Opaque Attack on Three-Party Quantum Secret Sharing Based on Entanglement

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Security of the three-party quantum secret sharing (QSS) schemes based on entanglement and a collective eavesdropping check is analyzed in the case of considerable quantum channel losses. An opaque attack scheme is presented for the dishonest agent to eavesdrop the message obtained by the other agent freely, which reveals that these QSS schemes are insecure for transmission efficiencies lower than 50%, especially when they are used to share an unknown quantum state. Finally, we present a general way to improve the security of QSS schemes for sharing not only a private key but also an unknown quantum state.

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In a secret sharing [1], a sender, say Alice, has two agents, Bob and Charlie who are at remote places. Alice hopes that the agents can carry out her instruction (the message $M_A$), but she suspects that one of them (not more than one) may be dishonest and the dishonest one will do harm to her belongings if he can deal with it independently. Moreover, Alice does not know who the dishonest one is. For the security of the secret message $M_A$, Alice splits it into two pieces, $M_B$ and $M_C$, and then sends them to Bob and Charlie, respectively. The two agents can read out the message $M_A = M_B \oplus M_C$ if and only if they cooperate, otherwise none can obtain a useful information. Quantum secret sharing (QSS) is the generalization of classical secret sharing into quantum scenario and has progressed quickly in recent years. It provides a secure way for sharing not only a classical information [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] but also a quantum information [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] if and only if they cooperate, otherwise none can obtain a useful information. Quantum secret sharing (QSS) is the generalization of classical secret sharing into quantum scenario and has progressed quickly in recent years. It provides a secure way for sharing not only a classical information [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] but also a quantum information [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In the latter, the sender Alice will send an unknown quantum state to her agents, and one of them can recover it with the help of the others [12, 13, 14, 15]. Now, QSS has also been studied in experiment [16, 17, 18, 19]. In general, QSS is far more complex than quantum key distribution (QKD) [20] as the dishonest agent, a powerful eavesdropper, has a chance to hide his eavesdropping by cheating the others.

Almost all the QSS schemes existing can be attributed to one of the two types, the collective eavesdropping-check one and the individual eavesdropping-check one. The feature of the QSS schemes based on a collective eavesdropping check is that the procedure of the eavesdropping check can be completed only when the sender Alice gets the cooperation of the dishonest agent. That is, Alice should require him to publish his outcomes of the measurements on the samples chosen randomly for checking eavesdropping. Otherwise, she and the honest agent cannot accomplish the task. The typical models are those in the Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In contrast, in the individual eavesdropping-check QSS schemes, the eavesdropping check between Alice and the honest agent does not resort to the information published by the dishonest one. In other words, Alice and the honest agent can analyze the error rate of their samples independent of the dishonest one, and determine whether the quantum channel between them is secure or not. Typical such QSS schemes are the ones presented in Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

There is a class of one-way QSS schemes whose security is based on entanglement and a collective eavesdropping check (ECEC), called them one-way ECEC QSS, such as those in Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], including the Hillery-Bužek-Berthiaume (HBB) scheme [3] and the Karlsson-Koashi-Imoto (KKI) scheme [2], and the experimental demonstrations for sharing a classical message [17, 18, 19] or a qubit [16]. Although there are differences among particular schemes, almost all of them realized the following scenario. First, the sender Alice prepares two photons in an entangled state $|\xi^\pm\rangle_{BC} = \frac{1}{\sqrt{2}}(|\alpha\rangle_B|\beta\rangle_C \pm |\bar{\alpha}\rangle_B|\bar{\beta}\rangle_C)$, and sends the photon $B$ to Bob and the photon $C$ to Charlie (here $|\alpha\rangle$ and $|\bar{\alpha}\rangle$ are the two eigenvectors of a two-level quantum system, so do $|\beta\rangle$ and $|\bar{\beta}\rangle$). Secondly, Bob and Charlie choose at least two nonorthogonal measuring bases (MBs) to measure their photons independently. Thirdly, Alice tells Bob and Charlie which photons are chosen as the samples for checking eavesdropping, and then the three participants analyze the error rate of the samples collectively for determining whether the quantum channel is secure or not. Finally, if Alice deems that the quantum channel is secure, she and her agents distill a private key from the outcomes for which Bob and Charlie have two correlated MBs. Let us emphasize here two properties of the one-way ECEC QSS. First, the quantum systems are sent from Alice to her agents in one way. Second, its test eavesdropping procedure depends on the entanglement correlation of the quantum systems and the information published by all the participants.

As the security of a QSS scheme is based on the public statistical analysis of the error rate of the samples chosen randomly by the three participants including the poten-
entially dishonest agent, it must be guaranteed against the dishonest agent with unlimited computing power whose technology is limited only by the laws of quantum mechanics \[21\]. Without loss of generality, we assume that the dishonest agent is Bob. The aim of this paper is to present an opaque attack scheme for Bob to obtain Alice’s message (a classical one or a quantum one) without the cooperation of the other agent Charlie, provided that quantum channel losses are high enough. The superiority of Bob over current technology is restricted to the possibility of near lossless photon transmission, the storage of quantum states, and the capability of preparing and manipulating two-photon Bell state. Same as Ref. \[22\], our scheme considers the opportunity of eavesdropping arising due to a separation of two procedures, namely, the eavesdropping-check procedure and the message sharing procedure (including the one for sharing an unknown quantum state).

Now, let us elaborate the principle of the one-way ECEC QSS with the example of the KKI scheme \[2\]. Alice encodes a randomly binary bit string \(S\), selecting for instance two sets of states \({|\psi^+\rangle, |\phi^-\rangle}\) \(\equiv\{0\}, \{1\}\) and \({|\Psi^+\rangle, |\Phi^-\rangle}\) \(\equiv\{0'\}, \{1'\}\) which are defined as follows.

\[
|\psi^+_BC\rangle = \frac{1}{\sqrt{2}}(|+z\rangle_B|-z\rangle_C|−z\rangle_B|+z\rangle_C),
|\phi^-_BC\rangle = \frac{1}{\sqrt{2}}(|+z\rangle_B|+z\rangle_C|−z\rangle_B|−z\rangle_C),
|\Psi^+_BC\rangle = \frac{1}{\sqrt{2}}(|+z\rangle_B|+x\rangle_C|−z\rangle_B|−x\rangle_C),
|\Phi^-_BC\rangle = \frac{1}{\sqrt{2}}(|+z\rangle_B|−x\rangle_C|−z\rangle_B|+x\rangle_C),
\]

where the subscripts \(B\) and \(C\) represent the two photons in an entangled state, and \(|±z\rangle\) and \(|±x\rangle\) are the eigenstates of the z-spin (basis \(Z\)) and the x-spin (basis \(X\)), respectively,

\[
|±x\rangle = \frac{1}{\sqrt{2}}(|±\rangle).
\]

The KKI QSS scheme works with three steps below.

1. Alice prepares two photons randomly in one of the four states \({|\psi^+\rangle, |\phi^-\rangle, |\Psi^+\rangle, |\Phi^-\rangle}\), and sends the photon \(B\) to Bob and the photon \(C\) to Charlie. The two agents Bob and Charlie locally measure their photons with the two bases \(Z\) and \(X\) randomly. They repeat this procedure for obtaining an enough large set of outcomes \(S_a\) for distilling their private key \(K_A = K_B \oplus K_C\). Here \(K_A, K_B\) and \(K_C\) are the keys kept by Alice, Bob and Charlie in secret sharing, respectively.

2. Bob and Charlie declare a set of outcomes for a test of eavesdropping first and then the measurement bases for all the outcomes \(S_a\). As claimed by the authors in KKI QSS scheme \[2\], this order is very important for testing eavesdropping. Even they invented a refined order to declare the information for the test bits although the quantum information carriers are three-particle Greenberger-Horne-Zeilinger states.

\[
(G) = \frac{1}{2}(|+x\rangle_A(|+x\rangle_B|+x\rangle_C|−x\rangle_B|−x\rangle_C) + |−x\rangle_A(|−x\rangle_B|−x\rangle_C|−x\rangle_B|+x\rangle_C) = \frac{1}{2}(|+y\rangle_A(|+x\rangle_B|−y\rangle_C|−x\rangle_B|+y\rangle_C) + |−y\rangle_A(|−x\rangle_B|+y\rangle_C|−x\rangle_B|−y\rangle_C),
\]

where \(|±y\rangle = \frac{1}{\sqrt{2}}(|±\rangle ± i|−\rangle\)\). When Alice measures her photon \(A\) with the basis \(X\), the two photons \(B\) and \(C\) are in one of the two states \({|\Psi^+\rangle, |\Phi^-\rangle}\) if the basis \(Y\) is chosen by Alice. Here the two bases of an entangled photon pair \(BC\) are defined as

\[
|\psi^{+}\rangle \equiv \frac{1}{\sqrt{2}}(|+x\rangle_B|−x\rangle_C|−x\rangle_B|+x\rangle_C),
|\phi^{-}\rangle \equiv \frac{1}{\sqrt{2}}(|+x\rangle_B|+x\rangle_C|−x\rangle_B|−x\rangle_C),
|\Psi^{+}\rangle \equiv \frac{1}{\sqrt{2}}(|+x\rangle_B|+y\rangle_C|−x\rangle_B|−y\rangle_C),
|\Phi^{-}\rangle \equiv \frac{1}{\sqrt{2}}(|+x\rangle_B|−y\rangle_C|−x\rangle_B|+y\rangle_C).
\]

It is obvious that these two bases can be transformed into each other by the transformations \(|−x\rangle_C ←|+y\rangle_C\) and \(|+x\rangle_C ←|−y\rangle_C\), as same as the KKI scheme \[2\] in which \(|−z\rangle_C ←|+x\rangle_C\) and \(|+z\rangle_C ←|−x\rangle_C\). It is not difficult to prove that the other one-way ECEC QSS schemes \[4, 5, 6\] are equivalent to the KKI scheme with or without a little of modification.

Our opaque attack scheme on the KKI scheme includes two procedures, the one for checking eavesdropping and the one for generating a private key (or for sharing quantum information). Let us first describe the attack on eavesdropping check. It includes the following two steps.

(a) The dishonest agent Bob first intercepts the photon \(C\) sent from Alice to Charlie, and replaces it with a fake photon \(C'\) which is one particle in the Bell state \(|\phi^+\rangle_{BC'} = \frac{1}{\sqrt{2}}(|+z\rangle_B|+z\rangle_C|−z\rangle_B|−z\rangle_C)\) prepared by Bob himself. Moreover, he uses a near lossless

(3) After the information about the outcomes and the bases for the test bits have been released, Alice tells Bob and Charlie which of two bases the state was sent, but not which state. For completing the error rate analysis of the test bits, Alice publishes which states were sent for them. In this way, half of the outcomes \(S_a\) are useful as Bob and Charlie choose such two correlated bases for their measurements that they can deduce the states sent by Alice after she announces her bases, and the other outcomes will be discarded.

In essence, the KKI scheme \[2\] is a typical model for the one-way ECEC QSS. The HBB scheme \[3\] is equivalent to the KKI scheme if the three participants exploit the refined order to declare the information for the test bits although the quantum information carriers are three-particle Greenberger-Horne-Zeilinger states.
quantum channel to receive the photons $B$ and $C$ sent from Alice, and stores them with a quantum memory.

(b) When and only when Bob gets the information that the photon $B$ is chosen as the sample for detecting eavesdropping, he performs a Bell-state measurement on the photons $C$ and $B'$. If he obtains the result $|\phi^+\rangle_{BC} = \frac{1}{\sqrt{2}}(|+\rangle_B|-\rangle_C + |-\rangle_B|+\rangle_C)$, Bob measures the photon $B$ with the basis $Z$ or the basis $X$, the same as that in the original KKI scheme, and announces the outcome of the single-photon measurement. If Bob gets the result $|\psi^-\rangle_{BC} = \frac{1}{\sqrt{2}}(|+\rangle_B|-\rangle_C - |-\rangle_B|+\rangle_C)$, he first takes the operation $i\sigma_y = |+\rangle\langle -| - |-\rangle\langle +| \text{ on the photon } B$ and then manipulates it as same as the case with the result $|\phi^+\rangle_{BC}$. If he gets the other two Bell states $|\phi^-\rangle_{BC} = \frac{1}{\sqrt{2}}(|+\rangle_B|+\rangle_C - |+\rangle_B|-\rangle_C)$ and $|\psi^+\rangle_{BC} = \frac{1}{\sqrt{2}}(|+\rangle_B|+\rangle_c + |+\rangle_B|-\rangle_C)$, he declares he did not receive that particular bit. As he knows the fact that his eavesdropping will introduce errors in the outcomes when Bob obtains $|\phi^+\rangle_{BC}$ or $|\psi^+\rangle_{BC}$, his cheating will erase the errors in the test bits. Of course, Bob’s cheating will in principle induce 50% losses of the outcomes in an ideal condition.

Suppose that the honest agent Charlie obtains the outcome $|R_{BC}\rangle = a|+\rangle + b|-\rangle$ when he measures the photon $C'$ with the basis $Z$ or $X$. The photon $B'$ will collapse to the state $|R_{B'}\rangle = |R_{BC}\rangle$ as well. After the Bell-state measurement was done by Bob on the photons $B'C$ and the result $|\phi^+\rangle_{B'C}$ was obtained, the photon $B$ will collapse to the the states (without being normalized) $|a|+\rangle - |b|-, a|+\rangle + |b|+, (a-b)|+\rangle + (a+b)|-, (a-b)|+\rangle + (a+b)|-$ when the entangled photon pair $BC$ is prepared by Alice in the states $|\phi^+\rangle_{BC}$, $|\psi^+\rangle_{BC}$, $|\Phi^+\rangle_{BC}$, and $|\Psi^+\rangle_{BC}$, respectively. Table I shows the states obtained by the three participants Alice, Charlie and Bob. Alice’s states are given in the columns and Charlie’s outcomes are given in the rows. Bob’s states before he measures the photon $B$ with a correlated basis appear in the boxes. They are the same as those in the original KKI QSS scheme. The difference between the results $|\phi^+\rangle_{B'C}$ and $|\psi^+\rangle_{B'C}$ is just the unitary operation $i\sigma_y$. That is to say, Bob’s attack will introduce no error in the test bits for detecting eavesdropping if he obtains the result $|\phi^+\rangle_{B'C}$ or $|\psi^+\rangle_{B'C}$ no matter what the state of the original two photons $B$ and $C$ prepared by Alice is and the bases chosen by the two agents for their photons. If he gets the other two Bell states, which takes place with the probability 50%, he will introduce about 50% error rate in the test bits, but he can hide these errors with cheating. That is, for these outcomes, he tells Alice and Charlie that he gets nothing when he measures the photon $B$ because of the quantum channel losses. So Alice and Charlie cannot detect this eavesdropping if the losses aroused by the quantum channel is more than 50% no matter what the order of the information published do the three participants choose, not the case claimed in the original KKI QSS scheme.

Table I: The states obtained by the three participants Alice, Charlie and Bob.

| Alice  | $|\phi^+\rangle$ | $|\psi^+\rangle$ | $|\Phi^+\rangle$ | $|\Psi^+\rangle$ |
|--------|-----------------|-----------------|-----------------|-----------------|
| Charlie| $|+\rangle$      | $|+\rangle$      | $|-\rangle$      | $|-\rangle$      |
|        | $|-\rangle$      | $|+\rangle$      | $|-\rangle$      | $|-\rangle$      |
|        | $|+\rangle$      | $|-\rangle$      | $|-\rangle$      | $|-\rangle$      |
|        | $|-\rangle$      | $|-\rangle$      | $|-\rangle$      | $|-\rangle$      |

For the other instances used for creating a private key (not for checking eavesdropping), Bob need only perform a Bell-state measurement on the two photons $B$ and $C$ after Alice announces their basis, and then he can get all the information about Alice’s key $K_A$. On the other hand, in the period of declaring bases for the key bits, Bob can announce a fake information about the sequence of the bases $Z$ and $X$, and then measure the photon $B'$ with the same basis as chosen by Charlie and obtain the key $K_C$ after Charlie publishes her bases for the outcomes. In this way, the eavesdropping done by Bob cannot be detected.

So far, we have presented our opaque attack scheme for the dishonest agent to eavesdrop the one-way ECEC QSS fully and freely with the typical model, the KKI QSS scheme. This attack scheme works for other one-way ECEC QSS schemes $\text{2, 3, 4, 5, 6, 7, 8, 13, 14, 15, 16, 17, 18}$, including those for sharing an unknown quantum state $\text{2, 3, 7, 14, 16, 17, 18}$, with or without a little of modification as Bob only measures the photons used for the test bits, not for message before he gets all other information published by Alice and Charlie. Bob does not introduce errors in the test bits but only produces losses. The losses induced by Bob can be, however, hidden in the channel losses. Same as Ref. $\text{22}$, let us suppose that Alice and her agents use an original quantum channel with a total transmission efficiency $\eta$ not exceeding 50%. Typical values of $\eta$ for long-distance experimental quantum communication well fit this range $\text{22, 23}$. For eavesdropping, Bob replaces the original quantum channel by a better one with the transmission efficiency $\eta'$. The probability $P_e$ that Bob can eavesdrop the one-way ECEC QSS freely with this opaque cheating attack scheme is $P_e = \frac{2(\eta' - \eta)}{\eta'}$ when $\eta' < 2\eta$; otherwise, $P_e = 100\%$. In order to keep the transmission efficiency $\eta$ between Alice and Charlie, Bob should filter out $(1 - \eta) \times 100\%$ of the fake photon $C'$ reaching Charlie. In this way the information about the eavesdropping is completely erased from the outcomes obtained by Charlie in the detecting eavesdropping procedure and the message sharing procedure. This means that Bob can eavesdrop fully and freely the information in the one-way ECEC QSS if he replaces the original quantum channel with a better quantum channel whose total transmission efficiency is double of that of original one and controls the balance of transmission efficiencies.

Let us now consider how to improve the one-way ECEC QSS to make it secure. Its insecurity over a lossy quan-
quantum channel in principle arises from two factors. One is that the dishonest agent Bob always controls a photon which is entangled with the other one received by the honest agent before the participants do their eavesdropping check. The other one is that the test eavesdropping procedure is a collective one, i.e., dependent on the information published by the dishonest agent. The first factor gives Bob the chance that he can obtain a correlated outcome with a certain probability in the eavesdropping check procedure and also get correctly all the information obtained by all the participants subsequently (especially in the case for sharing an unknown quantum state). The second factor provides Bob the tools to erase the errors produced by his attack in the test eavesdropping procedure. For improving the security of one-way ECEC QSS, the boss Alice can disentangle the two photons sent from her to her two agents for the outcomes in the test bits. In this way, the eavesdropping check procedures between Alice and Bob, and Alice and Charlie are as same as the Bennett-Brassard 1984 (BB84) QKD scheme[24], which is proven secure for generating a private key[25]. We should confess that the modified one-way ECEC QSS for sharing a private key, to some extent, is not better than two BB84 QKD processes as the latter provides a simple way for three participants to share a private key efficiently without resorting to entanglement at the expense of exchanging a little more classical information. As for the one-way ECEC QSS for sharing an unknown quantum state, the two photons prepared by the boss Alice and sent to her two agents for checking eavesdropping should be in a product state.

In conclusion, we have presented an undetectable opaque cheat attack scheme for the dishonest agent in one-way ECEC QSS to eavesdrop the outcomes obtained by the other agent. This scheme works on the realistic implementation of one-way ECEC QSS if the quantum channel losses cannot be ignored. The dishonest agent first intercepts the photons sent from the sender and stores them. He sends the other agent a fake photon in a Bell state and exploits the quantum losses to hide his action when the fake photon is chosen for detecting eavesdropping. As he can control the efficiency of transmission, he can eavesdrop the QSS freely. Obviously, this attack is efficient in the case there are more than two agents in a QSS scheme with a little modification as well. We also suggest a general way for improving the security of one-way ECEC QSS. It works especially when the QSS schemes are used to share an unknown quantum state.

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Note added- The anonymous referee thinks that there is no security loophole in one-way ECEC QSS schemes if a standard key sifting procedure, in which the events where no photon was detected are removed before the participants perform the common eavesdropping check, is performed. This is true when the QSS schemes are used for sharing a classical information because all the participants measured all the particles including those used for distilling a private key, which will give no chance for the dishonest agent to get the outcomes obtained by the other agent perfectly. However, this way does not work when the QSS schemes are used for sharing an unknown quantum state as the dishonest agent need not measure the particles for sharing the unknown quantum state (he need only measure the particles used for checking eavesdropping), which means that they are insecure in this time even though the participants exploit a standard key sifting procedure to check eavesdropping.

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