COMPATIBLE ACTIONS AND COHOMOLOGY OF CRYSTALLOGRAPHIC GROUPS

ALEJANDRO ADEM∗ , JIANQUAN GE, JIANZHONG PAN, AND NANSEN PETROSYAN

Abstract. We compute the cohomology of crystallographic groups \( \Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}/p \) with holonomy of prime order by establishing the collapse at \( E_2 \) of the spectral sequence associated to their defining extension. As an application we compute the group of gerbes associated to many six–dimensional toroidal orbifolds arising in string theory.

1. Introduction

Given a finite group \( G \) and an integral representation \( L \) for \( G \) (i.e. a homomorphism \( G \to GL_n(\mathbb{Z}) \), where \( L \) is the underlying \( \mathbb{Z}G \)–module), we can define the semi–direct product \( \Gamma = L \rtimes G \). Calculating the cohomology of these groups is a problem of intrinsic algebraic interest; indeed if the representation is faithful then these groups can be thought of as crystallographic groups (see [7], page 74).

From the geometric point of view, the action on \( L \) gives rise to a \( G \)–action on the \( n \)–torus \( X = \mathbb{T}^n \); this approach can be used to derive important examples of orbifolds, known as toroidal orbifolds (see [2]). In the case when \( n = 6 \) these are of particular interest in string theory (see [4], [11]).

Given the split group extension \( 0 \to L \to \Gamma \to G \to 1 \), the basic problem which we address is that of providing conditions which imply the collapse (without extension problems) of the associated Lyndon–Hochshild–Serre spectral sequence. The conditions which we establish are representation–theoretic, namely depending solely on the structure of the integral representation \( L \). This can be a difficult problem (see [15] for further background) and there are well–known examples where the spectral sequence does not collapse.

Our approach is to systematically apply the methods used in [2]. The key idea is to construct a free resolution \( F \) for the semidirect product \( L \rtimes G \) such that the Lyndon–Hochshild–Serre spectral sequence of the group extension collapses at \( E_2 \). This requires

Date: February 1, 2008.

Key words and phrases. spectral sequence, group cohomology.

∗The first author was partially supported by the NSF and by NSERC.
a chain–level argument, more specifically the construction of a compatible $G$–action on a certain free resolution for the torsion–free abelian group $L$ (see §2 for details). We concentrate on the case of $G = \mathbb{Z}/p$, a cyclic group of prime order $p$, as the representation theory is well–understood. Our main algebraic result is the following

**Theorem 1.1.** Let $G = \mathbb{Z}/p$, where $p$ is any prime. If $L$ is any finitely generated $\mathbb{Z}G$–lattice\(^1\) and $\Gamma = L \rtimes G$ is the associated semi–direct product group, then for each $k \geq 0$

$$H^k(\Gamma, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, \wedge^j(L^*))$$

where $\wedge^j(L^*)$ denotes the $j$–th exterior power of the dual module $L^* = \text{Hom}(L, \mathbb{Z})$.

Expressed differently: these results imply a complete calculation for the integral cohomology of crystallographic groups $\mathbb{Z}^n \rtimes \mathbb{Z}/p$ where $p$ is prime. These calculations can be made explicit.

The theorem has an interesting geometric application:

**Theorem 1.2.** Let $G = \mathbb{Z}/p$, where $p$ is any prime. Suppose that $G$ acts on a space $X$ homotopy equivalent to $(S^1)^n$ with $X^G \neq \emptyset$, then for each $k \geq 0$

$$H^k(EG \times_G X, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, H^j(X, \mathbb{Z})) \cong H^k(\Gamma, \mathbb{Z})$$

where $\Gamma = \pi_1(X) \rtimes G$.

On the other hand, the explicit computation for torsion–free crystallographic groups with holonomy of prime order was carried out long ago by Charlap and Vásquez (see [8], page 556). Combining the two results we obtain a complete calculation:

**Theorem 1.3.** Let $\Gamma$ denote a crystallographic group with holonomy of prime order $p$, expressed as an extension

$$1 \to L \to \Gamma \to \mathbb{Z}/p \to 1$$

where $L$ is a free abelian group of finite rank.

1. If $\Gamma$ is torsion–free, then $L \cong N \oplus \mathbb{Z}$ (it splits off a trivial direct summand) and

$$H^k(\Gamma, \mathbb{Z}) \cong H^0(\mathbb{Z}/p, \wedge^k(L^*)) \oplus H^1(\mathbb{Z}/p, \wedge^{k-1}(N^*))$$

for $0 \leq k \leq rk(L)$; $H^k(\Gamma, \mathbb{Z}) = 0$ for $k > rk(L)$.

2. If $\Gamma$ is not torsion–free, then $H^*(\Gamma, \mathbb{Z})$ can be computed using Theorem 1.1.

---

\(^1\)A $\mathbb{Z}G$–lattice is a $\mathbb{Z}G$–module which happens to be a free abelian group.
In this paper we also consider the situation for the cyclic group of order four; some partial results are obtained but a general collapse has not been established. However, based on these and other computations we conjecture that for any cyclic group, the spectral sequence associated to a semi–direct product of the form \( \mathbb{Z}^n \rtimes G \) must collapse at \( E_2 \).

In the last section we give an application of our methods to calculations for six–dimensional toroidal orbifolds, showing that among the 18 inequivalent \( N = 1 \) supersymmetric string theories on symmetric orbifolds of \((2,2)\)-type without discrete background, only two of them cannot be analyzed using our methods i.e. we cannot show the existence of compatible actions for the associated modules. If \( X = [\mathbb{T}^6/G] \) is an orbifold arising this way, then our results provide a complete calculation for its associated group of gerbes \( \text{Gb}(X) \cong H^3(EG \times_G \mathbb{T}^6, \mathbb{Z}) \) (see [2] for more details).

2. Preliminary Results

The notion of a \textit{compatible action} was first introduced in [9]. If such an action exists it allows one to construct practical projective resolutions and from these to compute the cohomology of the group. We will give the basic definition and the main theorem that follows. More details can be found in [2].

Let \( \Gamma = L \rtimes_G G = L \rtimes G \) be the semidirect product of a finite group \( G \) and a finite dimensional \( \mathbb{Z} \)-lattice \( L \) via a representation \( \rho : G \to GL(L) \). \( G \) acts on the group \( L \) by the homomorphism \( \rho \), and this extends linearly to an action on the group algebra \( R[L] \), where \( R \) denotes a commutative ring with unit. We write \( l^g \) for \( \rho(g)l \) where \( l \in R[L], g \in G \).

In the rest of this paper \( R \) will represent \( \mathbb{Z} \) (the integers) or \( \mathbb{Z}(p) \) (the ring of integers localized at a fixed prime \( p \)).

\textbf{Definition 2.1}. Given a free resolution \( \epsilon : F \to R \) of \( R \) over \( R[L] \), we say that it admits an action of \( G \) compatible with \( \rho \) if for all \( g \in G \) there is an augmentation-preserving chain map \( \tau(g) : F \to F \) such that

1. \( \tau(g)[l \cdot f] = l^g \cdot [\tau(g)f] \) for all \( l \in R[L] \) and \( f \in F \),
2. \( \tau(g)\tau(g') = \tau(gg') \) for all \( g, g' \in G \),
3. \( \tau(1) = 1_F \).

The following two lemmas (see [2]) reduce the construction of compatible actions to the case of faithful indecomposable representations.

\textbf{Lemma 2.2}. If \( \epsilon_i : F_i \to R \) is a projective \( R[L_i] \)-resolution of \( R \) for \( i = 1, 2 \), then \( \epsilon_1 \otimes \epsilon_2 : F_1 \otimes F_2 \to R \) is a projective \( R[L_1 \times L_2] \)-resolution of \( R \). Furthermore, if \( G \) acts...
compatibly on $F_i$ by $\tau_i$ for $i = 1, 2$, then a compatible action of $G$ on $\epsilon_1 \otimes \epsilon_2 : F_1 \otimes F_2 \to R$ is given by $\tau(g)(f_1 \otimes f_2) = \tau_1(g)(f_1) \otimes \tau_2(g)(f_2)$.

**Lemma 2.3.** If $L$ is a $R[G_1]$-module, $\pi : G_2 \to G_1$ a group homomorphism, and $\epsilon : F \to R$ is a $R[L]$-resolution of $R$ such that $G_1$ acts compatibly on it by $\tau'$, then $G_2$ also acts compatibly on it by $\tau(g)f = \tau'(\pi(g))f$ for any $g \in G$.

If a compatible action exists, we can give $F$ a $\Gamma$-module structure as follows. An element $\gamma \in \Gamma$ can be expressed uniquely as $\gamma = lg$, with $l \in L$ and $g \in G$. We set $\gamma \cdot f = (lg) \cdot f = l \cdot \tau(g)f$. Note that given any $G$–module $M$, this inflates to a $\Gamma$–action on $M$ via the projection $\Gamma \to G$.

We can always construct a special free resolution $F$ of $R$ over $L$, characterized by the property that the cochain complex $\text{Hom}_L(F, R)$ for computing the cohomology $H^*(L, R)$ has all coboundary maps zero (more details will be provided in the next section). Using this fact, the following was proved in [2]:

**Theorem 2.4.** (Adem-Pan) Let $\epsilon : F \to R$ be a special free resolution of $R$ over $L$ and suppose that there is a compatible action of $G$ on $F$. Then for all integers $k \geq 0$, we have

$$H^k(L \rtimes G, R) = \bigoplus_{i+j=k} H^i(G, H^j(L, R)).$$

This result can be interpreted as saying that the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q(L, R)) \Rightarrow H^{p+q}(L \rtimes G, R)$$

collapses at $E_2$ without extension problems. Note that this is not always the case; in fact there are examples of semi–direct products of the form $\mathbb{Z}^n \rtimes (\mathbb{Z}/p)^2$ where the associated spectral sequence has non–trivial differentials (see [15]). This will be discussed in §5.

### 3. Construction of Compatible Actions

Let $R[L]$ denote the group ring of $L$, a free abelian group with basis $\{x_1, \ldots, x_n\}$. Then the elements $x_1 - 1, \ldots, x_n - 1$ form a regular sequence in $R[L]$, hence the Koszul complex $K_* = K(x_1 - 1, \ldots, x_n - 1)$ is a free resolution of the trivial module $R$. It has the additional property of being a differential graded algebra (or DGA). We briefly recall how it looks. There are generators $a_1, \ldots, a_n$ in degree one, and the graded basis for $K_*$ can be identified with the usual basis for the exterior algebra they generate. The differential is given by the following formula: if $a_{i_1 \ldots i_p} = a_{i_1} \ldots a_{i_p}$ is a basis element in $K_p$, then
\[ d(a_{i_1\ldots i_p}) = \sum_{j=1}^p (-1)^{j-1}(x_{i_j} - 1)a_{i_1\ldots \hat{i}_j\ldots i_p}. \]

Now the cohomology of the free abelian group \( L \) is precisely an exterior algebra on \( n \) one-dimensional generators, which in fact can be identified with the dual elements \( a_1^*, \ldots, a_n^* \). In particular we see that the cochain complex \( \text{Hom}_{R[L]}(K_*, R) \) has zero differentials, and hence \( K_* \) is a special free resolution of \( R \) over \( R[L] \) (this resolution also appears in [7] pp. 96–97).

We now consider how to construct a compatible \( G \)-action on \( K_* \), given a \( G \)-module structure on \( L \).

**Theorem 3.1.** If \( G \) acts on the lattice \( L \), let \( K_* = K(x_1 - 1, \ldots, x_n - 1) \) denote the special free resolution of \( R \) over \( R[L] \) defined using the Koszul complex associated to the elements \( x_1 - 1, \ldots, x_n - 1 \), where \( \{x_1, \ldots, x_n\} \) form a basis for \( L \). Suppose that there is a homomorphism \( \tau : G \to \text{Aut}(K_1) \) such that for every \( g \in G \) and \( a \in K_1 \) it satisfies

\[ d\tau(g)(a) = d(a)^g \]

where \( d : K_1 \to K_0 \) is the usual Koszul differential, and \( d(a)^g \in K_0 = R[L] \). Then \( \tau \) extends to \( K_* \) using its DGA structure and so defines a compatible \( G \)-action on \( K_* \).

**Proof.** First we observe that \( \tau(g) \) acts on \( K_0 = R[L] \) via the original \( G \)-action, i.e. \( \tau(g)(x) = x^g \) for any \( x \in K_0 \). Next we define the action on the basis of \( K_* \) as a graded \( R[L] \)-module, namely:

\[ \tau(g)(a_{i_1} \ldots a_{i_p}) = \tau(g)(a_{i_1}) \ldots \tau(g)(a_{i_p}). \]

If \( \alpha \in R[L] \) and \( u \in K_* \), we define \( \tau(g)(\alpha u) = \alpha^g \tau(g)(x) \). By linearity and the DGA structure of \( K_* \), this will define \( \tau : G \to \text{Aut}(K_*) \), with the desired properties. \( \square \)

Generally speaking it can be quite difficult to construct a compatible action; however there is an important special case where it is quite straightforward.

**Theorem 3.2.** Let \( \phi : G \to \Sigma_n \) denote a group homomorphism, where \( \Sigma_n \) denotes the symmetric group on \( n \) elements. Let \( G \) act on \( \mathbb{Z}^n \) via this homomorphism. Then the associated Koszul complex \( K_* \) admits a compatible \( G \)-action.

\(^2\)Such a module is called a permutation module.
Proof. By Lemma 2.3 we can assume that $G$ is a subgroup of $\Sigma_n$, hence it will suffice to prove this for $\Sigma_n$ itself. If we take generators $a_1, \ldots, a_n$ for the Koszul complex corresponding to the elements $x_1, \ldots, x_n$ in the underlying module $L$, then we can define $\tau$ as follows:

$$\tau(\sigma)(a_i) = a_{\sigma(i)}.$$  

This obviously defines a permutation representation on $K_1$, and compatibility follows from the fact that for all $a_i, 1 \leq i \leq n$ and $\sigma \in \Sigma$ we have

$$d\tau(\sigma)(a_i) = d(a_{\sigma(i)}) = x_{\sigma(i)} - 1 = (x_i - 1)^\sigma.$$  

This completes the proof. \(\square\)

Aside from permutation representations, it is difficult to construct general examples of compatible actions. However if $G$ is a cyclic group then we can handle an important additional type of module.

**Proposition 3.3.** Let the cyclic group $G = \langle t | t^n = 1 \rangle$ act on $\mathbb{Z}^{n-1}$ by:

$$\xi_1 : t \mapsto \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ -1 & -1 & -1 & \ldots & -1 & -1 \end{pmatrix} \in GL_{n-1}(\mathbb{Z}).$$

If $x_1, \ldots, x_n$ is the canonical basis under which the action is represented by the matrix above, then the free resolution $K_* = K_*(x_1-1, \ldots, x_n-1)$ admits an action of $G$ compatible with $\xi_1$, which can be defined by:

$$\tau(t)(a_1) = -x_{n-1}^{-1}a_{n-1} \quad \tau(t)(a_k) = -x_{n-1}^{-1}(a_{n-1} - a_{k-1}), \quad 1 < k \leq n - 1.$$  

Proof. The proof is a straightforward calculation verifying that $\tau$ defines a compatible action. First we verify that $\tau^n = 1$. For this we observe that if $A$ is the matrix in $GL_{n-1}(\mathbb{Z})$ representing the generator $t$, then expressed in terms of the basis $\{a_1, \ldots, a_n\}$ we have that $\tau(t) = x_{n-1}^{-1}A$. If we iterate this action and use the fact that $\tau(g)(au) = a^\theta \tau(g)(u)$ then we obtain

$$\tau(t)^n = (x_{n-1}^{-1})^{t^n-1}(x_{n-1}^{-1})^{t^{n-2}} \cdots (x_{n-1}^{-1})^t(x_{n-1}^{-1})A^n = 1.$$
This follows from the fact that the characteristic polynomial of $A$ is the cyclotomic polynomial $p(z) = 1 + z + \cdots + z^{n-1}$, hence we have that $p(A) = 0$ on the underlying module $L$ and so in multiplicative notation we have that $u^t \cdot u^{t-1} \cdot \cdots \cdot u^1 \cdot u = 1$ for any $u \in L$.

Next we verify compatibility:

$$\tau(t)d(a_1) = \tau(t)(x_1 - 1) = x_{n-1}^{-1} - 1$$

$$d\tau(t)(a_1) = d(-x_{n-1}^{-1}a_{n-1}) = -x_{n-1}^{-1}(x_{n-1} - 1) = x_{n-1}^{-1} - 1.$$  

Similarly for all $1 < k \leq n - 1$ we have that:

$$\tau(t)d(a_k) = \tau(t)(x_k - 1) = x_{k-1}x_{n-1}^{-1} - 1$$

$$= -x_{n-1}^{-1}(x_{n-1} - 1 - x_{k-1} - 1) = d(-x_{n-1}^{-1}(a_{n-1} - a_{k-1})) = d\tau(t)(a_k).$$

□

For $G = \mathbb{Z}/n$, the module which gives rise to the matrix in 3.3 is the augmentation ideal $IG$, which has rank equal to $n - 1$. The following proposition is an application of the results in this section.

**Proposition 3.4.** Let $G = \mathbb{Z}/n$, and assume that $L$ is a $\mathbb{Z}G$–lattice such that

$L \cong M \oplus IG^t$

where $M$ is a permutation module. Then, for any coefficient ring $R$, the special free resolution $K_*$ over $R[L]$ admits a compatible $G$–action.

**Proof.** This follows from applying Lemma 2.2 to Theorem 3.2 and Proposition 3.3. □

For our cohomology calculations it will be practical to use the coefficient ring $R = \mathbb{Z}(p)$, where $p$ is a prime. In this situation, for $G = \mathbb{Z}/p$ (see [10]) there are only three distinct isomorphism classes of indecomposable $RG$–lattices, namely $R$ (the trivial module), $IG$ (the augmentation ideal) and $RG$, the group ring. Moreover, if $L$ is any finitely generated $\mathbb{Z}G$–lattice, we can construct a $\mathbb{Z}G$-homomorphism $f : L' \to L$ such that

- $L' \cong \mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t$
- $f$ is an isomorphism after tensoring with $R$.

We shall call $L'$ a representation of type $(r, s, t)$. 
4. Applications to Cohomology

We are now ready to prove our main result.

**Theorem 4.1.** Let $G = \mathbb{Z}/p$, where $p$ is any prime. If $L$ is any finitely generated $\mathbb{Z}G$–lattice, and $\Gamma = L \rtimes G$ is the associated semi–direct product group, then for each $k \geq 0$

$$H^k(\Gamma, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, \wedge^j(L^*))$$

where $\wedge^j(L^*)$ denotes the $j$-th exterior power of the dual module $L^* = \text{Hom}(L, \mathbb{Z})$.

**Proof.** First, let us prove the analogous result for the cohomology with coefficients in $R = \mathbb{Z}_{(p)}$. We make the assumption that $L$ is a module of type $(r, s, t)$. We need to verify:

$$H^k(\Gamma, R) \cong \bigoplus_{i+j=k} H^i(G, \wedge^j(L^*_R))$$

where $L^*_R = L^* \otimes R$. In fact, we see that in the associated Lyndon-Hochschild-Serre spectral sequence for the extension $0 \to L \to \Gamma \to G \to 1$ with $E^{i,j}_2 = H^i(G, H^j(L, R)) \Rightarrow H^k(\Gamma, R)$

there are no differentials and no extension problems. This follows from applying Theorem [2.4](#) and the fact that the module $L$ gives rise to a special resolution with a compatible action by Proposition [3.4](#).

Now let us consider the case when $L$ is not of type $(r, s, t)$. As observed previously, we can construct a $\mathbb{Z}G$–lattice $L'$ and a map $f : L' \to L$ such that $L'$ is of type $(r, s, t)$ and $f$ is an isomorphism after tensoring with $R$. Under these conditions $f$ will induce a map between the spectral sequences with $R$–coefficients for the extensions corresponding to $L$ and $L'$. However by our hypotheses, $\wedge^k(L^*_R)$ and $\wedge^k(L^*_R)$ are isomorphic as $RG$–modules for all $k \geq 0$, with the isomorphism induced by $f$. Hence the corresponding $E_2$–terms are isomorphic, and so the spectral sequences both collapse and the result follows.

It now remains to prove the result with coefficients in the integers $\mathbb{Z}$. Note that by the universal coefficient theorem, we have $H^*(\Gamma, \mathbb{Z}_{(p)}) \cong H^*(\Gamma, \mathbb{Z}) \otimes \mathbb{Z}_{(p)}$ hence the only relevant discrepancy between $H^*(\Gamma, \mathbb{Z}_{(p)})$ and $H^*(\Gamma, \mathbb{Z})$ might arise from the presence of torsion prime to $p$ in the integral cohomology of $\Gamma$. However, a quick inspection of the spectral sequence of the extension $0 \to L \to \Gamma \to G \to 1$ with $\mathbb{Z}_{(p)}$ coefficients shows that there is no torsion prime to $p$ in the cohomology, as $L$ is free abelian and $G$ is a $p$–group. This completes our proof. \qed
In [2], Corollary 3.3, it was observed that the spectral sequence for the extension $L \rtimes G$ satisfies a collapse at $E_2$ without extension problems if the same is true for all the restricted extensions $L \rtimes G_p$, where the $G_p \subset G$ are the $p$–Sylow subgroups of $G$. We obtain the following

**Corollary 4.2.** Let $G$ denote a finite group of square–free order, and $L$ any finitely generated $\mathbb{Z}G$–lattice. Then for all $k \geq 0$ we have

$$H^k(L \rtimes G, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, \wedge^j(L^*)).$$

We now consider a more geometric situation. Suppose that the group $G = \mathbb{Z}/p$ acts on a space $X$ which has the homotopy type of a product of circles.

**Theorem 4.3.** Let $G = \mathbb{Z}/p$, where $p$ is any prime. Suppose $G$ acts on a space $X$ homotopy equivalent to $(S^1)^n$ with $X^G \neq \emptyset$, then for each $k \geq 0$

$$H^k(EG \times_G X, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, H^j(X, \mathbb{Z})) \cong H^k(\Gamma, \mathbb{Z})$$

where $\Gamma = \pi_1(X) \rtimes G$.

**Proof.** The space $EG \times_G X$ fits into a fibration $X \hookrightarrow EG \times_G X \rightarrow BG$ which has a section due to the fact that $X^G \neq \emptyset$. Let $\Gamma$ denote the fundamental group of $EG \times_G X$. The long exact sequence for the homotopy groups of the fibration gives rise to a split extension

$$1 \rightarrow \pi_1(X) \rightarrow \Gamma \rightarrow G \rightarrow 1.$$  

Since $\pi_1(X) \cong L$, a $\mathbb{Z}G$–lattice, this shows that $\Gamma \cong L \rtimes G$, where the $G$ action is induced on $L$ via the action on the fiber. Note that $EG \times_G X$ is an Eilenberg-MacLane space of type $K(\Gamma, 1)$. Hence, $H^*(EG \times_G X, \mathbb{Z}) \cong H^*(\Gamma, \mathbb{Z})$ and the result follows from Theorem 4.1.

\[\square\]

Note that a special case of this result was proved in [1], namely for actions where $\pi_1(X) \otimes \mathbb{Z}(p)$ is isomorphic to a direct sum of indecomposables of rank $p - 1$. The terms $H^i(\mathbb{Z}/p, \wedge^j(L^*))$ can be computed if $L$ is known up to isomorphism. In fact, all we need is to know $L$ up to $\mathbb{Z}/p$ cohomology, as this will determine its indecomposable factors (at least up to $\mathbb{Z}(p)$–equivalence).

As we mentioned in the introduction, our results complete the calculation for the cohomology of crystallographic groups with prime order holonomy when combined with previous work on the torsion–free case (Bieberbach groups). The terms appearing in the formulas in Theorem 1.3 can be explicitly computed, as was observed in [8].
5. Extensions to Other Groups

In this section we explore to what degree our results can be extended to other groups. In the case of the cyclic group of order four, the indecomposable integral representations are easy to describe, so it is a useful test case.

Let $G = \mathbb{Z}/4$, from [3] we can give a complete list of all (nine) indecomposable pair-wise nonequivalent integral representations by the following adopted table, where $a$ is a generator for $\mathbb{Z}/4$:

$$
\begin{align*}
\rho_1 : a &\to 1; \\
\rho_2 : a &\to -1; \\
\rho_3 : a &\to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \\
\rho_4 : a &\to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \\
\rho_5 : a &\to \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}; \\
\rho_6 : a &\to \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \\
\rho_7 : a &\to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \\
\rho_8 : a &\to \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix}; \\
\rho_9 : a &\to \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix};
\end{align*}
$$

**Theorem 5.1.** Let $G = \mathbb{Z}/4$ and $L$ a finitely generated $\mathbb{Z}G$-lattice. If $L$ is a direct sum of indecomposables of type $\rho_i$ for $i \leq 7$, and $i \neq 6$, then there is a compatible action and

$$
H^k(L \rtimes G, \mathbb{Z}) = \bigoplus_{i+j=k} H^i(G, H^j(L, \mathbb{Z})).
$$

**Proof.** For the indecomposables $\rho_1, \rho_2, \rho_3, \rho_4$, compatible actions are known to exist on the associated resolutions by the results in [2]. The same is true\(^3\) for $\rho_5$ and $\rho_7$ by 3.3 and 3.2. Hence if $L$ is any integral representation expressed as a direct sum of $\rho_i$, $i \leq 7$ and $i \neq 6$, the result follows from 2.2 and 2.4. \hfill \Box

\(^3\)In fact $\rho_5$ corresponds to the dual module $IG^*$, but for cyclic groups there is an isomorphism $IG \cong IG^*$. 
In the case of $\rho_6, \rho_8$ and $\rho_9$, a compatible action is not known to exist. However, via an explicit computation done in [14], we can establish the collapse of the spectral sequence for the extension $\Gamma_6$ associated to $\rho_6$, yielding

$$H^i(\Gamma_6, \mathbb{Z}) = \begin{cases} 
  \mathbb{Z} & \text{if } i = 0, 1 \\
  \mathbb{Z}/4 \oplus \mathbb{Z} & \text{if } i = 2 \\
  \mathbb{Z}/2 \oplus \mathbb{Z} & \text{if } i = 3 \\
  \mathbb{Z}/4 \oplus \mathbb{Z}/2 & \text{if } i \geq 4 
\end{cases}$$

which verifies the statement analogous to 5.1 for the cohomology of $\Gamma_6$.

Indeed, for all the examples of semidirect products we have considered so far, there is a collapse at $E_2$ in the Lyndon-Hochschild-Serre spectral sequence of the group extension $0 \to L \to G \rtimes L \to G \to 1$ and therefore we can make the following:

**Conjecture 5.2.** Suppose that $G$ is a finite cyclic group and $L$ a finitely generated $\mathbb{Z}G$–lattice; then for any $k \geq 0$ we have

$$H^k(G \rtimes L, \mathbb{Z}) = \bigoplus_{i+j=k} H^i(G, H^j(L, \mathbb{Z})).$$

In [15] examples were given of semi-direct products of the form $L \rtimes (\mathbb{Z}/p)^2$ where the associated mod $p$ Lyndon-Hochschild-Serre spectral sequence has non–zero differentials. This relies on the fact that for $G = \mathbb{Z}/p \times \mathbb{Z}/p$, there exist $\mathbb{Z}G$–modules $M$ which are not realizable as the cohomology of a $G$–space. These are the counterexamples to the Steenrod Problem given by G.Carlsson (see [5]), where $M$ can be identified with $L^*$. No such counterexamples exist for finite cyclic groups $\mathbb{Z}/N$, which means that disproving our conjecture will require a different approach.

We should also mention that by using results due to Nakaoka (see [12], pages 19 and 50) we know that the spectral sequence for a wreath product $\mathbb{Z}^n \rtimes G$, where $G \subset \Sigma_n$ (the symmetric group) acting on $\mathbb{Z}^n$ via permutations will always collapse at $E_2$, without extension problems. This can be interpreted as the fact that a strong collapse theorem holds for all permutation modules and all finite groups $G$. A simple proof of this result can be obtained by applying Theorem 2.4 to Proposition 3.2.

As suggested by [15], the results here can be considered part of a very general problem, which is both interesting and quite challenging:

**Problem:** Given a finite group $G$, find suitable conditions on a $\mathbb{Z}G$–lattice $L$ so that the spectral sequence for $L \rtimes G$ collapses at $E_2$. 
6. Application to Computations for Toroidal Orbifolds

Interesting examples arise from calculations for six–dimensional orbifolds, where the usual spectral sequence techniques become rather complicated. Here our methods provide an important new ingredient that allows us to compute rigorously beyond the known range. An important class of examples in physics arises from actions of a cyclic group $G = \mathbb{Z}/N$ on $T^6$. In our scheme, these come from six-dimensional integral representations of $\mathbb{Z}/N$. However, the constraints from physics impose certain restrictions on them (see [11], [16]). If $\theta \in GL_6(\mathbb{Z})$ is an element of order $N$, then it can be diagonalized over the complex numbers. The associated eigenvalues, denoted $\alpha_1, \alpha_2, \alpha_3$, should satisfy $\alpha_1\alpha_2\alpha_3 = 1$, and in addition all of the $\alpha_i \neq 1$. The first condition implies that the orbifold $T^6 \to T^6/G$ is a Calabi–Yau orbifold, and so admits a crepant resolution. These more restricted representations have been classified in [11], where it is shown that there are precisely 18 inequivalent lattices of this type.

It turns out that calculations are focused on computing the equivariant cohomology $H^*(EG \times_G T^6, \mathbb{Z})$ (see [2] and [4] for more details). As was observed in [2], we can compute the group of gerbes associated to the orbifold $\mathcal{X} = [T^6/G]$ via the isomorphism

$$Gb(\mathcal{X}) \cong H^3(EG \times_G T^6, \mathbb{Z}) \cong H^3(\mathbb{Z}^6 \times G, \mathbb{Z}),$$

whence our methods can be used to obtain some fairly complete results in this setting. Before proceeding we recall that as in Corollary 4.2 the collapse of the spectral sequence for an extension $L \times G$ will follow from the existence of compatible $Syl_p(G)$ actions on the Koszul complex $K_*$ for every prime $p$ dividing $|G|$. If these exist we shall say that $K_*$ admits a local compatible action.

**Theorem 6.1.** Among the 18 inequivalent integral representations associated to the six–dimensional orbifolds $T^6/\mathbb{Z}/N$ described above, only two of them are not known to admit (local) compatible actions. Hence for those 16 examples there is an isomorphism

$$H^k(E\mathbb{Z}/N \times_{\mathbb{Z}/N} T^6, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(\mathbb{Z}/N, H^j(T^6, \mathbb{Z}))$$

**Proof.** Consider the defining matrix of an indecomposable action of $\mathbb{Z}/N$ on $\mathbb{Z}^n$ with determinant one, expressed in canonical form as

\footnote{In the language of physics, they show that there exist 18 inequivalent $N = 1$ supersymmetric string theories on symmetric orbifolds of $(2, 2)$–type without discrete background.}
\[ \theta = \begin{pmatrix} 0 & \ldots & 0 & v_1 \\ 1 & 0 & \ldots & 0 & v_2 \\ 0 & 1 & 0 & \ldots & 0 & v_3 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & v_n \end{pmatrix} \]

where \( v_1 = \pm 1 \).

In \cite{11} it was determined that the matrices that specify the indecomposable modules appearing as summands for the \( N = 1 \) supersymmetric \( \mathbb{Z}/N \)-orbifolds can be given as follows, where the vectors represent the values \((v_1, v_2, \ldots, v_n)\):

**Indecomposable matrices relevant for \( N = 1 \) supersymmetry**

| \( n = 1 \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) |
|---|---|---|---|
| \( \mathbb{Z}/2^{(1)} : (-1) \) | \( \mathbb{Z}/3^{(2)} : (-1,-1) \) | \( \mathbb{Z}/4^{(3)} : (-1,-1,-1) \) | \( \mathbb{Z}/6^{(4)} : (-1,0,-1,0) \) |
| \( \mathbb{Z}/4^{(2)} : (-1,0) \) | \( \mathbb{Z}/6^{(3)} : (-1,0,0) \) | \( \mathbb{Z}/8^{(4)} : (-1,0,0,0) \) |
| \( \mathbb{Z}/6^{(2)} : (-1,1) \) | | | \( \mathbb{Z}/12^{(4)} : (-1,0,1,0) \) |

| \( n = 5 \) | \( n = 6 \) |
|---|---|
| \( \mathbb{Z}/6^{(5)} : (-1,-1,-1,-1,-1) \) | \( \mathbb{Z}/7^{(6)} : (-1,-1,-1,-1,-1,-1) \) |
| \( \mathbb{Z}/8^{(5)} : (-1,-1,0,0,-1) \) | \( \mathbb{Z}/8^{(6)} : (-1,0,-1,0,-1,0) \) |
| | \( \mathbb{Z}/12^{(6)} : (-1,-1,0,1,0,-1) \) |

We will show that all of these, except possibly \( \mathbb{Z}/8^{(5)} \) and \( \mathbb{Z}/12^{(6)} \), admit local compatible actions. The examples of rank two or less were dealt with in \cite{2}; for \( N = 2, 3, 6, 7 \) the result follows directly from \cite{4.1 and 4.2}. The case \( \mathbb{Z}/4^{(3)} \) was covered in \cite{5.1}. We will deal explicitly with the cases \( \mathbb{Z}/8^{(4)}, \mathbb{Z}/12^{(4)} \) and \( \mathbb{Z}/8^{(6)} \).

(1) The group \( \mathbb{Z}/8 \) acts on \( \mathbb{Z}^4 \) with generator represented by the matrix:

\[ T = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]
We define a compatible action by the following formulas:
\[ \tau(t)(a_1) = -x^{-1}_4 a_4, \tau(t)(a_2) = a_1, \tau(t)(a_3) = a_2, \tau(t)(a_4) = a_3. \]

(2) The group \( \mathbb{Z}/12 \) acts on \( \mathbb{Z}^4 \) with generator represented by the matrix:
\[
T = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

For this example it suffices to construct a compatible action for \( \mathbb{Z}/4 \) (with generator represented by the matrix \( T^3 \)) as we already know that a compatible action exists restricted to \( \mathbb{Z}/3 \). Now we have that \( \mathbb{Z}/4 \) acts on \( \mathbb{Z}^4 \) with
\[
T^3 = \begin{pmatrix}
0 & -1 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

which is a matrix whose square is \(-I\). This implies that the module is a sum of two copies of the faithful rank two indecomposable (see §5), for which a compatible action is known to exist (as explained in Theorem 5.1), and so this case is taken care of.

(3) The group \( \mathbb{Z}/8 \) acts on \( \mathbb{Z}^6 \) with generator represented by the matrix:
\[
T = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

The formulas for a compatible action are given by
\[ \tau(t)(a_1) = -x^{-1}_6 a_6, \tau(t)(a_2) = a_1, \tau(t)(a_3) = x^{-1}_6 (a_2 - a_6) \]
\[ \tau(t)(a_4) = a_3, \tau(t)(a_5) = x^{-1}_6 (a_4 - a_6), \tau(t)(a_6) = a_5. \]

We have shown that (local) compatible actions exist for all representations constructed using indecomposables other than \( \mathbb{Z}/8^{(5)} \) and \( \mathbb{Z}/12^{(6)} \). However, these indecomposables can only appear once in the list due to dimensional constraints, namely in the form \( \mathbb{Z}/8^{(5)} \oplus \mathbb{Z}/2^{(1)} \) and \( \mathbb{Z}/12^{(6)} \) itself. Thus our proof is complete. \( \square \)
COMPATIBLE ACTIONS AND COHOMOLOGY OF CRYSTALLOGRAPHIC GROUPS

References

[1] Adem, A., \( \mathbb{Z}/p\mathbb{Z} \) actions on \((\mathbb{S}^n)^k\), Trans. Amer. Math. Soc. \textbf{300} (1987), no. 2, 791–809.

[2] Adem, A. and Pan, J., Toroidal Orbifolds, Gerbes and Group Cohomology, Trans. Amer. Math. Soc. \textbf{358} (2006), 3969-3983.

[3] Berman, S. and Gudikov, P. Indecomposable representations of finite groups over the ring of \( p \)-adic integers, Izv. Akad. Nauk. SSSR \textbf{28} (1964), 875–910.

[4] de Boer, J., Dijkgraaf, R., Hori, K., Keurentjes, A., Morgan, J., Morrison, D. and Sethi, S., Triples, Fluxes, and Strings, Adv. Theor. Math. Phys. \textbf{4} (2000), no. 5, 995–1186.

[5]Carlsson, G., A counterexample to a conjecture of Steenrod, Inv. Math. \textbf{64} (1981), no. 1, 171–174.

[6]Cartan, H. and Eilenberg, S., Homological Algebra, Oxford University Press, Oxford, 1956.

[7]Charlap, L., Bieberbach Groups and Flat Manifolds, Universitext, Springer–Verlag, Berlin, 1986.

[8]Charlap, L. and Vasquez, A., Compact Flat Riemannian Manifolds \( II \): the Cohomology of \( \mathbb{Z}_p \)–manifolds, Amer. J. Math. \textbf{87} (1965), 551–563. Trans, Amer. Math. Soc.

[9]Brady, T., Free resolutions for semi-direct products, Tohoku Math. J. (2) \textbf{45} (1993), no. 4, 535–537.

[10] Curtis, C.W. and Reiner, I., Representation Theory of Finite Groups and Associative Algebras Wiley-Interscience (1987).

[11]Erl er, J. and Klemm, A., Comment on the generation number in orbifold compactifications, Comm. Math. Phys. \textbf{153} (1993), 579–604.

[12]Evens, L., Cohomology of Groups, Oxford Mathematical Monographs, Oxford University Press (1991).

[13]Joyce, D., Deforming Calabi-Yau orbifolds, Asian J. Math. \textbf{3} (1999), no. 4, 853–867.

[14]Petrosyan, N. Jumps in cohomology of groups, periodicity, and semi–direct products, Ph.D. Dissertation, University of Wisconsin–Madison (2006).

[15]Totaro, B., Cohomology of Semidirect Product Groups, J. of Algebra \textbf{182} (1996), 469–475.

[16]Vafa, C. and Witten, E., On orbifolds with discrete torsion, J. Geom. Phys. \textbf{15} (1995), no. 3, 189–214.

Department of Mathematics, University of British Columbia, Vancouver BC V6T 1Z2, Canada

E-mail address: adem@math.ubc.ca

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

E-mail address: gejq04@mails.tsinghua.edu.cn

Institute of Mathematics, Academia Sinica, Beijing 100080, China

E-mail address: pjz@amss.ac.cn

Department of Mathematics, Indiana University, Bloomington IN 47405, USA

E-mail address: nanpetro@indiana.edu