Quaternionic metamonogenic functions in the unit disk

J. Morais$^a$ and R. Michael Porter$^b$

$^a$Department of Mathematics, ITAM, Río Hondo #1, Col. Progreso Tizapán, 01080 Mexico City, Mexico.
E-mail: joao.morais@itam.mx

$^b$Department of Mathematics, CINVESTAV-Querétaro, Libramiento Norponiente #2000, Fracc. Real de Juriquilla. Santiago de Querétaro, C.P. 76230 Mexico

Abstract

We construct a set of quaternionic metamonogenic functions (that is, in $\text{Ker}(D + \lambda)$ for diverse $\lambda$) in the unit disk, such that every metamonogenic function is approximable in the quaternionic Hilbert module $L^2$ of the disk. The set is orthogonal except for the small subspace of elements of orders zero and one. These functions are used to express time-dependent solutions of the imaginary-time wave equation in the polar coordinate system.

Keywords: Quaternionic analysis, Bessel functions, quaternionic functions, Moisil-Teodorescu operator.

1 Introduction

We consider the first-order partial differential quaternionic operator (sometimes called the Moisil-Teodorescu operator) $D + \lambda$ ($\lambda \in \mathbb{R} \setminus \{0\}$) in planar domains, where

$$D = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j. \quad (1)$$

Here $i$ and $j$ are two of the three basic quaternionic units, and the Dirac operator $D$ acts on the left on smooth quaternion-valued functions of a complex variable $z = x + iy$. This is a special case of Clifford-type operators for which there is a vast literature covering functions defined on spaces of diverse dimensions, beginning with W. Hamilton [14], continued by R. Fueter [7, 8] and followed by many others including [1, 2, 4, 5, 6, 12, 13, 19].

Functions defined in $\mathbb{R}^2$ taking values in Clifford algebras of dimension $> 2$ have been relatively less investigated. The case $\mathbb{R}^2 \rightarrow \mathbb{R}^4 \cong \mathbb{H}$ was considered
in [17], where a detailed investigation of $D + \lambda$ was carried out for quaternion-valued functions in the particular situation of elliptical domains. In [20], the authors constructed a one-parameter family of reduced quaternion-valued ($\mathbb{R}^3$-valued) functions of a pair of real variables lying in an ellipse, termed $\lambda$-metamonogenic Mathieu functions. Returning to the context $\mathbb{R}^2 \rightarrow \mathbb{H}$, we consider here the case of functions defined in the unit disk employing Bessel functions of the first kind in place of Mathieu functions. We produce a set of metamonogenic functions (that is, in $\text{Ker}(D + \lambda)$ for diverse $\lambda$), which is orthogonal in the unit disk for orders $\geq 2$ and in a certain sense complete in $\text{Ker}(D + \lambda) \cap L^2$ for every $\lambda$. As an application, in the final section we use these functions to express time-dependent solutions of the imaginary-time wave equation in the disk.

2 Preliminaries

2.1 Metamonogenic functions

We consider the quaternionic operator $D$ defined by (1). This is interpreted as follows, in fairly standard notation and terminology, in which $z = x + iy$ is a complex number, and a quaternion $a = a_0 + a_1i + a_2j + a_3k$. Here $a_m \in \mathbb{R}$ and $i, j, k$ are the quaternionic imaginary units satisfying $i^2 = j^2 = k^2 = ijk = -1$. The set of real quaternions $\mathbb{H} = \mathbb{H}(\mathbb{R})$ is naturally identified with $\mathbb{R}^4$, which determines the usual component-wise addition and also induces the absolute value on $\mathbb{H}$. Thus $D$ acts on $\mathbb{H}$-valued functions

$$f(x, y) = f_0(x, y) + f_1(x, y)i + f_2(x, y)j + f_3(x, y)k,$$

defined in domains in the complex plane $\mathbb{C}$ applying the quaternionic multiplication rules, in principle, on the left or right, giving $Df$ or $fD$. We will only consider the operator acting from the left, as the other case is analogous.

Let $\Omega$ be a domain in $\mathbb{R}^2$ (open and connected). Let $L^2(\Omega) = L^2(\Omega, \mathbb{H})$ denote the space of all $\mathbb{H}$-valued functions $f: \Omega \rightarrow \mathbb{H}$ such that the components $f_m$ ($m = 0, 1, 2, 3$) are in the usual $L^2(\Omega, \mathbb{R})$. It is easily seen that $L^2(\Omega)$ is naturally a right $\mathbb{H}$-linear module and admits the $\mathbb{H}$-valued right inner product

$$\langle f, g \rangle_{\mathbb{H}} = \int_{\Omega} f(x, y) g(x, y) dxdy$$

(2)
for \( f, g \in L^2(\Omega) \). Thus \( L^2(\Omega) \) is a quaternionic right Hilbert module with the associated norm \( \|f\|_2 = \langle f, f \rangle_H^{1/2} = \langle f, f \rangle_R^{1/2} \), where \( \langle f, g \rangle_R = \text{Sc} \langle f, g \rangle_H \) coincides with the usual \( L^2 \)-norm for \( f \), viewed as an \( \mathbb{R}^4 \)-valued function in \( \Omega \) \[10, 11\].

For functions taking values in the 2-dimensional subspace \( \mathbb{R}i \oplus \mathbb{R}j \subseteq \mathbb{H} \), \( D \) echoes the classical Cauchy-Riemann operator \( 2\partial/\partial z = \partial/\partial x + i\partial/\partial y \), but it sends such functions to the complementary subspace \( \mathbb{R} \oplus \mathbb{R}k \subseteq \mathbb{H} \).

As usual, \( \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 \) will denote the Laplace operator in \( \mathbb{R}^2 \).

**Definition 2.1.** Let \( \Omega \subseteq \mathbb{R}^2 \). Given \( \lambda \in \mathbb{R} \setminus \{0\} \), a function \( f \in C^2(\Omega, \mathbb{R}) \) is said to be \( \lambda \)-metaharmonic when

\[(\Delta + \lambda^2)f = 0.\]

When \( f \in C^1(\Omega, \mathbb{H}) \) and

\[(D + \lambda)f = 0,\]

one says that \( f \) is \( \textit{left} \ \lambda \)-metamonogenic.

We thus have the spaces of left \( \lambda \)-metamonogenic functions

\[\mathcal{M}(\Omega; \lambda) = \text{Ker}(D + \lambda) \subseteq C^1(\Omega, \mathbb{H})\]

and

\[\mathcal{M}_2(\Omega; \lambda) = \mathcal{M}(\Omega; \lambda) \cap L^2(\Omega).\]

It is well known [23] that metaharmonic functions are of class \( C^\infty \), and so by the following factorization of the Laplacian via \( D \) (cf. [14, Section CVII] and [9]), metamonogenic functions are of class \( C^\infty \) also.

**Proposition 2.2.** \((D + \lambda)(D - \lambda) = -(\Delta + \lambda^2) \) for \( \lambda \in \mathbb{R} \setminus \{0\} \).

In polar coordinates \( x = \rho \cos \theta, \ y = \rho \sin \theta \), one has

\[D_{r,\theta} = \left( \cos \theta \frac{\partial}{\partial \rho} + \frac{\sin \theta}{\rho} \frac{\partial}{\partial \theta} \right)i + \left( \sin \theta \frac{\partial}{\partial \rho} - \frac{\cos \theta}{\rho} \frac{\partial}{\partial \theta} \right)j, \tag{3}\]

and the Helmholtz operator in polar coordinates is

\[\Delta_{\rho,\theta} + \lambda^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \lambda^2. \tag{4}\]
From Proposition 2.2, it is clear that the components of any $f \in \mathcal{M}(\Omega; \lambda)$ are $\lambda$-metaharmonic. The equation $(D + \lambda)f = 0$ is equivalent to the system of partial differential equations

\[
\begin{align*}
\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} - \lambda f_0 &= 0, \\
-\frac{\partial f_0}{\partial x} - \frac{\partial f_3}{\partial y} - \lambda f_1 &= 0, \\
\frac{\partial f_3}{\partial x} - \frac{\partial f_0}{\partial y} - \lambda f_2 &= 0, \\
-\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y} - \lambda f_3 &= 0.
\end{align*}
\]

(5)

From this the following is immediate.

**Proposition 2.3.** Given any two scalar $\lambda$-metaharmonic functions $f_1, f_2$ in any domain $\Omega \subseteq \mathbb{C}$, there are unique ($\lambda$-metaharmonic) functions $f_0, f_3$ such that $f_0 + f_1i + f_2j + f_3k$ is $\lambda$-metamonogenic.

Indeed, take

\[
f_0 = \frac{1}{\lambda} \left( \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} \right), \quad f_3 = \frac{1}{\lambda} \left( \frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} \right),
\]

and observe that the system (5) is satisfied. Similarly, given $f_1, f_2$ one has unique functions $f_0, f_3$ completing the components of a $\lambda$-metamonogenic function. One may think of the relationship of pairs $(f_1, f_2)$ and $(f_0, f_3)$ as a generalization of the notion of harmonic conjugates.

### 3 Quaternionic metamonogenic functions

This section introduces a family of $\lambda$-metamonogenic functions in the real Hilbert space $L^2$ of the unit disk, which is the object of study of this paper.

The factorization of Proposition 2.2 suggests that quaternionic $\lambda$-metamonogenic functions should play a role for the Laplace operator $\Delta$, similar to the usual metaharmonic functions in two variables for the corresponding Helmholtz operator [22].


3.1 A class of $\lambda$-metamonogenic functions

First we define a continuous family of quaternionic metamonogenic functions. Let $J_n(z), \ z \in \mathbb{C}$ denote the $n$-th Bessel function of the first kind, $n = 0, 1, 2, \ldots$ \[15\]. We recall that $J_0(0) = 1, J_n(0) = 0 \ (n \neq 0)$, $J_1'(0) = 1, J_n'(0) = 0 \ (n \neq 1).$

Definition 3.1. Let $n \geq 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$. For $z = x + iy = \rho e^{i\theta} \in \mathbb{C}$, the $n$-th standard $\lambda$-metamonogenic function is

$$F_n[\lambda](z) = \left(J'_n(\lambda\rho) + \frac{n}{\lambda\rho} J_n(\lambda\rho)\right) \cos(n-1)\theta + (J_n(\lambda\rho) \cos n\theta) \ i + (J_n(\lambda\rho) \sin n\theta) \ j - ((J'_n(\lambda\rho) + \frac{n}{\lambda\rho} J_n(\lambda\rho))) \sin(n - 1)\theta \ k \quad \text{if} \ z \neq 0$$

and for $z = 0$ the limiting value,

$$F_0[\lambda](0) = i, \quad F_1[\lambda](0) = 1, \quad F_n[\lambda](0) = 0 \ (n \geq 2).$$

In particular, $F_0[\lambda](z) = -J_1(\lambda\rho) \cos \theta + J_0(\lambda\rho) \ i - J_1(\lambda\rho) \sin \theta \ k$ because of the second of the recurrence relations

$$\frac{2n}{\zeta} J_n(z) = J_{n-1}(z) + J_{n+1}(z),$$
$$2J'_n(z) = J_{n-1}(z) - J_{n+1}(z),$$

with $J_{-1}(z) = -J_1(z)$.

Note that the $i$ and $j$ components of $F_n[\lambda](z)$ are the classical solutions $J_n(\lambda\rho) \cos n\theta, J_n(\lambda\rho) \sin n\theta$ for the Helmholtz equation in polar coordinates, which are indeed complete in the space of all solutions in $L^2(\Omega_0, \mathbb{R})$, where $\Omega_0 = \{z \in \mathbb{C}: \ |z| < 1\}$ denotes the unit disk in the complex plane \[23\]. It follows directly from Proposition \[2.2\] that all $F_n[\lambda](z)$ are $\lambda$-metamonogenic.

We also note that $F_n[\lambda]$ may be constructed as

$$F_n[\lambda] = F_n^+[\lambda] i + F_n^-[\lambda] j,$$

5
in terms of the reduced-quaternionic valued functions

\[ F_n^\pm[\lambda](z) = J_n(\lambda \rho)\Phi_n^\pm(z) - (\cos \theta i + \sin \theta j)J_n'(\lambda \rho)\Phi_n^\pm(z) \]

\[ \mp \frac{n}{\lambda \rho} (\sin \theta i - \cos \theta j)J_n(\lambda \rho)\Phi_n^\pm(z), \]

where we write \( \Phi_n^+(z) = \cos n\theta \), \( \Phi_n^-(z) = \sin n\theta \).

### 3.2 Basic metamonogenics

Next we introduce a special subset of the \( \lambda \)-metamonogenic functions defined in the previous section. It is well known [15] that \( J_n \) has a countable collection of simple real zeros \( j_{n,m} \),

\[ 0 < j_{n,1} < j_{n,2} < \cdots. \]

The basic metamonogenic functions are defined by

\[ F_{n,m}(z) = F_n[j_{n,m}](z). \]  

(9)

for \( n \geq 0 \), \( m \geq 1 \). Thus

\[ (D + j_{n,m})F_{n,m} = 0. \]  

(10)

Some examples of \( F_{n,m} \) in \( \Omega_0 \) are given in Figure [1]. Our main result is as follows.

**Theorem 3.2.** (i) Let \((n_1, m_1) \neq (n_2, m_2)\). If \( \{n_1, n_2\} \neq \{0, 1\} \), then

\[ \langle F_{n_1,m_1}, F_{n_2,m_2} \rangle_{\mathbb{H}} = 0, \]

(11)

while

\[ \langle F_{0,m_1}, F_{1,m_2} \rangle_{\mathbb{H}} = -2\pi \frac{J_1(j_{0,m_1})J_0(j_{1,m_2})}{j_{0,m_1} - j_{1,m_2}} i, \]

(12)

\[ \langle F_{1,m_1}, F_{0,m_2} \rangle_{\mathbb{H}} = -2\pi \frac{J_0(j_{1,m_1})J_1(j_{0,m_2})}{j_{1,m_1} - j_{0,m_2}} i. \]

(13)

(ii) The norms of these metamonogenic functions are given by

\[ \|F_{n,m}\|_2^2 = 2\pi J_{n-1}^2(j_{n,m}) = 2\pi J_{n+1}^2(j_{n,m}). \]

(14)
Figure 1: The functions $F_{n,m}$ for assorted values of $(n, m)$. The scalar parts are shown in the left column. The symmetries due to the presence of the functions $\Phi^\pm_n$ are clearly visible.

(iii) Let $f \in \mathcal{M}_2(\Omega_0; \lambda)$ where $\lambda \in \mathbb{R}\setminus\{0\}$. Then $f$ is in the closed subspace of the right quaternionic Hilbert module $L^2(\Omega_0)$ spanned by $\{F_{n,m}: n \geq 0, m \geq 1\}$; that is, there are $c_{n,m} \in \mathbb{H}$ such that

$$f(z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} F_{n,m}(z) c_{n,m}.$$

Proof. Since $F_{n,m}$ is continuous in fact in the whole plane, it is clearly in $L^2(\Omega_0)$. The proof divides naturally into parts.
(i) Orthogonality. We must show that
\[ \int_{\Omega_0} \overline{F_{n_1,m_1}(z)} F_{n_2,m_2}(z) \, dx \, dy = 0 \]
whenever \((n_1, m_1) \neq (n_2, m_2)\) and \(\{n_1, n_2\} \neq \{0, 1\}\). We break down the integrand into quaternionic components as follows,
\[
\overline{F_{n_1,m_1}(z)} F_{n_2,m_2}(z) \\
= (A_1 - B_1 i - C_1 j - D_1 k)(A_2 + B_2 i + C_2 j + D_2 k) \\
= A_1 A_1 + B_1 B_2 + C_1 C_2 + D_1 D_2 + (A_1 B_2 - B_1 A_2 - C_1 D_2 + D_1 C_2)i \\
+ (A_1 C_2 - C_1 A_2 + B_1 D_2 - D_1 B_2)j + (-B_1 C_2 + C_1 B_2 + A_1 D_2 - D_1 A_2)k.
\]

For convenience, let us write \(j_1 = j_{n_1,m_1}, \ j_2 = j_{n_2,m_2}\). With this notation, one finds after a great deal of cancellation that
\[
A_1 A_2 = \left( J'_{n_1}(j_1 \rho) J'_{n_2}(j_2 \rho) + \frac{n_2}{j_2 \rho} J'_{n_1}(j_1 \rho) J_{n_2}(j_2 \rho) + \frac{n_1}{j_1 \rho} J_{n_1}(j_1 \rho) J'_{n_2}(j_2 \rho) \\
+ \frac{n_1 n_2}{j_1 j_2 \rho^2} J_{n_1}(j_1 \rho) J_{n_2}(j_2 \rho) \right) \left( (\cos^2 \theta) \Phi^+_{n_1} \Phi^+_{n_2} + (\cos \theta \sin \theta) \Phi^+_{n_1} \Phi^-_{n_2} \right. \\
\left. + (\sin \theta \cos \theta) \Phi^-_{n_1} \Phi^+_{n_2} + (\sin^2 \theta) \Phi^-_{n_1} \Phi^-_{n_2} \right),
\]
\[
B_1 B_2 = J_{n_1}(j_1 \rho) J_{n_2}(j_2 \rho) \Phi^+_{n_1} \Phi^+_{n_2},
\]
\[
C_1 C_2 = J_{n_1}(j_1 \rho) J_{n_2}(j_2 \rho) \Phi^-_{n_1} \Phi^-_{n_2},
\]
\[
D_1 D_2 = \left( J'_{n_1}(j_1 \rho) J'_{n_2}(j_2 \rho) + \frac{n_2}{j_2 \rho} J'_{n_1}(j_1 \rho) J_{n_2}(j_2 \rho) + \frac{n_1}{j_1 \rho} J_{n_1}(j_1 \rho) J'_{n_2}(j_2 \rho) \\
+ \frac{n_1 n_2}{j_1 j_2 \rho^2} J_{n_1}(j_1 \rho) J_{n_2}(j_2 \rho) \right) \left( (\sin^2 \theta) \Phi^+_{n_1} \Phi^+_{n_2} - (\sin \theta \cos \theta) \Phi^+_{n_1} \Phi^-_{n_2} \right. \\
\left. - (\cos \theta \sin \theta) \Phi^-_{n_1} \Phi^+_{n_2} + (\cos^2 \theta) \Phi^-_{n_1} \Phi^-_{n_2} \right).
\]

Now it is best to group the parts as follows: using \(\Phi^+_{n_1} \Phi^+_{n_2} - \Phi^-_{n_1} \Phi^-_{n_2} = \Phi^+_{n_1-n_2}\), first
\[
A_1 A_2 + D_1 D_2 = \left( J'_{n_1}(j_1 \rho) J'_{n_2}(j_1 \rho) + \frac{n_2}{j_2 \rho} J'_{n_1}(j_1 \rho) J_{n_2}(j_2 \rho) \\
+ \frac{n_1}{j_1 \rho} J_{n_1}(j_1 \rho) J'_{n_2}(j_2 \rho) + \frac{n_1 n_2}{j_1 j_2 \rho^2} J_{n_1}(j_1 \rho) J_{n_2}(j_2 \rho) \right) \Phi^+_{n_1-n_2},
\]
\[
B_1 B_2 + C_1 C_2 = J_{n_1}(j_1 \rho) J_{n_2}(j_2 \rho) \Phi^+_{n_1-n_2},
\]
and then we can integrate,

\[
\langle F_{n_1, m_1}, F_{n_2, m_2} \rangle_R = \int_0^{2\pi} \int_0^1 (A_1 A_1 + B_1 B_2 + C_1 C_2 + D_1 D_2) \rho \, d\rho \, d\theta \\
= \int_0^1 (\rho J'_{n_1}(j_{1}\rho)J'_{n_2}(j_{2}\rho) + \rho J_{n_1}(j_{1}\rho)J_{n_2}(j_{2}\rho) \\
+ \frac{n_2}{j_2} J'_{n_1}(j_{1}\rho)J_{n_2}(j_{2}\rho) + \frac{n_1}{j_1} J_{n_1}(j_{1}\rho)J'_{n_2}(j_{2}\rho) \\
+ \frac{n_1 n_2}{j_1 j_2} \rho^{-1} J_{n_1}(j_{1}\rho)J_{n_2}(j_{2}\rho)) \, d\rho \int_0^{2\pi} \Phi_{n_1 - n_2}^+ \, d\theta. \tag{15}
\]

For \( n_1 \neq n_2 \) the \( \theta \)-integral is zero, and so is the scalar product. Now suppose \( n_1 = n_2 = n \), and consider the \( \rho \)-integral. We will use the following orthogonality property [15] for Bessel functions scaled by distinct zeros,

\[
\int_0^1 J_n(j_{n,m_1}\rho)J_n(j_{n,m_2}\rho) \, \rho \, d\rho = 0 \tag{16}
\]

when \( m_1 \neq m_2 \), as well as (7).

Within the \( \rho \)-integral at the end of (15) we find

\[
\int_0^1 (r J'_{n}(j_{1}\rho)J'_{n}(j_{2}\rho) + \frac{n^2}{j_1 j_2} \rho^{-1} J_{n}(j_{1}\rho)J_{n}(j_{2}\rho)) \, d\rho,
\]

which by (7) is equal to

\[
= \frac{1}{4} \int_0^1 ((J_{n-1}(j_{1}\rho) - J_{n+1}(j_{1}\rho))(J_{n-1}(j_{2}\rho) - J_{n+1}(j_{2}\rho)) \\
+ (J_{n-1}(j_{1}\rho) + J_{n+1}(j_{1}\rho))(J_{n-1}(j_{2}\rho) + J_{n+1}(j_{2}\rho)) \, \rho \, d\rho \\
= \frac{1}{2} \int_0^1 (J_{n-1}(j_{1}\rho)J_{n-1}(j_{2}\rho) + J_{n+1}(j_{1}\rho)J_{n+1}(j_{2}\rho) \, \rho \, d\rho.
\]

Also in (15) we have

\[
\int_0^1 \left( \frac{n}{j_2} J'_{n}(j_{1}\rho)J_{n}(j_{2}\rho) + \frac{n}{j_1} J_{n}(j_{1}\rho)J'_{n}(j_{2}\rho) \right) \, d\rho \\
= \frac{1}{2} \int_0^1 ((J_{n-1}(j_{1}\rho)J_{n-1}(j_{1}\rho) - J_{n+1}(j_{1}\rho)J_{n+1}(j_{1}\rho)) \, \rho \, d\rho,
\]

9
again with the help of (7). Combining these we find that

$$
\langle F_{n,m_1}, F_{n,m_2} \rangle = 2\pi \int_0^1 (J_n(j_1 \rho)J_n(j_2 \rho) + J_{n-1}(j_1 \rho)J_{n-1}(j_2 \rho)) \, \rho \, d\rho,
$$

which is zero by (16) as we are assuming $m_1 \neq m_2$.

Similarly, straightforward computations yield that the vector part of the quaternionic inner product is

$$
\text{Vec} \langle F_{n_1,m_1}, F_{n_2,m_2} \rangle = \left( \int_0^1 (J_{n_1-1}(j_1 \rho)J_{n_2}(j_2 \rho) - J_{n_1}(j_1 \rho)J_{n_2-1}(j_2 \rho)) \, \rho \, d\rho \int_0^{2\pi} \Phi_{n_1+n_2-1}^+ \, d\theta \right) i \\
+ \left( \int_0^1 (J_{n_1-1}(j_1 \rho)J_{n_2}(j_2 \rho) + J_{n_1-1}(j_1 \rho)J_{n_2-1}(j_2 \rho)) \, \rho \, d\rho \int_0^{2\pi} \Phi_{n_1+n_2-1}^- \, d\theta \right) j \\
+ \left( \int_0^1 (J_{n_1-1}(j_1 \rho)J_{n_2}(j_2 \rho) - J_{n_1}(j_1 \rho)J_{n_2-1}(j_2 \rho)) \, \rho \, d\rho \int_0^{2\pi} \Phi_{n_1-n_2}^- \, d\theta \right) k.
$$

Analogous computations involving the remaining three components show that $\text{Vec} \langle F_{n_1,m_1}, F_{n_2,m_2} \rangle = 0$ whenever $(n_1,m_1) \neq (n_2,m_2)$ and $\{n_1,n_2\} \neq \{0,1\}$, which establishes (11), while (12)–(13) follow directly from (17)–(18).

(ii) Norms. In [15, p. 303] we have for any $\lambda$ that

$$
2 \int_{\rho_1}^{\rho_2} J_n(\lambda \rho)^2 \, \rho \, d\rho = \rho^2(J_n(\lambda \rho)^2 - J_{n-1}(\lambda \rho)J_{n+1}(\lambda \rho)) |_{\rho_1}^{\rho_2}.
$$

Thus with $\lambda = j_{n,m}$, (17)–(18) specialize to

$$
\|F_{n,m}\|^2 = 2\pi \int_0^1 (\rho J_n(j_{n,m} \rho)^2 + \rho J_{n-1}(j_{n,m} \rho)^2) \, d\rho.
$$

which gives (14).

(iii) Completeness. Now fix $\lambda$ and suppose that $f = f_0 + f_1 i + f_2 j + f_2 k \in \mathcal{M}_2(\Omega_0; \lambda)$ is orthogonal to every $F_{n,m}$ in the sense of (2). Since every well defined function in $\Omega_0$ is periodic in $\theta$ in polar coordinates, it follows from
Definition 3.1 that

\[ 0 = \langle f, F_{n,m} \rangle_{\mathbb{R}} \]

\[ = \frac{1}{\lambda} \int_0^{2\pi} \int_0^1 \left( \frac{\partial f_1}{\partial \rho} \rho J_n'(j_{n,m} \rho) \Phi_n^+ + \frac{n}{j_{n,m}} \frac{\partial f_1}{\partial \rho} J_n(j_{n,m} \rho) \Phi_n^+ - \frac{\partial f_1}{\partial \theta} J_n'(j_{n,m} \rho) \Phi_n^- \right. \]
\[ - \frac{n}{j_{n,m}} \rho^{-1} \frac{\partial f_1}{\partial \theta} J_n(j_{n,m} \rho) \Phi_n^- + \frac{\partial f_2}{\partial \rho} \rho J_n'(j_{n,m} \rho) \Phi_n^- \]
\[ + \frac{n}{j_{n,m}} \frac{\partial f_2}{\partial \rho} J_n(j_{n,m} \rho) \Phi_n^- + \frac{\partial f_2}{\partial \theta} J_n'(j_{n,m} \rho) \Phi_n^+ \]
\[ + \frac{n}{j_{n,m}} \rho^{-1} \frac{\partial f_2}{\partial \theta} J_n(j_{n,m} \rho) \Phi_n^+ \right) \, d\rho \, d\theta \]
\[ + \int_0^{2\pi} \int_0^1 f_1 J_n(j_{n,m} \rho) \Phi_n^+ \rho \, d\rho \, d\theta + \int_0^{2\pi} \int_0^1 f_2 J_n(j_{n,m} \rho) \Phi_n^- \rho \, d\rho \, d\theta. \quad (19) \]

We apply integration by parts to the second and third terms of the first integral:

\[ \int_0^{2\pi} \int_0^1 \frac{n}{j_{n,m}} \frac{\partial f_1}{\partial \rho} J_n(j_{n,m} \rho) \Phi_n^+ \, d\rho \, d\theta - \int_0^{2\pi} \int_0^1 \frac{\partial f_1}{\partial \theta} J_n'(j_{n,m} \rho) \Phi_n^- \, d\rho \, d\theta \]
\[ = \frac{1}{j_{n,m}} \int_0^1 J_n(j_{n,m} \rho) \left( \int_0^{2\pi} \frac{\partial f_1}{\partial \rho} (\Phi_n^-)' \, d\theta \right) \, d\rho - \int_0^{2\pi} \Phi_n^- \left( \int_0^1 \frac{\partial f_1}{\partial \rho} J_n'(j_{n,m} \rho) \, d\rho \right) \, d\theta \]
\[ = -\frac{1}{j_{n,m}} \int_0^1 J_n(j_{n,m} \rho) \frac{\partial^2 f_1}{\partial \rho^2} \Phi_n^- \, d\rho \, d\theta + \frac{1}{j_{n,m}} \int_0^1 J_n(j_{n,m} \rho) \frac{\partial^2 f_1}{\partial \rho \partial \theta} \Phi_n^- \, d\rho \, d\theta \]
\[ = 0 \]

when \( n > 0 \) by (6), and for the sixth and seventh terms,

\[ \int_0^{2\pi} \int_0^1 \frac{n}{j_{n,m}} \frac{\partial f_2}{\partial \rho} J_n(j_{n,m} \rho) \Phi_n^- \, d\rho \, d\theta + \int_0^{2\pi} \int_0^1 \frac{\partial f_2}{\partial \theta} J_n'(j_{n,m} \rho) \Phi_n^+ \, d\rho \, d\theta = 0 \]
also when \( n > 0 \). Integrating the remaining integrals by parts, we have

\[
0 = \langle f, F_{n,m} \rangle_{\mathbb{R}} = \frac{1}{j_{n,m}^2} \int_0^{2\pi} \Phi_n^+ \left( \int_0^1 (\Delta_{\rho,\theta} f_1) J_n(j_{n,m}\rho) \rho \, d\rho \right) d\theta \\
- \frac{1}{j_{n,m}^2} \int_0^{2\pi} \Phi_n^- \left( \int_0^1 (\Delta_{r,\theta} f_2) J_n(j_{n,m}\rho) \rho \, d\rho \right) d\theta \\
+ \int_0^{2\pi} \int_0^1 f_1 J_n(j_{n,m}\rho) \Phi_n^+ \rho \, d\rho \, d\theta \\
+ \int_0^{2\pi} \int_0^1 f_2 J_n(j_{n,m}\rho) \Phi_n^- \rho \, d\rho \, d\theta
\]

when \( n > 0 \).

Since \( f_1, f_2 \) are \( \lambda \)-metaharmonic,

\[
(\lambda^2 + j_{n,m}^2) \left( \int_0^{2\pi} \int_0^1 f_1 J_n(j_{n,m}\rho) \Phi_n^+ \rho \, d\rho \, d\theta + \int_0^{2\pi} \int_0^1 f_2 J_n(j_{n,m}\rho) \Phi_n^- \rho \, d\rho \, d\theta \right) = 0
\]

when \( n > 0 \). Similar arguments using the \( i, j, k \) components enable one to show that in fact

\[
\int_0^{2\pi} \int_0^1 f_1 J_n(j_{n,m}\rho) \Phi_n^- \rho \, d\rho \, d\theta = 0,
\]

\[
\int_0^{2\pi} \int_0^1 f_2 J_n(j_{n,m}\rho) \Phi_n^+ \rho \, d\rho \, d\theta = 0
\]

for \( n > 0 \). By the completeness of the set \( \{ J_n(j_{n,m}\rho) \Phi_n^\pm \} \) in \( L^2(\Omega_0, \mathbb{R}) \) it follows that \( f_1 \) and \( f_2 \) are in the linear span of \( \{ J_0(j_{0,m}\rho) \Phi_0^\pm \} \). Since \( \Phi_0^+ = 1 \), \( \Phi_0^- = 0 \), we have the series representations

\[
f_1 = \sum_{m=1}^{\infty} c_{1,m} J_0(j_{0,m}\rho), \quad f_2 = \sum_{m=1}^{\infty} c_{2,m} J_0(j_{0,m}\rho), \quad (20)
\]
converging in $L^2$ for real constants $c_{1,m}, c_{2,m}$. By Proposition 2.3

$$f_0 = \frac{1}{\lambda} \left( \frac{\partial}{\partial x} \sum_{m=1}^{\infty} c_{1,m} J_0(j_0, m \rho) + \frac{\partial}{\partial y} \sum_{m=1}^{\infty} c_{2,m} J_0(j_0, m \rho) \right)$$

$$= \sum_{m=1}^{\infty} \frac{j_0, m}{\lambda} J_0'(j_0, m \rho)(c_{1,m} \cos \theta + c_{2,m} \sin \theta),$$

$$f_3 = \frac{1}{\lambda} \left( \frac{\partial}{\partial y} \sum_{m=1}^{\infty} c_{1,m} J_0(j_0, m \rho) - \frac{\partial}{\partial x} \sum_{m=1}^{\infty} c_{2,m} J_0(j_0, m \rho) \right)$$

$$= \sum_{m=1}^{\infty} \frac{j_0, m}{\lambda} J_0'(j_0, m \rho)(-c_{2,m} \cos \theta + c_{1,m} \sin \theta).$$

Let $m' \geq 1$. Using these series representations, first we look at the scalar part of the hypothesis

$$0 = \langle F_{0,m'}, f \rangle_H$$

$$= \langle -J_1(j_{0,m'} \rho) \cos \theta + J_0(j_{0,m'} \rho) \mathbf{i} - J_1(j_{0,m'} \rho) \sin \theta \mathbf{k},$$

$$\sum_{m=1}^{\infty} \frac{j_0, m}{\lambda} J_0'(j_0, m \rho)(c_{1,m} \cos \theta + c_{2,m} \sin \theta) + f_1 \mathbf{i} + f_2 \mathbf{j}$$

$$+ \left( \sum_{m=1}^{\infty} \frac{j_0, m}{\lambda} J_0'(j_0, m \rho)(-c_{2,m} \cos \theta + c_{1,m} \sin \theta) \right) \mathbf{k} \rangle_H = 0.$$  

By $L^2$ convergence,

$$0 = -\sum_{m} c_{1,m} \frac{j_0, m}{\lambda} \int_{0}^{1} \rho J_1(j_0, m' \rho) J_0'(j_0, m \rho) d\rho \int_{0}^{2\pi} \cos^2 \theta d\theta$$

$$-\sum_{m} c_{2,m} \frac{j_0, m}{\lambda} \int_{0}^{1} \rho J_1(j_0, m' \rho) J_0'(j_0, m \rho) d\rho \int_{0}^{2\pi} \cos \theta \sin \theta d\theta$$

$$-\int_{0}^{2\pi} \int_{0}^{1} J_0(j_0, m' \rho) f_1 \rho d\rho d\theta$$

$$-\sum_{m} c_{2,m} \frac{j_0, m}{\lambda} \int_{0}^{1} \rho J_1(j_0, m' \rho) J_0'(j_0, m \rho) d\rho \int_{0}^{2\pi} \sin \theta \cos \theta d\theta$$

$$+\sum_{m} c_{1,m} \frac{j_0, m}{\lambda} \int_{0}^{1} \rho J_1(j_0, m' \rho) J_0'(j_0, m \rho) d\rho \int_{0}^{2\pi} \sin^2 \theta d\theta.$$
Thus $f_1$ is orthogonal to $J_{0,m'}$ and hence is orthogonal in fact to all $J_{n,m}\Phi^\pm_n$, which implies $f_1 = 0$. When one expands the $k$ component of the inner product it is seen similarly that $f_2 = 0$. In consequence, $f = 0$ identically as required.

The information (12)–(13) permits one to orthogonalize (say via the Gram-Schmidt process) the subspace generated by \{\(F_{0,m}, F_{1,m}: m \geq 1\}\}, which by (11) will combine with the remaining $F_{n,m}$ to give a full orthogonal basis. The resulting functions are not particularly interesting, so we will omit the details.

4 Time-dependent solutions

Consider the partial differential equation

\[(\Delta + K^2 \frac{\partial^2}{\partial t^2})v = 0\]  

for $v(z,t) \in \mathbb{H}, z \in \Omega_0, t \geq 0$. This can be interpreted as a wave equation using imaginary time $it$. (cf. the Wick transformation \[\text{[3]}\]).

We consider the natural quaternionic extensions of the real-valued solutions of (21). Since

\[(D + K \frac{\partial}{\partial t})(D - K \frac{\partial}{\partial t}) = -(\Delta^2 + K^2),\]  

we are led to consider the companion equation

\[(D + K \frac{\partial}{\partial t})v = 0.\]  

Since the operator $\Delta + K^2(\partial^2/\partial t^2)$ has only real ingredients, it operates independently on each component of $v = v_0 + v_1i + v_2j + v_3k$.

Because of (10), a time-dependent function given by a series of the form

\[v(z,t) = \sum_{n=0}^\infty \sum_{m=1}^\infty F_{n,m}(z) c_{n,m} e^{jn,m t}\]  

converging in $L^2(\Omega_0)$ clearly satisfies (23). One may propose a boundary value problem for this equation with an initial condition given by an arbitrary $v(z) \in \mathcal{M}_2(\Omega_0; \lambda)$, whose coefficients $c_{n,m} \in \mathbb{H}$ are given according to
Theorem 3.2

\[ v(z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} F_{n,m}(z)c_{n,m}. \]  

(25)

(In fact, one may prescribe only \( v_0 \) and \( v_3 \) in \( \text{Ker}(\Delta + \lambda^2) \) according to Proposition 2.3). It is not difficult to show by means of the Cauchy-Kovalevskaya theorem [21] that this is the only real-analytic solution \( z \) satisfying the initial conditions \( v(z,0) = g(z) \) and

\[ \frac{\partial}{\partial t} \bigg|_{t=0} v_0(z,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} j_{n,m} F_{n,m}(z)c_{n,m}. \]  

(26)

Figure 2: The initial condition \( (t = 0, \text{top row}) \) contains high order terms which are not visible in the graphics until approximately \( t > 0.3 \), when the exponential terms in time in the become sufficiently large.
(A similar result for reduced-quaternion-valued functions in elliptical domains is worked out in detail in [20]).

An example of the evolution of a wave function [25] is given in Figure 2.

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