Scalar induced gravitational waves from primordial black hole Poisson fluctuations in Starobinsky inflation

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Abstract. The gravitational potential of a gas of initially randomly distributed primordial black holes (PBH) can induce a stochastic gravitational-wave background through second-order gravitational effects. This gravitational-wave background can be abundantly generated in a cosmic era of domination of ultralight primordial black holes, with masses $m_{\text{PBH}} < 10^{9}g$, which evaporate before Big Bang Nucleosynthesis. Hence, the condition to avoid overproduction of gravitational waves at PBH evaporation time, can act as a novel method to extract constraints on cosmological models and gravitational theories. We consider $f(R)$ gravity as the underlying gravitational theory and we study its effect at the level of the gravitational potential of Poisson distributed primordial black holes. After the general analysis we focus on Starobinsky $R^2$ gravity model and we extract strong constraints on the involved mass parameter, denoted as $M$, as a function of the initial primordial black hole abundance, $\Omega_{\text{PBH},f}$ and the black hole mass, $m_{\text{PBH}}$. In particular, one finds that in general $5 \times 10^{-14} \lesssim \frac{M_{\text{min}}}{M_{\text{Pl}}} \lesssim 10^{-5}$, and only in the extreme possible regime where $\Omega_{\text{PBH},f} > 10^{-3}$ we get that $10^{-5} \lesssim \frac{M_{\text{min}}}{M_{\text{Pl}}} \lesssim 10^{-1}$.

Keywords: Primordial Black Holes, Gravitational Waves, Modified Gravity, Starobinsky Inflation, $f(R)$ gravity
1 Introduction

Primordial black holes (PBHs) are formed in the early universe before the birth of stars, out of the collapse of enhanced energy density perturbations. These ultracompact objects were firstly proposed in the early '70s [1–3] and are currently attracting increasing attention since they can address a number of issues of modern cosmology. According to recent arguments, they can potentially account for a part or all of the dark matter content of the Universe [4], and additionally they can offer an explanation for the large-scale structure formation through Poisson fluctuations [5, 6]. Furthermore, they can provide seeds for the supermassive black holes residing in the centre of galaxies [7, 8], as well as constitute viable candidates for the progenitors of the black-hole merging events recently detected by the LIGO/VIRGO collaboration [9] through the emission of
gravitational waves (GWs). Other evidence in favor of the PBH scenario can be found in [10].

Due to the significance of PBHs and the huge progress achieved in the field of gravitational-wave astronomy, there have been many attempts connecting PBHs and GWs [11]. Firstly, a large amount of research has been devoted to the GW background signals associated to PBH merging events [12–17]. Moreover, extensive research has been also performed regarding the PBH Hawking radiated-graviton background [18, 19] as well as concerning the scalar induced GWs connected to the primordial high curvature perturbations which gave rise to PBHs [20–23] (for a recent review see [24]). However, apart from the aforementioned GW signals, it has been recently noted in [25], and further studied in [26, 27], that the Poisson fluctuations of a gas of randomly distributed PBHs can induce second-order GWs at distances much larger than the PBH mean separation scale. These GWs are not induced by the primordial curvature perturbations, which gave rise to PBHs, but instead by the PBH density fluctuations themselves.

At the same time, there are many reasons indicating that one should construct modified gravitational theories. At the theoretical level, gravitational modifications are known to be able to improve the renormalizability issues of general relativity [28, 29]. At the phenomenological level, modified gravity can offer an alternative way to explain the two phases of the Universe’s accelerated expansion, namely the early-time, inflationary one [30, 31], and/or the late-time, dark-energy one [32–34]. In all cases, these modified gravitational theories possess general relativity as a particular limit, but in general they have a richer structure and extra degrees of freedom that can describe the Universe’s evolution.

One of the simplest classes of modified gravity is \( f(R) \) gravity, which is obtained through the extension of the Einstein-Hilbert Lagrangian to an arbitrary function of the Ricci scalar [35]. Apart from its general cosmological application, in the inflationary framework the particular subclass of the theory known as Starobinsky, or \( R^2 \) gravity [36], proves to be one of the best-fitted models to the cosmological data [37]. Hence, due to its success, \( f(R) \) gravity has been extensively studied in the literature. In particular, in such investigations one is in general interested in extracting the corrections on various observational signals, induced by the \( f(R) \) modifications on top of the corresponding general-relativity predictions (see [38–60] and references therein).

Therefore, in the present work we are interested in investigating the GW signal induced by PBH Poisson fluctuations, in the framework of \( f(R) \) gravity. In particular, since all the relevant studies up to now have been performed in the framework of general relativity, apart from [61, 62] where the authors study the primordial scalar induced GWs in modified gravity constructions, in the following we calculate the effect of \( f(R) \) corrections on the PBH gravitational potential power spectrum and subsequently on the associated scalar induced GW background. In this way, one may use it as an extra and novel method to constrain possible \( f(R) \) modifications, as well as an independent test of general relativity.

The plan of the work is as follows. In Sec. 2, we review the PBH gravitational potential in general relativity, and in Sec. 3 we perform the extended analysis, extracting
the PBH gravitational potential in the framework of $f(R)$ gravity. In Sec. 4, we focus on Starobinsky gravity as the underlying $f(R)$ theory and we extract the relevant scalar induced gravitational wave signal. Then, in Sec. 5 we obtain lower bounds on the mass parameter $M$ of Starobinsky gravity. Finally, Sec. 6 is devoted to the conclusions.

2 The primordial black hole gravitational potential in general relativity

In the context of general relativity (GR), the action is written as follows:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda \right) + \int d^4x \sqrt{-g} \mathcal{L}_m,$$

with $G$ being the gravitational Newton constant (throughout this paper we work in units where $c = 1$), $R$ the Ricci scalar, $\Lambda$ the cosmological constant, $\mathcal{L}_m$ the total matter Lagrangian density (radiation, baryonic and dark matter) of the Universe and $T^m_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}}$ the corresponding total matter energy-momentum tensor. Varying the action (2.1) with respect to the metric $g^{\mu\nu}$ we obtain the usual Einstein field equations, namely

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T^m_{\mu\nu}.$$

(2.2)

Note that the Bianchi identity $\nabla_{\mu}G_{\nu}^{\mu} = 0$ implies the conservation of the total energy-momentum tensor.

2.1 Background evolution

Proceeding to a cosmological setup, we consider a flat Friedmann - Lemaître - Robertson -Walker (FLRW) background metric of the form

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,$$

(2.3)

where $a(t)$ is the scale factor. By adopting this background metric and assuming that the total matter content of the universe is described by the perfect fluid energy-momentum tensor $T^m_{\mu\nu} = \text{diag}(-\bar{\rho}, \bar{p}, \bar{p}, \bar{p})$, where $\bar{\rho}$ and $\bar{p}$ are the total matter (i.e. including radiation, baryonic and dark matter) energy density and pressure respectively, the GR field equations give rise to the two Friedmann equations:

$$H^2 = \frac{8\pi G}{3} \bar{\rho} + \frac{\Lambda}{3} \equiv \frac{8\pi G}{3} \bar{\rho}_{\text{tot}}$$

(2.4)

$$\dot{H} + H^2 = -\frac{4\pi G}{3} (\bar{\rho} + 3\bar{p}) + \frac{\Lambda}{3} \equiv -\frac{4\pi G}{3} (\bar{\rho}_{\text{tot}} + 3\bar{p}_{\text{tot}}),$$

(2.5)

where $H = \dot{a}/a$ is the Hubble parameter, with dots denoting derivatives with respect to the cosmic time $t$. In the above expressions $\bar{\rho}_{\text{tot}}$ and $\bar{p}_{\text{tot}}$ correspond to the total background energy density and pressure of the Universe, i.e the total matter sector as well as the cosmological constant term, which is interpreted as a dark energy fluid with
whose energy-momentum tensor is \( T^\text{de}_{\mu\nu} = \text{diag}(\rho_{\text{de}}, p_{\text{de}}, p_{\text{de}}, p_{\text{de}}) \).

Nevertheless, since in this work we focus on the early-time matter (i.e. PBH) dominated era, the contribution of the cosmological constant or effective dark energy at the background level can be neglected.

Lastly, it proves convenient to introduce the conformal time \( \eta \) defined through \( \frac{dt}{a} = d\eta \), and similarly the conformal Hubble parameter defined as \( H \equiv \frac{a'}{a} = aH \), where primes denote derivatives with respect to \( \eta \). Hence, the above two Friedmann equations become simply

\[
H^2 = \frac{8\pi G a^2}{3} \bar{\rho}_{\text{tot}} \tag{2.6}
\]

\[
H' = -\frac{4\pi G a^2}{3} (\bar{\rho}_{\text{tot}} + 3\bar{p}_{\text{tot}}). \tag{2.7}
\]

### 2.2 Scalar perturbations

Let us now refer to the perturbation evolution. Focusing on scalar perturbations, the perturbed FLRW metric in the Newtonian gauge reads as

\[
ds^2 = a^2(\eta) \left\{ -(1 + 2\Psi) d\eta^2 + [(1 - 2\Phi)\delta_{ij}] dx^i dx^j \right\}, \tag{2.8}
\]

where for convenience we perform the calculations using the conformal time \( \eta \). In the above ansatz, \( \Psi \) and \( \Phi \) stand for the Bardeen potentials \([63]\), which are first order quantities in cosmological perturbation theory.

Further, we allow perturbations around the background stress-energy tensor of the total matter content of the Universe (matter and radiation) which we write as follows:

\[
T^0_0 = -(\bar{\rho} + \delta\rho)
\]

\[
T^i_i = (\bar{\rho} + \bar{p})\upsilon_i, \quad \upsilon_i \equiv a\delta u_i
\]

\[
T^0_j = \bar{p}(\delta^0_j + \Pi^0_j), \tag{2.9}
\]

where \( \delta \equiv \delta\rho/\bar{\rho} \) is the relative energy density perturbation, \( \delta u_i \equiv \upsilon_i/a \) is the velocity perturbation and \( \Pi^0_j \) is the (dimensionless) anisotropic stress. The evolution of \( \Phi \) and \( \Psi \) is governed by the perturbed Einstein equations, which are \([64]\):

\[
3H(\Phi' + H\Psi) - \nabla^2 \Phi = -4\pi G a^2 \delta \rho \tag{2.10}
\]

\[
(\Phi' + H\Psi)_{,i} = 4\pi G a^2 (\bar{\rho} + \bar{p})\upsilon_{,i} \tag{2.11}
\]

\[
\Phi'' + H(\Phi' + 2\Psi') + (H^2 + 2H')\Phi + \nabla^2(\Phi - \Psi)/3 = 4\pi Ga^2 \delta p \tag{2.12}
\]

\[
\Phi - \Psi = 8\pi Ga^2 \bar{p} \Pi. \tag{2.13}
\]

Since during the time period we are concerned with the anisotropic stress of the Universe is negligible, from (2.13) we see that \( \Phi \approx \Psi \), which we will adopt from now on. This potential can actually be identified with the PBH gravitational potential, whose behavior will be derived in the following analysis.\footnote{The first-order gravitational potential due to the primordial energy density perturbations is ignored here as we concentrate on the induced GW signal due to the PBH energy density perturbations. This contribution can be added to the contribution calculated in our work, if we desire to include the primordial scalar induced GWs \([24]\) too.}
We proceed by defining the total entropy perturbation as
\[ S \equiv H \left( \frac{\delta p}{\bar{p}'} - \frac{\delta \rho}{\bar{\rho}'} \right). \]  
(2.14)
Since the (total) energy-momentum tensor is conserved, the background continuity equation holds, namely \( \bar{\rho}' = -3H(\bar{\rho} + \bar{\rho}) \). Therefore, from (2.14) we acquire:
\[ \delta p = c_s^2[\delta \rho - 3(\bar{\rho} + \bar{\rho})S], \]  
(2.15)
where \( w \equiv \bar{p}/\bar{\rho} \) is the equation-of-state parameter and \( c_s^2 \equiv \bar{p}'/\bar{\rho}' \) is the sound speed square of the total matter content of the Universe. Finally, one can combine (2.10) with (2.12) and (2.15) to get the following equation governing the behavior of the gravitational potential \( \Phi \):
\[ \Phi'' + 3H (1 + c_s^2) \Phi' - c_s^2 \nabla^2 \Phi + 3 (c_s^2 - w) H^2 \Phi = -\frac{9}{2} c_s^2 (1 + w) H^2 S. \]  
(2.16)

2.3 The Power Spectrum of the PBH Gravitational Potential

Having extracted above the background and the perturbation equations for the PBH gravitational potential, we derive here the corresponding power spectrum following closely [25]. The common assumption concerning PBH formation is that PBHs are formed in the radiation-dominated era. Hence, considering PBHs as a matter fluid, their formation process can be regarded as a transition of a fraction of the radiation energy density into PBHs. Thus, assuming that PBHs are randomly distributed in space at formation time, their energy density is inhomogeneous while the total energy density of the background is homogeneous. Consequently, the PBH energy density perturbation can be viewed as an isocurvature Poisson fluctuation. As it was found in [25], the Poissonian power spectrum for the PBH density contrast at PBH formation era, assuming monochromatic PBH mass function [65], reads as
\[ P_\delta(k) = \frac{k^3}{2\pi^2} P_\delta = \frac{2}{3\pi} \left( \frac{k}{k_{UV}} \right)^3 \Theta(k_{UV} - k), \]  
(2.17)
where \( k_{UV} \equiv a/\bar{r} \) is a UV cut-off scale related to the mean PBH separation scale. This UV cut-off scale is introduced here since at scales smaller than the mean PBH separation scale the PBH fluid description is not valid. In particular, at these scales one probes the granularity of the PBH energy density field entering the non-linear regime where \( P_\delta(k) > 1 \).

The next step is to relate the above power spectrum of the PBH energy density perturbations to a power spectrum for the PBH gravitational potential \( \Phi \). In order to achieve this we should have in mind that since in the radiation-dominated era, \( \Omega_{\text{PBH}} \equiv \frac{\rho_{\text{PBH}}}{\rho_{\text{tot}}} \propto a \), if the initial abundance of PBHs is large enough, then PBHs can potentially dominate the universe energy budget. Consequently, the isocurvature PBH energy density perturbation in the radiation-dominated era will be converted to an adiabatic curvature perturbation in the subsequent PBH dominated era [66, 67], which will be related to a gravitational potential \( \Phi \).
To derive now $\Phi$ from $\delta_{\text{PBH}}$, we use as an intermediate variable the uniform-energy density curvature perturbation of a fluid, $\zeta$, which is related with the Bardeen potential $\Phi$ and the respective energy density perturbation by the following definition [68]:

$$\zeta \equiv -\Phi - \mathcal{H} \frac{\delta \rho}{\bar{\rho}}. \tag{2.18}$$

If the energy-momentum tensor of this fluid is conserved, the (background) continuity equation $\dot{\bar{\rho}} = -3\mathcal{H}(\bar{\rho} + \bar{p})$ holds, and thus $\zeta$ is expressed as

$$\zeta \equiv -\Phi + \frac{\delta}{3(1 + w)}, \tag{2.19}$$

where $w \equiv \bar{p}/\bar{\rho}$ is the equation-of-state parameter of the total matter content of the Universe. In our case, since the energy-momentum tensors of radiation and PBH-matter are separately conserved, we can use (2.19) for $\zeta_r$ and $\zeta_{\text{PBH}}$ and acquire:

$$\zeta_r = -\Phi + \frac{1}{4} \delta_r, \tag{2.20}$$

$$\zeta_{\text{PBH}} = -\Phi + \frac{1}{3} \delta_{\text{PBH}}. \tag{2.21}$$

Finally, we introduce the isocurvature perturbation defined as:

$$S = 3(\zeta_{\text{PBH}} - \zeta_r) = \delta_{\text{PBH}} - \frac{3}{4} \delta_r. \tag{2.22}$$

On superhorizon scales, $\zeta_r$ and $\zeta_{\text{PBH}}$ are conserved separately [68], like the isocurvature perturbation $S$. Thus, in the PBH-dominated era, $\zeta \simeq \zeta_{\text{PBH}} = \zeta_r + S/3 \simeq S/3$. Since $S$ is conserved, it can be calculated at formation time $t_f$. Therefore, neglecting the adiabatic contribution associated to the radiation fluid at the PBH formation time, since it is negligible for the scales considered here, from Eq. (2.22) we obtain that $S = \delta_{\text{PBH}}(t_f)$. Hence, we finally find

$$\zeta \simeq \frac{1}{3} \delta_{\text{PBH}}(t_f) \quad \text{if} \quad k \ll \mathcal{H}. \tag{2.23}$$

Using now the fact that $\zeta \simeq -\mathcal{R}$ on superhorizon scales (see e.g. [68]), where $\mathcal{R}$ is the comoving curvature perturbation defined by

$$\mathcal{R} = \frac{2 \Phi'}{3} \frac{\mathcal{H} + \Phi}{1 + w} + \Phi, \tag{2.24}$$

one gets straightforwardly that in the PBH-matter dominated era, where $w = 0$ and $\Phi$ is constant in time [68],

$$\Phi \simeq -\frac{1}{5} \delta_{\text{PBH}}(t_f) \quad \text{if} \quad k \ll \mathcal{H}. \tag{2.25}$$

On sub-Hubble scales, one can determine the evolution of $\delta_{\text{PBH}}$ by solving the evolution equation for the matter density perturbations, namely the Mészáros growth
equation [69], which, in the case of a Universe with radiation and PBH-matter, takes the form:

$$\frac{d^2 \delta_{PBH}}{ds^2} + \frac{2 + 3s}{2s(s+1)} \frac{d\delta_{PBH}}{ds} - \frac{3}{2s(s+1)} \delta_{PBH} = 0. \quad (2.26)$$

By solving the above equation one can find that the the dominant solution deep in the PBH-dominated era can be written as \(\delta_{PBH} \simeq 3s \delta_{PBH}(t_i)/2\). Now, the relation between the Bardeen potential and the density contrast is dictated by the Poisson equation, and in a matter-dominated era takes the form

$$\delta_{PBH} = -\frac{2}{3} \left( \frac{k}{\mathcal{H}} \right)^2 \Phi. \quad (2.27)$$

Therefore, plugging the solution for \(\delta_{PBH}\) into the aforementioned formula, one obtains

$$\Phi \simeq -\frac{9}{4} \left( \frac{\mathcal{H}_d}{k} \right)^2 \delta_{PBH}(t_i) \quad \text{if} \quad k \gg \mathcal{H}_d, \quad (2.28)$$

where \(\mathcal{H}_d\) is the conformal Hubble function at PBH domination time. Finally, making an interpolation between (2.28) and (2.25), and using (2.17) one obtains that

$$P_\Phi(k) \equiv \frac{k^3}{2\pi^2} P_\Phi = \frac{2}{3\pi} \left( \frac{k}{k_{UV}} \right)^3 \left( 5 + \frac{4}{9} \frac{k^2}{k_d^2} \right)^{-2}, \quad (2.29)$$

where \(k_d \equiv \mathcal{H}_d\) is the comoving scale exiting the Hubble radius at PBH domination time. From Eq. (2.29), one can see that \(P_\Phi\) has a broken power-law behavior: when \(k \ll k_d\) we have that \(P_\Phi \propto k^3\), while when \(k \gg k_d\) we acquire \(P_\Phi \propto 1/k\). We mention that it reaches its maximum when \(k \sim k_d\), where \(P_\Phi\) is of order \((k_d/k_{UV})^3\).

3 The primordial black hole gravitational potential in \(f(R)\) gravity

In the previous section we presented the calculation of the PBH gravitational potential power spectrum in the framework of general relativity. In this section we proceed to the bulk of our analysis, which is to perform the same calculation but in the case of \(f(R)\) modified gravity, extracting the corresponding corrections.

We consider a modified action of the form [35]

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) + \int d^4x \sqrt{-g} \mathcal{L}_m, \quad (3.1)$$

where \(f(R)\) is a general function of the Ricci scalar \(R\). Variation of the action (3.1) with respect to the metric \(g^{\mu\nu}\) yields the following field equations:

$$FR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) F = 8\pi G T^m_{\mu\nu}, \quad (3.2)$$

where we have set \(F \equiv df(R)/dR\). One characteristic feature of the richer structure of \(f(R)\) gravity is the existence of an additional propagating degree of freedom, the
so-called scalaron field [70]. Its equation can be obtained by taking the trace of (3.2), which yields:

\[
\Box F(R) = \frac{1}{3} [2f(R) - F(R)R + 8\pi G T^m] \equiv \frac{dV}{dF}, \tag{3.3}
\]

where \( T^m \) is the trace of the energy-momentum tensor of the (total) matter content of the Universe. As we observe, equation (3.3) is a wave equation for \( \phi_{sc} \equiv F(R) \) whose mass is given by \( m_{sc}^2 \equiv d^2V/dF^2 \), which reads:

\[
m_{sc}^2 = \frac{1}{3} \left( \frac{F}{F_{,R}} - R \right), \tag{3.4}
\]

where \( F_{,R} \equiv dF/dR = d^2f/dR^2 \). An alternative way to see this is by performing a conformal transformation to the Einstein frame [35]. Amongst others, the presence of this additional degree of freedom induces an extra polarization mode for the gravitational waves [35], as we will see in the next section.

For our purposes, we shall formulate \( f(R) \) gravity in terms of an effective curvature-induced fluid. Specifically, we shall express the equations (3.2) as the corresponding ones in GR (2.2), with the addition of the following energy-momentum tensor \([71]\) instead of the one induced by the cosmological constant:

\[
T^{(R)}_{\mu\nu} \equiv (1 - F) R^\mu_{\nu} + \frac{1}{2} \delta^\mu_{\nu} (f - R) - (\delta^\mu_{\nu} \Box - \nabla^\mu \nabla_\nu) F. \tag{3.5}
\]

Similarly to the GR case, we will first examine the evolution at the background and perturbation levels, and then we will calculate the power spectrum of the PBH gravitational potential.

### 3.1 Background evolution

Applying \( f(R) \) gravity to a cosmological framework, namely using the FLRW metric (??), we extract the Friedmann equations, which in terms of the conformal time are written as

\[
\mathcal{H}^2 = \frac{8\pi G \a^2}{3} \bar{\rho}_{tot}, \tag{3.6}
\]

\[
\mathcal{H}' = -\frac{4\pi G \a^2}{3} (\bar{\rho}_{tot} + 3\bar{p}_{tot}). \tag{3.7}
\]

We mention that these equations acquire the same form as in the GR case, with the only difference being that in the total content of the Universe we need to take into account the contribution of the \( f(R) \) curvature-induced effective fluid, whose energy density and pressure are given by [35]:

\[
\bar{\rho}_{(R)} \equiv -T^{(R)}_{00} = \frac{1}{8\pi G \a^2} \left( 3\mathcal{H}^2 - \frac{1}{2} a^2 f + 3F \mathcal{H}' - 3F' \right) \tag{3.8}
\]

\[
\bar{p}_{(R)} \equiv \frac{T^{(R)}_{i i}}{3} = \frac{1}{8\pi G \a^2} \left( -2\mathcal{H}' - \mathcal{H}^2 + \frac{1}{2} a^2 f - F \mathcal{H}' - 2F' \mathcal{H}^2 + F'' + F F' \right). \tag{3.9}
\]
3.2 Scalar perturbations

In order to describe the evolution of scalar perturbations, we shall use again the metric (2.8) and the perturbed form of the (total) energy-momentum tensor (2.9). The perturbed field equations are similar in form with the corresponding ones of GR, with the addition of $\delta \rho_{f(R)}, \delta p_{f(R)}, v_{f(R)}$ and $\Pi_{f(R)}$. They are provided explicitly in Appendix A. Therefore, we need to take into account the contribution of the $f(R)$ curvature-induced effective fluid to the expressions introduced in subsection 2.2.

In this context, we define the total entropy perturbation as:

$$S_{\text{tot}} \equiv H \left( \frac{\delta p_{\text{tot}}}{\bar{p}_{\text{tot}}} - \frac{\delta \rho_{\text{tot}}}{\bar{\rho}_{\text{tot}}} \right).$$

(3.10)

Again the total energy-momentum tensor is conserved, so the background continuity equation holds, namely $\bar{\rho}'_{\text{tot}} = -3H(\bar{\rho}_{\text{tot}} + \bar{p}_{\text{tot}})$, so from (3.10) we acquire:

$$\delta p_{\text{tot}} = c^2_{\text{tot}}[\delta \rho_{\text{tot}} - 3(\bar{\rho}_{\text{tot}} + \bar{p}_{\text{tot}}) S_{\text{tot}}],$$

(3.11)

where $w_{\text{tot}} \equiv \bar{p}_{\text{tot}}/\bar{\rho}_{\text{tot}}$ is the total equation-of-state parameter and $c^2_{\text{tot}} \equiv \bar{p}'_{\text{tot}}/\bar{\rho}'_{\text{tot}}$ is the sound speed square of the total content of the Universe. By combining (A.1) with (A.3) and (3.11) to get the following equation governing the behavior of the gravitational potential $\Phi$:

$$\Phi'' + 3H \left( 1 + c^2_{\text{tot}} \right) \Phi' - c^2_{\text{tot}} \nabla^2 \Phi + 3 \left( c^2_{\text{tot}} - w_{\text{tot}} \right) H^2 \Phi = -\frac{9}{2} c^2_{\text{tot}} (1 + w_{\text{tot}}) H^2 S_{\text{tot}}.$$

(3.12)

3.3 The Power Spectrum of the PBH Gravitational Potential in $f(R)$ gravity

We can now repeat the procedure of subsection 2.3 but in the context of $f(R)$ gravity. Hence, we need to take into account the presence of the $f(R)$ curvature-induced effective fluid. Therefore, on top of the usual $\zeta_r$ and $\zeta_{\text{PBH}}$, we have $\zeta_{f(R)}$, too. Since by construction its energy-momentum tensor (3.5) is conserved, we can use (2.19) to get:

$$\zeta_{f(R)} = -\Phi + \frac{1}{3(1 + w_{f(R)})} \delta_{f(R)},$$

(3.13)

where $w_{f(R)} \equiv \bar{p}_{f(R)}/\bar{\rho}_{f(R)} = -a^2 f^2 + 2 \left( (1+2F)H^2 - 2HF' + (2+2F)F' + H'F'' \right)$ is the equation-of-state parameter of the effective fluid. We will study how these curvature perturbations evolve on super-Hubble ($k \ll H$) and sub-Hubble ($k \gg H$) scales.

On super-Hubble scales, $\zeta_r$ and $\zeta_{\text{PBH}}$ are separately conserved [68], as is the isocurvature perturbation between them, which is defined by

$$S = 3 (\zeta_{\text{PBH}} - \zeta_r).$$

(3.14)

However, the total curvature perturbation is not conserved and is equal to

$$\zeta = -\Phi + \frac{\delta_{\text{tot}}}{3(1 + w_{\text{tot}})} = \frac{4}{3} \bar{\rho}_r \zeta_r + \bar{\rho}_{\text{PBH}} \zeta_{\text{PBH}} + (1 + w_{f(R)}) \bar{\rho}_{f(R)} \zeta_{f(R)}.$$

(3.15)
At this point it is reasonable to assume that \( w_{\Omega(R)} \approx -1 \) at the time period we are interested in, since the scenario should not differ significantly form \( \Lambda \text{CDM} \) cosmology at the background level. Hence, \( \zeta \) becomes

\[
\zeta = \frac{4}{4+3s} \zeta_r + \frac{3s}{4+3s} \zeta_{\text{PBH}},
\]

with \( s \equiv \frac{a}{a_d} \), where \( a_d \) denotes the value of the scale factor \( a \) at the time PBHs start to dominate. From this expression, we can see that \( \zeta \) evolves from its initial value \( \zeta_r \), deep in the radiation era, to \( \zeta_{\text{PBH}} \), deep in the PBH era. As a consequence, in the PBH-dominated era, \( \zeta \simeq \zeta_{\text{PBH}} = \zeta_r + S/3 \). Since \( S \) is conserved on super-Hubble scales, it can be evaluated at formation time \( t_f \). Furthermore, the isocurvature perturbation can be identified with \( \delta_{\text{PBH}}(t_f) \), which will be calculated in the following subsection, assuming implicitly a uniform radiation energy density in the background. Indeed, in the following we will focus on the PBH contribution and we will ignore the usual adiabatic contribution (associated to the radiation fluid), which is negligible at the scales we are interested in, hence we simply have

\[
\zeta \simeq \frac{1}{3} \delta_{\text{PBH}}(t_f) \quad \text{if} \quad k \ll H.
\]

Concerning the super-Hubble scales, as we show in Appendix B, in \( f(R) \) gravity one can also use the property \( \zeta \simeq -R \) (see e.g. [68]), where \( R \) is the comoving curvature perturbation defined in (2.24), by requiring that \( \delta F \approx 0 \) for \( k \ll H \) which ensures (in addition to the usual assumption that the anisotropic stress of radiation is negligible at these scales) that \( \Psi \approx \Phi \). During a matter-dominated era, such as the one driven by PBHs, \( \Phi' \) can be neglected since it is proportional to the decaying mode, thus we obtain \( R = -\zeta = (5/3)\Phi \). Finally, combining with (3.17), this implies that

\[
\Phi \simeq -\frac{1}{5} \delta_{\text{PBH}}(t_f) \quad \text{if} \quad k \ll H.
\]

Let us now focus on sub-Hubble scales. One can determine the evolution of \( \delta_{\text{PBH}} \) by solving the evolution equation of the matter density perturbations, namely the Meszaros equation [69], in a Universe where we have radiation, matter in form of PBHs, and an effective dark energy fluid due to the \( f(R) \) gravity modulations, which should however be negligible before Big Bang Nucleosynthesis (BBN) time where one expects a subdominant energetic contribution from the dark energy sector.

At the background level, the Friedman equation (3.6) can be expressed as

\[
H^2 \simeq H^2_0 \Omega_{\text{PBH},f} \left( \frac{1}{s} + \frac{1}{s_f^2} \right),
\]

where \( s \equiv a/a_d \) and \( a_d \) denotes the time at the transition from the radiation to the PBH domination era, and where we have assumed that \( \Omega_{r,f} = 1 \) since PBHs are considered to
be formed in the radiation era [25]. Note that the scale factor is normalised at one at formation time, i.e. $a_f = 1$.

At the perturbation level, we can use the standard cosmological perturbation theory at subhorizon scales, where the matter perturbations obey the growth equation [72, 73]:

$$\ddot{\delta}_m + \mathcal{H}\dot{\delta}_m - 4\pi Ga^2 \bar{\rho}_m \delta_m = 0.$$  \hfill (3.20)

Treating the gas of PBHs as a matter fluid and accounting for the screening of the gravitational constant due to $f(R)$ gravity modification, one should replace in the above equation $\delta_m$ with $\delta_{\text{PBH}}$ and $G$ with $G_{\text{eff}}$ where [74]

$$G_{\text{eff}} \equiv G F \left( \frac{1 + 4 k^2 F R_{F}}{1 + 3 k^2 F R_{F}} \right).$$  \hfill (3.21)

Hence, assembling everything, and using $s$ as the time variable, the growth equation (3.20) can be recast in the following form:

$$\frac{d^2 \delta_{\text{PBH}}}{ds^2} + \frac{2 + 3s}{2s(s+1)} \frac{d\delta_{\text{PBH}}}{ds} - \frac{3}{2s(s+1)} F \frac{1 + 4 k^2 F R_{F}}{1 + 3 k^2 F R_{F}} \delta_{\text{PBH}} = 0.$$  \hfill (3.22)

We proceed by relating our solution for $\delta_{\text{PBH}}$ from (3.22) with $\Phi$, via the sub-Hubble scale approximation of the time-time field equation in $f(R)$ gravity for the PBH dominated era (equations (A.1) and (A.7) of Appendix A), which is:

$$\delta_{\text{PBH}} = - \frac{2}{3} \left( \frac{k}{\mathcal{H}} \right)^2 F \left( 1 + 3 \frac{k^2 F R_{F}}{F} \right) \frac{1 + 4 k^2 F R_{F}}{1 + 3 k^2 F R_{F}} \Phi.$$  \hfill (3.23)

Hence, making an interpolation between Eq. (3.18) and Eq. (3.23) as in the case of GR, and using the expression for the PBH matter power spectrum in Eq. (2.17) we straightforwardly extract the following PBH gravitational potential power spectrum:

$$P_\Phi(k) \equiv \frac{k^3}{2\pi^2} P_\Phi(k) = \frac{2}{3\pi} \left( \frac{k}{k_{\text{UV}}} \right)^3 \left[ 5 + 2 \left( \frac{k}{\mathcal{H}} \right)^2 F \left( 1 + 3 \frac{k^2 F R_{F}}{F} \right) \xi(a) \right]^{-2}.$$  \hfill (3.24)

In the above expression, $\xi(a)$ is defined as

$$\xi(a) \equiv \frac{\delta_{\text{PBH}}(a)}{\delta_{\text{PBH}}(a_f)},$$  \hfill (3.25)

where $\delta_{\text{PBH}}(a)$ is the solution of Eq. (3.22). As checked numerically, $\xi(a)$ has a mild dependence on on the comoving scale $k$, and thus for practical reasons we will consider $\xi(a)$ as $k$ independent. Lastly, note that in the case of GR we have $F = 1$ and $\xi(a) \propto a/a_d$, and thus we recover the result of (2.29).
4 Scalar induced gravitational waves in Starobinsky $R^2$ gravity

In the previous section we derived the power spectrum of the gravitational potential of initially Poisson-distributed PBHs, and thus in this section we are able to extract the stochastic gravitational wave background induced at second order from the PBH Poisson fluctuations. Since we will perform specific calculations we have to specify our $f(R)$ form. As we mentioned in the Introduction, one of the most studied cases, which can also give rise to an inflationary scenario with a very efficient agreement with observations, is the Starobinsky or $R^2$ gravity [36], in which

$$f(R) = R + \frac{1}{6M^2}R^2,$$  \hspace{1cm} (4.1)

with $M$ being the model parameter with dimensions of mass.

Before deriving the GW spectrum induced from a gas of PBHs, it is important to stress out here a major issue emerging from the study of induced GWs at second order. In particular, while the tensor modes are gauge invariant at first order, this is not valid at second order [75–79]. This implies that, a priori, one needs to specify in which gauge the gravitational waves are observed. However, in this work we explore a GW backreaction problem without paying attention to observational predictions. In particular, if the energy density associated to the induced gravitational waves overcomes the one of the background, one expects perturbation theory to break down in any gauge. Hence, it is legitimate to assume that our findings bear little dependence on the gauge choice.

4.1 Tensor Perturbations

Having clarified the gauge choice issue, we continue by studying the tensor perturbations $h_{ij}$ induced by the gravitational potential $\Phi$. In particular, the perturbed metric, assuming as usual zero anisotropic stress and $\delta F/F \approx 0$, in the Newtonian gauge is written as

$$ds^2 = a^2(\eta) \left\{ -(1 + 2\Phi)d\eta^2 + \left[ (1 - 2\Phi)\delta_{ij} + \frac{h_{ij}}{2} \right] dx^i dx^j \right\},$$  \hspace{1cm} (4.2)

where we have multiplied by a factor $1/2$ the second order tensor perturbation as is standard in the literature. Then, by Fourier transforming the tensor perturbations and taking into account the three polarization modes of the GWs in $f(R)$ gravity, namely the $\times$ and the $+$ as in GR and the scalaron one, denoted with sc, the equation of motion for the tensor modes $h_{lk}$ reads as

$$h_{lk}'' + 2Hh_{lk}' + (k^2 - \lambda m_{sc}^2)h_{lk} = 4S_{lk}^s,$$  \hspace{1cm} (4.3)

where $\lambda = 0$ when $s = (+), (\times)$ and $\lambda = 1$ when $s = (sc)$. The scalaron mass term, $m_{sc}^2$, is given by equation (3.4), and thus in the case of the Starobinsky model it becomes

2The contribution from the first-order tensor perturbations is not considered here since we concentrate on gravitational waves induced by scalar perturbations at second order.
simply $m_{sc}^2 = M^2$. The source function $S_k^s$ is given by

$$S_k^s = \int \frac{d^3q}{(2\pi)^{3/2}} e_{ij}^s(k)q_i q_j \left[ 2\Phi_q \Phi_{k-q} + \frac{4}{3(1+w_{tot})} (\mathcal{H}^{-1} \Phi_q + \Phi_q)(\mathcal{H}^{-1} \Phi_{k-q} + \Phi_{k-q}) \right],$$

(4.4)

where $s = (+), (\times), (sc)$. The polarization tensors $e_{ij}^s(k)$ are defined as [33]

$$e_{ij}^{(+)}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{ij}^{(\times)}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{ij}^{(sc)}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. $$

(4.5)

As we have mentioned above, since we focus on second-order effects, in this work we assume that the background evolution is close to that of ΛCDM scenario, and since in the time period we are investigating the Universe is matter (i.e. PBH) dominated, we have $e_{tot}^2 \approx w_{tot} \approx w_{PBH} = 0$. Hence, for the time evolution of the potential given by Eq. (3.12), we obtain

$$\Phi_k'' + \frac{6(1+w_{tot})}{1+3w_{tot}} \frac{1}{\eta} \Phi_k' + w_{tot} k^2 \Phi_k = 0. $$

(4.6)

The above equation accepts a solution with one constant and one decaying mode on super sound-horizon scales. In the late-time limit, one can neglect the decaying mode, and write the solution for the Fourier transform of $\Phi$ as $\Phi_k(\eta) = T_\Phi(\eta) \phi_k$, where $\phi_k$ is the value of the gravitational potential at some initial time (which here we consider it to be the time at which PBHs dominate the energy content of the Universe, $x_d$) and $T_\Phi(\eta)$ is a transfer function, defined as the ratio of the dominant mode between the times $x$ and $x_d$. Consequently, Eq. (4.4) can be written in a more compact form as

$$S_k^s = \int \frac{d^3q}{(2\pi)^{3/2}} e^s(k, q) F(q, k - q, \eta) \phi_q \phi_{k-q}, $$

(4.7)

where

$$F(q, k - q, \eta) \equiv 2T_\Phi(q\eta)T_\Phi(|k - q|\eta) + \frac{4}{3(1+w)} [\mathcal{H}^{-1} q T_\Phi(q\eta) + T_\Phi(q\eta)]$$

$$\cdot \left[ \mathcal{H}^{-1} |k - q| T_\Phi(|k - q|\eta) + T_\Phi(|k - q|\eta) \right],$$

(4.8)

and the contraction $e_{ij}^s(k)q_i q_j \equiv e^s(k, q)$ can be expressed in terms of the spherical coordinates $(q, \theta, \varphi)$ of the vector $q$ as

$$e^s(k, q) = \begin{cases} \frac{1}{\sqrt{2}} q^2 \sin^2 \theta \cos 2\varphi \text{ for } s = (+) \\ \frac{1}{\sqrt{2}} q^2 \sin^2 \theta \sin 2\varphi \text{ for } s = (\times) \\ \frac{1}{\sqrt{2}} q^2 \cos^2 \theta \text{ for } s = (sc) \end{cases}. $$

(4.9)

Finally, the solution of Eq. (4.3) for the tensor modes $h_k^s$ can be obtained using the Green’s function formalism where one can write for $h_k^s$ that

$$a(\eta) h_k^s(\eta) = 4 \int_{\eta_d}^\eta d\tilde{\eta} G_k^s(\eta, \tilde{\eta}) a(\tilde{\eta}) S_k^s(\tilde{\eta}) ,$$

(4.10)
where the Green’s function \( G^s_k(\eta, \bar{\eta}) \) is the solution of the homogeneous equation

\[
G^{s''}_k(\eta, \bar{\eta}) + \left( k^2 - \lambda m^2_{sc} - \frac{a''}{a} \right) G^s_k(\eta, \bar{\eta}) = \delta(\eta - \bar{\eta}),
\]

(4.11)

with the boundary conditions \( \lim_{\eta \to \bar{\eta}} G^s_k(\eta, \bar{\eta}) = 0 \) and \( \lim_{\eta \to \bar{\eta}} G^{s'}_k(\eta, \bar{\eta}) = 1 \).

Having extracted above the tensor perturbations, the next step is to derive the tensor power spectrum, \( P_h(\eta, k) \) for the different polarization modes, which is defined as the equal time correlator of the tensor perturbations through the following relation:

\[
\langle h^r_k(\eta) h^{s'}_{k'}(\eta) \rangle \equiv \frac{2\pi^2}{k^3} P^s_{h}(\eta, k),
\]

(4.12)

where \( s = (\times) \) or (\( + \)) or (sc). At the end, after a straightforward but rather long calculation one acquires that

\[
P^{(\times)}_h(\eta, k) = 4 \int_0^\infty dv \int_{1-v}^{1+v} du \frac{[4v^2 - (1 + v^2 - u^2)^2]}{4uv} I^2(u, v, x) P_{\Phi}(kv) P_{\Phi}(ku),
\]

(4.13)

whereas for the scalaron polarization one obtains that

\[
P^{(sc)}_h(\eta, k) = 8 \int_0^\infty dv \int_{1-v}^{1+v} du \frac{(1 + v^2 - u^2)^2}{4uv} I^2(u, v, x) P_{\Phi}(kv) P_{\Phi}(ku).
\]

(4.14)

The two auxiliary variables \( u \) and \( v \) are defined as \( u \equiv |k - q|/k \) and \( v \equiv q/k \), and the kernel function \( I(u, v, x) \) is given by

\[
I(u, v, x) = \int_{x_1}^x dx \frac{a(\bar{x})}{a(x)} k G^s_k(x, \bar{x}) F_k(u, v, \bar{x}).
\]

(4.15)

In the above expressions, \( x = k\eta \) and we use the notation \( F_k(u, v, \eta) \equiv F(k, |k - q|, \eta) \) since the function \( F(q, k - q, \eta) \) depends only on the modulus of its first two arguments. Finally, note also that the power spectrum of the PBH gravitational potential should be calculated at a reference initial time, which here is considered to be the PBH domination time.

4.2 The gravitational wave energy density spectrum

Since we have extracted the power spectrum of the tensor perturbations, we can now calculate the energy density associated to the scalar induced GWs. We focus only on subhorizon scales, in which one does not feel the curvature of spacetime and hence he can use a flat spacetime approximation. Consequently, after a straightforward but lengthy calculation the GW energy density can be recast as [84]

\[
\rho_{GW}(\eta, x) = \frac{M_{\text{Pl}}^2}{32\pi^2} \left[ \partial_\eta h_{\alpha\beta} \partial_\eta h^{\alpha\beta} + \partial_i h_{\alpha\beta} \partial^i h^{\alpha\beta} \right],
\]

(4.16)
which is simply the sum of a kinetic term and a gradient term. The overall bar stands for an oscillation averaging on sub-horizon scales, which is performed to deduce only the envelope of the gravitational-wave spectrum. The GW spectral abundance is just the GW energy density per logarithmic comoving scale, i.e.

$$\Omega_{\text{GW}}(\eta, k) = \frac{1}{\bar{\rho}_{\text{tot}}} \frac{d\rho_{\text{GW}}(\eta, k)}{d \ln k}. \quad (4.17)$$

Let us now focus on a matter-dominated era driven by PBHs, where $w = 0$. Under these conditions, the transfer function $T_\Phi$ is constant in time, and we normalise it to one at PBH domination time, namely $T_\Phi(x_d) = 1$. This forces the source term $S_s^k$ to be constant in time and as a consequence at sub-horizon scales, where $k \gg H$, from Eq. (4.3) we acquire that $h^s_k \simeq \frac{4S^s_k}{k^2}$. Consequently, the tensor modes have a mild dependence on time and therefore the kinetic term in relation (4.16) gives a negligible contribution to the GW energy density. Therefore, we straightforwardly obtain that

$$\langle \rho_{\text{GW}}(\eta, x) \rangle \simeq \langle \rho_{\text{GW,grad}}(\eta, x) \rangle = \sum_{s=+,-,\times,\cdot,\cdot} \frac{M_{\text{Pl}}^2}{32a^2} \left( \frac{\Delta h_{\alpha\beta}^s}{k} \right)^2$$

$$= \frac{M_{\text{Pl}}^2}{32a^2 (2\pi)^3} \sum_{s=+,-,\times,\cdot,\cdot} \int d^3k_1 \int d^3k_2 k_1 k_2 \left( h_{k_1}^s(\eta) h_{k_2}^{s,\ast}(\eta) \right) e^{i(k_1-k_2) \cdot x}, \quad (4.18)$$

where the brackets stand for an ensemble average. At the end, by combining Eq. (4.18), Eq. (4.17) and Eq. (4.12) and taking into account from Eq. (4.13) that the $(\times)$ and $(\cdot)$ polarization modes give an equal contribution, we find that

$$\Omega_{\text{GW}}(\eta, k) \simeq \frac{1}{\bar{\rho}_{\text{tot}}} \frac{d\rho_{\text{GW,grad}}(\eta, k)}{d \ln k} = \frac{1}{96} \left( \frac{k}{\mathcal{H}(\eta)} \right)^2 \left[ 2\mathcal{P}_h^{(\times)}(\eta, k) + \mathcal{P}_h^{(\cdot)}(\eta, k) \right]. \quad (4.19)$$

5 **Constraints on Starobinsky $R^2$ gravity**

In this section, we use the previous results in order to derive constraints on Starobinsky $R^2$ modified gravity, namely on its mass parameter $M$, by studying the associated scalar induced GW signal generated due to PBH Poisson fluctuations in the context of $f(R)$ gravity. In particular, we investigate and extract the scalar-induced GW spectrum produced during a cosmic era driven by PBHs. In order to achieve this we treat the PBHs as a matter fluid, and thus with zero equation-of-state parameter, an approximation which is justifiable for scales larger than the mean PBH separation scales where $k < k_{\text{UV}}$ (see the discussion in subsection 2.3).

5.1 **The theoretical parameters involved**

Before going into the investigation of the GW signal let us discuss the relevant theoretical parameters involved in the problem at hand. These parameters are actually the mass
of the PBH $m_{\text{PBH}}$, the initial PBH abundance at formation time $\Omega_{\text{PBH},f}$, and the dimensionless parameter $\alpha$ defined as the ratio of the Hubble parameter at PBH formation time over the energy scale parameter of Starobinsky (or $R^2$) gravity $M$

$$\alpha \equiv H_f / M.$$  

(5.1)

Regarding the PBH mass range we assume that the PBHs considered here are formed after the end of inflation and evaporate before BBN time. Accounting therefore for the current Planck upper bound on the tensor-to-scalar ratio for single-field slow-roll models of inflation, i.e. $\rho_{\text{inf}}^{1/4} < 10^{16}\text{GeV}$ [85], taking $\rho_{\text{BBN}}^{1/4} \simeq 1\text{MeV}$, and considering the fact that the mass of a PBH is roughly equal to the mass inside the Hubble volume at PBH formation time, $m_{\text{PBH}} = 4\pi \rho f H_f^{-3}/3$, we can straightforwardly show that the relevant PBH mass range is given by

$$10^g < m_{\text{PBH}} < 10^9g,$$  

(5.2)

where moreover we have used the fact that the Hawking evaporation time of a black hole scales with the mass $m_{\text{PBH}}$ as $t_{\text{evap}} = \frac{160}{g_{\text{eff}}} \frac{m_{\text{PBH}}^3}{M_{\text{Pl}}^4}$ [86], where $g_{\text{eff}}$ is the effective number of relativistic degrees of freedom. In our numerical applications, we take $g_{\text{eff}} = 100$ since it is the order of magnitude predicted by the Standard Model before the electroweak phase transition [87].

Concerning now the range of $\Omega_{\text{PBH},f}$ in order to have a transient PBH domination era, this can be set by demanding that the PBH evaporation time $t_{\text{evap}}$, is larger than the PBH domination time $t_d$. In particular, knowing that during a radiation domination era $\Omega_{\text{PBH}} = \rho_{\text{PBH}}/\rho_d \propto a^{-3}/a^{-4} \propto a$, then the PBHs dominate the energy budget of the universe when $\Omega_{\text{PBH}} = 1$, from which we find that $a_d = a_f/\Omega_{\text{PBH},f}$. Thus, knowing that during radiation domination era $H \simeq 1/(2t)$, and demanding that $t_{\text{evap}} > t_d$, we obtain that

$$\Omega_{\text{PBH},f} > 10^{-15} \sqrt{g_{\text{eff}}/100} \frac{10^g}{m_{\text{PBH}}}.$$  

(5.3)

Finally, regarding the dimensionless parameter $\alpha$, knowing that in Starobinsky gravity $M \geq H_{\text{inf}}$, and that PBHs are formed after inflation, i.e. $H_{\text{inf}} \geq H_f$, we get that $M \geq H_f$. Consequently, the relevant range for $\alpha$ can be recast as

$$0 \leq \alpha \leq 1.$$  

(5.4)

In the limit, $\alpha \to 0$ we recover GR.

5.2 Gravitational waves from an era driven by primordial black holes

Having introduced in the previous subsection the relevant parameters involved we derive here the GW spectrum during an era of PBH domination. To do so, the first step is to calculate the kernel function $I(u,v,x)$ defined in Eq. (4.15). Since we are in a matter
(i.e. PBHs) dominated era, namely with \( w = 0 \), in the subhorizon limit, i.e. \( x \gg 1 \), \( I(u, v, x) \) reads as (see Appendix C)

\[
I^2(x) = \frac{100}{9} \times \begin{cases} 
1 & \text{if } s = (\times), (+) \\
\frac{k^4}{M^4} & \text{if } s = (sc)
\end{cases}.
\]

(5.5)

As one may notice from expression (5.5), we have a suppression factor of the order \( k^4 / M^4 \), which suppresses the scalaron contribution. This factor will dominate in the region close to the UV-cutoff, where \( k^4 / M^4 \sim O(1) \). This can be seen from Fig. 2, where in the case of \( m_{\text{PBH}} = 10^5 \) g and \( \Omega_{\text{PBH}} = 10^{-3} \) one can notice a bump close to the UV cut-off region, a fact which enhances the GW signal by at least two orders of magnitude, however only in the extreme regimes where \( H_f \sim M \), or equivalently \( \alpha \sim 1 \).

In the following, we aim to compute the integrated GW signal in order to extract a lower bound for the mass parameter of Starobinsky inflation. In our considerations, we neglect the scalaron contribution underestimating the GW signal in the regions where \( H_f \sim M \), which decreases the lower bound of \( M \) given the fact that the GW spectrum is a decreasing function of \( M \) as it can be seen from Fig. 2. In this sense, the lower bounds derived below regarding \( M \) are rather conservative, however still very strong. Nevertheless, if one wishes to search for observational signatures of this signal, they should include the scalaron contribution, which makes the \( f(R) \) GW signal distinctive from the GR one through the aforementioned bump.

Under this approximation, the GW spectrum (4.19) can be recast in the following form:

\[
\Omega_{\text{GW}}(\eta, k) = \frac{4}{\sqrt{\pi}} \left( \frac{k}{aH} \right)^2 \left( \frac{k}{k_{\text{UV}}} \right)^6 \mathcal{F}(y, \Omega_{\text{PBH}}, f, \alpha),
\]

where

\[
\mathcal{F}(y, \Omega_{\text{PBH}}, f, \alpha) = \int_0^{\Lambda_{\text{UV}}} dv \int_{\left[ 1-v \right]}^{\left[ \Lambda_{\text{UV}}, 1-v \right]} du \left[ \frac{4v^2 - (1 + v^2 - u^2)^2}{4 \left( 3 + \frac{2\Xi(\alpha, \Omega_{\text{PBH}}, f)}{k} y^2 v^2 \right) \left( 3 + \frac{2\Xi(\alpha, \Omega_{\text{PBH}}, f)}{k} y^2 u^2 \right)} \right]^2 uv,
\]

with \( y = k / (a_d H_d) \). \( \Lambda_{\text{UV}} \) is the upper bound of the integral in \( v \) due to the UV cut-off scale, discussed in subsection 2.3, and is defined as [25]

\[
\Lambda_{\text{UV}} = \frac{k_{\text{UV}}}{k}.
\]

Finally, the function \( \Xi(\alpha, \Omega_{\text{PBH}}, f) \) is defined as

\[
\Xi(\alpha, \Omega_{\text{PBH}}, f) = F(a_d) \left[ 1 + \frac{3k^2}{2} \frac{F(\alpha_d)}{F(a_d)} \right] \xi(\alpha, \Omega_{\text{PBH}}, f) \simeq \xi(\alpha, \Omega_{\text{PBH}}, f),
\]

(5.9)

where \( \xi \) is the ratio of the PBH density contrast over the PBH density contrast at PBH formation given in (3.25). \( \Xi(\alpha, \Omega_{\text{PBH}}, f) \simeq \xi(\alpha, \Omega_{\text{PBH}}, f) \) since as as we have verified
numerically \( F(a_d) = 1 + \alpha^2 \Omega_{PBH,f}^2 \sim O(1) \) and 
\[
\left( 1 + 3 \frac{k^2 F_{R(a_d)}}{F(a_d)} \right) \left( 1 + 2 \frac{k^2 F_{R(a_d)}}{F(a_d)} \right) \sim O(1).
\]
Note that we have dropped the argument \( a_d \) from \( \xi \) in order not to have a heavy notation and we will keep this convention throughout the paper.

In Fig. 1 we depict the function \( \xi(\alpha, \Omega_{PBH,f}) \) as a function of \( \alpha \), taking different values of \( \Omega_{PBH,f} \). As we observe, \( \xi(\alpha, \Omega_{PBH,f}) \) is a decreasing function of \( \alpha \), with a plateau behaviour for small values of \( \alpha \). For relatively small \( \Omega_{PBH} \) values we can also infer that \( \xi(\alpha, \Omega_{PBH,f}) \) depends slightly on \( \Omega_{PBH,f} \).

Consequently, having calculated \( \xi(\alpha, \Omega_{PBH,f}) \), we can insert it in expression (5.6) and extract the GW spectrum. In Fig. 2 we show the GW spectral abundance at PBH evaporation time, \( \Omega_{GW}(\eta_{evap}, k) \), namely at the end of the PBH-dominated era, for different values of the parameter \( \alpha = H_f/M \). As one may see, as \( \alpha \) increases we have a departure from the GR limit. In particular, the increase of \( \alpha \) enhances the GW signal.

**Figure 1.** The ratio of the PBH density contrast computed at PBH domination time over the PBH density contrast at PBH formation \( \xi(\alpha, \Omega_{PBH,f}) \) given in (3.25), as a function of \( \alpha \), for fixed \( m_{PBH} = 10^5 \text{g} \) and for various values of \( \Omega_{PBH,f} \).
due to the fact that for fixed $\Omega_{\text{PBH},f}$ $\xi(\alpha, \Omega_{\text{PBH},f})$ is a decreasing function of $\alpha$, as it can be seen from Fig. 1.

Figure 2. The GW spectral abundance $\Omega_{\text{GW}}(\eta_{\text{evap}}, k)$ at PBH evaporation time, for various values of the parameter $\alpha = H_f/M$, in the case where $m_{\text{PBH}} = 10^5 g$ and $\Omega_{\text{PBH},f} = 10^{-3}$. The dashed black line represents the GR limit.

5.3 Gravitational wave backreaction constraints

Interestingly enough, according to the above analysis we deduce that for some values of the involved parameters one is met with an overproduction of gravitational waves at PBH evaporation time, which is something unphysical. Therefore, in order to avoid this GW backreaction issue we demand that $\Omega_{\text{GW},\text{tot}}(\eta_{\text{evap}}) < 1$. Hence, this condition will lead to bounds for the relevant parameters of the problem at hand.

In particular, one can extract analytical bounds for $m_{\text{PBH}}, \Omega_{\text{PBH},f}$ and $\alpha$. In order to achieve this, one can expand $\mathcal{F}$ in the regimes $y \ll 1$ and $y \gg 1$. Following the
procedure described in Appendix B of [25] one obtains that

\[
\mathcal{F}(y, \Omega_{\text{PBH},f}) \simeq \begin{cases} 
\frac{125}{38} \sqrt{\frac{5}{6}} \frac{\pi}{625 \pi^2} y^2 & \text{for } y \ll 1 \text{ and } \Omega_{\text{PBH},f} \ll 1 \\
\frac{128}{5}(\alpha, \Omega_{\text{PBH},f}) y^6 & \text{for } y \gg 1
\end{cases}.
\] (5.11)

Then, inserting the above expression into (5.6) we acquire

\[
\Omega_{GW}(\eta_{\text{evap}}, k \ll \mathcal{H}_d) \simeq \frac{8 \sqrt{2}}{3} \frac{1}{\pi^{7/2}(\alpha, \Omega_{\text{PBH},f})} \left( \frac{g_{\text{eff}}}{100} \right)^{2/3} k \left( \frac{m_{\text{PBH}}}{M_{\text{Pl}}} \right)^{4/3} \Omega_{\text{PBH},f}^{16/3},
\] (5.12)

\[
\Omega_{GW}(\eta_{\text{evap}}, k \gg \mathcal{H}_d) \simeq 50 \left( \frac{3}{5} \right)^{3/2} \frac{1}{\xi^4(\alpha, \Omega_{\text{PBH},f})} \left( \frac{g_{\text{eff}}}{100} \right)^{-2/3} \left( \frac{m_{\text{PBH}}}{M_{\text{Pl}}} \right)^{4/3} \Omega_{\text{PBH},f}^{16/3}.
\] (5.13)

Finally, by integrating over \( \ln k \) we obtain the total amount of GWs produced during the PBH domination era, namely

\[
\Omega_{GW,\text{tot}}(\eta_{\text{evap}}) = \int d\ln k \, \Omega_{GW}(\eta_{\text{evap}}, k).
\] (5.14)

Specifically, by replacing (5.12) and (5.13) into (5.14), \( \Omega_{GW,\text{tot}}(\eta_{\text{evap}}) \) is written as

\[
\Omega_{GW,\text{tot}}(\eta_{\text{evap}}) = \mu [\kappa - \ln(\Omega_{\text{PBH},f})] \Omega_{\text{PBH},f}^{16/3},
\] (5.15)

with

\[
\mu = \frac{20}{\pi^{7/2}(\alpha, \Omega_{\text{PBH},f})} \left( \frac{3}{5} \right)^{1/2} \left( \frac{g_{\text{eff}}}{100} \right)^{-2/3} \left( \frac{m_{\text{PBH}}}{M_{\text{Pl}}} \right)^{4/3}
\] (5.16)

and

\[
\kappa = \frac{2 \sqrt{2}}{9} \frac{\sqrt{\xi(\alpha, \Omega_{\text{PBH},f})}}{\pi} + \frac{3}{2} \ln 2.
\] (5.17)

As a last step, let us extract the bounds for the parameters \( m_{\text{PBH}}, \Omega_{\text{PBH},f} \) and \( \alpha \). To do so, we need to solve the equation \( \Omega_{GW,\text{tot}}(\eta_{\text{evap}}) = 1 \). This equation can be solved in terms of the Lambert function [88], obtaining

\[
\Omega_{\text{PBH},f}^{\text{max}} = \left[ -\frac{3\mu}{16} W_{-1} \left( -\frac{16}{3\mu} e^{-\frac{16}{3\mu}} \right) \right]^{-3/16},
\] (5.18)

where \( W_{-1} \) is the \( \text{“-1”-branch of the Lambert function} \). Given the fact that \( m_{\text{PBH}} > 10^{16} \) [see Eq. (5.2)], we find that \( \mu \gg 1 \), while \( \kappa \) is of order one. Consequently, the argument

\[
\mathcal{F}(y, \Omega_{\text{PBH},f}) \quad \text{independently of } \Omega_{\text{PBH},f} \text{ is given by}
\]

\[
\mathcal{F}(y \ll 1, \Omega_{\text{PBH},f}) = 500 \xi^{7/2}(\alpha, \Omega_{\text{PBH},f}) \left[ \sqrt{30} \tan \left( \frac{2}{15} \sqrt{\frac{\xi(\alpha, \Omega_{\text{PBH},f})}{\Omega_{\text{PBH},f}^{2/3}}} \right) \right. \\
\left. - 6 \sqrt{\xi(\alpha, \Omega_{\text{PBH},f})} \left( 1250 \xi_{\text{PBH},f}^{16/3} + 44 \xi^2(\alpha, \Omega_{\text{PBH},f}) \xi_{\text{PBH},f}^{16/3} + 400 \xi(\alpha, \Omega_{\text{PBH},f}) \xi_{\text{PBH},f}^{16/3} \right) \right]^{-2/3}. (5.10)
\]
of the Lambert function is close to zero, and in this regime it can be approximated by a logarithmic function, i.e. $W_{-1} \left( -\frac{16}{3} e^{-\frac{16}{3}} \right) \simeq |\ln \left( -\frac{16}{3} e^{-\frac{16}{3}} \right)|$. Now taking into account the mild dependence of the logarithmic function on its argument, for our numerical purposes we will choose a central value in PBH mass range, namely $m_{\text{PBH}} = 10^5 \text{g}$, and we will consider the logarithm as constant. Concerning the value of $\xi(\alpha)$, given the fact that for $\Omega_{\text{PBH}}, f \leq 0.01$ it varies between 2.2 and 2.5 (see Fig. 1) we will take it equal to 2.4.

![Figure 3](image_url)

**Figure 3.** The minimum of the mass parameter $M$ of Starobinsky gravity in terms of the reduced Planck mass $M_{\text{Pl}}$ (y axis) as a function of $\Omega_{\text{PBH}, f}$ (x axis) and $m_{\text{PBH}}$ (color-bar axis). The values of $m_{\text{PBH}}$ are chosen such that PBHs form after inflation and evaporate before Big Bang Nucleosynthesis, see (5.2), whereas the displayed values of $\Omega_{\text{PBH}, f}$ correspond to regimes where PBHs dominate the energy budget of the universe for a transient period, see (5.3) and their gravitational potential does not lead to an overproduction of induced GWs, see (5.19). For the numerical applications we have used $g_{\text{eff}} = 100$.

At the end, we straightforwardly obtain that

$$
\Omega_{\text{PBH}, f} \leq 10^{-4} \left( \frac{10^9 \text{g}}{m_{\text{PBH}}} \right)^{1/4} \xi^{21/32}(M, \Omega_{\text{PBH}, f}),
$$

(5.19)
where $\xi(M, \Omega_{PBH}, f)$ is expressed in terms of the mass parameter of Starobinsky gravity.

Let us now provide the constraints of the minimum of $M$ in terms of the reduced Planck mass $M_{Pl}$, i.e. on the maximum possible deviation from general relativity. In Fig. 3 we present the allowed $M_{\text{min}}/M_{Pl}$ as a function of $\Omega_{PBH}, f$ and $m_{PBH}$. As we observe, we may discriminate between two regions. The trapezoidal region where more or less $\Omega_{PBH,f} < 10^{-3}$, and the lateral region on the right where $\Omega_{PBH,f} > 10^{-3}$. In particular, in the trapezoidal region the inequality (5.19) is always satisfied for every value of $\alpha \in [0,1]$, and as a consequence $\alpha_{\text{max}} = 1$ there. In terms of $M_{\text{min}}$ this is equivalent to $M_{\text{min}} = H_f = \frac{\sqrt{\alpha} M^2_{Pl}}{m_{PBH}}$. For this reason, in this region for a fixed value of $m_{PBH}$, $M_{\text{min}}/M_{Pl}$ does not vary with $\Omega_{PBH}, f$.

On the other hand, in the lateral region on the right of Fig. 3, $\Omega_{PBH,f}$ can reach its maximum value given by the r.h.s. of inequality (5.19), and therefore in order to find the maximum value of $\alpha$, or equivalently the minimum value of $M$, one needs to solve the equality in (5.19) and find $M_{\text{min}}$ as a function of $\Omega_{PBH,f}$ and $m_{PBH}$. Interestingly enough, in the region where $\Omega_{PBH,f} > 10^{-3}$ and the PBH mass is very small, i.e. $m_{PBH} \sim (10g)$, as it can be seen in the upper right region of Fig. 3, one can find that the minimum value of $M$ can reach values up to $0.1M_{Pl}$, very close to the GUT scale.

In summary, we found that in order to avoid an overproduction of gravitational waves at PBH evaporation time, the deviation of Starobinsky gravity from general relativity should be constrained to strong bounds. In particular, for almost all possible initial PBH abundance at formation time $\Omega_{PBH,f}$ and all possible PBH masses $m_{PBH}$, we find that $5 \times 10^{-14} \lesssim \frac{M_{\text{min}}}{M_{Pl}} \lesssim 10^{-5}$, and only in the extreme possible regime where $\Omega_{PBH,f} > 10^{-3}$ we find that $10^{-5} \lesssim \frac{M_{\text{min}}}{M_{Pl}} \lesssim 10^{-4}$. Note that these constraints have been extracted through the novel procedure of scalar induced gravitational waves from PBH Poisson fluctuations, and the fact that they are in the same direction but stronger than those extracted through the completely independent procedures of BBN [89, 90] or cosmological (inflationary) confrontation [37], is a verification of the analysis.

6 Conclusions

Primordial black holes are of great significance, since they may constitute a part or all of the dark matter sector, they may provide an explanation for the large-scale structure formation through Poisson fluctuations, and moreover they can offer the seeds for the progenitors of the black-hole merging events as well as for the supermassive black holes formation. Their effect on the GW background signals, and in particular the second-order GWs induced by the gravitational potential of Poisson-distributed PBHs, has been studied only in the framework of general relativity. Hence, in this work we extended the analysis of the literature in the case of $f(R)$ gravity. We focused on Starobinsky $R^2$ gravity, one of the most favored inflationary models from the observational side, however our formalism is applicable for every model in the context of $f(R)$ gravity.

Firstly, we calculated the effect of $f(R)$ modification on the PBH gravitational potential power spectrum and we extracted the associated scalar-induced GW spectrum during an era driven by ultralight PBHs ($m_{PBH} < 10^9 g$), which evaporate before BBN.
In particular, we found its dependence on the relevant parameters involved, namely the PBH mass $m_{\text{PBH}}$, the initial PBH abundance at formation time $\Omega_{\text{PBH,f}}$, and the mass parameter of the Starobinsky gravity $M$ by accounting as well for the three polarization states of GWs in $f(R)$ gravity, namely the $(\times)$, the $(+)$ and the scalaron one.

Concerning the dependence of the GW spectrum on $M$, we found that by fixing $m_{\text{PBH}}$ and $\Omega_{\text{PBH,f}}$ then decreasing the mass parameter $M$ leads to an enhancement of the GW spectrum. Specifically, for values of the Hubble parameter at PBH formation time, $H_f$, close to $M$ we found that there is a characteristic bump close to the UV cut-off region, which is due to the extra scalaron polarization state in $f(R)$ gravity, leading therefore the associated GW signal to exhibit a characteristic profile compared to what is expected in GR.

Interestingly, in some region of our parameter space ($m_{\text{PBH}}, \Omega_{\text{PBH,f}}, M$) we found regimes where the overall energy density of the induced GWs at PBH evaporation time becomes greater than the total energy density of the universe, which is unphysical and thus needs to be avoided. Thus, following closely [25], in order to avoid this GW back-reaction problem we demanded that the overall energy density contribution of the GWs at evaporation time is less than one, $\Omega_{\text{GW, tot}(\eta_{\text{evap}})} < 1$. Hence, this condition allows us to extract an upper bound on $\Omega_{\text{PBH,f}}$ as a function of the PBH mass and the mass parameter $M$. Intriguingly, this upper bound is the respective GR bound screened by a function of $M$, namely

$$\Omega_{\text{PBH,f}} \leq 10^{-4} \left( \frac{10^9 \text{g}}{m_{\text{PBH}}} \right)^{1/4} \xi_{21/32}(M, \Omega_{\text{PBH,f}}).$$  \hspace{1cm} (6.1)

By saturating the above inequality we extracted a lower bound on the mass parameter of Starobinsky gravity $M$ in terms of the reduced Planck mass $M_{\text{Pl}}$, as a function of $\Omega_{\text{PBH,f}}$ and $m_{\text{PBH}}$, i.e we extracted the maximum possible deviation from general relativity. In particular, for almost all possible initial PBH abundances at formation time $\Omega_{\text{PBH,f}}$ and all possible PBH masses $m_{\text{PBH}}$, we found that $5 \times 10^{-14} \lesssim \frac{M_{\text{min}}}{M_{\text{Pl}}} \lesssim 10^{-5}$, and only in the extreme possible regime where $\Omega_{\text{PBH,f}} > 10^{-3}$ and the PBH mass is very small, i.e. $m_{\text{PBH}} \sim (10 \text{g})$, we found that $10^{-5} \lesssim \frac{M_{\text{min}}}{M_{\text{Pl}}} \lesssim 10^{-1}$, very close to the GUT scale.

We close this work by making a comment on the aforementioned procedure. By making use of the cosmological perturbation theory we extracted the power spectrum of the gravitational potential, and by imposing the UV cut-off scale we ensured that we are well within the perturbative regime. This is very important since it is $\Phi$ that induces the second-order gravitational waves. Nevertheless, from the point of view of the energy density perturbation $\delta$, as it is well established in the context of GR, during matter domination $\delta$ grows linearly with the scale factor. A similar picture was found here too, namely $\delta$ grows with the scale factor although non linearly. Thus, there will be scales where $\delta$ can acquire values larger than one, entering into the non-linear regime although $\Phi$ remains much smaller than one. Therefore, in order to clarify the status of these scales one should follow the full virialisation dynamics [25, 27] something which is beyond the scope of this work. However, we may speculate that a growth of $\delta$ will enhance the power spectrum above the Poissonian value, which in turn will lead to an
even larger signal than that extracted above. In that sense, the bounds obtained here in particular regarding the initial abundances of PBHs, [see Eq. (6.1)] can be considered as conservative ones.

In summary, through the above analysis we showed that the condition to avoid an overproduction of gravitational waves at PBH evaporation time can act as a novel method to extract constraints on cosmological models and gravitational theories, independent from other methods such as the BBN or cosmological confrontations. Hence, by the combined application of all these approaches we have an improved tool to constrain proposed scenarios and test possible deviations from general relativity.

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A Scalar perturbation equations in $f(R)$ gravity

In the case of $f(R)$ gravity, in the Newtonian gauge one extracts the following scalar perturbed field equations [35]:

\[
3\dot{H}(\Phi' + \mathcal{H}\Psi) + k^2\Phi = -4\pi G a^2 \delta \rho_{\text{tot}}, \tag{A.1}
\]

\[
\Phi' + \mathcal{H}\Psi = 4\pi G a^2 (\bar{\rho}_{\text{tot}} + \bar{p}_{\text{tot}}) v_{\text{tot}}, \tag{A.2}
\]

\[
\Phi'' + \mathcal{H}(\Phi' + 2\Psi') + (\mathcal{H}^2 + 2\mathcal{H}')\Phi - k^2(\Phi - \Psi)/3 = -4\pi G a^2 \delta \rho_{\text{tot}}, \tag{A.3}
\]

\[
\Phi - \Psi = 8\pi G a^2 \bar{p}_{\text{tot}} \Pi_{\text{tot}}, \tag{A.4}
\]

where

\[
(\bar{\rho}_{\text{tot}} + \bar{p}_{\text{tot}}) v_{\text{tot}} \equiv \sum_{l=m,r,f} (\bar{\rho}^l + \bar{p}^l) v^l, \tag{A.5}
\]

and

\[
\bar{p}_{\text{tot}} \Pi_{\text{tot}} \equiv \sum_{l=m,r,f} \bar{p}^l \Pi^l. \tag{A.6}
\]

Additionally, the perturbed energy density and pressure of the effective fluid arising from $f(R)$ mortification, are written respectively as

\[
\delta \rho_{f(R)} = -\delta T^{(R)}_0 = -\frac{1}{8\pi G a^2} \left\{ (1 - F)[ -6\mathcal{H}'\Psi + k^2\Psi - 3\mathcal{H}(\Phi' + \Psi') - 3\Phi'' ] \\
- 3\mathcal{H}' \delta F + a^2 \delta f/2 - k^2\Psi + 2k^2\Phi + 6(\mathcal{H}' + \mathcal{H}^2)\Psi + 3\Phi'' + 3\mathcal{H}(\Phi' + 3\Phi') \\
+ k^2\delta F + 3\mathcal{H}\delta F' - 3F'(\Phi' + 2\mathcal{H}\Psi) \right\}, \tag{A.7}
\]
\[ \delta p_{(R)}^i \equiv \frac{\delta T_{(R)}^{(R)} i}{3} = \frac{1}{8\pi G a^2} \left\{ - (\mathcal{H}' + 2\mathcal{H}^2) \delta F + a^2 \delta f/2 + k^2 (2\Phi - \Psi) + 3\mathcal{H}(\Psi' + 3\Phi') + 3\Phi'' + 6(\mathcal{H}' + \mathcal{H})\Psi + \delta F'' + 2k^2 \delta F/3 + \mathcal{H}\delta F' - F'(2\Phi + 2\mathcal{H}\Psi + \Psi') - 3\Psi F'' + (1 - F) \left[ - k^2 \Phi - \Phi'' - 3\mathcal{H}(5\Phi' + \Psi') - (2\mathcal{H}' + 4\mathcal{H}^2)\Psi - k^2 (\Phi - \Psi)/3 \right] \right\}. \]  

(A.8)

Finally, we have

\[ (\bar{\rho}_{(R)} + \bar{P}_{(R)}) v_{(R)}^{(R)} \equiv - \delta T_{(R)}^{(R) i} = \frac{1}{8\pi G} \left\{ 2(1 - F)(\Phi' + \mathcal{H}\Psi)_i + \delta F'_i + F'\Psi_i - \mathcal{H}\delta F_i \right\}, \]

and

\[ \Pi_{ij}^{(R)} \bar{P}_{(R)} \equiv \delta T_{(R)}^{(R) i} = \frac{1}{8\pi G a^2} \left\{ (1 - F)(\Phi - \Psi)_{ij} + \delta F_{ij} \right\}, \quad i \neq j. \]  

(A.9)

In this context, we can define the (total) comoving curvature perturbation in the usual manner, namely

\[ \mathcal{R} = -\Phi - \mathcal{H} \nu_{\text{tot}}. \]  

(A.11)

**B Super-Hubble scales in \( f(R) \) gravity**

On super-Hubble scales, Eq. (A.1) becomes

\[ 3\mathcal{H}(\Phi' + \mathcal{H}\Psi) = -4\pi G a^2 \delta \rho_{\text{tot}}, \]

and thus together with Eq. (A.2) yields:

\[ \mathcal{R} = -\Phi + \frac{\delta \rho_{\text{tot}}}{3(1 + w_{\text{tot}})} \xrightarrow{(3.15)} -\zeta, \quad k \ll \mathcal{H}. \]  

(B.1)

Furthermore, from Eq. (A.11) and Eq. (A.2) we can write:

\[ \mathcal{R} = \Phi + \frac{\mathcal{H}(\Phi' + \mathcal{H}\Psi)}{4\pi G a^2 \rho_{\text{tot}}(1 + w_{\text{tot}})^2} \xrightarrow{\mathcal{H}^2 = 8\pi G a^2 \rho_{\text{tot}}/3} \Phi + \frac{2\Psi'/\mathcal{H} + \Psi}{3} + \frac{1 + w_{\text{tot}}}{1 + w_{\text{tot}}}. \]  

(B.2)

Moreover, from Eq. (A.4) and Eq. (A.10) we see that \( \Phi - \Psi = 8\pi G a^2 \bar{P}^r \Pi^r + \delta F/F \) and hence by assuming that at super-Hubble modes \( \Pi^r \approx 0 \) and \( \delta F/F \approx 0 \), we deduce that \( \Phi \approx \Psi \). Therefore, under these assumptions and for \( k \ll \mathcal{H} \) we can write for \( \mathcal{R} \):

\[ \mathcal{R} = \frac{2\Psi'/\mathcal{H} + \Phi}{3} + \Phi. \]  

(B.3)

**C The kernel function \( I(u, v, x) \)**

In this Appendix we derive the kernel function \( I(u, v, x) \) defined in Eq. (4.15) for all the three polarization modes, namely the \((\times)\), the \((+)\) and the scalaron one. In order to achieve this we firstly extract the Green function \( G_k(\eta, \bar{\eta}) \) by solving Eq. (4.11).
In particular, Eq. (4.11) accepts an analytic solution in the case where \( w = 0 \), which depending on the GW polarization reads as

\[
kG^{(\times)\text{or} (+)}_{k}(\eta, \bar{\eta}) = \frac{1}{x \bar{x}} \left[(1 + x \bar{x}) \sin(x - \bar{x}) - (x - \bar{x}) \cos(x - \bar{x})\right], \tag{C.1}
\]

\[
kG^{(sc)}_{k}(\eta, \bar{\eta}) = \frac{k^3}{x \bar{x} (M^2 - k^2)^{3/2}} \left\{ \frac{\sqrt{M^2 - k^2}}{k} (x - \bar{x}) \cosh \left[ \frac{\sqrt{M^2 - k^2}}{k} (x - \bar{x}) \right] \right. \\
+ \frac{(M^2 - k^2)x \bar{x} - k^2}{k^2} \sinh \left[ \frac{\sqrt{M^2 - k^2}}{k} (x - \bar{x}) \right]. \tag{C.2}
\]

The associated \( I(u, v, x) \) function for the (\( \times \)) and (+) polarization modes can be recast as

\[
I^2(x) = \frac{100}{9} \left[ 1 + \cos(x - x_d) \left( \frac{3}{x^2} - \frac{3x_d}{x^2} - \frac{x_d^2}{x^2} \right) - \sin(x - x_d) \left( \frac{3}{x^3} + \frac{3x_d}{x^3} - \frac{x_d^2}{x^3} \right) \right]^2, \tag{C.3}
\]

and as we can see it does not depend on \( u \) and \( v \). Similarly, for the scalaron polarization we have

\[
I^2(x) = \frac{100k^4}{9(M^2 - k^2)^6} \left\{ (M^2 - k^2)^2 \left[ x^3 + M^2 xx_d^2 - k^2 (3x_d + x(x_d^2 - 3)) \right] \right. \\
\times \cosh \left[ \frac{\sqrt{M^2 - k^2}}{k} (x - x_d) \right] + k \sqrt{M^2 - k^2} \left[ M^2 x_d (x_d - 3x) \right] \\
\left. + k^2 \left( 3 + 3xx_d - x_d^2 \right) \right] \sinh \left[ \frac{\sqrt{M^2 - k^2}}{k} (x - x_d) \right] \right\}^2, \tag{C.4}
\]

which is independent of \( u \) and \( v \) too. Taking now into account the fact that \( k_{\text{UV}} = \mathcal{H}_f \Omega_{\text{PBH}, f}^{1/3} \) and that roughly \( k < k_{\text{UV}} \) as well as that \( \mathcal{H}_f \leq M \), then one can easily see that \( k/M < \Omega_{\text{PBH}, f}^{1/3} \ll 1 \). Consequently, the above functions in a PBH dominated era and in the subhorizon limit, i.e. \( x \gg 1 \), become

\[
I^2(x) = \frac{100}{9} \times \begin{cases} 1 & \text{if } s = (\times), (+) \\ \frac{k^4}{M^7} & \text{if } s = (sc) \end{cases}. \tag{C.5}
\]

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