CONFINEMENT OF COLOR: A REVIEW

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Abstract

The status of our understanding of the mechanisms of color confinement is reviewed, in particular the results of numerical simulations on the lattice.

1 Introduction

Quarks and gluons are visible at short distances. They have never been observed as free particles. Search for quarks started in 1963, when first Gell-Mann introduced them as fundamental constituents of hadrons: the signature is their fractional charge \( q = \pm 1/3, \pm 2/3 \).

The upper limit to the ratio of quark abundance to proton abundance is \( n_q/n_p < 10^{-27} \), to be compared with the expectation in the Standard Cosmological Model \( n_q/n_p = 10^{-12} \). The experimental limit on the production cross section in nucleon collisions is \( \sigma < 10^{-40}\text{cm}^2 \) to be compared with the expected value \( \sigma \sim 10^{-25}\text{cm}^2 \) in the absence of confinement. The only natural explanation of these small numbers is that these ratios are exactly zero, or that confinement is an absolute property due to some symmetry.

A transition, however, can occur at high temperature to a phase in which quarks and gluons are deconfined and form a quark-gluon plasma. Big experiments at heavy ions colliders aim to detect such a phase transition, even if no clear signature for it is known. A number of theoretical ideas exist on the mechanism of confinement, which we shall briefly review below.

Lattice is a unique tool to investigate the existence of the deconfining phase transition, to check the theoretical ideas and possibly to give indications on what to look at in experiments.

2 Lattice investigations

The partition function of a field theory at temperature \( T \) is equal to the euclidean Feynman integral extending in time from 0 to \( 1/T \), with periodic boundary conditions in time for bosons, antiperiodic for fermions.

\[
Z \equiv \text{Tr} \exp(-H/T) = \int [d\phi] \exp \left[ - \int_0^{1/T} dt \int d^3x L(\vec{x}, t) \right] \quad (1)
\]
Finite temperature QCD is simulated on a lattice $N_s^3 \times N_t$ with $N_s \gg N_t$. The temperature $T$ is given by

$$T = \frac{1}{N_t a(\beta)}$$

where $a$ is the lattice spacing in physical units, which by renormalization group arguments in the weak coupling regime is given by

$$a(\beta) = \frac{1}{\Lambda} \exp(-\beta/b_0)$$

with $\beta = 2N/g^2$; $-b_0$ the lowest order coefficient of the beta function, which is negative because of asymptotic freedom. It follows then from eq.(2) that the strong coupling region corresponds to low temperatures, weak coupling to high temperatures.

If a deconfining phase transition exists at some temperature, how can it be detected, or what is the criterion for confinement?

For pure gauge theories a reasonable answer exists, which consists in looking at the static potential acting between a quark and an antiquark at large distances: if it is positive and diverging there is confinement by definition. In principle this criterion does not insure that no colored particles exist as an asymptotic state, but certainly means that heavy quarks are confined. The static potential is related to the correlator $D(\vec{x})$ of Polyakov lines $D(\vec{x}) = \langle L(\vec{x})L(0) \rangle$ as follows

$$V(x) = -T \ln D(\vec{x})$$

It can be shown by use of the cluster property that at large distances

$$D(\vec{x}) \sim \exp\left(-\frac{\sigma}{T}|\vec{x}|\right) + |\langle L \rangle|^2$$

A temperature $T_c$ is found in numerical simulations such that

- for $T < T_c$ $\langle L \rangle = 0$ and hence $V(r) = \sigma \cdot r$ (confinement)
- for $T > T_c$ $\langle L \rangle \neq 0$, $V(r) \sim$ const (deconfinement)

both for $SU(2)$ and for $SU(3)$ pure gauge theories. $\langle L \rangle$ is an order parameter for confinement, and $Z_3$ is the corresponding symmetry.

Of course no real phase transition can take place on a finite lattice, so that the transition from 0 to 1 of $\langle L \rangle$ is smooth on a finite lattice, and becomes steeper and steeper as the size goes large. The steepness is measured by the susceptibility $\chi_L$,

$$\chi_L = \int d^3x \langle L(x)L^\dagger(0) - L(0)L^\dagger(0) \rangle$$
which diverges with some critical index $\gamma$ at the critical point

$$\chi_L \simeq \tau^{-\gamma} \quad \tau \equiv \left(1 - \frac{T}{T_c}\right)$$

(8)

Other relevant critical indices are the index $\nu$ of the correlation length $\xi$ of the order parameter

$$\xi \propto \tau^{-\nu}$$

(9)

and the index $\alpha$ of the specific heat

$$C_v - C_v^0 \propto \tau^{-\alpha}$$

(10)

$\alpha, \gamma$ and $\nu$ identify the universality class and/or the order of the phase transition. A weak first order transition is a limiting case $\alpha = 1$, $\gamma = 1$ and $\nu = 1/d$ ($d$ the number of spatial dimensions i.e. 3).

The critical indices are determined from the dependence of susceptibilities on the spatial size of the system, by use of a technique known as finite size scaling[6]. The result is that for quenched SU(2) the transition is second order and belongs to the universality class of the 3d ising model[7], for SU(3) it is weak 1rst order [8].

In the presence of dynamical quarks $Z_3$ is not a symmetry and therefore the Polyakov line cannot be an order parameter. Moreover the string breaks due to the instability for production of dynamical quark pairs and the potential at large distances is not growing with the distance, even if there is confinement. Another symmetry exists at zero quark masses, the chiral symmetry. At $T=0$ it is spontaneously broken, the pseudoscalar bosons being the Goldstone particles, but it is restored at $T \approx 170$ Mev. The corresponding order parameter is $\langle \bar{\psi} \psi \rangle$. It is not clear what exactly chiral symmetry has to do with confinement: in any case it is explicitely broken by quark masses, and therefore it cannot be the symmetry responsible for confinement discussed in sect 1. For a theory with $N_f = 2$, $m_u = m_d = m$, which is a model approximation of reality, the situation is schematically represented in fig 1. The critical line $T_c(m)$ is defined by the maxima of the susceptibilities $\chi_L, \chi_{\bar{\psi} \psi}, \chi_{C_v}$, which coincide within errors [9][10], and as an empirical definition the region below the line is assumed to be confined, the region above it to be deconfined. Theoretical ideas are needed to understand the symmetry pattern of the system.

As for the chiral transition a renormalization group analysis and the assumption that the pions are the relevant degrees of freedom gives the following predictions[11]. If the U(1) axial symmetry is restored below the chiral transition, the transition is first order and such is the critical line at $m \neq 0$. If instead the anomaly persists below $T_c$ the transition is second order and the critical line at $m \neq 0$ is a crossover.

3 Theoretical ideas

A number of theoretical models of deconfinement exist in the literature.
There is a Gribov model, which is a clever picture of the chiral phase transition\cite{12}. It does not apply, however, to quenched theory. In the spirit of the $N_c \to \infty$ approach the mechanism of confinement should be the same for quenched and unquenched.

Confinement could be produced by the condensation of vortices\cite{13}. The model corresponds to a well defined symmetry in 2+1 dimensions and quenched theory, but in any case the $Z_3$ symmetry does not survive the introduction of dynamical quarks.

A most appealing idea is dual superconductivity of the vacuum\cite{14}: chromoelectric charges are confined by dual Meissner effect, which squeezes the chromoelectric field acting between colored particles into Abrikosov flux tubes, in the same way as magnetic charges are confined in ordinary superconductors.

A number of pioneering papers on this mechanism were based on the definition and the counting of monopoles\cite{17}. We shall instead concentrate on the symmetry patterns involved. Dual superconductivity means that the vacuum is a Bogolubov-Valatin superposition of states with different monopole charge (monopole condensation).

In order to define monopoles a magnetic U(1) gauge symmetry must be identified in QCD, which has to be a color singlet if monopoles condense without breaking the color symmetry.

The procedure to identify such magnetic U(1) is known as Abelian Projection\cite{15}. We shall present the abelian projection in a form which will prove useful for what follows\cite{16}. Let $G_{\mu \nu} = T^i G^i_{\mu \nu}$ be the gauge field strength with $T^i$ the gauge group generators in the fundamental representation, and $\Phi = T^i \Phi^i$ any

\figure{The phase diagram of $N_f = 2$ QCD.}{fig1}{width=0.5\textwidth}
operator in the adjoint representation. Define

\[ F_{\mu\nu} = Tr \{ \phi G_{\mu\nu} \} - \frac{i}{g} Tr \{ \phi [D_\mu \phi, D_\nu \phi] \} \] (11)

\( F_{\mu\nu} \) is gauge invariant and color singlet, and such are separately the two terms in its definition.

**Theorem[16].** A necessary and sufficient condition for the cancellation of bilinear terms \( A_\mu A_\nu \) between the two terms in the right hand side of eq.(11) is that

\[ \Phi = \Phi^a = U(x) \Phi^a_{\text{diag}} U^\dagger(x) \] (12)

with \( U(x) \) an arbitrary gauge transformation and

\[ \Phi^a_{\text{diag}} = \text{diag} \left( \begin{array}{cccc} N-a & N-a & \cdots & N-a \\ N & N & \cdots & N \\ \vdots & \vdots & \ddots & \vdots \\ N & N & \cdots & N \end{array} \right) \] (13)

For any choice of the form eq(12) \( F_{\mu\nu} \) obeys Bianchi identities and the identity holds

\[ F^a_{\mu\nu} = \partial_\mu Tr \{ \phi^a A_\nu \} - \partial_\nu Tr \{ \phi^a A_\mu \} - \frac{i}{g} Tr \{ \phi^a [\partial_\mu \phi^a, \partial_\nu \phi^a] \} \] (14)

\( F_{\mu\nu} \) is gauge invariant and can be computed in the gauge in which \( \Phi^a \) is diagonal. In that gauge

\[ F^a_{\mu\nu} = \partial_\mu Tr (\Phi^a A_\nu) - \partial_\nu Tr (\Phi^a A_\mu) \] (15)

has an abelian form. By developing \( A^a_{\text{diag}} \) in terms of roots \( A_\mu = \alpha^i A^i_\mu \)

\[ \alpha^i = \text{diag}(0, 0, 0 \ldots 1, -1, 0 \ldots 0) \] (16)

with

\[ Tr(\alpha^i \Phi^j) = \delta^{ij} \]

\[ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \] (17)

The gauge transformation \( U(x) \) which brings to the unitary gauge is called abelian projection.

A magnetic current can be defined as

\[ J^a_\nu = \partial_\mu F^{a*}_{\mu\nu} \] (18)

This current is identically zero due to Bianchi identities, but can be non zero in compact formulations like lattice regularization , in which Dirac strings are invisible. In any case \( J^a_\mu \) is identically conserved

\[ \partial_\mu J^a_\mu = 0 \] (19)
and defines a magnetic U(1) conserved charge. This magnetic U(1) symmetry can either be Higgs broken, and then the system is a magnetic (dual) superconductor, or it can be realized à la Wigner, and then magnetic charge is superselected. For any choice of the field \( \Phi^a \) in eq (12) a magnetic U(1) symmetry is defined.

To detect dual superconductivity the vev of a magnetically charged operator can be used as an order parameter. Such an operator has been constructed \(^18\), and is magnetically charged and U(1) gauge invariant \(^19\). The continuum version of the construction goes as follows. Define

\[
\mu^a(x,t) = e^{i \int d^3 y Tr(\phi^a(\vec{y},t) \vec{b}_\perp(x-\vec{y}))}
\]

(20)

where \( \phi^a \) is defined by eq (12) and \( \vec{E}(\vec{x},t) \) is the chromoelectric field operator \( E_i = G_{0i} \) and

\[
\vec{\nabla} b_\perp = 0, \quad \vec{\nabla} \wedge b_\perp = \frac{2\pi}{g} \vec{r} + \text{Dirac string}
\]

(21)

\( \mu^a \) is gauge invariant by construction if \( \Phi^a \) transforms in the adjoint representation. In the abelian projected gauge, where \( \Phi^a = \Phi^a_{\text{diag}} \) it assumes the form

\[
\mu^a(x,t) = \exp \left\{ i \int d^3 y E^a_\perp(\vec{y},t) b_\perp(x-\vec{y}) \right\}
\]

(22)

where \( E^a_\perp \) is the component of the electric field along the residual U(1) direction as defined by eqs. (16),(17), and only the transverse part survives in the convolution with \( b_\perp \). In any quantization procedure \( E^a_\perp \) is the conjugate momentum to \( A^a_\perp \) so that \( \mu^a \) is nothing but the translation operator of \( A^a_\perp \) and

\[
\mu^a(x,t)|A^a_\perp(\vec{y},t)\rangle = |A^a_\perp(\vec{y},t) + b_\perp(x-\vec{y})\rangle
\]

(23)

\( \mu^a \) creates a magnetic monopole.

In the confined phase, in which monopoles condense and the ground state is not an eigenstate of the magnetic charge \( \langle \mu^a \rangle \neq 0 \) can signal dual superconductivity. In the deconfined phase \( \langle \mu^a \rangle = 0 \) All this refers to a given choice of the abelian projection, i.e. of the gauge transformation \( U(x) \) defining \( F^a_{\mu
u} \).

To explore \(^20\), \(^21\) how physics depends on the choice of the abelian projection let us go back to eq (20). By use of the cyclic invariance of the trace \( \mu^a \) can be rewritten

\[
\mu^a(x,t) = e^{i \int d^3 y Tr(\phi^a_{\text{diag}} U(\vec{y},t) \vec{E}(\vec{y},t) U(\vec{y},t) \vec{b}_\perp(x-\vec{y}))}
\]

(24)

In computing the correlation functions of \( \mu^a \)'s a change of variables can be performed in the Feynman integral corresponding to a gauge transformation generated by \( U(x) \). If \( U(x) \) is independent of the field configuration the jacobian of the transformation is 1, the operator \( \mu^a \) assumes to all effects the form eq. (24) so that the correlators, and in particular the one point function \( \langle \mu^a \rangle \) are independent of \( U(x) \). If \( U(x) \) depends on the field configuration, as happens e.g. for the max abelian gauge or for any gauge in which a specific field
dependent operator is diagonalized, then the jacobian can be different from 1 and the correlators depend on the abelian projection. However, if the number density of monopoles is finite, the gauge transformation which connects two abelian projections is continuous everywhere except in a finite number of points and preserves topology: the operator $\mu^a$ defined by eq(22) will then create a monopole in all abelian projections. If $\langle \mu^a \rangle \neq 0$ it signals dual superconductivity in all abelian projections.

An extensive investigation of the density of monopoles in different abelian projections has been performed, and indeed the number density of monopoles is finite. Fig 2 illustrates the method, and refers specifically to the abelian projection in which the Polyakov line operator is diagonal. The eigenvalues of that (unitary) operator have the form

$$L_i = e^{i\phi_i}, \quad i = 1, 2, 3$$

(25)

and in defining the abelian projection are ordered in decreasing order of $\phi_i$. A monopole singularity in a point x implies that two eigenvalues are equal, eg $\phi_1$ and $\phi_2$. Fig 2 shows the distribution of the difference of the first two eigenvalues on the lattice sites of 1000 field configurations of quenched SU(3) on a $16^4$ lattice. In no site there is a monopole. Repeating the determination on a finer lattice gives similar results. As a consequence one can state that the number density of monopoles is finite and the dual superconductivity (or non) is an intrinsic property, independent of the abelian projection which defines the monopoles.

An extensive analysis on the lattice [18] shows that the vacuum is indeed a dual superconductor in the confined phase, and goes to normal in the deconfined

Figure 2: An example of probability distribution of the difference of the two highest eigenvalues of the phase $\Phi$ of the Polyakov line $e^{i\Phi}$, at the lattice sites. $SU(3)$ gauge group, $\beta = 6.4$, lattice $16^4, 10^3$ configurations.
one. A finite size scaling analysis of the susceptibility $\rho$ defined as

$$\rho^a = \frac{d}{d\beta} \ln \langle \mu^a \rangle$$  \hspace{1cm} (26)$$

gives that in the quenched case $\langle \mu^a \rangle$ is strictly zero above the critical temperature defined by the Polyakov line order parameter, is different from zero below it. The behaviour around $T_c$ allows to determine the critical indices, which are consistent with those determined by use of the Polyakov line. This is clear evidence that dual superconductivity is a mechanism for confinement.[18]

4 The case of full QCD

The order parameters $\langle \mu^a \rangle$ can equally well be defined in the presence of dynamical quarks (Full QCD) [22] and have the same physical meaning of creators of monopoles. One can then ask if a criterion for confinement could be provided by $\langle \mu^a \rangle$’s, i.e. by dual superconductivity (or absence of). One should prove that in the dual superconducting phase no colored asymptotic states exist, which is of course non trivial. However this is not much different from the situation in quenched theory with the Polyakov line criterion, as discussed in sect 2. It has in fact been checked [22] that QCD vacuum is a dual superconductor in the phase below the critical line of fig1, $\langle \mu^a \rangle \neq 0$, and is normal in the region above it $\langle \mu^a \rangle = 0$. The finite size scaling analysis in this case goes as follows. For dimensional reasons the order parameter $\langle \mu^a \rangle$ has the form

$$\langle \mu^a \rangle = \Phi(\frac{a}{\xi}, \frac{N_s}{\xi}, mL^{y_h})$$  \hspace{1cm} (27)$$

where $\xi$ is the correlation length, $a$ the lattice spacing, $m$ the quark mass and $y_h$ the corresponding anomalous dimension. Near the critical line $\xi$ goes large compared to $a$ and the dependence on $a/\xi$ can be neglected. (Scaling) The problem has two scales. If $y_h$ is known one can choose different values of the mass and of the spacial size $N_s$ such that $mN_s$ is constant, and then

$$\langle \mu^a \rangle \sim \frac{N_s}{\xi} f(\frac{N_s}{\xi})$$  \hspace{1cm} (28)$$

By use of eq(9) the variable $N_s\xi$ can be traded with $\tau N_s^{1/\nu}$ and the scaling law follows

$$\rho^a = N_s^{1/\nu} f(\tau N_s^{1/\nu}), \quad \tau = (1 - \frac{T}{T_c})$$  \hspace{1cm} (29)$$

whence $\nu$ can be extracted and the order of the transition can be determined. The result is $\nu = .33$ compatible with a first order transition.

A cross check is obtained by studying the scaling of the maximum of the specific heat, which for the same choices of $m$ and $N_s$ should scale as

$$C_v - C_v^0 \sim N_s^{\alpha/\nu}$$  \hspace{1cm} (30)$$
If the critical indices determined through $\langle \mu^a \rangle$ coincide with those resulting from the analysis of the specific heat, this would be additional evidence for dual superconductivity as a mechanism of confinement, implying that $\langle \mu^a \rangle$ can be the order parameter.

The situation is described in ref.\[23\], and is presently at the stage of indication that this is indeed the case. Numerical work is on the way which will definitely clarify the problem.

In conclusion Confinement is a fundamental but difficult problem. Some understanding has been reached on the symmetry patterns involved. Lattice is a unique tool to address the problem.

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