Beyond the Einstein Equation of State: Wald Entropy and Thermodynamical Gravity

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We show that the classical equations of gravity follow from a thermodynamic relation, $\delta Q = T \delta S$, where $S$ is taken to be the Wald entropy, applied to a local Rindler horizon at any point in spacetime. Our approach works for all diffeomorphism-invariant theories of gravity. This suggests that classical gravity may be thermodynamic in origin.

INTRODUCTION

Black holes have often provided insights into the nature of quantum gravity and the structure of spacetime. For example, the holographic principle emerged through Gedanken-experiments that took a property of black holes — the sub-extensive scaling of Bekenstein-Hawking entropy — and applied it to arbitrary gravitational systems.

Another potentially profound insight comes from a paper by Jacobson [1]. Jacobson considered the puzzling fact that the laws of black hole mechanics, derived in classical general relativity, seem mysteriously to anticipate the laws of black hole thermodynamics, derived in semi-classical gravity. Rather than trying to explain how classical laws could “know about” quantum-mechanical ones, Jacobson reversed the logic, regarding the thermodynamics to be a premise rather than a consequence. Quite remarkably, by assigning the thermodynamic properties of black hole horizons to local light-cones in spacetime (not necessarily near a black hole), the Einstein equation re-appears as an equation of state. This seems to suggest, as indeed many theorists believe, that gravity is not a fundamental theory but originates in some kind of thermodynamic approximation.

The question arises whether this alluring result is somehow an artifact of Einstein gravity, or whether the connection between thermodynamics and gravity goes deeper, persisting also in general, higher-curvature theories of gravity. But extending the original derivation to higher-curvature theories is nontrivial, in part because that derivation makes use of the Raychaudhuri equation, whose usefulness is obscured in higher-curvature theories: the Ray-
The Raychaudhuri equation relates the derivative of the expansion of the horizon to the Ricci tensor, but a simple relation between the Ricci tensor and the stress tensor holds only for Einstein gravity. Moreover, in generic theories of gravity, the entropy is not simply proportional to the area.

In this paper, we obtain the classical gravitational equations from thermodynamics without making use of the Raychaudhuri equation. Specifically, we show that the classical equations of gravity follow directly from the Clausius relation, \( \delta S = \delta Q/T \). Here for \( S \) we use Wald’s definition of entropy, which is the entropy (in place of \( A/4 \)) that satisfies the first law of thermodynamics in higher-curvature theories. Our result suggests that classical gravity might have a quite intriguing thermodynamic origin.

**GENERAL THEORIES OF GRAVITY**

Consider a general diffeomorphism-invariant theory of gravity in any number of dimensions. For convenience, we will assume that the Lagrangian is a polynomial in the Riemann tensor but does not involve its derivatives. One may regard the Lagrangian formally as dependent on both the metric and the Riemann tensor even though of course the Riemann tensor, depends on the metric \([2, 3]\). Specifically, let the action be

\[
I = \frac{1}{16\pi} \int d^D x \sqrt{-g} L(g_{ab}, R_{abcd}) .
\]

We have set Newton’s constant to unity. Define

\[
P_{abcd} = \frac{\partial L}{\partial R_{abcd}} .
\]

\(P_{abcd}\) has the same algebraic symmetries as the Riemann tensor, including cyclicity. One then finds that the equation of motion that follows from (1) (supplemented by appropriate generalizations of Gibbons-Hawking-like boundary terms) is

\[
P_{a}{}^{cde} R_{bced} - 2 \nabla^c \nabla^d P_{acdb} - \frac{1}{2} L g_{ab} = 8\pi T_{ab} .
\]

For example, when the Lagrangian is \( L = f(R) \), we find

\[
P_{abcd} = \frac{1}{2} f'(R) \left( g^{ac} g^{bd} - g^{ad} g^{bc} \right).
\]

Thus, the equation of motion is

\[
f'(R) R_{ab} - \nabla_a \nabla_b f'(R) + \left( \Box f'(R) - \frac{1}{2} f(R) \right) g_{ab} = 8\pi T_{ab} .
\]
This reduces to Einstein’s equation when \( f(R) = R \).

Another example is Lovelock gravity \([4, 5]\), the most general extension of Einstein gravity for which the equations of motion do not contain derivatives of the Riemann tensor. The Lagrangian is \( L = \sum_{m=0}^{m_{\text{max}}} c_m L_m \), where \( c_m \) are constants of dimension \((\text{length})^{2m-2}\), which are arbitrary as far as gravity is concerned, and \( m_{\text{max}} = (D-2)/2 \) for even \( D \) dimensions and \( m_{\text{max}} = (D-1)/2 \) for odd \( D \). Each term \( L_m \) is made up of contractions of products of the Riemann tensor:

\[
L_m = \frac{1}{2^m} \delta_{i_1 \ldots i_{2m}}^{j_1 \ldots j_{2m}} R_{i_1 j_2}^{j_1 j_2} \ldots R_{i_{2m-1} i_{2m}}^{j_{2m-1} j_{2m}} ,
\]

Here the \( \delta \) symbol is the generalized Kronecker delta, defined as the sum over signed permutations of products of ordinary Kronecker deltas. The Einstein-Hilbert action with a cosmological constant is just a special case of the Lovelock action with \( c_1 = 1 \) and \( c_0 = -2\Lambda \). When \( D \leq 4 \), there are no other possible terms; the next term appears for \( D \geq 5 \). It is \( L_2 = R^2 - 4 R^{ab} R_{ab} + R^{abcd} R_{abcd} \), known as Gauss-Bonnet gravity, which appears in the low-energy effective action of certain string theories \([6, 7]\); its coefficient in ten-dimensional heterotic string theory is \( c_2 = +\alpha'/4 \). The Gauss-Bonnet action is a topological invariant in four dimensions, just as the Einstein-Hilbert action is a topological invariant in two dimensions. It is convenient to write (5) in the form

\[
L_m = Q^{abcd}_{(m)} R_{abcd} .
\]

Then \( P^{abcd} = mQ^{abcd}_{(m)} \), which has the nice property that \( \nabla_a P^{abcd} = 0 \). The equation of motion for Lovelock theory is

\[
\sum_{m=0}^{m_{\text{max}}} c_m \left( m \ Q_{(m)}^{acde} R^{b} _{cde} - \frac{1}{2} L_m g^{ab} \right) = 8\pi T^{ab} ,
\]

which follows easily from (3).

In each of these theories, one can associate an entropy with Killing or black hole horizons. For example, in place of \( A/4 \), the entropy in \( f(R) \) gravity is

\[
S_f = f'(R) \frac{A}{4} ,
\]

while for Gauss-Bonnet gravity, black holes have an entropy of

\[
S_{\text{G-B}} = \frac{1}{4} \int d^{D-2}x \sqrt{\sigma} \left( 1 + 2c_2 (D-2) R \right) ,
\]
where \((D-2)R\) is the scalar curvature of (the cross-section of) the horizon. We will show below that, as in Jacobson’s derivation of Einstein’s equation from \(S = A/4\ [1]\), varying these entropies and imposing the Clausius relation, \(\delta Q = T\delta S\), leads directly to the equations of classical gravity.

**Wald Entropy**

Wald [2, 3] and other authors [9, 10] have developed a powerful and elegant Lagrangian-based method for determining the entropy of a black hole with a Killing horizon. Wald’s method works for any diffeomorphism-invariant theory in any number of dimensions and does not require Euclideanization. Here we adopt a simplified version of the formalism [11].

Consider a generally covariant Lagrangian, \(L\), that depends on the Riemann tensor but does not contain derivatives of the Riemann tensor. Under the diffeomorphism \(x^a \rightarrow x^a + \xi^a\) the metric changes via \(\delta g_{ab} = -\nabla_a \xi_b - \nabla_b \xi_a\). By diffeomorphism-invariance, the change in the action, when evaluated on-shell, is given only by a surface term. This leads to a conservation law, \(\nabla_a J^a = 0\), for which we can write \(J^a = \nabla_b J_{ab}\), where \(J_{ab}\) defines (not uniquely) the antisymmetric Noether potential associated with the diffeomorphism \(\xi^a\) [2].

For a Lagrangian of the type \(L = L(g_{ab}, R_{abcd})\) direct computation shows that \(J_{ab}\) is given by (see [11])

\[
J_{ab} = -2P_{abcd}\nabla_c \xi_d + 4\xi_d \left(\nabla_c P_{abcd}\right),
\]

(10)

with \(P_{abcd} = \partial L/\partial R_{abcd}\). The Noether charge associated with a rigid diffeomorphism \(\xi^a\) is defined by integrating the Noether potential over a closed spacelike surface \(S\):

\[
Q = \int_S J_{ab} dS_{ab}.
\]

(11)

When \(\xi^a\) is a timelike Killing vector (the one whose norm vanishes at the Killing horizon), it turns out [2, 3] that the corresponding Noether charge is precisely the entropy, \(S\), associated with the horizon, apart from a few factors:

\[
S = \frac{1}{8\kappa} \int_S dS_{ab} J^{ab}.
\]

(12)

Here \(\kappa\) is the surface gravity of the black hole horizon. The integral for this “Wald entropy” can be evaluated over any spacelike cross-section of the Killing horizon [9]. In fact we can formally define the quantity \(S\) on any closed spacelike surface, \(S\), of codimension two (such
as a section of a stretched horizon), and only at the end take the limit in which that $S$
approaches a section of the Killing horizon. It can be shown, for example, that both (8) and
(9) are just special cases of Wald entropy.

**GRAVITATION FROM THERMODYNAMICS**

Now let us show how the classical equations of gravity, (3), arise thermodynamically.
(That the equations look thermodynamical has been shown for spherically-symmetric Love-
lock gravity [12].) The set-up is as follows [1]. Take any spacetime point $p$ and pick any
future-directed null vector $k^a$ emanating from $p$. In the vicinity of $p$, the plane orthogonal to
$k^a$ defines a local acceleration, or Rindler, horizon, $H$. Let $B_1$ be any spacelike neighborhood
of $p$ of codimension two that locally lives on the Rindler plane, and let $B_2$ be some further
section of the Rindler plane along $k^a$. Next, let $\xi^a$ be a future-directed approximate timelike
Killing vector that generates boosts and asymptotically approaches $k^a$. The orbits of $\xi^a$ and
the plane orthogonal to the acceleration vector of $\xi^a$ define a stretched horizon, $\Sigma$. As in
the membrane paradigm [13, 14], points on $H$ and points on $\Sigma$ can be put in one-to-one
correspondence by, say, ingoing null rays that pierce both surfaces. Let $S_i$ be the images of
$B_i$ on $\Sigma$ via this correspondence. See Fig. 1.

Let $\xi^a\xi_a = -\alpha^2$, where the norm $\alpha$ (which turns out to be a lapse) is taken to be constant
over $\Sigma$. This norm vanishes at $H$, a Killing horizon. Let $u^a$ be the proper velocity of a fiducial
observer moving along the orbit of $\xi^a$ i.e. $u^a = \left(\frac{d}{d\tau}\right)^a = \frac{1}{\alpha}\xi^a$, where $\tau$ is the proper time. Let
$n^a$ be the spacelike unit normal to $\Sigma$, pointing in the direction of increasing $\alpha$. Both $u^a$ and
$n^a$ map to $k^a$ in the limit that $\alpha \to 0$, for which $\Sigma \to H$.

After these preliminaries, we are ready to deduce the classical equations of gravity from
thermodynamics. The key idea [1] is to assign black hole thermodynamic properties to local
Rindler horizons [15]. The stretched horizon is assigned a local temperature, $T_{loc} = \kappa/2\pi\alpha$, as
well as the Wald entropy appropriate to the given theory of gravity.

By (10) and (12), the Wald entropy associated with a stretched horizon at time $\tau$ is

$$S = -\frac{1}{4\kappa} \int_{S(\tau)} dS_{ab} \left( F^{abcd} \nabla_c \xi_d - 2\xi_d \nabla_c F^{abcd} \right).$$

(13)
FIG. 1: The local Rindler horizon, $H$, of an arbitrary spacetime point $p$ is defined by a null vector, $k^a$. A stretched horizon, $\Sigma$, is defined by a timelike approximate Killing vector $\xi^a$ and has a normal vector field $n^a$. $B_i$ and $S_i$ are spacelike patches of codimension two that inhabit the planes of the Rindler and stretched horizons in the directions orthogonal to the figure, with $p$ contained in $B_1$.

Next, we vary the entropy along the timelike congruence. The entropy change is

$$\delta S = S(\tau_2) - S(\tau_1) = \frac{1}{4\kappa} \left[ \int_{S(\tau_2)} dS_{ab} \left( P^{abcd} \nabla_c \xi_d - 2\xi_d \nabla_c P^{abcd} \right) - \int_{S(\tau_1)} dS_{ab} \left( P^{abcd} \nabla_c \xi_d - 2\xi_d \nabla_c P^{abcd} \right) \right]$$

$$= \frac{1}{4\kappa} \int_{\Sigma} d\Sigma_a \nabla_b A^{ab} - \oint_{\Sigma} dS_{ab} A^{ab}, \quad (14)$$

In the last step, we have used Stokes’ theorem for an antisymmetric tensor field $A^{ab}$:

$$\int_{\Sigma} d\Sigma_a \nabla_b A^{ab} = - \oint_{\Sigma} dS_{ab} A^{ab}, \quad (15)$$

where our $\Sigma$ has the boundary $S = S(\tau_1) \cup S(\tau_2)$, and the minus sign comes about because $\Sigma$ is timelike. (To be explicit, our conventions here are $d\Sigma_a = n_a dA d\tau$ and $dS_{ab} = \frac{1}{2}(n_a u_b - u_a n_b) dA$, where the normal $n^a$ to the stretched horizon points outwards, away from the true horizon.) Recall that $P^{abcd}$ has the same algebraic symmetries as the Riemann tensor, including cyclicity. Using those symmetries, we find that

$$\delta S = \frac{1}{4\kappa} \int_{\Sigma} \left[ -\nabla_b \left( P^{abcd} + P^{acbd} \right) \nabla_c \xi_d + P^{abcd} \nabla_b \nabla_c \xi_d - 2\xi_d \nabla_b \nabla_c P^{abcd} \right] d\Sigma_a \quad (16)$$

So far we have not used any properties of $\xi^a$. Now we will assume that $\xi^a$ is an approximate timelike Killing vector. An exact Killing vector satisfies Killing’s equation $\nabla_b \xi_c + \nabla_c \xi_b = 0$ from which it follows that $\nabla_a \nabla_b \xi_c = R^d_{abc} \xi_d$. An approximate Killing vector indeed satisfies Killing’s equation locally. We will also assume the applicability of the second equation
within our local Rindler patch, a point we will discuss in the next section. The terms in parentheses drop out by symmetry. Using our assumption, we find

\[ T_{\text{loc}} \delta S = \frac{1}{8\pi \alpha} \int_{\Sigma} (P_{abcd} R_{dcbe} \xi^e - 2 \xi_d \nabla_b \nabla_c P^{abcd}) n_a d\tau dA \]  

(17)

On the other hand, the locally-measured energy or heat flux into the stretched horizon is

\[ \delta Q = + \int_{\Sigma} d\Sigma_a T^a u^e = \frac{1}{\alpha} \int_{\Sigma} dA d\tau n_a T_a^e \xi^e . \]  

(18)

Now \( S \) is not yet the entropy of the true horizon, \( H \), since we still have to take the limit in which the stretched horizon becomes null. Then both \( n^a \) and \( u^a \) become proportional to the null vector \( k^a \) (with the same proportionality constant). Equating \( \delta Q \) and \( T_{\text{loc}} \delta S \) and taking the null limit, we obtain

\[ (P_{cde} R_{bcde} - 2 \nabla^c \nabla^d P_{acdb}) k^a k^b = 8\pi T_{ab} k^a k^b . \]  

(19)

Since this holds for all null vectors \( k^a \) at \( p \), we infer that

\[ P_{a}^{cde} R_{bcde} - 2 \nabla^c \nabla^d P_{acdb} + \varphi g_{ab} = 8\pi T_{ab} , \]  

(20)

for some scalar function \( \varphi \). By demanding conservation of the stress tensor and using the Bianchi identities, we find that \( \varphi = -\frac{1}{2} L + \Lambda \), where \( \Lambda \) is an integration constant. Thus we see that imposing \( T_{\text{loc}} \delta S = \delta Q \) at any point in spacetime necessarily implies that

\[ P_{a}^{cde} R_{bcde} - 2 \nabla^c \nabla^d P_{acdb} - \frac{1}{2} L g_{ab} + \Lambda g_{ab} = 8\pi T_{ab} . \]  

(21)

With the cosmological constant appearing as an integration constant, this is precisely the classical equation of motion, (3), for our theory of gravity.

**DISCUSSION**

We have shown that the equations of classical gravity follow from thermodynamics. Our derivation did not require the Raychaudhuri equation. Moreover, since we started with the Wald entropy, we could go beyond the Einstein equation to the equations of motion of general theories of gravity. Since these were obtained from the Clausius relation, they can be regarded as equations of state — relations between thermodynamic state variables.
Satisfying as this is, there remain some loose ends. One observation \[16\] is that the derivation of the Wald entropy itself relies on the equations of motion being obeyed. Although our approach never explicitly invokes the equations of motion, it is still unclear whether any derivation, including Jacobson’s original calculation, that begins with an on-shell expression which agrees with Wald entropy (such as $A/4$) is implicitly assuming the answer, or whether that is simply self-consistency. Another technical concern is our use of the equation $\nabla_a \nabla_b \xi_c = R_{abcd} \xi^d$. This equation is obviously true when $\xi^a$ is an exact Killing vector but to what extent can one trust it when $\xi^a$ is an approximate Killing vector? For a general spacetime, Riemann normal coordinates can be applied to any local patch. In such coordinates an approximate boost Killing vector looks like $x \partial_t + t \partial_x$. However, such a vector does not obey $\nabla_a \nabla_b \xi_c = R_{abcd} \xi^d$ in general, so our assumption cannot be satisfied through coordinate choices alone. In this light, it is interesting that for $f(R)$ theories, previous work has found that the Clausius relation does not give the equations of motion but also has additional terms \[17\]; these have been interpreted as non-equilibrium effects. Perhaps the failure of $\nabla_a \nabla_b \xi_c = R_{abcd} \xi^d$ to hold may be traced to such effects. In that case, our derivation may indicate how to determine potential non-equilibrium terms for a general theory of gravity. (On the other hand, we also make use of this equation for Einstein gravity where there are no such terms, so perhaps it is an innocuous assumption.) It would be interesting to understand this better, as well as to connect our method to previous approaches \[17\].

While this paper was being prepared, the preprint \[18\] appeared, claiming similar conclusions; unfortunately, among other things, their starting formula for entropy (equation 9 in \[18\]) is manifestly incorrect, leaving the result in doubt.

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