Solitary and periodic wave solutions of the loaded modified Benjamin-Bona-Mahony equation via the functional variable method

Abstract. In this article, we establish new travelling wave solutions for the loaded Benjamin-Bona-Mahony and the loaded modified Benjamin-Bona-Mahony equation by the functional variable method. The performance of this method is reliable and effective and gives the exact solitary wave solutions and periodic wave solutions. All solutions of these equations have been examined and three dimensional graphics of the obtained solutions have been drawn by using the Matlab program. We get some travelling wave solutions, which are expressed by the hyperbolic functions and trigonometric functions. This method is effective to find exact solutions of many other similar equations.

Key words: loaded Benjamin-Bona-Mahony equation, loaded modified Benjamin-Bona-Mahony equation, hyperbolic functions, trigonometric functions, periodic wave solutions, solitary wave solutions, functional variable method

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1. Introduction

Benjamin-Bona-Mahony (BBM) equation is well known in the analysis of the surface waves of long wavelength in liquids, hydro magnetic waves in cold plasma, acoustic-gravity waves in compressible fluids, and acoustic waves in harmonic crystals and it describes the model for propagation of long waves which incorporates nonlinear and dissipative effects [1]. In the last two decades, various versions of the BBM equation have been investigated in the literature [2].

In 1972, Benjamin, Bona, and Mahony formulated model equation for the unidirectional propagation of small-amplitude long waves on the surface of water in a channel [5]. A general form of the BBM equation is

\[ u_x + u_t - \alpha uu_x - u_{txx} = 0, \quad (1) \]

where \( u(x,t) \) is an unknown function, \( x \in \mathbb{R}, t \geq 0 \), \( \alpha \) is any constant.

The BBM equation has been investigated as a regularized version of the KdV equation for shallow water waves [3]. In certain theoretical investigations the equation is studied as a model for long waves and from the standpoint of existence and stability, the equation offers considerable technical advantages over the KdV equation [4]. In addition to shallow water waves, the equation is applicable to the study of drift waves in plasma or the Rossby waves in rotating fluids. Under certain conditions, it also provides a model of onedimensional transmitted waves.

The modified Benjamin-Bona-Mahony equation is a special type of the BBM equation. By changing nonlinear term of the form \( \alpha u^nu_x (n = 2) \), the new modified form is obtained as follows:

\[ u_x + u_t - \alpha u^2u_x - u_{txx} = 0, \quad (2) \]

BBM equation can be solved by many methods. This equation is solved by \((G'/G)\)-expansion method [6], exp-function method [7, 8], homotopy perturbation method [9, 10] and the variation iteration method [11]. Zabusky and Kruskal investigated the interaction of solitary waves and the recurrence of initial states [12]. The Adomian decomposition method is another method to design some of the exact solitary wave solutions of the generalized form of the BBM equation [13]. Besides the analytical and exact solutions of the BBM equation, many numerical techniques from different families are developed and implemented for the numerical solutions to various evolution problems for the BBM equation [14, 15].

The existence of the solutions of initial value problems for the modified BBM equation has been considered in [16]. Yusufoğlu and Bekir used the tanh and the sine-cosine methods to obtain exact solutions of the modified BBM equation [17]. By the Exp-function method, Yusufoğlu obtained new solitatory solutions for the modified BBM equations [18]. Layeni and Akinola used the
hyperbolic auxiliary function method and reported some new exact solutions of the modified BBM equation [19]. Omrani used fully discrete Galerkin approximations for the BBM equation and discussed the convergence of the method [20]. Fakhar et al. used the homotopy analysis method to obtain approximate explicit solutions of nonlinear Benjamin-Bona-Mahony-Burgers equations [21].

In this article, we consider the following the loaded BBM equation and the modified BBM equation

$$u_x + u_t - \alpha uu_x - u_{txx} + \gamma(t)u(0,t)u_x = 0,$$

(3)

$$u_x + u_t - \alpha u^2 u_x - u_{txx} + \gamma(t)u(0,t)u_x = 0,$$

(4)

where $u(x,t)$ is an unknown function, $x \in \mathbb{R}$, $t \geq 0$, $\alpha$ is constant, $\gamma(t)$ is the given real continuous function.

We construct exact travelling wave solutions of the loaded BBM equation and modified BBM equation by the functional variable method. All solutions of these equations have been examined and three dimensional graphics of the obtained solutions have been drawn by using the Matlab program. We get some traveling wave solutions, which are expressed by the hyperbolic functions and trigonometric functions. The functional variable method is flexible, reliable and straightforward to find solutions of some nonlinear evolution equations arising in engineering and science.

Nowadays in connection with intensive research of problems optimal management of the agroecosystem, for example, the problem of long-term forecasting and regulation of the level of groundwater and soil moisture, there has been a significant increase in interest in loaded equations. Among the works devoted to loaded equations, one should especially note the works of A. Kneser [22], L. Lichtenstein [23], A. M. Nakhushev [24, 25], and others. It is known that the loaded differential equations contain some of the traces of an unknown function. In [26, 27, 28, 29], the term of "loaded equation" was used for the first time, the most general definitions of the loaded differential equation were given and also a detailed classifications of the differential loaded equations as well as their numerous applications were presented. A complete description of solutions of the nonlinear loaded equations and their applications can be found in papers [30, 31, 32, 33, 34, 35].

2. Description of the functional variable method

Consider nonlinear evolution equations with independent variables $x$, $y$ and $t$ is of the form

$$F(u, u_x, u_y, u_t, u_{xx}, u_{tt}, u_{yy}, u_{xy}, u_{xt}, u_{yt}, \ldots) = 0,$$

(5)

where $F$ is a polynomial in $u = u(x, y, t)$ and its partial derivatives. Zerarka and others in [36, 37] have summarized the functional variable method in the following.
Step 1. We use the wave transformation
\[ \xi = px + qy - kt, \]
where \( p \) and \( q \) are constants, \( k \) is the speed of the traveling wave.

Next, we can introduce the following transformation for a travelling wave solution of eq. (5)
\[ u(x, y, t) = u(\xi), \]
and the chain
\[ \frac{\partial u}{\partial x} = p \frac{du}{d\xi}, \quad \frac{\partial u}{\partial y} = q \frac{du}{d\xi}, \quad \frac{\partial u}{\partial t} = -k \frac{du}{d\xi}. \]

Using eq. (7) and (8), the nonlinear partial differential eq. (5) can be transformed into an ordinary differential equation of the form
\[ P(u, u', u'', u''', ...) = 0, \]
where \( P \) is a polynomial in \( u(\xi) \) and its total derivatives, \( u' = \frac{du}{d\xi} \).

Step 2. Then we make a transformation in which the unknown function \( u \) is considered as a functional variable in the form
\[ u' = F(u), \]
then, the solution can be found by the relation
\[ \int \frac{du}{F(u)} = \xi + \xi_0, \]
here \( \xi_0 \) is a constant of integration which is set equal to zero for convenience. Some successive differentiations of \( u \) in terms of \( F \) are given as
\[ u'' = \frac{1}{2} \frac{d^2 F^2(u)}{du^2}, \]
\[ u''' = \frac{1}{2} \frac{d^3 F^2(u)}{du^3} + \frac{d^2 F^2(u)}{du^2} \frac{dF^2(u)}{du}, \]
\[ u'''' = \frac{1}{2} \left[ \frac{d^4 F^2(u)}{du^4} F^2(u) + \frac{d^3 F^2(u)}{du^3} \frac{dF^2(u)}{du} \right]. \]

Step 3. The ordinary differential eq. (9) can be reduced in terms of \( u, F \) and its derivatives upon using the expressions of eq. (12) into eq. (5) gives
\[ H(u, \frac{dF(u)}{du}, \frac{d^2 F(u)}{du^2}, \frac{d^3 F(u)}{du^3}, ...) = 0. \]

The key idea of this particular form eq. (13) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, eq. (13) provides the expression of \( F \) and this, together with eq. (10), give appropriate solutions to the original problem.
3. Solutions of the loaded Benjamin-Bona-Mahony equation via the functional variable method

Using the wave variable
\[ u(x,t) = u(\xi), \quad \xi = px - kt, \] (14)
that will convert eq. (3) to an ordinary differential equation
\[ pu' - ku' - \alpha pu' + kp^2 u'' + p_\gamma(t)u(0,t)u' = 0, \] (15)
integrating once eq. (15) with respect to \( \xi \), we have
\[ u'' = \frac{1}{kp^2} \left( \frac{\alpha p}{2} u^2 + (k - p - p_\gamma(t))u(0,t)u \right). \] (16)

Following eq. (12), it is easy to deduce from eq. (16) an expression for the function \( F(u) \)
\[ \frac{1}{2} \frac{d}{du} \left( F^2(u) \right) = \frac{1}{kp^2} \left( \frac{\alpha p}{2} u^2 + (k - p - p_\gamma(t))u(0,t)u \right). \] (17)

Integrating eq. (17) and setting the constant of integration to zero yields
\[ F(u) = u \sqrt{\frac{\alpha}{3kp}} \left( \sqrt{u + \eta(t)} \right). \] (18)

where \( \eta(t) = \frac{3(k - p - p_\gamma(t))u(0,t)}{\alpha p} \).

From eq. (10) and eq. (18) we deduce that
\[ \frac{du}{u\sqrt{u + \eta(t)}} = \sqrt{\frac{\alpha}{3kp}} \, d\xi. \] (19)

After integrating eq. (19), with zero constant of integration, we have following exact solution
\[ u(x,t) = \frac{12}{\alpha p} \left( k - p - p_\gamma(t)u(0,t) \right) e^{\sqrt{\frac{k - p - p_\gamma(t)u(0,t)}{kp^2} (px - kt)}} \left( 1 - e^{\sqrt{\frac{k - p - p_\gamma(t)u(0,t)}{kp^2} (px - kt)}} \right)^2. \] (20)

It is obvious that the function \( u(0,t) \) can be easily found based on expression eq. (20).

From (20) we obtain two types of travelling solutions of the loaded BBM equation (3), i.e.:
1) when \( \frac{k - p - p_\gamma(t)u(0,t)}{kp^2} > 0 \), we have the solitary wave solutions
\[ u(x,t) = \frac{3}{\alpha p} (k - p - p_\gamma(t)u(0,t)). \]
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\[ \left( \text{ctgh}^2 \left( \sqrt{\frac{k - p - p\gamma(t)u(0,t)}{kp^2}} \frac{px - kt}{2} \right) - 1 \right), \]  

(21)

2) when \( \frac{k - p - p\gamma(t)u(0,t)}{kp^2} < 0 \), we have the periodic wave solutions

\[ u(x,t) = \frac{3}{\alpha p} (k - p - p\gamma(t)u(0,t)) \cdot \left( \text{ctgh}^2 \left( \sqrt{\frac{k - p - p\gamma(t)u(0,t)}{kp^2}} \frac{px - kt}{2} \right) + 1 \right). \]

(22)

Now, by choosing free parameters we will write the simple form of solitary and periodic wave solutions of the loaded BBM equation which can be used for the graphical illustrations.

If \( k = -1, p = -1, \alpha = 12 \) and \( \gamma(t) = -t^2 \), then we have

\[ u(x,t) = \frac{1}{4t^2} \ln^2 (\frac{t^2 + 2 + t\sqrt{t^2 + 4}}{2}) \left( \text{ctgh}^2 \left( \frac{t - x}{2t} \ln (\frac{t^2 + 2 + t\sqrt{t^2 + 4}}{2}) \right) - 1 \right). \]  

(23)

If \( k = -1, p = -1, \alpha = 3 \) and \( \gamma(t) = t^2 \), then we have

\[ u(x,t) = 4 \left( 1 + \text{ctgh}^2 (t - x) \right). \]  

(24)

4. Graphical representation of the loaded Benjamin-Bona-Mahony equation

We have presented some graphs of solitary and periodic waves constructed by taking suitable values of the involved unknown parameters to visualize the underlying mechanism to the original physical phenomena. Using mathematical software Matlab, three-dimensional plots of the obtained solutions have been shown in Figures 1 and Figures 2.

![Fig. 1. Solitary wave solution of the loaded BBM equation (3) for \( k = -1, p = -1, \alpha = 12 \) and \( \gamma(t) = -t^2 \)]
5. Solutions of the loaded modified Benjamin-Bona-Mahony equation by the functional variable method

Assume that eq. (4) has an exact solution in the form of a travelling wave

$$u(x, t) = u(\xi), \xi = px - kt,$$

the eq. (4) can be converted to an ordinary differential equation

$$pu' - ku' - \alpha pu^2 u' + kp^2 u''' + p\gamma(t)u(0, t)u' = 0. \quad (26)$$

Once integrating eq. (26), setting the constant of integrating to zero, we obtain

$$u'' = \frac{1}{kp^2} \left( \frac{\alpha p}{3} u^3 + (k - p - p\gamma(t)u(0, t)) u \right). \quad (27)$$

Following eq. (12), it is easy to deduce from eq. (27) an expression for the function $F(u)$

$$\frac{1}{2} \frac{d(F^2(u))}{du} = \frac{1}{kp^2} \left( \frac{\alpha p}{3} u^3 + (k - p - p\gamma(t)u(0, t)) u \right). \quad (28)$$

Integrating eq. (28) and setting the constant of integration to zero yields

$$F(u) = u \sqrt{\frac{\alpha}{6kp} \left( \sqrt{u^2 - \mu(t)} \right)}, \quad (29)$$

where $\mu(t) = \frac{6(p + p\gamma(t)u(0, t) - k)}{\alpha p}$

From eq. (10) and eq. (29) we deduce that

$$\frac{du}{u\sqrt{u^2 - \mu(t)}} = \sqrt{\frac{\alpha}{6kp}} d\xi. \quad (30)$$

After integrating eq. (30), with zero constant of integration, we have following exact solution

$$u(x, t) = \sqrt{\frac{6}{\alpha p}} \frac{\sqrt{(p + p\gamma(t)u(0, t) - k)}}{\cos \sqrt{(p + p\gamma(t)u(0, t) - k)p}} \frac{px - kt}{p}. \quad (31)$$
It is obvious that the function \( u(0, t) \) can be easily found based on expression eq. (31).

1) When \( \sqrt{\left(\frac{p + p \gamma(t) u(0,t) - k}{k}\right)} > 0, k > 0 \) we have the periodic wave solutions

\[
u(x, t) = \sqrt{\frac{6}{\alpha p}} \cos \sqrt{\left(\frac{p + p \gamma(t) u(0,t) - k}{k}\right) \frac{px - kt}{p}}.\]

(32)

2) When \( \sqrt{\left(\frac{p + p \gamma(t) u(0,t) - k}{k}\right)} < 0, k < 0 \) we have the solitary wave solutions

\[
u(x, t) = \sqrt{\frac{6}{\alpha p}} \cosh \sqrt{\left(\frac{p + p \gamma(t) u(0,t) - k}{k}\right) \frac{px - kt}{p}}.\]

(33)

If \( k = 1, p = 1, \alpha = 6 \) and \( \gamma(t) = t^2 \), then we have

\[
u(x, t) = \frac{t \sqrt{u(0,t)}}{\cos(t(x - t) \sqrt{u(0,t)})},\]

(34)

where

\[
u(0,t) = \frac{3 \sqrt{27t^3} + 3 \sqrt{3} \sqrt{27t^6} - 8}{3t^2} + \frac{2}{t^2 \sqrt{27t^3} + 3 \sqrt{3} \sqrt{27t^6} - 8}.\]

(35)

If \( k = -1, p = -1, \alpha = -6 \) and \( \gamma(t) = -t^2 \), then we have

\[
u(x, t) = \frac{t \sqrt{u(0,t)}}{\cosh(t(t - x) \sqrt{u(0,t)})},\]

(36)

where

\[
u(0,t) = \frac{3 \sqrt{27t^3} + 3 \sqrt{3} \sqrt{27t^6} + 8}{3t^2} + \frac{2}{t^2 \sqrt{27t^3} + 3 \sqrt{3} \sqrt{27t^6} + 8}.\]

(37)

6. Physical interpretations of the loaded modified Benjamin-Bona-Mahony equation

This section aims to present graphical illustrations of the obtained traveling wave solutions of Gardner equation. Using mathematical software Matlab, three-dimensional plots of the obtained solutions have been shown in Figure 3 and Figure 4. In the concept of mathematical physics, a soliton or solitary wave is defined as a self-reinforcing wave packet that upholds its shape. At the same time, it propagates at a constant amplitude and velocity. Solitary and periodic wave solutions represent an important type of solutions for nonlinear partial differential equations as many nonlinear partial differential equations have been found to have a variety of solitary wave solutions.
7. Conclusion

Some new traveling wave solutions have been successfully used to obtain several traveling wave solutions of the loaded Benjamin-Bona-Mahony and the loaded modified Benjamin-Bona-Mahony equation by the functional variable method. The advantage of method is give more solution functions such as periodic solutions and hyperbolic solutions than other popular analytical methods. We have shown that, this method can provide a useful way to efficiently find the exact structures of solutions to a variety of nonlinear wave equations. The solution procedure can be easily implemented in Matlab program. The functional variable method is flexible, reliable and straightforward to find solutions of some nonlinear evolution equations arising in engineering and science.

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