Shear viscosity of hot QCD from transport theory and thermal field theory in real time formalism

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Abstract

We study shear viscosity in weakly coupled hot pure gauge field QCD theory basing on transport theory and the Kubo formula using the closed time path formalism (CTP) of real time finite temperature field theory. We show that the viscosity can be obtained as the integral of a retarded three-point function. Non-perturbative corrections to the bare one loop result can be obtained by solving a Schwinger-Dyson type integral equation for this vertex. This integral equation represents the resummation of an infinite series of ladder diagrams which all contribute to the leading-log order result. We show that this integral equation has exactly the same form as the linearized Boltzmann equation and explain the reason behind this formal equality.

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I. INTRODUCTION

Fluctuations occur in a system slight perturbed away from equilibrium. The responses to these fluctuations are described by transport coefficients which characterize the dynamics of long wavelength, low frequency fluctuations in the medium [1,2]. The investigation of transport coefficients in high temperature gauge theories is important in cosmological applications such as electroweak baryogenesis [3], and in the context of heavy ion collisions [4,5].

There are two basic methods to calculate transport coefficients: transport theory and response theory [6–12]. Using the transport theory method one starts from a local equilibrium form for the distribution function and performs an expansion in the gradient of the four velocity field. The coefficients of this expansion are determined from the classical Boltzman equation [12,13]. In the response theory approach one divides the Hamiltonian into a bare piece and a perturbative piece that is linear in the gradient of the velocity field. One uses standard perturbation theory to obtain the Kubo formula for the viscosity in terms of retarded Green functions [7,13]. These Green functions are then evaluated using equilibrium quantum field theory. As is typical in finite temperature field theory, it is not sufficient to calculate perturbatively in the coupling constant even for the weak coupling case. There are certain infinite sets of diagrams that contribute at the same order in perturbation theory and have to be resummed [13–15].

In this paper, we want to compare these two methods. The response theory approach allows us to calculate transport coefficients from first principles using the well understood methods of quantum field theory. On the other hand, the transport theory approach involves the use of the Boltzman equation which is itself derived from some more fundamental theory using, among other things, the quasiparticle approximation. In this sense, the response theory approach is more fundamental than the transport theory method. However, the response theory approach can be quite difficult to implement, even for a high temperature weakly coupled scalar theory, because of the need to resum infinite sets of diagrams [13,14]. These considerations motivate us to understand more precisely the connection between the more practical transport theory method, and the more fundamental response theory approach.

Some progress has already been made in this direction. It has been shown that keeping only terms which are linear in the gradient expansion in the transport theory calculation is equivalent to using the linear response approximation to obtain the usual Kubo formula for the shear viscosity in terms of a retarded two-point function, and calculating that two-point function using standard equilibrium quantum field theory techniques to resum an infinite set of ladder diagrams for scalar fields [13,15–17]. This result is not surprising since it has been known for some time that ladder diagrams give large contributions to $n$-point functions with ultra soft external lines [18–22]. Some studies on transport coefficients and ladder resummation for hot gauge theory using Kubo formula within the imaginary time formalism were reported in [23–25]. Due to the need for resummation, the diagrammatic approach to transport coefficients in hot field theories, based on Kubo formulae, has developed a reputation for being extremely cumbersome. This is particularly true for the ITF formalism with its need for analytical continuation and the corresponding complicated cutting rules [24] As a result, an alternative approach based on the Boltzmann equation in kinetic theory has recently become more popular [12]. With this approach, the transport coefficients in high
temperature theory has recently been studied up to leading order in [12]. But these calculations are not trivial either, and they cannot fully replace the kind of intuitive insight into the underlying physical mechanisms which is provided by a diagrammatic analysis. Our goal is therefore to calculate the shear viscosity of hot gauge theory from both transport method and diagrammatic method in real time formalism, and thus understand the connection of these two methods.

This paper will be organized as follows. In section II we define shear viscosity using a hydrodynamic expansion of the energy-momentum tensor. In section III we calculate shear viscosity using the transport theory method by performing a gradient expansion on the Boltzmann equation and obtaining a linearized Boltzmann equation. In section IV we calculate the shear viscosity using the Kubo formula which relates the shear viscosity to the two-point retarded Green function of the viscous-shear stress tensor. Starting from the Kubo formula we calculate the shear viscosity at leading-log approximation using standard techniques of finite temperature quantum field theory in real time formalism. We show that the shear viscosity can be obtained as an integral over a three-point vertex. This three-point vertex satisfies an integral equation, which represents the resummation of an infinite series of ladder diagrams. We show that this integral equation representing ladder resummations has the same form as the linearized Boltzmann equation and explain the reason behind this formal equality. We discuss our results and present our conclusions section V.

II. VISCOSITY

In a system that is out of equilibrium, the existence of gradients in thermodynamic parameters like the temperature and the four dimensional velocity field give rise to thermodynamic forces. These thermodynamic forces lead to deviations from the equilibrium expectation value of the energy momentum tensor which are characterized by transport coefficients like the thermal conductivity and the shear and bulk viscosities. In order to separate these different physical processes we decompose the energy-momentum tensor as,

\[ T^{\mu\nu} = \epsilon u^\mu u^\nu - p \Delta^{\mu\nu} + P^\mu u^\nu + P^\nu u^\mu + \pi^{\mu\nu}; \quad \Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu. \] (2.1)

The quantities \( \epsilon \), \( p \), \( P_\mu \) and \( \pi_{\mu\nu} \) have the physical meanings of internal energy density, pressure, heat current and viscous shear stress, respectively. The four vector \( u_\mu(x) \) is the four dimensional velocity field which satisfies \( u_\mu(x)u_\mu(x) = 1 \). The expansion coefficients are given by

\[ \begin{align*}
\epsilon &= u_\alpha u_\beta T^{\alpha\beta}, \\
p &= -\frac{1}{3} \Delta_{\alpha\beta} T^{\alpha\beta}, \\
P_\mu &= \Delta_{\mu\alpha} u_\beta T^{\alpha\beta}, \\
\pi_{\mu\nu} &= (\Delta_{\mu\alpha} \Delta_{\nu\beta} - \frac{1}{3} \Delta_{\nu\mu} \Delta_{\alpha\beta}) T^{\alpha\beta}.
\end{align*} \] (2.2)

The viscosity is obtained from the expectation value of the viscous shear stress part of the energy momentum tensor. We expand in gradients of the velocity field and write,
\[
\delta(\pi_{\mu\nu}) = \eta H_{\mu\nu} + \cdots \\
H_{\mu\nu} = \partial_{\mu} u_\nu + \partial_{\nu} u_\mu - \frac{2}{3} \Delta_{\mu\nu} \Delta_{\rho\sigma} \partial^{\rho} u^{\sigma}
\]

where \( \eta \) is the coefficient of the term that is linear in the gradient of the four velocity, defined as shear viscosity.

Throughout this paper we work with weak-coupled \( SU(3) \) Yang-Mills theory. The Lagrangian for this theory is

\[
\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \\
F^a_{\mu\nu} = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} - ig f^{abc} A^b_{\mu} A^c_{\nu}
\]

where the coupling constant \( g \ll 1 \). In local rest frame, the traceless spatial part of the energy-momentum tensor is

\[
\pi_{ij} = F^a_{i\mu} F^a_{j\mu} - \frac{1}{3} \delta_{ij} F^a_{\alpha\mu} F^a_{\alpha\mu}
\]

### III. SHEAR VISCOSITY FROM KINETIC THEORY

Kinetic theory and the Boltzmann equation can be used to study transport properties of dilute many-body systems. One assumes that, except during brief collisions, the system can be considered as being composed of classical particles with well defined position, energy and momentum. This picture is valid when the mean free path is large compared with the Compton wavelength of the particles. At high temperature, the typical mean free path of thermal excitations in hot QCD is \( \mathcal{O}(1/g^2 T \ln g^{-1}) \) and is always larger than the typical Compton wavelength of effective thermal oscillations which is \( \mathcal{O}(1/g^2 T) \). We introduce a phase space distribution function \( f(x, k) \) which describes the evolution of the phase space probability density for the fundamental particles comprising a fluid. In this expression and in the following equations the underlined momenta are on shell.

The Boltzmann equation of pure gauge field QCD describing the evolution of the gluon distribution function \( f(x, k) \) has the form:

\[
\left[ \frac{\partial}{\partial t} + v_k \cdot \frac{\partial}{\partial x} + \mathbf{F}_{\text{ext}} \cdot \frac{\partial}{\partial k} \right] f(k, x, t) = C[f].
\]

The external force \( \mathbf{F}_{\text{ext}} \) term will only be relevant in discussing the electrical conductivity. For calculations to leading logarithmic order in \( g \), and for the transport coefficient under consideration, it will be sufficient to include in the collision term \( C[f] \) only two-body scattering processes, so that

\[
C[f] = \frac{1}{2} \int_{123} d \Gamma_{12 \rightarrow 3k} [f_1 f_2 (1 + f_3) (1 + f_k) - (1 + f_1) (1 + f_2) f_3 f_k]
\]

with \( f_i := f(x, p_i) \), \( f_k := f(x, k) \). The symbol \( d \Gamma_{12 \rightarrow 3k} \) represents the differential transition rate for particles of momentum \( P_1 \) and \( P_2 \) to scatter into momenta \( P_3 \) and \( K \) and is given by...
\[ d\Gamma_{12\to3k} := \frac{1}{2\omega_k} |\mathcal{T}(p_1, p_2, p_3 | p_i \rangle \Pi_{i=1}^3 \frac{d^3p_i}{(2\pi)^32\omega_{p_i}}(2\pi)^4\delta(p_1 + p_2 - p_3 - k) \] (3.3)

where \( \mathcal{T} \) is the multiparticle scattering amplitude.

The form for \( f(x, k) \) in local equilibrium is,

\[ f^{(0)} = \frac{1}{e^{\beta(x)u(x)k} - 1} := n_k; \quad N_k := 1 + 2n_k \] (3.4)

We study the Boltzmann equation in the hydrodynamic regime where we consider times which are long compared to the mean free time and describe the relaxation of the system in terms of long wavelength fluctuations in locally conserved quantities. For a simple fluid without any additional conserved charges, the only locally conserved quantities are energy and momentum. To solve the Boltzman equation in this near equilibrium hydrodynamic regime, we expand the distribution function around the local equilibrium form using a gradient expansion. We go to a local rest frame in which we can write \( \bar{u}(x) = 0 \). Note that this does not imply that gradients of the form \( \partial_iu_j \) must be zero. In the local rest frame (2.3) becomes,

\[ \delta\langle\pi_{ij}\rangle = -\eta H_{ij} + \cdots \] (3.5)

\[ H_{ij} = \partial_iu_j + \partial_ju_i - \frac{2}{3}\delta_{ij}(\bar{\partial} \cdot \bar{u}) \]

In all of the following expressions we keep only linear terms that contain one power of \( H_{ij} \) and quadratic terms that contain two powers of \( H_{ij} \), since these are the only terms that contribute to the shear viscosity we are trying to calculate.

We write,

\[ f = f^{(0)} + f^{(1)} + \cdots \] (3.6)

with, \( f^{(1)} \sim k_\mu \partial^\mu f^{(0)} \). Using (3.4) we obtain,

\[ k_\mu \partial^\mu f^{(0)} = \beta n_k(1 + n_k) \frac{1}{2}I_{ij}(k)H_{ij} \]

\[ f^{(1)} := -n_k(1 + n_k)\phi_k; \quad \phi_k = \beta \frac{1}{2}B_{ij}(k)H_{ij} \] (3.7)

where we define

\[ \hat{I}_{ij}(k) = (\hat{k}i\hat{k}j - \frac{1}{3}\delta_{ij}) ; \quad I_{ij}(k) = k^2\hat{I}_{ij}(k) \] (3.8)

and write,

\[ B(\vec{k})_{ij} = \hat{I}_{ij}(k)B(\vec{k}) \] (3.9)

The viscous shear stress part of the energy momentum tensor is given by

\[ \langle\pi_{ij}\rangle = \int \frac{d^3k}{(2\pi)^32\omega_k} f(x, \vec{k}) (k_i k_j - \frac{1}{3}\delta_{ij}k^2) \] (3.10)
Using the expansion (3.6) and rotation invariance
\[ k_i k_j B(k) \rightarrow \frac{1}{3} \delta_{ij} k^2 B(k) \]
\[ k_i \hat{k}_l k_j \hat{k}_m B(k) \rightarrow \frac{1}{15} (\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) k^2 B(k) \]
we obtain the linear contribution from (3.7)
\[ \delta \langle \pi_{ij} \rangle = -\frac{\beta}{15} \int \frac{d^3k}{(2\pi)^3} n(1 + n) k^2 B(k) H_{ij}. \] (3.12)
The lowest order term gives zero. Comparing with (3.5) we have,
\[ \eta = \frac{\beta}{15} \int \frac{d^3k}{(2\pi)^3} n(1 + n) k^2 B(k) \] (3.13)
Thus we have shown that the shear viscosity can be obtained from the function \( B(k) \).

IV. VISCOSITY AND LADDER RESUMMATION IN REAL TIME FORMASIM

The Kubo formulae allow us to use quantum field theory to calculate nonequilibrium transport coefficients. The results should be the same as those obtained in the previous section using transport theory. The Kubo formula expresses the shear viscosity in terms of a retarded two-point Green function [7,13,15]
\[ \eta = \frac{1}{10} \frac{d}{dq_0} \text{Im}[\lim_{\vec{q} \to 0} G_R(Q)]|_{q_0=0} \]  

(4.1)

where \( G_R \) is the retarded two-point Green function of the viscous shear stress

\[ G_R(x, t; x', t'') = -i\theta(t - t'') [\pi_{ij}(x, t), \pi_{ij}(x', t'')] \]  

(4.2)

We obtain a diagramatic expansion for the viscosity from (4.1). We use the closed time path formulation of finite temperature field theory, and work in the Keldysh representation. Several reviews of this technique are available in the literature [27–31]. The closed time path integration contour involves two branches, one running from minus infinity to positive infinity just above the real axis, and one running back from positive infinity to negative infinity just below the real axis [29,31]. All fields can take values on either branch of the contour and thus there is a doubling of the number of degrees of freedom. It is straightforward to show that this doubling of degrees of freedom is necessary to obtain finite Green functions.

The two-point function or the propagator can be written as a \( 2 \times 2 \) matrix of the form

\[ D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \]  

(4.3)

where \( D_{11} \) is the propagator for fields moving along \( C_1 \), \( D_{12} \) is the propagator for fields moving from \( C_1 \) to \( C_2 \), etc.

The retarded and advanced propagators are given by the combinations,

\[ D_R = D_{11} - D_{12} \]
\[ D_A = D_{11} - D_{21}. \]  

(4.4)

The 1PI part of the two-point function, or the polarization insertion, is obtained by truncating legs. The retarded and advanced parts are given by,

\[ \Pi_R = \Pi_{11} + \Pi_{12} \]
\[ \Pi_A = \Pi_{11} + \Pi_{21}. \]  

(4.5)

The situation is similar for higher \( n \)-point functions. For example, the three-point function which is retarded with respect to the first leg is given by

\[ \Gamma_{R1} = \Gamma_{111} + \Gamma_{112} + \Gamma_{121} + \Gamma_{122}. \]  

(4.6)

The other three-point vertices that we will need are:

\[ \Gamma_{R2} = \Gamma_{111} + \Gamma_{112} + \Gamma_{211} + \Gamma_{212} \]
\[ \Gamma_{R3} = \Gamma_{111} + \Gamma_{121} + \Gamma_{211} + \Gamma_{221} \]
\[ \Gamma_F = \Gamma_{111} + \Gamma_{121} + \Gamma_{212} + \Gamma_{222} \]

In our case, we want to obtain a perturbative expansion for the correlation functions of composite operators \( D_R(X, Y) \) which appear in (4.1). We define the vertices \( \Gamma_{ij} \) as the vertex obtained by truncating external legs from the following connected vertex:
\[
\Gamma_{ij,\mu\nu}^{(ab)} = \langle T_c \pi_{ij}(X) A^a_{\mu}(Y) A^b_{\nu}(Z) \rangle 
\] (4.7)

where \( T_c \) is the operator that time orders along the closed time path contour. From this definition, one can draw the skeleton diagram of the two-point correlation function of the viscous shear stress tensors with a full vertex and full propagators as

![Diagram](image)

Fig. 1: Skeleton diagram of the two-point correlation function for shear viscosity from linear response. The dashed external line represents the composite operator \( \pi_{ij} \)

The coupling vertex between the operator \( \pi^{ij} \) and two gluons with incoming momenta \( P, K \) and indices \((\mu, a), (\nu, b)\) respectively can be read

\[
\Gamma_{\mu\nu,ij}^{(0)}(P, K) = -\delta^{ab}(\delta^{\mu\nu} \left( p^i k^j + k^i p^j - \frac{2}{3} \delta^{ij} p \cdot k \right) + P \cdot K \left( \delta^{ij} \delta^{\mu\nu} + \delta^{\mu\nu} \delta^{ij} - \frac{2}{3} \delta^{ij} \delta^{\mu\nu} \right) \\
- \left( p^i K^\mu \delta^{ij} + p^j K^\nu \delta^{ij} + P^\mu k^j \delta^{ij} + P^\nu k^i \delta^{ij} \right) - \frac{2}{3} \delta^{ij} (P^\mu k^\nu \delta^{ij} + K^\mu p^\nu \delta^{ij}) \right),
\]

(4.8)

where \( P = (p_0, p) \), \( P \cdot K = p_0 k_0 - p \cdot k \) and \( p_0 \) is the energy of the corresponding gluon.

For \( Q = 0 \), \( P = -K \) and on shell gluons \( K^2 = 0 \), \( \Gamma(K, Q, P) \equiv \Gamma(K) \), we have:

\[
P_{\mu\nu}^T(K) \Gamma_{\mu\nu,ij}^{(0)}(K) = -2\delta^{ab} I_{ij}(k),
\]

\[
P_{\mu\alpha}^T(K) \Gamma_{\mu\nu,ij}^{(0)}(K) P_{\nu\beta}^T(K) = -2\delta^{ab} I_{ij}(k) P_{\mu\alpha}^T(K) P_{\nu\beta}^T(K)
\]

(4.9)

where \( P_{\mu\nu}^T(K) \) is the transverse projector defined as \( P_{00}^T = P_{0i}^T = 0 \), and \( P_{ij}^T(k) = \delta_{ij} - \hat{k}_i \hat{k}_j \). \( \hat{k}_i \) represents unit vector of \( k \). Rotation invariance leads for the full vertex:

\[
P_{\mu\nu}^T(K) \Gamma_{ij,\mu\nu}^{(ab)}(K) = -2\delta^{ab} \Gamma(K) I_{ij}(k),
\]

\[
P_{\mu\alpha}^T(K) \Gamma_{ij,\alpha\beta}^{(ab)}(K) P_{\nu\beta}^T(K) = -2\delta^{ab} I_{ij}(k) P_{\mu\alpha}^T(K) P_{\nu\beta}^T(K) \Gamma(K)
\]

(4.10)

Since for the hard gluons, only the transverse part dominates the leading-log contribution, we can replace the full gluon propagator by its transverse part.

\[
D^{\alpha\mu}(K) \rightarrow P_{ij,\mu\nu}^T D(K)
\]

(4.11)

with
\[ D(K) = \frac{1}{K^2 - \Pi_T(K)} \]  

(4.12)

where \( \Pi_T(K) \) represents the transverse gluon self-energy.

Using the Feynman rules of the closed time path formalism (CTP) of real time finite temperature field theory, we write the two-point correlation function of the composite operators \( \pi_{ij} \) as

\[ G_{ab}(Q) = i \int dK \Gamma_{c a d}^{ij,\mu
u}(K, Q, -K - Q) i D_{bc}^{\alpha\mu}(K) i D_{db}^{\nu\beta}(K + Q) \Gamma^{(0)ij,\alpha\beta}(p) \tau_b \]  

(4.13)

shown diagramatically in Fig.1. Where \( \int dK \equiv \int \frac{d^4 k}{(2\pi)^4} \), and the indices \( \{a, b, c, d\} \) are Keldysh indices and take values \( \{1, 2\} \). The gluon propagators here are full resummed propagators. We perform the sum over Keldysh indices using the Mathematica program described in [32].

After performing the sums over Keldysh indices and making use of Eqs (4.9), (4.10), we find imaginary part of the retarded two-point function

\[ \text{Im} G_R(Q) = -\frac{4}{3} \int dK k^2 (N_{k+q} - N_k) [\Gamma_{R2}(K, Q, -K - Q) D_A(K) D_R(K + Q) + \text{h.c}] \]  

(4.14)

In obtaining this result we have used the fact that we will eventually take the limit \( Q \to 0 \) to obtain the viscosity (4.1). This limit gives rise to what is known as the pinch effect: using an obvious notation we write the pairs of propagators \( D_A(K) D_R(K + Q) \) as \( a_k r_{k+q} \) or \( r_k a_{k+q} \). Thus terms proportional to \( a_k a_{k+q} \) or \( r_k r_{k+q} \) can be dropped.

This is the so-called pinch limit, because the large terms occur when the contour is “pinched” between the poles of the two propagators, which gives rise to this enhancement factor in the denominator that is proportional to the imaginary part of the self-energy. We regulate the pinching singularity with the imaginary part of the self-energy and obtain \([15]\),

\[ r_k a_{k+q} \to -\frac{\rho_k}{2i \text{Im} \Pi^T_k}; \quad \rho_k = i(r_k - a_k) \]  

(4.15)

where \( \Pi^T \) is the retarded part of the transverse gluon self-energy.

Now we expand in \( q_0 \) and keep only the term proportional to \( q_0 \) since this term is the only one that contributes to (4.1). There is an over all thermal factor of the form \( N_k - N_{k+q} \) in (4.14). The expansion of this thermal factor is straightforward:

\[ N_k - N_{k+q} = 2q_0 \beta n_k (1 + n_k) + \cdots \]  

(4.16)

Consider the behaviour of the vertices when \( q_0 \to 0 \). We introduce the notation \( \Gamma(K, 0, -K) := \Gamma(K) \). It is straightforward to show that

\[ \Gamma_{R2}(K, Q, -K - Q) \to \text{Re} \Gamma_{R2}(K) \]  

(4.17)

Using these results to simplify (4.14) and substituting into (4.1) we obtain,
\[
\eta = \frac{\beta}{15} \int dK k^2 \rho_k n_k (1 + n_k) \left[ \frac{\text{Re} \Gamma_{R2}(K)}{\text{Im} \Pi_k^T} \right] \quad (4.18)
\]

Comparing with (3.13) we see that the results are identical if we use (3.9) and identify

\[
B(k) = \frac{\text{Re} \Gamma_{R2}(k)}{\text{Im} \Pi_k^T} \quad (4.19)
\]

where momentum \( K \) must be on new mass shell: \( \delta(K^2 - \text{Re} \Gamma_{R2}(k)) \)

It has been known for some time that in scalar theory, the set of diagrams which give the dominate contributions to the viscosity are the ladder diagrams. These diagrams contribute to the viscosity to the same order in perturbation theory as the bare one loop graph, and thus need to be resummed [13,26,15]. Following the similar power counting rule one can show that the set of ladder diagrams shown in Fig.2 give the dominate contributions to the viscosity in pure gauge field QCD theory at high temperature. This effect occurs for the following reason. It appears that the ladder graphs are suppressed relative to the bare vertex by extra powers of the coupling, which come from the extra vertex factors that one obtains when one adds rungs (vertical lines). However, these extra factors of the coupling are compensated for by a kinematical factor. This factor arises through the pinch effect which is described above. The addition of an additional rung in a ladder graph always produces an extra pair of propagators of the form \( a_k r_{k+q} \) or \( r_k a_{k+q} \). Products of this form contribute a factor which produces an enhancement. This factor occurs when the contour is “pinched” between the poles of the two propagators which gives rise to a contribution in the denominator that is proportional to the imaginary part of the self-energy.

In order to include those ladder diagrams, which dominate the contribution to the viscosity, we obtain the vertex \( \Gamma_{ij}(P, Q, -P - Q) \) as the solution to the integral equation shown in Fig.2.

Fig.2: Integral equation for ladder resummation, the blob stands for the full vertex \( \Gamma \). Thick lines represent for hard transverse gluons, where a finite width is needed to regularize the pinch singularity. While thin lines represent soft gluons, where HTLs resummed propagators are required.

Since we only calculate up to the leading-log order, in order to simplify the calculations, several points are in order:

1) We consider weak coupled degrees of freedom \( (g \ll 1) \);
2) Hard degrees of freedom with the momentum of order \( T \) for the rails dominate the contribution;
3) Small momentum transfer \((S \ll gT)\), the channel dominates the contribution. Namely, the momenta on the rungs (vertical lines) are soft of order \(gT\).

In order to get rid of color indices and Lorentz indices, we contract \(\delta^{ab} P^{\mu \nu}_T (K)\) with both sides of above equation and make use of

\[
\delta^{ab} \delta^{cd} f^{e'f} g^{e'f'} \delta^{g'd} f^{g'f'} = \delta^{ab} \delta^{cd} f^{e'f} g^{e'f'} = N_c^2
\]

and

\[
P^T_{\alpha \beta} (K) \gamma^\rho \gamma^\tau \gamma^\alpha P^T_{\rho \sigma} (P) \gamma^\sigma \gamma^{\tau'} = 8 P_P P_{P'}
\]

\[
P^T_{\lambda \delta} (R) \gamma^\kappa \gamma^\lambda P^T_{\kappa \delta'} (R) \gamma^{\delta \delta'} = 8 R_R R_{R'}
\]

where \(\gamma_{\alpha, \beta, \gamma} (P, T, K) = (P - T)_\gamma \delta_{\alpha \beta} + (T - K)_\alpha \delta_{\beta \gamma} + (K - P)_\beta \delta_{\alpha \gamma}\) comes from the triple-gluon vertex.

After some algebra calculations, we obtain a matrix integral equation in Keldysh space

\[
\Gamma_{\alpha \beta \kappa \lambda} (K, Q, -K - Q) = \Gamma^{(0), \alpha \beta \kappa \lambda} - 64 N_c^2 \int dP dR \Gamma^{\alpha \beta \kappa \lambda}_{d e c} (P, Q, -P - Q) D_{a d} (P) D_{e c} (P + Q)
\]

\[
\cdot D_{a c} (R + K - P) \tau^a_3 \gamma^\rho \gamma^\kappa \gamma^\lambda \gamma^\rho R_R D_{\tau^\kappa} (T) P_{P'} R_{P'} D_{\tau^\kappa'} (T) \tau^a_3
\]

(4.22)

We perform the sums over the Keldysh indices using the Mathematica program in [32] and simplify the result by taking \(Q\) to zero, keeping only the pinching terms, and using (4.15). We obtain,

\[
\Gamma^{R2}_{\alpha \beta \kappa \lambda} (K) = k_m k_l - \frac{1}{3} \delta_{m l} k^2 + \frac{3 N_c^2}{2} \int dP dR dR' (2\pi)^4 \delta^4 (P + R - R' - K)
\]

\[
|T|^2 \rho P \rho R R' \Gamma^{R2}_{\alpha \beta \kappa \lambda} (P) \frac{1}{\Im \Pi^{R2}_{R2} (P)} (1 + n_p) (1 + n_R) n_{R'} / (1 + n_k)
\]

(4.23)

where \(T^2 \equiv 16 g^4 |P_P r_{T S}^\kappa R_K|^2\), which is recognized as the matrix element squared for the collision of the hard gluon in \(t\) channel as shown in Fig. 3(a). Here \(r_{T S}^\kappa\) is the hard thermal loops resummed retarded propagator for the soft transfer momentum \(S\).

Fig. 3: (a) Two-body gluon-gluon elastic scattering in \(t\) channel. (b) Self-energy describing collisions in the (resummed) Born approximation. The thick lines refer to the hard colliding gluons. The thin lines with a blob denote soft gluon propagators dressed by the HTL.
Note that this integral equation is decoupled: the only three point vertex that appears is $\Gamma^{\text{lm}}_{R_2}$. To simplify this expression further we write $\Gamma_{ij} = \hat{I}_{ij}(p)\Gamma(P)$, use (4.19) and symmetrize the integral on the right hand side over the three integration variables $P$, $R'$ and $R$. We multiply and divide the left hand side by $\text{Im}\Pi^T_R(P)$ and replace this expression in the numerator by the HTL result [21,33,34]

$$\text{Im}\Pi^T_R(K) = \frac{1}{2} \left( \frac{1}{1 + n_k} \right) \int dP dR dR' (2\pi)^4 \delta^4(P + R - R' - K)|T|^2 \frac{\rho_p\rho_r\rho_{R'}}{(1 + n_p)(1 + n_{R'})n_R}$$

(4.24)

Rearranging we obtain [15],

$$I(k, k)_{lm} = \frac{N^2}{2} \int dP dR dR' (2\pi)^4 \delta(P + R - R' - K)|T|^2 \frac{(1 + n_p)(1 + n_{R'})n_R}{(1 + n_k)} \rho_p\rho_r\rho_{R'} \left[ B_{lm}(R') + B_{lm}(K) - B_{lm}(P) - B_{lm}(R) \right]$$

(4.25)

where we have used (4.17) and (4.19). When the delta functions are used to do the frequency integrals, this equation has exactly the same form as the equation obtained from the linearized effective Boltzmann equation (3.17), with a new mass shell describing thermal excitations. Comparing (3.13) and (3.17) with (4.18) and (4.25) we conclude that calculating shear viscosity of the hot pure gauge field QCD using effective transport theory by keeping only first order terms in the gradient expansion, is equivalent to using the Kubo formula obtained from response theory, with a three point vertex obtained by resumming ladder graphs.

Now we have established an interesting connection between kinetic theory and diagrammatic linear response theory for the nonabelian gauge theory. We obtain formally the same expressions for the viscosity in both approaches. It is desirable to understand the deeper reasons behind this formal equality.

The viscosity is obtained from the expectation value of the viscous shear stress tensor. There are two approaches to calculate the expectation value. The first method is from kinetic theory. With this approach, we use a distribution function $f$ to evaluate the the expectation value of the viscous shear stress tensor (see Eq.(3.10}). This distribution function is expanded around a local equilibrium function $f(0)$ up to the first order $f(1)$ of the gradient expansion. The second approach is by linear response theory. The expectation value is calculated from the non-equilibrium density matrix $\rho$ [7,17], which can be expanded around the local equilibrium density matrix $\rho_0$ up to the linear response order in gradient expansion, $\rho = \rho_0 + \rho_1$. The expectation value of a physics quantity does not depend on the approaches one uses, either from the kinetic theory by the distribution function, or from the linear response theory by the density matrix under the same approximation in gradient expansion. Therefore $B(k)$ from (3.13) and from (4.19) should be the same although they seem to describe completely two different things! The former one comes from the first-order distribution function $f^{(1)}$; The later, though related to a retarded 3-point function, is resulted from the first-order matrix density $\rho_1$. Physically, they both describe how far the system is perturbated away from the local equilibrium. In fact, both methods describe the first-order deviation of the system away from the local equilibrium, one of them using the description of the distribution function, while the other one using the density matrix.
On the other hand, both approaches also contain the same physics processes (diagrams), namely two-body collisions. In order to understand this point, let’s recall that the Boltzmann equation can be derived from the Schwinger-Dyson equation of the full 2-point functions in real-time formalism under certain approximations, like Wigner transformation and truncation schemes etc [21]. Different truncation schemes (or self-energy insertions) lead to different collision terms. The Boltzmann equation we used only contains $2 \to 2$ collision terms, which come from the cut of the self-energy diagrams containing $2 \to 2$ processes as shown in Fig.3. The cutting rules tell us that the cuts of self-energy diagrams give physics scattering amplitudes. The diagrammatic method from the integral equation of the full vertex is essentially like a resummation for the full 2-point correlation diagram of the stress tensor. Again in this resummation, only the diagrams containing $2 \to 2$ processes are included.

Because of these reasons above, we expect these two approaches should be able to give the same result for any theories under the same approximation.

V. SUMMARY AND CONCLUSIONS

We have studied shear viscosity of the nonabelian pure gauge theory using two different methods. The first method uses standard transport theory. We start from a local equilibrium form for the distribution function and perform a gradient expansion. We calculate the shear viscosity by expanding the Boltzmann equation and obtaining a linearized Boltzmann equation that can be solved consistently. The second technique uses the Kubo formula for linear response, which allows us to calculate the shear viscosity from the retarded two-point Green function of the viscous shear stress tensor. The transport theory calculation involves the use of the Boltzmann equation, which is itself obtained from some more fundamental theory. The response theory calculation uses the well known methods of equilibrium finite temperature quantum field theory and is, in this sense, more fundamental. However, the calculation is complicated by the need to resum infinite sets of diagrams at finite temperature. At leading order, it is well known that a correct calculation of the linear response coefficient in scalar theory involves the resummation of ladder graphs.

We have identified precisely which diagrams in hot pure gauge QCD need to be resummed by studying the connection between the transport theory calculation and the response theory calculation in real time formalism. We have shown that calculating shear viscosity using transport theory by keeping terms that are linear in the gradient of the velocity field in the expansion of the Boltzmann equation, is equivalent to calculating the same shear viscosity from quantum field theory at finite temperature using the linear response Kubo formula with a vertex given by a specific integral equation. This integral equation is the same as that recently found by Arnold, Moore and Yaffe [12] by analyzing the infrared divergences of the linearized collision integrals without screened interactions. This integral equation is also the same as the linearized Boltzmann equation derived by Blaizot and Iancu from the Kadanof-Baym equations by using a gradient expansion and keeping only linear terms [21]. Thus, this fact constitutes a non trivial check of the formalism we have used. This integral equation shows that the complete set of diagrams that need to be resummed includes infinite ladder graphs. We explain the reasons behind this formal equality between this two approaches. First, both approaches describe the first-order deviation of the system away from the local
equilibrium, one of them using the description of the distribution function, while the other one using the density matrix. Secondly both methods involve the same physics processes or diagramms (two body collisions). Therefore, We expect this interesting connection between these two approaches should exist for any other theories under the same approximations.

There are several directions for future work. The study here is only valid up to leading-log approximation of the transport coefficients. Going beyond to this approximation, one needs to include certain inelastic LPM suppressed splitting processes and thus needs to resum more complicated diagrams [12,35]. It would be interesting to generalize this work to the full leading order of the shear viscosity with real QCD with quark flavors in real time formalism.

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