ON THE MOTIVIC CLASS OF AN ALGEBRAIC GROUP

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Abstract. We give an example of a torus $G$ over a finitely generated field extension $F$ of $\mathbb{Q}$ whose classifying stack $BG$ is stably rational and such that $\{BG\} \neq \{G\}^{-1}$ in the Grothendieck ring of algebraic stacks over $F$. We also give an example of a finite étale group scheme $A$ such that $BA$ is stably rational and $\{BA\} \neq 1$.

1. Introduction

Let $F$ be a field. The Grothendieck ring of algebraic stacks $K_0(\text{Stacks}_F)$ was introduced by Ekedahl in [6], following up on earlier works [1], [9], [19]. It is a variant of the Grothendieck ring of varieties $K_0(\text{Var}_F)$. By definition, $K_0(\text{Stacks}_F)$ is generated as an abelian group by the equivalence classes $\{X\}$ of all algebraic stacks $X$ of finite type over $F$ with affine stabilizers. These classes are subject to the scissor relations $\{X\} = \{Y\} + \{X \setminus Y\}$ for every closed substack $Y \subseteq X$, and the relations $\{E\} = \{A^n \times X\}$ for every vector bundle $E$ of rank $n$ over $X$. The product is defined by $\{X\} \cdot \{Y\} := \{X \times Y\}$, and extended by linearity.

In particular, given a linear algebraic group $G$ over $F$, we may consider the class $\{BG\}$ of its classifying stack in $K_0(\text{Stacks}_F)$. The problem of computing $\{BG\}$ appears to be related to the problem of the stable rationality of $BG$, although no direct implications are known. Recall that $BG$ is stably rational if for one (equivalently, every) generically free representation $V$ of $G$, the rational quotient $V/G$ is stably rational. In other words, $BG$ is stably rational when the Noether problem for stable rationality has a positive solution for $G$, in the sense of [8, §3]. The case of a finite (constant) group $G$ was considered in [5]: it frequently happens that $\{BG\} = 1$ (notably for the symmetric groups, see [5, Theorem 4.3]), although there are examples of finite groups $G$ for which $\{BG\} \neq 1$; see [5, Corollary 5.2, Corollary 5.8]. The Ekedahl invariants, a new kind of invariant of $G$ defined in [5] in terms of the class $\{BG\}$ in $K_0(\text{Stacks}_F)$, are the essential ingredient in the proof that $\{BG\} = 1$, and have been further studied in [12, 13] and [14]. So far, all the known examples of finite group schemes $G$ for which $\{BG\} \neq 1$ are also counterexamples to the Noether problem. This suggests the following question.

Question 1.1. (cf. [5 §6]) Is it true that, for a finite group scheme $G$, the following two conditions are equivalent?

- $BG$ is stably rational;
- $\{BG\} = 1$ in $K_0(\text{Stacks}_F)$.

Now let $G$ be a connected linear algebraic group. Recall that $G$ is special if every $G$-torsor is Zariski-locally trivial. For example, $GL_n, SL_n$ and $Sp_n$ are special; see [3]. It was shown by Ekedahl that if $P \to S$ is a torsor under the special group $G$, then $\{P\} = \{G\}\{S\}$. This is immediate if $S$ is a scheme, but less obvious
when $S$ is a stack; see [2, Corollary 2.4]. Applying this to the universal $G$-torsor $\text{Spec} F \to BG$, one obtains $\{BG\}\{G\} = 1$.

The equality $\{BG\} = \{G\}^{-1}$ appears to be the analogue for connected groups of the relation $\{BG\} = 1$ for finite group schemes. In [2], these equalities are referred to as expected class formulas, and there is a sense in which they are "almost" true. In [6, §2] Ekedahl defines a generalized Euler characteristic

$$\chi_c : K_0(\text{Stacks}_F) \to K_0(\text{Coh}_F)$$

taking values in the Grothendieck ring $K_0(\text{Coh}_F)$ of Galois representations over $F$. If $G$ is a finite group scheme, the equality $\chi_c(\{BG\}) = 1$ always holds [5, Proposition 3.1]. On the other hand, if $G$ is connected, then $\chi_c(\{BG\}\{G\}) = 1$; see [2, §2.2]. Since $\{BG\} \neq 1$ for some finite groups $G$, the following question naturally arises.

**Theorem 1.5.** Let $F$ be a finitely generated field extension of $\mathbb{Q}$. There exists a torus $G$ of rank 3 such that

- $BG$ is stably rational,
- $\{BG\} \neq \{G\}^{-1}$ in $K_0(\text{Stacks}_F)$.

Questions 1.1, 1.2 and 1.4 remain open in the case, where the base field $F$ is assumed to be algebraically closed. Our arguments do not shed any new light in this setting.
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The remainder of this paper is structured as follows. In Section 2 we review well known computations of motivic classes for non-split tori. In Section 3 we give explicit formulas for the motivic classes of $G$ and $BG$. In Section 4 we prove Theorem 1.5. In fact, we will present two proofs; the first one is self-contained, and the second makes use of the results of [17] on the $\lambda$-ring structure of $K_0(\text{Var}_F)$. Finally, in Section 5 we prove Theorem 1.6.

2. Preliminaries

We will denote by $F$ a field of char $F \neq 2$. We will write $L$ for the class $\{A^1\}$ in $K_0(\text{Var}_F)$ or $K_0(\text{Stacks}_F)$. If $E$ is an étale algebra over $F$, we will denote by $\{E\}$ the class $\{\text{Spec } E\}$ in $K_0(\text{Var}_F)$ or $K_0(\text{Stacks}_F)$. We will write $C_2$ for the cyclic group of two elements, and $S_n$ for the symmetric group on $n$ symbols.

The following observations will be repeatedly used during the proof of Theorem 1.5.

Lemma 2.1. Let $X$ be a scheme over $F$, $E$ an étale algebra of degree $n$ over $F$, $\alpha \in H^1(F, S_n)$ the class corresponding to $E/F$.

(a) Let $S_n$ act on the disjoint union $\coprod_{i=1}^n X$ by permuting the $n$ copies of $X$. Then $\alpha(\coprod_{i=1}^n X) \sim X_E$.

(b) Let $S_n$ act on $X^n$ by permuting the $n$ factors. Then $\alpha(X^n) \sim R_{E/F}(X)$.

Proof. (a) Let $Y := \coprod_{i=1}^n X$, and let $S_n$ act on $Y$ by permuting the copies of $X$. By definition, $\alpha Y = (Y \times \text{Spec } E)/S_n \cong (Y \times_X X_E)/S_n$, where $S_n$ acts diagonally. This shows that $\alpha Y$ is the twist of $X_E$ by the trivial $S_n$-torsor $Y \to X$ in the category of $X$-schemes, which implies $\alpha Y \cong X_E$. □

(b) See the bottom of page 5 in [7]. □

Lemma 2.2. Let $1 \to N \to G \to H \to 1$ be an exact sequence of group schemes over $F$, and assume that $G$ is special. Then $\{BN\} = \{H\}/\{G\}$.

Proof. See [2, Proposition 2.9]. □

Lemma 2.3. Let $T$ be an algebraic torus over $F$, and let $T'$ be its dual. Assume that $T$ is stably rational. Then

(a) $BT'$ is stably rational;

(b) $\{BT'\}\{T\} = 1$ in $K_0(\text{Stacks}_F)$.

Proof. Since $T$ is stably rational, by [20, §4.7, Theorem 2] there is a short exact sequence

(2.4) $1 \to T_1 \to T_2 \to T \to 1$

where $T_1$ and $T_2$ are quasi-split (that is, of the form $R_{\text{A}/F}(\mathbb{G}_m)$ for some étale algebra $\text{A}/F$). Since quasi-split tori are isomorphic to their dual, the sequence dual to (2.4),

(2.5) $1 \to T' \to T_2 \to T_1 \to 1$, 

is an exact sequence of group schemes over $F$, and hence $\{BT'\}\{T\} = 1$ in $K_0(\text{Stacks}_F)$, as desired. □
shows that $T'$ embeds in $T_2$. We may view $T_2$ as a maximal torus inside $GL_n$, where $n = \text{rank} T_2$. This gives a faithful representation of $T'$ with quotient birational to $T_1$. Since quasi-split tori are rational, it follows that $BT'$ is stably rational.

Quasi-split tori are special, so we may apply Lemma 2.2 to (2.4) and (2.5). We obtain $\{T\} = \{T_2\}/\{T_1\}$ and $\{BT'\} = \{T_1\}/\{T_2\}$, so $\{BT'\}\{T\} = 1$. \hfill \Box

**Lemma 2.6.** Let $E := F(\sqrt{m})$ be a separable quadratic field extension, and let $\alpha$ denote the class of $E/F$ in $H^1(F, C_2)$. Then:

(a) $R_{E/F}(G_m) \cong R_{E/F}(G_m)/G_m$.
(b) Let $\text{Gal}(E/F)$ act on $\mathbb{P}^1$ via $z \mapsto z^{-1}$. Then $\alpha \mathbb{P}^1 \cong \mathbb{P}^1$.
(c) $R_{E/F}(G_m)/G_m$ is rational and

$\{R_{E/F}(G_m)/G_m\} = \{BR_{E/F}(G_m)/G_m\}^{-1} = \mathbb{L} - \{E\} + 1$.

(d) $\{R_{E/F}(G_m)\} = \{BR_{E/F}(G_m)\}^{-1} = (\mathbb{L} - 1)(\mathbb{L} - \{E\} + 1)$.

(e) $\{R_{E/F}(\mathbb{P}^1)\} = \mathbb{L}^2 + \{E\}\mathbb{L} + 1$.

**Proof.** (a) Both tori correspond to the unique non-trivial $\text{Gal}(E/F)$-lattice of rank 1. Hence, $\alpha \mathbb{P}^1 \cong \mathbb{P}^1$.

(b) The $C_2$-action on $\mathbb{P}^1$ has a fixed point $z = 1$. Thus, $\alpha \mathbb{P}^1 \cong \mathbb{P}^1$.

(c) Let $T := R_{E/F}(G_m) \cong R_{E/F}(G_m)/G_m$. The open embedding $G_m \hookrightarrow \mathbb{P}^1$ as the complement of $Z := \{0, \infty\}$ is equivariant under the $C_2$-action on $G_m$ and $\mathbb{P}^1$ given by $z \mapsto z^{-1}$. Twisting by $\alpha$, we obtain by (b) an open embedding of $T$ in $\mathbb{P}^1$ as the complement of $\alpha Z$. In particular, $T$ is rational. By Lemma 2.1(a) $\alpha Z \cong \text{Spec} E$, so $\{T\} = \{\mathbb{P}^1\} - \{\alpha Z\} = \mathbb{L} + 1 - \{E\}$.

Now (c) follows from Lemma 2.3(b).

(d) The first equality holds because $R_{E/F}(G_m)$ is special. Consider the short exact sequence

$$1 \to G_m \to R_{E/F}(G_m) \to T \to 1.$$ 

Since $R_{E/F}(G_m)$ is special, Lemma 2.2 yields

$$\{R_{E/F}(G_m)\} = (\mathbb{L} - 1)\{BT\}^{-1},$$

thus (d) follows from (c).

(e) Write $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, and consider the $C_2$-equivariant decomposition

$$(\mathbb{P}^1)^2 = (\mathbb{A}^1)^2 \amalg (\mathbb{A}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{A}^1) \amalg \{(\infty, \infty)\}.$$ 

By Hilbert 90 Theorem and Lemma 2.1(b), twisting the action by $\alpha$ gives

$$R_{E/F}(\mathbb{P}^1) = \mathbb{A}^2 \amalg \mathbb{A}^1 \amalg \text{Spec} F,$$

thus $\{R_{E/F}(\mathbb{P}^1)\} = \mathbb{L}^2 + \{E\}\mathbb{L} + 1$. \hfill \Box

3. The classes of $G$ and $BG$

Assume that there exists a separable biquadratic extension

$$K := F(\sqrt{m_1}, \sqrt{m_2})$$

of $F$. As we shall see in the next section (see Lemma 4.1), $K$ exists when $F$ is finitely generated over $\mathbb{Q}$. Let

$$E_1 := F(\sqrt{m_1}), \quad E_2 := F(\sqrt{m_2}), \quad E_{12} := F(\sqrt{m_1m_2}), \quad E := E_1 \times E_2.$$
We define the torus
\[ G := R_{E/F}^1(\mathbb{G}_m) \]
and let
\[ G' := R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m \]
be the dual torus of \( G \). Let \( \sigma_1 \) and \( \sigma_2 \) be generators for \( \text{Gal}(K/F) \cong C_2^2 \) such that \( E_1 = K^{\sigma_1} \) and \( E_2 = K^{\sigma_2} \). Consider the \( \text{Gal}(K/F) \)-action on \( \mathbb{G}_m^2 \), where 
\[ \sigma_1(u, v) = (v^{-1}, u^{-1}) \] and \( \sigma_2(u, v) = (v, u) \), and set
\[ T := \alpha \mathbb{G}_m^2, \]
where \( \alpha \in H^1(F, C_2^2) \) corresponds to the extension \( K/F \).

**Lemma 3.1.** We have
\[ \{ T \} = \mathbb{L}^2 + \{ \{ E_{12} \} - \{ K \} \} \mathbb{L} + 1. \]

**Proof.** The embedding of \( \mathbb{G}_m \) in \( \mathbb{P}^1 \) as the complement of \( Z := \{ 0, \infty \} \) gives an open embedding \( \mathbb{G}_m^2 \hookrightarrow (\mathbb{P}^1)^2 \) such that the Galois action on \( \mathbb{G}_m^2 \) extends to \( (\mathbb{P}^1)^2 \). By definition,
\[ \alpha(\mathbb{P}^1)^2 = ((\mathbb{P}^1)^2 \times \text{Spec } K)/\text{Gal}(K/F), \]
where \( \text{Gal}(K/F) = \langle \sigma_1, \sigma_2 \rangle \) acts diagonally. We first take the quotient by the subgroup \( \langle \sigma_1 \sigma_2 \rangle \). Since \( \sigma_1 \sigma_2(u, v) = (u^{-1}, v^{-1}) \) and \( E_{12} = K^{\sigma_1 \sigma_2} \), by Lemma 2.6(b)
\[ \alpha(\mathbb{P}^1)^2 = ((\mathbb{P}^1)^2 \times \text{Spec } E_{12})/C_2, \]
where \( C_2 \) acts on \( (\mathbb{P}^1)^2 \) by switching the two factors. Here we are using the fact that every automorphism of \( (\mathbb{P}^1)^2 \) must respect the ruling (because it respects the intersection form), and so \( \text{Aut}((\mathbb{P}^1)^2) = (\text{Aut}(\mathbb{P}^1))^2 \rtimes C_2 \), where \( C_2 \) switches the two factors. By Lemma 2.6(b) we deduce that \( \alpha(\mathbb{P}^1)^2 \cong R_{E_{12}/F}(\mathbb{P}^1) \), so by Lemma 2.6(e)
\[ \{ \alpha(\mathbb{P}^1)^2 \} = \mathbb{L}^2 + \{ \{ E_{12} \} \} \mathbb{L} + 1. \]

We may partition \( (\mathbb{P}^1)^2 \setminus \mathbb{G}_m^2 \) in two strata
\[ Z_1 := Z \times Z, \quad Z_2 := (Z \times \mathbb{G}_m) \sqcup (\mathbb{G}_m \times Z). \]
Both \( Z_1 \) and \( Z_2 \) have 4 connected components, and \( \text{Gal}(K/F) \) acts on \( Z_1 \) and \( Z_2 \) by transitively permuting the components as the Klein subgroup of \( S_4 \). By Lemma 2.6(a) \( \alpha Z_1 = \text{Spec } K \) and \( \alpha Z_2 = \mathbb{G}_m \times \text{Spec } K \). By Lemma 2.6
\[ \{ T \} = \{ \alpha(\mathbb{P}^1)^2 \} - \{ \alpha Z_1 \} - \{ \alpha Z_2 \}
= \mathbb{L}^2 + \{ E_{12} \} \mathbb{L} + 1 - \{ K \} - \{ K \} \mathbb{L} - 1
= \mathbb{L}^2 + \{ \{ E_{12} \} \} \mathbb{L} + 1. \]

We now give two short exact sequences involving \( G \). Each of them is the starting point for a different proof of Theorem 1.3(b).

**Proposition 3.3.** There are short exact sequences of tori
\[ 1 \to \mathbb{G}_m \to G \to T \to 1, \]
where \( T \) is a torus of rank 2, and
\[ 1 \to R_{E_{12}/F}(\mathbb{G}_m) \to R_{K/F}(\mathbb{G}_m) \to G \to 1. \]
The third term \( Z \) is the trivial \( \text{Gal}(K/F) \)-lattice of rank 1. The map \( \pi \) is defined by \( \pi(a, b, c, d) = a + b - c - d \), and \( P := \text{Ker} \pi \). A basis for \( P \) is given by \( v_1 := (1, 0, 1, 0) \) and \( v_2 := (1, 0, 0, 1) \). In these coordinates, \( \sigma_1(a, b) = (-b, -a) \) and \( \sigma_2(a, b) = (b, a) \). It is clear that \( P \) is the character lattice of \( T \), so (3.4) follows.

Let \( N := M^\vee \). We may identify \( N \) with
\[
N = \{(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 : \sum a_i = 0\}.
\]

Under this identification, \( \text{Gal}(K/F) = C_2^3 \) acts on \( N \) by permuting the subscripts of the coordinates as the subgroup \( \langle (12), (34) \rangle \) of \( S_4 \). We denote by \( e_i \) the \( i \)-th element of the standard basis of \( \mathbb{Z}^4 \). Let \( Q \) be the free \( \mathbb{Z} \)-module generated by the symbols \( e_{13}, e_{23}, e_{14}, e_{24} \). We have a short exact sequence
\[
0 \to \mathbb{Z}^\pm \xrightarrow{\varphi} Q \xrightarrow{\pi} N \to 0
\]
where \( \pi \) is defined by \( \pi(e_{ij}) = e_i - e_j \) and \( \varphi \) by \( \varphi(1) = e_{13} - e_{23} - e_{14} + e_{24} \). The group \( C_2^3 \) acts on \( Q \) by permuting the indices of the coordinates as the subgroup \( \langle (12), (34) \rangle \) of \( S_4 \), and the image of \( \varphi \) is stable under the action. The induced action on \( \mathbb{Z}^\pm \) is the restriction to \( \langle (12), (34) \rangle \) of the sign representation of \( S_4 \). Notice that \( Q \cong \mathbb{Z}[C_2^3] \) is a permutation module, generated by \( e_{13} \). Dualization of (3.7) gives
\[
0 \to M \to \mathbb{Z}[C_2^3] \to \mathbb{Z}^\pm \to 0.
\]
The corresponding exact sequence of tori is (3.5). \( \Box \)

**Proposition 3.8.**
(a) \( BG \) is stably rational.
(b) \( \{BG\}\{G'\} = 1 \) in \( K_0(\text{Stacks}_F) \).

**Proof.** Consider the sequence
\[
1 \to \mathbb{G}_m \to G' \to (R_{E_1/F}(\mathbb{G}_m)/\mathbb{G}_m) \times (R_{E_2/F}(\mathbb{G}_m)/\mathbb{G}_m) \to 1,
\]
which exhibits \( G' \) as a \( \mathbb{G}_m \)-torsor over a rational variety, by Lemma 2.4(c). Thus \( G' \) is rational, and now (a) and (b) follow from Lemma 2.3. \( \Box \)

**Proposition 3.10.** We have
\[
\{G\} = (L - 1)(L^2 + (\{E_{12}\} - \{K\})L + 1)
\]
and
\[
\{BG\}^{-1} = (L - 1)(L - \{E_1\} + 1)(L - \{E_2\} + 1)
\]
in \( K_0(\text{Stacks}_F) \).

**Proof.** By (3.4), \( G \) is a \( \mathbb{G}_m \)-torsor over \( T \). Since \( \mathbb{G}_m \) is special, \( \{G\} = (L - 1)\{T\} \). The class of \( T \) was determined in Lemma 3.1.

By Proposition 3.8(b), \( \{BG\}^{-1} = \{G'\} \). Since \( \mathbb{G}_m \) is special, by (3.9), \( \{G'\} = (L - 1)\{R_{E_1/F}(\mathbb{G}_m)\}\{R_{E_2/F}(\mathbb{G}_m)\} \). Now (3.12) follows from Lemma 2.4(c). \( \Box \)
4. Proof of Theorem 1.5

Lemma 4.1. Let $F$ be a finitely generated field extension of $\mathbb{Q}$. Then $F$ admits infinitely many distinct quadratic extensions.

Proof. Assume by contradiction that $F$ admits at most finitely many distinct quadratic extensions. Denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$, and let $L := F \cap \overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $L$. Then $L$ also admits at most finitely many distinct quadratic extensions. Since every subfield of $F$ is also finitely generated over $\mathbb{Q}$, $L$ is finitely generated; see [11, Chapter VIII, Exercise 4]. A finitely generated algebraic extension is finite, so $L$ is a number field. Denote by $\mathcal{P}$ the set of prime integers. For every $p \in \mathcal{P}$, the extension $L(\sqrt{p})/L$ is either trivial or quadratic. By assumption all but finitely many of these must be trivial, so $L(\sqrt{p} : p \in \mathcal{P})$ is also a number field. This is absurd, because $\mathbb{Q}(\sqrt{p} : p \in \mathcal{P})$ is contained in $L(\sqrt{p} : p \in \mathcal{P})$, and by standard field theory $[\mathbb{Q}(\sqrt{p} : p \in \mathcal{P}) : \mathbb{Q}] = +\infty$. Thus $F$ admits infinitely many quadratic extensions, as desired. \[\square\]

Theorem 1.5(a) was proved in Proposition 4.3(a), so we will focus on Theorem 1.5(b). Recall that by assumption $F$ is finitely generated over $\mathbb{Q}$. By Lemma 4.1 there exists a biquadratic extension of $F$, so the assumptions of Section 3 hold. We maintain the notations given at the beginning of Section 3.

First proof of Theorem 1.5(b). Consider $G := R_{E/F}^{\text{alg}}(\mathbb{G}_m)$. We claim that $G$ does not satisfy (1.3). Assume on the contrary that it does. Then by Proposition 3.10

$$(L - 1)(L - \{E_1\} + 1)(L - \{E_2\} + 1) = (L - 1)(L^2 + (\{E_{12}\} - \{K\})L + 1)$$

in $K_0(\text{Stacks}_F)$. Since $L - 1$ is invertible in $K_0(\text{Stacks}_F)$, this implies

$$L^2 + (2 - \{E_1\} - \{E_2\})L + (1 - \{E_1\})(1 - \{E_2\}) = L^2 + (\{E_{12}\} - \{K\})L + 1,$$

that is

$$\varphi(L) := (2 - \{E_1\} - \{E_2\} - \{E_{12}\} + \{K\})L + \{K\} - \{E_1\} - \{E_2\} = 0.$$

Recall that $K_0(\text{Stacks}_F)$ is the localization of $K_0(\text{Var}_F)$ at $L$ and the cyclotomic polynomials in $L$; see [3, Theorem 1.2]. It follows that $\varphi(L)f(L) = 0$ holds in $K_0(\text{Var}_F)$, for some monic polynomial $f(\mathcal{L}) \in \mathbb{Z}[\mathcal{L}]$.

Since $F$ is a finitely generated field extension of $\mathbb{Q}$, the image of the $\ell$-adic cyclotomic character is infinite, so by [16, Lemma 4.3] the coefficients of $\varphi(L)f(L)$ must vanish in $K_0(\text{Var}_F)$. In particular, the leading coefficient must be zero, that is,

$$(1.4) \quad \{K\} - \{E_1\} - \{E_2\} - \{E_{12}\} + 2\{F\} = 0$$

in $K_0(\text{Var}_F)$. By [17, §3] motivic classes of distinct field extensions of $F$ are linearly independent in $K_0(\text{Var}_F)$, a contradiction. \[\square\]

Second proof of Theorem 1.5(b). By (3.5), Lemma 2.2 and Lemma 2.4(c)

$$(1.5) \quad \{R_{K/F}(\mathbb{G}_m)\} = G/[BR_{E_{12}/F}^{\text{alg}}(\mathbb{G}_m)] = \{G\}(L - \{E_{12}\} + 1)$$

in $K_0(\text{Stacks}_F)$. Assume that $\{BG\} = \{G\}^{-1}$. Then, combining (3.12) and (1.5),

$$(1.6) \quad \{R_{K/F}(\mathbb{G}_m)\} = (L - 1)(L - \{E_1\} + 1)(L - \{E_2\} + 1)(L - \{E_{12}\} + 1).$$

By [17, (1.1)],

$$\{R_{K/F}(\mathbb{G}_m)\} = L^4 + a_1L^3 + a_2L^2 + a_3L + a_4$$
for some integer combinations of classes of spectra of étale algebras $a_i$. More precisely, there is a $\lambda$-ring structure on $K_0(\text{Var}_F)$, and $a_i = (-1)^i \lambda^i \langle \{K\} \rangle$; see [17, §2]. In any $\lambda$-ring $\lambda^i(x) = x$, so $a_1 = -\{K\}$. Both sides of (4.4) are thus monic polynomials of degree 4 in $L$. We subtract $L^4$ from both sides of (4.4), and denote by $\varphi_1(L)$ and $\varphi_2(L)$ the left and the right hand side of the new equation, respectively. They are cubic polynomials in $L$ with leading coefficients

$$-\{K\}, \quad 2 - \{E_1\} - \{E_2\} - \{E_{12}\},$$

respectively. Since $F$ is a finitely generated field extension of $\mathbb{Q}$, the image of the $\ell$-adic cyclotomic character is infinite, so by [16, Lemma 4.3] the coefficients of $\varphi_1(L)f(L)$ and $\varphi_2(L)f(L)$ must agree in $K_0(\text{Var}_F)$. In particular, their leading coefficients must coincide. In other words, (4.2) holds, and we once again obtain a contradiction, as in the first proof.

**Remark 4.5.** By [20, §4.9 Example 7] every torus of rank 2 is rational, so by [16, §4.9] $G$ is rational. By Lemma 4.3 $BG'$ is stably rational and $\{BG'\} = \{G\}^{-1}$. By Proposition 4.3(b) $\{BG\} = \{G'\}^{-1}$, so $\{BG'\} \{G'\} = \{BG\} \{G\}^{-1} \{G\}^{-1}$. Since $\{BG\} \{G\} \neq 1$, the conclusions of Theorem 1.5(a) hold for $G'$ as well.

### 5. Proof of Theorem 1.6

We keep the notation of the previous section.

**Proposition 5.1.** Let $m \geq 1$ be an integer, and let $A$ be the $2m$-torsion subgroup of $G'$. Then $BA$ is stably rational and $\{BA\} = \{BG\}^{-1} \{G\}^{-1}$.

**Proof.** We identify $\text{Gal}(K/F)$ with the subgroup $\langle (12), (34) \rangle \cong C_2^2$ of $S_4$ in such a way that $\sigma_1$ corresponds to (12) and $\sigma_2$ to (34). Let

$$M := \{x \in \mathbb{Z}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

be the character lattice of $R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$. We have a surjection $\varphi : P \to M$, where $P \cong \mathbb{Z}[\langle (12), (34) \rangle]$ has a basis given by $e_1, e_2, e_3, e_4$, $\langle (12), (34) \rangle$ acts on the $e_{ij}$ by permuting the indices, and $\varphi(e_{ij}) := e_i - e_j$. Let $N$ be the kernel of $P \to M \to M/nM$, so that the sequence

$$0 \to N \to P \to M/nM \to 0 \quad (5.2)$$

is exact. It corresponds to a short exact sequence

$$1 \to A \to R_{K/F}(\mathbb{G}_m) \to S \to 1 \quad (5.3)$$

for some torus $S$ whose character lattice is $N$. By Lemma 2.2

$$\{BA\} = \{S\}/\{R_{K/F}(\mathbb{G}_m)\}$$

The lattice $N$ has a basis given by

$$v_{13} := ne_{13}, \quad v_{23} := ne_{23}, \quad v_{14} := ne_{14}, \quad v := e_{13} - e_{23} - e_{14} + e_{24}.$$

We have an exact sequence

$$0 \to N' \to N \xrightarrow{\pi} \mathbb{Z} \to 0 \quad (5.2)$$

where $Z$ is the trivial lattice of rank 1, $\pi(a v_{13} + b v_{23} + c v_{14} + d v) = a + b + c$, and $N' := \text{Ker} \pi$. The corresponding exact sequence of tori is

$$1 \to \mathbb{G}_m \to S \to S' \to 1.$$
where \( S' \) is a torus having \( N' \) as character lattice. Since \( G_m \) is special, we deduce that \( S \) is birational to \( S' \times G_m \). Moreover, \( \{ S \} = (\mathbb{L} - 1)\{ S' \} \), so
\[
\{ BA \} = (\mathbb{L} - 1)\{ S' \}/\{ R_{K/F}(G_m) \}.
\]

A basis for \( N' \) is given by
\[
v, \quad v_{13} - v_{23}, \quad v_{13} - v_{14}.
\]

With respect to this basis, the action of (12) and (34) is given by the matrices
\[
(5.4) \quad \rho_{(12)} := \begin{pmatrix} -1 & 0 & 2m \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_{(34)} := \begin{pmatrix} -1 & 2m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Let
\[
\tau := \begin{pmatrix} 1 & m & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau^{-1} = \begin{pmatrix} 1 & -m & -m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Then
\[
(5.6) \quad \tau^{-1}\rho_{(12)}\tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau^{-1}\rho_{(34)}\tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

This shows that \( S' \cong R_{E_1/F}(G_m) \times R_{E_2/F}(G_m) \times R_{E_{12}/F}(G_m) \), so by Lemma 2.4.1 we deduce that \( S' \) is rational and
\[
(5.7) \quad \{ S' \} = (\mathbb{L} - \{ E_1 \} + 1)(\mathbb{L} - \{ E_2 \} + 1)(\mathbb{L} - E_{12} + 1).
\]

Since \( S \) is birational to \( S' \times G_m \), \( S \) is also rational, so by (5.3) we obtain that \( BA \) is stably rational. Moreover, by (5.4) and (5.7)
\[
\{ BA \} = (\mathbb{L} - 1)(\mathbb{L} - \{ E_1 \} + 1)(\mathbb{L} - \{ E_2 \} + 1)(\mathbb{L} - E_{12} + 1)/\{ R_{K/F}(G_m) \}.
\]

Using (3.5) and (3.11), we conclude that \( \{ BA \} = \{ BG \}_1 \{ G \}^{-1} \). \hfill \Box

**Proof of Theorem 5.6.** Let \( m \geq 1 \) be an integer, and let \( A \) be the \( 2m \)-torsion subgroup of \( G' \). By Proposition 5.1 \( BA \) is stably rational and \( \{ BA \} = \{ BG \}_1 \{ G \}^{-1} \) in \( K_0(\text{Stacks}_F) \). By assumption, \( F \) is finitely generated over \( \mathbb{Q} \), so we may apply Theorem 1.5. Thus \( \{ BG \} \neq \{ G \}^{-1} \), and \( \{ BA \} \neq 1 \). \hfill \Box

**Remark 5.8.** Let \( L/F \) be an étale algebra, and let \( A \) be the \( n \)-torsion subgroup of \( R_{L/F}(G_m) \). We have a short exact sequence
\[
1 \rightarrow A \rightarrow R_{L/F}(G_m) \rightarrow R_{L/F}(G_m) \rightarrow 1
\]
given by the \( n \)-th power map. Since \( R_{L/F}(G_m) \) is special, by [2] Proposition 2.9
\[
\{ BA \} = \{ R_{L/F}(G_m) \}/\{ R_{L/F}(G_m) \} = 1.
\]

Let now \( A' \) be the \( n \)-torsion subgroup of \( R_{L/F}(G_m) \). The embedding of \( R_{L/F}(G_m) \) in \( R_{L/F}(G_m) \) gives a short exact sequence
\[
1 \rightarrow A' \rightarrow R_{L/F}(G_m) \rightarrow T \rightarrow 1
\]
for some torus \( T \). Since \( R_{L/F}(G_m) \) is special, by [2] Proposition 2.9
\[
\{ BA' \} = \{ T \}/\{ R_{L/F}(G_m) \}.
\]
The image of the composition $\mathbb{G}_m \to R_{L/F}(\mathbb{G}_m) \to T$ is a subtorus of $T$ isomorphic to $\mathbb{G}_m$, and we have an exact sequence

$$1 \to \mathbb{G}_m \to T \to R_{L/F}(\mathbb{G}_m)/\mathbb{G}_m \to 1.$$ 

Therefore $\{T\} = \{\mathbb{G}_m\}\{R_{L/F}(\mathbb{G}_m)/\mathbb{G}_m\} = \{R_{L/F}(\mathbb{G}_m)\}$. Thus $\{BA'\} = 1$.

By taking $L = E$, we see that the formula $\{BG\} \{G\} = 1$ fails for $G$ (see Theorem 1.5), while the formula $\{BA'\} = 1$ holds for every torsion subgroup $A' = G[n]$ of $G$.

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REFERENCES

[1] Kai Behrend and Ajneet Dhillon. On the motivic class of the stack of bundles. Advances in Mathematics, 212(2):617–644, 2007.

[2] Daniel Bergh. Motivic classes of some classifying stacks. Journal of the London Mathematical Society, 93(1):219–243, 2015.

[3] Séminaire Claude Chevalley and JP Serre. Espaces fibrés algébriques. Séminaire Claude Chevalley, 3:1–37, 1958.

[4] Ajneet Dhillon, Matthew B Young, et al. The motive of the classifying stack of the orthogonal group. The Michigan Mathematical Journal, 65(1):189–197, 2016.

[5] Torsten Ekedahl. A geometric invariant of a finite group. arXiv preprint arXiv:0903.3148, 2009.

[6] Torsten Ekedahl. The Grothendieck group of algebraic stacks. arXiv preprint arXiv:0903.3143, 2009.

[7] Mathieu Florence and Zinovy Reichstein. On the rationality problem for forms of moduli spaces of stable marked curves of positive genus. arXiv preprint arXiv:1709.05696, 2017.

[8] Mathieu Florence and Zinovy Reichstein. The rationality problem for forms of $M_{0,n}$. Bulletin of the London Mathematical Society, 50(1):148–158, 2018.

[9] Dominic Joyce. Motivic invariants of Artin stacks and stack functions. The Quarterly Journal of Mathematics, 58(3):345–392, 2007.

[10] B. É. Kunyavskii. Three-dimensional algebraic tori. In Investigations in number theory (Russian), pages 90–111. Saratov. Gos. Univ., Saratov, 1987. Translated in Selecta Math. Soviet. 9 (1990), no. 1, 1–21.

[11] Serge Lang. Algebra, volume 211 of Graduate texts in mathematics. Springer-Verlag, New York., 2002.

[12] Ivan Martino. Ekedahl Invariants, Veronese Modules and Linear Recurrence Varieties. PhD thesis, Department of Mathematics, Stockholm University, 2014.

[13] Ivan Martino. The Ekedahl invariants for finite groups. Journal of Pure and Applied Algebra, 220(4):1294–1309, 2016.

[14] Ivan Martino. Introduction to the Ekedahl Invariants. Mathematica Scandinavica, 120(2):211–224, 2017.

[15] Roberto Pirisi and Mattia Talpo. On the motivic class of the classifying stack of $G_2$ and the spin groups. To appear in International Mathematics Research Notices. arXiv preprint arXiv:0903.3143.

[16] Karl Rökaeus. The computation of the classes of some tori in the Grothendieck ring of varieties. arXiv preprint arXiv:0708.4396, 2007.

[17] Karl Rökaeus. The class of a torus in the Grothendieck ring of varieties. American Journal of Mathematics, 133(4):939–967, 2011.

[18] Mattia Talpo and Angelo Vistoli. The motivic class of the classifying stack of the special orthogonal group. Bulletin of the London Mathematical Society, 49(5):818–823, 2017.

[19] Bertrand Toën. Grothendieck rings of Artin n-stacks. arXiv preprint math/0509098, 2005.
[20] Valentin Evgen’evich Voskresenskii and Boris Kunyavski. *Algebraic groups and their birational invariants*, volume 179. American Mathematical Soc., 2011.