ALMOST PERIODIC SOLUTIONS AND STABLE SOLUTIONS FOR
STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we discuss the relationships between stability and almost periodicity for solutions of stochastic differential equations. Our essential idea is to get stability of solutions or systems by some inherited properties of Lyapunov functions. Under suitable conditions besides Lyapunov functions, we obtain the existence of almost periodic solutions in distribution.

1. Introduction

In 1924–1926, Bohr founded the theory of almost periodic functions [5, 6, 7]. Roughly speaking, an almost periodic function means that it is periodic up to any desired level of accuracy. Since many differential equations arising from physics and other fields admit almost periodic solutions, almost periodicity becomes an important property of dynamical systems and is extensively studied in the area of differential equations and dynamical systems. We refer the reader to the books, e.g. Amerio and Prouse [1], Fink [12], Levitan and Zhikov [15], Yoshizawa [28] etc, for an exposition.

For deterministic differential equations, the existence of almost periodic solutions was studied under various stability assumptions. Markov [17] defined a kind of stability which implies almost periodicity. Deyssach and Sell [11] assumed that there exists one bounded uniformly stable solution. Miller [18] assumed the existence of one bounded totally stable solution. Seifert [22] proposed the so-called Σ-stability, while Sell [23, 24] proposed the stability under disturbance from the hull; actually, these two concepts of stability are equivalent. Coppel [9] sharpened Miller’s result without the uniqueness of solutions by using the properties of asymptotically almost periodic functions; Yoshizawa [27] developed the idea of Coppel and improved all the results mentioned above. On the other hand, the Lyapunov’s second method was employed to investigate the existence of almost periodic solutions: Hale [14] and Yoshizawa [26] assumed the existence of Lyapunov functions for pairs of solutions to conclude the uniform asymptotic stability in the large of the bounded solution.

However, the various stability assumptions mentioned above are not easily verified directly in practice. It is known that some stabilities, such as uniform stability and uniform asymptotic stability, can be characterized by Lyapunov functions. So it seems that it is a good idea to give some explicit conditions on Lyapunov functions to study the existence of almost periodic solutions, as Hale and Yoshizawa did in [14, 26]. This is exactly what we are to do in the present paper for stochastic differential equations (SDE).

For the stochastically perturbed semilinear equations, almost periodic solutions were studied by assuming that the linear part of these equations satisfies the property of exponential
dichotomy; see Halanay [13], Morozan and Tudor [19], Da Prato and Tudor [10], and Arnold and Tudor [2], among others. For general SDEs, Várvara [25] studied asymptotical almost periodic (weaker than almost periodic) solutions by assuming that the stochastic system is total stable. Very recently, Liu and Wang [16] investigated the almost periodic solutions to SDEs by the separation method.

This paper is organized as follows. Section 2 is a preliminary section. Section 3 contains main results of this paper, in which we study almost periodic solutions for SDEs by mainly the Lyapunov function method. In Section 4, we illustrate our results by some applications.

2. Preliminaries

Assume that $(M,d)$ is a complete metric space. Here is the definition of $M$-valued almost periodic and uniform almost periodic functions in the sense of Bohr:

Definition 2.1. (i) Assume $\varphi(\cdot) : \mathbb{R} \to M$ is continuous. We say set $A \subset \mathbb{R}$ is relatively dense in $\mathbb{R}$ if for any given $\epsilon > 0$, there exists $l = l(\epsilon) > 0$, such that for every $a \in \mathbb{R}$, $(a,a+l) \cap A \neq \emptyset$. If there is a set $T(\epsilon,\varphi)$ relatively dense such that for any $\tau \in T(\epsilon,\varphi)$ we have

$$\sup_{t \in \mathbb{R}} \rho(\varphi(t + \tau),\varphi(t)) < \epsilon,$$

then we say that the function $\varphi$ is almost periodic.

(ii) A continuous function $f(\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is almost periodic in $t$ uniformly on compact sets if for every compact set $S \subset \mathbb{R}^d$ there exists a relatively dense set $T(\epsilon,f,S)$ such that for every $\tau \in T(\epsilon,f,S)$ we have:

$$\sup_{(t,x) \in \mathbb{R} \times S} |f(t + \tau,x) - f(t,x)| < \epsilon.$$

We also say such $f(t,x)$ is uniformly almost periodic for short.

Bochner [3, 4] gave an equivalent condition to Bohr’s almost periodicity. The above definition of uniform almost periodicity can be found in Yoshizawa’s book [28]; Seifert and Fink made another definition of uniform almost periodicity (see Definition 2.1 in [12]).

For sequence $\alpha = \{\alpha_n\}$, we denote $\lim_{n \to +\infty} \varphi(\cdot + \alpha_n)$ as $T_{\alpha}\varphi(\cdot)$ if it exists, and the mode of convergence will be specified at each use; the similar notation will be used for $T_{\alpha}f(\cdot,\cdot) = \lim_{n \to +\infty} f(\cdot + \alpha_n,\cdot)$. For simplicity, we also denote $\varphi(\cdot + a)$ by $\varphi(\cdot)$ and $f(\cdot + a,\cdot)$ by $f_a(\cdot,\cdot)$ for given $a \in \mathbb{R}$.

For $\mathbb{R}^d$-valued uniformly almost periodic function $f(t,x)$, we denote

$$H(f) := \{g(t,x); \text{there is sequence } \alpha \text{ such that } T_{\alpha}f = g$$

uniformly on $\mathbb{R} \times S$ for each compact set $S \subset \mathbb{R}^d\},$$
as the hull of $f$. The hull has the following properties:

Proposition 2.2. Let $f(t,x)$ be uniformly almost periodic. Then:

(i) any $g \in H(f)$ is also uniformly almost periodic and $H(g) = H(f)$;
(ii) for any $g \in H(f)$, there exists a sequence $\alpha$ with $\alpha_n \to +\infty$ (or $\alpha_n \to -\infty$) such that $T_{\alpha}f = g$ uniformly on $\mathbb{R} \times S$ for any compact $S \subset \mathbb{R}^d$;
(iii) for any sequence $\alpha'$, there exists a subsequence $\alpha \subset \alpha'$ such that $T_{\alpha}f$ exists uniformly on $\mathbb{R} \times S$ for any compact $S \subset \mathbb{R}^d$.

We respectively denote $[0, +\infty)$ and $(-\infty, 0]$ as $\mathbb{R}_+$ and $\mathbb{R}_-$, and recall the definition of asymptotically almost periodic function valued in $M$ as follows.
Definition 2.3. Suppose that function \( f(\cdot) : \mathbb{R}_+ \to M \) is continuous and there exists an almost periodic function \( \eta(\cdot) : \mathbb{R} \to M \), such that
\[
\lim_{t \to +\infty} d(f(t), \eta(t)) = 0.
\] (2.3)
Then we say \( f(t) \) is asymptotically almost periodic (a.a.p. in short) on \( \mathbb{R}_+ \). The \( \eta(t) \) in (2.3) is called the almost periodic part of \( f \). The function \( f \) being a.a.p. on \( \mathbb{R}_- \) can be defined similarly.

Remark 2.4 (See [15], Chapter 1). If \( f \) is a.a.p. on \( \mathbb{R}_+ \) or \( \mathbb{R}_- \), then its almost periodic part is unique.

Lemma 2.5. The following statements are equivalent to \( f \) being asymptotic almost periodic on \( \mathbb{R}_+ \):

(i) For any sequence \( \alpha' = \{\alpha'_n\} \) such that \( \alpha'_n \to +\infty \), there exists suitable subsequence \( \alpha \subset \alpha' \) such that \( T_\alpha f(t) \) uniformly exists on \( \mathbb{R}_+ \).

(ii) For any sequence \( \alpha' = \{\alpha'_n\} \) such that \( \alpha'_n \to +\infty \), there exists a subsequence \( \alpha \subset \alpha' \) and a constant \( \sigma = \sigma(\alpha) > 0 \) such that \( T_\alpha f \) exists pointwise on \( \mathbb{R}_+ \) and if sequences \( \delta > 0, \beta \subset \alpha, \gamma \subset \alpha \) are such that
\[
T_{\delta+\beta} f = h_1 \quad \text{and} \quad T_{\delta+\gamma} f = h_2
\]
exist pointwisely on \( \mathbb{R}_+ \), then either \( h_1 \equiv h_2 \) or \( \inf_{t \in \mathbb{R}_+} d(h_1(t), h_2(t)) \geq 2\sigma \).

The similar results hold when \( f \) is asymptotic almost periodic on \( \mathbb{R}_- \).

In this paper, we study the SDE:
\[
(2.4) \quad dX(t) = f(t, X(t))dt + g(t, X(t))dW(t),
\]
where \( f(t, x) \) is an \( \mathbb{R}^d \)-valued continuous function, \( g(t, x) \) is a \((d \times m)\)-matrix-valued continuous function, and \( W \) is a standard \( m \)-dimensional Brownian motion. And we usually assume the coefficients are uniformly almost periodic. Note that almost periodicity is defined on the whole \( \mathbb{R} \), but the Brownian motions in SDEs usually defined on \( \mathbb{R}_+ \). So we need to introduce two-sided Brownian motion: for two independent Brownian motions \( W_1(t), W_2(t) \) on the probability space \((\Omega, \mathcal{F}, P)\), let
\[
W(t) = \begin{cases} 
W_1(t), & \text{for } t \geq 0, \\
-W_2(-t), & \text{for } t \leq 0.
\end{cases}
\]

Then \( W \) is a two-sided Brownian motion defined on the filtered probability space \((\Omega, \mathcal{F}, P, \mathcal{F}_t)\) with \( \mathcal{F}_t = \sigma\{W(u) : u \leq t\}, t \in \mathbb{R} \).

Furthermore, we always assume (2.4)’s coefficients satisfy the following condition:

(H) The functions \( f, g \) are uniformly almost periodic. And there exists a constant \( K > 0 \) such that, for every \( t \in \mathbb{R} \) and \( x, y \in \mathbb{R}^d \),
\[
|f(t, x) - f(t, y)| \vee |g(t, x) - g(t, y)| \leq K|x - y|,
\]
where \( a \vee b = \max\{a, b\} \) for \( a, b \in \mathbb{R} \).

For SDE (2.4) satisfying condition (H), if there exists a sequence \( \alpha \) such that \( T_{\alpha} f = \tilde{f} \) and \( T_{\alpha} g = \tilde{g} \), we denote the SDE with coefficients \((T_{\alpha} f, T_{\alpha} g)\) as \((\tilde{f}, \tilde{g}) \in H(f, g)\) or \( T_{\alpha}(f, g) = (\tilde{f}, \tilde{g}) \) for short. Besides, by the definition of uniform almost periodic function, if coefficients of (2.4) satisfy the condition (H), they must satisfy the global linear growth condition, that is, there is some constant \( \tilde{K} > 0 \), such that
\[
|f(t, x)| \vee |g(t, x)| \leq \tilde{K}(1 + |x|^2), \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^d.
\]
For \( \mathbb{R}^d \)-valued random variable \( X \) on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), we denote \( \mathcal{L}(X) \) as the distribution (or law) of \( X \) on \( \mathbb{R}^d \). We denote by \( \mathcal{P}(\mathbb{R}^d) \) the space of all Borel probability measures on \( \mathbb{R}^d \). For an \( \mathbb{R}^d \)-valued random variable \( X \) or stochastic process \( Y(t) \), we define the following norms:

\[
\|X\|_2 := \left( \int_{\Omega} |X(\omega)|^2 d\mathbb{P}(\omega) \right)^{\frac{1}{2}}, \quad \|Y(t)\|_{\infty} := \sup_t \|Y(t)\|_2.
\]

In what follows, we denote:

\[
L^2(P, \mathbb{R}^d) := \{X : \|X\|_2 < \infty\}, \quad B_r := \{X \in L^2(P, \mathbb{R}^d) : \|X\|_2 \leq r\},
\]

\[
\mathcal{D}_r := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \exists X \in B_r \text{ such that } \mathcal{L}(X) = \mu\},
\]

\[
\mathcal{B}^{(f,g)} := \{X(\cdot) : (X, W) \text{ weakly solves equation } (f, g) \text{ on } \mathbb{R}\}
\]

on some filtered probability space for some \( W \) and \( \|X\|_{\infty} \leq r \),

\[
\mathcal{D}^{(f,g)} := \{\mu : \mu(\cdot) = \mathcal{L}(X(\cdot)) \text{ for some } X \in \mathcal{B}^{(f,g)}\},
\]

\[
\mathcal{B}^{(f,g)} := \bigcup_{r>0} \mathcal{B}^{(f,g)}, \quad \mathcal{D}^{(f,g)} := \bigcup_{r>0} \mathcal{D}^{(f,g)}.
\]

We focus on the almost periodicity of distributions of SDEs’ solutions instead of solutions themselves. It’s well known that \( \mathcal{P}(\mathbb{R}^d) \) can be metrized with some distance (which we denote as \( \rho(\cdot, \cdot) \)), such that the convergence under distance \( \rho(\cdot, \cdot) \) is equivalent to the convergence under the weak-* topology of \( \mathcal{P}(\mathbb{R}^d) \), and \( \mathcal{P}(\mathbb{R}^d) \) is a complete metric space under \( \rho(\cdot, \cdot) \) (see [20] Theorem 2.6.2 for details).

For a \( \mathcal{P}(\mathbb{R}^d) \)-valued continuous function \( f \), one of the necessary conditions of the almost periodicity of \( f \) is that, the set \( \{f(t) : t \in \mathbb{R}\} \) is contained in some compact set. Naturally we need to consider distributions of solutions for SDEs within some compact set. We get compactness on the space \( \mathcal{P}(\mathbb{R}^d) \) by \( L^2 \)-boundedness (see [21] for details).

We define the uniform stability of distributions of solutions for SDEs as follows:

**Definition 2.6.** \( \forall t_0 \in \mathbb{R} \), we say element \( \mu(t) \in \mathcal{D}^{(f,g)} \) is uniformly stable on \([t_0, +\infty)\) within \( \mathcal{D}^{(f,g)} \) if for every \( \epsilon > 0 \), there exists \( \delta = \delta(\epsilon) > 0 \) such that for any \( t_1 \geq t_0 \) and any other element \( \eta(t) \in \mathcal{D}^{(f,g)} \) satisfying

\[
\rho(\mu(t_1), \eta(t_1)) < \delta,
\]

we have

\[
\sup_{t \in [t_1, +\infty)} \rho(\mu(t), \eta(t)) < \epsilon.
\]

If \( \mu(t) \) is uniformly stable on \([t_1, +\infty)\) for every \( t_1 \in \mathbb{R} \), we call it uniformly stable for short.

In what follows, we get the stability of solutions’ distributions mainly by Lyapunov functions, which satisfy the following condition:

\((\text{L})\) Assume that \( V(\cdot, \cdot) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}_+ \) is a function \( C^2 \) in \( t \in \mathbb{R} \), \( C^3 \) in \( x \in \mathbb{R}^d \). Assume that the differentials \( D^iV \) of \( V \) for \( i = 0, 1, 2 \) and the derivatives \( V_{tx_ix_j}, V_{x_ix_jx_k} \) for \( i,j,k = 1,2,\cdots,d \) are bounded on \( \mathbb{R} \times S \) for every compact set \( S \subset \mathbb{R}^d \). Furthermore,

\[
\inf_{t \in \mathbb{R}} V(t, x) > 0 \text{ for each } x \neq 0, \text{ and } V(t, 0) = 0 \text{ for all } t \in \mathbb{R}.
\]

3. **Main Results**

In this paper, we need following results from [16] for further discussion:

**Proposition 3.1.** ([16], Theorem 3.1). Consider the following family of Itô stochastic equations on \( \mathbb{R}^d \)

\[
dX = f_n(t, X)dt + g_n(t, X)dW, \quad n = 1, 2, \cdots,
\]
where \( f_n \) are \( \mathbb{R}^d \)-valued, \( g_n \) are \((d \times m)\)-matrix-valued, and \( W \) is a standard \( m \)-dimensional Brownian motion. Assume that \( f_n, g_n \) satisfy condition (H). Assume further that \( f_n \to f \), \( g_n \to g \) point-wise on \( \mathbb{R} \times \mathbb{R}^d \) as \( n \to \infty \), and that \( X_n(t) \in \mathcal{B}_r^{(f_n, g_n)} \) for some constant \( r > 0 \), independent of \( n \). Then there is a subsequence of \( \{X_n\} \) which converges in distribution, uniformly on compact intervals, to some \( X(t) \in \mathcal{B}_r^{(f, g)} \).

**Proposition 3.2** ([16], Lemma 4.1). Consider SDE \((2.4)\) with coefficients satisfying condition (H). If SDE \((2.4)\) admits an \( L^2 \)-bounded solution \( X(t) \) on \( \mathbb{R} \) which is asymptotically almost periodic in distribution on \( \mathbb{R} \), then SDE \((2.4)\) admits a solution \( Y \) on \( \mathbb{R} \) which is almost periodic in distribution such that

\[
\lim_{t \to +\infty} \rho(\mathcal{L}(X(t)), \mathcal{L}(Y(t))) = 0 \quad \text{and} \quad \sup_{t \in \mathbb{R}} \mathbb{E}|Y(t)|^2 \leq \sup_{t \in \mathbb{R}} \mathbb{E}|X(t)|^2.
\]

In particular, \( \mathcal{L}(Y) \) is the almost periodic part of \( \mathcal{L}(X) \). The similar result holds when \( X \) is asymptotically almost periodic in distribution on \( \mathbb{R}^- \).

Consider \((2.4)\) and let \( V \) satisfy condition (L). For \( t \in \mathbb{R} \) and \( x, y \in \mathbb{R}^d \), denote

\[
\mathcal{L}V(t, x - y) := \frac{\partial V}{\partial t}(t, x - y) + \sum_{i=1}^{d} \frac{\partial V}{\partial x_i}(t, x - y)(f_i(t, x) - f_i(t, y))
\]

\[
+ \frac{1}{2} \sum_{l=1}^{m} \sum_{i,j=1}^{d} (g_{il}(t, x) - g_{il}(t, y)) \frac{\partial^2 V}{\partial x_i \partial x_j}(t, x - y) \cdot (g_{ij}(t, x) - g_{ij}(t, y)).
\]

(3.2)

Now we give a sufficient condition to the uniform stability in distribution we defined in Definition 2.6.

**Theorem 3.3.** Suppose that \((2.4)\)'s coefficients satisfy condition (H) and there is a function \( V(\cdot, \cdot) \) satisfying condition (L). Assume that there exists some constant \( b > 0 \) such that for all \((t, x) \in \mathbb{R} \times \mathbb{R}^d \),

\[
V(t, x) \leq b|x|^2,
\]

(3.3)

\[
\mathcal{L}V(t, x - y) \leq 0.
\]

(3.4)

Then if \( D_{r, 2.4} \neq \emptyset \) for some \( r > 0 \), all the elements of it are uniformly stable within \( D_{r, 2.4} \); i.e., all of these elements are almost periodic.

**Proof.** **Step 1. Uniform stability.** If there is some \( \mu(t) \in D_{r, 2.4} \) which is not uniformly stable on \([t_0, +\infty)\) within \( D_{r, 2.4} \) for some \( t_0 \in \mathbb{R} \), then there is a sequence \( \mu_n(t) \in D_{r, 2.4} \) such that \( \rho(\mu_n(t_0), \mu(t_0)) \to 0 \) and there are \( t_n \geq t_0 \) such that

\[
\inf_n \rho(\mu_n(t_n), \mu(t_n)) \geq \epsilon_0.
\]

(3.5)

By Skorohod representation theorem, there exist suitable random variables \( \hat{X}_n, \hat{X} \) such that \( \mathcal{L}(\hat{X}_n) = \mu_n(t_0) \), \( \mathcal{L}(\hat{X}) = \mu(t_0) \) and \( \hat{X}_n \overset{a.s.}{\to} \hat{X} \). By the global Lipschitz condition of coefficients, there exist strong solutions \( X_n(t), X(t) \in \mathcal{B}_r^{(f_n, g_n)} \) for given Brownian motion \( W \) such that \( X_n(t_0) = \hat{X}_n, X(t_0) = \hat{X}, \) and \( \mathcal{L}(X(t)) = \mu(t) \), \( \mathcal{L}(X_n(t)) = \mu_n(t) \).

We want to prove that \( \rho(\mu_n(t_n), \mu(t_n)) \to 0 \) and hence get contradiction to \( (3.5) \). It suffices to prove that \( X_n(t) \) uniformly converge to \( X(t) \) in probability on \([t_0, +\infty)\), that is, for every \( \epsilon > 0 \), when \( n \) is large enough,

\[
\mathbb{P}\{\sup_{t \geq t_0} |X_n(t) - X(t)| \geq \epsilon\} < \epsilon.
\]

(3.6)
Firstly, we prove that \( V(t, X_n(t) - X(t)) \) is a supermartingale on \([t_0, +\infty)\) for each \( n \). For \( t \geq t_0 \), we have
\[
X_n(t) - X(t) = \hat{X}_n - \hat{X} + \int_{t_0}^{t} f(s, X_n(s)) - f(s, X(s))ds + \int_{t_0}^{t} g(s, X_n(s)) - g(s, X(s))dW(s).
\]
For every \( \epsilon > 0 \), let
\[
(3.7) \quad V_\epsilon := \inf_{|x| \leq \epsilon, t \geq t_0} V(t, x).
\]
By (3.6) we can see \( V_\epsilon > 0 \). For \( t_0 \leq s < t < +\infty \), and every \( k, n \in \mathbb{N} \), we define a stopping time
\[
\tau^n_k := \inf\{t \geq s : |X_n(t)| \vee |X(t)| > k\}.
\]
By Itô’s formula,
\[
\begin{align*}
V(\tau^n_k \wedge t, X_n(\tau^n_k \wedge t) - X(\tau^n_k \wedge t)) &= V(s, X_n(s) - X(s)) + \int_{s}^{\tau^n_k \wedge t} \mathcal{L}V(u, X_n(u) - X(u))du \\
&+ \int_{s}^{\tau^n_k \wedge t} \sum_{i=1}^{d} \sum_{j=1}^{d} (g_{ji}(u, X_n(u)) - g_{ji}(u, X(u))) \frac{\partial V}{\partial x_j}(u, X_n(u) - X(u))dW_i(u).
\end{align*}
\]
Then we have
\[
\mathbb{E}\left( \int_{s}^{\tau^n_k \wedge t} \sum_{i=1}^{d} \sum_{j=1}^{d} (g_{ji}(u, X_n(u)) - g_{ji}(u, X(u))) \frac{\partial V}{\partial x_j}(u, X_n(u) - X(u))dW_i(u) \right) = 0 \text{ a.s.}
\]
By (3.4),
\[
\begin{align*}
\mathbb{E}(V(\tau^n_k \wedge t, X_n(\tau^n_k \wedge t) - X(\tau^n_k \wedge t))|\mathcal{F}_s) &= \mathbb{E}(V(s, X_n(s) - X(s))|\mathcal{F}_s) + \mathbb{E}\left( \int_{s}^{\tau^n_k \wedge t} \mathcal{L}V(u, X_n(u) - X(u))du |\mathcal{F}_s \right) \leq \mathbb{E}(V(s, X_n(s) - X(s))|\mathcal{F}_s) \leq V(s, X_n(s) - X(s)), \text{ a.s.}
\end{align*}
\]
Because \( V(t, x) \) is \( C^2 \) in \( t, \tau^n_k \xrightarrow{a.s.} +\infty \) as \( k \to +\infty \) for every \( n \), by Fatou’s lemma we have:
\[
\begin{align*}
\mathbb{E}(V(t, X_n(t) - X(t))|\mathcal{F}_s) &= \mathbb{E}(\liminf_{k \to +\infty} V(\tau^n_k \wedge t, X_n(\tau^n_k \wedge t) - X(\tau^n_k \wedge t))|\mathcal{F}_s) \\
&\leq \liminf_{k \to +\infty} \mathbb{E}(V(\tau^n_k \wedge t, X_n(\tau^n_k \wedge t) - X(\tau^n_k \wedge t))|\mathcal{F}_s) \\
&\leq V(s, X_n(s) - X(s)), \text{ a.s.}
\end{align*}
\]
So \( V(t, X_n(t) - X(t)) \) is a supermartingale on \([t_0, +\infty)\).

Now we want to prove that \( \mathbb{E}\sqrt{V(t_0, \hat{X}_n - \hat{X})} \) is sufficiently small when \( n \) is large enough. By Jensen’s inequality and (3.3) we have
\[
(3.10) \quad \mathbb{E}(\sqrt{V(t, X_n(t) - X(t))}|\mathcal{F}_s) \leq \sqrt{\mathbb{E}(V(t, X_n(t) - X(t))|\mathcal{F}_s) \leq \sqrt{V(s, X_n(s) - X(s)), \text{ a.s.}}}
\]
That is, \( \sqrt{V(t, X_n(t) - X(t))} \) is a supermartingale. So by the martingale inequality we have
\[
P\left\{ \sup_{t \in [t_0, +\infty)} |X_n(t) - X(t)| \geq \epsilon \right\} \leq P\{ \sup_{t \in [t_0, +\infty)} \sqrt{V(t, X_n(t) - X(t))} \geq \sqrt{V_\epsilon} \}
\]
\[
(3.11)
\]
Note that \( \hat{X}_n \xrightarrow{a.s.} \hat{X} \) and \( \sup_n E|\hat{X}_n|^2 \leq r^2 \), we have (cf. [8, Theorems 4.5.2, 4.5.4]):
\[
E|\hat{X}_n| \to E|\hat{X}|, \text{ as } n \to +\infty,
\]
and
\[
\lim_{n \to +\infty} E|\hat{X}_n - \hat{X}| = 0.
\]
By (3.13), we have
\[
E\sqrt{V(t_0, \hat{X}_n - \hat{X})} \leq \sqrt{E|\hat{X}_n - \hat{X}|} \leq \frac{\sqrt{E|\hat{X}_n|}}{\sqrt{V_\epsilon}}
\]
which implies that, if \( n \) is large enough such that
\[
E|\hat{X}_n - \hat{X}| < \frac{\epsilon \sqrt{V_\epsilon}}{\sqrt{b}},
\]
we will have (3.6). Thus
\[
\sup_{t \geq t_0} \rho(\mu_n(t), \mu(t)) \to 0,
\]
which is contradictory to (3.5). Thus each element of \( D_{f,T} \) is uniformly stable within \( D_{f,T} \).

**Step 2. Inherited property and a.a.p.** Now we want to prove that the consequence of step 1 is also valid for all the hull equations.

Let the sequence \( \alpha' \) be such that \( (T_{\alpha'f}, T_{\alpha'g}) \) uniformly exists on \( \mathbb{R} \times S \) for any compact set \( S \subset \mathbb{R} \). Since \( V, V_t, V_{x_i} \) are bounded on \( \mathbb{R} \times S \), \( V(t + \alpha'_n, x) \) are uniformly bounded and equi-continuous on \( I \times S \) for any compact interval \( I \subset \mathbb{R} \). By Arzela-Ascoli’s theorem, there is suitable subsequence \( \alpha \subset \alpha' \) such that \( T_\alpha V(t, x) \) exists uniformly on \( I \times S \). By the diagonalization argument, the \( \alpha \) could be chosen such that \( T_\alpha V \) exists uniformly on any compact subset of \( \mathbb{R} \times \mathbb{R}^d \).

Similarly we can extract further subsequence from \( \alpha \), which we still denote by \( \alpha \) itself, such that \( T_\alpha V_t, T_\alpha V_{x_i}, T_\alpha V_{x_i x_j} \) exist uniformly on compact subsets of \( \mathbb{R} \times \mathbb{R}^d \). More precisely, we have
\[
(3.12) \quad \frac{\partial T_\alpha V}{\partial t} = T_\alpha V_t, \quad \frac{\partial T_\alpha V}{\partial x_i} = T_\alpha V_{x_i}, \quad \frac{\partial^2 T_\alpha V}{\partial x_i \partial x_j} = T_\alpha V_{x_i x_j}, \text{ for } i, j = 1, \ldots, d, \text{ on } \mathbb{R} \times \mathbb{R}^d.
\]
So we have
\[
(3.13) \quad T_\alpha V(t, x) \leq b|x|^2,
\]
\[
\mathcal{L}T_\alpha V(t, x - y) = \frac{\partial T_\alpha V}{\partial t}(t, x - y) + \sum_{i=1}^d \frac{\partial T_\alpha V}{\partial x_i}(t, x - y)(T_\alpha f_i(t, x) - T_\alpha f_i(t, y))
\]
\[
(3.14) \quad \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^d \left( T_\alpha g_{ij}(t, x) - T_\alpha g_{ij}(t, y) \right) \frac{\partial^2 T_\alpha V}{\partial x_i \partial x_j}(t, x - y) \leq 0
\]
for all \( (t, x, y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \). Repeating Step 1, we obtain that all the elements of \( D_{f,T}^{(T_{\alpha f}, T_{\alpha g})} \) are uniformly stable within \( D_{f,T}^{(T_{\alpha f}, T_{\alpha g})} \).
By the uniform stability and the finiteness of the set $D^{(f,g)}_r$, there is a separating constant $d(f,g)$, depending only on $(f,g)$ but independent of $\mu \in D_r^{(f,g)}$, such that for any two different elements $\eta(t), \mu(t) \in D_r^{(f,g)}$ we have

$$\inf_{t \in \mathbb{R}_-} \rho(\eta(t), \mu(t)) > d(f,g). \tag{3.15}$$

By Proposition $2.2$ (ii), we may assume with loss of generality that the above sequence $\alpha$ satisfies $\lim_{n \to \infty} \alpha_n = -\infty$, so it follows from (3.15) that

$$\inf_{t \in \mathbb{R}_-} \rho(T_{\alpha}\eta(t), T_{\alpha}\mu(t)) \geq d(f,g).$$

On the other hand, it follows from Proposition $3.1$ that $T_{\alpha}\mu(t) \in D_r^{(T_{\alpha}f,T_{\alpha}g)}$, so $D_r^{(T_{\alpha}f,T_{\alpha}g)}$ has no less elements than $D_r^{(f,g)}$ does.

Conversely, by Proposition $2.2$ (i), $(T_{\alpha}f,T_{\alpha}g)$ is uniformly almost periodic and $(f,g) \in H(T_{\alpha}f,T_{\alpha}g)$. So, by the same symmetric argument as above, $D_r^{(f,g)}$ also has no less elements than $D_r^{(T_{\alpha}f,T_{\alpha}g)}$ does and the separating constant $d(T_{\alpha}f,T_{\alpha}g) \leq d(f,g)$. That is, all the equations in the hull $H(f,g)$ share the same number of elements as $D_r^{(f,g)}$ and the same separating constant $d(f,g)$.

Now we prove that all the elements of $D_r^{(f,g)}$ are a.a.p. For the above sequence $\alpha$ with $\alpha_n \to -\infty$ and given sequence $\delta = \{\delta_n\}$ with $\delta_n < 0$, by Proposition $2.2$ (iii) there exist suitable subsequences which we denote as themselves such that $(T_{\alpha+\delta}f,T_{\alpha+\delta}g)$ exists uniformly on $\mathbb{R} \times S$ for any compact set $S \subset \mathbb{R}^d$. By Arzela-Ascoli’s theorem there are subsequences $\beta, \gamma \subset \alpha$ such that $T_{\beta+\delta}\mu(t), T_{\gamma+\delta}\mu(t)$ exist uniformly on compact intervals (see the proof of Theorem 3.1 for details). By Proposition 3.1 $T_{\beta+\delta}\mu(t), T_{\gamma+\delta}\mu(t) \in D_r^{(T_{\alpha+\delta}f,T_{\alpha+\delta}g)}$, then by the separating property obtained above we have

$$T_{\beta+\delta}\mu(t) \equiv T_{\gamma+\delta}\mu(t) \text{ or } \inf_{t \in \mathbb{R}_-} \rho(T_{\beta+\delta}\mu(t), T_{\gamma+\delta}\mu(t)) \geq d(f,g).$$

Then it follows from Lemma 2.3 that all the elements of $D_r^{(f,g)}$ are all a.a.p. on $\mathbb{R}_-$. By Proposition 3.2 there is some $\hat{\mu}(t) \in D_r^{(f,g)}$, which is almost periodic and satisfies

$$\lim_{t \to -\infty} \rho(\mu(t), \hat{\mu}(t)) = 0.$$  

By the separating property, $\hat{\mu}(t) = \mu(t)$, which implies that each element of $D_r^{(f,g)}$ is almost periodic. The proof is complete. \hfill \Box

The following result, which limits the number of $D_r^{(f,g)}$’s elements to be one, is an important special case of Theorem 3.3 and is more convenient for use in applications.

**Theorem 3.4.** Suppose that $2.3$’s coefficients satisfy condition (H). Assume that there is a function $V(\cdot,\cdot)$ satisfying condition (L), and there are constants $a, b > 0$ such that

$$a|x|^2 \leq V(t,x) \leq b|x|^2 \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{3.16}$$

Assume that there is some positively definite function $c(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ which is convex, increasing on $\mathbb{R}_+$, and

$$\frac{\partial V(t,x-y)}{\partial y} \leq -c(|x-y|^2) \quad \forall t \in \mathbb{R}, \forall x,y \in \mathbb{R}^d. \tag{3.17}$$

Then if $D_r^{(f,g)} \neq \emptyset$, it has a unique element which is almost periodic.

**Proof.** We prove the uniqueness by contradiction. If there are two elements $\mu(t), \eta(t) \in D_r^{(f,g)}$, then there’s some $r > 0$ such that $\mu(t), \eta(t) \in D_r^{(f,g)}$. Assume that $X(t), Y(t)$ are two strong
\( \mathbb{L}^2 \)-bounded solutions of (2.31) for given Brownian motion \( W(t) \) such that \( \mathcal{L}(X(t)) = \mu(t) \), \( \mathcal{L}(Y(t)) = \eta(t) \).

For given \( \epsilon > 0 \), let \( T(\epsilon) = 2br^2/\epsilon + 1 \). Firstly, we prove that for every \( t \in \mathbb{R} \) there is \( t_1 \in [t, t + T(\epsilon)] \) such that

\[
(3.18) \quad \mathbb{E}|X(t_1) - Y(t_1)|^2 < \epsilon.
\]

If this is not true, then there is \( \bar{t} \in \mathbb{R} \) and \( \epsilon_0 > 0 \) such that

\[
\inf_{t \in [\bar{t}, \bar{t} + T(\epsilon_0)]} \mathbb{E}|X(t) - Y(t)|^2 \geq \epsilon_0.
\]

Similar to the proof of Theorem 3.3, for given \( s \in \mathbb{R} \), we define

\[
\tau_k := \inf\{t \geq s : |Y(t)| \vee |X(t)| > k\}.
\]

Then it follows from Ito’s formula that for \( t \geq s \),

\[
V(\tau_k \wedge t, X(\tau_k \wedge t) - Y(\tau_k \wedge t)) = \mathbb{E}V(s, X(s) - Y(s)) + \int_{s}^{\tau_k \wedge t} \mathbb{E}\mathcal{L}V(u, X(u) - Y(u))du
\]

\[
+ \int_{s}^{\tau_k \wedge t} \sum_{i=1}^{m} \sum_{j=1}^{d} [g_{ji}(u, X_n(u)) - g_{ji}(u, X(u))] \frac{\partial V}{\partial x_j}(u, X_n(u) - X(u))dW_i(u).
\]

Since

\[
\sup_{t \in \mathbb{R}} \mathbb{E}|X(t) - Y(t)|^2 \leq 2r^2,
\]

by (3.16), (3.17) we have

\[
\mathbb{E}V(\tau_k \wedge t, X(\tau_k \wedge t) - Y(\tau_k \wedge t)) \leq b\mathbb{E}|X(s) - Y(s)|^2 - \mathbb{E} \int_{s}^{\tau_k \wedge t} c(|X(u) - Y(u)|^2)du
\]

\[
\leq 2br^2 - \mathbb{E} \int_{s}^{\tau_k \wedge t} c(|X(u) - Y(u)|^2)du.
\]

Because \( c(r) \) is convex, increasing on \( \mathbb{R}_+ \), by Jensen’s inequality we have:

\[
\mathbb{E} \int_{s}^{\tau_k \wedge t} c(|X(u) - Y(u)|^2)du \geq \int_{s}^{\tau_k \wedge t} c(\mathbb{E}|X(u) - Y(u)|^2)du \geq c(\epsilon_0)(\tau_k \wedge t - s).
\]

So

\[
(3.19) \quad \mathbb{E}V(\tau_k \wedge t, X(\tau_k \wedge t) - Y(\tau_k \wedge t)) \leq 2br^2 - c(\epsilon_0)(\tau_k \wedge t - s).
\]

Noting that \( \tau_k \xrightarrow{a.s.} +\infty \) as \( k \to +\infty \), by Fatou’s lemma and (3.19) we have

\[
\mathbb{E}V(t, X(t) - Y(t)) = \mathbb{E}(\liminf_{k \to +\infty} V(\tau_k \wedge t, X(\tau_k \wedge t) - Y(\tau_k \wedge t)))
\]

\[
\leq \liminf_{k \to +\infty} \mathbb{E}[2br^2 - c(\epsilon_0)(\tau_k \wedge t - s)]
\]

\[
\leq 2br^2 - c(\epsilon_0)(t - s).
\]

Letting \( s = \hat{t} \) and \( t = \hat{t} + T(\epsilon_0) \), we have

\[
0 \leq \mathbb{E}(\hat{t} + T(\epsilon_0), X(\hat{t} + T(\epsilon_0)) - Y(\hat{t} + T(\epsilon_0))) \leq 2br^2 - c(\epsilon_0)T(\epsilon_0) = -c(\epsilon_0) < 0,
\]

a contradiction. Thus there is \( t_1 \in [t, t + T(\epsilon)] \) such that (3.18) is valid.

For given \( s \in \mathbb{R} \), assume \( t_1 \in [s, s + T(\epsilon)] \) fulfills (3.18). By (3.16)–(3.18), for any \( t \geq t_1 \), we have:

\[
a\mathbb{E}|X(t) - Y(t)|^2 \leq \mathbb{E}V(t, X(t) - Y(t)) \leq \mathbb{E}V(t_1, X(t_1) - Y(t_1)) \leq b\epsilon.
\]
Note that \( s \in \mathbb{R} \) is arbitrarily chosen and \( T(\epsilon) \) is only determined by \( \epsilon > 0 \), so we actually have proved \( \mathbb{E}[X(t) - Y(t)]^2 \leq \epsilon \) for all \( t \in \mathbb{R} \).

Thus \( X(t) = Y(t) \) for all \( t \in \mathbb{R} \) almost surely, which implies that \( \mu(t) = \eta(t) \) for all \( t \in \mathbb{R} \). That is, \( \mathcal{D}_\mathcal{E}(\mathcal{A}) \) has a unique element if it is not empty. Finally, it follows from Theorem 3.3 that this unique element is almost periodic. The proof is complete. \( \square \)

Now we give another result for the existence of \( \mathcal{D}_\mathcal{E}(\mathcal{A}) \)'s almost periodic elements without the information of the number of its elements.

**Theorem 3.5.** Assume that \( \mathcal{E}_\mathcal{A} \)‘s coefficients satisfy condition (H), and there exists a function \( V(\cdot, \cdot) \) satisfying condition (L). Suppose that there is some constant \( b > 0 \) such that (3.3) is valid on \( \mathbb{R}_+ \times \mathbb{R}^d \) and for all \( t \in \mathbb{R}_+, s_1, s_2 \in \mathbb{R}_+ \) and \( x, y \in \mathbb{R}^d \),

\[
\mathcal{L}_{s_1, s_2} V(t, x - y) := \frac{\partial V}{\partial t}(t, x - y) + \sum_{i=1}^d \frac{\partial V}{\partial x_i}(t, x - y) (f_i(t + s_1, x) - f_i(t + s_2, y))
\]

\[
+ \frac{1}{2} \sum_{l=1}^m \sum_{i,j=1}^d (g_{ii}(t + s_1, x) - g_{ii}(t + s_2, y)) \frac{\partial^2 V}{\partial x_i \partial x_j}(t, x - y)
\]

\[
\cdot (g_{ij}(t + s_1, x) - g_{ij}(t + s_2, y)) \leq 0.
\]

Then if \( \mathcal{E}_\mathcal{A} \) has \( L^2 \)-bounded solutions, the distributions of these solutions are a.a.p. on \( \mathbb{R}_+ \) and \( \mathcal{A} \) admits at least one solution with almost periodic distribution.

**Proof.** For sequence \( \alpha = \{\alpha_n\} \) such that \( \alpha_n \to +\infty \), assume that \( (T_{\alpha f}, T_{\alpha g}) \) exist uniformly on \( \mathbb{R} \times S \) for any compact set \( S \subset \mathbb{R}^d \), and \( T_{\alpha \mu}(t) \) exists uniformly on compact intervals (see, again, the proof of [10] Theorem 3.1 for details). For \( r > 0 \) and every \( \mu(t) \in \mathcal{D}_\mathcal{E}(\mathcal{A}) \), we what to prove that \( T_{\alpha \mu}(t) \) uniformly exists on \( \mathbb{R}_+ \).

By Skorohod representation theorem, there are suitable random variables \( \hat{X}_n, \hat{X} \) such that \( \hat{X}_n \xrightarrow{a.s.} \hat{X} \) as \( n \to +\infty \) and \( \mathcal{L}(\hat{X}_n) = \mu(\alpha_n), \mathcal{L}(\hat{X}) = T_{\alpha \mu}(0) \). By the global Lipschitz condition of coefficients, for given Brownian motion \( W(t) \), there are strong solutions \( X_n(t) \in \mathcal{B}_r(f_{\alpha_n g_{\alpha_n}}) \) such that \( X_n(0) = \hat{X}_n \), and \( \mathcal{L}(X_n(t)) = \mu(t + \alpha_n) \). And for every \( n, p \in \mathbb{N} \), we have

\[
X_{n+p}(t) - X_n(t) = \hat{X}_{n+p} - \hat{X}_n + \int_0^t (f(u + \alpha_{n+p}), X_{n+p}(u)) - f(u + \alpha_n, X_n(u))du
\]

\[
+ \int_0^t (g(u + \alpha_{n+p}, X_{n+p}(u)) - g(u + \alpha_n, X_n(u)))dW(u).
\]

We now show that \( V(t, X_{n+p}(t) - X_n(t)) \) is a supermartingale on \( \mathbb{R}_+ \) for given \( n \) and \( p \). For every \( 0 < s < t \), we define stopping times \( \tau_k^{n,p} := \inf\{t \geq s; |X_n(t)| \vee |X_{n+p}(t)| > k\} \), for every \( k, n, p \in \mathbb{N} \).

By Itô’s formula we have

\[
V(\tau_k^{n,p} \wedge t, X_{n+p}(\tau_k^{n,p} \wedge t) - X_n(\tau_k^{n,p} \wedge t))
\]

\[
= V(s, X_{n+p}(s) - X_n(s)) + \int_s^{\tau_k^{n,p} \wedge t} \mathcal{L}_{\alpha_{n+p}, \alpha_n} V(u, X_{n+p}(u) - X_n(u))du
\]

\[
+ \int_s^{\tau_k^{n,p} \wedge t} \sum_{i=1}^m \sum_{j=1}^d [g_{ii}(u + \alpha_{n+p}, X_{n+p}(u)) - g_{ii}(u + \alpha_n, X_n(u))] \frac{\partial V}{\partial x_j}(u, X_{n+p}(u) - X_n(u))dW_i(u).
\]
Since
\[
\mathbb{E}\left( \int_s^{t_n,p} \sum_{i=1}^d \sum_{j=1}^m [g_{ji}(u + \alpha_{n+p}, X_{n+p}(u)) - g_{ji}(u + \alpha_n, X_n(u))] \cdot \frac{\partial V}{\partial x_j}(u, X_{n+p}(u) - X_n(u)) dW_i(u) | \mathcal{F}_s \right) = 0, \text{ a.s.}
\]
it follows from (3.20) that
\[
\mathbb{E}(V(t, X_{n+p}(t)) - X_n(t)) | \mathcal{F}_s) = \mathbb{E}(\lim_{k \to +\infty} V(t, X_{n+p}(\tau_{k}^{n,p} \land t) - X_n(\tau_{k}^{n,p} \land t)) | \mathcal{F}_s)
\]
\[
\leq \lim_{k \to +\infty} \mathbb{E}(V(t, X_{n+p}(s) - X_n(s)) | \mathcal{F}_s)
\]
\[
= V(s, X_{n+p}(s) - X_n(s)), \text{ a.s.}
\]
Noting that \( \tau_{k}^{n,p} \overset{a.s.}{\to} +\infty \) as \( k \to +\infty \) for every \( n, p \), we have by Fatou’s lemma (similar to (3.9)):
\[
\mathbb{E}(V(t, X_{n+p}(t) - X_n(t)) | \mathcal{F}_s)
\leq \mathbb{E}(V(t, X_{n+p}(t) - X_n(t)) | \mathcal{F}_s)
\]
\[
\leq V(s, X_{n+p}(s) - X_n(s)), \text{ a.s.}
\]
That is, \( V(t, X_{n+p}(t) - X_n(t)) \) is a supermartingale for given \( p \) and \( n \). Similar to (3.10), by Jensen’s inequality,
\[
\mathbb{E}\left( \sqrt{V(t, X_{n+p}(t) - X_n(t))} | \mathcal{F}_s \right) \leq \sqrt{\mathbb{E}(V(t, X_{n+p}(t) - X_n(t)) | \mathcal{F}_s)}
\]
\[
\leq \sqrt{V(s, X_{n+p}(s) - X_n(s)), \text{ a.s.}}
\]
So \( \sqrt{V(t, X_{n+p}(t) - X_n(t))} \) is also a supermartingale.

For any \( \epsilon > 0 \), we define \( V_\epsilon > 0 \) as the one in (3.7). Then by the martingale inequality, we have
\[
P\left\{ \sup_{t \in \mathbb{R}_+} |X_{n+p}(t) - X_n(t)| \geq \epsilon \right\} \leq \mathbb{E}\left( \sup_{t \in \mathbb{R}_+} \sqrt{V(t, X_{n+p}(t) - X_n(t))} \geq \sqrt{V_\epsilon} \right)
\]
(3.21)

Because \( \mathbb{E}|\hat{X}_n|^2 \leq r^2 \) and \( \hat{X}_n \overset{a.s.}{\to} \hat{X} \), we have (see Theorems 4.5.2, 4.5.4),
\[
\mathbb{E}|\hat{X}_n| \to \mathbb{E}|\hat{X}|, \text{ as } n \to +\infty,
\]
and
\[
\lim_{n \to +\infty} \mathbb{E}|\hat{X}_n - \hat{X}| = 0.
\]
So
\[
\lim_{n \to +\infty} \sup_{p \in \mathbb{N}} \mathbb{E}|\hat{X}_{n+p} - \hat{X}_n| = 0.
\]
When \( n \) is large enough, by (3.3) we have
\[
\sup_{p \in \mathbb{N}} \mathbb{E}\sqrt{V(0, \hat{X}_{n+p} - \hat{X}_n)} \leq \sqrt{\mathbb{E}|\hat{X}_{n+p} - \hat{X}_n|} < \epsilon \sqrt{V_\epsilon}.
\]
This together with (3.21) implies
\[ \sup_{p \in \mathbb{N}} \mathbb{P} \{ \sup_{t \in \mathbb{R}_{+}} |X_{n+p}(t) - X_n(t)| \geq \epsilon \} < \epsilon. \]

By Theorem 4.1.3 in [8], there exists a suitable stochastic process \( \bar{X}(t) \) such that \( X_n(t) \overset{p}{\to} \bar{X}(t) \) uniformly on \( \mathbb{R}_{+} \). Thus \( \mu(t + \alpha_n) \) uniformly converges to some \( T_{\alpha(t)}(t) \) on \( \mathbb{R}_{+} \).

By Lemma 2.3, we can see that each \( \mu(t) \in \mathcal{D}_{\mathbb{R}}^{\mathbb{R}} \) is a.a.p. on \( \mathbb{R}_{+} \). So the distribution of any \( \mathbb{L}^2 \)-bounded solution is a.a.p. on \( \mathbb{R}_{+} \). By Proposition 3.2, there exists some \( \mathbb{L}^2 \)-bounded solution of (2.4) with almost periodic distribution. The proof is complete. \( \square \)

To discuss the almost periodicity of SDE’s solutions, we need to find ways to obtain \( \mathbb{L}^2 \)-bounded solutions on \( \mathbb{R} \), which may reduce to finding \( \mathbb{L}^2 \)-bounded solutions on \( \mathbb{R}_{+} \):

**Proposition 3.6** (cf. [16], Theorem 4.7). Assume that (2.4)’s coefficients satisfy condition (H), (2.4) admits a solution \( \varphi \) on \([t_0, +\infty)\) for some \( t_0 \in \mathbb{R} \), and \( \sup_{t \geq t_0} ||\varphi(t)||_2 \leq M \) for some constant \( M > 0 \), then (2.4) has a solution \( \tilde{\varphi} \) on \( \mathbb{R} \) with \( ||\tilde{\varphi}(t)||_\infty \leq M \).

We now conclude this section by giving a sufficient condition for the existence of \( \mathbb{L}^2 \)-bounded solutions via Lyapunov functions:

**Theorem 3.7.** Assume that (2.4)’s coefficients satisfy condition (H), and there is a function \( V \) satisfying condition (L) such that for some constant \( R > 0 \)
\[ a|x|^2 \leq V(t, x) - b(t)|x|^2 + c(t), \text{ when } |x| \leq R, \]
where constant \( a > 0 \), \( b(\cdot) \), \( c(\cdot) \) are positive functions on \( \mathbb{R} \). Assume further that
\[ LV(t, x) := \frac{\partial V}{\partial t} + \sum_{i=1}^{d} \frac{\partial V}{\partial x_i} f_i + \sum_{l=1}^{m} \sum_{i,j=1}^{d} g_{il} \frac{\partial^2 V}{\partial x_i \partial x_j} g_{jl} \leq 0, \text{ when } |x| \geq R. \]

Then if \( X(t) \) is a solution of (2.4) with initial condition \( \mathbb{E}|X(t_0)|^2 < +\infty \), \( X(t) \) is \( \mathbb{L}^2 \)-bounded on \([t_0, +\infty)\).

**Proof.** Suppose that \( X(t) \) is the solution of (2.4) with \( \mathbb{L}^2 \)-bounded initial value at \( t_0 \). Since the coefficients satisfy condition (H), \( X(t) \) exists on \([t_0, +\infty)\). We define a sequence of stopping times:
\[ \tau_n := \inf \{ t \geq t_0 : |X(t)| \geq n, \text{ or } |X(t)| \leq R \}, \]
and
\[ \tau := \inf \{ t \geq t_0 : |X(t)| \leq R \}. \]

Then \( \tau_n \overset{a.s.}{\to} \tau \) as \( n \to +\infty \).

Denote \( B_R \) as the close ball \( \{ x \in \mathbb{R}^d : |x| \leq R \} \). When \( X(t_0) \) is supported on \( \mathbb{R}^d - B_R \), by Itô’s formula, for \( t \geq t_0 \),
\[ \mathbb{E}V(t \wedge \tau_n, X(t \wedge \tau_n)) = \mathbb{E}V(t_0, X(t_0)) + \mathbb{E} \int_{t_0}^{t \wedge \tau_n} LV(u, X(u))du \]
\[ \leq \mathbb{E}V(t_0, X(t_0)) \leq c(t_0) + b(t_0)\mathbb{E}|X(t_0)|^2. \]

Then Fatou’s lemma implies that
\[ (3.22) \quad \mathbb{E}V(t \wedge \tau, X(t \wedge \tau)) \leq \mathbb{E}V(t_0, X(t_0)) \leq c(t_0) + b(t_0)\mathbb{E}|X(t_0)|^2, \]
by letting \( n \to +\infty \).
When $X(t_0)$ is supported on $\mathbb{R}^d$, denote $\bar{M}$ as a bound of $V(t,x)$ for $|x| \leq R$, then by (3.22) we have for $t \geq t_0$

$$
EV(t, X(t)) \leq P(\tau^R \geq t) \cdot \int_{\{X(t_0) > R\}} V(t_0, X(t_0), \omega) dP(\omega)
$$

(3.23)

$$
+ P(\tau^R < t) \cdot \left[ \bar{M} + \int\left\{X(t_0) > R\right\} V(t_0, X(t_0), \omega) dP(\omega) \right]
$$

$$
\leq 2[c(t_0) + b(t_0)E|X(t_0)|^2] + M.
$$

Note that in (3.23) either $|X(t)| \leq R$ or $a|X(t)|^2 \leq V(t, X(t))$, so $X(t)$ is $L^2$-bounded on $[t_0, +\infty)$.

4. Applications

In this section, we illustrate our theoretical results by several examples. Firstly we consider the simplest case of almost periodic SDEs.

**Example 4.1.** Consider one-dimensional SDE

$$
dX(t) = f(t, X(t))dt + g(t, X(t))dW(t),
$$

where $f, g$ satisfy condition (H) and are $C^1$ in $x$. Assume that for some constant $c > 0$,

$$
\sup_{t,x} |\partial g(t,x)|^2 \leq c, \quad \sup_{t,x} |\partial f(t,x)| \leq -c.
$$

Then if $D \not= 0$, it has a unique element which is almost periodic.

**Proof.** Let $V(t,x) = |x|^2$. Then it’s easy to see that $V$ satisfies condition (L), and

$$
\frac{\partial V}{\partial t}(t,x) = 0, \quad \frac{\partial V}{\partial x}(t,x) = 2x, \quad \frac{\partial^2 V}{\partial x^2}(t,x) = 2.
$$

By (1.2) and mean value theorem, for every $x, y \in \mathbb{R}$ and every $t \in \mathbb{R}$, if $x \neq y$, there exist $\hat{\xi} = \hat{\xi}(t,x,y)$, $\hat{\xi} = \hat{\xi}(t,x,y)$ such that $\hat{\xi}, \hat{\xi} \in (x \land y, x \lor y)$, and

$$
(f(t,x) - f(t,y))(x-y) = \frac{\partial f}{\partial x}(t,\hat{\xi})(x-y)^2 \leq -c(x-y)^2,
$$

$$
(g(t,x) - g(t,y))^2 = (x-y)^2 |\partial g(t,\hat{\xi})|^2 \leq c(x-y)^2.
$$

So

$$
\mathcal{L}V(t,x,y) = 2(f(t,x) - f(t,y))(x-y) + (g(t,x) - g(t,y))^2 \leq -c(x-y)^2 = -c|x-y|^2.
$$

By Theorem 3.4 we can easily get the required result. \hfill \Box

Now let us consider some two-dimensional applications.

**Example 4.2.** Consider two-dimensional SDE:

$$
\begin{cases}
    dX_1(t) = [f_1(t, X_1(t)) + \sigma X_2(t)]dt + [A_1(t)X_1(t) + g_1(t)]dW_1(t), \\
    dX_2(t) = [f_2(t, X_2(t)) - \sigma X_1(t)]dt + [A_2(t)X_2(t) + g_2(t)]dW_2(t),
\end{cases}
$$

where $f_i(t, x)$ are $C^1$ in $x$ and satisfy condition (H) for $i = 1, 2$. $\sigma \not= 0$ is a constant. Assume that $A_i, g_i$ are almost periodic and $f_i(t,0) \equiv 0$, $i = 1, 2$. Denote $a(t) := \max_{i=1,2} \{A_i^2(t), g_i^2(t)\}$. Assume further that for $t, x \in \mathbb{R}$,

$$
\frac{\partial f_i}{\partial x}(t, x) \leq -2a(t) - 1, \quad i = 1, 2.
$$
Then $\mathcal{D}^{(1)}$ has a unique element which is almost periodic.

Proof. Let $V(\cdot, \cdot): \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, $V(t, x) = |x|^2 = x_1^2 + x_2^2$. Then $V(t, x)$ satisfies condition (L), and for $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, $i, j = 1, 2$,

$$\frac{\partial V}{\partial x_i}(t, x) = 2x_i, \quad \frac{\partial^2 V}{\partial x_i \partial x_j}(t, x) = 2, \quad \frac{\partial^2 V}{\partial x_i \partial x_j}(t, x) = 0, \text{ when } i \neq j.$$

By (4.4) and mean value theorem, for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, there are $\xi_i = \xi_i(t, x_i, y_i) \in (x_i \wedge y_i, x_i \vee y_i)$, $i = 1, 2$, such that

$$\mathcal{L}V(t, x - y) = 2 \sum_{i=1, 2} (f_i(t, x_i) - f_i(t, y_i)(x_i - y_i)) + \sum_{i=1, 2} A_i^2(t)(x_i - y_i)^2$$

$$\leq \sum_{i=1, 2} [a(t) + 2\frac{\partial f_i}{\partial x_i}(t, \xi_i)(x_i - y_i)^2]$$

$$\leq (-3a(t) - 2)(x_i - y_i)^2 \leq -2|x - y|^2.$$

Since $f_i(t, 0) = 0$, for every $x_i$, $t$, there exist $\hat{\xi}_i = \hat{\xi}_i(t, x_i) \in (x_i \wedge 0, x_i \vee 0)$ such that

$$f_i(t, x_i)x_i = \frac{\partial f_i}{\partial x_i}(t, \hat{\xi}_i)x_i \leq -(2a(t) + 1)x_i^2.$$

So

$$LV(t, x) = 2 \sum_{i=1, 2} f_i(t, x_i)x_i + \sum_{i=1, 2} [A_i(t)x_i + g_i(t)]^2$$

$$\leq \sum_{i=1, 2} [2A_i^2(t)x_i^2 + 2g_i^2(t) + 2\frac{\partial f_i}{\partial x_i}(t, \hat{\xi}_i)x_i^2]$$

$$\leq \sum_{i=1, 2} [-2(a(t) + 1)x_i^2 + 2a(t)].$$

Obviously $LV(t, x) \leq 0$ when $|x| \geq \sqrt{2}$. By the global Lipschitz condition of the coefficients, we can see that (1.3) must have $L^2$-bounded solutions from Proposition 3.6 and Theorem 3.7. By Theorem 3.3, we can get the result required.

Example 4.3. Consider two-dimensional SDE:

$$\begin{align*}
\mathrm{d}X_1(t) &= \left[-(A_1^2(t) + A_2^2(t) + 1)X_1(t) + 2A_1^2(t)X_2(t)\right]dt \\
&\quad + A_1(t)(X_1(t) - X_2(t))dW_1(t), \\
\mathrm{d}X_2(t) &= \left[-(A_2^2(t) + A_1^2(t) + 1)X_2(t) + 2A_2^2(t)X_1(t)\right]dt \\
&\quad + A_2(t)(X_1(t) - X_2(t))dW_2(t).
\end{align*}$$

(4.5)

If $A_i(t)$ are almost periodic for $i = 1, 2$, then (4.5) has $L^2$-bounded solutions, and all the $L^2$-bounded solutions of (1.3) have the same distribution which is almost periodic.

Proof. Similar to the proof of Example 4.2, let $V(t, x) = x_1^2 + x_2^2$. For $t \in \mathbb{R}$, $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we have

$$\mathcal{L}V(t, x - y) = 2 \sum_{i=1, 2} \left[-(A_1^2(t) + A_2^2(t) + 1)(x_i - y_i)^2 + A_i^2(t)(x_1 - y_1)(x_2 - y_2)\right]$$

$$\quad + \sum_{i=1, 2} [(A_1^2(t) + A_2^2(t))(x_i - y_i)^2] - 2(A_1^2(t) + A_2^2(t))(x_1 - y_1)(x_2 - y_2)$$

$$\leq -2|x - y|^2.$$
and
\[
LV(t, x) = 2 \sum_{i=1,2} \left[ -(A_1^2(t) + A_2^2(t) + 1)x_i^2 + 2A_1^2(t)x_1x_2 \right] + \sum_{i=1,2} A_i^2(t)(x_1 - x_2)^2
\leq -2|x|^2 \leq 0.
\]

By Proposition 3.6 and Theorem 3.7, (4.5) has $L^2$-bounded solutions. By Theorem 3.4, $D(4.5)$ has a unique element which is almost periodic.

\[\square\]

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