Large monochromatic components and long monochromatic cycles in random hypergraphs

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Abstract

We extend results of Gyárfás and Füredi on the largest monochromatic component in \( r \)-colored complete \( k \)-uniform hypergraphs to the setting of random hypergraphs. We also study long monochromatic loose cycles in \( r \)-colored random hypergraphs. In particular, we obtain a random analog of a result of Gyárfás, Sárközy, and Szemerédi on the longest monochromatic loose cycle in 2-colored complete \( k \)-uniform hypergraphs.

1 Introduction

It is known, due to Gyárfás [9], that for any \( r \)-coloring of the edges of \( K_n \), there is a monochromatic component of order at least \( n/(r-1) \) and this is tight when \( r-1 \) is a prime power and \( n \) is divisible by \((r-1)^2\). Later, Füredi [7] introduced a more general method which implies the result of Gyárfás. More recently, Mubayi [18] and independently, Liu, Morris, and Prince [17], gave a simple proof of a stronger result which says that in any \( r \)-coloring of the edges of \( K_n \), there is either a monochromatic component on \( n \) vertices or a monochromatic double star of order at least \( n/(r-1) \). Recently, Bal and DeBiasio [1] and independently, Dudek and Prałat [6], showed that the Erdős-Rényi random graph behaves very similarly with respect to the size of the largest monochromatic component.

Recall that the random graph \( G(n,p) \) is the random graph \( G \) with vertex set \([n]\) in which every pair \( \{i,j\} \in \binom{[n]}{2} \) appears independently as an edge in \( G \) with probability \( p \). An event in a probability space holds asymptotically almost surely (or a.a.s.) if the probability that it holds tends to 1 as \( n \) goes to infinity. More precisely, it was shown in [1] and [6] that that a.a.s. for any \( r \)-coloring of the edges of \( G(n,p) \), there is a monochromatic component of order \((\frac{1}{r-1} - o(1))n\), provided that \( pn \to \infty \) (that means the average degree tends to infinity). As before, this result is clearly best possible.

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In this paper we study a generalization of these results to $k$-uniform hypergraphs (each edge has order $k$). As in the $k = 2$ case, our results hold even for very sparse random hypergraphs; that is, we only assume that the average degree, $pn^{k-1}$, tends to infinity together with $n$.

1.1 Large components

We say that a hypergraph $H = (V, E)$ is connected if the shadow graph of $H$ (that is, the graph with vertex set $V(H)$ and edge set $\{\{x, y\} : \{x, y\} \subseteq e$ for some $e \in E(H)\}$) is connected. A component of a hypergraph is a maximal connected subgraph. Let $r$ be a positive integer and $H$ be a hypergraph. Let $\chi_r : E(H) \to [r]$ be a coloring of the edges of $H$. Denote by $mc(H, \chi_r)$ the order of the largest monochromatic component under $\chi$ and let

$$mc_r(H) = \min_{\chi_r} mc(H, \chi_r).$$

For hypergraphs much less is known; however, Gyárfás [9] (see also Füredi and Gyárfás [8]) proved the following result. Let $K^k_n$ denote the complete $k$-uniform hypergraph of order $n$.

**Theorem 1.1** (Gyárfás [9]). For all $n \geq k \geq 3$,

(i) $mc(k)(K^k_n) = n$, and

(ii) $mc(k+1)(K^k_n) \geq \frac{k}{k+1} n$. Furthermore, this is optimal when $n$ is divisible by $k + 1$.

The optimality statement is easy to see. Indeed, split the vertex set into $k + 1$ parts $V_1, \ldots, V_{k+1}$ each of size $n/(k+1)$ and color the edges so that color $i$ is not used on any edge which intersects $V_i$. Clearly each monochromatic component has size $kn/(k+1)$.

Füredi and Gyárfás [8] and Gyárfás and Haxell [10] proved a number of other results regarding the value of $mc_r(K^k_n)$; however, in general, the value of $mc_r(K^k_n)$ is not known (see Section 7 for more details).

Our main theorem shows that in order to prove a random analog of any such result about $mc_r(K^k_n)$ it suffices to prove a nearly complete (or large minimum degree) analog of such a result. Let the random hypergraph $H^{(k)}(n, p)$ be the $k$-uniform hypergraph $H$ with vertex set $[n]$ in which every $k$-element set from $\binom{n}{k}$ appears independently as an edge in $H$ with probability $p$.

**Theorem 1.2.** Suppose a function $\varphi = \varphi(r, k)$ satisfies the following condition: for all $k \geq 2$, $r \geq 1$, and $\alpha^* > 0$ there exists $\epsilon^* > 0$ and $t_0 > 0$ such that if $G$ is a $k$-uniform hypergraph on $t \geq t_0$ vertices with $e(G) \geq (1 - \epsilon^*)(\binom{t}{k})$, then $mc_r(G) \geq (\varphi - \alpha^*)t$.

Then for any $k \geq 2$, $r \geq 1$, $\alpha > 0$, and $p = p(n)$ such that $pn^{k-1} \to \infty$ we have that a.a.s. $mc_r(H^{(k)}(n, p)) \geq (\varphi - \alpha)n$.

As an application, we prove a version of Theorem 1.1 for nearly complete hypergraphs (Sections 4 and 5) and then obtain a version for random hypergraphs as an immediate corollary (Section 3.2).

**Theorem 1.3.** For all $\alpha > 0$ and $k \geq 3$ there exists $\epsilon > 0$ and $n_0$ such that if $G$ is a $k$-uniform hypergraph on $n \geq n_0$ vertices with $e(G) > (1 - \epsilon)(\binom{n}{k})$, then
Figure 1: A 4-uniform loose cycle (a) and a 5-uniform diamond (b).

(i) $mc_k(G) \geq (1 - \alpha)n$, and
(ii) $mc_{k+1}(G) \geq \left( \frac{k}{k+1} - \alpha \right)n$.

(We give an explicit bound on $\alpha$ in terms of $\epsilon$ in the proof of Theorem 4.1 and Corollary 5.2.)

Corollary 1.4. Let $k \geq 3$, let $\alpha > 0$, and let $p = p(n)$ be such that $pn^{k-1} \rightarrow \infty$. Then a.a.s.

(i) $mc_k(H^{(k)}(n,p)) \geq (1 - \alpha)n$, and
(ii) $mc_{k+1}(H^{(k)}(n,p)) \geq \left( \frac{k}{k+1} - \alpha \right)n$.

1.2 Loose-cycles

We say that a $k$-uniform hypergraph $(V,E)$ is a loose cycle if there exists a cyclic ordering of the vertices $V$ such that every edge consists of $k$ consecutive vertices and every pair of consecutive edges intersects in a single vertex (see Figure 1a). Consequently, $|E| = \frac{|V|}{(k-1)}$.

Let $G$ be a $k$-uniform hypergraph. A connected loose cycle packing on $t$ vertices is a connected sub-hypergraph $H \subseteq G$ and a vertex disjoint collection of loose cycles $C_1, \ldots, C_\ell \subseteq H$ such that $\sum_{i=1}^\ell |V(C_i)| = t$. A connected diamond matching on $t$ vertices is a connected loose cycle packing on $t$ vertices such that every cycle consists of 2 edges and so has exactly $2k - 2$ vertices (see Figure 1b). When $k = 2$ we consider an edge to be a cycle with 2 vertices.

Gyárfás, Sárközy and Szemerédi [11] proved that for all $k \geq 3$ and $\eta > 0$, there exists $n_0$ such that if $n \geq n_0$, then every 2-edge-coloring of $K_n^k$ contains a monochromatic cycle of length $(1 - \eta)\frac{2k-2}{2k-1}n$. (The $k = 3$ case was previously proved by Haxell, Luczak, Peng, Rödl, Ruciński, Simonovits, and Skokan [13].) Their proof follows from a more basic result which says that for all $k \geq 3$ and $\eta > 0$, there exists $\epsilon > 0$ and $n_0$ such that if $G$ is a $k$-uniform hypergraph on $n \geq n_0$ vertices with $e(G) \geq (1 - \epsilon)\binom{n}{k}$, then every 2-edge-coloring of $G$ contains a monochromatic connected diamond matching on $(1 - \eta)\frac{2k-2}{2k-1}n$ vertices.
We provide a result which reduces the problem of finding long monochromatic loose cycles in random hypergraphs to the problem of finding large monochromatic connected loose cycle packing in nearly complete hypergraphs (Section 3.3). Then applying the result of [11], we get an asymptotically tight result for 2-colored random $k$-uniform hypergraphs.

**Theorem 1.5.** Suppose a function $\psi = \psi(r,k)$ satisfies the following condition: for all $k \geq 2$, $r \geq 1$, and $\alpha^* > 0$ there exists $\epsilon^* > 0$ and $t_0 > 0$ such that if $G$ is a $k$-uniform hypergraph on $t \geq t_0$ vertices with $e(G) \geq (1 - \epsilon^*)(\binom{t}{k})$, then every $r$-coloring of the edges of $G$ contains a monochromatic connected loose cycle packing on at least $(\psi - \alpha^*)t$ vertices.

Then for any $k \geq 2$, $r \geq 1$, $\alpha > 0$, and $p = p(n)$ such that $pn^{k-1} \to \infty$ we have that a.a.s. every $r$-edge-coloring of $\mathcal{H}^{(k)}(n,p)$ contains a monochromatic loose cycle on at least $(\psi - \alpha)n$ vertices.

Using a result of Gyárfás, Sárközy, and Szemerédi [11] about large monochromatic connected diamond matchings in nearly complete hypergraphs, we get the following corollary.

**Corollary 1.6.** Let $k \geq 2$ and $\alpha > 0$. Choose $p = p(n)$ such that $pn^{k-1} \to \infty$. Then a.a.s. there exists a monochromatic loose cycle on at least $\left(\frac{2k-2}{2k-1} - \alpha\right)n$ vertices in any 2-coloring of the edges of $\mathcal{H}^{(k)}(n,p)$.

## 2 Notation and definitions

Let $G$ be a $k$-uniform hypergraph. Let $v \in V(G)$ and $U \subseteq V(G)$. Define

$$d(v, U) = \left| \left\{ S \in \binom{U}{k-1} : S \cup \{v\} \in E(G) \right\} \right|.$$ 

Furthermore, let $\delta(G) = d(v, V(G))$, which is just the minimum degree of $G$. Suppose now that $v \notin U$ and define the restricted link graph of $v$, denoted by $L(v, U)$, with vertex set $U$ and edge set $\left\{ S \in \binom{U}{k-1} : S \cup \{v\} \in E(G) \right\}$. We call $L(v) = L(v, V(G) \setminus \{v\})$ the link graph of $v$, which clearly is a $(k-1)$-uniform hypergraph on $n - 1$ vertices.

Also recall that the 1-core of $G$ is the largest induced subgraph of $G$ with no isolated vertices.

Now suppose that the edges of $G$ are $r$-colored. We follow the convention that any edge $e$ of the link graph of a vertex $v$ inherits the color of the edge $e \cup \{v\}$ in $G$.

For expressions such as $n/t$ (for example the size of a cluster in the regularity lemma) that are supposed to be an integer, we always assume that $n/t \in \mathbb{Z}$ by rounding appropriately without affecting the argument.

## 3 Random hypergraphs

### 3.1 Sparse weak hypergraph regularity

First, we will provide a few definitions.
Let $H = (V, E)$ be a $k$-uniform hypergraph. Given pairwise-disjoint sets $U_1, \ldots, U_k \subseteq V$, and $p > 0$, we define $e_H(U_1, \ldots, U_k)$ to be the total number of edges $e = v_1 \ldots v_k$ in $H$ such that $v_i \in U_i$ for all $1 \leq i \leq k$. Also, we define

$$d_{H,p}(U_1, \ldots, U_k) = \frac{e_H(U_1, \ldots, U_k)}{p|U_1| \cdot \ldots \cdot |U_k|}.$$ 

When the host graph $H$ is clear from context, we may refer to $e_H(U_1, \ldots, U_k)$ as $e(U_1, \ldots, U_k)$ $d_{H,p}(U_1, \ldots, U_k)$ as just $d_p(U_1, \ldots, U_k)$.

For $\epsilon > 0$, we say the $k$-tuple $(U_1, \ldots, U_k)$ of pairwise-disjoint subsets of $V$ is $(\epsilon, p)$-regular if

$$|d_p(W_1, \ldots, W_k) - d_p(U_1, \ldots, U_k)| \leq \epsilon$$

for all $k$-tuples of subsets $W_1 \subseteq U_1, \ldots, W_k \subseteq U_k$ satisfying $|W_1| \cdot \ldots \cdot |W_k| \geq \epsilon |U_1| \cdot \ldots \cdot |U_k|$.

We say that $H$ is a $(\eta, p, D)$-upper uniform hypergraph if for any pairwise disjoint sets $U_1, \ldots, U_k$ with $|U_1| \geq \ldots \geq |U_k| \geq \eta |V(H)|$, $d_p(U_1, \ldots, U_k) \leq D$.

The following theorem is a sparse version of weak hypergraph regularity which will be the workhorse used to prove the main result for the random hypergraph. The sparse version of the regularity lemma [21] for graphs was discovered independently by Kohayakawa [15], and Rödl (see, for example, [4]), and subsequently improved by Scott [20]. The following is a straightforward generalization of their result for hypergraphs, which we state here without proof.

**Theorem 3.1.** For any given integers $k \geq 2$, $r \geq 1$, and $t_0 \geq 1$, and real numbers $\epsilon > 0$, $D \geq 1$, there are constants $\eta = \eta(k,r,\epsilon,t_0,D) > 0$, $T_0 = T_0(k,r,\epsilon,t_0,D) \geq t_0$, and $N_0 = N_0(k,r,\epsilon,t_0,D)$ such that any collection $H_1, \ldots, H_r$ of $k$-uniform hypergraphs on the same vertex set $V$ with $|V| \geq N_0$ that are all $(\eta, p, D)$-upper-uniform with respect to density $0 < p \leq 1$ admits an equipartition (i.e. part sizes differ by at most 1) of $V$ into $t$ parts with $t_0 \leq t \leq T_0$ such that all but at most $\epsilon(k)$ of the $k$-tuples induce an $(\epsilon, p)$-regular $k$-tuple in each $H_i$.

### 3.2 Large components

We will use the following lemma to turn an $(\epsilon, p)$-regular $k$-tuple of some color into a large monochromatic subgraph.

**Lemma 3.2.** Let $0 < \epsilon < 1/3$, let $k \geq 2$, and let $G = (V, E)$ be a $k$-partite hypergraph with vertex partition $V = V_1 \cup V_2 \cup \ldots \cup V_k$. If for all $U_1, \ldots, U_k$, with $|U_i| \geq \epsilon |V_i|$ and $U_i \subseteq V_i$ for all $i \in [k]$, there exists an edge $\{u_1, \ldots, u_k\}$ with $u_i \in U_i$ for all $i \in [k]$, then $G$ contains a connected component $H$ such that $|V(H) \cap V_i| \geq (1-\epsilon)|V_i|$ for all $i \in [k]$.

**Proof.** Start by choosing for each $i \in [k]$ a set $X_i \subseteq V_i$ with $|X_i| = \lfloor 3\epsilon |V_i| \rfloor$ and let $G'$ be the hypergraph induced by $\{X_1, \ldots, X_k\}$. Suppose that no component of $G'$ has intersection of size at least $\epsilon |V_i|$ with any $X_i$. Choose $t \geq 2$ as small as possible so that there exist components $H_1, \ldots, H_t$ of $G'$ such that for some $\ell \in [k]$, $|X_\ell \cap \bigcup_{i \in [\ell]} V(H_i)| \geq \epsilon |V_\ell|$. By the minimality of $t$, we have that for all $j \in [\ell]$, $|X_j \cap \bigcup_{i \in [\ell]} V(H_i)| < 2\epsilon |V_j|$. So let $X_\ell = X_\ell \cap \bigcup_{i \in [\ell]} V(H_i)$ and for all $j \in [k] \setminus \{\ell\}$, let $X_j = X_j \setminus \bigcup_{i \in [\ell]} V(H_i)$. For all $i \in [k]$,
we have \( |X'_i| \geq \epsilon |V_i| \); however, by the construction of the sets \( X'_1, \ldots, X'_k \), there is no edge in \( G'[X'_1 \cup \cdots \cup X'_k] \) which violates the hypothesis.

So we may assume that some component \( H' \) of \( G' \) intersects, say \( V_1 \), in at least \( \epsilon |V_1| \) vertices. We will show that for each \( i \), \( |V(H') \cap X_i| \geq \epsilon |V_i| \). Indeed, let us assume for a moment that there exists some \( j \) such that \( |V(H') \cap X_j| < \epsilon |V_j| \) and define for each \( i \in [k] \) the set \( Y_i \) according to the following rule. If \( |V(H') \cap X_i| \geq \epsilon |V_i| \), then set \( Y_i = V(H') \cap X_i \); otherwise set \( Y_i = X_i \setminus V(H') \). Clearly, \( |Y_i| \geq \epsilon |V_i| \). By the hypothesis, there is an edge in the hypergraph \( G'[Y_1, \ldots, Y_k] \) induced by \( \{Y_1, \ldots, Y_k\} \). But this cannot happen since \( Y_1 \subseteq V(H') \) and \( Y_j \) for \( j \neq 1 \) is disjoint from \( V(H') \). Therefore, for each \( i \), \( |V(H') \cap X_i| \geq \epsilon |V_i| \).

Now let \( H \) be the largest component of \( G \) that contains \( H' \). Suppose to the contrary and without loss of generality that \( |V(H) \cap V_i| < (1 - \epsilon) |V_k| \). But as there must be an edge in \( G[V(H') \cap V_1, \ldots, V(H') \cap V_{k-1}, V_k \setminus V(H)] \), this is a contradiction. So \( H \) is the desired component.

We will also make use of the following version of the Chernoff bound (see, e.g., inequality (2.9) in [13]). For a binomial random variable \( X \) and \( \gamma \leq 3/2 \), we have

\[
\Pr(|X - \mathbb{E}(X)| \geq \gamma \mathbb{E}(X)) \leq 2 \exp\left(-\gamma^2 \mathbb{E}(X)/3\right). \tag{1}
\]

Proof of Theorem 1.2. Let \( r, k, \alpha, \) and \( \varphi = \varphi(r, k) \) be given. Set \( \alpha^* = \alpha/2 \). Let \( \epsilon^* \) and \( t_0 \) be the constants guaranteed by the values of \( k, r, \) and \( \alpha^* \). Let \( \epsilon < \min\{1/(2r), \alpha/(4\varphi), \epsilon^*\} \). Thus, if \( \Gamma \) is a \( k \)-uniform hypergraph on \( t \geq t_0 \) vertices with \( \mathbb{E}(\Gamma) \geq (1 - \epsilon)(\binom{t}{k}) > (1 - \epsilon^*)(\binom{t}{k}) \), then \( mc_r(\Gamma) \geq (\varphi - \alpha/2)t \). Let \( H = (V, E) = \mathcal{H}(k)(n, p) \).

First observe that for any fixed positive \( \eta \) any sub-hypergraph \( H' \) of \( H \) is \( (\eta, p, 2) \)-upper uniform. Indeed, let \( U_1, \ldots, U_k \subseteq V \) with \( |U_1| \geq \cdots \geq |U_k| \geq \eta n \) be given. Then the expected number of edges in \( H \) having exactly one vertex in each \( U_i \) is \( |U_1| \cdot \cdots |U_k| |p| \geq \eta^k n^k p \). Thus (1), applied with \( \gamma = 1 \), implies that

\[
\Pr(\epsilon(U_1, \ldots, U_k) \geq 2|U_1| \cdot \cdots |U_k| |p| \leq 2 \exp\left(-|U_1| \cdot \cdots |U_k| |p|/3\right) \\
\leq 2 \exp\left(-\eta^k n^k p/3\right) = o(2^{-kn}),
\]

since \( pm^{k-1} \to \infty \). Consequently, a.a.s. the number of edges in \( H' \) is at most \( 2|U_1| \cdot \cdots |U_k| |p| \). Finally, since the number of choices for \( U_i \)'s is trivially bounded from above by \( (2^n)^k = 2^{kn} \), the union bound yields that a.a.s. \( H' \) is \( (\eta, p, 2) \)-upper uniform.

Apply Theorem 3.1 with \( k, r, t_0, \epsilon \), as above, and \( D = 2 \). Let \( \eta, T_0 \) and \( N_0 \) be the constants that arise and assume that \( n \geq N_0 \). Let \( c \) be an \( r \)-coloring of the edges of \( H \) and let \( H_i \) be the subgraph of \( H \) induced by the \( i \)-th color, that means, \( V(H_i) = V \) and \( E(H_i) = \{e \in E \mid c(e) = i\} \). Then let \( t \) be the constant guaranteed by Theorem 3.1 for graphs \( H_i \) and let \( V_1 \cup \cdots \cup V_t \) be the \( (\epsilon, p) \)-regular partition of \( V \).

Let \( R \) be the \( k \)-uniform cluster graph with vertex set \([t]\) where \( \{i_1, \ldots, i_k\} \) is an edge if and only if \( V_{i_1}, \ldots, V_{i_k} \) form an \( (\epsilon, p) \)-regular \( k \)-tuple. Color \( \{i_1, \ldots, i_k\} \) in \( R \) by a majority color in the \( k \)-partite graph \( H[V_{i_1}, \ldots, V_{i_k}] \). Let \( H' \) be the sub-hypergraph colored by this color in \( H[V_{i_1}, \ldots, V_{i_k}] \). Observe that \( d_{H', \rho}(V_{i_1}, \ldots, V_{i_k}) \geq 1/(2r) \). Indeed, the Chernoff bound (1), applied with \( \gamma = 1/2 \), implies that a.a.s. \( \epsilon(V_{i_1}, \ldots, V_{i_k}) \geq |V_{i_1}| \cdot \cdots |V_{i_k}| |p|/2 \). Thus, \( d_{H', \rho}(V_{i_1}, \ldots, V_{i_k}) \geq |V_{i_1}| \cdot \cdots |V_{i_k}| |p|/(2r) \), as required. Furthermore, since \( \epsilon < 1/(2r) \), we also get that \( d_{H', \rho}(V_{i_1}, \ldots, V_{i_k}) \geq \epsilon \).
Clearly, we also have $|E(R)| > (1 - \epsilon)(k^t)$, so by the assumption, there is a monochromatic, say red, component of size at least $(\varphi - \alpha/2)t$ in the cluster graph. Let us assume that $\{i_1, \ldots, i_k\}$ is a red edge in $R$. Thus, $(V_{i_1}, \ldots, V_{i_k})$ is an $(\epsilon, p)$-regular $k$-tuple in $H'$ and so for all $k$-tuples $U_{i_1} \subseteq V_{i_1}, \ldots, U_{i_k} \subseteq V_{i_k}$ satisfying $|U_{i_1}| \cdots |U_{i_k}| \geq \epsilon|V_{i_1}| \cdots |V_{i_k}|$ we have

$$|d_{H',p}(U_{i_1}, \ldots, U_{i_k}) - d_{H',p}(V_{i_1}, \ldots, V_{i_k})| \leq \epsilon,$$

which implies that

$$d_{H',p}(U_{i_1}, \ldots, U_{i_k}) > d_{H',p}(V_{i_1}, \ldots, V_{i_k}) - \epsilon \geq 1/(2r) - \epsilon > 0.$$

Consequently, there exists an edge $u_{i_1} \ldots u_{i_k}$ in $H'$ such that $u_{ij} \in U_{i_j}$ for each $1 \leq j \leq k$. Once again, this is true for any $U_{ij}$'s satisfying $|U_{i_1}| \cdots |U_{i_k}| \geq \epsilon|V_{i_1}| \cdots |V_{i_k}|$. But the latter clearly implies that $|U_{i_j}| \geq \epsilon|V_{i_j}|$. Hence, by Lemma 3.2 there exists a component in $H'$ that contains at least $(1 - \epsilon)|V_{i_j}|$ vertices from each $V_{i_j}$. In other words, if $i_j$ is contained in a red edge in the cluster graph $R$, at least $(1 - \epsilon)|V_i|$ vertices inside $V_i$ are in the same red component. Furthermore, since $(1 - \epsilon)|V_i| > \frac{1}{2}|V_i|$, any two red edges that intersect in $R$ will correspond to red connected subgraphs that intersect in $H$, and thus are in the same red component in $H$. So, the $(\varphi - \alpha/2)t$ vertices in the largest monochromatic component in $R$ corresponds to a monochromatic component in $H$ of order at least

$$(\varphi - \alpha/2)t(1 - \epsilon)|V_i| \geq (\varphi - \alpha/2)t(1 - \epsilon)(1 - \epsilon)^n_t$$

$$= (\varphi - \alpha/2)(1 - 2\epsilon + \epsilon^2)n \geq (\varphi - \alpha/2 - 2\epsilon\varphi)n \geq (\varphi - \alpha)n,$$  

where the last inequality uses $\epsilon < \alpha/(4\varphi)$.

\[ \square \]

3.3 Loose cycles

We first generalize a result explicitly stated by Letzter [16, Corollary 2.1], but independently proved implicitly by Dudek and Prałat [5] and Pokrovskiy [19].

Lemma 3.3. Let $H$ be a $k$-uniform $k$-partite graph with partite sets $X_1, \ldots, X_k$ such that $|X_2| = \ldots = |X_{k-1}| = m$ and $|X_1| = |X_k| = m/2$ for some $m$. Then for all $0 \leq \zeta \leq 1$ if there are no sets $U_1 \subseteq X_1, \ldots, U_k \subseteq X_k$ with $|U_1| = \ldots = |U_k| \geq \zeta m$ such that $H[U_1, \ldots, U_k]$ is empty, then there is a loose path on at least $(1 - 4\zeta)m - 2$ edges.

Proof. The proof is based on the depth first search algorithm. A similar idea for graphs was first noticed by Ben-Eliezer, Krivelevich and Sudakov [2, 3].

Let $\zeta$ be given. We will proceed by a depth first search algorithm with the restriction that the vertices of degree 2 in the current path are in $X_1 \cup X_k$. We will let $X^*_1$ be the current set of unexplored vertices in $X_1$. We will let $X'_i$ be the current set of vertices which were added to the path at some point, but later rejected. So, initially we have $X^*_1 = X_1$ and $X'_1 = \emptyset$. Start the algorithm by removing any vertex from $X^*_1$ and adding it to the path $P$.

Suppose $P$ is the current path. First consider the case where $P$ has been reduced to a single vertex, say without loss of generality $P = x_1 \in X_1$. If there are no edges $\{x_1, \ldots, x_k\}$ where $x_i \in X^*_i$ is unexplored for all $2 \leq i \leq k$, then we move $x_1$ out of the path and back into $X^*_1$ and start the algorithm again by choosing an edge in $G[X^*_1, \ldots, X^*_k]$ if we can
and stopping the algorithm otherwise. If there is an edge \( \{x_1, \ldots, x_k\} \) where \( x_i \in X_i^* \) is unexplored for all \( 2 \leq i \leq k \), then we add \( \{x_1, \ldots, x_k\} \) to \( P \) and remove \( x_i \) from \( X_i^* \) for all \( 2 \leq i \leq k \) and continue the algorithm from \( x_k \).

Now assume \( P \) has at least one edge. Let the last edge of \( P \) be \( \{x_1, \ldots, x_k\} \) and assume without loss of generality that \( x_1 \) has degree 1 in \( P \) (i.e. is the current endpoint of the path). If there are no edges \( \{x_1, y_2, \ldots, y_k\} \) where \( y_i \in X_i^* \) is unexplored for all \( 2 \leq i \leq k \), then we move \( x_i \) from the path to \( X_i' \) for all \( 1 \leq i \leq k-1 \) and continue the algorithm from \( x_k \). If there is an edge \( \{x_1, y_2, \ldots, y_k\} \) where \( y_i \in X_i^* \) is unexplored for all \( 2 \leq i \leq k \), then we add \( \{x_1, y_2, \ldots, y_k\} \) to \( P \) and remove \( y_i \) from \( X_i^* \) for all \( 2 \leq i \leq k \) and continue the algorithm from \( y_k \).

Note that during every stage of the algorithm, there is no edge \( \{x_1, \ldots, x_k\} \) where \( x_1 \in X_1' \) and \( x_i \in X_i^* \) for all \( 2 \leq i \leq k \) and no edge \( \{y_1, \ldots, y_k\} \) where \( y_k \in X_k' \) and \( y_i \in X_i^* \) for all \( 2 \leq i \leq k \). We also know that at every stage of the algorithm, \( |X'_2| = \ldots = |X'_{k-1}| = |X'_1| + |X'_k| \) since every time a vertex from either \( X_1 \) or \( X_k \) gets rejected, so do vertices from each \( X_i \), and at no step does a vertex from both \( X_1 \) and \( X_k \) get rejected. Furthermore,

\[
|X'_2| = \ldots = |X'_{k-1}| = |X'_1| + |X'_k| + 1
\]

at every step where \( P \neq \emptyset \) since each time a vertex is removed from any \( X_i^* \), for \( 2 \leq i \leq k-1 \), exactly one vertex from \( X_i^* \cup X'_k \) is removed as well, or one vertex is added to \( X_i^* \cup X'_k \) (when \( P \) is a single vertex that cannot be extended) then two are removed, with the exception of when the initial edge is selected. Note that at each step \( X_i = X_i^* \cup X'_i \cup (P \cap X_i) \) where the unions are disjoint.

Notice that this algorithm cannot stop while each \( |X_i^*| \geq \zeta m \), since otherwise the \( X_i^* \) sets would violate the hypothesis.

After each step of the algorithm, either the difference between \( |X_i^*| \) and \( |X'_i| \) decreases by 1, or the difference between \( |X'_k| \) and \( |X_k^*| \) decreases by 1. So there is a stage in the algorithm where either \( |X'_1| = |X_1^*| \) or \( |X'_k| = |X_k^*| \) but not both. Suppose without loss of generality, we are at a stage where \( |X'_1| = |X_1^*| \) and \( |X'_k| < |X_k^*| \). Since \( P \) always contains almost the same number of vertices in \( X_1 \) as it does vertices in \( X_k \) (off by at most one), we have that the sums \( |X'_1| + |X'_k| \) and \( |X_1^*| + |X_k^*| \) differ by at most 1. This yields that

\[
2|X'_1| = |X'_1| + |X'_k| \leq |X_1^*| + |X_k^*| + 1 < 2|X_k^*| + 1
\]

and so \( |X'_1| \leq |X_k^*| \). Thus, \( |X'_2| = \ldots = |X_{k-1}^*| = |X_1^*| + |X_k^*| + 1 \geq 2|X_1^*| \) for all \( 1 \leq i \leq k \). Thus, if \( |X'_1| \geq \zeta m \), and consequently, each \( |X_i^*| \geq |X'_1| \geq \zeta m \), we are done since \( H[X_1^*, X_2^*, \ldots, X_k^*] \) has no edges, a contradiction. Otherwise, if \( |X'_1| < \zeta m \), then

\[
|P \cap X_1| = |X_1| - |X_1^*| - |X'_1| = m/2 - 2|X'_1| \geq m/2 - 2\zeta m.
\]

But by our construction, each vertex in \( P \cap X_1 \) corresponds to two edges in the path, except at most two vertices, so we have that there are at least \( m - 4\zeta m - 2 \) edges in \( P \).

For a graph \( G \), and hypergraph \( F \) with \( V(G) \subseteq V(F) \), we say \( F \) is a Berge-\( G \) if there is a bijection \( f : E(G) \to E(F) \) such that \( e \subseteq f(e) \) for all \( e \in E(G) \). A Berge path \( (E_1, \ldots, E_{\ell-1}) \) is a Berge-\( P_{\ell} \) where \( P_{\ell} = (e_1, \ldots, e_{\ell-1}) \), and \( E_i = f(e_i) \). We say \( F \) contains a Berge-\( G \) if \( F \) contains a sub-hypergraph that is a Berge-\( G \).
we have that

Proof of Theorem 1.5. Let 

Observe that 

Note that a connected hypergraph has the property that between any two vertices, there is a Berge path.

Let 

be such that 

Replace 

Further, 

Let us use induction on 

Thus, if \( \Gamma \) is a \( t \)-uniform hypergraph on \( k \) vertices with \( e(\Gamma) \geq (1-\epsilon)(\frac{k}{\alpha})^t \),

Figure 2: A Berge-\( P_4 \) path \( P = (E_1, E_2, E_3, E_4) \) between \( V_1 \) and \( V_{10} \) with \( P_4 = (\{V_1, V_2\}, \{V_4, V_3\}, \{V_5, V_7\}, \{V_7, V_{10}\}) \). Observe that \( P \) contains no shorter Berge path between \( V_1 \) and \( V_{10} \).

Lemma 3.4. Let \( P = (E_1, \ldots, E_\ell) \) be a Berge path in the cluster graph \( R \) of \( H \) with \( |V(R)| = m \) and edges of density \( \epsilon \). Assume that \( P \) contains no shorter Berge path that connects the two endpoints of \( P \), which we will call \( V_1 \) and \( V_s \). Since there is no shorter Berge path, any two nonconsecutive edges \( E_i \) and \( E_j \) must be disjoint, and so there is a labeling \( V_1, \ldots, V_s \) of the vertices of \( V(P) \) such that \( E_1 = \{V_1, \ldots, V_k\} \), \( E_\ell = \{V_{s-k+1}, \ldots, V_s\} \), and for each \( i \) all the vertices of \( E_i \setminus E_{i+1} \) come before the vertices of \( E_i \cap E_{i+1} \), which come before the vertices of \( E_{i+1} \setminus E_i \) (see Figure 2).

Let \( P = (E_1, \ldots, E_\ell) \) be a Berge path in the cluster graph \( R \) of \( H \) with \( |V(R)| = m \) and edges of density \( \epsilon \). Assume that \( P \) contains no shorter Berge path that connects the two endpoints of \( P \), which we will call \( V_1 \) and \( V_s \). Since there is no shorter Berge path, any two nonconsecutive edges \( E_i \) and \( E_j \) must be disjoint, and so there is a labeling \( V_1, \ldots, V_s \) of the vertices of \( V(P) \) such that \( E_1 = \{V_1, \ldots, V_k\} \), \( E_\ell = \{V_{s-k+1}, \ldots, V_s\} \), and for each \( i \) all the vertices of \( E_i \setminus E_{i+1} \) come before the vertices of \( E_i \cap E_{i+1} \), which come before the vertices of \( E_{i+1} \setminus E_i \) (see Figure 2).

Then, there is a loose path in \( H \) that goes from a vertex in \( V_1 \) to a vertex in \( V_s \) using only vertices from \( \bigcup_{i=1}^s U_i \).

Proof. Let us use induction on \( \ell \). For \( \ell = 1 \) then the statement follows from \( \epsilon \)-regularity. Now assume the statement holds for \( \ell - 1 \) and consider such a path \( P \) of length \( \ell \). Note that by construction of the sequence of vertices \( V_1, \ldots, V_s \), we have \( V_{s-k+1} \in E_{\ell-1} \cap E_\ell \). Further, \( P - E_\ell \) is a shortest Berge path connecting \( V_1 \) to \( V_{s-k+1} \) and \( U_1, \ldots, U_{s-k+1} \) satisfy the requirements of the theorem, so by the inductive hypothesis, there is a loose path from \( V_1 \) to \( V_{s-k+1} \) contained in \( \bigcup_{i=1}^{s-k+1} U_i \). Let \( v_1 \in V_{s-k+1} \) be the last vertex in this path. If we replace \( U_{s-k+1} \) with \( U_{s-k+1} \setminus \{v_1\} \), then we can apply the inductive hypothesis again and find another loose path from \( V_1 \) to \( V_{s-k+1} \) that does not use \( v_1 \). Since \( |U_{s-k+1}| \geq 2\sqrt{\epsilon} m \), we can repeat this process to find at least \( \sqrt{\epsilon} m \) loose paths from \( V_1 \) to \( V_{s-k+1} \) which all have distinct endpoints in \( V_{s-k+1} \). Let \( X \) be the set of vertices in these at least \( \sqrt{\epsilon} m \) paths.

Observe that \( |X \cap U_s| = 0 \) since \( P - E_\ell \) does not reach \( U_s \), and for each \( s-k+2 \leq i \leq s-1 \) we have that \( |X \cap U_i| \leq \sqrt{\epsilon} \) since each path hits at most one vertex from each cluster, so for all \( s-k+2 \leq i \leq s \), \( |U_i \setminus X| \geq \sqrt{\epsilon} m \) and \( |U_{s-k+1} \cap X| \geq \sqrt{\epsilon} m \). So there is an edge in \( H[U_{s-k+1} \cap X, U_{s-k+2} \setminus X, \ldots, U_s \setminus X] \). This edge, along with whichever path from \( V_1 \) to \( V_{s-k+1} \) is incident to it gives us the desired loose path from \( V_1 \) to \( V_s \).

Now we prove the main result of this section.

Proof of Theorem 1.5 Let \( r, k, \alpha, \) and \( \psi = \psi(r, k) \) be given. Set \( \alpha^* = \alpha/2 \). Let \( \epsilon^* \) and \( t_0 \) be the constants guaranteed by the values of \( k, r, \) and \( \alpha^* \). Let \( \epsilon < \min\{1/(2r), ((\alpha/(14\psi))^k, \epsilon^*) \} \).

Thus, if \( \Gamma \) is a \( k \)-uniform hypergraph on \( t \geq t_0 \) vertices with \( e(\Gamma) \geq (1-\epsilon)(\frac{k}{\alpha})^t \),
then $\Gamma$ contains a monochromatic connected loose cycle packing on at least $(\psi - \alpha/2)t$ vertices for every $r$-edge-coloring.

Apply Theorem 3.1 with $k$, $r$, $t_0$, $\epsilon$, as above, and $D = 2$. Let $\eta$, $T_0$ and $N_0$ be the constants that arise. Let $n \geq N_0$ be large. Let $H = H^{(k)}(n, p)$. Color the edges of $H$ with $r$ colors. Let $H_i$ be the sub-hypergraph of $H$ colored $i$ for $1 \leq i \leq r$. As in the proof of Theorem 1.2 it follows from the Chernoff bound that each $H_i$ is ($\eta$, $p$, $D$)-uniform. Then Theorem 3.1 gives us that $H_1, \ldots, H_r$ admits an $(\epsilon, p)$-regular partition.

Let $R$ be the $k$-uniform cluster graph of $H$ with clusters $V_1, \ldots, V_t$ each of size $m = n/t$. Color each edge $\{V_{i_1}, \ldots, V_{i_k}\}$ of $R$ with a majority color in the $k$-partite graph $H[V_{i_1}, \ldots, V_{i_k}]$. Let $H'$ be the sub-hypergraph colored by this majority color in $H[V_{i_1}, \ldots, V_{i_k}]$. Then again as in the proof of Theorem 1.2 we have $d_{H', p}(V_{i_1}, \ldots, V_{i_k}) \geq \epsilon$.

Since $t \geq t_0$, and $|E(R)| \geq (1 - \epsilon)(\eta)$, $R$ contains a monochromatic connected loose cycle packing, $C_1, \ldots, C_t$, on at most $(\psi - \alpha/2)t$ clusters. Let $f = \sum_i |E(C_i)| \geq (\psi - \alpha/2)t/(k - 1)$ be the number of edges in the packing. Let $E_1, \ldots, E_f$ be an enumeration of these edges. There is a bijection between the edges in the cycle packing and the vertices of degree $2$ in the packing. We may assume without loss of generality that $V_i$ is the vertex corresponding to $E_i$ for each $1 \leq i \leq f$. Partition $V_i$ into two equal sized sets $V_{i,1}$ and $V_{i,2}$ for each $1 \leq i \leq f$.

Fix $1 \leq i \leq f$ for a moment. Let $V_j \neq V_i$ be the second vertex of degree $2$ in $E_i$. Set $X_1 = V_{i,1}$, $X_k = V_{j,2}$ and let $X_2, \ldots, X_{k-1}$ be the clusters in $E_i \\setminus \{V_i, V_j\}$. Set $\zeta = \sqrt[4]{\epsilon}$. Then Lemma 3.3 gives us that there is a loose path, call it $P_i$, on exactly $(1 - 4\sqrt[4]{\epsilon})m - 2$ edges in $H'[X_1, \ldots, X_k]$. We can do this for each $1 \leq i \leq f$ to find $f$ loose paths that we call long. Let $U_{i,1}$ and $U_{i,2}$ be the first and last $\sqrt[4]{\epsilon}m$ vertices along $P_i$ in $V_i \cap V(P_i)$ respectively.

For each $V_i$, let $\tau(V_i)$ denote the set of vertices in $V_i$ that are not in any of the $f$ long paths. Notice that if $V_i$ is in the cycle packing, $\tau(V_i) \geq 4\sqrt[4]{\epsilon}m$ by the length of each $P_i$ and if $V_i$ is not in the cycle packing, $\tau(V_i) = |V_i| = m \geq 4\sqrt[4]{\epsilon}m$. Let $\mathcal{L}$ be the monochromatic component of $R$ containing the cycle packing.

We now show how to find a loose path that connect the end of the path $P_i$ to the beginning of $P_{i+1}$ for each $1 \leq i \leq f$. Assume we have found vertex disjoint loose paths $Q_1, \ldots, Q_{i-1}$ in $H'$ such that $Q_j$ connects a vertex in $U_{j,2} \subseteq V_j$ with a vertex in $U_{j+1,1} \subseteq V_{j+1}$ and does not intersect any other vertices in $\bigcup_{j=1}^{i-1} V(P_j)$. Furthermore, let $V_i^{-1} = \bigcup_{j=1}^{i-1} V(Q_j)$ and assume that $|\mathcal{V}_{i-1} \cap \tau(V_j)| \leq 2(i - 1)$ for each $1 \leq j \leq t$. Now we will construct a path $Q_i$ as follows.

Let $\Pi$ be the shortest Berge path from $V_i$ to $V_{i+1}$ in $\mathcal{L}$. Let $s$ be the number of clusters in $\Pi$. Let $V_{j_1} = V_i$ and $V_{j_k} = V_{i+1}$ and let $V_{j_2}, \ldots, V_{j_{s-1}}$ be the other $s - 2$ clusters in $\Pi$. Let $U_{j_1} = U_{i,2}$, $U_{j_s} = U_{i+1,1}$ and $U_{j_2} = \tau(V_{j_2}) \\setminus V_{i-1}, \ldots, U_{j_{s-1}} = \tau(V_{j_{s-1}}) \\setminus V_{i-1}$. Then we have $|U_{j_1}|, |U_{j_s}| \geq \sqrt[4]{\epsilon}m$ (by definition of $U_{i,2}$ and $U_{i+1,1}$) and for each $1 \leq \ell \leq s - 1,$

$$|U_{j_\ell}| \geq \tau(V_{j_\ell}) - 2f \geq 4\sqrt[4]{\epsilon}m - 2\sqrt[4]{\epsilon}m = 2\sqrt[4]{\epsilon}m,$$

since $f \leq \sqrt[4]{\epsilon}m$ for sufficiently large $n$ (and so $m$). Hence, by Lemma 3.4 we have a loose path $Q_i$ from $U_{i,2}$ to $U_{i+1,1}$ that is disjoint from each short loose path found previously and that does not intersect any of the long paths except $P_i$ and $P_{i+1}$ at a single vertex in $U_{i,2}$ and $U_{i+1,1}$, respectively. Finally, observe that for $V_i = \bigcup_{j=1}^{i} V(Q_j)$ we still have that

$$|\mathcal{V}_i \cap \tau(V_j)| = |\mathcal{V}_{i-1} \cap \tau(V_j)| + |V(Q_i) \cap \tau(V_j)| \leq 2(i - 1) + 2 = 2i$$
for each $1 \leq j \leq t$, since if $Q_i$ contained three or more vertices from a single cluster $V_j$, the corresponding Berge path $\Pi$ would not have been a shortest Berge path.

Notice then these short paths $Q_i$’s together with the long paths $P_i$’s give us a long loose cycle, $C$. By our choice for each $U_{i,1}$ and $U_{i,2}$, our cycle uses all the edges from each $P_i$ except at most $\frac{\sqrt{\epsilon}m}{k}$ from each the beginning of the path and the end. There are $f$ long paths, so the total number of edges in our cycle is at least

$$|E(C)| \geq \sum_{i=1}^{f}(|E(P_i)| - 2\frac{\sqrt{\epsilon}m}{k}) \geq f(1 - 7\frac{\sqrt{\epsilon}}{k})m \geq \frac{(\psi - \alpha/2)t}{k - 1}(1 - 7\frac{\sqrt{\epsilon}}{k})m \geq \frac{\psi - \alpha}{k - 1}n,$$

where the last inequality uses $\epsilon < (\alpha/(14\psi))^k$. This immediately implies that $|V(C)| \geq (\psi - \alpha)n$, as required. 

The following result was proved by Gyárfás, Sárközy, and Szemerédi [11].

**Theorem 3.5** ([11]). Suppose that $k$ is fixed and the edges of an almost complete $k$-uniform hypergraph on $n$ vertices are 2-colored. Then there is a monochromatic connected diamond matching $cD_k$ such that $|V(cD_k)| = (1 + o(1))\frac{2k-2}{2k-1}n$.

Combining Theorem 3.5 with Theorem 1.5, we immediately get a proof of Corollary 1.6, which is best possible as observed in [11].

**Observation 3.6.** There exists a 2-coloring of the edges of the complete $k$-uniform hypergraph $K^k_n = (V, E)$ such that the longest loose cycle covers no more than $\frac{2k-2}{2k-1}n$ vertices.

**Proof.** Choose any set $S$ of $\frac{2k-2}{2k-1}n$ vertices, and color all the edges completely inside $S$ red, and the rest of the edges blue. Clearly the longest red loose cycle has no more than $\frac{2k-2}{2k-1}n$ vertices. For the longest blue path, notice that each vertex in $V \setminus S$ can be in at most two edges, and each edge must contain one such vertex. Thus the longest possible cycle is of length $\frac{2}{2k-1}n$ or order $\frac{2k-2}{2k-1}n$. 

4 \hspace{1cm} $k$-uniform hypergraphs colored with $k$ colors

**Theorem 4.1.** Let $k \geq 3$ and let $0 < \epsilon < 16^{-k}$. Then there exists some $n_0$ such that for any $k$-uniform hypergraph $G$ on $n > n_0$ vertices with $e(G) > (1 - \epsilon)\binom{n}{k}$, we have $mc_k(G) \geq (1 - 8\sqrt{\epsilon})n$.

**Proof.** Let $G$ be a $k$-uniform hypergraph on $n$ vertices with $e(G) > (1 - \epsilon)\binom{n}{k}$ and suppose that $mc_k(G) < (1 - 8\sqrt{\epsilon})n$. Let $G$ be $k$-colored in such a way that the largest monochromatic component of $G$ is of size $mc_k(G)$.

First, for each $i \in [k]$, we find a partition $\{A_i, B_i\}$ of $V(G)$ such that $(1 - 4\sqrt{\epsilon})n \geq |A_i|, |B_i| \geq 4\sqrt{\epsilon}n$ and there are no edges of color $i$ which are incident with both a vertex in $A_i$ and a vertex in $B_i$. To find such a partition, let $C_i$ be the largest component of color $i$. By assumption $|V(C_i)| < (1 - 8\sqrt{\epsilon})n < (1 - 4\sqrt{\epsilon})n$. If $|V(C_i)| \geq 4\sqrt{\epsilon}n$, then we can let $A_i = V(C_i)$ and $B_i = V(G) \setminus V(C_i)$. If $|V(C_i)| \leq 4\sqrt{\epsilon}n$, then some union of components of color $i$, call it $C^*_i$, will have the property that $4\sqrt{\epsilon}n \leq |V(C^*_i)| \leq 4\sqrt{\epsilon}n + 4\sqrt{\epsilon}n \leq n/2$, since $\epsilon < 1/16^k$. (We can always choose $C^*_i$ such that $|V(C^*_i)| \geq 4\sqrt{\epsilon}n$, since we treat
vertices that are isolated with respect to color $i$ as separate components.) Here we can let $A_i = V(C_i^*)$ and $B_i = V(G) \setminus V(C_i^*)$.

We can also assume that $|A_1 \cap A_2| \geq 2 \sqrt[n]{2n} \cap \ell \geq 2 \sqrt[n]{2n}$. Indeed, if $|A_1 \cap A_2| < 2 \sqrt[n]{2n}$, then $|B_1 \cap B_2| > 2 \sqrt[n]{2n}$. In this case, we can switch the roles of $A_2$ and $B_2$ and now we have $|B_1 \cap B_2| > 2 \sqrt[n]{2n}$ and $|A_1 \cap A_2| = |A_1| - |A_1 \cap B_2| > |A_1| - 2 \sqrt[n]{2n} \geq 2 \sqrt[n]{2n}$. Similarly, if $|B_1 \cap B_2| < 2 \sqrt[n]{2n}$, switching the roles of $A_2$ and $B_2$ gives us the desired inequalities. Thus we can always assume that $|A_1 \cap A_2| \geq 2 \sqrt[n]{2n}$ and $|B_1 \cap B_2| \geq 2 \sqrt[n]{2n}$.

Furthermore, we can assume that $|A_1 \cap A_2 \cap \cdots \cap A_k| \geq 2 \sqrt[n]{2n/k^2}$. Indeed, we already have that $|A_1 \cap A_2| \geq 2 \sqrt[n]{2n}$. Now for a fixed $i$ satisfying $3 \leq i \leq k$ assume that $|A_1 \cap \cdots \cap A_{i-1}| \geq 2 \sqrt[n]{2n} / 2^{i-3}$. Since $\{A_1 \cap (A_1 \cap \cdots \cap A_{i-1}), B_i \cap (A_1 \cap \cdots \cap A_{i-1})\}$ is a partition of $A_1 \cap \cdots \cap A_{i-1}$, if $A_i$ does not cover at least half of vertices in $A_1 \cap \cdots \cap A_{i-1}$, then $B_i$ does, so switching the roles of $A_i$ and $B_i$ if necessary will give us that $|A_1 \cap \cdots \cap A_i| \geq 2 \sqrt[n]{2n} / 2^{i-2}$. Thus, we can recursively choose $A_i$ such that $|A_1 \cap A_2 \cap \cdots \cap A_k| \geq 2 \sqrt[n]{2n/k^2}$.

Now observe that the number of $k$-tuples of distinct vertices $(x_1, \ldots, x_k)$ where $x_1 \in A_1 \cap A_2 \cap \cdots \cap A_k$, $x_2 \in B_1 \cap B_2$, and $x_i \in B_i$ for all $3 \leq i \leq k$, is at least

$$2 \sqrt[n]{2n/k^2} \cdot (2 \sqrt[n]{2n} - 1) \cdot (4 \sqrt[n]{2n} - 2) \cdot (4 \sqrt[n]{2n} - k - 1) = 2^k n^k + O(n^{k-1}).$$

(2)

Note that any such $k$-tuple intersects every set $A_1, \ldots, A_k, B_1, \ldots, B_k$ and thus no such $k$-tuple can be an edge of $G$ since if it were an edge of $G$, it would be colored, say by color $i$, and thus cannot contain vertices from both $A_i$ and $B_i$. Since there are less than $\epsilon n^k$ non-edges in $G$, there are less than $\epsilon n^k$ ordered $k$-tuples of distinct vertices $(x_1, \ldots, x_k)$ where $\{x_1, \ldots, x_k\}$ is not an edge, which contradicts (2) for $n$ sufficiently large.

\section{5 $k$-uniform hypergraphs colored with $k+1$ colors}

The following proof makes use of some technical lemmas which can all be found in Section \[\[\]

\begin{thm}
Let $k \geq 3$ and $0 < \epsilon < \frac{1}{k^{k+2}n}$. If $G$ is a $k$-uniform hypergraph on $n$ vertices with $\delta(G) \geq (1 - \epsilon)\binom{n-1}{k-1}$, then $mc_{k+1}(G) \geq \left(\frac{k}{k+1} - \sqrt{k\epsilon}\right)n$.
\end{thm}

\begin{proof}
Suppose that the edges of $G$ are $(k+1)$-colored. First note for all $v \in V(G)$, the link graph of $v$ is a $(k+1)$-colored $(k-1)$-uniform hypergraph on $n-1$ vertices with at least $(1-\epsilon)\binom{n-1}{k-1}$ edges and thus by Lemma \[\[\] applied with $k-1$ in place of $k$ and $\ell = 2$, we get a monochromatic 1-core in the link graph of $v$ of order at least $\left(\frac{k-1}{k+1} - \sqrt{k\epsilon}\right)(n-1)$. Since all these vertices are connected via $v$, this gives us a monochromatic component containing $v$ of order at least $\left(\frac{k-1}{k+1} - \sqrt{k\epsilon}\right)(n-1)+1 \geq \left(\frac{k-1}{k+1} - \sqrt{k\epsilon}\right)n$. Now let $C_1$ be the largest monochromatic component in $G$, and note that by the calculation above,

$$|C_1| \geq \left(\frac{k-1}{k+1} - \sqrt{k\epsilon}\right)n.$$

For ease of reading, set $\alpha = k\epsilon$. For any vertex $v \in V(G) \setminus C_1$, we have

$$d(v,C_1) \geq \left(\frac{|C_1|}{k-1}\right) - \epsilon \left(\frac{n-1}{k-1}\right) \geq \left(1 - \frac{k\epsilon}{(k+1)\left(\sqrt{k\epsilon}\right)}\right) \geq (1-\alpha)\left(\frac{|C_1|}{k-1}\right).$$

(3)

\end{proof}
where the second inequality holds by Observation 6.1 (applied with $U = C_1$ and $\lambda = \frac{k-1}{k+1} - \sqrt{\epsilon}$) and the last inequality holds since $\frac{k-1}{k+1} - \sqrt{\epsilon} \geq \frac{1}{2}$.

Now choose a monochromatic component $C_2$ so that $|C_1 \cap C_2|$ is maximized. We claim that

$$|C_1 \cap C_2| \geq \left( \frac{k-1}{k} - \sqrt{\alpha} \right) |C_1|.$$ 

To see this, let $v \in V(G) \setminus C_1$ and note that the link graph of $v$ restricted to $C_1$ is a $(k-1)$-uniform hypergraph on $|C_1|$ vertices with at least $(1 - \alpha)\binom{|C_1|}{k-1}$ edges (by (3)). Furthermore, note that the the link graph of $v$ restricted to $C_1$ only uses $k$ colors since the color of $C_1$ cannot show up by our choice of $v \notin C_1$. Thus by Lemma 6.3 (with $k$ replaced by $k - 1$ and $\ell = 1$) there is a monochromatic component $C_2$ containing $v$ such that $|C_1 \cap C_2| \geq \left( \frac{k-1}{k} - \sqrt{\alpha} \right) |C_1|$, thus proving the claim.

Now if $V \setminus C_1 \subseteq C_2$, then we have

$$|C_1| \geq |C_2| = n - |C_1| + |C_1 \cap C_2| \geq n - |C_1| + \left( \frac{k-1}{k} - \sqrt{\alpha} \right) |C_1|,$n

which implies

$$|C_1| \geq \frac{n}{k+1} + \sqrt{\alpha} \geq \left( \frac{k}{k+1} - \sqrt{\alpha} \right) n;$$

and we are done, so suppose not.

We now show that $|C_1 \setminus C_2|$ must be small (in other words, $|C_1 \cap C_2|$ must be big); more precisely, we will show that

$$|C_1 \setminus C_2| \leq 8^{2(k-\sqrt{\alpha})}|C_1|. \quad (4)$$

Let $v \in V(G) \setminus (C_1 \cup C_2)$ and consider the link graph $L'$ obtained from $L(v, C_1)$ by deleting all edges entirely contained in $C_1 \setminus C_2$.

Notice that $L(v, C_1)$ is a $(k-1)$-uniform hypergraph with at least $(1 - \alpha)\binom{|C_1|}{k-1}$ edges (by (3)). Furthermore, $L'$ is $(k-1)$-colored since it cannot contain any edges whose color matches $C_1$, and if there was an edge in $L(v, C_1)$ whose color matched $C_2$, it would need to be contained in $C_1 \setminus C_2$, but such edges do not exist in $L'$. Thus by Lemma 6.4 (applied with $G' = L(v, C_1)$, $G = L'$, $A = C_1 \cap C_2$ and $B = C_1 \setminus C_2$) either there is a monochromatic component containing $v$ and at least $(1 - 8^{2(k-\sqrt{\alpha})})|C_1|$ vertices of $C_1$, in which case (4) holds since $C_2$ was assumed to be the component that maximizes $|C_1 \cap C_2|$, or there is a monochromatic component containing $v$ and at least

$$|C_1 \cap C_2| - \sqrt{\alpha}|C_1| + \left( \frac{k-2}{k-1} - \sqrt{k-1} \frac{4(k-1)\sqrt{\alpha}}{2^{k-2}} \right) |C_1 | \setminus C_2| \quad (5)$$

vertices of $C_1$. Observe that it cannot be the case that

$$\left( \frac{k-2}{k-1} - \sqrt{k-1} \frac{4(k-1)\sqrt{\alpha}}{2^{k-2}} \right) |C_1 | \setminus C_2| > \sqrt{\alpha}|C_1|,$$

since this and (5) would imply that we have a monochromatic component containing $v$ and more than $|C_1 \cap C_2|$ vertices of $C_1$, which contradicts the choice of $C_2$. So it must be that
\[
\left(\frac{k-2}{k-1} - \frac{\sqrt{k-1} \sqrt[4]{1-\sqrt[4]{\frac{1}{k-2}}}}{2^{k-2}}\right) |C_1 \setminus C_2| \leq \sqrt{\alpha}|C_1|,
\]
which implies
\[
|C_1 \setminus C_2| \leq \frac{\sqrt{\alpha}|C_1|}{\frac{k-2}{k-1} - \frac{\sqrt{k-1} \sqrt[4]{1-\sqrt[4]{\frac{1}{k-2}}}}{2^{k-2}}}.
\]
(6)

We further claim that
\[
\frac{\sqrt{\alpha}|C_1|}{\frac{k-2}{k-1} - \frac{\sqrt{k-1} \sqrt[4]{1-\sqrt[4]{\frac{1}{k-2}}}}{2^{k-2}}} < 8 \cdot 2^{(k-1)\sqrt{\alpha}|C_1|}.
\]
(7)

Indeed, first note that since \(\alpha < 2^{-8k}\) and \(k \geq 3\), we get \(\sqrt[4]{1-\sqrt[4]{\frac{1}{k-2}}} < 2^{-2k/(k-1)} \leq 1/8\), \(\sqrt{k-1} \leq 1\) and \(\frac{k-2}{k-1} \geq 1/2\). Consequently,
\[
\frac{k-2}{k-1} - \frac{\sqrt{k-1} \sqrt[4]{1-\sqrt[4]{\frac{1}{k-2}}}}{2^{k-2}} \geq \frac{1}{2} - \frac{1}{8} > \frac{1}{8},
\]
which together with \(\sqrt{\alpha} \leq 2^{(k-1)\sqrt{\alpha}}\) yields (7). Thus by (6) and (7),
\[
|C_1 \setminus C_2| < 8 \cdot 2^{(k-1)\sqrt{\alpha}|C_1|}
\]
as desired.

Now that we have established (4), we will show that any vertex in \(V(G) \setminus C_1\) is in some monochromatic component that has large intersection with \(C_1\). More precisely, for all \(v \in V(G) \setminus C_1\), there exist some monochromatic component \(C \ni v\) such that \(|C \cap C_1| \geq (1 - 9 \cdot 2^{(k-1)\sqrt{\alpha}})|C_1|\). Indeed, let \(v \in V(G) \setminus C_1\). If \(v \in C_2\), then (4) implies that \(C = C_2\) satisfies. Otherwise, as in (4) and (5) (with \(C_2\) replaced by \(C\)) we get that there is a monochromatic component containing \(v\) and at least
\[
|C_1 \cap C| - \sqrt{\alpha}|C_1| = |C_1| - |C_1 \setminus C| - \sqrt{\alpha}|C_1| \\
\geq (1 - 8 \cdot 2^{(k-1)\sqrt{\alpha}} - \sqrt{\alpha})|C_1| \geq (1 - 9 \cdot 2^{(k-1)\sqrt{\alpha}})|C_1|
\]
vertices of \(C_1\). Also observe that since \((1 - 9 \cdot 2^{(k-1)\sqrt{\alpha}})|C_1| > |C_1|/2\), any two such components \(C\) intersect with each other.

Since there are only \(k\) possible colors for edges between \(C_1\) and \(V(G) \setminus C_1\), there must exist at least \((n - |C_1|)/k\) vertices in \(V(G) \setminus C_1\) that are all in some monochromatic component \(C^*\) together along with at least \(1 - 9 \cdot 2^{(k-1)\sqrt{\alpha}}|C_1|\) vertices of \(C_1\). Thus,
\[
|C_1| \geq |C^*| \geq \frac{n - |C_1|}{k} + (1 - 9 \cdot 2^{(k-1)\sqrt{\alpha}})|C_1|,
\]
where the first inequality holds by the maximality of \(C_1\). So solving for \(C_1\) we finally obtain that
\[
|C_1| \geq \frac{n}{k \left(\frac{1}{k} + 9 \cdot 2^{(k-1)\sqrt{\alpha}}\right)} > \left(\frac{k}{k+1} - \sqrt{\alpha}\right)n,
\]
where the final inequality follows from the upper bound on \(\epsilon\) (and so on \(\alpha\)).

\textbf{Corollary 5.2.} Let \(k \geq 3\) and \(0 < \epsilon < \frac{1}{2k \cdot 2^{k+1}}\). If \(G\) is a \(k\)-uniform hypergraph on \(n\) vertices with \(\epsilon(G) \geq (1-\epsilon)(\binom{n}{k})\), then \(\text{mc}_{k+1}(G) \geq \left(\frac{k}{k+1} - \sqrt{\epsilon}\right)n\).
In particular, if get that all but at most \(|V'| \geq (1 - \sqrt{\epsilon})n\) and \(\delta(G') \geq (1 - 2k\sqrt{\epsilon})\left(\frac{1}{k} - 1\right)\). Now by Theorem 5.1 we have

\[
mc_{k+1}(G) \geq mc_{k+1}(G') \geq \left(\frac{k}{k+1} - \sqrt{2k \frac{k+1}{2}\sqrt{\epsilon}}\right)(1 - \sqrt{\epsilon})n \geq \left(\frac{k}{k+1} - 2k^{k+1}\sqrt{\epsilon}\right)n.
\]

\[\square\]

6 Technical lemmas

The first two observations are simple counting arguments, but it is convenient for us to state them explicitly.

Observation 6.1. Let \(n \geq k \geq 2\), let \(0 < \epsilon, \lambda \leq 1\). If \(|U| = \lambda n \geq k^2\), then

\[
\left(\frac{|U| - 1}{k - 1}\right) - \epsilon\left(\frac{n - 1}{k - 1}\right) \geq \left(1 - \frac{k\epsilon}{\lambda^{k-1}}\right)\left(\frac{|U| - 1}{k - 1}\right).
\]

Proof. First we show that \(\lambda n \geq k^2\) implies that \(\frac{n - (k - 1)}{\lambda n - (k - 1)} \leq \frac{k - \sqrt{k}}{\lambda}\). By Bernoulli’s inequality, \((1 + x)^r \geq 1 + xr\) for all real numbers \(r \geq 1\) and \(x \geq -1\), (see, e.g., inequality 58 in [12]) we get that

\[
k^{k/(k-1)} = (1 + (k - 1))^{k/(k-1)} \geq 1 + k,
\]

and equivalently, \(\frac{k - \sqrt{k}}{\lambda} \geq 1 + 1/k\). Thus,

\[
k^{k/(k-1)} - \frac{1}{k} = 1 + \frac{k - 1}{k(k - 1)} \geq 1 + \frac{k - 1}{k^2 - k + 1}
\]

\[
\geq 1 + \frac{k - 1}{\lambda n - k + 1} = \frac{\lambda n}{\lambda n - (k - 1)} \geq \frac{\lambda n - \lambda (k - 1)}{\lambda n - (k - 1)}.
\]

Now,

\[
\left(\frac{|U| - 1}{k - 1}\right) - \epsilon\left(\frac{n - 1}{k - 1}\right) = \left(\frac{|U| - 1}{k - 1}\right) - \epsilon\left(\frac{|U| - 1}{k - 1}\right)\cdot \left(\frac{n - 1}{k - 1}\right) = \left(\frac{|U| - 1}{k - 1}\right) - \epsilon\left(\frac{|U| - 1}{k - 1}\right)\left(\frac{n - (k - 1)}{\lambda n - (k - 1)}\right)^{k-1}
\]

\[
\geq \left(1 - \frac{k\epsilon}{\lambda^{k-1}}\right)\left(\frac{|U| - 1}{k - 1}\right).
\]

\[\square\]

Observation 6.2. Let \(k \geq 2\), let \(0 < \eta < 1\), and let \(0 < \epsilon \leq (1 - k\sqrt{1/k})^{1/(1-\eta)}\). Let \(G = (V, E)\) be a \(k\)-uniform hypergraph on \(n\) vertices with \(e(G) \geq (1 - \epsilon)(\binom{n}{k})\). For all \(U \subseteq V\) all but at most \(\epsilon^{1-\eta}n\) vertices have

\[
d(v, U) \geq \left(\frac{|U| - 1}{k - 1}\right) - \eta\left(\frac{n - 1}{k - 1}\right).
\]

In particular, if \((1 - \epsilon^{1-\eta})n \geq k^2\), then there exists an induced subgraph \(G' = (V', E')\) with \(|V'| \geq (1 - \epsilon^{1-\eta})n\) and \(\delta(G') \geq (1 - 2k\epsilon^\eta)\left(\frac{|V'| - 1}{k - 1}\right)\).

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Proof. Let \( V^* \) = \( \{ v \in V : d(v) < (\binom{n-1}{k-1}) - e(\binom{n-1}{k-1}) \} \). We have
\[
|V^*| \leq e(G) \leq \binom{n}{k},
\]
which implies \( |V^*| \leq \epsilon^{1-\eta}n \). So let \( U \subseteq V \) with \( |U| = \lambda n \geq k^2 \) and note that for all \( v \in V \setminus V^* \) we have
\[
d(v, U) \geq \left( \frac{|U| - 1}{k - 1} \right) - \epsilon^{1-\eta} \binom{n}{k-1},
\]
as required.

To see the final statement, let \( V' = V \setminus V^* \) and note that \( |V'| \geq (1 - \epsilon^{1-\eta})n \) by the calculation of \( |V^*| \) above. From (9) (applied with \( U = V' \)) we have
\[
\delta(G[V']) \geq \left( \frac{|V'| - 1}{k - 1} \right) - \epsilon^{1-\eta} \binom{n}{k-1}.
\]
Now Observation 6.1 with \( \lambda = 1 - \epsilon^{1-\eta} \) yields
\[
\delta(G[V']) \geq \left( 1 - \frac{k \epsilon^{1-\eta}}{1 - \epsilon^{1-\eta}} \right) \left( \frac{|V'| - 1}{k - 1} \right) \geq (1 - 2k \epsilon^{1-\eta}) \binom{n}{k-1},
\]
where the last inequality holds since \( 1 - \epsilon^{1-\eta} \geq k - \sqrt{1/2} \).

\[ \square \]

Lemma 6.3. Let \( k \geq 2, \ell \geq 1, \) and \( 0 < \epsilon < 256^{-k} \). Then there exists \( n_0 > 0 \), a sufficiently large integer, such that if \( G = (V, E) \) is a \((k + \ell)\)-colored \( k \)-uniform hypergraph on \( n \geq n_0 \) vertices with \( e(G) > (1 - \epsilon)(\binom{n}{k}) \), then \( G \) contains a monochromatic 1-core on at least \( (\frac{k}{k+\ell} - \sqrt{\epsilon})n \) vertices.

Proof. By (5) in Observation 6.2 (applied with \( \eta = 1/2, \lambda = 1, \) and \( U = V \)) at least \( (1 - \sqrt{\epsilon})n \) vertices have degree at least \( (1 - \sqrt{\epsilon}) \binom{n}{k-1} \). Call these vertices \( V' \).

Suppose that the 1-core of each of the \( k + \ell \) colors has order less than \( (\frac{k}{k+\ell} - \sqrt{\epsilon})n \); that is, for each color \( i \in [k + \ell] \), more than \( \left( \frac{\ell}{k+\ell} + \sqrt{\epsilon} \right) n \) vertices are not incident with an edge of color \( i \). Let \( X \) be the set of vertices that are not incident with at least \( \ell + 1 \) colors. We claim that \( |X| \geq \frac{k+\ell}{k} \sqrt{\epsilon}n \geq \sqrt{\epsilon}n \). Indeed, let \( T \subseteq V \times [k + \ell] \) be the set of ordered pairs where \((v, \ell) \in T \) if the vertex \( v \) is not incident with any edge of color \( \ell \). Since for each \( i \in [k + \ell] \), there are more than \( \left( \frac{\ell}{k+\ell} + \sqrt{\epsilon} \right) n \) vertices that are not incident with an edge of color \( i \), we have that
\[
|T| \geq (k + \ell) \left( \frac{\ell}{k+\ell} + \sqrt{\epsilon} \right) n = \ell n + \sqrt{\epsilon}(k + \ell)n.
\]

Now each vertex of \( V \setminus X \) contributes no more than \( \ell \) ordered pairs to the set \( T \), and each vertex in \( X \) can contribute no more than \( k + \ell \), so we have that
\[
|T| \leq (n - |X|)\ell + |X|(k + \ell) = \ell n + k |X|.
\]

\

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Combining (10) and (11) gives us that \( \ell n + k|X| \geq \ell n + \sqrt{e}(k + \ell)n \), which implies that 
\[ |X| \geq \frac{k + \ell}{k} \sqrt{en} \geq \sqrt{en}, \] 
which proves the claim.

By the bounds on \(|X|\) and \(|V'|\) there exists \( v \in V' \cap X \), and the link graph of \( v \) is a 
\((k - 1)\)-uniform hypergraph on \( n - 1 \) vertices with at least \((1 - \sqrt{e})(n-1)^{k-1}\) edges which is 
colored with at most \( k - 1 \) colors and thus by Theorem 4.1 there exists a monochromatic component on at least 
\[
(1 - 8 \sqrt{\epsilon})(n - 1) \geq \left( \frac{k}{k + \ell} - \sqrt{\epsilon} \right) n
\]
vertices, where the above inequality holds by the choice of \( \epsilon \). So, the 1-core of some color 
must have order at least \((k / (k + \ell) - \sqrt{\epsilon})n\).

Lemma 6.4. Let \( k \geq 2 \) and \( 0 < \epsilon \leq 512^{-k} \). Then there exists a sufficiently large integer \( n_0 \) 
such that the following holds. Let \( G \) be a hypergraph obtained from a \( k \)-uniform hypergraph 
\( G' \) on \( n \geq n_0 \) vertices with \( e(G') \geq (1 - \epsilon)n^k \) by letting \( \{A, B\} \) be a partition (chosen 
arbitrarily) of \( V(G') \) with \(|A| > \sqrt{en}\), and deleting all edges which lie entirely inside \( B \). 
Then for any \( k \)-edge-coloring of \( G \) there exists a monochromatic \( 1 \)-core in \( G \) on at least 
\[
\min \left\{ (1 - 8 \sqrt{\epsilon})n, |A| - \sqrt{en} + \left( \frac{k - 1}{k} - \frac{\sqrt{1 - 4 \sqrt{\epsilon}}}{2k-1} \right)|B| \right\}
\]
vertices.

Proof. By [8] in Observation 6.2 (applied to \( G' \) with parameters \( \eta = 1/2, \lambda = 1 \), and \( U = V \)) 
we get that all but at most \( \sqrt{en} \) vertices in \( G' \) have degree at least \((1 - \sqrt{\epsilon})(n-1)^{k-1}\). Let \( A' \) 
be the vertices in \( A \) which have degree at least \((1 - \sqrt{\epsilon})(n-1)^{k-1}\) in \( G \) and note that since the 
degrees of the vertices in \( A \) are the same in \( G \) as in \( G' \), we have that \(|A'| \geq |A| - \sqrt{en} > 0 \). 
First suppose that there exists a color \( i \in [k] \) and a vertex \( v \in A' \) such that \( v \) is not 
contained in the 1-core of color \( i \). Then the link graph of \( v \) in \( G \) is a \((k - 1)\)-colored 
\((k - 1)\)-graph on \( n - 1 \) vertices with at least \((1 - \sqrt{\epsilon})(n-1)^{k-1}\) edges. By Theorem 4.1 
we have a monochromatic component in the link graph of \( v \) of order at least \((1 - 8 \sqrt{\epsilon})(n-1)\) 
which together with \( v \) gives a monochromatic component (and so a 1-core) in \( G \) of order 
\((1 - 8 \sqrt{\epsilon})(n-1) + 1 \geq (1 - 8 \sqrt{\epsilon})n \). Otherwise every vertex in \( A' \) is contained in the 1-core of 
every color. Now if \(|A'| \geq (1 - 8 \sqrt{\epsilon})n\), then we have a 1-core of the desired size, so 
suppose \(|A'| < (1 - 8 \sqrt{\epsilon})n\), which implies 
\[
|B| = n - |A| \geq n - |A'| - \sqrt{en} \geq 4 \sqrt{\epsilon}n.
\]
Let \( v \in A' \) and consider the link graph of \( v \) restricted to \( B \), which is a \((k - 1)\)-uniform 
hypergraph on \(|B| \) vertices colored with \( k \) colors which, by Observation 6.3 (applied with 
\( U = B \), \( \lambda = |B|/n \geq 4 \sqrt{\epsilon} \)), has at least 
\[
\left( \frac{|B|}{k - 1} \right) - \sqrt{\epsilon} \left( \frac{n - 1}{k - 1} \right) \geq \left( 1 - \frac{k \sqrt{\epsilon}}{\lambda^{k-1}} \right) \left( \frac{|B|}{k - 1} \right) \geq \left( 1 - \frac{k \sqrt{\epsilon}}{4 k^{k-1}} \right) \left( \frac{|B|}{k - 1} \right) = \left( 1 - \frac{1}{4^{k-1}} \right) \left( \frac{|B|}{k - 1} \right)
\]
\[
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\]
edges. Now by Lemma 6.3 (applied to the link graph of $v$ with $k$ replaced by $k - 1$ and $\ell = 1$), there is a monochromatic 1-core of color $i$ which contains at least $\left( \frac{k - 1}{k} - \frac{\sqrt{k + \epsilon}}{2k - 1} \right) |B|$ vertices of $B$, and since every vertex in $A'$ is in the 1-core of color $i$, the total size of the 1-core of color $i$ is at least

$$|A'| + \left( \frac{k - 1}{k} - \frac{\sqrt{k + \epsilon}}{2k - 1} \right) |B| \geq |A| - \sqrt{\epsilon}n + \left( \frac{k - 1}{k} - \frac{\sqrt{k + \epsilon}}{2k - 1} \right) |B|.$$

\qed

7 Conclusion

The most satisfactory results of this paper are for $k$-colored or $(k + 1)$-colored $k$-uniform hypergraphs. In order to obtain them, we extended Theorem 1.1 to almost complete hypergraphs. However, for complete hypergraphs more is known.

**Theorem 7.1** (Füredi and Gyárfás [8]). Let $k, r \geq 2$ and let $q$ be the smallest integer such that $r \leq q^{k-1} + q^{k-2} + \cdots + q + 1$. Then $mc_r(K^k_n) \geq \frac{n}{q}$. This is sharp when $q^k$ divides $n$, $r = q^{k-1} + q^{k-2} + \cdots + q + 1$, and an affine space of dimension $k$ and order $q$ exists.

**Theorem 7.2** (Gyárfás and Haxell [10]). $mc_5(K^3_n) \geq 5n/7$ and $mc_6(K^3_n) \geq 2n/3$. Furthermore these are tight when $n$ is divisible by 7 and 6 respectively.

It would be interesting to extend these results to the nearly complete setting, from which the random analogs would then follow by applying Theorem 1.4. It is worth noting that these results follow by applying Füredi’s fractional transversal method which does not seem to extend to non-complete graphs in a straightforward manner.

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