Critical behavior of charged black holes in Gauss-Bonnet gravity’s rainbow

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Following an earlier study regarding Gauss-Bonnet-Maxwell black holes in the presence of gravity’s rainbow \cite{Hendi:2015}, in this paper, we will consider all constants as energy dependent ones. The geometrical and thermodynamical properties of this generalization are studied and the validation of the first law of thermodynamics is examined. Next, through the use of proportionality between cosmological constant and thermodynamical pressure, van der Waals-like behavior of these black holes in extended phase space is investigated. An interesting critical behavior for sets of rainbow functions in this case is reported. Also, the critical behavior of uncharged and charged solutions is analyzed and it is shown that the generalization to a charged case puts an energy dependent restriction on values of different parameters.

I. INTRODUCTION

Supergravity can be obtained as a low energy field theory approximation of string theory \cite{Polchinski:1998}. The leading order correction to this theory depends on the string used to obtain the low energy effective field theory expansion. In the heterotic string theory, the lowest corrections are described by a Gauss-Bonnet (GB) term \cite{Witten:1985}. Furthermore, it is known from the renormalization group flow that the constants depend on the scale at which a theory is probed \cite{Banks:1998}. So, we expect that the different constants in general relativity and GB gravity also depend on the scale at which these theories are measured. The renormalization group flow for supergravity solutions \cite{Gates:1997} and GB gravity \cite{Hendi:2015} has been analyzed. In fact, a renormalization group flow has been used for measuring the flow of the cosmological constant \cite{EslamPanah:2015} and Newton constant \cite{EslamPanah:2016}. Now, as the scale at which a theory is measured depends on the energy of the probe, it is expected that these constants will also depend on the energy. Therefore, in this paper, we will use gravity’s rainbow \cite{Hendi:2015} for analyzing GB black holes with the energy dependent constants. It may be noted that the initial motivation of gravity’s rainbow came by analyzing the target space metric of string theory. This was done by regarding string theory as a two dimensional theory, and considering the target space metric as a matrix of coupling constants. Then using renormalization group flow, this matrix of coupling constants would flow and depend on the scale at which this theory is measured. This would in turn make them dependent on the energy that is used for probing this theory \cite{Hendi:2015}. Thus, the geometry would also be energy dependent.

Such a modification of a geometry at high energy scale can be viewed as a UV completion. It may be noted that just like the Horava-Lifshitz gravity \cite{Horava:2008}, the gravity’s rainbow \cite{Hendi:2015}, has also been viewed as a UV completion of general relativity. This is because both of these approaches are based on a modification of the usual energy-momentum dispersion relation in the UV limit. Such a modification of the usual energy-momentum dispersion relation in the UV limit occurs in a large number of approaches to quantum gravity, such as the discrete spacetime \cite{Arkani-Hamed:2000}, models based on string field theory \cite{Nambu:1992}, spin-network in loop quantum gravity \cite{Rovelli:1993}, non-commutative geometry \cite{Connes:1996}, spacetime foam \cite{Hooft:1996} and ghost condensation \cite{Alvarez-Gaume:1997}. It may be noted that even the Greisen-Zatsepin-Kuzmin limit (GZK limit) suggests that the usual energy-momentum relation could get modified in the UV limit \cite{Greisen:1974, Zatsepin:1971}. The GZK limit can be used for analyzing quantum gravitational effects as an upper limit on the energy of cosmic ray. The Pierre Auger Collaboration and the High Resolution Fly’s Eye experiment have reconfirmed earlier results of the GZK cutoff \cite{PierreAuger:2008}. All these observations suggest that there is a strong experimental motivation for a UV modification of the usual energy-momentum dispersion relation. Motivated by the Horava-Lifshitz gravity, the geometries occurring in the types IIA \cite{Horava:2009} and IIB string theory \cite{Horava:2009} have been also modified in the UV limit. Different Lifshitz scaling for space and time have also been used for analyzing certain aspects of the AdS/CFT correspondence \cite{Aharony:2008,_registry:2008}. It may be noted that for a suitable choice of the rainbow functions, the Horava-Lifshitz gravity can be related to the gravity’s rainbow \cite{Hendi:2015}. This is because both of these approaches are based on a modification of general relativity in the UV.

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limit such that the general relativity is obtained in the IR limit. Basically, Horava-Lifshitz gravity has been used to study the UV completion of various interesting geometries. Motivated by the close relation between the gravity’s rainbow and Horava-Lifshitz gravity, the UV completion of many black hole solutions has been recently obtained by using the formalism of gravity’s rainbow [32].

One of the most interesting effects of the UV completion of geometries is the modification of the black hole thermodynamics at the last stage of the evaporation of the black holes. As the gravity’s rainbow reduces to the general relativity in the IR limit, the black hole thermodynamics in gravity’s rainbow reduces to the usual black hole thermodynamics for very large black holes. However, as the black holes evaporate and reduce in size, the black hole thermodynamics in gravity’s rainbow showing deviation from the usual black hole thermodynamics. This deviation becomes significant at the end stage of the evaporation of black hole in gravity’s rainbow [33]. It has been observed that the temperature of black holes reaches a maximum value, and then it starts to reduce beyond this maximum value. At a critical value the temperature of black holes becomes zero, and the black holes do not radiate Hawking radiation at that stage. So, a black hole remnant forms in gravity’s rainbow. This formation of a black hole remnant has important phenomenological consequences for the detection of mini black holes at the LHC [34]. It is known that the usual uncertainty principle has to be modified to the generalized uncertainty principle to have an upper bound on the energy of a particle. As this particle acts as a probe for the geometry of the black hole, it also fixes the energy scale of the gravity’s rainbow. Thus, this bound on the energy of a particle emitted in the Hawking radiation can be used as energy scale in the rainbow functions. This modifies the thermodynamics of black holes [33, 35]. Such a modification in the black hole thermodynamics has also been observed for black rings [36]. The temperature of black rings also reaches a maximum value and reduces beyond that value to form a remnant. It has been argued that a remnant will form for all black objects [32]. This has been explicitly demonstrated for Kerr black holes, Kerr-Newman black holes in de Sitter space, charged anti-de Sitter (AdS) black holes, higher dimensional Kerr-AdS black holes and black saturn [32]. The geometric and thermodynamic properties of the charged dilatonic black holes in gravity’s rainbow have also been investigated [57].

The effect of gravity’s rainbow on the thermodynamics of black holes in GB gravity coupled to Maxwell’s theory has been studied [38]. In this analysis, it was observed that even though the thermodynamics of the black holes get modified in the GB gravity’s rainbow, the first law of thermodynamics still holds for this modified thermodynamics. However, in this analysis, different constants were not made to depend on the energy of the probe. As we expect the constant to depend on the energy of the probe, in this paper, we will generalize this analysis and make the constants energy dependant. We will also investigate the critical behavior of black hole solutions in this theory. It may be noted that the pressure-volume (PV) criticality has been studied for black holes in GB gravity using extended phase space thermodynamics [33, 42]. In the extended phase space thermodynamics, the cosmological constant is viewed as a thermodynamic pressure, and so it is possible to define a thermodynamic volume conjugate to this thermodynamic pressure [43–47]. It has been observed that along with the usual black hole phase transition, a new phenomenon of reentrant phase transitions occurs for rotating AdS black holes in extended phase space [48]. In these reentrant phase transitions a monotonic variation of the temperature yields two phase transitions, and this situation is similar to that which is seen in multi-component liquids. The PV criticality has also been studied in quasi-topological gravity [49]. In this paper, we study the effect of the UV completion of general relativity on the PV criticality in extended phase space. Also, we analyze the effects of rainbow functions on PV criticality in GB gravity’s rainbow. In addition, we analyze the effect of energy dependent constants on thermodynamic properties of black hole solutions. We also study the effects of such modification on the critical values and van der Waals like behavior of the system.

II. PROBE ENERGY DEPENDENT CONSTANTS: EXACT SOLUTIONS

As it was mentioned before, we expect that all the constants depend on the energy of the probe. Such a dependency of the constants on the energy of the probe can be explicitly analyzed using gravity’s rainbow. This is an advantage of using the gravity’s rainbow. Now, following earlier work on Einstein-GB-Maxwell black holes in gravity’s rainbow, we will generalize solutions to the case where different constants are dependent on the energy of the probe. Our first step is analyzing the effect of this energy dependency on the field equations.

The $d$-dimensional action of GB-Maxwell gravity with negative cosmological constant can be written as

$$I = -\frac{1}{2\kappa} \int d^d x \sqrt{-g} \left[ \alpha_0 \mathcal{L}_0 + \alpha_1 \mathcal{L}_1 + \alpha_2 \mathcal{L}_2 + \kappa \mathcal{L}(F) \right],$$

(1)

where $\kappa = 8\pi G(E)$ and we set $\alpha_0 = \alpha_1 = 1$ and $\alpha_2 = \alpha(E)$ in which the last one is the so-called GB coefficient. In this paper, we will only consider positive values of GB coefficient. In addition, $\mathcal{L}_i$’s are the first three terms of Lovelock Lagrangian which are corresponding to the cosmological constant, Einstein and GB Lagrangian, with the
following explicit forms

\[
\mathcal{L}_0 = -2\Lambda(E), \\
\mathcal{L}_1 = \mathcal{R}, \\
\mathcal{L}_2 = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + \mathcal{R}^2.
\]

The \( \mathcal{L}(\mathcal{F}) \) is the Lagrangian of electrodynamics in which we choose linear Maxwell Lagrangian, \( \mathcal{L}(\mathcal{F}) = -\mathcal{F} \), where \( \mathcal{F} = F_{\mu\nu}F^{\mu\nu} \) is the Maxwell invariant.

Variation of the action \( (1) \) with respect to the metric tensor \( g_{\mu\nu} \) and the Faraday tensor \( F_{\mu\nu} \), leads to the following field equations

\[
G_{\mu\nu}^0 + G_{\mu\nu}^1 + \alpha_2(E) G_{\mu\nu}^2 = 8\pi G(E)T_{\mu\nu},
\]

\[
\nabla_\mu F^{\mu\nu} = 0,
\]

where

\[
T_{\mu\nu} = 2F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{2}g_{\mu\nu}\mathcal{F},
\]

\[
G_{\mu\nu}^0 = -\frac{1}{2}g_{\mu\nu}\mathcal{L}_0,
\]

\[
G_{\mu\nu}^1 = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{L}_1,
\]

\[
G_{\mu\nu}^2 = -2 \left( 2R_{\mu\sigma\nu\rho}R_{\sigma\rho} - R^{\rho\sigma\lambda}R_{\mu\rho\sigma\lambda} - RR_{\mu\nu} \right) - \frac{1}{2}g_{\mu\nu}\mathcal{L}_2.
\]

Following Ref. [38], one finds the spherical symmetric metric governing the gravity’s rainbow which has dependency on rainbow functions in following form

\[
d\tau = -ds^2 = \frac{\Psi(r)}{f(E)^2}dt^2 - \frac{1}{g(E)^2} \left[ \frac{dr^2}{\Psi(r)} + r^2 \left( d\theta^2 + \prod_{i=2}^{d-1} \sin^2 \theta_j d\theta_j^2 \right) \right].
\]

Using Eqs. (2) - (7) with metric (8), we can find following electromagnetic field tensor and metric function

\[
F_{tr} = \frac{q(E)}{r^{d_1}},
\]

\[
\Psi(r) = 1 + \frac{r^2}{2\alpha'(E)q^2(E)} \left( 1 - \sqrt{\Theta(r)} \right),
\]

\[
\Theta(r) = 1 + \frac{8\alpha'(E)q(E)}{d_1d_2} \left( \Lambda(E) + \frac{d_1d_2m(E)}{2r^{d_1}} - \frac{8d_1d_3\pi G(E)q^2(E)g^2(E)f^2(E)}{r^{2d_2}} \right),
\]

where \( d_i = d - i \). Also, \( q(E) \) and \( m(E) \) are integration constants which are, respectively, related to the total electric charge and total mass of the solutions, and \( \alpha'(E) = d_3d_4\alpha(E) \).

For the simplicity and in order to find the contribution of each parameters, we consider following notations

\[
q(E) = h_1^2(E)q, \quad \Lambda(E) = h_2^2(E)\Lambda, \quad G(E) = h_3^2(E)G, \quad \alpha'(E) = h_4^2(E)\alpha', \quad m(E) = h_5^2(E)m,
\]

and since we are working in natural units, we set \( 8\pi G = 1 \).

Here, we would like to make some remarks regarding the properties of the solutions. Calculations show that there is a curvature singularity at \( r = 0 \), which can be covered with an event horizon. Hence, it will be interesting to see the effects of the gravity’s rainbow on the singularity and asymptotic behavior of the solutions. In order to study these effects, we use series expansion of the Kretschmann scalar for small and large values of radial coordinate. By doing so, one can find following relations

\[
\lim_{r \to 0} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{-4d_3(d^2 - 7d + 13)q(E)^{d_3}f(E)^2q^2h_4^2(E)}{\alpha'h_2^2(E)r^{2d_2}},
\]
\[
\lim_{r \to \infty} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = g(E)^{d_1} \left[ \frac{8d_{-1} \Lambda_{eff}^2}{d_1^2 d_2} - \frac{4\Lambda_{eff}}{d_1 d_2} \right],
\]

where
\[
\Lambda_{eff} = \frac{d_1 d_2}{4\alpha' h_2(E)g(E)^2} \left[ \sqrt{1 + \frac{8\alpha' \Lambda_{eff}^2}{d_1 d_2} - 1} \right].
\]

Interestingly, the obtained relations indicate that the rainbow functions affect the strength of singularity as well as asymptotical behavior of the solutions. In other words, the asymptotical behavior of the solutions is AdS with an effective cosmological constant, \(\Lambda_{eff}\). It is clear that the GB parameter and rainbow function can modify \(\Lambda_{eff}\). It means that \(\Lambda_{eff}\) reduces to \(\Lambda(E)\) for \(\alpha(E) \to 0\) and \(g(E) \to 1\), simultaneously.

### III. THERMODYNAMICAL QUANTITIES

Now, we are in a position to study thermodynamical quantities of the solutions. Using the concept of the surface gravity, it is a matter of calculation to obtain temperature as

\[
T = \frac{d_2 g(E) (d_3 r_+^2 + \alpha' d_5 h_3^2(E) g^2(E) - 2\Lambda h_3^2(E) r_+^4) r_+^{2d_1} - 2q^2 d_2 \alpha' h_3^2(E) h_3^2(E) g^2(E) f^2(E) r_+^2}{4\pi d_1 (r_+^2 + 2\alpha' h_3^2(E) g^2(E) g(E) f(E) r_+^{2d_1 - 1/2}).
\]

One of the conditions for having physical solutions is imposed by positive temperature. Since we have considered positive values of GB parameter, the denominator of the temperature is always positive and only numerator has contribution to negativity of the solutions. For AdS black holes, only charge term in numerator of the temperature contribute to negativity of it. Taking closer look at the numerator, one can see that energy dependency of constants plays a crucial role in domination of different terms. Therefore, the conditions for having physical solutions (positive temperature) is highly sensitive to energy variation of different constants.

Since we are working in the context of higher derivative gravity, it is not allowed to use the area law for calculating entropy. We use the Wald formula with the following result

\[
S = \frac{r_+^{d_2} \left( 1 + \frac{2\alpha' d_5 h_3^2(E) g^2(E)}{d_1 r_+^2} \right)}{4\pi g^{d_2}(E)}.
\]

In addition, the total electric charge of the solutions is obtained through the use of Gauss’s law as

\[
Q = \frac{q d_3 h_3^2(E) h_3^2(E) f(E)}{4\pi g^{d_3}(E)}.
\]

For the electric potential, we have

\[
U = A_\mu \chi^\mu \bigg|_{r \to \infty} - A_\mu \chi^\mu \bigg|_{r \to r_+} = \frac{h_3^2(E) q}{r_+^{d_3}}.
\]

The total mass of the black holes could be obtained through counter-term method or the Arnowitt-Deser-Misner approach with the following unique form

\[
M = \frac{1}{16\pi} \frac{h_3^2(E) d_2 m}{f(E) g^{d_1}(E)},
\]

where by evaluating metric function on outer horizon, one can find total finite mass as

\[
M = \frac{\left[ 2d_1 d_2 g^2(E) (r_+^2 + \alpha' h_3^2(E) g^2(E)) - \Lambda r_+^{d_1} h_3^4(E) \right] r_+^{d_1} + q^2 d_1 d_3 h_3^2(E) h_3^2(E) g^2(E) f^2(E) r_+^{2d_2}}{8\pi d_1 f(E) g^{d_1}(E)}.
\]

Before we proceed, it is worthwhile to mention a few remarks about conserved and thermodynamical quantities. The modification of GB gravity when the constants are not energy dependent has already been analyzed [38]. Thermodynamical quantities of the solutions in [38] differ from those obtained here which highlights the effects of dependency.
of constants on the probe energy. Especially, the electric potential was shown to be independent of energy function in Ref. [38], while here, it depends on the energy of the probe. Strictly speaking, these two types of black holes are phenomenologically different. Recently, it was pointed out that most of constants in physics are not actually constant. In fact, a measurement of their expectation values leads to the result that they should be varying parameters. Here, we have taken such consideration into account and shown that thermodynamically speaking, black holes will be modified in such consideration.

Now, we are in a position to check the validation of the first law of black hole’s thermodynamics. Using total mass of these black holes (21) with the obtained entropy (17) and electric charge (18) as extensive parameters, one can define following intensive parameters

\[
T = \left( \frac{\partial M}{\partial r_+} \right)_q \left( \frac{\partial r_+}{\partial S} \right)_q \quad \text{and} \quad U = \left( \frac{\partial M}{\partial q} \right)_{r_+},
\]

where by evaluating these equations, one can confirm that the obtained temperature and electric potential in Eq. (22) coincide with those extracted from Eqs. (16) and (19). Therefore, despite the modifications in thermodynamical quantities by consideration of energy dependant constants, the first law of thermodynamics is valid for these black holes.

IV. PROBE ENERGY DEPENDENT CONSTANTS: CRITICAL QUANTITIES AND VAN DER WAALS LIKE BEHAVIOR

The extended phase space expression comes from consideration of the cosmological constant not as a fixed quantity but a thermodynamical variable which is known as pressure. Although thermodynamical pressure is generally proportional to the negative cosmological constant with a proportionality constant \(\frac{-1}{8\pi}\), there are some cases in which we should modify it [37, 50]. In this paper, it is observed that the metric function and asymptotical behavior of the system have been modified due to the existence of rainbow functions. Therefore, it is necessary to check the possible effects of rainbow functions on thermodynamical pressure. To do so, we evaluate the energy-momentum tensor. It is straightforward to obtain the following relations for different components of energy-momentum tensor

\[
T_{t}^i = T_{r}^r = -f^2(E)g^2(E)F_{t r}^2 - \frac{\Lambda(E)}{8\pi},
\]

\[
T_{\theta}^\theta_i = f^2(E)g^2(E)F_{t r}^2 - \frac{\Lambda(E)}{8\pi}.
\]

In these relations, the first term is due to the existence of electromagnetic field (which is coupled with rainbow functions). Surprisingly, the \(\Lambda(E)\) term is not coupled with any rainbow functions of the metric. In other words, although rainbow functions of the metric modify \(\Lambda(E)\) term in the metric function and asymptotical behavior of the solutions, they do not affect thermodynamical pressure which is related to the cosmological constant. Therefore, in studying the critical behavior of the system through the analogy between \(\Lambda\) and \(P\), one can use following relation

\[
P = -\frac{\Lambda(E)}{8\pi} = -\frac{h_2^2(E)\Lambda}{8\pi}.
\]

Thermodynamically speaking, the conjugating thermodynamical variable corresponding to the pressure is thermodynamical volume which in the context of enthalpy is calculated by

\[
V = \left( \frac{\partial H}{\partial P} \right)_{s, Q}.
\]

It is worthwhile to mention that in order to have a well-defined vacuum solution with \(m = q = 0\), the pressure \(P\) has to satisfy the following constraint [41, 51, 52]

\[
0 \leq \frac{64\pi \alpha' h_2^2(E) P}{d_1 d_2} \leq 1,
\]

in which it puts a restriction on the pressure as maximal pressure

\[
P \leq P_{\text{max}} = \frac{d_1 d_2}{64\pi \alpha' h_2^2(E)},
\]
which indicates that only for sufficiently small pressures, the solution (10) has an asymptotic AdS region.

Now, remembering that we are working in extended phase space, the total mass of the black holes will not play the role of internal energy. Instead, it is interpreted as enthalpy. With this consideration, one can find the Gibbs free energy as

$$G = H - TS = M - TS.$$  \hspace{1cm} (29)

As for the volume of these black holes, by using Eqs. (21), (25) and (26), we obtain

$$V = \frac{r_+^{d_1}}{d_1 g^{d_1}(E) f(E)},$$  \hspace{1cm} (30)

which is modified in the presence of the rainbow functions. In other words, in the presence of gravity’s rainbow, thermodynamical volume of the black holes is a function of the rainbow functions, and therefore, the final form of these black holes is determined by the model of rainbow functions under consideration. On the other hand, even by consideration of the dependency of different constants on the probe energy, the volume of these black holes is same as that of probe energy independent constants. This behavior is expected, since the abstract form of metric (5) is free of any constant.

The equation of state and Gibbs free energy of these solutions could be found by using Eqs. (10), (25) and (29), which result into

$$P = \frac{d_2 \left[ r_+^2 + 2\alpha h_4^2(E)g^2(E) \right] g(E)f(E)}{4r_+^4} + \frac{d_2^2 h_4^2(E) h_3^2(E) g^2(E) f^2(E) q^2}{8\pi r_+^{2d_2}} - \frac{d_2 \left[ d_3 r_+^2 + \alpha d_5 h_4^2(E) g^2(E) \right] g^2(E)}{16\pi r_+^4},$$  \hspace{1cm} (31)

and

$$G = \frac{1}{d_4 g(E)f(E) \left( r_+^2 + 2\alpha h_4^2(E)g^2(E) \right) r_+^4} \left\{ \frac{d_4 \left[ \frac{d_4}{16\pi g^{d_4}(E)} - \frac{\alpha^\prime}{8\pi} \left( \frac{d_8 r_+^{d_8-1}}{2g^{d_8}(E)} + \frac{\alpha^\prime d_2 h_4^2(E) r_+^{d_1}}{g^{d_4}(E)} \right) g_4^2(E) \right]}{2\pi d_2} \left[ \frac{d_4 d_5}{2g^{d_4}(E)r_+^{d_4}} + \frac{\alpha^\prime d_2 d_7 h_4^2(E)}{g^{d_4}(E)r_+^{d_4}} \right] \right\}.$$  \hspace{1cm} (32)

There are several approaches toward calculating critical values. In this paper, we will employ the properties of the inflection points in $P - r_+$ diagrams. In this method, one can follow the relations for calculating critical volume which in case of these black holes, it will be proportional to the horizon radius

$$\left( \frac{\partial P}{\partial r_+} \right)_T = \left( \frac{\partial^2 P}{\partial r_+^2} \right)_T = 0,$$  \hspace{1cm} (33)

where by using Eqs. (31) and (33), one obtains

$$4q d_2^2 h_4^2(E) h_3^2(E) g(E) f^2(E) \left[ d_5/2 + 6\alpha^\prime h_4^2(E) d_7/2 g^2(E) r_+^2 \right] - d_4 g(E) r_+^4 + 12\alpha^\prime h_4^2(E) g^3(E) \left[ r_+^2 - \alpha^\prime d_5 h_4^2(E) g^2(E) \right] = 0.$$  \hspace{1cm} (34)

As a special case, we consider 5-dimensional solutions in the absence of electric charge to obtain critical horizon radius, temperature and pressure. So, we can write

$$r_c = \sqrt{6\alpha^\prime h_4(E) g(E)}, \hspace{0.5cm} & \hspace{0.5cm} T_c = \frac{1}{2\pi h_4(E) f(E) \sqrt{6\alpha^\prime}}, \hspace{0.5cm} & \hspace{0.5cm} P_c = \frac{1}{48 h_4^2(E) \alpha^\prime \pi},$$  \hspace{1cm} (35)

which lead to the following ratio

$$\frac{P_c r_c}{T_c} = \frac{f(E) g(E)}{4}.$$  \hspace{1cm} (36)

It is notable that in the absence of rainbow functions ($f(E) = g(E) = 1$), this ratio reduces to 1/4, and therefore, Eq. (36) indicates that consideration of the gravity’s rainbow can modify this near universal ratio. Interestingly, for
neutral solutions \((q = 0)\), the critical pressure is independent of metric’s rainbow functions while the critical horizon radius and the critical temperature are functions of one of rainbow functions.

Next, we are going to consider the charged GB black holes in the presence of gravity’s rainbow in 5—dimensions. The critical horizon radius in this case is given as

\[
r_{c-q} = \sqrt[6]{\frac{6B^{1/3} \left[ B^{2/3} + 3\alpha' h_3^2(E)g^2(E)B^{1/3} + 15 q^2 h_1^4(E) h_3^2(E)f^2(E) + 9\alpha'^2 h_4^4(E)g^4(E) \right]}{3B^{1/3}}}.
\]

where

\[
B = 189\alpha'^2 h_1^4(E) h_3^2(E) h_3^2(E) g^2(E) f^2(E) + 27\alpha'^2 h_4^6(E) g^6(E) + 3 q h_1^2(E) h_3(E) f(E) \times \\
\sqrt{729\alpha'^4 h_3^8(E) g^8(E) - 375 q^2 h_1^8(E) h_3^4(E) f^4(E) + 3294\alpha'^2 q^2 h_1^4(E) h_3^2(E) h_3^2(E) g^4(E) f^2(E)}.
\]

It is worthwhile to emphasize that, in this paper, we consider the positive values of GB parameter \((\alpha' > 0)\). The negative value of \(\alpha'\) enforces other set of conditions for having positive critical parameters. Furthermore, \(T_{c-q}\) will be given by

\[
T_{c-q} = \frac{g(E) \left[ r_{c-q}^4 - 4 q^2 h_1^4(E) h_3^2(E) f^2(E) \right]}{2 \pi \left( r_{c-q}^2 + 6 \alpha' h_4^2(E) g^2(E) \right)} f(E) r_{c-q}^3,
\]

and \(P_{c-q}\) is

\[
P_{c-q} = \frac{g^2(E) \left( 3r_{c-q}^6 - 4 q^2 h_1^4(E) h_3^2(E) f^2(E) \left[ 5 r_{c-q}^2 - 6 \alpha' h_4^2(E) g^2(E) \right] - 6 \alpha' h_4^2(E) g^2(E) r_{c-q}^4 \right]}{8 \pi \left( r_{c-q}^2 + 6 \alpha' h_4^2(E) g^2(E) \right) r_{c-q}^6}.
\]

The obtained relation for \(T_{c-q}\) imposes a restriction for having positive critical temperature. Since denominator of \(T_{c-q}\) is positive, the restriction for having positive \(T_{c-q}\) comes from the numerator of this relation with the following explicit form

\[
r_{c-q}^4 - 4 q^2 h_1^4(E) h_3^2(E) f^2(E) > 0.
\]

It is evident that the positivity of the critical temperature depends on the energy variations of constants, the electric charge and energy function of the metric. As for the critical pressure, similarly, only its numerator may yield negative values. In other words, the numerator of the critical pressure imposes following condition for having positive values of \(P_{c-q}\)

\[
3r_{c-q}^6 - 4 q^2 h_1^4(E) h_3^2(E) f^2(E) \left[ 5 r_{c-q}^2 + 6 \alpha' h_4^2(E) g^2(E) \right] - 6 \alpha' h_4^2(E) g^2(E) r_{c-q}^4 > 0.
\]

Here, the restriction is rooted in two parts of the action; one is related to the generalization of GB gravity and the other one is related to the electrodynamic part. In other words, by cancelling the contributions of GB gravity and electromagnetic field, the pressure will always be positive.

Now, we are in a position to calculate \(\frac{P_{c-q} r_{c-q}}{T_{c-q}}\). Using Eqs. \((41)\), \((42)\) and \((39)\), we obtain

\[
\frac{P_{c-q} r_{c-q}}{T_{c-q}} = \frac{g(E) f(E) \left( 3r_{c-q}^6 - 4 q^2 h_1^4(E) h_3^2(E) f^2(E) \left[ 5 r_{c-q}^2 - 6 \alpha' h_4^2(E) g^2(E) \right] - 6 \alpha' h_4^2(E) g^2(E) r_{c-q}^4 \right]}{8 \left( r_{c-q}^2 - 4 q^2 h_1^4(E) h_3^2(E) f^2(E) \right) r_{c-q}^6}.
\]

First of all, contrary to the absence of charge, in this case, all critical values are functions of both of the rainbow functions of metric. This emphasizes the fact that thermodynamical structure of the charged black holes in gravity’s rainbow is completely different from the neutral ones.

The critical horizon radius must be real valued and positive. Therefore, there is a restriction

\[
729\alpha'^4 h_3^8(E) g^8(E) - 375 q^4 h_1^8(E) h_3^4(E) f^4(E) + 3294\alpha'^2 q^2 h_1^4(E) h_3^2(E) h_3^2(E) g^4(E) f^2(E) > 0.
\]

We should note that this restriction is due to the existence of charge. In other words, generalization from neutral to charged solutions, put restrictions on values that rainbow functions and GB parameter can acquire.

The obtained critical parameters indicate that in general, these black holes with consideration of energy dependent constants, enjoy second order phase transition in their phase space. Although the presence of second order phase
FIG. 1: $P - r_+$ (left), $T - r_+$ (middle) and $G - T$ (right) diagrams for $\alpha' = 5$, $q = 0$ and $d = 5$.
$g(E/E_p) = \sqrt{1 - \eta(E/E_p)}$, $f(E/E_p) = 1$, $h_i(E) = 0.9$, $E = 1$, $E_p = 5$, $n = 2$, $\eta = 1$ (continuous line), $\eta = 10$ (dotted line) and $\eta = 20$ (dashed line).
$P - r_+$ diagram for $T = T_c$, $T - r_+$ diagram for $P = P_c$ and $G - T$ diagram for $P = 0.5 P_c$.

FIG. 2: $P - r_+$ (left), $T - r_+$ (middle) and $G - T$ (right) diagrams for $\alpha' = 5$, $q = 0$ and $d = 5$.
$g(E/E_p) = 1$, $f(E/E_p) = e^{\frac{E}{E_p}} - 1$, $h_i(E) = 0.9$, $E = 1$, $E_p = 5$, $\beta = 0.02$ (continuous line), $\beta = 0.2$ (dotted line) and $\beta = 2$ (dashed line).
$P - r_+$ diagram for $T = T_c$, $T - r_+$ diagram for $P = P_c$ and $G - T$ diagram for $P = 0.5 P_c$.

FIG. 3: $P - r_+$ (left), $T - r_+$ (middle) and $G - T$ (right) diagrams for $\alpha' = 5$, $q = 0$ and $d = 5$.
$f(E/E_p) = g(E/E_p) = \frac{1}{1 + \lambda \frac{E}{E_p}}$, $h_i(E) = 0.9$, $E = 1$, $E_p = 5$, $\lambda = 1$ (continuous line), $\lambda = 2$ (dotted line) and $\lambda = 3$ (dashed line).
$P - r_+$ diagram for $T = T_c$, $T - r_+$ diagram for $P = P_c$ and $G - T$ diagram for $P = 0.5 P_c$. 
FIG. 4:  $P - T$ diagrams (left: $g(E/E_p) = \sqrt{1 - \eta(E/E_p)^n}$, $f(E/E_p) = 1$), (middle: $g(E/E_p) = 1$, $f(E/E_p) = \frac{e^{\beta E/E_p} - 1}{\beta E/E_p}$) and (right: $f(E/E_p) = g(E/E_p) = 1$) for $\alpha' = 5$, $q = 0$, $d = 5$, $h_i(E) = 0.9$, $E = 1$ and $E_p = 5$.

For $g(E/E_p) = \sqrt{1 - \eta(E/E_p)^n}$, $f(E/E_p) = 1$: $n = 2$, $\eta = 1$ (continuous line), $\eta = 10$ (dotted line) and $\eta = 20$ (dashed line).

For $g(E/E_p) = 1$, $f(E/E_p) = \frac{e^{\beta E/E_p} - 1}{\beta E/E_p}$: $\beta = 0.02$ (continuous line), $\beta = 0.2$ (dotted line) and $\beta = 2$ (dashed line).

For $f(E/E_p) = g(E/E_p) = 1$: $\lambda = 1$ (continuous line), $\lambda = 2$ (dotted line) and $\lambda = 3$ (dashed line).

FIG. 5:  $P - r_+$ (left), $T - r_+$ (middle) and $G - T$ (right) diagrams for $\alpha' = 5$, $q = 6$ and $d = 5$.

$g(E/E_p) = \sqrt{1 - \eta(E/E_p)^n}$, $f(E/E_p) = 1$, $h_i(E) = 0.9$, $E = 1$, $E_p = 5$, $n = 3$, $\eta = 1$ (continuous line), $\eta = 10$ (dotted line) and $\eta = 20$ (dashed line).

$P - r_+$ diagram for $T = T_c$, $T - r_+$ diagram for $P = P_c$ and $G - T$ diagram for $P = 0.5 P_c$.

FIG. 6:  $P - r_+$ (left), $T - r_+$ (middle) and $G - T$ (right) diagrams for $\alpha' = 5$, $q = 6$ and $d = 5$.

$g(E/E_p) = 1$, $f(E/E_p) = \frac{e^{\beta E/E_p} - 1}{\beta E/E_p}$, $h_i(E) = 0.9$, $E = 1$, $E_p = 5$, $\beta = 1$ (continuous line), $\beta = 2$ (dotted line) and $\beta = 3$ (dashed line).

$P - r_+$ diagram for $T = T_c$, $T - r_+$ diagram for $P = P_c$ and $G - T$ diagram for $P = 0.5 P_c$. 
FIG. 7: $P - r_+$ (left), $T - r_+$ (middle) and $G - T$ (right) diagrams for $\alpha' = 5$, $q = 6$ and $d = 5$. $f(E/E_p) = g(E/E_p) = \frac{1}{1 - \lambda E/E_p}$, $h_i(E) = 0.9$, $E = 1$, $E_p = 5$, $\lambda = 1$ (continuous line), $\lambda = 2$ (dotted line) and $\lambda = 3$ (dashed line).

$P - r_+$ diagram for $T = T_c$, $T - r_+$ diagram for $P = P_c$ and $G - T$ diagram for $P = 0.5 P_c$.

FIG. 8: $P - T$ diagrams (left: $g(E/E_p) = \sqrt{1 - \eta(E/E_p)}$, $f(E/E_p) = 1$), (middle: $g(E/E_p) = 1$, $f(E/E_p) = \frac{e^{\beta E/E_p} - 1}{\beta E/E_p}$) and (right: $f(E/E_p) = g(E/E_p) = \frac{1}{1 - \lambda E/E_p}$, $\lambda = 1$ (continuous line), $\lambda = 2$ (dotted line) and $\lambda = 3$ (dashed line).

transition is restricted to satisfy specific conditions, in general the second order phase transition is a part of thermodynamical properties of these black holes. Now, in order to elaborate the existence of second order phase transition, we choose specific examples for free parameters and energy functions to plot three set of diagrams which are $T - r_+$, $P - r_+$ and $G - T$. The existences of subcritical isobars for critical pressure in $T - r_+$ diagram, the region of phase transition (reflection point) for critical temperature in $P - r_+$ diagram and swallow-tail for pressures smaller than critical pressure in $G - T$ diagram, indicate that a second order phase transition is taking place for the obtained critical values.

Taking into account the rainbow functions, we regard three known models for considering their effects. The first model is motivated from loop quantum gravity and non-commutative geometry, in which rainbow functions are [53]

\[ f(E/E_p) = g(E/E_p) = \sqrt{1 - \eta(E/E_p)} \]  \hspace{1cm} (43)

The second model comes from the hard spectra of gamma-rays motivation with the following form [54]

\[ f(E/E_p) = g(E/E_p) = \frac{e^{\beta E/E_p} - 1}{\beta E/E_p} \]  \hspace{1cm} (44)

Taking the constancy of the velocity of the light into account, one can find following relations for the rainbow functions
as third model

\[ f \left( \frac{E}{E_p} \right) = g \left( \frac{E}{E_p} \right) = \frac{1}{1 - \lambda E / E_p}. \] (45)

Now, using Eqs. (16), (31) and (32) with the obtained critical values (Eqs. (37) - (39)), one can plot the following \( T - r_+ \), \( P - r_+ \) and \( G - T \) diagrams for different models of rainbow functions in special cases (Eqs. (43) - (45)) (Figs. 1, 3 and 5 - 7).

It is evident from plotted graphs that the obtained critical values are representing the second order phase transition points (due to the characteristic behaviors of the different phase diagrams). In some classes of rainbow functions, for neutral case, variations of the parameters of rainbow functions have no effect (see right and left panels of Figs. 1 and 2, respectively). This specific behavior was not observed in the charged solutions.

By excluding right panel of Fig. 1 in other cases, the critical temperature, the size of swallow-tail and differences between energy of different phases are decreasing functions of the parameters of rainbow functions (\( \eta, \beta \) and \( \lambda \)) (see right panels of Figs. 2, 3 and 5 - 7). Meanwhile, by excluding left panel of Fig. 2 the critical pressure is a decreasing function of \( \eta \) and \( \beta \) (see left panels of Figs. 1, 5 and 6) and an increasing function of \( \lambda \) (see left panels of Figs. 3 and 7). As for subcritical isobar, except for the neutral case of loop motivated rainbow functions (see middle panel of Fig. 1), its length is an increasing function of the parameters of rainbow functions (see middle panels of Figs. 2, 3 and 5 - 7).

In the absence of charge, presence of the rainbow functions provides interesting effects. Taking a closer look at the Fig. 1 shows that while we are varying rainbow function which results into modification of the \( P - r_+ \) diagram, the critical temperature remains fixed (Fig. 1 middle and right panels), and interestingly, the total behavior of the Gibbs free energy versus temperature is not affected by this variation either. Same behavior is observed in Fig. 2 left panel. This property enables us to modify different critical values while a specific one of them remains unchanged.

Using the fact that the Gibbs free energy, temperature, and pressure of the system are constant during the phase transition, we have plotted the coexistence line of the black holes (Figs. 4 and 8). The small and large black holes have identical temperature and pressure along the coexistence line, and the critical points are located at the end of the coexistence line where above these points the phase transition does not occur. Fig. 4 (left panel) shows that the coexistence lines for variation of \( \eta \) are completely identical, because \( g \left( \frac{E}{E_p} \right) \) has no effect on the critical pressure and temperature. On the other hand, Fig. 8 (left panel) indicates that variation of \( \eta \) does not have significant effect on the coexistence lines. According to properties of the coexistence lines, one can conclude that for these black holes the reentrant of phase transition does not take place.

V. CLOSING REMARKS

In this paper, the dependency of all constants on energy functions was considered, and charged GB black holes in the presence of gravity’s rainbow were studied. Thermodynamical and geometrical properties of these black holes were investigated and it was shown that the power of the singularity, asymptotical behavior and thermodynamical quantities of these black holes were modified due to the dependency of constants on energy.

Next, using the concepts of extended phase space, van der Waals like behavior and the second order phase transition of these black holes were studied.

First of all, we found that in the presence of gravity’s rainbow, thermodynamical volume of the black holes is modified and it is rainbow function dependent. In other words, the total behavior of the volume is determined by rainbow functions as well as dimensions.

Next, we found that, for neutral case, the obtained critical radius and critical temperature were functions of rainbow functions of the metric, whereas the critical pressure was independent of them and was only dependent on energy variation of the constants, which in return resulted in specific behaviors in phase diagrams in special cases.

Interestingly, in the presence of the electric charge, a limitation was found for having real positive critical horizon radius. In other words, the presence of charge puts restriction on values that different parameters can acquire. Contrary to the neutral case, in the case of charged solutions, all critical values (the critical temperature, the horizon radius and the pressure) were dependent on rainbow functions of the metric. In other words, the critical behavior of the system was modified due to the presence of gravity’s rainbow. It is worth mentioning that the presence of rainbow functions was observed in ratio of \( \frac{r_+}{r_c} \) for both charged and neutral cases, which is a variation of a similar ratio in van der Waals system of liquid/gas. In addition, the coexistence lines indicated that for this type of black holes, the reentrant of phase transition does not happen.

The specific behaviors and results of the paper motivate one to analyze new interesting phenomenology for such black hole thermodynamics. It will be worthwhile to study the effects of non-linear electrodynamics on critical behavior of GB gravity’s rainbow. It will be interesting to generalize the obtained static solutions to a case of dynamical ones and
investigate the cosmological consequences of such solutions. The behavior of the Hawking radiation near the critical point is another subject of interest.

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