ON THE BROWNIAN MEANDER AND EXCURSION CONDITIONED TO HAVE A FIXED TIME AVERAGE

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Abstract. We study the density of the time average of the Brownian meander/excursion over the time interval \([0, 1]\). Moreover we give an expression for the law of the Brownian meander/excursion conditioned to have a fixed time average.

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1. Introduction

Let \((m_t, t \in [0, 1])\) be a standard Brownian meander and set \(\langle m, 1 \rangle := \int_0^1 m_r dr\), average of \(m\). In this note we answer two questions:

1. what is the density of the random variable \(\langle m, 1 \rangle\)?
2. what is the law of \((m_t, t \in [0, 1])\) conditionally on the value of \(\langle m, 1 \rangle\)?

We provide an expression for both objects, based on suitable Girsanov transformations.

We recall that \(m\) is equal in law to a Brownian motion \(B\) conditioned to be non-negative on \([0, 1]\). Using this result, we compute first the law of \(B\) conditioned to have a fixed average, then we write a Girsanov transformation and finally we condition this process to be non-negative.

We answer similar questions for the normalized Brownian excursion. We refer to [3] for details about the Brownian meander and the Brownian excursion.

1.1. The result. Let \(m\) be a Brownian meander on \([0, 1]\) and \(B\) a standard Brownian motion such that \(\{m, B\}\) are independent and let \(c \geq 0\) be a constant. We introduce the continuous processes:

\[
\begin{align*}
  u_t &:= \begin{cases} 
    \frac{1}{\sqrt{2}} m_{2t}, & t \in [0, 1/2] \\
    \frac{1}{\sqrt{2}} m_1 + B_{t-\frac{1}{2}}, & t \in [1/2, 1],
  \end{cases} \\
  U^c_t &:= \begin{cases} 
    u_t, & t \in [0, 1/2] \\
    u_t + (12 t (2 - t) - 9) \left( c - \int_0^1 u \right), & t \in [1/2, 1].
  \end{cases}
\end{align*}
\]

Notice that \(\int_0^1 U^c_t dt = c\).

Theorem 1.1. For all bounded Borel \(\Phi : C([0, 1]) \to \mathbb{R}\) and \(f : \mathbb{R} \to \mathbb{R}\):

\[
\mathbb{E} \left[ \Phi(m) f(\langle m, 1 \rangle) \right] = \int_0^\infty \sqrt{\frac{24}{\pi}} \mathbb{E} \left[ \Phi(U^c) e^{-12 (\int_0^{1/2} (U^c_t + U^c_{1/2}) dr - c)^2} 1_{\{U^c_t \geq 0, \forall t \in [0, 1]\}} \right] f(c) dc.
\]

(1.1)
Corollary 1.2. If we set for all \( c \geq 0 \)
\[
p_{(m,1)}(c) := \sqrt{\frac{24}{\pi}} E\left[ e^{-12\left(\int_0^{1/2}(U^c_r+U^c_{1/2}) dr-c\right)^2} 1\{U^c_r \geq 0, \forall t \in [0,1]\} \right],
\]
then \( p_{(m,1)} \) is the density of \( \langle m, 1 \rangle \), i.e.
\[
P(\langle m, 1 \rangle \in dc) = p_{(m,1)}(c) 1_{\{c \geq 0\}} dc.
\]
Moreover \( p_{(m,1)} \) is continuous on \([0, \infty)\).

Notice that a.s. \( U^c_r \geq 0 \) for all \( t \in [0,1/2] \), since a.s. \( m \geq 0 \): therefore a.s.
\[
\{U^c_r \geq 0, \forall t \in [0,1]\} = \{U^c_r \geq 0, \forall t \in [1/2,1]\}.
\]

The probability of this event is positive for all \( c > 0 \) while it is 0 for \( c = 0 \), since \( \int_0^1 U^0_t \, dt = 0 \). In particular \( p_{(m,1)}(0) = 0 \). Moreover for all \( c > 0 \) we can define the probability measure on \( C([0,1]) \) given by
\[
E[\Phi(m) | \langle m, 1 \rangle = c] := \frac{1}{Z_c} E\left[ \Phi(U^c) e^{-12\left(\int_0^{1/2}(U^c_r+U^c_{1/2}) dr-c\right)^2} 1\{U^c_r \geq 0, \forall t \in [0,1]\} \right],
\]
where \( \Phi : C([0,1]) \mapsto \mathbb{R} \) is bounded Borel and \( Z_c > 0 \) is a normalization factor.

Corollary 1.3. \( (\mathbb{P}[m \in \cdot | \langle m, 1 \rangle = c], c > 0) \) is a regular conditional distribution of \( m \) given \( \langle m, 1 \rangle \), i.e.
\[
\mathbb{P}(m \in \cdot, \langle m, 1 \rangle \in dc) = \mathbb{P}[m \in \cdot | \langle m, 1 \rangle = c] \, p_{(m,1)}(c) 1_{\{c > 0\}} dc.
\]
Moreover \( (0, \infty) \ni c \mapsto \mathbb{P}[m \in \cdot | \langle m, 1 \rangle = c] \) is continuous in the weak topology.

In section 4 below we give similar results for the Brownian excursion, see Theorem 4.1.

The results of this note are of interest in connection with a class of Stochastic Partial Differential Equations, studied in [1] and [4], which have the property of conservation of the space average. For instance, the stochastic equation considered in [4] admits as invariant measure the Brownian excursion conditioned to have a fixed average. Moreover, the Brownian meander conditioned to have a fixed average and the density \( p_{(m,1)} \) appear in an infinite-dimensional integration by parts formula in [1] Corollary 6.2.

2. An absolute continuity formula

Let \((X_t)_{t \in [0,1]}\) be a continuous centered Gaussian process with covariance function \( q_{t,s} := E[X_t X_s] \). We have in mind the case of \( X \) being a Brownian motion or a Brownian bridge. In this section we consider two processes \( Y \) and \( Z \), both defined by linear transformations of \( X \), and we write an absolute continuity formula between the laws of \( Y \) and \( Z \).

For all \( h \) in the space \( M([0,1]) \) of all signed measures with finite total variation on \([0,1]\] we set:
\[
Q : M([0,1]) \mapsto C([0,1]), \quad Q\lambda(t) := \int_0^1 q_{t,s} \lambda(ds), \quad t \in [0,1].
\]
We denote by \( \langle \cdot, \cdot \rangle : C([0,1]) \times M([0,1]) \mapsto \mathbb{R} \) the canonical pairing,
\[
\langle h, \mu \rangle := \int_0^1 h_t \mu(dt).
\]
where a continuous function \( k \in C([0,1]) \) is identified with \( k_t \, dt \in M([0,1]) \). We consider \( \lambda, \mu \in M([0,1]) \) such that:
\[
\langle Q\lambda, \mu \rangle = 0, \quad \langle Q\lambda, \lambda \rangle + \langle Q\mu, \mu \rangle = 1. \tag{2.1}
\]
We set for all $\omega \in C([0, 1])$:

$$
\gamma(\omega) := \int_0^1 \omega_s \lambda(ds), \quad \Lambda_t := Q\lambda(t), \quad t \in [0, 1], \quad I := \langle Q\lambda, \lambda \rangle,
$$

$$
a(\omega) := \int_0^1 \omega_s \mu(ds), \quad M_t := Q\mu(t), \quad t \in [0, 1], \quad 1 - I = \langle Q\mu, \mu \rangle,
$$

and we notice that $\gamma(X) \sim N(0, I)$, $a(X) \sim N(0, 1 - I)$ and $\{\gamma(X), a(X)\}$ are independent by (2.1). We fix a constant $\kappa \in \mathbb{R}$ and if $I < 1$ we define the continuous processes

$$
Y_t := X_t + (\Lambda_t + M_t)(\kappa - a(X) - \gamma(X)), \quad t \in [0, 1],
$$

$$
Z_t := X_t + \frac{1}{1 - I} M_t(\kappa - a(X) - \gamma(X)), \quad t \in [0, 1].
$$

**Lemma 2.1.** Suppose that $I < 1$. Then for all bounded Borel $\Phi : C([0, 1]) \mapsto \mathbb{R}$:

$$
E[\Phi(Y)] = E[\Phi(Z) \rho(Z)],
$$

where for all $\omega \in C([0, 1])$:

$$
\rho(\omega) := \frac{1}{\sqrt{1 - I}} \exp \left(-\frac{1}{2} \frac{1}{1 - I} (\gamma(\omega) - \kappa)^2 + \frac{1}{2} \kappa^2 \right).
$$

We postpone the proof of Lemma 2.1 to section 5.

### 3. The Brownian meander

In this section we prove Theorem 1.1. In the notation of section 2 we consider $X = (B_t, t \in [0, 1])$, standard Brownian motion. It is easy to see that for all $t \in [0, 1]$:

$$
E[B_t \int_0^1 B_r \, dr] = \frac{t (2 - t)}{2}, \quad E\left[\left( \int_0^1 B_r \, dr \right)^2 \right] = \frac{1}{3}.
$$

Therefore, it is standard that for all $c \in \mathbb{R}$, $B$ conditioned to $\int_0^1 B = c$ is equal in law to the process:

$$
B^c_t := B_t + \frac{3}{2} t (2 - t) \left( c - \int_0^1 B \right), \quad t \in [0, 1].
$$

**Lemma 3.1.** Let $c \in \mathbb{R}$. For all bounded Borel $\Phi : C([0, 1]) \mapsto \mathbb{R}$:

$$
E\left[\Phi(B) \mid \int_0^1 B = c \right] = E[\Phi(B^c)] = E[\Phi(S) \rho(S)],
$$

where

$$
S_t := \begin{cases} 
B_t, & t \in [0, 1/2] \\
B_t + (12 t (2 - t) - 9) \left( c - \int_0^1 B \right), & t \in [1/2, 1] \end{cases}
$$

$$
\rho(\omega) := \sqrt{3} \exp \left(-12 \left( \int_0^{1/2} (\omega_r + \omega_{1/2}) \, dr - c \right)^2 + \frac{3}{2} c^2 \right), \quad \omega \in C([0, 1]).
$$
Proof. We are going to show that we are in the setting of Lemma 2.1 with $X = B$, $Y = B^c$ and $Z = S$. We denote the Dirac mass at $\theta$ by $\delta_\theta$. In the notation of section 2, we consider: 

$$\lambda(dt) := \sqrt{3} \left( 1_{[0,1/2]}(t) dt + \frac{1}{2} \delta_{1/2}(dt) \right), \quad \mu(dt) := \sqrt{3} \left( 1_{[1/2,1]}(t) dt - \frac{1}{2} \delta_{1/2}(dt) \right),$$

and $\kappa := \sqrt{3} c$. Then:

$$\gamma(\omega) = \sqrt{3} \int_0^{1/2} (\omega_r + \omega_{1/2}) \, dr, \quad a(\omega) = \sqrt{3} \int_{1/2}^1 (\omega_r - \omega_{1/2}) \, dr,$n

$$\gamma(\omega) + a(\omega) = \sqrt{3} \int_0^1 \omega_r \, dr, \quad I = 3 \int_0^{1/2} (1-r)^2 \, dr = \frac{7}{8}.$$n

$$\Lambda_t = \begin{cases} \sqrt{3} t \left( 1 - \frac{t}{2} \right), & t \in [0, 1/2] \\ \frac{3\sqrt{3}}{8}, & t \in [1/2, 1] \end{cases}, \quad M_t = \begin{cases} 0, & t \in [0, 1/2] \\ \sqrt{3} t \left( 1 - \frac{t}{2} \right) - \frac{3\sqrt{3}}{8}, & t \in [1/2, 1] \end{cases}.$$n

Tedious but straightforward computations show that with these definitions we have $X = B$, $Y = B^c$ and $Z = S$ in the notation of Lemma 2.1 and (2.1) holds true. Then the thesis of Lemma 3.1 follows from Lemma 2.1. \qed

Proof of Theorem 1.1. Recall that $m$ is equal in law to $B$ conditioned to be non-negative (see [2] and [4.11] below). We want to condition $B$ first to be non-negative and then to have a fixed time average. It turns out that Lemma 3.1 allows to compute the resulting law by inverting the two operations: first we condition $B$ to have a fixed average, then we use the absolute continuity between the law of $B^c$ and the law of $S$ and finally we condition $S$ to be non-negative.

We set $K_\varepsilon := \{ \omega \in C([0,1]) : \omega \geq -\varepsilon \}, \varepsilon \geq 0$. We recall that $B$ conditioned on $K_\varepsilon$ tends in law to $m$ as $\varepsilon \to 0$, more generally for all $s > 0$ and bounded continuous $\Phi : C([0,s]) \mapsto \mathbb{R}$, by the Brownian scaling:

$$\lim_{\varepsilon \to 0} \mathbb{E} \left[ \Phi(B_t, t \in [0,s]) \mid B_t \geq -\varepsilon, \forall t \in [0,s] \right] = \mathbb{E} \left[ \Phi \left( \sqrt{s} m_{t/s}, t \in [0,s] \right) \right], \quad (3.1)$$

and this is a result of [2]. By the reflection principle, for all $s > 0$:

$$\mathbb{P}(B_t \geq -\varepsilon, \forall t \in [0,s]) = \mathbb{P}(|B_s| \leq \varepsilon) \sim \sqrt{\frac{2}{\pi s}} \varepsilon, \quad \varepsilon \to 0. \quad (3.2)$$

In particular for all bounded $f \in C(\mathbb{R})$

$$\mathbb{E} \left[ \Phi(m) f((m,1)) \right] = \lim_{\varepsilon \to 0} \sqrt{\frac{\pi}{2}} \frac{1}{\varepsilon} \mathbb{E} \left[ \Phi(B) 1_{K_\varepsilon}(B) f((B,1)) \right].$$

We want to compute the limit of $\frac{1}{2} \mathbb{E} \left[ \Phi(B^c) 1_{K_\varepsilon}(B^c) \right]$ as $\varepsilon \to 0$. Notice that $S$, defined in Lemma 3.1 is equal to $B$ on $[0, 1/2]$. Therefore, by (3.1) and (3.2) with $s = 1/2$:

$$\sqrt{\frac{\pi}{2}} \frac{1}{\varepsilon} \mathbb{E} \left[ \Phi(B^c) 1_{K_\varepsilon}(B^c) \right] \to \sqrt{2} \mathbb{E} \left[ \Phi(U^c) \rho(U^c) 1_{K_0}(U^c) \right].$$
Comparing the last two formulae for all \( f \in C(\mathbb{R}) \) with compact support:
\[
\sqrt{\frac{2}{\pi}} \frac{1}{2} \mathbb{E} \left[ \Phi(B) |K_e(B) f(\langle B, 1 \rangle) \right] = \int_{\mathbb{R}} \sqrt{\frac{2}{\pi}} \frac{1}{2} \mathbb{E} \left[ \Phi(B^c) |K_e(B^c) f(c) N(0, 1/3)(dc) \right] \\
\rightarrow \int_{0}^{\infty} \sqrt{\frac{24}{\pi}} \mathbb{E} \left[ \Phi(U^c) e^{-12 \int_{0}^{1/2} (U^c_t + U^c_{1/2}) dr - c} \right] 1_{K_0(U^c)} f(c) dc = \mathbb{E} [\Phi(m) f(\langle m, 1 \rangle)]
\]
and \((1.1)\) is proven. \(\square\)

4. THE BROWNIAN EXCURSION

Let \((e_t, t \in [0,1])\) be the normalized Brownian excursion, see \[3\], and \((\beta_t, t \in [0,1])\) a Brownian bridge between 0 and 0. Let \(\{m, \hat{m}, b\}\) be a triple of processes such that:

1. \(m\) and \(\hat{m}\) are independent copies of a Brownian meander on \([0,1]\)
2. conditionally on \(\{m, \hat{m}\}\), \(b\) is a Brownian bridge on \([1/3,2/3]\) from \(\frac{1}{\sqrt{3}} m_1\) to \(\frac{1}{\sqrt{3}} \hat{m}_1\)

We introduce the continuous processes:
\[
v_t := \begin{cases} 
\frac{1}{\sqrt{3}} m_{3t}, & t \in [0,1/3] \\
\beta_t, & t \in [1/3,2/3], \\
\frac{1}{\sqrt{3}} \hat{m}_{1-3t}, & t \in [2/3,1], 
\end{cases}
\]
\[
V_t^c := \begin{cases} 
v_t, & t \in [0,1/3] \cup [2/3,1] \\
v_t + 18 (9 t (1-t) - 2) \left( c - \int_{0}^{1} v \right), & t \in [1/3,2/3].
\end{cases}
\]

Notice that \(\int_{0}^{1} V_t^c dt = c\). We set for all \(\omega \in C([0,1]):\)
\[
\rho^c(\omega) := \exp \left\{ -162 \left( \int_{0}^{1/3} (\omega_r + \omega_{1-r}) dr + \frac{\omega_1 + \omega_2}{6} - c \right)^2 - \frac{3}{2} (\omega_3 - \omega_4)^2 \right\}.
\]

**Theorem 4.1.** For all bounded Borel \(\Phi : C([0,1]) \mapsto \mathbb{R}\) and \(f : \mathbb{R} \mapsto \mathbb{R}\):
\[
\mathbb{E} [\Phi(e) f((e,1))] = \int_{0}^{\infty} 27 \sqrt{\frac{6}{\pi^3}} \mathbb{E} [\Phi(V^c) \rho^c(V^c) 1_{K_0(V^c)}] f(c) dc \quad (4.1)
\]

**Corollary 4.2.** If we set
\[
p_{(e,1)}(c) =: 27 \sqrt{\frac{6}{\pi^3}} \mathbb{E} \left[ \rho^c(V^c) 1_{\{V_t^c \geq 0, \forall t \in [0,1]\}} \right].
\]
then \(p_{(e,1)}\) is the density of \(\langle e, 1 \rangle\) on \([0, \infty)\), i.e.
\[
\mathbb{P}(\langle e, 1 \rangle \in dc) = p_{(e,1)}(c) 1_{\{c \geq 0\}} dc.
\]

Moreover \(p_{(e,1)}\) is continuous on \([0, \infty)\).

Notice that a.s. \(V_t^c \geq 0\) for all \(t \in [0,1/3] \cup [2/3,1]\), since a.s. \(m \geq 0\): therefore a.s. \(\{V_t^c \geq 0, \forall t \in [0,1]\} = \{V_t^c \geq 0, \forall t \in [1/3,2/3]\}\).
The probability of this event is positive for all $c > 0$ while it is 0 for $c = 0$, since $\int_0^1 V_t^0 \, dt = 0$. In particular $p(e, 1)(0) = 0$. Moreover for all $c > 0$ we can define the probability measure on $C([0, 1])$ by

$$\mathbb{E}\left[ \Phi(e) \mid \langle e, 1 \rangle = c \right] := \frac{1}{Z_c} \mathbb{E}\left[ \Phi \left( V^c \right) \rho^c(V^c) \mathbb{1}_{\{V^c \geq 0, \forall t \in [0, 1]\}} \right]$$

where $\Phi : C([0, 1]) \mapsto \mathbb{R}$ is bounded Borel and $Z_c > 0$ is a normalization factor.

**Corollary 4.3.** $(\mathbb{P} [e \in \cdot \mid \langle e, 1 \rangle = c], c > 0)$ is a regular conditional distribution of $e$ given $\langle e, 1 \rangle$, i.e.

$$\mathbb{P}(e \in \cdot, \langle e, 1 \rangle \in dc) = \mathbb{P}[e \in \cdot \mid \langle e, 1 \rangle = c] \ p(e, 1)(c) \ 1_{\{c > 0\}} \ dc.$$ 

Moreover $(0, \infty) \ni c \mapsto \mathbb{P}[e \in \cdot \mid \langle e, 1 \rangle = c]$ is continuous in the weak topology.

Notice that for all $t \in [0, 1]$:

$$\mathbb{E}\left[ \beta_t \int_0^1 \beta_r \, dr \right] = \frac{t(1 - t)}{2}, \quad \mathbb{E}\left[ \left( \int_0^1 \beta_r \, dr \right)^2 \right] = \frac{1}{12}.$$ 

Therefore, for all $c \in \mathbb{R}$, $\beta$ conditioned to $\int_0^1 \beta = c$ is equal in law to the process:

$$\beta^c_t := \beta_t + 6 t (1 - t) \left( c - \int_0^1 \beta \right), \quad t \in [0, 1].$$

**Lemma 4.4.** Let $c \in \mathbb{R}$. For all bounded Borel $\Phi : C([0, 1]) \mapsto \mathbb{R}$:

$$\mathbb{E}\left[ \Phi(\beta) \mid \int_0^1 \beta = c \right] = \mathbb{E}[\Phi(\beta^c)] = \mathbb{E}\left[ \Phi \left( \Gamma^c \right)^\beta \right] \rho_1(\Gamma^c)$$

where for all $\omega \in C([0, 1])$

$$\Gamma^c_t = \begin{cases} \omega_t, & t \in [0, 1/3) \cup [2/3, 1] \\ \omega_t + 18 \left( 9 t (1 - t) - 2 \right) \left( c - \int_0^1 \omega \right), & t \in [1/3, 2/3] \end{cases}$$

$$\rho_1(\omega) := \sqrt{27} \exp \left( -162 \left( \int_0^1 (\omega_r + \omega_{1-r}) \, dr + \frac{\omega_1 + \omega_2}{6} - c \right)^2 + 6 c^2 \right).$$

**Proof.** Similarly to the proof of Lemma 3.1, we are going to show that we are in the situation of Lemma 2.1 with $X = \beta$, $Y = \beta^c$ and $Z = \Gamma^c$. In the notation of Lemma 2.1 we consider

$$\lambda(dt) := \sqrt{12} \left( 1_{[0, 1/3) \cup [2/3, 1]}(t) \, dt + \frac{\delta_{1/3}(dt) + \delta_{2/3}(dt)}{6} \right),$$

$$\mu(dt) := \sqrt{12} \left( 1_{[1/3, 2/3]}(t) \, dt - \frac{\delta_{1/3}(dt) + \delta_{2/3}(dt)}{6} \right),$$

and $\kappa := \sqrt{12} c$. Then:

$$\gamma(\beta) = \sqrt{12} \int_0^{1/3} \left( \beta_r + \frac{1}{2} \beta_{1/3} \right) \, dr + \sqrt{12} \int_{1/3}^{2/3} \left( \beta_r + \frac{1}{2} \beta_{2/3} \right) \, dr, \quad I = \frac{26}{27}.$$
Such that:

We recall now that

We set:

\[ \Lambda_t = 1_{[0, \frac{1}{2})} \cup [\frac{1}{2}, 1](t) \sqrt{3} t(1-t) + 1_{(\frac{1}{2}, 1)}(t) \frac{2\sqrt{3}}{9}, \quad M_t = 1_{[\frac{1}{4}, \frac{3}{4}]}(t) \sqrt{3} t(1-t). \]

Again the thesis follows by direct computations and from Lemma 2.1.

**Proof of Theorem 4.1** We follow the proof of Theorem 1.1. Define \( \{B, b, \hat{B}\} \), processes such that:

1. \( B \) and \( \hat{B} \) are independent copies of a standard Brownian motion over \([0, 1/3]\),
2. conditionally on \( \{B, \hat{B}\} \), \( b \) is a Brownian bridge over \([1/3, 2/3]\) from \( B_{1/3} \) to \( \hat{B}_{1/3} \).

We set:

\[
    r_t := \begin{cases} 
    B_t & t \in [0, 1/3] \\
    b_t & t \in [1/3, 2/3] \\
    \hat{B}_{1-t} & t \in [2/3, 1]. 
    \end{cases}
\]

Moreover we set, denoting the density of \( N(0, t)(dy) \) by \( p_t(y) \):

\[ \rho_2(\omega) := \frac{p_{\frac{1}{2}}(\omega_{\frac{1}{2}} - \omega_{\frac{1}{2}})}{p_0(0)} = \sqrt{3} \exp \left( -\frac{3}{2}(\omega_{\frac{1}{2}} - \omega_{\frac{1}{2}})^2 \right), \quad \omega \in C([0, 1]). \]

By the Markov property of \( \beta \):

\[ \mathbb{E}[\Phi(r) \rho_2(r)] = \mathbb{E}[\Phi(\beta)]. \]

Then, recalling the definition of \( \rho^c \) above, by Lemma 2.1 and Lemma 4.4:

\[ \mathbb{E}[\Phi(\beta^c)] = \mathbb{E}\left[ \Phi\left( \Gamma^\beta \right) \rho_1 \left( \Gamma^\beta \right) \right] = \mathbb{E}[\Phi(\Gamma^r) \rho_1(\Gamma^r) \rho_2(\Gamma^r)] = 9 \mathbb{E}[\Phi(\Gamma^r) \rho^c(\Gamma^r)] e^{6c^2}. \]

We recall now that \( \mathbb{P}(\beta \in K_{\varepsilon}) = 1 - \exp(-2\varepsilon^2) \sim 2\varepsilon^2 \) as \( \varepsilon \to 0 \), where \( K_{\varepsilon} = \{\omega \in C([0, 1]) : \omega \geq -\varepsilon\} \). We want to compute the limit of \( \frac{1}{2\varepsilon^2} \mathbb{E}[\Phi(\beta^c) 1_{K_{\varepsilon}}(\beta^c)] \) as \( \varepsilon \to 0 \). On the other hand \( \mathbb{P}(B_t \geq -\varepsilon, \forall t \in [0, 1/3]) \sim \sqrt{\frac{2}{3}} \varepsilon \) by (3.2). Then by (3.1) and (3.2):

\[
    \frac{1}{2\varepsilon^2} \mathbb{E}[\Phi(\beta^c) 1_{K_{\varepsilon}}(\beta^c)] \to \frac{27}{\pi} \mathbb{E}[\Phi(V^c) \rho^c(V^c) 1_{K_0}(V^c)] e^{6c^2}, \quad \frac{27}{\pi} = \frac{1}{2} \sqrt{\frac{6}{\pi}} \sqrt{\frac{6}{\pi}} 9.
\]

On the other hand, \( \beta \) conditioned on \( K_{\varepsilon} \) tends in law to the normalized Brownian excursion \((e_t, t \in [0, 1])\), as proven in [2]. Then we have for all bounded \( f \in C(\mathbb{R}) \):

\[
    \frac{1}{2\varepsilon^2} \mathbb{E}[\Phi(\beta) 1_{K_{\varepsilon}}(\beta) f(\langle \beta, 1 \rangle)] \to \mathbb{E}[\Phi(e) f(\langle e, 1 \rangle)]
\]

Comparing the two formulae for all \( f \in C(\mathbb{R}) \) with compact support:

\[
    \frac{1}{2\varepsilon^2} \mathbb{E}[\Phi(\beta) 1_{K_{\varepsilon}}(\beta) f(\langle \beta, 1 \rangle)] = \int_\mathbb{R} \frac{1}{2\varepsilon^2} \mathbb{E}[\Phi(\beta^c) 1_{K_{\varepsilon}}(\beta^c)] f(c) N(0, 1/12)(dc)
\]

\[
    \to \int_0^\infty 27 \sqrt{\frac{6}{\pi^2}} \mathbb{E}[\Phi(V^c) \rho^c(V^c) 1_{K_0}(V^c)] f(c) dc = \mathbb{E}[\Phi(e) f(\langle e, 1 \rangle)]
\]

and (4.1) is proven. \( \square \)
5. Proof of Proposition 2.1

The thesis follows if we show that the Laplace transforms of the two probability measures in (2.2) are equal. Notice that $Y$ is a Gaussian process with mean $\kappa (\Lambda + M)$ and covariance function:
\[
q_{t,s}^Y = \mathbb{E} [(Y_t - \kappa (\Lambda_t + M_t)) (Y_s - \kappa (\Lambda_s + M_s))] = q_{t,s} - (\Lambda_t + M_t) (\Lambda_s + M_s),
\]
for $t, s \in [0,1]$. Therefore, setting for all $h \in C([0,1]): Q_Y h(t) := \int_0^1 q_{t,s}^Y h_s ds$, $t \in [0,1]$, the Laplace transform of the law of $Y$ is:
\[
\mathbb{E} \left[ e^{(Y,h)} \right] = e^{\kappa (h, \Lambda + M) + \frac{1}{2} (Q_Y h, h)}.
\]
Recall now the following version of the Cameron-Martin Theorem: for all $h \in M([0,1])$
\[
\mathbb{E} \left[ \Phi(X) e^{(X,h)} \right] = e^{\frac{1}{2} (Qh, h)} \mathbb{E} [\Phi(X + Qh)].
\]
Notice that $\gamma(Z) = \gamma(X)$, by (2.1). Therefore $\rho(Z) = \rho(X)$. We obtain, setting $\overline{h} := h - \frac{1}{1-t} (M, h)(\lambda + \mu)$:
\[
\mathbb{E} \left[ e^{(Z,h)} \rho(Z) \right] = e^{\kappa (M, h)^{-1}} \mathbb{E} \left[ e^{(X, \overline{h})} \rho(X) \right] = e^{\kappa (M, h)^{-1} + \frac{1}{2} (Q\overline{h}, \overline{h})} \mathbb{E} [\rho (X + Q\overline{h})] =
\]
\[
e^{\kappa (M, h)^{-1} + \frac{1}{2} (Q\overline{h}, \overline{h})} \frac{1}{\sqrt{1-t}} \mathbb{E} \left[ e^{-\frac{1}{2} \frac{1}{1-t} (\gamma(X) + (\overline{h}, \Lambda) - \kappa)^2 + \frac{1}{2} \kappa^2} \right].
\]
By the following standard Gaussian formula for $\alpha \sim N(0, \sigma^2)$, $\sigma \geq 0$ and $c \in \mathbb{R}$:
\[
\mathbb{E} \left[ e^{-\frac{1}{2} (\alpha^2 + c^2)} \right] = \frac{1}{\sqrt{1+\sigma^2}} e^{-\frac{1}{2} \frac{c^2}{1+\sigma^2}},
\]
we have now for $\gamma(X) \sim N(0, I)$:
\[
\mathbb{E} \left[ e^{-\frac{1}{2} \frac{1}{1-t} (\gamma(X) + (\overline{h}, \Lambda) - \kappa)^2} \right] = \frac{1}{\sqrt{1+\sigma^2}} e^{-\frac{1}{2} \frac{1}{1-t} \frac{1}{1-t} (\overline{h}, \Lambda) - \kappa)^2} = \sqrt{1-t} e^{-\frac{1}{2} (\overline{h}, \Lambda) - \kappa)^2}
\]
Therefore, recalling the definition of $\overline{h} := h - \frac{1}{1-t} (M, h)(\lambda + \mu)$, we obtain after some trivial computation:
\[
\log \mathbb{E} \left[ e^{(Z,h)} \rho(Z) \right] = \kappa \frac{1}{1-t} (M, h) + \frac{1}{2} (Q\overline{h}, \overline{h}) - \frac{1}{2} \frac{1}{1-t} (\overline{h}, \Lambda) - \kappa)^2 + \frac{1}{2} \kappa^2 
\]
\[
= \kappa (\Lambda + M, h) + \frac{1}{2} (Qh, h) - \frac{1}{2} (\Lambda + M, h)^2 = \kappa (h, \Lambda + M) + \frac{1}{2} (QY h, h). \quad \square
\]

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