FRACTIONAL INTEGRALS AND FOURIER TRANSFORMS

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This paper gives a short survey of some basic results related to estimates of fractional integrals and Fourier transforms. It is closely adjoint to our previous survey papers [32] and [34]. The main methods used in the paper are based on nonincreasing rearrangements. We give alternative proofs of some results.

We observe also that the paper represents the mini-course given by the author at Barcelona University in October, 2014.

1. Nonincreasing rearrangements

Denote by $S_0(\mathbb{R}^n)$ the class of all measurable and almost everywhere finite functions $f$ on $\mathbb{R}^n$ such that for each $y > 0$

$$
\lambda_f(y) \equiv |\{x \in \mathbb{R}^n : |f(x)| > y\}| < \infty.
$$

A non-increasing rearrangement of a function $f \in S_0(\mathbb{R}^n)$ is a non-increasing function $f^*$ on $\mathbb{R}_+ \equiv (0, +\infty)$ such that for any $y > 0$

$$
|\{t \in \mathbb{R}_+ : f^*(t) > y\}| = \lambda_f(y).
$$

We shall assume in addition that the rearrangement $f^*$ is left continuous on $(0, \infty)$. Under this condition it is defined uniquely by

$$
f^*(t) = \inf\{y > 0 : \lambda_f(y) < t\}, \quad 0 < t < \infty.
$$

Besides, we have the equality

$$
f^*(t) = \sup_{|E|=t} \inf_{x \in E} |f(x)|.
$$

The following relation holds

$$
\sup_{|E|=t} \int_E |f(x)|dx = \int_0^t f^*(u)du. \quad (1.1)
$$

In what follows we denote

$$
f^{**}(t) = \frac{1}{t} \int_0^t f^*(u)du.
$$

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By \((1.1)\), the operator \(f \mapsto f^{**}\) is subadditive,
\[(f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t).\]
Moreover, this operator is bounded in \(L^p\) for \(p > 1\),
\[\|f^{**}\|_p \leq p'\|f\|_p, \quad 1 < p \leq \infty.\]
This estimate follows from Hardy’s inequality.

**Hardy-type inequalities**

First, we have the classical Hardy’s inequalities (see, e.g., [56, p. 196]).

**Lemma 1.1.** Let \(\alpha > 0\) and \(1 \leq p < \infty\). Then for any non-negative measurable on \((0, \infty)\) function \(\varphi\),
\[
\left(\int_0^\infty \left(\int_0^t \varphi(u)du\right)^p t^{-\alpha - 1} dt\right)^{1/p} \leq \frac{p}{\alpha} \left(\int_0^\infty \left(t\varphi(t)\right)^p t^{-\alpha - 1} dt\right)^{1/p}
\]
and
\[
\left(\int_0^\infty \left(\int_t^\infty \varphi(u)du\right)^p t^{-\alpha - 1} dt\right)^{1/p} \leq \frac{p}{\alpha} \left(\int_0^\infty \left(t\varphi(t)\right)^p t^{-\alpha - 1} dt\right)^{1/p}.
\]

We say that a measurable function \(f\) on \((0, \infty)\) is quasi-decreasing if there exists a constant \(c > 0\) such that \(f(t_1) \leq cf(t_2)\), whenever \(0 < t_2 < t_1 < \infty\).

We will need also a Hardy-type inequality for quasi-decreasing functions in the case \(0 < p < 1\).

**Lemma 1.2.** Let \(f\) be a non-negative, quasi-decreasing function on \((0, \infty)\). Suppose also that \(\alpha > 0, \beta > -1\) and \(0 < p < 1\). Then
\[
\int_0^\infty u^{-\alpha - 1} \left(\int_0^u f(t) t^\beta dt\right)^p du \leq A \int_0^\infty u^{-\alpha - 1} \left(f(u) u^{\beta + 1}\right)^p du
\]
and
\[
\int_0^\infty u^{\alpha - 1} \left(\int_0^u f(t) t^\beta dt\right)^p du \leq A \int_0^\infty u^{\alpha - 1} \left(f(u) u^{\beta + 1}\right)^p du.
\]

Observe that the best constants in these inequalities were found by Bergh, Burenkov and Persson [8].

Further, we shall use the following Hardy – Littlewood inequality.

**Theorem 1.3.** Let \(f, g \in S_0(\mathbb{R}^n)\). Then
\[
\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq \int_0^\infty f^*(t)g^*(t)dt.
\]
Proof. Applying Fubini’s theorem, we get

\[ \int_{\mathbb{R}^n} |f(x)g(x)| dx = \int_{\mathbb{R}^n} dx \int_0^\infty du \int_0^\infty dv \left| \{x : |f(x)| > u, |g(x)| > v\} \right| du dv \]

\[ \leq \int_0^\infty \int_0^\infty \min(\lambda_f(u), \lambda_g(v)) du dv \]

\[ = \int_0^\infty \int_0^\infty \left| \{t : f^*(t) > u, g^*(t) > v\} \right| du dv \]

\[ = \int_0^\infty f^*(t) g^*(t) dt. \]

\[ \square \]

Let \( 0 < p, r < \infty \). A function \( f \in S_0(\mathbb{R}^n) \) belongs to the Lorentz space \( L_{p,r}(\mathbb{R}^n) \) if

\[ \|f\|_{p,r} \equiv \left( \int_0^\infty \left( t^{1/p} f^*(t) \right)^r dt \right)^{1/r} < \infty. \]

For \( 0 < p < \infty \), the space \( L_{p,\infty}(\mathbb{R}^n) \) is defined as the class of all \( f \in S_0(\mathbb{R}^n) \) such that

\[ \|f\|_{p,\infty} \equiv \sup_{t>0} t^{1/p} f^*(t) < \infty. \]

We have that \( \|f\|_{p,p} = \|f\|_p \). Further, for a fixed \( p \), the Lorentz spaces \( L_{p,r} \) increase as the secondary index \( r \) increases. That is, we have the strict embedding \( L_{p,r} \subset L_{p,s} \) for \( r < s \); in particular,

\[ L_{p,r} \subset L^{p,p} \equiv L^p, \quad 0 < r < p \]

(see [7, p. 217]).

Let \( f \in S_0(\mathbb{R}^n) \). The spherically symmetric rearrangement of \( f \) is defined by

\[ f_s^*(x) = f^*(v_n|x|^{n}), \quad x \in \mathbb{R}^n, \]

where \( v_n = \pi^{n/2} \Gamma(n/2 + 1) \) is the measure of the \( n \)-dimensional unit ball. The function \( f_s^* \) is equimeasurable with \( f \), it possesses the spherical symmetry and decreases monotonically as \( |x| \) increases.

The classical Pólya-Szegö principle states that for any \( f \in C_0^\infty(\mathbb{R}^n) \) and any \( 1 \leq p \leq \infty \)

\[ \|\nabla f_s^*\|_p \leq \|\nabla f\|_p. \]

Another remarkable result on spherically symmetric rearrangements is the following theorem.
Theorem 1.4. Let $f$, $g$, and $h$ be nonnegative functions in $S_0(\mathbb{R}^n)$. Then
\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)g(x-y)h(x)dxdy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f^*_s(y)g^*_s(x-y)h^*_s(x)dxdy. \]

This inequality first was proved for sequences by Hardy and Littlewood. Afterwards, it was proved for functions by F. Riesz in the dimension $n = 1$ and by Sobolev for any $n \in \mathbb{N}$ (see [5], [21], [36], [52]).

The basic properties of rearrangements can be found in the books [7], [15], [21], [35], [36], [56].

2. Convolutions

The convolution of two functions $f$ and $g$ is defined as
\[ f \ast g(x) = g \ast f(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \quad (2.1) \]

If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ for $1 \leq p \leq \infty$, then by Hölder’s inequality the integral (2.1) is absolutely convergent for every $x \in \mathbb{R}^n$. Furthermore, in this case the convolution is a continuous function on $\mathbb{R}^n$. At the same time, there are other conditions that implies almost everywhere convergence of (2.1). For example, if $f, g \in L^1(\mathbb{R}^n)$, then
\[ \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} |f(x-y)g(y)|dy = \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} |f(x-y)g(y)|dx = \|f\|_1 \|g\|_1 < \infty \]
and whence $f \ast g \in L^1(\mathbb{R}^n)$. This implies in particular that $f \ast g(x) < \infty$ for a.e. $x \in \mathbb{R}^n$.

The following theorem was proved by W.H. Young.

Theorem 2.1. Let $1 \leq p, q \leq \infty$, $1 \leq r \leq \infty$ and
\[ \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \]

If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then the integral (2.1) converges absolutely for a.e. $x \in \mathbb{R}^n$ and
\[ \|f \ast g\|_r \leq \|f\|_p \|g\|_q \quad (2.2) \]

Proof. We may suppose that $f, g \geq 0$. Let $\varphi = f^p$ and $\psi = g^q$. Observe that $1/p \geq 1/r$ and $1/q \geq 1/r$. We have
\[ f(x-y)g(y) = \varphi(x-y)^{1/p} \psi(y)^{1/q} = [\varphi(x-y)\psi(y)]^{1/r} \varphi(x-y)^{1/p-1/r} \psi(y)^{1/q-1/r}. \]
Let
\[ \frac{1}{\lambda} = \frac{1}{p} - \frac{1}{r}, \quad \frac{1}{\mu} = \frac{1}{q} - \frac{1}{r}. \]

Then
\[ \frac{1}{r} + \frac{1}{\lambda} + \frac{1}{\mu} = 1. \]

Using Hölder’s inequality (with three terms) we have
\[ \int_{\mathbb{R}^n} f(x - y)g(y)dy \leq \left( \int_{\mathbb{R}^n} \varphi(x - y)\psi(y)dy \right)^{1/r} \| f \|_{p}^{1-p/r} \| g \|_{q}^{1-q/r}. \]

This easily implies (2.2).

Generally, this inequality is not sharp. The best constant in Young’s inequality was found by Beckner [5] in 1975 and independently by Bras-camp and Lieb [13] in 1976 (see also [4]).

**Theorem 2.2.** Let \( f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n), 1 \leq p, q \leq \infty \), and \( 1/p + 1/q = 1 + 1/r \) (\( r \geq 1 \)). Then
\[ \| f \ast g \|_r \leq \left( \frac{A_pA_q}{A_r} \right)^n \| f \|_p \| g \|_q, \quad (2.3) \]

where
\[ A_s = \left[ s^{1/s}(s')^{-1/s'} \right]^{1/2}. \]

We shall give some comments.

1. The constant in (2.3) is optimal. It is easy to check that when \( n = 1 \) and \( p, q \neq 1 \), there is equality in (2.3) for \( f(x) = \exp(-p'x^2) \) and \( g(x) = \exp(-q'x^2) \).

2. For the convolution norm we have
\[ C = \sup \frac{\| f \ast g \|_r}{\| f \|_p \| g \|_q} = \sup \frac{\| f \ast g \ast h \|_\infty}{\| f \|_p \| g \|_q \| h \|_r}. \]

3. The convolution norm for the dimension \( n \) is \( C^n \), where \( C \) is the one-dimensional norm.

4. Convolution takes radial functions to radial functions (since it is true for the Fourier transform).

5. The following rearrangement inequality was used in the proof of Theorem 2.2. This inequality was derived from the Riesz-Sobolev Theorem [1,4].
Theorem 2.3. Let \( h, f_1, \ldots, f_m \) be nonnegative functions in \( S_0(\mathbb{R}^n) \). Then

\[
\int_{\mathbb{R}^n} h(x)(f_1 \ast \cdots \ast f_m)(x)dx \leq \int_{\mathbb{R}^n} h_{\ast}^n(x)((f_1)_{\ast}^n \ast \cdots \ast (f_m)_{\ast}^n)(x)dx.
\]

Rearrangement estimate for convolutions

The following theorem was proved by O'Neil [42].

Theorem 2.4. Let \( f, g \in S_0(\mathbb{R}^n) \) be nonnegative functions and \( h = f \ast g \). Then for all \( t > 0 \)

\[
h^{**}(t) \leq tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(u)g^*(u)du. \tag{2.4}
\]

Proof. Fix \( t > 0 \) and choose a measurable set \( E_t \) of measure \( |E_t| = t \) such that

\[
\{ x \in \mathbb{R}^n : f(x) > f^*(t) \} \subset E_t \subset \{ x \in \mathbb{R}^n : f(x) \geq f^*(t) \}.
\]

Let \( f_1(x) = (f(x) - f^*(t))\chi_{E_t}(x) \) and \( f_2(x) = f(x) - f_1(x) \). For any measurable set \( A \subset \mathbb{R}^n \) of measure \( |A| = t \) we have

\[
\int_A (f_1 \ast g)(x)dx = \int_A dx \int_{\mathbb{R}^n} f_1(x - y)g(y)dy
= \int_{\mathbb{R}^n} f_1(y) \left( \int_A g(x - y)dx \right)dy
\leq t g^{**}(t) \int_{E_t} f(y)dy - t^2 f^*(t)g^{**}(t)
\leq t^2 f^{**}(t)g^{**}(t) - t^2 f^*(t)g^{**}(t).
\]

Notice that \( f_2(x) = f(x) \) if \( x \notin E_t \) and \( f_2(x) = f^*(t) \) for \( x \in E_t \). Clearly \( f_2^*(u) = f^*(t) \) for \( 0 < u < t \) and \( f_2^*(u) = f^*(u) \) for \( u > t \). Thus

\[
\int_A (f_2 \ast g)(x)dx = \int_A dx \int_{\mathbb{R}^n} f_2(x - y)g(y)dy
\leq \int_A \left( \int_0^\infty f_2^*(u)g^*(u)du \right)dx
= \int_A \left( f^*(t) \int_0^t g^*(u)du + \int_t^\infty f^*(u)g^*(u)du \right)dx
= t^2 f^*(t)g^{**}(t) + t \int_0^\infty f^*(u)g^*(u)du.
\]

It follows that

\[
\frac{1}{t} \int_A (f \ast g)(x)dx \leq tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(u)g^*(u)du
\]
for any measurable set \( A \) with measure \(|A| = t\), which completes the proof. \( \Box \)

**Convolution inequalities in Lorentz spaces**

The following theorem was proved by O’Neil [42] in 1963 (and later by Hunt [25] in 1966).

**Theorem 2.5.** Let

\[
\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \quad (1 < p, q, r < \infty)
\]

and

\[
\frac{1}{\mu} + \frac{1}{\nu} = \frac{1}{s} \quad (0 < \mu, \nu, s \leq \infty).
\]

Then

\[
||f * g||_{r,s} \leq c ||f||_{p,\mu} ||g||_{q,\nu}.
\]

If \( s = r \), we can take \( \mu = p(r + 1), \nu = q(r + 1) \), and we obtain a stronger inequality, than Young’s inequality.

However, the sharp constants in the case of Lorentz norms are unknown.

To prove Theorem 2.5 it is sufficient to apply the rearrangement inequality for convolutions and Hardy-type inequalities.

### 3. Fractional integrals

Let \( f \) be a \( 2\pi \)-periodic integrable on \([0, 2\pi]\) function with the Fourier series

\[
f(x) \sim \sum_{n \in \mathbb{Z}} c_n e^{inx}.
\]  

(3.1)

Assume that \( c_0 = 0 \). Set \( F(x) = \int_0^x f(t) dt \). Then \( F(0) = F(2\pi) = 0 \) and

\[
F(x) = \sum_{n \in \mathbb{Z}, n \neq 0} \frac{c_n}{in} e^{inx}
\]

(the convergence is uniform).

Let now \( \alpha > 0 \). Set

\[
(in)^\alpha = |n|^\alpha \exp \left( \frac{1}{2} \pi i \alpha \text{sign } n \right) \quad (n \neq 0)
\]

and consider the series

\[
\sum_{n \in \mathbb{Z}, n \neq 0} \frac{c_n}{(in)^\alpha} e^{inx}.
\]  

(3.2)
If $\alpha \in \mathbb{N}$, then the series (3.2) is obtained by integration $\alpha$ times of the series (3.1). It can be shown (see [58, Ch. 12]), that for any $\alpha > 0$ the series (3.2) converges almost everywhere, it is the Fourier series of its sum $f_\alpha$, and satisfies the equality

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1} dt.$$ 

The function $f_\alpha$ is called the integral of order $\alpha$ of the function $f$ (Weyl).

The following theorem was proved by Hardy and Littlewood [19].

**Theorem 3.1.** Let $f \in L^p[0, 2\pi]$ ($1 < p < \infty$). Assume that

$$\int_{0}^{2\pi} f(x)dx = 0.$$ 

Let $0 < \alpha < 1/p$, $p^* = p/(1-\alpha p)$. Then $f_\alpha \in L^{p^*}[0, 2\pi]$ and

$$||f_\alpha||_{p^*} \leq c||f||_p.$$ 

Note that this theorem is not true for $p = 1$, $0 < \alpha < 1$. Indeed, let

$$f(x) = \frac{1}{x} \left( \ln\left( \frac{\pi}{x} \right) \right)^{\alpha-2}, \quad 0 < x \leq \pi, \quad f(x) = 0 \quad \text{for} \quad \pi < x \leq 2\pi,$$

and let $f$ be extended periodically with the period $2\pi$ to the whole real line. We have $f \in L^1[0, 2\pi]$. On the other hand,

$$f_\alpha(x) \geq \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} f(t) dt \geq \frac{c}{x^{1-\alpha}} \left( \ln\left( \frac{\pi}{x} \right) \right)^{\alpha-1}, \quad 0 < x \leq \pi \quad (c > 0),$$

and thus $f_\alpha \notin L^{1/(1-\alpha)}[0, 2\pi]$.

Theorem 3.1 is one of those results which formed the basis for the development of the general Embedding Theory of function spaces. In 1938 Sobolev [52] (see also [53]) extended this theorem to the Riesz potentials of functions of several variables and obtained embedding theorems for the spaces $W^1_p$ he introduced.

**Riesz potentials**

Let $n \in \mathbb{N}$ and $0 < \alpha < n$. The Riesz kernel in $\mathbb{R}^n$ is defined by

$$K_\alpha(x) = \frac{|x|^{\alpha-n}}{\gamma_n(\alpha)}, \quad \gamma_n(\alpha) = \frac{\pi^{n/2} \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}.$$ 

The Riesz potential of a function $f$ is defined as the convolution

$$I_\alpha f(x) = K_\alpha * f(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy.$$
The coefficient $1/\gamma_n(\alpha)$ is determined by the equation
\[ \hat{I}_\alpha f(\xi) = (2\pi|\xi|)^{-\alpha} \hat{f}(\xi), \quad \xi \in \mathbb{R}^n, \xi \neq 0. \]
This equality is understood in the sense of distributions, that is
\[ \int_{\mathbb{R}^n} I_\alpha f(x)g(x)dx = \int_{\mathbb{R}^n} (2\pi|\xi|)^{-\alpha} \hat{f}(\xi)\hat{g}(\xi)d\xi, \]
whenever $f$ and $g$ belong to the Schwartz class $S$.

The value of the coefficient $1/\gamma_n(\alpha)$ is also important for the validity
of the Riesz composition formula
\[ I_\alpha * I_\beta = I_{\alpha+\beta}, \quad \alpha > 0, \beta > 0, \alpha + \beta < n. \]

Assume that $0 < \alpha < n$, $1 \leq p < n/\alpha$, and $f \in L^p(\mathbb{R}^n)$. It is easy to show that the integral defining $I_\alpha f(x)$ converges absolutely for almost all $x \in \mathbb{R}^n$.

The following Hardy-Littlewood-Sobolev theorem [19], [52] gives an extension of Theorem 3.1.

**Theorem 3.2.** Let $n \in \mathbb{N}$, $0 < \alpha < n$, $1 < p < n/\alpha$, and $p^* = np/(n-\alpha p)$. If $f \in L^p(\mathbb{R}^n)$, then
\[ \|I_\alpha f\|_{p^*} \leq c\|f\|_p. \]

**Remark 3.3.** The condition
\[ \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \quad (1 < p < \frac{n}{\alpha}) \]
is not only sufficient, but also necessary for the boundedness of the operator $f \mapsto I_\alpha f$ from $L^p$ to $L^q$. It follows easily by dilation arguments. Indeed, assume that for some $p, q \in (1, \infty)$
\[ \|I_\alpha f\|_q \leq A\|f\|_p \quad (3.3) \]
for any $f \in L^p(\mathbb{R}^n)$. Let $f$ be a characteristic function of the unit ball in $\mathbb{R}^n$. Set $f_\delta(x) = f(\delta x)$ for $\delta > 0$. Then $\|f_\delta\|_p = \delta^{-n/p}\|f\|_p$. Further, $I_\alpha f_\delta(x) = \delta^{-\alpha}I_\alpha f(\delta x)$ and $\|I_\alpha f_\delta\|_q = \delta^{-\alpha - n/q}\|I_\alpha f\|_q$. Thus, by (3.3),
\[ \delta^{-\alpha - n/q}\|I_\alpha f\|_q \leq A\delta^{-n/p}\|f\|_p. \]
It follows that $\delta^{n/p-n/q-\alpha} \leq C$ for any $\delta > 0$. It is possible if and only if $n/p - n/q - \alpha = 0$.

**Remark 3.4.** Theorem fails to hold for $p = 1$ and $p = n/\alpha$. 

FRACTIONAL INTEGRALS
Remark 3.5. The classical Sobolev theorem states that for \( n \geq 2, 1 \leq p < n \)

\[
||f||_{p^*} \leq c||\nabla f||_p, \quad p^* = \frac{np}{n-p}.
\]

We stress that, in contrast to Theorem 3.2, the latter inequality is true for \( p = 1 \), too.

**Hedberg’s approach**

Given any locally integrable function \( f \) on \( \mathbb{R}^n \), denote by \( Mf \) the Hardy – Littlewood maximal function of \( f \) defined by

\[
Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,
\]

where \( B(x,r) \) stands for the ball centered at \( x \) and having radius \( r \).

By the Hardy – Littlewood maximal theorem,

\[
||Mf||_p \leq A_p ||f||_p, 1 < p \leq \infty.
\]

Moreover, \((Mf)^*(t) \asymp f^{**}(t)\) (F. Riesz, C. Herz; see [7, p. 122]).

Hedberg [23] in 1972 gave a short and elegant proof of Theorem 3.2 basing on the following lemma.

**Lemma 3.6.** If \( 0 < \alpha < n \), then for all \( x \in \mathbb{R}^n \) and any \( \delta > 0 \),

\[
\int_{|x-y| \leq \delta} |f(y)||x-y|^{\alpha-n} dy \leq A\delta^\alpha Mf(x).
\]

**Proof.** We have

\[
\begin{align*}
&\int_{|x-y| \leq \delta} |f(y)||x-y|^{\alpha-n} dy \\
&= \sum_{k=0}^{\infty} \int_{2^{-k-1}\delta < |x-y| \leq 2^{-k}\delta} |f(y)||x-y|^{\alpha-n} dy \\
&\leq A \sum_{k=0}^{\infty} (2^{-k}\delta)^{\alpha-n} \int_{|x-y| \leq 2^{-k}\delta} |f(y)| dy \\
&\leq A\delta^\alpha Mf(x) \sum_{k=0}^{\infty} 2^{-k\alpha} = A\delta^\alpha Mf(x).
\end{align*}
\]

□

**Proposition 3.7.** For any \( 0 < \alpha < n \), \( 1 \leq p < n/\alpha \), and any measurable function \( f \geq 0 \) for all \( x \in \mathbb{R}^n \)

\[
I_\alpha f(x) \leq A||f||_{p^{\alpha/n}} Mf(x)^{1-\alpha/p^n}.
\] (3.4)
Proof. Let $\delta > 0$. By Hölder’s inequality,

$$
\int_{|x-y| \geq \delta} |f(y)||x-y|^{\alpha-n} dy \leq A\delta^{\alpha-n/p}||f||_p.
$$

By Lemma 3.6

$$
\int_{|x-y| \leq \delta} |f(y)||x-y|^{\alpha-n} dy \leq A\delta^\alpha Mf(x).
$$

Choosing

$$
\delta = \left( \frac{||f||_p}{Mf(x)} \right)^{p/n},
$$

we obtain (3.4).

□

Proof of Theorem 3.2. Let $1/p - 1/q = \alpha/n$. Then by (3.4)

$$(I_\alpha f)(x) \leq A^q||f||_p^{\alpha q/n} Mf(x)^p.$$

Thus,

$$||I_\alpha f||_q \leq A||f||_p^{\alpha q/n} ||Mf||_p^{p/q}.$$

Since $p/q = 1 - \alpha p/n$ and $||Mf||_p \leq A_p||f||_p$, we obtain that

$$||I_\alpha f||_q \leq A'||f||_p.$$

Fractional maximal functions and Riesz potentials

For $0 < \alpha < n$, set

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |f(y)|dy.$$

If $f \geq 0$, then $I_\alpha f$ is estimated pointwise by the fractional maximal function from below. Indeed,

$$
\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \geq \int_{B(x,r)} \frac{f(y)}{|x-y|^{n-\alpha}} dy \\
\geq \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |f(y)| dy
$$

for any $r > 0$ and any $x \in \mathbb{R}^n$. Thus,

$$I_\alpha f(x) \geq cM_\alpha f(x) \quad (c > 0).$$

The opposite inequality is in general false (for example, take $f(y) = |y|^{-\alpha}$ and $x = 0$).

Nevertheless, Muckenhoupt and Wheeden \[40\] (see also \[2\]) proved the following theorem.
Theorem 3.8. Let $1 < p < \infty$ and $0 < \alpha < n$. Then
\[ ||I_\alpha f||_p \leq c ||M_\alpha f||_p.\]
Actually this inequality was proved for $L^p_w$, where $w$ is an $A_\infty$ weight.

Estimates of rearrangements

As it was mentioned above, for the classical Hardy – Littlewood maximal function we have (F. Riesz, C. Herz)
\[ c^{-1} f^{**}(t) \leq (Mf)^*(t) \leq cf^{**}(t) \quad (c > 0). \]
Observe that for $\varphi(x) = 1/|x| \ (x \in \mathbb{R}^n)$ we have $\varphi^*(t) = (v_n/t)^{1/n}$, where $v_n$ is the measure of the $n$–dimensional unit ball. Indeed,
\[ \{x : \frac{1}{|x|} > y\} = \frac{v_n}{y^n} \quad (y > 0); \]
and we find $y = \varphi^*(t)$ from the equation $t = v_n y^{-n}$. Thus, for the Riesz kernel $K_\alpha(x) = |x|^{\alpha-n}/\gamma_n(\alpha) \ (0 < \alpha < n)$ we have
\[ K_n^*(t) = \frac{c}{t^{1-\alpha/n}} \quad \text{and} \quad K_n^{**}(t) = \frac{c'}{t^{1-\alpha/n}}. \]
Applying O’Neil’s inequality, we obtain (0 < $\alpha$ < $n$)
\[ (I_\alpha f)^**(t) \leq c \left[ \frac{1}{t^{\alpha/n}} f^{**}(t) + \int_t^\infty s^{\alpha/n-1} f^*(s) ds \right]. \]
It follows that the operator $I_\alpha$ is bounded from $L^1$ into $L^{n/(n-\alpha),\infty}$ and from $L^{n/\alpha,1}$ into $L^\infty$. These statements are sharp.

Cianchi, Kerman, Opic, and Pick [16] obtained a sharp rearrangement inequality for $M_\alpha f$.

Theorem 3.9. Let $n \in \mathbb{N}$ and $0 < \alpha < n$. Then there exists a constant $C > 0$, depending only on $n$ and $\alpha$, such that
\[ (M_\alpha f)^*(t) \leq C \sup_{t \leq s < \infty} s^{\alpha/n} f^{**}(s), \quad t > 0, \]
for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Proof. Denote
\[ \Phi(x, r) = |B(x, r)|^{\alpha-n-1} \int_{B(x, r)} |f(y)| dy \]
and
\[ F(x) = \sup_{r > 0} \Phi(x, r). \]
We have that
\[ \Phi(x, r) \leq [v_n^{1/n} r]^\alpha \min[Mf(x), f^{**}(v_n r^n)]. \]
Fix $t > 0$. Set

$$F_1(x) = \sup_{0 < r < t^{1/n}} \Phi(x, r), \quad F_2(x) = \sup_{r \geq t^{1/n}} \Phi(x, r).$$

We have

$$F_1(x) \leq v_n^{\alpha/n} t^{\alpha/n} M f(x).$$

Thus,

$$F_1^*(t) \leq ct^{\alpha/n} f^{**}(t).$$

On the other hand,

$$F_2(x) \leq \sup_{r \geq t^{1/n}} [v_n^{1/n} r]^{\alpha} f^{**}(v_n r^n).$$

It follows that

$$\|F_2\|_\infty \leq c \sup_{s \geq t} s^{\alpha/n} f^{**}(s).$$

If $\alpha = 0$, we get the Riesz estimate $(Mf)^*(t) \leq Cf^{**}(t)$.

**Corollary 3.10.** The operator $M_\alpha$ is bounded from $L^{n/\alpha, \infty}$ to $L^\infty$.

**Refinement of the Hardy-Littlewood-Sobolev Theorem**

The following theorem was obtained by O’Neil [42].

**Theorem 3.11.** Let $n \in \mathbb{N}$, $0 < \alpha < n$, $1 < p < n/\alpha$, and $p^* = np/(n - \alpha p)$. If $f \in L^p(\mathbb{R}^n)$, then $I_\alpha f \in L^{p^*, p}/\mathbb{R}^n$, and

$$\|I_\alpha f\|_{p^*, p} \leq c \|f\|_p.$$

Indeed, we have seen that, by O’Neil inequality,

$$(I_\alpha f)^*(t) \leq c \left[ t^{\alpha/n} f^{**}(t) + \int_t^\infty s^{\alpha/n - 1} f^*(s) ds \right].$$

It remains to apply Hardy’s inequalities.

Since we have a strict embedding $L^{p^*, p}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, Theorem 3.11 provides a refinement of the Hardy-Littlewood-Sobolev Theorem.

**The limiting case** $p = n/\alpha$, $0 < \alpha < n$

The following theorem was proved by D. Adams [1].

**Theorem 3.12.** Let $n \in \mathbb{N}$, $0 < \alpha < n$, $p = n/\alpha$, and let $f \in L^p(\mathbb{R}^n)$. Assume that $\text{supp } f \subset B(0, R)$ and that $\|f\|_p = 1$. Then there is a constant $A = A(n, \alpha)$ such that

$$\int_{B(0,R)} \exp \left( \beta_0 |I_\alpha f(x)|^{p'} \right) dx \leq AR^n,$$
where $\beta_0 = \beta_0(n, \alpha) = \gamma_n(\alpha)n/\omega_{n-1}$ ($\omega_{n-1}$ is the area of the $n-$dimensional unit sphere).

This theorem has an interesting history. Moser [39] in 1971 proved that
\[
\int_{B(0,R)} \exp \left( \beta |u(x)|^n \right) dx \leq AR^n
\]
for all $0 < \beta \leq \beta_0 = n\omega_{n-1}^{1/(n-1)}$ and all $u \in W^1_n(B(0, R))$ with
\[
\int_{B(0,R)} |\nabla u(x)|^n dx \leq 1
\]
($W^1_n(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^1_n(\Omega)$). It was known before that (3.5) holds for some $\beta > 0$.

Theorem of Adams implies Moser’s theorem. Indeed, the identity
\[
g(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \nabla g(y) \cdot (x-y) \frac{dy}{|x-y|^n}
\]
yields that
\[
|g(x)| \leq \frac{\gamma_n(1)}{\omega_{n-1}} I_1(|\nabla g|)(x).
\]
Observe that the opposite isn’t true.

Finishing this section, we mention the paper by Eiichi Nakai [41] which contains a short but comprehensive survey of the results on Riesz potentials.

4. BESSEL POTENTIALS

The Riesz potentials have a lot of important applications. However, the kernels $K_\alpha(x)$ go slowly to zero as $|x|$ tends to $\infty$. This creates some difficulties in the use of Riesz potentials. In particular, the operator $I_\alpha$ is not bounded on $L^p$.

It is natural to replace $K_\alpha$ with a function that has the same singularity at 0 but a more rapid decay at $\infty$.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i2\pi x \cdot \xi} dx, \quad \xi \in \mathbb{R}^n.
\]

The Bessel kernel of order $\alpha > 0$ on $\mathbb{R}^n$ is defined as the function for which the Fourier transform equals
\[
\hat{G}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}, \quad \xi \in \mathbb{R}^n.
\]
The following equality holds (see [44, 8.1])

\[ G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2} \Gamma(\alpha/2)} \int_0^\infty z^{(\alpha-n)/2-1} e^{-\pi|x|^2/z-\pi z/(4\pi)} \, dz. \]

We have

\[ \int_{\mathbb{R}^n} G_\alpha(x) \, dx = 1, \]

\[ G_\alpha + \beta = G_\alpha \ast G_\beta \quad (\alpha, \beta > 0). \]

Further, for \( 0 < \alpha < n \),

\[ G_\alpha(x) = K_\alpha(x) + o(|x|^{\alpha-n}), \quad x \to 0. \]

Thus, the local behaviour of the kernel \( G_\alpha(x) \) as \( x \to 0 \) is the same as that of \( K_\alpha \). The advantage of the Bessel kernels is that they decrease sufficiently fast on the infinity,

\[ G_\alpha(x) = O(e^{-|x|/2}), \quad x \to \infty. \]

If \( f \in L^p(\mathbb{R}^n) \) \( (1 \leq p \leq \infty) \), then the Bessel potential of order \( \alpha > 0 \) for the function \( f \) is defined by the equality

\[ J_\alpha f(x) = G_\alpha \ast f(x). \]

By Minkowski’s inequality,

\[ ||J_\alpha f||_p \leq ||G_\alpha||_1 ||f||_p = ||f||_p, \quad 1 \leq p \leq \infty. \]

If \( \alpha \geq n \), then \( G_\alpha \in L^p(\mathbb{R}^n) \) for any \( 1 \leq p \leq \infty \). In the case \( 0 < \alpha < n \) we have \( G_\alpha \in L^s(\mathbb{R}^n) \) for all \( 1 \leq s < n/(n-\alpha) \). Applying Young’s inequality, we have

\[ ||J_\alpha f||_q \leq c||f||_p, \quad 1 \leq p \leq q \leq \infty, \quad 1/p - 1/q < \alpha/n. \]

By the Hardy – Littlewood – Sobolev theorem on Riesz potentials,

\[ ||I_\alpha f||_{p^*} \leq c||f||_p, \quad 1 < p < \frac{n}{\alpha}, \quad 0 < \alpha < n, \quad p^* = \frac{np}{n-\alpha p}. \]

Since \( G_\alpha \) is majorized by \( K_\alpha \) on the unit ball and \( G_\alpha(x) = O(e^{-|x|/2}) \) at infinity, we obtain that

\[ ||J_\alpha f||_{p^*} \leq c||f||_p, \quad 1 < p < \frac{n}{\alpha}, \quad 0 < \alpha < n. \]

**Relations between Bessel and Riesz potentials**

Let \( \alpha > 0 \) and \( 1 < p < \infty \). Let \( F = G_\alpha \ast f \), where \( f \in L^p(\mathbb{R}^n) \). Then \( F \) can be represented as \( F = K_\alpha \ast g \), where \( g \in L^p(\mathbb{R}^n) \).

Indeed,

\[ \hat{f}(\xi) = (1 + 4\pi^2|\xi|^2)^{\alpha/2} \hat{F}(\xi). \]
From here,
\[
\left( \frac{4\pi^2|\xi|^2}{1 + 4\pi^2|\xi|^2} \right)^{\alpha/2} \hat{f}(\xi) = (2\pi|\xi|)^{\alpha/2} \hat{F}(\xi). \tag{4.1}
\]

We show that the left-hand side of (4.1) is the Fourier transform of some function \( g \in L^p(\mathbb{R}^n) \).

We shall use the binomial expansion
\[
(1 - t)^{\alpha/2} = 1 + \sum_{m=1}^{\infty} A_{m,\alpha} t^m, \quad |t| < 1,
\]
where
\[
A_{m,\alpha} = (-1)^m \frac{\alpha(\alpha - 1) \cdots (\alpha - m + 1)}{m!}.
\]
All the \( A_{m,\alpha} \) have the same signs for \( m \) sufficiently large. Thus,
\[
\sum_{m=1}^{\infty} |A_{m,\alpha}| < \infty.
\]

Taking \( t = (1 + 4\pi^2|\xi|^2)^{-1} \), we get
\[
\left( \frac{4\pi^2|\xi|^2}{1 + 4\pi^2|\xi|^2} \right)^{\alpha/2} = 1 + \sum_{m=1}^{\infty} A_{m,\alpha} (1 + 4\pi^2|\xi|^2)^{-m}
\]
\[
= 1 + \sum_{m=1}^{\infty} A_{m,\alpha} \hat{G}_{2m}(\xi).
\]

We observe that
\[
G_\beta(x) \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^n} G_\beta(x) dx = 1, \quad \beta > 0.
\]

Setting
\[
h(x) = \sum_{m=1}^{\infty} A_{m,\alpha} G_{2m}(x),
\]
we have that \( h \in L^1(\mathbb{R}^n) \) and
\[
\left( \frac{4\pi^2|\xi|^2}{1 + 4\pi^2|\xi|^2} \right)^{\alpha/2} \hat{f}(\xi) = \hat{f}(\xi) + \hat{f} * \hat{h}(\xi).
\]

We see that the left-hand side of (4.1) is the Fourier transform of the function \( g = f + f * h \). Obviously, \( g \in L^p(\mathbb{R}^n) \). Finally, (4.1) yields that \( F = K_\alpha * g \).

Clearly, the converse is not true - a function represented as a Riesz potential may not be represented as a Bessel potential. Indeed, the operator \( I_\alpha \) is not bounded in \( L^p \).
Fractional Sobolev spaces

Let $1 \leq p \leq \infty$ and $r \in \mathbb{N}$. Denote by $W^r_p(\mathbb{R}^n)$ the Sobolev space of functions $f \in L^p(\mathbb{R}^n)$ for which all weak derivatives $D^s f$ ($s = (s_1, \ldots, s_n)$) of order $|s| = s_1 + \cdots + s_n \leq r$ exist and belong to $L^p(\mathbb{R}^n)$. The norm in $W^r_p$ is defined by

$$||f||_{W^r_p} = \sum_{|s| \leq r} ||D^s f||_p.$$

A natural extension of the Sobolev spaces to fractional values of $r$ give the spaces of Bessel potentials (Sobolev – Liouville spaces; Aronszajn and Smith; Calderón, 1961).

Let $1 \leq p \leq \infty$ and $\alpha > 0$. We say that a measurable on $\mathbb{R}^n$ function $f$ belongs to the space $L^\alpha_p(\mathbb{R}^n)$, if there exists a function $g \in L^p(\mathbb{R}^n)$ such that $f(x) = G_\alpha * g(x)$ for almost all $x \in \mathbb{R}^n$ (this function is unique). The norm in $L^\alpha_p(\mathbb{R}^n)$ is defined by $||f||_{L^\alpha_p} = ||g||_p$. We observe that $L^\alpha_p(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, $L^\alpha_p(\mathbb{R}^n) \subset L^\beta_p(\mathbb{R}^n)$ ($\alpha > \beta$, $1 \leq p \leq \infty$).

The following theorem was proved independently by A. Calderón [14] and Lizorkin [37].

**Theorem 4.1.** Let $r \in \mathbb{N}$ and $1 < p < \infty$. Then

$$W^r_p(\mathbb{R}^n) = L^r_p(\mathbb{R}^n),$$

and the norms are equivalent.

The relations between $W^r_p(\mathbb{R}^n)$ and $L^r_p(\mathbb{R}^n)$ in the extreme cases $p = 1$ and $p = \infty$ are the following.

(i) When $n = 1$, then $W^r_p(\mathbb{R}) = L^r_p(\mathbb{R})$, if $r$ is even and $p = 1$, $p = \infty$.

(ii) When $n \geq 2$, then $W^r_p(\mathbb{R}^n) \subset L^r_p(\mathbb{R}^n)$, if $r$ is even and $p = 1$, $p = \infty$; the reverse inclusion fails for both $p = 1$ and $p = \infty$.

(iii) For all $n$, if $r$ is odd, then no one of the spaces $W^r_p(\mathbb{R})$ and $L^r_p(\mathbb{R})$ is contained in the other.

Observe that the proof of (i) is quite simple. Indeed,

$$\hat{f}^{(\mu)}(\xi) = -4\pi^2 \xi^2 \hat{f}(\xi).$$

On the other hand, the equality $f = G_2 * g$ is equivalent to

$$\hat{g}(\xi) = (1 + 4\pi^2 \xi^2) \hat{f}(\xi).$$

Hence, $\hat{g}(\xi) = \hat{f}(\xi) - \hat{f}''(\xi)$ and $g = f - f''$. Applying induction on $r$, we obtain (i).
Also, we shall briefly describe the relations between fractional Sobolev spaces $L^\alpha_p$ and other spaces of fractional smoothness - Besov spaces.

Let a function $f$ be given on $\mathbb{R}^n$. For $r \in \mathbb{N}$ and $h \in \mathbb{R}^n$ we set

$$\Delta^r(h)f(x) = \sum_{i=0}^r (-1)^{r-i} \binom{r}{i} f(x + ih).$$

If $f \in L^p(\mathbb{R}^n)$, then the function

$$\omega^r(f; \delta)_p = \sup_{0 \leq |h| \leq \delta} ||\Delta^r(h)f||_p$$

is called the modulus of continuity of order $r$ of the function $f$ in $L^p$.

Let $\alpha > 0$ and $1 \leq p, q < \infty$. Assume that $r > \alpha$, $r \in \mathbb{N}$. A function $f \in L^p(\mathbb{R}^n)$ belongs to the class $B^\alpha_{p,q}(\mathbb{R}^n)$ if

$$||f||_{B^\alpha_{p,q}} \equiv ||f||_p + \left( \int_0^\infty \left( t^{-\alpha} \omega^r(f; t)_p^q \frac{dt}{t} \right)^{1/q} \right) < \infty.$$

The Nikol’skii space $H^\alpha_p(\mathbb{R}^n) \equiv B^\alpha_{p,\infty}(\mathbb{R}^n)$ is defined as the class of all functions $f \in L^p(\mathbb{R}^n)$ such that

$$\omega^r(f; t)_p = O(\delta^\alpha), \quad r > \alpha.$$

Denote also $B^\alpha_{p,p} \equiv B^\alpha_p$.

The following relations hold.

$$B^\alpha_{p,q}(\mathbb{R}^n) \subset L^\alpha_p(\mathbb{R}^n), \quad 1 \leq q \leq \min(p, 2)$$

and

$$L^\alpha_p(\mathbb{R}^n) \subset B^\alpha_{p,q}(\mathbb{R}^n), \quad q \geq \max(p, 2).$$

If $1 \leq p \leq 2$, then

$$B^\alpha_p \subset L^\alpha_p \subset B^\alpha_{p,2}.$$

If $2 \leq p < \infty$, then

$$B^\alpha_{p,2} \subset L^\alpha_p \subset B^\alpha_{p}.$$

For $1 \leq p \leq \infty$

$$B^\alpha_{p,1} \subset L^\alpha_p \subset B^\alpha_{p,\infty}.$$
5. Anisotropic spaces

Let \( r \in \mathbb{N}, 1 \leq p < \infty \), and \( 1 \leq j \leq n \). Denote by \( W^r_{p,j}(\mathbb{R}^n) \) the Sobolev space of all functions \( f \in L^p(\mathbb{R}^n) \) for which there exists the weak partial derivative \( D^r_j f \in L^p(\mathbb{R}^n) \). Set also

\[
W^r_{p,1,\ldots,r_n}(\mathbb{R}^n) = \bigcap_{j=1}^n W^r_{p,j}(\mathbb{R}^n) \quad (r_j \in \mathbb{N}, 1 \leq p < \infty).
\]

The norm in \( W^r_{p,1,\ldots,r_n}(\mathbb{R}^n) \) is defined by

\[
||f||_{W^r_{p,1,\ldots,r_n}} = ||f||_p + \sum_{j=1}^n ||D^r_j f||_p.
\]

We observe that

\[
W^r_{p,1,\ldots,r_n}(\mathbb{R}^n) = W^r_p(\mathbb{R}^n), \quad 1 < p < \infty.
\]

It is a consequence of the following theorem (K. Smith [51]).

**Theorem 5.1.** Let \( 1 < p < \infty \) and \( r \in \mathbb{N} \). Then for any multi-index \( s = (s_1, \ldots, s_n) \) with non-negative integer components such that

\[
|s| = \sum_{i=1}^n s_i = r;
\]

the weak derivative \( D^s f \) exists and

\[
|||D^s f|||_p \leq c \sum_{i=1}^n |||D^r_i f|||_p.
\]

For \( p = 1 \) and \( p = \infty \) this theorem fails. Namely, Ornstein [45] constructed a function \( f \) on \( \mathbb{R}^2 \) such that \( |||D^{1,1} f|||_1 \) cannot be estimated by \( C(||D^1_1 f||_1 + ||D^1_2 f||_1) \). For \( p = \infty \) a counterexample is given by the function which equals to \( xy \ln |\ln(x^2 + y^2)| \) in some neighborhood of the origin (see [9, Ch. 3]).

We will also consider the fractional Sobolev spaces. These spaces were introduced and studied in the sixties by Lizorkin.

Recall that the Bessel kernel of order \( \alpha > 0 \) in \( \mathbb{R} \) is defined as the function with Fourier transform

\[
\hat{G}_\alpha(\xi) = (1 + 4\pi^2 \xi^2)^{-\alpha/2}, \quad \xi \in \mathbb{R}.
\]

Let \( 1 \leq p \leq \infty \), \( \alpha > 0 \), and \( 1 \leq j \leq n \). Let \( f \) be a measurable function on \( \mathbb{R}^n \). We say that \( f \) belongs to the space \( L^p_{\alpha,j}(\mathbb{R}^n) \) if there
exists a function $f_j \in L^p(\mathbb{R}^n)$ such that for almost all $x \in \mathbb{R}^n$

$$f(x) = \int_{\mathbb{R}} G_\alpha(x_j - t) f_j(t, \hat{x}_j) \, dt.$$  \hfill (5.1)

It is not difficult to prove that the equality (5.1) determines the function $f_j$ uniquely, up to its values on a set of $n$–dimensional Lebesgue measure zero. We have

$$||f||_p \leq ||f_j||_p.$$

We call $f_j$ the Bessel derivative of the function $f$ of order $\alpha$ with respect to $x_j$. We denote it by $J^\alpha_{x_j} f$. If $\alpha$ is fractional, we use also the standard notation $D^\alpha_{x_j} f$. However, for integer $\alpha$ we keep the latter notation for the usual weak derivative.

The following equality holds

$$L^\alpha_{p;j}(\mathbb{R}^n) = W^\alpha_{p;j}(\mathbb{R}^n), \quad (1 < p < \infty, \ \alpha \in \mathbb{N})$$

and the norms are equivalent.

If $1 \leq p \leq \infty$ and $\alpha > 0$, we set $\tilde{L}^\alpha_{p;j} = L^\alpha_{p;j}$ if $\alpha$ is fractional, and $\tilde{L}^\alpha_{p;j} = W^\alpha_{p;j}$ if $\alpha$ is integer.

Let $\alpha_j > 0 \ (j = 1, \ldots, n)$ and $1 \leq p \leq \infty$. Set

$$L^{\alpha_1, \ldots, \alpha_n}_p(\mathbb{R}^n) = \cap_{j=1}^n \tilde{L}^{\alpha_j}_{p;j}(\mathbb{R}^n)$$

and

$$||f||_{L^{\alpha_1, \ldots, \alpha_n}_p} = \sum_{j=1}^n ||f||_{\tilde{L}^{\alpha_j}_{p;j}}$$

We shall call $L^{\alpha_1, \ldots, \alpha_n}_p(\mathbb{R}^n)$ a fractional Sobolev space or a Sobolev-Liouville space. For integer $\alpha_j$ and $1 \leq p \leq \infty$

$$W^{\alpha_1, \ldots, \alpha_n}_p(\mathbb{R}^n) = L^{\alpha_1, \ldots, \alpha_n}_p(\mathbb{R}^n).$$

We observe that Lizorkin defined $L^{\alpha_1, \ldots, \alpha_n}_p(\mathbb{R}^n)$ as the intersection

$$\cap_{j=1}^n L^{\alpha_j}_{p;j}(\mathbb{R}^n)$$

(that is, he used only Bessel’s derivatives). Our definition differs from this only in the case when $p = 1$ or $p = \infty$ and at least one of the $\alpha_j$ is an odd integer.

We have the equality

$$L^{\alpha, \ldots, \alpha}_p(\mathbb{R}^n) = L^{\alpha}_p(\mathbb{R}^n), \quad (\alpha > 0, \ 1 < p < \infty),$$

and the norms are equivalent (Lizorkin [38], Strichartz [57]).
It is possible to show that $L_1^\alpha(R^n) \nsubseteq L_1^{\alpha,...,\alpha}(R^n)$ for $n \geq 2$. We have an open problem: is it true that

$$L_1^{\alpha,...,\alpha}(R^n) \subset L_1^\alpha(R^n), \quad n \geq 2?$$

For $W-$ spaces it is obviously true.

**Embeddings**

Using O’Nei’s inequality for Riesz potentials, we have for $1 < p < n/\alpha$, $p^* = np/(n - \alpha p)$

$$L_1^\alpha(R^n) \subset L^{p^*,p}(R^n).$$

We stress that this embedding fails for $p = 1$.

In the anisotropic case Lizorkin [38] proved the following Sobolev type embedding.

**Theorem 5.2.** Let $\alpha_j > 0$ ($j = 1,...,n$),

$$\alpha = n \left( \sum_{j=1}^{n} \frac{1}{\alpha_j} \right)^{-1}, \quad 1 < p < \frac{n}{\alpha}, \quad \text{and} \quad p^* = \frac{np}{n - \alpha p}.$$ 

Then for every function $f \in L_1^{\alpha_1,...,\alpha_n}(R^n)$

$$||f||_{p^*,p} \leq c ||f||_{L_1^{\alpha_1,...,\alpha_n}}.$$ 

Applying estimates of rearrangements, we proved [32] a refinement of this theorem.

**Theorem 5.3.** Assume that $1 < p < \infty$, $n \geq 1$ or $p = 1$, $n \geq 2$. Let $\alpha_j > 0$ ($j = 1,...,n$) and let

$$\alpha \equiv n \left( \sum_{j=1}^{n} \frac{1}{\alpha_j} \right)^{-1} < \frac{n}{p}.$$ 

Set $p^* = np/(n - \alpha p)$. Then for every function $f \in L_1^{\alpha_1,...,\alpha_n}(R^n)$

$$||f||_{p^*,p} \leq c \sum_{j=1}^{n} ||D_1^{\alpha_j} f||_p.$$ 

(5.2)

Emphasize that, in contrast to the case $n = 1$, for $n \geq 2$ Theorem 5.3 is true for $p = 1$, too.

Since $||f||_{p^*} \leq c ||f||_{p^*,p}$, Theorem 5.3 provides a refinement of Theorem 5.2. We see also that for Lizorkin spaces Theorem 5.2 holds in the case when $p = 1$, $n \geq 2$, and all $\alpha_j$ are non-integer.
Let $\alpha_1 = \cdots = \alpha_n = \alpha$. If $1 < p < \infty$, then $L_p^{\alpha,\ldots,\alpha} = L_p^\alpha$, and we obtain embedding

$$L_p^\alpha(\mathbb{R}^n) \subset L_p^{n^\alpha}(\mathbb{R}^n) \quad \left(1 < p < \frac{n}{\alpha}, \ p^* = \frac{np}{n - \alpha p}\right)$$

mentioned above. For $p = 1$ it doesn’t hold. However, Theorem 5.3 holds for $p = 1$, $n \geq 2$, too. That is, we have

$$L_1^{\alpha,\ldots,\alpha}(\mathbb{R}^n) \subset L_p^{n/(n - \alpha)}(\mathbb{R}^n), \quad 0 < \alpha < n, \ n \geq 2.$$

We proved also estimates of difference norms (embeddings to the Besov spaces).

Let a function $f$ be given on $\mathbb{R}^n$. We have already defined the modulus of continuity $\omega_r^r(f; \delta)_p$. Namely, for $r \in \mathbb{N}$ and $h \in \mathbb{R}^n$ we set

$$\Delta_r^r(h) f(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x + ih),$$

If $f \in L^p(\mathbb{R}^n)$, then the function

$$\omega_r^r(f; \delta)_p = \sup_{0 \leq |h| \leq \delta} ||\Delta_r^r(h) f||_p$$

is called the modulus of continuity of order $r$ of the function $f$ in $L^p$.

Now we define the partial moduli of continuity. Let $r \in \mathbb{N}$, $1 \leq j \leq n$, and $h \in \mathbb{R}$. Set

$$\Delta_j^r(h) f(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x + ih e_j),$$

where $e_j$ is the unit coordinate vector in $\mathbb{R}^n$. If $f \in L^p(\mathbb{R}^n)$, then the function

$$\omega_j^r(f; \delta)_p = \sup_{0 \leq |h| \leq \delta} ||\Delta_j^r(h) f||_p$$

is called the partial modulus of continuity of order $r$ of the function $f$ with respect to the variable $x_j$ in $L^p$. If $r = 1$, then we omit the superscript in this notation.

If $f$ has the weak derivative $D_j^r f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\Delta_j^r(h) f(x) = \int_0^h \cdots \int_0^h D_j^r f(x + (u_1 + \ldots + u_r) e_j) du_1 \ldots du_r$$

for almost all $x$.

For $r > 0$, let $\bar{r}$ be the least integer such that $r \leq \bar{r}$. We have the following theorem (see [29], [30], [33]).
Theorem 5.4. Let \( r_1, \ldots, r_n \) be positive numbers and let

\[
r = n \left( \sum_{j=1}^{n} \frac{1}{r_j} \right)^{-1}, \quad 1 \leq p < q < \infty, \quad \kappa = 1 - \frac{n}{r} \left( \frac{1}{p} - \frac{1}{q} \right) > 0.
\]

Set \( \alpha_j = \kappa r_j \) (\( j = 1, \ldots, n \)). If \( 1 < p < \infty \) and \( n \geq 1 \), or \( p = 1 \) and \( n \geq 2 \), then for every function \( f \in L^r_{r_1, \ldots, r_n}(\mathbb{R}^n) \)

\[
\sum_{j=1}^{n} \left( \int_0^\infty \left[ h^{-\alpha_j} \omega_j^q(f; h) \right]^p \frac{dh}{h} \right)^{1/p} \leq c \sum_{j=1}^{n} \| D^{r_j}_{\alpha} f \|_p.
\]

Nikolskii-Besov and Lipschitz spaces

Let \( f \in L^p(\mathbb{R}^n) \) (\( 1 \leq p \leq \infty \)), \( \alpha > 0 \), and \( 1 \leq j \leq n \). Let \( r \) be the least integer such that \( r > \alpha \). The function \( f \) belongs to the class \( H^\alpha_{r} (\mathbb{R}^n) \) if

\[
\| f \|_{H^\alpha_{r}} \equiv \| f \|_{p} + \sup_{\delta > 0} \frac{\omega_j^r(f; \delta)}{\delta^\alpha} < \infty.
\]

(5.3)

Emphasize that if \( \alpha \in \mathbb{N} \), then in (5.3) we take the modulus of continuity of the order \( r = \alpha + 1 \).

If \( \alpha_j > 0 \) (\( j = 1, \ldots, n \)) and \( 1 \leq p \leq \infty \), the Nikolskii space \( H^\alpha_{p_1, \ldots, p_n}(\mathbb{R}^n) \) is defined by

\[
H^\alpha_{p_1, \ldots, p_n}(\mathbb{R}^n) = \bigcap_{j=1}^{n} H^\alpha_{p_j}(\mathbb{R}^n).
\]

Assume now that \( \alpha > 0 \), \( 1 \leq p, q < \infty \), and \( 1 \leq j \leq n \). As above, let \( r \) be the least integer such that \( r > \alpha \). A function \( f \in L^p(\mathbb{R}^n) \) belongs to the class \( B^\alpha_{p,q} (\mathbb{R}^n) \) if

\[
\| f \|_{B^\alpha_{p,q}} \equiv \| f \|_{p} + \left( \int_0^\infty \left( t^{-\alpha} \omega_k^q(f; t) \right)^p \frac{dt}{t} \right)^{1/q} < \infty.
\]

Denote also \( B^\alpha_{p,\infty} \equiv B^\alpha_{p} \).

Let \( \alpha_j > 0 \) (\( j = 1, \ldots, n \)) and \( 1 \leq p, q < \infty \). Then we set

\[
B^\alpha_{p,q} (\mathbb{R}^n) = \bigcap_{j=1}^{n} B^\alpha_{p,q} (\mathbb{R}^n) \quad (B^\alpha_{p_1, \ldots, p_n} \equiv B^\alpha_{p_1, \ldots, p_n}).
\]

It is easy to see that

\[
\| f \|_{H^\alpha_{p,j}} = \lim_{\theta \to +\infty} \| f \|_{B^\alpha_{p,\theta_j}}.
\]

This is why we set, by definition, \( B^\alpha_{p,\infty} (\mathbb{R}^n) = H^\alpha_{p,j} (\mathbb{R}^n) \).
It is also well known that 

\[ B^\alpha_{p,\theta \eta} \subset B^\alpha_{p,\eta \eta} \quad \text{if} \quad 1 \leq \theta < \eta \leq \infty. \]

Stress again that in the definition of the Nikol’skii space \( H^\alpha_{p,j}(\mathbb{R}^n) \) the order \( r \) of the modulus of continuity is strictly greater than the smoothness exponent \( \alpha \). If \( \alpha \in \mathbb{N} \), it is also natural to admit the value \( r = \alpha \). However, it leads to completely different spaces – Lipschitz type spaces.

Assume that \( \alpha > 0 \) and denote by \( \bar{\alpha} \) the least integer \( s \geq \alpha \). Let \( 1 \leq p < \infty \) and \( 1 \leq j \leq n \). Denote by \( \Lambda^\alpha_{p,j}(\mathbb{R}^n) \) the class of all functions \( f \in L^p(\mathbb{R}^n) \) such that

\[ ||f||_{\Lambda^\alpha_{p,j}} \equiv \sup_{\delta > 0} \frac{\omega^\bar{\alpha}(f; \delta)_p}{\delta^\alpha} < \infty. \]

Set also \( ||f||_{\Lambda^\alpha_{p,j}} = ||f||_p + ||f||_{\alpha_{p,j}} \).

Clearly, \( ||f||_{\Lambda^\alpha_{p,j}} = ||f||_{H^\alpha_{p,j}} \) if \( \alpha \not\in \mathbb{N} \). If \( \alpha \in \mathbb{N} \), then we have the strict embedding \( \Lambda^\alpha_{p,j} \subset H^\alpha_{p,j} \). Moreover, by the Hardy–Littlewood theorem, if \( \alpha \in \mathbb{N} \), then

\[ \Lambda^\alpha_{p,j}(\mathbb{R}^n) = W^\alpha_{p,j}(\mathbb{R}^n) \quad \text{for} \quad 1 < p < \infty. \]

The strict embedding holds for \( 1 \leq p \leq \infty \)

\[ L^\alpha_{p,j}(\mathbb{R}^n) \subset \Lambda^\alpha_{p,j}(\mathbb{R}^n) = H^\alpha_{p,j}(\mathbb{R}^n), \quad \alpha \not\in \mathbb{N} \]

If \( \alpha \in \mathbb{N} \), then

\[ L^\alpha_{1,j}(\mathbb{R}^n) \equiv W^\alpha_{1,j}(\mathbb{R}^n) \subset \Lambda^\alpha_{1,j}(\mathbb{R}^n) \]

and

\[ L^\alpha_{p,j}(\mathbb{R}^n) \equiv W^\alpha_{p,j}(\mathbb{R}^n) = \Lambda^\alpha_{p,j}(\mathbb{R}^n) \quad (1 < p < \infty). \]

If \( \alpha_j > 0 \) \( (j = 1, \ldots, n) \) and \( 1 \leq p < \infty \), we set

\[ \Lambda^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n) = \bigcap_{j=1}^n \Lambda^{\alpha_j}_{p,j}(\mathbb{R}^n). \]

We shall call \( \Lambda^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n) \) a Lipschitz space.

If all \( \alpha_j \) are non-integer, then \( \Lambda^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n) = H^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n) \). If all \( \alpha_j \) are integer and \( 1 < p < \infty \), then

\[ \Lambda^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n) = L^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n) = W^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n). \]

The most interesting (and the most difficult) case is when among the numbers \( \alpha_j \) there are integers, but not all of them are integers. In this case \( \Lambda^{\alpha_1,\ldots,\alpha_n} \) inherits partly properties of the Sobolev spaces, and partly - properties of the Nikol’skii spaces.
We will discuss the problem of embedding with limiting exponent for Lipschitz classes.

For any $\alpha_j > 0$ and $1 \leq p \leq \infty$ we have the following embeddings

$$L_p^{\alpha_1, \ldots, \alpha_n}(\mathbb{R}^n) \subset \Lambda_p^{\alpha_1, \ldots, \alpha_n}(\mathbb{R}^n) \subset H_p^{\alpha_1, \ldots, \alpha_n}(\mathbb{R}^n). \quad (5.4)$$

For $1 \leq p \leq \infty$, the right embedding in (5.4) becomes equality if and only if $\alpha_j \notin \mathbb{N}$ for all $j = 1, \ldots, n$. In the left embedding equality takes place if and only if $1 < p \leq \infty$ and $\alpha_j \in \mathbb{N}$, $j = 1, \ldots, n$.

Let $n \geq 2$. Set

$$\alpha \equiv n \left( \sum_{j=1}^{n} \frac{1}{\alpha_j} \right)^{-1}.$$

Assume that $1 \leq p < \infty$ and $\alpha < np/(n - \alpha p)$. Let $p^* = np/(n - \alpha p)$. Then

$$L_p^{\alpha_1, \ldots, \alpha_n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \text{for all } p < q \leq p^*$$

and

$$H_p^{\alpha_1, \ldots, \alpha_n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \text{for all } p < q < p^*,$$

but for $q = p^*$ the latter embedding does not hold. The problem arises: what can be said about the embedding

$$\Lambda_p^{\alpha_1, \ldots, \alpha_n}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)?$$

The first result in this problem was obtained in our work [28] for $0 < \alpha_j \leq 1$.

**Theorem 5.5.** Let $1 \leq p < \infty$, $0 < \alpha_j \leq 1$, and

$$\alpha \equiv n \left( \sum_{j=1}^{n} \frac{1}{\alpha_j} \right)^{-1} < \frac{n}{p}.$$

Let $p^* = np/(n - \alpha p)$. Let $\nu$ be the number of $\alpha_j$ that are equal to 1. The embedding

$$\Lambda_p^{\alpha_1, \ldots, \alpha_n}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n) \quad (5.5)$$

holds if and only if

$$\nu \geq \frac{n}{\alpha} - p.$$

**Remark 5.6.** It follows that, in contrast to the Sobolev-Liouville and Nikol’skii spaces, the embedding $\Lambda_p^{\alpha_1, \ldots, \alpha_n} \subset L^q$ is not uniquely determined by the value of the harmonic mean $\alpha$. Roughly speaking, this means that for the spaces $\Lambda_p^{\alpha_1, \ldots, \alpha_n}$ the contribution of the variable $x_k$ is not proportional to $1/\alpha_k$. 
Theorem 5.5 was extended by Netrusov [43] to arbitrary values of $\alpha_k > 0$. Moreover, Netrusov proved a theorem on embedding of $\Lambda^{\alpha_1, \ldots, \alpha_n}$ into Lorentz spaces. He proposed another approach based on a modification of the method of integral representations. However, his proof was also long and complicated, and it did not work in the case $p = 1$. Applying rearrangements, we obtained [32] a new proof of these results, including the case $p = 1$.

**Theorem 5.7.** Let $n \geq 2$ and $\alpha_j > 0$ ($j = 1, \ldots, n$). Let

$$\alpha = n \left( \sum_{j=1}^{n} \frac{1}{\alpha_j} \right)^{-1}, \quad 1 \leq p < \frac{n}{\alpha}, \quad \text{and} \quad p^* = \frac{np}{n - \alpha p}.$$ 

Assume that there is an integer among the numbers $\alpha_j$. Let

$$\alpha' = \left( \sum_{j: \alpha_j \in \mathbb{N}} \frac{1}{\alpha_j} \right)^{-1} \quad \text{and} \quad s = \frac{n\alpha' p}{\alpha}.$$ 

Then for every function $f \in \Lambda^{\alpha_1, \ldots, \alpha_n}(\mathbb{R}^n)$ we have

$$\|f\|_{p^*, s} \leq c \sum_{j=1}^{n} \|f\|_{\chi^{\alpha_j}_{p^*}}. \quad (5.6)$$

It was also proved by Netrusov that the index $s$ in this theorem cannot be replaced by a smaller one. Note that for a given value of the mean index $\alpha$, the bigger is the number of the integers among $\alpha_j$ the smaller is the index $s$. If there are no integers $\alpha_j$ at all, then $s = \infty$. In the other extreme case, if all $\alpha_j$ are integers, we have $s = p$ and Theorem 5.7 coincides with embedding theorem with limiting exponent for anisotropic Sobolev spaces $W^{\alpha_1, \ldots, \alpha_n}_p$ [30].

If $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, n$, then we have $s = np/(\nu \alpha)$, where $\nu$ is the number of $\alpha_j$ that are equal to 1. We have $s \leq p^*$ if and only if $\nu \geq n/\alpha - p$. This is exactly the necessary and sufficient condition for the embedding (5.5) (see Theorem 5.5).

We observe that direct estimates of rearrangements in terms of partial moduli of continuity are unknown; it would be very interesting to find such estimates (see [32], [34]).

6. Fourier transforms

**Definition of the Fourier transform**
Let $f$ be a function in $L^1(\mathbb{R}^n)$. The Fourier transform of $f$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i2\pi x \cdot \xi} \, dx, \quad \xi \in \mathbb{R}^n.$$  

The Convolution theorem states that if $f, g \in L^1(\mathbb{R}^n)$, then

$$\hat{f} \ast \hat{g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

We have also the classical Plancherel’s theorem.

**Theorem 6.1.** If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\hat{f} \in L^2(\mathbb{R}^n)$, and $||\hat{f}||_2 = ||f||_2$.

Thus, the map $f \mapsto \hat{f}$ is a bounded linear operator defined on a dense subset $L^1 \cap L^2$ of the space $L^2(\mathbb{R}^n)$; moreover, it is an isometry. If $f \in L^2(\mathbb{R}^n)$ (but $f \not\in L^1(\mathbb{R}^n)$), there exists a sequence $\{f_k\}$ of functions in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $||f - f_k||_2 \to 0$. We have

$$||\hat{f}_j - \hat{f}_k||_2 = ||f_j - f_k||_2$$

and hence $\{\hat{f}_k\}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. Thus, $\{\hat{f}_k\}$ converges to some function in $L^2(\mathbb{R}^n)$, which we call $\hat{f}$. Obviously, $\hat{f}$ does not depend on the choice of a sequence $\{f_k\}$. Moreover,

$$||\hat{f}||_2 = \lim_{k \to \infty} ||\hat{f}_k||_2 = \lim_{k \to \infty} ||f||_2 = ||f||_2.$$

We observe also that $f \mapsto \hat{f}$ is not only an isometry but it is a unitary transformation of $L^2(\mathbb{R}^n)$ onto itself. It follows from the Fourier inversion theorem.

**Theorem 6.2.** For any $f \in L^2(\mathbb{R}^n)$

$$f(x) = \lim_{k \to \infty} \int_{|\xi| \leq k} \hat{f}(\xi)e^{i2\pi x \cdot \xi} \, d\xi \quad (\text{convergence in } L^2(\mathbb{R}^n)).$$

Thus, $f$ is the Fourier transform of $g(\xi) = \hat{f}(-\xi)$.

We have defined the Fourier transform for functions in $L^1(\mathbb{R}^n)$ and functions in $L^2(\mathbb{R}^n)$. Now, let $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, that is,

$$f = f_1 + f_2, \quad f_1 \in L^1(\mathbb{R}^n), \quad f_2 \in L^2(\mathbb{R}^n).$$

(6.1)

Set $\hat{f} = \hat{f}_1 + \hat{f}_2$. In the case $f \in L^1 \cap L^2$ this definition coincides with the previous one. It is easy to show that definition does not depend on the choice of decomposition (6.1). Since $L^p(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ for $1 \leq p \leq 2$, the Fourier transform is defined for all $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$.

The Hausdorff – Young inequality
Theorem 6.3. Let $f \in L^p(\mathbb{R}^n), \; 1 \leq p \leq 2$. Then

$$||\hat{f}||_{p'} \leq ||f||_p.$$  \hfill (6.2)

This theorem was proved by W. Young in 1913 in the case when $p'$ is an even integer. Namely, Young observed that in this case the Fourier transform inequality can be obtained from the convolution inequality. For example, for $p = 4/3, \; p' = 4$

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^4 d\xi = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^n} |\hat{f} \ast \hat{f}(\xi)|^2 d\xi = ||f \ast f||_2^2 \leq ||f||_{4/3}^4.$$  \hfill (6.2)

In 1923 Hausdorff proved inequality (6.2) for all $p \in [1, 2]$.

In the periodic case the Hausdorff – Young theorem states the following.

Theorem 6.4. Let $f \in L^p[0, 2\pi], \; 1 \leq p \leq 2$, and

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \quad (n \in \mathbb{Z}).$$

Then

$$\left( \sum_{n \in \mathbb{Z}} |c_n|^{p'} \right)^{1/p'} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right)^{1/p}.$$  \hfill (6.3)

In 1923 F. Riesz extended this theorem to an arbitrary uniformly bounded orthonormal system. In 1926 M. Riesz gave an alternative proof of this result as one of applications of his convexity theorem.

For $p > 2$ Theorems 6.3 and 6.4 fail. For example, let $p > 2$ and let a sequence $\{c_k\}$ of positive numbers be such that

$$\sum_{k=1}^{\infty} c_k^2 < \infty, \quad \text{but} \quad \sum_{k=1}^{\infty} c_k^{p'} = \infty.$$  \hfill (6.3)

Then the function

$$f(x) = \sum_{k=1}^{\infty} c_k \cos 2^k x$$

belongs to $L^p[0, 2\pi]$, but inequality (6.3) doesn’t hold.

Inequality (6.3) is sharp for all $1 \leq p \leq 2$. Indeed, it becomes equality for functions $f(x) = Ae^{i2\pi mx}$. Hardy and Littlewood proved that equality in (6.3) is attained only for such exponential functions.
However, inequality (6.2) for $1 < p < 2$ can be improved. First it was proved by K. Babenko [3] in 1961 for $p' = 2, 4, 6, ...$. For all $p \in (1, 2)$ Beckner [5] in 1975 proved the inequality

$$||\hat{f}||_{p'} \leq A_p^p ||f||_p,$$

where

$$A_p = \left( \frac{p^{1/p}}{(p')^{1/p'}} \right)^{1/2}$$

For the gaussian $\exp(-\pi|x|^2)$ (6.4) becomes equality.

**Hardy – Littlewood – Paley inequality**

First, the following Hardy – Littlewood type theorem holds.

**Theorem 6.5.** If $f \in L^p(\mathbb{R}^n)$, $1 < p \leq 2$, then

$$\left( \int_{\mathbb{R}^n} |\xi|^{n(p-2)} |\hat{f}(\xi)|^p d\xi \right)^{1/p} \leq c ||f||_p.$$  

(6.5)

A stronger inequality is given by the Hardy – Littlewood – Paley theorem.

**Theorem 6.6.** If $f \in L^p(\mathbb{R}^n)$, $1 < p \leq 2$, then

$$\left( \int_0^\infty t^{p-2} \hat{f}^*(t)^p dt \right)^{1/p} \leq c ||f||_p.$$  

(6.6)

This theorem gives a refinement of Theorem 6.5. Indeed, as we have already observed, for the function $\varphi(x) = 1/|x|$ ($x \in \mathbb{R}^n$) we have $\varphi^*(t) = (v_n/t)^{1/n}$. Thus, by the Hardy – Littlewood inequality,

$$\int_{\mathbb{R}^n} |\xi|^{n(p-2)} |\hat{f}(\xi)|^p d\xi \leq \int_0^\infty t^{p-2} \hat{f}^*(t)^p dt.$$  

Further, the left-hand side of (6.7) is exactly the Lorentz norm $||\hat{f}||_{p',p}$. Recall that we have a strict embedding $L^{p',p} \subset L^{p'}$, $1 < p \leq 2$. Thus, inequality (6.6) implies the Hausdorff – Young inequality, but with additional constant on the right-hand side (which blows up as $p \to 1$).

We observe that initially Theorems 6.5 and 6.6 were proved by Hardy and Littlewood [20], [22] (1927, 1931) for the trigonometric Fourier series. In particular, in the periodic case Theorem 6.5 states the following.

**Theorem 6.7.** Let $f \in L^p[0,2\pi]$, $1 < p \leq 2$, and

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx} dx \quad (n \in \mathbb{Z}).$$
Then
\[
\left( \sum_{n \in \mathbb{Z}} |c_n|^p (|n| + 1)^{p-2} \right)^{1/p} \leq A \| f \|_p.
\] (6.7)

In 1931 Paley [46] extended this theorem to the Fourier series with respect to arbitrary uniformly bounded orthonormal system \{\varphi_n\} on [0, 1]. From this result he derived a rearrangement inequality for the Fourier coefficients \(a_n\)
\[
\left( \sum_{n=1}^{\infty} (a_n^*)^p n^{p-2} \right)^{1/p} \leq A \| f \|_p.
\]

Of course, these theorems fail for \(p = 1\). For example, the series
\[
\sum_{n=1}^{\infty} \frac{\cos nx}{\log(n + 1)}
\]
is the Fourier series of some integrable function \(f\), but
\[
\sum_{n=1}^{\infty} \frac{a_n}{n} = \sum_{n=1}^{\infty} \frac{1}{n \log(n + 1)} = \infty.
\]

In 1937, Pitt [50] proved the following theorem.

**Theorem 6.8.** Let \(1 < p \leq q < \infty\), \(0 \leq \alpha < 1/p'\), and \(\lambda = 1/q + 1/p - 1 + \alpha \geq 0\). Let \(f \in L^1[0, 2\pi]\) and
\[
c_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(x) e^{-inx} dx \quad (n \in \mathbb{Z}).
\]

Then
\[
\left( \sum_{n \in \mathbb{Z}} |c_n|^q (|n| + 1)^{-\lambda q} \right)^{1/q} \leq A \left( \int_{-\pi}^{\pi} |f(x)|^p |x|^\alpha dx \right)^{1/p}.
\] (6.8)

In 1956, Stein [51] extended this result to arbitrary uniformly bounded systems on [0, 1]. Under the same conditions on \(p, q,\) and \(\alpha\), he obtained the following rearrangement inequality
\[
\left( \sum_{n=1}^{\infty} (a_n^*)^q n^{-\lambda q} \right)^{1/q} \leq \left( \int_{0}^{1} f^*(x)^p x^\alpha dx \right)^{1/p}.
\]

Suppose now that \(f \in L^{p,r}(\mathbb{R}^n) \ (1 < p < 2, 1 \leq r \leq \infty)\). Then \(f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)\) and hence, \(\hat{f}\) is defined. Moreover, the following inequality holds.
Theorem 6.9. If \( f \in L^{p,r}(\mathbb{R}^n) \) \((1 < p < 2, 1 \leq r \leq \infty)\), then \( \hat{f} \in L^{p',r}(\mathbb{R}^n) \), and
\[
||\hat{f}||_{p',r} \leq c||f||_{p,r}.
\]

This theorem was obtained by Herz [24] in 1968 with the use of the Marcinkiewicz interpolation theorem.

Let us consider the case \( p = 2 \). In this case, Herz [24] showed that the Fourier transform maps \( L^{2,r} \) continuously into \( L^{2,q} \) if and only if \( r \leq 2 \leq q \). The positive part of this statement is obvious. By duality, to obtain the negative part, it is sufficient to show that, whatever be \( 1 \leq q < 2 \), the Fourier transform doesn’t map \( L^{2,1} \) continuously into \( L^{2,q} \). To prove it, Herz used the so called lacunary functions. For the simplicity, we consider the periodic case. We show that for any \( 1 \leq q < 2 \) there exists a function \( f \in L^{2,1} \) such that the sequence of its Fourier coefficients doesn’t belong to \( l^{2,q} \). Let \( 1 \leq q < 2 \). Set
\[
f(x) = \sum_{k=2}^{\infty} \cos \frac{2kx}{\sqrt{k(\ln k)^{1/q}}}.
\]

Denote for \( n \in \mathbb{N} \)
\[
a_n = \begin{cases} (\sqrt{k(\ln k)^{1/q}})^{-1} & \text{if } n = 2^k, \ k \geq 2, \\ 0 & \text{otherwise.} \end{cases}
\]

Since the sequence \( \{a_n\} \) belongs to \( l^2 \), the function \( f \) belongs to \( L^p \) for any \( 1 < p < \infty \). However, \( \{a_n\} \) doesn’t belong to \( l^{2,q} \).

We mention also the paper by Bochkarev [10]. In this paper, the Fourier coefficients with respect to uniformly bounded orthonormal systems were studied. First, it was proved the following

Theorem 6.10. Let \( \{\varphi_n\} \) be an orthonormal system of functions on \([0,1]\) such that
\[
||\varphi_n||_\infty \leq M, \ n \in \mathbb{N}.
\]
Let \( f \in L^1[0,1] \) and let \( c_n \) be Fourier coefficients of \( f \) with respect to \( \{\varphi_n\} \). Then for any \( 2 < q \leq \infty \) and \( n \geq 2 \),
\[
\sum_{k=1}^{n} (c_k^*)^2 \leq AM^2(\log n)^{1-2/q}||f||_2^2 \tag{6.9}
\]
Bochkarev showed that inequality (6.9) cannot be improved.

We observe that John Benedetto and Hans Heining [6] studied Fourier transform inequalities in weighted Lorentz spaces. Their approach was based on estimates of rearrangements.
Rearrangement inequalities for Fourier transforms

The following theorem was proved by Jurkat and Sampson [27] (1984).

**Theorem 6.11.** Let \( f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n) \). Then

\[
\left( \int_0^t \hat{f}^*(u)^2 du \right)^{1/2} \leq t^{1/2} \int_0^\tau f^*(s) ds + \left( \int_\tau^\infty f^*(s)^2 ds \right)^{1/2} \tag{6.10}
\]

for any \( t, \tau > 0 \).

**Proof.** Choose a set \( E \subset \mathbb{R}^n \) of measure \( \tau \) such that
\[
\{ x : |f(x)| > f^*(\tau) \} \subset E \subset \{ x : |f(x)| \geq f^*(\tau) \}.
\]
Let \( g = f \chi_E, \ h = f - g \). Then
\[
||g||_1 = \int_0^\tau f^*(s) ds, \quad ||h||_2 = \left( \int_\tau^\infty f^*(s)^2 ds \right)^{1/2}. \tag{6.11}\]

We have also
\[
||\hat{g}||_{\infty} \leq ||g||_1 \quad \text{and} \quad ||\hat{h}||_2 = ||h||_2.
\]
Observe that \( \hat{f}^*(u) \leq ||\hat{g}||_{\infty} + \hat{h}^*(u) \). Thus,
\[
\left( \int_0^t \hat{f}^*(u)^2 du \right)^{1/2} \leq t^{1/2} ||\hat{g}||_{\infty} + ||\hat{h}||_2.
\]
Applying (6.11), we obtain (6.10). \( \square \)

**Corollary 6.12.** Let \( f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n) \). Then
\[
\int_0^t \hat{f}^*(u)^2 du \leq c \int_0^t \left( \int_0^{1/u} f^*(s) ds \right)^2 du = c \int_1^\infty f^{**}(u)^2 du \tag{6.12}\]
for any \( t > 0 \).

From these results Jurkat and Sampson derived some weighted upper estimates for \( \hat{f} \) (in particular, inequalities in Lorentz norms). For this, they used the following convexity principle (which represents a generalization of Hardy, Littlewood and Pólya theorem).

**Theorem 6.13.** Assume that \( \Psi(s) \geq 0 \) is increasing and convex for \( s \geq 0 \), and \( \Phi(s) \geq 0 \) is increasing and concave for \( s \geq 0 \). Let \( U(t), G(t), H(t) \) be nonnegative and measurable for \( t > 0 \). If \( G \) decreases, then
\[
\int_0^t G(s) U(s) ds \leq \int_0^t H(s) U(s) ds \text{ for all } t > 0. \tag{6.13}\]
implies
\[ \int_0^t \Psi(G(s))U(s)ds \leq \int_0^t \Psi(H(s))U(s)ds \quad \text{for all } t > 0. \]

If \( H \) increases, then (6.13) implies
\[ \int_0^t \Phi(G(s))U(s)ds \leq \int_0^t \Phi(H(s))U(s)ds \quad \text{for all } t > 0. \]

Rearrangement inequality (6.12) actually was proved in 1971 by Jodeit and Torchinsky \[26\]. More exactly, they proved the following theorem.

**Theorem 6.14.** Let \( T \) be a sublinear operator. Then the operator \( T \) is of type \((1, \infty)\) and \((2, 2)\) if and only if for some constant \( K \) and each \( f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n) \)
\[ \int_0^t (Tf)^*(u)^2 du \leq K \int_{1/t}^\infty f^{**}(u)^2 du, \quad t > 0. \]

**Estimates of Fourier transforms in Sobolev spaces**

Let \( r, n \in \mathbb{N} \) and \( 1 < p \leq 2 \). Assume that \( f \in W^r_p(\mathbb{R}^n) \). We have
\[ |\hat{D}^r f(\xi)| = (2\pi|\xi|)^r |\hat{f}(\xi)| \]
and thus
\[ |\hat{f}(\xi)| \asymp |\xi|^{-r} \sum_{k=1}^n |\hat{D}^r_k f(\xi)|. \quad (6.14) \]

By the Hardy-Littlewood inequality, we have
\[ \left( \int_{\mathbb{R}^n} |\xi|^{pr+n(p-2)} |\hat{f}(\xi)|^p d\xi \right)^{1/p} \leq c \sum_{k=1}^n ||D^r_k f||_p. \quad (6.15) \]

If \( f, g \in S_0(\mathbb{R}^n) \), then \((fg)^*(t) \leq f^*(t/2)g^*(t/2)\) (see, e.g., \[35\]). Further, recall that the rearrangement of the function \( \varphi(\xi) = 1/|\xi| \) is equal to \((v_n/t)^{1/n}\). Thus, using (6.14) and applying Hardy-Littlewood-Paley inequality to the derivatives, we obtain
\[ \left( \int_0^\infty t^{pr/n+p-2} \hat{f}^*(t)^p dt \right)^{1/p} \leq c \sum_{k=1}^n ||D^r_k f||_p. \quad (6.16) \]

We observe that the right-hand sides of (6.15) and (6.16) contain only the norms of non-mixed derivatives. But we know (Theorem \[5.1\] that
for $1 < p < \infty$

$$\sum_{|s|=r} ||D^s f||_p \leq c \sum_{k=1}^{n} ||D_k^r f||_p.$$  

Stress that this estimate fails for $p = 1$.

For $p = 1$ the Hardy-Littlewood inequality does not hold and it is impossible to use the reasonings described above. In fact, for the Fourier transforms of functions in $W^1_1(\mathbb{R})$, the only available estimates are those based upon the obvious inequality $||\hat{g}||_\infty \leq ||g||_1$.

On the other hand, by Hardy’s inequality, for any $f \in H^1(\mathbb{R}^n)$ ($n \in \mathbb{N}$)

$$\int_{\mathbb{R}^n} \frac{|\hat{f}(\xi)|}{|\xi|^n} d\xi \leq c ||f||_{H^1}.$$  

It was first discovered by Bourgain [11] in 1981 that for $n \geq 2$ the Fourier transforms of the derivatives of the functions in Sobolev space $W^1_1(\mathbb{R}^n)$ satisfy Hardy’s inequality. More exactly, Bourgain considered the periodic case. His studies were continued by Pelczyński and Wojciechowski [47]. First, we have the following theorem (Bourgain; Pelczyński and Wojciechowski).

**Theorem 6.15.** Let $f \in W^r_1(\mathbb{R}^n)$ ($n \geq 2$, $r \in \mathbb{N}$). Then

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)||\xi|^r d\xi \leq c \sum_{|s|=r} ||D^s f||_1.$$  

Equivalently,

$$\sum_{|s|=r} \int_{\mathbb{R}^n} \frac{|\hat{D^s f}(\xi)|}{|\xi|^n} d\xi \leq c \sum_{|\alpha|=r} ||D^\alpha f||_1.$$  

This is Hardy type inequality. Bourgain [12] in 1985 and Pelczyński and Wojciechowski [47] in 1993 proved also the following theorem.

**Theorem 6.16.** If $f \in W^1_1(\mathbb{R}^n)$ ($n \geq 2$), then

$$\sum_{\nu \in \mathbb{Z}} \left( \int_{D_{\nu}} |\hat{f}(\xi)|^n d\xi \right)^{1/n} \leq c \sum_{k=1}^{n} ||D_k^r f||_1,$$  

where

$$D_{\nu} = \{ \xi \in \mathbb{R}^n : 2^{\nu-1} < |\xi| \leq 2^\nu \} \quad (\nu \in \mathbb{Z}).$$

We observe that by Sobolev-Gagliardo-Nirenberg theorem,

$$W^1_1(\mathbb{R}^n) \subset L'^n(\mathbb{R}^n), \quad n \geq 2.$$
Applying Hausdorff-Young inequality, we obtain that for any \( f \in W_{1}^{1}(\mathbb{R}^{n}) \) its Fourier transform belongs to \( L^{n}(\mathbb{R}^{n}) \). Of course, (6.18) gives a stronger statement.

We return to Theorem 6.15. In comparison with inequality (6.15), the right-hand side of (6.17) contains all the derivatives of order \( r \). The \( L^{1} \)-norms of mixed derivatives cannot be estimated by the \( L^{1} \)-norms of pure derivatives. Nevertheless, we proved \([31]\) that the right-hand side of inequality (6.17) can be replaced by the sum of non-mixed derivatives of the same order.

**Theorem 6.17.** Let \( r_{1}, \ldots, r_{n} \in \mathbb{N} \) \((n \geq 2)\) and let

\[
    r = n \left( \sum_{k=1}^{n} \frac{1}{r_{k}} \right)^{-1}. 
\]

If \( f \in W_{1}^{r_{1}, \ldots, r_{n}}(\mathbb{R}^{n}) \) \((n \geq 2)\), then

\[
    \int_{\mathbb{R}^{n}} |\hat{f}(\xi)| \left( \sum_{k=1}^{n} |\xi_{k}|^{r_{k}} \right)^{1-n/r} d\xi \leq c \sum_{k=1}^{n} \|D_{k}^{r_{k}} f\|_{1}. \tag{6.19}
\]

In the isotropic case \( r_{1} = \cdots = r_{n} = r \) inequality (6.19) has the form

\[
    \int_{\mathbb{R}^{n}} |\hat{f}(\xi)||\xi|^{-n} d\xi \leq c \sum_{k=1}^{n} \|D_{k}^{r} f\|_{1}. \tag{6.20}
\]

In contrast to (6.17), the right-hand side of (6.20) contains only the norms of non-mixed derivatives.

For any \( s = (s_{1}, \ldots, s_{n}) \) with \( |s| = r \)

\[
    |\hat{D}^{s} f(\xi)| = (2\pi)^{r} |\hat{f}(\xi)| \prod_{j=1}^{n} |\xi_{j}|^{s_{j}} \leq (2\pi |\xi|)^{r} |\hat{f}(\xi)|. 
\]

Thus, we obtain

**Theorem 6.18.** Let \( f \in C_{0}^{\infty}(\mathbb{R}^{n}) \) \((n \geq 2)\). Then for any \( r \in \mathbb{N} \)

\[
    \sum_{|s|=r} \int_{\mathbb{R}^{n}} \frac{|\hat{D}^{s} f(\xi)|}{|\xi|^{n}} d\xi \leq c \sum_{k=1}^{n} \|D_{k}^{r} f\|_{1}. 
\]

In \([31]\), we proved also the following rearrangement inequality.

**Theorem 6.19.** Let \( r_{1}, \ldots, r_{n} \in \mathbb{N} \) \((n \geq 2)\) and let

\[
    r = n \left( \sum_{k=1}^{n} \frac{1}{r_{k}} \right)^{-1} < n. 
\]
If \( f \in W^{r_1,\ldots,r_n}(\mathbb{R}^n) \) \((n \geq 2)\), then
\[
||\hat{f}||_{n/r,1} \leq c \sum_{k=1}^{n} ||D^{r_k}_{k}f||_1.
\tag{6.21}
\]

Observe that in conditions of this theorem we have
\[
||f||_{n/(n-r),1} \leq c \sum_{k=1}^{n} ||D^{r_k}_{k}f||_1 \tag{6.22}
\]
(it was proved in our work \[30\]). If \( n/(n-r) < 2 \) (for example, if \( n \geq 3 \) and \( r = 1 \)), then \((6.21)\) follows immediately from \((6.22)\), if we apply the inequality
\[
||\hat{f}||_{p',r} \leq ||f||_{p,r} \quad (1 < p < 2, \ 1 \leq r \leq \infty).
\]

These arguments cannot be applied, for example, to the space \( W^1_1(\mathbb{R}^2) \). We have \( W^1_1(\mathbb{R}^2) \subset L^{2,1}(\mathbb{R}^2) \), and this embedding is sharp. As we know, the condition \( f \in L^{2,1}(\mathbb{R}^2) \) does not imply that \( \hat{f} \in L^{2,1}(\mathbb{R}^2) \). However, by Theorem 6.19 for functions in \( W^1_1(\mathbb{R}^2) \) their Fourier transforms belong to \( L^{2,1}(\mathbb{R}^2) \).

In our work \[33\] we extended Theorems 6.17 and 6.19 to the fractional Sobolev spaces \( L^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n) \). In this work we proved also the following theorem.

**Theorem 6.20.** Assume that \( r_j \in \mathbb{N} \) \((j = 1, \ldots, n; \geq 2)\). Then
\[
\left( \int_{\mathbb{R}^n} |\hat{f}(\xi)|^n \prod_{j=1}^{n} |\xi_j|^{r_j-1} \ d\xi \right)^{1/n} \leq c \prod_{j=1}^{n} ||D^{r_j}_{j} f||_1
\]
for each \( f \in W^{r_1,\ldots,r_n}(\mathbb{R}^n) \).

For \( n = 2 \) this result was obtained by other methods by Pelczyński and Senator \[48\].

The proofs of these results can be subdivided in two parts. The main (and much more involved part) is the following theorem which we have already formulated above.

For \( r > 0 \), let \( \bar{r} \) be the least integer such that \( r \leq \bar{r} \).

**Theorem 6.21.** Let \( r_1, \ldots, r_n \) be positive numbers and let
\[
r = n \left( \sum_{j=1}^{n} \frac{1}{r_j} \right)^{-1}, \ 1 \leq p < q < \infty, \ \varkappa = 1 - \frac{n}{r} \left( \frac{1}{p} - \frac{1}{q} \right) > 0.
\]
Set \( \alpha_j = \kappa r_j \) \((j = 1, \ldots, n)\). If \( 1 < p < \infty \) and \( n \geq 1 \), or \( p = 1 \) and \( n \geq 2 \), then for every function \( f \in L^r_{r_1,\ldots,r_n}(\mathbb{R}^n) \)

\[
\sum_{j=1}^{n} \left( \int_{0}^{\infty} \left[ h^{-\alpha_j} \omega_j^p(f; h) \right]^q \frac{dh}{h} \right)^{1/p} \leq c \sum_{j=1}^{n} ||D^{r_j} f||_p.
\]

For \( r_1 = \ldots = r_n = 1 \) this theorem was proved in our work [29] in 1988. Anisotropic case was studied in our works [30] and [33].

The other part of the proofs was based on the estimates via moduli of continuity.

**Estimates of Fourier transforms in terms of moduli of continuity**

Apparently the first estimates of this type were obtained by S. Bernstein in 1914. Bernstein proved that for each periodic function \( f \in \text{Lip} \alpha \) \((\alpha > 1/2)\) the sequence of its Fourier coefficients belongs to \( l^1 \), and this property fails for \( \alpha = 1/2 \).

The following lemma was proved in our paper [31].

**Lemma 6.22.** Let \( r_1, \ldots, r_n \in \mathbb{N} \) \((n \in \mathbb{N})\),

\[
r = n \left( \sum_{j=1}^{n} \frac{1}{r_j} \right)^{-1}, \quad \text{and} \quad s_j = \frac{r}{nr_j}.
\]

Let \( f \in L^p(\mathbb{R}^n), 1 \leq p \leq 2 \). Then

\[
\hat{f}^*(t) \leq ct^{1/p-1} \sum_{j=1}^{n} \omega_j^{r_j}(f; t^{-s_j}) \quad (t < 0).
\]

**Proof.** Set \( \varphi_{h,j}(x) = \Delta_j^r(h)f(x) \) \((h > 0)\). Let \( E \subset \mathbb{R}^n \) be a set of measure \( t \) such that

\[
|\hat{f}(\xi)| \geq \hat{f}^*(t) \quad \text{for any} \quad \xi \in E.
\]

Further, let

\[
A_j = \{ \xi : |\xi_j| \geq \frac{t^{s_j}}{2} \}, \quad j = 1, \ldots, n.
\]

We have \( \sum_{j=1}^{n} s_j = 1 \). Thus,

\[
\left| \left( \bigcup_{j=1}^{n} A_j \right)^c \right| = \left| \bigcap_{j=1}^{n} A_j^c \right| = \frac{t}{2^n}.
\]
This implies that \(|E \cap (\cup_{j=1}^{n} A_j)| \geq t/2\). Hence, there exists \(j = j(t)\) such that \(|E \cap A_j| \geq t/(2n)\). Set \(Q = E \cap A_j\). We have
\[
\widehat{\varphi_{h,j}}(\xi) = \hat{f}(\xi)\sigma(h\xi_j), \quad \text{where} \quad \sigma(u) = (e^{2\pi u} - 1)^{r_j}.
\]

Set \(\delta = r_j t^{-s_j}\). We show that
\[
\frac{1}{\delta} \int_{0}^{\delta} |\sigma(h\xi_j)| \, dh > \frac{1}{2} \quad \text{for any} \quad \xi \in Q.
\]

Indeed,
\[
|\sigma(u)| \geq (1 - \cos 2\pi u)^{\nu_j} \geq 1 - r_j \cos 2\pi u.
\]

Thus, if \(|\lambda| \geq t^{\nu_j}/2\), then
\[
\frac{1}{\delta} \int_{0}^{\delta} |\sigma(\lambda h)| \, dh \geq 1 - \frac{r_j \sin 2\pi \lambda \delta}{2\pi \lambda \delta} \geq 1 - \frac{1}{\pi}.
\]

Now we have
\[
\frac{1}{\delta} \int_{0}^{\delta} d\lambda \int_{Q} |\widehat{\varphi_{h,j}}(\xi)| \, d\xi = \frac{1}{\delta} \int_{0}^{\delta} \int_{Q} |\hat{f}(\xi)| \sigma(h\xi_j) \, d\xi 
\]
\[
\geq \frac{1}{2} |Q| \hat{f}(t) \geq \frac{t}{4n} \hat{\text{widehat} f^*(t)}.
\]

On the other hand, by the Hausdorff-Young inequality,
\[
\int_{Q} |\widehat{\varphi_{h,j}}(\xi)| \, d\xi \leq |Q|^{1/p} \left( \int_{Q} |\widehat{\varphi_{h,j}}(\xi)|^{p'} \, d\xi \right)^{1/p'} 
\]
\[
\leq |Q|^{1/p} ||\varphi_{h,j}||_p \leq t^{1/p} ||\varphi_{h,j}||_p.
\]

Therefore,
\[
\hat{f}^*(t) \leq c t^{1/p-1} \frac{1}{\delta} \int_{0}^{\delta} ||\varphi_{h,j}||_p \, dh \leq c' t^{1/p-1} \omega_j^{r_j}(f; t^{-s_j})_p.
\]

Assume that \(\alpha_j > 0\) and \(1 \leq p, \theta < \infty\). Let \(r_j\) be the least integer such that \(r_j > \alpha_j\). Let
\[
||f||_{\alpha_1, \ldots, \alpha_n} = \sum_{j=1}^{n} \left( \int_{0}^{\infty} (t^{-\alpha_j} \omega_j^{r_j}(f; t)_p) \theta \frac{dt}{t} \right)^{1/\theta}
\]
denote the homogenous Besov norm.

Applying Lemma \(6.22\) we immediately obtain
Theorem 6.23. Let $\alpha_j > 0$ ($j = 1, ..., n$), $1 \leq p \leq 2$, $1 \leq \theta \leq \infty$, 

$$\alpha = n \left( \sum_{j=1}^{n} \frac{1}{\alpha_j} \right)^{-1}, \quad \text{and} \quad q = \left( \frac{\alpha}{n} + \frac{1}{p'} \right)^{-1}.$$

If $f \in B^{\alpha_1,\ldots,\alpha_n}_{p,\theta}(\mathbb{R}^n)$, then $\widehat{f} \in L^{q,\theta}(\mathbb{R}^n)$, and 

$$||\widehat{f}||_{q,\theta} \leq c ||f||_{b^{\alpha_1,\ldots,\alpha_n}_{p,\theta}}.$$

In the case $\alpha_1 = \ldots = \alpha_n = \alpha$ this result was obtained by other methods by A. Pietsch [49, p. 270] (1980). Taking $\theta = 1$ and $\alpha = n/p$, we obtain the Bernstein-Szasz theorem: if $f \in B^{n/p,1}_{p,1}(\mathbb{R}^n)$ ($1 \leq p \leq 2$), then $\widehat{f} \in L^1(\mathbb{R}^n)$.

Return to the inequality

$$||\widehat{f}||_{n/r,1} \leq c \sum_{k=1}^{n} ||D^{r_k} f||_1, \quad r = n \left( \sum_{k=1}^{n} \frac{1}{r_k} \right)^{-1} < n. \quad (6.24)$$

As we have already mentioned, the proof of this inequality was obtained by the combination of estimates of the rearrangement $\widehat{f}^*(t)$ via moduli of continuity in some $L^p$, $1 < p < 2$, and estimates of these moduli of continuity via norms of partial derivatives in $L^1$. It would be interesting to find direct estimates of $\widehat{f}^*(t)$ involving rearrangements of the derivatives (and, maybe, rearrangement of $f$) and yielding $(6.24)$.

We observe that Jurkat-Sampson inequality

$$\int_0^t \hat{\varphi}^*(u)^2 du \leq c \int_0^t \left( \int_0^{1/u} \varphi^*(s) ds \right)^2 du = c \int_1^{\infty} \varphi^{**}(u)^2 du$$

applied to $\varphi = D^{r_k} f$ cannot work since at the right-hand side we have the operator $\varphi^{**}$ which is unbounded in $L^1$.

In our joint work with J. García-Cuerva [17] we obtained sharp weighted rearrangement estimates of Fourier transforms in terms of moduli of continuity (we studied isotropic case).

Finally, we observe that the relations between moduli of smoothness of functions and growth properties of their Fourier transforms were studied in many works, especially in the last 20 years. We refer to the works of Benedetto and Heinig [6] and Gorbachev and Tikhonov [18] and references therein.
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