We study the bootstrap method in harmonic oscillators in one-dimensional quantum mechanics. We find that the problem reduces to the Dirac’s ladder operator problem and is exactly solvable. Thus, harmonic oscillators allow us to see how the bootstrap method works explicitly.
1 Introduction

Recently, the bootstrap analysis in zero \[1, 2\] and one-dimensional systems \[3\] have been proposed, and they are actively studied in various models \[4, 5, 6, 7, 8, 9\]. This method works even in matrix models at \(N = \infty\), which is not possible in Monte-Carlo computations \[1, 2, 3, 4\]. Besides, the method may work even if the action contains imaginary terms which cause sign problems \[7\]. Hence, this method might play a complementary role to the Monte-Carlo method.

In this note, we study a harmonic oscillator in one-dimensional quantum mechanics by using the bootstrap method \[3, 5\]. Since harmonic oscillators are simple, it might provide us an insight how the bootstrap method works in quantum mechanics. Not only that, since harmonic oscillators are essential building blocks of quantum field theories, studying harmonic oscillators may help us to find a way to apply the bootstrap method to QFTs.

We find that the bootstrap problem in harmonic oscillators reduces to the Dirac’s ladder operator problem and is exactly solvable. This result suggests that the bootstrap method might be regarded as a generalization of the ladder operator method, and it might explain why the numerical bootstrap method works in various quantum mechanics problems \[3, 5, 6, 7, 8, 9\].

Note: When we are finalizing this work, a related study appeared \[5\]. There, a harmonic oscillator was solved by using the numerical bootstrap method. Since they chose different operators in the bootstrap matrix, the exact result was not found. See Sec. 2.1 for the details.

2 Bootstrap method in Harmonic oscillator

We start from a one-dimensional harmonic oscillator,

\[
H = \frac{1}{2} (P^2 + X^2) = a^d a + \frac{1}{2},
\]

where we have taken \(\hbar = 1\) and defined the ladder operators,

\[
a = \frac{1}{\sqrt{2}} (X + iP), \quad a^d = \frac{1}{\sqrt{2}} (X - iP).
\]

To employ the bootstrap method, we consider the following operator

\[
\hat{O} = \sum_{n=0}^{K} c_n a^n = c_0 + c_1 a + c_2 a^2 + \cdots + c_K a^K,
\]
where \( \{c_n\} \) are constants, and \( K \) is a positive integer. Since \( \langle O^\dagger O \rangle \geq 0 \) is satisfied for any states in this system for arbitrary well-defined operators \( O \),

\[
\langle \tilde{O}^\dagger \tilde{O} \rangle \geq 0
\]  

(2.4)
is satisfied for any constants \( \{c_n\} \). Hence, the following \((K + 1) \times (K + 1)\) matrix \( \mathcal{M} \) has to be positive-semidefinite \( [3] \),

\[
\mathcal{M} := \begin{pmatrix}
1 & \langle a \rangle & \langle a^2 \rangle & \cdots & \langle a^K \rangle \\
\langle (a^\dagger)^2 \rangle & \langle (a^\dagger)^2 a \rangle & \langle (a^\dagger)^2 a^2 \rangle & \cdots & \langle (a^\dagger)^2 a^K \rangle \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\langle (a^\dagger)^K \rangle & \langle (a^\dagger)^K a \rangle & \langle (a^\dagger)^K a^2 \rangle & \cdots & \langle (a^\dagger)^K a^K \rangle
\end{pmatrix} \succeq 0.
\]  

(2.5)

This strongly constrains possible expectation values of the operators. We call \( \mathcal{M} \) as a bootstrap matrix. Note that, as \( K \) increases, the constraint would become stronger.

From now, we focus on an energy eigenstate with an energy eigenvalue \( E \), and we assume that the expectation values are those for this state. We investigate the possible values of \( E \) which are consistent with the condition \( \mathcal{M} \succeq 0 \). Then, the energy eigenstate has to satisfy the following two additional conditions,

\[
\langle [H, O] \rangle = 0, \quad \langle HO \rangle = E \langle O \rangle,
\]  

(2.6)

for any well-defined operator \( O \). By taking \( O = (a^\dagger)^m a^n \) in these two equations, we obtain

\[
(m - n)\langle (a^\dagger)^m a^n \rangle = 0, \quad \langle (a^\dagger)^{m+1} a^{n+1} \rangle = \left( E - m - \frac{1}{2} \right) \langle (a^\dagger)^m a^n \rangle.
\]  

(2.7)

Here, the first equation implies that \( \langle (a^\dagger)^m a^n \rangle = 0 \), if \( m \neq n \). Thus, the bootstrap matrix \( \mathcal{M} \) (2.5) becomes diagonal,

\[
\mathcal{M} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \langle a^\dagger a \rangle & 0 & \cdots & 0 \\
0 & 0 & \langle (a^\dagger)^2 a \rangle & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \langle (a^\dagger)^K a^K \rangle
\end{pmatrix},
\]  

(2.8)

and the condition \( \mathcal{M} \succeq 0 \) reduces to

\[
\langle (a^\dagger)^m a^n \rangle \geq 0, \quad n = 1, \cdots, K.
\]  

(2.9)
Besides, the second equation in (2.7) implies
\[ \langle (a^\dagger)^n a^n \rangle = \prod_{k=0}^{n-1} \left( E - k - \frac{1}{2} \right). \] (2.10)

Obviously, if \( K = \infty \), the conditions (2.9) and (2.10) are equivalent to the Dirac’s ladder operator problem and the solution is given as
\[ E = n + \frac{1}{2}, \quad (n = 0, \cdots, K - 1), \quad \text{or} \quad E \geq K + \frac{1}{2}. \] (2.11)

Therefore, the bootstrap method reproduces the exact result when we take \( K \to \infty \).

2.1 Numerical Bootstrap in Harmonic Oscillator

You may wonder that our derivation is a “fine tuning” of the operators in the bootstrap matrix (2.5), and, if we choose different operators, the results might change. Indeed, the results depend on the choice of the operators. Let us consider the following two sets of operators \( \{X^n\} \) and \( \{X^mP^n\} \). In order to construct the corresponding two bootstrap matrices, we define
\[ \hat{O}_X := \sum_{n=0}^{K} c_n X^n, \quad \hat{O}_{XP} := \sum_{m=0}^{K} \sum_{n=0}^{K} c_{mn} X^m P^n. \] (2.12)

Then, through the condition (2.4), we obtain the bootstrap matrices
\[ \mathcal{M}_X := \begin{pmatrix} 1 & \langle X \rangle & \langle X^2 \rangle & \cdots \\ \langle X \rangle & \langle X^2 \rangle & \langle X^3 \rangle & \cdots \\ \langle X^2 \rangle & \langle X^3 \rangle & \langle X^4 \rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{M}_{XP} := \begin{pmatrix} 1 & \langle X \rangle & \langle P \rangle & \cdots \\ \langle X \rangle & \langle X^2 \rangle & \langle XP \rangle & \cdots \\ \langle P \rangle & \langle PX \rangle & \langle P^2 \rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \] (2.13)

By substituting \( \{X^mP^n\} \) into the equations (2.6), we obtain,
\[ \begin{align*}
0 &= -m(m-1)\langle X^{m-2}P^n \rangle - 2im\langle X^{m-1}P^{n+1} \rangle + n(n-1)\langle X^m P^{n-2} \rangle + 2im\langle X^{m+1} P^{n-1} \rangle \\
0 &= \langle X^{m+2}P^n \rangle + \langle X^m P^{n+2} \rangle - 2E\langle X^m P^n \rangle - 2im\langle X^{m-1} P^{n+1} \rangle - m(m-1)\langle X^{m-2}P^n \rangle \\
\end{align*} \] (2.14)
Figure 1: Energy spectra of the harmonic oscillator through the numerical bootstrap. We take $K = 8$ for $\mathcal{M}_X$ and $K_x = K_p = 2$ for $\mathcal{M}_{XP}$ in (2.12). In the case of $\mathcal{M}_{XP}$, we obtain the exact result for the first two eigenstates.

We can solve these equations\footnote{From these equations, we obtain a closed expression for the operators $\{X^n\}$ \[3, 5\],}

\[ -4(n + 1)\langle X^{n+1} \rangle + 8nE\langle X^{n-1} \rangle + n(n - 1)(n - 2)\langle X^{n-3} \rangle = 0. \tag{2.15} \]

and all the operator $\{X^n P^n\}$ are described as functions of energy $E$. Then, the bootstrap matrices (2.13) become,

\[
\mathcal{M}_X := \begin{pmatrix}
1 & 0 & E & \cdots \\
0 & E & 0 & \cdots \\
E & 0 & \frac{3}{2} E^2 + \frac{3}{8} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\mathcal{M}_{XP} := \begin{pmatrix}
1 & 0 & 0 & \cdots \\
0 & E & \frac{i}{2} & \cdots \\
0 & -\frac{i}{2} & E & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} . \tag{2.16}
\]

Now, we numerically investigate possible values of $E$ at which the bootstrap matrices become positive-semidefinite\footnote{We use Mathematica package FindMinimum in our numerical computations.} Here, we take $K = 8$ for $\mathcal{M}_X$ and $K_x = K_p = 2$ for $\mathcal{M}_{XP}$ in (2.12). Note that the size of the bootstrap matrix $\mathcal{M}_X$ and $\mathcal{M}_{XP}$ are both nine in this case. The results are summarized in Fig. 1

We find that the first two energy eigenvalues are exactly derived from $\mathcal{M}_{XP}$ while we cannot obtain the exact ones from $\mathcal{M}_X$. In the case of $\mathcal{M}_{XP}$, since the ladder operators (2.2) can be constructed from the linear combinations of $X$ and $P$, the bootstrap method may correctly capture the exact result discussed above. Hence, we do not need a fine tuning. Just considering both $P$ and $X$ in the bootstrap matrix would be enough to obtain the exact result.

On the other hand, in the case of $\mathcal{M}_X$, since we cannot construct the ladder operators only from $X$, the constraint $\mathcal{M}_X \succeq 0$ is not as strong as $\mathcal{M}_{XP} \succeq 0$. (If we take the size
of the bootstrap matrix sufficiently large, the allowed region of $E$ are strongly constrained, and it becomes point like, asymptotically.)

**Acknowledgements** The authors would like to thank Takehiro Azuma for valuable discussions and comments. A part of numerical computation in this work was carried out at the Yukawa Institute Computer Facility. The work of T. M. is supported in part by Grant-in-Aid for Scientific Research C (No. 20K03946) from JSPS.

**References**

[1] Peter D. Anderson and Martin Kruczenski. Loop Equations and bootstrap methods in the lattice. *Nucl. Phys. B*, 921:702–726, 2017.

[2] Henry W. Lin. Bootstraps to strings: solving random matrix models with positivity. *JHEP*, 06:090, 2020.

[3] Xizhi Han, Sean A. Hartnoll, and Jorrit Kruthoff. Bootstrapping Matrix Quantum Mechanics. *Phys. Rev. Lett.*, 125(4):041601, 2020.

[4] Vladimir Kazakov and Zechuan Zheng. Analytic and Numerical Bootstrap for One-Matrix Model and "Unsolvable" Two-Matrix Model. 8 2021.

[5] David Berenstein and George Hulsey. Bootstrapping Simple QM Systems. 8 2021.

[6] Jyotirmoy Bhattacharya, Diptarka Das, Sayan Kumar Das, Ankit Kumar Jha, and Moulindu Kundu. Numerical Bootstrap in Quantum Mechanics. 8 2021.

[7] Yu Aikawa, Takeshi Morita, and Kota Yoshimura. Application of Bootstrap to $\theta$-term. 9 2021.

[8] David Berenstein and George Hulsey. Bootstrapping More QM Systems. 9 2021.

[9] Serguei Tchoumakov and Serge Florens. Bootstrapping Bloch bands. 9 2021.