We present a systematic theoretical study of the BCS-BEC crossover in two-dimensional Fermi gases with Rashba spin-orbit coupling (SOC). By solving the exact two-body problem in the presence of an attractive short-range interaction we show that the SOC enhances the formation of the bound state: the binding energy $E_b$ and effective mass $m_B$ of the bound state grows along with the increase of the SOC. For the many-body problem, even at weak attraction, a dilute Fermi gas can evolve from a BCS superfluid state to a Bose condensation of molecules when the SOC becomes comparable to the Fermi momentum. The ground-state properties and the Berezinskii-Kosterlitz-Thouless (BKT) transition temperature are studied, and analytical results are obtained in various limits. For large SOC, the BKT transition temperature recovers that for a Bose gas with an effective mass $m_B$. We find that the condensate and superfluid densities have distinct behaviors in the presence of SOC: the condensate density is generally enhanced by the SOC due to the increase of the molecule binding, the superfluid density is suppressed because of the non-trivial molecule effective mass $m_B$.

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It has been widely believed for a long time that a smooth crossover from Bardeen–Cooper–Schrieffer (BCS) superfluidity to Bose–Einstein condensation (BEC) of molecules could be realized in an attractive Fermi gas [1–3]. This BCS-BEC crossover phenomenon has been successfully demonstrated in ultracold fermionic atoms by means of the Feshbach resonance [4]. Some recent experimental efforts in generating synthetic non-Abelian gauge field has opened up the opportunity to study the spin-orbit coupling (SOC) effect in cold atomic gases [5]. For fermionic atoms [6], it provides an alternative way to study the BCS-BEC crossover [7] according to the theoretical observation that novel bound states in three dimensions can be induced by a non-Abelian gauge field even though the attraction is weak [8, 9].

Recently, the anisotropic superfluidity in 3D Fermi gases with Rashba SOC has been intensively studied [10–12]. Two-dimensional (2D) fermionic systems with Rashba SOC is more interesting for condensed matter systems [13] and topological quantum computation [14]. By applying a large Zeeman splitting, a non-Abelian topologically superconducting phase and Majorana fermionic modes can emerge in spin-orbit coupled 2D systems [15]. In the absence of SOC, the BCS-BEC crossover and Berezinskii-Kosterlitz-Thouless (BKT) transition temperature in 2D attractive fermionic systems were investigated long ago [15, 16](see [17] for a review), which provides a possible mechanism for pseudogap formation in high-temperature superconductors [18].

In this Letter we present a systematic study of 2D attractive Fermi gases in the presence of Rashba SOC. The main results are summarized as follows: (i) The SOC enhances the dimerization bound states in 2D. At large SOC, even for weak intrinsic attraction, the many-body ground state is a Bose-Einstein condensate of bound molecules. In the presence of a harmonic trap, the atom cloud shrinks with increased SOC. (ii) The BKT transition temperature is enhanced by the SOC at weak attraction, and for large SOC it tends to the critical temperature for a gas of molecules with a nontrivial effective mass. The SOC effect therefore provides a new mechanism for pseudogap formation in 2D fermionic systems. (iii) In the presence of SOC, the superfluid ground state exhibits both spin-singlet and -triplet pairings, and the triplet one has a non-trivial contribution to the condensate density. In general, the condensate density is enhanced by the SOC due to the increase of the molecule binding. However, the superfluid density has entirely different behavior: it is suppressed by the SOC due to the increasing molecule effective mass.

Model and effective potential — A quasi-2D Fermi gas can be realized by arranging a one-dimensional optical lattice along the axial direction and a weak harmonic trapping potential in the radial plane, such that fermions are strongly confined along the axial direction and form a series of pancake-shaped quasi-2D clouds [13–22]. The strong anisotropy of the trapping potentials, namely $\omega_\perp \gg \omega_\parallel$ where $\omega_\perp (\omega_\parallel)$ is the axial (radial) frequency, allows us to use an effective 2D Hamiltonian to deal with the radial degrees of freedom.

The Hamiltonian of a spin-1/2 attractive Fermi gas with Rashba SOC is given by $H = - 2 \int d^2 \vec{r} \psi(\vec{r}) (\mathcal{H}_0 + \mathcal{H}_{\text{so}}) \psi(\vec{r}) - U \int d^2 \vec{r} \bar{\psi}_\uparrow(\vec{r}) \psi_\uparrow(\vec{r}) \bar{\psi}_\downarrow(\vec{r}) \psi_\downarrow(\vec{r})\psi$, where $\psi = [\psi_\uparrow, \psi_\downarrow]^T$ represents the two-component fermion fields, $\mathcal{H}_0 = - \frac{\hbar^2}{2m} \nabla^2 + \mu - h \sigma_z$ is the free single-particle Hamiltonian with $\mu$ being the chemical potential and $h$ the Zeeman splitting, and $\mathcal{H}_{\text{so}} = -ih\alpha(\sigma_\parallel \partial_\parallel - \sigma_\perp \partial_\perp)$ is the Rashba SOC term [22]. Here $\sigma_{x,y,z}$ are the Pauli matrices which act on the two-component fermion fields. The short range attractive interaction is modeled by a contact coupling $U$ [23]. In the following we use the natural units $\hbar = k_B = m = 1$.

In the functional path integral formalism, the partition function of the system is $Z = \int D\phi D\bar{\phi} \exp[-S[\psi, \bar{\psi}]]$, where $S[\psi, \bar{\psi}] = \int_0^{\beta} \! d\tau \int d^2 \vec{r} \bar{\psi}_\theta \partial_\tau \psi + H(\psi, \bar{\psi})$ with the inverse temperature $\beta = 1/T$. Introducing the auxiliary complex pairing field $\Phi(x) = -U\psi_\uparrow(x)\psi_\downarrow(x)$ [$x = (\tau, \vec{r})$] and applying the Hubbard-Stratonovich transformation, we arrive at $Z = \int D\Psi D\bar{\Psi} \exp \left[ \int \hspace{1cm} \int dx \bar{\Psi}(x) \mathcal{G}^{-1}(x, x') \Psi(x') \right] \int D\phi D\bar{\phi} \exp \left[ \int \hspace{1cm} d\tau \int d^2 \vec{r} \bar{\psi}_\theta \partial_\tau \psi + H(\psi, \bar{\psi}) \right]$.
The inverse single-particle Green function \( G^{-1}(x, x') \) is given by

\[
G^{-1}(x, x') = \begin{pmatrix}
-\partial_x - \mathcal{H}_0 - \mathcal{H}_{so} & i\sigma_\tau \Phi(x) \\
-i\sigma_\tau \Phi^*(x) & -\partial_x + \mathcal{H}_0 - \mathcal{H}_{so}
\end{pmatrix} \delta(x - x').
\]

Integrating out the fermion fields, we obtain \( Z = \int \mathcal{D}\Phi \mathcal{D}\Phi^* \exp \{-S_{eff}[\Phi, \Phi^*]\} \), where the effective action reads \( S_{eff}[\Phi, \Phi^*] = U^{-1} \int dx [\Phi(x)]^2 - \frac{i}{4} \text{Tr} \ln(G^{-1}(x, x')) \).

Two-body problem — The exact two-body problem at vanishing density can be studied by considering the Green function \( \Gamma(Q) \) of the fermion pairs, where \( Q = (i\nu_n, \mathbf{q}) \) with \( \nu_n = 2\pi n T \) (n integer) being the bosonic Matsubara frequency. In the present formalism, \( \Gamma^{-1}(Q) \) can be obtained from its coordinate representation defined as \( \Gamma^{-1}(x, x') = (B\nu)^{-1} \delta^2 S_{eff}[\Phi, \Phi^*]/[\delta \Phi^*(x) \delta \Phi(x')] \big|_{\nu=0} \). For \( \Phi = 0 \), the single-particle Green function reduces to its non-interacting form \( G_0(K) = \text{diag}[g_+(K), g_-(K)] \) with \( g_\pm(K) = [i\omega_n + (\xi_k - h \mp \xi_0)] \pm i(\sigma_\tau k_\tau \mp \sigma_\tau k_\tau)^{-1} \), where \( K = (i\omega_n, \mathbf{k}) \) with \( \omega_n = (2n + 1)\pi T \) being the fermionic Matsubara frequency. Here \( \xi_k = \xi_k - \mu \) and \( \epsilon_k = k^2/2 \). The single-particle spectrum generally has two branches: \( \omega^\pm_k = \xi_k \pm \sqrt{\lambda_k^2 \xi^2 + h^2} \).

After the analytical continuation \( i\nu_n \to \omega + i0^+ \), the real part of \( \Gamma^{-1}(Q) \) takes the form

\[
\Gamma^{-1}(\omega, \mathbf{q}) = \frac{1}{U} - \sum_{\alpha, \gamma = \pm} \frac{1 - f(\omega^\alpha_k) - f(\omega^\gamma_k)}{4(\omega^\alpha_k + \omega^\gamma_k - \omega)} (1 + \alpha\gamma T_{k\mathbf{q}}),
\]

where \( f(E) = 1/(e^{E_T} + 1) \) is the Fermi-Dirac distribution function, and \( T_{k\mathbf{q}} = (h \sigma_\tau \mathbf{k} \cdot \mathbf{p} + h^2)/\sqrt{(h^2 \mathbf{k}^2 + h^2)(h^2 \mathbf{p}^2 + h^2)} \) with \( \mathbf{p} = \mathbf{k} + \mathbf{q} \). \( \Gamma^{-1} \) takes the form similar to that of the relativistic systems [24], due to the fact that \( \mathcal{H}_{so} \) behaves like a Dirac Hamiltonian. Since in 2D the bound state forms for arbitrarily small attraction [25], the contact coupling \( U \) can be regularized by the two-body problem at vanishing SOC, \( U^{-1} = \sum_k (2\epsilon_k + \epsilon_0) \) [15, 17], where \( \epsilon_0 \) is the binding energy at vanishing SOC. This equation recovers the exponential behavior \( \epsilon_0 = 2\Lambda \exp(-4\pi/\Lambda) \) in 2D [26], where \( \Lambda \gg \epsilon_0 \) is an energy cutoff. All physical equations are finally UV convergent in terms of \( \epsilon_0 \) and we set \( \Lambda \to \infty \) in the dilute limit.

From now on we consider the case \( h = 0 \). The binding energy \( E_B \) at nonzero SOC is determined by the solution of \( \omega + 2\mu = -E_B \) for \( \Gamma^{-1}(\omega, \mathbf{q}) = 0 \). From the imaginary part of \( \Gamma^{-1}(Q) \), the bound state corresponds to the solution in the regime \(-\infty < \omega + 2\mu < -\lambda^2 \) and hence \( E_B > \lambda^2 \). Completing the momentum integrals analytically, we obtain a simple algebraic equation for \( E_B [27] \).

\[
\ln \frac{E_B}{\epsilon_B} = \frac{2\lambda}{\sqrt{E_B - \lambda^2}} \arctan \frac{\lambda}{\sqrt{E_B - \lambda^2}}.
\]

The solution can be generally expressed as \( E_B = \epsilon_B + 4\eta/(\eta^2/\epsilon_B) \) where \( \eta = \lambda^2/2 \). For \( \eta \ll \epsilon_B \), we have \( \epsilon_B \approx 1 \) and \( E_B \) is well given by \( E_B \approx \epsilon_B + 2\lambda^2 \). For \( \eta/\epsilon_B \to \infty \), the solution approaches very slowly to the asymptotic result \( E_B \approx \lambda^2 \). In general, \( E_B \) increases with increased SOC, as shown in Fig 1. It is straightforward to show that the bound state contains both spin singlet and triplet components [8].

For small nonzero \( \eta \), the solution for \( \omega \) can be written as \( \omega + 2\mu = -E_B + \xi^2/(2m) \), where \( m_B \) is the molecule effective mass. Substituting this dispersion into the equation \( \Gamma^{-1}(\omega, \mathbf{q}) = 0 \) we obtain [27]

\[
\frac{2m_B}{m_B} = 1 - \frac{1}{2\lambda} \frac{2\sqrt{k} - 1 - (k - 2)(\xi^2 - \arctan \frac{\xi^2}{\lambda^2})}{2\sqrt{k} - 1 + (\xi^2 - \arctan \frac{\xi^2}{\lambda^2})}.
\]

where \( k = E_B/\lambda^2 \). For \( \lambda \to 0 \), we obtain the usual result \( m_B \to 2m \). For \( \lambda \to \infty \), we have \( E_B \to \lambda^2 \) and \( m_B \) approaches the asymptotic result \( 4m \). In general, \( m_B \) is larger than \( 2m \), as shown in Fig 1. Together with the result for \( E_B \), we conclude that a novel bound state (referred to as rashbon [10]) forms. It would have significant impact on the many-body problem discussed in the following.

Ground state — For the many-body problem, we consider a homogeneous Fermi gas with fixed fermion density \( n = N/V \). For convenience, we define the Fermi momentum \( n = k_F/(2\pi) \) and Fermi energy by \( \epsilon_F = k_F^2/(2m) \). The ground state \( (T = 0) \) can be studied in the self-consistent mean-field theory, where we replace the pairing field \( \Phi \) by its expectation value \( \langle \Phi \rangle = \Delta \). Without loss of generality, we set \( \Delta = \epsilon_0 \) to be real.

The mean-field ground-state energy \( \Omega = S_{eff}[\Delta, \Lambda] / (\beta V) \) can be evaluated as \( \Omega = \Delta^2/U + (1/2) \sum_k (2\epsilon_k - \epsilon^0_k) \) [15, 17], where \( \epsilon^0_k = [(\xi^2_k + \Delta^2)^{1/2}] \) are the quasiparticle excitation energies with \( \epsilon^0_k = \epsilon_k \mp \lambda \). According to the equation that \( E_B \) satisfies, \( \Omega \) can be evaluated as \( \Omega = \Omega_{2D}(\Delta, \mu, \epsilon_B) + \Omega_1 \), where \( \Omega_{2D}(\Delta, \mu, \epsilon_B) = (\lambda^2/2\pi) \ln \left( \sqrt{\mu^2 + \Delta^2 - \mu} / \mu \right) - 1/2 - \mu / (\sqrt{\mu^2 + \Delta^2 - \mu}) \) is formally the ground-state energy for vanishing SOC [15, 17], and \( \Omega_1 = -\lambda(2\pi)^{1/2} \int d\mathbf{k} [\sqrt{\xi^2_k - \eta^2} + \Delta^2 - (\xi_k - \eta)] \) is the contribution due to the SOC effect.

From the explicit form of the ground-state energy, the gap and number equations can be expressed as

\[
\left[ \mu + \Delta^2 \right]^{1/2} - \mu = \epsilon_0 \exp \left[ 2 I_1 (\mu / \eta, \Delta / \eta) \right],
\]

\[
\left[ \mu^2 + \Delta^2 \right]^{1/2} + \mu = 2\epsilon_F - 2\eta \left[ 1 - I_2 (\mu / \eta, \Delta / \eta) \right],
\]

where \( I_1, I_2 \) are the modified Bessel functions of order 1 and 2, respectively.
respectively. Here the functions $I_1$ and $I_2$ are defined as $I_1(a, b) = \int_0^1 dx [(x^2 - 1 - \alpha)^2 + b^2]^{-1/2}$ and $I_2(a, b) = \int_0^1 dx [(x^2 - 1 - \alpha)^2 + b^2]^{-1/2}$. $I_1$, $I_2$ and $\Omega_1$ can be analytically evaluated using the elliptic functions. For vanishing SOC, we recover the well-known analytical results, $\Delta = \sqrt{2 E_F} \xi(k)$ and $\mu = -E_B/2$ with $E_B$ calculated from Eq. [5].

Now let us start from weak attraction, $\epsilon_B \ll \epsilon_F$. For sufficiently small SOC, we have $I_1 \to 0$ and $I_2 \to -1$, and the solution is well approximated by $\Delta \approx \sqrt{2 E_F} \epsilon_F$ and $\mu \approx \epsilon_F - \epsilon_B/2 - 2\eta$, which indicates a BCS superfluid state. For large SOC, we expect that $\mu$ becomes negative and $|\mu| \gg \Delta$. Substituting this into the gap equation, we find $\mu \approx -E_B/2$, which indicates a Bose-Einstein condensate of molecules with binding energy $E_B$. Then expanding the number equation in powers of $\Delta/|\mu|$ and keeping the leading order, we obtain $\Delta \approx \sqrt{2 E_F} \epsilon_F \zeta(k)$, where $\zeta(k) = 2k^{-1}(k - 1)^{3/2}(2 \sqrt{k-1} + \frac{3}{2} - \arctan \left(\frac{3}{2 \sqrt{k-1}}\right))^{-1}$. This is a transparent formula to show that the pairing gap $\Delta$ increases with increased SOC, consistent with the perturbative approach [28]. These analytical results are in good agreement with the numerical results shown in Fig. 2 for intermediate $\lambda/k_F$ [29].

Using the fermion Green function $G(K)$, we can show that the fermion momentum distribution $n(k)$ is isotropic and can be expressed as $n(k) = (1/4) \sum_{\sigma} \left(1 - \xi_k^\sigma/E_k^\sigma\right)^2$ [27]. As shown in Fig. 3 with increased SOC, the distribution broadens, which indicates a BCS-BEC crossover. The new feature here is that the distribution generally displays nonmonotonic behavior. The peak in the distribution is just located at $k = \lambda$.

The pair wave functions $\phi_{\sigma\sigma'}(k) = \langle \psi_{k\sigma'}|\psi_{-k\sigma'} \rangle$ can be evaluated as $\phi_1(k) = -(\Delta/4)e^{i\epsilon_F} \sum_{\sigma} \alpha/E_k^\sigma$ and $\phi_1(k) = -(\Delta/4) \sum_{\sigma} 1/E_k^\sigma$ [27], where $e^{i\epsilon_F} = (k_x + i k_y)/|k|$. Therefore, the superfluid state exhibits both singlet and triplet pairings for nonzero SOC. The numerical results for the ratio $|\phi_1(k)|/|\phi_1(k)|$ displayed in Fig. 3 show that the triplet pairing spreads to wider momentum regime with increased SOC. According to the general formula for the condensate number of fermion pairs [30], $N_0 = \frac{1}{2} \sum_{\sigma, \sigma'} \int d^2r d^2\tau |\langle \psi_{\sigma}(r)|\psi_{\sigma'}(r') \rangle|^2$, the condensate density reads $n_0 = \sum_k [|\phi_1(k)|^2 + |\phi_1(k)|^2]$. The triplet pairing amplitude contributes, in contrast to the fermionic superfluids with only singlet pairing [31]. For large SOC, we find analytically that $2 N_0/N = 1 - O(\frac{\lambda}{k_F}) \to 1$ (see also Fig. 3), which indicates the Bose-Einstein condensation of weakly interacting rashbons.

In the presence of a trap potential $V(r) = \frac{1}{2} \omega^2 r^2$, the chemical potential becomes $\mu(r) = \mu_0 - V(r)$ and the density distribution $n(r)$ can be solved from the constraint $N = 2\pi \int dr n(r)$ in the local density approximation. As shown in Fig. 4 the atom cloud shrinks with increased SOC, which can be viewed as a preliminary experimental signal of the BCS-BEC crossover.

**BKT transition temperature** — At finite temperature in 2D we should rewrite the complex ordering field $\Phi(x)$ in terms of its modulus $\Delta(x)$ and phase $\theta(x)$, i.e., $\Phi(x) = \Delta(x) \exp[i\theta(x)]$. Since the random fluctuations of the phase $\theta(x)$ forbid long-
the Pauli matrices in the Nambu-Gor'kov space.

\[ \rho \text{our model can be evaluated as} \]

\[ \rho \text{der remains.} \]

less [34] showed that below a critical temperature \( T_{\text{c}}/n_{\text{eq}} = 1 \) \[ \text{range order in 2D, we have} \langle \Phi(x) \rangle = 0 \text{ but} \langle \Delta(x) \rangle \neq 0 \text{ at} \]

\[ \text{T} \neq 0, \text{ However, Berezinskii} \] [33] and Kosterlitz and Thouless [34] showed that below a critical temperature \( T_{\text{BKT}} \), there exist bound vortex-antivortex pairs and quasi-long-range order remains.

To determine the BKT transition temperature, we derive an effective action for the U(1) phase field \( \theta(x) \).

To this end we make a gauge transformation \( \psi(x) = \exp \left[i \theta(x)/2 \right] \psi(x) \) [16, 17]. Then we arrive at the expression \[ Z = \int \Delta D\Delta D\theta \exp \left[ -\beta U_{\text{eff}}[\Delta(x), \theta(x)] \right], \]

where the effective action \[ \beta U_{\text{eff}}[\Delta(x), \theta(x)] = U^{-1} \int dx \Delta^{2}(x) - \frac{1}{2} \text{Tr}_{\lambda} S^{-2}[\Delta(x), \theta(x)] \] now depends on the modulus-phase variables. The Green function of the initial (charged) fermions takes a new form \[ S^{-1}(\Delta(x), \theta(x)) = G^{-1}(\Delta(x)) - \Sigma[\theta(x)]. \]

Here \[ G^{-1}(\Delta(x)) = G^{-1}(\Delta(x), \Delta(x)) \] is the Green function of the neutral fermion, and \[ \Sigma[\theta] = \tau_{i}[\theta_{i}/2 + (\nabla \theta)^{2}/8 - \tilde{\rho}(\nabla \theta)^{2}/4 + i \nabla \theta \cdot \nabla \theta/2] + (1/2)[\tau_{i} \sigma_{i} \partial_{j} \theta - i \sigma_{i} \partial_{j} \theta], \]

where \( \tau_{i}(i = 1, 2, 3) \) are the Pauli matrices in the Nambu-Gor’kov space.

Since the low-energy dynamics for \( \Delta \neq 0 \) is governed by long-wavelength fluctuations of \( \theta(x) \), we neglect the amplitude fluctuations and treat \( \Delta \) as its saddle point value [16, 17]. Then the effective action can be decomposed as \[ U_{\text{eff}}[\Delta(x), \theta(x)] = U_{\text{eff}}[\Delta, \theta(x)] + U_{\text{pol}}[\Delta]. \]

The potential part reads \[ U_{\text{pol}}[V = \Delta^{2}/U + \sum_{i} \langle \xi_{i} \rangle - \tilde{W}(E_{c}^{+}) - \tilde{W}(E_{c}^{-}) \rangle \]

where \( \tilde{W}(E) = E/2 + T \ln(1 + e^{-\beta E}) \). The kinetic part can be obtained by the derivative expansion \[ \beta U_{\text{kin}}[\Delta, \theta(x)] = \sum_{n=1}^{\infty} \int \frac{d}{(G \Sigma)^{n}}. \]

Keeping only lowest-order derivatives of \( \theta(x) \), we find that the kinetic term \( U_{\text{kin}} \) coincides with the classical spin XY-model, which has the continuum Hamiltonian \( H_{\text{XY}} = \xi \int d^{2}r \left[ \nabla \theta^{2}(r) \right]^{2} \]

where the phase stiffness \( \bar{\mathcal{J}} = \frac{\xi}{m_{\text{B}}} \) and \( \rho_{s} \) is the superfluid density [35]. The superfluid density in our model can be evaluated as \[ \rho_{s} = n_{\text{eq}} = \rho_{s}, \]

where \[ \rho_{s}(\lambda/8\pi) = \sum_{s \neq 0} ^{\infty} d_{s} \left( \xi_{s}^{+} \xi_{s}^{-} \right) [1 - 2 f(E_{c}^{+})/E_{c}^{+}] \]

and \[ \rho_{s} = -1/4 \pi \sum_{s \neq 0} ^{\infty} d_{s} \left( \xi_{s}^{+} \xi_{s}^{-} \right) \]

For sufficiently small \( \xi_{0} \) and SOC, \( \Delta \) is correspondingly small and \( T_{\text{BKT}} \) recovers the mean-field result \( T_{\Delta} \). On the other hand, for large \( \xi_{0} \) and/or SOC, \( \rho_{s} \) can be well approximated by its zero-temperature value for \( T \sim T_{\text{BKT}} \). We are interested in the case with small \( \xi_{0} \) and large SOC. For large SOC, using the fact \( \Delta \ll |\mu| \), we find analytically that [27]

\[ \rho_{s}(T = T_{\Delta}) \approx \frac{2m}{m_{\text{B}}}, \] \[ \mathcal{J}(T = T_{\Delta}) \approx \frac{m_{\text{B}}}{m_{\text{B}}}. \]
For an inter-atomic potential described by a 2D circularly symmetric well, the binding energy $\epsilon_B$ is given by $\epsilon_B = 1/(2r_0^2) \exp[-2/(\pi r_0^2)]$ in the dilute limit $\nu r_0^2 \to 0$. For quasi-2D cold atoms confined by an axial trapping frequency $\omega_z$, the binding energy is given by $\epsilon_B = (C\omega_z/\pi) \exp[\sqrt{2\pi}/a_s]$, where $a_s$ is the 3D s-wave scattering length, $l_s = \sqrt{\hbar/\omega_z}$, and $C \approx 0.915$. See D. S. Petrov and G. V. Shlyapnikov, Phys. Rev. A64, 042705 (2001).

See Supplemental Material for details of the derivation.

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[26] For an inter-atomic potential described by a 2D circularly symmetric well of radius $r_0$ and depth $\nu_0$, the binding energy $\epsilon_B$ is given by $\epsilon_B = 1/(2r_0^2) \exp[-2/(\pi r_0^2)]$ in the dilute limit $\nu r_0^2 \to 0$. For quasi-2D cold atoms confined by an axial trapping frequency $\omega_z$, the binding energy is given by $\epsilon_B = (C\omega_z/\pi) \exp[\sqrt{2\pi}/a_s]$, where $a_s$ is the 3D s-wave scattering length, $l_s = \sqrt{\hbar/\omega_z}$, and $C \approx 0.915$. See D. S. Petrov and G. V. Shlyapnikov, Phys. Rev. A64, 042705 (2001).

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Appendix: In this supplementary material, we present the derivation details of some results in the main text.

(A) Two-Body Problem: Binding Energy and Effective Mass

Using the free fermion propagators $g_{\pm}(K)$, $\Gamma^{-1}(Q)$ can be expressed as

$$\Gamma^{-1}(Q) = \frac{1}{U} - \frac{1}{2} \sum_k \text{Tr} \left[ g_+(K + Q) \sigma_i g_-(K) \sigma_i \right].$$

(7)

Completing the Matsubara frequency sum, we obtain Eq. (2) of the text. For the two-body problem, we discard the Fermi-Dirac distribution function and define the solution for $\Gamma^{-1}(\omega, q) = 0$ as $E_q = -(\omega + 2\mu)$. The two-body equation becomes

$$\sum_k \left( \frac{2}{k^2 + \epsilon_B} - \frac{2E_{kq}}{E_{kq}^2 - 4\lambda^2 k^2 - 4\rho(2\epsilon^2 - \lambda^2 k^2)} \right) = 0.$$ 

(8)

Here $\varphi$ is the angle between $k$ and $q$, and $E_{kq} = E_q + \epsilon_{k+q/2} + \epsilon_{k-q/2} = E_q + k^2 + q^2/4$.

For zero center-of-mass momentum $q$, the above equation reduces to $\int_0^\infty dk [2(k^2 + \epsilon_B)^{-1} - \sum_{\omega=\pm}(k^2 + 2\epsilon \lambda k + E_B)^{-1}] = 0$. The integral can be carried out directly. The easiest way is to use the trick $\int_0^\infty dk \frac{k^2}{(k^2 + \epsilon_B)^2 - 4\lambda^2 k^2}$.

For nonzero center-of-mass momentum $q$, we write $E_q \approx E_B - q^2/(2m_B)$ for small $q^2$ and expand Eq. (8) to the order $O(q^2)$, then we obtain

$$\left(1 - \frac{2m}{m_B}\right) \int_0^\infty dk \frac{(k^2 + E_B)^2 + 4\lambda^2 k^2}{(k^2 + E_B)^2 - 4\lambda^2 k^2} = \int_0^\infty dk \frac{8\lambda^4 k^2}{(k^2 + E_B)^2 - 4\lambda^2 k^2}.\]$$

(10)

Defining $\kappa = E_B/\lambda^2$, this equation becomes

$$1 - \frac{2m}{m_B} = 2\int_0^\infty dx \frac{x}{(x + \kappa)[(x + \kappa)^2 - 4x]} \left[ \int_0^\infty \frac{d\xi}{[(x + \kappa)^2 - 4x]} \right]^{-1}.\]$$

(11)

Completing the integrals analytically, we obtain Eq. (4) of the text.

(B) Derivation of the Ground-State Energy

In the mean-field approximation, the ground-state energy can be expressed as

$$\Omega = \frac{\Delta^2}{U} - \frac{1}{2} \beta \sum_k \ln \text{det} \Gamma^{-1}(i\omega_n, k),$$

(12)

where the inverse fermion Green function reads

$$\Gamma^{-1}(i\omega_n, k) = \begin{pmatrix} i\omega_n - \xi_k + \hbar \sigma_z - \lambda(k, \sigma_x - k, \sigma_y) & i\sigma_z \Delta \\ -i\sigma_x \Delta & i\omega_n + \xi_k - \hbar \sigma_z - \lambda(k, \sigma_x + k, \sigma_y) \end{pmatrix}.$$ 

(13)

Using the formula for block matrix, we first work out the determinant and obtain

$$\text{det} \Gamma^{-1}(i\omega_n, k) = \left[ (i\omega_n)^2 + \hbar^2 - \xi_k^2 - \lambda^2 k^2 - \Delta^2 \right]^{-1} - 4\hbar^2 (i\omega_n)^2 - 4\lambda^2 k^2 (\xi_k^2 - \hbar^2).$$

(14)

Then completing the Matsubara frequency sum and taking $T = 0$ we obtain $\Omega = $ and $\Sigma_k (2\epsilon_k - E^+_k - E^-_k)$ where the term $\sum_k \xi_k$ is added to recover the correct ground state energy for the normal state ($\Delta = 0$). The quasiparticle dispersions are given by the positive roots of the equation $\text{det} \Gamma^{-1} = 0$, i.e.,

$$E^+_k = \left[ \xi_k^2 + \Delta^2 + \lambda^2 k^2 + \hbar^2 + 2\sqrt{\xi_k^2(\lambda^2 k^2 + \hbar^2) + \hbar^2 \Delta^2} \right]^{1/2}.\]$$

(15)
For \( h = 0 \), they reduces to \( E^2 = (\xi_k + \lambda k)^2 + \Delta^2 \). At finite temperature, the thermodynamic potential reads \( \Omega = U_{\text{pot}}/V = \Delta^2 / U + \sum_{k} [W(E^{\uparrow}_{k}) - W(E^{\downarrow}_{k})] \) where \( W(E) = E^2 / 2 + T \ln (1 + e^{-\beta E}) \).

For \( T = 0 \) and \( h = 0 \), the ground-state energy can be expressed in terms of \( E_B \) as \( \Omega = (\Delta^2 / 4\pi) \sum_{\sigma=\pm} \int_{0}^{\infty} dk [2E_k + 2\alpha k + E_B]^{-1} - (E^2_k + \xi_k^2)^{-1} \). Since the integrals are convergent, we can use the trick \( k^2 + 2\alpha k = (k + \lambda)^2 - \lambda^2 \) and convert the integration variables to \( k + \lambda \). After a straightforward calculation, we obtain

\[
\Omega = \Omega_{2D}(\Delta, \mu, E_B) + \frac{\Delta^2}{4\pi} \frac{2\lambda}{\sqrt{E_B - \lambda^2}} \arctan \frac{\lambda}{\sqrt{E_B - \lambda^2}} + \Omega_J. \tag{16}
\]

Noticing the fact that \( E_B \) satisfies Eq. (3) of the text, we obtain \( \Omega = \Omega_{2D}(\Delta, \mu, E_B) + \Omega_J \).

(C) Solution of the Gap and Number Equations at Large SOC

The original forms of the gap and number equations at \( T = 0 \) are

\[
\frac{1}{U} = \frac{1}{2} \sum_{k} \left( \frac{1}{2E^{\uparrow}_{k}} + \frac{1}{2E^{\downarrow}_{k}} \right), \quad n = \sum_{k} \left( 1 - \frac{\xi_k^2}{2E^{\uparrow}_{k}} - \frac{\xi_k^2}{2E^{\downarrow}_{k}} \right). \tag{17}
\]

For large SOC, we expect \( \mu < 0 \) and \( \Delta \ll |\mu| \). Therefore, we can expand the equations in powers of \( \Delta / |\mu| \) and keep only the leading order terms. The gap equation becomes

\[
\int_{0}^{\infty} dk \left( \frac{2}{k^2 + \epsilon_B} - \sum_{\sigma=\pm} \frac{1}{k^2 + 2\alpha \lambda k - 2\mu} \right) = 0. \tag{18}
\]

We obtain \( \mu = -E_B / 2 \). Substituting this into the number equation, we obtain

\[
n = \frac{\epsilon_f}{\pi} = \frac{\Delta^2}{2\pi} \sum_{\sigma=\pm} \int_{0}^{\infty} dk \left( \frac{2}{k^2 + \epsilon_B + E_B^2} \right) = \frac{\Delta^2}{4\pi} \int_{0}^{\infty} dk \frac{(k^2 + E_B^2) + 4\lambda^2 k^2}{(k^2 + E_B^2)^2 - 4\lambda^2 k^2}. \tag{19}
\]

We notice that the integral also appears in Eq. (10). Completing the integral analytically, we obtain \( \Delta = \sqrt{2E_B \epsilon_f \zeta(\kappa)} \) where \( \zeta(\kappa) \) is defined in the text.

(D) The Fermion Green Function and Related Quantities

The explicit form of the fermion Green function \( G(i\omega_n, k) \) can be evaluated using the formula for block matrix. For \( h = 0 \), we find that the matrix elements (in the Nambu-Gor'kov space) can be expressed as

\[
G_{11} = \mathcal{A}_{11} + \frac{k_1 \sigma_x - k_2 \sigma_y}{k} \mathcal{B}_{11}, \quad G_{22} = \mathcal{A}_{22} + \frac{k_1 \sigma_x + k_2 \sigma_y}{k} \mathcal{B}_{22}, \quad G_{12} = -i\sigma_y \left[ \mathcal{A}_{12} + \frac{k_1 \sigma_x + k_2 \sigma_y}{k} \mathcal{B}_{12} \right], \quad G_{21} = i\sigma_y \left[ \mathcal{A}_{21} + \frac{k_1 \sigma_x - k_2 \sigma_y}{k} \mathcal{B}_{21} \right]. \tag{20}
\]

Here \( \mathcal{A}_{ij} \) and \( \mathcal{B}_{ij} \) take the forms

\[
\mathcal{A}_{11} = \frac{1}{2} \sum_{\alpha=\pm} \frac{i\omega_n + \xi_k^\alpha}{(i\omega_n)^2 - (E^\alpha_k)^2}, \quad \mathcal{A}_{22} = \frac{1}{2} \sum_{\alpha=\pm} \frac{i\omega_n - \xi_k^\alpha}{(i\omega_n)^2 - (E^\alpha_k)^2}, \quad \mathcal{A}_{12} = \frac{1}{2} \sum_{\alpha=\pm} \frac{\Delta}{(i\omega_n)^2 - (E^\alpha_k)^2}, \quad \mathcal{A}_{21} = \mathcal{A}_{12}, \tag{21}
\]

and

\[
\mathcal{B}_{11} = \frac{1}{2} \sum_{\alpha=\pm} \frac{\Delta}{i\omega_n)^2 - (E^\alpha_k)^2}, \quad \mathcal{B}_{22} = -\frac{1}{2} \sum_{\alpha=\pm} \frac{i\omega_n - \xi_k^\alpha}{(i\omega_n)^2 - (E^\alpha_k)^2}, \quad \mathcal{B}_{12} = -\frac{1}{2} \sum_{\alpha=\pm} \frac{\Delta}{(i\omega_n)^2 - (E^\alpha_k)^2}, \quad \mathcal{B}_{21} = -\mathcal{B}_{12}. \tag{22}
\]
Using the matrix elements of the Green function, we can calculate various quantities. First, the momentum distribution can be evaluated as

\[ n(k) \equiv \langle \tilde{\psi}_k \psi_k \rangle = \frac{1}{\beta} \sum_n \mathcal{A}_{11}(i\omega_n, k)e^{i\omega_n0^+}. \] (23)

Second, the singlet and triplet pairing amplitudes can be expressed as

\[ \phi_{11}(k) \equiv \langle \psi_k \psi_{-k} \rangle = \frac{1}{\beta} \sum_n \mathcal{A}_{21}(i\omega_n, k), \quad \phi_{1\uparrow}(k) \equiv \langle \psi_k \psi_{-k} \rangle = -\frac{1}{\beta} 2 \sum_n \mathcal{B}_{21}(i\omega_n, k), \]

\[ \phi_{1\downarrow}(k) \equiv \langle \psi_k \psi_{-k} \rangle = \frac{k^2 - 4k}{k} \sum_n \mathcal{B}_{21}(i\omega_n, k), \quad \phi_{1\downarrow}(k) \equiv \langle \psi_k \psi_{-k} \rangle = \frac{k^2 + 4k}{k} \sum_n \mathcal{B}_{21}(i\omega_n, k). \] (24)

Therefore, we have the relations \( \phi_{11}(k) = -\phi_{1\downarrow}(k) \) and \( \phi_{1\uparrow}(k) = -\phi_{1\downarrow}(k) \).

According to Leggett’s definition \( \Sigma \), the condensate number of fermion pairs is given by

\[ N_0 = \frac{1}{2} \sum_{\sigma, \sigma'} \int \int d^2r d^2r' |\langle \psi_\sigma(r) \psi_{\sigma'}(r') \rangle|^2. \] (25)

For systems with only singlet pairing, this recovers the usual result \( N_0 = \int \int d^2r d^2r' |\langle \psi_\sigma(r) \psi_{\sigma'}(r') \rangle|^2 \). Converting this to the momentum space, we find that the condensate density \( n_0 = N_0/V \) should be a sum of all absolute squares of the pairing amplitudes. The final result for \( T = 0 \) is

\[ n_0 = \frac{1}{8} \sum_k \left[ |\phi_{11}(k)|^2 + |\phi_{1\uparrow}(k)|^2 + |\phi_{1\downarrow}(k)|^2 \right] \]

\[ = \frac{1}{8} \sum_k \left[ \frac{\Delta^2}{(E_k^+)^2} + \frac{\Delta^2}{(E_k^-)^2} \right]. \] (26)

For large attraction and/or SOC, we expect \( \Delta \ll |\mu| \). Using the number equation \( \Sigma \) and expanding all terms in powers of \( \Delta/|\mu| \), we can show that \( 2N_0/N = 1 - O(\Delta^4/|\mu|^4) \). Therefore, the condensate fraction approaches unity at large attraction and/or SOC.

**(E) Effective Action of the Phase Field**

To obtain the effective action for the phase field \( \theta(x) \) to the order \( \nabla \theta^2 \), we notice that the available operators in \( \Sigma[\theta \theta] \) are \( \Sigma_1 = \tau_3(\nabla \theta)^2/8, \Sigma_2 = -i\nabla \theta \cdot \nabla /2 \) and \( \Sigma_3 = (\lambda/2)(\tau_3 \sigma \partial_i \theta - i\sigma \partial_i \theta) \). According to the derivative expansion, we have carefully checked that there are four types of nonzero contributions:

\[ U_1 \sim \text{Tr}(\Sigma_1), \quad U_2 \sim \text{Tr}(\Sigma_2 \Sigma_2), \quad U_3 \sim \text{Tr}(\Sigma_2 \Sigma_3), \quad U_4 \sim \text{Tr}(\Sigma_3 \Sigma_3). \] (27)

Since the superfluid state is isotropic, the phase stiffness should also be isotropic. We have carefully checked that all anisotropic terms vanish exactly. Completing the trace in the Nambu-Gor’kov and spin spaces, we finally obtain the following expressions for the four types of contributions:

\[ U_1 = \frac{1}{2} \sum_n \sum_k \int \int d^2r (\nabla \theta)^2 \]

\[ U_2 = \frac{1}{2} \sum_n \sum_k \int \int d^2r (\mathcal{A}_{11}^2 + \mathcal{B}_{11}^2 + \mathcal{A}_{22}^2 + \mathcal{B}_{22}^2 + 2\mathcal{A}_{21}^2 + 2\mathcal{B}_{21}^2) \]

\[ U_3 = \frac{1}{2} \sum_n \sum_k \int d^2r (\mathcal{A}_{11}^2 + \mathcal{A}_{22}^2 + 2\mathcal{A}_{21}^2) \]

\[ U_4 = \frac{1}{2} \sum_n \sum_k \int d^2r (\mathcal{A}_{11}^2 + \mathcal{B}_{22}^2 + 2\mathcal{A}_{21}^2 \mathcal{B}_{21}) \] (28)
Collecting all terms, the effective action is reduced to a spin XY-model Hamiltonian $H_{XY} = \frac{1}{2} \int d^2 r [\nabla \theta (r)]^2$, where the phase stiffness $\mathcal{J}$ is given by

$$
\mathcal{J} = \frac{1}{\beta} \sum_n \sum_k \left[ \frac{1}{4} (A_{11} e^{i\omega_0} - A_{22} e^{-i\omega_0}) + \frac{k^2}{8} \left( A_{11}^2 + B_{11}^2 + A_{22}^2 + B_{22}^2 + 2A_{21}^2 + 2B_{21}^2 \right) \right]
$$

Completing the Matsubara frequency sum we then obtain the expression given in the text.

\textbf{(F) Properties of the Superfluid Density}

First, setting $\Delta = 0$, we find that $\rho_s = 0$. Therefore $\rho_s$ vanishes exactly in the normal state, as expected. Second, for vanishing SOC, the expressions of $\rho_s$ and $\mathcal{J}$ recover the well known form given in \cite{17}. Here we will examine the behavior of $\rho_s$ for large SOC at $T = 0$. At zero temperature, the superfluid density reduces to

$$
\rho_s = n - \rho_A, \quad \rho_A = \frac{\lambda}{8\pi} \int_0^\infty dk \left[ \frac{1}{\xi_k^+} + \frac{1}{\xi_k^-} \right] = \frac{\Delta^2}{\pi} \int_0^\infty d\xi_k \frac{\xi_k^+}{\xi_k^-} \left[ 1 + \frac{2}{\xi_k^+} - \frac{1}{\xi_k^-} \right].
$$

Therefore, even at $T = 0$, the superfluid stiffness does not recover the result $\rho_s = n$ for ordinary fermionic superfluids. Let us show what happens at large $\lambda$. In this case $\mu \approx -E_B/2$ and $\Delta \ll |\mu|$. Therefore, we can expand the expression in powers of $\Delta/|\mu|$ and keep only the leading order terms. Doing so, we obtain (see Eq. \ref{19})

$$
n \approx \frac{\Delta^2}{8\pi} \lambda \int_0^\infty dk \frac{1}{(\xi_k^+)^2 + (\xi_k^-)^2} \approx \frac{\Delta^2}{\pi} \int_0^\infty d\xi_k \frac{(k^2 + E_B)^2 + 4\lambda^2 k^2}{(k^2 + E_B)^2 - 4\lambda^2 k^2}.
$$

and

$$
\rho_A \approx \frac{\Delta^2}{8\pi} \lambda \int_0^\infty dk \left\{ \frac{1}{\xi_k^+} - \frac{1}{\xi_k^-} - \frac{1}{2} \left( \frac{1}{(\xi_k^+)^2} - \frac{1}{(\xi_k^-)^2} \right) \right\} \approx \frac{\Delta^2}{\pi} \int_0^\infty d\xi_k \frac{8\lambda^4 k^2}{(k^2 + E_B)^2 [(k^2 + E_B)^2 - 4\lambda^2 k^2]^2}.
$$

Comparing the above results with Eq. \ref{10}, we find that $\rho_A/n = 1 - 2m/m_B$. Therefore, for large SOC, the superfluid density and the phase stiffness are reduced to

$$
\rho_s = \frac{2m}{m_B} n, \quad \mathcal{J} = \frac{2m}{m_B} \frac{n}{4m} = \frac{n_B}{m_B}
$$

where $n_B = n/2$ is the density of rashbons. This means that, at large SOC, the phase stiffness self-consistently recovers that for a rashbon gas.