Generating uniform random vectors in $\mathbb{Z}_p^k$: the general case

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Abstract

This paper is about the rate of convergence of the Markov chain $X_{n+1} = AX_n + B_n \pmod{p}$, where $A$ is an integer matrix with nonzero eigenvalues and $\{B_n\}$ is a sequence of independent and identically distributed integer vectors, with support not parallel to a proper subspace of $\mathbb{Q}^k$ invariant under $A$. If $|\lambda_i| \neq 1$ for all eigenvalues $\lambda_i$ of $A$, then $n = O((\ln p)^2)$ steps are sufficient and $n = O(\ln p)$ steps are necessary to have $X_n$ sampling from a nearly uniform distribution. Conversely, if $A$ has the eigenvalues $\lambda_i$ that are roots of positive integer numbers, $|\lambda_1| = 1$ and $|\lambda_i| > 1$ for all $i \neq 1$, then $O(p^2)$ steps are necessary and sufficient.

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1 Introduction

In this paper we generalize some results obtained in the paper [2] about Markov chains on \( \mathbb{Z}^k \) of the form

\[
X_{n+1} = AX_n + B_n \pmod{p},
\]

where \( X_0 = x_0 \in \mathbb{Z}^k \), \( A \in GL_k(\mathbb{Q}) \cap M_k(\mathbb{Z}) \), \( p \) is an integer and \( \{B_n\}_n \) is a sequence of independent and identically distributed integer vectors.

If \( k = 1 \) and \( B_n \) is a fixed integer \( b \), for particular values of \( p \) this recursion is used to produce pseudorandom numbers on computers (see, for example, Knuth’s book [9]).

In the paper [3], the constant term \( b \) is chosen with a fixed probability at each step and the authors study the following Markov chain:

\[
X_{n+1} = aX_n + B_n \pmod{p},
\]

where \( a \) is a positive integer. This randomness is introduced in order to produce uniformly distributed random numbers on the set \( \{0, 1, \ldots, p - 1\} \). In the cited paper, it is shown that, for \( a = 2 \), \( n = O(\ln p \ln \ln p) \) steps are sufficient to sample \( X_n \) from an almost uniform distribution. On the other hand, if \( a = 1 \) then \( n = O(p^2) \) steps are necessary and sufficient to achieve randomness. A further generalization is described in [6], where the integer \( a \) is also allowed to vary.

The extension of the previous results to the higher-dimensional case is due to Asci [2] and next to Hildebrand and McCollum [8], with the study of some particular cases of the recursion (1). In [2], the distribution of \( B_n \) is the most general (the support of the distribution cannot be parallel to any proper subspace of \( \mathbb{Q}^k \) invariant under \( A \)), but the matrix \( A \) has only integer eigenvalues. In [8], \( A \) is arbitrary, but only a specific distribution for \( B_n \) is considered.

The general case is studied in this paper, with the condition \( \|B_n\|_\infty \in L^2 \) and some further conditions on \( p \). We find two different types of behaviour for the sequence (1), depending on the size of the complex eigenvalues of \( A \). If \( |\lambda_i| \neq 1 \) for all eigenvalues \( \lambda_i \), then \( n = O((\ln p)^2) \) steps are sufficient and \( n = O(\ln p) \) steps are necessary to reach the uniform distribution (theorems 3.1 and 3.13). In particular, for a matrix \( A \) with eigenvalues
that are roots of positive integers and $|\lambda_i| > 1$, we can show that $n = O(\ln p \ln \ln p)$ steps are sufficient (theorem 3.7). On the other hand, if the eigenvalue $\lambda_i$ are roots of positive integers, $|\lambda_1| = 1$ and $|\lambda_i| > 1$ for all $i \neq 1$, then $O(p^2)$ steps are necessary and sufficient (theorems 3.12 and 3.14). These theorems agree with the one-dimensional case studied in [3] and with the results in [2] and [8].

In Section 2, we recall some general results about random walk on groups and the preliminary lemmas proved in [2]. The main results of our work can be found in Section 3.

### 2 Preliminary results

Consider the sequence (1) and observe that we can suppose $X_n \in \mathbb{Z}_p^k$.

Set $P_n(x) = p(X_n = x), \forall x \in \mathbb{Z}_p^k$, and $\mu(x) = p(B_n = x), \forall x \in \mathbb{Z}^k, \forall n \in \mathbb{N}$; moreover, denote by $U$ the uniform distribution on $\mathbb{Z}_p^k$. Define:

$$V = \{x \in \mathbb{Z}^k : x = h - k, \text{ where } h, k \in \text{supp } \mu\}.$$

Indicate by $d$, where $d \leq k$, the degree of the minimum polynomial of $A$. By definition:

$$\prod_{i=1}^d (A - \lambda_i I) = \prod_{i=1}^d (t - \lambda_i I) = 0 \in M_k(\mathbb{Z}), \quad \lambda_i \in \{\lambda_1, ..., \lambda_d\}, \forall i = d+1, ..., k,$$

where $\lambda_1, ..., \lambda_d, ..., \lambda_k$ are the eigenvalues of $A$. Finally, set:

$$V^{d-1} = \{A^m x : x \in V, m = 0, 1, ..., d - 1\}.$$

In order to show that the distribution $P_n$ tends to the uniform distribution $U$, as $n \to +\infty$, use the Fourier analysis (see Diaconis’ monograph [4] and [5]). Define the variation distance between $P_n$ and $U$ in the following way:

$$\|P_n - U\| = \frac{1}{2} \sum_{\alpha \in \mathbb{Z}_p^k} |P_n(\alpha) - U(\alpha)|.$$

It is possible to prove that

$$\|P_n - U\| = \frac{1}{2} \sup_{f \in F} |E_{P_n}(f) - E_U(f)| = \max_{A \subset \mathbb{Z}_p^k} |P_n(A) - U(A)|,$$
where $F \equiv \{ f : \mathbb{Z}_p^k \rightarrow \mathbb{C} : \|f\| \leq 1 \}$.

Henceforth, our purpose will be to find a bound for $\|P_n - U\|$ in terms of $n$ and $p$.

Observe that, if we indicate with $\{Y_n\}_n$ the sequence defined by (1) and the condition $X_0 = 0$, we have $X_n = \varphi_n(Y_n)$, where the one to one function $\varphi_n : \mathbb{Z}_p^k \rightarrow \mathbb{Z}_p^k$ is defined by $\varphi_n(x) = A^n x_0 + x$. Moreover:

$$\|P_n - U\| = \|P_n \circ \varphi_n - U\|,$$

then we can consider $X_0 = 0$.

Let $f : \mathbb{Z}^k \rightarrow \mathbb{C}$; define the Fourier transform $\hat{f} : \mathbb{R}^k \rightarrow \mathbb{C}$ by:

$$\hat{f}(\alpha) = \sum_{h \in \mathbb{Z}^k} \exp \left( \frac{2\pi i}{p} \langle h, \alpha \rangle \right) f(h).$$

We have the following four results, whose proofs are similar to the proofs of the lemmas 2.5, 3.1, 3.3 and 3.4 in [2]: the only difference is that $\alpha$ ranges in $\mathbb{R}^k$ instead of in $\mathbb{Z}_p^k$.

Lemma 2.1 is also proved in [4], in a more general case.

**Lemma 2.1 (Upper bound lemma).**

$$\|P_n - U\| \leq \frac{1}{4} \sum_{\alpha \in \mathbb{Z}_p^k \setminus \{0\}} |\hat{P}_n(\alpha)|^2.$$

**Lemma 2.2.** Suppose $\gcd(\det(A), p) = 1$, $X_0 = 0$, $\alpha \in \mathbb{R}^k$; then:

1) $$\hat{P}_n(\alpha) = \prod_{j=0}^{n-1} \hat{\mu} \left( \alpha A^j \right).$$

2) $$|\hat{P}_n(\alpha)|^2 \leq \prod_{j=0}^{n-1} \left( \sum_{h, i \in \mathbb{Z}_p^k} \mu(h) \mu(i) \cos \left( \frac{2\pi}{p} \langle h - i, A^j \alpha \rangle \right) \right)$$

$$\leq \prod_{j=0}^{n-1} \left( 1 - 2\mu(u) \mu(v) + 2\mu(u) \mu(v) \cos \left( \frac{2\pi}{p} \langle u - v, A^j \alpha \rangle \right) \right),$$

$\forall u, v \in \text{supp } \mu$.

**Lemma 2.3.** Let $\alpha \in \mathbb{Z}_p^k \setminus \{0\}$; then:

$$\|P_n - U\| \geq \frac{1}{2} |\hat{P}_n(\alpha)|.$$
Lemma 2.4. Suppose that the support of \( \mu \) is not parallel to a proper subspace of \( Q^k \) invariant under \( A \). Then, there exists a basis \( \{y_1, ..., y_k\} \subset V^{d-1} \) of \( Q^k \). Furthermore, for all \( p \in \mathbb{N} \) such that \( \gcd(\det(y_1, ..., y_k), p) = 1 \), for all \( \alpha \in \mathbb{R}^k - (p\mathbb{Z})^k \), there exists \( i \in \{1, ..., k\} \) such that \( \langle y_i, \alpha \rangle \neq 0 \mod p \). In particular, if the support of \( \mu \) is not parallel to a proper subspace of \( Q^k \), we have \( y_1, ..., y_k \in V, \langle y_i, \alpha \rangle \neq 0 \mod p \), for some \( i \in \{1, ..., k\} \).

Henceforth, we will indicate by \( B \) the matrix \((y_1 ... y_k)\), where the vectors \( y_1, ..., y_k \) are defined by Lemma 2.4.

Lemma 2.5. \( \forall \; e, j \in \mathbb{N} \), we have:

1) \( t^e A^e = \prod_{i=1}^{e} (t^e A - \lambda_i I) + \sum_{s=0}^{e-1} \left( \lambda_{s+1} t^{e-s-1} \prod_{i=1}^{s} (t^e A - \lambda_i I) \right) \).

2) \( t^e A^e \prod_{i=1}^{e} (t^e A - \lambda_i I) = \sum_{h=e+1}^{d} \sum_{k_1, ..., k_{d-h+1}: k_m \geq 0, \forall m = 1, ..., d-h+1, \sum_{m=1}^{d-h+1} k_m = j - d + h} \prod_{m=h}^{d} \lambda_m^{k_{d-m+1}} \prod_{n=h+1}^{d} (t^e A - \lambda_n I) \prod_{i=1}^{e} (t^e A - \lambda_i I) \).

Proof.

1) The proof is equal to the proof of Lemma 3.2 in [2].

2) We can suppose \( e \leq d - 1 \), since otherwise the two members of 2) are equal to the null matrix. Set:

\[
H_{h,j} \equiv \left\{ k_1, ..., k_{d-h+1} : k_m \geq 0, \forall m = 1, ..., d - h + 1, \sum_{m=1}^{d-h+1} k_m = j - d + h \right\},
\]

\[
f(\lambda_1, ..., \lambda_d, j - d + h) \equiv \sum_{H_{h,j}} \prod_{m=h}^{d} \lambda_m^{k_{d-m+1}}.
\]

Proceed by induction on \( j \); if \( j = 0 \), \( \forall \; e = 0, 1, ..., d - 1 \), the thesis is true.
Suppose that the thesis is true for $j$; then, for $j + 1$, $\forall e = 0, 1, \ldots, d - 1$:

$$t^A j+1 \prod_{i=1}^{e}(t^A - \lambda_i I) = t^A \left( t^A \prod_{i=1}^{e}(t^A - \lambda_i I) \right)$$

$$= \sum_{h=e+1}^{d} (t^A - \lambda_h I + \lambda_h I) f(\lambda_h, \ldots, \lambda_d, j + d + h) \prod_{n=h+1}^{d} (t^A - \lambda_n I) \prod_{i=1}^{e}(t^A - \lambda_i I)$$

$$= (t^A - \lambda_{e+1} I) f(\lambda_{e+1}, \ldots, \lambda_d, j + d + e + 1) \prod_{n=e+2}^{d} (t^A - \lambda_n I) \prod_{i=1}^{e}(t^A - \lambda_i I)$$

$$+ \sum_{h=e+1}^{d-1} \left( \lambda_h f(\lambda_h, \ldots, \lambda_d, j + d + h) \prod_{n=h+1}^{d} (t^A - \lambda_n I) \right)$$

$$+ (t^A - \lambda_{h+1} I) f(\lambda_{h+1}, \ldots, \lambda_d, j + 1 - d + h) \prod_{n=h+2}^{d} (t^A - \lambda_n I) \prod_{i=1}^{e}(t^A - \lambda_i I)$$

$$+ \lambda_d \prod_{i=1}^{e}(t^A - \lambda_i I).$$

Observe that

$$\lambda_h f(\lambda_h, \ldots, \lambda_d, j + d + h) + f(\lambda_{h+1}, \ldots, \lambda_d, j + 1 - d + h) = f(\lambda_h, \ldots, \lambda_d, j + 1 - d + h),$$

since $f(\lambda_h, \ldots, \lambda_d, j + 1 - d + h)$, a homogeneous polynomial of degree $j + 1 - d + h$ in the variables $\lambda_h, \ldots, \lambda_d$, can be obtained by multiplying $f(\lambda_h, \ldots, \lambda_d, j - d + h)$, a homogeneous polynomial of degree $j - d + h$ in the variables $\lambda_h, \ldots, \lambda_d$, by the variable $\lambda_h$ and by summing up $f(\lambda_{h+1}, \ldots, \lambda_d, j + 1 - d + h)$, a homogeneous polynomial of degree $j + 1 - d + h$ in the variables $\lambda_{h+1}, \ldots, \lambda_d$. Then:

$$t^A j+1 \prod_{i=1}^{e}(t^A - \lambda_i I) = f(\lambda_{e+1}, \ldots, \lambda_d, j - d + e + 1) \prod_{i=1}^{d}(t^A - \lambda_i I)$$

$$+ \sum_{h=e+1}^{d-1} f(\lambda_h, \ldots, \lambda_d, j + 1 - d + h) \prod_{n=h+1}^{d} (t^A - \lambda_n I) \prod_{i=1}^{e}(t^A - \lambda_i I) + \lambda_d^{j+1} \prod_{i=1}^{e}(t^A - \lambda_i I)$$

$$= \sum_{h=e+1}^{d} \sum_{m=h}^{d} \lambda_m^{k_{d-m+1}} \prod_{n=h+1}^{d} (t^A - \lambda_n I) \prod_{i=1}^{e}(t^A - \lambda_i I),$$

since $\prod_{i=1}^{d}(t^A - \lambda_i I)$ is equal to the null matrix. □
3 Main results

Theorem 3.1. Assume that $A$ has eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{C}^*$, $|\lambda_i| \neq 1$, for $i = 1, \ldots, k$, and assume that the support of $\mu$ is not parallel to a proper subspace of $Q^k$ invariant under $A$. Then, there exists $c \in \mathbb{R}^+$ such that, for all $p \in \mathbb{N}$ such that $\gcd(\det(A), p) = \gcd(\det(B), p) = 1$, and for all $n \geq c(\ln p)^2$, we have:

$$\|P_n - U\| \leq \varepsilon(p), \quad \text{where } \lim_{p \to \infty} \varepsilon(p) = 0.$$

Proof. From the lemmas 2.1 and 2.2:

$$\|P_n - U\|^2 \leq \frac{1}{4} \sum_{\alpha \in \mathbb{Z}_p^k - \{0\}} |\hat{P}_n(\alpha)|^2; \quad (2)$$

$$|\hat{P}_n(\alpha)|^2 = \prod_{j=0}^{n-1} |\hat{\mu}\left(^tA^j\alpha\right)|^2 = \prod_{j=0}^{n-1} \left( \sum_{u,v \in \mathbb{Z}^k} \mu(u)\mu(v) \cos \left( \frac{2\pi}{p} \langle u - v, ^tA^j\alpha \rangle \right) \right). \quad (3)$$

In order to estimate $\prod_{j=0}^{n-1} |\hat{\mu}\left(^tA^j\alpha\right)|^2$, for $\delta \in (0, \frac{1}{2})$ set:

$$L_\delta = \bigcup_{i=1}^k \left( [0, 1]^{i-1} \times [\delta, 1 - \delta] \times [0, 1]^{k-i} \right), \quad \xi_j = \frac{^tA^j\alpha}{p},$$

and indicate by $\{\xi_j\}$ the vector whose components are the fractional parts of the components of $\xi_j$. Consider the vectors $y_1, \ldots, y_k$ defined by Lemma 2.4; then, $\forall m = 1, \ldots, k$,

$$y_m = A^{z_m}x_m, \quad \text{where } x_m \in V, \ z_m \in \{0, 1, \ldots, d - 1\}.$$ 

Finally set $z = \max_{m=1,\ldots,k} z_m$. The following results hold:

Lemma 3.2. There exists $b \in (0, 1)$ such that, if $\{\xi_i\} \in L_\delta$, then $\prod_{j=0}^{i+z} |\hat{\mu}\left(^tA^j\alpha\right)|^2 \leq b$.

Proof. Consider the function $g : [0, 1]^k \to [0, 1]$ defined by:

$$g(t) = \prod_{j=0}^{i+z} |\hat{\mu}\left(^tA^jpt\right)|^2 = \prod_{j=0}^{i+z} \left( \sum_{u,v \in \mathbb{Z}^k} \mu(u)\mu(v) \cos \left( \frac{2\pi}{p} \langle u - v, ^tA^j t \rangle \right) \right).$$

If $t_l \in (0, 1)$ for some $l \in \{1, \ldots, k\}$, then $pt \not\equiv 0 \pmod{p}$ and, by Lemma 2.4, $\exists m \in \{1, \ldots, k\}$ such that $\langle x_m, ^tA^mpt \rangle \not\equiv 0 \pmod{p}$; then $\langle x_m, ^tA^m t \rangle \not\equiv \mathbb{Z}$ and, by definition of
Since \( g \) is continuous and \( L_\delta \) is closed and bounded, then \( g(t) \leq b < 1 \), \( \forall \ t \in L_\delta \); in particular, if \( \{\xi_i\} \in L_\delta \):

\[
\prod_{j=0}^{i+z} |\hat{\mu}(t^{j}A^i\alpha)|^2 = \prod_{j=0}^{i-1} |\hat{\mu}(t^{j}A^i\alpha)|^2 \prod_{l=0}^{z} |\hat{\mu}(t^{l+i}A^i\alpha)|^2 \leq g(\{\xi_i\}) \leq b. \quad \square
\]

**Lemma 3.3.** There exist \( \delta, \bar{\alpha} \in \mathbb{R}^+, \delta \in (0, \frac{1}{2}) \), and \( \bar{\tau} \in \mathbb{N}^* \), \( \bar{\tau} \leq \bar{\alpha} \ln p \), such that, for any \( p \in \mathbb{N} \) sufficiently large and for any \( \alpha \in \mathbb{Z}_p^k - \{0\} \), \( \{\xi_{\bar{\tau}}\} \) has a component in \([\delta, 1 - \delta]\).

**Proof.** The proof follows from Lemma 3 in [8]. \( \square \)

Let \( c > -\frac{k\bar{\alpha}}{\ln b} \) and suppose \( n \geq c(\ln p)^2 \); then, for \( p \) sufficiently large, we have \( n \geq rt \), where

\[
t = [\bar{\tau}\ln p] + d, \quad r = [\bar{\alpha}\ln p],
\]

for some \( \bar{\tau} > -\frac{k}{\ln b} \).

Let \( \bar{\tau} \in \mathbb{N}^* \) defined by Lemma 3.3. If \( \alpha \in \mathbb{Z}_p^k - \{0\} \), from Lemma 3.2 and Lemma 3.3, since \( \{\xi_{\bar{\tau}}\} \in L_\delta \):

\[
\prod_{j=0}^{\bar{\tau}+z} |\hat{\mu}(t^{j}A^i\alpha)|^2 \leq b < 1.
\]

Since \( \bar{\tau} + z \leq t - 1 \), from (3):

\[
|\hat{P}_t(\alpha)|^2 = \prod_{j=0}^{t-1} |\hat{\mu}(t^{j}A^i\alpha)|^2 \leq b.
\]

By repeating the previous arguments with \( t^{A^i\alpha} \) instead of \( \alpha, \forall i = 0, 1, ..., r - 1 \), we have:

\[
|\hat{P}_{rt}(\alpha)|^2 = \prod_{i=0}^{r-1} \prod_{j=0}^{t-1} |\hat{\mu}(t^{j+i}A^i\alpha)|^2 \leq b^r.
\]

Then, from (3):

\[
\sum_{\alpha \in \mathbb{Z}_p^k - \{0\}} |\hat{P}_n(\alpha)|^2 \leq \sum_{\alpha \in \mathbb{Z}_p^k - \{0\}} |\hat{P}_{rt}(\alpha)|^2 \leq b^r p^k \\
\leq \frac{b^{\bar{\alpha}\ln p} p^k}{b} = \frac{\exp \left( (\bar{\alpha}\ln b + k) \ln p \right)}{b}.
\]

Since \( \lim_{p \to \infty} \frac{\exp \left( (\bar{\alpha}\ln b + k) \ln p \right)}{b} = 0 \), by definition of \( \bar{\alpha} \), from (2) we obtain the statement of the theorem. \( \square \)
Theorem 3.4. Assume that $A$ has eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ such that $\lambda_{l_i}^i \in \mathbb{N}^* - \{1\}$, for some $l_i \in \mathbb{N}^*$, for any $i = 1, \ldots, k$, and assume that the support of $\mu$ is not parallel to a proper subspace of $\mathbb{Q}^k$ invariant under $A$. Then, there exists $c \in \mathbb{R}^+$ such that, for all $p \in \mathbb{N}$ such that $\gcd(\det(A), p) = \gcd(\det(B), p) = 1$, and for all $n \geq c \ln p \ln \ln p$, we have:

$$
\|P_n - U\| \leq \varepsilon(p), \quad \text{where } \lim_{p \to \infty} \varepsilon(p) = 0.
$$

Proof. Set

$$
l = \text{lcm}(l_i : i = 1, \ldots, k), \quad C = A^l, \quad \sigma_i = \lambda_{l_i}^i, \forall i = 1, \ldots, k. \quad (4)
$$

Since $\prod_{i=1}^d (tA - \lambda_i I) = 0$, we have $\prod_{i=1}^d (tC - \sigma_i I) = 0$. Moreover, $\forall E \subseteq \{1, \ldots, d\}$, define:

$$
Y_E = \left\{ \alpha \in \mathbb{Z}_k^p - \{0\} : \prod_{i \in E} (tC - \sigma_i I) \alpha \neq 0 \text{ (mod } p), \prod_{i \in T} (tC - \sigma_i I) \alpha = 0 \text{ (mod } p), \forall T \subset \{1, \ldots, d\} : |E| + 1 \leq |T| \leq d \right\}.
$$

The following relation holds:

$$
\mathbb{Z}_p^k - \{0\} = \bigcup_{E \subseteq \{1, \ldots, d\}} Y_E. \quad (5)
$$

In fact, observe that, $\forall \alpha \in \mathbb{Z}_p^k - \{0\}$, we can define:

$$
e = \max \left\{ t \in \{0, 1, \ldots, d - 1\} : \prod_{i \in E} (tC - \sigma_i I) \alpha \neq 0 \text{ (mod } p), \quad \text{for some } E \subseteq \{1, \ldots, d\} \text{ such that } |E| = t \right\};
$$

then $\alpha \in Y_E$, for some $E \subseteq \{1, \ldots, d\}$ such that $|E| = e$, and this implies (5).

Moreover:

$$
Y_E = \bigcup_{\emptyset \neq S \subset \{1, \ldots, k\}} Y_{S,E},
$$

where, if the vectors $y_1, \ldots, y_k$ are defined by Lemma 2.4:

$$
Y_{S,E} = \left\{ \alpha \in Y_E : \left\langle y_m, \prod_{i \in E} (tC - \sigma_i I) \alpha \right\rangle \neq 0 \text{ (mod } p), \forall m \in S, \right\}
$$

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\[
\left\langle y_m, \prod_{i \in E} \left(t \cdot (C - \sigma_i I) \right) \alpha \right\rangle = 0 \pmod{p}, \ \forall \ m \notin S \right\}\] .

Then:
\[
\sum_{\alpha \in \mathbb{Z}_k^d - \{0\}} |\tilde{P}_n(\alpha)|^2 = \sum_{E \subseteq \{1, ..., d\} \not\subseteq \mathbb{Z}_k^d} \sum_{\emptyset \not\subseteq \{1, ..., k\}} \sum_{Y_{S,E}} |\tilde{P}_n(\alpha)|^2.
\]

If \( \emptyset \not\subseteq S \subseteq \{1, ..., k\} \), reorder the set \( S \) in the following manner:
\[
S = \{m_{1,S}, ..., m_{|S|,S}\}, \quad \text{where } m_{i,S} < m_{j,S} \iff i < j.
\]

Then, \( \forall \ h = 1, ..., |S| \):
\[
y_{m,h,S} = A^{z_{m,h,S}}x_{m,h,S}, \quad \text{where } x_{m,h,S} \in V, \ z_{m,h,S} \in \{0, 1, ..., d - 1\}.
\]

Set \( \bar{x}_{h,S} = x_{m,h,S}, \ \bar{y}_{h,S} = y_{m,h,S}, \ z_{h,S} = z_{m,h,S} \) and let \( \bar{u}_{h,S}, \ \bar{v}_{h,S} \) the vectors of the support of \( \mu \) such that \( \bar{u}_{h,S} - \bar{v}_{h,S} = \bar{x}_{h,S} \). From Lemma 2.2, \( \forall \ \alpha \in \mathbb{Z}_p^k - \{0\} \), we have:
\[
|\tilde{P}_n(\alpha)|^2 \leq \left| \frac{|n - 1|}{n} \right|^2 \sum_{j=0}^{|S|} |\tilde{\mu}(\frac{t \cdot (C^j \alpha)}{p})| \leq \prod_{h=1}^{|S|} \prod_{j \in M_{h,S}} \left(1 - 2\mu(\bar{u}_{h,S})\mu(\bar{v}_{h,S}) + 2\mu(\bar{u}_{h,S})\mu(\bar{v}_{h,S}) \cos \left(\frac{2\pi}{p} \langle \bar{y}_{h,S}, \frac{t \cdot (C^j \alpha)}{p} \rangle \right) \right),
\]

where \( M_{h,S} \equiv \left\{(h - 1) \left\lfloor \frac{n - 1}{|S|} \right\rfloor, ..., h \left\lfloor \frac{n - 1}{|S|} \right\rfloor - z_{h,S} - 1\right\}, \quad \forall \ h = 1, ..., |S| \).

If \( \alpha \in Y_{S,E} \), for some \( \emptyset \not\subseteq S \subseteq \{1, ..., k\} \) and \( E \not\subseteq \{1, ..., d\} \), reorder the numbers \( \sigma_1, ..., \sigma_d \) so that the first \( |E| \) correspond to the set \( \{\sigma_i : i \in E\} \). Moreover, \( \forall \ j, n \in \mathbb{N}, \ \forall \ h = 1, ..., |S|, \) set:
\[
a_{h,S,E} = \left\langle \bar{y}_{h,S}, \prod_{i=1}^{|E|} \left(t \cdot (C - \sigma_i I) \right) \alpha \right\rangle \in \mathbb{Z}_p - \{0\}, \quad \xi_{j,n,h,S} = \left\langle \bar{y}_{h,S}, \left(t \cdot \prod_{i=1}^n \left(C^j - \sigma_i I\right) \right) \frac{\alpha}{p} \right\rangle
\]

(the fractional part of \( \left\langle \bar{y}_{h,S}, \left(t \cdot \prod_{i=1}^n \left(C^j - \sigma_i I\right) \right) \frac{\alpha}{p} \right\rangle \)). Observe that, from Lemma 2.5, 1):
\[
\xi_{j+|E|,0,h,S} = \left\langle \bar{y}_{h,S}, \left(t \cdot \prod_{i=1}^{|E|} \left(C^j - \sigma_i I\right) \right) \frac{\alpha}{p} \right\rangle + \sum_{m=0}^{|E| - 1} \sigma_{m+1} \xi_{j+|E| - m - 1, m,h,S}.
\]
Moreover, use Lemma 2.5, 2) and the definition of $Y_E$; $\forall \alpha \in Y_{S,E}$, $\forall j \in \mathbb{N}$, in the right member of 2) multiplied by $\alpha$ (with $e = |E|$, $A = C$ and $\lambda_i = \sigma_i$), only $\sigma_j^{|E|} \prod_{i=1}^{|E|} (|C - \sigma_i|) \alpha$ is different from 0 (corresponding to $h = m = d$, $k_1 = j$). Then, $\forall h = 1, \ldots, |S|$: 

$$t \prod_{i=1}^{|E|} (|C - \sigma_i|) \alpha = \prod_{i=1}^{|E|} (|C - \sigma_i|) \alpha \Rightarrow \langle \mathbf{Y}_{h,S}, t \prod_{i=1}^{|E|} (|C - \sigma_i|) \alpha \rangle = \frac{a_{h,S,E}}{p}$$

$$\Rightarrow \xi_{j+|E|,0,h,S} = \left\{ \sigma_d^j \frac{a_{h,S,E}}{p} + \sum_{m=0}^{|E|-1} \sigma_m \xi_{j+|E|-m-1,m,h,S} \right\}.$$

(8)

Let $g_{h,S} : [0,1] \rightarrow [0,1]$ the function defined by:

$$g_{h,S}(t) = 1 - 2\mu(\mathbf{u}_{h,S})\mu(\mathbf{v}_{h,S}) + 2\mu(\mathbf{u}_{h,S})\mu(\mathbf{v}_{h,S}) \cos(2\pi t).$$

From (7), we have:

$$\sum_{Y_{S,E}} |\hat{P}_n(\alpha)|^2 \leq \sum_{a_{h,S,E} \in \mathbb{Z}_p - \{0\}, h=1, \ldots, |S|} \prod_{j=1}^{|S|} g_{h,S}(\xi_{j,0,h,S})$$

$$= \prod_{h=1}^{|S|} \sum_{a_{h,S,E} \in \mathbb{Z}_p - \{0\}} \prod_{j=1}^{|S|} g_{h,S}(\xi_{j,0,h,S}).$$

(9)

Set $L = \left[ -\frac{1}{2^d \sigma d+1}, 1 - \frac{1}{2^d \sigma d+1} \right]$, where $\sigma = \max_{i=1,\ldots,k} \sigma_i$.

Observe that $g_{h,S}$ is continuous, $g_{h,S}(t) = 1 \Leftrightarrow t \in \{0,1\}$; then, $g_{h,S}$ has a maximum $b_{h,S} < 1$ in $L$, since $L$ is closed and bounded; in particular:

$$g_{h,S}(\xi_{j,0,h,S}) \leq \begin{cases} b_{h,S} & \text{if } \xi_{j,0,h,S} \in L \\ 1 & \text{otherwise} \end{cases}.$$

(10)

The following result follows:

**Lemma 3.5.** Suppose $\xi_{j,0,h,S}, \xi_{j+1,0,h,S}, \ldots, \xi_{j+e-1,0,h,S} \notin L$, for some $j \in \mathbb{N}^+$, $e \in \mathbb{N}$; then, for any $s \in \{0,1,\ldots,e-1\}$ and for any $r \in \{j, j+1, \ldots, j+e-s-1\}$:

$$\xi_{r,s,h,S} \in \left[ -\frac{1}{2^d \sigma d+s+1}, \frac{1}{2^d \sigma d+s+1} \right] (\text{mod } \mathbb{Z}).$$

(11)

In particular:

$$\xi_{j+e-s-1,0,h,S} \in \left[ -\frac{1}{2^d \sigma d+3}, \frac{1}{2^d \sigma d+3} \right] (\text{mod } \mathbb{Z}).$$

(12)
Proof. Prove the lemma by induction on \( s \); by hypothesis, if \( s = 0 \) and \( r \in \{ j, j + 1, \ldots, j + e - 1 \} \), the thesis is true.

Suppose that the thesis is true for \( s = n \); then, for \( s = n + 1 \), \( \forall r = j, j + 1, \ldots, j + e - n - 2 \), we have:

\[
^t C^n r \prod_{i=1}^{n+1} (^t C - \sigma_i I) = ^t C^n r \prod_{i=1}^{n} (^t C - \sigma_i I) - \sigma_{n+1} ^t C^n r \prod_{i=1}^{n} (^t C - \sigma_i I).
\]

By the inductive hypothesis:

\[
\xi_{r,n+1,h,S} \in 2\sigma \left[ -\frac{1}{2^d-n\sigma_d} \cdot \frac{1}{2^d-n\sigma_d} \right] \pmod{Z}
\]

Thus, we have \((\mathbb{I})\). In particular, since \( d - s \geq 2 \), \((\mathbb{I})\) follows. \( \square \)

In order to finish the proof of Theorem 3.4, we will borrow some arguments from the papers [2], [3] and [6].

Fix \( h, S, E \), set \( a = a_{h,S,E} \) and consider the expansion of \( \frac{a}{p} \) in base \( \sigma_d \):

\[
\frac{a}{p} = 0.a_1a_2a_3\ldots \quad \text{Define:}
\]

\[
t = \left\lfloor \log_{\sigma_d} p \right\rfloor \quad \text{then,} \quad \sigma_d^{t-1} < p < \sigma_d^t,
\]

\[
r_{h,S} = \left\lfloor \frac{|M_{h,S}|}{t} \right\rfloor = \left\lfloor \left[ \left( \frac{n-1}{|S|} \right) - \sigma_{h,S} \right] / t \right\rfloor.
\]

Moreover, recall that a "generalized alternation" between two consecutive digits \( a_j a_{j+1} \) of the expansion is defined as either the case \( a_j \neq a_{j+1} \) or the case \( a_j = a_{j+1} \notin \{0, \sigma_d - 1\} \).

Lemma 3.6. Suppose that there is a generalized alternation between the digits \( j + 1, j + 2 \) - th of the expansion of \( \frac{a}{p} \); then \( \xi_{j+i,0,h,S} \in L \), for some \( i \in \{0, 1, \ldots, |E| \} \).

Proof. The assumption imply:

\[
\left\{ \frac{\sigma_j a}{\sigma_p} \right\} \in \left[ \frac{1}{\sigma_d}, 1 - \frac{1}{\sigma_d} \right] \subset \left[ \frac{1}{\sigma^2}, 1 - \frac{1}{\sigma^2} \right].
\]
If $\xi_{j,0,h,S}, \xi_{j+1,0,h,S}, \ldots, \xi_{j+|E|-1,0,h,S} \not\in L$, then, from (8) and Lemma 3.5:

$$
\xi_{j+|E|,0,h,S} \in \left( \frac{1}{\sigma^2}, 1 - \frac{1}{\sigma^2} \right) + \left[ -\sum_{m=0}^{\sigma m+1} \frac{\sigma m+1}{2d-m} \cdot \sum_{m=0}^{\sigma m+1} \frac{\sigma m+1}{2d-m} - 1 \right] \subset \left( \frac{1}{2d\sigma^2}, 1 - \frac{1}{2d\sigma^2} \right) \subset L.
$$

Consider the first $rt$ integer numbers of the set $M_{h,S}$ and partition such numbers into $r$ disjoint sets $M_i = M_{i,h,S}$, $1 \leq i \leq r$, each of length $t$, such that, if $i < j$, $x \in M_i$ and $y \in M_j$, then $x < y$. Moreover, $\forall i = 1, \ldots, r$, consider the block of digits

$$B_{a,i} = B_{a,i,h,S} = \{a_j : j \in M_i\}
$$

and, $\forall B \subset \{a_1, a_2, \ldots\}$, $B$ made up of consecutive digits, indicate by $A(B)$ the number of generalized alternations in $B$. Finally, $\forall D \subset \mathbb{N}$, set:

$$C(D) = C(D,h,S) = |\{j \in D : \xi_{j,0,h,S} \in L\}|.
$$

Suppose $A(\{a_j : j \in M_{h,S}\}) = md$, for some $m \in \mathbb{N}$. Since $|E| \leq d - 1$, from Lemma 3.6 we deduce $C(M_{h,S}) \geq m$; in general:

$$C(M_{h,S}) \geq \left[ \frac{A(\{a_j : j \in M_{h,S}\})}{d} \right] \geq \frac{\sum_{i=1}^{r} A(B_{a,i})}{d} - 1. \quad (13)
$$

It is possible to prove the following two results:

**Lemma 3.7.** $\forall i \in \{1, \ldots, r\}$, as $a$ ranges in $\mathbb{Z}_p - \{0\}$, the blocks $B_{a,i}$ are distinct and have at least one generalized alternation. Moreover, $\forall i, j \in \{1, \ldots, r\}$:

$$\{B_{a,i} : a \in \mathbb{Z}_p - \{0\}\} = \{B_{a,j} : a \in \mathbb{Z}_p - \{0\}\}.$$

**Lemma 3.8.**

$$\sum_{j=1}^{s} \prod_{i=1}^{r} a_{\pi_i(j)} \leq \sum_{j=1}^{s} a_j,$$
where, $\forall i = 1, ..., r, \forall j = 1, ..., s$, $\pi_i$ is a permutation of $\{1, ..., s\}$ and $a_j \geq 0$.

By utilizing the formulas (9), (10), (13) and the lemmas 3.7 and 3.8, we have:

$$
\sum_{\forall h, S} |\hat{P}_n(\alpha)|^2 \leq \prod_{h=1}^{S} \sum_{a \in \mathbb{Z}_p-\{0\}} \prod_{j \in M_{h, S}} g_{h, S}(\xi_{j, 0, h, S}) \leq \prod_{h=1}^{S} \sum_{a \in \mathbb{Z}_p-\{0\}} b_{h, S}^{C(M_{h, S})} \\
\leq \frac{1}{b_S} \prod_{h=1}^{S} \sum_{a \in \mathbb{Z}_p-\{0\}} \prod_{i=1}^{r} f_{h, S}^{A(B_{a, i})} \quad \text{(where } b_S = \prod_{h=1}^{S} b_{h, S}, \ f_{h, S} = \sqrt{b_{h, S}} < 1) \\
\leq \frac{1}{b_S} \prod_{h=1}^{S} \sum_{l \in \{0\}} f_{h, S}^{rA(B_{a, 1})} \leq \frac{1}{b_S} \prod_{h=1}^{S} \sum_{A(B) > 0} f_{h, S}^{rA(B)}. \quad \text{(14)}
$$

Indicate with $M(j)$ the number of blocks of length $t$ with $A(B) = j$; then:

$$
M(j) \leq \frac{t-1}{j} \sigma_d^{j+1} \leq \binom{t}{j} \sigma_d^{j+1}
$$

$$
\Rightarrow \sum_{\text{length } B=t, A(B)>0} f_{h, S}^{rA(B)} \leq \sum_{j=1}^{t} M(j) f_{h, S}^{rj} \leq \sigma_d \sum_{j=1}^{t} \binom{t}{j} (\sigma_d f_{h, S})^j
$$

$$
\leq \sigma_d \left( (1 + \sigma_d f_{h, S})^t - 1 \right) \leq \sigma_d \left( \exp(\sigma_d f_{h, S}) - 1 \right). \quad \text{(15)}
$$

Suppose $c_{h, S} > -\frac{\ln |S|}{\ln f_{h, S} \ln \sigma_d}$, $n \geq c_{h, S} \ln p \ln \ln p$. Since $t = \left\lceil \frac{\ln p}{\ln \sigma_d} \right\rceil \geq \frac{\ln p}{\ln \sigma_d} + 1$, there exists $\tilde{c}_{h, S} \in \left( \frac{\ln |S|}{\ln f_{h, S} \ln \sigma_d}, c_{h, S} \right)$ such that, for $p$ sufficiently large:

$$
r = \left\lceil \left( \frac{\ln |S|}{\ln f_{h, S} \ln \sigma_d} - \tilde{c}_{h, S} \right) / t \right\rceil \geq \frac{\ln \sigma_d c_{h, S}}{|S|} \ln \ln p, \quad t \leq \frac{2 \ln p}{\ln \sigma_d}
$$

$$
\Rightarrow \sigma_d f_{h, S} \leq 2\sigma_d \exp \left( \frac{\ln f_{h, S} \ln \sigma_d \tilde{c}_{h, S}}{|S|} \right) \ln \ln p \equiv \gamma_{h, S}(p),
$$

where $\lim_{p \to \infty} \gamma_{h, S}(p) = 0$, by definition of $\tilde{c}_{h, S}$. Finally:

$$
\text{(15)} \leq \sigma_d \left( \exp(\gamma_{h, S}(p)) - 1 \right) \equiv \varepsilon_{h, S}(p), \quad \text{where } \lim_{p \to \infty} \varepsilon_{h, S}(p) = 0.
$$

From the formulas (2), (6), (14) and (15), we obtain the thesis of the theorem, with $c = \max h_j c_{h, S}$. $\square$
Theorem 3.9. Assume that $A$ has eigenvalues $\lambda_1, \ldots, \lambda_d, \ldots, \lambda_k \in \mathbb{C}^*$ such that $\lambda_i^l \in \mathbb{N}^*$ for some $l_i \in \mathbb{N}^*$, for any $i = 1, \ldots, k$, where $|\lambda_1| = 1$ and $|\lambda_i| > 1 \ \forall \ i = 2, \ldots, d$, and assume that the support of $\mu$ is not parallel to a proper subspace of $Q^k$ invariant under $A$. Then, there exist $\alpha, c \in \mathbb{R}^+$ and $N \in \mathbb{N}$ such that, for all $p \in \mathbb{N}$ such that $p > N$, $\gcd(\det(A), p) = \gcd(\det(B), p) = 1$, and for all $n \geq cp^2$, we have:

$$\|P_n - U\| \leq 2^{k-1} \exp\left(-\frac{\alpha n}{p^2}\right).$$

Proof. Define $l, C$ and $\sigma_i, \forall \ i = 1, \ldots, k$, as in (4); then, $d \prod_{i=1}^{d} (tC - \sigma_i I) = 0$. Suppose $d > 1$ and define, $\forall \ \{1\} \subset E \subset \{1, \ldots, d\}$:

$$Z = \left\{ \alpha \in \mathbb{Z}_p^k - \{0\} : (tC - I)\alpha \neq 0 \ (\text{mod} \ p) \right\},$$

$$Z_E = \left\{ \alpha \in Z : \prod_{i \in E} (tC - \sigma_i I)\alpha \neq 0 \ (\text{mod} \ p), \right\}$$

The following relation is analogous to (5):

$$Z = \bigcup_{\{1\} \subset E \subset \{1, \ldots, d\}} Z_E.$$  \hspace{1cm} (16)

Then:

$$\sum_{\alpha \in \mathbb{Z}_p^k - \{0\}} |\hat{P}_n(\alpha)|^2 = \sum_{\{1\} \subset E \subset \{1, \ldots, d\}} \sum_{Z_E} |\hat{P}_n(\alpha)|^2 + \sum_{Z^c} |\hat{P}_n(\alpha)|^2. \hspace{1cm} (17)$$

Analogously to the proof of Theorem 3.4, where we have valued $\sum_{Y_{S,E}} |\hat{P}_n(\alpha)|^2$, by utilizing the fact that $\sigma_i > 1, \forall \ i = 2, \ldots, d$, we can prove that $\exists \ c_1, c_2 \in \mathbb{R}^+$ such that:

$$\sum_{\{1\} \subset E \subset \{1, \ldots, d\}} \sum_{Z_E} |\hat{P}_n(\alpha)|^2 \leq \varepsilon_1(p, n), \ \text{where} \ \varepsilon_1(p, n) = c_1 \ln p \exp\left(-\frac{n}{c_2 \ln p}\right). \hspace{1cm} (18)$$

Indeed, we must estimate only the sum $\sum_{Z^c} |\hat{P}_n(\alpha)|^2$. Observe that

$$Z^c = \bigcup_{\emptyset \neq S \subset \{1, \ldots, k\}} \mathbb{Z}_S,$$
where, if the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_k$ are defined by Lemma 2.4:

$Z_S = \{ \alpha \in \mathbb{Z}^c : \langle \mathbf{y}_m, \alpha \rangle \neq 0 \pmod{p}, \, \forall \, m \in S \}, \langle \mathbf{y}_m, \alpha \rangle = 0 \pmod{p}, \, \forall \, m \notin S \}.

Then:

$$
\sum_{\mathbb{Z}^c} |\hat{P}_n(\alpha)|^2 = \sum_{\emptyset \neq S \subset \{1, \ldots, k\}} \sum_{\mathbb{Z}^S} |\hat{P}_n(\alpha)|^2. \tag{19}
$$

If $\emptyset \neq S \subset \{1, \ldots, k\}$ and $h \in \{1, \ldots, |S|\}$, define $\mathbf{x}_{h,S}, \mathbf{y}_{h,S}, \mathbf{z}_{h,S}, \mathbf{u}_{h,S}$ and $\mathbf{v}_{h,S}$ as in the proof of Theorem 3.4. Moreover, set:

$$a_{h,S} = \left \langle \mathbf{y}_{h,S}, \alpha \right \rangle.$$

Observe that, $\forall \, j \in \mathbb{N}$ and $\forall \, \alpha \in \mathbb{Z}^c$, we have:

$$\left \langle \mathbf{y}_{h,S}, t^j \right \rangle = a_{h,S} \pmod{p}.$$

Then, from the formula (7), $\forall \, \alpha \in \mathbb{Z}^c$:

$$|\hat{P}_n(\alpha)| \leq \left \lfloor \frac{n - 1}{|S|} \right \rfloor - \mathbf{z}_{h,S} \geq n - 1 - (d - 1) \geq \frac{n - (dk + 1)}{lk}. \quad \text{Then:}
$$

$$\sum_{\mathbb{Z}^S} |\hat{P}_n(\alpha)|^2 \leq \sum_{a_{h,S} \in \mathbb{Z}_p - \{0\}} \prod_{h=1}^{|S|} \left ( 1 - 2\mu(\mathbf{u}_{h,S})\mu(\mathbf{v}_{h,S}) + 2\mu(\mathbf{u}_{h,S})\mu(\mathbf{v}_{h,S}) \cos \left ( \frac{2\pi}{p} a_{h,S} \right ) \right )^{M_{h,S}}, \tag{20}
$$

where $|M_{h,S}| = \left \lfloor \frac{n - 1}{|S|} \right \rfloor - \mathbf{z}_{h,S} \geq n - 1 - (d - 1) \geq \frac{n - (dk + 1)}{lk}$. Then:

$$\sum_{\mathbb{Z}^S} |\hat{P}_n(\alpha)|^2 \leq \sum_{a_{h,S} \in \mathbb{Z}_p - \{0\}} \prod_{h=1}^{|S|} \left ( 1 - 2\mu(\mathbf{u}_{h,S})\mu(\mathbf{v}_{h,S}) + 2\mu(\mathbf{u}_{h,S})\mu(\mathbf{v}_{h,S}) \cos \left ( \frac{2\pi}{p} a_{h,S} \right ) \right )^{M_{h,S}} \leq \prod_{h=1}^{|S|} \sum_{a_{h,S} \in \mathbb{Z}_p - \{0\}} \left ( 1 - 2\mu(\mathbf{u}_{h,S})\mu(\mathbf{v}_{h,S}) + 2\mu(\mathbf{u}_{h,S})\mu(\mathbf{v}_{h,S}) \cos \left ( \frac{2\pi}{p} a_{h,S} \right ) \right )^{M_{h,S}}. \tag{21}
$$

Note that $-1 + \cos x \leq -\frac{2}{p^2} x^2, \, \forall \, x \in [-\pi, \pi]$. Furthermore, if $a_{h,S} \in \mathbb{Z}_p - \{0\}$, we can suppose:

$$a_{h,S} \in \mathbb{Z}^* \cap \left [ -\frac{p - 1}{2}, \frac{p - 1}{2} \right ] \Rightarrow \frac{2\pi}{p} a_{h,S} \in [-\pi, \pi].$$
Then, $\forall h = 1, \ldots, |S|$: 
\[
\sum_{a_{h,S} \in \mathbb{Z}_p - \{0\}} \left( 1 - 2\mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S}) + 2\mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S}) \cos \left( \frac{2\pi}{p} a_{h,S} \right) \right)^{|M_{h,S}|} \\
\leq 2 \sum_{a_{h,S} \in (\mathbb{Z} \cap [1, \frac{p}{2}])} \left( 1 - \frac{16}{p^2} \mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S}) a_{h,S}^2 \right)^{|M_{h,S}|} \\
\leq 2 \sum_{a_{h,S} \in \mathbb{N}^*} \exp \left( -\frac{16}{lk^p \overline{p}^2} \mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S})(n - (dlk + 1)) a_{h,S}^2 \right). 
\] (22)

Let $\tau \in (0, 1)$ such that $2\tau^3 + \tau^2 = 1$ ($\Leftrightarrow \frac{\tau^3}{1 - \tau^2} = \frac{1}{2}$) and set:
\[
t_{h,S} = \exp \left( -\frac{16}{lk^p \overline{p}^2} \mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S})(n - (dlk + 1)) \right), \quad \overline{c} = \max_{h,S} \frac{-lk\tau^n}{16\mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S})}.
\]

Let $c > \tau$, $p$ sufficiently large and $n \geq cp^2$; then:
\[
n - (dlk + 1) \geq \overline{c}p^2 \Rightarrow t_{h,S} \leq \tau,
\]
from which
\[
(22) = 2 \left( \sum_{a_{h,S} \in \mathbb{N}^*} t_{h,S}^2 \right) = 2 \left( t_{h,S} + \sum_{a_{h,S} \geq 2} t_{h,S}^2 \right) \leq 2 \left( t_{h,S} + \sum_{a_{h,S} \geq 2} a_{h,S}^2 \right) \\
= 2 \left( t_{h,S} + \frac{t_{h,S}}{1 - t_{h,S}} \right) \leq 2t_{h,S} \left( 1 + \frac{\tau^3}{1 - \tau^2} \right) = 3t_{h,S},
\]
by definition of $\tau$. Then:
\[
\sum_{a_{h,S} \in \mathbb{Z}_p - \{0\}} \left( 1 - 2\mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S}) + 2\mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S}) \cos \left( \frac{2\pi}{p} a_{h,S} \right) \right)^{|M_{h,S}|} \\
\leq 3\exp \left( -\frac{16}{lk^p \overline{p}^2} \mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S})(n - (dlk + 1)) \right)
\]

\[
\Rightarrow \prod_{h=1}^{|S|} \sum_{a_{h,S} \in \mathbb{Z}_p - \{0\}} \left( 1 - 2\mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S}) + 2\mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S}) \cos \left( \frac{2\pi}{p} a_{h,S} \right) \right)^{|M_{h,S}|} \\
\leq 3^{|S|} \exp \left( -\frac{2\alpha_S(n - (dlk + 1))}{p^2} \right), \quad \text{where} \quad \alpha_S = \frac{8}{lk} \sum_{h=1}^{|S|} \mu(\overline{u}_{h,S})\mu(\overline{v}_{h,S}).
\]
Moreover, from (19) and (21):

\[ \sum_{|S|} |\hat{P}_n(\alpha)|^2 \leq \exp\left( -\frac{2\alpha(n-(dlk+1))}{p^2} \right) \sum_{|S|=1}^k \binom{k}{|S|} 3^{|S|} \]

\[ = (4^k - 1) \exp\left( -\frac{2\alpha(n-(dlk+1))}{p^2} \right), \]

where \( \alpha = \min_{\emptyset \subset S \subset \{1, \ldots, k\}} \alpha_S = \frac{8}{\pi} \min_{h,S} \mu(h,S) \mu(v,h,S) = \frac{-\ln \tau}{c}. \) Then, from (2), (17) and (18):

\[ \|P_n - U\|^2 \leq \frac{1}{4} \left( \varepsilon_1(p,n) + (4^k - 1) \exp\left( -\frac{2\alpha(n-(dlk+1))}{p^2} \right) \right). \]

Observe that \( \lim_{p \to \infty} \frac{2\alpha(dlk+1)}{p^2} = 0 \) and, if \( n \geq cp^2, \lim_{p \to \infty} \exp\left( -\frac{2\alpha(n)}{p^2} \right) = 0. \) Then, if \( p \) is sufficiently large:

\[ \|P_n - U\|^2 \leq \frac{1}{4} \left( 4^k \exp\left( -\frac{2\alpha n}{p^2} \right) \right), \]

from which

\[ \|P_n - U\| \leq \frac{\eta(p)}{2}, \]

Theorem 3.10. Suppose that the matrix \( A \) has an eigenvalue \( \lambda \in \mathbb{C}, |\lambda| > 1 \) (hence, the matrix \( t^A \) too), that the support of \( \mu \) is not parallel to a proper subspace of \( \mathbb{Q}^k \) invariant under \( A \) and that \( \|B_n\|_\infty \in L^2 \), for all \( n \in \mathbb{N} \). Then, there exist \( c \in \mathbb{R}^+ \) and \( N \in \mathbb{N} \) such that, for all \( p \in \mathbb{N} \) such that \( p > N \), \( \gcd(\det(A),p) = \gcd(\det(B),p) = 1 \), and for all \( n \leq c \ln p \), we have:

\[ \|P_n - U\| \geq \frac{1}{2} \eta(p), \quad \text{where} \quad \lim_{p \to \infty} \eta(p) = 1. \]

Consequently, \( O(\ln p) \) steps are needed to reach the uniform distribution.

Proof. If \( t^A \) has an eigenvalue \( \lambda \in \mathbb{C}, |\lambda| > 1 \), then:

\[ \|t^A\|_\infty = \sup_{x \in \mathbb{C}^k - \{0\}} \frac{\|t^A x\|_\infty}{\|x\|_\infty} \geq |\lambda| > 1. \]

Let \( \alpha \in \mathbb{Z}_p^k - \{0\} \); from the lemmas 2.2 and 2.3:

\[ \|P_n - U\| \geq \frac{1}{2} \left| \hat{P}_n(\alpha) \right| \]

\[ = \frac{1}{2} \prod_{j=0}^{n-1} \left( \sum_{h,i \in \mathbb{Z}_p^k} \mu(h) \mu(i) \cos \left( \frac{2\pi}{p} \langle h - i, t^A \alpha \rangle \right) \right)^{1/2}. \]
Since \( \cos x \geq 1 - \frac{x^2}{2} \forall x \in \mathbb{R} \), we have:

\[
\|P_n - U\| \geq \frac{1}{2} \prod_{j=0}^{n-1} \left( 1 - \frac{\rho \|tA\|_{2j}^2}{p^2} \right)^{1/2}, \quad \text{where } \rho = 2\pi^2 k^2 \|\alpha\|_\infty^2 \sum_{h,i \in \mathbb{Z}^k} \mu(h) \mu(i) \|h - i\|_\infty^2 \in \mathbb{R}^+.
\]

(23)

Moreover, \( \exists d \in \mathbb{R}^+ \) such that \( 1 - x \geq \exp(-2x) \), \( \forall x \in [0, d] \).

Suppose \( n \leq c \ln p \), where \( c < \frac{1}{\ln \|tA\|_\infty} \). Then, \( \forall j = 0, 1, \ldots, n - 1 \):

\[
\|tA\|_\infty^{2j} < \|tA\|_\infty^n \leq \|tA\|_{c\ln p} = p^{c\ln\|tA\|_\infty} \Rightarrow \frac{\|tA\|_{2j}^2}{p^2} < p^{2(c\ln\|tA\|_\infty - 1)}.
\]

Since \( \lim_{p \to \infty} p^{2(c\ln\|tA\|_\infty - 1)} = 0 \), for sufficiently large \( p \) we can suppose \( p\|tA\|_\infty^{2j} \in [0, d] \); hence:

\[
\|P_n - U\| \geq \frac{1}{2} \prod_{j=0}^{n-1} \exp \left( -\frac{\rho \|tA\|_{2j}^2}{p^2} \right) = \frac{1}{2} \exp \left( -\frac{\rho}{p^2} \sum_{j=0}^{n-1} (\|tA\|_\infty^2)^j \right)
\]

\[
= \frac{1}{2} \exp \left( -\frac{\rho}{\|tA\|_\infty^2 - 1} \cdot \frac{\|tA\|_\infty^{2n} - 1}{p^2} \right)
\]

\[
\geq \frac{1}{2} \exp \left( -\frac{\rho}{\|tA\|_\infty^2 - 1} \cdot p^{2(c\ln\|tA\|_\infty - 1)} + \frac{\rho}{\|tA\|_\infty^2 - 1} \right) \equiv \frac{1}{2} \eta(p).
\]

By definition of \( c \), we have the thesis. \( \square \)

**Theorem 3.11.** Suppose that the matrix \( A \) has an eigenvalue \( \lambda \in \mathbb{C} \) such that \( \lambda^l = 1 \), for some \( l \in \mathbb{N}^\ast \) (hence, the matrix \( t^\ast A \) too), that the support of \( \mu \) is not parallel to a proper subspace of \( \mathbb{Q}^k \) invariant under \( A \) and that \( \|B_n\|_\infty \in L^2 \), for all \( n \in \mathbb{N} \). Then, there exist \( \gamma \in \mathbb{R}^+ \) and \( N \in \mathbb{N} \) such that, for all \( p \in \mathbb{N} \) such that \( p > N \), \( \gcd(\det(A), p) = \gcd(\det(B), p) = 1 \), we have:

\[
\|P_n - U\| \geq \frac{1}{2} \exp \left( -\frac{\gamma n}{p^2} \right).
\]

Consequently, \( O(\frac{p^2}{\gamma}) \) steps are needed to reach the uniform distribution.

**Proof.** The assumption on \( \lambda \) implies \( t^\ast A^l x = x \), for some \( x \in \mathbb{C}^k - \{0\} \), and so \( (t^\ast A^l - I)x = 0 \), which implies \( x \in \mathbb{Q}^k - \{0\} \); then \( \exists \alpha \in \mathbb{Z}^k - \{0\} \) such that \( t^\ast A^l \alpha = \alpha \).
∀ \( p > \| \alpha \|_\infty \), we can suppose \( \alpha \in \mathbb{Z}^k - \{0\}, \; ^tA^i \alpha = \alpha \pmod{p} \); then, \( \forall \; j \in \mathbb{N}, \; \exists \; i \in \{0, 1, ..., l - 1\} \) such that \( ^tA^i \alpha = ^tA^j \alpha \pmod{p} \).

By proceeding as in the proof of the previous theorem, we obtain the following formula, analogous to (23):

\[
\|P_n - U\| \geq \frac{1}{2} \left(1 - \frac{\gamma}{p^2}\right)^{n/2},
\]

where \( \gamma = 2\pi^2 k^2 \| \alpha \|_\infty^2 \max_{i \in \{0, 1, ..., l - 1\}} \| ^tA \|_\infty^{2i} \sum_{h, i \in \mathbb{Z}^k} \mu(h) \mu(i) \| h - i \|_\infty^2 \in \mathbb{R}^+ \). Finally, \( \forall \; p \) sufficiently large:

\[
\|P_n - U\| \geq \frac{1}{2} \exp \left(-\frac{\gamma n}{p^2}\right). \quad \square
\]

4 Problems for further study

A natural problem to study is the generalization of the recursion (11) to the analogous recursion in \( \mathbb{R}^k \) reduced modulo \( c \), for some real number \( c \). In this case, the idea is to use the Fourier transform on \( \mathbb{R}^k \) instead of in \( \mathbb{Z}^k \) and then generalize the lemmas in Section 2. The expectation is to prove the convergence in law of the Markov chain and to estimate the rate of convergence to the uniform distribution on some subset of \( \mathbb{R}^k \): the set where the chain ranges. However, this set can be different from \( \mathbb{R}^k \pmod{c} \) and it can be also countable. In order to establish it and to develop the theory, some changes of the results in Section 2 are needed: for example, Lemma 2.1 (upper bound lemma) is not valid in the continuous context and it must be modified. Conversely, results as Lemma 2.5 seem useful also in the study of the high powers of the real matrix \( A \) in the modified recursion (1).

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