Dynamics of directed graphs: the world-wide Web

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We introduce and simulate a growth model of the world-wide Web based on the dynamics of outgoing links that is motivated by the conduct of the agents in the real Web to update outgoing links (re)directing them towards constantly changing selected nodes. Emergent statistical correlation between the distributions of outgoing and incoming links is a key feature of the dynamics of the Web. The growth phase is characterized by temporal fractal structures which are manifested in the hierarchical organization of links. We obtain quantitative agreement with the recent empirical data in the real Web for the distributions of in- and out-links and for the size of connected component. In a fully grown network of $N$ nodes we study the structure of connected clusters of nodes that are accessible along outgoing links from a randomly selected node. The distributions of size and depth of the connected clusters with a giant component exhibit supercritical behavior. By decreasing the control parameter—average fraction $\beta$ of updated and added links per time step—towards $\beta_c(N) < 10\%$ the Web can resume a critical structure with no giant component in it. We find a different universality class when the updates of links are not allowed, i.e., for $\beta \equiv 0$, corresponding to the network of science citations.

I. INTRODUCTION

Understanding the evolution of complex networks remains a challenging problem of modern statistical physics. Modeling the dynamic structure of these networks has significant practical applications, for instance in the design of search and control algorithms and in predicting emergence of new phenomena in information networks. An evolving network consists of increasing number of nodes, which are connected by links according to certain rules specific to each network. Although the spatial distribution of nodes in a network is random, a different structure can be observed when the distribution of links is considered. In this respect two major groups are: (1) random networks, with the number of links attached to a node fluctuating around an average value, and (2) networks with hierarchical structure of links, manifested in the power-law decay of the distributions of node ranks. The structure of connections has immediate impact on the accessibility of nodes and on functional stability of the network. For instance, the hierarchically connected networks are shown to be robust to large failure rates, however, they are vulnerable to removal of a few 'key nodes'. The inhomogeneous structure of connections has been found in metabolic cycles, in various social networks, including the science citation network, and the Internet and the world-wide Web.

The world-wide Web is not a separate physical networks. Rather it is a subset of the Internet that uses a specific protocol (hypertext transfer protocol) and directed links to access a variety of data, representing an example of distributed client/server computing. Based on these properties a hierarchical structure of links emerges that is peculiar to the world-wide Web. Recently large-scale measurements in the world-wide Web that involved $10^8$ nodes reveal that: (a) Both the distributions of incoming links and distribution of outgoing links follow a power-law behavior with different exponents; (b) The distribution of size of connected component—number of linked nodes in the web crawls—also exhibits scaling behavior; and (c) The Web shows a very intricate structure of connections, with a large strongly connected component in the center.

Recently a great deal of effort was devoted to understand emergence of the scale-free structure of links in various complex networks. An important ingredient of the dynamics—preferential attachment was shown to lead to the power-law distribution for incoming links (the outgoing links are fixed by the rules of the model). The model of preferential attachments for incoming links has been generalized to include initial attractiveness of nodes in Ref. [1], where the authors proved analytically the existence of the scaling region for large evolution times. Recently it was demonstrated in Ref. [2] how a giant connected component can occur in the model of random directed graphs assuming arbitrary and statistically independent distributions of incoming and outgoing links. The predictions of the random graph theory, however, show substantial quantitative differences compared to the properties of the real world-wide Web.

In this work we simulate growth and response of a directed graph with the dynamic rules that are motivated by prominent features of the world-wide Web. These dynamic rules yield the statistically correlated distributions of outgoing and incoming links. Another important feature of the model is that the links between pairs of nodes are not fixed in time, but may vary on the time scale of the network’s evolution (updates of links). This property, that may be found in some metabolic cycles, is not shared...
with many other social networks, whose physical links are either fixed or vary on much slower time scale. We argue that the (at present high) frequency of updates of the outgoing links, which is peculiar to the agents in the Web, is essential for the observed inhomogeneous structure of connections. We find a consistent agreement with the recent empirical data on the real world-wide Web both in the structure of links and in the response of the network.

The paper is organized as follows: In Sec. II we introduce the growth rules of the network and determine the emergent rank distributions of outgoing and incoming links. In Sec. III we study the temporal fractal structures in the growth phase of the network and determine the corresponding distributions of the first-return links. Sec. IV is devoted to investigation of the connected components on the fully grown network and fractal properties of noise. A short summary and the discussion of the results is given in Sec. V.

II. GROWTH MODEL AND RANK DISTRIBUTIONS

We suggest the following simplified set of rules to incorporate basic features pertinent to the real world-wide Web: (i) Directed nature of linking. The world-wide Web represents a directed graph with nodes corresponding to Web pages, and arcs corresponding to hyperlinks between pages \[\text{[5][13]}\]. (j) Growth and rearrangements at unique time scale. At each time unit \(t\) a new node \(i = t\) is added to the network \((\text{growth})\) and a number \(M(t)\) of new links are distributed among the nodes \((\text{update of links})\) following two rules specified below. A fraction \(f_0(t) \equiv \alpha M(t)\) of new links are outgoing links from the new added node \(i = t\), whereas the rest \(f_1(t) \equiv (1 - \alpha)M(t)\) are the updated links at other nodes in the network. Hence, the relevant parameter in the model is the ratio of updated and added links at each time step, i.e., \(\beta \equiv f_1(t)/f_0(t) = (1 - \alpha)/\alpha\), which is independent of the actual number of links \(M(t)\). Thus, in the model \(M(t)\) represents a net increment of the number of outgoing links at time step \(t\). Variations in \(M(t)\) are, in principle, not restricted, being caused by adding new connections between old nodes (resulting in the increase of the number of links), and/or by removing some earlier links (causing decrease of the number of links). We assume that variations in \(M(t)\) are such that we can define an average value \(\bar{M} \equiv \overline{M(t)}\), which can be considered as a constant in first approximation. In practice, the number of nodes \(n\) and the number of links in the network increases with time, so that reasonable values for the average \(\bar{M}\) are positive. For consistency, we keep \(\bar{M} = 1\) throughout this work. Different values of \(\bar{M}\) do not affect the universal properties in the scaling region \((\text{i.e., for large evolution times})\). Taking much larger \(\bar{M}\) values, however, considerably increases computation time.

(k) Preferential update \((k1)\) and preferential attachment \((k2)\). These properties represent a paradigm of social behavior, for instance preferential attachments are driven by “popularity” of a node, as discussed in earlier models of evolving social networks. In addition, here we have that not all nodes are getting updated at every time step, rather only a few nodes update at a time. Moreover, some of the nodes update outgoing links more frequently than others. This features can be formulated in terms of probabilities as follows: Where do the links come from? We assume that apart from the new node, updating at time \(t\) occurs with larger probability at most active nodes \((\text{preferential activity})\), i.e., an outgoing link from the node \(k \leq i\) appears with the probability

\[
\text{Prob}_1 = \frac{\alpha M + q_{\text{out}}(k, t)}{(1 + \alpha)M + i}.
\]

A new link goes to the node \(n\) (whose age is \(t - n\)) with the probability

\[
\text{Prob}_2 = \frac{\alpha M + q_{\text{in}}(n, t)}{(1 + \alpha)M + i},
\]

i.e., “a popular” node attracts even more links. Here \(q_{\text{out}}(n, t)\) and \(q_{\text{in}}(n, t)\) are the dynamically varying number of outgoing and incoming links, respectively, at the node \(n\) at current time \(t\). It is assumed that at the time of addition of a node \(i\) to the network \(q_{\text{out}}(i, i) = q_{\text{in}}(i, i) = 0\). Note that the second rule in Eq. \((2)\) is formally equivalent to the model of preferential attachment for incoming links \([1]\). However, in our model the role of the parameter \(\alpha\) in this equation is precisely determined through the rule in Eq. \((1)\), which regulates outgoing links: (1) For \(\alpha = 1/(1 + \beta) < 1\) a fraction \((1 - \alpha)M\) of outgoing links in Eq. \((1)\) refers to updates at earlier nodes; (2) For \(\alpha = 1\) (equivalent to \(\beta = 0\)) no updates are allowed. New links originate only from the new added node; (3) Formally one can apply Eqs. \((1)\) \((2)\) for \(\alpha\) strictly larger than unity, however, in this region \(\beta\) becomes negative, suggesting that the number of links in the network is not conserved. Therefore only the incoming links \((2)\) can be counted correctly when \(\alpha\) is in the interval \(\infty > \alpha > 1\); (4) In the mathematical limit \(\alpha \rightarrow \infty\), or in practical cases when \(\alpha \gg \bar{q}/M\), the effects of the dynamical quantities \(q_{\text{out}}(k, t)\) and \(q_{\text{in}}(n, t)\) become negligible, and we recover the case of fully random network.

In what follows we would like to demonstrate that the features \((i), (j), (k1)\) and \((k2)\) explained above are essential for the behavior observed in the Web. Earlier models \([1][13]\) concentrated only on the preferential attachment rule \((k2)\), while largely neglecting properties \((i), (j)\) and \((k1)\). The most complete account of the preferential attachment model is given in Ref. \([1]\). On the other side, the random graph model \([3]\) considered the directed nature of network \((i)\), while neglecting the details of the linking rules \((j), (k1)\) and \((k2)\).
Using the rules in Eqs. (1)-(2) we grow a large network of $N = 10^6$ nodes. We measure the ranking of nodes in this network by the distributions of outgoing links and incoming links, $P(q_{\text{out}})$ and $P(q_{\text{in}})$, respectively, which are shown in Fig. 1 for parameter $\beta = 3$, corresponding to $\alpha = 0.25$ in Eqs. (1)-(3). For comparison we also simulate the distributions of outgoing and incoming links in the case of fully random directed network. For a range of values of their arguments the cumulative distributions for finite $\beta$ values exhibit a power-law behavior according to

$$P(q_{\text{out}}) \sim q_{\text{out}}^{-(\tau_{\text{out}} - 1)}; \quad P(q_{\text{in}}) \sim q_{\text{in}}^{-(\tau_{\text{in}} - 1)}. \quad (3)$$

By increasing the parameter $\alpha$ in Eqs. (1)-(3) in the range $(0, 1)$, corresponding to decrease of the relative fraction of updated links $\beta$ in the interval $(\infty, 0)$, the slopes of the distributions increase. The scaling exponents $\tau_{\text{in}}$ and $\tau_{\text{out}}$ given in the inset to Fig. 1 are parametrized by $\alpha$. Our numerical results for $\tau_{\text{in}}$ agree with the exact solution $\tau_{\text{in}} = 2 + \alpha$ for the incoming links. For the distribution of outgoing links no analytical results are available. Our numerical results in the inset to Fig. 1 can be well approximated with the linear dependence $\tau_{\text{out}} \approx 2 + 3\alpha$. Therefore, the difference between the two exponents $\Delta \equiv \tau_{\text{out}} - \tau_{\text{in}} \approx 2\alpha$ persists for any finite $0 < \alpha < 1$. Comparing these results with the available empirical data we estimate the parameter $\alpha = 0.22 \pm 0.1$. Here we used the following values as the experimental estimates for the exponents $\tau_{\text{in}} = 2.16 \pm 0.1$, obtained as the average value from the data in [1] and last reference in [1], and $\tau_{\text{out}} = 2.62 \pm 0.1$ from data in [1], by fitting only the straight part of the curves while avoiding the noisy tails and the curvature at small $q$. It is clear that these values are not definitive. More careful fits of the data and further measurements are necessary in order to reduce the experimental error bars. Hence, the approximative value of the control parameter $\beta$ in the current state of the Web is $\beta \approx 3$, i.e., in the average three updated links come to one added link at each evolution step. We now discuss the properties of the network for this particular value of the control parameter $\beta$.

### III. DYNAMIC FRACTAL STRUCTURES

In the dynamical systems with large-scale organization the occurrence of algebraically decaying distributions can be linked to the formation of certain fractal structures [11] in the system. In the case of complex evolving networks, however, the existence of such structures is less clear, due to the spatial randomness of the graph. To show that the power-law decay of the distributions of node ranks, shown in Fig. 1, is associated with certain fractal structures on the network we examine the temporal pattern of linking between nodes. A part of the pattern is shown in Fig. 2 (top panel). Clustering in the patterns indicate frequently active nodes, whereas voids of various sizes indicate lesser activity at corresponding nodes. Visually heterogeneous picture suggests that both pattern of origins of links and pattern of targets have fractal structure. The fractality of these patterns can be quantified, for instance, by computing the distribution of time intervals between two successive linking at a given node (return time). The distribution of time intervals $\Delta t$ between two successive linking to a selected node in the network $P(\Delta t)$ measured for the network of 10,000 nodes and the parameter $\beta = 3$ is given in Fig. 2 (lower panel). The algebraic decay of the distribution is a signature of the underlying fractal structure of the pattern. The distribution of time intervals between two successive links from a selected node also shows a power-law tail for large $\Delta t$. At small intervals $\Delta t$ the pattern is nearly random. The two distributions coincide, indicating mutual correlations in the patterns, for time intervals $\Delta t > 100$ (that are accessible for large evolution times $t$). To emphasize the relation to Fig. 1 we recall that the number of links of either kind at a node accumulates with time. Hence, the regions of large time intervals between successive linking at a node, $\Delta t$ in Fig. 2, correspond to small number of links at that node, $q$ in Fig. 1, and vice versa.

### IV. DYNAMIC CRITICAL STATES

The scaling behavior of the rank distributions can be examined in terms of the potential dynamic critical states by measuring the statistics of triggered avalanches [16] on the network. We consider an analog of the avalanche size in the networks: the size of a cluster of nodes which are physically accessible along directed links starting from a random node in the network. A network is grown using the rules in Eqs. (1)-(2). We then select a random node and make a list of nodes which are connected by outgoing links from that node. In the next step we make a new list following outgoing links from the nodes on the previous list, keeping only the nodes which are different in the two lists, and so on. The process ends up when no new nodes can be reached—the list is empty. The number of different lists before an empty list occurs can be recognized as the depth of the connected component. Technically, we preserve ranks $q_{\text{in}}$ and $q_{\text{out}}$ for each node from the growth phase and, in the case of small networks, the exact physical links between the nodes. In large networks only ranks are preserved and links are searched using the rule in Eq. (1). For a large ensemble of networks the results are expected to be statistically the same. In this way the size of a cluster of linked nodes represents a response of the system on the random excitation. The process is reminiscent of the invasion percolation on a random graph, where a new edge $j$, which already does not belong to the graph, is invaded along an outgoing hyperlink $i \rightarrow j$. 

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from the node $i$ if the node rank $q_{out}(i)$ exceeds unity. The probability to add the edge $j$ to the graph is given by the preference rule in Eq. (1). In a more familiar case of the invasion percolation in physical wetting on the lattice [17] that probability is given by the least resistance rule. To our knowledge the problem of invasion percolation on random graphs has not been studied so far [20].

The differential distribution of size of connected clusters of nodes is given in Fig. 3 for different values of the parameter $\beta$. The distribution of depth of the same clusters is given in the inset to Fig. 3. In the case $\beta = 3$ we find that small clusters follow a power-law distribution with the exponent $\tau_s = 2.79 \pm 0.12$, comparable with the measured [8] value $\tau_s = 2.52$. (Note that in the case of Internet the corresponding exponent is estimated [8] as $\tau_s = 1.90$.) In Fig. 3 the occurrence of the peak in the distributions at large clusters $S_0 \approx 10^2$ indicates that a large number of the avalanches have approximately same size. This is a signature of the existence of a large subset of strongly connected nodes (a giant component): Once a 'crawl' enters the subset of strongly connected nodes, it explores it entirely. By taking a larger number of nodes $N$ the position of the peak moves towards larger values approximately as $S_0 \sim N/7$.

Before discussing the distributions in Fig. 3, we demonstrate that the network with the hierarchical structure of links is characterized by a fractal noise. In the search of a connected cluster described above we examine detailed variation of the number of nodes added to the cluster at each step of investigation. We obtain the noisy signal shown Fig. 4. Properties of the signal vary with the control parameter of the dynamics $\beta$. Considering the number of steps as a total elapsed time of investigation, the Fourier spectrum of the signal is shown in the top panel in Fig. 4. It shows a region of correlated behavior with power-law decay between the upper cut off at high frequencies and lower cut off (due to finite size of the network). In addition, the spectrum in the case of supercritical network exhibits a peak at a characteristic frequency $f_0$, which is absent in the case of the critical parameter $\beta = \beta_c(N)$ (cf. Fig. 4).

V. DISCUSSION AND CONCLUSIONS

In the theory of critical states [20,11], appearance of the peak in the distribution of avalanches, such as the distributions shown in Fig. 3, indicates that the system is supercritical. In the critical state the absence of any characteristic scale is manifested in the purely algebraic decay of the distribution of cluster size until a cut-off, which depends on the size of the network $N$. On the other hand, the form of distribution in Fig. 3 suggests that a critical point exists that can be reached by varying a relevant parameter of the dynamics [11]. In fact, by decreasing the parameter $\beta$ we see in Fig. 3 that the slope $\tau_s$ decreases and the peak eventually disappears at a critical value [20] of the parameter $\beta = \beta_c \approx 0.081$. Beyond the critical value of the control parameter the scaling behavior of the size and depth of connected components is entirely lost. In the critical state the scaling exponents are (cf. inset to Fig. 1): $\tau_{in} \approx 2.925$, $\tau_s = \tau_d = 1$. The distribution of outgoing links shows very sharp decay, the exponent $\tau_{out}$ is difficult to measure. It should be stressed that the absence of the peak in the critical state suggests that no giant component can be formed. Rather, the network consists here of many small groups of well interconnected nodes and percolating directed links between these groups.

It should be stressed that the special limit of our model $\beta = 0$ (i.e., $\alpha = 1$), corresponding to the original model of preferential attraction of incoming links proposed by Barabasi, Albert and Jeong [10], belongs to an entirely different class of scaling behavior as regards the distribution of size and depth of connected clusters: these distributions are flat between lower and upper cutoffs (see Fig. 3). The only distribution which enjoys a power law in this limit is the distribution of incoming links, which has the exact [11] exponent $\tau_{in} = 3$ (see also inset to Fig. 1). In this case ($\beta = 0$) updates at already existing nodes are no longer possible. This feature—freezing of the outgoing links—as well as quantitative disagreement in the exponent $\gamma$ for distribution of incoming links, makes the model of Ref. [10] inappropriate as a model of the world-wide Web dynamics. A realistic network with frozen outgoing links is the network of scientific citations [3], where physical links, corresponding to the cited references in already published papers, remain fixed in time.

In conclusion, we have demonstrated that certain salient features of the dynamics of the world-wide Web require more careful modeling, compared to models of a generic complex evolving network with preferential attraction of links [11] and models of random graphs with rewiring [13]. The dynamic structure and functioning of the world-wide Web is deeply rooted in the activity of the agents who are creating the outgoing links. The hierarchical structure of outgoing links, which is documented by measurements in the real world-wide Web [5], is related to the bias activity of the agents in the rule [10] of our model. A random selection of the active agent (see curve (c) in Fig. 1) fails to describe the distribution of outgoing links in the real Web. Temporal variation of the outgoing links inside the network, i.e., updates of links, has twofold consequences on the global structure of the network: (i) When the updates of the outgoing links are allowed (i.e., when $\beta$ strictly larger than zero), the structure of incoming links qualitatively changes, the exponent $\tau_{in}(\beta) < 3$. The distribution of incoming links in our model coincides with the analytical results of Dorogovtsev et al. [3] in the scaling region. In addition, the distribution of size and depth of connected clusters becomes hierarchical when $\beta > 0$, in agreement with results
found in the real Web, which is not the case when no updates are allowed ($\beta = 0$). (ii) Varying the frequency of updates $\beta$ implies changes in the Web structure, both in the outgoing and incoming links. This has immediate impact on the accessibility of nodes. The correlations between the outgoing and incoming links suggests that the local structure of the network is qualitatively different compared to the case without updates. Here we did not study in detail the probability that two nodes are linked (clustering coefficient). Rather we studied the physical properties of the network which make the background of the observed behavior: fractal temporal patterns and number of nodes that join the cluster at one time unit.

By comparison of the simulated results and the data obtained in the real world-wide Web we obtained a systematic agreement both for the distributions of outgoing and incoming links and for the size of connected components when a single control parameter $\beta$ is fixed to $\beta = 3$ within error bars of the data. To our knowledge no previous model claimed to describe the world-wide Web achieved such degree of consistency. This makes us believe that the present simplified model takes into account some basic features of the dynamics of Web. It would be interesting to estimate $\beta$ by directly measuring the average number of updated links in the world-wide Web relative to the number of added links originating from each added node.

The structure of links shown by the distributions in Figs. 1 and 2 is closely related to the evolution of the number of links at a given node and to other local properties of the network. A detailed study of these properties within the present model requires additional work that remains to be done in the future. Here we expect, based on the analytical results for the incoming links of Ref. [11], that in our model the average number of links $< q_{\text{in}}(i,t) >$ at a node $i$ will decay in time $t \gg i$ as $< q_{\text{in}}(i,t) > \sim (i/t)^{-\gamma_{\text{in}}}$, where the exponent $\gamma_{\text{in}}$ is given by the exact scaling relation $\gamma_{\text{in}} = 1/(\tau_{\text{in}} - 1)$, leading to $\gamma_{\text{in}} = 1/(1 + \alpha) \approx 0.86$. Assuming that the density of the outgoing links at a given node also exhibits scaling behavior, as Figs. 1-2 suggest, we can predict a slower decay for the average number of outgoing links at a given node. The expected exponent is $\gamma_{\text{out}} \approx 1/(1 + 3\alpha) \approx 0.6$.

The simulation results that we reported here suggest that the emergent structure of links in the world-wide Web is strongly related to the updating policy of the agents: who updates and how often. It remains to understand the potentially more intricate reverse effect: how the amount of information currently stored in the Web influences the conduct of the agents, with implications on self-tuning of the control parameter $\beta$. In our model the structure of the network at $\beta > \beta_c(N)$ appears to be supercritical, where $\beta_c(N) < 0.1$ in a large network. Therefore, it is plausible to expect that the evolutionary selected values of $\beta$ will be much smaller than the currently estimated value $\beta \approx 3$, if the scenario of self-tuning of the control parameter is active in the real Web. In this respect the current state of the real Web can be regarded as a transient rather than a stationary state.

Finally, our results support the conclusion that in the directed graphs two growth rules are necessary to describe the dynamics of the outgoing and incoming links, respectively. In the case of the world-wide Web the statistically correlated distributions of outgoing and incoming links appear as a fundamental feature of the evolution of the Web. This conclusion may serve as a starting point for the future modeling of the real world-wide Web in terms of the master equations.

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[20] Our results suggest that $0 < \beta_c(N) < 0.1$ varying with $N$, but it is strictly larger than zero for any finite $N$ accessible in the simulations. An estimate of $\beta_c$ by employing different system sizes and a finite-size scaling analysis remains out of the scope of this paper.

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![Diagram](image1)

**FIG. 1.** Cumulative distributions of outgoing links (a) and incoming links (b) for the network of $N = 1,000,000$ nodes and average ratio $\beta = 3$ of updated respective to added links per time unit. For comparison we have included the corresponding distributions (c) and (d) in the case of fully random directed graph. Fitted slopes of the straight sections of the curves (a) and (b) are compatible with the scaling exponents in Eq. (3) as $\tau_{\text{out}} = 2.75$ and $\tau_{\text{in}} = 2.25$, respectively. Inset: Variation of the scaling exponents $\tau_{\text{out}} - 1$ (diamonds) and $\tau_{\text{in}} - 1$ (triangles) with the control parameter $\alpha \equiv 1/(\beta + 1)$ in the physical range $(0, 1)$. Dotted line: exact solution for the case of incoming links from Ref. [11].

![Diagram](image2)

**FIG. 2.** Top panel: Part of the temporal pattern of linking in the growth phase of network with $N = 10,000$ nodes. One link per time step is considered. Dots represent nodes from which the link originates, whereas crosses are target nodes. Lower panel: Distribution of time intervals $\Delta t$ between two successive linking (a) from a given node, and (b) to a given node. The two patterns become correlated for the time intervals $\Delta t$ in the range $100 < \Delta t < 3,000$, corresponding to the scaling region of the rank distributions $2000 > q > 200$ in Fig. 1. Distributions are averaged over 100 samples and logarithmically binned. Both distributions are normalized to the total number of time steps. Note that the events with $\Delta t = 0$ that correspond to the outgoing links from new added nodes, contributing to the distribution (a), are not shown.
FIG. 3. Differential distribution of size of connected clusters in the network grown with the rules of Eqs. (1)-(2) for different values of the control parameter $\beta = 3$ (diamonds), 1 (circles), 0.081 (triangles), and zero (crosses). We employ 1,000 avalanches in each of 100 samples of the network with $N = 1,000$ nodes. Inset: Corresponding distributions of depth of the connected clusters.

FIG. 4. (Lower panel) Number of nodes $n(t)$ added to a connected cluster at one investigation step vs cumulative number of steps $t$. Size of one connected cluster is represented by the surface enclosed between two consecutive drops of the signal $n(t)$ to the base line $n(t) = 1$. Parameters are: $N = 1,000$ and $\beta = 1$. (Top panel) Fourier spectrum of the same signal. Also shown is the spectrum of the corresponding signal at the critical value of the parameter $\beta_{c}(N) = 0.081$. Data are logarithmically binned. Straight lines are power-law fits with the slopes $\phi = 1.45 \pm 0.06$ (for $\beta = 1$) and $\phi = 1.01 \pm 0.06$ (for $\beta = 0.081$).