Associativity, Jacobi, Bremner, and All That

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Abstract. I discuss various aspects of multi-linear algebras related to associativity.

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1. Introduction
Nambu introduced a multilinear operator $N$-bracket in the context of a novel formulation of mechanics [15]:

$$ [A_1 A_2 \cdots A_N] = \sum_{\sigma \in S_N} \text{sgn} (\sigma) \ A_{\sigma_1} \cdots A_{\sigma_N} ,$$

where the sum is over all $N!$ permutations of the operators. For example, the operator 3-bracket is

$$ [ABC] = ABC - ACB + BCA - BAC + CAB - CBA .$$

The operator product here is assumed to be associative.

The same construction independently appeared in the mathematical literature more than 50 years ago [12, 13]. The theory of such multi-operator products, as well as their “classical limits” in terms of multivariable Jacobians, has been studied extensively [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 14, 16, 17, 18, 19, 20].

From an algebraic point of view, it is natural to seek the analogue of the Jacobi identity for operator $N$-brackets. For the case of even $N$-brackets, the obvious generalization where one $N$-bracket acts on another leads to a true identity. However, for odd $N$-brackets this usually does not work. For instance, it is almost always true that

$$ [[ABC] DE] - [[ADE] BC] - [A [BDE] C] - [AB [CDE]] \neq 0 .$$

That is to say, the so-called FI (“fundamental identity”) fails. There is one especially notable exception: $su(2)$, as described by Nambu [15].

Fortunately, even brackets are not odd. They need only act twice to yield an identity. Namely [6, 11],

$$ [B_1 \cdots B_{N-1} [B_N \cdots B_{2N-1}]] = 0 \quad \text{for} \ N \ \text{even.}$$

Here, total antisymmetrization of all the $B$s is understood. When $N = 2$ this is the familiar Jacobi identity. The proof is by direct calculation and follows as a consequence of associativity.

Unfortunately, an odd $N$-bracket acting on just one other odd $N$-bracket does not vanish even when totally antisymmetrized over all entries, but rather produces a $(2N - 1)$-bracket [6, 5]. Therefore the simplest identity obeyed by odd brackets of only one type, that does not introduce higher-order brackets, requires that they act at least thrice.
**Bremner Identity and GBIs**

Bremner [1] proved an identity (henceforth the “BI”) for associative operator 3-brackets acting thrice.

\[
[ [A \ [bcd] \ e] \ f \ g] = [ [A] \ [def] \ g] ,
\]

where it is understood that all lower case entries are totally antisymmetrized by implicitly summing over all \(6! = 720\) signed permutations of them.

The BI can be proven through a resolution of both LHS and RHS as a series of canonically ordered words. By direct calculation we find

\[
[ [A \ [bcd] \ e] \ f \ g] = 24 \ Abcdefg - 36 \ bAcdefg + 36 \ bcAdefg - 24 \ bcdAefg + 36 \ bcdeAfg - 36 \ bcdefAg + 24 \ bcdefgA .
\]

The same expansion holds for \([ [Abc] \ [def] \ g]\), again by direct calculation. That is to say, both \([ [A \ [bcd] \ e] \ f \ g]\) and \([ [Abc] \ [def] \ g]\) can be rendered as a 7-bracket plus another 3-bracket containing 3-brackets:

\[
[ [A \ [bcd] \ e] \ f \ g] = \frac{1}{20} [ Abcdefg ] - \frac{1}{6} [ A [bcd] [efg] ] = [ [Abc] \ [def] \ g] .
\]

Thus the BI amounts to the combinatorial statement that there are two distinct ways to write a 7-bracket in terms of nested 3-brackets.

Xiang Jin, Luca Mezincescu, and I proved that a similar identity holds for any odd-order bracket acting thrice [4]. For odd \(N = 2L + 1\), this generalized BI ("GBI") is

\[
[ \ [AB_1 \cdots B_{2L}] \ [B_{2L+1} \cdots B_{4L+1}] \ B_{4L+2} \cdots B_{6L} ] \\
= [ \ [A \ [B_1 \cdots B_{2L+1}] \ B_{2L+2} \cdots B_{4L}] \ B_{4L+1} \cdots B_{6L} ] .
\]

Again, this identity is a consequence of only associativity. Thus all odd brackets built from associative products of operators need only act thrice to yield an identity. Given an hypothesized closed algebra of odd \(N\)-brackets, the GBI provides the simplest test for consistency with an underlying associative product.

To prove the GBI, we again expanded in terms of canonically ordered words. By direct calculation,

\[
[[AB_1 \cdots B_{2L+1} \ B_{2L+2} \cdots B_{4L}] \ B_{4L+1} \cdots B_{6L}] = \sum_{n=0}^{6L} (-1)^n m_n B_1 \cdots B_n A \ B_{n+1} \cdots B_{6L} ,
\]

where it is implicit that one is to totally antisymmetrize over all the Bs. All the coefficients in the resolution are integers. Explicitly,

\[
m_n = (2L + 1)! \ (2L)! \ (2L - 1)! \times c_n ,
\]

\[
c_n = \begin{cases} 
(n + 1) \ (4L - n) / 2 & \text{for} \ 0 \leq n \leq 2L \\
10L^2 - 6Ln + L + n^2 & \text{for} \ 2L + 1 \leq n \leq 3L \\
6L - n & \text{for} \ 3L + 1 \leq n \leq 6L
\end{cases}.
\]

The same expansion holds for \([ [AB_1 \cdots B_{2L}] \ [B_{2L+1} \cdots B_{4L+1}] \ B_{4L+2} \cdots B_{6L}]\), again by direct calculation. Hence the GBI.
For example, \( L = 1 \) gives the previous coefficients (6), while \( L = 2 \) gives

\[
\begin{pmatrix}
m_0 \\
m_1 \\
m_2 \\
m_3 \\
m_4 \\
m_5 \\
m_6 \\
m_7 \\
m_8 \\
m_9 \\
m_{10} \\
m_{11} \\
m_{12}
\end{pmatrix}
= 5!4!3! \times
\begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5 \\
c_6 \\
c_7 \\
c_8 \\
c_9 \\
c_{10} \\
c_{11} \\
c_{12}
\end{pmatrix}
= 5!4!3! \times
\begin{pmatrix}
4 \\
7 \\
9 \\
10 \\
10 \\
7 \\
6 \\
7 \\
10 \\
7 \\
10 \\
7 \\
4
\end{pmatrix}
= (5!4!3!) \times
\begin{pmatrix}
69 120 \\
120 960 \\
155 520 \\
172 800 \\
120 960 \\
103 680 \\
120 960 \\
172 800 \\
172 800 \\
155 520 \\
120 960 \\
69 120
\end{pmatrix}.
\]

(11)

For simplicity, I emphasize the 3-bracket case in the following.

**3-Algebras**

For a 3-algebra with linearly independent operators \( T_a \) that obey

\[
[T_aT_bT_c] = i \, F_{abc} \, T_d,
\]

the BI becomes, with implicit total antisymmetrization of the six \( b_j \) indices,

\[
F_{ab_1b_2} \, F_{b_3b_4b_5} \, F_{x_6y_7z_8} = F_{b_1b_2b_3} \, F_{a_4b_4x} \, F_{y_5z_6y_7}. \tag{13}
\]

Alternatively, after renaming and cycling indices,

\[
F_{b_1b_2b_3} \left( F_{axb_4} \, F_{y_5z_6y_7} - F_{ayb_5} \, F_{xz_6y_7} \right) = 0. \tag{14}
\]

This *trilinear relation* is a condition on the structure constants required by an underlying associativity for any posited 3-algebra.

**Exercise** (25 points; show all details; due Friday): Use (14) to prove a classification theorem for 3-algebras.

To be more specific, consider now any *closed bilinear algebra* where all commutators and anticommutators are also elements of the algebra, as given by

\[
[T_aT_b] = i f_{ab} \, T_c, \quad \{T_aT_b\} = g_{ab} \, T_c. \tag{15}
\]

For example, for \( u(N) \) with the \( T_a \) given by \( N \times N \) matrices, the second RHS involves the well-known \( d_{abc} \) symbol, as well as Kronecker delta terms. Or, with a bit of freedom of interpretation, one may think of the operator product expansion of any CFT in this way.

For a bilinear algebra of this form, the corresponding 3-algebra is also completely determined, or “induced.” This follows from

\[
2 \times [ABC] = \{[AB] \, C\} + \{[BC] \, A\} + \{[CA] \, B\}. \tag{16}
\]

The induced 3-algebra structure constants are given in terms of the \( f \) and \( g \) symbols by

\[
2 \, F_{abc} \, x = f_{ab} \, u \, g_{uc} \, x + f_{bc} \, u \, g_{wa} \, x + f_{ca} \, u \, g_{ab} \, x. \tag{17}
\]
Thus the BI conditions on the induced 3-algebra structure constants can be re-expressed in terms of \( f \) and \( g \). Again with implicit antisymmetrizations, the BI conditions become

\[
f_{b_1 b_2}^u g_{ab_3} \left( F_{axb_2}^y F_{yb_3b_1}^z - F_{axb_3}^y F_{yb_2b_1}^z \right) = 0 .
\]  

(18)

These conditions are indeed obeyed when the \( f \) and \( g \) symbols satisfy the conditions wrought by associativity.

The Jacobi identity (JI),

\[
\{A [BC]\} + \{B [CA]\} + \{C [AB]\} = 0 ,
\]

(19)

is a consequence of associativity but it is not equivalent to it, even when augmented with the super Jacobi identity (SJI),

\[
\{\{AB\} C\} = \{\{A [BC]\}\} + \{B [AC]\} .
\]

(20)

Here, we have used the usual (anti)commutator notation, sans commas, \( \{AB\} = AB + BA \) and \( [AB] = AB - BA \).

However, there is another trilinear identity which, when paired with the SJI, is equivalent to associativity. Namely,

\[
\{A [BC]\} = \{\{AB\} C\} - \{\{AC\} B\} .
\]

(21)

For want of a more compelling name, we will refer to this third relation as the “super-duper Jacobi identity” (SDJI)\(^1\). Note that the JI follows from the SDJI.

For a closed bilinear algebra the SJI and SDJI identities require the following conditions to be obeyed by the structure constants:

\[
g_{ab} \ f_{uc} \ x = g_{bu} \ f_{uc} \ x + g_{ub} \ f_{ac} \ x , \quad f_{bc} \ u \ f_{au} \ x = g_{ub} \ g_{uc} - g_{ac} \ g_{ub} \ x .
\]

(22)

The more familiar JI conditions on the bilinear algebra structure constants follow from the second of these.

As I mentioned already, the BI conditions for the induced 3-algebra structure constants, expressed in terms of \( f \) and \( g \), follow from these two conditions. Moreover, the structure constants \( F \) for any induced \( N \)-bracket algebra can be expressed in terms of \( f \) and \( g \) for such closed bilinear algebras, and it can be shown that the conditions on \( F \) imposed by associativity are indeed satisfied as a consequence of these same two conditions on \( f \) and \( g \).

As an aside, there are deformed versions of these identities involving the “quommutators”

\[
[AB]_{\lambda} = \lambda AB - \lambda^{-1} BA .
\]

(23)

These naturally lead to 3-brackets. For example, the “Jaquobi identity”:

\[
[[AB]_{\lambda} C]_{\mu} + [[BC]_{\lambda} A]_{\mu} + [[CA]_{\lambda} B]_{\mu} = \frac{1}{2} \left( \lambda + \frac{1}{\lambda} \right) \left( \mu - \frac{1}{\mu} \right) [ABC] + \frac{1}{2} \left( \lambda - \frac{1}{\lambda} \right) \left( \mu - \frac{1}{\mu} \right) \{ABC\} .
\]

(24)

Etc.\(^2\) Here, we have also used the totally symmetrized 3-bracket: \( \{ABC\} = ABC + ACB + BCA + BAC + CAB + CBA \).

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1 Apologies to Irving Berlin.

2 The ultimate Jaquobi identity would involve six complex parameters, \( \lambda_j \) and \( \mu_j \), \( j = 1, 2, 3 \), as in:

\[
\left[ [AB]_{\lambda_1} C \right]_{\mu_1} + \left[ [BC]_{\lambda_2} A \right]_{\mu_2} + \left[ [CA]_{\lambda_3} B \right]_{\mu_3} .
\]

Requiring that this vanish gives six equations for the parameters.

Assuming \( \mu_1 \neq 0 \), \( \lambda_3 \neq 0 \), \( \mu_3 \neq 0 \), the generic solution is:

\[
\lambda_1 = \mu_1 \lambda_3 \mu_3 , \quad \lambda_2 = \mu_1 \lambda_3 , \quad \text{and} \quad \mu_2 = \frac{1}{\mu_1 \lambda_3} .
\]

So the solution manifold has complex dimension three, including an overall complex scale. For fixed scale, it is in fact a geometrically ruled surface, and it must contain the usual Jacobi, the super Jacobi, and the super-duper Jacobi identities. Indeed, it does. The symmetric group is a symmetry of the solution manifold, so other solutions are given by permutations of 1, 2, 3. There is also parity. Another solution is obtained from the generic one just by flipping the signs of all the parameters.
Classical Manifolds

There are also interesting questions for classical manifold theory that arise in this context. A classical 3-bracket is defined by

$$[A, B, C] = \omega^{abc} \partial_a A \partial_b B \partial_c C,$$

with antisymmetric but otherwise arbitrary 3-tensor \( \omega^{abc} \). The combination that constitutes the so-called FI is

$$[E, F, [A, B, C]] - [[E, F, A], B, C] - [A, [E, F, B], C] - [A, B, [E, F, C]] = \left( \omega^{abc} \omega^{def} - \omega^{dbc} \omega^{aef} - \omega^{adc} \omega^{bef} - \omega^{abd} \omega^{cef} \right) \partial_d \left( \partial_a A \partial_b B \partial_c C \partial_e E \partial_f F \right)$$

\[+ \left( \partial_a A \partial_b B \partial_c C \partial_e E \partial_f F \right) \left( \omega^{def} \partial_d \omega^{abc} - \omega^{dbc} \partial_d \omega^{aef} - \omega^{adc} \partial_d \omega^{bef} - \omega^{abd} \partial_d \omega^{cef} \right). \tag{26} \]

In the literature, when \( \omega \) is such that this vanishes, this is called a Nambu-Poisson manifold. This gives two types of bilinear constraints on \( \omega \), obviously.

But, to conclude this talk, it seems more reasonable (to me at least) that one should impose, instead of the FI, a classical analogue of the BI. For a classical N-bracket involving \( n \geq N \) (odd) independent variables, with antisymmetric but otherwise arbitrary N-tensor \( \omega^{a_1 \cdots a_N} \),

$$[B_{i_1}, B_{i_2}, \ldots, B_{i_N}] = \omega^{a_1 \cdots a_N} \partial_{a_1} B_{i_1} \cdots \partial_{a_N} B_{i_N},$$

we define a Bremer-Poisson manifold as one for which the BI holds. This leads to requirements on the \( \omega \) tensor that differ from those imposed by the FI. So defined, Bremer-Poisson and Nambu-Poisson manifolds are different, in general. We will discuss this in more detail elsewhere.

Perhaps N-brackets and algebras have an important role to play in physics, as originally suggested by Nambu. Recently there has been considerable interest in N-brackets, especially 3-brackets, as expressed in the physics literature (see [2] and references therein). These ideas await further development.

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