Quantum adaptation of noisy channels

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Probabilistic quantum filtering is proposed to properly adapt sequential independent quantum channels in order to stop sudden death of entanglement. In the adaptation, the quantum filtering does not distill or purify more entanglement, it rather properly prepares entangled state to the subsequent quantum channel. For example, the quantum adaptation probabilistically eliminates the sudden death of entanglement of two-qubit entangled state with isotropic noise injected into separate amplitude damping channels. The result has a direct application in quantum key distribution through noisy channels.

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I. INTRODUCTION

Quantum entanglement propagating through noisy quantum channel is crucial for modern application of quantum information science, for example, for security of quantum key distribution [1] or cluster-state quantum computing [2]. The quantum channel represents a real physical system used to transmit, store or operate quantum states. Recently, it has been recognized that the entanglement can be even lost if a noisy entangled state is inserted into a channel which is entanglement preserving for all the channel parameters (such as the amplitude and phase damping channel) [3, 4, 5, 6]. Such the behavior is called the sudden death of entanglement and it can dramatically reduce the security of the key distribution or the efficiency of cluster-state quantum computing. Therefore, it is interesting how to stop the sudden death of entanglement, deterministically or even probabilistically, at a cost of success rate of the transmission of entanglement.

The sudden death of entanglement has been reported as a property of a given non-maximally entangled state passing through the quantum channel representing a finite-time continuous interaction with a reservoir. A state preparation of the non-maximally entangled state is considered to be independent from the subsequent noisy channel. Let us be even more abstract way. Such the non-maximally entangled state can be generally understand as an output from some previous independent channel applied on this maximally entangled state. Thus we have a composite of two independent channels. The reservoirs corresponding to these two channels are therefore considered to be independent. It very well corresponds to a broad class of realistic physical situations in which the reservoirs of two channels are not interacting. Optimizing over input represented by maximally entangled states it can be recognized whether the composite channel, exhibiting sudden death of entanglement, is actually entanglement breaking channel [7]. For the entanglement breaking channel, no entanglement propagates through the channel. Also no entanglement can be distilled after the channel. Below, the sudden death of entanglement will be understand rather as a property of a composition of channels with independent reservoirs. It is not always possible to split a given quantum channel into independent sub-channels. Then the reservoirs corresponding to sub-channels are not independent and their exact dynamics and coupling have to be studied in a detail. We will focus just on the case of independent channels.

We study two-qubit entanglement undergoing local unitary quantum dynamics. This means that each qubit interacts just with its own reservoir resulting in channels for which the Kraus decomposition is valid [8]. We concern only about the cases when the sudden death of entanglement in the composite channel breaks entanglement completely. To stop the sudden death of entanglement, single-copy distillation [9, 10] can be sometimes simply placed between the channels. Then distillation increases entanglement before the subsequent channel and any construction of the distillation is only optimized with respect to the state after the previous channel.

In this paper, the adaptation of quantum channels is proposed to prevent the sudden death of entanglement. The probabilistic adaptation differs from the single-copy entanglement distillation and purification [9, 10]. In the adaptation, even in the case that the entanglement could not be increased by single-copy distillation after first channel, still the entangled state can be better prepared to the subsequent channel to preserve entanglement. Basically, the proper adaptation depends on the subsequent noisy channel. As will be demonstrated, it can help to stop the sudden death of entanglement when the single-copy distillation is inefficient. It is rather complex problem to find generally optimal adaptation between the channels. Therefore we rather discuss realistic examples of the sudden death of entanglement to demonstrate a potential power of the adaptation. First, we concentrate on a simple example of two subsequent single-qubit non-unitary channels with an anisotropic noise. We show by optimal unitary adaptation of such the channels, the sudden death of entanglement is completely canceled for all the channel parameters. Second, probabilistic adaptation between the channel with an isotropic noise and the subsequent amplitude damping channel is analyzed. In this case, the sudden death of entanglement cannot be
stopped by the unitary adaptation. But if the probabilistic adaptation is applied then the sudden death of entanglement vanishes completely. It demonstrates, that the sudden death of entanglement can be partially or even fully caused by the improper adaptation of the noisy channels. To find whether the sudden death of entanglement is really presented, it is necessary to discuss the optimal adaptation and then analyze if the entanglement passing through the adapted channels will be broken or not. Such the results have a direct application in quantum key distribution through composite realistic channels and in a multi-qubit version, also in the cluster-state preparation for quantum computing.

In Sec. II we introduce our method of quantum adaptation for composition of independent quantum channels. We use the Kraus decomposition for the representation of the channels and give arguments for its validity. We accent two different configurations of quantum channels, asymmetrical and symmetrical one and explain what is a non-trivial feature of the sudden death of entanglement. We introduce the single-copy quantum filtering operations between the channels to help to stop the sudden death of entanglement. In Sec. III we analyze simple example of quantum adaptation for the asymmetrical configuration of simple quantum channels. For this case the sudden death of entanglement can be undone by simply performing an appropriate unitary transform between the channels. In Sec. IV we show that in symmetrical configuration of quantum channels a unitary transform is not enough to undo the break of entanglement. Introducing quantum filters between the channels can help to stop the entanglement breaking for the symmetrical configuration. We conclude in Sec. V.

II. QUANTUM ADAPTATION OF INDEPENDENT CHANNELS

From our point of view, the sudden death of entanglement is an entanglement breaking property of a composition of independent quantum channels if at least one is not entanglement breaking at all. We assume any particle from entangled state evolves locally unitarily and reacts just with its own reservoir. Reservoirs are mutually independent. For such a case we may describe the evolution of the entangled state by using independent quantum channels in the form of Kraus decomposition

\[ \chi(\rho) = \sum_{i=1}^{N} A_i \rho A_i^\dagger, \]  

where \( \sum_i A_i^\dagger A_i = \mathbb{I} \). An important class of the channels are unital channels (for example, depolarization channel or phase damping channel), which preserves the isotropic noise. Such the channels transform maximal entangled state only to a mixture of maximally entangled states. The channel is entanglement breaking if no entanglement remains although any maximally entangled state \( |\Psi\rangle_{AB} \) is passing through the channel \[7\]. Mathematically, for single-qubit channel it corresponds to a condition based on positive partial transposition criterion \[12\], explicitly \[\chi_B(\rho_{AB})]^T_A \geq 0\], where \( \rho_{AB} = |\Psi\rangle_{AB} \langle \Psi| \). After such the channel no entanglement can be distilled even by multi-copy distillation \[12\].

The basic asymmetrical and symmetrical configurations of the propagation of two-qubit maximally entangled state through independent noisy channels are depicted on Fig. 1. In the first case, a single qubit from maximally entangled two-qubit state \( \rho_{AB} = |\Psi\rangle_{AB} \langle \Psi| \) is propagating through two independent channels \( \chi_{B1} \) and \( \chi_{B2} \) having mutually independent reservoirs. In the second case, both the qubits are symmetrically propagating through independent (but simply identical) consecutive channels \( \chi_{A1}, \chi_{A2} \) and \( \chi_{B1}, \chi_{B2} \) all having mutually independent reservoirs. The maximally entangled state \( \rho_{AB} \) passing through the asymmetrical composition of two channels \( \chi_{B2} \circ \chi_{B1}(\rho_{AB}) \) can be described as

\[ \chi_{B2} \circ \chi_{B1}(\rho_{AB}) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} B_{2j} B_{1i} \rho_{AB} B_{1i}^T B_{2j}^T, \]  

and the composite channel is entanglement breaking if

\[ [\chi_{B2} \circ \chi_{B1}(\rho_{AB})]^T_A \geq 0. \]  

For the symmetrical configuration, the maximally entangled state going through the symmetrical both-side channels is transformed to

\[ \chi_{A2} \circ \chi_{A1} \circ \chi_{B2} \circ \chi_{B1}(\rho_{AB}) = \sum_{i,k=1}^{N_1} \sum_{j,l=1}^{N_2} B_{2j} B_{1j} A_{2k} A_{1i} \rho_{AB} A_{1i}^T A_{2k}^T B_{2j}^T B_{1j}^T, \]  

and the composite channel is entanglement breaking if

\[ [\chi_{A2} \circ \chi_{A1} \circ \chi_{B2} \circ \chi_{B1}(\rho_{AB})]^T_A \geq 0. \]  

For the asymmetrical situation, the entanglement breaking property does not depend on a kind of maximally entangled state \[7\], but for the symmetrical configuration it is not generally true \[7\]. The channel is sequentially entanglement preserving if

\[ [\chi_{B1}(\rho_{AB})]^T_A < 0, \quad [\chi_{A1}(\rho_{AB})]^T_A < 0 \]  

and only such the channels will be taken into consideration. If the channel is not sequentially entanglement preserving then the entanglement cannot be successfully transmitted with a help of any adaptation method. Also the composite channels preserving entanglement will be simply omitted in the following discussion. It is obvious that two consecutive channels can break entanglement although they do not break it separately. But this is not non-trivial effect of the sudden death of entanglement a non-trivial feature of the ESD \[3, 4, 5, 6\].
that each channel is entanglement preserving separately but the whole concatenation (whole channel) is entanglement breaking. Physically, we have different independent channels preserving entanglement for any value of finite channel parameters. But their combination exhibiting the sudden death of entanglement can give entanglement breaking channel for some values of channels parameters. Since it is impossible improve the composite channel just by the operations before and after the channel, it is non-trivial case of the ESD.

To stop the sudden death of entanglement in the composite channel, a single-copy quantum filter can be generally placed between the individual channels. The quantum filter on single-qubit state can be described by the transformation

$$\rho' = \frac{F \rho F^\dagger}{\text{Tr}(F \rho F^\dagger)},$$

where $F^\dagger F \leq \mathbb{1}$. Generally, the quantum filtering can be decomposed as $F = UF_0V$, where $U, V$ are single-qubit unitary operations and $F_0 = \text{diag}(1, \sqrt{r}), 0 \leq r \leq 1$. If $r = 1$ the filtering is reduced just to unitary operation. It was theoretically described that such the local filtration applied after the noisy channel can increase entanglement \cite{[9]}. Recently, the local filtering has been experimentally demonstrated for entangled pair of photons generated from spontaneous parametric down-conversion \cite{[10]}. The result of the optimal local filtration always approaches the mixture of Bell diagonal states, from which more entanglement cannot be further distilled by any single-copy local filters. It automatically excludes a chance to stop the entanglement sudden death for a composition of the unital channels. On the other hand, for the non-unital channels the single copy filtration could be able to increase entanglement or perform proper adaptation of the channels.

With the filters between the channels, after the asymmetrical channels in the configuration (A) the maximally entangled state is

$$\chi_{B2} \circ F_B \circ \chi_{B1}(\rho_{AB}) = \frac{1}{N} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} B_{2j} F_B B_{1i} \rho_{AB} B_{1i}^\dagger F_B^\dagger B_{2j}^\dagger,$$

where the success rate is $S = \sum_{i=1}^{N_1} \text{Tr}(F_{1i} \rho_{AB} B_{1i}^\dagger F_{1i}^\dagger)$. On the other hand, the composition symmetrical channels (B) with the intermediate filtration produces

$$\chi_{A2} \circ F_A \circ \chi_{A1} \circ \chi_{B2} \circ F_B \circ \chi_{B1}(\rho_{AB}) = \frac{1}{S} \times$$

$$\sum_{i,k=1}^{N_1} \sum_{j,l=1}^{N_2} B_{2l} F_B B_{1j} A_{2k} F_A A_{1i} \rho_{AB} A_{1i}^\dagger F_A^\dagger A_{2k}^\dagger B_{1j}^\dagger F_B^\dagger B_{2l}^\dagger,$$

where the success rate is

$$S = \sum_{j,l=1}^{N_2} \text{Tr}(F_{A} A_{1l} F_B B_{1j} \rho_{AB} B_{1j}^\dagger F_{B}^\dagger A_{1l}^\dagger F_{A}^\dagger).$$

The task considered here is, by the application of quantum filters, transmit the entanglement through the composite channel which is entanglement breaking. It means to find if the composite channel, satisfying \cite{[9]} or \cite{[10]}, will preserve entanglement, i.e. will fulfill the conditions

$$[\chi_{B2} \circ F_B \circ \chi_{B1}(\rho_{AB})]^T_A < 0$$

or

$$[\chi_{A2} \circ F_A \circ \chi_{A1} \circ \chi_{B2} \circ F_B \circ \chi_{B1}(\rho_{AB})]^T_A < 0$$

at a cost of success rate of the filtration. It is impossible to solve such the complex task analytically for all the compositions of any channels, even for some specific channels is sophisticated. Therefore, rather then general answer we will analyze some physically interesting examples (for example, previously published in Ref. \cite{[9]}) to demonstrate that adaptation by single-copy filtration or even just unitary operation can be powerful tool to stop the sudden death of entanglement.

### III. ASYMMETRICAL EXAMPLE

The simplest example of the adaptation for the asymmetrical configuration (A) is following. In the asymmetrical configuration, the channel $\chi_{B1}$ ($\chi_{B2}$) transmits any qubit state either perfectly with a probability $p_1$ ($p_2$) or, with probability $1 - p_1$ ($1 - p_2$), the qubit is completely lost and another qubit in the pure state $|0\rangle$ ($|1\rangle$) occurs.
in the channel. Here, for simplicity, we use just two orthogonal states but similar analysis can be done for two general mixed states. It is easily to check that both the channels are entanglement preserving for \( p_1, p_2 > 0 \). If such the channels are combined, the maximally entangled state \( |\Psi_{-}\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \) is transformed to the mixture

\[
\rho = p_1 p_2 |\Psi_{-}\rangle \langle \Psi_{-}| + \frac{p_2(1 - p_1)}{2} |0\rangle \langle 0| + \frac{1 - p_2}{2} |1\rangle \langle 1|,
\]

which is entangled (using partial transposition criterion \( \text{(12)} \) if and only if \( p_1, p_2 \neq 0 \) satisfy

\[
p_2 > \frac{1 - p_1}{1 - p_1 + p_1^2}. \tag{14}
\]

Then outgoing two-qubit state has the concurrence \( \text{(14)} \)

\[
C(\rho) = p_1 p_2 - \sqrt{(1 - p_1)(1 - p_2)p_2} \tag{15}
\]

Otherwise the composite of these two channels is entanglement breaking. This is the sudden death of entanglement which apparently accompanied by the entanglement breaking. But fortunately, if the unitary transformation making \( |0\rangle \leftrightarrow |1\rangle \) is simply applied between the channels, then the density matrix changes to

\[
\rho' = p_1 p_2 |\Phi_{-}\rangle \langle \Phi_{-}| + \frac{1 - p_1 p_2}{2} |1\rangle \langle 1|,
\]

where \( |\Phi_{-}\rangle = (|00\rangle - |11\rangle)/\sqrt{2} \), which is always entangled for any \( p_1, p_2 \neq 0 \). All the entanglement breaking channels vanish just simply by the unitary operation. If two general states are considered instead of \( |0\rangle \) and \( |1\rangle \), the unitary transformation depends on both the channels. The same result can be obtained for any maximally entangled state entering into the channels. The amount of entanglement is simply given by the concurrence \( C'(\rho) = p_1 p_2 \). As a result, practically, the composite channel is not entanglement breaking channel at all (for \( p_1 p_2 > 0 \)). The entanglement can be further enhanced using local filtering after the composite channel to approach maximal concurrence \( C''(\rho) = \sqrt{p_1 p_2} \). The optimal filtering is \( |1\rangle \rightarrow \sqrt{p_1 p_2} |1\rangle \) and \( |0\rangle \rightarrow \epsilon |0\rangle \), where \( \epsilon \rightarrow 0 \). It is evidently a simple witness that even unitary adaptation between the different channels can stop the sudden death of entanglement.

**IV. SYMMETRICAL EXAMPLE**

In the following example, we show that the sudden death of entanglement can depend on the input maximally entangled state in the symmetrical configuration and the adaptation by the unitary operation is then insufficient to reduce the break of entanglement. But using quantum filters, the break of entanglement can be completely eliminated. Remind, if the filter is placed between the channels to improve the transmission, it can just increase the entanglement from the first channel at the maximum and then it can be send through the second channel. But such the cases cannot be understand as the adaptation, it is exactly the known single-copy distillation \( \text{(10)} \). The adaptation means that the filter depends also on the parameters of the subsequent channel. Since the filtering cannot increase the entanglement of the mixture of the Bell states, the states with isotropic noise \( \text{(15)} \)

\[
\rho_1' = p |\Psi_{-}\rangle \langle \Psi_{-}| + \frac{1 - p}{4} |1\rangle \langle 1|,
\]

\[
\rho_2' = p |\Phi_{-}\rangle \langle \Phi_{-}| + \frac{1 - p}{4} |1\rangle \langle 1|, \tag{17}
\]

where \( |\Phi_{-}\rangle = (|00\rangle - |11\rangle)/\sqrt{2} \), after the first channels are considered good candidates to see an effect of the adaptation. For both the states, the entanglement is preserved if \( p > 1/3 \). The states \( \text{(17)} \) can outcome from the depolarizing channel acting on single (or both) of the qubits. The depolarizing channel on single qubit is represented by the set of the Kraus operators

\[
D_3 = \sqrt{\frac{1 + 3p}{4}}, \quad D_1 = \sqrt{\frac{1 - p}{4}} \sigma_i \tag{18}
\]

where \( i = 1, 2, 3 \) and \( \sigma_i \) are the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}
\]
The depolarization is common physical decoherence process, it can arise from the random isotropic unitary changes of the state in the channel.

Such the states [17] are then locally processed by the filters $F_A, F_B$ and then entry into the identical amplitude damping channels acting symmetrically on both the qubits. The amplitude damping channel is non-unital channel described by the Kraus matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \quad (20)$$

It often arises from a resonant interaction of qubit system with zero-temperature reservoir characterized by a Hamiltonian $H_A = \sum_k g_k \sigma^z a_k^\dagger + g_k \sigma^+ a_k$, where $\sigma^\pm = \frac{1}{2} (\sigma_1 \pm i \sigma_2)$ are operators of the two-level system, $\sigma_k, a_k^\dagger$ are the reservoir operators, $g_k$ is coupling constant and the averaging is over the modes of reservoir, for review [16]. Physically, the source of decoherence is then just the spontaneous emission of the two-level system. This non-inital amplitude damping channel will not break the entanglement for any $\gamma \in (0, 1)$, corresponding to finite time dynamics. This channel has been used in Ref. [3], where non-trivial sudden death of entanglement has been recognized.

For the symmetrical configuration, the sudden death of entanglement depends on the input maximally entangled state. In the Fig. 2 for channel parameters in the union of crosses and dots the sudden death of entanglement occurs for the input state $|\Psi_\rangle$ whereas for the input state $|\Psi_-\rangle$ sudden death of entanglement appears just in the area of crosses. The unitary adaptation is able to make conversion between these cases without any reduction of the break of entanglement. But the same effect can be obtained if the input maximally entangled state is changed. Therefore, in the region of dots the sudden death of entanglement is not accompanied by the break of entanglement. Thus unitary adaptation cannot help to stop the break of entanglement at all, contrary to the previous example.

Interestingly, quantum filtering can help to adapt the channels each to other. Even for this specific example, it is complex to find the optimal filter analytically. From this reason, the numerical genetic algorithm for function optimization has been used [17]. The optimization has been performed in a net of the points and at the end, the optimized filters have been used to check their ability to stop the sudden death of entanglement. There has been included, beside quantum filters, also unitary operations into the optimization routine. The results of numerical calculations are depicted in Fig. 2. In all the numerically analyzed cases, the sudden death of entanglement is corrected (denoted by crosses). From the numerical optimization, it is also possible to find that a quantum filtering is sufficient even taking both the filters in the basis of the amplitude damping and identical thus $F_A, F_B = \text{diag}(1, \sqrt{r})$ can be simply used. From the partial transposition criterion [12], it is possible to derive a sufficient condition

$$0 < \sqrt{r} < \frac{2\sqrt{p(1+p)} - (1+p)}{\gamma (1-p)} \quad (21)$$

for the quantum filters to stop the break of entanglement. The success rate of the filtration is then $S = p(1-\sqrt{r})^2 + (1+\sqrt{r})^2$. To find optimal filter and the concurrence after the adaptation, the numerical optimization still has to be performed. The results are depicted on Fig. 3, where the optimal parameter $r$ of the filter and maximal value of the concurrence are plotted as functions of the channel parameters $\gamma$ and $p$. Evidently, in all the cases, the filtering completely eliminates the sudden death of entanglement caused by the symmetrical amplitude-damping channels.

\section{Conclusion}

In Conclusion, single-copy quantum adaptation of channels has been proposed to stop non-trivial sudden death of entanglement arising in a composite of the independent channels. The adaptation differs from the en-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{The plot of optimal parameter $r$ of a diagonal filter depending on the parameters $\gamma$ and $p$ of the channels (top figure) and the plot of concurrence depending on the parameters $\gamma$ and $p$ of the channels (bottom figure). Both plots are results of the quantum adaptation for the sequence of depolarizing and amplitude damping channel. We used diagonal filters $F_A, F_B = \text{diag}(1, \sqrt{r})$ for the adaptation.}
\end{figure}
tanglement distillation, it rather prepares the entangled states for the transmission through the subsequent channel. A power of both the unitary operation and quantum filters to completely reduce the sudden death of entanglement has been demonstrated. The presented results have direct application for the quantum key distribution through noisy channel and, in an extended multi-qubit version, also for the preparation of cluster states for quantum computing.

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