COVERING THE BAIRE SPACE BY FAMILIES WHICH ARE NOT FINITELY DOMINATING

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Abstract. It is consistent (relative to ZFC) that each union of \(\max\{b, g\}\) many families in the Baire space \(\omega^\omega\) which are not finitely dominating is not dominating. In particular, it is consistent that for each nonprincipal ultrafilter \(U\), the cofinality of the reduced ultrapower \(\omega^\omega/U\) is greater than \(\max\{b, g\}\). The model is constructed by oracle chain condition forcing, to which we give a self-contained introduction.

1. Introduction

The undefined terminology used in this paper is as in [9, 2]. A family \(Y \subseteq \omega^\omega\) is finitely dominating if for each \(g \in \omega^\omega\) there exist \(k\) and \(f_1, \ldots, f_k \in Y\) such that \(g(n) \leq \max\{f_1(n), \ldots, f_k(n)\}\) for all but finitely many \(n\). The additivity number for classes \(\mathcal{Y} \subseteq \mathcal{Z} \subseteq \mathcal{P}(\omega^\omega)\) with \(\bigcup \mathcal{Y} \notin \mathcal{Z}\) is

\[
\text{add}(\mathcal{Y}, \mathcal{Z}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{Y} \text{ and } \bigcup \mathcal{F} \notin \mathcal{Z}\}.
\]

Let \(\mathcal{D}\) (respectively, \(\mathcal{D}_{\text{fin}}\)) be the collection of all subsets of \(\omega^\omega\) which are not dominating (respectively, finitely dominating). Define

\[
\text{cov}(\mathcal{D}_{\text{fin}}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{D}_{\text{fin}} \text{ and } \bigcup \mathcal{F} = \omega^\omega\}.
\]

It is easy to see that \(\text{add}(\mathcal{D}_{\text{fin}}, \mathcal{D}) = \text{cov}(\mathcal{D}_{\text{fin}})\), so we will use this shorter notation.

In [8] it is pointed out that

\[
\max\{b, g\} \leq \text{cov}(\mathcal{D}_{\text{fin}}),
\]

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the inequality \( b \leq \text{cov}(\mathcal{D}_{\text{fin}}) \) being immediate from the definitions, and the inequality \( g \leq \text{cov}(\mathcal{D}_{\text{fin}}) \) having been implicitly proved in [5, Theorem 2.2]. (For the reader's convenience, we give a short proof for this in Corollary 2.3). In [8] it is shown that in all “standard” forcing extensions (e.g., those appearing in [2, §11]), equality holds. It is conjectured in [8] that this equality is not provable. We prove this conjecture. In fact, we prove a stronger result: Let \( \mathcal{M} \) denote the ideal of meager sets of real numbers.

**Theorem 1.1.** It is consistent (relative to ZFC) that \( \aleph_1 = \text{non}(\mathcal{M}) = g < \text{cov}(\mathcal{D}_{\text{fin}}) = \text{cov}(\mathcal{M}) = c = \aleph_2. \)

The statement of Theorem 1.1 determines the values of almost all standard cardinal characteristics of the continuum in the model witnessing it: If \( \mathcal{N} \) is the ideal of null sets of real numbers, then by provable inequalities (see [9, 2]), we have that \( p, t, h, \text{add}(\mathcal{N}), \text{add}(\mathcal{M}), b, s, \text{cov}(\mathcal{N}), \) and \( \text{non}(\mathcal{M}) \) are all equal to \( \aleph_1 \), and \( \text{cov}(\mathcal{M}), \text{non}(\mathcal{N}), r, d, u, i, \text{cof}(\mathcal{M}), \) and \( \text{cof}(\mathcal{N}) \) are all equal to \( \aleph_2 \) in this model.

In [8] it is shown that for each nonprincipal ultrafilter \( U \) on \( \omega \), \( \text{cov}(\mathcal{D}_{\text{fin}}) \leq \text{cof}(\omega/\omega_U) \).

**Corollary 1.2.** It is consistent (relative to ZFC) that for each nonprincipal ultrafilter \( U \) on \( \omega \), \( \max\{b, g\} < \text{cof}(\omega/\omega_U) \).

This corollary partially extends the closely related Theorems 3.1 and 3.2 of [7], which are proved using the same machinery: Oracle chain condition forcing.

2. **Making \( \text{cov}(\mathcal{D}_{\text{fin}}) \) and \( \text{cov}(\mathcal{M}) \) Large**

From now on, by ultrafilter we always mean a nonprincipal ultrafilter on \( \omega \). We will use the following convenient characterization. For functions \( f, g \in \omega^\omega \) and an ultrafilter \( U \) we write \( f \leq_U g \) for \( \{n : f(n) \leq g(n)\} \in U \).

**Lemma 2.1** ([8]). For each cardinal number \( \kappa \), the following are equivalent:

1. \( \kappa < \text{cov}(\mathcal{D}_{\text{fin}}) \);
2. For each \( \kappa \)-sequence \( \langle (U_\alpha, g_\alpha) : \alpha < \kappa \rangle \) with each \( U_\alpha \) an ultrafilter and each \( g_\alpha \in \omega^\omega \) there exists \( g \in \omega^\omega \) such that for each \( \alpha < \kappa \), \( g_\alpha \leq_U g \).

We first show how this characterization easily implies an assertion made in the introduction.

**Definition 2.2.** For \( A \in [\omega]^\omega \), define the function \( A^+ \in \omega^\omega \) by \( A^+(n) = \min\{k \in A : n < k\} \) for all \( n \).
Corollary 2.3 ([5]). \( g \leq \text{cov}(\mathcal{D}_{\text{fin}}) \).

Proof. We use Lemma 2.1. Assume that \( \kappa < g \), and \( (U_\alpha, g_\alpha) \), \( \alpha < \kappa \), are given with each \( U_\alpha \) an ultrafilter and each \( g_\alpha \in \omega \omega \). We must show that there exists \( g \in \omega \omega \) such that for each \( \alpha < \kappa \), \( g_\alpha \leq U_\alpha \) \( g \). We will use the following "morphism".

Lemma 2.4. For each \( f \in \omega \omega \) and each ultrafilter \( U \),

\[
\mathcal{G}_{U, f} = \{ A \in [\omega]^\omega : f \leq U A^+ \}
\]

is groupwise dense.

Proof. Clearly, \( \mathcal{G}_{U, f} \) is closed under taking almost subsets. Assume that \( \{ [a_n, a_{n+1}) : n \in \omega \} \) is an interval partition of \( \omega \). By merging consecutive intervals we may assume that for each \( n \), and each \( k \in [a_n, a_{n+1}) \), \( f(k) \leq a_{n+2} \).

Since \( U \) is an ultrafilter, there exists \( \ell \in \{0, 1, 2\} \) such that

\[
A_\ell = \bigcup_n [a_{3n+\ell}, a_{3n+\ell+1}) \in U
\]

Take \( A = A_{\ell+2 \mod 3} \). For each \( k \in A_\ell \), let \( n \) be such that \( k \in [a_{3n+\ell}, a_{3n+\ell+1}) \). Then \( f(k) \leq a_{3n+\ell+2} = A^+(k) \). Thus \( A \in \mathcal{G}_{U, f} \).

Thus, we can take \( A \in \bigcap_{\alpha < \kappa} \mathcal{G}_{U_\alpha, g_\alpha} \) and \( g = A^+ \).

How are we going force a large value for \( \text{cov}(\mathcal{D}_{\text{fin}}) \)? If \( \text{cov}(\mathcal{D}_{\text{fin}}) = \aleph_1 \), then by Lemma 2.1 this is witnessed by a sequence \( \langle (U_\alpha, g_\alpha) : \alpha < \aleph_1 \rangle \). To refute a single such witness, we will use the following forcing notion, where \( A_\alpha \in U_\alpha \) for each \( \alpha < \aleph_1 \).

Definition 2.5. Fix an ordinal \( \gamma \). Assume that \( A_\alpha \in [\omega]^\omega \) and \( g_\alpha \in \omega \omega \) for \( \alpha < \gamma \). Define a forcing notion

\[
Q = Q(A_\alpha, g_\alpha : \alpha < \gamma) = \{ (n, h, F) : n \in \omega, h \in [n, \omega], F \in [\gamma]^{\leq n}, \text{ and } (\forall \alpha \in F_1) (\forall n \in [n_1, n_2) \cap A_\alpha) g_\alpha(n) \leq h_2(n) \}
\]

with \( (n_1, h_1, F_1) \leq (n_2, h_2, F_2) \) if \( n_1 \leq n_2, h_2 \upharpoonright n_1 = h_1, F_1 \subseteq F_2, \) and

\[
(\forall \alpha \in F_1) (\forall n \in [n_1, n_2) \cap A_\alpha) g_\alpha(n) \leq h_2(n).
\]

Observe that \( Q \) is \( \sigma \)-centered. \( Q \) is a restricted variant of the Hechler forcing. Advanced readers are recommended to skip the proof of the following lemma, which is the same as for the Hechler forcing.

Lemma 2.6. Assume that \( A_\alpha \in [\omega]^\omega \cap V \) and \( g_\alpha \in \omega \omega \cap V \) for each \( \alpha < \gamma \). Then for \( Q = Q(A_\alpha, g_\alpha : \alpha < \gamma) \), \( V^Q \models (\exists g \in \omega \omega) (\forall \alpha < \gamma) A_\alpha \subseteq^* \{ n : g_\alpha(n) \leq g(n) \} \).
Proof. Assume that $G$ is a $\mathbb{Q}$-generic filter over $V$. Let $g = \bigcup \pi_2[G]$, where $\pi_2$ denotes the projection on the second coordinate. Clearly, $g$ is a partial function from $\omega$ to $\omega$. By density arguments, we have that $g$ is as required. To see this, consider first the sets

$$D_m = \{(n, h, F) \in \mathbb{Q} : m \leq n\}$$

for $m \in \omega$. Each $D_m$ is dense in $\mathbb{Q}$. Assume that $(n, h, F) \in \mathbb{Q}$. If $m \leq n$ then $\lfloor n, m \rfloor = \emptyset$; therefore $(n, h, F) \subseteq (n, h, F \cup \{\alpha\}) \in D_m$. Otherwise, define $h' : m \to \omega$ by $h'(k) = h(k)$ for $k < n$, and $h'(k) = \max\{f_\beta(k) : \beta \in F\}$ for $k \in [n, m)$. Then $(m, h', F)$ is a member of $D_{m,\alpha}$ extending $(n, h, F)$. The density of the sets $D_m$ implies that $\text{dom}(g) = \omega$. Moreover, for each $\alpha < \gamma$ the set

$$E_\alpha = \{(n, h, F) \in \mathbb{Q} : \alpha \in F\}$$

is dense in $\mathbb{Q}$ (for each condition $(n, h, F)$, $(n, h, F \cup \{\alpha\})$ is a stronger condition which belongs to $E_\alpha$). Now fix $\alpha < \gamma$ and choose an element $(n_0, h_0, F_0) \in G \cap E_\alpha$. For each $n \in A_\alpha \setminus n_0$ choose an element $(n_1, h_1, F_1) \in G \cap D_{n+1}$, and a common extension $(n_2, h_2, F_2)$ of $(n_0, h_0, F_0)$ and $(n_1, h_1, F_1)$. As $\alpha \in F_0$ and $n \in [n_0, n_2) \cap A_\alpha$, we have that $g_\alpha(n) \leq g(n)$. Since this holds for each $n \geq n_0$, we have that $A_\alpha \subseteq^* \{n : g_\alpha(n) \leq g(n)\}$. \hfill \Box

Consequently, doing an iteration of forcing notions with the above forcing used cofinally often, with $\gamma = \aleph_1$ and an appropriate bookkeeping will increase $\text{cov}(\mathcal{D}_{\text{fin}})$. We will be more precise in the proof of Theorem 2.9.

Observe that the sets $A_\alpha$ played no special role and in fact we could take $A_\alpha = \omega$ for each $\alpha$ (in this case we obtain a dominating real). However, this freedom to choose $A_\alpha$ will play a crucial role in the sequel, where we would like to make sure that $b$ (or $\text{non}(\mathcal{M})$) and $g$ remain small while we increase $\text{cov}(\mathcal{D}_{\text{fin}})$.

We now make some easy observations concerning our planned forcing. We will construct our model by a finite support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \aleph_2 \rangle$ of c.c.c. forcing notions $\mathbb{Q}_\alpha$ which add reals for cofinally many $\alpha < \aleph_2$. Consequently, $V^{\mathbb{P}}$ satisfies $\mathfrak{c} \geq \aleph_2$, where $\mathbb{P} = \mathbb{P}_{\aleph_2} = \bigcup_{\alpha < \aleph_2} \mathbb{P}_\alpha$. The model $V$ we begin with will satisfy $V = L$ (in fact, $\diamondsuit_{\aleph_1}$ and $\diamondsuit_{\aleph_2}(\text{Sat}^\kappa_1)$, with $\text{Sat}^\kappa_1 = \{\alpha < \aleph_2 : \text{cf}(\alpha) = \aleph_1\}$, are enough). Consequently, $V$ satisfies $|\mathbb{P}| = \aleph_2 = 2^{\aleph_1}$. Since $\mathbb{P}$ satisfies the c.c.c., (nice) $\mathbb{P}$-names for reals are countable and therefore there are at most $|\mathbb{P}|^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ names for reals in $\mathbb{P}$, so $V^{\mathbb{P}} \models \mathfrak{c} = \aleph_2$.

Since we are using a finite support iteration, Cohen reals are introduced cofinally often along the iteration, and this is well known to
imply $\text{cov}(M) \geq \aleph_2$ in the final model (briefly: Each meager set in the final model is contained in an $F_\sigma$, thus Borel, meager set. Each Borel set is coded by a real, and every real appears at a stage $\alpha < \aleph_2$, so Cohen reals added later will not belong to the Borel meager set which is the interpretation of this code, and since this property is absolute, they will not belong to the interpretation in the final model. Since $\aleph_2$ is regular, the codes for $\aleph_1$ many Borel meager sets all appear at an intermediate stage, so their union does not contain Cohen reals added later).

**Corollary 2.7.** In the final model, $\text{cov}(M) = c = \aleph_2$ holds.

Now we show how to impose some more constraints on our iteration $\langle P_\alpha, Q_\alpha : \alpha < \aleph_2 \rangle$ so that in $V^{P_{\aleph_2}}$, $\text{cov}(D_{\text{fin}}) = \aleph_2$. Our exposition follows closely the treatment of names given in [4].

**Choice 2.8.** We fix a $\diamondsuit_{\aleph_2}(S^2_{11})$-sequence $\langle S_\delta : \delta \in S^2_{11} \rangle$ in the ground model. The idea is that stationarily often $S_\delta$ will guess a function

$$f : (\aleph_1 \times \aleph_2) \cup \aleph_1 \to ([\aleph_2]^{\aleph_0})^{\aleph_0}. \quad (\text{So for each } \delta < \aleph_2 \text{ of cofinality } \aleph_1, S_\delta : (\aleph_1 \times \delta) \cup \aleph_1 \to ([\delta]^{\aleph_0})^{\aleph_0}.)$$

We identify $\aleph_2$ with the partial order $P_{\aleph_2}$ we are about to build. Then $[\aleph_2]^{\aleph_0}$ contains all of the maximal antichains. Thus $([\aleph_2]^{\aleph_0})^{\aleph_0}$ contains a name for each subset of $\omega$ (which corresponds to an element of $^{\omega} \omega$). Now any sequence

$$\langle (U_\alpha : g_\alpha) : \alpha < \aleph_1 \rangle$$

in the extension has a ground model function $f : (\aleph_1 \times \aleph_2) \cup \aleph_1 \to ([\aleph_2]^{\aleph_0})^{\aleph_0}$, such that $f(\alpha)$ is a name for $g_\alpha$ and $f(\alpha, \cdot)$ is a name for an enumeration of the elements of $U_\alpha$.

For each $f$ as in Equation (1),

$$\{ \delta \in S^2_{11} : S_\delta = f \upharpoonright \delta \}$$

is stationary in $\aleph_2$. We will inductively define an $\aleph_2$-stage finite support iteration and an injection function $F_\delta : P_\delta \to \aleph_2$ for $\delta < \aleph_2$ such that the range of each $F_\delta$ is an initial segment of $\aleph_2$ which includes $\delta$, and for $\varepsilon < \delta < \aleph_2$, $F_\varepsilon \subseteq F_\delta$.

For $\delta < \aleph_2$ we will denote by $\text{name}(S_\delta)$ the sequence of $\aleph_1$ sets of reals $U_\alpha$ and of $\aleph_1$ reals $g_\alpha$ of the form

$$\langle (\bigcup_{n \in \omega} \{ n \} \times F_\delta^{-1}(S_\delta(\alpha, \xi)(n)) : \xi < \delta \}, \bigcup_{n \in \omega} \{ n \} \times F_\delta^{-1}(S_\delta(\alpha)(n)) : \alpha < \aleph_1 \rangle.$$
At stage $\delta \in S^2_1$ in the construction, if $\models_{\mathbb{P}_\delta} \text{"name}(S_\delta)$ is a sequence of $\aleph_1$ ultrafilters and $\aleph_1$ functions”, then we can take $\mathbb{P}_\delta$-names $A_\alpha$, $\alpha < \aleph_1$, such that $\models_{\mathbb{P}_\delta} A_\alpha \in (U_\alpha) \upharpoonright \delta$, which means $\models_{\mathbb{P}_\delta} \text{"} A_\alpha$ is in the first component of name$(S_\delta)$”.

**Theorem 2.9.** Let $V \models \mathcal{D}(S^2_1)$ and let $\mathbb{P}_{\aleph_2}$ be any forcing as in Choice 2.8. Then $V_{\mathbb{P}_{\aleph_2}} \models \text{cov} (\mathcal{D}_{\text{fin}}) = \aleph_2$.

**Proof.** If $\models_{\mathbb{P}_{\aleph_2}} \text{"} \langle (U_\alpha, g_\alpha) : \alpha < \aleph_1 \rangle$ is a sequence of functions and ultrafilters”, then at club many stages $\delta$ the restriction of the names to $\delta$ is also forced to be a sequence of ultrafilters in $V_{\mathbb{P}_\delta}$. For a proof of this (even in the countable support proper scenario) see [1]. But the restriction of the name to $\delta$ is guessed by name$(S_\delta)$ for stationarily many $\delta$’s in this club. So at such a stage $\delta$ the forcing $\mathbb{Q}_\delta$ adds a function $h$ such that $g_\alpha \leq_{U_\alpha} h$ for all $\alpha < \aleph_1$ and this shows that the sequence was not a witness for $\text{cov} (\mathcal{D}_{\text{fin}}) = \aleph_1$. $\square$

3. Interlude: Oracle chain condition forcing

Usually, the major difficulty in forcing inequalities between combinatorial cardinal characteristics of the continuum is to make sure that those which are required to be smaller (non$(\mathcal{M})$ and $g$ in our case) indeed remain small in the generic extension. In this section we describe one such method, which is suitable for our purposes: Oracle chain condition forcing [6, Chapter IV] (see also [3, 4]).

Oracle chain condition forcing is a method for forcing with $\aleph_2$-stage finite support iteration, in such a way that some prescribed intersections of $\aleph_1$ many (descriptively nice) sets which are empty in an intermediate model remain empty in the final model.

**Definition 3.1.** An oracle (or $\aleph_1$-oracle) is a sequence $\bar{\mathcal{M}} = \langle M_\delta : \delta \text{ limit } < \aleph_1 \rangle$ of countable transitive models of a sufficiently large finite portion of ZFC (henceforth denoted ZFC*), such that for each $\delta, \delta \in M_\delta$ is countable in $M_\delta$, and for each $A \subseteq \aleph_1$, the set

$$\text{Trap}_{\bar{\mathcal{M}}}(A) = \{ \delta < \aleph_1 : \delta \text{ is a limit ordinal, and } A \cap \delta \in M_\delta \}$$

is a stationary subset of $\aleph_1$.

Clearly, $\Diamond$ implies the existence of an oracle. The sets $\text{Trap}_{\bar{\mathcal{M}}}(A)$ generate a filter $\text{Trap}_{\bar{\mathcal{M}}}$, which is normal and proper. Moreover, for each $A, B \subseteq \aleph_1$, there exists $C \subseteq \aleph_1$ such that $\text{Trap}_{\bar{\mathcal{M}}}(C) = \text{Trap}_{\bar{\mathcal{M}}}(A) \cap \text{Trap}_{\bar{\mathcal{M}}}(B)$.

**Notation 3.2.** Assume that $\mathbb{P} \subseteq \mathbb{Q}$ are forcing notions, and $N$ is a set. Then $\mathbb{P} \lessdot_N \mathbb{Q}$ means: Every predense subset of $\mathbb{P}$ which belongs to $N$ is predense in $\mathbb{Q}$. 
Lemma 3.3.

(1) $<_N$ is transitive,

(2) If $N \subseteq N'$, then $P <_N Q$ implies $P <_N Q$;

(3) If $Q = \bigcup_{\alpha<\beta} Q^\alpha$ and $P <_N Q^\alpha$ for each $\alpha$, then $P <_N Q$. \hfill \Box

Definition 3.4. Assume that $\bar{M}$ is an oracle. A forcing notion $P$ satisfies the $\bar{M}$-chain condition if there exists an injection $\iota : P \to \aleph_1$, such that

$$\{ \delta < \aleph_1 : \delta \text{ is a limit ordinal, and } \iota^{-1}[\delta] <_{M_{\iota,\delta}} P \} \in \text{Trap}_{\bar{M}},$$

where $M_{\iota,\delta} = \{ \iota^{-1}[A] : A \subseteq \delta \text{ and } A \in M_{\delta} \}$.

Thus each countable forcing notion satisfies the $\bar{M}$-chain condition, and if $P$ satisfies the $\bar{M}$-chain condition, then $P$ has the c.c.c., and $|P| \leq \aleph_1$. The definition of the $\bar{M}$-chain condition can be extended to forcing notions of cardinality $\aleph_2$ [6, IV.1.5]; however this is not needed here.

Proving the $\bar{M}$-chain condition according to Definition 3.4 is rather inconvenient. We give a useful method to verify the $\bar{M}$-chain condition.

Proposition 3.5. Assume that $\bar{M}$ is an oracle, $P = \bigcup_{\alpha<\aleph_1} P^\alpha$, for each $\alpha < \aleph_1$, $\iota_\alpha$ is a bijection from $P^\alpha$ onto a countable ordinal, and $\langle N_\alpha : \alpha < \aleph_1 \rangle$ is a sequence of countable transitive models of ZFC*, such that the following conditions hold:

(1) For each $\alpha < \beta < \aleph_1$,
   (a) $P^\alpha \subseteq P^\beta$ with $P^\beta \setminus P^\alpha$ countably infinite,
   (b) $\iota_\alpha \subseteq \iota_\beta$; and
   (c) $N_\alpha \subseteq N_\beta$.

(2) For each (large enough) $\alpha < \aleph_1$,
   (a) $\iota_\alpha : P^\alpha \to \omega \alpha$ is bijective,
   (b) $M_{\iota_\alpha}, \langle P^\alpha, \leq_P \rangle, \iota_\alpha \in N_\alpha$; and
   (c) $P^\alpha <_{N_\alpha} P^{\alpha+1}$.

Then $P$ satisfies the $\bar{M}$-chain condition.

Proof. Using Lemma 3.3, we get by induction on $\beta$ that for each $\alpha \leq \beta < \aleph_1$, $P^\alpha <_{N_\alpha} P^\beta$. In particular, $P^\alpha <_{N_\alpha} P$ for each $\alpha$. Define $\iota = \bigcup_{\alpha<\aleph_1} \iota_\alpha$. Then $\iota : P \to \aleph_1$ is an injection.

Assume that $\delta < \aleph_1$ is a (large enough) limit ordinal, and let $\alpha$ be such that $\delta = \omega \alpha$. Then

$$\iota^{-1}[\delta] = \iota^{-1}[\omega \alpha] = \iota_\alpha^{-1}[\omega \alpha] = P^\alpha.$$

Assume that $A \subseteq \delta$, $A \in M_\delta$, and $\iota^{-1}[A] = \iota_\alpha^{-1}[A]$ is predense in $P^\alpha$.

As $\iota_\alpha \in N_\alpha$, $\iota_\alpha^{-1}[A] \in N_\alpha$. As $P^\alpha <_{N_\alpha} P$, $\iota_\alpha^{-1}[A]$ is predense in $P$. 

This shows that for all (large enough) limit ordinals \( \delta < \aleph_1 \), \( \iota^{-1}[\delta] < \mathcal{M}_{\delta} \). Obviously, this implies the requirement in Definition 3.4. \( \square \)

Proposition 3.5 gives us a recipe for verifying the \( \mathcal{M} \)-chain condition: Construct \( \mathbb{P} \) by inductively constructing \( \mathbb{P}^\beta \), such that (1)(a) holds. If \( \beta \) is a limit, take \( \mathbb{P}^\beta = \bigcup_{\alpha < \beta} \mathbb{P}^\alpha \). Otherwise \( \beta = \alpha + 1 \) and \( \mathbb{P}^\alpha \) is defined. Then there exists \( \iota^\beta \) such that (1)(b) and (2)(a) hold. Choose \( N_\alpha \) as in (1)(c) and (2)(b) (and containing some other elements if needed), and use \( N_\alpha \) to define \( \mathbb{P}^\alpha + 1 \) such that (2)(c) holds (this is the only tricky part in the construction). We can simplify the last step in this recipe a bit further.

Lemma 3.6. Assume that \( N \) is a transitive model of \( \text{ZFC}^* \), such that \( \langle \mathbb{P}, \leq \rangle \in N \). Then: \( \mathbb{P} \prec N \mathbb{Q} \) if, and only if, each open dense subset of \( \mathbb{P} \) which belongs to \( N \) is predense in \( \mathbb{Q} \).

Proof. We need to prove (\( \Leftarrow \)). Assume that \( I \in N \) is predense in \( \mathbb{P} \). Then \( I^* = \{ p \in \mathbb{P} : (\exists q \in I) p \geq q \} \in N \), and is open and dense in \( \mathbb{P} \). Thus, \( I^* \) is predense in \( \mathbb{Q} \), and therefore \( I \) is predense in \( \mathbb{Q} \) as well. \( \square \)

Corollary 3.7. (2)(c) in Proposition 3.5 can be replaced by:

(2)(c'') Each open dense subset of \( \mathbb{P}^\alpha \) which belongs to \( N_\alpha \) is predense in \( \mathbb{P}^\alpha + 1 \).

The following theorem exhibits the importance of the oracle chain condition for a single step forcing.

Theorem 3.8 ([6, IV.2.1]). Assume that \( V \models \Diamond \), and \( \varphi_\alpha(x) \), \( \alpha < \aleph_1 \), are \( \Pi^1_2 \) formulas\(^1\) (possibly with real parameters), and

\[
V \models \neg (\exists x) (\forall \alpha < \aleph_1) \varphi_\alpha(x).
\]

If this continues to hold when we add a Cohen real to \( V \), then there exists an oracle \( \bar{M} \) such that for each forcing notion \( \mathbb{P} \) satisfying the \( \mathcal{M} \)-chain condition, \( V^\mathbb{P} \models \neg (\exists x) (\forall \alpha < \aleph_1) \varphi_\alpha(x) \).

The following consequence can be derived from Theorem 3.8.

Lemma 3.9 ([6, IV.2.2]). Assume that \( \Diamond \) holds in \( V \). There is an oracle \( \bar{M} \) in \( V \) such that for each \( \mathbb{P} \) satisfying the \( \mathcal{M} \)-c.c., if, in \( V \), \( A \) is a nonmeager set of reals, then \( A \) is nonmeager in \( V^\mathbb{P} \). Consequently, \( V^\mathbb{P} \models \text{non}(\bar{M}) = \aleph_1 \).

Oracle chain condition can (and is intended to) be used with finite support iterations.

\(^1\)That is, formulas of the form \((\forall a \in \mathbb{R}) (\exists b \in \mathbb{R}) \psi\), where \( \psi \in L_{\aleph_1, \aleph_0} \) (\( L_{\aleph_1, \aleph_0} \) is the extension of the first order language by allowing countable conjunctions).
Lemma 3.10 ([6, IV:3.2–3.3]). Assume that $\bar{M}$ is an oracle.

(1) For a finite support iteration $\langle P_\alpha, Q_\alpha : \alpha < \gamma \rangle$, if each $P_\alpha$ satisfies the $M$-chain condition, then so does $P_\gamma = \bigcup_{\alpha < \gamma} P_\alpha$.

(2) If $|P| = \aleph_1$, and $P$ satisfies the $M$-chain condition (in $V$), then in $V^P$ there is an oracle $\bar{M}^*$ such that for each $Q \in V^P$ satisfying the $\bar{M}^*$-chain condition, $P \ast Q$ satisfies the $\bar{M}$-chain condition (in $V$).

Consider a finite support iteration $\langle P_\alpha, Q_\alpha : \alpha < \aleph_2 \rangle$ of forcing notions, and let $P = \bigcup_{\alpha < \aleph_2} P_\alpha$. Assume that we wish to use Theorem 3.8 for $P$. Then by Lemma 3.10(1), it suffices to make sure that each $P_\alpha$ satisfies the $\bar{M}$-chain condition. By Lemma 3.10(2), this amounts to choosing each $Q_\alpha$ in a way that it satisfies the oracle chain condition for the oracle $\bar{M}^*$ corresponding to the oracle $\bar{M}$ given in Theorem 3.8 for $P_\alpha$.

The nice thing is that we need not worry what exactly are these oracles, as long as we can make sure that for any prescribed oracle $\bar{M}$, the forcing notion $Q_\alpha$ used in the iteration can be chosen so that it satisfies the $M$-chain condition.

We sometimes have to make more than one oracle commitment. In fact, we may wish to add new commitments cofinally often along the iteration (indeed, we do that in the proof of Theorem 5.11). This can be achieved by coding all of the oracles of interest (those introduced in earlier stages of the iteration as well as the new ones required in the current iteration) in a single oracle. Since the length of the iteration is $\aleph_2$, the following lemma tells that this is possible.

Lemma 3.11 ([6, IV.3.1]). If $\bar{M}_\alpha, \alpha < \aleph_1$, are oracles in $V$, then there exists a single oracle $\bar{M}$ such that for each $P$ satisfying the $\bar{M}$-chain condition, $P$ satisfies the $\bar{M}_\alpha$-chain condition for each $\alpha$.

4. Keeping $\text{non}(\mathcal{M})$ small

The main lemma needed to carry out our constructions is the following.

Lemma 4.1. Assume that $\bar{M}$ is an oracle, and for each $\alpha < \aleph_1$, $\mathcal{U}_\alpha$ is an ultrafilter and $g_\alpha \in \omega^\omega$. Then there exist sets $A_\alpha \in \mathcal{U}_\alpha$, $\alpha < \aleph_1$, such that $Q = Q(A_\alpha, g_\alpha : \alpha < \aleph_1)$ (Definition 2.5) satisfies the $\bar{M}$-chain condition.

Proof. We use Proposition 3.5 and the remarks following it (with $P$ replaced by $Q$ everywhere). We choose $A_\alpha$ by induction on $\alpha$. At stage $\alpha$ we define

$$Q^\alpha = Q(A_\beta, g_\beta : \beta < \alpha)$$
We now turn to the problem of keeping Lemma 5.1. Assume that \( \max \{ q \} \) such that:

\[
\forall \alpha \in \mathbb{N}, \exists n, h, F \in \mathbb{Q}^{+} \ni f(n, h, F) = a_{k} = f^{k}(0).
\]

Since \( \mathcal{U}_{\alpha} \) is an ultrafilter, there exists \( \ell \in \{0, 1\} \) such that

\[
A_{\alpha} := \bigcup_{k \in \omega} \{a_{2k+\ell}, a_{2k+1+\ell}\} \in \mathcal{U}_{\alpha}.
\]

It remains to show that this definition guarantees (2)(c), that is, \( \mathbb{Q}^{\alpha} <_{\mathbb{N}} \mathbb{Q}^{\alpha+1} \). We will use Corollary 3.7 for that. Assume that \( D \in \mathbb{N}_{\alpha} \) is an open dense subset of \( \mathbb{Q}^{\alpha} \), and \( p = (n, h, F) \in \mathbb{Q}^{\alpha+1} \setminus \mathbb{Q}^{\alpha} \) (so \( \alpha \in F \)). Define, for each \( m > n \), \( h_{m} : m \to \omega \) by

\[
h_{m}(k) = \begin{cases} h(k) & k < n \\ \max\{g_{\beta}(k) : \beta \in F\} & n \leq k \end{cases}
\]

Then \( (n, h, F) \leq (m, h_{m}, F) \), and in particular \( (n, h, F \setminus \{\alpha\}) \leq (m, h_{m}, F \setminus \{\alpha\}) \). Note that the mapping \( m \mapsto h_{m} \) belongs to \( \mathbb{N}_{\alpha} \).

Define \( f : \omega \to \omega \) by letting \( f(k) \) be the minimal \( m \) such that there exists an element \( (m, h, F) \in D \) which extends \( (k, h_{k}, F \setminus \{\alpha\}) \). Then \( f \in \mathbb{N}_{\alpha} \), so there exists \( k \) such that \( m := f(a_{2k+\ell-1}) < a_{2k+\ell} \).

Let \( q_{0} = (a_{2k+\ell-1}, h_{a_{2k+\ell-1}}, F \setminus \{\alpha\}) \). By the definition of \( f \), there exists \( q_{1} := (m, h, F) \in D \) which extends \( q_{0} \). Let \( q_{2} = (m, h, F \cup \{\alpha\}) \in \mathbb{Q}^{\alpha+1} \).

Then \( q_{1} \leq q_{2} \) since they share the same domain. Since \( q_{1} \in D \), it remains to show that \( (n, h, F) \leq q_{2} \). \( (n, h, F \setminus \{\alpha\}) \leq q_{0} \leq q_{1} \); thus \( (n, h, F \setminus \{\alpha\}) \leq q_{2} \), and hence it suffices to show that for each \( i \in [m, n] \cap A_{\alpha}^{+} \), \( g_{\alpha}(i) \leq h(i) \).

But since \( A_{\alpha} \cap [a_{2k+\ell-1}, a_{2k+\ell}] = \emptyset \), \( [m, n] \cap A_{\alpha} \subseteq [n, a_{2k+\ell-1}] \), and if \( i \in [n, a_{2k+\ell-1}] \), then \( h(i) = h_{a_{2k+\ell-1}}(i) = \max\{g_{\beta}(i) : \beta \in F\} \geq g_{\alpha}(i) \), since \( \alpha \in F \), and we are done.

By Lemma 3.10, Lemma 4.1 will enable us to keep \( \mathbb{N}(\mathcal{M}) \) small. We now turn to the problem of keeping \( \mathfrak{g} \) small.

5. Keeping \( \mathfrak{g} \) small

First we state a sufficient condition for \( \mathfrak{g} \) being small.

**Lemma 5.1.** Assume that \( \{Y_{\zeta} : \zeta < c\} \subseteq [\omega]^{\omega} \), and \( \kappa \) is a cardinal such that:

1. For each meager set \( B \subseteq [\omega]^{\omega} \), \( |\{\zeta : Y_{\zeta} \notin B\}| = c \).
2. For each \( B \in [\omega]^{\omega} \), \( \{|\zeta < c : B \subseteq^{*} Y_{\zeta}\}| < \kappa \).
Then \( g \leq \kappa \).

**Proof.** By a result of Blass [2], \( g \leq \text{cf}(c) \), so we can assume that \( \kappa \leq \text{cf}(c) \). We now define \( \kappa \) sets and then show that they are groupwise dense and that their intersection is empty.

Let \( \langle \tilde{n}^\zeta : \zeta < c \rangle \) list all strictly increasing sequences of natural numbers, each sequence appearing cofinally often. By induction on \( \zeta < c \) we choose \( \varepsilon_\zeta \leq \kappa, \gamma_\zeta < c \) and \( C_\zeta \in [\omega]^{\omega} \) as follows.

If there is some \( \varepsilon < \kappa \) such that for each \( \xi < \zeta \) with \( \varepsilon_\xi = \varepsilon \) we have \([n_1^\xi, n_{i+1}^\xi] \not\subseteq C_\xi\) for all but finitely many \( i \), then we take as \( \varepsilon_\zeta \) the minimal such \( \varepsilon \). By the assumption (1), we can choose \( \gamma_\zeta \) to be the minimal \( \gamma < c \) such that \( \gamma \neq \gamma_\xi \) for all \( \xi < \zeta \) and there are infinitely many \( i \) such that \([n_1^\xi, n_{i+1}^\xi] \subseteq Y_\gamma \). In this case we set \( C_\zeta = \bigcup\{n_1^\xi, n_{i+1}^\xi : i \in \omega, [n_1^\xi, n_{i+1}^\xi] \subseteq Y_\gamma \} \). Otherwise we set \( \varepsilon_\zeta = \kappa \) and \( C_\zeta = \omega \).

For each \( \xi < \kappa \), define

\[
\mathcal{G}_\xi = \{ B \in [\omega]^{\omega} : (\exists \zeta < c) \varepsilon_\zeta \geq \xi \text{ and } B \subseteq^* C_\zeta \}.
\]

We show that each \( \mathcal{G}_\zeta \) is groupwise dense. Clearly, it is closed under almost subsets. Let an increasing sequence \( \tilde{n} \) be given. Then for each \( \nu < \xi \), there is by our construction some \( \zeta(\nu) < c \) such that \( \varepsilon_{\zeta(\nu)} = \nu \) and \([n_1^{\zeta(\nu)}, n_{i+1}^{\zeta(\nu)}] \subseteq C_{\zeta(\nu)} \) for infinitely many \( i \). As \( \kappa \leq \text{cf}(c) \), \( \zeta(*) = \sup\{\zeta(\nu) : \nu < \xi \} < c \). By the choice of \( \langle \tilde{n}^\zeta : \zeta < c \rangle \) there is some \( \beta \in (\zeta(*), c) \) such that \( \tilde{n}^\beta = \tilde{n} \). So \( \varepsilon_\beta \geq \xi \), and \( \bigcup\{n_1^\beta, n_{i+1}^\beta : [n_1^\beta, n_{i+1}^\beta] \subseteq Y_{\gamma_\beta}\} = C_\beta \in \mathcal{G}_\xi \).

To see that \( \bigcap \{ \mathcal{G}_\zeta : \zeta < \kappa \} = \emptyset \), assume that \( B \) is infinite and for each \( \xi, B \in \mathcal{G}_\xi \). Then for each \( \xi < \kappa \), there is \( \beta_\xi < c \) such that \( \varepsilon_{\beta_\xi} = \xi \) and \( B \subseteq^* C_{\beta_\xi} \subseteq Y_{\gamma_{\beta_\xi}} \). Since \( \kappa \) is regular, we can thin out and assume that if \( \xi_1 < \xi_2 \), then \( \varepsilon_{\beta_{\xi_1}} \neq \varepsilon_{\beta_{\xi_2}} \). Thus we have that for \( \xi_1 < \xi_2, \beta_{\xi_1} \neq \beta_{\xi_2} \), and hence \( \gamma_{\beta_{\xi_1}} \neq \gamma_{\beta_{\xi_2}} \). Consequently, \( |\{ \gamma_{\beta_\xi} : \xi < \kappa \}| = \kappa \).

But \( \{ \gamma_{\beta_\xi} : \xi < \kappa \} \subseteq \{ \zeta < c : B \subseteq^* Y_\zeta \} \), contradicting the assumption (2).

As we already stated in the previous sections, we shall use a finite support iteration \( \langle \mathbb{P}_\delta, \mathbb{Q}_\delta, : \delta < \aleph_1 \rangle \) of c.c.c. forcing notions, and choose constant or increasing oracles \( M^\delta \), such that \( \mathbb{P}_\delta \) has the \( M^\delta \)-chain condition for each \( \delta \). We start with a ground model satisfying \( \diamond_{\aleph_1} \) and \( \diamond_{\aleph_1}(S_1^0) \). Let \( \langle S_\delta : \delta \in S_1^0 \rangle \) be a \( \diamond_{\aleph_1}(S_1^0) \)-sequence.

There are three possibilities for \( \mathbb{Q}_\delta \). If \( \text{cf}(\delta) = \aleph_0 \) or if \( \delta \) is a successor, then \( \mathbb{Q}_\delta \) is the Cohen forcing.

If \( \text{cf}(\delta) = \aleph_1 \) and \( \models_{\mathbb{P}_\delta} \) “name(\( S_\delta \)) is a sequence of ultrafilters \( U_\alpha \) and of functions \( g_\alpha, \alpha < \aleph_1 \)”, then we choose \( A_\alpha, \alpha < \aleph_1 \) as in Lemma 4.1 but with additional provisos and force with \( \mathbb{Q}_\delta = \mathbb{Q}(\langle A_\alpha, g_\alpha : \alpha < \aleph_1 \rangle) \).
For the premise of this sentence we shortly say: $S_\delta$ guesses $\langle (U_\alpha, g_\alpha) : \alpha < \aleph_1 \rangle$. Otherwise, we set $Q_\delta = \{0\}$.

**Definition 5.2.** For $\gamma \leq \aleph_2$ we consider the class $\mathcal{K}_\gamma$ of $\gamma$-approximations

$$\langle (\mathbb{P}_\delta, Q_\delta, M_\delta^\delta, W_1, W_2) : \delta < \gamma \rangle$$

with the following properties:

(a) $\langle \mathbb{P}_\delta, Q_\delta : \delta < \gamma \rangle$ is a finite support iteration of partial orders such that for each $\delta < \gamma$, $|\mathbb{P}_\delta| \leq \aleph_1$.

(b) $\langle M_\delta^\delta : \delta < \gamma \rangle$ is a constant sequence of oracles such that for all $\delta$, $\mathbb{P}_\delta$ satisfies the $M_\delta$-chain condition and for $\delta + 1 < \gamma$, $\mathbb{P}_\delta \models "Q_\delta$ satisfies the $(M_\delta^{\delta+1})^{-\text{c.c.}}"$ (as in Lemma 3.10(2)). The constant value of the oracle sequence is some oracle $M$ as in Lemma 3.9, keeping $\text{cov}(\mathcal{M}) = \aleph_1$.

(c) $W_1, W_2 \subseteq \aleph_2 \setminus S_1^2$, $W_1$ and $W_2$ are disjoint and if $\gamma$ is a limit of cofinality $\aleph_1$, then $W_1 \cap \gamma, W_2 \cap \gamma$ are both cofinal in $\gamma$.

(d) If $\beta \in (W_1 \cup W_2) \cap \gamma$ then $Q_\beta$ is the Cohen forcing adding the real $\gamma_\beta \in \omega_2$.

(e) If $\delta \in S_1^2 \cap \gamma$ and $S_\delta$ guesses $\langle (U_\alpha(\delta), g_\alpha(\delta)) : \alpha < \aleph_1 \rangle$, then there is some strictly increasing enumeration $\langle \zeta_\alpha(\delta) : \alpha < \aleph_1 \rangle$ of a cofinal part of $W_2 \cap \delta$, and for every $\alpha < \aleph_1$ there is $\ell_{\zeta_\alpha(\delta)} \in \{0, 1\}$ such that $Y_{\zeta_\alpha(\delta)}^{\ell_{\zeta_\alpha(\delta)}} := r^{-1}_{\zeta_\alpha(\delta)}(\{\ell_{\zeta_\alpha(\delta)}\}) \in U_\alpha$, and $Q_\delta = \mathbb{Q}(Y_{\zeta_\alpha(\delta)}^{\ell_{\zeta_\alpha(\delta)}}, g_\alpha(\delta) : \alpha < \aleph_1)$.

(f) For all $\delta \leq \gamma$, $\mathbb{P}_\delta \models \{\beta \in W_1 \cap \delta : A \subseteq \gamma_\beta \} \text{ is at most countable."}^3$ Here, for $\delta = \gamma$ limit, $\mathbb{P}_\gamma$ is the direct limit of $\langle \mathbb{P}_\beta : \beta < \gamma \rangle$, and for $\delta = \gamma = \beta + 1$, $\mathbb{P}_\gamma = P_\beta \ast Q_\beta$.

With the help of several lemmas we will prove the following.

**Theorem 5.3.** If $V \models \diamondsuit_{\aleph_1}$ and $\diamondsuit_{\aleph_2}(S_1^2)$, then for each $\gamma \leq \aleph_2$, $\mathcal{K}_\gamma$ is not empty.

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2The $\zeta_\alpha(\delta)$, $\alpha < \aleph_1$, chosen here do not have to be coherent when regarding different $\delta$’s and we index them with $\delta$ because we need it. Strictly speaking the $\ell_{\zeta_\alpha(\delta)}$ is a function $\ell_{\zeta_\alpha(\delta)}(\delta)$. And also strictly speaking we should index by $\gamma$ as well, but we are suppressing this because we are anyway only working with end extensions when increasing $\gamma$.

3Here it is $W_1$. We use the Cohens in $W_2$ to build the forcings of type $Q_\delta = \mathbb{Q}(Y_{\zeta_\alpha(\delta)}^{\ell_{\zeta_\alpha(\delta)}}, g_\alpha(\delta) : \alpha < \aleph_1)$ and the Cohens $Y_\zeta^1, \zeta \in W_1$, to build the $Y_\zeta$’s as in Lemma 5.1.
Let $V$ fulfill the premises and let $\mathbb{P}_{\aleph_2}$ be the direct limit of the first components of an $\aleph_2$-approximation. If $G$ is a $\mathbb{P}_{\aleph_2}$-generic filter and $Y_\zeta^1[\mathbb{P}_{\aleph_2}] = Y_\zeta$ for $\zeta \in \mathcal{W}_1$, then we have in the final model a sequence $\langle Y_\zeta : \zeta < \aleph \rangle$ as in Lemma 5.1 with $\kappa = \aleph_1$.

**Corollary 5.4.** $V^\mathbb{P}_{\aleph_2} \models \text{cov}(\mathcal{M}) = q = \aleph_1 < \text{cov}(\mathcal{D}_{\text{fin}}) = \aleph_2$.

We prove Theorem 5.3 by induction on $\gamma$ and we shall work with end extensions. For some $\gamma$’s, one has to work to show item (e). We will do this in our first lemma. For all $\gamma$’s but maybe the successor steps of points not in $S^2_1$, one has to work to show that item (f) can be preserved in the induction. This will be done in the last three lemmas.

**Lemma 5.5.** Consider a successor $\gamma = \delta + 1$, $\delta \in S^2_1$. Given any $\aleph_1$-oracle $(\bar{M}^{\delta+1})^*$, the sequence $\langle \zeta_\alpha(\delta) : \alpha < \aleph_1 \rangle$ can be chosen as in (e) so that the forcings given in item (e) have the $(\bar{M}^{\delta+1})^*$-c.c.

**Proof.** This is a variation of Lemma 4.1. We suppress some of the $\delta$’s. We choose $\langle \zeta_\alpha : \alpha < \aleph_1 \rangle$ enumerating $W_2 \cap \delta$ so that, given the oracle $(\bar{M}^{\delta+1})^* = \langle N_\alpha : \alpha < \aleph_1 \rangle$, the Cohen real $r_{\zeta_\alpha}$ is generic over $N_\alpha$. For this it suffices that the countable model $N_\alpha \in V^{\mathbb{P}_{\aleph_1}}$, which means that $\zeta_\alpha$ just has to be sufficiently large. Let the $a_k$ be chosen as in the proof of Lemma 4.1. Then there are infinitely many $k$ such that

$$r_{\zeta_\alpha}^{-1}(\{\ell_{\zeta_\alpha}\}) \cap [a_{2k+\ell-1}, a_{2k+\ell}) = \emptyset,$$

and as in the proof of Lemma 4.1 this suffices. 

**Choice 5.6.** We start with $\bar{M}$ as described. By Lemma 3.10, all the $\mathbb{P}_\delta$, $\delta \leq \aleph_2$, have the $\bar{M}$-chain condition as soon as we can arrange that all the $\mathbb{Q}_\delta$ have the $(\bar{M})^*$-chain condition in $V^{\mathbb{P}_\delta}$. The Cohen forcing has the $\bar{M}$-chain condition for any $\bar{M}$. The $\mathbb{Q}_\delta$ in the steps $\delta \in S^2_1$ can be chosen by the previous lemma so that they have the $(\bar{M})^*$-c.c.

**Lemma 5.7.** If $\delta \in S^2_1$, $\mathbb{Q}_\delta$ is chosen as in Lemma 5.5, and $\mathbb{P}_\delta$ satisfies (f) of Definition 5.2, then $\mathbb{P}_{\delta+1}$ has the property stated in item (f).

**Proof.** Suppose that $p \Vdash_{\mathbb{P}_{\delta+1}} \text{“} A \in [\omega]^\omega \text{ and } \{\zeta \in \mathcal{W}_1 \cap \delta : A \subseteq \bar{Y}_\zeta^\delta\} = \aleph_1^\omega \text{”}$. and w.l.o.g. $p \Vdash_{\mathbb{P}_{\delta+1}} \text{“} A \in [\omega]^\omega \text{ and } \{\zeta \in \mathcal{W}_1 \cap \delta : A \subseteq \bar{Y}_\zeta^\delta\} \text{ is increasingly enumerated by } \{\zeta_\alpha : \alpha < \aleph_1\} = W_1(A) \text{”}.

We take for $n \in \omega$ a maximal antichain $\{p_{n,i} : i \in \omega\}$ above $p$ deciding the statements $\tilde{n} \in A$ with truth value $t_{n,i}$. Let $C_{n,i} = \{\varepsilon \leq \delta : p_{n,i}(\varepsilon) \neq 1\}$. For $\varepsilon \in C_{n,i} \cap S^2_1$ with $\mathbb{Q}_\varepsilon \neq \{0\}$, let $p_{n,i}(\varepsilon) = (m_{n,i}(\varepsilon), h_{n,i}(\varepsilon), F_{n,i}(\varepsilon))$. Let $F_{n,i}^*(\varepsilon) = \{\zeta_\alpha(\varepsilon) : \alpha \in F_{n,i}(\varepsilon)\}$. We assume that all these are objects not just names. For $\varepsilon \in C_{n,i} \setminus S^2_1$ let
\[ p_{n,i}(\varepsilon) = h_{n,i}(\varepsilon), \quad m_{n,i}(\varepsilon) = |h_{n,i}(\varepsilon)| \] and set the other two components for simplicity zero. Set \( m_{n,i} = \max\{m_{n,i}(\varepsilon) : \varepsilon \in C_{n,i}\} \). Set
\[
\mathcal{C} = \langle (m_{n,i}(\varepsilon), h_{n,i}(\varepsilon), F_{n,i}(\varepsilon), F'_{n,i}(\varepsilon), \langle g_\alpha(\varepsilon) \mid m_{n,i} : \alpha \in F_{n,i}(\varepsilon)\rangle) : \varepsilon \in C_{n,i} \rangle : n, i \in \omega \).

For each \( \beta \in \mathbb{N}_1 \), let \( p_\beta \supseteq \varepsilon \) such that for each \( (F) \beta < \varepsilon \), we thin out further and assume that there are \( (m_{n,i}(\varepsilon), h_{n,i}(\varepsilon), F_{n,i}(\varepsilon), F'_{n,i}(\varepsilon), \langle g_\alpha(\varepsilon) \mid m_{n,i} : \alpha \in F_{n,i}(\varepsilon)\rangle) : \varepsilon \in C_{n,i} \rangle : n, i \in \omega \).

We thin out further and assume that there are \( (m_{n,i}(\varepsilon), h_{n,i}(\varepsilon), F_{n,i}(\varepsilon), F'_{n,i}(\varepsilon), \langle g_\alpha(\varepsilon) \mid m_{n,i} : \alpha \in F_{n,i}(\varepsilon)\rangle) : \varepsilon \in C_{n,i} \rangle : n, i \in \omega \).

These are finite arrays of finite sets.

Now we thin out: First we assume that for some \( k \in \omega \) for all \( \beta < \mathbb{N}_1 \), \( |C_\beta| = k \), \( s_\beta \leq k \). We apply the delta system lemma to \( C_\beta, \beta \in \mathbb{N}_1 \), to get a root \( \tilde{C} \). We assume that \( \delta \in C \), as this is the difficult case. We apply the delta system lemma for each \( \varepsilon \in C \) to the \( F(\varepsilon), \beta \in \mathbb{N}_1 \), and to get a root \( F(\varepsilon) \), and to \( F'_\beta(\varepsilon), \beta \in \mathbb{N}_1 \), and get a root \( F'(\varepsilon) \). We further assume that for each \( \beta \) in the delta system and for all \( \varepsilon \in C \), all \( F'_\beta(\varepsilon) \setminus F(\varepsilon) \) are above \( \max(\cup \varepsilon \in C(\varepsilon)'(\varepsilon)) \cup (C \setminus \delta) \) and same for the primed ones. We thin out further and assume that there are \( (m(\varepsilon), h(\varepsilon), F(\varepsilon)) \) such that for all \( \beta < \mathbb{N}_1 \), for all \( \varepsilon \in C \), \( m_\beta(\varepsilon) = m(\varepsilon), \quad h_\beta(\varepsilon) = h(\varepsilon) \in m(\varepsilon)\omega, \) and for the \( \varepsilon \in C_\beta \setminus C \), the increasingly enumerated \( \varepsilon \)’s in \( C_\beta = \{\varepsilon_i : i < k\} \), are isomorphic to the lexicographically first \( \{\varepsilon_i : i < k\} \), i.e., \( m_\beta(\varepsilon_i) = m(\varepsilon_i), \quad h_\beta(\varepsilon_i) = h(\varepsilon_i) \in m(\varepsilon_i)\omega, \) and we use a delta system argument on the \( F(\varepsilon_i) \) giving a root \( F(\varepsilon_i) \) and again impose on the parts \( F(\varepsilon_i) \setminus F(\varepsilon_i) \), that they have to lie above \( \bigcup_{i < k} F(\varepsilon_i) \) and are all of the same size. The analogous thinning out is done for the primed parts, that have to lie above \( \max(\bigcup_{i < k}(F'(\varepsilon_i)) \cup (C \setminus \delta)) \), by the same \( \beta \) (but depending on \( i \)), and all of the \( \langle F(\varepsilon_i) : i < k \rangle \) shall have the same \( \leq \) or \( \geq \)-relations with the members of \( C_\beta(\varepsilon_i) \). Moreover, if \( \varepsilon \) is a Cohen coordinate in \( C_\beta \), then \( p_\beta(\varepsilon) \) does not depend on \( \beta \).

We let \( m_{max} \) be the the maximum of the \( m(\varepsilon) \) and of the lengths of all the finitely many Cohen coordinates for all \( \beta \) in the delta system. Let \( \prec \) denote the initial segment relation for finite sequences. We thin out further and assume that all the \( R_\beta(m_{max}) \) have the same quantifier free
(\langle \aleph_1, \rangle\text{-type over } \text{Ran}(\mathcal{C}) \cup \text{Ran}(\mathcal{C})). \text{ Speaking about components of five tuples } (m, h, F, F', g) \text{ separately is allowed as well as evaluating } g \text{ and the members of all involved finite sets. There are only countably many quantifier types in this language that can be fulfilled by a (finite) sequence } R_\beta(m_{\max}) \text{ in our delta system.}

Let } G_\delta \text{ be a subset of } \mathbb{P}_\delta \text{ that is generic over } V \text{ such } W^* = \{ \gamma \in W_1(A) \cap \delta : p_\gamma \upharpoonright \delta \in G_\delta \} \text{ is uncountable.}

For } \gamma \in W^*, \text{ let in } V[G_\delta],

\[ B_\gamma = \{ n \in \omega : \exists \bar{p}' \in \mathbb{P}_{\delta+1}, p' \geq p_\gamma, p' \upharpoonright \delta \in G_\delta, \text{ and } p' \vDash \mathbb{P}_{\delta+1} n \in A \}. \]

\[ B_\gamma \subseteq^* \gamma^\xi_\alpha[G], \text{ and the latter is fully evaluated by } G, \text{ because } \xi_\alpha \in W_1 \subseteq \delta + 1 \text{ for } \alpha < \aleph_1, \text{ and } \delta \not\in W_1. \]

We shall show that for } \beta, \gamma \in W^*, B_\beta \cap [\kappa, \infty) = B_\gamma \cap [\kappa, \infty) = B \in V[G]. \text{ Then } B \text{ is a counterexample to } ((\mathbb{P}, Q_\beta, M^\varepsilon, W_1, W_2) : \varepsilon \leq \delta, \beta < \delta) \in K_\delta. \]

Let } ||_{\mathbb{P}_{\delta+1}} \text{ denote the compatibility relation in } \mathbb{P}_{\delta+1}. \text{ If } n \in B_\beta, \text{ then } p_\beta ||_{\mathbb{P}_{\delta+1}} p_{n,i} \text{ for the one } i \text{ such that } p_{n,i} \in G, \text{ and for this } i \text{ we have } t_{n,i} = \text{true}. \text{ The same holds for } n \not\in B_\beta \text{ with false. So our claim that } B_\beta \cap [\kappa, \infty) = B_\gamma \cap [\kappa, \infty) \text{ for all } \beta, \gamma \in W^* \text{ now follows from}

**Claim 5.8.** For all } \beta, \gamma \text{ in } W^*:

\[ p_\beta ||_{\mathbb{P}_{\delta+1}} p_{n,i} \text{ iff } p_\gamma ||_{\mathbb{P}_{\delta+1}} p_{n,i}. \]

**Proof.** The point is the coordinate } \delta, \text{ since the restrictions to } \delta \text{ are in } G_\delta, \text{ and hence compatible. Assume } p_{n,i}(\delta) = (m_{n,i}, h_{n,i}, F_{n,i}), p_\beta(\delta) = (m_\beta, h_\beta, F_\beta), p_\gamma(\delta) = (m_\gamma, h_\gamma, F_\gamma). \text{ We do not write the } \delta \text{ at these points, but will not suppress it completely. We assume that } p_\beta(\delta) \text{ is compatible with } p_{n,i}(\delta).

First case: } m_\beta \geq m_{n,i}. \text{ Then } p_\beta ||_{p_{n,i}} \text{ means } h_\beta \triangleright h_{n,i} \text{ and for all } \alpha \in F_\beta \cup F_{n,i} \text{ for all } m \in [m_{n,i}, m_\beta) \cap Y^\xi_\alpha(\delta), (h_\beta(m) \geq g_\alpha(\delta)(m)). \text{ We have to show that the same holds for } p_\gamma. \text{ First, by our thinning out } m_\beta = m_\gamma, h_\beta = h_\gamma, \text{ and hence } h_\beta \triangleright h_{n,i}, \text{ and } F_\beta \cap F_{n,i} = F_\gamma \cap F_{n,i} .

1 a) We have to show: For all } \alpha \in F_{n,i} \text{ for all } m \in [m_{n,i}, m_\gamma) \cap Y^{\xi_\alpha(\delta)}(h_\gamma(m) \geq g_\alpha(\delta)(m)). \text{ And since } h_\beta = h_\gamma, \text{ for all } \alpha \in F_{n,i} \text{ for all } m \in [m_{n,i}, m_\gamma) \cap Y^{\xi_\alpha(\delta)}(h_\gamma(m) \geq g_\alpha(\delta)(m)). \text{ We have to show that the same holds for } p_\gamma. \text{ First, by our thinning out } m_\beta = m_\gamma, h_\beta = h_\gamma, \text{ and hence } h_\beta \triangleright h_{n,i}, \text{ and } F_\beta \cap F_{n,i} = F_\gamma \cap F_{n,i} .

1 b) We also have to show: For all } \alpha \in F_\gamma \text{ for all } m \in [m_{n,i}, m_\gamma) \cap Y^{\xi_\alpha(\delta)}(h_\gamma(m) \geq g_\alpha(\delta)(m)). \text{ For } \alpha \in F_\gamma \cap F_\beta \text{ the latter requirement is clearly fulfilled, as } h_\beta = h_\gamma. \text{ For the part } F_\gamma \setminus F(\delta) \text{ we need to look closer: Suppose some condition in } p_\gamma \text{ forced something about } Y^{\xi_\alpha(\delta)}. \]
Then $p_\gamma(\zeta_\alpha(\delta)) \neq 1$ and hence $\zeta_\alpha(\delta) \in C_\gamma \cap W_2$. But then because of the indiscernibility over $m_\gamma = m_\beta \leq m_{\max}$ (which is a component of $C$), $\zeta_\alpha(\delta) \in C_\beta$ and hence it is in the root $C$. So $p_\beta$ forced by our thinning out same fact about $Y^k_{\zeta_\alpha(\delta)} \cap m_{\max}$. Hence, for all $\alpha \in F_\gamma$ for all $m \in [m_{n,i}, m_\gamma) \cap Y^k_{\zeta_\alpha(\delta)}$, $(h_\gamma(m) \geq g_\alpha(\delta)(m))$. So, taking 1 a) and 1 b) together, $p_\gamma \| p_{n,i}$.

Second case: $m_\beta \leq m_{n,i}$. Then $h_\beta \triangleleft h_{n,i}$, and $p_\beta \| p_{n,i}$ means that for all $\alpha \in F_\beta \cup F_{n,i}$ for all $m \in [m_\beta, m_{n,i}) \cap Y^k_{\zeta_\alpha(\delta)}$, $(h_\gamma(m) \geq g_\alpha(\delta)(m))$. This latter statement does hold also for $F_\gamma$ instead of $F_\beta$ and $m_\gamma$ instead of $m_\beta$, because $m_\gamma = m_\beta$ and $(F_\beta, (g_\alpha(\delta) \upharpoonright m_{n,i} : \alpha \in F_\beta))$ and $(F_\gamma, (g_\alpha(\delta) \upharpoonright m_{n,i}) : \alpha \in F_\gamma))$ are part of $R_\beta(m_{\max})$ and $R_\gamma(m_{\max})$ and hence indiscernible over $h_{n,i}$ for arguments $m \in Y^k_{\zeta_\alpha(\delta)}$, as for these $m$'s, that are forced to be in a Cohen part, $\zeta_\alpha(\delta) \in C$ and hence by our thinning out we have $m_{\max} \geq m$. Also $h_\gamma \triangleleft h_{n,i}$, and hence $p_\gamma \| p_{n,i}$.

So the claim is proved and with it also Lemma 5.7.

**Lemma 5.9.** (1) If $\cf(\gamma) = \aleph_1$ and $Q$ and $\bar{M}^\gamma$ are as in the previous lemma and if $\langle \mathbb{P}_\beta, Q_\beta, \bar{M}_\beta, W_1, W_2 : \beta < \gamma \rangle \in \mathcal{K}_\gamma$, then

$\langle \mathbb{P}_\beta, Q_\beta, \bar{M}_\beta, W_1, W_2 : \beta < \gamma \rangle \upharpoonright \mathbb{P}_\gamma, Q_\gamma, \bar{M}_\gamma \rangle \in \mathcal{K}_{\gamma+1}$.

(2) If $\cf(\gamma) = \aleph_0$ and if $\langle \mathbb{P}_\beta, Q_\beta, \bar{M}_\beta, W_1, W_2 : \beta < \gamma \rangle \in \mathcal{K}_\gamma$, then

$\langle \mathbb{P}_\beta, Q_\beta, \bar{M}_\beta, W_1, W_2 : \beta < \gamma \rangle \upharpoonright \mathbb{P}_\gamma, C, \bar{M}_\gamma \rangle \in \mathcal{K}_{\gamma+1}$.

(3) If $\cf(\gamma) = \aleph_0$ and if $\langle \mathbb{P}_\beta, Q_\beta, \bar{M}_\beta, W_1, W_2 : \beta < \gamma \rangle \upharpoonright \beta \in \mathcal{K}_\beta$ for each $\beta < \gamma$, then $\langle \mathbb{P}_\beta, Q_\beta, \bar{M}_\beta, W_1, W_2 : \beta < \gamma \rangle \in \mathcal{K}_\gamma$.

(4) If $\cf(\gamma) = \aleph_1$ or $\gamma = \aleph_2$, and if $\langle \mathbb{P}_\beta, Q_\beta, \bar{M}_\beta, W_1, W_2 : \beta < \gamma \rangle \upharpoonright \beta \in \mathcal{K}_\beta$ for each $\beta < \gamma$, then $\langle \mathbb{P}_\beta, Q_\beta, \bar{M}_\beta, W_1, W_2 : \beta < \gamma \rangle \in \mathcal{K}_\gamma$.

**Proof.** (1) This was proved in Lemma 5.7.

(2) If $\mathcal{A}$ is an almost subset of uncountably many $Y_\zeta$'s, then there is some $\gamma_0 < \gamma$ that there are uncountably many such $\zeta$ below $\gamma_0$. $\mathcal{A}$ is possibly a name using the last, new forcing. But this is just Cohen forcing. So there is some finite part of a Cohen condition forcing that $\mathcal{A}$ is in uncountably many $Y_\zeta$'s. But then also the forcing $\mathbb{P}_\gamma$ already contains a name for some infinite $B \subseteq \omega$ almost contained in the intersection of uncountably many $Y_\zeta$'s with $\zeta < \gamma_0$. So $P_\gamma$ does not fulfill property (f) and hence the induction hypothesis is not fulfilled.

(3) First we use the pigeonhole principle for the $Y_\zeta$'s as in the previous item. Then we use the following
Lemma 5.10. Assume

(a) \( \langle \mathbb{P}_n : n \in \omega \rangle \) is a \( \triangleleft \)-increasing sequence of c.c.c. forcing notions with union \( \mathbb{P} \),

(b) \( \mathcal{Y} \) is a set of \( \mathbb{P}_0 \)-names of infinite subsets of \( \omega \),

(c) for \( n \in \omega \) we have \( \Vdash \mathbb{P}_n \kappa = \text{cf}(\kappa) > |\{ Y \in \mathcal{Y} : B \subseteq^* Y \}| \),

whenever \( B \) is a \( \mathbb{P}_n \)-name of an infinite subset of \( \omega \).

Then condition (c) holds for \( \mathbb{P} \) too.

Proof. Since \( \mathbb{P} \) is a c.c.c. forcing notion, also in \( V^\mathbb{P} \) we have \( \kappa \) is a regular cardinal.

If the desired conclusion fails, then we can find a \( \mathbb{P} \)-name \( B \) of an infinite subset of \( \omega \) and a sequence \( \langle (p_\alpha, Y_\alpha, m_\alpha) : \alpha < \kappa \rangle \) such that

(\( \alpha \)) \( m_\alpha \in \omega \),

(\( \beta \)) \( Y_\alpha \in \mathcal{Y} \) without repetitions,

(\( \gamma \)) \( p_\alpha \in \mathbb{P}, p_\alpha \Vdash \mathbb{P} B \setminus m_\alpha \subseteq Y_\alpha \).

Since \( \text{cf}(\kappa) > \aleph_0 \), for some \( n(*) \), \( m(*) \in \omega \) the set \( S = \{ \alpha < \kappa : p_\alpha \in \mathbb{P}_{n(*)}, m_\alpha = m(*) \} \) has cardinality \( \kappa \). We identify it with \( \kappa \).

Now for every large enough \( \alpha \in S \) we have

\[ p_\alpha \Vdash \mathbb{P} \kappa = |\{ \beta \in S : p_\beta \in G_{\mathbb{P}_{n(*)}} \}|. \]

Why? Else for an end segment of \( \alpha < \kappa \) there is \( q_\alpha \geq p_\alpha \) such that for all but \( < \kappa \) many \( \beta \in S \), \( q_\beta \notin G_{\mathbb{P}_{n(*)}} \). That means that for an end segments of \( \alpha < \kappa \), w.l.o.g., for all \( \alpha \in \kappa \), \( \text{Perp}_\alpha := \{ \beta \in S : q_\beta \perp q_\alpha \} \) contains an end segment of \( S \). Then we take the diagonal intersection \( D \) of all these end segments of \( S \). Since \( \kappa \) is regular, \( D \) contains a club in \( \kappa \). But then \( \{ q_\beta : \beta \in D \} \) is an antichain in \( \mathbb{P}_{n(*)} \) of size \( \kappa \).

Contradiction.

Let \( G_{n(*)} \) be a subset of \( \mathbb{P}_{n(*)} \) generic over \( V \), and let \( S_* := \{ \beta \in S : p_\beta \in G_{n(*)} \} \). We choose \( G_{n(*)} \), such that \( |S_*| = \kappa \). We let \( B' = \cap \{ Y_\beta \setminus m(*) : \beta \in S_* \} \). Then in \( V[G_{n(*)}] \), \( B' \) is an infinite subset of \( \omega \) included in \( \kappa \) members of \( \mathcal{Y} \), contradicting the assumption. So Lemma 5.10 is proved.

(4) If \( \mathbb{P}_\delta \) adds some \( A \), then this already comes earlier, say in \( V^{\mathbb{P}_\varepsilon} \), \( \varepsilon < \delta \), because \( A \subseteq \omega \) and because of the c.c.c. If \( A \subseteq^* Y_\zeta \) is forced, then \( \zeta < \varepsilon \). This contradicts the induction hypothesis for \( \mathbb{P}_\varepsilon \). This completes the proof of Lemma 5.9.

The lemmas together give that there is an \( \aleph_2 \)-approximation, and the proof of Theorem 5.3 is completed.
With some extra care our proof can be modified to yield the following (cf. [7, 4]).

**Theorem 5.11.** It is consistent (relative to ZFC) that all of the following assertions hold:

1. Each unbounded set of $\omega$ contains an unbounded subset of size $\aleph_1$,
2. Each nonmeager subset of $\omega$ contains a nonmeager subset of size $\aleph_1$,
3. $\mathfrak{g} = \aleph_1$; and
4. $\text{cov}(\mathcal{D}_{\aleph_0}) = \text{cov}(\mathcal{M}) = \mathfrak{c} = \aleph_2$.

**Proof.** This time we work with a version of $\mathcal{K}_\gamma$ with increasing oracles, which means that the $\mathcal{M}^\varepsilon$-chain condition implies $\mathcal{M}^\delta$-chain condition for $\varepsilon > \delta$ and that $\mathbb{P}_\delta \Vdash \text{"$\mathbb{P}[\delta,\varepsilon)$ has the $\mathcal{M}^{\delta+1}$-c.c."}$, though the initial segment need not yet fulfill it, and the name for this new oracle may not yet have an evaluation in an initial segment $\mathbb{P}_\gamma$, $\gamma < \delta$. The new parts of the oracles take care of the unbounded and the nonmeager families that appear later in the iteration and that are frozen by the next step if their intersection with $V^{|\mathbb{P}_\delta|}$ is guessed by the diamond sequence and happens to be unbounded or nonmeager at the current stage $\delta$: The conservation of the unboundedness and nonmeagerness of the intersection is written into all the oracles from $\delta$ onwards. \hfill $\square$

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