ANALYSIS OF THE MINIMAL REPRESENTATION OF $\text{Sp}(r, \mathbb{R})$

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Abstract The minimal representations of $\text{Sp}(r, \mathbb{R})$ can be realized on a Hilbert space of holomorphic functions. This is the analogue of the Brylinski-Kostant model. It can also be realized on a Hilbert space of $L^2$ functions on $\mathbb{R}^n$. This is the Schrödinger model. We will describe the two realizations and a transformation which maps one model to the other. It involves the classical Bargmann transform.

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Introduction. In the paper [A11] a general construction for a simple complex Lie algebra $\mathfrak{g}$ and of a real form $\mathfrak{g}_\mathbb{R}$ has been proposed, starting from a pair $(V, Q)$ where $V$ is a simple Jordan algebra of rank $r$ and $Q$ a polynomial on $V$, homogeneous of degree $2r$. The Lie algebra $\mathfrak{g}$ is of Hermitian type. In the paper [A12b], the manifolds $\Xi$ and $\Xi^\sigma$ are the orbits of the linear form $\tau_{c_{\lambda}}$ and its conjugate $\tau_{c_{\lambda}}^\sigma = \kappa(\sigma)\tau_{c_{\lambda}}$, for an idempotent $c_{\lambda}$, under the structure group $\text{Str}(V)$ acting by the restriction of an irreducible representation $\kappa$ of the conformal group $\text{Conf}(V, Q)$. The spaces $\mathcal{F}(\Xi)$ and $\mathcal{F}(\Xi^\sigma)$ are Hilbert spaces of holomorphic functions on the complex manifolds $\Xi$ and $\Xi^\sigma$. The minimal representations of $\mathfrak{g}_\mathbb{R}$ are realized in $\mathcal{F}(\Xi)$ and $\mathcal{F}(\Xi^\sigma)$. In this paper we consider the special case where $V = \text{Sym}(r, \mathbb{C})$, $Q$ is the square of the determinant and the construction leads to the Lie algebra $\mathfrak{g} = \text{sp}(r, \mathbb{C})$ and to a real form $\mathfrak{g}_\mathbb{R} = \text{Ad}(g_0)(\text{sp}(r, \mathbb{R}))$ for some $g_0 \in \text{Sp}(r, \mathbb{C})$. The two minimal representations $\rho$ and $\rho^\sigma$ of $\mathfrak{g}_\mathbb{R}$ are respectively realized in $\mathcal{F}(\Xi)$ and $\mathcal{F}(\Xi^\sigma)$, which turn out to be the classical Fock space $\mathcal{F}(\mathbb{C}^r)$. They integrate to representations $T$ and $T^\sigma$ of the complex group $\text{Mp}(r, \mathbb{C})$ and their restriction to $\text{Mp}(r, \mathbb{R})$ realize unitary representations in $T(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))$ and $T^\sigma(g_0^{-1})(\mathcal{F}(\mathbb{C}^r))$. On another hand, we consider the two minimal representations $\mathcal{R}$ and $\mathcal{R}^\sigma$, i.e. the Segal-Shale-Weil representations, of $\text{Mp}(r, \mathbb{R})$ on $L^2(\mathbb{R}^r)$. We also give unitary integral operators $\mathcal{B}$ from the space $L^2(\mathbb{R}^r)$ onto the space $T(g_0^{-1})(\mathcal{F}(\Xi))$ which intertwins $\rho = dT$ and $d\mathcal{R}$ of $\text{sp}(r, \mathbb{R})$, and $\mathcal{B}^\sigma$ from the space $L^2(\mathbb{R}^r)$ onto the space $T^\sigma(g_0^{-1})(\mathcal{F}(\Xi^\sigma))$ which intertwins $\rho^\sigma = dT^\sigma$ and $d\mathcal{R}^\sigma$ of $\text{sp}(r, \mathbb{R})$. 

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1. The analogue of the Brylinski-Kostant model. General case

Let $V$ be a simple complex Jordan algebra with rank $r$ and dimension $n$ and $Q$ the homogeneous polynomial of degree $2r$ on $V$ given by $Q(v) = \Delta(v)^2$ where $\Delta$ is the Jordan algebra determinant. The structure group of $V$ is defined by:

$$\text{Str}(V) = \{ g \in \text{GL}(V) \mid \exists \chi(g) \in \mathbb{C}, \Delta(gz) = \chi(g)\Delta(z) \}$$

The conformal group $\text{Conf}(V)$ is the group of rational transformations $g$ of $V$ generated by: the translations $z \mapsto z + a$ ($a \in V$), the dilations $z \mapsto \ell z$ ($\ell \in \text{Str}(V)$), and the conformal inversion $\sigma : z \mapsto -z^{-1}$ (see [M78]).

Let $p$ be the space of polynomials on $V$ generated by the polynomials $Q(z - a)$ of $Q$, with $a \in V$. Let $\kappa$ be the cocycle representation of $K = \text{Conf}(V)$, defined in [A11] and [AF12] as follows:

$$(\kappa(g)p)(z) = \mu(g^{-1}, z)p(g^{-1}z),$$

$$(\mu(g, z) = \chi((Dg(z)^{-1}) \quad (g \in K, z \in V).$$

The function $\kappa(g)p$ belongs actually to $p$ (see [FG96], Proposition 6.2). The cocycle $\mu(g, z)$ is a polynomial in $z$ of degree $\leq \deg Q$ and

$$(\kappa(\tau_a)p)(z) = p(z - a) \quad (a \in V),$$

$$(\kappa(\ell)p)(z) = \chi(\ell)p(\ell^{-1}z) \quad (\ell \in L),$$

$$(\kappa(\sigma)p)(z) = Q(z)p(-z^{-1}).$$

Let $L = \text{Str}(V)$. It is established in [A11] that the element $H_0 \in \mathfrak{z}(\mathfrak{l})$, given by $\exp(tH_0) = \mathcal{l}_{e^{-t}} : z \in V \mapsto e^{-t}z$, defines a grading of $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{p}_{-r} \oplus \mathfrak{p}_{-r+1} \oplus \ldots \oplus \mathfrak{p}_0 \oplus \ldots \oplus \mathfrak{p}_{r-1} \oplus \mathfrak{p}_r,$$

$$\mathfrak{p}_j = \{ p \in \mathfrak{p} \mid d\kappa(H_0)p = j\mathfrak{p} \}$$

is the set of polynomials in $\mathfrak{p}$, homogeneous of degree $j + r$. Furthermore $\kappa(\sigma) : \mathfrak{p}_j \rightarrow \mathfrak{p}_{-j}$, and $\mathfrak{p}_{-r} = \mathbb{C}$, $\mathfrak{p}_r = \mathbb{C}Q$, $\mathfrak{p}_{r-1} \simeq V$, $\mathfrak{p}_{-r+1} \simeq V$. Observe that $\mathfrak{p}_{r+1} = \{ \kappa(\sigma)p \mid p \in \mathfrak{p}_{r+1} \}$ and that $\mathfrak{p}_{-r+1}$ is the space of linear forms on $V$.

Assume $r \neq 1$ and denote by $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^\sigma$ with $\mathcal{V} = \mathfrak{p}_{-r+1}$, $\mathcal{V}^\sigma = \mathfrak{p}_{r-1}$. Let’s consider the linear form $\tau : z \mapsto \text{tr}(z) = \text{trace}(z)$ and its image $\tau^\sigma$ given by $\tau^\sigma(z) = Q(z)\tau(-z^{-1})$. Denote by $E = \tau$ and $F = \tau^\sigma$ and let $X_E \in \mathfrak{t}_1$ and $X_F \in \mathfrak{t}_{-1}$ such that $E = d\kappa(X_E)1$ and $F = d\kappa(X_F)Q$. Put $[1, Q] = H_0$, $[E, Q] = -X_E$, $[F, 1] = X_F$. Let $\lambda_0$ be a bilinear form on $\mathcal{V} \times \mathcal{V}^\sigma$. Then $\mathfrak{g} = \mathfrak{t} \oplus \mathcal{W}$ carries a unique simple Lie algebra structure such that

(i) $[X, X'] = [X, X']_t \quad (X, X' \in \mathfrak{t}),$

(ii) $[X, p] = d\kappa(X)p \quad (X \in \mathfrak{t}, p \in \mathcal{W}),$

(iii) $[E, F] = \lambda_0(E, F)H_0 + [X_E, X_F].$
We recall also the real form \( \mathfrak{g}_V \) of \( \mathfrak{g} \) which will be considered in the sequel. We fix a Euclidean real form \( V_\mathbb{R} \) of the complex Jordan algebra \( V \), denote by \( z \mapsto \bar{z} \) the conjugation of \( V \) with respect to \( V_\mathbb{R} \), and then consider the involution \( g \mapsto \bar{g} \) of \( K \) given by: \( \bar{g}z = \bar{g}\bar{z} \). The involution \( \alpha \) defined by \( \alpha(g) = \sigma\bar{g}\sigma^{-1} \) is a Cartan involution of \( K \) and \( K_\mathbb{R} = \{ g \in K \mid \alpha(g) = g \} \) is a compact real form of \( K \) and it follows that \( L_\mathbb{R} = L \cap K_\mathbb{R} \) is a compact real form of \( L \). Observe that, since for \( g \in \text{Str}(V) \), \( \sigma \circ g \circ \sigma = g' \), the adjoint of \( g \) with respect to the symmetric form \( (w, w') = \tau(w\bar{w}') \), then \( L_\mathbb{R} = \{ l \in L \mid ll' = id_V \} \). Let \( \mathfrak{u} \) be the compact real form of \( \mathfrak{g} \) such that \( l \cap \mathfrak{u} = \mathfrak{i}_\mathbb{R} \), the Lie algebra of \( L_\mathbb{R} \). Denote by \( \mathcal{W}_\mathbb{R} = \mathfrak{u} \cap (iu) \). Then, the real Lie algebra defined by \( \mathfrak{g}_\mathbb{R} = \mathfrak{i}_\mathbb{R} + \mathcal{W}_\mathbb{R} \) is a real form of \( \mathfrak{g} \) and this decomposition is its Cartan decomposition.

Since the complexification of the Cartan decomposition of \( \mathfrak{g}_\mathbb{R} \) is \( \mathfrak{g} = \mathfrak{l} + \mathcal{W} \) and since \( \mathcal{W} = V + V^\sigma = d\kappa(U(l))E + d\kappa(U(l))F \) is a sum of two simple \( l \)-modules, it follows that the simple real Lie algebra \( \mathfrak{g}_\mathbb{R} \) is of Hermitian type. One can show that \( \mathcal{W}_\mathbb{R} = \{ p \in \mathcal{W} \mid \beta(p) = p \} \) where we defined for a polynomial \( p \in \mathcal{W} \), \( \bar{p} = p(\bar{z}) \), and considered the antilinear involution \( \beta \) of \( \mathcal{W} \) given by \( \beta(p) = \kappa(\sigma)p \).

Consider the isomorphisms \( v \in V \mapsto \tau_v \in \mathcal{V} \) and \( v \in V \mapsto \tau_v^\sigma \in \mathcal{V}^\sigma \) where \( \tau_v \) is the linear form on \( V \) given by \( \tau_v(v') = \tau(vv') \) and \( \tau_v^\sigma = \kappa(\sigma)\tau_v \).

For a suitable choice of a bilinear form \( \lambda_0 : \mathcal{V} \times \mathcal{V}^\sigma \to \mathbb{C} \), \( \mathfrak{g} \) and \( \mathfrak{g}_\mathbb{R} \) are isomorphic to matrix Lie algebras ([A12b]). Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{l} \), which contains \( H_0 \). Since the Lie algebra \( \mathfrak{g} = \mathfrak{l} + \mathcal{W} \) is of Hermitian type, then \( \mathfrak{h} \) is also a Cartan subalgebra of \( \mathfrak{g} \). Let \( \Delta^+ (\mathfrak{g}, \mathfrak{h}) \) be a suitably chosen positive system of roots and let \( \lambda \) be the highest root. Denote by \( \mathfrak{g}_\lambda \) and \( \mathfrak{g}_{-\lambda} \) the root spaces corresponding to the highest and lowest root. One knows that the minimal adjoint nilpotent orbit of \( \mathfrak{g} \) is given by \( O_{\min} = G \cdot (\mathfrak{g}_\lambda \setminus \{0\}) \), where \( G = \text{Int}(\mathfrak{g}) \) is the group of inner automorphisms of \( \mathfrak{g} \). Recall from [A12b], Lemma 2.1, that \( \mathfrak{g}_\lambda \subset \mathcal{V} \setminus \{0\} \) and \( \mathfrak{g}_{-\lambda} = \{ \kappa(\sigma)T \mid T \in \mathfrak{g}_\lambda \} \subset \mathcal{V}^\sigma \setminus \{0\} \). Denote by \( \tau_\lambda \) the linear form given by \( \tau_\lambda(v) = \tau_{c_\lambda}(v) = \tau(c_\lambda v) \), and by \( \tau_\lambda^\sigma = \kappa(\sigma)\tau_\lambda \). They are nilpotent elements of \( \mathfrak{g}_\lambda \), \( \tau_\lambda \in \mathfrak{g}_\lambda \) and \( \tau_\lambda^\sigma \in \mathfrak{g}_{-\lambda} \). The orbits \( \Xi \) of \( \tau_\lambda \) and \( \Xi^\sigma \) of \( \tau_\lambda^\sigma \) under the group \( L \) acting on \( \mathfrak{p} \) by the representation \( \kappa \), are conical varieties related by \( \Xi^\sigma = \kappa(\sigma)\Xi \). They are the minimal nilpotent \( L \)-orbits in \( \mathcal{W} \).

Polynomials \( \xi \in \Xi \) and \( \xi^\sigma = \kappa(\sigma)\xi \in \Xi^\sigma \) can be written

\[
\xi(v) = \kappa(l)\tau_\lambda(v) = \chi(l)\tau(((l')^{-1}c_\lambda)v),
\]

\[
\xi^\sigma(v) = \kappa(\sigma)\xi(v) = \chi(l)Q(v)\tau(-(l')^{-1}c_\lambda)v^{-1}),
\]

where \( l' \in \text{Str}(V) \) is the adjoint of \( l \) for the inner product \( (x, y) = \text{tr}(xy) \). Then \( \Xi \) and \( \Xi^\sigma \) are realized as a \( L \)-orbit of \( c_\lambda \) in \( V \) and have explicit coordinate systems.
We consider in this paper the case of \(V_\mathbb{R} = \text{Sym}(r, \mathbb{R}), \ V = \text{Sym}(r, \mathbb{C}), \text{Str}(V_\mathbb{R}) = \text{GL}(r, \mathbb{R}), \text{Str}(V) = \text{GL}(r, \mathbb{C})\) acting on \(V_\mathbb{R}\) or on \(V\) by \(gx = g \cdot x \cdot g^t\). Then \(c_\lambda = \text{diag}(1,0,\ldots,0)\), the orbits \(\Xi\) and \(\Xi^\sigma\) are both identified to

\[
\Gamma(\Xi) = \{g \cdot c_\lambda \cdot g^t \mid g \in \text{GL}(r, \mathbb{C})\} = \{\xi_z = zz^t \mid z \in \mathbb{C}^r\}.
\]

The group \(L\) acts on the spaces \(O(\Xi)\) and \(O(\Xi^\sigma)\) of holomorphic functions on \(\Xi\) and \(\Xi^\sigma\) respectively by:

\[
(\pi_\alpha(l)f)(\xi) = \chi(l)^\alpha f(\kappa(l)^{-1} \xi) \quad \text{and} \quad (\pi^\sigma_\alpha(l)f)(\xi^\sigma) = \chi(l)^\alpha f(\kappa(l)^{-1} \xi^\sigma).
\]

For \(\xi = Xz \in \Gamma(\Xi)\) and for every function \(f \in O(\Xi)\), we write \(f(\xi) = \phi(z)\). In these coordinates, the representations \(\pi_\alpha\) and \(\pi^\sigma_\alpha\) are given by

\[
\pi_\alpha(l)\phi(z) = \chi(l)^\alpha \phi(l_1^{-1} \cdot z) \quad \text{for} \ l = (l_1, (l_1^t)^{-1})
\]

and

\[
\pi^\sigma_\alpha(l)\phi(z) = \chi(l)^{-\alpha} \phi(l_1^t \cdot z) \quad \text{for} \ l = (l_1, (l_1^t)^{-1}).
\]

For \(m \in \mathbb{Z}\), let \(O_m(\Xi)\) and \(O_m(\Xi^\sigma)\) be the the spaces of holomorphic functions \(f\) on \(\Xi\) and \(f^\sigma\) on \(\Xi^\sigma\) respectively such that for every \(w \in \mathbb{C}^*\),

\[
f(w\xi) = w^m f(\xi) \quad \text{and} \quad f^\sigma(w\xi^\sigma) = w^m f^\sigma(\xi^\sigma).
\]

These spaces are respectively invariant under \(\pi_\alpha\) and \(\pi^\sigma_\alpha\). For \(f \in O_m(\Xi), h \in O_{m+\frac{1}{2}}(\Xi)\) and \(f^\sigma \in O_m(\Xi^\sigma), h^\sigma \in O_{m+\frac{1}{2}}(\Xi^\sigma)\) their corresponding functions \(\phi, \psi\) and \(\phi^\sigma, \psi^\sigma\) on \(\mathbb{C}^r\) satisfy for \(\mu > 0\),

\[
\phi(\mu \cdot (z)) = \mu^{2m} \phi(z), \quad \psi(\mu \cdot (z)) = \mu^{2m+1} \psi(z),
\]

\[
\phi^\sigma(\mu \cdot (z)) = \mu^{2m} \phi^\sigma(z), \quad \psi^\sigma(\mu \cdot (z)) = \mu^{2m+1} \psi^\sigma(z),
\]

and the correspondences \(f(\xi) \mapsto \phi(z)\) and \(f^\sigma(\xi^\sigma) \mapsto \phi^\sigma(z)\) map the spaces \(O_m(\Xi), O_m(\Xi^\sigma)\) and \(O_{m+\frac{1}{2}}(\Xi), O_{m+\frac{1}{2}}(\Xi^\sigma)\), to respectively

\[
O_{2m}(\mathbb{C}^r) = \{\phi \in O(\mathbb{C}^r) \mid \phi(\mu z) = \mu^{2m} \phi(z)\},
\]

\[
O_{2m+1}(\mathbb{C}^r) = \{\phi \in O(\mathbb{C}^r) \mid \phi(\mu z) = \mu^{2m+1} \phi(z)\}.
\]

Let \(\tilde{O}_{2m}(\mathbb{C}^{d_0})\), \(\tilde{O}_{2m+1}(\mathbb{C}^{d_0})\) and \(\tilde{O}_{2m}(\mathbb{C}^{d_0})\), \(\tilde{O}_{2m+1}(\mathbb{C}^{d_0})\) be the sets of such functions \(\phi, \phi^\sigma, \psi\) and \(\psi^\sigma\) corresponding to the functions \(f \in O_m(\Xi)\), \(f^\sigma \in O_m(\Xi^\sigma)\), \(h \in O_{m+\frac{1}{2}}(\Xi)\), \(h^\sigma \in O_{m+\frac{1}{2}}(\Xi^\sigma)\).

Denote by \(\pi_{\alpha,m}, \pi_{\alpha,m+\frac{1}{2}}\) and \(\pi^\sigma_{\alpha,m}, \pi^\sigma_{\alpha,m+\frac{1}{2}}\) the restrictions of the representations \(\pi_\alpha\) and \(\pi^\sigma_\alpha\) to the spaces \(O_m(\Xi), O_{m+\frac{1}{2}}(\Xi)\) and \(O_m(\Xi^\sigma), O_{m+\frac{1}{2}}(\Xi^\sigma)\) and also by \(\pi_{\alpha,2m}, \pi^\sigma_{\alpha,2m}, \pi_{\alpha,2m+1}, \pi^\sigma_{\alpha,2m+1}\) the corresponding representations on \(\tilde{O}_{2m}(\mathbb{C}^r), \tilde{O}_{2m+1}(\mathbb{C}^r), \tilde{O}_{2m}^\sigma(\mathbb{C}^r), \tilde{O}_{2m+1}^\sigma(\mathbb{C}^r)\). It follows from Theorem 3.1. in [A.12b] that these spaces consist in polynomials and are finite dimensional, \(O_m(\Xi) = \{0\}\) and \(O_m(\Xi^\sigma) = \{0\}\) for \(m < 0\), and \(\pi_{\alpha,2m}, \pi_{\alpha,2m+1}, \pi^\sigma_{\alpha,2m}, \pi^\sigma_{\alpha,2m+1}\) are irreducible.

Also, \(L_\mathbb{R}\)-invariant norms on \(O_m(\Xi), O_{m+\frac{1}{2}}(\Xi)\) and \(O_m(\Xi^\sigma), O_{m+\frac{1}{2}}(\Xi^\sigma)\) are defined by:
\[ \|\phi\|_m^2 = \frac{1}{a_m} \int_{C^r} |\phi(z)|^2 H(z)^{-2m} m_0(dz), \]
\[ \|\phi\|_{m+\frac{1}{2}}^2 = \frac{1}{a_{m+\frac{1}{2}}} \int_{C^r} |\phi(z)|^2 H(z)^{-2m+1} m_0(dz), \]

where
\[ H(z) = \tau(\frac{1}{r} I_r + z \bar{z}) = 1 + \text{tr}(z \bar{z}), \]
and \( m_0(d(z)) = H(z)^{-(r+1)} m(d(z)) \), is the \( L \)-invariant measure, \( p_m \) and \( p_{m+\frac{1}{2}} \) are suitable integers, and
\[ a_m = \int_{C^r} H(z)^{-2m} m_0(dz), \quad a_{m+\frac{1}{2}} = \int_{C^r} H(z)^{-(2m+1)} m_0(dz). \]

These spaces become invariant Hilbert spaces with reproducing kernels:
\[ K_m(\xi,\xi') = \Phi(\xi,\xi')^{2m}, \quad K_{m+\frac{1}{2}}(\xi,\xi') = \Phi(\xi,\xi')^{2m+1}, \]
with
\[ \Phi(\xi,\xi') = \tau(\frac{1}{r} I_r + z \bar{z}') \] for \( \xi = \xi_z, \xi' = \xi_{z'} \).

They are the irreducible \( L_R \)-invariant subspaces of \( O(\Xi) \) and \( O(\Xi^\sigma) \). The Fock spaces \( F(\Xi) \) and \( F(\Xi^\sigma) \) are Hilbert subspaces of \( O(\Xi) \) and \( O(\Xi) \) which are \( L \)-invariant, they therefore decompose
\[ F(\Xi) = \sum_{m=0}^\infty O_m(\Xi) + \sum_{m=0}^\infty O_{m+\frac{1}{2}}(\Xi), \]
\[ F(\Xi^\sigma) = \sum_{m=0}^\infty O_m(\Xi^\sigma) + \sum_{m=0}^\infty O_{m+\frac{1}{2}}(\Xi^\sigma). \]

The Hilbert norms on \( F(\Xi) \) and \( F(\Xi^\sigma) \) are of the following form: for
\[ f = \sum_{m=0}^\infty f_m + \sum_{m=0}^\infty f_{m+\frac{1}{2}} \]
then
\[ \|f\|_F^2 = \sum_{m=0}^\infty \frac{1}{c_m} \|f_m\|^2 + \sum_{m=0}^\infty \frac{1}{c_{m+\frac{1}{2}}} \|f_{m+\frac{1}{2}}\|^2. \]

The sequences \( (c_m) \), \( (c_{m+\frac{1}{2}}) \), are determined in such a way that the representation \( \rho_\alpha \) of \( g_R \) and \( \rho_\alpha^\sigma \) are unitary (see [A12b], Theorem 5.1). One gets
\[ c_m = (2m)!^r, \quad c_{m+\frac{1}{2}} = (2m+1)!^r, \]
and then \( F(\Xi) \) and \( F(\Xi^\sigma) \) turn out to be the classical Fock space \( F(\mathbb{C}^r) \).
For the representations $\rho_\alpha$ and $\rho^\sigma_\alpha$ of $\mathfrak{g}$, the elements $\omega \in \mathfrak{l}$ act by
\[
\rho_\alpha(\omega) = d\pi_\alpha(\omega) - \frac{1}{2}d\pi_\alpha(H_0)
\]
and
\[
\rho^\sigma_\alpha(\omega) = d\pi^\sigma_\alpha(\omega) - \frac{1}{2}d\pi^\sigma_\alpha(H_0).
\]

For $\rho_\alpha$, the elements $p \in \mathcal{V}$ act by multiplication and the elements $p^\sigma \in \mathcal{V}^\sigma$ act by differentiation, and, for $\rho^\sigma_\alpha = \pi(\sigma)\rho_\alpha(\sigma)$, the elements $p \in \mathcal{V}$ act by differentiation and the elements $p^\sigma \in \mathcal{V}^\sigma$ act by multiplication.

The representation $\rho_\alpha$ is determined by the operators $\rho_\alpha(E)$ which involves the multiplication operator $\tau(z^2)$, and $\rho_\alpha(F) = -\rho_\alpha(E)^*$. The representation $\rho^\sigma_\alpha$ is determined by the operators $\rho^\sigma_\alpha(E)$ which involves the differential operator $\tau(\frac{\partial^2}{\partial z^2})$, and $\rho_\alpha(F) = -\rho_\alpha(E)^*$.

The operators $\rho_\alpha(E), \rho_\alpha(F), \rho_\alpha(H_0)$ and $\rho^\sigma_\alpha(E), \rho_\alpha(\sigma)(F), \rho^\sigma_\alpha(H_0)$ are given by
\[
\rho_\alpha(E)\phi(z) = \frac{i}{4}\tau(z^2)\phi(z),
\]
\[
\rho^\sigma_\alpha(E)\phi(z) = \frac{i}{4}\tau(\frac{\partial^2}{\partial z^2})\phi(z),
\]
\[
\rho_\alpha(F)\phi(z) = \frac{i}{4}\tau(\frac{\partial^2}{\partial z^2})\phi(z),
\]
\[
\rho^\sigma_\alpha(F)\phi(z) = \frac{i}{4}\tau(z^2)\phi(z),
\]
\[
\rho_\alpha(H_0)\phi(z, z') = (1 - r)(-\alpha r\phi(z, z') + \frac{1}{2}\mathcal{E}\phi(z, z')), 
\]
\[
\rho^\sigma_\alpha(H_0)\phi(z, z') = (1 - r)(\alpha r\phi(z, z') + \frac{1}{2}\mathcal{E}\phi(z, z')), 
\]
where $\alpha = -\frac{1}{4}$ and $\mathcal{E}$ is the Euler operator given by
\[
(\mathcal{E}\phi)(z) = \frac{d}{ds}_{|s=1} \phi(sz).
\]
2. Some harmonic analysis

We consider on $V = \text{Sym}(r, \mathbb{C})$ the homogeneous polynomial
$$Q(x) = \det(x)^2.$$  
Then the structure group is
$$L = \text{Str}(V, \Delta) = \text{GL}(r, \mathbb{C})$$
(quotiented by $\{\pm I_r\}$). The orbit $\Xi$ of $\tau_\lambda$ under $L$ has dimension $r$ and can be identified to
$$\Gamma(\Xi) = \{\xi_z = zz^t \mid z \in \mathbb{C}^r\}.$$  
Let $\mathcal{Y}_m(\mathbb{R}^r)$ be the space of spherical harmonics of degree $m$ on $\mathbb{R}^r$: harmonic polynomials which are homogeneous of degree $m$ on $\mathbb{R}^r$. The map
$$\mathcal{Y}_m(\mathbb{R}^r) \to \mathcal{O}_m(\Xi), \quad \Phi \mapsto f,$$
given by
$$f(\xi_z) = \int_S <z, x>^m \Phi(x)s(dx),$$
where
$$<z, x> = \sum_{j=1}^r z_j x_j,$$
is an isomorphism which intertwins the representations of $O(r)$ on both spaces. ($S$ is the unit sphere in $\mathbb{R}^r$, and $s(dx)$ is the uniform measure on $S$ with total measure equal to one (see [F.15], section 2).

For a holomorphic function $a$ on $\mathbb{C}$ we define the integral operator $A$ from $\mathcal{C}(S)$ into $\mathcal{O}(\Xi)$:
$$Af(\xi_z) = \int_S a(<z, x>)f(x)s(dx).$$
The operator $A$ is equivariant with respect to the action of $O(r)$ and maps $\mathcal{Y}_m(\mathbb{R}^r)$ into $\mathcal{O}_m(\Xi)$. 

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3. The Lie algebra $\mathfrak{g}$ and its isomorphism with $\mathfrak{sp}(r, \mathbb{C})$

The Lie algebra $\mathfrak{g}$ is isomorphic to the matrix Lie algebra

$$\tilde{\mathfrak{g}} = \left\{ \begin{pmatrix} \omega_1 & v \\ u & -\omega_1^t \end{pmatrix} \mid u, v \in \text{Sym}(r, \mathbb{C}), \ \omega_1 \in \mathfrak{gl}(r, \mathbb{C}) \right\},$$

and the isomorphism is given by

$$\tau_u + \omega + \tau^\sigma_v \in \mathfrak{g} = \mathcal{V} \oplus \mathcal{I} \oplus \mathcal{V}^\sigma \mapsto \tilde{\tau}_u + \tilde{\omega} + \tilde{\tau}^\sigma_v \in \tilde{\mathfrak{g}}$$

with $\omega = (\omega_1, -\omega_1^t)$,

$$\tilde{\omega} = \begin{pmatrix} \omega_1 & 0 \\ 0 & -\omega_1^t \end{pmatrix} + \text{tr}(\omega_1) \begin{pmatrix} -I_r & 0 \\ 0 & I_r \end{pmatrix},$$

$$\tilde{\tau}_u = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}, \ \tilde{\tau}^\sigma_v = \frac{1}{2} \begin{pmatrix} 0 & v^t \\ 0 & 0 \end{pmatrix}.$$

Then, $\mathfrak{g}$ is isomorphic to $\mathfrak{sp}(r, \mathbb{C})$. In fact this follows from Proposition 1.1. in [A12b]: every $\omega = (\omega_1, -\omega_1^t) \in \mathcal{I}$ acts on $\mathcal{V}$ by $\omega x = \omega_1 \cdot x + x \cdot \omega_1^t$.

It follows that for every $\omega, \omega' \in \mathcal{I}$, one has $[\tilde{\omega}, \tilde{\omega}'] = [\omega, \omega']$. Moreover, for $\omega = (\omega_1, -\omega_1^t) \in \mathcal{I}$, one has

$$[\omega, \tau_u] = d\kappa(\omega)\tau_u = \tau_{-\omega_1^t u - u\omega_1 + 2\text{tr}(\omega_1)u}$$

and

$$[\tilde{\omega}, \tilde{\tau}_u] = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -\omega_1^t u - u\omega_1 + 2\text{tr}(\omega_1)u & 0 \end{pmatrix}$$

then

$$[\tilde{\omega}, \tilde{\tau}_u] = [\tilde{\omega}, \tilde{\tau}_u].$$

One has also

$$[\omega, \tau^\sigma_v] = d\kappa(\omega)\tau^\sigma_v = \tau_{\omega_1^t v + v\omega_1^t + 2\text{tr}(\omega_1) v}$$

and

$$[\tilde{\omega}, \tilde{\tau}^\sigma_v] = \frac{1}{2} \begin{pmatrix} 0 & v^t \omega_1^t + \omega_1 v^t + 2\text{tr}(\omega_1)v^t \\ 0 & 0 \end{pmatrix}$$

then

$$[\tilde{\omega}, \tilde{\tau}^\sigma_v] = [\tilde{\omega}, \tilde{\tau}^\sigma_v].$$

The matrices corresponding to $E = \tau$, $F = \tau^\sigma$ and $H_0$ (in (i) and (iii)) are

$$\tilde{E} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ I_r & 0 \end{pmatrix}, \ \tilde{F} = \frac{1}{2} \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix}, \ \tilde{H}_0 = (1 - r) \begin{pmatrix} -\frac{1}{2} I_r & 0 \\ 0 & \frac{1}{2} I_r \end{pmatrix}.$$

then $[\tilde{E}, \tilde{F}] = \frac{1}{2(1 - r)} \tilde{H}_0$ and, since $[E, F] = (\lambda_0(E, F) + \frac{1}{2})H_0 = (\frac{1}{2} - \frac{r}{2(r - 1)})H_0$, then $[\tilde{E}, \tilde{F}] = [\tilde{E}, \tilde{F}]$. Observe that

$$[\tilde{H}_0, \tilde{E}] = -(r - 1)\tilde{E} \text{ and } [\tilde{H}_0, \tilde{F}] = (r - 1)\tilde{F},$$

i.e.

$$[\tilde{H}_0, \tilde{E}] = [\tilde{H}_0, \tilde{E}] \text{ and } [\tilde{H}_0, \tilde{F}] = [\tilde{H}_0, \tilde{F}].$$

This proves that we have obtained an explicit Lie algebra isomorphism from $\mathfrak{g}$ to $\tilde{\mathfrak{g}} = \mathfrak{sp}(r, \mathbb{C})$. 

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Now, let’s consider the image \( \tilde{g}_R \) of the real form \( g_R \) by this isomorphism. Since \( g_R \) is given by \( g_R = l_R + \mathcal{W}_R \), where \( l_R \) is the compact real form of \( l \) and \( \mathcal{W}_R \) is generated by the elements \( \tau_u + \tau_u^\sigma \) and \( i(\tau_u - \tau_u^\sigma) \) for \( u \in V_R = \text{Sym}(\mathbb{R}) \), then \( \tilde{g}_R \) is given by \( \tilde{g}_R = l_R + \mathcal{W}_R \) with

\[
\tilde{l}_R = \left\{ \begin{pmatrix} \omega_1 & 0 \\ 0 & -\omega_1^t \end{pmatrix} \right\} | \omega_1 = w_1 + iw'_1, w_1 \in \text{Skew}(r, \mathbb{R}), w'_1 \in \text{Sym}(r, \mathbb{R}) \},
\]

\[
\tilde{\mathcal{W}}_R = \left\{ \begin{pmatrix} 0 & u - iv \\ u + iv & 0 \end{pmatrix} \right\} | u, v \in \text{Sym}(r, \mathbb{R}) \},
\]

i.e.

\[
\tilde{g}_R = \left\{ \begin{pmatrix} \omega_1 & u - iv \\ u + iv & -\omega_1^t \end{pmatrix} \right\} | \omega_1 = w_1 + iw'_1 \in \mathfrak{su}(r), u, v \in \text{Sym}(r, \mathbb{R}) \}.
\]

Let \( g_0 \) be the element of \( \text{Sp}(r, \mathbb{C}) \) given by

\[
g_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} iI_r & I_r \\ -I_r & -iI_r \end{pmatrix}.
\]

Since

\[
\text{Ad}(g_0) \left( \begin{array}{cc} w_1 & w'_1 \\ -w'_1 & w_1 \end{array} \right) = \left( \begin{array}{cc} w_1 + iw'_1 & 0 \\ 0 & w_1 - iw'_1 \end{array} \right)
\]

and

\[
\text{Ad}(g_0) \left( \begin{array}{cc} v & -u \\ -u & v \end{array} \right) = \left( \begin{array}{cc} 0 & u - iv \\ u + iv & 0 \end{array} \right),
\]

then

\[
\text{Ad}(g_0^{-1})(\tilde{l}_R) = \left\{ \begin{pmatrix} w_1 & w'_1 \\ -w'_1 & w_1 \end{pmatrix} \right\} | w_1 \in \text{Skew}(r, \mathbb{R}), w'_1 \in \text{Sym}(r, \mathbb{R}) \},
\]

and

\[
\text{Ad}(g_0^{-1})(\tilde{\mathcal{W}}_R) = \left\{ \begin{pmatrix} v & -u \\ -u & -v \end{pmatrix} \right\} | u, v \in \text{Sym}(r, \mathbb{R}) \},
\]

in such a way that

\[
\text{Ad}(g_0^{-1})(\tilde{g}_R) = \{ \begin{pmatrix} w_1 + v & w'_1 - u \\ -w'_1 - u & w_1 - v \end{pmatrix} \} | w_1 \in \text{Skew}(r, \mathbb{R}), w'_1, u, v \in \text{Sym}(r, \mathbb{R}) \} = \mathfrak{sp}(r, \mathbb{R}).
Furthermore, it is well known that the minimal nilpotent $\tilde{G}$-orbit in $\tilde{g}$ is
\begin{equation*}
\tilde{O}_{\min} = \text{Ad}(\tilde{G})(e_{11})
\end{equation*}
where $E_{11}$ is the diagonal matrix $E_{11} = \text{diag}(1, 0, \ldots, 0)$, and $e_{11} = \begin{pmatrix} 0 & E_{11} \\ 0 & 0 \end{pmatrix} = 2\tau_{\sigma}^C$ is the highest root. It follows that
\begin{equation*}
\tilde{O}_{\min} \cap \tilde{W} = \tilde{O}_{\min} \cap \tilde{V} \cup \tilde{O}_{\min} \cap \tilde{V}^\sigma
\end{equation*}
with
\begin{equation*}
\tilde{O}_{\min} \cap \tilde{V}^\sigma = \text{Ad}(\tilde{L})(e_{11})
\end{equation*}
\begin{equation*}
= \{ \text{Ad}(l)(e_{11}) \mid l = \begin{pmatrix} l_1 & 0 \\ 0 & (l_1^t)^{-1} \end{pmatrix} \mid l_1 \in \text{GL}(r, \mathbb{C}) \}
\end{equation*}
\begin{equation*}
= \{ \begin{pmatrix} 0 & l_1 E_{11} l_1^t \\ 0 & 0 \end{pmatrix} \mid l_1 \in \text{GL}(r, \mathbb{C}) \}
\end{equation*}
\begin{equation*}
= \{ \begin{pmatrix} 0 & zz^t \\ 0 & 0 \end{pmatrix} \mid z \in \mathbb{C}^r \}
\end{equation*}
and
\begin{equation*}
\tilde{O}_{\min} \cap V = \text{Ad}(\tilde{L})(\text{Ad}(J)e_{11})
\end{equation*}
\begin{equation*}
= \{ \text{Ad}(l)(\text{Ad}(J)e_{11}) \mid l = \begin{pmatrix} l_1 & 0 \\ 0 & (l_1^t)^{-1} \end{pmatrix} \mid l_1 \in \text{GL}(r, \mathbb{C}) \}
\end{equation*}
\begin{equation*}
= \{ \begin{pmatrix} 0 & l_1 E_{11} l_1^t \\ 0 & 0 \end{pmatrix} \mid l_1 \in \text{GL}(r, \mathbb{C}) \}
\end{equation*}
\begin{equation*}
= \{ \begin{pmatrix} 0 & 0 \\ zz^t & 0 \end{pmatrix} \mid z \in \mathbb{C}^r \}
\end{equation*}
It follows that $\tilde{O}_{\min} \cap \tilde{V}^\sigma$ and $\tilde{O}_{\min} \cap V$ are respectively the images of the orbits $\Xi$ and $\Xi^\sigma$ by the isomorphism $\mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ and are diffeomorphic to
\begin{equation*}
\Gamma(\Xi) = \{ \xi_z = zz^t \mid z \in \mathbb{C}^r \}
\end{equation*}
and that the map $\kappa(\sigma)$ corresponds here to $X \mapsto \text{Ad}(J)X$.
Moreover,
\begin{equation*}
\tilde{O}_{\min} \cap \text{sp}(r, \mathbb{R}) = \tilde{O}_{\min} \cap \text{Ad}(g_0^{-1})(\tilde{\mathfrak{g}}) = Y^+ \cup Y^-.
\end{equation*}
4. The Schrödinger model.

Let $\Gamma_\mathbb{R}$ be the open cone in $\mathbb{R}^r$ and $S$ be the unit sphere given respectively by:

$$\Gamma_\mathbb{R} = \{ x \in \mathbb{R}^r : |x| \neq 0 \},$$

and

$$S = \{ x \in \mathbb{R}^r : |x| = 1 \}.$$

The group $L_\mathbb{R} = GL(r, \mathbb{R})$ acts on $\mathbb{R}^r$ by the natural representation. This action stabilizes the cone $\Gamma_\mathbb{R}$. The multiplicative group $\mathbb{R}^*_+$ acts on $\Gamma_\mathbb{R}$ as a dilation and the quotient space $M = \Gamma_\mathbb{R}/\mathbb{R}^*_+$ is identified with $S$. This defines an action of $L_\mathbb{R}$ on $S$, which leads to a $L_\mathbb{R}$-equivariant principal $\mathbb{R}^*_+$-bundle: $\Gamma_\mathbb{R} \to S, x \mapsto \frac{x}{|x|}$.

For $\lambda \in \mathbb{C}$, let $\mathcal{E}_\lambda(\Gamma_\mathbb{R})$ and $\mathcal{E}_\lambda^\sigma(\Gamma_\mathbb{R})$ be the spaces of $C^\infty$-functions on $\Gamma_\mathbb{R}$ homogeneous of degree $\lambda$:

$$\mathcal{E}_\lambda(\Gamma_\mathbb{R}) = \mathcal{E}_\lambda^\sigma(\Gamma_\mathbb{R}) = \{ f \in C^\infty(\Gamma_\mathbb{R}) \mid f(tx) = t^\lambda f(x), \quad x \in \Gamma, t > 0 \},$$

The group $L_\mathbb{R} = GL(r, \mathbb{R})$ acts naturally on $\mathcal{E}_\lambda(\Gamma_\mathbb{R})$, and, under the action of the subgroup $O(r)$, the space $\mathcal{E}_\lambda(\Gamma_\mathbb{R})$ decomposes as:

$$\mathcal{E}_\lambda(\Gamma_\mathbb{R}) \mid_S \cong \bigoplus_{k=0}^{\infty} \mathcal{Y}_k(\mathbb{R}^r).$$

These representations extend to representations $R$ and $R^\sigma$ of the metaplectic group $Mp(r, \mathbb{R})$ on the Hilbert space $L^2(\mathbb{R}^r)$ as follows: denote by

$$g(l_1) = \begin{pmatrix} l_1 & 0 \\ 0 & (l_1^t)^{-1} \end{pmatrix} \text{ for } l_1 \in GL(r, \mathbb{R}),$$

$$t(u) = \exp(2\tau_u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \quad t(u)^\sigma = \exp(2\tau^\sigma_u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad (u \in \text{Sym}(r)),$$

$$J_r = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}.$$

It is well known that the elements $g(l_1), t(u)$ and $J_r$, for $l_1 \in GL(r, \mathbb{R}), u \in \text{Sym}(r, \mathbb{R})$, generate the group symplectic group $Sp(r, \mathbb{R})$. One considers the two representations $R$ and $R^\sigma$ of the metaplectic group $Mp(r, \mathbb{R})$ on $L^2(\mathbb{R}^r)$, determined by:
\[ R(g(l_1))f(x) = (\det l_1)^{-\frac{1}{2}}f(l_1^*x), \]
\[ R(t(u)^\sigma)f(x) = e^{\frac{-i}{4}\tau_u(x^2)}f(x), \]
\[ R(J_r)f(x) = a_0 \int_{\mathbb{R}^r} e^{i\tau(xy)} f(y) dy. \]

and

\[ R^\sigma(g(l_1))f(x) = (\det l_1)^{-\frac{1}{2}}f(l_1^*x), \]
\[ R^\sigma(t(u))f(x) = e^{\frac{-i}{4}\tau_u(x^2)}f(x), \]
\[ R^\sigma(J_r)f(x) = a_0 \int_{\mathbb{R}^r} e^{i\tau(xy)} f(y) dy. \]

Then
\[ dR(\tau_u^\sigma)f(x) = -\frac{i}{4}\tau_u(x^2)f(x), \quad dR(\tilde{\tau}_u)f(x) = -\frac{i}{4}\tau_u(\frac{\partial^2}{\partial x^2})f(x) \]

and
\[ dR^\sigma(\tau_u)f(x) = -\frac{i}{4}\tau_u(x^2)f(x), \quad dR^\sigma(\tilde{\tau}_u)f(x) = -\frac{i}{4}\tau_u(\frac{\partial^2}{\partial x^2})f(x) \]

In particular
\[ dR(\tilde{E})f(x) = -\frac{i}{4}\tau\left(\frac{\partial^2}{\partial x^2}\right)f(x), \quad dR(\tilde{F})f(x) = -\frac{i}{4}\tau(x^2)f(x) \]

and
\[ dR^\sigma(\tilde{E})f(x) = -\frac{i}{4}\tau\left(\frac{\partial^2}{\partial x^2}\right)f(x), \quad dR^\sigma(\tilde{F})f(x) = -\frac{i}{4}\tau(x^2)f(x) \]

Denote by
\[ L^2(\mathbb{R}^r)_{\text{even}} = \{ f \in L^2(\mathbb{R}^r) \mid f(-x) = f(x) \} \]

and
\[ L^2(\mathbb{R}^r)_{\text{odd}} = \{ f \in L^2(\mathbb{R}^r) \mid f(-x) = -f(x) \}. \]

The following facts are well-known:

1) (Irreducibility) The representations \((R, L^2(\mathbb{R}^r)_{\text{even}}), (R, L^2(\mathbb{R}^r)_{\text{odd}}), (R^\sigma, L^2(\mathbb{R}^r)_{\text{even}}), (R^\sigma, L^2(\mathbb{R}^r)_{\text{odd}})\) of \(\text{Mp}(r, \mathbb{R})\) are irreducible.

2) (\(K\)-type decomposition) The underlying \((\mathfrak{g}, K)\)-modules, \((R)_{K^L}\) and \((R^\sigma)_{K^L}\), for \(K = K^L = O(r, \mathbb{C})\), have the following \(K\)-type formulas
\[
(R)_{K^L} = \bigoplus_{m=0}^{\infty} \mathcal{Y}_{2m}(\mathbb{R}^r) + \bigoplus_{m=0}^{\infty} \mathcal{Y}_{2m+1}(\mathbb{R}^r), \\
(R^\sigma)_{K^L} = \bigoplus_{m=0}^{\infty} \mathcal{Y}_{2m}(\mathbb{R}^r) + \bigoplus_{m=0}^{\infty} \mathcal{Y}_{2m+1}(\mathbb{R}^r). 
\]

3) (Unitarity) The representations \(R\) and \(R^\sigma\) of \(\text{Mp}(r, \mathbb{R})\) on \(L^2(\mathbb{R}^r)\) are unitary.
5. The intertwining operator

Recall from section 3 that

\[ \text{Ad}(g_0^{-1})(\widetilde{g}_R) = \{ \begin{pmatrix} w_1 + v & w'_1 - u \\ -w'_1 - u & w_1 - v \end{pmatrix}, w_1 \in \text{Skew}(r, \mathbb{R}), w'_1, u, v \in \text{Sym}(r, \mathbb{R}) \} = \mathfrak{sp}(r, \mathbb{R}) \]

where

\[ g_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} iI_r & I_r \\ -I_r & -iI_r \end{pmatrix}. \]

Denote by \( \widetilde{G} = \text{Sp}(r, \mathbb{C}) \). It is generated by the elements \( g(l_1), t(u) \) and \( J_r \) (for \( l_1 \in \text{GL}(r, \mathbb{C}), u \in \text{Sym}(r, \mathbb{C}) \)).

The representations \( \rho := \rho_\alpha \) and \( \rho^\sigma := \rho^\alpha_\sigma \) in section 1 (i.e with \( \alpha = -\frac{1}{2} \)) 'integrate' to the representations \( T \) and \( T^\sigma \) of \( \widetilde{G} = \text{Mp}(r, \mathbb{C}) \) on \( \mathcal{O}_{\text{fin}}(\Xi) \) given by

\[ T(g(l_1))\phi(z) = (\det l_1)^{2\alpha}\phi(l_1^{-1}z), \]

\[ T(t(u))\phi(z) = e^{\frac{i}{2} \tau_u(z^2)}\phi(z), \]

\[ T(J)\phi(z) = a_0 \int_{\mathbb{R}^r} e^{i\tau(zy)}\phi(y)dy \]

and

\[ T^\sigma(g(l_1))\phi(z) = (\det l_1)^{-2\alpha}\phi(l_1^1z), \]

\[ T^\sigma(t(u)^\sigma)\phi(z) = e^{\frac{i}{2} \tau_u(z^2)}\phi(z), \]

\[ T^\sigma(J)\phi(z) = a_0 \int_{\mathbb{R}^r} e^{i\tau(zy)}\phi(y)dy \]

in such a way that

\[ dT(\tau_u)\phi(z) = \frac{i}{4} \tau_u(z^2)\phi(z), \quad dT(\tau_u^\sigma)\phi(z) = \frac{i}{4} \tau_u(\frac{\partial^2}{\partial z^2})\phi(z), \]

\[ dT^\sigma(\tau_u)\phi(z) = \frac{i}{4} \tau_u(z^2)\phi(z), \quad dT^\sigma(\tau_u^\sigma)\phi(z) = \frac{i}{4} \tau_u(\frac{\partial^2}{\partial z^2})\phi(z) \]

(where we precise that the exponent \( 2\alpha \) arises from \( \chi(l)^\alpha = (\det l_1)^{2\alpha} \)).
Denote by
\[ \tilde{G}_{r} = \text{Ad}(g_{0})(\text{Mp}(r, \mathbb{R})). \]
and consider the Hilbert spaces \( T_{g_{0}^{-1}(-\frac{\boldsymbol{1}}{2})}(f(\mathbb{C}')) \) and \( T_{0}^{-1}(f(\mathbb{C}')) \), equipped with the norms
\[ \|\psi\|_{T_{g_{0}^{-1}(f(\mathbb{C}'))}} = \|T_{0}\psi\|_{f(\mathbb{C}')} \]
and
\[ \|\psi\|_{T_{0}^{-1}(f(\mathbb{C}'))} = \|T_{0}\psi\|_{f(\mathbb{C}')} \].
Then, \( (T, T_{g_{0}^{-1}}(f(\mathbb{C}'))) \) and \( (T_{\sigma}, T_{\sigma}(g_{0}^{-1})(f(\mathbb{C}'))) \) are unitary representations of the metaplectic group \( \text{Mp}(r, \mathbb{R}) \). In fact, for \( g \in \text{Mp}(r, \mathbb{R}) \), there is \( g' \in \tilde{G}_{r} \) such that
\[ g = g_{0}^{-1}g'g_{0}, \]
then
\[ T(g) = T(g_{0}^{-1}T(g')T(g_{0})) \]
and \( T_{\sigma}(g) = T_{\sigma}(g_{0}^{-1}T_{\sigma}(g')T_{\sigma}(g_{0})). \)
It follows that for \( \psi \in T_{g_{0}^{-1}(f(\mathbb{C}'))} \), \( \psi' \in (T_{g_{0}^{-1}}(f(\mathbb{C}'))) \) and for \( g \in \text{Mp}(r, \mathbb{R}) \),
\[ \|T(g)\psi\|_{T_{g_{0}^{-1}}(f(\mathbb{C}'))} = \|T(g_{0}^{-1})T(g')T(g_{0})\psi\|_{T_{g_{0}^{-1}}(f(\mathbb{C}'))} \]
\[ = \|T(g')T(g_{0})\psi\|_{f(\mathbb{C}')} = \|T(g_{0})\psi\|_{f(\mathbb{C}')} = \|\psi\|_{T_{g_{0}^{-1}}(f(\mathbb{C}'))} \]
and similarly,
\[ \|T_{\sigma}(g)\psi'\|_{T_{\sigma}(g_{0}^{-1})f(\mathbb{C}')} = \|T_{\sigma}(g_{0}^{-1})T_{\sigma}(g')T_{\sigma}(g_{0})\psi'\|_{T_{\sigma}(g_{0}^{-1})f(\mathbb{C}')} \]
\[ = \|T_{\sigma}(g')T_{\sigma}(g_{0})\psi'\|_{f(\mathbb{C}')} = \|T_{\sigma}(g_{0})\psi'\|_{f(\mathbb{C}')} = \|\psi'\|_{T_{\sigma}(g_{0}^{-1})f(\mathbb{C}')} \].
Recall that the Bargmann transform
\[ \mathcal{B} : L^{2}(\mathbb{R}^{r}) \rightarrow f(\mathbb{C}') \]
is a unitary operator given by the integral formula
\[ (\mathcal{B}f)(z) = \pi^{-\frac{r}{2}}\int_{\mathbb{R}^{r}} e^{-\frac{1}{2}(\tau(x^{2})+\tau(z^{2}))+\sqrt{2}\tau(zx)} f(x)dx \]
where \( dx \) is Lebesgue measure, and \( \tau \) is the usual symmetric bilinear form in \( r \) variables.
Theorem 5.1—

(i) The unitary representations \( (R, L^2(\mathbb{R}^r)) \) and \( (T^\sigma, T^\sigma(g_0^{-1})(\mathcal{F}(\mathbb{C}^r)) \) of the group \( \text{Mp}(r, \mathbb{R}) \) are unitarily equivalent. The intertwining operator is given by \( \mathcal{B}_0 = T(g_0^{-1}) \circ \mathcal{B} \).

(ii) The unitary representations \( (R^\sigma, L^2(\mathbb{R}^r)) \) and \( (T, T(g_0^{-1})(\mathcal{F}(\mathbb{C}^r)) \) of the group \( \text{Mp}(r, \mathbb{R}) \) are unitarily equivalent. The intertwining operator is given by \( \mathcal{B}_0^\sigma = T^\sigma(g_0^{-1}) \circ \mathcal{B} \).

**Proof.**

In fact, if an operator \( \mathcal{B}_0 : L^2(\mathbb{R}^r) \to T^\sigma(g_0^{-1})(\mathcal{F}(\mathbb{C}^r) \) intertwines \( R \) and \( T^\sigma \), then for every \( g = g_0^{-1}g'g_0 \in \text{Mp}(r, \mathbb{R}) \), one has
\[
T^\sigma(g)\mathcal{B}_0 = \mathcal{B}_0R(g),
\]
i.e.
\[
T^\sigma(g_0^{-1})T^\sigma(g')T^\sigma(g_0)\mathcal{B}_0 = \mathcal{B}_0R(g)
\]
which means
\[
T^\sigma(g')(T^\sigma(g_0)\mathcal{B}_0) = (T^\sigma(g_0)\mathcal{B}_0)R(g)
\]
i.e.
\[
T^\sigma(g')\tilde{\mathcal{B}} = \tilde{\mathcal{B}}R(g) \text{ with } \tilde{\mathcal{B}} = (T^\sigma(g_0)\mathcal{B}_0).
\]

Similarly, if an operator \( \mathcal{B}_0^\sigma : L^2(\mathbb{R}^r) \to T(g_0^{-1})(\mathcal{F}(\mathbb{C}^r) \) intertwines \( R^\sigma \) and \( T \), then for every \( g = g_0^{-1}g'g_0 \in \text{Mp}(r, \mathbb{R}) \), one has
\[
T(g)\mathcal{B}_0^\sigma = \mathcal{B}_0^\sigma R^\sigma(g),
\]
i.e.
\[
T(g_0^{-1})T(g')T(g_0)\mathcal{B}_0^\sigma = \mathcal{B}_0^\sigma R^\sigma(g)
\]
which means
\[
T(g')(T(g_0)\mathcal{B}_0^\sigma) = (T(g_0)\mathcal{B}_0^\sigma)R^\sigma(g)
\]
i.e.
\[
T(g')\tilde{\mathcal{B}}^\sigma = \tilde{\mathcal{B}}^\sigma R^\sigma(g) \text{ with } \tilde{\mathcal{B}}^\sigma = (T(g_0)\mathcal{B}_0^\sigma).
\]

In what follows, we will see that \( \tilde{\mathcal{B}} = \tilde{\mathcal{B}}^\sigma = \mathcal{B} \), the classical Bargmann transform. In fact, let \( b \) and \( b^\sigma \) be functions in one complex variable and let \( \tilde{\mathcal{B}} : L^2(\mathbb{R}^r) \to \mathcal{F}(\mathbb{C}^r) \) and \( \tilde{\mathcal{B}}^\sigma : L^2(\mathbb{R}^r) \to \mathcal{F}(\mathbb{C}^r) \) be the integral operators given by : for \( x \in S \), \( \xi_z = zz^t \in \Gamma(\Xi) \),
\[
(\tilde{\mathcal{B}}f)(\xi_z) = \int_S b(z, x)f(x)s(dx)
\]
and
\[
(\tilde{\mathcal{B}}^\sigma f)(\xi) = \int_S b^\sigma(z, x)f(x)s(dx).
\]
They map $\mathcal{C}^\infty(S)$ into $\mathcal{O}(\Gamma(\Xi))$, are $O(r)$-equivariant, and map $\mathcal{Y}_k(\mathbb{R}^r)$ onto $\mathcal{O}_k(\mathbb{C}^r)$.

The 'intertwining' relation for $\tilde{B}^\sigma$:

$$T(g')\tilde{B}^\sigma = \tilde{B}^\sigma R^\sigma(g)$$

leads in particular to the 'intertwining' relations

$$dT(\tilde{E} + \tilde{F}) \circ \tilde{B}^\sigma = \tilde{B}^\sigma \circ dR^\sigma(\text{Ad}(g_0^{-1})(\tilde{E} + \tilde{F})), $$

and

$$dT(i(\tilde{E} - \tilde{F})) \circ \tilde{B}^\sigma = \tilde{B}^\sigma \circ dR^\sigma(\text{Ad}(g_0^{-1})(i(\tilde{E} - \tilde{F})).$$

But, since

$$\tilde{E} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ I_r & 0 \end{pmatrix}, \quad \tilde{F} = \frac{1}{2} \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix},$$

then

$$\tilde{E} + \tilde{F} = \frac{1}{2} \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \quad \text{and} \quad i(\tilde{E} - \tilde{F}) = \frac{1}{2} \begin{pmatrix} 0 & -iI_r \\ iI_r & 0 \end{pmatrix},$$

and

$$\text{Ad}(g_0^{-1})(\tilde{E} + \tilde{F}) = \frac{1}{2} \begin{pmatrix} 0 & -I_r \\ -I_r & 0 \end{pmatrix} = -(\tilde{E} + \tilde{F}),$$

$$\text{Ad}(g_0^{-1})(i(\tilde{E} - \tilde{F})) = \frac{1}{2} \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix} =: \frac{1}{2} \omega(I_r).$$

The 'intertwining' relations become

$$dT(\tilde{E}) \circ \tilde{B}^\sigma + dT(\tilde{F}) \circ \tilde{B}^\sigma = -\tilde{B}^\sigma \circ dR^\sigma(\tilde{E}) - \tilde{B}^\sigma \circ dR^\sigma(\tilde{F}),$$

and

$$idT(\tilde{E}) \circ \tilde{B}^\sigma - idT(\tilde{F}) \circ \tilde{B}^\sigma = \tilde{B}^\sigma \circ dR^\sigma(\frac{1}{2} \omega(I_r)).$$

It follows that

$$2dT(\tilde{F}) \circ \tilde{B}^\sigma = \tilde{B}^\sigma \circ (\frac{1}{2} dR^\sigma(\omega(I_r))) + dR^\sigma(-\tilde{E}) + dR^\sigma(-\tilde{F})) \quad (*)\text{.}$$

Finally, using the integral form for the operator $\tilde{B}^\sigma$, and the formulas

$$dR^\sigma(\omega(I_r)f(x)) = dR^\sigma\left(\left(\begin{array}{cc} I_r & 0 \\ 0 & -I_r \end{array}\right)f(x)\right) = -\frac{r}{2} f(x) + \mathcal{E} f(x) \text{ for } f \in L^2(\mathbb{R}^r),$$

$$dR^\sigma(\tilde{E})f(x) = -\frac{i}{4} \tau(\frac{\partial^2}{\partial x^2})f(x), \quad dR^\sigma(\tilde{F})f(x) = -\frac{i}{4} \tau(x^2)f(x) \text{ for } f \in L^2(\mathbb{R}^r),$$

and

$$dT(\tilde{E})\phi(z) = \frac{i}{4} \tau(z^2)\phi(z), \quad dT(\tilde{F})\phi(z) = \frac{i}{4} \tau(\frac{\partial^2}{\partial z^2})\phi(z) \text{ for } \phi \in \mathcal{F}(\mathbb{C}^r),$$

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one can deduce from (*) that
\[ i \frac{\tau}{2} \left( \frac{\partial^2}{\partial z^2} \right) \int_{\mathbb{R}^r} b^\sigma(z, x) f(x) \, dx = \int_{\mathbb{R}^r} b^\sigma(z, x) \left( -\frac{i r}{4} + i \frac{\tau(x \frac{\partial}{\partial x})}{2} + i \frac{\tau}{4} \left( \frac{\partial^2}{\partial x^2} \right) + i \frac{\tau(x^2)}{4} \right) f(x) \, dx \]
\[ = \int_{\mathbb{R}^r} \left( -\frac{i r}{4} - i \frac{\tau(x \frac{\partial}{\partial x})}{2} + i \frac{\tau}{4} \left( \frac{\partial^2}{\partial x^2} \right) + i \frac{\tau(x^2)}{4} \right) b^\sigma(z, x) f(x) \, dx \]
which leads to the following differential equation for the function \( b^\sigma \):
\[ -\tau \left( \frac{\partial^2}{\partial z^2} \right) b^\sigma(z, x) = \left( \frac{r}{2} + \tau(x \frac{\partial}{\partial x}) - \frac{1}{2} \tau(x^2) - \frac{1}{2} \tau(z^2) \right) b^\sigma(z, x). \]
Observe that the solution of this equation is given by:
\[ b^\sigma(z, x) = e^{-\frac{1}{2}(\tau(x^2)+\tau(z^2)) + \sqrt{2} \tau(zx)}. \]
In fact, one has
\[ \frac{\partial^2}{\partial z_i^2} b(z, x) = \frac{\partial}{\partial z_i} \left( -z_i + \sqrt{2} x_i \right) b^\sigma(z, x) = (-1 + (-z_i + \sqrt{2} x_i)^2) b^\sigma(z, x) = (-1 + z_i^2 + 2x_i^2 - 2\sqrt{2} x_i z_i) b^\sigma(z, x). \]
Then
\[ -\tau \left( \frac{\partial^2}{\partial z_i^2} \right) b^\sigma(z, x) = \left( r - \tau(z^2) - 2\tau(x^2) + 2\sqrt{2} \tau(xz) \right) b^\sigma(z, x). \]
Similarly, one gets
\[ -\tau \left( \frac{\partial^2}{\partial z_i^2} \right) b^\sigma(z, x) = \left( r - \tau(x^2) - 2\tau(z^2) + 2\sqrt{2} \tau(xz) \right) b^\sigma(z, x). \]
On another part,
\[ x_i \frac{\partial}{\partial x_i} b^\sigma(z, x) = -x_i^2 + \sqrt{2} x_i z_i. \]
Then
\[ \tau \left( x \frac{\partial}{\partial x} \right) b^\sigma(z, x) = \left( -\tau(x^2) + \sqrt{2} \tau(xz) \right) b(z, x). \]
It follows that
\[ \left( \frac{r}{2} + \tau(x \frac{\partial}{\partial x}) - \frac{1}{2} \tau \left( \frac{\partial^2}{\partial x^2} \right) - \frac{1}{2} \tau(x^2) \right) b^\sigma(z, x) \]
\[ = \left( \frac{r}{2} - \tau(x^2) + \sqrt{2} \tau(xz) + \frac{r}{2} - \frac{1}{2} \tau(x^2) - \tau(z^2) + \sqrt{2} \tau(xz) - \frac{1}{2} \tau(x^2) \right) b^\sigma(z, x) \]
\[ = (r - 2\tau(x^2) - \tau(z^2) + 2\sqrt{2} \tau(xz)) b^\sigma(z, x). \]
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