General quantum antibrackets

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Abstract

The recently introduced quantum antibracket is further generalized allowing for the defining odd operator Q to be arbitrary. We give exact formulas for higher quantum antibrackets of arbitrary orders and their generalized Jacobi identities. Their applications to BV-quantization and BFV-BRST quantization are then reviewed including some new aspects.

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1 Introduction

In [1] we introduced new quantum objects called quantum antibrackets, which are operator mappings from classical antibrackets exactly like commutators are mappings from Poisson brackets. Classical antibrackets were introduced in [2, 3] and have mainly been used in the Lagrangian BV-quantization of gauge theories [3]. Apart from providing an operator version of the BV-quantization [1], the quantum antibrackets have been used to give a new kind of quantum master equation for generalized quantum Maurer-Cartan equations for arbitrary open groups [4]. Remarkably enough the quantum antibrackets used in these master equations were generalized ones, which neither satisfy the Jacobi identities nor Leibniz’ rule. For such brackets one has to use a hierarchy of higher quantum antibrackets defined in a definite way. In their first form they were introduced already in [1] which is valid for the restricted case when all operators commute. A more general form valid for operators in arbitrary involutions were given in [1]. (In [1] it was also shown that all these results have, at the classical level, a dual version in terms of new types of generalized Poisson brackets.) In [4] the quantum antibrackets were generalized to Sp(2)-brackets defined in analogy to the Sp(2)-antibrackets used in the Sp(2)-version of BV-quantization [7, 8, 9] and most of the above results were then generalized.

Here we review these results with some natural further developments. We start from the most general quantum antibracket as defined in [1] and give then systematically all their main properties. Finally we review the applications considered so far. The quantum antibrackets and their exact properties are given in sections 2 and 3, and the main part of the presented results are new. In section 4 we consider then the restriction to ordinary quantum antibrackets which satisfy all properties as required by the conventional classical antibrackets. In sections 5 and 6 we review the applications to BV-quantization and BFV-BRST quantization respectively including some new aspects. Finally, in section 7 we give some remarks on the corresponding properties and applications of quantum Sp(2)-antibrackets.

2 General quantum antibrackets

The general quantum antibracket is defined by the expression [1]

\[(f, g)_Q \equiv \frac{1}{2} \left( [f, [Q, g]] - [g, [Q, f]](-1)^{(\varepsilon_f+1)(\varepsilon_g+1)} \right), \tag{2.1} \]

where \(f\) and \(g\) are any operators with Grassmann parities \(\varepsilon(f) \equiv \varepsilon_f\) and \(\varepsilon(g) \equiv \varepsilon_g\) respectively. \(Q\) is an odd operator, \(\varepsilon(Q) = 1\). The commutators on the right-hand side is the graded commutator defined by

\[[f, g] \equiv fg - gf(-1)^{\varepsilon_f \varepsilon_g}, \quad \forall f, g. \tag{2.2} \]

The quantum antibracket (2.1) satisfies the properties:

1) Grassmann parity

\[\varepsilon((f, g)_Q) = \varepsilon_f + \varepsilon_g + 1. \tag{2.3} \]
2) Symmetry

\[(f, g)_Q = -(g, f)_Q (-1)^{(\varepsilon_f+1)(\varepsilon_g+1)}. \tag{2.4}\]

3) Linearity

\[(f + g, h)_Q = (f, h)_Q + (g, h)_Q, \quad (\text{for } \varepsilon_f = \varepsilon_g). \tag{2.5}\]

4) If one entry is an odd/even parameter \(\lambda\) we have

\[(f, \lambda)_Q = 0 \quad \text{for any operator } f. \tag{2.6}\]

All these properties agree exactly with the corresponding properties of the classical antibracket \((f, g)\) for functions \(f\) and \(g\). However, the classical antibracket satisfies in addition

5) the Jacobi identities

\[(f, (g, h))(-1)^{(\varepsilon_f+1)(\varepsilon_h+1)} + \text{cycle}(f, g, h) \equiv 0, \tag{2.7}\]

6) and Leibniz’ rule

\[(fg, h) - f(g, h) - (f, h)g(-1)^{\varepsilon_g(\varepsilon_h+1)} = 0. \tag{2.8}\]

These properties are not satisfied by the quantum antibracket \((2.1)\) in its general form. Instead we have

\[
(f, (g, h)Q)_Q(-1)^{(\varepsilon_f+1)(\varepsilon_h+1)} + \text{cycle}(f, g, h) =
\]

\[
= \frac{1}{6}(-1)^{\varepsilon_f+\varepsilon_g+\varepsilon_h}\left\{\left([f, [g, [h, Q^2]]] + \frac{1}{2}[f, [g, [h, Q]]], Q\right) (-1)^{\varepsilon_f\varepsilon_h} +
\right.
\]

\[
+ \left([f, [h, [g, Q^2]]] + \frac{1}{2}[f, [h, [g, Q]]], Q\right) (-1)^{\varepsilon_h(\varepsilon_f+\varepsilon_g)}\} + \text{cycle}(f, g, h), \tag{2.9}\]

and

\[
(fg, h)_Q - f(g, h)_Q - (f, h)_Qg(-1)^{\varepsilon_g(\varepsilon_h+1)} =
\]

\[
= \frac{1}{2} \left([f, h][g, Q](-1)^{\varepsilon_h(\varepsilon_g+1)} + [f, Q][g, h](-1)^{\varepsilon_g}\right). \tag{2.10}\]

In our previous treatments we have only considered nilpotent \(Q\)-operators which is a natural restriction from the point of view of BV-quantization. Here we leave \(Q\) unrestricted in the general treatment. One may then notice that any odd operator \(Q\) satisfies the algebra

\[[Q, Q] = 2Q^2, \quad [Q^2, Q^2] = [Q^2, Q] = 0, \tag{2.11}\]

which directly follows from the definition \((2.2)\). In terms of the quantum antibracket \((2.1)\) \(Q\) satisfies the algebra

\[(Q, Q)_Q = (Q^2, Q^2)_Q = (Q^2, Q)_Q = 0. \tag{2.12}\]

Notice also the relation

\[(f, Q)_Q = \frac{3}{2} [f, Q^2], \quad \forall f. \tag{2.13}\]
A nonzero $Q^2$ complicates some formulas. For instance, instead of (20) in [1] we have here

\[
[Q, (f, g) Q] = ([Q, f], [g, Q]) Q + (f, [Q, g]) Q (-1)^{\varepsilon f + 1} - (-1)^{\varepsilon f} \left( [f, [g, Q^2]] + [g, [f, Q^2]] (-1)^{\varepsilon f g} \right) = [(Q, f), [Q, g]] - \frac{1}{2} (-1)^{\varepsilon f} \left( [f, [g, Q^2]] + [g, [f, Q^2]] (-1)^{\varepsilon f g} \right) + 2 \left( -1 \right)^{\varepsilon f} \sqrt{\varepsilon f + 1} - 2 \left( -1 \right)^{\varepsilon f} \varepsilon f \varepsilon g.
\]\n
(2.14)

In our derivation of the generalized Jacobi identities (2.9) we have used the relation

\[
(f, (g, h) Q) Q = [[Q, f], (g, h) Q] + \frac{1}{2} [Q, [f, (g, h) Q]] (-1)^{\varepsilon f},
\]

(2.15)

which directly follows from the definition (2.1) (further details are given below).

3 Hierarchy of general higher order antibrackets and generalized Jacobi identities.

Although the quantum antibracket (2.1) in its general form does not satisfy the Jacobi identities and Leibniz' rule, we may provide for a systematic description of its algebraic properties in terms of higher order quantum antibrackets.

In [1, 4] it was shown that higher order quantum antibrackets in terms of operators in involution may be defined in terms of a generating operator and that the generating operator determine the Jacobi identities. This construction may in fact be generalized to such an extent that it allows us to define general higher order quantum antibrackets in terms of arbitrary operators. For the general case we need then the following two generating operators (In refs. [1, 4] $Q$ was considered to be nilpotent, $Q^2 = 0$.)

\[
Q(\lambda) \equiv e^{-A} Q e^A, \quad Q^2(\lambda) \equiv e^{-A} Q^2 e^A,
\]

(3.1)

where $A$ is an even operator defined by

\[
A = f_a \lambda^a,
\]

(3.2)

where $f_a$, $a = 1, 2, \ldots$, are arbitrary operators with Grassmann parities $\varepsilon_a \equiv \varepsilon(f_a)$ and where $\lambda^a$ are parameters with Grassmann parities $\varepsilon(\lambda^a) = \varepsilon_a$. (In ref. [1] $\{f_a\}$ was a set of commuting operators.) We have the equalities

\[
Q(\lambda) = \sum_{n=0}^{\infty} Q_n, \quad Q^2(\lambda) = \sum_{n=0}^{\infty} Q^2_n,
\]

(3.3)

where

\[
Q_0 \equiv Q, \quad Q_n \equiv \frac{1}{n!} \cdots [Q, A], \ldots, A]_n \text{ for } n \geq 1,
\]

\[
Q^2_0 \equiv Q^2, \quad Q^2_n \equiv \frac{1}{n!} \cdots [Q^2, A], \ldots, A]_n \text{ for } n \geq 1,
\]

(3.4)

where the last index $n$ indicates that the expression involves $n$ commutators. We define the general higher order quantum antibrackets by (cf. [1])

\[
(f_a_1, \ldots, f_a_n) Q \equiv - Q(\lambda) \tilde{\partial}_{a_1} \tilde{\partial}_{a_2} \cdots \tilde{\partial}_{a_n} (-1)^{E_n} \bigg|_{\lambda = 0} =
\]

3
\[
\begin{align*}
\text{operators in terms of symmetrized multiple commutators with an arbitrary odd } & Q: \\
\text{To the lowest orders we get explicitly:} & \\
(f_{a_1}, \ldots, f_{a_n})_Q = -\frac{1}{n!}(-1)^{E_n} \sum_{\text{sym}} \ldots [[Q, f_{a_1}], \ldots, f_{a_n}],
\end{align*}
\]

where the sum is over all possible orders of \( a_1, \ldots, a_n \) (\( n! \) terms) with appropriate sign factors obtained from the corresponding reordering of the monomial \( \lambda^{a_1} \cdots \lambda^{a_n} \) (this follows from the last equality in (3.5)). The higher antibrackets satisfy the properties

\[
(\ldots, f_a, f_b, \ldots)_Q = (-1)^{(\varepsilon_{a+1})(\varepsilon_{b+1})}(\ldots, f_b, f_a, \ldots)_Q,
\]

where the expression for \( n = 2 \) is exactly the antibracket (2.1). For nilpotent \( Q \) and commuting operators \( f_{a_k} \) (3.6) reduces to the higher quantum antibrackets in [1] (eq. (33)). If furthermore \( f_{a_k} \) are functions of some coordinates and \( Q \) a nilpotent differential operator then (3.6) on unity yields the classical higher antibrackets considered in [10].

The higher order quantum antibrackets may also be expressed recursively in terms of the next lower ones. The precise relations may be obtained from the recursion relation

\[
Q_n = \frac{1}{n}[Q_{n-1}, A].
\]

This inserted into (3.5) yields

\[
(f_{a_1}, \ldots, f_{a_n})_Q = \frac{1}{n} \sum_{k=1}^n [(f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_{k+1}}, \ldots, f_{a_n})_Q, f_{a_k}][-1]^{B_{k,n}} = \\
= -\frac{1}{n} \sum_{k=1}^n [f_{a_k}, (f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_{k+1}}, \ldots, f_{a_n})_Q][-1]^{C_{k,n}},
\]

where

\[
\begin{align*}
B_{k,n} & \equiv \varepsilon_{a_k}(\varepsilon_{a_{k+1}} + \ldots + \varepsilon_{a_n}) + \sum_{s=2\lfloor \frac{k}{2} \rfloor + 1}^{n} \varepsilon_{a_s}, \\
C_{k,n} & \equiv \varepsilon_{a_k}(\varepsilon_{a_1} + \ldots + \varepsilon_{a_{k-1}}) + \varepsilon_{a_k} + \sum_{s=2\lfloor \frac{k}{2} \rfloor + 1}^{n} \varepsilon_{a_s},
\end{align*}
\]

(3.11)
To the lowest order we have explicitly (this expression was also given in [4])

\[
(f_a, f_b, f_c)Q = \frac{1}{3}(-1)^{(ε_a+1)(ε_c+1)} \left( ([f_a, f_b]Q, f_c)(-1)^{ε_c+(ε_a+1)(ε_c+1)} + \text{cycle}(a, b, c) \right) = \frac{1}{3}(-1)^{(ε_a+1)(ε_c+1)} \left( [f_a, (f_b, f_c)Q](-1)^{ε_b+ε_a(ε_c+1)} + \text{cycle}(a, b, c) \right). \quad (3.12)
\]

In order to derive generalized Jacobi identities we notice first that the definition (2.1)
of the antibracket yields the relation (cf (2.15))

\[
(f_k, (f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_k+1}, \ldots, f_{a_n})Q)Q = [[f_k, Q], (f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_k+1}, \ldots, f_{a_n})Q] + \frac{1}{2} [Q, [f_k, (f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_k+1}, \ldots, f_{a_n})Q](-1)^{ε_a_k}. \quad (3.13)
\]

This combined with the recursion relation (3.10) implies then

\[
\sum_{k=1}^{n} (f_{a_k}, (f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_{k+1}}, \ldots, f_{a_n})Q)Q (-1)^{D_{k,n}} = \sum_{k=1}^{n} [[f_k, Q], (f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_{k+1}}, \ldots, f_{a_n})Q](-1)^{D_{k,n}} - \frac{n}{2} [Q, (f_{a_1}, \ldots, f_{a_n})Q],
\]

\[
D_{k,n} = ε_{a_k} + C_{k,n} = ε_{a_k} (ε_{a_1} + \ldots + ε_{a_{k-1}}) + \sum_{s=2^{k+1}}^{n} ε_{a_s}. \quad (3.14)
\]

The first sum on the right-hand side may be expressed in terms of higher antibrackets by means of the identities (cf [1])

\[
\left( [Q(λ), Q(λ)] - 2Q^2(λ) \right) \bigg|_{λ=0} = 0, \quad (3.15)
\]

which are equivalent to

\[
\left( \sum_{k=0}^{n} [Q_k, Q_{n-k}] - 2Q^2_n \right) \bigg|_{λ=0} = 0, \quad (3.16)
\]

where \(Q^2_n\) is defined in (3.14). For \(n = 0, 1, 2, 3\) we have explicitly

\[
\begin{align*}
n = 0 : & \quad [Q, Q] - 2Q^2 = 0, \\
n = 1 : & \quad [Q, [Q, f_a]] - [Q^2, f_a] = 0, \\
n = 2 : & \quad [Q, (f_a, f_b)Q] - [[Q, f_a], [Q, f_b]] + \frac{1}{2}[[Q^2, f_a], f_b](-1)^{ε_a} + \frac{1}{2}[[Q^2, f_b], f_a](-1)^{ε_b(ε_a+1)} = 0, \\
n = 3 : & \quad [[Q, (f_a, f_b, f_c)Q](-1)^{(ε_a+1)(ε_c+1)} + ([f_a, Q], (f_b, f_c)Q](-1)^{ε_b+ε_a(ε_c+1)} + cycle(a, b, c)) - \frac{1}{6} \sum_{\text{sym}} [[[Q^2, f_a], f_b], f_c](-1)^{ε_aε_c} = 0. \quad (3.17)
\end{align*}
\]
The first is just a trivial identity, the second is a Jacobi identity, and the third is exactly (2.14). For \( n \geq 3 \) (3.16) multiplied by \((-1)^E_n\) becomes

\[
-[Q, (f_a, \ldots, f_{a_n})] + \sum_{k=1}^{n} [[f_{a_k}, Q], (f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_{k+1}}, \ldots, f_{a_n})](-1)^{D_{k,n}} +
\]

\[
+ R_n - (-1)^{E_n} \frac{1}{n!} \sum_{\text{sym}} \cdots [[Q^2, f_{a_1}], f_{a_2}], \ldots, f_{a_n}] = 0, \quad (3.18)
\]

where

\[
R_n \equiv \frac{1}{2} \sum_{k=2}^{n-2} [Q_k, Q_{n-k}] \partial_{a_1} \partial_{a_2} \cdots \partial_{a_n} (-1)^E_n =
\]

\[
= \frac{1}{2} \sum_{k=2}^{n-2} \sum_{\text{sym}} ([f_{a_1}, \ldots, f_{a_k}), (f_{a_{k+1}}, \ldots, f_{a_n}])Q](-1)^{F_{k,n}}, \quad F_{k,n} = \sum_{r=1}^{(n,k)} \varepsilon_{a_r}, \quad (3.19)
\]

where \((n,k) \equiv n\) for \( k \) odd, and \((n,k) \equiv k\) for \( k \) even. The symmetrized sum is over all different orders with additional sign factors \((-1)^{E_n+E_n+A_n}\) where \(E_n\) is \(E_n\) for the new order and \(A_n\) from the reordering of the monomial \(\lambda^{a_1} \cdots \lambda^{a_n}\).

The expression (3.18) inserted into (3.14) leads then to the generalized Jacobi identities

\[
\sum_{k=1}^{n} (f_{a_k}, (f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_{k+1}}, \ldots, f_{a_n})Q](-1)^{D_{k,n}} = -\frac{n-2}{2} [Q, (f_{a_1}, \ldots, f_{a_n})] -
\]

\[
- R_n + (-1)^{E_n} \frac{1}{n!} \sum_{\text{sym}} \cdots [[Q^2, f_{a_1}], f_{a_2}], \ldots, f_{a_n}] = 0 \quad (3.20)
\]

where \(R_n\) is the sum of commutators of antibrackets of orders between 2 and \(n-2\) given by (3.19). For \( n = 3 \) we have in particular \((R_3 = 0)\)

\[
(f_a, (f_b, f_c)Q](-1)^{(\varepsilon_a+1)(\varepsilon_c+1)} + \text{cycle}(a, b, c) = -\frac{1}{2}[[f_a, f_b, f_c)Q](-1)^{(\varepsilon_a+1)(\varepsilon_c+1)}, Q] -
\]

\[
-\frac{1}{6}(-1)^{\varepsilon_a+\varepsilon_b+\varepsilon_c} \left( [[[Q^2, f_a], f_b], f_c](-1)^{\varepsilon_a\varepsilon_c} + [[[Q^2, f_c], f_b], f_a](-1)^{\varepsilon_b\varepsilon_c} +
\]

\[
+ \text{cycle}(a, b, c) \right), \quad (3.21)
\]

which may be rewritten as (2.3).

## 4 Ordinary quantum antibrackets

It is natural to impose a restriction such that the quantum antibrackets satisfy the Jacobi identities (2.7) and Leibniz’ rule (2.8). Such antibrackets we call ordinary quantum antibrackets. This requires of course that all higher order quantum antibrackets vanish (from \( n = 3 \)). The precise conditions are best extracted from the explicit expressions (2.9) and (2.10). From (2.10) we find that the quantum antibracket (2.1) satisfies Leibniz’ rule (2.8), i.e.

\[
(fg, h)Q - f(g, h)Q - (f, h)Qg(-1)^{\varepsilon_f(\varepsilon_h+1)} = 0, \quad (4.1)
\]
for two classes of operators: the class of commuting operators or the class of operators commuting with $Q$. Since the antibracket is zero in the latter case we consider only maximal sets of commuting operators denoted $\mathcal{M}$ from now on.

The Jacobi identities (2.7) are satisfied provided the operator $Q$ satisfies the condition

$$[f, [g, [h, Q^2]]] + \frac{1}{2}[[f, [g, h]], Q] = 0, \quad \forall f, g, h \in \mathcal{M}.$$  \hfill (4.2)

(This condition is trivially satisfied for the class of operators commuting with $Q$.) First we notice that for the class of commuting operators, $\mathcal{M}$, the quantum antibracket (2.1) reduces to

$$(f, g)_Q = [f, [Q, g]] = [[f, Q], g], \quad \forall f, g \in \mathcal{M}.$$  \hfill (4.3)

In order to use this expression repeatedly like in the Jacobi identities, we must require

$$(f, g)_Q \in \mathcal{M}, \quad \forall f, g \in \mathcal{M},$$  \hfill (4.4)

which is equivalent to

$$[f, [g, [h, Q]]] = 0, \quad \forall f, g, h \in \mathcal{M}.$$  \hfill (4.5)

However, this together with (4.2) requires then

$$[f, [g, [h, Q^2]]] = 0, \quad \forall f, g, h \in \mathcal{M}.$$  \hfill (4.6)

Thus, ordinary nontrivial quantum antibrackets are defined for a class of commuting operators $\mathcal{M}$. It is naturally defined by (4.3) where $Q$ must satisfy the conditions (4.5) and (4.6).

In order to give explicit solutions we consider like in [1, 5] a supersymmetric manifold of dimension $(2n, 2n)$ spanned by the canonical coordinates $\{x^a, x^*_a, p_a, p^*_a\}, \ a = 1, \ldots, n$, with Grassmann parities $\varepsilon_a \equiv \varepsilon(x^a) = \varepsilon(p_a)$ and $\varepsilon(x^*_a) = \varepsilon(p^*_a) = \varepsilon_a + 1$. Their canonical commutation relations have the nonzero part

$$[x^a, p_b] = i\hbar \delta^a_b, \quad [x^*_a, p^*_b] = i\hbar \delta^b_a.$$  \hfill (4.7)

Let now the class of commuting operators $\mathcal{M}$ be all functions of $x^a$ and $x^*_a$. The following $Q$-operator satisfies then the conditions (1.3) and (4.6):

$$Q = p_a p^*_a (-1)^{\varepsilon_a}.$$  \hfill (4.8)

In terms of this $Q$ the quantum antibracket (2.1) becomes

$$(ih)^{-2}(f, g)_Q = f \overset{\leftarrow}{\partial}_a \partial^*_a g - g \overset{\leftarrow}{\partial}_a \partial^*_a f (-1)^{(\varepsilon_f + 1)(\varepsilon_g + 1)},$$  \hfill (4.9)

which exactly agrees with the standard classical antibracket in which $x^a$ and $x^*_a$ are fields and antifields [3]. The $Q$-operator (4.8) is nilpotent ($Q^2 = 0$). However, the $Q$-operator is not uniquely determined by the quantum antibracket. In fact, the quantum antibracket (1.3) is also obtained from the $Q$-operator

$$Q = p_a p^*_a (-1)^{\varepsilon_a} + ih f^a(x, x^*) p_a(-1)^{\varepsilon_a} + ih f^*_a(x, x^*) p^*_a(-1)^{\varepsilon_a + 1} - (ih)^2 g(x, x^*)$$  \hfill (4.10)
for any functions $f^a, f^a_*$ and $g$ of the commuting operators $x^a$ and $x^a_*$. (The normalization chosen in (4.10) is appropriate for applications to BV-quantization.) The $Q$-operator (4.10) is in general not nilpotent.

In terms of general coordinates $X^A$, $A = 1, \ldots, 2n$, the solution of the condition (4.5) is

$$Q = -\frac{1}{2} E^{AB}(X) P_B P_A (-1)^{\varepsilon_B} + i\hbar F^A(X) P_A (-1)^{\varepsilon_A} - (i\hbar)^2 G(X),$$

(4.11)

where $P_A (\varepsilon(P_A) = \varepsilon(X^A) \equiv \varepsilon_A)$ is the conjugate momentum operator to $X^A ([X^A, P_B] = i\hbar \delta^A_B)$. In order to also satisfy condition (4.6), $Q^2$ must be at most quadratic in the conjugate momenta. This requires

$$E^{AD} \partial_D E^{BC} (-1)^{(\varepsilon_A + 1)(\varepsilon_C + 1)} + \text{cycle}(A, B, C) = 0.$$

(4.12)

The $Q$-operator (4.11) inserted into (2.1) for the class of operators which are functions of $X^A$ yields the quantum antibracket

$$(i\hbar)^{-2} (f, g)_Q = f \left. \frac{\partial}{\partial A} E^{AB} \partial_B g \right|, \quad \forall f, g \in \mathcal{M},$$

(4.13)

where $\partial_A \equiv \partial/\partial X^A$. One may easily check that the Jacobi identities of (4.13) requires (4.12) (cf [13]). The operator (4.11) is like (4.10) in general not nilpotent. (In fact, (4.10) is (4.11) in Darboux coordinates $x^a$ and $x^a_*$.) Notice that if $Q$ has terms involving third or higher powers of $P_A$ then even the general condition (4.2) does not allow for any solutions since the first term in (4.2) involves first order derivatives of $f, g, h$ while the second term involves second derivatives.

## 5 Operator version of BV-quantization

The Lagrangian BV-quantization of general gauge theories is formulated within the path integral formulation. The classical antibrackets play a crucial role in this formulation. Here we show that the quantum antibrackets provide for the appropriate algebraic tools in a corresponding operator formulation. (These results were given in [1, 6, 13].) In this construction it is the ordinary quantum antibrackets that provide for the relevant framework. We have therefore to restrict ourselves to a maximal commuting set of operators. This is just a polarization into fields and their momenta. The field operators are then the maximal set to be chosen. The crucial ingredient in the BV-quantization is the quantum master equation

$$\Delta e^{\frac{i\hbar}{\hbar} W} = 0,$$

(5.1)

where $\Delta$ is an odd, second order differential operator. $W$ is the master action. Within the operator formulation (5.1) is replaced by

$$Q |W\rangle = 0,$$

(5.2)

where $Q$ is the odd operator that enters the definition of the quantum antibracket. The BV-formulation requires $Q$ to be

1. maximally quadratic in the momenta
2. nilpotent \( (Q^2 = 0) \)

3. hermitian

4. reduce to the differential operator \( \Delta \) in the Schrödinger picture which should have no \( \hbar \)-dependence.

The first and last condition requires \( Q \) to be of the form \( (4.10) \) or \( (4.11) \) with \( \hbar \)-independent functions. Starting from the general \( Q \)-operator \( (4.11) \) we find then the general hermitian, nilpotent solution

\[
Q \equiv -\frac{1}{2}\rho^{-1/2}P_A\rho E^{AB}P_B\rho^{-1/2}(-1)^{\varepsilon_B} - (ih)^2G(X),
\]

(5.3)

where \( \rho(X) \) is the volume density operator. \( Q \) is hermitian if \( \rho, X^A, G \), and \( E^{AB} \) are hermitian, and \( P^A = (-1)^{\varepsilon_A}P_A \). The \( \hbar \)-dependence in \( Q \) is chosen such that

\[
\langle X|Q|W \rangle = -(ih)^2\Delta\langle X|W \rangle,
\]

(5.4)

where

\[
\Delta = \Delta_0 + G(X), \quad \Delta_0 \equiv \frac{1}{2}\rho^{-1}\partial_A \rho E^{AB}\partial_B(-1)^{\varepsilon_A},
\]

(5.5)

where in turn \( X^A \) now are classical fields. Notice that \( P_A = -i\hbar\rho^{-1/2}\partial_A \rho^{1/2}(-1)^{\varepsilon_A} \) since the eigenstates \( |X \rangle \) are normalized according to

\[
\int |X\rangle \rho(X)dX\langle X| = 1.
\]

(5.6)

The most general \( \Delta \)-operator considered so far is \( \Delta_0 \) \([1]\). We do not know to what extent a nonzero \( G(X) \) may be used. Notice that the nilpotence of \( \Delta \) requires \( \Delta_0 \) to be nilpotent and \( G \) to satisfy

\[
\Delta_0G = 0, \quad \Delta_0\frac{1}{2}(-1)^{\varepsilon_A}\rho^{-1}\left(\partial_A \rho E^{AB}\partial_B - G\partial_A E^{AB}\right) = 0.
\]

(5.7)

The equivalence between (5.1) and (5.2) follows if we choose

\[
\langle X|W \rangle = \exp\left\{\frac{i}{\hbar}W(X)\right\} \Leftrightarrow |W\rangle = \exp\left\{-\frac{i}{\hbar}W(X)\right\}\rho^{1/2}|0\rangle_P.
\]

(5.8)

Notice the normalization \( \langle X|\rho^{1/2}|0\rangle_P = 1 \).

The partition function \( Z \), i.e. the path integral of the gauge fixed action, is given by \( Z = \langle X|W \rangle \), where \( |W \rangle \) is the master state and \( |X \rangle \) a gauge fixing state both satisfying the quantum master equation (5.2). \( |W \rangle \) and \( |X \rangle \) have the general form

\[
|W \rangle = \exp\left\{\frac{i}{\hbar}W(X)\right\}\rho^{1/2}|0\rangle_P, \quad |X \rangle = \exp\left\{-\frac{i}{\hbar}X^\dagger(X, \lambda)\right\}\rho^{1/2}|0\rangle_P, \quad (5.9)
\]

where \( \lambda^\alpha \) are Lagrange multipliers and \( \pi_\alpha \) their conjugate momenta. The vacuum state \( |0\rangle_P \otimes |0\rangle_\pi \) satisfies \( P_A|0\rangle_P = \pi_\alpha|0\rangle_P = 0 \). By means of the extended eigenstates \( |X, \lambda \rangle \) satisfying the completeness relations

\[
\int |X, \lambda\rangle \rho(X)dX d\lambda \langle X, \lambda| = 1,
\]

(5.10)
and \((X, \lambda | \rho^{1/2} | 0)_{P, \lambda} = 1\), the partition function \(Z = \langle \mathcal{X} | \mathcal{W} \rangle\) becomes explicitly

\[
Z = \langle \mathcal{X} | \mathcal{W} \rangle = \int \rho(X) dX d\lambda \exp \left\{ \frac{i}{\hbar} \left[ \mathcal{W}(X) + \mathcal{X}(X, \lambda) \right] \right\}
\]  

(5.11)

where \(\mathcal{W}\) and \(\mathcal{X}\) in the path integral denote the master action and gauge fixing actions, \((X, \lambda | \mathcal{W}(X) \rho^{1/2} | 0)_{P, \lambda}\) and \((X, \lambda | \mathcal{X}(X, \lambda) \rho^{1/2} | 0)_{P, \lambda}\) respectively, which both satisfy the quantum master equation (5.1). This agrees with the results in [11, 9]. The path integral (5.11) is invariant under the anticanonical transformation

\[
\delta X^A = (X^A, -\mathcal{W} + \mathcal{X}) \mu,
\]  

(5.12)

where \(\mu\) is an odd constant. It is also invariant under changes of the gauge fixing function \(\mathcal{X}\) in accordance with the general invariance of the quantum master equation. Infinitesimally we have invariance under

\[
\delta \mathcal{X} = (\mathcal{X}, f) - i\hbar \Delta f,
\]  

(5.13)

where \(f\) is an odd function.

In order to illustrate the above generalized BV-quantization we give a detailed explicit treatment in terms of the Darboux coordinates \(x^a\) and \(x^*_a\) [1]. In the BV-quantization they are viewed as fields and antifields. The nilpotent and hermitian \(Q\)-operator in conventional BV-quantization is given by (4.8). Notice the relation

\[
\langle x, x^* | Q | S \rangle = -(i\hbar)^2 \Delta \langle x, x^* | S \rangle,
\]  

(5.14)

where

\[
\Delta = (-1)^{\varepsilon_a} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^*_a}
\]  

(5.15)

is the well-known nilpotent operator in the BV-quantization. The quantum master equation is

\[
Q | S \rangle = 0 \iff \Delta \exp \left\{ \frac{i}{\hbar} S(x, x^*) \right\} = 0,
\]  

(5.16)

if

\[
| S \rangle \equiv \exp \left\{ \frac{i}{\hbar} S(x, x^*) \right\} |0\rangle_{pp^*}.
\]  

(5.17)

The path integral of the gauge fixed action, \(Z\), is given by \(Z = \langle \Psi | S \rangle\), where \(| S \rangle\) is the master state (5.17) and \(|\Psi\rangle\) a gauge fixing state also satisfying the quantum master equation (5.16), i.e. \(Q \langle\Psi\rangle = 0\). In the standard case we have explicitly

\[
|\Psi\rangle = \exp \left\{ \hbar^{-2} [Q, \Psi(x)] \right\} |0\rangle_{px^*},
\]  

(5.18)

where the operators have the hermiticity properties

\[
(x^a)^\dagger = x^a, \quad (x^*_a)^\dagger = -x^*_a, \quad (p_a)^\dagger = p_a(-1)^{\varepsilon_a}, \quad (p^*_a)^\dagger = p^*_a(-1)^{\varepsilon_a}, \quad \Psi^\dagger = \Psi.
\]  

(5.19)
and where the vacuum states satisfy
\[ p_a |0\rangle_{pp^*} = p^*_a |0\rangle_{pp^*} = 0, \quad p_a |0\rangle_{px^*} = x^*_a |0\rangle_{px^*} = 0, \quad Q |0\rangle_{pp^*} = Q |0\rangle_{px^*} = 0. \]
\[ (5.20) \]

Note that \( S(x, x^*) \) and \( \Psi(x) \) belong to the class of commuting operators. Note also that \( |\Psi\rangle \) satisfies
\[ (x^*_a - \partial_a \Psi(x)) |\Psi\rangle = 0, \quad \left( p_a + p^*_b \partial_b \Psi(x) \right) |\Psi\rangle = 0, \]
\[ (5.21) \]
which fixes \( p_a \) and \( x^*_a \). The explicit form of the gauge fixed partition function is then
\[ Z = \langle \Psi | S \rangle = p_{x^*} |0\rangle \exp \{ \hbar^{-2} [Q, \Psi(x)] \} \exp \{ i\hbar^{-1} S(x, x^*) \} |0\rangle_{pp^*} = \int DxDx^* \exp \{ i\hbar^{-1} S(x, x^*) \} \delta(x^*_a - \partial_a \Psi(x)), \]
\[ (5.22) \]
where the last equality is obtained by inserting the completeness relations
\[ \int |x, x^*\rangle DxDx^* \langle x, x^*| = \int |x, p^*_x\rangle DxDp^* \langle x, p^*_x| = 1, \]
\[ (5.23) \]
and the properties
\[ p_{x^*} \langle x^*_a | x^*\rangle_{pp^*} = \langle x^*_a | x^*\rangle_{pp^*} = 1, \quad \langle p^*_x | x^*_a \rangle = (2\pi\hbar)^{-n_-/2} |^{-n_+} \exp \left\{ -i\hbar^{-1} p^*_a x^*_a \right\}, \]
\[ (5.24) \]
where \( n_+ (n_-) \) is the number of bosons (fermions) among the \( x^a \) operators. Eq.\( (5.22) \) agrees with the standard BV quantization \[3\]. The independence of the gauge fixing operator \( \Psi \) follows from
\[ \delta Z = \langle \Psi | \int_0^1 d\alpha \exp \{ -\alpha \hbar^{-2} [Q, \Psi] \hbar^{-2} [Q, \delta \Psi] \exp \{ \alpha \hbar^{-2} [Q, \Psi] \} |S\rangle = 0, \]
\[ (5.25) \]
since \( |S\rangle \) and \( |\Psi\rangle \) satisfy the master equation \( (5.16) \).

6 Quantum antibrackets within general BFV-BRST quantization

Within the Hamiltonian formulation of general gauge theories there are first class constraints \( \theta_a = 0 \) where \( \theta_a \) by definition are variables in arbitrary involution with respect to the Poisson bracket on the considered symplectic manifold, \textit{i.e.}
\[ \{ \theta_a, \theta_b \} = U_{ab}^c \theta_c, \]
\[ (6.1) \]
where the structure coefficients \( U_{ab}^c \) may be arbitrary functions. In \[13\] it was shown that the algebra \( \{ \theta_a, \theta_b \} \) on a ghost extended manifold always may be embedded in one single real, odd function \( \Omega \), the BFV-BRST charge, in such a way that \( \{ \Omega, \Omega \} = 0 \) in terms of the extended Poisson bracket. The corresponding quantum theory is consistent if the corresponding odd, hermitian operator \( \Omega \) is nilpotent, \textit{i.e.} \( \Omega^2 = 0 \). For a finite number of

\footnote{Note that for odd \( n \) the states in \( (5.23), (5.24) \) do not have a definite Grassmann parity \[12\].}
degrees of freedom such a solution always exists and is of the form \[ \Omega = \sum_{i=0}^{N} \Omega_i, \] (6.2)

\[ \Omega_0 \equiv C^a \theta_a, \quad \Omega_i \equiv \Omega_{a_1 \ldots a_{i+1}}^b (\mathcal{P}_{b_1} \cdots \mathcal{P}_{b_i} C^{a_{i+1}} \cdots C^{a_1})_{\text{Weyl}}, \quad i = 1, \ldots, N, \] (6.3)

where we have introduced the ghost operators \( C^a \) and their conjugate momenta \( \mathcal{P}_a \) satisfying

\[ [C^a, \mathcal{P}_b] = i \hbar \delta^a_b, \quad (C^a)^\dagger = C^a, \quad \mathcal{P}_a^\dagger = -(-1)^{\varepsilon_a} \mathcal{P}_a. \] (6.4)

The Grassmann parities are

\[ \varepsilon_a \equiv \varepsilon(\theta_a), \quad \varepsilon(C^a) = \varepsilon(\mathcal{P}_a) = \varepsilon_a + 1. \] (6.5)

The operators \( \Omega_{a_1 \ldots a_{i+1}}^b \) in (6.3), whose explicit form we do not give here, contain the original operators and are such that \( \Omega \) is hermitian and nilpotent. (In (6.3) the ghost operators are Weyl ordered which means that \( \Omega_i \) are all hermitian.) If the ghosts \( C^a \) are assigned ghost number one and \( \mathcal{P}_a \) ghost number minus one, \( \Omega \) in (6.2) has ghost number one. Notice the relation

\[ [G, \Omega] = i \hbar \Omega, \quad G \equiv -\frac{1}{2} (\mathcal{P}_a C^a - C^a \mathcal{P}_a (-1)^{\varepsilon_a}), \] (6.6)

where \( G \) is the ghost charge.

The BRST charge \((6.2)+(6.3)\) is the BRST charge in the minimal sector. In order to construct gauge fixed theories in the most general way we need to extend the manifold further with Lagrange multipliers and antighosts. The complete BRST charge \( Q \), which also is hermitian and nilpotent, contains \( \Omega \) and additional terms involving the Lagrange multipliers and antighosts \( [14] \). The gauge fixing is then performed by means of an odd operator \( \Psi \) such that the effective gauge fixed Hamiltonian contains the operator \([Q, \Psi]\). \( \Psi \) seems always possible to be chosen to be nilpotent in such a way that it may be identified with a coBRST charge \( [16] \).

### 6.1 Equations of motion in terms of antibrackets

The equations of motion in a gauge fixed theory is determined by an effective Hamiltonian operator of the form \( H = \mathcal{H} + (i \hbar)^{-1}[Q, \Psi] \) where \( \mathcal{H} \) commutes with \( Q \). Now any gauge theory may be expressed in terms of an equivalent reparametrization invariant theory in which the Hamiltonian is of the form \( H = (i \hbar)^{-1}[Q, \Psi] \) where the new \( Q \) contains a term with a new ghost multiplying the constraint variable \( P_0 + \mathcal{H} \), where \( P_0 \) is the conjugate momentum to time, and where the new gauge fixing fermion \( \Psi \) has a corresponding new gauge fixing term. The effective equations of motion becomes then

\[ \dot{A} = (i \hbar)^{-1}[A, H] = (i \hbar)^{-2}[A, [Q, \Psi]] = (i \hbar)^{-2}(A, [Q, \Psi]) - (i \hbar)^{-2} \frac{1}{2} [Q, [\Psi, A]], \] (6.7)
where the quantum antibracket is the general one defined by (2.1). The last equality implies
\[
\langle \phi \mid (i\hbar)^2 \dot{A} - (A, \Psi)_Q \mid \phi' \rangle = 0,
\]
where \( |\phi\rangle, |\phi'\rangle \) are physical states annihilated by \( Q \). The equation of motion (6.7) may also be expressed entirely in terms of antibrackets if we make use of the symmetric combination of both \( Q \)- and \( \Psi \)-antibrackets. We have the relation
\[
(i\hbar)^2 \ddot{A} = [A, [Q, \Psi]] = \frac{2}{3} (A, \Psi) + (A, Q)\phi.
\]

6.2 Quantum master equation for generalized Maurer-Cartan equations.

In [4] we proposed a quantum master equation for generalized Maurer-Cartan equations of operators in arbitrary involutions. Such operators are encoded in the hermitian, nilpotent BFV-BRST charge operator, which in the minimal sector is given by \( \Omega \) in (6.2)-(6.3). Finite group transformations on operators are determined by the Lie equations
\[
A(\phi) \nabla_a = A(\phi) \frac{\partial}{\partial a} - (i\hbar)^{-1} [A(\phi), Y_a(\phi)] = 0,
\]
where \( \partial_a \) is a derivative with respect to the parameter \( \phi^a \), \( \varepsilon(\phi^a) = \varepsilon_a \). (\( \phi^a \) may also be viewed as a new set of commuting operators.) On states the corresponding Lie equations are
\[
\langle A(\phi) \mid \frac{\partial}{\partial a} - (i\hbar)^{-1} Y_a(\phi) \rangle = 0.
\]
The operator \( Y_a \) in both these relations depends on \( \phi^a \) and must satisfy the integrability conditions
\[
Y_a \frac{\partial}{\partial b} Y_b (\phi) - Y_b \frac{\partial}{\partial a} (\phi) (-1)^{\varepsilon_a \varepsilon_b} = (i\hbar)^{-1} [Y_a, Y_b],
\]
which in turn are integrable without further conditions. In order for the Lie equations (6.10) to be connected to the integration of the quantum involution encoded in \( \Omega \), \( Y_a(\phi) \) has to be of the form
\[
Y_a(\phi) = \lambda^b_a(\phi) \theta_b (1-1)^{\varepsilon_a \varepsilon_b} + \{ \text{possible ghost dependent terms} \}, \quad \lambda^b_a(0) = \delta^b_a.
\]
where \( \lambda^b_a(\phi) \) are operators in general. One may note that in a ghost independent scheme the generators of the finite transformations are \( \theta_a \). However, within our BRST framework, \([\Omega, P_a] = \theta_a + \{ \text{ghost dependent terms} \} \) are the appropriate generators which motivates the form (6.13). Here we take one step further and define \( Y_a \) to be of the general form
\[
Y_a(\phi) = (i\hbar)^{-1} [\Omega, \Omega_a(\phi)], \quad \varepsilon(\Omega_a) = \varepsilon_a + 1,
\]
where \( Y_a \) has ghost number zero and \( \Omega_a \) ghost number minus one. This form implies that \([Y_a(\phi), \Omega] = 0 \), so that if \([A(0), \Omega] = 0 \) then \([A(\phi), \Omega] = 0 \), i.e. a BRST invariant operator remains BRST invariant when transformed according to (6.10). Equations (6.13) and (6.14) together with (6.2) and (6.3) imply that \( \Omega_a \) in (6.14) must be of the form
\[
\Omega_a(\phi) = \lambda^b_a(\phi) P_b + \{ \text{possible ghost dependent terms} \}, \quad \lambda^b_a(0) = \delta^b_a.
\]
One may note that the Lie equations (6.10) also may be defined in terms antibrackets. From (6.7) we have

\[ A(\phi) \gamma_a - (ih)^{-2}(A, \Omega_a) = -ih^{-2} \frac{1}{2}[[A, \Omega_a], \Omega] = 0, \]  \hspace{1cm} (6.16)

where we have introduced the general quantum antibracket defined in accordance with (2.1). The integrability condition (6.12) for \( Y_a \) leads by means of (6.14) to the following equivalent equation for \( \Omega_a \)

\[ \Omega_a \gamma_b - \Omega_b \gamma_a (-1)^{\varepsilon_a \varepsilon_b} = (ih)^{-2}(\Omega_a, \Omega_b) - \frac{1}{2}(ih)^{-1}[\Omega_{ab}, \Omega], \]  \hspace{1cm} (6.17)

which also involves the \( \Omega \)-antibracket in (6.16). Due to the form (6.14) of \( \Omega_a \), eq. (6.17) are generalized Maurer-Cartan equations for \( \chi_a^b(\phi) \). The integrability conditions of (6.17) lead to equivalent first order equations for \( \Omega_{ab} \) and so on. Thus, \( Y_a \) is replaced by a whole set of operators, and the integrability conditions (6.12) for \( Y_a \) are replaced by a whole set of integrability conditions.

In [4] we proposed that all these integrability conditions are embedded in one single quantum master equation given by

\[ (S, S)_\Delta = ih[\Delta, S], \]  \hspace{1cm} (6.18)

where \( \Delta \) is an extended nilpotent BFV-BRST charge given by

\[ \Delta \equiv \Omega + \eta^a \pi_a (-1)^{\varepsilon_a}, \quad \Delta^2 = 0, \quad [\phi^a, \pi_b] = ih\delta^a_b, \]  \hspace{1cm} (6.19)

where in turn \( \pi_a \) are conjugate momenta to \( \phi^a \), now turned into operators, and \( \eta^a, \varepsilon(\eta^a) = \varepsilon_a + 1 \), are new ghost variables to be treated as parameters. The operator \( S(\phi, \eta) \) in the master equation (6.18) is an even operator defined by

\[ S(\phi, \eta) = G + \eta^a \Omega_a(\phi) + \frac{1}{2} \eta^b \eta^a \Omega_{ab}(\phi)(-1)^{\varepsilon_b} + \]  \[ + \frac{1}{6} \eta^c \eta^b \eta^a \Omega_{abc}(\phi)(-1)^{\varepsilon_b + \varepsilon_a \varepsilon_c} + \ldots \]  \[ \ldots + \frac{1}{n!} \eta^a_1 \ldots \eta^a_n \Omega_{a_1 \ldots a_n}(\phi)(-1)^{\varepsilon_n} + \ldots, \]  \[ \varepsilon_n \equiv \sum_{k=1}^{[\frac{n}{2}]} \varepsilon_{a_{2k-1}} + \sum_{k=1}^{[\frac{n}{2}]} \varepsilon_{a_{2k-1}} \varepsilon_{a_{2k}}, \]  \hspace{1cm} (6.20)

where \( G \) is the ghost charge operator in (6.6). (In [4] we made another choice for \( \varepsilon_n \).) The operators \( \Omega_{a_1 \ldots a_n}(\phi) \) in (6.20) satisfy the properties

\[ \varepsilon(\Omega_{a_1 \ldots a_n}) = \varepsilon_{a_1} + \ldots + \varepsilon_{a_n} + n, \quad [G, \Omega_{a_1 \ldots a_n}] = -nih\Omega_{a_1 \ldots a_n}. \]  \hspace{1cm} (6.21)

The last relation implies that \( \Omega_{a_1 \ldots a_n} \) has ghost number minus \( n \). If we assign ghost number one to \( \eta^a \) then \( \Delta \) has ghost number one and \( S \) has ghost number zero. Our main conjecture is that the operators \( \Omega_{a_1 \ldots a_n}(\phi) \) in (6.20) may be identified with \( \Omega_a, \Omega_{ab} \) in (6.17) and all the \( \Omega \)'s in their integrability conditions in a particular manner.

The antibracket \( (S, S)_\Delta \) in (6.18) is the quantum antibracket defined by (2.1) with \( Q \) replaced by \( \Delta \). Thus, we have

\[ (S, S)_\Delta = [[S, \Delta], S]. \]  \hspace{1cm} (6.22)
By means of this relation it is easily seen that the consistency of (6.18) requires \([\Delta, S]\) to be nilpotent. We have

\[
[\Delta, (S, S)_\Delta] = 0 \quad \Leftrightarrow \quad [\Delta, S]^2 = 0. \tag{6.23}
\]

The explicit form of \([S, \Delta]\) is to the lowest orders in \(\eta^a\)

\[
[S, \Delta] = i\hbar \Omega + \eta^a[\Omega_a, \Omega] + \eta^b \eta^a \Omega_a \partial_b i\hbar (-1)^{\epsilon_b} + \frac{1}{2} \eta^b \eta^a [\Omega_{ab}, \Omega] (-1)^{\epsilon_b} +
\]

\[
+ \frac{1}{2} \eta^c \eta^b \eta^a \Omega_{ab} \partial_c i\hbar (-1)^{\epsilon_b + \epsilon_c} + \frac{1}{6} \eta^c \eta^b \eta^a [\Omega_{abc}, \Omega] (-1)^{\epsilon_b + \epsilon_a + \epsilon_c} + O(\eta^4). \tag{6.24}
\]

To zeroth and first order in \(\eta^a\) the master equation (6.18) is satisfied identically. However, to second order in \(\eta^a\) it yields exactly (6.17). At third order in \(\eta^a\) it yields

\[
\partial_a \Omega_{bc}(-1)^{\epsilon_a \epsilon_c} + \frac{1}{2} (i\hbar)^{-2} [\Omega_a, \Omega_{bc}] (-1)^{\epsilon_a \epsilon_c} + \text{cycle}(a,b,c) =
\]

\[
= -(i\hbar)^{-3} [\Omega_a, \Omega_b, \Omega_c] (-1)^{\epsilon_a \epsilon_c} - \frac{2}{3} (i\hbar)^{-1} [\Omega'_{abc}, \Omega],
\]

\[
\Omega'_{abc} \equiv \Omega_{abc} - \frac{1}{8} \left\{ (i\hbar)^{-1} [\Omega_{ab}, \Omega_c] (-1)^{\epsilon_a \epsilon_c} + \text{cycle}(a,b,c) \right\}, \tag{6.25}
\]

where we have introduced the higher quantum \(\Omega\)-antibrackets of order 3 defined by (3.6) and operators \(\Omega\) with still more indices. We conjecture that these equations agree exactly with the integrability conditions of (6.25).

Comparing equation (6.25) and the integrability conditions of (6.17) we find exact agreement. We have also checked that the consistency condition (6.23) yields exactly (6.17) to second order in \(\eta^a\), which is consistent with (6.17) as it should. Similarly we have checked that (6.23) to third order in \(\eta^a\) yields a condition which is consistent with (6.25), exactly like (6.17) is consistent with (6.17).

The master equation (6.18) yields at higher orders in \(\eta^a\) equations involving still higher quantum \(\Omega\)-antibrackets defined by (3.6) and operators \(\Omega_{abc...}\) with still more indices. We expect that these equations agree exactly with the integrability conditions of (6.25). For a rank-\(N\) theory we expect that there exists a solution of the form (6.20) to the master equation (6.18), which terminates just at the maximal order \(\eta^N\). In the appendix we treat quasigroup first rank theories in detail.

We end this subsection with some transformation formulas of the master equation (6.18). Let us define the transformed operators \(S(\alpha)\) and \(\Delta(\alpha)\) by

\[
S(\alpha) \equiv e^{i\alpha F} S e^{-i\alpha F}, \quad \Delta(\alpha) \equiv e^{i\alpha F} \Delta e^{-i\alpha F}, \tag{6.26}
\]

where \(\alpha\) is a parameter and \(F\) an arbitrary even operator. If \(S\) and \(\Delta\) satisfy the master equation (6.18) then \(S(\alpha)\) and \(\Delta(\alpha)\) satisfy the transformed master equation

\[
(S(\alpha), S(\alpha))_{\Delta(\alpha)} = i\hbar [\Delta(\alpha), S(\alpha)]. \tag{6.27}
\]

If \(F\) in (6.26) is restricted to satisfy the master equation (6.18), i.e.

\[
(F, F)_{\Delta} = i\hbar [\Delta, F], \tag{6.28}
\]

then \(\Delta(\alpha)\) in (6.26) reduces to

\[
\Delta(\alpha) = \Delta + (i\hbar)^{-1} [\Delta, F](1 - e^{-\alpha}). \tag{6.29}
\]
(Notice that $(6.28)$ implies $\Delta''(\alpha) + \Delta'(\alpha) = 0$.) For $F = S$ we have in particular that $S$ satisfies the master equation $(6.18)$ with $\Delta$ replaced by $\Delta(\alpha)$ in $(6.29)$ where $F$ is replaced by $S$.

There are also transformations on $S$ leaving $\Delta$ unaffected for which the master equation $(6.18)$ is invariant. The natural automorphism of $(6.18)$ is

$$S \rightarrow S' \equiv \exp \left\{ -(i\hbar)^{-2}[\Delta, \Psi] \right\} S \exp \left\{ (i\hbar)^{-2}[\Delta, \Psi] \right\},$$

(6.30)

where $\Psi$ is an arbitrary odd operator. It is easily seen that $S'$ also satisfies the master equation $(6.18)$. For infinitesimal transformations we have

$$\delta S = (i\hbar)^{-2}[S, [\Delta, \Psi]],$$
$$\delta_{21} S \equiv (\delta_2 \delta_1 - \delta_1 \delta_2) S = (i\hbar)^{-2}[S, [\Delta, \Psi_{21}]],$$
$$\Psi_{21} = (i\hbar)^{-2}(\Psi_2, \Psi_1)_{\Delta}.$$  

(6.31)

Analogously to the equivalent forms of the general equations of motion in $(6.7)$ and $(6.9)$ we have also

$$\delta S = (i\hbar)^{-2}\left( (S, \Delta) - \frac{1}{2}[\Delta, [\Psi, S]] \right) =$$
$$= (i\hbar)^{-2}\frac{2}{3}\left( (S, \Delta) + (S, [\Delta, \Psi]) \right).$$

(6.32)

If the transformation $(6.30)$ connects any solutions of the master equation $(6.18)$ then the general solution is

$$S = \exp \left\{ -(i\hbar)^{-2}[\Delta, \Psi] \right\} G \exp \left\{ (i\hbar)^{-2}[\Delta, \Psi] \right\},$$

(6.33)

where $\Psi$ depends on all variables including $\phi^a$ and $\eta^a$ but not on $\pi_a$. $\Psi$ is only required to have total ghost number minus one since $S$ has total ghost number zero. The explicit form $(6.20)$ of $S$ is then reproduced. Notice that $S = G$ is a trivial solution of the master equation $(6.18)$.

We would also like to mention that there is a possibility to extend the formalism to $\eta$-dependent states and operators which satisfy the Lie equations $(6.10)$-$(6.11)$ with the $\eta$-dependent connections $\tilde{Y}_a(\phi, \eta) \equiv (i\hbar)^{-1}[\Delta, \tilde{\Omega}_a(\phi, \eta)]$ where $\tilde{\Omega}_a(\phi, \eta)$ is determined by the equation

$$\tilde{S} \overset{\leftarrow}{\partial_a} - (i\hbar)^{-1}[S, \tilde{Y}_a] = 0.$$  

(6.34)

These $\eta$-extended states, $|\tilde{A}\rangle$, and operators, $\tilde{A}$, satisfy the equations

$$S|\tilde{A}\rangle = i\hbar g(|A\rangle)|\tilde{A}\rangle, \quad [S, \tilde{A}] = i\hbar g(A)\tilde{A},$$

(6.35)

where $|A\rangle = |\tilde{A}\rangle|_{\eta=0}$ and $A = \tilde{A}|_{\eta=0}$. The details will be given elsewhere.
7 Generalization to the Sp(2)-case

There is an Sp(2)-extended version of BV-quantization \cite{3, 5} which in its most general form is called triplectic quantization \cite{8}. In this formalism there are two generalized antibrackets which are called Sp(2)-antibrackets. Also these brackets may be mapped on operators. The quantum Sp(2)-antibrackets are defined by \cite{1, 6} \((a, b, c, \ldots = 1, 2, \text{Sp}(2)\text{-indices})\)

\[
(f, g)_{Q}^{0} = \frac{1}{2} \left( [f, [Q^{a}, g]] - [g, [Q^{a}, f]] (-1)^{(e_{f}+1)(e_{g}+1)} \right),
\]

(7.1)

where \(Q^{a}\) are two odd operators. The corresponding classical antibrackets satisfy the properties (2.3)-(2.8) except that the Jacobi identities are valid for symmetrized Sp(2)-indices. The quantum Sp(2)-antibrackets (7.1) satisfy (2.3)-(2.6). However, instead of the Jacobi identities we have

\[
(f, (g, h)_{Q}^{a,b})_{Q}^{c} (-1)^{e_{f}e_{h}} + \text{cycle}(f, g, h) =
\]

\[
= \frac{1}{6} (-1)^{e_{f}+e_{g}+e_{h}} \left\{ \left( [f, [g, [h, [Q^{a}, Q^{b}]]]] + \frac{1}{2} [f, [g, [h, Q^{a}]], Q^{b}]] \right) (-1)^{e_{f}e_{h}} + \right.
\]

\[
+ \left( [f, [h, [g, [Q^{a}, Q^{b}]]]] + \frac{1}{2} [f, [h, [g, Q^{a}]], Q^{b}]] \right) (-1)^{e_{h}e_{g}} \right\} + \text{cycle}(f, g, h),
\]

(7.2)

and instead of Leibniz’ rule we have

\[
(f g, h)_{Q}^{a,b} - (f, g)_{Q}^{a,b} h - (f, h)_{Q}^{a,b} g (-1)^{e_{g}+1} =
\]

\[
= \frac{1}{2} \left( [f, h][g, Q^{a}] (-1)^{e_{h}e_{g}+1} + [f, Q^{a}][g, h] (-1)^{e_{f}} \right).
\]

(7.3)

The relation (2.13) generalizes e.g. to

\[
(f, Q^{a}Q^{b}) = \frac{3}{2} [f, [Q^{a}, Q^{b}]].
\]

(7.4)

Higher order quantum Sp(2)-antibrackets are defined by (3.5) with \(Q\) replaced by \(Q^{a}\). The properties (3.6)-(3.12) are then valid by the same replacement. The relations (3.13)-(3.14) are valid with two Sp(2)-indices. To obtain the generalized Jacobi identities we need the corresponding identities to (3.13). However, since they involve the trivial equalities

\[
e^{-A}Q^{a}e^{A}, e^{-A}Q^{b}e^{A} - e^{-A}[Q^{a}, Q^{b}]e^{A} = 0,
\]

(7.5)

which are symmetric in the Sp(2)-indices, the generalized Jacobi identities corresponding to (3.20) will also be symmetric in the Sp(2)-indices. Corresponding to the treatment in section 4 we may also define ordinary quantum Sp(2)-antibrackets by a restriction to a maximal set of commuting operators. If these operators are functions of commuting coordinate operators then \(Q^{a}\) and \([Q^{a}, Q^{b}]\) must be maximally quadratic in the momenta in order for the Sp(2)-antibrackets to strictly satisfy the Jacobi identities and Leibniz’ rule.

In the considered applications of the quantum Sp(2)-antibrackets (7.1) to the Sp(2)-version of the BV-quantization and to the Sp(2)-version of BFV-BRST quantization the two odd operators \(Q^{a}\) were required to satisfy \cite{3}

\[
Q^{a}Q^{b} + Q^{b}Q^{a} = [Q^{a}, Q^{b}] = 0.
\]

(7.6)
The appearance of two nilpotent defining operators is natural since the Sp(2)-versions are directly related to the so called BRST-antiBRST quantization [17]. In the Sp(2)-version of BV-quantization we have to consider commuting operators which are functions of a maximal commuting set of coordinate operators. The operators $Q^a$ are then maximally quadratic in the momenta so that the quantum Sp(2)-antibrackets are ordinary ones. In the Schrödinger representation $Q^a$ are then equal to the two nilpotent differential operators, $\Delta^a$, exactly like $Q$ was represented by $\Delta$ in section 5. The quantum master equations are here

$$Q^a|\mathcal{W}\rangle = 0, \quad a = 1, 2. \quad (7.7)$$

In [6] this formalism was shown to provide for an operator version of the Sp(2)-extended BV-quantization corresponding to the what we had in section 5 for the ordinary BV-quantization. For instance, the master equations and the gauge fixed partition function in triplectic quantization were shown to follow from (7.7) in analogy to what we had in section 5. The quantum master equations for generalized quantum Maurer-Cartan equations for arbitrary open groups given in section 6 were also shown to be possible to formulate in terms of the Sp(2)-brackets (7.1) in [6].

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Appendix A

Application of the quantum master equation to quasigroup first rank theories.

As an illustration of the formulas in subsection 6.2 we consider now constraint operators $\theta_a$ forming a rank one theory in which case we have (we consider $\mathcal{CP}$-ordered operators here)

$$\Omega = \mathcal{C}^a \theta'_a + \frac{1}{2} \mathcal{C}^b \mathcal{C}^a \mathcal{U}^e_{ab} \mathcal{P}_c (-1)^{\varepsilon_c + \varepsilon_b}, \quad \theta'_a \equiv \theta_a + \frac{1}{2} i \hbar \mathcal{U}^b_{ab} (-1)^{\varepsilon_b}. \quad (A.1)$$

The nilpotence of $\Omega$ requires

$$[\theta'_a, \theta'_b] = i \hbar \mathcal{U}^c_{ab} \theta'_c, \quad (i \hbar \mathcal{U}^d_{ab} \mathcal{U}^e_{de} + [\mathcal{U}^e_{ab}, \theta'_c] (-1)^{\varepsilon_c \varepsilon_e}) (-1)^{\varepsilon_a \varepsilon_c} + \text{cycle}(a, b, c) = 0, \quad (-1)^{\varepsilon_a \varepsilon_c} \left( [\mathcal{U}^e_{ab}, \mathcal{U}^f_{cd}] (-1)^{\varepsilon_c \varepsilon_e + \varepsilon_a} \right) - [\mathcal{U}^f_{ab}, \mathcal{U}^e_{cd}] (-1)^{\varepsilon_c (\varepsilon_a + \varepsilon_d) + \varepsilon_a \varepsilon_f} + \text{cycle}(a, b, c) = 0. \quad (A.2)$$

The last conditions are certainly satisfied if

$$[\mathcal{U}^c_{ab}, \mathcal{U}^f_{de}] = 0, \quad [[\theta_a, \mathcal{U}^c_{ab}], \mathcal{U}^g_{ef}] = 0. \quad (A.3)$$
which corresponds to quasigroups \[\text{\cite{13}}\]. In this case \(\Omega_a\) may be chosen to be

\[
\Omega_a(\phi) = \lambda^a_b(\phi)P_b, \quad \lambda^b_a(0) = \delta^b_a, 
\]

\[(A.4)\]

where we assume that

\[
[\lambda^b_a, \lambda^d_c] = 0 \quad \Rightarrow \quad [\Omega_a, \Omega_b] = 0.
\]

\[(A.5)\]

The quantum \(\Omega\)-antibracket is then given by

\[
(\Omega_a, \Omega_b) \Omega = [\Omega_a, [\Omega_b, \Omega]_\Omega] = -(ih)^2\lambda^a_bU^d_cP_d(-1)^{\varepsilon_a+\varepsilon_b+\varepsilon_c} + \\
+ih\left(\lambda^a_c[\lambda^d_b, \lambda^c_a] - \lambda^d_b[\lambda^a_c, \lambda^c_a](-1)^{\varepsilon_a}\right)P_d(-1)^{\varepsilon_c} - \\
-ih\left(\lambda^a_cC^e[U^c_{ef}, \lambda^d_b](-1)^{\varepsilon_b(\varepsilon_e+1)} - \lambda^d_bC^e[U^c_{ef}, \lambda^a_c](-1)^{\varepsilon_a(\varepsilon_e+1)+\varepsilon_a}\right)P_dP_c - \\
-C^e[\lambda^a_c, \lambda^d_b]P_dP_c(-1)^{(\varepsilon_a+1)(\varepsilon_e+1)_{(\varepsilon_e+1)} + (\varepsilon_b+1)(\varepsilon_e+1)}.
\]

\[(A.6)\]

If we also require

\[
(\Omega_a, \Omega_b, \Omega_c) = 0 \quad \Leftrightarrow \quad [(\Omega_a, \Omega_b)_\Omega, \Omega_c] = 0,
\]

\[(A.7)\]

then \((\Omega_a, \Omega_b)_\Omega\) in \[(A.6)\] satisfies the Jacobi identities which makes \[(6.17)\] integrable if \(\Omega_{ab} = 0\). This condition is satisfied if we impose

\[
[\lambda^b_a, U^e_{de}] = 0, \quad [\lambda^b_a, [\lambda^d_c, \theta_e]] = 0.
\]

\[(A.8)\]

Eq.\[(6.17)\] may now be written as

\[
\partial_a\tilde{X}^c_b - \partial_b\tilde{X}^c_a(-1)^{\varepsilon_a\varepsilon_b} = \tilde{X}^c_a\tilde{U}^c_{de}(-1)^{\varepsilon_b\varepsilon_c\varepsilon_d+\varepsilon_e}.
\]

\[(A.9)\]

where \(\tilde{X}^b_a \equiv V\lambda^b_aV^{-1}\) and \(\tilde{U}^c_{ab} \equiv VU^c_{ab}V^{-1}\) where in turn the operator \(V(\phi)\) is determined by the equation

\[
(ih)\partial_aV = V\lambda^b_a\theta'_b(-1)^{\varepsilon_b}.
\]

\[(A.10)\]

Eq.\[(A.7)\] and \(\Omega_{ab} = 0\) make all higher integrability conditions identically zero. One may note that

\[
Y_a(\phi) = (ih)^{-1}[\Omega, \Omega_a] = \lambda^b_a\theta'_b(-1)^{\varepsilon_a+\varepsilon_b} + \\
+\lambda^b_cC^dU^c_{de}P_e(-1)^{\varepsilon_a+\varepsilon_c} + (ih)^{-1}C^d[\theta_b, \lambda^c_a]\Omega_c.
\]

\[(A.11)\]

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