COMPACTNESS OF TRANSFER OPERATORS AND SPECTRAL REPRESENTATION OF RUELLE ZETA FUNCTIONS FOR SUPER-CONTINUOUS FUNCTIONS

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Abstract. Transfer operators and Ruelle zeta functions for super-continuous functions on one-sided topological Markov shifts are considered. For every super-continuous function, we construct a Banach space on which the associated transfer operator is compact. Using this Banach space, we establish the trace formula and spectral representation of Ruelle zeta functions for a certain class of super-continuous functions. Our results include, as a special case, the classical trace formula and spectral representation for the class of locally constant functions.

1. Introduction. Let $N \geq 2$ be an integer and $A$ an $N \times N$ zero-one matrix. We say that $A$ is aperiodic if there exists a positive integer $k$ such that all entries of $A^k$ are positive. In this paper, we always assume that $A$ is aperiodic. We set

$$\Sigma_A^+ = \{ \omega = (\omega_m)_{m \in \mathbb{N} \cup \{0\}} \in \{1, \ldots, N\}^{\mathbb{N} \cup \{0\}} : A(\omega_m \omega_{m+1}) = 1, \ m \in \mathbb{N} \cup \{0\} \}$$

and equip $\Sigma_A^+$ with the product topology. Then, $\Sigma_A^+$ is a compact topological space.

We define the shift map $\sigma_A : \Sigma_A^+ \to \Sigma_A^+$ by

$$(\sigma_A \omega)_m = \omega_{m+1}, \ m \in \mathbb{N} \cup \{0\}.$$ 

Then, $\sigma_A$ is a continuous mapping. We call the dynamical system $(\Sigma_A^+, \sigma_A)$ a one-sided topological Markov shift.

For $\phi : \Sigma_A^+ \to \mathbb{C}$ and $m \in \mathbb{N} \cup \{0\}$, we write

$$\text{var}_m(\phi) = \sup_{\omega, \omega'} |\phi(\omega) - \phi(\omega')|,$$

where the $\sup_{\omega, \omega'}$ is taken over all $\omega, \omega' \in \Sigma_A^+$ with $\omega_k = \omega'_k$, $0 \leq k \leq m - 1$. If $\text{var}_m(\phi)^{1/m} \to 0$ as $m \to \infty$, then we call $\phi$ a super-continuous function. (This term is taken from [11]. See also Remark 2.3 below.) We set

$$V = \{ \phi : \Sigma_A^+ \to \mathbb{C} : \phi \text{ is a super-continuous function} \}.$$

For $m \in \mathbb{N} \cup \{0\}$, we also set

$$L_m = \{ \phi \in V : \text{var}_m(\phi) = 0 \}.$$
We call an element of $\bigcup_{m \geq 0} L_m$ a \textit{locally constant function} on $\Sigma^+$. Note that $L_0$ is the set of constant functions and that $L_0 \subset L_1 \subset \cdots$. Moreover, for $m \in \mathbb{N} \cup \{0\}$, $L_m$ is a finite-dimensional linear subspace of $V$ and $\dim L_m \leq N^m$. There exists a natural topology of $V$. Indeed, for $\theta \in (0,1)$, we define the metric $d_\theta$ on $\Sigma^+$ by

$$d_\theta(\omega, \omega') = \theta^m, \quad m_0 = \min\{m \in \mathbb{N} \cup \{0\} : \omega_m \neq \omega'_m\}$$

and denote by $F_\theta$ the set of complex-valued $d_\theta$-Lipschitz continuous functions on $\Sigma^+$. Then, $V = \bigcap_{\theta \in (0,1)} F_\theta$ (see Lemma 2.4 below). We denote by $\| \cdot \|$ the Lipschitz norm with respect to the metric $d_\theta$. We equip $V$ with the topology induced by the family of norms $\{\| \cdot \|_\theta\}_{\theta \in (0,1)}$.

Let $f \in V$. The \textit{Ruelle transfer operator} $\mathcal{L}_f : V \to V$ of $f$ is defined as follows:

$$(\mathcal{L}_f \phi)(\omega) = \sum_{\omega' \in \Sigma^+ : \sigma \omega' = \omega} e^{f(\omega')}(\omega').$$

We set

$$\Lambda_f = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda \text{ is an eigenvalue of } \mathcal{L}_f : V \to V\}.$$  

From [10, Theorem 1], $\Lambda_f$ is a discrete subset of $\mathbb{C} \setminus \{0\}$ and each eigenvalue has finite multiplicity. Hence, the structure of $\Lambda_f$ is similar to that of the spectrum of a compact operator on a Banach space. In fact, if $f$ is locally constant, then there exists a Banach space $E \subset V$ with $\mathcal{L}_f(E) \subset E$ such that $\mathcal{L}_f : E \to E$ is compact and the discrete structure of $\Lambda_f$ comes from the compactness. More precisely, the following assertion holds.

**Theorem A** ([10, Section 3]). Let $m \geq 2$ and $f \in L_m$. Then, $\mathcal{L}_f(L_{m-1}) \subset L_{m-1}$ and $\Lambda_f = \sigma(\mathcal{L}_f : L_{m-1} \to L_{m-1}) \setminus \{0\}$. Moreover, for $\lambda \in \Lambda_f$, the multiplicity of $\lambda$ as an eigenvalue of $\mathcal{L}_f : V \to V$ coincides with that as an eigenvalue of $\mathcal{L}_f : L_{m-1} \to L_{m-1}$.

Here, for a bounded linear operator $T : E \to E$ on a Banach space $E$, we denote by $\sigma(T : E \to E)$ the spectrum of $T$.

The first aim of this paper is to extend the above theorem to all super-continuous $f$. Let $\{\theta_m\}_{m \in \mathbb{N}}$ satisfy

$$\theta_1 \geq \theta_2 \geq \cdots \geq 0, \quad \lim_{m \to \infty} \theta_m = 0. \quad (2)$$

We set

$$B(\{\theta_m\}) = \{\phi \in V : \text{there exists } C \geq 0 \text{ such that } \text{var}_m(\phi) \leq C \theta_{m+1}^m \text{ for } m \in \mathbb{N} \cup \{0\}\} \quad (3)$$

and write, for $\phi \in B(\{\theta_m\})$,

$$\|\phi\|_{B(\{\theta_m\})} = \|\phi\|_{\infty} + \inf \{C \geq 0 : \text{var}_m(\phi) \leq C \theta_{m+1}^m \text{ for } m \in \mathbb{N} \cup \{0\}\}. \quad (4)$$

It is easy to see that $(B(\{\theta_m\}), \| \cdot \|_{B(\{\theta_m\})})$ is a Banach space. The first main result of this paper is as follows:

**Theorem 1.1.** Let $f \in V$ and let $\{\theta_m\}$ satisfy (2). Assume that

$$\text{var}_m(f)^{1/m} \leq \theta_m, \quad m \in \mathbb{N}. \quad (5)$$

We write $B = B(\{\theta_m\})$. Then the following two assertions hold:

(i) $\mathcal{L}_f(B) \subset B$ and $\mathcal{L}_f : B \to B$ is compact.

(ii) $\Lambda_f = \sigma(\mathcal{L}_f : B \to B) \setminus \{0\}$. Moreover, for $\lambda \in \Lambda_f$, the multiplicity of $\lambda$ as an eigenvalue of $\mathcal{L}_f : V \to V$ coincides with that as an eigenvalue of $\mathcal{L}_f : B \to B$. 


We write
\[ \theta_m(f) = \sup_{k \geq m} \text{var}_k(f)^{1/k} \]
for \( f \in V \) and \( m \in \mathbb{N} \). It is easy to see that the sequence \( \theta_m = \theta_m(f) \), \( m \in \mathbb{N} \) satisfies (2) and (5). Moreover, if \( m_0 \geq 2 \) and \( f \in L_{m_0} \setminus L_{m_0-1} \), then \( B(\{\theta_m(f)\}) = L_{m_0-1} \).

Thus, Theorem 1.1 is an extension of Theorem A to all super-continuous \( \zeta \) for any \( r \in f \) of Theorem B by (6). We call (6) the spectral representation zercoincides with the multiplicity of the corresponding eigenvalue.

Let Theorem 1.2.

(\cite[Section 3]{Section 3}) Theorem C canonical product form of \( f \) is an exponential of a formal power series defined by
\[ \zeta_f(z) = \exp \left( \sum_{q=1}^{\infty} \frac{z^q}{q} \sum_{\omega \in \text{Per}_q(\sigma_A)} e^{S_q(\omega)} \right), \quad z \in \mathbb{C}, \]
where, for \( q \in \mathbb{N} \), \( \text{Per}_q(\sigma_A) \) denotes the set of \( \omega \in \Sigma_A^+ \) with \( \sigma_A^q \omega = \omega \) and \( S_q(\omega) = \sum_{k=0}^{q-1} f(\sigma_A^k \omega) \). It is well known that the radius of convergence of the formal power series is not less than \( e^{-P(\mathcal{R}f)} \), where \( P(\mathcal{R}f) \) denotes the topological pressure of the real part \( \mathcal{R}f \) of \( f \). Let
\[ \lambda_1(f), \lambda_2(f), \ldots \]
be the sequence of non-zero eigenvalues of \( \mathcal{L}_f : V \to V \), where each eigenvalue is counted according to its multiplicity and \( |\lambda_n(f)| \geq |\lambda_{n+1}(f)| \) holds for \( n \in \mathbb{N} \). (If the number of the eigenvalues is finite, say, \( M \), then we put \( \lambda_n(f) = 0 \) for \( n > M \).)

The following theorem is an immediate consequence of \cite[Corollary 6]{Corollary 6}.

**Theorem B.** Let \( f \in V \). Then, \( \zeta_f(z)^{-1} \) admits a holomorphic extension to \( \mathbb{C} \) and its zeros are exactly \( \{\lambda_n(f)^{-1} : n \in \mathbb{N}, \lambda_n(f) \neq 0\} \). Moreover, the order of each zero coincides with the multiplicity of the corresponding eigenvalue.

On the other hand, if \( f \) is locally constant, then we have the next Weierstrass canonical product form of \( \zeta_f \).

**Theorem C** ([\cite[Section 3]{Section 3}]). Let \( f \in V \) be locally constant. Then, \( \{n \in \mathbb{N} : \lambda_n(f) \neq 0\} \) is a finite set and
\[ \zeta_f(z)^{-1} = \prod_{n=1}^{\infty} (1 - z\lambda_n(f)). \quad (6) \]

Equation (6) means that the entire function \( \prod_{n=1}^{\infty} (1 - z\lambda_n(f)) \) is a holomorphic extension of \( \zeta_f(z)^{-1} \) to \( \mathbb{C} \). Thus, for a locally constant \( f \), we also obtain an analog of Theorem B by (6). We call (6) the spectral representation of \( \zeta_f(z) \).

The second aim of this paper is to establish the representation (6) for a wider class of \( f \in V \). To this end, we consider the following condition for \( f \in V \) and \( r \in (0, 1) \):
\[ \text{var}_m(f)^{1/m} = O(r^m) \quad \text{as} \quad m \to \infty, \quad (7) \]
that is, \( \limsup_{m \to \infty} \text{var}_m(f)^{1/m}/r^m < \infty \). If \( f \) is locally constant, then (7) is valid for any \( r \in (0, 1) \). (For each \( r \in (0, 1) \), we give an example of non-locally constant \( f \in V \) satisfying (7) in Example 5.5 below.)

Here is the second main result of this paper.

**Theorem 1.2.** Let \( f \in V \) and let \( r \in (0, 1) \) satisfy (7). Then, for \( p > 0 \) with
\[ r^{2p\varphi_{h_{	ext{top}}}(\sigma_A)} < 1, \quad (8) \]
the following three assertions hold:
(i) $\sum_{n=1}^{\infty} |\lambda_n(f)|^p < \infty$.

(ii) For $q \in \mathbb{N}$ with $q \geq p$, we have

$$\sum_{n=1}^{\infty} \lambda_n(f)^q = \sum_{\omega \in \text{Per}_q(\sigma_A)} e^{S_{\omega}f(\omega)}.$$  

(iii) Let $k_0$ be the smallest $k \in \mathbb{N} \cup \{0\}$ such that $\sum_{n=1}^{\infty} |\lambda_n(f)|^{k+1} < \infty$ and (9) holds for $q \in \mathbb{N}$ with $q > k$. We set $E(z, k_0) = (1 - z) \exp(\sum_{k=1}^{k_0} z^k/k), z \in \mathbb{C}$. Then, the infinite product $\prod_{n=1}^{\infty} E(z\lambda_n(f), k_0)$ converges uniformly on any compact set of $\mathbb{C}$ and we have

$$\zeta_f(z)^{-1} = \exp \left( -\sum_{q=1}^{k_0} \frac{z^q}{q} \sum_{\omega \in \text{Per}_q(\sigma_A)} e^{S_{\omega}f(\omega)} \right) \prod_{n=1}^{\infty} E(z\lambda_n(f), k_0).$$  

Note that if $p \leq 1$, then $k_0 = 0$, and hence, (10) yields the spectral representation (6) of $\zeta_f(z)$.

Equations like (9), which give a connection between the poles of a zeta function and the spectrum of the associated transfer operator, are often called trace formulas. The trace formulas for dynamical zeta functions are widely studied in differentiable dynamical systems; see, e.g., [4, 6, 12, 13].

To prove Theorem 1.2 above, we introduce the operator ideal $\mathfrak{L}_p^a(E)$. Let $E$ be a Banach space. We denote by $\mathfrak{L}(E)$ the set of bounded linear operators on $E$. For $p > 0$, we set

$$\mathfrak{L}_p^a(E) = \left\{ T \in \mathfrak{L}(E) : \sum_{n=1}^{\infty} a_n(T)^p < \infty \right\},$$

where, for $n \in \mathbb{N}$, $a_n(T)$ denotes the $n$-th approximation number of $T$ defined by

$$a_n(T) = \inf\{\|T - A\| : A \in \mathfrak{L}(E), \text{rank} A < n\}.$$  

(11)

It is easy to see that any element of $\mathfrak{L}_p^a(E)$ is a compact operator and $\mathfrak{L}_p^a(E)$ is a left and right ideal of $\mathfrak{L}(E)$, that is, $\mathfrak{L}_p^a(E)$ is closed under addition and scalar multiplication, and $AT, TA \in \mathfrak{L}_p^a(E)$ for $A \in \mathfrak{L}(E)$ and $T \in \mathfrak{L}_p^a(E)$. Moreover, for $T \in \mathfrak{L}_p^a(E)$, the following two assertions hold (see [9, Theorems 3.6.3 and 4.2.26] for the proof):

(I) Let $\lambda_1(T), \lambda_2(T), \ldots$ be the sequence of non-zero eigenvalues of $T$, where each eigenvalue is counted according to its multiplicity and $|\lambda_n(T)| \geq |\lambda_{n+1}(T)|$ holds for $n \in \mathbb{N}$. (If the number of the eigenvalues is finite, say, $M$, then we put $\lambda_n(T) = 0$ for $n > M$.) Then, $\sum_{n=1}^{\infty} |\lambda_n(T)|^p < \infty$.

(II) Let $p = 1$ and we write

$$\|\|T\|\| = \sum_{n=1}^{\infty} a_n(T), \quad \tau(T) = \sum_{n=1}^{\infty} \lambda_n(T).$$

Let $T_1, T_2, \ldots \in \mathfrak{L}_1^a(E)$. If $\lim_{m \to \infty} \|T_m - T\| = 0$, then $\lim_{m \to \infty} \tau(T_m) = \tau(T)$.

Applying the theory of $\mathfrak{L}_p^a(E)$ to $E = B$, we prove Theorem 1.2 in Section 5.

This paper is organized as follows. In Section 2, we give preliminary definitions and basic facts. In Section 3, we prove some estimates for the proofs of the main results, i.e., Theorems 1.1 and 1.2. In Section 4, we prove Theorem 1.1, and in Section 5, we prove Theorem 1.2. In Appendix A, we study the properties of
transfer operators acting on \( V \). It is natural to hope that the transfer operator 
\( \mathcal{L}_f : V \to V \) is a compact operator. However, in Appendix A, we prove that this is 
not the case for any \( f \in V \). Moreover, we give an example of \( f \in V \) such that 
\( \sum_{n=1}^{\infty} |\lambda_n(f)| = \infty \). In Appendix B, we study the properties of \( V \) itself. We will 
see that \( V \) is naturally a nuclear space. Moreover, we prove that \( V \) has many non-
trivial (i.e., non-locally constant) elements. More precisely, we prove that the set of 
non-locally constant elements of \( V \) is a residual subset of \( V \). In Appendix C, we 
study the asymptotic behavior of eigenvalues of transfer operators. Using the Weyl 
inequality in Banach spaces (see, e.g., [7, Theorem 2.a.6]), we obtain an asymptotic 
study the asymptotic behavior of eigenvalues of transfer operators. Using the Weyl 
result of Demuth et al. [3].

2. Preliminaries. An element of \( \bigcup_{m \in \mathbb{N} \cup \{0\}} \{1, \ldots, N\}^m \) is called a word. For \( m \in \mathbb{N} \cup \{0\} \) and a word \( w \in \{1, \ldots, N\}^m \), we write \( |w| = m \). Moreover, we write \( w = w_0 \cdots w_{|w|-1} \) for a word \( w \), where \( w_k \in \{1, \ldots, N\} \), \( 0 \leq k \leq |w| - 1 \). The empty word is the unique word with \( |w| = 0 \). A word \( w \) with \( |w| \geq 2 \) is self-avoiding if \( w_k \neq w_l \) for \( 0 \leq k \neq l \leq |w| - 1 \). We define the new word \( vw \) for two words \( v, w \) by \( vw = v_0 \cdots v_{|v|-1}w_0 \cdots w_{|w|-1} \). Moreover, for a word \( w \) with \( |w| \geq 1 \), we define \( w^* = w \cdots w \). A word \( w \) is said to be \( A \)-admissible if \( |w| \geq 2 \) and \( A(w_kw_{k+1}) = 1 \) for \( 0 \leq k \leq |w| - 1 \). For \( \omega \in \Sigma_A^+ \) and \( m \in \mathbb{N} \cup \{0\} \), we define the word \( \omega|m \in \{1, \ldots, N\}^m \) by

\[
\omega|m = \begin{cases} 
\omega_0 \cdots \omega_{m-1} & (m \geq 1), \\
\text{the empty word} & (m = 0).
\end{cases}
\]

For \( m \in \mathbb{N} \cup \{0\} \) and \( w \in \{1, \ldots, N\}^m \), we set

\[
[w] = \{ \omega \in \Sigma_A^+ : \omega|m = w \}.
\]

A point \( \omega \in \Sigma_A^+ \) is said to be periodic if \( \sigma_A^q \omega = \omega \) for some \( q \in \mathbb{N} \). For a periodic point \( \omega \), its period is the smallest \( q \in \mathbb{N} \) such that \( \sigma_A^q \omega = \omega \). We denote by \( \text{Per}_q(\Sigma_A^+) \) the set of periodic points \( \omega \in \Sigma_A^+ \) with \( \sigma_A^q \omega = \omega \).

We recall that the \( N \times N \) zero-one matrix \( A \) is assumed to be aperiodic, that is, all entries of \( A^k \) are positive for some positive integer \( k \). The following lemma is needed in Appendices A and B.

**Lemma 2.1.** At least one row of \( A \) has more than two entries which are equal to one, and similarly for columns.

**Proof.** Assume that every row has just one entry which is equal to 1. Then there exists a permutation \( \tau \) of the set \( \{1, \ldots, N\} \) such that \( A(ij) = 1 (j = \tau(i)) = 0 (j \neq \tau(i)) \). Thus, \( A^k(ij) = 1 (j = \tau^k(i)) = 0 (j \neq \tau^k(i)) \) for \( k \geq 1 \), and hence, \( A \) is not aperiodic. The transpose \( A^T \) of \( A \) is also aperiodic. Hence, the assertion for columns also holds. \( \square \)

We recall from Section 1 that, for \( \theta \in (0, 1) \), \( F_\theta \) denotes the set of complex-valued functions on \( \Sigma_A^+ \) that are Lipschitz continuous with respect to the metric \( d_\theta \) on \( \Sigma_A^+ \). The Lipschitz norm \( \|\phi\|_\theta \) for \( \phi \in F_\theta \) is defined by \( \|\phi\|_\theta = \|\phi\|_\infty + \|\phi\|_\theta \), where

\[
\|\phi\|_\infty = \max_{\omega \in \Sigma_A^+} |\phi(\omega)|, \quad [\phi]_\theta = \sup_{\omega, \omega' \in \Sigma_A^+ : \omega \neq \omega'} \frac{|\phi(\omega) - \phi(\omega')|}{d_\theta(\omega, \omega')}.
\]
Then, \((F_\theta, \| \cdot \|_\theta)\) is a Banach space. Moreover, we easily see that if \(\theta < \theta'\), then \(\| \phi \|_{\theta'} \leq \| \phi \|_\theta\) for \(\phi \in F_\theta\), and hence, \(F_\theta \subset F_{\theta'}\).

Let us recall the following definition of a super-continuous function.

**Definition 2.2.** A super-continuous function on \(\Sigma_+^\dagger\) is a function \(\phi : \Sigma_+^\dagger \to \mathbb{C}\) such that \(\text{var}_m(\phi)^{1/m} \to 0\) as \(m \to \infty\).

We denote by \(V\) the set of all super-continuous functions on \(\Sigma_+^\dagger\).

**Remark 2.3.** A super-continuous function on a topological Markov shift was first defined by Quas and Siefken in [11] as follows: \(\phi : \Sigma_+^\dagger \to \mathbb{C}\) is called a super-continuous function if there exists a positive and non-increasing sequence \(\{ A_m \}_{m \in \mathbb{N}}\) such that \(\text{var}_m(\phi) \leq A_m\) for \(m \in \mathbb{N}\) and \(A_{m+1}/A_m \to 0\) as \(m \to \infty\). Let \(V'\) be the set of super-continuous functions in the sense of [11]. Then, \(V = V'\). Indeed, \(V' \subset V\) is obvious. Let \(\phi \in V\). We may assume that \(0 < \text{var}_m(\phi) < 1\) for any \(m \in \mathbb{N}\). We set \(A_m = \theta_m^m\) for \(m \in \mathbb{N}\), where \(\theta_m = \inf\{ \theta \in (0, 1) : \text{var}_k(\phi) \leq \theta^k, k \geq m\}\). Then, \(\{ A_m \}_{m \in \mathbb{N}}\) is positive and non-increasing. Moreover, \(\text{var}_m(\phi) \leq A_m\) for \(m \in \mathbb{N}\) and \(A_{m+1}/A_m \to 0\) as \(m \to \infty\). Thus, \(\phi \in V'\).

**Lemma 2.4.** We have \(V = \bigcap_{\theta \in (0, 1)} F_\theta\).

**Proof.** Let \(\phi \in V\). Fix \(\theta \in (0, 1)\). For sufficiently large \(m\), we have \(\text{var}_m(\phi)^{1/m} \leq \theta\), and hence, we have \(\text{var}_m(\phi) \leq \theta^m\). This implies \(\phi \in F_\theta\).

Let \(\phi \in \bigcap_{\theta \in (0, 1)} F_\theta\). For \(\theta \in (0, 1)\), there exists \(C > 0\) such that \(\text{var}_m(\phi) \leq C\theta^m\) for \(m \in \mathbb{N} \cup \{ 0 \}\), and hence, \(\lim_{m \to \infty} \text{var}_m(\phi)^{1/m} \leq \theta\). Letting \(\theta \to 0\), we obtain \(\lim_{m \to \infty} \text{var}_m(\phi)^{1/m} = 0\).

Recall from Section 1 that \(V\) is equipped with the topology induced by the family of norms \(\{ \| \cdot \|_\theta \}_{\theta \in (0, 1)}\). By the definition of the topology of \(V\), we easily see that \(\mathcal{L}_f : V \to V\) is continuous for \(f \in V\). Moreover, since \(\| \cdot \|_{\theta'} \leq \| \cdot \|_\theta\) for \(\theta < \theta'\), we see that the topology of \(V\) coincides with that induced by the countable subfamily \(\{ \| \cdot \|_{1/(m+1)} \}_{m \in \mathbb{N}}\). Hence, we obtain the following proposition:

**Proposition 2.5.** \(V\) is a Fréchet space.

Fix a Borel probability measure \(\mu\) on \(\Sigma_+^\dagger\) such that \(\mu(G) > 0\) for every non-empty open set \(G\) of \(\Sigma_+^\dagger\). (The Gibbs measure for a real-valued function in \(F_\theta\) satisfies this condition; see, e.g., [8, Chapter 3].) Let \(C(\Sigma_+^\dagger)\) be the set of complex-valued continuous functions on \(\Sigma_+^\dagger\). For \(m \in \mathbb{N}\), we define a finite-rank operator \(E_m : C(\Sigma_+^\dagger) \to L_m\) by

\[
(E_m \phi)(\omega) = \frac{1}{\mu([\omega|m]\}) \int_{[\omega|m]} \phi d\mu. \quad (12)
\]

Notice that \(E_m \phi = \phi\) for \(\phi \in L_m\). In addition to (1), we write \(V_{\theta}^m(\phi) = \sup_{k \geq m} \frac{\text{var}_k(\phi)}{\theta^k}\) for \(\phi : \Sigma_+^\dagger \to \mathbb{C}, m \in \mathbb{N} \cup \{ 0 \}\) and \(\theta \in (0, 1)\). Notice that \([\phi]_{\theta} = V_{\theta}^m(\phi)\) for \(\phi \in F_\theta\).

The next lemma will be used in Sections 3–5 and Appendix B.

**Lemma 2.6.** Let \(\phi \in C(\Sigma_+^\dagger), m \in \mathbb{N}\) and \(\theta, \theta' \in (0, 1)\).

(i) If \(\phi\) is real-valued, then so is \(E_m \phi\) and \(\max E_m \phi \leq \max \phi\).

(ii) For \(k \in \mathbb{N} \cup \{ 0 \}\), \(\text{var}_k(E_m \phi) \leq \text{var}_k(\phi)\).
(iii) If \( \phi \in F_\theta \), then \( \| \phi - E_m \phi \|_\infty \leq V^\theta_m(\phi)\theta^m \).

(iv) If \( \phi \in F_\theta \) and \( \theta < \theta' \), then \( \| \phi - E_m \phi \|_{\omega'} \leq 3V^\theta_m(\phi)(\theta/\theta')^m \).

Proof. (i) is obvious.

(ii) Let \( \omega, \omega' \in \Sigma^+_A \) satisfy \( \omega k = \omega' k \). We show that \( |(E_m \phi)(\omega) - (E_m \phi)(\omega')| \leq \text{var}_k(\phi) \). We first assume \( k \geq m \). Then, \( (E_m \phi)(\omega) = (E_m \phi)(\omega') \). We next assume \( k < m \). Then,

\[
|E_m \phi(\omega) - E_m \phi(\omega')| \leq \frac{1}{\mu([\omega|m]) \mu([\omega'|m])} \int_{[\omega|m]} \left( \int_{[\omega'|m]} |\phi(\xi) - \phi(\xi')| \mu(d\xi') \right) \mu(d\xi).
\]

Let \( \xi \in [\omega|m] \) and \( \xi' \in [\omega'|m] \). Then, \( \xi k = \omega k = \omega' k = \xi' k \) since \( k < m \), and hence, \( |\phi(\xi) - \phi(\xi')| \leq \text{var}_k(\phi) \). Thus, \( |E_m \phi(\omega) - E_m \phi(\omega')| \leq \text{var}_k(\phi) \).

(iii) Let \( \omega \in \Sigma^+_A \). We have

\[
|\phi(\omega) - (E_m \phi)(\omega)| \leq \frac{1}{\mu([\omega|m])} \int_{[\omega|m]} |\phi(\omega) - \phi(\omega')| \mu(d\omega').
\]

If \( \omega' \in [\omega|m] \), then \( |\phi(\omega) - \phi(\omega')| \leq \text{var}_m(\phi) \leq V^\theta_m(\phi)\theta^m \). Thus, (iii) follows.

(iv) By (iii), we have \( \| \phi - E_m \phi \|_\infty \leq V^\theta_m(\phi)(\theta/\theta')^m \). Therefore, it is enough to show that

\[
\text{var}_k(\phi - E_m \phi)(\theta')^k \leq 2V^\theta_m(\phi)(\theta/\theta')^m(\theta')^k,
\]

and hence, \( \text{var}_k(\phi - E_m \phi)/\theta'^k \leq V^\theta_m(\phi)(\theta/\theta')^m \). Next we assume \( k < m \). From (iii), \( \text{var}_k(\phi - E_m \phi) \leq 2\| \phi - E_m \phi \|_\infty \leq 2V^\theta_m(\phi)\theta^m \), and hence, \( \text{var}_k(\phi - E_m \phi)(\theta')^k \leq 2V^\theta_m(\phi)(\theta/\theta')^m(\theta')^m-k \leq 2V^\theta_m(\phi)(\theta/\theta')^m \). Combining, we obtain (13).

We note the next easy inequality:

\[
|e^z - e^{z'}| \leq 3e^{\max(\Re z, \Re z')}|z - z'|, \quad z, z' \in \mathbb{C}.
\]

The following Lasota-Yorke type inequality is well known and a key tool for the proofs of the main results.

**Lemma 2.7.** For \( f, \phi \in C(\Sigma^+_A) \) and \( k \in \mathbb{N} \), we have

\[
\text{var}_k(\mathcal{L}_f \phi) \leq N e^{\max \Re f} \{ 3 \text{var}_{k+1}(f) \| \phi \|_\infty + \text{var}_{k+1}(\phi) \}.
\]

Proof. Let \( \omega, \omega' \in \Sigma^+_A \) satisfy \( \omega k = \omega' k \). Since \( \omega_0 = \omega'_0 \), we have \( \{ i : A(i\omega_0) = 1 \} = \{ i : A(i\omega'_0) = 1 \} \). Therefore,

\[
|(\mathcal{L}_f \phi)(\omega) - (\mathcal{L}_f \phi)(\omega')| \leq \sum_{i : A(i\omega_0) = 1} \{ |e^{f(i\omega)}| |\phi(i\omega) - \phi(i\omega')| + |e^{f(i\omega)} - e^{f(i\omega')}| |\phi(i\omega')| \}.
\]

Let \( A(i\omega_0) = 1 \). We easily have \( |e^{f(i\omega)}| |\phi(i\omega) - \phi(i\omega')| \leq e^{\max \Re f} \text{var}_{k+1}(\phi) \). Moreover, by (14), we have \( |e^{f(i\omega)} - e^{f(i\omega')}| |\phi(i\omega')| \leq 3e^{\max \Re f} \text{var}_{k+1}(f) \| \phi \|_\infty \). Thus, the desired inequality holds.
3. Some estimates for the proofs of the main results. Fix a sequence \(\{\theta_m\}_{m \in \mathbb{N}}\) satisfying (2). Recall from (3) and (4) the definitions of the space \(\mathcal{B}(\{\theta_m\})\) and the norm \(\|\cdot\|_{\mathcal{B}(\{\theta_m\})}\), respectively. In this section, we write \(\mathcal{B} = \mathcal{B}(\{\theta_m\})\) and \(\|\cdot\| = \|\cdot\|_{\mathcal{B}(\{\theta_m\})}\) for the sake of simplicity. We set

\[ C = \sup_{m \in \mathbb{N}} \theta_m. \quad (15) \]

We begin with the following easy lemma (we omit the proof):

**Lemma 3.1.** For \(m \in \mathbb{N}\), the following three assertions hold:

(i) If \(\theta_m = 0\), then \(\mathcal{B} \subset L_{m-1}\).
(ii) If \(\theta_m > 0\), then \(L_{m-1} \subset \mathcal{B}\).
(iii) If \(m \geq 2\), \(\theta_m = 0\) and \(\theta_{m-1} > 0\), then \(\mathcal{B} = L_{m-1}\).

Let \(E_m\) be as in (12).

**Corollary 3.2.** For \(m \in \mathbb{N}\), we have \(E_m(\mathcal{B}) \subset \mathcal{B}\).

**Proof.** First, we consider the case in which \(\theta_1 = 0\). Then, \(\mathcal{B} = L_0\) from Lemma 3.1 (i), and hence, \(E_m(L_0) = L_0\).

Next, we consider the case in which \(\theta_1 > 0\) and \(\theta_m = 0\) for some \(m \geq 2\). Take \(m_0 \geq 2\) so that \(\theta_{m_0} = 0\) and \(\theta_{m_0-1} > 0\). From Lemma 3.1 (iii), \(\mathcal{B} = L_{m_0-1}\). Thus, if \(m \geq m_0 - 1\), then \(E_m(L_{m_0-1}) = L_{m_0-1}\), and if \(m < m_0 - 1\), then \(E_m(L_{m_0-1}) \subset L_m \subset L_{m_0-1}\).

Finally, we consider the case in which \(\theta_m > 0\) for all \(m \geq 2\). From Lemma 3.1 (ii), \(\bigcup_{m \geq 0} L_m \subset \mathcal{B}\). Thus, \(E_m(\mathcal{B}) \subset L_m \subset \mathcal{B}\). \(\square\)

**Lemma 3.3.** Let \(m \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}\) and \(\phi \in \mathcal{B}\).

(i) \(\|\phi - E_m \phi\|_\infty \leq \|\phi\| \theta_{m+1}^m\).
(ii) \(\text{var}_k(\phi - E_m \phi) \leq 2 \|\phi\| \theta_{m+1}^m\). Moreover, if \(k \geq m\), then \(\text{var}_k(\phi - E_m \phi) \leq \|\phi\| \theta_{k+1}^m\).
(iii) If \(\theta_{m+1} \leq 1\), then \(\|I - E_m\|_{\mathcal{B} \to \mathcal{B}} \leq 3\).

**Proof.** (i) If \(\omega, \omega' \in \Sigma_\mathcal{A}\) satisfy \(\omega|m = \omega'|m\), then \(\|\phi(\omega) - \phi(\omega')\| \leq \text{var}_m(\phi) \leq \|\phi\| \theta_{m+1}^m\). Thus, (i) follows from the same argument as that in the proof of Lemma 2.6 (iii).

(ii) From (i), \(\text{var}_k(\phi - E_m \phi) \leq 2 \|\phi - E_m \phi\|_\infty \leq 2 \|\phi\| \theta_{m+1}^m\). If \(k \geq m\), then \(\text{var}_k(\phi - E_m \phi) = \text{var}_k(\phi) \leq \|\phi\| \theta_{k+1}^m\).

(iii) Take \(\phi \in \mathcal{B}\) so that \(\|\phi\| \leq 1\). By (i) and \(\theta_{m+1} \leq 1\), we have \(\|\phi - E_m \phi\|_\infty \leq 1\). Hence, it is enough to show that \(\text{var}_k(\phi - E_m \phi) \leq 2 \theta_{k+1}^m\) for \(k \in \mathbb{N} \cup \{0\}\). If \(k \geq m\), then, from the latter part of (ii), \(\text{var}_k(\phi - E_m \phi) \leq \theta_{k+1}^m\). If \(k < m\), then, from the former part of (ii) and \(\theta_{m+1} \leq 1\), \(\text{var}_k(\phi - E_m \phi) \leq 2 \theta_{m+1}^m \leq 2 \theta_{k+1}^m\). \(\square\)

Take \(b_1, b_2 > 0\). We consider the following condition for \(g \in V\):

\[ e^{\max \mathcal{R}g} \leq b_1 \quad \text{and} \quad \text{var}_k(g) \leq b_2 \theta_k^g \quad \text{for} \quad k \in \mathbb{N} \cup \{0\}. \quad (16) \]

**Lemma 3.4.** There exists \(C_1 > 0\), depending only on \(b_1\) and \(b_2\), such that the following inequality holds for \(k \in \mathbb{N} \cup \{0\}, \phi \in \mathcal{B}\) and \(g \in V\) satisfying (16):

\[ \text{var}_k(\mathcal{L}_g \phi) \leq C_1 \|\phi\| \theta_{k+1}^g. \]

**Proof.** We have \(\text{var}_0(\mathcal{L}_g \phi) \leq 2 \|\mathcal{L}_g \phi\|_\infty \leq 2 N e^{\max \mathcal{R}g} \|\phi\|_\infty \leq 2 N b_1 \|\phi\|_\infty\). Let \(k \in \mathbb{N}\). By Lemma 2.7, we have \(\text{var}_k(\mathcal{L}_g \phi) \leq N b_1 \{3 \text{var}_{k+1}(g) \|\phi\|_\infty + \text{var}_{k+1}(\phi)\}\). We also have \(\text{var}_{k+1}(g) \leq b_2 \theta_{k+1}^g \leq C b_2 \theta_{k+1}^g\) and \(\text{var}_{k+1}(\phi) \leq \|\phi\| \theta_{k+1}^g\). Hence, the assertion holds for \(C_1 = \max\{2N b_1, N b_1 (3Cb_2 + C)\}\). \(\square\)
For $g \in V$ and $m, q \in \mathbb{N}$, we define the two operators $K_{g, m}, K_{g, m}^{(q)}$ by

$$K_{g, m} = \mathcal{L}_g \circ E_m, \quad K_{g, m}^{(q)} = \mathcal{L}_g^q - (\mathcal{L}_g - K_{g, m})^q.$$ 

Notice that $K_{g, m}^{(1)} = K_{g, m}$.

**Lemma 3.5.** Let $g \in V$. If there exists $b > 0$ such that $\text{var}_k(g) \leq b \theta_k^k$ for $k \in \mathbb{N}$, then the following three assertions hold:

(i) $\mathcal{L}_g(B) \subset B$.

(ii) $K_{g, m}^{(q)}(B) \subset B$ for $m, q \in \mathbb{N}$.

(iii) $\text{rank} K_{g, m}^{(q)} \leq q \text{rank} E_m$ for $m, q \in \mathbb{N}$.

**Proof.** (i) follows from Lemma 3.4 immediately.

(ii) Thanks to (i), it is enough to show that $K_{g, m}(B) \subset B$. From Corollary 3.2 and (i), we have $K_{g, m}(B) = \mathcal{L}_g(E_m(B)) \subset \mathcal{L}_g(B) \subset B$.

(iii) We have $K_{g, m}^{(q)} = \sum_{k=0}^{q-1} \mathcal{L}_g^{q-1-k} K_{g, m}(\mathcal{L}_g - K_{g, m})^k$. For $k \in \{0, \ldots, q - 1\}$, $\text{rank} \mathcal{L}_g^{q-1-k} K_{g, m}(\mathcal{L}_g - K_{g, m})^k \leq \text{rank} K_{g, m} \leq \text{rank} E_m$. Thus, we obtain (iii). □

The following lemma plays a key role in the proof of Theorem 1.1.

**Lemma 3.6.** There exists $C_2 > 0$, depending only on $b_1$ and $b_2$, such that the following two inequalities hold for $m, q \in \mathbb{N}$ with $\theta_{m+1} \leq 1$ and $g \in V$ satisfying (16):

$$\| \mathcal{L}_g \|_{\mathcal{B} \to \mathcal{B}} \leq C_2,$$

$$\| \mathcal{L}_g^q - K_{g, m}^{(q)} \|_{\mathcal{B} \to \mathcal{B}} \leq C_2 \theta_{m+1}^q. \tag{18}$$

**Proof.** We prove (17). Let $C_1$ be as in Lemma 3.4. Take $\phi \in \mathcal{B}$ so that $\| \phi \| \leq 1$. Then, $\| \mathcal{L}_g \phi \|_{\infty} \leq N e^{\max R_g} \| \phi \|_{\infty} \leq N b_1 1 \| \phi \| \leq N b_1$. Moreover, from Lemma 3.4, $\text{var}_k(\mathcal{L}_g \phi) \leq C_1 \theta_{k+1}^k$ for $k \in \mathbb{N} \cup \{0\}$. Thus, (17) holds for $C_2 = C_1 + 1$.

We prove (18). Since $\mathcal{L}_g^q - K_{g, m}^{(q)} = (\mathcal{L}_g - K_{g, m})^q$, we may prove (18) only for $q = 1$. Take $\phi \in \mathcal{B}$ so that $\| \phi \| \leq 1$.

First, from Lemma 3.3 (i), $\| \mathcal{L}_g(\phi - E_m \phi) \|_{\infty} \leq N e^{\max R_f} \| \phi - E_m \phi \|_{\infty} \leq N b_1 \theta_{m+1}^m$. This and $\theta_{m+1} \leq 1$ imply

$$\| \mathcal{L}_g(\phi - E_m \phi) \|_{\infty} \leq N b_1 \theta_{m+1}^m. \tag{19}$$

Next, we show that

$$\text{var}_k(\mathcal{L}_g(\phi - E_m \phi)) \leq N b_1 (3 C b_2 + 2) \theta_{m+1}^k \theta_{k+1}^k, \quad k \in \mathbb{N} \cup \{0\}. \tag{20}$$

From (19), $\text{var}_0(\mathcal{L}_g(\phi - E_m \phi)) \leq 2 \| \mathcal{L}_g(\phi - E_m \phi) \|_{\infty} \leq 2 N b_1 \theta_{m+1}$. Let $k \in \mathbb{N}$. From Lemmas 2.7 and 3.3 (i),

$$\text{var}_k(\mathcal{L}_g(\phi - E_m \phi))$$

$$\leq N e^{\max R_g} \left\{ 3 \text{var}_{k+1}(\phi - E_m \phi) + \text{var}_{k+1}(\phi - E_m \phi) \right\}$$

$$\leq N b_1 \{3 C b_2 \theta_{k+1}^k \theta_{m+1} + \text{var}_{k+1}(\phi - E_m \phi)\}.$$ 

Thus, it is enough to show that

$$\text{var}_{k+1}(\phi - E_m \phi) \leq 2 \theta_{m+1}^k \theta_{k+1}^k. \tag{21}$$

From Lemma 3.3 (ii), if $k + 1 \geq m$, then $\text{var}_{k+1}(\phi - E_m \phi) \leq \theta_{k+1}^k + \theta_{k+1}^k \leq \theta_{m+1}^k \theta_{k+1}^k$, and if $k + 1 < m$, then $\text{var}_{k+1}(\phi - E_m \phi) \leq 2 \theta_{m+1}^m \leq 2 \theta_{m+1}^m \theta_{k+1}^k \leq 2 \theta_{m+1}^m \theta_{k+1}^k$. Thus, (21) holds.

By (19) and (20), we conclude that (18) holds for $C_2 = 3 N b_1 (C b_2 + 1)$.
Lemma 3.7. There exists $C_g > 0$, depending only on $b_1$ and $b_2$, such that $\mathcal{L}_{g,m}(B) \subset B$ and the following inequality holds for $m \in \mathbb{N}$ with $\theta_m \leq 1$ and $g \in V$ satisfying (16):

$$\|\mathcal{L}_g - \mathcal{L}_{g,m}\|_{B \to B} \leq C_3 \theta_m,$$

where $g_m = E_m g$.

Proof. From (i) and (ii) in Lemma 2.6, $g_m$ also satisfies (16). Thus, $\mathcal{L}_{g,m}(B) \subset B$ follows from Lemma 3.5 (i).

We show that the inequality holds for $C_3 = 3N b_1 b_2 (C + 3)$. Take $\phi \in B$ so that $\|\phi\| \leq 1$. It is enough to show the following two inequalities:

$$\|(L_g - L_{g,m})\phi\|_\infty \leq 3 N b_1 b_2 \theta_m,$$

and

$$\text{var}_k((L_g - L_{g,m})\phi) \leq 3 N b_1 b_2 (C + 2) \theta_m \theta_{k+1}, \quad k \in \mathbb{N} \cup \{0\}. \quad (23)$$

To prove (22) and (23), we first show that

$$\|e^g - e^{g_m}\|_\infty \leq 3 b_1 b_2 \theta_m^m. \quad (24)$$

From (14) and Lemma 2.6 (i), $\|e^g - e^{g_m}\|_\infty \leq 3 b_1 \|g - g_m\|_\infty$. If $\omega, \omega' \in \Sigma^+$ satisfy $\omega|m = \omega'|m$, then $|g(\omega) - g(\omega')| \leq b_2 \theta_m^m$, and hence, (24) follows from the same argument as that in the proof of Lemma 2.6 (iii).

We prove (22). Let $\omega \in \Sigma^+$. Then, by (24), we have

$$|((L_g - L_{g,m})\phi)(\omega)| \leq \sum_{i : A(i \omega_0) = 1} |e^{g(i \omega)} - e^{g_m(i \omega)}| \|\phi(i \omega)\| \leq 3 N b_1 b_2 \theta_m^m.$$

Moreover, $\theta_m^m \leq \theta_m$ since $\theta_m \leq 1$. Thus, (22) holds.

We prove (23). From (22), $\text{var}_0((L_g - L_{g,m})\phi) \leq 2\|(L_g - L_{g,m})\phi\|_\infty \leq 6 N b_1 b_2 \theta_m$. Let $k \in \mathbb{N}$ and let $\omega, \omega' \in \Sigma^+$ satisfy $\omega|k = \omega'|k$. Since $\omega_0 = \omega'_0$, we have $\{i : A(i \omega_0) = 1\} = \{i : A(i \omega'_0) = 1\}$. For $i$ with $A(i \omega_0) = 1$, we set $I_1(i) = |e^{g(i \omega)} - e^{g_m(i \omega)}| |\phi(i \omega) - \phi(i \omega')|$, and $I_2(i) = |e^{g(i \omega)} - e^{g_m(i \omega)} - (e^{g(i \omega')} - e^{g_m(i \omega')})| \|\phi(i \omega)\|$. Then,

$$|((L_g - L_{g,m})\phi)(\omega) - ((L_g - L_{g,m})\phi)(\omega')| \leq \sum_{i : A(i \omega_0) = 1} I_1(i) + I_2(i).$$

By (24), we have

$$I_1(i) \leq 3 b_1 b_2 \theta_m^m \theta_{k+1}^k \leq 3 C b_1 b_2 \theta_m^m \theta_{k+1}^k.$$

On the other hand, if $k + 1 \geq m$, then, by (14), $I_2(i) = |e^{g(i \omega)} - e^{g(i \omega')}| \leq 3 \theta_m^m \theta_{k+1}^k \leq 3 b_1 b_2 \theta_m^m \theta_{k+1}^k$, and if $k + 1 < m$, then, by (24), $I_2(i) \leq 2 |e^g - e^{g_m}|_\infty \leq 6 b_1 b_2 \theta_m^m \leq 6 b_1 b_2 \theta_m^m \theta_{k+1}^k$. Thus,

$$I_2(i) \leq 6 b_1 b_2 \theta_m^m \theta_{k+1}^k,$$

and hence, $I_1(i) + I_2(i) \leq 3 b_1 b_2 (C + 2) \theta_m \theta_{k+1}^k$. Thus, (23) holds. \hfill \Box

4. Proof of Theorem 1.1. In this section, we prove Theorem 1.1, which is the main result of this paper.

Proof of Theorem 1.1. (i) $\mathcal{L}_f(B) \subset B$ follows from Lemma 3.5 (i). We show the compactness of $\mathcal{L}_f : B \to B$. By (18), we have $\lim_{m \to \infty} \|\mathcal{L}_f - K_{f,m}\|_{B \to B} = 0$. Since $K_{f,m}$ is a finite-rank operator, $\mathcal{L}_f$ is a compact operator.
(ii) Without loss of generality, we may assume that \( \theta_m > 0 \) for any \( m \in \mathbb{N} \). Then, from Lemma 3.1 (ii),

\[
\bigcup_{m \geq 0} L_m \subset \mathcal{B}.
\]  

(25)

It is enough to show the following claim:

**Claim 4.1.** Let \( \rho > 0 \). Then, \( \Lambda_f \cap \{ |\lambda| > \rho \} = \sigma(\mathcal{L}_f : \mathcal{B} \to \mathcal{B}) \cap \{ |\lambda| > \rho \} \). Moreover, for \( \lambda \in \Lambda_f \) with \( |\lambda| > \rho \), the corresponding multiplicities coincide with each other.

For \( \theta \in (0, 1) \), we denote by \( V_\theta \) the completion of \( V \) by the norm \( \| \cdot \|_\theta \). Clearly, \( V_\theta \subset F_\theta \). Hence, from Lemma 2.4, \( V = \bigcap_{\theta \in (0, 1)} V_\theta \). Moreover, \( \mathcal{L}_f(V_\theta) \subset V_\theta \) since \( \mathcal{L}_f(V) \subset V \). Therefore, it is enough to show the following claim:

**Claim 4.2.** Let \( \rho > 0 \). For \( \theta \in (0, 1) \) with \( \theta e^{P(R)} < \rho \) and \( \lambda \in \sigma(\mathcal{L}_f : V_\theta \to V_\theta) \) with \( |\lambda| > \rho \), \( \lambda \) is an eigenvalue of \( \mathcal{L}_f : \mathcal{B} \to \mathcal{B} \), and hence, \( \lambda \) is that of \( \mathcal{L}_f : V_\theta \to V_\theta \). Moreover, the corresponding multiplicities coincide with each other.

From [10, Theorem 1], \( r_{\text{ess}}(\mathcal{L}_f : V_\theta \to V_\theta) \leq r_{\text{ess}}(\mathcal{L}_f : F_\theta \to F_\theta) = \theta e^{P(R)} < \rho \). Here, for a Banach space \( E \) and a bounded linear operator \( T \) on \( E \), \( r_{\text{ess}}(T : E \to E) \) denotes the essential spectral radius of \( T \), that is,

\[
r_{\text{ess}}(T : E \to E) = \inf \{ r \geq 0 : \text{any } \lambda \in \sigma(T : E \to E) \text{ with } |\lambda| > r \text{ is an isolated eigenvalue with finite multiplicity} \}.
\]

From Lemma 2.6 (iv), \( \bigcup_{m \geq 0} L_m \) is dense in \( V_\theta \). Therefore, from (25), \( \mathcal{B} \) is dense in \( V_\theta \). Thus, Claim 4.2 follows from [1, Lemma A.1].

5. **Proof of Theorem 1.2.** In this section, we prove Theorem 1.2, which is the second main result of this paper.

Let \( f \in V \) and let \( r \in (0, 1) \) satisfy (7). Take \( D > 0 \) so that

\[
\var_m(f)^{1/m} \leq Dr^m, \quad m \in \mathbb{N}.
\]

In this section, we set

\[
\theta_m = Dr^m, \quad m \in \mathbb{N}.
\]  

(26)

When \( \theta_m \) is of the form (26), we can obtain a slightly better estimate than (18).

**Lemma 5.1.** There exists \( C_4 > 0 \), depending only on \( b_1 \) and \( b_2 \), such that the following inequality holds for \( m, q \in \mathbb{N} \) with \( m \geq 2, Dr^{m+1} \leq 1 \) and \( g \in V \) satisfying (16):

\[
\| \mathcal{L}_g^q - K_{g,m}^q \|_{\mathcal{B} \to \mathcal{B}} \leq C_4^q(r^{2m})^q.
\]

**Proof.** The outline of the proof is the same as that of the proof of (18). We may prove the inequality only for \( q = 1 \). Notice that \( C \) in (15) is equal to \( Dr \).

Take \( \phi \in \mathcal{B} \) so that \( \| \phi \| \leq 1 \). From Lemma 3.3 (i), \( \| \mathcal{L}_g(\phi - E_m \phi) \|_\infty \leq Nb_1 \theta_m^m \). Since \( m \geq 2 \) and \( \theta_{m+1} = Dr^{m+1} \leq 1 \), we have \( \theta_m^m \leq \theta_{m+1}^2 = D^2 r^{2m+2} \), and hence,

\[
\| \mathcal{L}_g(\phi - E_m \phi) \|_\infty \leq D^2 Nb_1 r^{2m+2}.
\]  

(27)

Thus, we may show that

\[
\var_k(\mathcal{L}_g(\phi - E_m \phi)) \leq D Nb_1 (3D^2 r^{3b_2} + \max(1, 2Dr^2))r^{2m} \theta_{k+1}^k, \quad k \in \mathbb{N} \cup \{0\}.
\]
From (27), \( \varphi_0(\mathcal{L}_g(\phi - E_m\phi)) \leq 2\|\mathcal{L}_g(\phi - E_m\phi)\|_\infty \leq 2D^2Nb_1r^{2m+2} \). Let \( k \in \mathbb{N} \). From Lemmas 2.7 and 3.3 (i),
\[
\varphi_k(\mathcal{L}_g(\phi - E_m\phi)) \leq Nb_1\{3D^3r^3b_2r^{2m}\theta_{k+1}^2 + \varphi_{k+1}(\phi - E_m\phi)\}.
\]
Thus, it is enough to show that
\[
\varphi_{k+1}(\phi - E_m\phi) \leq D\max(1,2D^2)r^{2m}\theta_{k+1}^3.
\]
(28)
We first assume \( k + 1 \geq m \). From Lemma 3.3 (i), \( \varphi_{k+1}(\phi - E_m\phi) \leq \theta_{k+1}^{m+1} \).
Moreover,
\[
\theta_{k+1}^{k+1} = \left( \frac{\theta_{k+2}}{\theta_{k+1}} \right)^{k+1} \theta_{k+1}^{k+1} = Dr^{2(k+1)}\theta_{k+1}^k \leq Dr^{2m}\theta_{k+1}^k.
\]
Thus, (28) holds for \( k + 1 \geq m \). We next assume \( k + 1 < m \). Then, from Lemma 3.3 (ii), \( \varphi_{k+1}(\phi - E_m\phi) \leq 2\theta_{m+1}^{m-k}\theta_{m+1}^k \leq 2\theta_{m+1}^{k}\theta_{m+1}^k = 2D^2r^{2m+2}\theta_{k+1}^k \). Thus, (28) holds for \( k + 1 < m \).

For \( x \in \mathbb{R} \), \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \). Recall, from (11) in Section 1, the definition of approximation numbers of a bounded linear operator acting on a Banach space. We estimate the approximation numbers of transfer operators acting on the Banach space \( B \).

**Lemma 5.2.** Let \( C_1 \) be as in Lemma 5.1. For \( q \in \mathbb{N} \) and \( R > e^{h_{\top}(\sigma A)} \), there exists \( M_1 \in \mathbb{N} \), depending only on \( q \) and \( R \), such that the following inequality holds for \( m \geq M_1 \) and \( g \in V \) satisfying (16):
\[
a_{m(R^m) + 1}(\mathcal{L}_g) \leq C_1^2(r^{2m})^q.
\]

**Proof.** The equation rank \( E_m = \#\{w : A\text{-admissible and } |w| = m\} \) implies that (rank \( E_m \))\(^{1/m} \to e^{h_{\top}(\sigma A)} \) as \( m \to \infty \). Thus, there exists \( M_1 \geq 2 \) such that \( q \text{rank } E_m \leq R^m \) and \( D_r^{m+1} \leq 1 \) for \( m \geq M_1 \). Let \( m \geq M_1 \). From Lemma 3.5 (iii), \( \text{rank } K_{g,m}^{q} = R^m < [R^m] + 1 \). Hence, by Lemma 5.1, we have \( a_{m(R^m) + 1}(\mathcal{L}_g) \leq ||\mathcal{L}_g - K_{g,m}^q||_{B \to B} \leq C_1^2(r^{2m})^q \), as desired.

**Corollary 5.3.** Let \( C_4 \) be as in Lemma 5.1. For \( q \in \mathbb{N} \) and \( R > e^{h_{\top}(\sigma A)} \), there exists \( N_1 \in \mathbb{N} \), depending only on \( q \) and \( R \), such that the following inequality holds for \( n \geq N_1 \) and \( g \in V \) satisfying (16):
\[
a_n(\mathcal{L}_g) \leq \left( \frac{C_4}{r^2} \right)^q \left( \frac{1}{n - 1} \right)^{-\frac{2q}{\log R - \pi}}.
\]

**Proof.** Let \( M_1 \) be as in Lemma 5.2. Take \( N_1 \in \mathbb{N} \) so that \( |\log R(n-1)| \geq M_1 \) for \( n \geq N_1 \). Let \( n \geq N_1 \). Then, \( n \geq |R^{|\log R(n-1)}| + 1 \) since \( n > n - 1 \geq R^{|\log R(n-1)|} \). Put \( m = |\log R(n-1)| \). From Lemma 5.2, \( a_n(\mathcal{L}_g) \leq C_1^2(r^{2m})^q = (C_4/r^2)^q(r^{2m+1})^q \). Since \( m + 1 \geq \log R(n-1) \), we have \( r^{2(m+1)} \leq (n-1)^{2/\log R} \). Thus, the desired result follows.

Recall from Section 1 the definition of the operator ideal \( \mathcal{L}_p^\alpha(E) \). The following corollary plays a key role in the proof of Theorem 1.2.

**Corollary 5.4.** Let \( g \in V \) satisfy (16). For \( p > 0 \) with (8) and \( m, q \in \mathbb{N} \) with \( p \geq q \), the following two assertions hold (we write \( g_m = E_m g \)):
(i) \( \mathcal{L}_g, \mathcal{L}_{g_m} \in \mathcal{L}_p^\alpha(B) \).
(ii) \( \mathcal{L}_g, \mathcal{L}_{g_m} \in \mathcal{L}_1^\alpha(B) \) and \( \sum_{n=1}^{\infty} a_n(\mathcal{L}_g - \mathcal{L}_{g_m}) \to 0 \) as \( m \to \infty \).
Proof. We prove (i) and the former part of (ii). From (i) and (ii) in Lemma 2.6, \(g_m\) also satisfies (16). Thus, it is enough to prove the assertions only for \(g\). Take \(R > e^{\epsilon_{n_0}(\sigma_\Lambda)}\) with \(r^2R < 1\). Then,
\[
-\frac{2q}{\log r R} \geq -\frac{2p}{\log r R} > 1.
\]
(29)
Hence, \(A_f \in \mathcal{A}_1^\bullet(B)\) and \(A_f^q \in \mathcal{L}_1^\bullet(B)\) follow from Corollary 5.3.

We prove the latter part of (ii). Let \(N_1 \in \mathbb{N}\) be as in Corollary 5.3. Let \(\epsilon > 0\). By (29), we can take \(n_0 \geq N_1\) so that \(\sum_{n \geq n_0} (C_4/r^2)^q(n-1)^2q/\log r R < \epsilon\). Moreover, by Lemma 3.7, we can take \(m_0 \in \mathbb{N}\) so that \(\theta_{m+1} \leq 1\) and \(n_0\|A_f^q - A_f^q\|_{B \rightarrow B} < \epsilon\) for \(m \geq m_0\).

Let \(m \geq m_0\). We show \(\sum_{n=1}^{\infty} n_0(A_f^q - A_f^q) < 6\epsilon\). We easily have
\[
\sum_{n=1}^{\infty} n_0(A_f^q - A_f^q) \leq 2n_0a_1(A_f^q - A_f^q) = 2n_0\|A_f^q - A_f^q\|_{B \rightarrow B} < 2\epsilon.
\]
On the other hand, we have
\[
\sum_{n \geq 2n_0} n_0(A_f^q - A_f^q) \leq \sum_{l \geq n_0} \sum_{m=2l}^{\infty} n_0(A_f^q - A_f^q) \leq 2\sum_{l \geq n_0} a_{2l}(A_f^q - A_f^q).
\]
By [9, Theorem 2.3.3], \(a_{2l}(A_f^q - A_f^q) \leq a_{2l-1}(A_f^q - A_f^q) \leq a_l(A_f^q) + a_l(A_f^q).
\]
Thus, by Corollary 5.3, we have
\[
\sum_{n \geq 2n_0} n_0(A_f^q - A_f^q) \leq 2\left\{ \sum_{l \geq n_0} a_l(A_f^q) + \sum_{l \geq n_0} a_l(A_f^q) \right\} < 4\epsilon,
\]
as desired. \(\square\)

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. (i) follows from Corollary 5.4 (i) and (I) in Section 1.
(ii) Let \(f_m = E_m f\). Then, \(\sum_{\omega \in \text{Per}_\omega(\sigma_\Lambda)} e^{S_\omega f(\omega)} = \lim_{m \rightarrow \infty} \sum_{\omega \in \text{Per}_\omega(\sigma_\Lambda)} e^{S_\omega f_m(\omega)}\).
Since \(f_m\) is locally constant, \(\sum_{\omega \in \text{Per}_\omega(\sigma_\Lambda)} e^{S_\omega f_m(\omega)} = \sum_{n=1}^{\infty} \lambda_n(f_m)^q\).
Moreover, \(\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} \lambda_n(f_m)^q = \sum_{n=1}^{\infty} \lambda_n(f)^q\) follows from Corollary 5.4 (ii) and (II) in Section 1. Thus, (9) holds.

(iii) From [2, Theorem 2.6.5], the infinite product \(P(z) = \prod_{n=1}^{\infty} E(z \lambda_n(f), k_0)\) converges uniformly on any compact set of \(\mathbb{C}\). Therefore, it is enough to show the following equality for \(z \in \mathbb{C}\) with sufficiently small \(|z|\):
\[
P(z) = \exp\left(-\sum_{q=k_0+1}^{\infty} \frac{z^q}{q} \sum_{\omega \in \text{Per}_\omega(\sigma_\Lambda)} e^{S_\omega f(\omega)}\right).
\]
We denote by \(\text{Log}\) the principle branch of the complex logarithm. By \(\log(1 - z) = -\sum_{n \geq 1} z^n/n\) for \(|z| < 1\), we have \(P(z) = \exp(-\sum_{q=k_0+1}^{\infty} \frac{z^q}{q} \lambda_n(f)^q)\) for \(z \in \mathbb{C}\) with sufficiently small \(|z|\). Since \(\sum_{n=1}^{\infty} |\lambda_n(f)|^{k_0+1} < \infty\), we can exchange the order of the summation and we have \(P(z) = \exp(-\sum_{q=k_0+1}^{\infty} \frac{z^q}{q} \sum_{n=1}^{\infty} \lambda_n(f)^q)\) = \(\exp(-\sum_{q=k_0+1}^{\infty} \frac{z^q}{q} \sum_{\omega \in \text{Per}_\omega(\sigma_\Lambda)} e^{S_\omega f(\omega)})\), as desired. \(\square\)

We show that, for each \(r \in (0, 1)\), there exists a non-locally constant \(f \in V\) satisfying (7).
Example 5.5. Let $A = \left\{ \frac{1}{2} \right\}$. Take two sequences $\{\theta^{(1)}_m\}_{m \in \mathbb{N} \cup \{0\}}, \{\theta^{(2)}_m\}_{m \in \mathbb{N} \cup \{0\}}$ so that $\theta^{(1)}_m > \theta^{(2)}_m > 0, \ m \in \mathbb{N} \cup \{0\}$, and

$$1 > \theta^{(1)}_m \downarrow 0, \quad 1 > \theta^{(2)}_m \downarrow 0, \quad \limsup_{m \to \infty} \frac{\theta^{(2)}_m}{\theta^{(1)}_m} < 1.$$

We define $f: \Sigma^+_A \to \mathbb{R}$ by $f(\omega) = \sum_{m \geq 0} \{\theta^{(\omega_m)}_m\}^{1/m}$. Then, it is easy to see that $\text{var}_m(f)^{1/m} \simeq \theta^{(1)}_m$, that is, there exists $C > 0$ such that $C^{-1} \leq \text{var}_m(f)^{1/m}/\theta^{(1)}_m \leq C$ for $m \in \mathbb{N}$. Hence, for $r \in (0, 1), \theta^{(1)}_m = r^m$ and $\theta^{(2)}_m = (1/2)\theta^{(1)}_m$, $f$ is a real-valued, non-locally constant function in $V$ such that $\text{var}_m(f)^{1/m} \asymp r^m$.

Appendix A. Transfer operators on $V$. A metrizable topological vector space is said to be complete if every Cauchy sequence converges. Note that our space $V$ is metrizable and complete since $V$ is a Fréchet space (see Proposition 2.5). We recall the following definition of a compact operator on a metrizable and complete topological vector space.

Definition A.1. Let $X$ be a metrizable and complete topological vector space and $T$ a continuous linear operator on $X$. We say that $T$ is a compact operator if the closure $T(N)$ of the image $T(N)$ is compact for some neighborhood $N$ of zero.

In this appendix, we prove the following two theorems:

Theorem A.2. $\mathcal{L}_f: V \to V$ is not a compact operator for any $f \in V$.

Theorem A.3. If $A = \left\{ \frac{1}{2} \right\}$, then there exists a real-valued $f \in V$ such that $\sum_{n \geq 1} |\lambda_n(f)| = \infty$.

First, we prove Theorem A.3.

Proof of Theorem A.3. [6, Proposition 4.1] implies that there exists a real-valued $f \in V$ such that $\zeta_f(z)^{-1} = 1 - 2z - z(1 - z)\sin z$. For $x \in \mathbb{R}$, we write $F(x) = (1 - 2x)/(x(1 - x))$ and $G(x) = \sin x - F(x)$. Since $F(x) > 0$ for $x > 1$ and $\lim_{x \to \infty} F(x) = 0$, we find that $G(2n\pi) < 0$ and $G(2n\pi + (\pi/2)) > 0$ for sufficiently large $n \in \mathbb{N}$. Thus, for sufficiently large $n \in \mathbb{N}$, there exists $\eta_n \in (2n\pi, 2n\pi + \frac{\pi}{2})$ such that $G(\eta_n) = 0$. Therefore, we have

$$\sum_{n \geq 1} |\lambda_n(f)| \geq \sum_{n: \text{large}} \frac{1}{\eta_n} \geq \sum_{n: \text{large}} \frac{1}{2n\pi + \frac{\pi}{2}} = \infty,$$

as desired $\square$

Next, we prove Theorem A.2. For the sake of simplicity, we write

$$\text{var}^\theta_m(\phi) = \frac{\text{var}_m(\phi)}{\theta^m}$$

for $\theta \in (0, 1), \phi \in F_\theta$ and $m \in \mathbb{N} \cup \{0\}$.

To prove Theorem A.2, we need the following lemma:

Lemma A.4. Let $A \subset V$ be a neighborhood of zero. Then there exists an open neighborhood $G \subset A$ of zero satisfying the following two conditions:

$$\phi \frac{1}{2}1_i \in G \quad \text{for} \quad \phi \in G \quad \text{and} \quad i \in \{1, \ldots, N\}, \quad (30)$$

$$\text{there exists} \ \theta \in (0, 1) \ \text{such that} \ \sup_{\phi \in G} \|\phi\|_\theta = \infty. \quad (31)$$
Proof. The collection of the sets of the form \( \{ \phi \in V : \|\phi\|_\theta < \epsilon \} \), \( \theta \in (0, 1) \), \( \epsilon > 0 \), is a fundamental system of neighborhoods of zero. Thus, there exist \( \theta_0 \in (0, 1) \) and \( \epsilon > 0 \) such that the set \( G = \{ \phi \in V : \|\phi\|_{\theta_0} < \epsilon \} \) is contained in \( A \). We prove (30) and (31) for \( G \).

Let \( \phi \in G \) and \( i \in \{1, \ldots, N\} \). If \( \omega_0 = \omega_0' \), then \( \phi 1[i](\omega) - \phi 1[i](\omega') = \phi(\omega) - \phi(\omega') \) \( (\omega_0 = i) \), and hence, for \( m \in \mathbb{N} \), we have \( \vartheta_m^n(\phi 1[i]) \leq [\phi]_{\theta_0} \).

Also, if \( \omega_0 \neq \omega_0' \), then \( \phi 1[i](\omega) - \phi 1[i](\omega') = \phi(\omega) \) \( (\omega_0 = i) \), \( = 0 \) \( (\omega_0 \neq i \text{ and } \omega_0' \neq i) \), and hence, we have \( \vartheta_m^n(\phi 1[i]) \leq \|\phi\|_{\infty} \). Combining, we have

\[
\|\phi 1[i]\|_{\theta_0} = \|\phi 1[i]\|_{\infty} + \sup_{m \geq 0} \vartheta_m^n(\phi 1[i]) \leq \|\phi\|_{\infty} + \max(\|\phi\|_{\infty}, [\phi]_{\theta_0}) \leq 2\|\phi\|_{\theta_0}.
\]

Thus, \( \|\phi \|_{\theta_0} < \epsilon \), and hence, \( (\phi/2)1[i] \in G \). Therefore, (30) follows.

Lemma 2.1 ensures that there exist \( i_1, i_2 \in \{1, \ldots, N\} \) such that \( i_1 \neq i_2 \) and \( A(i_1) = A(i_2) = 1 \). Moreover, there exists \( j \in \{1, \ldots, N\} \) such that \( A(j) = 1 \). Take an \( A \)-admissible word \( w \) such that \( w_0 = i \) and \( w_{|w|-1} = j \). For \( n \in \mathbb{N} \), we define

\[
\phi_n = \frac{\theta_0^n|w|+1}{2} 1_n[w \cdots w_{ij}]\theta_0^n|w|+1.
\]

For \( \theta \in (0, 1) \), we have \( \vartheta_{|n|+1}(1_n[w \cdots w_{ij}]\theta_0^n|w|+1) = 1/\theta_0^n|w|+1 \), and hence,

\[
\|\phi_n\|_{\theta} = \frac{\theta_0^n|w|+1}{2} \left(1 + \frac{1}{\theta_0^n|w|+1}\right).
\]

In particular, we have \( \|\phi_n\|_{\theta_0} < \epsilon \). Therefore, \( \phi_n \in G \). On the other hand, for \( \theta \in (0, \theta_0) \), we see that \( \|\phi_n\|_{\theta} \geq (\epsilon/2)(\theta_0/\theta)^n|w|+1 \to \infty \) as \( n \to \infty \), hence \( \sup_{\phi \in G} \|\phi\|_{\theta} \geq \sup_{n \in \mathbb{N}} \|\phi_n\|_{\theta} = \infty \). Thus, (31) follows.

Proof of Theorem A.2. Fix a neighborhood \( A \subset V \) of zero. We show that the following assertion holds:

there exists \( \{\phi_n\}_{n \in \mathbb{N}} \subset A \) such that \( \{\mathcal{L}^f \phi_n\}_{n \in \mathbb{N}} \) has no convergent subsequence.

Without loss of generality, we may assume that \( A \) satisfies both (30) and (31) with \( G \) replaced by \( A \).

First, we consider the case in which \( \sup_{\phi \in A} \|\phi\|_{\infty} = \infty \). The inequality \( \|\phi\|_{\infty} \leq \sum_{i=1}^{N} \|\phi 1[i]\|_{\infty} \) implies that there exists \( i \in \{1, \ldots, N\} \) such that \( \sup_{\phi \in A} \|\phi 1[i]\|_{\infty} = \infty \). Thus, we can take \( \{\psi_n\}_{n \in \mathbb{N}} \subset A \) so that \( \lim_{n \to \infty} \|\psi_n(2)/2\theta_0^n|w|+1\|_{\infty} = \infty \).

We write \( \phi_n = (\psi_n/2)1[i] \). It is clear that \( \{\phi_n\}_{n \in \mathbb{N}} \subset A \). Take \( \omega^{(n)} \in \Sigma_A \) so that \( \|\phi_n\|_{\infty} = |\phi_n(\omega^{(n)})| \). Since \( \|\phi_n\|_{\infty} > 0 \) implies \( \omega_0^{(n)} = i \), we have

\[
(L^f \phi_n)(\sigma \omega^{(n)}) = \sum_{j:A(\omega_j^{(n)})=1} e^{(i(\sigma A \omega^{(n)}))}\phi_n(j(\sigma A \omega^{(n)}))1[i](j(\sigma A \omega^{(n)}))
\]

\[
= e^{(i(\sigma A \omega^{(n)}))}\phi_n(i(\sigma A \omega^{(n)})) = e^{(i(\omega^{(n)}))}\phi_n(\omega^{(n)})
\]

for sufficiently large \( n \in \mathbb{N} \). Hence,

\[
\|L^f \phi_n\|_{\infty} \geq |(L^f \phi_n)(\sigma A \omega^{(n)})| = e^{Rf(\omega^{(n)})}|\phi_n(\omega^{(n)})| \geq e^{\min Rf} \|\phi_n\|_{\infty} \to \infty
\]

as \( n \to \infty \). Therefore, (32) holds.
Next, we consider the case in which sup_{φ ∈ A} \|φ\|_∞ < ∞. We see that there exists θ ∈ (0, 1) such that sup_{φ ∈ A} \|φ\|_θ = ∞. Since \|φ\|_θ = \|φ\|_∞ + |φ|_θ and sup_{φ ∈ A} \|φ\|_∞ < ∞, we have sup_{φ ∈ A} |φ|_θ = ∞. Moreover, for m ∈ N, we have

\[
∞ = sup_{φ ∈ A} |φ|_θ ≤ \left( \sup_{φ ∈ A} sup_{0 < k ≤ m} var^0_k(φ) \right) + \left( \sup_{φ ∈ A} sup_{k > m} var^0_k(φ) \right),
\]

and hence, we have sup_{φ ∈ A} sup_{k > m} var^0_k(φ) = ∞. Therefore, for k ∈ N, there exist ω(k), ˜ω(k) ∈ Σ^+_A and ψ_k ∈ A such that

\[
ω(k) \neq ˜ω(k), \quad \lim_{k→∞} d_θ(ω(k), ˜ω(k)) = 0, \quad \lim_{k→∞} \frac{|ψ_k(ω(k)) − ψ_k(˜ω(k))|}{d_θ(ω(k), ˜ω(k))} = 0.
\]

We can choose i ∈ \{1, \ldots, N\} so that

\[
\mathcal{N} = \{k ∈ N : ω^{(k)}_0 = ˜ω^{(k)}_0 = i\}
\]

is an infinite set. We write \mathcal{N} = \{k_1, k_2, \ldots\}, where k_n < k_{n+1} for n ∈ N. Notice that, for n ∈ N, d_θ(σ_A^ω(k_n), σ_A^˜ω(k_n)) > 0, and hence, d_θ(ω(k_n), ˜ω(k_n)) = θd_θ(σ_A^ω(k_n), σ_A^˜ω(k_n)).

We write φ_n = (ψ_{k_n}/2^1)i. It is clear that \{φ_n\}_{n ∈ N} ⊂ A. By a calculation similar to (33), we have

\[
( \mathcal{L}_f φ_n)(σ_A^ω(k_n)) − ( \mathcal{L}_f φ_n)(σ_A^˜ω(k_n)) = e^f(ω(k_n))\{φ_n(ω(k_n)) − φ_n(˜ω(k_n))\} − φ_n(˜ω(k_n))\{e^f(ω(k_n)) − e^f(˜ω(k_n))\},
\]

and thus, we have

\[
\| \mathcal{L}_f φ_n \| ≥ \frac{|( \mathcal{L}_f φ_n)(σ_A^ω(k_n)) − ( \mathcal{L}_f φ_n)(σ_A^˜ω(k_n))|}{d_θ(σ_A^ω(k_n), σ_A^˜ω(k_n))}
\]

\[
≥ θ\left\{ \frac{|e^f(ω(k_n))\{φ_n(ω(k_n)) − φ_n(˜ω(k_n))\}|}{d_θ(ω(k_n), ˜ω(k_n))} − \frac{|φ_n(˜ω(k_n))\{e^f(ω(k_n)) − e^f(˜ω(k_n))\}|}{d_θ(ω(k_n), ˜ω(k_n))} \right\}
\]

\[
≥ \frac{θe^{min \|f\|}}{2} \frac{|ψ_{k_n}(ω(k_n)) − ψ_{k_n}(˜ω(k_n))|}{d_θ(ω(k_n), ˜ω(k_n))} − θ \left( sup_{φ ∈ A} \|φ\|_∞ \right) \frac{∥e^f∥_θ → ∞}{\}
\]

as n → ∞. Therefore, (32) holds. □

Appendix B. Some properties of V. We recall the following definitions of a nuclear operator and a nuclear space.

**Definition B.1.** Let E, F be Banach spaces and T : E → F a bounded linear operator. We say that T is a nuclear operator if T can be written in the form

\[
Tx = \sum_{n=1}^{∞} λ_n(x, x'_n)y_n,
\]

where the sequence \{λ_n\}_{n ∈ N} ⊂ C is summable and both of the two sequences \{x_n\}_{n ∈ N} ⊂ E' and \{y_n\}_{n ∈ N} ⊂ F are bounded. (E' denotes the dual Banach space of E.)

**Definition B.2.** Let X be a locally convex Hausdorff topological vector space. We say that X is a nuclear space if, for every continuous seminorm p on X, there exists a continuous seminorm q on X such that p ≤ q and the natural embedding X_q → X_p is a nuclear operator. Here, X_p denotes the completion of X/ker p by p.
Let $E,F$ be Banach spaces and $T : E \to F$ a bounded linear operator. We extend the definition (11) of the approximation numbers of $T$ to the case $E \neq F$. For $n \in \mathbb{N}$, the $n$-th approximation number $a_n(T : E \to F)$ of $T$ is defined by

$$a_n(T : E \to F) = \inf\{\|T - A\| : A : E \to F \text{ is a finite-rank operator with rank } A < n\}.$$

In this appendix, we prove the following two theorems:

**Theorem B.3.** Let $R$ be the set of $\phi \in V$ such that $\phi$ is not cohomologous with any locally constant function, that is,

$$R = \{\phi \in V : \phi - \varphi \neq \psi \circ \sigma_A - \psi \text{ for any locally constant function } \varphi \text{ and continuous function } \psi\}.$$  

Then, $R$ is a residual subset of $V$. In other words, there exists a sequence $V_1, V_2, \ldots$ of open and dense subsets of $V$ such that $R \supset \bigcap_{m \in \mathbb{N}} V_m$.

To prove Theorem B.3, we need the following lemma:

**Lemma B.5.** Let $\theta < \theta'/N$, then the natural embedding $\iota : F_\theta \to F_{\theta'}$ is nuclear.

**Proof.** By [9, Proposition 2.3.11], it is enough to show that $\sum_{n \geq 1} a_n(\iota : F_\theta \to F_{\theta'})$ converges. For $m \in \mathbb{N}$, Lemma 2.6 (iv) and the inequality $E_m \leq N^m$ imply that $a_{N^m+1}(\iota : F_\theta \to F_{\theta'}) \leq 3(\theta/\theta')^m$. Thus, we have

$$\sum_{n \geq N} a_n(\iota : F_\theta \to F_{\theta'}) \leq \sum_{m \geq 1} \sum_{N^m < n \leq N^{m+1}} a_{N^m+1}(\iota : F_\theta \to F_{\theta'}) \leq 3(N - 1) \sum_{m \geq 1} \left(\frac{N\theta}{\theta'}\right)^m < \infty,$$

as desired. \hfill \Box

**Proof of Theorem B.3.** For $\theta \in (0,1)$, $V_\theta$ denotes the completion of $V$ by the norm $\|\cdot\|_\theta$. If $0 < \theta < \theta'/1$, then

$$F_\theta \subset V_{\theta'} \subset F_{\theta'}.$$  \hfill (34)

Indeed, $V_{\theta'} \subset F_{\theta'}$ is obvious and $F_\theta \subset V_{\theta'}$ follows from Lemma 2.6 (iv).

Let $p$ be a continuous seminorm on $V'$ and $\tilde{V}_p$ the completion of $V/\ker p$ by $p$. There exist $\theta' \in (0,1)$ and $C > 0$ such that $p(\cdot) \leq C\|\cdot\|_{\theta'}$. We set $\theta' = \theta'' / 2, \theta = \theta'' / (N + 1)$. By (34), the natural embedding $\iota : V_\theta \to \tilde{V}_p$ can be decomposed as follows:

$$\iota : V_\theta \xrightarrow{\iota_1} (F_\theta, \|\cdot\|_\theta) \xrightarrow{\iota_2} (F_{\theta'}, \|\cdot\|_{\theta'}) \xrightarrow{\iota_3} (F_{\theta'}, \|\cdot\|_{\theta''}) \xrightarrow{\iota_4} \tilde{V}_p.$$

Here, all $\iota_1, \iota_2, \iota_3, \iota_4$ are the natural embeddings. Note that $\iota_2$ is nuclear from Lemma B.5 and $\iota_1, \iota_3, \iota_4$ are continuous. Thus, $\iota : V_\theta \to \tilde{V}_p$ is nuclear from [14, Proposition III.47.1], and hence, $V$ is a nuclear space. \hfill \Box

Recall from Section 1 that, for $q \in \mathbb{N}$, $\phi : \Sigma_A^+ \to \mathbb{C}$ and $\omega \in \Sigma_A^+$, we write $S_q \phi(\omega) = \sum_{k=0}^{q-1} \phi(\sigma_k^A \omega)$.

To prove Theorem B.4, we need some lemmas.
Lemma B.6. Let $m \in \mathbb{N}$. Let $\mathbf{A}$-admissible words $w, v$ satisfy $|w| \geq m$, $w_0 = v_0$ and $\mathbf{A}(w_{|w|-1}w_0) = \mathbf{A}(v_{|v|-1}v_0) = 1$. Then, for $\varphi \in L_m$, we have

$$S_{2|w|+|v|}\varphi((wvv)^*) = S_{|w|}\varphi(w^*) + S_{|w|+|v|}\varphi((wvv)^*).$$

Proof. By $S_{p+q}\varphi(\omega) = S_p\varphi(\omega) + S_q\varphi(\sigma^p_{\mathbf{A}}\omega)$, we have

$$S_{2|w|+|v|}\varphi((wvv)^*) = S_{|w|}\varphi((wvv)^*) + S_{|v|}\varphi((wvv)^*).$$

Notice that $S_{|w|}\varphi((wvv)^*) = \sum_{k=0}^{|w|-1} \varphi(w_k\cdots w_{|w|-1}w_0w_1\cdots w_{|w|}v^k)$. Since $\varphi \in L_m$ and $(w_k\cdots w_{|w|-1}w_0w_1\cdots w_{|w|}v^k)m = (w_k\cdots w_{|w|-1}w^k)m$ for $0 \leq k \leq |w| - 1$, we have $\varphi(w_k\cdots w_{|w|-1}w_0w_1\cdots w_{|w|}) = \varphi(w_k\cdots w_{|w|-1}w^k)$ for $0 \leq k \leq |w| - 1$. Hence, $S_{|w|}\varphi((wvv)^*) = S_{|w|}\varphi(w^*)$. Similarly, $S_{|v|}\varphi((wvv)^*) = S_{|v|}\varphi((wvv)^*)$ and $S_{|w|}\varphi(v(wvv)^*) = S_{|w|}\varphi(\sigma^{|w|}_{\mathbf{A}}(wvv)^*)$. Therefore, we have

$$S_{2|w|+|v|}\varphi((wvv)^*) = S_{|w|}\varphi(w^*) + S_{|v|}\varphi((wvv)^*) + S_{|w|}\varphi(\sigma^{|w|}_{\mathbf{A}}(wvv)^*),$$

as desired. \(\square\)

Here is a key lemma.

Lemma B.7. There exist $\mathbf{A}$-admissible words $\tilde{w}, v$ such that the following two assertions hold:

(i) $\tilde{w}_0 = v_0$, $\mathbf{A}(\tilde{w}_{|\tilde{w}|-1}\tilde{w}_0) = \mathbf{A}(v_{|v|-1}v_0) = 1$ and $|\tilde{w}| \geq |v|$.

(ii) $v$ is self-avoiding and $v_{|v|-1} \notin \{\tilde{w}_k : 0 \leq k \leq |\tilde{w}|-1\}.$

Proof. From Lemma 2.1, there exists $i \in \{1, \ldots, N\}$ such that the $i$-th column has more than two entries which are equal to one. Take an $\mathbf{A}$-admissible word $\tilde{w}$ so that $\tilde{w}_0 = \tilde{w}_{|\tilde{w}|-1} = i$ and $|\tilde{w}| = \min\{|w| : w \text{ is } \mathbf{A}\text{-admissible}, w_0 = w_{|w|-1} = i\}$. \(35\)

Let $j_1 = |\tilde{w}|-2$. There exists $j_2 \in \{1, \ldots, N\}$ such that $j_2 \neq j_1$ and $\mathbf{A}(j_2) = 1$. We prove the following assertion:

$$j_2 \notin \{\tilde{w}_k : 0 \leq k \leq |\tilde{w}|-2\}. \tag{36}$$

First, $j_2 \neq |\tilde{w}|-2$ since $|\tilde{w}|-2 = j_1$. Next, we show that $j_2 = j_0$. Recall $\tilde{w}_0 = i$. We assume $j_2 \neq j_0$. Then, $\mathbf{A}(i) = 1$, and hence, from (35), $\tilde{w} = i$. Therefore, we have the contradiction $j_1 = i = j_2$. Finally, we show that $j_2 \neq \tilde{w}_k$ for any $k \in \{1, \ldots, |\tilde{w}|-3\}$. We assume $j_2 = \tilde{w}_k$ for some $k \in \{1, \ldots, |\tilde{w}|-3\}$. Then, the word $w = w_0\tilde{w}_1\tilde{w}_2\cdots \tilde{w}_k$ is $\mathbf{A}$-admissible and satisfies $v_0 = w_{|w|-1} = i$. Moreover, $|w| = k + 2 < |\tilde{w}|$. Thus, by (35), we have a contradiction, and (36) follows.

We prove the lemma. Take an $\mathbf{A}$-admissible word $v$ so that $v$ is self-avoiding and $v_0 = i, v_{|v|-1} = j_2$. Moreover, take $N \in \mathbb{N}$ so that $N(|\tilde{w}|-1) \geq |v|$ and set

$$\tilde{w} = (\tilde{w}_0\cdots \tilde{w}_{|\tilde{w}|-2})(\tilde{w}_0\cdots \tilde{w}_{|\tilde{w}|-2})\cdots (\tilde{w}_0\cdots \tilde{w}_{|\tilde{w}|-2}).$$

Then, these $\tilde{w}, v$ satisfy the desired properties. \(\square\)

Let $\tilde{w}, v$ be as in Lemma B.7. For $m \in \mathbb{N}$, we write

$$w^{(m)} = \underbrace{\tilde{w} \tilde{w} \cdots \tilde{w}}_{m \text{ times}}.$$
Lemma B.8. Let $m \in \mathbb{N}$ and let $\phi : \Sigma^+_{\lambda} \to \mathbb{C}$ be cohomologous with an $m$-locally constant function. Then, we have

$$S_{[w_m]}>[v_m] \phi((w_m^m)v^m)^* = S_{[w_m]}>[v_m] \phi((w_m^m)^*) + S_{[w_m]}>[v_m] \phi((w_m^m)v^m)^*.$$  

(37)

Proof. We write $w = w_m^m$ for the sake of simplicity. Let $\phi$ be cohomologous with $\varphi \in L_m$. Lemma B.7 (ii) implies that the periodic point $(wwv)^*$ has period $2|w| + |v|$. Thus, we have $S_{[w_m]+[v_m]} \phi((wwv)^*) = S_{[w_m]+[v_m]} \phi((wwv)^*)$. Similarly, we have $S_{[w_m]} \phi(w^m) = S_{[w_m]} \phi(v^m)$ and $S_{[w_m]+[v_m]} \phi((wwv)^*) = S_{[w_m]+[v_m]} \phi((wwv)^*)$. Thus, we obtain (37) by Lemma B.6.

We are ready to prove Theorem B.4.

Proof of Theorem B.4. For $m \in \mathbb{N}$, we set

$$V_m = \{ \phi \in V \mid \phi \text{ does not satisfy (37)} \}.$$  

Lemma B.8 implies that $R \supset \bigcap_{m \in \mathbb{N}} V_m$. Thus, it is enough to show that, for $m \in \mathbb{N}$, $V_m$ is an open and dense subset of $V$.

We write $w = w_m^m$. Then, the two maps $\phi \mapsto S_{[w_m]+[v_m]} \phi((wwv)^*)$ and $\phi \mapsto S_{[w_m]} \phi(w^m) + S_{[w_m]+[v_m]} \phi((wwv)^*)$ from $V$ to $\mathbb{C}$ are continuous, and hence, $V_m$ is open.

We prove the denseness. Let $\phi \in V \setminus V_m$. We write $e_n = (1/n)1_{[wwv]}$ for $n \in \mathbb{N}$. Then, for $\theta \in (0, 1)$, we have $\|e_n\| \leq (1/n)(1 + 2/\theta^2|w|+|v|)$, and hence, $\||\phi + e_n) - \phi\|_\theta \to 0$ as $n \to \infty$. Thus, $\phi + e_n \to \phi$ in $V$ as $n \to \infty$. Therefore, it is enough to prove

$$\phi + e_n \in V_m, \quad n \in \mathbb{N}.$$  

(38)

We easily have $S_{[w_m]}e_n(w^m) = 0$. We show that $S_{[w_m]+[v_m]}e_n((wwv)^*) = 0$. It is enough to show that

$$\sigma_{\lambda}^k((wwv)^*)[2|w| - 1] \neq wwv, \quad 0 \leq k \leq |w| + |v| - 1.$$  

(39)

Let $0 \leq k \leq |w| + |v| - 1$. Then, $(\sigma_{\lambda}^k((wwv)^*))_{[w_m]+[v_m]} - 1 = (wwv)^*_{[w_m]+[v_m]} - 1 = v_{[w]+v}|w| - 1$. On the other hand, $|w| + |v| - 1 - k \leq 2|w| - 1$ from $|w| \geq |v| \geq |v|$, and hence, $(wwv)^*_{[w_m]+[v_m]} - 1 \neq v_{[w]} - 1$. Thus, (39) holds.

We prove (38). We have $S_{[w_m]+[v_m]}((\phi + e_n)(wwv)^*) = S_{[w_m]+[v_m]}((wwv)^*)$ since $S_{[w_m]+[v_m]}e_n((wwv)^*) \geq 1/n > 0$. Lemma B.8 and $S_{[w_m]}e_n((wwv)^*) \neq S_{[w_m]+[v_m]}((wwv)^*)$ imply $S_{[w_m]+[v_m]}((\phi + e_n)(wwv)^*) = S_{[w_m]}((\phi + e_n)(w^m)) + S_{[w_m]+[v_m]}((\phi + e_n)(wwv)^*)$. Thus, $\phi + e_n$ does not satisfy (37), and we obtain (38).

Appendix C. Asymptotic behavior of the eigenvalues. In this appendix, we give the following asymptotic behavior of eigenvalues of transfer operators:

Theorem C.1. Let $\{\theta_m\}$ satisfy (2) and assume that $\theta_m > 0$ for any $m \in \mathbb{N}$. Let $C_2$ be as in Lemma 3.6. Let $\alpha > 0$ and $R > e^{\nu_{c+\alpha}(\sigma(A))}$. Then, there exist $C_5 > 0$ and $M \in \mathbb{N}$, depending only on $b_1, b_2, \alpha$ and $R$, such that the following inequality holds for $m \geq M$ and $g \in V$ satisfying (16):

$$\#\{n \in \mathbb{N} : |\lambda_n(g)| > (C_2 + 1)\theta_m \} \leq C_5\theta_m^{-\alpha}R^{-m-1}.$$  

Proof. We write $B = B(\{\theta_m\})$. Take $M \in \mathbb{N}$ so that $\theta_m \leq 1$ and rank $E_{m-1} \leq R^{m-1}$ for $m \geq M$. For $s > 0$, we denote by $N_m(s)$ the number of eigenvalues $\lambda$ of $L_g : B \to B$ with $|\lambda| > s$, where each $\lambda$ is counted according to its multiplicity. Then, by Theorem 1.1 (ii), $\#\{n \in \mathbb{N} : |\lambda_n(f)| > (C_2 + 1)\theta_m \} = N_m((C_2 + 1)\theta_m)$. 

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We see by (18) that $(C_2 + 1)\theta_m - \|L_g - K_{g,m-1}\|_{\mathcal{B} \to \mathcal{B}} \geq \theta_m$. Thus, by [3, Corollary 4.3], there exists $C(\alpha) > 0$ such that the following inequality holds:

$$N_m((C_2 + 1)\theta_m) \leq C(\alpha)(C_2 + 1)\theta_m^\alpha \sum_{k=1}^\infty a_k(K_{g,m-1})^\alpha.$$ 

Since $a_k(K_{g,m-1}) = 0$ for $k > \text{rank } K_{g,m-1}$ and $\text{rank } K_{g,m-1} \leq \text{rank } E_{m-1}$, we see that $\sum_{k=1}^\infty a_k(K_{g,m-1})^\alpha \leq R^{m-1}\|K_{g,m-1}\|_{\mathcal{B} \to \mathcal{B}}^\alpha$. From (17) and Lemma 3.3 (iii), $\|K_{g,m-1}\|_{\mathcal{B} \to \mathcal{B}} \leq 4C_2$. Thus, the assertion holds for $C_5 = (C_2 + 1)(4C_2)^\alpha C(\alpha)$. \[\square\]

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