Research Article

On Zero Left Prime Factorizations for Matrices over Unique Factorization Domains

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In this paper, zero prime factorizations for matrices over a unique factorization domain are studied. We prove that zero prime factorizations for a class of matrices exist. Also, we give an algorithm to directly compute zero left prime factorizations for this class of matrices.

1. Introduction

Multidimensional linear systems theory has a wide range of applications in circuits, systems, control of networked systems, signal processing, and other areas (see, e.g., [1, 2]). Multivariate polynomial matrix theory is a well-established tool for these systems, since many problems in the analysis and synthesis of control systems can be well solved using multivariate polynomial matrix techniques [1–3].

In recent years, n-D polynomial matrix factorizations have been widely studied [4–10]. In [11, 12], the zero left prime factorization problem was raised. This problem has been solved in [4–6]. The minor left prime factorization problem has been solved in [7, 10]. In the algorithms given in [7, 10], a fitting ideal of some module over the multivariate (n-D) polynomial ring needs to be computed. It is a little complicated.

It is well known that a multivariate polynomial ring over a field is a unique factorization domain. Then, the following problem is interesting.

Problem 1. How to decide if a matrix with full row rank over a unique factorization domain has a zero left prime factorization?

In this paper, we will give a partial solution to this problem.

2. Preliminaries

Let R be a unique factorization domain. The set of all $l \times m$ matrices with entries from R is denoted by $R^{l \times m}$. Let $F \in R^{l \times m}$. We denote the greatest common divisor of all $l \times l$ minors of F by $d(F)$. Let $C \in R^{l \times l}$ be a submatrix of F. By deleting C from F, we get a submatrix of F. This submatrix is denoted by $F \setminus C$.

Let $C \in R^{m \times m}$, $\text{adj}(C)$ denotes the adjoint matrix of C. $\text{acof}_{ij}(C)$ denotes the $i, j$th algebraic cofactor of C.

Definition 1. Let $F \in R^{l \times m}$, and let $C \in R^{l \times l}$ be a submatrix of F. A minor of F consisting of $l - 1$ columns from C and one column from $F \setminus C$ is said to be a related minor of C. The following definition is from the multidimensional systems theory [13].

Definition 2. Let $F \in R^{l \times m}$ be of full row rank. Then, F is said to be zero left prime (ZLP) if the $l \times l$ minors of F generate the unit ideal $R$. Suppose $F$ has a factorization $F = CF_1$, where $C \in R^{l \times l}$ and $F_1 \in R^{l \times m}$. If $F_1$ is ZLP, then this factorization is said to be a zero left prime factorization.

3. Main Results

First, we need a lemma.

Lemma 1. Let $F = (C, \overline{C}) \in R^{l \times m}$, where $C \in R^{l \times l}$ and $\overline{C} \in R^{l \times (m-l)}$. Then, the elements of $\text{adj}(C) \cdot \overline{C}$ are just all related minors of C (up to a sign).
Proof. Let $C = (c_{ij})_{l \times l}$ and $\overline{C} = (\overline{c}_{ij})_{(m-l) \times (m-l)}$. Let $\text{adj} C \cdot \overline{C} = (b_{ij})_{(m-l) \times (m-l)}$. Then,

$$b_{ij} = \text{acof}_{ij}(C)\overline{c}_{ij} + \cdots + \text{acof}_{il}(C)\overline{c}_{lj}$$

$$= \det \begin{pmatrix} 
    c_{i1} & \cdots & c_{i-1} & c_{il} \\
    \vdots & \vdots & \vdots & \vdots \\
    c_{il} & \cdots & c_{i-1} & c_{i1} 
\end{pmatrix}$$

(1)

by Laplace Theorem. Thus, $b_{ij}$ is a related minor of $C$ (up to a sign). It is clear that they are just all related minors of $C$ (up to a sign).

Now, we prove the main theorem of this paper. □

**Theorem 1.** Let $F \in \mathbb{R}^{l \times m}$ such that there exists an invertible matrix $F \in \mathbb{R}^{l \times m}$ such that $F = CF_1$ and $F_1$ is ZLP, then there exists an $l \times l$ submatrix $C$ of $F$ such that $\text{det}C = d(F)$. We can give an algorithm to directly compute the ZLP factorization of $F$.

**Algorithm 1**

(i) Compute all $l \times l$ minors of $F$ and $d(F)$.

(ii) Find an $l \times l$ submatrix $C$ of $F$ such that $\text{det}C = d(F)$.

(iii) Compute invertible matrix $Q$ such that $FQ = (C, \overline{C})$.

(iv) Let $Q_1 = \begin{pmatrix} I_l & -C^{-1}\overline{C} \\
0 & I_{m-l} \end{pmatrix}$ and $F_1 = (I_l, O)Q_1^{-1}Q^{-1}$.

Then, $F = CF_1$.

Now, we give an example to illustrate this algorithm.

**Example 1.** Let $R = \mathbb{Z}[x, y]$, and let

$$F = \begin{pmatrix} 
6x^2y + 2xy & 2x & 2xy \\
6x^2y + 6x^2y + 2xy^2 + 5xy & 2xy + 2x & 2xy + 2xy + 2xy + y 
\end{pmatrix}.$$  

(5)

Then, $d(F) = 2xy$. Let

$$C = \begin{pmatrix} 
2x & 2xy \\
2xy + 2x & 2xy + 2xy + y 
\end{pmatrix}.$$  

(6)

Then, $C$ is a $2 \times 2$ submatrix of $F$ and $\text{det}C = d(F)$. Let

$$Q = \begin{pmatrix} 
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 
\end{pmatrix}.$$  

(7)

Then, $FQ = (C, \overline{C})$, where

$$\overline{C} = \begin{pmatrix} 
6x^2y + 2xy \\
6x^2y + 6x^2y + 2xy^2 + 5xy 
\end{pmatrix}.$$  

(8)

Thus, $-C^{-1}\overline{C} = \begin{pmatrix} 
-y & -3x \\
1 & 0 & -y \\
0 & 1 & -3x \\
0 & 0 & 1 
\end{pmatrix}$. Let

$$Q_1 = \begin{pmatrix} 
0 & 1 & -y \\
1 & 0 & -y \\
0 & 1 & -3x \\
0 & 0 & 1 
\end{pmatrix}.$$  

(9)

Then,

$$Q_1^{-1}Q^{-1} = \begin{pmatrix} 
y & 1 & 0 \\
3x & 0 & 1 \\
1 & 0 & 0 
\end{pmatrix}.$$  

(10)

Let

$$F_1 = \begin{pmatrix} 
y & 1 & 0 \\
3x & 0 & 1 
\end{pmatrix}.$$  

(11)

Then, $F = CF_1$.  

Proof. Let $C = (c_{ij})_{l \times l}$ and $\overline{C} = (\overline{c}_{ij})_{(m-l) \times (m-l)}$. Let $\text{adj} C \cdot \overline{C} = (b_{ij})_{(m-l) \times (m-l)}$. Then,

$$\text{det} \begin{pmatrix} 
    c_{i1} & \cdots & c_{i-1} & c_{il} \\
    \vdots & \vdots & \vdots & \vdots \\
    c_{il} & \cdots & c_{i-1} & c_{i1} 
\end{pmatrix} = \det \begin{pmatrix} 
    \overline{c}_{i1} & \cdots & \overline{c}_{i-1} & \overline{c}_{il} \\
    \vdots & \vdots & \vdots & \vdots \\
    \overline{c}_{il} & \cdots & \overline{c}_{i-1} & \overline{c}_{i1} 
\end{pmatrix}$$

(1)

by Laplace Theorem. Thus, $b_{ij}$ is a related minor of $C$ (up to a sign). It is clear that they are just all related minors of $C$ (up to a sign).

Now, we prove the main theorem of this paper. □
Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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