Nonextensive models for earthquakes

R. Silva
Observatório Nacional, Rua Gal. José Cristino 77, 20921-400 Rio de Janeiro - RJ, Brasil and Universidade do Estado do Rio Grande do Norte, 59610-210, Mossoró, RN, Brasil

G. S. França and C. S. Vilar
Departamento de Física, Universidade Federal do Rio Grande do Norte, 59072-970, Natal, RN, Brasil

J. S. Alcaniz
Observatório Nacional, Rua Gal. José Cristino 77, 20921-400 Rio de Janeiro - RJ, Brasil (Dated: March 23, 2022)

We have revisited the fragment-asperity interaction model recently introduced by Sotolongo-Costa and Posadas (Physical Review Letters 92, 048501, 2004) by considering a different definition for mean values in the context of Tsallis nonextensive statistics and introducing a new scale between the earthquake energy and the size of fragment $\epsilon \propto r^3$. The energy distribution function (EDF) deduced in our approach is considerably different from the one obtained in the above reference. We have also tested the viability of this new EDF with data from two different catalogs (in three different areas), namely, NEIC and Bulletin Seismic of the Revista Brasileira de Geofísica. Although both approaches provide very similar values for the nonextensive parameter $q$, other physical quantities, e.g., the energy density differs considerably, by several orders of magnitude.

PACS numbers: 89.75.Da; 91.30.Bi; 91.30.Dk

I. INTRODUCTION

Over the last two decades, a great deal of attention has been paid to the so-called nonextensive Tsallis entropy, both from theoretical and observational viewpoints. This particular nonextensive formulation seems to present a consistent theoretical tool to investigate complex systems in their nonequilibrium stationary states, systems with multifractal and self-similar structures, systems dominated by long-range interactions, anomalous diffusion phenomena, among others. Some recent applications of Tsallis entropy $S_q \neq 1$ to a number of complex scenarios is now providing a more definite picture of the kind of physical problems to which this $q$-formalism can in fact be applied.

In this regard, systems of interest in geophysics has also been studied in light of this nonextensive formalism. In this particular context, the very first investigation was done by Abe who showed that the statistical properties of three-dimensional distance between successive earthquakes follow a $q$-exponential function with the nonextensive parameter lying in the interval $[0, 1]$. Since then, other geophysical analyses have been performed as, for instance, the statistics of the calm time, which indicates a scale-free nature for earthquake phenomena and corresponds to a $q$-exponential distribution with $q > 1$, and models for temperature distributions and radon emission of volcanos. More recently, a very interesting model for earthquakes dynamics related to Tsallis nonentensive framework has been proposed by Sotolongo-Costa and Posadas (SCP Model). Such a model consists basically of two rough profiles interacting via fragments filling the gap between them, where the fragments are produced by local breakage of the local plates. By using the nonextensive formalism the authors of Ref. not only showed the influence of the size distribution of fragments on the energy distribution of earthquakes but also deduced a new energy distribution function (EDF), which gives the well-known Gutenberg-Richter law as a particular case.

However, in dealing with this nonextensive framework, a particular attention must be paid to the possible definitions for mean values, which play a fundamental role within the domain of this nonextensive statistics. In this concern, recent studies of the properties of the relative entropy and the Shore-Johnson theorem for consistent minimum cross-entropy principle, revealed the necessity of the so-called $q$-expectation value in studies involving this nonextensive statistical mechanics (see for details). Thus, by introducing this $q$-definition of mean value we re-analyzed the fragment-asperity interaction model of Sotolongo-Costa and Posadas. Moreover, a new scale law between the released relative energy $\epsilon$ and the 3-dimensional size of fragments has also been introduced. By using the standard method of entropy maximization we also deduced a new energy distribution function, which differs considerably from the one obtained in Ref. In order to test the viability of our approach we used data taken from two seismic catalogs, namely, NEIC and Bulletin Seismic of the Revista Brasileira de Geofísica. It is shown that although both
approaches provide very similar values for the nonextensive parameter $q$, the other physical quantity, e.g., the energy density differences diver by several orders of magnitude.

This paper is organized as follows. In Sec. II, the standard formalism of nonextensive statistical mechanics is reexamined, as well as the theoretical basis of the SCP model. In Sec III, a new EDF is analytically calculated through extremization of Tsallis’ entropy under the constraints of the $q$-expectation value and normalization condition. In Sec IV, we test this new EDF with data from two different catalogs and estimate the best-fit values for the nonextensive parameter $q$ and the proportionality constant between the released relative energy $\varepsilon$ and the volume of the fragments $r^3$, i.e., the energy density, $a$. We end this paper by emphasizing the main results in the conclusion Section.

II. NON-EXTENSIVE FRAMEWORK AND SCP MODEL

In this Section, we recall the nonextensive theoretical basis of the SCP model. As widely known, the Tsallis’ statistics generalizes the Boltzmann-Gibbs statistics in what concerns the concept of entropy. Such formalism is based on the parametric class of entropies given by

$$S_{q \neq 1} = -k_B \int p^q(\sigma) \ln p(\sigma) d\sigma,$$

(1)

where $k_B$ is the Boltzmann constant. In the SCP model, $p(\sigma)$ stands for the probability of finding a fragment of relative surface $\sigma$ (which is defined as a characteristic surface of the system), $q$ is the nonextensive parameter and the $q$-logarithmic function above is defined by

$$\ln_q p = (1 - q)^{-1}(p^{1-q} - 1), \quad (p > 0)$$

(2)

which recovers the standard Boltzmann-Gibbs entropy $S_1 = -k_B \int p \ln p d^3p$ in the limit $q \to 1$. It is worth mentioning that most of the experimental evidence supporting Tsallis proposal are related to the power-law distribution associated with $S_{q \neq 1}$ description of the classical $N$-body problem.

The SCP model is a simple approach for earthquakes dynamics revealing a very interesting application of the Tsallis’ framework. Indeed, the fundamental idea consists in the fact that the space between faults is filled with the residues of the breakage of the tectonic plates. In this regard, the authors studied the influence of the size distribution of fragments on the energy distribution of earthquakes. The theoretical motivation follows from the fragmentation phenomena in the context of the geophysics systems. In this latter work, Englaman et al showed that the standard Boltzmann-Gibbs formalism, although useful, cannot account for an important feature of fragmentation process, i.e., the presence of scaling in the size distribution of fragments, which is one of the main ingredients of the SCP approach. Thus, a nonextensive formalism is not only justified in SCP model but also necessary since the process of violent fractioning is very probably a nonextensive phenomenon, leading to long-range interactions among the parts of the object being fragmented (see, e.g., Ref. [13]). In reality, such an influence was earlier emphasized in other investigations [14]. In general lines, the SCP model follows similar arguments to those presented in Refs. [15] being, however, a more realistic seismic model than the one proposed in Ref. [16]. In particular, the theoretical ingredients reads:

- the mechanism of relative displacement of fault plates is the main cause of earthquakes;
- the surfaces of the tectonic plates are irregular and the fragments filling the space between them are very diverse and have irregular shapes;
- the mechanism of triggering earthquakes is established through the combination of irregularities of the fault planes and the distribution of fragments between them;
- the fragment distribution function and consequently the EDF emerges naturally from a nonextensive framework.

From the above arguments, the EDF deduced in Ref. [1] is given by

$$\log(N_{>m}) = \log N + \left(\frac{2 - q}{1 - q}\right) \times$$

$$\times \log \left[1 + a(q - 1)(2 - q)^{(\frac{q - 2}{q}} - 1) \times 10^{2m}\right].$$

According to Ref. [1], the above expression describes very well the energy distribution in all detectable range of magnitudes, unlike the empirical formula of Gutenberg-Richter [5].

III. NEW APPROACH

Now, let us discuss the standard method of maximization of the Tsallis entropy. Here and hereafter, the Boltzmann constant is set equal to unity for the sake of simplicity. Thus, the functional entropy to be maximized is

$$\delta S_q^* = \delta \left(S_q + \alpha \int_0^\infty p(\sigma)d\sigma - \beta \sigma_q\right) = 0,$$

(4)

where $\alpha$ and $\beta$ are the Lagrange multipliers. The constrains used above are the normalization of the distribution

$$\int_0^\infty p(\sigma)d\sigma = 1$$

(5)

and the $q$-expectation value

$$\sigma_q = \langle \sigma \rangle_q = \int_0^\infty \sigma P_q(\sigma)d\sigma$$

(6)
FIG. 1: The relative cumulative number of earthquakes [Eq. (12)] as a function of the magnitude $m$. In all panels, the data points correspond to earthquakes lying in the interval $3 < m < 8$. (a): Samambaia fault - Brazil: 100 events from Bulletin Seismic of the Revista Brasileira de Geofísica. (b): New Madrid fault - USA: 173 data points taken from NEIC catalog. (c): Anatolian fault - Turkey: 8980 events from NEIC catalog. The best-fit values for the parameters $q$ and $a$ are shown in the respective Panel. A summary of this analysis is also shown in Table I.

with the escort distribution [17] given by

$$P_q = \frac{p^q(\sigma)}{\int_0^\infty p^q(\sigma)d\sigma}.$$  (7)

By considering the same physical arguments of Ref. [1], we derive, after some algebra, the following expression for the fragment size distribution function

$$p(\sigma) = \left[1 - \frac{(1 - q)}{(2 - q)}(\sigma - \sigma_q)\right]^\frac{1}{q-1},$$  (8)

which corresponds to the area distribution for the fragments of the fault plates. Here, however, differently from Ref. [1], which assumes $\varepsilon \sim r$, we use a new energy scale $\varepsilon \sim r^3$. Thus, the proportionality between the released relative energy $\varepsilon$ and $r^3$ ($r$ is the size of fragments) is now given by $\sigma - \sigma_q = (\varepsilon/a)^{2/3}$, where $\sigma$ scales with $r^2$ and $a$ (the proportionality constant between $\varepsilon$ and $r^3$) has dimension of volumetric energy density. In particular, this new scale is in accordance with the standard theory of seismic rupture, the well-known seismic moment scaling with rupture length (see, for instance [18]).

The new EDF of earthquakes is, therefore, obtained by changing variables from $\sigma - \sigma_q$ to $(\varepsilon/a)^{2/3}$. From [8] it is straightforward to show that

$$p(\varepsilon)d\varepsilon = \frac{C\varepsilon^{-\frac{2}{3}}d\varepsilon}{[1 + C'\varepsilon^{2/3}]^{q-1}},$$  (9)

which has also a power-law form with $C$ and $C'$ given by

$$C = \frac{2}{3a^{2/3}} \quad \text{and} \quad C' = -\frac{(1 - q)}{(2 - q)a^{2/3}}.$$  (10)

In the above expression, the energy probability is written as $p(\varepsilon) = n(\varepsilon)/N$, where $n(\varepsilon)$ corresponds to the number of earthquakes with energy $\varepsilon$ and $N$ total number earthquakes.

IV. TESTING THE NEW EDF WITH THE CUMULATIVE NUMBER OF EARTHQUAKES

In order to test the viability of the new EDF above derived [Eq. (12)] we introduce the cumulative number of earthquakes, given by integral [1]

$$\frac{N(\varepsilon)}{N} = \int_{\varepsilon}^\infty p(\varepsilon)d\varepsilon,$$  (11)
TABLE I: Limits to $q$ and $a$

| Fault                  | Ref.       | $q$          | $a$          |
|-----------------------|------------|--------------|--------------|
| California - USA      | [1]        | 1.65         | $5.73 \times 10^{-6}$ |
| Iberian Peninsula - Spain | [1]        | 1.64         | $3.37 \times 10^{-6}$ |
| Andalucía - Spain     | [1]        | 1.60         | $3.0 \times 10^{-6}$  |
| Samambaia - Brazil    | This Paper | 1.60         | $1.3 \times 10^{10}$  |
| New Madrid - USA      | This Paper | 1.63         | $1.2 \times 10^{10}$  |
| Anatolian - Turkey    | This Paper | 1.71         | $2.8 \times 10^{10}$  |

where $N_{e>}$ is the number of earthquakes with energy larger than $\varepsilon$. Now, substituting $\varepsilon$ into Eq. (11), and considering $m = \frac{1}{q} \log \varepsilon$ (m stands for magnitude) it is possible to calculate the above expression. In reality, note that depending on the value of $q$ the limits of the integral (11) presents a cutoff on the maximum value allowed for energy $\varepsilon$, which is given by $\varepsilon_{\text{max}} = \sqrt{q^2/3(2-q)/(1-q)}$ for the intervals $q < 1$ and $q > 2$, while for $1 < q < 2$ the cutoff is absent in the distribution. Note also that in the limit $q \to 1$, $\varepsilon_{\text{max}} \to \infty$ and $p(\varepsilon)$ goes to the exponential function. As matter of fact, the calculation of the integral (11) for $q \neq 1$ leads to the general expression

\[
\log(N_{>m}) = \log N + \left(\frac{2-q}{1-q}\right) \times \log \left[1 - \left(\frac{1-q}{2-q}\right) \times \left(\frac{10^{2m}}{a^{2/3}}\right)\right],
\]

which, similarly to the modified Gutenberg-Richter law (See, e.g., Refs. [12] for more details), describes appropriately the energy distribution in a wider detectable range of magnitudes.

Figure 1 shows the relative cumulative number of earthquakes ($G_{m>}/N_{m>}/N$) as a function of the magnitude $m$. The data points, corresponding to earthquakes events lying in the interval $3 < m < 8$, were taken from two different catalogs, namely, Bulletin Seismic of the Revista Brasileira de Geofísica (left panels) and NEIC (central and right panels). The left, central and right Panels show the results of our analysis for the Samambaia fault, Brazil (100 events), New madrid fault, USA (173 events), and Anatolian fault, Turkey (8980 events), respectively. We note that, similarly to original version of SCP model, our approach, represented by Eqs. (12), provide a very good fit to the experimental data of the two catalogs here considered. It is worth emphasizing, however, that the energy density differ by several orders of magnitude from our model to the original SCP model. Therefore, we expect that other independent estimates of the parameter $a$ may indicate which approach is more physically realistic. The estimates of the parameters $q$ and $a$ obtained in this paper and in Ref. [1] are summarized in Table 1.

V. CONCLUSION

In Ref. [10], what seems to be the correct definition for expectation values within the Tsallis nonextensive statistical mechanics was rediscussed. Based on properties of the generalized relative entropies and the Shore-Johnson theorem, it was shown that the expectation value of any physical quantity in this extended framework converges to the normalized $q$-expectation value, instead of to the ordinary definition.

In this paper, by considering this necessity of $q$-expectation values in Tsallis nonextensive framework, we have revisited the fragment-asperity interaction model for earthquakes, as introduced in Ref. [1]. A new energy distribution function has been calculated, which allowed us to determine the relative cumulative number of earthquakes as a function of the magnitude. Additionally, a new scale law between the released relative energy $\varepsilon$ and the volume of fragments $r^3$ has also been introduced, i.e., in agreement with the so-called seismic moment scaling with rupture length. As discussed earlier, although our analysis and the one presented in Ref. [1] provide very similar values for the nonextensive parameter $q$, the other physical quantity, e.g., the energy density differ by several orders of magnitude. It would be interesting, therefore, if we could have experimental estimates for these quantities in order to compare the predictions of the models. Finally, it is worth mentioning that the estimates for the nonextensive parameter from the two catalogs here considered (Fig. 1) are consistent with the upper limit $q < 2$, obtained from several independent studies involving the Tsallis nonextensive framework [20].

Acknowledgments: The authors thank the anonymous referees for their valuable suggestions and comments. We also thank the partial support by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq - Brazil), JSA is supported by CNPq (307860/2004-3) and CNPq (475835/2004-2). CSV is supported by FAPEM.

[1] O. Sotolongo-Costa, and A. Posadas, Phys. Rev. Lett. 92, 048501 (2004).
[2] C. Tsallis, J. Stat. Phys. 52, 479 (1988); See also [http://tsallis.cat.cbpf.br/biblio.htm](http://tsallis.cat.cbpf.br/biblio.htm) for an updated bibliography.
[3] Braz. J. Phys. 29, 1 (1999), Special Issue on Nonextensive Statistical Mechanics and Thermodynamic, edited by S. R. A. Salinas and C. Tsallis; Nonextensive EntropyInterdisciplinary Applications, edited by M. German and C. Tsallis (Oxford University Press, New York, 2004).
[4] S. Abe, and N. Suzuki, J. Geophys. Res. 108 (B2), 2113 (2003).
[5] S. Abe, Y. Okamoto (Eds.), Nonextensive Statistical Me-
mechanics and Its Applications, Springer, Heidelberg, 2001.

[6] S. Abe, and N. Suzuki, Physica A, 350, 588 (2005).

[7] G. Gervino et al., Physica A, 340, 402 (2004).

[8] B. Guttenberg, C. F. Richter, Bull. Deismol. Soc. Am. 34, 185 (1944).

[9] C. Tsallis, R. S. Mendes, and A. Plastino, Physica A, 261, 534 (1998).

[10] S. Abe, and G. B. Bagci, Phys. Rev. E, 71, 016139 (2005).

[11] R. Silva, A. R. Plastino and J. A. S. Lima, Phys. Lett. A 249, 401 (1998); A. R. Plastino and A. Plastino, Braz. Journ. Phys. 29, 79 (1999); J. A. S. Lima, R. Silva A. R. Plastino, Phys. Rev. Lett. 86, 2938 (2001); G. Kaniadakis, Physica A 296, 405 (2001); J. A. S. Lima, R. Silva and J. Santos, Phys. Rev E 61, 3260 (2000); E. M. F. Curado, F. D. Nobre, Phys. Rev. E 67, 021107 (2003); M. Shiino, Phys. Rev. E 67, 056118 (2003); B. Dybiec and E. Gudowska-Nowak, Phys Rev E 69, 016105 (2004).

[12] R. Englman, N. Rivier, and Z. Jaeger, Philos. Mag. B, 56, 751 (1987).

[13] O. Sotolongo-Costa, A. H. Rodriguez, and G. J. Rodgers, Entropy 02, 172 (2000).

[14] H. Sauler, C. G. Sammis and D. Sornette, J. Geophys. Res. 101, 17661 (1996).

[15] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. 57, 341 (1967); H. Olami, J. S. Feder and K. Christensen, Phys. Rev. Lett. 68, 1244 (1992); V. De Rubeis, R. Hallgas, V. Loreto, G. Paladin, L. Pietronero and P. Tosi, Phys. Rev. Lett. 76, 2599 (1996).

[16] H. J. Herrmann, G. Mantica and D. Bessis, Phys. Rev. Lett. 65, 3223 (1990).

[17] S. Abe, Phys. Rev E, 68, 031101 (2003).

[18] Thorne Lay and Terry C. Wallace, Modern Global Seismology, Academic Press (1995).

[19] V.F. Pisarenko and D. Sornette, Pure and Applied Geophysics 160, 2343 (2003); 161, 839 (2004); V. Pisarenko, D. Sornette, and M. Rodkin, Computational Seismology 35, 138 (2004).

[20] B. M. Boghosian, Braz. Journ. Phys. 29, 91(1999); I. V. Karlin, M. Grmela and A. N. Gorban, Phys. Rev. E 65, 036128 (2002); R. Silva and J. S. Alcaniz, Phys. Lett. A 313, 393 (2003); R. Silva and J. S. Alcaniz, Physica A 341, 208 (2004); G. Kaniadakis, M. Lissia and A. M. Scarfone, Phys. Rev. E 71, 046128 (2005); S.H. Hansen,D. Egli,L. Hollenstein and C. Salzmann, New Astronomy 10, 379 (2005); R. Silva, J. S. Alcaniz and J. A. S. Lima, Physica A 356, 500 (2005); R. Silva, and J. A. S. Lima, Phys. Rev E 72, 057101 (2005)