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MAPPINGS THAT PRESERVE CAUCHY SEQUENCES

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Summary. Mappings preserving Cauchy sequences and their relation to continuous and uniformly continuous mappings are investigated.

Keywords: Cauchy sequences, continuous mappings, uniformly continuous mappings.

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Let \((X, d_x), (Y, d_Y)\) be metric spaces. A sequence \(S\) in \(X\) is a mapping of the set \(N\) of all positive integers into \(X\); \(R\) denotes the set of reals and \(\text{cl } A\) denotes the closure of \(A\). Let \(F_X\) denote the set of all Cauchy sequences in \(X\). Further, let \(C(X, Y)\) and \(U(X, Y)\) denote the set of all continuous and uniformly continuous mappings \(f: X \to Y\), respectively. Let \(F(X, Y)\) be the set of all mappings \(f: X \to Y\) preserving Cauchy sequences, i.e.

\[ F(X, Y) = \{ f: X \to Y : S \in F_X \Rightarrow f \circ S \in F_Y \} . \]

R. F. Snipes in [2] has investigated the properties of \(F(X, Y)\). He has shown that

\[ U(X, Y) \subset F(X, Y) \subset C(X, Y) . \]

We recall some properties of \(F(X, Y)\) by [2].

Let \((X, d_x), (Y, d_Y), (Z, d_Z)\) be metric spaces. Then

(1) if \(f \in F(X, Y)\) and \(g \in F(Y, Z)\), then \(g \circ f \in F(X, Z)\);

(2) if \(f \in F(X, Y)\) and \(A \subset X\), then \(f|_A \in F(A, Y)\);

(3) if \((X, d_x)\) is a complete metric space, then \(F(X, Y) = C(X, Y)\);

(4) if \((Y, d_Y)\) is a complete metric space, \(A\) a subset of \(X\) and \(f \in F(A, Y)\), then there is \(g \in F(\text{cl } A, Y)\) such that \(g|_A = f\);

(5) if \((Y, d_Y)\) is a complete metric space, \(A\) a subset of \(X\) and \(f \in F(A, Y)\), then there is \(g \in F(\text{cl } A, Y)\) such that \(g|_A = f\);

(6) if \(Y\) is a normed linear space, then \(F(X, Y)\) is a linear space.

In this paper we investigate on what assumptions the equalities in (1) hold.

We recall that a subset \(A\) of a metric space \((X, d)\) is discrete if for every \(x \in A\) there is a positive \(\varepsilon\) such that \(d(x, y) > \varepsilon\) whenever \(y \in A\) and \(y \neq x\). A set \(A\) is uniformly discrete if \(\varepsilon\) does not depend on \(x\).
Definition 1. Let \((X, d)\) be a metric space. A set \(A \subset X\) is called F-discrete if every totally bounded subset \(B\) of \(A\) is finite.

Remark 1. Every uniformly discrete set is F-discrete and every F-discrete set is discrete. In fact, let \(A\) be a uniformly discrete set. Then there is a positive \(\varepsilon\) such that \(d(x, y) > \varepsilon\) for any \(x, y \in A, x \neq y\). Let \(B \subset A\) be a totally bounded set. Then there is a finite set \(C \subset A\) such that \(B \subset \bigcup_{x \in C} S(x, \varepsilon)\). By virtue of the uniform discreteness of \(A\) we have \(B = C\).

If \(A\) is not discrete, then there is a point \(a \in A\) and a sequence \(S\) in \(A\) converging to \(a\) such that \(\{S(n): n \in \mathbb{N}\}\) is infinite. But \(\{S(n): n \in \mathbb{N}\}\) is a totally bounded set and hence \(A\) is not F-discrete.

Definition 2. We say that a metric space \((X, d)\) has the property \((V)\) if there is a sequence \(S\) in \(X\) such that

(7) the sequence \(S\) has no Cauchy subsequence,

(8) for every positive \(\varepsilon\) there are positive integers \(m, n\) such that \(0 < d(S(m), S(n)) < \varepsilon\).

Remark 2. Evidently any totally bounded space does not possess the property \((V)\).

Lemma 1. Let \((X, d)\) be a metric space. Then the following three conditions are equivalent:

(9) \((X, d)\) is a complete metric space;

(10) if \(A\) and \(B\) are disjoint closed subsets of \(X\), then there is a function \(f \in F(X, R)\) such that \(f|_A = 0\) and \(f|_B = 1\);

(11) every closed discrete set in \(X\) is F-discrete.

Proof. (9) \(\Rightarrow\) (10): In view of the normality of \(X\) there is a function \(f \in C(X, R)\) such that \(f|_A = 0\) and \(f|_B = 1\). According to (4), \(f \in F(X, R)\).

(10) \(\Rightarrow\) (11): Let us assume that there is a closed discrete set \(M\) in \(X\) which is not F-discrete. Therefore there is an infinite totally bounded set \(P \subset M\). Hence there is a one-to-one Cauchy sequence \(S\) in \(P\).

From the closedness and discreteness of \(M\) we obtain that the sequence \(S\) does not converge. Then any subsequence of \(S\) does not converge, either. Let us denote \(A = \{S(2n - 1): n \in \mathbb{N}\}\) and \(B = \{S(2n): n \in \mathbb{N}\}\). Then \(A\) and \(B\) are closed disjoint sets in \(X\). Let \(f: X \to R\) be a function such that \(f|_A = 0\) and \(f|_B = 1\). Then \(S \in F_X\), but \(f \circ S \notin F_R\); i.e. \(f \notin F(X, R)\).

(11) \(\Rightarrow\) (9): Let us assume that \((X, d)\) is not a complete metric space. Then there is a Cauchy sequence \(S\) in \(X\) which does not converge. Then any subsequence of \(S\)
does not converge, either. It is easy to verify that \( A = \{ S(n): n \in \mathbb{N} \} \) is a closed, discrete, totally bounded and infinite set. Therefore \( A \) is not \( F \)-discrete.

**Lemma 2.** Let \((X, d)\) be a metric space. Then \((X, d)\) does not possess the property (V) if and only if every \( F \)-discrete subset of \( X \) is uniformly discrete.

**Proof.** Necessity. Let \((X, d)\) fail to have the property (V). Let us assume that there is an \( F \)-discrete set \( A \) in \( X \) which is not uniformly discrete. Then for \( n \in \mathbb{N} \) there are \( x_n, y_n \in A \) such that

\[
0 < d(x_n, y_n) < \frac{1}{n}.
\]

Let \( S \) be a sequence defined by \( S(2n - 1) = x_n \) and \( S(2n) = y_n \). Then \( S \) satisfies (8) and hence \( S \) does not satisfy (7). Therefore there is a Cauchy subsequence \( T \) of \( S \). Then \( B = \{ T(n): n \in \mathbb{N} \} \) is a totally bounded subset of \( A \). In view of the \( F \)-discreteness of \( A \) the set \( B \) is finite. Thus there is \( \varepsilon > 0 \) such that \( d(T(m), T(n)) \notin (0, \varepsilon) \) for all \( m, n \in \mathbb{N} \), which contradicts (12).

**Sufficiency.** Let \((X, d)\) have the property (V). Then there is a sequence \( S \) in \( X \) satisfying (7) and (8). Let \( A = \{ S(n): n \in \mathbb{N} \} \). Then in view of (8), \( A \) is not a uniformly discrete set. If \( B \) is a totally bounded subset of \( A \), then by (7) \( B \) must be a finite set. Therefore \( A \) is a \( F \)-discrete set.

**Lemma 3.** Let \((X, d)\) be a metric space. Let \( A \) be a subset of \( X \) and \( f: A \rightarrow \langle 0, 1 \rangle \) a function such that \( f \in F(A, \langle 0, 1 \rangle) \). Then there is a function \( g: X \rightarrow \langle 0, 1 \rangle \) such that \( g \in F(X, \langle 0, 1 \rangle) \) and \( g\|_A = f \).

**Proof.** Let \((X^*, d_{X^*})\) be the completion of \((X, d_X)\). Therefore \((X^*, d_{X^*})\) is a complete metric space and there is a one-to-one mapping \( j: X \rightarrow X^* \) such that \( j \in U(X, j(X)) \) and \( j(X) \) is dense in \( X^* \). By (1) and (2) we have \( f \circ j^{-1} \in F(j(A), \langle 0, 1 \rangle) \). According to (5) there is \( h \in F(\text{cl}(j(A)), \langle 0, 1 \rangle) \) such that \( h\|_{j(A)} = f \circ j^{-1} \).

By Tietze's theorem there is a continuous function \( h^*: X^* \rightarrow \langle 0, 1 \rangle \) such that \( h^*\|_{\text{cl}(j(A))} = h \). In view of (4), \( h^* \in F(X, \langle 0, 1 \rangle) \).

Let \( g = h^* \circ j \). Then \( g \in F(X, \langle 0, 1 \rangle) \) by (3) and (2). It is easy to see that \( g\|_A = f \).

Now let \( Y \) be a normed linear space. If \( Y = \{ 0 \} \) then obviously the equality in (1) holds. Hence we shall assume that \( Y \neq \{ 0 \} \).

**Theorem 1.** Let \((X, d)\) be a metric space and let \((Y, \| \cdot \|)\), \( Y \neq \{ 0 \} \), be a normed linear space. Then \( F(X, Y) = C(X, Y) \) if and only if \((X, d)\) is a complete metric space.

**Proof.** Necessity. Let \((X, d)\) fail to be a complete metric space. Then by Lemma 1 there is a closed discrete set \( M \) in \( X \) which is not \( F \)-discrete. Hence there is a one-to-one Cauchy sequence \( S \) in \( M \). Let \( g: M \rightarrow \langle 0, 1 \rangle \) be defined by

\[
g(x) = 1 \text{ if } x = S(2n) \text{ for some } n \in \mathbb{N} \text{ and } g(x) = 0 \text{ otherwise}.
\]
In view of the discreteness of $M$, $g$ is a continuous function. Hence by Tietze's theorem there is a continuous function $g^*: X \rightarrow \langle 0, 1 \rangle$ such that $g^*|_{M} = g$. Let $y \in Y$, $y \neq 0$. It is easy to see that $h: \langle 0, 1 \rangle \rightarrow Y$, defined by $h(t) = ty$, is a uniformly continuous mapping. Hence the mapping $f = h \circ g^*$ is continuous. However, $f \notin F(X, Y)$, because $S \in F_X$ and $f \circ S \notin F_Y$.

Sufficiency follows by (4).

**Theorem 2.** Let $(X, d)$ be a metric space and let $(Y, \| \cdot \|)$, $Y \neq \{0\}$, be a normed linear space. Then $F(X, Y) = U(X, Y)$ if and only if $(X, d)$ fails to have the property $(V)$.

**Proof.** Necessity. Let $(X, d)$ have the property $(V)$. Then by Lemma 2 there is an $F$-discrete set $M$ in $X$ which is not uniformly discrete. It is easy to see that there are sequences $S$, $T$ in $M$ such that

(13) \[ 0 < d(S(n), T(n)) < 1/n \] for all $n \in N$

and

(14) \[ S(i) \neq T(j) \] for each $i, j \in N$.

Let us define a function $g: M \rightarrow \langle 0, 1 \rangle$ as follows:

\[
g(x) = 1 \quad \text{if} \quad x = S(n) \quad \text{for some} \quad n \in N \quad \text{and} \quad g(x) = 0 \quad \text{otherwise}.
\]

Then obviously $g(T(n)) = 0$ for $n \in N$.

Let $P$ be a Cauchy sequence in $M$. Then $\{P(n): n \in N\}$ is a totally bounded subset of $M$ and hence it is finite. The sequence $P$ is thus eventually constant and hence $f \circ P$ is an eventually constant sequence, too. Hence $g \circ P \in F_Y$ and $g \in F(M, \langle 0, 1 \rangle)$. By Lemma 3 there exists $g^* \in F(X, \langle 0, 1 \rangle)$ such that $g^*|_{M} = g$. Let $y \in Y$, $y \neq 0$, and let $h: \langle 0, 1 \rangle \rightarrow Y$ be defined by $h(t) = ty$. Then (2) implies $f = h \circ g^* \in F(X, Y)$. However, $f$ is not uniformly continuous because for any $n \in N$ we have

\[
d(S(n), T(n)) < 1/n \quad \text{and} \quad \|f(S(n)) - f(T(n))\| = \|y\|.
\]

Sufficiency. Let $(X, d)$ fail to have the property $(V)$. We will assume that there is $f \in F(X, Y) - U(X, Y)$. Then there are sequences $S$, $T$ in $X$ and $\varepsilon > 0$ such that

(15) \[ 0 < d(S(n), T(n)) < 1/n \]

and

(16) \[ \|f(S(n)) - f(T(n))\| \geq \varepsilon. \]

The set $A = \{S(n): n \in N\} \cup \{T(n): n \in N\}$ is not uniformly discrete. Hence according to Lemma 2, the set $A$ is not $F$-discrete. Therefore there is an infinite totally bounded set $B \subset A$. Let e.g. the set $B \cap \{S(n): n \in N\}$ be infinite. Then the sequence $S$
has a Cauchy subsequence $P$, i.e. there is an increasing function $u: \mathbb{N} \to \mathbb{N}$ such that $P = S \circ u$. Let $Q$ be a sequence in $M$ defined by

$$Q(2n) = S(u(n)) \quad \text{and} \quad Q(2n - 1) = T(u(n)) \quad \text{for all} \quad n \in \mathbb{N}.$$ 

Then it is easy to verify (by (15)) that $Q$ is a Cauchy sequence. Hence $f \circ Q$ is a Cauchy sequence, too. However, this contradicts (16).

**Corollary.** Let $(X, d)$ be a metric space and let $(Y, \|\cdot\|)$, $Y \neq \{0\}$, be a normed linear space. Then $U(X, Y) = C(X, Y)$ if and only if every closed discrete subset of $X$ is uniformly discrete.

Let $X$ be a set and $(Y, \|\cdot\|)$ a normed linear space. We recall that a mapping $f: X \to Y$ is bounded if there is a positive $k$ such that $\|f(x)\| \leq k$ for each $x \in X$. If $P(X, Y) \subset Y^X$, then $bP(X, Y)$ denotes all bounded mappings belonging to $P(X, Y)$. If $P(X, Y)$ is a linear space, then $bP(X, Y)$ is a linear normed space with the sup-norm.

**Lemma 4.** Let $X$ be a set and $(Y, \|\cdot\|)$ a normed linear space. Let $P(X, Y)$, $Q(X, Y) \subset Y^X$ be linear spaces. Let $P(X, Y)$ be a closed subset of $Q(X, Y)$ with respect to the topology of uniform convergence. If $bP(X, Y) = bQ(X, Y)$, then $P(X, Y)$ is a nowhere dense set in $Q(X, Y)$.

**Proof.** The uniform convergence in $Q(X, Y)$ is metrizable with the metric

$$d(f, g) = \min \{1, \sup \{|f(x) - g(x)| : x \in X\}\} \quad \text{for} \quad f, g \in Q(X, Y).$$

In view of the closedness of $P(X, Y)$, it is sufficient to show that $P(X, Y)$ is a boundary set.

Let $\varepsilon > 0$ and $f \in Q(X, Y)$. We shall show that there is a $g \in Q(X, Y) - P(X, Y)$ such that $d(f, g) < \varepsilon$. Since $bP(X, Y)$ is a closed linear subspace of $bQ(X, Y)$ and $bP(X, Y) = bQ(X, Y)$, according to [1] the set $bP(X, Y)$ is nowhere dense in $bQ(X, Y)$. Hence there is $h \in bQ(X, Y) - bP(X, Y)$ such that $\|h(x)\| < \varepsilon/2$ for all $x \in X$. If $f \not\in P(X, Y)$, then we put $g = f$. If $f \in P(X, Y)$, then we put $g = f + h$. Then $g \in Q(X, Y) - P(X, Y)$ and $d(f, g) < \varepsilon$.

**Lemma 5.** Let $(X, d_X)$, $(Y, d_Y)$ be metric space. Let $F$, $F(n) = f_n$, be a sequence of functions belonging to $F(X, Y)$ which uniformly converges to $f$. Then $f$ belongs to $F(X, Y)$.

**Proof.** Let $f_n \in F(X, Y)$ for all $n \in \mathbb{N}$ and let $F$ converge uniformly to $f$. We shall show that $f \in F(X, Y)$. Let $S \in F_X$ and $\varepsilon > 0$. Then there is $m \in \mathbb{N}$ such that $d_Y(f_m(x), f(x)) < \varepsilon/3$ for each $x \in X$.

Since $f_m \circ S \in F_Y$ by the hypotheses, there is $p \in \mathbb{N}$ such that $d_Y(f_m(S(i)), f_m(S(j))) < \varepsilon/3$ for $i, j \geq p$. Then for any $i, j \geq p$ we have

$$d_Y(f(S(i)), f(S(j))) \leq d_Y(f(S(i)), f_m(S(i))) + d_Y(f_m(S(i)), f(S(j))) + d_Y(f(S(i)), f_m(S(j))) + d_Y(f_m(S(j)), f(S(j))) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$
Therefore $f \circ S \in F_Y$, i.e. $f \in F(X, Y)$.

**Theorem 3.** Let $(X, d)$ be a metric space and $(Y, \|\cdot\|)$, $Y \neq \{0\}$ a normed linear space. If $(X, d)$ is not a complete metric space, then $F(X, Y)$ is a nowhere dense set in $C(X, Y)$ with respect to the topology of uniform convergence. If $(X, d)$ has the property $(V)$, then $U(X, Y)$ is a nowhere dense set in $F(X, Y)$ with respect to the topology of uniform convergence.

**Proof.** Obviously $C(X, Y)$, $U(X, Y)$ and (by (6)) $F(X, Y)$ are linear spaces. The set $U(X, Y)$ is closed in $F(X, Y)$ and by Lemma 5 also $F(X, Y)$ is closed in $C(X, Y)$. If $(X, d)$ is not a complete metric space, then by Theorem 2 and its proof and Lemma 4 we see that $F(X, Y)$ is a nowhere dense set in $C(X, Y)$. Similarly, if $(X, d)$ has the property $(V)$, then by Theorem 2 and its proof and Lemma 4 we see that $U(X, Y)$ is a nowhere dense set in $F(X, Y)$.

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**Súhrn**

ZOBRAZENIA, KTORÉ ZACHOVÁVAJÚ FUNDAMENTÁLNE POSTUPNOSTI

Ján Borsík

V práci sa vyšetrujú zobrazenia, ktoré zachovávajú fundamentálne postupnosti a ich vzťah ku spojitým a rovnomerne spojitým zobrazeniam.

**Резюме**

ОТОБРАЖЕНИЯ, СОХРАНЯЮЩИЕ ПОСЛЕДОВАТЕЛЬНОСТИ КОШИ

Ján Borsík

В статье исследуется отношение отображений, сохраняющих последовательности Коши, к непрерывным и равномерно непрерывным отображениям.

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