Discrimination between evolution operators

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Under broad conditions, evolutions due to two different Hamiltonians are shown to lead at some moment to orthogonal states. For two spin-$\frac{1}{2}$ systems subject to precession by different magnetic fields the achievement of orthogonalization is demonstrated for every scenario but a special one. This discrimination between evolutions is experimentally much simpler than procedures proposed earlier based on either sequential or parallel application of the unknown unitaries. A lower bound for the orthogonalization time is proposed in terms of the properties of the two Hamiltonians.

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I. INTRODUCTION

Quantum information is predicated on the preparation and identification of evolving quantum states; consequently, the determination of quantum states is of prime importance ([1], Chapter 6). Perfect determination is possible only for states that are known to be orthogonal [2, 3]; in the absence of such knowledge the efficiency of imperfect determination can be improved by making non-projective and/or collective measurements whose outcomes are associated with entangled states, or by exploiting the quantum mechanical interference between states [4]. Still, for perfect determination of non-orthogonal (pure or mixed) states an infinite number of state-copies are needed. Recently a probabilistic measure of the discriminatory process was proposed [5].

The above picture changes when discrimination is made between unitary operations; for these a finite number of runs of the unknown gates suffice [6]-[8], a fact which is by itself surprising, since the number of independent parameters entering the unitary operations is of the order of the square of the number of states (assuming the latter to be finite). What balances this circumstance, is that there is a choice of states on which the unitaries can be tested. It now seems safe to state that while in certain cases the entanglement property of this chosen state can aid in the discrimination, perfect discrimination is also possible without entanglement [8].

The present work addresses unitaries in the framework of the class of evolution operators and, in particular, those with time-independent Hamiltonians $H$, so that their form is $e^{-iHt}$. Here the time $t$ takes over in many senses the role of the (discrete) number $N$ of the unitary gates featured in the above mentioned works [6]-[8], but in terms of practical effort the automatically time-evolving unitary is much preferable to repeated subjection to gates. The aim of this work is thus the determination of a minimal discrimination time between two different evolution operators, with liberty in the choice of the starting state. We identify the discrimination time with the time (to be denoted by $t_\perp$) that two differently evolving states first become orthogonal. An algebraic treatment of the opposite issue, the recurrence of states under pulsed and other time dependent perturbations, was given in [9] and [10].

The relationship of this topic to time-orthogonality of states and to the rate of sweeping the Hilbert space is evident, the latter issue having already been solved for some states in [11]-[13] and endowed with geometrical interpretation in [14]-[15]. In these, lower bound expressions are given in terms of the average energy $\mathcal{E}$ and the mean deviation of the energy $\Delta\mathcal{E}$ of the state. The states under present consideration are multi-component pure states, for which the existence of orthogonality times is studied in section III. The obvious difference (noted in section IV) with the previous references is that we have two different (in general, non-commuting) Hamiltonians, rather than a single one. A single-qubit state example is fully worked out in section V. The lower bounds $t_{\parallel}$ for $t_\perp$ derived in section 6 by two different methods are supported by the solved example (of section V). The fastest orthogonalizing pair of Hamiltonians turns out to be a rather trivial couple (section VI C). A discussion in section VII is mainly devoted to entanglement.

II. TWO HAMILTONIANS

We consider two Hamiltonians $H_a$ and $H_b$ that give rise to the evolution operators $e^{-A(t)}$ and $e^{-B(t)}$. Here $A(t) = iH_at$ and $B(t) = iH_bt$, where $t$ is time. Physically, $H_a$ can be made different from $H_b$ by adding different excitation operators to the common system Hamiltonian (say, $H_0$). One can discriminate between the two Hamiltonians (or excitation mechanisms) if, for some initial state $\psi$, $e^{-B(t)}\psi$ is orthogonal to $e^{-A(t)}\psi$ at some orthogonality time (or times) $t = t_\perp$ [7]-[8]. This means

$$\langle \psi | e^{B(t_\perp)} e^{-A(t_\perp)} | \psi \rangle = 0.$$  (1)
III. DERIVATION OF ORTHOGONALITY

Since \( e^{-A(t)} \) and \( e^{B(t)} \) are both unitary, their product
\[
e^{C(t)} = e^{B(t)} e^{-A(t)}
\]
is also unitary and can therefore be brought to the form
\[\text{diag}(e^{\theta_1(t)}, e^{\theta_2(t)}, \ldots, e^{\theta_d(t)})\]
by a unitary transformation matrix \( M(t) \) in which all the \( \theta \)'s are real (\( d \) is the dimension of the system’s Hilbert space). When \( A(t) \) and \( B(t) \) are different (i.e. \( A(t) \) is not of the form: \( B(t) + \) a Hilbert-space scalar, which is possibly time dependent), not all the \( \theta \)'s are the same and \( M(t) \) differs from the unit matrix. (In the converse case, when \( A(t) \) and \( B(t) \) are not different, all \( \theta(t) \)'s are the same and \( M(t) \) is the unit matrix.) Suppose now that two \( \theta \)'s that differ have indices \( i \) and \( j \).

We now construct the state
\[
\psi(t) = M^{\dagger}(t) \psi(0, i, j; \alpha) / \sqrt{2}
\]
in which \( \psi(i, 0; j, \alpha) \) is a \( d \)-dimensional column vector whose \( i \)'th element is \( e^{i \alpha} = 1 \) and whose \( j \)'th element is \( e^{i \alpha} \) (\( \alpha \) being real), all other entries in the column being zero.

We now form the bracket
\[
\langle \psi | e^{B(t)} e^{-A(t)} | \psi \rangle = \langle \psi | M^{\dagger}(t) \text{diag}(e^{i \theta_1(t)}, e^{i \theta_2(t)}, \ldots, e^{i \theta_d(t)})
\times M(t) | \psi \rangle = \frac{1}{2}(e^{i \theta_i(t)} + e^{i \theta_j(t)})
\]

Since the \( \theta \)'s differ (i.e., when \( A(t) \) and \( B(t) \) are different), it is expected that for some \( t = t_\perp \) they will differ by \( \pi \) and the bracket will vanish.

The reason that the state \( \psi \) has only two non-zero components can be explained as follows. Since
\[
\langle \psi | \text{diag}(e^{i \theta_j(t)}) | \psi \rangle = \sum_j e^{i \theta_j(t)} | \psi_j \rangle^2
\]
and the numbers \( | \psi_j \rangle^2 \) form a discrete probability distribution, the set of possible values of \( \langle \psi | \text{diag}(e^{i \theta_j(t)}) | \psi \rangle \) forms a convex set in the complex plane, with the numbers \( e^{i \theta_j} \) being extremal points. Initially, this set is the singleton \( \{1\} \). With increasing time, the size of the set increases. Orthogonality is first achieved when the set includes the point 0. This happens when 0 hits one of the edges of the convex set. The edges correspond to \( \psi \) with at most two non-zero components.

IV. EVOLUTION-OPERATOR ANALOGY

Writing the (unitary) exponential \( e^{C(t)} \) introduced in equation (2) as \( e^{-i[t - \frac{C(t)}{2}]_+} \), the quantity in the square bracket is not time independent. This is seen by recalling the Campbell-Baker-Hausdorff formula for \( C(t) \equiv \ln[e^{B(t)} e^{-A(t)}] \) that contains commutators of \( A(t) \) and \( B(t) \) to arbitrary order. However, to the zeroth order of commutators, or equivalently for the linear-in-\( t \) approximation for \( C(t) \), this quantity is a "Hamiltonian", it being the difference of two Hamiltonians. To this order, therefore, \( e^{C(t)} \) is an evolution operator. The speed at which a Hamiltonian evolution operator develops a state into one or several orthogonal states has been studied previously (e.g., [11] and references therein). One result is that the time \( t_\perp \) at which an initial state (starting at zero energy) evolves to an orthogonal state (in units for which \( \hbar = 1 \)) is given by
\[
t_\perp \geq \frac{\pi}{2 \bar{E}}
\]
where \( \bar{E} \) is the average energy.

The use of this (approximate) measure can be employed in various situations, for bipartite and multipartite systems (entangled and otherwise). We shall also use it in the following illustrative example.

V. ILLUSTRATION: A QUBIT

Let us write out the Hamiltonians for the evolutions \( a \) and \( b \) explicitly,
\[
H_a = r_0 a I + \omega_a \sum_{k=x,y,z} r_{ia} \sigma_k,
\]
\[
H_b = r_0 b I + \omega_b \sum_{k=x,y,z} r_{ib} \sigma_k.
\]

Then according to Sec. II, we can write \( A(t) = iH_a t \) and \( B(t) = iH_b t \), where \( t \) denotes time. The formula which we intend to verify is Eq. (1), namely, for any given pair of \( H_a \) and \( H_b \) we choose \( t = t_\perp \) appropriately so that Eq. (1) is fulfilled. Next we omit the \( r_0 \) part from Eq. (8) since this induces merely a global phase shift in the evolution of the state, which has no effect on the left hand side of Eq. (1).

The problem is an analog of a spin-1/2 particle precessing with frequency \( \omega_a \) for a time \( t = t_\perp \) in a magnetic field pointing in the direction \( \vec{r}_a = [r_{xa}, r_{ya}, r_{za}] \) (where the corresponding Larmor frequency \( \omega_a \) is proportional to the magnetic field strength) and subsequently precessing about \( \vec{r}_b = -[r_{xb}, r_{yb}, r_{zb}] \) for the same time interval \( t = t_\perp \) with a frequency \( \omega_b \). \( |r_{a}^\perp| = |r_{b}^\perp| = 1 \). The question is, whether the final state \( \psi(t) \equiv \psi(t = 2t_\perp) \) becomes orthogonal to the initial state \( \psi \equiv \psi(t = 0) \) for a suitable \( t_\perp \). Visualizing this situation within the Poincare-sphere description, we ask whether the resulting state \( \psi \) points to an antipodal point on the sphere, meaning \( \langle \psi_f | \psi_i \rangle = 0 \). Thus the problem can be converted to a problem of purely geometrical origin.

Nevertheless, let us remain at the quantum-mechanical level, and express the rotation matrices \( \exp(-iH_u t) \) and \( \exp(iH_b t) \) by Pauli matrices, by applying the following
identity:

\[ R_r (\theta) \equiv \exp(-i \theta \vec{r} \vec{\sigma}/2) = \cos(\theta/2) \mathbb{I} - i \sin(\theta/2)(r_x \sigma_x + r_y \sigma_y + r_z \sigma_z), \quad (9) \]

where the operator \( R_r (\theta) \) in the Poincare picture defines a rotation by \( \theta \) about the axis \( \vec{r} \).

After some algebra we can express the product of the two operators \( R_{\vec{r}_a}(\theta_a) \) and \( R_{\vec{r}_b}(-\theta_a) \) by one operator \( R_{\vec{r}_{ab}}(\theta_a) = R_{\vec{r}_a}(\theta_a)R_{\vec{r}_b}(-\theta_a) \) [16], where the overall angle \( \theta_{ab} \) and the new axis \( \vec{r}_{ab} \) are given by

\[ \cos \left( \frac{\theta_{ab}}{2} \right) = \cos(-\theta_a/2) \cos(\theta_b/2) - \sin(-\theta_a/2) \sin(\theta_b/2) \vec{r}_a \cdot \vec{r}_b, \quad (10) \]

\[ \sin \left( \frac{\theta_{ab}}{2} \right) \vec{r}_{ab} = \sin(-\theta_a/2) \cos(\theta_b/2) \vec{r}_a + \cos(-\theta_a/2) \sin(\theta_b/2) \vec{r}_b - \sin(-\theta_a/2) \sin(\theta_b/2) \vec{r}_a \times \vec{r}_b. \quad (11) \]

Let us relabel the Poincare-sphere so that \( \vec{r}_{ab} \) points to the north pole. In this case, the imaginary part of equation (9) is simply \(-i \sin(\theta_a/2) r_{ab} \sigma_z \) and in order to satisfy the imaginary part of the orthogonality condition in the form

\[ \langle \psi | \sigma_z | \psi \rangle = 0, \quad (12) \]

Then \( \langle \psi | \sigma_z | \psi \rangle = 0 \).

We now turn to the second task: to nullify the contribution from the real part of equation (9). Let us call \( \gamma \) the angle between \( \vec{r}_a \) and \( \vec{r}_b \). Then we have in equation (10) \( \vec{r}_a \cdot \vec{r}_b = \cos \gamma \). With equation (10) in hand we may write the real part of the orthogonality condition in the form

\[ 0 = \cos(-\theta_a/2) \cos(\theta_b/2) - \sin(-\theta_a/2) \sin(\theta_b/2) \cos(\gamma). \quad (13) \]

With further algebra and observing that in our case \( \theta_a = 2 \omega_a t \) and \( \theta_b = 2 \omega_b t \), we arrive at the form

\[ 0 = \cos^2 \frac{\gamma}{2} \cos(\omega_a - \omega_b) t + \sin^2 \frac{\gamma}{2} \cos(\omega_a + \omega_b) t. \quad (14) \]

Let us pick from the right hand side (RHS) of this equation the term with greater amplitude. At \( t = 0 \) the RHS is equal to 1. Then if \( \cos^2 (\gamma/2) > \sin^2 (\gamma/2) \), at \( t = t_{\perp}^{\max} = |\pi/(\omega_a - \omega_b)| \) or else (by the converse inequality) at \( t = t_{\perp}^{\max} = \pi/(\omega_a + \omega_b) \) the expression on the right must be negative. Thus by continuity argument there must be a point \( (t_{\perp}) \) between \( t = 0 \) and \( t = t_{\perp}^{\max} \) where the RHS of Eq. (14) becomes zero. However if \( \cos^2 (\gamma/2) > \sin^2 (\gamma/2) \) that is \( \gamma < \pi/2 \) and \( \omega_a = \omega_b \), \( t_{\perp}^{\max} \) becomes infinite, and the condition \( \langle \psi | \psi \rangle = 0 \) is not guaranteed to be fulfilled.

This result has the following implications. If \( \omega_a \neq \omega_b \), then the Hamiltonians \( a \) and \( b \) could be perfectly discriminated. However, if \( \omega_a = \omega_b \) and the angle \( \gamma \) between \( \vec{r}_a \) and \( \vec{r}_b \) is smaller than \( \pi/2 \) then Eq. (1) cannot be fulfilled for any finite time \( t_{\perp} \). When \( \omega_a \neq \omega_b \) but \( \vec{r}_a = \vec{r}_b \), the two Hamiltonian operators are different (by the definition given) and they can still be perfectly distinguished within a finite time \( t \), although they commute. In [7, 8] it is the run number \( N \), which is broadly equivalent to the orthogonality time in the present work, that tends to infinity when the two unitaries become the same. [A formal expansion of equation (14) for short times gives

\[ t^2 = \frac{8}{\omega_a^2 + \omega_b^2 - 2 \omega_a \omega_b \vec{r}_a \cdot \vec{r}_b} \quad (15) \]

and shows this tendency (but the short time expansion is no longer valid)]

A. Evolution time approach

The difference Hamiltonian, which as noted previously, can be viewed as approximately guiding the initial state to an orthogonal state, is

\[ H_a - H_b = (\omega_a \vec{r}_a - \omega_b \vec{r}_b) \cdot \vec{\sigma}. \quad (16) \]

This can be diagonalized to give the average energy \( \bar{E} \) of the two eigen-states (with the lower state being placed at zero energy) given by

\[ \bar{E} = \sqrt{\omega_a^2 + \omega_b^2 - 2 \omega_a \omega_b \vec{r}_a \cdot \vec{r}_b}. \quad (17) \]

We first show results for a case when the evolution time approach is expected to give orthogonality times identical to the unitary operator orthogonality times. This will
be the case when the two magnetic fields are perfectly aligned:

$$\vec{r}_a \cdot \vec{r}_b \equiv \cos \gamma = 1, \quad \gamma = 0. \quad (18)$$

The fields’ Larmor frequencies still differ,

$$\frac{\omega_a - \omega_b}{\omega_a + \omega_b} \equiv r \neq 0 \quad (19)$$

so that while the $a$ and $b$ Hamiltonians commute, the evolution operators are still different. The computed orthogonality times ($t_\perp$), obtained from solution of equation (14), as function of the normalized frequency differences $r$ are shown in Figure 1. When we compare them to the approximate (lower bound) expression given in the RHS of the inequality (5), using the above formula for $\tilde{E}$, we obtain perfect agreement (without adjustment of a proportionality factor).

In the next figure (Fig. 2) we show (with the dotted curve) the computed orthogonality times, again obtained from solution of equation (14), as a function of the alignment angle $\gamma$, with the frequency difference fixed (at $r = 0.5$) and compare this to the approximate (lower bound) expression shown in equation (7) (the full curve in Fig. 2). There is still a reasonable agreement between the two curves. However, at non-perfect alignment angles ($|\gamma| > 0$) this agreement gets spoiled for frequency differences smaller than $r = 0.5$ or $\frac{w_a}{w_b} < 3$.

**VI. LOWER BOUND ON THE ORTHOGONALITY TIME**

**A. Derivation from the Anandan-Aharonov relation [17]**

We derive a lower bound on the orthogonality time $t = t_\perp$ for general $d$-dimensional systems, introduced in Section II. For that purpose we invoke the geometric quantity [17]

$$s = 2 \int \Delta E(t) dt, \quad (20)$$

where $\Delta E(t)$ is the energy uncertainty in state $\psi$ defined by

$$\Delta E = \sqrt{\langle H^2 \rangle - \langle \psi | H | \psi \rangle^2}. \quad (21)$$

According to [17], the quantity $s$ is the distance along a given curve $C$ in the projective Hilbert space $P$ as measured by the Fubini-Study metric. In Section II, equation (1) can be interpreted as state $\psi$ evolving for a time interval $t_\perp$ under Hamiltonian $H_a$ and, following that, for the same interval $t_\perp$ under Hamiltonian $-H_b$ to an orthogonal state $\psi_f$. Thus one is allowed to write the right hand side of equation (20) as two separate time integrals, whose sum is $2\Delta E_a t_\perp + 2\Delta E_b t_\perp$ in our particular case. On the other hand, the shortest distance between orthogonal states can be achieved along a geodesic curve where $s = \pi$, hence in a generic evolution we have the relation $s \geq \pi$. Summing up these facts and substituting them into equation (20) we obtain the lower bound

$$t_\perp \geq \frac{\pi}{2(\Delta E_a + \Delta E_b)} \equiv t_{lb} \quad (22)$$

on the orthogonality time needed to discriminate perfectly between two evolution operators $e^{-A(t)}$ and $e^{-B(t)}$ defined in Section II.

Now let us suppose that we have knowledge only about the eigenvalues of the Hamiltonians $H_a$ and $H_b$ in question. Having the eigenvalues of a $d$-dimensional Hamiltonian $H'$ (where $d$ can not be infinite but can be arbitrarily large) gives rise to the upper bound $2\Delta E' \leq E'^{max} - E'^{min} = 2\omega'$, where $E'^{max}$ and $E'^{min}$ denote, respectively, the highest and lowest eigenvalues of the Hamiltonian $H'$. The preceding upper bound on insertion into equation (22) yields the inequality

$$t_\perp \geq \frac{\pi}{2\omega_a + 2\omega_b} \quad (23)$$

for the time $t_\perp$ to discriminate between the pair of evolution operators $e^{-A(t)}$ and $e^{-B(t)}$. Note, that this bound on $t_\perp$ is sharp in the sense that, given $\omega_a$ and $\omega_b$, we can always construct appropriate Hamiltonians $H_a$ and $H_b$ from them so that the bound (23) for a given initial state $\psi$ would be saturated. This can be achieved by choosing,
in particular, the Hamiltonians
\begin{align}
H_a &= \omega_a|E_+\rangle\langle E_+| - \omega_a|E_-\rangle\langle E_-|,
H_b &= -\omega_b|E_+\rangle\langle E_+| + \omega_b|E_-\rangle\langle E_-|,
\end{align}
and letting the initial state $|\psi\rangle = \frac{1}{\sqrt{2}}(|E_+\rangle + e^{i\alpha}|E_-\rangle)$ for some $\alpha \in [0, 2\pi]$. We obtain for the special studied case $d = 2$ in Section V, that the optimal $t_1$ is obtained for the alignment angle $\gamma = \pi$, a choice that indeed generates a pair of Hamiltonians of the kind corresponding to equation (24).

B. Derivation through Brody et al’s formula \cite{14, 15}

Actually, in order to obtain the bound (23) we do not need to resort to the Anandan-Aharonov relation (20), as we will show in the following by using formulas from Refs. \cite{14} and \cite{15}, whose derivation only requires elementary trigonometry. The problem discussed in these works is the following: Consider a pair of initial and final states $\psi_i$ and $\psi_F$ in a d-dimensional Hilbert space. The task is to find the Hamiltonian $H$ on this Hilbert space which takes $\psi_i$ into $\psi_F$ in the shortest possible time $\tau$. It has been found by elementary considerations (e.g., Eq. (5) in Ref. \cite{15}), that
\begin{equation}
\tau = \frac{2 \arccos |\langle \psi_i | \psi_F \rangle|}{2\omega}, \tag{25}
\end{equation}
where $2\omega = E^{\text{max}} - E^{\text{min}}$ denotes the difference of the largest and the smallest eigenvalues of $H$. In applying the above formula (25) for our situation we need to involve an intermediate state $\psi_m$ so that $|\psi_m\rangle = e^{-iH_mt} |\psi\rangle$ and $|\psi_F\rangle = e^{iH_xt} |\psi_m\rangle$. Rearranging equation (25) we obtain two equations involving our three states,
\begin{align}
\alpha_a &= 2 \arccos |\langle \psi_i | \psi_m \rangle| = t_1 2\omega_a,
\alpha_b &= 2 \arccos |\langle \psi_m | \psi_F \rangle| = t_1 2\omega_b. \tag{26}
\end{align}

If we add up the two angles, $\alpha_a$ and $\alpha_b$, on the left hand side of equation (26), since $|\langle \psi | \psi_F \rangle| = 0$, the sum can minimally take up the value $\pi$. Then we have $t_1 (2\omega_a + 2\omega_b) = \alpha_a + \alpha_b \geq \pi$. That is, $t_1 \geq \pi/(2\omega_a + 2\omega_b)$; thus we are back to the inequality (23) derived earlier with the aid of the Anandan-Aharonov relation. However, from the present derivation it is more transparent that in order to achieve the smallest value for $t_1$, it is necessary to express $\psi_m$, using some combination of the initial and final states $\psi$ and $\psi_F$ (otherwise the angle $\alpha_a + \alpha_b$ would be greater than $\pi$). Since for a generic pair of Hamiltonians $H_a$ and $H_b$ with dimensions $d > 2$ the condition that $\psi_m$ is in the two-dimensional subspace spanned by $\{|\psi\rangle, |\psi_F\rangle\}$ is a very severe condition to meet, the orthogonality time $t_1$ would in general be much larger than the value given by the lower bound (23).

C. Finding Hamiltonians for the lower bound

This subsection generalizes the argument leading to equation (24).

We consider two Hamiltonians $H_a$ and $H_b$ that give rise to the evolution operators $e^{-A(t)}$ and $e^{-B(t)}$. As before, $A(t) = iH_at$ and $B(t) = iH_bt$. We write $e^{B(t)}e^{-A(t)} \equiv e^{-C(t)}$ and $e^{-iH_c(t)}$. We assume to know orthogonality lower bounds $t_i^2 = t_{i(b)}(E_a)$ for $H_a$ alone (and similarly $t_{i(b)}(E_b)$ for $H_b$) as given in previous work \cite{8}-\cite{11}, expressed in terms of some energy $E_{a,b}$ (e.g., the span of the energy spectrum, or the energy uncertainty, as in equation (20)). Having the Hamiltonian $H_c$ we now try to find another Hamiltonian $H_b$ such that will achieve the lower bound of the orthogonality time. [We shall also trivially find the upper bound $(\infty)$ orthogonality time].

We recall from the Campbell-Baker-Hausdorff expansion that one can express $-C(t) = \ln[e^{B(t)}e^{-A(t)}]$ or $-itH_c(t) = \ln[e^{iH_c}e^{-iH_c}]$ in terms of a hierarchy of commutators, the zero order term being $H_c(t) \approx H_a - H_b \approx H_c$. We also assume that $H_a$ is given and is such that $TrH_a = 0$ (which should be possible to construct for Hamiltonians having a finite basis) and consider only traceless $H_a$ ($TrH_a = 0$). We further set $Tr(H_c^2) = K^2 Tr(H_a^2)$, where $K$ is an arbitrary constant. There may be some terms in $H_a$ and $H_b$ that are common to both of them and commute with both (e.g., kinetic energy operators that commute with spin variables, where only the latter constitute the differing parts in $H_a$ and $H_b$). We discount these terms.

Then we conjecture that $Tr(H_c^2(t))$ is maximal when $H_b = KH_a$, where $K$ is negative. In equation (24), $K = -\frac{\omega_b}{\omega_a}$. (A proof for this conjecture, provided to us by a referee, is given with slight modifications in the Appendix.)

Then $H_c = (1 + |K|)H_a = H_a - H_b$, $Tr(H_c^2(t)) = (1 + |K|^2) Tr(H_a^2)$, and the lower bound for $c$ is $t_i^2 = t_{i(b)}(E_a + |E_b|) \equiv t_{i(b)}(E_c)$. (We write the modulus of $E_b$, since only positive energies enter in the expressions $t_i(E)$, whereas with $K$ negative $E_b$ could be misinterpreted as a negative term.)

Also, trivially, when $K = 1$, $H_c = 0$ and the resulting bra-ket is always unity, never zero. This choice (not necessarily unique) achieves the infinite orthogonality time upper bound.

VII. SUMMARY AND CONCLUSIONS

This paper treated the following: Two different Hamiltonians ($H_a$ and $H_b$) are given and it is desired to be able to distinguish between them. The method of this paper is to form unitary operators out of these Hamiltonians and apply them on an initial state. Should the states become orthogonal, this indicates the distinctness of the Hamiltonians. However, two issues emerge: First, there is no guarantee that for any chosen initial state differently evolving states ever become orthogonal. Secondly
(which has some practical implications), what are the times (possibly minimal times) within which one can expect orthogonality to be obtained? By constructing some special states out of \( H_a \) and \( H_b \), as in section III and shown in equation (3), one can expect to reach orthogonality. The construction is feasible for systems with arbitrary Hilbert-space dimensionality (and for any number of components), but the achievement of orthogonality is not guaranteed in this work. However, the illustration worked out in section V for a qubit supports the existence of orthogonality, except for effectively identical Hamiltonians (such as when the non-scalar parts of \( H_a \) and \( H_b \) are proportional to each other). Our formal apparatus has indicated such special cases by infinitely long orthogonality times, as in the text after equation (14).

This paper, which has questioned discrimination between two Hamiltonians, can be regarded as an intermediate between past works treating state-orthogonalizations and unitary-discriminations. We have found ways to establish orthogonalization times \( t_\perp \) in general pure-state situations and calculated \( t_\perp \) in a single qubit model. Lower bounds \( t_0 \) of \( t_\perp \) were proposed in terms of quantities arising from the two Hamiltonians.

It is not clear whether the orthogonality times can be shortened by starting with entangled states (or whether infinite \( t_\perp \)'s can be reduced to finite values with entangled states). From the way that maximal distinguishability is sharpened by starting with entangled states (or whether arising from the two Hamiltonians. We have found ways to establish orthogonalization times \( t_\perp \) in general pure-state situations and calculated \( t_\perp \) in a single qubit model. Lower bounds \( t_0 \) of \( t_\perp \) were proposed in terms of quantities arising from the two Hamiltonians.

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Appendix: Proof of Conjecture

The following theorem immediately establishes the assertion in section VI C of the text that with \( iH_c(t)t = \ln(e^{iH_at}e^{-iH_at}) \) and the ratio \( \text{Tr}(H_2^c(t))/\text{Tr}(H_2^c(t)) \) fixed, \( \text{Tr}(H_2^c(t)) \) is maximal when \( H_b = KH_a \), where \( K \) is negative.

Theorem 1. Let the complex logarithm be defined on \( C \setminus R^- \), i.e., with a cut along along the negative real axis. For all unitaries \( U \) and \( V \)

\[
\| \ln(UV) \|_2 \leq \| \ln U \|_2 + \| \ln V \|_2,
\]

where \( \| \cdot \|_2 \) denotes the square root of the trace of the matrix squared (the Frobenius norm). An alternative statement of the theorem is, in terms of Hermitian \( A \) and \( B \) with spectrum in the half-open interval \( (-\pi, \pi] \),

\[
\| \ln(e^{iA}e^{iB}) \|_2 \leq \| A \|_2 + \| B \|_2.
\]

In the notation of the conjecture, Theorem 1 states that for \( U = e^{iH_at} \), \( V = e^{-iH_at} \) and \( H_c(t) \) as above,

\[
\| H_c(t) \|_2 \equiv \| \ln(e^{iH_at}e^{-iH_at}) \|_2 \leq \| H_at \|_2 + \| H_b \|_2.
\]

But when \( H_b = KH_a \), with \( K = -\| H_b \|/\| H_a \| \),

\[
\| H_c(t) \|_2 \equiv (1 - K)\| H_a \|_2 = \| H_at \|_2 + \| H_b \|_2,
\]

which satisfies the equality option in equation (29), showing that the choice made, \( H_b = KH_a \), maximizes \( T\text{r}(H_2^c(t)) \). In the proof of the theorem (which is done by induction) the time \( t \) is irrelevant, so we replace the three Hamiltonians in the text by three related Hermitian matrices designated as \( X, Y, Z(1) \). These are connected through

\[
e^{iZ(s)} = e^{iX}e^{isY} (0 \leq s \leq 1)
\]

with the eigenvalues of \( Z(s) \) restricted to the half-open interval \( (-\pi, \pi] \). In addition, \( Z(s) \) is constrained to be continuous over \( 0 \leq s \leq 1 \), which requires that \( e^{iX}e^{isY} \) has no eigenvalue equal to \(-1 \) for \( 0 \leq s \leq 1 \), i.e., no eigenvalue that crosses the cut in the complex plane. (It may be possible to remove this constraint, but the Hamiltonians in the text satisfy it.)

We first show that for infinitesimal \( \Delta s \), \( \| Z(s+\Delta s) \|_2 \leq \| Z(s) \|_2 + \Delta s \| Y \|_2 \). We write

\[
iZ(s+\Delta s) = \ln(e^{iZ(t)}(1 + i\Delta sY)) \]

(32) to first order in \( \Delta s \). Without loss of generality we can consider all matrices in a basis in which \( Z(s) \) is diagonal, \( Z(s) = \text{diag}(\theta_1, \theta_2, \ldots, \theta_d) \), for a \( d \)-dimensional \( Z(s) \), with \(-\pi < \theta_j \leq \pi \). For diagonal \( G \) the (Fréchet) derivative of the matrix logarithm is given by [18]

\[
\frac{\partial}{\partial s_{s \to 0}} \ln(G + sH) = \ln^{[1]}(G) \circ H,
\]

where \( \circ \) signifies the entrywise matrix product and \( \ln^{[1]}(G) \) is the matrix of divided differences defined as

\[
\ln^{[1]}(G)_{jk} = \frac{\ln G_{jk} - \ln G_{kk}}{G_{jj} - G_{kk}}, \quad j \neq k
\]

\[
\ln^{[1]}(G)_{jj} = 1/G_{jj}.
\]

Then, putting \( e^{iZ(s)} = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_d}) \), for infinitesimal \( \Delta s \),

\[
iZ(s+\Delta s) = \ln(e^{iZ(s)} + i\Delta se^{iZ(s)}Y) = iZ(s) + i\Delta s \ln^{[1]}(e^{iZ(s)}) \circ e^{iZ(s)}Y
\]

(36)
Substituting for \( h^{[1]}(e^{iZ(s)}) \) as
\[
(h^{[1]}(e^{iZ(s)}))_{jk} = i(\theta_j - \theta_k), \quad (j \neq k)
\]
\[
(h^{[1]}(e^{iZ(s)}))_{jj} = e^{-i\theta_j}
\]
we get for the two-norm
\[
\|Z(s + \Delta s)\|^2_2 = \sum_j |\theta_j + \Delta s Y_{jj}|^2 + (\Delta s)^2 \sum_{j \neq k} |i(\theta_j - \theta_k)e^{i\theta_j} Y_{jk}|^2.
\]
(37)

To first order in \( \Delta s \)
\[
\|Z(s + \Delta s)\|^2_2 = \sum_j |\theta_j|^2 + 2\Delta s \sum_j |\theta_j Y_{jj}| \leq \|Z(s)\|^2_2 + 2\Delta s \|Z(s)\|_2 \|Y\|_2
\]
\[
\leq (\|Z(s)\|_2 + \Delta s \|Z(s)\|_2 \|Y\|_2)^2.
\]
(39)

where equation (40) follows from equation (39) by the Cauchy-Schwarz inequality. Thus for infinitesimal \( \Delta s \) we have
\[
\|Z(s + \Delta s)\|_2 \leq \|Z(s)\|_2 + \Delta s \|Y\|_2.
\]
(42)

The procedure can be repeated for \( \|Z(s)\|_2 \) and so forth, finally giving
\[
\|Z\|_2 \leq \|Z(0)\|_2 + s \|Y\|_2
\]
\[
\equiv \|X\|_2 + \|Y\|_2
\]
(43)

which was to be proved.

(The referee pointed out that the conditions on the Hamiltonians being traceless are not necessary.)