Abstract

We show how the fully resummed thermal pressure is rendered ultraviolet finite by standard zero-temperature renormalisation. The analysis is developed in a 6-dimensional scalar model that mimics QED and has $N$ flavours. The $N \to \infty$ limit of the model can be calculated completely. At a critical temperature, one of the degrees of freedom has vanishing screening mass like the transverse gauge bosons in four-dimensional finite-temperature perturbation theory. The renormalised nonperturbative interaction pressure of this model is evaluated numerically.

1 Introduction

The perturbation series for the pressure in finite-temperature QCD suffers from severe infrared problems. In principle, these may be cured by a resummation technique. This resummation is most simply carried out before renormalisation. It goes without saying that, after renormalisation, the pressure must be finite if it is to make physical sense, but it is far from obvious how the mathematics takes care of this. In this paper we study this in a model that may be regarded as a simplified mimic of QED, in which there are $N$ particles each having the same mass and “charge”. Our analysis makes use of standard techniques for renormalising composite operators. We find that indeed the usual renormalisation, carried out purely at the zero-temperature level and therefore introducing no new quantities needing to be determined by experiment, renders the pressure finite.

For simplicity, our discussion begins with the $N \to \infty$ limit of the model, which can be calculated exactly. The model is richer than the large-$N$ $\phi^4$ theory which we have studied previously, in that now the self-energies in the large-$N$ limit vary with momentum, and wave-function renormalisation is needed. After renormalising the expression for the pressure and going some way towards evaluating it analytically, we complete the calculation of the large-$N$ case numerically.

Such a calculation is potentially useful even when $N$ is not large. The formula for the resummed pressure involves the thermal self-energies of the fields, which inevitably are calculated in some approximation from a finite number of Feynman graphs. The resummation then effectively converts this
finite set of graphs to a contribution to the pressure from an infinite number of graphs. While the exact form of the pressure must be ultraviolet finite, it is not obvious that it is still finite when only a partial set of graphs is included in the self-energy. One way of selecting a consistent approximation is to use in the finite-\(N\) case only the set of graphs that would survive to some given order in \(N^{-1}\) if one were to take the large-\(N\) limit. In a theory in which there are a number of real unrenormalised fields \(\phi_r\), the pressure at temperature \(T\) is calculated\[1\] from the thermal averages of the composite operators \(\phi^2_r\):

\[
\frac{\partial}{\partial m_{0r}^2} P(T) = -\frac{1}{2} \left( \langle \phi_r(x)\phi_r(0) \rangle_T - \langle 0|\phi_r(x)\phi_r(0)|0 \rangle \right)_{x=0}
\]

or, equivalently,

\[
\frac{\partial}{\partial m_{0r}^2} P(T) = -\frac{1}{2} \left( \langle T\phi_r(x)\phi_r(0) \rangle_T - \langle 0|T\phi_r(x)\phi_r(0)|0 \rangle \right)_{x=0}
\]

Here, the differentiation is with respect to the unrenormalised mass of the field \(\phi_r\), with all the other unrenormalised parameters kept fixed. One way to integrate this to give \(P(T)\) is to write

\[
m_{0r}^2 = x m_{0r}^2, \quad r = 1, 2, \ldots
\]

Then integrate with respect to \(x\) from 1 to \(\infty\) and insert the boundary condition that the pressure should vanish when \(x = \infty\), that is when all the masses are infinite*. However if, as is the case in QED, it happens that taking just one of the masses — \(m_{0s}\) say — to infinity switches off all the interaction, there is a simpler method:

\[
P(T) = \int_{m_{0s}^2}^{\infty} dm_{0s}^2 \frac{\partial}{\partial m_{0s}^2} P(T) + P_0(T)
\]

where \(P_0(T)\) is the contribution to the thermal pressure from all the fields except \(\phi_s\), with the interaction between them switched off.

The thermal averages of the composite operators that appear in (1.1) may be expressed as integrals over thermal Green’s functions:

\[
\langle \phi_r(x)\phi_r(0) \rangle_T \bigg|_{x=0} = \int \frac{d^nq}{(2\pi)^n} D^{12}_{rT}(q)
\]

\[
\langle T\phi_r(x)\phi_r(0) \rangle_T \bigg|_{x=0} = \int \frac{d^nq}{(2\pi)^n} D^{11}_{rT}(q)
\]

The notation \(D^{12}_{rT}, D^{11}_{rT}\) is that of real-time thermal field theory in the Keldysh formalism\[4\] with a time path with \(\sigma = 0\): they are elements of the matrix propagator

\[
D_{rT}(q) = M_T(q^0) \begin{bmatrix} D_{rT}(q) & 0 \\ 0 & D_{rT}(q) \end{bmatrix} M_T(q^0)
\]

where

\[
D_{rT}(q) = \frac{i}{q^2 - m_{0r}^2 - \Pi_{rT}(q^0, q^2)}
\]

* Our methods resemble those of a renormalisation-flow analysis\[3\].
\[
M_T(q^0) = \left[ \frac{1}{e^{q^0/T} - 1} \right]^{1/2} \begin{bmatrix}
e^{\frac{1}{2}q^0/T} & e^{-\frac{1}{2}q^0/T} \\
\frac{1}{2}q^0/T & e^{\frac{1}{2}q^0/T}
\end{bmatrix}
\]

(1.5b)

Here, \( \Pi_{rT} \) is defined in terms of the thermal self-energy matrix

\[
-i \Pi_{rT} = M_T^{-1}(q^0) \begin{bmatrix}
0 \\
(-i \Pi_{rT}(q))^* 
\end{bmatrix} M_T^{-1}(q^0)
\]

(1.5c)

Inserting (1.5) into (1.1a) gives

\[
\frac{\partial}{\partial m_{0r}^2} P(T) = \frac{1}{2} \text{Im} \int \frac{d^n q}{(2\pi)^n} \left\{ \frac{1 + 2n(q)}{q^2 - m_{0r}^2 - \Pi_{rT}(q^0, q^2)} - \frac{1}{q^2 - m_{0r}^2 - \Pi_r(q^2)} \right\}
\]

(1.1c)

where \( \Pi_r(q^2) \) is the zero-temperature self energy and \( n(q) \) is the Bose distribution \( (e^{q^0/T} - 1)^{-1} \). The version (1.1b) of our basic formula gives instead

\[
\frac{\partial}{\partial m_{0r}^2} P(T) = \frac{1}{2} \int \frac{d^n q}{(2\pi)^n} \left\{ \text{Im} \frac{2n(q)}{q^2 - m_{0r}^2 - \Pi_{rT}(q^0, q^2)} + \frac{i}{q^2 - m_{0r}^2 - \Pi_r(q^2)} - \frac{i}{q^2 - m_{0r}^2 - \Pi_{rT}(q^0, q^2)} \right\}
\]

(1.1d)

The equivalence of (1.1c) and (1.1d) may be seen from the fact* that each of the last two terms in the integrand of (1.1d) is the analytic continuation in \( q^0 \) of the real-time propagator, and by making a Wick rotation, so verifying that the integral over each of the last two terms is real.

We apply the formula (1.1) to a mock electron-photon interaction in which, for simplicity, the fields are scalar; its unrenormalised form is

\[
\mathcal{L}^{\text{INT}} = -\lambda_0 A^a \psi^\dagger_r \tau^a \psi_r
\]

(1.6)

The masses are \( m_{01} \) for the electrons, and \( m_{02} \) for the photon. In proper QED, \( C \)-parity or spin conservation removes one-photon-reducible graphs from the pressure; here we achieve this instead by making the photon an isovector, and the electrons isodoublets, so that \( a \) runs over 3 values with \( \tau^a \) the Pauli matrices. We take space-time to be 6-dimensional, so that \( \lambda_0 \) is dimensionless and the divergences are similar to those of proper QED. In intermediate steps we use dimensional regularisation with \( n = 6 - 2\epsilon \).

There are \( N \) identical electrons: the index \( r \) runs over \( N \) values. In the next section we set

\[
\lambda_0 = \frac{g_0}{\sqrt{N}}
\]

(1.7)

and consider the case of large \( N \). The more general case, where \( N \) is not necessarily large, is the subject of section 3. In section 4 we return to the large-N version of the model and evaluate the leading term in the interaction pressure completely. The thermal “photon” spectrum turns out to involve negative corrections to the mass such that there is a critical temperature, where screening disappears while keeping the plasmon mass nonzero. Right at the critical temperature, where the nonperturbative interaction pressure is still well-defined, the spectrum of our model is even rather similar to that of perturbative four-dimensional gauge theories in that it has a vanishing screening mass like the magnetostatic modes. Section 5 is a summary and discussion.

* Formula (2.68) of reference [3]
\( g_0^2 \pi \)

\[ \text{(a)} \quad \text{and} \quad \text{(b)} \]

Figure 1: self-energy graphs

2 Large \( N \)

In the large-\( N \) limit, the free-field pressure is linear in \( N \), and the correction from the interaction is of order \( N^0 \). To calculate this we need the leading terms in the photon self-energy \( g_0^2 \pi \delta^{ab} \) and the electron self-energy, which are respectively of order \( N^0 \) and \( N^{-1} \) and correspond to the graphs of figure 1. In figure 1b, the photon propagator is the Dyson-resummed propagator with the photon self-energy \( g_0^2 \pi \) of figure 1a.

If the photon mass \( m_{02} \rightarrow \infty \) the interaction is switched off, and so we can use the version (1.3) of the formula for the pressure. With (1.1a) and (1.5) this reads

\[
P(T) = P_0^{(1)}(T, m_{01}^2) - \frac{i}{2} C \text{ Im} \int_{m_{02}^2}^{\infty} \frac{d m_{02}^2}{2 \pi} \frac{q^6 - 2 \epsilon q}{(2 \pi)^6 - 2 \epsilon} \left\{ \frac{1 + 2n(q)}{q^2 - m_{02}^2 - g_0^2 \pi T(q', q^2)} - \frac{1}{q^2 - m_{02}^2 - g_0^2 \pi(q^2)} \right\}
\]

(2.1)

where \( C = \frac{1}{2} \text{tr} \tau^2 = 3 \) is the number of “photon” field components in our model. The integration over the photon mass \( m_{02}^\prime \) is with the coupling \( g_0 \) and the electron mass \( m_{01} \) fixed. \( P_0^{(1)}(T, m_{01}^2) \) is the contribution to the pressure from the electrons when the interaction is switched off. The \( N \) doublet fields \( \psi_r \) correspond to \( 4N \) real fields, thus:

\[
P_0^{(1)}(T, m_{01}^2) = -2N \text{ Im} \int_{m_{01}^2}^{\infty} \frac{d m_{01}^2}{2 \pi} \frac{q^6 - 2 \epsilon q}{(2 \pi)^6 - 2 \epsilon} \left\{ \frac{1 + 2n(q)}{q^2 - m_{01}^2} - \frac{1}{q^2 - m_{02}^2} \right\}
\]

(2.2)

\[ = 4N \pi \int \frac{q^6 - 2 \epsilon q}{(2 \pi)^6 - 2 \epsilon} n(q) \theta(q^2 - m_{01}^2) \]

Here, and elsewhere if needed, we assume the usual \( i \epsilon \) prescription.

The next step is to express (2.1) and (2.2) in terms of renormalised quantities. We introduce a renormalised photon mass \( m_2^\prime \) for each value of the bare mass \( m_{02}^\prime \). The most appropriate scheme is on-shell renormalisation, in which the renormalised photon mass is given by

\[
m_2^2 = m_{02}^2 + g_0^2 \pi(m_2^2)
\]

(2.3)

The electron mass \( m_{01} \) is renormalised similarly, though its renormalisation vanishes as \( N^{-1} \) when \( N \rightarrow \infty \).

We are going to calculate the pressure up to terms of order \( N^0 \), so that in the calculation of \( \pi(q^2) \) and \( \pi_T(q', q^2) \) we need make no self-energy insertions in the internal electron lines; these functions depend just on \( q \) and on the unrenormalised electron mass \( m_{01} \), which we may equate to its renormalised value \( m_1 \). However, the renormalisation of \( m_{01} \) will be important below. To leading order in \( N \), the renormalised coupling is

\[
\bar{g}^2(m_2^2) = Z_2(m_2^2)g_0^2
\]

(2.4a)

\[
Z_2(m_2^2) = \frac{1}{1 - \frac{\partial m_2^2}{\partial m_{02}^2}}
\]
where $\pi'(m^2)$ denotes $\partial \pi(q^2)/\partial q^2$ evaluated at $q^2 = m^2$. Thus, for fixed $g_0^2$,

$$
\bar{g}^2(m_2^2) = \frac{g^2}{1 + g^2(\pi'(m_2^2) - \pi'(m_2^2))}
$$

$$
g^2 = \bar{g}^2(m_2^2)
$$

(2.4b)

The equations (2.4) give pathologies reminiscent of those of large-$N$ $\phi^4$ theory. Similar pathologies were discussed for an exactly solvable model a long time ago. If we insist that $g_0^2 > 0$, then we find from (2.4a) that $g^2 \to 0$ as $\epsilon \to 0$. However, if we take the view that the value of $g_0^2$ is irrelevant for physics, and simply choose $g^2$ to have some positive value, we find that $\bar{g}^2(m_2^2)$ has a Landau pole. But provided that $g^2 < 192\pi^3$, the pole is at such a large value $M^2$ of $m_2^2$ that it is physically irrelevant: $M^2 \sim m_2^2 \exp(192\pi^3/g^2)$. Then our thermal field theory is a sensible theory provided we restrict ourselves to temperatures $T \ll M$.

The curly bracket in (2.1) is

$$
Z_2(m_2^2) \left\{ \frac{1 + 2n(q)}{q^2 - m_2^2 - \bar{g}^2(m_2^2)\pi_T(q^0, q^2, m_2^2)} + \frac{1}{q^2 - m_2^2 - \bar{g}^2(m_2^2)\tilde{\pi}(q^2, m_2^2)} \right\}
$$

(2.5)

Here, $\bar{\pi}_T$ and $\bar{\pi}$ are convergent functions in 6 dimensions, when $\epsilon \to 0$:

$$
\bar{\pi}_T(q^0, q^2, m_2^2) = \pi_T(q^0, q^2) - (q^2 - m_2^2)\pi'(m_2^2) - \pi(m_2^2)
$$

$$
\tilde{\pi}(q^2, m_2^2) = \pi(q^2) - (q^2 - m_2^2)\pi'(m_2^2) - \pi(m_2^2)
$$

(2.6)

Of course, both $\bar{\pi}_T$ and $\bar{\pi}$ depend also on $m_1^2$, but this is kept fixed at its physical value in both these functions. We may use (2.4) to change the integration variable from $m_0^2$ to $m_2^2$. Because we are taking $g_0^2 < 0$ and $\bar{g}^2 > 0$, when $m_0^2$ increases from $m_0^2$ to $\infty$ we find that $m_2^2$ decreases from $m_2^2$ to $-\infty$. These negative squared masses are just a calculational device and they do not enter in the final result but, provided that $m_2 < 2m_1$, they mean that the squared renormalised mass and coupling defined in (2.3) and (2.4) are real throughout the integration, and the integral in (2.1) is

$$
-\frac{1}{2} C \text{ Im} \int_{m_2^2}^{-\infty} \frac{d^6-2\epsilon q}{(2\pi)^{6-2\epsilon}} \left\{ \frac{1 + 2n(q)}{q^2 - m_2^2 - \bar{g}^2(m_2^2)\pi_T(q^0, q^2, m_2^2)} - \frac{1}{q^2 - m_2^2 - \bar{g}^2(m_2^2)\tilde{\pi}(q^2, m_2^2)} \right\}
$$

(2.7)

We may perform the mass integration, because from (2.4)

$$
\frac{\partial}{\partial m_2^2} \bar{g}^2(m_2^2) = \bar{g}^4(m_2^2) \pi''(m_2^2)
$$

(2.8a)

and from (2.6)

$$
\frac{\partial}{\partial m_2^2} \bar{\pi}_T(q^0, q^2, m_2^2) = \frac{\partial}{\partial m_2^2} \bar{\pi}(q^2, m_2^2) = -(q^2 - m_2^2)\pi''(m_2^2)
$$

(2.8b)

From (2.8a), (2.8b) we find that (2.7) is

$$
C \pi \int \frac{d^6-2\epsilon q}{(2\pi)^{6-2\epsilon}} n(q) \theta(q^2 - m_2^2)
$$
\[-\frac{1}{2} C \Im \int \frac{d^{6-2\epsilon} q}{(2\pi)^{6-2\epsilon}} \left[ 2n(q) \log \frac{g^2 \pi_T(q^0, q^2, m_2)}{m_2^2 - q^2} + \log \frac{g^2 \pi_T(q^0, q^2, m_2^2) + m_2^2 - q^2}{g^2 \pi(q^0, q^2, m_2^2) + m_2^2 - q^2} \right] \quad (2.9)\]

This integral diverges when \(\epsilon \to 0\). The divergence must be cancelled by a similar one in \(P_{0}^{(1)}(T, m_0^2)\), given in (2.2), which we rewrite as

\[P_{0}^{(1)}(T, m_0^2) = P_{0}^{(1)}(T, m_1^2) + B(T)\]

\[B(T) = -4N\pi \int \frac{d^{6-2\epsilon} k}{(2\pi)^{6-2\epsilon}} n(k) \left[ \theta(k^2 - m_1^2) - \theta(k^2 - m_0^2) \right] \quad (2.10)\]

Evidently \(P_{0}^{(1)}(T, m_1^2)\) is convergent, but \(B(T)\) is not. Because \(m_1^2 - m_0^2\) goes to zero as \(N^{-1}\) as \(N\) becomes large,

\[B(T) \sim 2N(m_1^2 - m_0^2) \int \frac{d^{6-2\epsilon} k}{(2\pi)^{6-2\epsilon}} n(k) 2\pi \delta(k^2 - m_1^2) \quad (2.11a)\]

Figure 1b gives

\[(m_1^2 - m_0^2)N = C g^2 \int \frac{d^{6-2\epsilon} q}{(2\pi)^{6-2\epsilon}} \frac{1}{q^2 - m_2^2 - g^2 \pi(q^2, m_2^2) (k - q)^2 - m_1^2} \bigg|_{k^2 = m_1^2} \quad (2.11b)\]

So

\[B(T) = -\frac{1}{2} C g^2 \Im \int \frac{d^{6-2\epsilon} q}{(2\pi)^{6-2\epsilon}} \frac{1}{q^2 - m_2^2 - g^2 \pi(q^2, m_2^2) \bar{\pi}_T(q^0, q^2)} \quad (2.11c)\]

\[\bar{\pi}_T(q^0, q^2) = 4 \int \frac{d^{6-2\epsilon} k}{(2\pi)^{6-2\epsilon}} n(k) 2\pi \delta(k^2 - m_1^2) \frac{1}{(k - q)^2 - m_1^2} \quad (2.11d)\]

(We have used the fact that \(B(T)\) is real.

We must show that the divergent parts of (2.9) and (2.11c) cancel when \(\epsilon \to 0\). From the graph of figure 1a,

\[\pi(q^2) = 2i \int \frac{d^{6-2\epsilon} k}{(2\pi)^{6-2\epsilon}} \frac{1}{k^2 - m_1^2} \frac{1}{(q - k)^2 - m_1^2} \quad (2.12a)\]

Also

\[\bar{\pi}^{11}_T(q^0, q^2, m_2^2) - \bar{\pi}(q^2, m_2^2) = \frac{2}{(2\pi)^{6-2\epsilon}} \int d^{6-2\epsilon} k_1 d^{6-2\epsilon} k_2 \delta(k_1 + k_2 - q) \left\{ \frac{n(k_1) 2\pi \delta(k_2^2 - m_1^2)}{k_2^2 - m_1^2} + \frac{n(k_2) 2\pi \delta(k_1^2 - m_1^2)}{k_1^2 - m_1^2} - in(k_1) n(k_2) 2\pi \delta(k_1^2 - m_1^2) 2\pi \delta(k_2^2 - m_1^2) \right\} \quad (2.12b)\]

and from this we may calculate \(\bar{\pi}_T(q^0, q^2, m_2^2)\) because (1.5c) tells us that

\[\text{Re} \; \Pi_{\epsilon T} = \text{Re} \; \Pi^{11}_{\epsilon T}\]

\[\text{Im} \; \Pi_{\epsilon T} = \frac{\text{Im} \; \Pi^{11}_{\epsilon T}}{1 + 2n(q)} \quad (2.13)\]

From (2.11d), (2.12b) and (2.13) it is immediate that

\[\text{Re} \; \bar{\pi}_T(q^0, q^2) = \text{Re} \{\bar{\pi}_T(q^0, q^2, m_2^2) - \bar{\pi}(q^2, m_2^2)\} \quad (2.14a)\]

It is familiar\(^6\) that \(\text{Im} \; \pi(q^2)\) is even in \(q^0\) and

\[\text{Im} \; \pi(q^2) = -\theta(q^2 - 4m_2^2) \frac{1}{(2\pi)^{6-2\epsilon}} \int d^{6-2\epsilon} k_1 d^{6-2\epsilon} k_2 \delta(k_1 + k_2 - q) 2\pi \delta(k_1^2 - m_1^2) 2\pi \delta(k_2^2 - m_1^2) \quad (2.15a)\]

From (2.12b) we see that
\[
\text{Im } \{\tilde{\pi}_T^{11}(q^0, q^2, m_2^2) - \tilde{\pi}(q^2, m_2^2)\} = \]
\[-\frac{1}{(2\pi)^{n-2\epsilon}} \int d^{n-2\epsilon} k_1 d^{n-2\epsilon} k_2 \delta(k_1 + k_2 - q) 2\pi \delta(k_1^2 - m_1^2) 2\pi \delta(k_2^2 - m_1^2) \{n(k_1) + n(k_2) + 2n(k_1)n(k_2)\} \] (2.15b)

This is nonzero for both \(q^2 > 4m_1^2\) and \(q^2 < 0\). For \(q^2 > 4m_1^2\) the \(\delta\)-functions demand that both \(k_1^0\) and \(k_2^0\) are positive or both negative, so that
\[
n(k_1) + n(k_2) + 2n(k_1)n(k_2) = \{1 + 2n(q)\} \{1 + n(k_1) + n(k_2)\} - 1 \] (2.16a)
and we find that the relation (2.14a) is true also for the imaginary parts of the functions involved. But for \(q^2 < 0\) the \(\delta\)-functions require \(k_1^0\) and \(k_2^0\) to have opposite signs, and instead
\[
n(k_1) + n(k_2) + 2n(k_1)n(k_2) = \epsilon(q^0) \epsilon(k_1^0) \{1 + 2n(q)\} \{n(k_2) - n(k_1)\} \] (2.16b)

In consequence,
\[
\text{Im } \hat{\pi}_T(q^0, q^2) = \text{Im } \{\tilde{\pi}_T(q^0, q^2, m_2^2) - \tilde{\pi}(q^2, m_2^2)\} - \theta(-q^2) \phi_T(q^0, q^2) \]
\[
\phi_T(q^0, q^2) = \theta(q^0) \phi_T(q^0, q^2) + \theta(-q^0) \phi_T(-q^0, q^2) \]
\[
\theta(q^0) \phi_T(q^0, q^2) = 4\theta(q^0) \frac{1}{(2\pi)^{n-2\epsilon}} \int d^{n-2\epsilon} k_1 d^{n-2\epsilon} k_2 \delta(k_1 + k_2 - q) 2\pi \delta(k_1^2 - m_1^2) 2\pi \delta(k_2^2 - m_1^2) \theta(k_1^0)n(k_1) \] (2.14b)

So finally, when the free-field pressure is subtracted off, the interaction pressure is
\[
P(T)^{\text{int}} = -\frac{1}{2} C \text{ Im } \int \frac{d^{n-2\epsilon} q}{(2\pi)^{n-2\epsilon}} \left[2n(q) \log \frac{g^2 \tilde{\pi}_T(q^0, q^2, m_2^2) + m_2^2 - q^2}{m_2^2 - q^2}\right. \]
\[+ \log \frac{g^2 \tilde{\pi}_T(q^0, q^2, m_2^2) + m_2^2 - q^2}{g^2 \tilde{\pi}(q^2, m_2^2) + m_2^2 - q^2} + g^2 \frac{\tilde{\pi}_T(q^0, q^2, m_2^2) - \tilde{\pi}(q^2, m_2^2) - i\theta(-q^2) \phi_T(q^0, q^2)}{q^2 - m_2^2 - g^2 \tilde{\pi}(q^2, m_2^2)} \left.\right] \] (2.17)

For \(\epsilon \to 0\), that is in 6 dimensions, the integrand in (2.17) is finite, so it remains to check that its high-\(q\) behaviour is such that the integral is finite. From (2.12a)
\[
\pi(q^2) = -\frac{2}{(4\pi)^{3-\epsilon}} \Gamma(-1 + \epsilon) \int_0^1 dx [m_1^2 - q^2 x(1-x)]^{1-\epsilon} \] (2.12c)
so that
\[
\tilde{\pi}(q^2, m_2^2) = -\frac{2}{(4\pi)^{3-\epsilon}} \Gamma(-1 + \epsilon) \int_0^1 dx \left\{[m_1^2 - q^2 x(1-x)]^{1-\epsilon} - [m_1^2 - m_2^2 x(1-x)]^{1-\epsilon} + \right. \]
\[\left. (q^2 - m_2^2) x(1-x)(1-\epsilon)[m_1^2 - m_2^2 x(1-x)]^{-\epsilon}\right\} \] (2.12d)

When \(q^2\) becomes large, \(\tilde{\pi}(q^2, m_2^2)\) contains terms of order \(q^2\) and \(q^{-2\epsilon}\), while the integral shown in (2.12b) that gives the difference between \(\pi_{T}^{11}\) and \(\tilde{\pi}\) is only of order \(1/q^2\), so that \(\tilde{\pi}_T(q^0, q^2, m_2^2) \sim \tilde{\pi}(q^2, m_2^2)\) and
\[
\log \frac{g^2 \tilde{\pi}_T(q^0, q^2, m_2^2) + m_2^2 - q^2}{g^2 \tilde{\pi}(q^2, m_2^2) + m_2^2 - q^2} \sim -g^2 \frac{\tilde{\pi}_T(q^0, q^2, m_2^2) - \tilde{\pi}(q^2, m_2^2)}{q^2 - m_2^2 - g^2 \tilde{\pi}(q^2, m_2^2)} + O(1/q^8) \] (2.18)
On the other hand one can see from (2.14b) that when \(|q|\) is large, whether or not \(q^0\) also is large, \(\phi_T(q^0, q^2)\) is exponentially small. Hence when \(q^2\) is large the last two terms in the integrand of (2.17) combine to make the integral over \(q\) convergent.

The central result of this section is our formula (2.17) for the interaction pressure to order \(N^0\). The reader may worry about some of the steps in its derivation, for instance an integration over negative renormalised squared masses for the photon in (2.7). Thus in intermediate steps we considered tachyonic photons! In the next section we will discuss the general renormalisation problem for the pressure. This will lead us in the \(1/N\) expansion to another derivation of (2.17) which avoids these problems.

3 Renormalisability of the pressure

We now return to the interaction (1.6) and show how it leads in 6 dimensions to a finite expression for the pressure even when we do not take the large-\(N\) limit. The form (1.6) is designed to simulate QED without introducing the complications that arise from spin, and so our analysis follows closely the familiar renormalisation of zero-temperature QED, such as is described in the book of Bjorken and Drell[7]. As we shall see, the task of expressing the derivatives of the pressure with respect to the masses in terms of renormalised propagator and vertex functions leads us to a problem of overlapping divergences. This turns out to be similar to the overlapping-divergence problem for the vacuum polarisation in QED, where, following Dyson, one introduces a certain electron-positron scattering kernel (see chapter 19 of [7]). We shall follow a similar road here.

Our notation will be as follows. We use the labels \(\alpha, \beta, \ldots\) and \(a, b, \ldots\) to denote components of isodoublets and isotriplets, respectively, and \(r, s, \ldots\) for flavour labels. We also write the unrenormalised fields of the “electrons” and “photons” together as

\[
\phi_{1r} = \psi_r
\]
\[
\phi_{2}^a = A^a
\]  

(3.1)

These are the unrenormalised fields. As in the last section, their unrenormalised masses are \(m_{0i}\), \(i = 1, 2\). Where it does not cause confusion, we will not explicitly write the isospin and flavour labels. The renormalised fields will be labelled with an additional suffix \(R\); their masses are \(m_{i}\), which are related to the unrenormalised masses by

\[
m^2_i - m^2_{0i} - \Pi_i(q^2) \big|_{q^2=m^2_i} = 0 \quad i = 1, 2
\]  

(3.2a)

Here, \(\Pi_1(q^2)\) and \(\Pi_2(q^2)\) are the self energies with the various Kronecker deltas factored off. From them, we also construct the two wave-function renormalisation constants

\[
Z_i = \left[1 - \frac{\partial \Pi_i(q^2)}{\partial q^2}\right]_{q^2=m^2_i}^{-1}
\]  

(3.2b)

and hence the renormalised fields

\[
\phi_{iR} = Z_i^{-1/2} \phi_i
\]  

(3.2c)

We are assuming that \(0 \leq m^2_2 < 4m^2_1\), so that the renormalisations are real. The propagators before and after renormalisation are

\[
D_i(q^2) = i[q^2 - m_i^2 - \Pi_i(q^2)]^{-1} \cdot 1
\]
\[ D_{iR}(q^2) = Z_i^{-1}D_i(q^2) = i[q^2 - m_i^2 - \Pi_i(q^2)]^{-1} \cdot 1 \]  
(3.2d)

where the two functions
\[ \Pi_i(q^2) = Z_i[\Pi_i(q^2) - \Pi_i(m_i^2)] - (Z_i - 1)(q^2 - m_i^2) \]  
i = 1, 2  
(3.2e)

are finite and have value zero and zero derivative at \( q^2 = m_i^2 \). In (3.2d) \( \mathbb{1} \) is to be read as \( \delta_{\alpha\beta}\delta_{rs} \) for \( i = 1 \) and as \( \delta_{ab} \) for \( i = 2 \).

We introduce the one-particle-irreducible vertex function \( \gamma(p', p) \) which couples a pair of electrons of momenta \( p, p' \) to a photon. Renormalise it and the coupling, so that
\[ \lambda_i \gamma_{R}(p', p) = \lambda_0 Z_1 Z_2^{1/2} \gamma(p', p). \]  
(3.3a)

We choose to fix \( \gamma_R(p', p) \), and so define the renormalised coupling \( \lambda_i \), by imposing the condition
\[ \gamma_R \bigg|_{SP} = 1, \]  
(3.3b)

where
\[ \gamma_R \bigg|_{SP} = \gamma_R(p, p') \]  
evaluated for \( p^2 = p'^2 = (p - p')^2 = -M^2 \)  
(3.3c)

for some fixed mass \( M \). This renormalisation absorbs, for example, a divergence from the one-loop triangle graph (which we could neglect in the large-\( N \) limit).

Note that the coupling (1.6) is invariant under the following \( C \)-transformation
\[ C : \phi_r \rightarrow \epsilon (\phi_r^T)^T (r = 1, ..., N), \]  
\[ \phi_2 \rightarrow -\phi_2 \]  
(3.3d)

where
\[ \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]  
(3.3e)

This forbids a nonzero vertex function for three “photons” in our model, similarly to Furry’s theorem in QED. By the usual power counting arguments (compare [6]), we see then that the mass, wave function and coupling renormalisations (3.2), (3.3a)–(3.3c) make the (perturbative) theory finite.

We need to introduce a \( 2 \rightarrow 2 \)-body connected scattering amplitude \( T \). This is a \( 2 \times 2 \) matrix connecting the channels

1: \( \phi_1 + \phi_1 \)  
2: \( \phi_2 + \phi_2 \)  
(3.4)

Each element of \( T \) is also a matrix in isospin and flavour space. We define \( T \) to be amputated – there are no single-particle poles in its external legs – and we exclude from it all terms which are one-particle reducible in the \( s \)-channel, though \( T_{11} \) does have a \( t \)-channel photon pole and \( T_{12} \) and \( T_{21} \) have \( t \) and \( u \) channel electron poles. We also introduce the two-particle-irreducible kernel \( K \) associated with \( T \). It has no \( s \)-channel two-particle intermediate states and is related to \( T \) by
\[ T = K + KPT = K + TPK \]  
(3.5a)

Here the \( 2 \times 2 \) matrix \( P \) is diagonal; one diagonal element \( P_{11} \) is the tensor product of two electron propagators, and the other \( P_{22} \) is half the tensor product of two photon propagators. The \( 12 \) element of the first matrix equation in (3.5a) is drawn in figure 2, together with the definition of the matrix
The factor $1/2$ in the definition of $\mathbf{P}$ takes account of the symmetry of channel 2 under interchange of the two photons. Because of the relation (3.2d) between the renormalised and unrenormalised propagators,

$$\mathbf{P}_R = Z^{-1} \mathbf{P} Z^{-1}$$

$$Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}$$

(3.5b)

On the other hand, an amputated scattering amplitude on renormalisation acquires a factor $Z^{1/2}$ for each external leg, so the renormalised versions of $\mathbf{T}$ and $\mathbf{K}$ are

$$\mathbf{T}_R = Z \mathbf{T} Z$$

$$\mathbf{K}_R = Z \mathbf{K} Z$$

(3.5c)

Hence after renormalisation

$$\mathbf{T}_R = \mathbf{K}_R + \mathbf{K}_R \mathbf{P}_R \mathbf{T}_R = \mathbf{K}_R + \mathbf{T}_R \mathbf{P}_R \mathbf{K}_R$$

(3.5d)

Note that $\mathbf{T}$ and $\mathbf{K}$ have skeleton expansions that express them in a unique way in terms of integrals over the unrenormalised propagators and vertex functions in (3.2) and (3.3), while $\mathbf{T}_R$ and $\mathbf{K}_R$ have similar expansions in terms of the renormalised functions.

We now define the composite operators constructed from the electron and photon fields

$$\chi_1(x) = \phi_1^\dagger(x) \phi_1(x) - \langle 0 | \phi_1^\dagger(x) \phi_1(x) | 0 \rangle$$

$$\chi_2(x) = \frac{1}{2} [ \phi_2^\dagger(x) - \langle 0 | \phi_2^\dagger(x) | 0 \rangle ]$$

(3.6)

Here, a sum over flavour and isospin indices is implied. Define also the one-particle irreducible Green's functions that couple these to the channels 1 and 2 defined in (3.4); they form a $2 \times 2$ matrix $\Gamma$, shown
graphically in figure 3. Each element of the matrix is a function of the momenta \( p \) and \( p' \) of the final-state particles. In fact for our purposes we need only consider the case \( p = p' \), so that \( \Gamma \) is a function just of the single variable \( p^2 \). To zeroth order in perturbation theory, \( \Gamma = \Gamma_0 \) which, apart from Kronecker deltas for flavour and isospin indices is just the unit matrix:

\[
\Gamma_0 = \begin{pmatrix}
\delta_{rs} \delta_{\alpha\beta} & 0 \\
0 & \delta_{ab}
\end{pmatrix}
\]  

(3.7a)

The complete \( \Gamma \) has flavour and isospin structure

\[
\Gamma = \begin{pmatrix}
\hat{\Gamma}_{11}(p^2)\delta_{rs} \delta_{\alpha\beta} & \hat{\Gamma}_{12}(p^2)\delta_{ab} \\
\hat{\Gamma}_{21}(p^2)\delta_{rs} \delta_{\alpha\beta} & \hat{\Gamma}_{22}(p^2)\delta_{ab}
\end{pmatrix}
\]  

(3.7b)

and it satisfies a Dyson equation

\[
\Gamma = \Gamma_0 + \Gamma_0 \Gamma_{\text{PT}} = \Gamma_0 + \Gamma_{\text{PK}}
\]  

(3.7c)

where we have used (3.5a). We define renormalised composite fields by

\[
\chi_R = Z^{-1}_\chi \chi
\]  

(3.8a)

where the matrix \( Z_\chi \) is to be chosen; then

\[
\Gamma_R = Z^{-1}_\chi \Gamma Z = Z^{-1}_\chi \Gamma_0 Z + \Gamma_R \Gamma_{\text{PK}}
\]  

(3.8b)

The isospin and flavour structure of \( \Gamma_R \) is the same as that of \( \Gamma \), that is (3.7b) but with scalar functions \( \hat{\Gamma}_{ijR}(p^2) \). Using the usual power-counting argument and the skeleton expansion for \( K_R \) we see that indeed \( \Gamma_R \) can be made finite by suitable choice of \( Z_\chi \), in such a way that the divergence from the last term in (3.8b) is absorbed in each order of perturbation theory (compare for example chapter 19 of [7]). We also see from (3.8b) that any \( Z_\chi \) that makes the elements of \( \Gamma_R \) finite at one (but not necessarily the same) point in momentum space is an acceptable choice for this renormalisation matrix.

In order to choose \( Z_\chi \) appropriately, we return to the relation (3.2a) between the renormalised and unrenormalised masses. Remembering that \( \Pi_i(q^2) \) depend also on \( m_{01}^2 \) and \( m_{02}^2 \), we have

\[
\frac{\partial m_{ij}^2}{\partial m_{0i}^2} - \delta_{ij} - \frac{\partial}{\partial m_{0i}^2} \Pi_j(q^2) \bigg|_{q^2=m_j^2} - \frac{\partial}{\partial q^2} \Pi_j(q^2) \bigg|_{q^2=m_j^2} \frac{\partial m_{ij}^2}{\partial m_{0i}^2} = 0
\]  

(3.9a)

This gives, with the definition (3.2b) of \( Z_i \),

\[
\frac{\partial m_{ij}^2}{\partial m_{0i}^2} Z_j^{-1} = \delta_{ij} + \frac{\partial}{\partial m_{0i}^2} \Pi_j(q^2) \bigg|_{q^2=m_j^2}
\]  

(3.9b)

The differentiation with respect to \( m_{0i}^2 \) is with fixed bare coupling \( \lambda_0 \). When applied to any Feynman graph for \( \Pi_j \), it gives a sum of terms in which each internal line of type \( i \) in turn is doubled. We recognise therefore that the quantity on the right-hand side of (3.9b) is just

\[
\delta_{ij} + \frac{\partial}{\partial m_{0i}^2} \Pi_j(q^2) = \hat{\Gamma}_{ij}(q^2)
\]  

(3.9c)
evaluated at \( q^2 = m_j^2 \). Hence, if we define

\[
[Z_\chi]_{ij} = \frac{\partial m_j^2}{\partial m_{0i}^2}
\]  

(3.9d)

we see from (3.5b) and (3.8b) that

\[
\hat{\Gamma}_{ijR}(q^2) \big|_{q^2=m_j^2} = \delta_{ij}
\]

(3.10a)

Therefore the elements of \( \mathbf{\Gamma}_R(q^2) \) are finite when the final-state particles \( q \) are on shell, that is at one point in momentum space. From the arguments given above it follows that they are also finite when the particles are off shell.

We are going to calculate the derivative of the pressure with respect to the renormalised masses at fixed bare coupling. It is convenient to introduce a renormalisation prescription in which, unlike in (3.2) and (3.3), fixed bare coupling corresponds to fixed renormalised coupling while the renormalised masses vary. In this modified prescription we replace the bare masses \( m_{0i} \) in (3.2a) with fixed bare masses \( \mu_{0i} \), though keep the same bare coupling \( \lambda_0 \). Then (3.2b), (3.2c) and (3.3) lead to a modified renormalised coupling \( \lambda \). We claim that \( \lambda \) is a finite function of \( \lambda \). To show this, differentiate the definition (3.3) of \( \lambda \) with respect to the bare masses at fixed \( \lambda_0 \):

\[
\frac{\partial \lambda}{\partial m_{0j}^2} = \lambda_0 Z_1 Z_2 \frac{\partial \gamma(p',p)|_{\text{SP}}}{\partial m_{0j}^2} + \lambda \frac{\partial}{\partial m_{0j}^2} \log Z_1 + \frac{1}{2} \lambda \frac{\partial}{\partial m_{0j}^2} \log Z_2
\]

(3.11a)

Now

\[
\lambda_0 \frac{\partial \gamma(p',p)}{\partial m_{0j}^2} = \gamma_j(p',p)
\]

(3.11b)

is the unrenormalised amputated Green’s function for the fields \( \chi_j, \phi_1, \phi_2 \) at zero four momentum for the \( \chi_j \)-field. Its superficial degree of divergence is \(-2\), so it is rendered finite by the renormalisation

\[
\gamma_j(p',p)Z_1Z_2^{\frac{1}{2}} = \sum_{k=1,2}[Z_\chi]_{jk} \gamma_{kR}(p',p)
\]

(3.11c)

Differentiating the definition (3.2b) of the \( Z_i \) gives

\[
\frac{\partial}{\partial m_{0j}^2} \log Z_i = \sum_{k=1,2} \left\{ \delta_{ki} \left( \frac{\partial^2 \Pi_i(q^2)}{(dq^2)^2} + \frac{\partial}{\partial q^2} \hat{\Gamma}_{iR}(q^2) \right) \right\} \bigg|_{q^2=m_i^2} \frac{\partial m_i^2}{\partial m_{0j}^2}
\]

(3.11d)

Then (3.11a), together with (3.9d), gives

\[
\frac{\partial \lambda}{\partial m_1^2} = \gamma_{1R}(p',p)|_{\text{SP}} + \lambda \left( \frac{\partial^2 \Pi_1(q^2)}{(dq^2)^2} + \frac{\partial}{\partial q^2} \hat{\Gamma}_{1R}(q^2) \right) \bigg|_{q^2=m_1^2} + \frac{1}{2} \lambda \frac{\partial}{\partial q^2} \hat{\Gamma}_{12R}(q^2) \bigg|_{q^2=m_1^2}
\]

\[
\frac{\partial \lambda}{\partial m_2^2} = \gamma_{2R}(p',p)|_{\text{SP}} + \lambda \frac{\partial}{\partial q^2} \hat{\Gamma}_{2R}(q^2) \bigg|_{q^2=m_2^2} + \frac{1}{2} \lambda \left( \frac{\partial^2 \Pi_2(q^2)}{(dq^2)^2} + \frac{\partial}{\partial q^2} \hat{\Gamma}_{22R}(q^2) \right) \bigg|_{q^2=m_2^2}
\]

(3.11e)

This expresses, in terms of renormalised quantities only, how \( \lambda \) changes when the renormalised masses are changed, keeping \( \lambda_0 \) fixed. Thus, as we have claimed above, two renormalised couplings for different renormalised masses but the same \( \lambda_0 \) differ by a finite amount. The definition (3.3a,b) of \( \lambda \) expresses it as a function of \( \lambda_0 \) and the bare masses \( m_{0i} \), which in turn can be considered from (3.2a) as functions
of $\lambda_0$ and the renormalised masses $m_i$. Then from $\lambda$ considered as function of $\lambda_0$ and the masses $m_i$ and the corresponding definition of $\tilde{\lambda}$ as function of $\lambda_0$ and the masses $\mu_i$ we get two equations from which, in principle, we may eliminate $\lambda_0$ and so express $\tilde{\lambda}$ as a function of $\lambda$ (though in practice this will be a nontrivial task). However, from its definition it is clear that $\tilde{\lambda}$ remains fixed when the masses $m_i$ vary in differentiations with fixed $\lambda_0$.

We now return to formula (1.1). We choose the version (1.1b) and write it in the form

$$\frac{\partial}{\partial m_{01}^2} P(T) = -\sum_j \int \frac{d^n q}{(2\pi)^n} \text{tr} \left[ \Gamma_{0ij} 2^{1-j} \Delta_{IT}^{11}(q) \right] \quad i = 1, 2 \quad (3.12a)$$

Here $\Gamma_0$ is defined in (3.7a) and

$$\Delta_{IT}^{11}(q) = D_{IT}^{11}(q) - D_j(q^2) = \Delta_{IT}^{11}(q) \cdot 1 \quad (3.12b)$$
is the difference of the unrenormalised thermal 11-propagator and the zero temperature propagator for the field $\phi_j \ (j = 1, 2)$. The trace is with respect to the isospin and flavour indices.

The renormalisation of (3.12a) leads to a problem of overlapping divergences. To see this we consider the diagrammatic expansion of $\partial P(T)/\partial m_{01}^2$, for example, shown in figure 4. The second term of figure 4 can be interpreted as a correction either to the propagator on the right or to the vertex function of the composite operator $\chi_1$ on the left. This is analogous to the case of the vacuum polarisation function in QED (compare chapter 19 of [7]), as will be our methods for a proper renormalisation of (3.12a).

We express $\Delta_{IT}^{11}$ in terms of $\Delta_{IT}^{11R}$, which is defined in a similar way, but with the renormalised fields replacing the bare fields so that

$$\Delta_{IT}^{11R}(q) = Z_i^{-1} \Delta_{IT}^{11} \quad (3.12b)$$

We use the fact that $\Gamma_0$ is a constant matrix, but can be expressed from (3.7c) as

$$\Gamma_0 = \Gamma - \Gamma PK \quad (3.12ba)$$

where the individual terms on the right-hand side are momentum-dependent. Now we choose as momentum argument on the right-hand side just the integration variable $q$ of (3.12a). With this and the definition (3.9d) of the matrix $[Z \chi]_{ij}$, we have

$$\frac{\partial}{\partial m_i^2} P(T) = -\sum_j \int \frac{d^n q}{(2\pi)^n} \text{tr} \left\{ \left[ Z^{-1} (\Gamma - \Gamma PK) Z \right]_{ij} 2^{1-j} \Delta_{IT}^{11} \right\} \quad (3.12c)$$
or

$$\frac{\partial}{\partial m_i} P(T) = -\sum_j \int \frac{d^n q}{(2\pi)^n} \text{tr} \left\{ \left[ (\Gamma_R - \Gamma_R P R K_R) \right]_{ij} 2^{1-j} \Delta_{IT}^{11} \right\} \quad (3.12d)$$
In using the formula, we must remember that the partial derivative is with the bare coupling \( \lambda_0 \) fixed, so the appropriate renormalised coupling is \( \tilde{\lambda} \).

The right-hand side of the expression (3.12d) is shown graphically in figure 5. The renormalised vertex, kernel and propagator functions \( \Gamma_R, K_R, D_R, \Delta^1_{TR} \) are, of course, finite for \( n = 6 \). But we still have three loop integrals to do, which we call \( L_1, L_2 \) and \( L_3 \), as indicated in figure 5.

Let us first determine the degree of divergence for \( n = 6 \) of the loop integrations \( L_1 \) and \( L_3 \). To this end, we write an operator-product expansion:

\[
\phi_i R(x)\phi^\dagger_i R(0) = C_i(x^2)\mathbb{1} + \sum_j C_{ij}(x^2)\chi_j R(0) + \ldots
\]

(3.13a)

As the \( \chi_j R \) have the same dimension as the left-hand side of (3.13a), the small-distance behaviour of \( C_{ij}(x^2) \) is \( (x^2)^b \), up to possible logarithms. The thermal propagator \( D^1_{TR}(q) \) is the Fourier transform of the vacuum expectation value of (3.13a). Because \( \Delta^1_{TR}(q) \) is the difference between the temperature \( T \) and temperature 0 propagators, the first term on the right-hand side of (3.13a) does not contribute to it and the leading power behaviour for \( q \to \infty \) is obtained from the second term on the right-hand side of (3.13a).

Thus, at worst,

\[
\Delta_{iTR}(q) \sim (q^2)^{-3} \quad \text{for} \quad q \to \infty.
\]

(3.13b)

Power counting shows that the \( L_1 \) integration is logarithmically divergent. For the second term in figure 5, we choose to do the loop integration \( L_3 \) first. Superficially, it is convergent. It really does converge if there are no divergent subintegrations. Of course, no divergent subintegrations occur in \( K_R \) and \( \Delta_{TR} \). Thus we could get a divergent subintegration only\(^[8] \) if the loop \( L_3 \) is closed in \( K_R \) directly on one single skeleton vertex. But this cannot happen because \( K_R \) contains no \( s \)-channel 1-particle-reducible diagrams. Thus we conclude that the \( L_3 \) loop is convergent; it gives a high-\( q^2 \) behaviour \( 1/q^2 \) to that loop. From this, we see that the \( L_2 \) integration diverges logarithmically.

So both terms in figure 5 are logarithmically divergent. It remains to show that these divergences cancel, leaving a finite result. From the discussion above it is clear that we will have to consider only the part of \( \Delta_{iTR}(q) \) proportional to \( (q^2)^{-3} \) for \( q \to \infty \). The higher terms in the operator product expansion (3.13a) lead to convergent contributions in all loops \( L_1, L_2, L_3 \). Thus, for the discussion of the convergence we can replace \( \Delta^1_{TR} \) in (3.12d) and figure 5 by any expression having the same \( (q^2)^{-3} \) behaviour for \( q \to \infty \). We choose the following 4-point Green’s functions:

\[
J_{4ik}(q,p) = \int d^n x \int d^n z_1 \int d^n z_2 e^{iqx} e^{-ipz_1} e^{ipz_2} \left\{ \langle 0|T(\phi_{iR}(x)\phi_{iR}^\dagger(0)\phi_{kR}^\dagger(z_1)\phi_{kR}(z_2))|0\rangle \\
- \langle 0|T(\phi_{iR}(x)\phi_{iR}^\dagger(0))|0\rangle \langle 0|T(\phi_{kR}^\dagger(z_1)\phi_{kR}(z_2))|0\rangle \right\}
\]

(3.16)

where \( p \) is an arbitrary fixed momentum. For simplicity, we continue not to write explicitly the isospin and flavour indices of the fields. Those attached to \( \phi_{iR} \) and \( \phi_{iR}^\dagger \) are carried by \( J_{4ik} \), while those attached to \( \phi_{kR} \) and \( \phi_{kR}^\dagger \) are equal to each other and summed. This summation excludes \( s \)-channel one particle reducible diagrams from \( J_{4ik} \).

Inserting here the operator product expansion
Let us then insert $J$ in (3.16) which have linearly independent contributions from the thermal expectation values of $\chi_{jR}(0)$ in the operator product expansion (3.13a), where $j = 1, 2$. Thus, in order to prove convergence, we choose two “trial” terms $J_4$ corresponding to the index $k = 1, 2$ in (3.16) which have linearly independent contributions from the $\chi_{jR}(0)$ ($j = 1, 2$) in the operator product expansion. The structure of $J_{4ik}$ is as follows (compare (3.5d) and figure 6):

$$J_{4ik}(q, p) = \text{tr}_p \left[ 2^{i-1} \left[ \mathbf{P}_R(p) \right]_{ik} (2\pi)^n |\delta(q - p) + \delta_{k2}\delta(q + p)| + 2^{i-1} \left[ \mathbf{P}_R(q) \mathbf{T}_R(q, p) \mathbf{P}_R(p) \right]_{ik} 2^{k-1} \right]$$

(3.17)

where $\text{tr}_p$ stands for the summation over the isospin and flavour quantum numbers connected with the $p$-lines in figure 6.

Let us then insert $J_4$ in (3.12d):

$$\sum_j \int \frac{d^n q}{(2\pi)^n} \text{tr}_q \left\{ \left[ \mathbf{G}_R - \mathbf{G}_R \mathbf{K}_R \right]_{ij} 2^{1-j} J_{4jk} \right\}$$

$$= \sum_j \frac{d^n q}{(2\pi)^n} \text{tr}_q \left\{ \left[ \mathbf{G}_R - \mathbf{G}_R \mathbf{P}_R \mathbf{K}_R \right]_{ij} (q) \right.$$  

$$\text{tr}_p \left[ \mathbf{P}_R(p) \mathbf{j}_k (2\pi)^n (\delta(q - p) + \delta_{k2}\delta(q + p)) + \left[ \mathbf{P}_R(q) \mathbf{T}_R(q, p) \mathbf{P}_R(p) \right]_{jk} 2^{k-1} \right] \right\}$$

$$= \text{tr}_p \left\{ \left[ \mathbf{G}_R(p) \mathbf{P}_R(p) - (\mathbf{G}_R \mathbf{P}_R \mathbf{K}_R \mathbf{P}_R)(p) + (\mathbf{G}_R \mathbf{P}_R \mathbf{T}_R \mathbf{P}_R)(p) \right.$$  

$$- (\mathbf{G}_R \mathbf{P}_R \mathbf{K}_R \mathbf{P}_R \mathbf{T}_R \mathbf{P}_R)(p))_{ik} 2^{k-1} \right\}.$$  

(3.18)

Using now (3.5d) we see that the last three terms on the right-hand side (3.18) cancel, leaving us with the first term which is obviously finite, being the product of the renormalised quantities $\mathbf{G}_R(p)$ and $\mathbf{P}_R(p)$.

This concludes our discussion of the renormalisation of the derivatives of the pressure with respect to the masses. We have shown that in (3.12d) all divergences cancel leaving us with an ultraviolet finite result.

It remains to make the $1/N$ expansion of the general formula (3.12d), where only renormalised quantities appear. This is straightforward but turns out to be a lengthy calculation. We give some details in the Appendix. As can be seen from there, starting from (3.12d) the unrenormalised masses and coupling are never encountered any more and no integration over negative renormalised squared mass is needed. The differential of the pressure with respect to both squared masses is obtained from (A.5), (A.6), (A.23) as:

$$\sum_j dm_j^2 \frac{\partial}{\partial m_j^2} P(T, m_1^2, m_2^2) = \sum_j dm_j^2 \frac{\partial}{\partial m_j^2} \left\{ P_0^{(1)}(T, m_1^2) + P_0^{(2)}(T, m_2^2) + P^{\text{int}}(T, m_1^2, m_2^2) \right.$$  

$$+ \mathcal{O}(1/N) \right\},$$

(3.19)
\[ P^{(1)}_0(T, m^2_1) = 4N\pi \int \frac{d^nq}{(2\pi)^n} n(q)\theta(q^2 - m^2_1), \]
\[ P^{(2)}_0(T, m^2_2) = C\pi \int \frac{d^nq}{(2\pi)^n} n(q)\theta(q^2 - m^2_2). \]  

\[ P^{(1)}_0(T, m^2_1) \] is the pressure from the free electrons of renormalised mass \( m_1 \), \( P^{(2)}_0(T, m^2_2) \) similarly for the photons and \( P^{\text{int}}(T, m^2_1, m^2_2) \) is the interaction pressure to order \( N^0 \) as in eq. (2.17). (To be consistent with the notation of this section we write here \( P^{\text{int}}_0 \) for the expression (2.17).) The mass dependences are indicated explicitly, and we remind the reader that always the coupling constant \( \tilde{g} \) is kept fixed.

To summarise: We have calculated the derivatives of the pressure \( \partial P(T, m^2_1, m^2_2) / \partial m^2_j \) to order \( N \) and \( N^0 \) for all masses of photons and electrons satisfying \( 0 \leq m^2_2 < 4m^2_1 \). Starting from one point \( (m^2_1, m^2_2) \) in this physical region we can integrate along any path \( C_1 \) in the \( m^2_1 - m^2_2 \) plane (figure 7) running to infinite masses, but always staying in the physical region.

Using then as boundary condition that at infinite masses the pressure vanishes, we get:

\[ P(T, m^2_1, m^2_2) = - \int_{C_1} \sum_j \frac{d^{n-2}_m}{dm^2_j} \frac{\partial}{\partial m^2_j} P(T, m^2_1, m^2_2) \]
\[ = P^{(1)}_0(T, m^2_1) + P^{(2)}_0(T, m^2_2) + P^{\text{int}}(T, m^2_1, m^2_2) + O(1/N) \]  

On the other hand, in the simpler calculation performed in section 2 we had to integrate along \( C_2 \), that is at fixed \( m^2_1 = m_1^2 \) with \( m^2_2 \) running from \( m^2_2 \) to \( -\infty \). Clearly, \( C_2 \) leaves the physical region. Nevertheless the result obtained in (2.17) is correct, as we have shown here how it can be obtained without going to unphysical values for the masses.

Thus the final result for the pressure to order \( N^1 \) and \( N^0 \) is exactly as described in section 2: To order \( N^1 \) we get the pressure of free “electrons” and “positrons” of renormalised mass \( m_1 \), to order \( N^0 \) we get the free pressure of “photons” with renormalised mass \( m_2 \) plus the interaction pressure \( P^{\text{int}}(T, m^2_1, m^2_2) \) of (2.17).

### 4 Evaluation of the large-\( N \) limit

We now turn to the evaluation of the exact result (2.17) which we obtained in the limit \( N \rightarrow \infty \). Since (2.17) is UV finite, we can put \( \epsilon \rightarrow 0 \). All formulae in this section thus refer to \( n = 6 \) dimensions.
Equation (2.17) can now be evaluated by a number of nested numerical integrations which involve the following functions as building blocks. Firstly, the zero-temperature contributions to the “photon” self-energy:

\[
\text{Re } \tilde{\pi}(q^2) = -\frac{1}{32\pi^3} \left( \int_0^1 dx \left[ m_1^2 - q^2 x(1-x) \right] \log \left| \frac{m_1^2 - q^2 x(1-x)}{m_1^2 - m_2^2 x(1-x)} \right| + \frac{1}{6} (q^2 - m_2^2) \right) \tag{4.1}
\]

\[
\text{Im } \tilde{\pi}(q^2) = -\frac{1}{32\pi^2} \int_0^1 dx \left[ q^2 x(1-x) - m_1^2 \right] \theta(q^2 x(1-x) - m_1^2) = -\frac{1}{192\pi^2} \theta(q^2 - 4m_1^2) \frac{(q^2 - 4m_1^2)^{3/2}}{\sqrt{q^2}} \tag{4.2}
\]

which can be expressed in terms of elementary functions, though we have done so only for the imaginary part.

Secondly, the thermal contributions. The real part of the thermal self-energy is given by

\[
\text{Re } (\tilde{\pi}_T(q^0, q^2) - \tilde{\pi}(q^2)) = \frac{1}{16\pi^3} \int_0^\infty dk \frac{k^3 n(\omega_k)}{\omega_k} \left\{ \begin{array}{l}
1 - \left( \frac{2\omega_k q_0 - q^2}{2k|q|} \right)^2 \log \left| \frac{\omega_k q_0 - k|q| + \frac{1}{2} q^2}{\omega_k q_0 + k|q| - \frac{1}{2} q^2} \right| \\
+ \left( 1 - \left( \frac{2\omega_k q_0 + q^2}{2k|q|} \right)^2 \log \left| \frac{\omega_k q_0 + k|q| + \frac{1}{2} q^2}{\omega_k q_0 - k|q| - \frac{1}{2} q^2} \right| \\
+ \frac{2q^2}{k|q|} \right\} \right. \tag{4.3}
\]

where \( \omega_k = \sqrt{k^2 + m_1^2} \). The imaginary part is most easily obtained from \( \pi_T^{12} \) via

\[
\text{Im } \pi_T = -\frac{1}{2} i \epsilon(q_0)(1 - e^{\beta q_0}) \pi_T^{12}
\]

and using \( f(x)f(y) = f(x+y)[1 + f(x) + f(y)] \) for \( f(x) = 1/(e^x - 1) \), yielding

\[
\text{Im } (\tilde{\pi}_T(q^0, q^2) - \tilde{\pi}(q^2)) = -\frac{\epsilon(q_0)}{16\pi^2} \int_0^\infty dk \frac{k^3 n(\omega_k)}{\omega_k} \left\{ \begin{array}{l}
1 - \left( \frac{2\omega_k q_0 - q^2}{2k|q|} \right)^2 \theta \left( 1 - \left| \frac{2\omega_k q_0 - q^2}{2k|q|} \right| \right) \epsilon(q_0 - \omega_k) \\
+ \left( 1 - \left( \frac{2\omega_k q_0 + q^2}{2k|q|} \right)^2 \theta \left( 1 - \left| \frac{2\omega_k q_0 + q^2}{2k|q|} \right| \right) \right) \epsilon(q_0 + \omega_k) \right\} \tag{4.4}
\]

In (2.17) we also need \( \tilde{\pi}_T \). Its real part is identical to (4.3), whereas its imaginary part differs from (4.4) by replacing all the sign functions \( \epsilon \) with 1.

In the limit \( |q| \to 0 \), the integral in (4.4) can be done analytically with the result

\[
\text{Im } \pi_T(q^0, 0) = \text{Im } \tilde{\pi}_T(q^0, 0) = \left( 1 + 2n(\frac{1}{2} q_0) \right) \text{Im } \tilde{\pi}(q^2 = q_0^2) \tag{4.5}
\]

Another limiting case which can be solved is the high-temperature limit \( |q_0|, |q|, m_1 \ll T \), which gives

\[
\tilde{\pi}_T(q^0, q^2) = -\frac{T^2}{24\pi} \left( 1 - \frac{q_0^2}{q^2} \right) \left[ 1 - \frac{q_0}{2|q|} \log \frac{q_0 + |q|}{q_0 - |q|} \right] + O(T) \tag{4.6}
\]
Remarkably, this is basically the same function that appears in the longitudinal component of the polarization tensor of hot QED and QCD\[4\], except that here it comes with a reversed over-all sign.*

As a consequence, the spectrum as read from the analytic structure of the full thermal “photon” propagator \(1/(q^2 + m_2^2 - g^2\bar{\pi}_T(q^0, q^2))\) is rather unusual.

In the case of initially massless “photons”, the full thermal propagator still has singularities at the light-cone, because (4.3) vanishes at \(q^2 = 0\); our massless “photons” do not acquire thermal masses. However, there are nontrivial corrections to the residues of the poles at \(q^2 = 0\) according to

\[
Z(q^2) = \lim_{q_0 \to \pm q} (1 - g^2 \frac{\partial}{\partial q_0} \bar{\pi}_T)^{-1} = \left(1 + \frac{g^2}{4\pi^3 q^2} \int_0^\infty dk \frac{k^2 n(\omega_k)}{\omega_k} \left[ \frac{\omega_k \log (\omega_k + k)}{2k} \frac{\omega_k}{\omega_k - k} - 1 \right] \right)^{-1} < 1
\]

This vanishes as \(|q|/T \to 0\) and also as \(m_1/T \to 0\). In the high-temperature limit it is very small for momenta that are not at least comparable with \(T\) in magnitude. So while no mass gap is generated, as the temperature increases thermal effects progressively remove the modes with larger and larger momenta. In the infinite-temperature limit the residue of the pole becomes zero, so that then there are no propagating plasmons at all.

On the other hand, for non-zero or non-neglige “photon” mass, there are always simple poles in the photon propagator. In this case there are thermal mass corrections, but they are negative, towards lighter (but nonzero) effective masses. At the same time, the residues of the corresponding poles are diminished.

At \(q_0 = 0\), \(\bar{\pi}_T\) normally gives the screening mass-squared for static fields, which in gauge theories is the Debye mass. While screening corresponds to poles in the propagator for imaginary values of the spatial momentum, in our model we have a pole at real spatial momentum if the temperature is larger than some critical temperature \(T_{\text{crit}}\). For \(T \gg m_1, m_2\) this pole is located at \(q^2 = g^2T^2/(24\pi)\), according to (4.6). A similar behaviour has been found in the gravitational polarization tensor of ultrarelativistic plasmas when evaluated on a flat-space background \[10\]. There the value of \(|q|\) at the pole at \(q_0 = 0\) is identified with the so-called Jeans mass characterizing the gravitational instability of the plasma. In our case, such a pole seems to reflect the fact that the potential of our model is unbounded from below, so that when the “photon” mass is small enough, thermal fluctuations can lead to a run-away symmetry breaking without the need of tunneling.

For non-zero \(q_0\), there are no poles at space-like momenta, because for those (4.6) has a large imaginary part proportional to \(T^2\) corresponding to Landau damping. So there are no propagating thermal tachyons.

To summarize, the spectrum of our model in the limit of large \(N\) as read from the thermal propagators is the following. The only thermal corrections occur in the photon spectrum, which for temperatures sufficiently small compared to the photon mass \(m_2\) consist of negative (momentum-dependent) corrections to \(m_2\). The latter are largest at low momenta and tend to zero for very high momenta. The corresponding dispersion law is depicted in figure 8a for \(g = 10\) and \(m_1 = m_2 = m = T < T_{\text{crit}} \approx 1.236m\). Up to the critical temperature, the static fields have finite screening length, which becomes infinite at \(T_{\text{crit}}\), whereas the plasma frequency (the long-wavelength limit of the dynamical mass) remains nonzero. Right at the critical temperature, the spectrum is thus very similar to that of the transverse vector bosons in the high-temperature limit of 4-dimensional gauge theories: a vanishing screening

* This abnormal sign in \(\phi_0^3\)-theories has previously been noted in reference\[9\].
mass together with a nonzero plasmon mass. For temperatures above $T_{\text{crit}}$, there is a pole at space-like momentum $q_0 = 0$ and $q^2 = m_J^2$ signalling a Jeans-type instability. As shown in figure 8b for $T = 1.5m > T_{\text{crit}}$, the real part of the inverse photon propagator has zeros for space-like momenta with $q^2 < m_J^2$, which gives rise to a pole of the propagator where also the imaginary part vanishes, which is at $q_0 = 0$.

The presence of a pole at $q_0 = 0$ and $|q| = m_J > 0$ causes the interaction pressure

$$P(T)^{\text{int}} = -\frac{1}{2} C \int \frac{d^6q}{(2\pi)^6} \left[ 2n(q) \arg \frac{g^2 \pi_T(q^0, q^2, m_2^2) + m_2^2 - q^2}{m_2^2 - q^2} + \arg \frac{g^2 \pi_T(q^0, q^2, m_2^2) + m_2^2 - q^2}{g^2 \pi(q^2, m_2^2) + m_2^2 - q^2} + g^2 \Im \frac{\pi_T(q^0, q^2, m_2^2)}{q^2 - m_2^2 - g^2 \pi(q^2, m_2^2)} \right]$$

(2.17)

to become IR singular for all $T > T_{\text{crit}}$. In the first term, for $|q_0|$ smaller than some finite number, the real part of $g^2 \pi_T + m_2^2 - q^2$ changes sign from positive to negative as $|q|$ is changed from some large value to a sufficiently small one (see figure 8b). The imaginary part on the other hand is negative throughout except for $|q_0| = 0$, where it vanishes. The argument in the first term therefore approaches a step function $-\pi \theta(m_J - |q|)$ as $|q_0| \to 0$. This causes the first term in (2.17) to diverge logarithmically in the IR when $T > T_{\text{crit}}$.

For finite photon mass and $T \leq T_{\text{crit}}$, the pole at $q_0 = 0$ and $|q| = m_J > 0$ is absent and (2.17') appears to be well-defined.

Turning finally to the numerical evaluation of (2.17), we can distinguish three different regions of integration depending on the appearance of imaginary parts.
The inverse photon propagator \( g^2 \pi_T + m_2^2 - q^2 \) has imaginary parts only for \( q^2 > 4m_1^2 \) (pair creation) and for \( q^2 < 0 \) (Landau damping). The quasi-particles described by the time-like poles of the photon propagator are therefore undamped and stable, provided \( m_2 < 2m_1 \). We denote their position by \( \omega_T(q^2) \). For \( |q_0| < \omega_T(q^2) \) the real part of \( g^2 \pi_T + m_2^2 - q^2 \) is positive and for \( |q_0| > \omega_T(q^2) \) it is negative.

Correspondingly, we have:

I — spacelike momenta, \( q^2 < 0 \): \( \pi \) is real in this region, but \( \pi_T \) has imaginary parts corresponding to Landau damping, so that

\[
P(T)_{I}^{\text{int}} = -C \int \frac{d^5 q}{(2\pi)^6} \int_{|q|}^{q_0} dq_0 \left\{ (2m(q_0) + 1) \arg[g^2 \pi_T(q_0^0, q^2, m_2^2) + m_2^2 - q^2] + g^2 \frac{\Im \pi_T(q_0^0, q^2, m_2^2)}{q^2 - m_2^2 - g^2 \pi(q^2, m_2^2)} \right\} \tag{4.8a}
\]

II — timelike momenta below threshold, \( 0 < q^2 < 4m_1^2 \): the first two terms in (2.17) contribute only for \( \sqrt{q^2 + m_2^2} \geq |q_0| \geq \omega_T(q^2) \), whereas the last term has a pole in this range with unit residue thanks to our on-shell renormalization scheme. The integration over \( q_0 \) can be carried out with the result

\[
P(T)_{II}^{\text{int}} = -C \int \frac{d^5 q}{(2\pi)^6} (-\pi) \left\{ 2 \log \frac{e^{|q^2 + m_2^2|} - 1}{e^{\omega_T(q^2)} - 1} + \omega_T(q^2) - \sqrt{q^2 + m_2^2} \right. \]
\[
+ g^2 \frac{\Re \pi_T(q_0^0, \sqrt{q^2 + m_2^2}, q^2, m_2^2)}{2\sqrt{q^2 + m_2^2}} \right\} \tag{4.8b}
\]

III — timelike momenta above threshold, \( q^2 > 4m_1^2 \):

\[
P(T)_{III}^{\text{int}} = -C \int \frac{d^5 q}{(2\pi)^6} \int_{\sqrt{q^2 + 4m_1^2}}^{\infty} dq_0 \left\{ 2n(q_0) \left( \arg \left( g^2 \pi_T(q_0^0, q^2, m_2^2) + m_2^2 - q^2 \right) + \pi \right) \right.
\]
\[
+ \arg \left( g^2 \pi_T(q_0^0, q^2, m_2^2) + m_2^2 - q^2 \right) - \arg \left( g^2 \pi(q_0^0, q^2, m_2^2) + m_2^2 - q^2 \right) + g^2 \frac{\Im \pi_T(q_0^0, q^2, m_2^2)}{q^2 - m_2^2 - g^2 \pi(q^2, m_2^2)} \right\} \tag{4.8c}
\]

The numerical evaluation of these expressions is quite challenging because of large cancellations among the individual contributions so that rather high working precision is needed. Our results for the nonperturbative interaction pressure are given in figure 9 for \( m = m_1 = m_2 \) and three different values of the coupling, \( g = 1, \sqrt{10}, \) and 10, and these are compared with the strictly perturbative contribution proportional to \( g^2 \), which also has to be computed numerically because we do not restrict ourselves to the high-temperature limit and so the full momentum dependence of the thermal self-energy enters there too.

For temperatures \( T \ll m \) there is rather little difference between the perturbative and the nonperturbative results. As the temperature increases, the latter grow bigger than the former until they abruptly end in a singularity at the critical temperature. For sufficiently small coupling and for \( m_1 \sim m_2 \), the critical temperature above which (2.17) ceases to exist can be estimated from the high-temperature expansion of the thermal photon propagator following from (4.6): \( T_{\text{crit}} \approx \sqrt{24\pi m_2/g} \approx 9m_2/g \). The actual values for \( g = 1, \sqrt{10}, 10 \) are \( T_{\text{crit}} \approx 9.133, 3.164, 1.236 \) times \( m \), respectively. At these temperatures the thermal pressure ceases to exist, because there is sort of a phase transition to a run-away and therefore inexistent broken phase.
At exactly $T = T_{\text{crit}}$, we have a situation which is closest to gauge theories in 4 dimensions, because there $m_J = 0$, corresponding to a vanishing screening mass as is the case for the magnetostatic modes in perturbation theory. The interaction pressure is still well-defined and is given by the end-points of the various curves in figure 9. A conspicuous difference from the results of 4-dimensional gauge theories is that the interaction pressure is positive, which is related to the abnormal sign of all thermal mass corrections in our model.

5 Conclusions

We have pursued two purposes in our study of our six-dimensional scalar model. Firstly, we have investigated how precisely the nonperturbative formula for the thermal pressure proposed in reference 1 is rendered finite by standard zero-temperature renormalisation.

The renormalisation of the derivatives of the pressure with respect to the masses leads us to a problem of overlapping divergences which we solved in a manner analogous to Dyson’s method for QED, introducing a certain $2 \times 2$ scattering kernel. We derived the renormalised Dyson-type equations, which turned out to be essential for our discussion of renormalisability. Finally, we had to invoke Wilson’s operator product expansion. We think that we can draw the lesson from this that in more complicated theories like QCD things will not be simpler and one will again have to deal with overlapping divergence problems. On the other hand, having an expression for the pressure in terms of renormalised Green’s
functions as given for our model in (3.12d) and knowing the essential equations these Green’s functions must satisfy in order to have ultraviolet finiteness, may help to devise consistent approximation schemes leading to finite results in all orders. With these methods we should also be able to study explicitly the effects occurring when a particle—in our case the “photon”—becomes unstable. In the Green’s functions for \( T=0 \) this amounts simply to a pole moving from the first to the second Riemann sheet and thus we expect that our result for the pressure, which is expressed in terms of these functions, should not change drastically.

Besides these general aspects, we have investigated the large \( N \) limit of our model which diagrammatically is similar to QED in the limit of large flavour numbers. In this limit the leading contribution to the interaction pressure comes from a ring resummation of the photon polarization function, while the electron lines remain undressed. When \( N \) is not large, keeping only this contribution corresponds to what is known as random-phase approximation (RPA)\(^{[11]}\) in many-body physics. In contrast to the simpler ring resummation of the Debye screening mass\(^{[12]}\), one has to deal with a resummation of a momentum-dependent quantity. In practice, however, one usually aims at an (improved) perturbative scheme and uses this resummation only as far as needed to extract the next-to-leading order term in the interaction pressure, which because of the infrared singularities in the usual series is nonanalytic\(^{[13]}\) \(^{[14]}\) in \( \epsilon^2 \). Here we have found that the RPA can be interpreted as the leading term in a large-\( N \)-expansion and we have retained the full nonperturbative information that it incorporates.

In our six-dimensional scalar model we have in fact encountered rather drastic resummation effects, because this model has a critical temperature above which an instability similar to the gravitational Jeans instability occurs. So despite the diagrammatic similarity with QED, this theory is rather different from it. But below the critical temperature we were able to obtain a nonperturbative expression for the interaction pressure, and evaluate it numerically. Right at the critical temperature, where the nonperturbative interaction pressure is still well-defined, the spectrum of our model is even rather similar to that of perturbative four-dimensional gauge theories in that it has a vanishing screening mass like the magnetostatic modes.

The computation of the nonperturbative interaction pressure in more realistic theories such as ordinary QED in the limit of large flavour numbers would be technically not too different from what we have done here. Similar simplifications seem to be of interest even in QCD in the small-\( N_c \) large-\( N_f \) limit\(^{[15]}\).

We plan to investigate those theories along the above lines in a separate work.

**Appendix**

In this appendix we show how the \( 1/N \) expansion of the general formula (3.12d) leads us to the results of section 2. In the first step we perform all the traces implicit in (3.12d). For this we define:

\[
\hat{K}_{jLR}(q,p) = i2^{2-j-l}(\text{tr}_q \text{tr}_p)(K_{jLR}(q,p)), \tag{A.1}
\]
where \( \text{tr}_q (\text{tr}_p) \) means the trace over the internal indices of the lines where the momentum \( q \) (\( p \)) flows (figure 10). We also define the differential forms:

\[
\omega_j(p) = \sum_l \frac{d^m p}{d m_l^2} \hat{\Gamma}_{ij R}(p^2),
\]

which can be expressed through the self-energy functions using the renormalised version of (3.9c):

\[
\omega_j(p) = \sum_l \left\{ -\frac{d^m p}{d m_l^2} \left[ p^2 - m_j^2 - \Pi_j(p^2) \right] + \left[ p^2 - m_j^2 - \Pi_j(p^2) \right] \frac{d^m p}{d m_l^2} \log \frac{Z_j}{Z_j} \right\} \tag{A.3}
\]

Here \( \tilde{Z}_j \) are the wave function renormalisation constants for the theory with the coupling \( \lambda \), but masses \( \mu_j \) (compare the paragraph following (3.11e)). We find then for the pressure from (3.12d)

\[
\sum_j \frac{d^m p}{d m_j^2} P(T) = -\int (2\pi)^n \left\{ \omega_1(q) 2N \Delta_{1TR}^{11}(q) + \omega_2(q) \frac{C}{2} \Delta_{2TR}^{11}(q) \right. \\
- \left. i \int \frac{d^m p}{(2\pi)^n} \sum_{j,l} \omega_j(q) \left[ q^2 - m_j^2 - \Pi_j(q^2) \right]^{-2} \hat{K}_{jIR}(q,p) \Delta_{1TR}^{11}(p) \right\} \tag{A.4}
\]

This is the starting point for the \( 1/N \) expansion. For any (scalar) quantity \( F \) we write \( F = \sum_r (r)F \) with \( (r)F \) the term of order \( (1/N)^r \). We get then from (A.4) for the term of order \( N \)

\[
\sum_j \frac{d^m p}{d m_j^2} \left( -1 \right) P(T) = -2N \int \frac{d^m q}{(2\pi)^n} \omega_1(q) \cdot \Delta_{1TR}^{11}(q) \tag{A.5}
\]

\[
= -2N \omega_1(q) \int \frac{d^m q}{(2\pi)^n} 2\pi \delta(q^2 - m_j^2).
\]

Therefore, to order \( N \) we get the pressure from the free electrons and positrons of mass \( m_1 \), that is \( P^{(1)}_0(T, m_1^2) \) of (2.10):

\[
(-1) P(T) = P^{(1)}_0(T, m_1^2). \tag{A.6}
\]

To order \( N^0 \) we get from (A.4) after some work, and using the fact that the leading contributions to \( K \) are single-particle exchanges:

\[
\sum_j \frac{d^m p}{d m_j^2} \left( 0 \right) P(T) = -\int \frac{d^m q}{(2\pi)^n} \left\{ 2N \left[ -\sum_j \frac{d^m p}{d m_j^2} \Pi_1(m_1^2) \right. \\
+ \left. \frac{d^m p}{d m_j^2} \right] \Delta_{1TR}^{11}(q) \right. \\
+ \left. 2N \Delta_{1TR}^{11}(q) - \frac{1}{2} C \Delta_{2TR}^{11}(q) \right\} \tag{A.7}
\]
where, with the notation of section 2,

\[ (1) \Pi_1(q^2) = \frac{i g^2}{N} C \int \frac{d^nk}{(2\pi)^n} \left[ (q - k)^2 - m_1^2 \right]^{-1} \left[ k^2 - m_2^2 - g^2 \pi (k^2, m_2^2) \right]^{-1}, \]  
(A.8)

\[ (0) \Pi_2(q^2) = g_0^2 \pi (q^2), \]  
(A.9)

\[ (0) Z_2 = \left[ 1 - g_0^2 \pi' (m_2^2) \right]^{-1}. \]  
(A.10)

The unrenormalised quantities \( \Pi_{1,2}, \ Z_2, \ g_0 \) occurring in (A.7) are to be understood as functions of the renormalised parameters. Of course the individual terms in (A.7) contain divergences, which must cancel in the sum as we know from sections 2,3 and as we will again see explicitly below.

Now we use the general sum rule

\[ \int \frac{dq^0}{q^2} \left[ \hat{\Delta}^{11}_{jTR}(q) - \hat{\Delta}^{12}_{jTR}(q) \right] = 0 \]  
(A.11)

which we already mentioned in connection with (1.1d), to express the integrals over \( \hat{\Delta}^{11} \) in (A.7) by integrals over \( \hat{\Delta}^{12} \) which in turn is related to the imaginary part of the thermal propagators. In this way we obtain

\[ \sum_j dm_j^2 \frac{\partial}{\partial m_j^2} (0) P(T) = \eta_1 + \eta_2 + \eta_3, \]  
(A.12)

where the differential forms \( \eta_{1,2,3} \) are given by

\[ \eta_1 = 2N \int \frac{d^nq}{(2\pi)^n} \left[ \sum_j dm_j^2 \frac{\partial}{\partial m_j^2} (1) \Pi_1(k^2) \right]_{k^2 = m_1^2} n(q)2\pi\delta(q^2 - m_1^2) \]  
(A.13)

\[ \eta_2 = 2N dm_1^2 \int \frac{d^nq}{(2\pi)^n} \text{Im} \left\{ \frac{1 + 2n(q)}{(q^2 - m_1^2)^2} (1) \Pi(T)(q) - \frac{1}{(q^2 - m_1^2)^2} (1) \Pi_1(q^2) \right\} \]  
(A.14)

\[ \eta_3 = \frac{1}{2} C \left[ dm_2^2 - g^2 \frac{\partial \pi(m_3^2)}{\partial m_1^2} dm_2^2 \right] \int \frac{d^nq}{(2\pi)^n} \text{Im} \left[ \frac{1 + 2n(q)}{q^2 - m_2^2 - g^2 \pi T(q, m_2^2)} - \frac{1}{q^2 - m_2^2 - g^2 \pi (q^2, m_2^2)} \right] \]  
(A.15)

with

\[ (1) \Pi_1(q^2) = (1) \Pi_1(q^2) - (1) \Pi_1(m_1^2) - (q^2 - m_1^2)(1) \Pi'_1(m_1^2) \]  
(A.16)

\[ (1) \Pi(T)(q) = (1) \Pi_1(q^2) \]

\[ + \frac{g^2 C}{N} \int \frac{d^nk}{(2\pi)^n} \left\{ \frac{i}{(q - k)^2 - m_1^2} + n(q - k)2\pi\delta((q - k)^2 - m_1^2) \right\} \left[ k^2 - m_2^2 - g^2 \pi T(k, m_2^2) \right]^{-1} \]

\[ - \frac{i}{(q - k)^2 - m_1^2} \left[ k^2 - m_2^2 - g^2 \pi(k^2, m_2^2) \right]^{-1} \]

\[- i \left[ - \frac{in(k)}{(q - k)^2 - m_1^2} + 2\pi\delta((q - k)^2 - m_1^2) \left\{ n(k) \left( \theta(q^0)\theta(k^0 - q^0) + \theta(-q^0)\theta(q^0 - k^0) \right) + n(q - k) \left( \theta(q^0)\theta(-k^0) + \theta(-q^0)\theta(k^0) \right) \right\} \right] 2 \text{ Im} \left[ k^2 - m_2^2 - g^2 \pi T(k, m_2^2) \right]^{-1} \]  
(A.17)
Here \((1)\bar{\Pi}_1(q^2)\) is the renormalised electron self-energy function to order \(1/N\) and \((1)\bar{\Pi}_{1T}(q)\) is the corresponding thermal function.

The forms \(\eta_1\) and \(\eta_3\) have already a simple structure. The form \(\eta_2\) which arises from the \(1/N\) term \((1)\Delta_{1 FR}^{11}\) of the electron propagator in (A.7) is more difficult to handle. The strategy is to insert (A.16) and (A.17) in (A.14), which leads to integrals over \(q\) and \(k\), and then to perform first the \(q\)-integration. In this way we get after some nontrivial calculations:

\[
\eta_2 = 2N \int \frac{d^4k}{(2\pi)^4} n(k)2\pi\delta(k^2 - m_1^2) \\
\quad + \frac{C}{2} g^2 \int \frac{d^4k}{(2\pi)^4} \text{Im} \left\{ (1 + 2n(k)) \left[ k^2 - m_2^2 - g^2\bar{\pi}_T(k,m_2^2) \right]^{-1} \frac{\partial}{\partial m_1^2} \bar{\pi}_T(k) \right\}
\]

Putting now everything together, we arrive at:

\[
\sum_j d^4m_j \frac{\partial}{\partial m_j^2} P(T) = -\sum_j d^4m_j \frac{\partial}{\partial m_j^2} \left[ (1)\Pi_1(m_1^2) \frac{\partial P_0^{(1)}(T,m_1^2)}{\partial m_1^2} + C \right]
\]

In the first term on the right hand side of (A.18a) we can use:

\[
(1)\Pi_1(m_1^2) \frac{\partial P_0^{(1)}(T,m_1^2)}{\partial m_1^2} = \frac{1}{2} \left[ \frac{\partial P_0^{(1)}(T,m_1^2)}{\partial m_1^2} + \text{c.c.} \right]
\]

This is how the function \(\bar{\pi}_T(q)\) (2.11d) appears in the present way of calculating the pressure.

The last relation we need is the one between the coupling constants \(g\) and \(\tilde{g} = \lambda \sqrt{N}\) (compare the paragraph before (3.11a).) To order \(N^0\) we have

\[
g^2 = \tilde{g}^2 \left[ 1 + \tilde{g}^2 \left\{ \Pi'(\mu_2^2) - \pi'(m_2^2) \right\} \right]^{-1},
\]

where \(\pi(q^2)\) is defined as \(\pi(q^2)\) in (2.12a), but with mass \(\mu_1\) instead of \(m_1\). From (A.19) we find for the derivatives of \(g\) with respect to \(m_j^2\) and \(\tilde{g}\) fixed:

\[
\frac{\partial g^2}{\partial m_j^2} = (g^2)^2 \frac{\partial}{\partial m_j^2} \left[ \pi'(m_j^2) \right], \quad (j = 1, 2).
\]
From this we derive easily
\[
\frac{\partial}{\partial m_1^2} \left[ g^2 \tilde{\pi}_T(k, m_2^2) \right] = g^2 \frac{\partial}{\partial m_1^2} \left[ \pi_T(k) - \pi(m_2^2) \right]
- \left[ k^2 - m_2^2 - g^2 \tilde{\pi}_T(k, m_2^2) \right] g^2 \frac{\partial}{\partial m_1^2} \pi'(m_2^2),
\]
(\textit{A}.21)
\[
\frac{\partial}{\partial m_2^2} \left[ g^2 \tilde{\pi}_T(k, m_2^2) \right] = -g^2 \left[ k^2 - m_2^2 - g^2 \tilde{\pi}_T(k, m_2^2) \right] \pi''(m_2^2)
\]
(\textit{A}.22)

Collecting everything together we get now in a straightforward way:
\[
\sum_j dm_j^2 \frac{\partial}{\partial m_j^2} P^{(0)}(T) = \sum_j dm_j^2 \frac{\partial}{\partial m_j^2} P^{(0)}(T) \text{int}
- C \frac{dm_2^2}{2} \int \frac{dm k}{(2\pi)^n} n(k) 2\pi \delta(k^2 - m_2^2),
\]
(\textit{A}.23)

where \(P^{(0)}(T)\text{int}\) is the interaction pressure as in (2.17) and the last term on the right-hand side is
\(dm_2^2\) times the derivative with respect to \(m_2^2\) of the pressure of the free photons of renormalised mass \(m_2\). In deriving (\textit{A}.23) we have made use of the relations
\[
\text{Im} \, \pi''(m_2^2) = 0,
\]
\[\text{Im} \, \frac{\partial}{\partial m_2^2} \pi'(m_2^2) = 0
\]
(\textit{A}.24)

which are valid for stable photons and electrons, that is for
\[
0 \leq m_2^2 < 4 m_1^2
\]
(\textit{A}.25)

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