Abstract In this work, we construct new Bailey pairs for the integral pentagon identity in terms of q-hypergeometric functions. The pentagon identity considered here represents the equality of the partition functions of certain three-dimensional supersymmetric dual theories. It can be also interpreted as the star-triangle relation for the Ising-type integrable lattice model.

We also adopt the following convention

\[(a, b; q)_\infty := (a; q)_\infty (b; q)_\infty.\]  \hspace{1cm} (1.2)

Theorem 1.1 Let \(a_1, a_2, a_3, b_1, b_2, b_3, q \in \mathbb{C}\) and integers \(m_i, n_i \in \mathbb{Z}\). Then

\[
\sum_{m \in \mathbb{Z}} \frac{dz}{2\pi iz} \left( -q^\frac{1}{2} \right)^3 \sum_{i=1}^{3} \frac{[m_i + m]}{2} \frac{[n_j - m]}{2} \frac{z}{n_j} \frac{1}{n_i} \frac{q^{1 + \frac{[m_j + m]}{2}}}{a_i} \frac{q^{1 + \frac{[n_j - m]}{2}}}{b_j} (a_i z; q)_\infty (b_j z; q)_\infty
\]

\[
= (-q^\frac{1}{2})^{\sum_{i,j=1}^{3} \frac{[m_i + n_j]}{2}} \prod_{i,j=1}^{3} (a_i b_j)^{-\frac{[m_i + n_j]}{2}} \frac{1}{a_i b_j} (q^{-\frac{1}{2}} a_i b_j; q)_\infty, \]  \hspace{1cm} (1.3)

where the balancing conditions are

\[
\prod_{i=1}^{3} a_i b_i = q, \]  \hspace{1cm} (1.4)

\[
\sum_{i=1}^{3} m_i + n_i = 0, \]  \hspace{1cm} (1.5)

and \(T\) represents the positively oriented unit circle.

1 Introduction

Bailey’s lemma \([1,2]\) is a powerful tool to derive hypergeometric identities (ordinary, trigonometric, and elliptic type). In this work, we construct new integral Bailey pairs for the pentagon identity in terms of q-hypergeometric functions. The pentagon identity can be interpreted as a Pachner’s 3-2 move for triangulated three-dimensional manifolds. Such identities also play a role in the study of supersymmetric gauge theories, integrable models, knot theory, etc.\(^1\)

Let \(q, z \in \mathbb{Z}\) with \(|q| < 1\). We define the infinite q-product

\[(z; q)_\infty := \prod_{k=0}^{\infty} (1 - z q^k). \]  \hspace{1cm} (1.1)

We refer to some recent works \([3–11]\).

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1\(^1\) See some recent works \([3–11]\).

\(^2\) In this case parameters \(a_i\) and \(b_i\) stand for the flavor symmetry and \(z\) is the fugacity for the \(U(1)\) gauge group.
ten as the star-triangle relation \(^3\) for some integrable model of statistical mechanics.

The proof of the form above is given in [8] for the balancing conditions

\[
\prod_{i=0}^{3} a_i = \prod_{i=0}^{3} b_i = q^{\frac{1}{3}}, \quad \sum_{i=0}^{3} m_i = \sum_{i=0}^{3} n_i = 0. \tag{1.6}
\]

The absolute values can be eliminated by the identity \([12]\)

\[
\frac{(q^{1+\frac{|m|}{2}}/z; q)_{\infty}}{(q^{\frac{1}{2}} z; q)_{\infty}} = (-q^{\frac{1}{2}} z \prod_{i=0}^{3} (q^{1+\frac{|m|}{2}}/z; q)_{\infty} / (q^{\frac{1}{2}} z; q)_{\infty}), \tag{1.7}
\]

and one ends up with the following \(q\)-hypergeometric sum/integral identity \([6–8]\)

\[
\sum_{m \in \mathbb{Z}} \int_{\frac{1}{2} \pi i \mathbb{Z}}^{\frac{3}{2} \pi i \mathbb{Z}} \prod_{i=1}^{3} \left( \frac{q^{1+\frac{|m|}{2}}}{a_i z}, q^{\frac{1}{2}} z \right)_{\infty} \frac{1}{a_i b_i z^{\frac{3}{2}}}, q_{\infty} \zeta \frac{dz}{2\pi i} = \frac{1}{\prod_{i=1}^{3} a_i b_i a_i^{m_i} b_i^{n_i}}, \tag{1.8}
\]

\section{Integral pentagon identity}

In \([6–8]\) it was shown that the identity (1.3) can be written as an integral pentagon identity

\[
\sum_{m \in \mathbb{Z}} \int_{\frac{1}{2} \pi i \mathbb{Z}}^{\frac{3}{2} \pi i \mathbb{Z}} \prod_{i=1}^{3} B[a_i, n_i + m; b_i z^{-1}, m_i - m] = B[a_1 b_2, n_1 + m_1; a_3 b_1; n_3 + m_3]
\times B[a_2 b_1, n_2 + m_1; a_3 b_2; n_3 + m_2], \tag{2.1}
\]

where we define the following function as

\[
B_m[a, n; b, m] = (-q^{\frac{1}{2}} z)^{\frac{|n|}{2} + \frac{|n|}{3} a - \frac{|n|}{2} b} \frac{(ab)^{\frac{|n|}{2}}}{(q^{1+\frac{|n|}{2}} a, q^{\frac{1}{2}} z b, q^{1+\frac{|n|}{2}} ab, q)^{\infty}} \frac{(q^{1+\frac{|m|}{2}} a^{-1}, q^{1+\frac{|m|}{2}} b^{-1}, q^{\frac{|m|}{2}} ab)^{\infty}}{q^{\frac{|m|}{2} a, q^{\frac{1}{2}} z b, q^{1+\frac{|m|}{2}} ab, q^{1+\frac{|m|}{2}} ab^{-1}))(q^{|m|} z)^{\infty}}. \tag{2.2}
\]

In a general sense, any algebraic relation for operators \(B\)

\[
BBB = BB \tag{2.3}
\]

which can be interpreted as a 2–3 Pachner move of a triangulated three-dimensional manifold is called a pentagon relation \([4,5]\). Note that the integral pentagon identity (2.1) for the \(N = 2\) supersymmetric \(S^2 \times S^1\) partition functions is supposed to be related to some topological invariant of corresponding 3-manifold via 3d–3d correspondence \([12,13]\) that connects three-dimensional \(N = 2\) supersymmetric theories and triangulated 3-manifolds. There are several examples of pentagon identities arising from supersymmetric gauge theory computations, see, e.g. [6–15].

\section{3 Bailey pairs}

Rogers–Ramanujan type identities are being continuously used in the solution of the integrable models, namely to derive the Yang–Baxter and the pentagon identities. In fact, a well-known example of this usage is conducted during the investigations of the hard hexagon model by Baxter. It turns out that Bailey discovered a systematic way to derive these types of identities \([1,2,16,17]\). As generalized by Andrews \([18,19]\), there exists an iterative scheme to derive infinitely many of these identities if one pair, called a Bailey pair is known. This forms the so-called Bailey chain. The induction step of generating the particular Bailey pairs is referred to as the Bailey lemma for the chain we consider.

A generalization of the Bailey pair approach to the integral identities is firstly done by Spiridonov in \([20,21]\). The construction of integral Bailey pairs yields new powerful verifications of various supersymmetric dualities \([22,23]\), generating solutions to the Yang-Baxter equation \([24–27]\), etc.

Accordingly, the generalized version of the Bailey chain is a couple of infinite sequences of holomorphic functions \(\{a_n^{(i)} \}_{n \geq 0}\) and \(\{b_n^{(i)} \}_{n \geq 0}\) such that there exists an identity independent of \(i\) which connect \(a_n^{(i)}\) and \(b_n^{(i)}\) as

\[
\beta_n^{(i)} = F_n(a_0^{(i)}, a_1^{(i)}, \ldots, a_n^{(i)}), \quad (3.1)
\]

where \(F\) can be an operator which may now include sum or integrals. Here, \(a_n^{(i)}\) and \(b_n^{(i)}\) are constructed according to

\[
a_n^{(i)} = G(a_0^{(i)}, a_1^{(i)}, \ldots, a_n^{(i)}), \quad (3.2)
\]

\[
\beta_n^{(i)} = H(b_0^{(i)}, b_1^{(i)}, \ldots, b_n^{(i)}), \quad (3.3)
\]

where \(G\) and \(H\) represent integral-sum operators.

\begin{definition}
Let \(\{a_m(z; t) \}_{m \in \mathbb{Z}}\) and \(\{b_m(z; t) \}_{m \in \mathbb{Z}}\) be two sequences of functions. They are said to form a Bailey pair with respect to the parameter \(t\) iff

\[
\beta_m(w; t) = \sum_{n \in \mathbb{Z}} \int dz B[tewz^{-1}, m - n + n_t] a_n(z; t). \tag{3.4}
\]
\end{definition}
Lemma 3.1 If \(\{\alpha_m(z; t)\}_{m \in \mathbb{Z}}\) and \(\{\beta_m(z; t)\}_{m \in \mathbb{Z}}\) form a Bailey pair with respect to \(t\), then the following sequences
\[
\alpha'_n(w; st) = B\{tw, n + n_u + n_s, 2n_s\} \alpha_n(w; t) \tag{3.5}
\]
\[
\beta'_n(w; st) = \sum_{m \in \mathbb{Z}} \int \frac{dx}{2\pi i x} B\{swx^{-1}, -m + n + n_s; ux, n_u + m\} B\{st^2 uw, n + 2n_t + n_s, sw^{-1} x, -n + m + n_s\} \beta_m(x; t) \tag{3.6}
\]
form a Bailey pair with respect to the parameter \(st\).

Proof We have to show that
\[
\beta'_n(w, st) = \sum_{p \in \mathbb{Z}} \int B\{stwy^{-1}, n - p + n_s + n_t, sty^{-1} x, -n + p + n_s + n_t, \alpha'_p(y, st)\} dy \tag{3.7}
\]
Inserting (3.4) in (3.6), we first calculate the left-hand side of the equality (3.7)
\[
\beta'_n(w; st) = \sum_{m \in \mathbb{Z}} \sum_{p \in \mathbb{Z}} \int \frac{dx}{2\pi i x} B\{swx^{-1}, -m + n + n_s; ux, m + n_u\}
\times B\{st^2 uw, n + n_u + 2n_t + n_s, sw^{-1} x, -n + m + n_s\}
\times \sum_{p \in \mathbb{Z}} \int \frac{dx}{2\pi i x} B\{txy^{-1}, m - p\}
\times \alpha_p(y, t) \psi_h dx \tag{3.8}
\]
Hence, by regrouping the terms accordingly, we obtain\(^5\)
\[
\sum_{p \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \int \frac{dx}{2\pi i x} \left(\frac{x}{2\pi i x} \left[\sum_{n \in \mathbb{Z}} B\{tw, n + n_u + n_s, 2n_s\} \alpha_n(w; t) \times B\{st^2 uw, n + n_u + 2n_t + n_s, sw^{-1} x, -n + m + n_s\} \times \frac{1}{\pi i x} \int \frac{dx}{2\pi i x} \right]ight) \times \alpha_p(y, t) \psi_h dx = 0 \tag{3.9}
\]
where we required the sum of the powers of \(x\) to vanish, namely
\[
n_u + n_s + n_t = 0 \tag{3.10}
\]
Upon renaming the variables as
\[
\begin{align*}
a_1 &= u ightarrow m_1 = n_u \\
b_1 &= sw ightarrow n_1 = n_s \\
a_2 &= s u^{-1} ightarrow m_2 = -n + n_s \\
b_2 &= q s^{-1} t^{-1} u^{-1} ightarrow n_2 = n_u \\
a_3 &= t y^{-1} ightarrow m_3 = -p + n_t \
b_3 &= t x ightarrow n_3 = p + n_t
\end{align*} \tag{3.11}
\]
we identify the integral relation (1.3). Also, observe that the constraint (3.10) resulted in the balancing condition (1.5). We hence get upon simplification and regrouping of the terms
\[
\sum_{p \in \mathbb{Z}} \int \frac{dx}{2\pi i x} \left(\frac{x}{2\pi i x} \left[\sum_{n \in \mathbb{Z}} B\{tw, n + n_u + n_s, 2n_s\} \alpha_n(w; t) \times B\{st^2 uw, n + n_u + 2n_t + n_s, sw^{-1} x, -n + m + n_s\} \times \frac{1}{\pi i x} \int \frac{dx}{2\pi i x} \right]ight) \times \alpha_p(y, t) \psi_h dx = 0 \tag{3.12}
\]
\[
\sum_{p \in \mathbb{Z}} \int \frac{dx}{2\pi i x} \left(\frac{x}{2\pi i x} \left[\sum_{n \in \mathbb{Z}} B\{tw, n + n_u + n_s, 2n_s\} \alpha_n(w; t) \times B\{st^2 uw, n + n_u + 2n_t + n_s, sw^{-1} x, -n + m + n_s\} \times \frac{1}{\pi i x} \int \frac{dx}{2\pi i x} \right]ight) \times \alpha_p(y, t) \psi_h dx = 0 \tag{3.13}
\]
\[
\sum_{p \in \mathbb{Z}} \int \frac{dx}{2\pi i x} \left(\frac{x}{2\pi i x} \left[\sum_{n \in \mathbb{Z}} B\{tw, n + n_u + n_s, 2n_s\} \alpha_n(w; t) \times B\{st^2 uw, n + n_u + 2n_t + n_s, sw^{-1} x, -n + m + n_s\} \times \frac{1}{\pi i x} \int \frac{dx}{2\pi i x} \right]ight) \times \alpha_p(y, t) \psi_h dx = 0 \tag{3.14}
\]
\[^5\text{For convenience } q \text{ of the } q\text{-product is omitted.}\]
which is the desired operator equality

\[
\sum_{p \in \mathbb{Z}} \int d\mathcal{B}[stw^{-1}, n_s+n_t+n-p, stw^{-1}y, n_s+n_t-n+p, stw^{-1}y, n_s+n_t-n+p] = \sum_{p \in \mathbb{Z}} \int d\mathcal{B}[stw^{-1}, n_s+n_t+n-p, stw^{-1}y, n_s+n_t-n+p, stw^{-1}y, n_s+n_t-n+p].
\]

\[\Box\]

4 Conclusions

In this work, we have constructed a new integral Bailey pair for the pentagon identity in the form of \( q \)-hypergeometric functions. One can use this Bailey construction to obtain new supersymmetric dualities for \( \text{linear quiver theories.} \) Namely, any relation between Bailey pairs \( \alpha^{(n)} \) and \( \beta^{(n)} \) gives integral identities corresponding to the equality of partition functions of certain dual linear quivers, see e.g. [22, 23].

We would like to mention that the pentagon identity presented here can also be written as the star-triangle relation for some integrable lattice model of statistical mechanics. It would be interesting to construct the Bailey pairs corresponding to the star-triangle form of the same integral identity.

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Data availability This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This is a theoretical study and no experimental data has been listed.]

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