Optimal control of a delayed smoking model with immigration

Caixia Sun* and Jianwen Jia*

School of Mathematics and Computer Science, Shanxi Normal University, Linfen, People's Republic of China

ABSTRACT
In this paper, we formulate a smoking model with delay and immigration, and the possibility of optimal controls both over the susceptible and heavy problem smoker subjects is assumed. The existence of the optimal control pair is also proved. We derive the optimality condition via the Pontryagin’s maximum principle with delay and obtain the existence of the optimal solution. Numerical simulation is given. Numerical simulations are made to support the main results.

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1. Introduction
Drinking and smoking are increasingly serious problem of social epidemic. Nowadays many researchers investigate the behaviours by constructing mathematical models [1,3,12,17,18,33]. We have known that smoking is one of the major health problems in the world. More than 5 million deaths are caused by the effect of smoking in different organs of human body in the world. The study of epidemiological process such as spread of nonfatal diseases in a population has been widely used for the spread of smoking habit [11]. Smoking is also known as a slow killer. Cardiopathy is 70 percent more in smokers compared to the persons who are not smoking. Smokers have a 10 percent higher incidence rate of carcinoma of the lungs than that of nonsmokers. Bad breath, stained teeth, high blood pressure, coughing are the main effects of short-term smoking. Smoking always cause disease, disability and death. The World Health Organization reported that smoking causes 250 million people deaths, and predicted that 10 million people will die of smoking-related diseases every year by 2030 [32]. Worldwide, there are approximately 1 billion people smoke cigarettes in the world. Cigarette smoking persists in part because long-term smoking cessation rates are modest on existing treatments [13]. Therefore, smoking is considered as a significant global public health problem. Similar with the spread of many infectious diseases, smoking can spread through social contact. Mathematical modelling has been used to investigate the dynamics of smoking. The first mathematical model for giving up smoking was proposed by Castullo-Garsow et al. in [3], they divided the population into three classes: potential smokers (P), smokers (S) and quit smokers (Q). Subsequently, Sharomi
and Gumel [29] divided the quit smokers into temporarily quit smoker ($Q_t$) and permanently quit smoker ($Q_p$). To understand the dynamical behaviour and other features on giving up smoking model in more detail, the readers can read the monographs [1,12,18,22] and the references cited.

The people have different living in the different cities, including smoking behaviour, beliefs and so on. Immigration will bring these habits to the resettlement areas and transmit these habits to the people around them. On the other hand, we need to persuade immigrants who have problems in smoking to stop smoking. Hence, immigrants who have problems in smoking have an impact on the smoking environment. They make the work of giving up smoking more difficulty to do. So it is meaningful to study the model with immigration.

It is well known that delay epidemic models have been studied widely [2,16,27,31], where delay plays an important role in these models. However, there are few papers concerning the influence of delay and immigration on the dynamics of smoking model.

Motivated by the above consideration, we divide the whole population into four compartments. The nonsmokers who has never smoked (susceptible individuals) ($S(t)$), the light problem smokers who smoked less than 20 cigarettes per day ($A(t)$), the heavy problem smokers who smoked more than 20 cigarettes per day ($I(t)$) and the recovered individuals who had smoked in the past ($R(t)$).

We get the following smoking model with time delay

$$
\begin{align*}
\dot{S}(t) &= \Lambda + (1 - q_1 - q_2) \prod S(t)(\beta A(t) + \gamma I(t)) - \mu S(t) + \eta e^{-\mu \tau} R(t - \tau), \\
\dot{A}(t) &= S(t)(\beta A(t) + \gamma I(t)) - pA(t) - \mu A(t) - mA(t) + q_1 \prod, \\
\dot{I}(t) &= pA(t) - \delta I(t) - \mu I(t) + q_2 \prod, \\
\dot{R}(t) &= \delta I(t) + mA(t) - \mu R(t) - \eta e^{-\mu \tau} R(t - \tau),
\end{align*}
$$

(1)

where $\Lambda$ is the recruitment rate of susceptible individuals who do not smoke or smoke only moderately. We assume that if a susceptible individual is infected by smokers, he will enter into light problem smokers. $\beta$ is the transmission coefficient of infection for the susceptible individuals from the light problem smokers. $\gamma$ is the transmission coefficient of infection for the susceptible individuals from the heavy problem smokers. A light problem smokers will become a heavy problem smokers at rate $p$ due to excessive smoking. A light problem smokers will become a recovered individual at rate $m$ due to treatment. A heavy problem smokers will become a recovered individual at rate $\delta$ due to treatment. A recovered individual becomes a susceptible individual at rate $\eta$. $\mu$ is the natural death rate. $\eta e^{-\mu \tau} R(t - \tau)$ represents that an individual has survived natural death in recovery compartment before becoming susceptible again, where $\tau > 0$ is a constant representing the time lag of the immunity against smoking. $\Pi$ represents the total number of population immigrants. $\Pi$ is divided into three parts, that is, immigrants who enter into the compartment of nonsmokers, immigrants who enter into the compartment of light smokers and immigrants who enter into the compartment of heavy smokers. $q_1$ and $q_2$ are the proportion of $\Pi$ who enter into light smokers and heavy smokers, respectively. The parameters are all positive.
In this work, we focus on the question about how to optimally combine education and media campaign strategies such that the cost of the implementation of the two interventions is minimized while reducing the use of tobacco in the population.

The organization of this paper is as follows. In Section 2, we formulate the optimal control problem and we use the Pontryagin’s Maximum principle with delay given in Göllmann et al. [10] to characterize it. In Section 3, we give the numerical method and the simulation results. We end Section 4 with conclusion.

2. The optimal control problems

In this section, we investigate the optimal control strategy of model (1). The optimal control theory is one of the mature branch of mathematics, which is widely used in engineering and other branches of science. This branch of mathematics mainly concerns with exploring a control law for a system so that a particular criterion is attained [19–21,29]. To do this, an effective strategy will be investigated to control the smoking in the community. In order to control the smokers from smoking we will investigate model (1). The control variable $u_1, u_2$ to be used in the present model represents the education campaigns and media campaigns. Various practices and theoretical studies show that the educational campaigns are effective in lowering the level of smoking use. The authors of [6] found that educational campaigns could boost quitting tobacco use among adults and decrease adult smoking rates. The report released by [14] showed that sufficiently funded anti-smoking media campaigns could decrease smoking among young generation. The knowledge about smoking-related diseases and exposure to anti-tobacco use messages on media could enhance the intention to give up smoking [4,23]. The appropriate use of the control strategy can play a vital role in reducing the smoking. Physical interpretation of the control variable is that it needs to attain the minimum level of chain smokers, lightsmokers and potential smokers. While in case of inattention about education campaigns, thenumber of various categories of tobacco users will increase with time and the number of quit smokers will decrease. The perfect and comprehensive use of the control will bring down the level of smokers, which will reduce the use of tobacco in the population.

Now, the basic model (1) is generalized by incorporating two control interventions. The controls are $u_1, u_2$, which represents the percentage of education campaigns and media campaigns. We get the following equations:

$$
\dot{S}(t) = \Lambda + (1 - q_1 - q_2) \prod -S(t)(\beta A(t) + \gamma I(t)) - \mu S(t)
+ \eta e^{-\mu t} R(t - \tau) - u_1(t) S(t),
$$

$$
\dot{A}(t) = S(t)(\beta A(t) + \gamma I(t)) - p A(t) - \mu A(t) - m A(t) + q_1 \prod,
$$

$$
\dot{I}(t) = p A(t) - \delta I(t) - \mu I(t) + q_2 \prod -u_2(t) I(t),
$$

$$
\dot{R}(t) = \delta I(t) + m A(t) - \mu R(t) - \eta e^{-\mu t} R(t - \tau) + u_1(t) S(t) + u_2(t) I(t),
$$

with the bounds $u_{i\min} \leq u_i(t) \leq u_{i\max}$.

For biological reasons, we assume that the initial conditions of (2) are given as follows:

$$(\psi_1(\theta), \psi_2(\theta), \psi_3(\theta), \psi_4(\theta)) \in \mathbb{C}_+ = \mathbb{C}([-\tau, 0], \mathbb{R}_+^4), \quad \psi_i(0) > 0, i = 1, 2, 3, 4,$$
where
\[ \mathbb{R}^4_+ = \{ (S, A, I, R) : S, A, I, R \geq 0 \} . \]

The total number of population at time \( t \) is given by
\[ N(t) = S(t) + A(t) + I(t) + R(t). \]

According to model (1) or (2), one get
\[ (S + A + I + R)' = \Lambda + \prod -\mu (S + A + I + R), \]
so, \( S + A + I + R \leq \Lambda + \prod / \mu \).

Therefore, we will study model (2) in the following set:
\[ \Omega = \left\{ (S, A, I, R) : 0 \leq S, A, I, R \leq \frac{\Lambda + \prod}{\mu} \right\}. \]

It is easy to see that there exists a unique solution \((S(t), A(t), I(t), R(t))\) of system (2) with initial conditions (3).

Our optimal control problem is to minimize the objective functional given by
\[ J(u_1, u_2) = \int_0^T \left[ A_1 S(t) + A_2 I(t) + \frac{1}{2} B_1 u_1^2(t) + \frac{1}{2} B_2 u_2^2(t) \right] dt \quad (4) \]

Subject to the differential equation (2), the first two terms in the objective functional represent the benefit of education campaigns and media campaigns that we wish to reduce, and the parameters \( A_1 \) and \( A_2 \) are positive constants to keep a balance in the size of \( S(t) \) and \( I(t) \), respectively. We use in the second term in the objective functional, where \( B_1 \) and \( B_2 \) are positive weight parameter which is associated with the control \( u_i(t)(i = 1, 2) \) and the square of the control variable reflects the severity of the side effects of education campaigns and media campaigns.

Our goal is to minimize the objective functional defined in (4) by decreasing the number of the heavy problem smokers and susceptible individuals, i.e. by using possible minimal control variables \( u_1(t), u_2(t) \in U_{ad} \).

where \( U_{ad} \) is the control set defined by
\[ U_{ad} = \{ u = (u_1, u_2) | u_i(t) \text{ measurable, } 0 \leq u_i(t) \leq u_i(t)^{\text{max}} \leq 1, \]
\[ t \in [0, T], i = 1, 2 \}, \]

where \( u_i^{\text{max}} \) is the maximum attainable value for \( u_1 \) and \( u_2^{\text{max}} \) is the maximum attainable value for \( u_2 \). \( T \) is the terminal time.

2.1. Existence of an optimal control pair

To prove the existence of the optimal control, we will use an approach of [9,28].

**Theorem 2.1:** There exists optimal control functions \( u_1^*, u_2^* \) and a set of corresponding solution \( S^*(t), A^*(t), I^*(t), R^*(t) \) so that \( J(u_1^*, u_2^*) = \min J(u_1, u_2), u_1, u_2 \in U_{ad} \).

**Proof:** The theorem can be proved by the Cesari Theorem [9], which satisfies the following arguments:

(1) The set of controls and state variables is non-empty.
(2) The control space is closed and convex.
(3) The right side of the system (2) is bounded by a linear function with the state and control.
(4) The integrand in the objective function is convex with respect to input controls $u_1$ and $u_2$.
(5) There exists a constant $D_1 > 1$ and positive numbers $D_2$ and $D_3$ such that the integrand of the objective functional satisfies

$$A_1S(t) + A_2I(t) + \frac{1}{2}B_1u_1^2(t) + \frac{1}{2}B_2u_2^2(t) \geq D_2 \left(|u_1(t)|^2 + |u_2(t)|^2\right)^{D_1/2} - D_3.$$  

In order to verify these conditions, we use a result by Lukes in [28] to give the existence of solutions of system (2) with bounded coefficients, which proofs condition 1. Our control set satisfies condition 2. Condition 3 can be proved according to the following discussion (similar classical arguments in [8]). Condition 4 is verified by the definition.

The system (2) can be rewritten as follows:

$$G(x) = Cx + F(x),$$

where $x(t) = [S(t), A(t), I(t), R(t)]^T$ is the vector of the state variables, matrix $C$ and the nonlinear function $F(x)$ are defined as follows, respectively:

$$C = \begin{pmatrix}
-\mu - u_1 & 0 & 0 & \eta e^{-\mu \tau} \\
0 & -p - \mu - m & 0 & 0 \\
0 & p & -\delta - \mu - u_2 & 0 \\
u_1 & m & \delta + u_2 & -\mu - \eta e^{-\mu \tau}
\end{pmatrix}$$

$$F(x) = \begin{pmatrix}
\Lambda + (1 - q_1 - q_2)\Pi - \beta SA - \gamma SI \\
\beta SA + \gamma SI + q_1\Pi \\
q_2\Pi \\
0
\end{pmatrix}$$

From the Hölder inequality [15], setting $x_1 = (S_1(t), A_1(t), I_1(t), R_1(t))$ and $x_2 = (S_2(t), A_2(t), I_2(t), R_2(t))$, then

$$F(x_1) - F(x_2) = \begin{pmatrix}
-\beta S_1A_1 + \beta S_2A_2 - \gamma S_1I_1 + \gamma S_2I_2 \\
\beta S_1A_1 - \beta S_2A_2 + \gamma S_1I_1 - \gamma S_2I_2 \\
0 \\
0
\end{pmatrix}$$

Therefore

$$|F(x_1) - F(x_2)| = | - \beta S_1A_1 + \beta S_2A_2| + | - \gamma S_1I_1 + \gamma S_2I_2|$$

$$+ |\beta S_1A_1 - \beta S_2A_2| + |\gamma S_1I_1 - \gamma S_2I_2|$$

$$\leq 2\beta |S_1A_1 - S_2A_2| + 2\gamma |S_1I_1 - S_2I_2|$$

$$= 2\beta |A_1(S_1 - S_2) + S_2(A_1 - A_2)| + 2\gamma |I_1(S_1 - S_2) + S_2(I_1 - I_2)|$$
and the Hamiltonian
\[
H(S, A, I, R, u_1, u_2, \lambda \dddot{i}, t) = \left( A_1 S(t) + A_2 I(t) + \frac{1}{2} B_1 u_1^2(t) + \frac{1}{2} B_2 u_2^2(t) \right)
+ \lambda_1 \left( \Lambda + (1 - q_1 - q_2) \prod -S(t)(\beta A(t) + \gamma I(t)) - \mu S(t) \right)
+ \eta e^{-\mu t} R(t - \tau) - u_1(t) S(t)
+ \lambda_2 \left( S(t)(\beta A(t) + \gamma I(t)) - p A(t) - \mu A(t) - m A(t) + q_1 \prod \right)
+ \lambda_3 \left( p A(t) - \delta I(t) - \mu I(t) + q_2 \prod -u_2(t) I(t) \right)
+ \lambda_4 \left( \delta I(t) + m A(t) - \mu R(t) - \eta e^{-\mu t} R(t - \tau) + u_1(t) S(t) + u_2(t) I(t) \right),
\]
where \( \lambda_i, i = 1, 2, 3, 4 \) are the adjoint functions to be determined suitably.

To obtain the adjoint variables, we followed the Pontryagins Maximum principle with delay[10], we have
Theorem 2.2: Given optimal controls $u_1^*(t)$, $u_2^*(t)$ and solutions $S^*(t)$, $A^*(t)$, $I^*(t)$ and $R^*(t)$ of the corresponding conditions system (4) and (2), there exists adjoint variables $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ that satisfy

$$
\frac{d\lambda_1(t)}{dt} = -A_1 + \lambda_1(\beta A^* + \gamma I^* + \mu + u_1^*) - \lambda_2(\beta A^* + \gamma I^*) - \lambda_4 u_1^*,
$$

$$
\frac{d\lambda_2(t)}{dt} = \lambda_1 \beta S^* - \lambda_2(\beta S^* - \rho - \mu - m) - \lambda_3 \rho - \lambda_4 m,
$$

$$
\frac{d\lambda_3(t)}{dt} = -A_2 + \lambda_1 \gamma S^* - \lambda_2 \gamma S^* + \lambda_3(\delta + \mu + u_2^*) - \lambda_4(\delta + u_2^*),
$$

$$
\frac{d\lambda_4(t)}{dt} = \lambda_4 \mu - \chi_{[0,T]}(\lambda_1(t + \tau) \eta e^{-\mu \tau} - \lambda_4(t + \tau) \eta e^{-\mu \tau}).
$$

(6)

$\lambda_i(T) = 0, i = 1, 2, 3, 4$.  

(7)

$$
u_1^*(t) = \max \left( \min \left( \frac{(\lambda_1(t) - \lambda_4(t)) S^*(t)}{B_1}, u_1^{\max} \right), 0 \right).
$$

(8)

$$
u_2^*(t) = \max \left( \min \left( \frac{(\lambda_3(t) - \lambda_4(t)) I^*(t)}{B_2}, u_2^{\max} \right), 0 \right).
$$

(9)

Proof: Using the Pontryagins Maximum principle with delay, we differentiate the Hamiltonian $H$ with respect to each state and obtain the adjoint equations:

$$
\dot{\lambda}_1(t) = -\frac{\partial H(t)}{\partial s},
$$

$$
\dot{\lambda}_2(t) = -\frac{\partial H(t)}{\partial A},
$$

$$
\dot{\lambda}_3(t) = -\frac{\partial H(t)}{\partial I},
$$

$$
\dot{\lambda}_4(t) = -\frac{\partial H(t)}{\partial R} - \chi_{[0,T-\tau]} \frac{\partial H(t + \tau)}{\partial S_\tau}.
$$

Here $\chi_{[0,T-\tau]}$ is the characteristic function in the interval $[0, T - \tau]$, which is defined [7] by

$$
\chi_{[[0,T-\tau]]} = \begin{cases} 
1, & \text{if } t \in [[0,T-\tau]], \\
0, & \text{otherwise.}
\end{cases}
$$

With transversality conditions

$$
\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = 0,
$$

(10)

by using the optimality we obtain

$$
\frac{\partial H}{\partial u_1} = B_1 u_1^*(t) - \lambda_1(t) S^*(t) + \lambda_4(t) S^*(t) = 0, \quad \text{at } u_1 = u_1^*(t),
$$

$$
\frac{\partial H}{\partial u_2} = B_2 u_2^*(t) - \lambda_3(t) I^*(t) + \lambda_4(t) I^*(t) = 0, \quad \text{at } u_2 = u_2^*(t),
$$
which we find
\[ u_1^*(t) = \frac{(\lambda_1(t) - \lambda_4(t))S^*(t)}{B_1} \quad \text{and} \quad u_2^*(t) = \frac{(\lambda_3(t) - \lambda_4(t))I^*(t)}{B_2}. \]

Furthermore, the control function \( u_1^*(t), u_2^*(t) \) is given by
\[
u_1^*(t) = \begin{cases} 
0, & \text{if } \frac{(\lambda_3(t) - \lambda_4(t))S^*(t)}{B_1} \leq 0, \\
\frac{(\lambda_1(t) - \lambda_4(t))S^*(t)}{B_1}, & \text{if } 0 < \frac{(\lambda_1(t) - \lambda_4(t))S^*(t)}{B_1} < u_1^\text{max}, \\
\max \left( \frac{(\lambda_1(t) - \lambda_4(t))S^*(t)}{B_1}, u_1^\text{max} \right), & \text{if } \frac{(\lambda_1(t) - \lambda_4(t))S^*(t)}{B_1} \geq u_1^\text{max},
\end{cases}
\]
\[
u_2^*(t) = \begin{cases} 
0, & \text{if } \frac{(\lambda_3(t) - \lambda_4(t))I^*(t)}{B_2} \leq 0, \\
\frac{(\lambda_3(t) - \lambda_4(t))I^*(t)}{B_2}, & \text{if } 0 < \frac{(\lambda_3(t) - \lambda_4(t))I^*(t)}{B_2} < u_2^\text{max}, \\
\max \left( \frac{(\lambda_3(t) - \lambda_4(t))I^*(t)}{B_2}, u_2^\text{max} \right), & \text{if } \frac{(\lambda_3(t) - \lambda_4(t))I^*(t)}{B_2} \geq u_2^\text{max}.
\end{cases}
\]

So the optimal control pair is characterized as (8) and (9).

Therefore, our optimality system is given by
\[
\frac{dS^*(t)}{dt} = \Lambda + (1 - q_1 - q_2) \prod - S^*(\beta A^* + \gamma I^*) - \mu S^* + \eta e^{-\mu \tau} R^*(t - \tau)
\]
\[
- \max \left( \min \left( \frac{(\lambda_1(t) - \lambda_4(t))S^*(t)}{B_1}, u_1^\text{max} \right), 0 \right) S^*,
\]
\[
\frac{dA^*(t)}{dt} = S^*(\beta A^* + \gamma I^*) - pA^* - \mu A^* - mA^* + q_1 \prod
\]
\[
- \max \left( \min \left( \frac{(\lambda_3(t) - \lambda_4(t))I^*(t)}{B_2}, u_2^\text{max} \right), 0 \right) I^*,
\]
\[
\frac{dI^*(t)}{dt} = pA^* - \delta I^* - \mu I^* + q_2 \prod
\]
\[
- \max \left( \min \left( \frac{(\lambda_3(t) - \lambda_4(t))I^*(t)}{B_2}, u_2^\text{max} \right), 0 \right) I^*,
\]
\[
\frac{dR^*(t)}{dt} = \delta I^* + mA^* - \mu R^* - \eta e^{-\mu \tau} R^*(t - \tau)
\]
\[
+ \max \left( \min \left( \frac{(\lambda_1(t) - \lambda_4(t))S^*(t)}{B_1}, u_1^\text{max} \right), 0 \right) S^*
\]
\[
+ \max \left( \min \left( \frac{(\lambda_2(t) - \lambda_3(t))S^*(t)}{B_2}, u_2^\text{max} \right), 0 \right) I^*,
\]
\[
\frac{d\lambda_1(t)}{dt} = - A_1 + \lambda_1(\beta A^* + \gamma I^* + \mu + u_1^*) - \lambda_2(\beta A^* + \gamma I^*) - \lambda_4 u_1^*,
\]
\[
\frac{d\lambda_2(t)}{dt} = \lambda_1 \beta S^* - \lambda_2(\beta S^* - p - \mu - m) - \lambda_3 p - \lambda_4 m,
\]
\[
\frac{d\lambda_3(t)}{dt} = - A_2 + \lambda_1 \gamma S^* - \lambda_2 \gamma S^* + \lambda_3(\delta + \mu + u_2^*) - \lambda_4(\delta + u_2^*),
\]
\[
\frac{d\lambda_4(t)}{dt} = \lambda_4 \mu - \chi_{[0, T - \tau]}(\lambda_1(t + \tau) \eta e^{-\mu \tau} - \lambda_4(t + \tau) \eta e^{-\mu \tau}),
\]

(11)
\begin{align*}
u_1^*(t) &= \max \left( \min \left( \frac{(\lambda_1(t) - \lambda_4(t))S(t)}{B_1}, u_1^{\text{max}} \right), 0 \right), \\
u_2^*(t) &= \max \left( \min \left( \frac{(\lambda_3(t) - \lambda_4(t))I(t)}{B_2}, u_2^{\text{max}} \right), 0 \right),
\end{align*}

with \(S(0) = S_0, A(0) = A_0, I(0) = I_0, R(0) = R_0, \lambda_i(T) = 0, i = 1, 2, 3, 4.\)

It is easy to show the uniqueness, hence the following theorem.

\section{Numerical simulations}

In this section, based on \([5,24–26]\), we give a numerical method to solve the optimality system (12).

Let there exists a step size \(h > 0\) and integers \((n, b) \in \mathbb{N}^2\) with \(\tau = bh\) and \(T - t_0 = nh\). For reasons of programming, we consider \(b\) knots to left of \(t_0\) and right of \(T\), and we obtain the following partition:

\[\Delta = (t_{-b} = -\tau < \cdots < t_{-1} \cdots t_0 = 0 < t_1 < \cdots < t_n = T < t_{n+1} < \cdots < t_{n+b}).\]

Then we have \(t_i = t_0 + ih(-b \leq i \leq n + b)\). Next, we define the state and adjoint variables \(S(t), A(t), I(t), R(t), \lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t)\) and the controls \(u_1(t), u_2(t)\) in terms of nodal points \(S_i, A_i, I_i, R_i, \lambda_1^i, \lambda_2^i, \lambda_3^i, \lambda_4^i, u_1^i\) and \(u_2^i\).

Now we give some numerical simulations in order to illustrate the theoretical results obtained in this paper. Let the initial values be \((2, 0.2, 0.1, 0.5)\), and we use the parameter values given in Table 1. The weight constant values in the objective functional are \(A_1 = 10, A_2 = 10, B_1 = 1\) and \(B_2 = 1\).

To examine the impact of each control on reducing the susceptible individuals and heavy smokers, we used the following strategy:

\begin{table}[h]
\centering
\caption{Parameter values used for numerical simulations}
\begin{tabular}{|l|l|l|l|}
\hline
Parameters & Descriptions & Values & Source \\
\hline
\(\Lambda\) & The recruitment rate of susceptible individuals & 0.8 & Assumed \\
\(\beta\) & The transmission coefficient of infection for the susceptible individuals from the light problem smokers & 0.17 & \([30]\) \\
\(\gamma\) & The transmission coefficient of infection for the susceptible individuals from the heavy problem smokers & 0.15 & \([30]\) \\
\(p\) & The rate of light problem smokers become a heavy problem smokers & 0.1 & Assumed \\
\(m\) & The rate of light problem smokers become a recovered individual & 0.9 & Assumed \\
\(\delta\) & The rate of heavy problem smokers become a recovered individual & 0.7 & Assumed \\
\(\eta\) & The rate of A recovered individual becomes a susceptible & 0.4 & Assumed \\
\(\mu\) & all individuals death rate & 0.25 & \([30]\) \\
\(\Pi\) & The total number of population immigrants & 0.7 & Assumed \\
\(q_1\) & The proportion of \(\Pi\) who enter into light smokers & 0.1 & Assumed \\
\(q_2\) & The proportion of \(\Pi\) who enter into heavy smokers & 0.21 & Assumed \\
\(u_1\) & efficacy of vaccination & 0–1 & Assumed \\
\(u_2\) & efficacy of treatment & 0–1 & Assumed \\
\(\tau\) & The time lag of the immunity against smoking & 1 & Assumed \\
\hline
\end{tabular}
\end{table}
Algorithm 1

Step 1

for \( i = -b, \ldots, 0 \), do

\( S_i = S_0, A_i = A_0, I_i = I_0, R_i = R_0, u_1^i = 0 \) and \( u_2^i = 0 \)

end for

for \( i = n, \ldots, n + b \), do

\( \lambda_1^i = 0, \lambda_2^i = 0, \lambda_3^i = 0, \lambda_4^i = 0 \)

end for

Step 2

\( S_{i+1} = S_i + h[\Lambda + (1 - q_1 - q_2)\Pi - S_i(\beta A_i + \gamma I_i) - \mu S_i + \eta e^{-\mu \tau} R_i - b - u_1^i S_i], \)

\( A_{i+1} = A_i + h[S_i(\beta A_i + \gamma I_i) - p A_i - \mu A_i + q_1 \Pi], \)

\( I_{i+1} = I_i + h[p A_i - \delta I_i - \mu I_i + q_2 \Pi - u_2^i I_i], \)

\( R_{i+1} = R_i + h[\delta I_i + m A_i - \mu R_i - \delta - e^{-\mu \tau} R_i - b + u_1^i S_i + u_2^i I_i], \)

\( \lambda_1^{n-i} = \lambda_1^{n-i} - h[-A_1 + \lambda_1^{n-i}(\beta A_i + \gamma I_i) - \mu + u_1^i] - \lambda_2^{n-i}(\beta A_i + \gamma I_i) - \lambda_4^{n-i} u_1^i \}

\( \lambda_2^{n-i} = \lambda_2^{n-i} - h[\lambda_2^{n-i} S_i - \lambda_2^{n-i}(\beta S_i - p - \mu - m) - \lambda_3^{n-i} S_i + \lambda_4^{n-i} p - \lambda_4^{n-i} m] \}

\( \lambda_3^{n-i} = \lambda_3^{n-i} - h[-A_2 + \lambda_1^{n-i} \gamma S_i - \lambda_2^{n-i} \gamma S_i - \lambda_3^{n-i}(\delta + \mu + u_2^i) - \lambda_4^{n-i}(\delta + u_2^i)] \}

\( \lambda_4^{n-i} = \lambda_4^{n-i} - h[\lambda_4^{n-i} \mu - \chi(0,T-\tau)(t_{n-i})(\lambda_4^{n-i} b + \eta e^{-\mu \tau} - \lambda_4^{n-i} b + \eta e^{-\mu \tau})], \)

\( \theta_1^{i+1} = \frac{\lambda_1^{n-i} - \lambda_4^{n-i} S_{i+1}}{b_1} \)

\( \theta_2^{i+1} = \frac{\lambda_3^{n-i} - \lambda_4^{n-i} S_{i+1}}{b_2} \)

\( u_1^i = \max(\min(\theta_1^{i+1}, u_1^{\max}), 0) \)

\( u_2^i = \max(\min(\theta_2^{i+1}, u_2^{\max}), 0) \)

Step 3

for \( i = 1, \ldots, n \), do

\( S_i = S_i, A_i = A_i, I_i = I_i, R_i = R_i, u_1^i = u_1^i \) and \( u_2^i = u_2^i \)

end for

3.1. Only media campaigns control \((u_1 = 0, u_2 \neq 0)\)

With this strategy, implementing only the media campaigns control \( u_1 \) as intervention.

In Figure 1, we observe that there is a significant decrease in the number of heavy smokers controlled compared with those not controlled, we also see that there is a significant increase in the number of susceptible individuals and light problem smokers.

3.2. Only education campaigns control \((u_1 \neq 0, u_2 = 0)\)

With this strategy, applying prevention only the education campaigns control \( u_1 \) as intervention.

In Figure 1, we observe that there is a significant decrease in the number of susceptible individuals controlled compared with those not controlled, we also see that there is a slight increase in the number of heavy smokers.

3.3. Combined education campaigns and media campaigns strategy

We use both the education campaigns control \( u_1 \) and media campaigns control \( u_2 \) to optimize the objective function \( J(u) \).
Figure 1. Evolution of different classes of individuals with both controls $u_1 \neq 0, u_2 = 0$ (marked by the solid lines) and without controls (marked by the dotted lines).

Figure 2. Evolution of different classes of individuals with both controls (marked by the solid lines) and without controls (marked by the dotted lines).

In Figure 2, we observe that there is a significant decrease in the number of susceptible individuals and heavy smokers controlled compared with those not controlled.

The values of optimal control variable $u_1(t)$ and $u_2(t)$ are represented in Figure 3, which make ensure that by introducing an optimal control into an SAIR model can have a obvious effect on the elimination of susceptible individuals and heavy smokers.
4. Conclusion

We have established a delay smoking model with immigration. An optimal control problem for the model is investigated. The two control functions $u_1$ and $u_2$ represent the percentage of education campaigns and media campaigns for susceptible individuals and heavy smokers, respectively. We showed the existence for the optimal control pair, Pontryagins maximum principle with delay is used to characterize these optimal controls, and the optimality system is derived. Our numerical results show that the two control practices are very effective in reducing the susceptible individuals and heavy smokers.

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Disclosure statement

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