BERRY–ESSEEN BOUNDS FOR GENERALIZED U STATISTICS

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Abstract

In this paper, we establish optimal Berry–Esseen bounds for the generalized U-statistics. The proof is based on a new Berry–Esseen theorem for exchangeable pair approach by Stein’s method under a general linearity condition setting. As applications, an optimal convergence rate of the normal approximation for subgraph counts in Erdös–Rényi graphs and graphon-random graph is obtained.

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1 INTRODUCTION

Let $X = (X_1, \ldots, X_n) \in \mathcal{X}^n$ and $Y = (Y_{i,j}, 1 \leq i < j \leq n) \in \mathcal{Y}^{n(n-1)/2}$ be two families of i.i.d. random variables; moreover, $X$ and $Y$ are also mutually independent and we set $Y_{j,i} = Y_{i,j}$ for $j > i$. For $k \geq 1$, let $f : \mathcal{X}^k \times \mathcal{Y}^{k(k-1)/2} \to \mathbb{R}$ be a function and we say $f$ is symmetric if the value of the function $f(X_{i_1}, \ldots, X_{i_k}; Y_{i_1, i_2}, \ldots, Y_{i_{k-1}, i_k})$ remains unchanged for any permutation of indices $1 \leq i_1 \neq i_2 \neq \ldots \neq i_k \leq n$. In this paper, we consider the generalized U-statistic defined by

$$S_{n,k}(f) = \sum_{\alpha \in \mathcal{I}_{n,k}} f(X_{\alpha(1)}, \ldots, X_{\alpha(k)}; Y_{\alpha(1), \alpha(2)}, \ldots, Y_{\alpha(k-1), \alpha(k)}),$$

(1.1)

where for every $\ell \geq 1$ and $n \geq \ell$,

$$\mathcal{I}_{n,\ell} = \{\alpha = (\alpha(1), \ldots, \alpha(\ell)) : 1 \leq \alpha(1) < \cdots < \alpha(\ell) \leq n\}.$$  

(1.2)

We note that every $\alpha \in \mathcal{I}_{n,\ell}$ is an $\ell$-fold ordered index.

As a generalization of the classical U-statistic, generalized U-statistics have been widely applied in the random graph theory as a count random variable. Janson and Nowicki (1991) studied the limiting behavior of $S_{n,k}(f)$ via a projection method. Specifically, the function $f$ can be represented as an orthogonal sum of terms indexed by subgraphs of the complete graph with $k$ vertices. Janson and Nowicki (1991) showed that the limiting behavior of $S_{n,k}(f)$ depends on topology of the principle support graphs (see more details in Subsection 2.1) of $f$. In particular, the random variable $S_{n,k}(f)$ is asymptotically normally distributed if the principle support graphs are all connected. However, the convergence rate is still unknown.
The main purpose of this paper is to establish a Berry–Esseen bound for $S_n$ by using Stein’s method. Stein’s method is a powerful tool to estimating convergence rates for distributional approximation. Since introduced by Stein (1972) in 1972, Stein’s method has shown to be a powerful tool to evaluate distributional distances for dependent random variables. One of the most important techniques in Stein’s method is the exchangeable pair approach, which is commonly taken in computing the Berry–Esseen bound for both normal and nonnormal approximations. We refer to Stein (1986); Rinott and Rotar (1997); Chatterjee and Shao (2011) and Shao and Zhang (2016) for more details on Berry–Esseen bound for bounded exchangeable pairs. It is worth mentioning that Shao and Zhang (2019) obtained a Berry–Esseen bound for unbounded exchangeable pairs.

Let $W$ be the random variable of interest, and we say $(W, W')$ is an exchangeable pair if $(W, W') \overset{d.}= (W', W)$. For normal approximation, it is often to assume the following condition holds:

$$E\{W - W' | W\} = \lambda (W + R), \quad (1.3)$$

where $\lambda > 0$ and $R$ is a random variable with a small $E|R|$. The condition (1.3) can be understood as a linear regression condition. Although an exchangeable pair can be easily constructed, it may be not easy to verify the linearity condition (1.3) in some applications.

In this paper, we aim to establish an optimal Berry–Esseen bound for the generalized $U$-statistics by developing a new Berry–Esseen theorem for exchangeable pair approach by assuming a more general condition than (1.3). More specifically, we replace $W - W'$ in (1.3) by a random variable $D$ that is an antisymmetric function of $(X, X')$. The new result is given in Section 4. There are several advantages of our result. Firstly, we propose a new condition more general than (1.3) that may be easy to verify. For instance, the condition can be verified by constructing an antisymmetric random variable by the Gibbs sampling method, embedding method, generalized perturbative approach and so on. Secondly, the Berry–Esseen bound often provides an optimal convergence rate for many practical applications.

The rest of this paper is organized as follows. In Section 2, we give the Berry–Esseen bounds for $S_{n,k}(f)$. Applications to subgraph counts in $\kappa$-random graphs are given in Section 3. The new Berry–Esseen theorem for exchangeable pair approach under a new setting is established in Section 4. We give the proofs of our main results in Section 5. The proofs of other results are postponed to Section 6.

## 2 MAIN RESULTS

Let $(X, Y)$, $f$ and $S_{n,k}(f)$ be defined in Section 1. For any $\ell \geq 1$, $[\ell] = \{1, \ldots, \ell\}$ and $[\ell]_2 = \{(i, j) : 1 \leq i, j \leq \ell\}$. Let $A \subset [\ell]$ and let $B \subset [\ell]_2$, and let $X_A = (X_i : i \in A)$ and $Y_B = (Y_{i,j} : (i, j) \in B)$. Specially, we can simply write $f(X_1, \ldots, X_k; Y_1, \ldots, Y_{k-1}, k)$ as $f(X_{[k]}; Y_{[k]_2})$. Let $G_{A,B}$ be the graph with vertex set $A$ and edge set $B$, and let $v_{A,B}$ be the number of nodes in $G_{A,B}$.
By the Hoeffding decomposition, we have
\[ f(X[k]; Y[k]) = \sum_{A \subset [k], B \subset [k]} f_{A,B}(X_A; Y_B), \]
where \( f_{A,B} : \mathcal{X}^{|A|} \times \mathcal{Y}^{|B|} \to \mathbb{R} \) is defined as
\[ f_{A,B}(x_A; y_B) = \sum_{(A',B') : A' \subset A, B' \subset B} (-1)^{|A'| + |B'| - |A'| - |B'|} \times \mathbb{E}\{ f(X_1, \ldots, X_k; Y_{1,2}, \ldots, Y_{k-1,k}) \mid X_{A'} = x_{A'}, Y_{B'} = y_{B'} \}, \quad (2.1) \]
where \( x_A = \{x_i : i \in A\} \) and \( y_B = \{y_{i,j} : (i,j) \in B\} \) for \( A \subset [k] \) and \( B \subset [k] \). We remark that if \( A = \emptyset \) and \( B = \emptyset \), then \( f_{\emptyset,\emptyset}(X_{\emptyset}; Y_{\emptyset}) = \mathbb{E}\{ f(X[k]; Y[k]) \} \). For \( \ell = 0, 1, \ldots, k \), let
\[ f_{(\ell)}(X[k]; Y[k]) = \begin{cases} \mathbb{E}\{ f(X[k]; Y[k]) \} & \text{if } \ell = 0, \\ \sum_{v_{A,B} = \ell} f_{A,B}(X_A; Y_B) & \text{if } \ell \geq 1, \end{cases} \quad (2.2) \]
where \( v_{A,B} \) is the number of nodes in \( G_{A,B} \). Let \( d = \min\{\ell > 0 : f_{(\ell)} \neq 0\} \), and we call \( d \) the principal degree of \( f \). We say \( f_{(d)} \) is the principal part of \( f \). Moreover, we say the subgraphs \( G_{A,B} \) such that \( v_{A,B} = d \) and \( f_{A,B} \neq 0 \) are the principal support graphs of \( f \).

The central limit theorems for \( S_{n,k}(f) \) is proved by Janson and Nowicki (1991). Let \( \sigma_{A,B} = \|f_{A,B}(X_A; Y_B)\| \), and let \( \mathcal{G}_{f,d} = \{G_{A,B} : \sigma_{A,B} \neq 0, v_{A,B} = d\} \) be the set of principal index graph. We remark that if \( f \) has the principal degree \( d \), then \( \text{Var}(S_{n,k}(f)) \) is of order \( n^{2k-d} \), see Lemmas 2 and 3 in Janson and Nowicki (1991). Janson and Nowicki (1991) proved that if all graphs in \( \mathcal{G}_f \) are connected, then
\[ \frac{S_{n,k}(f) - \mathbb{E}\{ S_{n,k}(f) \}}{\sqrt{\text{Var}(S_{n,k}(f))}} \overset{d}{\to} \mathcal{N}(0,1). \]
Note that if not all principal support graphs are connected, then the limiting distribution of the scaled version of \( S_{n,k} \) is nonnormal (see Theorems 2 and 3 in Janson and Nowicki (1991)), and we will consider this case in another paper.

Now, assume that \( f \) is a symmetric function having principal degree \( d \) (\( 1 \leq d \leq k \)). In this subsection, we give a Berry–Esseen bound for \( S_{n,k}(f) \). For \( x \in \mathcal{X} \), let
\[ f_1(x) := f_{(1)}_{,\emptyset}(x) = \mathbb{E}\{ f(X[k]; Y[k]) \mid X_1 = x \} - \mathbb{E}\{ f(X[k]; Y[k]) \}. \]
If \( \|f_1(X_1)\|_2 > 0 \), then it follows that \( d = 1 \). Here and in the sequel, we denote by \( \|Z\|_p := (\mathbb{E}|Z|^p)^{1/p} \) for \( p > 0 \) and we denote by \( \Phi(\cdot) \) the distribution function of \( \mathcal{N}(0,1) \). The following theorem provides the Berry–Esseen bound for \( S_{n,k}(f) \) in the case where \( \|f_1(X_1)\|_2 > 0 \).

**Theorem 2.1.** If \( \sigma_1 := \|f_1(X_1)\|_2 > 0 \), then
\[ \sup_{z \in \mathbb{R}} \left| \mathbb{P}\left[ \frac{S_{n,k}(f) - \mathbb{E}\{ S_{n,k}(f) \}}{\sqrt{\text{Var}(S_{n,k}(f))}} \leq z \right] - \Phi(z) \right| \leq \frac{12k\|f(X[k]; Y[k])\|_2^2}{\sqrt{n}\sigma_1^2}. \quad (2.3) \]
Remark 2.2. We remark that \( \text{Var}(S_{n,k}(f)) = O(n^{2k-1}) \) as \( n \to \infty \). Typically, the right hand side of (2.3) is of order \( n^{-1/2} \). Specially, if \( f(X_{[k]}, Y_{[k]}) = h(X_{[k]}) \) for some symmetric function \( h : \mathcal{X}^k \to \mathbb{R} \), then \( S_{n,k} \) is the classical \( U \)-statistic. In this case, Chen and Shao (2007) obtained a Berry–Esseen bound of order \( n^{-1/2} \) under the assumption that \( \|h(X_{[k]})\|_3 < \infty \).

If \( \sigma_1 = 0 \), then \( d \geq 2 \), that is, the principal degree of \( f \) is at least 2. We have the following theorem.

Theorem 2.3. Let \( \tau := \|f(X_{[k]}; Y_{[k]})\|_4 < \infty \) and let \( \sigma_{\text{min}} := \min(\sigma_{A,B} : G_{A,B} \in \mathcal{G}_{f,d}) \). Assume that \( f \) is a symmetric function having principal degree \( d \) for some \( 2 \leq d \leq k \), and assume further that for all graphs in \( \mathcal{G}_{f,d} \) are connected. Then, we have

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left( \frac{(S_{n,k}(f) - \mathbb{E}(S_{n,k}(f)))}{\sqrt{\text{Var}(S_{n,k}(f))}} \leq z \right) - \Phi(z) \right| \leq Cn^{-1/2},
\]

where \( C > 0 \) is a constant depending only on \( k, d, \sigma_{\text{min}} \), and \( \tau \).

If we further assume that the function \( f \) does not depend on \( X \), i.e., \( f(X; Y) = g(Y) \) for some symmetric \( g : \mathcal{Y}^{k(k-1)/2} \to \mathbb{R} \), we obtain a sharper convergence rate. To give the theorem, we first introduce some more notation. Let \( G^{(r)} \) be the graph generated from \( G \) by deleting the node \( r \) and all the edges connecting to the node \( r \). We say \( G \) is strongly connected if \( G^{(r)} \) is connected or empty for all \( r \in V(G) \). We note that all strongly connected graphs are also connected. The following theorem provides a sharper Berry–Esseen bound than that in Theorem 2.3.

Theorem 2.4. Assume that \( f(X_{[k]}; Y_{[k]}) = g(Y_{[k]}) \) almost surely for some symmetric \( g : \mathcal{Y}^{k(k-1)/2} \to \mathbb{R} \). Let \( \tau \) and \( \sigma_{A,B} \) be defined in Theorem 2.3. Assume that the conditions in Theorem 2.3 are satisfied and assume further that all graphs in \( \mathcal{G}_{f,d} \) are strongly connected. Then,

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left( \frac{(S_{n,k}(g) - \mathbb{E}(S_{n,k}(g)))}{\sqrt{\text{Var}(S_{n,k}(g))}} \leq z \right) - \Phi(z) \right| \leq Cn^{-1},
\]

where \( C > 0 \) is a constant depending on \( k, d, \sigma_{\text{min}} \), and \( \tau \).

3 APPLICATIONS

3.1 Subgraphs counts in random graphs generated from graphons

A symmetric Lebesgue measurable function \( \kappa : [0,1]^2 \to [0,1] \) is called a graphon, which was firstly introduced by Lovász and Szegedy (2006) to represent the graph limit. Given a graphon \( \kappa \) and \( n \geq 2 \), the \( \kappa \)-random graph \( \mathcal{G}(n, \kappa) \) can be generated as follows: Let \( n \geq 1 \) and let \( X = (X_1, \ldots, X_n) \) be a vector of independent uniformly distributed random variables on \([0,1]\). Given \( X \), we generate the graph \( \mathcal{G}(n, \kappa) \) by connecting the node pair \((i,j)\) independently with probability \( \kappa(X_i, X_j) \). This construction was firstly introduced
Subgraph counts are important statistics in estimating graphons. As a special case, when \( \kappa \equiv p \) for some \( p \in (0, 1) \), the \( \kappa \)-random graph model becomes the classical Erdős–Rényi model \( \mathrm{ER}(p) \). The study of asymptotic properties of subgraph counts in \( \mathrm{ER}(p) \) dates back to Nowicki (1989), Barbour, Karoński and Ruciński (1989), Janson and Nowicki (1991) for more details. Recently, Krokowski, Reichenbachs and Thäle (2017), Röllin (2017) and Privault and Serafin (2018) applied Stein’s method to obtain an optimal Berry–Esseen bound for triangle counts in \( \mathrm{ER}(p) \). For subgraph counts in \( \kappa \)-random graph, Kaur and Röllin (2020) proved an upper bound of the Kolmogorov distance for multivariate normal approximations for centered subgraph counts with order \( n^{-1/(p+2)} \) for some \( p > 0 \). However, the Berry–Esseen bounds for subgraph counts of \( \kappa \)-random graph is still unknown so far. In this subsection, we apply Theorems 2.3 and 2.4 to prove sharp Berry–Esseen bounds for subgraph counts statistics.

Let \( \Xi = (\xi_{i,j})_{1 \leq i \leq j \leq n} \) be the adjacency matrix of \( G(n, \kappa) \), where for each \((i, j)\), the binary random variable \( \xi_{i,j} \) indicates the connection of the graph. Formally, let \( Y = (Y_{1,1}, \ldots, Y_{n-1,n}) \) be a vector of independent uniformly distributed random variables that is also independent of \( X \), and then we can write \( \xi_{i,j} = NY_{i,j} \leq \kappa(X_i, X_j) \). For any nonrandom simple \( F \) with \( v(F) = k \), the (injective) subgraph counts and induced subgraph counts in \( G(n, \kappa) \) are defined by

\[
T_F^{\text{inj}} := T_F^{\text{inj}}(G(n, \kappa)) = \sum_{\alpha \in I_{n,k}} \varphi_F^{\text{inj}}(\xi_{\alpha(1),\alpha(2)}, \ldots, \xi_{\alpha(k-1),\alpha(k)}),
\]

\[
T_F^{\text{ind}} := T_F^{\text{ind}}(G(n, \kappa)) = \sum_{\alpha \in I_{n,k}} \varphi_F^{\text{ind}}(\xi_{\alpha(1),\alpha(2)}, \ldots, \xi_{\alpha(k-1),\alpha(k)}),
\]

respectively, where for \((x_{1,1}, \ldots, x_{k-1,k}) \in \mathbb{R}^{k(k-1)/2},\)

\[
\varphi_F^{\text{inj}}(x_{1,1}, \ldots, x_{k-1,k}) = \sum_{H: H \supseteq F} \prod_{(i,j) \in E(H)} x_{i,j},
\]

\[
\varphi_F^{\text{ind}}(x_{1,2}, \ldots, x_{k-1,k}) = \sum_{H: H \supseteq F} \prod_{(i,j) \in E(H)} x_{i,j} \prod_{(i,j) \not\in E(H)} (1 - x_{i,j}).
\]

Here, the summation \( \sum_{H: H \supseteq F} \) ranges over the subgraphs with \( v(F) \) nodes that are isomorphic to \( F \) and thus contains \( v(F)! / |\text{Aut}(F)| \) terms, where \( |\text{Aut}(F)| \) is the number of automorphisms of \( F \). Moreover, we note that both \( \varphi_F^{\text{inj}} \) and \( \varphi_F^{\text{ind}} \) are symmetric. For example, if \( F \) is the 2-star, then \( k = 3 \), \( |\text{Aut}(F)| = 2 \) and

\[
\varphi_F^{\text{inj}}(\xi_{1,2}, \xi_{1,3}, \xi_{2,3}) = \xi_{1,2}\xi_{1,3} + \xi_{1,2}\xi_{2,3} + \xi_{1,3}\xi_{2,3},
\]

\[
\varphi_F^{\text{ind}}(\xi_{1,2}, \xi_{1,3}, \xi_{2,3}) = \xi_{1,2}\xi_{1,3}(1 - \xi_{2,3}) + \xi_{1,2}\xi_{2,3}(1 - \xi_{1,3}) + \xi_{1,3}\xi_{2,3}(1 - \xi_{1,2}).
\]

If \( F \) is a triangle, then \( |\text{Aut}(F)| = 6 \) and

\[
\varphi_F^{\text{inj}}(\xi_{1,2}, \xi_{1,3}, \xi_{2,3}) = \varphi_F^{\text{ind}}(\xi_{1,2}, \xi_{1,3}, \xi_{2,3}) = \xi_{1,2}\xi_{1,3}\xi_{2,3}.
\]
Let
\[
T_F(\kappa) = \int_{[0,1]^k} \prod_{(i,j) \in E(F)} \kappa(x_i, x_j) \prod_{i \in V(F)} dx_i,
\]
\[
i_F^{\text{ind}}(\kappa) = \int_{[0,1]^k} \prod_{(i,j) \in E(F)} \kappa(x_i, x_j) \prod_{(i,j) \notin E(F)} (1 - \kappa(x_i, x_j)) \prod_{i \in V(F)} dx_i.
\]
Then, we have
\[
\mathbb{E}\{\varphi_F^{\text{inj}}(\xi_1, \ldots, \xi_{k-1}, k)\} = \frac{k!}{|\text{Aut}(F)|} t_F(\kappa),
\]
\[
\mathbb{E}\{\varphi_F^{\text{ind}}(\xi_1, \ldots, \xi_{k-1}, k)\} = \frac{k!}{|\text{Aut}(F)|} i_F^{\text{ind}}(\kappa).
\]
As \(\xi_{i,j} = \text{NY}_{i,j} \leq \kappa(X_i, X_j)\), let
\[
f_F^{\text{inj}}(X_{[k]}; Y_{[k]}) = \varphi_F^{\text{inj}}(\xi_1, \ldots, \xi_{k-1}, k).
\]
Now, as random variables \((\xi_{i,j})_{1 \leq i < j \leq n}\) are conditionally independent given \(X\), we have
\[
\mathbb{E}\{f_F^{\text{inj}}(X_{[k]}; Y_{[k]}) \mid X\} = \sum_{H \subseteq F} \prod_{(i,j) \in E(H)} \kappa(X_i, X_j),
\]
\[
\mathbb{E}\{f_F^{\text{ind}}(X_{[k]}; Y_{[k]}) \mid X\} = \sum_{H \subseteq F} \prod_{(i,j) \in E(H)} \kappa(X_i, X_j) \prod_{(i,j) \notin E(H)} (1 - \kappa(X_i, X_j)).
\]
Let
\[
f_1^{\text{inj}}(x) = \mathbb{E}\{f_F^{\text{inj}}(X_{[k]}; Y_{[k]}) \mid X_1 = x\}
\]
\[
= \sum_{H \subseteq F} \mathbb{E}\left\{\prod_{(i,j) \in E(H)} \kappa(X_i, X_j) \mid X_1 = x\right\},
\]
and similarly, let
\[
f_1^{\text{ind}}(x) = \mathbb{E}\{f_F^{\text{ind}}(X_{[k]}; Y_{[k]}) \mid X_1 = x\}
\]
\[
= \sum_{H \subseteq F} \mathbb{E}\left\{\prod_{(i,j) \in E(H)} \kappa(X_i, X_j) \prod_{(i,j) \notin E(H)} (1 - \kappa(X_i, X_j)) \mid X_1 = x\right\}.
\]
We have the following theorem, which follows from Theorem 2.1 directly.

**Theorem 3.1.** Let \(\sigma_1^{\text{inj}} = \|f_1^{\text{inj}}(X_1) - \mathbb{E}\{f_1^{\text{inj}}(X_1)\}\|_2\) and \(\sigma_1^{\text{ind}} = \|f_1^{\text{ind}}(X_1) - \mathbb{E}\{g_1^{\text{ind}}(X_1)\}\|_2\).

Assume that \(\sigma_1^{\text{inj}} > 0\), then
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left[\frac{\sqrt{n}}{k \sigma_1^{\text{inj}}} \binom{n}{k}^{-1} \left( T_F^{\text{inj}} - \mathbb{E}\{T_F^{\text{inj}}\} \right) \leq z \right] - \Phi(z) \right| \leq Cn^{-1/2}.
\]
Moreover, assume that \(\sigma_1^{\text{ind}} > 0\), then
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left[\frac{\sqrt{n}}{k \sigma_1^{\text{ind}}} \binom{n}{k}^{-1} \left( T_F^{\text{ind}} - \mathbb{E}\{T_F^{\text{ind}}\} \right) \leq z \right] - \Phi(z) \right| \leq Cn^{-1/2}.
\]
If $\kappa \equiv p$ for a fixed number $0 < p < 1$, then the random variables $(\xi_{i,j})_{1 \leq i < j \leq n}$ are i.i.d. and the functions $\varphi_F^{\text{inj}}$ and $\varphi_F^{\text{ind}}$ do not depend on $X$. We have the following theorem:

**Theorem 3.2.** Let $\kappa \equiv p$ for $0 < p < 1$. Then

$$
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \frac{T_F^{\text{inj}} - \mathbb{E}\{T_F^{\text{inj}}\}}{\left(\text{Var}\{T_F^{\text{inj}}\}\right)^{1/2}} \leq z \right] - \Phi(z) \right| \leq Cn^{-1}.
$$

**Remark 3.3.** For the $L_1$ bound, Barbour et al. (1989) proved the same order of $O(n^{-1})$ in the case that $p$ is a constant. For the Berry–Esseen bound, Privault and Serafin (2018) proved a general Berry–Esseen bound for subgraph counts for Erdős–Rényi random graph using a different method. Specially, if $p$ is a constant, then Theorem 3.2 provides the same result as in Privault and Serafin (2018).

For induced subgraph counts, we need to consider some separate cases. Let $s(F)$ and $t(F)$ denote the number of 2-stars and triangles in $F$, respectively. If any of the following conditions holds, then it has been proven by Janson and Nowicki (1991) that

$$
\frac{T_F^{\text{ind}} - \mathbb{E}\{T_F^{\text{ind}}\}}{\left(\text{Var}\{T_F^{\text{ind}}\}\right)^{1/2}} \text{ converges to a standard normal distribution:}
$$

1. (G1) if $e(F) \neq p\left(\frac{e(F)}{2}\right)$;
2. (G2) if $e(F) = p\left(\frac{e(F)}{2}\right)$, $s(F) \neq 3p^2\left(\frac{e(F)}{3}\right)$;
3. (G3) if $e(F) = p\left(\frac{e(F)}{2}\right)$, $s(F) = 3p^2\left(\frac{e(F)}{3}\right)$ and $t(F) \neq p^3\left(\frac{e(F)}{3}\right)$.

The following theorem gives the Berry–Esseen bounds for induced subgraph counts.

**Theorem 3.4.** Let $\kappa \equiv p$ for $0 < p < 1$. If (G1) or (G3) holds, then

$$
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \frac{T_F^{\text{ind}} - \mathbb{E}\{T_F^{\text{ind}}\}}{\left(\text{Var}\{T_F^{\text{ind}}\}\right)^{1/2}} \leq z \right] - \Phi(z) \right| \leq Cn^{-1}. \tag{3.1}
$$

If (G2) holds, then

$$
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \frac{T_F^{\text{ind}} - \mathbb{E}\{T_F^{\text{ind}}\}}{\left(\text{Var}\{T_F^{\text{ind}}\}\right)^{1/2}} \leq z \right] - \Phi(z) \right| \leq Cn^{-1/2}. \tag{3.2}
$$

4 A NEW BERRY–ESSEEN BOUND FOR EXCHANGEABLE PAIR APPROACH

4.1 Berry–Esseen bound

In this section, we establish a new Berry–Esseen theorem for exchangeable pair approach under a new setting. Let $X \in \mathcal{X}$ be a random variable valued on a measurable space and let $W = \varphi(X)$ be the random variable of interest where $\varphi : \mathcal{X} \to \mathbb{R}$. Assume that $\mathbb{E}\{W\} = 0$ and $\mathbb{E}\{W^2\} = 1$. We propose the following condition:
Let \((X, X')\) be an exchangeable pair and let \(F : \mathcal{X} \times \mathcal{X} \to \mathbb{R}\) be an antisymmetric function. Assume that \(D := F(X, X')\) satisfies the following condition:

\[
\mathbb{E}[D|X] = \lambda(W + R),
\]

where \(\lambda > 0\) is a constant and \(R\) is a random variable.

We remark that the operator of antisymmetric functions was firstly mentioned by Holmes and Reinert (2004), and the condition (A) was considered by Chatterjee (2007), who applied the exchangeable pair approach to prove concentration inequalities.

The following theorem provides a uniform Berry–Esseen bound for exchangeable pair approach under the assumption (A).

**Theorem 4.1.** Let \((X, X')\) and \(D\) satisfy the condition (A). Let \(W' = \varphi(X')\) and \(\Delta = W - W'\). Then,

\[
\sup_{z \in \mathbb{R}} |\mathbb{P}[W \leq z] - \Phi(z)| \leq \mathbb{E} \left| 1 - \frac{1}{2\lambda} \mathbb{E}[D\Delta|X] \right| + \frac{1}{\lambda} \mathbb{E}[|D^*\Delta|] + \mathbb{E}|R|, \tag{4.2}
\]

provided that \(D^* := F^*(X, X') \geq |D|\), where \(F^*\) is a symmetric function.

**Remark 4.2.** Assume that (1.3) is satisfied. Then, we can choose \(D = \Delta = W - W'\), and the right hand side of (4.2) reduces to

\[
\mathbb{E} \left| 1 - \frac{1}{2\lambda} \mathbb{E}[\Delta^2|W] \right| + \frac{1}{\lambda} \mathbb{E}[|\Delta^*\Delta|] + \mathbb{E}|R|,
\]

where \(\Delta^* := \Delta^*(W, W')\) is a symmetric function for \(W\) and \(W'\) such that \(\Delta^* \geq |\Delta|\). Thus, Theorem 4.1 recovers to Theorem 2.1 in Shao and Zhang (2019).

The following corollary is useful for random variables that can be decomposed as a sum of \(W\) and a remainder term. Specifically, let \(T := T(X)\) be a random variable such that \(T = W + U\), where \(W = \varphi(X)\) is as defined at the beginning of this section, and \(U := U(X)\) is a remainder term. The following corollary gives a Berry–Esseen bound for \(T\).

**Corollary 4.3.** Let \((X, X') \in \mathcal{X} \times \mathcal{X}\) be an exchangeable pair and let \(D := F(X, X')\) where \(F : \mathcal{X} \times \mathcal{X} \to \mathbb{R}\) is antisymmetric. Assume that

\[
\mathbb{E}[D|X] = \lambda(W + R), \tag{4.3}
\]

for some \(\lambda > 0\) and some random variable \(R\). Let \(U' := U(X')\) and \(\Delta = \varphi(X) - \varphi(X')\). Then, we have

\[
\sup_{z \in \mathbb{R}} |\mathbb{P}[T \leq z] - \Phi(z)| \leq \mathbb{E} \left| 1 - \frac{1}{2\lambda} \mathbb{E}[D\Delta|X] \right| + \frac{1}{\lambda} \mathbb{E}[|D^*\Delta|] + \frac{3}{2\lambda} \mathbb{E}|D(U - U')| + \mathbb{E}|R| + \mathbb{E}|U|,
\]

provided that \(D^* := D^*(X, X')\) is any symmetric function of \(X\) and \(X'\) such that \(D^* \geq |D|\).
Remark 4.4. Assume that \( X = (X_1, \ldots, X_n) \) is a family of independent random variables. Let \( W = \sum_{i=1}^{n} \xi_i \) be a linear statistic, where \( \xi_i = h_i(X_i) \) and \( h_i \) is a nonrandom function, such that \( \mathbb{E}\{\xi_i\} = 0 \) and \( \sum_{i=1}^{n} \mathbb{E}\{\xi_i^2\} = 1 \), and let \( U = U(X_1, \ldots, X_n) \in \mathbb{R} \) be a random variable. Let \( T = W + U \), \( \beta_2 = \sum_{i=1}^{n} \mathbb{E}\{|\xi_i|^{2N}|\xi_i| > 1\} \) and \( \beta_3 = \sum_{i=1}^{n} \mathbb{E}\{|\xi_i|^{3N}|\xi_i| \leq 1\} \). Chen and Shao (2007) (see also Shao and Zhou (2016)) proved the following result:

\[
\sup_{z \in \mathbb{R}} |P[T \leq z] - \Phi(z)| \leq 17(\beta_2 + \beta_3) + 5 \mathbb{E}|U| + 2 \sum_{i=1}^{n} \mathbb{E}|\xi_i(U - U^{(i)})|,
\]

(4.4)

where \( U^{(i)} \) is any random variable independent of \( \xi_i \).

The Berry–Esseen bound in Corollary 4.3 improves Chen and Shao (2007)’s result in the sense that the random variable \( W \) in our result is not necessarily a partial sum of independent random variables, and our result in Corollary 4.3 can be applied to a general class of random variables.

4.2 Proof of Theorem 4.1

In this subsection, we prove Theorem 4.1 by Stein’s method. The proof is similar to that of Theorem 2.1 in Shao and Zhang (2013). To begin with, we need to prove the following lemma, which is useful in the proof of Theorem 4.1.

Lemma 4.5. Let \( f \) be a nondecreasing function. Then,

\[
\frac{1}{2\lambda} \mathbb{E}\left\{ D \int_{-\Delta}^{0} \left( f(W + u) - f(W) \right) du \right\} \leq \frac{1}{2\lambda} \mathbb{E}\{D^* \Delta f(W)\},
\]

where \( D^* \) is as defined in Theorem 4.1.

Proof of Lemma 4.5. Since \( f(\cdot) \) is nondecreasing, it follows that

\[
\Delta(f(W) - f(W')) \geq 0
\]

and

\[
0 \geq \int_{-\Delta}^{0} (f(W + u) - f(W)) du \\
\geq -\Delta(f(W) - f(W'))
\]

which yields

\[
-\mathbb{E}\left\{DND > 0\Delta(f(W) - f(W'))\right\} \leq \mathbb{E}\left\{ D \int_{-\Delta}^{0} \left( f(W + u) - f(W) \right) du \right\} \\
\leq -\mathbb{E}\left\{DND < 0\Delta(f(W) - f(W'))\right\}.
\]
Recalling that $W = \varphi(X)$, $D = F(X, X')$ is antisymmetric and $D^* = F^*(X, X')$ is symmetric, as $(X, X')$ is exchangeable, we have

$$E\{DN > 0\Delta\{f(W) - f(W')\}\} = -E\{DN < 0\Delta(f(W) - f(W'))\},$$

and

$$E\{D^*ND > 0\Delta f(W)\} = -E\{D^*ND < 0\Delta f(W')\}.$$ Moreover, as $E\{D^*\Delta ND = 0\Delta f(W)\} \geq 0$ and $E\{D^*ND = 0\Delta f(W)\} = -E\{D^*ND = 0\Delta \varphi(W')\}$, it follows that

$$E\{D^*\Delta ND = 0\Delta f(W)\} \geq 0.$$

Therefore,

$$\frac{1}{2\lambda} \left| E\left\{ D \int_{-\Delta}^{0} \{f(W + u) - f(W)\} \, du \right\} \right| \leq \frac{1}{2\lambda} E\{DN < 0\Delta(f(W) - f(W'))\} \leq \frac{1}{2\lambda} E\{D^*ND < 0\Delta(f(W) - f(W'))\} = \frac{1}{2\lambda} E\{D^*\Delta ND > 0 + ND < 0\} \{1 - \Phi(z)\} f(W) \leq \frac{1}{2\lambda} E\{D^*\Delta f(W)\}. \quad \Box$$

**Proof of Theorem 4.1.** We apply some ideas of Theorem 2.1 in [Shao and Zhang (2019)](http://example.com) to prove the desired result. Let $z \geq 0$ be a fixed real number, and $f_z$ the solution to the Stein equation:

$$f'(w) - wf(w) = \mathbb{N}w \leq z - \Phi(z), \quad (4.5)$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution. It is well known that (see, e.g., [Chen, Goldstein and Shao (2011)](http://example.com))

$$f_z(w) = \begin{cases} \sqrt{2\pi}e^{w^2/2}\Phi(w)\{1 - \Phi(z)\} & \text{if } w \leq z, \\ \sqrt{2\pi}e^{w^2/2}\Phi(z)\{1 - \Phi(w)\} & \text{otherwise}, \end{cases} \quad (4.6)$$

Since $E\{D|W = \lambda(W + R)\}$, and $D = F(X, X')$ is antisymmetric, it follows that, for any absolutely continuous function $f$,

$$0 = E\{D(f(W) + f(W'))\} = 2E\{Df(W)\} - E\{D(f(W) - f(W'))\} = 2\lambda E\{(W + R)f(W)\} - E\left\{ D \int_{-\Delta}^{0} f'(W + u) \, du \right\}. \quad 10$$
Rearranging the foregoing equality, we have
\[
\mathbb{E}\{W f(W)\} = \frac{1}{2\lambda} \mathbb{E}\left\{ D \int_{-\Delta}^{0} f'(W + u) \, du \right\} - \mathbb{E}\{R f(W)\}. \tag{4.7}
\]
By (4.7),
\[
\mathbb{E}\{W f_z(W)\} = \frac{1}{2\lambda} \mathbb{E}\left\{ D \int_{-\Delta}^{0} f'_z(W + u) \, du \right\} - \mathbb{E}\{R f_z(W)\},
\]
and thus,
\[
\mathbb{P}(W > z) - \{1 - \Phi(z)\} = \mathbb{E}\{f'_z(W) - W f_z(W)\} = J_1 - J_2 + J_3, \tag{4.8}
\]
where
\[
J_1 = \mathbb{E}\left\{ f'_z(W) \left( 1 - \frac{1}{2\lambda} \mathbb{E}\{D \Delta \mid W\} \right) \right\},
\]
\[
J_2 = \frac{1}{2\lambda} \mathbb{E}\left\{ D \int_{-\Delta}^{0} (f'_z(W + u) - f'_z(W)) \, du \right\},
\]
\[
J_3 = \mathbb{E}\{R f_z(W)\}.
\]
We now bound \(J_1, J_2\) and \(J_3\), separately. By Chen et al. (2011, Lemma 2.3), we have
\[
\|f_z\| \leq 1, \quad \|f'_z\| \leq 1, \quad \sup_{z \in \mathbb{R}} |w f(w)| \leq 1. \tag{4.9}
\]
Therefore,
\[
|J_1| \leq \mathbb{E}\left|1 - \frac{1}{2\lambda} \mathbb{E}\{D \Delta \mid W\}\right|,
\]
\[
|J_3| \leq \mathbb{E}|R|.
\]
For \(J_2\), observe that \(f'_z(w) = w f(w) - N w > z + \{1 - \Phi(z)\}\), and both \(w f_z(w)\) and \(N w > z\) are increasing functions (see, e.g. Chen et al. (2011, Lemma 2.3)), by Lemma 4.5,
\[
|J_2| \leq \frac{1}{2\lambda} \mathbb{E}\left| D \int_{-\Delta}^{0} \{ (W + u)f_z(W + u) - W f'_z(W)\} \, du \right| \]
\[
+ \frac{1}{2\lambda} \mathbb{E}\left| D \int_{-\Delta}^{0} \{N W + u > z - N W > z\} \, du \right| \tag{4.11}
\]
\[
\leq \frac{1}{2\lambda} \mathbb{E}\left\{ \mathbb{E}\{D^* \Delta \mid W\}\left|\left| W f_z(W)\right| + N W > z\right\}\right\}
\]
\[
\leq J_{21} + J_{22},
\]
where
\[
J_{21} = \frac{1}{2\lambda} \mathbb{E}\left\{ \mathbb{E}\{D^* \Delta \mid W\}\mid W f_z(W)\right\},
\]
\[
J_{22} = \frac{1}{2\lambda} \mathbb{E}\left\{ \mathbb{E}\{D^* \Delta \mid W\}\mid N W > z\right\}.
\]
Then, by (4.3), \(|J_2| \leq \frac{1}{\lambda} \mathbb{E}\left|\mathbb{E}\{D^* \Delta \mid W\}\right|\). This proves Theorem 4.1 together with (4.10).

\[\square\]
4.3 Proof of Corollary 4.3

In this subsection, we apply Theorem 4.1 to prove Corollary 4.3. By (4.3), we have

\[ \mathbb{E}\{D \mid X\} = \lambda(T + U - R). \]

Let \( T' = \varphi(X') + U(X') \), then we have \((T, T')\) is exchangeable. Then, by Theorem 4.1, we have

\[
\sup_{z \in \mathbb{R}} \mathbb{P}[T \leq z] - \Phi(z)
\leq E\left[1 - \frac{1}{2\lambda} \mathbb{E}\{D(T - T') \mid X\}\right]
+ \frac{1}{\lambda} \mathbb{E}\mathbb{E}\left[D^*(T - T') \mid X\right] + \mathbb{E}|U| + \mathbb{E}|R|
\leq E\left[1 - \frac{1}{2\lambda} \mathbb{E}\{\varphi(X) - \varphi(X') \mid X\}\right]
+ \frac{1}{\lambda} \mathbb{E}\mathbb{E}\left[D^*(\varphi(X) - \varphi(X')) \mid X\right] + \mathbb{E}|U| + \mathbb{E}|R| + \frac{3}{2\lambda} \mathbb{E}|D(R - R'|).
\]

This completes the proof by recalling that \( \Delta = \varphi(X) - \Phi(X') \).

5 PROOFS OF THEOREMS 2.1, 2.3 AND 2.4

In this section, we give the proofs of Theorems 2.1, 2.3 and 2.4.

5.1 Proof of Theorem 2.1

Without loss of generality, we assume that \( n \geq \max(2, k^2) \), otherwise the inequality is trivial. We use Corollary 4.3 to prove this theorem. For each \( \alpha = (\alpha(1), \ldots, \alpha(k)) \in \mathcal{I}_{n,k} \), let

\[
r(X_{\alpha(1)}, \ldots, X_{\alpha(k)}; Y_{\alpha(1),\alpha(2)}, \ldots, Y_{\alpha(k-1),\alpha(k)})
= f(X_{\alpha(1)}, \ldots, X_{\alpha(k)}; Y_{\alpha(1),\alpha(2)}, \ldots, Y_{\alpha(k-1),\alpha(k)}) \sum_{j=1}^{k} f_1(X_{\alpha(j)}). \tag{5.1}
\]

Let \( \sigma_n = \sqrt{\text{Var}\{S_{n,k}(f)\}} \), and

\[
T = \frac{1}{\sigma_n}(S_{n,k}(f) - \mathbb{E}\{S_{n,k}(f)\}) = W + U,
\]

where

\[
W = \frac{1}{\sigma_n} \sum_{i=1}^{n} f_1(X_i),
\]

\[
U = \frac{1}{\sigma_n} \sum_{\alpha \in \mathcal{I}_{n,k}} r(X_{\alpha(1)}, \ldots, X_{\alpha(k)}; Y_{\alpha(1),\alpha(2)}, \ldots, Y_{\alpha(k-1),\alpha(k)}).
\]
By orthogonality we have \( \text{Cov}(W, U) = 0 \), and thus
\[ \sigma_n^2 \geq \text{Var}(W) = \left( \frac{n - 1}{n - k} \right)^2 \text{Var} \left( \sum_{j=1}^{n} f_1(X_j) \right) = \left( \frac{n}{k} \right)^2 \frac{k^2 \sigma_1^2}{n}. \] (5.2)

Let \((X'_1, \ldots, X'_n)\) be an independent copy of \((X_1, \ldots, X_n)\). For each \(i = 1, \ldots, n\), define \(X^{(i)} = (X^{(i)}_1, \ldots, X^{(i)}_n)\) where
\[ X^{(i)}_j = \begin{cases} X_j & \text{if } j \neq i, \\ X'_i & \text{if } j = i, \end{cases} \]
and let
\[ U^{(i)} = \frac{1}{\sigma_n} \sum_{\alpha \in I, k} r(X^{(i)}_{\alpha(1)}, \ldots, X^{(i)}_{\alpha(k)}, Y_{\alpha(1),\alpha(2)}, \ldots, Y_{\alpha(k-1),\alpha(k)}). \]
The following lemma provides the upper bounds of \( \mathbb{E}\{R_1^2\} \) and \( \mathbb{E}\{(R_1 - R_1^{(i)})^2\} \).

**Lemma 5.1.** For \( n \geq 2 \) and \( k \geq 2 \),
\[ \mathbb{E}\{U^2\} \leq \frac{(k - 1)^2 \tau^2}{2(n - 1)\sigma_1^2}. \] (5.3)
\[ \mathbb{E}\{(U - U^{(i)})^2\} \leq \frac{2(k - 1)^2 \tau^2}{n(n - 1)\sigma_1^2}. \] (5.4)
The proof of Lemma 5.1 is put in the appendix.

Now, we apply Corollary 4.3 to prove the Berry–Esseen bound for \( T \). To this end, let \( \xi_i = \sigma_n^{-1} f_1(X_i) \) for each \( 1 \leq i \leq n \). Let \( I \) be a random index uniformly distributed over \( \{1, \ldots, n\} \), which is independent of all others. Let
\[ D = \Delta = \frac{1}{\sigma_n} \left( \frac{n - 1}{n - k} \right) (f_1(X_I) - f_1(X'_I)), \]
then it follows that
\[ \mathbb{E}\{D \mid W\} = \frac{1}{n} W. \]
Thus, (4.3) is satisfied with \( \lambda = 1/n \) and \( R = 0 \). Moreover, we have
\[ \frac{1}{2\lambda} \mathbb{E}\{D \Delta \mid X\} = \frac{1}{2\sigma_n} \left( \frac{n - 1}{n - k} \right)^2 \sum_{i=1}^{n} (f_1(X_i) - f_1(X'_i))^2, \]
\[ \frac{1}{\lambda} \mathbb{E}\{|D\Delta \mid X\} = \frac{1}{\sigma_n} \left( \frac{n - 1}{n - k} \right)^2 \sum_{i=1}^{n} (f_1(X_i) - f_1(X'_i)) |f_1(X_i) - f_1(X'_i)|. \]
Also,
\[
\frac{1}{2\lambda} \mathbb{E}\{D\Delta\} = \mathbb{E}\{W^2\} = 1 - \mathbb{E}\{U^2\}, \quad \mathbb{E}\{|D\Delta|\} = 0.
\]

Therefore, by the Cauchy inequality and Lemma 5.1 we have for \( n \geq \max(2, k^2) \),
\[
\mathbb{E}\left| \frac{1}{2\lambda} \mathbb{E}\{D\Delta \mid X\} - 1 \right| \\
\leq \mathbb{E}\left| \frac{1}{2\lambda} \mathbb{E}\{D\Delta \mid X\} - \frac{1}{2\lambda} \mathbb{E}\{D\Delta\} \right| + \mathbb{E}\{U^2\} \\
\leq \frac{1}{2\sigma_n^2} \left( \frac{n-1}{n-k} \right)^2 \left( \text{Var}\left\{ \sum_{i=1}^{n} (f_1(X_i) - f_1(X_i'))^2 \right\} \right)^{1/2} + \frac{(k-1)^2 \tau^2}{2(n-1)\sigma_1^2} \\
\leq \frac{2\tau^2}{\sqrt{n}\sigma_1^2} + \frac{(k-1)\tau^2}{\sqrt{n}\sigma_1^2} \leq \frac{(k+1)\tau^2}{\sqrt{n}\sigma_1^2},
\]
where we used (5.2) in the last line. Using the same argument, we have for \( n \geq \max\{2, k^2\} \),
\[
\mathbb{E}\left| \frac{1}{\lambda} \mathbb{E}\{|D\Delta \mid X\}\right| \\
\leq \frac{1}{\sigma_n^2} \left( \frac{n-1}{n-k} \right)^2 \left( \text{Var}\left\{ \sum_{i=1}^{n} (f_1(X_i) - f_1(X_i'))^2 \right\} \right)^{1/2} \\
\leq \frac{4\tau^2}{\sqrt{n}\sigma_1^2}.
\]

Now we give the bounds for \( U \) and \( U^{(i)} \). We have two cases. For the case where \( k = 1 \), then it follows that \( U = 0 \) and \( U^{(i)} = 0 \). As for \( k \geq 2 \), noting that \( (n-1)^{-1/2} \leq 2n^{-1/2} \) for \( n \geq 2 \), by Lemma 5.1 and the Cauchy inequality, we have
\[
\mathbb{E}|U| \leq \frac{0.71(k-1)\tau}{(n-1)^{1/2}\sigma_1} \leq \frac{2(k-1)\tau}{\sqrt{n}\sigma_1},
\]
and
\[
\sum_{i=1}^{n} \mathbb{E}\{|(\xi_i - \xi_i')(U - U^{(i)})|\} \leq \frac{2.84(k-1)\tau}{(n-1)^{1/2}\sigma_1} \leq \frac{6(k-1)\tau}{\sqrt{n}\sigma_1}.
\]

By Corollary 4.3 and noting that \( \sigma_1^2 \leq \mathbb{E}\{f(X_{(\alpha)}; Y_{(\alpha)})^2\} \leq \tau^{1/2} \), we have
\[
\sup_{z \in \mathbb{R}} |\mathbb{P}[T \leq z] - \Phi(z)| \leq \frac{(k+5)\tau^2}{\sqrt{n}\sigma_1^2} + \frac{11(k-1)\tau}{\sqrt{n}\sigma_1} \\
\leq \frac{12k\tau^2}{\sqrt{n}\sigma_1^2}.
\]

This proves (2.3).
5.2 Proof of Theorem 2.2

We first prove a proposition for the Hoeffding decomposition.

Proposition 5.2. For \( A \subset [n], B \subset [n] \) such that \((A, B) \neq (\emptyset, \emptyset)\), and for any \( \tilde{A}, \tilde{B} \) such that \( \tilde{A} \subset A \) and \( \tilde{B} \subset B \) but \((\tilde{A}, \tilde{B}) \neq (A, B)\), we have

\[
\mathbb{E}\{f_{A, B}(X_A; Y_B) \mid X_{\tilde{A}}, Y_{\tilde{B}}\} = 0.
\]  \hspace{1cm} (5.5)

Proof. If \(|A| + |B| = 1\), then for \((\tilde{A}, \tilde{B}) = (\emptyset, \emptyset)\), by definition,

\[
\mathbb{E}\{f_{A, B}(X_A; Y_B) \mid X_{\tilde{A}}, Y_{\tilde{B}}\} = \mathbb{E}\{f_{A, B}(X_A; Y_B)\} = \mathbb{E}\{f(X_k; Y_{[k]}2)\} - \mathbb{E}\{f(X_k; Y_{[k]}2)\} = 0.
\]

We prove the proposition by induction. Assume that (5.5) holds for \(1 \leq |A| + |B| \leq m\). Let \( \mathcal{A}_{\tilde{A}, \tilde{B}} = \{(A', B') : A' \subset A, B' \subset B\} \) and let \( \mathcal{A}_{\tilde{A}, \tilde{B}}^c = \{(A', B') : A' \subset A, B' \subset B, (A', B') \neq (A, B)\} \setminus \mathcal{A}_{\tilde{A}, \tilde{B}}^c\). Reordering (2.1) by the inclusive-exclusive formula we have

\[
f_{A, B}(X_A; Y_B) = \mathbb{E}\{f(X_k; Y_{[k]}2)\mid X_A, Y_B\} - \sum_{|A'| + |B'| < |A| + |B|} f_{A', B'}(X_{A'}; Y_{B'}),
\]

\[
= \mathbb{E}\{f(X_k; Y_{[k]}2)\mid X_A, Y_B\} - \sum_{(A', B') \in \mathcal{A}_{\tilde{A}, \tilde{B}}} f_{A', B'}(X_{A'}; Y_{B'})
\]

\[
- \sum_{(A', B') \in \mathcal{A}_{\tilde{A}, \tilde{B}}^c} f_{A', B'}(X_{A'}; Y_{B'}).
\]

By the induction assumption, we have

\[
\sum_{(A', B') \in \mathcal{A}_{\tilde{A}, \tilde{B}}^c} \mathbb{E}\{f_{A', B'}(X_{A'}; Y_{B'})\mid X_{\tilde{A}}, Y_{\tilde{B}}\} = 0.
\]

Then, the desired result follows. \( \square \)

Let

\[
\mathcal{A}_{n, \ell} = \{\alpha = (\alpha(1), \ldots, \alpha(\ell)) : 1 \leq \alpha(1) \neq \ldots \neq \alpha(\ell) \leq n\}.
\]

Then, \( \mathcal{I}_{n, \ell} \subset \mathcal{A}_{n, \ell} \). For \( A \subset [\ell] \) and \( B \subset [\ell]_2 \) and \( \alpha = (\alpha(1), \ldots, \alpha(\ell)) \in \mathcal{A}_{n, \ell} \), write

\[
\alpha(A) = (\alpha(i))_{i \in A}, \quad \alpha(B) = ((\alpha(i), \alpha(j)))_{(i, j) \in B},
\]

\[
X_{\alpha(A)} = (X_i)_{i \in \alpha(A)}, \quad Y_{\alpha(B)} = (Y_{i,j})_{(i, j) \in \alpha(B)}.
\]

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Moreover, for any \( \alpha \in I_{n,\ell} \) and \( f_{A,B} : X^{|A|} \times Y^{|B|} \to \mathbb{R} \), let
\[
\tilde{S}_{n,\ell}(f_{A,B}) = \sum_{\alpha \in A_{n,\ell}} f_{A,B}(X_{\alpha(A)}; Y_{\alpha(B)}),
\]
and similarly, \( S_{n,\ell}(f_{A,B}) \) can be represented as \( \sum_{\alpha \in I_{n,\ell}} f_{A,B}(X_{\alpha(A)}; Y_{\alpha(B)}) \).

Let \((Y_{1,1}',\ldots,Y_{n-1,n})\) be an independent copy of \(Y=(Y_{1,1},\ldots,Y_{n-1,n})\). For any \((i,j)\in A_{n,2}\), let \(Y^{(i,j)} = (Y_{1,1}',\ldots,Y_{n-1,n}')\) with
\[
Y_{p,q}^{(i,j)} = \begin{cases} 
Y_{p,q} & \text{if } \{p,q\} \neq \{i,j\}, \\
y_{p,q}' & \text{if } \{p,q\} = \{i,j\},
\end{cases}
\]
for \((p,q)\in I_{n,2}\).

Then, it follows that for each \((i,j)\in A_{n,2}\), \(((X,Y),(X,Y')^{(i,j)})\) is an exchangeable pair. For any \(B \subset [n]_2\), let \(Y^{(i,j)}_B = (Y^{(i,j)}_{p,q})_{(p,q) \in B}\). For any \(A \subset [\ell]\), \(B \subset [\ell]_2\), \(\alpha = (\alpha(1),\ldots,\alpha(\ell)) \in I_{n,\ell}\) and \(f_{A,B} : X^{|A|} \times Y^{|B|} \to \mathbb{R}\), define
\[
Y^{(i,j)}_{\alpha(B)} = (Y^{(i,j)}_{\alpha(p),\alpha(q)})_{(p,q) \in B},
\]
\[
S_{n,\ell}^{(i,j)}(f_{A,B}) = \sum_{\alpha \in I_{n,\ell}} f_{A,B}(X_{\alpha(A)}; Y^{(i,j)}_{\alpha(B)}),
\]
\[
\tilde{S}_{n,\ell}^{(i,j)}(f_{A,B}) = \sum_{\alpha \in A_{n,\ell}} f_{A,B}(X_{\alpha(A)}; Y^{(i,j)}_{\alpha(B)}).
\]

Let \(f_{(\ell)}\) be defined as in (2.2), and it follows that
\[
f = \sum_{\ell=0}^{k} f_{(\ell)}, \quad f_{(0)} = \mathbb{E}\{f(X_{[k]}; Y_{[k]})\}, \quad S_{n,k}(f_{(0)}) = \mathbb{E}\{S_{n,k}(f)\}.
\]

Moreover, by assumption, as \(f\) has principal degree \(d\), and it follows that \(f_{(\ell)} \equiv 0\) for \(\ell = 1,\ldots,d-1\). Let \(s_n = (\text{Var}(S_{n,k}(f)))^{1/2}\) and \(s_{n,\ell} = (\text{Var}(S_{n,k}(f_{(\ell)})))^{1/2}\). The next lemma estimates the upper and lower bounds of \(s_n^2\) and \(s_{n,d}^2\). The proof is similar to that of Lemma 4 of [Janson and Nowicki (1991)], and we omit the details.

**Lemma 5.3.** We have for each \((i,j)\in A_{n,2}\) and \(d \leq \ell \leq k\),
\[
\sigma_{n,\ell}^2 = \sum_{(A,B): v_{A,B}=\ell} \frac{n!(n-\ell)!\sigma_{A,B}^2}{(n-k)^2(k-\ell)!|\text{Aut}(G_{A,B})|} \leq Cn^{2k-\ell-2}, \tag{5.6}
\]
\[
\sigma_n^2 = \sum_{\ell=d}^{k} \sum_{(A,B): v_{A,B}=\ell} \frac{n!(n-\ell)!\sigma_{A,B}^2}{(n-k)^2(k-\ell)!|\text{Aut}(G_{A,B})|} \leq Cn^{2k-d-2}, \tag{5.7}
\]
\[
\mathbb{E}\{(S_{n,k}(f_{(\ell)}) - S_{n,k}^{(i,j)}(f_{(\ell)}))^2\} \leq Cn^{2k-\ell-2}, \tag{5.8}
\]
and
\[
\sigma_n^2 \geq \sigma_{n,d}^2 \geq cn^{2k-d}\sigma_{\text{min}}^2, \tag{5.9}
\]
where \(|\text{Aut}(G)|\) is the number of the automorphisms of \(G\), and \(c,C > 0\) are some absolute constant.
For any $A \subset [k]$ and $B \subset [k]_2$, let

$$
\mu_{A,B} := \frac{1}{|\text{Aut}(G_{A,B})||B|} \binom{n-v_{A,B}}{n-k},
$$

$$
\nu_{A,B} := |B| \times \mu_{A,B} = \frac{1}{|\text{Aut}(G_{A,B})|} \binom{n-v_{A,B}}{n-k},
$$

and for any $\alpha \in A_{n,\ell} (\ell = 1, \ldots, k)$, let

$$
\xi_{\alpha(A,B)}^{(i,j)} = f_{A,B}(X_\alpha(A); Y_\alpha(B)) - f_{A,B}(X_{\alpha(A)}; Y_{\alpha(B)}).
$$

Recall that $G_{A,B}$ is the graph generated by $(A, B)$. For any $(A_j, B_j)$ for $j = 1, 2$, we simply write $v_j = v_{A_j, B_j}$ as the number of nodes of the graph $G_{A_j, B_j}$. Recall that $G_{f,d} = \{(A, B) : A \subset [d], B \subset [d]_2, \sigma_{A,B} > 0, v_{A,B} = d\}$ and we similarly define $G_{f,d+1} = \{(A, B) : A \subset [d+1], B \subset [d+1]_2, \sigma_{A,B} > 0, v_{A,B} = d + 1\}$. We have the following lemmas.

**Lemma 5.4.** For all $(A_1, B_1), (A_2, B_2) \in G_{f,d}$ such that $G_{A_1,B_1}$ and $G_{A_2,B_2}$ are connected, we have

$$
\text{Var}\left\{ \sum_{(i,j) \in A_{n,2}} \left( \sum_{\alpha_1 \in A_{n,d}^{(i,j)}} \xi_{\alpha_1(A_1,B_1)}^{(i,j)} \right) \left( \sum_{\alpha_2 \in A_{n,d}^{(i,j)}} \xi_{\alpha_2(A_2,B_2)}^{(i,j)} \right) \right\} \leq C n^{2d-1} \tau^4.
$$

**Lemma 5.5.** Assume that $k \geq d + 1$. For all $(A_1, B_1), (A_2, B_2) \in G_{f,d} \cup G_{f,d+1}$, we have

$$
\text{Var}\left\{ \sum_{(i,j) \in A_{n,2}} \left( \sum_{\alpha_1 \in A_{n,d,v_1}^{(i,j)}} \xi_{\alpha_1(A_1,B_1)}^{(i,j)} \right) \left( \sum_{\alpha_2 \in A_{n,d,v_2}^{(i,j)}} \xi_{\alpha_2(A_2,B_2)}^{(i,j)} \right) \right\} \leq C n^{2 \max\{v_1, v_2\} - 2} \tau^4.
$$

We are now ready to give the proof of **Theorem 2.3**.

Proof of **Theorem 2.3** We assume that $n \geq \max\{k, 2\}$ without loss of generality, otherwise the result is trivial. Recall that $f_{(d)}$ is defined in (2.2). Write $T = \sigma_n^{-1} (S_{n,k}(f) - \mathbb{E}\{S_{n,k}(f)\})$, and

$$
W = \sigma_n^{-1} S_{n,k}(f_{(d)}), \quad U = T - W = \sigma_n^{-1} \sum_{\ell = d+1}^k S_{n,k}(f_{(\ell)}).
$$

(5.10)

Here, if $d + 1 > k$, then set $\sum_{\ell = d+1}^k S_{n,k}(f_{(\ell)}) = 0$. With a slight abuse of notation, we write $(A, B) \in G_{f,d}$ if $G_{A,B} \in G_{f,d}$. We have

$$
W = \frac{1}{\sigma_n} \sum_{\alpha \in A_{n,d}} \sum_{(A,B) \in G_{f,d}} \binom{n-d}{k-d} \frac{f_{A,B}(X_\alpha(A); Y_\alpha(B))}{|\text{Aut}(G_{A,B})|}.
$$

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\[
\frac{1}{\sigma_n} \sum_{\alpha \in A_{n,d}} \sum_{(A,B) \in \mathcal{G}_{f,d}} \begin{pmatrix} n - d \\ k - d \end{pmatrix} \frac{f_{A,B}(X_{\alpha(A)}; Y_{\alpha(B)})}{|\text{Aut}(G_{A,B})|},
\]

because by assumption, \( f_{A,B} \equiv 0 \) for all \((A, B) \in \mathcal{G}_{f,d} \).

For each \((i, j) \in A_{n,2} \), let

\[
W^{(i,j)} = \frac{1}{\sigma_n} s_{n,k}^{(i,j)}(f(d)), \quad U^{(i,j)} = \sigma_n^{-1} \sum_{\ell = d+1}^k s_{n,k}^{(i,j)}(f(\ell)).
\]

Let \((I, J)\) be a random 2-fold index uniformly chosen in \( A_{n,2} \), which is independent of all others. Then, \(((X, Y), (X, Y^{(I, J)}))\) is an exchangeable pair. Let

\[
\Delta = W - W^{(I, J)} = \frac{1}{\sigma_n} \sum_{\alpha \in A_{n,d}} \sum_{(A,B) \in \mathcal{G}_{f,d}} \nu_{A,B} \xi_{\alpha(A,B)}^{(I, J)}.
\]

Also, define

\[
D = \frac{1}{\sigma_n} \sum_{\alpha \in A_{n,d}} \sum_{(A,B) \in \mathcal{G}_{f,d}} \mu_{A,B} \xi_{\alpha(A,B)}^{(I, J)}.
\]

Then, we have \( D \) is antisymmetric with respect to \((X, Y)\) and \((X, Y^{(I, J)})\).

Let \( A_{n,d}^{(i,j)} = \{\alpha \in A_{n,d} : \{i, j\} \subseteq \{\alpha\}\} \). Then,

\[
\mathbb{E}\{D \mid X, Y\} = \frac{1}{n(n-1)e_n} \sum_{(i,j) \in A_{n,2}} \sum_{\alpha \in A_{n,d}^{(i,j)}} \sum_{(A,B) \in \mathcal{G}_{f,d}} \mu_{A,B} \mathbb{E}\{\xi_{\alpha(A,B)}^{(i,j)} \mid X, Y\}.
\]

By (5.5),

\[
\mathbb{E}\{f_{A,B}(X_{\alpha(A)}; Y_{\alpha(B)}) \mid X, Y\} = \mathbb{E}\{f_{A,B}(X_{\alpha(A)}; Y_{\alpha(B)}) \mid X_A, Y_B \setminus \{Y_{i,j}\}\}
= \begin{cases} 
0 & \text{if } (i, j) \in B, \\ f_{A,B}(X_{\alpha(A)}; Y_{\alpha(B)}) & \text{otherwise}.
\end{cases}
\]

Moreover, note that for \(\alpha \in A_{n,d}\),

\[
\sum_{(i,j) \in A_{n,2}} N(i, j) \in \alpha(B) = 2|B|,
\]

and thus

\[
\mathbb{E}\{D \mid X, Y\}
\]
\[
\begin{align*}
&= \frac{1}{n(n-1)} \sigma_n \sum_{\alpha \in A_{n,d}} \sum_{(A,B) \in \mathcal{G}_{f,d}} \mu_{A,B} f_{A,B}(X_{\alpha(A)}; Y_{\alpha(B)}) \\
&\quad \times \sum_{(i,j) \in A_{n,2}} \mathbb{N}(i, j) \in \alpha(B) \\
&= \frac{2}{n(n-1)} \sigma_n \sum_{\alpha \in A_{n,d}} \sum_{(A,B) \in \mathcal{G}_{f,d}} \nu_{A,B} f_{A,B}(X_{\alpha(A)}; Y_{\alpha(B)}) \\
&= \frac{2}{n(n-1)} W. \tag{5.12}
\end{align*}
\]

Thus, (1.3) is satisfied with \( \lambda = 2/(n(n-1)) \) and \( R = 0 \). Moreover, by exchangeability,

\[
\mathbb{E}\{D\Delta\} = 2 \mathbb{E}\{DW\} = 2 \lambda \sigma_n^2 / \sigma_n^2. \tag{5.13}
\]

Then, we have

\[
\frac{1}{2\lambda} \mathbb{E}\{D\Delta \mid X, Y, Y'\} = \frac{1}{4 \sigma_n^2} \sum_{(A_1,B_1) \in \mathcal{G}_{f,d}} \sum_{(A_2,B_2) \in \mathcal{G}_{f,d}} \mu_{A_1,B_1} \nu_{A_2,B_2} \\
\times \sum_{(i,j) \in A_{n,2}} \left( \sum_{\alpha \in A_{n,d}^{(i,j)}} \xi_{\alpha(A_1,B_1)}^{(i,j)} \right) \left( \sum_{\alpha \in A_{n,d}^{(i,j)}} \xi_{\alpha(A_2,B_2)}^{(i,j)} \right).
\]

Now, by the Cauchy inequality, (5.13) and Lemmas 5.3 and 5.4, we have

\[
\mathbb{E}\left[ \frac{1}{2\lambda} \mathbb{E}\{D\Delta \mid X, Y, Y'\} - 1 \right] \\
\leq \mathbb{E}\left[ \frac{1}{2\lambda} \mathbb{E}\{D\Delta \mid X, Y, Y'\} - \frac{1}{2\lambda} \mathbb{E}\{D\Delta\} \right] + \frac{\sigma_n^2 - \sigma_{n,d}^2}{\sigma_n^2} \\
\leq \frac{1}{4 \sigma_n^2} \sum_{(A_1,B_1) \in \mathcal{G}_{f,d}} \sum_{(A_2,B_2) \in \mathcal{G}_{f,d}} \mu_{A_1,B_1} \nu_{A_2,B_2} \\
\times \left( \text{Var} \left\{ \sum_{(i,j) \in A_{n,2}} \left( \sum_{\alpha \in A_{n,d}^{(i,j)}} \xi_{\alpha(A_1,B_1)}^{(i,j)} \right) \left( \sum_{\alpha \in A_{n,d}^{(i,j)}} \xi_{\alpha(A_2,B_2)}^{(i,j)} \right) \right\} \right)^{1/2} \\
+ \frac{\sigma_n^2 - \sigma_{n,d}^2}{\sigma_n^2} \\
\leq Cn^{-1/2}.
\]

Taking \( D^* = |D| \), by Lemma 5.5,

\[
\frac{1}{\lambda} \mathbb{E}|\mathbb{E}\{D^*\Delta \mid X, Y, Y'\}|
\]
\[
\frac{1}{\lambda} \mathbb{E}[D^* \Delta | X, Y, Y'] \\
\leq \frac{1}{4\sigma_n^2} \sum_{(A_1, B_1) \in \mathcal{G}_{f,d}} \sum_{(A_2, B_2) \in \mathcal{G}_{f,d}} \mu_{A_1, B_1} \nu_{A_2, B_2} \\
\times \left( \text{Var}\left\{ \sum_{(i,j) \in A_{n,2}} \sum_{\alpha \in A_{\alpha(i,j), n, d}} \xi_{\alpha(A_1, B_1)}^{(i,j)} \left( \sum_{\alpha \in A_{\alpha(i,j), n, d}} \xi_{\alpha(A_2, B_2)}^{(i,j)} \right) \right\} \right)^{1/2} \\
\leq C n^{-1}.
\]

Now, by (5.10) and Lemma 5.3 we have

\[
\mathbb{E}[U]^2 \leq C \sigma_n^{-2} (k - d) \sum_{\ell = d + 1}^{k} \mathbb{E}(S_{n, k}^2(f(\ell))) \leq C n^{-1},
\]

\[
\mathbb{E}(U - U^{(i,j)})^2 \leq C \sigma_n^{-2} (k - d) \sum_{\ell = d + 1}^{k} \mathbb{E}\left\{ (S_{n, k}(f(\ell)) - S_{n, k}^{(i,j)}(f(\ell)))^2 \right\} \leq C n^{-3},
\]

\[
\mathbb{E}(W - W^{(i,j)})^2 \leq C \sigma_n^{-2} \mathbb{E}\left\{ (S_{n, k}(f(d)) - S_{n, k}^{(i,j)}(f(d)))^2 \right\} \leq C n^{-2}.
\]

Thus,

\[
\frac{1}{\lambda} \mathbb{E}[U] \leq C n^{-1/2},
\]

\[
\frac{1}{\lambda} \mathbb{E}[\Delta(U - U^{(i,j)})] = \sum_{i \in Z_{n,2}} \mathbb{E}[|W - W^{(i,j)}|(U - U^{(i,j)})]|]
\]

\[
\leq C n^{-1/2}.
\]

Applying Corollary 4.3 we obtain the desired result. \(\square\)

### 5.3 Proof of Theorem 2.4

The proof of Theorem 2.4 is similar to that of Theorem 2.3. Without loss of generality, we assume that \(k \geq d + 1\), otherwise the proof is even simpler.

For any \(A \subset [k]\) and \(B \subset [k]_2\), recall that

\[
\mu_{A,B} := \frac{1}{|\text{Aut}(G_{A,B})||B|} \binom{n - v_{A,B}}{n - k},
\]

\[
\nu_{A,B} := |B| \mu_{A,B} = \frac{1}{|\text{Aut}(G_{A,B})|} \binom{n - v_{A,B}}{n - k}.
\]

By Proposition 5.2 we have there exists a Hoeffding decomposition of \(g\) as follows:

\[
g(y) = \sum_{B \subset [k]_2} g_B(y_B),
\]
where \( y = (y_{1,2}, \ldots, y_{k-1,k}) \) and \( y_B = (y_{i,j} : (i,j) \in B) \). Also, for any \( B \subset [k]_2 \) and \( \alpha \in \mathcal{A}_{n,\ell} (\ell = 1, \ldots, k) \), let \( \eta_{\alpha(B)}^{(i,j)} = g_B(Y_{\alpha(B)}) - g_B(Y_{\alpha(B)}^{(i,j)}) \).

For any \( B \in [k]_2 \), let \( V_B \) be the node set of the graph with edge set \( B \). For any \( r \in V_B \), let \( B^{(r)} = \{ (i,j) : (i,j) \in B, i \neq r, j \neq r \} \). Recall that \( \tilde{G}_{f,d+1} = \{(A,B) : A \subset [k], B \subset [k]_2, v_{A,B} = d+1, \sigma_{A,B} > 0\} \) and \( \tilde{G}_{f,d} = \{(A,B) \in G_{f,d} : G_{A,B} \) is strongly connected.}\}

We need to apply the following lemma in the proof of Theorem 2.4.

**Lemma 5.6.** Assume that \( k \geq d + 1 \). For all \( (A_j, B_j) \in \tilde{G}_{f,d} \cup \tilde{G}_{f,d+1} \) and let \( v_j = v_{A_j,B_j} \) for \( j = 1, 2 \), we have

\[
\text{Var} \left\{ \sum_{(i,j) \in A_{n,2}} \left( \sum_{\alpha_1 \in \mathcal{A}_{n,v_1}} \eta_{\alpha_1(B_1)}^{(i,j)} \right) \left( \sum_{\alpha_2 \in \mathcal{A}_{n,v_2}} \eta_{\alpha_2(B_2)}^{(i,j)} \right) \right\} \leq C n^{2d-2} r^4.
\]

**Proof of Theorem 2.4** Again, write \( T = \sigma_n^{-1}(S_{n,k}(g)) - \mathbb{E}(S_{n,k}(g)) \), and let

\[
W = \sigma_n^{-1}(S_{n,k}(g_{(d)})) + S_{n,k}(g_{(d+1)})), \quad U = \sigma_n^{-1} \sum_{\ell=d+2}^k S_{n,k}(g_{(\ell)}).
\]

(5.14)

Here, if \( d + 1 > k \), then set \( \sum_{\ell=d+1}^k S_{n,k}(g_{(\ell)}) = 0 \). Then, \( T = W + U \). Now we apply Corollary 4.3 again to prove the desired result. To this end, we need to construct an exchangeable pair. For each \( (i,j) \in A_{n,2} \), let

\[
W^{(i,j)} = \frac{1}{\sigma_n} (S_{n,k}^{(i,j)}(g_{(d)})) + S_{n,k}^{(i,j)}(g_{(d+1)})), \quad U^{(i,j)} = \sigma_n^{-1} \sum_{\ell=d+2}^k S_{n,k}^{(i,j)}(g_{(\ell)}).
\]

By assumption, we have

\[
W = \frac{1}{\sigma_n} \sum_{(A,B) \in \tilde{G}_{f,d} \cup \tilde{G}_{f,d+1} \subset A_{n,\ell}(G)} \nu_{A,B} g_{A,B}(X_{\alpha(A)}; Y_{\alpha(B)}).
\]

Let \( (I, J) \) be a random 2-fold index uniformly chosen in \( A_{n,2} \), which is independent of all others. Then, \( ((X,Y), (X,Y(I,J))) \) is an exchangeable pair. Let

\[
\Delta = W - W^{(I,J)} = \frac{1}{\sigma_n} \left( \sum_{(A,B) \in \tilde{G}_{f,d} \cup G_{A,B}} \nu_{A,B} g_{A,B}^{(I,J)}(X_{\alpha(A)}; Y_{\alpha(B)}) \right).
\]

Also, define

\[
D = \frac{1}{\sigma_n} \left( \sum_{(A,B) \in \tilde{G}_{f,d} \cup G_{A,B}} \mu_{A,B} g_{A,B}^{(I,J)}(X_{\alpha(A)}; Y_{\alpha(B)}) \right).
\]
Then, \( D \) is antisymmetric with respect to \((X, Y)\) and \((X, Y^{(I,J)})\).

Following a similar argument leading to (5.12),
\[
\mathbb{E}\{D | X, Y\} = \frac{2}{n(n-1)}W.
\] (5.15)

Thus, (4.3) is satisfied with \( \lambda = 2/(n(n-1)) \) and \( R = 0 \). Moreover, by exchangeability,
\[
\mathbb{E}\{D\Delta\} = 2\mathbb{E}\{DW\} = 2\lambda\mathbb{E}\{W^2\} = 2\lambda(\sigma_{n,d}^2 + \sigma_{n,d+1}^2)/\sigma_n^2.
\] (5.16)

Now, by the Cauchy inequality, (5.16) and Lemmas 5.3 and 5.6, we have
\[
\mathbb{E}\left| \frac{1}{2\lambda} \mathbb{E}\{D\Delta | X, Y, Y'\} - 1 \right| \leq \mathbb{E}\left| \frac{1}{2\lambda} \mathbb{E}\{D\Delta | X, Y, Y'\} - \frac{1}{2\lambda} \mathbb{E}\{D\Delta\} \right| + \frac{\sigma_n^2 - \sigma_{n,d}^2 - \sigma_{n,d+1}^2}{\sigma_n^2} \leq Cn^{-1}.
\]

With \( D^* = |D| \), and by Lemma 5.3 again,
\[
\frac{1}{\lambda} \mathbb{E}\left| \mathbb{E}\{D^*\Delta | X, Y, Y'\} \right| \leq Cn^{-1}.
\]

Now, by (5.14) and Lemma 5.3, we have
\[
\mathbb{E}|U|^2 \leq C\sigma_n^{-2}(k - d) \sum_{\ell=d+2}^{k} \mathbb{E}(S_{n,k}^2(g(\ell))) \leq Cn^{-2},
\]
\[
\mathbb{E}(U - U^{(i,j)})^2 \leq C\sigma_n^{-2}(k - d) \sum_{\ell=d+2}^{k} \mathbb{E}\{(S_{n,k}(g(\ell)) - S_{n,k}^{(i,j)}(g(\ell)))^2\} \leq Cn^{-4},
\]
\[
\mathbb{E}(W - W^{(i,j)})^2 \leq C\sigma_n^{-2}\left\{\|S_{n,k}(g(d)) - S_{n,k}^{(i,j)}(g(d))\|_2^2 + \|S_{n,k}(g(d+1)) - S_{n,k}^{(i,j)}(g(d+1))\|_2^2\right\} \leq Cn^{-2}.
\]

Thus,
\[
\mathbb{E}|U| \leq Cn^{-1},
\]
\[
\frac{1}{\lambda} \mathbb{E}\left| \Delta(U - U^{(i,j)}) \right| \leq C \sum_{(i,j) \in \mathcal{A}_{n,2}} \mathbb{E}[(W - W^{(i,j)})(U - U^{(i,j)})] \leq Cn^{-1}.
\]

Applying Corollary 4.3 we obtain the desired result. \( \square \)
6 PROOF OF OTHER RESULTS

6.1 Proof of Theorem 3.2

As \( f_{F}^{\text{inj}} \) does not depend on \( X \) if \( \kappa \equiv p \) for some \( 0 < p < 1 \). Fix \( F \). Define

\[
g^{\text{inj}}(Y) = f_{F}^{\text{inj}}(X; Y)
\]

and by Proposition 5.2, we have \( g^{\text{inj}} \) has the following decomposition:

\[
g^{\text{inj}}(Y) = \sum_{B \subseteq \kappa} g^{\text{inj}}_B(Y_B).
\]

(6.1)

By Janson and Nowicki (1991, p. 361), we have

\[
g_{\{(1,2)\}}^{\text{inj}}(y_{1,2}) = \frac{2e(F)(v(F) - 2)!}{|\text{Aut}(G)|} p^{e(F) - 1}(y_{1,2} - p) \neq 0.
\]

Therefore, by Theorem 2.4 with \( d = 2 \), we complete the proof.

6.2 Proof of Theorem 3.3

Again, let

\[
g^{\text{ind}}(Y) = f_{F}^{\text{ind}}(X; Y),
\]

and similar to (6.1), we have

\[
g^{\text{ind}}(Y) = \sum_{B \subseteq \kappa} g^{\text{ind}}_B(Y_B).
\]

Recall that \( e(F) \) is the number of 2-stars in \( F \) and \( t(F) \) is the number of triangles in \( F \). Let

\[
\bar{e}(F) = \binom{v(F)}{2}^{-1} e(F), \quad \bar{s}(F) = \binom{v(F)}{3}^{-1} s(F) - \frac{1}{3}, \quad \bar{t}(F) = \binom{v(F)}{3}^{-1} t(F).
\]

Let

\[
N(F) = \frac{v(F)!}{|\text{Aut}(F)|} p^{e(F)}(1 - p)^{(v(F))^2 - e(F)}.
\]

By Janson and Nowicki (1991), letting \( B_1 = \{(1,2)\}, B_2 = \{(1,2),(1,3)\} \) and \( B_3 = \{(1,2),(1,3),(2,3)\} \), we have

\[
g_{B_1}^{\text{ind}}(y) = \frac{N(F)}{p(1 - p)}(\bar{e}(F) - p)(y - p),
\]

\[
g_{B_2}^{\text{ind}}(y_{12}, y_{13}) = \frac{N(F)}{p^2(1 - p)^2}(\bar{s}(F) - 2p\bar{e}(F) + p^2).
\]

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\[ g_{B_3}^{\text{ind}}(y_{12}, y_{13}, y_{23}) = \frac{N(F)}{\left(p^3(1-p)^3\right)} \left( \bar{t}(F) - 3p\bar{s}(F) + 3p^2\bar{e}(F) - p^3 \right) \times (y_{12} - p)(y_{13} - p)(y_{23} - p). \]

We now consider the following three cases.

Case 1. If \( e(F) \neq p\left(\frac{v(F)}{2}\right) \). In this case, we have \( g_{B_1}^{\text{ind}} \neq 0 \). Then, by Theorem 2.4, we have (3.1) holds.

Case 2. If \( \bar{e}(F) = p \) and \( \bar{s}(F) \neq p^2 \). In this case, we have \( g_{B_1}^{\text{ind}} \equiv 0 \), \( g_{B_2}^{\text{ind}} \neq 0 \).

However, the graph generated by \( B_2 \) is a 2-star, which is not strongly connected. Then, by Theorem 2.3, we have (3.2) holds.

Case 3. If \( \bar{e}(F) = p \), \( \bar{s}(F) = p^2 \) and \( \bar{t}(F) \neq p^3 \). In this case, we have
\[ g_{B_1}^{\text{ind}} \equiv 0, \quad g_{B_2}^{\text{ind}} \equiv 0, \quad g_{B_3}^{\text{ind}} \neq 0. \]

Because the graph generated by \( B_3 \) is a triangle, which is strongly connected. Then, by Theorem 2.4, we have (3.1) holds.

A PROOFS OF SOME LEMMAS

A.1 Proof of Lemma 5.1

Proof of Lemma 5.1. We write \( \{\alpha\} = \{\alpha(1), \ldots, \alpha(k)\} \) for any \( \alpha = (\alpha(1), \ldots, \alpha(k)) \in A_{n,k} \). Also, write \( r_{\alpha} = r(\alpha(1), \ldots, \alpha(k)), Y_{\alpha(1),\alpha(2)}, \ldots, Y_{\alpha(k-1),\alpha(k)} \).

Now, observe that
\[
\text{Var} \left\{ \sum_{\alpha \in \mathcal{I}_{n,k}} r_{\alpha} \right\} = \frac{1}{\sigma_n^2} \sum_{\alpha \in \mathcal{I}_{n,k}} \sum_{\alpha' \in \mathcal{I}_{n,k}} \text{Cov}(r_{\alpha}, r_{\alpha'}). \quad (A.1)
\]

Note that if \( \{\alpha\} \cap \{\alpha'\} = \emptyset \), then \( r_{\alpha} \) and \( r_{\alpha'} \) are independent, then clearly it follows that
\[
\text{Cov}(r_{\alpha}, r_{\alpha'}) = 0 \quad (A.2)
\]

if \( \{\alpha\} \cap \{\alpha'\} = \emptyset \). If there exists \( i \in \{1, \ldots, n\} \) such that \( \{\alpha\} \cap \{\alpha'\} = \{i\} \), then
\[
\text{Cov}(r_{\alpha}, r_{\alpha'}) = \mathbb{E} \left\{ \text{Cov}(r_{\alpha}, r_{\alpha'} \mid X_i) \right\} + \text{Cov} \left( \mathbb{E}(r_{\alpha} \mid X_i), \mathbb{E}(r_{\alpha'} \mid X_i) \right). \quad (A.3)
\]

By independence, we have the first term of (A.3) is 0. For the second term, note that for any \( i \in \{\alpha\} \), then \( \mathbb{E}(r_{\alpha} \mid X_i) = 0 \), and thus the second term of (A.3) is also 0. Therefore,
\[
\text{Cov}(r_{\alpha}, r_{\alpha'}) = 0, \quad \text{if } |\{\alpha\} \cap \{\alpha'\}| = 1. \quad (A.4)
\]
For any $\alpha$ and $\alpha'$ such that $|\{\alpha\} \cap \{\alpha'\}| \geq 2$, by the Cauchy inequality, we have

$$\text{Cov}(r_\alpha, r_{\alpha'}) \leq \text{Var}(r_\alpha).$$

Recall that $r_\alpha$ and $g(X_j)$ are orthogonal for every $j \in \{\alpha\}$. By (5.1), we have

$$\text{Var}(r_\alpha) = \text{Var}(f(X_{\alpha(1)}, \ldots, X_{\alpha(k)}; Y_{\alpha(1),\alpha(2)}, \ldots, Y_{\alpha(k-1),\alpha(k)})) - \sum_{j \in \{\alpha\}} \mathbb{E}\{f_j(X_j)^2\} \leq \tau^2.$$

Thus, it follows that

$$|\text{Cov}(r_\alpha, r_{\alpha'})| \leq \tau^2, \text{ if } |\{\alpha\} \cap \{\alpha'\}| \geq 2. \quad (A.5)$$

Combining (5.2), (A.1), (A.2), (A.4) and (A.5), we have

$$\mathbb{E}\{U^2\} \leq \frac{\tau^2}{\sigma_n^2} \sum_{\alpha \in T_{n,k}} \sum_{\alpha' \in T_{n,k}} \mathbb{N}|\{\alpha\} \cap \{\alpha'\}| \geq 2 \leq \frac{n\tau^2}{k^2\sigma_1^2} \binom{n}{k}^{-1} \binom{k}{2} \binom{n-k}{k-2} \quad (A.6)$$

This proves (5.3).

Now we prove (5.4). Let $T_{n,k}^{(i)} = \{\alpha = \{\alpha(1), \ldots, \alpha(k)\} : \alpha(1) < \cdots < \alpha(k), i \in \{\alpha\}\}$. Note that

$$U - U^{(i)} = \frac{1}{\sigma_n} \sum_{\alpha \in T_{n,k}^{(i)}} r_\alpha^{(i)}.$$

where

$$r_\alpha^{(i)} = r_\alpha - r(X_{\alpha(1)}^{(i)}, \ldots, X_{\alpha(k)}^{(i)}; Y_{\alpha(1),\alpha(2)}, \ldots, Y_{\alpha(k-1),\alpha(k)}).$$

For each $\alpha$, by independence, we have

$$\text{Var}(r_\alpha^{(i)}) = 2 \mathbb{E}\{\text{Var}(r_\alpha \mid X_j, j \in \{\alpha\} \setminus \{i\}, Y_{\alpha(1),\alpha(2)}, \ldots, Y_{\alpha(k-1),\alpha(k)})\} \leq 2 \text{Var}(r_\alpha) \leq 2\tau^2.$$

Similar to (A.6), we have

$$\mathbb{E}\{(U - U^{(i)})^2\} = \frac{1}{\sigma_n^2} \sum_{\alpha \in T_{n,k}^{(i)}} \sum_{\alpha' \in T_{n,k}^{(i)}} \text{Cov}(r_\alpha^{(i)}, r_{\alpha'}^{(i)}).$$
Without loss of generality, let \( r \in \mathbb{Z}_n \). Therefore, we have

\[
\frac{2n^2 \tau^2}{k^2 \sigma^2} \sum_{\alpha \in I_{n,k}} \sum_{\alpha' \in I_{n,k}} \mathbb{N}(|\{\alpha\} \cap \{\alpha'\}| \geq 2)
\]

\[
\frac{2(n-k)^2 \tau^2}{k^2 \sigma^2} \left( \frac{n}{k} \right)^{-2} \left( n-1 \right) \left( n-k \right) \left( k-1 \right) \left( k-2 \right)
\]

\[
\frac{2(n-1)^2 \tau^2}{n(n-1) \sigma^2}
\]

This completes the proof.

\( \square \)

### A.2 Proof of Lemma 5.3

Recall that \( \{\alpha\} = \{\alpha(1), \ldots, \alpha(\ell)\} \) for \( \alpha \in A_{n,d} \). To prove Lemma 5.3, we need the following lemma.

**Lemma A.1.** Let \((A_1, B_1), (A_2, B_2) \in G_{f,d}, (i, j), (i', j') \in A_{n,d} \), \( \alpha_1, \alpha_2 \in A_{n,d}^{(i,j)} \) and \( \alpha_1', \alpha_2' \in A_{n,d}^{(i', j')} \). Let

\[
s = |\{\alpha_1\} \cap \{\alpha_2\}|, \quad t = |\{\alpha_1'\} \cap \{\alpha_2'\}|
\]

If \(|\{\alpha_1\} \cup \{\alpha_2\} \cap \{\alpha_1'\} \cap \{\alpha_2'\}| \leq 2d - (s + t)\), then

\[
\text{Cov}\left\{e^{(i,j)}_{\alpha_1(A_1,B_1)}, e^{(i,j)}_{\alpha_2(A_2,B_2)}, e^{(i', j')}_{\alpha_1'(A_1,B_1)}, e^{(i', j')}_{\alpha_2'(A_2,B_2)}\right\} = 0. \quad (A.7)
\]

**Proof of Lemma A.1.** Let

\[
V_0 = \{\alpha_1\} \cap \{\alpha_2\}, \quad V_1 = \{\alpha_1\} \setminus V_0, \quad V_2 = \{\alpha_2\} \setminus V_0, \quad s = |V_0|,
\]

\[
V_0' = \{\alpha_1'\} \cap \{\alpha_2'\}, \quad V_1' = \{\alpha_1'\} \setminus V_0', \quad V_2' = \{\alpha_2'\} \setminus V_0', \quad t = |V_0'|
\]

Then, we have \( V_1 \cap V_2 = \emptyset, V_1' \cap V_2' = \emptyset, 2 \leq s, t \leq d \). Without loss of generality, assume that \( s \leq t \).

If \( 2d - (s + t) = 0 \), which is equivalent to \( s = d, t = d \), then \( \{\alpha_1\} = \{\alpha_2\} \) and \( \{\alpha_1'\} = \{\alpha_2'\} \). If \( \{\alpha_1\} \cap \{\alpha_1'\} = \emptyset \), then \((\xi^{(i,j)}_{\alpha_1(A_1,B_1)}), (\xi^{(i,j)}_{\alpha_2(A_2,B_2)})\) and \((\xi^{(i', j')}_{\alpha_1'(A_1,B_1)}), (\xi^{(i', j')}_{\alpha_2'(A_2,B_2)})\) are independent, which implies that (A.7) holds.

If \( 2d - (s + t) > 0 \) and \(|\{\alpha_1\} \cup \{\alpha_2\} \cap \{\alpha_1'\} \cup \{\alpha_2'\}| < 2d - (s + t) \), then there exists \( r \in [n] \) such that \( r \in (V_1' \cup V_2') \setminus \{\alpha_1, \alpha_2\} \). Now, assume that \( r \in V_2' \setminus \{\alpha_1, \alpha_2\} \) without loss of generality. Let

\[
\mathcal{F}_r = \sigma(X_p, Y_{p,q}, p, q \in [n] \setminus \{r\}) \cup \sigma(Y_{r,j}). \quad (A.9)
\]

Therefore, we have \( e^{(i,j)}_{\alpha_1(A_1,B_1)}, e^{(i,j)}_{\alpha_2(A_2,B_2)}, e^{(i', j')}_{\alpha_1'(A_1,B_1)} \in \mathcal{F}_r \). Then, by (5.5),

\[
\mathbb{E}\{e^{(i', j')}_{\alpha_2'(A_2,B_2)} \mid \mathcal{F}_r\}
\]

\[
= \mathbb{E}\{f_{G_1}(X_{\alpha_2(A_1,B_1)}; Y_{\alpha_1'(A_1,B_1)}) - f_{G_1}(X_{\alpha_2'(A_1,B_1)}; Y_{\alpha_2'(A_1,B_1)}) \mid \mathcal{F}_r\}
\]

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Combining (A.11) and (A.12) we prove that (A.7) holds for
and
which further implies that

\[ \mathbb{E}\left\{ \xi^{(i',j')}_{\alpha_{1}'(A_1,B_1)} \xi^{(i',j')}_{\alpha_{2}'(A_2,B_2)} \mid \mathcal{F}_r \right\} = 0, \]

which further implies that

\[ \mathbb{E}\left\{ \xi^{(i',j')}_{\alpha_{1}'(A_1,B_1)} \xi^{(i',j')}_{\alpha_{2}'(A_2,B_2)} \mid \mathcal{F}_r \right\} = 0, \]

and

\[
\text{Cov}\left\{ \xi^{(i,j)}_{\alpha_{1}(A_1,B_1)} \xi^{(i,j)}_{\alpha_{2}(A_2,B_2)} \xi^{(i',j')}_{\alpha_{1}'(A_1,B_1)} \xi^{(i',j')}_{\alpha_{2}'(A_2,B_2)} \right\} \\
= \mathbb{E}\left\{ \xi^{(i,j)}_{\alpha_{1}(A_1,B_1)} \xi^{(i,j)}_{\alpha_{2}(A_2,B_2)} \xi^{(i',j')}_{\alpha_{1}'(A_1,B_1)} \xi^{(i',j')}_{\alpha_{2}'(A_2,B_2)} \mid \mathcal{F}_r \right\} \\
= \mathbb{E}\left\{ \mathbb{E}\left\{ \xi^{(i,j)}_{\alpha_{1}(A_1,B_1)} \xi^{(i,j)}_{\alpha_{2}(A_2,B_2)} \mid \mathcal{F} \right\} \mathbb{E}\left\{ \xi^{(i',j')}_{\alpha_{1}'(A_1,B_1)} \xi^{(i',j')}_{\alpha_{2}'(A_2,B_2)} \mid \mathcal{F} \right\} \right\} \\
= 0.
\]

If $2d - (s + t) > 0$ and $\left(\{\alpha_1\} \cup \{\alpha_2\}\right) \cap \left(\{\alpha_1'\} \cup \{\alpha_2'\}\right) = 0$, then either the following two conditions holds: (a) there exists $r \in V'_1 \cup V'_2 \setminus \left(\{\alpha_1\} \cup \{\alpha_2\}\right)$ or (b) $V_0 \cap V'_0 = \emptyset$. If (a) holds, then following a similar argument that leading to (A.11), we have (A.7) holds.

If (b) is true, letting $\mathcal{F} = \sigma(X, \{Y_{p,q} : p, q \in V_1 \cup V_2 \cup V'_1 \cup V'_2\})$, we have conditional on $\mathcal{F}$, $\left(\xi^{(i,j)}_{\alpha_{1}(A_1,B_1)}, \xi^{(i,j)}_{\alpha_{2}(A_2,B_2)}\right)$ is conditionally independent of $(\xi^{(i',j')}_{\alpha_{1}'(A_1,B_1)}, \xi^{(i',j')}_{\alpha_{2}'(A_2,B_2)})$, and thus,

\[
\text{Cov}\left\{ \xi^{(i,j)}_{\alpha_{1}(A_1,B_1)} \xi^{(i,j)}_{\alpha_{2}(A_2,B_2)} \xi^{(i',j')}_{\alpha_{1}'(A_1,B_1)} \xi^{(i',j')}_{\alpha_{2}'(A_2,B_2)} \right\} \\
= \text{Cov}\left\{ \mathbb{E}\left\{ \xi^{(i,j)}_{\alpha_{1}(A_1,B_1)} \xi^{(i,j)}_{\alpha_{2}(A_2,B_2)} \mid \mathcal{F} \right\}, \mathbb{E}\left\{ \xi^{(i',j')}_{\alpha_{1}'(A_1,B_1)} \xi^{(i',j')}_{\alpha_{2}'(A_2,B_2)} \mid \mathcal{F} \right\} \right\}.
\]

Without loss of generality, we assume that $V_1 \cup V_2 \cup V'_1 \cup V'_2 \neq \emptyset$, otherwise the argument is even simpler. Moreover, we may assume that $V_1 \neq \emptyset$. Let $\mathcal{F}_0 = \sigma(Y_{p,q} : p, q \in V_0)$, and we have $\xi^{(i,j)}_{\alpha_{1}(A_1,B_1)}$ and $\xi^{(i,j)}_{\alpha_{2}(A_2,B_2)}$ are conditionally independent given $\mathcal{F} \vee \mathcal{F}_0$. Moreover, by (B.5), $\mathbb{E}\left\{ \xi^{(i,j)}_{\alpha_{1}(A_1,B_1)} \mid \mathcal{F} \vee \mathcal{F}_0 \right\} = \mathbb{E}\left\{ \xi^{(i,j)}_{\alpha_{2}(A_2,B_2)} \mid \mathcal{F} \vee \mathcal{F}_0 \right\} = 0$, and thus

\[
\mathbb{E}\left\{ \xi^{(i,j)}_{\alpha_{1}(A_1,B_1)} \xi^{(i,j)}_{\alpha_{2}(A_2,B_2)} \mid \mathcal{F} \right\} = 0.
\]

Therefore, we have under the condition (b),

\[
\text{Cov}\left\{ \xi^{(i,j)}_{\alpha_{1}(A_1,B_1)} \xi^{(i,j)}_{\alpha_{2}(A_2,B_2)} \xi^{(i',j')}_{\alpha_{1}'(A_1,B_1)} \xi^{(i',j')}_{\alpha_{2}'(A_2,B_2)} \right\} = 0. \quad \text{(A.12)}
\]

Combining (A.11) and (A.12) we prove that (A.7) holds for $\left|\{\alpha_1, \alpha_2\} \cap \{\alpha_1', \alpha_2'\}\right| = 2d - (s + t)$. This completes the proof. \( \blacksquare \)
Proof of Lemma 5.4. In this proof, we denote by $C$ a constant depending on $k$ and $d$, which may take different values in different places. Note that $2 \leq s, t \leq d$, and

$$
\text{Var}\left\{ \sum_{(i,j) \in A_{n,2}} \left( \sum_{a_1 \in A_{n,d}^{(i,j)}} \xi_{\alpha_1(A_1,B_1)}^{(i,j)} \right) \left( \sum_{a_2 \in A_{n,d}^{(i,j)}} \xi_{\alpha_2(A_2,B_2)}^{(i,j)} \right) \right\}
$$

$$
= \sum_{(i,j) \in A_{n,2}} \sum_{\alpha_1 \in A_{n,d}^{(i,j)}} \sum_{\alpha_2 \in A_{n,d}^{(i,j)}} \text{Cov}\left\{ \xi_{\alpha_1(A_1,B_1)}^{(i,j)} \xi_{\alpha_2(A_2,B_2)}^{(i,j)} \right\}
$$

$$
= \sum_{s,t=0}^{d} \sum_{(i,j) \in A_{n,2}} \sum_{\alpha_1 \in A_{n,d}^{(i,j)}} \sum_{\alpha_2 \in A_{n,d}^{(i,j)}} \sum_{\alpha_1' \in A_{n,d}^{(i',j')}} \sum_{\alpha_2' \in A_{n,d}^{(i',j')}} \text{Cov}\left\{ \xi_{\alpha_1(A_1,B_1)}^{(i,j)} \xi_{\alpha_2(A_2,B_2)}^{(i,j)} \xi_{\alpha_1'(A_1,B_1)}^{(i',j')} \xi_{\alpha_2'(A_2,B_2)}^{(i',j')} \right\} \text{NO}_{s,t},
$$

where $O_{s,t} = \{ \{\alpha_1\} \cap \{\alpha_2\} = s \} \cap \{ \{\alpha_1'\} \cap \{\alpha_2'\} = t \}$. If $||\{\alpha_1\} \cup \{\alpha_2\}|| \cap \{\{\alpha_1'\} \cap \{\alpha_2'\}|| \leq 2d - (s + t)$, by (A.7) in Lemma A.1 we have

$$
\text{Cov}\left\{ \xi_{\alpha_1(A_1,B_1)}^{(i,j)} \xi_{\alpha_2(A_2,B_2)}^{(i,j)} \xi_{\alpha_1'(A_1,B_1)}^{(i',j')} \xi_{\alpha_2'(A_2,B_2)}^{(i',j')} \right\} = 0.
$$

If $||\{\alpha_1\} \cup \{\alpha_2\}|| \cap \{\{\alpha_1'\} \cap \{\alpha_2'\}|| > 2d - (s + t)$, then, recalling that $(\xi_{\alpha_1(A_1,B_1)}^{(i,j)}) d_{\alpha_2(A_2,B_2)}^{(i,j)} (\xi_{\alpha_1'(A_1,B_1)}^{(i',j')} \xi_{\alpha_2'(A_2,B_2)}^{(i',j')})$, we have

$$
|\text{Cov}\left\{ \xi_{\alpha_1(A_1,B_1)}^{(i,j)} \xi_{\alpha_2(A_2,B_2)}^{(i,j)} \xi_{\alpha_1'(A_1,B_1)}^{(i',j')} \xi_{\alpha_2'(A_2,B_2)}^{(i',j')} \right\}|
$$

$$
\leq \mathbb{E}\{ (\xi_{\alpha_1(A_1,B_1)}^{(i,j)})^2 (\xi_{\alpha_2(A_2,B_2)}^{(i,j)})^2 \}
$$

$$
\leq C \left( \mathbb{E}\{ f_{A_1,B_1}^{k} (X_{\alpha_1(A_1,B_1)}; Y_{\alpha_1(A_1,B_1)}) \} + \mathbb{E}\{ f_{A_2,B_2}^{k} (X_{\alpha_2(A_2,B_2)}; Y_{\alpha_2(A_2,B_2)}) \} \right)
$$

$$
\leq C \tau^4.
$$

(A.15)

Therefore, with

$$
O_1 = \{ \{\alpha_1\} \cup \{\alpha_2\} || \{\{\alpha_1'\} \cup \{\alpha_2'\}|| > 2d - (s + t) \},
$$

we have

$$
\text{Var}\left\{ \sum_{(i,j) \in A_{n,2}} \left( \sum_{a \in A_{n,d}^{(i,j)}} \xi_{\alpha(A_1,B_1)}^{(i,j)} \right) \right\}
$$

$$
\leq C \tau^4 \sum_{s,t=0}^{d} \sum_{(i,j) \in A_{n,2}} \sum_{\alpha \in A_{n,d}^{(i,j)}} \sum_{\alpha' \in A_{n,d}^{(i',j')}} \text{NO}_{s,t} \cap O_{s,t}
$$

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and by anti-symmetry again,

\[ C r^4 \sum_{s,t=0}^d \eta^{(2d-s)-(2d-t)-(2d-s-t+1)} \]

\[ \leq C n^{d-1-1} r^4. \]

**Proof of Lemma 5.5.** If \( k < d + 1 \), then it follows that \( \xi_{\alpha(G)} = 0 \) for all \( G \in \Gamma_{d+1} \) and \( \alpha \in \mathcal{A}_{n,d+1} \). Therefore, we assume \( k \geq d + 1 \) without loss of generality.

Observe that

\[
\text{Var}\left\{ \sum_{(i,j) \in A_{n,2}} \left( \sum_{\alpha_1 \in \mathcal{A}_{n,v_1}} \xi^{(i,j)}_{\alpha_1(A_1,B_1)} \right) \sum_{\alpha_2 \in \mathcal{A}_{n,v_2}} \xi^{(i,j)}_{\alpha_2(A_2,B_2)} \right\} = \sum_{(i,j) \in A_{n,2}} \sum_{(i',j') \in A_{n,2}} \text{Cov}\left\{ \left( \sum_{\alpha_1 \in \mathcal{A}_{n,v_1}} \xi^{(i,j)}_{\alpha_1(A_1,B_1)} \right) \sum_{\alpha_2 \in \mathcal{A}_{n,v_2}} \xi^{(i,j)}_{\alpha_2(A_2,B_2)} \right\},
\]

Letting

\[ \mathcal{F}_1 = \sigma(X) \vee \sigma(Y_{p,q}, Y'_{p,q} : \{p, q\} \neq \{i, j\}), \]

and noting that

\[
\left( \sum_{\alpha_1 \in \mathcal{A}_{n,v_1}} \xi^{(i,j)}_{\alpha_1(A_1,B_1)} \right) \sum_{\alpha_2 \in \mathcal{A}_{n,v_2}} \xi^{(i,j)}_{\alpha_2(A_2,B_2)}
\]

is anti-symmetric with respect to \((Y_{i,j}, Y'_{i,j})\), we have

\[
\mathbb{E}\left\{ \left( \sum_{\alpha_1 \in \mathcal{A}_{n,v_1}} \xi^{(i,j)}_{\alpha_1(A_1,B_1)} \right) \sum_{\alpha_2 \in \mathcal{A}_{n,v_2}} \xi^{(i,j)}_{\alpha_2(A_2,B_2)} \right\} = 0.
\]

Now, we consider the following two cases. First, if \( \{i, j\} \neq \{i', j'\} \), we have

\[
\left( \sum_{\alpha_1' \in \mathcal{A}_{n,v_1}} \xi^{(i',j')}_{\alpha_1'(A_1,B_1)} \right) \sum_{\alpha_2' \in \mathcal{A}_{n,v_2}} \xi^{(i',j')}_{\alpha_2'(A_2,B_2)} \text{ is } \mathcal{F}_1 \text{ measurable}
\]

and by anti-symmetry again,

\[
\mathbb{E}\left\{ \left( \sum_{\alpha_1 \in \mathcal{A}_{n,v_1}} \xi^{(i,j)}_{\alpha_1(A_1,B_1)} \right) \sum_{\alpha_2 \in \mathcal{A}_{n,v_2}} \xi^{(i,j)}_{\alpha_2(A_2,B_2)} \right\} = 0.
\]

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Therefore,

\[
\text{Cov}\left\{ \left( \sum_{\alpha_1 \in A_{n,v_1}} \xi_{\alpha_1}^{(i,j)}(A_1,B_1) \right) \left| \sum_{\alpha_2 \in A_{n,v_2}} \xi_{\alpha_2}^{(i,j)}(A_2,B_2) \right. \right\} = 0 \quad (A.17)
\]

for \( \{i,j\} \neq \{i',j'\} \).

It suffices to consider the case where \( \{i,j\} = \{i',j'\} \). Observe that

\[
\text{Cov}\left\{ \left( \sum_{\alpha_1 \in A_{n,v_1}} \xi_{\alpha_1}^{(i,j)}(A_1,B_1) \right) \left| \sum_{\alpha_2 \in A_{n,v_2}} \xi_{\alpha_2}^{(i,j)}(A_2,B_2) \right. \right\} = E\left\{ \left( \sum_{\alpha_1 \in A_{n,v_1}} \xi_{\alpha_1}^{(i,j)}(A_1,B_1) \right) \left| \sum_{\alpha_2 \in A_{n,v_2}} \xi_{\alpha_2}^{(i,j)}(A_2,B_2) \right\} \left( \sum_{\alpha_2' \in A_{n,v_2}} \xi_{\alpha_2'}^{(i,j)}(A_2,B_2) \right) \right\} = 0 \quad (A.18)
\]

Let \( H_1 = \{\alpha_1\} \setminus \{\alpha'_1\} \) and \( H'_1 = \{\alpha'_1\} \setminus \{\alpha_1\} \). Let \( t = |\alpha_1 \cap \alpha'_1| \), and then we have \( 2 \leq t \leq v_1 \). Now, as

\[
\sum_{\alpha_2 \in A_{n,v_2}} \xi_{\alpha_2}^{(i,j)}(A_2,B_2) = \sum_{\alpha_2, \alpha'_2 \in A_2} \xi_{\alpha_2}^{(i,j)}(A_2,B_2)
\]

where \( A_1 = \{\alpha_2, \alpha'_2 \in A_{n,v_2} : (H_1 \cup H'_1) \setminus \{\alpha_2, \alpha'_2\} \neq \emptyset \} \) and \( A_2 = \{\alpha_2, \alpha'_2 \in A_{n,v_2} : (H_1 \cup H'_1) \setminus \{\alpha_2, \alpha'_2\} = \emptyset \} \). If there exists \( r \in (H_1 \cup H'_1) \setminus \{\alpha_2, \alpha'_2\} \), letting \( F_r = \sigma(X_p, Y_{p,q}, Y_{p,q} : p,q \in [n] \setminus \{r\}) \), then we have

\[
\sum_{\alpha_2, \alpha'_2 \in A_1} \xi_{\alpha_2}^{(i,j)}(A_2,B_2) \xi_{\alpha'_2}^{(i,j)}(A_2,B_2) \in F_r,
\]

and by orthogonality, we have

\[
E\{ \xi_{\alpha_1}^{(i,j)}(A_1,B_1) | F_r \} = 0.
\]

Therefore, we have

\[
E\left\{ \xi_{\alpha_1}^{(i,j)}(A_1,B_1) \xi_{\alpha'_1}^{(i,j)}(A_1,B_1) \left| \sum_{\alpha_2, \alpha'_2 \in A_1} \xi_{\alpha_2}^{(i,j)}(A_2,B_2) \xi_{\alpha'_2}^{(i,j)}(A_2,B_2) \right. \right\} = 0.
\]
and let abuse of notation, $F$ or $j$

By (A.16), (A.17) and (A.19), we complete the proof.

Therefore, we have

$$
|A_{X} - \mathbb{E}[A_{X}|X]| \
\leq C\tau^2 \sqrt{\mathbb{E}\left\{ \left| \sum_{\alpha_2,\alpha_2' \in A_{2}} \xi^{(i,j)}_{\alpha_2(A_{2},B_{2})} \xi^{(i,j)}_{\alpha_2' (A_{2},B_{2})} \right|^2 \right\}}.
$$

Following the similar argument in the proof of Lemma 5.4 and recalling that $\{\alpha_1 \cap \alpha_1'\} = 0$ and $|A_2| \leq Cn^{2(t-2)}(n^{v_2-v_1} \lor 1)$, we have

$$
\mathbb{E}\left\{ \left| \sum_{\alpha_2,\alpha_2' \in A_{2}} \xi^{(i,j)}_{\alpha_2(A_{2},B_{2})} \xi^{(i,j)}_{\alpha_2' (A_{2},B_{2})} \right|^2 \right\} \leq Cn^{2(t-2)}(n^{v_2-v_1} \lor 1)\tau^4.
$$

Therefore, we have

$$
\left| \mathbb{E}\left\{ \sum_{(i,j) \in A_{n,2}} \xi^{(i,j)}_{\alpha_1(A_{1},B_{1})} \xi^{(i,j)}_{\alpha_1' (A_{1},B_{1})} \left| \sum_{\alpha_2,\alpha_2' \in A_{n,2}} \xi^{(i,j)}_{\alpha_2(A_{2},B_{2})} \xi^{(i,j)}_{\alpha_2' (A_{2},B_{2})} \right| \right\} \right| \leq Cn^{2(t-2)}(n^{v_2-v_1} \lor 1)\tau^4.
$$

Substituting the foregoing inequality to (A.18), we have

$$
\sum_{(i,j) \in A_{n,2}} \text{Cov}\left\{ \sum_{\alpha_1 \in A_{n,2}} \xi^{(i,j)}_{\alpha_1(A_{1},B_{1})} \mid \sum_{\alpha_2 \in A_{n,2}} \xi^{(i,j)}_{\alpha_2(A_{2},B_{2})} \right\} \leq Cn^{2\max\{v_1,v_2\} - 2}\tau^4. \tag{A.19}
$$

By (A.16), (A.17) and (A.19) we complete the proof.

\[ \square \]

### A.3 Proof of Lemma 5.6

Lemma 5.6 follows from a similar argument as that in the proof of Lemma 5.4 and the following lemma. Let $\tilde{G}_{f,\ell} = \{(A, B) \in G_{f,\ell} : G_{A, B} \text{ is strongly connected}\}$. Now, as the function $g$ does not depend on $X$, we set $A_m = \emptyset$ in the following lemma. With a slight abuse of notation, for $j = 1, 2$ and for $B_m \subset [k]_2$, let $G_m$ be the graph generated by $B_m$ and let $v_m$ be the number of nodes of $G_m$, and we write $B_m \in \mathcal{G}$ if $G_m \in \mathcal{G}$.
Lemma A.2. Let $B_m \in \tilde{G}_{f,d} \cup \tilde{G}_{f,d+1}$ for $m = 1, 2$. Let $(i, j), (i', j') \in \mathcal{A}_{n,2}$, and let $\alpha_m \in \mathcal{A}^{(i,j)}_{n,m}, \alpha'_m \in \mathcal{A}^{(i', j')}_{n,m}$ for $m = 1, 2$. Let $s = |\{\alpha_1\} \cap \{\alpha_2\}|$ and $t = |\{\alpha'_1\} \cap \{\alpha'_2\}|$. For $m = 1, 2$, let $\gamma_m$ indicate that $B_m \in \tilde{G}_{f,d} \cup \tilde{G}_{f,d+1}$. Then
\[
\text{Cov}\left\{\eta^{(i,j)}_{\alpha_1(B_1)}\eta^{(i,j)}_{\alpha_2(B_2)}, \eta^{(i', j')}_{\alpha'_1(B_1)}\eta^{(i', j')}_{\alpha'_2(B_2)}\right\} = 0 \tag{A.20}
\]
for $|\{\alpha_1, \alpha_2\} \cap \{\alpha'_1, \alpha'_2\}| < v_1 + v_2 + \gamma_1 + \gamma_2 - (s+t)$.

Proof. The proof is similar to that of Lemma A.1.

Let $V_0, V'_0, V_1, V'_1, V_2, V'_2$ be defined as in (A.8). Note that if $G_B$ has isolated nodes, then $\eta_{\alpha(B)} = 0$ for all $\alpha \in \mathcal{A}_{n,2}$, where $v_B$ is the number of nodes of the graph generated by the index set $B$. If $v_1 + v_2 = s + t$, then it follows that $\{\alpha_1\} = \{\alpha_2\}$ and $\{\alpha'_1\} = \{\alpha'_2\}$. If $|\{\alpha_1\} \cap \{\alpha'_1\}| < 2$, then $\eta^{(i,j)}_{\alpha_1(B_1)}\eta^{(i,j)}_{\alpha_2(B_2)}$ and $\eta^{(i', j')}_{\alpha'_1(B_1)}\eta^{(i', j')}_{\alpha'_2(B_2)}$ are independent, which further implies that (A.20) holds.

Now we consider the case where $v_1 + v_2 > s + t$. If $|\{\alpha_1, \alpha_2\} \cap \{\alpha'_1, \alpha'_2\}| < v_1 + v_2 - (s + t)$, then following the same argument as that leading to (A.11), we have (A.20) holds.

If $G_1$ is connected and $|\{\alpha_1, \alpha_2\} \cap \{\alpha'_1, \alpha'_2\}| = v_1 + v_2 - (s + t)$, then either of the following two conditions holds: (a) there exists $r \in V'_2 \setminus (\{\alpha_1\} \cup \{\alpha_2\} \cup V'_0 \cup V'_1)$ or (b) $V_0 \cap V'_0 = \emptyset$. If (a) holds, then following a similar argument as before, we have (A.20) holds. Now we consider the case where (b) holds. Let $H_1 = \{p, q : p \in V_0, q \in V'_1\}$ and $\mathcal{F}_1 = \sigma(Y_{p,q}, Y'_{p,q} : \mathcal{A}_{n,2} \setminus H_1)$.

By orthogonality, we have $\mathbb{E}\{\eta^{(i,j)}_{\alpha_2(B_2)} \mid \mathcal{F}_1\} = 0$.

Note that $\eta_{\alpha_2(B_2)}, \eta^{(i,j)}_{\alpha_1(B_1)}, \eta^{(i,j)}_{\alpha'_2(B_2)} \in \mathcal{F}_1$, we have
\[
\mathbb{E}\left\{\eta^{(i,j)}_{\alpha_1(B_1)}\eta^{(i,j)}_{\alpha_2(B_2)}\right\} = \mathbb{E}\left\{\eta^{(i,j)}_{\alpha_2(B_2)} \mathbb{E}\left\{\eta^{(i,j)}_{\alpha_1(B_1)} \mid \mathcal{F}_1\right\}\right\} = 0,
\]
\[
\text{Cov}\left\{\eta^{(i,j)}_{\alpha_1(B_1)}\eta^{(i,j)}_{\alpha_2(B_2)}, \eta^{(i', j')}_{\alpha'_1(B_1)}\eta^{(i', j')}_{\alpha'_2(B_2)}\right\} = \mathbb{E}\left\{\mathbb{E}\left\{\eta^{(i,j)}_{\alpha_1(B_1)}\eta^{(i,j)}_{\alpha_2(B_2)} \eta^{(i', j')}_{\alpha'_1(B_1)}\eta^{(i', j')}_{\alpha'_2(B_2)} \mid \mathcal{F}_1\right\} \mid \mathcal{F}_1\right\}\right\} = 0.
\]
This proves (A.20) for the case where $|\{\alpha_1, \alpha_2\} \cap \{\alpha'_1, \alpha'_2\}| = v_1 + v_2 - (s + t)$.

Now, we further assume that $\gamma_1 = \gamma_2 = 1$. If $G_1$ or $G_2$ is a graph containing one single edge, then the proof is even simpler. Without loss of generality, we now assume that $G^{(r)}_m$ is connected for every $r \in [n]$ for $m = 1, 2$. We then prove that (A.20) holds when $|\{\alpha_1, \alpha_2\} \cap \{\alpha'_1, \alpha'_2\}| = v_1 + v_2 - (s + t) + 1$. Under this condition, additional to (a) and (b), there is still another event that may happen: (c) there exists $r \in [n]$ such that $\{r\} = V_0 \cap V'_0$. As the cases (a) and (b) have been discussed, we only need to prove that (A.20) holds under (c).
As \( \{i, j\} \subset V_0 \), we have \( s \geq 2 \), and \( V_0 \setminus \{r\} \) is not empty. Let

\[
\mathcal{F}_2 = \sigma \{ Y_{p,q}, Y'_{p,q} : p \in V_1 \cup V_2 \cup V'_1 \cup V'_2, q \in V_1 \cup V_2 \cup V'_1 \cup V'_2 \cup \{r\} \}.
\]

Then, conditional on \( \mathcal{F}_2 \), we have \( \eta^{(i,j)}_{\alpha_1(B_1)} \eta^{(i,j)}_{\alpha_2(B_2)} \) and \( \eta^{(i',j')}_{\alpha_1'(B_1')} \eta^{(i',j')}_{\alpha_2'(B_2')} \) are conditionally independent. Hence,

\[
\text{Cov}\{ \eta^{(i,j)}_{\alpha_1(B_1)} \eta^{(i,j)}_{\alpha_2(B_2)}, \eta^{(i',j')}_{\alpha_1'(B_1')} \eta^{(i',j')}_{\alpha_2'(B_2')} \} = \text{Cov}\{ \mathbb{E}\{ \eta^{(i,j)}_{\alpha_1(B_1)} \eta^{(i,j)}_{\alpha_2(B_2)} | \mathcal{F}_2 \}, \mathbb{E}\{ \eta^{(i',j')}_{\alpha_1'(B_1')} \eta^{(i',j')}_{\alpha_2'(B_2')} | \mathcal{F}_2 \} \}.
\]

Letting

\[
\mathcal{F}_3 = \sigma \{ Y_{p,q}, Y'_{p,q} : p \in V_0 \setminus \{r\}, q \in V_2 \cup \{r\} \}.
\]

Now, if \( G_1^{(r)} \) is connected for every \( r \in [n] \), there is at least one edge in \( G_1 \) connecting \( V_0 \setminus \{r\} \) and \( V_1 \), and thus

\[
\mathbb{E}\{ \eta^{(i,j)}_{\alpha_1(B_1)} \eta^{(i,j)}_{\alpha_2(B_2)} | \mathcal{F}_2 \cup \mathcal{F}_3 \} = \eta^{(i,j)}_{\alpha_2(B_2)} \mathbb{E}\{ \eta^{(i,j)}_{\alpha_1(B_1)} | \mathcal{F}_2 \cup \mathcal{F}_3 \} = 0,
\]

where the last equality follows from orthogonality. Noting that \( \mathcal{F}_2 \subset \mathcal{F}_3 \), then \( \mathbb{E}\{ \eta^{(i,j)}_{\alpha_1(B_1)} \eta^{(i,j)}_{\alpha_2(B_2)} | \mathcal{F}_2 \} = 0 \) and thus (A.20) holds.

\[\Box\]

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**REFERENCES**

A. D. Barbour, M. Karoński and A. Ruciński (1989). A central limit theorem for decomposable random variables with applications to random graphs. *J. Comb. Theory Ser. B* 47, 125–145.

B. Bollobás, S. Janson and O. Riordan (2007). The phase transition in inhomogeneous random graphs. *Random Structures & Algorithms* 31, 3–122.

S. Chatterjee and Q.-M. Shao (2011). Nonnormal approximation by stein’s method of exchangeable pairs with application to the curie–weiss model. *Ann. Appl. Probab.* 21, 464–483.

S. Chatterjee (2007). Stein’s method for concentration inequalities. *Probab. Theory Relat. Fields* 138, 305–321.

L. H. Y. Chen, L. Goldstein and Q.-M. Shao (2011). *Normal Approximation by Stein’s Method*. Probability and Its Applications. Springer, Heidelberg, New York.

L. H. Chen and Q.-M. Shao (2007). Normal approximation for nonlinear statistics using a concentration inequality approach. *Bernoulli* 13, 581–599.

P. Diaconis and D. Freedman (1981). On the statistics of vision: The Julesz conjecture. *J. of Math. Psycho.* 24, 112–138.

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S. Holmes and G. Reinert (2004). Stein’s method for the bootstrap. In Stein’s Method: Expository Lectures and Applications 46 93–132. Institute of Mathematical Statistics, Hayward, CA.

S. Janson and K. Nowicki (1991). The asymptotic distributions of generalized u-statistics with applications to random graphs. Probab. Theory Relat. Fields 90, 341–375.

G. Kaur and A. Röllin (2020). Higher-order fluctuations in dense random graph models. Available at ArXiv 200615805.

K. Krokowski, A. Reichenbachs and C. Thäle (2017). Discrete Malliavin–Stein method: Berry–Esseen bounds for random graphs and percolation. Ann. Probab. 45, 1071–1109.

L. Lovász (2012). Large networks and graph limits, volume 60. American Mathematical Soc.

L. Lovász and B. Szegedy (2006). Limits of dense graph sequences. J. Comb. Theory Ser. B 96, 933–957.

K. Nowicki (1989). Asymptotic normality of graph statistics. J. Stat. Plann. Inference 21, 209–222.

N. Privault and G. Serafin (2018). Normal approximation for sums of discrete U-statistics - application to Kolmogorov bounds in random subgraph counting. Available at arXiv 1806.05339.

Y. Rinott and V. Rotar (1997). On coupling constructions and rates in the clt for dependent summands with applications to the antivoter model and weighted u-statistics. Ann. Appl. Probab. pages 1080–1105.

A. Röllin (2017). Kolmogorov bounds for the normal approximation of the number of triangles in the Erdos-Renyi random graph. Available at arXiv:1704.00410.

Q.-M. Shao and Z.-S. Zhang (2016). Identifying the limiting distribution by a general approach of Stein’s method. Sci. China. Math. 59, 2379–2392.

Q.-M. Shao and Z.-S. Zhang (2019). Berry–Esseen bounds of normal and nonnormal approximation for unbounded exchangeable pairs. Ann. Probab. 47, 61–108.

Q.-M. Shao and W.-X. Zhou (2016). Cramér type moderate deviation theorems for self-normalized processes. Bernoulli 22, 2029–2079.

C. Stein (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Probability Theory, pages 583–602, Berkeley, Calif. University of California Press.

C. Stein (1986). Approximate computation of expectations. Institute of Mathematical Statistics Lecture Notes—Monograph Series, 7. Institute of Mathematical Statistics, Hayward, CA.