Electromagnetism on Anisotropic Fractals

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Abstract
We derive basic equations of electromagnetic fields in fractal media which are specified by three independent fractal dimensions $\alpha_i$ in the respective directions $x_i$ ($i = 1, 2, 3$) of the Cartesian space in which the fractal is embedded. To grasp the generally anisotropic structure of a fractal, we employ the product measure, so that the global forms of governing equations may be cast in forms involving conventional (integer-order) integrals, while the local forms are expressed through partial differential equations with derivatives of integer order but containing coefficients involving the $\alpha_i$'s. First, a formulation based on product measures is shown to satisfy the four basic identities of vector calculus. This allows a generalization of the Green-Gauss and Stokes theorems as well as the charge conservation equation on anisotropic fractals. Then, pursuing the conceptual approach, we derive the Faraday and Ampère laws for such fractal media, which, along with two auxiliary null-divergence conditions, effectively give the modified Maxwell equations. Proceeding on a separate track, we employ a variational principle for electromagnetic fields, appropriately adapted to fractal media, to independently derive the same forms of these two laws. It is next found that the parabolic (for a conducting medium) and the hyperbolic (for a dielectric medium) equations involve modified gradient operators, while the Poynting vector has the same form as in the non-fractal case. Finally, Maxwell’s electromagnetic stress tensor is reformulated for fractal systems. In all the cases, the derived equations for fractal media depend explicitly on fractal dimensions and reduce to conventional forms for continuous media with Euclidean geometries upon setting the dimensions to integers.

1 Introduction
In a recent article (Li & Ostoja-Starzewski, 2009) we formulated a continuum mechanics of anisotropic fractal elastic media using two independent approaches: a mechanical (Newtonian) one and a variational (Lagrangian-Hamiltonian) one. Proceeding separately on both paths, we derived the identical equations of
wave motion in one-, two- and three-dimensional media, and this provided a verification of the consistency of the product measure capable of grasping the anisotropic fractal media as well as the verification of the final equations. The question arises whether one can use the same strategy to derive the field equations of electromagnetism in fractal media, be they isotropic or anisotropic. Various studies generalizing the classical, quantum or relativistic field theories to fractal spaces (Stillinger, 1977; Svozil, 1987; Palmer & Stavrinos, 2004; Not-tale, 2010) as well as the electromagnetism to fractal media appeared over the past two decades (Tarasov, 2006, 2010). They have been formulated either through a conceptual path – in the case of electromagnetism, by developing the Gauss, Faraday and Ampère laws, so as to obtain the Maxwell equations – or through variational principles. However, a derivation obtained on one path (say, conceptual) has never been verified through another, separate derivation (respectively, variational).

In the following we proceed under the premise that the conceptual derivation of Maxwell’s equations for anisotropic fractal media should yield the same results as the derivation based on a variational principle, both derivations being based on the same dimensional regularization. To grasp the generally anisotropic structure of a fractal, we employ the recently introduced product measure, so that the global forms of governing equations may be cast in forms involving conventional (integer-order) integrals, while the local forms are expressed through partial differential equations with derivatives of integer order but containing coefficients involving fractal dimensions \( \alpha_i \) in three respective directions \( x_i \), \( i = 1, 2, 3 \).

We first show that the formulation based on product measures satisfies the four basic identities of vector calculus. This readily leads to a generalization of the Green-Gauss and Stokes theorems involving the fractal gradient and curl operators, and hence to the charge conservation equation on anisotropic fractals. Then, pursuing the conceptual approach, we derive the Faraday and Ampère laws for such fractal media, which, along with two auxiliary null-divergence conditions, effectively give the modified Maxwell equations.

Proceeding on a separate track, we employ a variational principle for electromagnetic fields, appropriately adapted to fractal media, to independently derive the same forms of these two laws. We next examine whether the parabolic (for a conducting medium) and the hyperbolic (for a dielectric medium) equations involve fractal gradient operators, and also whether the Poynting vector has the same form as in the non-fractal case. Finally, Maxwell’s electromagnetic stress tensor is reformulated for fractal systems. In all the cases, the derived equations for fractal media depend explicitly on fractal dimensions and reduce to conventional forms for continuous media with Euclidean geometries upon setting the dimensions to integers. In order to make the presentation clear, in some places we give the tensorial relations in the symbolic as well as the index notations.
2 Background

2.1 Anisotropic Fractal Distributions

The distribution of charges in an electrically and magnetically conducting isotropic fractal medium embedded in the Euclidean space \( E^3 \) obeys a power law

\[
Q(R) \sim R^D, \quad D < 3,
\]

where \( D \) is the fractal dimension and \( R \) is the spatial resolution. However, in an anisotropic fractal medium (generally denoted by \( W \)) we write

\[
Q(x_1, x_2, x_3) \sim x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3},
\]

where each \( \alpha_k \) plays the role of a fractal dimension along each axis \( x_k \) \((i = 1, 2, 3) \) of the Cartesian space in which the fractal is embedded. In other directions, the fractal dimension is not necessarily the sum of projected fractal dimensions, however, as noted by Falconer (2003), "Many fractals encountered in practice are not actually products, but are product-like." Hence, we expect the equality between the fractal dimension \( D \) of the total charge and \( \alpha_1 + \alpha_2 + \alpha_3 \) to hold for fractals encountered in practice. When \( D \to 3 \), with each \( \alpha_i \to 1 \), the conventional concept of charge distribution is recovered.

By analogy to the formulation of continuum mechanics on anisotropic fractals (Li & Ostoja-Starzewski, 2009), we now express (2.1) through a product measure

\[
Q(W) = \int_W \rho(x_1, x_2, x_3) dl_{\alpha_1}(x_1) dl_{\alpha_2}(x_2) dl_{\alpha_3}(x_3),
\]

where \( \rho \) is the density of distribution. In (2.3), the length measure in each coordinate is given in terms of the transformation coefficients \( c^{(k)}_i \) by

\[
dl_{\alpha_k}(x_k) = c^{(k)}_1(\alpha_k, x_k) dx_k, \quad k = 1, 2, 3 \quad \text{(no sum)},
\]

where \( l_k \) is the total length (integral interval) along \( x_k \) and \( l_{k0} \) is the characteristic length in the given direction, like the mean pore size. The relation (2.4) implies that the infinitesimal fractal volume element, \( dV_D \), is related to the cubic volume element, \( dV_3 = dx_1 dx_2 dx_3 \) by

\[
dV_D = dl_{\alpha_1}(x_1) dl_{\alpha_2}(x_2) dl_{\alpha_3}(x_3) = c_1^{(1)} c_1^{(2)} c_1^{(3)} dx_1 dx_2 dx_3 = c_3 dV_3,
\]

with \( c_3 = c_1^{(1)} c_1^{(2)} c_1^{(3)} \).

Similarly, the infinitesimal fractal surface element, \( dS_d \), is related to the planar surface element, \( dS^{(k)}_d = dx_i dx_j \) with the normal vector along \( x_k \), according to

\[
dS^{(k)}_d = c_2 dS_2,
\]

with \( c_2^{(k)} = c_1^{(i)} c_1^{(j)} = c_3 / c_1^{(k)} \), \( i \neq j \), \( i, j \neq k \).
The sum
\[ d(k) = \alpha_i + \alpha_j, \quad i \neq j, \quad i, j \neq k, \]
is the fractal dimension of the surface \( S_d(k) \) along the diagonals \( |x_i| = |x_j| \) in \( S_d(k) \). Again, this equality is not necessarily true elsewhere, but is expected to hold for fractals encountered in practice (Falconer, 2003).

The transformation coefficients \( c_1^{(k)} \) showing up in (2.4) can be represented in terms of the modified Riemann-Liouville fractional integral of Jumarie (2005, 2009), thus
\[ c_1^{(k)} = \alpha_k \left( \frac{l_k - x_k}{l_k \alpha} \right)^{\alpha_k - 1}, \quad k = 1, 2, 3, \quad \text{(no sum)}, \]
although further developments will not depend on this explicit form.

2.2 Vector calculus on anisotropic fractals

Following (Li & Ostoja-Starzewski, 2009), we will extensively employ the fractal derivative (fractal gradient) operator \( \nabla D \)
\[ \nabla D \phi = e_k \nabla_k D \phi \quad \text{or} \quad \nabla_k D \phi = \frac{1}{c_1^{(k)}} \frac{\partial \phi}{\partial x_k} \quad \text{(no sum on} \ k) \]
where \( e_k \) are base vectors. Hence, the fractal divergence of a vector field
\[ \text{div} f = \nabla D \cdot f \quad \text{or} \quad \nabla_k D f_k = \frac{1}{c_1^{(k)}} \frac{\partial f_k}{\partial x_k}. \]

Note that this leads to a fractal curl operator of a vector field
\[ \text{curl} f = \nabla D \times f \quad \text{or} \quad \nabla_k D \times f_i = e_{jki} \frac{1}{c_1^{(k)}} \frac{\partial f_i}{\partial x_k}. \]

We observe that the four fundamental identities of the conventional vector calculus carry over to these new operators:

(i) The divergence of the curl of a vector field \( f \):
\[ \text{div} \cdot \text{curl} f = \nabla_k D \times f = \frac{1}{c_1^{(j)}} \frac{\partial f}{\partial x_j} \left[ e_{jki} \frac{1}{c_1^{(k)}} \frac{\partial f_i}{\partial x_k} \right] = e_{jki} \frac{1}{c_1^{(j)}} \frac{1}{c_1^{(k)}} \frac{\partial f_i}{\partial x_j \partial x_k} = 0. \]

(ii) The curl of the gradient of a scalar field \( \phi \):
\[ \text{curl} \times (\text{grad} \phi) = \nabla D \times (\nabla_k D \phi) = e_{ijk} \frac{1}{c_1^{(j)}} \frac{\partial \phi}{\partial x_j} \left[ \frac{1}{c_1^{(k)}} \frac{\partial \phi}{\partial x_k} \right] = e_{jki} \frac{1}{c_1^{(j)}} \frac{1}{c_1^{(k)}} \frac{\partial f_i}{\partial x_j \partial x_k} = 0. \]

In both cases above we can pull \( 1/c_1^{(k)} \) in front of the gradient because the coefficient \( c_1^{(k)} \) is independent of \( x_j \).
(iii) The divergence of the gradient of a scalar field \( \phi \) is written in terms of the fractal gradient as

\[
\text{div} \cdot (\text{grad} \phi) = \nabla^D \cdot \left( \frac{1}{c_1^{(j)}} \frac{\partial}{\partial x_j} \left[ \frac{1}{c_1^{(j)}} \frac{\partial \phi}{\partial x_j} \right] \right) = \frac{1}{c_1^{(j)}} \left[ \frac{\partial \phi_j}{c_1^{(j)}} \right], \tag{13}
\]

which gives an explicit form of the fractal Laplacian.

(iv) The curl of the curl operating on a vector field \( f \):

\[
\text{curl} \times (\text{curl} f) = \nabla^D \times (\nabla^D \times f) = \epsilon_{prj} \frac{1}{c_1^{(r)}} \frac{\partial}{\partial x_r} \left[ \frac{1}{c_1^{(k)}} \frac{\partial f_i}{\partial x_k} \right] = \epsilon_{prj} \frac{1}{c_1^{(r)}} \frac{\partial}{\partial x_r} \left[ \frac{1}{c_1^{(k)}} \frac{\partial f_i}{\partial x_k} \right] = \epsilon_{prj} \frac{1}{c_1^{(r)}} \frac{\partial}{\partial x_r} \left[ \frac{1}{c_1^{(k)}} \frac{\partial f_i}{\partial x_k} \right] = \nabla^D \left( \nabla^D \cdot f \right) - \nabla^D \nabla^D f \tag{14}
\]

As further background, we now extend two integral theorems to generally anisotropic fractals.

### 2.3 Stokes and Green-Gauss theorems for anisotropic fractals

**Stokes theorem.** We begin with \( \int_A (\nabla^D \times f) \cdot ndS_d \) and proceed in the index notation:

\[
\int_{\partial W} n_k \nabla^D f_i dS_d = \int_{\partial W} n_k \epsilon_{kji} \frac{1}{c_1^{(j)}} f_{i,j} dS_d = \int_{\partial W} n_k \epsilon_{kji} \frac{1}{c_1^{(j)}} f_{i,j} c_2^{(k)} dS_2 = \int_{\partial W} n_k \epsilon_{kji} f_{i,j} c_2^{(k)} dS_2 = \int_{\partial W} n_k \epsilon_{kji} f_{i,j} c_1^{(i)} dS_2 = \int_{\partial W} n_k \epsilon_{kji} f_{i,j} c_1^{(i)} dS_2 = \int_A f_i c_1^{(i)} dS_2 = \int_A f_i dS_D \tag{15}
\]

In the second equality above we employed the dimensional regularization, followed by taking note of (2.6), including the fact that \( c_1^{(i)} \) is independent of \( x_j \). Next, by the Stokes theorem in the Euclidean space \( \mathbb{E}^3 \), we have

\[
\int_{\partial W} n_k \epsilon_{kji} \left( f_i c_1^{(i)} \right) dS_2 = \int_{\partial W} f_i c_1^{(i)} dl_i = \int_A f_i dl_i D \tag{16}
\]

where, again, we used (2.4). Thus, we arrive at the *Stokes theorem for fractals*

\[
\int_A n \cdot \text{curl} f \ dS_d = \int_A f \cdot dl_1 \tag{17}
\]

Formally, the above agrees with Tarasov’s (2006) result in the case of isotropy, although our curl operator also grasps anisotropy.
**Green-Gauss theorem.** We begin with $\int_A f \otimes n \, dS$ and proceed in the index notation:

$$
\int_{\partial W} f_i n_k \, dS_d = \int_{\partial W} f_i n_k c_2^{(k)} \, dS_2 = \int_W \left( f_i c_2^{(k)} \right)_k \, dV_2
$$

$$
= \int_W \left( f_i c_2^{(k)} \right)_k c_3^{-1} \, dV_D = \int_W f_i n_k c_2^{(k)} c_3^{-1} \, dV_D = \int_W f_i n_k \frac{1}{c_3^{(k)}} \, dV_D,
$$

whereby in the second equality we employed the dimensional regularization, followed by taking note of (2.5), then followed by employing the Green-Gauss theorem in $\mathbb{E}^3$, and in turn followed by noting that $c_2^{(k)}$ is independent of $x_k$. Thus, we arrive at the *Green-Gauss theorem for fractals* as

$$
\int_{\partial W} f \cdot n \, dS_d = \int_W \nabla^D f \, dV_D
$$

(19)

The same comment as that following (2.7) applies here.

### 2.4 Charge conservation on anisotropic fractals

Let $J$ be the current density, $n$ be the direction normal to the surface, and $\eta$ be the charge density on the fractal $W$. Then we have

$$
\int_{\partial W} J \cdot n \, dS_d = - \int_W \eta \, dV_D
$$

(20)

On account of (2.18), we obtain

**global form**

$$
\int_W \nabla^D \cdot J \, dV_D = - \frac{d}{dt} \int_W \eta \, dV_D
$$

**local form**

$$
\nabla^D \cdot J = - \frac{\partial}{\partial t} \eta
$$

(21)

The fact that $\nabla^D$ automatically appears in the above will be exploited below.

Focusing on a linear electromagnetic response, with $\sigma$ being the electric conductivity tensor, Ohm’s law for an anisotropic medium reads

$$
J = \sigma \cdot E \quad \text{or} \quad J_i = \sigma_{ij} E_j
$$

(22)

the isotropic form following for $\sigma_{ij} = \delta_{ij} \sigma$. The fact that the constitutive form above carries over directly from the non-fractal case is motivated by the analogy to elastic media, where Hooke’s law is unchanged when going from non-fractal to fractal media (Li & Ostoja-Starzewski, 2009); that result ensured the consistency of the Newtonian and Hamiltonian approaches to the derivation of governing equations - recall the beginning of the Introduction.
3 Formulation of Maxwell’s equations via Stokes and Green-Gauss theorems

3.1 Faraday’s law

First, note that the Faraday law holds for a fractal $W$ embedded in $\mathbb{E}^3$

$$\frac{d}{dt} \int_A \mathbf{B} \cdot \mathbf{n} dA_d = - \int_I \mathbf{E} \cdot d\mathbf{l}_D \quad (1)$$

On account of the Stokes theorem for fractals (2.17), we obtain

$$\frac{d}{dt} \int_A B_k n_k dA_d = - \frac{d}{dt} \int_A e_{kji} \nabla_D^j E_i n_k dA_d = - \frac{d}{dt} \int_A e_{kji} \frac{1}{c_1^{(j)}} E_{i,j} n_k dA_d \quad (2)$$

which, by localization leads, to

$$0 = \frac{\partial}{\partial t} B_k + e_{kji} \nabla_D^j E_i \equiv \frac{\partial}{\partial t} B_k + e_{kji} \frac{1}{c_1^{(j)}} E_{i,j}$$

or

$$0 = \frac{\partial}{\partial t} \mathbf{B} + \nabla^D \times \mathbf{E} \quad (3)$$

3.2 Ampère’s law

First, note that the Ampère law holds for a fractal $W$ embedded in $\mathbb{E}^3$

$$\int_A \mathbf{C} \cdot \mathbf{n} dA_d = \int_I \mathbf{H} \cdot d\mathbf{l}_D = \int_I \mathbf{n} \cdot (\nabla^D \times \mathbf{H}) dA_d \quad (4)$$

Noting

$$\mathbf{C} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

we obtain

$$\int_A \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} - \nabla^D \times \mathbf{H} \right) dA_d = 0 \quad (5)$$

which, again by focusing on the integrand, yields

$$\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} - \nabla^D \times \mathbf{H} = 0 \quad (6)$$

On account of the speed of light being related to the dielectric ($\epsilon_0$) and magnetic ($\mu_0$) constants by the well known relation $c = 1/\sqrt{\epsilon_0 \mu_0}$, the above is equivalent to

$$\frac{1}{\epsilon_0} \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} - c^2 \nabla^D \times \mathbf{B} = 0 \quad (7)$$

$$\frac{1}{\epsilon_0} \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} - c^2 \nabla^D \times \mathbf{B} = 0 \quad (8)$$
The equations (3.7) and (3.8) are subject to the constraints

\[ \begin{align*}
\text{Gauss law for magnetism} & : \nabla^D \cdot \mathbf{B} = 0 \\
\text{Gauss law} & : \nabla^D \cdot \mathbf{E} = 0
\end{align*} \tag{9} \]

where the presence of the fractal (rather than the classical) divergence operator \( \nabla^D \) has clearly been suggested by (2.21).

4 Derivation from variational principle

The variational principle for the Maxwell equations in conventional (non-fractal) setting \( \mathbb{E}^3 \)

\[ \begin{align*}
\nabla \cdot \mathbf{E} & = 0, \\
\nabla \cdot \mathbf{B} & = 0, \\
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} & = 0, \\
\frac{\partial \mathbf{E}}{\partial t} - c^2 \nabla \times \mathbf{B} + \frac{1}{\epsilon_0} \mathbf{J} & = 0,
\end{align*} \tag{1} \]

is (Seliger & Whitham, 1968)

\[ \delta \int \int L dV dt = 0, \]

with

\[ L = \epsilon_0 \left[ \frac{1}{2} c^2 B_i B_i - \frac{1}{2} E_i E_i + A_i \left( \frac{\partial E_i}{\partial t} - c^2 e_{ijk} B_k B_j + \frac{1}{\epsilon_0} J_i \right) + \chi E_i \right]. \tag{2} \]

Here the vector potential \( \mathbf{A} \) and the scalar potential \( \chi \) are the Lagrange multipliers, while the integrand of (4.2) is slightly extended to include the current density \( \mathbf{J} \). By varying (4.2) with respect to \( \mathbf{E}, \mathbf{B}, \mathbf{A}, \) and \( \chi \), gives, respectively,

\[ \begin{align*}
\delta \mathbf{E} : & \quad \mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla \chi \\
\delta \mathbf{B} : & \quad \mathbf{B} = \nabla \times \mathbf{A} \\
\delta \mathbf{A} : & \quad \frac{\partial \mathbf{E}}{\partial t} - c^2 \nabla \times \mathbf{B} + \frac{1}{\epsilon_0} \mathbf{J} = 0 \tag{3} \\
\delta \chi : & \quad \nabla \cdot \mathbf{E} = 0
\end{align*} \]

It is clear that (4.3) satisfy the first pair of Maxwell’s equations (4.1) identically. Henceforth, our goal is to obtain the Maxwell equations (4.1) for an anisotropic fractal, by using the same type of a variational approach.

For a fractal \( \mathcal{W} \) embedded in \( \mathbb{E}^3 \), the variational principle for electromagnetic fields is written as

\[ \delta \int_{t_1}^{t_2} \int_{\mathcal{W}} L dV dt = 0 \tag{4} \]
with $L$ given by (4.2).

Varying (4.4) with respect to $E$, we obtain

$$E = -\partial A / \partial t - \nabla^D (\chi c_1) \quad \text{or} \quad E_i = -\partial A_i / \partial t - \frac{1}{c_1^{(i)}} \left( \chi c_1^{(i)} \right)_i \equiv \nabla^D_i \left( A_i c_1^{(i)} \right)$$

while varying it with respect to $B$, we obtain the Gauss law for magnetism on fractals

$$B = \nabla^D \times (A c_1) \quad \text{or} \quad B_k = e_{ijk} \frac{1}{c_1^{(j)}} \left( A_i c_1^{(j)} \right)_j \equiv \nabla^D_j \left( A_i c_1^{(j)} \right)$$

(5)

In both cases, the fractal gradient operator, $\nabla^D$, shows up due to the presence of $dV_D$ in (4.4); recall (2.5).

The next step is to verify whether substituting (4.5) into (3.3) would give a zero vector. Thus,

$$0 = \frac{\partial}{\partial t} B_k + e_{kji} \frac{1}{c_1^{(j)}} E_{i,j} = \partial B_k / \partial t + e_{kji} \frac{1}{c_1^{(j)}} \left[ -\frac{\partial}{\partial t} \left( A_i c_1^{(j)} \right)_j - \left( \chi c_1^{(j)} \right)_j \right]$$

$$= \partial B_k / \partial t + e_{kji} \frac{1}{c_1^{(j)}} \left[ -\frac{\partial}{\partial t} \left( A_i c_1^{(j)} \right)_j - \left( \chi c_1^{(j)} \right)_j \right]$$

$$= \frac{\partial}{\partial t} \left[ B_k - e_{kji} \frac{1}{c_1^{(j)}} \left( A_i c_1^{(j)} \right)_j \right] - e_{kji} \frac{1}{c_1^{(j)}} \left( \chi c_1^{(j)} \right)_j$$

$$\equiv \frac{\partial}{\partial t} \left[ B_k - \nabla^D_j \times (A_i c_1^{(j)}) \right] - \nabla^D_j \times \left( \chi c_1^{(j)} \right)_j$$

(7)

The first term in the square brackets in the last line of the above agrees with (4.6), while the last term vanishes, providing the fractal is isotropic ($c_1^{(i)} = c_1^{(j)}$, $i \neq j$), in which case (4.7) is satisfied identically. Thus, for an anisotropic fractal, there appears a source/disturbance — an observation consistent with Tarasov’s formulation, though his explicit form differs from ours.

Also, substituting (4.6) into (3.9) yields

$$0 = \frac{\partial}{\partial t} B_k + e_{kji} \frac{1}{c_1^{(j)}} \left( A_i c_1^{(j)} \right)_j \equiv \nabla^D_k \nabla^D_j \times (A_i c_1^{(j)})$$

(8)

which vanishes providing the fractal is isotropic ($c_1^{(i)} = c_1^{(j)}$, $i \neq j$).

Varying (4.4) with respect to $A$, we obtain

$$\partial E / \partial t - c^2 \nabla^D \times B + \frac{1}{\epsilon_0} J = 0 \quad \text{or} \quad \partial E_i / \partial t - c^2 \nabla^D_j \left( B_k c_1^{(j)} \right) + \frac{1}{\epsilon_0} J_i = 0$$

(9)

which agrees with (3.8) perfectly.

Varying (4.4) with respect to $\chi$, we obtain the Gauss law

$$\nabla^D \cdot E = 0 \quad \text{or} \quad \nabla^D_i E_i = \frac{1}{c_1^{(i)}} E_{i,i} = 0$$

(10)
which perfectly agrees with (3.9)\textsubscript{2}. Thus, we have derived by an entirely independent route the same set of Maxwell’s equations modified to fractals as those in Section 2. In analogy to (Li & Ostoja-Starzewski, 2009) which treated the continuum mechanics of fractal elastic media independently by Newtonian and Lagrangian approaches, this shows that the formulation of continuum physics equations based on the product measures is consistent. In the case of isotropy, our equations do not reduce to those of Tarasov (2010).

5 Second order differential equations of electromagnetism

In Gaussian units Equs (3.9), (3.3) and (3.8) are

\begin{align}
\nabla^D \cdot \mathbf{E} &= 0, \\
\nabla^D \cdot \mathbf{B} &= 0, \\
\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} + \nabla^D \times \mathbf{E} &= 0, \\
\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \nabla^D \times \mathbf{B} + \frac{4\pi}{c} \mathbf{J} &= 0.
\end{align}

Taking derivatives gives the second-order Maxwell’s equations for a fractal

\begin{align}
\nabla^D : (\nabla^D \mathbf{E}) - \nabla^D (\nabla^D \cdot \mathbf{E}) &= \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{4\pi}{c^2} \frac{\partial \mathbf{J}}{\partial t} \\
\text{or} \\
\frac{1}{c_1^{(p)}} \left[ E_{k,p}/c_1^{(p)} \right]_p - \frac{1}{c_1^{(l)}} \left[ E_{k,l}/c_1^{(l)} \right]_p &= \frac{1}{c^2} \frac{\partial^2 E_k}{\partial t^2} + \frac{4\pi}{c} \frac{\partial J_k}{\partial t}.
\end{align}

Now, consider two special cases:

Conductor. With Ohm’s law for anisotropic media (2.22), the electrodynamics equations simplify to

\begin{align}
\nabla^D \cdot \mathbf{D} &= 0, \\
\nabla^D \cdot \mathbf{B} &= 0, \\
\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} + \nabla^D \times \mathbf{E} &= 0, \\
\frac{\sigma}{c} \frac{\partial \mathbf{E}}{\partial t} - \nabla^D \times \mathbf{H} &= 0,
\end{align}
Upon using the fractal curl operation, we obtain a parabolic type equation

\[ \nabla^D \cdot \nabla^D H = \sigma \c^2 \cdot \frac{\partial H}{\partial t} \]

or

\[ \frac{1}{c_1(p)^2} \left( H_{k+p} / c_1^{(p)} \right) \cdot p = \frac{\sigma_{km}}{c^2} \frac{\partial H_m}{\partial t} \]  

**Dielectric.** From (5.3) we find a hyperbolic type equation

\[ \nabla^D \cdot \nabla^D E = \sigma \c^2 \cdot \frac{\partial^2 E}{\partial^2 t} \]

or

\[ \frac{1}{c_1(p)^2} \left( E_{k+p} / c_1^{(p)} \right) \cdot p = \frac{\sigma_{km}}{c^2} \frac{\partial^2 E_m}{\partial^2 t} \]

6 Electromagnetic Energy and Stress

**Pontying vector.** Now, we return to (3.3) and (3.7). Multiplying the first of these equations by \( H \), and the second one by \( E \), leads to

\[ \nabla^D \times E \cdot H = -\frac{\partial B}{\partial t} \cdot H \]  

(1)

\[ \nabla^D \times H \cdot E = J \cdot E + \frac{\partial}{\partial t} D \cdot E \]  

(2)

Subtracting (6.2) from (6.1), on account of

\[ H \cdot \nabla^D \times E + E \cdot \nabla^D \times H = \nabla^D \cdot (E \times H), \]  

(3)

we obtain

\[ \nabla^D \cdot (E \times H) + J \cdot E = -\frac{\partial}{\partial t} D \cdot E - \frac{\partial}{\partial t} B \cdot H \]

or

\[ \epsilon_{ijk} \frac{1}{c_1^{(j)}} (E_k H_i)_{(j)} + J_i E_i = -E_i \frac{\partial}{\partial t} D_i - H_i \frac{\partial}{\partial t} B_i \]

Integrating (6.4) over the fractal’s volume and applying the Green-Gauss theorem, yields

\[ \int_S G \cdot n \, dS_d + \int_W J \cdot E \, dV_D = -\int_W \left[ \frac{\partial}{\partial t} D \cdot E - \frac{\partial}{\partial t} B \cdot H \right] dV_D \]  

(5)
where
\[ G = E \times H \] (6)
is identified as the Poynting vector, which is seen to have the same form as in non-fractal media.

Now, the electric and magnetic force densities are
\[ f^E = \rho E + P \cdot \nabla D \]
\[ f^M = J^M \times B + M \cdot (B^D \nabla) = \frac{1}{c} J^E \times B + M \cdot (B \nabla D) \] (7)

This has to be accompanied by an expression of the Ampère law in electrostatic units
\[ \nabla D \times H = \frac{4\pi}{c} J + \frac{1}{c} \dot{D} = \frac{4\pi}{c} (J + \frac{1}{c} \dot{D}) \] (8)

### 7 Conclusion

We determine the equations governing electromagnetic fields in generally anisotropic fractal media using the dimensional regularization together with the recently introduced product measure allowing an independent characterization of anisotropy in three Cartesian directions of the Euclidean space in which an anisotropic fractal is embedded. This measure also gives rise to fractal gradient, divergence and curl operators, which are shown to satisfy the four fundamental identities of the vector calculus. Next, the conservation of the electric charge on a fractal shows that the fractal divergence has to be used. With this Ansatz, in the first place, we obtain Maxwell equations modified to generally anisotropic fractals using two independent approaches: a conceptual one (involving generalized Faraday and Ampère laws), and the one directly based on a variational principle for electromagnetic fields. In both cases the resulting equations are the same, thereby providing a self-consistent verification of our derivations.

Just as in the previous works of Tarasov, we find that the presence of anisotropy in the fractal structure leads to a source/disturbance as a result of generally unequal fractal dimensions in various directions. However, our modified Maxwell equations do not coincide with those of Tarasov (2006, 2010). The modifications due to fractal geometry carry over to the parabolic (for conductor) and hyperbolic (for dielectric) equations, while the Poynting vector is found to have the same form as in the non-fractal case. Finally, the electromagnetic stress tensor is reformulated for fractal systems. Overall, all the derived relations depend explicitly on three fractal dimensions \( \alpha_i \) in the respective Cartesian directions \( x_i, i = 1, 2, 3 \), as well as the spatial resolution; upon setting all \( \alpha_i = 1 \), these relations reduce to conventional forms of Maxwell equations.

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