Structures Preserved by Consistently Graded Lie Superalgebras

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Abstract

Dual Pfaff equations (of the form $\bar{D}^a = 0$, $\bar{D}^a$ some vector fields of degree $-1$) preserved by the exceptional infinite-dimensional simple Lie superalgebras $\mathfrak{sle}(5|10)$, $\mathfrak{vle}(3|6)$ and $\mathfrak{mb}(3|8)$ are constructed, yielding an intrinsic geometric definition of these algebras. This leads to conditions on the vector fields, which are solved explicitly. Expressions for preserved differential form equations (Pfaff equations), brackets (similar to contact brackets) and tensor modules are written down. The analogous construction for the contact superalgebra $\mathfrak{f}(1|m)$ (a.k.a. the centerless $N = m$ superconformal algebra) is reviewed.
1 Introduction

The simple Lie algebras of vector fields (v.f.) were classified by Cartan in 1909 [3]. The classification consists of the four infinite series:

- \( \text{vect}(n) \) general v.f. in \( n \) dimensions,
- \( \text{svect}(n) \) divergence-free v.f.,
- \( \mathfrak{h}(n) \) Hamiltonian v.f. (\( n \) even),
- \( \mathfrak{t}(n) \) contact v.f. (\( n \) odd).

The corresponding problem in the super case was only recently settled [5, 8, 13, 17], building on earlier work in [1, 2, 4, 6, 7, 11, 12, 14, 15, 16, 18]. The classification consists of ten infinite series:

- \( \text{vect}(n|m) \) arbitrary v.f. in \( n|m \) dimensions,
- \( \text{svect}(n|m) \) divergence-free v.f.,
- \( \mathfrak{h}(n|m) \) Hamiltonian v.f. (\( n \) even),
- \( \mathfrak{le}(n) \) odd Hamiltonian or Leitesian v.f. \( \subset \text{vect}(n|m) \),
- \( \mathfrak{sle}(n) \) divergence free Leitesian v.f.,
- \( \mathfrak{tl}(n|m) \) contact v.f. (\( n \) odd),
- \( \mathfrak{to}(n) \) odd contact v.f. \( \subset \text{vect}(n|m+1) \),
- \( \mathfrak{stle}_\beta(n) \) a deformation of div-free odd contact v.f.,
- \( \mathfrak{sle}(n) \) a deformation of \( \mathfrak{sl}(n) \),
- \( \mathfrak{sto}(n) \) a deformation of \( \mathfrak{so}(n) \).

A geometric way to describe these algebras is by stating what structures they preserve, or what other conditions the vector fields obey:

| Algebra | Basis          | Description/structure preserved |
|---------|----------------|--------------------------------|
| \( \text{vect}(n|m) \) | \( u^i, \theta^a \) | -                               |
| \( \text{svect}(n|m) \) | \( u^i, \theta^a \) | \( \text{vol} \)                |
| \( \mathfrak{h}(n|m) \) | \( u^i, \theta^a \) | \( \omega_{ij} du^i du^j + g_{ab} d\theta^a d\theta^b \) |
| \( \mathfrak{le}(n) \) | \( u^i, \theta_i \) | \( du^i d\theta_i \)         |
| \( \mathfrak{sle}(n) \) | \( u^i, \theta_i \) | \( du^i d\theta_i, \text{vol} \) |
| \( \mathfrak{tl}(n+1|m) \) | \( t, u^i, \theta^a \) | \( dt + \omega_{ij} u^i du^j + g_{ab} d\theta^a d\theta^b = 0 \) |
| \( \mathfrak{to}(n) \) | \( \tau, u^i, \theta_i \) | \( d\tau + u^i d\theta_i + \theta_i du^i = 0 \) |
| \( \mathfrak{stle}_\beta(n) \) | \( \tau, u^i, \theta_i \) | \( M_f \in \mathfrak{to}(n) : \text{div}_\beta M_f = 0 \) |
| \( \mathfrak{sle}(n) \) | \( u^i, \theta_i \) | \( (1 + \theta_1 \ldots \theta_n) X, X \in \mathfrak{sle}(n) \) |
| \( \mathfrak{sto}(n) \) | \( \tau, u^i, \theta_i \) | \( (1 + \theta_1 \ldots \theta_n) X, X \in \mathfrak{sto}_{n+1}(n) \) |

In this table, \( u^i \) denotes bosonic variables, \( \theta^a \) and \( \theta_i \) fermionic variables, and \( t (\tau) \) is an extra bosonic (fermionic) variable. The indices range over the dimensions indicated: \( i = 1, \ldots, n \) and \( a = 1, \ldots, m \). \( \omega_{ij} = -\omega_{ji} \) and \( g_{ab} = g_{ba} \).
are structure constants. The notation $\alpha = 0$ implies that it is this Pfaff equation that is preserved, not the form $\alpha$ itself; a vector field $X = X^\mu(x)\partial_\mu$ acts on $\alpha$ as $\mathcal{L}_X \alpha = f_X \alpha$, $f_X$ some polynomial function. vol denotes the volume form; vector fields preserving $\text{vol}$ satisfy $\text{div}X \equiv (-)^X \partial_\mu X^\mu = 0$.

In the mathematics literature, the exceptional Lie superalgebras are described in terms of Cartan prolongation. Unfortunately, this method is defined recursively in a way which does not exhibit the geometric content. The purpose of the present paper is to describe the structures preserved by three of the four consistently graded exceptions. Conversely, knowledge of these structures gives an intrinsic geometric definition of the algebras themselves. I also write down explicit equations satisfied by the vector fields in these algebras, give general solutions to these equations, explicitly write down the brackets (analogous to Poisson or contact brackets), and construct the tensor modules. This is listed as the first open problem in \cite{17}. The present paper also gives a partial answer to the third item in Kac' vision list for the new millenium \cite{9}.

Let $g \subset \text{vect}(n|m)$ be an algebra of polynomial vector fields acting on $\mathbb{C}^{n|m}$. It has a Weisfeiler $\mathbb{Z}$-grading of depth $d$ if it can be written as

$$ g = g_{-d} + \ldots + g_{-1} + g_0 + g_1 + \ldots, $$

where the subspace $g_k$ consists of vector fields that are homogeneous of degree $k$, $g_0$ acts irreducibly on $g_{-1}$, and $g_{-k} = g_{k+1}$. However, it is not the usual kind of homogeneity, because we do not assume that all directions are equivalent. Denote the coordinates of $n|m$-dimensional superspace by $x^\mu$ and let $\partial_\mu$ be the corresponding derivatives. Then we define the grading by introducing positive integers $z_\mu$ such that $\deg x^\mu = z_\mu$ and $\deg \partial_\mu = -z_\mu$. The operator which computes the Weisfeiler grading is $Z = \sum_\mu z_\mu x^\mu \partial_\mu$, and $g_k$ is the subspace of vector fields $X$ satisfying $[Z, X] = kX$. If we only considered $g$ as a graded vector space, we could of course make any choice of integers $z_\mu$, but we also want $g$ to be graded as a Lie algebra: $[g_i, g_j] = g_{i+j}$. The depth $d$ is identified with the maximal $z_\mu$. We write $g_- = g_{-d} + \ldots + g_{-1}$ and $g_+ = g_1 + g_2 + \ldots$; $g_-$ is a nilpotent superalgebra and a $g_0$ module.
Cartan prolongation is defined as follows:

1. Start with a realization for the non-positive part $g_0 \ltimes g_-$ of $g$ in $n|m$-dimensional superspace.

2. Define $g_k$ recursively for positive $k$ as the maximal subspace of $\text{vect}(n|m)$ satisfying $[g_k, g_-] \subset g_{k-1}$.

In this paper I will use the following alternative method to construct the prolong:

1. Again start with a realization for $g_0 \ltimes g_-$.

2. Determine the set of structures preserved by this algebra.

3. Define the Cartan prolong $g_\ast = (g_d, \ldots, g_1, g_0)$, as the full subalgebra of $\text{vect}(n|m)$ preserving the same structures.

Clearly, the set of vector fields that preserve some structures automatically define a subalgebra of $\text{vect}(n|m)$, so the problem is to find the right set. If the conditions imposed are too strong, the solution may be a finite-dimensional algebra, and if they are too weak, the resulting algebra may not be simple. In analogy with the known cases listed above, it is natural to assume that such a structure is either some differential form, or an equation satisfied by forms (Pfaff equation), or a system of Pfaff equations. However, although such Pfaff equations can be constructed (and are so in this paper), it is simpler to consider the set of dual Pfaff equations, which are of the form $\tilde{D}^a = 0$, where $\tilde{D}^a$ is some vector field in $\text{vect}(n|m)$ of degree $-1$ (necessarily not in $g$). Denote the space spanned by such $\tilde{D}^a$ by $\tilde{g}_- \subset \text{vect}(n|m)$. We have $[Z, \tilde{D}^a] = -\tilde{D}^a$ and $[\tilde{D}^a, \tilde{D}^b] = 0$ for all $\tilde{D}^a \subset \tilde{g}_-$. Since $g_\ast = g_k$, this implies $[\tilde{g}_-, \tilde{g}_-] = 0$. Moreover, defining $\tilde{g}_- = g^\perp_1$ and $g_- = \bigoplus_k \tilde{g}_k$, we see that $[\tilde{g}_-, g_-] = 0$, so $g_-$ and $\tilde{g}_-$ form two commuting subalgebras of $\text{vect}(n|m)$. At least in all cases considered in this paper, these two subalgebras are isomorphic nilpotent algebras. Now define the prolong $g = \{X \in \text{vect}(n|m) : [\tilde{D}^a, X] = f^a_b \tilde{D}^b\}$, for each $\tilde{D}^a \in \tilde{g}_-$, where $f^a_b$ are some polynomial functions, depending on $X$.

To see that the second definition implies the first, let $D \in g_-$ and $X \in g_k$. $Y = [D, X]$ satisfies $[\tilde{D}^a, Y] = [\tilde{D}^a, [D, X]] = [D, [\tilde{D}^a, X]] = [D, f^a_b \tilde{D}^b]$, so $Y \in g_{k-1}$ and $[g_{k-1}, g_k] \subset g_{k-1}$.

Vector fields that preserve the dual Pfaff equation $\tilde{g}_- = 0$ must also preserve the higher order equations $\tilde{g}_- = 0$ and, for $\mathbf{mb}(3|8)$, $\tilde{g}_- = 0$. This gives rise to further conditions obeyed by the vector fields, but these additional relations are identities which follow from $\tilde{g}_- = 0$. A new feature in the exceptions is the appearance of certain symmetry conditions. These are genuinely new constraints which have no counterpart in the contact algebra.
The algebras under consideration in this paper have the following description as Cartan prolongs:

\[
\begin{align*}
\mathfrak{k}_{5|10} &= (\mathfrak{5}, \mathfrak{5}^* \wedge \mathfrak{5}^*, \mathfrak{sl}(5))_s, \\
\mathfrak{v}_{3|6} &= (\mathfrak{3} \boxtimes \mathfrak{1}, \mathfrak{3}^* \boxtimes \mathfrak{2}, \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1))_s, \\
\mathfrak{m}_{3|8} &= (\mathfrak{1} \boxtimes \mathfrak{2}, \mathfrak{3} \boxtimes \mathfrak{1}, \mathfrak{3}^* \boxtimes \mathfrak{2}, \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1))_s, \\
\mathfrak{k}_{1|6} &\subset \mathfrak{k}(1|6) = (\mathfrak{1}, \mathfrak{6}, \mathfrak{so}(6) \oplus \mathfrak{gl}(1))_s.
\end{align*}
\]

Here \(\boxtimes\) denotes the smash product, \(\mathfrak{n}\) is the \(n\)-dimensional representation of \(\mathfrak{sl}(n)\) or \(\mathfrak{so}(n)\) and \(\mathfrak{n}^*\) its dual. In addition, I apply the same technique to construct the contact algebras \(\mathfrak{k}(1|m) = (\mathfrak{1}, \mathfrak{m}, \mathfrak{so}(m) \oplus \mathfrak{gl}(1))_s\). Although the results for the contact algebras are not new, they are of interest for several reasons:

1. \(\mathfrak{k}_{5|10}\), \(\mathfrak{v}_{3|6}\) and \(\mathfrak{m}_{3|8}\) are conceptually similar to \(\mathfrak{k}(1|m)\), albeit more complicated, so this is a good place to develop the machinery necessary for the exceptions.

2. The exception \(\mathfrak{k}_{1|6}\) is a subalgebra of \(\mathfrak{k}(1|6)\), so \(\mathfrak{k}_{1|6}\) satisfies the conditions described in section 2, together with some additional relations.

3. \(\mathfrak{k}(1|m)\) is consistently graded. In fact, the only simple Lie superalgebras with consistent gradings are \(\mathfrak{k}(1|m)\), \(\mathfrak{k}_{5|10}\), \(\mathfrak{v}_{3|6}\), \(\mathfrak{m}_{3|8}\) and \(\mathfrak{k}_{1|6}\) \(\square\). The present paper thus describes every consistently graded simple Lie superalgebra, although no significant results are obtained for \(\mathfrak{k}_{1|6}\).

4. For small values of \(m\), the Laurent polynomial version of \(\mathfrak{k}(1|m)\) has a central extension, known in physics as the \(N = m\) superconformal algebra. The exceptional algebras do not admit central extensions, but being subalgebras of \(\mathfrak{vect}(n|m)\) they clearly have non-central Virasoro-like extensions.

For \(\mathfrak{v}_{3|6}\) and \(\mathfrak{m}_{3|8}\), the degree zero subalgebra \(\mathfrak{g}_0 = \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)\), i.e. the non-compact form of the symmetries of the standard model in particle physics. This suggests that these algebras may have important applications to physics \(\square\). To my knowledge, this is the only place where the standard model algebra arises naturally and unambiguously in a mathematically deep context. Note that \(\mathfrak{g}_0\) is not just any subalgebra of \(\mathfrak{g}\), but that \(\mathfrak{g}\) is completely determined by \(\mathfrak{g}_0 \ltimes \mathfrak{g}_-\). Moreover, there is a 1-1 correspondence between \(\mathfrak{g}_0\) and \(\mathfrak{g}\) irreps. E.g., the \(\mathfrak{vect}(n) = (\mathfrak{n}, \mathfrak{gl}(n))_s\) modules are tensor fields and closed forms, corresponding to \(\mathfrak{gl}(n)\) tensors.

\(^1\) Note that the definition of \(\mathfrak{m}_{3|8}\) in \(\square\) is flawed.
Therefore, one may speculate that a $g$ symmetry may be mistaken experimentally for a $g_0$ symmetry.

Throughout this paper I use tensor calculus notation. $A^i$ denotes a contravariant vector and $B_j$ a covariant vector. Repeated indices, one up and one down, are implicitly summed over (Einstein convention). Derivatives are denoted by various types of d's ($d$, $\partial$ and $\partial$). $\epsilon^{ab}$, $\epsilon^{ijk}$, $\epsilon^{ijklm}$ and $\epsilon^{abcdef}$ denote the totally anti-symmetric constant symbols in $\mathbb{C}^2$, $\mathbb{C}^3$, $\mathbb{C}^5$ and $\mathbb{C}^6$, respectively. When dealing with the non-positive subalgebra $g_0 \ltimes g_-$, all parities are known and it is convenient to explicitly distinguish between anti-symmetric (straight) and symmetric (curly) brackets: $[A,B] = -[B,A]$ and $\{A,B\} = \{B,A\}$. For general vector fields $X = X^\mu(x)\partial_\mu$ (always assumed to be homogeneous in parity), the (straight) brackets are graded in the usual way: $[X,Y] = -(\cdots)^{XY}[Y,X]$, where the symbol $(\cdots)^X$ is $+1$ on bosonic components and $-1$ on fermionic ones. The sign convention is that $X$ acts as

$$\mathcal{L}_X \partial_\nu = -(-)^{\nu X} \partial_\nu X^\mu \partial_\mu,$$

$$\mathcal{L}_X dx^\mu = (-)^{(X + \mu + \nu)\nu} \partial_\nu X^\mu dx^\nu. \quad (1.3)$$

The exceptional algebras are denoted by their names designed by Shchepochkina [17]. Kac and collaborators instead use the names $E(5|10) = \kappa(5|10)$, $E(3|6) = \nu(3|6)$, $E(3|8) = \mu(3|8)$, $E(1|6) = \kappa(1|6)$, and $K(1|m) = \kappa(1|m)$. There are two reasons to choose Shchepochkina’s convention. She was the one who discovered the exceptions, and Kac’ notation does not exhibit the family structure of regradings. E.g., $\mu(3|8)$ have three regradings (realizations on superspaces of different dimensions): $\mu(3|8)(4|5) : 2$, $\mu(3|8)(5|6) : 2$, $\mu(3|8) : 3$, where the number after the colon indicates depth.

2 $\kappa(1|m)$

Consider $\mathbb{C}^{1|m}$ with basis spanned by one even coordinate $t$ and $m$ odd coordinates $\theta_a$, $a = 1, 2, ..., m$. Let $\deg \theta_a = 1$ and $\deg t = 2$. The graded Heisenberg algebra has the non-zero relations

$$\{d^a, \theta_b\} = \delta^a_b, \quad [\partial_0, t] = 1, \quad (2.1)$$

where $d^a = \partial/\partial \theta_a$ and $\partial_0 = \partial/\partial t$. Introduce a constant metric $g_{ab} = g_{ba}$ with inverse $g^{ab}$. The metric and its inverse are used to raise and lower indices, so $\theta^a = g^{ab}\theta_b$ and $\theta_a = g_{ab}\theta^b$, etc.

The contact algebra $\kappa(1|m)$ is generated by vector fields of the form

$$K_f = (2 - \theta_a d^a) f \partial_0 + (-)^{f} d_a f d^a + \partial_0 f \theta_a d^a, \quad (2.2)$$
where \( f = f(\theta, t) \) is a function on \( \mathbb{C}^{1|m} \). These vector fields satisfy the algebra \([K_f, K_g] = K_{[f,g]_K}\), where the contact bracket reads

\[
[f, g]_K = (2 - \theta_a d^a) f \partial_0 g - \partial_0 f (2 - \theta_a d^a) g + (-)^f d_a f d^a g. \tag{2.3}
\]

By expanding \( f(\theta, t) \) in a power series in \( \theta \) we obtain more explicit descriptions of the contact bracket. For small \( m = 0, 1, 2, 3, 4 \), the resulting (Laurent polynomial) algebras are well known in physics, under the names centerless Virasoro and \( N = m \) superconformal algebras, respectively. Note that (2.3) is well defined also for \( m \geq 5 \).

The non-positive part of \( \mathfrak{g}(1|m) \subset \mathfrak{vect}(1|m) \) is spanned by the vector fields

| \( \text{deg} \) | \( f \) | \text{vector field} |
|---|---|---|
| \( -2 \) | \( \frac{1}{2} \) | \( E = \partial_0 \) |
| \( -1 \) | \( -\theta_a \) | \( D^a = d^a - \theta^a \partial_0 \) |
| \( 0 \) | \( -\theta^a \theta^b \) | \( J^{ab} = \theta^a d^b - \theta^b d^a \) |
| \( 0 \) | \( t \) | \( Z = 2t \partial_0 + \theta_a d^a \) |

The non-zero bracket in \( \mathfrak{g}_- \) reads

\[
\{ D^a, D^b \} = -2 g^{ab} E. \tag{2.5}
\]

One notes that (2.5) defines a Clifford algebra, with \( D^a \) playing the role of gamma matrices and \( E \) that of the unit operator. The analogous nilpotent algebras for the exceptions, (4.4), (5.9) and (6.4), constitute interesting generalizations of Clifford algebras.

A basis for \( \tilde{\mathfrak{g}}_{-1} \) is given by

\[
\tilde{D}^a = d^a + \theta^a \partial_0 = D^a + 2\theta^a \partial_0. \tag{2.6}
\]

which satisfy

\[
\{ \tilde{D}^a, \tilde{D}^b \} = 2 g^{ab} E, \quad \{ D^a, \tilde{D}^b \} = 0. \tag{2.7}
\]

Any vector field in \( \mathcal{C}^{1|m} \) has the form

\[
X = Q \partial_0 + P_a d^a = \tilde{Q} \partial_0 + P_a \tilde{D}^a, \tag{2.8}
\]

where

\[
\tilde{Q} = Q + (-)^X \theta^a P_a. \tag{2.9}
\]
\( X \) preserves the dual Pfaff equation \( \tilde{D}^a = 0 \), i.e.
\[
[X, \tilde{D}^a] = -(-)^X \tilde{D}^a P_b \tilde{D}^b,
\]
provided that
\[
\tilde{D}^a \tilde{Q} = 2(-)^X P^a.
\]
Compatibility between
\[
[E, X] = \partial_0 \tilde{Q} E + \partial_0 P_a \tilde{D}^a
\]
and (2.7) in the form \( E = \frac{1}{2} g_{ab} \{ \tilde{D}^a, \tilde{D}^b \} \), implies that
\[
\partial_0 \tilde{Q} - \frac{2}{m} (-)^X \tilde{D}^a P_a = 0.
\]
However, this is an identity which follows from (2.11) by considering \( \tilde{D}_a \tilde{D}^a \tilde{Q} \), so no new independent conditions on the vector fields arise.

The pairing \( \langle \tilde{D}^a, \alpha \rangle = 0 \) gives
\[
\alpha = dt + \theta^a d\theta_a,
\]
or more explicitly \( \alpha = dt + \rho_{ab} \theta_a d\theta_b \). We now show that \( \alpha \) satisfies a Pfaff equation:
\[
\begin{align*}
\mathcal{L}_X \theta_a &= P_a, \\
\mathcal{L}_X t &= Q, \\
\mathcal{L}_X d\theta_a &= (-)^X d^b P_a d\theta_b + \partial_0 P_a dt, \\
\mathcal{L}_X dt &= -(-)^X d^b Q d\theta_b + \partial_0 Q dt.
\end{align*}
\]
In particular,
\[
\begin{aligned}
Et &= 1, & E\theta_a &= Edt = Ed\theta_a = 0, \\
D^a t &= -\theta^a, & D^a dt &= -d\theta^a, \\
D^a \theta_b &= \delta^a_b, & D^a d\theta_b &= 0, \\
J^{ab} t &= 0, & J^{ab} dt &= 0, \\
J^{ab} \theta_c &= \delta_c^b \theta^a - \delta^a_c \theta^b, & J^{ab} d\theta_c &= \delta_c^b d\theta^a - \delta^a_c d\theta^b, \\
Z t &= 2t, & Z dt &= 2dt, \\
Z \theta_a &= \theta_a, & Z d\theta_a &= d\theta_a.
\end{aligned}
\]
One checks that
\[
J^{ab} \alpha = D^a \alpha = E\alpha = 0.
\]
Thus the Pfaff equation $\alpha = 0$ is preserved by $g_-$ and $g_0$ and therefore by all of $\mathfrak{g}(1|m) \subset \text{vect}(1|m)$.

An explicit calculation yields

$$\mathcal{L}_X \alpha = \partial_b \tilde{Q} dt + \left( -(-)^X d^k \tilde{Q} + 2P^b \right) d\theta_b. \quad (2.18)$$

Since the Pfaff equation $\alpha = 0$ is preserved, we have $\mathcal{L}_X \alpha = f\alpha = f dt + f\theta^a d\theta_a$ for $f = \partial_b \tilde{Q}$. Substitution into the formula above shows that $X$ must satisfy (2.11).

More generally, consider $\mathbb{C}^{2n+1|m}$ with basis spanned by $2n$ even coordinates $u^i$, $i = 1, 2, ..., 2n$, $m$ odd coordinates $\theta^a$, $a = 1, 2, ..., m$, and one additional even coordinate $t$. Let $\omega_{ij} = -\omega_{ji}$ be anti-symmetric structure constants and let $g_{ab} = g_{ba}$, as before. $\mathfrak{g}(2n+1|m)$ is the subalgebra of $\text{vect}(2n+1|m)$ which preserves the Pfaff equation $\alpha = 0$, where

$$\alpha = dt + \omega_{ij} u^i du^j + g^{ab} \theta_a d\theta_b. \quad (2.19)$$

Hence $\mathfrak{g}(1|m)$ has two natural classes of tensor modules, with bases $\alpha$ and $\gamma^a$ and module action

$$\mathcal{L}_X \alpha = \partial_b \tilde{Q} \alpha, \quad \mathcal{L}_X \gamma^a = -(-)^X \tilde{D}^a P_b \gamma^b. \quad (2.20)$$

Assuming that $\alpha$ and $\gamma^a$ are fermions, we can now construct the volume form $v_\gamma = \epsilon_{a_1 a_2 \ldots a_m} \gamma^{a_1} \gamma^{a_2} \ldots \gamma^{a_m}$, transforming as

$$\mathcal{L}_X v_\gamma = -(-)^X \tilde{D}^a P_a v_\gamma. \quad (2.21)$$

Since $\alpha$ has degree +2 and $v_\gamma$ has degree $-m$, the form $v_\gamma^2 \alpha^2$ is invariant, which implies the relation (2.13). For $m = 2$, the invariant form is the volume form and (2.13) becomes $\text{div} X = 0$.

From the vector density $\gamma^a$ with weight $-1$ we can construct a vector $\bar{\gamma}^a$ of zero weight, transforming as

$$\mathcal{L}_X \bar{\gamma}^a = -(-)^X (\tilde{D}^a P_b - \frac{1}{m} \delta^a_b \tilde{D}^c P_c) \bar{\gamma}^b$$

$$= \frac{1}{4} \tilde{D}^a \tilde{D}^b \tilde{Q} (g^{ac} \bar{\gamma}^b - g^{bc} \bar{\gamma}^c). \quad (2.22)$$

Hence we obtain the explicit realization

$$\mathcal{L}_X = X + \frac{1}{4} \tilde{D}^a \tilde{D}^b \tilde{Q} J^{ab} + \frac{1}{2} \partial_b \tilde{Q} \tilde{Z}, \quad (2.23)$$

where $X$ acts trivially on $\bar{\gamma}^a$ and

$$\tilde{J}^{ab} \bar{\gamma}^c = g^{ac} \bar{\gamma}^b - g^{bc} \bar{\gamma}^a, \quad (2.24)$$

$$\tilde{Z} \bar{\gamma}^c = 0.$$
Thus \( \tilde{J}^{ab} \) and \( \tilde{Z} \) generate the Lie algebra \( \mathfrak{g}_0 = so(m) \oplus gl(1) \). However, it is clear that \( \mathcal{L}_X \) will satisfy the same algebra for every representation of \( \mathfrak{g}_0 \). Substitution of irreducible \( \mathfrak{g}_0 \) modules into (2.23) gives the tensor modules for \( \mathfrak{k}(1|m) \). In particular, the expressions for \( \mathfrak{g}_0 \) becomes \( \mathcal{L}_\mathfrak{g} = J^{ab} + J^{ab}, \mathcal{L}_Z = Z + \tilde{Z}, \) i.e. two commuting copies of \( \mathfrak{g}_0 \).

To obtain an explicit expression for the vector fields in \( \mathfrak{k}(1|m) \), we set \( \tilde{Q} = f(\theta, u) \), an arbitrary polynomial function. From (2.9) and (2.11) we obtain \( P_a = \frac{1}{2}(-)^\chi \tilde{D}_a f, \) and

\[
X = \frac{1}{2}K_f = f \partial_0 + \frac{1}{2}(-)^f \tilde{D}_a f \tilde{D}_a.
\] (2.25)

We have \([X_f, X_g] = X_{[f,g]_\mathfrak{k}} \), where

\[
[f, g]_\mathfrak{k} = f \partial_0 g - (-)^fg \partial_0 f + \frac{1}{2}(-)^f \tilde{D}_a f \tilde{D}_a g.
\] (2.26)

One checks that this contact bracket is related to (2.3) by \([f, g]_\mathfrak{k} = \frac{1}{2}[f, g]_\mathfrak{k} \).

3 \( \mathfrak{tas}(1|6) \)

\( \mathfrak{tas}(1|6) \) contains the exceptional simple subalgebra \( \mathfrak{tas}(1|6) \). Clearly every vector field in \( \mathfrak{tas}(1|6) \) preserves the Pfaff equation (2.14), but in addition there is a condition coming from degree +1. The description here closely follows [1].

Let \( J^{ab} = -\theta^a \theta^b \), so \( K_{j^{ab}} = J^{ab} \) generate \( so(m) \). We have \([J^{ab}, \theta^c]_\mathfrak{k} = g^{bc} \theta^a - g^{ac} \theta^b \) and

\[
[j^{ab}, \theta^c \theta^d \theta^e]_\mathfrak{k} = g^{bc} \theta^d \theta^e - g^{ac} \theta^b \theta^e + 4 \text{ cyclic terms}.
\] (3.1)

Therefore, \( \omega^{abc}_+ = g^{bc} \theta^a \theta^c + \epsilon^{abcdef} \theta_d \theta_e \theta_f \) and \( \omega^{abc}_- = \theta^a \theta^b \theta^c - \epsilon^{abcdef} \theta_d \theta_e \theta_f \) transform independently under \( so(6) \); recall that the metric \( g_{ab} \) is used to lower indices. Denote the corresponding \( so(6) \) modules \( V_+ \) and \( V_- \), respectively, and let \( M \) be the \( so(6) \) module corresponding to \( f = t \theta^a \). \( \mathfrak{tas}(1|6) \) is obtained by requiring that \( \mathfrak{g}_1 = M \oplus V_+ \) (or equivalently \( \mathfrak{g}_1 = M \oplus V_- \)). In contrast, \( \mathfrak{g}_1 = M \oplus V_+ \oplus V_- \) for the contact algebra \( \mathfrak{k}(1|6) \).

The contact vector fields in \( \mathfrak{k}(1|m) \) at degree +1 are

| \text{deg} | f | \mathcal{K}_f |
|---|---|---|
| 1 | \( t \partial_a \) | \( A_a = t \theta_a \partial_0 - t \partial_a + \theta_a \theta_0 \partial_b \) |
| 1 | \( -\theta_a \theta_0 \theta_c \) | \( B_{abc} = \theta_a \theta_b \theta_c \partial_0 + \theta_a \theta_b \partial_c + \theta_b \theta_c \partial_a + \theta_c \theta_0 \partial_b \) |

(3.2)

For \( m = 6 \), we can define the dual of \( B_{abc} \) as

\[
B_{abc}^* = \epsilon_{abcdef} B^{def}.
\] (3.3)

\( \mathfrak{tas}(1|6) \) is obtained by requiring that \( B_{abc} \) be self-dual, i.e. \( B_{abc} = B_{abc}^* \).
A similar construction can be carried out for \( m = 4 \); the corresponding subalgebra of \( \mathfrak{f}(1|4) \) is finite-dimensional \[ \mathfrak{f}(1|4) \]. The \( \mathfrak{f}(1|2n) \) subspace \( \mathfrak{g}_n \) contains two subspaces that are isomorphic as \( \mathfrak{so}(2n) \) modules, but it is not possible to eliminate one of them when \( 2n > 6 \).

I have not been able to formulate \( \mathfrak{fas}(1|6) \) as an algebra of vector fields preserving some Pfaff equation or dual Pfaff equation.

4 \( \mathfrak{fsle}(5|10) \)

Let \( u = (u^i) \in \mathbb{C}^5 \), \( \partial_i = \partial/\partial u^i \), \( i = 1, 2, 3, 4, 5 \). Let \( \xi = \xi^i(u)\partial_i \) be a divergence-free vector field: \( \partial_i \xi^i = 0 \), and let \( \omega = \omega_{ij}(u)du^i \wedge du^j \) be a closed two-form: \( \omega_{ji} = -\omega_{ij} \) and \( \partial_i \omega_{jk} + \partial_j \omega_{ki} + \partial_k \omega_{ij} = 0 \). \( \mathfrak{fsle}(5|10) \) has generators \( \mathbb{L}_\xi = \mathbb{L}_i(\xi^i) \) and \( \mathbb{G}_\omega = \mathbb{G}^{ij}(\omega_{ij}) \). The brackets read \[ \begin{align*}
[\mathbb{L}_\xi, \mathbb{L}_\eta] &= \mathbb{L}_k(\xi^k \partial_j \eta^j - \eta^j \partial_j \xi^k), \\
[\mathbb{L}_\xi, \mathbb{G}_\omega] &= \mathbb{G}^{jk}(\xi^i \partial_j \omega_{ik} + \partial_j \xi^i \omega_{ik} + \partial_k \xi^i \omega_{ij}), \\
{\mathbb{G}_\omega, \mathbb{G}_\eta} &= \epsilon^{ijklm} \mathbb{L}_m(\omega_{ij} \eta_{kl}).
\end{align*} \] (4.1)

Consider \( \mathbb{C}^{5|10} \) with basis spanned by five even coordinates \( u^i \), \( i = 1, 2, 3, 4, 5 \) and ten odd coordinates \( \theta_{ij} = -\theta_{ji} \). Let \( \deg \theta_{ij} = 1 \) and \( \deg u^i = 2 \). The graded Heisenberg algebra has the non-zero relations \[ [\partial_j, u^i] = \delta^i_j, \quad \{d^{ij}, \theta_{kl}\} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k, \] (4.2)
where \( \partial_i = \partial/\partial u^i \) and \( \partial_i = -d^{ji} = \partial/\partial \theta_{ij} \). The non-positive part of \( \mathfrak{fsle}(5|10) \subset \mathfrak{vect}(5|10) \) is spanned by the vector fields

| deg | vector field |
|-----|-------------|
| -2  | \( E_k = \partial_k \) |
| -1  | \( D^{kl} = d^{kl} - \frac{1}{2} \epsilon^{klmni} \theta_{mn} \partial_k \) |
| 0   | \( I^k_l = u^k \partial_l - \theta_{ij} d^{kj} - \frac{1}{2} \delta^k_i (u^i \partial_l - \theta_{ij} d^{ij}) \) |

Note that \( Z = 2u^i \partial_i + \frac{1}{2} \theta_{ij} d^{ij} \), which is the operator that computes the Weisfeiler grading, is not part of \( \mathfrak{g}_0 \). However, for convenience we will also consider the non-simple algebra \( \mathfrak{fsle}(5|10) = (\mathfrak{5} \ast \mathfrak{5} \ast, \mathfrak{g}l(5)_s) \), obtained by adjoining \( Z \) to \( \mathbb{L}^3 \). It was shown in \[ \mathbb{L} \] that \( \mathfrak{fsle}(5|10) = \mathfrak{fsle}(5|10) \oplus \mathbb{C} Z \).

The non-zero brackets in \( \mathfrak{g}_- \) read

\[ \{D^{ij}, D^{kl}\} = -2 \epsilon^{ijklm} E_m. \] (4.4)

A basis for \( \mathfrak{g}_{-1} \) is given by

\[ \tilde{D}^{ij} = d^{ij} + \frac{1}{2} \epsilon^{ijklm} \theta_{kl} \partial_m = D^{ij} + \epsilon^{ijklm} \theta_{kl} \partial_m, \] (4.5)
which satisfy
\[
\{ \tilde{D}^{ij}, \tilde{D}^{kl} \} = 2\epsilon^{ijklm}E_m, \\
\{ D^{ij}, \tilde{D}^{kl} \} = 0.
\] (4.6)

Any vector field in \( \text{vect}(5|10) \) has the form
\[
X = Q^i \partial_i + \frac{1}{2} P_{ij} d^{ij} = \tilde{Q}^i \partial_i + \frac{1}{2} P_{ij} \tilde{D}^{ij},
\] (4.7)
where the \( \frac{1}{2} \) is necessary to avoid double counting and
\[
\tilde{Q}^i = Q^i + \frac{1}{4} (-)^X \epsilon^{ijklm} \theta_{jk} P_{lm}.
\] (4.8)

\( X \) preserves the dual Pfaff equation \( \tilde{D}^{ij} = 0 \), i.e.
\[
[X, \tilde{D}^{ij}] = -\frac{1}{2} (-)^X \tilde{D}^{ij} P_{kl} \tilde{D}^{kl},
\] (4.9)
provided that
\[
\tilde{D}^{ij} \tilde{Q}^k = (-)^X \epsilon^{ijklm} P_{lm}.
\] (4.10)

In particular, we have the symmetry relations
\[
\tilde{D}^{ij} \tilde{Q}^k = \tilde{D}^{jk} \tilde{Q}^i = -\tilde{D}^{ik} \tilde{Q}^j.
\] (4.11)

Compatibility between
\[
[E_i, X] = \partial_i \tilde{Q}^j E_j + \frac{1}{2} \partial_i P_{jk} \tilde{D}^{jk}
\] (4.12)
and (4.6) in the form \( E_i = \frac{1}{3!} \epsilon_{ijklm} \{ \tilde{D}^{jk}, \tilde{D}^{lm} \} \), implies that
\[
\partial_i \tilde{Q}^j = \frac{1}{6} (-)^X (\tilde{D}^{kl} P_{kl} \tilde{Q}^j - 2\tilde{D}^{jk} P_{ik}).
\] (4.13)

In particular,
\[
\partial_i \tilde{Q}^i - \frac{1}{2} (-)^X \tilde{D}^{ij} P_{ij} \equiv \text{div } X = 0.
\] (4.14)

However, (4.13) is an identity which follows from (4.10) by considering \( \epsilon_{ijklm} \tilde{D}^{ij} \tilde{D}^{kl} Q^m \), so no new independent conditions on the vector fields arise.

The pairing \( \langle \tilde{D}^{ij}, \alpha^k \rangle = 0 \) gives
\[
\alpha^i = du^i + \frac{1}{4} \epsilon^{ijklm} \theta_{jk} d\theta_{lm}.
\] (4.15)

We now show that \( \alpha^i \) satisfies a Pfaff equation:
\[
\mathcal{L}_X \theta_{ij} = P_{ij}, \\
\mathcal{L}_X u^i = Q^i, \\
\mathcal{L}_X d\theta_{ij} = \frac{1}{2} (-)^X d^{kl} P_{ij} d\theta_{kl} + \partial_k P_{ij} d\theta^k, \\
\mathcal{L}_X du^i = -\frac{1}{2} (-)^X d^{kl} Q^i d\theta_{kl} + \partial_k Q^i d\theta^k.
\] (4.16)
In particular,

\[ E_k u^i = \delta^i_k, \quad E_k u^i = E_k \theta_{ij} = E_k \theta_{ij} = 0, \]
\[ D^{kl} u^i = -\frac{1}{2} \epsilon^{kmni} \theta_{mn}, \quad D^{kl} d u^i = -\frac{1}{2} \epsilon^{kmni} d \theta_{mn}, \]
\[ D^{kl} \theta_{ij} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k, \quad D^{kl} d \theta_{ij} = 0, \]
\[ I^i_k u^i = \delta^i_j u^k - \frac{1}{5} \delta^i_k u^j, \quad I^i_k d u^i = \delta^i_j d u^k - \frac{1}{5} \delta^i_k d u^j, \]
\[ I^i_k \theta_{ij} = -\delta^i_j \theta_{ik} - \frac{2}{5} \delta^i_k \theta_{ij}, \quad I^i_k d \theta_{ij} = -\delta^i_j d \theta_{ik} - \frac{2}{5} \delta^i_k d \theta_{ij}, \]
\[ Zu^i = 2u^i, \quad Z d u^i = 2 d u^i, \quad Z d \theta_{ij} = d \theta_{ij}. \]

One checks that

\[ I^i_k \alpha^i = \delta^i_k \alpha^k - \frac{1}{5} \delta^i_k \alpha^j, \quad D^{kl} \alpha^i = E_k \alpha^i = 0. \] (4.18)

An explicit calculation yields

\[ \mathcal{L}_X \alpha^i = \partial_j \tilde{Q}^j d u^i + \left( -\frac{1}{2} (-)^X d^{np} \tilde{Q}^j + \frac{1}{2} \epsilon^{ijklm} P_{jk} d \theta_{mn} \right) \] (4.19)

Since the Pfaff equation \( \alpha^i = 0 \) is preserved, we have \( \mathcal{L}_X \alpha^i = f^i_j \alpha^j = \gamma^{ij} (d u^i + \frac{1}{2} \epsilon^{ijklm} \theta_{kl} d \theta_{mn}) \) for \( f^i_j = \partial_j \tilde{Q}^j \). Substitution into the formula above yields the condition (4.10).

Hence the (5|10) has two natural classes of tensor modules, with bases \( \alpha^i \) and \( \gamma^{ij} \) and module action

\[ \mathcal{L}_X \alpha^i = \partial_j \tilde{Q}^j \alpha^i, \quad \mathcal{L}_X \gamma^{ij} = -\frac{1}{2} (-)^X \tilde{D}^{ij} P_{kl} \gamma^{kl}. \] (4.20)

Assuming that \( \alpha^i \) and \( \gamma^{ij} \) are fermions, we can now construct the volume forms \( v_\alpha = \alpha^1 \alpha^2 \alpha^3 \alpha^4 \alpha^5 \) and \( v_\gamma = \gamma^{12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45} \), transforming as

\[ \mathcal{L}_X v_\alpha = \partial_j \tilde{Q}^j v_\alpha, \quad \mathcal{L}_X v_\gamma = -\frac{1}{2} (-)^X \tilde{D}^{ij} P_{ij} v_\gamma. \] (4.21)

Since \( v_\alpha \) has degree +10 and \( v_\gamma \) has degree −10, the form \( v_\alpha v_\gamma \) is invariant, which implies the relations (4.14).

From (4.20) we deduce the transformation laws for a scalar density \( v \) of weight +1, an \( sl(5) \) vector \( \tilde{X} \) and an \( sl(5) \) bivector \( \gamma^{ij} \):

\[ \mathcal{L}_X v = \frac{1}{10} \partial_j \tilde{Q}^j v = \frac{1}{20} (-)^X \tilde{D}^{ij} P_{ij} v, \]
\[ \mathcal{L}_X \tilde{X}^i = \frac{1}{15} \left( \partial_j \tilde{D}^{kl} P_{kl} - 5 \tilde{D}^{ij} P_{jk} \right) \tilde{X}^i \]
\[ = \left( \partial_j \tilde{Q}^j - \frac{1}{5} \delta^j_i \partial_k \tilde{Q}^k \right) \tilde{X}^i, \]
\[ \mathcal{L}_X \gamma^{ij} = \frac{1}{20} \left( \delta^i_k \delta^j_l \tilde{D}^{mn} P_{mn} - 10 \tilde{D}^{ij} P_{kl} \right) \gamma^{kl}. \] (4.22)
Hence we obtain the explicit realization

\[ \mathcal{L}_X = X - \frac{i}{2}(-)^X \tilde{D}^{ik} P_{jk} \tilde{P}^l_i + \frac{1}{20}(-)^X \tilde{D}^{ij} P_{ij} \tilde{Z} \]

\[ = X + \partial_j \tilde{Q}^i \tilde{P}^j_i + \frac{1}{10} \partial_i \tilde{Q}^i \tilde{Z}, \]

(4.23)

where \( \tilde{P}^i_j \) and \( \tilde{Z} \) generate the Lie algebra \( \tilde{g}_0 = gl(5) \) and \( X \) commutes with \( \tilde{g}_0 \). This expression holds for the non-simple algebra \( \tilde{f}\tilde{s}\tilde{e}\tilde{c}(5|10) \); for \( \tilde{f}\tilde{s}\tilde{e}\tilde{c}(5|10) \) \( \tilde{Z} = 0 \). By introducing canonical conjugate oscillators \( \tilde{a}_+, \tilde{a}_- \) and \( v^* \), subject to the Heisenberg algebra \([ \tilde{a}_I, \tilde{a}_J^* ] = \delta^I_J, [v^*, v] = 1 \), we obtain the explicit expression for the \( \tilde{g}_0 \) generators: \( \tilde{I}_j = \tilde{a}_j^* \tilde{a}_j^* - \frac{1}{2} \delta^I_J \tilde{a}_j^* \tilde{a}_k^* \) and \( \tilde{Z} = vv^* \). However, it is clear that \( \mathcal{L}_X \) will satisfy the same algebra for every representation of \( gl(5) \). Substitution of irreducible \( \tilde{g}_0 \) modules into (4.23) gives the tensor modules for \( \tilde{f}\tilde{s}\tilde{e}\tilde{c}(5|10) \). In particular, the expressions for \( \tilde{g}_0 \) become \( \mathcal{L}_ij = \tilde{I}_i^j + \tilde{I}_j^i, \mathcal{L} = \tilde{Z} + \tilde{Z} \), i.e. two commuting copies of \( \tilde{g}_0 \).

To obtain an explicit expression for the vector fields in \( \tilde{f}\tilde{s}\tilde{e}\tilde{c}(5|10) \), we set \( \tilde{Q}^i = f^i(\theta, u) \), five arbitrary polynomial functions. From (4.8) and (4.10) we obtain \( P_{ij} = \frac{1}{12}(-)^X \epsilon_{ijklm} \tilde{D}^{kl} f^m \) and

\[ X = U_f = f^i \partial_i + \frac{1}{24}(-)^f \epsilon_{ijklm} \tilde{D}^{ij} f^k \tilde{D}^{lm}, \]

(4.24)

where \( f = f^i(\theta, u) \partial_i \) is a vector field acting on \( \mathbb{C}^{5|10} \) and \( (-)^f = +1 \) if \( f \) is an even vector field, i.e. \( f^i \) is an even function. Due to the symmetry condition (4.11), the components \( f^i \) are not independent, but subject to the relation

\[ \tilde{D}^{ij} f^k = \tilde{D}^{jk} f^i = -\tilde{D}^{ik} f^j, \]

(4.25)

This condition may alternatively be written as \( 2 \tilde{D}^{ij} f^k = \tilde{D}^{jk} f^i + \tilde{D}^{ki} f^j \).

There is one additional relation, which singles out \( f^i \in \tilde{f}\tilde{s}\tilde{e}\tilde{c}(5|10) \subset \tilde{f}\tilde{s}\tilde{e}\tilde{c}(5|10) \):

\[ \partial_i \tilde{Q}^i = 0 \Rightarrow \partial_i f^i = 0. \]

(4.26)

This extra condition is clearly satisfied by all operators in (4.23), except for \( Z \).

A \( \tilde{f}\tilde{s}\tilde{e}\tilde{c}(5|10) \) bracket is defined by \([ U_f, U_g ] = U_{[f,g]_{\tilde{f}\tilde{s}\tilde{e}\tilde{c}}} \), where

\[
[f, g]_{\tilde{f}\tilde{s}\tilde{e}\tilde{c}} = f^i \partial_i g^j \partial_j - \partial_j f^i g^j \partial_i + (-)^f H^p_{ijklmn} \tilde{D}^{ij} f^k \tilde{D}^{lm} g^m \partial_p
- (-)^g f^p H^p_{ijklmn} \tilde{D}^{ij} f^k \tilde{D}^{lm} g^m \partial_p
= \frac{1}{24} \epsilon_{ijklm} \delta^p_q - \frac{1}{240} \epsilon_{ijklm} \epsilon^{qrst} \epsilon^{pqrst}.
\]

(4.27)
Finally we list the vector fields $f$ and $U_f$ for $\mathfrak{g}_0 \ltimes \mathfrak{g}_-$:

| deg | $f$ | $U_f$ |
|-----|-----|------|
| $-2$ | $\partial_i$ | $E_i$ |
| $-1$ | $-\epsilon^{ijklm} \theta_{kl} \partial_m$ | $D^{ij}$ |
| $0$ | $u^i \partial_j - \frac{1}{6} \delta^i_j u^k \partial_k + \frac{1}{2} \epsilon^{ijklm} \theta_{jk} \theta_{lm} \partial_n$ | $I^i$ |
| $0$ | $2u^i \partial_i$ | $Z$ |

(4.28)

Other vector fields $f$ of non-positive degree are not compatible with the symmetry condition \[1.23\]. Clearly, the vector field corresponding to $U_f = Z$ does not satisfy \[1.26\].

5 \textbf{vtc}(3|6)

Let $u = (u^i) \in \mathbb{C}^3$, $i = 1, 2, 3$. Let $\xi = \xi^i(u) \partial_i / \partial u^i$ be a vector field, let $X = X^b(u) T^b_a$ be an sl(2)-valued current, and let $\omega = \omega_{ia}(u) du^i dz^a$ be a $\mathbb{C}^2$-valued one-form. Here $T^b_a \in sl(2)$: $[T^b_a, T^c_d] = \delta^b_c T^a_d - \delta^a_c T^b_d$, and $X^a_0 = 0$. \textbf{vtc}(3|6) has generators $L_\xi = L_i(\xi^i)$, $J_X = J^b_a(X^a_b)$ and $G_\omega = G^{ia}(\omega_{ia})$, and the brackets read \[3\]

$$
[L_\xi, L_\eta] = L_k(\xi^i \partial_i \eta^k - \eta^j \partial_j \xi^k),
$$

$$
[L_\xi, J_X] = J^b_a(X^a_b),
$$

$$
[J_X, J_Y] = J^b_a(Y^a_c X^c_b - X^a_c Y^c_b),
$$

$$
[L_\xi, G_\omega] = G^{ia} (\xi^i \partial_i \omega_{ja} + \partial_j \xi^i \omega_{ia} + \frac{1}{2} \partial^i \xi^j \omega_{ja}),
$$

$$
[J_X, G_\omega] = G^{ia} (X^a_\omega \omega_{ib}),
$$

$$
\{G_\omega, G_\eta\} = \epsilon^{ijk} e^{ab} L_k (\omega_{ia} \nu_{jb}) + \epsilon^{ijk} \epsilon^{ac} J^b_a (\partial_k \omega_{ic} \nu_{jb} - \omega_{ib} \partial_k \nu_{jc}).
$$

In this and the next section we use some special relations valid in two dimensions only:

$$
\phi^a = e^{ab} \phi^b, \quad \phi_a = e_{ab} \phi^b,
$$

$$
e^{ab} e_{bc} = \delta^a_c, \quad e^{ab} e_{bc} = -\delta^a_c, \quad (5.2)
$$

$$
\phi^a \psi_a = -\phi_a \psi^0, \quad (5.3)
$$

$$
e^{ab} \phi^c + e^{bc} \phi^a + e^{ca} \phi^b = 0, \quad (5.4)
$$

$$
\phi^a \phi^b - \phi^b \phi^a = -e^{ab} \phi^c \psi_c. \quad (5.5)
$$

(5.6)

Our convention is $\epsilon^{12} = \epsilon_{21} = 1$, $\epsilon^{21} = \epsilon_{12} = -1$. The constants $e^{ab}$ and $e_{ab}$ can be used to raise and lower sl(2) indices.

Consider $\mathbb{C}^{3|6}$ with basis spanned by three even coordinates $u^i, i = 1, 2, 3$ and six odd coordinates $\theta_{ia}$. Let $\deg \theta_{ia} = 1$ and $\deg u^i = 2$. The graded Heisenberg algebra has the non-zero relations

$$
[\partial_j, u^i] = \delta^i_j, \quad \{d^a, \theta_{jb}\} = \delta^a_j \delta^b_0, \quad (5.7)
$$
where $\partial_i = \partial/\partial u^i$ and $d^a_i = \partial/\partial \theta_{ia}$. The non-positive part of $\mathfrak{vle}(3|6) \subset \text{vect}(3|6)$ is spanned by the vector fields

| deg | vector field |
|-----|--------------|
| $-2$ | $E_i = \partial_i$ |
| $-1$ | $D^{ia} = d^a_i - \varepsilon^{ijk} \theta^a_j \partial_k$ |
| $0$ | $I^k = u^k \partial_k - \theta_{ia} d^{ka} - \frac{1}{3} \delta^k_l (u^i \partial_i - \theta_{ia} d^a_i)$ |
| $0$ | $J^{ia} = -\theta_{iad} d^{ie} + \frac{1}{2} \delta^{ka}_{ia} \theta_{ia} d^a_i$ |
| $0$ | $Z = 2u^i \partial_i + \theta_{ia} d^a_i$ |

The non-zero brackets in $\mathfrak{g}_-$ are

\[ \{ D^{ia}, D^{jb} \} = -2 \varepsilon^{ijk} \varepsilon^{ab} E_k. \]  \hspace{1cm} (5.9)

A basis for $\mathfrak{g}_{-1}$ is given by

\[ \tilde{D}^{ia} = d^a_i + \varepsilon^{ijk} \theta^a_j \partial_k = D^{ia} + 2 \varepsilon^{ijk} \theta^a_j \partial_k, \]  \hspace{1cm} (5.10)

which satisfy

\[ \{ \tilde{D}^{ia}, \tilde{D}^{jb} \} = 2 \varepsilon^{ijk} \varepsilon^{ab} E_k, \]  \hspace{1cm} (5.11)

\[ \{ D^{ia}, \tilde{D}^{jb} \} = 0. \]

Any vector field in $\text{vect}(3|6)$ has the form

\[ X = Q^i \partial_i + P_{ia} d^a_i = \tilde{Q}^i \partial_i + P_{ia} \tilde{D}^{ia}, \]  \hspace{1cm} (5.12)

where

\[ \tilde{Q}^i = Q^i - (-)^X \varepsilon^{ijk} \theta^a_j P_{ka}. \]  \hspace{1cm} (5.13)

$X$ preserves the dual Pfaff equation $\tilde{D}^{ia} = 0$, i.e.

\[ [X, \tilde{D}^{ia}] = -(-)^X \tilde{D}^{ia} P_{ja} \tilde{D}^{jb}, \]  \hspace{1cm} (5.14)

provided that

\[ \tilde{D}^{ia} \tilde{Q}^j = -2 (-)^X \varepsilon^{ijk} P^{a}_{ka}. \]  \hspace{1cm} (5.15)

In particular, we have the symmetry relations

\[ \tilde{D}^{ia} \tilde{Q}^j = -\tilde{D}^{ja} \tilde{Q}^i. \]  \hspace{1cm} (5.16)

Compatibility between

\[ [E_i, X] = \partial_i \tilde{Q}^j E_j + \partial_i P_{ja} \tilde{D}^{jb}. \]  \hspace{1cm} (5.17)
and (5.11) in the form $E_i = \frac{1}{8} \epsilon_{ijk} \epsilon_{ab} \{ \tilde{D}^{ia}, \tilde{D}^{kb} \}$, implies that
\[
\partial_i \tilde{Q}^j = \frac{1}{2} (-)^X (\tilde{D}^{ka} P_{ka} \delta_i^j - \tilde{D}^{ja} P_{ia}).
\] (5.18)

In particular,
\[
\partial_i \tilde{Q}^j - (-)^X \tilde{D}^{ja} P_{ia} \equiv \text{div} X = 0.
\] (5.19)

However, (5.18) is an identity which follows from (5.15) by considering $\epsilon_{ijl} \epsilon_{ab} \tilde{D}^{ia} \tilde{D}^{jb} \tilde{Q}^k$, so no new independent conditions on the vector fields arise.

The pairing $\langle \tilde{D}^{ia}, \alpha \rangle = 0$ gives
\[
\alpha^i = du^i - \epsilon^{ijk} \theta^j a d\theta_{ka}.
\] (5.20)

We now show that $\alpha^i$ satisfies a Pfaff equation:
\[
\mathcal{L}_X \theta_{ia} = P_{ia},
\mathcal{L}_X u^i = Q^i,
\mathcal{L}_X d\theta_{ia} = (-)^X d^{jb} P_{ia} d\theta_j + \partial_j P_{ia} du^j,
\mathcal{L}_X du^i = -(-)^X d^{jb} Q^i d\theta_j + \partial_j Q^i du^j.
\] (5.21)

In particular,
\[
\begin{align*}
E_k u^i &= \delta_i^k, & E_k d\theta_{ia} &= E_k \theta_{ia} = E_k d\theta_{ia} = 0, \\
D^{kc} u^i &= -\epsilon^{kli} \theta_c^i, & D^{kc} d\theta_{ia} &= -\epsilon^{kli} d\theta_c^i, \\
D^{kc} \theta_{ia} &= \delta_a^i \delta^c_k, & D^{kl} d\theta_{ia} &= 0, \\
I^k u^i &= \delta^i_k u^k - \frac{1}{3} \delta^i_k u^i, & I^k d\theta_{ia} &= \delta^i_k d\theta_a^k - \frac{1}{3} \delta^i_k d\theta_{ia}, \\
I^k \theta_{ia} &= -\delta^i_k \theta_{ia} + \frac{1}{3} \delta^i_k \theta_a^i, & I^k d\theta_{ia} &= -\delta^i_k d\theta_{ia} + \frac{1}{3} \delta^i_k d\theta_{ia}, \\
J^k d\theta_{ia} &= 0, \\
J^k d\theta_{ia} &= 0, \\
J^k \theta_{ia} &= -\delta^i_k \theta_{ia} + \frac{1}{2} \delta^i_k \theta_a^i, & J^k d\theta_{ia} &= -\delta^i_k d\theta_{ia} + \frac{1}{2} \delta^i_k d\theta_{ia}, \\
Z u^i &= 2u^i, & Z d\theta_{ia} &= 2d\theta_{ia}, \\
Z \theta_{ia} &= \theta_{ia}, & Z d\theta_{ia} &= d\theta_{ia}.
\end{align*}
\] (5.22)

One checks that
\[
I^k \alpha^i = \delta^i_k \alpha^k - \frac{1}{3} \delta^i_k \alpha^i,
Z \alpha^i = 2\alpha^i,
J^k \alpha^i = D^{kl} \alpha^i = E_k \alpha_i = 0.
\] (5.23)

An explicit calculation yields
\[
\mathcal{L}_X \alpha^i = \partial_j \tilde{Q}^j du^j + (-(-)^X d^{jb} \tilde{Q}^i - 2\epsilon^{lij} P_{jb}^{il}) d\theta_{ib}.
\] (5.24)
Since the Pfaff equation $\alpha^i = 0$ is preserved, we have $L_{\chi} \alpha^i = f_j^i \alpha^j = f_j^i (du^j - e^{kl} \theta^k_0 d\theta_0^l)$ for $f_j^i = \partial_j \tilde{Q}^i$. Substitution into the formula above yields the conditions (5.13).

Hence we obtain the explicit realization of $\tilde{\alpha}^i$ and $\tilde{\gamma}^{ia}$ and module action

$$L_{\chi} \alpha^i = \partial_j \tilde{Q}^j \alpha^j,$$

$$L_{\chi} \tilde{\gamma}^{ia} = -(-)^X \tilde{D}^{ia} P_{jb} \tilde{\gamma}^{jb}. \quad (5.25)$$

Assuming that $\alpha^i$ and $\tilde{\gamma}^{ia}$ are fermions, we can now construct the volume forms $v_\alpha = \epsilon_{ijk} \alpha^i \alpha^j \alpha^k$ and $v_\gamma = \epsilon_{ijk} \epsilon_{lmn} \epsilon_{abc} \epsilon_{ef} \gamma^{ja} \gamma^{jb} \gamma^{kc} \gamma^{ld} \gamma^{mef} \gamma^{nf}$, transforming as

$$L_{\chi} v_\alpha = \partial_i \tilde{Q}^i v_\alpha,$$

$$L_{\chi} v_\gamma = -(-)^X \tilde{D}^{ia} P_{ja} v_\gamma. \quad (5.26)$$

Since $v_\alpha$ has degree +6 and $v_\gamma$ has degree −6, the form $v_\alpha v_\gamma$ is invariant, which implies the relation (5.13).

From (5.26) we deduce the transformation laws for a scalar density $v$ of weight +1, an $sl(3)$ vector $\tilde{\alpha}^i$ and an $sl(3) \oplus sl(2)$ vector $\tilde{\gamma}^{ia}$:

$$L_{\chi} v = \frac{1}{6} \partial_i \tilde{Q}^i v = \frac{1}{6} (-)^X \tilde{D}^{ia} P_{ia} v,$$

$$L_{\chi} \tilde{\alpha}^i = \frac{1}{6} (-)^X (\tilde{\delta}^i_j \tilde{D}^{ka} P_{ka} - 3 \tilde{D}^{ia} P_{ja}) \tilde{\alpha}^j = (\partial_j \tilde{Q}^j - \frac{1}{3} \tilde{\delta}^i_j \partial_k \tilde{Q}^k) \tilde{\alpha}^i, \quad (5.27)$$

$$L_{\chi} \tilde{\gamma}^{ia} = \frac{1}{6} (-)^X (\tilde{\delta}^i_j \tilde{D}^{ka} P_{ka} - 6 \tilde{D}^{ia} P_{ja}) \tilde{\gamma}^{jb}. \quad (5.28)$$

I have not obtained any explicit expression for an $sl(2)$ vector $\tilde{\beta}^a$, but in view of (5.27) it is natural to assume that it transforms as

$$L_{\chi} \tilde{\beta}^a = \frac{1}{6} (-)^X (\tilde{\delta}^a_b \tilde{D}^{ic} P_{ kc} - 2 \tilde{D}^{ia} P_{jb}) \tilde{\beta}^b. \quad (5.29)$$

Hence we obtain the explicit realization

$$L_{\chi} = X + \frac{1}{6} (-)^X \tilde{D}^{ia} P_{jb} (-3 \delta^b_a \tilde{J}^i_j - 2 \delta^j_i \tilde{J}^b_a + \delta^j_i \delta^b_a \tilde{Z})$$

$$= X + \partial_j \tilde{Q}^i \tilde{J}^i_j - \frac{1}{6} (-)^X \tilde{D}^{ia} P_{ja} \tilde{J}^b + \frac{1}{6} \partial_i \tilde{Q}^i \tilde{Z}, \quad (5.29)$$

where $\tilde{J}^i_j$ and $\tilde{Z}$ generate the Lie algebra $\tilde{g}_0 = sl(3) \oplus sl(2) \oplus gl(1)$ and $X$ commutes with $\tilde{g}_0$. Note that the term multiplying $\tilde{J}^b_a$ is somewhat uncertain, since (5.28) is a conjecture. By introducing canonical conjugate oscillators $\tilde{\alpha}^*_i$, $\tilde{\beta}^*_a$ and $\tilde{\gamma}^*$, subject to the Heisenberg algebra $[\tilde{\alpha}^*_i, \tilde{\alpha}^*_j] = \tilde{\delta}^*_i_j$, $[\tilde{\beta}^*_a, \tilde{\beta}^*_b] = \tilde{\delta}^*_a_b$, $[\tilde{\gamma}^*, \tilde{\gamma}^*] = 1$, we obtain the explicit expression for the $\tilde{g}_0$ generators: $\tilde{J}^i_j = \tilde{\alpha}^*_i \tilde{\alpha}^*_j - \frac{1}{3} \tilde{\delta}^*_i_j \tilde{\alpha}^*_k \tilde{\alpha}^*_k, \tilde{J}^a_b = \tilde{\beta}^*_a \tilde{\beta}^*_b - \frac{1}{2} \tilde{\delta}^*_a_b \tilde{\beta}^*_c \tilde{\beta}^*_c$ and $\tilde{Z} = \tilde{\gamma}^* \tilde{\gamma}^*$. However, it is clear that $L_{\chi}$ will satisfy the same algebra for every representation of
sl(3) ⊕ sl(2) ⊕ gl(1). Substitution of irreducible \( \mathfrak{g}_0 \) modules into (5.29) gives the tensor modules for \( \mathfrak{ve}(3|6) \). In particular, the expressions for \( \mathfrak{g}_0 \) become

\[
\mathcal{L}_I^a = I_j^a + \bar{I}_j^a, \quad \mathcal{L}_{J_b} = J_a^b + \bar{J}_b^a \quad \text{and} \quad \mathcal{L}_Z = Z + \bar{Z},
\]

i.e. two commuting copies of \( \mathfrak{g}_0 \).

To obtain an explicit expression for the vector fields in \( \mathfrak{ve}(3|6) \), we set \( \bar{Q}^i = f^i(\theta, u) \), three arbitrary polynomial functions. From (5.13) and (5.15) we obtain

\[
P_i^a = \frac{1}{4}(-)^i \epsilon_{ijk} D^k a f^j \]

where \( f = f^i(\theta, u) \partial_i \) is a vector field acting on \( \mathbb{C}^3|6 \) and \((-)^i = +1 \) if \( f \) is an even vector field, i.e. \( f \) is an even function. Due to the symmetry condition (5.16), the components \( f^i \) are not independent, but subject to the relation

\[
\bar{D}^a_i f^i = - \bar{D}^a_i f^i,
\]

A \( \mathfrak{ve}(3|6) \) bracket is defined by \([V_f, V_g] = V_{[f,g]_{\mathfrak{ve}}} \). We find

\[
[f,g]_{\mathfrak{ve}} = f^i \partial_i g^j \partial_j - (-)^i f^j g^j f^i \partial_i
\]

\[+ (-)^j H^m_{ik|jl} \bar{D}^i a f^k \bar{D}^j b g^l \partial_m - (-)^j f^g \bar{D}^j a g^l \bar{D}^i b f^k \partial_m,
\]

\[
H^m_{ik|jl} = \frac{1}{4} \epsilon_{ikj} \delta^m_l + \frac{1}{16} \epsilon_{ijk} \delta^m_l + \frac{1}{16} \epsilon_{ijkl} \delta^m_l. \tag{5.32}
\]

Finally we list the vector fields \( f \) and \( V_f \) for \( \mathfrak{g}_0 \times \mathfrak{g}_- \):

| \text{deg} | f | V_f |
|----------|-----------------|------------------|
| -2       | \partial_i      | \bar{E}_i         |
| -1       | -2 \epsilon^{ij} \theta_j^a \partial_k | \bar{D}^a_i |
| 0        | u^i \partial_j - \frac{1}{3} u^{ik} \partial_k \delta_j^i - \epsilon^{ikl} \theta_j^a \theta_k^a \partial_l | \bar{I}_j^a |
| 0        | \bar{J}_b^a     | \bar{J}_b^a |
| 0        | 2 \epsilon^{ij} \theta_j^a \theta_k^a \partial_k | Z |

Other vector fields \( f \) of non-positive degree are not compatible with the symmetry condition (5.31).

6 \ \mathbf{mb}(3|8)

Let \( u = (u^i) \in \mathbb{C}^3 \), \( i = 1, 2, 3 \), and let \( z = (z^a) \in \mathbb{C}^2 \), \( a = 1, 2 \). Let \( \xi = \xi^i(u) \partial_i / \partial u^i \) be a vector field, let \( X = X_b^a(u) T_b^a \) be an \( sl(2) \)-valued current, let \( \omega = \omega_b^a(u) \partial / \partial u^i dz^a \) be a \( \mathbb{C}^2 \)-valued vector field, and let \( \sigma = \sigma^a(u) \partial / \partial z^a \) be a \( \mathbb{C}^2 \)-valued function. Here \( T_b^a \in sl(2) \): \([T_b^a, T_d^a] = \delta_b^d T_a^a - \delta_d^a T_b^a \), and
$X_0^a = 0$. $\mathfrak{m} \mathfrak{b}(3|8)$ has generators $\mathcal{L}_i = L_i(\xi^i)$, $\mathcal{J}_X = \mathcal{J}^b_a(X_b^a)$, $\mathcal{G}_\omega = \mathcal{G}^b_a(\omega^b_a)$ and $\mathcal{S}_\sigma = \mathcal{S}_a(\sigma^a)$. The brackets read

\[
\begin{align*}
[\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_k(\xi^i \partial_i \eta^k - \eta^i \partial_j \xi^k), \\
[\mathcal{L}_\xi, \mathcal{J}_X] &= \mathcal{J}_k(\xi^i \partial_i X_b^a), \\
[\mathcal{J}_X, \mathcal{J}_Y] &= \mathcal{J}_k(Y_c^a X_b^c - X_c^a Y_b^c), \\
[\mathcal{L}_\xi, \mathcal{G}_\omega] &= \mathcal{G}^b_j(\xi^i \partial_i \omega_a^b + \frac{1}{2} \partial_i \xi^i \omega_a^b - \partial_i \xi^i \omega_a^b), \\
[\mathcal{L}_\xi, \mathcal{S}_\sigma] &= \mathcal{S}_a(\xi^i \partial_i \sigma^a + \frac{1}{2} \partial_i \xi^i \sigma^a), \\
[\mathcal{J}_X, \mathcal{G}_\omega] &= \mathcal{G}^b_j(X_b^a \omega^b_a), \\
[\mathcal{J}_X, \mathcal{S}_\sigma] &= -\mathcal{S}_a(X_b^a \sigma^b), \\
\{\mathcal{G}_\omega, \mathcal{G}_\nu\} &= 0, \\
\{\mathcal{G}_\omega, \mathcal{S}_\sigma\} &= \mathcal{L}_i(\omega_a^b \sigma^b) + \mathcal{J}^b_a(\partial_i \omega_b^a \sigma^a - \omega_b^a \partial_i \sigma^a), \\
\{\mathcal{S}_\sigma, \mathcal{S}_\tau\} &= \epsilon^{ijk} \epsilon_{abc} \mathcal{L}_k(i \sigma^a \partial_j \tau^b).
\end{align*}
\]

Consider $\mathbb{C}^3 \mathfrak{8}$ with basis spanned by three even coordinates $u^i$, $i = 1, 2, 3$, six odd coordinates $\theta_{ia}$, and two more odd coordinates $\vartheta^a$. Let $\deg \theta_{ia} = 1$, $\deg u^i = 2$ and $\deg \vartheta^a = 3$. The graded Heisenberg algebra has the non-zero relations

\[
\{\bar{\partial}_b, \vartheta^a\} = \delta^a_b, \quad [\partial_j, u^i] = \delta^i_j, \quad \{d^i_a, \theta_{jb}\} = \delta^i_j \delta^a_b,
\]

where $\partial_i = \partial/\partial u^i$, $d^a = \partial/\partial \theta_{ia}$, and $\bar{\partial}_a = \partial/\partial \vartheta^a$. The non-positive part of $\mathfrak{m} \mathfrak{b}(3|8) \subset \text{vect}(3|8)$ is spanned by the vector fields

| deg | vector field |
|-----|-------------|
| -3  | $F_a = \bar{\partial}_a$ |
| -2  | $E_i = \partial_i + \theta^a_i \bar{\partial}_a$ |
| -1  | $D_i^a = d^i_a + 3 \epsilon^{ijk} \theta^a_j \partial_k + \epsilon^{ijk} \theta^a_j \partial_b^k \bar{\partial}_b + u^i \bar{\partial}_a$ |
| 0   | $I_k^a = u^k \partial_i - \theta_{ia} d^k_a - \frac{1}{3} \delta^k_i (u^i \partial_i - \theta_{ia} d^i_a)$ |
| 0   | $J_d^c = \partial_i \bar{\partial}_d - \theta_{ic} d^i_a - \frac{1}{2} \delta^c_{j} (\partial_i \bar{\partial}_d - \theta_{ic} d^i_a)$ |
| 0   | $Z = 3 \partial_a \bar{\partial}_a + 2 u^i \partial_i + \theta_{ia} d^i_a$ |

The non-zero brackets in $\mathfrak{g}_-$ are

\[
\begin{align*}
\{D_i^a, D_j^b\} &= 6 \epsilon^{ijk} \epsilon_{abc} E_k, \\
[D_i^a, E_j] &= -2 \delta^a_j F^a.
\end{align*}
\]

To verify these brackets one must make use of (5.6), and the Jacobi identity $\{\{D_i^a, D_j^b\}, D_k^c\} + \text{cyclic} = 0$ requires (5.5).
A basis for \( \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_{-2} \) is given by
\[
\tilde{D}^i_a = d^i_a - 3 \epsilon^{ijk} \theta_j^a \partial_k + \epsilon^{ijk} \theta^a \theta^b \partial_b - \alpha^i \partial_a, \\
\tilde{E}_i = \partial_i - \theta^a \partial_a,
\]
which satisfy
\[
\{ \tilde{D}^i_a, \tilde{D}^j_b \} = -6 \epsilon^{ijk} \epsilon^{ab} \tilde{E}_k, \\
[\tilde{D}^i_a, \tilde{E}_j] = 2 \delta^i_j F^a, \\
\{ \tilde{D}^i_a, \tilde{D}^j_b \} = [\tilde{D}^i_a, \tilde{E}_j] = [\tilde{D}^i_a, E_j] = 0.
\]

Any vector field in \( \text{vect}(3|8) \) has the form
\[
X = R^a \partial_a + Q^i \partial_i + P_a d^a = \tilde{R}^a \partial_a + \tilde{Q}^i \tilde{E}_i + P_a \tilde{D}^i_a, 
\]
where
\[
\tilde{Q}^i = Q^i + 3(\epsilon^{ijk} \theta_j^a P_k a), \\
\tilde{R}^a = R^a + (-X) X \theta^a Q^i + 2 \epsilon^{ijk} \theta^a \theta^b P_{kb} - u^i P^a.
\]

\( X \) preserves the dual Pfaff equation \( \tilde{D}^i a = 0 \), i.e.
\[
[X, \tilde{D}^i a] = -(-X) \tilde{D}^i a P_{jb} \tilde{D}^j b, 
\]
provided that
\[
\tilde{D}^i a \tilde{Q}^j = 6(-X) \epsilon^{ijk} P^a_k, \\
\tilde{D}^i a \tilde{R}^b = -2(-X) \epsilon^{ab} \tilde{Q}^i.
\]

In particular, we have the symmetry relations
\[
\tilde{D}^i a \tilde{Q}^j = -\tilde{D}^j a \tilde{Q}^i, \\
\tilde{D}^i a \tilde{R}^b = -\tilde{D}^b a \tilde{R}^i.
\]

Compatibility between
\[
[E_i, X] = (\tilde{E}_i \tilde{R}^a + 2 P^a_i) F_a + \tilde{E}_i \tilde{Q}^j \tilde{E}_j + \tilde{E}_i P_{jb} \tilde{D}^j b
\]
and (6.6) in the form \( \tilde{E}_i = \frac{1}{2 \epsilon} \epsilon^{ijk} \epsilon_{ab} \{ \tilde{D}^j a, \tilde{D}^{kb} \} \), implies that
\[
\tilde{E}_i \tilde{R}^a = -2 P^a_i, \\
\tilde{E}_i \tilde{Q}^j = \frac{1}{2} (-X) (\tilde{F}^{ka} P_{ka} \delta^j_i - \tilde{D}^j a P_{ia}).
\]

In particular,
\[
\tilde{E}_i \tilde{Q}^i = (-X) \tilde{D}^i a P_{ia}.
\]
Hence (6.12) can be written as

\[ [\tilde{E}_i, X] = \frac{1}{2}(-)^X (\tilde{D}^i a P_{ja} \tilde{E}_i - \tilde{D}^j a P_{ia} \tilde{E}_j) + \tilde{E}_i P_{jb} \tilde{D}^j b. \]  

(6.15)

However, (6.13) are identities which follow from (6.10) by considering \( \epsilon_{ijl} \epsilon_{ab} \tilde{D}^i a \tilde{D}^j b Q^k \) and \( \epsilon_{ijk} \epsilon_{ab} \tilde{D}^i a \tilde{D}^j b \tilde{R}^c \), so no new independent conditions on the vector fields arise.

Since \( \mathfrak{m} b (3|8) \) is of depth 3, there are further relations. Compatibility between

\[ [F_a, X] = \partial_a \tilde{R}^b F_b + \partial_a \tilde{Q}^j \tilde{E}_j + \partial_a P_{jb} \tilde{D}^j b \]  

(6.16)

and (6.4) in the form \( F^a = \frac{1}{6} [\tilde{D}^ia, \tilde{E}_i] \) leads to

\[
\begin{align*}
6 \partial_a \tilde{Q}^i & = 6(-)^X \epsilon^{ijk} \tilde{E}_j P_{ka} + \tilde{D}^i a \tilde{E}_j \tilde{Q}^j, \\
3 \partial_b \tilde{R}^a & = (-)^X \delta^a_b \tilde{E}_i \tilde{Q}^i - \tilde{D}^i b P_{ia}.
\end{align*}
\]

(6.17)

In view of (6.14), this means in particular

\[ (-)^X \partial_a \tilde{R}^a = \tilde{E}_i \tilde{Q}^i = (-)^X \tilde{D}^i a P_{ia}. \]  

(6.18)

Also (6.17) are identities which follow from (6.6), (6.10) and (6.13), so no further independent conditions on the vector fields arise.

The pairings \( \langle \tilde{D}^i a, \alpha^j \rangle = \langle \tilde{D}^i a, \beta^b \rangle = 0 \) give

\[
\begin{align*}
\alpha^i & = du^i + 3 \epsilon^{ijk} \theta^a_j d\theta_{ka}, \\
\beta^a & = d\theta^a - u^i d\theta^i_0 + \theta^a_0 d\theta^i + 2 \epsilon^{ijk} \theta^a_i \theta^b_j d\theta_{kb}.
\end{align*}
\]

(6.19)

We now show that \( \alpha^i \) and \( \beta^a \) satisfy Pfaff equations:

\[
\begin{align*}
\mathcal{L}_X \theta_{ia} & = P_{ia}, \\
\mathcal{L}_X u^i & = Q^i, \\
\mathcal{L}_X \theta^a & = R^a, \\
\mathcal{L}_X d\theta_{ia} & = (-)^X d^j b P_{ia} d\theta_{jb} + \partial_j P_{ia} du^j + (-)^X \partial_b P_{ia} d\theta^b, \\
\mathcal{L}_X du^i & = -(-)^X d^j b Q^j d\theta_{jb} + \partial_j Q^i d\theta^j - (-)^X \partial_b Q^i d\theta^b \\
\mathcal{L}_X d\theta^a & = (-)^X d^j b R^a d\theta_{jb} + \partial_j R^a d\theta^j + (-)^X \partial_b R^a d\theta^b.
\end{align*}
\]

(6.20)
In particular,

\[ F_c \vartheta^a = \delta^a_\vartheta, \quad F_c \vartheta^a = 0, \]

\[ F_c \vartheta_{ia} = F_c d \vartheta_{ia} = 0, \quad F_c u^i = F_c du^i = 0, \]

\[ E_k \vartheta^a = \theta^a_k, \quad E_k d \vartheta^a = d \theta^a_k, \]

\[ E_k \vartheta^i = \delta^i_k, \quad E_k du^i = 0, \]

\[ E_k \vartheta_{ia} = 0, \quad E_k d \vartheta_{ia} = 0, \]

\[ D^{kc} \vartheta^a = \varepsilon^{kij} \vartheta^c_i \vartheta^a_j + \varepsilon^a \vartheta_k u^k, \]

\[ D^{kc} u^i = 3 \varepsilon^{ikl} \vartheta^c_l, \]

\[ D^{kc} \vartheta_{ia} = \delta^c_i \delta^a_k, \quad D^{kc} d \vartheta_{ia} = 0, \]

\[ I^k_t \vartheta^a = 0, \quad I^k_t d \vartheta^a = 0, \quad (6.21) \]

\[ I^k_t u^i = \delta^i_k \vartheta^a_d, \quad I^k_t du^i = \delta^i_d \vartheta^a_k - \frac{1}{3} \delta^k \vartheta^a_d, \]

\[ I^k_t \vartheta_{ia} = - \delta^i_k \vartheta_{ia} + \frac{1}{3} \delta^k \vartheta_{ia}, \quad I^k_t d \vartheta_{ia} = - \delta^i_k d \vartheta_{ia} + \frac{1}{3} \delta^k d \vartheta_{ia}, \]

\[ J^c_d \vartheta^a = \delta^a_d \vartheta^c, \quad J^c_d du^i = \delta^a_d d \vartheta^c - \frac{1}{2} \delta^a_d d \vartheta^c, \]

\[ J^c_d u^i = 0, \quad J^c_d \vartheta_{ia} = - \delta^c_d \vartheta_{id} + \frac{1}{2} \delta^c_d \vartheta_{ia}, \]

\[ Z \vartheta^a = 3 \vartheta^a, \quad Z d \vartheta^a = 3 d \vartheta^a, \]

\[ Z u^i = 2 u^i, \quad Z du^i = 2 du^i, \]

\[ Z \vartheta_{ia} = \vartheta_{ia}, \quad Z d \vartheta_{ia} = d \vartheta_{ia}. \]

One checks that

\[ I^k_t \alpha^i = \delta^i_k \alpha^k - \frac{1}{3} \delta^k \alpha^i, \quad I^k_t \beta^a = 0, \]

\[ J^c_d \beta^a = \delta^a_d \beta^c, \quad J^c_d \beta^a = - \frac{1}{2} \delta^a_d \beta^c, \]

\[ Z \alpha^i = 2 \alpha^i, \quad Z \beta^a = 3 \beta^a, \]

\[ D^{kc} \alpha^i = E_k \alpha^i = F_c \alpha^i = 0, \quad D^{kc} \beta^a = E_k \beta^a = F_c \beta^a = 0. \]

Thus the systems of Pfaff equations \( \beta^a = 0 \) and \( \alpha^i = \beta^a = 0 \) are preserved by \( g_- \) and \( g_0 \) and therefore by all of \( \mathfrak{mb}(3|8) \). A long calculation gives

\[ \mathcal{L}_X \alpha^i = \tilde{E}_j \tilde{Q}^i \alpha^j - (-)^X \tilde{\partial}_a \tilde{Q}^i \beta^a, \]

\[ \mathcal{L}_X \beta^a = (-)^X \tilde{\partial}_b \tilde{F}^a \beta^b, \]

provided that the conditions (6.10) and (6.13) hold.

Hence \( \mathfrak{mb}(3|8) \) has three natural classes of tensor modules, with bases
\( \alpha^i, \beta^a \) and \( \gamma^{ia} \), and module action

\[
\mathcal{L}_X \alpha^i = -\partial_i Q^i - (-)^X \delta_a Q^i \beta^a, \\
\mathcal{L}_X \beta^a = (-)^X \delta_b R^a \beta^b, \tag{6.24} \\
\mathcal{L}_X \gamma^{ia} = -(-)^X \bar{D}^{ia} P_{jb} \gamma^{jb}.
\]

One notes that the homogeneous part of the first equation must define a module action by itself, since \( \beta^a \) transforms homogeneously. Hence we set

\[
\mathcal{L}_X \alpha^i = \bar{E}_j Q^j \alpha^i. \tag{6.25}
\]

Let us assume that \( \alpha^i \) and \( \beta^a \) are fermionic. The volume forms \( v_\alpha = \alpha^1 \alpha^2 \alpha^3 \), \( v_\beta = \beta^1 \beta^2 \) and \( v_\gamma = \gamma^{11} \gamma^{12} \gamma^{21} \gamma^{22} \gamma^{31} \gamma^{32} \) transform as

\[
\mathcal{L}_X v_\alpha = \bar{E}_i Q^j v_\alpha, \\
\mathcal{L}_X v_\beta = (-)^X \delta_a R^a v_\beta, \tag{6.26} \\
\mathcal{L}_X v_\gamma = -(-)^X \bar{D}^{ia} P_{ia} v_\gamma.
\]

Since \( v_\alpha \) and \( v_\beta \) have degree +6 each, and \( v_\gamma \) has degree -6, the forms \( v_\alpha v_\gamma \) and \( v_\beta v_\gamma \) are invariant, which implies the relations (6.18).

From (6.24) we deduce the transformation laws for a scalar density \( v \) of weight +1, an \( sl(3) \) vector \( \bar{\alpha}^i \), an \( sl(2) \) vector \( \bar{\beta}^a \), and an \( sl(3) \oplus sl(2) \) vector \( \bar{\gamma}^{ia} \):

\[
\mathcal{L}_X v = \frac{1}{6} \partial_i \bar{Q}^i v = \frac{1}{6} (-)^X \bar{D}^{ia} P_{ia} v, \\
\mathcal{L}_X \bar{\alpha}^i = \frac{1}{6} (-)^X (\delta^i_j \bar{D}^{ja} P_{ka} - 3 \bar{D}^{ia} P_{ja}) \bar{\alpha}^j, \\
\mathcal{L}_X \bar{\beta}^a = \frac{1}{6} (-)^X (\delta^a_b \bar{D}^{bc} P_{ce} - 2 \bar{D}^{ia} P_{jb}) \bar{\beta}^b, \tag{6.27} \\
\mathcal{L}_X \bar{\gamma}^{ia} = \frac{1}{6} (-)^X (\delta^i_j \delta^j_k \bar{D}^{kc} P_{ke} - 6 \bar{D}^{ia} P_{jb}) \bar{\gamma}^{jb}.
\]

Hence we obtain the explicit realization

\[
\mathcal{L}_X = X + \frac{1}{6} (-)^X \bar{D}^{ia} P_{jb} (-3 \delta^b_a \bar{J}^j_i - 2 \delta^j_i \bar{J}^b_a + \delta^j_i \delta^b_a Z) \\
= X + \bar{E}_j \bar{Q}^j \bar{P}_i + (-)^X \delta_b R^a \bar{J}^b_a + \frac{1}{6} \bar{E}_i Q^i Z, \tag{6.28}
\]

where \( \bar{J}^j_i \) and \( \bar{Z} \) generate the Lie algebra \( \bar{g}_0 = sl(3) \oplus sl(2) \oplus gl(1) \) and \( X \) commutes with \( \bar{g}_0 \). By introducing canonical conjugate oscillators \( \bar{\alpha}^*_i, \bar{\beta}^*_a \) and \( v^* \), subject to the Heisenberg algebra \( [\bar{\alpha}^*_i, \bar{\alpha}^j] = \delta^i_j, \ [\bar{\beta}^*_a, \bar{\beta}^b] = \delta^a_b, \ [v^*, v] = 1 \), we obtain the explicit expression for the \( \bar{g}_0 \) generators: \( \bar{J}^j_i = \bar{\alpha}^i \bar{\alpha}^*_j - \frac{1}{2} \bar{\delta}^j_i \bar{\alpha}^k \bar{\alpha}^*_k \), \( \bar{J}^b_a = \bar{\beta}^a \bar{\beta}^*_b - \frac{1}{2} \bar{\delta}^b_a \bar{\beta}^c \bar{\beta}^*_c \) and \( \bar{Z} = vv^* \). However, it is clear that \( \mathcal{L}_X \) will satisfy the same algebra for every representation of \( sl(3) \oplus sl(2) \oplus gl(1) \). Substitution
of irreducible $\mathfrak{g}_0$ modules into $[6.28]$ gives the tensor modules for $\mathfrak{mb}(3|8)$. In particular, the expressions for $\mathfrak{g}_0$ become $L_{I_{ij}} = I_{ij} + \bar{I}_{ij}$, $L_{J_{ka}} = J_{ka} + \bar{J}_{ka}$ and $L_Z = Z + \bar{Z}$, i.e. two commuting copies of $\mathfrak{g}_0$.

To obtain an explicit expression for the vector fields in $\mathfrak{mb}(3|8)$, we set $\bar{R}^a = f^a(\theta, u, \vartheta)$, two arbitrary polynomial functions. From (6.8) and (6.10) we obtain

\[
\bar{Q}^i = \frac{1}{4}(-)^X \bar{D}^ia f_a,
\]
\[
P^a = \frac{1}{3!} \epsilon_{ijk} \bar{D}^ia \bar{D}^{jb} f_b,
\]

i.e.

\[
X \equiv M_f = f^a \partial_a + \frac{1}{4}(-)^f \bar{D}^ia f_a \bar{E}_i + \frac{1}{3!} \epsilon_{ijk} \bar{D}^ia \bar{D}^{jb} f_b \bar{D}^{ka},
\]

where $f = f^a(\theta, u, \vartheta)\partial_a$ is a vector field acting on $\mathbb{C}^{3|8}$ and $(-)^f = +1$ if $f$ is an even vector field, i.e. $f^a$ is an odd function. Due to the symmetry condition (6.11), the components $f^a$ are not independent, but subject to the relations

\[
\bar{D}^ia f^b = -\bar{D}^{ib} f^a, \\
\bar{D}^ia \bar{D}^{ib} f_b = -\bar{D}^{ja} \bar{D}^{ib} f_b.
\]

$\mathfrak{mb}(3|8)$ contains an $\mathfrak{vtc}(3|6)$ subalgebra. We see from (6.23) that if $\partial_b \bar{Q}^a = 0$, then the Pfaff equation $\alpha^i = 0$ is preserved, not just the combined Pfaff equations $\alpha^i = \beta^a = 0$. This condition defines a subalgebra since additional structure is preserved. $\bar{Q}^i$ and $P^a$ can now be expressed in terms of three $\vartheta$-independent functions

\[
f^i(\theta, u) = \frac{1}{4}(-)^X \bar{D}^ia f_a(\theta, u, \vartheta),
\]

as $\bar{Q}^i = f^i$ and $P^a = \frac{1}{3!} \epsilon_{ijk} \bar{D}^ja f^k$. It is clear from (6.7) that when $\bar{Q}^i$ and $P^a$ do not depend on $\vartheta$, the bracket between two vector fields is completely determined by the part $X' = Q^i \partial_i + P_a \bar{D}^ia$, and hence the subalgebra is $\mathfrak{vtc}(3|6)$. Note that $f^a$ still depends on $\vartheta$, in the way specified by (6.32).

An $\mathfrak{mb}(3|8)$ bracket is defined by $[M_f, M_g] = M_{[f,g]}_{\mathfrak{mb}}$. We find

\[
[f, g]_{\mathfrak{mb}} = f^a \partial_a g^b \partial_b - (-)^fg^b \partial_b f^a \partial_a \\
+ \frac{1}{4}(-)^f \bar{D}^ia f_a \bar{E}_i g^b \partial_b - \frac{1}{4}(-)^fg^b g^d \bar{D}^{jb} g_e \bar{E}_j f^a \partial_a \\
+ \epsilon_{ijk} H_{abed}^{ce} \bar{D}^ja \bar{D}^{kc} f_e \bar{D}^{jb} \bar{D}^{ic} f_d \partial_c,
\]

\[
H_{abed}^{ce} = -\frac{1}{3!} \epsilon_{abc} \delta^d_e - \frac{1}{3!} \epsilon_{bd} \delta^a_e,
\]

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Finally we list some of the vector fields \( f \) and \( M_f \) for \( \mathfrak{g}_0 \times \mathfrak{g}_- \):

| deg | \( f \) | \( M_f \) |
|-----|--------|--------|
| -3  | \( \partial_a \) | \( F_a \) |
| -2  | \( 2 \theta^a \partial_a \) | \( E_i \) |
| -1  | \( 2u^i \partial^a + 6 \epsilon^{ijk} \theta^a \theta^b \partial_b \) | \( D^{ia} \) |
| 0   | \( 3\theta^a \partial_a + u^i \theta^a \partial^a \) | \( Z \) |

To compute the vector fields \( f \) which correspond to \( I^a_j \) and \( J^a_b \) is quite tedious and has not been attempted. However, since \( \epsilon^{ijk} \theta^a \theta^b \theta^c = 0 \) it is clear that \( M_f = I^a_j \) for \( f \) some linear combination of \( u^i \theta^a \partial_a \) and \( \epsilon^{ikl} \theta^a \theta^b \theta^c \partial_b \), and \( M_f = J^a_b \) for \( f \) some linear combination of \( \theta^a \partial_b \), \( u^i \theta^a \partial^a \), and \( \epsilon^{ijk} \theta^a \theta^b \theta^c \partial_c \); the right combinations are found by demanding that the symmetry conditions (6.31) hold. Other vector fields of non-positive degree are not compatible with (6.31).

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