A Fast and Robust Method for Global Topological Functional Optimization

Elchanan Solomon*  Alexander Wagner*  Paul Bendich
Department of Mathematics, Duke University  Department of Mathematics, Duke University  Department of Mathematics, Duke University
Durham, USA  Durham, USA  Durham, USA
yitzchak.solomon@duke.edu  alexander.wagner@duke.edu  paul.bendich@duke.edu

Abstract—Topological statistics, in the form of persistence diagrams, are a class of shape descriptors that capture global structural information in data. The mapping from data structures to persistence diagrams is almost everywhere differentiable, allowing for topological gradients to be backpropagated to ordinary gradients. However, as a method for optimizing a topological functional, this backpropagation method is expensive, unstable, and produces very fragile optima. Our contribution is to introduce a novel backpropagation scheme that is significantly faster, more stable, and produces more robust optima. Moreover, this scheme can also be used to produce a stable visualization of dots in a persistence diagram as a distribution over critical, and near-critical, simplices in the data structure.

I. INTRODUCTION

In its early days, topological data analysis (TDA) was viewed as being in competition with other methods and models in data science, and much of TDA research proceeded independently from the state-of-the-art in the machine learning space. In recent years, however, the role of TDA as a component in a larger data analysis pipeline has come to the forefront. One can divide the literature into the following streams:

1) Using TDA to extract features from data that are then fed into standard machine learning or statistics pipelines. Cf. Bendich et al. [BMM+16], Brown and Knudson [BK09], and Gamble and Heo [GH10].

2) Using topological signatures as measures of model complexity. Cf. Gebhart et al. [GSH19], Guss and Salakhutdinov [GS18], Carneu et al. [CMEM19], and Rieck et al. [RTB+18].

3) Designing neural network architectures that can handle topological signatures. Cf. the PersLay architecture of Carrière et al. [CCI+19].

4) Incorporating topological terms into classical loss functions. Cf. Chen et al. [CNBW18] and Hu et al. [HLSC19].

As this work belongs to the final stream above, let us consider the prior work in greater depth. In [CNBW18], Chen et al. propose adding a regularizer term to the loss function of a complex model that penalizes the topological complexity of the decision boundary. They introduce an efficient algorithm for computing the gradients of this topological penalty, implement it in conjunction with a standard kernel classifier, and demonstrate improved results for both synthetic and real-world datasets. The computational tractability of their approach comes from the fact that the homological dimension of interest is zero, where persistence computations are particularly fast.

In [HLSC19], Hu et al. propose a novel framework for building a neural network that maps an image to its segmentation. They introduce a topological loss into the training phase by asking that the model output approximate the ground truth segmentation in a metric that combines cross-entropy and the 2-Wasserstein distance on persistence diagrams. To alleviate the instability of topological backpropagation and the relatively expensive computational cost of persistent homology, their framework works with one single, small patch of the image at a time. Experiments on natural and biomedical image datasets demonstrate that the incorporation of topology provides quantitatively superior results across a host of measures.

The focus of this paper is not classification or segmentation, but topological functional optimization. That is, our goal is to optimize a functional on the space of images that has both a classical, machine-learning component (approximating a fixed, input image in mean squared error (MSE), cross entropy, etc.) and a topological component, e.g. \( \alpha \Phi(PD(f)) + (1 - \alpha) \text{MSE}(f, f_0) \). Our proposed framework is naturally unsupervised and can accept a wide variety of user-specified functionals. Here is a sample list of useful image optimization tasks covered by our framework:

- **Topologically accurate signal downsampling.** It is often prohibitively expensive to transmit large signals. One can cast downsampling as the problem of mapping a signal into a lower-dimensional space (either a shorter signal or a signal belonging to a simple parametrized family) while minimizing some measure of distortion. By incorporating a topological loss into this optimization task, we can ensure that our downsampling preserves key structural
• Image simplification. Topological data can be used as a measure of image complexity. By defining a functional that penalizes small-scale topological features, we obtain a scheme for removing topological noise from images. Cf. Edelsbrunner et al. [EMP06].

• Enforcing correct topology. In contrast with the prior example, there are settings in which we believe an image ought to exhibit particular topology at a given scale, such as a certain number of connected components, the existence of a cycle or void, etc. By defining a functional that penalizes distance from this prescribed topology, we can produce a modified image with the correct topology.

The challenges of topological functional optimization are three-fold: (1) topological gradients are very expensive to compute, (2) the mapping from topological gradients to ordinary gradients (defined on the image space) is extremely unstable, and (3) topology can be made or broken by changing individual pixels, so the optima produced via straightforward gradient descent are fragile. To address these difficulties, we introduce a novel topological backpropagation scheme that is faster, more stable, and produces more robust optima than traditional methods.

A. Outline of Paper

Section II reviews the literature on topological backpropagation and demonstrates how gradients in the space of persistence diagrams can be pulled back to produce gradients on the original data structure. This is followed by an analysis that explains why the traditional method is unstable, slow to compute, and produces undesirable optima. In Section III, we introduce our novel approach to topological backpropagation via smearing the topological loss. This smearing procedure requires computing the topological statistics of many approximates of our data. We then introduce STUMP, our scheme for quickly generating these approximates, combining their gradients, and further stabilizing the result. In Section IV, we consider three synthetic optimization tasks and compare the results of “vanilla” topological optimization with our new, smearing approach. We perform a robustness and speed analysis, demonstrating that our method outperforms the traditional one in both metrics. In Section V we discuss how our pipeline can be used to provide stable visualizations of dots in a persistence diagram and a simple generalization of our pipeline to point cloud data.

B. Persistent Homology

The content of this paper assumes familiarity with the concepts and tools of persistent homology. Interested readers can consult the articles of Carlsson [Car09] and Ghrist [Ghr08] and the textbook of Edelsbrunner and Harer [EH10].

II. TOPOLOGICAL BACKPROPAGATION

The incorporation of persistent homology into model training is based on three properties of persistent homology:

• (semantic) Topological data can be used to measure a host of important, yet abstract, concepts in data analysis: noise, connectivity, consistency, shape, boundary, scale, dimension, etc.

• (computational) Persistent homology can be computed efficiently.

• (submersion) The differential of the map from model parameters to persistence diagrams has full rank almost everywhere. In the language of differential topology, such a map is called a submersion.

It is this final, submersion property that allows for the backpropagation of topological gradients to gradients in the parameter space of the model. We now make this backpropagation scheme precise. Let our model $g$ be parameterized by a set of parameters $\alpha_j$, let $\{b_i, d_i\}_{i \in T}$ be the set of birth-death times of the persistence diagram $PD(g)$, and let $\Phi$ be a functional on the space of persistence diagrams. In order to optimize $\Phi$ as a function of the model parameters $\alpha_j$, we need to be able to compute the partial derivatives:

$$\frac{\partial b_i}{\partial \alpha_j} \quad \text{and} \quad \frac{\partial d_i}{\partial \alpha_j}.$$

To compute these derivatives, one can take advantage of the pairing between birth and death times in $PD(g)$ and critical simplices of $g$ (this pairing is well-defined in the generic setting that all critical simplices have distinct function values). When using a lower-star filtration, one can further simplify this pairing by choosing, for each critical simplex, the vertex whose additional to the filtration implied the addition of the critical simplex (there is a unique such vertex under the generic assumption that all vertex values are distinct). Let us therefore write $\pi$ to identify this mapping from birth or death times to vertices of our discretized domain $D$. Note, crucially, that $\pi$ is locally constant with respect to perturbations of the function $g$.

With such a pairing, we can write the partial derivatives above in a more tractable form:

$$\frac{\partial b_i}{\partial \alpha_j} = \frac{\partial g(\pi(b_i))}{\partial \alpha_j} = \frac{\partial g}{\partial \alpha_j}(\pi(b_i))$$

$$\frac{\partial d_i}{\partial \alpha_j} = \frac{\partial g(\pi(d_i))}{\partial \alpha_j} = \frac{\partial g}{\partial \alpha_j}(\pi(d_i))$$

We can thus use the chain rule to deduce:

$$\frac{\partial \Phi}{\partial \alpha_j} = \sum_{i \in I} \frac{\partial \Phi}{\partial b_i} \frac{\partial b_i}{\partial \alpha_j} + \frac{\partial \Phi}{\partial d_i} \frac{\partial d_i}{\partial \alpha_j}$$

$$= \sum_{i \in I} \frac{\partial \Phi}{\partial b_i} \frac{\partial g}{\partial \alpha_j}(\pi(b_i)) + \frac{\partial \Phi}{\partial d_i} \frac{\partial g}{\partial \alpha_j}(\pi(d_i))$$

The partial derivatives $\frac{\partial \Phi}{\partial b_i}$ and $\frac{\partial \Phi}{\partial d_i}$ must be computed explicitly for the functional $\Phi$ of interest. In [PSO18], closed-forms of these derivatives are given for the special cases when $\Phi$ is the bottleneck or 2-Wasserstein distance to a target persistence diagram.

Lastly, it is worth noting that topological backpropagation gives the user the freedom to treat birth and death simplices separately. This is often desired in practice, as will be seen in the experiments in Section V.
A. Instability of Topological Backpropagation

The method of topological backpropagation via persistence dot-critical vertex pairings has a number of limitations. The most crucial is that of instability. It is a well-known result in applied topology that persistence diagrams themselves are stable to perturbations of the underlying function, cf. Cohen-Steiner et al. [CSEH07] and Chazal et al. [CCSG+09]. However, no such stability applies to the location of critical vertices. This failure of stability was investigated by Bendich et al. [BBWT19] and presents itself as a challenge to many topological inverse problems. In our setting, when implementing topological backpropagation in functional optimization, this translates into unstable gradients.

B. Computational Cost of Topological Backpropagation

Software like GUDHI [Dlo20] provides for fast calculation of persistence dot-critical vertex pairings. In principle, computing this pairing is no more expensive than computing persistence. To see why, let \( f : D \to \mathbb{R} \) be our function, and let us assume that we are in the generic setting that \( f \) is injective on the vertices of \( D \). Define \( F : D \to \mathbb{N} \) to map every vertex to a natural number representing the ordinal position in which it appears in the filtration induced by \( f \). Thus, the vertex with the lowest \( f \)-value is mapped to zero, the vertex with the second \( f \)-value is mapped to one, and so forth. The birth and death times of the dots in the resulting persistence diagram \( PD(F) \) give the indices of their critical vertices. To transform \( PD(F) \) into \( PD(f) \), replace the birth and death indices with the \( f \)-values of the corresponding vertices.

Morozov [Mor05] showed that the worst-case complexity of computing persistence is cubic in the number of simplices. When our simplicial complex \( D \) is a triangulation of a \( k \)-dimensional manifold, the number of simplices grows exponentially in \( k \) with the resolution of the triangulation. Thus, even for \( k = 2 \) and \( k = 3 \), the computation of persistence homology and the persistence dot-critical vertex pairings scales poorly in the resolution of the image. From a computational perspective, it is therefore ideal to compute as few high-resolution persistence diagrams as possible.

C. Robustness of Optima

When the output of topological optimization is perturbed, either unintentionally (in lossy communications) or intentionally (as in an adversarial attack), the topology of the image changes. A local optima is robust if the value of the topological function can only be increased by adding a substantial amount of noise, as measured in MSE. Because topological backpropagation pins the responsibility for a given topological feature on a single pixel, it tends to introduce or destroy topology in a very fragile way, as will become clear in the Experiments section.

III. Smearing Topological Optimization

Consider a topological optimization task where the loss is of the form \( L(f) = \alpha \Phi(PD(f)) + (1 - \alpha) \text{MSE}(f, f_0) \). We can make this loss function more robust by associating to every function \( f \) a set of approximate functions \( S(f) \), equipped with a measure \( \mu \), and replacing the loss with:

\[
L_S(f) = \alpha \int_{S(f)} \Phi(PD(g))d\mu(g) + (1 - \alpha) \text{MSE}(f, f_0).
\]

Thus, the topological term of our loss function measures the “average” topology of a set of approximates to \( f \), weighted via \( \mu \). Informally, we call this smearing the topological loss over the set of approximates.

A. Generating Approximates

We now introduce a general scheme for constructing sets of approximates, given a function \( f : D \to \mathbb{R} \) defined on a simplicial complex \( D \). The first ingredient is an open cover \( \mathcal{U} \) of \( D \). We downsample \( D \) by considering the Čech complex \( \check{C}(\mathcal{U}, D) \), see Figure III.1. Each vertex of \( \check{C}(\mathcal{U}, D) \) corresponds to an open set \( U_i \in \mathcal{U} \). In order to produce a function on the Čech complex, we need a rule for averaging the set of values \( \{ f(v) \mid v \in U_i^0 \} \) for each open set \( U_i \). To that end, we associate to each open set \( U_i \) the probability simplex \( \Delta_i \) on its set of vertices, \( U_i^0 \). That is, an element \( \omega_i \in \Delta_i \) is an assignment of nonnegative weights to the vertices in \( U_i^0 \) such that \( \sum_{v \in U_i^0} \omega_i(v) = 1 \). We write \( \omega = \{ \omega_i \} \) to denote a choice of element in \( \Delta_i \), for each \( i \), which thus gives rise to a function \( f_\omega \) on \( \check{C}(\mathcal{U}, D) \):

\[
f_\omega([U_i]) = \sum_{v \in V(U_i)} \omega_i(v)f(v).
\]

See Figure III.2. The value of \( f_\omega \) on a non-vertex simplex \( \sigma \in \check{C}(\mathcal{U}, D) \) is defined to be the maximum value of \( f_\omega \) on the vertices of \( \sigma \). To specify how the elements \( \omega_i \in \Delta_i \) are chosen, we pick a set of measures \( \mu = \{ \mu_i \} \), one for each \( \Delta_i \):

1) If \( \mu_i \) is an atomic measure, concentrated on the center of the simplex \( \Delta_i \), the resulting downsample associates to each open set \( U_i \) the average of the values of \( f \) on its vertices.

2) If \( \mu_i \) is a uniform measure on the zero-skeleton \( \Delta_i^0 \), a downsample \( \omega \) is obtained by randomly picking a vertex \( v \in U_i^0 \) and setting \( f_\omega([U_i]) = f(v) \).

3) If \( \mu_i \) is a uniform measure on the entirety of \( \Delta_i \), a downsample corresponds to taking a random, normalized linear combination of the values of \( f \) on the open set \( U_i \).

The set of approximates \( S(f) \) is the set of all downsampled functions \( f_\omega \) on the Čech complex, with the measure as chosen. Since the Čech complex is smaller than the original complex, the computation of individual persistence diagrams is accelerated.

B. The Smear Gradient

For a fixed weighting \( \omega \), the map \( f \to f_\omega \) is linear. The chain rule then implies that:

\[
\frac{d\Phi(PD(f_\omega))}{df} = \frac{d\Phi(PD(f_\omega))}{d\omega} \circ \omega.
\]

1The experiments in Section IV are actually computed using cubical complexes but are equivalent to the formalism here via the Freudenthal triangulation.
Under mild technical assumptions that allow us to move the gradient under the integral sign, we therefore have for $\alpha = 1$:

$$\frac{dL_S(f)}{df} = \int_\omega \left( \frac{d\Phi(PD(f_\omega))}{df_\omega} \circ \omega \right) d\mu(\omega).$$

This provides a simple formula for computing the gradient with respect to $f$ in terms of the gradients of the downsamples $f_\omega$ but is not exactly computable in practice, due to the high-dimensionality of the set $S(f)$ over which we must integrate. To approximate this integral, we can, at each step of the optimization, sample finitely many $f_\omega$ and compute an empirical average. An even faster approach, which we implement in practice, is to mirror stochastic gradient descent by considering a single downsampled $f_\omega$ at each descent step and mixing the gradients via momentum using Adam [KB14]. Taken altogether, we call our pipeline STUMP: Stochastic Topological Updates via Momentum and Pooling.

![Fig. III.1](image1.png)

**Fig. III.1:** On the left, we see a simplicial complex $D$ with an open cover $\mathcal{U}$. The corresponding Čech complex is shown on the right. When the open sets of $\mathcal{U}$, and all their possible intersections, are contractible, the topological type of $D$ is the same as that of the Čech complex; this is the well-known Nerve Theorem.

![Fig. III.2](image2.png)

**Fig. III.2:** This figure demonstrates how the values of the approximate function $f_\omega$ are obtained. For a given open set $U_i$, a $\mu_i$-randomly chosen weighting $\omega_i \in \Delta_i$ prescribes a linear combination of values in $U_i$.

### C. Clarke Subdifferentials

The robustness of the optimization scheme can be further improved by considering perturbations of the initial function $f$. Here we give two heuristic motivations for adding explicit noise. The first is the qualitative effect on results of optimization. For instance, the picture on the left of Figure III.3 is the result of an optimization procedure that added explicit noise to the input before stochastic downsampling while the picture on the right is the result in the absence of explicit noise.

The second argument involves the gradient sampling methodology [BLO05]. When minimizing an unstable function, a more robust search direction can be obtained by considering the minimum norm element of the convex hull of gradients of nearby points. More precisely, Lemma 2.1 of [BLO05] states that if $G$ is a compact convex subset of $\mathbb{R}^d$ and $g^* \in G$ is a minimum norm element of $G$, then $d^* = -g^*/\|g^*\|$ solves $\inf_{\|g\| \leq 1} \sup_{g \in G} \langle g, d \rangle$. In other words, $d^*$ is a minimax update direction.

![Fig. III.3](image3.png)

**Fig. III.3:** The effect of explicitly adding noise in generating approximates.

The space of perturbations that will be used in the experiments in Section IV is the cube $[-\epsilon, \epsilon]^d$, so $G$ will be the convex hull of gradients of points in $x + [-\epsilon, \epsilon]^d$. As a proxy for finding the minimum norm element of $G$, one can sample points $x_1, \ldots, x_m \in G$, compute gradients $g_i = \nabla \Phi(PD(x_i, \omega_i))$ at each of these points, and find the minimum norm element of $\text{conv}(g_1, \ldots, g_m)$, i.e.

$$\min \| \sum_{i=1}^m c_i g_i \|^2 \text{ subject to } c \in \Delta^{m-1}.$$ 

If the $g_i$ are pairwise orthogonal, the problem above has the simple solution $c_i := \|g_i\|^{-2}/(\sum_{i=1}^m \|g_i\|^{-2})$. If in addition the norm of each $g_i$ is equal, then each $c_i$ would equal $1/m$. In other words, under these two extreme assumptions, we may approximate a robust update direction by simply averaging nearby gradients. We tested the validity of these assumptions for a particular example, the starting point of the smear optimization for the blobs experiment in Section IV.

Figure III.4 shows the Gram matrix on the left and the values of the $c_i$’s defined above on the right when $m = 100$. Note that the Gram matrix is somewhat diagonal and the values of the $c_i$ fluctuate very tightly around $0.01 = 1/m$. The degree of orthogonality among the $g_i$’s corresponds to the degree of instability of persistence dot-critical vertex pairings. On the other hand, the stability of the $c_i$’s reflects the stability of persistence diagrams to perturbation.
Fig. III.4: The $ij$-entry of the matrix on the left is $\langle g_i, g_j \rangle$. The graph on the right shows the values of $c_1, \ldots, c_{100}$.

IV. EXPERIMENTAL RESULTS

We now consider a number of synthetic topological optimization tasks and compare the results with and without smearing. Our goal is to demonstrate that smearing greatly speeds up topological optimization, producing more robust optima. We have three experiments:

1) Double well: The image consists of two depressions, or wells, that have some overlap. The goal is to increase $H_0$ persistence and separate the wells. This is done by applying topological backpropagation to the critical vertices responsible for deaths in $H_0$, i.e. we create $H_0$ by raising a wall between the two wells, rather than making the wells lower.

2) Sampled circle: The image consists of a sum of Gaussians centered at points sampled from a circle. The goal is to increase $H_1$ and fill in the circle. This is done by applying topological backpropagation to the critical vertices responsible for births in $H_1$, i.e. we want to create $H_1$ by making the circle appear earlier, rather than raising the center of the circle.

3) Blobs: The image consists of some amorphous blobs connected by bridges at middling height. The goal is to decrease $H_0$, thereby better connecting the blobs. This is done by applying backpropagation to the critical vertices responsible for deaths in $H_0$, i.e. we want to decrease $H_0$ by deepening the bridges between them, rather than raising and flattening the blobs out.

For our three experiments: (a) The persistence region of interest was $[-\infty, \infty, 50, \infty]$ in birth-lifetime space, (b) The weighting $\alpha$ in the mixed-loss is $(1 - 1/P)$, where $P$ is the total number of pixels in the image. This balances the topological loss, whose gradient is supported on a relatively sparse set of pixels, and the MSE, whose gradient is supported on every pixel, (c) The learning rate is $5 \times 10^{-2}$, (d) We used the Adam optimizer [KB14] with 10000 steps, (e) Each pixel was perturbed independently by adding uniform noise in the range $[-\epsilon, \epsilon]$. The level of noise $\epsilon$ was 50 for both the well and blobs experiment and 100 for the circle experiment, (f) For smeared loss, the 1-Wasserstein norm was used to define the functional, although the 2-Wasserstein norm also gives good results. For vanilla topological backpropagation, the 2-Wasserstein norm was used, as the 1-Wasserstein optima were very poor, and tended not to adjust the topology at all, (g) GUDHI [Dlo20] was used for all persistence computations. (h) Downsampling was done using method 3 described in Section III-A.

The results can be seen in Figure IV.1. We see in all three examples that the optima produced by STUMP look more stable and match closely with our intuition for what the goal of the optimization task should be. What remains is to compare the robustness and speed of vanilla and smeared topological backpropagation.

Fig. IV.1: A comparison of vanilla and STUMP optima for three optimization tasks. The images go from 0 (black) to 255 (yellow) in value.

A. Robustness

One way to test robustness is to randomly perturb our optima and see how the topology changes. For each of our optima, we consider 50 perturbations obtained by adding pixel-wise uniform noise, with the resulting image clipped, so that pixels lie in the range $[0, 255]$. We then compute the $W^1$ and $W^2$ norms of the persistence dots in the region $[-\infty, \infty, 50, \infty]$, computing both the average and worst-case results at each noise level. See Figure IV.2.

As noise is introduced in the wells experiment, the non-essential connected component (the well on the right) is born earlier. In the vanilla optima, the wall separating the connected components is cut through, and the death time falls by the same amount, so that the overall persistence remains constant. In the smeared optima, the death time remains roughly constant, and hence the total persistence goes up, even in the worst-case perturbation.
For the circle experiment, we see that both optima have similar topological loss at $\epsilon = 0$. As noise is introduced, the death time of the persistent feature doesn’t change, as the max value of the image is clipped at 255. In the vanilla optima, the birth time drops as the bridges closing the circle are cut, causing a drop in total persistence. The smear optima is more robust, and the total persistence remains stable, even in the worst-case perturbation.

Lastly, in the blobs experiment, due to clipping the pixel intensities below at 0, the birth times of dots in the region $[-\infty, \infty, 50, \infty]$ do not go down. What drives up persistence, therefore, is a delayed death time due to fragile connections between regions. We see that at all noise levels greater than zero, the vanilla optima has more persistence than the smeared optima, demonstrating a lack of robustness.

**B. Speed**

In all the preceding examples, downsampling was performed by considering adjacent $k \times k$ patches of the original image and, for each patch, applying a random element $\omega$ of $\Delta^{k^2-1}$, chosen uniformly. Hence, the downsampled image contained $k^{-2}$ as many pixels as the original image. In the associated optimization, this replaces the computation of $\nabla \Phi(PD(f))$ with the faster computation of $\nabla \Phi(PD(f_\omega))$. Because of this, the vanilla wells, circle, and blobs experiments took 5015, 2169, and 3576 seconds, respectively, while STUMP took 202, 106, and 195 seconds.

Replacing the original gradient with a downsampled gradient certainly speeds up each step compared to vanilla optimization, but it remains to show that the loss function is reduced more rapidly. To this end, we now return to the third experiment regarding connecting blobs. Since the loss for this experiment consists of two non-negative terms, the mean squared error and the total persistence in a region, we may reasonably compare how quickly various types of optimization reduce the starting loss. In Figure IV.3, we consider four types and plot the percentage of the original loss reduced by each optimization procedure as a function of time. In red and green, we show vanilla topological optimization where total persistence is measured using $W^1$ and $W^2$, respectively. We then consider the addition of stochastic downsampling in orange, where total persistence is measured using $W^1$. Finally, in blue, we add explicit noise before downsampling the image. The graph on the left corresponds to the blobs image shown in Figure IV.1. Within four minutes, both versions of our procedure have reduced the loss by about 90% while the vanilla methods only manage to reduce around 25% of the loss after 10 minutes.

One possible explanation for the dramatic increase in loss reduction in the blobs experiment is the large degree of homogeneity of this image. The second graph in Figure IV.3 corresponds to an identical optimization scheme for a different image. This new image was generated by sampling uniform noise between 0 and 255. For this experiment, we see a less extreme increase in loss reduction afforded by our procedure over vanilla optimization.

**V. Extensions**

The methodology of smearing, and the STUMP pipeline, can also be applied to other settings and purposes.

**A. Critical Smears**

Strictly speaking, the method of topological optimization via smearing the loss function does not accomplish the task of topological backpropagation. That is, it works by considering gradients on many different persistence diagrams, as opposed to working with the gradient of the persistence diagram of the original function $f$. However, there is a way to use the ideas of smearing to this end as well, which we call **critical smearing**.

In critical smearing, we compute the gradient of the original topological loss $\Phi(PD(f))$, giving rise to a gradient on the
persistence diagram $\text{PD}(f)$. We then compute the persistence diagrams of many different functions of the form $(f + h)_{\omega}$, and transfer the gradient from $\text{PD}(f)$ to gradients on these approximate diagrams. We then pull back these transferred gradients to gradients on the Čech complex via persistence dot-critical vertex pairings, and finally back to gradients on $D$ via $\omega$, where the gradients are averaged to give a smeared gradient. If the initial gradient on $\text{PD}(f)$ is supported on a single dot, the resulting smeared gradient can be thought of as a fuzzy assignment of critical vertices for this dot.

There are many possible ways to define gradient transfer between persistence diagram. We propose that this step be accomplished via finding a matching between the dots of two persistence diagrams, and having points in one diagram inherit the gradients of the points they are matched with.

Fast matchings can be computed via the Sliced Wasserstein approximation of the Wasserstein distance (cf. Carrière et al. [CCO17]), and that is the approach we adopt here. Consider again the circle in Figure [V.1] first column, second row. When we add uniform noise in $[-50, 50]$ and subsequently downsample using $5 \times 5$ blocks, we obtain images as in Figure [V.1]. If we set our loss function $\Phi$ to penalize dots in the persistence diagram with lifetime greater than 30, add uniform noise in $[-50, 50]$, downsample via $5 \times 5$ blocks, sample 1000 times, and transfer gradients via Sliced Wasserstein (with 20 projections), the critical smear can be seen in Figure [V.2].

![Fig. V.1](image1)

**Fig. V.1:** Left: Circle with uniform noise added pixelwise. Right: Noisy image after pooling with $5 \times 5$ blocks.

**B. Point Clouds**

It is relatively straightforward to adjust the above pipeline for topological backpropagation on point clouds. Downsampling can be accomplished by randomly sampling a subset of points, and error can be modeled by randomly perturbing the location of each point independently.

**VI. CONCLUSION**

Our novel pipeline for topological optimization, STUMP, produces optima that are empirically more robust, and visually more intuitive, than the traditional method and with a considerably shorter computation time. The generalizability and parallelizability of gradient smearing opens the way to a host of promising interactions between applied topology and machine learning.

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![Visualizing Smear](image2)

**Fig. V.2:** Visualization of the critical smear corresponding to the underlying 1-dimensional circular feature. The birth cells are in red, and the death cells are in blue.
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