Parts formulas involving the Fourier–Feynman transform associated with Gaussian paths on Wiener space

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Abstract
Park and Skoug established several integration by parts formulas involving analytic Feynman integrals, analytic Fourier–Feynman transforms, and the first variation of cylinder-type functionals of standard Brownian motion paths in Wiener space $C_0[0, T]$. In this paper, using a very general Cameron–Storvick theorem on the Wiener space $C_0[0, T]$, we establish various integration by parts formulas involving generalized analytic Feynman integrals, generalized analytic Fourier–Feynman transforms, and the first variation (associated with Gaussian processes) of functionals $F$ on $C_0[0, T]$ having the form

$$F(x) = f(\langle \alpha_1, x \rangle, \ldots, \langle \alpha_n, x \rangle)$$

for scale-invariant almost every $x \in C_0[0, T]$, where $\langle \alpha, x \rangle$ denotes the Paley–Wiener–Zygmund stochastic integral $\int_0^T \alpha(t) dx(t)$, and $\{\alpha_1, \ldots, \alpha_n\}$ is an orthogonal set of nonzero functions in $L_2[0, T]$. The Gaussian processes used in this paper are not stationary.

Keywords Cameron–Storvick theorem · Gaussian process · Generalized analytic Feynman integral · Generalized analytic Fourier–Feynman transform · First variation

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1 Introduction

The theory of the Fourier–Wiener transform suggested by Cameron and Martin [2,5,7] about 75 years ago now is playing more and more significant role in infinite dimensional analysis, Feynman integration theory, and applications in mathematical physics. The Fourier–Wiener transform and several analogies which are more exquisite have been improved in various research fields on infinite dimensional Banach spaces. For instance, the analytic Fourier–Feynman transform [1,8,14,27,28,30,31,36,38,39,47,48], the sequential Fourier–Feynman transform [9,10,13], and the integral transform [15,22,37,40,41] are developed by many authors. Most of the topics are concentrated on classical and abstract Wiener spaces.

Let $C_0\left[0, T\right]$ denote the one-parameter Wiener space, that is, the space of all real-valued continuous functions $x$ on $\left[0, T\right]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0\left[0, T\right]$ and let $m$ denote the Wiener measure. Then, as it is well-known, $(C_0\left[0, T\right], \mathcal{M}, m)$ is a complete measure space.

In the theory of infinite dimensional analysis, the integration by parts formula is also one of the fundamental tools to analyze the integration of functionals on the infinite dimensional spaces. In [3], Cameron derived an integration by parts formula for the Wiener measure $m$. This is the first infinite dimensional integration by parts formula. In [25], Donsker also established this formula using a different method, and applied it to study Fréchet–Volterra differential equations. The integration by parts formula on $C_0\left[0, T\right]$ introduced in [3] was improved in [11,46,47] to study the parts formulas involving the analytic Feynman integral and the analytic Fourier–Feynman transform (henceforth FFT). Since then the parts formula for the analytic Feynman integral is called the Cameron–Storvick theorem by many mathematicians.

The concept of the generalized Wiener integral (namely, the Wiener integral associated with Gaussian paths) and the generalized analytic Feynman integral (namely, the analytic Feynman integral associated with Gaussian paths) on $C_0\left[0, T\right]$ were introduced by Chung, Park and Skoug [24]. See [44,45] for further work involving the generalized Wiener integral associated with the Gaussian processes. In [24,44,45], the generalized Wiener integral was defined by the Wiener integral

$$\int_{C_0\left[0, T\right]} F(Z_h(x, \cdot))m(dx),$$

where $Z_h(x, \cdot)$ is a Gaussian path given by the stochastic integral

$$Z_h(x, t) = \int_0^t h(s)dx(s) \text{ with } h \in L_2[0, T].$$

For a precise definition of this stochastic integral, see Sect. 2 below. Also the concept of the generalized analytic Feynman integral and the generalized analytic FFT (henceforth GFFT) were more developed based on the generalized Wiener integral (1.1), see [12,17–21,23,29]. If we choose $h \equiv 1$ in (1.2), as a constant function, the generalized Wiener integral (1.1) reduces an ordinary Wiener integral, i.e.,
Park and Skoug [46] established several integration by parts formulas involving analytic Feynman integrals, analytic Fourier–Feynman transforms, and the first variation of cylinder-type functionals of standard Brownian motion paths in the Wiener space $C_0[0, T]$. The purpose of this paper is to establish various integration by parts formulas involving the generalized analytic Feynman integral and the GFFT of functionals in non-stationary Gaussian paths $Z_h(x, \cdot)$ given by (1.2). In Sect. 3 below we illustrate the importance of this topic and the motivation of this paper.

The Wiener process used in [1–3,5,7–11,13–15,22,27,28,30–35,37,39,46–48] is a stationary process. However, the stochastic process $Z(1.2)$, the process used in this paper are non-stationary processes. However, by choosing $h$ this paper are quite a lot more complicated because the Gaussian processes used in this paper are non-stationary processes. However, by choosing $h(t) \equiv 1$ on $[0, T]$ in (1.2), the process $Z_1$ reduces to an ordinary Wiener process on $C_0[0, T]$, and so the expected results on $C_0[0, T]$ are immediate corollaries of the results in this paper.

2 Preliminaries

In this section we first present a brief background and some well-known results about the Wiener space $C_0[0, T]$.

A subset $B$ of $C_0[0, T]$ is said to be scale-invariant measurable [32] provided $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set $N$ is said to be scale-invariant null provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional $F$ is said to be scale-invariant measurable provided $F$ is defined on a scale-invariant measurable set and $F(\rho \cdot)$ is Wiener-measurable for every $\rho > 0$. If two functionals $F$ and $G$ are equal s-a.e., we write $F \approx G$.

The Paley–Wiener–Zygmund (henceforth PWZ) stochastic integral [42] plays a key role throughout this paper. Let $\{\phi_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $L_2[0, T]$, each of whose elements is of bounded variation on $[0, T]$. Then for each $v \in L_2[0, T]$, the PWZ stochastic integral $\langle v, x \rangle$ is defined by the formula

$$\langle v, x \rangle = \lim_{n \to \infty} \int_0^T \sum_{j=1}^n (v, \phi_j)_2 \phi_j(t) dx(t)$$

for all $x \in C_0[0, T]$ for which the limit exists, where $(\cdot, \cdot)_2$ denotes the $L_2$-inner product. For each $v \in L_2[0, T]$, the limit defining the PWZ stochastic integral $\langle v, x \rangle$ is essentially independent of the choice of the complete orthonormal set $\{\phi_n\}_{n=1}^{\infty}$ and it exists for s-a.e. $x \in C_0[0, T]$. If $v$ is of bounded variation on $[0, T]$ then $\langle v, x \rangle$ equals the Riemann–Stieltjes integral $\int_0^T v(t) dx(t)$ for s-a.e. $x \in C_0[0, T]$, and for each $v$ in $L_2[0, T]$, $\langle v, \cdot \rangle$ is a Gaussian random variable on $C_0[0, T]$ with mean zero and variance $\|v\|_2^2$. If $\{\alpha_1, \ldots, \alpha_n\}$ is an orthogonal set of functions in $L_2[0, T]$, then the
random variables, $\langle \alpha_j, x \rangle$'s, are independent. For a more detailed study of the PWZ stochastic integral, see [34,43].

Throughout this paper we let

$$\text{Supp}_2[0, T] = \{ h \in L_2[0, T] : m_L(\text{supp}(h)) = T \} = \{ h \in L_2[0, T] : h \neq 0 \text{ m}_L\text{-a.e. on } [0, T] \}$$

and

$$\text{Supp}_\infty[0, T] = \{ h \in L_\infty[0, T] : h \neq 0 \text{ m}_L\text{-a.e. on } [0, T] \}$$

where $m_L$ denotes Lebesgue measure on $[0, T]$. We note that $\text{Supp}_\infty[0, T] \subset \text{Supp}_2[0, T]$, and for any $h \in \text{Supp}_2[0, T]$, $\|h\|_2 > 0$.

Given any function $h$ in $\text{Supp}_2[0, T]$, let $Z_h : C_0[0, T] \times [0, T] \to \mathbb{R}$ be the stochastic process given by

$$Z_h(x, t) = \langle h \chi_{[0,t]}, x \rangle,$$

(2.1)

where $\chi_{[0,1]}$ denotes the indicator function of the set $[0, t]$. Next, let $\beta_h(t) = \int_0^t h^2(u)du$. The stochastic process $Z_h$ on $C_0[0, T] \times [0, T]$ is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0,T]} Z_h(x, s)Z_h(x, t)\text{m}(dx) = \beta_h(\min\{s, t\}).$$

In addition, by [50, Theorem 21.1], $Z_h(\cdot, t)$ is stochastically continuous in $t$ on $[0, T]$. Also, for any $h_1, h_2 \in \text{Supp}_2[0, T]$,

$$\int_{C_0[0,T]} Z_{h_1}(x, s)Z_{h_2}(x, t)\text{m}(dx) = \int_0^{\min\{s,t\}} h_1(u)h_2(u)du.$$

Of course, as discussed in Sect. 1 above, if $h(t) \equiv 1$ on $[0, T]$, then the process $W$ on $C_0[0, T] \times [0, T]$ given by $(w, t) \mapsto W_t(x) = Z_1(x, t) = x(t)$ is a Wiener process (standard Brownian motion). We note that the coordinate process $Z_1$ is stationary in time, whereas the stochastic process $Z_h$ generally is not. For more detailed studies on the stochastic process $Z_h$, see [12,17–21,23,24,29,44,45].

If $h \in \text{Supp}_2[0, T] \cap BV[0, T]$, then for all $x \in C_0[0, T]$, $Z_h(x, t)$ is continuous in $t$. From the definition of the PWZ stochastic integral, it follows that for each $v \in L_2[0, T]$ and each $h \in \text{Supp}_\infty[0, T]$,

$$\langle v, Z_h(x, \cdot) \rangle = \langle vh, x \rangle$$

(2.2)

for s-a.e. $x \in C_0[0, T]$. Thus, throughout this paper, we require $h$ to be in $\text{Supp}_\infty[0, T]$ rather than simply in $\text{Supp}_2[0, T]$.

Let $\mathbb{C}$, $\mathbb{C}_+$, and $\mathbb{C}_+$ denote the set of complex numbers, complex numbers with positive real part and nonzero complex numbers with nonnegative real part, respectively.
For each $\lambda \in \mathbb{C}$, $\lambda^{1/2}$ denotes the principal square root of $\lambda$; i.e., $\lambda^{1/2}$ is always chosen to have positive real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$ for all $\lambda \in \mathbb{C}_+$.

**Definition 2.1** Let $h$ be a function in $\text{Supp}_2[0, T]$ and let $F$ be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_0[0, T]$ such that

$$
\int_{C_0[0,T]} F(\lambda^{-1/2} \mathcal{Z}_h(x, \cdot)) \, m(dx) = J(h; \lambda)
$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(h; \lambda)$ analytic on $\mathbb{C}_+$ such that $J^*(h; \lambda) = J(h; \lambda)$ for all $\lambda > 0$, then $J^*(h; \lambda)$ is defined to be the generalized analytic Wiener integral (associated with the Gaussian paths $\mathcal{Z}_h(x, \cdot)$) of $F$ over $C_0[0, T]$ with parameter $\lambda$, and for $\lambda \in \mathbb{C}_+$ we write

$$
\int_{C_0[0,T]} \text{anf}_\lambda F(\mathcal{Z}_h(x, \cdot)) \, m(dx) = J^*(h; \lambda).
$$

Let $q \neq 0$ be a real number and let $F$ be a functional such that the generalized analytic Wiener integral, $\int_{C_0[0,T]} \text{anf}_q F(\mathcal{Z}_h(x, \cdot)) \, m(dx)$, exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral (associated with the Gaussian paths $\mathcal{Z}_h(x, \cdot)$) of $F$ with parameter $q$ and we write

$$
\int_{C_0[0,T]} \text{anf}_q F(\mathcal{Z}_h(x, \cdot)) \, m(dx) = \lim_{\lambda \to -iq} \int_{C_0[0,T]} \text{anf}_\lambda F(\mathcal{Z}_h(x, \cdot)) \, m(dx).
$$

Next (see [17,19–21,23,29]) we state the definition of the $L_p$ analytic $\mathcal{Z}_h$-GFFT (namely, the analytic GFFT associated with the Gaussian paths $\mathcal{Z}_h(x, \cdot)$).

**Definition 2.2** Let $h$ be a function in $\text{Supp}_2[0, T]$. For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0, T]$, let

$$
T_{\lambda,h}(F)(y) = \int_{C_0[0,T]} F(y + \mathcal{Z}_h(x, \cdot)) \, m(dx).
$$

For $p \in (1, 2]$ we define the $L_p$ analytic $\mathcal{Z}_h$-GFFT, $T_{q,h}^{(p)}(F)$ of $F$, by the formula,

$$
T_{q,h}^{(p)}(F)(y) = \lim_{\lambda \to -iq} T_{\lambda,h}(F)(y)
$$

if it exists; i.e., for each $\rho > 0$,

$$
\lim_{\lambda \to -iq} \int_{C_0[0,T]} |T_{\lambda,h}(F)(\rho y) - T_{q,h}^{(p)}(F)(\rho y)|^p \, m(dy) = 0
$$
where $1/p + 1/p' = 1$. We define the $L_1$ analytic $Z_{h}$-GFFT, $T_{q,h}^{(1)}(F)$ of $F$ by the formula

$$T_{q,h}^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_{\lambda,h}(F)(y)$$

for s.a.e. $y \in C_0[0, T]$ whenever this limit exists.

We note that for $p \in [1, 2]$, $T_{q,h}^{(p)}(F)$ is defined only s.a.e.. We also note that if $T_{q,h}^{(p)}(F)$ exists and if $F \approx G$, then $T_{q,h}^{(p)}(G)$ exists and $T_{q,h}^{(p)}(G) \approx T_{q,h}^{(p)}(F)$. One can see that for each $h \in \text{Supp}_2[0, T]$, $T_{q,h}^{(p)}(F) \approx T_{q,-h}^{(p)}(F)$, since

$$\int_{C_0[0, T]} F(x) m(dx) = \int_{C_0[0, T]} F(-x) m(dx).$$

Remark 2.3 Note that if $h(t) \equiv 1$ on $[0, T]$, then $Z_{h}(x, t) = x(t)$ for all $x \in C_0[0, T]$. In this case the generalized analytic Feynman integral given by equation (2.3) above and the $L_p$ analytic $Z_{x}$-GFFT, $T_{q,1}^{(p)}(F)$, agree with the previous definitions of the analytic Feynman integral and the analytic FFT, $T_{q}^{(p)}(F)$, see [1,8,11,14,27,28,30,31,33–35,38,39,46–48].

Next we give the definition of the first variation $\delta F$ of a functional $F$. The following definition is due to by Chang et al. [12].

**Definition 2.4** Let $h_1$ and $h_2$ be nonzero functions in $\text{Supp}_2[0, T]$, let $F$ be a Wiener measurable functional on $C_0[0, T]$, and let $w \in C_0[0, T]$. Then

$$\delta_{h_1,h_2} F(x|w) \equiv \delta F(Z_{h_1}(x, \cdot)|Z_{h_2}(w, \cdot))$$

$$= \frac{\partial}{\partial \mu} F(Z_{h_1}(x, \cdot) + \mu Z_{h_2}(w, \cdot)) \bigg|_{\mu=0}$$

(2.4)

(if it exists) is called the first variation of $F$ in the direction $w$.

Remark 2.5 Setting $h_1 = h_2 \equiv 1$ on $[0, T]$, our definition of the first variation reduces to the first variation studied in [3,11,14,35,37,39,46–48]. That is,

$$\delta_{1,1} F(x|w) = \delta F(x|w).$$

Throughout this paper we shall always choose $w$ to be an element of $C_0'[0, T]$ where

$$C_0'[0, T] = \{ w \in C_0[0, T] : w \text{ is absolutely continuous on } [0, T] \text{ with } w' \in L_2[0, T] \}. $$

(2.5)
3 A short survey of Cameron–Storvick theorems

In [3], Cameron introduced the first variation (a kind of Gâteaux derivative) of functionals on \( C_0[0, T] \) and obtained a formula involving the Wiener integral of the first variation. In [11], Cameron and Storvick established a similar result for the analytic Feynman integral of functionals on \( C_0[0, T] \). They also applied their celebrated result to establish the existence of the analytic Feynman integral of unbounded functionals on \( C_0[0, T] \). We start this section by stating the original Cameron–Storvick theorem and the parts formula for the analytic Feynman integral of functionals \( F \) on \( C_0[0, T] \). To do this, in this section we consider the ordinary ‘analytic Feynman integral’ (namely, the Feynman integral associated with the Gaussian paths \( Z_1(x, \cdot) \)), \( \int_{C_0[0,T]} \delta F(x|m(dx)) \), and the Cameron–Storvick’s first variation, \( \delta F \equiv \delta_{1,1} F \), for functionals \( F \) on \( C_0[0, T] \).

Theorem 3.1 Let \( z \in L^2[0, T] \) and let \( w(t) = \int_0^t z(s)ds \). For each \( \rho > 0 \), let \( F(\rho x) \) be Wiener integrable on \( C_0[0, T] \) and let \( F(\rho x) \) have a first variation \( \delta F(\rho x|\rho w) \) for all \( x \in C_0[0, T] \) such that for some positive function \( \eta(\rho) \),

\[
\sup_{|h| \leq \eta(\rho)} |\delta F(\rho x + \rho h w|\rho w)|
\]
is Wiener integrable. Then if either member of the following equation exists, both analytic Feynman integrals below exist, and for each \( q \in \mathbb{R} \setminus \{0\} \),

\[
\int_{C_0[0,T]} \delta F(x|m(dx)) = -iq \int_{C_0[0,T]} \langle z, x \rangle F(x)m(dx).
\]

(3.1)

Remark 3.2 In [11], Cameron and Storvick require \( z \) to be “essentially of bounded variation”, but as was pointed by Cameron [3, p.915] this requirement can be replaced by the requirement that \( z \) be “of class \( L^2[0, T] \)” since all of our Stieltjes integrals are interpreted as PWZ stochastic integrals.

The following integration by parts formula and further applications are investigated in many previous researches. For instance, see [35,46,47].

Theorem 3.3 Let \( w \) be a function in \( C_0[0, T] \), and let \( F \) and \( G \) be scale-invariant measurable functionals on \( C_0[0, T] \). Assume that the first variations in the following equations all exist. Then it follows that

\[
\int_{C_0[0,T]} \left[ F(x)\delta G(x|w) + \delta F(x|w)G(x) \right]m(dx) \equiv -iq \int_{C_0[0,T]} \langle z, x \rangle F(x)G(x)m(dx)
\]

(3.2)

where by \( \equiv \) we mean that if either side exists, both side exist and equality holds.
Using a heuristic use of the Cameron–Storvick theorem (namely equation (3.1)), equation (3.2) is a simple consequence, because

$$\delta(FG)(x|w) = F(x)\delta(G(x|w) + \delta(F(x|w)G(x)$$

for almost all functionals $F$ and $G$ on $C_0[0, T]$. Thus, to establish the equality in (3.2), the authors of the papers [35,46,47] guaranteed the existences of the Feynman integrals and the first variations in the corresponding formulas to the equation (3.2). But the singularities of the Wiener measure [4,6,32], and the unusual behaviors of the analytic Feynman integral and the analytic FFT [8,31,33,34] of functionals on $C_0[0, T]$ make establishing various integration by parts formulas involving the Feynman integral and the FFT very difficult. These are due to the fact that the Wiener measure $m$ is not a quasi-invariant probability measure. It is well known that there is no quasi-invariant measure on infinite dimensional linear spaces (see [49]). Thus the translation theorem (Cameron–Martin theorem) and the Girsanov theorem on infinite dimensional Banach spaces have been studied in the literature. An essential structure hidden in the proof of the Cameron–Storvick theorem is based on the Cameron–Martin translation theorem on Wiener space $C_0[0, T]$.

We consider the class of the (ordinary) $L_p$ analytic FFTs, $\{T_q^{(p)}\}_{q \in \mathbb{R}}$, where the FFT $T_0^{(p)}$ with parameter $q = 0$ denotes the identity transform, i.e., $T_0^{(p)}(F) = F$ for functionals $F$ on $C_0[0, T]$. Then the class $\{T_q^{(p)}\}_{q \in \mathbb{R}}$ of the $L_p$ analytic FFTs forms a commutative group acting on various large classes of functionals on $C_0[0, T]$. We refer to the article [30] for a more detailed study of this topic. In fact, in [30], Huffman, Park and Skoug presented the results with the class of the $L_1$ analytic FFTs, $\{T_q^{(1)}\}_{q \in \mathbb{R}}$, to furnish simple illustrations of the algebraic structure of the classes of FFTs. But, as commended in [30], most of the results hold for the class of the $L_p$ FFTs with $p \in [1, 2]$.

On the other hand, in [17,23], Chang, Choi and Skoug discovered new algebraic structures of the classes of the GFFT associated Gaussian processes. Furthermore, in [20,21], the authors investigated various relationships between the GFFT and the corresponding convolution products. There are many improvements and applications of subjects involving the concepts of the GFFT. As a natural consequence work, it would be interesting to determine if the relationship between the ordinary FFT and the first variation could be extended to the case of the relationship between the GFFT and the general first variation defined by (2.4). Thus, in this paper we also study other properties of the GFFT together with the generalized first variation.

As discussed above, the essential structure of parts formulas on Wiener space is based on the Cameron–Storvick theorem. Thus, to establish our parts formulas involving the generalized Feynman integral and the GFFT, we will present a more general Cameron–Storvick theorem using the above notation.

**Theorem 3.4 ([18])** Let $h_1$ and $h_2$ be functions in $\text{Supp}_2[0, T]$ and given $z \in \text{BV}[0, T]$, let $w_{zh_1} \in C_0[0, T]$ be defined by

$$w_{zh_1}(t) = \int_0^t z(s)h_1(s)ds.$$  \hspace{1cm} (3.3)
Let $F$ be a functional on $C_0[0, T]$ such that $F(Z_{h_1}(x, \cdot))$ is Wiener integrable over $C_0[0, T]$. Furthermore assume that for each $\rho > 0$,

$$\int_{C_0[0,T]} |\delta F(\rho Z_{h_1}(x, \cdot) | \rho Z_{h_2}(w_{z h_1}, \cdot))| m(dx) < +\infty. \quad (3.4)$$

Then, if either member of the following equation exists, both generalized analytic Feynman integrals below exist, and for each $q \in \mathbb{R} \setminus \{0\}$,

$$\int_{C_0[0,T]}^{\text{anf}_q} \delta F(Z_{h_1}(x, \cdot)| Z_{h_2}(w_{z h_1}, \cdot)) m(dx)$$

$$= -i q \int_{C_0[0,T]}^{\text{anf}_q} \langle z, Z_{h_2}(x, \cdot) \rangle F(Z_{h_1}(x, \cdot)) m(dx).$$

**Remark 3.5** In [18], Chang and Choi require $z$ to be “of bounded variation”, but this requirement also can be replaced by the requirement that $z$ be “of class $L_2[0, T]$” since all of our Stieltjes integrals are interpreted as PWZ stochastic integral. Also, the condition (3.4) above can be replaced with the condition: for some $\eta > 0$,

$$\sup_{|h| \leq \eta(\rho)} |\delta F(\rho Z_{h_1}(x, \cdot) + \rho h Z_{h_2}(w_{z h_1}, \cdot))|$$

is Wiener integrable as a function of $x$. Thus, setting $h_1 \equiv 1$ and $h_2 \equiv 1$ in (3.4) yields the formula (3.1). For related work on these conditions, see [16,47].

### 4 Cylinder functionals

Functionals that involve PWZ stochastic integrals are quite common. A functional $F$ on $C_0[0, T]$ is called a cylinder functional if there exists a linearly independent set $\mathcal{V} = \{v_1, \ldots, v_m\}$ of nonzero functions in $L_2[0, T]$ such that

$$F(x) = \psi((v_1, x), \ldots, (v_m, x)), \quad x \in C_0[0, T], \quad (4.1)$$

where $\psi$ is a complex-valued Lebesgue measurable function on $\mathbb{R}^m$.

It is easy to show that for the functional $F$ of the form (4.1), there exists an orthogonal set $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ of nonzero functions in $L_2[0, T]$ such that $F$ is expressed as

$$F(x) = f((\alpha_1, x), \ldots, (\alpha_n, x)), \quad x \in C_0[0, T], \quad (4.2)$$

where $f$ is a complex-valued Lebesgue measurable function on $\mathbb{R}^n$. Thus, there is no loss of generality in assuming that every cylinder functional on $C_0[0, T]$ is of the form (4.2).
For \( h \in \text{Supp}_\infty[0, T] \), let \( \mathcal{Z}_h \) be the Gaussian process given by (2.1) above and let \( F \) be given by equation (4.2). Then by equation (2.2),

\[
F(\mathcal{Z}_h(x, \cdot)) = f(\langle \alpha_1, \mathcal{Z}_h(x, \cdot) \rangle, \ldots, \langle \alpha_n, \mathcal{Z}_h(x, \cdot) \rangle) = f(\langle \alpha_1h, x \rangle, \ldots, \langle \alpha_nh, x \rangle).
\]

**Remark 4.1** Even though the set \( \mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \) of nonzero functions in \( L_2[0, T] \) is orthogonal, the subset \( \mathcal{A}h \equiv \{\alpha h : \alpha \in \mathcal{A}\} \) of nonzero functions in \( L_2[0, T] \) need not be orthogonal. Given an orthogonal set \( \mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \) of nonzero functions in \( L_2[0, T] \), let \( \mathcal{O}_{\text{Supp}_\infty}(\mathcal{A}) \) be the class of all functions \( h \in \text{Supp}_\infty[0, T] \) such that \( \mathcal{A}h \) is orthogonal in \( L_2[0, T] \). Since \( \dim L_2[0, T] = \infty \), infinitely many functions \( h \) exist in \( \mathcal{O}_{\text{Supp}_\infty}(\mathcal{A}) \).

**Example 4.2** For any \( h \in \mathbb{R} \setminus \{0\}, h \in \mathcal{O}_{\text{Supp}_\infty}(\mathcal{A}) \), as a constant function on \( [0, T] \).

**Example 4.3** For any orthogonal set \( \mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \) of nonzero functions in \( L_2[0, T] \), let \( L(S) \) be the subspace of \( L_2[0, T] \) which is spanned by

\[
S = \{\alpha_i\alpha_j : 1 \leq i < j \leq n\},
\]

and let \( L(S)^\perp \) be the orthogonal complement of \( L(S) \). Let

\[
\mathcal{P}_{\text{Supp}_\infty}(\mathcal{A}) = \{h \in \text{Supp}_\infty[0, T] : h^2 \in L(S)^\perp\}.
\]

Since \( \dim L(S) \) is finite, and \( \text{Supp}_\infty[0, T] \) is dense in \( L_2[0, T] \) (\( \text{Supp}_\infty[0, T] \) contains all polynomials on \( [0, T] \)), \( \dim(L(S)^\perp \cap \text{Supp}_\infty[0, T]) = \infty \) and so \( \mathcal{P}_{\text{Supp}_\infty}(\mathcal{A}) \) has infinitely many elements.

Let \( h \) be an element of \( \mathcal{P}_{\text{Supp}_\infty}(\mathcal{A}) \). It is easy to show that \( \|\alpha_jh\|^2_2 > 0 \) for all \( j \in \{1, \ldots, n\} \). From the definition of the \( \mathcal{P}_{\text{Supp}_\infty}(\mathcal{A}) \), we see that for \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \),

\[
(\alpha_i h, \alpha_j h)_2 = \int_0^T \alpha_i(t)\alpha_j(t)h^2(t)dt = 0.
\]

From these, we see that \( \mathcal{A}h \) is an orthogonal set of functions in \( L_2[0, T] \) for any \( h \) in \( \mathcal{P}_{\text{Supp}_\infty}(\mathcal{A}) \), i.e., \( \mathcal{P}_{\text{Supp}_\infty}(\mathcal{A}) \subset \mathcal{O}_{\text{Supp}_\infty}(\mathcal{A}) \).

The following lemma is very useful in order to establish our parts formulas in this paper.

**Lemma 4.4** *(Wiener Integration Theorem)* Let \( G = \{g_1, \ldots, g_n\} \) be an orthogonal set of nonzero functions in \( \text{Supp}_2[0, T] \). Let \( f : \mathbb{R}^n \to \mathbb{C} \) be a Lebesgue measurable function. Then for any \( \rho > 0 \),

\[
\int_{\text{C}_0[0, T]} f(\rho \langle g_1, x \rangle, \ldots, \rho \langle g_n, x \rangle)m(dx) = \left( \prod_{j=1}^n 2\pi \rho^2 \|g_j\|_2^2 \right)^{-1/2} \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) \exp \left\{ -\sum_{j=1}^n \frac{u_j^2}{2\rho^2 \|g_j\|_2^2} \right\} du_1 \cdots du_n.
\]

(4.3)
where by \(\Rightarrow\) we mean that if either side exists, both sides exist and equality holds.

Let \(n\) be a positive integer (fixed throughout this paper) and let \(\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}\) be an orthogonal set of nonzero functions from \((L_2[0, T], \| \cdot \|_2)\). Let \(m\) be a nonnegative integer. Then for \(1 \leq p < +\infty\), let \(\mathcal{B}_A(p; m)\) be the space of all functionals of the form (4.2) for \(\text{s-a.e. } x \in C_0[0, T]\) where all of the \(k\)th-order partial derivatives

\[
 f_{j_1, \ldots, j_k}(u_1, \ldots, u_n) = f_{j_1, \ldots, j_k}(u)
\]

of \(f : \mathbb{R}^n \to \mathbb{R}\) are continuous and in \(L_p(\mathbb{R}^n)\) for \(k \in \{0, 1, \ldots, m\}\) and each \(j_i \in \{1, \ldots, n\}\). Also, let \(\mathcal{B}_A(\infty; m)\) be the space of all functionals of the form (4.2) for \(\text{s-a.e. } x \in C_0[0, T]\) where for \(k \in \{0, 1, \ldots, m\}\), all of the \(k\)th-order partial derivatives \(f_{j_1, \ldots, j_k}(u)\) of \(f\) are in \(C_0(\mathbb{R}^n)\), the space of bounded continuous functions on \(\mathbb{R}^n\) that vanish at infinity.

## 5 Integration by parts formulas for the generalized analytic Feynman integral

In this section we establish integration by parts formulas for the generalized analytic Feynman integral. We start this section with the existence of the generalized analytic Feynman integral associated with Gaussian paths \(\mathcal{Z}_h\) of functionals \(F\) in \(\mathcal{B}_A(p; m)\).

**Theorem 5.1** Let \(\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}\) be an orthogonal set of nonzero functions in \(L_2[0, T]\), let \(p \in [1, +\infty)\) be given, let \(m\) be a nonnegative integer, and let \(F \in \mathcal{B}_A(p; m)\) be given by equation (4.2). Then for any \(h \in \mathcal{O}_{\text{supp } \mathcal{A}}(\mathcal{A})\) and all \(q \in \mathbb{R} \setminus \{0\}\), the generalized analytic Feynman integral associated with the Gaussian paths \(\mathcal{Z}_h(x, \cdot)\) of \(F\) with parameter \(q\) exists and is given by the formula

\[
 \int_{C_0[0, T]}^{\text{ant}_q} F(\mathcal{Z}_h(x, \cdot)) \, \text{m}(dx) = \left( \prod_{j=1}^n \frac{-iq}{2\pi \| \alpha_j h \|_2^2} \right)^{1/2} \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) \exp \left\{ \frac{iq}{2} \sum_{j=1}^n \frac{u_j^2}{\| \alpha_j h \|_2^2} \right\} du_1 \cdots du_n.
\]

**Proof** Using (4.2), (2.2), and (4.3) with \(G\) replaced with \(Ah\), it follows that for all \(\lambda > 0\),

\[
 J_F(h; \lambda) = \int_{C_0[0, T]} f(\lambda^{-1/2} \langle \alpha_1, \mathcal{Z}_h(x, \cdot) \rangle, \ldots, \lambda^{-1/2} \langle \alpha_n, \mathcal{Z}_h(x, \cdot) \rangle) \, \text{m}(dx)
\]

\[
 = \int_{C_0[0, T]} f(\lambda^{-1/2} \langle \alpha_1 h, x \rangle, \ldots, +\lambda^{-1/2} \langle \alpha_n h, x \rangle) \, \text{m}(dx)
\]

\[
 = \left( \prod_{j=1}^n \frac{\lambda}{2\pi \| \alpha_j h \|_2^2} \right)^{1/2} \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n \frac{u_j^2}{\| \alpha_j h \|_2^2} \right\} du_1 \cdots du_n.
\]
For \((\lambda, \mathbf{u}) \in \mathbb{C}_+ \times \mathbb{R}^n\), let
\[
H_{Ah}(\lambda; \mathbf{u}) = \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^{n} \frac{u_j^2}{\|\alpha_j h\|_2^2} \right\}
\]
and for \(\lambda \in \mathbb{C}_+\), let
\[
J_{F}^*(h; \lambda) = \left( \prod_{j=1}^{n} \frac{\lambda}{2\pi \|\alpha_j h\|_2^2} \right)^{1/2} \int_{\mathbb{R}^n} f(\mathbf{u}_1, \ldots, \mathbf{u}_n)H_{Ah}(\lambda; \mathbf{u})\, d\mathbf{u}_1 \cdots d\mathbf{u}_n. \tag{5.2}
\]

Then we see that
(i) for all \(\lambda > 0\), \(J_{F}^*(h; \lambda) = J_{F}(h; \lambda)\),
(ii) for each \(\lambda \in \mathbb{C}_+\), \(H_{Ah}(\lambda; \mathbf{u})\), as a function of \(\mathbf{u}\), is an element of \(L^p(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)\) for all \(p \in [1, +\infty]\), and
(iii) \(|H_{Ah}(\lambda; \mathbf{u})| \leq 1\) for all \((\lambda, \mathbf{u}) \in \tilde{\mathbb{C}}_+ \times \mathbb{R}^n\).

Thus using Hölder’s inequality, we can see that \(|f(\mathbf{u})|H_{Ah}(\lambda; \mathbf{u})\), as a function of \(\mathbf{u}\), is an element of \(L^1(\mathbb{R}^n)\) whenever \(f \in L^p(\mathbb{R}^n)\) for every \(p \in [1, +\infty]\). Hence, using the dominated convergence theorem, it follows that \(J_{F}^*(h; \lambda)\) is a continuous function of \(\lambda\) on \(\mathbb{C}_+\). Clearly, \(H_{Ah}(\lambda; \mathbf{u})\) is analytic on \(\mathbb{C}_+\) as a function of \(\lambda\). Hence, by the Fubini theorem and the Cauchy theorem, we obtain that
\[
\int_{\Delta} J_{F}^*(h; \lambda)\, d\lambda = \left( \prod_{j=1}^{n} \frac{\lambda}{2\pi \|\alpha_j h\|_2^2} \right)^{1/2} \int_{\mathbb{R}^n} f(\mathbf{u}) \int_{\Delta} H_{Ah}(\lambda; \mathbf{u})\, d\lambda\, d\mathbf{u} = 0
\]
for any rectifiable simple closed curve \(\Delta\) lying in \(\mathbb{C}_+\). Thus by the Morera theorem, \(\int_{\mathbb{C}_+[0,T]} F(\mathcal{Z}_h(x, \cdot))m(dx) = J_{F}^*(h; \lambda)\) given by (5.2) is an analytic function of \(\lambda\) throughout \(\mathbb{C}_+\). Finally, by the dominated convergence theorem, it follows equation (5.1).

The following observations are very useful to complete the proof of our main theorems (i.e., Theorems 5.5, 6.5, and 6.6 below).

(1) If we choose \(z \in L^2[0, T]\) and define
\[
w_z(t) = \int_{0}^{t} z(s)\, ds \tag{5.3}
\]
for \(t \in [0, T]\), then \(w_z\) is an element of \(C'_0[0, T]\), see equation (2.5), \(Dw_z = z m_L\)-a.e. on \([0, T]\), where \(Dw(s) = \frac{dw}{ds}(s)\), and for all \(v \in L^2[0, T]\),
\[
\langle v, w_z \rangle = \langle v, Dw_z \rangle = \langle v, z \rangle, \tag{5.4}
\]
where of course \(\langle v, z \rangle_2 = \int_{0}^{T} v(s)z(s)\, ds\).
(2) Let \( h_1 \) and \( h_2 \) be functions in \( \text{Supp}_\infty[0, T] \). Given \( z \in L_2[0, T] \), let \( w_{zh_1} \in C'_0[0, T] \) be given by \((3.3)\) above. In this case, using \((2.2)\) and \((5.4)\), we then also see that

\[
(v, Z_{h_2}(w_{zh_1}, \cdot)) = (vh_2, w_{zh_1}) = (vh_2, zh_1)_2.
\]

(3) Given an orthogonal set \( A = \{\alpha_1, \ldots, \alpha_n\} \) of nonzero functions in \( L_2[0, T] \), let \( p \in [1, +\infty] \) be given, let \( m \) be a positive integer, let \( F \in \mathcal{B}_A(p; m) \). Then, using \((4.2)\) and \((2.2)\), we see that for any function \( h \in \mathcal{O}_{\text{Supp}_\infty}(A) \), \( F(Z_h(x, \cdot)) \) belongs to the space \( \mathcal{B}_{Ah}(p; m) \).

Our next lemma follows directly from the definitions of \( \delta_{h_1, h_2}F \) and \( \mathcal{B}_A(p; m) \).

**Lemma 5.2** Let \( A = \{\alpha_1, \ldots, \alpha_n\} \) be an orthogonal set of nonzero functions in \( L_2[0, T] \), let \( p \in [1, +\infty] \) be given, let \( m \) be a positive integer, let \( F \in \mathcal{B}_A(p; m) \) be given by equation \((4.2)\), and let \( w_z \in C'_0[0, T] \) be given by \((5.3)\). Then for any functions \( h_1 \in \mathcal{O}_{\text{Supp}_\infty}(A) \) and \( h_2 \in \text{Supp}_\infty[0, T] \),

\[
\delta_{h_1, h_2}F(x|w_z) \equiv \delta F(Z_{h_1}(x, \cdot)|Z_{h_2}(w_z, \cdot)) = \sum_{j=1}^n (\alpha_jh_2, z)_2 f_j(\langle \alpha_1h_1, x \rangle, \ldots, \langle \alpha_nh_1, x \rangle) \tag{5.5}
\]

for s-a.e. \( x \in C_0[0, T] \). Furthermore, \( \delta_{h_1, h_2}F(\cdot|w_z) \in \mathcal{B}_{Ah}(p; m - 1) \).

**Lemma 5.3** Let \( A, p, m, F, \) and \( w_z \) be as in Lemma 5.2. For \( p' \in [1, +\infty] \) with \((1/p) + (1/p') = 1\), let \( G \in \mathcal{B}_A(p'; m) \) be given by

\[
G(x) = g(\langle \alpha_1, x \rangle, \ldots, \langle \alpha_n, x \rangle). \tag{5.6}
\]

Define \( R(x) = F(x)G(x) \) for \( x \in C_0[0, T] \). Then \( R \in \mathcal{B}_A(1; m) \), and for any functions \( h_1 \in \mathcal{O}_{\text{Supp}_\infty}(A) \) and \( h_2 \in \text{Supp}_\infty[0, T] \), \( \delta_{h_1, h_2}R(\cdot|w_z) \in \mathcal{B}_{Ah}(1; m - 1) \), as a function of \( x \).

**Proof** Note that \( R(x) = r(\langle \alpha_1, x \rangle, \ldots, \langle \alpha_n, x \rangle) \) where

\[
r(u_1, \ldots, u_n) = f(u_1, \ldots, u_n)g(u_1, \ldots, u_n).
\]

We see that \( R \) is an element of \( \mathcal{B}_A(1; m) \) since all the \( k \)-th order partial derivatives of \( r \) are continuous and in \( L_1(\mathbb{R}^n) \) for \( k \in \{0, 1, \ldots, m\} \) by Hölder’s inequality. The fact that \( \delta R_{h_1, h_2}(x|w_z) \), as a function of \( x \), is an element of \( \mathcal{B}_{Ah}(1; m - 1) \) now follows from Lemma 5.2. \( \square \)

**Remark 5.4** Given \( z \in L_2[0, T] \) and \( h_1 \in \mathcal{O}_{\text{Supp}_\infty}(A) \), let \( w_{zh_1} \in C'_0[0, T] \) be given by \((3.3)\). Then equation \((5.5)\) with \( w_z \) replaced with \( w_{zh_1} \) can be rewritten as
\[ \delta_{h_1,h_2} F(x|w_{zh_1}) = \delta F(Z_{h_1}(x,\cdot)|Z_{h_2}(w_{zh_1},\cdot)) \]
\[ = \sum_{j=1}^{n} (\alpha_j h_2, zh_1) f_j ((\alpha_1 h_1, x), \ldots, (\alpha_n h_1, x)) \quad (5.7) \]

for s-a.e. \( x \in C_0[0,T] \).

In our next theorem we obtain an integration by parts formula for the generalized analytic Feynman integral.

**Theorem 5.5** Let \( A, \rho, m, F, \) and \( G \) be as in Lemma 5.3. Given \( z \in L_2[0,T] \) and \( h_1 \in \mathcal{O}_{\text{Supp}_{\infty}}(A) \), let \( w_{zh_1} \in C_0[0,T] \) be given by (3.3). Then for any function \( h_2 \) in \( \text{Supp}_{\infty}[0,T] \), and all \( q \in \mathbb{R} \setminus \{0\} \), it follows that

\[ \int_{C_0[0,T]}^{\text{anf}_q} \left[ F(Z_{h_1}(x,\cdot)) \delta G(Z_{h_1}(x,\cdot)|Z_{h_2}(w_{zh_1},\cdot)) \right. \\
+ \left. \delta F(Z_{h_1}(x,\cdot)|Z_{h_2}(w_{zh_1},\cdot)) G(Z_{h_1}(x,\cdot)) \right] m(dx) \]
\[ = -iq \int_{C_0[0,T]}^{\text{anf}_q} \langle z, Z_{h_2}(x,\cdot) \rangle F(Z_{h_1}(x,\cdot)) G(Z_{h_1}(x,\cdot)) m(dx). \quad (5.8) \]

**Proof** Again define \( R(x) \equiv F(x)G(x) \) for \( x \in C_0[0,T] \) and let \( r(u_1,\ldots,u_n) \equiv f(u_1,\ldots,u_n)g(u_1,\ldots,u_n) \). Then as noted in Lemma 5.3 and its proof, \( R \in \mathcal{B}_{A}(1;m) \), \( h_1,h_2, R(|w_{zh_1}) \in \mathcal{B}_{Ah_1}(1;m-1) \), and all the \( k \)-th order partial derivatives of \( r \) are continuous and in \( L_1(\mathbb{R}^n) \) for \( k \in \{0,1,\ldots,m\} \). Hence \( R(\rho x) \) is Wiener integrable on \( C_0[0,T] \) for each \( \rho > 0 \). In addition, applying (5.7), it follows that for s-a.e. \( x \in C_0[0,T] \),

\[ \delta_{h_1,h_2} R(x|w_{zh_1}) \equiv \delta R(Z_{h_1}(x,\cdot)|Z_{h_2}(w_{zh_1},\cdot)) \]
\[ = F(Z_{h_1}(x,\cdot)) \delta G(Z_{h_1}(x,\cdot)|Z_{h_2}(w_{zh_1},\cdot)) \]
\[ + \delta F(Z_{h_1}(x,\cdot)|Z_{h_2}(w_{zh_1},\cdot)) G(Z_{h_1}(x,\cdot)) \]
\[ = f((\alpha_1 h_1, x), \ldots, (\alpha_n h_1, x)) \sum_{j=1}^{n} (\alpha_j h_2, zh_1) g_j((\alpha_1 h_1, x), \ldots, (\alpha_n h_1, x)) \]
\[ + g((\alpha_1 h_1, x), \ldots, (\alpha_n h_1, x)) \sum_{j=1}^{n} (\alpha_j h_2, zh_1) f_j((\alpha_1 h_1, x), \ldots, (\alpha_n h_1, x)). \quad (5.9) \]

Now since \( fg_j \) and \( gf_l \) are all continuous and in \( L_1(\mathbb{R}^n) \) for \( j,l \in \{1,2,\ldots,n\} \), it is quite easy to see that \( \delta R(\rho Z_{h_1}(x,\cdot)|\rho Z_{h_2}(w_{zh_1},\cdot)) \), as a function of \( x \), is Wiener integrable for all \( \rho > 0 \). In addition, \( \delta R(Z_{h_1}(x,\cdot)|Z_{h_2}(w_{zh_1},\cdot)) \) is analytic Feynman integrable which can be seen by integrating the right-hand side of (5.9) term by term.
For example, applying (4.3), it follows that for any $j \in \{1, \ldots, n\}$,
\[
\int_{C_0[0,T]} \frac{\an f_q}{\sqrt{2\pi}} \left( \prod_{j=1}^{n} \frac{-i q}{2\pi \|\alpha_j h_1\|^2} \right) f(\langle \alpha_1 h_1, x \rangle, \ldots, \langle \alpha_n h_1, x \rangle)(\alpha_j h_2, z h_1) m(dx)
\]
\[
= (\alpha_j h_2, z h_1) \left( \prod_{j=1}^{n} \frac{-i q}{2\pi \|\alpha_j h_1\|^2} \right)^{1/2}
\times \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) g_j(u_1, \ldots, u_n) \exp \left\{ \frac{iq}{2} \sum_{j=1}^{n} \frac{u_j^2}{\|\alpha_j h_1\|^2} \right\} du_1 \cdots du_n
\]
since $f(u_1, \ldots, u_n) g_j(u_1, \ldots, u_n)$ is continuous and in $L_1(\mathbb{R}^n)$. Thus (5.8) follows from Theorem 3.4 above.

By choosing $p = 2$ and $F = G$ in Theorem 5.5, we obtain the following corollary.

**Corollary 5.6** Let $A$, $m$, $z$, $h_1$, and $w_{zh_1}$ be as in Theorem 5.5. Let $F \in B_A(2; m)$ be given by (4.2). Then for any function $h_2$ in $\text{Supp} \infty[0,T]$, and all $q \in \mathbb{R} \backslash \{0\}$, it follows that
\[
\int_{C_0[0,T]} \frac{\an f_q}{\sqrt{2\pi}} F(\mathcal{Z}_{h_1}(x, \cdot)) \delta F(\mathcal{Z}_{h_1}(x, \cdot) \mathcal{Z}_{h_2}(w_{zh_1}, \cdot)) m(dx)
\]
\[
= -\frac{iq}{2} \int_{C_0[0,T]} \frac{\an f_q}{\sqrt{2\pi}} \langle z, \mathcal{Z}_{h_2}(x, \cdot) \rangle [F(\mathcal{Z}_{h_1}(x, \cdot))]^2 m(dx).
\]

### 6 Integration by parts formulas involving generalized analytic Fourier–Feynman transforms

In this section we establish integration by parts formulas involving the $\mathcal{Z}_h$-GFTTs. We start this section with the existence theorem of the $L_p$ analytic $\mathcal{Z}_h$-GFFT of functionals $F$ in $B_A(p; m)$.

**Theorem 6.1** Let $A = \{\alpha_1, \ldots, \alpha_n\}$ be an orthogonal set of nonzero functions in $L_2[0,T]$, let $p \in [1, 2]$ be given, let $m$ be a nonnegative integer, and let $F \in B_A(p; m)$ be given by equation (4.2). Then for any $k \in O_{\text{Supp} \infty}(A)$, and all $q \in \mathbb{R} \backslash \{0\}$, the $L_p$ analytic $\mathcal{Z}_k$-GFFT $T_{q,k}^{(p)}(F)$ of $F$ exists and is given by the formula
\[
T_{q,k}^{(p)}(F)(y) = \left( \prod_{j=1}^{n} \frac{-i q}{2\pi \|\alpha_j k\|^2} \right)^{1/2} \int_{\mathbb{R}^n} f(u_1, \ldots, u_n)
\times \exp \left\{ \frac{iq}{2} \sum_{j=1}^{n} \frac{[u_j - \langle \alpha_j, y \rangle]^2}{\|\alpha_j k\|^2} \right\} du_1 \cdots du_n
\]
\[
= \left( \prod_{j=1}^{n} \frac{-i q}{2 \pi \| \alpha_j k \|_2^2} \right)^{1/2} \int_{\mathbb{R}^n} f(u_1 + (\alpha_1, y), \ldots, u_n + (\alpha_n, y)) \\
\times \exp \left\{ i q \frac{1}{2} \sum_{j=1}^{n} \frac{u_j^2}{\| \alpha_j k \|_2^2} \right\} du_1 \cdots du_n
\] 

(6.1)

for s-a.e. \( y \in C_0[0, T] \). Furthermore, \( T_{q, k}^{(p)}(F) \in B_{\mathcal{A}}(p'; m) \) where \((1/p) + (1/p') = 1\).

**Proof** First, in the case \( m = 0 \), the proof given in [19, Theorems 4.7 and 4.8] with the current hypotheses on \( F \) and \( k \) also works here. Now let an orthogonal set \( \mathcal{A} = \{ \alpha_1, \ldots, \alpha_n \} \) of nonzero functions in \( L_2[0, T] \), \( p \in [1, 2], m \in \{1, 2, \ldots\} \) be given and let \( F \in B_{\mathcal{A}}(p; m) \). Since \( B_{\mathcal{A}}(p; m) \subset B_{\mathcal{A}}(p; 0) \), we know that \( T_{q, k}^{(p)}(F) \) exists and is given by equation (6.1). The proof that \( T_{q, k}^{(p)}(F) \) behaves similarly to the proof in [19] for the case \( m = 0 \).

In view of Theorems 5.5 and 6.1, we get the following corollary.

**Corollary 6.2** Let \( \mathcal{A} = \{ \alpha_1, \ldots, \alpha_n \} \) be an orthogonal set of nonzero functions in \( L_2[0, T] \), let \( m \) be a positive integer, and given \( z \in L_2[0, T] \) and \( h_1 \in \mathcal{O}_{\text{Supp}_\infty}(\mathcal{A}), \), let \( w_{z_1} \in C_0[0, T] \) be given by (3.3). Let \( F \) and \( G \) in \( B_{\mathcal{A}}(2; m) \) be given by (4.2) and (5.6), respectively. Then for any functions \( k_1, k_2 \in \mathcal{O}_{\text{Supp}_\infty}(\mathcal{A}), h_2 \in \text{Supp}_\infty[0, T], \) and any \( q_1, q_2, q_3 \in \mathbb{R} \setminus \{0\}, \) it follows that

\[
\int_{C_0[0, T]} \mathcal{T}_{q, k}^{(2)}(F)(Z_{h_1}(x, \cdot)) \delta \mathcal{T}_{q, k}^{(2)}(G)(Z_{h_1}(x, \cdot)) \mathbb{m}(dx) \\
+ \mathcal{T}_{q, k}^{(2)}(F)(Z_{h_1}(x, \cdot)) \mathcal{T}_{q, k}^{(2)}(G)(Z_{h_1}(x, \cdot)) \mathbb{m}(dx) \\
= -i q_3 \int_{C_0[0, T]} \mathcal{T}_{q, k}^{(2)}(F)(Z_{h_1}(x, \cdot)) \mathcal{T}_{q, k}^{(2)}(G)(Z_{h_1}(x, \cdot)) \mathbb{m}(dx).
\]

In our next theorem we show that the transform with respect to the first argument of the variation equals the variation of the transform.

**Theorem 6.3** Let \( \mathcal{A} = \{ \alpha_1, \ldots, \alpha_n \} \) be an orthogonal set of nonzero functions in \( L_2[0, T] \), let \( p \in [1, 2] \) be given, let \( m \) be a positive integer, and let \( F \in B_{\mathcal{A}}(p; m) \) be given by equation (4.2). Also, let \( w_z \in C_0[0, T] \) be given by (5.3) above. Then for any functions \( h_1 \in \mathcal{O}_{\text{Supp}_\infty}(\mathcal{A}), \) \( (h_2, k) \subset \text{Supp}_\infty[0, T] \) with \( kh_1 \in \mathcal{O}_{\text{Supp}_\infty}(\mathcal{A}) \) (or \( k \in \mathcal{O}_{\text{Supp}_\infty}(\mathcal{A}h_1) \)), all \( q \in \mathbb{R} \setminus \{0\}, \) and s-a.e. \( y \in C_0[0, T], \) it follows that

\[
T_{q, k}^{(p)}(\delta_{h_1, h_2} F(\cdot|w_z))(y) = \delta_{h_1, h_2} T_{q, k}^{(p)}(F)(y|w_z)
\] 

(6.2)
which, as a function of \( y \), is an element of \( B_{A_{h_1}}(p'; m - 1) \). Also, both expressions in (6.2) are given by the expression

\[
\left( \prod_{j=1}^{n} \frac{-iq}{2\pi \| \alpha_j kh_1 \|^2} \right)^{1/2} \int_{\mathbb{R}^n} \sum_{j=1}^{n} (\alpha_j h_2, z)_2 \\
\times f_j(u_1 + \langle \alpha_1 h_1, y \rangle, \ldots, u_n + \langle \alpha_n h_1, y \rangle) \exp \left\{ \frac{iq}{2} \sum_{j=1}^{n} \frac{u_j^2}{\| \alpha_j kh_1 \|^2} \right\} du_1 \cdots du_n.
\]

(6.3)

**Proof** First, using (2.4) with \( F \), \( x \) and \( w \) replaced with \( T_{q,kh_1}^{(p)}(F) \), \( y \) and \( w_z \), and (6.1) with \( k \) and \( y \) replaced with \( kh_1 \) and \( Z_{h_1}(y, \cdot) \), respectively, (2.2), and (5.4), we obtain that

\[
\delta_{h_1,h_2} T_{q,kh_1}^{(p)}(F)(y|w_z) = \delta T_{q,kh_1}^{(p)}(F)(Z_{h_1}(y, \cdot)\|Z_{h_2}(w_z, \cdot))
\]

\[
= \frac{\partial}{\partial \mu} T_{q,kh_1}^{(p)}(F)(Z_{h_1}(y, \cdot) + \mu Z_{h_2}(w_z, \cdot)) \bigg|_{\mu=0}
\]

\[
= \left( \prod_{j=1}^{n} \frac{-iq}{2\pi \| \alpha_j kh_1 \|^2} \right)^{1/2} \int_{\mathbb{R}^n} \frac{\partial}{\partial \mu} f(u_1 + \langle \alpha_1 h_1, y \rangle + \mu \langle \alpha_1 h_2, w_z \rangle, \ldots, u_n + \langle \alpha_n h_1, y \rangle + \mu \langle \alpha_n h_2, w_z \rangle) \bigg|_{\mu=0}
\]

\[
\times \exp \left\{ \frac{iq}{2} \sum_{j=1}^{n} \frac{u_j^2}{\| \alpha_j kh_1 \|^2} \right\} du_1 \cdots du_n
\]

(6.4)

The second equality of (6.4) follows from the fact that \( T_{q,k}^{(p)}(F) \) is in \( B_{A}(p'; m) \), and Theorem 2.27 in [26].

Next, using (5.5), it follows that

\[
T_{q,k}^{(p)}(\delta_{h_1,h_2} F(\cdot|w_z))(y)
\]

\[
= \int_{C_0[0,T]} \delta_{h_1,h_2} F(y + Z_k(x, \cdot)|w_z)m(dx)
\]

\[
= \int_{C_0[0,T]} \delta F(Z_{h_1}(y + Z_k(x, \cdot, \cdot)|Z_{h_2}(w_z, \cdot))m(dx)
\]
A close examination of the proof of Theorem 6.3 shows that \( \delta_{h_1, h_2} T_{q, k h_1}^{(p)}(F)(\cdot|w) \) is an element of \( \mathcal{B}_{Ah_1}(p'; m - 1) \), since \( T_{q, k h_1}^{(p)}(F)(\cdot|w) \) is an element of \( \mathcal{B}_{A}(p'; m) \).

**Remark 6.4** A close examination of the proof of Theorem 6.3 shows that \( \delta_{h_1, h_2} T_{q, k h_1}^{(p)}(F)(\cdot|w) \) is an element of \( \mathcal{B}_{Ah_1}(p'; m - 1) \) and is given by the expression (6.3) with \( k h_1 \) replaced with \( k \) for s.a.e. \( y \in C_0[0, T] \).

In our next theorems we obtain part formulas involving GFFTs.

**Theorem 6.5** Let \( A, m, z, h_1, w_{zh_1}, F, \) and \( G \) be as in Corollary 6.2. Then for any functions \( h_2, k \in \text{Supp}_{\infty}[0, T] \) with \( kh_1 \in \mathcal{O}_{\text{Supp}_{\infty}(A)} \) (or \( k \in \mathcal{O}_{\text{Supp}_{\infty}(Ah_1)} \)), and all \( q_1, q_2, q_3 \in \mathbb{R} \setminus \{0\} \), it follows that

\[
\int_{C_0[0, T]}^{\text{anf}_{q_3}} \left[ T_{q_1, k h_1}^{(2)}(F)(Z_{h_1}(x, \cdot)) \delta T_{q_2, k h_1}^{(2)}(G)(Z_{h_1}(x, \cdot)\mid Z_{h_2}(w_{zh_1}, \cdot)) \\
+ \delta T_{q_1, k h_1}^{(2)}(F)(Z_{h_1}(x, \cdot)\mid Z_{h_2}(w_{zh_1}, \cdot)) T_{q_2, k h_1}^{(2)}(G)(Z_{h_1}(x, \cdot)) \right] \text{d}(dx)
\]

\[
= -i q_3 \int_{C_0[0, T]}^{\text{anf}_{q_3}} (z, Z_{h_2}(x, \cdot)) T_{q_1, k h_1}^{(2)}(F)(Z_{h_1}(x, \cdot)) T_{q_2, k h_1}^{(2)}(G)(Z_{h_1}(x, \cdot)) \text{d}(dx).
\]

(6.5)

**Proof** For \( x \in C_0[0, T] \), let \( R(x) = T_{q_1, k h_1}^{(2)}(F)(x) T_{q_2, k h_1}^{(2)}(G)(x) \). Then by Theorem 6.1, \( T_{q_1, k h_1}^{(2)}(F) \) and \( T_{q_2, k h_1}^{(2)}(G) \) are in \( \mathcal{B}_{A}(2; m) \). Therefore, by Lemma 5.3, \( R \) is in \( \mathcal{B}_{A}(1; m) \). Moreover, by Lemma 5.2, \( \delta_{h_1, h_2} R(x|w_{zh_1}) \), as a function of \( x \), is an element of \( \mathcal{B}_{Ah_1}(1; m - 1) \). Thus equation (6.5) follows from Theorem 5.5 with \( F \) and \( G \) replaced with \( T_{q_1, k h_1}^{(2)}(F) \) and \( T_{q_2, k h_1}^{(2)}(G) \), respectively.

**Theorem 6.6** Let \( A, p, m, F, G, z, h_1, \) and \( w_{zh_1} \) be as in Theorem 5.5. Then for any functions \( h_2, k \in \text{Supp}_{\infty}[0, T] \) with \( kh_1 \in \mathcal{O}_{\text{Supp}_{\infty}(A)} \) (or \( k \in \mathcal{O}_{\text{Supp}_{\infty}(Ah_1)} \)), and all \( q_1, q_2 \in \mathbb{R} \setminus \{0\} \), it follows that

\[
\int_{C_0[0, T]}^{\text{anf}_{q_2}} \left[ F(Z_{h_1}(x, \cdot)) \delta T_{q_1, k h_1}^{(p)}(G)(Z_{h_1}(x, \cdot)\mid Z_{h_2}(w_{zh_1}, \cdot)) \\
+ \delta F(Z_{h_1}(x, \cdot)\mid Z_{h_2}(w_{zh_1}, \cdot)) T_{q_1, k h_1}^{(p)}(G)(Z_{h_1}(x, \cdot)) \right] \text{d}(dx)
\]

\[
= -i q_2 \int_{C_0[0, T]}^{\text{anf}_{q_2}} (z, Z_{h_2}(x, \cdot)) F(Z_{h_1}(x, \cdot)) T_{q_1, k h_1}^{(p)}(G)(Z_{h_1}(x, \cdot)) \text{d}(dx).
\]

(6.6)
Proof For $x \in C_0[0, T]$, let $R(x) = F(x)T_{q_1,kh_1}^{(p)}(G)(x)$. Again, since $T_{q_1,kh_1}^{(p)}(G)$ is in $B_A(p'; m)$ and $F$ is in $B_A(p; m)$, it follows that $R$ belongs to $B_A(1; m)$, and as a function of $x$, $\delta_{h_1,h_2}R(x|w_{zh_1})$ belongs to $B_A(1; m - 1)$. Thus equation (6.6) follows from Theorem 5.5 with $G$ replaced with $T_{q_1,kh_1}^{(p)}(G).$ □

Choosing $F = G$ in Theorem 6.5 above, we obtain the following corollary.

Corollary 6.7 Let $A, m, z, h_1, w_{zh_1},$ and $F$ be as in Corollary 6.2. Then for any functions $h_2, k \in \text{Supp}_\infty[0, T]$ with $kh_1 \in \mathcal{O}_{\text{Supp}_\infty}(A)$ (or $k \in \mathcal{O}_{\text{Supp}_\infty}(Ah_1)$), and all $q_1, q_2 \in \mathbb{R} \setminus \{0\}$, it follows that

$$
\int_{C_0[0,T]} \text{anf}_{q_2} T_{q_1,kh_1}^{(2)}(F)(\mathcal{Z}_{h_1}(x, \cdot)) \delta T_{q_1,kh_1}^{(2)}(F)(\mathcal{Z}_{h_1}(x, \cdot)) \mathcal{Z}_{h_2}(w_{zh_1}, \cdot) \mathcal{m}(dx)
$$

$$\quad = -\frac{i q_2}{2} \int_{C_0[0,T]} \langle z, \mathcal{Z}_{h_2}(x, \cdot) \rangle \left[ T_{q_1,kh_1}^{(2)}(F)(\mathcal{Z}_{h_1}(x, \cdot)) \right]^2 \mathcal{m}(dx).
$$

Choosing $F = G$ in Theorem 6.6 above, we also obtain the following corollary.

Corollary 6.8 Let $A, p, m, F,$ and $w_{zh_1}$ be as in Theorem 6.6. Then for any functions $h_2, k \in \text{Supp}_\infty[0, T]$ with $kh_1 \in \mathcal{O}_{\text{Supp}_\infty}(A)$ (or $k \in \mathcal{O}_{\text{Supp}_\infty}(Ah_1)$), and all $q_1, q_2 \in \mathbb{R} \setminus \{0\}$, it follows that

$$
\int_{C_0[0,T]} \text{anf}_{q_2} \left[ F(\mathcal{Z}_{h_1}(x, \cdot)) \delta T_{q_1,kh_1}^{(p)}(F)(\mathcal{Z}_{h_1}(x, \cdot)) \mathcal{Z}_{h_2}(w_{zh_1}, \cdot) \right]
\quad + \delta F(\mathcal{Z}_{h_1}(x, \cdot)) \mathcal{Z}_{h_2}(w_{zh_1}, \cdot) T_{q_1,kh_1}^{(p)}(F)(\mathcal{Z}_{h_1}(x, \cdot)) \mathcal{m}(dx)
$$

$$\quad = -i q_2 \int_{C_0[0,T]} \langle z, \mathcal{Z}_{h_2}(x, \cdot) \rangle F(\mathcal{Z}_{h_1}(x, \cdot)) T_{q_1,kh_1}^{(p)}(F)(\mathcal{Z}_{h_1}(x, \cdot)) \mathcal{m}(dx).
$$

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