On $\Psi$-Laplace transform method and its applications to $\Psi$-fractional differential equations

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Abstract
Motivated by some recent developments in $\Psi$-fractional calculus, in this paper some new properties and the uniqueness of $\Psi$-Laplace transform in the settings of $\Psi$-fractional calculus are established. The final goal of this research is to demonstrate the effectiveness of $\Psi$-Laplace transform for solving $\Psi$-fractional ordinary and partial differential equations.

Keywords: Fractional calculus; generalized Laplace transform; generalized fractional operators; $\Psi$-convolution structure

1. Introduction

The birth of fractional calculus finds its roots in the last years of the seventeenth century, when Newtons work along with Leibnizs served basis for the inception of classical calculus. Leibniz devised the notation $\frac{d^n}{dx^n} f(x)$ to denote the nth-order derivative of the function $f$. When he communicated this to de l’Hospital, the later asked what the meaning of the said notation would be if $n = \frac{1}{2}$. This communication in the present day is unanimously considered to be the foundation of fractional calculus. At present, this field has become a matter of deep interest for many researchers.

In fractional calculus, the Riemann-Liouville fractional derivative is note-worthy but it has certain disadvantages when trying to model physical problems because of its inappropriate physical conditions. Caputo made a significant contribution by affirming the definition of fractional derivative which is suitable for physical conditions [2]. Moreover, several other families of fractional operators have been introduced until now, out of which Liouville,

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Erdlyi-Kober, Hadamard, Grunwald-Letnikov, Hilfer are just a few to mention \cite{3, 10}. Due to a large number of definitions of fractional operators, it was important to establish the generalized fractional operators for which the classical ones are particular cases. One of the extensions of Riemann-Liouville fractional operators are the so-called fractional operators of a function by another function $\Psi$ ($\Psi$-RL fractional operators) which can be seen in \cite{7, 9}. In \cite{11} Almeida proposed the Caputo version of fractional derivative of a function by another function $\Psi$ ($\Psi$-C fractional derivative) and studied some useful properties of the fractional calculus. Inspired by the definitions of $\Psi$-RL and Hilfer \cite{10} fractional derivatives, the authors introduced the so-called $\Psi$-Hilfer fractional derivative which unifies a large class of fractional operators \cite{12}.

The passion in writing this paper is largely due to compulsive use of fractional differential equations (FDEs) in physics, economics, engineering and other branches of sciences \cite{2–8}. Since no such method exists in literature which applies in general for solving every FDE analytically, so developing some suitable method to find analytic solutions to some classes of FDEs is one of the most challenging tasks. In the past few years, the researchers have been showed their interest in introducing fractional interpretations of the classical integral transforms, namely, the Laplace and Fourier transforms \cite{1, 13–21}. In \cite{1, 3, 6, 7, 14, 22}, it can be seen that integral transforms like Laplace, Fourier, generalized Laplace and $\rho$-Laplace were considered as effective tools for obtaining analytic solutions to some classes of FDEs.

For the reasons best known to us hitherto, no attempts have been made to use integral transforms for obtaining analytic solutions to FDEs in the settings of $\Psi$-Hilfer fractional derivatives. In this particular paradigm we use generalized Laplace transform for finding analytic solutions to some classes of FDEs involving $\Psi$-RL, $\Psi$-C and $\Psi$-Hilfer fractional derivatives.

The article is organized as follows. Section 2 contains preliminary definitions from classical and fractional calculus. In Section 3, we prove some new properties and uniqueness of the generalized Laplace transform. The $\Psi$-Laplace transforms of the $\Psi$-Hilfer fractional derivatives are obtained in Section 4. In Section 5, the applicability of the $\Psi$-Laplace transform is discussed. Finally, Section 6 is devoted to the applications.

2. Preliminaries

Prior to introducing the $\Psi$-Laplace transform, we first recall some definitions from the classical and fractional calculus.
2.1. \(\Psi\)-RL, \(\Psi\)-C and \(\Psi\)-Hilfer fractional operators

In view of the fact that there is a large class of fractional operators available in literature which makes choosing the appropriate approach a difficult task while dealing with a given problem. So it is important to introduce the generalizations of classical fractional operators to overcome the issue of choosing a suitable operator. In this subsection we invoke some generalized definitions of fractional integrals and derivatives.

Definition 2.1. Let \(\mu\) be a real number such that \(\mu > 0\), \(-\infty \leq a < b \leq \infty\), \(m = \lfloor \mu \rfloor + 1\), \(f\) be an integrable function defined on \([a, b]\) and \(\Psi \in C^1([a, b])\) be an increasing function such that \(\Psi'(t) \neq 0\) for all \(t \in [a, b]\). Then, the \(\Psi\)-RL fractional integral and \(\Psi\)-RL fractional derivative of a function \(f\) of order \(\mu\) are defined as

\[
\mathcal{I}_a^\mu,\Psi f(t) := \frac{1}{\Gamma(\mu)} \int_a^t \left(\Psi(t) - \Psi(s)\right)^{\mu-1} \Psi'(s)f(s)ds \quad (2.1)
\]

and

\[
\mathcal{D}_a^\mu,\Psi f(t) := \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^m \mathcal{I}_a^{m-\mu,\Psi} f(t) \quad (2.2)
\]

respectively.

It is to be noted that for \(\Psi(t) \to t\), \(\mathcal{I}_a^\mu,\Psi f(t) \to \mathcal{I}_a^\mu f(t)\) which is the standard Riemann-Liouville integral. Moreover for \(\Psi(t) \to \ln(t)\) the integral defined in (2.1) approaches to the Hadamard fractional integral.

Inspired by Caputo’s concept [23] of fractional derivative, Almeida [11] presents the following Caputo version of (2.2) and studies some important properties of fractional calculus.

Definition 2.2. Let \(\mu\) be a real number such that \(\mu > 0\), \(-\infty \leq a < b \leq \infty\), \(m = \lfloor \mu \rfloor + 1\), \(f, \Psi \in C^m([a, b])\) be the functions such that \(\Psi\) is increasing and \(\Psi'(t) \neq 0\) for all \(t \in [a, b]\). Then, the \(\Psi\)-C fractional derivative of a function \(f\) of order \(\mu\) is defined as

\[
\mathcal{C}D_a^\mu,\Psi f(t) := \mathcal{I}_a^{m-\mu,\Psi} \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^m f(t). \quad (2.3)
\]

Taking \(\Psi(t) \to \ln(t)\) and \(\Psi(t) \to t\), we get the Caputo-type Hadamard fractional derivative [24] and Caputo fractional derivative [7] respectively.

Motivated by the definitions of \(\Psi\)-RL and Hilfer fractional derivatives, Sousa and Oliveira [12] introduce the \(\Psi\)-Hilfer fractional derivative which we recall in the following definition.
Definition 2.3. \[\text{Let } \mu \text{ be a real number such that } \mu > 0, -\infty < a < b < \infty, m = \lfloor \mu \rfloor + 1, f, \Psi \in C^m([a, b], \mathbb{R}) \text{ be the functions such that } \Psi \text{ is increasing and } \Psi'(t) \neq 0 \text{ for all } t \in [a, b]. \]

Then, the \(\Psi\)-Hilfer fractional derivative of a function \(f\) of order \(\mu\) and type \(0 \leq \nu \leq 1\) is given by

\[
D_{a^+}^{\mu, \nu, \Psi} f(t) := \mathcal{I}_a^{\nu(m-\mu), \Psi} \left( \frac{1}{\Psi'(t)} \right)^m \mathcal{I}_a^{(1-\nu)(m-\mu), \Psi} f(t). \tag{2.4}
\]

2.2. Laplace and Fourier transforms

Definition 2.4. Assume that the function \(f\) is defined for \(t \geq 0\). Then the Laplace transform of a function \(f\), denoted by \(\mathcal{L} \{f\}\), is defined by the improper integral

\[
\mathcal{L} \{f(t)\} := \int_0^\infty e^{-st} f(t) dt \tag{2.5}
\]

provided that the integral in (2.5) exists for all \(s\) larger than or equal to some \(s_0\).

Definition 2.5. Assume that \(f\) is a piecewise smooth, continuous and absolutely integrable function. Then the Fourier transform of a function \(f\), denoted by \(\mathcal{F} \{f\}\) or \(\tilde{f}(k)\), is defined by

\[
\mathcal{F} \{f(t)\} := \int_{-\infty}^{\infty} e^{-ikt} f(t) dt \tag{2.6}
\]

where \(k\) is the Fourier transform variable.

The inverse Fourier transform of \(\mathcal{F} \{f(t)\}\) is defined by

\[
\mathcal{F}^{-1} \{\mathcal{F} \{f(t)\}\} := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikt} \tilde{f}(k) dk. \tag{2.7}
\]

2.3. Some special functions

There are several special functions which are considered to be helpful for finding the solutions of FDEs. In the following definitions, we state a few of them.

Definition 2.6. The entire function

\[
W(z, \mu, \nu) := \sum_{j=0}^{\infty} \frac{z^j}{j! \Gamma(\mu j + \nu)}, \text{ where } \mu > -1, \ \nu \in \mathbb{C} \tag{2.8}
\]

which is valid in the whole complex plane, is known as the Wright function. It appeared for the first time in \([25, 26]\) in connection with E. M. Wrights investigations in the asymptotic theory of partitions.
In [32], Gosta Mittag-Leffler introduced the well-known Mittag-Leffler function $E_\mu(z)$, given by

$$
E_\mu(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\mu j + 1)}, \quad \mu \in \mathbb{C}, \quad \text{Re}(\mu) > 0. \tag{2.9}
$$

Later on, a natural generalization of $E_\mu(z)$ was discussed by Wiman in [33]. He introduced the function $E_{\mu,\nu}(z)$ as

$$
E_{\mu,\nu}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\mu j + \nu)}, \quad \mu, \nu \in \mathbb{C}, \quad \text{Re}(\mu) > 0. \tag{2.10}
$$

If we consider $\nu = 1$ in (2.10), we obtain the Mittag-Leffler function (2.9). In [27], Prabhakar presented the more generalized version of (2.9)-(2.10) which we recall in the following definition.

**Definition 2.7.** The Prabhakar function is defined by the series representation

$$
E_{\gamma,\mu,\nu}(z) := \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{\infty} \frac{\Gamma(\gamma + j)z^j}{j!\Gamma(\mu j + \nu)}, \quad \mu, \nu, \gamma \in \mathbb{C}, \quad \text{Re}(\mu) > 0. \tag{2.11}
$$

It is an entire function of order $1/\text{Re}(\mu)$, which is also known as three parameter Mittag-Leffler function. This function plays a necessary role in the explanation of the anomalous dielectric properties in heterogeneous systems. Some important properties of this function can be seen in [28–31].

### 3. The Ψ-Laplace transform

In this section, we discuss a generalized integral transform introduced by Jarad and Abdeljawad [1] which can be used to solve linear FDEs in the frame of Ψ-RL, Ψ-C and Ψ-Hilfer fractional derivatives. This new integral transform is the natural generalization of classical Laplace transform. Throughout this paper, we call it the Ψ-Laplace transform. Some new properties and the uniqueness of Ψ-Laplace transform in the settings of Ψ-fractional calculus form the part of this section.

**Definition 3.1.** Let $f : [0, \infty) \to \mathbb{R}$ be a real valued function and $\Psi$ be a non-negative increasing function such that $\Psi(0) = 0$. Then the Ψ-Laplace transform of $f$ is denoted by $L_\Psi \{f\}$ and is defined by

$$
F(s) := L_\Psi \{f(t)\} := \int_{0}^{\infty} e^{-s\Psi(t)}\Psi'(t)f(t)dt \tag{3.1}
$$

for all $s$. 

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Definition 3.2. A function \( f : [0, \infty) \to \mathbb{R} \) is of \( \Psi \)-exponential order \( c > 0 \) if there exist positive constant \( M \) such that for all \( t > T \)

\[
|f(t)| \leq Me^{c\Psi(t)}.
\]

Symbolically, we write

\[
f(t) = O(e^{c\Psi(t)}) \quad \text{as} \quad t \to \infty.
\]

Lemma 3.3. In this Lemma, we see the \( \Psi \)-Laplace transforms of some elementary functions.

(a) \( \mathcal{L}_\Psi \{ (\Psi(t))^\mu \} = \frac{\Gamma(\mu+1)}{s^{\mu+1}} \), for \( s > 0 \).

(b) \( \mathcal{L}_\Psi \{ e^{a\Psi(t)} \} = \frac{1}{s-a} \), for \( s > a \).

(c) \( \mathcal{L}_\Psi \{ E_{\mu} \left( \lambda(\Psi(t))^\mu \right) \} = \frac{s^{\mu-1}}{s^{\mu}-\lambda s^\mu} \), for \( \text{Re}(\mu) > 0 \) and \( \left| \frac{\lambda}{s^\mu} \right| < 1 \).

(d) \( \mathcal{L}_\Psi \{ (\Psi(t))^{\mu-1} E_{\mu,\nu} \left( \lambda(\Psi(t))^\mu \right) \} = \frac{1}{s^{\mu-\lambda}} \), for \( \text{Re}(\mu) > 0 \) and \( \left| \frac{\lambda}{s^\mu} \right| < 1 \).

Example 3.4. Assume that \( \text{Re}(\mu) > 0 \) and \( \left| \frac{\lambda}{s^\mu} \right| < 1 \). If \( f(t) = (\Psi(t))^{\nu-1} E_{\mu,\nu}^\gamma \left( \lambda(\Psi(t))^\mu \right) \) where \( E_{\mu,\nu}^\gamma \) denotes the Prabhakar function (2.11), then by Definition 3.1 and Binomial series, we have

\[
\mathcal{L}_\Psi \{ (\Psi(t))^{\nu-1} E_{\mu,\nu}^\gamma \left( \lambda(\Psi(t))^\mu \right) \} = \mathcal{L}_\Psi \left\{ \sum_{i=0}^{\infty} \frac{\lambda^i \Gamma(\gamma+\nu)}{i! \Gamma(\mu i + \nu)} (\Psi(t))^{\mu i + \nu - 1} \right\}
\]

\[
= \sum_{i=0}^{\infty} \frac{\lambda^i \Gamma(\gamma+\nu)}{i! \Gamma(\mu i + \nu)} \mathcal{L}_\Psi \{ (\Psi(t))^{\mu i + \nu - 1} \}
\]

\[
= \sum_{i=0}^{\infty} \frac{\lambda^i \Gamma(\gamma+\nu)}{i! \Gamma(\mu i + \nu)} \frac{\Gamma(\mu i + \nu)}{s^{\mu i + \nu}}
\]

\[
= \frac{1}{s^\nu} \sum_{i=0}^{\infty} \frac{\Gamma(\gamma+\nu)}{i!} \left( \frac{\lambda}{s^\mu} \right)^i = \frac{s^{\mu \gamma - \nu}}{(s^\mu - \lambda)\gamma}.
\]

Now we state the sufficient conditions for the existence of \( \Psi \)-Laplace transform of a function.

Theorem 3.5. If \( f : [0, \infty) \to \mathbb{R} \) is a piecewise continuous function and is of \( \Psi \)-exponential order, then the \( \Psi \)-Laplace transform of \( f \) exists for \( s > c \).
Proof. We have
\[ |L_\Psi \{ f(t) \} | = \left| \int_0^\infty e^{-s\Psi(t)} \Psi'(t) f(t) dt \right| \leq \int_0^\infty e^{-s\Psi(t)} \Psi'(t) |f(t)| dt \leq M \int_0^\infty e^{-s\Psi(t)} \Psi'(t) e^{\Psi(t)} dt = \frac{M}{s-c}, \text{ for } s > c. \tag{3.2} \]
Thus, the prove of the Theorem 3.5 is complete. \qed

Remark 3.6. From (3.2), it follows that \( \lim_{s \to \infty} |L_\Psi \{ f(t) \} | = 0 \), i.e., \( \lim_{s \to \infty} L_\Psi \{ f(t) \} = 0 \). This property can be named as the limiting property of the \( \Psi \)-Laplace transform.

In following Theorems, we state the \( \Psi \)-Laplace transforms of the \( \Psi \)-RL and \( \Psi \)-C fractional operators \([1]\).

Theorem 3.7. Let \( \mu > 0 \) and \( f \) be of \( \Psi \)-exponential order, piecewise continuous function over each finite interval \([0, T]\). Then
\[ L_\Psi \left\{ (T_0^\mu \Psi f)(t) \right\} = s^{-\mu} L_\Psi \{ f(t) \}. \tag{3.3} \]

Theorem 3.8. Assume that \( \mu > 0 \), \( m = [\mu] + 1 \), and \( f(t) \), \( T_0^{m-\mu, \Psi} f(t) \), \( D^1, \Psi T_0^{m-\mu, \Psi} f(t) \), ..., \( D^{m-1, \Psi} T_0^{m-\mu, \Psi} f(t) \) where \( D^j, \Psi = \left( \frac{1}{\Psi(t)} \frac{d}{dt} \right)^j \), are continuous on \((0, \infty)\) and of \( \Psi \)-exponential order, while \( D_0^{\mu, \Psi} f(t) \) is piecewise continuous on \([0, \infty)\). Then
\[ L_\Psi \left\{ D_0^{\mu, \Psi} f(t) \right\} = s^\mu L_\Psi \{ f(t) \} - \sum_{i=0}^{m-1} s^{m-i-1} (T_0^{m-i-\mu, \Psi} f)(0). \]

Theorem 3.9. If \( \mu > 0 \), \( m = [\mu] + 1 \), and \( f(t) \), \( D^1, \Psi f(t) \), \( D^2, \Psi f(t) \), ..., \( D^{m-1, \Psi} f(t) \) are continuous on \([0, \infty)\) and of \( \Psi \)-exponential order, while \( C D_0^{\mu, \Psi} f(t) \) is piecewise continuous on \([0, \infty)\). Then
\[ L_\Psi \left\{ C D_0^{\mu, \Psi} f(t) \right\} = s^\mu L_\Psi \{ f(t) \} - \sum_{i=0}^{m-1} s^{\mu-i-1} (D^i, \Psi f)(0). \]

Definition 3.10. \([1]\) Let \( f \) and \( g \) be of \( \Psi \)-exponential order, piecewise continuous functions over each finite interval \([0, T]\). Then, we define the \( \Psi \)-convolution of \( f \) and \( g \) by
\[ (f *_{\Psi} g)(t) := \int_{t=\Psi^{-1}(\Psi(t))}^{t=\Psi^{-1}(\Psi(\tau))} f\left( \Psi^{-1}(\Psi(t) - \Psi(\tau)) \right) g(\tau) \Psi'(\tau) d\tau. \tag{3.4} \]
In the following Theorem, we discuss the commutativity, associativity and distributivity of the Ψ-convolution of two functions.

**Theorem 3.11.** Let \( f \) and \( g \) be of Ψ-exponential order, piecewise continuous functions over each finite interval \([0, T]\). Then

(a) \( f \ast_\Psi g = g \ast_\Psi f \).

(b) \((f \ast_\Psi g) \ast_\Psi h = f \ast_\Psi (g \ast_\Psi h)\).

(c) \( f \ast_\Psi (ag + bh) = af \ast_\Psi g + bf \ast_\Psi h \).

**Proof.** Proof of (a) can be seen in [1]. For (b), consider the left hand side and using (3.4) we have

\[
\{f \ast_\Psi g\}(t) \ast_\Psi h(t) = \int_0^t \left( \int_0^s f(u)g\left( \Psi^{-1}(\Psi(t) - \Psi(s)) - \Psi^{-1}(\Psi(u)) \right) \Psi'(u)du \right) h\left( \Psi^{-1}(\Psi(t) - \Psi(s)) \right) \Psi'(s)ds
= \int_0^t \left( \int_0^s f(u)g\left( \Psi^{-1}(\Psi(s) - \Psi(u)) \right) \Psi'(u)du \right) h\left( \Psi^{-1}(\Psi(t) - \Psi(s)) \right) \Psi'(s)ds
= \int_0^t \int_0^t f(u)g\left( \Psi^{-1}(\Psi(s) - \Psi(u)) \right) \Psi'(u) \times h\left( \Psi^{-1}(\Psi(t) - \Psi(u)) \right) \Psi'(s)dsdu.
\]

By setting \( v = \Psi^{-1}(\Psi(s) - \Psi(u)) \), we get

\[
\{f \ast_\Psi g\}(t) \ast_\Psi h(t) = \int_0^t f(u)\Psi'(u) \int_0^t g\left( \Psi^{-1}(\Psi(t) - \Psi(u) - \Psi(v)) \right) dvdu
= \int_0^t f(u)\Psi'(u) \{f \ast_\Psi g\}(t)du = f(t) \ast_\Psi \{g \ast_\Psi h\}(t).
\]

The proof of (c) is easy. So we omit the straightforward details.

**Remark 3.12.** Consider a set \( A \) of all Ψ-Laplace transformable functions then \( A \) forms a commutative semi-group with respect to the binary operation \( \ast_\Psi \). Moreover \( A \) does not form a group because \( f^{-1} \ast_\Psi g \) is not Ψ-Laplace transformable in general.

In the following theorem, we prove the uniqueness of Ψ-Laplace transform.
Theorem 3.13. Assume that \( f \) and \( g \) are piecewise continuous functions on \([0, \infty)\) and of \( \Psi \)-exponential order \( c > 0 \). If \( F(s) = G(s) \) for \( s > a \), then \( f(t) = g(t) \) for all \( t \geq 0 \).

Proof. Since \( F(s) = G(s) \), so \( \mathcal{L}_\Psi \{f - g\} = 0 \). Thus, we will prove that if \( \mathcal{L}_\Psi \{f(t)\} (s) = 0 \) for all \( s > a \) then \( f(t) = 0 \) for all \( t \geq 0 \).

Fixing \( s_0 > a \) and making substitution \( u = e^{-\Psi(t)} \) in (3.1), then for \( s = s_0 + n + 1 \) we get

\[
0 = F(s) = \int_0^\infty e^{-s_0 \Psi(t)} e^{-s \Psi(t)} \Psi'(t) f(t) dt = \int_0^1 u^n \left\{ u^{s_0} f \left( \Psi^{-1}(-\ln u) \right) \right\} du \tag{3.5}
\]

where \( n = 0, 1, 2, \ldots \) Assume that \( r(u) = u^{s_0} f \left( \Psi^{-1}(-\ln u) \right) \) which is a piecewise continuous function on \((0, 1]\) and

\[
\lim_{u \to 0} r(u) = \lim_{t \to \infty} e^{-s_0 \Psi(t)} f(t) = 0.
\]

If we consider \( r(0) = 0 \), then \( h \) is a piecewise continuous function satisfying

\[
\int_0^1 p(u) r(u) du = 0 \tag{3.6}
\]

where \( p \) is any polynomial. Thus, if \( \hat{r} \) has a power series expansion which converges uniformly on \([0, 1]\), then Eq. (3.6) can be rewritten as

\[
\int_0^1 \hat{r}(u) r(u) du = 0. \tag{3.7}
\]

On contrary, suppose that \( r \) is not a zero function then we can find a point \( u_0 \in (0, 1) \), an interval \( I = [u_0 - c_0, u_0 + c_0] \subset [0, 1] \) and a constant \( c \) such that \( r(u) \geq c > 0 \) for all \( u \in I \).

If we set \( \hat{r}(u) = e^{-(u-u_0)^2} \), then clearly Eq. (3.7) holds. Thus for \( x = u - u_0 \), we have

\[
J_1 = \int_{u_0 - c_0}^{u_0 + c_0} \hat{r}(u) du = \int_{-c_0}^{c_0} e^{-x^2} dx\]

and

\[
J_2 = \int_{u_0 + c_0}^{u_0 + c_0} \hat{r}(u) du = \int_{c_0}^{1-u_0} e^{-x^2} dx\]

and

\[
J_3 = \int_{u_0 - c_0}^{u_0 - c_0} \hat{r}(u) du = \int_{-c_0}^{-u_0} e^{-x^2} dx.
\]

If we set \( l = \int_{-\infty}^{\infty} e^{-x^2} dx \), then clearly \( l > 0 \) and for a given \( \epsilon > 0 \), we deduce

\[
J_1 \geq \frac{l}{2}, \quad 0 \leq J_2 \leq \epsilon, \quad 0 \leq J_3 \leq \epsilon.
\]
Since \( r(u) \geq c > 0 \) for all \( u \in I \) and \( |h| < n_0 \) where \( n_0 \in \mathbb{N} \), we have
\[
\int_I \hat{r}(u) r(u) du \geq \frac{lc}{2} > 0, \quad \left| \int_{[0,1]} \hat{r}(u) r(u) du \right| \leq 2n_0 \epsilon
\]
and hence
\[
\int_0^1 \hat{r}(u) r(u) du \geq \frac{lc}{2} - 2n_0 \epsilon > 0
\]
provided \( \epsilon < \frac{lc}{4n_0} \), contradicting Eq. (3.7). Thus, \( r \) is the zero function which implies that \( f \) is the zero function and this completes the proof. \( \square \)

4. The \( \Psi \)-Laplace transform of the \( \Psi \)-Hilfer fractional derivative

In this section, we compute the \( \Psi \)-Laplace transform of \( \Psi \)-Hilfer fractional derivative.

**Theorem 4.1.** If \( \mu > 0, m = \lfloor \mu \rfloor + 1, 0 \leq \nu \leq 1 \), and \( f(t), \mathcal{D}^j_{\Psi} \mathcal{I}_{0}^{(1-\nu)(m-\mu), \Psi} f(t) \in C [0, \infty) \) and of \( \Psi \)-exponential order for \( j = 0, 1, 2, \ldots, m-1 \), while \( \mathcal{D}^0_{\mu, \nu, \Psi} f(t) \) is piecewise continuous on \( [0, \infty) \). Then
\[
\mathcal{L}_\Psi \left\{ \mathcal{D}^\mu_{0, \nu, \Psi} f(t) \right\} = s^\mu \mathcal{L}_\Psi \{ f(t) \} - \sum_{i=0}^{m-1} s^{m(1-\nu)+\nu-i-1} \left( \mathcal{I}_{0}^{(1-\nu)(m-\mu)-i, \Psi} f(0) \right).
\]

**Proof.** From the definition of integral operator \( \mathcal{D}^\mu_{0, \nu, \Psi} f \) and (3.1), we have
\[
\mathcal{L}_\Psi \left\{ \left( \mathcal{D}^\mu_{0, \nu, \Psi} f(t) \right) \right\} = \mathcal{L}_\Psi \left\{ \mathcal{I}_{a}^{\nu(m-\mu), \Psi} \left( \frac{1}{\Psi(t)} \frac{d}{dt} \right)^m \mathcal{I}_{a}^{(1-\nu)(m-\mu)\Psi} f(t) \right\}.
\]
Using Theorem 3.8 and 3.7, we get
\[
\mathcal{L}_\Psi \left\{ \left( \mathcal{D}^\mu_{0, \nu, \Psi} f(t) \right) \right\} = s^{-\nu(m-\mu)} \mathcal{L}_\Psi \left\{ \left( \frac{1}{\Psi(t)} \frac{d}{dt} \right)^m \mathcal{I}_{a}^{(1-\nu)(m-\mu)\Psi} f(t) \right\}
\]
\[
= s^{-\nu(m-\mu)} \left[ s^m \mathcal{L}_\Psi \left\{ (\mathcal{I}_{0}^{(1-\nu)(m-\mu)\Psi} f(t)) \right\} 
\right.
\]
\[
- \sum_{i=0}^{m-1} s^{m-i-1} \left( \mathcal{D}^i_{\nu, \Psi} \mathcal{I}_{0}^{(1-\nu)(m-\mu)\Psi} f(0) \right) \right]
\]
\[
= s^{-\nu(m-\mu)} \left[ s^m s^{-(1-\nu)(m-\mu)} \mathcal{L}_\Psi \left\{ (\mathcal{I}_{0}^{(1-\nu)(m-\mu)\Psi} f(t)) \right\} 
\right.
\]
\[
- \sum_{i=0}^{m-1} s^{m-i-1} (\mathcal{I}_{0}^{(1-\nu)(m-\mu)-i, \Psi} f(0)) \right]
\]
\[
= s^{\mu} \mathcal{L}_\Psi \{ f(t) \} - \sum_{i=0}^{m-1} s^{m(1-\nu)+\nu-i-1} (\mathcal{I}_{0}^{(1-\nu)(m-\mu)-i, \Psi} f(0)).
\]
This completes the proof. \( \square \)
5. Effectiveness of the $\Psi$-Laplace transform method for solving fractional-order differential equations

In this section, we examine the effectiveness of the $\Psi$-Laplace transform method for solving fractional-order differential equations of the following type

$$
C^\mu_0 \Psi y(t) = Ay(t) + g(t), \quad 0 < \mu < 1, \quad t \geq 0, \quad (5.1)
$$
$$
y(0) = \eta, \quad (5.2)
$$

where $C^\mu_0 \Psi$ is the Caputo-type fractional differential operator, $A$ is $n \times n$ constant matrix and $g(t)$ is $n-$dimensional continuous function.

**Theorem 5.1.** Let (5.1) − (5.2) has a unique and continuous solution $y(t)$. Assume that $g(t)$ is continuous on $[0, \infty)$ and $\Psi$-exponentially bounded, then $y(t)$ and $C^\mu_0 \Psi y(t)$ are both $\Psi$-exponentially bounded.

**Proof.** It can be noticed that (5.1) − (5.2) is equivalent to the Volterra equation given below

$$
y(t) = \eta + \frac{1}{\Gamma(\mu)} \int_0^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) \{Ay(\tau) + g(\tau)\} d\tau, \quad 0 \leq t < \infty. \quad (5.3)
$$

By assumption, $g(t)$ is $\Psi$-exponentially bounded, so there exist positive constants $c$, $M$ and large enough $T$ such that $||g(t)|| \leq Me^{c\Psi(t)}$ for all $t \geq T$. For $t \geq T$, (5.3) can be written as

$$
y(t) = \eta + \frac{1}{\Gamma(\mu)} \int_0^T (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) \{Ay(\tau) + g(\tau)\} d\tau
$$
$$
+ \frac{1}{\Gamma(\mu)} \int_T^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) \{Ay(\tau) + g(\tau)\} d\tau.
$$

Since, $y(t)$ is unique and continuous solution of (5.1) − (5.2) on $[0, \infty)$, thus $Ay(t) + g(t)$ is bounded on $[0, T]$ that is there exists constant $l > 0$ such that $||Ay(t) + g(t)|| < l$. So, we get

$$
||y(t)|| \leq ||\eta|| + \frac{l}{\Gamma(\mu)} \int_0^T (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) d\tau
$$
$$
+ \frac{1}{\Gamma(\mu)} \int_T^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau)||A|| ||y(\tau)|| d\tau
$$
$$
+ \frac{1}{\Gamma(\mu)} \int_T^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau)||g(\tau)|| d\tau.
$$
Using $e^{-c\Psi(t)} \leq e^{-c\Psi(T)}$, $e^{-c\Psi(t)} \leq e^{-c\Psi(\tau)}$, $||g(t)|| \leq Me^{c\Psi(t)}$ and multiplying the above inequality by $e^{-c\Psi(t)}$, we find

$$
||y(t)||e^{-c\Psi(t)} \leq ||\eta||e^{-c\Psi(t)} + \frac{le^{-c\Psi(t)}}{\Gamma(\mu)} \int_0^T (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) d\tau \\
+ \frac{e^{-c\Psi(t)}}{\Gamma(\mu)} \int_0^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) ||A|| ||y(\tau)|| d\tau \\
+ \frac{e^{-c\Psi(t)}}{\Gamma(\mu)} \int_0^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) ||g(\tau)|| d\tau \\
\leq ||\eta||e^{-c\Psi(T)} + \frac{le^{-c\Psi(T)}}{\mu\Gamma(\mu)} \left((\Psi(t))^{\mu} - (\Psi(t) - \Psi(T))^{\mu}\right) \\
+ \frac{||A||}{\Gamma(\mu)} \int_0^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) ||\Psi(t)||e^{-c\Psi(\tau)} d\tau \\
+ \frac{M}{\Gamma(\mu)} \int_0^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau)e^{c(\Psi(\tau) - \Psi(t))} d\tau \\
\leq ||\eta||e^{-c\Psi(T)} + \frac{l(\Psi(T))^{\mu}e^{-c\Psi(T)}}{\mu\Gamma(\mu)} + \frac{M}{\Gamma(\mu)} \int_0^\infty e^{-cs} s^{\mu-1} ds \\
+ \frac{||A||}{\Gamma(\mu)} \int_0^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) ||y(\tau)||e^{-c\Psi(\tau)} d\tau \\
\leq ||\eta||e^{-c\Psi(T)} + \frac{l(\Psi(T))^{\mu}e^{-c\Psi(T)}}{\mu\Gamma(\mu)} + \frac{M}{c\mu} \\
+ \frac{||A||}{\Gamma(\mu)} \int_0^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) ||y(\tau)||e^{-c\Psi(\tau)} d\tau.
$$

Assume that

$$a = ||\eta||e^{-c\Psi(T)} + \frac{l(\Psi(T))^{\mu}e^{-c\Psi(T)}}{\mu\Gamma(\mu)} + \frac{M}{c\mu}, \quad b = \frac{||A||}{\Gamma(\mu)}, \quad r(t) = ||y(t)||e^{-c\Psi(t)},$$

then, we have

$$r(t) \leq a + b \int_0^t (\Psi(t) - \Psi(\tau))^{\mu-1} \Psi'(\tau) ||y(\tau)||e^{-c\Psi(\tau)} d\tau.$$

Using Gronwall-inequality [36], we deduce

$$r(t) \leq a\mathcal{E}_\mu \left(||A|| (\Psi(t) - \Psi(\tau))^{\mu}\right) \leq a\mathcal{E}_\mu \left(||A|| (\Psi(t))^{\mu}\right). \quad (5.4)$$

For $0 < \mu < 1$, $u > 0$, $t \geq 0$, the following inequality can easily be proved

$$\mathcal{E}_\mu \left(u (\Psi(t))^{\mu}\right) \leq Ce^{u^{1/\mu} \Psi(t)}, \text{ where } C > 0. \quad (5.5)$$

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From (5.4) and (5.5), we have
\[ r(t) \leq aCe^{(||A||)^{1/\mu}\Psi(t)}, \]
and finally, we get
\[ ||y(t)|| \leq aCe^{\left\{\left(||A||\right)^{1/\mu}+c\right\}\Psi(t)}. \]
Thus, \( y(t) \) is \( \Psi \)-exponentially bounded. Moreover, from Eq. (5.1), we have
\[ ||C^D_{0}^{\mu,\Psi}y(t)|| \leq ||A|| ||y(t)|| + ||g(t)|| \]
\[ \leq a||A||Ce^{\left\{\left(||A||\right)^{1/\mu}+c\right\}\Psi(t)} + Me^{\Psi(t)} \]
\[ \leq \left(a||A||C + M\right)e^{\left\{\left(||A||\right)^{1/\mu}+c\right\}\Psi(t)}. \]
Thus, \( C^D_{0}^{\mu,\Psi}y(t) \) is also \( \Psi \)-exponentially bounded and this completes the proof.

Similar results can be proved for fractional-order differential equations in the settings of \( \Psi \)-RL and \( \Psi \)-Hilfer fractional derivatives.

6. Applications

In this section, by using the \( \Psi \)-Laplace transformation method, we state and find solutions of different classes of linear FDEs with constant coefficients, in the settings of \( \Psi \)-RL, \( \Psi \)-C and \( \Psi \)-Hilfer fractional derivatives. We now divide this section into the following subsections.

6.1. Solutions of some non-homogeneous linear \( \Psi \)-RL and \( \Psi \)-C FDEs

In this subsection, we use the \( \Psi \)-Laplace transformation method to solve the ordinary FDEs in the frame of \( \Psi \)-RL and \( \Psi \)-C fractional derivatives.

**Theorem 6.1.** [1] The FDE
\[ D_{0}^{\mu,\Psi}y(t) - \lambda y(t) = f(t), \ 0 < \mu \leq 1, \ \lambda \in \mathbb{R}, \]
with initial condition
\[ (I_{0}^{1-\mu,\Psi})y(0) = c, \ c \in \mathbb{R}, \]
has the solution
\[ y(t) = c(\Psi(t))^{\mu-1}E_{\mu,\mu}\left(\lambda(\Psi(t))^{\mu}\right) + (\Psi(t))^{\mu-1}E_{\mu,\mu}\left(\lambda(\Psi(t))^{\mu}\right) *_{\Psi} f(t). \]
Theorem 6.2. [1] The FDE
\[ C^\mu_0 \Psi y(t) - \lambda y(t) = f(t), \quad 0 < \mu \leq 1, \quad \lambda \in \mathbb{R}, \] (6.3)
with initial condition
\[ y(0) = c, \quad c \in \mathbb{R}, \] (6.4)
has the solution
\[ y(t) = c \mathcal{E}_\mu \left( \lambda(\Psi(t))^\mu \right) + (\Psi(t))^{\mu-1} \mathcal{E}_{\mu,\mu} \left( \lambda(\Psi(t))^\mu \right) *_{\Psi} f(t). \] (6.5)

Remark 6.3. Sometimes FDEs are more appropriate to model natural states. For example, if we consider \( \Psi(t) = t \) and \( f(t) = 0 \) in (6.3) then the resultant FDE is more suitable in modeling the population growth than the ordinary differential equation [37]. Moving one step forward, Almeida [11] showed that a population growth model could be reproduced more accurately by considering different \( \Psi \)’s.

Corollary 6.4. Consider a special case of initial value problem (6.3)-(6.4)
\[ C^\mu_0 \Psi y(t) - y(t) = 1, \quad 0 < \mu \leq 1, \] (6.6)
\[ y(0) = 1. \] (6.7)
Then
(a) \( y(t) = \mathcal{E}_\mu (t^{\frac{\mu}{2}}) + t^{\frac{\mu}{2}} \mathcal{E}_{\mu,\mu+1}(t^{\frac{\mu}{2}}), \) for \( \Psi(t) = \sqrt{t}. \)
(b) \( y(t) = \mathcal{E}_\mu (t^\mu) + t^\mu \mathcal{E}_{\mu,\mu+1}(t^\mu), \) for \( \Psi(t) = t. \)
(c) \( y(t) = \mathcal{E}_\mu (t^{2\mu}) + t^{2\mu} \mathcal{E}_{\mu,\mu+1}(t^{2\mu}), \) for \( \Psi(t) = t^2. \)

Proof. (b) From (3.3) and (6.5), we have
\[ y(t) = \mathcal{E}_\mu (t^\mu) + \int_0^t t^{\mu-1} \mathcal{E}_{\mu,\mu}(t^\mu) d\tau = \mathcal{E}_\mu (t^\mu) + \int_0^t \sum_{k=0}^{\infty} \frac{\tau^k}{\Gamma(\mu k + \mu + 1)} d\tau \]
\[ = \mathcal{E}_\mu (t^\mu) + \sum_{k=0}^{\infty} \frac{t^{\mu k + \mu}}{\Gamma(\mu k + \mu + 1)} = \mathcal{E}_\mu (t^\mu) + t^\mu \mathcal{E}_{\mu,\mu+1}(t^\mu). \]
Similarly, one can prove part (a) and (c). Plots of solutions (a), (b) and (c) are given in Figure 1 (a), (b) and (c) respectively.
Theorem 6.5. The fractional diffusion equation

\[ \frac{\partial^\mu u}{\partial t^\mu} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad \text{where} \quad 0 < \mu \leq 1, \]

(6.8)

with initial and boundary conditions

\[ u(x, t) \to 0 \quad \text{as} \quad |x| \to \infty, \]

(6.9)

\[ (I_0^{1-\mu, \Psi})u(x, t) \bigg|_{t=0} = f(x), \quad x \in \mathbb{R}, \]

(6.10)

has the solution

\[ u(x, t) = \int_{-\infty}^{\infty} G(x - \eta, t)f(\eta)d\eta, \]

where

\[ G(x, t) = \frac{1}{2\sqrt{\kappa}}\left(\Psi(t)\right)^{\frac{\mu}{2}}W\left(-\frac{|x|}{\sqrt{\kappa}(\Psi(t))^{\frac{\mu}{2}}}, -\frac{\mu}{2}, \frac{\mu}{2}\right). \]
Proof. Applying the Fourier transform to both sides of (6.8) and (6.10) with respect to $x$, and using (6.9), we have

$$D^\mu_{0}\tilde{u}(k, t) = -\kappa k^2 \tilde{u}(k, t), \quad (6.11)$$

$$\left. (I_{0}^{-\mu}) \tilde{u}(k, t) \right|_{t=0} = \tilde{f}(k). \quad (6.12)$$

Applying the $\Psi$-Laplace transform to both sides of (6.11) with respect to $t$, and using (6.12), we get

$$\mathcal{L}_\Psi \{ \tilde{u}(k, t) \} = \frac{\tilde{f}(k)}{(s^\mu + \kappa k^2)}$$

$$= \mathcal{L}_\Psi \left\{ \tilde{f}(k)(\Psi(t))^{\mu-1} \mathcal{E}_{\mu, \mu} \left( - \kappa k^2 (\Psi(t))^\mu \right) \right\},$$

and from above equality, we find

$$\tilde{u}(k, t) = \tilde{f}(k)(\Psi(t))^{\mu-1} \mathcal{E}_{\mu, \mu} \left( - \kappa k^2 (\Psi(t))^\mu \right). \quad (6.13)$$

The inverse Fourier transform of (6.13) gives

$$u(x, t) = \int_{-\infty}^{\infty} G(x - \eta, t) f(\eta) d\eta \quad (6.14)$$

where

$$G(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} (\Psi(t))^{\mu-1} \mathcal{E}_{\mu, \mu} \left( - \kappa k^2 (\Psi(t))^\mu \right) \cos(kx) dk.$$ 

The above integral can be evaluated by using the $\Psi$-Laplace transform of $G(x, t)$ with respect to $t$ as

$$\mathcal{L}_\Psi \{ G(x, t) \} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(kx)}{(s^\mu + \kappa k^2)} dk$$

$$= \frac{1}{2\sqrt{\kappa}} s^{-\frac{\mu}{2}} e^{-\frac{i\kappa}{\sqrt{\kappa}}} \left\{ \frac{1}{2\sqrt{\kappa}(\Psi(t))^{\mu-1}} W \left( - \frac{|x|}{\sqrt{\kappa}(\Psi(t))^{\frac{\mu}{2}}}, -\mu, \frac{\mu}{2} \right) \right\}.$$

Finally,

$$G(x, t) = \frac{1}{2\sqrt{\kappa}}(\Psi(t))^{\frac{\mu}{2}-1} W \left( - \frac{|x|}{\sqrt{\kappa}(\Psi(t))^{\frac{\mu}{2}}}, -\mu, \frac{\mu}{2} \right).$$

Thus, this completes the proof. \qed

It can be noted that for $\Psi(t) = t$ and $\mu = 1$, the Cauchy problem (6.8)-(6.10) reduces to the classical diffusion problem and solution (6.14) reduces to the classical fundamental solution.
6.2. Solutions of some general $\Psi$-Hilfer FDEs

Assume that

$$0 < \mu_1 \leq \mu_2 < 1, \ 0 \leq \nu_j \leq 1, \ a_j \in \mathbb{R} \quad \text{for } j = 1, 2.$$ 

Consider the $\Psi$-Hilfer FDE

$$a_1 \mathcal{D}_{0}^{\mu_1,\nu_1}y(t) + a_2 \mathcal{D}_{0}^{\mu_2,\nu_2}y(t) + a_3 y(t) = f(t), \quad (6.15)$$

with initial conditions

$$(I_{0}^{1-\nu_j}(1-\mu_j))y(0) = b_j \quad \text{for } j = 1, 2. \quad (6.16)$$

For dielectric relaxation in glasses, an equation of the form (6.15) was introduced by Hilfer in [34]. In the space of Lebesgue integrable functions, Tomovski et al. [35] obtained the solution of a particular case of Cauchy problem (6.15)-(6.16) when $\Psi(t) = t$.

In the following Theorem, using $\Psi$-Laplace transform we find the general solution of initial value problem (6.15)-(6.16).

**Theorem 6.6.** The initial value problem (6.15)-(6.16) has the solution

$$y(t) = \frac{1}{a_2} \sum_{i=0}^{\infty} \left( - \frac{a_1}{a_2} \right)^i \left[ \left( \Psi(t) \right)^{(\mu_2-\mu_1)i+\mu_2-1} \mathcal{L}_{\nu_2,\mu_2-\mu_1}^{i+1} \left( - \frac{a_3}{a_2} (\Psi(t))^{\mu_2} \right) * f(t) \right. \\
+ a_2 b_2 \left( \Psi(t) \right)^{(\mu_2-\mu_1)i+\mu_2+\nu_2(1-\mu_2)-1} \mathcal{L}_{\nu_2,\mu_2-\mu_1}^{i+1} \left( - \frac{a_3}{a_2} (\Psi(t))^{\mu_2} \right) \\
+ a_1 b_1 \left( \Psi(t) \right)^{(\mu_2-\mu_1)i+\mu_2+\nu_1(1-\mu_1)-1} \mathcal{L}_{\nu_2,\mu_2-\mu_1}^{i+1} \left( - \frac{a_3}{a_2} (\Psi(t))^{\mu_2} \right).$$

**Proof.** Applying $\Psi$-Laplace transform to both sides of (6.15) and using initial conditions (6.16), we have

$$\mathcal{L}_{\Psi} \{y(t)\} = \mathcal{L}_{\Psi} \{f(t)\} + a_2 b_2 \frac{\mathcal{L}_{\Psi} \{f(t)\} \left( s^{\nu_2(\mu_2-1)} \right)}{a_1 s^{\mu_1} + a_2 s^{\mu_2} + a_3} + a_1 b_1 \frac{\mathcal{L}_{\Psi} \{f(t)\} \left( s^{\nu_2(\mu_2-1)} \right)}{a_1 s^{\mu_1} + a_2 s^{\mu_2} + a_3}. \quad (6.17)$$

Moreover for $j = 1, 2$ we have

$$\frac{s^{\nu_j(\mu_j-1)}}{a_1 s^{\mu_1} + a_2 s^{\mu_2} + a_3} = \frac{1}{a_2} \left( \frac{s^{\nu_j(\mu_j-1)}}{s^{\mu_2} + \frac{a_3}{a_2}} \right) \left( \frac{1}{1 + \frac{a_1}{a_2} \left( \frac{s^{\mu_1}}{s^{\mu_2} + \frac{a_3}{a_2}} \right)} \right)$$
and finally we have

\[ Y(s) = \frac{1}{a_2} \sum_{i=0}^{\infty} \left( -\frac{a_1}{a_2} \right)^i \frac{s^{\mu_1 i + \nu_j \mu_j - \nu_j}}{(s^{\mu_2} + \frac{a_3}{a_2})^{i+1}} \]

Thus, from Equation (6.17) we find

\[ Y(s) = \frac{1}{a_2} \sum_{i=0}^{\infty} \left( -\frac{a_1}{a_2} \right)^i \frac{s^{\mu_1 i}}{(s^{\mu_2} + \frac{a_3}{a_2})^{i+1}} \mathcal{L}_\Psi \{ f(t) \} \]

and

\[ \mathcal{L}_\Psi \{ f(t) \} = \frac{1}{a_1 s^{\mu_1} + a_2 s^{\mu_2} + a_3} \mathcal{L}_\Psi \{ f(t) \} \]

Thus, from Equation (6.17) we find

\[ Y(s) = \frac{1}{a_2} \sum_{i=0}^{\infty} \left( -\frac{a_1}{a_2} \right)^i \mathcal{L}_\Psi \left[ (\Psi(t))^{(\mu_2 - \mu_1)i + \mu_2 - 1} \mathcal{E}^{i+1}_{\mu_2,(\mu_2 - \mu_1)i + \mu_2} \left( -\frac{a_3}{a_2}(\Psi(t))^{\mu_2} \right) * \Psi f(t) \right] \]

and finally we have

\[ y(t) = \frac{1}{a_2} \sum_{i=0}^{\infty} \left( -\frac{a_1}{a_2} \right)^i \left[ (\Psi(t))^{(\mu_2 - \mu_1)i + \mu_2 - 1} \mathcal{E}^{i+1}_{\mu_2,(\mu_2 - \mu_1)i + \mu_2} \left( -\frac{a_3}{a_2}(\Psi(t))^{\mu_2} \right) * \Psi f(t) \right] \]

\[ + a_2 b_2 (\Psi(t))^{(\mu_2 - \mu_1)i + \mu_2 + \nu_2(1-\mu_2) - 1} \mathcal{E}^{i+1}_{\mu_2,(\mu_2 - \mu_1)i + \mu_2 + \nu_2(1-\mu_2)} \left( -\frac{a_3}{a_2}(\Psi(t))^{\mu_2} \right) \]

\[ + a_1 b_1 (\Psi(t))^{(\mu_2 - \mu_1)i + \mu_2 + \nu_1(1-\mu_1) - 1} \mathcal{E}^{i+1}_{\mu_2,(\mu_2 - \mu_1)i + \mu_2 + \nu_1(1-\mu_1)} \left( -\frac{a_3}{a_2}(\Psi(t))^{\mu_2} \right) . \]

**Theorem 6.7.** Assume that \( 0 < \mu_1 \leq \mu_2 \leq \mu_3 < 1, 0 \leq \nu_j \leq 1 \) and \( a_j \in \mathbb{R} \) for \( j = 1, 2, 3 \). Then the initial value problem

\[ a_1 D_{0^+}^{\mu_1,\nu_1,\Psi} y(t) + a_2 D_{0^+}^{\mu_2,\nu_2,\Psi} y(t) + a_3 D_{0^+}^{\mu_3,\nu_3,\Psi} y(t) + a_4 y(t) = f(t), \quad (6.18) \]
\( (T_0^{(1-\nu_j)(1-\mu_j)} \Psi) y(0) = b_j \quad \text{for} \quad j = 1, 2, 3. \) (6.19)

has the solution

\[
y(t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{a_3^{i+1}} \sum_{k=0}^{i} \binom{i}{k} a_1^k a_2^{i-k} (\Psi(t))^{(\mu_3-\mu_2)i+(\mu_2-\mu_1)k+\mu_3-1} \\
\times \left[ a_1 b_1 (\Psi(t))^{\nu_1(1-\mu_1)} \mathcal{E}_{\mu_3, (\mu_3-\mu_2)i+(\mu_2-\mu_1)k+\mu_3+\nu_1(1-\mu_1)} \left( -\frac{a_4}{a_3} (\Psi(t))^{\mu_3} \right) \\
+ a_2 b_2 (\Psi(t))^{\nu_2(1-\mu_2)} \mathcal{E}_{\mu_3, (\mu_3-\mu_2)i+(\mu_2-\mu_1)k+\mu_3+\nu_2(1-\mu_2)} \left( -\frac{a_4}{a_3} (\Psi(t))^{\mu_3} \right) \\
+ a_3 b_3 (\Psi(t))^{\nu_3(1-\mu_3)} \mathcal{E}_{\mu_3, (\mu_3-\mu_2)i+(\mu_2-\mu_1)k+\mu_3+\nu_3(1-\mu_3)} \left( -\frac{a_4}{a_3} (\Psi(t))^{\mu_3} \right) \\
+ \mathcal{E}_{\mu_3, (\mu_3-\mu_2)i+(\mu_2-\mu_1)k+\mu_3} \left( -\frac{a_4}{a_3} (\Psi(t))^{\mu_3} \right) \ast \Psi f(t) \right].
\]

**Proof.** Making use of the technique demonstrated in the previous result, it is easy to derive the solution. So we omit the straightforward but tedious details. \( \square \)

**Remark 6.8.** The author in [34] noticed that a special case of the FDE (6.18) when

\[
\Psi(t) = t, \quad a_4 = 1, \quad f(t) = 0, \quad \mu_j = \nu_j = 1 \quad \text{for} \quad j = 1, 2, 3
\]

illustrates the process of dielectric relaxation in glycerol over 12 decades in frequency. Moreover, in the space of Lebesgue integrable functions, . Tomovski et al. [35] found the solution of a special case of Cauchy problem (6.18)-(6.19) when \( \Psi(t) = t. \)

**References**

[1] Jarad F, Abdeljawad T. Generalized fractional derivatives and Laplace transform. Discrete and Continuous Dynamical Systems-S. 2019:17751786.

[2] Diethelm K. The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo type. Berlin, Heidelberg: Springer; 2010.

[3] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Amsterdam, London, New York: Elsevier (North-Holland) Science Publishers; 2006 (North-Holland Mathematical Studies; Vol. 204).
Herzallah MAE, El-Sayed AMA, Baleanu D. On the Fractional-Order Diffusion-Wave Process. Rom J Phys. 2010;55(3-4):274-284.

Hilfer R. Applications of fractional calculus in Physics. New Jersey: World Scientific; 2000.

Podlubny I. Fractional differential equations. New York: Acad. Press; 1999.

Samko SG, Kilbas AA, Marichev OI. Fractional Integrals and Derivatives: Theory and Applications. New York: Gordon and Breach; 1993.

Tarasov VE. Fractional dynamics: application of fractional calculus to dynamics of particles, fields and media. HEP: Springer; 2011.

Osler TJ. The fractional derivatives of a composite function. SIAM J Math Anal 1970;1:288293.

Hilfer R, Luchko Y, Tomovski Z. Operational method for solution of the fractional differential equations with the generalized Riemann-Liouville fractional derivatives. Fract. Calc. Appl. Anal. 2009;12:299318.

Almeida R. A Caputo fractional derivative of a function with respect to another function. Commun Nonlinear Sci. 2017;44:460-481.

Sousa JVD, de Oliveira EC. On the Ψ-Hilfer fractional derivative. Commun Nonlinear Sci. 2018;60:72-91.

Jumarie G. Laplace’s transform of fractional order via the Mittag-Leffler function and modified Riemann-Liouville derivative. Appl Math Lett. 2009;22(11):1659-1664.

Jarad F, Abdeljawad T. A modified Laplace transform for certain generalized fractional operators. Results Nonlinear Anal. 2018;2018(2):8898.

Bultheel A, Martinez-Sulbaran H. Recent developments in the theory of the fractional Fourier and linear canonical transforms. B Belg Math Soc-Sim. 2006;13(5):971-1005.

Kerr FH. Namias Fractional Fourier-Transforms on L2 and Applications to Differential-Equations. J Math Anal Appl. 1988;136(2):404-418.
[17] Zayed AI. A class of fractional integral transforms: A generalization of the fractional Fourier transform. Ieee T Signal Proces. 2002;50(3):619-627.

[18] Zayed AI. Fractional Fourier transform of generalized functions. Integr Transf Spec F. 1998;7(3-4):299-312.

[19] Ozaktas HM, Zalevsky Z, Kutay MA. The Fractional Fourier Transform with applications in optics and signal processing. UK: Wiley; 2001.

[20] Namias V. The fractional order Fourier-Transform and its application to Quantum-Mechanics. J I Math Appl. 1980;25(3):241-265.

[21] McBride AC, Kerr FH. On Namias’s Fractional Fourier-Transforms. Ima J Appl Math. 1987;39(2):159-175.

[22] Silva FS, Moreira DM, Moret MA. Conformable Laplace Transform of Fractional Differential Equations. Axioms. 2018;7(3).

[23] Caputo M. Linear model of dissipation whose q is almost frequency independent-II. Geophys J R Astron Soc. 1967;13:529-539.

[24] Gambo YY, Jarad F, Baleanu D, Abdeljawad T. On Caputo modification of the Hadamard fractional derivatives. Adv Differ Equ-Ny. 2014;2014(10).

[25] Wright EM. On the coefficients of power series having exponential singularities. Journal of the London Mathematical Society. 1933;8:7179.

[26] Wright EM. The generalized bessel function of order greater than one. Quart. J. Math., Oxford Ser. 1940;11:3648.

[27] Prabhakar TR. A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Math. J. 1971;19(1):715.

[28] Kilbas AA, Saigo M, Saxena RK. Generalized Mittag-Leffler function and generalized fractional calculus operators. Integr Transf Spec F. 2004;15(1):31-49.

[29] Nigmatullin RR, Khamzin AA, Baleanu D. On the Laplace integral representation of multivariate Mittag-Leffler functions in anomalous relaxation. Math Method Appl Sci. 2016;39(11):2983-2992.

21
[30] Saxena RK, Saigo M. Certain properties of fractional calculus operators associated with generalized Mittag-Leffler function. Fract. Calc. Appl. Anal. 2005;8(2):141154.

[31] Garra R, Garrappa R. The Prabhakar or three parameter Mittag-Leffler function: Theory and application. Commun Nonlinear Sci. 2018;56:314-329.

[32] Mittag-Leffler GM. Sur la nouvelle fonction $E_\alpha(x)$. C R Acad Sci. 1903;137:554558.

[33] Wiman A. Ueber den Fundamentalsatz in der Theorie der Funktionen $E_\alpha(x)$. Acta. Math. 1905;29:191201.

[34] Hilfer R. Experimental evidence for fractional time evolution in glass forming materials. Chem Phys. 2002;284(1-2):399-408.

[35] Tomovski Z, Hilfer R, Srivastava HM. Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions. Integr Transf Spec F. 2010;21(11):797-814.

[36] Vanterler da C. Sousa J, Capelas de Oliveira E. A Gronwall inequality and the Cauchy-type problem by means of $\psi$-Hilfer operator, arXiv: 1709. 03634 [math.CA] (2017).

[37] Almeida R, Bastos NR, Monteiro MTT. Modeling some real phenomena by fractional differential equations. Math Methods Appl Sci. 2016;39(16):48464855.