Global Smooth Effects and Well-Posedness for the Derivative Nonlinear Schrödinger Equation with Small Rough Data

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Abstract

We obtain the global smooth effects for the solutions of the linear Schrödinger equation in anisotropic Lebesgue spaces. Applying these estimates, we study the Cauchy problem for the generalized elliptical and non-elliptical derivative nonlinear Schrődinger equations (DNLS) and get the global well posedness of solutions with small data in modulation spaces $M^{3/2}_{2,1}(\mathbb{R}^n)$. Noticing that $B^{s+n/2}_{2,1} \subset M^{s}_{2,1} \subset B^{s}_{2,1}$ are optimal inclusions, we have shown the global well posedness of DNLS with a class of rough data.

Keywords. Derivative nonlinear Schrödinger equation, global smooth effects, global well posedness, small data.

MSC: 35 Q 55, 46 E 35, 47 D 08.

1 Introduction

This paper is a continuation of our earlier work [28] and we study the Cauchy problem for the generalized derivative nonlinear Schrödinger equation (gDNLS)

$$iu_t + \Delta_\pm u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad u(0, x) = u_0(x),$$

where $u$ is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\Delta_\pm u = \sum_{i=1}^{n} \varepsilon_i \partial^2_{x_i}, \quad \varepsilon_i \in \{1, -1\}, \quad i = 1, \ldots, n,$$
\( \nabla = (\partial_{x_1}, \ldots, \partial_{x_n}) \), \( F : \mathbb{C}^{2n+2} \to \mathbb{C} \) is a polynomial,

\[
F(z) = P(z_1, \ldots, z_{2n+2}) = \sum_{m+1 \leq |\beta| \leq M+1} c_\beta z^\beta, \quad c_\beta \in \mathbb{C}, \quad (1.3)
\]

\( 2 + 4/n \leq m \leq M, \ m, M \in \mathbb{N} \).

A large amount of work has been devoted to the study of the local and global well posedness of (1.1), see Bejenaru and Tataru [2], Chihara [3, 4], Kenig, Ponce and Vega [11, 12], Klainerman [15], Klainerman and Ponce [16], Ozawa and Zhai [18], Shatah [19], B. Wang and Y. Wang [28]. When the nonlinear term \( F \) satisfies some energy structure conditions, or the initial data suitably decay, the energy method, which went back to the work of Klainerman [15] and was developed in [3, 4, 16, 18, 19], yields the global existence of (1.1) in the elliptical case \( \Delta_\pm = \Delta \).

Recently, Ozawa and Zhai obtained the global well posedness in \( H^s(\mathbb{R}^n) \) \((n \geq 3, s > 2 + n/2, m \geq 2)\) with small data for (1.1) in the elliptical case, where an energy structure condition on \( F \) is still required.

By setting up the local smooth effects for the solutions of the linear Schrödinger equation, Kenig, Ponce and Vega [11, 12] were able to deal with the non-elliptical case and they established the local well posedness of Eq. (1.1) in \( H^s \) with \( s \gg n/2 \). Recently, the local well posedness results have been generalized to the quasi-linear (ultrahyperbolic) Schrödinger equations, see [13, 14].

In one spatial dimension, B. Wang and Y. Wang [28] showed the global well posedness of gDNLS (1.1) for small data in critical Besov spaces \( \dot{B}^{1+n/2-2/m}_{2,1} \cap \dot{B}^{1+n/2-1/M}_{2,1} (\mathbb{R}) \), \( m \geq 4 \). In higher spatial dimensions \( n \geq 2 \), by using Kenig, Ponce and Vega’s local smooth effects and establishing time-global maximal function estimates in space-local Lebesgue spaces, B. Wang and Y. Wang [28] showed the global well posedness of gDNLS (1.1) for small data in Besov spaces \( B^{s}_{2,1}(\mathbb{R}^n) \) with \( s > n/2 + 3/2, m \geq 2 + 4/n \).

Wang and Huang [27] obtained the global well posedness of (1.1) in one spatial dimension with initial data in \( M^{1+1/m}_{2,1} \), \( m \geq 4 \). In this paper, we will use a new way to study the global well posedness of (1.1) and show that (1.1) is globally well posed in \( M^s_{2,1}(\mathbb{R}^n) \) with \( s \geq 3/2, m \geq 2 \) and \( m > 4/n \) for the small Cauchy data. Our starting point is the smooth effect estimates for the linear Schrödinger equation in one spatial dimension (cf. [7, 10, 11, 20, 29]), from which we get a series of linear estimates in higher dimensional anisotropic Lebesgue spaces, including the global smooth effect estimates, the maximal function estimates and their relations to the Strichartz estimates. The maximal function estimates follow an idea as
These estimates together with the frequency-uniform
decomposition method yield the global well posedness of solutions in
modulation spaces $M^s_{2,1}$, $s \geq 3/2$.

1.1 $M^s_{2,1}$ and $B^s_{2,1}$

In this paper, we are mainly interested in the cases that the initial
data $u_0$ belongs to the modulation space $M^s_{2,1}$ for which the norm can be
equivalently defined in the following way (cf. [8, 25, 26, 27]):

$$\|f\|_{M^s_{2,1}} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\mathcal{F}f\|_{L^2(Q_k)}, \quad (1.4)$$

where $\langle k \rangle = 1 + |k|$, $Q_k = \{\xi : -1/2 \leq \xi_i - k_i < 1/2, \ i = 1,...,n\}$. For simplicity, we write $M^s_{2,1} = M^0_{2,1}$. Since only the modulation space $M^s_{2,1}$ will be used in this paper, we will not state the definition of the general modulation spaces $M^s_{p,q}$, one can refer to Feichtinger [8]. Modulation spaces $M^s_{2,1}$ are related to the Besov spaces $B^s_{2,1}$ for which the norm is defined as follows:

$$\|f\|_{B^s_{2,1}} = \|\mathcal{F}f\|_{L^2(B(0,1))} + \sum_{j=1}^{\infty} 2^{sj} \|\mathcal{F}f\|_{L^2(B(0,2^{-j}) \setminus B(0,2^{j-1}))}, \quad (1.5)$$

where $B(x_0, R) := \{\xi \in \mathbb{R} : |\xi - x_0| \leq R\}$. It is known that there holds the following optimal inclusions between $B^{n/2+s}_{2,1}$, $M^s_{2,1}$ and $B^s_{2,1}$ (cf. [23, 22, 27]):

$$B^{n/2+s}_{2,1} \subset M^s_{2,1} \subset B^s_{2,1}. \quad (1.6)$$

So, comparing $M^s_{2,1}$ with $B^{s+n/2}_{2,1}$, we see that $M^s_{2,1}$ contains a class of functions $u$ satisfying $\|u\|_{M^s_{2,1}} = \infty$ but $\|u\|_{B^{s+n/2}_{2,1}} \ll 1$. On the other hand, we can also find a class of rough functions $u$ satisfying $\|u\|_{B^s_{2,1}} = \infty$ but $\|u\|_{M^s_{2,1}} \ll 1$. Another important inclusion between $M_{2,1}$ and $L^\infty$ is that $M_{2,1} \subset L^\infty$ and this embedding is also optimal, see Figure 1.

1.2 Main Results

For the definitions of the anisotropic Lebesgue spaces $L^{p_1}_{\alpha_1}L^{p_2}_{\alpha_2}(\mathbb{R}^{1+n})$ and the frequency-uniform decomposition operators $\{\square_k\}_{k \in \mathbb{Z}^n}$, one can refer to Section 1.3. We have
Figure 1: Optimal inclusions: $B_{2,1}^{n/2} \subset M_{2,1} \subset L^\infty \cap L^2$.

**Theorem 1.1** Let $n \geq 2$, $2 \leq m \leq M < \infty$, $m > 4/n$. Assume that $u_0 \in M_{2,1}^{3/2}$ and $\|u_0\|_{M_{2,1}^{3/2}} \leq \delta$ for some small $\delta > 0$. Then (1.1) has a unique global solution $u \in C(\mathbb{R}, M_{2,1}^{s/2}) \cap X$, where

$$
\|u\|_X = \sum_{\alpha=0,1} \sum_{i=1}^n \sum_{\ell=1}^n \sum_{k \in \mathbb{Z}^n, |k_i|>4} \langle k \rangle^{\alpha} \|\partial^\alpha_{x_i} \Box_k u\|_{L^\infty_t L^2_{(x_j)_{j \neq i}}(\mathbb{R}^{1+n})} 
+ \sum_{\alpha=0,1} \sum_{i=1}^n \sum_{\ell=1}^n \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2-1/m} \|\partial^\alpha_{x_i} \Box_k u\|_{L^m_t L^\infty_{(x_j)_{j \neq i}}(\mathbb{R}^{1+n})} 
+ \sum_{\alpha=0,1} \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \|\partial^\alpha_{x_i} \Box_k u\|_{L^\infty_t L^2_x \cap L^{2+m}_{x,t}(\mathbb{R}^{1+n})},
$$

where $k = (k_1, \ldots, k_n)$. Moreover, $\|u\|_X \lesssim \delta$.

In Theorem 1.1 if $u_0 \in M_{2,1}^s$ with $s > 3/2$, then we have $u \in C(\mathbb{R}, M_{2,1}^s)$. When the nonlinearity $F$ has a simple form, say,

$$
iu_t + \Delta_{\pm} u = \sum_{i=1}^n \lambda_i \partial_{x_i} (u^{\kappa_i+1}), \quad u(0, x) = u_0(x),
$$

we obtained in [27] the global well posedness of the DNLS (1.8) for the small data in modulation spaces $M_{2,1}^{1/k_1}$ in one spatial dimension. In higher spatial dimensions $n \geq 2$, we have
Theorem 1.2 Let $n \geq 2$, $\kappa_i \geq 2$, $\kappa_i > 4/n$, $\kappa_i \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $\kappa = \min_{1 \leq i \leq n} \kappa_i$. Assume that $u_0 \in M_{2,1}^{1/2}$ and $\|u_0\|_{M_{2,1}^{1/2}} \leq \delta$ for some small $\delta > 0$. Then (1.8) has a unique global solution $u \in C(\mathbb{R}, M_{2,1}^{1/2}) \cap X$, where

$$
\|u\|_{X_1} = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n, |k_i| > 4} \langle k_i \rangle \|\Box_k u\|_{L^2_{\infty,i}L^2_{xj,j \neq i}^{1}(\mathbb{R}^{1+n})} \\
+ \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2-1/k} \|\Box_k u\|_{L^2_{i}L^\infty_{xj,j \neq i}^{1}(\mathbb{R}^{1+n})} \\
+ \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \|\Box_k u\|_{L^\infty_{i}L^2_{xj,j \neq i}^{1}(\mathbb{R}^{1+n})}.
$$

(1.9)

Moreover, $\|u\|_{X_1} \lesssim \delta$.

We remark that in Theorem 1.2, the same result holds if the nonlinear term $\partial_{x_i}(u^{\kappa_i+1})$ is replaced by $\partial_{x_i}(|u|^{\kappa_i}u)$ ($\kappa_i \in 2\mathbb{N}$).

Corollary 1.3 Let $n \geq 2$, $s > (n+1)/2$. Let $X$ and $X_1$ be as in Theorems 1.1 and 1.2, respectively. We have the following results.

(i) Let $2 \leq m \leq M < \infty$, $m > 4/n$. Assume that $u_0 \in H^{s+1}$ and $\|u_0\|_{H^{s+1}} \leq \delta$ for some small $\delta > 0$. Then (1.1) has a unique global solution $u \in X$.

(ii) Let $\kappa_i \geq 2$, $\kappa_i > 4/n$, $\kappa_i \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$. Assume that $u_0 \in H^s$ and $\|u_0\|_{H^s} \leq \delta$ for some small $\delta > 0$. Then (1.8) has a unique global solution $u \in X_1$.

When $m = 1$, Christ [5] showed the ill posedness of (1.8) in any $H^s$ for one spatial dimension case. For general nonlinearity in (1.1), we do not know what happens in the case $m = 1$ in higher spatial dimensions.

1.3 Notations

The following are some notations which will be frequently used in this paper: $\mathbb{C}, \mathbb{R}, \mathbb{N}$ and $\mathbb{Z}$ will stand for the sets of complex number, reals, positive integers and integers, respectively. $c \leq 1$, $C > 1$ will denote positive universal constants, which can be different at different places. $a \lesssim b$ stands for $a \leq Cb$ for some constant $C > 1$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We write $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. We denote by $p'$ the dual number of $p \in [1, \infty]$, i.e., $1/p + 1/p' = 1$. We will use Lebesgue spaces $L^p := L^p(\mathbb{R}^n)$, $\| \cdot \|_p := \| \cdot \|_{L^p}$, Sobolev spaces $H^s = (I - \Delta)^{-s/2}L^2$. 

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Some properties of these function spaces can be found in [1, 24]. We will use the function spaces $L^p_t L^p_x(\mathbb{R}^{n+1})$ and $L^p_x L^p_t(\mathbb{R}^{n+1})$ for which the norms are defined by

$$
\|f\|_{L^p_t L^p_x(\mathbb{R}^{n+1})} = \|\|f\|_{L^p_x(\mathbb{R}^n)}\|_{L^p_t(\mathbb{R})}, \quad \|f\|_{L^p_x L^p_t(\mathbb{R}^{n+1})} = \|\|f\|_{L^p_t(\mathbb{R})}\|_{L^p_x(\mathbb{R}^n)};
$$

$L^p_{x,t}(\mathbb{R}^{n+1}) := L^p_t L^p_x(\mathbb{R}^{n+1})$. We denote by $L^p_{x_1} L^p_{x_2} L^p_{x_3} := L^p_{x_1} L^p_{x_2} L^p_{x_3}(\mathbb{R}^{1+n})$ the anisotropic Lebesgue space for which the norm is defined by

$$
\|f\|_{L^p_{x_1} L^p_{x_2} L^p_{x_3}} = \left\| \left\| \left\| f \right\|_{L^p_{x_1}(\mathbb{R})} \right\|_{L^p_{x_2}(\mathbb{R}^n)} \right\|_{L^p_{x_3}(\mathbb{R})}.
$$

It is also convenient to use the notation $L^p_{x_1} L^p_{x_2} L^p_{x_3} L^p_{x_4} := L^p_{x_1} L^p_{x_2} L^p_{x_3} L^p_{x_4}(\mathbb{R}^{1+n})$. For any $1 < k < n$, we denote by $\mathcal{F}_{x_1,\ldots,x_k}$ the partial Fourier transform:

$$
(\mathcal{F}_{x_1,\ldots,x_k} f)(\xi_1,\ldots,\xi_k, x_{k+1},\ldots,x_n) = \int_{\mathbb{R}^k} e^{-i(x_1\xi_1+\ldots+x_k\xi_k)} f(x)dx_1\ldots dx_k
$$

and by $\mathcal{F}_{-x_1,\ldots,-x_k}$ the partial inverse Fourier transform, similarly for $\mathcal{F}_{t,x}$ and $\mathcal{F}_{-t,\xi}$. Denote $\mathcal{F} := \mathcal{F}_{x_1,\ldots,x_n}$, $\mathcal{F}^{-1} := \mathcal{F}_{-x_1,\ldots,-x_n}$, $D^s_{x_i} := (-\partial_{x_i}^2)^{s/2} = \mathcal{F}^{-1}_{\xi_i} |\xi|^s \mathcal{F}_{x_i}$ expresses the partial Riesz potential in the $x_i$ direction. $\partial_{x_i}^s = \mathcal{F}_{\xi_i}^{-1}(i\xi_i)^{-s} \mathcal{F}_{x_i}$. We will use the Bernstein multiplier estimate; cf. [1, 24]. For any $r \in [1, \infty]$,

$$
\|\mathcal{F}^{-1} \varphi \mathcal{F} f\|_r \leq C \|\varphi\|_{H^s} \|f\|_r, \quad s > n/2.
$$

We will use the frequency-uniform decomposition operators (cf. [25, 26, 27]). Let $\{\sigma_k\}_{k \in \mathbb{Z}^n}$ be a function sequence satisfying

$$
\begin{cases}
\sigma_k(\xi) \geq c, & \forall \xi \in Q_k, \\
\text{supp}\sigma_k \subseteq \{\xi : |\xi - k| \leq \sqrt{n}\}, \\
\sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) \equiv 1, & \forall \xi \in \mathbb{R}^n, \\
|D^\alpha \sigma_k(\xi)| \leq C_m, & \forall \xi \in \mathbb{R}^n, |\alpha| \leq m \in \mathbb{N}.
\end{cases}
$$

Denote

$$
\Upsilon = \{\{\sigma_k\}_{k \in \mathbb{Z}^n} : \{\sigma_k\}_{k \in \mathbb{Z}^n} \text{ satisfies (1.13)}\}.
$$

Let $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon$ be a function sequence and

$$
\Box_k := \mathcal{F}^{-1}(i\xi)^{-1} \mathcal{F}, \quad k \in \mathbb{Z}^n,
$$

which are said to be the frequency-uniform decomposition operators. One may ask the existence of the frequency-uniform decomposition operators. Indeed, let $\rho \in \mathcal{S}(\mathbb{R}^n)$ and $\rho : \mathbb{R}^n \to [0,1]$ be a smooth radial bump function adapted to the
ball $B(0, \sqrt{n})$, say $\rho(\xi) = 1$ as $|\xi| \leq \sqrt{n}/2$, and $\rho(\xi) = 0$ as $|\xi| \geq \sqrt{n}$. Let $\rho_k$ be a translation of $\rho$: $\rho_k(\xi) = \rho(\xi - k)$, $k \in \mathbb{Z}^n$. We write $\eta_k(\xi) = \rho_k(\xi) \left( \sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \right)^{-1}$, $k \in \mathbb{Z}^n$. (1.16)

We have $\{\eta_k\}_{k \in \mathbb{Z}^n} \in \Upsilon$. It is easy to see that for any $\{\eta_k\}_{k \in \mathbb{Z}^n} \in \Upsilon$, we have

$$\|f\|_{M^s_t} \sim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\Box_k f\|_{L^2(\mathbb{R}^n)}.$$

(1.17)

We will use the function space $\ell^s_1(L^p_t L^r_x(I \times \mathbb{R}^n))$ which contains all of the functions $f(t, x)$ so that the following norm is finite:

$$\|f\|_{\ell^s_1(L^p_t L^r_x(I \times \mathbb{R}^n))} := \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\Box_k f\|_{L^p_t L^r_x(I \times \mathbb{R}^n)}.$$

For simplicity, we write $\ell^s_1(L^p_t L^r_x(I \times \mathbb{R}^n)) = \ell^{s,0}_1(L^p_t L^r_x(I \times \mathbb{R}^n))$.

This paper is organized as follows. In Section 2 we show the global smooth effect estimates of the solutions of the linear Schrödinger equation in anisotropic Lebesgue spaces. In Sections 3 and 4 we consider the frequency-uniform localized versions for the global maximal function estimates, the global smooth effects, together with their relations to the Strichartz estimates. In Sections 5 and 6 we prove our Theorems 1.2 and 1.1, respectively. In the Appendix we generalize the Christ-Kiselev Lemma to the anisotropic Lebesgue spaces in higher dimensions.

## 2 Anisotropic global smooth effects

In this section, we always denote

$$S(t) = e^{it\Delta_z} = \mathcal{F}^{-1} e^{it\sum_{j=1}^n \epsilon_j \xi_j^2} \mathcal{F}, \quad \mathcal{A} f(t, x) = \int_0^t S(t - \tau) f(\tau, x) d\tau.$$

### Proposition 2.1

For any $i = 1, \ldots, n$, we have the following estimate:

$$\|\partial_{x_1} \mathcal{A} f\|_{L^1_{t,x}(L^2_{(x_j)_{j \neq 1}} \cap L^2_{(R^{1+n})})} \lesssim \|f\|_{L^4_{t,x}(L^2_{(x_j)_{j \neq 1}} \cap L^2_{(R^{1+n})})}. \quad (2.1)$$

**Proof.** We have

$$\partial_{x_1} \mathcal{A} f = c \mathcal{F}^{-1}_{t,x} \frac{\xi_1}{|\xi|^2 - \tau} \mathcal{F}_{t,x} f. \quad (2.2)$$
We can assume, without loss of generality that $|\xi|^2 = \xi_1^2 + \varepsilon_2 \xi_2^2 + \ldots + \varepsilon_n \xi_n^2 := \xi_1^2 + |\xi|^2_\perp$. By Plancherel’s identity,

$$
\| \partial_x A f \|_{L^\infty_x L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})} \\
= \| \mathcal{F}^{-1}_{\xi_1} \frac{\xi_1}{\xi_1^2 + |\xi|^2_\perp - \tau} \mathcal{F}_t f \|_{L^\infty_x L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})} \\
\leq \| \mathcal{F}^{-1}_{\xi_1} \frac{\xi_1}{\xi_1^2 + |\xi|^2_\perp - \tau} \mathcal{F}_t f \|_{L^2_{x_2,\ldots,x_n} L^\infty_x L^2_t(\mathbb{R}^{1+n})}. 
$$

(2.3)

By changing the variable $\tau \to \mu + \|\xi\|^2_\perp$, we have

$$
\| \mathcal{F}^{-1}_{\mu,\xi} \frac{\xi}{\xi^2 - \tau} \mathcal{F}_t f \|_{L^\infty_x L^2_t(\mathbb{R}^{1+n})} \lesssim \|f\|_{L^1_x L^2_t(\mathbb{R}^{1+n})},
$$

(2.5)

we have from (2.3), (2.4) and (2.5) that

$$
\| \partial_x A f \|_{L^\infty_x L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})} \lesssim \| \exp^{it|\xi|^2_\perp} \mathcal{F}_{x_2,\ldots,x_n} f \|_{L^2_{x_2,\ldots,x_n} L^\infty_x L^2_t(\mathbb{R}^{1+n})}. 
$$

(2.6)

Using Minkowski’s inequality and Plancherel’s equality, we immediately have

$$
\| \partial_x A f \|_{L^\infty_x L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})} \lesssim \|f\|_{L^1_x L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})}. 
$$

(2.7)

The other cases can be shown in a similar way. \(\square\)

**Proposition 2.2** For any $i = 1,\ldots,n$, we have the following estimate:

$$
\| D^{1/2}_{x_i} S(t) u_0 \|_{L^\infty_x L^2_{\mathcal{F}(x_i) \neq i} L^2_t(\mathbb{R}^{1+n})} \lesssim \| u_0 \|_2.
$$

(2.8)

**Proof.** By Plancherel’s equality and Minkowski’s inequality,

$$
\| S(t) u_0 \|_{L^\infty_x L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})} = \| \mathcal{F}^{-1}_{\xi_1} \exp^{it\xi_1 \xi_1^2} \mathcal{F}_{x_1} (\mathcal{F}_{x_2,\ldots,x_n} u_0) \|_{L^\infty_x L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})} \\
\leq \| \mathcal{F}^{-1}_{\xi_1} \exp^{it\xi_1 \xi_1^2} \mathcal{F}_{x_1} (\mathcal{F}_{x_2,\ldots,x_n} u_0) \|_{L^2_{x_2,\ldots,x_n} L^\infty_x L^2_t(\mathbb{R}^{1+n})}. 
$$

(2.9)
Recall the half-order smooth effect of $S(t)$ in one spatial dimension (cf. [10]),
\[
\left\| \mathcal{F}_\xi^{-1} e^{it\xi^2} \mathcal{F}_x u_0 \right\|_{L_t^\infty L_x^2(R^{1+1})} \lesssim \left\| D_x^{-1/2} u_0 \right\|_{L^2(R)}.
\] (2.10)

Hence, in view of (2.9) and (2.10), using Plancherel's equality, we immediately have
\[
\left\| S(t) u_0 \right\|_{L_t^\infty L_x^2(\mathbb{R}^{1+n})} \lesssim \left\| D_x^{-1/2} u_0 \right\|_{L^2(\mathbb{R}^n)},
\] (2.11)

which implies the result, as desired. □

The dual version of (2.8) is

**Proposition 2.3** For any $i = 1, \ldots, n$, we have the following estimate:
\[
\left\| \partial_{x_i} \mathcal{A} f \right\|_{L_t^\infty L_x^2(\mathbb{R}^{1+n})} \lesssim \left\| D_x^{1/2} f \right\|_{L_t^1 L_x^2(\mathbb{R}^{1+n})} \left\| \mathcal{A} \right\|_{L_t^1 L_x^2(\mathbb{R}^{1+n})}.
\] (2.12)

**Proof.** Denote $\mathbb{R}_+ = [0, \infty)$. By Proposition 2.2,
\[
\left| \int_{\mathbb{R}_+} \left( (\mathcal{A} \partial_{x_1} f)(t), \psi(t) \right) dt \right|
= \left| \int_{\mathbb{R}_+} \left( f(\tau), \int_\tau^\infty S(\tau - t) \partial_{x_1} \psi(t) dt \right) d\tau \right|
\leq \left\| D_x^{1/2} f \right\|_{L_t^1 L_x^2(\mathbb{R}^{1+n})} \left\| \partial_{x_1} S(\tau - t) D_x^{-1/2} \psi(t) \right\|_{L_t^\infty L_x^2(\mathbb{R}^{1+n})} dt
\lesssim \left\| D_x^{1/2} f \right\|_{L_t^1 L_x^2(\mathbb{R}^{1+n})} \left\| \psi \right\|_{L_t^1 L_x^2(\mathbb{R}^{1+n})}.
\] (2.13)

By duality, we have the result. □

# 3 Linear estimates with $\square_k$-decomposition

In this section we consider the smooth effect estimates, the maximal function estimates, the Strichartz estimates and their interaction estimates for the solutions of the linear Schrödinger equations by using the frequency-uniform decomposition operators. For convenience, we will use the following function sequence $\{\sigma_k\}_{k \in \mathbb{Z}^n}$:

**Lemma 3.1** Let $\eta_k : \mathbb{R} \rightarrow [0, 1]$ $(k \in \mathbb{Z})$ be a smooth-function sequence satisfying condition (1.13). Denote
\[
\sigma_k(\xi) := \eta_{k_1}(\xi_1) \cdots \eta_{k_n}(\xi_n), \quad k = (k_1, \ldots, k_n).
\] (3.1)

Then we have $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon$. 9
Recall that in [26], we established the following Strichartz estimates in a class of function spaces by using the frequency-uniform decomposition operators.

**Lemma 3.2** Let \( 2 \leq p < \infty, \gamma \geq 2 \lor \gamma(p) \),

\[
\frac{2}{\gamma(p)} = n\left(\frac{1}{2} - \frac{1}{p}\right).
\]

Then we have

\[
\|S(t)\varphi\|_{L^1_t(L^\gamma_x(\mathbb{R}^{1+n}))} \lesssim \|\varphi\|_{M^2_1(\mathbb{R}^n)},
\]

\[
\|\mathcal{A}f\|_{L^1_t(L^\gamma_x(\mathbb{R}^{1+n}))} \lesssim \|f\|_{L^\infty_t(L^2_x(\mathbb{R}^n))}.
\]

In particular, if \( 2 + 4/n \leq p < \infty \), then we have

\[
\|S(t)\varphi\|_{L^1_t(L^p_x(\mathbb{R}^{1+n}))} \lesssim \|\varphi\|_{M^2_1(\mathbb{R}^n)},
\]

\[
\|\mathcal{A}f\|_{L^1_t(L^p_x(\mathbb{R}^{1+n}))} \lesssim \|f\|_{L^\infty_t(L^2_x(\mathbb{R}^n))} \cap L^1_t(L^\infty_x(L^2(\mathbb{R}^n))).
\]

The next lemma is essentially known, see [24, 25].

**Lemma 3.3** Let \( \Omega \subset \mathbb{R}^n \) be a compact set with \( \text{diam } \Omega < 2R, 0 < p \leq q \leq \infty \).

Then there exists a constant \( C > 0 \), which depends only on \( p, q \)

\[
\|f\|_q \leq CR^{n(1/p - 1/q)} \|f\|_p, \quad \forall f \in L^p_{\Omega},
\]

where \( L^p_{\Omega} = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \text{supp } \hat{f} \subset \Omega, \|f\|_p < \infty \} \).

In Lemma 3.3 we emphasize that the constant \( C > 0 \) is independent of the position of \( \Omega \) in frequency spaces, say, in the case \( \Omega = B(k, \sqrt{n}), k \in \mathbb{Z}^n \), Lemma 3.3 uniformly holds for all \( k \in \mathbb{Z}^n \).

**Lemma 3.4** We have for any \( \sigma \in \mathbb{R} \) and \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \) with \( |k_i| \geq 4 \),

\[
\|\Box_k D^\sigma_{x_i} u\|_{L^p_{x_1}L^{p_2}_{x_2} \ldots \ldots L^{p_n}_{x_n}L^2(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^\sigma \|\Box_k u\|_{L^p_{x_1}L^{p_2}_{x_2} \ldots \ldots L^{p_n}_{x_n}L^2(\mathbb{R}^{1+n})}.
\]

Replacing \( D^\sigma_{x_i} \) by \( \partial^\sigma_{x_i} \) (\( \sigma \in \mathbb{N} \)), the above inequality holds for all \( k \in \mathbb{Z}^n \).

**Proof.** Using Lemma 3.1 one has that

\[
\Box_k D^\sigma_{x_i} u = \sum_{\ell = -1}^1 \int_{\mathbb{R}} \left( \mathcal{F}_{\xi_i}^{-1}(\eta_{k_i + \ell}(\xi_i)|\xi_i|^\sigma) \right)(y_i)(\Box_k u)(x_i - y_i)dy_i.
\]
It follows that
\[ \left\| \mathcal{L}_{k} D_{x_{i}}^{\alpha}u \right\|_{L_{t}^{p_{1}}L_{x_{2}}^{p_{2}}...L_{x_{n}}^{p_{2}}(\mathbb{R}^{1+n})} \leq \sum_{\ell=-1}^{1} \left\| \mathcal{F}_{\xi_{i}}^{-1}(\eta_{k_{i}+\ell}(\xi_{i})|\xi_{i}|^{\alpha}) \right\|_{L^{1}(\mathbb{R})} \left\| \mathcal{L}_{k} u \right\|_{L_{t}^{p_{1}}L_{x_{2}}^{p_{2}}...L_{x_{n}}^{p_{2}}(\mathbb{R}^{1+n})} \]
\[ \leq \left\langle k_{i} \right\rangle^{\alpha} \left\| \mathcal{L}_{k} u \right\|_{L_{t}^{p_{1}}L_{x_{2}}^{p_{2}}...L_{x_{n}}^{p_{2}}(\mathbb{R}^{1+n})}. \]

The result follows. \( \square \)

Ionescu and Kenig [9] showed the following maximal function estimates in higher spatial dimensions \( n \geq 3 \):
\[ \left\| \Delta_{k}S(t)u_{0} \right\|_{L_{x}^{2}(\mathbb{R}^{1+n})} \lesssim 2^{(n-1)k/2} \left\| \Delta_{k}u_{0} \right\|_{L^{2}(\mathbb{R}^{n})}. \] \( \tag{3.2} \)

We partially resort to their idea to obtain the following

**Proposition 3.5** Let \( 4/n < q \leq \infty, q \geq 2 \). Then we have
\[ \left\| \mathcal{L}_{k}S(t)u_{0} \right\|_{L_{x}^{q}(\mathbb{R}^{1+n})} \lesssim \left\langle k_{i} \right\rangle^{1/q} \left\| \mathcal{L}_{k} u \right\|_{L^{2}(\mathbb{R}^{n})}. \] \( \tag{3.3} \)

**Proof.** For convenience, we write \( \bar{x} = (x_{1},...,x_{n-1}) \). By duality, it suffices to show that for any \( \varphi \in L_{x_{1}}^{q}L_{x,t}^{1}(\mathbb{R}^{1+n}) \cap L^{\infty}(\mathbb{R}^{1+n}) \) with \( \varphi(t) = \pm \varphi(-t) \),
\[ \int_{\mathbb{R}} (\mathcal{L}_{k}S(t)u_{0}, \varphi(t))dt \lesssim \left\langle k_{i} \right\rangle^{1/q} \left\| \mathcal{L}_{k} u \right\|_{L^{2}(\mathbb{R}^{n})} \left\| S(t)\varphi \right\|_{L_{x}^{q}L_{x,t}^{1}(\mathbb{R}^{1+n})}. \] \( \tag{3.4} \)

By duality, we have
\[ \int_{\mathbb{R}} (\mathcal{L}_{k}S(t)u_{0}, \varphi(t))dt \lesssim \left\| u_{0} \right\|_{L^{2}(\mathbb{R}^{n})} \left\| \int_{\mathbb{R}} \mathcal{L}_{k}S(-t)\varphi(t)dt \right\|_{L^{2}(\mathbb{R}^{n})}. \] \( \tag{3.5} \)

We have from Lemma 3.4 that
\[ \left\| \int_{\mathbb{R}} \mathcal{L}_{k}S(-t)\varphi(t)dt \right\|_{L^{2}(\mathbb{R}^{n})}^{2} \lesssim \left\| S(t)\varphi \right\|_{L_{x}^{q}L_{x,t}^{1}(\mathbb{R}^{1+n})} \int_{\mathbb{R}} \left\| \mathcal{L}_{k}S(2t-\tau)\varphi(\tau) \right\|_{L_{x}^{q}L_{x,t}^{1}(\mathbb{R}^{1+n})}d\tau. \] \( \tag{3.6} \)

In view of Lemma 3.1, we can write \( \mathcal{L}_{k} = \mathcal{F}^{-1} \eta_{k_{1}}(\xi_{1})...\eta_{k_{n}}(\xi_{n})\mathcal{F} := \mathcal{F}^{-1} \eta_{k_{1}}(\xi_{1})\eta_{k}(\xi)\mathcal{F} \).

By Minkowski’s and Young’s inequalities,
\[ \left\| \int_{\mathbb{R}} \mathcal{L}_{k}S(2t-\tau)\varphi(\tau) \right\|_{L_{x}^{q}L_{x,t}^{1}(\mathbb{R}^{1+n})} \]
Hence, it suffices to show that

\[ \langle k_1 \rangle \lesssim \left\| \mathcal{F}^{-1} e^{ix_1 \xi^2} \eta_k(\xi) \eta_{k_1}(\xi_1) \mathcal{F} \int_{\mathbb{R}} S(-\tau) \varphi(\tau) d\tau \right\|_{\dot{L}_{x,t}^{q/2} L_{x,t}^{\infty}(\mathbb{R}^{n+1})} \]

\[ \lesssim \left\| \mathcal{F}^{-1} e^{ix_1 \xi^2} \eta_k(\xi) \mathcal{F} \int_{\mathbb{R}} S(-\tau) \varphi(\tau) d\tau \right\|_{\dot{L}_{x,t}^{q/2} L_{x,t}^{\infty}(\mathbb{R}^{n+1})} \]

\[ \lesssim \left\| \mathcal{F}^{-1} e^{ix_1 \xi^2} \eta_{k_1}(\xi_1) \mathcal{F} \int_{\mathbb{R}} S(-\tau) \varphi(\tau) d\tau \right\|_{\dot{L}_{x,t}^{q/2} L_{x,t}^{\infty}(\mathbb{R}^{n+1})} \]

Hence, for \(|x_1| > 1|

\[ \left\| \mathcal{F}^{-1} e^{ix_1 \xi^2} \eta_{k_1}(\xi_1) \right\|_{L_{x,t}^{q/2} L_{x,t}^{\infty}(\mathbb{R})} \lesssim \langle k_1 \rangle^{2/q} \]

In view of the decay of \( \Box_k S(t) \), we see that (cf. [26])

\[ \left\| \mathcal{F}^{-1} e^{ix_1 \xi^2} \eta_k(\xi) \left\|_{L_x^{\infty}(\mathbb{R}^{n+1})} \lesssim (1 + |t|)^{-(n-1)/2}, \right\| \mathcal{F}^{-1} e^{ix_1 \xi^2} \eta_{k_1}(\xi_1) \right\|_{L_x^{\infty}(\mathbb{R})} \lesssim (1 + |t|)^{-1/2}. \]

On the other hand, integrating by part, one has that for \(|x_1| > 4 |t| \langle k_1 \rangle \)

\[ \left| \mathcal{F}^{-1} e^{ix_1 \xi^2} \eta_{k_1}(\xi_1) \right| \lesssim |x_1|^{-2}. \]

Hence, for \(|x_1| > 1|

\[ \left| \mathcal{F}^{-1} e^{ix_1 \xi^2} \eta_k(\xi) \right| \lesssim (1 + |x_1|)^{-2} + \langle k_1 \rangle^{n/2} \langle k_1 \rangle^{1/2} |x_1|^{-n/2}. \]

So, we have

\[ \left\| \mathcal{F}^{-1} e^{ix_1 \xi^2} \eta_{k_1}(\xi_1) \right\|_{L_{x,t}^{q/2} L_{x,t}^{\infty}(\mathbb{R})} \lesssim 1 + \langle k_1 \rangle^{n/2} \langle k_1 \rangle^{1/2} \langle k_1 \rangle^{-2/2} \]
By Proposition 3.5, we have
\[ \sup_{|x_1| \leq c k_1, t, x_2, \ldots, x_n} \left| \mathcal{F}^{-1}(e^{i \xi_2 \eta_1^2} \eta_1^2)(\xi_1)(x_1) \prod_{i=2}^{n} \mathcal{F}^{-1}(e^{i \xi_2 \eta_0^2} \eta_0^2)(\xi_i)(x_i) \right|^q \ dx_1. \]

Taking \( t = -x_1/2 \varepsilon_1 k_1 \) and \(|x_i| < c, i = 2, \ldots, n\), we easily see that
\[ \left| \mathcal{F}^{-1}(e^{i \xi_2 \eta_1^2} \eta_1^2)(\xi_1)(x_1) \right| \gtrsim 1, \]
\[ \left| \mathcal{F}^{-1}(e^{i \xi_2 \eta_0^2} \eta_0^2)(\xi_i)(x_i) \right| \gtrsim 1. \]

Therefore, we have
\[ \|\square_k S(t) u_0\|^q_{L_{t}^{q} L_{x_{2}, \ldots, x_{n}}^{\infty}(\mathbb{R}^{1+n})} \gtrsim k_1. \]

The dual version of Proposition 3.5 is the following

**Proposition 3.7** Let \( 2 \leq q \leq \infty, q > 4/n \). Then we have for any \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \),
\[ \left\| \square_k \int_{\mathbb{R}} S(t - \tau) f(\tau) d\tau \right\|_{L_{t}^{q} L_{x}^{\infty}(\mathbb{R}^{1+n})} \lesssim (k_i)^{1/q} \left\| \square_k f \right\|_{L_{x}^{q} L_{t}^{1}((x)_{j \neq i}) L_{t}^{1}(\mathbb{R}^{1+n})}. \]  

**Proof.** Denote
\[ \square_k = \sum_{\ell \in \Lambda} \square_{k+\ell}, \quad \Lambda = \{ \ell \in \mathbb{Z}^n : \text{supp } \sigma_k \cap \text{supp } \sigma_{k+\ell} \neq \emptyset \}. \]

Write
\[ L_k(f, \psi) := \left| \int_{\mathbb{R}} \left( \square_k \int_{\mathbb{R}} S(t - \tau) f(\tau) d\tau, \psi(t) \right) dt \right| \]  

By Proposition 3.5
\[ L_k(f, \psi) = \left| \left( \square_k f(\tau), \square_k \int_{\mathbb{R}} S(t - \tau) \psi(t) dt \right) d\tau \right| \]
\[ \leq \left\| \square_k f \right\|_{L_{x}^{q} L_{t}^{1}((x)_{j \neq i}) L_{t}^{1}(\mathbb{R}^{1+n})} \left\| \square_k \int_{\mathbb{R}} S(t - \tau) \psi(t) dt \right\|_{L_{x}^{q} L_{t}^{1}((x)_{j \neq i}) L_{t}^{1}(\mathbb{R}^{1+n})} \]
\[ \leq \left\| \square_k f \right\|_{L_{x}^{q} L_{t}^{1}((x)_{j \neq i}) L_{t}^{1}(\mathbb{R}^{1+n})} (k_i)^{1/q} \left\| \square_k \psi \right\|_{L_{x}^{q} L_{t}^{1}(\mathbb{R}^{1+n})}. \]

By duality, we have the result, as desired. \( \square \)

In view of Propositions 2.1 and 2.3 we have

**Proposition 3.8** We have for any \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \),
\[ \left\| \square_k \mathcal{A}_i f \right\|_{L_{t}^{q} L_{x}^{2}((x)_{j \neq i}) L_{t}^{2}(\mathbb{R}^{1+n})} \lesssim \left\| \square_k f \right\|_{L_{x}^{q} L_{t}^{2}((x)_{j \neq i}) L_{t}^{2}(\mathbb{R}^{1+n})}, \]  

\[ \left\| \square_k \mathcal{A}_i f \right\|_{L_{t}^{q} L_{x}^{2}(\mathbb{R}^{1+n})} \lesssim (k_i)^{1/2} \left\| \square_k f \right\|_{L_{x}^{q} L_{t}^{2}((x)_{j \neq i}) L_{t}^{2}(\mathbb{R}^{1+n})}. \]
Proof. By Proposition 2.1 we immediately have (3.12). In view of Proposition 2.3 and Lemma 3.4 we have (3.13) in the case $|k_i| \geq 3$. If $|k_i| \leq 2$, in view of Proposition 2.3

$$\|k_\mathcal{A}_x \partial_{x_i} f\|_{L^\infty_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})} \lesssim \|D^{1/2}_x k_\mathcal{A}_x f\|_{L^\infty_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})} \lesssim \|k f\|_{L^1_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})},$$

which implies the result, as desired. \qed

By the duality, we also have the following

**Proposition 3.9** Let $2 \leq q \leq \infty$ and $q > 4/n$. Then we have

$$\|k_\mathcal{A}_x \partial_{x_i} f\|_{L^\infty_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^{1/2 + 1/q} \|k f\|_{L^1_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})}. \quad (3.14)$$

**Proof.** By Propositions 3.7, 3.8 and Lemma 3.4

$$L_k(\partial_{x_i} f, \psi) = \left| \left( k \int_{\mathbb{R}} S(-\tau) \partial_{x_i} f(\tau) d\tau, k \int_{\mathbb{R}} S(-t) \psi(t) dt \right) \right| \leq \left| \left( k \int_{\mathbb{R}} S(-\tau) \partial_{x_i} f(\tau) d\tau \right) \right|_{L^2(\mathbb{R}^n)} \left| \left( k \int_{\mathbb{R}} S(-t) \psi(t) dt \right) \right|_{L^2(\mathbb{R}^n)} \lesssim \langle k_1 \rangle^{1/2} \|k f\|_{L^1_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})} \langle k_1 \rangle^{1/2} \|k_\psi\|_{L^1_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})} \lesssim \langle k_1 \rangle^{1/2 + 1/q} \|k f\|_{L^1_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})} \|\psi\|_{L^1_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})}. \quad (3.15)$$

Again, by duality, it follows from (3.15) and Christ-Kiselev’s Lemma that (3.14) holds. \qed

**Proposition 3.10** Let $2 \leq r < \infty$, $2/\gamma(r) = n(1/2 - 1/r)$ and $\gamma > \gamma(r) \lor 2$. We have

$$\|k_\mathcal{A} S(t)u_0\|_{L^\gamma_\mathcal{A} L^r_\mathcal{A}(\mathbb{R}^{1+n})} \lesssim \|k u_0\|_{L^2(\mathbb{R}^n)}, \quad (3.16)$$

$$\|k_\mathcal{A} \mathcal{A} f\|_{L^\infty_\mathcal{A} L^2_\mathcal{A} \cap L^\gamma_\mathcal{A} L^r_\mathcal{A}(\mathbb{R}^{1+n})} \lesssim \|k f\|_{L^1_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})}, \quad (3.17)$$

$$\|k_\mathcal{A} \partial_{x_i} f\|_{L^\infty_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^{1/2} \|k f\|_{L^1_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})}, \quad (3.18)$$

$$\|k_\mathcal{A} \partial_{x_i} f\|_{L^\infty_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^{1/2} \|k f\|_{L^1_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})}, \quad (3.19)$$

and for $2 \leq q < \infty$, $q > 4/n$, $\alpha = 0, 1$,

$$\|k_\mathcal{A} \partial_{x_i}^\alpha f\|_{L^\infty_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^{\alpha + 1/q} \|k f\|_{L^1_\mathcal{A} L^2_\mathcal{A}(\mathbb{R}^{1+n})}, \quad (3.20)$$
Proof. From Lemma 3.2 it follows that (3.16) and (3.17) hold. We now show (3.18). We use the same notations as in Proposition 3.9. By Lemmas 3.2, 3.4 and Proposition 3.8

\[ L_k(\partial_x f, \psi) \lesssim \langle k_i \rangle^{1/2} \| k f \|_{L^1_t L^2_{x_1} L^\infty_{x_2,\ldots,x_n} L^1_t(\mathbb{R}^{1+n})} \| \partial_x \psi \|_{L^q_t L^r_t(\mathbb{R}^{1+n})} \]
\[ \lesssim \langle k_i \rangle^{1/2} \| k f \|_{L^1_t L^2_{x_1} L^\infty_{x_2,\ldots,x_n} L^1_t(\mathbb{R}^{1+n})} \| \psi \|_{L^q_t L^r_t(\mathbb{R}^{1+n})}. \]  
(3.21)

By duality, it follows from (3.28) and Christ-Kiselev’s Lemma that (3.18) holds. Exchanging the roles of \( f \) and \( \psi \), we immediately have (3.19). By Lemmas 3.2, 3.4 and Proposition 3.7 we have (3.20). \( \square \)

Corollary 3.11  
Let \( 4/n \leq p < \infty, 2 \leq q < \infty, q > 4/n \). We have

\[ \| D^{1/2}_{x_1} k S(t) u_0 \|_{L^\infty_t L^2_{x_1} \ldots L^2_{x_n} L^1_t(\mathbb{R}^{1+n})} \lesssim \| k u_0 \|_{L^2(\mathbb{R}^n)}, \] 
(3.22)

\[ \| k S(t) u_0 \|_{L^q_t L^\infty_{x_1} \ldots L^\infty_{x_n} L^2(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^{1/q} \| k u_0 \|_{L^2(\mathbb{R}^n)}, \] 
(3.23)

\[ \| k S(t) u_0 \|_{L^{2+p}_{t,x} \cap L^\infty_{t,x} L^2(\mathbb{R}^{1+n})} \lesssim \| k u_0 \|_{L^2(\mathbb{R}^n)}, \] 
(3.24)

By duality, it follows from (3.28) and Christ-Kiselev’s Lemma that (3.18) holds. Exchanging the roles of \( f \) and \( \psi \), we immediately have (3.19). By Lemmas 3.2, 3.4 and Proposition 3.7 we have (3.20). \( \square \)

4  
Linear estimates with derivative interaction

In view of (3.25) in Corollary 3.11 the operator \( \mathcal{A} \) in the space \( L^\infty_t L^2_{x_1} \ldots L^2_{x_n} L^1_t(\mathbb{R}^{1+n}) \) has succeed in absorbing the partial derivative \( \partial_{x_1} \). However, it seem that \( \mathcal{A} \) can not deal with the partial derivative \( \partial_{x_2} \) in the space \( L^\infty_t L^2_{x_1} \ldots L^2_{x_n} L^1_t(\mathbb{R}^{1+n}) \). So, we need a new way to handle the interaction between \( L^\infty_t L^2_{x_1} \ldots L^2_{x_n} L^1_t(\mathbb{R}^{1+n}) \) and \( \partial_{x_2} \). We have the following
Proposition 4.1 Let $i = 2, ..., n$, $2 \leq q \leq \infty$, $q > 4/n$. Let $2 \leq r < \infty$, $2/\gamma(r) = n(1/2 - 1/r)$, $\gamma > 2 \lor \gamma(r)$. Then we have

\begin{align}
\| \Box_k \partial_x \mathcal{A} f \|_{L^q_x L^{1/2}_{t,x_2, ..., x_n} L^2_t(\mathbb{R}^{1+n})} & \lesssim \| \partial_{x_1} \partial_{x_1}^{-1} \Box_k f \|_{L^1_x L^2_{t,x_2, ..., x_n} L^2_t(\mathbb{R}^{1+n})}, \\
\| \Box_k \partial_x \mathcal{A} f \|_{L^q_x L^{1/2}_{t,x_2, ..., x_n} L^2_t(\mathbb{R}^{1+n})} & \lesssim \| \partial_x D^{-1/2}_{x_1} \Box_k f \|_{L^r_t L^\gamma'_t(\mathbb{R}^{1+n})}, \\
\| \Box_k \partial_x \mathcal{A} f \|_{L^q_x L^{1/2}_{t,x_2, ..., x_n} L^2_t(\mathbb{R}^{1+n})} & \lesssim \langle k_i \rangle^{1/2} \langle k_1 \rangle^{1/q} \| \Box_k f \|_{L^1_x L^2_{t,x_2, ..., x_n} L^2_t(\mathbb{R}^{1+n})}, \\
\| \Box_k \partial_x \mathcal{A} f \|_{L^q_x L^{1/2}_{t,x_2, ..., x_n} L^2_t(\mathbb{R}^{1+n})} & \lesssim \langle k_i \rangle \langle k_1 \rangle^{1/q} \| \Box_k f \|_{L^r_t L^\gamma'_t(\mathbb{R}^{1+n})}.
\end{align}

Proof. \((4.1)\) is a straightforward consequence of Proposition 2.1. We have

\[
\mathcal{L}(\partial_x f, \psi) := \left| \int_{\mathbb{R}} \left( \int_{\mathbb{R}} S(t - \tau) \partial_x f(\tau) d\tau, \psi(t) \right) dt \right| 
\leq \left\| \int_{\mathbb{R}} S(-\tau) \partial_x D^{-1/2}_{x_1} f(\tau) d\tau \right\|_{L^2_x} \left\| \int_{\mathbb{R}} S(-t) \psi(t) dt \right\|_{L^2_t(\mathbb{R}^n)}.
\]

By the Strichartz inequality and Proposition 2.3

\[
\mathcal{L}(\partial_x f, \psi) \lesssim \| \partial_x D^{-1/2}_{x_1} f \|_{L^r_t L^\gamma'_t(\mathbb{R}^{1+n})} \| \psi \|_{L^1_x L^2_{t,x_2, ..., x_n} L^2_t(\mathbb{R}^{1+n})}.
\]

By duality, \((4.6)\) implies \((4.2)\). Similarly, in view of Propositions 2.3, 3.7 and Lemma 3.4

\[
\mathcal{L}(\partial_x \Box_k f, \psi) \leq \left\| \int_{\mathbb{R}} S(-\tau) D^1_{x_2} \Box_k f(\tau) d\tau \right\|_{L^2_x} \left\| \Box_k \int_{\mathbb{R}} S(-t) \psi(t) dt \right\|_{L^2(t(\mathbb{R}^n))} 
\lesssim \langle k_2 \rangle^{1/2} \| \Box_k f \|_{L^1_x L^2_{t,x_2, ..., x_n} L^2_t(\mathbb{R}^{1+n})} \langle k_1 \rangle^{1/q} \| \Box_k \psi \|_{L^r_t L^\gamma'_t(\mathbb{R}^{1+n})} 
\lesssim \langle k_2 \rangle^{1/2} \langle k_1 \rangle^{1/q} \| \Box_k f \|_{L^1_x L^2_{t,x_2, ..., x_n} L^2_t(\mathbb{R}^{1+n})} \| \psi \|_{L^r_t L^\gamma'_t(\mathbb{R}^{1+n})}.
\]

By duality, \((4.3)\) follows from \((4.7)\). Finally,

\[
\mathcal{L}(\partial_{x_2} \Box_k f, \psi) \leq \left\| \int_{\mathbb{R}} S(-\tau) \partial_{x_2} \Box_k f(\tau) d\tau \right\|_{L^2_x} \left\| \Box_k \int_{\mathbb{R}} S(-t) \psi(t) dt \right\|_{L^2(t(\mathbb{R}^n))} 
\lesssim \langle k_2 \rangle \langle k_1 \rangle^{1/q} \| \psi \|_{L^r_t L^\gamma'_t(\mathbb{R}^{1+n})} \| \Box_k f \|_{L^r_t L^\gamma'_t(\mathbb{R}^{1+n})},
\]

which implies \((4.4)\), as desired. \(\square\)

Lemma 4.2 Let $\psi : [0, \infty) \to [0, 1]$ be a smooth bump function satisfying $\psi(x) = 1$ as $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 2$. Denote $\psi(x) = \psi(\xi_2/2\xi_1)$, $\psi_2(x) = 1 - \psi(\xi_2/2\xi_1)$, $\xi \in \mathbb{R}^n$. Then we have for $\sigma \geq 0$,

\[
\sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^\sigma \| \mathcal{F}^{-1}_{\xi_1, \xi_2} \psi_1 \mathcal{F}_{\xi_2, \xi_2} \partial_{x_2} \mathcal{A} f \|_{L^q_x L^{1/2}_{t,x_2, ..., x_n} L^2_t(\mathbb{R}^{1+n})}
\]
and for $\sigma \geq 1$,

$$\sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k_1 \rangle^\sigma \| \Box_k f \|_{L^2_{x_1}L^2_{x_2},...,L^2_n(\mathbb{R}^{1+n})}, \quad (4.9)$$

$$\sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k_1 \rangle^\sigma \| \mathcal{F}^{-1}_{\xi_1,\xi_2} \mathcal{F}_{x_1, x_2} \Box_k \partial_{x_2} \mathcal{A} f \|_{L^\infty_{x_1}L^2_{x_2},...,L^2_n(\mathbb{R}^{1+n})} \quad (4.10)$$

Proof. For simplicity, we denote

$$I = \| \mathcal{F}^{-1}_{\xi_1,\xi_2} \psi_1 \mathcal{F}_{x_1, x_2} \Box_k \partial_{x_2} \mathcal{A} f \|_{L^\infty_{x_1}L^2_{x_2},...,L^2_n(\mathbb{R}^{1+n})},$$

$$II = \| \mathcal{F}^{-1}_{\xi_1,\xi_2} \psi_2 \mathcal{F}_{x_1, x_2} \Box_k \partial_{x_2} \mathcal{A} f \|_{L^\infty_{x_1}L^2_{x_2},...,L^2_n(\mathbb{R}^{1+n})}.$$ Let $\eta_\ell$ be as in Lemma $3.1$. For $k \in \mathbb{Z}^n, |k| > 4$, applying the almost orthogonality of $\Box_k$, we have

$$I \lesssim \sum_{|\ell_1|,|\ell_2| \leq 1} \left\| \mathcal{F}^{-1}_{\xi_1,\xi_2} \psi_1 \left( \frac{\xi_2}{2\xi_1} \right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \mathcal{F}_{x_1, x_2} \Box_k \partial_{x_1} \mathcal{A} f \right\|_{L^\infty_{x_1}L^2_{x_2},...,L^2_n(\mathbb{R}^{1+n})}.$$ \quad (4.11)

Denote

$$(f \otimes_{12} g)(x) = \int_{\mathbb{R}^2} f(t, x_1 - y_1, x_2 - y_2, x_3, ..., x_n)g(t, y_1, y_2)dy_1dy_2. \quad (4.12)$$

We have for any Banach function space $X$ defined on $\mathbb{R}^{1+n}$,

$$\| f \otimes_{12} g \|_X \leq \| g \|_{L^1_{y_1,y_2}(\mathbb{R}^2)} \sup_{y_1,y_2} \| f(\cdot, \cdot - y_1, \cdot - y_2, \cdot, ..., \cdot) \|_X. \quad (4.13)$$

Hence, by (4.11) and (4.13),

$$I \lesssim \sum_{|\ell_1|,|\ell_2| \leq 1} \left\| \mathcal{F}^{-1}_{\xi_1,\xi_2} \psi_1 \left( \frac{\xi_2}{2\xi_1} \right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \right\|_{L^1(\mathbb{R}^2)} \| \Box_k \partial_{x_1} \mathcal{A} f \|_{L^\infty_{x_1}L^2_{x_2},...,L^2_n(\mathbb{R}^{1+n})} \quad (4.14)$$

Using Bernstein’s multiplier estimate, for $|k| > 4$, we have

$$\left\| \mathcal{F}^{-1}_{\xi_1,\xi_2} \psi_1 \left( \frac{\xi_2}{2\xi_1} \right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \right\|_{L^1(\mathbb{R}^2)}$$

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By Proposition 3.8 (4.14) and (4.15), we have

\[ I \lesssim \| \Box_k f \|_{L^1_x L^2_t L^2_{r,n}} L^2(R^{1+n}), \quad |k_1| \geq 4. \]  

(4.16)

Next, we consider the estimate of \( II \). Using Proposition 4.1

\[ II \lesssim \| \mathcal{F}_{\xi_1,\xi_2}^{-1} (\xi_2/\xi_1) \psi_2 \mathcal{F}_{x_1,x_2} \Box_k f \|_{L^1_x L^2_t L^2_{r,n}} L^2(R^{1+n}) \]

\[ \lesssim \sum_{|k_1|,|k_2| \leq 1} \left\| \mathcal{F}_{\xi_1,\xi_2}^{-1} \left( 1 - \psi \left( \frac{\xi_2}{2\xi_1} \right) \right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \right\|_{L^1(R^d)} \]

\[ \times \| \Box_k f \|_{L^1_x L^2_t L^2_{r,n}} L^2(R^{1+n}) \cdot \]

(4.17)

Notice that \( \text{supp} \psi_2 \subset \{ \xi : |\xi_2| \geq 2|\xi_1| \} \). If \( |k_1| \geq 4 \), we have \( |k_2| > 6 \) and \( |k_2| \geq |k_1| \) in the summation of the left-hand side of (4.10). So, \( \sum_{k \in \mathbb{Z}^n, |k_1| \geq 4} \langle k_1 \rangle^\sigma II \leq \sum_{k \in \mathbb{Z}^n, |k_1| \geq 4} \langle k_2 \rangle^\sigma \langle k_1 \rangle^{-1} \).

\[ \boxed{\langle k_2 \rangle \langle k_1 \rangle^{-1}.} \]

(4.18)

(4.17) and (4.18) yield the estimate of \( II \), as desired.

\[ \square \]

**Conjecture 4.3** Using a similar way as in the proof of (4.11), we can show that

\[ \| k |D_k f(x,t) \|_{L^2(R^d)} \lesssim \| D_{x_2}^{1/2} D_{x_1}^{-1/2} k f \|_{L^1_x L^2_t L^2_{r,n}} L^2(R^{1+n}) \cdot \]

So, we can conjecture that

\[ \| k |D_k f(x,t) \|_{L^2(R^d)} \lesssim \| D_{x_2}^{1/2} D_{x_1}^{-1/2} k f \|_{L^1_x L^2_t L^2_{r,n}} L^2(R^{1+n}) \cdot \]

(4.19)

Since we do not know if the Christ-Kiselev Lemma holds in the endpoint case, it is not clear for us if (4.19) is true.

If (4.19) is true, repeating the proof above, we can show that (4.9) holds for all \( \sigma \geq 1/2 \). We can improve the results of Theorems 1.1 and 1.2 by assuming that \( u_0 \in M^{1+1/m}_{2,1} \) and \( u_0 \in M^{1/m}_{2,1} \), respectively.
Lemma 4.4 Let $2 \leq q \leq \infty$, $q > 4/n$ and $(\gamma, r)$ be as in Proposition 4.1. Let $k = (k_1, \ldots, k_n)$, $k_{\text{max}} := \max_{1 \leq i \leq n} |k_i|$. Then we have
\begin{equation}
\| \Box_k \partial_{x_i} \mathcal{A} f \|_{L^q_{1 \times 1} L^\infty_{x_2, \ldots, x_n} L^\infty_{t} (\mathbb{R}^{1+n})} \lesssim \langle k \rangle^{1+1/q} \| \Box_k f \|_{L^q_{1} L^\infty_{x} L^\infty_{t} (\mathbb{R}^{1+n})}. \tag{4.20}
\end{equation}

Proof. It follows from (4.1) that (4.20) holds.\hfill \square

Lemma 4.5 Let $k = (k_1, \ldots, k_n)$, $k_{\text{max}} := \max_{1 \leq i \leq n} |k_i|$ and $q$ be as in Lemma 4.4. Then we have for $\sigma \geq 0$ and $i, \alpha = 1, \ldots, n$,
\begin{equation}
\sum_{k \in \mathbb{Z}^n, |k_\alpha| = k_{\text{max}} > 4} \langle k \rangle^{\sigma} \| \Box_k \partial_{x_i} \mathcal{A} f \|_{L^q_{1 \times 1} L^\infty_{x_2, \ldots, x_n} L^\infty_{t} (\mathbb{R}^{1+n})} \lesssim \sum_{k \in \mathbb{Z}^n, |k_\alpha| > 4} \langle k \rangle^{\sigma+1/2+1/q} \| \Box_k f \|_{L^q_{1 \times 1} L^2_{x_2, \ldots, x_n} L^2_{t} (\mathbb{R}^{1+n})}. \tag{4.21}
\end{equation}

Proof. First, we consider the case $\alpha = 1$. In view of (3.26) and $|k_1| = k_{\text{max}} > 4$,
\begin{equation}
\| \Box_k \partial_{x_i} \mathcal{A} f \|_{L^q_{1 \times 1} L^\infty_{x_2, \ldots, x_n} L^\infty_{t} (\mathbb{R}^{1+n})} \lesssim \sum_{|\ell_1|, |\ell_2| \leq 1} \| \mathcal{F}^{-1} \left( \frac{\xi}{\xi_1} \eta_{k_1+\ell_1}(\xi) \eta_{k_1+\ell_1}(\xi_1) \right) \|_{L^1(\mathbb{R}^2)}
\times \| \Box_k \partial_{x_i} \mathcal{A} f \|_{L^q_{1 \times 1} L^\infty_{x_2, \ldots, x_n} L^\infty_{t} (\mathbb{R}^{1+n})} \lesssim \langle k_1 \rangle^{-1} \langle k \rangle^{1/2+1/q} \| \Box_k f \|_{L^q_{1} L^2_{2} \ldots L^2_{n} L^2_{t} (\mathbb{R}^{1+n})} \lesssim \langle k \rangle^{1/2+1/q} \| \Box_k f \|_{L^q_{1} L^2_{2} \ldots L^2_{n} L^2_{t} (\mathbb{R}^{1+n})}. \tag{4.22}
\end{equation}

(4.22) implies the result, as desired. Next, we consider the case $\alpha = 2$. Notice that $|k_2| = \max_{1 \leq i \leq n} |k_i| > 4$. By (4.3),
\begin{equation}
\| \Box_k \partial_{x_i} \mathcal{A} f \|_{L^q_{1 \times 1} L^\infty_{x_2, \ldots, x_n} L^\infty_{t} (\mathbb{R}^{1+n})} \lesssim \sum_{|\ell_2|, |\ell_1| \leq 1} \| \mathcal{F}^{-1} \left( \frac{\xi}{\xi_2} \eta_{k_2+\ell_2}(\xi) \eta_{k_2+\ell_2}(\xi_2) \right) \|_{L^1(\mathbb{R}^2)}
\times \| \Box_k \partial_{x_i} \mathcal{A} f \|_{L^q_{1 \times 1} L^\infty_{x_2, \ldots, x_n} L^\infty_{t} (\mathbb{R}^{1+n})} \lesssim \langle k_2 \rangle^{-1} \langle k \rangle^{1/2+1/q} \| \Box_k f \|_{L^q_{1} L^2_{2} \ldots L^2_{n} L^2_{t} (\mathbb{R}^{1+n})} \lesssim \langle k \rangle^{1/2+1/q} \| \Box_k f \|_{L^q_{1} L^2_{2} \ldots L^2_{n} L^2_{t} (\mathbb{R}^{1+n})}. \tag{4.23}
\end{equation}
The other cases $\alpha = 3, \ldots, n$ is analogous to the case $\alpha = 2$ and we omit the details of the proof.\hfill \square

Remark 4.6 From the proof of Lemma 4.5, we easily see that
\begin{equation}
\sum_{k \in \mathbb{Z}^n, |k_\alpha| = k_{\text{max}} > 4} \langle k \rangle^{\sigma} \| \Box_k \partial_{x_i} \mathcal{A} f \|_{L^q_{1 \times 1} L^\infty_{x_2, \ldots, x_n} L^\infty_{t} (\mathbb{R}^{1+n})} \lesssim \sum_{k \in \mathbb{Z}^n, |k_\alpha| > 4} \langle k \rangle^{\sigma+1/2+1/q} \| \Box_k f \|_{L^q_{1 \times 1} L^2_{x_j, \ldots, x_n} L^2_{t} (\mathbb{R}^{1+n})}. \tag{4.24}
\end{equation}
5 Proof of Theorem 1.2

Now we briefly indicate the proof of Theorem 1.2. We assume that the nonlinear term takes the form

$$F(u, \nabla u) = \partial_{x_1}(u^{\kappa_1+1}) + \partial_{x_2}(u^{\kappa_2+1}).$$

In order to handle the nonlinear term $\partial_{x_i}(u^{\kappa_i+1})$, we use the space $L_x^\infty L^2_{(x_j)_{j \neq i}} L^2_t(\mathbb{R}^{1+n})$ to absorb the derivative $\partial_{x_i}$. Hence, we introduce the following semi-norms to treat the nonlinearity:

$$\|u\|_{Y_i} = \sum_{k \in \mathbb{Z}^n, |k_i| > 4} \langle k \rangle \|\Box_k u\|_{L_x^\infty L^2_{(x_j)_{j \neq i}} L^2_t(\mathbb{R}^{1+n})}, \quad i = 1, 2.$$

Since (3.12) is a worse estimate in the case $|k_i| \lesssim 1$, we throw away the low frequency part in the $\xi_i$-direction in the definition of $\|u\|_{Y_i}$. To handle the low frequency part, we use the Strichartz norm:

$$\|u\|_S = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \|\Box_k u\|_{L_x^\infty L^{2+\kappa}_x L^2_t(\mathbb{R}^{1+n})}.$$

We emphasize that the Strichartz inequalities (3.24) and (3.30) are better estimates than the smooth effects in (3.22) and (3.25) for the low frequency part, respectively.

Using the integral equation

$$u(t) = S(t)u_0 - i\mathcal{A}(\partial_{x_1}u^{\kappa_1+1} + \partial_{x_2}u^{\kappa_2+1}),$$

we have

$$\|u\|_{Y_1} \leq \|S(t)u_0\|_{Y_1} + \|\mathcal{A}(\partial_{x_1}u^{\kappa_1+1})\|_{Y_1} + \|\mathcal{A}(\partial_{x_2}u^{\kappa_2+1})\|_{Y_1}.$$

In view of (3.22), $\|S(t)u_0\|_{Y_1}$ is bounded by $\|u_0\|_{M^{1/2}}$. $\|\mathcal{A}(\partial_{x_1}u^{\kappa_1+1})\|_{Y_1}$ can be handled by using the linear estimates obtained in Section 3. Noticing that

$$u^{\kappa_1+1} = \left( \sum_{k^{(1)} \in \mathbb{Z}^n} + \sum_{k^{(1)} \in \mathbb{Z}^n \setminus \mathcal{S}_1} \right) \Box_{k^{(1)}} u^{\kappa_1+1},$$

where $\mathcal{S}_1 = \{k^{(1)} \in \mathbb{Z}^n : |k^{(1)}_1| \vee \ldots \vee |k^{(1)}_{\kappa_1+1}| > 4\}$, (3.25) and (3.28) in Corollary 3.11 yield,

$$\|\mathcal{A}(\partial_{x_1}u^{\kappa_1+1})\|_{Y_1} \lesssim \sum_{k \in \mathbb{Z}^n, |k_i| > 4} \langle k \rangle \sum_{\mathcal{S}_1} \|\Box_k \Box_{k^{(1)}} u^{\kappa_1+1} \|_{L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n}}.$$
By performing a nonlinear mapping estimate, we have

\[ \| \mathcal{A}(\partial_{x_i} u^{\kappa_i+1}) \|_{Y_1} \lesssim \| u \|_{Y_1} \| u \|_{Z_i}^{\kappa_i} + \| u \|_{S}^{\kappa_i+1}, \quad (5.2) \]

where

\[ \| u \|_{Z_i} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2 - 1/\kappa} \| \Box_k u \|_{L_t^\infty L_x^{\infty}(\mathbb{R}^{1+n})}, \quad i = 1, 2. \]

Unfortunately, \( \| \mathcal{A}(\partial_{x_2} u^{\kappa_2+1}) \|_{Y_1} \) contains the interaction between the working space \( L_\infty x_1 L_2^{2} x_2, \ldots, x_n L_2^{2}(\mathbb{R}^{1+n}) \) and the derivative \( \partial_{x_2} \), which is out of the control of the smooth effect (3.25). So, we look for another way to estimate \( \| \mathcal{A}(\partial_{x_2} u^{\kappa_2+1}) \|_{Y_1} \). Roughly speaking, our idea is to use the following estimates (see Lemma 4.2):

\[ \| \mathcal{F}^{-1} \chi_E(\xi; |\xi_2| \leq |\xi_1|) \mathcal{A}(\partial_{x_2} f) \|_{Y_1} \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle \| \Box_k f \|_{L_t^1 L_x^1 L_2^{2} x_2, \ldots, x_n L_2^{2}(\mathbb{R}^{1+n})}, \]

\[ \| \mathcal{F}^{-1} \chi_E(\xi; |\xi_1| \leq |\xi_2|) \mathcal{A}(\partial_{x_2} f) \|_{Y_1} \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle \| \Box_k f \|_{L_t^1 L_x^1 L_2^{2} x_2, \ldots, x_n L_2^{2}(\mathbb{R}^{1+n})}, \]

where \( \chi_E \) denotes the characteristic function on the set \( E \). So, \( \| \mathcal{A}(\partial_{x_2} u^{\kappa_2+1}) \|_{Y_1} \) has similar bound to \( \| \mathcal{A}(\partial_{x_i} u^{\kappa_i+1}) \|_{Y_1} \) as in (5.1). Eventually, we have

\[ \| \mathcal{A}(\partial_{x_2} u^{\kappa_2+1}) \|_{Y_1} \lesssim (\| u \|_{Y_1} + \| u \|_{Z_1}) \| u \|_{Z_1}^{\kappa_2} + \| u \|_{S}^{\kappa_2+1}. \quad (5.3) \]

By using the integral equation, we need to further bound \( \| \mathcal{A} \partial_{x_i} u^{\kappa_i+1} \|_{Z_1 \cap S}, i = 1, 2 \). For instance, for the estimate of \( \| \mathcal{A} \partial_{x_2} u^{\kappa_2+1} \|_{Z_1} \), we resort to the above idea and consider the following interaction estimate:

\[ \| \Box_k \partial_{x_2} \mathcal{A} f \|_{L_\infty x_1 L_2^{2} x_2, \ldots, x_n L_\infty^{2}(\mathbb{R}^{1+n})} \lesssim \langle k \rangle^{1/2} \langle k \rangle^{1/\kappa} \Box_k f \|_{L_t^1 L_x^1 L_2^{2} x_2, \ldots, x_n L_2^{2}(\mathbb{R}^{1+n})}, \]

which leads to that we can bound \( \| \mathcal{A} \partial_{x_2} u^{\kappa_2+1} \|_{Z_1} \) by an analogous version of the right-hand side of (5.1), so, by (5.2) (see Lemma 4.3).

Finally, using (3.27) and (3.30), we can get the same estimate of \( \| \mathcal{A} \partial_{x_i} u^{\kappa_i+1} \|_{S} \) as in (5.2) and (5.3), respectively.

**Proof of Theorem 1.2.** We now give the details of the proof of Theorem 1.2. Denote

\[ \rho_1(u) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n, |k_i| > 4} \langle k_i \rangle \| \Box_k u \|_{L_t^\infty L_x^2(x_j) \neq k_i L_2^{2}(\mathbb{R}^{1+n})}, \]

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\[
\rho_2(u) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{2-1/\kappa} \| \Box_k u \|_{L^{\kappa}_t L^{\infty}_x (\mathbb{R}^{1+n})},
\]
\[
\rho_3(u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \| \Box_k u \|_{L^{\kappa}_t L^2_x \cap L^{2+\kappa}_t (\mathbb{R}^{1+n})}.
\]

Put
\[
X := \left\{ u \in \mathcal{H}'(\mathbb{R}^{1+n}) : \| u \|_X := \sum_{i=1}^{3} \rho_i(u) \leq \delta_0 \right\}.
\]

We consider the following mapping:
\[
\mathcal{T} : u(t) \rightarrow S(t)u_0 - i \mathcal{A} \left( \sum_{i=1}^{n} \lambda_i \partial_{x_i} u^{\kappa_i+1} \right).
\]

For convenience, we denote
\[
\| u \|_{Y_i} = \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \| \Box_k u \|_{L^{\kappa}_t L^{2}_x (\mathbb{R}^{1+n})}.
\]

In order to estimate \( \rho_1(u) \), it suffices to control \( \| \cdot \|_{Y_1} \). By (2.8) and Plancherel’s identity, we have
\[
\| S(t)u_0 \|_{Y_1} \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k_1 \rangle \| \Box_k D_{x_1}^{-1/2} u_0 \|_{L^2(\mathbb{R}^n)}
\]
\[
\lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \| \Box_k u_0 \|_{L^2(\mathbb{R}^n)}.
\]

By (3.3), Lemma 3.2 we have
\[
\rho_i(S(t)u_0) \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \| \Box_k u_0 \|_{L^2(\mathbb{R}^n)}, \quad i = 2, 3.
\]

Denote
\[
S_{i,1}^{(i)} := \{ (k^{(1)}, \ldots, k^{(\kappa_i+1)}) \in (\mathbb{Z}^n)^n : |k_i^{(1)}| \vee \ldots \vee |k_i^{(\kappa_i+1)}| > 4 \},
\]
\[
S_{i,2}^{(i)} := \{ (k^{(1)}, \ldots, k^{(\kappa_i+1)}) \in (\mathbb{Z}^n)^n : |k_i^{(1)}| \vee \ldots \vee |k_i^{(\kappa_i+1)}| \leq 4 \}.
\]

Using the frequency-uniform decomposition, we have
\[
u^{\kappa_i+1} = \sum_{k^{(1)}, \ldots, k^{(\kappa_i+1)} \in \mathbb{Z}^n} \Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_i+1)}} u
\]
\[
= \sum_{S_{i,1}^{(i)}} \Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_i+1)}} u + \sum_{S_{i,2}^{(i)}} \Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_i+1)}} u. \quad (5.4)
\]

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Using (3.25) and (3.28), we obtain that
\[
\|\mathcal{A}_t u^{n+1}\|_{Y_1} \lesssim \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \sum_{g^{(1)}_{1_1}} \|\Box_k (\Box_k(u) \ldots \Box_k(u))\|_{L^1_{t_x} L^2_{x_2} \ldots \ldots L^2_{x_n} (\mathbb{R}^{1+n})} \nonumber \\
+ \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^{3/2} \sum_{g^{(1)}_{1_2}} \|\Box_k (\Box_k(u) \ldots \Box_k(u))\|_{L^{(2+n)/(1+n)}_{t,x} (\mathbb{R}^{1+n})} 
\]
\[:= I + II. \quad (5.5)\]

In view of the support property of \(\Box_k u\), we see that
\[
\Box_k (\Box_k(u) \ldots \Box_k(u)) = 0, \text{ if } |k - k^{(1)} - \ldots - k^{(n+1)}| \geq C. \quad (5.6)
\]

Hence, by Lemma 3.4, we have
\[
I \lesssim \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \sum_{g^{(1)}_{1_1}} \|\Box_k (\Box_k(u) \ldots \Box_k(u))\|_{L^1_{t_x} L^2_{x_2} \ldots \ldots L^2_{x_n} (\mathbb{R}^{1+n})} \nonumber \\
\leq \|\Box_k (u)\|_{L^1_{t_x} L^2_{x_2} \ldots \ldots L^2_{x_n} (\mathbb{R}^{1+n})} \prod_{i=2}^{n+1} \|\Box_k (u)\|_{L^2_{t_x} L^2_{x_2} \ldots \ldots L^2_{x_n} (\mathbb{R}^{1+n})} \cap L^\infty (\mathbb{R}^{1+n}).
\]

Since \(|k - k^{(1)} - \ldots - k^{(n+1)}| \leq C\) implies that \(|k^{(1)} - \ldots - k^{(n+1)}| \leq C\), we see that \(|k_1| \leq C \max_{i=1, \ldots, n+1} |k^{(i)}_1|\). We may assume that \(|k^{(i)}_1| = \max_{i=1, \ldots, n+1} |k^{(i)}_1|\) in the summation \(\sum_{g^{(1)}_{1_1}}\) in (5.7) above. So,
\[
I \lesssim \sum_{k^{(1)} \in \mathbb{Z}^n, |k^{(1)}| > 4} \langle k^{(1)} \rangle \|\Box_k (u)\|_{L^1_{t_x} L^2_{x_2} \ldots \ldots L^2_{x_n} (\mathbb{R}^{1+n})} \nonumber \\
\times \sum_{k^{(2)}, \ldots, k^{(n+1)} \in \mathbb{Z}^n} \prod_{i=2}^{n+1} \|\Box_k (u)\|_{L^2_{t_x} L^2_{x_2} \ldots \ldots L^2_{x_n} (\mathbb{R}^{1+n})} \cap L^\infty (\mathbb{R}^{1+n}) 
\lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^{n+1}. \quad (5.8)
\]

In view of (5.6) we easily see that \(|k_1| \leq C\) in II of (5.5). Hence,
\[
II \lesssim \sum_{k \in \mathbb{Z}^n, |k| > 4} \sum_{g^{(1)}_{1_2}} \|\Box_k (u) \ldots \Box_k(u)\|_{L^{(2+n)/(1+n)}_{t,x} (\mathbb{R}^{1+n})} \chi_{|k - k^{(1)} - \ldots - k^{(n+1)}| \leq C}
\]

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Hence, we have

\[
\|\mathcal{S}_x u^{k_1+1}\|_{Y_1} \lesssim \rho_1(u) (\rho_2(u) + \rho_3(u))^{k_1} + \rho_3(u)^{1+k_1}.
\]  
(5.10)

Next, we estimate \(\|\mathcal{A} \partial_x u^{k_2+1}\|_{Y_1}\). Let \(\psi\) be as in Lemma 4.2. For convenience, we write

\[
P_i = \mathcal{F}^{-1}_{\xi_1, \xi_2} \psi_i, \mathcal{F}_{x_1, x_2}, \quad i = 1, 2.
\]  
(5.11)

We have

\[
\|\mathcal{A} \partial_x u^{k_2+1}\|_{Y_1} \lesssim \left\| P_i \partial_x \mathcal{A} u^{k_2+1}\right\|_{Y_1} + \left\| P_2 \partial_x \mathcal{A} u^{k_2+1}\right\|_{Y_1} := III + IV.
\]  
(5.12)

Using the decomposition (5.4),

\[
III \leq \left\| P_1 \partial_x \mathcal{A} \sum_{S_{2,1}} (\boxtimes_{1}^{(k_1)} u \ldots \boxtimes_{1}^{(k_2+1)} u) \right\|_{Y_1}
+ \left\| P_1 \partial_x \mathcal{A} \sum_{S_{2,2}} (\boxtimes_{1}^{(k_1)} u \ldots \boxtimes_{1}^{(k_2+1)} u) \right\|_{Y_1} := III_1 + III_2.
\]  
(5.13)

Applying Lemma 4.2 and then following the same way as in the estimate to (5.7),

\[
III_1 \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4, |k_2| \leq |k_1|} \left\| \boxtimes_k (\boxtimes_{1}^{(k_1)} u \ldots \boxtimes_{1}^{(k_2+1)} u) \right\|_{L^1_t L^2_x L^2_{x_2} \ldots L^2_{x_n} L^2_z (\mathbb{R}^{1+n})}
\lesssim \rho_1(u) (\rho_2(u) + \rho_3(u))^{k_2}.
\]  
(5.14)

For the estimate of \(III_2\), noticing the fact that \(\text{supp} \psi_1 \subset \{\xi : |\xi_2| \leq 4|\xi_1|\}\) and using the multiplier estimate, then applying (4.2), we have

\[
III_2 \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4, |k_2| \leq |k_1|} \left\| \boxtimes_k (\boxtimes_{1}^{(k_1)} u \ldots \boxtimes_{1}^{(k_2+1)} u) \right\|_{L^1_t L^2_x (\mathbb{R}^{1+n})}
\lesssim \rho_3(u)^{1+k_2}.
\]  
(5.15)

We need to further control \(IV\). Using the decomposition (5.4),

\[
IV \leq \left\| P_2 \partial_x \mathcal{A} \sum_{S_{2,1}} (\boxtimes_{1}^{(k_1)} u \ldots \boxtimes_{1}^{(k_2+1)} u) \right\|_{Y_1}
\]
\[ + \left\| P_2 \partial_{x_2} \mathcal{A} \sum_{\mathcal{S}_2} (\Box_k(1) u \ldots \Box_k(\kappa_2+1) u) \right\|_{y_1} := IV_1 + IV_2. \quad (5.16) \]

By Lemma 4.2,
\[ IV_1 \lesssim \sum_{k \in \mathbb{Z}^n, \ |k| > 4} \langle k \rangle \sum_{\mathcal{S}_2} \left\| \Box_k (\Box_k(1) u \ldots \Box_k(\kappa_2+1) u) \right\|_{L_{x_1}^1 L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})}. \quad (5.17) \]

By symmetry of \( k^{(1)}, \ldots, k^{(\kappa_2+1)} \), we can assume that \( |k_2^{(1)}| = \max_{1 \leq i \leq \kappa_2+1} |k_2^{(i)}| \) in \( \mathcal{S}_2 \). Using the same way as in the estimate of \( I \), we have
\[ IV_1 \lesssim \sum_{\mathcal{S}_2, \ |k_2^{(1)}| > 4} \langle k_2^{(1)} \rangle \left\| \Box_k(1) u \ldots \Box_k(\kappa_2+1) u \right\|_{L_{x_1}^1 L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})}. \quad (5.18) \]

By Hölder’s inequality,
\[
\left\| \Box_k(1) u \ldots \Box_k(\kappa_2+1) u \right\|_{L_{x_1}^1 L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})} \\
\lesssim \left\| \Box_k(1) u \right\|_{L_x^\infty L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})}^{1/2} \left\| \Box_k(2) u \right\|_{L_x^\infty L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})}^{1/2} \\
\times \left\| \Box_k(1) u \ldots \Box_k(\kappa_2+1) u \right\|_{L_{x_1}^1 L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})} \\
\lesssim \left\| \Box_k(1) u \right\|_{L_x^\infty L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})} \prod_{i=2}^{\kappa_2+1} \left\| \Box_k(i) u \right\|_{L_{x_1}^1 L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})}. \quad (5.19) \]

In view of the inclusion \( L_{x_1}^\kappa L_{x_2}^\infty \ldots L_{x_n}^\infty \cap L_{x_1}^\infty L_{x_2}^\infty \ldots L_{x_n}^\infty \), we immediately have
\[ IV_1 \lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_2}. \quad (5.20) \]

Noticing the fact that \( \text{supp} \psi_2 \subset \{ \xi : |\xi_2| \geq 2|\xi_1| \} \) and applying (4.2), we have
\[ IV_2 \lesssim \sum_{k \in \mathbb{Z}^n, \ |k| > 4} \langle k_2 \rangle^{3/2} \sum_{\mathcal{S}_2} \left\| \Box_k (\Box_k(1) u \ldots \Box_k(\kappa_2+1) u) \right\|_{L_{t,x}^{(2+n)/(1+n)} (\mathbb{R}^{1+n})} \\
\lesssim \sum_{k \in \mathbb{Z}^n, \ |k| > 4} \sum_{\mathcal{S}_2} \left\| \Box_k (\Box_k(1) u \ldots \Box_k(\kappa_2+1) u) \right\|_{L_{t,x}^{(2+n)/(1+n)} (\mathbb{R}^{1+n})} \\
\lesssim \rho_3(u)^{1+\kappa_2}. \quad (5.21) \]

The other terms in \( \rho_1(\cdot) \) can be bounded in a similar way. So, we have shown that
\[ \rho_1 \left( \mathcal{A} \left( \sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1} \right) \right) \lesssim \sum_{i=1}^n \left( \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i} \right). \quad (5.22) \]
We estimate $\rho_2(\cdot)$. Denote
\[
\|u\|_{Z_i} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2-1/\kappa} \|\Box_k u\|_{L_t^\infty L_x^\infty(\mathbb{R}^{1+n})}, \tag{5.23}
\]
We have
\[
\rho_2 \left( \mathcal{A} \left( \sum_{j=1}^n \lambda_j \partial_{x_j} u^{\kappa_j+1} \right) \right) \leq \sum_{i=1}^n \left\| \mathcal{A} \left( \sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1} \right) \right\|_{Z_i} . \tag{5.24}
\]
Due to the symmetry of $Z_1, \ldots, Z_n$, it suffices to consider the estimate of $\| \cdot \|_{Z_1}$. Recall that $k_{\text{max}} = |k_1| \vee \ldots \vee |k_n|$. We have
\[
\|v\|_{Z_1} \leq \left( \sum_{k \in \mathbb{Z}^n, k_{\text{max}} > 4} + \sum_{k \in \mathbb{Z}^n, k_{\text{max}} \leq 4} \right) \langle k \rangle^{1/2-1/\kappa} \|\Box_k v\|_{L_t^\infty L_x^\infty(\mathbb{R}^{1+n})} \\
\quad := \Gamma_1(v) + \Gamma_2(v) . \tag{5.25}
\]
In view of Lemma 4.4 and Hölder’s inequality,
\[
\Gamma_2 \left( \mathcal{A} \left( \sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1} \right) \right) \leq \sum_{k \in \mathbb{Z}^n, k_{\text{max}} \leq 4} \left\| \Box_k \mathcal{A} \left( \sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1} \right) \right\|_{L_t^\infty L_x^\infty(\mathbb{R}^{1+n})} \\
\quad \lesssim \sum_{i=1}^n \sum_{k^{(1)}, \ldots, k^{(n)} \in \mathbb{Z}^n} \left\| \Box_k^{(1)} u \ldots \Box_k^{(n)} u \right\|_{L_t^2 L_x^\infty(\mathbb{R}^{1+n})} \\
\quad \lesssim \sum_{i=1}^n \sum_{k^{(1)}, \ldots, k^{(n)} \in \mathbb{Z}^n} \left\| \Box_k^{(1)} u \right\|_{L_t^2 L_x^\infty(\mathbb{R}^{1+n})} \cdots \left\| \Box_k^{(n)} u \right\|_{L_t^2 L_x^\infty(\mathbb{R}^{1+n})} \\
\quad \lesssim \sum_{i=1}^n \rho_3(u)^{\kappa_i+1} . \tag{5.26}
\]
It is easy to see that
\[
\Gamma_1(v) \leq \left( \sum_{k \in \mathbb{Z}^n, |k_1| = k_{\text{max}} > 4} + \ldots + \sum_{k \in \mathbb{Z}^n, |k_n| = k_{\text{max}} > 4} \right) \langle k \rangle^{1/2-1/\kappa} \|\Box_k v\|_{L_t^\infty L_x^\infty(\mathbb{R}^{1+n})} \\
\quad := \Gamma_1^1(v) + \ldots + \Gamma_1^n(v) . \tag{5.27}
\]
Using (5.4), Lemmas 4.4 and 4.5 we have
\[
\Gamma_1^1 \left( \mathcal{A} \left( \sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1} \right) \right) \]
Using the same way as in (5.8) and (5.9), one easily sees that
\[ \Gamma_2 \leq \left( \sum_{i=1}^{n} \lambda_i (\partial_x u^{\kappa_i+1}) \right) \lesssim \sum_{i=1}^{n} (\rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i}). \] (5.29)

We estimate \( \Gamma_1(\cdot) \). By Lemmas 4.4 and 4.5,
\[ \Gamma_1 \left( \mathcal{A} \left( \sum_{i=1}^{n} \lambda_i (\partial_x u^{\kappa_i+1}) \right) \right) \lesssim \sum_{i=1}^{n} (\rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i}). \] (5.31)

This reduces the same estimate as \( \Gamma_1(\cdot) \). We easily see that \( \Gamma_1(\cdot) \) for \( 3 \leq i \leq n \) can be controlled in a similar way as \( \Gamma_2(\cdot) \). Hence, we have shown that
\[ \left\| \mathcal{A} \left( \sum_{i=1}^{n} \lambda_i (\partial_x u^{\kappa_i+1}) \right) \right\| \lesssim \sum_{i=1}^{n} (\rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i}). \] (5.32)

For the estimates of \( \rho_3(\mathcal{A} \partial_x u^{\kappa_i+1}) \), we have from (3.17) and Lemma 3.4 that
\[ \left\| \Box_k \mathcal{A} \partial_x f \right\|_{L^2_t L^2_x} \lesssim \left\| \Box_k \partial_x f \right\|_{L^{2+\kappa_i+(1+\kappa_i)}(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle \left\| \Box f \right\|_{L^{2+\kappa_i+(1+\kappa_i)}(\mathbb{R}^{1+n})}. \] (5.33)

Hence, using (5.4), (3.30) and (3.27), we obtain that can be controlled by the right hand side of (5.28).
\[ \rho_3(\mathcal{A} \partial_x u^{\kappa_i+1}) \lesssim \sum_{k \in \mathbb{Z}^n, |k| \leq 4} \langle k \rangle^{3/2} \sum_{k(1),\ldots,k(n+1) \in \mathbb{Z}^n} \left\| \Box_k \partial_x f \right\|_{L^{2+\kappa_i+(1+\kappa_i)}(\mathbb{R}^{1+n})}. \]
\[ + \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \sum_{S^{(1)}_{1,1}} \| \Box_k (\Box_k(u) \ldots \Box_k(u)) \|_{L^2_t L^2_{x_1} \ldots L^2_{x_n} L^2_1}\]

\[ + \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^{3/2} \sum_{S^{(1)}_{1,2}} \| \Box_k (\Box_k(u) \ldots \Box_k(u)) \|_{L^{(2+\kappa)/(1+\kappa)}_t L^2_x} \]

(5.33)

By $(5.8)$ and $(5.9)$, we have

\[ \rho_3(A \partial_x u_{\kappa+1}) \lesssim \sum_{i=1}^n \left( \rho_1(u) (\rho_2(u) + \rho_3(u)^{\kappa}) + \rho_3(u)^{1+\kappa} \right). \]  

(5.34)

Hence, we have shown that

\[ \| T u \|_X \lesssim \| u_0 \|_{M^{1/2}_{x,1}} + \sum_{i=1}^n \| u \|^{1+\kappa_i}_X. \]  

(5.35)

Using a standard contraction mapping argument, we can finish the proof of Theorem 1.2. \hfill  \Box

6 Proof of Theorem 1.1

Roughly speaking, we will prove our Theorem 1.1 by following some ideas as in the proof of Theorem 1.2. However, due to the nonlinearity contains $u^{\kappa+1}$, and $(\nabla u)^\nu$ and $u^\kappa (\nabla u)^\nu$ as special cases, the proof of Theorem 1.2 cannot be directly applied. We construct the space $X$ as follows. Denote

\[ \varrho_1^{(i)}(u) = \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \| \Box_k u \|_{L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n} L^2_t(\mathbb{R}^{1+n})}; \]

\[ \varrho_2^{(i)}(u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2 - 1/m} \| \Box_k u \|_{L^m_{x_1} L^\infty_{x_2} \ldots L^\infty_{x_n} L^\infty_t(\mathbb{R}^{1+n})}; \]

\[ \varrho_3^{(i)}(u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \| \Box_k u \|_{L^{2+\kappa +m} \cap L^\infty_t L^2_x(\mathbb{R}^{1+n})}. \]

Put

\[ X := \left\{ u \in \mathcal{S}'(\mathbb{R}^{1+n}) : \| u \|_X := \sum_{\ell=1}^3 \sum_{\alpha=0,1} \sum_{i,j=1}^n \varrho_\ell^{(i)}(\partial^\alpha_{x_j} u) \leq \delta \right\}. \]

Considering the following mapping:

\[ \mathcal{T} : u(t) \rightarrow S(t)u_0 - i\mathcal{A}F(u, \bar{u}, \nabla u, \nabla \bar{u}), \]
we will show that $\mathcal{F} : X \to X$ is a contraction mapping.

Since $\|u\|_X = \|\bar{u}\|_X$, we may assume, without loss of generality that

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) = F(u, \nabla u) := \sum_{m+1 \leq \kappa + |\nu| \leq M+1} c_{\kappa \nu} u^\kappa (\nabla u)^\nu,$$

where $(\nabla u)^\nu = u_1^{\nu_1} \ldots u_n^{\nu_n}$. For the sake of convenience, we denote

$$v_1 = \ldots = v_\kappa = u, \quad v_{\kappa + 1} = \ldots = v_{\kappa + \nu_\kappa} = u_{x_1}, \ldots, v_{\kappa + |\nu| - \nu_{\kappa + 1}} = \ldots = v_{\kappa + |\nu|} = u_{x_n}.$$

By (2.8), for $\alpha = 0, 1$,

$$\varrho_1^{(i)} (\partial_x^\alpha S(t) u_0) \lesssim \sum_{k \in \mathbb{Z}^n, |k_\kappa| > 4} \langle k_i \rangle^{1/2} \langle k_j \rangle \| \Box_k u_0 \|_{L^2(\mathbb{R}^n)} \leq \| u_0 \|_{M^{3/2}}.$$

By (3.23), (3.24), we have for $\alpha = 0, 1$,

$$\varrho_2^{(i)} (\partial_x^\alpha S(t) u_0) + \varrho_3^{(i)} (\partial_x^\alpha S(t) u_0) \lesssim \| u_0 \|_{M^{3/2}}.$$

Hence,

$$\| u \|_X \lesssim \| u_0 \|_{M^{3/2}}.$$

In order to estimate $\varrho_1^{(i)} (\mathcal{A} \partial_x^\alpha (v_1 \ldots v_{\kappa + |\nu|}))$, $i, j = 1, \ldots, n$, it suffices to estimate $\varrho_1^{(i)} (\mathcal{A} \partial_x^\alpha (v_1 \ldots v_{\kappa + |\nu|}))$ and $\varrho_1^{(1)} (\mathcal{A} \partial_x^\alpha (v_1 \ldots v_{\kappa + |\nu|}))$. Similarly as in (5.4), we will use the decomposition

$$\Box_k (v_1 \ldots v_{\kappa + |\nu|}) = \Box_k \left( \sum_{S_{1}^{(i)}} (\Box_k (v_1 \ldots v_{\kappa + |\nu|})) + \sum_{S_{2}^{(i)}} (\Box_k (v_1 \ldots v_{\kappa + |\nu|})) \right),$$

where

$$S_{1}^{(i)} := \left\{ (k_1^{(1)}, \ldots, k_{(\kappa + |\nu|)}) : |k_i^{(1)}| \lor \ldots \lor |k_i^{(\kappa + |\nu|)}| > 4 \right\},$$

$$S_{2}^{(i)} := \left\{ (k_1^{(1)}, \ldots, k_{(\kappa + |\nu|)}) : |k_i^{(1)}| \lor \ldots \lor |k_i^{(\kappa + |\nu|)}| \leq 4 \right\}.$$

In view of (3.12) and (3.19),

$$\varrho_1^{(1)} (\mathcal{A} \partial_x^\alpha (v_1 \ldots v_{\kappa + |\nu|})) \lesssim \sum_{k \in \mathbb{Z}^n, |k_\kappa| > 4} \langle k_i \rangle \sum_{S_{1}^{(i)}} \| \Box_k \left( \Box_k (v_1 \ldots v_{\kappa + |\nu|}) \right) \|_{L^1_{x_1} L^2_{x_2} \ldots \ldots x_n L^2(\mathbb{R}^{1+n})}$$
Hölder's inequality and Lemma (3.3),

We give the estimate of \( \tilde{\kappa} \) as in Lemma 4.2 and hence, using a similar way as in (5.9),

By Hölder's inequality and Lemma 3.3

\[
\| \Box_k v_i \|_{L^{\kappa + |\nu|}} \leq \| \Box_k v_i \|_{L_x^2 L_t^\infty L^{\kappa + |\nu|}} \leq \| \Box_k v_i \|_{L_x^2 L_t^\infty (\mathbb{R}^{1+n})}.
\]

(6.4)

Hence, noticing that \( v_i = u \) or \( v_i = u_{x_1} \), we have from (6.3) and (6.4),

\[
I \lesssim \| u \|_{X}^{\kappa + |\nu|}.
\]

(6.5)

Similar to (5.9), we see that \( |k_1| \leq C \) in the summation of II. Again, in view of Hölder's inequality and Lemma 3.3,

\[
\| \Box_k v_1 \cdots \Box_k v_{\kappa + |\nu|} \|_{L_x^{\kappa + |\nu|}} \leq \prod_{i=1}^{\kappa + |\nu|} \| \Box_k v_i \|_{L_x^{\kappa + |\nu| + 1} L_t^\infty (\mathbb{R}^{1+n})}.
\]

(6.6)

Hence, using a similar way as in (5.9),

\[
II \lesssim \| u \|_{X}^{\kappa + |\nu|}.
\]

(6.7)

We now give the estimate of \( q_1^{(1)}(\mathcal{A} \partial_{x_2}^n (v_1 \cdots v_{\kappa + |\nu|})) \). Since we have obtained the estimate in the case \( \alpha = 0 \), it suffices to consider the case \( \alpha = 1 \). Let \( \psi_i (i = 1, 2) \) be as in Lemma 1.2 and \( P_i = \mathcal{F}^{-1} \psi_i \mathcal{F} \). We have

\[
q_1^{(1)}(\mathcal{A} \partial_{x_2}^n (v_1 \cdots v_{\kappa + |\nu|}))
\]

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\[
\leq \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \| P_1 \square_k (\mathcal{A} \partial_{x_2} (v_1 \ldots v_{\kappa + |\nu|})) \|_{L^2_{x_2} \ldots x_n L^2_t} \\
+ \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \| P_2 \square_k (\mathcal{A} \partial_{x_2} (v_1 \ldots v_{\kappa + |\nu|})) \|_{L^2_{x_1} L^2_{x_2} \ldots x_n L^2_t} \\
:= III + IV.
\] (6.8)

Using the decomposition (6.1),
\[
III \leq \sum_{S_1^{(1)}, k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \| \square_k (\mathcal{A} \partial_{x_2} (\square_k (v_1 \ldots \square_k (v_{\kappa + |\nu|})) v_{\kappa + |\nu|})) \|_{L^1_{x_1} L^2_{x_2} \ldots x_n L^2_t} \\
+ \sum_{S_2^{(1)}, k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \| \square_k (\mathcal{A} \partial_{x_2} (\square_k (v_1 \ldots \square_k (v_{\kappa + |\nu|})) v_{\kappa + |\nu|})) \|_{L^1_{x_1} L^2_{x_2} \ldots x_n L^2_t} \\
:= III_1 + III_2.
\] (6.9)

By Lemma 4.2,
\[
III_1 \lesssim \sum_{S_1^{(1)}, k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \| \square_k (\square_k (v_1 \ldots \square_k (v_{\kappa + |\nu|})) v_{\kappa + |\nu|}) \|_{L^1_{x_1} L^2_{x_2} \ldots x_n L^2_t}.
\] (6.10)

By symmetry, we may assume \(|k_1^{(1)}| = \max(|k_1^{(1)}|, \ldots, |k_1^{(\kappa + |\nu|)}|)\) in \(S_1^{(1)}\). Hence,
\[
III_1 \lesssim \sum_{S_1^{(1)}, |k_1| > 4} \langle k_1^{(1)} \rangle \| \square_1 (v_1) \|_{L^\infty_{x_1} L^2_{x_2} \ldots x_n L^2_t} \prod_{i=2}^{\kappa + |\nu|} \| \square_i (v_i) \|_{L^\infty_{x_1} L^2_{x_2} \ldots x_n L^2_t} \\
\lesssim \varphi_1^{(1)} (v_1) \prod_{i=2}^{\kappa + |\nu|} (\varphi_2^{(1)} (v_i) + \varphi_3^{(1)} (v_i)) \lesssim \| u \|_{X}^{\kappa + |\nu|}.
\] (6.11)

Applying (4.2) and using a similar way as in (5.15),
\[
III_2 \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4, |k_2| \leq |k_1|} \langle k \rangle^{3/2} \sum_{S_2^{(1)}} \| \square_k (\square_k (v_1 \ldots \square_k (v_{\kappa + |\nu|})) v_{\kappa + |\nu|}) \|_{L^1_{x_1} L^2_{x_2} \ldots x_n L^2_t} / \prod_{i=1}^{\kappa + |\nu|} \varrho_3^{(1)} (v_i) \leq \| u \|_{X}^{\kappa + |\nu|}.
\] (6.12)

So, we have shown that
\[
III \lesssim \| u \|_{X}^{\kappa + |\nu|}.
\] (6.13)

Now we estimate IV. Using the decomposition (6.1),
\[
IV \leq \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \| P_2 \square_k (\mathcal{A} \partial_{x_2} (\square_k (v_1 \ldots \square_k (v_{\kappa + |\nu|})) v_{\kappa + |\nu|})) \|_{L^\infty_{x_1} L^2_{x_2} \ldots x_n L^2_t}
\]
\[ + \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k_1 \rangle \sum_{S(2)^{(2)}} \| P_2 \Box_k (\mathcal{A} \partial_{x_2} (\Box_{k(1)} v_1 \ldots \Box_{k(\kappa+|\nu|)} v_{\kappa+|\nu|})) \|_{L_{t,x}^p L_{x_1}^2 \ldots L_{x_n}^2} \]
\]
\[ := IV_1 + IV_2. \quad (6.14) \]

By Lemma 4.2,
\[ IV_1 \lesssim \sum_{S(2)^{(2)}} \sum_{k \in \mathbb{Z}^n, |k_2| > 4} \langle k_2 \rangle \| \Box_k (\Box_{k(1)} v_1 \ldots \Box_{k(\kappa+|\nu|)} v_{\kappa+|\nu|}) \|_{L_{t,x}^1 L_{x_1}^2 \ldots L_{x_n}^2}. \quad (6.15) \]

In view of the symmetry, one can bound \( IV_1 \) by using the same way as that of \( III_1 \) and as in (5.17)–(5.20):
\[ IV_1 \lesssim \| u \|_{X}^{\kappa+|\nu|}. \quad (6.16) \]

For the estimate of \( IV_2 \), we apply (4.2),
\[ IV_2 \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k_1 \rangle^{1/2} \langle k_2 \rangle \sum_{S(2)^{(2)}} \| P_2 \Box_k (\Box_{k(1)} v_1 \ldots \Box_{k(\kappa+|\nu|)} v_{\kappa+|\nu|}) \|_{L_{t,x}^{(2+m)/(1+m)} (\mathbb{R}^{1+n})} \]
\[ \lesssim \sum_{S(2)^{(2)}} \| \Box_{k(1)} v_1 \ldots \Box_{k(\kappa+|\nu|)} v_{\kappa+|\nu|} \|_{L_{t,x}^{(2+m)/(1+m)} (\mathbb{R}^{1+n})} \lesssim \| u \|_{X}^{\kappa+|\nu|}. \quad (6.17) \]

Hence, in view of (6.16) and (6.17), we have
\[ IV \lesssim \| u \|_{X}^{\kappa+|\nu|}. \quad (6.18) \]

Collecting (6.5), (6.7), (6.13), (6.18), we have shown that
\[ \sum_{\alpha=0,1} \sum_{i,j=1}^n \| \mathcal{G}^{(i)}_1 (\mathcal{A} \partial_{x_j} (u^\kappa (\nabla u)^\nu)) \| \lesssim \| u \|_{X}^{\kappa+|\nu|}. \quad (6.19) \]

**Lemma 6.1** Let \( s \geq 0, 1 \leq p, p_i, \gamma, \gamma_i \leq \infty \) satisfy
\[ \frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_N}, \quad \frac{1}{\gamma} = \frac{1}{\gamma_1} + \ldots + \frac{1}{\gamma_N}. \quad (6.20) \]

Then
\[ \sum_{k \in \mathbb{Z}^n} \langle k_1 \rangle^s \| \Box_k (u_1 \ldots u_N) \|_{L_{t,x}^p (\mathbb{R}^{1+n})} \lesssim \prod_{i=1}^N \left( \sum_{k \in \mathbb{Z}^n} \langle k_1 \rangle^s \| \Box_k u_i \|_{L_{t,x}^{\gamma_i} L_{x_1}^{p_i} (\mathbb{R}^{1+n})} \right). \quad (6.21) \]

**Proof.** See [26], Lemma 7.1. \( \square \)

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Next, we consider the estimates of $\varrho_2^{(1)}(\mathcal{A}(u^n(\nabla u)\nu))$ and $\varrho_3^{(1)}(\mathcal{A}(u^n(\nabla u)\nu))$. In view of (3.30) and (3.20),

$$
\sum_{j=2,3} \varrho_j^{(1)}(\mathcal{A}(u^n(\nabla u)\nu)) \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \| \square_k (u^n(\nabla u)\nu) \|_{L^{2+m}_{L_{t,x}^{4}}(\mathbb{R}^{1+n})}^{2+m}.
$$

(6.22)

We use Lemma 6.1 to control the right hand side of (6.22):

$$
\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \| \square_k (v_1 \ldots v_{n+|\nu|}) \|_{L^{2+m}_{L_{t,x}^{4}}(\mathbb{R}^{1+n})}^{2+m} \\
\lesssim \prod_{i=1}^{m+1} \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \| \square_k v_i \|_{L^{2+m}_{t,x}^{4}(\mathbb{R}^{1+n})} \right)^{k+|\nu|} \\
\prod_{i=m+2}^{m+1} \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \| \square_k v_i \|_{L^{\infty}_{t,x}^{2}(\mathbb{R}^{1+n})} \right)^{k+|\nu|} \\
\lesssim \prod_{i=1}^{m+1} \varrho_3^{(1)}(v_i) \leq \| u \|_{X^{k+|\nu|}}^{k+|\nu|}.
$$

(6.23)

We estimate $\varrho_2^{(1)}(\mathcal{A} \partial_1 x (u^n(\nabla u)\nu))$. Recall that $k_{\text{max}} = |k_1| \lor \ldots \lor |k_n|$.

$$
\varrho_2^{(1)}(\mathcal{A} \partial_1 x (u^n(\nabla u)\nu)) \\
\lesssim \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} \langle k \rangle^{1/2 - 1/m} \| \square_k \mathcal{A} \partial_1 x (v_1 \ldots v_{n+|\nu|}) \|_{L^{m}_{t} L^{\infty}_{t} \ldots L^{\infty}_{t}(\mathbb{R}^{1+n})} \\
+ \sum_{k \in \mathbb{Z}^n, k_{\text{max}} \leq 4} \langle k \rangle^{1/2 - 1/m} \| \square_k \mathcal{A} \partial_1 x (v_1 \ldots v_{n+|\nu|}) \|_{L^{m}_{t} L^{\infty}_{t} \ldots L^{\infty}_{t}(\mathbb{R}^{1+n})} \\
:= V + VI.
$$

(6.24)

By (5.20) and Lemma 6.1 we have

$$
VI \lesssim \sum_{k \in \mathbb{Z}^n} \| \square_k (v_1 \ldots v_{n+|\nu|}) \|_{L^{2+m}_{L_{t,x}^{4}}(\mathbb{R}^{1+n})} \lesssim \| u \|_{X^{k+|\nu|}}^{k+|\nu|}.
$$

(6.25)

It is easy to see that

$$
V \lesssim \left( \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} + \ldots + \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} \right) \langle k \rangle^{1/2 - 1/m} \\
\times \| \square_k \mathcal{A} \partial_1 x (v_1 \ldots v_{n+|\nu|}) \|_{L^{m}_{t} L^{\infty}_{t} \ldots L^{\infty}_{t}(\mathbb{R}^{1+n})} := Y_1(u) + \ldots + Y_n(u).
$$

(6.26)

Applying the decomposition (6.1) and Lemmas 4.4 and 4.5, we obtain that

$$
Y_1(u) \lesssim \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle \sum_{g_1^{(1)}} \| \square_k (\square_k(v_1 \ldots \square_k(v_{n+|\nu|}))) \|_{L^{m}_{t} L^{\infty}_{t} \ldots L^{\infty}_{t}(\mathbb{R}^{1+n})}
$$
which reduces to the case α = 1 in (6.2). So,

\[ \Upsilon_1(u) \lesssim \|u\|_{X}^{\kappa+\nu}. \]  

(6.28)

Again, in view of Lemmas 4.4 and 4.5

\[ \Upsilon_2(u) \lesssim \sum_{k \in \mathbb{Z}^n, |k|>4} \langle k_2 \rangle \sum_{S_2^{(1)}} \| \Box_k (\Box_{k(1)} v_1 \ldots \Box_{k(\kappa+\nu)} v_{\kappa+\nu}) \|_{L^2_t L^2_x; x_2, x_3, \ldots, x_n L^{2'}_t (\mathbb{R}^{1+n})} + \sum_{k \in \mathbb{Z}^n, |k|>4} \langle k_2 \rangle^{3/2} \sum_{S_2^{(2)}} \| \Box_k (\Box_{k(1)} v_1 \ldots \Box_{k(\kappa+\nu)} v_{\kappa+\nu}) \|_{L^2_t L^2_x; x_2, x_3, \ldots, x_n L^{2'}_t (\mathbb{R}^{1+n})}, \]  

(6.29)

which reduces to the same estimate as \( \Upsilon_1(u) \). Using the same way as \( \Upsilon_2(u) \), we can get the estimates of \( \Upsilon_3(u), \ldots, \Upsilon_n(u) \). So,

\[ \rho_2^{(1)} (\mathcal{A} \partial_{x_1} (v_1 \ldots v_{\kappa+\nu})) \lesssim \|u\|_{X}^{\kappa+\nu}. \]  

(6.30)

We need to further bound \( \rho_2^{(1)} (\mathcal{A} \partial_{x_i} (v_1 \ldots v_{\kappa+\nu})) \), \( i = 2, \ldots, n \), which is essentially the same as \( \rho_2^{(1)} (\mathcal{A} \partial_{x_1} (v_1 \ldots v_{\kappa+\nu})) \). Indeed, it is easy to see that (6.24) holds if we substitute \( \partial_{x_1} \) with \( \partial_{x_i} \). Moreover, using Lemmas 6.1, 4.4 and 4.5 we easily get that

\[ \rho_2^{(1)} (\mathcal{A} \partial_{x_i} (v_1 \ldots v_{\kappa+\nu})) \lesssim \|u\|_{X}^{\kappa+\nu}. \]  

(6.31)

By Lemma 3.1 (3.17), we see that

\[ \| \Box_k \mathcal{A} \partial_{x_1} f \|_{L^\infty_t L^2_x \cap L^{2+\nu}_{t,x} (\mathbb{R}^{1+n})} \lesssim \langle k_1 \rangle \| \Box_k f \|_{L^{2+\nu}_{t,x} (\mathbb{R}^{1+n})}. \]  

(6.32)

Hence, in view of (3.28) and (3.18), repeating the procedure as in the estimates of \( \rho_3(u) \) in Theorem 1.2, \( \rho_2^{(1)} (\mathcal{A} \partial_{x_1} (v_1 \ldots v_{\kappa+\nu})) \) can be controlled by the right hand side of (6.27) and (6.25). Summarizing the estimates as in the above, we have shown that

\[ \| \mathcal{F} u \|_X \lesssim \|u_0\|_{M^{3/2}} + \sum_{m+1 \leq \kappa+\nu \leq M+1} \|u\|_{X}^{\kappa+\nu}. \]  

(6.33)

Applying a standard contraction mapping argument, we can prove our result.
A Appendix

In this section, we generalize the Christ-Kiselev Lemma [6] to anisotropic Lebesgue spaces. Our idea follows Molinet and Ribaud [17], and Smith and Sogge [21]. Denote

\[ T f(t) = \int_{-\infty}^{\infty} K(t,t') f(t') dt', \quad T_{re} f(t) = \int_{0}^{t} K(t,t') f(t') dt'. \]  

(A.1)

If \( T : Y_1 \to X_1 \) implies that \( T_{re} : Y_1 \to X_1 \), then \( T : Y_1 \to X_1 \) is said to be a well restriction operator.

**Proposition A.1** Let \( T \) be as in (A.1). We have the following results.

1. If \( \wedge_{i=1}^{3} p_i > (\vee_{i=1}^{3} q_i) \vee (q_1 q_3 / q_2) \), then \( T : L_{x_1}^{q_1} L_{x_2}^{q_2} L^{q_3}_t (\mathbb{R}^3) \to L_{x_1}^{p_1} L_{x_2}^{p_2} L^{p_3}_t (\mathbb{R}^3) \) is a well restriction operator.

2. If \( p_1 > (\vee_{i=1}^{3} q_i) \vee (q_1 q_3 / q_2) \), then \( T : L_{x_1}^{q_1} L_{x_2}^{q_2} L^{q_3}_t (\mathbb{R}^3) \to L_{x_1}^{p_1} L_{x_2}^{p_2} L^{p_3}_t (\mathbb{R}^3) \) is a well restriction operator.

3. If \( q_1 < \wedge_{i=1}^{3} p_i \), then \( T : L_{x_1}^{q_1} L_{x_2}^{q_2} L^{q_3}_t (\mathbb{R}^3) \to L_{x_1}^{p_1} L_{x_2}^{p_2} L^{p_3}_t (\mathbb{R}^3) \) is a well restriction operator.

4. If \( \wedge_{i=1}^{3} p_i > (\vee_{i=1}^{3} q_i) \vee (q_1 q_3 / q_2) \), then \( T : L_{x_1}^{q_1} L_{x_2}^{q_2} L^{q_3}_t (\mathbb{R}^3) \to L_{x_2}^{p_2} L_{x_1}^{p_1} L^{p_3}_t (\mathbb{R}^3) \) is a well restriction operator.

Let \( f \in L_{x_1}^{q_1} L_{x_2}^{q_2} L^{q_3}_t (\mathbb{R}^3) \) so that \( \|f\|_{L_{x_1}^{q_1} L_{x_2}^{q_2} L^{q_3}_t (\mathbb{R}^3)} = 1 \). Define \( F : \mathbb{R} \to [0, 1] \) by

\[ F(t) := \left\| \left( \int_{-\infty}^{t} |f(s,x)|^{q_3} ds \right)^{1/q_3} \right\|_{L_{x_1}^{q_1} L_{x_2}^{q_2} t}^{q_1}. \]  

(A.2)

**Lemma A.2** Let \( I \subset [0, 1] \) is an interval, then it holds:

\[ \|X_{F^{-1}(I)} f\|_{L_{x_1}^{q_1} L_{x_2}^{q_2} L^{q_3}_t (\mathbb{R} \times \mathbb{R}^2)} \leq \|I\|^{\frac{q_2}{q_1} \wedge \frac{1}{q_3}} \wedge \frac{1}{q_2} \wedge \frac{1}{q_3}. \]  

(A.3)

**Proof.** For any \( I = (A, B) \subset [0, 1] \), there exist \( t_1, t_2 \in \mathbb{R} \) satisfying

\[ A = \left\| \left( \int_{-\infty}^{t_1} |f(s,x)|^{q_3} ds \right)^{1/q_3} \right\|_{L_{x_1}^{q_1} L_{x_2}^{q_2}} \quad B = \left\| \left( \int_{-\infty}^{t_2} |f(s,x)|^{q_3} ds \right)^{1/q_3} \right\|_{L_{x_1}^{q_1} L_{x_2}^{q_2}} \]

and \( F^{-1}(I) = (t_1, t_2) \). For \( x = (x_1, x_2) \), we define \( J(t, x) \) and \( E(t, x_1) \) by:

\[ J(t, x) = \left( \int_{-\infty}^{t} |f(s,x)|^{q_3} ds \right)^{1/q_3}, \quad E(t, x_1) = \left( \int J(t, x_1)^{q_2} dx_2 \right)^{1/q_2}. \]  

(A.4)
It is well known that for \( a \geq b > 0 \),
\[
r^a - s^a \leq C(r^b - s^b)(r^{a-b} + s^{a-b}), \quad 0 \leq s \leq r, \quad (A.5)
\]
and for \( 0 < a \leq b \),
\[
r^a - s^a \leq (r^b - s^b)^{a/b}, \quad 0 \leq s \leq r. \quad (A.6)
\]

We divide the proof into the following four cases.

Case 1. \( q_3 \geq q_2 \geq q_1 \). From (A.5) we have
\[
\|\chi_{F^{-1}(t)} f(\cdot, x)\|_{L_{t_1}^{q_3}}^{q_3} \lesssim (J(t_2, x)^{q_2} - J(t_1, x)^{q_2}) J(\infty, x)^{q_3 - q_2} \quad (A.7)
\]
Recalling the assumption \( \|f\|_{L_{t_1}^{q_1} L_{t_2}^{q_2} L_{t_3}^{q_3}(\mathbb{R} \times \mathbb{R}^2)} = 1 \), by (A.7), (A.4), (A.5), and Hölder inequality, we have
\[
\int \left( \int \|\chi_{F^{-1}(t)} f(\cdot, x)\|_{L_{t_1}^{q_3}}^{q_3} dx_2 \right)^{q_1 \over q_2} dx_1 \leq \int \left( \int (J(t_2, x)^{q_2} - J(t_1, x)^{q_2})^{q_3 \over q_2} J(\infty, x)^{(q_3 - q_2) \over q_3} dx_2 \right)^{q_1 \over q_2} dx_1 \\
\leq \int \left( \int (J(t_2, x)^{q_2} - J(t_1, x)^{q_2})^{q_3 \over q_2} \left\| J(\infty, x)^{(q_3 - q_2) \over q_3} \right\|_{L_{t_2}^{1/(1-q_2/q_3)}}^{q_1 \over q_2} dx_1 \\
= \int \left( E(t_2, x_1)^{q_2} - E(t_1, x_1)^{q_2} \right)^{q_3 \over q_2} \left( E(\infty, x_1)^{(q_3 - q_2) \over q_3} \right) dx_1 \\
\leq \int \left( E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{q_3 \over q_3} \left( E(\infty, x_1)^{(q_3 - q_1) \over q_3} \right) dx_1 \\
\leq \left( F(t_2) - F(t_1) \right)^{q_3 \over q_3} F(\infty)^{1-q_2/q_3} \leq |I|^{q_3 \over q_3}. \quad (A.8)
\]

Case 2. \( q_3 \geq q_2, q_2 < q_1 \). From (A.8) and (A.6), we have
\[
\int \left( \int \|\chi_{F^{-1}(t)} f(\cdot, x)\|_{L_{t_1}^{q_3}}^{q_3} dx_2 \right)^{q_1 \over q_2} dx_1 \leq \int \left( E(t_2, x_1)^{q_2} - E(t_1, x_1)^{q_2} \right)^{q_3 \over q_2} \left( E(\infty, x_1)^{(q_3 - q_2) \over q_3} \right) dx_1 \\
\leq \int \left( E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{q_3 \over q_3} \left( E(\infty, x_1)^{(q_3 - q_1) \over q_3} \right) dx_1 \\
\leq \left( F(t_2) - F(t_1) \right)^{q_3 \over q_3} F(\infty)^{1-q_2/q_3} \leq |I|^{q_3 \over q_3}. \quad (A.11)
\]
Case 3. $q_3 < q_2 \leq q_1$. From (A.6), we have

$$
\| \chi_{F^{-1}(I)}f(\cdot, x) \|_{L_t^{q_3}}^{q_3} \leq (J(t_2, x)^{q_2} - J(t_1, x)^{q_2})^{q_3/q_2}
$$

(A.12)

Using (A.6) again, we have

$$
\int \left( \int \| \chi_{F^{-1}(I)}f(\cdot, x) \|_{L_t^{q_3}}^{q_2} \, dx_2 \right)^{\frac{q_3}{q_2}} \, dx_1
\leq \int \left( \int (J(t_2, x)^{q_2} - J(t_1, x)^{q_2}) \, dx_2 \right)^{\frac{q_3}{q_2}} \, dx_1
\leq \int \left( E(t_2, x_1)^{q_2} - E(t_1, x_1)^{q_2} \right)^{\frac{q_3}{q_2}} \, dx_1
\leq \int \left( E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{\frac{q_3}{q_2}} \, dx_1
= \int \left( E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{\frac{q_3}{q_2}} \, dx_1
\leq \int \left( E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{\frac{q_3}{q_2}} E(\infty, x_1)^{\frac{q_1(q_2-q_1)}{q_2}} \, dx_1
= (F(t_2) - F(t_1))^{\frac{q_3}{q_2}} = |I|^{\frac{q_3}{q_2}}.
$$

(A.13)

$$
\int \left( \int \| \chi_{F^{-1}(I)}f(\cdot, x) \|_{L_t^{q_3}}^{q_2} \, dx_2 \right)^{\frac{q_3}{q_2}} \, dx_1
\leq \int \left( \int (J(t_2, x)^{q_2} - J(t_1, x)^{q_2}) \, dx_2 \right)^{\frac{q_3}{q_2}} \, dx_1
\leq \int \left( E(t_2, x_1)^{q_2} - E(t_1, x_1)^{q_2} \right)^{\frac{q_3}{q_2}} \, dx_1
\leq \int \left( E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{\frac{q_3}{q_2}} E(\infty, x_1)^{\frac{q_1(q_2-q_1)}{q_2}} \, dx_1
\leq (F(t_2) - F(t_1))^{rac{q_3}{q_2}} = |I|^{rac{q_3}{q_2}}.
$$

(A.14)

Case 4. $q_3 < q_2, q_2 > q_1$. From (A.13), (A.15) and Hölder inequality we have

$$
\int \left( \int \| \chi_{F^{-1}(I)}f(\cdot, x) \|_{L_t^{q_3}}^{q_2} \, dx_2 \right)^{\frac{q_3}{q_2}} \, dx_1
\leq \int \left( \int (J(t_2, x)^{q_2} - J(t_1, x)^{q_2}) \, dx_2 \right)^{\frac{q_3}{q_2}} \, dx_1
\leq \int \left( E(t_2, x_1)^{q_2} - E(t_1, x_1)^{q_2} \right)^{\frac{q_3}{q_2}} \, dx_1
\leq \int \left( E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{\frac{q_3}{q_2}} E(\infty, x_1)^{\frac{q_1(q_2-q_1)}{q_2}} \, dx_1
\leq (F(t_2) - F(t_1))^{rac{q_3}{q_2}} = |I|^{rac{q_3}{q_2}}.
$$

(A.15)

From (A.10), (A.11), (A.14) and (A.15) we get

$$
\| \chi_{F^{-1}(I)}f \|_{L_t^{q_1}L_x^{q_2}L_1^{q_3}(\mathbb{R}^2)} \leq C |I|^{rac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}} (\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}),
$$

(A.16)

which yields (A.3), as desired.

We will use Whitney’s decomposition to the triangle $\{(x, y) \in [0, 1]^2 : x < y\}$ (see Figure 2). First, we divide $[0, 1]^2$ into four congruent squares, consider the square with side-length 1/2 in the triangle region and decompose it into four dyadic squares with side-length 1/4, then remove the left-upper three ones in the triangle region. Secondly, considering the remaining region, we can find three squares with side-length 1/4 in the triangle. We decompose each square into four dyadic squares in the same way as in the first step. Repeating the procedure above to the end. So, we
Figure 2: Whitney’s decomposition in the triangle.

have decomposed the triangle region into infinite squares with dyadic border. Let $I$ and $J$ be the dyadic subintervals of $[0, 1]$ in the horizontal and perpendicular axes, respectively. We say that $I \sim J$ if they can consist the horizontal border and perpendicular border of a square described above, respectively. From the decomposition above we see that

(i) $|I| = |J|$ and $\text{dist}(I, J) \geq |I|$ for $I \sim J$.

(ii) The squares in $\{(x, y) \in [0, 1]^2 : x < y\}$ are pairwise disjoint.

(iii) For any dyadic subinterval $J$, there are at most two $I$ with $I \sim J$.

**Proof of Proposition A.1** First, we show the result of (1). We have

$$T_{re}f(t, x) := \int_{-\infty}^{t} K(t, t') f(t') dt' = \sum_{\{I, J : I \sim J\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f). \quad (A.17)$$

It follows that

$$\|T_{re}f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} L_{t}^{p_3}(\mathbb{R}^3)} \leq \sum_{j=1}^{\infty} \left\| \sum_{\{I, J : I \sim J, |I| = 2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f) \right\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} L_{t}^{p_3}(\mathbb{R}^3)} \quad (A.18)$$

For any $p \geq 1$, we easily see the following fact:

$$\left\| \sum_{\{I, J : I \sim J, |I| = 2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f) \right\|_{L_{t}^{p}(\mathbb{R})}^p$$
\[ \leq 2 \sum_{J_1:|J_1|=2^{-j}} \int_{\mathbb{R}} \chi_{F^{-1}(J_1)}(T(\chi_{F^{-1}(J_1)}f))^p dt. \] (A.19)

Hence, in view of (A.18) and (A.19) we have

\[ \|T_{ef}\|_{L^p_{L^1_1}L^p_{L^2_2}L^p_{L^3_3}(\mathbb{R}^3)} \leq \sum_{j=1}^{\infty} \left( \sum_{|I|=2^{-j}} \|T(\chi_{F^{-1}(I)}f)\|_{L^p_{L^1_1}L^p_{L^2_2}L^p_{L^3_3}(\mathbb{R})} \right)^{1/p_3}. \] (A.20)

If \( p \leq q \), by Minkowski’s inequality, we have

\[ \left( \sum_{j} \|a_j(x,y)\|_{L^p_y}^p \right)^{1/p} \leq \left( \sum_{j} \|a_j(x,y)\|_{L^p_y}^p \right)^{1/p} ; \] (A.21)

If \( p > q \), in view of \((a + b)^\theta \leq a^\theta + b^\theta\) for any \(0 \leq \theta \leq 1, a, b > 0\), we have

\[ \left( \sum_{j} \|a_j(x,y)\|_{L^p_y}^p \right)^{1/p} \leq \left( \sum_{j} \|a_j(x,y)\|_{L^q_y}^q \right)^{1/q} . \] (A.22)

We divide our discussion into the following three cases.

Case 1. \( p_1, p_2 \geq p_3 \). By (A.20), using (A.21) twice, we have

\[ \|T_{ef}\|_{L^p_{L^1_1}L^p_{L^2_2}L^p_{L^3_3}(\mathbb{R}^3)} \leq \sum_{j=1}^{\infty} \left( \sum_{|I|=2^{-j}} \|T(\chi_{F^{-1}(I)}f)\|_{L^p_{L^1_1}L^p_{L^2_2}L^p_{L^3_3}(\mathbb{R}^3)} \right)^{1/p_3}. \] (A.23)

Case 2. \( p_1 \leq p_2 \leq p_3 \). By (A.20), using (A.22) twice, we have

\[ \|T_{ef}\|_{L^p_{L^1_1}L^p_{L^2_2}L^p_{L^3_3}(\mathbb{R}^3)} \leq \sum_{j=1}^{\infty} \left( \sum_{|I|=2^{-j}} \|T(\chi_{F^{-1}(I)}f)\|_{L^p_{L^1_1}L^p_{L^2_2}L^p_{L^3_3}(\mathbb{R}^3)} \right)^{1/p_1}. \] (A.24)

Case 3. \( p_2 \leq p_1 \leq p_3 \). By (A.20) and (A.22), then applying (A.21), we have

\[ \|T_{ef}\|_{L^p_{L^1_1}L^p_{L^2_2}L^p_{L^3_3}(\mathbb{R}^3)} \leq \sum_{j=1}^{\infty} \left( \sum_{|I|=2^{-j}} \|T(\chi_{F^{-1}(I)}f)\|_{L^p_{L^1_1}L^p_{L^2_2}L^p_{L^3_3}(\mathbb{R}^3)} \right)^{1/p_2}. \] (A.25)

Denote \( p_{\text{min}} = \min(p_1, p_2, p_3) \). It follows from (A.23)–(A.25) that

\[ \|T_{ef}\|_{L^p_{L^1_1}L^p_{L^2_2}L^p_{L^3_3}(\mathbb{R}^3)} \leq \sum_{j=1}^{\infty} \left( \sum_{|I|=2^{-j}} |I|^{\frac{p_{\text{min}}}{q_1} \wedge \frac{p_{\text{min}}}{q_3} \wedge \frac{p_{\text{min}}}{q_2}} \right)^{1/p_{\text{min}}}. \]
The proof of (4) is almost the same as that of (1) and we omit the details of the proof.

Next, we prove (2). We have

\[
\|T re f\|_{L^p_t L^p_1 L^\infty_{x_1} L^\infty_{x_2} (\mathbb{R} \times \mathbb{R}^2)} \lesssim \sum_{j=1}^{\infty} \left\| \sum_{\{I, J: |I| = 2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f) \right\|_{L^p_t L^p_1 L^p_2 L^\infty_3 (\mathbb{R}^3)} 
\lesssim 2 \sum_{j=1}^{\infty} \left\| \sum_{\{I: |I| = 2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f) \right\|_{L^p_t L^p_1 L^p_2 L^p_3 (\mathbb{R}^3)}.
\]

Using the same way as in (A.19),

\[
\|T re f\|_{L^p_t L^p_1 L^p_2 L^\infty_{x_1} L^\infty_{x_2} (\mathbb{R}^3)} \lesssim \sum_{j=1}^{\infty} \left( \sum_{\{I, J: |I| = 2^{-j}\}} \| \chi_{F^{-1}(J)} f \|_{L^q_{x_1} L^q_{x_2} L^q_3 (\mathbb{R}^3)} \right)^{1/p_1}.
\]

So, we can control $\|T re f\|_{L^p_t L^p_1 L^p_2 L^\infty_{x_1} L^\infty_{x_2} (\mathbb{R}^3)}$ by the right-hand side of (A.26) in the case $p_{\min} = p_1$.

Finally, we prove (3). We define $F_1(t)$ as follows.

\[
F_1(t) := \int_{-\infty}^{t} \| f(s, x_1, x_2) \|_{L^q_{x_1} L^q_{x_2} (\mathbb{R} \times \mathbb{R}^2)} ds.
\]

From the definition of $F_1(t)$, it is easy to see that

\[
\left\| \chi_{F_1^{-1}(I)} (s) f(s) \right\|_{L^q_{x_1} L^q_{x_2} L^q_3 (\mathbb{R} \times \mathbb{R}^2)} = |I|^{1/q_1}.
\]

Hence, replacing (A.3) with (A.28), we can use the same way as in the proof of (1) to get the result, as desired.

We can generalized this result to $n$ dimensional spaces:

**Lemma A.3** Let $T$ be as in (A.1). We have the following results.

1. If $\min(p_1, p_2, p_3) > \max(q_1, q_2, q_3, q_1 q_3 / q_2)$, then $T : L^{q_1}_{x_1} L^{q_2}_{x_2} ... L^{q_n}_{x_n} (\mathbb{R}^{n+1}) \rightarrow L^{p_1}_{x_1} L^{p_2}_{x_2} ... L^{p_3}_{x_3} (\mathbb{R}^{n+1})$ is a well restriction operator.

2. If $p_0 > (\sqrt{\sum_{i=1}^{n} q_i}) \vee (q_1 q_3 / q_2)$, then $T : L^{q_1}_{x_1} L^{q_2}_{x_2} ... L^{q_n}_{x_n} (\mathbb{R}^{n+1}) \rightarrow L^{p_0}_{x_1} L^{p_1}_{x_2} ... L^{p_n}_{x_n} (\mathbb{R}^{n+1})$ is a well restriction operator.
(3) If \( q_0 < \min (p_1, p_2, p_3) \), then \( T : L_{t_1}^{q_0} L_{x_1}^{q_1} \cdots L_{x_n}^{q_n} (\mathbb{R}^{n+1}) \rightarrow L_{t_1}^{p_1} L_{x_1}^{p_2} \cdots L_{x_n}^{p_3} (\mathbb{R}^{n+1}) \) is a well restriction operator.

(4) If \( \min(p_1, p_2, p_3) > \max(q_1, q_2, q_3, q_1 q_3 / q_2) \), then \( T : L_{x_2}^{q_1} L_{x_1, x_3, \ldots, x_n}^{q_2} L_t^{q_3} (\mathbb{R}^{n+1}) \rightarrow L_{x_1}^{p_1} L_{x_2}^{p_2} \cdots L_t^{p_3} (\mathbb{R}^{n+1}) \) is a well restriction operator.

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