Kelvon-roton instability of vortex lines in dipolar Bose-Einstein condensates

M. Klawunn1, R. Nath1, P. Pedri2 and L. Santos1
1Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstr. 2, D-30167, Hannover, Germany
2Laboratoire de Physique Théorique et Modèles Statistiques, Université Paris Sud, 91405 Orsay Cedex, France

The physics of vortex lines in dipolar condensates is studied. Due to the nonlocality of the dipolar interaction, the 3D character of the vortex plays a more important role in dipolar gases than in typical short-range interacting ones. In particular, the dipolar interaction significantly affects the stability of the transverse modes of the vortex line. Remarkably, in the presence of a periodic potential along the vortex line, a roton minimum may develop in the spectrum of transverse modes. We discuss the appropriate conditions at which this roton minimum may eventually lead to an instability of the straight vortex line, opening new scenarios for vortices in dipolar gases.

Introduction.- When rotated at sufficiently large angular frequency, a superfluid develops vortex lines of zero density [1,2], around which the circulation is quantized due to the single-valued character of the corresponding wavefunction $\psi$. Quantized vortices constitute indeed one of the most important consequences of superfluidity, playing a fundamental role in various physical systems, as superconductors [3] and superfluid Helium [4]. Due to the enormous progress in the control and manipulation of ultra cold gases, Bose-Einstein condensates (BEC) offer an extraordinarily controllable system for the analysis of superfluidity, and in particular quantized vortices [6]. Vortices and even vortex lattices have been created in BEC in a series of milestone experiments [7,8,9].

Vortex lines are indeed 3D structures, which, resembling strings, may present transverse excitations, which were studied for classical vortices by Lord Kelvin already in 1880, and are therefore known as Kelvin modes [10]. These excitations have been also studied in the context of quantized vortices in superfluids by Pitaevskii [11]. Interestingly, the dispersion law for Kelvin modes at small wave vector $k$ follows a characteristic dependence $\varepsilon(k) \sim -k^2 \ln k \xi$, where $\xi$ is the healing length. Kelvin modes play an important role in various physical systems, as superconductors [3] and superfluid Helium [4].

In this Letter, we analyze the role that the long-range character of the DDI plays in the physics of vortex lines in dipolar BEC. Due to this long-range character, different parts of the vortex line interact via DDI, and hence the 3D character of the vortices plays a much more important role in dipolar gases than in usual short-range interacting ones. This has two main consequences. On one hand, for a fixed density the vortex core depends on the dimensionality of the problem, and can hence have in trapped gases a non-trivial dependence on the trap geometry. Additionally, depending on the dipole orientation, the DDI may significantly enhance or reduce the stiffness of the line against transverse excitations. Even more interestingly, under appropriate conditions, discussed in this Letter, the DDI may induce a minimum in the dispersion law for the transverse modes, i.e. a Kelvon-roton spectrum. For sufficiently large DDI this minimum may reach zero energy, and hence the DDI may destabilize the straight-vortex configuration, opening the possibility for achieving other ground-state vortex configurations.

Effective model.- In the following, we consider a dipolar BEC of particles with mass $m$ and electric dipole $d$...
where \( V_\text{f}(\vec{r}) = \alpha d^2 (1 - 3 \cos^2 \theta)/r^3 \), where \( \theta \) is the angle formed by the vector joining the interacting particles and the dipole direction. The coefficient \( \alpha \) can be tuned within the range \(-1/2 \leq \alpha \leq 1\) by rotating the external field that orients the dipoles much faster than any other relevant time scale in the system \([31]\). At sufficiently low field that orients the dipoles much faster than any other and the dipole direction. The coefficient \( \phi \) is the recoil energy and \( q \) is the laser wave vector.

In the tight-binding regime (sufficiently strong lattice) we can write \( \Psi(\vec{r}, t) = \sum_j f(z - b_j) \psi_j(\vec{r}, t) \), where \( b = \pi/q, \vec{b} = \{x, y\} \) and \( f(z) \) is the Wannier function associated to the lowest energy band. Substituting this Ansatz in Eq. (1) we obtain a discrete non-linear Schrödinger equation. We may then return to a continuous equation, where the lattice is taken into account in an effective mass along the lattice direction and in the renormalization of the coupling constant \([33, 34]\):

\[
i\hbar \frac{\partial \Psi(\vec{r})}{\partial t} = \left\{ \frac{-\hbar^2 \nabla^2}{2m} - \frac{\hbar^2 \beta^2}{2m^*} + \tilde{g}|\Psi(\vec{r})|^2 + \int d\vec{r}' |\Psi(\vec{r})|^2 V_d(\vec{r} - \vec{r}') \right\} \Psi(\vec{r}), \tag{2}
\]

where \( \tilde{g} = b\gamma \int f(z)^4 dz + g_a C \), with \( g_a = \alpha \pi \alpha^2 d^3/3, m^* = \hbar^2 \beta^2/2m \) is the effective mass, and \( J = \int f(z)[-\hbar^2/2m] \delta_x^2 + V_a(z)] f(z + b) dz \). Note that the discreteness of the lattice is unimportant, i.e. Eq. (2) is valid, if \( k \ll \pi/b \), being \( k \) the z-momentum. In the following we use the convenient dimensionless parameter \( \beta = ga/\tilde{g} \), that characterizes the strength of the DDI compared to the short-range interaction. It will be also useful in the following the Fourier transform of the DDI: \( \tilde{V}_d(\vec{k}) = g_a [3 \cos^2 \theta_k - 1]/2 \), with \( \cos^2 \theta_k = k_x^2/|k|^2 \).

**Homogeneous solution.** - We consider first an homogeneous solution of the form \( \Psi_\text{f}(\vec{r}, t) = \sqrt{\rho} \exp[-i\beta t/h] \), where \( \rho \) denotes the condensate density, and \( \mu = (g + \tilde{V}_d(0))/\tilde{n} \) is the chemical potential. From the corresponding Bogoliubov-de Gennes (BdG) equations one obtains that the energy \( \epsilon(\vec{k}) \) corresponding to an excitation of wave number \( \vec{k} \) fulfills: \( \epsilon(\vec{k})^2 = E_{\text{kin}}(\vec{k})[E_{\text{kin}}(\vec{k}) + E_{\text{int}}(\vec{k})] \) where \( E_{\text{kin}}(\vec{k}) = \hbar^2 k^2/2m + \hbar k^2/2m^* \) is the kinetic energy, and \( E_{\text{int}}(\vec{k}) = 2(g + \tilde{V}_d(\vec{k}))/\tilde{n} \). Stable phonons (i.e. excitations at low \( k \)) are only possible if \( E_{\text{int}} > 0 \) for all \( \vec{k} \), i.e. if \( 2 + \beta(3 \cos^2 \theta_k - 1) > 0 \). If \( g_d > 0 \) phonons with \( \vec{k} \) lying on the \( xy \) plane are unstable if \( \beta > 2 \), while for \( g_d < 0 \) phonons with \( \vec{k} \) along \( z \) are unstable if \( \beta < -1 \). Hence absolute phonon stability demands \(-1 < \beta < 2 \).

**Vortex solution.** - We consider at this point a condensate with a straight vortex line along the z-direction. The corresponding ground-state wavefunction is of the form \( \Psi_\text{g}(\vec{r}, t) = \phi_0(\rho) \exp(i\phi) \exp[-i\beta t/h] \), where \( \phi \) is the polar angle on the \( xy \) plane. The function \( \phi_0(\rho) \) fulfills

\[
\mu \phi_0(\rho) = \frac{\hbar^2}{2m} \left( -D_\rho + \frac{1}{\rho^2} \right) \phi_0(\rho) + \tilde{g} \phi_0(\rho) \phi_0(\rho), \tag{3}
\]

where \( D_\rho = \frac{1}{\rho} \partial_\rho \rho \partial_\rho \) and \( \tilde{g} = g - g_d/2 \). Note that since \( \Psi_\text{g} \) is independent of \( z \), the DDI just leads to a regularization of the contact interaction. The density of the vortex core is given by \( |\phi_0(\rho)|^2 \), which goes to zero at \( \rho = 0 \), and becomes equal to the bulk density \( \tilde{n} \) at distances larger than the corresponding healing length \( \xi = h/\sqrt{\rho \mu} \). Note that due to the regularization of the contact interaction, the size \( \xi \) of the vortex core depends on the DDI. In particular, this dependence is exactly the opposite as that expected for 2D vortices, since in 2D similar arguments provide \( \tilde{g} = g + g_d \). Hence, even for equal densities, and due to the long-range character of the DDI, the cores of a 2D vortex and a 3D vortex line can be remarkably different in a dipolar gas, differing significantly from the behavior of short-range interacting gases, where both 2D and 3D vortices would have the same core size.
The effects of the long-range character of the DDI become even more relevant in the physics of Kelvin modes. In order to analyze these modes, we consider a perturbation of the straight vortex solution of the form \( \Psi(\vec{r}, t) = \Psi_0(\vec{r}, t) + \chi(\vec{r}, t) \exp[i(\varphi - \mu t / \hbar)], \) where \( \chi(\vec{r}, t) = \sum_l [\psi_l(\rho) \exp[i(ql + gz - \epsilon_l t / \hbar)] - \nu_l(\rho)^* \exp[-i(ql + gz - \epsilon^* t / \hbar)]. \) Introducing this Ansatz into Eq. (2) and linearizing in \( \chi, \) one obtains the corresponding BdG equations:

\[
\begin{align*}
\epsilon u_l(\rho) &= \left[ \frac{\hbar^2}{2m} \left( -D_\rho + \frac{(l + 1)^2}{\rho^2} + \frac{m}{m^*} q^2 \right) - \mu + 2\bar{g}\psi_0(\rho)^2 \right] u_l(\rho) - \bar{g}\psi_0(\rho)^2 v_l(\rho) \\
&+ \frac{3\beta}{2 - \beta} \bar{g} \int_0^\infty d\rho' \rho' \psi_0(\rho') \psi_0(\rho) \left[ u_l(\rho') - v_l(\rho') \right] F(k\rho, k\rho') \\
\epsilon v_l(\rho) &= -\left[ \frac{\hbar^2}{2m} \left( -D_\rho + \frac{(l - 1)^2}{\rho^2} + \frac{m}{m^*} q^2 \right) - \mu + 2\bar{g}\psi_0(\rho)^2 \right] v_l(\rho) + \bar{g}\psi_0(\rho)^2 u_l(\rho) \\
&+ \frac{3\beta}{2 - \beta} \bar{g} \int_0^\infty d\rho' \rho' \psi_0(\rho') \psi_0(\rho) \left[ u_l(\rho') - v_l(\rho') \right] F_l(k\rho, k\rho'),
\end{align*}
\]

with \( F_l(x, x') = I_l(x_<) K_l(x_>) \), where \( I_l \) and \( K_l \) are modified Bessel functions, and \( x_> = \max(x, x') \), \( x_< = \min(x, x') \). For every \( q \) we determine the lowest eigenenergy \( \epsilon(q) \) that provides the dispersion law discussed below. The first line at the rhs of Eqs. (4) and (5) is exactly the same as that expected for a vortex in a short-range interacting BEC [11], but with the regularized value \( \bar{g} \). The last term at the rhs of both equations is directly linked with the long-range character of the DDI and, as we show below, leads to novel phenomena in the physics of Kelvin modes (\( l = 1 \)) in dipolar BECs.

In absence of DDI (or equivalently from Eqs. (4) and (5) without the last integral term) the dispersion law at low momenta \( (q\xi < 1) \) is provided by the well-known expression \( \epsilon(q) = -(\hbar^2 q^2 / 2m^*) \ln [(m/m^*)^{1/2} q\xi] \). The integral terms of Eqs. (4) and (5) significantly modify the Kelvin spectrum in a different way depending whether \( \beta \) is positive or negative. In order to isolate the effect of the DDI on the core size with respect to the effect of the integral terms in Eqs. (4) and (5) we fix \( \beta \) and change the parameter \( \beta \) which is proportional to the dipole-dipole coupling constant. Fig. 1 shows that for increasing \( \beta > 0 \) the excitation energy clearly increases, i.e. the vortex line becomes stiffer against transverse modulations.

In order to obtain an intuitive picture of why this is so, one may sketch the vortex core as a 1D chain of dipolar holes. Dipolar holes interact in exactly the same way as dipolar particles, and hence maximally attract each other when aligned along the dipole direction, i.e. the \( z \)-direction. In this sense, the configuration of minimal dipolar energy is precisely that of a straight vortex line along \( z \). A wiggling of the line produces a displacement of the dipolar holes to the side, and hence an increase of the dipolar energy. As a consequence, the DDI leads to an enhanced stiffness of the vortex line. From this intuitively transparent picture, we can easily understand that exactly the opposite occurs when \( \beta < 0 \). In that case, the dipolar holes maximally repel each other when aligned along the \( z \)-direction. Hence it is expected that the dipolar energy is minimized when departing from the straight vortex configuration. This results in a reduced energy of the excitations for \( \beta < 0 \) (Fig. 1), i.e. it becomes easier to wiggle the vortex line.

In principle, a sufficiently large DDI could even destabilize the straight vortex line. However, in the absence of an additional optical lattice along the \( z \)-direction, the destabilization would occur for values of \( \beta < -1 \), i.e. in the regime of phonon instability in which the whole dipolar BEC is unstable against local collapses. An increase of the potential depth of the additional lattice leads to a reduction of the role of the kinetic energy term \( m\dot{q}^2 / m^* \) in Eqs. (4) and (5) that enhances the effect of the dipolar interaction on the dispersion law. As a consequence, as shown in Fig. 2, in addition to the \( -k^2 \ln(q\xi) \) dependence at low momenta, a minimum eventually appears at intermediate momenta, i.e. the spectrum develops a Kelvin-roton character. Note that, typically, the dispersion law presents a relatively abrupt change in its curvature for a given value of \( q \) (\( \simeq 1.4 \) in Fig. 2). Interestingly this feature is given by an avoided level-crossing of the
lowest eigenvalues $\epsilon(q)$. As a consequence the character of the lowest Bogoliubov modes changes at this value of $q$. For a sufficiently small $(m/m^*)$, the roton minimum eventually reaches zero energy, and the straight vortex line becomes unstable against a novel type of instability. Fig. 3 shows as a function of $(m/m^*)$ and $\beta$, the different regimes for the straight-vortex line.

Roton minima occur in the dispersion law of superfluid Helium [1], and have been also predicted in trapped dipolar gases under a strong 1D and 2D confinement [20]. In those cases, a phonon-roton spectrum occurs related to those cases, a phonon-roton spectrum occurs related to 3D character of the vortices is much more crucial in dipolar BECs than in short-range interacting ones. The size of the vortex core depends significantly on the system dimensionality, which indicates that, although we have restricted to homogeneous gases, vortex lines in trapped dipolar BECs may have a non-trivial dependence on the trap geometry. On the other hand, and depending on the dipole orientation, the DDI may significantly increase or decrease the stiffness of the line against Kelvin excitations. Interestingly, under appropriate conditions (additional optical lattice along the vortex line), a Kelvon-roton dispersion law occurs. For sufficiently large DDI a roton instability of the straight vortex line develops, which may preclude the appearance of a new type of helicoidal ground-state configuration of the vortex line.

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