Ekedahl-Oort and Newton strata for Shimura varieties of PEL type

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Abstract

We study the Ekedahl-Oort stratification for good reductions of Shimura varieties of PEL type. These generalize the Ekedahl-Oort strata defined and studied by Oort for the moduli space of principally polarized abelian varieties (the “Siegel case”). They are parameterized by certain elements $w$ in the Weyl group of the reductive group of the Shimura datum. We show that for every such $w$ the corresponding Ekedahl-Oort stratum is smooth, quasi-affine, and of dimension $\ell(w)$ (and in particular non-empty). Some of these results have previously been obtained by Moonen, Vasiu, and the second author using different methods. We determine the closure relations of the strata.

We give a group-theoretical definition of minimal Ekedahl-Oort strata generalizing Oort’s definition in the Siegel case and study the question whether each Newton stratum contains a minimal Ekedahl-Oort stratum. As an interesting application we determine which Newton strata are non-empty. This criterion proves conjectures by Fargues and by Rapoport generalizing a conjecture by Manin for the Siegel case. We give a necessary criterion when a given Ekedahl-Oort stratum and a given Newton stratum meet.

Introduction

Starting point: The Siegel case

Fix a prime $p > 0$. For $g \geq 1$ let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension $g$ in characteristic $p$. We consider two natural and well studied stratifications on $\mathcal{A}_g$, the Newton and the Ekedahl-Oort stratification. The Newton stratification is the stratification corresponding to the isogeny class of the underlying $p$-divisible groups (with quasi-polarization) $(A, \lambda)[p^\infty]$ of the points $(A, \lambda)$ of $\mathcal{A}_g$. By Dieudonné theory the set of isogeny classes is parameterized by the attached symmetric (concave) Newton polygons. The Newton stratum in $\mathcal{A}_g$ attached to a symmetric Newton polygon $\nu$ is denoted by $N_\nu$. The Ekedahl-Oort stratification is defined by the isomorphism class of the $p$-torsion $(A, \lambda)[p]$ or, equivalently, the isomorphism class of the first de Rham cohomology endowed with its structure of an $F$-zip ([MW]). The set of strata is either parameterized by so-called elementary sequences (by Oort [Oo1]) or by certain elements in the Weyl group of the group $GSp_{2g}$ of symplectic similitudes (by
Moonen [Mo1], see also [MW]). By the pioneering work of Oort a lot is known about these two stratifications (see [Oo1] – [Oo5]), for instance:

(1) The Ekedahl-Oort strata are smooth, quasi-affine, and equi-dimensional and their dimension can be easily calculated in terms of the corresponding elementary sequence. The closure of an Ekedahl-Oort stratum is a union of Ekedahl-Oort strata – although the question which Ekedahl-Oort strata appear in the closure of a given one remained open.

(2) Oort defines the notion of a minimal $p$-divisible group and shows that it is already determined by its $p$-torsion (up to isomorphism). The corresponding Ekedahl-Oort strata are called minimal as well. Due to his definition of minimality it is clear that every Newton stratum contains a unique minimal Ekedahl-Oort stratum.

(3) For every symmetric Newton polygon $\nu$ the corresponding Newton stratum $N_\nu$ is non-empty (“Manin conjecture”). One even has that every principally polarized $p$-divisible group is the $p$-divisible group of a principally polarized abelian variety.

(4) The closure of a Newton stratum $N_\nu$ is the union of those Newton strata $N_{\nu'}$ whose Newton polygon $\nu'$ lies below $\nu$, where we normalize Newton polygons such that they are concave (“Grothendieck conjecture”).

Note that this is not a complete list. For some of these results generalizations are known for arbitrary PEL-data using different methods. Others have been generalized to other special cases of PEL-data, see below for details.

**Main results**

The goal of this article is to generalize and study the above notions for good reductions of general Shimura varieties of PEL type.

Let $\mathcal{D}$ denote a Shimura PEL-datum unramified at a prime $p > 0$, let $G$ be the attached reductive group scheme over $\mathbb{Z}_p$, and let $\mu$ be the conjugacy class of one-parameter subgroups defined by $\mathcal{D}$ (see Section 1.1). We assume that the fibers of $G$ are connected, i.e., we exclude the case (D) in Kottwitz’ terminology (recalled in Remark 1.1). Let $\mathcal{M}$ be the corresponding moduli space of PEL type defined by Kottwitz. We denote its reduction modulo $p$ by $\mathcal{M}_0$. It is defined over a finite field $\kappa$ determined by $\mathcal{D}$ and it classifies tuples $(A, \iota, \lambda, \eta)$, where $A$ is an abelian variety, where $\iota$ is an action on $A$ of an order in a semi-simple $\mathbb{Q}$-algebra $B$ given by $\mathcal{D}$, where $\lambda$ is a similitude class of prime-to-$p$ polarizations on $A$, and where $\eta$ is a prime-to-$p$-level structure (see Section 1.2 for details). For $B = \mathbb{Q}$ one obtains the Siegel case, i.e., the moduli space of principally polarized abelian varieties.

The Ekedahl-Oort stratification is defined by attaching to every $(A, \iota, \lambda, \eta)$ as above the first de Rham homology $H^1_{\text{DR}}(A)$ (which is by definition the dual of the first hypercohomology of the de Rham complex) endowed with its $F$-zip structure and the additional structure induced by $\iota$ and $\lambda$. For the sake of brevity we call such an object a $\mathcal{D}$-zip (see Section 2.1). Over a perfect field, covariant Dieudonné theory and a result of Oda show that it is equivalent to give the datum of the $\mathcal{D}$-zip of $(A, \iota, \lambda, \eta)$ or the $p$-torsion of $A$ together with the structure induced by $\iota$ and $\lambda$ (Example 2.2). The moduli space of all $\mathcal{D}$-zips is a smooth Artin stack of finite type over the finite field $\kappa$, $\kappa = \bar{\kappa}$.
which we denote by $\mathcal{D} - \mathcal{Z} ip$. The above construction defines a morphism

$$\zeta: \mathcal{A}_0 \to \mathcal{D} - \mathcal{Z} ip.$$ 

Let $\bar{\kappa}$ be an algebraic closure of $\kappa$. We consider the isomorphism class $w$ of a $\bar{\kappa}$-valued point of $\mathcal{D} - \mathcal{Z} ip$ (or, equivalently, the isomorphism class of a $\mathcal{D}$-zip over $\bar{\kappa}$) as a point of the underlying topological space of $\mathcal{D} - \mathcal{Z} ip \otimes \bar{\kappa}$. The corresponding Ekedahl-Oort stratum $\mathcal{A}_0^w$ is then defined as $\zeta^{-1}(w)$.

Isomorphism classes of $\mathcal{D}$-zips can be described as follows. Let $(W, I)$ be the Weyl group of the reductive group $G$ together with its set of simple reflections. To the one-parameter subgroup $\mu$ we attach a subset $J \subseteq I$ of simple reflections (see last subsection of Section 1.1). By the main result of [MW], the underlying topological space of $\mathcal{D} - \mathcal{Z} ip \otimes \bar{\kappa}$ is in bijection to $J W$, where

$$J W := \{ w \in W ; \ell(sw) > \ell(w) \text{ for all } s \in J \}$$

is the set of representatives of minimal length of $W J \backslash W$ (Appendix A.1 and A.2). Similar classification results have been obtained by B. Moonen ([Mo1]) and, more generally, by A. Vasiu ([Va3]).

By transport of structure we obtain a topology on $J W$ which can be described combinatorially by results of Pink, Ziegler and the second author ([PWZ]) via a partial order $\preceq$ on $J W$ which was introduced by He ([He]) in his study of the spherical completions of reductive groups (see Definition 4.8). This description can also be derived from results of the first author ([Vi2]). In particular the closures and the codimension of points are known. These results are recalled in Section 4.2.

The first goal of this paper is the study of the Ekedahl-Oort stratification and the morphism $\zeta$. We show (Theorem 5.1, Theorem 9.1):

**Theorem 1.** The morphism $\zeta$ is faithfully flat.

For the proof of the flatness we use a valuative criterion for universal openness (Proposition 5.2). Our proof that $\zeta$ satisfies this valuative criterion relies on the theory of Dieudonné displays developed by Zink ([Zi]) and Lau ([Lau1], [Lau2]), recalled in Section 3.1.

The theorem shows in particular that $\zeta$ is open and surjective and we deduce from the known facts on the topology of $J W$ the following result (Theorem 9.1, Corollary 9.2, Theorem 6.1):

**Theorem 2.** (1) The Ekedahl-Oort stratum $\mathcal{A}_0^w$ is non-empty for all $w \in J W$.

(2) $\mathcal{A}_0^w$ is equi-dimensional of dimension $\ell(w)$.

(3) The closure of $\mathcal{A}_0^w$ is the union of the Ekedahl-Oort strata $\mathcal{A}_0^{w'}$ for those $w' \in J W$ such that $w' \preceq w$.

In particular there exists a unique Ekedahl-Oort stratum of dimension 0 (corresponding to $w = 1$), which we call the superspecial Ekedahl-Oort stratum.

Assertion (2) had already been proved by Moonen ([Mo2]) for $p > 2$ and under the assumption of Assertion (1). Assertion (3) is new even in the Siegel case. In fact for the
The smoothness was already shown by Vasiu [Va2] and we include only a quick alternative proof for $p > 2$.

Our second goal is the generalization and group-theoretic reformulation of Oort’s notion of a minimal $p$-divisible group or a minimal Ekedahl-Oort stratum. For an abelian variety with endomorphism, polarization and level structure $(A, \iota, \lambda, \eta)$ as above, $\iota$ and $\lambda$ induce additional structures on the $p$-divisible group $A[p^\infty]$ of $A$. We obtain the notion of a $p$-divisible group with $D$-structure (see Definition 1.11). Let $k$ be an algebraically closed extension of the field of definition $\kappa$ of $A_0$. Let $K := \text{Frac} W(k)$ be the field of fractions of the ring of Witt vectors endowed with the Frobenius, denoted by $\sigma$. Let $K = G(W(k))$. Covariant Dieudonné theory yields a bijection

$$\begin{align*}
\left\{ \text{isomorphism classes of } p\text{-divisible groups with } D\text{-structure over } k \right\} \leftrightarrow C_k(G, \mu) := \left\{ K\text{-}\sigma\text{-conjugacy classes in } K\mu(p)K \right\}
\end{align*}$$

(see Section 7.2 for details). We define minimality of a $p$-divisible group with $D$-structure $X$ by a group-theoretic criterion for the corresponding element in $C_k(G, \mu)$ (Definition 8.2) which is equivalent to the condition that $X$ is up to isomorphism already determined by its $p$-torsion (see Remark 8.1). To study the notion of minimality we also define the technical condition for $X$ to be fundamental (Definition 8.7) which is related to the definition of fundamental elements in the affine Weyl group $\tilde{W}$ of $G$ given by Görtz, Haines, Kottwitz, and Reuman in [GHKR2] (although we slightly deviate from their definition). If $G$ is split (i.e., a product of groups of the form $GL_n$ and $GSp_{2g}$), we show the following result (Section 8.1).

**Proposition 4.** Assume that $G$ is split. Let $X$ be a $p$-divisible group with $D$-structure over an algebraically closed field.

1. If $X$ is minimal in the sense of Oort, then it is fundamental.
2. If $X$ is fundamental, then it is minimal (in our sense).

The proof of the second assertion relies essentially on a result of [GHKR2]. Both assertions together give in particular a new group-theoretic proof of the main result of [Oo4]. In fact, Theorem 0.2 of [Oo5] can then be used to show that all three conditions are equivalent if $G = GL_n$. For $G = GSp_{2g}$ the same assertion follows by using uniqueness of polarizations ([Oo3 Corollary (3.8)]). Thus all three conditions are equivalent if $G$ is split.

The third goal of this paper is to relate Ekedahl-Oort strata and Newton strata. The group-theoretic definition of Newton strata is due to Rapoport and Richartz ([RR]). Similarly as above it is well-known that there is an injection from the set of isogeny
classes of $p$-divisible groups with $\mathcal{D}$-structure over an algebraically closed field $k$ into the set $B(G)$ of $G(L)$-σ-conjugacy classes in $G(L)$. By Kottwitz' description of $B(G)$ this set is independent of $k$ and by attaching to $(A, i, \lambda, \eta)$ the isogeny class of the associated $p$-divisible group with $\mathcal{D}$-structure we obtain a map $N_t: \mathcal{A} \to B(G)$ whose image is known to be contained in a certain finite subset $B(G, \mu)$. The set $B(G)$ (and hence its subset $B(G, \mu)$) is endowed with a partial order which is the group-theoretic reformulation of the partial order on the set of (concave) Newton polygons of “one lying above the other” (see Section 7.2 for details). Then the sets $\mathcal{N}_b := N_t^{-1}(b)$ for $b \in B(G, \mu)$ are called the Newton strata of $\mathcal{A}$. They are locally closed in $\mathcal{A}$. The stratum corresponding to the minimal element of $B(G, \mu)$ is called the basic stratum.

We show (Theorem 8.10, Proposition 8.17, Theorem 8.18):

**Theorem 5.** (1) If $G$ is split, then every Newton stratum $\mathcal{N}_b$ contains a unique fundamental Ekedahl-Oort stratum $\mathcal{A}_w^0(b)$ (where an Ekedahl-Oort stratum $\mathcal{A}_w^0$ is called fundamental if the $p$-divisible group with $\mathcal{D}$-structure attached to one (or, equivalently, all) points $(A, i, \lambda, \eta)$ of $\mathcal{A}_w^0$ is fundamental).

(2) In general, the basic Newton stratum contains the superspecial Ekedahl-Oort stratum, and this stratum is minimal. Furthermore, for each Newton stratum $\mathcal{N}_b$ there is a Newton stratum $\mathcal{N}_b'$ associated with the same $b \in B(G)$ but possibly in a moduli space associated with a different $\mu$ which contains a minimal Ekedahl-Oort stratum.

The uniqueness assertion made for split groups does not hold for general $G$. An example for Hilbert-Blumenthal varieties is given in 8.14. We do not know whether the existence statement of (1) holds in general.

We use this theorem to prove “Manin’s conjecture” (i.e., the non-emptiness of all Newton strata) for the PEL-case and also an integral version of it:

**Theorem 6.** (1) For every $b \in B(G, \mu)$ the Newton stratum $\mathcal{N}_b$ is non-empty.

(2) For every algebraically closed field extension $k$ of $\kappa$ and for every $p$-divisible group $X$ with $\mathcal{D}$-structure over $k$ there exists a $k$-valued point of $\mathcal{A}_w$ whose attached $p$-divisible group with $\mathcal{D}$-structure is isomorphic to $X$.

This was conjectured by Fargues ([Far] Conjecture 3.1.1), and by Rapoport ([Ra] Conjecture 7.1). “Manin problems” have also been considered by Vasiu in [Vas] using a different language.

We use Theorem 5 to show in the split case two results about the intersection of Newton and Ekedahl-Oort strata. In the Siegel case the first one (Theorem 8.20) had been conjectured by Oort ([Oo3]) and proved by Harashita ([Har2]):

**Theorem 7.** Assume that $G$ is split. Then for all $w \in J^W$ with $\mathcal{A}_w^0 \cap \mathcal{N}_b \neq \emptyset$ one has $w(b) \leq w$ where $w(b)$ is defined as in Theorem 5.

The second result is a generalization of a theorem of Harashita ([Har2]) for the Siegel case which describes the Newton polygon of the generic point of some irreducible component of a given Ekedahl-Oort stratum (see Proposition 8.23 for a precise statement).

Finally we deduce from Theorem 7 the following result (Corollary 8.22).
Proposition 8. Let $G$ be split. For $b, b' \in B(G, \mu)$ with $b' \leq b$ the minimal Ekedahl-Oort stratum $\mathcal{A}_0^{w(b')}$ is contained in the closure of $\mathcal{A}_0^{w(b)}$.

In the Siegel case this has been shown by Harashita ([Har1], Corollary 3.2) and is used as a step in his proof of Theorem 7. This gives in particular a new proof of the “weak Grothendieck conjecture” ([Ra] Conjecture (7.3)(1)) in the split case:

Corollary 9. Let $G$ be split. For $b, b' \in B(G, \mu)$ with $b' \leq b$ the Newton stratum $\mathcal{N}_{b'}$ meets the closure of $\mathcal{N}_b$.

Beyond the Siegel case some of the results above were known for special cases of Shimura varieties of PEL type. The Ekedahl-Oort stratification for Hilbert-Blumenthal varieties (i.e. with the notation of Section 1.1 $(B, \ast, V) = (F, \text{id}, F^2)$, where $F$ is a totally real field extension of $\mathbb{Q}$ of degree $g$) has been studied by Goren and Oort ([GO]). In this case $W = (S_2)^g$ (where $S_2 = \{\text{id}, \tau\}$ is the symmetric group of two elements), $J = \emptyset$, the order $\succeq$ equals the Bruhat order which is the $g$-fold product of the order on $S_2$ with $\text{id} < \tau$. They show Theorem 2, Theorem 3 and the analogue of Proposition 8.23 below in this case.

The “linear case of signature $(1, n-1)$” (in the notation of Section 1.1 the derived group of $G_R$ is isomorphic to a product of SU$(1, n-1)$ and compact groups, and the prime $p$ is chosen such that only the case (AL) in the classification in Remark 1.3 occurs) has been studied by Harris and Taylor ([HT]) for their proof of the local Langlands conjecture. In this case the Ekedahl-Oort and the Newton stratification coincide, and all Ekedahl-Oort strata are minimal.

The “unitary case of signature $(1, n-1)$” (in the notation of Section 1.1 the derived group of $G_R$ is isomorphic to SU$(1, n-1)$, and the prime $p$ is chosen such that only the case (AU) in the classification in Remark 1.3 occurs) has been studied by Büttel and the second author ([BW], see also [VW]). In this case $W = S_n$, $J = \{\tau_2, \ldots, \tau_{n-1}\}$ (where $\tau_i$ denotes the transposition of $i$ and $i+1$), the partial order $\succeq$ on $^J W$ coincides with the Bruhat order and it is a total order. Every non-basic Newton stratum is a minimal Ekedahl-Oort stratum, and the basic Newton stratum consists of $[(n-1)/2]$ Ekedahl-Oort strata.

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Notation. Throughout the paper we use the following notation. Let $G$ be a group, $X \subset G$ a subset and $g \in G$. Then we set $gX = gXg^{-1}$. 
For a commutative ring $R$, an $R$-scheme $X$, and a commutative $R$-algebra $R'$ we denote by $X_{R'} := X \otimes_R R' := X \times_{\text{Spec } R} \text{Spec } R'$ the base change to $R'$.

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1 Good Reductions of Shimura varieties

1.1 Shimura data of PEL type

In this section we recall the notion of Shimura-PEL-data and their attached moduli spaces. Our main reference is Kottwitz [Ko2].

Shimura datum

Let $\mathcal{D} = (B, *, V, \langle \ , \rangle, O_B, \Lambda, h)$ denote a Shimura-PEL-datum, integral and unramified at a prime $p > 0$. Let $G$ be the associated reductive group over $\mathbb{Q}$, and denote by $[\mu]$ the associated conjugacy class of cocharacters of $G$. By this we mean the following data.

- $B$ is a finite-dimensional semi-simple $\mathbb{Q}$-algebra, such that $B_{\mathbb{Q}_p}$ is isomorphic to a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$.
- $*$ is a $\mathbb{Q}$-linear positive involution on $B$.
- $V$ is a finitely generated faithful left $B$-module.
- $\langle \ , \rangle : V \times V \to \mathbb{Q}$ is a symplectic form on $V$ such that $\langle bv, w \rangle = \langle v, b^* w \rangle$ for all $v, w \in V$ and $b \in B$.
- $O_B$ is a $*$-invariant $\mathbb{Z}(p)$-order of $B$ such that $O_B \otimes \mathbb{Z}_p$ is a maximal $\mathbb{Z}_p$-order of $B \otimes \mathbb{Q}_p$.
- $\Lambda$ is an $O_B$-invariant $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$, such that $\langle \ , \rangle$ induces a perfect pairing $\Lambda \times \Lambda \to \mathbb{Z}_p$.
- $G$ is the $\mathbb{Q}$-group of $B$-linear symplectic similitudes of $(V, \langle \ , \rangle)$, i.e., for any $\mathbb{Q}$-algebra $R$ we have

$$G(R) = \{ g \in \text{GL}_B(V \otimes R) \mid \langle gv, gw \rangle = c(g) \cdot \langle v, w \rangle \text{ for some } c(g) \in R^\times \} ;$$

- $h : \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m, \mathbb{C}}) \to G_\mathbb{R}$ is a homomorphism that defines a Hodge structure of type $(-1,0) + (0,-1)$ on $V \otimes \mathbb{R}$ such that there exists a square root $\sqrt{-1}$ of $-1$ such that $2\pi \sqrt{-1} \langle \ , \rangle$ is a polarization form; let $\mu_h : \mathbb{G}_{m, \mathbb{C}} \to G_\mathbb{C}$ be the cocharacter such that $\mu_h(z)$ acts on $V^{(-1,0)}$ (resp. $V^{(0,-1)}$) via $z$ (resp. via 1).
- $[\mu]$ is the $G(\mathbb{C})$-conjugacy class of the cocharacter $\mu_h$ associated with $h$. Then $V_{\mathbb{C}}$ has only weights 0 and 1 with respect to any $\mu \in [\mu]$. 

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Let $B = \prod_i B_i$ be a decomposition of $B$ into simple $\mathbb{Q}$-algebras. Because of the positivity of the involution, $^*$ induces an involution on each simple factor $B_i$.

**Remark 1.1.** Let $B$ be simple. Let $F$ be the center of $B$ and set $F_0 := \{ a \in F ; a^* = a \}$. Then $F_0$ is a totally real field extension of $\mathbb{Q}$. Let $C := \text{End}_B(V)$ and let $n := [F : F_0] \dim_F(C)^{1/2}/2$ (which is an integer by the existence of $h$). Then $C$ is a central simple $F$-algebra and $C \otimes_\mathbb{Q} \mathbb{R}$ is isomorphic to one of the following algebras

(A) the product of $[F_0 : \mathbb{Q}]$ copies of $M_2(\mathbb{C})$.
(B) the product of $[F_0 : \mathbb{Q}]$ copies of $M_{2n}(\mathbb{R})$.
(C) the product of $[F_0 : \mathbb{Q}]$ copies of $M_n(\mathbb{H})$, where $\mathbb{H}$ denotes the Hamilton quaternions.

In this article we make the assumption that $G$ is connected. This is equivalent to the assumption that there is no simple factor of $B$ such that we are in case (D).

**Remark 1.2.** We recall some facts about the derived group $G^{\text{der}}$ and the quotient $D := G/G^{\text{der}}$ from [Ko2] §7. Let $G'$ be the kernel of the multiplicator homomorphism $c : G \to G_{m,\mathbb{Q}}$. Then $G' = \text{Res}_{F_0/\mathbb{Q}}(G_0)$, where $\text{Res}_{F_0/\mathbb{Q}}(\cdot)$ denotes restriction of scalars from $F_0$ to $\mathbb{Q}$ and where $G_0$ is the $F_0$-group of $B$-linear symplectic automorphisms of $(V,(\cdot,\cdot))$. Let us again assume that $B$ is simple.

In case (C), $G_0$ is an (automatically inner) form of a split symplectic group over $F_0$. In particular $G_0$ is simply connected. Hence $G' = G^{\text{der}}$ is simply connected and $D \cong G_{m,\mathbb{Q}}$.

In case (A), $G_0$ is an inner form of the quasi-split unitary group over $F_0$ attached to an $(F/F_0)$-hermitian space $(V_0,(\cdot,\cdot))$. Thus $G^{\text{der}}_0$ is simply connected and hence $G^{\text{der}}$ is simply connected. Moreover, as passing from $G$ to an inner form does not change $D$, we see that $D$ is the subtorus of $G_{m,\mathbb{Q}} \times \text{Res}_{F_0/\mathbb{Q}} G_{m,F}$ of elements $(t,x)$ such that $N_{F/F_0}(x) = t^n$, where $n := \dim_F(V_0)$.

If $n$ is even, say $n = 2k$, then we have an isomorphism

$$D \cong G_{m,\mathbb{Q}} \times \text{Res}_{F_0/\mathbb{Q}} D_0, \quad (t,x) \mapsto (tx^{-k}),$$

where $D_0 := \text{Ker}(N_{F/F_0} : \text{Res}_{F_0/\mathbb{Q}} G_{m,F} \to G_{m,F_0})$. If $n$ is odd, say $n = 2k + 1$, then $(t,x) \mapsto t^{-k}x$ yields an isomorphism of $D$ with the $\mathbb{Q}$-torus of elements $y$ in $\text{Res}_{F_0/\mathbb{Q}} G_{m,F}$ such that $N_{F/F_0}(y)$ belongs to $G_{m,\mathbb{Q}}$. In particular we have an exact sequence

$$1 \to \text{Res}_{F_0/\mathbb{Q}} D_0 \to D \to G_{m,\mathbb{Q}} \to 1.$$
Note that the classifications given here and given in Remark 1.1 match. Indeed, if \( B \) is simple (and under our assumption that case (D) is excluded) we are in case (C) (as in Remark 1.1) if and only if \(*\) is trivial on \( F \) which is again equivalent to \((O_B \otimes \mathbb{Z}_p, \ast)\) being isomorphic to a product of \( \mathbb{Z}_p\)-algebras of type (C) in this classification.

**Reflex field and the determinant condition**

Let \( E \) be the reflex field associated with \( \mathcal{D} \), i.e. the field of definition of \([\mu]\). It is a finite extension of \( \mathbb{Q} \). We set \( O_{E,(p)} := O_E \otimes \mathbb{Z}_p \). If we choose \( \mu \in [\mu] \) and denote by \( V_{\mathbb{C}} = V_0 \oplus V_1 \) the weight decomposition corresponding to \( \mu \), then \( E \) is the field of definition of the isomorphism class of the complex representation of \( B \) on \( V_1 \).

We will now formulate Kottwitz’s determinant condition following Rapoport and Zink ([RZ] 3.23(a)). Choose a model \( W \) of (the isomorphism class of) the \( B \)-representation \( V_1 \) over \( E \) and let \( \Gamma \) be an \( O_B \)-invariant \( O_{E,(p)} \)-lattice in \( W \). Let

\[
O_B := \mathcal{V}(O'_B \otimes_{\mathbb{Z}_p} O_{E,(p)})
\]

be the geometric vector bundle over \( O_{E,(p)} \) corresponding to the free \( \mathbb{Z}_p \)-module \( O_B \) and let \( \delta_{\mathcal{D}} : O_B \to \mathbb{A}_E^1 \) be the morphism of \( O_{E,(p)} \)-schemes given on \( R \)-valued points (\( R \) any \( O_{E,(p)} \)-algebra) by

\[
(1.3) \quad \delta_{\mathcal{D}}(R) : O_B(R) = O_B \otimes_{\mathbb{Z}_p} R \to R, \quad b \mapsto \det(b \mid \Gamma \otimes_{O_E(p)} R).
\]

As \( \delta_{\mathcal{D}} \) is determined by its restriction to the generic fiber, it is independent of the choice of \( \Gamma \).

Let \( T \) be an \( O_{E,(p)} \)-scheme and let \( L \) be a finite locally free \( \mathcal{D}_T \)-module endowed with an \( O_B \)-action \( O_B \to \text{End}_{\mathcal{D}_T}(L) \). As above we obtain a morphism of \( T \)-schemes

\[
\delta_L : O_B \times_{O_{E,(p)}} T \to \mathbb{A}_T^1.
\]

**Definition 1.4.** We say that \( L \) satisfies the determinant condition (with respect to \( \mathcal{D} \)) if the morphisms \( \delta_L \) and \( \delta_{\mathcal{D}} \otimes \text{id}_T \) of \( T \)-schemes are equal.

If \( L \) satisfies the determinant condition, then its rank is equal to \( \dim_{\mathbb{C}}(V_1) = \dim_{\mathbb{Q}}(V) / 2 \).

**Remark 1.5.** As morphisms of schemes can be glued for the fpqc topology, it suffices to check the determinant condition locally for the fpqc topology.

Moreover, as \( O_B \) is unramified over \( \mathbb{Z}_p \), the determinant condition on \( L \) can be translated (possibly after some faithfully flat quasi-compact base change) into a condition on the rank of certain direct summands (see [RZ] 3.23(b) for details). This shows that the determinant condition on \( L \) is open and closed in the unramified case. More precisely, there exists an open and closed subscheme \( T_0 \) of \( T \) such that a morphism \( f : T' \to T \) factors through \( T_0 \) if and only if \( f^*L \) satisfies the determinant condition.
The reductive group scheme $G$ and the set of simple reflections $J$

As before let $\mathcal{D} = (B, V, \langle , \rangle, G, O_B, \Lambda, h, [\mu])$ denote a Shimura PEL-datum (see Section [1.1]). Let $G$ be the $\mathbb{Z}_p$-group scheme of $O_B$-linear symplectic similitudes of $\Lambda$. This is a reductive group scheme over $\mathbb{Z}_p$ whose generic fiber is $G_{\mathbb{Q}_p}$. It is quasi-split (Appendix [A.3]). We denote the special fiber of $G$ by $\bar{G}$.

We denote by $(W, I)$ the Weyl group of $G$ (or of $\bar{G}$) together with its set of simple reflections (Appendix [A.5]). The Frobenius endomorphism of $G$ induces an automorphism $\varphi$ of $(W, I)$. Let $J \subset I$ be the type of the conjugacy class of one-parameter subgroups $[\mu^{-1}]$ (Appendix [A.7]). It is defined over $\mathbb{K}$ (i.e. $\varphi^n(J) = J$ where $n = [\mathbb{K} : \mathbb{F}_p]$).

Remark 1.6. The reductive group scheme $G$ sits in an exact sequence

$$1 \to G' \to G \to \mathbb{G}_{m, \mathbb{Z}_p} \to 1,$$

where $c$ is the multiplicator homomorphism and $G'$ is the reductive $\mathbb{Z}_p$-group scheme of $O_B$-linear symplectic automorphisms of $\Lambda$. Every decomposition $(O_{B_1}, ^*) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p = (O_{B_2}, ^*) \times (O_{B_3}, ^*)$ yields a corresponding decomposition $G' = G'_1 \times G'_2$. Thus to describe the structure of $G'$ (and of $G$), we may assume that $(B, ^*)$ is simple and thus that we are in one of the cases of Remark [1.3] Then $G'$ has the following form (where $O_K$ is the ring of integers of a finite unramified extension of $\mathbb{Q}_p$).

- (AL) $G' \cong \text{Res}_{O_K/\mathbb{Z}_p} \text{GL}_{n, O_K}$ $(n \geq 1)$. In this case [1.4] is split.
- (AU) $G' \cong \text{Res}_{O_K/\mathbb{Z}_p} G_0$, where $G_0$ is the unitary group over $K$ attached to a perfect $(O_L/O_K)$-hermitian module $(V, ( , ))$, where $L$ is quadratic imaginary field extension of $K$.
- (C) $G' \cong \text{Res}_{O_K/\mathbb{Z}_p} \text{Sp}_{2g, O_K}$ $(g \geq 1)$.

Remark 1.7. Some of our results only hold if $G$ is split. From the explicit description of $G_{\mathbb{Q}_p}$ (e.g. in [Wd]) it follows that $G$ (or, equivalently, $G_{\mathbb{Q}_p}$) is split if and only if only the case (AL) and (C) with $K = \mathbb{Q}_p$ occur (notations of Remark [1.3] or Remark [1.6]).

1.2 Moduli spaces of abelian varieties with $\mathcal{D}$-structure

The integral model of the Shimura variety

Let $\mathcal{A}_p$ be the ring of finite adeles of $\mathbb{Q}$ with trivial $p$-th component and let $\mathcal{D}^{\mathbb{P}} \subset \mathbb{G}(\mathcal{A}_p)$ be a compact open subgroup. We denote by $\mathcal{A} = \mathcal{A}_{p, CV}$ the moduli space defined by Kottwitz [Ko2] §5. More precisely, $\mathcal{A}$ is the category fibered in groupoids over the category of $O_{E, (p)}$-schemes whose fiber category over an $O_{E, (p)}$-scheme $S$ is the category of tuples $(A, i, \lambda, \eta)$ satisfying the following properties.

- $A$ is an abelian scheme over $S$.
- $i: O_B \to \text{End}(A) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p$ is a $\mathbb{Z}_p$-algebra homomorphism; this induces an $O_B$-action on the dual abelian scheme $A^\vee$ by $b \mapsto i(b^*)^\vee$.
- $\lambda$ is a $\mathbb{Z}_p$-equivalence class of an $O_B$-linear polarization of order prime to $p$ (this means $\lambda$ is considered equivalent to $\alpha \lambda$, where $\alpha: S \to \mathbb{Z}_p^\times$ is a locally constant map).
- $\eta$ is a level structure of type $\mathcal{D}^{\mathbb{P}}$ on $A$ (in the sense of loc. cit.).
Then $\iota$ induces by functoriality a homomorphism $\iota: O_B \to \text{End}_{O_S}(\text{Lie}A)$. We require that the $O_S$-module $\text{Lie}(A)$ with this $O_B$-action satisfies the determinant condition (Definition 1.4).

A morphism of two tuples $(A_1, \iota_1, \lambda_1, \eta_1)$ and $(A_2, \iota_2, \lambda_2, \eta_2)$ in the fiber category over $S$ is an $O_B$-linear quasi-isogeny $f: A_1 \to A_2$ of degree prime to $p$ such that $f^*\lambda_2 = \lambda_1$ and such that $f^*\eta_2 = \eta_1$.

Then $\mathcal{A}$ is an algebraic Deligne-Mumford stack which is smooth over $O_{E,(p)}$. If $C^p$ is sufficiently small, $\mathcal{A}$ is representable by a smooth quasi-projective scheme over $O_{E,(p)}$ (see loc. cit. or [Lan] 1.4.1.11 and 1.4.1.13).

Fix an embedding of the algebraic closure $\overline{Q}$ of $Q$ in $C$ into some fixed algebraic closure $\overline{Q}_p$ of $Q_p$. Via this embedding we can consider $[\mu]$ as a $G(\overline{Q}_p)$-conjugacy class of cocharacters. Denote by $v|p$ the place of $E$ given by the chosen embedding $\overline{Q} \hookrightarrow \overline{Q}_p$ and write $E_v$ for the $v$-adic completion of $E$. Let $\kappa = \kappa(v)$ be its residue class field. Finally let $\bar{\kappa}$ be the residue field of the ring of integers of $\overline{Q}_p$. This is an algebraic closure of $\kappa$.

We denote by

$$\mathcal{A}_0 := \mathcal{A}_{\mathcal{D},C^p,0} := \mathcal{A}_{\mathcal{D},C^p} \otimes_{O_{E,(p)}} \kappa$$

the special fiber of $\mathcal{A}_{\mathcal{D},C^p}$ at $v$.

**Modules and $p$-divisible groups with $\mathcal{D}$-structure**

For a $T$-valued point $(A, \iota, \lambda, \eta)$ of $\mathcal{A}_{\mathcal{D},C^p}$ the $O_B$-action and the polarization induce additional structures on the $p$-divisible group and the cohomology of $A$. Let us make this more precise.

We call two perfect pairings $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ on a finite locally free $\mathcal{O}_T$-module $M$ similar if there exists an open affine covering $T = \bigcup_j V_j$ and for all $j$ a unit $c_j \in \Gamma(V_j, \mathcal{O}^*_V)$ such that $(m, m')_2 = c_j(m, m')_1$ for all $m, m' \in \Gamma(V_j, M)$.

**Definition 1.8.** Let $T$ be a $\mathbb{Z}_p$-scheme. A $\mathcal{D}$-module over $T$ is a locally free $\mathcal{O}_T$-module $M$ of rank $\dim_Q(V)$ endowed with an $O_B$-action and the similitude class of a symplectic form $\langle \cdot, \cdot \rangle$ such that $\langle bm, m' \rangle = \langle m, b^*m' \rangle$ for all $b \in O_B$ and local sections $m, m'$ of $M$.

An isomorphism $M_1 \sim M_2$ of $\mathcal{D}$-modules over $T$ is an $\mathcal{O}_T \otimes O_B$-linear symplectic similitude.

For a morphism $f: T' \to T$ of $\mathbb{Z}_p$-schemes and a $\mathcal{D}$-module $M$ there is the obvious notion of a pull back $f^*M = M_{T'}$ of $M$ to a $\mathcal{D}$-module on $T'$.

The $\mathbb{Z}_p$-module $\Lambda$ together with the induced $O_B$-action and the induced pairing is a $\mathcal{D}$-module over $\mathbb{Z}_p$. The following lemma shows in particular that all $\mathcal{D}$-modules are étale locally isomorphic to the $\mathcal{D}$-module $\Lambda_T$.

**Lemma 1.9.** Let $T$ be a $\mathbb{Z}_p$-scheme and let $M_1$ and $M_2$ be two $\mathcal{D}$-modules over $T$. Then locally for the étale topology on $T$, $M_1$ and $M_2$ are isomorphic as $\mathcal{D}$-modules. Moreover the scheme of isomorphisms $\text{Isom}_{\mathcal{D}}(M_1, M_2)$ of $\mathcal{D}$-modules is a smooth affine $T$-scheme.

**Proof.** This is a special case of [RZ] Theorem 3.16. \qed

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Example 1.10. Let $T$ be an $O_{E_v}$-scheme and let $(A, \iota, \lambda, \eta) \in \mathcal{A}_{\mathcal{D}, C_F}(T)$. We denote by $H^1_{\text{DR}}(A/T)$ the first de Rham homology, i.e. $H^1_{\text{DR}}(A/T)$ is the $\mathcal{O}_T$-linear dual of the first de Rham cohomology $R^1 f_*(\Omega^1_A)$ of $A$. This is a locally free $\mathcal{O}_T$-module of rank $\text{dim}_{\mathbb{Q}}(V)$. As $A \to H^1_{\text{DR}}(A/T)$ is a covariant functor, we obtain an $O_B$-action on $H^1_{\text{DR}}(A/T)$. The $\mathbb{Z}^{\times}_{(p)}$-equivalence class $\lambda$ yields a similitude class of perfect alternating forms $(\cdot, \cdot)$ on $H^1_{\text{DR}}(A/T)$. Thus $H^1_{\text{DR}}(A/T)$ is a $\mathcal{D}$-module over $T$.

Definition 1.11. Let $T$ be a $\mathbb{Z}_p$-scheme. A $p$-divisible group with $\mathcal{D}$-structure over $T$ is a triple $(X, \iota, \lambda)$ consisting of a $p$-divisible group $X$ over $T$ of height $\text{dim}_{\mathbb{Q}}(V)$, an $O_B$-action $\iota : O_B \to \text{End}(X)$ and a $\mathbb{Z}^{\times}_{(p)}$-equivalence class of an $O_B$-linear isomorphism $\lambda : X \to X^\vee$ such that $\lambda^\vee = -\lambda$. Furthermore Lie$(X)$ with the induced $O_B$-action is assumed to satisfy the determinant condition (Definition 1.4).

Here we denote by $X^\vee$ the dual $p$-divisible group. If $X$ is endowed with an $O_B$-action $\iota : O_B \to \text{End}(X)$, we endow $X^\vee$ with the $O_B$ action $b \mapsto \iota(b^*)^\vee$.

Example 1.12. Let $T$ be an $O_{E_v}$-scheme and let $(A, \iota, \lambda, \eta) \in \mathcal{A}_{\mathcal{D}, C_F}(T)$. Via functoriality $\iota$ and $\lambda$ induce the structure of a $p$-divisible group with $\mathcal{D}$-structure on $A[p^\infty]$.

Let $(X, \iota, \lambda)$ and $(X', \iota', \lambda')$ be $p$-divisible groups with $\mathcal{D}$-structure. An isomorphism (resp. an isogeny) of $p$-divisible groups with $\mathcal{D}$-structure is an $O_B$-linear isomorphism (resp. an isogeny) $f : X \to X'$ such that $f^*\lambda' = \lambda$ (resp. $f^*\lambda' = p^e \lambda$ for some integer $e \geq 0$), where we set $f^*X' := f^\vee \circ \lambda' \circ f$.

Reformulation of the determinant condition

Proposition 1.13. Let $T$ be an $O_{E_v}$-scheme and let $M$ be a $\mathcal{D}$-module over $T$. Let $H \subset M$ be a totally isotropic $O_B \otimes \mathcal{O}_T$-submodule which is locally on $T$ a direct summand as an $\mathcal{O}_T$-module. Then the following assertions are equivalent:

(i) $M/H$ satisfies the determinant condition.

(ii) Locally for the étale topology on $T$ there exists an isomorphism $\alpha : M \to \Lambda_T$ of $\mathcal{D}$-modules such that the stabilizer of $\alpha(H)$ in $G_T$ is a parabolic of type $J$.

Proof. Both conditions are local for the étale topology. Therefore we may assume that $T$ is the spectrum of a strictly henselian ring $A$. By Lemma 1.8 there exists an isomorphism $\alpha : M \to \Lambda_T := \Lambda \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_T$ of symplectic $\mathcal{O}_T \otimes O_B$-modules. Thus we may assume that $M = \Lambda_T$. The stabilizer $P$ of $H$ is a smooth subgroup scheme of $G_T$. The properties that $P$ is a parabolic subgroup of $G$ and that $H$ satisfies the determinant condition can be checked on the geometric fibers (Remark 1.3). Thus we may assume that $T$ is the spectrum of an algebraically closed field. Then going through the cases in Remark 1.3 the claim follows from the explicit description of the determinant condition in [Wall].

1.3 The Siegel case as an example

As an example, we will consider the case that $B = \mathbb{Q}$ and $O_B = \mathbb{Z}_{(p)}$, the Siegel case. Then $G$ is the group of symplectic similitudes of a symplectic $\mathbb{Q}$-vector space $V$ of
dimension $2g$ and $G$ is the group of symplectic similitudes of a symplectic $\mathbb{Z}_p$-module $\Lambda$ of rank $2g$. This is a split reductive group. The conjugacy class $[\mu]$ is the unique one such that the decomposition $V_\mathbb{C} = V_0 \oplus V_1$ is a decomposition into totally isotropic subspaces. It is defined already over $\mathbb{Q}$, i.e. $E = \mathbb{Q}$.

The field $\kappa$ equals $\mathbb{F}_p$ and $\mathcal{G}$ is the group of symplectic similitudes of a symplectic $\mathbb{F}_p$-vector space. Its Weyl group $W$ is described in Appendix A.3. With the notations introduced there the subset $J$ of the set of simple elements is just \{ $s_1, \ldots, s_{g-1}$ \}.

Every element $b \in B = \mathbb{Q}$ acts on the $g$-dimensional vector space $V_1$ as a scalar and therefore

$$\delta_g : \mathcal{O}_B = \mathbb{A}^{1}_{\mathbb{Z}(p)} \rightarrow \mathbb{A}^{1}_{\mathbb{Z}(p)}, \quad b \mapsto b^g.$$ 

Therefore the condition that for a $\mathbb{Z}(p)$-scheme $S$ a finite locally free $\mathcal{O}_S$-module $\mathcal{L}$ satisfies the determinant condition just means that the rank of this module is equal to $g$.

Fix an integer $N \geq 1$ prime to $p$ and set $C^p := \{ g \in \mathbb{G}((\mathbb{A}_f)) : g \equiv 1 \bmod N \}$. Then $\mathcal{A}_{g,N} \subset \mathcal{A}_g$ is the moduli space $\mathcal{A}_{g,N}$ over $\mathbb{Z}(p)$ of $g$-dimensional abelian varieties endowed with a principal polarization and a symplectic full level-$N$ structure.

In this case, a $\mathcal{D}$-module over a $\mathbb{Z}(p)$-scheme $T$ is a locally free $\mathcal{O}_T$-module $M$ of rank $2g$ endowed with a similitude class of symplectic forms. An $\mathcal{O}_T$-submodule $H$ satisfies the conditions of Proposition 1.13 if and only if $H$ is Lagrangian (Appendix A.3). A $p$-divisible group with $\mathcal{D}$-structure is a $p$-divisible group $X$ of height $2g$ together with a $\mathbb{Z}_p$-equivalence class of isomorphisms $\lambda : X \sim X^\vee$ such that $\lambda^\vee = -\lambda$.

## 2 Definition of the Ekedahl-Oort stratification

### 2.1 $\mathcal{D}$-zips attached to $S$-valued points of $\mathcal{A}_0$

Let $S$ be a $\kappa$-scheme and let $(A, \iota, \lambda, \eta)$ be an $S$-valued point of $\mathcal{A}_0$. We set $M := H^1_{\text{DR}}(A/S)$ endowed with its structure of a $\mathcal{D}$-module (Example 1.10). The de Rham cohomology $H^1_{\text{DR}}(A/S)$ is endowed with two locally direct summands, namely $f_*\Omega^1_{A/S}$ given by the Hodge spectral sequence and $R^1f_*(\mathcal{H}^0(\Omega^{\bullet}_{A/S}))$ given by the conjugate spectral sequence. We obtain locally direct summands of $M$ by defining

$$C := (f_*\Omega^1_{A/S})^\perp, \quad D := (R^1f_*(\mathcal{H}^0(\Omega^{\bullet}_{A/S})))^\perp.$$ 

The formation of the de Rham cohomology endowed with these two spectral sequences is functorial in $A$ in a contravariant way. The $\mathcal{O}_S$-linear submodules $C$ and $D$ of $M$ are then functorial, $O_B$-invariant and locally on $S$ direct summands of rank equal to $\dim(A/S)$. We also recall that there is an isomorphism $f_*\Omega^1_{A/S} \sim \text{Lie}(A)^\vee$ which is functorial in $A$. Thus we obtain a functorial identification

$$\text{(2.1)} \quad C = \text{Ker}(H^1_{\text{DR}}(A/S) \rightarrow \text{Lie}(A)).$$

Moreover, $D$ and $C$ are totally isotropic with respect to $\langle \cdot, \cdot \rangle$ (e.g., see [DP] 1.6).

For every $\mathcal{O}_S$-module $X$ we set $X^{(p)} := \text{Frob}_S^\sharp X$, where $\text{Frob}_S : S \rightarrow S$ is the absolute Frobenius. Dualizing the construction in [MW] Section 7.5 shows that the Cartier
isomorphism induces isomorphisms

\[ \varphi_0 : (M/C)^{(p)} \xrightarrow{\sim} D, \quad \varphi_1 : C^{(p)} \xrightarrow{\sim} M/D, \]

The functoriality in \( A \) of the Cartier isomorphism shows that \( \varphi_0 \) and \( \varphi_1 \) are \( O_B \)-linear and the tuple \((M, C, D, \varphi_0, \varphi_1)\) is a \( \mathcal{D} \)-zip over \( S \) in the following sense.

**Definition 2.1.** Let \( S \) be a \( \kappa \)-scheme. A \( \mathcal{D} \)-zip is a tuple \( \underline{M} = (M, C, D, \varphi_0, \varphi_1) \) satisfying the following conditions.

(a) \( M \) is a \( \mathcal{D} \)-module over \( S \).

(b) \( C \) and \( D \) are \( O_B \)-invariant totally isotropic \( \mathcal{O}_S \)-submodules of \( M \) which are locally on \( S \) direct summands such that \( M/C \) satisfies the determinant condition. This implies that \( C \) and \( D \) both have rank \((1/2) \dim_C(V)\) and \( C^\perp = C \) and \( D^\perp = D \).

(c) \( \varphi_0 : (M/C)^{(p)} \xrightarrow{\sim} D \) and \( \varphi_1 : C^{(p)} \xrightarrow{\sim} M/D \) are \( O_B \otimes \mathcal{O}_S \)-linear isomorphisms such that for some representative \( \langle \cdot, \cdot \rangle \) in the similitude class of symplectic forms on \( M \) the following diagram commutes

\[
\begin{array}{ccc}
(M/C)^{(p)} & \xrightarrow{\varphi_0} & D \\
\downarrow & & \downarrow \\
(C^\vee)^{(p)} & \xrightarrow{\varphi_1^\vee} & (M/D)^\vee,
\end{array}
\]

where the vertical maps are the isomorphisms induced by the isomorphism \( M \xrightarrow{\sim} M^\vee \) given on local sections by \( m \mapsto \langle - , m \rangle \).

An isomorphism of \( \mathcal{D} \)-zips over \( S \) is an isomorphism of \( \mathcal{D} \)-modules preserving the submodules \( C \) and \( D \) and commuting with \( \varphi_0 \) and \( \varphi_1 \). We obtain the category of \( \mathcal{D} \)-zips over \( S \) (where every morphism is an isomorphism). Moreover we have the obvious notion of a pullback \( f^* \) of a \( \mathcal{D} \)-zip for a morphism \( f : S' \to S \) of \( \kappa \)-schemes. Thus we obtain a category \( \mathcal{D} - \mathcal{Z}ip \) fibered in groupoids over the category of \( \kappa \)-schemes. Clearly this is a stack for the fpqc-topology. Moreover it is easy to see that \( \mathcal{D} - \mathcal{Z}ip \) is an algebraic stack in the sense of Artin.

**Example 2.2.** Let \( S = \text{Spec} \ k \), where \( k \) is a perfect field, and let \( (A, \iota, \lambda, \eta) \in \mathcal{A}_0(k) \).

Let \( \mathcal{M} := H^1_{\text{cris}}(A/W(k))^\vee \) the covariance Dieudonné module of the \( p \)-divisible group of \( A \). It is endowed with a \( \sigma \)-linear endomorphism \( F \) such that \( V := F^{-1}p \) exists on \( \mathcal{M} \). Thus \( M = M(A) := \mathcal{M}/p\mathcal{M} \) is a \( k \)-vector space endowed with a \( \text{Frob}_k \)-linear map \( F : M \to M \) and a \( \text{Frob}_k^{-1} \)-linear map \( V : M \to M \) such that \( \text{Ker}(F) = \text{Im}(V) \) and \( \text{Ker}(V) = \text{Im}(F) \). The formation of \( (M, F, V) \) is functorial in \( A \) and we obtain an \( O_B \)-action and a similitude class of symplectic forms on \( M \) induced by \( \iota \) and \( \lambda \), respectively. We set \( C := \text{Ker}(F) \) and \( D := \text{Ker}(V) \) and denote by \( \varphi_0 : (M/C)^{(p)} \xrightarrow{\sim} D \) the isomorphism induced by \( F \) and by \( \varphi_1 : C^{(p)} \xrightarrow{\sim} M/D \) the isomorphism induced by \( V^{-1} \). Then \((M, C, D, \varphi_0, \varphi_1)\) is a \( \mathcal{D} \)-zip. By [Oda Corollary 5.11 there is a functorial isomorphism \( M(A) \xrightarrow{\sim} H^1_{\text{DR}}(A/k) \) which is an isomorphism of \( \mathcal{D} \)-zips.

Attaching to \((A, \iota, \lambda, \eta)\) the tuple \((M, \langle \cdot, \cdot \rangle, C, D, \varphi_0, \varphi_1)\) defines a morphism of algebraic stacks

\[(2.2) \quad \zeta : \mathcal{A}_0 \to \mathcal{D} - \mathcal{Z}ip.\]
We will see in Section 4.2 that there are only finitely many isomorphism classes of \( \mathcal{D} \)-zips over any fixed algebraically closed extension \( k \) of \( \kappa \) and that the set of isomorphism classes does not depend on \( k \).

**Definition 2.3.** Let \( [M] \) be the isomorphism class of a \( \mathcal{D} \)-zip over the algebraic closure \( \bar{k} \) of \( \kappa \). We consider \( [M] \) as a (locally closed) point of the underlying topological space of \( \mathcal{D} - \text{Zip} \circledast _{\kappa} \bar{k} \). We call \( \mathcal{A}_{0}^{[M]} := \zeta^{-1}([M]) \subseteq \mathcal{A}_{0} \circledast \bar{k} \) the Ekedahl-Oort stratum attached to \( [M] \).

\( \mathcal{A}_{0}^{[M]} \) is locally closed in the underlying topological space of \( \mathcal{A}_{0} \circledast \bar{k} \) and we endow it with its reduced structure. A \( \bar{k} \)-valued point \( (A, \iota, \lambda, \eta) \) of \( \mathcal{A}_{0} \) is in \( \mathcal{A}_{0}^{[M]}(\bar{k}) \) if and only if the attached \( \mathcal{D} \)-zip is isomorphic to \( [M] \).

In Section 4 we recall the classification of \( \mathcal{D} \)-zips which yields a different index set for the Ekedahl-Oort strata.

### 2.2 Ekedahl-Oort strata in the Siegel case I

We continue the example of the Siegel case of Section 1.3. Via the construction in Example 2.2 a \( \mathcal{D} \)-zip over \( \bar{k} \) corresponds to a tuple \( \underline{M} = (M, F, V, \langle \;, \; \rangle) \), where \( M \) is a \( \bar{k} \)-vector space of dimension \( 2g \) together with a \( \text{Frob}_{\bar{k}} \)-linear map \( F: M \rightarrow M \), a \( \text{Frob}_{\bar{k}}^{-1} \)-linear map \( V: M \rightarrow M \), and a similitude class of symplectic pairings \( \langle \; , \; \rangle \) such that \( \text{Ker}(F) = \text{Im}(V) \), \( \text{Ker}(V) = \text{Im}(F) \) and such that

\[
\langle Fx, y \rangle = \langle x, Vy \rangle^p,
\]

\( x, y \in M \).

We call such a tuple \( \underline{M} \) a *symplectic Dieudonné space*.

If \( \mathcal{A}_{0} = \mathcal{A}_{0,N} \) is the moduli space of principally polarized abelian varieties \( (A, \lambda) \) of dimension \( g \) with full level \( N \) structure \( \eta \), then \( \mathcal{A}_{0}^{[M]} \) is the reduced locally closed substack of \( \mathcal{A}_{0} = \mathcal{A}_{0,N} \) such that \( \mathcal{A}_{0}^{[M]}(\bar{k}) \) consists of those \( \bar{k} \)-valued points \( (A, \lambda, \eta) \) of \( \mathcal{A}_{0,N} \) such that the covariant Dieudonné module \( (M^{'}, F^{'}, V^{'}) \) of \( A[p] \) together with the similitude class of symplectic pairings \( \langle \; , \; \rangle^{'} \) induced by \( \lambda \) is isomorphic to \( [M] \). Thus we obtain in this case the Ekedahl-Oort stratification as defined in [Oo1].

### 2.3 Smooth coverings of \( \mathcal{A}_{0} \)

We define two smooth coverings \( \mathcal{A}_{0}^\# \) and \( \tilde{\mathcal{A}}_{0} \) of \( \mathcal{A}_{0} \) as follows:

For every \( \kappa \)-scheme \( S \) the \( S \)-valued points of \( \mathcal{A}_{0}^\# \) are given by tuples \( (A, \iota, \lambda, \eta, \alpha) \) where \( (A, \iota, \lambda, \eta) \in \mathcal{A}_{0}(S) \) and where \( \alpha \) is an \( O_{B}/pO_{B} \)-linear symplectic similitude \( H_{1}^{\text{DR}}(A/S) \xrightarrow{\sim} \Lambda \otimes _{\mathbb{Z}_p} \mathcal{O}_{S} \).

Therefore \( \mathcal{A}_{0}^\# \) is a \( \tilde{G} \)-torsor over \( \mathcal{A}_{0} \) for the étale topology. Here \( \tilde{G} \) is the smooth group scheme obtained as the special fiber of \( G \).

The \( S \)-valued points of \( \tilde{\mathcal{A}}_{0} \) are given by tuples \( (A, \lambda, \iota, \eta, \alpha, C, D) \) with \( (A, \lambda, \iota, \eta, \alpha) \in \mathcal{A}_{0}^\# \) and where \( C \) and \( D \) are \( O_{B}/pO_{B} \)-invariant totally isotropic complements of \( \alpha(C) \) and of \( \alpha(D) \) in \( \Lambda \otimes _{\mathbb{Z}_p} \mathcal{O}_{S} \), respectively.
Lemma 3.1.

Proof.

In this section we always denote by \((R, \mathfrak{m})\) a complete local noetherian ring with perfect residue field of characteristic \(p\). We also assume that \(p \geq 3\) or \(pR = 0\).

In this section we will endow Dieudonné displays in the sense of Zink \([Zi]\) with additional structure. We start by recalling some facts on Dieudonné displays.

The Zink ring

We use the notations and terminology of Lau’s paper \([Lau2]\). In the sense of \([Lau2]\) Definition 1.2 the ring \(R\) is admissible topological. In particular we have the Zink ring \(\mathbb{W}(R)\) which is endowed with a Frobenius \(\sigma\) and a Verschiebung \(\tau\) (denoted by \(f\) and \(v\) in loc. cit.). By loc. cit. it has the following properties.

1. The Zink ring \(\mathbb{W}(R)\) is a subring of the Witt ring \(W(R)\) stable under \(\sigma\) and \(\tau\).
2. The kernel of \(\mathbb{W}(R) \to W(k)\) is \(\mathbb{W}(\mathfrak{m})\), the set of elements \(x = (x_0, x_1, \ldots) \in W(\mathfrak{m})\) such that the sequence \((x_i)_i\) converges to zero for the \(\mathfrak{m}\)-adic topology.
3. There exists a unique ring homomorphism \(s: W(k) \to \mathbb{W}(R)\) which is a section of the projection \(\mathbb{W}(R) \to W(k)\). Thus \(\mathbb{W}(R) = s(W(k)) \oplus \mathbb{W}(\mathfrak{m})\).
4. \(\mathbb{W}(R) = \lim \mathbb{W}(R/\mathfrak{m}^n)\) is a local \(p\)-adically complete ring with residue field \(k\).
5. If \(R\) is Artinian, then \(\mathbb{W}(\mathfrak{m})\) consists only of nilpotent elements. In particular \(\mathbb{W}(R)_{\text{red}} = W(k)\).

The first two properties characterize \(\mathbb{W}(R)\). Let \(\mathbb{I}_R\) be the kernel of \(w_0: \mathbb{W}(R) \to R\). Then \(\mathbb{I}_R\) is the image of \(\tau\). We denote by \(\sigma_1: \mathbb{I}_R \to \mathbb{W}(R)\) the inverse of \(\tau\).

For every \(\mathbb{W}(R)\)-module \(M\) we set \(M^\sigma = \mathbb{W}(R) \otimes_{\mathbb{W}(R)} M\). For two \(\mathbb{W}(R)\)-modules \(M\) and \(N\) we identify \(\sigma\)-linear maps \(M \to N\) with linear maps \(M^\sigma \to N\). If \(M\) is of the form \(M = \Lambda \otimes_{\mathbb{Z}_p} \mathbb{W}(R)\) for some \(\mathbb{Z}_p\)-module \(\Lambda\) we have a canonical isomorphism \(M^\sigma \cong M\) which we use to identify these two \(\mathbb{W}(R)\)-modules.

Let \(X\) be any \(\mathbb{W}(R)\)-scheme. Then the ring endomorphism \(\sigma\) of \(\mathbb{W}(R)\) induces a map \(\sigma: X(\mathbb{W}(R)) \to X(\mathbb{W}(R))\). We will use this notation in particular for \(X = G \otimes_{\mathbb{Z}_p} \mathbb{W}(R)\) and for the scheme of parabolics of \(G \otimes_{\mathbb{Z}_p} \mathbb{W}(R)\).

We will use the following result.

Lemma 3.1. Let \(R\) be a ring as above and let \(k\) be its (perfect) residue field. For any smooth \(W(k)\)-scheme \(X\) the canonical map

\[c_X: X(\mathbb{W}(R)) \to X(R) \times_{X(k)} X(W(k))\]

is surjective.

Proof. Let \(\mathbb{W}_n(R) = R \times_{R/\mathfrak{m}^n} \mathbb{W}(R/\mathfrak{m}^n)\). Then \(\mathbb{W}(R) = \lim \mathbb{W}_n(R)\). The kernels of the transition maps \(\mathbb{W}_n(R) \to \mathbb{W}_{n-1}(R)\) consist of nilpotent elements. The formal smoothness of \(X\) implies that \(X(\mathbb{W}(R)) = \lim X(\mathbb{W}_n(R))\) maps surjectively to \(X(\mathbb{W}_1(R)) = X(R \times_k W(k)) = X(R) \times_{X(k)} X(W(k))\).
Definition of Dieudonné displays

Recall ([Lau2] Definition 2.6 and Section 2.8) that a Dieudonné display over $R$ is a tuple $(\mathcal{P}, \mathcal{Q}, F, F_1)$ where $\mathcal{P}$ is a finitely generated projective $\mathbb{W}(R)$-module, $\mathcal{Q}$ is a submodule such that there exists a decomposition $\mathcal{P} = S \oplus T$ of $\mathbb{W}(R)$-modules with $\mathcal{Q} = S \oplus \mathbb{I}_R T$ (called a normal decomposition), and where $F: \mathcal{P} \to \mathcal{P}$ and $F_1: \mathcal{Q} \to \mathcal{P}$ are $\sigma$-linear maps of $\mathbb{W}(R)$-modules such that

\begin{equation}
F_1(ax) = \sigma_1(a)F(x), \quad \text{for all } a \in \mathbb{I}_R, x \in \mathcal{P}
\end{equation}

and such that $F_1(\mathcal{Q})$ generates the $\mathbb{W}(R)$-module $\mathcal{P}$.

These axioms imply

\begin{equation}
F(x) = pF_1(x), \quad \text{for all } x \in \mathcal{Q}.
\end{equation}

Furthermore, as $\mathbb{W}(R)$ is local, $\mathcal{P}$ is in fact a free $\mathbb{W}(R)$-module.

**Remark 3.2.** Let $\mathcal{S}$ and $\mathcal{T}$ be finitely generated projective $\mathbb{W}(R)$-modules. Set $\mathcal{P} := \mathcal{S} \oplus \mathcal{T}$ and $\mathcal{Q} := \mathcal{S} \oplus \mathbb{I}_R \mathcal{T}$. Using (3.1) and (3.2) it follows that there is a bijection between the set of pairs $(F, F_1)$ such that $(\mathcal{P}, \mathcal{Q}, F, F_1)$ is a Dieudonné display and the set of isomorphisms $\Psi: \mathcal{P}^\sigma \cong \mathcal{P}$ of $\mathbb{W}(R)$-modules given by

\[ (F: \mathcal{P}^\sigma \to \mathcal{P}, F_1: \mathcal{Q}^\sigma \to \mathcal{P}) \mapsto \Psi := F_1|_{\mathcal{S}^\sigma} \oplus F|_{\mathcal{T}^\sigma} : (\mathcal{S} \oplus \mathcal{T})^\sigma \to \mathcal{P}. \]

We call $(\mathcal{S}, \mathcal{T}, \Psi)$ a split Dieudonné display over $R$.

Duality for Dieudonné displays

We recall Lau’s duality for Dieudonné displays ([Lau1]). A bilinear form between Dieudonné displays $\mathcal{P} = (\mathcal{P}, \mathcal{Q}, F, F_1)$ and $\mathcal{P}' = (\mathcal{P}', \mathcal{Q}', F', F'_1)$ is a $\mathbb{W}(R)$-bilinear map $\beta: \mathcal{P} \times \mathcal{P}' \to \mathbb{W}(R)$ such that $\beta(\mathcal{Q}, \mathcal{Q}') \subseteq \mathbb{I}_R$ and such that

\begin{equation}
\beta(F_1x, F'_1x') = \sigma_1 \beta(x, x')
\end{equation}

for $x \in \mathcal{Q}$ and $x' \in \mathcal{Q}'$. There exists a Dieudonné display $\mathcal{P}^\vee = (\mathcal{P}', \mathcal{Q}', F', F'_1)$, the dual Dieudonné display of $\mathcal{P}$, such that for every Dieudonné display $\mathcal{P}'$ there exists a functorial isomorphism

\[ \text{Hom}(\mathcal{P}', \mathcal{P}^\vee) = \{ \text{Bilinear forms between } \mathcal{P} \text{ and } \mathcal{P}' \}. \]

The tautological pairing $\gamma$ between $\mathcal{P}$ and $\mathcal{P}^\vee$ is perfect and $\mathcal{P}^\vee$ is the $\mathbb{W}(R)$-linear dual of $\mathcal{P}$. Given a normal decomposition $\mathcal{P} = \mathcal{S} \oplus \mathcal{T}$ for $\mathcal{P}$ there is a unique normal decomposition $\mathcal{P}^\vee = \mathcal{S}' \oplus \mathcal{T}'$ for $\mathcal{P}^\vee$ such that $\gamma(\mathcal{S}, \mathcal{S}') = \gamma(\mathcal{T}, \mathcal{T}') = 0$. The attached linear operators $\Psi$ and $\Psi^\vee$ (Remark 3.2) satisfy

\begin{equation}
\gamma(\Psi(x), \Psi^\vee(x')) = \gamma(x, x')
\end{equation}

for $x \in \mathcal{P}^\sigma$ and $x' \in \mathcal{P}'^\sigma$. 

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Dieudonné displays and $p$-divisible groups

By [Zi] and [Lau1] (see also [Lau2] Theorem 3.24 and Corollary 3.26) we have:

Theorem 3.3. There exists an equivalence $\Phi_R$ between the category of $p$-divisible groups over $R$ and the category of Dieudonné displays over $R$. This equivalence is compatible with base change by continuous ring homomorphisms $R \to R'$ and with duality.

Moreover, let $A$ be an abelian scheme over $R$ and let $\mathcal{D} = (\mathcal{P}, \mathcal{Q}, F, F_1)$ be the Dieudonné display of the $p$-divisible group of $A$. Then by the construction in [Lau2] §3 (relating the Dieudonné display and the (covariant) Dieudonné crystal) and by [BBM] Chap. 5, we have a functorial isomorphism of exact sequences

\[
\begin{array}{cccccc}
0 & \to & (R^1 f_* \mathcal{O}_A)^\vee & \to & H^1_{DR} (A/R) & \to & \text{Lie}(A) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{Q}/\mathfrak{I}_R \mathcal{P} & \to & \mathcal{P}/\mathfrak{I}_R \mathcal{P} & \to & \mathcal{P}/\mathcal{Q} & \to & 0.
\end{array}
\]

Note that $(R^1 f_* \mathcal{O}_A)^\vee \cong (\text{Lie } A^\vee)^\vee$.

3.2 $\mathcal{D}$-Dieudonné displays

We continue to assume that $R$ is a complete local noetherian ring with perfect residue field of characteristic $p$ such that $p \geq 3$ or $pR = 0$. We will now endow Dieudonné displays with a “$\mathcal{D}$-structure”. For our purpose it suffices to do this only for Dieudonné displays with a fixed normal decomposition and whose underlying $\mathcal{W}(R)$-module is $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{W}(R)$. The general definition is similar, but rather tedious and thus omitted.

Definition 3.4. Assume that $R$ is in addition an $O_{E_0}$-algebra. We set $M := \Lambda \otimes_{\mathbb{Z}_p} \mathcal{W}(R)$. Then $M$ carries a symplectic form $\langle \cdot, \cdot \rangle$ and an $O_B$-action. A split $\mathcal{D}$-Dieudonné display over $R$ consists of a tuple $(\mathcal{S}, \mathcal{T}, \Psi)$ where $\mathcal{S}$ and $\mathcal{T}$ are totally isotropic $O_B \otimes \mathcal{W}(R)$-submodules of $M$ such that $\mathcal{S} \oplus \mathcal{T} = M$ and where $\Psi : M^\sigma \to M$ is a symplectic $\mathcal{W}(R) \otimes O_B$-linear similitude. We assume that $M/\mathcal{S}$ satisfies the determinant condition.

Assume in addition that $R$ is a $\kappa$-algebra. We will now attach to an $R$-valued point $(A, \lambda, \epsilon, \eta, \alpha)$ of $\mathfrak{A}_0^\#$ (Section 23) a split $\mathcal{D}$-Dieudonné display. Let $(\mathcal{P}, \mathcal{Q}, F, F_1)$ be the Dieudonné display associated with the $p$-divisible group of $A$ (Theorem 3.3). Then $\lambda$ induces by functoriality a similitude class of perfect alternating forms $\langle \cdot, \cdot \rangle$ on the free $\mathcal{W}(R)$-module $\mathcal{P}$. The $O_B$-action by $\iota$ induces an $O_B$-action on $\mathcal{P}$ and we obtain a $\mathcal{D}$-module $\mathcal{P}$ over $\mathcal{W}(R)$. Moreover $O_B$ acts by homomorphisms of Dieudonné displays and $\langle \cdot, \cdot \rangle$ is a bilinear form of Dieudonné displays. Therefore $\mathcal{Q}$ is a $\mathcal{W}(R) \otimes O_B$-submodule of $\mathcal{P}$, and $F$ and $F_1$ are $O_B$-linear.

As $(\mathcal{W}(R), \mathfrak{I}(R))$ is henselian, Lemma 1.9 implies that there is a $\mathcal{W}(R) \otimes O_B$-linear symplectic similitude

$$\tilde{\alpha} : \mathcal{P} \xrightarrow{\sim} \Lambda \otimes \mathcal{W}(R)$$

whose reduction modulo $\mathfrak{I}(R)$ is equal to $\alpha$. We use $\tilde{\alpha}$ to identify $\mathcal{P}$ and $\Lambda \otimes \mathcal{W}(R)$. 

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By (3.5) we have a split exact sequence of free $R$-modules

$$0 \to C \to P/I_R P \to P/Q \to 0$$

where $C$ is as in Section 2.1. In the category of $\mathbb{W}(R) \otimes O_B$-modules we choose a totally isotropic direct summand $S$ of $M$ such that its reduction modulo $I_R$ is equal to $C$ and a totally isotropic complement $T$ of $S$ in $M$. Let $\Psi: P^\sigma \xrightarrow{\sim} P$ be the isomorphism of $O$-modules attached to $(F, F_1)$ (Remark 3.2). Then $(S, T, \Psi)$ is a split $O$-Dieudonné display.

4 Group-theoretic reformulation

4.1 Group-theoretic description of $O$-displays

We are now going to give a group-theoretic reformulation of the split Dieudonné displays with additional structure of Definition 3.4. We continue to assume that $R$ is a complete local noetherian $O_{E, v}$-algebra with perfect residue field $k$ of characteristic $p$ and that $p \geq 3$ or $p R = 0$.

**Definition 4.1.** For a subset $J \subseteq I$ defined over $\kappa$ let $\tilde{Y}_J$ be the $O_{E, v}$-scheme such that for each $O_{E, v}$-scheme $X$ the $X$-valued points of $\tilde{Y}_J$ are triples $(P, L, g)$, where $P$ is a parabolic subgroup of $G_X$ of type $J$, where $L$ is a Levi subgroup of $P$, and where $g \in G(X)$. By Appendix A.4 $\tilde{Y}_J$ is a quasi-projective smooth scheme.

**Construction 4.2.** Let $(S, T, \Psi)$ be a split $O$-Dieudonné display over $R$ (Definition 3.4). We construct an associated element $(\tilde{P}, \tilde{L}, \tilde{g})$ in $\tilde{Y}_J(\mathbb{W}(R))$ as follows. We define $P$ as the stabilizer of the flag $0 \subset S \subset M := \Lambda \otimes \mathbb{W}(R)$ in $G_{\mathbb{W}(R)}$. Then $\tilde{P}$ is a parabolic of the associated type $J$ by Proposition 1.13. Furthermore, $\tilde{L}$ is defined as the stabilizer of the decomposition $M = S \oplus T$ in $G_{\mathbb{W}(R)}$. Then $\tilde{L}$ is a Levi subgroup of $\tilde{P}$. Finally let $\tilde{g}$ be the composition

$$M \xrightarrow{\sim} M^\sigma \xrightarrow{\Psi} M.$$

**Lemma 4.3.** The construction above defines a bijection between the set of all split $O$-Dieudonné displays over $R$ and the set $\tilde{Y}_J(\mathbb{W}(R))$.

**Proof.** We construct an inverse. Let $(\tilde{P}, \tilde{L}, \tilde{g})$ be in $\tilde{Y}_J(\mathbb{W}(R))$. We let $S$ be the unique $O_B$-invariant totally isotropic direct summand of $M$ whose stabilizer is equal to $\tilde{P}$. Then $M/S$ satisfies the determinant condition. Let $T$ be the unique $O_B$-invariant complement of $S$ in $M$ such that the stabilizer of the decomposition $S \oplus T$ is equal to $\tilde{L}$. Further let $\Psi$ be the $\sigma$-linear map attached to $\tilde{g}$. Then $(S, T, \Psi)$ is a split $O$-Dieudonné display. This construction defines an inverse.

**Remark 4.4.** Let $(S, T, \Psi)$ be a split $O$-Dieudonné display and $(\tilde{P}, \tilde{L}, \tilde{g}) \in \tilde{Y}_J(\mathbb{W}(R))$ the corresponding triple. Let $(P, Q, F, F_1)$ be the Dieudonné display associated with $(S, T, \Psi)$. By definition we have $P = S \oplus T = \Lambda \otimes \mathbb{W}(R)$ and $Q = S \oplus I_R T$. By (3.2) and Remark 3.2 the $\sigma$-linear map $F: P \to P$ is given by $\tilde{g} \circ (p \text{id}_S \oplus \text{id}_T) \circ (\text{id}_\Lambda \otimes \sigma)$.
4.2 Group-theoretic description of $\mathcal{D}$-zips

We recall (a special case of) the definition of the schemes studied in [MW] and, more generally, in [PWZ]. Recall that $\mathcal{O}$ denotes the fiber of $G$ over $\mathbb{F}_p$. Denote by $F$ the Frobenius on $G$ given by the $p$-th power. Then $F$ induces an automorphism $\bar{\varphi}: (W,I) \to (W,I)$ of Coxeter systems. We denote by $\Gamma_\kappa = \text{Gal}(\overline{\kappa}/\kappa)$ the Galois group of $\kappa$. It acts on $(W,I)$ via the group generated by $\bar{\varphi}^n$ where $n = [\kappa : \mathbb{F}_p]$.

Let $w_0$ be the element of maximal length in $W$, set $K := w_0\bar{\varphi}(J)$ and let $x \in K^W\bar{\varphi}(J)$ be the element of minimal length in $W_Kw_0W_{\bar{\varphi}(J)}$. Thus

$$x = w_{0,K}w_0 = w_0w_{0,\bar{\varphi}(J)},$$

where $w_{0,K}$ and $w_{0,\bar{\varphi}(J)}$ denote the longest elements in $W_K$ and in $W_{\bar{\varphi}(J)}$, respectively. Then $x$ is the unique element of maximal length in $K^W\bar{\varphi}(J)$.

The $\mathcal{O}$-scheme $X_J$

We denote by $X_J = X_{J,F,x}$ the functor on $\kappa$-schemes which is the Zariski-sheafification of the functor $X_J^\flat$ which associates with a $\kappa$-scheme $S$ the set of triples $(P,Q,U_{gF(P)})$ where $P \subset G_S$ is a parabolic of type $J$, where $Q \subset G_S$ is a parabolic of type $K$ and where $g \in G(S)$ is an element such that $\text{relpos}(Q,gF(P)) = x$ (here $U_R$ denotes the unipotent radical of a parabolic subgroup $R$ and $\text{relpos}$ the relative position, see [SGA3]).

By [MW] Corollary 4.3 this functor is representable by a $\kappa$-scheme. Note that strictly speaking, in [MW] the functor, where $F$ is the $\kappa$-Frobenius, is considered. However, the representability of the functor considered here follows from the same proof. For any affine scheme $S$ we have $X_J(S) = X_J^\flat(S)$ (use $H^1(S,U) = 0$ if $U$ is the unipotent radical of a parabolic group of a reductive group scheme and if $S$ is affine, e.g., see [SGA3] Exp. XXVI, Corollaire 2.2).

**Remark 4.5.** As $K := x\bar{\varphi}(J)$ and $x$ is the longest element in $K^W\bar{\varphi}(J)$, we have $\text{relpos}(Q,gF(P)) = x$ if and only if $Q \cap gF(P)$ is a common Levi subgroup of $Q$ and $gF(P)$, i.e. $Q$ and $gF(P)$ are opposite parabolic subgroups ([Lu] §8).

**Remark 4.6.** The forgetful morphism which is defined on points by $(P,Q,[g]) \mapsto (P,Q)$ is a morphism $X_J \to \text{Par}_J \times \text{Par}_K$ which is a torsor under the base change to $\text{Par}_J \times \text{Par}_K$ of the Levi quotient of the universal parabolic subgroup over $\text{Par}_J$ ([MW] Lemma 4.2). This reductive group scheme over $\text{Par}_J \times \text{Par}_K$ is of relative dimension $\dim(P/U_P)$ for any parabolic of $G$ of type $J$. In particular $X_J$ is a geometrically connected smooth $\kappa$-scheme whose dimension equals $\dim(\text{Par}_J) + \dim(\text{Par}_K) + \dim(P/U_P) = \dim(G)$.

The $\kappa$-group scheme $G_\kappa = G \otimes_{\mathbb{Z}_p} \kappa$ acts on $X_J$ by

$$h \cdot (P,Q,[g]) = (hP,^hQ,[hgF(h)^{-1}]).$$

The arguments in [MW] Section 6 show:
Proposition 4.7. The algebraic stack $\mathmathcal{D} - \mathmathcal{Z}ip$ and the algebraic quotient stack $[G_\kappa \backslash X_J]$ are isomorphic. In particular, there is a bijection between isomorphism classes of $\mathmathcal{D}$-zips over $\bar{\kappa}$ and $G(\bar{\kappa})$-orbits on $X_J(\bar{\kappa})$.

Theorem 12.17 of [PWZ] yields a bijection of the set of $G(\bar{\kappa})$-orbits of $X_J(\bar{\kappa})$ with the set $JW$. In particular, there are only finitely many orbits and hence only finitely many isomorphism classes of $\mathmathcal{D}$-zips over $\bar{\kappa}$. The bijection is given by

\begin{equation}
JW \ni w \mapsto G(\bar{\kappa}) \text{-orbit of } (P_J, \dot{w}P_K, [\dot{w}\dot{x}]).
\end{equation}

Here we have chosen a Borel $B$ of $G$, a maximal torus $T$ of $B$ (yielding an identification of $W$ with the Weyl group of $T$), and for all $w \in W$ a representative $\dot{w}$ in $\text{Norm}_G(T)(\bar{\kappa})$. Moreover $P_J$ (resp. $P_K$) denotes the unique parabolic subgroup $G_\kappa$ of type $J$ (resp. $K$) containing $B_\kappa$.

Definition 4.8. We endow $JW$ with a relation $\leq$, where we define $w' \leq w$ if there exists $y \in W_J$ with $yw'(y^{-1})x^{-1} \leq w$ with respect to the Bruhat order. Here $x = w_0w\varphi(J)$ is the element defined above.

Lemma 4.9. The relation $\leq$ is a partial order on $JW$ compatible with the Galois action by $\Gamma_\kappa$ (i.e., $w' \leq w$ if and only if for one (or, equivalently by Lemma 4.9, for all) $\gamma \in \Gamma_\kappa$).

Proof. It is shown in [PWZ] Corollary 6.3 that $\leq$ is a partial order. Let $n = [\kappa : \mathbb{F}_p]$. As the group of automorphisms of $(W, I)$ induced by the Galois action is generated by $\varphi^n$, it remains to show that $w' \leq w$ implies $\varphi^n(w') \leq \varphi^n(w)$. But this follows from $\varphi^n(J) = J$ which implies $\varphi^n(W_J) = W_J$ and $\varphi^n(x) = x$.

Proposition 4.10. The underlying topological space of $[G_\kappa \backslash X_J] \otimes_\kappa \bar{\kappa}$ is homeomorphic to the topological space $(JW)_{\text{top}}$ attached to the partially ordered set $(JW, \leq)$.

The description of the underlying topological space of a quotient stack for an action with finitely many orbits and the notion of the topological space attached to a partially ordered set are recalled in Section A.9.

Proof. For $w, w' \in JW$ let $O$ and $O'$ be the $G(\bar{\kappa})$-orbits of $X_J(\bar{\kappa})$ corresponding to $w$ and $w'$, respectively. As explained in Section A.9, it suffices to show that the closure of $O'$ contains $O$ if and only if $w \leq w'$. This is shown in [PWZ] Theorem 12.15.

In particular we obtain for every algebraically closed extension $k$ of $\kappa$ a bijection

\begin{equation}
\{\text{isomorphism classes of } \mathmathcal{D}\text{-zips over } k\} \leftrightarrow JW.
\end{equation}

Remark 4.11. Proposition 4.10 shows that the underlying topological space of $[G_\kappa \backslash X_J]$ is homeomorphic to the set of $\Gamma_\kappa$-orbits on $JW$ endowed with the quotient topology (Section A.9). In other words, it is homeomorphic to the topological space attached to the partially ordered set $\Gamma_\kappa \backslash JW$ of $\Gamma_\kappa$-orbits on $JW$, where we set $\Gamma_\kappa w' \geq \Gamma_\kappa w$ if and only if for one (or, equivalently by Lemma 4.9 for all) $v' \in \Gamma_\kappa w'$ there exists $v \in \Gamma_\kappa w$ such that $v' \leq v$.  

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Remark 4.12. Let \(k\) be an algebraically closed extension of \(\kappa\), \(w \in j^*W\), and let \(O_w \subset X_J(k)\) be the corresponding \(G(k)\)-orbit. Let \(\bar{w}\) be the image of \(w\) under the canonical map \(j^*W = W_J \setminus W \to W_J \setminus W_K = j^*W_K\). Then for all \((P,Q,[g]) \in O_w\) one has \(\text{relpos}(P,Q) = \bar{w}\) ([MW] Section 4.6).

The smooth cover \(\tilde{X}_J\) of \(X_J\)

We denote by \(\tilde{X}_J\) the scheme which represents the functor on \(\kappa\)-schemes which associates with a \(\kappa\)-scheme \(S\) the set of triples \((P,Q,g)\) where \(P \subset G_S\) is a parabolic of type \(J\), where \(Q \subset G_S\) is a parabolic of type \(K\) and where \(g \in G(S)\) is an element such that \(\text{relpos}(Q,gF(P)) = x\).

The smooth cover \(\tilde{X}_J\) of \(X_J\) is then \(G_\kappa\)-equivariant.

4.3 Group-theoretic definition of \(\zeta\) and the Ekedahl-Oort stratification

We relate the reductions of Shimura varieties and the schemes \(\tilde{Y}_J\) (Definition 4.11), \(\tilde{X}_J\), and \(X_J\). We first construct a morphism

\[
\psi: \tilde{Y}_J \otimes \kappa \to \tilde{X}_J
\]

as follows. For every \(S\)-valued point \((P,L,g)\) of \(\tilde{Y}_J\) we define \(Q\) as the unique opposite parabolic such that \(g^{-1}Q \cap F(P) = F(L)\) ([SGA3] Exp. XXVI, 4.3.). Then \(Q\) is of type \(K = w_0^J\varphi(J)\), and \((P,Q,g) \in \tilde{X}_J(S)\) by Remark 4.5.

Now define a morphism

\[
\tilde{\theta}: \mathcal{A}_0 \to \tilde{Y}_J \otimes \kappa
\]

as follows: Let \(R\) be a \(\kappa\)-algebra and let \((A,\iota,\lambda,\eta,\alpha, C', D')\) be an \(R\)-valued point of \(\mathcal{A}_0\). Let \((M,C,D,\varphi_0,\varphi_1)\) be the \(\mathcal{D}\)-zip attached to \((A,\iota,\lambda)\) (Section 2.1). Then \(\alpha\) is by definition an isomorphism of \(\mathcal{D}\)-modules \(M \cong \Lambda \otimes R\). Let \(P\) be the stabilizer of \(\alpha(C)\) in \(G_R\), let \(L\) be the stabilizer of the decomposition \(\alpha(C) \oplus C' = \Lambda_R\), and let \(g\) be the composition

\[
\Lambda_R \cong \Lambda_R^{(p)} = \alpha(C)^{(p)} \oplus C'^{(p)} \xrightarrow{\varphi_1^{(p)} \oplus \varphi_0^{(p)}} \alpha(D) \oplus D' = \Lambda_R
\]

where \(\varphi_1\) and \(\varphi_0\) are the morphisms induced by \(\varphi_1\) and \(\varphi_0\) via the isomorphism \(\alpha\). We obtain an \(R\)-valued point \((P,L,g)\) of \(\tilde{Y}\). This construction is functorial in \(R\) and defines the morphism \(\tilde{\theta}\).

The composition \(\psi \circ \tilde{\theta}\) is a morphism \(\zeta: \mathcal{A}_0 \to \tilde{X}_J\). It induces morphisms

\[
\zeta^\#: \mathcal{A}_0^\# \to X_J, \\
\zeta: \mathcal{A}_0 \to [G_\kappa \setminus X_J].
\]

Via the isomorphism \(\mathcal{D} - \mathcal{Z}ip \cong [G_\kappa \setminus X_J]\) (Proposition 4.7) the morphism \(\zeta\) defined here is identified with the morphism \(\zeta\) in (2.2).
By Proposition 4.10 we know that the points of the underlying topological space of $[G_{\kappa} \setminus X_J] \otimes \bar{\kappa}$ are in natural bijection with $^JW$. For $w \in ^JW$ set
\[ \mathcal{A}_0^w := \zeta^{-1}(\{w\}). \]
This is a locally closed subset of the underlying topological space of $\mathcal{A}_0 \otimes \bar{\kappa}$.

**Definition 4.13.** We call the corresponding reduced locally closed substack of $\mathcal{A}_0 \otimes \bar{\kappa}$ the Ekedahl-Oort stratum corresponding to $w \in ^JW$. It is again denoted by $\mathcal{A}_0^w$.

**Remark 4.14.** The residue gerbe ([LM] (11.1)) $\mathcal{G}(w)$ of the point $w$ in the underlying topological space of $[G_{\kappa} \setminus X_J] \otimes \bar{\kappa}$ is a locally closed smooth algebraic substack. Then the locally closed substack $\mathcal{A}_0(w) := \zeta^{-1}(\mathcal{G}(w))$ of $\mathcal{A}_0$ has the same underlying topological space as $\mathcal{A}_0^w$. It can be shown that $\mathcal{A}_0(w) = \mathcal{A}_0^w$ (at least for $p > 2$). As we do not need this in the sequel, we decided to define the structure of a substack just by taking the reduced substack.

Let $\text{Spec} \bar{\kappa} \rightarrow [G_{\kappa} \setminus X_J]$ be a representative of the point $w$ and let $M$ be the corresponding $\mathcal{D}$-zip over $\bar{\kappa}$. Then $\mathcal{A}_0(w) = \mathcal{A}_0^{[M]}$ with the notation in Definition 2.3. Let $(A, \iota, \lambda, \eta)$ be the universal family restricted to $\mathcal{A}_0(w)$ and let $\mathcal{M}$ be the attached $\mathcal{D}$-zip on $\mathcal{A}_0(w)$. Then by the definition of the residue gerbe, a morphism of $\bar{\kappa}$-schemes $f : T \rightarrow \mathcal{A}_0 \otimes \bar{\kappa}$ factors through $\mathcal{A}_0(w)$ if and only if $f^*\mathcal{M}$ is fppf-locally on $T$ isomorphic to the pullback $M_T$ of $M$ to $T$.

**Remark 4.15.** Instead of working with points in $[G_{\kappa} \setminus X_J] \otimes \bar{\kappa}$ to define Ekedahl-Oort strata one could also work with points in $[G_{\kappa} \setminus X_J]$ (considered as locally closed substacks of $[G_{\kappa} \setminus X_J]$ defined over $\kappa$) to define $\kappa$-rational Ekedahl-Oort strata indexed by $\Gamma_{\kappa}$-orbits $\Gamma_{\kappa}w$ on $^JW$ (Remark 4.11). We denote them by $\mathcal{A}_0^{\Gamma_{\kappa}w}$. Then
\[ \mathcal{A}_0^{\Gamma_{\kappa}w} \otimes \bar{\kappa} = \bigcup_{w' \in \Gamma_{\kappa}w} \mathcal{A}_0^{w'}. \]
Conversely, $\mathcal{A}_0^w$ for $w \in ^JW$ is already defined over the finite extension $\kappa(w)$ of $\kappa$ in $\bar{\kappa}$ such that $\text{Gal}(\bar{\kappa}/\kappa(w)) = \{ \gamma \in \Gamma_{\kappa} : \gamma(w) = w \}$.

**Example 4.16.** The partially ordered set $(^JW, \preceq)$ has a unique minimal element, namely 1, and a unique maximal element, $^\mu w := w_{0,J}w_0$, where $w_0$ is the longest element in $W$ and $w_{0,J}$ is the longest element in $W_J$. We call the corresponding Ekedahl-Oort strata $\mathcal{A}_0^1$ and $\mathcal{A}_0^w$ the superspecial and the $^\mu$-ordinary Ekedahl-Oort stratum, respectively. As the $\Gamma_{\kappa}$-action preserves the order, it fixes 1 and $^\mu w$. Therefore both strata are defined over $\kappa$.

### 4.4 Ekedahl-Oort strata in the Siegel case II

We continue the example of the Siegel case from Sections 1.3 and 2.2. As explained in 1.8 $^JW$ can be identified with $\{0,1\}^g$. Moreover, the Frobenius acts trivially on $W$, i.e. $\bar{\varphi} = \text{id}$, and $K = J$ because $w_0 = -1$ is central. As the Galois group of $\kappa = \mathbb{F}_p$ acts trivially on $(W,I)$, all Ekedahl-Oort strata are already defined over $\mathbb{F}_p$ (Remark 4.11).
By Section 2.2 and (4.3) we have for every algebraically closed extension \( k \) of \( \mathbb{F}_p \) a bijection
\[
\{ \text{isomorphism classes of symplectic Dieudonné spaces over } k \} \leftrightarrow \{0, 1\}^g.
\] (4.6)

For \( w = (\epsilon_i)_i \in \{0, 1\}^g \) let \( M = (M, F, V, \langle , \rangle) \) be a symplectic Dieudonné space over \( k \) in the isomorphism class corresponding to \( w \). Combining (A.6) and Remark 4.12 we obtain
\[
a(M) := \dim M/(FM + VM) = g - \# \{ i ; \epsilon_i = 1 \}.
\] (4.7)

Therefore the longest element \( \mu w = (1, 1, \ldots, 1) \) in \( J^W \) corresponds to the unique isomorphism class of \( M \) with \( a(M) = 0 \), which means that the associated truncated \( p \)-divisible group is ordinary. Thus in the Siegel case the \( \mu \)-ordinary Ekedahl-Oort stratum is equal to the stratum of ordinary principally polarized abelian varieties in \( \mathcal{A}_0 \).

The shortest element \( 1 = (0, \ldots, 0) \) in \( J^W \) corresponds to the isomorphism class of \( M \) with \( a(M) = g \), which means that the associated truncated \( p \)-divisible group is superspecial. Thus in the Siegel case the superspecial Ekedahl-Oort stratum is equal to the stratum of principally polarized abelian varieties in \( \mathcal{A}_0 \) such that the underlying abelian variety of a geometric point is isomorphic to a product of supersingular elliptic curves.

5 Flatness of \( \zeta \)

**Theorem 5.1.** The morphism \( \zeta \) is flat.

By definition the following diagram is cartesian
\[
\xymatrix{\mathcal{A}_0^\# \ar[r] & \mathcal{A}_0 \ar[d]^\zeta \ar[d]_{\zeta^*} \\
X_J \ar[r] & [G \backslash X_J],}
\]

where the horizontal morphisms are \( G \)-torsors and in particular smooth and surjective. We will show that \( \zeta^\# \) is universally open. As \( X_J \) and \( \mathcal{A}_0^\# \) are both regular, it then follows from [EGA] IV, (15.4.2), that \( \zeta^\# \) is flat. By faithfully flat descent this shows that \( \zeta \) is flat.

To show that \( \zeta^\# \) is universally open, we use the following criterion.

**Proposition 5.2.** Let \( Y \) be a locally noetherian scheme, let \( X \) be a scheme, and let \( f : X \to Y \) be a morphism of finite type. Assume that for every commutative diagram
\[
\xymatrix{\text{Spec}(k) \ar[r]^h & X \ar[d]^f \\
\text{Spec}(R) \ar[r]^g & Y}
\] (5.1)

25
where $R$ is a complete discrete valuation ring with algebraically closed residue field $k$, there exists a surjective morphism $\text{Spec}(\tilde{R}) \to \text{Spec}(R)$, where $\tilde{R}$ is a local integral domain and a morphism $\tilde{g}: \text{Spec}(\tilde{R}) \to X$ such that the following diagram commutes

\[
\begin{array}{ccc}
\text{Spec}(\tilde{R} \otimes_R k) & \to & \text{Spec}(k) \\
\downarrow & & \downarrow \\
\text{Spec}(\tilde{R}) & \to & \text{Spec}(R)
\end{array}
\]

Then $f$ is universally open.

It is easy to see that such a property already characterizes universally open morphisms. We will not need this remark in the sequel.

**Proof.** By [EGA] IV (8.10.2) it suffices to show that the base change $f_{A^n}: A^n_X \to A^n_Y$ is open for all $n \geq 0$. The affine space $A^n_Y$ is again locally noetherian and the hypothesis for $f$ implies the same property for $f_{A^n_X}$. Thus we may replace $Y$ by $A^n_Y$ and $X$ by $A^n_X$.

Therefore it suffices to show that $f$ is open. The question is local on $Y$ and we may thus assume that $Y$ is noetherian.

Let $U \subset X$ be an open subset. By Chevalley’s theorem $f(U)$ is constructible. Therefore it suffices to show that $f(U)$ is stable under generization (e.g., [GW] Lemma 10.17). Let $x_0 \in U$ and $y_0 = f(x_0) \in f(U)$ and let $y_1 \in Y$ be a generization with $y_1 \neq y_0$. By Lemma 5.3 below, there exists a diagram like in 5.1 such that $g(s) = y_0$, $g(\eta) = y_1$ (where $s$ (resp. $\eta$) is the special (resp. generic) point of $\text{Spec}(R)$) and such that the image of $h$ is $x_0$. We apply the hypothesis and find a morphism $\tilde{g}: \text{Spec}(\tilde{R}) \to X$ such that $f \circ \tilde{g}$ is the composition

$$\text{Spec}(\tilde{R}) \to \text{Spec}(R) \to Y.$$ 

The image $x_1$ of the generic point of $\text{Spec}(\tilde{R})$ under $\tilde{g}$ is a generization of $x_0$ and hence lies in $U$ as $U$ is open. Therefore $y_1 = f(x_1) \in f(U)$.

**Lemma 5.3.** Let $Y$ be a locally noetherian scheme and let $f: X \to Y$ be a morphism of schemes. Let $x_0 \in X$, $y_0 := f(x_0)$ and let $y_1 \neq y_0$ be a generization of $y_0$. Then there exists a commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(\kappa) & \to & X \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \to & Y
\end{array}
\]

where $R$ is a complete discrete valuation ring with algebraically closed residue field $\kappa$, and where $i: \text{Spec}(\kappa) \hookrightarrow \text{Spec}(R)$ is the inclusion such that the image of the generic (resp. special) point of $\text{Spec}(R)$ under $g$ is $y_1$ (resp. $y_0$) and such that the image of $h$ is $x_0$.
Proof. There exists a morphism $g' : \text{Spec}(R') \to Y$ where $R'$ is a discrete valuation ring such that $g'(s') = y_0$ and $g'(s') = y_1$ where $s'$ (resp. $s'$) is the closed (resp. generic) point of $\text{Spec}(R')$.

Let $m'$ be the maximal ideal of $R'$ and let $\kappa$ be an algebraically closed field extension of $\kappa(y_0)$ such that there exist $\kappa(y_0)$-embeddings $\kappa(x_0) \to \kappa$ and $\kappa(s') \to \kappa$ and let $R' \to R$ be a flat local homomorphism of $R'$ into a complete discrete valuation ring $R$ with residue field $\kappa$ such that $m'R$ is the maximal ideal of $R$ (this exists by [EGA] 0169.3). We set $g$ as the composition
\[
\text{Spec}(R) \to \text{Spec}(R') \xrightarrow{g'} Y
\]
and $h$ as the composition
\[
\text{Spec}(\kappa) \to \text{Spec}(\kappa(x_0)) \to X. \quad \square
\]

Proof of the universal openness of $\zeta^\#$. Every equi-characteristic complete discrete valuation ring $R$ with residue field $k$ is isomorphic to the ring of formal power series $k[[\varepsilon]]$. Therefore by Proposition 6.2 it suffices to show the following lemma. \□

Lemma 5.4. Let $k$ be an algebraically closed field of characteristic $p$, let $R = k[[\varepsilon]]$ be the ring of formal power series in one variable $\varepsilon$ and set $R_1 = k[[\varepsilon^{1/p}]]$. We denote by
\[
\pi : \text{Spec}(R_1) \to \text{Spec}(R)
\]
the natural morphism.

Let $x = (A, \iota, \lambda, \eta, \alpha)$ be a $k$-valued point of $\mathfrak{A}_0^\#$. Let $(P, Q, [g]) \in X_J(k)$ be the image of $x$ under $\zeta^\#$. Denote by $(P_\varepsilon, Q_\varepsilon, [g_\varepsilon]) \in X_J(R)$ any deformation of $(P, Q, [g])$ to $R$ (which exists because $X_J$ is smooth). Then there exists a deformation $x_1 = (A_1, \iota_1, \lambda_1, \eta_1, \alpha_1) \in \mathfrak{A}_0^\#(R_1)$ of $x$ such that $\zeta^#(x_1) = \pi^*(P_\varepsilon, Q_\varepsilon, [g_\varepsilon]).$

Note that the smoothness of $X_J$ implies that a deformation $(P_\varepsilon, Q_\varepsilon, [g_\varepsilon])$ as in the lemma always exists.

Proof. Let $\mathcal{P} = (P, Q, F, F')$ be the Dieudonné display of the $p$-divisible group of the abelian variety $A$. The free $W(k)$-module $\mathcal{P}$ is equipped with a perfect alternating form via $\lambda$ and an $O_B$-action. Moreover, we can fix an $O_B$-linear symplectic similitude $\bar{\alpha}$ of $\mathcal{P}$ with $\Lambda \otimes W(k)$ which lifts the isomorphism $\alpha$ (Lemma 4.9). Set $\mathcal{P}_\varepsilon = \Lambda \otimes \mathcal{W}(R)$ and $\mathcal{P}_{\varepsilon,1} = \Lambda \otimes \mathcal{W}(R_1)$.

We choose a normal decomposition $\mathcal{P} = S \oplus T$ as in Definition 3.4 and obtain a split $\mathcal{P}$-display $(S, T, \Psi)$. Let $(\tilde{P}, \tilde{L}, \tilde{g})$ be the associated element in $\tilde{Y}_J(W(k))$ (Construction 4.2).

Then the reduction of $\tilde{g}$ modulo $p$ is an element $g \in [g]$ by the definition of $\zeta^\#$. Let $g_\varepsilon \in [g_\varepsilon]$ be an element such that the reduction of $g_\varepsilon$ modulo $\varepsilon$ is equal to $g$. Set $P_{\varepsilon,1} := P_\varepsilon \otimes R_1$, $Q_{\varepsilon,1} := P_\varepsilon \otimes R_1$ and let $g_{\varepsilon,1}$ be the element $g_\varepsilon$ considered as an $R_{1\varepsilon}$-valued point of $G$.

Let $L_{\varepsilon,1}$ be the Levi subgroup of $P_{\varepsilon,1}$ such that
\[
F(L_{\varepsilon,1}) = (g_{\varepsilon,1}^{-1})Q_{\varepsilon,1} \cap F(P_{\varepsilon,1}).
\]
We now apply Lemma 3.1 to the smooth scheme $\tilde{Y}_J$ and the ring $R_1$. We see that there exists an element

$$\left( \tilde{P}_{\varepsilon,1}, \tilde{L}_{\varepsilon,1}, \tilde{g}_{\varepsilon,1} \right) \in \tilde{Y}_J(W(R_1))$$

whose reduction to $R_1$ equals $(P_{\varepsilon,1}, L_{\varepsilon,1}, g_{\varepsilon,1})$ and whose reduction to $W(k)$ equals $(\tilde{P}, \tilde{L}, \tilde{g})$.

Let $(S_{\varepsilon,1}, T_{\varepsilon,1}, \Psi_{\varepsilon,1})$ be the split $D$-display associated with $(\tilde{P}_{\varepsilon,1}, \tilde{L}_{\varepsilon,1}, \tilde{g}_{\varepsilon,1})$ (Construction 4.2) and $(X_1, \iota_1, \lambda_1)$ be the $p$-divisible group with $D$-structure corresponding to $(S_{\varepsilon,1}, T_{\varepsilon,1}, \Psi_{\varepsilon,1})$ (Theorem 3.3). By Serre-Tate theory we obtain the desired point $x_1 \in \mathfrak{a}_0^\#(R_1)$.

### 6 Closures of Ekedahl-Oort strata

From now on we fix a maximal torus $T$ of $G$ and a Borel group $B$ containing $T$, both defined over $\mathbb{Z}_p$. This is possible by [1,4]. Let $(X^*(T), \Phi, X_*(T), \Phi^\vee, \Delta)$ be the corresponding based root datum. Its Weyl system is $(W, I)$. The based root datum and its Weyl system are endowed with an action by $\Gamma = \pi_1(\text{Spec} \mathbb{Z}_p, \text{Spec} \bar{k}) = \text{Gal}(\bar{k}/\mathbb{F}_p)$.

We consider the conjugacy class $[\mu]$ defined by the Shimura datum (Section 1.1) as a $W$-orbit in $X_*(T)$. The representative in $X^*(T)$ that is dominant with respect to $B$ is denoted by $\mu$. It is defined over $W(k)$ or, if we consider only the reduction modulo $p$ of $(G, B, T)$, over $k$.

**Theorem 6.1.** The closure of an Ekedahl-Oort stratum $\mathfrak{a}^w_0$ ($w \in J^W$) is a union of Ekedahl-Oort strata. More precisely,

$$\overline{\mathfrak{a}^w_0} = \bigcup_{w' \preceq w} \mathfrak{a}^{w'}_0,$$

where $\preceq$ is the partial order defined in Definition 4.8.

For $p > 2$ the first assertion has been shown in [Wr12] (6.8). For the Siegel case, Oort has given a different parametrization of the Ekedahl-Oort strata (in terms of elementary sequences, see Section 4.4) and also proved the first assertion ([O01]). The description which strata appear in the closure of a given stratum is new even in the Siegel case.

**Proof.** By Proposition 4.10 the closure of $\{w\}$ in the underlying topological space of $[G \setminus X_J] \otimes \bar{k}$ is $\{ w' \in J^W ; w' \preceq w \}$. As $\zeta$ is universally open, we have $\overline{\mathfrak{a}^w_0} = \zeta^{-1}(\{w\})$.

### Comparison to loop groups

For a field $k$ and a linear algebraic group $H$ we denote by $LH$ the loop group. It is the group ind-scheme over $k$ representing the sheaf for the fpqc-topology

$$(k\text{-algebras}) \to (\text{groups}), \quad R \mapsto LH(R) := H(R(\mathbb{A}^1)),$$

see [Fal] Definition 1. We denote by $LG$ the loop group of $G_{\mathbb{F}_p}$.
Recall that $\mu \in X_\lambda(T)$ is the dominant coweight determined by the PEL Shimura datum $\mathcal{D}$, defined over $K$. Let $k$ be an algebraically closed extension of $K$, set $G := G(k[z])$, and let $K_1$ be the kernel of the reduction modulo $z$ map $K \rightarrow G(k)$. We denote by $\mu(z) \in LG(k)$ the image under $\mu: \mathbb{G}_m \rightarrow T$ of $z \in \mathbb{G}_m(k((z)))$. For all $w \in W$ we have a chosen representative $\tilde{w} \in G(k) \rightarrow G(k((z)))$. As in [Vi1] we consider for each $w \in J$ the corresponding truncation stratum in the loop group of $LG$ which is defined as the locally closed reduced subscheme of $LG \otimes_{\mathbb{F}_p} \kappa(w)$ (where $\kappa(w)$ is the field of definition of $w$, see Remark 4.15) with

$$S_{\lambda} = \{ k^{-1}w \tilde{x}_\mu(z)k_1 \sigma(k) ; k \in K, k_1 \in K_1 \},$$

where $x_\mu(z) := w_0 \omega_\mu(z) = x$ is the element already considered in (4.1). Remark 7.1 below explains why this can be considered as an “Ekedahl-Oort stratum for loop groups”.

**Corollary 6.2.** A stratum $\mathcal{S}_{\lambda}^0$ is contained in the closure of $\mathcal{S}_{\lambda}^0$ if and only if $S_{\lambda} \subset S_{\lambda}$ is contained in the closure of $S_{\lambda}$ in $LG$.

**Proof.** This follows from the explicit descriptions of the closure relations in Theorem 6.1 and in [Vi1], Corollary 4.7. \qed

### 7 The Newton stratification

#### 7.1 Affine Weyl groups

To be able to compare the moduli space $\mathcal{M}_0$ with the loop group $LG$ we make the following definitions. We consider two cases in which $O_L$ denotes either $W(k)$ or $k[z]$, where $k$ is an algebraically closed field of characteristic $p$. Let $L$ be the field of fractions of $O_L$. By $\epsilon$ we denote the uniformizer $p$ or $z$.

By $I$ we denote the inverse of $B$ under the projection $G(O_L) \rightarrow G(k)$.

Let $\tilde{W} = N_T(L)/T(O_L) \cong W \ltimes X(T)$ denote the extended affine Weyl group of $G$. It has a decomposition $W \cong \Omega \ltimes W_{aff}$. Here $\Omega$ is the subset of elements of $\tilde{W}$ which stabilize the chosen Iwahori subgroup $I$. The second factor $W_{aff}$ is the affine Weyl group of $G$, an infinite Coxeter group generated by the simple reflections together with the simple affine reflection defined by $I$. If $M$ is a Levi subgroup of $G$ containing $T$, let $I_M = I \cap M(O_L)$. Let $W_M$ and $W_M$ be the Weyl group and extended affine Weyl group for $M$. They are canonically subgroups of $W$ and $W_M$. Let $\Omega_M$ be the subgroup of elements of $\tilde{W}_M$ which stabilize $I_M$.

For $\lambda \in X(T)$ we have the representative $\lambda(\epsilon) \in T(L)$. Together with the chosen representatives of $W$ we have representatives $\tilde{x}$ in $G(L)$ for all elements $x \in \tilde{W}$. By the Bruhat-Tits decomposition (see [Ti]) each element of $G(L)$ is contained in a double coset $I \tilde{x} I$ for a unique $x \in \tilde{W}$.

Let $a = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and let $a_M$ be the subset of elements which are centralized by $M$. Finally we denote by $\mathfrak{a}$ the unique alcove in the antidominant Weyl chamber of the standard apartment of the Bruhat-Tits building of $G$ whose closure contains the origin. Then $I$ is the Iwahori subgroup fixing $\mathfrak{a}$. We call $\mathfrak{a}$ the opposite base alcove.
7.2 Crystals and isocrystals with \( \mathcal{D} \)-structure attached to points in \( \mathcal{A}_0 \)

The map \( \Upsilon \)

Let \( k \) be an algebraically closed field extension of \( \kappa \). By (covariant) Dieudonné theory, isomorphism classes of \( p \)-divisible groups \( (X, \iota, \lambda) \) with \( \mathcal{D} \)-structure over \( k \) are in bijection with isomorphism classes of their Dieudonné modules \( (P, F) \). Here \( P \) is a \( \mathcal{D} \)-module over \( W(k) \) and \( F : P^\sigma \to P \) is an injective \( W(k) \otimes O_B \)-linear map which preserves the symplectic form up to the scalar \( p \in W(k) \). Thus after tensoring with \( L := \text{Frac } W(k) \), \( F \) is an isomorphism of \( \mathcal{D} \)-modules over \( L \).

By Lemma 1.9 we can choose an isomorphism \( \alpha : P \sim \to \Lambda \otimes_{\mathbb{Z}_p} W(k) \) of \( \mathcal{D} \)-modules. By transport of structure with \( \alpha \) we obtain \( F = b(\text{id} \Lambda \otimes \sigma) \) for some \( b \in G(L) \). Let \( K = G(W(k)) \). Then for a different trivialization \( \alpha' \) we have \( \alpha' = g \circ \alpha \) for some \( g \in K \) and \( b \) is replaced by \( g^{-1} b \sigma(g) \).

If we denote by \( C(G) = C_k(G) \) the set of \( K \)-\( \sigma \)-conjugacy classes

\[
[b] := \{ g^{-1} b \sigma(g) : g \in K \}
\]

of elements \( b \in G(L) \), we therefore obtain an injective map

\[
(7.1) \quad \{ \text{isomorphism classes of } p \text{-divisible groups with } \mathcal{D} \text{-structure over } k \} \hookrightarrow C_k(G).
\]

The image consists of the subset \( C(G, \mu) = C_k(G, \mu) \) of \( K \)-\( \sigma \)-conjugacy classes of elements \( g \in K \mu(p)K \).

Let \( (A, \iota, \lambda, \eta) \) be a \( k \)-valued point of \( \mathcal{A}_0 \). Then the \( p \)-divisible group \( A[p^\infty] \) carries a \( \mathcal{D} \)-structure and thus we can attach an element \( [b] \in C(G) \) which we denote by \( \Upsilon(A, \iota, \lambda, \eta) \). We obtain a map

\[
(7.2) \quad \Upsilon : \mathcal{A}_0(k) \to C(G, \mu).
\]

**Remark 7.1.** For a \( p \)-divisible group with \( \mathcal{D} \)-structure the (covariant) Dieudonné module of its \( p \)-torsion carries the structure of a \( \mathcal{D} \)-zip (cf. Example 2.2). This induces a map on isomorphism classes, i.e., by (7.1) and (4.3) a map

\[
(7.3) \quad \text{tc} : C(G, \mu) \longrightarrow \mathcal{D} W,
\]

which we call truncation map (at level 1).

This map is surjective: It follows from Remark 4.4 and (4.2) that for \( w \in \mathcal{D} W \) a pre-image is given by \( \llbracket \dot{w} x \mu(p) \rrbracket \) where \( x \) is as in (4.1).

As explained in the introduction of [VI], \( \text{tc} \) maps \( [g] \) and \( [g'] \) of \( C(G, \mu) \) to the same element if and only if

\[
(7.4) \quad \exists k_1, k_1' \in K_1 := \text{Ker}(K \to G(k)) : \llbracket k_1 g' k_1' \rrbracket = [g].
\]

As \( K_1 \subset K \), this condition is also equivalent to the existence of \( k_1 \in K_1 \) such that \( [g' k_1] = [g] \).
The sets $B(G)$ and $B(G, \mu)$

Again let $k$ be an algebraically closed extension of $\kappa$ and consider the two cases $L = \text{Frac} W(k)$ and $L = k((z))$. Let $O_L$ be its ring of integers and $\epsilon$ the uniformizer $p$ or $z$. Let $B(G)$ be the set of $G(L)$-\(\sigma\)-conjugacy classes $\{g^{-1}b_0\sigma(g) : g \in G(L)\}$ of elements $b_0 \in G(L)$. For $L = \text{Frac} W(k)$ there is a canonical surjection $C(G) \rightarrow B(G)$.

The elements of $B(G)$ are classified (in much greater generality) by Kottwitz [Ko]. In [Ko] only the case $L = \text{Frac} W(k)$ is considered. However, the same arguments show the analogous classification for the other case. There is a map $\nu: B(G) \rightarrow (X_*(T) \otimes \mathbb{Q})^\Gamma$ where $\Gamma = \text{Gal}(\tilde{k}/\mathbb{F}_p)$. We call $\nu(b)$ the Newton polygon of $b$. For $G = GL_n$ it coincides with the usual Newton polygon of $\mathbb{F}$-isocrystals.

Furthermore Kottwitz [Ko] defines a map $\kappa_G: G(L) \rightarrow \pi_1(G)^\Gamma$ (to be precise, the reformulation using $\pi_1(G)^\Gamma$ is due to Rapoport and Richartz, [RR], Section 1). Here $\pi_1(G)$ is the quotient of $X_*(T)$ by the coroot lattice and $\pi_1(G)^\Gamma$ denotes the coinvariants under the Galois group $\Gamma$. In our situation this map has the following simplified description: Let $b \in G(L)$ be a representative of $b$ and let $\lambda \in X_*(T)$ be the unique dominant cocharacter with $b \in G(O_L) \mu(\epsilon) G(O_L)$. Then $\kappa(b)$ is the image of $\lambda$ under the projection $X_*(T) \rightarrow \pi_1(G)^\Gamma$. Indeed, the maps $\kappa_G$ considered by Kottwitz are invariant under $\sigma$-conjugation, they are group homomorphisms, and are natural transformations in $G$. For $G = \mathbb{G}_m$ we have $\kappa_G(b) = \nu_\epsilon(b)$. In our case the properties of $\kappa_G$ mentioned above show that $\kappa_G$ is trivial on $K(k)$ as each such element is $\sigma$-conjugate to $1$ (by a version of Lang’s theorem), and that the torus element $p^\mu$ is mapped to its image in $\pi_1(G)$. Together we obtain the explicit description of $\kappa_G$ given above.

An element $b \in B(G)$ is determined by $\nu(b)$ and $\kappa(b)$. Note that $(X_*(T) \otimes \mathbb{Q})^\Gamma \times \pi_1(G)^\Gamma$ is the same for both cases of $L$. Furthermore, the explicit description of the image of the maps $\nu$ and $\kappa$ (as for example in [RR], Remark 1.18) shows that it also does not depend on these cases and that it is independent of the choice of the algebraically closed field $k$. In this way we obtain a bijection between $B(G)$ for $L = \text{Frac} W(k)$ and $L = k((z))$.

Let $X_*(T)_{\text{dom}}$ be the cone of $B$-dominant cocharacters in $X_*(T)$ and let $\mu \in X_*(T)_{\text{dom}}$ be the dominant representative of the conjugacy class $[\mu]$ from the Shimura datum $\mathcal{D}$. Then $\Gamma = \text{Gal}(\tilde{k}/\mathbb{F}_p)$ acts on $X_*(T)_{\text{dom}}$. Let $\Gamma_{\mu}$ be the stabilizer of $\mu$ in $\Gamma$ and set

$$\bar{\mu} := [\Gamma : \Gamma_{\mu}]^{-1} \sum_{\tau \in \Gamma/\Gamma_{\mu}} \tau(\mu) \in (X_*(T) \otimes \mathbb{Q})_{\text{dom}}^\Gamma.$$ 

On $X_*(T) \otimes \mathbb{Q}$ one considers the standard order which is given by $\nu' \leq \nu$ if and only if $\nu - \nu'$ is a non-negative linear combination of positive coroots. It induces a partial ordering on $B(G)$ by setting $b \leq b'$ if and only if $\nu(b) \leq \nu(b')$ and $\kappa(b) = \kappa(b')$. We define

$$(7.5) \quad B(G, \mu) := \{ b \in B(G) : b \leq [\mu] \}.$$ 

This is a finite set. By work of Kottwitz and Rapoport [RR], Lucarelli [L] and Gashi [Ga], $b \in B(G, \mu)$ if and only if $b \cap G(O_L) \mu(\epsilon) G(O_L) \neq \emptyset$. The same arguments show the analogous assertion for $L = k((z))$. Note that this condition for non-emptiness of the
intersection is a purely group-theoretic condition in terms of $b \in B(G)$ and $\mu \in X_*(T)$, in particular, we obtain a canonical bijection between the sets $B(G, \mu)$ for the two cases.

The Newton stratification of $\mathcal{A}_0$

The composition of $\Upsilon$ (7.2) with $C(G) \to B(G)$ is given by the map that attaches to a $k$-valued point $(A, \iota, \lambda, \eta)$ of $\mathcal{A}_0$ the isogeny class of its $p$-divisible group with $\mathcal{D}$-structure. We call its image in $B(G)$ the Newton point of $(A, \iota, \lambda, \eta)$. For $s \in \mathcal{A}_0$, let $\mathcal{O}_{\mathcal{A}_0,s}$ be the local ring of $\mathcal{A}_0$ as Deligne-Mumford stack (if $\mathcal{A}_0$ is a scheme, then $\mathcal{O}_{\mathcal{A}_0,s}$ is a strict henselization of the usual local ring at the point $s$). Choose an algebraically closed extension $k$ of the residue field $\kappa(s)$ of $\mathcal{O}_{\mathcal{A}_0,s}$. We denote the Newton point of the composition $\text{Spec} k \to \text{Spec} \kappa(s) \to \mathcal{A}_0$ by $N_t(s)$. The independence of $B(G)$ of $k$ shows that $N_t(s)$ does not depend on the choice of $k$. Thus we obtain a map

$$N_t : \mathcal{A}_0 \to B(G, \mu).$$

For $b \in B(G, \mu)$ we denote by $\mathcal{N}_b$ the set of points in $s \in \mathcal{A}_0$ such that $N_t(s) = b$. By $\text{RR}$, $\mathcal{N}_b$ is a locally closed subset of the underlying topological space of $\mathcal{A}_0$ and there is an inclusion of closed subsets

$$(7.6) \quad \overline{\mathcal{N}_b} \subseteq \bigcup_{b' \leq b} \mathcal{N}_{b'}.$$

This is a group-theoretic formulation of Grothendieck’s specialization theorem. We endow $\mathcal{N}_b$ with its reduced structure of a locally closed substack (a reduced subscheme if $\mathcal{A}_0$ is a scheme).

Definition 7.2. $\mathcal{N}_b$ is called the Newton stratum attached to $b \in B(G, \mu)$.

Example 7.3. There is a unique maximal element $b_\mu$ and a unique minimal element $b_{\text{basic}}$ in $B(G, \mu)$ characterized by $\nu(b_\mu) = \bar{\mu}$ and by $\nu(b_{\text{basic}}) \in X_*(Z)\mathbb{Q}$, where $Z \subset T$ is the center of $G$. The corresponding Newton strata $\mathcal{N}_{b_\mu}$ and $\mathcal{N}_{b_{\text{basic}}}$ are called the $\mu$-ordinary Newton stratum and the basic Newton stratum, respectively.

By (7.6) $\mathcal{N}_{b_{\text{basic}}}$ is closed in $\mathcal{A}_0$. Moreover, (7.6) shows that $\mathcal{N}_{b_{\text{basic}}}$ is the complement of the (finite) union of the closures of the non-$\mu$-ordinary Newton strata. Therefore $\mathcal{N}_{b_{\text{basic}}}$ is open in $\mathcal{A}_0$.

8 Minimal Ekedahl-Oort strata

8.1 Minimal Ekedahl-Oort strata and fundamental elements

For $p$-divisible groups over an algebraically closed field of positive characteristic Oort defines within each isogeny class of $p$-divisible groups a single isomorphism class which he calls minimal (Ood). His definition is recalled in Remark 8.12 below. It is equivalent to the condition that the ring of endomorphisms of the Dieudonné module of a minimal $p$-divisible group is a maximal order in the endomorphisms of its isocrystal. Oort proves
(Oo4 and Oo5) that a $p$-divisible group $X$ is minimal if and only if for any $p$-divisible group $X'$ one has

$$X[p] \cong X'[p] \Rightarrow X \cong X'.$$

Similar results for $p$-divisible groups with a principal polarization $\lambda: X \rightarrow X'$ follow by using uniqueness of polarizations (Oo3 Corollary (3.8)). For general $p$-divisible groups with $\mathcal{D}$-structure we take the group-theoretic analogue of (8.1) as a definition for minimality.

**Remark 8.1.** Let $k$ be an algebraically closed extension of $\kappa$, $L := \text{Frac} W(k)$, and recall that $K = G(W(k))$ and $K_1 = \text{Ker}(K \rightarrow G(k))$. Let $(X, \iota, \lambda)$ be a $p$-divisible group with $\mathcal{D}$-structure over $k$ and let $x \in G(L)$ such that the isomorphism class of $(X, \iota, \lambda)$ is given by the $K$-$\sigma$-conjugacy class $[x] \in C_k(G, \mu)$ via (7.1). Then the following conditions are equivalent.

(i) For all $y \in K_1 x K_1$ there exists a $g \in K$ such that $g y \sigma(g)^{-1} = x$.

(ii) For all $x' \in [x]$ and for all $y \in K_1 x' K_1$ there exists a $g \in K$ such that $g y \sigma(g)^{-1} = x'$.

(iii) Any $p$-divisible group with $\mathcal{D}$-structure $(X', \iota', \lambda')$ with $(X', \iota', \lambda)[p] \cong (X, \iota, \lambda)[p]$ is isomorphic to $(X, \iota, \lambda)$.

The equivalence of (i) and (ii) is immediate because $K_1$ is a normal subgroup of $K$ and the equivalence of (ii) and (iii) follows from (7.4). Clearly the equivalent conditions above depend (for a fixed field $k$) only on the isomorphism class of $(X, \iota, \lambda)[p]$, i.e., on $\text{tc}([x]) \in J W (7.3)$. They can be expressed by the condition that $\text{tc}^{-1}(\text{tc}([x]))$ consists of a single element in $C_k(G, \mu)$.

**Definition 8.2.** (1) Let $k$, $L$ and $K_1$ be as in Remark 8.1. An element $x \in G(L)$ is called *minimal* if the equivalent conditions of Remark 8.1 hold. Then each element of $[x] \in C_k(G)$ is also minimal and we call $[x]$ *minimal*. A $p$-divisible group with $\mathcal{D}$-structure over $k$ is called *minimal* if its class in $C_k(G, \mu)$ is minimal.

(2) An element $w \in J W$ (or its corresponding Ekedahl-Oort stratum $\mathcal{O}_0^w$) is called *minimal* if there exists a minimal $[x] \in C_k(G, \mu)$ such that $\text{tc}([x]) = w$.

**Remark 8.3.** If $x \in G(L)$ is minimal, then $\sigma(x)$ is also minimal. This shows that if $w \in J W$ is minimal, then each element in the same $\Gamma_k$-orbit of $w$ (Remark 4.15) is also minimal.

**Example 8.4.** Moonen ([Mo3], Theorem in Section 0.3) has shown that the $\mu$-ordinary Ekedahl-Oort stratum is minimal in the sense of Definition 8.2.

Our next goal is to study the question whether each Newton stratum contains a minimal Ekedahl-Oort stratum. We first translate this question into group-theoretic language. We obtain our results using a variant of the notion of fundamental elements introduced by Görtz, Haines, Kottwitz, and Reuman in [GHKR2], and generalized by Viehmann to unramified groups in [V11].

To construct abelian varieties with additional structures we will use the following principle. Let $\mathcal{D} = (B, *, V, \langle \cdot, \cdot \rangle, O_B, \Lambda, h)$ and $\mathcal{D}' = (B', *, V, \langle \cdot, \cdot \rangle', O_B, \Lambda, h')$ be two unramified Shimura-PEL-data which agree except for the data $\langle \cdot, \cdot \rangle$ and $h$ and such that
Lemma 8.5. Let \( k \) be an algebraically closed extension of \( k \) and of \( k' \), let \((A',\iota',X',\eta')\in\mathcal{A}_0'(k)\) and let \(X':=(X',\iota',\lambda',\eta')\). Let \(X=(X,\iota,\lambda)\) be a \(p\)-divisible group with \(\mathcal{D}\)-structure and let \(\rho: X'\to X\) be an \(O_B\)-linear isogeny of \(p\)-divisible groups compatible with the polarizations on \(X'\) and \(X\) up to a power of \(p\). Then there exist a \(k\)-valued point \((A,\iota,\lambda,\eta)\) of the moduli space \(\mathcal{A}_0\) associated with \(\mathcal{D}\) and an \(O_B\)-linear isogeny \(f: A'\to A\) with \(f^*\lambda=p^e\lambda\) for some \(e\geq 0\) such that \((A,\iota,\lambda,\eta)[p^\infty]=(X,\iota,\lambda)\) (up to isomorphism of \(p\)-divisible groups with \(\mathcal{D}'\)-structure) and such that \(f[p^\infty]=\rho\).

It is not claimed that \(f\) satisfies any compatibility with the level structures.

Proof. Dividing \(A'\) by \(C:=\text{Ker}(\rho)\) yields an isogeny \(f: A'\to A:=A'/C\) between \(A'\) and an abelian variety \(A\) with \(A[p^\infty]=X\). Furthermore, the \(\mathcal{D}'\)-structure (resp. the level structure) on \(A'\) induces an \(O_B\)-action and a polarization \((\iota,\lambda)\) on \(A\) (resp. a level structure \(\eta\) on \(A\)) whose restrictions to \(X\) are the given ones. The determinant condition for \(\mathcal{D}\) holds for \(A\) because it holds for \(X\).

Lemma 8.6. Let \((A,\iota,\lambda,\eta)\) be a \(k\)-valued point of \(\mathcal{A}_0\) and let \(b\) be its image in \(B(G,\mu)\). There exists a minimal \(k\)-valued point \((A_1,\iota_1,\lambda_1,\eta_1)\) which is isogenous to \((A,\iota,\lambda,\eta)\) (i.e., there exists an \(O_B\)-linear isogeny \(f: A\to A_1\) such that \(f^*\lambda_1=p^e\lambda\) for some \(e\geq 0\) and \(f^*\eta_1=\eta\)) if and only if the following condition holds.

\[
K_{\mu}(p)K\cap b \subseteq G(L) \text{ contains a minimal element.}
\]

Proof. The condition is clearly necessary. Conversely, let \(g\in K_{\mu}(p)K\cap b\) be a minimal element. Let \(X=(X,\iota,\lambda)\) be the \(p\)-divisible group with \(\mathcal{D}\)-structure of \((A,\iota,\lambda,\eta)\) and let \(b_0\in G(L)\) be a representative of the corresponding class in \(C(G,\mu)\). Let \(h\in G(L)\) with \(g=h^{-1}b_0\sigma(h)\). Let \(X_1\) be a \(p\)-divisible group with \(\mathcal{D}\)-structure whose corresponding class is \([g]\in C(G,\mu)\). Then \(X_1\) is minimal and \(h\) induces an isogeny \(\rho: X\to X_1\) of \(p\)-divisible groups with \(\mathcal{D}\)-structure. Thus the lemma follows from Lemma 8.3 for \(\mathcal{D} = \mathcal{D}'\).

Definition 8.7. (1) Let \(F\) be a finite unramified extension of \(Q_p\) and let \(P\) be a semi-standard parabolic subgroup of \(G_{O_F}\), i.e. a parabolic subgroup containing \(T_{O_F}\) but not necessarily \(B_{O_F}\). Let \(N\) be its unipotent radical and let \(M\) be the Levi factor containing \(T_{O_F}\). Let \(\overline{N}\) be the unipotent radical of the opposite parabolic. Let \(\mathcal{I}_M=\mathcal{I}\cap M(O_L)\) and similarly \(\mathcal{I}_N=\mathcal{I}\cap N(O_L)\) and \(\mathcal{I}_N^\perp=\mathcal{I}\cap \overline{N}(O_L)\). Then an element \(x\in \text{W}\) is called \(P\)-fundamental if \(\sigma(\hat{x}\mathcal{I}_M\hat{x}^{-1})=\mathcal{I}_M\), \(\sigma(\hat{x}\mathcal{I}_N\hat{x}^{-1})\subseteq \mathcal{I}_N\), and \(\sigma(\hat{x}\mathcal{I}_N^\perp\hat{x}^{-1})\supseteq \mathcal{I}_N^\perp\).

(2) We call \(x\in \text{W}\) fundamental if it is \(P\)-fundamental for some finite unramified extension \(F\) and some semi-standard parabolic subgroup \(P\) of \(G_{O_F}\).
(3) We call a class \( c \in C(G) \) \textit{fundamental} if there exists a fundamental element \( x \in \tilde{W} \) and a representative \( \hat{x} \) such that \( c = [\hat{x}] \). If \( c \in C(G, \mu) \) is fundamental, we also call the corresponding isomorphism class of \( p \)-divisible groups with \( \mathcal{D} \)-structure \textit{fundamental}.

This definition generalizes the notion of \( P \)-fundamental elements in [GHKR2] from split groups to unramified groups, see also [VI], Definition 6.1. In [GHKR2], fundamental elements in \( W \) are defined by a condition which is a priori weaker than ours (i.e. as elements for which the first assertion of Remark 8.8 holds). We did not consider the question whether elements in \( \tilde{W} \) which are fundamental in their sense are also fundamental in our sense.

\textbf{Remark 8.8.} For a fundamental element \( x \in \tilde{W} \) and in the equi-characteristic case, Görtz, Haines, Kottwitz, and Reuman show in [GHKR2] Proposition 6.3.1 that every element of \( IxI \) is \( I \)-\( \sigma \)-conjugate to \( \hat{x} \). The same argument still shows this property in our more general situation.

This implies in particular that every element of \( K_1 \hat{x} K_1 \) (where \( K_1 := \text{Ker}(K \rightarrow G(k)) \subset I \)) is \( K \)-\( \sigma \)-conjugate to \( \hat{x} \) (where \( K = G(W(k)) \subset I \)). Hence if \( x \in \tilde{W} \) is a fundamental element, then every representative \( \hat{x} \) of \( x \) is minimal. In particular we see that fundamental \( p \)-divisible groups with \( \mathcal{D} \)-structure are minimal.

\subsection{The split case}

If \( G \) is split, we use results of [GHKR2] to show the following proposition:

\textbf{Proposition 8.9.} Assume that \( G \) is split and that \( \mu \) is minuscule. Then each element \( b \) of \( B(G) \) contains the representative of a fundamental element of \( \tilde{W} \). Any two such fundamental elements in \( \tilde{W} \) are conjugate under \( W \). In particular, \( b \) contains a unique fundamental element \( c_b \) of \( C(G) \). If \( b \in B(G, \mu) \), then \( c_b \in C(G, \mu) \).

\textbf{Proof.} By [GHKR2] Corollary 13.2.4 each element \( b \) of \( B(G) \) contains a fundamental element \( x_b \), and [VI], Lemma 5.3 and 6.11 show that any two fundamental elements in \( b \) are conjugate under \( W \). To be precise, [GHKR2] consider the equi-characteristic case, but the same proof shows the analogous result in our situation. It remains to show the last assertion. Thus it is enough to show that for each \( b \in B(G, \mu) \) there exists a fundamental element \( x \in W \mu W \) with \( \hat{x} \in b \). We have already seen that there is a fundamental element \( x_b \) in \( b \). Let \( \mu' \in X_*(T) \) be dominant with \( x_b \in W \mu' W \). We want to show that \( \mu' = \mu \). As \( \mu \) is minuscule, it is enough to show that \( \mu' \leq \mu \). Note that this is a statement purely in terms of the affine Weyl group and of the given element of \( B(G, \mu) \rightarrow X_*(T)_Q \times \pi_1(G) \). In particular, we can prove this assertion using equi-characteristic methods, i.e. replacing \( W(k) \) by \( k[z] \). As \( b \in B(G, \mu) \) there is an element \( g \in G(k((z))) \) with \( g \in G(k[z])\mu(z)G(k[z]) \) and in the \( \sigma \)-conjugacy class in \( G(k((z))) \) associated with \( b \). By [VI], Proposition 5.5 \( \hat{x}_b \) lies in the closure of the double coset \( G(k[z])\mu(z)G(k[z]) \). In particular, \( \mu' \leq \mu \). \bbox

The preceding result and its proof are valid for an arbitrary split reductive group scheme. From now on we assume again that \( G \) is associated with a PEL-Shimura-datum.
Theorem 8.10. Assume that \( G \) is split. Then for every \( b \in B(G, \mu) \) there exists a unique Ekedahl-Oort stratum \( \mathcal{A}_0^{aw(b)} \) that corresponds to a representative of a fundamental element and such that \( \mathcal{A}_0^{aw(b)} \subseteq \mathcal{N}_b \).

By Remark 8.8, \( \mathcal{A}_0^{aw(b)} \) is a minimal Ekedahl-Oort stratum.

Proof. By Proposition 8.9 there exists a fundamental element \( x \in \tilde{W} \) such that \( \tilde{x} \) is a \( b \) and \( [\tilde{x}] \in C(G, \mu) \). Moreover \( x \) is unique up to \( W \)-conjugation (which is the same as \( W \)-\( \sigma \)-conjugation because \( G \) is split) in \( \tilde{W} \). This shows that \( [\tilde{x}] \) depends only on \( b \). We call its image under the truncation map \( (7.3) \) \( w(b) \). By definition, the corresponding Ekedahl-Oort stratum \( \mathcal{A}_0^{aw(b)} \) meets the Newton stratum \( \mathcal{N}_b \). But by Remark 8.8 \( \mathcal{A}_0^{aw(b)} \) is minimal and thus is contained in \( \mathcal{N}_b \).

By Lemma 8.6 we obtain

Corollary 8.11. Let \( \mathcal{A}_0 \) be a moduli space of abelian varieties with \( \mathcal{D} \)-structure such that the associated group \( G \) is split. Let \( (A, \iota, \lambda, \eta) \) be a \( k \)-valued point of \( \mathcal{A}_0 \). Then there exists a minimal \( k \)-valued point \( (A_1, \iota_1, \lambda_1, \eta_1) \) which is isogenous to \( (A, \iota, \lambda, \eta) \).

Remark 8.12. We compare fundamental elements for \( G = GL_h \) and minimal \( p \)-divisible groups in the sense of Oort (\cite{Oo4}) (for short we will call them Oort-minimal). Let us first recall the definition of Oort-minimal \( p \)-divisible groups over an algebraically closed field \( k \) of positive characteristic. By definition there is a unique isomorphism class of \( G \)-minimal \( p \)-divisible groups in each isogeny class of \( p \)-divisible groups over \( k \). It is given as follows. Let \( N \) be the isocrystal corresponding to the isogeny class. If \( X \) is minimal in the given class, its Dieudonné module is isomorphic to a Dieudonné module of the following form. There is a \( \mathbb{Q}_p \)-rational decomposition \( N = \bigoplus_{i=1}^l N_i \) of \( N \) into simple isocrystals \( N_i \) such that \( M = \bigoplus_{i=1}^l M_i \cap N_i \) and such that \( M_i \cap N_i \) has a basis \( e_1, \ldots, e_h \) with \( F(e_j) = e_{j+n_i} \). Here \( \lambda_i = n_i/h_i \) with \( (n_i, h_i) = 1 \) is the slope of \( N_i \) and we use the notation \( e_{j+n_i} = pe_j \).

Let now \( f_j = e_{j+1} \). Then \( f_1, \ldots, f_h \) is a basis of \( N \). Let \( h := \dim N \). One easily checks that if we write \( F = b \sigma \) for \( b \in GL_h(L) \) with respect to the given basis, then \( b \) is contained in the Levi subgroup \( M \) given by the decomposition \( N = \bigoplus_{i=1}^l N_i \). We fix the diagonal torus \( T \) in \( G = GL_h \) and the Borel subgroup of upper triangular matrices containing it. If \( \mu \) denotes the \( M \)-dominant Hodge polygon of \( b \) (with respect to these choices), then \( \mu \in \{ 0, 1 \}^h \subset \mathbb{Z}^h = X(T) \) is minuscule and \( b \) satisfies \( b I_M b^{-1} = I_M \).

By \cite{Vii}, Lemma 6.11 this implies that the Dieudonné module also has a trivialization \( (M, F) \cong (W(k)^n, b \sigma) \) such that \( b \in G(L) \) is a representative of a fundamental element in \( \tilde{W} \). Thus an Oort-minimal \( p \)-divisible group of height \( h \) (or its corresponding class in \( C(G) \)) corresponds to a representative of a fundamental element.

The same argument works for \( G = GSp_{2g} \) by working with isocrystals where simple isocrystals of slope \( \lambda \) occur with the same multiplicity as isocrystals of slope \( 1 - \lambda \).

Thus we see that Oort-minimal \( p \)-divisible groups (without any additional structure) are fundamental, and fundamental \( p \)-divisible groups are minimal by Remark 8.8. We obtain a new proof of the main result of \cite{Oo4}:
Corollary 8.13. Oort-minimal $p$-divisible groups are minimal in the sense of Definition 8.2.

If $G$ is split (i.e. isomorphic to a product of reductive groups of the form $GL_n$ or $GSp_{2g}$), then – as explained above – Oort has shown in a series of papers that each Newton stratum contains a unique minimal Ekedahl-Oort stratum. This then shows that the notion of Oort-minimal, fundamental and minimal $p$-divisible groups coincide in this case.

In general the existence of minimal Ekedahl-Oort stratum within a given Newton stratum is not clear. Lemma 8.9 shows that for $b \in B(G, \mu)$ the corresponding Newton stratum $\mathcal{N}_b$ contains a minimal Ekedahl-Oort stratum if the following group-theoretical condition holds.

(8.3) There is a fundamental element in $W_{\mu}W$ whose image in $B(G)$ is equal to $b$.

Example 8.14. The uniqueness of minimal Ekedahl-Oort strata within a given Newton stratum does not hold in general. One example of this phenomenon is the “unramified inert Hilbert-Blumenthal case”, i.e., the case where we have an exact sequence

$$1 \to \text{Res}_{K/Q_p} SL_{2,K} \to G_{Q_p} \to G_m \to 1,$$

where $K$ is a unramified extension of $Q_p$ of some degree $g$. We give an explicit example that for $g \geq 6$ there exist minimal Ekedahl-Oort strata which are not in the same Galois orbit, but which are in the same Newton stratum. We consider $G_{Q_p}$ as a subgroup of $\text{Res}_{K/Q_p} GL_{2,K}$. Let $T$ and $B$ be the maximal torus and the Borel subgroup of $G$ consisting of diagonal matrices and of upper triangular matrices, respectively. We identify $\Delta = \text{Gal}(K/Q_p)$ with $\mathbb{Z}/g\mathbb{Z}$. The Weyl group $W$ is isomorphic to $S_2^\Delta$ where $S_2 = \{1, s\}$ is the symmetric group of two elements. For $\tau \in \Delta$ we denote the non-trivial element in the $\tau$th factor by $s_\tau$. We have

$$X_s(T) \cong \{(g r_1, g r_2) \tau \in (\mathbb{Z}^2)^\Delta : \exists c \in \mathbb{Z}: g_{r_1} + g_{r_2} = c \text{ for all } \tau \}.$$

We have $\mu = ((1,0), \ldots, (1,0))$. We consider two particular elements $\phi_1 = (1,0)$ and $\phi_2 = (0,1)s$ in $\tilde{W}_{GL_2} \cong S_2 \ltimes \mathbb{Z}^2$. Let $x, x' \in \tilde{W}$ with decomposition $x = \prod x_\tau$ and $x' = \prod x'_\tau$. Let $x_5 = x_6 = x'_4 = x'_6 = \phi_1$ and all other $x_\tau$ and $x'_\tau$ equal to $\phi_2$ (viewed as an element of the corresponding component of $\tilde{W}$). Then $x$ and $x'$ are not in the same Galois orbit in $\tilde{W}$. We have $x = y^{-1} x' \sigma(y)$ where $y = s_3 s_4 (-1,1)_4$, and thus $x, x'$ are in the same Newton stratum.

We define two parabolic subgroups $P, P'$ of $G_{Q_p}$ as intersections of $G_{Q_p}$ with the following parabolic subgroups $\tilde{P}, \tilde{P}'$ of $\text{Res}_{K/Q_p} GL_{2,K}$. Let $\tilde{P}_2, \tilde{P}_4, \tilde{P}_2', \tilde{P}_4'$ be the subgroup of lower triangular matrices, and let all other $\tilde{P}_r$ and $\tilde{P}_r'$ be the subgroups of $GL_2$ of upper triangular matrices. Let $\tilde{P}_K = \prod \tilde{P}_r$ and similarly for $\tilde{P}'$. Then an explicit calculation shows that $x$ is $P$-fundamental and $x'$ is $P'$-fundamental. Thus they induce two minimal Ekedahl-Oort strata contained in the same Newton stratum.

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8.3 The basic case

Although we do not know whether in general an isogeny class of \( p \)-divisible groups with \( \mathcal{D} \)-structure contains a minimal or even a fundamental \( p \)-divisible group with \( \mathcal{D} \)-structure, we will now explain that this is true in the \( \mu \)-ordinary and in the basic case.

Remark 8.15. Moonen ([Mo3], Theorem in Section 0.3) has shown that the \( \mu \)-ordinary Newton stratum is equal to the \( \mu \)-ordinary Ekedahl-Oort stratum and that this is minimal.

Proposition 8.16. All basic classes \( b \in B(G) \) contain a fundamental element of \( C(G) \). If \( b \in B(G, \mu) \) is basic, then it contains a fundamental element of \( C(G, \mu) \).

Proof. We use the notation from Subsection 7.1. Let \( x \in \Omega \) be such that its image under the surjection \( \Omega \to \pi_1(G) \to \pi_1(G)_\Gamma \) is equal to \( \kappa(b) \). Then \( \bar{x} \in B(G) \) is by definition basic and has the same image in \( \pi_1(G)_\Gamma \) as \( b \). Thus \( \bar{x} = b \). Furthermore \( x \in \Omega \) implies that \( x \) is \( P \)-fundamental for \( P = G \).

Let \( b \in B(G, \mu) \) for some \( \mu \) correspond to the isogeny class of a \( p \)-divisible group with \( \mathcal{D} \)-structure. Then one can choose \( x \in \Omega \) to be the unique element whose image in \( \pi_1(G) \) is equal to the image of \( \mu \in X_1(T) \) under the projection to \( \pi_1(G) \). Then \( \hat{x} \in K\mu(p)K \).

Note that this a purely group-theoretic result and its proof works for any reductive group scheme \( G \) over \( \mathbb{Z}_p \). For basic Newton strata one has the following explicit description of a corresponding minimal Ekedahl-Oort stratum.

Proposition 8.17. The Ekedahl-Oort stratum \( \mathcal{A}_0^1 \) for \( w = 1 \) is minimal and non-empty. If \( b \in B(G, \mu) \) is the basic element, then \( \mathcal{A}_0^1 \subseteq N_b \).

Proof. By [Far] 3.1.8 or [Ko2] §18, the basic Newton stratum is non-empty. Let \( (A, \iota, \lambda, \eta) \) be a \( k \)-valued point of \( N_b \). Let \( (X, \iota, \lambda) \) be a \( p \)-divisible group with \( \mathcal{D} \)-structure whose associated \( \mathcal{D} \)-zip is of type \( w = 1 \). We want to show that \( (X, \iota, \lambda) \) is minimal and isogenous to the \( p \)-divisible group with \( \mathcal{D} \)-structure of \( (A, \iota, \lambda, \eta) \). Indeed, then the non-emptiness of the stratum \( \mathcal{A}_0^1 \) follows as in the proof of Lemma 8.5 and the inclusion \( \mathcal{A}_0^1 \subseteq N_b \) follows from the definition of minimality. Thus by Remark 7.1 it is enough to show that \( x := w_0w_0,\mu \in \hat{W} \) is fundamental and that its image in \( B(G, \mu) \) is basic. We show that \( xIx^{-1} = I \), where \( I \) is the chosen Iwahori subgroup. Then \( x \) is \( P \)-fundamental for \( P = M = G \) and \( x \in \Omega \) and hence its image in \( B(G, \mu) \) is basic. To prove \( xIx^{-1} = I \) we use the decomposition of \( I \) into \( T \cap I = T(O_L) \) and \( U_\alpha \cap I \) for all root subgroups \( U_\alpha \). We have \( x(T \cap I)x^{-1} = T \cap I \). Furthermore

\[
U_\alpha \cap I = \begin{cases} 
U_\alpha(O_L) & \text{if } \alpha > 0 \\
\{g \in U_\alpha(O_L) \mid g \equiv 1 \pmod{p}\} & \text{if } \alpha < 0.
\end{cases}
\]

We have to show

\[
(8.4) \quad w_0w_0,\mu(U_\alpha(L) \cap I)(w_0w_0,\mu)^{-1} = w_0w_0,\mu U_\alpha(L)(w_0w_0,\mu)^{-1} \cap I
\]
for every $\alpha$. Let $P_\mu$ be the standard parabolic subgroup associated with $\mu$ and let $M_\mu$ and $N_\mu$ be its Levi factor containing $T$ and its unipotent radical. Let $N_\mu^{\geq 0}$ be the unipotent radical of the opposite parabolic. For roots $\alpha$ of $T$ in $M_\mu$ conjugation by $\mu$ acts trivially on $U_\alpha$, and conjugation by $w_0 w_{0,\mu}$ maps positive roots in $M_\mu$ to positive roots (not necessarily in $M_\mu$) and negative roots in $M_\mu$ to negative roots. Thus (8.4) holds for roots of $T$ in $M$. For roots $\alpha$ of $T$ in $N_\mu$ we have $\langle \mu, \alpha \rangle = 1$, hence conjugation by $\mu$ maps $U_\alpha \cap I = U_\alpha(O_L)$ to $\{ g \in U_\alpha(O_L) \mid g \equiv 1 \pmod{p} \}$. Furthermore $w_0 w_{0,\mu}$ maps these roots to $N_\mu^{\geq 0}$, hence to negative roots. Together with a similar consideration for the roots in $N_\mu$ we obtain that $w_0 w_{0,\mu} (U_\alpha \cap I) (w_0 w_{0,\mu})^{-1} = w_0 w_{0,\mu} U_\alpha (w_0 w_{0,\mu})^{-1} \cap I$ for every $\alpha$.

\section{The general case}

In general we do not know whether an isogeny class of $p$-divisible groups with $D$-structure contains a fundamental $p$-divisible group with $D$-structure. However, we show in this subsection the slightly weaker statement that each isogeny class of $p$-divisible groups with $O_B$-action and polarization (but without requiring a determinant condition) contains a fundamental $p$-divisible group with this additional structure. For non-split groups, fundamental $p$-divisible groups in a given isogeny class are in general not unique, compare Example 8.14.

\textbf{Theorem 8.18.} Let $G$ and $\mu$ be the reductive $\mathbb{Z}_p$-group and the minuscule cocharacter associated with a Shimura PEL-datum and let $b \in B(G_{\mathbb{Q}_p}, \mu)$. Then there exists an $x \in \tilde{W}$ such that $x \in b$ is fundamental and such that $x \in W\mu'$ for some minuscule coweight $\mu'$.

In particular, $x$ satisfies condition (8.2). The theorem and its proof are a refinement of a corresponding but slightly weaker statement for split groups, cf. [VII], Theorem 6.5.

For the proof of the theorem we may assume that we are in one of the cases (AL), (AU), or (C) of Remark 1.3. As a preparation of the proof we construct an explicit representative of $b$. Let $\nu \in X_*(T)^{\mathbb{Q}_p, \text{dom}}$ be the dominant Newton point of $b$ and let $M_b$ be its centralizer. There is a standard Levi subgroup $M$ of $G_{\mathbb{Q}_p}$ which is minimal among those Levi subgroups of $G_{\mathbb{Q}_p}$ that are defined over $\mathbb{Q}_p$ and contain an element $b' \in M(L) \cap b$ whose $M$-dominant Newton point is equal to $\nu$. In other words, $b$ is superbasic in $M$. For properties of superbasic elements compare [GHKR1], Section 5.9, see also [CKV]. The adjoint group $M^{\text{ad}}$ is then isomorphic to a product of groups of the form $\text{Res}_{F_i/\mathbb{Q}_p} \text{PGL}_{n_i}$ for some finite unramified extension $F_i$ of $\mathbb{Q}_p$ and some $n_i$. Using the explicit form of the groups $G$ associated with Shimura PEL-data (Remark 1.4) one obtains that $M$ is of the form

$$M \cong M'_0 \times \prod_{i=1}^t M_i$$

where $M_i = \text{Res}_{F_i/\mathbb{Q}_p} \text{GL}_{n_i}$ for some unramified extension $F_i$ of $\mathbb{Q}_p$ and some $n_i \geq 1$ and where $M'_0$ is either isomorphic to $\mathbb{G}_{m, \mathbb{Q}_p}$ (in Case (AL) or if in the isocrystal

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corresponding to $\rho(b)$ ($\rho: G_{\mathbb{Q}_p} \to \text{GL}(V_{\mathbb{Q}_p})$ the standard representation) the multiplicity of the simple isocrystal with slope 1/2 is even) or where $M'_0$ sits in a non-split exact sequence

\[(8.5) \quad 0 \to \text{Res}_{F_0/\mathbb{Q}_p} SL_2 \to M'_0 \to \mathbb{G}_m \to 0,\]

where $F_0$ is again unramified over $\mathbb{Q}_p$.

The image $b'_i$ of $b'_i$ in each factor $M'_0$ or $M_i = \text{Res}_{F_i/\mathbb{Q}_p} GL_{n_i}$ is basic and the valuation $q_i$ of its determinant is coprime to $n_i$. If $M'_0$ is as in (8.5) then $q_0 = [F_0 : \mathbb{Q}_p]$. If $M'_0$ is as in (8.5) let $M_0 := \text{Res}_{F_0/\mathbb{Q}_p} GL_2$ and $n_0 = 2$, otherwise let $M_0 := M'_0$, $F_0 := \mathbb{Q}_p$, and $n_0 = 1$. For $i = 0, \ldots, t$ we let $I_i$ be the set of $\mathbb{Q}_p$-embeddings $F_i \to \overline{\mathbb{Q}}_p$. Fixing one such embedding we identify $I_i$ with $\mathbb{Z}/d_i\mathbb{Z}$ or $\{1, \ldots, d_i\}$, where $d_i := [F_0 : \mathbb{Q}_p]$.

Over $L$, the group $M_i$ decomposes into a product $\prod_{I_i} GL_{n_i,L}$. For each $\tau \in I_i$ let $(e_{\tau,l})_{1 \leq l \leq n_i}$ denote the standard basis of $L^{n_i}$. For $l \in \mathbb{Z}$ we define $e_{\tau,l}$ by $e_{\tau,l+n_i} = p^{e_{\tau,l}}.$

We want to define an element $x$ of $M(L)$ by defining its projection $x_i$ to each factor $M'_0(L)$ and $M_i(L) = \prod_{I_i} GL_{n_i,L}$. For $i = 0, \ldots, t$ define $x_i \in M_i(L)$ by

$$(x_i(e_{\tau,l}) := e_{\tau,l+n_i}^\frac{q_i}{d_i} \cdot \frac{(r-1)^{q_i}}{d_i^r}. \right)$$

Then $\det x_i = \pm p^{n_i}$. Thus if $i = 0$ and $M'_0$ is as in (8.5), then $\det x_i \in \mathbb{Q}_p^\times$ and we obtain that indeed $x_0 \in M'_0(L)$.

Furthermore, $(x_i\sigma)^{d_i n_i} = p^{n_i}$, hence $x_i$ is basic and thus $x$ is contained in the given $\sigma$-conjugacy class $b$. Using the explicit form of $x$ one can see that we obtained a representative $x$ of $b$ with the following properties.

(1) $x$ is contained in the image of the embedding of $\tilde{W}_M$ into $G(L)$.

(2) Let $\nu_x = \nu$ and $\mu_x$ be the $M$-dominant Newton- and Hodge-polygon of $x$. Then for every root $\alpha$ of $T$ in $G$ we have

$$|\langle \alpha, \mu_x - \nu_x \rangle| < 2.$$ 

(3) $\mu_x$ is minuscule in $G$.

(4) Let $N$ be the unipotent radical of $P = MB$, in particular $\langle \alpha, \nu \rangle \geq 0$ for each root $\alpha$ of $T$ with $U_\alpha \subseteq N$. Then the previous property implies that $\langle \alpha, \mu_x \rangle$ (being an integer) is at least $-1$.

(5) Properties (1), (2) and (4) also hold for $x \sigma(x) \cdots \sigma^m(x)$ for all $m \geq 0$.

**Proof of Theorem 8.18**

Let $M, N, P$, and $x$ be as above. We denote by $\phi_x$ the morphism $g \to \sigma(x^{-1} gx)$ on $G(L)$.

**Claim.** There is an Iwahori subgroup $\tilde{I} \subseteq K$ with $\tilde{I} \cap N = I \cap M$ and $\phi_x(\tilde{I} \cap N) \subseteq \tilde{I} \cap N$.

We first show that this claim implies the theorem. As $x \in \Omega_M$ we have $\phi_x(I_M) = \sigma(I_M) = I_M$. By [VII], Lemma 6.6 we have $\phi_x(\tilde{I} \cap N) \supseteq \tilde{I} \cap N$. As $\tilde{I} \subseteq K$ there is a $y \in W$ with $y^{-1}\tilde{I}y = I$. Let $M' = \text{y}^{-1} My$ and $P' = \text{y}^{-1} Py$ with unipotent radical $N'$.
and opposite $\overrightarrow{N}$. Then $y^{-1}I_M y = I_{M'}$. Let $x' = \sigma^{-1}(y)^{-1}xy \in b$. We have

$$
\sigma(x'I_{M'}(x'))^{-1} = \sigma(x'y^{-1}I_M y(x')^{-1}) = y^{-1}\sigma(x\tilde{I}_M x^{-1})y = I_{M'}
$$

and similar translations for $N'$ and $\overrightarrow{N}$. Hence $x'$ is a $P'$-fundamental alcove in $b$. From Property (3) of $x$ and the definition of $x'$ we obtain that the Hodge polygon of $x'$ is also minuscule. The theorem follows.

It remains to show that the claim is true. Let $r > 0$ be such that $G$ is split over some unramified extension of $\mathcal{O}_F$ of degree $r$. In particular, $\sigma^r$ then acts trivially on the root system of $G$ and on $\tilde{W}$. Let $c := \sigma(x)\sigma^2(x)\cdots \sigma^r(x) \in \tilde{W}$. Applying the decomposition $\tilde{W} = W \times X_\alpha(T)$ we obtain $c = w_c w_c$. Let $n_c$ be the order of $w_c$ in $W$. Replacing $r$ by $n_c r$ and using $\sigma(x)\sigma^2(x)\cdots \sigma^{n_c}(x) = c^{n_c} \in X_\alpha(T)$ we may assume that we already have $c \in X_\alpha(T)$. From the explicit definition of $x$ we see that $c = r\nu$ as elements of $X_\alpha(T)$. In particular, $c$ is dominant in $G$ and central in $M$.

We define $\mathcal{I}'$ as the image of the standard Iwahori subgroup $\mathcal{I}$ under conjugation by $w_0 M w_0$. It satisfies $I_M = \mathcal{I}' \cap M = \mathcal{I} \cap M$, $I_N' = \mathcal{I}' \cap N = K_1 \cap N$ and $I_N = \mathcal{I}' \cap \overrightarrow{N} = K \cap \overrightarrow{N}$.

The conjugation action of $c \in X_\alpha(T)$ on root subgroups can be described as follows. Let $\alpha$ be a root and $U_\alpha$ the corresponding root subgroup. Then $p^c U_\alpha(\epsilon)p^{-c} = U_\alpha(p^{\alpha,c} \epsilon)$. Using the fact that $\sigma^r$ acts trivially on $\tilde{W}$ we obtain that $\sigma^r(c^{n_c} I_{M} c^{-1}) = \tilde{c}^{n_c} I_{M} c^{-1} = I_{M}$ and $\sigma^r(c^{n_c} I_{N} c^{-1}) = c^{n_c} I_{N} c^{-1} \subseteq I_{N}$. Hence $c$ itself is a $P$-fundamental alcove for the $\sigma^r$-conjugacy class of $c$ in $G$. Let $\tilde{\mathcal{I}}$ with $\tilde{\mathcal{I}} \cap M = I_M$ be unique Iwahori subgroup of $G(L)$ which has in addition the property that $\tilde{\mathcal{I}} \cap N$ is minimal containing $I_N, \phi_x(I_N), \ldots, \phi_x^{-1}(I_N)$, cf. [VII, Lemma 6.8]. Then $\phi_x(\tilde{\mathcal{I}})$ is again an Iwahori subgroup. It satisfies $\phi_x(\tilde{\mathcal{I}}) \cap M = \phi_x(\tilde{\mathcal{I}} \cap M) = I_M$ and the analogous minimality property for $\phi_x(I_N), \ldots, \phi_x^{-1}(I_N)$. We have $\phi_x^c(I_N) = \sigma^r(c^{n_c} I_{N} c^{-1}) \subseteq I_N$. Thus $\phi_x^c(\tilde{\mathcal{I}} \cap N) \subseteq \tilde{\mathcal{I}} \cap N$. It remains to show that $\tilde{\mathcal{I}} \subseteq K$. This is equivalent to $\tilde{\mathcal{I}}_M, \tilde{\mathcal{I}}_N, \tilde{\mathcal{I}}_\overrightarrow{N} \subseteq K$. We have $\tilde{\mathcal{I}}_M = I_M \subseteq K$. The other two containments are equivalent to $K_1 \cap N \subseteq \tilde{\mathcal{I}}_N \subseteq K \cap N = \mathcal{I} \cap N$. As $K_1 \cap N = I_N$, the first of these inclusions follows from the definition of $\tilde{\mathcal{I}}_N$. For the second we have to show that $I_N, \phi_x(I_N), \ldots, \phi_x^{-1}(I_N)$ are all contained in $\mathcal{I} \cap N$. We decompose $I_N$ into root subgroups. For each root $\alpha$ of $T$ in $N$ we have to show that $\phi_x^c(U_\alpha(p\mathcal{O}_L)) \subseteq \mathcal{I} \cap N$ for $n = 0, \ldots, r - 1$. We write $x_n = \sigma(x)^{\alpha^2(x)} \cdots \sigma^{\alpha^n(x)}$. By properties 4. and 5. of $x$ it is of the form $w_n \mu_n$ for some $w_n \in W_M$ and $\mu_n$ satisfying $\langle \beta, \mu_n \rangle \geq -1$ for each root $\beta$ of $T$ in $N$. Hence

$$
\phi_x^c(U_\alpha(p\mathcal{O}_L)) = x_n \sigma^n(U_\alpha(p\mathcal{O}_L)) x_n^{-1} = w_n \sigma^n(U_{\sigma^n(\alpha)}(p\mathcal{O}_L)) \mu_n^{-1} w_n^{-1} = w_n U_{\sigma^n(\alpha)}(p^{1+\sigma^n(\alpha)\mu_n}) \mathcal{O}_L w_n^{-1} \subseteq \mathcal{I} \cap N
$$

which finishes the proof of the claim and of the theorem. □
8.5 Generic Newton strata in Ekedahl-Oort strata in the split case

We now determine the set of Newton strata which intersect the closure of a given Ekedahl-Oort stratum in a moduli space of abelian varieties with \( \mathcal{P} \)-structure where the associated group \( G \) is split.

The following lemma shows that the existence of fundamental alcoves implies a combinatorial criterion for non-emptiness of intersections of Iwahori-double cosets and Newton strata. In particular, it can be used to compare these intersections for the two cases \( L = \text{Frac}(W(k)) \) and \( L = k((z)) \), compare for example the proof of Theorem 8.20.

**Lemma 8.19** ([GHKR2], Proposition 13.3.1). Let \( x_b \in \hat{W} \) be a fundamental alcove associated with an element of \( B(G) \). Let \( b \) be the associated \( \sigma \)-conjugacy class in \( G(L) \) (for \( L = k[z] \) or \( L = \text{Frac}(W(k)) \)). Let \( \mathcal{I} \) be the standard Iwahori subgroup defined as the inverse image of \( B(k) \) under the projection \( G(O_L) \to G(k) \). Then

\[
\{ x \in \hat{W} ; I \hat{x} \mathcal{I} \cap b \neq \emptyset \} = \{ \hat{x} \in I \hat{y}^{-1} \hat{x}_b \mathcal{I} \sigma(\hat{y}) \mathcal{I} \text{ for some } \hat{y} \in \hat{W} \}.
\]

**Proof.** To prove the containment \( \subseteq \) let \( b_0 \in b \) and assume that \( \hat{x} = g^{-1} b_0 \sigma(g) \in I \hat{x} \mathcal{I} \) for some \( g \in G(L) \). Using the Bruhat-Tits decomposition we have \( g \in I \hat{y} \mathcal{I} \) for some \( y \in \hat{W} \). Hence \( x \) is contained in the right hand side. For the other inclusion assume that \( I \hat{x} \mathcal{I} \subseteq I \hat{y}^{-1} \hat{x}_b \mathcal{I} \sigma(\hat{y}) \mathcal{I} \). Then \( I \hat{x} \mathcal{I} \) meets \( \hat{y}^{-1} \hat{x}_b \mathcal{I} \sigma(\hat{y}) \). By Remark 8.8 every element of \( I \hat{x}_b \mathcal{I} \) can be written as \( i^{-1} \hat{x}_b \sigma(i) \) for some \( i \in \mathcal{I} \). Hence there is an element in \( I \hat{x} \mathcal{I} \) of the form \( \hat{y}^{-1} i^{-1} \hat{x}_b \sigma(i) \). Thus \( I \hat{x} \mathcal{I} \cap b \neq \emptyset \). \( \Box \)

**Theorem 8.20.** Let \( \mathcal{P} \) be such that the associated group \( G \) is split. Let \( w \in \mathcal{P} \) and \( b \in B(G, \mu) \) such that \( \mathcal{A}_0^w \cap N_b \neq \emptyset \). Let \( \mathcal{A}_0^{w'} \) be the unique Ekedahl-Oort stratum containing fundamental elements associated with \( b \). Then \( \mathcal{A}_0^{w'} \subseteq \mathcal{A}_0^w \).

For the Siegel moduli space this criterion has been conjectured by Oort ([Oo3], Conjecture 6.9) and has been shown by Harashita in [Har2] using different methods.

**Proof.** By Theorem 6.1 we have to show that there is a \( y \in W_J \) with \( y^{-1} w' x_\mu \sigma(y) x_\mu^{-1} \leq w \) with respect to the Bruhat order.

**Claim.** Let \( \nu(b) \in X_\ast(T)_{Q, \text{dom}} \) be the Newton point of \( b \in B(G, \mu) \). Then we have \( \mathcal{A}_0^w \cap N_b \neq \emptyset \) if and only if the \( \sigma \)-conjugacy class in the loop group \( LG \) corresponding to \( \nu(b) \in X_\ast(T)_{Q, \text{dom}} \) intersects \( S_{w, \mu} \) non-trivially.

This claim implies the theorem. Indeed, in the context of loop groups, Theorem 1.4, Proposition 5.5, and Lemma 6.11 of [Vil] together imply the following. Let \( w \in \mathcal{P} \) and \( b \in B(G, \mu) \) such that \( S_{w, \mu} \) intersects the \( \sigma \)-conjugacy class \( b \). Let \( S_{w', \mu} \) be the truncation stratum associated with a fundamental alcove for \( b \). Then \( S_{w', \mu} \subseteq S_{w, \mu} \). Then the claim together with Corollary 6.2 translate this assertion into the one we want to prove.

It remains to prove the claim. By Remark 7.1 we see that \( \mathcal{A}_0^{w'} \cap N_b \neq \emptyset \) holds if and only if \( K_1 w_\mu x_\mu(p) K_1 \cap b \neq \emptyset \). The same proof as for [Vil], Theorem 1.1(2) shows that this condition is equivalent to the same condition with \( K_1 \) replaced by \( \mathcal{I} \). Here \( \mathcal{I} \) denotes the chosen Iwahori subgroup for \( O_L = W(k) \), i.e. it is the inverse image of \( B(k) \) under \( K \to G(k) \). By Lemma 8.19 this condition can be expressed purely
Proof. The analogous statement for loop groups is shown in [Vi1], Proposition 5.7 and Corollary 8.22. Then (8.6) is minimal (or, equivalently in the split case, fundamental). We show below that
\[ \nu(b,\kappa(b)) \begin{cases} \nu(b,\kappa(b)) \in J \subseteq J^W, & b \mapsto w(b) \\ \end{cases} \]
whose image consists of the \( w \in J^W \) such that the corresponding Ekedahl-Oort stratum \( \mathcal{A}_0^w \) is minimal (or, equivalently in the split case, fundamental). We show below that for \( b,b' \in B(G,\mu) \) one has \( b' \leq b \) if and only if \( w(b') \preceq w(b) \).

**Remark 8.21.** Theorem 8.20 shows that \( w(b) \) for \( b \in B(G,\mu) \) can also be described as the unique minimal element (with respect to the partial order \( \preceq \)) in the set
\[ J^W_b := \{ w \in J^W ; \mathcal{A}_0^w \cap N_b \neq \emptyset \} . \]
Equivalently, it is the unique element of minimal length in \( J^W_b \).

**Corollary 8.22.** Let \( G \) be split and let \( b,b' \in B(G,\mu) \) be two elements with \( b' \leq b \). Then
\[ \mathcal{A}_0^{w(b')} \subseteq \overline{\mathcal{A}_0^w} . \]
In particular, for \( b,b' \in B(G,\mu) \) one has \( b' \leq b \) if and only if \( w(b') \preceq w(b) \).

Although the last assertion is purely group-theoretical we do not know of any purely group-theoretic proof of this result.

**Proof.** The analogous statement for loop groups is shown in [Vi1], Proposition 5.7 and Lemma 5.3. The assertion we are interested in thus follows from Corollary 6.2.

**Proposition 8.23.** Let \( G \) be split. Let \( w \in J^W \) and let
\[ \text{Min}(w) := \{ w' \in J^W ; \mathcal{A}_0^{w'} \text{ is minimal and } \mathcal{A}_0^{w'} \subseteq \mathcal{A}_0^w \} . \]
Let \( b \in B(G,\mu) \) be a maximal element in \( \{ b' \in B(G,\mu) ; w(b') \in \text{Min}(w) \} \) (i.e., in the set of Newton points such that their corresponding Newton strata contain a minimal Ekedahl-Oort stratum \( \mathcal{A}_0^{w'} \) with \( \mathcal{A}_0^{w'} \subseteq \mathcal{A}_0^w \)). Then the Newton stratum \( N_b \) contains the generic point of some irreducible component of \( \mathcal{A}_0^w \).

In the Siegel case this result is shown by Harashita in [Har2]. In this case Ekedahl and van der Geer have shown for \( p > 2 \) that each Ekedahl-Oort stratum which is not contained in the supersingular locus is irreducible ([EvG], Theorem 11.5). In particular, there is a unique generic Newton polygon in each Ekedahl-Oort stratum \( \mathcal{A}_0^w \) of \( \mathcal{A}_g \). Then the above result determines this Newton polygon.
Proof. The sets of generic Newton points of irreducible components of $\mathcal{A}_0^w$ and of $\mathcal{A}_0^w$ coincide. By Theorem 8.20, the set of Newton points of elements of $\mathcal{A}_0^w(\kappa)$ is equal to the set of Newton points of points $\xi \in \mathcal{A}_0^w$ which are in a minimal Ekedahl-Oort stratum. By the definition of minimality all $p$-divisible groups in one minimal Ekedahl-Oort stratum are isomorphic and in particular in the same Newton stratum. It is equal to the $\sigma$-conjugacy class $[\dot{x}]$ of one (equivalently, any) representative $\dot{x}$ of a fundamental element $x \in \tilde{W}$ corresponding to this minimal Ekedahl-Oort stratum. Let now $b \in B(G, \mu)$ be maximal among these $\sigma$-conjugacy classes $[\dot{x}]$, and let $Y$ be an irreducible component of $\mathcal{A}_0^w$ containing a point of $\mathcal{N}_b$. By the Grothendieck specialization theorem (7.6) the generic Newton stratum in $Y$ is greater or equal to $b$. By maximality of $b$ they have to be equal.

9 Properties of Ekedahl-Oort strata

9.1 Dimensions of Ekedahl-Oort strata

Theorem 9.1. Every Ekedahl-Oort stratum $\mathcal{A}_0^w \subseteq \mathcal{A}_0$ for $w \in J^W$ is non-empty.

Proof. It is enough to show that the morphism

$$\zeta \otimes \text{id}_\kappa : \mathcal{A}_0 \otimes \kappa \to [G_\kappa \setminus X_J] \otimes \kappa$$

is surjective. As it is open, this follows as soon as we know that the unique closed point of $[G_\kappa \setminus X_J] \otimes \kappa$ is in the image. The underlying topological space of $[G_\kappa \setminus X_J] \otimes \kappa$ is $J^W$ with the topology induced by the partial order $\preceq$ (Proposition 8.10). Its unique closed point is the superspecial element $w = 1$. By Proposition 8.17 this is in the image.

In [Wd2] it is already shown that all strata are nonempty if $B$ is a totally real field extension of $\mathbb{Q}$.

Corollary 9.2. Each Ekedahl-Oort stratum $\mathcal{A}_0^w$ is equi-dimensional of dimension $\ell(w)$.

For $p > 2$ this has also been proved by Moonen [Mo2] using the results of [Mo1] and [Wd2] and under the assumption that the stratum is non-empty. In the Siegel case, this corollary has been shown by Oort in [Oo1], where we use (A.5) to translate his expression for the dimension in terms of elementary sequences into the length of $w$.

Proof. We know that the $G(\kappa)$-orbit $O^w$ corresponding to $w \in J^W$ has codimension $\dim(\text{Par}_J) - \ell(w)$ in $X_J \otimes \kappa$ ([MW]). Therefore, the same holds for the substack $[G \otimes \kappa \setminus O^w]$ of $[G_\kappa \setminus X_J] \otimes \kappa$. As $\zeta$ is universally open, it respects codimension and all its fibers $\mathcal{A}_0^w$ are equi-dimensional of dimension

$$\dim(\mathcal{A}_0^w) = \dim(\mathcal{A}_0) - \text{codim}(\mathcal{A}_0^w, \mathcal{A}_0) = \dim(\text{Par}_J) - (\dim(\text{Par}_J) - \ell(w)) = \ell(w).$$
9.2 Ekedahl-Oort strata are smooth and quasi-affine

For the sake of brevity, we call a level-$n$-truncated $p$-divisible group with $\mathcal{D}$-structure a $\mathcal{D}$-BT$_n$.

**Proposition 9.3.** Each Ekedahl-Oort stratum $A^0_w$ is smooth.

This result has been proved by Vasiu (for arbitrary $p$) in [Va1] Theorem 5.3.1. For the convenience of the reader we include a quick proof for $p > 2$.

**Proof of Proposition 9.3 for $p > 2$.** Let $\mathcal{A}_0 \to \mathcal{D}$ be the algebraic stack of $\mathcal{D}$-BT$_1$s over $\kbar$ (see [Wd2] (1.7) and (6.3) for its definition). For $w \in \mathcal{J}W$ let $M^w$ by a $\mathcal{D}$-zip over $\kbar$ whose isomorphism class corresponds to $w$ under the bijection (4.3). Let $X^w$ be the corresponding $\mathcal{D}$-BT$_1$. Then $X^w$ defines a point $\text{Spec} \kbar \to \mathcal{D}$. Its residue gerbe (in the sense of [LM] (11.1)) is the smooth locally closed sub stack $\mathcal{A}_0 \to \mathcal{D}$ classifying $\mathcal{D}$-BT$_1$s that are locally for the fpf-topology isomorphic to $X^w$.

We attach to a point $(A, \iota, \lambda, \eta)$ of $A^0_0$ the $p$-torsion of $A$ endowed with the $O_B$-action induced by $\iota$ and the isomorphism $A[p] \tilde{\to} A[p]^\vee$ induced by $\lambda$. This yields a morphism $\Psi: A^0_0 \to \mathcal{A}_0$ which is smooth if $p > 2$ (Wd2 (6.4)). Therefore $\Psi^{-1}(\mathcal{A}_0)$ is smooth for $p > 2$. Now $A^0_w$ and $\Psi^{-1}(\mathcal{A}_0)$ have the same underlying topological space because they have the same $k$-valued points for every perfect field extension $k$ of $\kbar$. As $A^0_w$ is reduced by definition, we obtain that $A^0_w = \Psi^{-1}(\mathcal{A}_0)$ is smooth.

**Remark 9.4.** Assume that $(O_B, *)$ consists only of factors of type (AL) (Remark [1.3]). In this case the formal smoothness of morphisms between deformations of truncated $p$-divisible groups with $\mathcal{D}$-structure in [Wd2], which is used for the smoothness of $\Psi$ in the proof of Proposition 9.3, can be reformulated into a statement about formal smoothness of morphisms between deformations of truncated $p$-divisible groups with $O_K$-action ($K$ an unramified extension of $\mathbb{Q}_p$) but without polarization. In this case the argument in [Wd2] does not use $p > 2$ and the morphism $\Psi$ in the proof of Proposition 9.3 is also smooth for $p = 2$.

We prove now that every Ekedahl-Oort stratum is quasi-affine. In the Siegel case this has been shown by Oort in [Oo1]. Our argument in the general case is similar but instead of working with canonical filtrations we use a result of Vasiu. We start with the following lemma.

**Lemma 9.5.** Let $g: X \to Y$ be a finite locally free surjective morphism of schemes. Then $X$ is quasi-affine if and only if $Y$ is quasi-affine.

**Proof.** The condition on $Y$ is clearly sufficient. Now assume that $X$ is quasi-affine. As $X$ is quasi-compact and separated and as $g$ is surjective and universally closed, $Y$ is quasi-compact and separated. Moreover $\mathcal{O}_X = g^* \mathcal{O}_Y$ is ample by hypothesis. Therefore $\mathcal{O}_Y$ is ample because $g$ is finite locally free surjective (EGA II (6.6.3)). This shows that $Y$ is quasi-affine.

**Theorem 9.6.** Assume that $\mathcal{A}_0$ is a scheme. Then every Ekedahl-Oort stratum $\mathcal{A}_0^w$ ($w \in \mathcal{J}W$) is quasi-affine.
**Proof.** Let \((A, \iota, \lambda, \eta)\) be the restriction of the universal family over \(\mathcal{A}_0\) to \(\mathcal{A}_0^w\) and let \(M = (M, C, D, \varphi_0, \varphi_1)\) be its attached \(\mathcal{D}\)-zip. By [Val] Theorem 5.3.1, we find a finite locally free surjective morphism \(g: S \to \mathcal{A}_0^w\) such that \(g^*M\) is isomorphic to a constant \(\mathcal{D}\)-zip. In particular, \(g^*M\) and \(g^*C\) are globally free \(\mathcal{O}_S\)-modules. By the definition of \(C\) this shows that \(g^*f_1\Omega_{A/\mathcal{A}_0}^1\) is a globally free \(\mathcal{O}_S\)-module, where \(f: A \to \mathcal{A}_0^w\) denotes the structure morphism. Setting \(\omega_A := \det(f_1\Omega_{A/\mathcal{A}_0}^1)\) we see that \(g^*\omega_A\) is a trivial line bundle. On the hand we know by [Lan] Theorem 7.2.4.1 that \(\omega_A\) is ample. Thus \(g^*\omega_A\) is ample and \(S\) is quasi-affine. Therefore \(\mathcal{A}_0^w\) is quasi-affine by Lemma 9.6.

**Remark 9.7.** In [Val] Theorem 5.3.1 it is shown that the inclusion \(\mathcal{A}_0^w \hookrightarrow \mathcal{A}_0\) is quasi-affine. This also follows from Theorem 9.6 or from [PWZ] Theorem 12.7. In [NVW] Theorem 6.3 (for \(p \geq 5\)) the much stronger result is shown that each Ekedahl-Oort stratum \(\mathcal{A}_0^w\) is pure in \(\mathcal{A}_0\) (i.e., the inclusion \(\mathcal{A}_0^w \hookrightarrow \mathcal{A}_0\) is an affine morphism). This result neither implies nor is implied by Theorem 9.6.

### 10 Non-emptiness of Newton strata

The previous results allow us to deduce the non-emptiness of all Newton strata which has been conjectured by Fargues ([Far] Conjecture 3.1.1) and Rapoport ([Ra] Conjecture 7.1).

**Theorem 10.1.** For all \(b \in B(G, \mu)\) the Newton stratum \(N_b\) in \(\mathcal{A}_0\) is non-empty.

**Theorem 10.2.** For every algebraically closed field extension \(k\) of \(\kappa\) and every \(p\)-divisible group with \(\mathcal{D}\) structure \((X_0, \iota_0, \lambda_0)\) there exists \((A, \iota, \lambda, \eta) \in \mathcal{A}_0(k)\) such that \((X_0, \iota_0, \lambda_0)\) is isomorphic to \((A, \iota, \lambda)[p^\infty]\) (as \(p\)-divisible groups with \(\mathcal{D}\) structure).

We first show that both theorems are equivalent.

**Proof of the equivalence of Theorem 10.1 and Theorem 10.2.** The map \(C(G) \to B(G)\) induces a surjection \(C(G, \mu) \to B(G, \mu)\) (Section 7.2). Thus the non-emptiness of all Newton strata means that for every \(p\)-divisible group \((X_0, \iota_0, \lambda_0)\) with \(\mathcal{D}\)-structure over \(\kappa\) there exists \((A, \iota, \lambda, \eta) \in \mathcal{A}_0(\kappa)\) such that \((A, \iota, \lambda)[p^\infty]\) and \((X_0, \iota_0, \lambda_0)\) are isogenous (as \(p\)-divisible groups with \(\mathcal{D}\)-structure). Thus Theorem 10.2 implies Theorem 10.1.

Conversely, let \((X_0, \iota_0, \lambda_0)\) be a \(\mathcal{D}\)-structure and let \(b \in B(G, \mu)\) be its Newton point. If Theorem 10.1 holds, there exists \((A_1, \iota_1, \lambda_1, \eta_1) \in \mathcal{A}_0(b)\) and an isogeny of \(p\)-divisible groups with \(\mathcal{D}\)-structure \(f: (X_0, \iota_0, \lambda_0) \to (A_1, \iota_1, \lambda_1)[p^\infty]\). Then Theorem 10.2 follows by Lemma 8.5.

For the proof of Theorem 10.1 we need some preparations. Let \(K\) be a number field and let \(H\) be a reductive (and hence connected) group over \(K\). Let \(\Sigma\) be a finite set of places of \(K\) containing all archimedean places. We let \(H^\Sigma(K, H)\) be the kernel of the canonical map \(H^1(K, H) \to \prod_{v \in \Sigma} H^1(K_v, H)\), where \(v\) runs through all places of \(K\) which are not in \(\Sigma\). Let \(\lambda = \lambda_H\) be the restriction of the canonical map \(H^1(K, H) \to \prod_{v \in \Sigma} H^1(K_v, H)\) to \(H^\Sigma(K, H)\). Consider the following condition.
The map

$$\lambda = \lambda_H : \Pi\Sigma(K, H) \to \prod_{v|\infty} H^1(K_v, H)$$

is surjective.

If $\Sigma$ and $\Sigma'$ are finite sets of places as above with $\Sigma \subseteq \Sigma'$ and $(H, \Sigma)$ satisfies Condition (S), then $(H, \Sigma')$ satisfies Condition (S).

Lemma 10.3. Let $H$ be a reductive group over a number field $K$ and let $\Sigma$ be a finite set of places containing all archimedean places.

1. Let $L$ be a finite extension of $K$, let $H = \text{Res}_{L/K} H_0$ be the restrictions of scalars from a reductive algebraic group $H_0$ over $L$ and let $\Sigma_0$ be the set of all places of $L$ lying over some place in $\Sigma$. Then $(H_0, \Sigma_0)$ satisfies Condition (S) if and only if $(H, \Sigma)$ satisfies Condition (S).

2. If $H$ is simply connected, then Condition (S) is satisfied for all $\Sigma$.

3. Let the derived group $H^{\text{der}}$ of $H$ be simply connected and set $D := H/H^{\text{der}}$. If $(D, \Sigma)$ satisfies Condition (S), then $(H, \Sigma)$ satisfies Condition (S).

Proof. The first assertion follows from Shapiro’s lemma. Assertion (2) follows from the fact that for every reductive group $H$ over a number field $K$ the canonical morphism

$$(10.1)\quad H^1(K, H) \to \prod_{v|\infty} H^1(K_v, H)$$

is surjective ([PR] Prop. 6.17) and that for simply connected groups $H^1(K_v, H) = 0$ for all finite places $v$ of $K$ ([PR] Theorem 6.4) which implies $\Pi\Sigma(K, H) = H^1(K, H)$. In fact, as simply connected groups also satisfy the Hasse principle, $\lambda_H$ is bijective if $H$ is simply connected.

To show assertion (3) we consider the following commutative diagram

All rows are exact sequences of pointed sets (for the third row this follows from the exactness of the second row and the equality $\Pi\Sigma(K, H^{\text{der}}) = H^1(K, H^{\text{der}})$). Moreover, for all places $v$ of $K$ it is known that the local cohomology groups $H^1(K_v, H)$ carry abelian group structures functorial in $H$, thus the first and the last row are in fact
exact sequences of abelian groups. By \cite{Bor} Theorem 5.7 the map \( \pi \) is surjective. This implies together with the injectivity of \( i \) that \( \pi \Sigma \) is surjective.

Now let \( y \in \prod_{v | \infty} H^1(K_v, H) \). As \( \pi \Sigma \) and \( \lambda_D \) are surjective, there exists \( x \in \prod \Sigma(K, H) \) such that \( \lambda(x) - y = (\rho \circ \lambda_{H, \text{det}})(z_0) \) for some \( z_0 \in \prod \Sigma(K, H^\text{der}) \). Let \( z \in \prod \Sigma(K, H) \) be the image of \( z_0 \). It remains to find an element \( w \in \prod \Sigma(K, H) \) such that \( \lambda(w) = \lambda(x) - \lambda(z) \). For this we choose a maximal torus \( T \) of \( H \) such that \( x \) and \( z \) are both in the image of \( \eta \); \( H^1(K, T) \to H^1(K, H) \) (this is always possible by \cite{Bor} Theorem 5.11), say \( x = \eta(x') \) and \( z = \eta(z') \). Define \( w := \eta(x' - z') \). As both compositions

\[
H^1(K, T) \to \prod_{v \notin \Sigma} H^1(K_v, T) \to \prod_{v \notin \Sigma} H^1(K_v, H),
\]

\[
H^1(K, T) \to \prod_{v | \infty} H^1(K_v, T) \to \prod_{v | \infty} H^1(K_v, H)
\]

are homomorphisms of groups, one has \( w \in \prod \Sigma(K, H) \) and \( \lambda(w) = \lambda(x) - \lambda(z) \).

Recall that we fixed an integral Shimura-PEL-datum \( \mathcal{D} = (B, *, V, \langle , , \rangle, O_B, \Lambda, h) \) unramified at \( p \). We denote by \( G = G(\mathcal{D}) \) the attached \( \mathbb{Q} \)-group and we let \( [\mu] = [\mu(\mathcal{D})] \) be the attached \( G(\overline{\mathbb{Q}}_p) \)-conjugacy class of minuscule cocharacters of \( G_{\overline{\mathbb{Q}}_p} \). For the proof of Theorem \ref{thm:main} we may assume and do assume from now on that \( \tilde{B} \) is a simple \( \mathbb{Q} \)-algebra. Let \( F \) be the center of \( B \) and let \( F_0 = F^{* = \text{id}} \). Then \( F_0 \) is a totally real number field and either \( F = F_0 \) (Case C) or \( F \) is a quadratic imaginary extension of \( F_0 \) (Case A). Let \( \Sigma \) be the set of places of \( \mathbb{Q} \) consisting of the infinite place and of all finite places \( v \) such that there exists a place of \( F_0 \) over \( v \) which is ramified in \( F \). As \( \mathcal{D} \) is unramified at \( p \), we have \( p \notin \Sigma \).

**Lemma 10.4.** Let \( [\mu'] \) be any \( G(\overline{\mathbb{Q}}_p) \)-conjugacy class of minuscule cocharacters of \( G_{\overline{\mathbb{Q}}_p} \). Then there exists an integral Shimura-PEL-datum \( \mathcal{D}' \) unramified at \( p \) of the form \( \mathcal{D}' = (B, *, V, \langle , , \rangle, O_B, \Lambda, h') \) such that for every place \( v \notin \Sigma \) the \( (B, *) \)-skew hermitian spaces \( (V, \langle , , \rangle) \) and \( (V, \langle , , \rangle') \) are isomorphic over \( \overline{\mathbb{Q}}_v \) and such that \( [\mu(\mathcal{D}')] = [\mu'] \).

**Proof.** We identify the set of \( G(\overline{\mathbb{Q}}_p) \)-conjugacy class of cocharacters of \( G_{\overline{\mathbb{Q}}_p} \) with the set of \( G(\mathbb{C}) \)-conjugacy classes of cocharacters of \( G_{\mathbb{C}} \). The classification of PEL data over \( \mathbb{R} \) (\cite{Ko} §4) shows that every conjugacy class of a minuscule cocharacter occurs as the conjugacy class attached to a real PEL datum of the form \( (B_{\mathbb{R}}, *, V_{\mathbb{R}}, \langle , , \rangle', h') \). Now skew-hermitian \( (B_{\mathbb{R}}, *) \)-spaces (resp. \( (B, *) \)-spaces) of the same dimension as \( V \) are classified by \( H^1(\mathbb{R}, G) \) (resp. by \( H^1(\mathbb{Q}, G) \)). Thus the lemma is shown if we show that the reductive \( \mathbb{Q} \)-group \( G \) and \( \Sigma \) satisfy the Condition (S).

By Lemma \ref{lem:condition} it suffices to show that \( D := G/G^\text{der} \) and \( \Sigma \) satisfy Condition (S). We use the notation of Remark \ref{rem:notation} where we studied \( D \). In case (C) one has \( D = G_m, \mathbb{Q} \) and thus Condition (S) is trivially satisfied by Hilbert 90. In case (A) let \( \Sigma_0 \) be the set of places of \( F_0 \) consisting of the archimedean places and the places which are ramified in \( F \). Let us assume for a moment that \( (D_0, \Sigma_0) \) satisfies Condition (S). Then by Lemma \ref{lem:factorization} the \( \mathbb{Q} \)-torus \( D' := \text{Res}_{F_0/\mathbb{Q}} D_0 \) and \( \Sigma \) satisfy Condition (S).
If $n$ is even, then (1.1) shows that $(D, \Sigma)$ satisfies Condition (S). If $n$ is odd, we use the exact sequence (1.2) and obtain a commutative diagram

$$
\begin{align*}
\text{III}^\Sigma(\mathbb{Q}, D') & \longrightarrow \text{III}^\Sigma(\mathbb{Q}, D) \\
\downarrow \lambda_{D'} & \downarrow \lambda_D \\
H^1(\mathbb{R}, D') & \longrightarrow H^1(\mathbb{R}, D).
\end{align*}
$$

By our assumption, $\lambda_{D'}$ is surjective and again by Hilbert 90 the lower horizontal map is surjective. Thus $\lambda_D$ is surjective.

It remains to show that $D_0 = \text{Ker}(N_{F/F_0} : \text{Res}_{F/F_0} \mathbb{G}_{m,F} \rightarrow \mathbb{G}_{m,F_0})$ and $\Sigma_0$ satisfy Condition (S). For every field extension $K$ of $F_0$ one has

$$H^1(K, D_0) = K^\times / N_{F\otimes F_0 K/K}((F \otimes F_0 K)^\times).$$

We thus have $\prod_{v|\infty} H^1((F_0)_v, D_0) = \prod_{v} \mathbb{R}^\times / \mathbb{R}^{>0}$, where $\iota$ runs through the set $I$ of embeddings of $F_0$ into $\mathbb{R}$. For $x \in F_0^\times$ we set $\text{sgn}(x) := (\text{sgn}(\iota(x))_{\iota \in I} \in \{\pm 1\}^I$. Now choose for $\epsilon \in \{\pm 1\}^I$ some unit $x \in O_{F_0}^\times$ with $\text{sgn}(x) = \epsilon$. Then by local class field theory $x$ is a norm for the extension $F_v := F \otimes_{F_0} (F_0)_v$ of $(F_0)_v$ for all unramified places $v$ of $F_0$.

**Proof of Theorem 10.1.** Let $b \in B(G, \mu)$. By Theorem 8.18 there is a minuscule $\mu'$ with the following property. Let $\mathcal{D}'$ be a corresponding Shimura-PEL-datum, unramified at $p$, as in Lemma 10.4 (in particular $[\mu(\mathcal{D}')] = [\mu']$). Then the Newton stratification $N'_{\mathcal{D}'}$ in $\mathcal{A}_{\mathcal{D}'}$ contains a fundamental Ekedahl-Oort stratum. This Ekedahl-Oort stratum is nonempty by Theorem 9.1. Let $(A, \iota, \lambda, \eta)$ be the abelian variety with $\mathcal{D}'$-structure associated with a $k$-valued point in $N'_{\mathcal{D}'}$. Let $(X, \iota, \lambda)$ be a $p$-divisible group with $\mathcal{D}'$-structure in the isogeny class determined by $b$. Then there is a quasi-isogeny between $X$ and $A[p^\infty]$ compatible with the $O_B$-action and respecting the polarizations up to a scalar. Lemma 8.3 thus implies that there is an abelian variety with $\mathcal{D}'$-structure whose $p$-divisible group is $(X, \iota, \lambda)$, which proves the theorem.  

### A Appendix on Coxeter groups, reductive group schemes and quotient stacks

In this appendix we fix some conventions and recall results on Coxeter groups, on reductive group schemes and on quotient stacks that are used in the main text.

#### A.1 Coset representatives of Coxeter groups

Let $W$ be a Coxeter group and $I$ its generating set of simple reflections. Let $\ell$ denote the length function on $W$.

Let $J$ be a subset of $I$. We denote by $W_J$ the subgroup of $W$ generated by $J$ and by $W^J$ (respectively $J^W$) the set of elements $w$ of $W$ which have minimal length in their coset $wW_J$ (respectively $W_Jw$). Then every $w \in W$ can be written uniquely...
as \( w = w^J w_J = w_J^J w \) with \( w_J, w_J' \in W_J, w^J \in W^J \) and \( Jw \in JW \), and \( \ell(w) = \ell(w_J) + \ell(w^J) = \ell(w_J') + \ell(Jw) \) (see [DDPW], Proposition 4.16). In particular, \( W^J \) and \( JW \) are systems of representatives for \( W/W_J \) and \( W_J \backslash W \) respectively.

Furthermore, if \( K \) is a second subset of \( I \), let \( JW^K \) be set of \( w \in W \) which have minimal length in the double coset \( W_J w W_K \). Then \( JW^K = JW \cap W^K \) and \( JW^K \) is a system of representatives for \( W_J \backslash W/W_K \) (see [DDPW] (4.3.2)).

A.2 Bruhat order

We let \( \leq \) denote the Bruhat order on \( W \). This natural partial order is characterized by the following property: For \( x, w \in W \) we have \( x \leq w \) if and only if for some (or, equivalently, any) reduced expression \( w = s_{i_1} \cdots s_{i_n} \) as a product of simple reflections \( s_i \in I \), one gets a reduced expression for \( x \) by removing certain \( s_i \)'s from this product.

More information about the Bruhat order can be found in [BB], Chapter 2. The set \( JW \) can be described as

\[
JW = \{ w \in W : w < sw \text{ for all } s \in J \}
\]

(see [BB], Definition 2.4.2 and Corollary 2.4.5).

A.3 Reductive group schemes, maximal tori, and Borel subgroups

Let \( S \) be a scheme. A reductive group scheme over \( S \) is a smooth affine group scheme \( G \) over \( S \) such that for every geometric point \( s \in S \) the geometric fiber \( G_{\bar{s}} \) is a connected reductive algebraic group over \( \kappa(\bar{s}) \).

Let \( G \) be a reductive group scheme over \( S \). A maximal torus of \( G \) is closed subtorus \( T \) of \( G \) such that \( T_{\bar{s}} \) is a maximal element in the set of subtori of \( G_{\bar{s}} \) for all \( s \in S \). By a theorem of Grothendieck ([SGA3] Exp. XIV, 3.20) maximal tori of \( G \) exist Zariski locally on \( S \). A Borel subgroup of \( G \) is a closed smooth subgroup scheme \( B \) of \( G \) such that for all \( s \in S \) the geometric fiber \( B_{\bar{s}} \) is a Borel subgroup of \( G_{\bar{s}} \) in the usual sense (i.e., a maximal smooth connected solvable subgroup). A reductive group scheme over \( S \) is called split if there exists a maximal torus \( T \) of \( G \) such that \( T \cong \mathbb{G}_m^r \) for some integer \( r \geq 0 \).

A.4 Parabolic subgroups and Levi subgroups

A smooth closed subgroup scheme \( P \) of \( G \) is called parabolic subgroup of \( G \) if the fpqc quotient \( G/P \) is representable by a smooth projective scheme or, equivalently by [SGA3] Exp. XXII (5.8.5), if \( G_{\bar{s}}/P_{\bar{s}} \) is proper for all \( s \in S \). Every Borel subgroup of \( G \) is a parabolic subgroup. The unipotent radical of \( P \), denoted by \( U_P \), is the largest smooth normal closed subgroup scheme with unipotent and connected fibers. It exists by [SGA3] Exp. XXII, (5.11.4). If \( P \) contains a maximal torus \( T \) of \( G \), there exists a unique reductive closed subgroup scheme \( L \) of \( P \) containing \( T \) such that the canonical homomorphism \( L \to P/UP \) is an isomorphism (loc. cit.). Any such subgroup \( L \) is called a Levi subgroup of \( P \).
The functor that sends an $S$-scheme $T$ to the set of Borel (resp. parabolic) subgroups of $G \times_S T$ is representable by a smooth projective $S$-scheme by [SGA3] Exp. XXVI, 3. We call the representing scheme $Bor_G$ (resp. $Par_G$). The functor that attaches to an $S$-scheme $T$ the set of pairs $(P, L)$, where $P$ is a parabolic subgroup of $G \times_S T$ and $L$ is a Levi subgroup of $P$ is representable by a smooth quasi-projective $S$-scheme (loc. cit.).

If $S$ is local, $G$ is called quasi-split if there exists a Borel subgroup of $G$. Every split reductive group scheme is quasi-split. If $S = \text{Spec } R$, where $R$ is a henselian local ring whose residue field $k$ has cohomological dimension $\leq 1$ (e.g., if $k$ is finite), then $G$ is quasi-split. Indeed, the special fiber $G_k$ has a Borel subgroup $B_k$ by [Se] III, §2.3. As the scheme of Borel subgroups $Bor_G$ is smooth, there exists a lifting of $B_k$ to a Borel subgroup of $G$.

A.5 Weyl groups and the type of a parabolic subgroup

Let $G$ be a reductive group over an algebraically closed field, let $B$ be a Borel subgroup of $G$, and let $T$ be a maximal torus of $B$. Let $W(T) := \text{Norm}_G(T)/T'$ denote the associated Weyl group, and let $I(B, T) \subset W(T)$ denote the set of simple reflections defined by $B$. Then $W(T)$ is a Coxeter group with respect to the subset $I(B, T)$.

A priori this data depends on the pair $(B, T)$. However, any other such pair $(B', T')$ is obtained by conjugating $(B, T)$ by some element $g \in G$ which is unique up to right multiplication by $T$. Thus conjugation by $g$ induces isomorphisms $W(T) \sim W(T')$ and $I(B, T) \sim I(B', T')$ that are independent of $g$. Moreover, the isomorphisms associated to any three such pairs are compatible with each other. Thus $W := W_G := W(T)$ and $I := I(B, T)$ for any choice of $(B, T)$ can be viewed as instances of “the” Weyl group and “the” set of simple reflections of $G$, in the sense that up to unique isomorphisms they depend only on $G$.

Now let $G$ be a quasi-split reductive group scheme over a connected scheme $S$. Then we obtain for any geometric point $\bar{s} \to S$ the Weyl group and the set of simple reflections $(W_{\bar{s}}, I_{\bar{s}})$ of $G_{\bar{s}}$. The algebraic fundamental group $\pi_1(S, \bar{s})$ acts naturally on $W_{\bar{s}}$ preserving $I_{\bar{s}}$ (because $G$ is quasi-split), and every étale path $\gamma$ from $\bar{s}$ to another geometric point $\bar{s}'$ of $S$ yields an isomorphism of $(W_{\bar{s}}, I_{\bar{s}}) \sim (W_{\bar{s}'}, I_{\bar{s}'})$ that is equivariant with respect to the isomorphism $\pi_1(S, \bar{s}) \sim \pi_1(S, \bar{s}')$ induced by $\gamma$ (cf. [SGA3] Exp. XII, 2.1). In particular $(W_{\bar{s}}, I_{\bar{s}})$ together with its action by $\pi_1(S, \bar{s})$ is independent of the choice of $\bar{s}$ up to isomorphism. We call it the Weyl system of $G$.

A.6 Types and relative positions of parabolic subgroups

If $P$ is a parabolic subgroup of $G$ and $s \in S$, the type $J(\bar{s}) \subset I$ of the parabolic subgroup $P_{\bar{s}}$ of $G_{\bar{s}}$ is independent of $s \in S$ ([SGA3] Exp. XXVI, 3) and we call $J := J(\bar{s})$ the type of $P$. For a subset $J$ of $I$ we denote by $\text{Par}_J$ the open and closed subscheme of $\text{Par}$ parameterizing parabolic subgroups of type $J$. If $S$ is a semi-local scheme, $J$ and $\text{Par}_J$ are defined over a finite étale covering of $S$ ([SGA3] Exp. XXIV, 4.4.1).

For simplicity assume that $S$ is local. Let $J, K \subset I$ be subsets and let $S_1 \to S$ be the finite étale extension over which $J$ and $K$ are defined. Let $w \in J^I K$. For every $S_1$-scheme $S'$ and for every parabolic subgroup $P$ of $G_{S'}$ of type $J$ and every parabolic
subgroup $Q$ of $G_{S'}$ of type $K$ we write

$$\text{relpos}(P, Q) = w$$

if there exists an fpqc-covering on $S'' \to S'$, a Borel subgroup $B$ of $G_{S''}$ and a split maximal torus $T$ of $B$ such that $P_{S''}$ contains $B$ and $Q_{S''}$ contains $\dot{w}B$, where $\dot{w} \in \text{Norm}_{G_{S''}}(T)(S'')$ is a representative of $w \in \text{Norm}_{G_{S''}}(T)(S'')/T(S'')$.

If $S' = \text{Spec } k$ for an algebraically closed field, then $(P, Q) \mapsto \text{relpos}(P, Q)$ yields a bijection between $G(k)$-orbits on $\text{Par}_J(k) \times \text{Par}_K(k)$ and the set $J W^K$.

### A.7 One-parameter subgroups and their type

Let $G$ be a reductive group scheme over a connected scheme $S$. A one-parameter subgroup of $G$ is by definition a homomorphism of group schemes $\lambda: \mathbb{G}_m, S \to G$, where $\mathbb{G}_m, S$ denotes the multiplicative group over $S$. We identify the character group of $\mathbb{G}_m, S$ with $\mathbb{Z}$. Composing $\lambda$ with the adjoint representation $G \to \text{GL}(\text{Lie}(G))$ yields a decomposition $\text{Lie}(G) = \bigoplus_{n \in \mathbb{Z}} g_n$. There is a unique parabolic subgroup $P(\lambda)$ of $G$ and a unique Levi subgroup $L(\lambda)$ of $P(\lambda)$ such that

$$\text{Lie} P(\lambda) = \bigoplus_{n \geq 0} g_n, \quad \text{Lie} L(\lambda) = g_0.$$  

Indeed, because of the uniqueness assertion we may show this locally for the étale topology and hence can assume that the image of $\lambda$ lies in a split maximal torus. Then the claim follows from [SGA3] Exp. XXVI, 1.4 and 4.3.2. By loc. cit. we may define $L(\lambda)$ also as the centralizer of $\lambda$.

The type of $P(\lambda)$ is also called the type of $\lambda$. It depends only on the conjugacy class of $\lambda$.

### A.8 The example of the symplectic group

As an example consider a symplectic space $(V, \langle , \rangle)$ of dimension $2g$ over a field $k$ and denote by $G = \text{GSp}(V, \langle , \rangle)$ the group of symplectic similitudes of $(V, \langle , \rangle)$. Let $S_{2g}$ denote the symmetric group of the set $\{1, \ldots, 2g\}$. Then the Weyl system $(W, I)$ of $G$ is given by

$$W = \{ w \in S_{2g} : w(i) + w(2g + 1 - i) = 2g + 1 \text{ for all } i = 1, \ldots, g \},$$

$$I = \{ s_1, \ldots, s_g \}, \quad \text{where} \quad s_i = \begin{cases} \tau_i \tau_{2g-i}, & \text{for } i = 1, \ldots, g-1; \\ \tau_g, & \text{for } i = g. \end{cases}$$

(A.2)

Here $\tau_j$ denotes the transposition of $j$ and $j + 1$. The length of an element $w \in W$ can be computed as follows

$$\ell(w) = \# \{ (i, j) : 1 \leq i < j \leq g, w(i) > w(j) \} + \# \{ (i, j) : 1 \leq i < j \leq g, w(i) + w(j) > 2g + 1 \}.$$
For every $d$-tuple $(x_1, \ldots, x_d)$ of real numbers we denote by $(x_1, \ldots, x_d)\uparrow$ the $d$-tuple $(x_{\sigma(1)}, \ldots, x_{\sigma(d)})$ with $\sigma \in S_d$ such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(d)}$. Then the Bruhat order is given by

$$w' \leq w \iff \forall 1 \leq i \leq g : (w'(1), \ldots, w'(i))\uparrow \leq (w(1), \ldots, w(i))\uparrow,$$

where we compare tuples componentwisely.

To study the Siegel case we consider the subset $J := \{s_1, \ldots, s_{g-1}\}$ of $I$. Then $W_J$ consists of those permutations $w \in W$ such that $w(\{1, \ldots, g\}) = \{1, \ldots, g\}$. The map

$$W_J \to S_g, \quad w \mapsto w|_{\{1, \ldots, g\}}$$

is a group isomorphism. The set $JW$ consists in this case of those elements $w \in W$ such that $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(g)$. Of course, this implies $w^{-1}(g+1) < \cdots < w^{-1}(2g)$.

It is convenient to give an alternative description of $JW$. To $w \in JW$ we attach $\epsilon = (\epsilon_i)_{1 \leq i \leq g} \in \{0,1\}^g$ by

$$\epsilon_i := \begin{cases} 0, & \text{if } i \in \{w^{-1}(1), \ldots, w^{-1}(g)\}; \\ 1, & \text{otherwise}, \end{cases} \quad i = 1, \ldots, g.$$

This yields a bijection $JW \leftrightarrow \{0,1\}^g$. The length of such an element $\epsilon = (\epsilon_1, \ldots, \epsilon_g)$ is equal to

(A.3) \quad $\ell(\epsilon) = \sum_{i=1}^{g} i\epsilon_{g+1-i}$.

The set $JW/J \cong W_J\backslash W/W_J$ corresponds to the set of $W_J$-orbits of $\{0,1\}^g$, where $W_J = S_g$ acts by permuting the entries of $(\epsilon_i)_i \in \{0,1\}^g$. Thus $(\epsilon_i)_i \mapsto \#\{i : \epsilon_i = 1\}$ yields a bijection

(A.4) \quad $JW/J \leftrightarrow \{0, \ldots, g\}$.

The set $JW$ can also be described by elementary sequences in the sense of Oort (see [EvG] §2; note that in loc. cit. the set $W^J$ is considered which we identify with $JW$ via $w \mapsto w^{-1}$): For $w \in JW$ we define

$$\varphi_w : \{0, \ldots, g\} \to \mathbb{Z}_{\geq 0}, \quad i \mapsto i - \#\{1 \leq a \leq g : w^{-1}(a) \leq i\}.$$ 

Then $\varphi_w$ is an elementary sequence, i.e., a map $\varphi : \{0, \ldots, g\} \to \mathbb{Z}_{\geq 0}$ such that $\varphi(0) = 0$ and such that $\varphi(i-1) \leq \varphi(i) \leq \varphi(i-1)+1$ for all $i = 1, \ldots, g$. The map $w \mapsto \varphi_w$ yields a bijection between $JW$ and the set $S_g$ of elementary sequences. The tuple $(\epsilon_i)_i \in \{0,1\}^g$ corresponding to an element $w \in JW$ can be described via the elementary sequence $\varphi_w$ by $\epsilon_i = \varphi_w(i) - \varphi_w(i-1)$. Using (A.3) an easy calculation shows

(A.5) \quad $\ell(w) = \sum_{i=1}^{g} \varphi_w(i)$.

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Finally, for $w, w' \in JW$ we have $w' \leq w$ if and only if $\varphi_{w'}(i) \leq \varphi_w(i)$ for all $i = 1, \ldots, g$.

Let us now describe parabolic subgroups of $G$ of type $J$. Let $S$ be any $k$-scheme. Then $V_S = V \otimes_k \mathcal{O}_S$ is a free $\mathcal{O}_S$-module of rank $2g$, and the base change of $\langle , \rangle$ is a perfect alternating form on $V_S$. An $\mathcal{O}_S$-submodule $\mathcal{L}$ of $V_S$ is called Lagrangian if $\mathcal{L}$ is locally on $S$ a direct summand of $V_S$ of rank $g$ and if $\mathcal{L}$ is totally isotropic. Attaching to a Lagrangian $\mathcal{L}$ its stabilizer in $G_S$ yields a bijection between the set of Lagrangians in $V_S$ and the set of parabolic subgroups $P$ of $G_S$ of type $J$ (with $J = \{s_1, \ldots, s_{g-1}\}$).

Let $L$ and $L'$ be Lagrangians in $V_S$ and let $P$ and $P'$ be the corresponding parabolic subgroups of $G_S$ of type $J$. Then for $d \in \{0, \ldots, g\} = JW$ \eqref{eq:A.4} we have

\begin{equation}
\mathrm{relpos}(P, P') = d \iff V_S/(L + L') \text{ is locally free of rank } g - d.
\end{equation}

Here the condition that $V_S/(L + L')$ is locally free (of some rank) is equivalent to the condition that fppf-locally $P$ and $Q$ contain a common maximal torus \cite{MW} Section 3.5).

\section{The underlying topological space of a quotient stack}

We recall some – probably well known – facts on quotient stacks. For lack of a better reference we refer to \cite{Wd2} (4.2)-(4.4). Let $k$ be a field, $\bar{k}$ an algebraic closure, $\Gamma := \text{Aut}(\bar{k}/k)$ the profinite group of $k$-automorphisms of $\bar{k}$. Let $X$ be a $k$-scheme of finite type and let $H$ be a smooth affine group scheme over $k$ that acts on $X$. We assume that there are only finitely many $H(\bar{k})$-orbits in $X(\bar{k})$.

Let $[H/X]$ be the algebraic quotient stack. To describe its underlying topological space we first recall that if $\leq$ is a partial order on a set $Z$, we can define a topology on $Z$ by defining a subset $U$ of $Z$ to be open if for all $u \in U$ and $z \in Z$ with $u \leq z$ one has $z \in U$.

Let $\Xi$ be the set of $H(\bar{k})$-orbits in $X(\bar{k})$. For $O, O' \in \Xi$ we set $O' \leq O$ if the closure of $O$ contains $O'$. This defines a partial order and hence a topology on $\Xi$. The group $\Gamma$ acts on $\Xi$ and we denote the set of $\Gamma$-orbits on $\Xi$ by $\Xi$. We endow $\Xi$ with the quotient topology. Then $\Xi$ is homeomorphic to the underlying topological space of $[H/X]$.

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