A CONVERSE THEOREM FOR DEGREE 2 ELEMENTS OF THE SELBERG CLASS WITH RESTRICTED GAMMA FACTOR.

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Abstract. We prove a converse theorem for a family of $L$-functions of degree 2 with gamma factor coming from a holomorphic cuspform. We show these $L$-functions coincide with either those coming from a newform or a product of $L$-functions arising from Dirichlet characters. We require some analytic data on the Euler factors, but don’t require anything on the shape. We also suppose that the twisted $L$-functions satisfy expected functional equations. We incorporate the ideas from [3] so that the non-trivial twists are allowed to have arbitrary poles.

1. Introduction

In [2], Booker proves a version of Weil’s converse theorem without needing knowledge of the root number. We extend on his work, by not assuming a specific shape of the Euler factors. However we do still need to have some analytic data on the Euler factors, namely holomorphy and non-vanishing on and to the right of the line $\Re(s) = 1/2$. This is a natural restriction to make when looking from the perspective of the Selberg class (see [10] for details on the Selberg class). Our result also makes partial progress of extending the result of Kaczorowski, Perelli in [6] to general conductor. In their paper, they get their result for conductor 5 without any information on twists through the theory of non-linear twists of $L$-functions. Combining our theorem with a result classifying the gamma factors of degree 2 elements, we could prove a more general theorem.

Remark. In [2], Booker proves a version of Weil’s converse theorem without needing knowledge of the root number. We extend on his work, by not assuming a specific shape of the Euler factors. However we do still need to have some analytic data on the Euler factors, namely holomorphy and non-vanishing on and to the right of the line $\Re(s) = 1/2$. This is a natural restriction to make when looking from the perspective of the Selberg class (see [10] for details on the Selberg class). Our result also makes partial progress of extending the result of Kaczorowski, Perelli in [6] to general conductor. In their paper, they get their result for conductor 5 without any information on twists through the theory of non-linear twists of $L$-functions. Combining our theorem with a result classifying the gamma factors of degree 2 elements, we could prove a more general theorem.

Let $S^\text{new}_k(\Gamma_0(N), \xi)$ denote the newforms of weight $k$, level $N$ and nebentypus character $\xi$. We prove the following theorem.

Theorem 1.1. Let $\{a_n\}_{n=1}^\infty$ be a multiplicative sequence of complex numbers satisfying $a_n = O(n^\lambda)$ for some $\lambda \in \mathbb{R}_{>0}$ and such that $\sum_{j=0}^\infty a_p p^{-js}$ is analytic and non-vanishing for $\Re(s) \geq 1/2$ and every prime $p$. Fix positive integers $k, N$. For any primitive Dirichlet character $\chi$ of conductor $q$ coprime to $N$, define

$$\Lambda_\chi(s) = \Gamma_C \left( s + \frac{k - 1}{2} \right) \sum_{n=1}^\infty a_n \chi(n) n^{-s}$$

for $\Re(s) > 1 + \lambda$, where $\Gamma_C(s) = 2(2\pi)^{-s}\Gamma(s)$. Suppose, for every such $\chi$, that $\Lambda_\chi(s)$ continues to a meromorphic function on $\mathbb{C}$ and satisfies the functional equation

$$\Lambda_\chi(s) = \varepsilon_\chi(Nq^{1/2} - s) \Lambda_\chi(1 - s),$$

for some $\varepsilon_\chi \in \mathbb{C}$. Let $\textbf{1}$ denote the character of modulus 1 and suppose there is a nonzero polynomial $P$ such that $P(s)\Lambda_\textbf{1}(s)$ continues to an entire function of finite order.

Then one of the following holds:

(i) $k = 1$ and there are primitive characters $\xi_1$ (mod $N_1$) and $\xi_2$ (mod $N_2$) such that $N_1N_2 = N$ and $\sum_{n=1}^\infty a_n n^{-s} = L(s, \xi_1)L(s, \xi_2)$, where $L(s, \xi_1), L(s, \xi_2)$ are the usual Dirichlet $L$-functions.

(ii) $\sum_{n=1}^\infty a_n n^{-s} e(nz) \in S^\text{new}_k(\Gamma_0(N), \xi)$ for some Dirichlet character $\xi$ of conductor dividing $N$.

Remark. If we suppose that for every $\chi$ of conductor $q \equiv 1$ (mod $N$) that $\Lambda_\chi(s)$ is entire, then we don’t require (i) for characters of conductor not congruent to 1 (mod $N$). If $\Lambda_\chi$ is not entire then we use this extra information in Lemma 2.1.

Remark. A similar result should be true for $L$ functions with gamma factors coming from Maass forms, but would be more difficult. It requires analysis of hypergeometric functions as in [9].

From now on we shall assume the hypothesis of theorem 1.1.
2. Analysis of Euler factors.

We shall assume for now that $\Lambda_1(s)$ is entire and deal with the meromorphic case at the end. By the conditions in theorem $[11]$ have the following lemma to constrain the poles of $\Lambda_\chi$, shown in the proof of $[3]$ Theorem 1.1).

**Lemma 2.1.** The function $\Lambda_\chi(s)$ is entire of finite order for every primitive character $\chi$ of prime conductor $q \nmid N$.

For $\chi$ a primitive character of conductor $q \nmid N$, both $\Lambda_1(s)$ and $\Lambda_\chi(s)$ are entire of finite order. The Phragmén-Lindelöf convexity principle means they are bounded in vertical strips.

Let $\mathbb{H} = \{ z \in \mathbb{C} : \Im(z) > 0 \}$ denote the upper half plane. Let $e(s) := e^{2\pi is}$ for $s \in \mathbb{C}$. For $z \in \mathbb{H}$, set

$$ f_n = a_n n^{\frac{k-1}{2}}, \quad f(z) = \sum_{n=1}^{\infty} f_n e(nz), \quad \overline{f}(z) = \sum_{n=1}^{\infty} \overline{f_n} e(nz). $$

Recall the $k$-slash operator defined for any function $g : \mathbb{H} \to \mathbb{C}$ and any matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of positive determinant by

$$ (g|\gamma)(z) = (\det \gamma)^{k/2}(cz + d)^{-k} g \left( \frac{az + b}{cz + d} \right). $$

For matrices $\gamma, \gamma'$, we write $\gamma \simeq \gamma'$, if $g|\gamma = g|\gamma'$.

The functional equation $[11]$ for $\chi = 1$ and Hecke’s argument (see Theorem 4.3.5 from $[8]$) implies that $f|H_N = \epsilon_1 i^k \overline{f}$, where $H_N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $I_2$ be the identity matrix. Then since both $f$ and $\overline{f}$ are Fourier series, $P \simeq f I_2$ and $P \simeq \overline{f} I_2$.

Consider the additive twist by $\alpha$, defined by $F(s, \alpha) = \sum_{n=1}^{\infty} a_n n^{-s} e(n\alpha)$ and $\overline{F}(s, \alpha) = \sum_{n=1}^{\infty} \overline{a_n} n^{-s} e(n\alpha)$. Moreover denote $L_f(s) := F(s, 0)$ and $L_{\overline{f}}(s) := \overline{F}(s, 0)$. Let $q \equiv 1 \pmod{N}$ be a prime number. For $\chi$ a Dirichlet character mod $q$, the Gauss sum is defined by $\tau(\chi) = \sum_{\alpha=1}^{q} \chi(\alpha) e\left( \frac{\alpha^2}{q} \right)$. By Fourier analysis on $\mathbb{Z}/q\mathbb{Z}$,

$$ e\left( \frac{n}{q} \right) = 1 - \frac{q}{q-1} \chi_0(n) + \frac{1}{q-1} \sum_{\chi \pmod{q} \nmid \chi_0} \tau(\chi) \chi(n). $$

Multiplying by $a_n n^{-s}$ and summing, we get the following relationship between additive twists and multiplicative twists.

$$ F(s, \frac{1}{q}) = L_f(s) - \frac{q}{q-1} L_{\overline{f}}(s) + \frac{1}{q-1} \sum_{\chi \pmod{q} \nmid \chi_0} \tau(\overline{\chi}) F(s, \chi), $$

where $L_q(s) = \sum_{j=0}^{\infty} a_q q^{-js}$ and $F(s, \chi) = \sum_{n=1}^{\infty} a_n \chi(n)n^{-s}$.

We shall prove the following lemma.

**Lemma 2.2.** Let $q \equiv 1 \pmod{N}$ be a sufficiently large prime number. Then the function $F_q(s)^{-1}$ is a polynomial in $q^{-s}$ of degree at most 2.

Let $\Im(s) = t$. Because $F_q(s)^{-1}$ is $\frac{2\pi}{\log q}$-periodic, for the rest of this section we shall assume $t > 0$ is contained in the interval $\left[ \frac{2\pi}{\log q}, \frac{4\pi}{\log q} \right]$.

**Proof.** By Mellin transformations,

$$ F(s, \frac{1}{q}) = \int_{0}^{\infty} y^{s+\frac{1}{2}} f\left( \frac{1}{q} + iy \right) dy \overline{L_f(s)} (2\pi)^{-s} \frac{1}{\Gamma(s + \frac{1}{2}) \Gamma(s + \frac{1}{2})}. $$

This equation will allow us to meromorphically continue $F(s, \frac{1}{q})$. 

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We want to estimate \( f_0^\infty \frac{y^s + \frac{k-1}{2} - 1}{q} f\left(\frac{1}{q} + iy\right) dy \) as \( \Re(s) \to -\infty \). For \( y > 1 \), we can use the trivial estimate

\[
(4) \quad \left| f\left(\frac{1}{q} + iy\right) \right| \leq \sum_{n=1}^{\infty} |a_n| n^{k-1} e^{-2\pi ny} \ll \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} e^{-2\pi ny} \ll e^{-2\pi y}.
\]

When \( y \leq 1 \) we will use the modularity relation \( f|H_N = \varepsilon_1 f \) and the fact \( q \equiv 1 \pmod{M} \). Write \( q = MN + 1 \), where \( M \) is an integer. Then \( f|P^M H_N = \overline{\gamma} f \) since \( H_N^2 \simeq f \). We observe that,

\[
H_N P^M H_N = \begin{pmatrix} -N & 0 \\ MN^2 & -N \end{pmatrix} \cong f \begin{pmatrix} 1 & 0 \\ -MN & 1 \end{pmatrix}.
\]

Also this means, \( P^{-1} H_N P^M H_N \cong f \begin{pmatrix} q & -1 \\ 1-q & 1 \end{pmatrix} \).

Hence we get a relationship under the action of

\[
H_N P^{-1} H_N P^M H_N \cong f \begin{pmatrix} q-1 & -1 \\ Nq & -N \end{pmatrix} =: \gamma,
\]

namely

\[
\overline{\gamma} = \omega f
\]

for some \( \omega \) with \(|\omega| = 1\). This can be written explicitly in the form

\[
f(z) = \omega N^{\frac{k}{2}} (qz - 1)^{-1-k} f\left(\frac{(q-1)z - 1}{Nqz - N}\right).
\]

Using this identity, we have

\[
f\left(\frac{1}{q} + iy\right) \ll_q \left(\frac{1}{y}\right)^k \sum_{n=1}^{\infty} |a_n| n^{k-1} e^{-2\pi n q(1+O(y))}
\]

for \( y \leq 1 \) because

\[
\frac{(q-1)\left(\frac{1}{q} + iy\right) - 1}{Nq\left(\frac{1}{q} + iy\right) - N} = \frac{-1}{q^2 N (iy)} (1+O(y))
\]

for \( y \leq 1 \). Note that

\[
\sum_{n=1}^{\infty} |a_n| n^{k-1} e^{-2\pi n (1+O(y))} \ll e^{-2\pi y}.
\]

Let \( s = \sigma + it \) where \( \sigma < -\frac{k-1}{2} \). Then

\[
\int_0^{\infty} y^{s+\frac{k-1}{2}-1} f\left(\frac{1}{q} + iy\right) dy \ll_q \int_1^1 y^{\sigma+\frac{k-1}{2}-1} e^{-2\pi y} dy + \int_1^{\infty} y^{\sigma+\frac{k-1}{2}-1} e^{-2\pi y} dy
\]

\[
\ll_q \left(\frac{q^2 N}{2\pi}\right)^{|\sigma|} \int_0^{\infty} y^{-\sigma+\frac{k+1}{2}} e^{-y} dy + \frac{1}{|\sigma+\frac{k+1}{2}|}
\]

(5)

\[
\ll_q \left(\frac{q^2 N}{2\pi}\right)^{|\sigma|} \Gamma\left(-\sigma+\frac{k+1}{2}\right) + \frac{1}{|\sigma+\frac{k+1}{2}|}.
\]

The second term converges to zero as \( \sigma \to -\infty \). By Stirlings formula

\[
\left| \Gamma\left(-\sigma+\frac{k+1}{2}\right) \right| \ll \left(\frac{\sigma}{e}\right)^{|\sigma|+\frac{k}{2}}.
\]

By the functional equation (1) for \( \chi = 1 \), as \( \sigma \to -\infty \)

\[
\left| L_f(s) \Gamma(s + \frac{k-1}{2}) (2\pi)^{-s} \right| = \varepsilon_1 \frac{\sqrt{N}}{2\pi} \left(\frac{N}{2\pi}\right)^{-s} \Gamma(1-s + \frac{k-1}{2}) L_f(1-s)
\]

\[
\ll_q \left(\frac{N}{2\pi}\right)^{|\sigma|} \left(\frac{\sigma}{e}\right)^{|\sigma|+\frac{k}{2}},
\]

(6)
where we use the fact $t$ is contained in the interval $\left[\frac{2\pi}{\log q}, \frac{4\pi}{\log q}\right]$.

From (3), (5) and (6), as $\sigma \to -\infty$

$$\frac{F(s, \frac{1}{q})}{L_f(s)} \ll q^{2|\sigma|}.$$ 

Moreover, by the functional equation (1),

$$\frac{F(s, \chi)}{L_f(s)} = \frac{\Gamma(s + \frac{k-1}{2})}{\Gamma(s + \frac{k+1}{2})} \frac{F(s, \chi)}{L_f(s)} \ll q^{2|\sigma|},$$

as $\sigma \to -\infty$.

Let $E_q(q^{-s}) = \frac{1}{F_q(s)}$. Since $F_q(s) = \sum_{j=0}^{\infty} a_q q^{-j}$, we have a power series expansion for $E_q(z)$. By the hypotheses on $F_q(s)$ the radius of convergence of $E_q(z)$ is at least $q^{-1/2}$. To prove lemma 2.2, we need to show that $E_q(z)$ is a polynomial of degree $\leq 2$. We know

$$F(s, \frac{1}{q}) = \frac{(2\pi)^{\frac{k-1}{2}}}{(2\pi)^{-s}} \int_0^\infty y^{s-\frac{k-1}{2}} f(\frac{1}{q} + iy) dy$$

is entire thanks to the first line of (5) and since $\Gamma(s)$ has no zeroes. By (2), and our estimates above, $E_q(q^{-s})$ has a meromorphic continuation to $\mathbb{C}$, and satisfies

(7) 

$$E_q(q^{-s}) \ll q^{2|\sigma|}$$

uniformly in $t$ as $\sigma \to -\infty$. Thus

$$E_q(z) \ll q |z|^2$$

for $|z|$ sufficiently large. Once we have shown $E_q(z)$ is entire we have finished the proof of the lemma.

We shall show $E_q(q^{-s})$ is entire if $q$ is sufficiently large.

Thanks to our assumption on $F_q(s)$ in Theorem 1.1, $E_q(q^{-s})$ is analytic for $\Re(s) \geq 1/2$. Also, the estimate (7) means it is also analytic in some left half plane. By $t$-periodicity, for large $q$ it suffices to show $E_q(q^{-s})$ is analytic in some region of the form

$$D_K := \{ s \in \mathbb{C} : \Re(s) \in (K, \frac{1}{2}), \Im(s) \in [0,1]\},$$

where $K < \frac{1}{2}$. By Lemma 2.1, $F(s, \chi)$ is entire so the equation

$$E_q(q^{-s}) = \frac{q-1}{q} \left( 1 - \frac{F(s, \frac{1}{q})}{L_f(s)} + \frac{1}{q-1} \sum_{\chi \equiv \chi \pmod{q}} \tau(\chi) \frac{F(s, \chi)}{L_f(s)} \right)$$

implies the only possible poles of $E_q(q^{-s})$ must come from zeroes of $L_f$. There are only finitely many zeroes of $L_f$ in the region $D_K$. By $t$ periodicity, if $E_q(q^{-s})$ has a pole at $s$ in $D_K$, then $L_f$ must also have zeroes at $s + \frac{2\pi i k}{\log q}$ for $k \in \mathbb{Z}$. For large enough $q$ we would have too many zeroes of $L_f$ in the region $D_K$. Hence for large $q$, $E_q(q^{-s})$ cannot have any poles in $D_K$ as required.

$\square$

3. ANALYSIS OF THE COEFFICIENTS IN THE EULER FACTORS

We analyse the Euler factors using arguments from [2].

Let $q \equiv 1 \pmod{N}$ be a sufficiently large prime number (as in lemma 2.2), then $F_q(s)^{-1}$ can be expressed in the form $1 - \lambda q^{-s} + \mu q^{-2s}$ where $\lambda = a_q$, $\mu = a_q^2 - a_{q^2}$. Define

$$\epsilon = \begin{cases} \frac{\lambda}{\mu} & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0, \mu \neq 0, \\ 1 & \text{if } \lambda = 0, \mu = 0. \end{cases}$$

Remark. We shall show that the third case can’t happen so our choice here isn’t important.
Let \( r = 1 - e\pi \), then \( D_q(s) = r + q - 1 - q(1 - \lambda q^{-s} + \mu q^{-2s}) \) satisfies a functional equation of the form \( D_q(s) = e^{-2\pi s}D_q(1-s) \). Define \( \Lambda_{c+q}(s) = \Lambda_c(s) + r\Lambda_1(s) \), where \( \Lambda_c(s) = \gamma(s + k/2) \sum_{n=1}^{\infty} a_nc^n(n)n^{-s} \) and \( c_0(n) = \sum_{(a,q)=1} e\left(\frac{am}{q}\right) \) is a Ramanujan sum.

Since \( c_q(n) = q - 1 - q\chi_0(n) \) by Fourier analysis on \( \mathbb{Z}/q\mathbb{Z} \),

\[
D_q(s) = \frac{\Lambda_{c+q}(s)}{\Lambda_1(s)}.
\]

Hence we have the following relationship,

(8) \( \Lambda_{c+q+r}(s) = \epsilon\epsilon_1(Nq^2)^{-s}\Lambda_{c+q+r}(1-s) \).

For \( q \equiv 1 \pmod{N} \) and \( \chi \) a character mod \( q \), define

\[
f_\chi = \sum_{a \text{ (mod } q)} \chi(a)f_1 \left[ \begin{array}{cc}
1 & \frac{a}{q} \\
0 & 1
\end{array} \right] + \mathbb{1}_{\chi=\chi_0}rf \quad \text{and} \quad f_\chi = \sum_{a \text{ (mod } q)} \chi(a)f_1 \left[ \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right] + \mathbb{1}_{\chi=\chi_0}f_\tau.
\]

Also define

\[
C_\chi = \begin{cases}
\tau & \text{if } \chi \text{ is trivial,} \\
\chi(-N)e\chi_0\tau(\chi)/\tau(\chi) & \text{otherwise.}
\end{cases}
\]

By substituting the Fourier expansion of \( f \), we see \( f_\chi \) has \( n \)th Fourier coefficient

\[
f_\chi(c_q(n) + r) \quad \text{if } \chi \text{ is trivial,}
f_\chi(r\chi(\overline{\chi})) \chi(n) \quad \text{otherwise,}
\]

with a similar expression for \( f_\tau \). By (9) and the functional equation (11), Hecke’s argument implies we have the modularity relationship

(9) \( f_\chi \left| \begin{array}{cc}
0 & 1 \\
Nq^2 & -1
\end{array} \right| = \tau^k\chi(-N)e\chi_0C_\chi f_\tau \)

If \( \gamma, \gamma' \in \Gamma_0(N) \) have the same top row, then it is easy to check that \( \gamma'\gamma^{-1} \) is a power of \( \left( \begin{array}{cc}
1 & 0 \\
N & 1
\end{array} \right) \). As

\[
\left( \begin{array}{cc}
1 & 0 \\
N & 1
\end{array} \right) = H_NP^{-1}H_N^{-1},
\]

\( f|_{\gamma} \) depends only on the top row of \( \gamma \). Let \( \gamma_{q,a} \) denote any element of \( \Gamma_0(N) \) with top row \( (q, -a) \).

Let \( \gamma = \left( \begin{array}{cc}
q & -b \\
-Nm & r
\end{array} \right) \) be an arbitrary element of \( \Gamma_1(N) \). If \( m = 0 \), then \( \gamma \) is (up to sign if \( N \leq 2 \)) a power of \( P \), so \( f|_{\gamma} = f \). Otherwise, multiplying \( \gamma \) on the left by \( P^{-j} \) leaves \( f|_{\gamma} \) unchanged and replaces \( q \) by \( q + jmN \). By Dirichlet’s theorem, we may assume that \( q \equiv 1 \pmod{N} \) is a prime and is large enough so Lemma 22 holds.

Equation (11) implies the following.

\[
\sum_{a \text{ (mod } q)} C_\chi \chi(a)f_1 \left[ \begin{array}{cc}
1 & \frac{a}{q} \\
0 & 1
\end{array} \right] + \mathbb{1}_{\chi=\chi_0}rf = C_\chi f_\chi = \tau^k\epsilon_1 f_\tau \left| \begin{array}{cc}
0 & 1 \\
Nq^2 & -1
\end{array} \right|^{-1}
\]

\[
= \tau^k\epsilon_1 \left( \sum_{m \text{ (mod } q)} \chi(-Nm)f_\chi \left| \begin{array}{cc}
1 & \frac{m}{q} \\
0 & 1
\end{array} \right| \left( \begin{array}{cc}
0 & -1 \\
N & 0
\end{array} \right) \left( \begin{array}{cc}
0 & -1 \\
Nq^2 & 0
\end{array} \right)^{-1} + \mathbb{1}_{\chi=\chi_0}f_\tau \left| \begin{array}{cc}
0 & 0 \\
q^2 & 1
\end{array} \right|^{-1} \right)
\]

\[
= \sum_{m \text{ (mod } q)} \chi(-Nm)f_\chi \left| \begin{array}{cc}
0 & -1 \\
N & 0
\end{array} \right| \left( \begin{array}{cc}
0 & -1 \\
Nq^2 & 0
\end{array} \right)^{-1} + \mathbb{1}_{\chi=\chi_0}f_\tau \left| \begin{array}{cc}
0 & 0 \\
q^2 & 1
\end{array} \right|
\]

\[
(10) \quad = \sum_{a \text{ (mod } q)} \chi(a)f_\gamma \left| \begin{array}{cc}
1 & \frac{a}{q} \\
0 & 1
\end{array} \right| + \mathbb{1}_{\chi=\chi_0}f_\tau \left| \begin{array}{cc}
q^2 & 0 \\
0 & 1
\end{array} \right|.
\]
Fix a residue $b$ coprime to $q$. By orthogonality of Dirichlet characters and (10),

$$f|_{\gamma_q,b}(0 | 0 1) = \frac{1}{\varphi(q)} \sum_{\chi \mod q} \chi(b) \sum_{a \mod q \atop (a,q) = 1} \chi(a)f|_{\gamma_q,a}(0 | 0 1)$$

$$= \frac{1}{\varphi(q)} \left( \sum_{\chi \mod q} \chi(b)\Lambda \sum_{a \mod q \atop (a,q) = 1} \chi(a)f \left| \left( \begin{array}{c} 1 \\ 0 \\ \frac{a}{q} \\ 1 \end{array} \right) \right] + \tau f \left| \left( \begin{array}{c} q^2 \\ 0 \\ 0 \\ 1 \end{array} \right) \right] \right).$$

Replacing $a$ by $ab$ on the right hand side and writing

$$\tilde{C}_q(a) = \left\{ \begin{array}{ll} \frac{1}{\varphi(q)} \sum_{\chi \mod q} \chi(a)f & \text{if } (a,q) = 1, \\ \frac{q}{\varphi(q)} & \text{otherwise,} \end{array} \right.$$

we obtain

(11) $$f|_{\gamma_q,b} = \sum_{a=0}^{q-1} \tilde{C}_q(a)f \left| \left( \begin{array}{c} 1 \\ 0 \\ \frac{(a-1)b}{q} \\ 1 \end{array} \right) \right] - \frac{\tau f}{\varphi(q)} \left| \left( \begin{array}{c} q^2 \\ 0 \\ -bq \\ 1 \end{array} \right) \right].$$

Let

$$S_q(x) = \sum_{a=0}^{q-1} \tilde{C}_q(a)e \left( \frac{(a-1)x}{q} \right).$$

From (11), the $n$th Fourier coefficient of $f|_{\gamma_q,b}$ is $f_n S_q(bn) - \frac{\tau f}{\varphi(q)} q\overline{\chi}(n) f_n / q^2$.

We shall use the following Lemma.

**Lemma 3.1.** Let $q \equiv 1 \pmod{N}$ be a sufficiently large prime so Lemma 2.2 holds. For any $a$ such that $(a,q) = 1$, there exists an $n \equiv a \pmod{q}$ such that $f_n \neq 0$.

**Proof.** Proof of lemma 3.4.

Let $(a,q) = 1$ and suppose there does not exist any $n \equiv a \pmod{q}$ such that $a_n \neq 0$. Let $\chi_0$ be the trivial character mod $q$. By Fourier analysis, $\chi_0(n) = \frac{q-1-c_q(n)}{q}$, so

$$\mathbb{I}_{n \equiv a} = \frac{1}{q} - \frac{c_q(n) + r}{q(q-1)} + \frac{r}{q(q-1)} + \frac{1}{q-1} \sum_{\chi \mod q \atop \chi \neq \chi_0} \chi(n).$$

Multiplying by $a_n n^{-s}$ and summing implies

$$(q-1+r) \Lambda_1(s) = \Lambda_{c_q+r}(s) - q \sum_{\chi \mod q \atop \chi \neq \chi_0} \chi(a) \Lambda_\chi(s).$$

Using the functional equations (1) and (8),

$$(q-1+r) \epsilon_1 N^{1/2-s} \Lambda_1(1-\overline{s}) = \epsilon_1 (Nq^2)^{1/2-s} \Lambda_{c_q+r}(1-\overline{s}) - q(Nq^2)^{1/2-s} \sum_{\chi \mod q \atop \chi \neq \chi_0} \epsilon_\chi \Lambda_\chi(1-\overline{s}).$$

Multiplying by $\overline{\epsilon_1} (Nq^2)^{1/2-s}$, replace $s$ by $1-\overline{s}$ and conjugating, we get

$$(q-1+r) \overline{\epsilon_1} q^{1-2s} \Lambda_1(s) = \overline{\epsilon_1} \Lambda_{c_q+r}(s) - q \sum_{\chi \mod q \atop \chi \neq \chi_0} \overline{\epsilon_\chi} \Lambda_\chi(s).$$

Comparing Dirichlet coefficients at $q$, either $a_q = 0$ or $q-1+r = 0$. The second equation implies $|\mu| = q$, which cannot happen as $F_q(s)$ has no poles for $\Re(s) \geq 1/2$. Comparing at $q^2$ now, $q(q-1+r) = \overline{\epsilon_1} (q-1+r)a_q$. Since $a_q = 0$, $a_{q^2} = -\mu$, so this equation leads again to $|\mu| = q$, again giving a contradiction. □
Since \( q \equiv 1 \pmod{N} \), \(^2\) implies \( f|_{\gamma_1} = f \). Using Lemma \(^5\) and equating Fourier coefficients of \( f|_{\gamma_1} \) and \( f \) implies \( S_q(x) = 1 \) for all \( x \) coprime to \( q \). For \( n \) such that \( q \nmid n \), the \( n \)th Fourier coefficient of \( f|_{\gamma_1} \) is \( f_n \). If \( q|n \), the Fourier coefficient of \( f|_{\gamma_1} \) is independent of \( b \), so \( f|_{\gamma_1} \) has the same \( n \)th Fourier coefficient as \( f|_{\gamma_1} \). Hence \( f|_{\gamma_1} = f \) for all \( b \) coprime to \( q \). It is also easily shown in the case \( b \equiv 0 \pmod{q} \), so \( f \in M_q(\Gamma_1(N)) \).

Remark. For our proof we only need the fact \( f \in M_q(\Gamma_1(N)) \). The above calculations imply that in fact \( r = 0 \).

Notice that by definition \( S_q(0) = \tau \left( \frac{\phi(q)}{\varphi(q)} + 1 \right) \). Using Fourier analysis on \( \mathbb{Z}/q\mathbb{Z} \)

\[
\hat{C}_q(a + 1) = \frac{1}{q} \sum_{x=0}^{q-1} S_q(-ax/q) e(ax/q) = \mathbb{1}_{a=0} + \frac{S_q(0)-1}{q}.
\]

Hence \( \hat{C}_q(0) = \frac{S_q(0)-1}{q} = \frac{S_q(0)}{q} - \frac{1}{q} \). Suppose \( \lambda \neq 0 \), so \( f_q \neq 0 \). Then comparing the \( q \)th Fourier coefficient of \( f|_{\gamma_1} \) and \( f \) implies \( S_q(0) = 1 \), i.e \( r = 0 \). The fact \( S_q(0) = 1, r = 0 \) also shows \( \epsilon = 1 \) and \( |\mu| = 1 \), so \( \lambda \) is real in this case.

Now suppose \( \lambda = 0 \). Then \( f_q = -\mu q^{k-1}, f_1 = 1 \). Comparing coefficients at \( q^2 \) now,

\[
\mu q^{k-1} = \mu q^{k-1} \left( \frac{\tau q}{\varphi(q)} + 1 \right) + \frac{\tau q}{\varphi(q)},
\]

which implies \( |\mu| = 1 \). Hence \( \epsilon = \mu \), so \( r = 0 \) again.

4. PROOF OF THEOREM 1.1

Proof. We use the following argument from \(^2\). Let \( C \) denote the set of normalized Hecke eigenforms of weight \( k \) and conductor dividing \( N \), and for \( g \in C \) with Fourier expansion \( \sum_{n=1}^{\infty} g_n e(nz) \), let \( L_g(s) = \sum_{n=1}^{\infty} g_n n^{-s} e\left(\frac{n}{N}x\right) \).

Let \( X \) denote the set of pairs \((\xi_1, \xi_2)\) where \( \xi_1 \pmod{N_1}, \xi_2 \pmod{N_2} \) are primitive Dirichlet characters such that \( N_1 N_2 | N, \xi_1(-1) \xi_2(-1) = (-1)^k \) and if \( k = 1 \) then \( \xi_1(-1) = 1 \). Let

\[
L_{\xi_1, \xi_2}(s) = L(s + \frac{k-1}{2}, \xi_1)L(s - \frac{k-1}{2}, \xi_2)
\]

where the factors on the right are the usual Dirichlet \( L \) functions.

Since \( f \in M_k(\Gamma_1(N)) \), by newform theory and the description of Eisenstein series in \(^3\) Chapter 4.7, there are Dirichlet polynomials \( D_{\xi_1, \xi_2} \) and \( D_g \) such that

\[
L_f(s) = \sum_{(\xi_1, \xi_2) \in X} D_{\xi_1, \xi_2} L_{\xi_1, \xi_2}(s) + \sum_{g \in C} D_g(s)L_g(s)
\]

Furthermore the coefficients of each Dirichlet polynomial are supported on divisors of \( N \). Let us define \( F_{p,q}(s) \) to be the Euler factor of \( L_g(s) \) at \( p \), where \( g \in C \). Also, let \( F_{p, (\xi_1, \xi_2)}(s) \) be the Euler factor of \( L_{\xi_1, \xi_2}(s) \) at \( p \).

We will say that the Euler products of two \( L \) functions \( L_1, L_2 \) are equivalent if their Euler factors are the same except for finitely many primes and inequivalent otherwise. The Ranking–Selberg method (see \(^4\) Corollary 4.4) implies that the \( L \) functions on the right hand side of \(^1\) are pairwise inequivalent. Combining this with the linear independence result in \(^5\) Theorem 2\), we see the right hand side of \(^1\) has exactly one non-zero term. Hence \( L_f(s) = D_y(s)L_y(s) \) for some \( y \in C \cup X \).

In either case

\[
D_y(s) = \prod_{p | N} \frac{F_p(s)}{F_{p,y}(s)}
\]

and \( D_y \) satisfies a functional equation of the form \( D_y(s) = \varepsilon_y(N_y)^{\frac{s}{2}} D_y(1 - \overline{s}) \) where \( N_y \) is a positive integer, \( |\varepsilon_y| = 1 \). The \( Q \) linear independence of \( \log p \) for primes \( p \) and the fact \( D_y(s) \) is entire implies that \( \frac{F_{p,y}(s)}{F_p(s)} \) is entire for \( p | N \) and its zeroes are symmetric with respect to the line \( \Re(s) = \frac{1}{2} \).

Suppose \( y = (\xi_1, \xi_2) \in X \).

If \( k \geq 2 \), for \( p > N \), \( F_p(s) = F_{p,y}(s) \) has a pole on the line \( \Re(s) = \frac{k-1}{2} \geq \frac{1}{2} \). This is a contradiction to...
our assumptions on \( F_p(s) \) in the statement of the Theorem 12. Hence \( k = 1 \).

When \( k = 1, \) if \( p \mid N, \) \( \frac{F_p(s)}{F_p(0)} \) has no zeroes for \( \Re(s) \geq \frac{1}{2} \). The symmetry of the zeroes around \( \Re(s) = 1/2 \) then implies there are in fact no zeroes. In particular, \( D_y(s) \) has no zeroes and the fundamental theorem of algebra implies it must be constant. Equating coefficients of \( L_f(s) \) and \( L_y(s) \) implies \( D_y(s) = 1 \). The functional equation then implies \( N = N_1N_2 \) as required.

Suppose \( y = q \in C \).

Deligne showed that \( L_y \) satisfies the Ramanujan hypothesis, so \( L_y(s) \) lies in the Selberg class. Convergence of the logarithm in the definition of the Selberg class means \( F_p(s) \) has no zeroes and is analytic for \( \Re(s) \geq 1/2 \). Repeating the argument above, \( D_y(s) = 1 \) and the conductor of \( g \) equals \( N \) by the functional equation as required.

\[ \square \]

We now deal with the case that \( \Lambda_1(s) \) is not entire. We shall deal with this issue as in [2] by twisting away the poles.

Fix a prime \( q \mid N \) and a primitive character \( \chi \mod{q} \) and consider the twisted sequence \( a_n' = a_n\chi(n) \) in place of \( a_n \) and \( Nq^2 \) in place of \( N \). Then all the hypotheses of Theorem 12 are satisfied, however now our associated \( L \) function \( \Lambda_1(s) \) is entire. Hence, either there is a primitive cuspform \( f' \) of conductor \( Nq^2 \) with Fourier coefficients \( a_n' \equiv n \mod{q} \), or \( k = 1 \) and there are primitive characters \( \xi_1, \xi_2 \) such that

\[ a_n' = \sum_{d|n} \xi_1(n/d)\xi_2(d). \]

We deal with the cuspidal case first. By newform theory [1 Theorem 3.2], we can twist \( f' \) by \( \gamma \) to get a primitive cuspform \( f \) of conductor \( Nq^2 \), for some \( j \) with Fourier coefficients \( a_n'\chi(n/n)\chi^{-1}(q) = a_n\chi^{-1}(q) \) for every \( n \) coprime to \( q \). Applying this argument to two different choices of \( q, \) strong multiplicity one implies \( f \) has conductor \( N \) and Fourier coefficients \( a_n\chi^{-1}(q) \) as wanted.

In the non cuspidal case, let \( \xi_i \mod{N_i} \) for \( i = 1, 2 \) be the primitive character inducing \( \xi_i' \mod{N} \). Then

\[ (13) \quad a_p = \xi_1(p) + \xi_2(p) \quad \text{for all sufficiently large primes } p. \]

The characters \( \xi_1 \) and \( \xi_2 \) have opposite parity so we normalize \( \xi_1 \) to be even. Dirichlet’s theorem on primes in arithmetic progressions implies \( \xi_1, \xi_2 \) are uniquely defined by (13). Again, using two choices of \( q, \) we have \( N_1N_2 = N \) and \( a_n = \sum_{d|n} \xi_1(n/d)\xi_2(d) \). Hence we have completed the proof.

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