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| Author(s) | TANIYAMA, KOUKI                                                                                                                                  |
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Kyoto University
UNKNOTTING NUMBERS OF DIAGRAMS OF A GIVEN NONTRIVIAL KNOT ARE UNBOUNDED

KOUKI TANIYAMA (谷山公規)

概要
任意の非自明結び目 \( K \) と任意の自然数 \( n \) に対して、\( K \) のあるダイアグラム \( D \) が存在して \( D \) の結び目解消数は \( n \) 以上となる。\( K \) の結び目解消数の2倍が \( K \) の最小交点数から1引いたもの以下であることはよく知られている。ここで等式が成り立つのは \( K \) が \( (2,p) \)-トーラス結び目であるときに限る。

Let \( L \) be a link in the 3-sphere \( S^3 \) and \( D \) a diagram of \( L \) on the 2-sphere \( S^2 \). It is well known that by changing over/under information at some crossings of \( D \) we have a diagram of a trivial link. See for example [3]. Let \( u(D) \) be the minimal number of such crossing changes. Namely, there are some \( u(D) \) crossings of \( D \) such that changing them yields a trivial link diagram, and changing less than \( u(D) \) crossings never yields a trivial link diagram. We call \( u(D) \) the unlinking number of \( D \). In the case that \( D \) is a diagram of a knot \( u(D) \) is called the unknotting number of \( D \). The unlinking number \( u(L) \) of \( L \) is defined by the minimum of \( u(D) \) where \( D \) varies over all diagrams of \( L \). Namely we have the following equality.

\[
  u(L) = \min \{ u(D) \mid D \text{ is a diagram of } L \}.
\]

For a knot \( K \) \( u(K) \) is called the unknotting number of \( K \). Then it is natural to ask whether or not the set \( \{ u(D) \mid D \text{ is a diagram of } L \} \) is bounded above. In [1] Nakanishi showed that an unknotting number one knot \( 6_2 \) has an unknotting number two diagram. Then he showed the following theorem in [2].

Theorem 1 [2]. Let \( K \) be a nontrivial knot. Then \( K \) has a diagram \( D \) with \( u(D) \geq 2 \).

As an extension of Theorem 1, we have the following theorem.

Theorem 2. Let \( L \) be a nontrivial link. Then for any natural number \( n \) there exists a diagram \( D \) of \( L \) with \( u(D) \geq n \).
That is, the set \( \{ u(D) \mid D \text{ is a diagram of } L \} \) is unbounded above.

We note that Theorem 2 is an immediate consequence of the following proposition.

Proposition 3. Let \( L \) be a nontrivial link and \( D \) a diagram of \( L \). Then there exists a diagram \( D' \) of \( L \) with \( u(D') = u(D) + 2 \).

The proof of Proposition 3 is done by using a modification of diagram illustrated in Figure 1 that is essentially the same as that used in [2]. See [4] for the detail.
As an immediate consequence of Proposition 3 we have the following corollary.

**Corollary 4.** Let $L$ be a nontrivial link. Then the set $\{u(D) \mid D \text{ is a diagram of } L\}$ contains a set $\{u(L) + 2m \mid m \text{ is a non-negative integer}\}$.

**Question 5.** Let $L$ be a nontrivial link. Is the set $\{u(D) \mid D \text{ is a diagram of } L\}$ equals the set $\{u(L) + m \mid m \text{ is a non-negative integer}\}$?

The following proposition is a partial answer to Question 5.

**Proposition 6.** Let $L$ be an alternating link with $u(L) = 1$. Suppose that $L$ has an alternating diagram $D_0$ with $u(D_0) = 1$. Then the set $\{u(D) \mid D \text{ is a diagram of } L\}$ equals the set of natural numbers $\{u(L) + m \mid m \text{ is a non-negative integer}\}$.

Let $c(D)$ be the number of crossings in $D$. We call $c(D)$ the crossing number of $D$. Then the crossing number $c(L)$ of $L$ is defined by the minimum of $c(D)$ where $D$ varies over all diagrams of $L$. It is natural to ask the relation between $u(D)$ and $c(D)$, or $u(L)$ and $c(L)$. For a diagram $D$ of a knot $K$ other than a trivial diagram the following inequality is well-known. See for example [3].

$$u(K) \leq u(D) \leq \frac{c(D) - 1}{2}.$$  

In particular this inequality holds for a minimal crossing diagram $D$ of $K$ where $c(D) = c(K)$. Thus for any nontrivial knot $K$ we have the following inequality.

$$u(K) \leq \frac{c(K) - 1}{2}.$$  

It is also well known that the equality holds for $(2,p)$-torus knots. Conversely we have the following theorem.

**Theorem 7.** (1) Let $D$ be a diagram of a knot that satisfies the equality

$$u(D) = \frac{c(D) - 1}{2}.$$  

Then $D$ is one of the diagrams illustrated in Figure 2. Namely $D$ is a reduced alternating diagram of some $(2,p)$-torus knot, or $D$ is a diagram with just one crossing.
(2) Let $K$ be a nontrivial knot that satisfies the equality
\[ u(K) = \frac{c(K) - 1}{2}. \]

Then $K$ is a $(2,p)$-torus knot for some odd number $p \neq \pm 1$. Namely only 2-braid knots satisfy the equality.

Figure 2

For links the situation is somewhat different. Let $D$ be a diagram of a link. Then the following inequality is well-known.
\[ u(L) \leq u(D) \leq \frac{c(D)}{2}. \]

Thus for any link $L$ we have the following inequality.
\[ u(L) \leq \frac{c(L)}{2}. \]

The following theorem shows that not only $(2,p)$-torus links but some other links satisfy the equality.

**Theorem 8.** (1) Let $D = \gamma_1 \cup \cdots \cup \gamma_p$ be a diagram of a $p$-component link that satisfies the equality
\[ u(D) = \frac{c(D)}{2}. \]

Then each $\gamma_i$ is a simple closed curve on $S^2$ and for each pair $i, j$, the subdiagram $\gamma_i \cup \gamma_j$ is an alternating diagram or a diagram without crossings.

(2) Let $L$ be a $\mu$-component link that satisfies the equality
\[ u(L) = \frac{c(L)}{2}. \]

Then $L$ has a diagram $D = \gamma_1 \cup \cdots \cup \gamma_\mu$ such that each $\gamma_i$ is a simple closed curve on $S^2$ and for each pair $i, j$, the subdiagram $\gamma_i \cup \gamma_j$ is an alternating diagram or a diagram without crossings.

Two examples of such links are illustrated in Figure 3. We note that for a link described in Theorem 8 the unlinking number equals the sum of the absolute values of all pairwise linking numbers.

The detail will appear in [4].
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DEPARTMENT OF MATHEMATICS, SCHOOL OF EDUCATION, WASEDA UNIVERSITY, NISHI-WASEDA
1-6-1, SHINJUKU-KU, TOKYO, 169-8050, JAPAN

E-mail address: taniyama@waseda.jp