The hadronic light-by-light scattering contribution to the muon anomalous magnetic moment from lattice QCD

Thomas Blum,1,2 Norman Christ,3 Masashi Hayakawa,4,5 Taku Izubuchi,6,2 Luchang Jin,1,2,6 Chulwoo Jung,6 and Christoph Lehner7,6

1Physics Department, University of Connecticut, 2152 Hillside Road, Storrs, CT, 06269-3046, USA
2RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA
3Physics Department, Columbia University, New York, New York 10027, USA
4Department of Physics, Nagoya University, Nagoya 464-8602, Japan
5Nishina Center, RIKEN, Wako, Saitama 351-0198, Japan
6Physics Department, Brookhaven National Laboratory, Upton, New York 11973, USA
7Universität Regensburg, Fakultät für Physik, 93040, Regensburg, Germany

We report the first result for the hadronic light-by-light scattering contribution to the muon anomalous magnetic moment with all errors systematically controlled. Several ensembles using 2+1 flavors of physical mass Möbius domain-wall fermions, generated by the RBC/UKQCD collaborations, are employed to take the continuum and infinite volume limits of finite volume lattice QCD. We find \( a_{\text{HLbL}} = 7.20(3.98)_{\text{stat}}(1.65)_{\text{sys}} \times 10^{-10} \). Our value is consistent with previous model results and leaves little room for this notoriously difficult hadronic contribution to explain the difference between the Standard Model and the BNL experiment.

I. INTRODUCTION

The anomalous magnetic moment of the muon is providing an important test of the Standard Model. The current discrepancy between experiment and theory stands between three and four standard deviations. An ongoing experiment at Fermilab (E989) and one planned at J-PARC (E821) aim to reduce the uncertainty of the BNL E821 value [1] by a factor of four, and similar efforts are underway on the theory side [2–30]. A key part of the latter is to compute the hadronic light-by-light (HLbL) contribution from first principles using lattice QCD [31–37]. Such a calculation, with all errors under control, is crucial to interpret the anticipated improved experimental results.

The magnetic moment is an intrinsic property of a spin-1/2 particle, and is defined through its interaction with an external magnetic field \( \mathbf{B} \), \( H_{\text{int}} = -\mathbf{\mu} \cdot \mathbf{B} \). Here

\[
\mathbf{\mu} = -g \frac{e}{2m} \mathbf{S},
\]

where \( \mathbf{S} \) is the particle's spin, \( q \) and \( m \) are the electric charge and mass, respectively, and \( g \) is the Landé g-factor. The Dirac equation predicts that \( g = 2 \), exactly, so any difference from 2 must arise from interactions. Lorentz and gauge symmetries tightly constrain the form of the interactions,

\[
\langle \mu(p')|J_\nu(0)|\mu(p) \rangle = -e\bar{u}(p') \left( F_1(q^2)\gamma_\nu + i F_2(q^2) \frac{\gamma_\nu \cdot \gamma_\rho}{4m} q_\rho \right) u(p), \quad (2)
\]

where \( J_\nu \) is the electromagnetic current, and \( F_1 \) and \( F_2 \) are form factors, giving the charge and magnetic moment at zero momentum transfer (\( q = p' - p = 0 \)), or static limit. \( u(p) \) and \( \bar{u}(p) \) are Dirac spinors. The anomalous part of the magnetic moment is given by \( F_2(0) \) alone, and is known as the anomaly,

\[
a_\mu \equiv (g - 2)/2 = F_2(0). \quad (3)
\]

The desired matrix element in (2) is extracted in quantum field theory from a correlation function of fields as depicted in the Feynman diagrams shown in Fig. [1]. Here we work in coordinate (Euclidean) space and use Lattice QCD for the hadronic part which is intrinsically non-perturbative. QED is treated using the same discrete, finite, lattice as used for the hadronic part, while we remove the spatial zero modes of the photon propagator. This method is called QED\(_L\) [38]. It is perturbative with respect to QED, i.e., only diagrams where the hadronic part is connected to the muon by three photons enter the calculation.

II. QED\(_L\) METHOD

Here the muon, photons, quarks, and gluons are treated on a finite, discrete lattice. The method is described in

* ljin.luchang@gmail.com
Two additional, but related, parts of the method bear mentioning. First, the form dictated by the right hand side of Eq. \( (2) \) suggests the limit \( q \to 0 \) is unhelpful since the desired \( F_2 \) term is multiplied by 0. Second, in our Monte Carlo lattice QCD calculation the error on the \( F_2 \) contribution blows up in this limit. The former is avoided by evaluating the first moment with respect to \( x_{op} \) at the external vertex and noticing that an induced extra term vanishes exponentially in the infinite volume limit \( [32] \). This moment method allows the direct calculation of the correlation function at \( q = 0 \), and hence \( F_2(0) \). To deal with the second issue, we first recall that it is the Ward identity that guarantees the unwanted term to vanish in the moment method. We thus enforce the Ward identity exactly on a configuration-by-configuration basis \([32]\), i.e., before averaging over gauge fields by inserting the external photon at all possible locations on the quark loop in Fig. \( 2 \). This makes the factor of \( q \) in Eq. \( (2) \) exact for each measurement and not just in the average and reduces the error on \( F_2(0) \) significantly. Implementing the above techniques produces an order \( O(1000) \) fold improvement in the statistical error over the original non-perturbative QED method used to compute the hadronic light-by-light scattering contribution \([31]\).

The quark-disconnected diagrams that occur at \( O(\alpha^3) \) are shown Fig. \( 3 \). All but the upper-leftmost diagram vanish in the \( SU(3) \) flavor limit and are suppressed by powers of \( m_u, d - m_s \), depending on the number of quark loops with a single photon attached. For now we ignore them and concentrate on the leading disconnected diagram which is computed with a method similar to the one described in the previous section. To ensure the loops are connected by gluons, explicit vacuum subtraction is required. However, in the leading diagram the moment at \( x_{op} \) implies the left-hand loop in Fig. \( 3 \) vanishes due to parity symmetry, and the vacuum subtraction is done to reduce noise.

As for the connected case, two point sources (at \( y \) and \( z \) in Fig. \( 3 \)) are chosen randomly, and the sink points are summed over. \( M \) propagators are computed, and all \( M^2 \) combinations are used to perform the stochastic sum. This “\( M^2 \) trick” is crucial to bring the statistical fluctuations of the disconnected diagram under control.
III. LATTICE SETUP

The simulation parameters are given in Tab. I. All particles have their physical masses (isospin breaking for the up and down quark masses is not included). The discrete Dirac operator is known as the (Möbius) domain wall fermion ((M)DWF) operator. Similarly the discrete gluon action is given by the plaquette plus rectangle Iwasaki gauge action. Additionally, three ensembles with larger lattice spacing employ the dislocation-suppressing-determinant-ratio (DSDR) to soften explicit chiral symmetry breaking effects for MDWFs [39]. We use All Mode Averaging (AMA) [44] and Multi-grid Lanczos [45] techniques to speed up the fermion propagator generation.

The muons and photons take discrete free-field forms. The muons are DWFs with infinite size in the extra fifth dimension, and the photons are non-compact in the Feynman gauge. In the latter all modes with \( q = 0 \) are dropped, a finite volume formulation of QED known as QEDL [38].

### TABLE I. 2+1 flavors of MDWF gauge field ensembles generated by the RBC/UKQCD collaborations [38]. The lattice spacing \( a \), spatial extent \( L \), extra fifth dimension size \( L_s \), muon pion mass \( m_\pi \), and number of QCD configuration used for the connected and the disconnected diagrams.

|                  | 48I | 64I | 24D | 32D | 48D | 32Dfine |
|------------------|-----|-----|-----|-----|-----|---------|
| \( a^{-1} \) (GeV) | 1.730 | 2.359 | 1.015 | 1.015 | 1.015 | 1.378 |
| \( a \) (fm)     | 0.114 | 0.084 | 0.194 | 0.194 | 0.194 | 0.143 |
| \( L \) (fm)     | 5.47 | 5.38 | 4.67 | 6.22 | 9.33 | 4.58 |
| \( L_s \)        | 48 | 64 | 24 | 24 | 24 | 32 |
| \( m_\pi \) (MeV) | 139 | 135 | 142 | 142 | 142 | 144 |
| \( m_\mu \) (MeV) | 106 | 106 | 106 | 106 | 106 | 106 |
| \# meas con      | 65 | 43 | 157 | 70 | 8 | 55 |
| \# meas discon   | 104 | 44 | 156 | 69 | 0 | 55 |

IV. RESULTS

Before moving to the hadronic case, the method was tested in pure QED [32]. Results for several lattice spacings and box sizes are shown in Fig. 4. The systematic uncertainties are large, but under control. Note that the finite volume errors are polynomial in \( 1/L \) and not exponential, due to the photons which interact over a long range. The data are well fit to the form

\[
a_\mu(L, a) = a_\mu \left( \frac{b_2}{(m_\mu L)^2} + \frac{b_3}{(m_\mu L)^3} \right) \times \left( 1 - c_1(m_\mu a)^2 + c_2(m_\mu a)^4 \right).
\]

The continuum and infinite volume limit is \( F_2(0) = 1.730 \) GeV, Iwasaki ensemble listed in the first column of Tab. I, for which we found \( a_\mu^{\text{con}} = 11.60(0.96)_{\text{stat}} \times 10^{-10} \), \( a_\mu^{\text{discon}} = -6.25(0.80)_{\text{stat}} \times 10^{-10} \), and \( a_\mu^{\text{tot}} = \)

![Fig. 4. QED light-by-light scattering contribution from the muon loop to the muon anomaly. The lattice spacing decreases from bottom to top. Solid lines are from a fit using Eq. (4).](image)

![Fig. 5. Cumulative contributions to the muon anomaly, connected (upper) and disconnected (lower). \( r \) is the distance between the two sampled currents in the hadronic loop (the other two currents are summed exactly) and the horizontal axis is the cumulative contributions from \( r \) and below. \( 24^3 \) IDSDR (squares), \( 24^3 \) IDSDR (squares), \( 32^3 \) IDSDR (crosses), \( 48^3 \) Iwasaki (diamonds), and \( 64^3 \) Iwasaki (plusses).](image)

Our physical point calculation [33] started on the \( 48^3 \), \( a^{-1} = 1.730 \) GeV, Iwasaki ensemble listed in the first column of Tab. I, for which we found \( a_\mu^{\text{con}} = 11.60(0.96)_{\text{stat}} \times 10^{-10} \), \( a_\mu^{\text{discon}} = -6.25(0.80)_{\text{stat}} \times 10^{-10} \), and \( a_\mu^{\text{tot}} = \) consistent with the well known perturbative value [41], \( 46.5 \times 10^{-10} \).
5.35(1.35)_{\text{stat}} \times 10^{-10}$ for the connected, leading disconnected, and total HLbL contributions to the muon anomaly, respectively. The errors quoted are purely statistical. We have since improved the statistics on the leading disconnected diagram with measurements on 39 additional configurations, and the contribution becomes $-6.03(60) \times 10^{-10}$. Since then we have computed on several additional ensembles in order to take the continuum and infinite volume limits (see Tab. I). The systematic errors mostly result from the higher order discretization and finite volume effects which are not included in the fitting formula Eq. (5). We therefore estimate the errors through the change of the results after adding a corresponding term in the fitting formula. For $\mathcal{O}(1/L^3)$, we add another $1/(m_\mu L)^3$ term with the same coefficient as the $1/(m_\mu L)^2$ term. For $\mathcal{O}(a^4)$ effects, we add an $a^4$ term also for the Iwasaki ensembles with coefficient similar to the I-DSDR ensembles. For $\mathcal{O}(a^2 \log(a^2))$ effects, we multiply the discretization effect terms in Eq. (5) by $(1 - (\alpha_S/\pi) \log(a^2 \text{ GeV}))$. For $\mathcal{O}(a^2/L)$, we multiply the discretization effect terms in Eq. (5) by $(1 - 1/(m_\mu L))$. In addition, for the only two contributions which we have not included in the present HLbL calculation: (a) strange quark contribution to the connected diagrams; (b) sub-leading disconnected diagrams' contribution. We have performed lattice calculations with the QED$_\infty$ approach on the 24D ensemble to estimate the systematic errors. These systematic errors are added in quadrature and summarized in Tab. III.

While the large relative error on the total is a bit unsatisfactory, we emphasize that our result represents an important estimate on the hadronic light-by-light scattering contribution to the muon anomaly, with all systematic errors controlled. It appears that this contribution cannot bring the Standard Model and the E821 experiment in agreement.

| | con | discon | tot |
|---|---|---|---|
| $a_\mu$ | 23.76(3.96) | -17.12(3.46) | 6.80(4.65) |
| sys $\mathcal{O}(1/L^3)$ | 2.34(0.41) | 1.72(0.32) | 0.83(0.56) |
| sys $\mathcal{O}(a^4)$ | 0.88(0.53) | 0.83(0.46) | 1.08(0.98) |
| sys $\mathcal{O}(a^2 \log(a^2))$ | 0.21(0.18) | 0.28(0.14) | 0.06(0.21) |
| sys $\mathcal{O}(a^2/L)$ | 4.18(2.37) | 3.93(2.30) | 0.50(2.38) |
| sys strange con | 0.30 | 0 | 0.30 |
| sys sub-discon | 0 | 0.50 | 0.50 |
| sys all | 4.89(2.17) | 4.41(2.15) | 1.56(0.90) |

TABLE II. Central value and various systematic errors. Numbers in parentheses are statistical error for the corresponding values.

| | con | discon | tot |
|---|---|---|---|
| $a_\mu$ | 24.16(2.30) | -17.12(3.46) | 7.20(3.98) |
| sys hybrid $\mathcal{O}(a^2)$ | 0.20(0.45) | 0 | 0.20(0.45) |
| sys $\mathcal{O}(1/L^3)$ | 2.34(0.41) | 1.72(0.32) | 0.83(0.56) |
| sys $\mathcal{O}(a^4)$ | 0.93(0.32) | 0.83(0.46) | 1.07(0.97) |
| sys $\mathcal{O}(a^2 \log(a^2))$ | 0.23(0.08) | 0.05(0.16) | 0.05(0.16) |
| sys $\mathcal{O}(a^2/L)$ | 4.43(1.38) | 3.93(2.30) | 0.72(2.06) |
| sys strange con | 0.30 | 0 | 0.30 |
| sys sub-discon | 0 | 0.50 | 0.50 |
| sys all | 5.12(1.32) | 4.41(2.15) | 1.65(1.13) |

TABLE III. Central value and various systematic errors, use the hybrid continuum limit for the connected diagrams. Numbers in parentheses are statistical error for the corresponding values.
In fact we can do even a bit better with the data on hand. As seen in Fig. 5, which shows the cumulative sum of all contributions up to a given separation of the two sampled currents in the hadronic loop, the total connected contribution saturates at a distance of about 1 fm for all ensembles. This suggests the region $r \gtrsim 1$ fm adds mostly noise and little signal, and the situation gets worse in the limits. A more accurate estimate can be obtained by taking the continuum limit for the sum up to $r = 1$ fm, and above that by taking the contribution from the relatively precise $48^3$ ensemble. We include a systematic error on this long distance part since it is not extrapolated to $a = 0$. The infinite volume limit is taken as before. This hybrid procedure yields $\sigma_\mu^{\text{con}} = 24.16(2.30)_{\text{stat}}(5.12)_{\text{sys}} \times 10^{-10}$, with a statistical error that is roughly 2 times smaller and the additional $O(a^2)$ systematic error from the hybrid procedure is only $0.20 \times 10^{-10}$. Unfortunately a similar procedure for the disconnected diagram is not reliable, as can be seen in the lower panel of Fig. 5. The cumulative plots do not reach plateaus around 1 fm, but instead tend to fall significantly up to 2 fm, or more. Once the cut moves beyond 1 fm it is no longer effective. The different behavior between the two stems from the different sampling strategies used for each [22]. Using the improved connected result, we find our final result for $\sigma_{\mu}^{\text{tot}}$,

$$\sigma_{\mu}^{\text{tot}} = 7.20(3.98)_{\text{stat}}(1.65)_{\text{sys}} \times 10^{-10},$$

where the error is mostly statistical. We also include all systematic errors added in quadrature, including the hybrid $O(a^2)$ error of the connected diagram. The systematic errors are summarized in Tab. III.

V. SUMMARY AND OUTLOOK

We have presented results for the hadronic light-by-light scattering contribution to the muon $g - 2$ from Lattice QCD+QED calculations with all errors under control. Large discretization and finite volume corrections are apparent but under control, and the value in the continuum and infinite volume limits is compatible with previous model and dispersive treatments, albeit with a large statistical error. Despite the large error, which results after a large cancellation between quark-connected and disconnected diagrams, our calculation suggests that light-by-light scattering cannot be behind the approximately 3.7 standard deviation discrepancy between the Standard Model and the BNL experiment E821. Future calculations will reduce the error significantly. The calculations presented here strengthen the much anticipated test of the Standard Model from the new experiments at Fermilab and J-PARC, with the former planning to announce first results near the beginning of 2020.

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A. QED Test with different fitting forms

We test various fitting formulas from lattice calculation of the muonic leptonic light-by-light contribution to muon $g - 2$. The analytic result is known to be $a_\mu = 0.371 \times (\alpha/\pi)^3 = 46.5 \times 10^{-10}$ using the conventional perturbative calculation.

In Ref. [32], we have performed the lattice calculation with three different lattice volumes, and each with three different lattice spacings. The results are listed in Tab. IV. Then, the continuum limit is calculated for each lattice volume and then extrapolate to infinite volume limit. In this paper, we adopt a different strategy. We use one formula which include both discretization effects and finite volume effects to fit all the data points. In the fitting forms listed below, we studied the size of $O(a^4)$, $O(1/L^3)$, and $O(a^2/L)$, effects in addition to the leading $O(a^2)$, $O(1/L^2)$ effects.

$$a_\mu(a, L) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_3}{(m_\mu L)^3} \right) \times \left(1 - c_1(m_\mu a)^2 + c_2(m_\mu a)^4 \right)$$

(8)

The results of the fit is listed in the following table:

| Variable | Value     | Statistical Error |
|----------|-----------|-------------------|
| $a_\mu$  | 45.34372  | 0.13747           |
| $b_2$    | 4.44949   | 0.00982           |
| $c_1/10$ | 1.88140   | 0.01844           |
| $c_2/10^2$ | 1.59916  | 0.03796           |

- fit-product-form-2L-2a

$$a_\mu(a, L) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2} \right) \left(1 - c_1(m_\mu a)^2 \right)$$

(9)

The results of the fit is listed in the following table:

| Variable | Value     | Statistical Error |
|----------|-----------|-------------------|
| $a_\mu$  | 41.74717  | 0.05870           |
| $b_2$    | 4.44563   | 0.00981           |
| $c_1/10$ | 1.15123   | 0.00210           |

- fit-product-form-3L-4a

$$a_\mu(a, L) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_3}{(m_\mu L)^3} \right) - c_1(m_\mu a)^2 + c_2(m_\mu a)^4$$

(10)

The results of the fit are listed in the following table:

| Variable | Value     | Statistical Error |
|----------|-----------|-------------------|
| $a_\mu$  | 43.09059  | 0.13226           |
| $b_2$    | 4.91796   | 0.14967           |
| $b_3$    | 4.61329   | 0.41827           |
| $c_1/10$ | 1.51229   | 0.01716           |
| $c_2/10^2$ | 1.30464  | 0.03263           |

- fit-plus-form-3L-4a-cross

$$a_\mu(a, L) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_3}{(m_\mu L)^3} \right) - c_1(m_\mu a)^2 \left(1 - \frac{b'_1}{m_\mu L} \right) + c_2(m_\mu a)^4$$

The results of the fit is listed in the following table:
systematic errors are resulting from lacking the corre-
connected diagrams’ contribution. Many of the above
contribution to the connected diagrams, and sub-leading dis-
connected diagrams’ contribution. Many of the above
systematic errors are resulting from lacking the corre-

\[\begin{array}{|c|c|c|}
\hline
\text{Variable} & \text{Value} & \text{Statistical Error} \\
\hline
a_\mu & 46.21446 & 0.15636 \\
b_2 & 6.54789 & 0.13960 \\
b_3 & 7.06359 & 0.38837 \\
c_1/10 & 1.84720 & 0.01728 \\
c_2/10^2 & 1.19503 & 0.03024 \\
b'_1 & 1.07316 & 0.01327 \\
\hline
\end{array}\]

\[a_\mu(L, a^1, a^D) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2}\right) - c_1(a^1 \text{ GeV})^2 - c_1^D(a^D \text{ GeV})^2 + c_2^D(a^D \text{ GeV})^4 \]

\[= a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2}\right) - \left(c_1(a^1 \text{ GeV})^2 + c_1^D(a^D \text{ GeV})^2 - c_2^D(a^D \text{ GeV})^4\right) \times \left(1 - \frac{\alpha_S}{\pi} \log \left((a \text{ GeV})^2\right)\right) \]

- fit-plus-form-2L-2a-2ad-4ad-cross

\[a_\mu(L, a^1, a^D) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2}\right) - c_1^D(a^D \text{ GeV})^2 - c_2^D(a^D \text{ GeV})^4 \times \left(1 - \frac{1}{m_\mu L}\right) \]

- fit-product-form-2L-2a-2ad-4ad

\[a_\mu(L, a^1, a^D) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_2}{(m_\mu L)^4}\right) \times \left(1 - c_1(a^1 \text{ GeV})^2 - c_1^D(a^D \text{ GeV})^2 + c_2^D(a^D \text{ GeV})^4\right) \]

- fit-product-form-3L-2a-2ad-4ad

\[a_\mu(L, a^1, a^D) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2}\right) \times \left(1 - c_1(a^1 \text{ GeV})^2 - c_1^D(a^D \text{ GeV})^2 + c_2^D(a^D \text{ GeV})^4\right) \]

- fit-product-form-2L-2a-4a-2ad

\[a_\mu(L, a^1, a^D) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2}\right) - c_1(a^1 \text{ GeV})^2 + c_1^D(a^D \text{ GeV})^2 + c_2(a \text{ GeV})^4 \]

- fit-product-form-2L-2a-4a-4ad

\[a_\mu(L, a^1, a^D) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2}\right) \times \left(1 - c_1(a \text{ GeV})^2 + c_1^D(a^D \text{ GeV})^2 + c_2^D(a^D \text{ GeV})^4\right) \]

- fit-product-form-2L-2a-4a-4ad-lna

\[a_\mu(L, a^1, a^D) = a_\mu \left(1 - \frac{b_2}{(m_\mu L)^2}\right) - c_1(a \text{ GeV})^2 + c_1^D(a^D \text{ GeV})^2 + c_2^D(a^D \text{ GeV})^4 \]
FIG. 7. QED test with the following forms: fit-product-form-3L-4a, fit-product-form-2L-4a, fit-product-form-2L-2a, fit-plus-form-3L-4a, fit-plus-form-3L-4a-cross, in this order.

- **fit-product-form-2L-2a-2ad-4ad-lna**

\[ a_\mu(L, a^I, a^D) = a_\mu \left( 1 - \frac{b_2}{(m_\mu L)^2} \right) \]

\[ \times \left( 1 - \left( c_1^I (a^I \text{ GeV})^2 + c_1^D (a^D \text{ GeV})^2 - c_2^D (a^D \text{ GeV})^4 \right) \times \left( 1 - \frac{\alpha_s}{\pi} \log ((a \text{ GeV})^2) \right) \right) \]  

(22)

Again, in these formulas, \( a^I \) is the lattice spacing for the Iwasaki ensembles, 48I and 64I. We define it to be zero for I-DSDR ensembles. Similarly, \( a^D \) is the lattice spacing for I-DSDR ensembles, and it is zero for Iwasaki ensembles. With this notation, the lattice spacings for all our ensembles are always equal to \( a = a^I + a^D \). In two of the fitting forms, \( \alpha_s(a) \) is used to estimate the size of

- **fit-product-form-2L-2a-2ad-4ad-cross**
the $a^2 \log(a^2)$ contribution. We use the 2-loop $\alpha_s^{MS}$ with $\Lambda_{QCD} = 325$ MeV: 

\[
\frac{a^{-1}}{\text{GeV}} \quad \alpha_s \\
1.015 \quad 0.44 \\
1.378 \quad 0.34 \\
1.730 \quad 0.30 \\
2.359 \quad 0.26
\]

All these fit forms correct the $O(a^2)$ lattice artifacts presented in DWF lattice calculations for Iwasaki ensembles (48I and 64I). The results obtained has a relative large statistical error mostly because of the continuum extrapolation. In Section IV we introduced the hybrid continuum limit for the connected diagrams’ contribution, where we calculate the continuum limit for contributions from the region $r \leq 1$ fm and use the relatively precise 48I ensemble results without extrapolation for the $r > 1$ fm region. This hybrid procedure is possible due to the small size of the contribution from the region $r > 1$ fm. This is expected due to the exponential suppression from QCD mass-gap, and, in addition, for the connected diagrams, we label the three vertex locations on the quark loop that connect to the internal photons as $x$, $y$, $z$ and require $r = |x - y| \leq \min(|x - z|, |y - z|)$. This is a direct consequence of Eq. (3) in Ref. [32]. Operationally, this hybrid continuum limit for the connected diagrams is implemented by replacing the long distance region ($r > 1$ fm) of the 64I ensemble data with the corresponding 48I ensemble data:

\[
a_{\mu}^{64I}(r \leq 1 \text{ fm}) + a_{\mu}^{48I}(r > 1 \text{ fm}).
\]

We perform the above replacement under the super jackknife procedure before the global fit together with all the results from other ensembles.

To estimate the hybrid $O(a^2)$ systematic error resulting from the above replacement, we perform a slightly different replacement:

\[
a_{\mu}^{64I}(r \leq 1 \text{ fm}) + a_{\mu}^{48I,\text{con}}(r \leq 1 \text{ fm}) + a_{\mu}^{48I}(r > 1 \text{ fm}).
\]

The difference of the results obtained with these two replacements is used as the estimation of the hybrid $O(a^2)$ systematic error.

We then estimate systematic errors due to lacking higher order finite volume or discretization terms in our default fitting form $\text{fit-plus-form-2L-2a-2ad-4ad}$ by comparing the results between the fitting formulas:

- $O(1/L^3)$: Difference between $\text{fit-plus-form-2L-2a-2ad-4ad}$ and $\text{fit-plus-form-3L-2a-2ad-4ad};$
- $O(a^4)$: The maximum of the following two differences: (a) $\text{fit-plus-form-2L-2a-2ad-4ad}$ and $\text{fit-plus-form-2L-2a-2ad-4ad}$; (b) $\text{fit-plus-form-2L-2a-2ad-4ad}$ and $\text{fit-plus-form-2L-2a-4a-4ad};$
- $O(a^2\log(a^2))$: Difference between $\text{fit-plus-form-2L-2a-2ad-4ad}$ and $\text{fit-plus-form-2L-2a-4a-4ad};$
- $O(a^2/L)$: The maximum of the following two differences: (a) $\text{fit-plus-form-2L-2a-2ad-4ad}$ and $\text{fit-plus-form-2L-2a-2ad-4ad-cross};$ (b) $\text{fit-plus-form-2L-2a-2ad-4ad}$ and $\text{fit-product-form-2L-2a-2ad-4ad}.$

For the systematic errors due to not including (a) the strange quark contribution to the connected diagrams and (b) sub-leading disconnected diagrams’ contributions, we performed lattice calculation with the 24D ensemble using the QED$_{\infty}$ method [47]. The results are plotted in Fig. 8 and Fig. 9. We estimate the systematic error due to omitting these two contributions to be: $\pm 0.3 \times 10^{-10}$ and $\pm 0.5 \times 10^{-10}$ respectively.

![FIG. 8. Strange quark contribution to the connected diagrams calculated with QED$_{\infty}$. The strange quark contribution to the disconnected diagrams is already included in the present QED$_L$ calculation.](image-url)

We then add all these systematic effects in quadrature as our final systematic error. We also use these procedure per jackknife sample and calculate the statistical uncertainty of the systematic error. This procedure is repeated for each of the “con”, “discon”, and “tot” contributions. The “tot” contribution is calculated by perform a single fit with the sum of “con” and the “discon” contributions for each ensemble. Since we only calculated the “con” contribution for the 48D ensemble, 48D is not included in the fit for “tot”. As such, the “tot” fit is usually not exactly equal to the sum of the individual “con” and “discon” fits, which we label them as “sum”. The results are shown in the Tab. VII Statistical errors for central values and systematic errors are listed in the table. We use the “tot” in this table as our central value.

We did the same exercise for $\text{fit-product-form-2L-2a-2ad-4ad}$ as a comparison. The results are shown in Tab. VII.
FIG. 9. Sub-leading disconnected diagrams’ contribution. We only include the second diagram in Fig. 3. Only light quark contribution is calculated except for the tadpole part, where we reuse the calculation for the disconnected diagrams in HVP calculations described in Ref. [3] and both light quark and strange quark contribution is included. The remaining diagrams are equally or more suppressed.

Based on the experiences from the QED test, result from the \( \text{fit-product-form-}2L-2a-2ad-4ad \) fit the data better. However, in QCD calculations, \( \text{fit-plus-form-}2L-2a-2ad-4ad \) leads to smaller statistical error. At present level of accuracy, smaller statistical error is more important than potentially smaller systematic error. In particular, the systematic error is largely cancelled between the connected diagrams and the disconnected diagrams, while the statistical error does not. The fitting results with all the fitting forms are also summarized in Tab. VII.

We have also performed the calculation without the hybrid continuum limit. The results are shown in Tabs. VIII, IX, X.

Finally, the detailed results and plots for each fitting forms with or without the hybrid continuum limit are listed in remaining tables and plots.
TABLE V. Plus form fitting results, use the hybrid continuum limit for the connected diagrams.

| fit-form                  | con         | discon     | sum         | sum-sys $(a^2)$ | tot         | tot-sys $(a^2)$ |
|---------------------------|-------------|------------|-------------|-----------------|-------------|-----------------|
| fit-plus-form              |             |            |             |                 |             |                 |
| fit-plus-form-2L-2a-2ad-4ad| 24.16(2.30) | -17.12(3.46) | 7.04(3.97)  | 0.20(0.45)      | 7.20(3.98)  | 0.20(0.45)      |
| fit-plus-form-3L-2a-2ad-4ad| 26.51(2.53) | -18.84(3.56) | 7.67(4.20)  | 0.20(0.45)      | 8.03(4.23)  | 0.20(0.45)      |
| fit-plus-form-2L-2a-4a-2ad | 25.10(2.50) | -17.95(3.89) | 7.15(4.40)  | 0.23(0.51)      | 7.30(4.41)  | 0.23(0.51)      |
| fit-plus-form-2L-2a-4a-4ad | 24.85(2.43) | -16.69(3.34) | 8.15(3.93)  | 0.20(0.44)      | 8.28(3.94)  | 0.20(0.44)      |
| fit-plus-form-2L-2a-2ad-4ad-lna| 24.39(2.36) | -17.40(3.60) | 6.99(4.11)  | 0.21(0.47)      | 7.16(4.12)  | 0.21(0.47)      |

TABLE VI. Product form fitting results, use the hybrid continuum limit for the connected diagrams.

| fit-form                  | con         | discon     | sum         | sum-sys $(a^2)$ | tot         | tot-sys $(a^2)$ |
|---------------------------|-------------|------------|-------------|-----------------|-------------|-----------------|
| fit-plus-form              |             |            |             |                 |             |                 |
| fit-plus-form-2L-2a-2ad-4ad| 28.59(3.65) | -21.05(5.75) | 7.54(6.49)  | 0.33(0.73)      | 7.92(5.98)  | 0.31(0.70)      |
| fit-plus-form-3L-2a-2ad-4ad| 32.78(4.41) | -24.47(6.78) | 8.31(7.73)  | 0.37(0.83)      | 9.06(7.00)  | 0.36(0.80)      |
| fit-plus-form-2L-2a-4a-4ad | 30.10(4.02) | -22.45(6.47) | 7.65(7.25)  | 0.37(0.82)      | 8.06(6.68)  | 0.35(0.79)      |
| fit-plus-form-2L-2a-2ad-4ad-lna| 28.97(3.76) | -21.51(5.99) | 7.46(6.74)  | 0.34(0.75)      | 7.86(6.20)  | 0.33(0.73)      |

TABLE VII. Values for all fitting forms, use the hybrid continuum limit for the connected diagrams.
| fit-form | con  | discon | sum  | sum-sys \((a^2)\) | tot  | tot-sys \((a^2)\) |
|----------|------|--------|------|----------------|------|----------------|
| fit-plus-form-2L-2a-2ad-4ad | 23.76(3.96) | -17.12(3.46) | 6.64(4.64) | 0.00(0.00) | 6.80(4.65) | 0.00(0.00) |
| fit-plus-form-3L-2a-2ad-4ad | 26.10(4.10) | -18.84(3.56) | 7.27(4.84) | 0.00(0.00) | 7.63(4.87) | 0.00(0.00) |
| fit-plus-form-2L-2a-4a-2ad | 24.64(4.43) | -17.95(3.89) | 6.69(5.17) | 0.00(0.00) | 6.84(5.18) | 0.00(0.00) |
| fit-plus-form-2L-2a-4a-4ad | 24.45(3.69) | -16.69(3.34) | 7.75(4.31) | 0.00(0.00) | 7.88(4.32) | 0.00(0.00) |
| fit-plus-form-2L-2a-2ad-4ad-lna | 23.97(4.13) | -17.40(3.60) | 6.57(4.83) | 0.00(0.00) | 6.74(4.84) | 0.00(0.00) |
| fit-plus-form-2L-2a-2ad-4ad-cross | 25.67(4.91) | -18.68(4.32) | 6.99(5.73) | 0.00(0.00) | 7.03(5.73) | 0.00(0.00) |
| fit-product-form-2L-2a-2ad-4ad | 27.94(6.31) | -21.05(5.75) | 6.89(7.50) | 0.00(0.00) | 7.30(7.00) | 0.00(0.00) |
| fit-product-form-3L-2a-2ad-4ad | 32.03(7.36) | -24.47(6.78) | 7.56(8.83) | 0.00(0.00) | 8.34(8.11) | 0.00(0.00) |
| fit-product-form-2L-2a-2ad-4ad-lna | 28.29(6.59) | -21.51(5.99) | 6.78(7.82) | 0.00(0.00) | 7.20(7.30) | 0.00(0.00) |
| fit-product-form-2L-2a-2ad-4ad-cross | 31.00(8.24) | -23.55(7.66) | 7.45(9.84) | 0.00(0.00) | 7.59(8.52) | 0.00(0.00) |

TABLE X. Values for all fitting forms, do not use the hybrid continuum limit.
TABLE XI. Fit results for fit-plus-form-2L-2a-2ad-4ad with the hybrid continuum limit.

|   | \( a_\mu \) | \( b_2 \) | \( c_1^1 \) | \( c_1^D - c_1^1 \) | \( c_2^D \) |
|---|-------------|--------|-------------|-----------------|--------|
| con | 24.16(2.30) | 2.20(0.33) | 0.69(0.20) | 0.16(0.19) | 0.57(0.17) |
| discon | -17.12(3.46) | 2.15(0.52) | 1.19(0.40) | -0.14(0.14) | 0.71(0.27) |
| sum | 7.04(3.97) |         |            |                |        |
| sum-sys \((a^2)\) | 0.20(0.45) |        |            |                |        |
| tot | 7.20(3.98) | 2.46(1.70) | -0.49(1.85) | 0.84(0.95) | 0.19(1.01) |
| tot-sys \((a^2)\) | 0.20(0.45) |        |            |                |        |

FIG. 10. Fit plots for fit-plus-form-2L-2a-2ad-4ad with the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

TABLE XII. Fit results for fit-plus-form-3L-2a-2ad-4ad with the hybrid continuum limit.

|   | \( a_\mu \) | \( b_2 \) | \( c_1^1 \) | \( c_1^D - c_1^1 \) | \( c_2^D \) |
|---|-------------|--------|-------------|-----------------|--------|
| con | 25.10(2.53) | 4.20(0.57) | 0.63(0.19) | 0.17(0.17) | 0.54(0.16) |
| discon | -18.84(3.56) | 4.15(0.91) | 1.08(0.38) | -0.10(0.13) | 0.67(0.26) |
| sum | 7.67(4.20) |         |            |                |        |
| sum-sys \((a^2)\) | 0.20(0.45) |        |            |                |        |
| tot | 8.03(4.23) | 4.69(2.90) | -0.44(1.64) | 0.78(0.85) | 0.19(0.90) |
| tot-sys \((a^2)\) | 0.20(0.45) |        |            |                |        |

FIG. 11. Fit plots for fit-plus-form-3L-2a-2ad-4ad with the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

TABLE XIII. Fit results for fit-plus-form-2L-2a-4a-2ad with the hybrid continuum limit.

|   | \( a_\mu \) | \( b_2 \) | \( c_1^1 \) | \( c_2 \) | \( c_1^D - c_1^1 \) |
|---|-------------|--------|-------------|-------|-----------------|
| con | 25.10(2.50) | 2.12(0.33) | 0.99(0.25) | 0.62(0.18) | -0.06(0.15) |
| discon | -17.95(3.89) | 4.15(0.91) | 1.53(0.50) | 0.77(0.28) | -0.39(0.14) |
| sum | 7.15(4.40) |         |            |                |        |
| sum-sys \((a^2)\) | 0.23(0.51) |        |            |                |        |
| tot | 7.30(4.41) | 2.43(1.81) | -0.37(2.35) | 0.22(1.12) | 0.75(1.08) |
| tot-sys \((a^2)\) | 0.23(0.51) |        |            |                |        |
FIG. 12. Fit plots for fit-plus-form-2L-2a-4a-2ad with the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

|       | $a_\mu$     | $b_2$        | $c_3$      | $c_2^D - c_1^D$ |
|-------|-------------|--------------|------------|-----------------|
| con   | 24.85(2.43) | 2.14(0.35)   | 0.91(0.25) | 0.46(0.52)      |
| discon| -16.69(3.34)| 2.20(0.55)   | 1.00(0.42) | -0.43(0.44)     |
| sum   | 8.15(3.93)  |              |            |                 |
| sum-sys ($a^2$) | 0.20(0.44) |              |            |                 |
| tot-sys ($a^2$)  | 8.28(3.94)  | 2.14(1.51)   | 0.68(1.19)  | 2.16(2.12)      |
| tot   | 7.16(4.12)  | 2.48(1.75)   | -0.47(1.77)| 0.76(0.88)      |

TABLE XIV. Fit results for fit-plus-form-2L-2a-4a-4ad with the hybrid continuum limit.

FIG. 13. Fit plots for fit-plus-form-2L-2a-4a-4ad with the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

|       | $a_\mu$     | $b_2$        | $c_1$      | $c_2^D - c_1^D$ |
|-------|-------------|--------------|------------|-----------------|
| con   | 24.39(2.36) | 2.18(0.33)   | 0.65(0.19) | 0.15(0.17)      |
| discon| -17.40(3.60)| 2.20(0.55)   | 1.10(0.36) | -0.12(0.13)     |
| sum   | 6.99(4.11)  |              |            |                 |
| sum-sys ($a^2$) | 0.21(0.47) |              |            |                 |
| tot   | 7.16(4.12)  | 2.48(1.75)   | -0.47(1.77)| 0.76(0.88)      |
| tot-sys ($a^2$)  | 8.28(3.94)  | 2.14(1.51)   | 0.68(1.19)  | 2.16(2.12)      |

TABLE XV. Fit results for fit-plus-form-2L-2a-3ad-lna with the hybrid continuum limit.

FIG. 14. Fit plots for fit-plus-form-2L-2a-3ad-lna with the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).
TABLE XVI. Fit results for fit-plus-form-2L-2a-2ad-4ad-cross with the hybrid continuum limit.

|          | $a_{\mu}$ | $b_2$  | $c_1^I$ | $c_1^D - c_1^I$ | $c_2^D$ |
|----------|-----------|--------|---------|-----------------|---------|
| con      | 26.18(2.76) | 2.64(0.24) | 1.60(0.28) | 0.21(0.27) | 0.80(0.24) |
| discon   | -18.68(4.32) | 2.66(0.32) | 1.67(0.51) | -0.20(0.21) | 1.00(0.36) |
| sum      | 7.50(4.86)  |         |         |                 |         |
| sum-sys ($a^2$) | 0.26(0.57)  |         |         |                 |         |
| tot      | 7.54(4.87)  | 2.69(1.17) | -0.68(2.76) | 1.19(1.46) | 0.28(1.48) |
| tot-sys ($a^2$) | 0.26(0.57)  |         |         |                 |         |

TABLE XVII. Fit results for fit-product-form-2L-2a-2ad-4ad with the hybrid continuum limit.

|          | $a_{\mu}$ | $b_2$  | $c_1^I$ | $c_1^D - c_1^I$ | $c_2^D$ |
|----------|-----------|--------|---------|-----------------|---------|
| con      | 28.59(3.65) | 3.12(0.29) | 0.96(0.25) | 0.21(0.28) | 0.78(0.24) |
| discon   | -21.05(5.75) | 3.29(0.32) | 1.60(0.43) | -0.17(0.22) | 0.98(0.32) |
| sum      | 7.92(5.98)  | 2.92(1.13) | -0.62(2.69) | 1.11(1.43) | 0.26(1.46) |
| sum-sys ($a^2$) | 0.33(0.73)  |         |         |                 |         |
| tot      | 7.54(6.49)  |         |         |                 |         |
| tot-sys ($a^2$) | 0.31(0.70)  |         |         |                 |         |

TABLE XVIII. Fit results for fit-product-form-3L-2a-2ad-4ad with the hybrid continuum limit.

|          | $a_{\mu}$ | $b_2$  | $c_1^I$ | $c_1^D - c_1^I$ | $c_2^D$ |
|----------|-----------|--------|---------|-----------------|---------|
| con      | 32.78(4.41) | 5.79(0.49) | 0.96(0.25) | 0.25(0.27) | 0.81(0.23) |
| discon   | -24.47(6.78) | 6.11(0.52) | 1.60(0.43) | -0.13(0.21) | 1.01(0.31) |
| sum      | 8.31(7.33)  |         |         |                 |         |
| sum-sys ($a^2$) | 0.37(0.83)  |         |         |                 |         |
| tot      | 9.06(7.00)  | 5.50(1.90) | -0.62(2.68) | 1.16(1.43) | 0.30(1.42) |
| tot-sys ($a^2$) | 0.36(0.80)  |         |         |                 |         |
FIG. 17. Fit plots for \textit{fit-product-form-3L-2a-2ad-4ad} with the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

\begin{center}
\begin{tabular}{cccccc}
\hline
 & $a_{\mu}$ & $b_2$ & $c_1$ & $c_2$ & $c_1^D-c_1^I$ \\
\hline
con & 30.10(4.02) & 3.12(0.29) & 1.34(0.30) & 0.83(0.24) & -0.09(0.21) \\
discon & -22.45(6.47) & 3.29(0.32) & 2.04(0.52) & 1.04(0.32) & -0.51(0.19) \\
sum & 7.65(7.25) & & & & \\
sum-sys ($a^2$) & 0.37(0.82) & & & & \\
tot & 8.06(6.68) & 2.92(1.13) & 0.29(1.61) & 1.00(1.64) & \\
tot-sys ($a^2$) & 0.35(0.79) & & & & \\
\hline
\end{tabular}
\end{center}

TABLE XIX. Fit results for \textit{fit-product-form-2L-2a-4a-2ad} with the hybrid continuum limit.

FIG. 18. Fit plots for \textit{fit-product-form-2L-2a-4a-2ad} with the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

\begin{center}
\begin{tabular}{cccccc}
\hline
 & $a_{\mu}$ & $b_2$ & $c_1$ & $c_2$ & $c_1^D-c_1^I$ \\
\hline
con & 29.65(3.92) & 3.12(0.29) & 1.23(0.32) & 0.59(0.76) & -0.23(0.58) \\
discon & -20.41(5.55) & 3.29(0.32) & 1.38(0.48) & -0.52(0.69) & -1.47(0.59) \\
sum & 9.23(6.47) & & & & \\
sum-sys ($a^2$) & 0.32(0.72) & & & & \\
tot & 9.49(6.05) & 2.92(1.13) & 0.89(1.54) & 2.75(2.95) & 2.21(3.07) \\
tot-sys ($a^2$) & 0.31(0.69) & & & & \\
\hline
\end{tabular}
\end{center}

TABLE XX. Fit results for \textit{fit-product-form-2L-2a-4a-4ad} with the hybrid continuum limit.

FIG. 19. Fit plots for \textit{fit-product-form-2L-2a-4a-4ad} with the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).
\begin{table}[h]
\centering
\begin{tabular}{lcccccc}
& $a_\mu$ & $b_2$ & $c_1^L$ & $c_1^D - c_1^L$ & $c_2^D$ \\
\hline
con & 28.97(3.76) & 3.12(0.29) & 0.89(0.23) & 0.19(0.25) & 0.68(0.22) \\
\hline
discon & -21.51(5.99) & 3.29(0.32) & 1.48(0.39) & -0.14(0.20) & 0.87(0.28) \\
\hline
sum & 7.46(6.74) & & & & \\
sum-sys ($a^2$) & 0.34(0.75) & & & & \\
\hline
tot & 7.86(6.20) & 2.92(1.13) & -0.59(2.58) & 1.01(1.34) & 0.19(1.36) \\
tot-sys ($a^2$) & 0.33(0.73) & & & & \\
\end{tabular}
\caption{Fit results for fit-product-form-2L-2a-2ad-4ad-lna with the hybrid continuum limit.}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig20}
\caption{Fit plots for fit-product-form-2L-2a-2ad-4ad-lna with the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).}
\end{figure}

\begin{table}[h]
\centering
\begin{tabular}{lcccccc}
& $a_\mu$ & $b_2$ & $c_1^L$ & $c_1^D - c_1^L$ & $c_2^D$ \\
\hline
con & 31.86(4.75) & 3.61(0.25) & 1.51(0.39) & 0.22(0.46) & 1.13(0.41) \\
\hline
discon & -23.55(7.66) & 3.83(0.34) & 2.44(0.66) & -0.32(0.38) & 1.44(0.49) \\
\hline
sum & 8.31(8.57) & & & & \\
sum-sys ($a^2$) & 0.43(0.97) & & & & \\
\hline
tot & 8.37(7.40) & 3.17(1.19) & -0.89(4.08) & 1.59(2.31) & 0.34(2.27) \\
tot-sys ($a^2$) & 0.40(0.93) & & & & \\
\end{tabular}
\caption{Fit results for fit-product-form-2L-2a-2ad-4ad-cross with the hybrid continuum limit.}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig21}
\caption{Fit plots for fit-product-form-2L-2a-2ad-4ad-cross with the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).}
\end{figure}
TABLE XXIII. Fit results for \textit{fit-plus-form-2L-2a-2ad-4ad} without the hybrid continuum limit.

|        | $a_\mu$       | $b_2$       | $c_1$       | $c_1^D - c_1$ | $c_2^D$ |
|--------|---------------|-------------|-------------|---------------|---------|
| con    | 23.76(3.96)  | 2.24(0.45)  | 0.66(0.45)  | 0.16(0.21)    | 0.54(0.26) |
| discon | -17.12(3.46) | 2.15(0.52)  | 1.19(0.40)  | -0.14(0.14)   | 0.71(0.27) |
| sum    | 6.64(4.64)   |             |             |               |         |
| sum-sys ($a^2$) | 0.00(0.00)   |             |             |               |         |
| tot    | 6.80(4.65)   | 2.61(2.05)  | -0.70(2.67) | 0.89(1.22)    | 0.09(1.29) |
| tot-sys ($a^2$) | 0.00(0.00)   |             |             |               |         |

FIG. 22. Fit plots for \textit{fit-plus-form-2L-2a-2ad-4ad} without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

TABLE XXIV. Fit results for \textit{fit-plus-form-3L-2a-2ad-4ad} without the hybrid continuum limit.

|        | $a_\mu$       | $b_2$       | $c_1$       | $c_1^D - c_1$ | $c_2^D$ |
|--------|---------------|-------------|-------------|---------------|---------|
| con    | 26.10(4.10)  | 4.27(0.78)  | 0.60(0.41)  | 0.18(0.19)    | 0.52(0.24) |
| discon | -18.84(3.56) | 4.15(0.91)  | 1.08(0.38)  | -0.10(0.13)   | 0.67(0.26) |
| sum    | 7.27(4.84)   |             |             |               |         |
| sum-sys ($a^2$) | 0.00(0.00)   |             |             |               |         |
| tot    | 7.63(4.87)   | 4.94(3.46)  | -0.62(2.34) | 0.82(1.07)    | 0.10(1.14) |
| tot-sys ($a^2$) | 0.00(0.00)   |             |             |               |         |

FIG. 23. Fit plots for \textit{fit-plus-form-3L-2a-2ad-4ad} without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

TABLE XXV. Fit results for \textit{fit-plus-form-2L-2a-4a-2ad} without the hybrid continuum limit.

|        | $a_\mu$       | $b_2$       | $c_1$       | $c_2$       | $c_1^D - c_1$ |
|--------|---------------|-------------|-------------|-------------|---------------|
| con    | 24.64(4.43)  | 2.16(0.46)  | 0.94(0.54)  | 0.59(0.27)  | -0.04(0.23)   |
| discon | -17.95(3.89) | 2.05(0.53)  | 1.53(0.50)  | 0.77(0.28)  | -0.39(0.14)   |
| sum    | 6.69(5.17)   |             |             |             |               |
| sum-sys ($a^2$) | 0.00(0.00)   |             |             |             |               |
| tot    | 6.84(5.18)   | 2.59(2.24)  | -0.64(3.40) | 0.10(1.45)  | 0.85(1.56)    |
| tot-sys ($a^2$) | 0.00(0.00)   |             |             |             |               |

TABLE XXV. Fit results for \textit{fit-plus-form-2L-2a-4a-2ad} without the hybrid continuum limit.
FIG. 24. Fit plots for fit-plus-form-2L-2a-4a-2ad without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

|       | $a_\mu$  | $b_2$     | $c_1$     | $c^D_1 - c_1$ | $c^D_2 - c^D_1$ |
|-------|----------|-----------|-----------|---------------|-----------------|
| con   | 24.45(3.69) | 2.18(0.43) | 0.88(0.34) | 0.47(0.58)     | 0.12(0.60)      |
| discon| -16.69(3.34) | 2.20(0.55) | 1.00(0.42) | -0.43(0.44)    | 1.11(0.44)      |
| sum   | 7.75(4.31)  |           |           |               |                 |
| sum-sys ($a^2$) | 0.00(0.00)  |           |           |               |                 |
| tot   | 7.88(4.32)  | 2.25(1.68) | 0.57(1.39) | 2.28(2.67)     | -1.94(3.05)     |
| tot-sys ($a^2$)| 0.00(0.00)  |           |           |               |                 |

TABLE XXVI. Fit results for fit-plus-form-2L-2a-4a-4ad without the hybrid continuum limit.

FIG. 25. Fit plots for fit-plus-form-2L-2a-4a-4ad without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

|       | $a_\mu$  | $b_2$     | $c_1$     | $c^D_1 - c_1$ | $c^D_2 - c^D_1$ |
|-------|----------|-----------|-----------|---------------|-----------------|
| con   | 23.97(4.13) | 2.22(0.45) | 0.61(0.41) | 0.15(0.19)     | 0.48(0.23)      |
| discon| -17.40(3.60) | 2.11(0.52) | 1.10(0.36) | -0.12(0.13)    | 0.63(0.24)      |
| sum   | 6.57(4.83)  |           |           |               |                 |
| sum-sys ($a^2$) | 0.00(0.00)  |           |           |               |                 |
| tot   | 6.74(4.84)  | 2.64(2.15) | -0.66(2.57)| 0.81(1.14)     | 0.05(1.20)      |
| tot-sys ($a^2$)| 0.00(0.00)  |           |           |               |                 |

TABLE XXVII. Fit results for fit-plus-form-2L-2a-2ad-4ad-lna without the hybrid continuum limit.

FIG. 26. Fit plots for fit-plus-form-2L-2a-2ad-4ad-lna without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).
$\begin{array}{ccccccc}
  a_\mu & b_2 & c_1^L & c_1^D - c_1^L & c_2^D \\
  \text{con} & 25.67(4.91) & 2.66(0.28) & 0.95(0.60) & 0.21(0.30) & 0.77(0.35) \\
  \text{discon} & -18.68(4.32) & 2.66(0.32) & 1.67(0.51) & -0.20(0.21) & 1.00(0.36) \\
  \text{sum} & 6.99(5.73) & & & & \\
  \text{sum-sys} (a^2) & 0.00(0.00) & & & & \\
  \text{tot} & 7.03(5.73) & 2.76(1.34) & -1.00(4.08) & 1.29(1.95) & 0.12(1.92) \\
  \text{tot-sys} (a^2) & 0.00(0.00) & & & & 
\end{array}$

**TABLE XXVIII.** Fit results for fit-plus-form-2L-2a-2ad-4ad-cross without the hybrid continuum limit.

![Fig. 27](image_url)

**FIG. 27.** Fit plots for fit-plus-form-2L-2a-2ad-4ad-cross without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

$\begin{array}{ccccccc}
  a_\mu & b_2 & c_1^L & c_1^D - c_1^L & c_2^D \\
  \text{con} & 27.94(6.31) & 3.12(0.29) & 0.91(0.55) & 0.21(0.31) & 0.75(0.33) \\
  \text{discon} & -21.05(5.75) & 3.29(0.32) & 1.60(0.43) & -0.17(0.22) & 0.98(0.32) \\
  \text{sum} & 6.89(7.50) & & & & \\
  \text{sum-sys} (a^2) & 0.00(0.00) & & & & \\
  \text{tot} & 7.30(7.00) & 2.92(1.13) & -0.93(4.08) & 1.21(1.97) & 0.11(1.93) \\
  \text{tot-sys} (a^2) & 0.00(0.00) & & & & 
\end{array}$

**TABLE XXIX.** Fit results for fit-product-form-2L-2a-2ad-4ad without the hybrid continuum limit.

![Fig. 28](image_url)

**FIG. 28.** Fit plots for fit-product-form-2L-2a-2ad-4ad without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

$\begin{array}{ccccccc}
  a_\mu & b_2 & c_1^L & c_1^D - c_1^L & c_2^D \\
  \text{con} & 32.03(7.36) & 5.79(0.49) & 0.92(0.55) & 0.26(0.31) & 0.79(0.32) \\
  \text{discon} & -24.47(6.78) & 6.11(0.52) & 1.60(0.43) & -0.13(0.21) & 1.01(0.31) \\
  \text{sum} & 7.56(8.83) & & & & \\
  \text{sum-sys} (a^2) & 0.00(0.00) & & & & \\
  \text{tot} & 8.34(8.11) & 5.50(1.90) & -0.92(4.06) & 1.26(1.98) & 0.16(1.88) \\
  \text{tot-sys} (a^2) & 0.00(0.00) & & & & 
\end{array}$

**TABLE XXX.** Fit results for fit-product-form-3L-2a-2ad-4ad without the hybrid continuum limit.
FIG. 29. Fit plots for $\text{fit-product-form-3L-2a-2ad-4ad}$ without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

| $a_\mu$ | $b_2$ | $c_1^1$ | $c_2$ | $c_1^D - c_1^1$ |
|---|---|---|---|---|
| con | 29.37(7.09) | 3.12(0.29) | 1.28(0.65) | 0.81(0.34) | -0.07(0.31) |
| discon | -22.45(6.47) | 3.29(0.32) | 2.04(0.52) | 1.04(0.32) | -0.51(0.19) |
| sum | 6.92(8.41) | | | | |
| sum-sys ($a^2$) | 0.00(0.00) | | | | |
| tot | 7.35(7.85) | 2.92(1.13) | -0.86(5.17) | 0.12(2.17) | 1.16(2.51) |
| tot-sys ($a^2$) | 0.00(0.00) | | | | |

TABLE XXXII. Fit results for $\text{fit-product-form-2L-2a-4a-2ad}$ without the hybrid continuum limit.

FIG. 30. Fit plots for $\text{fit-product-form-2L-2a-4a-2ad}$ without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

| $a_\mu$ | $b_2$ | $c_1$ | $c_2^1$ | $c_2^D - c_2^1$ |
|---|---|---|---|---|
| con | 29.00(5.93) | 3.12(0.29) | 1.19(0.42) | 0.61(0.84) | -0.18(0.82) |
| discon | -20.41(5.55) | 3.29(0.32) | 1.38(0.48) | -0.52(0.69) | -1.47(0.59) |
| sum | 8.59(7.01) | | | | |
| sum-sys ($a^2$) | 0.00(0.00) | | | | |
| tot | 8.87(6.60) | 2.92(1.13) | 0.75(1.86) | 2.95(3.91) | 2.51(4.44) |
| tot-sys ($a^2$) | 0.00(0.00) | | | | |

TABLE XXXIII. Fit results for $\text{fit-product-form-2L-2a-4a-4ad}$ without the hybrid continuum limit.

FIG. 31. Fit plots for $\text{fit-product-form-2L-2a-4a-4ad}$ without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).
\begin{table}
\begin{tabular}{lcccc}
\hline
 & $a_\mu$ & $b_2$ & $c_1^I$ & $c_1^D - c_1^I$ & $c_2^D$
\hline
con & 28.29(6.59) & 3.12(0.29) & 0.85(0.51) & 0.20(0.28) & 0.66(0.30)
discon & -21.51(5.99) & 3.29(0.32) & 1.48(0.39) & -0.14(0.20) & 0.87(0.28)
sum & 6.78(7.82) & & & &
sum-sys ($a^2$) & 0.00(0.00) & & & &
tot & 7.20(7.30) & 2.92(1.13) & -0.89(3.95) & 1.11(1.86) & 0.06(1.81)
tot-sys ($a^2$) & 0.00(0.00) & & & &
\hline
\end{tabular}
\caption{Fit results for fit-product-form-2L-2a-2ad-4ad-lna without the hybrid continuum limit.}
\end{table}

FIG. 32. Fit plots for fit-product-form-2L-2a-2ad-4ad-lna without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).

\begin{table}
\begin{tabular}{lcccc}
\hline
 & $a_\mu$ & $b_2$ & $c_1^I$ & $c_1^D - c_1^I$ & $c_2^D$
\hline
con & 31.00(8.24) & 3.58(0.31) & 1.44(0.82) & 0.23(0.52) & 1.10(0.52)
discon & -23.55(7.66) & 3.83(0.34) & 2.44(0.66) & -0.32(0.38) & 1.44(0.49)
sum & 7.45(9.84) & & & &
sum-sys ($a^2$) & 0.00(0.00) & & & &
tot & 7.59(8.52) & 3.09(1.32) & -1.36(6.16) & 1.75(3.13) & 0.13(2.93)
tot-sys ($a^2$) & 0.00(0.00) & & & &
\hline
\end{tabular}
\caption{Fit results for fit-product-form-2L-2a-2ad-4ad-cross without the hybrid continuum limit.}
\end{table}

FIG. 33. Fit plots for fit-product-form-2L-2a-2ad-4ad-cross without the hybrid continuum limit. Connected (left), disconnected (middle), and total (right).