Field amplification and particle production by parametric resonance during inflation and reheating

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Abstract: Field amplification and particle production due to parametric resonance are highly nontrivial effects that couple to an oscillating source during inflation and reheating. Understanding this two effects is crucial for the connection between the resonance phenomenon and precise observational data. In this letter, we give a general and analytic analysis of parametric resonance of relevant field modes evolving during inflation and reheating by using the uniform asymptotic approximation. This analysis can provide a clear and quantitative explanation for the field amplification and particle production during the resonance. The potential applications of our results to several examples, including sound resonance during inflation, particle productions during reheating, and parametric resonance due to self-resonance potentials, have also been explored. The formalism developed in this letter is also applicable to parametric resonance in a broad areas of modern science.

I. INTRODUCTION

Parametric resonance is a resonance phenomenon that arises because some parameters of the system is varying periodically. In cosmology, it can occur in many scenarios during inflation and post-inflationary evolution. During inflation, the parametric resonance of inflationary perturbations can be induced by an effective oscillating sound speed which provides a mechanism for producing primordial black holes [1]. Similar resonance can also be trigged by an excited heavy field and produces features in the primordial spectrum and bispectrum [2–6]. After inflation, the oscillating inflaton field around its potential minimum can lead to resonance in fields coupled to it, giving rise to copious particle production in various fields including standard model particles [7–9], primordial magnetic fields [9], and gravitational waves [10]. In addition, a self-resonance potential can produce resonance in perturbations of inflaton itself and generate gravitational waves that could be within the forthcoming detections [11–14]. These complex and rich phenomena can be directly connected to precise observations, and a quantitative description and explanation of these resonance effects is a key ingredient for establishing it.

Normally, the important nontrivial effects due to parametric resonance are the amplification of the associated field modes and the corresponding particle productions at certain frequency ranges. Existing approaches for studying these two effects are either limited to numerical simulations or semi-analytic but qualitative which can be only applied to modes at resonance frequencies. Our purpose in this letter is to present a quantitative and general analysis of parametric resonance of relevant field modes evolving during inflation and reheating by applying the uniform asymptotic approximation, an approximation that has been verified to be powerful and robustness in calculating primordial spectra for various inflation models [15–30]. Our analysis does not only provide a quantitative derivation and explanation of the resonance conditions for the relevant modes, but also provide physical explanation for the field amplification and particle production rate which is valid at all frequencies. Applications of our results to sound resonance during inflation, particle productions during reheating, and parametric resonance due to self-resonance potential have also been explored. Details of the formalism and calculations will be reported elsewhere.

II. UNIFORM ASYMPTOTIC APPROXIMATION FOR MATHIEU EQUATION

The evolution of associated field mode $u_k$ during inflation and reheating can be formally put in the form of the so-called Mathieu equation [31]

$$d^2u_k/dx^2 + (A_k - 2q cos 2x)u_k = 0.$$  \hspace{1em} (1)

Here we consider the two parameters $A_k$ and $q$ as constant but as long as they are not varying too rapidly the assumption is reasonable. In order to apply the uniform asymptotic approximation, let us write the above equation into the standard form [16, 32, 33]

$$d^2u_k/dx^2 = \{g(x) + q(x)\}u_k$$  \hspace{1em} (2)

with $g(x) + q(x) = 2q cos 2x - A_k$. Since the combination $g(x) + q(x)$ is regular, we can always choose $q(x) = 0$

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1 The formalism developed in this letter can be easily extended to parametric resonance with other type equations, for example, to that with Lamé equation.
[16, 32, 33]. With this choice, we have

$$g(x) = 2q \cos 2x - A_k,$$

which is a periodic function, and in each oscillation, it has two turning points. Here we use $j$ to label the $j$-th oscillation, and the location of $\cos 2x = 1$ is at $x_j$ during this oscillation. Then the corresponding two turning points (i.e. $g(x) = 0$) are given by $x_j^\pm = \pm \frac{1}{2} \arccos \frac{2q}{A_k} + x_j$. Obviously, the two turning points could be both real and single ($A_k < 2q$), double ($A_k = 2q$ thus $x_j^- = x_j^+$), and complex conjugated ($A_k > 2q$ thus $x_j^\pm = (x_j^\pm)'$). We note that $x_j$ is the real part of $x_j^\pm$ when they are complex conjugated. Fig. 1 provides a schematic diagram for the $j$-th and $j + 1$-th oscillation for the case with $A_k < 2q$ where the corresponding two turning points in each oscillation are both real.

Since the function $g(x)$ is periodic, the general solution in each oscillation should take the same form. For this reason, we can only focus on the solution in the $j$-th oscillation, in which the function $g(x)$ has two turning points. Following [16, 33], we find that the general approximate solution of (2) in the $j$-th oscillation can be constructed in terms of parabolic cylinder function as

$$w_k = \left( \frac{\zeta_0^2 - \zeta^2}{-g(x)} \right)^\frac{1}{4} \left[ a_j W\left( \frac{\zeta_0^2}{2}, \sqrt{2}\zeta_j \right) + b_j W\left( \frac{\zeta_0^2}{2}, -\sqrt{2}\zeta_j \right) \right],$$

where $\zeta_0^2 = (4/\pi)\sqrt{2q - A_k E(\mathcal{X}, \mathcal{Y})}$ with $\mathcal{X} = (i/2) \ln((A_k - i\sqrt{4q^2 - A_k^2})/(2q))$, $\mathcal{Y} = 4q/(2q - A_k)$, and $E(\mathcal{X}, \mathcal{Y})$ denoting the incomplete elliptic integral of the second kind. The relation between variables $\zeta_j$ and $x$ during the $j$-th oscillation can be determined via

$$\int_{\text{Re}(x_j^\pm)}^{x} \sqrt{|g(x')|} dx' = \int_{\text{Re}(\zeta_j)}^{\zeta} \sqrt{|\zeta^2 - \zeta_0^2|} d\zeta'.$$

Obviously the sign of $\zeta_0^2$ is also sensitive to the nature of the turning points $x_j^+$ and $x_j^-$. $\zeta_0^2$ is positive when $x_j^-$ and $x_j^+$ are both real, $\zeta_0^2 = 0$ when $x_j^- = x_j^+$, and $\zeta_0^2$ is negative when $x_j^- = x_j^+$ are complex conjugated.

Connecting the two nearby oscillating regions, namely connecting $u_k^j$ and $u_k^{j+1}$, gives the recursion relation between the coefficients $(a_{j+1}, b_{j+1})$ and $(a_j, b_j)$,

$$\begin{pmatrix} a_{j+1} \\ b_{j+1} \end{pmatrix} = \frac{\kappa \sin \mathcal{B}}{\cos \mathcal{B}} \begin{pmatrix} \cos \mathcal{B} & - \kappa \sin \mathcal{B} \end{pmatrix} \begin{pmatrix} a_j \\ b_j \end{pmatrix},$$

where $\kappa = \sqrt{1 + e^{\pi \zeta_0^2}} - e^{\pi \zeta_0^2/2}$ and $\mathcal{B} = 2i \sqrt{2q - A_k E(\mathcal{X}, \mathcal{Y}) + \pi/2 + 2\phi(\zeta_0^2/2)}$ when $x_j^\pm$ are both real and $\mathcal{B} = 2i \sqrt{2q - A_k E(\mathcal{Y}) + \pi/2 + 2\phi(\zeta_0^2/2)}$ if $x_j^\pm$ are complex conjugated.

With this recursion relation, we are able to estimate the particle production rate during oscillations. Around the end of the $j + 1$-th oscillation (i.e. $x = x_{j+1} + \pi/2$), the WKB approximation is fulfilled and we employ the WKB approximate solution of mode function $u_k(x)$ as

$$u_k(x) = \frac{\alpha_{j+1}}{\sqrt{2(-g(x))^{1/4}}} e^{-i \int x_{j+1}^+ \frac{x}{2} \sqrt{-g(x')dx'}} + \frac{\beta_{j+1}}{\sqrt{2(-g(x))^{1/4}}} e^{i \int x_{j+1}^+ \frac{x}{2} \sqrt{-g(x')dx'}},$$

where $\alpha_{j+1}$ and $\beta_{j+1}$ are two Bogoliubov coefficients, which can be related to $a_{j+1}$ and $b_{j+1}$ by comparing both the uniform asymptotic approximate solution and WKB solution near the end of the $j + 1$-oscillation. With this procedure, the particle production $n_k^{j+1} \equiv |\beta_{j+1}|^2$ can be expressed as the recursion relation

$$n_k^{j+1} = e^{\pi \zeta_0^2} + (1 + 2e^{\pi \zeta_0^2})n_k^j + 2e^{\pi \zeta_0^2/2} \sqrt{1 + e^{\pi \zeta_0^2}} \times \sqrt{n_k^j(1 + n_k^j)} \cos (2\Theta + 2\mathcal{B} - \phi a_j + \phi b_j),$$

where $\Theta = -\mathcal{B}/2$. This result is general and accurate, which can be also approximately reduced to those given in [7, 34] with condition $\zeta_0^2 \lesssim 0$.

Using the recursion relation (6), one could relate the $(j + 1)$-th oscillation to $(j + 1 - N)$-th oscillation via

$$a_{j+1} + Y_\pm b_{j+1} = Z_N^N (a_{j-N} + Y_\pm b_{j-N}),$$

where $Y_\pm = -\sec \frac{\pi \mathcal{B}}{2k} \left[(\kappa^2 - 1) \sin \mathcal{B} \pm \sqrt{(\kappa^2 - 1)^2 \sin^2 \mathcal{B} - 4\kappa^2 \cos^2 \mathcal{B}} \right]$ and $Z_N = \frac{1}{2k} \left[(-1 - \kappa^2) \sin \mathcal{B} \pm \sqrt{(\kappa^2 - 1)^2 \sin^2 \mathcal{B} - 4\kappa^2 \cos^2 \mathcal{B}} \right]$ with $Y_+ Y_- = 1 = Z_+ Z_-$. This relation helps us to determine the coefficients $a_{j+1}$ and $b_{j+1}$ of the approximate solution (4) from the initial state. Fig. 2 shows the evolution of the field amplification $A(x) \equiv |u_k(x)/u_k(x_1 - \pi/2)|^2$ from analytical solution (4) with coefficients derived from (8) that fits numerical results very well. However, we have to mention that, the relation (8) is only valid when $N$ is not very large ($N \lesssim 30$ as shown by Fig. 2 for examples). If $N$ is large enough, the small error of the approximation can accumulate in each oscillation, thus it will becomes large enough to destroy the validity of (8). Formally, from (8) we can still derive both the field amplification.
\[ A_{j+1} = A(x_{j+1} + \frac{\pi}{2}) \] 

\[ n_k^{j+1} = \frac{[(Y_+ + \kappa^2 Y_-)Z_+^{j+1} - (Y_+ + \kappa^2 Y_+)Z_-^{j+1}]^2}{4(Y_+ - Y_-)^2\kappa^2} \] 

\[ A_{j+1} = \frac{(Y_+^2 + \kappa^2)(\sin \Theta + \kappa Y_+ \cos \Theta)^2}{\kappa^2(Y_+ - Y_-)^2} Z_+^{2j} \]

\[ + \frac{(Y_+^2 + \kappa^2)(\sin \Theta + \kappa Y_+ \cos \Theta)^2}{\kappa^2(Y_+ - Y_-)^2} Z_-^{2j} \]

\[ - 2\frac{(1 + \kappa^2)(\sin \Theta + \kappa Y_+ \cos \Theta)(\sin \Theta + \kappa Y_+ \cos \Theta)}{\kappa^2(Y_+ - Y_-)^2} \] 

Here we assume the oscillations start at \(x_1 - \frac{\pi}{2}\) and the field mode \(u_k\) is at the BD vacuum state at this point. As we mentioned, these formulas are accurate only when the number of oscillations \(j + 1\) is not very large. For large number of oscillations, these formulas will be not accurate enough but can show great insight qualitatively in the analysis of resonance as we shown later.

\[ |Z_+| > 1 \text{ or } |Z_-| > 1. \] (11)

Since \(Z_+Z_- = 1\), we observe that \(|Z_\pm| = 1\) when \(Z_\pm\) are complex conjugated. Thus for the field mode \(u_k\) growing in every oscillation, \(Z_\pm\) have to be both real, which in turn leads to the requirement of positivity of terms under the square root of \(Z_\pm\), that is

\[ \tan^2 \mathfrak{B} > e^{-\pi \xi^2}. \] (12)

This leads to

\[ n\pi + \arctan(e^{-\pi \xi^2}) < \mathfrak{B} < n\pi + \pi - \arctan(e^{-\pi \xi^2}), \]

where \(n\) is an integer which defines a series of instability bands labeled by \(n\). From this condition, we can derive the width of the \(\mathfrak{B}\) in each instability band, which is \(\Delta \mathfrak{B} = \pi - 2\arctan(e^{-\pi \xi^2} \frac{1}{2})\). This shows explicitly that the width \(\Delta \mathfrak{B}\) in each band essentially depends on \(\xi^2\) as shown in Fig. 3.

Depending on the value of \(\xi^2\), the instability bands can be divided into three different cases, the tachyonic resonance, the broad resonance, and the narrow resonance, as shown in Fig. 3. The evolution of \(A(x)\) with the three representative cases have been illustrated in Fig. 2 by comparing analytic and numerical results.

The tachyonic resonance corresponds to the modes with \(A_k < 2q\), thus \(g(x)\) has two real turning points in each oscillation and \(\xi^2 > 0\). These modes cross periodically the tachyonic region \((g(x) > 0)\) which leads to
exponentially growing for most of modes during each oscillation (c.f. the top panel of Fig. 2). For modes with \( e^{\pi c_0^2} \gtrsim 1 \), we find
\[
\begin{align*}
    n_k^{(j+1)} &\simeq e^{(j+1)\pi c_0^2} (4 \sin^2 \Omega)^j, \\
A_{j+1} &\simeq 4 e^{(j+1)\pi c_0^2} (4 \sin^2 \Omega)^j \sin^2 \Theta.
\end{align*}
\]
We observe that the particle production and field amplification are both exponentially enhanced which arises from two effects: the value of \( e^{\pi c_0^2} \) and the number of oscillations. The corresponding amplification becomes dramatically enhanced as \( e^{\pi c_0^2} \) increases.

The broad resonance corresponds to the modes with \( A_k \gtrsim 2q \), for which \( g(x) \) has two coalescing complex conjugated turning points and \( \zeta_0^2 \lesssim 0 \). For this case, the growth of the modes is weaker than that in the tachyonic instability band (c.f. the middle panel of Fig. 2). However, the amplification of the field modes can still be very efficient after a lot of oscillations.

Another interesting case is the narrow resonance which corresponds to \( A_k \gtrsim 2q \) and \( e^{\pi c_0^2} \ll 1 \). For this case, \( \mathcal{B} \) approximately lies in a very narrow region
\[
\left( n + \frac{1}{2} \right) \pi - e^{\pi c_0^2/2} < \mathcal{B} < \left( n + \frac{1}{2} \right) \pi + e^{\pi c_0^2/2},
\]
and its width can be approximately expressed as \( \Delta \mathcal{B} \simeq 2e^{\pi c_0^2/2} \). From (15), by dropping terms with \( e^{\pi c_0^2/2} \) we observe that the resonance bands approximately locate at \( \mathcal{B} \simeq n\pi + \frac{1}{2}\pi \), which gives \( A_k \simeq n^2 \). This implies the resonance only happens at the narrow ranges of \( A_k \) and \( q \) around \( A_k \simeq n^2 \). The particle production rate and field amplification for this case are given by
\[
\begin{align*}
    n_k^{(j+1)} &= \frac{1}{4} \left( \sqrt{1 + e^{2\pi c_0^2} + e^{\pi c_0^2/2}} \right)^{2j+2}, \\
A_{j+1} &\simeq \frac{1}{2} \left( \sqrt{1 + e^{2\pi c_0^2} + e^{\pi c_0^2/2}} \right)^{2j+2}.
\end{align*}
\]
Although \( e^{\pi c_0^2/2} \ll 1 \), the particle production and field amplification can still be enhanced if there are a large number of oscillations (c.f. the bottom panel of Fig. 2). For a fixed value of \( q \), \( \zeta_0^2 \) is monotonically decreasing with respect to \( A_k \). This indicates that for the same number of oscillation, the modes in the first band \((n = 1)\) is more significant than others.

**IV. APPLICATIONS TO PARAMETRIC RESONANCE DURING INFLATION AND REHEATING**

The formalism developed in the above can be applied to a lot of scenarios that are related to the inflationary cosmology. Here we consider its potential applications to three examples: the sound resonance during inflation [1], the particle production during reheating [7, 8], and the generations of oscillons for a self-resonance inflaton field [11, 12].

The sound resonance is studied recently in [1], which leads to a novel resonance mechanism for generations of large curvature perturbations \( \mathcal{R}_k \) during inflation. In this model, the sound speed takes the form \( c_s^2 = 1 - \xi(1 - \cos (2k \tau)) \), with \( \xi \) is the amplitude of the oscillation and \( k_s \), is the oscillation frequency. Then the equation of motion for the propagating mode \( u_k = z \mathcal{R}_k \) can be casted into the Mathieu equation (1) with \( A_k = k^2/k_s^2 + 2q - 4\xi, q = 2\xi - \xi k_s^2/k_s^2, \) and \( x = k_s \tau \). Applying the formulas of the amplification, we find that the primordial perturbation spectrum can be estimated by
\[
P_{\mathcal{R}}(k) \simeq A_{j+1} \times A_s (k/k_p)^{n_s-1},
\]
where \( A_s = \frac{\mu^2}{8\pi^2 M_p^2} \) is the amplitude of power spectrum as in the standard slow-roll inflation, \( n_s \) is the corresponding spectral index at pivot scale \( k_p \) and the field amplification \( A_{j+1} \) can be interpreted by (10) with the number of the oscillations given by \( j + 1 = \frac{k_s}{2\pi} (e^{\Delta N} - 1) \), where \( \Delta N \) denotes the number of e-folds from the beginning of oscillation to the horizon crossing for mode \( k \).

This, on the other hand, provides the second explanation of the significance of the first band \((n = 1)\) since it has more oscillations than other bands.

After inflation, the inflaton \( \phi(t) \) becomes oscillating around the minimum of its potential, i.e., \( \phi(t) \simeq \phi_0 + \sin (m_{\phi}t) \). This oscillating behavior leads to parametric resonance during the early stages of reheating, giving rise to copious particle production in fields coupled to it. For simplicity, we consider the coupling of the inflaton to the scalar field \( \chi \), through an interaction term of the form \((1/2)g^2\phi^2\chi^2\). Then the equation of motion for the Fourier modes \( \chi_k \) can be approximately described by (1) with \( u_k = a^{3/2}\chi_k, A_k = 2q + k^2/(m_{\phi}^2a^2), \) \( q = g^2\phi^2/(4m_{\phi}^2) \), and \( x = m_{\phi}t \). The particle production rate \( |n_k^{(j+1)}|^2 \) can be analyzed from (9) with the number of oscillations \( j + 1 = m_{\phi}t/\pi \). The strength of resonance depends on \( A_k \) and \( q \). Small \( q(\lesssim 1) \) corresponds to narrow resonance for which the width of the instability band is very small as shown in Fig. 3. For large \( q(\gtrsim 1) \) the broad resonance can occur for a wide range of the parameter space. The tachyonic resonance is also allowed during reheating process if one replaces the interaction \((1/2)g^2\phi^2\chi^2\) by \((1/2)g\phi^2\chi^2\) and allows \( g < 0 \). When \( |g| \) is large enough to make \( A_k < 2q \), then the tachyonic resonance occurs for which both the particle production rate and field amplification can be dramatically enhanced [35].

The oscillating inflaton can also become self-resonance if it has a self-resonance potential, with which the inflaton field perturbations \( \delta \phi_k \) obeys \( \delta \phi_k + (k^2/a^2 + V''(\phi) ) \delta \phi_k = 0 \) and can be amplified as the inflaton oscillates about the minimum of its potential. When the perturbation modes are initially displaced in the tachyonic region \((k^2/a^2 + V''(\phi) < 0) \) and then enter periodically into this region, these modes can be dramatically amplified due to the tachyonic resonance. These process have also been known as tachyonic preheating and tachyonic os-
citations as mentioned in [12]. For the modes with $k^2/a^2 + V''(\phi) > 0$ always satisfied, they belong to the parametric resonance with $A_k > 2q$ and the resonating amplification can only occur at certain ranges of frequencies that satisfy the resonance condition (12). Such amplifications have gained much attentions recently since it could provide interesting mechanisms for generating large GWs that could be detected by aLIGO-Virgo networks [11, 13].

Note that we have treated $A_k$ and $q$ as constants. In fact, they are decaying with the expansion of the Universe. In this case, the field modes in each oscillation can still be given by (4) but with $\zeta_0 \to \zeta_0^{(j)}$ being different in each oscillation. Similarly one needs to make substitute in each oscillation as $\mathfrak{A} \to \mathfrak{A}_j^{(j)}$, $Y_+ \to Y_+^{(j)}$, $Z_\pm \to Z_\pm^{(j)}$, and $\kappa \to \kappa_j$. The decaying of the oscillations also brings complications in determining the resonance condition. This is because the modes may only grow in some of oscillations, keeping non-growing in others. Therefore, among all of oscillations, the essential question now is to determine how many oscillations that the relevant modes can be amplified. We would like to address this issue in details in our future works.

V. SUMMARY AND OUTLOOK

To summarize, we have presented a quantitative and general analysis of parametric resonance of the relevant field modes evolving during inflation and reheating. This analysis gives a condition for the occurrence of the resonance and provides clear and quantitative explanation for the field amplification and particle productions. Further issues including the high-order approximations, effects of decaying oscillations, back reaction of quantum fluctuations on the background spacetime metric, observational predictions of the resonating field modes can also be studied based on our analysis. Our result is general and simple to use, and has applications beyond the inflation related context, for example, to the production of vector dark matter [36–38] by parametric resonance, to the production of electron-positron pairs from vacuum by a periodical laser pulses [39–41], and to nonequilibrium quantum field theory [42, 43].

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