Gravity from Lie algebroid morphisms

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Inspired by the Poisson Sigma Model and its relation to 2d gravity, we consider models governing morphisms from $T \Sigma$ to any Lie algebroid $E$, where $\Sigma$ is regarded as $d$-dimensional spacetime manifold. We address the question of minimal conditions to be placed on a bilinear expression in the 1-forms, $S^{ij}(X)A_iA_j$, so as to permit an interpretation as a metric on $\Sigma$. This becomes a simple compatibility condition of the $E$-tensor $S$ with the chosen Lie algebroid structure on $E$. For the standard Lie algebroid $E = TM$ the additional structure is identified with a Riemannian foliation of $M$, in the Poisson case $E = T^*M$ with a sub-Riemannian structure which is Poisson invariant with respect to its annihilator bundle. (For integrable image of $S$, this means that the induced Riemannian leaves should be invariant with respect to all Hamiltonian vector fields of functions which are locally constant on this foliation). This provides a huge class of new gravity models in $d$ dimensions, embedding known 2d and 3d models as particular examples.

I. INTRODUCTION

A. The problem—in the 2d Poisson setting

In the present paper we want to address essentially the following question: Given a manifold $M$ equipped with some 2-tensor $T$, say contravariant, and given a two-dimensional manifold $\Sigma$, which we call worldsheet or spacetime, under what conditions can we define a “reasonable” theory of gravity on $\Sigma$ with just these data? Certainly one needs to specify what one means by reasonable, and we will do this—in a rather minimalistic but also precise way—below. Moreover, we will address a likewise question also for higher dimensional spacetime manifolds $\Sigma$; but the formulation of the respective higher dimensional problem, including a replacement of $M$ the 2d Poisson setting

\[ L[X, A] := \int_\Sigma A_i \wedge dX^i + \frac{1}{2} P^{ij} A_i \wedge A_j \]  

such that $g$ constructed according to (2) satisfies $\det g \neq 0$ on all of $\Sigma$. This latter condition is also necessary for regarding $g$ as a metric tensor field. We may, however, not expect it to hold for all solutions to the field equations: For example, $A_i \equiv 0$, $X^i \equiv \text{const}$ is a solution to the field equations (3), (4), leading to $g \equiv 0$ irrespective of whatever $T$ is. Still, $S$ should not be e.g. vanishing, since then there would be no gravitationally interesting or admissible solutions; for a generic enough choice of $T$ this will not happen, however, and then there is no obstacle to regard (1) and (2) as defining some 2d theory of gravity.

Note that we did not require $L$ to be a functional of $g$ alone or a functional of $g$ at all. Applications, some of which we will recall in the body of the paper, show that this would be too restrictive: There may be some other “parent” action functional, which depends explicitly on $g$, or maybe only on some vielbeins from which $g$ results by the standard bilinear combination, and possibly on some other more or less physical fields and only after some (possibly involved) identifications, including (2), one ends up with the functional (1).

The initially posed question bears more structure[38], if we add an additional requirement, namely that the theory should be “topological”. This notion may be understood in various ways, cf e.g. [1]. Here we imply the following with it: in addition to the diffeomorphism invariance of the action functional $L$, we require its moduli space of classical solutions (for reasonable enough choice of $\Sigma$, such that its fundamental group has a finite rank) to be finite dimensional. For $\dim M > 2$ and generic

\[ A_i \wedge A_j \equiv A_i \otimes A_j \]
choice of \( \mathcal{P} \) this is not the case: For fixed topology of \( \Sigma \) the space of smooth solutions to (4) and (4) is infinite dimensional, even after identification of solutions differing by a gauge symmetry (here only diffeomorphisms of \( \Sigma \)). The theory described by \( \textbf{II} \) is topological, iff \( \textbf{2} \) the tensor \( \mathcal{P} \) satisfies

\[
\mathcal{P}^{ijkl} = 0,
\]

i.e., iff \( \mathcal{P} \) endows \( M \) with a Poisson structure where the Poisson bracket between functions \( f \) and \( g \) on \( M \) is defined by \( \{ f, g \} := f_\ast g \mathcal{P}^{ij} \).

Now there are much more gauge symmetries. Indeed, for any choice of functions \( \varepsilon_i(x) \) (\( \varepsilon \) corresponding to a section of \( \mathcal{X} \rightarrow T^* M \)), one verifies easily that if the fields are transformed infinitesimally according to

\[
\delta \varepsilon_i \cdot X^i = \varepsilon_j \mathcal{P}^{ji}(X), \quad \delta \varepsilon_i \cdot A_i = d \varepsilon_i + \mathcal{P}^{ijkl}(X) A_k \varepsilon_l,
\]

\( \mathcal{L} \) changes by a boundary term only. The diffeomorphism invariance of \( L \) now is a particular case of \( \textbf{0} \): a simple calculation shows that on-shell, i.e., by use of the field equations (3) and (4), the Lie derivative of the fields with respect to \( \varepsilon \), the Lie derivative of the fields with respect to \( \varepsilon \), is obtained upon the (field dependent) choice \( \varepsilon_i = A_{\mu i} \omega^\mu \).

In view of these additional symmetries a new question arises if one intends to identify \( \mathbf{2} \) with a metric on \( \Sigma \): How do these symmetries act on \( \Sigma \)? For instance, \( \Sigma \)—all the symmetries \( \mathbf{3} \), when acting on \( g \) obtained as non-degenerate \( (\det g \neq 0 \) everywhere) solution to the field equations, will boil down to diffeomorphisms of \( \Sigma \) only. Otherwise there is an additional local symmetry, which will be hard to interpret in general: There may be e.g. a flat spacetime metric \( g \) on \( \Sigma \) which would be "gauge equivalent" to a spacetime metric \( g \) describing some black hole. It is our intention to clarify the conditions to be placed on \( \Sigma \) which ensure that this rather unreasonable scenario is avoided.

To be precise:

**Definition 1** A Poisson Sigma Model (PSM) \( \mathbf{3}, \mathbf{4} \), i.e., the action functional \( \mathbf{4} \) where \( \mathcal{P} \) satisfies \( \mathbf{4} \), together with an identification \( \mathbf{2} \) for the metric on \( \Sigma \) is called a reasonable theory of 2d Poisson gravity, if the following two conditions hold (\( d = 2 \)):

- For any point \( x \in M \) there exist smooth solutions to the field equations on \( \Sigma \cong \mathbb{R}^d \) with a base map \( \mathcal{X} : \Sigma \rightarrow M \) such that the image of \( \mathcal{X} \) contains \( x \) and everywhere on \( \Sigma \) det \( g \neq 0 \).

- For any two gauge equivalent solutions \( (\mathcal{X}_1, A_1) \) and \( (\mathcal{X}_2, A_2) \) on a given manifold \( \Sigma \) which yield a non-degenerate metric, \( \det g_{ij}(x) \neq 0 \), \( \forall x \in \Sigma \), and which are maximally extended, there exists a diffeomorphism \( \mathcal{D} : \Sigma \rightarrow \Sigma \) such that \( g_{ij} = \mathcal{D}^\ast g_{ij} \).

We will add remarks explaining some of the technicalities of this definition after its generalization in subsection \( \textbf{II.C} \) below. Requiring \( \textbf{2} \) to define a reasonable theory of gravity will pose some simple geometrical restriction on \( \Sigma \). This will be seen to define a sub-Riemannian structure (cf. e.g. \( \textbf{5} \)) on \( M \), which has to be "compatible" with the Poisson structure \( \mathcal{P} \) in a particular manner (equation (40) or (46) below). From a certain perspective to become more transparent below this will be seen to correspond to a Poisson or Lie algebroid extension of the notion of a Riemannian foliation (cf. \( \textbf{2} \)), which may be also interesting mathematically in its own right. Moreover, by the above method we will be able to define a much wider class (also of just 2d gravity theories, the previously known ones—such as those in \( \mathbf{3}, \mathbf{7}, \mathbf{8}, \mathbf{9} \)—arising as a relatively small subclass.

As a straightforward application of this extension it will be immediate to construct a simple PSM, the gravity solution space of which coincides with the one of the exact string black hole, something excluded within the previously known models according to \( \textbf{10} \). We also demonstrate with an example that by these means one can construct gravitational models with solutions for the metric that have no local Killing vector field, a feature missing in the class of matterless 2d gravity models considered hitherto (cf. e.g. \( \mathbf{2}, \mathbf{3} \)). We also hope that by the above analysis much of the in part intricate global structure arising on \( \Sigma \) in the known gravity models, as classified in \( \textbf{11} \), \( \textbf{12} \), will find some explanation from the implicitly underlying geometrical structure \( (M, \mathcal{P}, \Sigma) \) on the target. In fact, already in the present work we want to highlight the emergence of a Killing vector field in these models as resulting from structures on \( M \). Finally, we may apply our considerations to the case of kappa deformed gravity \( \mathbf{13}, \mathbf{14} \), providing some coordinate independent grasp on the discussion which arose in that context.

**B. Generalizations**

Given a metric on \( \Sigma \), be it of the form \( \textbf{2} \) or not, and the data specified in the previous subsection, we may also add another term to \( \textbf{11} \):

\[
\frac{1}{2} \delta S^{ij} A_i \wedge \ast A_j.
\]

Here \( \ast : \Omega^p(\Sigma) \rightarrow \Omega^{2-p}(\Sigma) \) denotes the operation of taking the Hodge dual of a form with respect to \( g \). In fact, upon this addition to \( \textbf{11} \) one may obtain a theory that is at least classically equivalent to standard String Theory: Assuming the matrix \( T^{ij} \) to be invertible, we may integrate out the \( A \)-fields altogether, as they enter the action quadratically only; the result is the usual String Theory action in Polyakov form. This may be of particular interest also if \( \mathcal{P} \) is Poisson, since then one expects \( \mathcal{P} \) to govern the noncommutativity of the effective action induced on D-branes via the Kontsevich formula \( \textbf{17} \), which itself just results upon perturbative quantization of \( \textbf{11} \). For constant \( T \) this is verified easily \( \textbf{18} \).

We will not consider additions of the form \( \textbf{7} \) any further within the present work. Much closer to the present
subject are the following modifications of the problem posed in the previous subsection: Suppose that in addition to $T$ one is given also a covariant 2-tensor on $M$. We again may split it into symmetric and antisymmetric parts, say $G$ and $B$, respectively. Then we may add the pullback of $B = \frac{1}{2}G_{ij}\mathrm{d}X^i \wedge \mathrm{d}X^j$ with respect to $X : \Sigma \to M$ to $\mathcal{H}$ and the pullback of $G = \frac{1}{2}G_{ij}\mathrm{d}X^i \wedge \mathrm{d}X^j$ to the right hand side of $\mathcal{H}$. We then may repeat the questions of the previous subsection.

Requiring the resulting action functional to be topological now yields a modified condition $\mathcal{H}$, its right hand side consisting of the 3-form $\mathcal{H} = \mathrm{d}\mathcal{B}$ with all indices raised by contraction with $\mathcal{P}$ $\mathcal{H}$. This leads to the notion of a twisted Poisson or an $\mathcal{H}$-Poisson structure on $M$, cf. e.g. $\mathcal{H}$.

Now the second set of field equations receives an $\mathcal{H}$-dependent contribution. Likewise, there are still gauge symmetries for any $\epsilon_i$, while the transformation on $A_i$ in $\mathcal{H}$ receives some $\mathcal{H}$-dependent additions. Despite these evident changes, much of the standard PSM can still be transferred to the modified theory. Indeed, if $(1+BP)$ is invertible, one even may get rid of the additional term in $L$ by some redefinition of the $A$-fields (as seen best in the Hamiltonian formulation) leading to a modified, $\mathcal{P}$- and $\mathcal{B}$-dependent effective Poisson structure $\mathcal{P}'$.

(Such transformations were called gauge transformations in $\mathcal{H}$). A likewise statement holds for $g$: Using the first set of field equations, $\mathcal{H}$, yields a new effective tensor $S' = S + \mathcal{P}G\mathcal{P}$. Thus essentially everything of what will be said on the PSM with $\mathcal{P}$ will have a straightforward generalization to the twisted or HPSM with the modified identification for $g$ $\mathcal{P}$.

Finally, we may also be given a $(1,1)$-tensor field on $M$. Adding corresponding terms to $\mathcal{P}$ does not change much since again by use of the field equations one may reexpress everything in terms of a bilinear combination of $A$-fields. If one uses such a tensor field so as to redefine the first term in $\mathcal{H}$, on the other hand, replacing it e.g. by $A_i \wedge \mathrm{d}X^i C_i^j$, the situation changes if $C$ has a kernel. Let us mention here in parenthesis that such a modification of $\mathcal{H}$ arises if one considers 2d gravity models with additional scalar or fermionic matter fields, where in the first case there is a nontrivial kernel of $C$ considered as a map from $TM$ to $TM$ and in the second case a nontrivial kernel of the transposed map from $TM$ to itself, cf. $\mathcal{H}$ $\mathcal{H}$.

More generally we may replace $TM$ by some vector bundle $E$ over $M$. Together with a (coanchor) map $C : E \to TM$ and a covariant $E$-2-tensor $T_{IJ} = \mathcal{P}_{IJ} + S_{IJ}$ we may replace $\mathcal{H}$ by an integral over $A_i C_i^j \wedge \mathrm{d}X^i + \frac{1}{2}\mathcal{P}_{IJ} A^i \wedge A^j$. It may be interesting to clarify the conditions on such a more general action functional to be topological and then to address the question of when

$$g = \frac{1}{2}S_{IJ} A^i A^j$$

defines a reasonable notion of a metric on $\Sigma$.

Although this generalization of the PSM seems interesting also in its own right, we will not pursue it here any further. Instead we want to permit an even more drastic, but different change: We want to allow for any spacetime dimension of $\Sigma$. Moreover, the Poisson manifold $(M, \mathcal{P})$ may be viewed as a particular Lie algebroid structure on the vector bundle $TM$ over $M$; we will replace it by any Lie algebroid defined on a vector bundle $E$ over $M$.

First we note that the field equations $\mathcal{H}$ and $\mathcal{H}$ make perfectly sense also if $\Sigma$ is replaced by a manifold of some higher dimension, and likewise so for the symmetries $\mathcal{H}$. So, the questions addressed above do make sense also in $d$ spacetime dimensions.

The generalization of the target structure to Lie algebroids needs some extra explanations; before we can approach it, we want to recall the notion and the basic features of Lie algebroids in the next section, since we cannot assume that the average physics reader is familiar with it.

In the present work we will not focus on the action functional that produces the desired field equations and symmetries. Whenever an action functional which has field equations and symmetries containing the ones to be specified below, our analysis will apply. It is still comforting to know, however, that such action functionals can be constructed $\mathcal{H}$.

C. Structure of the paper

In the following section we will recall the definition of Lie algebroids over $M$ and provide the corresponding generalization of the field equations $\mathcal{H}$ and $\mathcal{H}$ as well as of the gauge symmetries $\mathcal{H}$ to this more general setting. In the subsequent section we then determine the conditions on $S$ which ensure that $g$ as defined in $\mathcal{H}$ provides a reasonable theory of gravity, as defined by the two marked conditions above or more precisely by Definition $\mathcal{H}$ below. This section provides the main result of the present work, summarized in Theorem $\mathcal{H}$.

Section $\mathcal{H}$ contains first illustrations of the general results, applying it to various particular Lie algebroids. Standard $(2+1)$-gravity is among the examples of the general framework developed here. Only in Section $\mathcal{H}$ we come back to the Poisson case—still permitting arbitrary dimension of $\Sigma$; in a further specialization we finally arrive back to the Poisson Sigma model. We then will show how previously known 2d gravity models fit into the present more general framework and provide some simple examples of new models. Finally we will address some more recent issues such as kappa deformation of 2d gravity theories. We complete the paper with an outlook including a list of open questions and possible further developments.
II. LIE ALGEBROIDS, GENERALIZED FIELD EQUATIONS AND SYMMETRIES

A. Lie algebroids, basic facts and examples

A Lie algebroid is a simultaneous generalization of a Lie algebra and a tangent bundle. Let us begin with a formal definition. (For further details cf. e.g. [22]).

Definition 2 A Lie algebroid \((E, \rho, [\cdot, \cdot])\) is a vector bundle \(\pi: E \to M\) together with a bundle map (“anchor”) \(\rho: E \to TM\) and a Lie algebra structure \([\cdot, \cdot]: \Gamma(E) \times \Gamma(E) \to \Gamma(E)\) satisfying the Leibniz identity

\[
[\psi, f \psi'] = f[\psi, \psi'] + (\rho(\psi)f)\psi'
\]

\(\forall \psi, \psi' \in \Gamma(E), \forall f \in C^\infty(M)\).

Let us provide some basic examples of Lie algebroids:

1. If \(M\) is a point, \(\rho\) is trivial and \(E\) is just an ordinary Lie algebra.

2. \(M\) arbitrary, but \(\rho \equiv 0\), mapping all elements of the fiber of \(E\) at any point \(X \in M\) to the zero vector in \(T_XM\), \(E\) is a bundle of Lie algebras, since then \(\rho\) provides a Lie bracket on each fiber of \(E\); \(M\) then may be viewed as a manifold of deformations of a Lie algebra.

3. \(E = TM, \rho = id_{TM}\), the bracket being the ordinary Lie bracket between vector fields. This is the so-called standard Lie algebroid.

4. \(E = T^*M, (M, \mathcal{P})\) a Poisson manifold. Here \(\rho = -\mathcal{P}\), \(\rho(\alpha) = -\alpha\mathcal{P}\), and the bracket \([df, dg] := d\{f, g\}\) between exact 1-forms is extended to all 1-forms by means of \([\psi, \psi']\).

We now add some remarks: Eq. restricts \(\rho\); using the Jacobi identity of the bracket \([\cdot, \cdot]\) between sections, it is possible to show [22] that \(\rho\) is a morphism of Lie algebras, \(\forall \psi, \psi' \in \Gamma(E)\):

\[
\rho(\psi, \psi') = [\rho(\psi), \rho(\psi')].
\]

As a consequence, the image of \(\rho\) is always an integrable distribution in \(TM\), defining the orbits of the Lie algebroid in \(M\). In the case of Poisson manifolds, these orbits coincide with the symplectic leaves of \((M, \mathcal{P})\).

Generalizing the observation in the second example above, \(\ker \rho \subset E\) always defines a bundle of Lie algebras. For different points on the same orbit, these Lie algebras are isomorphic, while they are not necessarily so for any two points in \(M\). For Poisson manifolds \(\ker \rho|_{X \in M}\) is the conormal bundle of the respective symplectic leaf \(L\) at a given point \(X \in M\). For regular points on Poisson manifolds these Lie algebras are abelian; the origin of a Lie Poisson manifold \(\mathfrak{g}^*\) provides an example for a nonregular point.

If \(\{b_I\}_{I=1}^n\) denotes a local basis of \(E\), the bracket and anchor give rise to structure functions \(c_{IJ}^K(X)\) and \(\rho^i_I(X)\), respectively:

\[
[b_I, b_J] = c_{IJ}^K b_K, \quad \rho(\partial_i) = \rho^i_I b_I.
\]

The compatibility conditions above then provide differential equations for them:

\[
c_{IJ}^S c_{KS}^L + c_{IJ}^L \rho^i_K + \text{cycl}(JKI) = 0, \quad c_{IJ}^K \rho_J^i - \rho^L_I \rho^i_J + \rho^j_L \rho^i_J = 0.
\]

While \(\rho^i_J\) behaves as a tensor with respect to a change of local basis and coordinates, \(c_{IJ}^K\) certainly does not; With \(b_I = F^I_i b_i\) and \(b_I = F^I_i b_i\) one has

\[
c_{KL}^I \rho_i^K = F^I_i F^L_j F^I_i c_{KL}^j + \rho^i_j F^I_i \rho^j_I - \rho^L_i F^I_i c_{KL}^I = 0.
\]

In the case of \(\rho = 0\), example 2 above, the second equation becomes trivial and the first one reduces to the standard Jacobi identity at any point \(X \in M\). For Poisson manifolds, on the other hand, \(b_I \sim dX^i, \rho^i_j \sim \mathcal{P}^i_j\), and \(c_{IJ}^K \sim \mathcal{P}^{ij} \mathcal{P}_k\). We see that this case is particular insofar as that the anchor and the structure functions are essentially one and the same object; Eq. reduces to the Jacobi identity and Eq. to its derivative.

A Lie algebroid structure on a vector bundle \(E\) permits a generalization of differential geometry to \(E\). In particular, there is a natural exterior \(E\)-derivative \(d_E: \Omega^p_E(M) \to \Omega^{p+1}_E(M)\), where \(\Omega^p_E(M) \equiv \Gamma(\Lambda^p E^*)\) and \(\Omega^0_E(M) = C^\infty(M)\). If \(\{b_I\}_{I=1}^n\) denotes the local basis of \(E^*\) dual to \(\{b_I\}_{I=1}^n\), it may be defined by means of

\[
d_E f := f \cdot \rho^i_I b_I, \quad d_E b^i := -\frac{1}{2} c_{JK}^I b^j \wedge b^K,
\]

extended to all of \(\Omega^p_E(M)\) by a graded Leibnitz rule. As a consequence of the Lie algebroid axioms, \(d_E^2 = 0\). (\(E\) together with a nilpotent \(d_E\) even provides an alternative definition of a Lie algebroid, cf. [22]).

Likewise there is an \(E\)-Lie derivative \(E\mathcal{L}: \Gamma(E) \times T^q_\mathcal{P}(E) \to T^q_\mathcal{P}(E)\), where \(T^q_\mathcal{P}(E) \equiv \Gamma\left(E^{\otimes q} \otimes (E^*)^\otimes q\right)\). For \(p = 1, q = 0\) it is defined by the ordinary bracket, \(E\mathcal{L}_\psi \psi' := [\psi, \psi']\), while for \(\alpha \in \Omega^p_E(M)\) one uses the formula (generalizing a well-known pendant in differential geometry resulting from \(E = TM\))

\[
E\mathcal{L}_\psi \alpha := (d_E \psi \cdot \alpha + \psi \cdot d_E) \alpha;
\]

it is extended to \(T^q_\mathcal{P}(E)\) by means of the ordinary Leibnitz rule. This concludes our short excursion into the mathematics of Lie algebroids.

B. Generalized field equations and symmetries

For a generalization we first need to interpret the field content of the PSM. The coordinates \(X^i\) define a map \(\Sigma: \Sigma \to M\), where \(\Sigma\) is our spacetime and \(M\) the
we then consider the following set of equations
\[ X \] is a morphism of Lie algebroids (cf. [25] for details).

It is straightforward to check that on behalf of the structural compatibility conditions, generalizing the Jacobi identity for the present setting it is, however, completely sufficient to know the gauge symmetries on-shell.

As a further consistency check we may verify that the complete set of field equations is covariant. This indeed follows quite easily by use of Eq. (14). Likewise so for the symmetries [20], keeping in mind their infinitesimal form, however; in particular,
\[ \delta(F^I_i A^I) = \delta(F^I_i) A^I + F^I_i \delta A^I \approx d\varepsilon^I + c_{K Li} A^K \varepsilon^L, \] where \( \varepsilon ^I \) and \( F^I_i \) is an arbitrary function of \( X^I(x) \), governing a change of basis in \( E \). (Although not also of \( x \) alone, as it would be permitted as a change of basis in the pullback bundle \( \chi ^* E \) at least if the structure functions \( c_{I J K} \) in [20] are understood as pullback of the structure functions on \( E \), as we want to reinterpret them).

As mentioned already in the Introduction, action functionals yielding field equations and symmetries containing \( \{ A, \varepsilon \} \) and \( \{ A, \varepsilon \} \) exist. One option, which works in any dimension \( d \), is to multiply the left hand sides of [18] and [19] by \( B_i \) and \( B_I \), a \((d-2)\)- and a \((d-1)\)-form, respectively (cf also [22] for more details).

C. Definition of a reasonable theory of \( E \)-gravity

We now are in the position to generalize Definition 1.

**Definition 3** A Lie algebroid \( E \) together with a symmetric section \( \mathcal{S} \) of \( TM(E) \) is said to define a reasonable theory of \( E \)-gravity, if the two marked items in Definition 1 hold true for \( g \) as defined in [8], where solutions refer to the set of field equations [18], [19] and the gauge symmetries are given by [20] (infinitesimally and on-shell).

We now add some remarks explaining the two items in the definition: In the first one, we required the existence of a (metric) non-degenerate solution in a neighborhood of the preimage of any point \( X \in M \). If this were not satisfied, but only for a submanifold \( U \subset M \), we could equally well take the subbundle \( E|_U \) as a new Lie algebroid, defined just over \( U \). Moreover, there would be no condition on \( \mathcal{S} \) for points \( X \in M \setminus U \). Thus, essentially without loss of generality, we require non-degeneracy for all points \( X \in M \).

Since the second item refers to a global gauge equivalence, while the symmetries [20] are given in infinitesimal form only, we need to add that within the present framework and in the context of [8] we regard two metric non-degenerate solutions \( (X^{(1)}, A^{(1)}) \) and \( (X^{(2)}, A^{(2)}) \) to the field equations as gauge equivalent if they may be connected by a flow of gauge transformations generated by [20] which does not enter a degenerate sector of the
theory, i.e. which does not pass through a solution not satisfying the first condition on \(g\).

This last specification is necessary because otherwise the set of theories would be empty: Even if the gauge transformations (20) when acting on non-degenerate solutions correspond merely to diffeomorphisms, they allow to connect non-degenerate solutions with degenerate ones. E.g. in the 2d Poisson case it has been shown that by such transformations, even if they are contained in the component of unity of the gauge group, one may generate non-degenerate gravity solutions with nontrivial kink number from non-degenerate ones with trivial kink number (if \(\pi_1(\Sigma) \neq 0\)), something that can never be achieved by a diffeomorphism of \(\Sigma\)—cf. [28] for an explicit example of this scenario within the Jackiw-Teitelboim model of 2d gravity [28, 51], and [31] for a generalization to arbitrary 2d dilaton gravity theories.

Finally, the addition that the solution should be maximally extended results from the following typical feature of a gravity theory: the “time-evolution”—or likewise the extension of a solution on \(\Sigma\) to one defined on a bigger manifold \(\bar{\Sigma} \supset \Sigma\)—is governed by the symmetries, even if these are on-shell diffeomorphisms only. Thus evolving e.g. some region \(U \subset \bar{\Sigma}\) in some locally defined time parameter \(t\) such that \(U_t \subset \Sigma\) and \(U_0 = U\), it may happen that for some large enough \(t = t_1\) one has \(U_1 \cap U = \emptyset, U_1 \equiv U_t,\) and that \(g\) on all of \(U = U_0\) is flat while on \(U_1\) it may be nowhere flat. Then \(U_0\) and \(U_1\) may be still gauge equivalent with respect to some symmetries which only on-shell reduce to the diffeomorphism, while certainly there does not exist a diffeomorphism relating \((U_0, g|_{U_0})\) and \((U_1, g|_{U_1})\). However, by construction of this example, there exists some extension \((\bar{\Sigma}, g)\) of the local solution defined on \(\Sigma \cong U_0\) such that \((U_0, g|_{U_0})\) and \((U_1, g|_{U_1})\) are part of this extended solution. (Note that, as a manifold, \(\bar{\Sigma}\) may still be diffeomorphic to \(\Sigma \cong U_0 \cong U_1;\) thus, extending a solution defined on \(\Sigma\) need not change its topology.) However, if two solutions are already maximally extendend, then we require that they are gauge equivalent only if they (or better their respective metrics) differ by some diffeomorphism from one another. Note that by our definition it is not excluded that two not gauge equivalent solutions still have diffeomorphic metrics. This can happen, if the model carries also some other physically interesting fields, such as e.g. non-abelian gauge fields. We do not want to exclude such possibilities.

Alternatively, in the second item we could have taken recourse to infinitesimal symmetries only. So, we could have required that for any metric non-degenerate solution \((\mathcal{X}, A)\), the infinitesimal variation of \(g\), as defined in \(\mathfrak{S}\), by a gauge transformation (20) may be expressed as the Lie derivative of some vector field \(\nu \in \Gamma(T\Sigma)\) acting on \(g\). Despite the technical complications, to our mind the global description is formulated closer to the desiderata.

Further illustration of the assumptions or requirements in the definition will be provided by the examples below.

### III. CONDITIONS FOR A REASONABLE THEORY OF GRAVITY

#### A. Metric non-degeneracy

A metric \(g\) on \(\Sigma\) is non-degenerate iff \(g(v, v')|_x = 0 \forall v \in T_x\Sigma\) implies \(v' = 0\). Likewise, it is non-degenerate iff the map \(g^*: T\Sigma \to T^*\Sigma, v \mapsto g(v, \cdot)\) is invertible. (To be precise, we would have to restrict to the fiber over a point \(x \in T\Sigma\) in the last sentence; this is to be understood, also analogously in what follows). Due to (3), we have \(g^* = \frac{1}{2} \varphi^* \circ \mathcal{S}^t \circ \varphi\), where \(\mathcal{S}^t: E \to E^*, a = a^t b \mapsto a^t S^t \gamma^t b^t\) and \(\varphi^*\) is the transpose map to \(\varphi\) (again, when restricted to \(\mathcal{X}(x)\), certainly). This implies that \(\ker \varphi\) has to be the zero vector. Moreover, we learn that the image of this map is not permitted to have a nontrivial intersection with the kernel of \(\mathcal{S}^t\). Thus we find:

**Lemma 1** For (metric) non-degenerate solutions to the field equations (18) and (19), one needs

\[
\ker \varphi = \{0\} \quad \text{and} \quad \operatorname{im} \varphi \cap \ker \mathcal{S}^t = \{0\}. \quad (23)
\]

As a simple corollary of this, we find that \(\dim \operatorname{im} \varphi = d\), where \(d\) denotes the dimension of \(\Sigma\), and that \(d \leq r\), where \(r = \operatorname{rk} E\) denotes the rank (fiber dimension) of \(E\) and that the dimension of \(\ker \mathcal{S}^t\) may not exceed the upper bound

\[
\dim \ker \mathcal{S}^t \leq r - d \iff \dim \operatorname{im} \mathcal{S}^t \geq d. \quad (24)
\]

If \(S\) has definite signature, (23) is also sufficient to ensure non-degeneracy of \(g\); and then (24) is sufficient for the existence of a non-degenerate solution \(\varphi\) at least in some neighborhood \(U \ni x\) of a given point \(x \in \Sigma\) due to the local integrability of the field equations for any given choice \(\varphi \sim (\mathcal{X}, A)|_x\) at that point. Choosing the trivial topology for \(\Sigma, \Sigma \approx \mathbb{R}^d\) at that point. Choosing the trivial topology for \(\Sigma, \Sigma \approx \mathbb{R}^d\), we can identify \(U\) with \(\Sigma\) and then fulfill the first of the two conditions for an \(E\)-gravity to be reasonable.

**Proposition 1** For semi-definite \(S\) or for any \(S\) with \(\dim \operatorname{im} \mathcal{S}^t = d\) the conditions (24) are sufficient to ensure that \(S\) is non-degenerate.

In general, however, (24) is not sufficient to ensure non-degeneracy of \(g\), as the following example may illustrate:

Let \(d = 2, r = 3, S = b_1 b_1 + b_2 b_2 - b_3 b_3\), such that \(\ker \mathcal{S}^t = \{0\}\), and let \(\varphi = (b_1, b_2 + b_3)\). The conditions in (24) are satisfied, but restricting \(S\) to the two-dimensional image of \(\varphi\), it reduces to \(S = b_1 b_1\), which is degenerate. To exclude such counter-examples, one may employ the following

**Proposition 2** For any \(W \subset E\),

\[
S|_W \text{ is non-deg. } \iff S|_{W \cap Z} \text{ is non-deg.}, \quad (25)
\]

where \(Z\) (or better \(Z|_{X \in M}\)) denotes the set (not a vector space) of null vectors, \(Z = \{V \in E_X | S(V, V) = 0\}\).
Proof: One direction is trivial, namely \( \Leftarrow \), since for any non-null vector \( V \), one already has \( S(V, V) \neq 0 \). The other direction is an exercise in elementary linear algebra.

So, in addition to (23), it suffices to check that for any vector \( V \in \text{im } \varphi \cap \mathbb{Z} \), there exists a vector \( V' \in \text{im } \varphi \cap \mathbb{Z} \) such that \( S(V, V') \neq 0 \) so as to ensure that the map \( \varphi \) (or solution) \( \varphi \sim (\mathcal{X}, A) \) leads to a non-degenerate metric \( g \) (upon usage of the defining relation (3)).

B. Transformation of the metric

We now come to the derivation of the key relation(s) of the present paper. Herein we first assume that \( \varphi \) is a solution to the field equations (38), (39) providing a non-degenerate metric \( g \). With the considerations of the previous subsection, it then will be easy to determine some minimal conditions on \( S \) such that this can be achieved at all, which was one of the two conditions placed on a reasonable theory of \( E \)-gravity. According to the second item (cf Definition 3), we are then left with finding conditions on \( S \) such that the symmetries \( \delta_{\varepsilon} \) coincide with diffeomorphisms of \( \Sigma \) on-shell. In other words, we need to ensure that for any \( \varepsilon \in \Gamma(\mathcal{X}^*E) \) one either has \( \delta_{\varepsilon}g \approx 0 \) or there has to exist a \( v \in \Gamma(T\Sigma) \) such that \( \delta_{\varepsilon}g \approx L_vg \). According to (24) we may, however, rewrite the right-hand side of the last equation as another gauge transformation, \( L_vg \approx \delta_{\varphi(v)}g \). Correspondingly, we find that

\[
\forall \varepsilon \in \Gamma(\mathcal{X}^*E) \exists v \in \Gamma(T\Sigma): \quad \delta_{\varepsilon - \varphi(v)}g \approx 0, \tag{26}
\]

where \( v \) may be also the zero vector field. This condition is sufficient and also necessary so as to ensure that the underlying assignment for \( g \) defines a reasonable theory of \( E \)-gravity.

We first compute \( \delta_{\varepsilon}g \). By a straightforward calculation, using the Leibnitz rule for the symmetries, one obtains

\[
\delta_{\varepsilon}g \approx \frac{1}{2} \mathcal{X}^*\left( \mathcal{E}L_{\varepsilon\lambda}S \right)_{IJ} \varepsilon^K A^I A^J + \mathcal{X}^*(S_{IJ}) \varepsilon^I A^J. \tag{27}
\]

If, moreover, \( \varepsilon \) is the pull back of a section \( \varepsilon \in \Gamma(E) \),

\[
\varepsilon^I = \mathcal{X}^*\varepsilon^I, \tag{28}
\]

then, using the field equations, Eq. (27) may be simplified further:

\[
\delta_{\varepsilon}g \approx \frac{1}{2} \mathcal{X}^*\left( \mathcal{E}L_{\varepsilon\lambda}\right)_{IJ} A^I A^J. \tag{29}
\]

For a general map \( \varphi \) we may choose \( \varepsilon \) of the form (28). On the other hand, for a given map \( \varphi \) this covers any choice of \( \varepsilon \), if the base map \( \mathcal{X}: \Sigma \to M \) is an embedding. Due to the first field equation (18) and due to (28), this requires

\[
\ker \rho \cap \im \varphi = \{0\}. \tag{30}
\]

Let us in the following assume for simplicity that \( S \) is semi-definite, i.e. e.g. \( S \geq 0 \). (We will soon relax this condition again; however, much of the discussion simplifies in this case and may at least serve as a first orientation). Then, due to (23), there is—for fixed, permitted \( \varepsilon \) and for any point \( x \in \Sigma \)—a unique orthogonal decomposition of \( E_{X(x)} \) according to

\[
E_{X(x)} = W_x \oplus W_x^\perp, \quad W_x \equiv \im \varphi |_x, \tag{31}
\]

where a possibly nonvanishing \( \ker S^0|_{X(x)} \) is part of \( W_x^\perp \) certainly. Note that this decomposition depends on \( x \) and not only on \( X(x) \), which may be a decisive difference, if (30) is not satisfied. Alternatively, we may formulate it as a decomposition of the pullback bundle:

\[
\mathcal{X}^*E = W \oplus W^\perp, \quad W = \im \varphi, \tag{32}
\]

where now \( W \) is a subbundle of \( E \) (restricted to the image of \( \mathcal{X} \)). If not otherwise stated, we will use \( 32 \).

We know that (26) has to be satisfied for any \( \varepsilon : \Sigma \to E \) covering \( \mathcal{X}: \Sigma \to M \). We now can uniquely decompose any \( \varepsilon \) into a part inside \( W \) and \( W^\perp \), calling it \( \varepsilon_W \) and \( \varepsilon_{\perp} \), respectively: \( \varepsilon = \varepsilon_W + \varepsilon_{\perp} \). One has to be careful: if \( \varepsilon \) satisfies (28), this in general does not imply a likewise decomposition of \( \varepsilon \)—since the decomposition (41) depends explicitly on \( x \) and not only on \( X(x) \). This changes, however, if (30) is satisfied; then also \( \varepsilon = \varepsilon_W + \varepsilon_{\perp} \). It is suggestive to assume that if (26) is to be satisfied, then \( \varepsilon_W \) needs to equal \( \varphi(v) \). This is, however, not mandatory: Although there is a unique decomposition of \( \varepsilon \) into the two orthogonal parts, it does not imply that also the respective variations of \( g \) are orthogonal (so that each of them would need to vanish separately). Instead we only know that there is some \( v' \) such that \( \varepsilon_W = \varphi(v') \) and then, with \( v - v' := w \), we can merely conclude from (26) that

\[
\delta_{\varepsilon_{\perp}g} \approx \delta_{\varphi(w)g} \quad \text{for some } w \in W. \tag{34}
\]

In the present paper we do not intend to explore this relation in full generality or even in examples; nor do we want to extend the general consideration to the case of an indefinite \( S \), where part of \( W \) may be contained in \( W^\perp \). Here we will content ourselves with solving the above condition for \( w := 0 \), i.e. we require

\[
\delta_{\varepsilon_{\perp}g} \approx 0 \quad \forall \varepsilon_{\perp} \in \Gamma(W^\perp). \tag{35}
\]

In subsection 4 below, however, we will also permit indefinite \( S \), but only for \( \dim \ker S^0 = r - d \). In this case, one may just replace \( W^\perp \) by \( \ker S^0 \) in the above. We will make this more explicit later on.
C. First compatibility conditions

Within this subsection we will first assume that (36) is satisfied. Note that this can be achieved only if $\ker \rho \subset E$ is not too big; in particular, due to the first condition in (24), it cannot be satisfied if $\dim \ker \rho > r - d$, $r \equiv \dim E$.

Thus here we may analyze (35) for the case of (28) and take (29) in combination with (36). We find

$$\left( E \mathcal{L}_{\psi} S \right) (\psi_2, \psi_3) = 0 \quad \forall \psi_1 \in \Gamma(W^\perp), \quad \psi_2, \psi_3 \in \Gamma(W).$$

This relation needs to hold for any point $X \in M$, because any $X$ can be in the image of $\mathcal{X}$', the base map of $\varphi$. And according to the above definition of a reasonable theory of $E$-gravity, by using all possible maps $\varphi$, we may also vary $W$ essentially at will: Eq. (36) has to hold

$$\forall W \subset E \text{ with } W \cap \ker S^\perp = \{0\} = W \cap \ker \rho, \text{ dim } W = d. \quad (37)$$

In summary:

**Proposition 3** In any Lie algebroid $E$ together with a symmetric, semi-definite section $S$ of $E^* \otimes E^*$ defining a reasonable theory of $E$-gravity (cf. Definition 5), the conditions (24) and (36) with (37) are satisfied.

The condition (36) with (37) becomes empty certainly if $\dim \ker \rho > r - d$. The following observations may be helpful in the present context:

**Corollary 1** Any semi-definite $S$ of a reasonable theory of $E$-gravity with $\dim(\ker \rho) \leq r - d$ satisfies

$$E \mathcal{L}_{\psi} S = 0 \quad \forall \psi \in \ker S^\perp. \quad (38)$$

**Proof:** Take $\psi_1 := \psi \in \ker S^\perp$ in (36), which is possible for any choice of $W$, since always $\ker S^\perp \subset W^\perp$. Next take $\psi_2$ in (36) as any vector in $E \cap \ker \rho \cap E \cap \ker S^\perp$. Finally, choose $\psi_3$ in such a way that the span of $\psi_2$ and $\psi_3$ can lie in some $W$ compatible with (37). Since $d \geq 2$, $\dim(\ker \rho) \leq r - d$, and $\dim(\ker S^\perp) \leq r - d$, the possible choices of $\psi_2$ and $\psi_3$ lie dense in $E$. By continuity we thus can conclude $\left( E \mathcal{L}_{\psi} S \right) (\psi_2, \psi_3) = 0$ for all $\psi_2, \psi_3 \in E$—and thus Eq. (38). \( \square \)

**Proposition 4** The conditions (36) and (38) (or (40) below) are $C^\infty (M)$-linear. Correspondingly it suffices to check them on a local basis.

**Proof:** In (36) we only need to check it for $\psi_1$. Using the simple-to-verify relation

$$E \mathcal{L}_{\psi} S = F E \mathcal{L}_{\psi} S + d_E F S (\psi, \cdot) \quad (39)$$

with $\psi = \psi_1$, we find that additional contributions vanish due to orthogonality of $\psi_1$ with $\psi_2$ and $\psi_3$. Likewise, in (38) an eventually additional term vanishes since $\psi \in \ker S^\perp$.

A simplification occurs, if (24) is satisfied:

**Corollary 2** For $\dim S^\perp = d$ and $\dim(\ker \rho) \leq r - d$, Eq. (30) reduces to Eq. (35).

**Proof:** In this case always $W^\perp = \ker S^\perp$. Thus (36) is automatically satisfied upon (38), which in turn was already found to be necessary. \( \square \)

The conditions established in the present section are necessary conditions (for satisfying (36) only), which, as remarked, in the case of $\ker \rho$ too big even become vacuous. On the other hand, when $\ker \rho$ is trivial—so that (36) is satisfied always, no matter what $\varphi$—the conditions in Proposition 3 are also sufficient to define a reasonable theory of $E$-gravity. This is quite restrictive, however: it implies that $E$ is isomorphic to some integrable distribution, $E \cong D \subset TM \text{ with } \left[ \Gamma(D), \Gamma(D) \right] \subset \Gamma(D)$. So not many Lie algebroids may be covered by that Proposition.

We will consider the case of injective anchor map in some detail in the examples. It will be seen by explicit inspection then that in this case it is necessary that $\ker S^\perp$ has maximal dimension so as to be compatible with (24). At least in this case it must be possible, therefore, to derive $\dim S^\perp = d$ from the conditions summarized in Proposition 3.

Moreover, in the present paper we were not able to provide examples of admissible $E$-gravity theories which do not saturate the bound. It may be conjectured, therefore, that this is even a necessary condition in general. In any case, the conditions (36) with (37) seem very restrictive, and it is plausible that stronger results can be derived from them.

D. Sufficient conditions

The main complication one encounters with generalizing the previous considerations to arbitrary Lie algebroids (also permitting $\dim(\ker \rho) > r - d$), and to possibly necessary and sufficient conditions to ensure (38), is the fact that in general the decomposition $S$ depends on $x \in \Sigma$ itself, and not just on its image $X(x)$ with respect to $\mathcal{X}: \Sigma \rightarrow M$. Correspondingly, in (27) it is not always possible to choose a basis $b_i$ in $E$ adapted to the decomposition of $S$ into its two parts $\mathcal{E} W$ and $\mathcal{E} \perp$. (Consider e.g. the extreme case where the image of $\mathcal{X}$ is just a point in $M$). This then can have the effect that $X^*(S_{IJ}) e^\perp_W A^I = 0$ (which is true by construction of $e^\perp_W$) does not imply $X^*(S_{IJ}) e^W A^I = 0$ (the second term in Eq. (27)). And indeed, as we will also illustrate by an explicit example in section IV B below, conditions of the form (36) turn out insufficient in general, no matter what $W$ is permitted to be.

In the following we will thus content ourselves with providing some sufficient conditions for an $E$-gravity theory to be reasonable. For this purpose we assume saturation of the general bound (24). This was seen to provide a simplification already in the previous subsection (cf. Corollary 2). We now state the main result of the present paper:
Theorem 1 Any Lie algebroid $E$ together with a symmetric section $S$ of $E^* \otimes E^*$ defines a reasonable theory of $E$-gravity (as defined in Definition 3), if $\dim \Sigma = d$ and

$$E \mathcal{L}_\psi S = 0 \quad \forall \psi \in \ker S^\sharp.$$  \hfill (40)

As before, $d = \dim \Sigma$ and $r = \text{rank } E$ and $E \mathcal{L}$ denotes the $E$-Lie derivative of the Lie algebroid as defined at the end of section 4.4.

Proof of Theorem 4 We remarked already previously that the symmetries are on-shell covariant, cf. Eq. 20. Correspondingly, the total expression (27) is independent of a choice of frame $b_I$ in $E$; the unwanted contribution from the first term cancels a likewise contribution from the second one on use of the field equations (18). Let us therefore choose a basis $b_I$ adapted to the kernel of $S^\sharp: E \to E^*$: \{ $b_I$ \}$^d_{I=1} = \{ (b^A_k \epsilon_A^I)_{A=1} \}$, where $\{ b_k^A \}$ spans $\ker S^\sharp$ and $\{ b_0^I \}$ some $d$-dimensional complement $W^0$. Note that this provides a fixed decomposition of $E$ according to

$$E = W^0 \oplus \ker S^\sharp.$$  \hfill (41)

This is in spirit quite different from the decomposition \cite{31}, where for the same point $X$ in $M$ there may be different decompositions into $W_x$ and $W^\perp_x$, depending on the choice of $x \in \Sigma$. In the present context, always $W_x^\perp = \ker S^\sharp_{X(x)}$, no matter what $x$. On the other hand, in general $W_x \neq W^0_{X(x)}$. Now we use the unique decomposition of $\varepsilon$ into its two parts corresponding to \cite{31}. For the calculation of the change of $g$ with respect to $\varepsilon$ we use the other decomposition \cite{31}. We thus need to express $\varepsilon_Y$ and $\varepsilon_\perp$ in terms of the adapted basis introduced above. While the first of these two quantities in general has contributions in directions of both $W_0$ and $\ker S^\sharp$, which, moreover, depend on $x$ (and not just on $X(x)$), $\varepsilon_\perp$ appearing in \cite{31} has a very simple decomposition:

$$\varepsilon_\perp = \varepsilon_\perp^A(x) b^A_k \ker S^\sharp_{X(x)}.$$  \hfill (33)

This has no non-vanishing components in the $W_0$ direction, and likewise also does not contribute along $\ker S^\sharp_{X(x)}$. Thus obviously (in the adapted basis chosen) the second term in \cite{27} vanishes identically for $\varepsilon = \varepsilon_\perp$. But also the first term vanishes on behalf of \cite{40} and the fact that in the sum over $K$ only contributions along $\ker S^\sharp_{X(x)}$ have non-zero coefficient $\varepsilon^K_\perp(x)$. This concludes the proof.

(For non-semidefinite $S$ one merely replaces $W_x^\perp$ in \cite{31} by $\ker S^\sharp_{X(x)}$, so as to guarantee uniqueness of the decomposition. Cf. also Proposition 11 for what concerns non-degeneracy of $g_\psi$.)

If we restrict our attention to \cite{35} (instead of the more general condition \cite{33}) then under the assumption on the dimension of $\ker S^\sharp$ the condition \cite{40} is also necessary for defining a reasonable theory of $E$-gravity. This is obvious from the above proof.

As mentioned in Proposition 11 the condition \cite{40} is $C^\infty$-linear. In fact, this goes even further \cite{40}.

Proposition 5 For any $S \in \Gamma(\mathbb{S}^2 E^*)$ of constant rank satisfying \cite{44} in a Lie algebroid $E$, $K := \ker S^\sharp$ defines a Lie subalgebroid.

Proof: Due to $[E \mathcal{L}_\psi, E \mathcal{L}_\varphi] = E \mathcal{L}_{[\psi, \varphi]}$ also the product of two sections $\psi, \varphi \in \Gamma(K)$ annihilate $S$. Assuming that $[\psi, \varphi] \notin \Gamma(K)$, one obtains a contradiction: $\exists \psi_3 \in \Gamma(E)$ such that $0 \neq S(E \mathcal{L}_\psi, \psi_3) = E \mathcal{L}_\psi (S(\psi, \psi_3)) - S(\psi_2, E \mathcal{L}_\psi, \psi_3) = 0$, since $\psi_2 \in \ker S^\sharp$. \hfill \Box

As an obvious consequence we find

Corollary 3 $\rho(K) \subset TM$ is an integrable distribution.

Under appropriate conditions, including that the foliation induced by $\rho(K)$ is a fibration, this may be used to define quotient bundles, to which $S$ projects as an otherwise arbitrary nondegenerate $E$-2-tensor. We intend to come back to this elsewhere.

IV. FIRST EXAMPLES

In this section we provide some elementary examples following the list after Definition 2.

A. Lie algebra case

In the case of an ordinary Lie algebra, $E = \mathfrak{g}$, only the second field equation is nontrivial. One then regards flat connections modulo gauge transformations on a $d$-dimensional spacetime $\Sigma$. In this way for $d = 3$ one may obtain standard $2+1$ gravity (with or without cosmological constant) in its Chern Simons formulation \cite{32,33}, to mention the most prominent example: If there is no cosmological constant, the Lie algebra is the one of $ISO(2,1)$, while in the presence of a cosmological constant $\lambda$ it is $so(3,1)$ and $so(2,2)$, depending on the sign of $\lambda$. So, in each of these cases, the “rank” $r$ of the bundle (here just over a point) is six. The rank of the bilinear form $S$ used in \cite{32,33} to construct the metric $g$ on $\Sigma$ is three, on the other hand, saturating our general bound \cite{21}. Equation \cite{40} just reduces to ad-invariance of that (degenerate) inner product on $\mathfrak{g}$ with respect to those elements of $\mathfrak{g}$ which are in its kernel $\mathfrak{t}$. According to Proposition 5 $\mathfrak{t}$ is a Lie subalgebra of $\mathfrak{g}$.

It is worthwhile to mention that all of these three Lie algebras permit an ad-invariant non-degenerate inner product. This is used to construct the action functional \cite{32,33}. However, this inner product cannot be used to construct the metric $g$—at least if we want to comply to the conditions in Theorem 4 where we need a degenerate inner product. For non-zero $\lambda$, the degenerate bilinear form $S$ of \cite{32,33} used to construct the metric is in fact not ad-invariant with respect to all of $\mathfrak{g}$, but only for $\mathfrak{t} < \mathfrak{g}$.
B. Bundles of Lie algebras

This is the case where \( E \) is some vector bundle over \( M \), equipped with a smoothly varying Lie algebra on the fiber. The field equations (35) and (39) then just imply that \( \Sigma \) has to be mapped to one (arbitrary) point \( X_0 \) in \( M \) and that the \( A \)-field is a flat connection of the Lie algebra in the fiber above \( X_0 \). Gauge transformations are again just of the standard Yang-Mills type, where the point \( X_0 \) cannot be changed (although there is a solution for any point \( X_0 \in M \)). The type of the Lie algebra (although not its dimension, since this is fixed by the rank of the bundle \( E \)) can change upon a different choice of \( X_0 \).

Let us further specialize to the case of abelian Lie algebras, i.e. all the fibers carry the same, trivial Lie algebra \([-,[,] = 0\). Then, irrespective of the choice of \( X_0 \), one finds as local solutions for the \( A \)-fields, \( A^I = df^I \), \( i = 1, \ldots, r \), for some functions \( f^I \). Gauge transformations correspond to \( A^I \sim A^I + dg^I \) for some arbitrary functions \( g^I \).

We want to work out this example in full detail, making a general ansatz for \( S \),

\[
S = \frac{1}{2} S^I_J (X) b_I b_J ,
\]

and then determine the conditions to be fulfilled such that the respective \( E \)-gravity becomes non-degenerate. This then will be compared with the general results obtained in the previous section. Note in the present case \( \ker \rho = E \), so that we cannot refer to the (necessary) conditions obtained in subsection III.C we can only refer to (III.B) which, however, had the drawback that it provided only sufficient conditions—due to the restriction on the rank of \( S \). In the present example we will find this restriction to be also necessary. In the case of bundles of abelian Lie algebras, moreover, the condition (III.D) (or also (III.C)) is empty, since the \( E \)-Lie derivative vanishes identically.

Let us now work out this example explicitly. With (8) we find that any solution for \( g \) is of the form

\[
g = \frac{1}{2} S^I_J (X_0) df^I df^J \]

in the present case. First we require the existence of non-degenerate solutions. For this \( S(X_0) \) needs to have at least rank \( d = \dim \Sigma \)—and in the spirit of the remark at the end of section III.A for all \( X_0 \in M \). This coincides with what we found in (24). (For simplicity we take \( S \) to have positive semi-definite signature so that this condition is also sufficient.)

To simplify the further discussion, let us in the following diagonalize the matrix \( S(X_0)_{ij} \): This can be done most easily by transition to new functions representing the solutions for \( A^I = df^I \), \( i = 1, \ldots, d \) and \( f^I = 0 \) for \( I > d \); then

\[
g = \frac{1}{2} dx^i dx^n ,
\]

which is just the standard flat metric on \( \mathbb{R}^d \).

Next we need to check the behavior of \( g \) under gauge transformations. We at once observe that with respect to gauge transformations (20) on a topologically trivial \( \Sigma \), any \( g \sim 0 \). A likewise statement holds also for 2+1 gravity in its Chern Simons formulation discussed above or 2d dilaton gravity in its Poisson Sigma formulation recalled below—and we thus do not want to exclude such a feature. However, restricting the gauge transformations to the non-degenerate sector, i.e. disregarding all solutions to the field equations which somewhere on \( \Sigma \) have \( \det g = 0 \), we want the gauge transformations to boil down to diffeomorphisms of \( \Sigma \), and to nothing else.

How or under what conditions is this realized in the present simple example? Indeed, suppose first that the rank \( m \) of \( S \) equals \( d \). Then any permitted \( g \) is of the form \( g = \frac{1}{2} \sum_{\mu=1}^d df^\mu df^\mu \) with \( \det g \neq 0 \), which is equivalent to \( df^I \wedge \ldots \wedge df^r \neq 0 \) in this case. Clearly, such a \( g \) is always flat, and any two such non-degenerate solutions are related to one another by a diffeomorphism of \( \Sigma \).

Now let us assume that \( m > d \). We clearly obtain the above flat solution \( g \) always, too, since we can just require all \( f^I \) with \( I > d \) to vanish. But then any other non-degenerate solution \( g \) on \( \mathbb{R}^d \) should be diffeomorphic to this flat solution (since the moduli space of solutions with respect to the symmetries (20) consists of one point here and in a reasonable theory of gravity these two notions coincide—up to the subtleties mentioned), i.e. be flat too. This, however, is not the case for \( m > r \): A straightforward calculation shows that the curvature of \( g_{\mu \nu} = \delta_{\mu \nu} + f_{,\mu} f_{,\nu} \) is non-zero (for sufficiently general second derivatives of \( f \)). This metric, however, is a gauge relative to (44) with respect to the symmetries (20), provided only \( m > r \) (and if \( f \) is kept sufficiently small, it never passes through a non-degenerate sector).

In summary we find in this simple case:

**Proposition 6:** For \( E \) being a bundle of abelian Lie algebras, \( \dim \text{im} S^I = d \) is necessary and sufficient for defining an admissible \( E \)-gravity theory.

Let us use the occasion to also illustrate the possible complications arising from the \( x \)-dependence of the decomposition (31). Let us choose \( \Sigma = \mathbb{R}^2 \), \( E = M \times \mathbb{R}^3 \), and \( S_{IJ} = \delta_{IJ} \). Consider \( A^I = dx^1 \), \( A^2 = dx^2 \), \( A^3 = df \) with \( f \in C^\infty(\mathbb{R}^2) \). Thus \( W_x = (\partial_1 + f,1(x)\partial_3, \partial_2 + f,2(x)\partial_3) \). Clearly, the following gauge parameter is in \( W^1_x \): \( \epsilon = \epsilon_1 = (f,1, f,3, -1) \) (the entries corresponding to the standard basis \( \{ \partial_i \}_{i=1}^2 \) in the fiber of \( E \)). We now turn to (27): the first term in (27) vanishes identically, whatever \( \epsilon \). In this example, the second term is already on-shell covariant by itself (due to \( df^1 \approx 0 \)). By construction, \( X^* S_{IJ} f_1^I A^J \approx 0 \). However, e.g. \( \delta_1 g_{11} \approx \delta_{11} (\epsilon_1)_{,1} A_1^1 = f_{,11} \neq 0 \) (for nonvanishing second derivatives of \( f \) with respect to \( x^1 \)). This
may be contrasted with the proof of Theorem 1 where the second term in \(24\) was found to vanish (upon an appropriate choice of a basis) no matter how complicated the \(x\)-dependence of the decomposition \(31\).

In the present example the resulting solutions for \(g\) were found to be flat always—for admissible theories. This will change in general, if the Lie algebras are taken non-abelian.

C. The standard Lie algebroid and integrable distributions

We first consider the illuminating case \(E = TM\). The field equations \(18\) reduce to \(dX^i = A^i\) in this case, while the second set of field equations, \(19\), becomes an identity. Thus, in the present case, using \(8\) we find

\[
g \approx \frac{1}{2} S_{ij} dX^i dX^j = \mathcal{X}^* S, \tag{45}
\]

\(g\) is just the pullback of the section \(S \in \Gamma(\sqrt{2} T^* M)\) to \(\Sigma\) by the map \(\mathcal{X}\).

Let us consider first \(n = d, n = \dim M\). Then by condition \(24\), \(S\) is non-degenerate and defines a metric on \(M\) and \(g\) is nothing but the pullback of this metric to \(\Sigma\). Note that this provides a simple example where the necessity for the addition of maximal extension of \(\Sigma\) in the second item of Definition 1 or 3 becomes quite transparent (cf. also the discussion at the end of subsection 11). \(\Sigma\) may be quite different for different regions in \(M\). After maximal extension, \(\mathcal{X}: \Sigma \rightarrow M\) is a (possibly branched) covering map (thus locally a diffeomorphism). If one does not permit the branched coverings (e.g. by requiring solutions to be either geodesically complete or that curvature invariants diverge towards the ideal boundary—assuming that this is satisfied for \(S\) on \(M\), so as to exclude constructions of solutions such as those in \(31\), the moduli space of classical solutions in this example is zero dimensional, independently of the topology of \(\Sigma\). It may consist, however, of several representative solutions, parametrized by the homotopy (covering) classes of the map \(\mathcal{X}\).

We now turn to \(d < n\). It is obvious that for \(\dim S^2 > d\) the conditions of a reasonable theory of \(E\)-gravity cannot be satisfied, since then we can embed some maximally extended \(\Sigma\) in various ways such that \(\mathcal{X}^* S\) will give non-equivalent (i.e. non-diffeomorphic) metrics on \(\Sigma\). Correspondingly, we find upon explicit inspection that in the case of \(TM\) we need saturation of the bound \(24\).

The condition \(38\) just becomes an ordinary Lie derivative on \(M\) in the present context. \(S\) has to be invariant with respect to motions in directions of its kernel. This kernel is integrable according to Corollary 3. \(S\) then provides a metric in the normal bundle of the leaves of the respective foliation. This metric is invariant along the foliation, moreover; it thus projects to locally defined quotient manifolds (we are dealing with regular foliations, since the rank of \(S\) is constant; thus locally the foliation is a fibration and a quotient manifold, the base of the fibration, can be defined), on which it is nondegenerate, moreover. Such a structure on \(M\) is called a Riemannian foliation, cf. e.g. \(R\) (or then probably pseudo-Riemannian foliation if the locally projected metric has indefinite signature).

Let us provide a simple example of such a feature on \(M = \mathbb{R}^3\) coordinatized by \((X, Y, Z)\): \(S := dX^2 + dY^2\). Clearly this is invariant with respect to \(\partial_2\), \(L_Z S = 0\), which spans its kernel; according to Proposition 4 and Corollary 2 this is already sufficient for a check. The foliation in this simple example is a fibration over \(\mathbb{R}^2 \ni (X, Y)\) with fiber \(\mathbb{R} \ni Z\). Thus the quotient manifold here exists even globally, and it is equipped with the non-degenerate metric \(dX^2 + dY^2\) (the projection of \(S\)).

In this example we assumed \(d = 2\), certainly. The representative metric \(g\) on \(\Sigma\) is the unique flat metric in this case (if it exists); whatever embedding of \(\Sigma\), for a non-degenerate solution one needs \(dX \neq 0\) and \(dY \neq 0\), and then \(X\) and \(Y\) can be taken as possible coordinates on \(\Sigma\). Note that in this example the topology of \(\Sigma\) permitting everywhere non-degenerate solutions for \(g\) is quite restricted; in fact, there is e.g. no compact Riemann surface for \(\Sigma\) with this property. Note that at least locally we can identify a solution on \(\Sigma\) with the quotient manifold constructed above. Different embeddings of \(\Sigma\) (provided homotopic along the \(Z\)-fibers) are just gauge equivalent; and this is also obvious from the explicit solutions since in any solution as constructed above \(g\) is independent of the function \(Z\).

In the present context one may certainly also easily obtain non-flat metrics \(g\) on \(\Sigma\); e.g., just multiply the above \(S\) by some non-trivial function \(f(X, Y)\).

As the standard Lie algebroid \(E = TM\) in general is one of the main models for a Lie algebroid, the guideline for it besides ordinary Lie algebras, it may be used also in the present context for further orientation in the general Lie algebroid case. Equipping a Lie algebroid \(E\) with an \(E\)-tensor \(S\) satisfying the conditions in Theorem 1 might thus be called equipping \(E\) with an \(E\)-Riemannian foliation.

Qualitatively, the situation does not change much when we regard integrable distributions \(D \subset TM\). First we observe that according to \(18\) \(\mathcal{X}\) maps \(\Sigma\) completely into a leaf \(L\) of the distribution. The moduli space of classical solutions will now consist of homotopy classes of such maps. But otherwise the discussion is essentially as before, just—for any fixed base map \(\mathcal{X}\) with \(TM\) replaced by \(D|_L = TL\). Note that now \(S\) is not just a covariant 2-tensor on \(M\). Rather, it is the equivalence class of such 2-tensors, where two 2-tensors are to be identified, if they coincide upon contraction with arbitrary vectors tangent to the distribution \(D\). As for an explicit example we may extend the previous one for \(TM\) by adding one or more further directions to \(M\), considering the previous \(M\) as a typical leaf \(L\).

**Proposition 7** For Lie algebroids with injective anchor map \(\rho\) the conditions in Theorem 1 are also necessary.
V. SPECIALIZING TO POISSON GRAVITY

A. Compatibility and Integrability

We first want to specialize \ref{10} to the Poisson case. Here \( E = T^*M, M \) the given Poisson manifold, and \( S \) becomes a symmetric, contravariant 2-tensor on \( M \), 
\( S = \frac{1}{2} S^{ij} \partial_i \partial_j \). Correspondingly, \( S^j; T^*M \to TM \) and \( \text{im} S^j \subset TM \) defines a distribution on \( M \). In contrast to the distribution of \( P^i \), which induces the symplectic foliation of \( M \), this distribution is not necessarily integrable. To provide a non-integrable example, just consider \( d = 2, M = \mathbb{R}^3 \) with the trivial Poisson bracket—so that the condition \ref{10} becomes empty—and take \( S = (\partial_1 + X^3 \partial_3) \partial_3 \).

The structure defined on \( M \) by \( S \) (with constant rank) is called a sub-Riemannian (or in the indefinite case then sub-pseudo-Riemannian) structure, cf. e.g. \ref{8}. According to Theorem \ref{11} it has to satisfy a particular compatibility condition with the Poisson structure on \( M \). We will now make this explicit under a further assumption, namely that \( \text{im} S^j \) is an integrable distribution. (In the context of sub-Riemannian structures this is usually assumed to not be the case; still, here we assume this for simplicity.) Then at least locally one may characterize the corresponding leaves, which we want to denote by \( R \), by the level set of some functions \( f^\alpha \in C^\infty(M) \), \( \alpha = 1 \ldots n - d \), where \( n \equiv \text{dim} M \). (Likewise the symplectic leaves will be denoted by \( L \) or \( (L, \Omega_L) \), where \( \Omega_L \) is the symplectic 2-form induced by the Poisson tensor \( P \) on \( M \); locally any leaf \( L \) can be characterized as the level set of some Casimir functions \( C^I, I = 1 \ldots n - k \) where \( k = \text{rank} P_X \) for any point \( X \in L \).

For any symmetric or antisymmetric (say contravariant) 2-tensor \( T \), the annihilator of its image \( \text{im} T^2 \subset TM \) is its kernel, \( \ker T^2 \subset TM \). In the above case we thus can span the kernel of \( S^j \) by \( df^\alpha \), \( \ker S^j = \langle df^\alpha \rangle_{\alpha=1} \). The quotient map \( S^j; T^*M/\langle df^\alpha \rangle_{\alpha=1} \to \text{im} S^j \) has an inverse. Since at any point \( T^*R \) may be identified with \( T^*M/\langle df^\alpha \rangle_{\alpha=1} \), this implies that the leaves \( R \) are equipped with some pseudo-Riemannian metric. Let us denote the latter by \( \mathcal{G}_R \).

So, in this scenario, \( M \) is foliated in two different ways, by symplectic leaves, \( M = \bigsqcup (L, \Omega_L) \), and by Riemannian ones, \( M = \bigsqcup (\mathcal{G}_R, \mathcal{G}_R) \). While the former foliation may be quite wild, the leaves of the latter ones all have at least the same dimension, namely \( d \equiv \text{dim} \Sigma \). (We will content ourselves here with specializing the conditions of Theorem \ref{11} to the Poisson case—under the additional assumption of integrable \( \text{im} S^j \).) The situation simplifies further if the pseudo-Riemannian foliation is even a fibration, such that in particular the quotient manifold \( M/R \) is well-defined (and leaves \( R \) lying dense in some higher dimensional submanifolds of \( M \) are excluded).

The two foliations (or, more precisely, the two structures induced on \( M \) by \( P \) and \( S \), respectively) are not independent from one another, due to condition \ref{10}:

\textbf{Proposition 8} Let \((M, P)\) be a Poisson manifold and \( S \in \Gamma(\sqrt{2}TM) \) define a \( d \)-dimensional fibration into pseudo-Riemannian leaves \((R, \mathcal{G}_R)\) as described above. Then \((M, P, S)\) define a reasonable theory of Poisson gravity, if

\[ \mathcal{L}_{v_f} S = 0 \quad \forall f \in C^\infty(M/R), \]  

where \( v_f \equiv \{ f, \cdot \} = P^{i\dot{j}} f_{,\dot{j}} \partial_i \) is the Hamiltonian vector field of the function \( f \) and \( \mathcal{L} \) denotes the usual Lie derivative.

To prove this statement it suffices to check that for closed 1-forms \( \alpha \in \Omega^1(M) \) one has \( \mathcal{L}\alpha = \mathcal{L}\rho(\alpha) \) since \( \rho(df) = v_f \). Due to Proposition \ref{9} it is sufficient to check \ref{10} for the set of \( n - d \) functions \( f^\alpha \) introduced above.

Let us add two remarks, valid in the simplified circumstances of the above Proposition. First, according to Proposition \ref{5} \( \ker S^j = \langle df^\alpha \rangle_{\alpha=1} \) defines a Lie subalgebroid. It is isomorphic to the Poisson type Lie algebroid \( T^*(M/R) \), where, by construction, the projection \( \pi: M \to M/R \) is a Poisson map (since \( f^\alpha \) can be thought of as pullback of at least locally defined functions on \( M/R \) —and since \( \ker S^j \) is a Lie subalgebroid of \( T^*M \) \ref{42}). Second, according to Corollary \ref{8} \( P^i(\ker S^j) \) defines an integrable distribution; it is spanned by the vector fields \( v_f \) appearing in Eq. \ref{10}. In general this foliation is distinct from the symplectic and from the Riemannian foliation. Clearly, its leaves always lie in symplectic leaves \( L \), however. Its relation to the Riemannian leaves is given by

\textbf{Proposition 9} The leaves of \( P^i(\ker S^j) \) lie inside the intersection \( L \cap R \) of symplectic and Riemannian leaves, iff \( R \) are coisotropic submanifolds of \((M, P)\).

\textbf{Proof:} \( v_{f,\alpha} \in \text{im} S^j \leftrightarrow \langle \ker S^j, v_{f,\alpha} \rangle = 0 \leftrightarrow \{ f^\alpha, f^\beta \} = 0. \)

\[ \square \]

As a rather obvious consequence of this we find

\textbf{Proposition 10} The (pseudo-)Riemannian structures on the leaves \( R \) may be chosen independently from one another (still varying smoothly certainly) iff the leaves \( R \) are coisotropic submanifolds of the Poisson manifold \( M \). In this case the metric on each leaf \( R \) has \( k \) independent Killing vector fields, where \( k = \text{dim} P^i(\ker S^j) \).

Note that in the present Poisson case this does not imply automatically that also the metric \( g \), defined on \( \Sigma \), has Killing vector fields. In fact, by the field equations, \( \Sigma \) is mapped into the symplectic leaves, which in general are different from the Riemannian ones; moreover, \( g \) is not just an ordinary pullback. Cf. also Section \ref{V.E} below for illustration.

In Proposition \ref{9} we assumed a fibration. More generally, for a foliation one has a likewise statement, if one uses for any \( X \in M \) sufficiently small neighborhoods \( U_X \) and replaces the above functions \( f \) by functions constant along connected components of \( R \cap U_X \) for any leaf \( R \).

We conclude this subsection with an obvious remark:
Corollary 4 If each leaf of the foliation induced by $S$ consists of a union of symplectic leaves (the leaves induced by $\mathcal{P}$), the condition (47) or (48) is fulfilled trivially.

This just corresponds to the particular case $k = 0$ in Proposition 10.

B. Examples in two dimensions

Most of the studied 2d gravity models result from the following simple choice of $(M, \mathcal{P}, S)$ \cite{8}: Take $M = \mathbb{R}^3$ with linear coordinates $(X^a, X^3)$, where the index $a$ runs over two values, which conventionally we denote by 1 and 2 when we consider Euclidean signature gravity and by $+$ and $-$ for Lorentzian signature; these indices will be raised and lowered by means of the standard flat Riemannian or Lorentzian metric $\eta_{ab}$, furthermore. The Poisson structure is then defined by $\{X^a, X^b\} = \varepsilon^{ab} W(X^c X_c, X^3)$

\begin{equation}
\{X^a, X^3\} = \varepsilon^{ab} X_b ,
\end{equation}

where $\varepsilon^{ab}$ are the (contravariant) components of the antisymmetric $\varepsilon$-tensor and $W$ is any smooth two-argument function. With an ansatz $S := \lambda^{ab}(X) \partial_a \partial_b$, ker $S^3 = \langle dX^3 \rangle$. Eq. (47) shows that $X^3$ generates (Lorentzian or Euclidean) rotations in the planes of constant $X^3$. According to Proposition 8, $S$ has to be invariant with respect to these “rotations”. As one may easily convince oneself, requiring $S$ to be smooth on all of $\mathbb{R}^3$ leaves only

\begin{equation}
S = \frac{1}{2} \gamma (X^c X_c, X^3) \eta^{ab} \partial_a \partial_b
\end{equation}

for some nonvanishing two-argument function $\gamma$. In most applications (cf. below) either $\gamma = 1$ or $\gamma$ depends on $X^3$ only.

In this example, the symplectic foliation is given by variation of the integration constant in the solution of the first order differential equation $d u / dv = W(u, v)$, replacing $u$ by $X^a X_a$ and $v$ by $X^3$ \cite{1}. E.g. for $W = V(X^3)$ one finds $X^a X_a = \int X^3 V(v) dv = \text{const.}$ as for the symplectic leaves. They are generally two-dimensional, except at simultaneous zeros of $X^a$ and $W(0, X^3)$, where the Poisson tensor vanishes altogether. On the two-dimensional leaves the symplectic form can be written as $\Omega_L = dX^a \wedge d\varphi$ where in the Euclidean case $\varphi$ is the standard azimuthal angular variable around the $X^3$ axis and in the Lorentzian case e.g. ln $X^+$. The pseudo-Riemannian leaves are characterized by $X^3 = \text{const}$ in the above ansatz and $\mathcal{G}_R = \frac{1}{2} \eta_{ab} dX^a dX^b$. Note that the intersection of the symplectic and the Riemannian leaves are rotationally invariant. So, e.g. for Euclidean signature they are generically circles in the $X^3$-planes around the origin $X^a = 0$.

C. Relation to 2d Lagrangians

We want to be rather brief here due to the rather extensive literature on this subject (cf. e.g. \cite{8} and \cite{13} for two recent reviews), deferring more details also to a separate work \cite{14}. Still it is illustrative to show how intricate the relation to some more established actions may be.

First of all we just mention that geometrical action functionals of the form $L[g] = \int_\Sigma d^2x \sqrt{|\det g|} f(R)$ or also

\begin{equation}
L[g, \tau^a] \sim L[e^a, \omega] = \int_\Sigma d^2x \sqrt{|\det g|} F(R, \tau^a \tau_a) \tag{49}
\end{equation}

may be covered by the choice in the previous subsection with $\gamma \equiv 1$. Here $R$ denotes the Ricci scalar (in two dimensions this contains all information about the curvature tensor), in the first case of the torsion-free Levi-Civita connection of $g$ and in the second case of the connection $\omega$ with torsion scalar $\tau^a$. The functions $f$ and $F$ are the Legendre transforms of the functions $V$ and $W$ of the preceding subsection, respectively. In these examples one identifies $A_a$ with a zweibein $e_a$ and $A_3$ with the single non-trivial component $\omega$ of the spin connection, using the action functional \cite{14} Poisson Sigma Model. We refer to \cite{8} for a detailed study of this relation, including a discussion of the situation when the functions $f$ or $F$ have Legendre transforms locally only.

Secondly, also dilaton-like Lagrangians

\begin{equation}
L[g, \phi] = \int_\Sigma d^2x \sqrt{|\det g|} \left[ \phi R + W((\nabla \phi)^2, \phi) \right] \tag{50}
\end{equation}

can be covered. At least if $W$ in (50) is at most linear in its first argument, there are various ways of doing so. In one of these, one usually performs a dilaton-dependent conformal transformation of $g$ and then identifies $A_3$ in (11) with the zweibein and spin connection of the new metric $\omega$ and $X^3$ with the dilaton $\phi$ (while $X^a$ become Lagrange multipliers for torsion zero; in this case $W_{\text{Poisson}}(X^a X_a, X^3) = W_{\text{above}}(0, X^3)$, cf. e.g. \cite{2} for further details). Closer to the present point of view is to regard this as a one-step-procedure, using the conformal factor as an $X^3$-dependent $\gamma$ in (48).

There is also another route to (50), which moreover works for general $W$ (which now agrees with the function introduced in the Poisson structure within the previous subsection), cf. e.g. \cite{2}. We believe that it is worthwhile to be a bit more explicit in this case. We first choose to identify $g$ with

\begin{equation}
g = \frac{1}{2} \eta^{ab} A_a A_b , \tag{51}
\end{equation}

corresponding to $\gamma = 1$ in (48). Then let $\omega(A)$ denote the solution of $dA^a + \varepsilon^{ab} \omega \wedge A_b = 0$—such a solution always exists and is even unique (interpreting $A_a$ as zweibein, this is the corresponding torsion-free connection 1-form). Next, shift $A_3$ by $\omega(A)$ in (11), $A_3 \rightarrow A_3 := A_3 + \omega(A)$. Then, with the previous choice of $\mathcal{P}$, the action functional
takes the form (again we replace $X^3$ by $\phi$)

\[
L[X,A] = \int_{\Sigma} (\omega(A) \wedge d\phi + \frac{1}{2} \varepsilon^{ab} A_a \wedge A_b W(X^c X_c, \phi) + A_3 \wedge (d\phi + \varepsilon_{abc} X^a A^b)).
\]

(52)

Classically it is always permitted to eliminate fields of an action functional by their own field equations. We apply this to $\tilde{A}_3$ and $X^a$: variation with respect to $\tilde{A}_3$ determines $X^a$ uniquely, and vice versa. Implementing this into (52), the last term vanishes, and $X^a$ is replaced by $-\varepsilon^{ab} A_b g^{\mu\nu} \partial_{\mu} \phi$. After a partial integration in the first term, this is now seen to coincide with (50). Note that the elimination of $X^a$ required that $A^a_{k\lambda}$ has an inverse (this is reflected by the use of $g^{\mu\nu}$ in the above explicit expression), which is legitimate after one identifies $g$ as in (61); the equivalence of action functionals then certainly may be expected also only on the metric non-degenerate solutions.

If in the first term of (50) $\phi$ is replaced by some function $U(\phi)$, one still may use the functional (1) along the above lines, where now $X^3 = U(\phi)$. Certainly this works only in regions of $\phi$ where $U$ is monotonic. If $U$ has several monotonic parts, one needs to use several action functionals to recover all the classical solutions. However, at least if $W$ in (50) is at most linear in its first argument, $W = V_1(\phi) + V_2(\phi)(\nabla^2 \phi)^2$, it may be shown (62) under rather mild conditions on $U$, $V_1$, and $V_2$, that for any solution with $\phi$ taking values within one of the monotonic sectors of $U$, the classically allowed values of $\phi$ remain in this sector.

D. Further 2d examples

Up to now, 2d models where considered merely with the 3d Poisson manifold $(M_3, \mathcal{P}(W))$ as provided in subsections V, VI—with only one exception, namely the inclusion of Yang-Mills fields [7]. In this case the total Poisson manifold has the form $(M, \mathcal{P}) = M_3 \times g^*$, where the function $W$ is permitted to depend also on the Casimir functions of the Lie Poisson manifold $g^*$. If $\gamma$ in (48) depends on $X^a X_a$, $X^3$ and again these Casimirs only, then the condition (10) in Proposition 8 is satisfied and $(M, \mathcal{P}, \mathcal{S})$ defines a reasonable theory of 2d Poisson gravity.

But certainly many more examples or models can be constructed by the general methods of the present paper, even in the realm of two dimensions. Let us use this fact to construct a simple PSM, the local solution space of which coincides with any given parametric family of 2d metrics, say $g(C_\alpha) = \frac{1}{2} g_{\mu\nu} (C_\alpha, x^1, x^2) dx^\mu dx^\nu$ in some local coordinate system $x^\mu$ and with $C_\alpha$, $\alpha = 1 \ldots N$ local parameters in some given $N$-dimensional manifold $\mathcal{M}$ of moduli. Just take $\mathcal{M} = M \times \mathbb{R}^2$, equipping the former factor with zero Poisson bracket and $\mathbb{R}^2$ with the standard one: $(q, p) \in \mathbb{R}^2$, $(q, p) = 1$, $(q, q) = 0 = \{p, p\}$.

Now choose $\mathcal{S}$ as

$$\mathcal{S} = \frac{1}{2} g_{11} \partial_q \partial_q + \frac{1}{2} g_{22} \partial_p \partial_p - g_{12} \partial_q \partial_p,$$

where all the coefficient functions $g_{\mu\nu}$ are evaluated at $(C_\alpha, q, p)$. Due to Corollary 4 this provides a reasonable theory of 2d gravity. With the simple field equations $dq + A_p = 0$, $dp - A_q = 0$, $dC^\alpha = 0$, following from (3) and the fact that locally the second set of field equations (3) may be satisfied without the addition of any further moduli parameters when using the symmetries (6) one obtains the desired result.

In (10) it was shown that for (30) one finds the (local) solution space of the exact string black hole for no choice of $W$: a model which does the job is trivially included in our framework, as just demonstrated for an arbitrary family of 2d metrics. (This can be easily adapted to also include the given parametric dependence of the dilaton field $\phi$). Likewise constructions also work in any dimension $d$, moreover.

In (13) and (14), on the other hand, the Poisson bracket on $\mathbb{R}^3$ recalled above was modified, in particular breaking (or “$\kappa$-deforming”) its rotational invariance. This is used hand in hand with the unmodified notion (31) for the metric. Correspondingly, the conditions for a theory to be admissible or “reasonable” in our sense and as made precise in Definition 3 are violated—with the corresponding features, too.

To provide an even simpler example of what happens when these conditions are violated, consider e.g. $\mathbb{R}^3 \ni (q, p, C)$ with $\mathcal{P} = \frac{1}{2} \partial_q \wedge \partial_p$ and $\mathcal{S} = \partial_q \partial_p + f(q, p) \partial_C \partial_C$ for some fixed, non-constant function $f$. Here ker $\mathcal{S}$ is trivial, and ker $\rho = (\partial_C)$. Now the necessary conditions of Proposition 4 are violated: Indeed, take $W = \langle dq, dp + dC \rangle$ such that $W^\perp = \langle dp - (C) \rangle$; now (30) is violated due to $(\mathcal{L}_{\partial_q} \mathcal{S}) (dp + dC, dp + dC) \propto \partial_q f \neq 0$. The general local solution for $g = \frac{1}{2} \mathcal{S}^{ij} A_i A_j$, on the other hand, takes the form $q = dq^2 + dp^2 + f d\lambda^2$, where $\lambda$ is an arbitrary function of $q$ and $p$ and pure gauge according to (30). Clearly, for $\lambda \equiv 0$ the metric is flat, while otherwise it is generically not. (Note that this example of a non-reasonable theory works also for $f = 1$, while then the necessary conditions found in Proposition 4 are too weak to exclude this case).

As we want to stress in the present note, the action functional (1) of the PSM alone certainly does not entail any notion of a metric $g$ on $\Sigma$; and the identification of $g$ implicitly introduces or requires further structures. These structures are identified as a symmetric contravariant 2-tensor on $M$ in the present paper. Moreover, the 2-tensor may not be chosen at will, but has to satisfy a compatibility condition with the chosen Poisson structure. The present framework is independent of coordinates on $M$, thus also permitting a coordinate independent discussion of compatibility (cf. 15).
E. Relation of Killing vectors to target geometry

It has been observed that all solutions of any gravity model resulting from the structures as defined in subsections 4.3 and 4.4—have at least one local Killing vector field $v \in \Gamma(T\Sigma)$. Moreover, along these lines, $X^3$ is constant; correspondingly, the Killing lines are apparently related to the one-dimensional intersection of the symplectic with the pseudo-Riemannian leaves. (Recall that classically only maps from $\Sigma$ into a symplectic leaf is compatible with the pseudo-Riemannian leaves are characterized by $X^3 = \text{const}$.) This is qualitatively different from the construction of models in subsection 2.3 for any given family of metrics $g$ on $\Sigma$, in particular also metrics without Killing symmetry: there the symplectic foliation agreed with the pseudo-Riemannian one is constant; correspondingly, the Killing lines are apparent.

We will make this relation explicit in the somewhat simplified scenario of Eq. (33), which, on behalf of the field equations (34), implies that locally we may identify $\Sigma$ with a symplectic leaf $L$ and we likewise may fall back on (28) and (29). In the present case $\ker S^2 = (dX^3)$. Let us denote a local Casimir function of the symplectic foliation by $C$, then $\rho = \ker \mathcal{P}^H = (dC)$. Now, for any point of $X \in \text{im} \mathcal{X} \subset M$ there is a unique $\lambda(X)$ such that $(\lambda dX^3 + dC)_{X(x)} \in \text{im} \varphi_x \subset T^*_{X(x)} M$. Thus, for any $\mu \in C^\infty(M)$ we have $\sigma := \mu(dX^3 + dC) \in \text{im} \varphi$. Let us calculate the variation of $g$ with respect to $X^3$:

$$\delta X^3 g \approx \frac{1}{2} v^* (E\mathcal{L}_v S)^{ij} A_i A_j = \frac{1}{2} v^* (E\mathcal{L}_\mu C S)^{ij} A_i A_j, \quad (53)$$

where we made use of Eq. (40). We know that the left hand side may be rewritten as the Lie derivative of a (fixed) vector field $v$ on $\Sigma$, since $v$ was in the image of $\text{im} \varphi$ and we then may employ (21). Thus, we have accomplished our task of showing the existence of a local Killing vector field provided that the right hand side of Eq. (156) can be made to vanish. Note that in this process the ambiguity in $\mu \in C^\infty(M)$ has to be cut down since in the space of Killing vector fields is a finite dimensional vector space (and not a module over the functions). For any $\alpha \in T^* M$, one has

$$E\mathcal{L}_\alpha S = \mathcal{L}_{\rho(\alpha)} S + \mathcal{P}^{iv}(d\alpha)_{st} S^{ij} \partial_i \partial_j. \quad (54)$$

Using $\alpha = \mu C$ and the fact that this $\alpha$ is in the kernel of $\rho$, we obviously only need to ensure that $d\alpha = d\mu \wedge dC = 0$ which is satisfied, iff $\mu = \mu(C)$. Since on the classical solutions $C = \text{const}$, this corresponds only to a constant rescaling of the Killing vector.

This concludes our proof for the case of (43). To show the appearance of this Killing vector field more generally, one proceeds similarly to the proof of Theorem 4, replacing (41) by $E = dC \oplus W^0 \oplus dX^3$ (which may be used at least locally on $M$ and for $dC \neq dX^3$).

VI. CONCLUSION

In the present paper we discussed conditions which can be used to endow a large class of topological models with some gravitational interpretation. Concerning the models we only specified their field equations and local symmetries in the present paper, so as to remain as general as possible (while still action functionals can be constructed for them, at least if one permits auxiliary fields). They could be assigned to any $d$-dimensional spacetime manifold $\Sigma$ and to any Lie algebroid $E$ used as target. On some more abstract level the field equations state that the maps $\varphi: T\Sigma \to E$ should be Lie algebroid (co)morphisms (i.e. the induced map $\Phi: \Omega^\bullet_{\mathcal{P}}(M) \to \Omega^c(M)$ should be a chain map), while the symmetries correspond to a notion of homotopy of such morphisms (cf. 22 for more details).

Our ansatz for the metric $g$ on $\Sigma$ was that it should be the image of a symmetric, covariant $E$-tensor $S$, $S \in \Gamma(\Lambda^2 E^*)$, with respect to the naturally extended map $\Phi$ just introduced above. ($g = \Phi(S)$; in explicit formulas this gives (35).

We attempted to clearly formulate the desiderata of what we want to call a reasonable theory of $E$-gravity (defined over $\Sigma$)—cf. Definition 3. Essentially it was the requirement that on solutions the gauge transformations of the model when applied to a non-degenerate metric $g$ should boil down to just diffeomorphisms of $\Sigma$.

In Theorem 4 we summarized our main result: If $S$ has a kernel of maximal rank so as to be still compatible with metric non-degeneracy (cf. Eq. (24)),

$$\dim \text{im} S^2 = d, \quad (55)$$

then it suffices to have $S$ invariant with respect to any section in its kernel, cf. Eq. (40). This last condition was found also to be necessary (to be precise, cf. Corollary 4) this was shown only under the assumption $\dim \ker \rho \leq r - d$ and by requiring the somewhat strengthened condition (155). We believe that Eq. (155) is necessary, too; however, in the present paper, we managed to show this only under the relatively strong additional assumption that $\ker \rho$ is trivial (cf. Proposition 4 or that it is all of $E$ and the bracket abelian (cf. section 4.3).

The invariance condition (40) may be reformulated as a kind of ad-invariance of the degenerate inner product induced by $S$: Let $(\psi_1, \psi_2) := (S, \psi_1 \otimes \psi_2)$, then Eq. (40) is equivalent to

$$\rho(\psi) \cdot (\psi_1, \psi_2) = ([\psi, \psi_1], \psi_2) + (\psi_1, [\psi, \psi_2]) \quad (56)$$

valid for $\psi \in \Gamma(\ker S^2)$ and for all sections $\psi_1, \psi_2$; here $\cdot$ denotes application of the vector field to the respective function to follow.

It is decisive that this condition has to hold only for the Lie subalgebroid ker $S^2$; if e.g. it were to hold for any section of $E$ and if the inner product were non-degenerate, then necessarily $\rho \equiv 0$, i.e. one would be left with a bundle of Lie algebras equipped with a fiberwise ad-invariant...
inner product. Moreover, as illustrated in section \[\text{LV}A\] and \[\text{LV}B\] even for  \(\rho \equiv 0\) a kernel of \(S\) is/will be required (for \(r > d\)) so as to really yield a reasonable (cf. Definition \[\text{A}\]) notion of a metric \(g\) on \(\Sigma\) by means of \(g = \Phi(S)\).

General ad-invariance of an inner product appears in a slight modification of the notion of a Lie algebroid, namely the Courant algebroid; modifying the bracket \([\cdot, \cdot]\) by a symmetric contribution stemming from the inner product, one can have ad-invariance even without \(\rho \equiv 0\) and despite non-degeneracy of the inner product, cf. e.g. \[\text{20, 33}\]. Let us remark in parenthesis that from a more recent perspective, cf. \[\text{37}\], it is also natural to drop the non-degeneracy condition: Lie algebroids then appear as completely degenerate Courant algebroids. In any case, it may be reasonable to generalize the considerations of the present paper to the realm of Courant algebroids.

Closer to the present structure are however possible quotient constructions. The guideline may be taken from the standard Lie algebroid \(E = TM\). Assuming \(\ker S^t\) to define a fibration, then \(S\) obviously is just the pullback (by the projection) of a non-degenerate and unrestricted metric on the base or quotient manifold. This generalizes: As one may show and as shall be made more explicit elsewhere, one may construct a reasonable theory of \(E\)-gravity just by means of a Lie algebroid morphism from \(E\) to some Lie algebroid \(E^\prime\) of rank \(d\), where the latter is equipped with an arbitrary non-degenerate fiber metric.

For \(E = TM\) the structure governing a gravity model was found to be the one of a Riemannian foliation (cf. section \[\text{IV}C\]). The general structure required then may be viewed as the corresponding Lie algebroid generalization, i.e. an \(E\)-Riemannian foliation. In the case of \(E = TM\), the metric \(g\) was found to locally coincide just with the metric on the locally defined quotient of the Riemannian foliation. In general, the relation between \(S\) and \(g\) is more subtle, and worth further investigations.

The usually considered gravity models in three spacetime dimensions result from \(E\) being a Lie algebra (cf. section \[\text{IV}A\]).

For \(E\) the cotangent bundle of a Poisson manifold, finally, a reasonable choice of \(S\) boils down to the choice of a sub-Riemannian structure, compatible with the given Poisson structure. Under the assumption that \(\text{im}S^t\) defines an integrable distribution, a sub-(pseudo)-Riemannian structure gives rise to a foliation of \(M\) by \(d\)-dimensional (pseudo)-Riemannian leaves. (This is not to be confused with the foliation generated by \(\rho \ker S^t\), cf. the discussion after Proposition \[\text{S}\].) The invariance condition \[\text{56}\] then was tantamount to requiring that \(S^t\)—or the degenerate inner product induced by it—is invariant with respect to the Hamiltonian vector fields of the (possibly only locally defined) functions characterizing the pseudo-Riemannian foliation (cf. \[\text{S}\]).

Known models of 2d gravity were seen to be a special case of the much more general construction of the present paper. Also we were able to relate the existence of a Killing vector of \(g\) in the previously known models to the invariance condition \[\text{56}\] together with the particularities of the two foliations defining these models. In view of Proposition \[\text{11}\] it may be rewarding to strive at a generalization of these results.

It is worthwhile also to generalize our consideration to the existence of a vielbein \(e^\alpha\) and a spin connection \(\omega^a_{\beta}\)—instead of just a metric \(g\), which results from the vielbein according to \(g = e^a e^b \eta_{ab}\), i.e. to determine the conditions such that some of the 1-form fields \(A^i\) permit identification with the Einstein-Cartan variables \((e^a, \omega^a_{\beta})\).

The present paper is meant to open a new arena for studying models of gravity. It should be possible to extend the in part extensive studies of lower dimensional models of gravity—on the classical and on the quantum level—to a much larger set of theories, including topological gravity models in arbitrary spacetime dimensions.

Finally, we mention again that within this paper we focused on the field equations and symmetries corresponding to Lie algebroids. Depending on the spacetime dimension and possibly additional structures chosen, there may be several different action functionals producing the same desired theory, at least in a subsector.

To give a simple example: If \(E\) is a Lie algebra \(\mathfrak{g}\), the field equations correspond just to flat connections and the symmetries to the standard non-abelian gauge transformations. This may be described without any further structure and restriction to spacetime by a BF-theory. For \(d = 3\) and with \(\mathfrak{g}\) permitting a non-degenerate, ad-invariant inner product, one may also take a Chern-Simons action; this even has the advantage of getting by without additional auxiliary fields—but on the other hand requires additional structures and restriction to a specific spacetime dimension.

This generalizes also to the present context. We intend to make this more explicit elsewhere \[\text{22}\].

Merely on the level of the field equations and symmetries, one may already address the interesting but also challenging question of observables. Since the theories constructed are topological, they are expected to capture important and interesting mathematical information—displaying an interplay between the topology of \(\Sigma\) and the target Lie algebroid and, if one also involves \(g\), the target \(E\)-Riemannian structure. As suggested by the particular case \(E = \mathfrak{g}\), one should, among others, introduce a generalization of the notion of a Wilson loop for this purpose.

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[38] This is true at least from the mathematical point of view. The condition imposed excludes propagating modes, which is not so desirable from the physical point of view. Still, many examples in two dimensions have this feature, and many more theories may be captured if the topological theory describes only one sector of a more extended one, cf. e.g. [8, 9] as well as the discussion further below.
[39] In fact, from the mathematical point of view, the more detailed study of the modified situation still may be quite interesting—and maybe also quite intricate (the non-degeneracy condition on \(g\), e.g., will have quite different implications than in the present note, cf. Lemma 1 below).
[40] This observation has been inspired by discussions with A. Weinstein, in particular those about the standard Lie algebroid, discussed in section IV below, in which he pointed out to me that in this case the structure induced by a compatible \(\mathcal{S}\) is the one of a Riemannian foliation.
[41] I am grateful to A. Kotov for discussions of this point. Cf. also Eq. (54) below.
[42] I am grateful to A. Weinstein for this remark.