A NOTE ON CAUCHY’S FORMULA

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Abstract. We use the correlation functions of vertex operators to give a proof of Cauchy’s formula

\[
\prod_{i=1}^{K} \prod_{j=1}^{N} (1 - x_i y_j) = \sum_{\mu \subseteq [K \times N]} (-1)^{|\mu|} s_{\mu}(x)s_{\mu'}(y).
\]

As an application of the interpretation, we obtain an expansion of \(\prod_{i=1}^{\infty} (1 - q^i)^{-1}\) in terms of half plane partitions.

1. INTRODUCTION

The Schur functions form a distinguished orthonormal basis in the ring of symmetric functions [13] with a number of applications, among which the most prominent is perhaps in the representation theory of both the symmetric and general linear groups [18]. One important identity in the theory is the Cauchy formula

\[
\prod_{i=1}^{K} \prod_{j=1}^{N} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y),
\]

where \(s_{\lambda}(x)\) is the Schur function in \(x\)'s and the sum is over all partitions \(\lambda\) with length \(l(\lambda) \leq \min\{K, N\}\). Foda, Wheeler and Zuparic [4] have used free fermions to study Schur functions and gave a physical interpretation of the limit of (1.1) using plane partitions, and the underlying algebraic structure is an infinite dimensional Heisenberg algebra with central charge 1. This is partly based on the vertex operator approach to symmetric functions [8,9].

In [11,17], charged free bosonic system provides a different Heisenberg algebra with central charge \(-1\): \(\mathcal{H} = \{h_n\}_{n \in \mathbb{Z}}\) with the commutation relation \([h_m, h_n] = -m\delta_{m,-n}\). On the Fock space \(\mathcal{V} \simeq \mathbb{C}[h_{-1}, h_{-2}, \ldots]\) (resp. the dual space \(\mathcal{V}^*\)) of the Heisenberg algebra \(\mathcal{H}\) generated by the vacuum vector \(|0\rangle\) (resp. dual vacuum \(\langle 0|\)), we can introduce the fermionic field \(\phi(z)\) to obtain a base \(\{|\lambda\rangle\}\) of \(\mathcal{V}\) (resp. \(\{\langle \lambda|\}\) of \(\mathcal{V}^*\)) that satisfies the skew-orthogonality:

\[
(|\lambda\rangle, |\mu\rangle) = \langle \lambda|\mu\rangle = (-1)^{|\lambda|} \delta_{\lambda,\mu'},
\]

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where $\mu'$ is the conjugate of partition $\mu$.

In this paper, we discuss an alternative approach to understand a related Cauchy identity by viewing the Schur functions as skew-orthogonal basis of the symmetric functions. With the help of the half-vertex operator $\phi^+(x) = \exp\left(\sum_{n=1}^{\infty} \frac{k-n}{n} x^n\right)$ we revisit/reprove a variant of Cauchy’s formula

\[
\prod_{i=1}^{K} \prod_{j=1}^{N} (1 - x_i y_j) = \sum_{\mu \in [K \times N]} (-1)^{|\mu|} s_{\mu}(x) s_{\mu'}(y),
\]

where the sum runs over all partitions $\mu$ with $\mu_1 \leq N$, $\mu'_1 \leq K$. The treatment is completely self-contained and offers new perspective to understand the dynamic procedure of the vertex operator action (see Prop. 2.4).

Plane partitions are two-dimensional analogues of ordinary partitions. They naturally appear in many problems of statistical physics and quantum field theory (see [1] and references therein). The well-known MacMahon generating function for plane partitions [13, 16] is

\[
\sum_{\pi} q^{||\pi||} = \prod_{i=1}^{\infty} \frac{1}{(1 - q^i)^i},
\]

where $\pi$ runs over all plane partitions. It has a physical interpretation via the Schur process and the KP hierarchy [14] as well as the free fermion system [4]. In the same spirit, we introduce half plane partitions, which form a special class of interlacing partition chains (lower triangular part of plane partitions), and use them to give combinatorial interpretations of $\prod_{i=1}^{K} \prod_{j=1}^{N} (1 - q^{i+j})$ as well as $\prod_{i=1}^{\infty} (1 - q^i)^{i-1}$. One of our results is that

\[
\sum_{\lambda} (-1)^{|\lambda|} \sum_{\{\lambda \rightarrow \pi\}} \sum_{\{\lambda' \rightarrow \tilde{\pi}\}} q^{||\pi||+||\tilde{\pi}||} = \prod_{i=1}^{\infty} (1 - q^i)^{i-1},
\]

where $\lambda \rightarrow \pi$ (resp. $\lambda' \rightarrow \tilde{\pi}$) runs through all interlacing chains associated with the half-plane partition $\pi$ (resp. $\tilde{\pi}$).

We remark that the vertex operator $\phi(z)$ is in fact a reformulated Bernstein operator (cf. [8, 20]) for the Schur functions. The action of the half-vertex operator $\phi^+(x)$ on Schur functions can be used to derive Macdonald’s skew Schur functions. Bernstein operator can also be formulated in plethystic manner [2, 3, 6, 12, 19], and another combinatorial formulation can be found in [7, 15].

The paper is organized as follows. In section 2, we consider the charged free bosonic system and study an infinite-dimensional Heisenberg algebra with negative central charge, which is different from the traditional treatment (cf. [8]). We then introduce the field operator $\phi(z)$ to obtain a base of the ring of symmetric functions. Through the dynamic action of the vertex operator, we show that the

\[\text{In the remainder, we use } \pi \text{ to denote a half plane partition.}\]
Cauchy identity follows naturally. In section 3, we use half plane partitions to express \( \prod_{i=1}^{\infty} (1 - q^i)^{i-1} \) by the identities from section 2.

2. charged free bosons and Cauchy’s identities

Let \( \varphi_i, \varphi_i^* \) (\( i \in \mathbb{Z} \)) be the charged free bosons satisfying the commutation relations:

\[
[\varphi_i, \varphi_j] = \delta_{i,-j}, \quad [\varphi_i, \varphi_j^*] = [\varphi_i^*, \varphi_j^*] = 0,
\]

where \([A, B] = AB - BA\) is the commutator. Their generating functions are

\[
\varphi(z) = \sum_{i \in \mathbb{Z}} \varphi_i z^{-i-1}, \quad \varphi^*(z) = \sum_{i \in \mathbb{Z}} \varphi_i^* z^{-i}.
\]

Let \( \mathcal{M} \) (resp. \( \mathcal{M}^* \)) be the (resp. dual) Fock space generated by the vacuum vector \( |0\rangle \) (resp. \( \langle 0| \)) defined by

\[
\varphi_i |0\rangle = \varphi_i^{*+1} |0\rangle = 0, \quad i \geq 0 \text{ (resp. } \langle 0| \varphi_i = \langle 0| \varphi_i^{*+1} = 0, \quad i < 0).\]

Define the bosonic operators \( h_n = \sum_{i = -\infty}^{\infty} : \varphi_{-i} \varphi_{i+n}^* : \), where the normal ordering \( : \) moves the factor annihilating \( |0\rangle \) to the right. Then \( \{h_n |n \in \mathbb{Z}\} \) generates the Heisenberg algebra \( \mathcal{H} \) with central charge \(-1\) [17, p7]:

\[
[h_m, h_n] = -m \delta_{m,-n}.
\]

For completeness, we verify (2.4) as follows. Note that \( h_n = \sum_{i \geq -n+1} \varphi_{-i} \varphi_{i+n}^* + \sum_{i \leq -n} \varphi_i^* \varphi_{i+n} \). It follows from \([AB, C] = A[B, C] + [A, C]B\) that

\[
[h_i, \varphi_j] = -\varphi_{i+j}, \quad [h_i, \varphi_j^*] = \varphi_{i+j}^*.
\]

Then we have that

\[
[h_m, h_n] = \sum_{i \geq -n+1} [h_m, \varphi_{-i} \varphi_{i+n}^*] + \sum_{i \leq -n} [h_m, \varphi_i^* \varphi_{i+n}] = -\sum_{i \geq -n+1} \varphi_{-i+m} \varphi_{i+n}^* + \sum_{i \geq -n+1} \varphi_{-i} \varphi_{i+m+n}^* + \sum_{i \leq -n} \varphi_{i+m+n}^* \varphi_{-i} - \sum_{i \leq -n} \varphi_i \varphi_{i+n}^* - \sum_{i \leq 0} \varphi_i^* \varphi_{m+n-i}
\]

\[
= -\sum_{i \geq 1} \varphi_{m+n-i} \varphi_i^* + \sum_{i \geq m+1} \varphi_{m+n-i} \varphi_i^* + \sum_{i \leq m} \varphi_i^* \varphi_{m+n-i} - \sum_{i \leq 0} \varphi_i \varphi_{m+n-i}
\]

\[
= -\sum_{i \geq 1} \varphi_{m+n-i} \varphi_i^* + \sum_{i \geq m+1} \varphi_{m+n-i} \varphi_i^* + \sum_{i \leq m} \varphi_i^* \varphi_{m+n-i} - \sum_{i \leq 0} \varphi_i \varphi_{m+n-i}
\]

(2.5)
We have used the convention that $\sum [\varphi_{m+1}, \varphi_i^+] = 0$.

The Fock space $V$, generated linearly by the left action of $\mathbb{C}[h_{-1}, h_{-2}, h_{-3}, \ldots]$ on $|0\rangle$, is a subspace of $M$. Similarly, $M^*$ has a subspace $V^* = \langle 0 | \mathbb{C}[h_1, h_2, h_3, \ldots]$. It is known that $V$ (or $V^*$) is the unique left (or right) irreducible representation of the Heisenberg algebra $H$. The following is clear.

**Proposition 2.1.** The charged free bosons carry an anti-involution $\omega$ defined by \( [11] \)

\begin{equation}
\omega(\varphi_i) = \varphi_i^+, \quad \omega(\varphi_i^+) = \varphi_i.
\end{equation}

Subsequently one has that $\omega(h_n) = h_{-n}$.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of weight $|\lambda| = \sum \lambda_i$ is a set of weakly decreasing nonnegative integers. Non-zero $\lambda_i$ are called parts of $\lambda$, and the number of parts is the length of $\lambda$, denoted by $l(\lambda)$. Sometimes we also list the parts in ascending order: $\lambda = (1^{m_1}2^{m_2} \cdots)$ and define $z_\lambda = \prod_i i^{m_i} m_i!$. The conjugate partition $\lambda'$ is defined by

\begin{equation}
\lambda'_i = \text{Card}\{j : \lambda_j \geq i\}.
\end{equation}

In particular, $\lambda'_1 = l(\lambda)$ and $|\lambda'| = |\lambda|$. Pick the rectangle $[N \times M]$ containing the Young diagram of $\lambda$, i.e., $\lambda_1 \leq M$, $\lambda'_1 \leq N$, for which we often write $\lambda \subseteq [N \times M]$. In particular, $\lambda \subset [N \times \infty]$ means the set of partitions $\lambda$ with $l(\lambda) \leq N$. Let $P$ be the set of partitions. A partition $\mu = (\mu_1, \ldots, \mu_{l+1})$ is said to interlace the partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, written as $\mu \succ \lambda$, if

\begin{equation}
\mu_i \geq \lambda_i \geq \mu_{i+1}
\end{equation}

for all $1 \leq i \leq l$. As a result $\mu \geq \lambda$ in the dominance order.

Choose the normalization $\langle 0 | 1 \rangle = 1$, and define the inner product of $x|0\rangle, y|0\rangle \in V$ via

\begin{equation}
(x|0\rangle, y|0\rangle) = \langle 0 | \omega(x) y | 0 \rangle,
\end{equation}
and extend bilinearly to the whole space. Thus \( (h_\lambda|0), h_\mu|0) = (-1)^{\ell(\lambda)} \delta_{\lambda,\mu} z_\lambda \), where \( h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots \).

Let \( \Lambda = \mathbb{Q}[x_1, x_2, \ldots]^{S_\infty} \) be the ring of symmetric functions in the \( x_n \). For each integer \( k \geq 0 \), we define the complete symmetric function \( s_k(x) \) \(^{[13]}\) in infinitely many variables \( x_1, x_2, \ldots \) by the generating function

\[
\sum_{k=0}^{\infty} s_k(x) z^k = \prod_{i=1}^{\infty} \frac{1}{1 - x_i z}.
\]

For convenience, set \( s_{-k}(x) = 0 \) for \( k > 0 \). To each partition \( \lambda \) we define the Schur function \( s_\lambda(x) \) \(^{[13]}\) by the Jacobi-Trudi formula

\[
s_\lambda(x) = \det(s_{\lambda_i - i+j}(x))_{1 \leq i, j \leq l(\lambda)}.
\]

It is well-known that

\[
\Lambda = \mathbb{Z}[s_1(x), s_2(x), \ldots] = \sum_{\lambda \in P} \mathbb{Z} s_\lambda(x).
\]

For the rest of the paper, we denote by \( s_\lambda\{x\} \) the Schur function in finitely many variables \( \{x\} = \{x_1, x_2, \cdots, x_K\} \). It is known that \(^{[13]} (3.1), (5.9), (5.11)\)

\[
(2.10) \quad s_\mu\{x\} = \sum_{\nu < \mu} s_\nu\{\bar{x}\} x_K^{\mu - \nu},
\]

\[
(2.11) \quad s_\mu\{x\} = 0, \quad l(\mu) > K,
\]

where \( \{x\} = \{x_1, \cdots, x_K\}, \{\bar{x}\} = \{x\}\backslash\{x_K\} \).

Introduce the vertex operator (cf. \(^{[5]}\) for general information)

\[
\phi(z) = \sum_{i \in \mathbb{Z}} \phi_i z^{-i} = \phi^+(z) \phi^-(z^{-1}) = \exp \left( \sum_{n=1}^{\infty} \frac{h_n}{n} z^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{h_n}{n} z^{-n} \right),
\]

where \( \phi^\pm(z) = \exp \left( \sum_{n=1}^{\infty} \frac{h_n}{n} z^n \right) \). Then by Prop. \(2.1\)

\[
\omega(\phi^\pm(z)) = \phi^\mp(z), \quad \omega(\phi(z)) = \phi(z^{-1}),
\]

i.e., \( \omega(\phi_i) = \phi_{-i} \). Clearly \( [\phi^\pm(z), \phi^\pm(w)] = 0 \). It follows from direct vertex operator calculation that for \( |zw| < 1 \)

\[
(2.12) \quad \phi^-(z) \phi^+(w) = (1 - zw) \phi^+(w) \phi^-(z).
\]

We also have

\[
(2.13) \quad \langle 0 | \phi_{-n} = \phi_n | 0 \rangle = 0, \quad n > 0.
\]
For partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, we denote $|\lambda\rangle = \phi_{-\lambda_1} \cdots \phi_{-\lambda_l}|0\rangle$ and $\langle \lambda | = \langle 0 | \phi_{\lambda_1} \cdots \phi_{\lambda_l}$. We also define the element $\chi_m$ by the generating function

$$\phi^+(z) = \exp \left( \sum_{n=1}^{\infty} \frac{h_n}{n} z^{n} \right) = \sum_{m=0}^{\infty} \chi_m z^m. \tag{2.14}$$

And for partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, we define the Schur element $\chi_\lambda|0\rangle \in \mathcal{V}$:

$$\chi_\lambda = \det(\chi_{\lambda_{i-j}})_{1 \leq i,j \leq l}. \tag{2.15}$$

Observe that $\chi_\lambda$ makes sense even if $\lambda$ is a composition. However $\chi_\lambda = 0$ if $\lambda + \delta = (\lambda_1 + l - 1, \lambda_2 + l - 2, \ldots, \lambda_l)$ has equal parts by the determinant property [10]. If $\lambda = \sigma(\mu + \delta) - \delta$ for a partition $\mu$, then $\chi_\lambda = \varepsilon(\sigma)\chi_\mu$.

We remark that $|\lambda\rangle$ (or $\langle \lambda |$) are Schur basis elements in $\mathcal{V}$ (or $\mathcal{V}^*$). In fact, $\Lambda \simeq \mathbb{Q}[h_{-1}, h_{-2}, \ldots]$ under the map $s_n \mapsto \chi_n$ [8,9]. Therefore $\Lambda_{\mathbb{C}} \simeq \mathcal{V}$ (or $\mathcal{V}^*$) under the identification, and $s_{\lambda} \simeq |\lambda\rangle$ (or $\langle \lambda |$). For more details on the vertex operator approach to symmetric functions, see [8]. Nevertheless, the following discussion is independent from this identification or motivation.

**Proposition 2.2.** One has that for $i, j \in \mathbb{Z}$

$$\phi_i \phi_j + \phi_{j+1} \phi_{i-1} = 0. \tag{2.16}$$

**Proof.** By (2.12) it follows that for $|z| > |w|

$$\phi(z) \phi(w) = (1 - \frac{w}{z})\phi^+(z)\phi^+(w)\phi^-(z^{-1})\phi^-(w^{-1}) \tag{2.17}$$

thus

$$z \phi(z) \phi(w) + w \phi(w) \phi(z) = 0. \tag{2.18}$$

The proposition follows by taking the coefficients. \hfill \Box

**Proposition 2.3.** For each partition $\lambda$, one has that

$$|\lambda\rangle = \chi_\lambda|0\rangle, \quad \langle \lambda | = \langle 0 | \omega(\chi_\lambda). \tag{2.19}$$

Moreover, $\{ |\lambda\rangle \}_{\lambda \in \mathcal{P}}$ and $\{ (-1)^{|\lambda|} |\lambda'\rangle \}_{\lambda \in \mathcal{P}}$ are (dual) bases of $\mathcal{V}$ and $\mathcal{V}^*$ respectively, i.e.,

$$\frac{\langle |\lambda\rangle |\mu\rangle}{\langle \lambda | \mu \rangle} = (\langle \lambda | \mu \rangle = (-1)^{|\lambda|} \delta_{\lambda', \mu}. \tag{2.20}$$

**Proof.** Using the method in [8], for any composition $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{Z}_+^l$, it follows from (2.17) and the Vandermonde determinant that

$$\phi_{-\lambda_1} \cdots \phi_{-\lambda_l}|0\rangle = \text{Res}_z z_1^{-\lambda_1-1} \cdots z_l^{-\lambda_l-1} \phi(z_1) \cdots \phi(z_l)|0\rangle$$
\begin{align*}
= & \text{Res}_z z_1^{-\lambda_1-t_1} z_2^{-\lambda_2-t_1+1} \cdots z_l^{-\lambda_l-t_1+1} \left( \prod_{1 \leq i < j \leq l} (z_i - z_j) \exp \left( \sum_{n=1}^{\infty} \frac{z_1^n + \cdots + z_l^n}{n} h_n \right) \right) |0\rangle \\
= & \text{Res}_z z_1^{-\lambda_1-t_1} z_2^{-\lambda_2-t_1+1} \cdots z_l^{-\lambda_l-t_1+1} \sum_{\sigma \in S_l} \varepsilon(\sigma) z_1^{\sigma(1)} \cdots z_l^{\sigma(l)} \exp \left( \sum_{n=1}^{\infty} \frac{z_1^n + \cdots + z_l^n}{n} h_n \right) |0\rangle \\
= & \text{Res}_z \sum_{\sigma \in S_l} \varepsilon(\sigma) z_1^{-\lambda_1+\sigma(l)-1} \cdots z_l^{-\lambda_l+\sigma(l-1)-1} \exp \left( \sum_{n=1}^{\infty} \frac{z_1^n + \cdots + z_l^n}{n} h_n \right) |0\rangle \\
= & \sum_{\sigma \in S_l} \varepsilon(\sigma) \chi_{\lambda_1-\sigma(l)+1} \chi_{\lambda_2-\sigma(l-1)+1} \cdots \chi_{\lambda_l-\sigma(1)+1} |0\rangle = \chi L |0\rangle,
\end{align*}

where \( \text{Res}_z f(z_1, \ldots, z_l) \) denotes the coefficient of \( z_1^{-1} \cdots z_l^{-1} \).

For two partitions \( \lambda = (\lambda_1, \ldots, \lambda_l) \) and \( \mu = (\mu_1, \ldots, \mu_k) \), we compute by using the Vandermonde determinant in variables \( z_1, \ldots, z_{k+l} \):

\[
\langle (\lambda), (\mu) \rangle = \langle 0| \phi_{\lambda_1} \phi_{\lambda_1-1} \cdots \phi_{\lambda_1} \phi_{1-\mu_1} \phi_{1-\mu_2} \cdots \phi_{1-\mu_k} |0\rangle \\
= & \text{Res}_z z_1^{\lambda_1-1} z_2^{\lambda_2-1} \cdots z_l^{\lambda_l-1} z_{l+1}^{\mu_1-1} z_{l+2}^{\mu_2-1} \cdots z_{l+k}^{\mu_k-1} |\varepsilon(\sigma)| |\phi(z_1) \cdots \phi(z_{l+k})| |0\rangle \\
= & \text{Res}_z \sum_{\sigma \in S_l} \varepsilon(\sigma) z_1^{\lambda_1+\sigma(l)-1} \cdots z_l^{\lambda_l+\sigma(l-1)-1} \prod_{1 \leq i < j \leq l+k} \left( 1 - \frac{z_j}{z_i} \right) \\
= & \varepsilon(\sigma) \delta_{\lambda_l, \mu_k} \cdots \delta_{\lambda_1, \mu_1} \delta_{\mu_1, \lambda_l+1} \cdots \delta_{\mu_k, \lambda_1} \\
= & \varepsilon(\sigma) \delta_{\sigma(l+k), k+l-\lambda_l} \cdots \delta_{\sigma(1), k+l-\lambda_l} \delta_{\mu_1, \lambda_l+1} \cdots \delta_{\mu_k, \lambda_1+1},
\]
for some \( \sigma \in S_{k+l} \). So the inner product is nonzero if and only if

\[
(2.22) \quad \{ k+l - \lambda_l, k+l - \lambda_{l-1}, \ldots, k+1 - \lambda_1, \mu_1+k, \ldots, \mu_k+1 \} \leftrightarrow \{ l+k, l+k-1, \ldots, 1 \}.
\]

We claim that \( (2.22) \) implies that \( \lambda = \mu' \) and \( \varepsilon(\sigma) = (-1)^{\lambda_1} \). Assume \( (2.22) \) holds. By summing the elements in both sets we have that \( |\lambda| = |\mu| \). Also it is easily seen that \( l \geq \mu_1 \) and \( k \geq \lambda_1 \). Clearly

\[
(2.23) \quad \lambda_i + \mu_j \not\equiv i+1 \not\equiv j
\]

for all \( 1 \leq i \leq l, 1 \leq j \leq k \). Suppose \( \lambda \not\equiv \mu' \), then there exists \( 1 \leq i \leq l \) such that \( \lambda_j = \mu'_{j'}, 1 \leq j \leq i-1 \) and \( \lambda_i \not\equiv \mu'_{i'} \). If \( \lambda_i > \mu'_{i'} \), then \( \mu_{\lambda_i} = i-1 \) by looking at the diagrams of \( \lambda \) and \( \mu' \), thus \( \lambda_i + \mu_{\lambda_i} = i + \lambda_i - 1 \), which contradicts \( (2.23) \). If \( \lambda_i < \mu'_{i'} \), then there exist \( i < j \) such that \( \lambda_s \not\equiv \mu'_{s'} \) for \( i \leq s < j \) and \( \lambda_j > \mu'_{j'} \) due to \( |\lambda| = |\mu'| \). Then \( \mu_{\lambda_j} = j-1 \), and \( \lambda_j + \mu_{\lambda_j} = j + \lambda_j - 1 \), which violates \( (2.23) \) again. Therefore \( \lambda = \mu' \).
Now we show that the coefficient of \( z_1^{-\lambda_1} \cdots z_l^{-\lambda_l} z_{l+1}^{\mu_1} \cdots z_{l+k}^{\mu_k} \) in \( \prod_{1 \leq i < j \leq l+k} (1 - \frac{z_j}{z_i}) \) is \((-1)^{|\lambda|}\). First, since \( \lambda_1 = \mu_1' = k \), for all \( l+1 \leq j \leq l+k \), \( -\frac{z_j}{z_i} \) appear in the expansion, and for all \( 1 \leq i \leq l-1 \), \( -\frac{z_j}{z_i} \) doesn’t exist in the product. Note that

\[
z_1^{-\lambda_1} \cdots z_l^{-\lambda_l} z_{l+1}^{\mu_1} \cdots z_{l+k}^{\mu_k} = z_1^{-\lambda_1} z_{l+1}^{\mu_1} \cdots z_{l+k}^{\mu_k} z_1^{-\lambda_l} \cdots z_{l-1}^{-\lambda_l} z_{l+1}^{\mu_1-1} \cdots z_{l+k}^{\mu_k-1}.
\]

Similarly, \( \lambda_2 = \mu_2' \) implies that for all \( l+1 \leq j \leq l+\mu_2' \), \( -\frac{z_j}{z_i} \) don’t appear in the expansion. Continuing the process, we see that the coefficient of \( z_1^{-\lambda_l} \cdots z_l^{-\lambda_l} z_{l+1}^{\mu_1} \cdots z_{l+k}^{\mu_k} \) in \( \prod_{1 \leq i < j \leq l+k} (1 - \frac{z_j}{z_i}) \) is \((-1)^{|\lambda|}\).

Using (2.21) and the remark after (2.15), \(|\lambda|\)’s \((\lambda \in \mathcal{P})\) span the space \( \mathcal{V} \) and the inner product result shows that \(|\lambda|\) is a base of \( \mathcal{V} \). \(\square\)

We remark that the coefficient of \( z_1^{\lambda_1} \cdots z_l^{\lambda_l} \) in the product \( \prod_{1 \leq i < j \leq l+k} (1 - \frac{z_j}{z_i}) \) equals to the sign of the shuffle (2.22), which is \((-1)^{|\lambda|}\). Also (2.22) generalizes the well-known combinatorial fact in [13, (1.7)] and our proof thus offers another algebraic one.

**Proposition 2.4.** For any partition \( \lambda \), one has that

\[
\phi^+(x)|\lambda\rangle = \sum_{\lambda < \mu \subseteq [(l(\lambda)+1) \times \infty]} x^{\mu - \lambda} |\mu\rangle,
\]

\[
\langle \lambda | \phi^-(x) \rangle = \sum_{\lambda < \mu \subseteq [(l(\lambda)+1) \times \infty]} x^{\mu - \lambda} \langle \mu |.
\]

**Proof.** It follows from (2.12) that

\[
\phi^+(x)\phi(z) = (1 - \frac{x}{z})^{-1} \phi(z)\phi^+(x).
\]

Note that \( \phi^+(x)|0\rangle = \phi(x)|0\rangle \), then

\[
\phi^+(x)\phi(z_1)\phi(z_2) \cdots \phi(z_l)|0\rangle = \prod_{i=1}^{l} (1 - \frac{x}{z_i})^{-1} \phi(z_1)\phi(z_2) \cdots \phi(z_l)\phi(x)|0\rangle.
\]

Now \( \phi^+(x)|\lambda\rangle \) is the coefficient \( C \) of \( z_1^{\lambda_1} \cdots z_l^{\lambda_l} \) in (2.27), and we claim that

\[
C = \sum_{n_1 \geq 0, 0 \leq n_i \leq \lambda_i - 1, 2 \leq i \leq l+1} x^{n_1 + \cdots + n_l + n_{l+1}} \phi_{-\lambda_1 - n_1} \phi_{-\lambda_2 - n_2} \cdots \phi_{-\lambda_l - n_l} \phi_{-n_{l+1}} |0\rangle,
\]

where \( \lambda_{l+1} = 0 \). By (2.16), for any fixed \( m, n \in \mathbb{Z} \) we have that

\[
\sum_{i \geq m, j \geq m+1} \phi_{-i} \phi_{-j} x^{i+j+n} = 0, \quad \sum_{i \geq m, j \geq m-1} \phi_i \phi_j x^{-i-j+n} = 0.
\]

\(^2\)Due to the fact [13], the action of \( \sum_{i \geq m, j \geq m+1} \phi_{-i} \phi_{-j} x^{i+j+n} \) on \(|\lambda\rangle\) is a finite sum.
For \( \lambda_i \geq \lambda_j \), we have that

\[
\sum_{n_i \geq 0, n_j \geq 0} \phi_{-\lambda_i - n_i} \phi_{-\lambda_j - n_j} x^{n_i + n_j} = \left( \sum_{n_i \geq 0, n_j \geq -\lambda_i - \lambda_j + 1} + \sum_{n_i \geq 0, 0 \leq n_j \leq -\lambda_i - \lambda_j} \right) \phi_{-\lambda_i - n_i} \phi_{-\lambda_j - n_j} x^{n_i + n_j}
\]

(2.30)

\[
= \sum_{n_i \geq 0, 0 \leq n_j \leq -\lambda_i - \lambda_j} \phi_{-\lambda_i - n_i} \phi_{-\lambda_j - n_j} x^{n_i + n_j}.
\]

In other words, the first identity of (2.20) can be used to trim the summation in (2.28). By definition \( C = \sum_{n_i \geq 0} x^{n_1 + \cdots + n_{l+1}} \phi_{-\lambda_1 - n_1} \cdots \phi_{-\lambda_{l+1} - n_{l+1}} |0\). Successive application of (2.30) to the factors from right to left implies (2.28). The summation indices of (2.28) satisfy that

(2.31)

\[
\lambda_i \geq \lambda_{i+1} + n_{i+1}, \quad 1 \leq i \leq l \quad (\lambda_{l+1} = 0).
\]

Then \((\lambda_i + n_i) - (\lambda_{i+1} + n_{i+1}) \geq n_i \geq 0\), and \(\mu = (\lambda_1 + n_1, \ldots, \lambda_l + n_l, n_{l+1})\) is a partition that interlaces \(\lambda: \lambda \prec \mu\). On the other hand, given \(\lambda \prec \mu\), then \(n_i = \mu_i - \lambda_i \geq 0\) corresponds to a term in (2.28). In summary we have shown that

(2.32)

\[
C = \sum_{\lambda \prec \mu} x^{\mu - \lambda} |\mu\rangle.
\]

\(\square\)

**Proposition 2.5.** One has the following equations:

(2.33)

\[
\phi^+(x_1) \phi^+(x_2) \cdots \phi^+(x_K) |0\rangle = \sum_{\mu \subseteq [K \times \infty]} s_{\mu} \{x\} |\mu\rangle,
\]

(2.34)

\[
|0\rangle \phi^-(y_1) \phi^-(y_2) \cdots \phi^-(y_N) = \sum_{\rho \subseteq [N \times \infty]} s_{\rho} \{y\} |\rho\rangle,
\]

where \(\{x\} = \{x_1, \ldots, x_K\}\) and \(\{y\} = \{y_1, \ldots, y_N\}\).

**Proof.** We argue by induction on \(K\). First (2.33) holds for \(K = 1\) by (2.24) with \(|\lambda\rangle = |0\rangle\). Assume (2.33) holds for \(K - 1\), then for \(\{x\} = \{x_1, \ldots, x_K\}\) and \(\{\bar{x}\} = \{x\} \setminus \{x_K\}\)

\[
\phi^+(x_1) \phi^+(x_2) \cdots \phi^+(x_K) |0\rangle = \phi^+(x_K) \phi^+(x_1) \cdots \phi^+(x_{K-1}) |0\rangle
\]

\[
= \sum_{\nu \subseteq [(K-1) \times \infty]} s_{\nu} \{\bar{x}\} \phi^+(x_K) |\nu\rangle
\]

\[
= \sum_{\nu \subseteq [(K-1) \times \infty]} s_{\nu} \{\bar{x}\} \sum_{\nu \prec \mu \subseteq [K \times \infty]} x^{\mu - \nu} |\mu\rangle
\]

\[
= \sum_{\mu \subseteq [K \times \infty]} s_{\mu} \{x\} |\mu\rangle,
\]

where we have used (2.10) in the last equation. \(\square\)
Combining Proposition 2.3 with Proposition 2.5, we obtain the following:

**Corollary 2.1.** The correlation function

\[ \langle 0 | \phi^-(x_1) \phi^-(x_2) \ldots \phi^-(x_K) \phi^+(y_1) \phi^+(y_2) \ldots \phi^+(y_N) | 0 \rangle \]

has the following two expressions

\[ \langle 0 | \phi^-(x_1) \phi^-(x_2) \ldots \phi^-(x_K) \phi^+(y_1) \phi^+(y_2) \ldots \phi^+(y_N) | 0 \rangle = \prod_{i=1}^{K} \prod_{j=1}^{N} (1 - x_i y_j), \]

(2.35)

\[ \langle 0 | \phi^-(x_1) \phi^-(x_2) \ldots \phi^-(x_K) \phi^+(y_1) \phi^+(y_2) \ldots \phi^+(y_N) | 0 \rangle = \sum_{\mu \subseteq [K \times N]} (-1)^{|\mu|} s_{\mu}(x) s_{\mu'}(y), \]

(2.36)

which immediately implies Cauchy’s formula

\[ \prod_{i=1}^{K} \prod_{j=1}^{N} (1 - x_i y_j) = \sum_{\mu \subseteq [K \times N]} (-1)^{|\mu|} s_{\mu}(x) s_{\mu'}(y). \]

(2.37)

Taking the limits \( N \to \infty, K \to \infty \), we obtain the Cauchy identity [13]:

\[ \prod_{i,j=1}^{\infty} (1 - x_i y_j) = \sum_{\mu \in \mathcal{P}} (-1)^{|\mu|} s_{\mu}(x) s_{\mu'}(y). \]

(2.38)

3. Half plane partitions and Cauchy’s identities

A half plane partition \( \pi \) is a set of finitely many nonzero integers \( \pi(i, j) \) that are weakly bi-decreasing: \( \pi(i, j) \geq \pi(i + 1, j), \pi(i, j) \geq \pi(i, j + 1) \) for all \( i \geq j \geq 1 \) with the additional condition

\[ \pi(i, j) = 0, \ i < j. \]

(3.1)

For convenience, one may add strings of zeros to \( \pi(i, j) \) for \( i \geq j \gg 0 \). The weight of \( \pi \) is \( |\pi| = \sum_{i,j \geq 1} \pi(i, j) \). The height \( h(\pi) \) of \( \pi \) is the maximal \( i \) such that \( \pi(i, 1) > 0 \). One also uses the notion of tableau, whereby the non-negative integer \( \pi(i, j) \) is placed in row \( i \) and column \( j \) for any \( i \geq j \geq 1 \).

![Tableau representation of a half plane partition.](image)

**Figure 1.** Tableau representation of a half plane partition. e.g. the top and third rows correspond to the boxes \((1, 1)\) and \((3, 1), (3, 2), (3, 3)\) respectively.

Let \( \pi \) be a half plane partition. For \( i \geq 0 \), define the partition \( \pi_i \), called a diagonal slice of \( \pi \), with the parts given by

\[ (\pi_i)_j = \pi(j + i, j), \quad j \geq 1. \]

(3.2)
For the half plane partition in Figure 1, the diagonal slices are given as follows:

\[(3.3) \quad \pi_0 = (5, 3, 1), \quad \pi_1 = (4, 2), \quad \pi_2 = (3), \quad \pi_3 = (2),\]

where \(|\pi| = 5 + 3 + 1 + 4 + 2 + 3 + 2 = 20, \quad h(\pi) = 4.\]

A half plane partition can be considered as a lower triangular part of a plane partition [13]. It is known that plane partitions and interlacing partitions are closely related. The following fact was due to Okounkov and Reshetikhin [14] for general plane partitions.

**Lemma 3.1.** Let \(\pi_i\) be the diagonal slices of the half plane partition \(\pi\). Then one has

\[(3.4) \quad \pi_i \succ \pi_{i+1}, \quad i \geq 0.\]

If \(\lambda\) is a partition, an interlacing partition chain of \(\lambda\) is a series of partitions starting from \(\lambda\) and ending at \(\emptyset\):

\[
\emptyset = \lambda^{(n)} \prec \cdots \prec \lambda^{(1)} \prec \lambda^{(0)} = \lambda.
\]

For a partition \(\lambda\), let \(\{\lambda \rightarrow T\}\) be the set of all interlacing partition chains \(T\) of \(\lambda\). By Lemma 3.1, each half plane partition \(\pi\) canonically gives rise to an interlacing partition chain of \(\pi_0\). For example, the interlacing partition chain of \(\pi_0 = (5, 3, 1)\) in Figure 1 is

\[(3.5) \quad \emptyset = \pi_4 \prec \pi_3 = (2) \prec \pi_2 = (3) \prec \pi_1 = (4, 2) \prec \pi_0 = (5, 3, 1).\]

Denote by \(\{\lambda \rightarrow \pi\}\) the set of half plane partitions initiating at \(\pi_0 = \lambda\), and \(\{\lambda \rightarrow \pi\}_n\) the set of half plane partitions with \(\pi_0 = \lambda\), \(\pi_n = \emptyset\). Thus we have

\[(3.6) \quad \{\lambda \rightarrow \pi\}_1 \subseteq \{\lambda \rightarrow \pi\}_2 \subseteq \{\lambda \rightarrow \pi\}_3 \subseteq \ldots\]

and \(\lim_{n \to \infty} \{\lambda \rightarrow \pi\}_n = \{\lambda \rightarrow \pi\}\).

Proposition 2.4 gives that

\[(3.7) \quad \phi^+(y_1) \ldots \phi^+(y_N)|0\rangle = \sum_{\emptyset = \pi_N < \cdots < \pi_0 \subseteq \mathbb{N} \times \infty}^{N} \prod_{i=1}^{N} y_i^{[\pi_{N-i} - |\pi_{N-i+1}|]} \langle \mu|,\]

\[(3.8) \quad \langle 0| \phi^-(x_1) \ldots \phi^-(x_K) = \sum_{\emptyset = \pi_K < \cdots < \pi_0 = \mathbb{N} \subseteq \mathbb{N}}^{K} \prod_{i=1}^{K} x_i^{[\pi_{K-i} - |\pi_{K-i+1}|]} \langle \lambda|.\]

Taking the \(q\)-specialization \(x_i = q^{K-i+1}\) and \(y_j = q^{N-j+1}\), one has

\[
\prod_{i=1}^{K} x_i^{[\pi_{K-i} - |\pi_{K-i+1}|]} = q^{[\pi_0 - |\pi_1|} q^2(|\pi_1 - |\pi_2|)} \ldots q^K(|\pi_{K-1} - |\pi_K|) = q^{[\pi_0] + [\pi_1] + \cdots + [\pi_K]} = q^{[\pi]},
\]

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\[ \prod_{i=1}^{N} y_i^{\pi_N-i-|\pi_N-i+1|} = q^{\pi_0-|\pi_1|} q^{2(|\pi_1|-|\pi_2|)} \ldots q^{N(|\pi_{N-1}-|\pi_N|)} = q^{\pi_0+|\pi_1|+\cdots+|\pi_N|} = q^{\pi}; \]

\[ \prod_{i=1}^{K} \prod_{j=1}^{N} (1 - x_i y_j) = \prod_{i=1}^{K} \prod_{j=1}^{N} (1 - q^{K+N-i-j+2}) = \prod_{i=1}^{K} \prod_{j=1}^{N} (1 - q^{i+j}), \]

where \( \pi_K = \emptyset, \pi_N = \emptyset \). Then by (2.20)

\[ \langle 0 | \phi^- (x_1) \ldots \phi^- (x_K) \phi^+ (y_1) \ldots \phi^+ (y_N) | 0 \rangle = \sum_{\lambda \subseteq [K \times \infty]} \sum_{\lambda \rightarrow \pi_K} q^{\pi} \sum_{\mu \subseteq [N \times \infty]} \sum_{\mu \rightarrow \pi_N} q^{\pi} \langle \lambda | \mu \rangle \]

\[ = \sum_{\lambda \subseteq [K \times N]} (-1)^{||\lambda||} \sum_{\lambda \rightarrow \pi_K} \sum_{\lambda' \rightarrow \pi_N} q^{\pi'}, \]

where \( \lambda' \) is the conjugate of \( \lambda \). Taking the limit \( K, N \to \infty \), we have the following result.

**Proposition 3.1.** One has the identity:

\[ \sum_{\lambda \in \mathcal{P}} (-1)^{||\lambda||} \sum_{\lambda \rightarrow \pi_K} q^{\pi} = \prod_{i=1}^{\infty} (1 - q^i)^{i-1}, \]

where \( \{ \lambda \rightarrow \pi \} \) (resp. \( \{ \lambda' \rightarrow \pi' \} \)) runs through all half plane partitions \( \pi \) (resp. \( \pi' \)) starting at the partition \( \lambda \) (resp. \( \lambda' \)).

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