Bose-Einstein Weights for Event Generators

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A simple new algorithm for the calculation of two-particle Bose-Einstein correlations from classical event generators is derived and discussed.

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For ultrarelativistic heavy-ion collisions, two-particle Bose-Einstein correlations between identical pions or kaons provide a unique possibility to reconstruct the geometry (size, temporal extension) and dynamics (collective expansion flow) of the source at the point of hadron freeze-out. (For a recent theoretical review see [1].) This reconstruction is, however, not completely model independent; it requires the use of “reasonable” source parametrizations (e.g. [2,3]) whose parameters are then fixed by a simultaneous analysis of single-particle momentum spectra and two-particle momentum correlations [4].

An invaluable help for the selection of “reasonable” source parametrization comes from microscopic event generators (e.g. VENUS [4], RQMD [5] or ARC [6]) which generate the phase-space distribution of hadrons at freeze-out from a dynamical Monte Carlo simulation of the (classical) kinetic phase-space evolution of the collision zone. Unfortunately, it was recently pointed out [7,8] that the direct computation of two-particle correlation functions from classical kinetic codes [4,5,6] is fraught with severe conceptual and practical problems. These can be simply understood by starting from the general relation (14) between the 2-particle correlation function \( C(q, K) \) and the (real) “emission function” (one-particle Wigner density at freeze-out) \( S(x, K) \).

\[
C(q, K) = 1 + \frac{\int d^4x S(x, K) e^{i q \cdot x}}{\int d^4x S(x, p_a) \int d^4y S(y, p_b)}.
\]

(1a)

\[ q = p_a - p_b, \quad q^0 = E_a - E_b, \quad K = \frac{p_a + p_b}{2}, \quad K^0 = \frac{E_a + E_b}{2}. \]

(1b)

\((p_a, E_a)\) and \((p_b, E_b)\) are the 4-momenta of the two observed particles. Eq. (1a) neglects final state interactions which we will leave out in this Letter in order to concentrate on the principal issues.

The numerator in the second term of Eq. (1a) can be rewritten as

\[
\text{Num}(q, K) = \int d^4x d^4y S(x, K) S(y, K) \times \cos(q \cdot (x - y)).
\]

(2)

The problem is the construction of the Wigner density \( S(x, K) \) from the output of the event generators. The latter consists of a set of phase-space points \((x_i, p_i)\) denoting the (on-shell) momenta \(p_i\) and points of last interaction \(x_i\) of the produced particles. According to (1b) \( K \) is the average of two on-shell momenta, but not itself on-shell, \( K^0 \neq \sqrt{m^2 + K^2} \). Therefore \( S(x, K) \) cannot be directly related to the phase-space density generated by the distribution of points \((x_i, p_i)\). To overcome this difficulty one usually imposes [11,12] the “smoothness assumption” \( S(x, K) \approx S(x, p_a) \approx S(x, p_b) \) and rewrites the expression (3) as

\[
\text{Num}(q, K) = \int d^4x d^4y S(x, p_a) S(y, p_b) \times \cos((p_a - p_b) \cdot (x - y)).
\]

(3)

One now identifies \( S(x, p) \) with the classical output distribution from the event generator,

\[
S_{\text{class}}(x, p) = \sum_{i=1}^{N} \delta^{(4)}(x - x_i) \delta^{(4)}(p - p_i),
\]

(4)

where \( N \) is the total number of pions of a given charge in the event. In the widely used Pratt algorithm [13] the expression which results after inserting (4) into (3) is simulated by the ad hoc prescription

\[
\text{Num}(q, K) \mapsto \sum_{i,j \in \text{bin}} \cos((p_i - p_j) \cdot (x_i - x_j)).
\]

(5)

Here “bin” denotes a small bin in \((q, K)\) with \(p_i - p_j \approx q\) and \((p_i + p_j)/2 \approx K\). (In practice the bin size depends on event statistics.)

The prescription (5) has two severe problems: first, the positivity of \( \text{Num}(q, K) \) got lost between Eqs. (4) and (5) when making the “smoothness assumption”. It was pointed out in Refs. [14] and practically demonstrated in Ref. [15] that for sources with strong \( x-p \) correlations (e.g. rapidly expanding sources) this can lead to unphysical oscillations of the simulated correlation function around unity. Second, the intuitive substitution law (5) is formally incorrect and results in a wrong selection of contributing pairs as well as an incorrect weighting factor for each pair.

To prove this last point let us write in (3)

\[
S(x, K) \cdot S(y, K) = \int d^4P_1 d^4P_2 S(x, P_1) S(y, P_2) \times \delta^{(4)}(P_1 - P_2) \delta^{(4)}\left(K - \frac{P_1 + P_2}{2}\right).
\]

(6)
Inserting the classical expression (4) one finds instead of (5)
\[
\text{Num}(q, K) = \sum_{i,j=1}^{N} d^{(4)}(p_i - p_j) d^{(4)}\left(K - \frac{p_i + p_j}{2}\right) \times \cos(q(x_i - x_j)) \quad (7a)
\]
\[
\longrightarrow \sum_{i,j\in\bin(K,\epsilon)} \cos(q(x_i - x_j)). \quad (7b)
\]
Here “bin(K, \epsilon)” denotes a small bin around K with width \epsilon in each of the four directions. The prescription (7) can be derived rigorously and directly by first generating from Eq. (4) a piecewise constant function (“histogram”) through “binning”,
\[
\bar{S}(x, K) = \int_{\text{bin}(K,\epsilon)} d^4p S_{\text{class}}(x, p) = \int_{K-\epsilon/2}^{K+\epsilon/2} d^4p S_{\text{class}}(x, p), \quad (8)
\]
and then inserting \bar{S}(x, K) into Eq. (9). Eq. (8) is a technical step required by finite event statistics; in practice, \epsilon should be chosen as small as technically possible. Note that the selection of pairs in (7) differs from the one in (9); for given K the algorithm (9) selects pairs with \( p_i \approx p_j \approx K \), independent of the value of q at which the correlation is to be evaluated. For different values of q at fixed K, the correlator is obtained by weighting the same set of pairs with different weight factors \( \cos(q(x_i - x_j)) \) which depend only on the spatial coordinates, but not on the momenta of the particles in the pair. This is consistent with the expectation from Eqs. (1,3,2) that the measured q-dependence of the correlator gives access to the distribution of relative distances \( x_i - x_j \) in the source (at fixed K). Since the steps from Eq. (9) to Eq. (8) involve only identical transformations (the difference between (7) and (8) arising only from a different choice of emission functions (9) resp. (8)), they preserve the positivity of the second term in (1) and is due to the smoothness approximation (8). – The two remaining lines in Fig. 1 show results from the same Monte Carlo simulation of (9) but reconstructing the correlator through the new algorithm (7) instead of (9). For the dot-dashed line the bin width \epsilon was chosen as \epsilon = 10 MeV, for the dotted line as \epsilon = 5 MeV. In both cases the simulated result deviates from the exact one (solid line); this is not a failure of the algorithm, but a result of the binning procedure (8) applied to the source (9) – a purely technical step required by finite event statistics. As seen, the discrepancy decreases with decreasing bin width \epsilon.

We have checked for the simple analytically solvable model presented in (9) that the algorithm (9) indeed allows to reconstruct from a classical Monte Carlo simulation the correct analytical expression for C(q, K); in particular it removes the unphysical oscillations found in (9). In Fig. 1 we show various approximations for the correlation function for a 1-dimensional source in z-direction with emission function
\[
S(z, t; K) = e^{-z^2/R^2} \delta(K - \alpha z) \delta(t), \quad (9)
\]
with R=10 fm and \alpha=0.02 GeV/fm. This classical source features perfect z-K-correlations, K=\alpha z, which are, of course, quantum mechanically forbidden (see below). For the source (9), the exact correlator (1) can be calculated analytically, yielding \( C(q) = 1 + \exp[q^2/(2\alpha^2 R^2)] \) (solid line in Fig. 1). The pathological rise of \( C(q) \) above the value 2 at q=0 is due to the violation of the uncertainty relation between z and \( p_z \) by the model (9); we selected this model because we believed that such a feature may be particularly difficult to reproduce in an event generator. Indeed, reconstructing the correlator from a Monte Carlo simulation of (9) via the Pratt prescription (9) yields the long-dashed line in Fig. 1 (9); it can be calculated analytically from Eq. (1) as \( C(q) = 1 + \cos(q^2/\alpha) \). This is always less than 2, but oscillates wildly around 1, becoming even 0 at regular q-intervals. This contradicts the positivity of the second term in (1) and is due to the smoothness approximation (8). – The two remaining lines in Fig. 1 show results from the same Monte Carlo simulation of (9) but reconstructing the correlator through the new algorithm (7) instead of (9). For the dot-dashed line the bin width \epsilon was chosen as \epsilon = 10 MeV, for the dotted line as \epsilon = 5 MeV. In both cases the simulated result deviates from the exact one (solid line); this is not a failure of the algorithm, but a result of the binning procedure (8) applied to the source (9) – a purely technical step required by finite event statistics. As seen, the discrepancy decreases with decreasing bin width \epsilon.

\[
\begin{array}{l}
\text{FIG. 1. Two-particle correlation function for the model source (9). Different curves show different approximations as described in the text. The statistical errors of the simulation are below the line widths.}
\end{array}
\]
minimum-uncertainty wave packets. In the remainder of this Letter we will discuss how such a procedure will modify the algorithm for calculating single-particle spectra and two-particle correlations from event generators, thereby rendering it quantum mechanically consistent.

Let us start from the folding relation for the emission function, derived in Ref. [13] within the covariant current formalism [13]:

$$S(x, K) = \int d^4 z d^4 Q \rho(x - z, Q) S_0(z, K - Q).$$  \hspace{1cm} (10)

Here

$$S_0(x, p) = \int d^4 v e^{-ip \cdot v} j_0^* (x + \frac{v}{2}) j_0 (x - \frac{v}{2})$$  \hspace{1cm} (11)

is the Wigner density associated with an elementary source current amplitude $j_0(x)$, taken below as a Gaussian wavepacket, and $(x, p)$ is a classical phase-space distribution for the centers $x_i$ and (average) momenta $p_i$ of these wave packets, here taken as

$$\rho(x, p) = \sum_{i=1}^{N} \delta^{(4)}(x - x_i) \delta^{(4)}(p - p_i).$$  \hspace{1cm} (12)

For the elementary source amplitude we make the ansatz

$$j_0(x) = N' \exp \left( - \frac{x^2}{2\sigma^2} \right) \delta(x^0);$$  \hspace{1cm} (13)

this source emits a Gaussian wave-packet with width parameter $\sigma$ at “freeze-out time” $x^0$. With this ansatz the elementary Wigner density $S_0$ becomes

$$S_0(x, p) = 8(\pi \sigma^2)^2 |N'|^2 \delta(x^0) \exp \left( - \frac{x^2}{\sigma^2} - \sigma^2 p^2 \right).$$  \hspace{1cm} (14)

Inserting this and (12) into (10) one finds

$$S(x, K) = N'' \sum_{i=1}^{N} \delta(x^0 - x_i^0) \exp \left( - \frac{(x - x_i)^2}{\sigma^2} \right) \times \exp \left( - \sigma^2 (K - p_i)^2 \right),$$  \hspace{1cm} (15)

with $N'' = 8(\pi \sigma^2)^2 |N'|^2$. This generalizes the classical ansatz (10) into a quantum mechanically consistent source Wigner density; with the free parameter $\sigma$ one can choose the relative degree of localization in coordinate space ($\sigma \to 0$) or momentum space ($\sigma \to \infty$), always preserving $\Delta x \cdot \Delta p = \hbar$.

The single particle spectrum, which occurs in the denominator of the second term of (14), is now given by

$$E_a \frac{dN}{dp_a} = \int d^4 x S(x, p_a) = N'' \sum_{i=1}^{N} v_i(p_a),$$  \hspace{1cm} (16a)

$$N'' = (2\pi \sigma^2)^2 |N'|^2,$$  \hspace{1cm} (16b)

$$v_i(p_a) = \exp \left( - \sigma^2 (p_a - p_i)^2 \right),$$  \hspace{1cm} (16c)

and similarly for $p_b$. It is normalized to the total number $N$ of pions in the event; this fixes the normalization constant $N'$ above. The exchange term (2) in the two-particle spectrum (with $q$ and $K$ defined in Eq. (11)) is similarly derived as

$$\text{Num}(q, K) = (N'')^2 \exp(-\frac{1}{2} \sigma^2 q^2) \sum_{i,j=1}^{N} w_{ij}(q, K),$$  \hspace{1cm} (17a)

$$w_{ij}(q, K) = v_i(K) v_j(K) \cos(q \cdot (x_i - x_j)).$$  \hspace{1cm} (17b)

The normalization $N''$ drops out in the correlator (14). Eq. (17) should be compared with the classical expressions (7) and (10). Like (10) (but contrary to (7)) it is positive definite and thus free of spurious oscillations around 0. The sum in (17a) is now over all pairs $(i, j)$; the sharp restriction to the bin “$\text{bin}(K, q)$” in (7) is replaced by the Gaussian weight factors $v_i(K)$ and $\exp(-\frac{1}{2} \sigma^2 q^2)$. Please note, however, that the former also occur in the new definition (16b) for the single particle spectrum, and must be kept in both places for consistency. Eq. (17b) shares with Eq. (7) the property (which was already discussed) that the cosine weight factor for each pair $(i, j)$ depends only on $x_i - x_j$, but not on $p_i$ and $p_j$. By combining Eqs. (10) and (17) it can be shown that now the correlator $C(q, K)$ is always between 1 and 2, i.e. that the pathological rise above 2 shown in Fig. 1 cannot happen for an emission function which respects the uncertainty relation. In Ref. [19] the results (10) and (17) (but not (7)) were derived with different methods, including finite multiplicity corrections.

Expressions (10) and (17) depend on one free parameter, the Gaussian width $\sigma$ of the wave-packets. It is instructive to discuss the two obvious limits, $\sigma \to 0$ and $\sigma \to \infty$. For $\sigma = 0$ the elementary sources are sharply localized in space (cf. Eq. (13)); as a result, the single-particle momentum spectrum (10) is completely flat. Furthermore, one sees from Eqs. (17) that the correlation function becomes $K$-independent, even for expanding sources of type (12) where the $x_i$ and $p_i$ are strongly correlated. Both features are clearly unrealistic. In the opposite limit, $\sigma \to \infty$, the Gaussian smearing factors in the single particle spectrum (10) disappear, and Eq. (16b) degenerates to a sum over $\delta$-functions; this is the usually employed algorithm for computing single-particle spectra from event generators with classical particle propagation (7). The two-particle correlation term (17), on the other hand, is then sharply concentrated at $q = 0$, i.e. the correlator $C(q, K)$ drops from 2 to 1 over a $q$-range of order 1/$\sigma$. This translates into a source radius $\sim \sigma$ and reflects the diverging spatial extension of the elementary wavepackets in this limit, irrespective of the (localized) spatial distribution $\rho(x, p)$ of their centers.

It is thus clear that in practice $\sigma$ must be kept finite, but non-zero. As pointed out in [13] this implies a broadening of the single-particle momentum distributions relative to the one derived from the classical
phase-space distribution $\rho(x, p)$ of freeze-out points. Using
the algorithms \cite{14,17,16} for a quantum mechanically
consistent computation of spectra and Bose-Einstein cor-
relations from classical event generators thus requires a
retuning of the codes to elementary $e^+e^-$ and $pp$ colli-
sions, using the same algorithms there.

This last step imposes rather restrictive limits for the
value of $\sigma$ \cite{15}. Since the pions from high energy $pp$
eu
and $e^+e^-$ collisions have an average transverse moment-
um $\langle p_\perp \rangle \simeq 0.35$ GeV, the Gaussian width in moment-
um space of the elementary wave packets must be be-
low this value. This implies $\sigma \gtrsim 0.5$ fm. On the other
hand, the effective source radii for such collisions extracted
from 2-particle correlations are of the order of
only $0.8 - 2$ fm \cite{20}. This implies $\sigma \lesssim 1$ fm. It was
already pointed out in the pioneering GGLP paper \cite{22}
that this value roughly agrees with the pion’s Compton
wavelength. This suggests the following interpretation
of the measured pion spectra and Bose-Einstein cor-
relations from elementary hadron-hadron and $e^+e^-$ colli-
sions: each elementary collision produces a small number
of elementary Gaussian wave packets with width $\sigma$, 0.5
fm $< \sigma < 1$ fm. For $\sigma = 0.5$ fm the width of the Gaussi-
sian momentum distribution in \cite{16} nearly exhausts the
measured $\langle p_\perp \rangle$: the measured single-pion spectra thus
reflect mostly the intrinsic momentum distribution of the
elementary pion wave packet. On the other hand,
the HBT radius $R_{HBT}^2 = \sigma^2/\sqrt{2}$ corresponding to the $q$-
Gaussian in \cite{17a} nearly exhausts the values extracted
from two-particle correlation measurements; the latter
thus mirror the intrinsic spatial width of the wave
packets. There is very little additional room in the data
for random (thermal) motion of the wave packets relative
to each other, nor for their spatial distribution over a larger
volume. If the total source is much bigger than 1 fm,
it must expand very rapidly, with homogeneity regions
which are not much larger than the size of an elementary
wave packet. In this sense, pion spectra from $e^+e^-$ and
$pp$ collision measure the smallest sources compatible with
the uncertainty relation.

In summary, we have derived a new algorithm for the
computation of single-particle spectra and two-particle
 correlations from classical event generators. The classical
algorithm \cite{5} removes the recently discovered defi-
ciences of the presently employed Pratt algorithm \cite{5}.
The quantum mechanical algorithm \cite{16,17} additionally
ensures that the uncertainty relation is not violated. We
also showed how the free parameter $\sigma$ in the latter algo-
rithm can be fixed from elementary $e^+e^-$ and hadron-
hadron collisions.

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