Research Article

Weighted Composition Operators from Besov Zygmund-Type Spaces into Zygmund-Type Spaces

Xiangling Zhu\(^1\) and Nanhui Hu\(^2,3\)

\(^1\)University of Electronic Science and Technology of China, Zhongshan Institute, 528402 Zhongshan, Guangdong, China
\(^2\)Department of Mathematics, Shantou University, Shantou, 515063 Guangdong, China
\(^3\)Department of Mathematics, Jiaying University, Meizhou, 514015 Guangdong, China

Correspondence should be addressed to Nanhui Hu; hunhjyu@163.com

Received 1 February 2020; Accepted 9 June 2020; Published 7 July 2020

Abstract

The boundedness, compactness, and essential norm of weighted composition operators from Besov Zygmund-type spaces into Zygmund-type spaces are investigated in this paper.

1. Introduction

Let \(D\) denote the open unit disk in the complex plane \(\mathbb{C}\) and \(H(D)\) the space of all analytic functions in \(D\). For an analytic self-map \(\varphi\) of \(D\) and \(u \in H(D)\), the weighted composition operator \(uC_{\varphi}\) is defined as follows:

\[
(uC_{\varphi}f)(z) = u(z)f(\varphi(z)), \quad f \in H(D), \quad z \in D. \tag{1}
\]

When \(u = 1\), \(uC_{\varphi}\) is just the composition operator, denoted by \(C_{\varphi}\). In the past several decades, composition operators and weighted composition operators have received much attention and appear in various settings in the literature (see, for example, [2–5, 8, 10, 13, 15, 16, 19]).

Let \(\alpha \in (0, \infty)\). The Bloch type space \(\mathcal{B}^\alpha\) consists of those functions \(f \in H(D)\) for which

\[
\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in D}(1 - |z|^2)^\alpha |f'(z)| < \infty. \tag{2}
\]

\(\mathcal{B}^\alpha\) is a Banach space under the above norm. It is known that when \(\alpha = 1\), \(\mathcal{B}^1 = \mathcal{B}\) is the classical Bloch space.

For \(0 < \beta < \infty\), an \(f \in H(D)\) is said to be in the Zygmund-type space \(\mathcal{Z}^\beta\), if

\[
\|f\|_{\mathcal{Z}^\beta} = |f(0)| + |f'(0)| + \sup_{z \in D}(1 - |z|^2)^\beta |f''(z)| < \infty. \tag{3}
\]

It is easy to check that \(\mathcal{Z}^\beta\) is a Banach space under the norm \(\|\cdot\|_{\mathcal{Z}^\beta}\). When \(\beta = 1\), \(\mathcal{Z}^1 = \mathcal{Z}\) is the Zygmund space. When \(\beta > 1\), \(\mathcal{Z}^\beta\) is just the Bloch type space \(\mathcal{B}^{\beta-1}\). In particular, when \(\beta = 2\), \(\mathcal{Z}^2\) is just the Bloch space \(\mathcal{B}\). Hence, the Zygmund space is the space of all \(f \in H(D)\) such that \(f' \in \mathcal{B}\) with norm

\[
\|f\|_{\mathcal{Z}} = |f(0)| + \|f'\|_{\mathcal{B}}. \tag{4}
\]

Let \(dA\) be the normalized area measure on \(D\). For \(1 < p < \infty\), the Besov space, denoted by \(B_p\), is the space of all \(f \in H(D)\) such that

\[
b_p(f) := \left( \int_D |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) \right)^{1/p} < \infty. \tag{5}
\]

This space is a Banach space with the following norm \(\|f\|_{B_p} = |f(0)| + b_p(f)\). In particular, \(B_2\) is the classical Dirichlet...
space. Besov spaces are Möbius invariant in the sense that $b_p(f \circ \psi) = b_p(f)$ for all $f \in B_p$ and $\psi \in \text{Aut}(D)$, the set of all Möbius maps of $D$ (see [1, 19]).

In [4], Colonna and Tjani introduced a new class type space $\mathcal{L}^p_{\beta,2}$, called the Besov Zygmund-type space, which consists of all $f \in H(D)$ such that $f' \in B_p$. Since the Besov space is contained in the Bloch space, it follows that the Besov Zygmund-type space is a subset of the Zygmund space, and hence, it is contained in the disk algebra.

Colonna and Li studied the boundedness and compactness of $uC_p : \mathcal{L}^p_{\beta,2} \rightarrow \mathcal{B}$ and related operators on the Zygmund space and results of composition operators, weighted composition operators, and related operators on the Zygmund spaces.) Colonna and Tjani characterized the boundedness and compactness of $uC_p : \mathcal{L}^p_{\beta,2} \rightarrow \mathcal{B}$ in [4].

Throughout the paper, we denote by $C$ a positive constant which may differ from one occurrence to the next. In addition, we say that $A \leq B$ if there exists a constant $C$ such that $A \leq CB$. The symbol $A \sim B$ means that $A \leq B \leq A$.

2. Main Results and Proofs

In this section, we formulate and prove our main results in this paper. For this purpose, we need the following lemmas.

Lemma 1. Suppose $1 < p < \infty$. Then, there exists a positive constant $C$ such that

$$|f'(z)| \leq C\|f\|_{\mathcal{L}^p_{\beta,2}} \left( \log \frac{2}{1 - |z|^2} \right)^{-1/(1-p)},$$

(6)

$$|f''(z)| \leq C\|f\|_{\mathcal{L}^p_{\beta,2}},$$

(7)

$$\|f\|_{\infty} \leq C\|f\|_{\mathcal{L}^p_{\beta,2}},$$

(8)

for every $f \in \mathcal{L}^p_{\beta,2}$.

Proof. For $f \in B_p$, it is well known that

$$|f(z)| \leq \|f\|_{B_p} \left( \log \frac{2}{1 - |z|^2} \right)^{-1/(1-p)}, \quad z \in D,$$

$$\|f\|_{B_p} \leq \|f\|_{L_p}.$$  

(9)

Then, the inequalities in (6) follow from the definition of the Besov Zygmund-type space. Since the Zygmund space is continuously embedded into $H^{\infty}$, as shown in Lemma 2.1 of [18], we get that $\|f\|_{L_p} \leq C\|f\|_{\mathcal{L}^p_{\beta,2}}$. The proof is complete.

Lemma 2 (see [4]). Let $1 < p < \infty$. Every sequence in $\mathcal{L}^p_{\beta,2}$ bounded in norm has a subsequence which converges uniformly in $D$ to a function in $\mathcal{L}^p_{\beta,2}$.

Lemma 3 (see [4]). Let $X$ be a Banach space that is continuously contained in the disk algebra, and let $Y$ be any Banach space of analytic functions on $D$. Suppose that

(i) the point evaluation functionals on $Y$ are continuous

(ii) for every sequence $\{f_n\}$ in the unit ball of $X$ that exists $f \in X$ and a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \rightarrow f$ uniformly on $D$

(iii) the operator $T : X \rightarrow Y$ is continuous if $X$ has the supremum norm and $Y$ is given the topology of uniform convergence on compact sets

Then, $T$ is a compact operator if and only if given a bounded sequence $\{f_j\}$ in $X$ such that $f_j \rightarrow 0$ uniformly on $D$, then the sequence $\|f_j\|_Y \rightarrow 0$ as $n \rightarrow \infty$.

The following result is a direct consequence of Lemmas 2 and 3.

Lemma 4. Let $1 < p < \infty$ and $0 < \beta < \infty$. If $T : \mathcal{L}^p_{\beta,2} \rightarrow \mathcal{L}^\beta$ is bounded, then $T$ is compact if and only if $\|Tf\|_{\mathcal{L}^\beta} \rightarrow 0$ as $k \rightarrow \infty$ for any sequence $\{f_j\}$ in $\mathcal{L}^p_{\beta,2}$ bounded in norm which converge to 0 uniformly in $D$.

The following estimates are fundamental in operator theory and function spaces on the unit disk (see ([19], Lemma 3.10)).

Lemma 5 (see [19]). Suppose that $z \in \mathbb{D}$ is real, $t > -1$, and

$$I_{t,\beta}(z) = \frac{(1 - |w|^2)^t}{|1 - wz|^{2 \beta + t}} dA(w).$$

(10)

(i) If $c < 0$, then as a function of $z$, $I_{t,\beta}$ is bounded on $\mathbb{D}$

(ii) If $c = 0$, then $I_{t,\beta}(z) = \log \left( \frac{1}{(1 - |z|^2)} \right)$, as $|z| \rightarrow 1$

(iii) If $c > 0$, then $I_{t,\beta}(z) = \frac{1}{(1 - |z|^2)^{c}}$, as $|z| \rightarrow 1$

Now, we are in a position to give the following characterization of bound composition operators from $\mathcal{L}^p_{\beta,2}$ to $\mathcal{L}^\beta$.

Theorem 6. Let $1 < p < \infty$, $0 < \beta < \infty$, $u \in H(D)$, and $\psi$ be an analytic self-map of $D$. Then, $uC_\psi : \mathcal{L}^p_{\beta,2} \rightarrow \mathcal{L}^\beta$ is bounded if and only if $u \in \mathcal{L}^\beta$, 

$$\sup_{z \in D} (1 - |z|^2)^{\beta} |2u'(z)\psi'(z)| + u(z)|\psi''(z)| \left( \log \frac{2}{1 - |\psi(z)|^2} \right)^{-1/(1-p)} < \infty,$$

(11)
\[
\sup_{z \in \mathcal{D}} \frac{(1 - |z|^2)^\beta |u(z)||\varphi'(z)|^2}{1 - |\varphi(z)|^2} < \infty. \tag{12}
\]

**Proof.** First, suppose that \( u \in \mathcal{L}^\beta \), (11) and (12) hold. For arbitrary \( z \in \mathcal{D} \) and \( f \in \mathcal{L}^p_{p-2} \), by Lemma 1, we have
\[
|\langle uC\varphi f \rangle(0) | \leq |u(0)||\|f\|_{\mathcal{L}^p_{p-2}}|,
\]
\[
\left| \langle uC\varphi f \rangle'(0) \right| \leq \left( \left| u'(0) \right| + |u(0)\varphi'(0) | \right)
\cdot \left( \log \frac{2}{1 - |\varphi(0)|^2} \right)^{1-(1/p)} \|f\|_{\mathcal{L}^p_{p-2}}.
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |\langle uC\varphi f \rangle''(z) |
\leq \sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u'(z)||\varphi'(z) ||f'(\varphi(z))|
\leq \sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
\]
\[
\sup_{z \in \mathcal{D}} (1 - |z|^2)^\beta |u''(z)||\|\varphi'(z) ||\|f''(\varphi(z))|,
By (14), we get
\[
\begin{aligned}
sup_{|z| \leq 1/2} (1 - |z|^2) \left| 2u'(w) \varphi' (w) \right| &+ u(w) \varphi''(w)\left| (\log \left( \sqrt{2} \, \varphi(w) \right) \right) \right|^{1/(1+p)} \\
&\leq \sup_{w \in \mathbb{D}} (1 - |w|^2) \left| 2u'(w) \varphi' (w) + u(w) \varphi''(w) \right| < \infty.
\end{aligned}
\]
(24)

From (23) and (24), we see that (11) holds. For $a \in \mathbb{D}$, define
\[
g_a(z) = \frac{1}{2} \left( \frac{(1 - |a|^2)^2}{(1 - az)^2} - \frac{(1 - |a|^2)^3}{(1 - az)^3} \right) + \frac{1}{2} \left( \frac{(1 - |a|^2)^4}{(1 - az)^4} \right).
\]
(25)

So,
\[
\begin{aligned}
g'_{a}(z) &= \frac{a}{2} \left( \frac{(1 - |a|^2)^2}{(1 - az)^2} - \frac{(1 - |a|^2)^3}{(1 - az)^3} \right) + \frac{3}{2} \left( \frac{(1 - |a|^2)^4}{(1 - az)^4} \right), \\
g''_{a}(z) &= a^2 \left( \frac{(1 - |a|^2)^2}{(1 - az)^2} - \frac{6a}{(1 - az)^3} \right) \left( \frac{(1 - |a|^2)^3}{(1 - az)^3} \right) + 6a^2 \left( \frac{(1 - |a|^2)^4}{(1 - az)^4} \right).
\end{aligned}
\]
(26)

By Lemma 5, we see that $g_a \in \mathcal{L}_p \to \mathcal{X}^\beta$ and $\sup_{a \in \mathbb{D}} \| g_a \|_{\mathcal{L}_p} < \infty$. By the boundedness of $uC_{\varphi} : \mathcal{L}_p \to \mathcal{X}^\beta$, we get $\sup_{a \in \mathbb{D}} \| uC_{\varphi} g_a \|_{\mathcal{X}^\beta} < \infty$. After a calculation, we have
\[
g_a(0) = 0, \\
g'_a(0) = 0, \\
|g''_a(a)| = \frac{|a|^2}{1 - |a|^2}.
\]
(27)

Hence, for any $w \in \mathbb{D}$,
\[
\begin{aligned}
(1 - |z|^2) \left| \varphi(w) \right| (u(w)) \left| \varphi'(w) \right| \left| (\log \left( \sqrt{2} \, \varphi(w) \right) \right) \right|^{1/(1+p)} \\
&\leq \| uC_{\varphi} g(w) \|_{\mathcal{X}^\beta} < \infty.
\end{aligned}
\]
(28)

On the one hand, from (28), we obtain
\[
\sup_{|z| \leq 1/2} (1 - |z|^2) \left| u(w) \right| \left| \varphi'(w) \right| < \infty.
\]
(29)

On the other hand, by (15), we get
\[
\begin{aligned}
\sup_{|z| < 1/2} (1 - |z|^2) \left| u(w) \right| \left| \varphi'(w) \right| \left| (\log \left( \sqrt{2} \, \varphi(w) \right) \right) \right|^{1/(1+p)} \\
&\leq \sup_{w \in \mathbb{D}} (1 - |w|^2) \left| u(w) \varphi'(w) \right| < \infty.
\end{aligned}
\]
(30)

From (29) and (30), we see that (12) holds. The proof is complete.

Next, we estimate the essential norm of $uC_{\varphi} : \mathcal{L}_p^{\infty} \to \mathcal{X}^\beta$. Recall that the essential norm of $uC_{\varphi} : \mathcal{L}_p^{\infty} \to \mathcal{X}^\beta$ is defined as the distance from $uC_{\varphi}$ to the set of compact operators $K : \mathcal{L}_p^{\infty} \to \mathcal{X}^\beta$, that is,
\[
\| uC_{\varphi} \|_{e, \mathcal{L}_p^{\infty} \to \mathcal{X}^\beta} = \inf \{ \| uC_{\varphi} - K \|_{\mathcal{L}_p^{\infty} \to \mathcal{X}^\beta} : K \text{ is a compact operator} \}.
\]
(31)

**Theorem 7.** Let $1 < p < \infty, 0 < \beta < \infty, u \in H(\mathbb{D})$, and $\varphi$ be an analytic self-map of $\mathbb{D}$ such that $uC_{\varphi} : \mathcal{L}_p^{\infty} \to \mathcal{X}^\beta$ is bounded. Then,
\[
\| uC_{\varphi} \|_{e, \mathcal{L}_p^{\infty} \to \mathcal{X}^\beta} = \max \{ E, G \}.
\]
(32)

Here,
\[
E = \limsup_{|z| \to 1} (1 - |z|^2) \varphi(z) (u(z)) \left| \varphi'(z) \right| \left| (\log \frac{2}{1 - |z|^2}) \right|^{1/(1+p)},
\]
\[
G = \limsup_{|z| \to 1} (1 - |z|^2) \left| u(z) \right| \left| \varphi'(z) \right| \left| (\frac{2}{1 - |z|^2}) \right|^{1/(1+p)}.
\]
(33)

**Proof.** First we prove that
\[
\| uC_{\varphi} \|_{e, \mathcal{L}_p^{\infty} \to \mathcal{X}^\beta} \geq \max \{ E, G \}.
\]
(34)

Let $\{ z_j \}_{j \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Define
\[
k_j(z) = \left( \frac{\varphi(z_j)z - 1}{\varphi(z_j)} \right) \left[ \left( 1 + \log \frac{2}{1 - \varphi(z_j)z} \right)^{1/(1+p)} + 1 \right]
\]
\[
\cdot \left( \log \frac{2}{1 - |\varphi(z_j)|^2} \right)^{-1/(1+p)} - 2 \int_0^z \log \frac{2}{1 - |\varphi(z_j)|} dw
\]
\[
\cdot \left( \log \frac{2}{1 - |\varphi(z_j)|^2} \right)^{-1/(1+p)}.
\]
(35)

\[
l_j(z) = \frac{1}{2} (1 - |\varphi(z_j)|^2)^{1/(1+p)} \left( 1 - |\varphi(z_j)|^2 \right)^{3/(1+p)} + \frac{1}{2} (1 - |\varphi(z_j)|^2)^{3/(1+p)}.
\]
From the proof of Theorem 6, we see that $k_j$ and $l_j$ belong to $L_p^p$. Moreover, $k_j$ and $l_j$ converge to 0 uniformly on $D$ as $j \to \infty$. Hence, for any compact operator $K : L_p^p \to L_p^p$, by Lemma 4, we obtain

\[
\|u_C - K\|_{L_p^p} \geq \limsup_{j \to \infty} \|u_C(k_j)\|_{L_p^p} - \limsup_{j \to \infty} \|K(k_j)\|_{L_p^p} \\
\geq \limsup_{j \to \infty} \left(1 - |z_j|^2\right)\beta u'\left(z_j\right) \phi'(z_j) / 1 - |\phi(z_j)|^2 + u(z) \phi'(z_j) / 1 - |\phi(z_j)|^2,
\]

as desired.

Next, we prove that

\[
\|u_C\|_{L_p^p} \leq \max \{ E, G \}.
\]

Let $r \in [0, 1)$. Define $K_r : H(D) \to H(D)$ by

\[
(K_r f)(z) = f_r(z) = f(rz), \quad f \in H(D).
\]

It is clear that $K_r$ is compact on $L_p^p$ and $\|K_r\|_{L_p^p} \leq 1$; moreover, $f_r - f \to 0$ uniformly on compact subsets of $D$ as $r \to 1$. Let $\{ r_j \} \subset (0, 1)$ such that $r_j \to 1$ as $j \to \infty$. Then, for each $j \in \mathbb{N}$, $u_C(K_{r_j}) : L_p^p \to L_p^p$ is compact. Hence,

\[
\|u_C\|_{L_p^p} \leq \limsup_{j \to \infty} \|u_C - u_C K_{r_j}\|_{L_p^p} \leq \max \{ E, G \}.
\]

Thus, we only need to show that

\[
\limsup_{j \to \infty} \left| u(0) f_r(\phi(0)) - u(0) f(\phi(0)) \right| = 0.
\]

For any $f \in L_p^p$ with $\|f\|_{L_p^p} \leq 1$, from the facts that

\[
\left| u(0) f_r(\phi(0)) - u(0) f(\phi(0)) \right| = 0,
\]

we have

\[
\limsup_{j \to \infty} \left| u(0) f_r(\phi(0)) - u(0) f(\phi(0)) \right| = 0.
\]

(42)
Now, we estimate $S_3$. Using Lemma 1 and the fact that $\|f\|_{L^p_{\beta}} \leq 1$, we have

$$S_3 = \limsup_{j \to \infty} \sup_{|z| \leq r_j} (1 - |z|^2)^\beta \left| u'(z)\phi'(z) + u(z)\phi''(z) + \frac{u(z)}{\rho(z)} \right| \leq \frac{u(z)}{\rho(z)}.$$ (47)

Taking the limit as $t \to \infty$, we get

$$S_3 \leq E.$$ (48)

Similarly, again by Lemma 1,

$$S_5 = \limsup_{j \to \infty} \sup_{|z| \leq r_j} (1 - |z|^2)^\beta \left| u'(z)\phi'(z) + u(z)\phi''(z) + \frac{u(z)}{\rho(z)} \right| \leq \frac{u(z)}{\rho(z)}.$$ (49)

Taking the limit as $t \to \infty$, we get

$$S_5 \leq G.$$ (50)

Hence, by (43), (44), (45), (46), (48), and (50), we get

$$\limsup_{j \to \infty} \left\| uC_{\phi} - uC_{\phi}K_j \right\|_{L^p_{\beta} \to L^p_{\beta}} \leq \max \{ E, G \},$$ (51)

which with (40) implies the desired result. The proof is complete.

From Theorem 7 and the result that $\|uC_{\phi}\|_{L^p_{\beta} \to L^p_{\beta}} = 0$ if and only if $uC_{\phi} : L^p_{\beta} \to L^p_{\beta}$ is compact, we get the following corollary.

**Corollary 8.** Let $1 < p < \infty$, $0 < \beta < \infty$, $u \in H(D)$, and $\phi$ be an analytic self-map of $D$ such that $uC_{\phi} : L^p_{\beta} \to L^p_{\beta}$ is bounded. Then, $uC_{\phi} : L^p_{\beta} \to L^p_{\beta}$ is compact if and only if

$$\limsup_{|\phi(z)| \to 1} \left( 1 - |z|^2 \right)^\beta \left| u(z)\phi''(z) \right| = 0,$$

$$\limsup_{|\phi(z)| \to 1} \left( 1 - |z|^2 \right)^\beta \left| u(z)\phi'(z) + \frac{u(z)}{\rho(z)} \right| = 0.$$ (52)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**

[1] S. D. Fisher, J. Arazy, and J. Peetre, "Möbius invariant function spaces," *Journal für die Reine und Angewandte Mathematik*, vol. 1985, no. 363, pp. 110–145, 1985.

[2] F. Colonna and S. Li, "Weighted composition operators from $H^1$ into the Zygmund spaces," *Complex Analysis and Operator Theory*, vol. 7, no. 5, pp. 1495–1512, 2013.

[3] F. Colonna and S. Li, "Weighted composition operators from the Lipschitz space into the Zygmund space," *Mathematical Inequalities & Applications*, vol. 17, no. 3, pp. 963–975, 2014.

[4] F. Colonna and M. Tjani, "Weighted composition operators from the Besov spaces into the weighted-type space $H^p_{\omega}$," *Journal of Mathematical Analysis and Applications*, vol. 402, no. 2, pp. 594–611, 2013.

[5] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[6] J. Du, S. Li, and Y. Zhang, "Essential norm of generalized composition operators on Zygmund type spaces and Bloch type spaces," *Annales Polonici Mathematici*, vol. 119, no. 2, pp. 107–119, 2017.

[7] J. Du, S. Li, and Y. Zhang, "Essential norm of weighted composition operators on Zygmund-type spaces with normal weight," *Mathematical Inequalities & Applications*, vol. 21, no. 3, pp. 701–714, 2018.

[8] K. Esmaeili and M. Lindström, "Weighted composition operators between Zygmund type spaces and their essential norms," *Integral Equations and Operator Theory*, vol. 75, no. 4, pp. 473–490, 2013.

[9] Q. Hu, S. Li, and Y. Zhang, "Essential norm of weighted composition operators from analytic Besov spaces into Zygmund type spaces," *Journal of Contemporary Mathematical Analysis*, vol. 54, no. 3, pp. 129–142, 2019.
[10] H. Li and X. Fu, “A new characterization of generalized weighted composition operators from the Bloch space into the Zygmund space,” *Journal of Function Spaces and Applications*, vol. 2013, article 925901, 6 pages, 2013.

[11] S. Li and S. Stević, “Volterra type operators on Zygmund spaces,” *Journal of Inequalities and Applications*, vol. 2007, Article ID 32124, 11 pages, 2007.

[12] S. Li and S. Stević, “Generalized composition operators on Zygmund spaces and Bloch type spaces,” *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1282–1295, 2008.

[13] S. Li and S. Stević, “Weighted composition operators from Zygmund spaces into Bloch spaces,” *Applied Mathematics and Computation*, vol. 206, no. 2, pp. 825–831, 2008.

[14] S. Li and S. Stević, “Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces,” *Applied Mathematics and Computation*, vol. 217, no. 7, pp. 3144–3154, 2010.

[15] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.

[16] S. Stević, “Weighted differentiation Composition Operators from the Mixed-Norm Space to the nth Weighed-Type Space on the Unit Disk,” *Abstract and Applied Analysis*, vol. 2010, Article ID 246287, 15 pages, 2010.

[17] S. Stević, “Weighted differentiation composition operators from $H^\infty$ and Bloch spaces to $n$th weighted-type spaces on the unit disk,” *Applied Mathematics and Computation*, vol. 216, no. 12, pp. 3634–3641, 2010.

[18] S. Ye and Q. Hu, “Weighted composition operators on the Zygmund space,” *Abstract and Applied Analysis*, vol. 2012, Article ID 462482, 18 pages, 2012.

[19] K. Zhu, *Operator Theory in Function Spaces*, vol. 138 of Math. Surveys and Monographs, American Mathematical Society, Providence, Rhode Island, 2nd edition, 2007.