved by each scheme for physical network coding.

II. NOTATIONS AND DEFINITIONS

We assume that there are only two communication flows, A → C and C → A, respectively. The relay B is neither a source nor a sink of any data in the system. All the nodes are half-duplex, such that a node can either transmit or receive at a given time. We use $x_U[m]$ to denote the $m$-th complex baseband transmitted symbol from node $U \in \{A, B, C\}$. A complex-valued vector is denoted by x. A packet of bits is denoted by D, and the number of bits in the packet is |D|. If only one node $U \in \{A, B, C\}$ is transmitting, then the $m$-th received symbol at the node $V \in \{A, B, C\} \setminus U$ is given by:

$$y_V[m] = h_UV x_U[m] + z_V[m]$$

where $h_UV$ is the complex channel coefficient between $U$ and $V$. $z_V$ is the complex additive white Gaussian noise $CV(0, N_0)$. The transmitted symbols have $E\{|x_U[m]|^2\} = 1$ and a normalized power $E\{|x_U|m|^2\} = 1$. Each node uses the same transmission power, which makes the links symmetric:

$$h_{AC} = h_{CA} = h_0; \quad h_{AB} = h_{BA} = h_1; \quad h_{CB} = h_{BC} = h_2$$

(2)

We consider time-invariant channels and $h_0, h_1, h_2$ are perfectly known by all nodes. This assumption allows us to find the two-way rates at which a reliable communication is possible. The bandwidth is normalized, such that we consider the following signal-to-noise ratios (SNRs):

$$\gamma_i = \frac{|h_i|^2}{N_0}, \quad i = 0, 1, 2$$

(3)

The bandwidth is normalized to 1 Hz, such that a link with SNR of $\gamma$ can reliably transfer up to:

$$C(\gamma) = \log_2(1 + \gamma) \text{ [bit/s]}$$

(4)

The time is measured in number of symbols, such that when a packet of $N$ symbols is sent at the data rate $r$, the packet contains $Nr$ bits. The packet lengths are sufficiently large, such that we can use codebooks that offer zero errors if the rate is chosen to be below the channel capacity.

Without loss of generality, we assume that

$$\gamma_2 \geq \gamma_1$$

(5)

The source-to-relay links are assumed better than the direct link [11]:

$$\gamma_1 > \gamma_0 \quad \gamma_2 > \gamma_0$$

(6)

If A and C transmit simultaneously, then B receives:

$$y_B[m] = h_1 x_A[m] + h_2 x_B[m] + z_B[m]$$

(7)

In this paper we will be interested in the two-way rate:

**Definition 1**: Let, during a time of $N$ symbols, A receive reliably $|D_{CA}|$ bits from C and C receive reliably $|D_{AC}|$ bits from A. Then the two-way rate is given by:

$$R_{A-C} = \frac{|D_{AC}| + |D_{CA}|}{N} \text{ [bit/s]}$$

(8)

We seek to maximize the two-way rate under the following two operational restrictions. First, in each round A and C transmit only fresh data, which is independent of any information exchange that took part in the previous rounds. Second, B is applying potentially suboptimal broadcast strategy, as we have not explicitly considered the broadcast strategies that achieve the full capacity region of the Gaussian broadcast channel [12]. Hence, the obtained two-way rates are lower bounds on the achievable rates in the two-way relay systems.

III. 3-STEP SCHEME

A single round in a 3-step scheme is (Fig. 1(a)):

**Step 1**: Node A transmits, nodes B and C receive.

**Step 2**: Node C transmits, nodes A and B receive.

**Step 3**: Node B transmits, nodes A and C receive. In this scheme, B should decode the data transmitted by node A (node C) in Step 1 (Step 2). The data transmitted by C in Step 2 is independent of the data received from A in Step 1. The data transmitted by the node B in Step 3 is a function of the data that was transmitted by A and C in Step 1 and 2, respectively, from the same round.

We first determine the size of the data broadcasted by B. If A is transmitting K symbols at a data rate $C(\gamma_1)$, then B receives reliably the packet $D_{AC}$ of $KC(\gamma_1)$ bits. At the same time, the total amount of information received at the node C is $KC(\gamma_0)$ bits, where $C(\gamma_0) < C(\gamma_1)$, due to (6). Hence, in the next step the relay needs to transmit at least:

$$|D_{BC}| = K[C(\gamma_1) - C(\gamma_0)]$$

(9)

bits to C in order to completely remove the uncertainty at C about the message transmitted by A. It is crucial to note that the node A knows the content of the packet $D_{BC}$. The argument to show this is that, after B receives $D_{AC}$, both A and B have the same information $D_{AC}$ and no information what has been received at C. Even then, the random binning technique [12] can be used to create the packet $D_{BC}$, such that $D_{BC}$ is uniquely and in advance determined for each $D_{AC}$.

Let the node A in Step 1 transmit a packet of $N(1-\theta)$ symbols at a rate $C(\gamma_1)$, where $0 < \theta < 1$. On successfully decoding $D_{AC}$, the relay node B prepares $D_{BC}$ that needs to be forwarded to C, with a packet size of:

$$|D_{BC}| = N(1-\theta) \log_2 [C(\gamma_1) - C(\gamma_0)] \text{ [bit]}$$

(10)

During the next $N\theta$ symbols, in Step 2, the node C transmits $D_{CA}$ at a rate $C(\gamma_2)$, out of which B creates $D_{BA}$ with:

$$|D_{BA}| = N\theta \log_2 [C(\gamma_2) - C(\gamma_0)] \text{ [bits]}$$

(11)
It follows from above that $A$ knows $\mathbf{D}_{BC}$ and $C$ knows $\mathbf{D}_{BA}$. In addition, the node $A$ does not know $\mathbf{D}_{BA}$, but it knows a priori the size of the packet $|\mathbf{D}_{BA}|$. The same is valid for $C$ and the packet size $|\mathbf{D}_{BC}|$. This is reasonable for the assumed time–invariant systems with fixed $h_0, h_1, h_2$.

**Theorem 1:** The maximal two–way rate for DF is

$$R_{DF} = C(\gamma_1) \frac{1 + \delta C(\gamma_2) - C(\gamma_0)}{1 + \delta C(\gamma_2) - C(\gamma_0)}$$

(12)

where $\delta = \frac{C(\gamma_1) - C(\gamma_0)}{C(\gamma_1) + C(\gamma_2) - 2C(\gamma_0)}$.

**Proof:** In Step 3, the node $B$ first compares the packet sizes $|\mathbf{D}_{BC}|$ and $|\mathbf{D}_{BA}|$. Two cases can occur:

1) Case 1: $|\mathbf{D}_{BC}| = |\mathbf{D}_{BA}|$: Using (10) and (11), we can translate this condition into inequality for $\theta$:

$$0 < \theta \leq \frac{C(\gamma_1) - C(\gamma_0)}{C(\gamma_1) + C(\gamma_2) - 2C(\gamma_0)}$$

(13)

The relay $B$ partitions the packet $\mathbf{D}_{BC}$ into $\mathbf{D}_{BC}^{(1)}$ and $\mathbf{D}_{BC}^{(2)}$:

$$|\mathbf{D}_{BC}^{(1)}| = |\mathbf{D}_{BA}| \quad |\mathbf{D}_{BC}^{(2)}| = |\mathbf{D}_{BC}| - |\mathbf{D}_{BA}|$$

(14)

$\mathbf{D}_{BC}^{(1)}$ consists of the first $|\mathbf{D}_{BA}|$ bits from $\mathbf{D}_{BC}$ and $\mathbf{D}_{BC}^{(2)}$ consists of the rest of the bits from $\mathbf{D}_{BC}$. Now $B$ creates:

$$\mathbf{D}_{B} = \mathbf{D}_{BC}^{(1)} \oplus \mathbf{D}_{BA}$$

(15)

where $\oplus$ is bitwise XOR. Due to the condition (5) and the fact that both $A$ and $C$ need to receive it, the packet $\mathbf{D}_{B}$ is transmitted at the lower rate $C(\gamma_1)$. After receiving $\mathbf{D}_{B}$, the node $A$ extracts the packet $\mathbf{D}_{BA}$ as $\mathbf{D}_{BA} = \mathbf{D}_{B} \oplus \mathbf{D}_{BC}^{(1)}$. This packet is then used together with the information that $A$ has received from node $C$ in Step 2 to decode the packet $\mathbf{D}_{CA}$.

On the other hand, after receiving $\mathbf{D}_{B}$, the node $C$ extracts $\mathbf{D}_{BC}^{(1)} = \mathbf{D}_{B} \oplus \mathbf{D}_{BA}$. Now $B$ transmits the packet $\mathbf{D}_{BC}^{(2)}$ to the node $C$ at a higher rate of $C(\gamma_2)$, as $A$ does not need to receive this information. With $\mathbf{D}_{BC}^{(2)}$ and $\mathbf{D}_{BC}^{(1)}$, the node $C$ creates $\mathbf{D}_{BC}$, which is further on used jointly with the information that $C$ has received in Step 1 to decode the packet $\mathbf{D}_{AC}$. The total duration of the three steps is $N_{1,DF}(\theta) = N(1 - \theta) + N\theta + \frac{M_{BC}^{(1)} + M_{BC}^{(2)}}{C(\gamma_2)}$, resulting in a two–way rate of:

$$R_{1,DF}(\theta) = \frac{|\mathbf{D}_{AC}| + |\mathbf{D}_{CA}|}{N_{1,DF}} \text{ [bits/s] }$$

(16)

where $|\mathbf{D}_{BC}|$ and $|\mathbf{D}_{BA}|$ are functions of $\theta$ and are given by (10) and (11), respectively. It can be proved that $R_{1,DF}(\theta)$ is monotonically increasing function of $\theta$, such that $R_{1,DF}(\theta)$ achieves its maximal value for the upper limiting value of $\theta$, given in (13). By applying $\theta = \frac{C(\gamma_1) - C(\gamma_0)}{C(\gamma_1) + C(\gamma_2) - 2C(\gamma_0)}$ into the terms of (16), we obtain the two–way rate given by (12).

2) Case 2: $|\mathbf{D}_{BC}| < |\mathbf{D}_{BA}|$: This is the region:

$$\frac{C(\gamma_1) - C(\gamma_0)}{C(\gamma_1) + C(\gamma_2) - 2C(\gamma_0)} < \theta \leq 1$$

(17)

The packet $\mathbf{D}_{BC}$ is padded with zeros to obtain the packet $\mathbf{D}_{BC}^p$ such that $|\mathbf{D}_{BC}^p| = |\mathbf{D}_{BA}|$. Since $A$ and $C$ know the size of $|\mathbf{D}_{BC}|$, they also know how many zeros are used for padding. The node $B$ creates the packet $\mathbf{D}_{B} = \mathbf{D}_{BC}^p \oplus \mathbf{D}_{BA}$. In Step 3 only the packet $\mathbf{D}_{B}$ is broadcasted at a transmission rate $C(\gamma_1)$. The node $A$ extracts $\mathbf{D}_{BA}$ as $\mathbf{D}_{BA} = \mathbf{D}_{BC}^p \oplus \mathbf{D}_{B}$ and uses the information received in Step 2 to decode $\mathbf{D}_{CA}$. Similarly, $C$ obtains $\mathbf{D}_{BC}$ from $\mathbf{D}_{B}$, removes the padding zeros and obtains $\mathbf{D}_{BC}$, which is then used jointly with the information from Step 1 to decode the packet $\mathbf{D}_{AC}$.

The total number of symbols is $N_{2,DF}(\theta) = N(1 - \theta) + N\theta + \frac{M_{BC}^p + M_{BA}}{C(\gamma_1)}$, and the two–way rate $R_{2,DF}(\theta)$ is again calculated by using the expression (16), by putting $N_{2,DF}$ instead of $N_{1,DF}$. It can be proved that $R_{2,DF}(\theta)$ decreases monotonically with $\theta$ and it reaches maximal value for the minimal $\theta$ in the region (17). Hence, the maximal two–way rate is again given by (12).

It can be seen that due to the condition (6), the two–way rate is $R_{DF} < C(\gamma_1)$. When $\gamma_1 = \gamma_2$, the obtained capacity expression is identical to what can be obtained from [7]. When $A$ and $C$ neglect the transmission over the direct link ($\gamma_0 = 0$), the two–way rate achieved by DF is:

$$R_{DF} = \frac{2C(\gamma_1)C(\gamma_2)}{C(\gamma_1) + 2C(\gamma_2)}$$

(18)

IV. 2–Step Schemes

In this section we deal with three schemes: Amplify–and Forward (AF), Joint Decode–and–Forward (JDF) and Denoise–and–Forward (DNF). The two steps are: Step 1: Nodes $A$ and $C$ transmit, node $B$ receives. Step 2: Node $B$ transmits, nodes $A$ and $C$ receive.

The transmission rates for $A$ and $C$ in Step 1 are denoted by $R_A$ and $R_C$, respectively. As we will see, the choice of $R_A$ and $R_C$ is a feature of each transmission scheme AF, JDF or DNF. Except for the selection of the rate pair $(R_A, R_C)$, the Step 1 is identical for all three schemes, where its duration is fixed to $N$ symbols and the $m$–th received symbol at node $B$ is given by (7).

A. Amplify–and–Forward (AF)

After Step 1, the node $B$ amplifies the received signal $y_B$ for a factor $\beta$ and broadcasts $x_B = \beta y_B$ to $A$ and $C$. As $x_B$ also consists of $N$ symbols, the total duration of the two steps is $2N$. The amplification factor $\beta$ is chosen as:

$$\beta = \frac{1}{\sqrt{|h_1|^2 + |h_2|^2 + N_0}}$$

(19)

to make the the average per–symbol transmitted energy at $B$ equal to 1 ($N_0$ is the noise variance). The $m$–th symbol received by $A$ in Step 2 is:

$$y_A[m] = \beta h_1 x_A[m] + z_A[m] = \beta h_1 x_A[m] + \beta h_1 h_2 x_C[m] + \beta h_1 z_B[m] + z_A[m]$$

Since $A$ knows $x_A[m], h_1, h_2$ and $\beta$, it can subtract $\beta h_1^2 x_A[m]$ from $y_A[m]$ and obtain:

$$r_A[m] = \beta h_1 h_2 x_C[m] + \beta h_1 z_B[m] + z_A[m]$$

(20)
which is a Gaussian channel for receiving $x_C[m]$ with SNR:
\[
\gamma_{C\rightarrow A}^{(AF)} = \frac{\beta^2 |h_1|^2 |h_2|^2}{(\beta^2 |h_1|^2 + 1)N_0} = \frac{\gamma_1 \gamma_2}{2\gamma_1 + 2\gamma_2 + 1}
\]
(21)

This notation denotes that $\gamma_{C\rightarrow A}^{(AF)}$ is the SNR that determines the rate $R_C$ at which $C$ can communicate to $A$. Similarly, we can find the SNR which determines the rate $R_A$:
\[
\gamma_{A\rightarrow C}^{(AF)} = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2 + 1}
\]
(22)

Hence, the rate pair $(R_A, R_C)$ used in Step 1 should be:
\[
R_A = C\left(\gamma_{A\rightarrow C}^{(AF)}\right) \quad R_C = C\left(\gamma_{C\rightarrow A}^{(AF)}\right)
\]
(23)

Finally, the two–way rate achieved by the AF scheme is:
\[
R_{AF} = \frac{NR_A + NR_C}{2N} = \frac{R_A + R_C}{2}
\]
(24)

B. Joint Decode–and–Forward (JDF)

Here the at rates $R_A$ and $R_C$ are chosen such that the node $B$ is able to decode both packets in Step 1. The rate pairs $(R_A, R_C)$ with such a property should lie inside the convex region [12] on Fig. 2. The sum–rate is maximized if the rate pair $(R_A, R_C)$ lies on the segment $L_AL_C$:
\[
R_A + R_C = C(\gamma_1 + \gamma_2)
\]
(25)

while $R_A + R_C < C(\gamma_1 + \gamma_2)$ in all other points of the region of achievable rates. The points $L_A$ and $L_C$ are determined as:
\[
R_A(L_A) = C(\gamma_1), \quad R_C(L_A) = C\left(\frac{\gamma_2}{1 + \gamma_1}\right)
\]
\[
R_A(L_C) = C\left(\frac{\gamma_1}{1 + \gamma_2}\right), \quad R_C(L_C) = C(\gamma_2)
\]
(26)

For the rate pair at $L_A$, the packet $x_C$ is decoded first, it is then subtracted from the received signal and then $x_A$ is decoded. At the point $L_C$, these operations are reversed. Any other point $L$ on the line $L_AL_C$ has rates
\[
R_A(\lambda) = C\left(\frac{\gamma_1}{1 + \gamma_2}\right) + \lambda \left(C(\gamma_1) - C\left(\frac{\gamma_1}{1 + \gamma_2}\right)\right)
\]
\[
R_C(\lambda) = C(\gamma_2) + \lambda \left(C\left(\frac{\gamma_2}{1 + \gamma_1}\right) - C(\gamma_2)\right)
\]
(27)

\[
R_{AF}(\lambda) = \frac{NR_A(\lambda) + NR_C(\lambda)}{N + NR_A(\lambda)/C(\gamma_1)} = C(\gamma_1) - \frac{C(\gamma_2)}{C(\gamma_1) + R_C(\lambda)}
\]
(28)

where $0 \leq \lambda \leq 1$ can be the time–sharing parameter, see [12].

**Theorem 2:** The maximal two–way rate for the joint decode–and–forward (JDF) scheme is
\[
R^*_{JDF} = \begin{cases} 
C(\gamma_1) - \frac{2C(\gamma_1 + \gamma_2)}{2C(\gamma_1 + \gamma_2) + C(\gamma_1 + \gamma_2)} & \text{if } \gamma_1 \leq \gamma_2 \leq \gamma_1 + \gamma_2^2 \\
C(\gamma_1) - \frac{2C(\gamma_1 + \gamma_2)}{2C(\gamma_1 + \gamma_2) + C(\gamma_1 + \gamma_2)} & \text{if } \gamma_2 > \gamma_1 + \gamma_2^2
\end{cases}
\]
(29)

**Proof:** The starting point is the fact that the line segment $L_AL_C$ contains at least one rate pair $(R_A, R_C)$ that maximizes the two–way rate. We omit this proof as it can be done in a similar way as the part of the proof that follows. We consider two different cases, one for each region of $\gamma_2$.  

![Fig. 2. The convex hull of the rate pairs $(R_A, R_C)$ that are decodable by $B$ in Step 1. The dashed line denotes the rate pairs with $R_A = R_C$.](image)

1) Case $\gamma_1 \leq \gamma_2 \leq \gamma_1 + \gamma_2^2$: In this region of values for $\gamma_1, \gamma_2$ there is a value $\lambda_0$, such that:
\[
R_A(\lambda_0) = R_C(\lambda_0)
\]
(30)
i. e. the dashed line on Fig. 2 intersects with the segment $L_AL_C$. The value of $\lambda_0$ is determined as:
\[
\lambda_0 = \frac{2C(\gamma_2) - C(\gamma_1 + \gamma_2)}{2C(\gamma_1) + 2C(\gamma_2) - 2C(\gamma_1 + \gamma_2)}
\]
(31)

There are two subcases:

**Subcase** $\lambda < \lambda_0$. Here $R_C(\lambda) > R_A(\lambda)$ and the packet $D_{CA}$ sent by node $C$ contains more bits than the packet $D_{AC}$. After decoding both packets, the node $B$ pads the packet $D_{AC}$ with zeros to obtain $D_{AC}^p$, with $|D_{AC}^p| = |D_{CA}|$ and creates:
\[
D_B = D_{AC}^p \oplus D_{CA}
\]
(32)

Note again that the nodes $A$ and $C$ know a priori how many padding zeros are used. Since $\gamma_1 \leq \gamma_2$, in Step 2 of the JDF scheme the node $B$ broadcasts $D_B$ at a rate $C(\gamma_1)$. After receiving $D_B$, the node $A$ obtains $D_{CA} = D_{AC}^p \oplus D_B$ and the node $C$ obtains $D_{AC}^p = D_{AC} \oplus D_B$ and hence obtains $D_{AC}$. The total number of symbols used in the two steps is $N_{1,JDF}(\lambda) = N + NR_C(\lambda)/C(\gamma_1)$, such that the two–way rate is:
\[
R_{1,JDF}(\lambda) = \frac{NR_A(\lambda) + NR_C(\lambda)}{N + NR_C(\lambda)/C(\gamma_1)} = C(\gamma_1) - \frac{C(\gamma_1 + \gamma_2)}{C(\gamma_1) + R_C(\lambda)}
\]
(33)

since (25) holds for each $\lambda$. As $R_C(\lambda)$ decreases with $\lambda$, the value $R_{1,JDF}(\lambda)$ is maximized for $\lambda = \lambda_0$, where $\lambda_0$ is given by (31), such that $R_{1,JDF}(\lambda_0) = C(\gamma_1) - \frac{2C(\gamma_1 + \gamma_2)}{2C(\gamma_1 + \gamma_2) + C(\gamma_1 + \gamma_2)}$.

**Subcase** $\lambda > \lambda_0$. Here $R_A(\lambda) > R_C(\lambda)$ and hence $|D_{AC}| > |D_{CA}|$. The proof uses similar line of argument as in case 1 of the proof of theorem [1] and therefore we briefly sketch it. The first part of the packet $D_{AC}$ is XOR-ed with the packet $D_{CA}$ and the resulting packet is broadcasted at rate $C(\gamma_1)$. Then, the rest of the packet $D_{AC}$ is broadcasted at a higher rate $C(\gamma_2)$. The total number of symbols in the two steps is:
\[
N_{2,JDF}(\lambda) = N + NR_C(\lambda)/C(\gamma_1) + NR_A(\lambda) - R_C(\lambda)/C(\gamma_2)
\]
(34)
This leads to two–way rate of
\[ R_{2,\text{DF}}(\gamma) = \frac{NC(\gamma_1 + \gamma_2)}{N_{2,\text{DF}}(\gamma)} \] (35)

It can be shown that \( N_{2,\text{DF}}(\gamma) \) is monotonically decreasing with \( \gamma \), while \( R_{2,\text{DF}}(\gamma_0) = R_{1,\text{DF}}(\gamma_0) \), which proves that the maximal rate is achieved at \( \gamma = \gamma_0 \).

2) Case \( \gamma_2 > \gamma_1 + \gamma_2' \). In this case for any \( \lambda \leq 1 \) it holds that \( R_{C}(\lambda) > R_{A}(\lambda) \). Hence, we can use the transmission method for the subcase \( \lambda \leq \lambda_0 \), discussed above. The obtained two–way rate is again given by (33), which is monotonically increasing with \( \lambda \) and attains the maximum for \( \lambda = 1 \). Hence, the maximal two–way rate is:
\[ R_{1,\text{DF}}(\lambda = 1) = C(\gamma_1) \frac{C(\gamma_1 + \gamma_2)}{C(\gamma_1)} = C(\gamma_1) \] (36)

It can be shown that there are other pairs \( R_{C}, R_{A} \) that achieve the maximal two–way rate. Those pairs lie on the segment \( L_{A}L_{E} \), where \( L_{E} \) is the point where \( R_{A} = R_{C} = C(\gamma_1) \). Note that \( R_{1,\text{DF}} < C(\gamma_1) \) when \( \gamma_2 < \gamma_1 + \gamma_2' \).

C. Denoise–and–forward (DFN)

In the first step of this scheme, the nodes \( A \) and \( C \) transmit the packets \( x_{A} \) and \( x_{C} \) at rates \( R_{A} \) and \( R_{C} \) but we do not require that the node \( B \) is able to decode the packets \( x_{A} \) and \( x_{C} \). During the \( N \) symbols of Step 1, \( B \) receives the \( N \)–dimensional complex vector \( y_{B} \), where the \( m \)–th symbol of \( y_{B} \) is given by (7). If the selected rate pair \( (R_{A}, R_{C}) \) is not achievable for the multiple access channel (i. e. lies outside the convex region on Fig. [2]), then \( B \) cannot find unique pair of codewords \( (x_{A}, x_{C}) \), such that the triplet \( (x_{A}, x_{C}, y_{B}) \) is jointly typical. The concept of joint typicality is rather a standard one in information theory and the reader is referred to [12] for precise definition. For our discussion it is sufficient to say that \( (x_{A}, x_{C}, y_{B}) \) is jointly typical when the codeword \( (x_{A}, x_{C}) \) is likely to produce \( y_{B} \) at \( B \). When the pair \( (R_{A}, R_{C}) \) is not achievable over the multiple–access channel, then, upon observing \( y_{B} \), the node \( B \) has a set of codeword pairs \( J(y_{B}) \) such that:
\[ J(y_{B}) = \{(x_{A}, x_{C})|(x_{A}, x_{C}, y_{B}) \text{ is jointly typical}\} \] (37)

Lemma 1: Let \( y_{B} \) be a typical sequence. Let \( (x_{A}^{1}, x_{C}^{1}) \) and \( (x_{A}^{2}, x_{C}^{2}) \) be two distinct codeword pairs in \( J(y_{B}) \). If \( R_{A} \leq C(\gamma_{1}) \) and \( R_{C} \leq C(\gamma_{2}) \), then \( A \) and \( C \) can always select the codebooks such that \( x_{A}^{1} \neq x_{A}^{2} \) and \( x_{C}^{1} \neq x_{C}^{2} \). (38)

Proof: If \( B \) knows packet of \( C \), then \( A \) can transmit to \( B \) reliably up to the rate \( C(\gamma_{1}) \). We prove the lemma by contradiction. Let us assume that the contrary is true: \( x_{A}^{1} \neq x_{A}^{2} \) and \( x_{C}^{1} = x_{C}^{2} \). Now, assume that, after receiving \( y_{B} \), the node \( B \) is told by a genie–helper which is the codeword \( x_{C}^{1} \). Then, \( B \) would still have ambiguity whether \( A \) has sent \( x_{A}^{1} \) or \( x_{A}^{2} \). But that contradicts the fact that \( A \) can communicate reliably to \( B \) at a rate \( \leq C(\gamma_{1}) \) if \( x_{C} \) is known a priori to \( B \). ■

From this lemma it follows that, if in Step 2 \( B \) manages to send the exact value \( y_{B} \) (with no additional noise) to \( A \) and \( C \), then \( A \) (\( C \)) will be able to retrieve the packet sent by \( C \) (\( A \)) in Step 1. In the DFN scheme the node \( B \) maps \( y_{B} \) to a discrete set of codewords and, in Step 2 it broadcasts the codeword to which \( y_{B} \) is mapped. Such a mapping to discrete codewords is referred to as denoising. Let \( Y_{B} \) denote the set of typical sequences \( y_{B} \), each of size \( N \). Let \( A \) be a set of denoising codewords \( \{w_{B}(1), w_{B}(2), \ldots w_{B}(|A|)\} \), where \(|A| \) is the cardinality of the set. The denoising is defined through the following mapping:
\[ D : Y_{B} \mapsto A \] (39)

The codewords in \( A \) are random i. e. selected in a manner that achieves the capacity of the associated Gaussian channel. Upon observing \( y_{B} \) in Step 1, in Step 2 the node \( B \) broadcasts the codeword \( D(y_{B}) \). The mapping \( D \) should have the following property:

Property 1: Given the codeword \( D(y_{B}) \) and with known codeword \( x_{A} (x_{C}) \), the other codeword \( x_{C} (x_{A}) \) can be retrieved unambiguously.

Such a property enables \( A \) and \( C \) to successfully decode each other’s packets after Step 2. The important question is: For given \( (R_{A}, R_{C}) \) from Step 1, what should be the minimal size \(|A| \), such that Property 1 is satisfied? Assume that \( R_{C} > R_{A} \), then there are \( 2^{NRC} \) possible codewords that \( C \) can send in Step 1 vs. \( 2^{NR_A} < 2^{NRC} \) sent by \( A \). Clearly, the cardinality should be at least \(|A| \leq 2^{NRC} \), because otherwise it is impossible for \( A \) to reconstruct the codeword sent by \( C \).

In this paper we conjecture, without proof, that it is always possible to design the denoising by using a set of minimal possible cardinality that can satisfy the Property 1:
\[ |A| = \max(2^{NR_A}, 2^{NRC}) \] (40)

Such a choice is guaranteed to offer an upper bound on the two–way rate of DNF and is equal to the achievable rate of DNF if the conjecture is valid.

Theorem 3: The upper bound on the two–way rate for denoise–and–forward (DFN) is
\[ R_{\text{DFN}}^1 = C(\gamma_1) \] (41)

where \( \gamma_1 \) is the SNR of the weaker link to the relay.

Proof: The rate \( R_{A} = C(\gamma_{1}) \) is maximal possible, while the rate \( R_{C} = C(\gamma_{2}) \), where \( \gamma_{1} \leq \gamma \leq \gamma_{2} \). After the Step 1, the node \( B \) maps the received sequence \( y_{B} \) according to the denoising to \( D(y_{B}) \). As there are \(|A| = 2^{NRC} \) denoising codewords, each one is represented by \( NR_{C} \) bits. Since both \( A \) and \( C \) need to receive it, the codeword \( D(y_{B}) \) needs to be sent at a rate \( C(\gamma_{1}) \). The total duration of the two steps is \( N_{\text{DFN}} = N + \frac{C(\gamma_{1})}{C(\gamma_{1})} \) which makes the two–way rate:
\[ R_{\text{DFN}}^1 = \frac{NC(\gamma_{1}) + NC(\gamma_{2})}{N + \frac{C(\gamma_{1})}{C(\gamma_{1})}} = C(\gamma_1) \] (42)

\[ \blacksquare \]
This result implies that the node $C$ does not need to “fully load” the channel by setting $R_C = C(\gamma_1)$ and any value of $R_C \geq C(\gamma_1)$ will result in the maximal two–way rate. Hence, the higher transmission rate $R_C$ does not improve the two–way rate, as it accumulates more data at $B$ which needs to be broadcasted at a low rate in Step 2. Finally, while the JDF scheme achieves a two–way rate of $C(\gamma_1)$ only when $\gamma_2 \geq \gamma_1 + \gamma_1^2$, the DNF scheme achieves it even for $\gamma_2 = \gamma_1$.

V. NUMERICAL ILLUSTRATION

Fig. 3 and Fig. 4 depict the two–way rate vs. the SNR $\gamma_1$. In both figures, the $DF$ scheme is evaluated for two different values of the SNR on the direct link, $\gamma_0 = 0$ and $\gamma_0 = 10$. Fig. 3 shows the results when the SNR of the link $B - C$ is $\gamma_2 = \gamma_1$. As expected, the upper bound $R_{\text{DNF}}$ is always highest for all $\gamma_1$. While $R_{\text{AF}}$ is lower than $R_{\text{DF}}$ for low SNRs, at high SNR the noise amplification loses significance and thus AF achieves higher two–way rate than JDF. Also, note that the improvement of the direct link $\gamma_0$ brings significant increase of the two–way rate in the DF scheme. Fig. 4 shows the results when $\gamma_2 = \gamma_1 + \gamma_1^2$, the lowest value for $\gamma_2$ at which the rate of JDF becomes equal to the upper bound for DNF. Clearly, the curve for DNF remains the same as in Fig. 3 while the increased $\gamma_2$ is reflected in improved two–way rates for AF and DF. The improvement is larger for AF, which now slightly outperforms DF with $\gamma_0 = \frac{1}{10}$ at higher SNRs.

VI. CONCLUSION

We have investigated several methods that implement physical network coding for two–way relay channel. We have grouped the physical network coding schemes into two generic groups of 3–step and 2–step schemes, respectively. The 3–step scheme is Decode–and–Forward (DF), while we consider are three 2–step schemes Amplify–and–Forward (AF), Joint Decode–and–Forward (JDF) and Denoise–and–Forward (DNF). We have derived the achievable rates for DF, AF, and JDF, as well as an upper bound on the achievable rate of DNF. The numerical results confirm that no scheme can achieve higher two–way rate than the upper bound of DNF. Nevertheless, there are certain SNR configurations of the source–relay links under which the maximal two–way rate of JDF is identical with the upper bound of DNF. As a future work, we are first going to provide a proof that the upper bound for DNF is achievable. Another important aspect is investigation of the impact that the efficient broadcasting schemes [12] can have on the DF and JDF scheme. It is interesting to investigate how to design a 3–step scheme when the direct link is better than one of the source–relay links. Although some practical DNF methods have been outlined in [6], it is important to investigate how to perform DNF when different modulation/coding methods are applied. Finally, a longer–term goal is to investigate how the physical network coding can be generalized to the scenarios with multiple communicating nodes and multiple relays.

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