Universal persistent currents in mesoscopic metal rings
due to long-range Coulomb interactions

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Abstract

We calculate the average persistent current in a mesoscopic metal ring threaded by a magnetic flux in the diffusive regime. It is shown that the classical electromagnetic energy leads to a universal average current of the order of \(\alpha e c / C_0\), where \(\alpha\) is the fine structure constant, \(-e\) is the charge of the electron, \(c\) is the velocity of light, and \(C_0\) is the classical capacitance of the ring. Striking similarities between persistent currents and universal conductance fluctuations are discovered. We suggest a simple experiment to test our theory.

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The experimental confirmation \[1,2\] of the existence of persistent currents in mesoscopic normal-metal rings threaded by a magnetic flux $\phi$ has stimulated many recent theoretical works \[3-8\]. However, a satisfactory explanation for the experiments has so far not been found. In all theoretical models the predicted current is smaller than the experimentally measured one. In the present work we shall resolve this discrepancy between theory and experiment. We show that in the diffusive regime the energy associated with long-wavelength and low-energy charge fluctuations is determined by classical charging energies of suitably defined capacitors. The flux dependence of these energies yields the dominant contribution to the persistent current. Moreover, we show that the average current is universal precisely in the same sense as the average variance of the conductance \[9\].

This article is divided into two parts. We first study the probability distribution of long-wavelength and low-energy density fluctuations in a disordered metal ring in the diffusive regime. Combining the insight gained from the first part with simple semiclassical arguments, we then derive the universal average current and compare it with experiments.

Consider a thin cylindrical metal ring with circumference $L$ and cross section $L^2_\perp$. The electrons inside the ring interact with the Coulomb potential, and are elastically scattered by impurities. The elastic mean free path $\ell$ is assumed to satisfy $L \gg \ell \geq L_\perp$. For a proper description of Coulomb effects it is crucial to take the neutralizing positive background charge density $\rho_+(r)$ of the ions into account. For our purpose it is sufficient to approximate $\rho_+(r)$ by a constant $\rho_+$ for $r$ inside the ring, and set $\rho_+(r) = 0$ if $r$ is outside the ring. Charge neutrality requires that the average electronic density equals the density of the positive background charge, $\overline{\rho(r)} = \rho_+(r)$, where the overbar denotes average over the disorder. In terms of the exact retarded/advanced Greens functions $G^{R/A}_\sigma(r, r', \epsilon)$ of spin-$\sigma$ electrons, the true electronic density at temperature $T$ is, for a given realization of the disorder, given by $\rho(r) = \int_{-\infty}^{\infty} d\epsilon f(\epsilon) \rho(r, \epsilon)$, where $f(\epsilon) = \left[e^{\epsilon/T} + 1\right]^{-1}$, and

$$\rho(r, \epsilon) = \frac{1}{2\pi i} \sum_{\sigma = \uparrow \downarrow} \left[G^A_\sigma(r, r, \epsilon) - G^R_\sigma(r, r, \epsilon)\right]. \quad (1)$$

We are interested in the probability distribution of the dimensionless random variables
\[ N_k = \int d\mathbf{r} e^{ik\theta} \int_{-E_r}^{E_r} d\epsilon f(\epsilon)\rho(\mathbf{r}, \epsilon), \quad k = 1, 2, \ldots, \]  

(2)

where the \( \mathbf{r} \)-integration is over the volume \( V = LL_\perp^2 \) of the ring, and \( \tan \theta = y/x \). We have chosen the coordinate system such that the ring lies in the \( xy \)-plane, with the origin at the center of the ring. The cutoff energy \( E_r \) is large compared with the Thouless energy \( E_c = \hbar D/L^2 \), but small compared with \( \hbar/\tau \). Here \( D \) is the diffusion coefficient, and \( \tau \) is the elastic lifetime. The \( N_k \) are the Fourier components of the electronic density due to states within an interval of width \( 2E_r \) around the Fermi energy. Below we show that only these states are responsible for the flux dependence of the energy. Because \( \overline{\rho(\mathbf{r}, \epsilon)} \) does not depend on \( \mathbf{r} \), the averages \( \overline{N_k} \) vanish for \( k \neq 0 \). More interesting are the variances

\[ P_k = \overline{\delta N_k \delta N_{-k}}, \]

(3)

where \( \delta X = X - \overline{X} \) for any random variable \( X \). We first calculate

\[ P(\mathbf{r}, \mathbf{r}', \epsilon, \epsilon') = \overline{\delta \rho(\mathbf{r}, \epsilon) \delta \rho(\mathbf{r}', \epsilon')}. \]

(4)

The \( P_k \) can then be obtained from Eqs.2 and 3. The essential observation is now that \( P(\mathbf{r}, \mathbf{r}', \epsilon, \epsilon') \) is for \( |\mathbf{r} - \mathbf{r}'| \gg \ell \) and \( |\epsilon|, |\epsilon'| \ll \hbar/\tau \) universal, and can be calculated quite reliably within a model of non-interacting electrons [10]. The behavior of this function is determined by the fact that, at large distances and low energies, density fluctuations in a disordered metal are governed by the diffusion equation. As long as inelastic processes do not drastically modify the diffusive behavior, their effect can be approximately taken into account on a phenomenological level by introducing a finite phase breaking energy in the cooperon- and diffuson propagators (see Eq.6 below) [10,4]. To justify this more rigorously, let us consider the diagrammatic calculation of \( P(\mathbf{r}, \mathbf{r}', \epsilon, \epsilon') \). A general diagram consists of two closed loops representing the exact spectral densities \( \rho(\mathbf{r}, \epsilon) \) and \( \rho(\mathbf{r}', \epsilon') \), which are connected in all possible ways by static impurity lines. Note that the product of the averages is subtracted in Eq.4, so that the loops must be connected together by at least one impurity line. The dominant flux dependent diagram in the absence of interactions is shown in Fig.4.
In the interacting system, Coulomb lines are not permitted to connect the loops. Precisely the same diagrammatic rules are encountered in the theory of conductance fluctuations [9]. The similarity of Fig. 1 with the diagrams shown in Fig. 5 of Lee et al. [9] is evident. Diagrams corresponding to interaction corrections to \( P(\mathbf{r}, \mathbf{r}', \epsilon, \epsilon') \) can be obtained from the diagrams shown in Figs. 9 and 10 of Ref. [9] by simply omitting the current vertices. The contribution of these diagrams to \( P(\mathbf{r}, \mathbf{r}', \epsilon, \epsilon') \) is negligible for exactly the same reasons as discussed in Ref. [9]. The close connection between fluctuations of the conductance and the density of states has also been pointed out in Ref. [10].

It follows that a reasonable approximation for the function \( P(\mathbf{r}, \mathbf{r}', \epsilon, \epsilon') \) in the presence of interactions can be obtained by performing the calculations for free electrons, and adding a phase breaking energy \( \Gamma \) to the poles of the cooperons and diffusons. The dominant flux dependent diagram is shown in Fig. 1. If \( \epsilon \) and \( \epsilon' \) approach the real axis from opposite sides, this diagram yields

\[
P(\mathbf{r}, \mathbf{r}', \epsilon, \epsilon') = 4[W(\mathbf{r}, \mathbf{r}', \epsilon - \epsilon')]^2, \tag{5}
\]

where \([\hbar^2/(2\pi\nu^2)]W(\mathbf{r}, \mathbf{r}', \epsilon)\) is the usual cooperon propagator. (\( \nu \) is the density of states at the Fermi energy for one spin species; the factor of 4 takes into account the spin degeneracy.) There exists a similar diagram involving two diffusons, which does not depend on the flux, and has been ignored in Eq. (5). It is understood that the \( P_k \) calculated below contain only the contribution from the cooperon channel. The Fourier transform of \( W(\mathbf{r}, \mathbf{r}', \epsilon) \) is for \( k \ll L/\ell \) and \( |\epsilon| \ll E_\tau \) given by

\[
\frac{1}{V} \int d\mathbf{r} \int d\mathbf{r}' e^{ik(\theta - \theta')}W(\mathbf{r}, \mathbf{r}', \epsilon) = \left[4\pi^2E_c(k + 2\varphi)^2 - i\epsilon + \Gamma\right]^{-1}, \tag{6}
\]

where the energy \( \Gamma \) models inelastic processes, such as Coulomb- and electron-phonon interactions [4, 10]. We have defined the dimensionless flux \( \varphi = \phi/\phi_0 \), where \( \phi_0 = h\epsilon/e \) is the flux quantum. The real time Fourier transform of \( W(\mathbf{r}, \mathbf{r}', \epsilon) \) is precisely the quasi-probability \( \tilde{W}_t(\mathbf{r}, \mathbf{r}') \) introduced in the semiclassical theory of weak localization by Chakravarty and
The origin of $\Gamma$ is most transparent in this approach: inelastic processes destroy the phase coherence between time reversed paths, and are taken into account phenomenologically by multiplying the quasi-probabilities of the non-interacting system by a factor of $e^{-\Gamma t/ \hbar}$.

Given Eqs. 5 and 6, it is straightforward to calculate the coefficients $P_k$ defined in Eq. 3. We find

$$ P_k = \frac{2}{\pi^2} \sum_{m=-\infty}^{\infty} \int_{0}^{E_r} d\epsilon \frac{\epsilon \coth [\epsilon/(2T)]}{4\pi^2 E_c (m + 2\varphi)^2 + \epsilon + \Gamma} \times \frac{1}{4\pi^2 E_c (m - k + 2\varphi)^2 + \epsilon + \Gamma}. \quad (7) $$

Evidently the $P_k$ are even periodic functions of $\varphi$ with period $1/2$, so that we may expand

$$ P_k = \frac{P_k^{(0)}}{2} + \sum_{m=1}^{\infty} P_k^{(m)} \cos (4\pi m \varphi). \quad (8) $$

After a simple contour integration we obtain

$$ P_k^{(m)} = \frac{1}{\pi^2} \int_{0}^{E_r/E_c} dx \frac{x^{1/2}}{(\pi k)^2 + x + \Gamma/E_c} \times \exp \left[ -m \frac{x + \Gamma/E_c}{1} \right] \coth \left[ \frac{x E_c}{(2T)} \right]. \quad (9) $$

Note that the $P_k^{(0)}$ are for $(\pi k)^2 + \Gamma/E_c \ll E_r/E_c$ proportional to $(E_r/E_c)^{1/2} = O(L/\ell)$. The cutoff dependence of all non-zero Fourier coefficients $P_k^{(m)}$, $m \geq 1$, is exponentially small for large $L/\ell$, so that we may let $E_r \to \infty$. In the limit $T, \Gamma \ll E_c$, the $P_k^{(m)}$ are for $m \geq 1$ universal numbers, determined by the shape of the system. It is important to stress that Eq. 8 includes the effect of inelastic processes, which are, however, negligible if the effective inelastic energy $\Gamma$ is small compared with $E_c$.

We now develop our semiclassical theory of persistent currents. Consider first the charge distribution on the ring associated with the first harmonic $N_1$ of the density fluctuation. Although $\overline{N_1} = 0$, the typical excess charge on one side of the ring is $e(\overline{N_1^2})^{1/2} = e(P_1)^{1/2} \propto e(L/\ell)^{1/2}$. Because by assumption $(L/\ell)^{1/2} \gg 1$, the associated electrostatic energy can be calculated classically, and is given by $e^2 P_1/(2C_1)$, where $C_1$ is the capacitance of a thin ring consisting of two oppositely charged halves. The contribution of modes with wavelengths
large compared with $\ell$ to the disorder averaged Hartree energy is simply the sum of the corresponding classical charging energies,

$$
\overline{\Omega}_\phi = \sum_{k=1}^{\Lambda} \frac{e^2}{2C_k} P_k ,
$$

(10)

where the integer $\Lambda$ is chosen such that $1 \ll \Lambda \ll L/\ell$, and the generalized capacitances $C_k$ are

$$
\frac{1}{C_k} = \frac{1}{V^2} \int dr \int dr' \frac{e^{ik(\theta - \theta')}}{|r - r'|} .
$$

(11)

The sum in Eq.(10) does not include the term $k = 0$, because $\int dr \delta\rho(r) = 0$ by charge neutrality. For $k \ll L/L_\perp$ we may approximate to leading logarithmic order $C_k \approx C_0 \approx L/[2 \ln(L/L_\perp)]$, which is the classical capacitance of the ring. The identification of Eq.(11) with classical capacitances is only valid for $L_\perp \ll L$. The possibility that charging energies could lead to large persistent currents has already been noticed by Imry and Altshuler [13]. Although they examined only the zero mode $C_0$, Imry speculated that local charge fluctuations in a globally neutral system might be the key to understand persistent currents.

Eq.(10) can also be derived formally. A rigorous theorem due to Hohenberg and Kohn [12] tells us that the exact Hartree energy of all positive and negative charges is a functional of the true electronic density $\rho(r)$,

$$
\Omega = \frac{e^2}{2} \int dr \int dr' \frac{[\rho(r) - \rho_+][\rho(r') - \rho_+]}{|r - r'|} .
$$

(12)

Although Eq.(12) contains explicitly only the bare Coulomb interaction, all higher order interaction processes are taken into account by demanding that $\rho(r)$ is, for a given disorder potential, the true electronic density of the interacting many-body system. In an infinite homogeneous system without disorder Eq.(12) vanishes, but for a mesoscopic disordered ring $\Omega > 0$. Charge neutrality implies $\rho_+ = \rho(r)$, so that the average $\overline{\Omega}$ is uniquely determined by the variance of the density. Note the essential role of the positive background charge to subtract the product of the average densities. To isolate the flux dependent part of $\overline{\Omega}$, it is sufficient to keep only the contribution from modes with wavelengths large compared
with $\ell$, involving states close to the Fermi energy. According to Eq. 7 only these modes are sensitive to the cooperon pole. Using the definitions 2, 3, and 11, we arrive precisely at Eq. 10. The physical reason for the absence of screening in Eq. 10 is that the random potential creates in the diffusive regime typical configurations where a large number of excess electrons is localized on segments of the ring. Therefore the ring behaves as if it were a classical capacitor.

The sequence of bubble diagrams retained in the random-phase approximation does not contribute to the variances $P_k$. In Fig. 2 we show the first bubble correction to Fig. 1. Because the poles of both Greens functions associated with the loop labeled by $\epsilon'$ lie in the same half-plane, the $r_5$-integration leads to the vanishing of the diagram. Furthermore, if the inner left loop in Fig. 2 is not connected by impurity lines to the other loops, the resulting diagram is exactly cancelled by another diagram of the same form, where the inner left loop represents the density of the positive background charge. This diagram arises from the term in the Hamiltonian that represents the interaction between electronic and ionic densities. For precisely the same reasons diagrams of this type can be ignored in the theory of conductance fluctuations [9]. An essential difference between our approach and Ref. [3] is that we have included the effect of the positive background from the very beginning in a non-perturbative way. The importance of charge neutrality has also been emphasized by Schmid [4].

From Eqs. 8, 9 and 10 we obtain the Hartree contribution to the average persistent current,

$$I = -e\partial \sum_{\phi} / \partial \phi = \sum_{m=1}^{\infty} I^{(m)} \sin(4\pi m \varphi),$$  \hspace{1cm} (13)

with

$$I^{(m)} = \frac{2\alpha}{\pi^2} \sum_{k=1}^{\infty} \frac{C_k}{C_k} \int_0^\infty dx \frac{x^2}{(\pi m k)^2 + x^2 + \gamma_m} \times \exp \left[ -\left( x^2 + \gamma_m \right)^{1/2} \right] \coth \left[ x^2 \beta_m / 2 \right],$$  \hspace{1cm} (14)

where $\beta_m = E_c / (m^2 T)$, $\gamma_m = m^2 \Gamma / E_c$, and $\alpha = e^2 / (hc) \approx 1 / 137$ is the fine structure constant. The summand in Eq. 14 vanishes as $k^{-2}$ for large $k$, so that the value of the sum is dominated
by the first few terms, and we have set $\Lambda = \infty$ in Eq. 10. Eq. 14 is the main result of this work. In the limit $T, \Gamma \ll E_c$, the current is independent of any microscopic details of the metal, and depends only on fundamental physical constants and generalized capacitances $C_k$, which are completely determined by the large-distance geometry of the system. The average variance of the conductance in the diffusive regime is universal in precisely the same sense.

Note that $\Gamma \lesssim E_c$ in the experiments [1, 2]. For $\Gamma \gg E_c$ the current would be exponentially small. To estimate the magnitude of $I^{(m)}$ for $\gamma_m \ll 1$, we may neglect $x^2$ compared with $(\pi m k)^2$ in the denominator of Eq. 14, because the factor $e^{-x}$ cuts off the integration at $x = O(1)$. In this approximation we obtain

$$\frac{I^{(m)}}{I_F} \approx -\frac{8\zeta(2) \ln(L/L_\perp)}{\pi^2 m^2} \frac{c}{137 v_F} g(\beta_m),$$

where $I_F = (-e)v_F/L$, $\zeta(2) \approx 1.64$, and

$$g(\beta) = \frac{1}{2} \int_0^\infty dx x^2 e^{-x} \coth \left[ \frac{x^2 \beta}{2} \right].$$

(16)

For $T \ll E_c/m^2$ we have $g(\beta_m) \approx g(\infty) = 1$. At finite temperature, the relevant energy scale where the current begins to deviate from its value at $T = 0$ is $E_c$. The temperature dependence of Eq. 15 is in agreement with Ambegaokar and Eckern [3]. As pointed out in Ref. [3], this temperature dependence seems to fit the experiment [1]. Moreover, $I^{(m)} \propto m^{-2}$, so that also the shape of our current agrees with Ref. [3]. However, our current is two orders of magnitude larger than the current calculated in all previous works. Assuming $L/L_\perp = 100$ and $T, \Gamma \ll E_c$, we obtain for the first harmonic $I^{(1)} \approx -0.9 \times I_F$. The amazing result is that $I^{(1)}$ has the same order of magnitude as in the ballistic regime [8].

Our theory can be tested experimentally by embedding the ring in an insulator with dielectric constant $\epsilon_r$. Because capacitances increase linearly with $\epsilon_r$, the current should then be reduced by a factor of $1/\epsilon_r$. The currents measured by Chandrasekhar et al. [2] in three different isolated gold rings are of order $I_F$. Although a disorder ensemble is not appropriate for a description of this experiment, the fact that the magnitude of the measured
current agrees with our calculation might indicate that the physics underlying our theory is also relevant for the experiment of Ref. [2]. Lévy et al. [1] have measured an average current of the order of $0.003 \times I_F$. The system studied in Ref. [1] consists of an array of $10^7$ copper rings with circumference $L \approx 2.2 \mu m$, occupying a $7mm^2$ area on a sapphire substrate. Note that the dielectric constant of sapphire is $\varepsilon_r \approx 10$, while in the experiment of Ref. [2] the substrate consists of Si, which has $\varepsilon_r \approx 3$. Taking the large dielectric constant into account and assuming $T = \Gamma = 0$, our theory predicts for the experiment of Lévy et al. $I \approx -0.1 \times I_F$, a factor of 30 larger than experimentally measured. This discrepancy can have many explanations. From Eq.14 it is clear that an effective $\Gamma \approx E_c$ can easily explain a reduction of $I$ by an order of magnitude. A second possibility is that inter-ring interactions and capacitances between rings are important. Note that in the experimental configuration of Ref. [1] the average distance between neighboring rings is smaller than $L$, so that the Coulomb interaction between rings is not negligible.

Our theory is based on the Hartree energy. Quantum mechanical exchange- and correlation effects have been ignored. We have explicitly verified that the current obtained from the first order exchange correction is at least a factor of $\ell/L$ smaller than the contribution of the Hartree term. Because the mechanism leading to the large current has a simple semiclassical interpretation, we believe that quantum mechanical correlation effects do not essentially modify our result.

In this paper we have developed a semiclassical theory of persistent currents in the diffusive regime. The only quantum mechanical ingredient in our theory are the time reversed paths of weak localization. The dominant contribution to the average persistent current is universal, and can be derived from the classical electromagnetic energy of the system.

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Figure Captions

**Fig.1:** Dominant flux dependent diagram determining $P(r, r', \epsilon, \epsilon')$. Solid lines with arrow denote disorder averaged Greens functions, dashed lines denote impurity scattering, and the maximally crossed ladders define cooperons.

**Fig.2:** First order bubble correction to Fig.1. The wavy line is the bare Coulomb interaction.