Strong and weak mean value properties on trees

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Abstract. We consider the mean value properties for finite variation measures with respect to a Markov operator in a discrete environment. We prove equivalent conditions for the weak mean value property in the case of general Markov operators and for the strong mean value property in the case of transient Markov operators adapted to a tree structure. In this last case, conditions for the equivalence between weak and strong mean value properties are given.

Keywords: Markov operators, harmonic functions, Martin boundary, mean value property, finite variation measures.

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1. Introduction

This paper deals with the discrete analogous of the theory of harmonic functions. The set of (discrete) harmonic functions, defined on an at most countable set $X$, can be constructed starting from a Markov operator $P$. More precisely, given a Markov operator $P$ acting on real functions defined on $X$ according to

$$(Pf)(x) = \sum_{y \in X} p(x, y)f(y)$$

(where $p : X \times X \to [0, 1]$ and $\sum_{y \in X} p(x, y) = 1$, for any $x \in X$) provided that $\sum_{y \in X} p(x, y)|f(y)|$ for all $x \in X$, then we call $f$ a $P$-harmonic function (or simply a harmonic function) if $Pf = f$ and we denote the set of harmonic functions by $\mathcal{H}(X, P)$.

A fundamental tool of this theory is represented by the Martin compactification (see [1]) which allows us to state and solve (under very general condition) the (discrete) Dirichlet problem. More generally, using this approach, one can obtain a useful representation of bounded harmonic functions (see [3] and [4]) $\mathcal{H}\infty(X, P)$. If $X$ is a vertex set of a tree and $P$ is adapted to the graph structure then a generalization of the previous arguments leads to a representation of any harmonic function (see [3]).

In this paper we study a mean value property (see Definition 3), which is actually the dual property of the one commonly known about harmonic functions.

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This problem was stated and partially studied in [2]. The main results of that paper are on radial trees (with respect to the centre of symmetry); here we use a completely different technique to obtain a generalization to the non-symmetric case. In particular we are interested to the relation between weak and strong MVPs.

Besides we consider the mean value property with respect to a set of (harmonic) functions $F = \{k(\cdot, \xi) : \xi \in F\}$ where $F$ is a suitable subset of the Martin boundary of harmonic measure 1 (see the next section for the definitions). We show that there are equivalent conditions to the weak and strong MVP in terms of MVPs with respect to such a family $F$.

We give a brief outline of the paper. In Section 2 we give some basic definitions and we fix the notation. Section 3 is a short introduction to the main representation theorem for harmonic functions on trees (which generalizes, in the case of trees, the well known Poisson-Martin representation theorem for bounded harmonic functions (see [1], Theorem 24.7)). Section 4 is devoted to the study of the relation between the weak MVP and the MVP with respect to a suitable subset of minimal harmonic functions $k(\cdot, \xi)$ for finite supported signed measure on a general graph (Theorem 8). The analogous case for the strong MVP on transient trees is considered in Section 5 (Theorem 9): in particular Corollary 12 guarantees, under very general conditions, the equivalence between weak and strong MVP on a wide class of random walk on trees. This property, which does not hold in general (Examples 10 and 11), is useful, since equivalent conditions for the weak mean value property for bounded variation signed measures (with bounded or unbounded support) on general graphs are known (see [2], Section 6). The results of Section 4 are extended to measures with unbounded support in Section 6. Finally, in Section 7 we briefly discuss some results in the case of general natural compactifications.

2. Preliminaries and basic definitions

In this section we state some basic definitions which will be widely used in the following.

Given any Markov operator $P$ on $X$ and any probability measure $\lambda_0$ on $P(X)$ it is well known that, according to Kolmogorov Theorem, there exists a probability space $(\Omega, \Sigma, P)$ and a (stationary) Markov chain $\{Z_n\}_{n \in \mathbb{N}}$ (called canonical Markov Chain) such that $P_{Z_0} \equiv \lambda_0$ and $\{p(x, y)\}_{x, y \in X}$ represent the transition probabilities of $\{Z_n\}_{n \in \mathbb{N}}$. In particular by $\{Z^x_n\}_{n \in \mathbb{N}}$ (given any fixed $x \in X$) we mean the canonical Markov chain corresponding to the Dirac initial distribution $\delta_x$. 
We denote by $G$ be the generating functions of the transition probabilities and by $F$ the generating function of the hitting probabilities (see [1], Chapter 1, Paragraph 1.B).

When we deal with the graph associated to $(X, P)$ we mean the graph generated by the random walk $(X, P)$ (i.e. $(X, E_P(X))$ where $E_P := \{(x, y) \in X \times X : p(x, y) > 0\}$). We consider only irreducible Markov operators (i.e. giving rise to irreducible random walks according to the usual definition); hence $(X, E_P)$ is a connected graph. In this case the properties of recurrence and transience ([1], Definition 1.14) does not depend on the particular state $x \in X$.

Given a topological Hausdorff space $(X, \tau)$, we say that $(\hat{X}, \hat{\tau}, i)$ is a compactification of $X$ if and only if $(\hat{X}, \hat{\tau})$ is a compact hausdorff space, $i : X \to \hat{X}$ is a homeomorphism between $X$ and $i(X)$ with the induced topology and $i(X)$ is dense in $\hat{X}$. In the case of a countable graph (with the discrete topology) then $i(X)$ is open and discrete. The boundary of the compactification is $\partial X := \hat{X} \setminus i(X)$; from now on, we identify $X$ with $i(X)$, hence the map $i$ will be the inclusion map and the compactification will be denoted simply by $(\hat{X}, \hat{\tau})$ (or shortly by $\hat{X}$).

If $\{Z_n\}$ is an irreducible Markov chain on $X$ and $\hat{X}$ is a compactification of the discrete set $X$, then we say that $\{Z_n\}$ converges a.c. to the boundary if and only if $\Pr(\lim_{n \to \infty} Z_n \in \partial X) = 1$; in this case it is well known that $Z_\infty := \lim_{n \to +\infty} Z_n$ defines a.e. a random variable with values in $\partial X$ (measurable with respect to the Borel $\sigma$-algebra on $\partial X$).

**DEFINITION 1.** Let $(X, P)$ be a random walk; a compactification $\hat{X}$ of the discrete set $X$ is called a natural compactification with respect to $P$ if and only if for any $x \in X$ the canonical Markov process $\{Z^x_n\}$ with initial distribution $\delta_x$ converges $\Pr_x$-a.c. to the boundary $\partial X$.

It is known (see [1]) that if $\hat{X}$ is a natural compactification, then the $\partial X$-valued random variables representing the limits to the boundary of $\{Z^x_n\}$ does not depend on $x$; hence there exists a $\partial X$-valued random variable $Z_\infty$ such that, for any $x \in X$, we have

$$\Pr_x(\lim_{n \to +\infty} Z^x_n = Z_\infty) = 1.$$  

We define the harmonic measures $\nu_x(A) := \Pr_x(Z_\infty^{-1}(A))$ for any borel set $A \subseteq \partial X$ and for every $x \in X$. Since

$$\nu_x = \sum_{y \in X} p(x, y) \nu_y,$$
it is obvious that these measures are mutually absolutely continuous, whence their supports coincides and
\[ L^\infty(\partial X, \nu_x) = L^\infty(\partial X, \nu_y) \subseteq L^p(\partial X, \nu_y) = L^p(\partial X, \nu_x), \]
\[ \forall x, y \in X, \forall p \in [1, +\infty), \]
where on $\partial X$ we consider the $\sigma$-algebra of Borel.

The Martin compactification (see [1] Chapter 4 Paragraph 24) is constructed by means of the Martin Kernel $k_o$ which is defined by $k_o(x, y) := F(x, y)/F(o, y)$; in particular it is the unique (up to homeomorphisms) smallest compactification of the discrete set $X$ such that all the functions $\{k_o(x, \cdot)\}_{x \in X}$ extend continuously to the boundary (denoted by $\mathcal{M}(X, P)$). The Martin compactification does not depend on the choice of $o$ (in the sense that all the compactifications obtained from different choices of the reference vertex are homeomorphic); we denote the extended function again by $k_o(x, \cdot)$ (for any $x \in X$). Theorem 24.10 of [1] shows that this is a natural compactification and that, for any pair of vertices $x, y \in X$, the Radon-Nikodym derivative of the harmonic measure $\nu_x$ with respect to $\nu_y$ is $k_y(x, \cdot)|_{\partial X}$ (the latter is the extended $\hat{X}$-valued Martin kernel restricted to the boundary $\partial X$).

A sequence of measures $\{\nu_n\}$ on a measurable space $(X, \sigma)$ (where $\sigma$ is the Borel $\sigma$-algebra generated by the topology $\tau$) is said to be weakly convergent to a measure $\nu$ if and only if for any $f \in C(X)$,
\[ \int_X f \, d\nu_n \xrightarrow{n \to +\infty} \int_X f \, d\nu. \]
A boundary point $\xi$ of a natural compactification of $(X, P)$ is said to be regular if and only if, for any sequence $\{x_n\}$ in $X$ convergent to $\xi$, we have that the corresponding sequence of harmonic measures $\{\nu_{x_n}\}$ converges weakly to $\delta_\xi$.

**DEFINITION 2.** A natural compactification $\hat{X}$ of $(X, P)$ is almost regular if and only if there exists $x \in X$ ($\Leftrightarrow$ for every $x \in X$) such that the set of regular point of $\partial X$ has $\nu_x$ measure 1.

According to a well known theorem (see Theorem 2.2 of [5] for instance) the Dirichlet problem for $P$-harmonic functions is solvable in the compactification $\hat{X}$ if and only if the compactification of $(X, P)$ is natural and every boundary point is regular.

From now on, if not otherwise explicitly stated, the compactification $\hat{X}$ of $X$ will be the Martin compactification and the boundary $\partial X$ will be $\mathcal{M}(X, P)$.

If $X$ is a tree, then for any couple of vertices $x, y \in X$ there exists a unique geodesic from $x$ to $y$ denoted by $\Pi[x, y]$. Let us denote by $X_{x,y}$
the set of vertices visited by the geodesic path $\Pi[x, y]$; hence for any $x, y, z \in X$, $X_{x,y} \cap X_{y,z} \cap X_{z,x}$ is a one point set (let us denote it by $p(x, y, z)$). Since $X_{x,y} = X_{y,x}$ and $X_{x,x} = \{x\}$ then $p(x, y, z)$ is invariant under permutation of $x, y, z$ and it is called the confluent $x \land z y$ of $x$ and $y$ with respect to the reference point $z$ (see [1] Chapter 1 Paragraph 6.B). Hence $x \land z y = y \land z x = z \land x y$; moreover the definition of confluent $\cdot \land_\circ \cdot$ with respect to a reference point $o \in X$ can be extended to the Martin compactification $\hat{X}$ and for the extended Martin kernel the following relation
\[
k_o(x, \xi) = \frac{k_o(z, y)k_z(x, y)}{k_o(z, x)}, \quad \forall x, y \in \hat{X},
\]
holds (indeed, in this case, $F(x, y) = F(x, z)F(z, y)$ for any $x, y \in X$ and any $z \in X_{x,y}$). By continuity and compactness we have that $\xi \mapsto k_o(x, \xi)$ is bounded (and positive) on $\hat{X}$ and
\[
\sup_{y \in X} k_o(x, y) = \max_{\xi \in \hat{X}} k_o(x, \xi) = 1/F(o, x);
\]
moreover since $F(x, y) = F(z, y)F(x, x \land z y)/F(z, x \land z y)$ for any given $x, y, z \in X$, we have that $k_o(x, y) = k_o(z, y)k_z(x, y)$ for any $x, y \in \hat{X}$ and any $z, o \in X$.

Given any couple of vertex $o, x \in T$ we call branching subtree of $x$ with respect to $o$ the set $T_{o,x} := \{y \in T : x \in \Pi[o, y]\}$ and we denote by $\partial T_{o,x}$ the set $\{\xi \in \partial X : x \in \Pi[o, \xi]\}$.

We turn our attention to the basic definition of mean value property.

**DEFINITION 3.** Let $(X, \Sigma_X)$ be a measurable space, $o \in X$ and let $\nu$ be a measure on $(X, \Sigma_X)$ with finite variation. If $\mathcal{F}$ is a family of functions in $L^1(|\nu|)$ then we say that $\nu$ has the mean value property (MVP) with respect to $\mathcal{F}$ and $o$ if
\[
L(h, \nu)(o) = 0, \quad \forall h \in \mathcal{F},
\]
where
\[
L(h, \nu)(x) := \int_X h d\nu - \nu(X)h(x), \quad \forall x \in X.
\]
In particular, if $(X, P)$ is a graph with an adapted random walk (with the $\sigma$-algebra of the set of subsets of $X$), we say that $\nu$ has the strong mean value property with respect to $o$ (resp. weak mean value property with respect to $o$) if equation 1 holds with $\mathcal{F} \equiv \mathcal{H}(X, P) \cap L^1(|\nu|)$ (resp. $\mathcal{F} \equiv \mathcal{H}^\infty(X, P) := \mathcal{H}(X, P) \cap l^\infty(X)$). If $X$ is a tree with root $o$ then a mean value property will always be with respect to $o$.

We stress that this one is a property of measures and not of functions as in the usual case.
REMARK 4. If we have an adapted random walk \((X,P)\) on the graph \(X\), \(o \in X\) and \(P\) is \(\Gamma_o\)-invariant (where \(\Gamma_o\) is the stabilizer of \(o\) in the automorphisms group \(\text{AUT}(X)\) of the graph \(X\)) then it is easy to prove that the generating functions \(G\) and \(F\) are \(\Gamma_o\)-invariant. Hence for any \(\gamma \in \Gamma_o\), the relation

\[ y \mapsto \lim_{n \to +\infty} \gamma(x_n) \]

(where \(x_n \to y \in \hat{X}\)) is a well defined map which extends \(\gamma\); this means that we can extend the subgroup \(\Gamma_o\) to a closed subgroup \(\tilde{\Gamma}_o\) of homeomorphic maps from \(\hat{X}\) onto itself.

It is quite obvious that, if \(\mu\) is a finite variation measure on \(X\), \(F\) a \(\Gamma_o\)-closed set of \(|\mu|\)-integrable maps (that is \(f \gamma \in F\) for any \(f \in F\) and for any \(\gamma \in \Gamma_o\)), then any \(f \in \Gamma\) is \(\mu \circ \gamma\)-integrable (for all \(\gamma \in \Gamma\)). As a consequence, if \(o \in X\) then for any \(\gamma \in \Gamma_o\), TFAE

(i) \(\mu\) has the MVP with respect to \(F\) and \(o\),

(ii) \(\mu \circ \gamma\) has the MVP with respect to \(F\) and \(o\);

in particular, the weak MVP property with respect to \(o\) can be defined for the equivalence classes of measure with respect to the equivalence relation \(\mu \equiv o \nu\) (\(\mu \equiv o \nu\) if and only if there exists \(\gamma \in \Gamma_o\) such that \(\mu = \nu \gamma\)).

Moreover the subset of all the regular boundary points is \(\tilde{\Gamma}_o\)-invariant.

3. Representation theorems for harmonic functions

The topics in this section are essentially those of [3] with the addition of some specific remarks and a result of [1]. This is intended to be a brief discussion about the main results of [3] which we report here for sake of completeness and to fix the notations.

Let \((S,E(S))\) be a connected graph with an adapted (nearest neighbour) random walk \(P\); chosen a reference vertex \(o \in S\), we denote by \(\hat{S}\) the Martin compactification of the random walk and by \(\partial S := \hat{S} \setminus S\) the space of ends. Moreover let \(\mathcal{F}\) the linear space of all the real-valued functions on \(S\), \(\mathcal{K}\) the linear subspace of \(\mathcal{F}\) containing the functions with finite support, \(\mathcal{H}\) the linear subspace of \(\mathcal{F}\) containing the harmonic functions and \(\mathcal{D}\) the algebra of the locally constant real-valued functions on \(\hat{S}\).

It is obvious that a function is locally constant if and only if the counterimage of every set is open; in particular a locally constant function is continuous and, if the domain is a compact topological space,
then its range is finite (this is our case). Note that the counterimage $f^{-1}(A)$ of any set $A$ by means of any locally constant function $f$ is open and closed.

If $(S, E(S))$ is a tree, $P$ an adapted transient random walk on $S$ and we consider the Martin Kernels $k_o : S \times \hat{S} \to \mathbb{R}$, defined in the previous section, then the relation

$$(K_o^* v)(x) := \sum_{y \in S} v(y)k_o(y, x), \quad \forall x \in S, \forall v \in K$$

defines a linear map which turns out to be an isomorphism from $K$ onto $D$ ([3], Proposition A.1). Differently stated, the set $\{ k_o,x : x \in S \}$ (where for every $x \in S$ the function $k_o,x$ is defined by $k_o,x(y) := k_o(x,y)$, $y \in \hat{S}$) is a basis of the linear space $D$.

We call *distribution* on $\hat{S}$ any linear functional on $D$ and we denote by $D^*$ the set of distributions. We denote also by $\beta$ the set of all subsets of $\hat{S}$ which are open and closed (this is trivially a $\sigma$-algebra). The set $\{ \chi_A : A \in \beta \}$ (where $\chi_A$ is the usual characteristic function of a set $A$) is a set of generators of $D$; for any $\lambda \in D^*$, we define a finitely additive measure $\mu_\lambda$ on $\beta$ by $\mu_\lambda(A) := \lambda(\chi_A)$ for every $A \in \beta$. The correspondence $\lambda \mapsto \mu_\lambda$ is seen to be an isomorphism from the space of distributions $D^*$ onto the linear space of finitely additive (signed) measure on $\beta$. This allows us to identify the distributions with the finite additive measures, the action of $\mu_\lambda$ on $v \in D$ is given by the equality

$$\lambda(v) = \int_{\hat{S}} v d\mu_\lambda = \sum_{\alpha \in \text{Rg}(v)} \alpha \mu_\lambda(f^{-1}(\alpha)).$$

We say that a distribution $\lambda$ is supported on a subset $E \subset \hat{S}$ if and only if $\lambda(f) = 0$ for any $f \in D$ such that $f_E \equiv 0$.

Moreover the following relations hold for any distribution $\lambda$ on $\hat{S}$ and for any closed subset $E \subset \hat{S}$,

$$\lambda(f) \geq 0, \forall f \in D \text{ such that } f \geq 0 \iff \mu_\lambda(A) \geq 0, \forall A \in \beta$$

if $f$ is supported on $E \iff \mu_\lambda(A) \geq 0, \forall A \in \beta, A \cap E = \emptyset$;

in the first case we write $\lambda \geq 0$.

**REMARK 5.** If $E \subset \hat{S}$, then $f|_E \equiv 0$ if and only if $\lambda(f) = 0$ for any distribution $\lambda$ supported on $E$.

Indeed the “only if” part is by definition; on the other hand, the Dirac measure $\delta_\xi$ is a $\sigma$-additive measure on $\mathcal{P}(\hat{S})$, the restriction to $\beta$ can be identified with the distribution $\lambda_\xi$ defined by $\lambda_\xi(f) := f(\xi)$ for any $\xi \in \hat{S}$. Hence for any $\xi \in E$, $\lambda_\xi$ is supported on $E$ and $0 = \lambda_\xi(f) = f(\xi).$
We conclude this section stating, without any proof, the Poisson Representation Theorem for harmonic functions on trees ([3], Proposition A.4) and discussing the Poisson representation for bounded harmonic functions on general set with general irreducible transition probabilities ([1], Theorem 24.12).

**Theorem 6.** Let \((S, P)\) be an irreducible, transient random walk on a tree; if \(o \in S\) and \(g \in \mathcal{F}\) then

(i) there exists a unique distribution \(\lambda_g\) on \(\hat{S}\) such that
\[
g(x) = \lambda(k_o(x, \cdot)), \quad \forall x \in S;
\]

(ii) \(g \in \mathcal{H}\) if and only if \(\lambda_g\) is supported on \(\partial X\);

(iii) \(g\) is superharmonic and positive if and only if \(\lambda_g \geq 0\).

We call the unique distribution \(\lambda_g\) the Martin potential of \(g\). An explicit expression for the map \(I : \mathcal{H} \to \mathcal{D}^*\) such that \(I(g) = \lambda_g\) was found in [4] (Theorem 2).

**Theorem 7.** Let \((X, P)\) be a general irreducible, transient random walk then the equation
\[
h(x) := \int_{\mathcal{M}} \varphi(\xi) k_o(x, \xi) d \nu_o(\xi) \equiv \int_{\mathcal{M}} \varphi(\xi) d \nu_x(\xi), \quad \forall x \in X \quad (3)
\]
defines a harmonic function; in particular the map \(\varphi \mapsto h\), given by equation 3, is a linear, bicontinuous operator from \(L^\infty(\mathcal{M}, \nu_o)\) onto \(\mathcal{H}^\infty(X, P)\).

### 4. The case of measures with finite support on general graphs

In this paragraph we characterize the weak MVP by means of the MVP with respect to the set of harmonic functions \(\{k_o(\cdot, \xi)\}_{\xi \in \partial X}\).

We know from [6] that the set of regular point of the boundary is a Borel set, hence it is measurable. Also in the transient case it could be an empty set; the Dirichlet problem is solvable if and only if every boundary point is regular (for the definition of solvability of the Dirichlet problem see [1], Chapter 4 Paragraph 20). In the next theorem we are mostly interested in the representation of bounded harmonic functions by means of the Martin compactification, for this...
reason it is crucial that the set of regular points is “sufficiently big” (not necessarily the whole boundary). When we say that a property is a.e. true on \( \partial X \) we mean w. r. to some harmonic measure (that is the same, w. r. to every harmonic measure).

**Theorem 8.** Let \((X, P)\) be an irreducible, transient random walk, \(o \in X\) and \(\mu\) a (signed) measure with finite support. Consider the following assertions

(i) \(\mu\) has the weak MVP with respect to \(o\);

(ii) \(L(k_o(\cdot, \xi), \mu)(o) = 0\) a. e. on \(\partial X\);

then (ii) \(\implies\) (i). In particular if the Martin compactification is almost regular then (i) \(\iff\) (ii).

**Proof.** (ii) \(\implies\) (i). It is Corollary 3.6 of [2].

(i) \(\implies\) (ii). If \(\xi_0\) is a regular boundary point then, defining

\[
 f_n(x) := \int_{\partial X} k_o(x_n, \xi) k_o(x, \xi) \, d\nu_o(\xi) = \int_{\partial X} k_o(x, \xi) \, d\nu_{x_n}, \quad \forall x \in X, \forall n \in \mathbb{N},
\]

where \(\{x_n\}\) is a sequence in \(X\) converging to \(\xi_0\) in the topology of \(\hat{X}\), we have that

\[
 f_n(x) \xrightarrow{n \to +\infty} k(x, \xi_0), \quad \forall x \in X
\]

because of Theorem 20.3 of [1]. Now \(f_n \in \mathcal{H}^\infty(X, P)\) for any \(n \in \mathbb{N}\) and the support of \(\mu\) is finite, thus we have

\[
 L(k_o(\cdot, \xi_0), \mu)(o) = \lim_{n \to +\infty} L_o(f_n, \mu)(o) = 0.
\]

\(\square\)

The previous theorem applies, in particular, if \((X, P)\) is a nearest neighbour, transient random walk on a tree \(X\) such that the Martin boundary contains at least two elements: in that case the set of regular point has \(\nu_o\)-measure 1 (see [5], Theorem 4.2). The previous theorem extends the equivalence between (i) and (iii) of Proposition 4.13 of [2].

5. The case of measures with finite support on trees

In this section we use of the Representation Theorem 6 we stated in Section 3. Our first aim is to extend the results of [2], regarding the finite variation measures with finite support on radial trees, to more general settings (i.e. general irreducible random walks on non-oriented trees).
THEOREM 9. Let $(T, P)$ an irreducible, transient random walk on a
tree with root $o$ and let $\mu$ a (signed) measure on $T$ with finite support.
Consider the following assertions:

(i) $\mu$ has the weak MVP with respect to $o$,

(ii) $\mu$ has the strong MVP with respect to $o$,

(iii) for any $\xi \in \partial T$ we have that $L(k_o(\cdot, \xi), \mu)(o) = 0$;

then (i) $\iff$ (ii) $\iff$ (iii).

Proof. (ii) $\iff$ (iii). We easily note that from Theorem 6, (ii) is
equivalent to

\[ \sum_{x \in \text{supp}(\mu)} < \lambda, k_o(x, \cdot) > \mu(x) = \sum_{x \in \text{supp}(\mu)} < \lambda, k_o(o, \cdot) > \mu(x) \]

for any distribution $\lambda$ on $\hat{T}$ with support in $\partial T$. Using the finiteness
of $\text{supp}(\mu)$ and since the space of distribution separates the points
(according to Remark 5), then the previous relation is equivalent to

\[ \sum_{x \in T} \mu(x)k_o(x, \cdot) = \mu(T)k_o(o, \cdot) \]

that is

\[ L(k_o(\cdot, \xi), \mu)(o) = 0, \quad \forall \xi \in \hat{T} \setminus T. \]

(ii) $\implies$ (i). It is trivial.

We proved it in [2] (Proposition 4.13) that, under the condition of
radiality, (i),(ii) and (iii) are equivalent. Before extending this result to
a bigger class of trees (see Corollary 12) we give some examples which
prove that the equivalence between (i) and (ii) does not hold in general
(even if $\mu$ has finite support and $T$ is a tree or a transient graph).

EXAMPLE 10. We construct an example of a finite supported measure
on a transient tree which has the weak mean value property but not
the strong one. Let us take any tree $T_1$ (which has transient simple
random walk) and a root $o$, and let us consider now the tree $T$ obtained
attaching to $T_1$ a straigh line (a copy of $\mathbb{N}$) to the root $o$ by indentifying
$o$ with the root $0 \in \mathbb{N}$ and let us call $P$ the simple random walk on $T$.

It is simple to note that given any pair of distinct vertices $x, y$ on
the straigh line (different from $o$) then for any $f \in \mathcal{H}(T, P)$ we have
$f|_{\mathbb{N}} \equiv \text{const}$ if and only if $f(x) = f(y)$. It is also easy to show that any
bounded harmonic function $f$ must be constant on $\mathbb{N}$. On such a tree
is not difficult to construct a harmonic function which is not constant (and hence not bounded) on \( \mathbb{N} \): let \( f_0 \) one of those functions. Let now consider the measure \( \mu \) defined by

\[
\mu(x) := \begin{cases} 
1 & \text{if } x = 1 \\
-1 & \text{if } x = 2 \\
0 & \text{if } x \notin \{1, 2\} 
\end{cases}
\]

such a measure has the weak MVP with respect to any point (since \( \mu(T) = 0 \) and for any bounded harmonic function \( f \) we have \( f(1) = f(2) \)), but it cannot have the strong MVP with respect to any vertex since \( L(f_0, \mu)(x) = f_0(1) - f_0(2) \neq 0 \). Note that \( T \) contains a recurrent branching subtree, namely the copy of \( \mathbb{N} \) (compare with Corollary 12).

**EXAMPLE 11.** Another example is provided by the simple random walk on the \( d \)-dimensional lattice \( \mathbb{Z}^d \); in this case the weak Liouville property holds (every bounded harmonic function is constant), meanwhile \( \{ f : \exists \alpha, \beta \in \mathbb{R} : f(i_1, \ldots, i_d) \equiv \alpha i_1 + \beta \} \subset \mathcal{H}(\mathbb{Z}^d, P) \). Whence if \( f \in \mathcal{H}(\mathbb{Z}^d, P) \) is a nonconstant function we cannot find a sequence \( \{ f_n \} \) in \( \mathcal{H}^\infty(\mathbb{Z}^d, P) \) converging to \( f \).

The equivalence between weak and strong MVP can be proved under certain conditions.

**COROLLARY 12.** Let \( (T, P) \) be an irreducible, transient random walk on a tree, \( o \in T \). Then TFAE:

(i) every infinite branching subtree is transient;

(ii) for any \( x, y \in T \) such that \( x \sim y \) and \( T_{x,y} \) is infinite we have \( F(y, x) < 1 \);

(iii) there exists \( x \in T \) (\( \Leftrightarrow \forall x \in T \)) such that \( \text{supp}(\nu_x) = \partial T \);

(iv) there exists \( x \in T \) (\( \Leftrightarrow \forall x \in T \)) such that there exists a flux from \( x \) to infinity, supported on each edge \( (z, y) \) such that \( T_{z,y} \) is infinite.

If the Martin boundary of \( T \) contains at least two elements, then any of the previous conditions is equivalent the following one:

(v) given any signed measure with finite support \( \mu \), then \( \mu \) has the weak MVP with respect to \( o \) if and only if it has the strong MVP with respect to \( o \)
Proof. The equivalence between (i), (ii) and (iii) follows by Theorem 7.5 of [7]. The equivalence between one of the previous with (iv) is straightforward (remember that any random walk on a tree is reversible).

(iii) \( \implies (v) \). Let \( \text{supp}(\nu_o) = \partial T \). If \( \xi \in \partial T \) and \( x \in B(o, n) \) such that \( \text{supp}(\mu) \subseteq B(o, n) \) and \( x \lor o \xi = x \) then \( \nu_o(\partial T_{o,x}) > 0 \). By Theorem 8 there exists \( \xi_1 \in \partial T_{o,x} \) such that \( L(k_o(\cdot, \xi_1), \mu)(o) = 0 \). Since \( k(y, \xi_1) = k(y, \xi_1 \lor o y) = k(y, \xi_1 \land o y) = k(y, \xi) \) for every \( y \in T \setminus T_{o,x} \supseteq \text{supp}(\mu) \) then we have \( k(\cdot, \xi_1) = k(\cdot, \xi) \mu \text{ a.e.} \) and hence

\[
0 = L(k_o(\cdot, \xi_1), \mu)(o) = L(k_o(\cdot, \xi), \mu)(o).
\]

By Theorem 9, (v) is proved.

(v) \( \implies (i) \). If \( T_0 \) is a recurrent, infinite branching subtree of the transient tree \( T \), hence \( \nu_x(\partial T_0) = 0 \) and every bounded harmonic function \( f \) is constant on \( T_0 \); now the measure \( \nu \) in the last statement can be constructed as in Example 10.

The previous corollary applies, in particular, if \( T \) is a locally finite tree with minimum degree 2 and with finite upper bound to the lengths of its unbranched geodesics (and \( P \) is the simple random walk): in this case the same property holds for any branching subtree (and this implies easily (i)).

Moreover it is an extension of Proposition 4.13 of [2] since for any irreducible, \( \Gamma_o \)-invariant random walk (see [2] Definition 2.2) on an \( o \)-radial tree the property (iii) of the previous corollary obviously holds.

6. The case of measures with unbounded support on general graphs

If we want to deal with measures with countable support then we must slightly modify the hypotheses of Theorem 8.

THEOREM 13. Let \((X, P)\) be an irreducible, transient random walk and \( \mu \) a (signed) measure.

a) For any \( x, y \in X \) we have that \( \sup_{z \in X} k_x(y, z) = \max_{z \in X} k_x(y, z) = 1/F(x, y) \) and for any \( x, y, z \in X \),

\[
\sup_{z \in X} k_x(\cdot, z) \in L^1(|\mu|) \iff \sup_{z \in X} k_y(\cdot, z) \in L^1(|\mu|).
\]
b) Let \( o \in X \) and \( \sup_{z \in X} k_o(\cdot, z) \in L^1(|\mu|) \) (i.e. \( 1/F(o, \cdot) \in L^1(|\mu|) \)) then TFAE

(i) \( \mu \) has the weak MVP with respect to \( o \);
(ii) \( L(k_o(\cdot, \xi), \mu)(o) = 0 \) a. e. on \( \partial X \);

then (ii) \( \implies \) (i). In particular if the compactification is almost regular then (i) \( \iff \) (ii).

Proof. (a) By definition \( k_x(y, z) = F(y, z)/F(x, z) \), since \( F(x, z) \geq F(x, y)F(y, z) \) and \( F(x, y) \leq 1 \), \( F(x, x) = 1 \) then by continuity of \( k_x \) we have that

\[
\sup_{z \in \hat{X}} k_x(y, z) = \max_{z \in X} k_x(y, z) = 1/F(x, y).
\]

Moreover, using again the previous inequality for \( F \), we have that, for any \( x, y \in X \),

\[
1/F(x, z) \leq 1/F(y, z) F(x, y), \quad \forall z \in X
\]

and hence, by equation 6, we obtain equation 5.

(b)

(ii) \( \implies \) (i). It is again Corollary 3.6 of [2].

(i) \( \implies \) (ii). The proof is quite similar to the analogous case of Theorem 8 and we just outline the differences. Since \( \sup_{x \in X} k_o(x, y) \equiv \sup_{\xi \in \hat{X}} k_o(x, \xi) < +\infty \) by the Lebesgue Bounded Theorem (or just using uniform convergence) we have that \( \xi \mapsto \int_X k_o(\cdot, \xi) \, d\mu \) is continuous on \( \hat{X} \) (and hence bounded by compactness) and the same holds for \( \xi \mapsto \int_X k_o(\cdot, \xi) \, d|\mu| \). If \( \xi_0 \) is a regular boundary point \( f_n \) is the sequence of functions defined by equation 4 from a sequence \( \{x_n\} \) converging to \( \xi_0 \), applying Fubini’s Theorem we have that \( k_o \in L^1(\nu_x \times |\mu|) \) and hence, using the Jordan decomposition of \( \mu \), we obtain

\[
L(f_n, \mu)(o) = \int_M L(k_o(\cdot, \xi), \mu)(o) \, d\nu_x.
\]

Now by the previous arguments \( L(k_o(\cdot, \xi), \mu)(o) \) is continuous on \( \partial X \) hence Theorem 20.3 of [1] yields again the conclusion. \( \square \)

It is important to note the difference between (iii) of Theorem 9 and (ii) of Theorems 8 and 13; in particular if \( T \) is as in Example 10, then the end point of the straight line \( N \) is not a regular point.
7. Some results for general compactifications

Throughout this whole section $\tilde{X}$ will always be a natural compactification and $\{\nu_x\}$ the relative family of harmonic measures. We state and prove two results which generalize those of the previous sections. In the first Theorem we give a sufficient condition for the MVP with respect a particular family of functions; in the case of the Martin compactification this is just the weak MVP.

THEOREM 14. Let $o \in X$ and $F := \{f : X \to \mathbb{R} : f(x) = \int_{\partial X} \varphi \, d\nu_x, \forall x \in X, \forall \varphi \in L^\infty(\partial X, \nu_o)\}$ and $\mu$ a finite variation (signed) measure on $X$ such that $L(\nu(A), \mu)(o) = 0$ for every borel set $A \subset \partial X$, then $\mu$ has the MVP with respect to $F$ and $o$.

Proof. We firstly note that, if $\varphi \in L^\infty(\partial X, \nu_o)$, then $h_\varphi(x) := \int_{\partial X} \varphi \, d\nu_x$ satisfies $h_\varphi \in L^\infty(X)$ and hence $h_\varphi \in L^1(|\mu|)$. If $s$ is a measurable simple function, $s = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}$, then

$$L(h_s, \mu)(o) = \sum_{i=1}^{\infty} \alpha_i L(\nu(A_i), \mu)(o) = 0.$$ 

Let $\varphi \in L^\infty(\partial X, \nu_o)$ and $\{s_n\}_n$ a sequence of measurable simple functions such that $s_n \to \varphi$ and $|s_n| \leq |\varphi|$, for any $n \in \mathbb{N}$, hence using the Bounded Convergence Theorem (on the measurable space $(\partial X, \nu_x)$), we have

$$h_{s_n}(x) := \int_{\partial X} s_n \, d\nu_x \to \int_{\partial X} \varphi \, d\nu_x =: h_\varphi(x), \quad \forall x \in X.$$ 

Moreover since $|h_{s_n}| \leq h_{|\varphi|}$, according to Bounded Convergence Theorem again (applied to $(X, \mu)$), we have

$$L(h_\varphi, \mu)(o) = \lim_{n \to +\infty} L(h_{s_n}, \mu)(o) = 0.$$ 

$\square$

In order to be able to guarantee the mean property of $\nu(A)$ of the previous Theorem one can check for the mean value property with respect to a subfamily of $F$.

THEOREM 15. Let $o \in X$, define $F_1 := \{f : X \to \mathbb{R} : f(x) = \int_{\partial X} \varphi \, d\nu_x, \forall x \in X, \forall \varphi \in C(\partial X)\}$ and consider $\mu$ a finite variation (signed) measure on $X$ which has the MVP with respect to $F_1$ and $o$ then $L(\nu(A), \mu)(o) = 0$ for every borel set $A \subset \partial X$. 
Proof. Since $\partial X$ is a Hausdorff, compact topological space and $\nu_o$ is regular (according to Theorem 2.18 of [8]), we have that $C_c(\partial X) \equiv C(\partial X)$ is a dense subset of $L^p(\partial X, \nu_o)$ (see [8] Theorem 3.14). Let us choose a function $g \in l^1(X)$, $g(x) > 0$ for any $x \in X$, and consider the measure $\overline{\nu}$ defined by $\overline{\nu}(A) := \sum_{x \in X} g(x) \nu_x(A)$ for every $A \subset \partial X$ borel subset. If $\mu$ satisfies the MVP, from the density property for any borel set $A \subset \partial X$, there exists a sequence $\{\varphi_n\}_n$ of continuous functions converging to $\chi_A$ in $L^1(\partial X, \overline{\nu})$ (and hence in $L^1(\partial X, \nu_x)$ for any $x \in X$) with the property $0 \leq \varphi_n \leq 1$. Moreover by Bounded Convergence Theorem (applied to $(X, \mu^+)$ and $(X, \mu^-)$) since $\int_{\partial X} \varphi_n \, d\nu_x \to \nu_x(A)$, bounded by 1 (which is $|\mu|$-integrable since it is a finite variation measure), then

$$L(\mu(A), \mu)(o) = \lim_{n \to +\infty} L(\varphi_n, \mu)(o) = 0.$$  

If $\mathcal{F}$ and $\mathcal{F}_1$ are as in Theorems 14 and 15 respectively and $o \in X$ then the MVP with respect to $\mathcal{F}_1$ and $o$ implies the MVP with respect to $\mathcal{F}$ and $o$. This means that in the case of the Martin compactification the weak MVP (and also the strong MVP if the hypotheses of Corollary 12 are satisfied) is equivalent to the MVP with respect to the family $\mathcal{F}_1$.

Moreover if $(X, P)$ is a transient tree then the arguments of Section 2 imply that the hypotheses of integrability of Theorem 7.2 is equivalent to $1/F(o, \cdot) \in L^1(|\mu|)$ and this is true for some $o \in X$ if and only if it is true for any $o \in X$. 

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