Elliptic canonical bases for toric hyper-Kähler manifolds

Tatsuyuki Hikita

Abstract

Lusztig defined certain involutions on the equivariant $K$-theory of Slodowy varieties and gave a characterization of certain bases called canonical bases. In this paper, we give a conjectural generalization of these involutions and $K$-theoretic canonical bases to conical symplectic resolutions which have good Hamiltonian torus actions and state several conjectures related to them which we check for toric hyper-Kähler manifolds. We also propose an elliptic analogue of these bar involutions. As a verification of our proposal, we explicitly construct elliptic lifts of $K$-theoretic canonical bases and prove that they are invariant under elliptic bar involutions for toric hyper-Kähler manifolds.

1 Introduction

Lusztig [20, 21] defined certain modules of affine Hecke algebras called periodic modules and defined the notion of canonical bases in them. In [22, 23], Lusztig gave a geometric construction of these modules in terms of equivariant $K$-theory of Springer resolutions or Slodowy varieties, and gave a geometric characterization of (signed) canonical bases using certain involution called bar involution. Basic properties of canonical bases including their existence and relation to modular representation theory of semisimple Lie algebras in large enough characteristic were also conjectured by Lusztig and proved by Bezrukavnikov-Mirković [6].

1.1 $K$-theoretic canonical bases

One of the aim of this paper is to give an analogue of the notion of bar involutions and canonical bases to equivariant $K$-theory of conical symplectic resolutions which have Hamiltonian torus actions with finitely many fixed points. For $ADE$ type quiver varieties, analogous bar involutions and canonical bases were proposed by Lusztig in [24], and constructed by Varagnolo-Vasserot in [42]. In all the previous works, bar involutions are defined by composing several automorphisms and it seems complicated at least to the author. Our main observation in this paper is that the bar involution for Springer resolutions (and conjecturally for other known cases satisfying our assumptions) have a very simple characterization by using the $K$-theoretic analogue of the stable bases introduced by Maulik-Okounkov [25].

In this introduction, we give a rough definition of the $K$-theoretic bar involution and the $K$-theoretic canonical bases. Let $X$ be a conical symplectic resolution with conical action of $S := \mathbb{C}^\times$. Throughout this paper, we always assume that the symplectic form has weight 2 and there exists a Hamiltonian torus action $H \acts X$ commuting with the $S$-action such that the fixed point set $X^H$ is finite.

The $K$-theoretic stable basis (see e.g. [32, 34]) depends on some data called chamber $\mathcal{C} \subset X^*_s(H) \otimes \mathbb{R}$, polarization $T^{1/2} \in K_{H \times \mathbb{S}}(X)$, and slope $s \in \text{Pic}(X) \otimes \mathbb{Z}$. Associated with these data, one can give a characterization of certain element $\text{Stab}_{K_{\mathcal{C},T^{1/2},s}}(p) \in K_{H \times \mathbb{S}}(X)$ in the equivariant $K$-theory of $X$ corresponding to each fixed point $p \in X^H$. These elements form a basis in the localized equivariant $K$-theory $K_{H \times \mathbb{S}}(X)_{\text{loc}} := K_{H \times \mathbb{S}}(X) \otimes_{K_{H \times \mathbb{S}(pt)}} \text{Frac}(K_{H \times \mathbb{S}}(pt))$.

Let $v \in K_{\mathbb{S}(pt)} \cong \mathbb{Z}[v, v^{-1}]$ be the element corresponding to the natural representation of $S$. We want to define $K_H(pt)$-linear and $K_{\mathbb{S}(pt)}$-antilinear (i.e. $\beta^K(v, \cdot) = v^{-1} \beta^K(\cdot)$) map $\beta^K_{X,T^{1/2},s} = \beta^K :$
The above β^K is an involution and preserves K(H×S)(X) ⊂ K(H×S)(X)_{loc}. Moreover, this does not depend on the choice of C.

We note that the K-theoretic bar involution does depend on the choice of polarization and slope. When X is a Springer resolution, we will check that this bar involution essentially coincides with the bar involution defined by Lusztig for a specific choice of polarization and slope.

Once we have defined the bar involution, we can give a characterization of the (signed) canonical basis following Lusztig. \[ \mathbb{B}_{X,T^{1/2},s}^{±} = \{ m \in K(H×S)(X) | β^K(m) = m, (m||m) \in 1 + v^{-1}K_H(pt)[v^{-1}] \} \].

We note that for any λ ∈ X^∗(H) and m ∈ B_{X,T^{1/2},s}^{±}, we have ±λm ∈ B_{X,T^{1/2},s}^{±}. In Proposition 3.21, we give a conjectural Kazhdan-Lusztig type algorithm to compute B_{X,T^{1/2},s}^{±}, and in particular, fix the sign in it which gives a subset B_{X,T^{1/2},s} ⊂ B_{X,T^{1/2},s}^{±} such that B_{X,T^{1/2},s}^{±} = B_{X,T^{1/2},s} \cup -B_{X,T^{1/2},s}. Another main conjecture in this paper is the following.

Conjecture 1.2. There exists an H×S-equivariant tilting bundle T^{1/2}_{s} on X such that B_{X,T^{1/2},s} coincides with the set of K-theory classes of indecomposable summands of T^{1/2}_{s} up to shifts by X^∗(H).

In particular, B_{X,T^{1/2},s} is a basis of K(H×S)(X) as a Z[v,v^{-1}]-module.

When X is a Springer resolution and for the specific choice of the data as above, this conjecture follows from a result of Bezrukavnikov-Mirković combined with our comparison result Proposition 2.3.

In this paper, we also check it for toric hyper-Kähler manifolds, see Corollary 5.36. We remark that this tilting bundle essentially coincides with the one constructed by McBreen-Webster and Špenko-Van den Bergh.

We remark that by the definition of K-theoretic canonical basis, the endomorphism ring A_{T^{1/2},s} := End(T^{1/2}_{s}) of the conjectural tilting bundle has nonnegative grading with respect to S. In particular, this is Koszul by the Kaledin’s argument in [6]. We will formulate a conjecture on the highest weight category structure on the equivariant module category of the Koszul dual of A_{T^{1/2},s}, see Conjecture 3.40.

A natural duality functor on it should lift the K-theoretic bar involution to an involution on the derived category D^b Coh_{H×S}(X) of H×S-equivariant coherent sheaves on X, see Conjecture 3.44.

We also note that by varying the slope parameters, we would obtain a family of tilting bundles and hence t-structures on D^b Coh_{H×S}(X). This should be a part of the data defining the real variations of stability conditions in the sense of Anno-Bezrukavnikov-Mirković, see Conjecture 3.47. We will give a conjecture on the wall-crossing of K-theoretic canonical bases, which is also given by a Kazhdan-Lusztig type algorithm, see Conjecture 3.49.
1.2 Elliptic bar involutions

The second and the main aim of this paper is to give an elliptic analogue of the $K$-theoretic bar involution. Since Aganagic-Okounkov [4] defined the elliptic analogue of the stable basis, it seems natural to consider the elliptic analogue of $[4]$ replacing $K$-theoretic stable bases by elliptic stable bases.

Let $\text{Stab}^{AO}_{T/\hat{X}}(p)$ be the elliptic stable basis associated with $p \in X^H$ in the sense of Aganagic-Okounkov [4]. This is a section of some line bundle on equivariant elliptic cohomology of $X$ extended by adding Kähler parameters. In order to obtain an involution, it seems natural to shift the Kähler parameters by $e^{\text{det} T^{1/2}}$ and then multiply it by $\Theta(N^{1,\ldots,-})$, the Thom class of the negative part of the tangent bundle of $X'$ at the fixed point $p'$ of $(X')^H$ corresponding to $p \in X^H$ under symplectic duality.

In order to consider its $K$-theory limits, we also twist it by $v' \cdot \sqrt{\text{det} T^{1/2} \cdot \text{det} T^{1/2}}^{-1}$. Let us denote by $\hat{\mathcal{S}}_{X,\mathcal{E}}(p)$ the resulting renormalized elliptic stable basis. Here, we omit the polarization from the notation but they depends on the choice.

We want to define the elliptic bar involution $\beta_{X}^{\text{ell}} = \beta_{X}^{\text{ell}}$ by the formula

$$\beta_{X}^{\text{ell}}(\hat{\mathcal{S}}_{X,\mathcal{E}}(p)) = (-1)^{\dim X} \hat{\mathcal{S}}_{X,\mathcal{E}}(-p(p))$$

for any $p \in X^H$. Here, $\beta_{X}^{\text{ell}}$ should be considered as an involution on some space of meromorphic functions on $\text{Spec}(K_{p,\mathcal{E}}(X) \otimes \mathbb{C}[\text{Pic}(X)^{\nu}])$. We also conjecture that $\beta_{X}^{\text{ell}}$ does not depend on the choice of $\mathcal{E}$ and hence it is an involution. We will check this for toric hyper-Kähler manifolds in Corollary 6.10.

We remark that in the above normalization, following symmetry under the symplectic duality is expected (cf. [36] [57]):

$$\hat{\mathcal{S}}_{X,\mathcal{E}}(p_1)|_{p_2} = \pm \hat{\mathcal{S}}_{X',\mathcal{E}}(p_2)|_{p_1}.$$ 

Here, we identified the equivariant parameters for $X$ and Kähler parameters for $X'$ and vice versa, as predicted by symplectic duality. In some sense, this symmetry explains the naturality of the above normalization. By taking the $K$-theory limits, this also explains the appearance of symplectic duality in the normalization of our $K$-theoretic bar involutions. For more detail, see section 4.3.

The main problem toward a definition of the notion of elliptic canonical basis is to generalize other conditions such as asymptotic norm one property or triangular property of $K$-theoretic canonical bases to the elliptic case. Although we do not know how to deal with this problem in general, we give a candidate for the elliptic canonical bases for toric hyper-Kähler manifolds by explicitly constructing an elliptic lift of $K$-theoretic canonical bases which are invariant under the elliptic bar involution.

1.3 Elliptic canonical bases for toric hyper-Kähler manifolds

Let us explain the main result of this paper briefly. Let $1 \to S \to T \to H \to 1$ be an exact sequence of algebraic tori. We will identify the character lattice $X^*(T) \cong \mathbb{Z}^n$ with its dual by taking the standard pairing $(-,-)$ on $\mathbb{Z}^n$. The toric hyper-Kähler manifold $X := \mu^{-1}(0)^{\mathbb{Z}^n}/S$ is defined by the Hamiltonian reduction of $T^* \mathbb{C}^n$ by $S$. Here, $\mu$ is the moment map for the $S$-action and the GIT stability parameter is taken generically. We assume that $X$ is smooth.

By considering the induced line bundles on the quotient, we obtain a natural homomorphism $\mathcal{L} : X^*(T) \to \text{Pic}^H_{T\times\hat{X}}(X)$. By considering $X_*(S) \subset X_*(T)$ as a subset of $X^*(T)$ by using the pairing, we define the provincial elliptic canonical basis by the following formula.

**Definition 1.3.** For each $\lambda \in X^*(T)$, we set

$$\Theta_X(\lambda) := (q; q)^{\text{rk} S} \sum_{\beta \in X_*(S)} (-1)^{(\kappa, \beta)} q^{\frac{1}{2} (\beta, \beta) + (\lambda, \kappa, \beta)} \mathcal{L}(\lambda + \beta) z^\beta.$$ 

Here, $(q; q)^{\infty} := \prod_{n \geq 1} (1 - q^n)$, $\kappa = (1, 1, \ldots, 1) \in \mathbb{Z}^n$, and $z^\beta$ is the Kähler parameter corresponding to $\beta$. 


One can easily check that for $\beta \in X_*(S)$ and $\alpha \in X^*(H) \subset X^*(T)$, we have
\[
\Theta_X(\lambda + \beta) = (-1)^{\langle \kappa, \beta \rangle} q^{-\frac{1}{2} \langle \beta, \beta \rangle} \Theta_X(\lambda), \\
\Theta_X(\lambda + \alpha) = a^\alpha \Theta_X(\lambda).
\]
Here, $a^\alpha$ is the equivariant parameter corresponding to $\alpha \in X^*(H)$. Therefore, the number of independent elements in $\{\Theta_X(\lambda)\}_{\lambda \in X^*(T)}$ is at most the number of elements of $\Xi := X^*(T)/(X_*(S) + X^*(H)) \cong X^*(S)/X_*(S) \cong X_*(H)/X^*(H)$, which turns out to be equal to the rank of $K(X)$. We also remark that the formula $6$ has the form of the theta function associated with the lattice $X_*(T)$, where the pairing is induced from $X^*(T)$. We point out that if we specialize all the equivariant parameters to $1$ in $6$, we get the character of irreducible module corresponding to $\lambda \in X^*(S)/X_*(S)$ for the lattice vertex operator superalgebra $V_{X_*(S)}$ associated with $X_*(S)$ for certain choice of conformal vector.

One can also define the elliptic canonical basis $\Theta_{X^!}(\lambda)$ for the symplectic dual $X^!$, which is defined analogously by the dual exact sequence of algebraic tori $1 \rightarrow H^! \rightarrow T^! \rightarrow S^! \rightarrow 1$. We note that we identify $X^*(T) \cong X^!(T^!)$ by using the pairing. We also remark that the choice of GIT parameter for $X^!$ is given by the choice of chamber for $X$. The main result of this paper is the following, see Theorem $6.9$ and Corollary $6.10$.

**Theorem 1.4.** In the above setting, we have $\beta_{X^!}^{X}(\Theta_{X}(\lambda)) = \Theta_{X}(\lambda)$ and $\beta_{X^!}^{X}(\Theta_{X^!}(\lambda)) = \Theta_{X^!}(\lambda)$ for any $\lambda \in X^*(T)$. We also have the following expansion of the elliptic stable bases into the elliptic canonical bases.
\[
\Theta_{X, e}^{\pm}(\mu) = \sum_{\lambda \in \Xi} (-1)^{\langle \lambda, \kappa \rangle} q^{\frac{1}{2} \langle \kappa, \lambda + \kappa \rangle} \Theta_{X^!}(\lambda)|\chi_{\mu}| \cdot \Theta_{X}(\lambda), \tag{6}
\]

We note that in the sum of $6$, each term does not depend on the choice of representatives for $\lambda$. We remark that this formula resembles the irreducible decomposition of the lattice vertex operator superalgebra $V_{X_*(S)}$ as a module for the commuting action of vertex operator subalgebras $V_{X_*(S)}$ and $V_{X^*(H)}$, which forms a dual pair in the sense that they are commutant to each other in $V_{X_*(S)}$. The author is not sure if this kind of phenomenon happens in general or not, but we hope that the results of this paper would give some hints for the investigation of elliptic canonical bases for other conical symplectic resolutions.

The plan of this paper is as follows. In section 2, we check that our definition of $K$-theoretic bar involution essentially coincides with Lusztig’s definition for Springer resolutions. In section 3, we propose the definition of $K$-theoretic bar involutions and state several conjectures related to $K$-theoretic canonical bases. In section 4, we propose the definition of elliptic bar involutions. In section 5, we specialize to the case of toric hyper-Kähler manifolds and calculate $K$-theoretic canonical bases. We also prove all the conjectures stated in section 3 in these cases. In section 6, we prove our main result on the elliptic canonical bases for toric hyper-Kähler manifolds.

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### 2 Reformulation

In this section, we reformulate Lusztig’s definition of the bar involution on the equivariant $K$-theory of Springer resolutions. We should remark that the Lusztig’s definition works for Slodowy varieties associated with any nilpotent element. If the nilpotent element is regular in some Levi algebras, then we can use our approach to define $K$-theoretic bar involutions. The results of this section are included
for motivational purposes and will not be used elsewhere in this paper. Hence we restrict ourselves to the case of Springer resolutions for simplicity. Comparison of our definition with Lusztig’s definition for other Slodowy varieties should be straightforward once some formulas for the $K$-theoretic stable bases analogous to Proposition 2.2 are available.

We first list some standard notations used in this section. Let $G$ be a semisimple algebraic group over $\mathbb{C}$ of adjoint type. We fix $B \subset G$ a Borel subgroup and $H \subset B$ a Cartan subgroup. We will denote by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{h}$ their Lie algebras. We use the convention that the nonzero $H$-weights appearing in $\mathfrak{b}$ is negative. We denote by $X^*(H)$ (resp. $X_*(H)$) the character lattice (resp. cocharacter lattice) of $H$. We set $h^+_\mathfrak{g} := X^*(H) \otimes_{\mathbb{Z}} \mathbb{R}$ and $h^-_\mathfrak{g} := X_*(H) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\{a_i\}_{i \in I}$ be the set of simple roots and $\{a'^i\}_{i \in I}$ be the set of simple coroots. Let $W$ be the Weyl group of $G$ and $s_i \in W$ be the simple reflection corresponding to $i \in I$. For $w \in W$, we denote by $l(w)$ the length of $w$. Let $w_0 \in W$ be the longest element and $e \in W$ be the identity element. We denote by $R_+$ (resp. $R'_+$) the set of positive roots (resp. positive coroots) and $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha \in h^+_\mathfrak{g}$ the half sum of positive roots.

2.1 Lusztig’s bar involution

We briefly recall Lusztig’s bar involution on the equivariant $K$-theory of Springer resolutions. For more details, we refer to [22, 23].

Let $\mathcal{B} := G/B$ be the flag variety and $X := T^*\mathcal{B} \cong \{(gB, y) \in \mathcal{B} \times \mathbb{A}^* \mid \text{Ad}(g)^{-1}(y) \in (\mathfrak{g}/\mathfrak{b})^*\}$ be the Springer resolution. The torus $T := H \times S$ acts naturally on $X$ by $(h, \sigma) \cdot (gB, y) = (hgB, \sigma^{-2} \text{Ad}(h)y)$ and this action preserves the subvariety $\mathcal{B}$. By the pushforward along zero section, we obtain an inclusion $K_T(\mathcal{B}) \subset K_T(X)$.

Let $\mathcal{H}$ be the affine Hecke algebra associated with the Langlands dual of $G$. I.e., $\mathcal{H}$ is the $\mathbb{Z}[v, v^{-1}]$-algebra generated by $T_w$ ($w \in W$) and $\theta_{\lambda}$ ($\lambda \in X^*(H)$) with the following relations:

- $(T_{w} - v)(T_{w} + v^{-1}) = 0$ ($\forall i \in I$),
- $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$ ($w, w' \in W$),
- $\theta_{\lambda} T_{w} - T_{w} \theta_{\lambda} = (v - v^{-1}) \theta_{\lambda - \bar{w}(\lambda)}$ ($\forall i \in I, \lambda \in X^*(H)$),
- $\theta_{\lambda} \theta_{\lambda'} = \theta_{\lambda + \lambda'}$ ($\forall \lambda, \lambda' \in X^*(H)$),
- $\theta_0 = 1$.

Here, for $\lambda \in X^*(H)$, we denote by $[\lambda] \in \mathbb{Z}[X^*(H)] \cong K_H(\text{pt})$ the corresponding element and for $c = \sum_{\lambda \in X^*(H)} c_{\lambda} [\lambda] \in K_H(\text{pt})$, we set $\theta_c = \sum_{\lambda \in X^*(H)} c_{\lambda} \theta_{\lambda}$. It is known (see e.g. [9, 22]) that $\mathcal{H}$ acts on $K_T(\mathcal{B})$ and $K_T(X)$, and the inclusion $K_T(\mathcal{B}) \subset K_T(X)$ is compatible with the $\mathcal{H}$-actions. We do not recall its construction, but in order to fix the convention, we write down its action on the fixed point basis. Our convention mainly follows that of [22].

The fixed points of $\mathcal{B}$ and $X$ with respect to the $H$-action are parametrized by $W$. For each $w \in W$, we also denote by $w := wB \in \mathcal{B}$ the corresponding fixed point. We denote by $\mathcal{O}_w \subset K_T(\mathcal{B}) \subset K_T(X)$ the $K$-theory class of the structure sheaf of the fixed point $w$. The $\mathcal{H}$-action on the fixed point basis is given by

- $T_w \mathcal{O}_w = \frac{v^{-1} - v}{1 - w(x_i)} \mathcal{O}_w + \frac{v - v^{-1}}{1 - w(x_i)} \mathcal{O}_{ws_i}$ ($\forall i \in I$)
- $\theta_{\lambda} \mathcal{O}_w = [w(\lambda)] \cdot \mathcal{O}_w$ ($\forall \lambda \in X^*(H)$)

We note that the action of $\theta_{\lambda}$ is given by tensor product of an equivariant line bundle $L_{\lambda} := (G \times C_{\lambda})/B$ on $\mathcal{B}$ (or its pullback to $X$), where the action of $B$ is given by $b \cdot (g, x) = (gb^{-1}, \lambda(b)x)$.

We fix a Lie algebra automorphism $\varpi : \mathfrak{g} \to \mathfrak{g}$ such that $\varpi(h) = -h$ for any $h \in \mathfrak{h}$. This induces an automorphism of $\mathcal{B}$ and $X$, which is denoted by the same letter. Lusztig’s bar involution $\beta^L : K_T(X) \to K_T(X)$ is then defined by

$$\beta^L := (-v)^{l(w_0)} T_{w_0}^{-1} \varpi^* D_X.$$
One can define an inner product \((-\|\|)_K: K_T(X) \times K_Z(X) \to \text{Frac}(K_T(\text{pt}))\) in the same way as replacing \(\beta^K\) by \(\beta^L\). In order to compute the image of \(K\)-theoretic stable bases by \(\beta^L\), the following lemma proved in [23] is useful.

**Lemma 2.1** ([23], Lemma 8.13). Let \(\mathcal{Z}_l := \text{pr}^{-1}(e) \subset X\). Then for any \(l, f, f' \in \mathcal{X}^*(H)\) and \(w, w' \in W\), we have
\[
(fT_w^{-1}O_{\mathcal{Z}_l}||f'T_{w'}^{-1}O_{\mathcal{Z}_l})_L = v^{-2l(w_0)}ff'\delta_{w,w'}.
\]

We remark that in [23], this formula is proved more generally for Slodowy varieties associated with nilpotent elements which are regular in some Levi subalgebras.

### 2.2 Stable bases

Next we recall a result of Su-Zhao-Zhong [41] describing the \(K\)-theoretic stable bases for Springer resolutions. For the definition and basic properties of \(K\)-theoretic stable bases used in this paper, we refer to section 3.2. In particular, the sign convention is slightly different from [32, 34].

We first fix the data defining the \(K\)-theoretic stable basis. For the chamber, we take the negative Weyl chamber \(\mathcal{C} = \{ x \in \mathfrak{h}_R \mid \forall i \in I, (x, \alpha_i) < 0 \}\). For the polarization \(T^{1/2}\), we take the pullback of the tangent bundle of \(B\) by \(X \to B\). For the slope, we take \(s \in \rho - A_0^+ \subset \text{Pic}(X) \otimes \mathbb{R} \cong \mathfrak{h}_R^*\), where \(A_0^+ = \{ x \in \mathfrak{h}_R^* \mid \forall \alpha^+ \in R_+^\alpha, 0 < \langle x, \alpha^+ \rangle < 1 \}\) is the fundamental alcove. We note that \(\rho\) corresponds to an actual (nonequivariant) line bundle on \(X\) and we take an \(\mathbb{T}\)-equivariant lift \(\mathcal{L}_\rho\). The following description of the \(K\)-theoretic stable basis is essentially proved in type \(A\) by Rimányi-Tarasov-Varchenko [35] and in general by Su-Zhao-Zhong [41].

**Proposition 2.2** ([41], Theorem 0.1). For any \(w \in W\), we have
\[
\text{Stab}^K_{\mathcal{C}, T^{1/2}, s}(w) = (-1)^{l(w)}[\rho - w\rho] \cdot T_w^{-1}O_{\mathcal{Z}_l}.
\]

**Proof.** We only need to compare the convention in [41] with ours. First, we note that the choice of the Borel subgroup in [41] is opposite to ours, hence the fixed point corresponding to \(w\) in [41] is \(ww_0\) in our notation. We also note that the line bundle corresponding to \(\lambda \in \mathcal{X}^*(H)\) is \(\mathcal{L}_{w_0, \lambda}\) in our notation. Hence the chamber is the same and the slope in [41] is taken in \(-A_0^+\). Moreover, the polarization is opposite to ours. Finally, one can check that the affine Hecke algebra action denoted by \(T_w\) in [41] is given by \(v^{l(w)}\mathcal{L}_\rho T_{ww_0}\mathcal{L}_\rho T_{w}\) in our notation. This does not depend on the choice of \(\mathcal{L}_\rho\).

One can easily check that
\[
(-1)^{l(w_0)}\text{Stab}^K_{\mathcal{C}, T^{1/2}, -\rho + s}(e) = (-v)^{l(w_0)}[2\rho] \cdot O_{\mathcal{Z}_l}.
\]

Therefore, Theorem 0.1 of [41] is
\[
(-1)^{l(w_0) - l(w)}\text{Stab}^K_{\mathcal{C}, T^{1/2}, -\rho + s}(w) = \mathcal{L}_\rho T_w^{-1}\mathcal{L}_\rho \cdot \mathcal{L}_\rho T_{ww_0}\mathcal{L}_\rho T_{w}^{-1}O_{\mathcal{Z}_l},
\]

in our notation. Here, the sign in the left hand side comes from our convention on the sign of \(K\)-theoretic stable basis. By using Lemma 3.7 and Lemma 3.8 we obtain
\[
\text{Stab}^K_{\mathcal{C}, T^{1/2}, s}(w) = v^{-l(w_0)}(\det T_w^{1/2})^{-1}i_w^*\mathcal{L}_\rho \cdot \mathcal{L}_\rho \otimes \text{Stab}^K_{\mathcal{C}, T^{1/2}, -\rho + s}(w)
\]
\[
= (-1)^{l(w)}[\rho - w\rho] \cdot T_w^{-1}O_{\mathcal{Z}_l},
\]

as required.

### 2.3 Comparison

We now prove the following formula which was our starting point of this work.
Proposition 2.3. For any \( w \in W \), we have
\[
\beta^L(\text{Stab}^K_{X,T^{1/2},s}(w)) = (-v)^{2l(w)} \text{Stab}^K_{\xi,T^{1/2},s}(w). \tag{7}
\]

Proof. By Lemma 2.1 and Proposition 2.2 we obtain
\[
(\text{Stab}^K_{\xi,T^{1/2},s}(w) : \mathbb{D}_X \beta^L \text{Stab}^K_{\xi,T^{1/2},s}(w')) = v^{-2l(w)} \delta_{w,w'}
\]
for any \( w, w' \in W \). On the other hand, Lemma 3.9 and Lemma 3.10 imply that
\[
(\text{Stab}^K_{\xi,T^{1/2},s}(w) : \mathbb{D}_X \text{Stab}^K_{\xi,T^{1/2},s}(w')) = (-v)^{l(w)} \delta_{w,w'}.
\]
By comparing the two formulas, the proposition follows since the pairing \((- : -)\) is perfect after localization and \(\{\text{Stab}^K_{\xi,T^{1/2},s}(w)\}_{w \in W}\) forms a basis of localized equivariant K-theory.

Our main observation in this paper is that except for the normalization, an analogue of the formula (7) makes sense if one can define K-theoretic stable bases in order to characterize certain antilinear map. Therefore, we try to make this kind of formula into a definition of K-theoretic bar involution. Our remaining task is to fix the normalization in some way, which turns out to be related to the notion of symplectic duality introduced by Braden-Licata-Proudfoot-Webster in [8].

3 Main conjectures

In this section, we propose a general definition of K-theoretic canonical bases for conical symplectic resolutions which have good Hamiltonian torus actions. We also formulate several conjectures about them which will be proved for toric hyper-Kähler manifolds in section 5.

3.1 Symplectic duality

First we briefly recall basic definitions on symplectic resolution and symplectic duality in the form we need later. In this paper, we mainly follow the setting of [8] for symplectic resolution. For symplectic duality, we partly follow the definition of 3d mirror symmetry in [27]. This is designed to give certain symmetry of elliptic stable bases under the duality, see Conjecture 4.4.

Let \( X \) be a connected smooth algebraic variety over \( \mathbb{C} \) with algebraic symplectic form \( \omega \) and an S-action. We assume that the S-weight of \( \omega \) is 2 and the S-action is conical, which means that S-weights appearing in the coordinate ring \( \mathbb{C}[X] \) are nonnegative and the weight 0 part consists only of constant functions. If the natural morphism \( \pi : X \to \text{Spec}(\mathbb{C}[X]) \) is a resolution of singularity, \( X = (X,\omega,S) \) is called conical symplectic resolution. We denote by \( o \in \text{Spec}(\mathbb{C}[X]) \) the unique S-fixed point and \( L := \pi^{-1}(o) \) the central fiber of \( \pi \).

In this paper, we always assume for simplicity that there does not exist another conical symplectic resolution \( X' \) and symplectic vector space \( V \neq 0 \) such that \( X \cong X' \times V \). We also assume that there exists an effective action of another algebraic torus \( H \) on \( X \) which is Hamiltonian and commute with the S-action such that the fixed point set \( X^H \) is finite. We always take maximal \( H \) among such torus actions. For \( p \in X^H \), we will denote by \( i_p : \{p\} \hookrightarrow X \) the natural inclusion. We set \( T := H \times S \).

For any \( p \in X^H \), we denote by \( \Phi(p) \) the multiset of \( H \)-weights appearing in the tangent space \( T_p X \) at \( p \). We call it the multiset of equivariant roots at \( p \). We also simply call an element in the union (as a set) \( \overline{\Phi} := \bigcup_{p \in X^H} \Phi(p) \) equivariant root for \( X \). An equivariant root \( \alpha \) defines a hyperplane in \( h_{\mathbb{R}} = X_*(H) \otimes_{\mathbb{Z}} \mathbb{R} \) by \( H_\alpha := \{ \xi \in h_{\mathbb{R}} \mid \langle \xi, \alpha \rangle = 0 \} \). A connected component \( \mathcal{C} \) in the complement \( h_{\mathbb{R}} \setminus \cup_{\alpha \in \overline{\Phi}} H_\alpha \) is called chamber, and it gives a decomposition of the tangent space \( T_p X = N_{p,+} \oplus N_{p,-} \) into attracting and repelling parts. We denote by \( \text{Attr}(p) := \{ x \in X | \lim_{t \to 0} \xi(t) \cdot x = p \} \) the attracting set of \( p \) with respect to \( \mathcal{C} \), where \( \xi \in X_*(H) \) is a one parameter subgroup of \( H \) contained in \( \mathcal{C} \). We note that this does not depend on the choice of \( \xi \) in \( \mathcal{C} \). The choice of chamber also gives a partial order \( \preceq_\mathcal{C} \) on \( X^H \) generated by \( p \in \text{Attr}(p') \implies p \preceq_\mathcal{C} p' \).
We set $P := \text{Pic}(X)$ and $P^\vee := \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$ its dual. We denote by $P_k := P \otimes \mathbb{R}$ and $\mathfrak{A} \subset P_k$ the ample cone of $X$. We will also need the notion of Kähler roots at each fixed point, but unfortunately, we do not know intrinsic definition of this notion. Our temporary definition is to take the equivariant roots at the corresponding fixed point for dual conical symplectic resolution. In this paper, we consider them as additional data and take a multiset $\Psi(p)$ of elements in $P^\vee$ for each $p \in X^H$. As a condition they should satisfy, we assume that the walls in the slope parameters where the $K$-theoretical stable basis $\text{Stab}^K_{\xi,T/s}(p)$ recalled in the next section jump are contained in the walls of the form $\{s \in P_k \mid \langle s, \beta \rangle \in \mathbb{Z}\}$ for some $\beta \in \Psi(p)$\footnote{In this paper, the notations for the bar involutions are always equipped with some upper index, and simple $\beta$ is used to denote a Kähler root. We hope this notation does not cause any confusions.}. We also assume that $\mathfrak{A}$ is a connected component of the complement of linear hyperplanes in $P_k$ defined by the Kähler roots in the union $\mathfrak{F} := \cup_{p \in X^H} \Psi(p)$.

**Remark 3.1.** For quiver varieties, one can read off the information about $\Psi(p)$ without using symplectic duality by a conjecture of Dinkins-Smirnov [14]. However, this does depend on the presentation of $X$ as a GIT quotient. Hence we need to allow some factor of symplectic vector spaces and modify the statement of symplectic duality to include some information about how to present the conical symplectic resolutions.

Since these differences only affect the overall constants in the $K$-theoretic bar involutions and canonical bases, we do not pursue this point further here. We only note that this freedom on the normalization is important for example when one try to compare our definition to the notion of global crystal bases of quantum affine algebras defined by Kashiwara [19]. For the canonical bases in equivariant $K$-theory of $ADE$ type quiver varieties defined by Varagnolo-Vasserot [42], this kind of comparison is given by Nakajima [31].

We also take an equivariant lift $\xi : P \to \text{Pic}^\xi(X)$, i.e., a section of the natural homomorphism $\text{Pic}^\xi(X) \to P$ given by forgetting the equivaraint structures. In this paper, when we consider symplectic resolutions, they are always equipped with the above additional data such as $\xi$, $\Psi(p)$, and $\mathfrak{L}$. We now formulate a notion of dual pair between conical symplectic resolutions $X = (\mathbf{X}, \mathfrak{C}, \mathfrak{A}, \Phi, \Psi, \mathfrak{L})$ and $X' = (\mathbf{X}', \mathfrak{C}', \mathfrak{A}', \Phi', \Psi', \mathfrak{L}')$ as follows.

**Definition 3.2.** We say that a pair of conical symplectic resolutions $X$ and $X'$ forms a dual pair if

- There exists an order reversing bijection $(X^H, \succeq_{\mathfrak{C}}) \cong ((X')^H, \succeq_{\mathfrak{C}'}).$ We denote by $p' \in (X')^H$ the fixed point corresponding to $p \in X^H$;
- There exist isomorphisms $\mathfrak{X}_* (H) \cong P'$ and $P \cong \mathfrak{X}_* (H')$ such that under this identification, we have $\mathfrak{C} = \mathfrak{C}'$, $\mathfrak{A} = \mathfrak{C}'$, $\Phi(p) = \Psi(p')$, and $\Psi(p) = \Phi(p')$;
- For any $\lambda \in P$, $\lambda' \in P'$, and $p \in X^H$, we have

\[
\langle \text{wt}_H i^*_p \mathfrak{L}(\lambda), \lambda' \rangle = -\langle \text{wt}_H i^*_p \mathfrak{L}'(\lambda'), \lambda \rangle, \tag{8}
\]
\[
\text{wt}_H i^*_p \mathfrak{L}(\lambda) = -\langle \text{wt}_H \det N_{p,-}^{\prime}, \lambda \rangle, \tag{9}
\]
\[
\text{wt}_H i^*_p \mathfrak{L}'(\lambda') = -\langle \text{wt}_H \det N_{p,-}, \lambda' \rangle; \tag{10}
\]
- For any $p \in X^H$, we have

\[
\text{wt}_H \det N_{p,-} + \frac{1}{2} \dim X = -\left( \text{wt}_H \det N_{p,-}^{\prime} + \frac{1}{2} \dim X' \right). \tag{11}
\]

Let $\{a_1, \ldots, a_c\}$ be a basis of $X^*_o (H)$ considered as elements of $K_H (pt)$ and $\{c_1, \ldots, c_c\}$ be the dual basis of $X_* (H) \cong P'$. Similarly, let $\{z_1, \ldots, z_c\}$ be a basis of $P^\vee \cong X^* (H')$ considered as elements of
$K_H^{\text{pt}}$ and \{l_1, \ldots, l_r\} be the dual basis of $P$. We also identify $\mathbb{S}^l = \mathbb{S}$ and their equivariant parameters by $v^i = v$. In these notations, the third condition in Definition 3.2 is equivalent to the following:

$$i_p^* \Sigma(\lambda) = \prod_{i=1}^{r} a_i^{-(i_p^* \Sigma(c_i), \lambda) + (\det N_p^1, -\lambda)},$$

$$i^*_p \Sigma^!(\lambda') = \prod_{i=1}^{r} z_i^{-(i_p^* \Sigma(l_i), \lambda') + (\det N_p, -\lambda')}.$$  \hspace{1cm} (12)

\textbf{Example 3.3.} Let $X = T^*(G/B)$ be the Springer resolution as in section 2. We recall that $G$ is of adjoint type. Let $G^\vee$ be the adjoint type semisimple algebraic group whose Lie algebra is the Langlands dual of $\mathfrak{g}$. We claim that the pair $X$ and $X^! = T^*(G^\vee/B^\vee)$ with certain choice of data forms a dual pair in the above sense. Here, we take a Borel subgroup $B^\vee \subset G^\vee$ as in section 2.

In this case, it is well-known that the Picard group $P$ of $X$ is given by the weight lattice and the ample cone is given by the positive Weyl chamber in our convention. For example, we take the chamber $\mathcal{C}$ to be the negative Weyl chamber as in section 2. Since the ample cone of $X^!$ is the positive Weyl chamber, we twist the isomorphism $P^! \cong \mathbb{X}_s(H)$ by $-1$ so that $\mathcal{C}$ and $\mathfrak{X}^!$ corresponds to each other. We take $\mathcal{C}^!$ to be the positive Weyl chamber and the isomorphism $P \cong \mathbb{X}_s(H^!)$ to be the natural one. We take $\Psi(w)$ to be the set of all coroots for any $w \in W$. The data $\Psi^!$ is chosen similarly. For $\mathfrak{L} : P \to \text{Pic}^+(X)$, we take $\mathfrak{L}(\lambda) = v^{(2\rho^\vee, \lambda)} \cdot \mathcal{L}_\lambda$ for any $\lambda \in P$, where $\mathcal{L}_\lambda$ is defined as in section 2.1 and $2\rho^\vee \in P^\vee$ is the sum of all positive coroots. We note that the shift $[-\lambda]$ is needed to make them $H$-equivariant. Similarly, we take $\mathfrak{L}^!(\lambda') = v^{(2\rho^\vee, \lambda')} \cdot \mathcal{L}^!$. Here, we consider $2\rho$ as an element of $(P^\vee)^!$.

We identify $X^H \cong W$ and $(X^!)^H \cong W$ by $w \leftrightarrow w^{-1}$. One can check that the partial order on $X^H$ given by $\mathfrak{C}$ is the Bruhat order and the partial order on $(X^!)^H$ given by $\mathfrak{C}^!$ is the opposite Bruhat order. Therefore, this gives an order reversing bijection. The second condition in Definition 3.2 is obvious from our choice. We note that $\det N_{w,-} = v^{-2l(w)(2\rho)}$ and $\det N_{w,-}^1 = v^{-2l(w)+2l(w)^!-2\rho^\vee}$. This easily implies (9), (10), and (11). Here, we note that $2\rho \in \mathbb{X}_s(H)$ is identified with $-2\rho$ in $(P^\vee)^!$. We note that $\operatorname{wt}_H i_w^* \Sigma(\lambda) = w\lambda - \lambda \in \mathbb{X}_s(H)$ and $\operatorname{wt}_H i_w^* \Sigma^!(\lambda') = w^{-1}\lambda' - \lambda' \in \mathbb{X}_s(H^!)$ for any $\lambda \in P$ and $\lambda' \in P^!$. Since $\lambda' \in P^!$ is identified with $-\lambda' \in \mathbb{X}_s(H)$, $\mathcal{S}$ follows from the obvious equation $(w\lambda - \lambda, -\lambda') = -(w^{-1}\lambda' - \lambda', \lambda)$.

We conjecture that if $X$ and $X^!$ are symplectic dual in the sense of Braden-Licata-Proudfoot-Webster [2], then they form a dual pair in the sense of Definition 3.2. We will check this for toric hyper-Kähler manifolds in Proposition 5.6 [2]. In this paper, we always assume that a symplectic resolution considered in this paper is equipped with a dual symplectic resolution in the above sense.

\textbf{Assumption 3.4.} For any choice of $\mathfrak{C}$, the conical symplectic resolution $X = (X, \mathfrak{C}, \mathfrak{X}, \ldots)$ has a dual conical symplectic resolution $X^! = (X^!, \mathfrak{C}^!, \mathfrak{X}^!, \ldots)$ such that $X$ and $X^!$ forms a dual pair in the sense of Definition 3.2.

We note that if $\mathcal{L} \in \text{Pic}^+(X)$ is a $\mathbb{T}$-equivariant line bundle and $l \in P$ is its underlying line bundle, then $\mathcal{L} \otimes \Sigma(l)^{-1}$ is a trivial as a non-equivariant line bundle on $X$. The Assumption 3.4 implies that

$$w(\mathcal{L}) := \operatorname{wt}_3 i_p^* \mathcal{L} - \sum_{\beta \in \Psi_+(p)} \langle \beta, l \rangle$$

does not depend on the choice of $p \in X^H$. Here, $\Psi_+(p)$ is the sub-multiset of $\Psi(p)$ consisting of Kähler roots at $p$ which are positive with respect to the ample cone $\mathfrak{X}$. We note that $w : \text{Pic}^+(X) \to \mathbb{Z}$ gives a homomorphism and the image of $\mathcal{L}$ lands in the kernel of $w$.

\subsection{K-theoretic standard bases}

In this section, we recall the definition of $K$-theoretic stable basis introduced in [32, 34]. In order to define this, we need to choose further data. For an element $\mathcal{F} \in K_T(X)$, we denote by
Lemma 3.9. For any \( X \), \( K \) we have:

\[ \lambda^{i} \in K_{i}(X) \]

where \( \lambda^{i} \) is the i-th exterior product of \( V \). We take an element \( T^{1/2} \in K_{i}(X) \) called polarization satisfying \( T^{1/2} + v^{-2}L^{\vee} = T_{\mu} \), where \( T_{\mu} \) is the K-theory class of the tangent bundle of \( X \). We note that for any \( G \in K_{i}(X) \), \( T_{\mu} = T^{1/2} - G + v^{-2}L^{\vee} \) and \( T_{\mu}^{op} = v^{-2}(T^{1/2})^{\vee} \) are also polarization for \( X \).

We also take a generic element \( s \in P_{K} \cup \beta_{s} \in \mathbb{Z} \) called slope. The map \( L \) naturally extends to give a fractional line bundle \( \mathcal{L}(s) \in P_{\mu,}^{\vee}(X) \) and each restriction at a fixed point \( \pi_{s}^{*} \mathcal{L}(s) \) gives an element of \( \mathcal{X}^{*}(\mathbb{T}) \). For an element of the form \( m = \sum_{\mu \in h_{K}} m_{\mu} \), we denote by \( \deg_{\mu}(m) \subset h_{K} \) the convex hull of \( \{ \mu \in h_{K} \mid m_{\mu} \neq 0 \} \).

Definition 3.5 (32 34]. A set of elements \( \{ \text{Stab}^{K}_{i,E,T^{1/2},s}(p) \} \) of \( K \) is called stable basis if it satisfies the following conditions:

- \( \text{Supp}(\text{Stab}^{K}_{i,E,T^{1/2},s}(p)) \subset \cup_{p'} \leq p \text{ Attr}^{E}(p'); \)
- \( \text{i}_{p}^{*} \text{Stab}^{K}_{i,E,T^{1/2},s}(p) = \sqrt{\det N_{p} - \det T^{1/2}_{T^{1/2}} \cdot \lambda^{i} N_{p} - \lambda^{i}} \)
- \( \deg_{H} \left( \text{i}_{p}^{*} \text{Stab}^{K}_{i,E,T^{1/2},s}(p') \cdot \text{i}_{p}^{*} \mathcal{L}(s) \right) \subset \deg_{H} \left( \text{i}_{p}^{*} \text{Stab}^{K}_{i,E,T^{1/2},s}(p') \cdot \text{i}_{p}^{*} \mathcal{L}(s) \right) \) for any \( p' \leq p \in X^{H} \).

Here, the square root appearing in the normalization is well-defined by 32 34. We note that our normalization is different from 32 34 by \( \sqrt{\det T^{1/2}_{T^{1/2}} - \det T^{1/2}_{T^{1/2}}} \). The existence of \( \sqrt{\det T^{1/2}_{T^{1/2}}} \) follows from the equation \( T^{1/2}_{T^{1/2}} + v^{-2}(T^{1/2})^{\vee} = 0 \). If the slope \( s \) is sufficiently generic so that \( wt_{H} \text{i}_{p}^{*} \mathcal{L}(s) - wt_{H} \text{i}_{p}^{*} \mathcal{L}(s) \notin X^{*}(H) \) for any \( p \neq p' \in X^{H} \), then the K-theoretic stable basis is unique if it exists by 32 Proposition 9.2.2]. The existence is claimed in some generality in 33 and proved for example when \( X \) is a toric hyper-Kähler manifold, quiver variety 11, or Springer resolution 11. In this paper, we assume that the \( K \)-theoretic stable bases exist uniquely for any \( X \) and the additional data we consider.

Assumption 3.6. The slope \( s \in P_{K} \) satisfies \( wt_{H} \text{i}_{p}^{*} \mathcal{L}(s) - wt_{H} \text{i}_{p}^{*} \mathcal{L}(s) \notin X^{*}(H) \) for any \( p \neq p' \in X^{H} \). Moreover, \( \text{Stab}^{K}_{i,E,T^{1/2},s}(p) \) exists for any \( p 

We next collect some basic results on the K-theoretic stable bases for our reference since we have changed the convention slightly.

Lemma 3.7 (32, Exercise 9.1.2). For any \( G \in K_{i}(X) \) and \( p \in X^{H} \), we have:

\[ \text{Stab}^{K}_{i,E,T^{1/2},s}(p) = v^{k}G \cdot \text{Stab}^{K}_{i,E,T^{1/2},s}(p). \]

Lemma 3.8. For any \( l \in P \) and \( p \in X^{H} \), we have:

\[ \text{Stab}^{K}_{i,E,T^{1/2},s+l}(p) = (\text{i}_{p}^{*} \mathcal{L}(l))^{-1} \cdot \mathcal{L}(l) \otimes \text{Stab}^{K}_{i,E,T^{1/2},s}(p). \]

Lemma 3.9 (34]. For any \( p \in X^{H} \), we have:

\[ \text{Stab}^{K}_{i,E,T^{1/2},s}(p) = (-v)^{-\frac{1}{2}} \dim X \text{Stab}^{K}_{i,E,T^{1/2},s-l}(p). \]

These formulas easily follow from the definition and the uniqueness of K-theoretic stable basis. Recall the inner product \( (\cdot : \cdot) \) defined in 2. The K-theoretic stable bases have the following orthogonality property with respect to \( (\cdot : \cdot) \).
Lemma 3.10 ([1], Proposition 1). For any $p, p' \in X^H$, we have
\[
\left( \text{Stab}_{\mathcal{E},T^1/2,s}^K(p) : \text{Stab}_{\mathcal{E},T^1/2,-s}^K(p') \right) = \delta_{p,p'}.
\]
In order to define $K$-theoretic bar involutions, we further renormalize the $K$-theoretic stable bases. For each $p \in X^H$, we set
\[
a_p(T^{1/2}, s) := \sum_{\beta \in \Psi^+(p)} \left( [\{s, \beta\}] + \frac{1}{2} \right) - \frac{1}{4} \dim X + \frac{1}{2} w(\det T^{1/2}) \in \frac{1}{2} \mathbb{Z}.
\]
By numerical experiments, we expect that $a_p(T^{1/2}, s) \in \mathbb{Z}$. This is equivalent to the following assumption which is also assumed in this paper.

Assumption 3.11. There exists a polarization $T^{1/2}$ on $X$ such that
\[
w(\det T^{1/2}) \equiv \frac{1}{2} \dim X + \frac{1}{2} \dim X' \mod 2.
\]
We note that if $T^{1/2}$ satisfies the above condition, then $T^{1/2}_G$ also satisfies the condition for any $G \in K_T(X)$. We set
\[
\mathbb{F} := \{ (\lambda, p) \mid \lambda \in X^*(H), p \in X^H \}.
\]
This set will label the $K$-theoretic standard bases and $K$-theoretic canonical bases.

Definition 3.12. For any $(\lambda, p) \in \mathbb{F}$, we set
\[
S_{\mathcal{E},T^{1/2},s}(\lambda, p) := (-1)^{\frac{1}{2} \dim X} \mathcal{E}_{s_p(T^{1/2}, s)}[\lambda] : \text{Stab}_{\mathcal{E},T^{1/2},s}^K(p).
\]
We call the set $\mathbb{E}_{\mathcal{E},T^{1/2},s}^\text{std} := \{ S_{\mathcal{E},T^{1/2},s}(\lambda, p) \mid (\lambda, p) \in \mathbb{F} \} \subset K_T(X)$ $K$-theoretic standard basis for $X$.

We will simply write $S_{\mathcal{E},T^{1/2},s}(p) := S_{\mathcal{E},T^{1/2},s}(0, p)$. The standard basis should be considered as a $\mathbb{Z}[v, v^{-1}]$-basis for the equivariant $K$-theory of the full attracting set $\cup_{p \in X^H} \text{Attr}(p)$. One explanation of the seemingly strange normalization in this definition will be given in section 4.3 by considering its elliptic analogue. Here, we list some basic properties of the $K$-theoretic standard bases for our reference.

Lemma 3.13. For any $G \in K_T(X)$ and $(\lambda, p) \in \mathbb{F}$, we have
\[
S_{\mathcal{E},T^{1/2},s}(\lambda, p) = S_{\mathcal{E},T^{1/2},s + \det G}(\lambda + \det \mathfrak{g}) \circ \mathcal{E}(p).
\]

Proof. This follows from Lemma 3.7 and $a_p(T^{1/2}, s) = a_p(T^{1/2}, s + \det G) - \det G - wt_\mathfrak{g} \circ \mathcal{E}(p)$.

Lemma 3.14. For any $l \in P$ and $(\lambda, p) \in \mathbb{F}$, we have
\[
S_{\mathcal{E},T^{1/2},s + l}(\lambda, p) = S_{\mathcal{E},T^{1/2},s}(\lambda - wt_H \mathcal{E}(l)) \circ \mathcal{E}(p).
\]

Proof. This follows from Lemma 3.8 and $a_p(T^{1/2}, s + l) = a_p(T^{1/2}, s) + \det \mathfrak{g} \circ \mathcal{E}(l)$.

Lemma 3.15. For any $(\lambda, p) \in \mathbb{F}$, we have
\[
S_{\mathcal{E},T^{1/2},s}(\lambda, p)^Y = (-v)^{\frac{1}{2} \dim X} S_{\mathcal{E},T^{1/2},-s}(-\lambda, p).
\]

Proof. This follows from Lemma 3.9 and $a_p(T^{1/2}) = -a_p(T^{1/2}, -s) - \dim X$.

Lemma 3.16. For any $p, p' \in X^H$, we have
\[
\left( S_{\mathcal{E},T^{1/2},s}(p) : S_{\mathcal{E},T^{1/2},-s}(p') \right) = v^{-\dim X} \delta_{p,p'}.
\]

Proof. This follows from Lemma 3.10 and $a_p(T^{1/2}) = -a_p(T^{1/2}, -s) - \dim X$.
3.3 K-theoretic bar involution

Now we define the $K$-theoretic bar involution. Since $\{S_{\xi,T^{1/2}_s}(p)\}_{p \in X^H}$ forms a basis of $K^*_\text{loc}(X)$ over $\text{Frac}(K_T(\text{pt}))$, we can define a $K_H(\text{pt})$-linear map $\beta^K = \beta^K_{\xi,T^{1/2}_s} : K^*_\text{loc}(X) \to K^*_\text{loc}(X)$ by the following conditions:

- $\beta^K(vm) = v^{-1}\beta^K(m)$ for any $m \in K^*_\text{loc}(X)$;
- $\beta^K(S_{\xi,T^{1/2}_s}(p)) = (-v)^{\dim X} S_{-\xi,T^{1/2}_s}(p)$ for any $p \in X^H$.

We note that $\beta^K_{-\xi,T^{1/2}_s} \circ \beta^K_{\xi,T^{1/2}_s} = \text{id}$. The following conjecture implies that $\beta^K$ is an involution, and hence we call it $K$-theoretic bar involution associated with the data $\xi$, $T^{1/2}_s$, and $s$.

**Conjecture 3.17.** The $K$-theoretic bar involution $\beta^K_{\xi,T^{1/2}_s}$ does not depend on the choice of $\xi$.

For toric hyper-Kähler manifolds, this conjecture is proved in Corollary 5.16. Assuming this conjecture, we will sometimes omit $\xi$ from the notation and write $\beta^K_{T^{1/2}_s} = \beta^K_{\xi,T^{1/2}_s}$. In this section, we prove certain triangular properties of $\beta^K$ with respect to the $K$-theoretic standard bases. We first define a partial order on the labeling set $\mathbb{F}$.

**Definition 3.18.** For $(\lambda, p), (\lambda', p') \in \mathbb{F}$, we write $(\lambda, p) \leq \xi, s (\lambda', p')$ if we have $\langle \lambda - \omega_H i_p^*\mathcal{L}(s), \xi \rangle \leq \langle \lambda' - \omega_H i_p^*\mathcal{L}(s), \xi \rangle$ for any $\xi \in \mathbb{C}$.

**Lemma 3.19.** The relation $\leq \xi, s$ defines a partial order on $\mathbb{F}$. Moreover, the number of elements $(\lambda, p) \in \mathbb{F}$ satisfying $(\lambda', p') \leq \xi, s (\lambda, p)$ is finite for any $(\lambda', p'), (\lambda''', p''') \in \mathbb{F}$, i.e., the partial order $\leq \xi, s$ is interval finite.

**Proof.** If $(\lambda, p) \leq \xi, s (\lambda', p')$ and $(\lambda', p') \leq \xi, s (\lambda, p)$, then we have $\lambda - \omega_H i_p^*\mathcal{L}(s) = \lambda' - \omega_H i_p^*\mathcal{L}(s)$. This implies that $\omega_H i_p^*\mathcal{L}(s)$ is interval finite. Hence $p = p'$ by Assumption 3.6. This proves the antisymmetry. The other properties are trivial to check.

The second claim follows from the compactness of the set $\bigcap_{\xi \in \mathbb{E}} \{\mu \in h^*_R \mid \langle \mu', \xi \rangle \leq \langle \mu, \xi \rangle \leq \langle \mu'', \xi \rangle\}$ for any $\mu', \mu'' \in h^*_R$.

Using this partial order, we define a $\mathbb{Z}[v, v^{-1}]$-module of formal sums

$$M_{\xi, s} := \left\{ \sum_{(\lambda, p) \in \mathbb{F}} f_{\lambda, p}(v) S_{\lambda, p} \bigg| f_{\lambda, p}(v) \in \mathbb{Z}[v, v^{-1}], \exists (\lambda_1, p_1), \ldots, (\lambda_m, p_m) \in \mathbb{F} \right. \text{ s.t. if } f_{\lambda, p} \neq 0, \text{ then } (\lambda, p) \geq \xi, s (\lambda_i, p_i) \text{ for some } i \right\}.$$

For any $\mathcal{F} \in K^*_\text{loc}(X)$, we have $\mathcal{F} = \sum_{p \in X^H} v^{\dim X} \mathcal{F} \cdot S_{-\xi,T^{1/2}_s}(p) \cdot S_{\xi,T^{1/2}_s}(p)$ by Lemma 3.16 and the possible denominators appearing in the inner product are of the form $\bigwedge^*_{\mathbb{P}}(T^*_{\mathbb{P}}X)$ for some $p \in X^H$. Therefore, by sending $S_{\xi,T^{1/2}_s}(\lambda, p)$ to $S_{\lambda, p}$ and expanding the rational function appearing in the coefficients into formal series in the positive or negative direction with respect to $\mathbb{F}$, we obtain two natural embeddings $i^{\xi}_{\mathbb{T}, T^{1/2}_s} : K^*_\text{loc}(X) \hookrightarrow M_{\mathbb{E}, \xi, s}$. The following lemma together with Lemma 3.19 implies that one can extend the $K$-theoretic bar involution to $M_{\mathbb{E}, \xi, s}$.

**Lemma 3.20.** For each $(\lambda, p) \in \mathbb{F}$, we have

$$i^{\xi}_{\mathbb{T}, T^{1/2}_s}((-v)^{\dim X} S_{-\xi,T^{1/2}_s}(\lambda, p)) \in S_{\lambda, p} + \sum_{(\lambda', p') \geq \xi, s (\lambda, p)} \mathbb{Z}[v, v^{-1}] \cdot S_{\lambda', p'}.$$

**Proof.** We first note that

$$v^{\dim X} \left( S_{-\xi,T^{1/2}_s}(p) : S_{-\xi,T^{1/2}_s}(p') \right) = \sum_{p'' \in X^H} i_{p''}^* \text{Stab}^K_{-\xi,T^{1/2}_s}(p) \cdot i_{p''}^* \text{Stab}^K_{-\xi,T^{1/2}_s}(p') \left/ \bigwedge^* (N_{p''}^+) \cdot \bigwedge^* (N_{p''}^-) \right. \tag{15}$$
If we write $N_{\rho''} = \sum_i w_i, w_i \in X^*(T)$, then we have
\[
i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, s}(p') \cdot i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, -s}(p'') = \frac{v}{\dim X} \det N_{\rho''} \cdot \bigwedge^\bullet \left( (N_{\rho''}^+)^* \right).
\]

If we expand this in positive (resp. negative) direction with respect to $\mathcal{C}$, then the expansion start from $(-v)^{\dim X}$ (resp. $(-v)^{-\dim X}$). This formula also implies that if we set $\rho_{\rho''} := wt_H \det N_{\rho''}$, then for any $\xi \in \mathcal{C}$, we have
\[
\langle \xi, \deg_H \left( i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, s}(p') \cdot i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, -s}(p'') \right) \rangle = \langle -\langle \xi, \rho_{\rho''} \rangle, \langle \xi, \rho_{\rho''} \rangle \rangle.
\]

On the other hand, by the definition of $K$-theoretic stable basis, we have
\[
\deg_H \left( i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, s}(p) \right) \subset \deg_H \left( i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, s}(p') \cdot i_{\rho''}^\bullet \mathcal{L}(s) \right),
\]
\[
\deg_H \left( i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, -s}(p') \right) \subset \deg_H \left( i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, -s}(p'') \cdot i_{\rho''}^\bullet \mathcal{L}^{-s} \right).
\]

Therefore, we obtain
\[
\langle \xi, \deg_H \left( i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, s}(p) \cdot i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, -s}(p') \right) \rangle \subset \left[ \langle -\langle \xi, \rho_{\rho''} \rangle + wt_H i_{\rho''}^\bullet \mathcal{L}(s) \rangle, \langle \xi, \rho_{\rho''} + wt_H i_{\rho''}^\bullet \mathcal{L}(s) \rangle \right].
\]

This implies that if $S_{\lambda, p''}$ appears in the expansion of $i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, s}(\lambda, p)$, then we have $\langle \xi, \lambda - \lambda' \rangle \geq \langle \xi, wt_H i_{\rho''}^\bullet \mathcal{L}(s) \rangle - \langle \xi, wt_H i_{\rho''}^\bullet \mathcal{L}(s) \rangle$ for any $\xi \in \pm \mathcal{C}$, which means $\langle \lambda, p \rangle \leq \pm \mathcal{C}$.

For the coefficient of $S_{\lambda, p}$, we note that when $p'' \neq p = p'$, the fractional shift appearing in (16) and (17) are opposite and not integral. Hence we have
\[
\langle \xi, \deg_H \left( i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, s}(p) \cdot i_{\rho''}^\bullet \text{Stab}_{\mathcal{C}, T^{1/2}, -s}(p') \right) \rangle \subset (-\langle \xi, \rho_{\rho''} \rangle, \langle \xi, \rho_{\rho''} \rangle).
\]

This implies that only $p'' = p = p'$ part in (15) contribute to the coefficient of $S_{\lambda, p}$.}

\[\square\]

### 3.4 $K$-theoretic canonical basis

Once we have defined the bar involution, we can now follow Lusztig [22, 23] to define the notion of signed canonical basis by imposing bar invariance and asymptotic norm one property. Recall the inner product $\langle \| \| \rangle : K_T(X) \times K_T(X) \rightarrow \text{Frac}(K_T(pt))$ defined in (3). We note that this depends on the choice of $\mathcal{C}, T^{1/2}$, and $s$ through $\beta^\mathcal{C}_{T^{1/2}}$, but we simply omit the dependence from the notation if there is no chance of confusion. Recall that $L \subset X$ is the central fiber. We set
\[
\mathcal{B}_{L, T^{1/2}, s}^\pm := \left\{ m \in K_T(L) \left\| \beta^\mathcal{C}_{T^{1/2}}(m) = v^{\dim X} m, (m||m) \in 1 + v^{-1} K_H(pt) [v^{-1}] \right\} \right.,
\]
\[
\mathcal{B}_{X, T^{1/2}, s}^\pm := \left\{ m \in K_T(X) \left\| \beta^\mathcal{C}_{T^{1/2}}(m) = m, (m||m) \in 1 + v^{-1} K_H(pt) [v^{-1}] \right\} \right.,
\]
where we expand the rational function in Laurent series of $v^{-1}$ with coefficients in $\text{Frac}(K_H(pt))$.

We note that for any $\lambda \in X^*(H)$ and $m \in \mathcal{B}_{X, T^{1/2}, s}^\pm$, we have $\pm \lambda \cdot m \in \mathcal{B}_{X, T^{1/2}, s}^\pm$. If we assume Conjecture 3.17, then this definition does not depend on the choice of $\mathcal{C}$. We call $\mathcal{B}_{L, T^{1/2}, s}^\pm$ and $\mathcal{B}_{X, T^{1/2}, s}^\pm$ signed $K$-theoretic canonical bases for $L$ and $X$.

If we assume that $\beta^\mathcal{C}$ is an involution, we can formally construct a family of bar invariant and asymptotic norm one elements by Kazhdan-Lusztig type algorithm. More precisely, we can prove the following, whose proof provides such an algorithm.
Proposition 3.21. Assume that $\beta^{K}_{\mathcal{E},T^{1/2},s}$ is an involution.

1. For any $(\lambda, p) \in \mathcal{F}$, there exists unique $C_{\lambda,p}^{T^{1/2},s} \in S_{\lambda,p} + \sum_{(\lambda', p') > (\lambda, p)} v^{-\dim X} \cdot \beta^{K}_{\mathcal{E},T^{1/2},s}(C_{\lambda,p}^{T^{1/2},s}) = C_{\lambda,p}^{T^{1/2},s}$.

2. For any $(\lambda, p) \in \mathcal{F}$, there exists unique $E_{\lambda,p}^{T^{1/2},s} \in S_{\lambda,p} + \sum_{(\lambda', p') < (\lambda, p)} v^{-\dim X} \cdot \beta^{K}_{\mathcal{E},T^{1/2},s}(E_{\lambda,p}^{T^{1/2},s}) = E_{\lambda,p}^{T^{1/2},s}$.

Proof. We only prove the first statement since the second statement can be proved similarly. We first prove the existence. We take any total order on $F$ refining $\leq_{s}$, and prove inductively that for any $(\lambda', p') \leq_{s} (\lambda, p)$, there exists an element of the form

$$C_{\lambda,p}^{\leq (\lambda', p')} = \sum_{(\lambda', p') \leq (\lambda, p)} f^{\lambda', p'}_{\lambda,p}(v)S_{\lambda', p'}$$

with $f^{\lambda,p}_{\lambda,p}(v) = 1$ and $f^{\lambda', p'}_{\lambda,p}(v) \in v^{-\dim X} \cdot \beta^{K}_{\mathcal{E},T^{1/2},s}(C_{\lambda,p}^{\leq (\lambda', p')})$ such that

$$v^{-\dim X} \cdot \beta^{K}_{\mathcal{E},T^{1/2},s}(C_{\lambda,p}^{\leq (\lambda', p')}) - C_{\lambda,p}^{\leq (\lambda', p')} \in \sum_{(\lambda', p') > (\lambda, p)} Z[v, v^{-1}] \cdot S_{\lambda', p'}.$$

For $(\lambda', p') = (\lambda, p)$, we can take $C_{\lambda,p}^{\leq (\lambda', p')} = S_{\lambda,p}$ by Lemma 3.20. Assume that we have constructed $C_{\lambda,p}^{\leq (\lambda', p')}$ and write

$$v^{-\dim X} \cdot \beta^{K}_{\mathcal{E},T^{1/2},s}(C_{\lambda,p}^{\leq (\lambda', p')}) - C_{\lambda,p}^{\leq (\lambda', p')} \in g(v)S_{\lambda', p'} + \sum_{(\lambda', p') > (\lambda, p)} Z[v, v^{-1}] \cdot S_{\lambda', p'}$$

for some $g(v) \in Z[v, v^{-1}]$. Since $\beta^{K}_{\mathcal{E},T^{1/2},s}$ is an involution, we also obtain

$$v^{-\dim X} \cdot \beta^{K}_{\mathcal{E},T^{1/2},s}(C_{\lambda,p}^{\leq (\lambda', p')}) - C_{\lambda,p}^{\leq (\lambda', p')} \in - g(v^{-1})S_{\lambda', p'} + \sum_{(\lambda', p') > (\lambda, p)} Z[v, v^{-1}] \cdot S_{\lambda', p'}.$$

By comparing the coefficient, we obtain $g(v) = - g(v^{-1})$. Hence there exists an element $f^{\lambda', p'}_{\lambda,p}(v) \in v^{-\dim X} \cdot \beta^{K}_{\mathcal{E},T^{1/2},s}(C_{\lambda,p}^{\leq (\lambda', p')})$ such that $g(v) = f^{\lambda', p'}_{\lambda,p}(v) - f^{\lambda, p'}_{\lambda,p}(v^{-1})$. Then it suffices to take $C_{\lambda,p}^{\leq (\lambda', p')} = C_{\lambda,p}^{\leq (\lambda', p')} + f^{\lambda', p'}_{\lambda,p}(v)S_{\lambda', p'}$. This proves the existence of $C_{\lambda,p}^{T^{1/2},s}$.

For the uniqueness, we assume that there is another $C_{\lambda,p}^{T^{1/2},s}$ satisfying the above conditions. Then we can expand

$$C_{\lambda,p}^{T^{1/2},s} = C_{\lambda,p}^{T^{1/2},s} + \sum_{(\lambda', p') > (\lambda, p)} g^{\lambda', p'}_{\lambda,p}(v)C_{\lambda', p'}^{T^{1/2},s},$$

with $g^{\lambda', p'}_{\lambda,p}(v) \in v^{-\dim X} \cdot \beta^{K}_{\mathcal{E},T^{1/2},s}(C_{\lambda,p}^{T^{1/2},s})$. We also obtain

$$C_{\lambda,p}^{T^{1/2},s} = C_{\lambda,p}^{T^{1/2},s} + \sum_{(\lambda', p') > (\lambda, p)} g^{\lambda', p'}_{\lambda,p}(v^{-1})C_{\lambda', p'}^{T^{1/2},s},$$

Hence we have $g^{\lambda', p'}_{\lambda,p}(v) = g^{\lambda', p'}_{\lambda,p}(v^{-1}) \in v^{-\dim X} \cdot \beta^{K}_{\mathcal{E},T^{1/2},s}(C_{\lambda,p}^{T^{1/2},s})$. This proves $C_{\lambda,p}^{T^{1/2},s} = C_{\lambda,p}^{T^{1/2},s}$. \qed

Conjecture 3.22. The $K$-theoretic bar involution $\beta^{K}_{\mathcal{E},T^{1/2},s}$ is an involution. For any $(\lambda, p) \in \mathcal{F}$, there exists $C_{\mathcal{E},T^{1/2},s}(\lambda, p) \in K_{T}(L)$ (resp. $\mathcal{E}_{\mathcal{E},T^{1/2},s}(\lambda, p) \in K_{T}(X)$) such that $i^{\mathcal{E}_{\mathcal{E},T^{1/2},s}}_{\mathcal{E},T^{1/2},s}(C_{\mathcal{E},T^{1/2},s}(\lambda, p)) = C_{\lambda,p}^{T^{1/2},s}$ (resp. $\mathcal{E}_{\mathcal{E},T^{1/2},s}(\mathcal{E}_{\mathcal{E},T^{1/2},s}(\lambda, p)) = C_{\lambda,p}^{T^{1/2},s}$). Moreover, $\mathcal{B}_{L,T^{1/2},s} := \{C_{\mathcal{E},T^{1/2},s}(\lambda, p) \}_{(\lambda, p) \in F}$ (resp. $\mathcal{B}_{X,T^{1/2},s} := \{\mathcal{E}_{\mathcal{E},T^{1/2},s}(\lambda, p) \}_{(\lambda, p) \in F}$) forms a basis of $K_{T}(L)$ (resp. $K_{T}(X)$) as a $Z[v, v^{-1}]$-module.
For toric hyper-Kähler manifolds, this conjecture is proved in Proposition 5.15, Lemma 5.17, and Corollary 5.37. We note that $C_{\lambda,p}^{T^{1/2},s}(\lambda,p) = |\lambda| \cdot C_{\lambda,T^{1/2},s}(p)$ and $E_{\lambda,p}^{T^{1/2},s}(\lambda,p) = |\lambda| \cdot E_{\lambda,T^{1/2},s}(p)$ for any $(\lambda,p) \in \mathcal{F}$, where we set $C_{\lambda,T^{1/2},s}(p) := \mathcal{E}_{\lambda,T^{1/2},s}(0,p)$ and $E_{\lambda,T^{1/2},s}(p) := \mathcal{E}_{\lambda,T^{1/2},s}(0,p)$. We call $\mathcal{B}_{L,T^{1/2},s}$ and $\mathcal{B}_{X,T^{1/2},s}$ $K$-theoretic canonical bases for $L$ and $X$ associated with the data $T^{1/2}$ and $s$. Conjecture 3.22 implies that the $K$-theoretic bar involutions preserve $K_T(L)$ and $K_T(X)$, which are already quite nontrivial from our definition. We should remark that $C_{\lambda,T^{1/2},s}(\lambda,p)$ and $E_{\lambda,T^{1/2},s}(\lambda,p)$ does depend on the choice of $C$, but as we will show, the sets $\mathcal{B}_{L,T^{1/2},s}$ and $\mathcal{B}_{X,T^{1/2},s}$ will not depend on it (possibly up to sign). If we change $C$, then the parametrization of the canonical basis by the fixed points will change.

We also note that since $L$ is contained in the full attracting sets, the conjecture implies that the sum in the definition of $C_{\lambda,p}^{T^{1/2},s}$ is actually a finite sum. In particular, the above Kazhdan-Lusztig type algorithm gives a way to calculate $K$-theoretic canonical bases for $L$ if we know some formula for the $K$-theoretic stable bases. The sum in the definition of $E_{\lambda,p}^{T^{1/2},s}$ is always an infinite sum except for the trivial cases, but as Proposition 5.31 shows, $\mathcal{B}_{L,T^{1/2},s}$ and $\mathcal{B}_{X,T^{1/2},s}$ are dual basis with respect to $(−||−)$, hence one can also calculate $\mathcal{B}_{X,T^{1/2},s}$ if we know $\mathcal{B}_{L,T^{1/2},s}$.

We also conjecture the following positivity property for the expansion of the $K$-theoretic canonical bases in terms of the $K$-theoretic standard bases. This is an analogue of the positivity of the Kazhdan-Lusztig polynomials.

**Conjecture 3.23.** For any $(\lambda,p) \in \mathcal{F}$, we have

\[
C_{\lambda,p}^{T^{1/2},s} \in \sum_{(\lambda',p') \geq (\lambda,p)} \mathbb{Z}_{\geq 0}[-v^{-1}] \cdot S_{\lambda',p'}
\]

\[
E_{\lambda,p}^{T^{1/2},s} \in \sum_{(\lambda',p') \leq (\lambda,p)} \mathbb{Z}_{\geq 0}[-v^{-1}] \cdot S_{\lambda',p'}.
\]

For toric hyper-Kähler manifolds, this conjecture is proved in Corollary 5.44.

### 3.5 Basic properties

In this section, we prove some basic properties of the $K$-theoretic canonical bases assuming Conjecture 3.22. We first check that $\mathcal{B}_{L,T^{1/2},s} = \mathcal{B}_{L,T^{1/2},s} \sqcup -\mathcal{B}_{L,T^{1/2},s}$ and $\mathcal{B}_{X,T^{1/2},s} = \mathcal{B}_{X,T^{1/2},s} \sqcup -\mathcal{B}_{X,T^{1/2},s}$. The asymptotic norm one property follows from the following lemma.

**Lemma 3.24.** For any $p, p' \in X^H$, we have

\[
(S_{\mathcal{E},T^{1/2},s}(p)||S_{\mathcal{E},T^{1/2},s}(p')) = \delta_{p,p'}.
\]

**Proof.** By Lemma 3.15 and Lemma 3.16, we have

\[
(S_{\mathcal{E},T^{1/2},s}(p)||S_{\mathcal{E},T^{1/2},s}(p')) = \left(S_{\mathcal{E},T^{1/2},s}(p) : \mathbb{D}_{X} \beta_{\mathcal{E},T^{1/2},s}(S_{\mathcal{E},T^{1/2},s}(p'))\right)
\]

\[
= (-v)^{\dim X} \cdot \left(S_{\mathcal{E},T^{1/2},s}(p) : S_{-\mathcal{E},T^{1/2},s}(p')\right)
\]

\[
= \delta_{p,p'}.
\]

We denote by $\dagger : K_T(pt) \to K_T(pt)$ the involution induced from the inverse for $H$.

**Corollary 3.25.** Assume Conjecture 3.22. For any $p, p' \in X^H$, we have

\[
(C_{\mathcal{E},T^{1/2},s}(p)||C_{\mathcal{E},T^{1/2},s}(p')) \in \delta_{p,p'} + v^{-1}K_H(pt)[v^{-1}],
\]

\[
(E_{\mathcal{E},T^{1/2},s}(p)||E_{\mathcal{E},T^{1/2},s}(p')) \in \delta_{p,p'} + v^{-1}K_H(pt)[v^{-1}],
\]
Proof. For the first statement, this follows from Lemma 3.24 since the expansion of $C_{\mathcal{E},T^{1/2},s}(p)$ in terms of the standard basis is a finite sum. For the second statement, let us write

$$
E \subset \mathcal{E}_{\mathcal{E},T^{1/2},s}(p) = \sum_{p' \in X^H} f_{p,p'} \cdot S_{E,\mathcal{T}^{1/2},s}(p')
$$

for some $f_{p,p'} \in \text{Frac}(K_{\mathcal{T}}(pt))$. The formal construction of $E_{\mathcal{T}^{1/2},s}$ implies that if we expand $f_{p,p'}$ in $v^{-1}$, we have $f_{p,p'} \in \delta_{p,p'} + v^{-1} \text{Frac}(K_{\mathcal{T}}(pt))[v^{-1}]$. On the other hand, since $X^S = L^2$ is smooth and proper, we have $f_{p,p'} \in K_{\mathcal{T}}(pt)(v^{-1})$. Hence we obtain $f_{p,p'} \in \delta_{p,p'} + v^{-1}K_{\mathcal{T}}(pt)[v^{-1}]$ and

$$(E \subset \mathcal{E}_{\mathcal{E},T^{1/2},s}(p) || E_{\mathcal{E},T^{1/2},s}(p')) = \sum_{p' \in X^H} f_{p,p'} \cdot f_{p,p'} \in \delta_{p,p'} + v^{-1}K_{\mathcal{T}}(pt)[v^{-1}]
$$

by Lemma 3.24.

We denote by $(-) : K_{\mathcal{T}}(pt) \to K_{\mathcal{T}}(pt)$ the involution induced from the inverse map for $S$.

**Corollary 3.26.** Assume Conjecture 3.22. We have

$$
\mathbb{B}^\pm_{L,T^{1/2},s} = \mathbb{B}_{L,T^{1/2},s} \cup -\mathbb{B}_{L,T^{1/2},s},
$$

$$
\mathbb{B}^\pm_{X,T^{1/2},s} = \mathbb{B}_{X,T^{1/2},s} \cup -\mathbb{B}_{X,T^{1/2},s}.
$$

**Proof.** We only prove the second statement since the proof is the same for the first. By Corollary 3.25, we have $\mathbb{B}_{X,T^{1/2},s} \cup -\mathbb{B}_{X,T^{1/2},s} \subset \mathbb{B}^\pm_{X,T^{1/2},s}$. Hence we only need to prove the other inclusion. Let $E \in \mathbb{B}^\pm_{X,T^{1/2},s}$. Since $\mathbb{B}^\pm_{X,T^{1/2},s}$ is a basis of $K_{\mathcal{T}}(X)$, one can write $E = \sum_{p \in X^H} f_{p}E_{\mathcal{E},T^{1/2},s}(p)$ for some $f_{p} \in K_{\mathcal{T}}(pt)$. By $\beta_{E,\mathcal{T}^{1/2},s}$-invariance, we obtain $f_{p} = f_{p}$. Let $N$ be the maximal degree in $v$ of $f_{p}$, $p \in X^H$, and write $f_{p} = \sum_{i=0}^{N} f_{p,i}v^{i}$ for some $f_{p,i} \in K_{\mathcal{T}}(pt)$. By Corollary 3.25, we obtain

$$(E \subset \mathcal{E}(E)) = \sum_{p \in X^H} f_{p,N}f_{p,N}v^{2N} + \cdots.
$$

By the asymptotic norm one property of $E$, we have $N = 0$ and $\sum_{p \in X^H} f_{p,0}f_{p,0} = 1$. This implies that $f_{p} = \pm[\lambda] \cdot \delta_{p,p'}$ for some $\lambda \in X^*(H)$ and $p' \in X^H$, hence $E \in \mathbb{B}^\pm_{X,T^{1/2},s}$. □

We note that Corollary 3.26 together with Conjecture 3.17 implies that the $K$-theoretic canonical bases $\mathbb{B}_{L,T^{1/2},s}$ and $\mathbb{B}_{X,T^{1/2},s}$ does not depend on the choice of $\mathcal{C}$ up to sign. Recall that under the assumption of Conjecture 3.22, we have $\beta_{E,\mathcal{T}^{1/2},s} = \beta_{E,\mathcal{T}^{1/2},s}$ and hence the signed $K$-theoretic canonical bases are the same for $\mathcal{C}$ and $-\mathcal{C}$. In particular, for each $(\lambda, p) \in \Phi$, there exists $(\lambda^-, p^-) \in \Phi$ such that $C_{\mathcal{E},T^{1/2},s}(\lambda, p) = \pm C_{\mathcal{E},T^{1/2},s}(\lambda^-, p^-)$. This implies that

$$
C_{\mathcal{E},T^{1/2},s}(\lambda, p) \in \pm S_{\mathcal{E},T^{1/2},s}(\lambda^-, p^-) + \sum_{(\lambda', p') < \epsilon_{s}(\lambda^-, p^-)} v^{-1}Z\{v^{-1}\} \cdot S_{\mathcal{E},T^{1/2},s}(\lambda', p').
$$

By the bar invariance, we also obtain

$$
C_{\mathcal{E},T^{1/2},s}(\lambda, p) \in \pm (-v)^{-\dim X}S_{\mathcal{E},T^{1/2},s}(\lambda^-, p^-) + \sum_{(\lambda', p') < \epsilon_{s}(\lambda^-, p^-)} v^{-\dim X+1}Z\{v\} \cdot S_{\mathcal{E},T^{1/2},s}(\lambda', p').
$$

If we further assume Conjecture 3.23 then the sign above must be positive. This proves the following statement.
Proposition 3.27. Assume Conjecture 3.22 For each \((\lambda, p) \in F\), there exists \((\lambda^-, p^-) \in F\) such that 
\[
\mathcal{C}_{\mathcal{E},T^{1/2,s}}(\lambda, p) = \sum_{(\lambda, p) \leq \lambda, (\lambda', p') \leq (\lambda^-, p^-)} P^{\lambda', p'}(v) \cdot \mathcal{S}_{\mathcal{E},T^{1/2,s}}(\lambda', p'),
\]
where \(P^{\lambda', p'}(v) = 1\), \(P^{\lambda', p'}(v) = (-v)^{\dim X}\), and
\[
P^{\lambda', p'}(v) \in v^{-1}Z[v^{-1}] \cap v^{-\dim X} Z[v]
\]
for \((\lambda, p) \leq (\lambda', p') \leq (\lambda^-, p^-)\). If we further assume Conjecture 3.23 then we have \(P^{\lambda', p'}(v) = (-v)^{-\dim X}\).

Using this, one can also give a characterization of \(\mathbb{B}_{L,T^{1/2,s}}\) which is similar to Kashiwara’s theory of global crystal bases.

Corollary 3.28. Assume Conjecture 3.22 We have
\[
K_T(L) \cap \left\{ \mathcal{S}_{\mathcal{E},T^{1/2,s}}(\lambda, p) + \sum_{(\lambda', p') \in F} \left(v^{-1}Z[v^{-1}] \cap v^{-\dim X} Z[v]\right) \cdot \mathcal{S}_{\mathcal{E},T^{1/2,s}}(\lambda', p') \right\} = \left\{ \mathcal{C}_{\mathcal{E},T^{1/2,s}}(\lambda, p) \right\}.
\]

Proof. Let \(C\) be an element of the LHS. Since \(\mathbb{B}_{L,T^{1/2,s}}\) is a \(Z[v, v^{-1}]\)-basis of \(K_T(L)\), we can write
\[
C = \sum_{(\lambda', p') \in F} f_{\lambda', p'}(v) \cdot \mathcal{C}_{\mathcal{E},T^{1/2,s}}(\lambda', p')
\]
for some \(f_{\lambda', p'}(v) \in Z[v, v^{-1}]\). By Proposition 3.27 the maximal and minimal degree of \(f_{\lambda', p'}(v)\) must be 0. By comparing the constant term of the coefficient of \(\mathcal{S}_{\mathcal{E},T^{1/2,s}}(\lambda', p')\), we obtain \(f_{\lambda', p'}(v) = \delta_{(\lambda, p), (\lambda', p')}\) and hence \(C = \mathcal{C}_{\mathcal{E},T^{1/2,s}}(p)\).

We next compute the pairing of \(\mathbb{B}_{L,T^{1/2,s}}\) and \(\mathbb{B}_{X,T^{1/2,s}}\). We first list some basic properties of the pairing. Recall that \(\dagger: K_T(pt) \to K_T(pt)\) is the involution induced from the inverse for \(H\) and \((-): K_T(pt) \to K_T(pt)\) is the involution induced from the inverse for \(S\).

Lemma 3.29. For any \(F, G \in K_T(X)_{loc}\), we have \((G||F)(F||G)\). If we denote by \((-||-)_{opp}\) the inner product defined by \(\beta_K^{E,T^{1/2,s}}\), then we have \((F^\dagger||G^\dagger)_{opp} = v^{\dim X} \cdot (F||G)\).

Proof. We only need to check the formulas for \(F = \mathcal{S}_{\mathcal{E},T^{1/2,s}}(p)\) and \(G = \mathcal{S}_{\mathcal{E},T^{1/2,s}}(p')\). By Lemma 3.24 this is obvious for the first one and the second one follows from Lemma 3.15.

Lemma 3.30. Assume that \(\beta_K^{E,T^{1/2,s}}\) is an involution. For any \(F, G \in K_T(X)_{loc}\), we have \((\beta_K(F)||\beta_K(G)) = v^{\dim X} \cdot (F||G)\).

Proof. We only need to check the formula for \(F = \mathcal{S}_{\mathcal{E},T^{1/2,s}}(p)\) and \(G = \mathcal{S}_{\mathcal{E},T^{1/2,s}}(p')\). By the assumption and Lemma 3.24 applied to the opposite chamber, we obtain \((\mathcal{S}_{\mathcal{E},T^{1/2,s}}(p)||\mathcal{S}_{\mathcal{E},T^{1/2,s}}(p')) = \delta_{p, p'}\). Then the statement follows from the definition of \(\beta_K\).

Proposition 3.31. Assume Conjecture 3.22 We have \((\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p)||\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p')) = \delta_{p, p'}\) for any \(p, p' \in X^H\).

Proof. By Conjecture 3.22, \((\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p)||\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p')) \in K_T(pt)\) since the support of \(\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p)\) is proper. By Proposition 3.21 and Lemma 3.24 we obtain \((\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p)||\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p')) = \delta_{p, p'} + v^{-1}K_H(pt)\). By Lemma 3.30 we also obtain \((\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p)||\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p')) = (\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p)||\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p'))\), hence we need to have \((\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p)||\mathcal{C}_{\mathcal{E},T^{1/2,s}}(p')) = \delta_{p, p'}\).

17
If we define the map \( \partial : K_{\tau}(pt) \to \mathbb{Z}[v, v^{-1}] \) by \( \partial(\sum_{\mu \in \mathbb{X}_{T}(H)} f_{\mu}(v) \cdot [\mu]) = f_{0}(v) \), then Proposition 3.31 implies that \( \mathbb{B}_{L,T^{1/2},s} \) and \( \mathbb{B}_{X,T^{1/2},s} \) forms a dual basis with respect to \( \partial(\cdot | \cdot) \). As a corollary, we also obtain the following result on the expansion of \( \mathbb{B}_{L,T^{1/2},s} \) in terms of \( \mathbb{B}_{X,T^{1/2},s} \).

**Corollary 3.32.** Assume Conjecture 3.22. For any \( (\lambda', p') \in \mathbb{F} \), we have

\[
S_{\epsilon,T^{1/2},s}(\lambda', p') = \sum_{(\lambda,p) \leq (\lambda',p')} P_{\lambda,p}^{\lambda',p'}(v) \cdot \mathcal{E}_{\epsilon,T^{1/2},s}(\lambda, p),
\]

where \( P_{\lambda,p}^{\lambda',p'}(v) \) is the same as Proposition 3.27 and \( P_{\lambda,p}^{\lambda',p'}(v) = 0 \) if \( (\lambda', p') \not\in \mathcal{E}_{s}(\lambda^{-}, p^{-}) \).

**Proof.** This follows from Proposition 3.27 and Proposition 3.31.

If we change the polarization or shift the slope, then the \( K \)-theoretic canonical bases change as follows.

**Lemma 3.33.** Assume Conjecture 3.22. We have \( C_{\epsilon,T^{1/2},s}(\lambda, p) = C_{\epsilon,T^{1/2},s+\det G}(\lambda + \text{wt} H \text{det } i^{*}_{p} G, p) \) and \( \mathcal{E}_{\epsilon,T^{1/2},s}(\lambda, p) = \mathcal{E}_{\epsilon,T^{1/2},s+\det G}(\lambda + \text{wt} H \text{det } i^{*}_{p} G, p) \) for any \( G \in K_{\tau}(X) \) and \( (\lambda, p) \in \mathbb{F} \). In particular, we have \( \mathbb{B}_{L,T^{1/2},s} = \mathbb{B}_{L,T^{1/2},s+\det G} \) and \( \mathbb{B}_{X,T^{1/2},s} = \mathbb{B}_{X,T^{1/2},s+\det G} \).

**Proof.** By Lemma 3.13, we obtain \( \beta^{K}_{\epsilon,T^{1/2},s} = \beta^{K}_{\epsilon,T^{1/2},s+\det G} \). Since \( (\lambda', p') >_{\epsilon,s} (\lambda, p) \) is equivalent to \( (\lambda' + \text{wt} H \text{det } i^{*}_{p} G, p') >_{\epsilon,s+\det G} (\lambda + \text{wt} H \text{det } i^{*}_{p} G, p) \) and

\[
C_{\epsilon,T^{1/2},s}(\lambda, p) \in S_{\epsilon,T^{1/2},s}(\lambda, p) + \sum_{(\lambda', p') >_{\epsilon,s}(\lambda, p)} v^{-1} \mathbb{Z}[v^{-1}] \cdot S_{\epsilon,T^{1/2},s}(\lambda', p')
\]

\[
= S_{\epsilon,T^{1/2},s+\det G}(\lambda + \text{wt} H \text{det } i^{*}_{p} G, p) + \sum_{(\lambda', p') >_{\epsilon,s}(\lambda, p)} v^{-1} \mathbb{Z}[v^{-1}] \cdot S_{\epsilon,T^{1/2},s+\det G}(\lambda' + \text{wt} H \text{det } i^{*}_{p} G, p'),
\]

\( C_{\epsilon,T^{1/2},s}(\lambda, p) \) also satisfies the characterizing properties of \( C_{\epsilon,T^{1/2},s+\det G}(\lambda + \text{wt} H \text{det } i^{*}_{p} G, p) \). The proof of \( \mathcal{E}_{\epsilon,T^{1/2},s}(\lambda, p) = \mathcal{E}_{\epsilon,T^{1/2},s+\det G}(\lambda + \text{wt} H \text{det } i^{*}_{p} G, p) \) is similar.

**Lemma 3.34.** Assume Conjecture 3.22. For any \( l \in P \) and \( (\lambda, p) \in \mathbb{F} \), we have \( C_{\epsilon,T^{1/2},s+\ell}(\lambda, p) = \mathcal{L}(l) \otimes C_{\epsilon,T^{1/2},s}(\lambda - \text{wt} H i^{*}_{p} \mathcal{L}(l), p) \) and \( \mathcal{E}_{\epsilon,T^{1/2},s+\ell}(\lambda, p) = \mathcal{L}(l) \otimes \mathcal{E}_{\epsilon,T^{1/2},s}(\lambda - \text{wt} H i^{*}_{p} \mathcal{L}(l), p) \). In particular, we have \( \mathbb{B}_{L,T^{1/2},s+\ell} = \mathcal{L}(l) \otimes \mathbb{B}_{L,T^{1/2},s} \) and \( \mathbb{B}_{X,T^{1/2},s+\ell} = \mathcal{L}(l) \otimes \mathbb{B}_{X,T^{1/2},s} \).

**Proof.** By Lemma 3.14, we obtain \( \beta^{K}_{\epsilon,T^{1/2},s+\ell} = \beta^{K}_{\epsilon,T^{1/2},s} \circ \mathcal{L}(l)^{-1} \), where \( \mathcal{L}(l) \) is identified with the automorphism of \( K_{\tau}(X) \) given by the tensor product of \( \mathcal{L}(l) \). Since \( (\lambda', p') >_{s+\ell}(\lambda, p) \) is equivalent to \( (\lambda' - \text{wt} H i^{*}_{p} \mathcal{L}(l), p') >_{s}(\lambda - \text{wt} H i^{*}_{p} \mathcal{L}(l), p) \) and

\[
C_{\epsilon,T^{1/2},s+\ell}(\lambda, p) \in S_{\epsilon,T^{1/2},s+\ell}(\lambda, p) + \sum_{(\lambda', p') >_{s}(\lambda, p)} v^{-1} \mathbb{Z}[v^{-1}] \cdot S_{\epsilon,T^{1/2},s+\ell}(\lambda', p')
\]

\[
= \mathcal{L}(l) \otimes \left( S_{\epsilon,T^{1/2},s}(\lambda - \text{wt} H i^{*}_{p} \mathcal{L}(l), p) + \sum_{(\lambda', p') >_{s}(\lambda, p)} v^{-1} \mathbb{Z}[v^{-1}] \cdot S_{\epsilon,T^{1/2},s}(\lambda' - \text{wt} H i^{*}_{p} \mathcal{L}(l), p') \right),
\]

\( \mathcal{L}(l)^{-1} \otimes C_{\epsilon,T^{1/2},s+\ell}(\lambda, p) \) also satisfies the characterizing properties of \( C_{\epsilon,T^{1/2},s}(\lambda - \text{wt} H i^{*}_{p} \mathcal{L}(l), p) \). The proof of \( \mathcal{E}_{\epsilon,T^{1/2},s+\ell}(\lambda, p) = \mathcal{L}(l) \otimes \mathcal{E}_{\epsilon,T^{1/2},s}(\lambda - \text{wt} H i^{*}_{p} \mathcal{L}(l), p) \) is similar.

For the behavior of \( K \)-theoretic canonical bases under the duality, we have the following.

**Lemma 3.35.** Assume Conjecture 3.22 and Conjecture 3.23. For any \( (\lambda, p) \in \mathbb{F} \), we have \( v^{-\dim X} C_{\epsilon,T^{1/2},s}(\lambda, p) \) \( = C_{\epsilon,T^{1/2},s}(\lambda^{-}, p^{-}) \) \( \) and \( \mathcal{E}_{\epsilon,T^{1/2},s}(\lambda, p) \) \( = \mathcal{E}_{\epsilon,T^{1/2},s}(\lambda^{-}, p^{-}) \). Here, \( (\lambda^{-}, p^{-}) \) is as in Proposition 3.27.

In particular, we have \( v^{-\dim X} \mathbb{B}_{L,T^{1/2},s} = \mathbb{B}_{L,T^{1/2},opp,s} \) and \( \mathbb{B}_{X,T^{1/2},s} = \mathbb{B}_{X,T^{1/2},opp,s} \).
Proof. By Lemma \ref{lem:dimFormula} and Proposition \ref{prop:WC}, one can check that $v^{-\dim X}C_{\mathcal{E}_{\mathbb{T}^{1/2},s}}(\lambda, p)^{\vee} \in K_{\mathbb{T}}(L)$ is contained in $S_{\mathbb{T},\mathbb{T}^{1/2},s}(-\lambda^{-}, p^{\tau}) + \sum_{(\lambda, p) \in \mathcal{P}} \left( v^{-\dim X} v^{-1}Z[v] \cap v^{-1}Z[v^{-1}] \right).$ By Corollary \ref{cor:WC}, this implies $v^{-\dim X}C_{\mathcal{E}_{\mathbb{T}^{1/2},s}}(\lambda, p)^{\vee} = C_{\mathcal{E}_{\mathbb{T}^{1/2},s}}(-\lambda^{-}, p^{\tau}).$ The second statement follows from the first by Lemma \ref{lem:Ext} and Proposition \ref{prop:WC}. \hfill \qed

Since the fixed point basis often has some representation theoretic and combinatorial meaning, we also give a formula for the transition matrix between the fixed point basis and $\mathbb{B}_{L,T^{1/2},s}$. For each $p \in X^{H}$, we denote by $O_{p} \in K_{\mathbb{T}}(X)$ the $K$-theory class of the skyscraper sheaf at $p$.

**Proposition 3.36.** Assume Conjecture \ref{conj:WC}. For any $p \in X^{H}$, we have

$$O_{p} = v^{\dim X} \sum_{p' \in X^{H}} \left( i_{p}^{*} \mathcal{E}_{\mathbb{T}^{1/2},s}(p') \right)^{\vee} \cdot C_{\mathcal{E}_{\mathbb{T}^{1/2},s}}(p').$$

**Proof.** This follows from Proposition \ref{prop:WC} and

$$(O_{p}||\mathcal{E}_{\mathbb{T}^{1/2},s}(p')) = (O_{p} : \mathbb{D}_{X} \mathcal{E}_{\mathbb{T}^{1/2},s}(p')) = v^{\dim X} \cdot \left( i_{p}^{*} \mathcal{E}_{\mathbb{T}^{1/2},s}(p') \right)^{\vee}.$$

**Remark 3.37.** For example, when $X$ is the Hilbert scheme of points in the affine plane and the slope is sufficiently close to 1, it turns out that $\mathcal{E}_{\mathbb{T}^{1/2},s}(p)$’s are given by the indecomposable summands of the Procesi bundle. If we identify $K_{\mathbb{T}}(L)$ and the space of symmetric functions as in \cite{17}, then $C_{\mathcal{E}_{\mathbb{T}^{1/2},s}}(p)$’s correspond to the Schur functions and $O_{p}$’s corresponds to the modified Macdonald polynomials. Transition matrix of these bases are given by the $q,t$-Kostka polynomials which are given by the characters of fibers of indecomposable summands of the Procesi bundle.

### 3.6 Categorical lifts

In this section, we state several conjectures about the categorical meaning of canonical and standard bases. We assume all the conjectures and assumptions stated in the previous sections without any comment.

We first recall the notion of tilting bundle on $X$. Let $\mathcal{T}$ be a vector bundle on $X$. We say that $\mathcal{T}$ is a tilting bundle on $X$ if it satisfies

- $\mathcal{T}$ is a weak generator for $D(QCoh(X))$, i.e., $RHom(\mathcal{T}, F) = 0$ implies $F \cong 0$ for $F \in D(QCoh(X))$.
- $Ext^{i}(\mathcal{T}, \mathcal{T}) = 0$ for $i \neq 0$.

When $X$ is a Slodowy variety, Bezrukavnikov-Mirković \cite{BZ} proved that there exists a $\mathbb{T}$-equivariant tilting bundle on $X$ such that Lusztig’s $K$-theoretic canonical basis for $X$ consists of indecomposable summands of the tilting bundle up to equivariant shifts. Moreover, the structure sheaf of $X$ is contained in the canonical basis. We also expect in general that $\mathbb{B}_{X,T^{1/2},s}$ is given by the classes of indecomposable summands of some tilting bundle on $X$.

**Conjecture 3.38.** For each $p \in X^{H}$, there exists a $\mathbb{T}$-equivariant vector bundle lifting $\mathcal{E}_{\mathbb{T}^{1/2},s}(p)$ (denoted by the same letter) such that $\mathcal{T}_{\mathcal{E}_{\mathbb{T}^{1/2},s}} := \bigoplus_{p \in X^{H}} \mathcal{E}_{\mathbb{T}^{1/2},s}(p)$ is a tilting bundle on $X$. Moreover, at least one of $\mathcal{E}_{\mathbb{T}^{1/2},s}(p)$ is a line bundle.

For toric hyper-Kähler manifolds, this conjecture is proved in Corollary \ref{cor:toric}. In particular, this conjecture implies that $\mathbb{B}_{X,T^{1/2},s}$ consists of $K$-theory classes of actual vector bundles on $X$ and hence our choice of the sign is geometrically natural.

**Corollary 3.39.** If we assume Conjecture \ref{conj:WC}, then the ring $\mathcal{A}_{\mathbb{T}^{1/2},s} := \text{End}(\mathcal{T}_{\mathbb{T}^{1/2},s})^{\text{opp}}$ is non-negatively graded with respect to the $S$-action and its degree 0 part is semisimple.
Proof. By the definition of tilting bundle, we have
\[
(E_{ε,T^{1/2},s}(p)||E_{ε,T^{1/2},s}(p')) = [\mathcal{R}G(\mathcal{E}_{ε,T^{1/2},s}(p) \otimes \mathcal{D}_X \mathcal{E}_{ε,T^{1/2},s}(p'))]
\]
\[
= [\mathcal{R}G(\mathcal{D}_X \mathcal{R}Hom(\mathcal{E}_{ε,T^{1/2},s}(p), \mathcal{E}_{ε,T^{1/2},s}(p')))]
\]
\[
= [\mathcal{Hom}(E_{ε,T^{1/2},s}(p), E_{ε,T^{1/2},s}(p'))'](v^{-1})
\]
as rational functions in equivariant parameters. Since the last term is also contained in \(K_H(pt)[(v^{-1})]\), we obtain \([\mathcal{Hom}(E_{ε,T^{1/2},s}(p), E_{ε,T^{1/2},s}(p'))] \in \delta_{p,p'} + vK_H(pt)[v]\) by Lemma 3.25. This implies that it is non-negatively graded with respect to \(S\) and the degree zero part is semisimple.

By Kaledin's argument in [6] section 5.5, this also implies that \(A_{ε,T^{1/2},s}\) is Koszul. We denote by \(B_{ε,T^{1/2},s}\) its Koszul dual. We note that \(A_{ε,T^{1/2},s}\) and \(B_{ε,T^{1/2},s}\) has a natural \(H\)-action. By the independence of \(B_{X,T^{1/2},s}\) on \(ε\), \(A_{ε,T^{1/2},s}\) and \(B_{ε,T^{1/2},s}\) does not depend on the choice of \(ε\) if we forget the \(H\)-actions. Since \(X\) is smooth, \(A_{ε,T^{1/2},s}\) has finite global dimension and hence \(B_{ε,T^{1/2},s}\) is finite dimensional.

By the standard properties of tilting bundle, there is a derived equivalence
\[
ψ_{ε,T^{1/2},s} : D^b\text{Coh}^\Gamma(X) \cong D^b(A_{ε,T^{1/2},s}^\text{gmod}H)
\]
given by the functor \(\mathcal{R}Hom(\mathcal{T}_{ε,T^{1/2},s}, -)\). Here, we denote by \(A_{ε,T^{1/2},s}^\text{gmod}H\) the category of finitely generated graded \(H\)-equivariant modules of \(A_{ε,T^{1/2},s}\). We denote the \(t\)-structure on \(D^b\text{Coh}^\Gamma(X)\) corresponding to the standard \(t\)-structure on \(D^b(A_{ε,T^{1/2},s}^\text{gmod}H)\) by \(τ_{T^{1/2},s}\). By construction, \(F \in D^b\text{Coh}^\Gamma(X)\) is contained in the heart if and only if \(R^\mathcal{R}Hom(\mathcal{T}_{ε,T^{1/2},s}, F) = 0\).

For \((λ,p) ∈ F\), we will write \(E_{ε,T^{1/2},s}(λ,p) := [λ] · \mathcal{E}_{ε,T^{1/2},s}(p)\) as in the \(K\)-theoretic one. Each \(ψ_{ε,T^{1/2},s}(E_{ε,T^{1/2},s}(λ,p))\) defines a graded \(H\)-equivariant indecomposable projective module of \(A_{ε,T^{1/2},s}\) denoted by \(P_{ε,T^{1/2},s}(λ,p)\). It has a unique one-dimensional simple quotient denoted by \(L_{ε,T^{1/2},s}(λ,p)\). We have
\[
δ(ψ_{ε,T^{1/2},s}^{-1}(L_{ε,T^{1/2},s}^A(λ,p))||E_{ε,T^{1/2},s}(λ',p')) = \left(\mathcal{R}Hom(E_{ε,T^{1/2},s}(λ',p'), v^{\dim X}ψ_{ε,T^{1/2},s}^{-1}(L_{ε,T^{1/2},s}^A(λ,p))[\dim X])\right)^H
\]
\[
= v^{\dim X}δ(λ,p),(λ',p').
\]
Hence the \(K\)-theory class of \(v^{-\dim X}ψ_{ε,T^{1/2},s}^{-1}(L_{ε,T^{1/2},s}^A(λ,p))\) coincides with \(C_{ε,T^{1/2},s}(λ,p)\) by Proposition 3.31. Using this lift, we sometimes regard \(K\)-theoretical canonical bases of \(K_T(L)\) as objects in \(D^b\text{Coh}^\Gamma(X)\).

By the Koszul duality (cf. [1] [24]), we also obtain the following derived equivalence
\[
\mathcal{X} : D^b(A_{ε,T^{1/2},s}^\text{gmod}H) \cong D^b(B_{ε,T^{1/2},s}^\text{gmod}H).
\]
(19)

We note that since \(B_{ε,T^{1/2},s}\) is finite dimensional, the Koszul duality equivalence preserves the boundedness by [4] Theorem 2.12.6. The standard \(t\)-structure on \(D^b(B_{ε,T^{1/2},s}^\text{gmod}H)\) induces a \(t\)-structure on \(D^b(A_{ε,T^{1/2},s}^\text{gmod}H)\) and its heart consists of linear complex of projective modules, that is, object quasi-isomorphic to a complex of the form
\[
0 → v^N P_N → v^{N-1} P_{N-1} → \cdots → v^M P_M → 0
\]
where each \(P_i\) sits in the \((-i)\)-th term and it is a direct sum of projective modules of the form \(P_{ε,T^{1/2},s}^A(λ,p)\). By the construction of \(\mathcal{X}\), we have \(\mathcal{X} \circ v[1] = v^{-1} \circ \mathcal{X}\) and \(L_{ε,T^{1/2},s}^B(λ,p) := \mathcal{X}(P_{ε,T^{1/2},s}^A(λ,p))\) is a graded \(H\)-equivariant one-dimensional simple module of \(B_{ε,T^{1/2},s}\). We note that any simple object in \(B_{ε,T^{1/2},s}^\text{gmod}H\) is of the form \(v^m L_{E,T^{1/2},s}^B(λ,p)\) for some \(m ∈ Z\) and \((λ,p) ∈ F\). Since \(A_{ε,T^{1/2},s}\) is Koszul, \(L_{ε,T^{1/2},s}^A(λ,p)\) is quasi-isomorphic to a linear complex of projective modules and hence we have \(L_{ε,T^{1/2},s}^B(λ,p) := \mathcal{X}(L_{ε,T^{1/2},s}^A(λ,p)) \in B_{ε,T^{1/2},s}^\text{gmod}H\). This is the injective hull of \(L_{ε,T^{1/2},s}^B(λ,p)\).

For a categorical lift of standard basis, we conjecture the following.
Conjecture 3.40. For each \((\lambda, p) \in X^H\), there exists an object \(\nabla_{\mathcal{E},T^1/2,s}(\lambda, p) \in \mathcal{B}_{\mathcal{E},T^1/2,s} \)-gmod\(^H\) such that \(\Delta_{\mathcal{E},T^1/2,s}^A(\lambda, p) \coloneqq \mathcal{K}^{-1}(\nabla_{\mathcal{E},T^1/2,s}(\lambda, p))\) is contained in the standard heart of \(D^b(\mathcal{A}_{\mathcal{E},T^1/2,s} \)-gmod\(^H\)) and the K-theory class of \(\nabla_{\mathcal{E},T^1/2,s}(\lambda, p) \coloneqq \psi_{\mathcal{E},T^1/2,s}(\Delta_{\mathcal{E},T^1/2,s}^A(\lambda, p)) \in D^b\text{Coh}\(\mathcal{X}\) coincides with \(\beta_{\mathcal{E},T^1/2,s}(\mathcal{S}_{\mathcal{E},T^1/2,s}(\lambda, p))\). If we set \(\Delta_{\mathcal{E},T^1/2,s}(\lambda, p) \coloneqq v^{\frac{\dim X}{2}}\nabla_{\mathcal{E},T^1/2,s}(\lambda, p)[\frac{\dim X}{2}] \in D^b\text{Coh}\(\mathcal{X}\), then we have

\[
\text{Hom}_{\mathcal{D}^b\text{Coh}\(\mathcal{X}\)}\left(v^i\Delta_{\mathcal{E},T^1/2,s}(\lambda, p), \nabla_{\mathcal{E},T^1/2,s}(\lambda', p')\right) = \left\{
\begin{array}{ll}
\mathbb{C} & \text{if } i = j = 0 \text{ and } (\lambda, p) = (\lambda', p'), \\
0 & \text{otherwise},
\end{array}
\right.
\] (20)

For toric hyper-Kähler manifolds, this conjecture is proved in Theorem 5.45. We note that the K-theory class of \(\Delta_{\mathcal{E},T^1/2,s}(\lambda, p)\) coincides with \(\mathcal{S}_{\mathcal{E},T^1/2,s}(\lambda, p)\) and the equation (20) lifts the orthonormality of K-theoretic standard bases in Lemma 3.24. We expect that \(\nabla_{\mathcal{E},T^1/2,s}(\lambda, p)\) and \(\Delta_{\mathcal{E},T^1/2,s}(\lambda, p)\) are essentially given by the theory of categorical stable envelope (c.f. [33]). Since \(\Delta_{\mathcal{E},T^1/2,s}(\lambda, p)\) is a Koszul module of \(\mathcal{A}_{\mathcal{E},T^1/2,s}\), we have a resolution of the form

\[
0 \to v^{\frac{\dim X}{2}}P_\text{deg X} \to \cdots \to vP_1 \to P_A^A(\lambda, p) \to \Delta_{\mathcal{E},T^1/2,s}(\lambda, p) \to 0
\] (21)
in \(\mathcal{A}_{\mathcal{E},T^1/2,s} \)-gmod\(^H\) by Corollary 3.32, where each \(P_i (i = 1, \ldots, \frac{\dim X}{2})\) is a direct sum of projective modules of the form \(P_{\mathcal{A},T^1/2,s}(\lambda', p)\) satisfying \((\lambda', p') <_{\mathcal{E},s} (\lambda, p)\). By the construction of \(\mathcal{K}\), it follows that \(\nabla_{\mathcal{E},T^1/2,s}(\lambda, p)\) is non-positively graded and we have an inclusion \(L_{\mathcal{E},T^1/2,s}(\lambda, p) \hookrightarrow \nabla_{\mathcal{E},T^1/2,s}(\lambda, p)\) such that any composition factor \(v^jL_{\mathcal{E},T^1/2,s}(\lambda', p')\) of the quotient \(\nabla_{\mathcal{E},T^1/2,s}(\lambda, p)/L_{\mathcal{E},T^1/2,s}(\lambda, p)\) satisfies \(j < 0\) and \((\lambda', p') <_{\mathcal{E},s} (\lambda, p)\).

Since the shift \(v[1]\) preserves the linear complex of projective modules, \(\psi_{\mathcal{E},T^1/2,s}(\Delta_{\mathcal{E},T^1/2,s}(\lambda, p))\) is quasi-isomorphic to a complex of the form

\[
0 \to v^{-\frac{\dim X}{2}}P_\text{deg X} \to \cdots \to vP_1 \to P_A^A(\lambda, p) \to \nabla_{\mathcal{E},T^1/2,s}(\lambda, p) \to 0
\]

where \(P_A^A(\lambda, p)\) sits in degree 0 and each \(P_i\) is the same as (21). If we write \(\Delta_{\mathcal{E},T^1/2,s}^B(\lambda, p) \coloneqq \mathcal{K}(\psi_{\mathcal{E},T^1/2,s}(\Delta_{\mathcal{E},T^1/2,s}(\lambda, p))) \in \mathcal{B}_{\mathcal{E},T^1/2,s} \)-gmod\(^H\), then this is non-negatively graded and we have a projection \(\nabla_{\mathcal{E},T^1/2,s}(\lambda, p) \to L_{\mathcal{E},T^1/2,s}(\lambda, p)\) such that any composition factor \(v^jL_{\mathcal{E},T^1/2,s}(\lambda', p')\) of the kernel satisfies \(j > 0\) and \((\lambda', p') <_{\mathcal{E},s} (\lambda, p)\). By (20), we obtain

\[
\text{Ext}_H^1(\mathcal{E},T^1/2,s)-\text{gmod}^H\left(v^j\Delta_{\mathcal{E},T^1/2,s}^B(\lambda, p), \nabla_{\mathcal{E},T^1/2,s}(\lambda', p')\right) = \left\{
\begin{array}{ll}
\mathbb{C} & \text{if } i = j = 0 \text{ and } (\lambda, p) = (\lambda', p'), \\
0 & \text{otherwise},
\end{array}
\right.
\] (22)

Next we show that \(\mathcal{B}_{\mathcal{E},T^1/2,s} \)-gmod\(^H\) has a structure of graded highest weight category assuming Conjecture 3.40. We first recall the definition of graded highest weight category following [10, 11]. Let \(\mathcal{C}\) be a \(\mathbb{C}\)-linear abelian category with a free \(\mathbb{Z}\)-action which is locally Artinian, contains enough injectives, and satisfies Grothendieck’s condition AB5. The action of \(j \in \mathbb{Z}\) on an object \(M \in \mathcal{C}\) is denoted by \(M \to M(j)\). For each \(M, N \in \mathcal{C}\), we set \(\text{Hom}_\mathcal{C}(M, N) \coloneqq \oplus_{j \in \mathbb{Z}} \text{Hom}_\mathcal{C}(M(j), N)\) and \(\text{Ext}_\mathcal{C}(M, N) \coloneqq \oplus_{j \in \mathbb{Z}} \text{Ext}_\mathcal{C}(M(j), N)\). For a simple object \(L\), we denote by \([M : L]\) the multiplicity of \(L\) in the composition series of \(M\).

Definition 3.41 ([10, 11]). A category \(\mathcal{C}\) as above is called graded highest weight category if there exists an interval finite poset \(\Lambda\) satisfying the following conditions:

- For each \(\lambda \in \Lambda\), we have a simple object \(L(\lambda)\) such that \(\{L(\lambda)(j)\}_{j \in \mathbb{Z}, \lambda \in \Lambda}\) is a complete set of non-isomorphic simple objects of \(\mathcal{C}\).

- For each \(\lambda \in \Lambda\), there is an object \(\nabla(\lambda)\) (called costandard object) with an inclusion \(\nabla(\lambda) \hookrightarrow \nabla(\lambda)\) such that any composition factor \(L(\mu)(j)\) of the quotient \(\nabla(\lambda)/L(\lambda)\) satisfies \(j < 0\) and \(\mu < \lambda\). Moreover, for each \(\lambda, \mu \in \Lambda\), \(\dim \text{Hom}_\mathcal{C}(\nabla(\lambda), \nabla(\mu)) + \sum_{j \in \mathbb{Z}} [\nabla(\lambda) : L(\mu)(j)]\) are finite.
• For each \( \lambda \in \Lambda \), injective hull \( I(\lambda) \) of \( L(\lambda) \) has an increasing filtration \( 0 = F_0(\lambda) \subset F_1(\lambda) \subset \cdots \) with \( \cup_i F_i(\lambda) = I(\lambda) \) such that \( F_i(\lambda) \cong \nabla(\lambda) \) and \( F_i(\lambda)/F_{i-1}(\lambda) \cong \nabla(\mu_i) \) for some \( j < 0 \) and \( \mu_i > \lambda \) if \( i > 1 \). Moreover, the set \( \{ i \mid \mu_i = \mu \} \) is finite for any \( \mu \in \Lambda \).

**Proposition 3.42.** Assume Conjecture 3.40 Then the category \( \mathcal{B}_\xi T^{1/2}, s \text{-gmod}^H \) has a structure of graded highest weight category with poset \((\mathcal{F}, \leq, s)\) and the costandard object parametrized by \((\lambda, p) \in \mathbb{F}\) is given by \( \nabla^\xi, T^{1/2}, s(\lambda, p) \).

**Proof.** Since \( \mathcal{B}_\xi T^{1/2}, s \) is finite dimensional over \( \mathbb{C} \), the category \( \mathcal{B}_\xi T^{1/2}, s \text{-gmod}^H \) is Artinian. The free \( \mathbb{Z} \)-action on \( \mathcal{B}_\xi T^{1/2}, s \text{-gmod}^H \) is given by the grading shifts \( M(j) := v^j M. \) The poset \( \mathcal{F} \) is interval finite by Lemma 3.19. For each \((\lambda, p) \in \mathcal{F}\), we associate the simple module \( L^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \). Then the first two condition in Definition 3.41 has been already checked above.

On the level of \( K \text{-theory} \), we have

\[
[I^\mathcal{B}_\xi T^{1/2}, s(\lambda, p)] = \sum_{(\lambda', p') \leq (\lambda, p)} P_{\lambda, p}^{\lambda', p'} (-v) \cdot [\nabla^\mathcal{B}_\xi T^{1/2}, s(\lambda', p')] \]

by Proposition 3.27. Since \( P_{\lambda, p}^{\lambda', p'}(-v) = 1 \) and \( P_{\lambda, p}^{\lambda', p'}(-v) \in v^{-1}\mathbb{Z}[v^{-1}] \) if \((\lambda', p') \neq (\lambda, p)\), it is enough to prove that \( I^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \) has a costandard filtration with \( F_1(\lambda, p) = \nabla^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \). This follows from the following lemma and its proof.

**Lemma 3.43.** An object \( M \in \mathcal{B}_\xi T^{1/2}, s \text{-gmod}^H \) has a costandard filtration if and only if

\[
\text{ext}^1_{\mathcal{B}_\xi T^{1/2}, s \text{-gmod}^H} \left( \Delta^\mathcal{B}_\xi T^{1/2}, s(\lambda, p), M \right) = 0 \tag{23}
\]

for any \((\lambda, p) \in \mathcal{F}\).

**Proof.** The only if part follows from (22). Let \( M \) be an object satisfying (23). Since \( M \) has finite length, we may prove the statement by induction on the length of \( M \). If \( M \neq 0 \), then one can take a minimal \((\lambda, p) \in \mathcal{F}\) such that \( \text{hom} \left( L^\mathcal{B}_\xi T^{1/2}, s(\lambda, p), M \right) \neq 0 \). For any \((\lambda', p') < s(\lambda, p)\), let \( K \) be the kernel of \( \Delta^\mathcal{B}_\xi T^{1/2}, s(\lambda', p') \rightarrow L^\mathcal{B}_\xi T^{1/2}, s(\lambda', p') \). By the discussion above, any composition factor \( v^1 L^\mathcal{B}_\xi T^{1/2}, s(\lambda'', p'') \) of \( K \) satisfies \((\lambda'', p'') < s(\lambda', p') < s(\lambda, p)\) and hence we have \( \text{hom}(K, M) = 0 \). By the exact sequence

\[
\text{hom}(K, M) \rightarrow \text{ext}^1 \left( L^\mathcal{B}_\xi T^{1/2}, s(\lambda', p'), M \right) \rightarrow \text{ext}^1 \left( \Delta^\mathcal{B}_\xi T^{1/2}, s(\lambda', p'), M \right)
\]

we obtain \( \text{ext}^1 \left( L^\mathcal{B}_\xi T^{1/2}, s(\lambda', p'), M \right) = 0 \) for any \((\lambda', p') < s(\lambda, p)\). Since the composition factors of \( \nabla^\mathcal{B}_\xi T^{1/2}, s(\lambda, p)/L^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \) are of the form \( v^1 L^\mathcal{B}_\xi T^{1/2}, s(\lambda', p') \) for \((\lambda', p') < s(\lambda, p)\), we obtain

\[
\text{ext}^1 \left( \nabla^\mathcal{B}_\xi T^{1/2}, s(\lambda, p)/L^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \right) = 0 \text{ and } \text{hom} \left( \nabla^\mathcal{B}_\xi T^{1/2}, s(\lambda, p)/L^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \right) = 0. \]

This implies that \( \text{hom} \left( \nabla^\mathcal{B}_\xi T^{1/2}, s(\lambda, p)/L^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \right) \cong \text{hom} \left( L^\mathcal{B}_\xi T^{1/2}, s(\lambda, p)/L^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \right) \), hence we can lift the inclusion \( L^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \hookrightarrow M \) to a homomorphism \( f : \nabla^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \rightarrow M \).

We claim that \( f \) is injective. Otherwise, there is a submodule \( v^j L^\mathcal{B}_\xi T^{1/2}, s(\lambda', p') \) in \( \nabla^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \). Then there is a nontrivial homomorphism between \( v^j \Delta^\mathcal{B}_\xi T^{1/2}, s(\lambda', p') \) and \( \nabla^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \), hence we must have \( j = 0 \) and \((\lambda', p') = (\lambda, p)\) by (22). This implies \( f(L^\mathcal{B}_\xi T^{1/2}, s(\lambda, p)) = 0 \) which contradicts the choice of \( f \).

Therefore, we obtain an inclusion \( \nabla^\mathcal{B}_\xi T^{1/2}, s(\lambda, p) \hookrightarrow M \). Let \( N \) be the cokernel of this inclusion. Then \( N \) also satisfies the condition (23) by (22) and the length of \( N \) is smaller than \( M \). By induction hypothesis, \( N \) has a costandard filtration and hence \( M \) does.

Finally, we also conjecture the following statement which lifts the \( K \)-theoretical bar involutions to the derived category. Let \( \mathcal{B}_\xi T^{1/2} = \oplus_{\lambda \in \chi^s(H)} \mathcal{B}_\xi T^{1/2}, s \) be the \( H \)-weight space composition.
Conjecture 3.44. There exists an anti-involution $\iota$ on $B_{\xi, T^{1/2}, s}$ which is identity on degree 0 part, compatible with the grading, and satisfies $\iota(B_{\xi, T^{1/2}, s}) = B_{\xi, T^{1/2}, s}^{-1}$ for any $\lambda \in \mathcal{X}(H)$.

For toric hyper-Kähler manifolds, this conjecture is proved in Corollary 5.40. Let $M = \oplus_{i \in \mathbb{Z}, \lambda \in \mathcal{X}(H)} M_i^\lambda$ be a graded $H$-equivariant module of $B_{\xi, T^{1/2}, s}$, where $M_i^\lambda$ has degree $i$ and $H$-weight $\lambda$. We define $DM = \oplus_{i \in \mathbb{Z}, \lambda \in \mathcal{X}(H)} (DM)^i_\lambda \subset B_{\xi, T^{1/2}, s}$-module structure through $\iota$. The properties of $\iota$ in Conjecture 3.44 implies that $DL_{\xi, T^{1/2}, s}^B(\lambda, p) \cong L_{\xi, T^{1/2}, s}^B(\lambda, p)$. In particular, $D$ induces an involution on $D^b\text{Coh}^T(X)$ which lifts the $K$-theoretic bar involution.

### 3.7 Wall-crossings

By varying the slope parameter, we obtain a family of $t$-structures $\tau_{T^{1/2}, s}$ on $\mathcal{D} := D^b\text{Coh}_L(X)$, the derived category of coherent sheaves on $X$ set-theoretically supported in $L$. It may be natural to expect that this is part of a data defining real variations of stability condition in the sense of Anno-Bezrukavnikov-Mirković [2]. We first recall the definition of real variations of stability conditions in our situation.

Let $\text{Alc}_K$ be the set of connected components of $P_K \setminus \cup_{\beta \in \mathcal{W}} \{ s \in P_K \mid \langle s, \beta \rangle \in \mathbb{Z} \}$. We call an element of $\text{Alc}_K$ Kähler alcove. For two Kähler alcoves $\mathcal{A}_-, \mathcal{A}_+ \in \text{Alc}_K$ sharing a codimension one face contained in $w_{\beta, w} := \{ s \in P_K \mid \langle s, \beta \rangle = m \}$ for some $m \in \mathbb{Z}$ and $\beta \in \mathcal{W}$ which is positive with respect to $\mathfrak{g}$, we say that $\mathcal{A}_+$ is above $\mathcal{A}_-$ if $\mathcal{A}_- \subset \{ s \in P_K \mid \langle s, \beta \rangle < m \}$ and $\mathcal{A}_+ \subset \{ s \in P_K \mid \langle s, \beta \rangle > m \}$. Let $\mathcal{Z} : P_K \rightarrow \text{Hom}_C(K(\mathcal{D}), \mathbb{R})$ be a polynomial map and $\tau$ be a map from $\text{Alc}$ to the set of bounded $t$-structures on $\mathcal{D}$. For $\mathcal{A} \in \text{Alc}_K$, let $\mathcal{A}_\mathcal{D}$ be the heart of the $t$-structure $\tau(\mathcal{A})$ on $\mathcal{D}$. For a hyperplane $w \subset P_K$ and $n \in \mathbb{Z}_{\geq 0}$, let $\mathcal{C}_{\mathcal{A}, w}^n \subset \mathcal{C}_{\mathcal{A}}$ be the full subcategory consisting of objects $M \in \mathcal{C}_{\mathcal{A}}$ such that the polynomial function on $P_K$ defined by $s \mapsto \langle \mathcal{Z}(s), [M] \rangle$ has zero of order at least $n$ on $w$. This is a Serre subcategory of $\mathcal{C}_{\mathcal{A}}$ and let $\mathcal{D}_{\mathcal{A}, w}^n := \{ F \in \mathcal{D} \mid \forall j, H^j(\mathcal{A})(F) \in \mathcal{C}_{\mathcal{A}, w}^n \}$ be a thick subcategory of $\mathcal{D}$. Here, $H^j(\mathcal{A})$ is the $j$-th cohomology functor with respect to the $t$-structure $\tau(\mathcal{A})$. We set $\text{gr}_{\mathcal{A}, w}^n(\mathcal{D}) := \mathcal{D}_{\mathcal{A}, w}^n/\mathcal{D}_{\mathcal{A}, w}^{n+1}$ and $\text{gr}_{\mathcal{A}}^n(\mathcal{C}) := \mathcal{C}_{\mathcal{A}}/\mathcal{C}_{\mathcal{A}}^{n+1}$.

Definition 3.45 (2). A data $(\mathcal{D}, \tau)$ as above is called real variation of stability conditions on $\mathcal{D}$ if it satisfies the following conditions:

- For any $\mathcal{A} \in \text{Alc}_K$ and nonzero $M \in \mathcal{C}_{\mathcal{A}}$, we have $\langle \mathcal{Z}(s), [M] \rangle > 0$ for any $s \in \mathcal{A}$.
- For any $\mathcal{A}_- \neq \mathcal{A}_+ \in \text{Alc}_K$ sharing a codimension one face contained in a hyperplane $w$ with $\mathcal{A}_+$ being above $\mathcal{A}_-$, we have $\mathcal{D}_{\mathcal{A}_-, w}^n = \mathcal{D}_{\mathcal{A}_+, w}^n$, and $\text{gr}_{\mathcal{A}_-}^n(\mathcal{C}_{\mathcal{A}_-}) = \text{gr}_{\mathcal{A}_+}^n(\mathcal{C}_{\mathcal{A}_+})$ if $\mathcal{D}_{\mathcal{A}_-, w}(\mathcal{D}) = \mathcal{D}_{\mathcal{A}_+, w}(\mathcal{D})$ for any $n \in \mathbb{Z}_{\geq 0}$.

The polynomial function $\mathcal{Z}$ is called central charge for the real variation of stability conditions.

Remark 3.46. A part of the conjecture of Bezrukavnikov-Okounkov stated in [2] claims that there exists a real variation of stability conditions on $\mathcal{D}$ as above (we do not need to assume that the fixed point set $X^H$ is finite). Moreover, the $t$-structures $\tau$ are given by quantization of $X$ in positive characteristic.

Let $\ell$ be a prime number and consider our conical symplectic resolutions over an algebraically closed field of characteristic $\ell$ temporarily. For $\lambda \in P$, one can consider Frobenius constant quantization of $O_X$ with quantization parameter $\lambda$ which gives a sheaf of Azumaya algebras $\mathcal{A}_\lambda$ on the Frobenius twist $X^{(1)}$ of $X$. Then the conjecture says that when $\ell$ is sufficiently large and $-\frac{\ell}{2} \notin \mathcal{A}$, the Azumaya algebra $\mathcal{A}_\lambda$ splits on the formal neighborhood of $L^{(1)}$ and if the splitting bundles are chosen in a compatible way, their $S$-equivariant lifts to $X \cong X^{(1)}$ gives a tilting bundle and its dual gives a $t$-structure compatible with $\tau(A)$ under base change to positive characteristic. See also [33] for another approach to this tilting bundle.

3 We change the sign here from [2] since we mainly use the slope parameters, which are essentially opposite to the quantization parameters.
As an analogue of Lusztig’s conjecture on modular representation theory, we expect that the set of vector bundles \( \{E_{\ell,T;1/2,σ}(p)\}_{p \in X^H} \) (considered in positive characteristic) give the set of indecomposable summands of the dual of a splitting bundle of \( \mathcal{A}_\lambda \) up to equivariant parameter twists when \( s = -\frac{1}{2} \) and \( ℓ \) is sufficiently large. We note that if we change the polarization \( T^{1/2} \) by \( T_0^{1/2} \) for some \( \mathcal{G} \in K_τ(X) \), then the \( K \)-theoretic canonical bases will change by tensor product of some line bundle by Lemma 3.33 and Lemma 3.34. Since a line bundle twist of a splitting bundle is also a splitting bundle, this change can be absorbed into the choice of splitting bundles.

Let \( \mathcal{P} \) be a vector bundle on \( X \) and \( \mathcal{L} : P_\mathcal{R} \to \text{Hom}_\mathbb{Z}(K(\mathcal{D}), \mathbb{R}) \) be a polynomial function satisfying

\[
(\mathcal{L}(p), F) = \frac{1}{\text{rk} \mathcal{P}} \chi(X, F) \otimes \mathcal{P}(p)
\]

for any line bundle \( \mathcal{L} \in \mathcal{P} \) and \( F \in \mathcal{D} \). We note that \( \mathcal{L}(p) = \mathcal{L} \) for any \( p \in \mathbb{Z} \). In this paper, we only conjecture the following weaker statement. We will check this for toric hyper-Kähler manifolds in Corollary 5.49.

**Conjecture 3.47.** There exists a vector bundle \( \mathcal{P} \) on \( X \) such that \( (\mathcal{L}, \tau) \) gives a real variation of stability conditions on \( \mathcal{D} \), where \( \tau \) is given by \( \tau(A) = τ_{T;1/2,σ} \) for \( s \in \mathcal{A} \).

**Remark 3.48.** The vector bundle \( \mathcal{P} \) should be a line bundle when \( X \) is a Slodowy variety by the result of [2], but it should be a vector bundle of higher rank in general. This vector bundle is expected to be given by looking at the asymptotic behavior under \( ℓ \to \infty \) of the multiplicity of indecomposable summands of the splitting bundle above for a fixed \( \lambda \). More precisely, fix a generic \( \lambda \) and let \( m_\ell(p) \) be the multiplicity of \( E_{\ell,T;1/2,σ}(p) \) in the splitting bundle of \( \mathcal{A}_\lambda \) (as non-equivariant vector bundles).

We set \( m_\ell := \lim_{\ell \to \infty} \frac{1}{\text{dim} X} \mathcal{L}(p) \) and take \( m = \mathbb{Z} \) such that \( m \cdot m_\ell \) is an integer for any \( p \in \mathcal{D} \). Let \( \mathcal{A} \in \text{Alc}_K \) be the alcove containing \( -\frac{1}{2} \) for any sufficiently large \( ℓ \) and take \( s \in \mathcal{A} \). Then we expect that one can take

\[
\mathcal{P} = \sum_{p \in \mathcal{D}^H} m \cdot m_\ell \cdot E_{\ell,T;1/2,σ}(p). 
\]

We note that the central charge \( \mathcal{L} \) does not depend on the choice of \( m \). We also expect that this does not depend on the choice of \( \lambda \) if we forget the equivariant structures.

We now describe the behavior of \( K \)-theoretic canonical bases under the wall-crossing of the slope parameters. As in the previous section, we expect that the information on the equivariant parameter \( v \) has some information on the cohomological shifts appearing in Definition 3.45. The following conjecture comes from numerical experiments.

**Conjecture 3.49.** Let \( \mathcal{A}_- \neq \mathcal{A}_+ \in \text{Alc}_K \) be two Kähler alcovs sharing a codimension one face contained in a hyperplane \( w \) with \( \mathcal{A}_- \) being above \( \mathcal{A}_+ \). For \( s_- \in \mathcal{A}_- \) and \( s_+ \in \mathcal{A}_+ \), there exists a sequence of integers \( 0 \leq n_0 < n_1 < \cdots < n_l \) and decompositions \( \mathbb{B}_{X,T;1/2,s_-} = \bigsqcup_{i=0}^l \mathbb{B}^{i}_{s_- w} \) and \( \mathbb{B}_{X,T;1/2,s_+} = \bigsqcup_{i=0}^l \mathbb{B}^{i}_{s_+ w} \) stable under equivariant parameter shifts for \( H \) such that for any \( \mathcal{E}' \in \mathbb{B}^{i}_{s_+ w} \), there exists \( \mathcal{E} \in \mathbb{B}^{i}_{s_- w} \) satisfying

\[
\mathcal{E}' = (-1)^{n_l-n_0-i}v^{n_l-1} \mathcal{E} + \sum_{j<i} f_{\mathcal{E},\mathcal{F}} \cdot \mathcal{F},
\]

where \( f_{\mathcal{E},\mathcal{F}} \) is a polynomial function in \( v^{n_i+1}Z[v] \) for any \( \mathcal{F} \in \mathbb{B}^{i}_{s_- H} \). Moreover, if we assume Conjecture 3.47 and let \( \mathcal{C} \in \mathbb{B}_{X,T;1/2,s_-} \) be the element dual to \( \mathcal{E} \), then \( n_i - n_0 - i \) is the order of vanishing at \( w \) of the polynomial function \( s \mapsto (\mathcal{L}(p), \mathcal{C}) \).
For toric hyper-Kähler manifolds, this conjecture follows from Proposition 3.20, Lemma 5.21, and Corollary 3.48. This conjecture implies that if one can calculate the behavior of $K$-theoretic canonical bases under wall-crossing, one can find the order of zero at various hyperplanes for the central charge of every elements in $\mathbb{B}_{L,T^1/2,s}$. This information is usually enough to determine $\mathcal{P}$ in practice.

One can also use this conjecture to compute $\mathbb{B}_{X,T^1/2,s_+}$ from the knowledge of $\mathbb{B}_{X,T^1/2,s_-}$ since this implies that $\beta_{X,T^1/2,s_+}^K$ is triangular with respect to $\mathbb{B}_{X,T^1/2,s_-}$ and the degree condition on $v$ enables us to calculate each $f_{\mathcal{E},\mathcal{F}}(v)$ by Kazhdan-Lusztig type algorithm as in the proof of Proposition 3.21. For example, when $X$ is the Hilbert scheme of $n$-points in the affine plane, then $\mathbb{B}_{X,T^1/2,s}$ is given by the indecomposable summands of the Procesi bundle up to equivariant parameter shifts when $s$ is sufficiently close to 1. Using this algorithm, one can calculate $\mathbb{B}_{X,T^1/2,s}$ for any $s$ in principle. We have checked the first part of Conjecture 3.49 (together with Conjecture 3.22) in this case when $n \leq 8$ by using computer.

4 Elliptic bar involutions

In this section, we discuss an elliptic analogue of the $K$-theoretic bar involutions. Since our definition of the $K$-theoretic bar involution only involves $K$-theoretic stable bases, it seems natural to define elliptic bar involution using elliptic stable bases defined by Aganagic-Okounkov [1]. We conjecture that under certain natural normalization (contrary to the seemingly ad hoc normalization in the $K$-theory case), this will give us an involution on a certain analytic completion of the localized (extended) equivariant $K$-theory.

In order to define an elliptic analogue of the $K$-theoretic canonical bases, we need to find an elliptic version of some conditions characterizing canonical bases other than bar invariance. In this paper, we do not investigate this direction further. Instead, we will construct a family of elliptic bar invariant elements which are “as simple as possible” and have some nice properties to be called elliptic canonical bases in the case of toric hyper-Kähler manifolds in section 6. We leave the problem of finding a characterization of these elements which makes sense in general for a future direction of research.

4.1 Elliptic standard bases

We first briefly recall the definition of elliptic stable basis defined by Aganagic-Okounkov. For more details, we refer to the original paper [1]. We follow the notations and assumptions of section 3.

We fix an elliptic curve $E := \mathbb{C}^\times/q^2$ over $\mathbb{C}$, where $q$ is a complex number satisfying $0 < |q| < 1$. Let $Ell_T(X)$ be the $\mathbb{T}$-equivariant elliptic cohomology of $X$ associated with $E$. Since odd cohomology of $X$ vanishes, this is a scheme which is affine over the abelian variety $E_T := \mathbb{X}_+(\mathbb{T}) \otimes_{\mathbb{Z}} E = Ell_T(pt)$. For a construction of equivariant elliptic cohomology, see for example [15]. We also set $E_P := P \otimes_{\mathbb{Z}} E$ and $E^*_P := \text{Pic}^2(X) \otimes_{\mathbb{Z}} E$. We note that there is an exact sequence

$$0 \to E^*_T \to E^*_P \to E_P \to 0,$$

where $E^*_T := X^*_{\mathbb{T}} \otimes_{\mathbb{Z}} E$. We define the extended equivariant elliptic cohomology for $X$ by $E(X) := Ell_T(X) \times E_P$, which is affine over $B_X := E_T \times E_P$. We note that since we have $\mathbb{X}_+(H) \cong P^1$ and $P \cong \mathbb{X}_+(H^t)$, we can identify $B_X \cong B_X^1$ by simply exchanging the equivariant and Kähler parameters. We also set $\check{E}(X) := Ell_T(X) \times E^*_P$, $\check{B}_X := E_T \times E^*_P$. The structure morphism $\check{E}(X) \to \check{B}_X$ will be denoted by $\check{\pi}$.

Let

$$\vartheta(x) := (x^{1/2} - x^{-1/2}) \prod_{m=1}^{\infty} (1 - q^m x)(1 - q^m x^{-1})$$

be a theta function. This is a multivalued holomorphic function on $\mathbb{C}^\times$ which satisfies $\vartheta(e^{2\pi i x}) = -\vartheta(x)$ and $\vartheta(qx) = -x^{-1}q^{-1/2}\vartheta(x)$ and can be seen as a section of degree 1 line bundle on $E$. Using this line
bundle, we identify $E$ and its dual abelian variety $E^\vee$. Then the Poincaré line bundle on $E \times E^\vee \cong E \times E$ can be defined by the quasi-periods of the function

$$(x, y) \mapsto \psi(x, y) := \frac{\vartheta(xy)}{\vartheta(x)d(y)},$$

i.e., it is single valued on $\mathbb{C} \times \mathbb{C}$ and satisfies $\psi(qx, y) = y^{-1}\psi(x, y)$ and $\psi(x, qy) = x^{-1}\psi(x, y)$. For $r \in \mathbb{Z}_{>0}$, we define a line bundle $\mathcal{O}(D)$ on $S' E$, the $r$-th symmetric product of $E$, by the factors of automorphism of the symmetric function $(x_1, \ldots, x_r) \mapsto \prod_{i=1}^r \vartheta(x_i)$.

Let $V$ be a $\mathbb{T}$-equivariant vector bundle on $X$. Its characteristic classes give us a morphism $c_V : \text{Ell}_\mathbb{T}(X) \to S' E$ and $\Theta(V) := c_V^* \mathcal{O}(D) \in \text{Pic}(\text{Ell}_\mathbb{T}(X))$ is called the Thom class of $V$. In particular, by considering the characteristic classes of line bundles, we obtain a morphism $\text{Ell}_\mathbb{T}(X) \to \text{Hom}_\mathbb{Z}(\text{Pic}^\mathbb{T}(X), E) \cong (E^\vee_T)^\vee$ and hence $\tilde{E}(X) \to E^\vee_T \times (E^\vee_T)^\vee$. We denote by $\mathcal{U}_X$ the line bundle on $\tilde{E}(X)$ defined by pulling back the Poincaré line bundle on $E^\vee_T \times (E^\vee_T)^\vee$.

For $\lambda \in \text{Pic}^\mathbb{T}(X) \cong \text{Hom}(E, E^\vee_T)$ and $\mu \in X^*(\mathbb{T}) \cong \text{Hom}(E_T, E)$, let $\tau(\lambda, \mu) : \tilde{B}_X \to \tilde{B}_X$ be the shift of Kähler parameters $(t, z) \mapsto (t, z + \lambda(\mu(t)))$, where $t \in E_T$ and $z \in E^\vee_T$. We denote by the same letter for the shift of Kähler parameters on $\tilde{E}(X)$.

For each fixed point $p \in X^H$, we obtain a natural morphism $i_p : \tilde{B}_X \cong \text{Ell}_\mathbb{T}(p) \times E^\vee_T \to \tilde{E}(X)$ coming from the inclusion $i_p : \{p\} \hookrightarrow X$ and the functoriality of elliptic cohomology. We set $\mathcal{U}_p := i_p^* \mathcal{U}_X$. Recall that we take a chamber $\mathcal{C}$ and a polarization $T^{1/2}$. Take a sufficiently generic $\xi \in \mathcal{C}$ and decompose $T^{1/2}_p = \text{ind}_p + \text{ind}_p - T^{1/2}_{p,0}$ into attracting, repelling, and fixed parts with respect to $\xi$, where we assume that $T^{1/2}_{p,0}$ coincides with the $H$-fixed part of $T^{1/2}_p$. We note that this decomposition might depends on the choice of $\xi$, but the definition of elliptic stable basis does not depend on this choice.

**Definition 4.1 (\[\text{\textbullet}\]).** For each $p \in X^H$, the elliptic stable basis $\text{Stab}^{\mathcal{C}}_{\mathcal{U}, T^{1/2}}(p)$ is a section of some line bundle on $\tilde{E}(X)$ characterized by the following conditions:

- $\text{Stab}^{\mathcal{C}}_{\mathcal{U}, T^{1/2}}(p)$ is a section of $\mathcal{U}_X \otimes \Theta(T^{1/2}) \otimes \tilde{\pi}(\det(\text{ind}_p, v^{-2})^* \mathcal{U}_p \otimes \Theta(T^{1/2})^{-1}) \otimes \ldots$, where $\ldots$ is a certain line bundle pulled back from $\tilde{B}_X / E_H$ and the section is allowed to be meromorphic on this factor. Here, $E_H$ acts on $\tilde{B}_X$ by the translation on the factor $E_T$.

- The support of $\text{Stab}^{\mathcal{C}}_{\mathcal{U}, T^{1/2}}(p)$ is contained in $\bigcup_{p' \subseteq p} \text{Attr}(p')$.

- We have $i_p^* \text{Stab}^{\mathcal{C}}_{\mathcal{U}, T^{1/2}}(p) = \vartheta(N_{p,-}) \in \Gamma(\tilde{B}_X, \Theta(N_{p,-}))$, where $\vartheta(N_{p,-}) = \prod_i \vartheta(w_i)$ if we write $N_{p,-} = \sum_i [w_i] \in K_T(p)$, $w_i \in X^*(\mathbb{T})$.

By \[\text{\textbullet}\], this is unique if it exists and the existence is proved for the case where $X$ is a toric hyper-Kähler manifold or a quiver variety. We assume the existence for the conical symplectic resolutions we consider in this paper. Moreover, this is constant on the $E^\vee_T$-orbits, hence defines a section of some line bundle on $E(X)$ which is also denoted by $\text{Stab}^{\mathcal{C}}_{\mathcal{U}, T^{1/2}}(p)$.

As in the case of $K$-theory, it might be better to change the normalization of the elliptic stable bases slightly for our purpose. Recall that we always assume the existence of dual conical symplectic X' = (X', E', A', ...) for X = (X, E, A, ...).

**Definition 4.2.** For each $p \in X^H$, we define the elliptic standard basis $\text{Stab}^{\mathcal{A}}_{\mathcal{U}}(p)$ by

$$\text{Stab}^{\mathcal{A}}_{\mathcal{U}}(p) = \vartheta(N_{p}', \ldots) \cdot \tau(\det T^{1/2}, v)^* (\text{Stab}^{\mathcal{C}}_{\mathcal{U}, T^{1/2}}(p)).$$

Next we describe the line bundle of which $\text{Stab}^{\mathcal{A}}_{\mathcal{U}}(p)$ defines a section. More precisely, we give a formula for the factors of automorphism of the restriction $\mathbf{S}_{X, p', p} := \mathbf{S}_{p', p} := i_{p'}^* \text{Stab}^{\mathcal{A}}_{\mathcal{U}}(p)$ for every $p, p' \in X^H$. We will consider $\mathbf{S}_{p', p}$ as a multivalued meromorphic function on $B_\mathcal{X}^\mathcal{A} \coloneqq (X_\mathcal{A}(\mathbb{T}) \times P) \otimes_\mathbb{Z} \mathbb{C}^\mathcal{A}$. As in section 3.1, we take a basis $\{a_1, \ldots, a_r\}$ of $X_\mathcal{A}(\mathbb{T})$ which will be considered as a system of coordinates on $X_\mathcal{A}(\mathbb{T}) \otimes_\mathbb{Z} \mathbb{C}^\mathcal{A}$. Similarly, we take a basis $\{z_1, \ldots, z_r\}$ of $P^\mathcal{A}$ which will be considered
as a system of coordinates on $P \otimes \mathbb{C}$. For each $\gamma \in \mathbb{X}(T) \times P$, we denote by $\theta_{X,p}(\gamma)$ the factor of automorphy of the function $\theta(\mathcal{N}_{p,-})$ under the translation by $q^\gamma := \gamma \otimes q \in \mathcal{B}_X$. We also set $\theta_{X,p',p}(\gamma) = \theta_{X,p',p}(\gamma) := \theta_{X,p}(\gamma) \cdot \theta_{X,p'}(\gamma)$. Assuming the existence of the dual pair in the sense of Definition 3.3, we prove the following.

**Proposition 4.3.** In the above situation, $S_{p',p}$ satisfies the following:

- For each $c \in \mathbb{X}(H)$, we have
  $$S_{p',p}(a \mapsto q^a) = i_p^* \mathcal{L}(t)^{-1} \cdot i_{p'}^* \mathcal{L}(t) \cdot \theta_{p',p}(c) \cdot S_{p',p}. \quad (24)$$

- For each $l \in P$, we have
  $$S_{p',p}(z \mapsto q^l z) = i_p^* \mathcal{L}(l) \cdot i_{p'}^* \mathcal{L}(l)^{-1} \cdot \theta_{p',p}(l) \cdot S_{p',p}. \quad (25)$$

- For $\delta \in \mathbb{X}(\mathbb{S})$ satisfying $(\delta, v) = 1$, we have
  $$S_{p',p}(v \mapsto qv) = \frac{\det \mathcal{N}_{p,-} \cdot \det \mathcal{N}_{p',-}^{\gamma}}{\det \mathcal{N}_{p',-} \cdot \det \mathcal{N}_{p,-}^{\gamma}} \cdot q \cdot \frac{\det \mathcal{N}_{p,-}}{\det \mathcal{N}_{p',-}^{\gamma}} \cdot \theta_{p',p}(\delta) \cdot S_{p',p}. \quad (26)$$

- $\sqrt{\det \mathcal{N}_{p,-} \cdot \det \mathcal{N}_{p',-}^{\gamma}} \cdot S_{p',p}$ is single valued on $\mathcal{B}_X$.

**Proof.** By Proposition 3.3 in [1] and $T_{p,-}^{\gamma} = N_{p,-} \cdot \mathrm{ind}_{p'} - v^{-2} \mathrm{ind}_{p'}^\gamma + T_{p',0}^{\gamma} = T_{p',0}^{\gamma}$, $S_{p',p}$ is a meromorphic section of the line bundle

$$\frac{\tau(\det T_{p,-}^{\gamma}, v)^\ast \mathcal{U}_p}{\tau(\det T_{p',0}^{\gamma}, v)^\ast \mathcal{U}_p} \cdot \frac{\Theta(\mathrm{ind}_{p'}) \Theta(v^2)^{\mathrm{rk} \mathrm{ind}_{p'}}}{\Theta(v^{-2} \cdot \mathrm{ind}_{p'})} \cdot \frac{\Theta(\mathcal{N}_{p,-}) \Theta(v^2)^{\mathrm{rk} \mathrm{ind}_{p'}}}{\Theta(\mathcal{N}_{p,-}^{\gamma}) \Theta(v^2)^{\mathrm{rk} \mathrm{ind}_{p'}}} \cdot \theta_{p',p}(\delta) \cdot S_{p',p}. \quad (27)$$

In particular, this implies the single valuedness of $\sqrt{\det \mathcal{N}_{p,-} \cdot \det \mathcal{N}_{p',-}^{\gamma}} \cdot S_{p',p}$. We now describe the factors of automorphy for each factors in $[27]$.

Let $\{l_1, \ldots, l_r\} \subset P$ be the dual basis of $\{z_1, \ldots, z_r\}$. We set $\mathcal{L}\Gamma := \mathcal{L}(l_i) \in \mathbb{Pic}^\mathbb{X}(X)$. Then $\{\mathcal{L} \Gamma, a_1, \ldots, a_r, v\}$ is a basis of $\mathbb{Pic}^\mathbb{X}(X)$. Let $\{z_1, \ldots, z_r, z_{a_1}, \ldots, z_{a_r}, z_v\} \subset \mathbb{Pic}^\mathbb{X}(X)^\ast$ be its dual basis. By definition, the line bundle $\mathcal{U}_p$ on $\mathcal{B}_X$ is characterized by the factors of automorphy of the function

$$\prod_{i=1}^r \psi(i_p^* \mathcal{L} \Gamma, z_{i}) \cdot \prod_{i=1}^c \psi(a_i, z_{a_i}) \cdot \psi(v, z_v)$$

for each $p \in \mathcal{X}^H$. Therefore, for $\mathcal{L} = \prod_{i=1}^r \mathcal{L} \Gamma \cdot \prod_{i=1}^c a_i^{n_{a_i}} \cdot v^{n_v} \in \mathbb{Pic}^\mathbb{X}(X)$, the line bundle $\tau(\mathcal{L}, v)^\ast \mathcal{U}_p$ is characterized by the factor of automorphy of the function

$$\prod_{i=1}^r \psi(i_p^* \mathcal{L} \Gamma, z_{i} v^{n_{a_i}}) \cdot \prod_{i=1}^c \psi(a_i, z_{a_i} v^{n_{a_i}}) \cdot \psi(v, z_v v^{n_v}).$$

By using the formula

$$\psi(q^m x, q^v y) = q^{-mn} x^{-n} y^{-m} \psi(x, y),$$

one can check that the factors of automorphy for $\tau(\mathcal{L}, v)^\ast \mathcal{U}_p$ are given as follows:

- For $a \mapsto q^a$, it is given by
  $$\prod_{i=1}^r z_i^{- (i_p^* \mathcal{L} \Gamma, c)} \cdot \prod_{i=1}^c z_{a_i}^{- (a_i, c)} \cdot v^{- (i_v^* \mathcal{L} \Gamma, c)};$$
For $z \mapsto q^l z$, it is given by $i_p^* \Omega(l)^{-1}$;

For $v \mapsto qv$, it is given by
$$z_v^{-1} \prod_{i=1}^r z_i^{-w_{ls} i_s^* \Xi_i} \cdot (qv)^{-w_{ls} i_s^* \Xi} \cdot i_p^* \Omega^{-1}.$$  

Hence the factors of automorphy for the first factor in (27) is given as follows:

- For $a \mapsto q^a a$, it is given by
  $$\prod_{i=1}^r \left( i_p^* \Omega_i(c) - (i_p^* \Omega_i(c)) \cdot v \cdot (\det T_p^{1/2}, c) - (\det T_p^{1/2}, c) - 2 (\det \text{ind}_p, c) \right);$$

- For $v \mapsto qv$, it is given by
  $$\prod_{i=1}^r z_i^{-w_{ls} i_s^* \Xi_i - w_{ls} i_s^* \Xi} \cdot (qv)^{w_{ls} \det T_p^{1/2} - w_{ls} \det T_p^{1/2} - 2 w_{ls} \det \text{ind}_p} \cdot \frac{\det T_p^{1/2}}{\det T_p^{1/2}} \cdot (\det \text{ind}_p)^{-2}.$$  

Since the second and the third factor in (27) does not depend on the Kähler parameters, the equation (25) follows.

One can also check that the factor of automorphy for the second factor in (27) is given as follows:

- For $a \mapsto q^a a$, it is given by $v^{2 (\det \text{ind}_{l'}, c)}$;

- For $v \mapsto qv$, it is given by $(qv)^2 w_{ls} \det \text{ind}_{l'} \cdot (\det \text{ind}_{l'})^2$.

Since we have
$$\det N_{p, -} = v^{-2 \text{rk ind}_p} \cdot \det T_p^{1/2} \cdot (\det \text{ind}_p)^{-2} \cdot (\det T_{p, 0}^{1/2})^{-1}$$
and the third factor in (27) does not depend on the equivariant parameters, the factor of automorphy of $S_{l', p}$ under $a \mapsto q^a a$ is given by
$$\prod_{i=1}^r \left( i_p^* \Omega_i(c) - (i_p^* \Omega_i(c)) \cdot v \cdot (\det N_{p, -}, c) - (\det N_{p', -}, c) \cdot \theta_{p', p}(c) \right).$$

This and (13) imply (24).

Since $T_{p, 0}^{1/2} + v^{-2 (T_{p, 0}^{1/2})^2} = 0$, we have $T_{p, 0}^{1/2} = \sum_i (v^{m_i} - v^{-2-m_i})$ for some $m_i \in \mathbb{Z}$. Using this, one can check that the factor of automorphy for the line bundle $\Theta(T_{p, 0}^{1/2})$ under $v \mapsto qv$ is given by
$$(qv)^2 w_{ls} \det T_{p, 0}^{1/2} + 2 \text{rk ind}_{l'} - w_{ls} \det T_{p, 0}^{1/2} - 2 \text{rk ind}_p$$
and hence the factor of automorphy of the third factor in (27) under $v \mapsto qv$ is given by
$$\prod_{i=1}^r z_i^{-w_{ls} i_s^* \Xi_i - w_{ls} i_s^* \Xi} \cdot (qv)^{w_{ls} \det N_{p, -} \cdot \text{det} N_{p', -} \cdot \theta_{p', p}(d)}.$$  

After some simplification using (28), one can check that the factor of automorphy of $S_{l', p}$ under $v \mapsto qv$ is given by
$$\prod_{i=1}^r z_i^{-w_{ls} i_s^* \Xi_i - w_{ls} i_s^* \Xi} \cdot (qv)^{w_{ls} \frac{\det N_{p, -} \cdot \text{det} N_{p', -} \cdot \theta_{p', p}(d)}{\det N_{p', -}}}.$$  

By (11), we obtain $w_{ls} \det N_{p, -} - w_{ls} \det N_{p', -} = w_{ls} \det N_{p', -}^l - w_{ls} \det N_{p', -}^l$ and hence (12) implies
$$\prod_{i=1}^r z_i^{-w_{ls} i_s^* \Xi_i - w_{ls} i_s^* \Xi} \cdot (qv)^{w_{ls} \frac{\det N_{p, -} \cdot \text{det} N_{p', -} \cdot \theta_{p', p}(d)}{\det N_{p', -}}}.$$  

This proves (26).
In particular, if we set $S_{p',p} := i_p^* \text{Stab}_{X}(p')$, then the line bundle on $B_X$ defined by the factors of automorphy of $S_{p',p}$ is the same as the line bundle on $B_X$ defined by $S_{p',p}$ under the identification $B_X \cong B_X$. Following [11, 35, 37], we conjecture that elliptic standard bases have certain symmetry under the symplectic duality.

**Conjecture 4.4** ([11, 35, 37]). For any $p, p' \in X^H$, $S_{p',p}$ is holomorphic and $S_{p',p} = \pm S_{p',p'}$.

For toric hyper-Kähler manifolds, this conjecture is proved in Corollary 6.3.

### 4.2 Flops

Before considering an elliptic analogue of the bar involution, we state some conjectures about the elliptic stable bases for the maximal flop of $X$. Recall that our conical symplectic resolution is always equipped with some additional data as in Section 3.1. For $X = (X, \mathcal{E}, \mathfrak{A}, \Phi, \Psi, \mathfrak{L})$, we define $-X := (X, -\mathfrak{A}, \mathfrak{E}, \Phi, \Psi, \mathfrak{L})$. We also assume that $-X^t$ has a dual conical symplectic resolution in the sense of Definition 3.2 which is denoted by $X_{\text{flop}} = (X_{\text{flop}}, \mathfrak{E}_{\text{flop}}, \mathfrak{A}_{\text{flop}}, \Phi_{\text{flop}}, \Psi_{\text{flop}}, \mathfrak{L}_{\text{flop}})$ and called a maximal flop of $X$. By definition, we have an identification $X_+(H) \cong P' \cong X_+(H') \cong P_{\text{flop}}$, and $(X^H, -\varepsilon) \cong (X_{\text{flop}}^H, -\varepsilon_{\text{flop}})$. We simply denote by $p \in X_{\text{flop}}^H$ the fixed point corresponding to $p \in X^H$.

Under this identification, we also have $\mathfrak{E}_{\text{flop}} = \mathfrak{E}$, $\mathfrak{A}_{\text{flop}} = -\mathfrak{A}$, $\mathfrak{A}_{\text{flop}}(p) = \Phi(p)$, and $\mathfrak{A}_{\text{flop}}(p) = \Psi(p)$. By (12), we also obtain $\gamma^t_{\text{flop}}(\lambda) = i_{p_{\text{flop}}}^* \Sigma(\lambda)$ for any $\lambda \in P$. The relation $\Phi_{\text{flop}}(p) = \Phi(p)$ implies that the tangent spaces $T_pX_{\text{flop}}$ and $T_pX$ have the same multiset of $H$-weights, and in particular, we have $\dim X_{\text{flop}} = \dim X$. For the $S$-weights, we expect the following.

**Conjecture 4.5.** For any $p \in X^H$, we have $T_pX_{\text{flop}} = v^2 \cdot T_pX$ as $\mathfrak{T}$-modules.

For toric hyper-Kähler manifolds, this conjecture is checked in Corollary 5.7. For toric hyper-Kähler manifolds, this conjecture is proved in Corollary 6.3. In particular, this conjecture implies $\dim X_{\text{flop}} = \dim X$. As in $K$-theory, the inverse for $S$ induces an involution $(\cdot) : \tilde{M}_X \to \tilde{M}_X$. We conjecture the following formula expressing $\tilde{S}_X$ geometrically.

**Conjecture 4.6.** $\tilde{S}_X = (1)_{\dim X} \cdot \tilde{S}_{-X_{\text{flop}}}.$

For toric hyper-Kähler manifolds, this conjecture is proved in Corollary 6.4. If we assume Conjecture 4.5 then we have $\mathfrak{A}_{p_{\text{flop}}} = -\mathfrak{A}_{p_{\text{flop}}} = (N_{p_{\text{flop}}}^0)^\gamma$ and hence for the diagonal entries, we obtain

$$\mathfrak{S}_{X,p} = \theta_N(N_{p_{\text{flop}}}^0) \partial(N_{p_{\text{flop}}}^0) = (1)_{\dim X} \cdot \mathfrak{S}_{X_{\text{flop}},p_{\text{flop}}} = \mathfrak{S}_{X_{\text{flop}},p_{\text{flop}}}^{-1}.$$

As another evidence for Conjecture 4.6, one can also check that each corresponding entry has the same factors of automorphy.

**Proposition 4.7.** Assume Conjecture 4.5. For any $p, p' \in X^H$, both $\mathfrak{S}_{X_{\text{flop}},p}$ and $\mathfrak{S}_{-X_{\text{flop}},p'}$ can be considered as a section of the same line bundle on $B_X \cong B_{-X_{\text{flop}}}.$

**Proof.** As above, one can check $\mathfrak{S}_{X_{\text{flop}},p}(\gamma) = \mathfrak{S}_{-X_{\text{flop}},p'}(\gamma)$ for any $\gamma \in X_+(H) \times P$ and $\mathfrak{S}_{X_{\text{flop}},p}(-\delta) = \mathfrak{S}_{-X_{\text{flop}},p'}(\delta)$ for $\delta \in X_+(\mathfrak{L})$. By Proposition 4.3, the coincidence of factors of automorphy under $a \mapsto q^z a$ and $z \mapsto q^z$ follows from $i_{p_{\text{flop}}^*}^\mathfrak{E}_{\text{flop}}(c) = i_{p_{\text{flop}}}^\mathfrak{E}(c)$ and $i_{p_{\text{flop}}}^\mathfrak{E}_{\text{flop}}(l) = i_{p_{\text{flop}}}^\mathfrak{E}(l)$.

For $v \mapsto qv$, we note that by (11) and (26), we have

$$\mathfrak{S}_{X_{\text{flop}},p}(v \mapsto qv) = \mathfrak{S}_{X_{\text{flop}},p}(v \mapsto q^{-1}v) = \frac{\det N_{p_{\text{flop}}}^0}{\det N_{p_{\text{flop}}}^0} \cdot q^w \frac{\det N_{p_{\text{flop}}}^0}{\det N_{p_{\text{flop}}}^0} \cdot \mathfrak{S}_{X_{\text{flop}},p}(\delta) \cdot \mathfrak{S}_{X_{\text{flop}},p}.$$
Therefore, the coincidence of the factor of automorphy under $v \mapsto qv$ together with the coincidence of monodromies follows from $\nabla_{p,-} = (\mathcal{N}^{\text{flop}}_{p,+})^v$. \qed

We now state the main conjecture in this section. This conjecture together with Conjecture 4.6 will imply that the elliptic bar involution defined in the next section is actually an involution.

**Conjecture 4.8.** The matrix $\mathbf{M}_X := \mathbf{S}_{X_{\text{flop}}'} \cdot \mathbf{S}_X^{-1}$ does not depend on the choice of the chamber $c$.

For toric hyper-Kähler manifolds, this conjecture is proved in Corollary 6.12. As another evidence for this conjecture, we prove that each entry of $\mathbf{M}_X$ can be considered as a section of some line bundle on $\mathbf{B}_X$ which does not depend on the choice of $c$. We write $\mathbf{U}_{p,p'}$ and $\mathbf{S}_{p,p'}^{\text{flop}}$ the $(p,p')$-entry of $\mathbf{S}_X'$ and $\mathbf{S}_{X_{\text{flop}}}$ respectively. We first calculate the factors of automorphy for $\mathbf{U}_{p,p'}$.

**Lemma 4.9.** $\mathbf{U}_{p,p'}$ satisfies the following:

- For each $c \in X_s(H)$, we have
  $$\mathbf{U}_{p,p'}(a \mapsto q^a) = i_p^* \mathbf{L}^c \cdot i_{p'}^* \mathbf{L}^{c^{-1}} \cdot \theta_{p,p'}(c)^{-1} \cdot \mathbf{U}_{p,p'};$$

- For each $l \in P$, we have
  $$\mathbf{U}_{p,p'}(z \mapsto q^l z) = i_p^* \mathbf{L}(l) \cdot i_{p'}^* \mathbf{L}(l) \cdot \theta_{p,p'}(l)^{-1} \cdot \mathbf{U}_{p,p'};$$

- For $\delta \in X_s(S)$ satisfying $(\delta, v) = 1$, we have
  $$\sqrt{\det N_{p,-} \det N_{p',-}} \cdot \mathbf{U}_{p,p'} = \frac{\det N_{p',-} \det N_{p,-}^{l}}{\det \mathbf{N}_{p,-} \det N_{p',-}^{l}} \cdot q^{\frac{\det N_{p',-} \det N_{p,-}^{l}}{\det \mathbf{N}_{p,-} \det N_{p',-}^{l}}} \cdot \theta_{p,p'}(\delta)^{-1} \cdot \mathbf{U}_{p,p'};$$

- $\sqrt{\det N_{p,-} \det N_{p',-}^{l}} \cdot \mathbf{U}_{p,p'}$ is single valued on $\mathbf{B}_X^N$.

**Proof.** Since $\mathbf{S}_{p,p'} = \mathbf{U}_{p,p'} = 0$ unless $p \preceq e p'$, we prove the statements by induction on the number of $p'' \in X^H$ satisfying $p \preceq e p'' \preceq e p'$. We note that by Proposition 4.3, the factor of automorphy of $\mathbf{S}_{p,p'}$ under the translation by $q^c$ is of the form $f_p(\gamma) f_{p'}(\gamma)^{-1} \theta_{p',p}(\gamma)$ for any $\gamma \in X_s(T) \times P$. It is enough to check that the factor of automorphy for $\mathbf{U}_{p,p'}$ is given by $f_p(\gamma)^{-1} f_{p'}(\gamma) \theta_{p',p}(\gamma)^{-1}$. If $p = p'$, then we have $\mathbf{U}_{p,p'} = (\mathbf{S}_{p,p'})^{-1}$ and the claim follows immediately.

If $p \preceq e p'$, then we have
$$\mathbf{U}_{p,p'} = -(\mathbf{S}_{p',p})^{-1} \cdot \sum_{p \preceq e p'' \preceq e p'} \mathbf{U}_{p,p''} \cdot \mathbf{S}_{p'',p'}.$$

By the induction hypothesis, the factor of automorphy of each term in the RHS is given by
$$\theta_{p',p''}(\gamma)^{-1} \cdot f_p(\gamma)^{-1} f_{p'}(\gamma) \theta_{p',p''}(\gamma)^{-1} \cdot f_p(\gamma) f_{p'}(\gamma)^{-1} \theta_{p',p''}(\gamma) = f_p(\gamma)^{-1} f_{p'}(\gamma) \theta_{p',p}(\gamma)^{-1}.$$

This proves the first three statements. The fourth statement can be proved similarly. \qed

**Proposition 4.10.** Each entry of $\mathbf{M}_X$ is a section of a line bundle on $\mathbf{B}_X$ which does not depend on the choice of $c$.

**Proof.** By Proposition 4.3 and Lemma 4.9, the factors of automorphy of $\mathbf{S}_{p,p'}^{\text{flop}} \cdot \mathbf{U}_{p',p'}$ for $p \preceq e p'' \preceq e p'$ are given as follows:

- For $a \mapsto q^a$, it is given by
  $$i_p^* \mathbf{L}(c) \cdot \theta_{X_{\text{flop}},p}(c);$$
For $z \mapsto q^l z$, it is given by

\[
\frac{i_{p'}^* \Sigma(l)}{i_{p'}^* \Sigma_{\text{flop}}(l)} \cdot \frac{i_{p'}^* \Sigma_{\text{flop}}(l)}{i_{p'}^* \Sigma(l)} \cdot \frac{\theta_{-X', p''}(l)}{\theta_{X', p''}(l)};
\]

(30)

For $v \mapsto qv$, it is given by

\[
\frac{\det N_{p',-}^{\text{flop}}}{\det N_{p,-}^{\text{flop}}} \cdot \frac{\det N_{p',+}^{\text{flop}}}{\det N_{p',-}^{\text{flop}}} \cdot q^{\text{wt}_z \det N_{p',-}^{\text{flop}}} \cdot \frac{\theta_{X, p'}(\delta)}{\theta_{X', p''}(\delta)} \cdot \frac{\det N_{p,-}^{\text{flop}}}{\det N_{p',-}^{\text{flop}}} \cdot \frac{\det N_{p',-}}{\det N_{p',+}^{\text{flop}}} \cdot q^{\text{wt}_z \det N_{p,-}^{\text{flop}}} \cdot \frac{\theta_{-X', p''}(\delta)}{\theta_{X', p''}(\delta)}.
\]

(31)

We first consider the case of $a \mapsto q^2 a$. Since (29) does not depend on $p''$, $M_{X,p,p'} = \sum_{q} S_{p,p'}^{\text{flop}} \cdot U_{p'',p'}$ has (29) as the factor of automorphy. In order to check the independence on $C$, it is sufficient to check that $i_{p'}^* \Sigma(c) \cdot \theta_{X, p'}(c)$ does not depend on the choice of $C$. By (13), the $H$-weight of $i_{p'}^* \Sigma(c)$ does not depend on the choice of $C$ and the $S$-weight is given by $-\langle \det N_{p,-}, c \rangle$. If we write $N_{p,-} = \sum_i w_i p^{m_i}$ for some $w_i \in \mathbb{X}^*(H)$ and $m_i \in \mathbb{Z}$, then it is enough to check the independence of

\[
u^{-\langle \det N_{p,-}, c \rangle} \cdot \theta_{X, p'}(c) = \prod_i (-1)^{\langle w_i, c \rangle} q^{-\frac{1}{2} \langle w_i, c \rangle^2} w_i^{-\langle w_i, c \rangle} v^{-(m_i+1)\langle w_i, c \rangle}
\]

on the choice of $C$. If we change $C$, then for some $i$, $w_i$ is replaced by $w_i^{-1}$ and $m_i$ is replaced by $-m_i - 2$ in the formula of $N_{p,-}$. Since this replacement does not change each factor of the RHS, the independence follows.

We next consider the case of $z \mapsto q^l z$. By (12), we have

\[
\frac{i_{p'}^* \Sigma_{\text{flop}}(l)}{i_{p'}^* \Sigma(l)} \cdot \frac{\theta_{-X', p''}(l)}{\theta_{X', p''}(l)} = v^{2\langle \det N_{p',-}, c \rangle} \cdot \theta_{X', p''}(l) = 1.
\]

Therefore, (30) does not depend on $p''$ and hence it is the factor of automorphy of $M_{X,p,p'}$. Its independence on $C$ is clear.

Now we consider the case of $v \mapsto qv$. We first check that (31) does not depend on $p''$. One can check that

\[
\frac{\theta_{-X', p''}(\delta)}{\theta_{X', p''}(\delta)} = (qv)^{-2\text{wt}_z \det N_{p',-}^{\text{flop}}} \cdot v^{-\text{dim} X'} \cdot (\text{det} N_{p',-}^{\text{flop}})^{-2}.
\]

On the other hand, we have

\[
\frac{\det N_{p',-}^{\text{flop}}}{\det N_{p,-}^{\text{flop}}} = v^{-\text{dim} X - 2\text{wt}_z \det N_{p,-}^{\text{flop}}} \cdot v^{\text{dim} X' + 2\text{wt}_z \det N_{p',-}^{\text{flop}}}.
\]

by (11), we have

\[
\frac{\det N_{p',-}^{\text{flop}}}{\det N_{p',-}^{\text{flop}}} = v^{2\text{dim} X'} \cdot (\text{det} N_{p',-}^{\text{flop}})^{2}.
\]

These equations imply that (31) does not depend on $p''$. In order to prove the independence of (31) on the choice of $C$, it is enough to check that

\[
\frac{\text{det} N_{p',-}^{\text{flop}}}{\text{det} N_{p,-}^{\text{flop}}} \cdot q^{-\text{wt}_z \det N_{p,-} \cdot \theta_{X, p}}.
\]

31
does not depend on $\mathcal{C}$. We note that the $H'$-weight of $\det N_{p,-}'$ does not depend on $\mathcal{C}$ and the $\mathbb{S}$-weight is $-\frac{\dim X'}{2} - \frac{\dim X}{2} - \wt g \det N_{p,-}$ by (11). If we write $N_{p,-} = \sum_i w_i v^{m_i}$ for some $w_i \in \mathbb{X}^*(H)$ and $m_i \in \mathbb{Z}$, then we have

\[
\frac{v^{\wt g} \det N_{p,-}'}{\det N_{p,-}} \cdot q^{-\wt g} \det N_{p,-} \cdot \theta_{X,p}(\delta) = v^{-\frac{\dim X'}{2} - \frac{\dim X}{2}} \cdot \prod_i (-1)^{m_i} (q v^2)^{-m_i} w_i^{-m_i - 1}.
\]

Since each factor in the RHS does not change under $w_i \mapsto w_i^{-1}$ and $m_i \mapsto -m_i - 2$, the independence on $\mathcal{C}$ follows.

Finally, independence of monodromies of $\sqrt{\det N_{p,-}}$ on $\mathcal{C}$ easily implies the independence of monodromies of $\mathcal{M}_{X,p,p'}$. \hfill $\square$

**Remark 4.11.** We note that the matrix $\mathcal{M}_X$ is closely related to the monodromy operator appearing in [1, Proposition 6.5] which intertwines the vertex function for $X$ and $X_{\text{flop}}$. We expect that the dependence on $\mathcal{C}$ in [1, Proposition 6.5] is eliminated by our choice of the normalization on the elliptic standard bases. Since the vertex functions do not depend on the choice of $\mathcal{C}$, one may hope that we kind of relations would imply Conjecture 4.8. We plan to investigate this approach in the future.

### 4.3 Elliptic bar involutions

In this section, we give a proposal for a definition of elliptic bar involution. Since our approach gives involution only after localization, we set $K(X)_{\text{loc}} := \bigoplus_{p \in X^H} \mathcal{M}_X$, where $\mathcal{M}_X$ is the field of (single-valued) meromorphic functions on $B_X$, and we will only work on $K(X)_{\text{loc}}$ in this paper. By restriction to the fixed points, we obtain a natural map $K_T(X) \otimes \mathbb{Z}[P'] \to K(X)_{\text{loc}}$. In order to consider $\text{Stab}_{\mathcal{H}}(\tau'(p))$ as an element of $K(X)_{\text{loc}}$ similarly, we need to kill the multi-valuedness. We use the data of polarization to fix this modification.

We choose a splitting $\text{Pic}^H(X) \cong P \oplus \mathbb{X}^*(H)$ by using the data $\mathcal{L} : P \to \text{Pic}^T(X)$ and take a polarization $T^{1/2}$ satisfying Assumption 3.11. Let $\kappa \in P \oplus \mathbb{X}^*(H)$ be the element corresponding to $\det T^{1/2} \in \text{Pic}^H(X)$. We also denote by $\mathcal{L} : P \oplus \mathbb{X}^*(H) \to \text{Pic}^T(X)$ the natural extension of $\mathcal{L}$. We note that $\mathcal{L}(\kappa) = v^{-w(\det T^{1/2})} \cdot \det T^{1/2}$. We also take similar data $T^{1/2,1}$ and $\kappa' \in P' \oplus \mathbb{X}^*(H')$ on the dual conical symplectic resolution $X'$.

**Definition 4.12.** For any $p \in X^H$, we set

\[
\mathcal{G}^\kappa_{X,p}(p) := \sqrt{\mathcal{L}(\kappa) \cdot i_p^* \mathcal{L}(\kappa')}^{-1} \cdot \text{Stab}_{\mathcal{H}}(\tau'(p)).
\]

**Lemma 4.13.** We have $\left(i_p^* \mathcal{G}^\kappa_{X,p}(p)\right)' \in K(X)_{\text{loc}}$.

**Proof.** By Assumption 3.11 for $T^{1/2}$ and $T^{1/2,1}$, we obtain $w(\det T^{1/2}) \equiv w(\det T^{1/2,1}) \mod 2$. Since we have $\sqrt{i_p^* \mathcal{L}(\kappa) \cdot i_p^* \mathcal{L}(\kappa')} = v^{-\frac{w(\det T^{1/2})}{2} - \frac{w(\det T^{1/2,1})}{2}} \sqrt{\det T^{1/2} \cdot \det T^{1/2,1}}$, the statement follows from Proposition 4.3. \hfill $\square$

We will identify $\mathcal{G}^\kappa_{X,p}(p)$ and $\left(i_p^* \mathcal{G}^\kappa_{X,p}(p)\right)' \in K(X)_{\text{loc}}$. By the triangular property of the elliptic stable bases, $\{\mathcal{G}^\kappa_{X,p}(p)\}_{p \in X^H}$ forms a basis of $K(X)_{\text{loc}}$ over $\mathcal{M}_X$.

**Definition 4.14.** We define the $\mathcal{M}_X$-semilinear map $\beta^\mathcal{H}_X = \beta^\mathcal{H}_{X,p} : K(X)_{\text{loc}} \to K(X)_{\text{loc}}$ by

\[
\beta^\mathcal{H}_X(\mathcal{G}^\kappa_{X,p}(p)) = (-1)^{\frac{\dim X}{2}} \mathcal{G}^\kappa_{X,p}(p)
\]

for any $p \in X^H$. Here, $\mathcal{M}_X$-semilinear means $\beta^\mathcal{H}_X(f \cdot m) = f \cdot \beta^\mathcal{H}_X(m)$ for any $f \in \mathcal{M}_X$ and $m \in K(X)_{\text{loc}}$. We also identified $P^\mathcal{H}_{\text{flop}} \oplus \mathbb{X}^*(H'_{\text{flop}})$ and $P^\mathcal{H} \oplus \mathbb{X}^*(H)$ in order to take $\kappa'$ for $X_{\text{flop}}$ in the RHS.
We note that $\beta_X^{\text{ell}}$ does not depend on the choice of $\kappa'$ because of the relation $\mathcal{L}_{\text{flop}}(\kappa') = \overline{\mathcal{L}}(\kappa')$. We also conjecture that $\beta_X^{\text{ell}}$ does not depend on the data $\mathcal{C}$. As an evidence for this conjecture, we check it by assuming Conjecture 4.6 and Conjecture 4.8.

**Proposition 4.15.** Assume Conjecture 4.6 and Conjecture 4.8. The map $\beta_X^{\text{ell}}$ does not depend on the choice of $\mathcal{C}$.

**Proof.** In this proof, we denote by $\beta_\mathcal{C} := \beta_X^{\text{ell}}$. We take another chamber $\mathcal{C}'$ and define $\beta_{\mathcal{C}'}$ similarly by using this chamber. We also set $\mathcal{S}_\mathcal{C} := \mathcal{S}_X$ and define $\mathcal{S}_{\mathcal{C}'}$ similarly by using $\mathcal{C}'$ instead of $\mathcal{C}$. Let us define the matrix $\mathcal{S}_\varepsilon := (i_{p,X}^*\mathcal{L}_{X}^\kappa(p'))_{p,p'\in X_u}$ and $\mathcal{S}_{\mathcal{C}'}$ similarly. If we identify $\mathcal{L}(\kappa)$ and the diagonal matrix $\text{diag}((\omega_{\varepsilon})_{p\in X_u})$, then we have

$$\mathcal{S}_\varepsilon = \mathcal{L}(\kappa)^{-\frac{1}{2}} \cdot \mathcal{S}_\varepsilon \cdot \mathcal{L}_\varepsilon'(\kappa')^{-\frac{1}{2}}.$$

Here, $\mathcal{L}_\varepsilon'(\kappa')$ is the diagonal matrix defined similarly as $\mathcal{L}(\kappa)$ by using $\mathcal{L}'(\kappa)$, but since it depends on the choice of $\mathcal{C}$, we put the index to indicate the dependence. We note that $\mathcal{L}_\varepsilon'(\kappa') = \overline{\mathcal{L}_\varepsilon}(\kappa')$. By definition, we have

$$\beta_\mathcal{C}(\mathcal{S}_\mathcal{C}) = (-1)^{\dim X} \cdot \mathcal{S}_{-\varepsilon},$$

where we understand that $\beta_\mathcal{C}$ is applied column by column. Since

$$\beta_{\mathcal{C}'}(\mathcal{S}_\mathcal{C}) = \beta_{\mathcal{C}'}(\mathcal{S}_{\mathcal{C}'}, \mathcal{S}_{-\mathcal{C}'}, \mathcal{S}_\varepsilon) = (-1)^{\dim X} \cdot \mathcal{S}_{-\varepsilon'} \cdot \mathcal{S}_{\varepsilon'},$$

the statement is equivalent to

$$\mathcal{S}_\varepsilon \cdot \mathcal{S}_{-\mathcal{C}' \varepsilon} = \overline{\mathcal{S}_\varepsilon} \cdot \mathcal{S}_{\varepsilon'}.\varepsilon$$

By (13) and Conjecture 4.6, we have

$$\mathcal{S}_\varepsilon \cdot \mathcal{S}_{-\mathcal{C}' \varepsilon} = \mathcal{L}(\kappa)^{-\frac{1}{2}} \cdot \mathcal{S}_\varepsilon \cdot \overline{\mathcal{L}_\varepsilon}(\kappa')^{-\frac{1}{2}} \cdot \mathcal{L}_\varepsilon'(\kappa')^{-\frac{1}{2}} \cdot \mathcal{S}_\varepsilon^{-\frac{1}{2}} \cdot \mathcal{L}(\kappa)^{\frac{1}{2}}$$

$$= (-1)^{\dim X} \cdot \dim X \cdot \mathcal{S}_{\text{flop}}(\kappa)^{-\frac{1}{2}} \cdot \mathcal{S}_{\text{flop}}(\kappa)^{-\frac{1}{2}}.$$

where we set $\mathcal{S}_{\text{flop}} = S_{-X_{\text{flop}}}$. Therefore, the statement follows from Conjecture 4.8.

We now explain the relation between elliptic and K-theoretic bar involutions. Take a generic slope $s \in P_\mathbb{R}$. By Proposition 3.6, elliptic stable bases and K-theoretic stable bases are related by certain limit under $q \to 0$. More precisely, we have

$$\lim_{q \to 0} \left( \sqrt{\det T^{1/2} \cdot \mathcal{S}_{-X_{1/2}}(p)} \right) = \mathcal{S}_{\text{ell}, T^{1/2}}(p),$$

where $z = q^{-s}$ means we specialize the Kähler parameter at $q^{-s} \in P \otimes_{\mathbb{Z}} \mathbb{C}$. We note that (32) is equivalent to

$$\beta_X^{\text{ell}} \left( \sqrt{\det T^{1/2} \cdot \mathcal{S}_{\text{ell}, T^{1/2}}(p)} \right) = (-1)^{\dim X} \cdot \overline{\det T^{1/2}} \cdot \mathcal{S}_{-X_{1/2}}(p)',$

where we set $\mathcal{S}_{-X_{1/2}}(p)' := \tau(\det T^{1/2}, v)^* \mathcal{S}_{-X_{1/2}}(p)$. By using Conjecture 4.6 and

$$\lim_{q \to 0} \frac{\vartheta(q^0 x)}{\vartheta(q^0)} = x^{-|\alpha| - \frac{1}{2}}$$
for \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \), we obtain
\[
\lim_{q \to 0} \left( \frac{\vartheta(N_{p^q}^{\text{top}})}{\vartheta(N_{p^q}^{\text{top}})} \right) = \prod_{\beta \in \Psi_+(p)} v^{2[(\alpha, \beta)]+1}.
\]

Since the limit in (33) does not change if we replace \( \text{Stab}_{\xi_{T^1/2}}^\text{AO}(p) \) by \( \text{Stab}_{\xi_{T^1/2}}^\text{AO}(p)' \), the \( K \)-theory limit \( \beta_{T^1/2,s}^K \) of \( \beta_{X}^H \) satisfies
\[
\beta_{T^1/2,s}^K(\text{Stab}_{\xi_{T^1/2},s}^K(p)) = (-v)^{\frac{d}{2}p} \cdot v^{2pT^{1/2},s} \cdot \text{Stab}_{\xi_{T^1/2},s}^K(p).
\]
This is equivalent to our definition of \( K \)-theoretic bar involution and explains seemingly ad hoc normalizations in our definition of the \( K \)-theoretic standard bases.

## 5 Toric hyper-Kähler manifolds

As an evidence for the main conjectures, we check all the conjectures stated in section 3 for the toric hyper-Kähler manifolds in this section. The conjectures stated in section 4 will be proved in the next section.

### 5.1 Preliminaries

In this section, we briefly recall basic facts about toric hyper-Kähler manifolds introduced by Bielawski-Dancer [7]. For more detail, see for example [7, 15, 29, 35]. We first prepare some notations used in the following sections. We fix an integer \( n \in \mathbb{Z}_{\geq 1} \) and set \( T := (\mathbb{C}^*)^n \). We consider an exact sequence of algebraic tori of the form
\[
1 \to S \to T \to H \to 1,
\]
where \( S \cong (\mathbb{C}^*)^r \) and \( H \cong (\mathbb{C}^*)^d \) for some \( r, d \in \mathbb{Z}_{\geq 0} \) with \( r + d = n \). Let
\[
0 \to X_*(S) \xrightarrow{\varphi} X_*(T) \xrightarrow{\pi} X_*(H) \to 0
\]
be the associated exact sequence. We fix an identification \( X_*(T) \cong \mathbb{Z}^n \) and take the standard basis \( \{\varepsilon_i^*\}_{i=1}^n \). We set \( a_i := a(\varepsilon_i) \) and assume that \( a_i \neq 0 \) for any \( i = 1, \ldots, n \). We also assume that \( a \) is unimodular, i.e., if \( \{a_1, \ldots, a_n\} \) is linearly independent, then they always generate \( X_*(H) \) over \( \mathbb{Z} \). We set
\[
\mathcal{B} := \{ I \subset \{1, \ldots, n\} \mid \{a_i\}_{i \in I} \text{ is a basis of } X_*(H) \}.
\]
We also consider the dual of the above exact sequence
\[
0 \to X^*(H) \xrightarrow{\varphi^*} X^*(T) \xrightarrow{\pi^*} X^*(S) \to 0.
\]
Let \( \{\varepsilon_i^*\}_{i=1}^n \subset X^*(T) \) be the dual basis of \( \{\varepsilon_i\}_{i=1}^n \) and set \( b_i := b(\varepsilon_i^*) \). We also assume that \( b_i \neq 0 \) for any \( i = 1, \ldots, n \). We note that the unimodularity of \( a \) is equivalent to the unimodularity of \( b \) and \( \{b_j\}_{j \in J} \) is a basis of \( X^*(S) \) if and only if \( J := \{1, \ldots, n\} \setminus J \in \mathcal{B} \).

A subset equipped with a decomposition \( C = C_+ \sqcup C_- \subset \{1, \ldots, n\} \) is called signed circuit if \( \{a_i\}_{i \in C} \) is a minimal linearly dependent subset of \( \{a_1, \ldots, a_n\} \) and \( \sum_{i \in C_+} a_i - \sum_{i \in C_-} a_i = 0 \). We note that by the unimodularity assumption, any minimal linear relations between \( a_i \)'s can be written in this form up to scalar multiplication. For a signed circuit \( C \), we denote by \( \beta_C = (\beta_1, \ldots, \beta_n) \in X_*(T) \) the element defined by
\[
\beta_i = \begin{cases} 0 & \text{if } i \notin C \\ 1 & \text{if } i \in C_+ \\ -1 & \text{if } i \in C_-. \end{cases}
\]
By definition, we have $\beta_C \in \text{Ker}(a) = X_*(S)$. We similarly define the notion of signed cocircuit using $b_i$ instead of $a_i$. For a signed cocircuit $C'$, we define $\alpha_{C'} \in X^*(H)$ in the same way as $\beta_C$.

We consider the natural $T$-action on $T^*C^n$ given by

$$(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n, y_1, \ldots, y_n) = (t_1 x_1, \ldots, t_n x_n, t_1^{-1} y_1, \ldots, t_n^{-1} y_n),$$

where $(t_1, \ldots, t_n) \in T$, $(x_1, \ldots, x_n)$ is a point in $C^n$, and $(y_1, \ldots, y_n)$ is a point in the cotangent fiber. This action is Hamiltonian with respect to the standard symplectic structure on $T^*C^n$ and its moment map is given by $\mu_n(x, y) = \sum_{i=1}^n x_i y_i$. Let $\mu(x, y) := \sum_{i=1}^n x_i y_i b_i \in \mathfrak{s}^*$ be the moment map for the $S$-action given by restriction, where $\mathfrak{s}$ is the Lie algebra of $S$. We note that in the coordinate ring $\mathbb{C}[T^*C^n] = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, the $S$-weights of $x_i$ and $y_i$ are given by $-b_i$ and $b_i$ respectively.

We fix a generic element $\eta \in X^*(S)$, where generic means for any circuit $C$, we have $\langle \eta, b_C \rangle \neq 0$. A point $(x, y) \in T^*C^n$ is called $\eta$-semistable if there exists a positive integer $m$ and a polynomial $f \in \mathbb{C}[x, y]$ such that $f$ has $S$-weight $-m\eta$ and $f(p) \neq 0$. Associated with these data, the toric hyper-Kähler manifold $X$ is defined by $X := \mu^{-1}(0)^{\eta-ss}/S$, where the superscript means the subset consisting of $\eta$-semistable points. We also define the Lawrence toric variety by $\mathcal{X} := (T^*C^n)^{\eta-ss}/S$. By the unimodularity assumption, these varieties are smooth and it is known that $\mathcal{X}$ is the universal Poisson deformation of $X$ in the sense of Namikawa, see [30].

For $I \subseteq \mathbb{B}$ and $j \in I^c$, we denote by $C^I_j$ the unique signed circuit contained in $I \cup \{j\}$ and $j \in C^I_{j,+}$ and we set $\beta^I_j := \beta_{C^I_j} \in X_*(S)$. We similarly define $\alpha^I_j \in X^*(H)$ for $I \subseteq \mathbb{B}$ and $i \in I$ using signed cocircuit contained in $I^c \cup \{i\}$. We note the following identity for any $i \in I$ and $j \in I^c$:

$$\langle \alpha^I_i, a_j \rangle = \langle -\beta^I_j, b_i \rangle. \tag{37}$$

We decompose $I = I_+ \cup I_-$ and $I^c = I^c_+ \cup I^c_-$, where we set $I_{\pm} := \{i \in I \mid \pm \langle \xi, \alpha_i \rangle > 0\}$ and $I^c_{\pm} := \{j \in I^c \mid \pm \langle \eta, \beta_j \rangle > 0\}$. We note that these decompositions depend on the choice of $\xi$ and $\eta$, but we omit the dependence from the notation.

**Lemma 5.1.** A point $(x, y) \in T^*C^n$ is $\eta$-semistable if and only if there exists $I \subseteq \mathbb{B}$ such that $x_j \neq 0 \quad (\forall j \in I_+^c)$ and $y_j \neq 0 \quad (\forall j \in I_-^c)$.

**Proof.** Let $p = (x, y)$ be a $\eta$-semistable point and $f \in \mathbb{C}[x, y]$ be a polynomial with $S$-weight $-m\eta$ and $f(p) \neq 0$ for some $m \in \mathbb{Z}_{>0}$. We may assume that $f = \sum x_i y_i^{a_i}$ is a monomial. Then we have $\sum_i (m_i - n_i) b_i = m\eta$ and Lemma 5.2 implies that there exists $I \subseteq \mathbb{B}$ such that $\pm (m_j - n_j) > 0$ for any $j \in I_{\pm}^c$. This implies that $x_j \neq 0$ for any $j \in I_+^c$ and $y_j \neq 0$ for any $j \in I_-^c$. Conversely, if $p \in T^*C^n$ satisfies the latter condition, then $f = \prod_{j \in I^c_+} x_j^{a_j} \prod_{j \in I^-} y_j^{-a_j}$ has $S$-weight $-\eta$ and $f(p) \neq 0$. \qed

**Lemma 5.2.** If we have $\sum m_i b_i = \eta$ for some $m_i \in \mathbb{R}$, then there exists an $I \subseteq \mathbb{B}$ such that $\pm m_j > 0$ for any $j \in I_{\pm}^c$.

**Proof.** We set $I_0 := \{i \mid m_i = 0\}$. If $\{a_i\}_{i \in I_0}$ is linearly dependent, then there exists a circuit $C \subseteq I_0$ and we obtain $\langle \eta, b_C \rangle = 0$, which contradicts the genericity of $\eta$. Hence $\{a_i\}_{i \in I_0}$ is linearly independent. If $I_0 \subseteq \mathbb{B}$, then we have $m_j = \langle \eta, \beta_j \rangle$ and the statement follows. If $I_0 \not\subseteq \mathbb{B}$, then there exists nonzero $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ such that $\langle \lambda, a_i \rangle = 0$ for any $i \in I_0$. Since we have $\sum (\lambda, a_i) b_i = 0$, we may replace $m_j$ by $m_j' := m_j + t(\lambda, a_i)$ for sufficiently small $t \in \mathbb{R}$ without changing the sign of $m_j$ for $j \in I_0$. By taking a minimal $t$ such that some $m_j'$ becomes 0, we obtain $t' \in \mathbb{R}$ such that $m_j + t' \geq 0$ for any $j$ with $m_j > 0$ and $I_1 := \{i \mid m_i' = 0\} \supseteq I_0$. By continuing this process, we obtain $I \subseteq \mathbb{B}$ such that $I \supset I_0$ and $\pm m_j > 0$ for any $j \in I_{\pm}^c$. \qed

For $I \subseteq \mathbb{B}$, we set $U_I := \{(x, y) \in T^*C^n \mid x_j \neq 0 (\forall j \in I_+), y_j \neq 0 (\forall j \in I_-^c)\}, U_t := \widetilde{U}/S \subset X$, and $U_I := (\mu^{-1}(0) \cap U_t)/S \subset X$. By Lemma 5.1, $\{U_I\}_{I \subseteq \mathbb{B}}$ and $\{U_I\}_{I \subseteq \mathbb{B}}$ are Zariski open affine coverings of $X$ and $X$ respectively. For $i \in I$, we set

$$x^I_i := \prod_{j \in I^c_+} x_j^{a_j} \prod_{j \in I^-} y_j^{-a_j}, \quad y^I_i := \prod_{j \in I^c_+} x_j^{a_j} \prod_{j \in I^-} y_j^{a_j}.$$
These are elements of $\mathbb{C}[U_i] = \mathbb{C}[U_i]^S$. It is easy to see that $\mathbb{C}[U_i] = \mathbb{C}[x_i U_j (j \in I^c), x_i^i, y_i^i (i \in I)]$ and $\mathbb{C}[U_i] = \mathbb{C}[x_i^i, y_i^i (i \in I)]$. In particular, we have $U_i \cong \mathbb{C}^{2d}$ and $\dim X = 2d$. Let $\mu_X : X \to S^*$ be the morphism induced from the moment map $\mu$. This is well-defined since $\mu$ is $S$-invariant. The above description of open coverings gives the following.

**Lemma 5.3.** The morphism $\mu_X$ is flat and $X \cong \mu_X^{-1}(0)$ as schemes.

Since $H$-weights of $x_i^i$ and $y_i^i$ (as functions on $U_i$) are $-\alpha_i^i$ and $\alpha_i^i$, respectively, $U_i$ has a unique $H$-fixed point which is denoted by $p_i$. In particular, we obtain a one-to-one correspondence between $B$ and $X^H$ given by $I \mapsto p_I$.

We consider the action of $S := \mathbb{C}^*$ on $X$ or $X$ induced by $\sigma \cdot (x, y) = (\sigma^{-1}x, \sigma^{-1}y)$, $(\sigma, y) \in T^* \mathbb{C}^n)$. With this $S$-action, it is known that $X$ is a conical symplectic resolution. As in previous sections, we set $T := H \times S$. We denote by $L$ the central fiber of $X \to \text{Spec}(\mathbb{C}[X])$.

We regard $\lambda \in X^*(T)$ as a 1-dimensional representation of $T \times S$ with trivial $S$-action and write the associated $T$-equivariant line bundle on the quotients $X$ or $X$ as $L(\lambda)$ or $\hat{L}(\lambda)$. We note that if $\lambda \in X^*(H)$, then $L(\lambda)$ is a trivial line bundle if we forget $H$-equivariant structure and the $H$-action is given by $\lambda$. This map gives an isomorphism $\text{Pic}(X) \cong X^*(S)$ under our assumption that $b_i \neq 0$. We also fix a splitting $i : X^*(S) \to X^*(T)$ of the natural surjection $X^*(T) \to X^*(S)$ and set $\mathcal{L}(l) := \mathcal{L}(i(l)) \in \text{Pic}(X)$ for any $l \in X^*(S)$. We note that $\mathcal{L}(\eta)$ is an ample line bundle relative to the projective morphism $X \to \text{Spec}(\mathbb{C}[X])$. For any $\lambda \in X^*(H)$, we write $a^\lambda \in K_H(\text{pt})$ the $K$-theory class corresponding to $\lambda$. The following result can be checked easily.

**Lemma 5.4.** For any $\lambda \in X^*(T)$ and $I \subseteq \emptyset$, we have the following identity in $K_T(\pi^i)$:

$$i^*_p \hat{L}(\epsilon^i)^j = v^{\pm 1}.$$ 

For $i \in I$, we have

$$i^*_p \hat{L}(\epsilon^i)^j = v^{-\langle a_i^i, \sum_{j \in I} a_j - \sum_{j \in I} a_j^i, a^j \rangle \cdot a^j}.$$ 

The equivariant $K$-theory class of the tangent bundle $T_X$ of $X$ is given by the following formula.

$$T_X = \sum_{\ell=1}^{n} v^{-1} \hat{L}(\epsilon^i_{\ell}) + \sum_{\ell=1}^{n} v^{-1} \hat{L}(-\epsilon^i_{\ell}) - r \cdot O_X - rv^{-2} \cdot O_X.$$ 

By using Corollary 5.5, we obtain the following corollary of Lemma 5.4.

**Corollary 5.5.** For $j \in I^c$, we have

$$i^*_p \hat{L}(\epsilon^i)^j = v^{\pm 1}.$$ 

The multisets of equivariant roots at $p_I$ is given by $\Phi(p_I) = \{\pm \alpha_{i}^i\}_{i \in I}$ and $\overline{\Phi} = \{\alpha_{C^c} \mid C^c : \text{signed cocircuit}\}$.

**Remark.** We fix $x \in X_{C}(H)$ satisfying $\langle \xi, \alpha_{C^c} \rangle \neq 0$ for any cocircuit $C^c$ and take the chamber $\mathcal{C}$ to be the connected component of $h_{\mathbb{R}} \setminus \cup_{\alpha \in \mathfrak{C}}\{x \in h_{\mathbb{R}} \mid \langle x, \alpha \rangle = 0\}$ containing $\xi$. With respect to this choice of chamber, we obtain

$$N_{\mu_{\mathcal{C}}^i, -} = \sum_{i \in I} v^{-1 - \langle a_i^i, \sum_{j \in I} a_j - \sum_{j \in I} a_j^i \rangle} \cdot a_{-i}^i + \sum_{i \in I} v^{-1 + \langle a_i^i, \sum_{j \in I} a_j - \sum_{j \in I} a_j^i \rangle} \cdot a_{-i}^i.$$ 

For the multisets of Kähler roots, we take $\Psi(p_I) := \{\pm \beta_{i}^i\}_{i \in I}$. We have $\overline{\Psi} = \{\beta_{C^c} \mid C : \text{signed circuit}\}$. It is known (see for example [20]) that the ample cone $\mathfrak{A} \subset \text{Pic}(X) \otimes \mathbb{R} \cong \mathfrak{a}_{\mathbb{R}}$ is given by the connected component of $\mathfrak{a}_{\mathbb{R}} \setminus \cup_{\beta \in \overline{\Psi}} \{x \in \mathfrak{a}_{\mathbb{R}} \mid \langle x, \beta \rangle = 0\}$ containing $\eta$. As in previous sections, $X$ is always equipped with these additional data $(X, \mathcal{C}, \mathfrak{A}, \Phi, \Psi, \mathcal{L})$. 


5.2 Dual pairs

In this section, we give a dual pair \((X^l, \mathcal{C}^l, \mathfrak{A}^l, \Phi^l, \Psi^l, \mathfrak{L}^l)\) for \((X, \mathcal{C}, \mathfrak{A}, \Phi, \Psi, \mathfrak{L})\). For \(X^l\), this is given by a symplectic dual of \((X, \mathcal{C})\) in the sense of Braden-Licata-Proudfoot-Webster [3]. I.e., \(X^l\) is the toric hyper-Kähler manifolds defined by the exact sequence of algebraic tori which is dual to (34):

\[
1 \to H^\vee \to T^\vee \to S^\vee \to 1.
\]  

(40)

Here, the GIT parameter \(\eta^l \in \mathbb{X}^* (H^\vee) \cong \mathbb{X}_* (H)\) for \(X^l\) is taken to be \(\xi\). We note that the exact sequence of cocharacter lattices associated with (40)

\[
0 \to \mathbb{X}_*(H^\vee) \xrightarrow{\delta^l} \mathbb{X}_*(T^\vee) \xrightarrow{\delta^l} \mathbb{X}_*(S^\vee) \to 0.
\]

is naturally isomorphic to the exact sequence (36) and the exact sequence of character lattices

\[
0 \to \mathbb{X}^*(S^\vee) \xrightarrow{\delta^l} \mathbb{X}^*(T^\vee) \xrightarrow{\delta^l} \mathbb{X}^*(H^\vee) \to 0.
\]

is isomorphic to (35). In particular, the roles of \(a_i\) and \(b_i\) are exchanged. Hence the set parametrizing the \(H^l := S^\vee\)-fixed points of \(X^l\) is given by

\[
\mathbb{B}^l := \{ J \subset \{1, \ldots, n\} \mid \{b_j\}_{j \in J} \text{ is a basis of } \mathbb{X}^* (S) \}.
\]

The map \(I \mapsto I^c\) gives a natural bijection \(\mathbb{B} \cong \mathbb{B}^l\) and hence gives a bijection \(X^H \cong (X^l)^{H^l}\). We denote by \(p^l_I\) the fixed point of \(X^l\) corresponding to \(I \in \mathbb{B}^l\) for any \(I \in \mathbb{B}\). Under the natural identification \(\mathbb{X}^* (H^l) \cong \mathbb{X}_* (S),\) we obtain \(\Phi^l(p^l_I) = \{ \pm \beta^l_j \}_{j \in I^c}\). For the chamber \(\mathcal{C}^l \subset \mathfrak{h}^l_\mathbb{R}\), we take \(\mathcal{C}^l := \mathfrak{A}\). For the Kähler roots, we take \(\Psi^l(p^l_I) := \{ \pm \alpha^l_i \}_{i \in I}^\vee\). By our choice of GIT parameter, the ample cone \(\mathfrak{A}^l \subset \mathfrak{h}^l_\mathbb{R}\) is given by \(\mathcal{C}\). Therefore, the second condition of Definition 3.2 is satisfied. The order reversing property of the bijection \(X^H \cong (X^l)^{H^l}\) will be checked in the next section, see Corollary 5.10.

As in the case of \(X\), we have a natural map \(\mathfrak{L}^l : \mathbb{X}^*(T^\vee) \to \text{Pic}^T (X^l)\). In order to define the lift \(\mathfrak{L}^l\), we need to take a splitting \(\iota^l : \text{Pic}^T (X^l) \cong \mathbb{X}_* (H) \to \mathbb{X}_* (T)\) which is compatible with the splitting \(\iota : \mathbb{X}^* (S) \to \mathbb{X}^* (T)\). Here, the compatibility means that for any \(\lambda \in \mathbb{X}^* (S)\) and \(\lambda^l \in \mathbb{X}_* (H)\), we have

\[
\langle \iota (\lambda), \iota^l (\lambda^l) \rangle = 0.
\]

(41)

Existence of such a splitting is clear. We define \(\mathfrak{L}^l : \mathbb{X}_* (H) \to \text{Pic}^T (X^l)\) by \(\mathfrak{L}^l (\lambda^l) = \mathfrak{L}^l (\iota (\lambda))\).

**Proposition 5.6.** The pair \((X, \mathcal{C}, \mathfrak{A}, \Phi, \Psi, \mathfrak{L})\) and \((X^l, \mathcal{C}^l, \mathfrak{A}^l, \Phi^l, \Psi^l, \mathfrak{L}^l)\) forms a dual pair in the sense of Definition 3.2.

**Proof.** We need to check (8), (9), (10), and (11) in our situation. By (39), we obtain

\[
\det N_{p^l_I} = v^{-d + \sum_{i \in I^+} a_i - \sum_{i \in I_-} a_i} \cdot v^\sum_{i \in I^+} a_i - \sum_{i \in I_-} a_i.
\]

In particular, we have

\[
\text{wt}_S \det N_{p^l_I} + \frac{\dim X}{2} = \left( \sum_{i \in I^+} a_i - \sum_{i \in I_-} a_i \right) + \left( \sum_{j \in I^+} b_j - \sum_{j \in I_-} b_j \right).
\]

Similarly, we obtain

\[
\text{wt}_S \det N'_{p^l_I} + \frac{\dim X^l}{2} = \left( \sum_{i \in I^+} a_i - \sum_{i \in I_-} a_i \right) + \left( \sum_{j \in I^+} b_j - \sum_{j \in I_-} b_j \right).
\]

Hence the equation (11) follows from (37).
Since we have
\[ \text{wt}_H \det N'_{p_{i^-}} = -\sum_{j \in I^+} \beta_j^i + \sum_{j \in I^-} \beta_j^i, \]
the equation (9) follows from Lemma 5.4. The equation (10) can be proved similarly. Now we check for any \( \lambda \in \mathbb{X}^* \), \( \lambda' \in \mathbb{X}_*(H) \), and \( I \in \mathbb{B} \). Since \( \{a_i\}_{i \in I} \) is a basis of \( \mathbb{X}_*(H) \), it suffices to check (42) for any \( \lambda' = a_i, \ i \in I \). Since we have \( \iota(\lambda) - \sum_{j \in I^+}(\lambda, \beta_j^i)\varepsilon_j^i \in \mathbb{X}^*(H) \), we can calculate LHS of (42) by using any lift of \( a_i \) to \( \mathbb{X}_*(T) \). Therefore, we obtain
\[ \langle \iota(\lambda) - \sum_{j \in I^+}(\lambda, \beta_j^i)\varepsilon_j^i, a_i \rangle = \langle \iota(\lambda) - \sum_{j \in I^+}(\lambda, \beta_j^i)\varepsilon_j^i, \varepsilon_i \rangle = \langle \iota(\lambda), \varepsilon_i \rangle. \]
On the other hand, we may replace \( \lambda \) by \( \iota(\lambda) \) in the RHS of (42) since \( \iota'(\lambda') - \sum_{i \in I^+}(\lambda', \alpha_i^J)\varepsilon_i \in \mathbb{X}_*(S) \). Therefore, we obtain
\[ -\langle \iota'(a_i) - \sum_{i \in I^+}(a_i, \alpha_i^J)\varepsilon_i, \lambda \rangle = -\langle \iota'(a_i), \iota(\lambda) \rangle + \langle \varepsilon_i, \iota(\lambda) \rangle. \]
Hence the equation (42) follows from (41). This proves (12). The proof of (13) is similar.

Proposition 5.6 implies that a maximal flop \( X_{\text{flop}} \) in the sense of section 4.2 is obtained in the same way as \( X \) by replacing \( \eta \) by \(-\eta\). Since this exchanges \( I^+_s \) and \( I^-_s \), the formula (38) implies Conjecture 4.5.

**Corollary 5.7.** Conjecture 4.5 holds for toric hyper-Kähler manifolds.

### 5.3 K-theoretic standard bases

In this section, we recall the description of K-theoretic stable bases for toric hyper-Kähler manifolds. In this paper, we always take the following polarization for the toric hyper-Kähler manifold \( X \):
\[ T^{1/2} := \sum_{i=1}^n v^{-1}L(\varepsilon_i^+) - r \cdot O_X. \]  
In particular, we have \( \det T^{1/2} = v^{-n} L(\varepsilon_1^+ + \cdots + \varepsilon_n^+) \) and hence we obtain \( w(\det T^{1/2}) = -n \). Therefore, Assumption 3.11 is satisfied. Since we will not use other polarization below, we will omit \( T^{1/2} \) from the notations. We set
\[ s_{\text{reg}}^* := \{ x \in s^*_\mathbb{R} \mid (x, \beta_C) \notin \mathbb{Z}, \forall C: \text{ circuit} \} \]
and we take a slope parameter \( s \in s_{\text{reg}}^* \). As in section 3.2, we consider the fractional line bundle \( L(s) \). By Lemma 5.4 we obtain
\[ \text{wt}_H \iota_{p_i}^* L(s) - \text{wt}_H \iota_{p_i}^* L(s) = \sum_{i \in I^+} \langle s, \beta_i^J \rangle \varepsilon_i^* - \sum_{j \in I^-} \langle s, \beta_j^J \rangle \varepsilon_j^* = - \sum_{j \in I^+ \cap J} \langle s, \beta_j^J \rangle \alpha_j^J \]  
for any \( I, J \in \mathbb{B} \). Here, we have used \( \beta_i^J = \sum_{j \in I^+}(\beta_i, \beta_j)\beta_j^J \) and \( \alpha_j^J = \varepsilon_j^* - \sum_{i \in I^+}(\beta_i^J, b_j)\varepsilon_i^* \) in the second equality. In particular, this is not contained in \( \mathbb{X}_*(H) \) if \( I \neq J \). This implies the first part of Assumption 3.6 and hence the uniqueness of K-theoretic stable bases. The existence of K-theoretic stable bases is proved in Proposition 5.11.

We note that the coordinate function \( x_i \) (resp. \( y_i \)) can be considered as a section of \( v^{-1}L(\varepsilon_i^+) \) (resp. \( v^{-1}L(-\varepsilon_i^+) \)) on \( X \). For \( I \in \mathbb{B} \), let \( L_I \) be the subvariety of \( X \) defined by the equations \( x_i = 0 \) (\( i \in I^- \)) and \( y_i = 0 \) (\( i \in I^+ \)). One can check that these defining equations form a regular sequence and the Koszul resolution gives the following.
Lemma 5.8. Let \( V_I := \bigoplus_{i \in I^-} v^{-1} L(e_i^+) \oplus \bigoplus_{i \in I^+} v^{-1} L(-e_i^+) \) be a vector bundle on \( X \). We have the following exact sequence

\[
0 \to \bigwedge^2 V_I' \to \cdots \to \bigwedge^2 V_I' \to V_I' \to \mathcal{O}_X \to \mathcal{O}_{L_I} \to 0.
\]

In particular, we have

\[
\mathcal{O}_{L_I} = \prod_{i \in I^-} (1 - v^2 \mathcal{L}(e_i^+)) \prod_{i \in I^+} (1 - v \mathcal{L}(e_i^+))
\]

in the equivariant \( K \)-theory of \( X \). Moreover, one can easily check the following.

Lemma 5.9. For any \( I \in \mathbb{B} \), we have \( L_I = \overline{\text{Attr}_E(p_I)} \). In particular, \( p_J \in \overline{\text{Attr}_E(p_I)} \) is equivalent to \( I^c_+ \cap J^c_+ = I^- \cap J^- = \emptyset \) for any \( I, J \in \mathbb{B} \).

Corollary 5.10. For any \( I, J \in \mathbb{B} \), \( p_J \preceq e \) \( p_I \) is equivalent to \( p_I^+ \preceq e \) \( p_J^+ \).

Proof. Lemma 5.9 implies that \( p_J \in \overline{\text{Attr}_E(p_I)} \) is equivalent to \( p_J^+ \in \overline{\text{Attr}_E(p_I^+)} \).

Now we give an explicit formula for the \( K \)-theoretic stable bases for \( X \). This is an explicit version of Exercise 9.1.15 in [22].

Proposition 5.11. For any \( I \in \mathbb{B} \), we have

\[
\text{Stab}_{E, s}^K(p_I) = v \sum_{i \in I^c_+} \mathcal{L}(e_i^+) \cdot \mathcal{L} \left( -\sum_{j \in I^c_+} \mathcal{L}(e_j^+) \right) \otimes \mathcal{O}_{L_I^+}. 
\]

Proof. Let us denote by \( \text{Stab}_I \) the RHS of (46). We check the three conditions in Definition 3.5 for \( \text{Stab}_I \). Since we have \( \text{Supp}(\text{Stab}_I) = L_I = \overline{\text{Attr}_E(p_I)} \) by Lemma 5.9, the first condition in Definition 3.5 is satisfied.

We note that by Corollary 5.10, we have

\[
T_I^{1/2} = \sum_{i \in I} v^{-1-\langle \alpha_i^+, \sum_{j \in I^c_+} \alpha_j \rangle - \sum_{j \in I} \alpha_j} \cdot a_i^+ + |I^c_+|(v^2-1).
\]

Hence by (39) and Corollary 5.5, we obtain

\[
\sqrt{\det N_{p_I}^-} = v^{I^c_+} \prod_{i \in I^c_+} v^{-\langle \alpha_i^+, \sum_{j \in I^c_+} \alpha_j \rangle - \sum_{j \in I^c_+} \alpha_j} \cdot a_i^{-\alpha_i^+} = v^{I^c_+} \cdot i^*_p \mathcal{O}_{L_I}.
\]

On the other hand, we have \( \Lambda^+_v(N_{p_I}^+) = i^*_p \mathcal{O}_{L_I} \) by (45). Therefore, the second condition in Definition 3.5 follows from Corollary 5.5.

Finally, we check the third condition of Definition 3.5. For \( I, J \in \mathbb{B} \), let us assume that \( i^*_p \text{Stab}_J \neq 0 \). By Lemma 5.9, we have \( I^c_+ \cap J^c_+ = I^- \cap J^- = \emptyset \). Hence up to the factor of \( v \) and sign, we obtain

\[
i^*_p \text{Stab}_J = \pm v^{\langle 1 - v^2 \rangle I^c_+ \cap J^c_+} \prod_{j \in I^c_+ \cap J^c_+} a_j^{\langle \alpha_j \rangle} \prod_{i \in I^c_+ \cap J^c} \left( 1 - v^{\langle 1 - 1 \rangle \langle \alpha_i \rangle - \sum_{j \in I^c} \alpha_j} \right) \cdot a_i^{-\alpha_i^+}.
\]

By (44), we obtain

\[
\deg_H \left( i^*_p \text{Stab}_J \right) = \deg_H \left( \sum_{j \in I^c \cap J} \langle \langle \alpha_j \rangle - \langle \beta_j \rangle \rangle \cdot \alpha_j^+ \right) = \sum_{j \in I^c \cap J} \deg_H(1 - a_j^+).
\]

Here, the sum means the Minkowski sum. On the other hand, we have

\[
\deg_H \left( i^*_p \text{Stab}_J \right) = \deg_H(1 - a_j^+) \quad \text{for each} \quad j \in I^c \cap J.
\]

Therefore, the third condition in Definition 3.5 follows from \( \langle \langle \alpha_j \rangle - \langle \beta_j \rangle \rangle \cdot \alpha_j^+ \in \deg_H(1 - a_j^+) \) for each \( j \in I^c \cap J \).
We next determine the $K$-theoretic standard bases. We note that in the notation of section 3.1, we have $\Psi_+(p_I) = \{\beta^I_j\}_{j \in I^+_s} \cup \{-\beta^I_j\}_{j \in I^-_s}$. Since we have $w(\det T^{1/2}) = -n = -r - d$, we obtain
\[
ap_I(s) = \sum_{j \in I^+_s} [\langle s, \beta^I_j \rangle] - \sum_{j \in I^-_s} [\langle s, -\beta^I_j \rangle] - d.
\]
Hence we obtain the following corollary of Proposition 5.11.

**Corollary 5.12.** For any $I \in \mathcal{B}$, the standard basis $\mathcal{S}_{\epsilon, s}(p_I)$ is given by
\[
\mathcal{S}_{\epsilon, s}(p_I) = (-v)^{-d} \cdot \mathcal{L} \left( - \sum_{i \in I^+_s} \epsilon_i^+ + \sum_{j \in I^-_s} [\langle s, \beta^I_j \rangle] \epsilon_j^+ \right) \otimes \mathcal{O}_{L_I}.
\]
By using (45), we also obtain
\[
\mathcal{S}_{\epsilon, s}(p_I) = \sum_{K \subseteq I} (-v)^{-|K|} \cdot \mathcal{L} \left( - \sum_{i \in I^+_s \cap K} \epsilon_i^+ - \sum_{i \in I^-_s \cap K} \epsilon_i^+ + \sum_{j \in I^-_s} [\langle s, \beta^I_j \rangle] \epsilon_j^+ \right).
\]
Here, the sum runs over all subsets of $I$. By exchanging $I_+$ and $I_-$, we also obtain
\[
\mathcal{S}_{-\epsilon, s}(p_I) = \sum_{K \subseteq I} (-v)^{-|K|} \cdot \mathcal{L} \left( - \sum_{i \in I^+_s \cap K} \epsilon_i^+ - \sum_{i \in I^-_s \cap K} \epsilon_i^+ + \sum_{j \in I^-_s} [\langle s, \beta^I_j \rangle] \epsilon_j^+ \right).
\]
We note that these formulas depend only on the Kähler alcove containing the slope $s$.

### 5.4 Alcove model

In this section, we reformulate Corollary 5.12 by using certain combinatorics of alcoves in $h^*_R$ and determine the $K$-theoretic canonical bases for toric hyper-Kähler manifolds.

We write $\iota(s) = (s_1, \ldots, s_n) \in X^*(T) \otimes \mathbb{Z} \cong \mathbb{R}^n$. Using this data, we consider a periodic hyperplane arrangement in $h^*_R$ defined by $H^*_i = H_{i,m} := \{ x \in h^*_R \mid \langle x, \alpha_i \rangle + s_i = m \}$ for any $i = 1, \ldots, n$ and $m \in \mathbb{Z}$. We note that if we use a different choice of the lift $\iota$, then the resulting hyperplane arrangement is given by a translation of the original one. We also remark that by the condition $s \in \mathfrak{a}^{\text{reg}}$, we have $\cap_{i \in C} H_{i,m} = \emptyset$ for any circuit $C$ and any choice of $m \in \mathbb{Z}$. We denote by $\text{Alc}_s$ the set of connected components of $h^*_R \setminus \cup_{i,m} H_{i,m}$. We simply call an element of $\text{Alc}_s$ alcove.

By using the data $\iota$, one can give a bijection between $\text{Alc}_s$ and $\mathcal{F}$, where $\mathcal{F}$ is defined in (14). For any $A \in \text{Alc}_s$, the closure $\overline{A}$ is a polytope and the linear form $\xi : h^*_R \to \mathbb{R}$ restricted to $\overline{A}$ takes its minimum at a vertex $x_A$ of the genericity of $\xi$. We set $H_{i,m}^s = H_{i,m}^s := \{ x \in h^*_R \mid \langle x, \alpha_i \rangle + s_i = m > 0 \}$. If we write $\{x_A\} = \cap_{i \in I} H_{i,m}$, then we have $I \in \mathcal{B}$ and $A \subseteq \cap_{i \in I} H_{i,m}^s \cap \cap_{i \in I} H_{i,m}^s$. We define a map $\varphi_{\epsilon, s} : \text{Alc}_s \to \mathcal{F}$ by $\varphi_{\epsilon, s}(A) := (\sum_{i \in I} m_i \alpha^I_i, p_I)$. We note that this does not depend on the choice of $\xi \in \mathfrak{c}$. It is easy to check that $\varphi_{\epsilon, s}$ is a bijection by the genericity of $\epsilon$. We denote by $\leq_{\epsilon}$ the partial order on $\text{Alc}_s$ induced from the partial order $\leq_{\text{lex}, s}$ on $\mathcal{F}$ via $\varphi_{\epsilon, s}$. By using Lemma 5.13, we obtain
\[
\sum_{i \in I} m_i \alpha^I_i - \text{wt}_H^s i^*_p s = x_A.
\]
This implies the following lemma.

**Lemma 5.13.** For any $A, B \in \text{Alc}_s$, $A \leq_{\epsilon} B$ if and only if $\langle x_A, \xi \rangle \leq \langle x_B, \xi \rangle$ for any $\xi \in \mathfrak{c}$. 

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Lemma 5.14. Following formula expressing Proposition 5.15. For any $S$, we note that by Lemma 5.13, we have $E$ by $K$. Since $x$ for some $x$ we have $E$. Therefore, we obtain $E$. We set $A$ and consider the $T$-equivariant line bundle $E(A) := \mathcal{L}(\mu_A)$. Note that this definition does not depend on the choice of $x \in A$ and also the choice of chamber $C$. If $\varphi_{\mathcal{L},s}(A) = \langle \sum_{i \in I} m_i, \alpha_i^+ \rangle$, then we have $A \subset \bigcap_{i \in I} H_i^+, \cap \bigcap_{i \in I} H_i^−$. Hence, we obtain $E(A) = \mathcal{L} \left( - \sum_{i \in I−} \epsilon_i^+ + \sum_{j \in I+} \langle \delta_j^+, \beta_j \rangle \right)$. Since $\langle x, a \rangle + s_i = m_i \in \mathbb{Z}$ for any $i \in I$ and $a_j = - \sum_{i \in I} (\beta_j^+)(b_i)a_i$ for any $j \in I'$, we have $\langle x, a \rangle + s_j = - \sum_{i \in I} (\beta_j^+)(b_i)a_i + s_j$ $= - \sum_{i \in I} m_i(\beta_j^+)(b_i) + s_j + \sum_{i \in I} s_i(\beta_j^+)(b_i)$ $= \sum_{i \in I} m_i(\alpha_i^+)(a_j) + \langle s, \beta_j^+ \rangle$. Therefore, we obtain $E(A) = \prod_{i \in I} a^m_i \alpha_i^+ \cdot \mathcal{L} \left( - \sum_{i \in I−} \epsilon_i^+ + \sum_{j \in I+} \langle \delta_j^+, \beta_j \rangle \epsilon_j^+ \right)$. (51)

We set $E_{\varphi_s}(p_I) := \mathcal{L} \left( - \sum_{i \in I−} \epsilon_i^+ + \sum_{j \in I'} \langle s, \beta_j^+ \rangle \epsilon_j^+ \right)$. Moreover, if $B \in \text{Alc}_s$ is an alcove such that $x_\mathcal{B} \in \mathbb{B}$ and $K \subset I$ is a subset such that $\{ \mathcal{H}_{k,m} \}_{k \in K}$ is the set of hyperplanes separating $A$ and $B$, then we have $E(B) = \prod_{i \in I} a^m_i \alpha_i^+ \cdot \mathcal{L} \left( - \sum_{i \in I−} \epsilon_i^+ - \sum_{i \in I− \cap K^c} \epsilon_i^+ + \sum_{j \in I'} \langle s, \beta_j^+ \rangle \epsilon_j^+ \right)$. (52)

In particular, the RHS of (52) is of the form $E_{\varphi_s}(p_I')$ for some $I' \in \mathbb{B}$ up to some $H$-equivariant parameter shift.

For any $A, B \in \text{Alc}_s$, we write $\ell(A, B)$ the number of hyperplanes separating $A$ and $B$. We set $N(A) := \{ B \in \text{Alc}_s \mid x_\mathcal{B} \in \mathbb{B} \}$ and define $S(A) \in K_T(X)$ by the following formula:

$S(A) := \sum_{B \in N(A)} (-v)^{-\ell(A, B)} E(B)$.

(53)

We note that by Lemma 5.13, we have $B \leq e A$ for any $B \in N(A)$. By (47) and (52), we obtain the following formula expressing $S_{\varphi_s}(\varphi_{\mathcal{L},s}(A))$ defined in Definition 3.12.

Lemma 5.14. For any $A \in \text{Alc}_s$, we have $S(A) = S_{\varphi_s}(\varphi_{\mathcal{L},s}(A))$.

We now prove $E(A) = E_{\varphi_s}(\varphi_{\mathcal{L},s}(A))$ in the notation of Conjecture 3.22. By (53), we obtain $E(A) \in S(A) + \sum_{B \leq e A} v^{-1} Z^{-1} \cdot S(B)$ (54) under certain completion as in section 3.3. Therefore, it is enough to check the following.

Proposition 5.15. For any $A \in \text{Alc}_s$, we have $\beta_{\mathcal{L},s}(E(A)) = E(A)$.

Proof. Since $\{ S_{\varphi_s}(p_I) \}_{I \in \mathbb{B}}$ is a basis of $K_T(X)_{\text{loc}}$ over $\text{Frac}(K_T(\mathcal{P}))$, $\{ E_{\varphi_s}(p_I) \}_{I \in \mathbb{B}}$ is also a basis of $K_T(X)_{\text{loc}}$ over $\text{Frac}(K_T(\mathcal{P}))$ by (51) and (53). We define $\text{Frac}(K_R(\mathcal{P}))$-linear involution $\beta'$ on $K_T(X)_{\text{loc}}$ by
\begin{itemize}
\item $\beta'(vm) = v^{-1} \beta'(m)$ for any $m \in K_{\mathbb{Z}}(X)_{\text{loc}}$.
\item $\beta'(\mathcal{E}_{\epsilon,s}(p_I)) = \mathcal{E}_{\epsilon,s}(p_I)$ for any $I \in \mathcal{B}$.
\end{itemize}

By (51), we have $\beta'(\mathcal{E}(A)) = \mathcal{E}(A)$ for any $A \in \text{Alc}_s$ and hence

\[ \beta'(\mathcal{S}_{\epsilon,s}(p_I)) = \sum_{\mathcal{K} \subset I} (-v)^{|\mathcal{K}|} \mathcal{L} \left( - \sum_{i \in \mathcal{L} \cap \mathcal{K}} v_i - \sum_{i \in \mathcal{I}_n \cap \mathcal{K}} \epsilon_i^* + \sum_{j \in \mathcal{I}_p} (\langle s, \beta_j^* \rangle \epsilon_j^* \right) \]

\[ = (-v)^d \sum_{\mathcal{K} \subset I} (-v)^{|\mathcal{K}|} \mathcal{L} \left( - \sum_{i \in \mathcal{L} \cap \mathcal{K}} v_i - \sum_{i \in \mathcal{I}_n \cap \mathcal{K}} \epsilon_i^* + \sum_{j \in \mathcal{I}_p} (\langle s, \beta_j^* \rangle \epsilon_j^* \right) \]

\[ = (-v)^d S_{-\epsilon,s}(p_I) \]

for any $I \in \mathcal{B}$ by (47), (48), and (52). This implies that $\beta' = \beta_{\mathcal{E},s}$ and hence $\mathcal{E}(A)$ is bar invariant. \hfill \Box

Since the set $\{\mathcal{E}(A)\}_{A \in \text{Alc}_s}$ does not depend on the choice of $\mathcal{C}$, we obtain Conjecture 3.17 for toric hyper-Kähler manifolds.

**Corollary 5.16.** The $K$-theoretic bar involution $\beta_{\mathcal{E},s}^K$ does not depend on the choice of $\mathcal{C}$.

Next, we determine $C_{\phi_{\mathcal{E},s}(A)}$ for any $A \in \text{Alc}_s$. Recall the map $\partial : K_{\mathbb{Z}}(pt) \rightarrow \mathbb{Z}[v, v^{-1}]$ defined in section 3.5. By Lemma 3.24 and Lemma 5.14 we have $\partial(S(A)||S(B)) = \delta_{A,B}$ for any $A, B \in \text{Alc}_s$. By induction, this and (54) implies that for any $A \in \text{Alc}_s$, there exists a unique element

\[ C(A) = \mathcal{S}(A) + \sum_{B \geq A} v^{-1} [v^{-1}] \cdot S(B) \]

such that $\partial(C(A)||\mathcal{E}(B)) = \delta_{A,B}$ for any $A, B \in \text{Alc}_s$. Since we have

\[ \partial(C(A)||S(B)) = \sum_{C \in \mathcal{N}(B)} (-v)^{\ell(C,B)} \partial(C(A)||\mathcal{E}(C)) \]

\[ = \begin{cases} (-v)^{\ell(A,B)} & \text{if } A \in \mathcal{N}(B) \\ 0 & \text{otherwise,} \end{cases} \]

we obtain

\[ C(A) = \sum_{B \in \mathcal{N}^{-}(A)} (-v)^{-\ell(A,B)} S(B) \quad (55) \]

where we set $\mathcal{N}^{-}(A) := \{ B \in \text{Alc}_s \mid A \in \mathcal{N}(B) \}$. We note that $\mathcal{N}^{-}(A)$ is a finite set.

**Lemma 5.17.** For any $A \in \text{Alc}_s$, we have $\beta_{\mathcal{E},s}^K(C(A)) = v^{\dim X} \cdot C(A)$.

**Proof.** By Lemma 3.30 and Proposition 5.15 we have

\[ \partial(v^{-\dim X} \cdot \beta_{\mathcal{E},s}^K(C(A))||\mathcal{E}(B)) = v^{-\dim X} \partial(\beta_{\mathcal{E},s}^K(C(A))||\beta_{\mathcal{E},s}^K(\mathcal{E}(B))) \]

\[ = \partial(C(A)||\mathcal{E}(B)) \]

\[ = \delta_{A,B} \]

for any $A, B \in \text{Alc}_s$. Hence we obtain $\beta_{\mathcal{E},s}^K(C(A)) = v^{\dim X} \cdot C(A)$. \hfill \Box

Following the notation in section 3.4, we set $\mathbb{B}_X,s := \{ \mathcal{E}(A) \}_{A \in \text{Alc}_s}$ and $\mathbb{B}_L,s := \{ C(A) \}_{A \in \text{Alc}_s}$ and call them $K$-theoretic canonical bases for $K_{\mathbb{Z}}(X)$ and $K_{\mathbb{Z}}(L)$. We will prove later (Corollary 5.37) that $\mathbb{B}_X,s$ (resp. $\mathbb{B}_L,s$) is actually a basis of $K_{\mathbb{Z}}(X)$ (resp. $K_{\mathbb{Z}}(L)$).
5.5 Wall-crossings

In this section, we study the behavior of $K$-theoretical canonical bases under the variation of $s \in \mathfrak{s}^\text{reg}$. We fix a signed circuit $C = C_+ \sqcup C_-$ satisfying $(\eta, \beta_C) > 0$ and consider a union of hyperplanes $w_C := \{ x \in \mathfrak{s}^\text{reg} \ | \ \langle x, \beta_C \rangle \in \mathbb{Z} \}$. We take a generic element $s_0 \in w_C$ such that $s_0$ does not lie in any $w_{C'}$ for some circuit $C' \neq C$. We consider two slopes $s_- , s_+ \in \mathfrak{s}^\text{reg}$ sufficiently close to $s_0$ such that they lie in the same connected component of $\mathfrak{s}^\text{reg} \setminus \bigcup_{C \neq C'} w_{C'}$ as $s_0$ and satisfy $(s_- , \beta_C) < (s_0 , \beta_C) < (s_+ , \beta_C)$. We study the difference between $K$-theoretical canonical bases $\mathbb{B}_{X,s_-}$ and $\mathbb{B}_{X,s_+}$. We fix a path $\gamma$ connecting $s_-$ and $s_+$ in a neighborhood of $s_0$ and passing through $s_0$.

For $(\lambda , p_I) \in \mathbb{F}$, we consider the vertex $x_{\lambda,I}(s)$ corresponding to $\varphi_{s_0}^{-1}(\lambda , p_I) \in \text{Alc}_s$ as in the previous section. By [49], we have $x_{\lambda,I}(s) = \lambda - w \eta_I i^*_s \mathcal{Q}(s)$. If $s$ goes to $s_0$, then it can happen that $\lim_{s \to s_0} x_{\lambda,I}(s) = \lim_{s \to s_0} x_{\mu,J}(s)$ for some $(\mu , p_J) \neq (\lambda , p_I) \in \mathbb{F}$. If this does not happen, then hyperplanes other than $\mathcal{H}_{\alpha,m}^i$ for $i \in I$ will be away from $x_{\lambda,I}(s)$ along $s \in \gamma$. Hence we obtain $\mathcal{E}_{s_-}(\lambda , p_I) = \mathcal{E}_{s_+}(\lambda , p_I)$.

**Lemma 5.18.** For $I \neq J \in \mathbb{B}$, $\lim_{s \to s_0} x_{\lambda,I}(s) = \lim_{s \to s_0} x_{\mu,J}(s)$ for some $\lambda , \mu \in \mathbb{X}^*(H)$ if and only if $|I^c \cap C| = |J^c \cap C| = 1$ and $I \cap C^c = J \cap C^c$.

**Proof.** Assume that $\lim_{s \to s_0} x_{\lambda,I}(s) = \lim_{s \to s_0} x_{\mu,J}(s)$ for some $(\mu , p_J) \neq (\lambda , p_I) \in \mathbb{F}$. By [44], this implies that $\sum_{j \in I \cap J} (s_0 , \beta_j^I) i^*_j \mathcal{Q}(s) \in \mathbb{X}^*(H)$ and hence $\langle s_0 , \beta_j^I \rangle \in \mathbb{Z}$ for any $j \in I \cap J$. By the choice of $s_0$, we must have $|I^c \cap J| = 1$ and $\beta_j^I = \pm \beta_C$ for $j \in I^c \cap J$. In particular, we have $J \cap C \subset I \cup \{ j \}$ and hence we obtain $|I \cap C| = 1$. By exchanging the role of $I$ and $J$, we also obtain $I \cap J^c = \{ i \}$ for some $i$ and $I \subset C \subset J \cap \{ i \}$. This implies that $I \cap C^c = (I \setminus \{ i \}) \cap C^c = (J \setminus \{ j \}) \cap C^c = J \cap C^c$.

Conversely, we assume that $I \neq J \in \mathbb{B}$ satisfy $|I^c \cap C| = |J^c \cap C| = 1$ and $I \cap C^c = J \cap C^c = K$. If we set $I^c \cap C = \{ j \}$ and $J^c \cap C = \{ i \}$, then we obtain $I = (C \setminus \{ j \}) \cup K$ and $J = (C \setminus \{ i \}) \cup K$ and hence $I \cap J^c = \{ i \}$ and $I \cap J^c = \{ j \}$. Since we have $j \in C \subset I \cup \{ j \}$ and $i \in C \subset J \cup \{ i \}$, we obtain $\beta_j^I = \pm \beta_C$ and $\beta_i^J = \pm \beta_C$. By [44], we obtain $\lim_{s \to s_0} x_{\lambda,I}(s) = \lim_{s \to s_0} x_{\mu,J}(s)$ for some $\lambda , \mu \in \mathbb{X}^*(H)$.

Now we study the behavior of $\mathcal{E}_{s}(\lambda , p_I)$ under the wall-crossing, where $I$ satisfies $|I^c \cap C| = 1$ and $I \cap C^c = K$ for a fixed subset $K \subset C^c$. We may assume that $\{ a_i \}_{i \in K}$ is linearly independent and $\{ a_i \}_{i \in C \cup K}$ spans $X_s(H)$. In this case, the number of such $I \in \mathbb{B}$ is given by $|C|$. We fix $m_i \in \mathbb{Z}$ for each $i \in C \cup K$ such that $\cap_{i \in C \cup K} \mathcal{H}_{a,m}^i := \{ s_0 \}$ is not empty. We note that this implies that $\sum_{i \in C} m_i - \sum_{i \in C^c} m_i = \langle s_0 , \beta_C \rangle$. We consider all $(\lambda , p_I) \in \mathbb{F}$ such that $\lim_{s \to s_0} x_{\lambda,I}(s) = x_0$. The number of such $(\lambda , p_I) \in \mathbb{F}$ is also given by $|C|$. By choosing $s_- , s_-$ sufficiently close to $s_0$, we may assume that there exists a convex neighborhood $U$ of $x_0$ in $\mathfrak{h}_C^\text{reg}$ containing all $x_{\lambda,I}(s)$ such that any hyperplane of the form $\mathcal{H}_{\alpha,m}^i$ does not intersect $U$ unless $i \in C \cup K$ and $m = m_i$ for any $s \in \gamma$. It is enough to consider the alcoves which intersect with $U$. We note that such an alcove $A$ is characterized by the sign $\epsilon : C \cup K \to \{ \pm \}$ such that $A \subset \cap_{i \in C \cup K} \mathcal{H}_{a,m}^i(\epsilon(i))$.

Let $\mathfrak{h}_C$ be the $\mathbb{R}$-span of $\{ a_i \}_{i \in C}$ and $\mathfrak{h}_K$ be the $\mathbb{R}$-span of $\{ a_i \}_{i \in K}$. By the choice of $K$, we obtain a decomposition $\mathfrak{h}_C \cong \mathfrak{h}_C \times \mathfrak{h}_K$ and this induces a decomposition $\mathfrak{h}_C^\text{reg} \cong \mathfrak{h}_C^\text{reg} \times \mathfrak{h}_K^\text{reg}$ such that $\langle \mathfrak{h}_C, \mathfrak{h}_K \rangle = 0$ and $\langle \mathfrak{h}_C, \mathfrak{h}_C \rangle = 0$. We may also assume that $U \cong U_C \times U_K$ for some convex open subsets $U_C \subset \mathfrak{h}_C^\text{reg}$ and $U_K \subset \mathfrak{h}_K^\text{reg}$. Since we have $\mathcal{H}_{a,m}^i = (\mathcal{H}_{a,m}^i \cap \mathfrak{h}_C) \times \mathfrak{h}_K$ for each $i \in C$ and $\mathcal{H}_{a,m}^i = \mathfrak{h}_C \times (\mathcal{H}_{a,m}^i \cap \mathfrak{h}_K)$ for each $i \in K$, they induce hyperplane arrangements on $\mathfrak{h}_C$ and $\mathfrak{h}_K$. For each alcove $A$ with $A \cap U \neq \emptyset$, there is an alcove $A_C$ in $\mathfrak{h}_C$ and an alcove $A_K$ in $\mathfrak{h}_K$ such that $A \cap U \cong (A_C \cap U_C) \times (A_K \cap U_K)$. In particular, we can consider alcoves for $\mathfrak{h}_C$ and $\mathfrak{h}_K$ separately in $U$.

Since $\{ a_i \}_{i \in K}$ is linearly independent, $\cap_{i \in K} \mathcal{H}_{a,m}^i \cap U_K$ is not empty and its volume is away from 0 along $s \in \gamma$ for any sign $\epsilon : C \to \{ \pm \}$. For each sign $\epsilon : C \to \{ \pm \}$, we set $\Delta_{\epsilon}(s) := \cap_{i \in C} \mathcal{H}_{a,m}^i(\epsilon(i)) \cap \mathfrak{h}_C^\text{reg}$. We define $\epsilon_{\pm} : C \to \{ \pm \}$ by $\epsilon(i) = \pm$ for any $i \in C_+$ and $\epsilon(i) = \mp$ for any $i \in C_-$.  

**Lemma 5.19.** If $\pm (s - s_0 , \beta_C) > 0$, then $\Delta_{\epsilon}(s) \neq \emptyset$ unless $\epsilon = \epsilon_{\pm}$. Moreover, the volume of $\Delta_{\epsilon}(s) \cap U_C$ is away from 0 along $s \in \gamma$ unless $\epsilon = \epsilon_{\pm}$ and the volume of $\Delta_{\epsilon}(s) \cap U_C$ is proportional to $(s - s_0 , \beta_C)|C|^{-1}$. 

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Proof. By definition, \( x \in \Delta_i(s) \) if and only if \( \epsilon(i) \cdot (\langle x, \alpha_i \rangle + s_i - m_i) > 0 \) for any \( i \in C \). Since we have
\[
\sum_{i \in C_+} \langle x, a_i \rangle + s_i - m_i - \sum_{i \in C_-} \langle x, a_i \rangle + s_i - m_i = \langle s - s_0, \beta_C \rangle,
\]
such an \( x \) does not exist for \( \epsilon = \epsilon_+ \). If \( \epsilon \neq \epsilon_+ \) and \( \epsilon \neq \epsilon_- \), then \( \Delta_i(s_0) \) is also not empty and the volume of \( \Delta_i(s_0) \cap U_C \) is positive. This implies the second statement.

If \( \epsilon = \epsilon_\pm \), then \( \Delta_{\epsilon_\pm}(s) \) is a \((|C| - 1)\)-dimensional simplex. By our choice of \( U \), every vertex of \( \Delta_{\epsilon_\pm}(s) \) is contained in \( U_C \) and hence \( \Delta_{\epsilon_\pm}(s) \subset U_C \). Since each edge of \( \Delta_{\epsilon_\pm}(s) \) has length proportional to \(|(s - s_0, \beta_C)|\), its volume is proportional to \(|(s - s_0, \beta_C)||C|^{-1} \).

For each sign \( \epsilon : C \cup K \to \{ \pm \} \) satisfying \( \cap_{i \in C \cup K} \mathcal{H}^{s_{\epsilon(i)}}_{\iota_i m_i} \neq \emptyset \), we denote by \( A_{\epsilon,x_0}(s) \in \text{Alc}_s \) the unique alcove which is contained in \( \cap_{i \in C \cup K} \mathcal{H}^{s_{\epsilon(i)}}_{\iota_i m_i} \) and intersect with \( U \). By definition, we obtain
\[
\mathcal{E}(A_{\epsilon,x_0}(s)) = \mathcal{L} \left( - \sum_{i \in \epsilon^{-1}(-)} \epsilon_i^* \right) \otimes \mathcal{L}_{x_0}
\]
for any \( s \in \gamma \), where we set
\[
\mathcal{L}_{x_0} := \mathcal{L} \left( \sum_{i \in C \cup K} m_i \epsilon_i^* + \sum_{i \in C \cup K} \langle (x_0, a_i) + \iota(s_0)_i \rangle \epsilon_i^* \right).
\]
Lemma 5.19 implies that for any subset \( C' \subset C \) and \( K' \subset K \), \( \mathcal{L}( - \sum_{i \in C \cup K} \epsilon_i^* ) \otimes \mathcal{L}_{x_0} \) is contained in \( \mathbb{B}_{X,s_{\epsilon}} \) if and only if \( C' \neq C \). In particular, \( \mathcal{L}( - \sum_{i \in C \cup K} \epsilon_i^* ) \otimes \mathcal{L}_{x_0} \) is contained in \( \mathbb{B}_{X,s_{\epsilon}} \) but not in \( \mathbb{B}_{X,s_{\epsilon}} \), and \( \mathcal{L}( - \sum_{i \in C \cup K} \epsilon_i^* ) \otimes \mathcal{L}_{x_0} \) is contained in \( \mathbb{B}_{X,s_{\epsilon}} \) but not in \( \mathbb{B}_{X,s_{\epsilon}} \). In summary, we obtained the following.

\textbf{Proposition 5.20.} For any element \( \mathcal{E} \in \mathbb{B}_{X,s_{\epsilon}} \), \( \mathcal{E} \) is not contained in \( \mathbb{B}_{X,s_{\epsilon}} \) if and only if the volume \( \text{Vol}(A(s)) \) of \( A(s) \) vanishes under the limit \( s \to s_0 \), where \( A(s) \in \text{Alc}_s \) is the alcove satisfying \( \mathcal{E}(A(s)) = \mathcal{E} \) for any \( s \in s_{\epsilon,\text{reg}} \) contained in the same Kähler alcove as \( s_0 \). If this holds, then the order of vanishing of \( \text{Vol}(A(s)) \) at \( s = s_0 \) is given by \(|C| - 1 \) and \( \mathcal{L}( \sum_{i \in C_- \cap C} \epsilon_i^* - \sum_{i \in C_+ \cap C} \epsilon_i^* ) \otimes \mathcal{E} \) is contained in \( \mathbb{B}_{X,s_{\epsilon}} \) but not in \( \mathbb{B}_{X,s_{\epsilon}} \). Moreover, \( \mathcal{L}( \sum_{i \in C \cap C_-} \epsilon_i^* - \sum_{i \in C \cap C_+} \epsilon_i^* ) \otimes \mathcal{E} \) is contained in both \( \mathbb{B}_{X,s_{\epsilon}} \) and \( \mathbb{B}_{X,s_{\epsilon}} \) for any subset \( \emptyset \neq C' \subset C \).

Now the wall-crossing formula relating \( \mathbb{B}_{X,s_{\epsilon}} \) and \( \mathbb{B}_{X,s_{\epsilon}} \) follows from the following lemma.

\textbf{Lemma 5.21.} For any signed circuit \( C \) satisfying \( \langle \eta, \beta_C \rangle > 0 \), there exists an exact sequence
\[
0 \to \mathcal{W}_{|C|} \to \cdots \to \mathcal{W}_1 \to \mathcal{W}_0 \to 0
\]
of vector bundles on \( X \), where
\[
\mathcal{W}_k := \bigoplus_{C' \subset C \atop |C'|=k} v^{k} \mathcal{L} \left( \sum_{i \in C' \subset C_-} \epsilon_i^* - \sum_{i \in C' \subset C_+} \epsilon_i^* \right).
\]

\textbf{Proof.} Since we have \( \langle \eta, \beta_C \rangle = \sum_{I \subset C \cup I} \langle \eta, \beta_I^0 \rangle \langle \beta_I, \beta_C \rangle > 0 \), we must have \( C_+ \cap I^\pm \neq \emptyset \) for \( C_\subset I^\pm \neq \emptyset \) for any \( I \in \mathbb{B} \). This implies that the subvariety of \( X \) defined by \( x_i = 0 \) for any \( i \in C_+ \) and \( y_i = 0 \) for any \( i \in C_- \) is empty. By considering the Koszul complex for these equations, we obtain an exact sequence of the form (56). \( \Box \)
In particular, Proposition \[5.20 \] and Lemma \[5.21 \] imply the first part of Conjecture \[3.49 \] by taking \( l = 1, n_0 = 0, n_1 = |C|, B_{X,s_i}^0 = B_{X,s_i} \cap B_{X,s_i}^+ \), and \( B_{X,s_i}^0 = B_{X,s_i} \setminus B_{X,s_i}^0 \). In section 5.11, we will show that the central charge of \( C(A(s)) \) is given by \( \text{Vol}(A(s)) \) for any \( A(s) \in \text{Alc}_s \). This implies the second part of Conjecture \[5.49 \] .

As another application of these results, we obtain the following.

**Corollary 5.22.** For any \( s \in s_{reg}^* \), the vector bundle \( \mathcal{T}_s := \bigoplus_{s \in \mathbb{B}} \mathcal{E}(s)(p_I) \) weakly generate \( D(Q\text{Coh}(X)) \).

**Proof.** We note that any line bundle of the form \( \mathcal{L}(\lambda) \) for \( \lambda \in X^*(T) \) is contained in \( B_{X,s'} \) for some \( s' \in s_{reg}^* \) by Lemma \[3.34 \] . By connecting \( s \) and \( s' \) by a generic path and applying Proposition \[5.20 \] and Lemma \[5.21 \] each time when the path crosses a wall, we obtain that \( \mathcal{L}(\lambda) \) is contained in the full triangulated subcategory of \( D^b(\text{Coh}(X)) \) generated by \( \{ \mathcal{E}(s)(p_I) \}_{I \in \mathbb{B}} \) for any \( \lambda \in X^*(T) \). In particular, \( RHom(\mathcal{T}_s, \mathcal{F}) = 0 \) implies that \( RHom(\mathcal{F} \otimes \mathcal{L}) = 0 \) for any sufficiently ample line bundles \( \mathcal{L} \). This implies \( \mathcal{F} \cong 0 \).

### 5.6 Linear programming

In this section, we collect some results about linear programming which will be used in the proof of Conjecture \[3.38 \] and Conjecture \[3.40 \] for toric hyper-Kähler manifolds. Our reference for the theory of linear programming is \[3 \].

We first prepare some notations about sign vectors. For \( x \in \mathbb{R} \), we set

\[
\sigma(x) := \begin{cases} 
+ & \text{if } x > 0 \\
- & \text{if } x < 0 \\
0 & \text{if } x = 0
\end{cases}
\]

and for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we set \( \sigma(x) := (\sigma(x_1), \ldots, \sigma(x_n)) \in \{+,-,0\}^n \). We will be interested in the sign patterns \( \sigma(V) := \{ \sigma(x) \mid x \in V \} \subset \{+,-,0\}^n \) for a vector subspace \( V \subset \mathbb{R}^n \).

We set \( E := \{1, \ldots, n\} \). For a sign vector \( Y \in \{+,-,0\}^E \), we define \( Y^+ := \{ i \in E \mid Y_i = + \} \), \( Y^- := \{ i \in E \mid Y_i = - \} \), and \( Y^0 := \{ i \in E \mid Y_i = 0 \} \). We define its support by \( \text{Supp}(Y) := Y^+ \cup Y^- \).

For a subset \( I \subseteq E \), we write \( Y_I \geq 0 \) if \( Y_i \geq 0 \) for any \( i \in I \), \( Y_I \leq 0 \) if \( Y_i \leq 0 \) for any \( i \in I \), and \( Y_I = 0 \) if \( Y_i = 0 \) for any \( i \in I \).

**Definition 5.23.** Two sign vectors \( Y, Z \in \{+,-,0\}^E \) are called **orthogonal** if

\[
(Y^+ \cap Z^+) \cup (Y^- \cap Z^-) \neq \emptyset \Leftrightarrow (Y^+ \cap Z^-) \cup (Y^- \cap Z^+) \neq \emptyset.
\]

This is denoted by \( Y \perp Z \). For a subset \( F \subseteq \{+,-,0\}^E \), the set

\[
F^\perp := \{ Y \in \{+,-,0\}^E \mid Y \perp Z \text{ for any } Z \in F \}
\]

is called the **orthogonal complement** of \( F \).

For a vector subspace \( V \subset \mathbb{R}^n \), we denote by \( V^\perp \subset \mathbb{R}^n \) the orthogonal complement of \( V \subset \mathbb{R}^n \) with respect to the standard inner product on \( \mathbb{R}^n \).

**Proposition 5.24.** For any vector subspace \( V \subset \mathbb{R}^n \), we have \( \sigma(V)^\perp = \sigma(V^\perp) \).

**Proof.** See Corollary 5.42 in \[3 \].

**Definition 5.25.** For a subset \( F \subset \{+,-,0\}^E \) and disjoint subsets \( I, J \subset E \), we set

\[
F \setminus (I/J) := \{ Y \in \{+,-,0\}^E \setminus (I/J) \mid \exists Z \in F \text{ s.t. } Z_i = 0 \text{ for } i \in I \text{ and } Z_e = Y_e \text{ for } e \in E \setminus (I \cup J) \}.
\]

This is called the **minor** of \( F \) obtained by deleting \( I \) and contracting \( J \).
Lemma 5.26. For any vector subspace $V \subset \mathbb{R}^n$ and disjoint subsets $I, J \subset E$, we have

$$\langle \sigma(V) \setminus I/J \rangle^\perp = \sigma(V)^\perp \setminus J/I.$$  

Proof. This follows from Proposition 5.24 See also Lemma 5.51 and Lemma 5.52 in [3].

Definition 5.27. For a subset $\mathcal{F} \subset \{+,-,0\}^E$, nonzero $Y \in \mathcal{F}$ is called elementary sign vector of $\mathcal{F}$ if $0 \neq \text{Supp}(Z) \subset \text{Supp}(Y)$ implies $\text{Supp}(Z) = \text{Supp}(Y)$ for any $Z \in \mathcal{F}$. The set of all elementary sign vectors of $\mathcal{F}$ is denoted by $\text{elem}(\mathcal{F})$.

Proposition 5.28. For any vector subspace $V \subset \mathbb{R}^n$, we have $\text{elem}(\sigma(V))^\perp = \sigma(V)^\perp$.

Proof. See Corollary 5.37 in [3].

Definition 5.29. For $Y, Z \in \{+,-,0\}^E$, we write $Y \preceq Z$ and say that $Y$ conforms to $Z$ if $Y^+ \subset Z^+$ and $Y^- \subset Z^-$. This relation defines a partial order $\preceq$ on $\{+,-,0\}^E$.

Lemma 5.30. For any vector subspace $V \subset \mathbb{R}^n$, $\text{elem}(\sigma(V))$ coincides with the set of minimal nonzero elements of $\sigma(V)$ with respect to the partial order $\preceq$.

Proof. See Lemma 5.30 in [3].

Proposition 5.31 (Minty’s Lemma). For any vector subspace $V \subset \mathbb{R}^n$ and every partition $E = R \sqcup G \sqcup B \sqcup W$ with $e \in R \cup G$, exactly one of the following holds:

- There exists $Y \in \sigma(V)$ such that $e \in \text{Supp}(Y)$, $Y_R \geq 0$, $Y_G \leq 0$, and $Y_W = 0$.
- There exists $Z \in \sigma(V^\perp)$ such that $e \in \text{Supp}(Z)$, $Z_R \geq 0$, $Z_G \leq 0$, and $Z_B = 0$.

Proof. See Proposition 5.12 in [3].

In the below, we will consider the case $V = \text{Ker}(b) \otimes \mathbb{Z} \subset X^*(T) \otimes \mathbb{Z} \cong \mathbb{R}^n$, where the identification $X^*(T) \otimes \mathbb{Z} \cong \mathbb{R}^n$ is given by the fixed basis $\{e_1^\circ, \ldots, e_n^\circ\}$. We note that the sign vectors $\sigma(V)$ does not change without tensoring $\mathbb{R}$. In this case, we have $V^\perp = \text{Ker}(a) \otimes \mathbb{Z}$ and hence $\text{elem}(\sigma(V^\perp)) = \{\sigma(\beta_C) \mid C = C_+ \cup C_- : \text{signed circuit}\}$.

5.7 Toric varieties

In this section, we recall a description of cohomology of line bundles on toric varieties. Since we only consider semi-projective toric varieties in this paper, we restrict our attention to these cases. Let $0 \leq r \leq N$ be nonnegative integers and consider an exact sequence of tori

$$1 \to T^r \to T^m \to T^{m-r} \to 1,$$

where $T^k := (\mathbb{C}^\times)^k$ for $k \in \mathbb{Z}_{\geq 0}$. Let

$$0 \to X_*(T^r) \xrightarrow{\mathcal{B}} X_*(T^m) \xrightarrow{\mathcal{A}} X_*(T^{m-r}) \to 0$$

be the associated exact sequence of cocharacter and character lattices. We set $a_i := A(e_i)$ and $b_i := B(e_i)$ for a fixed basis $\{e_1 \ldots, e_m\}$ of $X_*(T^m)$ and its dual basis $\{e_1^\circ, \ldots, e_m^\circ\}$ of $X^*(T^m)$. We fix a generic element $\eta \in \sum_{i=1}^m \mathbb{Z}_{\geq 0} b_i$ such that if $\eta$ is contained in a cone of the form $\mathbb{R}_{\geq 0} b_{i_1} + \cdots + \mathbb{R}_{\geq 0} b_{i_l}$ for $\{i_1, \ldots, i_l\} \subset \{1, \ldots, m\}$, then $\{b_{i_1}, \ldots, b_{i_l}\}$ generates $X^*(T^r) \otimes \mathbb{Z} \mathbb{R}$. We set

$$\Omega_\eta := \{I \subset \{1, \ldots, m\} \mid \eta \in \sum_{i \in I} \mathbb{R}_{\geq 0} b_i\}$$
and define a fan $\Sigma$ in $X_*(\mathbb{C}^m) \otimes_{\mathbb{Z}} \mathbb{R}$ by $\Sigma := \{ \sigma_I \mid I \in \Omega_\eta \}$, where $\sigma_I := \sum_{i \in I} a_i$. Let $X(\Sigma)$ be the toric variety associated with the fan $\Sigma$. By Theorem 2.4 in [13], $X(\Sigma)$ is isomorphic to the GIT quotient $(\mathbb{C}^m)^{ss} / T^\circ$. For simplicity, we assume that $X(\Sigma)$ is smooth, i.e., $\{ a_i \}_{i \in I}$ generates $X_*(\mathbb{C}^m) \otimes_{\mathbb{Z}} \mathbb{R}$ for any $I \in \Omega_\eta$ such that $|I| = r$. In this case, the action of $T^\circ$ on $(\mathbb{C}^m)^{ss}$ is free and hence one can define a $T^\circ$-equivariant line bundle $L(\lambda)$ associated with each character $\lambda \in X^*(\mathbb{C}^m)$.

We set $R(\Sigma) := \mathbb{C}[x_1, \ldots, x_m]$ with an $X^*(\mathbb{C}^m)$-grading given by $\text{deg}(x_i) = e_i^*$. Let $B(\Sigma) = (\prod_{i \in I} x_i \mid I \in \Omega_\eta)$ be a monomial ideal of $R(\Sigma)$. Since $B(\Sigma)$ is generated by homogeneous ideals, the local cohomology $H^i_{\mathbb{R}S}(R(\Sigma))$ of $R(\Sigma)$ with supports in $B(\Sigma)$ is also $X^*(\mathbb{C}^m)$-graded. We denote by $R(\Sigma)_\lambda$ and $H^i_{\mathbb{R}S}(R(\Sigma))_\lambda$ the weight $\lambda$ parts of $R(\Sigma)$ and $H^i_{\mathbb{R}S}(R(\Sigma))$ for any $\lambda \in X^*(\mathbb{C}^m)$.

One can relate them and the cohomology of the line bundle $L(\lambda)$ as follows.

**Lemma 5.32.** For any $\lambda \in X^*(\mathbb{C}^m)$ and $i \geq 1$, we have $H^i(X(\Sigma), L(\lambda))^{ss} \cong H^i_{\mathbb{R}S}(R(\Sigma))_\lambda$.

**Proof.** See for example Theorem 9.5.7 in [13].

Next we recall a description of $H^i_{\mathbb{R}S}(R(\Sigma))_\lambda$ in terms of simplicial cohomology due to Mustaţă [29]. For each $i = 1, \ldots, m$, let $\Delta_i := \{ I \subset \Omega_\eta \mid i \notin \cup_{j \in I} \} \} \}$ be a simplicial complex on $\Omega_\eta$. For a subset $M \subset \{ 1, \ldots, m \}$, we set $\Delta_M := \cup_{i \in M} \Delta_i$. Here, we understand that if $M = \emptyset$, then $\Delta_M$ is the void complex which has trivial reduced cohomology. For $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m$, we define $\text{neg}(\lambda) := \{ i \in \{ 1, \ldots, m \} \mid \lambda_i < 0 \}$. For a simplicial complex $\Delta$, we denote by $\tilde{H}^i(\Delta)$ the $i$-th reduced cohomology group of $\Delta$.

**Lemma 5.33 (29).** For each $\lambda \in \mathbb{Z}^m$ and $i \in \mathbb{Z}_{\geq 0}$, we have $H^i_{\mathbb{R}S}(R(\Sigma))_\lambda \cong \tilde{H}^{i-2}(\Delta_{\text{neg}(\lambda)})$.

**Proof.** See Theorem 2.1 in [29].

We will also need the following special case of Demazure vanishing theorem.

**Lemma 5.34.** We have $H^{i,0}(X(\Sigma), 0) = 0$.

**Proof.** Since $|\Sigma| = \sum_{i=1}^n \mathbb{R}_{\geq 0} a_i$ is convex, this follows from Demazure vanishing theorem, see for example Theorem 9.2.3 in [13].

For example, we may apply the above results for Lawrence toric varieties. For this, we take $m = 2n$ and change the index set $\{ 1, \ldots, m \}$ by $E_n := \{ \pm 1, \ldots, \pm n \}$. We take $b_{\pm i} = \pm b_i$ for $i = 1, \ldots, n$. In this case, we can take $a_i = (a_i, e_i) \in \mathbb{Z}^d \oplus \mathbb{Z}^n$ and $a_{-i} = (0, e_i) \in \mathbb{Z}^d \oplus \mathbb{Z}^n$, where $\{ e_1, \ldots, e_n \}$ is a basis of $\mathbb{Z}^n$. We also take the same $\eta$. By definition, the toric variety $X(\Sigma)$ associated with these data is the Lawrence toric variety $X$. By our assumption that $a_i \neq 0$ for any $i = 1, \ldots, n$, we have $E_n \setminus \{ \epsilon \} \in \Omega_\eta$ for any $\epsilon \in E_n$. In particular, the set of one dimensional cones $\Sigma(1)$ in $\Sigma$ is given by $\Sigma(1) = \{ \mathbb{R}_{\geq 0} \epsilon \}_{\epsilon \in E}$ and hence the ring $R(\Sigma) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ coincides with the Cox’s homogeneous coordinate ring [12] for the Lawrence toric variety $X$, where we write $y_i$ for the variable corresponding to $-i \in E_n$.

Therefore, we obtain a ring isomorphism

$$R(\Sigma) \cong \otimes_{\lambda \in X^*(\mathbb{C}^m)} \Gamma(X, \tilde{L}(\lambda))^{T^2n}$$

(57)

by Proposition 1.1 in [12]. If we restrict the torus action of $T^2n$ on $R(\Sigma)$ to $T$ via $T \to T^2n$ given by $t \mapsto (t, t^{-1})$, then the weight $\mu \in X^*(T)$ part of $R(\Sigma)$ is given by $\mathbb{C}[u_1, \ldots, u_n] \cdot x^\mu$, where we set $u_i := x_i y_i$ and $x^\mu := \prod_{i, \mu_i > 0} x_i^{\mu_i} \prod_{i, \mu_i < 0} y_i^{-\mu_i}$. Combined with (57), we obtain

$$\Gamma(X, \tilde{L}(\mu))^{T^2n} \cong \mathbb{C}[u_1, \ldots, u_n] \cdot x^\mu.$$  

(58)

We note that the degree of $x^\mu$ with respect to $S$-action is given by $\sum_i |\mu_i|$. For each $i = 1, \ldots, n$, the section $x_i \in \Gamma(X, \tilde{L}(e_i^*))$ gives a $\mathbb{C}[u_1, \ldots, u_n]$-module homomorphism $x_i : \Gamma(X, \tilde{L}(\mu - e_i^*)) \to \Gamma(X, \tilde{L}(\mu))^{T^2n}$ of degree 1. In terms of the isomorphism [58], we have

$$x_i : x^\mu \mapsto \begin{cases} x^\mu & \text{if } \mu_i > 0, \\ u_i \cdot x^\mu & \text{if } \mu_i \leq 0. \end{cases}$$  

(59)
Similarly, the section $y_i \in \Gamma(X, \tilde{L}(-\epsilon_i^*))^T$ gives a $\mathbb{C}[u_1, \ldots, u_n]$-module homomorphism $y_i : \Gamma(X, \tilde{L}(\mu + \epsilon_i^*))^T \to \Gamma(X, \tilde{L}(\mu))^T$ given by

$$y_i \cdot x^{\mu + \epsilon_i^*} = \begin{cases} x^{\mu} & \text{if } \mu_i < 0, \\ u_i \cdot x^{\mu} & \text{if } \mu_i \geq 0. \end{cases} \quad (60)$$

### 5.8 Tilting bundles

In this section, we check Conjecture 3.38 for toric hyper-Kähler manifolds, i.e., $\mathcal{T}_{\varepsilon, s} := \oplus_{t \in B} \mathcal{E}_{\varepsilon, s}(pt)$ is a tilting bundle on $X$. The main result of this section is proved via different methods by McBreen-Webster [28] and Špenko-Van den Bergh [40] independently. We give still another direct proof of it.

We note that for any $t, t' \in \mathbb{R}$, we have

$$[t] + [t'] \leq [t + t'] \leq [t] + [t'] + 1,$$

$$[t] - [t'] - 1 \leq [t - t'] \leq [t] - [t'].$$

These inequalities easily imply that for any $A \in \text{Alc}_a$ and signed circuit $C$, we have

$$|\langle s, \beta C \rangle| - |C| + 1 \leq \langle \mu_A, \beta C \rangle \leq |\langle s, \beta C \rangle| + |C| - 1,$$

where $\mu_A$ is defined as in (50). In particular, we have

$$-|C| + 1 \leq \langle \mu_A - \mu_{A'}, \beta C \rangle \leq |C| - 1 \quad (61)$$

for any $A, A' \in \text{Alc}_a$.

Recall that one can associate a $\mathbb{T}$-equivariant line bundle $\tilde{L}(\lambda)$ on the Lawrence toric variety $X$ for each $\lambda \in X^*(T)$. We define $\pi : X^*(T^2n) \to X^*(T)$ by $\pi(\lambda_1, \ldots, \lambda_n, \lambda_{-1}, \ldots, \lambda_{-n}) = \sum_{i=1}^n (\lambda_i - \lambda_{-i}) \epsilon_i^*$. By Lemma 5.32 and Lemma 5.33 we obtain

$$H^{>0}(X, \tilde{L}(\lambda))^T \cong \oplus_{\lambda \in \pi^{-1}(\lambda)} H^{>0}(\Delta_{\text{neg}(\lambda)}). \quad (62)$$

We first prove the vanishing of higher extensions on the level of Lawrence toric variety.

**Proposition 5.35.** For any $A, A' \in \text{Alc}_a$, we have $H^{>0}(X, \tilde{L}(\mu_A - \mu_{A'}))^T = 0$.

**Proof.** It is enough to prove $H^{>0}(X, \tilde{L}(\mu_A - \mu_{A'}))^T = 0$ since we have

$$H^{>0}(X, \tilde{L}(\mu_A - \mu_{A'}))^T = \bigoplus_{A \in \pi(X^*)} H^{>0}(X, \tilde{L}(\mu_A + \alpha - \mu_{A'}))^T$$

We set $\mu := \mu_A - \mu_{A'} \in \pi(X^*)$ and $Y := \sigma(\mu) \in \{+, -, 0\}^n$. We note that if $\lambda \in X^*(T)$ satisfies $b(\lambda) = 0$, then Lemma 5.34 and (62) imply that $H^{>0}(\Delta_{\text{neg}(\lambda)}) = 0$ for any lift $\hat{\lambda} \in \pi^{-1}(\lambda)$. Therefore, it is enough to prove that for any lift $\hat{\mu} \in \pi^{-1}(\mu)$, there exists $\lambda \in \text{Ker}(b)$ and $\hat{\lambda} \in \pi^{-1}(\lambda)$ such that $\text{neg}(\hat{\mu}) = \text{neg}(\hat{\lambda})$. If $\pm \mu_i > 0$, then the possibilities for the set $\text{neg}(\hat{\mu}_i, \hat{\mu}_{-i})$ is $\emptyset, \{i\}$, or $\{i, -i\}$. If $\mu_i = 0$, then the possibilities for the set $\text{neg}(\hat{\mu}_i, \hat{\mu}_{-i})$ is $\emptyset$ or $\{i, -i\}$, which is contained in the possibilities for $\mu_i \neq 0$. This implies that it is enough to prove the existence of $\lambda \in \text{Ker}(b)$ such that $\sigma(\mu_i) = \sigma(\lambda_i)$ if $\mu_i \neq 0$, i.e., $Y|_{\text{Supp}(Y)} \in \sigma(\text{Ker}(b))/Y^0$. By Proposition 5.24, Lemma 5.26 and Proposition 5.28, we have

$$\sigma(\text{Ker}(b))/Y^0 = \text{elem}(\sigma(\text{Ker}(a)) \setminus Y^0)$$

Since $\text{elem}(\sigma(\text{Ker}(a)) \setminus Y^0) = \{\sigma(\beta_C) | C : \text{signed circuit}, C \subset \text{Supp}(Y)\}$, if $Y|_{\text{Supp}(Y)} \notin \sigma(\text{Ker}(b))/Y^0$, then there exists a signed circuit $C \subset \text{Supp}(Y)$ such that $Y$ is not orthogonal to $\sigma(\beta_C)$. We may assume that $(Y^+ \cap C_+) \cup (Y^- \cap C_-) = \emptyset$. This and $C \subset \text{Supp}(Y)$ imply that for each $i \in C_\pm$, we have $\pm \mu_i \geq 1$ and hence $\langle \mu, \beta_C \rangle \geq \langle C \rangle$ which contradicts (61). $\Box$

**Corollary 5.36.** The vector bundle $\mathcal{T}_{\varepsilon, s}$ is a tilting bundle on $X$.  

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Proof. By Corollary 5.22 it is enough to prove $\text{Ext}^{>0}(T_{\ell},T_{\ell}) = 0$, i.e., $H^{>0}(X,L(\mu_A - \mu_{A'})) = 0$ for any $A, A' \in \text{Acl}_{\ell}$. Let $i: \{0\} \hookrightarrow \ast$ be the inclusion and recall the morphism $\mu_X: X \to \ast$ defined before Lemma 5.3. Since $\mu_X$ is flat, $i$ and $\mu_X$ are Tor independent and hence the base change formula implies

$$R\Gamma(X, L(\mu_A - \mu_{A'})) \cong Li^* R\mu_X \tilde{\ell}(\mu_A - \mu_{A'}).$$

(63)

Since $R\mu_X \tilde{\ell}(\mu_A - \mu_{A'})$ is concentrated on cohomological degree 0 by Proposition 5.35, the RHS of (63) has vanishing cohomology at positive degree. This proves $H^{>0}(X,L(\mu_A - \mu_{A'})) = 0$. \qed

Combined with Proposition 5.15 and Lemma 5.17 we obtain Conjecture 3.22 for toric hyper-Kähler manifolds.

Corollary 5.37. $B_{X,s}$ (resp. $B_{L,s}$) is a $\mathbb{Z}[v,v^{-1}]$-basis of $K_T(X)$ (resp. $K_T(L)$).

Proof. Corollary 5.36 implies that $B_{X,s}$ generates $K_T(X)$ over $\mathbb{Z}[v,v^{-1}]$ and hence it is a basis. Since the pairing $(- : -)$ defined in (2) induces a perfect pairing between $K_T(X)$ and $K_T(L)$, the pairing $(-||-)$ also gives a perfect pairing between $K_T(X)$ and $K_T(L)$. Therefore, the dual basis $B_{L,s}$ of $B_{X,s}$ is a $\mathbb{Z}[v,v^{-1}]$-basis of $K_T(L)$. \qed

As a module over $\mathbb{C}[u_1, \ldots, u_n] \cong \mathbb{C}[t^*]$, we have $R(\Sigma) \cong \oplus_{\mu \in X^*} \mathbb{C}[t^*] \cdot x^\mu$ and the $\mathbb{C}[t^*]$-module structure coming from the morphism $\mu_X: X \to \ast$ is the one induced from the natural inclusion $\mathbb{C}[t^*] \hookrightarrow \mathbb{C}[t^*]$. We also denote by $x^\mu \in \Gamma(X,L(\mu))^H$ the section coming from $x^\mu \in \Gamma(X,\tilde{\ell}(\mu))^T$. As a corollary of (63), we obtain the following.

Lemma 5.38. If $H^{>0}(X,\tilde{\ell}(\mu)) = 0$ for $\mu \in X^*(T)$, then we have $\Gamma(X,L(\mu))^H \cong \mathbb{C}[\hbar^*] \cdot x^\mu$.

As in section 3.6, we set $\mathcal{A}_{\ell,s} := \text{End}(T_{\ell,s})^{\text{op}}$. By Corollary 5.36, we obtain a derived equivalence

$$\psi_{\ell,s}: D^b\text{Coh}^T(X) \cong D^b(\mathcal{A}_{\ell,s}\text{-gmod}^H)$$

(64)

given by $\psi_{\ell,s}(\mathcal{F}) = R\text{Hom}(T_{\ell,s},\mathcal{F})$ for $\mathcal{F} \in D^b\text{Coh}^T(X)$.

We next give a presentation of the ring $\mathcal{A}_{\ell,s}$. We set

$$\mu_I := -\sum_{I \in I^-} e_+^I + \sum_{j \in I^+} \{s, \beta_j^I\} e_j^I$$

so that $T_{\ell,s} = \oplus_{I \in \mathbb{B}} L(\mu_I)$ for any $I \in \mathbb{B}$. Let $e_I \in \mathcal{A}_{\ell,s}$ be the idempotent corresponding to the identity map in $\text{Hom}(L(\mu_I),L(\mu_I))$. For $I,J \in \mathbb{B}$, the $H$-weight space of $e_I \mathcal{A}_{\ell,s} e_J$ of weight $\alpha \in X^*(H)$ is given by $\text{Hom}(L(\mu_I + \alpha),L(\mu_J))^{H} \cong \mathbb{C}[\hbar^*] \cdot x^H \cdot \mu^\alpha(J,J)$ by Lemma 5.38. We set $m_{I,J}^{\alpha,J,J} := x^\mu \cdot e_I \mathcal{A}_{\ell,s} e_J$. We note that $e_I = m_I^{I,I}$ and $\mathbb{C}[\hbar^*] \cong \mathbb{C}[X^H]$ is contained in the center of $\mathcal{A}_{\ell,s}$. For $I,J,J',K \in \mathbb{B}$ and $\alpha, \alpha' \in X^*(H)$, we obtain

$$m_{I,J}^{J,J'} \cdot m_{J,K}^{J,J'} = \delta_{J,J'} \prod_{i \mu < 0} \min\{i, |\mu|\} \cdot m_{I,K}^{\alpha + \alpha'}$$

(65)

where we set $\mu = \mu_J - \mu_I - \alpha$ and $\mu' = \mu_K - \mu_J - \alpha'$. In summary, we obtained the following.

Lemma 5.39. The $\mathbb{C}[\hbar^*]$-algebra $\mathcal{A}_{\ell,s}$ is isomorphic to

$$\bigoplus_{I,J \in \mathbb{B}, \alpha \in X^*(H)} \mathbb{C}[\hbar^*] \cdot m_{I,J}^{\alpha,J,J}$$

where the multiplication rule is given by (65). Moreover, $\mathfrak{h} \subset \mathbb{C}[\hbar^*]$ have $H$-weight 0 and degree 2, and $m_{I,J}^{\alpha,J,J}$ has $H$-weight $\alpha$ and degree $\sum_i |\mu_i|$, where $\mu = \mu_J - \mu_I - \alpha$. 49
For another presentation of the algebra $A_{c,s}$, which is apparently quadratic and a presentation of its Koszul dual $B_{c,s}$, see [28]. This presentation is enough to check Conjecture 3.44.

**Corollary 5.40.** The algebra $A_{c,s}$ and $B_{c,s}$ has an anti-involution which is identity on degree 0 part, compatible with the grading, and reversing the $H$-weights.

**Proof.** For the algebra $A_{c,s}$, we define the $\mathbb{C}[h^*]$-algebra anti-involution by sending $m^2_{ij}$ to $m^{-2}_{ji}$. It is easy to check that this preserves the relation (66) and satisfies the required conditions. It is also easy to check that this anti-involution induces a similar anti-involution on its quadratic dual which is isomorphic to $B_{c,s}$.

\[\nabla_{c,s}(A) := v^d \Delta_{c,s}(A)[d].\]

We set $\nabla_{c,s}(A) := v^d \Delta_{c,s}(A)[d]$. By definition and (51), we have

\[\nabla_{c,s}(A) = \mathcal{E}(A) \otimes O_{L_T},\]

where $L_T$ is the subvariety of $X$ defined by $x_i = 0$ ($i \in I_+$) and $y_i = 0$ ($i \in I$). For any subset $K \subseteq I$, we set $\mu_{A,K} := \mu_A - \sum_{i \in K \cap I_+} \epsilon_i^* + \sum_{j \in K \cap I_-} \epsilon_j^*$. We note that by (62), we have $\mathcal{L}(\mu_{A,K}) \in \mathcal{E}_{X,s}$ for any $K \subseteq I$. By Lemma 5.38, we have the following exact sequence:

\[0 \to v^d \mathcal{L}(\mu_{A,I}) \to \bigoplus_{K \subseteq I \atop |K| = d-1} v^{d-1} \mathcal{L}(\mu_{A,K}) \to \bigoplus_{i \in I} v \mathcal{L}(\mu_{A,I_i}) \to \mathcal{L}(\mu_A) \to \nabla_{c,s}(A) \to 0. \tag{67}\]

We recall that this is given by Koszul resolution.

**Proposition 5.41.** For any $A, B \in A_{c,s}$, we have $R^\mu_{A,B} \text{Hom}(\mathcal{E}(B), \nabla_{c,s}(A)) = 0$ and

\[\text{Hom}(\mathcal{E}(B), \nabla_{c,s}(A))^T = \begin{cases} \mathbb{C} \cdot x^\mu & \text{if } \pm \mu_i \leq 0 \text{ for any } i \in I_\pm, \\ 0 & \text{otherwise}, \end{cases} \tag{68}\]

where $\mu = \mu_A - \mu_B$ and $x^\mu$ is the image of $x^\mu$ in $\Gamma(X, \mathcal{L}(\mu))^T$ under the natural map $\Gamma(X, \mathcal{L}(\mu))^T \to \Gamma(X, \nabla_{c,s}(B) \otimes \mathcal{L}(-\mu_B))^T$ coming from (67).

**Proof.** By Proposition 5.35, we have $R^{>0}(X, \mathcal{L}(\mu_{A,K} - \mu_B)) = 0$ for any $K \subseteq I$. Therefore, Lemma 5.38 and (67) imply that $R\text{Hom}(\mathcal{E}(B), \nabla_{c,s}(A))^T$ is given by

\[0 \to \mathbb{C}[h^*] \cdot x^{\mu_{A,I} - \mu_B} \to \cdots \to \bigoplus_{i \in I_+} \mathbb{C}[h^*] \cdot x^{\mu_i} \oplus \bigoplus_{i \in I_-} \mathbb{C}[h^*] \cdot x^{\mu_i} \to \mathbb{C}[h^*] \cdot x^\mu \to 0,\]

where the complex is given by Koszul type complex for $x_i$ ($i \in I_+$) and $y_i$ ($i \in I$). If $\pm \mu_i \leq 0$ for any $i \in I$, then by (59) and (60), this complex is isomorphic to the Koszul complex of $\mathbb{C}[h^*]$ with respect to the regular sequence $[u_i]_{i \in I}$. This implies that its 0-th cohomology is one dimensional and other cohomologies vanish. If $\pm \mu_i > 0$ for some $i \in I_+$, then this complex is isomorphic to the Koszul complex of $\mathbb{C}[h^*]$ with respect to a sequence containing 1. Therefore, all the cohomologies vanish in this case.
Corollary 5.42. For any $A \in \text{Alc}_s$, $\psi_{\epsilon,s}(\nabla_{\epsilon,s}(A)) \in D^b(A_{\epsilon,s}\text{-gmod}^H)$ is a Koszul module of $A_{\epsilon,s}$.

Proof. Proposition 5.41 implies that $\psi_{\epsilon,s}(\nabla_{\epsilon,s}(A))$ is contained in the standard heart of $D^b(A_{\epsilon,s}\text{-gmod}^H)$. The exact sequence (67) implies that $\psi_{\epsilon,s}(\nabla_{\epsilon,s}(A))$ is a Koszul module of $A_{\epsilon,s}$. □

Next we give a formula expressing $S(A)$ (resp. $E(A)$) in terms of $\{C(B)\}_{B \in \text{Alc}_s}$ (resp. $\{S(B)\}_{B \in \text{Alc}_s}$). For $A \in \text{Alc}_s$, let $M(A)$ be the set of alcove $B$ such that for any hyperplane $H_{i,m}$ passing through $x_A$, $A$ and $B$ are on the same side with respect to $H_{i,m}$. We note that $B \in M(A)$ implies $B \geq A$. We also set $M^-(A) := \{B \in \text{Alc}_s \mid A \in M(B)\}$. In terms of the combinatorics of alcoves, the condition appearing in (68) can be written as $B \in M(A)$ or not. Moreover, the degree of $x^\mu$ is given by $\ell(A,B)$.

Corollary 5.43. For any $A \in \text{Alc}_s$, we have
\[
S(A) = \sum_{B \in M(A)} v^{-\ell(A,B)}C(B), \quad (69)
\]
\[
E(A) = \sum_{B \in M^-(A)} v^{-\ell(A,B)}S(B). \quad (70)
\]

Proof. As in the proof of Corollary 3.39, Proposition 5.41 implies that
\[
\partial(E(B)||S(A)) = [\text{Hom}(E(B),\nabla_{\epsilon,s}(A))]^\vee
\]
\[
= \begin{cases} v^{-\ell(A,B)} & \text{if } B \in M(A) \\ 0 & \text{if } B \notin M(A). \end{cases}
\]
This implies (69) and (70). □

Corollary 5.44. Conjecture 3.23 holds for toric hyper-Kähler manifolds.

Proof. This follows from (58) and (70). □

5.10 Ext-orthogonality

In this section, we prove the second half of Conjecture 3.40.

Theorem 5.45. For any $A, B \in \text{Alc}_s$, we have
\[
R\text{Hom}(\Delta_{\epsilon,s}(B), \nabla_{\epsilon,s}(A))^T \cong \begin{cases} \mathbb{C} & \text{if } A = B \\ 0 & \text{if } A \neq B \end{cases}
\]
as $\mathcal{S}$-modules, where $\mathbb{C}$ is considered as a trivial $\mathcal{S}$-module sitting in cohomological degree 0. In particular, Conjecture 3.40 holds for toric hyper-Kähler manifolds.

Proof. Let $A, B \in \text{Alc}_s$ be two alcoves with $\varphi_{\epsilon,s}(A) = (\lambda_A, p_I)$ and $\varphi_{\epsilon,s}(B) = (\lambda_B, p_J)$. We recall that $\mu_A = \lambda_A - \sum_{i \in I_+} \varepsilon_i^+ + \sum_{j \in J^+} \langle \beta_i^+ \rangle \varepsilon_j^+$ and $\mu_B = \lambda_B - \sum_{i \in I_-} \varepsilon_i^- + \sum_{j \in J^-} \langle \beta_i^- \rangle \varepsilon_j^-$. We set $\mu := \mu_A - \mu_B$ and $K := \{ k \in I \cap J \mid x_A, x_B \in H_{k,m_k} \text{ for some } m_k \in \mathbb{Z} \}$. We note that we have
\[
\mu_k = \begin{cases} 1 & \text{if } k \in K \cap I_+ \cap J_+ \\ -1 & \text{if } k \in K \cap I_- \cap J_+ \\ 0 & \text{if } k \in K \cap (I_+ \cap J_+ \cup I_- \cap J_-). \end{cases}
\]
By Lemma 5.8, $\Delta_{\epsilon,s}(B)$ is quasi-isomorphic to
\[
0 \to E(B) \to \bigoplus_{B' \in N(B), \ell(B,B')=1} v^{-1}E(B') \to \bigoplus_{B' \in N(B), \ell(B,B')=2} v^{-2}E(B') \to \cdots.
\]
Therefore, we obtain

\[ K \rightarrow R \nu \]

The condition \( B \rightarrow R K \) hence

\[ \text{Let } K \text{ and hence } \lambda \]

\[ -|\nabla| \cap \nu, \beta \]

\[ K \rightarrow R \nu \rightarrow 0, \quad \sum_{j \in K \cap K^0} C \cdot x^\mu | K_0 \cap (j) \rightarrow C \cdot x^\mu | K_0 \rightarrow 0. \quad (71) \]

Here, each map is induced from \( x_j : C[h^*] \cdot x^\mu K^0 - \epsilon_j^* \rightarrow C[h^*] \cdot x^\mu K^0 \) for \( j \in I_1 \cap K^0 \) and \( y_j : C[h^*] \cdot x^\mu K^0 + \epsilon_j^* \rightarrow C[h^*] \cdot x^\mu K^0 \) for \( j \in J \). By (59) and (60), the map \( x_j : C \cdot x^\mu - \epsilon_j^* \rightarrow C \cdot x^\mu \) is given by

\[ x_j : x^\mu - \epsilon_j^* = \begin{cases} x^\mu & \text{if } \mu_j^* > 0 \\ 0 & \text{if } \mu_j^* \leq 0 \end{cases} \]

and \( y_j : C \cdot x^\mu + \epsilon_j^* \rightarrow C \cdot x^\mu \) is given by

\[ y_j : x^\mu + \epsilon_j^* = \begin{cases} x^\mu & \text{if } \mu_j^* < 0 \\ 0 & \text{if } \mu_j^* \geq 0 \end{cases} \]

We set \( \nu := \mu K_0 \). If \( \nu_j > 0 \) for some \( j \in I_1 \cap K^0 \) or \( \nu_j < 0 \) for some \( j \in J \cap K^0 \), then the complex (71) is isomorphic to a Koszul complex of \( C \) with respect to a sequence containing 1 and hence it is acyclic. Therefore, if \( RHom(\Delta e_\nu(B), \nabla e_\nu(A))^T \neq 0 \), then we must have \( \pm \nu_j \geq 0 \) for any \( j \in I_1 \cap K^0 \). We also note that \( \nu_j = 0 \) for any \( k \in K \) and \( \pm \nu_j \leq 0 \) for any \( i \in I_1 \cap K^0 \). We claim that these conditions imply \( I = J \). If \( I = J \), then we have \( \nu_j = 0 \) for any \( i \in I \) and \( K_0 = \emptyset \). This implies that \( \mu_i = \nu_i = 0 \) for any \( i \in I \) and hence \( A = B \). In this case, \( K_1 = \emptyset \) and the complex (71) is isomorphic to \( C \) which sits in cohomological degree 0.

Now we assume \( I \neq J \). We first refine the inequalities (61). We note that

\[ \nu = \lambda_A - \lambda_B - \sum_{i \in I_1} \epsilon_i^* + \sum_{i \in I_1} \epsilon_i^* - \sum_{i \in I_0} \epsilon_i^* + \sum_{i \in I_1} [(s, \beta_j^{1'})] \epsilon_j^* - \sum_{j \in J} [(s, \beta_j^{1'})] \epsilon_j^* \]

Since \( K_0 \cap I_1 = K \cap I_1 \cap K^0 \) and \( K_0 \cap J_1 = K \cap J_1 \cap K^0 \), we have

\[ - \sum_{i \in I_1} \epsilon_i^* + \sum_{i \in I_1} \epsilon_i^* - \sum_{i \in I_0} \epsilon_i^* = - \sum_{i \in I_1} \epsilon_i^* + \sum_{j \in J} \epsilon_j^* \]

For any signed circuit \( C = C_+ \cup C_- \), we have \( \beta_C = \sum_{j \in C_+ \cap I_1} \beta_j^{1'} - \sum_{j \in C_- \cap I_1} \beta_j^{1'} \). This implies as in (61) that

\[ -[C_+ \cap I_1^0] + 1 + [(s, \beta_C)] \leq \sum_{j \in C_+ \cap I_1} [(s, \beta_j^{1'})] - \sum_{j \in C_- \cap I_1} [(s, \beta_j^{1'})] \leq [C_- \cap I_1^0] + [(s, \beta_C)]. \]

Therefore, we obtain

\[ \langle \nu, \beta_C \rangle \leq -[C_+ \cap I_1 \cap (J_1 + J_0^0) \cap K^0] + [C_- \cap I_1 \cap (J_1 + J_0^0) \cap K^0] + |C_- \cap I_1^0| \]

\[ + [C_+ \cap J_1 \cap (I_1 + I_0^0) \cap K^0] - [C_- \cap J_1 \cap (I_1 + I_0^0) \cap K^0] + |C_+ \cap J_1^0| - 1 \]

\[ \leq -[C_+ \cap I_1 \cap J^1] + [C_- \cap I_1 \cap J^1] + [C_+ \cap J_1 \cap J^1] + [C_- \cap J_1 \cap J^1] \]

\[ + [C_+ \cap J_1 \cap I_1^0] + [C_- \cap J_1 \cap I_1^0] - [C_- \cap J_1 \cap I_1^0] + |C_+ \cap J_1^0| - 1 \]

\[ = [C_- \cap (I_1 \cap J_1)] + [C_+ \cap (I_1 \cup J_1)] + [C_1 \cap (J_1 \cup L)] + [C_+ \cap (J \cup L_1)] - 1. \]

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In particular, if we assume the existence of a signed circuit $C$ such that $C_+ \subset I_- \cup J_+$ and $C_- \subset I_+ \cup J_-$, then the above inequality implies $(\nu, \beta_C) \leq 1$. On the other hand, we have $\nu_i \geq 0$ for any $i \in I_- \cup J_+$ and $\nu_i \leq 0$ for any $i \in I_+ \cup J_-$. This implies that $(\nu, \beta_C) \geq 0$ which gives a contradiction.

Therefore, it is enough to prove the existence of a signed circuit as above. We set $V = \text{Ker}(b) \otimes \mathbb{R}$ and $E = \{1, \ldots, n\}$ as in section 5.6. We also define a partition $E = R \sqcup G \sqcup B \sqcup W$ by $W = (I_- \cup J_+) \cap (I_+ \cup J_-) = (I_+ \cap J_+) \cup (I_- \cap J_-)$, $R = (I_- \cup J_+) \setminus W = (I_- \setminus J_-) \cup (J_+ \setminus J_-)$, $G = (I_+ \cup J_-) \setminus W = (I_+ \setminus J_+) \cup (J_- \setminus I_-)$, and $B = I^c \setminus J^c$. We note that the assumption $I \neq J$ implies $R \sqcup G \neq \emptyset$. By Lemma 5.30, it is enough to prove the existence of nonzero $Z \in \sigma(V^\perp)$ such that $Z_\mathbb{R} \geq 0$, $Z_G \leq 0$, and $Z_B = 0$. If we assume that such $Z$ does not exist, then Proposition 5.31 implies that there exists a nonzero $Y \in \sigma(V)$ such that $Y_\mathbb{R} \geq 0$, $Y_G \leq 0$, and $Y_W = 0$. By Lemma 5.30 again, there exists signed cocircuit $C^\vee = C^\vee_+ \cup C^\vee_-$ such that $C^\vee_+ \subset R \cup B$ and $C^\vee_- \subset G \cup B$. Since we have $I \cap C^\vee_+ \subset I_-$, $I \cap C^\vee_- \subset I_+$, and $I \cap C^\vee \neq \emptyset$, we obtain
\[
\langle \xi, \alpha_{C^\vee} \rangle = \sum_{i \in I \cap C^\vee_+} \langle \xi, \alpha_i \rangle - \sum_{i \in I \cap C^\vee_-} \langle \xi, \alpha_i \rangle < 0.
\]

On the other hand, $J \cap C^\vee_+ \subset J_+$, $J \cap C^\vee_- \subset J_-$, and $J \cap C^\vee \neq \emptyset$ imply that
\[
\langle \xi, \alpha_{C^\vee} \rangle = \sum_{i \in J \cap C^\vee_+} \langle \xi, \alpha_i \rangle - \sum_{i \in J \cap C^\vee_-} \langle \xi, \alpha_i \rangle > 0.
\]

This is a contradiction and hence the required circuit exists. This completes the proof of Theorem 5.45 and hence Conjecture 3.40 for toric hyper-Kähler manifolds.

5.11 Central charges

In this section, we prove Conjecture 3.42 and the second part of Conjecture 3.49. In order to prove them, we need to construct the central charge $Z : \mathfrak{s}_R^* \to \text{Hom}_{\mathbb{R}}(K(L), \mathbb{R})$. We claim that for a Kähler alcove $A$ and $s \in A$, the central charge of canonical bases $\mathcal{C}(A(s))$ corresponding to $A(s) \in \text{Alc}_s$ is given by the volume of the polytope $A(s)$ which is a polynomial function in $s$. In order to check that these polynomial functions do not depend on the choice of $A$, we consider their equivariant lifts.

Recall that we write $i(s) = (s_1, \ldots, s_n)$ for $s \in \mathfrak{s}_R^*$. For $I \in \mathcal{B}$, we set
\[
\square_I(s) := \{ x \in \mathfrak{h}_{\mathbb{R}}^* \mid 0 \leq \langle x, a_i \rangle + s_i \leq 1 \text{ for any } i \in I \}.
\]

Take a generic $c \in \mathfrak{h}$ such that $\langle c, \alpha \rangle \not\in \mathbb{Z}$ for any equivariant root $\alpha \in X^*(H)$. We consider $\mathbb{C}$ as a module over $K_H(pt)$ by $a^\alpha \mapsto e^{2\pi \sqrt{-1} c, \alpha}$ and define a map $Z_c : \mathfrak{s}_R^* \to \text{Hom}_{K_H(pt)}(K_H(L), \mathbb{C})$ by assigning
\[
\langle Z_c(s), O_{p_I} \rangle = \int_{\square_I(s)} e^{2\pi \sqrt{-1} c, x} dx = \prod_{i \in I} \frac{e^{2\pi \sqrt{-1} (c, a_i^\alpha)} - 1}{2\pi \sqrt{-1} (c, a_i^\alpha)} \cdot e^{-2\pi \sqrt{-1} s_i(c, a_i^\alpha)}
\]
for each $I \in \mathcal{B}$. Here, $O_{p_I}$ is the skyscraper sheaf at $p_I \in X^H$. Since $\{O_{p_I}\}_{I \in \mathcal{B}}$ is a basis of $K_H(L)$ after localization, the genericity of $c$ implies that this extends to a $K_H(pt)$-linear map $Z_c(s) : K_H(L) \to \mathbb{C}$. We note that these functions are analytic in $s$.

**Lemma 5.46.** For any $s \in \mathfrak{s}_R^*_{\text{reg}}$ and $A(s) \in \text{Alc}_s$, we have
\[
\langle Z_c(s), \mathcal{C}(A(s)) \rangle = \int_{A(s)} e^{2\pi \sqrt{-1} c, x} dx.
\]

**Proof.** We note that if we change $A(s)$ by $A(s) + \alpha$ for some $\alpha \in X^*(H)$, then both sides of (72) are multiplied by $e^{2\pi \sqrt{-1} c, \alpha}$. For any $A(s) \in \text{Alc}_s$, there exists unique $\alpha \in X^*(H)$ such that $A(s) + \alpha \in \square_I$.
Remark 5.47. I for any \( K \)-dimensional polytope is positive. Hence it is enough to check the second condition in Definition 3.45. This implies \( \langle \mathcal{Z}_c(s), \mathcal{O}_{pt} \rangle = \int_{\mathcal{A}(s)} e^{2\pi \sqrt{-1}(c,x)} dx \) for any \( I \in \mathbb{B} \). One can solve (73) to express \( \mathcal{O}(A(s)) \) in terms of \( \mathcal{O}_{pt} \). Since \( c \) is generic, one can also solve (74) to express \( \int_{\mathcal{A}(s)} e^{2\pi \sqrt{-1}(c,x)} dx \) in terms of \( \langle \mathcal{Z}_c(s), \mathcal{O}_{pt} \rangle \) in the same way. Therefore, \( \langle \mathcal{Z}_c(s), \mathcal{C}(A(s)) \rangle \) and \( \int_{\mathcal{A}(s)} e^{2\pi \sqrt{-1}(c,x)} dx \) should coincide for any \( A(s) \subset \boxtimes_I(s) \). This proves (72). \( \square \)

Remark 5.47. One can also consider an equivariant lift of \( \mathcal{Z} \) to \( KZ(L) \) by replacing the integral (72) by certain Euler type integrals appearing in the theory of Gelfand-Kapranov-Zelevinsky’s hypergeometric differential equations. We note that these differential equations come from quantum differential equations of toric hyper-Kähler manifolds by [27]. We do not pursue this direction further here since we do not need it.

Corollary 5.48. There exists a polynomial map \( \mathcal{Z}: s_{\mathbb{R}}^* \to \text{Hom}_{\mathbb{Z}}(K(L), \mathbb{R}) \) such that \( \langle \mathcal{Z}(s), \mathcal{C}(A(s)) \rangle = \text{Vol}(A(s)) \) for any \( s \in s_{\mathbb{R}}^* \) and \( A(s) \in \text{Alc}_s \). Moreover, there exists a vector bundle \( \mathcal{P} \) such that \( \mathcal{Z} = \mathcal{Z}_s \).

Proof. Since the RHS of (72) is holomorphic in \( s \) and \( \{\mathcal{C}(A(s))\}_{A(s) \in \text{Alc}_s} \) forms a basis of \( KH(L) \), \( \langle \mathcal{Z}(s), \mathcal{C} \rangle \) is holomorphic in \( s \) for any \( \mathcal{C} \in KH(L) \). Therefore, one can substitute \( c = 0 \) to obtain a map \( \mathcal{Z} = \mathcal{Z}_{c=0}: s_{\mathbb{R}}^* \to \text{Hom}_{\mathbb{Z}}(K(L), \mathbb{R}) \). Lemma 5.46 implies that for any \( A(s) \in \text{Alc}_s \), we have \( \langle \mathcal{Z}(s), \mathcal{C}(A(s)) \rangle = \text{Vol}(A(s)) \). Since this is a polynomial function in \( s \), \( \mathcal{Z} \) is also a polynomial function.

We next consider the value of the central charges at \( s = 0 \). We take \( s_0 \in s_{\mathbb{R}}^* \) in a neighborhood of \( 0 \) and consider \( \mathcal{P} := \sum_{A(s) \in \text{Alc}_s} \text{Vol}(A(0)) \cdot \mathcal{E}(A(s)) \), where \( A(0) \) is the limit of \( A(s) \) as \( s_0 \to 0 \) and \( m \in \mathbb{Z}_{>0} \) is taken so that \( m\text{Vol}(A(0)) \in \mathbb{Z} \) for any \( A(s) \in \text{Alc}_s \). We note that this does not depend on the choice of \( s_0 \) by Proposition 5.20. We note that \( \text{rk} \mathcal{P} = m \) since we have \( \sum_{A(s) \in \text{Alc}_s} \text{Vol}(A(s)) = \text{Vol}(\boxtimes_I(s)) = 1 \).

We note that by Proposition 5.31 we have \( \chi(X, \mathcal{C}(A) \otimes \mathcal{E}(A')) = \delta_{A,A'} \) for any \( A, A' \in \text{Alc}_s / \mathbb{X}^*(H) \). This implies \( \langle \mathcal{Z}_d(0), \mathcal{C}(A(s_0)) \rangle = \text{Vol}(A(0)) = \langle \mathcal{Z}_d(0), \mathcal{C}(A(s_0)) \rangle \) for any \( A(s_0) \in \text{Alc}_s \) and hence \( \mathcal{Z}_d(0) = \mathcal{Z}(0) \). For any \( l \in \text{Pic}(X) \) \( \cong \mathbb{X}^*(S) \), the periodic hyperplane arrangements in \( s_{\mathbb{R}}^* \) defined in section 5.4 does not change if we change \( s \) by \( s+l \). This and Lemma 5.34 imply that for any \( A(s_0) \in \text{Alc}_s \), there exists \( A(s_0 + l) \in \text{Alc}_{s_0+l} \) such that \( \text{Vol}(A(s_0)) = \text{Vol}(A(s_0 + l)) \) and \( \mathcal{C}(A(s_0 + l)) = \mathcal{C}(A(s_0)) \). Therefore, we obtain \( \langle \mathcal{Z}_d(l), \mathcal{C}(A(s_0 + l)) \rangle = \text{Vol}(l) = \text{Vol}(l + \mathcal{C}(A(s_0 + l))) \). This implies that \( \mathcal{Z}_d(l) = \mathcal{Z}_d(l) \) for any \( l \in \text{Pic}(X) \). Since both of them are polynomial functions in \( s \), we obtain \( \mathcal{Z}_d(s) = \mathcal{Z}_d(s) \) for any \( s \in s_{\mathbb{R}}^* \). \( \square \)

We now prove Conjecture 3.47 for toric hyper-Kähler manifolds. Recall that for any \( A \in \text{Alc}_K \), we associate a \( t \)-structure \( \tau(A) \) on \( \mathcal{D} := D^b \text{Coh}_L(X) \subset D^b \text{Coh}(X) \) defined by the tilting bundle \( \mathcal{T}_{\epsilon,s} \) for \( s \in A \).

Corollary 5.49. The pair \( (\mathcal{Z}, \tau) \) gives a real variation of stability conditions on \( \mathcal{Z} \).

Proof. The first condition in Definition 3.45 follows from Corollary 5.48 since the volume of a full dimensional polytope is positive. Hence it is enough to check the second condition in Definition 3.45.

For any \( s \in A \in \text{Alc}_K \) and a wall \( w = \{ x \in s_{\mathbb{R}}^* | \langle x, \beta_c \rangle = m \} \) of \( A \), we set \( \text{Alc}_{s,w} := \{ A(s) \in \text{Alc}_s \mid \text{Vol}(A(s)) \) does not vanish on \( w \} \) and \( \text{Alc}_{s,w}^0 := \text{Alc}_s \setminus \text{Alc}_{s,w} \). By Proposition 5.20 and Corollary 5.48
the Serre subcategories $\mathcal{C}_A$ of the heart $\mathcal{A}$ of $\tau(\mathcal{A})$ defined in section 3.7 are given by $\mathcal{C}_A$ if $n = 0$, generated by objects $\{ \mathcal{C}(A) \mid A \in \text{Alg}_{s,w}^1 \}$ if $0 < n < |C|$, and 0 if $n \geq |C|$. Here, we identified $\mathcal{C}(A) \in K_\tau(X)$ as an object of $\mathcal{D}$ as in section 3.6 by forgetting equivariant structures. We note that we have $\mathcal{C}_A = \mathcal{C}_A^{\mathbb{A}}_{A}^{-1} = \{ F \in \mathcal{A} \mid \text{Hom}(\mathcal{E}(A), F) = 0 \}$ for any $A \in \text{Alg}_{s,w}^0$. This easily implies that $\mathcal{D}_A = \mathcal{D}_A^{\mathbb{A}}_{A}^{-1} = \{ F \in \mathcal{D} \mid \text{RHom}(\mathcal{E}(A), F) = 0 \}$ for any $A \in \text{Alg}_{s,w}^0$. Let $A_+ \neq A_- \in \text{Alg}_K$ be two Kähler alcove sharing the same wall $w$ such that $A_+$ is above $A_-$. We take $s \in A_+$ since $\{ \mathcal{E}(A) \mid A \in \text{Alg}_{s,w}^0 \} = \{ \mathcal{E}(A) \mid A \in \text{Alg}_{s,w}^0 \}$. By Proposition 5.20 we obtain $\mathcal{D}_A \mathbb{A}_{s,w} = \mathcal{D}_A \mathbb{A}_{s,w}^{\mathbb{A}}$ for any $A \in \mathbb{Z}$.

First we recall the description of elliptic stable bases for toric hyper-Kähler manifolds given in [1, 39]. We will follow the notations of section 4 and 5. We prove some basic properties of them. As a corollary, we prove Conjecture 4.8 for toric hyper-Kähler manifolds. We will follow the notations of section 4 and 5.

6 Elliptic canonical bases for toric hyper-Kähler manifolds

In this final section, we define what we call elliptic canonical bases for toric hyper-Kähler manifolds and prove some basic properties of them. As a corollary, we prove Conjecture 4.8 for toric hyper-Kähler manifolds. We will follow the notations of section 4 and 5.

6.1 Elliptic standard bases

First we recall the description of elliptic stable bases for toric hyper-Kähler manifolds given in [1, 39]. Recall that we fix the polarization $T^{1/2}$ in [39] which satisfies $\det T^{1/2} = v^{-n} L(\kappa)$, where we set $\kappa := \varepsilon_1 + \cdots + \varepsilon_n \in \mathbb{X}^*(T)$. We also take similar polarization for $X'$ and $\kappa' := \varepsilon_1 + \cdots + \varepsilon_n \in \mathbb{X}_s(T)$.

Proposition 6.1 ([1, 39]). For any $A \in \mathbb{B}$, we have

$$\text{Stab}_{\mathcal{E}_{1/2}}(p_1) = \prod_{i \in I_+} \frac{\partial(v^{-1} L(-\varepsilon_i^+))}{\partial(v^{-1} L(\varepsilon_i^+))} \cdot \prod_{j \in I_-} \frac{\partial(v^{-1} L(-\varepsilon_j^-))}{\partial(v^{-1} L(\varepsilon_j^-))} \cdot \prod_{i \in I_+} \frac{\partial(v^{-1} L(-\varepsilon_i^+))}{\partial(v^{-1} L(\varepsilon_i^+))} \cdot \prod_{j \in I_-} \frac{\partial(v^{-1} L(-\varepsilon_j^-))}{\partial(v^{-1} L(\varepsilon_j^-))},$$

where $l_j := -\langle \beta_j^1, \kappa_i \rangle + \sum_{i \in I_+} b_i - \sum_{i \in I_-} b_i \geq 0$ for any $j \in I_+^0$.

Proof. By Theorem 5 in [39], we have

$$\text{Stab}_{\mathcal{E}_{1/2}}(p_1) = \prod_{i \in I_+} \frac{\partial(v^{-1} L(-\varepsilon_i^+))}{\partial(v^{-1} L(\varepsilon_i^+))} \cdot \prod_{j \in I_-} \frac{\partial(v^{-1} L(-\varepsilon_j^-))}{\partial(v^{-1} L(\varepsilon_j^-))} \cdot \prod_{i \in I_+} \frac{\partial(v^{-1} L(-\varepsilon_i^+))}{\partial(v^{-1} L(\varepsilon_i^+))} \cdot \prod_{j \in I_-} \frac{\partial(v^{-1} L(-\varepsilon_j^-))}{\partial(v^{-1} L(\varepsilon_j^-))},$$

for some $\beta_j \in \mathbb{X}_s(S)$ and $l_j \in \mathbb{Z}$ for $j \in I'$. We only need to determine $\beta_j$ and $l_j$. For any $l \in \mathbb{X}_s(S)$, the factor of automorphy of $i_{\tau}^* \text{Stab}_{\mathcal{E}_{1/2}}(p_1)$ under $z \mapsto q^z$ is given by $i_{\tau}^* \Sigma(l) \cdot i_{\tau}^* \Sigma(l)^{-1}$ by (25). On the other hand, the factor of automorphy of the RHS of (75) is given by $\prod_{j \in I'} i_{\tau}^* \Sigma(l_j) \cdot i_{\tau}^* \Sigma(l_j)^{-1}$ by Corollary 5.5. This implies that $\sum_{j \in I'} l_j \beta_j b_j = l = \sum_{j \in I'} l_j \beta_j b_j$ for any $l \in P$. Hence we have $\beta_j = \beta_j^l$ for any $j \in I'$.

We next consider the factor of automorphy of $\tau(\det T^{1/2}, v^*)^{-1}(i_{\tau}^* \text{Stab}_{\mathcal{E}_{1/2}}(p_1))$ under $a \mapsto q^a$ for $c \in \mathbb{X}_s(H)$. By (24) and Corollary 5.5, this is given by

$$v^{-\sum_{i \in I_+} c_i \beta_i^1} \cdot z^{-\sum_{j \in I'} l_j \beta_j b_j + \sum_{i \in I_+} c_i \beta_i b_i} \cdot \left( \sum_{j \in I'} -\sum_{c_i, \alpha_j} \right) \cdot \left( \sum_{j \in I'} -\sum_{c_i, \alpha_j} \right).$$
On the other hand, \(\text{Corollary 6.3.}\) implies that this should be equal to
\[
v^{-\langle c, \sum_{i \in I_+} \alpha_i^i \rangle + \langle c, \sum_{i \in I_-} \alpha_i^i \rangle - \langle c, \sum_{i \in I_+} (l_i^j - 1) \alpha_i^j \rangle - \langle c, \sum_{i \in I_-} (l_i^j + 1) \alpha_i^j \rangle} \cdot z^{-\sum_{j \in J} \langle c, \alpha_j^j \rangle \beta_j^j} \cdot i_{p_j}^* L \left( - \sum_{j \in J} \langle c, \alpha_j^j \rangle \epsilon_j^j \right)
\]
for any \(J \in \mathbb{B},\) where we set \(l_i^j = l_i + \langle \beta_j^j, \kappa \rangle.\) By comparing the exponent of \(v,\) we obtain
\[
- \sum_{i \in I_+} \alpha_i^i + \sum_{i \in I_-} \alpha_i^i - \sum_{j \in I_+} (l_i^j - 1) \alpha_i^j - \sum_{j \in I_-} (l_i^j + 1) \alpha_i^j = - \sum_{i \in I_+} \alpha_i^i + \sum_{i \in I_-} \alpha_i^i
\]
for any \(J \in \mathbb{B}.\) For each \(j \in I_{k_+}^N,\) taking \(J \in \mathbb{B}\) such that \(j \in J\) and considering the pairing with \(\alpha_j^j,\) we obtain \(l_i^j = \langle \sum_{i \in I_+} \alpha_i^i - \sum_{i \in I_-} \alpha_i^i, \alpha_j^j \rangle \pm 1.\) This implies \(l_i^j = -\langle \beta_j^j, \kappa \rangle + \sum_{i \in I_+} b_i - \sum_{i \in I_-} b_i \rangle \pm 1\) by \(\text{Corollary 5.5.}\)

\textbf{Corollary 6.2.} For any \(I \in \mathbb{B},\) we have
\[
\text{Stab}^{\mathcal{L}}(X, p_I) = (-1)^{|I_+| + |I_-|} \prod_{i=1}^n \vartheta(\mathcal{L}(\epsilon_i^i)) \cdot i_{p_i}^* \mathcal{L}(\epsilon_i^i).
\]

\textbf{Proof.} We note that by \text{Corollary 5.5.} applied for \(X^i,\) we have \(i_{p_j}^* \mathcal{L}(\epsilon_i^i) = v^{-\langle \beta_j^i, \sum_{i \in I_+} b_i - \sum_{i \in I_-} b_i \rangle} \cdot \beta_j^i \cdot \mathcal{L}(\epsilon_i^i)\) for any \(j \in I^c\) and \(i_{p_j}^* \mathcal{L}(\epsilon_i^i) = v^{\pm 1}\) for \(i \in I_k.\) This and Proposition 6.1 imply that
\[
\tau(\det T^{1/2}, v)^* \text{Stab}^{\mathcal{L}_{X,T^{1/2}}}(p_I) = (-1)^{|I_+| + |I_-|} \prod_{i=1}^n \vartheta(\mathcal{L}(\epsilon_i^i)) \cdot i_{p_i}^* \mathcal{L}(\epsilon_i^i) \cdot i_{p_j}^* \mathcal{L}(\epsilon_i^i)^{-1} \cdot \prod_{i \in I_+} \vartheta(v \cdot i_{p_j}^* \mathcal{L}(\epsilon_i^i))^{-1}.
\]
Now the statement of the corollary follows from
\[
\vartheta(N_{p_i}^\pm) = \prod_{i \in I_+} \vartheta(v^{-1} i_{p_j}^* \mathcal{L}(\epsilon_i^i)) \cdot \prod_{i \in I_-} \vartheta(v i_{p_j}^* \mathcal{L}(\epsilon_i^i))
\]
obtained by applying \(\text{Corollary 3.9.}\) to \(X^i.\)

In particular, this implies Conjecture 4.4 for toric hyper-Kähler manifolds. Recall that we wrote \(S_{X,p_I} = i_{p_I}^* \text{Stab}^{\mathcal{L}}(p_I).\)

\textbf{Corollary 6.3.} For any \(I, J \in \mathbb{B},\) we have \((-1)^{|I_+| + |I_-|} \cdot S_{X,p_I,p_J} = (-1)^{|J_+| + |J_-|} \cdot S_{X,p_J,p_I} \) and they are holomorphic sections of the line bundle on \(B_X \cong B_X^*\) described in Proposition 4.3.

Moreover, Corollary 6.2 also implies Conjecture 4.6 for toric hyper-Kähler manifolds.

\textbf{Corollary 6.4.} For any \(I, J \in \mathbb{B},\) we have \(S_{X,p_I,p_J} = (-1)^n \cdot S_{X_{\text{flap}},p_I,p_J} .\)

\textbf{Proof.} Let \(\mathcal{L}_{\text{flop}}(\lambda)\) be the \(T\)-equivariant line bundle on \(X_{\text{flop}}\) associated with \(\lambda \in X^*(T).\) By Corollary 6.2 and \(i_{p_j}^* \mathcal{L}_{\text{flop}}(\lambda) = i_{p_j}^* \mathcal{L}(\lambda),\) we obtain
\[
(-1)^{|I_+| + |I_-|} \cdot S_{X_{\text{flap}},p_I,p_J} = \prod_{i=1}^n \vartheta(i_{p_j}^* \mathcal{L}_{\text{flop}}(\epsilon_i^i)) \cdot i_{p_j}^* \mathcal{L}(\epsilon_i^i) = \prod_{i \in I_+} \vartheta(i_{p_j}^* \mathcal{L}(\epsilon_i^i)) \cdot i_{p_j}^* \mathcal{L}(\epsilon_i^i) = (-1)^{|I_+| + |I_-|} \cdot S_{X,p_I,p_J}.
\]
6.2 Elliptic canonical bases

Now we construct the elliptic canonical bases for toric hyper-Kähler manifolds. From now on, we identify $X^*(T) \cong \mathbb{Z}^n$ and $X_*(T) \cong \mathbb{Z}^n$ by using the standard inner product $(-,-) : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ given by $((\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n)) = \sum_{i=1}^n \lambda_i \mu_i$. We note that under this identification, we have $\kappa' = \kappa$. Using this identification, we may consider $X_*(S) \subset X^*(T)$ and $X^*(H) \subset X_*(T)$. We recall that $(q;q)_\infty = \prod_{m \geq 1} (1 - q^m)$.

Definition 6.5. For any $\lambda \in \mathbb{Z}^n$, we define

$$
\Theta_X(\lambda) := (q;q)_\infty^{d} \sum_{\beta \in X_*(S)} (-1)^{\langle \kappa, \beta \rangle} q^{\frac{1}{2} \langle \beta, \beta + \kappa \rangle + \langle \lambda, \beta \rangle} z^\beta,
$$

$$
\Theta_X(\lambda) := (q;q)_\infty^{d} \sum_{\alpha \in X^*(H)} (-1)^{\langle \kappa, \alpha \rangle} q^{\frac{1}{2} \langle \alpha, \alpha + \kappa \rangle + \langle \lambda, \alpha \rangle} a^\alpha.
$$

We can consider $\{\Theta_X(\lambda)\}_{\lambda \in \mathbb{Z}^n}$ and $\{\Theta_X^*(\lambda)\}_{\lambda \in \mathbb{Z}^n}$ as elements of $K(X)_\text{loc} \cong K(X_\text{reg})$ and call them elliptic canonical bases for $X$ and $X^\ast$ respectively.

We note that for any $\alpha \in X^*(H)$ and $\beta \in X_*(S)$, we have $\langle \alpha, \beta \rangle = 0$. Using this, one can easily check the following relations.

Lemma 6.6. For any $\alpha \in X^*(H)$ and $\beta \in X_*(S)$, we have

$$
\Theta_X(\lambda + \alpha) = a^\alpha \Theta_X(\lambda),
$$

$$
\Theta_X(\lambda + \beta) = (-1)^{\langle \kappa, \beta \rangle} q^{\frac{1}{2} \langle \beta, \beta + \kappa \rangle - \langle \lambda, \beta \rangle} z^{-\beta} \Theta_X(\lambda).
$$

In particular, the number of linearly independent elements in $\{\Theta_X(\lambda)\}_{\lambda \in \mathbb{Z}^n}$ over $\mathcal{M}_X$ is less than or equal to the number of elements of $\Xi := \mathbb{Z}^n/\langle X^*(H) + X_*(S) \rangle \cong \mathbb{Z}^n/\langle X^*(S) \rangle \cong \mathbb{Z}^n/\langle X^*(H) \rangle$. The unimodularity of $a$ and $b$ implies the following.

Lemma 6.7. We have $|\Xi| = |B|$.

Proof. By taking a basis of $X_*(H)$, we consider each $a_i \in X_*(H)$ as a column vector and $a = (a_1, \ldots, a_n)$ as a $(d \times n)$-matrix. We note that $|X_*(H)/X^*(S)| = |\det(a \cdot a)|$. For each subset $I = \{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$ with $|I| = d$, we denote by $a_I = (a_{i_1}, \ldots, a_{i_d})$ the $(d \times d)$-matrix obtained by removing certain columns from $a$. By the unimodularity of $a$, we have $\det(a_I) = \pm 1$ if $I \in B$ and $\det(a_I) = 0$ otherwise. Therefore, we obtain

$$
\det(a \cdot a) = \sum_{I \subset \{1, \ldots, n\}} \det(a_I, a_I) = \sum_{I \in B} \det(a_I)^2 = |B|.
$$

\[\blacksquare\]

Corollary 6.8. For any data $\mathcal{E}$ and $s \in \mathfrak{a}_{\text{reg}}$, the map $I \mapsto \mu_I$ gives a bijection $B \cong \Xi$, where $\mu_I$ is defined as in (6.5).

Proof. It is enough to prove that the map is injective. If there exists $\alpha \in X^*(H)$, $\beta \in X_*(S)$ and $I \neq J \in \mathbb{B}$ such that $\mu_J = \mu_I + \alpha + \beta$, then (6.1) implies that $(\beta, \beta_C) \leq |C| - 1$ for any signed circuit $C$. Since $\{\mathcal{L}(\mu_I)\}_{I \in \mathbb{B}}$ forms a basis of $K_T(X)$ over $K_T(\text{pt})$, we must have $\beta \neq 0$. By Lemma 5.30, there exists a signed circuit $C = C_+ \sqcup C_-$ such that $\pm \beta_i > 0$ for any $i \in C_\pm$. This implies that $(\beta, \beta_C) \geq |C|$ and hence gives a contradiction.

Now we prove the main result of this paper. Recall the Jacobi triple product formula:

$$
(q;q)_\infty \theta(x) = \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{m^2}{2}} x^{m+\frac{1}{2}}.
$$

We also recall that $\mathcal{G}_X(p_S) := \sqrt{\mathcal{L}(\kappa) \cdot \mathcal{L}^1(\kappa) - 1} \cdot \text{Stab}_{\chi}^{\text{ell}}(p_S)$.
**Theorem 6.9.** For any $I \in \mathbb{B}$, we have

$$(-1)^{|I_+|+|I_1^*|}, \mathcal{S}_X(p_I) = \sum_{\lambda \in \Xi} (-1)^{(\kappa,\lambda)} q^{\frac{1}{2}(\lambda+\lambda+\kappa)} i_{\beta_I}^* \Theta_{X'}(\lambda) \cdot \Theta_X(\lambda).$$  

(77)

Here, we fix a lift $\Xi \to \mathbb{Z}^n$ and consider $\lambda \in \Xi$ as an element of $\mathbb{Z}^n$.

**Proof.** We first note that by Lemma 6.6, each term in the RHS of (77) does not depend on the choice of a lift of $\lambda \in \Xi$ to $\mathbb{Z}^n$. By Corollary 6.2 and (76), we have

$$(-1)^{|I_1|+|I_1^*|}, (q; q)_\infty \text{Stab}^H_{\ell}(p_I) = \prod_{i=1}^n (q; q)_\infty \theta(\mathcal{L}_i^* \cdot i_{\beta_I}^* \mathcal{L}_i^* (e_i))$$

$$= \sum_{\mu \in \mathbb{Z}^n} (-1)^{(\mu,\kappa)} q^{\frac{1}{2}(\mu+\mu+\kappa)} i_{\beta_I}^* \mathcal{L}_i^* (\mu + \frac{1}{2} e_i) \cdot \mathcal{L}_i^* (\mu + \frac{1}{2} \kappa).$$

We note that any element of $\mathbb{Z}^n$ can be uniquely written as $\lambda + \alpha + \beta$ for $\lambda \in \Xi$, $\alpha \in \mathcal{X}^*(H)$, and $\beta \in \mathcal{X}_*(S)$. By using $\mathcal{L}(\alpha) = a^\alpha$, $\mathcal{L}_i^*(\beta) = z^\beta$, and $(\alpha, \beta) = 0$ for $\alpha \in \mathcal{X}^*(H)$ and $\beta \in \mathcal{X}_*(S)$, we obtain

$$(-1)^{|I_1|+|I_1^*|}, \mathcal{S}_X(p_I) = (q; q)_\infty \sum_{\lambda \in \Xi} (-1)^{(\lambda+\alpha+\beta,\kappa)} q^{\frac{1}{2}(\lambda+\alpha+\beta+\kappa)} i_{\beta_I}^* \mathcal{L}(\lambda + \alpha + \beta)$$

$$= \sum_{\lambda \in \Xi} (-1)^{(\kappa,\lambda)} q^{\frac{1}{2}(\lambda+\lambda+\kappa)} (q; q)_\infty \sum_{\alpha \in \mathcal{X}^*(H)} (-1)^{(\alpha,\alpha)} q^{\frac{1}{2}(\alpha+\alpha+\kappa)} i_{\beta_I}^* \mathcal{L}(\lambda + \alpha) a^\alpha$$

$$\times (q; q)_\infty \sum_{\beta \in \mathcal{X}_*(S)} (-1)^{(\kappa,\beta)} q^{\frac{1}{2}(\beta+\beta+\kappa)} \mathcal{L}(\lambda + \beta) z^\beta$$

$$= \sum_{\lambda \in \Xi} (-1)^{(\kappa,\lambda)} q^{\frac{1}{2}(\lambda+\lambda+\kappa)} i_{\beta_I}^* \Theta_{X'}(\lambda) \cdot \Theta_X(\lambda).$$

Since $\{\mathcal{S}_X(p_I)\}_{I \in \mathbb{B}}$ is a basis of $\mathcal{K}(X)_{\text{loc}}$ over $\mathcal{M}_X$. Lemma 6.7 and Theorem 6.9 implies that $\{\Theta_X(\lambda)\}_{\lambda \in \Xi}$ is also a basis of $\mathcal{K}(X)_{\text{loc}}$ over $\mathcal{M}_X$. By applying Theorem 6.9 for $-X$, we obtain

$$(-1)^{|I_-|+|I_-^*|}, \mathcal{S}_{-X}(p_I) = \sum_{\lambda \in \Xi} (-1)^{(\kappa,\lambda)} q^{\frac{1}{2}(\lambda+\lambda+\kappa)} i_{\beta_I}^* \Theta_{X_{\text{flop}}}(\lambda) \cdot \Theta_{-X}(\lambda)$$

$$= \sum_{\lambda \in \Xi} (-1)^{(\kappa,\lambda)} q^{\frac{1}{2}(\lambda+\lambda+\kappa)} i_{\beta_I}^* \Theta_{X'}(\lambda) \cdot \Theta_X(\lambda).$$

This and (77) implies that if we define $\mathcal{M}_X$-semilinear map $\beta_X : \mathcal{K}(X)_{\text{loc}} \to \mathcal{K}(X)_{\text{loc}}$ by

$$\beta_X(\Theta_X(\lambda)) = \Theta_X(\lambda)$$

(78)

for any $\lambda \in \Xi$, then we have $\beta_X(\mathcal{S}_X(p_I)) = (-1)^d \mathcal{S}_{-X}(p_I)$ for any $I \in \mathbb{B}$ i.e., $\beta_X = \beta_X^H$. This proves the following result which is the main observation in this paper and partly justify our definition of elliptic canonical bases for toric hyper-Kähler manifolds.

**Corollary 6.10.** For each $\lambda \in \mathbb{Z}^n$, we have $\beta_X^{\text{ell}}(\Theta_X(\lambda)) = \Theta_X(\lambda)$ and $\beta_X^{\text{ell}}(\Theta_{-X}(\lambda)) = \Theta_{-X}(\lambda)$. Moreover, the elliptic bar involution $\beta_X^{\text{ell}}$ does not depend on the choice of chamber $\mathfrak{C}$.

**Proof.** For the independence on $\mathfrak{C}$, it is enough to note that $\Theta_X(\lambda)$ does not depend on the choice of $\mathfrak{C}$ and (78) uniquely characterize the map $\beta_X^{\text{ell}}$. 

**Remark 6.11.** If one try to prove $\beta_X^{\text{ell}}(\Theta_X(\lambda)) = \Theta_X(\lambda)$ directly from the definition of $\beta_X^{\text{ell}}$, then certain nontrivial identities of various theta functions will be needed. Our proof mimics that of Proposition 5.15 and does not involve any nontrivial calculations. In fact, our definition of elliptic canonical bases is designed so that this kind of proof works nicely.
Corollary 6.12. Conjecture 4.8 holds for toric hyper-Kähler manifolds.

Proof. This follows from Corollary 6.10 by reversing the argument of the proof of Proposition 4.15.

This completes the proof of all the conjectures stated in section 3 and 4 in the case of toric hyper-Kähler manifolds.

6.3 K-theory limits

In this section, we check that the elliptic canonical bases for toric hyper-Kähler manifolds lift K-theoretic canonical bases for any slopes. This result gives another justification of our definition of elliptic canonical bases.

Let $\lambda \in \mathbb{Z}^n$ and $s \in s_{\text{reg}}$. Since $\Theta_X(\lambda)|_{\mathbb{Z}^{n-q}}$ might not have well-defined limit under $q \to 0$, we consider its leading term $LT_s(\Theta_X(\lambda)) = LT(\Theta_X(\lambda)|_{\mathbb{Z}^{n-q}})$. Here, we write $LT(f) := t_0$ for $f = \sum_{t \in \mathbb{Z}^n} f_t q^t$ and $t_0 := \min\{t \mid f_t \neq 0\}$ if it exists. In order to calculate $LT_s(\Theta_X(\lambda))$, we need to know when the function $\frac{1}{2}(\beta, \beta + \kappa) + (\lambda - s, \beta)$ on $\beta \in \mathcal{X}_+(S)$ takes its minimum. We note that $LT_s(\Theta_X(\lambda + \alpha)) = a^\alpha \cdot LT_s(\Theta_X(\lambda))$ and $LT_s(\Theta_X(\lambda + \beta)) = (-1)^{\langle c, \beta \rangle} LT_s(\Theta_X(\lambda))$ for any $\alpha \in \mathcal{X}^*(H)$ and $\beta \in \mathcal{X}_+(S)$. In particular, we only need to consider $LT_s(\Theta_X(\lambda))$ for $\lambda \in \mathcal{X}$. We will identify $\mathcal{X} = \{\mu_I\}_{I \in \mathcal{I}}$ by using Corollary 6.8.

We note that for any $t \in \mathbb{R}$, the function from $\mathbb{Z}$ to $\mathbb{R}$ defined by $\frac{1}{2} m(m+1) - tm$ for $m \in \mathbb{Z}$ takes its minimum at $m = |t|$ if $t \notin \mathbb{Z}$ and $m = t, t - 1$ if $t \in \mathbb{Z}$. Therefore, the function from $\mathbb{Z}^n$ to $\mathbb{R}$ given by

$$\frac{1}{2}(\mu, \mu + \kappa) - (s, \omega_H i^*_p \omega^i(\mu)) = \sum_{i=1}^n \frac{1}{2} \mu_i (\mu_i + 1) - \sum_{j \in I^c} \mu_j (s, \beta_j^i)$$

(79)

for $\mu \in \mathbb{Z}^n$ takes its minimum when $\mu_j = \langle (s, \beta_j^i) \rangle$ for any $j \in I^c$ and $\mu_i = 0, -1$ for any $i \in I$. We remark that for any such $\mu$, we have $\mathcal{L}(\mu) \in \mathcal{B}_{\mathcal{X}, s}$ by (52). In particular, (79) takes its minimum when $\mu = \mu_I$. Therefore, we have

$$\frac{1}{2}(\mu_I, \mu_I + \kappa) - (s, \omega_H i^*_p \omega^i(\mu_I)) \leq \frac{1}{2}(\mu_I + \beta, \mu_I + \beta + \kappa) - (s, \beta + \omega_H i^*_p \omega^i(\mu_I)) = \frac{1}{2}(\mu_I, \mu_I + \kappa) - (s, \omega_H i^*_p \omega^i(\mu_I)) + \frac{1}{2}(\beta, \beta + \kappa) + (\mu_I - s, \beta)$$

for any $\beta \in \mathcal{X}_+(S)$. I.e., the function $\frac{1}{2}(\beta, \beta + \kappa) + (\mu_I - s, \beta)$ on $\mathcal{X}_+(S)$ takes its minimum at $\beta = 0$. On the other hand, if this function takes its minimum at $0 \neq \beta \in \mathcal{X}_+(S)$, then the function (79) takes its minimum at $\mu_I + \beta$ and hence we obtain $\mathcal{L}(\mu_I), \mathcal{L}(\mu_I + \beta) \in \mathcal{B}_{\mathcal{X}, s}$. This contradicts the inequality (61) as in the proof of Corollary 6.8. Therefore, we obtain $LT_s(\Theta_X(\mu_I)) = \mathcal{L}(\mu_I) \in \mathcal{B}_{\mathcal{X}, s}$. In summary, we obtained the following formula.

Proposition 6.13. For any $\lambda \in \mathbb{Z}^n$, there exists unique $A \in \mathcal{A}_{\lambda}$ and $\beta \in \mathcal{X}_+(S)$ such that $\lambda = \mu_A + \beta$, where $\mu_A$ is defined as in (50). Moreover, we have $LT_s(\Theta_X(\lambda)) = (-1)^{\langle \kappa, \beta \rangle} \mathcal{E}(A)$.

Let us take $s_+, s_- \in s_{\text{reg}}$ as in section 5.5 which are separated by a wall $w = \{x \in s_\mathcal{X} \mid \langle x, \beta_C \rangle = m\}$ for some signed circuit $C$ and $m \in \mathbb{Z}$ with $(\eta, \beta_C) > 0$. For any $\lambda \in \mathbb{Z}^n$, there exists unique $A_\pm \in \mathcal{A}_{s_\mathcal{X}}$ and $\beta_\pm \in \mathcal{X}_+(S)$ such that $\lambda = \mu_A_\pm + \beta_\pm$ by Proposition 6.13. If $\mathcal{L}(\mu_A_\pm) \in \mathcal{B}_{\mathcal{X}, s_\mathcal{X}}$, then the uniqueness implies $\mu_A_\pm = \mu_A_\pm$ and $\beta_\pm = \beta_\pm$. Therefore, we obtain $LT_{s_\mathcal{X}}(\Theta_X(\lambda)) = LT_{s_\mathcal{X}}(\Theta_X(\lambda)) \in B_{s_\mathcal{X}}$. On the other hand, if $\mathcal{L}(\mu_A_\pm) \in B_{s_\mathcal{X}} \setminus B_{s_\mathcal{X}}$, then the uniqueness and Proposition 5.20 implies that we have $\mu_A_\pm = \mu_A_\pm - \beta_\pm$ and $\beta_\pm = \beta_\pm - \beta_C$. Therefore, Lemma 5.21 implies that $LT_{s_\mathcal{X}}(\Theta_X(\lambda)) = \mathcal{P}(C)$, i.e., $LT_{s_\mathcal{X}}(\Theta_X(\lambda))$ modulo lower terms spanned by $B_{s_\mathcal{X}}$. In some sense, the elliptic canonical bases organize part of wall-crossing phenomenon of K-theoretic canonical bases in Conjecture 3.49 in a beautiful way.
References

[1] M. Aganagic, A. Okounkov, Elliptic stable envelopes, arXiv:1604.00423

[2] R. Anno, R. Bezrukavnikov, I. Mirković, Stability conditions for Slodowy slices and real variations of stability, Mosc. Math. J. 15 (2015), no. 2, 187–203

[3] A. Bachem, W. Kern, Linear Programming Duality: An Introduction to Oriented Matroids, Universitext, Springer-Verlag, Berlin, 1992

[4] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996), no. 2, 473–527

[5] R. Bezrukavnikov, D. Kaledin, Fedosov quantization in positive characteristic, J. Amer. Math. Soc. 21 (2008), no. 2, 409–438

[6] R. Bezrukavnikov, I. Mirković, Representations of semisimple Lie algebras in prime characteristic and the noncommutative Springer resolution, Ann. of Math. (2) 178 (2013), no. 3, 835–919

[7] R. Bielawski, A. S. Dancer, The geometry and topology of toric hyperkähler manifolds, Comm. Anal. Geom. 8 (2000), no. 4, 727–760

[8] T. Braden, A. Licata, N. Proudfoot, B. Webster, Quantizations of conical symplectic resolutions, Astérisque No. 384 (2016)

[9] N. Chriss, V. Ginzburg, Representation theory and complex geometry, Birkhäuser Boston Inc., Boston, MA, 1997

[10] E. Cline, B. Parshall, L. Scott, Finite-dimensional algebras and highest weight categories, J. Reine Angew. Math. 391 (1988), 85–99

[11] E. Cline, B. Parshall, L. Scott, The homological dual of a highest weight category, Proc. London Math. Soc. (3) 68 (1994), 294–316

[12] D. A. Cox, The homogeneous coordinate ring of a toric variety, J. Algebraic Geom. 4 (1995), no. 1, 17–50

[13] D. A. Cox, J. B. Little, H. K. Schenck, Toric varieties, Graduate Studies in Mathematics, 124. American Mathematical Society, Providence, RI, 2011

[14] H. Dinkins, A. Smirnov, Characters of tangent spaces at torus fixed points and 3d-mirror symmetry, arXiv:1908.01199

[15] N. Ganter, The elliptic Weyl character formula, Compos. Math. 150 (2014), no. 7, 1196–1234

[16] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, Generalized Euler integrals and A-hypergeometric functions, Adv. Math. 84 (1990), no. 2, 255–271

[17] M. Haiman, Combinatorics, symmetric functions, and Hilbert schemes, Current developments in mathematics, 2002, 39–111, Int. Press, Somerville, MA, 2003

[18] T. Hausel, B. Sturmfels, Toric hyperKähler varieties, Doc. Math., 7 (2002), 495–534

[19] M. Kashiwara, On level-zero representations of quantized affine algebras, Duke Math. J. 112 (2002), no. 1, 117–175

[20] G. Lusztig, Hecke algebras and Jantzen’s generic decomposition patterns, Adv. in Math. 37 (1980), no. 2, 121–164
[21] G. Lusztig, Periodic W-graphs, Represent. Theory 1 (1997), 207–279
[22] G. Lusztig, Bases in equivariant K-theory, Represent. Theory 2 (1998), 298–369
[23] G. Lusztig, Bases in equivariant K-theory. II, Represent. Theory 3 (1999), 281–353
[24] G. Lusztig, Remarks on quiver varieties, Duke Math. J. 105 (2000), no. 2, 239–265
[25] D. Maulik, A. Okounkov, Quantum Groups and Quantum Cohomology, Astérisque No. 408 (2019)
[26] V. Mazorchuk, S. Ovsienko, C. Stroppel, Quadratic duals, Koszul dual functors, and applications, Trans. Amer. Math. Soc. 361 (2009), no. 3, 1129–1172
[27] M. McBreen, D. K. Shenfeld, Quantum cohomology of hypertoric varieties, Lett. Math. Phys. 103 (2013), no. 11, 1273–1291
[28] M. McBreen, B. Webster, Homological Mirror Symmetry for Hypertoric Varieties I, arXiv:1804.10646
[29] M. Mustaţă, Local cohomology at monomial ideals, J. Symbolic Comput. 29 (2000), no. 4–5, 709–720.
[30] T. Nagaoka, The universal Poisson deformation of hypertoric varieties and some classification results, arXiv:1810.02961
[31] H. Nakajima, Extremal weight modules of quantum affine algebras, Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., 40, Math. Soc. Japan, Tokyo, 2004, 343–369
[32] A. Okounkov, Lectures on K-theoretic computations in enumerative geometry, Geometry of moduli spaces and representation theory, 251–380, IAS/Park City Math. Ser., 24, Amer. Math. Soc., Providence, RI, 2017
[33] A. Okounkov, Enumerative geometry and geometric representation theory, Algebraic geometry: Salt Lake City 2015, 419–457, Proc. Sympos. Pure Math., 97.1, Amer. Math. Soc., Providence, RI, 2018
[34] A. Okounkov, A. Smirnov, Quantum difference equation for Nakajima varieties, arXiv:1602.09007
[35] N. Proudfoot, A survey of hypertoric geometry and topology, Contemp. Math., 460 (2008), 323–338
[36] R. Rimányi, A. Smirnov, A. Varchenko, Z. Zhou, 3d Mirror Symmetry and Elliptic Stable Envelopes, arXiv:1902.03677
[37] R. Rimányi, A. Smirnov, A. Varchenko, Z. Zhou, Three dimensional mirror self-symmetry of the cotangent bundle of the full flag variety, SIGMA Symmetry Integrability Geom. Methods Appl. 15 (2019), Paper No. 093
[38] R. Rimányi, V. Tarasov, A. Varchenko, Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety, J. Geom. Phys. 94 (2015), 81–119
[39] A. Smirnov, Elliptic stable envelope for Hilbert scheme of points in the plane, Selecta Math. (N.S.) 26 (2020), no. 1, Art. 3
[40] Š. Špenko, M. Van den Bergh, Tilting bundles on hypertoric varieties, arXiv:1805.05285
[41] C. Su, G. Zhao, C. Zhong, On the K-theory stable bases of the Springer resolution, arXiv:1708.08013

[42] M. Varagnolo, E. Vasserot, Canonical bases and quiver varieties, Represent. Theory 7 (2003), 227–258

[43] B. Webster, Coherent sheaves and quantum Coulomb branches I: tilting bundles from integrable systems, arXiv:1905.04623