Differential operators for Siegel-Jacobi forms

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Abstract For any positive integers $n$ and $m$, $H_{n,m} := H_n \times \mathbb{C}^{(m,n)}$ is called the Siegel-Jacobi space, with the Jacobi group acting on it. The Jacobi forms are defined on this space. We compute the Chern connection of the Siegel-Jacobi space and use it to obtain derivations of Jacobi forms. Using these results, we construct a series of invariant differential operators for Siegel-Jacobi forms. Also two kinds of Maass-Shimura type differential operators for $H_{n,m}$ are obtained.

Keywords connection, Jacobi form, differential operator

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1 Introduction

We first recall the concept of Jacobi forms and their differential operators. For given fixed positive integers $n, m$, let $H_n := \{ Z \in M_{n,n}(\mathbb{C}) \mid Z = Z^t, \text{Im}(Z) > 0 \}$ be the Siegel upper half plane of degree $n$, and $H_{n,m} := H_n \times \mathbb{C}^{(m,n)}$, the Siegel-Jacobi space. An element of $H_{n,m}$ can be written as $(Z, W)$ with $Z = Z^t = (z_{ij}) \in M_{n,n}$, $W = (w_{rs}) \in M_{m,n}$. Let $Y$ and $V$ be the imaginary part of the matrix $Z$ and $W$, respectively.

The Heisenberg group $H^{(n,m)}_{\mathbb{R}}$ is defined to be the set

$$H^{(n,m)}_{\mathbb{R}} := \left\{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu \lambda^t \text{ symmetric} \right\}$$

endowed with the multiplication law:

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda^t \mu' - \mu^t \lambda').$$

The Jacobi group of degree $n$, index $m$ is defined to be $G^J := \text{Sp}(n, \mathbb{R}) \ltimes H^{(n,m)}_{\mathbb{R}}$, endowed with the following multiplication law:

$$(\gamma, (\lambda, \mu; \kappa)) \cdot (\gamma', (\lambda', \mu'; \kappa')) = (\gamma \gamma', (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda \mu'^t - \mu \lambda'^t)),\)$$

where $\gamma, \gamma' \in \text{Sp}(n, \mathbb{R}); (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H^{(n,m)}_{\mathbb{R}}$ and $(\lambda, \mu) = (\lambda, \mu) \Gamma'$. The Jacobi group $G^J$ acts canonically on $H_{n,m}$ by

$$(\gamma, (\lambda, \mu; \kappa)) \cdot (Z, W) = (\gamma \cdot Z, (W + \lambda Z + \mu)(CZ + D)^{-1}),$$

(1.1)

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where $\gamma \cdot Z$ means the usual action of an element $\gamma = (A B \ C D) \in \text{Sp}(n, \mathbb{R})$ on the Siegel upper half plane by
\[
\gamma \cdot Z = (AZ + B)(CZ + D)^{-1}.
\]
See for example [13] for more details.

To define Jacobi forms, we need some discrete subgroups of $G^J$. Denote the discrete subgroup $\text{Sp}(n, \mathbb{Z}) \ltimes \mathbb{Z}^2$ of $G^J$ by $\Gamma^J$. We can also consider more general discrete subgroups, but now we concentrate on the $\text{Sp}(n, \mathbb{Z}) \ltimes \mathbb{Z}^2$ case.

We give the following formal definition of Jacobi forms of general degree.

**Definition 1.1.** Let $M$ be a positive definite half integer $m \times m$ matrix, a (holomorphic) Jacobi form $f$ of weight $k$ and index $M$, is a (holomorphic) function on $\mathbb{H}_{n,m}$, which satisfies the translation law of
\[
f(g(Z, W)) = \det(CZ + D)^k \cdot e^{2\pi \sqrt{-1} Tr((MW(CZ + D)^{-1}CW^{-1}M(AZ + B)(CZ + D)^{-1}) \cdot f(Z, W), \quad (1.2)}
\]
for $g = ((A \ B), (\lambda, \mu, \kappa)) \in \Gamma^J$.

This is the definition of Ziegler [18]. In this definition of Jacobi forms, we do not ask for any growth conditions, such as assuming they are holomorphic or eigenfunctions of some differential operators. Since our purpose is concentrated on differential operators, although these conditions are needed in practise, now we do not assume this. The set of Jacobi forms of weight $k$ and index $M$ is denoted by $J_{k,M}$, and the subset of holomorphic Jacobi forms are denoted by $J_{k,M}^{hol}$.

Jacobi forms of degree 1 are introduced and studied by Eicher and Zagier [10]. They are related to both modular forms and degree 2 Siegel modular forms and are useful in many aspects of number theory such as lifting problems. The non-holomorphic differential operators on Jacobi forms are well studied in [2]. There are several important differential operators for Jacobi forms of degree 1, including two raising operators and two lowering operators, which are
\[
Y_+ f = \sqrt{-1} \left( \frac{\partial f}{\partial w} + \frac{v}{y} 4\pi \sqrt{-1} M f \right),
\]
\[
Y_- f = y \frac{\partial f}{\partial w},
\]
\[
X_+ f = \sqrt{-1} \left( 2 \frac{\partial f}{\partial z} + 2 \frac{v}{y} \frac{\partial f}{\partial w} + 4\pi \sqrt{-1} M \left( \frac{v^2}{y^2} f + \frac{k}{y} f \right) \right),
\]
\[
X_- f = 2 \sqrt{-1} y \left( \frac{\partial f}{\partial z} + v \frac{\partial f}{\partial w} \right).
\]

Here $(z, w)$ is the local coordinate of $\mathbb{H}_{1,1}$, and $y$ and $v$ are their imaginary part respectively. Also there is a non-holomorphic heat operator which is related to the operator $\frac{\partial}{\partial z} - \sqrt{-1} \frac{v}{y} f$ in the correspondence of Jacobi forms and half integer modular forms. The heat operator is defined as (denoted as $D_+$ in [2])
\[
Lf := 8\pi M \sqrt{-1} \frac{\partial f}{\partial z} - \frac{\partial^2 f}{\partial w^2} - \frac{2M\pi - 4M\pi k}{y} f.
\]

These operators can help us understand Jacobi forms better and we generalize them to higher degree cases in Section 3. Although they are not linear, we will see later that this will help us construct invariant differential operators and Maass-Shimura operators.

We will consider their generalizations to general degree. Yang and Yin [15] investigated the derivative operators of Siegel modular forms. Similar to this work, we get the following differential operators.

**Theorem 1.2 (See Theorem 3.4).** For any $f \in J_{k,M}$, we have
(a) If $n = m$, $R_1(f) = \det(\frac{\partial f}{\partial t} + 4\pi \sqrt{-1} (Y^{-1}V^t M) f) \in J_{nk+1,nM}$.
(b) If $n = m$, $L_1(f) = \det(\frac{\partial f}{\partial w} Y) \text{ is in } J_{nk-1,nM}$.
(c) $R_2(f) = \det(\frac{\partial f}{\partial z} - \sqrt{-1} \frac{v}{y} fY^{-1} + 2\pi \sqrt{-1} Y^{-1} V^t M V Y^{-1} f + \frac{1}{2} \frac{\partial f}{\partial w} V Y^{-1} + \frac{1}{2} Y^{-1} V^t \frac{\partial f}{\partial w} Y^{-1} f) \in J_{nk+2,nM}$.
(d) $L_2(f) = \det(\frac{\partial f}{\partial z} Y^2 + \frac{1}{2} \frac{\partial f}{\partial w} V Y + \frac{1}{2} Y V^t \frac{\partial f}{\partial w} Y^{-1} f) \in J_{nk-2,nM}$. 
They are generalizations of $Y_+, Y_-, X_+$ and $X_-$, respectively. The proof of this theorem is contained in Subsections 3.1 and 3.2, using the following Theorem 1.3 of the Chern connections on the Siegel-Jacobi space. The authors are grateful to the reviewers for the advice of using Chern connections instead of the Levi-Civita connection in the following theorem.

**Theorem 1.3** (See Theorem 2.2). Let $\mathbb{D}$ be the Chern connection on the Hermitian manifold $\mathbb{H}_{n,m}$ associated with the invariant metric. Then $\mathbb{D}$ satisfies

$$
\mathbb{D}(dZ) = -\frac{\sqrt{-1}}{2A} (dZ, dW^t) \begin{pmatrix}
2\pi Y^{-1} + Y^{-1} V^t V Y^{-1} & -Y^{-1} V^t \\
-VY^{-1} & I
\end{pmatrix} \begin{pmatrix}
dZ \\
dW
\end{pmatrix},
$$

(1.3)

$$
\mathbb{D}(dW) = -\frac{\sqrt{-1}}{2A} VY^{-1} (dZ, dW^t) \begin{pmatrix}
Y^{-1} V^t V Y^{-1} & -Y^{-1} V^t \\
-VY^{-1} & I
\end{pmatrix} \begin{pmatrix}
dZ \\
dW
\end{pmatrix}
-\sqrt{-1} dW Y^{-1} dZ.
$$

(1.4)

We apply this theorem to invariant sections of Jacobi forms in Section 3. For details of this theorem, see Section 2. And the proof of this theorem is contained in Section 4.

Besides these, by considering the translation formula of the operators we obtained, we find a series of invariant differential operators for the Siegel-Jacobi forms of general degree. For Siegel modular space, Maass obtained the invariant differential operators in [13]. Similar to his result, we have the following theorem.

**Theorem 1.4** (See Theorem 3.6). The operator matrix $Y_+ Y_+$ is an invariant differential operator matrix on $\mathbb{H}_{n,m}$, thus each of the $(k,l)$ entries of this matrix is an invariant differential operator on $\mathbb{H}_{n,m}$, and the operators $H^j, T^j_{k,l}, U_{k,l}, V_{k,l}$ are all invariant differential operators.

The notation in this Theorem are explained in Subsection 3.3. We apply Maass’s method to prove this theorem.

Another main result of this paper is the construction of two Maass-Shimura type differential operators. In the Siegel case, Maass [13] and Shimura [14] developed a differential operator which transforms a weight $k$ Siegel modular form to a weight $k + 2$ non-holomorphic Siegel modular form. This operator is very useful in Shimura’s theory of nearly holomorphic Siegel modular forms. As for Siegel-Jacobi form case, we construct two such operators in Subsection 3.4.

**Theorem 1.5** (See Theorems 3.8 and 3.9). The operators

$$
H_{k,M} := \det(Y)^{\kappa - k - 1} \exp\{4\pi \text{Tr}(MVY^{-1}V^t)\} \det(X_+)(\det(Y)^{k+1-\kappa} \exp\{-4\pi \text{Tr}(MVY^{-1}V^t)\}),
$$

$$
L_{k,M} := \det(Y)^{\kappa' - k - 1} \exp\{-4\pi \text{Tr}(MVY^{-1}V^t)\} \det\left(-8\pi \sqrt{-1} \frac{\partial}{\partial Z} + \frac{\partial}{\partial W} M^{-1} \left(\frac{\partial}{\partial W}\right)^t\right)^t
\times (\det(Y)^{k+1-\kappa'} \exp\{-4\pi \text{Tr}(MVY^{-1}V^t)\}),
$$

both map from $J_{k,M}$ to $J_{k,M+2}$. Here $\kappa = \frac{n+1}{2}, \kappa' = \frac{n+m+1}{2}$.

In degree 1 case, the two operators become the operator $X_+$ and the heat operator, respectively. They may be used to research the nearly holomorphic theory for Jacobi forms.

Differential operators play an important role in the theory of automorphic forms. They are closely related to $L$-functions. There are many results on differential operators for various of automorphic forms. For the case of Siegel modular forms, holomorphic differential operators for Siegel modular forms and other automorphic forms are well studied by Ibukiyama [11]. In [5], Böcherer and Nagaoka used mod $p$ differential operators for Siegel modular forms to study mod $p$ properties. See also [4,9] for more work in these areas.

As for Jacobi forms on $\mathbb{H} \times \mathbb{C}^m$, linear differential operators are given by Conley and Raum [8]. Some invariant differential operators on the Siegel-Jacobi space are defined by Yang [17]. Also Choie and Ehlotz [7] studied the Rankin-Cohen brackets for Jacobi forms. In degree 1 case, Böcherer [3] obtained the Rankin-Cohen brackets by Maass-Shimura type operators. We hope that similar results
can be obtained by our Maass-Shimura type operator in general degree case. The first author thanks Professor Ibukiyama very much for pointing out this.

This paper is organized as follows. In Section 2, we recall the concept of connection and the metric in the Siegel-Jacobi space case, which will be used in our proof. The Chern connection $\mathbb{D}$ on $\mathbb{H}_{n,m}$ is given in Theorem 2.2. The explicit proof is quite complicated and will be given in Section 4. In Section 3, we consider the higher degree cases. Instead of the classical Hodge’s method, we will use connections computed above to obtain differentials of Jacobi forms. First of all we get some operators in the determinant form, including both raising and lowering operators, generalizing exactly the classical case. See Theorems 3.8 and 3.9 and their proofs. Section 4 contains the explicit proof of Theorem 1.3.

### 2 Connections of Siegel-Jacobi space

We recall some basic facts of Riemannian Geometry and Kähler geometry from [6,12]. Suppose that $M$ is a smooth manifold, $E$ is a vector bundle on $M$ and $\Gamma(E)$ is the set of global sections. A connection on $E$ is a map $\nabla: \Gamma(E) \to \Gamma(T(M^*) \otimes E)$ satisfying the following two conditions:

1. For any sections $s_1, s_2 \in \Gamma(E)$, $\nabla(s_1 + s_2) = \nabla(s_1) + \nabla(s_2)$.
2. For any section $s \in \Gamma(E)$ and any $\alpha \in C^\infty(M)$, $\nabla(\alpha s) = \alpha \nabla(s) + s \nabla(\alpha)$.

If $M$ is a Riemannian manifold with a Riemannian metric $g = \sum_{i,j} g_{ij} du^i du^j$, then there exists a unique torsion-free and metric-compatible connection, called the Levi-Civita connection whose Christoffel coefficients $\Gamma^k_{ij}$ satisfies

$$\Gamma^k_{ij} = \sum_{s} \frac{1}{2} g^{kl} \left( \frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right),$$

(2.1) where $g^{ij}$ means the $(i,j)$ factor of the inverse matrix of $(g_{ij})$. The connection matrix is defined to be $\omega = (\omega^i_j)$, where $\omega^i_j = \sum_k \Gamma^k_{ij} du^k$.

Now suppose that $M$ is a Hermitian manifold. A Hermitian manifold means a complex manifold with a Hermitian metric on its holomorphic tangent space. Suppose $z_1, \ldots, z_n$ is a complex local coordinate system in $M$. Then the Hermitian metric is given by the form $h = \sum_{\alpha, \beta} h_{\alpha\beta} dz_\alpha d\bar{z}_\beta$, where $h_{\alpha\beta}$ are the components of a positive-definite Hermitian matrix. The fundamental 2-form is given by $\Phi = \frac{i}{2} \sum_{\alpha, \beta} h_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta$. The manifold $M$ is Kähler if and only if the 2-form $\Phi$ is closed.

$M$ is also equipped with a unique torsion free connection, which is called the Chern connection. Its Christoffel coefficients satisfies $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$, $\Gamma^\alpha_{\bar{z}\gamma} = \Gamma^\alpha_{\gamma\bar{z}}$, and others are all 0.

The Christoffel coefficients of Chern connections can be computed as

$$\Gamma^\alpha_{\beta\gamma} = \sum_i h^{\alpha i} \frac{\partial h_{\beta i}}{\partial z_\gamma},$$

(2.2) where $(h^{\alpha i})$ means the inverse matrix of the metric matrix of $h$.

The Hermitian metric naturally defines a Riemannian metric. If the manifold is Kähler, the Chern connection can induce a connection on the corresponding Riemannian manifold, and that is just the Levi-Civita connection.

Actually, the connection is divided into two parts: the holomorphic part $\mathbb{D}^{1,0}$ and the non-holomorphic part $\mathbb{D}^{0,1}$. In general, the holomorphic and non-holomorphic part are conjugate to each other, and for simplicity, we will only consider the holomorphic part, and still write it as $\mathbb{D}$.

The Chern connection of the Siegel-Jacobi space is given in Theorem 2.2, whose proof is given in Section 4.

We first need to know the invariant metric of the Siegel-Jacobi space, in [16], Yang proved the following theorem:
Theorem 2.1. For any two positive real numbers $A$ and $B$, the following metric

$$ds_{n,m;A,B}^2 = A \cdot \text{Tr}(Y^{-1}dZY^{-1}dZ) + B\{\text{Tr}(Y^{-1}V^tVY^{-1}dZY^{-1}dZ) + \text{Tr}(Y^{-1}(dW)^t dW)\}$$

$$- B\{\text{Tr}(VY^{-1}dZY^{-1}(dW)^t) + \text{Tr}(Y^{-1}dZY^{-1}(dW)^t)\}$$

is a Riemannian metric on $H_{n,m}$, which is invariant under the action of the Jacobi group $G^J$.

The symbols $Y, V$ denote the imaginary parts of $Z$ and $W$, respectively, and

$$dZ = \begin{pmatrix}
  dz_{1,1} & dz_{1,2} & \cdots & dz_{1,n} \\
  dz_{2,1} & \ddots & \vdots & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  dz_{n,1} & \cdots & \cdots & dz_{n,n}
\end{pmatrix}, \quad dW = \begin{pmatrix}
  dw_{1,1} & dw_{1,2} & \cdots & dw_{1,n} \\
  dw_{2,1} & \ddots & \vdots & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  dw_{m,1} & \cdots & \cdots & dw_{m,n}
\end{pmatrix}.$$ 

In fact, this metric is Kähler and we can compute its Chern connection. The Chern connection associated with this metric $ds_{n,m;A,B}^2$, denoted by $D$, is invariant under the action of $G^J$, i.e., $gD = Dg$ for any $g \in G^J$. Especially, $D$ maps invariant sections to invariant sections, and we can use this fact to get differential operators for Jacobi forms as the Siegel modular case given in [15].

We first look at the case when $n = m = 1$. In this case, the invariant metric becomes

$$ds_{A,B}^2 = Ay^{-2}dzd\bar{z} + B(y^{-3}dzd\bar{z} + y^{-1}dwd\bar{w} - vy^{-2}dzd\bar{w} - vy^{-2}dw d\bar{z}).$$

So the metric matrix is written as

$$h = \begin{pmatrix}
  \frac{A}{y^2} + \frac{Bv}{y^3} & -\frac{By}{y^3} \\
  -\frac{By}{y^3} & \frac{B}{y}
\end{pmatrix},$$

and it is not hard to get

$$h^{-1} = \begin{pmatrix}
  \frac{y^2}{A} & \frac{vy}{A} \\
  \frac{vy}{A} & y^2 + \frac{A}{y}
\end{pmatrix}.$$ 

By the formula (2.2), we can get the connection matrix $w$ with $w_i^j = \sum_k \Gamma_{ik}^j dw^k$,

$$w = B\sqrt{-1}A^{-1}\left(\frac{\sqrt{1} + \frac{2}{A}y}{2Ay^3}\right)dz - \frac{x}{y} dw - \frac{1}{y}dz + (\frac{2}{y} + \frac{A}{2y}) dw.$$ 

Thus we get

$$D(z) = -\left(\frac{\sqrt{-1}}{y} + \sqrt{-1}Bv^2 \frac{2}{Ay^3}\right)dz^2 + \sqrt{-1}Bv \frac{2}{Ay} dz dw - \sqrt{-1}B \frac{2}{Ay} dw^2$$

$$\text{(c), } (dz, dw) = (z, w) \begin{pmatrix}
  \frac{-\sqrt{-1}}{y} & \frac{\sqrt{-1}Bv^2}{2Ay} \\
  \frac{\sqrt{-1}Bv^2}{2Ay} & -\sqrt{-1}B \frac{2}{Ay}
\end{pmatrix} \begin{pmatrix}
  dz \\
  dw
\end{pmatrix}.$$ 

$$D(w) = \frac{\sqrt{-1}Bv^3}{2Ay^3}dz^2 - \left(\frac{-\sqrt{-1}Bv^2}{2Ay} + \frac{\sqrt{-1}}{y}\right) dz dw - \sqrt{-1}Bv \frac{2}{Ay} dw^2$$

$$\text{(c), } (z, w) = (w, dz) \begin{pmatrix}
  \frac{-\sqrt{-1}Bv^3}{2Ay^3} & \frac{-\sqrt{-1}Bv^2}{2Ay} \\
  \frac{-\sqrt{-1}Bv^2}{2Ay} & -\sqrt{-1}Bv \frac{2}{Ay}
\end{pmatrix} \begin{pmatrix}
  dz \\
  dw
\end{pmatrix}.$$ 

For higher degree cases, we have the following results:
Let $D$ be the Chern connection on the manifold $H_{n,m}$ associated with the invariant metric in Theorem 2.1. Then $D$ satisfies

$$D(dZ) = -\frac{\sqrt{-1}B}{2A}(dZ,dW^t) \left( \begin{array}{cc} 2Y^{-1} + Y^{-1}V'YY^{-1} & -Y^{-1}V' \\ -V'Y^{-1} & I \end{array} \right) \left( \begin{array}{c} dZ \\ dW \end{array} \right),$$

$$D(dW) = -\frac{\sqrt{-1}B}{2A}VY^{-1}(dZ,dW^t) \left( \begin{array}{cc} Y^{-1}V'YY^{-1} & -Y^{-1}V' \\ -V'Y^{-1} & I \end{array} \right) \left( \begin{array}{c} dZ \\ dW \end{array} \right),$$

where $D(dZ)$ means

$$D(dz_{11}) \quad D(dz_{12}) \quad \cdots \quad D(dz_{1n})$$
$$D(dz_{21}) \quad \cdots \quad \cdots$$
$$\vdots \quad \cdots \quad \cdots$$
$$D(dz_{n1}) \quad \cdots \quad \cdots \quad D(dz_{nn})$$

and $D(dW)$ is similar.

The proof of the theorem by direct computation is given in the last section. We will use this theorem to compute differentials of Jacobi forms and get differential operators.

### 3 Differential operators on Siegel-Jacobi space

#### 3.1 Action of connection on invariant Jacobi forms

First, we will consider how the action of the Chern connection we obtained acts on an invariant function on $H_{n,m}$. Let $h \in \mathcal{J}_{0,0}$ be an invariant Jacobi form. Then its image under the connection map is also invariant under the action of $\Gamma^J$, which is

$$D(h) = \text{Tr} \left( \frac{\partial h}{\partial Z} dZ \right) + \text{Tr} \left( \frac{\partial h}{\partial W} dW \right).$$

Here we use the notation

$$\frac{\partial}{\partial W} := \left( \frac{\partial}{\partial w_{1,1}} \quad \frac{\partial}{\partial w_{1,2}} \quad \cdots \quad \frac{\partial}{\partial w_{1,n}} \right), \quad \frac{\partial}{\partial Z} := \left( \frac{\partial}{\partial z_{1,1}} \quad \frac{\partial}{\partial z_{1,2}} \quad \cdots \quad \frac{\partial}{\partial z_{1,n}} \right).$$

Now we consider the translation formula of $dZ$ and $dW$. Recall the action of an element $g = ((\lambda \mu \kappa), (\lambda \mu \kappa)) \in \Gamma^J$, we know that its action of $\mathbb{H}_{m,n}$ is given by the formula (1.1). Then for an element $(Z,W) \in \mathbb{H}_{m,n}$, denote its image under the action of $g$ by $(\check{Z}, \check{W})$. It is checked in [16, Section 2] that

$$d(\check{Z}) = ((CZ + D)^{-1})^t dZ(CZ + D)^{-1},$$
$$d(\check{W}) = dW(CZ + D)^{-1} + \{\lambda - (W + \lambda Z + \mu)(CZ + D)^{-1}C\}dZ(CZ + D)^{-1}. \quad (3.2)$$

Since the $dW$ term does not come from $d(\check{Z})$, considering the $dW$ term under the action of $\Gamma^J$, we can see the translation relation as follows:

$$\frac{\partial h((\check{Z}, \check{W}))}{\partial \check{W}} dW = (CZ + D) \frac{\partial h(Z, W)}{\partial W} dW. \quad (3.3)$$
If $m = n$, we can take determinant of the both sides, and then we get
\[
\det \left( \frac{\partial h}{\partial W} \right) = \det(CZ + D) \det \left( \frac{\partial h}{\partial W} \right).
\]
This means $\det(\frac{\partial h}{\partial W}) \in J_{1,0}$.

If $m > n$, we cannot take the determinant directly. But we can choose $n$ different rows of $\frac{\partial h}{\partial W}$, and take determinant of the new square matrixes so we can get $C^m_n$ operators in this form.

Next, we consider the twice differential $D^2(h)$, which is
\[
D^2(h) = \text{Tr} \left( \frac{\partial h}{\partial Z} D(dZ) + D \left( \frac{\partial h}{\partial Z} dZ \right) \right) + \text{Tr} \left( \frac{\partial h}{\partial W} D(dW) + D \left( \frac{\partial h}{\partial W} dW \right) \right).
\]
(3.4)

Here $D(\frac{\partial h}{\partial W})$ means the connection acts on every entry of the matrix.

This is invariant under the action of $\Gamma^J$. By the discussion for the translation formula of $dZ$ and $dW$ above, we can easily see that the terms consisting of $dw_i dw_j$ are also invariant. Applying the expression for $D(dZ)$ and $D(dW)$ in Theorem 2.2, we see that
\[
D^2(h) = - \text{Tr} \left( \frac{\sqrt{-1} B \partial h}{2A} \frac{\partial h}{\partial Z} (dZ, dW^t) \left( \begin{array}{cc} 2 \frac{\sqrt{2}}{B} Y^{-1} + Y^{-1}V^tVY^{-1} & -Y^{-1}V^t \\ -VY^{-1} & I \end{array} \right) \left( \frac{\partial h}{\partial Z} dZ \right) \right) + D \left( \frac{\partial h}{\partial Z} dZ \right)
\]
\[ - \text{Tr} \left( \frac{\sqrt{-1} B \partial h}{2A} \frac{\partial h}{\partial W} VY^{-1} (dZ, dW^t) \left( \begin{array}{cc} Y^{-1}V^tVY^{-1} & -Y^{-1}V^t \\ -VY^{-1} & I \end{array} \right) \left( \frac{\partial h}{\partial W} dZ \right) + \sqrt{-1} dwY^{-1} dZ \right)
\]
\[ - \text{Tr} \left( D \left( \frac{\partial h}{\partial W} \right) dW \right).
\]

Although the expression is complicated, we can see the $dw_i dw_j$ terms only show up in three parts in the above formula, which are
\[
\text{Tr} \left( \frac{\sqrt{-1} B \partial h}{2A} \frac{\partial h}{\partial Z} dW^t dW \right) + \text{Tr} \left( \frac{\sqrt{-1} B \partial h}{2A} \frac{\partial h}{\partial W} VY^{-1} dW^t dW \right) + \text{Tr} \left( D \left( \frac{\partial h}{\partial W} \right) dW \right).
\]

By the definition of Tr, we can express the above section as
\[
\text{Tr} \left( \frac{\sqrt{-1} B \partial h}{2A} \frac{\partial h}{\partial Z} dW^t dW \right) + \text{Tr} \left( \frac{\sqrt{-1} B \partial h}{2A} \frac{\partial h}{\partial W} VY^{-1} dW^t dW \right) + \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} d \left( \frac{\partial h}{\partial w_{ij}} \right) dw_{ij}. \quad (3.5)
\]

The last term can be revised again. Denote the $i$-row vector of $dW$ by $dW_i$. We have $d(W_i) = dW_i(CZ + D)^{-1} + T$, where $T$ is the $dW$ term. Thus by calculation the last term $\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} d \left( \frac{\partial h}{\partial w_{ij}} \right) dw_{ij}$ above can be rewritten as $\text{Tr} \left( \sum_{i \leq j \leq m} \frac{\partial}{\partial w_{ij}} \left( \frac{\partial h}{\partial w_{ij}} \right)^t dW_j dW_i \right)$, with
\[
\frac{\partial}{\partial W_i} \left( \frac{\partial h}{\partial W_j} \right)^t := \begin{pmatrix}
\frac{\partial^2 h}{\partial w_{ij} \partial w_{i1}} & \frac{\partial^2 h}{\partial w_{ij} \partial w_{i2}} & \ldots & \frac{\partial^2 h}{\partial w_{ij} \partial w_{in}} \\
\frac{\partial^2 h}{\partial w_{ij} \partial w_{21}} & \frac{\partial^2 h}{\partial w_{ij} \partial w_{22}} & \ldots & \frac{\partial^2 h}{\partial w_{ij} \partial w_{2n}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 h}{\partial w_{ij} \partial w_{n1}} & \frac{\partial^2 h}{\partial w_{ij} \partial w_{n2}} & \ldots & \frac{\partial^2 h}{\partial w_{ij} \partial w_{nn}}
\end{pmatrix}
\]
for any function $h$.

So now the formula (3.5) becomes
\[
\text{Tr} \left( \frac{\sqrt{-1} B \partial h}{2A} \frac{\partial h}{\partial Z} dW^t dW \right) + \text{Tr} \left( \frac{\sqrt{-1} B \partial h}{2A} \frac{\partial h}{\partial W} VY^{-1} dW^t dW \right) + \text{Tr} \left( \sum_{i \leq j} \frac{\partial}{\partial W_i} \left( \frac{\partial h}{\partial W_j} \right)^t dW_j^t dW_i \right). \quad (3.6)
\]

Moreover, from translation formula (3.3), and the translation of $dW$, we can easily deduce that the last term is invariant itself already, ignoring the extra $dZ$ part. Thus we obtain the section, whose $dW$ part is invariant
\[
\text{Tr} \left( \frac{\partial h}{\partial Z} dW^t dW \right) + \text{Tr} \left( \frac{\partial h}{\partial W} VY^{-1} dW^t dW \right). \quad (3.7)
\]

We will get derivatives of Jacobi forms from this section.
3.2 Action of connections on general Jacobi forms

Let $f \in J_{k,M}^{hol}$ be a holomorphic Jacobi form. It is known that $h := fh_1f$ is an invariant form on $\mathbb{H}_{n,m}$ (see [18, p. 202]), where

$$h_1 = \det(Y)^k \exp\{-4\pi \cdot \text{Tr}(MVY^{-1}V^t)\}.$$  

We will compute the explicit expression of $\det(\frac{\partial h}{\partial W})$ and of formula (3.7) for this $h$. First we will compute $\det(\frac{\partial Y}{\partial W})$. We have the following lemma.

**Lemma 3.1.** \(\frac{\partial (\text{Tr}(MVY^{-1}V^t))}{\partial W} = -\sqrt{-1}Y^{-1}V^tM.\)

**Proof.** Denote $Y^{-1}$ by $R$. As

$$\text{Tr}(MVY^{-1}V^t) = \sum_{a, b, r, s} M_{ab} V_{br} R_{rs} V_{as},$$

so

$$\frac{\partial \{\text{Tr}(MVY^{-1}V^t)\}}{\partial W_{ij}} = \frac{\partial \{\sum_{a, b, r, s} M_{ab} V_{br} R_{rs} V_{as}\}}{\partial W_{ij}}$$

$$= -\frac{\sqrt{-1}}{2} \sum_{b, r} M_{ib} V_{br} R_{rj} - \frac{\sqrt{-1}}{2} \sum_{a, s} M_{ai} R_{js} V_{as}$$

$$= -\sqrt{-1} \sum_{b, r} M_{ib} V_{br} R_{rj}.$$  

Since $M, Y$ are symmetric, we have $\frac{\partial (\text{Tr}(MVY^{-1}V^t))}{\partial W} = -\sqrt{-1}Y^{-1}V^tM.$ \(\square\)

Furthermore, as $h_1 = \det(Y)^k \exp\{-4\pi \text{Tr}(MVY^{-1}V^t)\}$, we have $\frac{\partial h_1}{\partial W} = 4\sqrt{-1}\pi Y^{-1}V^tMh_1$, and thus

$$\det \left( \frac{\partial h}{\partial W} \right) = \det \left( \frac{\partial f}{\partial W} h_1 f + \frac{\partial h_1}{\partial W} f h_1 f + \frac{\partial h_1 f}{\partial W} f \right)$$

$$= \det \left( \frac{\partial f}{\partial W} h_1 f + 4\sqrt{-1}(Y^{-1}V^tM)h_1 f f \right) \in J_{1,0}. $$

As $h$ is invariant, we see that $\det(\frac{\partial f}{\partial W} f + 4\sqrt{-1}Y^{-1}V^tM) f$ is in $J_{1,0}$. So

$$R_1(f) := \det \left( \frac{\partial f}{\partial W} f + 4\sqrt{-1}(Y^{-1}V^tM) f \right) \in J_{nk+1,nM}. \quad (3.8)$$

The method above works for holomorphic Jacobi forms only, but the results also hold for non-holomorphic Jacobi forms. Now we consider a not necessarily holomorphic Jacobi form $f$. As we discussed above $\frac{\partial (fh_1f)}{\partial W}$ and $\frac{\partial f}{\partial W} h_1 f + 4\sqrt{-1}(Y^{-1}V^tM)h_1 f f$ both satisfy the formula (3.3), and so is their difference $\frac{\partial f}{\partial W} h_1 f$. Then we know that $\det(\frac{\partial f}{\partial W} h_1 f) \in J_{1,0}$. Since $\det(Y)^{-2} \det(dZ) \det(d\bar{Z})$ is invariant under the Jacobi group (see [13]), dividing it in the conjugation of the above Jacobi form, we see that

$$\left( \det \left( \frac{\partial f}{\partial W} h_1 f \right) \right)^2 \det(Y^2) \det(dZ)^{-1}$$

is invariant under the action of $\Gamma^J$. Thus $L_1 := \det(\frac{\partial f}{\partial W} Y)$ is a Jacobi form in $J_{nk-1,nM}$.

The two operators $R_1$ and $L_1$ are generalization of $Y_+$ and $Y_-$ in the introduction, respectively. Next, we consider the explicit expression of formula (3.7). Now this becomes

$$\text{Tr} \left( \frac{\partial f}{\partial Z} h_1 f dW' dW \right) + \text{Tr} \left( \frac{\partial h_1 f f dW' dW}{\partial Z} \right) + \text{Tr} \left( \frac{\partial f}{\partial W} h_1 f VY^{-1} dW' dW \right)$$

$$+ \text{Tr} \left( \frac{\partial h_1 f f VY^{-1} dW' dW}{\partial W} \right). \quad (3.9)$$
This equals
\[ \text{Tr} \left( \left( \frac{\partial f}{\partial Z} - \frac{\sqrt{-1} k}{2} f Y^{-1} - 2\pi \sqrt{-1} Y^{-1} V^t M V Y^{-1} f \right) + 4\pi \sqrt{-1} Y^{-1} V^t M V Y^{-1} f + \frac{\partial f}{\partial W} V Y^{-1} \right) h_1 f dW dW' \right) \]
\[ = \text{Tr} \left( \left( \frac{\partial f}{\partial Z} - \frac{\sqrt{-1} k}{2} f Y^{-1} + 2\pi \sqrt{-1} Y^{-1} V^t M V Y^{-1} f + \frac{1}{2} \frac{\partial f}{\partial W} V Y^{-1} \right) h_1 f dW dW' \right), \]
by the following lemma.

**Lemma 3.2.** We have
\( (a) \frac{\partial R_{st}}{\partial Z_{kl}} = 2^{1/2} \delta(1-k) \sqrt{-1} (R_{kt} R_{sl} + R_{ks} R_{tl}), \) where \( R = Y^{-1}, \) and \( \delta(i, j) \) is the Dirac symbol.
\( (b) \frac{\partial (\text{Tr}(M V Y^{-1} V^t))}{\partial Z} = \sqrt{-1} Y^{-1} V^t M V Y^{-1}. \)
\( (c) \frac{\partial (\det(Y))}{\partial Z} = -\sqrt{-1} \det(Y) Y^{-1}. \)

**Proof.** For (a), \( R \) is uniquely determined by
\[ \sum_s Y_{is} R_{sl} = \delta(i, t), \]
where \( \delta \) is the Dirac symbol with
\[ \delta(i, t) = \begin{cases} 0, & \text{if } i \neq t, \\ 1, & \text{if } i = t. \end{cases} \]
Taking derivations on both sides, we get
\[ \sum_s \frac{\partial Y_{is}}{\partial Z_{kl}} R_{sl} + \sum_s Y_{is} \frac{\partial R_{st}}{\partial Z_{kl}} = 0, \]
i.e.,
\[ \begin{cases} \sum_s Y_{is} \frac{\partial R_{st}}{\partial Z_{kl}} = 0, & \text{if } i \neq k, l, \\
\frac{1}{\sqrt{-1}} R_{kt} + \sum_s Y_{ks} \frac{\partial R_{st}}{\partial Z_{kl}} = 0, & \text{if } i = k, \\
\frac{1}{\sqrt{-1}} R_{kt} + \sum_s Y_{ts} \frac{\partial R_{st}}{\partial Z_{kl}} = 0, & \text{if } i = l. \end{cases} \]

One can check that these equations determine \( \frac{\partial R_{st}}{\partial Z_{kl}} \) uniquely, and it is easily checked that if we take \( \frac{\partial R_{st}}{\partial Z_{kl}} = 2^{-1/2} \delta(1-k) \sqrt{-1} (R_{kt} R_{sl} + R_{ks} R_{tl}), \) then the equations above are all satisfied, which finishes the proof of (a).

For (b), by (a), we have
\[ \frac{\partial \text{Tr}(M V Y^{-1} V^t)}{\partial Z} = 2^{-1/2} \delta(1-k) \frac{\partial \left( \sum_{a,b,s,t} M_{ab} V_{bs} R_{st} V_{at} \right)}{\partial Z_{kl}} \]
\[ = \frac{1}{\sqrt{-1}} \sum_{a,b,s,t} M_{ab} V_{bs} (R_{kt} R_{sl} + R_{ks} R_{tl}) V_{at} \]
\[ = \frac{1}{\sqrt{-1}} \sum_{a,b,s,t} R_{kt} V_{at} M_{ab} V_{bs} R_{sl} \]
and the statement follows.

As for (c), it is easily checked that \( \frac{\partial (\det(Y))}{\partial Z_{ij}} = -\sqrt{-1} \delta_{ij} (Y_{i,j} + Y_{i,j}^*), \) with \( Y_{i,j}^* \) the cofactor of \( Y_{i,j} \).

So \( \frac{\partial (\det(Y))}{\partial Z_{ij}} = -\sqrt{-1} \det(Y) Y^{-1} \) follows from the definition of \( \frac{\partial}{\partial Z} \). \( \square \)
For any symmetric matrix function $A$, if we have $\text{Tr}(AdZ)$ invariant under the action of $\Gamma^J$, then we have
\[
\text{Tr}(\tilde{A}d(\tilde{Z})) = \text{Tr}(\tilde{A}((CZ + D)^{-1})^t dZ(CZ + D)^{-1}) = \text{Tr}((CZ + D)^{-1}\tilde{A}((CZ + D)^{-1})^t dZ) = \text{Tr}(AdZ).
\]
This means that
\[
\text{Tr}((\tilde{A} - (CZ + D)A(CZ + D)^t)dZ) = 0.
\]
Thus $(\tilde{A} - (CZ + D)A(CZ + D)^t)$ has to be 0 and $A$ satisfies the translation law of
\[
\tilde{A} = (CZ + D)A(CZ + D)^t
\]
with $\tilde{A} = g(A), g = ((A B), (\lambda, \mu, \kappa)) \in \Gamma^J$.

We also know that $(d\tilde{W}^tdW) = ((CZ + D)^{-1})^t(dW^tdW)(CZ + D)^{-1} + T$, where $T$ is some differential forms in coordinates of $dZ$. So the above argument works for $dW^tdW$ as well. More precisely, if we have $\text{Tr}(AdW^tdW)$ is invariant ignoring its $dZ$ part in the translation formula, then by the same argument as above, $A$ also have to satisfy the translation formula
\[
\tilde{A} = (CZ + D)A(CZ + D)^t.
\]

Applying this to (3.9), we see that
\[
\left(\frac{\partial f}{\partial Z} - \frac{\sqrt{-1}k}{2}fY^{-1} + 2\pi\sqrt{-1}Y^{-1}V^tMVY^{-1}f + \frac{1}{2}\frac{\partial f}{\partial W}VY^{-1} + \frac{1}{2}Y^{-1}V^t\frac{\partial f}{\partial W}^t\right)
\]
satisfies this translation formula.

Then taking determinant of the above matrix, we obtain that
\[
R_2(f) := \det\left(\frac{\partial f}{\partial Z} - \frac{\sqrt{-1}k}{2}fY^{-1} + 2\pi\sqrt{-1}Y^{-1}V^tMVY^{-1}f + \frac{1}{2}\frac{\partial f}{\partial W}VY^{-1} + \frac{1}{2}Y^{-1}V^t\frac{\partial f}{\partial W}^t\right)
\]
is an element in $J_{nk+2,nM}$.

**Remark 3.3.** Here we use the symmetric matrix of $\frac{\partial h}{\partial Z} + \frac{\partial h}{\partial W}VY^{-1}$. But actually one can show that $\frac{\partial h}{\partial Z} + \frac{\partial h}{\partial W}VY^{-1}$ works already. We choose the symmetric matrix because it seems more natural.

As we pointed out before, what we get are also right for non-holomorphic Jacobi forms. From this we see that $\text{Tr}((\frac{\partial f}{\partial Z}dW^tdW)fh_1 + \text{Tr}((\frac{\partial f}{\partial W})V^{-1}dW^tdW)fh_1$ is invariant ignoring the $dZ$ part. So the determinant
\[
\det\left(\frac{\partial f}{\partial Z}fh_1 + \frac{1}{2}\frac{\partial f}{\partial W}VY^{-1}fh_1 + \frac{1}{2}Y^{-1}V^t\frac{\partial f}{\partial W}^tfh_1\right) \in J_{2,0}.
\]
Taking conjugation and dividing $\det(Y)^{-2}\det(dZ)\det(d\tilde{Z})$ as we did before, we get
\[
L_2(f) := \det\left(\frac{\partial f}{\partial Z}Y^2 + \frac{1}{2}\frac{\partial f}{\partial W}VY + \frac{1}{2}Y^t\frac{\partial f}{\partial W}^t\right) \in J_{nk+2,nM}.
\]

Summarizing the results above, we have the following theorem.

**Theorem 3.4.**
(a) If $n = m$, for any $f \in J_{k,M}$, $R_n(f) := \det(\frac{\partial f}{\partial W} + 4\pi\sqrt{-1}(Y^{-1}V^tM)f) \in J_{nk+1,nM}$.
(b) If $n = m$, with the same $f$, $L_n(f) = \det(\frac{\partial f}{\partial W})$ is in $J_{nk-1,nM}$.
(c) $R_2(f) = \det(\frac{\partial f}{\partial Z} - \frac{\sqrt{-1}k}{2}fY^{-1} + 2\pi\sqrt{-1}Y^{-1}V^tMVY^{-1}f + \frac{1}{2}\frac{\partial f}{\partial W}VY^{-1} + \frac{1}{2}Y^{-1}V^t\frac{\partial f}{\partial W}^t) \in J_{nk+2,nM}$.
(d) $L_2(f) = \det(\frac{\partial f}{\partial Z}Y^2 + \frac{1}{2}\frac{\partial f}{\partial W}VY + \frac{1}{2}Y^t\frac{\partial f}{\partial W}^t) \in J_{nk+2,nM}$.

If $n$ and $m$ equal 1, then these four operators $R_1, L_1, R_2, L_2$ in this theorem are just the operators $Y_+, Y_-, X_+, X_-$ that we introduced in Section 1. Though they are not linear, we can use these to construct invariant differential operators and Maass-Shimura operators.
3.3 Invariant differential operators

In this subsection, we will study the invariant differential operators for the Siegel-Jacobi space, i.e., differential operators invariant under the action of the Jacobi group. For the space $\mathbb{H}_{1,m}$, things are clear, they just come from the composition of raising and lowering operators, as is shown in [8, Proposition 2.8]. While for higher degree cases, not much is known. In the Siegel modular space case, Maass has constructed invariant differential operators successfully. We can apply his method to the Siegel-Jacobi space case. Together with what we discussed above for the higher degree Jacobi forms, we are able to get invariant differential operators generalizing Maass’s results.

Set
\[
Y_+ = \frac{\partial}{\partial W}, \quad Y_{+,k} = \frac{\partial}{\partial W_k}, \quad Y_- = \frac{\partial}{\partial W} Y, \quad Y_{-,k} = \frac{\partial}{\partial W_k} Y,
\]
\[
X_+ = 2\sqrt{-1} \frac{\partial}{\partial Z} + \sqrt{-1} V^{-1} \frac{\partial}{\partial W} \left( \sqrt{-1}\partial Y^{-1} \frac{\partial}{\partial W} \right)^t,
\]
\[
X_- = Y(Y^{-1})^t,
\]
\[
K = 2\sqrt{-1} \frac{\partial}{\partial Z} + \sqrt{-1} V^t \frac{\partial}{\partial W} \left( Y^{-1} \frac{\partial}{\partial W} Y \right)^t = YX_+,
\]
\[
\Lambda = \sqrt{-1} \frac{\partial}{\partial Z} + \sqrt{-1} V^t \frac{\partial}{\partial W} \left( Y^{-1} \frac{\partial}{\partial W} Y \right)^t.
\]

Then from Subsections 3.1 and 3.2, we have the following proposition.

Proposition 3.5.
\[
\tilde{Y}_+ = (CZ + D)Y_+, \quad \tilde{Y}_- = Y_+(CZ + D)^{-1},
\]
\[
\tilde{Y}_{+,k} = (CZ + D)Y_{+,k}, \quad \tilde{Y}_{-,k} = Y_{-,k}(CZ + D)^{-1},
\]
\[
\tilde{X}_+ = (CZ + D)((CZ + D)X_+)^t, \quad \tilde{X}_- = ((CZ + D)^{-1})^t((CZ + D)^{-1}X_-)^t,
\]
\[
\tilde{K} = ((CZ + D)^{-1})^t((CZ + D)K^t)^t, \quad \tilde{\Lambda} = ((CZ + D)^{-1})^t((CZ + D)\Lambda^t)^t.
\]

Here $\tilde{}$ means after the action of
\[
g = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right), (\lambda, \mu, \kappa) \in \Gamma^J.
\]

Proof. The statement for $Y_+$ is obvious from (3.3), so
\[
\frac{\partial}{\partial W} = (C\bar{Z} + D)\frac{\partial}{\partial W}.
\]

We have
\[
\tilde{Y} = ((C\bar{Z} + D)^{-1})^t Y(C\bar{Z} + D)^{-1} = ((CZ + D)^{-1})^t Y(C\bar{Z} + D)^{-1}.
\]

Thus $\tilde{Y}_- = Y_-(CZ + D)^{-1}$. The statements for $Y_{+,k}$ and $Y_{-,k}$ can also be checked easily. In (3.7), we know that for any $h$ which is invariant under the action of $\Gamma^J$, $\text{Tr}(X_+(h)dZ)$ is also invariant. This means that
\[
\tilde{X}_+ = (CZ + D)((CZ + D)X_+)^t,
\]
thus
\[
\tilde{K} = ((C\bar{Z} + D)^{-1})^t((CZ + D)K^t)^t, \quad \tilde{\Lambda} = ((CZ + D)^{-1})^t((CZ + D)\Lambda^t)^t, \quad \tilde{X}_- = ((CZ + D)^{-1})^t((CZ + D)^{-1}X_-)^t.
\]

$\Box$
The operator matrix \( \Lambda K \) is not invariant, but we have the relation
\[
((CZ + D)\Lambda)^t = \Lambda((CZ + D)^t - \frac{n+1}{2}(Z - \overline{Z}))C^t,
\]
so we get
\[
\Lambda \overline{K} = ((CZ + D)^{-1})^t \left( \Lambda(CZ + D)^t - \frac{n+1}{2}(CZ + D)^t \right.
\]
\[
+ \left. \frac{n+1}{2}(CZ + D)^t \right)((CZ + D)^{-1})^t((CZ + D)\Lambda)^t.
\]
which can be written as
\[
\Lambda \overline{K} + \frac{n+1}{2} \overline{K} = ((CZ + D)^{-1})^t((CZ + D)(\Lambda K + \frac{n+1}{2} K))^t.
\]
Thus
\[
\text{Tr}\left( \Lambda \overline{K} + \frac{n+1}{2} \overline{K} \right) = \text{Tr}\left( \Lambda K + \frac{n+1}{2} K \right).
\]
Following Maass [13], we set
\[
A^{(1)} = \left( \Lambda K + \frac{n+1}{2} K \right),
\]
\[
A^{(j)} = A^{(1)}A^{(j-1)} - \frac{n+1}{2} \Lambda A^{(j-1)} + \frac{1}{2} \Lambda \text{Tr}(A^{(j-1)}) + \frac{1}{2}(Z - \overline{Z})((Z - \overline{Z})^{-1}(A^t A^{(j-1)})^t).
\]
By the same argument as in [13, pp. 111–116], one can show that
\[
\Lambda \overline{A}^{(j)} = ((CZ + D)^{-1})^t((CZ + D)A^{(j)}^t)^t,
\]
and
\[
H^j := \text{Tr}(A^{(j)})
\]
is invariant under the action of the Jacobi group.
Moreover, if we set
\[
T_{k,l}^j = \text{Tr}(Y_{-k}^t Y_{+}^t A^{(j)}), \quad U_{k,l} = \text{Tr}(Y_{-k}^t Y_{-l}^t X_{+}), \quad V_{k,l} = \text{Tr}(Y_{+k}^t Y_{+l}^t X_{-})
\]
then they are all invariant differential operators.

**Theorem 3.6.** The operator matrix \( Y_{-} Y_{+} \) is an invariant differential operator matrix on \( \mathbb{H}_{n,m} \). Thus each of the \((k,l)\) entries of this matrix is an invariant differential operator on \( \mathbb{H}_{n,m} \), and the operators \( H^j, T_{k,l}^j, U_{k,l}, V_{k,l} \) are all invariant differential operators.

**Remark 3.7.** The invariant operator \( Y_{-} Y_{+} \) has already been known in [17, Proposition 4.2]. We hope that all the invariant differential operators come as the combination of the operators in Theorem 3.6 as in the case of \( \mathbb{H}_{1,m} \).

### 3.4 Maass-Shimura type differential operators for Jacobi forms

The Maass-Shimura differential operators are operators defined for Siegel modular forms. Shimura [14] used this operator to study the properties of nearly holomorphic Siegel modular forms and obtained the algebraicity of values of Siegel modular forms. Let \( \mathbb{H}_n \) be the usual Siegel upper half plane with coordinate \( Z = (z_{ij}) \). The imaginary part of \( Z \) is denoted by \( Y \) as before. Let \( f \) be a Siegel modular form of weight \( k \) on \( \mathbb{H}_n \), the Maass-Shimura differential operator acts on \( f \) as
\[
\det(Y)^{\kappa-k-1} \left( \frac{\partial}{\partial Z} \right) (\det(Y)^{k+1-\kappa} f),
\]
where $\kappa$ equals $\frac{n+1}{2}$. The Maass-Shimura operator maps a weight $k$ modular form to a modular form of weight $k + 2$.

Now we can first define a similar differential operator for Jacobi forms by using the results above. Let

$$H_{k,M} := \det(Y)^{\kappa-k-1} \exp\{4\pi \text{Tr}(M_{Y^{-1}}V^t)\} \det(X_+)(\det(Y)^{k+1-\kappa} \exp\{-4\pi \text{Tr}(M_{Y^{-1}}V^t)\}).$$

Then we have the following theorem.

**Theorem 3.8.** $H_{k,M}$ is a differential operator mapping from $J_{k,M}$ to $J_{k+2,M}$.

**Proof.** We show that the theorem is true by comparing it with the Maass-Shimura operator.

First, consider the case $k = 0, M = 0$. Now the operator becomes

$$H_{0,0} = \det(Y)^{\kappa-1} \det(X_+)(\det(Y)^{1-\kappa}).$$

Recall that $X_+ = 2\sqrt{-1} \frac{\partial}{\partial \bar{w}} + \sqrt{-1} Y^{-1}V^t \frac{\partial}{\partial w} + (\sqrt{-1} Y^{-1}V^t \frac{\partial}{\partial w})^t$. Under the action of $G'$, we have

$$\bar{X}_+ = (CZ + D)((CZ + D)X_+)^t.$$

For Siegel modular forms, the classical Maass-Shimura operator is

$$\det(Y)^{\kappa-1} \det\left(\frac{\partial}{\partial Z}\right)(\det(Y)^{1-\kappa}),$$

and $\frac{\partial}{\partial Z}$ satisfies the translation law

$$\frac{\partial}{\partial Z} = (CZ + D)\left((CZ + D)\frac{\partial}{\partial Z}\right)^t,$$

which is formally the same as $X_+$. Since $X_+$ acts the same as $\frac{\partial}{\partial Z}$ on $CZ + D$, so $\det(X_+)$ and $\det(\frac{\partial}{\partial Z})$ also satisfy the same translation law. Moreover,

$$\bar{Y} = ((CZ + D)^{-1})^tY(CZ + D)^{-1},$$

so its composition with $\frac{\partial}{\partial W}$ does not change the translation formula. So $H_{0,0}$ and the Maass-Shimura operator satisfies the same translation law formally. This means that $H_{0,0}$ maps from $J_0$ to $J_{2,0}$, which is the basic case of our theorem.

For general weight and index case, recall that we have the invariant form

$$h = f \det(Y)^k \exp\{-4\pi \text{Tr}(M_{Y^{-1}}V^t)\} \bar{f}.$$

So applying $H_{0,0}$ to $h$, we get that

$$\det(Y)^{\kappa-1} \det(X_+)(\det(Y)^{1-\kappa} f \det(Y)^k \exp\{-4\pi \text{Tr}(M_{Y^{-1}}V^t)\} \bar{f})$$

is an element of $J_{2,0}$. Then multiplying $\frac{\partial}{\partial W}$, this becomes

$$H_{k,M} f = \det(Y)^{\kappa-k-1} \exp\{4\pi \text{Tr}(M_{Y^{-1}}V^t)\} \det(X_+)(\det(Y)^{k+1-\kappa} \exp\{-4\pi \text{Tr}(M_{Y^{-1}}V^t)\} f),$$

which is now a Jacobi form of weight $k + 2$ and index $M$. Thus we have proved the theorem.

This operator can be viewed as a generalization of the Maass-Shimura operator if we restrict a Jacobi form from $\mathbb{H}_{n,m}$ to $\mathbb{H}_n$. Also this is a generalization of the operator $X_+$ in the introduction.

By viewing the group $G'$ as a subgroup of $Sp(g,\mathbb{Z})$, we can define another analogue of the Maass-Shimura operator. In the degree 1 case, this is just the heat operator. It is defined as follows:

$$L_{k,M} := \det(Y)^{\kappa'-k-1} \exp\{-4\pi \text{Tr}(M_{Y^{-1}}V^t)\} \det\left(-8\pi \sqrt{-1} \frac{\partial}{\partial Z} + \frac{\partial}{\partial W} M^{-1}\left(\frac{\partial}{\partial W}\right)^t\right) \times (\det(Y)^{k+1-\kappa'} \exp\{-4\pi \text{Tr}(M_{Y^{-1}}V^t)\}).$$

where $\kappa' = \frac{n+m+1}{2}$.
Theorem 3.9. \( L_{k,M} \) is a differential operator mapping from \( J_{k,M} \) to \( J_{k+2,M} \).

Proof. Following from [13, p.317], Maass showed that for Siegel upper half plane of degree \( g \), we have the following transformation formula, by the action of an \( S = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in \text{Sp}(g, \mathbb{Z}) \),

\[
\begin{vmatrix}
\frac{\partial}{\partial Z} \\
\frac{\partial}{\partial W} \\
\end{vmatrix}
= |CZ + D|^{\frac{g+3}{2}} \begin{vmatrix}
\frac{\partial}{\partial Z} \\
\frac{\partial}{\partial W} \\
\end{vmatrix} |CZ + D|^{\frac{1-g}{2}},
\]

(3.11)

where \( \bar{Z} \) is the image of \( Z \) by the action of \( S \).

Now embed \( \mathbb{H}_{n,m} \) into \( \mathbb{H}_{n+m} \) by sending \((Z, W)\) to \((Z, W, W)^{1/2} \). Also, embed \( G^J = \text{Sp}(n, \mathbb{R}) \rtimes H_{\mathbb{R}}^{(n,m)} \) into \( \text{Sp}(n + m, \mathbb{R}) \) in the classical way by sending \((M, (\lambda, \mu; \kappa))\) to

\[
\begin{pmatrix}
A & 0 & B \\
\lambda & I & \mu \\
C & 0 & D \\
0 & 0 & I \\
\end{pmatrix},
\]

where \( M = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \).

We know the action of \( G^J \) on \( \mathbb{H}_{n,m} \) coincides with the action of \( \text{Sp}(n + m, \mathbb{R}) \) on \( \mathbb{H}_{n+m} \). Now fix a symmetric matrix \( M \). Applying (3.11), we see that

\[
\begin{vmatrix}
\frac{\partial}{\partial Z} \\
\frac{\partial}{\partial W} \\
\end{vmatrix}
= |CZ + D|^{\frac{g+3}{2}} \begin{vmatrix}
\frac{\partial}{\partial Z} \\
\frac{\partial}{\partial W} \\
\end{vmatrix} |CZ + D|^{\frac{1-g}{2}}.
\]

Combining this with \( |\bar{Y}| = |Y||CZ + D|^{-1}|C\bar{Z} + D|^{-1} \), we see that the operator

\[
l_{k,M} := |Y|^{\kappa'-1} \begin{vmatrix}
\frac{\partial}{\partial Z} \\
\frac{\partial}{\partial W} \\
\end{vmatrix} |Y|^{1-\kappa'}
\]

maps a \( \Gamma^J \)-invariant form to a weight 2 Jacobi form. Thus

\[
l_{k,M}(f \det(Y)^k \exp\{-4\pi \text{Tr}(MVY^{-1}V^t)\})
\]

is a weight 2, index 0 Jacobi form and by multiplying the invariant function, we see that so is the form

\[
l_{k,M}(f \det(Y)^k \exp\{-4\pi \text{Tr}(MVY^{-1}V^t)\})/(f \det(Y)^k \exp\{-4\pi \text{Tr}(MVY^{-1}V^t)\})
\]

we then deduce Theorem 3.9 by the following lemma.

Lemma 3.10. \( \begin{vmatrix}
\frac{\partial}{\partial Z} \\
\frac{\partial}{\partial W} \\
\end{vmatrix} \begin{vmatrix}
M \\
M \end{vmatrix} \) can be expressed as \( \frac{\partial}{\partial Z} + \frac{\partial}{\partial W} M^{-1} \frac{\partial}{\partial W} + \frac{\partial}{\partial W} M^{-1} \frac{\partial}{\partial W} \cdot |M| \).

Proof. Multiplying \((M, M^{-1} M, M^{-1} M^{-1})\) to \((\begin{smallmatrix} M & M \\ M & M \end{smallmatrix}) \). We get the result we need easily.

4 Computation of the connections

In this section, we give the proof of Theorem 2.2 by direct computation. The notation and ideas are similar to [15]. Set \( \Omega = \{(i, j) \mid 1 \leq i \leq j \leq n\}; \Omega' = \{(i', j') \mid 1 \leq i' \leq m, 1 \leq j' \leq n\} \), and fix the notation \( I = (i, j), J = (r, s), K = (p, q), L = (a, b) \in \Omega \); \( I' = (i', j'), J = (r', s'), K' = (p', q'), L' = (a', b') \in \Omega' \). In the following, we define \( Z_I := Z_{ij}, W_I := W_{ij} \).
Let \( R := (R_{ij})_{n \times n} = Y^{-1} \). Then the metric on the Siegel-Jacobi space in Theorem 2.1 is given by

\[
d^2 s_{n,m;A,B} = A \text{Tr}(RdZd\overline{Z}) + B(\text{Tr}(RdVd\overline{V}) - \text{Tr}(VdZd\overline{Z}))
\]

or in a more compact form

\[
d^2 s_{n,m;A,B} = A \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{m} R_{ir} R_{js} dZ_{ij} d\overline{Z}_{rs}
\]

where \( R_{ij} \) is a matrix of coefficients with respect to \( Z_{ij} \) and \( d\overline{Z}_{rs} \) is the volume form on \( \mathbb{C}^m \).

Proposition 4.1. The metric above is Kähler.

Proof. This proposition follows from the above expression of the metric and Lemma 3.2. See also [1] for more geometric properties of the manifold.

To prove the metric is Kähler, we have to prove that the closed form \( \omega \) associated with the metric is closed.

We first show that the \( dZ_{ij} \wedge dZ_{pq} \wedge d\overline{Z}_{rs} \) part of \( d\omega \) is 0. If we denote the coefficients of \( dZ_{ij} \) \( dZ_{rs} \) in \( d^2 s_{n,m;A,B} \) above by \( \phi(i,j,r,s) \), then this equals to say

\[
\frac{\partial \phi(i,j,r,s)}{\partial Z_{pq}} = \frac{\partial \phi(p,q,r,s)}{\partial Z_{ij}}.
\]  

(4.1)

Since \( \phi(i,j,r,s) \) obviously has two parts: \( 2^{1-\delta(i,j)-\delta(r,s)}(R_{ir} \delta_{js} + R_{jr} \delta_{is}) \) and

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{a=1}^{m} 2^{1-\delta(i,j)-\delta(r,s)}(R_{sk} V_{al} R_{li} R_{jr} + R_{sk} V_{al} R_{lj} R_{ir} + R_{r} V_{al} R_{lj} R_{js} + R_{r} V_{al} R_{lj} R_{js}),
\]

we first compute the partial derivative of \( 2^{1-\delta(i,j)-\delta(r,s)}(R_{ir} \delta_{js} + R_{jr} \delta_{is}) \) with respect to \( Z_{ij} \). By Lemma 3.2, this is

\[
2^{1-\delta(i,j)-\delta(r,s)} \delta(p,q) \sqrt{1}(R_{ip} R_{qr} R_{js} + R_{iq} R_{pr} R_{js} + R_{ir} R_{jq} R_{ps} + R_{ir} R_{jq} R_{ps})
\]

By a similar calculation, we see that the expression for the partial derivative of \( 2^{1-\delta(p,q)-\delta(r,s)}(R_{pr} R_{qs} + R_{qr} R_{ps}) \) with respect to \( Z_{ij} \) is also the formula above. So the first part for (4.1) holds.
For the other part, we will compute the partial derivative of

\[ \sum_{k=1}^{dW} \sum_{l=1}^{a} \sum_{s=1}^{dW} 2^{\delta(i,j)-\delta(r,s)} (R_{sk}V_{ak}V_{al}R_{li}R_{lj} + R_{sk}V_{ak}V_{al}R_{lj}R_{ir} + R_{rk}V_{ak}V_{al}R_{li}R_{js} + R_{rk}V_{ak}V_{al}R_{lj}R_{is}). \]

Using Lemma 3.2, we see that the derivative of the above formula with respect to \( Z_{pq} \) is

\[ \sum_{k=1}^{dW} \sum_{l=1}^{a} \sum_{s=1}^{dW} 2^{\delta(i,j)-\delta(r,s)} -\delta(p,q) -1 \sqrt{-1} V_{ak}V_{al}(R_{sp}R_{kq}R_{li}R_{lj} + R_{sq}R_{kp}R_{li}R_{lj} + R_{rk}R_{lp}R_{iq}R_{jr} + R_{rs}R_{kp}R_{lj}R_{is}). \]

Similarly, this also equals the partial derivative of the second part of \( \phi(p, q, r, s) \) with respect to \( Z_{ij} \). Thus we see that (4.1) holds and the \( dZ_{ij} \wedge dZ_{pq} \wedge d\bar{Z}_{rs} \) part is 0.

Next, we prove that the \( dZ_{ij} \wedge dW_{pq} \wedge dW_{rs} \) part is 0. One of these coefficients comes from the partial derivative of \( dW_{pq} \wedge dW_{rs} \), and this equals

\[ \delta(p, r)2^{-\delta(i,j)-1} \sqrt{-1} (R_{sp}R_{pq} + R_{sj}R_{qi}). \]

Others come from the partial derivative of \( dZ_{ij} \wedge dW_{rs} \) with respect to \( W_{pq} \). By computation, this part equals

\[ -\delta(p, r)2^{-\delta(i,j)-1} \sqrt{-1} (R_{sp}R_{pq} + R_{sj}R_{qi}). \]

So the \( dZ_{ij} \wedge dW_{pq} \wedge d\bar{W}_{rs} \) part is also 0.

Next, we consider the \( dZ_{ij} \wedge dZ_{pq} \wedge dW_{rs} \) part. This equaling 0 means that the partial derivative of \( \sum_{k=1}^{dW} 2^{-\delta(i,j)} V_{rk}(R_{ki}R_{js} + R_{kj}R_{is}) \) with respect to \( Z_{pq} \) equals the partial derivative of \( \sum_{k=1}^{dW} 2^{-\delta(p,q)} V_{rk}(R_{kp}R_{qs} + R_{kq}R_{ps}) \) with respect to \( Z_{ij} \). By computation, they both equal

\[ \sum_{k=1}^{dW} 2^{-\delta(i,j)-\delta(p,q)} -\delta(p,q) -1 \sqrt{-1} V_{rk}(R_{kp}R_{ql}R_{js} + R_{kq}R_{lp}R_{js} + R_{ki}R_{jp}R_{qs} + R_{kj}R_{iq}R_{sp}) \]

\[ + R_{kp}R_{qj}R_{is} + R_{kq}R_{ij}R_{is} + R_{kj}R_{ip}R_{qs} + R_{ki}R_{iq}R_{sp}). \]

So this part is also 0.

The next is the \( dZ_{ij} \wedge dW_{pq} \wedge d\bar{Z}_{rs} \) part. They come from the partial derivative of the \( dZ_{ij} \wedge d\bar{Z}_{rs} \) part and the \( dW_{pq} \wedge d\bar{Z}_{rs} \) part. By the same way of computation, they both equal

\[ 2^{-\delta(r,s)-\delta(s,r)} -\delta(p,q) -1 \sqrt{-1} \sum_{k} V_{lp}(R_{kq}R_{jr}R_{aq} + R_{kj}R_{is}R_{aq} + R_{kr}R_{jq}R_{is} + R_{ks}R_{jr}R_{iq}). \]

The \( dW_{ij} \wedge dW_{pq} \wedge d\bar{W}_{rs} \) part is obviously 0, so we only need to prove the \( dW_{ij} \wedge dW_{pq} \wedge d\bar{Z}_{rs} \) part is 0 now, other cases are just conjugations of the proved ones. This part comes from the partial derivative of \( dW_{ij} \wedge d\bar{Z}_{rs} \) and \( dW_{pq} \wedge d\bar{Z}_{rs} \), and it is easy to check that this part is also 0.

Combining all these, we have seen that \( d\omega = 0 \), and so the metric is Kähler. \( \square \)

Now the Hermitian-metric matrix associated with this metric is given by

\[ W = \begin{pmatrix} W^1 & W^2 & W^3 \\ W^2 & W^3 & W^1 \end{pmatrix}, \]

where \( W^1 = (W^1_{i,j})_{i,j \in \Omega} \), and

\[ W^1_{i,j} = A^22^{-\delta(i,j)-\delta(r,s)} \times (R_{ir}R_{js} + R_{jr}R_{is}) \]
Recall that

\[ \delta(I, J) := \begin{cases} 1, & \text{if } I = J \in \Omega, \\ 0, & \text{if } I \neq J \in \Omega. \end{cases} \]

To compute the connection, we have to know the inverse of \( W \). Let

\[ M^1 = (M^1_{I,J})_{I,J \in \Omega}, \quad M^2 = (M^2_{I,J'})_{I \in \Omega, J' \in \Omega'}, \quad \text{and} \quad M^3 = (M^3_{I,J'})_{I' \in \Omega', J' \in \Omega'}, \]

with

\[
M^1_{I,J} = \frac{1}{2A} Y_{ir}Y_{js} + \frac{1}{2A} Y_{jr}Y_{is}, \\
M^2_{I,J'} = \frac{1}{2A} V_{ir'}Y_{js'} + \frac{1}{2A} V_{jr'}Y_{is'}, \\
M^3_{I,J'} = \sum_{r=1}^{n} \sum_{s=1}^{m} \frac{1}{2A} V_{rk}R_{kl}V_{al}Y_{js} + \frac{1}{2A} V_{ri}Y_{js} + \frac{1}{B} \delta(i, r)Y_{js}.
\]

**Lemma 4.2.** Let \( M = (M^1 M^2 M^3) \), then \( M \) is the inverse matrix of \( W \).

**Proof.** (1) We use the Dirac symbol

\[
\delta(I, J) := \begin{cases} 1, & \text{if } I = J \in \Omega, \\ 0, & \text{if } I \neq J \in \Omega. \end{cases}
\]

Recall that \( I = (i, j), J = (r, s), \) and \( K = (p, q) \). We have

\[
\sum_{J \in \Omega} M^1_{I,J}W^1_{J,K} = \sum_{r<s} M^1_{(i,j)(r,s)} W^1_{(r,s)(p,q)} \\
= \delta(I, K) + \frac{B}{A} \sum_{r<s} (Y_{ir}Y_{js} + Y_{jr}Y_{is}) \left( \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{a=1}^{m} 2^{-\delta(p,q)-\delta(r,s)} \right) \\
\times \left( R_{rk}V_{ak}V_{al}R_{is}R_{rp} + R_{rk}V_{ak}V_{al}R_{ls}R_{rp} + R_{pk}V_{ak}V_{al}R_{ir}R_{rq} + R_{pk}V_{ak}V_{al}R_{ls}R_{rq} \right) \\
= \delta(I, K) + \frac{B}{A} \sum_{r=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{a=1}^{m} 2^{-\delta(p,q)} (Y_{ir}Y_{jr} (R_{rk}V_{ak}V_{al}R_{is}R_{rp} + R_{pk}V_{ak}V_{al}R_{ir}R_{rq})) \\
+ \frac{B}{A} \sum_{r<s} (Y_{ir}Y_{js} + Y_{jr}Y_{is}) \left( \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{a=1}^{m} 2^{-\delta(p,q)} \right) \\
\times \left( R_{rk}V_{ak}V_{al}R_{ir}R_{sp} + R_{rk}V_{ak}V_{al}R_{ls}R_{rp} + R_{pk}V_{ak}V_{al}R_{ir}R_{rq} + R_{pk}V_{ak}V_{al}R_{ls}R_{rq} \right) \\
= \delta(I, K) + \frac{B}{A} \sum_{r=1}^{n} \sum_{s=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{a=1}^{m} 2^{-\delta(p,q)} (Y_{ir}Y_{js} + Y_{jr}Y_{is}) \\
\times \left( R_{rk}V_{ak}V_{al}R_{is}R_{rp} + R_{pk}V_{ak}V_{al}R_{is}R_{rq} \right) \\
= \delta(I, K) + \frac{B}{A} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{a=1}^{m} 2^{-\delta(p,q)} V_{ak}V_{al} \\
\times \left( \delta(i, p)\delta(j, l)R_{qk} + \delta(i, q)\delta(j, l)R_{pk} + \delta(j, p)\delta(i, l)R_{qk} + \delta(j, q)\delta(i, l)R_{pk} \right),
\]
and

\[
\sum_{J \in \Omega'} M^2_{J'J} W^2_{K'} = \sum_{r'=1}^{m} \sum_{s'=1}^{n} M^2_{(i,j)(r',s')} W^2_{(p,q)(r',s')}
\]

\[
= - \frac{B}{A} \sum_{r'=1}^{m} \sum_{s'=1}^{n} (V_{r'j} Y_{j's'} + V_{r'j} Y_{is'}) \left( \sum_{k=1}^{n} 2^{-\delta(i,j)} V_{r'k} (R_{k} R_{j's'} + R_{kJ} R_{is'}) \right)
\]

\[
= - \frac{B}{A} \sum_{r'=1}^{m} \sum_{s'=1}^{n} 2^{-\delta(i,j)} (V_{r'j} V_{r'k} (\delta(j,q) R_{pk} + \delta(j,p) R_{qk}) + V_{r'j} V_{r'k} (\delta(i,p) R_{qk} + \delta(i,q) R_{pk})).
\]

Adding these equalities together, we can see that

\[
\sum_{J \in \Omega} M^1_{J} W^1_{J,K} + \sum_{J' \in \Omega'} M^2_{J'} W^2_{K,J'} = \delta(I, K).
\]  

(2) We will now compute

\[
\sum_{J \in \Omega} M^1_{J} W^2_{J,K^r} + \sum_{J' \in \Omega'} M^2_{J'} W^3_{J',K^r}.
\]

First we have

\[
\sum_{J \in \Omega} M^1_{J} W^2_{J,K^r} = \sum_{r \leq s} M^1_{(i,j)(r,s)} W^2_{(r,s)(p',q')}
\]

\[
= \frac{1}{A} \sum_{r \leq s} (Y_{r} Y_{j's} + Y_{r} Y_{is}) \left( - B \sum_{k=1}^{n} 2^{-\delta(r,s)} V_{r'k} (R_{k} R_{s'q'} + R_{sk} R_{q'}) \right)
\]

\[
= - \frac{B}{2A} \sum_{k=1}^{n} V_{r'k} (\delta(i,k) \delta(j,q') + \delta(j,k) \delta(i,q'))
\]

\[
= - \frac{B}{2A} V_{r'k} (\delta(j,q') + 2^{-1} V_{r'j} \delta(i,q'))
\]

and

\[
\sum_{J' \in \Omega'} M^2_{J'} W^3_{J',K^r} = \sum_{r=1}^{m} \sum_{s=1}^{n} M^2_{(i,j)(r',s')} W^3_{(p',q')}
\]

\[
= \frac{B}{2A} \sum_{r=1}^{m} \sum_{s=1}^{n} (V_{r'j} Y_{j's'} + V_{r'j} Y_{is'}) \delta(r', p') R_{s'q'}
\]

\[
= \frac{B}{2A} (V_{r'j} \delta(j,q') + V_{r'j} \delta(i,q')).
\]

So we have

\[
\sum_{J \in \Omega} M^1_{J} W^2_{J,K^r} + \sum_{J' \in \Omega'} M^2_{J'} W^3_{J',K^r} = 0.
\]

(3) We compute

\[
\sum_{J \in \Omega} M^2_{J} W^1_{J,K^r} + \sum_{J' \in \Omega'} M^3_{J'} W^2_{J',K^r}.
\]
which is more complicated. First we have

\[ \sum_{J \in \Omega} M_{J, l}^2 W_{J, l}^1 = \sum_{r \leq s} M_{(p', q')}^2 (r, s) W_{(r, s)}(s, t) \]

\[ = \frac{1}{A} \sum_{r \leq s} (V_{p', q'} V_{p' q'}) \delta (r, s) - \delta (r, s) \times (R_{t r} R_{s l} + R_{l t} R_{s r}) \]

\[ + \sum_{k=1}^{m} \sum_{l=1}^{n} 2^{-\delta (r, s)} (V_{p', q'} V_{p' q'}) \sum_{a=1}^{m} \sum_{r \leq s} (V_{p', q'} V_{p' q'}) \times \]

\[ \times (R_{s k} V_{a l} R_{l t} R_{s r} + R_{s k} V_{a l} R_{l t} R_{r s} + R_{s k} V_{a l} R_{l t} R_{s r}) \]

\[ = \frac{B}{A} \sum_{r \leq s} 2^{-\delta (r, s)} (V_{p', q'} V_{p' q'}) R_{r t} R_{s l} \]

\[ + \frac{B}{A} 2^{-\delta (r, s)} \sum_{k=1}^{m} \sum_{l=1}^{n} (V_{p', q'} V_{p' q'}) \sum_{a=1}^{m} \sum_{r \leq s} (V_{p', q'} V_{p' q'}) \]

\[ \times \left( \sum_{r \leq s} \sum_{k=1}^{m} \sum_{l=1}^{n} V_{a l} R_{l t} \delta (q', j) + V_{p', q'} R_{s l} \delta (q', i) \right) \]

\[ + \frac{B}{A} \sum_{r \leq s} 2^{-\delta (r, s)} \sum_{k=1}^{m} \sum_{l=1}^{n} V_{a l} R_{l t} \delta (q', j) + V_{p', q'} R_{s l} \delta (q', i) \right) \],

and similarly

\[ \sum_{J' \in \Omega'} M_{J', j}^2 W_{J', j}^2 = \sum_{r' \leq s' \leq 1} \sum_{r' \leq s' \leq 1} M_{(p', q')}^2 (r', s') W_{(r', s')} (s', t) \]

\[ = \frac{B}{A} \sum_{r' \leq s' \leq 1} \sum_{r' \leq s' \leq 1} \left( \sum_{k=1}^{m} \sum_{l=1}^{n} V_{a l} R_{l t} \delta (q', j) + V_{p', q'} R_{s l} \delta (q', i) \right) \]

Now it is easily checked that

\[ \sum_{J \in \Omega} M_{J, l}^2 W_{J, l}^1 + \sum_{J' \in \Omega'} M_{J', j}^2 W_{J', j}^2 = 0. \]  \hspace{1cm} (4.3)

(4) At last we calculate

\[ \sum_{J \in \Omega} M_{J, l}^2 W_{J, l}^2 + \sum_{J' \in \Omega'} M_{J', j}^2 W_{J', j}^3. \]
First,
\[ \sum_{J' \in \Omega'} M_{J'J}^2 W_{J'K'}^2 = \sum_{r \leq s} M_{(r,s)(r',s')}^2 W_{(r,s)(p',q')}^{2} \]

\[ = -\frac{B}{A} \sum_{r \leq s} (V_{i' r} Y_{s' j'} + V_{i' s} Y_{r' j'}) \sum_{k=1}^{n} 2^{-1 - \delta(r,s)} V_{r' k} (R_{kr} R_{sq'}) + R_{ks} R_{r' j'} \]

\[ = -\frac{B}{A} \sum_{r \leq s} \sum_{s=1}^{n} \sum_{k=1}^{n} V_{r' k} (V_{i' r} Y_{s' j'} + V_{i' s} Y_{r' j'}) R_{kr} R_{sq'} \]

\[ = -\frac{B}{2A} \sum_{l=1}^{n} \sum_{k=1}^{n} (V_{i' k} V_{i' l} R_{lk} \delta(q', j') + V_{i' k} V_{i' l} R_{iq'} \delta(k, j')) , \]

and
\[ \sum_{J' \in \Omega'} M_{J'J}^3 W_{J'K'}^3 = \sum_{r',s'} M_{(r',s')(r',s')}^3 W_{(r',s')(p',q')}^{3} \]

\[ = \frac{B}{2} \sum_{r' = 1}^{n} \sum_{s' = 1}^{n} \left( \frac{1}{A} \sum_{k=1}^{n} \sum_{l=1}^{n} V_{i' r' k} V_{i' l} Y_{s' r'} \right) + \frac{1}{2B} \delta(i', r') R_{r' s'} \]

\[ = \frac{B}{2A} \sum_{k=1}^{n} \sum_{l=1}^{n} \left( V_{i' k} V_{i' l} R_{kl} \delta(q', j') + \sum_{s'=1}^{n} V_{i' k} V_{i' s'} R_{s' q'} \right) + \delta(q', j') \delta(i', r'). \]

So
\[ \sum_{J \in \Omega} M_{JJ}^2 W_{JK}^2 + \sum_{J' \in \Omega'} M_{J'J}^3 W_{J'K'}^3 = \delta(I', K'). \quad (4.4) \]

Thus we have shown that \( WM = I \). \( \square \)

We will only prove the expression for \( D(dZ) \) in Theorem 2.2, the other part is the same. Similarly to [15, Subsection 3.2], we can compute \( \Gamma_{I,J}^K \) as

\[ \Gamma_{I,J}^K = \delta(I', K') \]

where \( \Gamma_{I,J}^K \) means the Christoffel symbol corresponding to the indices \( J, I, K \).

Before the calculation, we define
\[ \sigma_{(p,a)(r,s)} = \begin{cases} 1, & \text{if } Z_{pa} = Z_{rs}, \\ 0, & \text{if } Z_{pa} \neq Z_{rs}. \end{cases} \]

Then we have
\[ \sum_{L \in \Omega} W_{IL}^1 \frac{\partial M_{KL}^1}{\partial Z_J} = -\frac{\sqrt{-1}}{2} \sum_{a \leq b} A^2^{-\delta(i,j)-\delta(a,b)} \times (R_{ia} R_{jb} + R_{ja} R_{ib}) \]

\[ \times \frac{1}{A} \left( \sigma_{(p,a)(r,s)} Y_{q b} + \sigma_{(q,b)(r,s)} Y_{p a} + \sigma_{(q,a)(r,s)} Y_{p b} + \sigma_{(p,b)(r,s)} Y_{q a} \right) \]

\[ -\frac{\sqrt{-1}}{2} \sum_{a \leq b} \left( \sum_{k=1}^{n} \sum_{l=1}^{n} 2^{-1 - \delta(i,j)-\delta(a,b)} \times V_{r l} V_{s k} (R_{kl} R_{ia} R_{jb} + R_{ja} R_{kl} R_{ib} + R_{ib} R_{ja} R_{kl} + R_{ja} R_{ib} R_{kl}) \right) \]

\[ \times \frac{1}{A} \left( \sigma_{(p,a)(r,s)} Y_{q b} + \sigma_{(q,b)(r,s)} Y_{p a} + \sigma_{(q,a)(r,s)} Y_{p b} + \sigma_{(p,b)(r,s)} Y_{q a} \right) \]
\[ = -\sqrt{-1} \frac{1}{2} \sum_{a=1}^{n} \sum_{b=1}^{n} 2^{-\delta(i,j)} R_{ia} R_{jb}(Y_{gb} \sigma_{(p,a)}(r,s) + Y_{pa} \sigma_{(q,b)}(r,s)) + Y_{pa} \sigma_{(q,a)}(r,s) + Y_{qa} \sigma_{(p,b)}(r,s)) \\
\] 
\[ -\sqrt{-1} \frac{1}{2} \sum_{a=1}^{n} \sum_{b=1}^{n} B \frac{2^{-1-\delta(i,j)}}{A} \sum_{k=1}^{n} \sum_{l=1}^{n} V_{kl} V_{lt}((R_{ia} R_{jt} + R_{ja} R_{it}) \delta(q,l) \sigma_{(p,a)}(r,s) \\
+ (R_{it} R_{bl} + R_{jt} R_{al}) \delta(i,q) \sigma_{(p,b)}(r,s) \\
+ (R_{ia} R_{jt} + R_{ja} R_{it}) \delta(p,l) \sigma_{(q,a)}(r,s) + (R_{it} R_{bl} + R_{jt} R_{al}) \delta(i,p) \sigma_{(q,b)}(r,s)) \\
= -\sqrt{-1} \frac{1}{2} \sum_{a=1}^{n} \sum_{b=1}^{n} 2^{-\delta(i,j)} R_{ia} \sigma_{(q,j)}(r,s) + \delta(p,j) \sigma_{(q,a)}(r,s) \\
+ R_{ja} \sigma_{(p,i)}(r,s) + \delta(q,i) \sigma_{(p,a)}(r,s)) \\
+ \frac{1}{2} \sum_{a=1}^{n} \sum_{b=1}^{n} B \frac{2^{-1-\delta(i,j)}}{A} V_{kl} (R_{ia} R_{jt} + R_{ja} R_{it})(V_{kq} \sigma_{(p,a)}(r,s) + V_{kp} \sigma_{(q,a)}(r,s) \\
- \frac{1}{2} \sum_{b=1}^{n} \sum_{l=1}^{n} \sum_{k=1}^{m} B \frac{2^{-1-\delta(i,j)}}{A} V_{kt} V_{lt} (R_{it} R_{jl} \delta(j,p) + R_{jt} R_{bl} \delta(i,p)) \sigma_{(q,b)}(r,s), \\
\sum_{L' \in \Omega^r} W_{L}^{2} \partial M_{L}^{2} \partial Z_{j} = \frac{-\sqrt{-1} B}{2 A} \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{m} 2^{-1-\delta(i,j)} V_{a'k} (R_{ik} R_{jb'} + R_{jk} R_{i'b'}) \\
\times (\sigma_{(p,b')(r,s)} V_{a'b'} + \sigma_{(p,i')(r,s)} V_{a'i'}) \\
= \frac{-\sqrt{-1} B}{2 A} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{a=1}^{n} 2^{-1-\delta(i,j)} V_{kt} V_{lt} (R_{it} R_{ja} + R_{ja} R_{it})(V_{kq} \sigma_{(p,a)}(r,s) + V_{kp} \sigma_{(q,a)}(r,s)). \\
\] 
So adding these two parts,
\[ \Gamma_{j}^{j} = -\sqrt{-1} \left( B \sum_{k=1}^{m} \sum_{l=1}^{n} 2^{-\delta(i,j)} (R_{ia} \sigma_{(q,j)}(r,s) + \delta(p,j) \sigma_{(q,a)}(r,s)) \\
+ R_{ja} \sigma_{(p,i)}(r,s) + \delta(q,i) \sigma_{(p,a)}(r,s)) \right) \\
+ \frac{-\sqrt{-1} B}{2 A} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} 2^{-1-\delta(i,j)} V_{kt} V_{lt} (R_{it} R_{jl} \delta(j,p) + R_{jt} R_{bl} \delta(i,p)) \sigma_{(q,b)}(r,s). \\
\] 
If \( p = q \), then \( \Gamma_{j}^{j} \neq 0 \) only when \( Z_{ij} \) and \( Z_{rs} \) belong to the same row or column with \( Z_{pp} \).

If \( i = r = p \), \( \Gamma_{i}^{r} = -\sqrt{-1} R_{ij} - \sqrt{-1 T} B \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{i=1}^{n} V_{kl} V_{lt} R_{ji} R_{it} \).

If \( i = s = p \), \( \Gamma_{i}^{r} = -\sqrt{-1} R_{is} - \sqrt{-1 T} B \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{i=1}^{n} V_{kl} V_{lt} R_{is} R_{it} \).

If \( j = r = p \), \( \Gamma_{j}^{r} = -\sqrt{-1} R_{ij} - \sqrt{-1 T} B \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{i=1}^{n} V_{kl} V_{lt} R_{ji} R_{it} \).

If \( j = s = p \), \( \Gamma_{j}^{r} = -\sqrt{-1} R_{is} - \sqrt{-1 T} B \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{i=1}^{n} V_{kl} V_{lt} R_{is} R_{it} \).

If \( p < q \), we only consider the case when \( Z_{ij} \) belongs to the same row with \( Z_{pq} \), \( Z_{rs} \) belongs to the same column with \( Z_{pq} \), other cases are the same.

Consider the four cases, \( i = p = j, r = s = q; i = p = j, r < s = q; i = p < j, r = s = q; i = p = r, j = s = q \) respectively, it is not hard to see that in each of the cases, we always have
\[ \Gamma_{j}^{r} = -\sqrt{-1} R_{jr} - \frac{\sqrt{-1} B}{A} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{i=1}^{n} V_{kl} V_{lt} R_{ji} R_{it}. \]

If \( i = p < j, r < s = q, i \neq r, \) or \( j \neq q \), we have
\[ \Gamma_{j}^{r} = -\sqrt{-1} R_{jr} - \frac{\sqrt{-1} B}{A} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{i=1}^{n} V_{kl} V_{lt} R_{ji} R_{it}. \]
The other cases are similar. Then since we can compute $D(dZ)$ as

$$D(dZ_K) = - \sum_{I,J \in \Omega} \Gamma^K_{I,J} dZ_I dZ_J - \sum_{I' \in \Omega', J \in \Omega} \Gamma^K_{I',J} dZ'_I dZ_J - \sum_{I \in \Omega, J' \in \Omega'} \Gamma^K_{I,J'} dZ_I dZ'_{J'} - \sum_{I' \in \Omega', J' \in \Omega'} \Gamma^K_{I',J'} dZ'_I dZ'_{J'},$$

we can easily deduce that the $dZ dZ$ part in $D(dZ)$ can be written as

$$dZ \left( \sqrt{-1} Y^{-1} + \frac{\sqrt{-1} B}{2A} Y^{-1} V Y^{-1} \right) dZ.$$

The other parts can be got in the similar way and finally we can check that

$$D(dZ) = - \frac{\sqrt{-1} B}{2A} (dZ, dW^t) \begin{pmatrix} -2 \frac{4}{B} Y^{-1} - Y^{-1} V Y^{-1} & Y^{-1} V Y^{-1} & Y^{-1} V^{-1} - I \end{pmatrix} \begin{pmatrix} dZ \\ dW \end{pmatrix}.$$

Thus we have completed the theorem.

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