APPENDIX: THE GROMOV–GUTH–WHITNEY EMBEDDING THEOREM

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1. Summary

By using a different method of embedding manifolds in Euclidean space, the bound of Theorem A can be improved to achieve one tantalizingly close to Gromov’s linearity conjecture:

**Theorem A’.** Every closed smooth nullcobordant manifold of complexity $V$ has a filling of complexity at most $\varphi(V)$, where $\varphi(V) = o(V^{1+\varepsilon})$ for every $\varepsilon > 0$.

As with the original Theorem A, this holds for both unoriented and oriented cobordism.

Recall that the polynomial bound on the complexity of a nullcobordism follows from a quantitative examination of the method of Thom:

1. One embeds the manifold $M$ in $S^N$, with some control over the shape of a tubular neighborhood.
2. This induces a geometrically controlled map from $S^N$ to the Thom space of a Grassmannian; one constructs a controlled extension of this map to $D^{N+1}$.
3. Finally, from a simplicial approximation of this nullhomotopy, one can extract a submanifold of $D^{N+1}$ which fills $M$ and whose volume is bounded by the number of simplices in the approximation.

Part (2) is the result of the quantitative algebraic topology done to control Lipschitz constants of nullhomotopies. Abstracting away the method of embedding, we extract the following:

**Theorem.** Let $M^n$ be an oriented closed smooth nullcobordant manifold which embeds with thickness 1 in a ball in $\mathbb{R}^N$ of radius $R$; that is, there is an embedding whose exponential map on the unit ball normal bundle is also an embedding. Then $M$ has a filling of complexity at most $C(n,N)R^{N+1}$. (For unoriented cobordism, $C(n,N)R^N$ is sufficient.)

This is optimal in the sense that the asymptotics of the estimates in steps (2) and (3) cannot be improved. Then to prove Theorem A, we simply need the following estimate, which may also be of independent interest.

**Theorem B’.** Let $M$ be a closed Riemannian $n$-manifold of complexity $V$. Then for every $N \geq 2n + 1$, $M$ has a smooth 1-thick embedding $g : M \to \mathbb{R}^N$ into a ball of radius

$$R = C(n,N)\frac{1}{V^{1+\varepsilon}V^{1+\varepsilon}}(\log V)^{2n+2}.$$  

This then implies that for every $N$, $M$ has a filling of complexity at most

$$C(n,N)V^{1+\frac{1}{2n+2}}(\log V)^{(N+1)/(2n+2)},$$

proving Theorem A.

The embedding estimate is in turn derived from a similar estimate of Gromov and Guth [GG12] for piecewise linear embeddings of simplicial complexes. The combinatorial notion of thickness used in that paper does not immediately translate into a bound on the thickness of a smoothing. Rather, in order to prove our estimate we first prove a version of Gromov and Guth’s theorem, largely using their methods, with a stronger notion of thickness which controls what happens near every simplex. We then translate this into the smooth world using the following result.
**Theorem C'** (Corollary of [BDG17 Thm. 3]). Every Riemannian $n$-manifold of bounded geometry and volume $V$ is $C(n)$-bilipschitz to a simplicial complex with $C(n)V$ vertices with each vertex lying in at most $L(n)$ simplices. In particular, every smooth $n$-manifold of complexity $V$ has a triangulation with $C(n)V$ vertices and each vertex lying in at most $L(n)$ simplices.

**The PL picture.** In dimensions $< 8$ all PL manifolds are smoothable. Therefore Theorems A and C together imply that for $n \leq 6$, every PL nullcobordant manifold with $V$ vertices and at most $L$ simplices meeting at a vertex admits a PL filling with $C(n, L)\varphi(V)$ vertices and at most $L$ simplices meeting at a vertex, where $\varphi(V) = o(V^{1+\varepsilon})$ for every $\varepsilon > 0$. For $n = 3$, this complements the result of Costantino and D. Thurston [CT08] which gives bounded geometry fillings of quadratic volume without imposing restrictions on the local geometry of $M$.

On the other hand, in high dimensions the PL cobordism problem is still open, and poses interesting issues since unlike in the smooth category, $B_{PL}$ is not an explicit compact classifying space for PL structures. We hope to return to this in a future paper.

**So, is it linear?** Gromov’s linearity conjecture appears even more interesting now that we know that it is so close to being true. On the other hand, at least in the oriented case, linearity cannot be achieved by Thom’s method. Suppose that one could always produce “optimally space-filling” embeddings $M \hookrightarrow S^N$, that is, 1-thick embeddings in a ball of radius $V^{1/N}$. Even in this case, an oriented filling would have volume $C(n, N)V^{1+1/N}$.

Moreover, recent results of Evra and Kaufman [EK16] on high-dimensional expanders imply that, at least for simplicial complexes, the Gromov–Guth embedding bound is near optimal and space-filling embeddings of this type cannot be found. While $n$-manifolds are quite far from being $n$-dimensional expanders, it is possible that a similar or weaker but still nontrivial lower bound can be found. This would show that Thom’s method is not sufficient for constructing linear-volume unoriented fillings, either.

On the other hand, at the moment we cannot reject the possibility that it is possible to find linear fillings for manifolds by some method radically different from Thom’s. In particular, it is completely unclear how to go about looking for a counterexample to Gromov’s conjecture, although we believe that ideas related to expanders may play a key role.

2. PL embeddings with thick links

In [GG12], Gromov and Guth describe “thick” embeddings of $k$-dimensional simplicial complexes in unit $n$-balls, for $n \geq 2k+1$. They define the thickness $T$ of an embedding to be the maximum value such that disjoint simplices are mapped to sets at least distance $T$ from each other. [GG12 Thm. 2.1] gives a nearly sharp upper bound on the optimal thickness of such an embedding in terms of the volume and bounds on the geometry.

This condition is insufficient to produce smooth embeddings of bounded geometry, because as thickness decreases, adjacent 1-simplices of length $\sim 1$ may make sharper and sharper angles. In this section we show that Gromov and Guth’s construction can be improved to obtain embeddings that also have large angles. Recall that the link $lk \sigma$ of an $i$-simplex $\sigma$ inside a simplicial complex $X$ is the simplicial complex obtained by taking the locus of points at any sufficiently small distance $\varepsilon > 0$ from any point of $\sigma$ in all directions normal to $\sigma$. This complex contains an $(r - i)$-simplex for every $r$-simplex of $X$ incident to $p$. If $X$ is linearly embedded in $\mathbb{R}^n$, there is an obvious induced embedding $lk \sigma \rightarrow S^{n-i-1}$. We show the following:

**Theorem 2.1.** Suppose that $X$ is a $k$-dimensional simplicial complex with $V$ vertices and every vertex lying in at most $L$ simplices. Suppose that $n \geq 2k+1$. Then there are $C(n, L)$ and $\alpha(n, L) > 0$ and a subdivision $X'$ of $X$ which embeds linearly into the $n$-dimensional Euclidean ball of radius

$$R \leq C(n, L)V^{\frac{1}{2n}}(\log V)^{2k+2}$$
The linear extension to an embedding of $X$ and such that for any $i$-simplex $\sigma$ of $X'$, the induced embedding $\text{lk} \sigma \rightarrow S^{n-i-1}$ is $\alpha(n,L)$-thick.

**Proof.** The proof proceeds with the same major steps as in [GG12]. We first show that a random linear embedding which satisfies the condition that all links are thick, while not having the right thickness, is sparse in a weaker sense: most balls have few simplices crossing them. Gromov and Guth then show that the simplices can be bent locally, at a smaller scale, in order to thicken the embedding; this produces a linear embedding of a finer complex. We note that if the scale is small enough, this finer, bent embedding also has thick links.

We write $A \lesssim B$ for $A \leq C(n,L)B$ and $A \sim B$ to mean $B \lesssim A \lesssim B$. Following Gromov–Guth, we actually embed $X$ in a $V^{1-p}$-ball with thickness $\sim (\log V)^{(2k+2)}$; for simplicity, write $R = V^{1-p}$.

We start by choosing, uniformly at random, an assignment of the vertices of $X$ to points of $\partial B_R$ from those such that for some $\alpha_0(n,L) > 0$, the following hold:

1. Adjacent vertices are mapped to points at least distance $\alpha_0 R$ apart.
2. The linear extension to an embedding of $X$ has $\alpha_0$-thick links.

We call the resulting linear embedding $I_0(X)$. We can choose $\alpha_0$ so that this is possible since the thickness of the link of some vertex $v$ (and of incident higher-dimensional simplices) only depends on the placement of vertices at most distance 2 away. Moreover, this implies the following:

(*): The probability distribution of $v$ conditional on some prior distribution on the other vertices is pointwise $\lesssim$ the uniform distribution. This follows from the fact that this is true even when all vertices within distance 2 from $v$ are fixed.

This implies that given a $d$-simplex $\sigma$, the probability distribution of $\sigma$ (conditional on any distribution on the vertices outside $\sigma$) is likewise pointwise $\lesssim$ the uniform distribution where every vertex is mapped independently.

(†) If $d(v,w) \leq 2$, then $v$ and $w$ are mapped at least $c_0(n,L)R$ units apart. In particular, every embedded edge has length $\sim R$.

**Lemma 2.2.** With high probability, each unit ball $B_1(p) \subset B_R$ meets $\lesssim \log V$ simplices of $I_0(X)$.

**Proof.** By an argument of Gromov–Guth, the probability that a random $B_1(p)$ meets a fixed $d$-simplex $\sigma$ is $\lesssim V^{-1}$.

Therefore, the expected number of simplices hitting $B_1(p)$ is $\lesssim 1$. If each simplex hitting $B_1(p)$ was an independent event, then the probability that $S$ simplices meet $B_1(p)$ would be $\lesssim e^{-S}$; therefore, with high probability, for every $p$ the number of simplices hitting $B_1(p)$ would be $\lesssim \log V$. Indeed, complete independence is not necessary for this; the condition (*$*$) is sufficient.

This condition holds when the simplices have no common vertices. Therefore, we can finish with a coloring trick, as in Gromov–Guth. We color the simplices of $X$ so that any two simplices that share a vertex are different colors. This can be done with $(k+1)L$ colors. With high probability, the number of simplices of each color meeting $B_1(p)$ is $\lesssim \log V$. Since the number of colors is $\lesssim 1$, we are done. \hfill $\square$

Now we decompose each simplex into finer simplices, using the family of edgewise subdivisions due to Edelsbrunner and Grayson [EG00]. This is a family of subdivisions of the standard $d$-simplex with parameter $L$ which has the following relevant properties:

- All links of interior vertices are isometric, and all links of boundary vertices are isometric to part of the interior link.
- The subdivided simplices fall into at most $\frac{d}{2}$ isometry classes. In particular, all edges have length $\sim 1/L$. 

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When we apply the edgewise subdivision with parameter $L$, with the appropriate linear distortion, to $I_0(X)$, we get an embedding $I_0(X')$ of a subdivided complex $X'$ such that all edges have length $\sim 1$ by (1) and all links have thickness $\gtrsim \alpha_0$ and hence $\gtrsim 1$.

Now we use the following lemma of Gromov–Guth:

**Lemma 2.3.** For every $0 < \tau < 1$, there is a way to move the vertices of $X'$ by $\leq \tau$ such that the resulting embedding $I_\tau(X)$ is $\gtrsim \tau \cdot (\log V)^{(2k+2)}$-thick.

If we choose $\tau(n, L)$ sufficiently small compared to the edge lengths of $I(X')$, then there is an $\alpha(n, L)$ such that however we move vertices by $\leq \tau$, the links will still be $\alpha$-thick. Since these edge lengths are uniformly bounded below, this completes the proof.

3. **Thick smooth embeddings**

We now use Theorem 2.1 to build thick smooth embeddings of manifolds of bounded geometry.

**Theorem 3.1.** Let $M$ be a closed Riemannian $m$-manifold with $\text{geo}(M) \leq 1$ and volume $V$. Then for every $n \geq 2m + 1$, there is a smooth embedding $g : M \to \mathbb{R}^n$ such that

- $g(M)$ is contained in a ball of radius $R = C(m, n) V^{1/(m-n)} (\log V)^{2m+2}$.
- For every unit vector $v \in TM$,
  \[ K_0(m, n) R \leq |Dg(v)| \leq K_1(m, n) R. \]
- The reach of $g$ is greater than 1, that is, the extension of $g$ to the exponential map on the normal bundle of vectors of length $\leq 1$ is an embedding.

**Proof.** We prove this by reducing it to Theorem 2.1. That is, first we build a simplicial complex which is bilipschitz to $M$, with a bilipschitz constant depending only on $m$. We apply Theorem 2.1 to this complex to obtain a PL embedding, and then smooth it out, using the fact that PL embeddings in the Whitney range are always smoothable. The quantitative bound on the smoothing follows from the fact that the local behavior of the PL embedding comes from a compact parameter space, allowing us to choose from a compact parameter space of local smoothings.

Throughout this proof we write $A \lesssim B$ to mean $A \leq C(m, n)B$. This is different from the usage in the previous section.

The first step is achieved by the following result.

**Theorem 3.2.** There is a simplicial complex $X$ with at most $L = L(m)$ simplices meeting at each vertex and a homeomorphism $h : X \to M$ which is $\ell$-bilipschitz for some $\ell = \ell(m)$ when $X$ is equipped with the standard simplewise metric.

**Proof.** We start by constructing an $\varepsilon$-net $x_1, \ldots, x_V$ of points on $M$ for an appropriate $\varepsilon = \varepsilon(m) > 0$. We do this greedily: once we’ve chosen $x_1, \ldots, x_i$, we choose $x_{i+1}$ so that it is outside $\bigcup_{j=1}^i B_{\varepsilon}(x_i)$. In the end we get a set of points such that the $\frac{\varepsilon}{4}$-balls around them are disjoint and the $\varepsilon$-balls cover $M$.

Now, [BDG17, Theorem 3] in particular gives the following:

**Lemma 3.3.** If $\varepsilon(m)$ is small enough, there is a perturbation of $x_1, \ldots, x_V$ to $x'_1, \ldots, x'_V \in M$ and a simplicial complex $X$ with a bilipschitz homeomorphism $X \to M$ as well as the following properties:

- Its vertices are $x'_1, \ldots, x'_V$.
- It is equipped with the piecewise linear metric determined by edge lengths $d(x'_i, x'_j)$ which are geodesic distances in $M$.
- Its simplices have “thickness” $\geq C(m)$; this is defined to be the ratio of the least altitude of a vertex above the opposite face to the longest edge length. In particular, since the edge lengths are $\sim \varepsilon$, this means that each simplex is $C(m)$-bilipschitz to a standard one.
Moreover, suppose that \( g \) smooth embedding has \( \alpha \) \( C \) of possibilities which is also \( U \) \( k \) derivatives of its curvature tensor for every \( k \). This allows us, like in Lemma 6.1, to fix an atlas \( \mathcal{U} = \{ \phi_i : \mu \rightarrow M \} \) for \( M \), with the following properties:

1. The \( \phi_i (B_{\mu / 2}) \) also cover \( M \).
2. \( \mathcal{U} \) is the disjoint union of sets \( \mathcal{U}_1, \ldots, \mathcal{U}_r \) each consisting of pairwise disjoint charts.
3. The charts are uniformly bilipschitz, and the \( k \)th derivatives of all transition maps between charts are uniformly bounded depending only on \( m \) and \( n \).

Here \( \mu \) and \( r \) both depend only on \( m \) and \( n \). We construct our smoothing first on \( \mathcal{U}_1 \), then extend to \( \mathcal{U}_2 \), and so on by induction.

At each step of the induction, we use the following form of the weak Whitney embedding theorem \([\text{Hir76} \ \S2.2, \text{Thm. 2.13}]\): for \( s \geq 2r + 1 \), the set of smooth embeddings \( D^r \rightarrow \mathbb{R}^s \) is \( C^0 \)-dense in the set of continuous maps. Moreover, the set of smooth maps which restrict to some specific smooth map on a closed codimension zero submanifold is likewise dense in the set of such continuous maps \([\text{Hir76} \ \S2.2, \text{Ex. 4}]\).

The strategy is as follows. Note that the space of \( L \)-bilipschitz maps \( B_{\mu} \rightarrow \mathbb{R}^n \) up to translation is compact by the Arzelà–Ascoli theorem. At every stage, we also have a \( C^\infty \)-compact space of possible partial local smoothings. Then Whitney will allow us to choose an extension from a space of possibilities which is also \( C^\infty \)-compact.

We now give a detailed account of the inductive step. Suppose that we have defined a partial smooth embedding \( g : K \rightarrow \mathbb{R}^n \), where \( K \) is a compact codimension zero submanifold of \( M \) with

\[
(3.4) \quad \bigcup_{\phi \in \mathcal{U}_i \atop 1 \leq i < j} \phi \left( B_{\mu \cdot 2^{r-i-j}} \right) \subset K \subset \bigcup_{\phi \in \mathcal{U}_i \atop 1 \leq i < j} \phi (B_{\mu}).
\]

Moreover, suppose that \( g \) is \( \rho_{j-1} \)-close to \( f \) for some sufficiently small \( \rho_{j-1} \) depending on \( m \) and \( n \), and that for each \( \phi \in \mathcal{U}_i, \ i < j \), the partially defined function \( g \circ \phi \) is an element of a \( C^\infty \)-compact moduli space \( L_{j-1} \) of maps each from one of a finite set of subdomains of \( B_{\mu} \) to \( D^n \).

Fix a fine cubical mesh in \( B_{\mu} \); it should be fine enough that any transition function sends a distance of \( \mu / 2r \) to at least four times the diagonal of the cubes. The purpose of this mesh is to provide a uniformly finite set of subsets on which maps may be defined. Then, again by Arzelà–Ascoli, for any set \( K \) which is a union of cubes in this mesh, the space of potential transition maps \( K \rightarrow B_{\mu} \), satisfying the bounds on the covariant derivatives in all degrees is \( C^\infty \)-compact.

Fix \( \phi \in \mathcal{U}_j \). By the above, \( g|_{K \cap \phi(B_{\mu})} \circ \phi \), again restricted to the union of cubes on which it is fully defined (call this domain \( \hat{K} \subset B_{\mu} \)), is also chosen from a \( C^\infty \)-compact moduli space \( \mathcal{M}_1 \), whose elements are patched together from a bounded number of compositions of elements of \( L_{j-1} \) with transition maps as above. Of course, \( \mathcal{M}_1 \) consists of the unique map from the empty set.
Let $\mathcal{N}_j$ be the $C^0$-compact set of $L$-bilipschitz embeddings $B_{\mu(1-1/2r)} \to D^n$. Notice that the subset $\Delta \subset \mathcal{M}_j \times \mathcal{N}_j$ consisting of pairs whose $C^0$ distance is $\leq \rho_j-1$ is compact; this $\Delta$ contains the pair $(g|_{\hat{K}} \circ \phi, f \circ \phi)$.

Fix a smooth embedding $u : B_{\mu(1-1/2r)} \to D^n$. We say that $(\varphi, \psi) \in \Delta$ is $\varepsilon$-good for $u$, for some $\varepsilon > 0$, if:

- The $C^0$ distance between $u$ and $\psi$ is $< \rho_j$, where $\rho_j > \rho_j-1$ is fixed.
- The map interpolating between $\phi$ and $u$ via a bump function, only depending on $\hat{K}$, whose transition lies within the layer of cubes touching the boundary of $\hat{K}$, has reach $> \varepsilon$. (Here, we simply delete all boundary cubes outside of $B_{\mu(1-1/2r)}$ from the domain. Thus at this step the domain of $g$ actually recedes slightly; this is the motivation for the condition (3.4).)

For any fixed pair $(u, \varepsilon)$, these are both open conditions in $\Delta$, so there is an open set $V_{u, \varepsilon} \subseteq \Delta$ of good pairs $(\varphi, \psi)$. Moreover, since (by Whitney) we can always choose a $u$ which coincides on $\hat{K}$ with a given element of $\mathcal{M}_j$, these sets cover $\Delta$. Therefore we can take a finite subcover corresponding to a set of pairs $(u_i, \varepsilon_i)$. Taking a cover by compact subsets subordinate to this, we get a compact set of allowable extensions of elements of $\mathcal{M}_j$ to $B_{\mu(1-1/2r)}$; together with the modified sets of allowable maps on previous $U_i$’s (cut back so as to be defined on a domain of cubes) this makes $\mathcal{L}_j$.

We choose an extension of $g$ from the set of allowable extensions above. Doing this for every $\phi \in \mathcal{U}_j$ completes the induction step, giving some bound on the local geometry and reach by the compactness argument. Moreover, if we pick $\rho_j$ small enough compared to $\mu/2r$, then the embedding outside $\phi(B_{\mu})$ stays far enough away from the embedding inside. Nevertheless, all of these bounds become worse with every stage of the induction.

At the end of the induction, we have a smooth embedding of $M$. Every choice we made was from a compact set of local smoothings depending ultimately only on $m$ and $n$, which in turn controlled various bilipschitz and $C^k$ bounds. Thus the resulting submanifold $\hat{M} = g(M) \subset B_R$ has $\text{geo}(\hat{M}) \lesssim 1$. For the same reason, $g$ (as a map from $M$ with its original metric) has all directional derivatives $\sim R$. Moreover, since we didn’t move very far from $f$, points from disjoint simplices can’t have gotten too close to each other. This, together with the local conditions, shows that $\hat{M}$ has an embedded normal bundle of radius $\gtrsim 1$. By expanding everything by some additional $C(m, n)$ we achieve the bounds desired in the statement of the theorem.

Acknowledgements. The authors would like to thank Larry Guth and Sasha Berdnikov for stimulating correspondence during the course of this work, and two anonymous referees for helpful comments regarding the exposition.

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