Automation of the construction of the soliton solutions of nonlinear Schrödinger-type equations

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Abstract. An algorithm for constructing solitary wave solutions of nonlinear ordinary differential equations which is a variation of the simple equations method has been considered. The program was written in the Maple computer algebra system. The program has been tested on equations describing the propagation of pulses in an optical fiber.

1. Introduction
Many mathematical models that describe physical, chemical, biological, and other processes contain non-linear partial differential equations. If the Cauchy problem for such equations cannot be solved by the method of the inverse scattering problem [1], then one of the ways to find the solution is reduction to an ordinary differential equation, for example, using traveling wave variables or self-similar variables. If the obtained ordinary differential equation does not pass the Painlevé test [2], then it is non-integrable. For such equations, there are a number of methods for finding exact solutions, including the \( G'/G \) decomposition method [3], the tanh-method [4], the method of logistic function [5], the method of simplest equations [6] and others. In this paper, we consider an automation algorithm for the variation of the simplest equation method [7], which is effective for constructing optical solitons described by generalized Schrödinger equations.

2. Automation of the method of constructing soliton solutions
Let us consider an autonomous nonlinear ordinary differential equation of polynomial form

\[ E(y, y_z, y_{zz}, y_{zzz}, \ldots) = 0. \] (1)

As the simplest equation choose

\[ R_z^2 = R^2 (1 - \chi R^2), \] (2)

where \( \chi \) is the parameter. The equation (2) has the solution in the form

\[ R(z) = \frac{1}{a e^z + \frac{1}{4a} e^{-z}}, \] (3)

where \( a \) is an arbitrary constant. The algorithm for automatically constructing exact solutions is presented in Fig.1 and contains the following steps.
Figure 1. The flowchart of the algorithm of solitary wave solution construction

Step 1. The pole order of the solution of the equation

The pole order $p$ of the general solution of equation (1) is determined by substituting the expression $\frac{a_0}{z^p}$ in the leading terms of equation (1) and equating the degrees of $z$ of each term. For clarity, we use the Newton polygon corresponding to equation (1). In this case the pole order of the general solution of equation (1) is determined as the reciprocal of the coefficient of inclination of the edge corresponding to the leading terms of the differential equation. The paper [8] describes the algorithm of the ACNP program [9] for the automatic construction of Newton polygons of ordinary differential equations.
Step 2. Substitution of the simplest equation
Since the function $R(z)$ has the pole of the first order, to construct the exact solution of equation (1) we consider the following truncated expansion

$$y(z) = \sum_{k=0}^{p} A_k R^k(z).$$  \hfill (4)

Using the simplest equation (2), we calculate the derivatives of $R(z)$ in the form

$$R_{zz} = R - 2\chi R^3, R_{zzz} = R_z - 6\chi R^2 R_z, R_{zzzz} = R - 20\chi R^3 + 24\chi^2 R^5, \ldots.$$  \hfill (5)

We substitute expressions (4) and (5) in (1) and equate to zero the expressions at $R_z$ and various powers of $R$. We obtain an overdetermined system of equations for the coefficients of the truncated expansion (4) and the parameters of the differential equation (1).

Step 3. Finding solutions
We solve the overdetermined system of equations for $A_k$, $\chi$ and the parameters of the differential equation. When solving a system, it is often necessary to set the parameters that need to be found and impose additional restrictions on some parameters. If solutions of the system are found, it is possible to construct exact solutions of the equation (1) in the form (4) using the $R(z)$ function in the form (3).

Step 4. Checking solutions
Solutions are checked by substituting into the original differential equation.

Based on the considered algorithm, the program was written in the Maple computer algebra system for finding exact solutions of ordinary differential equations using the $R(z)$-function method.

3. Testing the program
The variation of the simplest equations method with $R(z)$ function was proposed in [7] and used in [10, 11, 12] to find exact solutions of equations describing the propagation of pulses in an optical fiber. The program was tested on equations from these works. The exact solutions found coincided with those presented in the works. To verify the program, we consider the hierarchy of the generalized nonlinear Schrödinger equation in the form

$$iq_t + i\sum_{n=1}^{N} a_{2n-1} q_{2n-1,x} + \sum_{n=1}^{N} a_{2n} q_{2n,x} + \sum_{n=1}^{N} b_n |q|^{2n} q = 0,$$  \hfill (6)

where $q(x,t)$ is the profile of pulse, $a_{2n-1}$, $a_{2n}$ and $b_n$ ($n = 1, \ldots, N$) are parameters of the mathematical model. The exact solutions of the eighth-order equation of the hierarchy (6) are presented in [10].

For large values of $N$, the results of the solution are cumbersome; therefore, we consider the equation (6) for $N = 5$

$$iq_t + i a_1 q_x + a_2 q_{xx} + i a_3 q_{3,x} + a_4 q_{4,x} + i a_5 q_{5,x} + a_6 q_{6,x} + i a_7 q_{7,x} + a_8 q_{8,x} + i a_9 q_{9,x} + a_{10} q_{10,x} + b_1 |q|^2 q + b_2 |q|^4 q + b_3 |q|^6 q + b_4 |q|^8 q + b_5 |q|^{10} q = 0.$$  \hfill (7)

We are looking for the solution of equation (7) in the form

$$q(x,t) = y(z) e^{i(k z - \omega t)}, z = x - C_0 t.$$  \hfill (8)
As a result of substituting (8) into (7), we obtain the system of equations

\[
(10k_{a_{10}} + a_{9}) y_{9,z} + (-120 k^3 a_{10} - 36 k^2 a_9 + 8 k a_8 + a_7) y_{7,z} + \\
+ (252 k^5 a_{10} + 126 k^4 a_9 - 56 k^3 a_8 - 21 k^2 a_7 + 6 k a_6 + a_5) y_{5,z} + \\
+ (-120 k^7 a_{10} - 84 k^6 a_9 + 56 k^5 a_8 + 35 k^4 a_7 - 20 k^3 a_6 - 10 k^2 a_5 - 4 k a_4 + a_3) y_{3,z} + \\
+ (10 k^9 a_{10} + 9 k^8 a_9 - 7 k^7 a_8 - 7 k^6 a_7 + 6 k^5 a_6 + 5 k^4 a_5 - 4 k^3 a_4 - 2 k^2 a_3 + 2 k a_2) y_{z} + \\
+ (a_1 - C_0) y_z = 0,
\]

\[
a_{10} y_{10,z} + b_5 y_{11} + b_4 y_9 + b_3 y_7 + b_2 y_5 + b_1 y^3 - (45 k^2 a_{10} + 9 k a_9 - a_8) y_{8,z} - \\
- (210 k^4 a_{10} - 84 k^3 a_9 + 28 k^2 a_8 + 7 k a_7 - a_6) y_{6,z} - \\
- (210 k^6 a_{10} - 70 k^5 a_9 - 35 k^4 a_7 + 15 k^3 a_6 + 5 k^2 a_5 - 6 k a_4 - a_3) y_{4,z} - \\
- (45 k^8 a_{10} - 36 k^7 a_9 + 28 k^6 a_8 + 21 k^5 a_7 - 15 k^4 a_6 + 10 k^3 a_5 + 6 k^2 a_4 + 3 k a_3 - a_2) y_{2,z} - \\
y(z) (k_{10} a_{10} + k^3 a_9 - k^2 a_8 - k^2 a_7 + k^2 a_6 + k^3 a_5 - k^4 a_4 - k^3 a_3 + k^2 a_2 + k a_1 - \omega) = 0.
\]

The first equation of system (9) holds if

\[
a_1 = -79360 k_{a_{10}} - 2176 k^7 a_9 - 96 k^5 a_6 - 8 k^3 a_4 - 2 k a_2 + C_0,
\]

\[
a_3 = -32640 k^3 a_{10} - 896 k^5 a_8 - 40 k^3 a_6 - 4 k a_4, a_5 = -4032 k^3 a_{10} - 112 k^5 a_8 - 6 k a_6,
\]

\[
a_7 = -240 k^3 a_{10} - 8 k a_8, a_9 = -10 k a_{10}.
\]

We are looking for exact solutions of the second equation of system (9) using the program for automatic construction of solitary wave solutions. Taking into account restrictions (10), we obtain the parameter values

\[
\omega = \frac{A_1^{10} k_{10} b_5}{362880 \chi^5} - \frac{11 A_1^{10} k^8 b_5}{241920 \chi^5} - \frac{A_1^{8} k^6 b_4}{40320 \chi^5} - \frac{209 A_1^{10} k^6 b_5}{86400 \chi^5} - \frac{A_1^{8} k^6 b_4}{480 \chi^4} - \\
\frac{17281 A_1^{10} k^6 b_5}{362880 \chi^5} - \frac{A_1^{8} k^6 b_4}{720 \chi^4} - \frac{47 A_1^{8} k^4 b_4}{960 \chi^4} - \frac{117469 A_1^{10} k^2 b_5}{40320 \chi^5} - \frac{7 A_1^{6} k^4 b_3}{144 \chi^3} - \\
\frac{3229 A_1^{8} k^2 b_4}{10080 \chi^4} - \frac{A_1^{4} k^4 b_2}{24 \chi^2} - \frac{63 b_5 A_1^{10}}{256 \chi^3} - \frac{259 A_1^{6} k^2 b_3}{720 \chi^3} - \frac{35 A_1^{4} k^4 b_2}{128 \chi^4} - \frac{5 A_1^{4} k^2 b_2}{12 \chi^2} - \\
\frac{5 A_1^{4} b_5}{16 \chi^3} - \frac{A_1^{2} k^2 b_3}{2 \chi} - \frac{3 A_1^{4} b_2}{8 \chi^2} - \frac{A_1^{2} b_1}{2 \chi} + k C_0,
\]

\[
a_1 = \frac{k^9 b_5 A_1^{10}}{362880 \chi^5} - \frac{117469 k_{b_5} A_1^{10}}{201600 \chi^5} - \frac{17281 k^3 b_5 A_1^{10}}{90720 \chi^5} - \frac{209 k^5 b_5 A_1^{10}}{14400 \chi^5} - \frac{5 k^5 A_1^{4} b_2}{6 \chi^2} - \\
\frac{k A_1^{7} b_1}{30240 \chi^5} - \frac{11 k^7 b_5 A_1^{10}}{5040 \chi^4} - \frac{k^7 A_1^{5} b_4}{80 \chi^3} - \frac{k^5 A_1^{6} b_3}{120 \chi^3} - \frac{7 k^3 A_1^{6} b_3}{36 \chi^3} - \frac{k^3 A_1^{4} b_2}{6 \chi^2} - \\
\frac{47 k^5 A_1^{8} b_4}{240 \chi^4} - \frac{3229 k A_1^{8} b_4}{5040 \chi^4} - \frac{259 k A_1^{6} b_3}{360 \chi^3} + C_0,
\]

\[
a_2 = \frac{A_1^{10} k_{b_5}}{80640 \chi^5} + \frac{11 A_1^{10} k^6 b_5}{86400 \chi^5} + \frac{A_1^{8} k^4 b_4}{1440 \chi^4} + \frac{209 A_1^{10} k^4 b_5}{5760 \chi^5} + \frac{A_1^{8} k^4 b_4}{32 \chi^4} + \frac{17281 A_1^{10} k^2 b_5}{60480 \chi^5} + \\
+ \frac{A_1^{6} k^4 b_3}{48 \chi^3} + \frac{47 A_1^{8} k^4 b_2}{160 \chi^4} + \frac{117469 b_5 A_1^{10}}{403200 \chi^5} + \frac{7 A_1^{6} k^2 b_3}{24 \chi^3} + \frac{3229 A_1^{8} b_4}{10080 \chi^4} + \frac{A_1^{4} k^2 b_2}{4 \chi^2} + \\
+ \frac{259 A_1^{8} b_3}{720 \chi^4} + \frac{5 A_1^{4} b_2}{12 \chi^2} + \frac{A_1^{2} b_1}{2 \chi},
\]
The solution of equation (7) has the form

\[ q(x, t) = \frac{4a A_1 e^{i(kx-\omega t)}}{4a^2 e^{-C_0 t} + \chi e^{-x+C_0 t}}, \tag{12} \]

where \( a, \chi, k, C_0, A_1 \) are arbitrary constants.

4. Conclusion
The algorithm for constructing exact solutions of ordinary differential equations using the variation of the simplest equation method with \( R(z) \) function has been considered. Based on the algorithm, the program was written in the Maple computer algebra system. The program was used to construct exact solutions of the tenth order equation from the hierarchy of the generalized nonlinear Schrödinger equation.

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\[ a_3 = \frac{k^7 b_5 A_1^{10}}{30240 \chi^5} + \frac{11 k^3 b_5 A_1^{10}}{4320 \chi^5} + \frac{k^5 A_1^8 b_4}{720 \chi^4} + \frac{209 k^3 b_5 A_1^{10}}{4320 \chi^4} + \frac{k^3 A_1^8 b_4}{24 \chi^4} + \frac{k^3 A_1 b_6 b_3}{36 \chi^3} + \frac{17281 k b_5 A_1^{10}}{90720 \chi^5} + \frac{47 k A_1^4 b_2}{240 \chi^4} + \frac{k A_1^6 b_1}{36 \chi^3} + \frac{k A_1^4 b_2}{6 \chi^2}, \]

\[ a_4 = \frac{A_1^{10} k^6 b_5}{17280 \chi^5} - \frac{11 A_1^{10} k^3 b_5}{3456 \chi^3} - \frac{A_1^8 k^4 b_4}{576 \chi^3} - \frac{209 A_1^{10} k^2 b_5}{5760 \chi^3} - \frac{A_1^8 k^2 b_4}{32 \chi^3} - \frac{17281 b_5 A_1^{10}}{362880 \chi^3} - \frac{A_1^6 k^2 b_3}{48 \chi^3} - \frac{47 A_1^8 b_4}{960 \chi^4} - \frac{7 A_1^6 b_3}{144 \chi^5} - \frac{A_1^8 b_4}{24 \chi^2}, \]

\[ a_5 = \frac{k^5 b_5 A_1^{10}}{14400 \chi^5} - \frac{11 k^3 b_5 A_1^{10}}{4320 \chi^5} - \frac{k^3 A_1^8 b_4}{720 \chi^4} - \frac{209 k b_5 A_1^{10}}{14400 \chi^5} - \frac{k A_1^8 b_4}{80 \chi^4} - \frac{k A_1^6 b_3}{120 \chi^3}, \]

\[ a_6 = \frac{A_1^{10} k^4 b_5}{17280 \chi^5} + \frac{11 A_1^{10} k^3 b_5}{8640 \chi^3} + \frac{A_1^8 k^2 b_4}{1440 \chi^4} + \frac{209 k b_5 A_1^{10}}{86400 \chi^5} + \frac{A_1^8 b_4}{480 \chi^2} + \frac{A_1^6 b_3}{720 \chi^5}, \]

\[ a_7 = \frac{k^3 b_5 A_1^{10}}{30240 \chi^5} + \frac{11 k b_5 A_1^{10}}{30240 \chi^4} + \frac{k A_1^8 b_4}{5040 \chi^4}, \quad a_8 = -\frac{1}{80640 \chi^5} - \frac{11 b_5 A_1^{10}}{241920 \chi^5} - \frac{A_1^8 b_4}{40320 \chi^4}, \]

\[ a_9 = -\frac{k b_5 A_1^{10}}{362880 \chi^5}, \quad a_{10} = \frac{b_5 A_1^{10}}{3628800 \chi^5}. \]