Decomposition Properties of Quantum Discord

Sunho Kim,† Jun-De Wu (武俊德),† and Minhyung Cho‡,†

1Department of Mathematics, Zhejiang University, Hangzhou 310027, China
2Department of Applied Mathematics, Kumoh National Institute of Technology, Kyungbuk, 730-701, Korea

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Abstract The quantum discord was introduced by Ollivier, Zurek, Henderson, and Vedral as an indicator of the degree of quantumness of mixed states. In this paper, we provide a decomposition condition for quantum discord. Moreover, we show that under the condition, the quantum correlations between the quantum systems can be captured completely by the entanglement measure. Finally, we present examples of our conclusions.

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1 Introduction and Preliminaries

In this article, we always assume that $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{K}_A$, and $\mathcal{K}_B$ are finite dimensional complex Hilbert spaces. Let $L(\mathcal{H}_A, \mathcal{K}_A)$ be the set of all linear operators from $\mathcal{H}_A$ to $\mathcal{K}_A$. A quantum state $\rho$ of some quantum system, described by $\mathcal{H}_A$, is a positive semi-definite operator of trace one, in particular, for any unit vector $|\psi\rangle \in \mathcal{H}_A$, the operator $\rho = |\psi\rangle \langle \psi|$ is said to be a pure state. We can identify the pure state $|\psi\rangle \langle \psi|$ with the unit vector $|\psi\rangle$. The set of all quantum states on $\mathcal{H}_A$ is denoted by $D(\mathcal{H}_A)$.

A quantum measurement is a set $\{M_x\}_{x \in \Sigma}$ of positive operators indexed by some classical label $x$ corresponding to the classical outcomes of the measurement. These operators form a resolution of the identity on the Hilbert space of the system that is being measured:[1–3]

$$\forall x: M_x \geq 0, \quad \sum_x M_x = \mathbb{1},$$

together with $\{A_x\}_{x \in \Sigma}$ such that $M_x = A_x^\dagger A_x$. In particular, when $\{M_x = \pi_x\}$ is a set of orthogonal projection operators, then $\{M_x = \pi_x\}$ is said to be a von Neumann measurement.

Given a quantum state $\rho \in D(\mathcal{H}_A)$, the quantum measurement $\{M_x\}$ induces a probability distribution $p = \{p_x\}_{x \in \Sigma}$, and the conditional state $\rho_{A|x}$ given outcome $x$ and the probability of this outcome is:

$$\rho_{A|x} = p(x)^{-1} A_x \rho A_x^\dagger, \quad p(x) = \text{Tr}(M_x \rho).$$

However, the following famous theorem told us that each quantum measurement can be seen as a von Neumann measurement on a larger quantum system, that is:

Theorem 1 (Neumark extension theorem[4–5]) Let $M = \{M_x\}_{x \in \Sigma}$ be a quantum measurement on $\mathcal{H}_A$ with $|\Sigma| = n$. Then there exist a Hilbert space $\mathcal{H}_E$ with dimension $\text{dim } \mathcal{H}_E = n$, a pure state $|\phi_0\rangle \in \mathcal{H}_E$, a von Neumann measurement $\{\pi_x^E\}$ on $\mathcal{H}_E$, and a unitary operator $U$ on $\mathcal{H}_A \otimes \mathcal{H}_E$ such that for each quantum state $\rho \in D(\mathcal{H}_A)$,

$$A_x \rho A_x^\dagger = \text{Tr}_E(\mathbb{1}_A \otimes \pi_x^E U \rho \otimes |\phi_0\rangle \langle \phi_0| U^\dagger \mathbb{1}_A \otimes \pi_x^E),$$

where $M_x = A_x^\dagger A_x$.

It follows from the theorem that[4]

$$M_x = A_x^\dagger A_x = \langle \phi_0| U^\dagger \mathbb{1}_A \otimes \pi_x^E U |\phi_0\rangle,$$

and the probability of the outcome $x$ read

$$p_x = \text{Tr}(M_x \rho) = \text{Tr}_E(U^\dagger \mathbb{1}_A \otimes \pi_x^E U \rho \otimes |\phi_0\rangle \langle \phi_0|).$$

Let $p = \{p_a\} \in \mathbb{R}^\Sigma$ be a probability distribution, the Shannon entropy $H(p)$ of $p$ is defined by[6]

$$H(p) = - \sum_{a \in \Sigma} p_a \log_2(p_a).$$

For each quantum state $\rho \in D(\mathcal{H}_A)$, the quantum analog of the Shannon entropy is the von Neumann entropy

$$S(\rho) = - \text{Tr}(\rho \log_2(\rho)).$$

An equivalent expression of von Neumann entropy is (Ref. [4], Chapter 6.1)

$$S(\rho) = \min_{\{\langle \psi_i | \rho \rangle\}} H(\{|\psi_i\rangle\}),$$

where the minimum is over all pure state convex decompositions of $\rho$.

Moreover, a pure state convex composition $\{|\psi^0_i\rangle\}$ of $\rho$ minimizes $H(\{|\psi_i\rangle\} : \{|\psi^0_i\rangle, p^0_i\})$ if and only if it is a spectral decomposition of $\rho$.

The identity can be generalized to get

$$S(\rho) \leq H(\{|\eta_i\rangle\}) + \sum_i \eta_i S(\rho_i)$$

for any quantum state ensemble $\{|\eta_i\rangle\}$, where $\{|\psi_i\rangle, \eta_i\}$ is a convex decomposition of $\rho$. Moreover, the equality
holds if and only if the quantum states \( \{ \rho_i \} \) have mutual orthogonal supports.

Let us consider two quantum systems \( \mathcal{H}_A \) and \( \mathcal{H}_B \), \( \rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \). In quantum information theory, the quantum mutual information

\[
I_{AB}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})
\]

of the quantum state \( \rho_{AB} \) is regarded as a measure of the total correlations between quantum systems \( \mathcal{H}_A \) and \( \mathcal{H}_B \) when the quantum system \( \mathcal{H}_A \otimes \mathcal{H}_B \) in the quantum state \( \rho_{AB} \), where \( \rho_A = \text{Tr}_B(\rho_{AB}) \) and \( \rho_B = \text{Tr}_A(\rho_{AB}) \) are the reduced states of \( \rho_{AB} \).

If we denote \( S(\rho_{B|A}) = S(\rho_{AB}) - S(\rho_A) \), then the quantum mutual information can be written in the following form:

\[
I_{AB}(\rho_{AB}) = S(\rho_B) - S(\rho_{B|A})
\]

One can prove that \( I_{AB}(\rho_{AB}) \geq 0 \) and \( I_{AB}(\rho) = 0 \) if and only if \( \rho_{AB} \) is a product state, that is \( \rho_{AB} = \rho_A \otimes \rho_B \).

Given a von Neumann measurement \( \{ \pi^A \} \) on quantum system \( \mathcal{H}_A \), let us define a conditional entropy on quantum system \( \mathcal{H}_B \) by

\[
S_{B|A}(\rho_{AB}|\{ \pi^A \}) = \sum_i \eta_i S(\rho_{B|i}),
\]

where

\[
\rho_{B|i} = \eta_i^{-1} \text{Tr}_A(\pi^A_i \otimes 1_B \rho_{AB}),
\]

\[
\eta_i = \text{Tr}(\pi^A_i \otimes 1_B \rho_{AB}).
\]

Denote

\[
\mathcal{J}\{ \pi^A \}(\rho_{AB}) = S(\rho_B) - S_{B|A}(\rho_{AB}|\{ \pi^A \}).
\]

In order to take a quantity which does not depend on the von Neumann measurements, one defines

\[
\mathcal{J}^{v.N.}_{B|A}(\rho_{AB}) = \max_{\{ \pi^A \}} \mathcal{J}\{ \pi^A \}(\rho_{AB})
\]

\[
= S(\rho_B) - \min_{\{ \pi^A \}} \left\{ \sum_i \eta_i S(\rho_{B|i}) \right\},
\]

it is interpreted as a measure of classical correlations between the quantum systems \( \mathcal{H}_A \) and \( \mathcal{H}_B \) when the quantum system \( \mathcal{H}_A \otimes \mathcal{H}_B \) in the quantum state \( \rho_{AB} \).

In general, \( I_{AB}(\rho_{AB}) \) may differ \( \mathcal{J}^{v.N.}_{B|A}(\rho_{AB}) \). Their difference

\[
\mathcal{D}^{v.N.}_{A|B}(\rho_{AB}) = I_{AB}(\rho_{AB}) - \mathcal{J}^{v.N.}_{B|A}(\rho_{AB})
\]

\[
= S(\rho_A) - S(\rho_B) + \min_{\{ \pi^A \}} \left\{ \sum_i \eta_i S(\rho_{B|i}) \right\}
\]

is interpreted as a measure of quantum correlations and is called quantum discord (Ref. [4], Chapter 10.1 and Refs. [7–9]). The minimum is achieved for some rank-one orthogonal projection measurement operators \( \{ \pi^A \} \).

Similarly, given a quantum measurement \( \{ M^A \} \) on \( \mathcal{H}_A \), let us define a conditional entropy on \( \mathcal{H}_B \) by

\[
S_{B|A}(\rho_{AB}|\{ M^A \}) = \sum_i \mu_i S(\rho_{B|i}),
\]

where

\[
\rho_{B|i} = \mu_i^{-1} \text{Tr}_A(M^A_i \otimes 1_B \rho_{AB}),
\]

\[
\mu_i = \text{Tr}(M^A_i \otimes 1_B \rho_{AB}).
\]

Denote

\[
\mathcal{J}\{ M^A \}(\rho_{AB}) = S(\rho_B) - S_{B|A}(\rho_{AB}|\{ M^A \}),
\]

\[
\mathcal{J}_{B|A}(\rho_{AB}) = \max_{\{ M^A \}} \mathcal{J}\{ M^A \}(\rho_{AB})
\]

\[
= S(\rho_B) - \min_{\{ M^A \}} \left\{ \sum_i \mu_i S(\rho_{B|i}) \right\},
\]

The corresponding discord \( \mathcal{D}_{A}(\rho_{AB}) \) is defined by[4]

\[
\mathcal{D}_{A}(\rho_{AB}) = I_{AB}(\rho_{AB}) - \mathcal{J}_{B|A}(\rho_{AB})
\]

\[
= S(\rho_A) - S(\rho_{AB}) + \min_{\{ M^A \}} \left\{ \sum_i \mu_i S(\rho_{B|i}) \right\}.
\]

As in the case of von Neumann measurements, the minimum is achieved for some rank-one measurement operators \( \{ M^A \} \).

That \( \mathcal{D}_{A}(\rho_{AB}) \leq \mathcal{D}^{v.N.}_{A|B}(\rho_{AB}) \) is clear. On the other hand, by Neumark extension theorem (1) and note that \( M^A = \{ (\epsilon_j) \} U^T \pi \otimes \pi F \epsilon \), we have

\[
\mathcal{D}_{A}(\rho_{AB}) = \mathcal{D}^{v.N.}_{A|B}(\rho_{AB} \otimes |\epsilon\rangle\langle \epsilon|). \quad (2)
\]

Given a pure state \( |\psi\rangle|\psi\rangle_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \), then \( S(\rho_A) = S(\rho_B) \).[10] The entanglement \( E(|\psi\rangle|\psi\rangle_{AB}) \) of \( |\psi\rangle|\psi\rangle_{AB} \) is defined by

\[
E(|\psi\rangle|\psi\rangle_{AB}) = S(\rho_A) = S(\rho_B).
\]

For any quantum state \( \rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \), the entanglement of formation \( E_f(\rho_{AB}) \) of \( \rho_{AB} \) is defined by[10–11]

\[
E_f(\rho_{AB}) = \min_{\{ |\psi\rangle\rangle_{AB} \}} \sum_i p_i E(|\psi_i\rangle|\psi_i\rangle_{AB}).
\]

where \( \{|\psi_i\rangle|\psi_i\rangle\rangle \in \Sigma \) is the pure state convex decomposition of \( \rho_{AB} \).

For any pure state \( |\psi\rangle|\psi\rangle_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \), we have[10–11]

\[
\mathcal{D}_{A}(|\psi\rangle|\psi\rangle_{AB}) = \mathcal{D}^{v.N.}_{A|B}(|\psi\rangle|\psi\rangle_{AB}) = E_f(|\psi\rangle|\psi\rangle_{AB})
\]

\[
= S(|\psi\rangle|\psi\rangle_{AB}) = S(|\psi\rangle|\psi\rangle_{AB}).
\]

Let \( \rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \), \( \rho_{AB} = \sum_{i\in \Sigma} p_i |\psi_i\rangle|\psi_i\rangle \) be its spectral decomposition. Generally, one has[4]

\[
0 \leq \min_{\{ M^A \}} \left\{ \sum_i \mu_i S(\rho_{B|i}) \right\} \leq \min_{\{ \pi^A \}} \left\{ \sum_i \eta_i S(\rho_{B|i}) \right\} \leq S(\rho_{AB}).
\]

Moreover,

\[
\min_{\{ \pi^A \}} \left\{ \sum_i \eta_i S(\rho_{B|i}) \right\} = S(\rho_{AB}),
\]

if and only if \( \sum_{i\in \Sigma} p_i |\psi_i\rangle|\phi_i\rangle \) is the spectral decomposition of \( \rho_{B|y} \), where

\[
p_{ij} = \frac{p_i \text{Tr}(\pi^A_i \otimes 1_B |u_i\rangle\langle u_i|)}{\eta_y},
\]

\[
|\phi_y\rangle = \frac{\text{Tr}_A(\pi^A_i \otimes 1_B |u_i\rangle\langle u_i|)}{\text{Tr}(\pi^A_i \otimes 1_B |u_i\rangle\langle u_i|)},
\]
the von Neumann measurement \( \{ \pi^A_i \} \) minimizes the conditiotnal entropy \( \sum_x \eta_x S(\rho_{B|x}) \), and \( Tr_B(|u_i\rangle\langle u_j|) = 0 \) when \( i \neq j \) and \( p_i p_j > 0 \).

In this paper, we give out the decomposition condition of quantum discords. Moreover, we show that under the condition, the quantum correlations between the quantum systems can be captured completely by the entanglement measure.

## 2 Decomposition of Quantum Discord

Firstly, we prove the following result.

**Theorem 2** Let \( \rho_{AB} \in D(H_A \otimes H_B) \), \( \rho_{AB} = \sum_{i \in \Sigma} p_i |u_i\rangle\langle u_i| \) be its spectral decomposition. Then

\[
\min_{\{\pi^A_i\}} \left\{ \sum_x \eta_x S(\rho_{B|x}) \right\} = 0, \tag{4}
\]

if and only if for any two \( i, j \in \Sigma \) and \( i \neq j \),

\[
\text{Tr}_A(|u_i\rangle\langle u_j|) = 0. \tag{5}
\]

**Proof** If for any two \( i, j \in \Sigma \) and \( i \neq j \), we have \( \text{Tr}_A(|u_i\rangle\langle u_j|) = 0 \), then \( \text{Tr}_B(|u_i\rangle\langle u_j|) = 0 \) and \( \text{Tr}_B(|u_i\rangle\langle u_j|) = 0 \) for any \( i, j \in \Sigma \).

We have \( \rho_{AB} = \sum_{i \in \Sigma} p_i |u_i\rangle\langle u_i| \) and \( \rho_{AB} = \sum_{i \in \Sigma} p_i |u_i\rangle\langle u_i| \).

Therefore, we have

\[
\min_{\{\pi^A_i\}} \left\{ \sum_x \eta_x S(\rho_{B|x}) \right\} = \min_{\{\pi^A_i\}} \left\{ \sum_k \eta_k S(\rho_{B|k}) \right\}.
\]

Moreover, for any pure state, it follows from the Schmidt decomposition that states \( \{ \rho_{B|k} \} \) are pure states for all \( i, k \) and thus have zero entropy, that is

\[
S(\rho_{B|k}) = 0 \quad \text{for all } i, k.
\]

Therefore, we have

\[
\min_{\{\pi^A_i\}} \left\{ \sum_x \eta_x S(\rho_{B|x}) \right\} = 0.
\]

On the other hand, if there exist \( i \neq j \) and \( p_i, p_j > 0 \) such that \( \text{Tr}_A(|u_i\rangle\langle u_j|) \neq 0 \), then for any von Neumann measurement \( \{ \pi^A_i \} \) on \( H_A \), there exists at least a \( \pi^A_k \) such that \( Tr_A(\pi^A_k \otimes 1_A |u_i\rangle\langle u_k|) \neq 0 \) for \( k = i, j \). Also, note that \( \sum_{i \in \Sigma} p_i |u_i\rangle\langle u_i| \) is the spectral decomposition of \( \rho \), we have \( \langle u_i^{(j)} | u_{i'}^{(j)} \rangle = 0 \) where \( Tr_A(\pi^A_i \otimes 1_A |u_i\rangle\langle u_k|) = 0 \) for \( k = i, j \). It is easy to show that \( S(\rho_{B|x}) > 0 \). This contradicts (4). \( \Box \)

**Theorem 3** Let \( \rho_{AB} \in D(H_A \otimes H_B) \), \( \rho_{AB} = \sum_{i \in \Sigma} p_i |u_i\rangle\langle u_i| \) be its spectral decomposition. If

\[
\text{Tr}_A(|u_i\rangle\langle u_j|) = 0, \tag{6}
\]

for any two \( i \neq j \) and \( p_i, p_j > 0 \), then

\[
D_{A}^{v.N.}(\rho_{AB}) = D_{A}^{v.N.}(\rho_{AB}) = E_f(\rho_{AB}) = \sum_{i \in \Sigma} p_i D_{A}^{v.N.}(\rho_{AB}). \tag{7}
\]

**Proof** If \( \text{Tr}_A(|u_i\rangle\langle u_j|) = 0 \), then we have

\[
S(\rho_A) = \sum_{i \in \Sigma} p_i S(\rho_{A|i}) + H(\{p_i\}),
\]

where \( \rho_{A|i} = Tr_B(|u_i\rangle\langle u_i|) \). And, it follows from the spectral decomposition \( \rho_{AB} = \sum_{i \in \Sigma} p_i |u_i\rangle\langle u_i| \) that \( S(\rho_{AB}) = H(\{p_i\}) \). Therefore, by the definition of quantum discord, Eq. (3) and Theorem 2, we have

\[
D_{A}^{v.N.}(\rho_{AB}) = D_{A}^{v.N.}(\rho_{AB}) = E_f(\rho_{AB}) = \sum_{i \in \Sigma} p_i D_{A}^{v.N.}(\rho_{AB}),
\]

where \( \rho_{A|i} = Tr_B(|u_i\rangle\langle u_i|) \). And, it follows from the spectral decomposition \( \rho_{AB} = \sum_{i \in \Sigma} p_i |u_i\rangle\langle u_i| \) that \( S(\rho_{AB}) = H(\{p_i\}) \). Therefore, by the definition of quantum discord, Eq. (3) and Theorem 2, we have

\[
D_{A}^{v.N.}(\rho_{AB}) = D_{A}^{v.N.}(\rho_{AB}) = E_f(\rho_{AB}) = \sum_{i \in \Sigma} p_i D_{A}^{v.N.}(\rho_{AB}).
\]

Moreover, it follows from Ref. [12] that

\[
E_f(\rho_{AB}) = \sum_i p_i E_f(|u_i\rangle\langle u_i|),
\]

thus, by Eq. (3) again, we have

\[
D_{A}^{v.N.}(\rho_{AB}) = D_{A}^{v.N.}(\rho_{AB}) = E_f(\rho_{AB}) = \sum_{i \in \Sigma} p_i D_{A}^{v.N.}(\rho_{AB}).
\]

This theorem means that if \( \sum_{i \in \Sigma} p_i |u_i\rangle\langle u_i| \) is a spectral decomposition of \( \rho_{AB} \) and

\[
\text{Tr}_A(|u_i\rangle\langle u_j|) = 0 \quad \text{where } i \neq j \text{ and } p_i, p_j > 0,
\]

then the quantum correlations between the quantum systems \( H_A \) and \( H_B \) can be captured completely by the entanglement measure.

**Remark 1** Note that an equivalent expression of von Neumann entropy is (Ref. [4], Chapter 6.1)

\[
S(\rho) = \min_{\{\psi(\cdot)|,p_i\}} H(\{p_i\}).
\]
where the minimum is over all pure state convex decompositions of $\rho$. Moreover, a pure state convex decomposition $\{|\psi_i\rangle , p_i\}$ of $\rho$ minimizes $\{H(\{|p_i\} : \{|\psi_i\rangle, p_i\})\}$ if and only if it is a spectral decomposition of $\rho$. Thus, the von Neumann entropy $S(\rho)$ of $\rho$ does not depend on the spectral decomposition form of $\rho$. Moreover, it follows from the proofs of Theorem 2 and Theorem 3 that they do not depend on the spectral decomposition forms of $\rho_{AB}$, too. For example, for Theorem 3, if it has another spectral decomposition $\rho_{AB} = \sum_{i\in\Sigma} q_i|v_i\rangle\langle u_i|$ of $\rho_{AB}$, satisfies that

$$\text{Tr}_A(|v_i\rangle\langle v_k|) = 0,$$

for any two $l \neq k$ and $q_l, q_k > 0$, then

$$D_A(\rho_{AB}) = E_f(\rho_{AB}) = \sum_{i \in \Sigma} p_i D_{\rho_{AB}}^{(i)} (\rho_{AB})$$

$$= \sum_{i \in \Sigma} p_i D_{\rho_{AB}}^{(i)} (|v_i\rangle\langle u_i|).$$

If we replace the condition of pure states in Eq. (5) with mixed states, we have the following conclusion:

**Theorem 4** Let $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$, $\rho_{AB} = \sum_{i \in \Sigma} p_i \rho_i$ be its an orthogonal decomposition. If for any two $i \neq j \in \Sigma$ and $p_i, p_j > 0$, then

$$\text{Tr}_A(\rho_i \rho_j) = 0,$$

then

$$D_A(\rho_{AB}) = \sum_{i \in \Sigma} p_i D_A(\rho_i).$$

**Proof** If $\text{Tr}_A(\rho_i \rho_j) = 0$ for any two $i, j \in \Sigma$ and $i \neq j$, then it is easy to show that $\text{Tr}_B(\rho_i)$ and $\text{Tr}_B(\rho_j)$ are orthogonal, it implies that there are subspaces $V_i^A, V_j^B \subseteq \mathcal{H}_A$ such that $\rho_i \in D(V_i^A \otimes \mathcal{H}_B)$ and $V_j^B \subseteq (V_j^A)^\perp$. If $\pi_i^A$ is the orthogonal projector onto $V_i^A$ for any $i$, and $\pi_m^A = 1_A - \sum_{i \in \Sigma} \pi_i^A$, we have

$$\pi_i^A \otimes 1_B \rho_{AB} \pi_i^A \otimes 1_B = p_i \rho_i.$$

Let $\{M_y\}_{y \in \Gamma}$ be the quantum measurement, which minimizes the conditional entropy $\sum_y p_y S(\rho_{B|y})$. Note that $\pi_i^A M_y \pi_i^A$ are positive operator for all $i, y$, and

$$\sum_{i, y} \pi_i^A M_y \pi_i^A = \sum_i \pi_i^A \left( \sum_y M_y \pi_i^A \right) = \sum_i \pi_i^A = 1_A,$$

thus, $\{M_y^{(i)} = \pi_i^A M_y \pi_i^A\}_{y \in \Gamma}$ is also a quantum measurement. By

$$\text{Tr}_A(M_y^{(i)} \otimes 1_B \rho_{AB}) = p_i \text{Tr}_A(M_y^{(i)} \otimes 1_B \rho_i),$$

for all $i, y$, we have

$$\sum_{y \in \Gamma} p_y S(\rho_{B|y}) \leq \sum_{y \in \Gamma} p_i p_y^{(i)} S(\rho_{B|y}^{(i)})$$

$$= \sum_{y \in \Gamma} p_i \left\{ \sum_{y \in \Gamma} p_y^{(i)} S(\rho_{B|y}^{(i)}) \right\},$$

where $\mu_y^{(i)} = p_i^{-1} \text{Tr}_A(M_y^{(i)} \otimes 1_B \rho_{AB})$ and $\rho_{B|y}^{(i)} = (p_i \mu_y^{(i)})^{-1} \text{Tr}_A(M_y^{(i)} \otimes 1_B \rho_{AB})$.

On the other side, it follows from $\mu_y \rho_{B|y} \geq \sum_i p_i \mu_y^{(i)} \rho_{B|y}^{(i)}$ and the concavity of von Neumann entropy that

$$\sum_{y \in \Gamma} p_y S(\rho_{B|y}) \geq \sum_{y \in \Gamma} p_i \left\{ \sum_{y \in \Gamma} p_y^{(i)} S(\rho_{B|y}^{(i)}) \right\}$$

$$= \sum_{y \in \Gamma} p_i \left\{ \sum_{y \in \Gamma} p_y^{(i)} S(\rho_{B|y}^{(i)}) \right\}.$$}

Therefore, by Inequality (11) and (12), we have

$$\min_{\{M_z\}} \sum_{z} \mu_z S(\rho_{B|z}) = \sum_{i \in \Sigma} p_i \min_{\{M_z^{(i)}\}} \left\{ \sum_{z} \mu_z^{(i)} S(\rho_{B|z}^{(i)}) \right\},$$

where $M_z^{(i)} = \pi_i^A M_z \pi_i^A$ for any $i, z$.

Next, if $\text{Tr}_A \rho_i \rho_j = 0$ for any two $i \neq j$ and $p_j, p_j > 0$, and $\rho_A = \sum_{i \in \Sigma} p_i \rho_{A|i}$, then, by Property (1) of entropy, we have

$$S(\rho_A) = \sum_{i \in \Sigma} p_i S(\rho_{A|i}) + H(\{p_i\}).$$

And, by the same reason as above, we also have

$$S(\rho) = \sum_{i \in \Sigma} p_i S(\rho_i) + H(\{p_i\}).$$

Therefore, by the definition of quantum discord, it follows that

$$D_A(\rho_{AB}) = S(\rho_A) - S(\rho) + \min_{\{M_z\}} \left\{ \sum_{z} \mu_z S(\rho_{B|z}) \right\}$$

$$= \sum_{i \in \Sigma} p_i S(\rho_{A|i}) + H(\{p_i\}) - \left\{ \sum_{i \in \Sigma} p_i S(\rho_i) + H(\{p_i\}) \right\} + \sum_{i \in \Sigma} p_i \min_{\{M_z^{(i)}\}} \left\{ \sum_{z} \mu_z^{(i)} S(\rho_{B|z}^{(i)}) \right\}$$

$$= \sum_{i \in \Sigma} p_i \left[ S(\rho_{A|i}) - S(\rho_i) + \min_{\{M_z^{(i)}\}} \left\{ \sum_{z} \mu_z^{(i)} S(\rho_{B|z}^{(i)}) \right\} \right] = \sum_{i \in \Sigma} p_i D_A(\rho_i).$$

$\square$

### 3 A Tripartite System

Let $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$. Taking a pure state $|\Psi\rangle\langle\Psi|_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ such that

$$\rho_{AB} = \text{Tr}_C(|\Psi\rangle\langle\Psi|_{ABC}),$$

$$\rho_{AB} = \sum_{i \in \Sigma} p_i |u_i\rangle\langle u_i|$$

is the spectral decomposition of $\rho_{AB}$. Now, we will prove that if $\rho_{AB}$ satisfies the condition of Theorem 3, then by the famous necessary and sufficient condition of zero discord in Ref. [13], $\mathcal{H}_B$ and $\mathcal{H}_C$ are not
entangled and even have vanishing discord by the local measurements on the system $\mathcal{H}_C$.

In fact, let

$$ Tr_A |u_i\rangle \langle u_j| = 0 $$

for any two $i \neq j$ and $p_i, p_j > 0$, $|\Psi\rangle_{ABC} = \sum_{i \in \Sigma} \sqrt{p_i} |u_i\rangle |v_i\rangle$

be the Schmidt decomposition of $|\Psi\rangle_{ABC}$. It follows from the condition that for any $i$, we have

$$ |u_i\rangle = \sum_{j \in \Sigma_i} \sqrt{q_{ij}} |\phi_{ij}\rangle |\psi_{ij}\rangle, $$

where $\{ |\phi_{ij}\rangle \}$ and $\{ |\psi_{ij}\rangle \}$ are orthonormal families of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, and $|\phi_{ij}\rangle |\phi_{ij}\rangle = 0$ for $i, k \in \Sigma, i \neq k$ and $j \in \Sigma_i, l \in \Sigma_k$. Thus,

$$ \rho_{BC} = Tr_A (|\Psi\rangle \langle \Psi|_{ABC}) $$

$$ = \sum_{k \in \Sigma} \sum_{i \in \Sigma_k} \langle \phi_{ik}| (\sum_{j \in \Sigma_i} \sqrt{p_{ij}} |\phi_{ij}\rangle |\psi_{ij}\rangle |v_i\rangle) (\sum_{j \in \Sigma_i} \sqrt{p_{ij}} |\phi_{ij}\rangle |\psi_{ij}\rangle |v_i\rangle)^* |\phi_{ik}\rangle $$

$$ = \sum_{i \in \Sigma} p_i \left( \sum_{j \in \Sigma_i} q_{ij} (|\phi_{ij}\rangle \langle \phi_{ij}|) \otimes |v_i\rangle \langle v_i| \right). $$

Therefore, $\rho_{BC}$ is a separable state, and note that $(v_i|v_k) = 0$ for all $i \neq k \in \Sigma$, we have $D_C(\rho_{BC}) = 0$.

The conclusion is proved.

4 Examples of Main Results

Finally, we present examples of Theorem 3 and Theorem 4.

**Example 1** Let $\mathcal{H}_A = C^6$, $\mathcal{H}_B = C^3$, and take three pure states on $\mathcal{H}_A \otimes \mathcal{H}_B$ for

$$ |u_0\rangle \equiv s_{0,0} |0\rangle_A |v_0\rangle_B + s_{0,1} |1\rangle_A |v_1\rangle_B, $$

$$ |u_1\rangle \equiv s_{1,0} |2\rangle_A |v_2\rangle_B + s_{1,1} |3\rangle_A |v_3\rangle_B, $$

$$ |u_2\rangle \equiv s_{2,0} |4\rangle_A |v_4\rangle_B + s_{2,1} |5\rangle_A |v_5\rangle_B, $$

where $\{ |k\rangle \}_{k=0}^5$ is a basis of $\mathcal{H}_A$, $|v_i\rangle \in \mathcal{H}_B$, $l = 1, 2, 3, s_{2,0}^2 + s_{2,1}^2 = 1, i = 0, 1, 2$. Then if $i \neq j, j, j \in \{ 0, 1, 2 \}$, we have $Tr_A (|u_i\rangle \langle u_j|) = 0$.

Moreover, for any probability distributions $p = (p_i)$, let $\rho_{AB} = \sum_{i=0}^2 p_i |u_i\rangle \langle u_i|$. Then the Schmidt decomposition of $|u_i\rangle$ is

$$ |u_0\rangle = r_{0,0} |\phi_{0,0}\rangle_A |\psi_{0,0}\rangle_B + r_{0,1} |\phi_{0,1}\rangle_A |\psi_{0,1}\rangle_B, $$

where $r_{0,0}^2 + r_{0,1}^2 = 1$, $\{ |\phi_{0,0}\rangle_A, |\phi_{0,1}\rangle_A, |2\rangle_A, |3\rangle_A, |4\rangle_A, |5\rangle_A \}$ are orthogonal. $|u_1\rangle$ and $|u_2\rangle$ have the Schmidt decompositions, too.

Now, by performing the von Neumann measurement $\Pi = \{ |\phi_{ij}\rangle \langle \phi_{ij}| : i = 0, 1, j = 0, 1, 2 \}$ on $\mathcal{H}_A$, we have

$$ \sum_x a_x S(\rho_{B|x}) = 0. $$

It implies that

$$ D^N_{A\rightarrow B}(\rho_{AB}) = \sum_{i=0}^2 p_i D^N_{A\rightarrow B}(|u_i\rangle \langle u_i|). $$

Moreover, for the quantum state $\rho_{AB}$, we have

$$ D_A(\rho_{AB}) = D^N_{A\rightarrow B}(\rho_{AB}) = E_f(\rho_{AB}). $$

**Example 2** Let $\mathcal{H}_A = C^6$, $\mathcal{H}_B = C^3$, take six pure states on $\mathcal{H}_A \otimes \mathcal{H}_B$ for

$$ |u_0\rangle \equiv s_{0,0} |0\rangle_A |v_0\rangle_B + s_{0,1} |1\rangle_A |v_0\rangle_B + s_{0,2} |2\rangle_A |v_0\rangle_B, $$

$$ |u_1\rangle \equiv s_{1,0} |1\rangle_A |v_1\rangle_B + s_{1,1} |2\rangle_A |v_1\rangle_B + s_{1,2} |3\rangle_A |v_1\rangle_B, $$

$$ |u_2\rangle \equiv s_{2,0} |4\rangle_A |v_2\rangle_B + s_{2,1} |5\rangle_A |v_2\rangle_B + s_{2,2} |6\rangle_A |v_2\rangle_B, $$

$$ |u_3\rangle \equiv s_{3,0} |3\rangle_A |v_3\rangle_B + s_{3,1} |4\rangle_A |v_3\rangle_B + s_{3,2} |5\rangle_A |v_3\rangle_B, $$

$$ |u_4\rangle \equiv s_{4,0} |1\rangle_A |v_4\rangle_B + s_{4,1} |2\rangle_A |v_4\rangle_B + s_{4,2} |3\rangle_A |v_4\rangle_B, $$

$$ |u_5\rangle \equiv s_{5,0} |5\rangle_A |v_5\rangle_B + s_{5,1} |6\rangle_A |v_5\rangle_B + s_{5,2} |7\rangle_A |v_5\rangle_B, $$

where $\{ |k\rangle \}_{k=0}^5$ is a basis of $\mathcal{H}_A$, $|v_i\rangle \in \mathcal{H}_B$ are random states in $\mathcal{H}_B$ and $\sum_m s_{i,m}^2 = 1$ for $l \in \{ 0, 1, 2 \}$.

Moreover, let $\rho_1 = \sum_{i=0}^2 p_i |u_i\rangle \langle u_i|$, $\rho_2 = \sum_{i=0}^2 q_i |u_i\rangle \langle u_i|$ for any probability distributions $p = (p_i), q = (q_i)$ and $\rho_{AB} = r\rho_1 + (1 - r)\rho_2, 0 < r < 1$. Then we have $Tr_A(\rho_1 \rho_2) = 0$, it follows from Theorem 4 that

$$ \min_{\rho_1, \rho_2} \sum_\Lambda \pi_\Lambda S(\rho_{B|\Lambda}) = \tau \min_{\rho_1, \rho_2} \left\{ \sum_x \mu_x^{(1)} S(\rho_{B|\Lambda}^{(1)}) \right\} + (1 - r) $$

$$ \times \min_{\rho_1, \rho_2} \left\{ \sum_y \mu_y^{(2)} S(\rho_{B|\Lambda}^{(2)}) \right\}, $$

where $\pi_1 = \sum_{i=0}^2 |i\rangle_A \langle i|, \pi_2 = \sum_{j=0}^5 |j\rangle_A \langle j|, M_x^{(1)} = \pi_1 M_x \pi_1, M_y^{(2)} = \pi_2 M_y \pi_2$.

Thus, by Eq. (13), we have

$$ D_A(\rho_{AB}) = rD_A(\rho_1) + (1 - r)D_A(\rho_2). $$
References

[1] B.E. Davies, *Quantum Theory of Open Systems*, Academic, New York (1976).

[2] A.S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland Publishing Company, Amsterdam (1982).

[3] K. Kraus, *States, Effects and Operations: Fundamental Notions of Quantum Theory*, Springer, Berlin (1983).

[4] D. Spehner, J. Math. Phys. 55 (2014) 075211.

[5] J. Watrous, *Theory of Quantum Information*, Institute for Quantum Computing, University of Waterloo (2008).

[6] C.E. Shannon, *A Mathematical Theory of Communication*, Bell Syst. Tech. J. 27 (1948) pp. 379-423; 623-656.

[7] H. Ollivier and W.H. Zurek, Phys. Rev. Lett. 88 (2001) 017901.

[8] L. Henderson and V. Vedral, J. Phys. A 34 (2001) 6899.

[9] V. Vedral, Phys. Rev. Lett. 90 (2003) 050401.

[10] C.H. Bennett, D.P. Di Vincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A 54 (1996) 3824.

[11] W.K. Wootters, Phys. Rev. Lett. 80 (1998) 2245.

[12] P. Horodecki, R. Horodecki, and M. Horodecki, Acta Phys. Slov. 48 (1998) 141.

[13] B. Dakić, V. Vedral, and Č. Brukner, Phys. Rev. Lett. 105 (2010) 190502.