A note for global existence of a two-dimensional chemotaxis–haptotaxis model with remodeling of non-diffusible attractant

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Abstract

In this paper, we study the following coupled chemotaxis–haptotaxis model with remodeling of non-diffusible attractant

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \\
    v_t &= \Delta v - v + u, \\
    w_t &= -v w + \eta w(1 - u - w)
\end{align*}
\]

in a bounded smooth domain \(\Omega \subseteq \mathbb{R}^2\) with zero-flux boundary conditions, where \(\chi, \xi\) and \(\eta\) are positive parameters. Under appropriate regularity assumptions on the initial data \((u_0, v_0, w_0)\), by developing some \(L^p\)-estimate techniques, we prove the global existence and uniqueness of classical solutions when \(\mu > 0\), where \(\mu\) is the logistic growth rate of cancer cells. This result removes the additional restriction on \(\mu\), where \(\mu\) is sufficiently large in Pang and Wang (2017 J. Differ. Equ. 263 1269–92) for the global existence of solutions.

Keywords: chemotaxis–haptotaxis, global existence, non-diffusible attractant

Mathematics Subject Classification numbers: 92C17, 35K55, 35K59, 35K20

1. Introduction

The oriented movement of biological cells or organisms in response to a chemical gradient is called chemotaxis (see Calvez and Carrillo [2], Fontelos et al [8], Hillen and Painter [13], Horstmann [15, 16], Jüger and Luckhaus [20], Kavallaris and Souplet [21], Nagai [29],...
Perthame [32], Sherratt [34], Winkler [50]). To describe chemotaxis of cell populations, in 1970, Keller and Segel (see [22]) proposed an important variant of the quasilinear chemotaxis model

\[
\begin{align*}
\nu_t &= \Delta \nu - \chi \nabla \cdot (\phi(u) \nabla u) - \chi \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\
v_t &= \Delta v + u - v, & x \in \Omega, t > 0.
\end{align*}
\] (1.1)

The interesting feature of models (1.1) is the possibility of blow-up of solutions in finite time, which strongly depends on the space dimension (see e.g. Horstmann et al [15, 17], Rascle and Ziti [33]). In fact, solutions of (1.1) may blow up in finite time when \( N \geq 2 \) (see Herrero and Velázquez [11], Osaki et al [30], Winkler [51]). In the higher-dimensional case when \( N \geq 3 \), small total mass of cells appears to be insufficient to rule out blow-up (see Winkler et al [18, 20]). However, it is shown by some recent results that the nonlinear diffusion function (see Ishida et al [9]) and the (generalized) logistic growth term (see Lankeit [24], Winkler et al [46, 49, 52], Zheng [54, 59, 60]) may prevent the blow-up of solutions.

In order to describe the processes of cancer invasion, one important extension of the classical Keller–Segel model to a more complex cell migration mechanism was proposed by Chaplain and Lolas (see Chaplain and Lolas [4]). In fact, the precise model suggested in [4] consists of two parabolic equations and a Bernoulli-type ordinary differential equation, as given by

\[
\begin{align*}
u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\
\tau v_t &= \Delta v + u - v, & x \in \Omega, t > 0, \\
w_t &= -\nu w + \eta w(1 - u - w), & x \in \Omega, t > 0, \\
\frac{\partial u}{\partial \nu} - \chi \frac{\partial u}{\partial \nu} - \xi \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
\nu(x, 0) &= u_0(x), \tau v(x, 0) = \tau v_0(x), w(x, 0) = w_0(x), & x \in \Omega,
\end{align*}
\] (1.2)

where \( \tau > 0, \eta \geq 0, \Omega \subseteq \mathbb{R}^N (N \geq 1) \) is the physical domain which we assume to be bounded with smooth boundary, \( \chi, \xi, \mu \) and \( \eta \) represent, respectively, the chemotactic sensitivities and haptotactic sensitivities, the proliferation rate of the cells and the remodeling rate of the extracellular matrix (ECM). Here the unknown quantities \( u = u(x, t), v = v(x, t) \) and \( w = w(x, t) \) denote the density of cancer cells, the concentration of enzyme and the density of healthy tissue, respectively.

The model (1.2) accounts for both chemotactic migration of cancer cells towards a diffusible matrix-degrading enzyme (MDE) secreted by themselves, and haptotactic migration towards a static tissue, also referred to as ECM (Chaplain et al [5, 10], Liotta and Clair [25]). On the one hand, it opens the new fields of applications to modeling approaches in the style pursued by Keller and Segel (see [22]); on the other hand, it gives rise to new mathematical challenges due to more involved couplings.

If \( \chi = 0 \), the PDE system (1.2) becomes the haptotaxis-only system (with remodeling of non-diffusible attractant)

\[
\begin{align*}
\nu_t &= \Delta u - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\
v_t &= \Delta v + u - v, & x \in \Omega, t > 0, \\
w_t &= -\nu w + \eta w(1 - u - w), & x \in \Omega, t > 0.
\end{align*}
\] (1.3)

Global existence and asymptotic behavior of solutions to (1.3) have been investigated in [6, 7, 26, 28, 35, 45, 47] and [36] for the case \( \eta = 0 \) and \( \eta > 0 \), respectively.
When $\eta = 0$, the PDE system (1.2) is reduced to the chemotaxis–haptotaxis system
\begin{equation}
\begin{aligned}
\begin{cases}
\tau u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\
\tau v_t = \Delta v + u - v, & x \in \Omega, t > 0, \\
\tau w_t = -\nabla w, & x \in \Omega, t > 0.
\end{cases}
\end{aligned}
\end{equation}

When $\tau = 0$, it denotes that the diffusion rate of the MDE is much greater than that of cancer cells (see Chaplain and Lolas [4], Winkler et al [1, 42]). In [39], Tao and Wang proved that model (1.4) possesses a unique global bounded classical solution for any $\mu > 0$ in 2D space, and for large $\mu > 0$ in 3D space; Tao and Winkler [43] studied global boundedness for model (1.4) with the condition $\mu > \frac{(N-2)^+}{N} \chi$, and also gave the exponential decay of $w$ in the large time limit for the additional explicit smallness on $w_0$; while, if $\tau = 1$ in (1.2), Tao and Wang [38] proved that model (1.2) possesses a unique global-in-time classical solution for any $\chi > 0$ in 1D space, and for small $\frac{2}{N} > 0$ in 2D space; Tao [35] improved the result of [38] for any $\mu > 0$ in 2D space; additionally, recent studies have shown that the solution behavior can be also impacted by the nonlinear diffusion (see Tao and Winkler [40], Wang et al [27, 48, 58]) and the (generalized) logistic damping (see Cao [3], Hillen et al [14], Zheng et al [57, 61]).

Compared with the chemotaxis-only system, haptotaxis-only system and the chemotaxis–haptotaxis system, the coupled chemotaxis–haptotaxis system with remodeling of non-diffusible attractant ($\eta > 0$ in (1.2)) is much less understood (Chaplain and Lolas [4], Pang and Wang [31], Tao and Winkler [44]). The main technical difficulty in their proof lies in the effects of the strong coupling in (1.2) on the spatial regularity of $u, v$ and $w$ when $\eta > 0$. When $\eta = 0$, one can build a one-sided pointwise estimate which connects $\Delta w$ to $v$ (see lemma 2.2 of [3] or (3.10) of [48]). Relying on such a pointwise estimate, one can derive two useful energy-type inequalities which can help us to bypass the term $\int_\Omega u^{p-1} \nabla \cdot (u \nabla w)$ (see lemma 3.2 of [58]). Using such information along with coupled estimate techniques and the boundedness of $\|\nabla v(\cdot, t)\|_{L^2(\Omega)}$, one can establish the estimates on $\int_\Omega u^p + \|\nabla v\|^q$ for any $p$ and $q > 1$ (see lemmas 3.3 and 3.4 of [58]), which combined with the standard regularity theory of parabolic equation and the Moser iteration procedure implies the boundedness of $u$ in $L^\infty(\Omega)$ (see lemma 3.5 of [58]). However, for the model (1.2) with $\eta > 0$, one needs to estimate the chemotaxis-related integral term $\int_\Omega a^p |\nabla v|^2$ (see (3.28) in [44]) or $\int_\Omega e^{-(r+1)(\tau+1)}a^p |\nabla v|^2$ (see (3.8) of [31]) with $a := w^{1-\mu}$, which requires much more technically demanding. In [31], assuming that $\mu > \eta \max\{\|u_0\|_{L^\infty(\Omega)}, 1\} + \mu^*(\chi^2, \xi)$ (the hypothesis can not be dropped (see the proof of lemma 3.2 of [31])), Pang and Wang [31] proved that the problem (1.2) admits a unique global solution $(u, v, w) \in (C^{2,1}(\Omega \times (0, \infty)))^3$. Moreover, $u$ is bounded in $\Omega \times (0, \infty)$. However, to the best of our knowledge, it is still an open problem to determine whether or not in the case $N = 2$ some unbounded solutions may exist in (1.2) with small $\mu > 0$. Indeed, as pointed by [1] (see also [37]), the hypothesis on $\mu > 0$ may yield the classical global solution. So, it is natural to ask whether the solution is globally exist when $\mu > 0$.

In this paper, we give a positive answer to this question.

Motivated by the aforementioned papers, the purpose of this work is to establish global solvability of (1.2). Our main result in this paper is the following.

**Theorem 1.1.** Let $\tau > 0, \chi > 0, \xi > 0$ and $\eta > 0$. Assume that $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with smooth boundary and the initial data $(u_0, v_0, w_0)$ is supposed to satisfy the following conditions
with some \( \vartheta \in (0,1) \). If \( \mu > 0 \), then there exists a triple \((u,v,w)\) in \( C^0(\bar{\Omega} \times [0,\infty)) \times C^{1,1}(\Omega \times (0,\infty))\) which solves (1.2) in the classical sense. Moreover, \( u \) and \( v \) are bounded in \( \Omega \times (0,\infty) \).

Remark 1.1.  

(i) If \( w \equiv 0 \) (the PDE system (1.2) is reduced to the chemotaxis-only system), it is not difficult to obtain that the solutions under the conditions of theorem 1.1 are uniformly bounded when \( N = 2 \), which coincides with the results of Osaki et al [30].

(ii) From theorem 1.1, we derive that solutions of model (1.2) are global and bounded for any \( \eta = 0, \mu > 0 \) and \( N \leq 2 \), which coincides with the result of Tao [37].

Without loss of generality, we may assume \( \tau = 1 \) in (1.2), since for \( \tau > 0 \) can be proved very similarly.

The plan of this paper is as follows. In section 2, we give some basic results and some preliminary lemmas as a preparation for the arguments in the later sections. In section 3, firstly, by using the technical lemmas (lemmas 3.2 and 3.3) and employing the variation-of-constants formula, we may establish the boundedness of \( \int_{\Omega} a p_0 \) (see lemma 3.4), where \( a = u e^{-\xi w} \) and \( p_0 > 1 \). In addition, we shall involve the variation-of-constants formula and \( L^p \)-estimate techniques to gain Finally, by using the Alikakos–Moser iteration, we establish the \( L^\infty(\Omega) \) bound of the function \( a \) (see the proof of theorem 1.1).

2. Preliminaries

Before formulating our main results, we first recall some preliminary lemmas used throughout this paper. Some basic properties of solution can be found in [18] (see also Winkler [50], Zhang and Li [53]).

Lemma 2.1 ([18]). For \( p \in (1,\infty) \), let \( A_p : = A_{p,p} \) denote the sectorial operator defined by

\[
A_p u := -\Delta u \quad \text{for all } u \in D(A_p) := \{ \varphi \in W^{2,p}(\Omega) | \frac{\partial \varphi}{\partial \nu} |_{\partial \Omega} = 0 \}.
\]

The operator \( A + 1 \) possesses fractional powers \( (A + 1)^\alpha (\alpha \geq 0) \), the domains of which have the embedding properties

\[
D((A + 1)^\alpha) \hookrightarrow W^{1,p}(\Omega) \text{ if } \alpha > \frac{1}{2}.
\]

If \( m \in \{0,1\} \), \( p \in [1,\infty) \) and \( q \in (1,\infty) \) with \( m - \frac{N}{p} < 2\alpha - \frac{N}{q} \), then we have

\[
\|u\|_{W^{1,p}(\Omega)} \leq C \| (A + 1)^\alpha u \|_{L^q(\Omega)} \quad \text{for all } u \in D((A + 1)^\alpha),
\]

where \( C \) is a positive constant. The fact that the spectrum of \( A \) is a p-independent countable set of positive real numbers \( 0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots \) entails the following consequences: for all \( 1 \leq p < q < \infty \) and \( u \in L^p(\Omega) \) the general \( L^p-L^q \) estimate
\[ \|(A + 1)^\alpha e^{-\Delta t} u\|_{L^p(\Omega)} \leq c t^{-\alpha - \frac{\gamma}{2}(1 - \frac{1}{p})} e^{(1 - \kappa)t}\|u\|_{L^p(\Omega)} \]

for any \( t > 0 \) and \( \alpha \geq 0 \) with some \( \kappa > 0 \).

In deriving some preliminary estimates for \( v \), we shall make use of the following property referred to as a variation of Maximal Sobolev regularity (see e.g. theorem 3.1 of [12] or [3]).

**Lemma 2.2.** Suppose \( \gamma \in (1, +\infty) \), \( g \in L^\gamma((0, T); L^\gamma(\Omega)) \). Let \( v \) be a solution of the following initial boundary value

\[
\begin{aligned}
\forall_t - \Delta v + v &= g, \quad (x, t) \in \Omega \times (0, T), \\
\frac{\partial v}{\partial \nu} &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
v(x, 0) &= v_0(x), \quad (x, t) \in \Omega.
\end{aligned}
\]

For each \( v_0 \in W^{2, \gamma}(\Omega) \) with \( \frac{\partial v_0}{\partial \nu} = 0 \) on \( \partial \Omega \), then there exists a unique solution \( v \in W^1(\Omega) \cap L^\gamma \) with \( \frac{\partial v}{\partial \nu} = 0 \). Furthermore, then there exists a positive constant \( C_\gamma \), such that

\[
\int_0^T e^{t\gamma} \|v(t, s)\|^2_{W^{2, \gamma}(\Omega)} ds \leq C_\gamma \left( \int_0^T e^{t\gamma}\|g(t, s)\|^2_{L^\gamma(\Omega)} ds + \|v_0\|^2_{W^{2, \gamma}(\Omega)} \right). 
\]

**Proof.** Set \( c(x, s) = e^{t\gamma} v(x, s) \). Then we derive that \( c \) satisfies

\[
\begin{aligned}
&c_t(x, s) - \Delta c(x, s) = f(x, s), \quad (x, s) \in \Omega \times (0, T), \\
&\frac{\partial c}{\partial \nu} = 0, \quad (x, s) \in \partial \Omega \times (0, T), \\
c(x, 0) = c_0(x), \quad (x, s) \in \Omega,
\end{aligned}
\]

where \( f(x, s) = e^{t\gamma} g(x, s) \). Applying the Maximal Sobolev regularity (see e.g. theorem 3.1 of [12]) to \( c \), we derive that

\[
\int_0^T \|c_t\|^2_{L^\gamma(\Omega)} ds + \int_0^T \|c\|^2_{L^\gamma(\Omega)} ds + \int_0^T \|c_s\|^2_{L^\gamma(\Omega)} ds 
\leq C_\gamma \left( \int_0^T \|f\|^2_{L^\gamma(\Omega)} ds + \|c_0\|^2_{L^\gamma(\Omega)} + \|\Delta c_0\|^2_{L^\gamma(\Omega)} \right).
\]

Substituting \( v \) into the above inequality and changing the variables imply that

\[
\int_0^T e^{t\gamma} (\|v(t, s)\|^2_{L^\gamma(\Omega)} + \|\Delta v(t, s)\|^2_{L^\gamma(\Omega)}) ds 
\leq C_\gamma \left( \int_0^T e^{t\gamma} \|g(t, s)\|^2_{L^\gamma(\Omega)} ds + \|c_0\|^2_{L^\gamma(\Omega)} + \|\Delta c_0\|^2_{L^\gamma(\Omega)} \right).
\]

On the other hand, by the elliptic \( L^p \)-estimate,

\[
\|v\|^2_{W^{2, \gamma}(\Omega)} \leq C_2 \gamma (\|\Delta v\|_{L^\gamma(\Omega)} + \|v\|_{L^\gamma(\Omega)}) \text{ for any } v \in W^{2, \gamma}(\Omega) \text{ with } \frac{\partial v}{\partial \nu} = 0.
\]

Consequently, combining (2.2) with (2.3), we can derive (2.1).
Lemma 2.3 ([23]). The Young inequality. Let \(1 < p, q < +\infty, \frac{1}{p} + \frac{1}{q} = 1\). Then for any positive constants \(a, \varepsilon\) and \(b\), we have
\[
ab \leq \varepsilon a^p + \frac{1}{q} (\varepsilon p)^{-\frac{1}{q}} b^q.
\]

In some parts of our subsequent analysis, we introduce the variable transformation (see Tao et al [39, 40, 44], Pang and Wang [31])
\[
a = u e^{-\xi w},
\]
on which (1.2) takes the form
\[
\begin{aligned}
a_t &= e^{-\xi w} \nabla \cdot (e^{\xi w} \nabla a) - \chi e^{-\xi w} \nabla \cdot (e^{\xi w} a \nabla v) + \xi a v w \\
\v_t &= \Delta v + a \nabla w - v, & x \in \Omega, t > 0, \\
w_t &= -v w + \eta w (1 - a e^{\xi w} w), & x \in \Omega, t > 0, \\
\frac{\partial a}{\partial \nu} &= \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
a(x, 0) &= a_0(x) = a_0(x) e^{-\xi w_0(x)}, v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega.
\end{aligned}
\]

The following lemma deals with local-in-time existence and uniqueness of a classical solution for the problem (1.2) (see [31]).
Lemma 2.4 ([31]). Assume that the nonnegative functions \(u_0, v_0\), and \(w_0\) satisfies (1.5) for some \(\vartheta \in (0, 1)\). Then there exists a maximal existence time \(T_{\text{max}} \in (0, \infty]\) and a triple of nonnegative functions
\[
\begin{aligned}
a \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
v \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
w \in C^{2,1}(\bar{\Omega} \times [0, T_{\text{max}}]),
\end{aligned}
\]
which solves (2.5) classically and satisfies
\[
0 \leq w \leq \rho := \max \{1, \|w_0\|_{L^\infty(\Omega)}\} \text{ in } \Omega \times (0, T_{\text{max}}).
\]
Moreover, if \(T_{\text{max}} < +\infty\), then
\[
\|a(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla w(\cdot, t)\|_{L^p(\Omega)} \rightarrow \infty \text{ as } t \nearrow T_{\text{max}}.
\]

3. Proof of the main result

In this section, we are going to establish an iteration step to develop the main ingredient of our result. Firstly, based on the ideas of lemma 3.1 in [31] (see also lemma 2.1 of [49]), we can derive the following properties of solutions of (1.2).
Lemma 3.1. Under the assumptions in theorem 1.1, we derive that there exists a positive constant \(C\) such that the solution of (1.2) satisfies
\[
\int_\Omega u(x, t) + \int_\Omega \nu^2(x, t) + \int_\Omega |\nabla v(x, t)|^2 \leq C \text{ for all } t \in (0, T_{\text{max}}).
\]
Lemma 3.2. Let
\[ A_1 = \frac{1}{\delta + 1} \left( \frac{\delta + 1}{\delta} \right)^{-\delta} \left[ \frac{\delta(\delta - 1)}{2} \right]^{2\delta+1} C_7 C_{\delta+1} \]
and \( H(y) = y + A_1 y^{-\delta} \) for \( y > 0 \). For any fixed \( \delta \geq 1, C_7, \chi, C_{\delta+1} > 0 \), then
\[ \min_{y > 0} H(y) = \frac{\delta(\delta - 1)}{2} \left( C_7 C_{\delta+1} \right)^{1/\delta}. \]

Proof. It is easy to verify that
\[ H'(y) = 1 - A_1 \delta y^{-\delta-1}. \]
Let \( H'(y) = 0 \), we have
\[ y = (A_1 \delta)^{1/\delta}. \]
On the other hand, by \( \lim_{y \to 0^+} H(y) = +\infty \) and \( \lim_{y \to +\infty} H(y) = +\infty \), we have
\[ \min_{y > 0} H(y) = H\left( (A_1 \delta)^{1/\delta} \right) = \frac{\delta(\delta - 1)}{2} \left( C_7 C_{\delta+1} \right)^{1/\delta}, \]
whereby the proof is completed.

Lemma 3.3. Let \( h(p) := \frac{\mu^2}{2} - \frac{\mu(1-1/p)}{2}(C_7 C_{p+1})^{1/p} - (p - 1)\xi\eta \rho \), where \( p \geq 1, \xi, \chi, \eta, \rho, \mu, C_7 \) and \( C_{p+1} \) are positive constants. Then there exists a positive constant \( p_0 > 1 \) such that
\[ h(p_0) > 0. \]

Proof. Since \( h(1) = \frac{\xi}{\rho} > 0 \), from the continuity of \( h \) it follows that for each \( \mu > 0 \), there is some \( p_0 > 1 \) such that (3.1) holds.

According to the local existence results of section 2 (see lemma 2.4), for any fixed \( s \in (0, T_{\text{max}}) \), it yields \( (u(\cdot, s), \bar{v}(\cdot, s), w(\cdot, s)) \in (C^2(\Omega))^3 \). Therefore, without loss of generality, we can assume that there exists a constant \( \beta > 0 \) such that
\[ \|u_0\|_{C^2(\Omega)} \leq \beta, \|v_0\|_{C^2(\Omega)} \leq \beta \text{ and } \|w_0\|_{C^2(\Omega)} \leq \beta. \]

Lemma 3.4. Let \( \mu, \chi, \eta \) and \( \xi \) be the positive constants. Assume that \( (u, \bar{v}, w) \) is a solution of (2.5) on \( (0, T_{\text{max}}) \). Then there exists a positive constant \( C = C(p_0, |\Omega|, \mu, \chi, \xi, \eta, \beta) \) such that
\[ \int_{\Omega} a^{\mu}(x, t) dx \leq C \text{ for all } t \in (0, T_{\text{max}}), \]
where \( p_0 > 1 \) is the same as in lemma 3.3.
Proof. By using (2.5) and integrations by parts, it yields
\[
\begin{align*}
\frac{d}{dt} \int_{\Omega} e^{\xi w} a^{p_0} + (p_0 + 1) \int_{\Omega} e^{\xi w} a^{p_0} \\
= & \xi \int_{\Omega} e^{\xi w} a^{p_0} \cdot \{ -w + \eta w (1 - a e^{\xi w} - w) \} \\
& + p_0 \int_{\Omega} e^{\xi w} a^{p_0 - 1} \cdot \{ e^{-\xi w} \nabla (e^{\xi w} \nabla a) - \chi e^{-\xi w} \nabla (e^{\xi w} a \nabla v) \} \\
& + a \xi w + a (\mu - \xi \eta w) (1 - a e^{\xi w} - w) + (p_0 + 1) \int_{\Omega} e^{\xi w} a^{p_0} \\
= & -p_0 (p_0 - 1) \int_{\Omega} e^{\xi w} a^{p_0 - 2} |\nabla a|^2 + p_0 (p_0 - 1) \chi \int_{\Omega} e^{\xi w} a^{p_0 - 1} \nabla a \cdot \nabla v \\
& + (p_0 - 1) \xi \int_{\Omega} e^{\xi w} a^{p_0 - 1} w + \int_{\Omega} e^{\xi w} a^{p_0} \{ (p_0 + 1) + (p_0 - 1) \xi \eta w (w - 1) + p_0 \mu (1 - w) \} \\
& + \int_{\Omega} e^{2 \xi w} a^{p_0 + 1} [(p_0 - 1) \xi \eta w - p_0 \mu] \\
:= & J_1 + J_2 + J_3 + J_4 + J_5 \quad \text{for all } t \in (0, T_{\max}).
\end{align*}
\] (3.4)

Now, in light of (2.6), (2.7) and the Young inequality (lemma 2.3), we derive that
\[
\begin{align*}
J_3 \leq & \varepsilon_1 \int_{\Omega} e^{2 \xi w} a^{p_0 - 1} + \frac{1}{p_0 + 1} (\varepsilon_1 \times \frac{p_0 + 1}{p_0})^{-p_0} [(p_0 - 1) \xi]^{p_0 + 1} \int_{\Omega} e^{\xi w (1 - p_0)} a^{p_0 - 1} \\
\leq & \varepsilon_1 \int_{\Omega} e^{2 \xi w} a^{p_0 + 1} + \frac{1}{p_0 + 1} (\varepsilon_1 \times \frac{p_0 + 1}{p_0})^{-p_0} [(p_0 - 1) \xi]^{p_0 + 1} \int_{\Omega} a^{p_0 + 1} \quad \text{for all } t \in (0, T_{\max}),
\end{align*}
\] (3.5)

\[
\begin{align*}
J_4 \leq & (p_0 + 1) + (p_0 - 1) \xi \eta w^2 + p_0 \mu \int_{\Omega} e^{\xi w} a^{p_0} \\
\leq & (p_0 + 1) [1 + \xi \eta w^2 + \mu] \int_{\Omega} e^{\xi w} a^{p_0} \\
\leq & \varepsilon_2 \int_{\Omega} e^{2 \xi w} a^{p_0 + 1} \\
& + \frac{1}{p_0 + 1} (\varepsilon_2 \times \frac{p_0 + 1}{p_0})^{-p_0} [(p_0 + 1) \xi w + 1 + \xi \eta w^2 + \mu] a^{p_0 + 1} \quad \text{for all } t \in (0, T_{\max})
\end{align*}
\] (3.6)

as well as
\[
J_5 \leq \int_{\Omega} e^{2 \xi w} a^{p_0 + 1} [(p_0 - 1) \xi \eta w - p_0 \mu] \quad \text{for all } t \in (0, T_{\max})
\] (3.7)

and
\[
\begin{align*}
J_2 \leq & \frac{p_0 (p_0 - 1)}{2} \int_{\Omega} e^{\xi w} a^{p_0 - 2} |\nabla a|^2 + \frac{p_0 (p_0 - 1)}{2} \chi^2 \int_{\Omega} e^{\xi w} a^{p_0 - 1} |\nabla v|^2 \\
\leq & \frac{p_0 (p_0 - 1)}{2} \int_{\Omega} e^{\xi w} a^{p_0 - 2} |\nabla a|^2 + \lambda_0 \int_{\Omega} e^{2 \xi w} a^{p_0 + 1} \\
& + \frac{1}{p_0 + 1} \left( \lambda_0 \times \frac{p_0 + 1}{p_0} \right) a^{-p_0} [(p_0 - 1) \xi^2]^{p_0 + 1} \int_{\Omega} e^{(1 - p_0) \xi w} |\nabla v|^2 (p_0 + 1) \\
\leq & \frac{p_0 (p_0 - 1)}{2} \int_{\Omega} e^{\xi w} a^{p_0 - 2} |\nabla a|^2 + \lambda_0 \int_{\Omega} e^{2 \xi w} a^{p_0 + 1} \\
& + \frac{1}{p_0 + 1} \left( \lambda_0 \times \frac{p_0 + 1}{p_0} \right) a^{-p_0} [(p_0 - 1) \xi^2]^{p_0 + 1} \int_{\Omega} |\nabla v|^2 (p_0 + 1) \quad \text{for all } t \in (0, T_{\max})
\end{align*}
\] (3.8)
and any small positive constants $\varepsilon_1, \varepsilon_2$ and $\lambda_0$.

Inserting (3.5)–(3.8) into (3.4), we derive that
\[
\frac{d}{dt} \int_{\Omega} e^{\varepsilon u} a^{p_0} + (p_0 + 1) \int_{\Omega} e^{\varepsilon u} a^{p_0} + \int_{\Omega} e^{2\varepsilon u} a^{p_0+1} [p_0 \mu - \varepsilon_1 - \varepsilon_2 - \lambda_0 - (p_0 - 1) \varepsilon \eta p]
\leq \frac{1}{p_0 + 1} \left( \lambda_0 \times \frac{p_0 + 1}{p_0} \right)^{-p_0} \left[ \frac{p_0(p_0 - 1)}{2} \right] \chi^{p_0+1} \int_{\Omega} |\nabla u|^2 (p_0+1)
\]
\[+ C_1(\varepsilon_1, \varepsilon_2) \text{ for all } t \in (0, T_{\max}).
\] (3.9)

where
\[
C_1(\varepsilon_1, \varepsilon_2) := \frac{1}{p_0 + 1} \left( \varepsilon_2 \times \frac{p_0 + 1}{p_0} \right)^{-p_0} (p_0 + 1)^{p_0+1} \left[ 1 + \varepsilon \eta p \right]^{p_0+1} \int_{\Omega} |\nabla u|^2 (p_0+1)
\]
\[+ \frac{1}{p_0 + 1} \left( \varepsilon_1 \times \frac{p_0 + 1}{p_0} \right)^{-p_0} [(p_0 - 1) \varepsilon]^{p_0+1}.
\] (3.10)

Next, from lemma 3.1, $N = 2$ and the Gagliardo–Nirenberg inequality, it follows that
\[
\|D(\cdot, t)\|_{L^p(\Omega)} \leq C_2 \text{ for all } t \in (0, T_{\max}).
\] (3.11)

This along with (3.10) follows
\[
C_1(\varepsilon_1, \varepsilon_2) \leq C_3(\varepsilon_1, \varepsilon_2)
\]
\[:= \frac{1}{p_0 + 1} \left( \varepsilon_2 \times \frac{p_0 + 1}{p_0} \right)^{-p_0} (p_0 + 1)^{p_0+1} \left[ 1 + \varepsilon \eta p \right]^{p_0+1} \int_{\Omega} |\nabla u|^2 (p_0+1)
\]
\[+ C_2 \frac{1}{p_0 + 1} \left( \varepsilon_1 \times \frac{p_0 + 1}{p_0} \right)^{-p_0} [(p_0 - 1) \varepsilon]^{p_0+1}.
\]

From this and (3.9) we also obtain
\[
\frac{d}{dt} \int_{\Omega} e^{\varepsilon u} a^{p_0} + (p_0 + 1) \int_{\Omega} e^{\varepsilon u} a^{p_0} + \int_{\Omega} e^{2\varepsilon u} a^{p_0+1} [p_0 \mu - \varepsilon_1 - \varepsilon_2 - \lambda_0 - (p_0 - 1) \varepsilon \eta p]
\leq \frac{1}{p_0 + 1} \left( \lambda_0 \times \frac{p_0 + 1}{p_0} \right)^{-p_0} \left[ \frac{p_0(p_0 - 1)}{2} \right] \chi^{p_0+1} \int_{\Omega} |\nabla u|^2 (p_0+1)
\]
\[+ C_3(\varepsilon_1, \varepsilon_2) \text{ for all } t \in (0, T_{\max}).
\]

Then for any $t \in (0, T_{\max})$, by means of the variation-of-constants representation for the above inequality, we can estimate
\[
\int_{\Omega} e^{\varepsilon u} a^{p_0}(\cdot, t) + [p_0 \mu - \varepsilon_1 - \varepsilon_2 - \lambda_0 - (p_0 - 1) \varepsilon \eta p] \int_{0}^{t} \int_{\Omega} e^{-(p_0-1)(t-s)} e^{2\varepsilon u} a^{p_0+1}
\leq \int_{\Omega} u_0^p + \frac{1}{p_0 + 1} \left( \lambda_0 \times \frac{p_0 + 1}{p_0} \right)^{-p_0} \left[ \frac{p_0(p_0 - 1)}{2} \right] \chi^{p_0+1} \int_{\Omega} |\nabla u|^2 (p_0+1)
\]
\[+ C_3(\varepsilon_1, \varepsilon_2) \text{ for all } t \in (0, T_{\max}).
\] (3.12)

Next, according to the Gagliardo–Nirenberg inequality, (3.11) and lemma 3.1, we can choose $C_4$ and $C_5$ such that
\[
\| \nabla v(\cdot, s) \|_{L^2(\Omega)}^{2(p_0+1)} \leq C_4 \| v(\cdot, s) \|_{W^{2, p_0+1}(\Omega)}^{p_0+1} \| \nabla v(\cdot, s) \|_{L^2(\Omega)}^{p_0+1} \\
\leq C_5 \| v(\cdot, s) \|_{W^{2, p_0+1}(\Omega)}^{p_0+1} \text{ for all } t \in (0, T_{\text{max}}).
\]

Therefore, with the help of (3.13), applying (2.1) of lemma 2.2 with \( \gamma = p_0 + 1 \), we obtain

\[
\frac{1}{p_0 + 1} (\lambda_0 \times \frac{p_0 + 1}{p_0}) \leq \int_0^t e^{-(p_0 - 1)(t - s)} \| \nabla v(\cdot, s) \|_{L^2(\Omega)}^p \leq C_4 \int_0^t e^{-(p_0 - 1)(t - s)} \| v(\cdot, s) \|_{W^{2, p_0+1}(\Omega)}^{p_0+1} + C_6
\]

\[
\leq \frac{1}{p_0 + 1} (\lambda_0 \times \frac{p_0 + 1}{p_0}) \int_0^t \int_\Omega e^{-(p_0 - 1)(t - s)} C_5 \| v(\cdot, s) \|_{W^{2, p_0+1}(\Omega)}^{p_0+1} + C_6
\]

\[
\leq \frac{1}{p_0 + 1} (\lambda_0 \times \frac{p_0 + 1}{p_0}) \int_0^t \int_\Omega e^{-(p_0 - 1)(t - s)} C_6 \| v(\cdot, s) \|_{W^{2, p_0+1}(\Omega)}^{p_0+1} + C_6
\]

(3.14)

for all \( t \in (0, T_{\text{max}}) \), where

\[
C_6 := \frac{1}{p_0 + 1} (\lambda_0 \times \frac{p_0 + 1}{p_0}) \int_\Omega e^{-(p_0 - 1)(t - s)} C_5 \| v(\cdot, s) \|_{W^{2, p_0+1}(\Omega)}^{p_0+1} \text{ and } C_7 := C_5 e^{(p_0 - 1)}.
\]

Substituting (3.14) into (3.12), we derive

\[
\int_\Omega e^{\varepsilon_1 \varepsilon_2 (p_0 + 1)} \int_0^t e^{-(p_0 - 1)(t - s)} C_5 \| v(\cdot, s) \|_{W^{2, p_0+1}(\Omega)}^{p_0+1} \text{ for all } t \in (0, T_{\text{max}}).
\]

where

\[
C_8(\varepsilon_1, \varepsilon_2) := C_3 \varepsilon_1 = C_7 + C_6.
\]

Choosing \( \lambda_0 = (A_1 p_0)^{\frac{1}{p_0 + 1}} \) in (3.15) and using lemma 3.2, we derive

\[
\int_\Omega e^{\varepsilon_1 \varepsilon_2 (p_0 + 1)} \int_0^t e^{-(p_0 - 1)(t - s)} C_5 \| v(\cdot, s) \|_{W^{2, p_0+1}(\Omega)}^{p_0+1} \text{ for all } t \in (0, T_{\text{max}}).
\]

Now, for the above positive constants \( \mu, \chi, \xi \) and \( \eta \), due to lemma 3.3, it has

\[
p_0 \mu - \frac{p_0(p_0 - 1)}{2} \chi^2 C_5 \| v(\cdot, s) \|_{W^{2, p_0+1}(\Omega)}^{p_0+1} > (p_0 - 1) \xi \eta > \frac{p_0 \mu}{2} > 0,
\]

thus, we can choose \( \varepsilon_1 \) and \( \varepsilon_2 \) appropriately small (e.g. \( \varepsilon_1 = \varepsilon_2 = \frac{p_0 \mu}{2} \)) such that
0 < \varepsilon_1 + \varepsilon_2 < p_0 \mu - \frac{p_0(p_0 - 1)\chi^2}{2}(C_7 C_p + 1) \frac{1}{p_0} - (p_0 - 1)\eta \rho. \quad (3.18)

Collecting (3.16) and (3.18), we derive that there exists a positive constant \(C_9\) such that

\[
\int_{\Omega} u^{p_0}(x,t) \, dx \leq C_9 \text{ for all } t \in (0, T_{\text{max}}).
\]

The proof of lemma 3.4 is completed. \(\Box\)

**Lemma 3.5.** Assume the hypothesis of lemma 3.4 holds. Then for all \(p > 1\), there exists a positive constant \(C = C(p, |\Omega|, \mu, \chi, \xi, \eta, \beta)\) such that

\[
\int_{\Omega} a^p(x,t) \, dx \leq C \text{ for all } t \in (0, T_{\text{max}}).
\]

**Proof.** Firstly from lemma 3.4 (see (3.3)) and (2.4), there exists a positive constant \(C_1\) such that

\[
\int_{\Omega} u^{p_0}(x,t) \, dx \leq C_1 \text{ for all } t \in (0, T_{\text{max}}), \quad (3.19)
\]

where \(p_0 > 1\) is the same as that in lemma 3.3. Next, we fix \(q < 2 p_0 \left(2 - p_0 \right) + 1\) and choose some \(\alpha > \frac{1}{2}\) such that

\[
q < \frac{1}{p_0} - \frac{1}{2} + \frac{\alpha}{2} \left(\alpha - \frac{1}{2}\right) \leq \frac{2 p_0}{(2 - p_0)^+}. \quad (3.20)
\]

Now, involving the variation-of-constants formula for \(v\), we have

\[
v(t) = e^{-(A+1)\sigma_0} + \int_0^t e^{-(t-s)(A+1)} u(s) \, ds, \ t \in (0, T_{\text{max}}). \quad (3.21)
\]

Hence, it follows from (3.2), (3.19)–(3.21) that

\[
\begin{align*}
&\| (A + 1)^{\alpha} v(t) \|_{L^q(\Omega)} \\
&\leq c \int_0^t (t-s)^{-\alpha - \frac{1}{2} \left(\alpha - \frac{1}{2}\right)} e^{-\kappa t} \| u(s) \|_{L^q(\Omega)} \, ds + ce^{-\kappa t} t^{-\alpha + \frac{1}{2}} \| u \|_{L^\infty(\Omega)} + C_2 \int_0^{t+\infty} \sigma^{-\alpha - \frac{1}{2} \left(\alpha - \frac{1}{2}\right)} e^{-\kappa \sigma} \, d\sigma + C_3 t^{-\alpha + \frac{1}{2}} \text{ for all } t \in (0, T_{\text{max}}),
\end{align*}
\]

where \(c > 0\) is given by lemma 2.1. Hence, in light of lemmas 2.1 and 2.4, due to (3.20) and (3.22), we have

\[
\int_{\Omega} |\nabla v(t)|^q \leq C_4 \text{ for all } t \in (0, T_{\text{max}}) \text{ and } q \in \left[1, \frac{2 p_0}{(2 - p_0)^+}\right] \quad (3.23)
\]

with some positive constant \(C_4\). Now, due to the Sobolev imbedding theorems and \(N = 2\), we conclude that

\[
\| v(\cdot, t) \|_{L^\infty(\Omega)} \leq C_5 \text{ for all } t \in (0, T_{\text{max}}). \quad (3.24)
\]
Applying the Young inequality, one obtains from \((2.6), (2.5)\) and \((3.24)\) that for any \(p > \max\{2, p_0 - 1\}\)
\[
\frac{d}{dt} \int_{\Omega} e^{\xi_0 a^p} + p(p - 1) \int_{\Omega} e^{\xi_0 a^{p-2}} |\nabla a|^2 + p\mu \int_{\Omega} e^{\xi_0 a^{p+1}}
\]
\[
= p(p - 1) \chi \int_{\Omega} e^{\xi_0 a^{p-1}} \nabla a \cdot \nabla v + (p - 1) \xi \int_{\Omega} e^{\xi_0 a^{p+1}}
\]
\[
+ \int_{\Omega} e^{\xi_0 a^p} \{(p + 1) + (p - 1) \xi w(w - 1) + p\mu(1 - w)\}
\]
\[
+ \int_{\Omega} e^{2\xi_0 a^{p+1}}(p - 1) \xi w
\]
\[
\leq \frac{p(p - 1)}{2} \int_{\Omega} e^{\xi_0 a^{p-2}} |\nabla a|^2 + \frac{p(p - 1)}{2} \chi^2 \int_{\Omega} e^{\xi_0 a^p} |\nabla v|^2 + (p - 1) \xi \int_{\Omega} e^{\xi_0 a^{p+1}}
\]
\[
+ \int_{\Omega} e^{\xi_0 a^p} \{(p + 1) + (p - 1) \xi w(w - 1) + p\mu(1 - w)\}
\]
\[
+ \int_{\Omega} e^{2\xi_0 a^{p+1}}(p - 1) \xi w
\]
\[
\leq \frac{p(p - 1)}{2} \int_{\Omega} e^{\xi_0 a^{p-2}} |\nabla a|^2 + \frac{p(p - 1)}{2} \chi^2 \int_{\Omega} e^{\xi_0 a^p} |\nabla v|^2 + C_6 \int_{\Omega} a^{p+1}
\]
\[
\leq \frac{p(p - 1)}{2} \int_{\Omega} e^{\xi_0 a^{p-2}} |\nabla a|^2 + \frac{p(p - 1)}{2} \chi^2 e^{\xi_0} \int_{\Omega} a^{p} |\nabla v|^2 + C_6 \int_{\Omega} a^{p+1} \text{ for all } t \in (0, T_{\text{max}}).
\]

Next, with the help of the Gagliardo–Nirenberg inequality (see e.g. [56]), it yields that
\[
C_6 \int_{\Omega} a^{p+1} = C_6 \|a^p\|^{\frac{2(p+1)}{p+1}}_{L^{(\frac{p+1}{p})}(\Omega)}
\]
\[
\leq C_7 (\|\nabla a^2\|_{L^2(\Omega)} \|a^2\|^{\frac{1}{2}}_{L^{\frac{p}{p_0}}(\Omega)} + \|a^2\|^{\frac{2(q_1)}{q_1}}_{L^{\frac{q_1}{q_1}}(\Omega)})^{\frac{2(p+1)}{p+1}}
\]
\[
\leq C_8 (\|\nabla a^2\|_{L^2(\Omega)} + 1)
\]
\[
= C_8 (\|\nabla a^2\|_{L^2(\Omega)} + 1)
\]

with some positive constants \(C_7, C_8\) and
\[
\mu_1 = \frac{p}{p_0} - \frac{p}{p+1} = \frac{p + 1 - p_0}{p} \in (0, 1).
\]

Since, \(p_0 > 1\) yields \(p_0 < \frac{2p_0}{\sigma(2-p_0)}\), in light of the Hölder inequality and \((3.23)\), we derive
\[
\frac{\chi^2 p(p - 1)}{2} e^{\xi_0} \int_{\Omega} a^p |\nabla v|^2 \leq \chi^2 p(p - 1) e^{\xi_0} \left( \int_{\Omega} a^{p+1} \right)^{\frac{1}{p+1}} \left( \int_{\Omega} |\nabla v|^{2p_0} \right)^{\frac{1}{p_0}}
\]
\[
\leq C_9 \|a^2\|^2_{L^{2\sigma p_0}(\Omega)},
\]

where \(C_9\) is a positive constant. Since \(p_0 > 1\) and \(p > p_0 - 1\), we have
\[
\frac{p_0}{p} \leq \frac{p_0}{p_0 - 1} < +\infty,
\]

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which together with the Gagliardo–Nirenberg inequality (see e.g. [56]) implies that
\[
C_9 \|a^2\|_{L^{\frac{m}{m-1}}(\Omega)} \leq C_{10}(\|\nabla a^2\|_{L^2(\Omega)}^m + \|a^2\|_{L^{2\mu}(\Omega)}^2)^2 
\leq C_{11}(\|\nabla a^2\|_{L^2(\Omega)}^{2\mu} + 1) 
= C_{11}(\|\nabla a^2\|_{L^2(\Omega)}^{2(\frac{p}{p-\mu_2} - 1)} + 1)
\]
with some positive constants $C_{10}$, $C_{11}$ and
\[
\mu_2 = \frac{p}{p_0} - \frac{p}{p_0 - 1}p \in (0, 1).
\]

Moreover, an application of the Young inequality shows that
\[
C_6 \int_\Omega a^{p+1} + \frac{\lambda^2 p(p-1)}{2} e^{\xi_0} \int_\Omega a^p \|\nabla a\|^2 \leq \frac{p(p-1)}{4} \int_\Omega a^{p-2} |\nabla a|^2 + C_{12} 
\leq \frac{p(p-1)}{4} \int_\Omega e^{\xi_0} a^{p-2} |\nabla a|^2 + C_{12,12}.
\]
Inserting (3.26) into (3.25), we conclude that
\[
\int_\Omega e^{\xi_0} a^p + \frac{p(p-1)}{4} \int_\Omega e^{\xi_0} a^{p-2} |\nabla a|^2 + \frac{p(p-1)}{4} \int_\Omega e^{\xi_0} a^{p+1} \leq C_{13}.
\]
Therefore, integrating the above inequality with respect to $t$ yields
\[
\|a(\cdot,t)\|_{L^p(\Omega)} \leq C_{14} \text{ for all } p \geq 1 \text{ and } t \in (0, T_{\text{max}})
\]
for some positive constant $C_{14}$. \qed

\textbf{Remark 3.1.} \ Since, in this paper, we only assume that $\mu > 0$ which is different from that in [31] (see the hypothesis of lemma 3.2 to [31]), firstly by using the technical lemma (see lemma 3.2), we could conclude the boundedness of $\int_\Omega a^p$ (for some $q_0 > 1$), then in light of the variation-of-constants formula and $L^q$-$L^p$ estimates for the heat semigroup, we may finally derive the boundedness of $\int_\Omega a^p$ (for any $p > 1$).

Our main result on global existence and boundedness thereby becomes a straightforward consequence of lemmas 2.4 and 3.5.

\textbf{The proof of theorem 1.1.}

\textbf{Proof.} \ The proof of theorem 1.1 consists of the following steps.

\textit{Step 1.} $\|a(\cdot,t)\|_{L^\infty(\Omega)}$: firstly, in light of (2.6), due to lemma 3.5, we derive that there exist positive constants $p_0 > 2$ and $C_1$ such that
\[
\|u(\cdot,t)\|_{L^N(\Omega)} \leq C_1 \text{ for all } t \in (0, T_{\text{max}}).
\]
Next, since $p_0 > 2$ and $N = 2$ yield to $+\infty = \frac{Np_0}{(N-p_0)p}$, therefore, by using lemma 2.1 (see also lemma 2.1 of [19]), we conclude that...
\[ \| \nabla v(t) \|_{L^\infty(\Omega)} \leq C_2 \text{ for all } t \in (0, T_{\text{max}}). \] (3.28)

Applying the Young inequality, in light of (2.6) and the first equation of (2.5), one obtains from (3.28) that for any \( \rho \geq 4 \)

\[ \frac{d}{dt} \int_\Omega e^{\xi \psi} a_p + p(p-1) \int_\Omega e^{\xi \psi} a^{p-2} |\nabla a|^2 + \int_\Omega e^{\xi \psi} a_p = \xi \int_\Omega e^{\xi \psi} a_p \cdot \{ -\psi w + \eta \psi(1 - \alpha e^{\xi \psi} - w) \} \\
+ p \int_\Omega e^{\xi \psi} a^{p-1} \cdot \{ e^{-\xi \psi} \nabla \cdot (e^{\xi \psi} \nabla a) - \chi e^{-\xi \psi} \nabla \cdot (e^{\xi \psi} a \nabla v) \} \\
+ a \xi \psi w + a(\mu - \xi \psi)(1 - \alpha e^{\xi \psi} - w) \} + p \int_\Omega e^{\xi \psi} a_p \]

\[ \leq \frac{p(p-1)}{4} \int_\Omega e^{\xi \psi} a^{p-2} |\nabla a|^2 + p(p-1) \chi^2 C_3 \int_\Omega e^{\xi \psi} a_p \\
+ (p-1) \xi \int_\Omega e^{\xi \psi} a^{p} \psi w + \int_\Omega e^{\xi \psi} a_p \{ \rho(p+1) + (p-1) \xi \psi(w-1) + p(1-w) \} \\
+ \int_\Omega e^{2 \xi \psi} a^{p+1} \{ (p-1) \xi \psi - p \rho \} \]

\[ \leq \frac{p(p-1)}{4} \int_\Omega e^{\xi \psi} a^{p-2} |\nabla a|^2 + C_4 \rho^2 (\int_\Omega a^{p+1} + 1) \text{ for all } t \in (0, T_{\text{max}}). \] (3.29)

where \( C_3 > 0 \) and \( C_4 > 0 \) are independent of \( p \). Here and throughout the proof of theorem 1.1, we shall denote by \( C_i (i \in \mathbb{N}) \) the several positive constants independent of \( p \). Therefore, (3.29) implies that

\[ \frac{d}{dt} \int_\Omega e^{\xi \psi} a_p + C_5 \int_\Omega |\nabla a_p|^2 + \int_\Omega e^{\xi \psi} a_p \leq C_4 \rho^2 (\int_\Omega a^{p+1} + 1) \text{ for all } t \in (0, T_{\text{max}}). \] (3.30)

Next, once more by means of the Gagliardo–Nirenberg inequality, we can estimate

\[ C_4 \rho^2 \int_\Omega a^{p+1} = C_4 \rho^2 \| a_p^2 \|_{L^{\frac{2(p+1)}{p}}(\Omega)} \\
\leq C_4 \rho^2 (\| \nabla a_p^2 \|_{L^{\frac{2(p+1)}{p}}(\Omega)} + \| a_p^2 \|_{L^{\frac{2(p+1)}{p}}(\Omega)}) \\
= C_4 \rho^2 (\| \nabla a_p^2 \|_{L^{\frac{2(p+1)}{p}}(\Omega)} + \| a_p^2 \|_{L^{\frac{2(p+1)}{p}}(\Omega)}) \\
\leq C_5 \| \nabla a_p^2 \|_{L^{\frac{2(p+1)}{p}}(\Omega)} + C_7 \rho^2 \| a_p^2 \|_{L^{\frac{2(p+1)}{p}}(\Omega)} + C_9 \rho^2 \| a_p^2 \|_{L^{\frac{2(p+1)}{p}}(\Omega)} \\
\leq C_5 \| \nabla a_p^2 \|_{L^{\frac{2(p+1)}{p}}(\Omega)} + C_7 \rho^2 \| a_p^2 \|_{L^{\frac{2(p+1)}{p}}(\Omega)}, \] (3.31)

where

\[ 0 < \gamma_1 = \frac{2}{1 - \frac{2}{2} + \frac{2}{p+1}} = \frac{p+2}{2(p+1)} < 1. \]
Here we have use the fact that \( \frac{d\nu}{d\tau} \geq 2 \). Therefore, inserting (3.31) into (3.30), we derive that
\[
\frac{d}{dt} \int_{\Omega} e^{\xi_2 a^2} + \int_{\Omega} e^{\xi_2 a^2} \leq C_4 p^{\frac{\nu}{2}} \|a^2\|_{L^{\infty}(\Omega)} + C_4 p^2
\]
\[
\leq C_4 p^{\frac{\nu}{2}} \left( \max\{1, \|a^2\|_{L^{\infty}(\Omega)}\} \right)^{\frac{\nu}{4}}.
\]
(3.32)

Now, choosing \( p_i = 2^{i+2} \) and letting \( M_i = \max\{1, \sup_{t \in (0, T)} \int_{\Omega} a^2 \} \) for \( T \in (0, T_{\text{max}}) \) and \( i = 1, 2, 3, \ldots \). We then obtain from (3.32) that
\[
\frac{d}{dt} \int_{\Omega} e^{\xi_2 a^2} + \int_{\Omega} e^{\xi_2 a^2} \leq C_4 p_i^{\frac{\nu}{2}} M_i^{\frac{\nu}{2}} (T),
\]
which together with the comparison argument entails that there exists \( \lambda > 1 \) independent of \( i \) such that
\[
M_i(T) \leq \max\{\lambda^2 M_{i-1}^\nu (T), e^{\xi_2 |\Omega|} \|a_0\|_{L^{\infty}(\Omega)} \}.
\]
(3.33)

Here we use the fact that \( \nu_i := \frac{20}{p_i - 2} \leq 4 \). Now, if \( \lambda^2 M_{i-1}^\nu (T) \leq e^{\xi_2 |\Omega|} \|a_0\|_{L^{\infty}(\Omega)} \) for infinitely many \( i \geq 1 \), we get
\[
\left( \sup_{t \in (0, T)} \int_{\Omega} a^p (\cdot, t) \right)^{\frac{1}{p-1}} \leq \left( \frac{e^{\xi_2 |\Omega|} \|a_0\|_{L^{\infty}(\Omega)}}{\lambda^2} \right)^{\frac{1}{p-1}}
\]
for such \( i \), which entails that
\[
\sup_{t \in (0, T)} \|a(\cdot, t)\|_{L^{\infty}(\Omega)} \leq \|a_0\|_{L^{\infty}(\Omega)}.
\]
(3.34)

Otherwise, if \( \lambda^2 M_{i-1}^\nu (T) > e^{\xi_2 |\Omega|} \|a_0\|_{L^{\infty}(\Omega)} \) for all sufficiently large \( i \), then by (3.33), we derive that
\[
M_i(T) \leq \lambda^2 M_{i-1}^\nu (T) \quad \text{for all sufficiently large} \quad i,
\]
and thus (3.35) is still valid for all \( i \geq 1 \) upon enlarging \( \lambda \) if necessary. That is,
\[
M_i(T) \leq \lambda^2 M_{i-1}^\nu (T) \quad \text{for all} \quad i \geq 1.
\]

Therefore, based on a straightforward induction (see e.g. lemma 3.12 of [44]) we have
\[
M_i(T) \leq \lambda^{i + \sum_{k=1}^{i} (j-1-|\kappa_k|)} M_{i-1}^{|\kappa_k|} M_0^{|\kappa_k|}
\]
(3.36)

for all \( i \geq 1 \), where \( \kappa_k = 2(1 + \varepsilon_k) \) satisfies \( \varepsilon_k = \frac{1}{k+2} \leq \frac{2^k}{2^k} \) for all \( k \geq 1 \) with some \( C_{11} > 0 \). Therefore, due to the fact that \( \ln(1 + x) \leq x(x \geq 0) \), we derive
\[
\Pi_i^{\nu_k \kappa_k} = 2^{i+1-|j|} C_{12}^{i+1} \ln(1 + \varepsilon_k)
\]
\[
\leq 2^{i+1-|j|} C_{12}^{|j|} \varepsilon_k
\]
\[
\leq 2^{i+1-|j|} C_{12}^{|j|} \quad \text{for all} \quad i \geq 1 \quad \text{and} \quad j \in \{1, \ldots, i\},
\]
which implies that
\[
\sum_{j=2}^{i} (j-1) \cdot \prod_{k=j}^{i} \frac{\kappa_k}{2^{i+2}} \leq \sum_{j=2}^{i} (j-1) \frac{2^{i+1-j}}{e^{C_{11}}} \\
\leq \frac{e^{C_{11}}}{2} \sum_{j=2}^{i} \frac{(j-1)}{2^j} \\
\leq \frac{e^{C_{11}}}{2} \left( \frac{1}{2} + \frac{1}{2^2} \right) \\
= \frac{3e^{C_{11}}}{8}.
\]

By the definition of \( p_i \), we easily deduce from (3.36) that
\[
M_{\pi_i}^h (T) \leq \lambda \sum_{j=2}^{i} \left( \frac{\gamma_{j-1}^{i-1} n_{j-1}^{i-1}}{2^{i+2}} \right) \frac{\gamma_j^{i+1}}{M_0^{\frac{e^{C_{11}}}{2}}} \\
\leq \lambda \sum_{j=2}^{i} \frac{\gamma_{j-1}^{i-1} n_{j-1}^{i-1}}{2^{i+2}} \frac{\gamma_j^{i+1}}{M_0^{\frac{e^{C_{11}}}{2}}},
\]
which after taking \( i \to \infty \) and \( T \nearrow T_{\max} \) readily implies that
\[
\| a(\cdot, t) \|_{L^\infty(\Omega)} \leq \lambda \frac{e^{C_{11}}}{M_0^{\frac{e^{C_{11}}}{2}}} \text{ for all } t \in (0, T_{\max}).
\] (3.37)

**Step 2 :** \( \| \nabla w(\cdot, t) \|_{L^1(\Omega)} \): Employing almost exactly the same arguments as that in the proof of lemmas 3.5 and 3.6 in [31] (the minor necessary changes are left as an easy exercise to the reader), and taking advantage of (3.28) and (3.37), we conclude the estimate for any \( T < T_{\max} \).
\[
\| \nabla w(\cdot, t) \|_{L^1(\Omega)} \leq C \text{ for all } t \in (0, T).
\]

Now, with the above estimate in hand, using (3.34) and (3.37), employing the extendibility criterion provided by lemma 2.4, we may prove theorem 1.1.

**Remark 3.2.** If \( \mu > \xi \eta \max \{ \| u_0 \|_{L^\infty(\Omega)}, 1 \} + \mu^*(\chi^0, \xi) \) (see the proof of lemma 3.4 to [31]), one only need to estimate \( C_2 \int_{\Omega} a^p \) other than \( C_2 \left( \int_{\Omega} a^{p+1} + 1 \right) \), which is different from this paper.

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