LECTURES ON THE ELSV FORMULA

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Abstract. The ELSV formula, first proved by Ekedahl, Lando, Shapiro, and Vainshtein, relates Hurwitz numbers to Hodge integrals. Graber and Vakil gave another proof of the ELSV formula by virtual localization on moduli spaces of stable maps to $\mathbb{P}^1$, and also explained how to simplify their proof using moduli spaces of relative stable maps to the pair $(\mathbb{P}^1, \infty)$. In this expository article, we explain what the ELSV formula is and how to prove it by virtual localization on moduli spaces of relative stable maps, following Graber-Vakil. This note is based on lectures given by the author at Summer School on “Geometry of Teichmüller Spaces and Moduli Spaces of Curves” at Center of Mathematical Sciences, Zhejiang University, July 14–20, 2008.

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1. Introduction

Deligne and Mumford introduced the notation of stable curves and constructed the moduli space $\mathcal{M}_g$ of genus $g$ stable curves. The moduli space $\mathcal{M}_{g,n}$ of $n$-pointed genus $g$ stable curves was constructed by Knudsen-Mumford and Knudsen. Since Mumford’s seminal paper in the early 1980s, the intersection theory of moduli spaces of stable curves has been studied extensively. Evaluations of Hodge integrals

$$\int_{\mathcal{M}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g}$$

are important and difficult problems in this subject. Here $\psi_i$ is the first Chern class of the line bundle $\mathbb{L}_i \to \mathcal{M}_{g,n}$ whose fiber at the moduli point $[(C, x_1, \ldots, x_n)]$ is the cotangent line $T_{x_i}C$ at the $i$-th marked point; $\lambda_i$ is the $i$-th Chern class of the Hodge bundle $\mathbb{E} \to \mathcal{M}_{g,n}$, which is a rank $g$ complex vector bundle whose fiber at $[(C, x_1, \ldots, x_n)]$ is $H^0(C, \omega_C)$, the space of sections of the dualizing sheaf $\omega_C$ of $C$ (see Section 2.2 for a review on Hodge integrals).

Using Mumford’s Grothendieck-Riemann-Roch calculations in, Faber proved, in [8], that general Hodge integrals can be uniquely reconstructed from the $\psi$ integrals (also known as descendant integrals):

$$\int_{\mathcal{M}_{g,h}} \psi_1^{j_1} \cdots \psi_h^{j_h}.$$

The descendant integrals can be computed recursively by Witten’s conjecture which asserts that the $\psi$ integrals (2) satisfy a system of differential equations known as the KdV equations. The KdV equations and the string equation determine all the $\psi$ integrals (2) from the initial value $\int_{\mathcal{M}_{0,3}} 1 = 1$.

The Witten’s conjecture was first proved by Kontsevich in [24]. This is one of the most striking and fundamental result in the intersection theory of moduli spaces of stable curves. By now, Witten’s conjecture has been reproved many times (Okounkov-Pandharipande [39], Mirzakhani [36], Kim-Liu [20], Kazarian-Lando [19], Chen-Li-Liu [3], Kazarian [18], Mulase-Zhang [37] ...). The ELSV formula, which relates Hurwitz numbers to Hodge integrals, plays a central role in several of the above proofs (Okounkov-Pandharipande, Kazarian-Lando, Kazarian, Mulase-Zhang ...). The ELSV formula is named after Ekedahl,
Lando, Shapiro, and Vainshtein, who first proved this formula in [6, 7]. Later, Graber and Vakil gave another proof by virtual localization on moduli spaces \( \overline{M}_{g,0}(\mathbb{P}^1, d) \) of genus \( g \) degree \( d \) stable maps to \( \mathbb{P}^1 \) [14]. Fantechi and Pandharipande proved a special case of the ELSV formula by virtual localization [10, Theorem 2]. Okounkov and Pandharipande’s paper [39] contains a detailed exposition of the proof of the ELSV formula by virtual localization on moduli spaces of stable maps to \( \mathbb{P}^1 \), following [10] and [14].

In [14, Section 5], Graber and Vakil explained how their proof could be much simplified using moduli spaces of relative stable maps to the pair \((\mathbb{P}^1, \infty)\). When Graber and Vakil wrote their paper [14], moduli spaces of relative stable maps had already been constructed in the symplectic category, by Li-Ruan [29] and by Ionel-Parker [16, 17]. However, Graber and Vakil needed such moduli spaces in the algebraic category, with desired properties (proper Deligne-Mumford stack with perfect obstruction theory, so that the virtual fundamental class exists, and the virtual localization is applicable). Jun Li constructed moduli spaces of relative stable maps with desired properties in the algebraic category [26, 27]. In this expository article, we explain what the ELSV formula is and how to prove it by virtual localization on moduli spaces of relative stable maps, following Graber-Vakil [14].

Virtual localization on moduli spaces of relative stable maps can be used to prove other Hodge integral identities. In [31] (resp. [32]), K. Liu, J. Zhou and the author used virtual localization on moduli spaces of relative stable maps to \( \mathbb{P}^1 \) relative to \( \infty \) (resp. to the toric blowup of \( \mathbb{P}^2 \) at two torus fixed points relative to the two exceptional divisors) to prove the Mariño-Vafa formula [35] (resp. a formula of two-partition Hodge integrals [43]), which relates certain generating function of Hodge integrals to the colored HOMFLY invariants of the unknot (resp. the Hopf link). (Okounkov and Pandharipande gave another proof of the Mariño-Vafa formula using virtual localization on moduli spaces of stable maps to \( \mathbb{P}^1 \) [40].) The ELSV formula can be obtained by taking certain limit of the Mariño-Vafa formula. See [33] for a survey of proofs and applications of the Mariño-Vafa formula and the formula of two-partition Hodge integrals.

We now give an overview of the remainder of this paper. In Section 2 we recall the definitions of Hurwitz numbers and Hodge integrals, and state the ELSV formula. In Section 3 we give a brief review of equivariant cohomology and localization. In Section 4, we interpret Hurwitz numbers as certain relative Gromov-Witten of the pair \((\mathbb{P}^1, \infty)\), and derive the ELSV formula by virtual localization on moduli spaces of relative stable maps to the pair \((\mathbb{P}^1, \infty)\).
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2. HURWITZ NUMBERS AND HODGE INTEGRALS

In this section, we review the definitions of Hurwitz numbers and Hodge integrals, and give the precise statement of the ELSV formula.

2.1. Hurwitz numbers. In this subsection, we give a brief review of the geometric and combinatorial definitions of Hurwitz numbers which count ramified covers of \( \mathbb{P}^1 \) with a given ramification type over \( \infty \in \mathbb{P}^1 \).

Let \( d \) be a positive integer. A partition of \( d \) is a sequence of positive integers \( \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_h > 0) \) such that \( \mu_1 + \cdots + \mu_h = d \). The sum of all components of \( \mu \), \( d \), is called the size of the partition \( \mu \), denoted \(|\mu|\); the number of components in \( \mu \), \( h \), is called the length of the partition \( \mu \), denoted \( \ell(\mu) \).

Given a nonnegative integer \( g \) and a partition \( \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_h > 0) \) of a positive integer \( d \), we consider ramified covers \( f : C \to \mathbb{P}^1 \) satisfying the following conditions:

(i) \( C \) is a connected compact Riemann surface of genus \( g \).
(ii) \( \deg f = d \).
(iii) \( f^{-1}(\infty) = \sum_{i=1}^{h} \mu_i x_i \) as Cartier divisors, where \( x_1, \ldots, x_h \) are distinct points in \( C \). (So \( \infty \) is a critical value of \( f \) if \( \ell(\mu) < d \).)
(iv) All other branch points of \( f \) (i.e. critical values of \( f \)) are simple. Namely, if \( b \in \mathbb{P}^1 - {\infty} \) is a critical value of \( f \) then there is a unique critical point \( x \in f^{-1}(b) \), and \( x \) is a nondegenerate critical point. So \( f^{-1}(b) \) consists of exactly \( d-1 \) distinct points.

Let \( b_1, \ldots, b_r \) be the branch points of \( f \) in \( \mathbb{C} = \mathbb{P}^1 - {\infty} \). Then the number \( r \) is determined by the genus \( g \) and the ramification type \( \mu \). To find \( r \), let \( B = \{b_1, \ldots, b_r, \infty\} \), and let \( C' = C - f^{-1}(B) \). Then \( f|_{C'} : C' \to \mathbb{P}^1 - B \) is an honest covering map of degree \( d \). We have

\[
\chi(C') = d \cdot \chi(\mathbb{P}^1 - B)
\]
where \( \chi(C') = 2 - 2g - (d - 1)r - h, \chi(\mathbb{P}^1 - B) = 1 - r \). We conclude that

\[
(3) \quad r = 2g - 2 + d + h = 2g - 2 + |\mu| + \ell(\mu).
\]

If we fix \( r \) distinct points \( b_1, \ldots, b_r \in \mathbb{P}^1 - \{\infty\} \) then there are only finitely many ramified covers \( f : C \to \mathbb{P}^1 \) satisfying (i)--(iv). Indeed, the domain Riemann surface \( C \) is determined by the monodromy \( \sigma_i \) around \( b_i \) which are transpositions in the permutation group \( S_d \) of \( \{1, 2, \ldots, d\} \). They satisfy

\[
(4) \quad \sigma_1 \cdots \sigma_r = \sigma_\infty
\]

where \( \sigma_\infty \) is the monodromy around \( \infty \). Let \( C_\mu \subset S_d \) be the conjugacy class which consists of products of \( h \) disjoint cycles of lengths \( \mu_1, \ldots, \mu_h \). Then \( \sigma_\infty \in C_\mu \).

The connected Hurwitz number \( H_{g,\mu} \) counts connected ramified covers \( f : C \to \mathbb{P}^1 \) satisfying (i)--(iv), weighted by \((\#\text{Aut}(f))^{-1}, \) where \( \text{Aut}(f) \) is the group of automorphisms of the map \( f \) and \( \#\text{Aut}(f) \) denotes the cardinality of the set \( \text{Aut}(f) \). If we fix \( \sigma_\infty \in C_\mu \) then

\[
(5) \quad H_{g,\mu} = \frac{1}{z_\mu} \# \left\{ (\sigma_1, \ldots, \sigma_r) \mid \sigma_i \text{ transpositions in } S_d, \sigma_1 \cdots \sigma_r = \sigma_\infty, \langle \sigma_1, \ldots, \sigma_r \rangle \text{ acts transitively on } \{1, 2, \ldots, d\} \right\},
\]

where \( d = |\mu|, r = 2g - 2 + |\mu| + \ell(\mu), \) and

\[
 z_\mu = \frac{d!}{\#C_\mu} = \mu_1 \cdots \mu_h \cdot \#\text{Aut}(\mu)
\]

is the cardinality of the centralizer of \( \sigma_\infty \).

The disconnected Hurwitz number \( H^\bullet_{\chi,\mu} \) counts possibly disconnected ramified covers \( f : C \to \mathbb{P}^1 \) satisfying (i)' \( \chi(C) = \chi \) and (ii), (iii), (iv), weighted by \((\#\text{Aut}(f))^{-1}. \) If we fix \( \sigma_\infty \in C_\mu \) then

\[
(6) \quad H^\bullet_{\chi,\mu} = \frac{1}{z_\mu} \# \left\{ (\sigma_1, \ldots, \sigma_r) \mid \sigma_i \text{ transpositions in } S_d, \sigma_1 \cdots \sigma_r = \sigma_\infty \right\},
\]

where \( d = |\mu| \) and \( r = -\chi + |\mu| + \ell(\mu). \)

Given a partition \( \mu, \) We introduce a generating function \( \Phi_\mu(\lambda) \) (resp. \( \Phi^\bullet_\mu(\lambda) \)) of connected (resp. disconnected) Hurwitz numbers

\[
\Phi_\mu(\lambda) = \sum_{g=0}^{\infty} \lambda^{2g-2+\ell(\mu)} H_{g,\mu}, \quad \Phi^\bullet_\mu(\lambda) = \sum_\chi \lambda^{-\chi+\ell(\mu)} H^\bullet_{\chi,\mu}.
\]
We now introduce variables $x_1, x_2, \ldots$ and let $p_i = x_1^i + x_2^i + \cdots$ be the Newton polynomials. Given a partition $\mu = (\mu_1 \geq \cdots \geq \mu_h > 0)$, define $p_\mu = p_{\mu_1} \cdots p_{\mu_h}$.

Then

$$
\exp \left( \sum_{\mu \neq \emptyset} \Phi_\mu(\lambda) p_\mu \right) = \sum_{\mu} \Phi_\mu^*(\lambda) p_\mu.
$$

where $\emptyset$ denotes the empty partition (the unique partition with zero size and zero length).

Identities (5) and (6) define Hurwitz numbers in terms of representations of the permutation group. Given a partition $\mu$ of $d > 0$, let $R_\mu$ denote the irreducible representation of $S_d$ associated to $\mu$, and let $\chi_\mu$ be the character of $R_\mu$.

**Theorem 1** (Burnside formula). Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_h > 0)$ be a partition of $d$. Then

$$
\Phi_\mu^*(\lambda) = \sum_{|\nu| = d} \frac{\chi_\nu(C_h)}{z_\mu} e^{x_\lambda} \frac{\dim R_\nu}{d!}.
$$

where $\kappa_\mu = \sum_{i=1}^h \mu_i(\mu_i - 2i + 1)$.

Indeed, Theorem 1 is a special case of the Burnside formula for general Hurwitz numbers $H_{g,\mu^1,\ldots,\mu^k}$, where $g, h$ are nonnegative integers, and $\mu^1, \ldots, \mu^k$ are partitions of the same positive integer $d$. The Hurwitz number $H_{g,\mu^1,\ldots,\mu^k}^h$ counts, with weight, degree $d$ ramified covers $f : C \to D$ of a fixed genus $h$ Riemann surface $D$ by a genus $g$ Riemann surface $C$, with prescribed ramification types $\mu^1, \ldots, \mu^k$ over $k$ fixed distinct points $q_1, \ldots, q_k$ in $D$. Theorem 1 corresponds to the special case where $h = 0, k = 1$.

2.2. Hodge Integrals. Let $\overline{M}_{g,h}$ be the moduli space of $h$-pointed, genus $g$ stable curves. (In this paper we always work over $\mathbb{C}$.) A point in $\overline{M}_{g,h}$ is represented by $[(C, x_1, \ldots, x_h)]$, where $C$ is a complex algebraic curve of arithmetic genus $g$ with at most nodal singularities, $x_1, \ldots, x_h$ are distinct smooth points on $C$, and $[(C, x_1, \ldots, x_h)]$ is stable in the sense that its automorphism group is finite. When $C$ is smooth, it can be viewed as a connected compact Riemann surface of genus $g$.

$\overline{M}_{g,h}$ is a proper smooth Deligne-Mumford stack (or a compact, complex, smooth orbifold) of (complex) dimension $3g - 3 + h$. It has a fundamental class $[\overline{M}_{g,h}] \in H_{2(3g-3+h)}(\overline{M}_{g,h}; \mathbb{Q})$. Given $\alpha_1, \ldots, \alpha_k \in H^*(\overline{M}_{g,h}; \mathbb{Q})$, we define their top intersection number to be

$$
\int_{\overline{M}_{g,h}} \alpha_1 \cdots \alpha_k := \langle [\overline{M}_{g,h}], \alpha_1 \cdots \alpha_k \rangle \in \mathbb{Q}
$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing of cycles with fundamental class.
where $\langle \ , \ \rangle$ is the pairing between the homology $H_*(\overline{\mathcal{M}}_{g,h};\mathbb{Q})$ and the cohomology $H^*(\overline{\mathcal{M}}_{g,h};\mathbb{Q})$.

The dualizing sheaf $\omega_C$ of a curve with at most nodal singularities is an invertible sheaf (line bundle). Near a smooth point, $\omega_C$ is generated by the local holomorphic differential $dz$, where $z$ is a local holomorphic coordinate; near a node, which is locally isomorphic to $(0,0) \in \{xy = 0 \mid (x,y) \in \mathbb{C}^2\}$, $\omega_C$ is generated by the meromorphic differential $dx/x = -dy/y$. The Hodge bundle $\mathbb{E}$ is a rank $g$ vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over the moduli point $[(C, x_1, \ldots, x_h)] \in \overline{\mathcal{M}}_{g,h}$ is $H^0(C,\omega_C)$. When $C$ is smooth, $\omega_C = \Omega^1_C$ is the sheaf of local holomorphic differentials on the compact Riemann surface $C$, and $H^0(C,\omega_C) = H^0(C,\Omega^1_C)$ is the space of holomorphic differentials on $C$. The $\lambda$ classes are defined by

$$\lambda_j = c_j(\mathbb{E}) \in H^{2g}(\overline{\mathcal{M}}_{g,h};\mathbb{Q}).$$

The cotangent line $T^*_xC$ of $C$ at the $i$-th marked point $x_i$ gives rise to a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The $\psi$ classes are defined by

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,h};\mathbb{Q}).$$

The $\lambda$ classes and $\psi$ classes lie in $H^*(\overline{\mathcal{M}}_{g,h};\mathbb{Q})$ instead of $H^*(\overline{\mathcal{M}}_{g,h};\mathbb{Z})$ because $\mathbb{E}$ and $\mathbb{L}_i$ are orbibundles on the compact orbifold $\overline{\mathcal{M}}_{g,h}$.

Hodge integrals are top intersection numbers of $\lambda$ classes and $\psi$ classes:

$$\int_{\overline{\mathcal{M}}_{g,h}} \psi_1^{j_1} \cdots \psi_h^{j_h} \lambda_1^{k_1} \cdots \lambda_g^{k_g} \in \mathbb{Q}.$$  

(8)

By definition, (8) is zero unless

$$j_1 + \cdots + j_r + k_1 + 2k_2 + \cdots + gk_g = 3g - 3 + h.$$  

A special class of Hodge integrals are linear Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,h}} \psi_1^{j_1} \cdots \psi_h^{j_h} \lambda_i$$  

(9)

where $i = 0, \ldots, g$. When $i = 0$, we have $\lambda_0 = 1$, so (9) reduces to top intersection of $\psi$ classes, known as descendent integrals:

$$\int_{\overline{\mathcal{M}}_{g,h}} \psi_1^{j_1} \cdots \psi_h^{j_h}.$$
2.3. The ELSV formula. The ELSV formula, first proved by Ekedahl, Lando, Shapiro, and Vainshtein [6, 7], relates the Hurwitz numbers $H_{g,\mu}$ to linear Hodge integrals.

**Theorem 2** (ELSV formula). Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_h > 0)$ be a partition of $d$. Then

$$H_{g,\mu} = \frac{(2g - 2 + d + h)!}{\# \text{Aut}(\mu)} \prod_{i=1}^{h} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\mathcal{M}_{g,h}} \Lambda^\vee(1) \prod_{i=1}^{h} (1 - \mu_i \psi_i)^{\mu_i},$$

where

$$\Lambda^\vee(1) = \sum_{i=0}^{g} (-1)^i \lambda_i, \quad \frac{1}{1 - \mu_i \psi_i} = \sum_{j=0}^{3g - 3 + h} (\mu_i \psi_i)^{j}.$$ 

3. Equivariant Cohomology and Localization

In this section, we give a brief introduction to equivariant cohomology and localization. See [11] for an excellent (and much more comprehensive) exposition of this subject.

3.1. Universal bundle. Let $G$ be a Lie group. Let $EG$ be a contractible topological space on which $G$ acts freely. (In this note, all the group actions are continuous.) Suppose that $G$ acts on the right of $EG$. The quotient $BG = EG/G$ is the classifying space of principal $G$-bundles, and the natural projection $EG \to BG$ is the universal principal $G$-bundle; $EG$ and $BG$ are defined up to homotopy equivalences.

**Example 3.** Let $G = \mathbb{C}^*$. Let $EG = \mathbb{C}^\infty - \{0\}$, which is contractible. Let $G = \mathbb{C}^*$ acts on the right of $EG = \mathbb{C}^\infty - \{0\}$ by

$$v \cdot \lambda = \lambda v, \quad \lambda \in \mathbb{C}^*, \quad v \in \mathbb{C}^\infty - \{0\}.$$

Then the $G$-action on $EG$ is free. The classifying space

$$BG = (\mathbb{C}^\infty - \{0\})/\mathbb{C}^* = \mathbb{CP}^\infty$$

is the infinite dimensional complex projective space.

In Example 3, $G = \mathbb{C}^*$ is a complex algebraic group, $EG = \mathbb{C}^\infty - \{0\}$ is an infinite dimensional complex manifold, and the $G$-action on $EG$ is holomorphic. So the classifying space $BG = \mathbb{CP}^\infty$ is a complex manifold, and the universal principal $\mathbb{C}^*$-bundle $EG \to BG$ is a holomorphic principal $\mathbb{C}^*$-bundle. The **tautological line bundle** $S \to \mathbb{CP}^\infty$ is the holomorphic line bundle associated to the universal principal $\mathbb{C}^*$-bundle. For any $k \in \mathbb{Z}$, let $\mathcal{O}_{\mathbb{CP}^\infty}(k) = S^{\otimes -k}$. (Strictly speaking, $\mathcal{O}_{\mathbb{CP}^\infty}(k)$ is the sheaf of local holomorphic sections of the holomorphic
line bundle $S^\otimes\rightarrow$, but we will not distinguish $O_{CP^\infty}(k)$ from $S^\otimes\rightarrow$ in this note.)

If $G = G_1 \times G_2$ then we may take $EG = EG_1 \times EG_2$, so that $BG = BG_1 \times BG_2$.

**Example 4.** If $G = (\mathbb{C}^\ast)^n$ then $BG = (B\mathbb{C}^\ast)^n = (\mathbb{C}P^\infty)^n$.

### 3.2. Equivariant cohomology

Let $G$ be a Lie group, and let $X$ be a topological space with a left $G$-action. Then $G$ acts on $EG \times X$ freely by

$$g \cdot (p, x) = (p \cdot g^{-1}, g \cdot x).$$

The homotopy orbit space $X_G$ is defined to be the quotient of $EG \times X$ by this free $G$-action. The projection $EG \times X \to EG$ to the first factor descends to a projection $\pi: X_G \to BG$, which is a fibration over $BG$ with fiber $X$.

The $G$-equivariant cohomology of the $G$-space $X$ is defined to be the ordinary cohomology of the homotopy orbit space $X_G$:

$$H^*_G(X; R) := H^*(X_G; R)$$

where $R$ is any coefficient ring. From now on we will assume $R = \mathbb{Q}$, the field of rational numbers, and write $H^*(\bullet)$ for $H^*(\bullet; \mathbb{Q})$. The following are some special cases.

1. If $X$ is point then $X_G = BG$, so $H^*_G(pt) = H^*(BG)$.
2. If $G$ acts on $X$ freely then $X_G$ is homotopically equivalent to the orbit space $X/G$, so $H^*_G(X) = H^*(X/G)$.
3. If $G$ acts on $X$ trivially then $X_G = BG \times X$. By K"unneth formula,

$$H^*_G(X) \cong H^*(X) \otimes \mathbb{Q} H^*(BG).$$

**Example 5.** If $G = \mathbb{C}^\ast$ then $H^*_G(pt) = H^*(\mathbb{C}P^\infty) \cong \mathbb{Q}[u]$, where $u \in H^2(X; \mathbb{Q})$ is the first Chern class of $O_{CP^\infty}(1)$.

**Example 6.** If $G = (\mathbb{C}^\ast)^n$ then $H^*_G(pt) = H^*((\mathbb{C}P^\infty)^n) \cong \mathbb{Q}[u_1, \ldots, u_n]$.

**Example 7.** Let $\mathbb{C}^\ast$ act on the $r$-dimensional complex projective space $\mathbb{P}^r$ by

$$t \cdot [z_0, \ldots, z_r] = [t^{a_0}z_0, \ldots, t^{a_r}z_n], \quad t \in \mathbb{C}^\ast, \quad [z_0, \ldots, z_r] \in \mathbb{P}^r,$$

where $a_0, \ldots, a_r \in \mathbb{Z}$. Then the fibration $\mathbb{P}_{\mathbb{C}^\ast} \to B\mathbb{C}^\ast$ can be identified with the $\mathbb{P}^r$-bundle

$$\mathbb{P}(O_{CP^\infty}(a_0) \oplus \cdots \oplus O_{CP^\infty}(a_r)) \to \mathbb{C}P^\infty.$$

To compute $H^*_G(\mathbb{P}^r) = H^*(\mathbb{P}_{\mathbb{C}^\ast})$, we recall the general formula for cohomology of a projective bundle. Let $E \to X$ be a rank $(r + 1)$
complex vector bundle over a topological space $X$, and let $\pi : \mathbb{P}(E) \to X$ be the projectivization of $E$, which is an $\mathbb{P}^r$-bundle over $X$. The cohomology $H^*(\mathbb{P}(E))$ of the total space $\mathbb{P}(E)$ is an $H^*(X)$-algebra generated by $H$ with a single relation

$$H^{r+1} + c_1(E)H^r + \cdots + c_{r+1}(E) = 0,$$

where $c_i(E)$ is the $i$-th Chern class of $E$, and $H$ is of degree 2.

In our case $E = \bigoplus_{i=0}^r \mathcal{O}_{\mathbb{C}P^r}(a_i)$, so the total Chern class of $E$ is given by

$$c(E) = \prod_{i=0}^r (1 + a_i u).$$

We have

$$H^*_c(\mathbb{P}^r) = H^*(\mathbb{P}^r) \cong \mathbb{Q}[u, H]/\langle \prod_{i=0}^r (H + a_i u) \rangle,$$

where $\mathbb{Q}[u, H]$ is the ring of polynomials in two variables $u, H$ with coefficients in $\mathbb{Q}$, and $\langle \prod_{i=0}^r (H + a_i u) \rangle$ is the principal ideal generated by $\prod_{i=0}^r (H + a_i u)$.

### 3.3. Equivariant vector bundle

A continuous map $f : X \to Y$ between $G$-spaces is called $G$-equivariant if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$.

Let $p : V \to X$ be a complex vector bundle over a $G$-space $X$. We say $p : V \to X$ is a $G$-equivariant complex vector bundle over $X$ if the following properties hold.

- **$V$** is a $G$-space.
- **$p$** is $G$-equivariant.
- **For every $g \in G$, define** $\tilde{\phi}_g : V \to V$ by $v \mapsto g \cdot v$, and $\phi_g : X \to X$ by $x \mapsto g \cdot x$. Then $\tilde{\phi}_g$ is a vector bundle map covering $\phi_g$:

\[
\begin{array}{ccc}
V & \xrightarrow{\tilde{\phi}_g} & V \\
p \downarrow & & \downarrow p \\
X & \xrightarrow{\phi_g} & X
\end{array}
\]

**Example 8.** When $X$ is a point, a complex vector bundle over $X$ is a complex vector space $V$, and a $G$-equivariant vector bundle over $X$ is a representation $\rho : G \to GL(V)$. 
3.4. **Equivariant Chern classes.** Let $\pi : V \to X$ be a $G$-equivariant vector bundle over a $G$-space $X$. Then $V_G$ is a complex vector bundle over $X_G$. The $k$-th $G$-equivariant Chern class of $V$ is defined to be the $k$-th Chern class of $V_G$:

$$(c_k)_G(V) := c_k(V_G) \in H^{2k}(X_G) = H^{2k}_G(X).$$

The $G$-equivariant Chern characters are defined similarly:

$$(\text{ch}_k)_G(V) := \text{ch}_k(V_G) \in H^{2k}_G(X).$$

The $G$-equivariant Euler class of $V$ is defined to be the Euler class (i.e. top Chern class) of $V_G$:

$$e_G(V) := e(V_G) = c_r(V_G) \in H^{2r}(X_G) = H^{2r}_G(X)$$

where $r = \text{rank}_C V$.

**Example 9.** For any $a \in \mathbb{Z}$, let $C_a$ be the 1-dimensional representation of $\mathbb{C}^*$ with character $t \mapsto t^a$. Then $C_a$ can be viewed as a $\mathbb{C}^*$-equivariant vector bundle over a point. We have

$$(C_a)_{\mathbb{C}^*} = \{(u, v) \in (\mathbb{C}^\infty - \{0\}) \times \mathbb{C} \}/t \cdot (u, v) \sim (t^{-1} u, t^a v) \cong \mathcal{O}_{\mathbb{CP}^\infty}(-a)

(c_1)_{\mathbb{C}^*}(C_a) = -au \in H^2_{\mathbb{C}^*}(pt) = \mathbb{Z}u.$$

3.5. **Atiyah-Bott localization formula.** Suppose that $T = (\mathbb{C}^*)^n$ acts on a complex manifold $X$, and that the fixed points set $X^T$ is a disjoint union of compact complex submanifolds $Z_1, \ldots, Z_N$ of $X$. Then the normal bundle $N_j$ of $Z_j$ in $X$ is a $T$-equivariant complex vector bundle over $Z_j$. The equivariant Euler class

$$e_T(N_j) \in H^*_T(Z_j) \cong H^*(Z_j) \otimes_{\mathbb{Q}} H^*(BT) = H^*(Z_j) \otimes \mathbb{Q}[u_1, \ldots, u_n]$$

is not a zero divisor. Let $R = \mathbb{Q}(u_1, \ldots, u_n)$. Then $e_T(N_j)$ is invertible in $H^*(Z_j) \otimes_{\mathbb{Q}} \mathbb{Q}(u_1, \ldots, u_n)$.

Recall that $\pi : X_T \to BT$ is a fibration with fiber $X$. When $X$ is compact, the fiber $\pi^{-1}(b)$ over a point $b \in BT$ represents a homology class $f \in H_{2n}(X_T)$, where $n = \dim_{\mathbb{C}} X$; this class is independent of choice of $b \in BT$. There is an additive map

$$\int_{X_T} : H^*_T(X) \to H^*_T(pt),$$

known as “integration along the fiber” or “push-forward to a point”; it sends $H^*_T(X)$ to $H_T^{2n-2n}(pt)$. (Intuitively, $\int_{X_T}$ is given by contraction with $f$.) Similarly, we have maps

$$\int_{(Z_j)_T} : H^*_T(Z_j) \to H^*_T(pt).$$
which send $H^q_T(Z_j)$ to $H^{q-2\dim_C Z_j}(pt)$. Here we will not give the precise definition of $\int_{X_T}$, but the maps $\int_{(Z_j)_T}$ can be described very explicitly, as follows. Any $\alpha \in H^*_T(Z_j) \cong H^*(Z_j) \otimes_\mathbb{Q} \mathbb{Q}[u_1, \ldots, u_n]$ is of the form

$$\alpha = \sum_{i=1}^m \alpha_i p_i$$

where $\alpha_i \in H^*(X)$ and $p_i \in \mathbb{Q}[u_1, \ldots, u_n]$. Then

$$\int_{(Z_j)_T} \alpha = \sum_{i=1}^m \langle [Z_j], \alpha_i \rangle p_i$$

where $[Z_j] \in H_{2\dim_C Z_j}(Z_j; \mathbb{Q})$ is the fundamental class, and $\langle , \rangle$ is the pairing between the homology $H_*(Z_j)$ and the cohomology $H^*(Z_j)$. We extend $\int_{(Z_j)_T}$ to

$$\int_{(Z_j)_T} : H^*_T(Z_j) \otimes_\mathbb{Q} \mathbb{Q}(u_1, \ldots, u_n) \to \mathbb{Q}(u_1, \ldots, u_n)$$

by taking $p_i \in \mathbb{Q}(u_1, \ldots, u_n)$.

**Theorem 10** (Atiyah-Bott localization formula [1])

$$\int_{X_T} \alpha = \sum_j \int_{(Z_j)_T} \frac{i_j^* \alpha}{c_T(N_j)}$$

where $i_j : Z_j \hookrightarrow X$ is the inclusion.

Let $X$ be a compact complex manifold with a holomorphic $T$-action. The constant map $X \to pt$ also induces an additive map between equivariant $K$-theories:

$$\pi ! : K_T(X) \to K_T(pt), \quad \mathcal{E} \mapsto \sum_i (-1)^i H^i(X, \mathcal{E})$$

where $\mathcal{E}$ is a $T$-equivariant holomorphic vector bundle over $X$, and $H^i(X, \mathcal{E})$ are the sheaf cohomology groups, which are representations of $T$.

A representation of $T$ is determined by its $T$-equivariant Chern character $ch_T$. We can compute $ch_T(\pi ! \mathcal{E})$ by Grothendieck-Riemann-Roch (GRR) theorem and the Atiyah-Bott localization formula. Applying GRR to the fibration $\pi : X_T \to BT$, we have

$$ch_T(\pi ! \mathcal{E}) = \int_{X_T} ch_T(\mathcal{E}) t_d T(TX)$$
where $\text{td}_T(TX)$ is the $T$-equivariant Todd class of the tangent bundle $TX$ of $X$. By localization,

$$
\int_{X_T} \text{ch}_T(\mathcal{E})\text{td}_T(TX) = \sum_{j=1}^N \int_{(Z_j)_T} i_j^* (\text{ch}_T(\mathcal{E})\text{td}_T(TX)) / e_T(N_j).
$$

We now specialize to the case where $Z_j$ are isolated points. We write $p_1, \ldots, p_N$ instead of $Z_1, \ldots, Z_N$. Let $m = \dim_{\mathbb{C}} X$, and let

$$
x_{j,1}, \ldots, x_{j,m} \in H^2_T(\text{pt}) = \bigoplus_{i=1}^n \mathbb{Q}u_i
$$

be the weights of the $T$-action on the tangent space $T_{p_j}X$ of $X$ at $p_j$. Then

$$
i_j^*\text{td}_T(TX) = \prod_{k=1}^m \frac{x_{j,k}}{1 - e^{-x_{j,k}}}, \quad e_T(N_j) = e_T(T_{p_j}X) = \prod_{k=1}^m x_{j,k}.
$$

Let $r = \text{rank}_{\mathbb{C}} \mathcal{E}$, and let

$$
y_{j,1}, \ldots, y_{j,r} \in H^2_T(\text{pt})
$$

be the weights of the $T$-action on the fiber $\mathcal{E}_{p_j}$ of $\mathcal{E}$ at $p_j$. Then

$$
i_j^*\text{ch}_T(\mathcal{E}) = \sum_{l=1}^r e^{y_{j,l}}.
$$

Therefore

$$
(11) \quad \text{ch}_T(\pi_! \mathcal{E}) = \sum_{j=1}^N \frac{\sum_{l=1}^r e^{y_{j,l}}}{\prod_{k=1}^m (1 - e^{-x_{j,k}})}.
$$

**Example 11.** Let $T = \mathbb{C}^*$ act on $\mathbb{P}^1$ by

$$
t \cdot [x, y] = [tx, y].
$$

The fixed points set consists of two isolated points $q^0 = [0, 1]$ and $q^1 = [1, 0]$. We have

$$
e_T(T_{q^0}\mathbb{P}^1) = u, \quad e_T(T_{q^1}\mathbb{P}^1) = -u.
$$

For an equivariant lifting of the line bundle $\mathcal{O}(k) \to \mathbb{P}^1$, we have

$$
e_T(\mathcal{O}(k)_{q^0}) = au, \quad e_T(\mathcal{O}(k)_{q^1}) = (a - k)u
$$

for some $a \in \mathbb{Z}$. 
By (11),

\[
\begin{align*}
\chi_T H^0(\mathbb{P}^1, \mathcal{O}(k)) - \chi_T H^1(\mathbb{P}^1, \mathcal{O}(k)) &= \frac{e^{au}}{1 - e^{-u}} + \frac{e^{(a-k)u}}{1 - e^u} = \frac{e^{au}(1 - e^{-(k+1)u})}{1 - e^{-u}} \\
&= \left\{ \begin{array}{ll}
\sum_{i=0}^{k} e^{(a-i)u}, & k \geq 0 \\
- \sum_{i=1}^{-k-1} e^{(a+i)u}, & k < 0
\end{array} \right.
\]

Indeed we have

\[
\begin{align*}
\chi_T H^0(\mathbb{P}^1, \mathcal{O}(k)) &= \left\{ \begin{array}{ll}
\sum_{i=0}^{k} e^{(a-i)u}, & k \geq 0 \\
0, & k < 0
\end{array} \right.
\] \\
\chi_T H^1(\mathbb{P}^1, \mathcal{O}(k)) &= \left\{ \begin{array}{ll}
0, & k \geq -1 \\
\sum_{i=1}^{-k-1} e^{(a+i)u}, & k < -1
\end{array} \right.
\]

Given \( a \in \mathbb{Z} \), let \( \mathbb{C}_a \) denote the irreducible representation of \( T = \mathbb{C}^* \) characterized by \( e_T(\mathbb{C}_a) = au \) (c.f. Example 9). Then

\[
\begin{align*}
H^0(\mathbb{P}^1, \mathcal{O}(k)) &= \left\{ \begin{array}{ll}
\oplus_{i=0}^{k} \mathbb{C}_{a-i}, & k \geq 0 \\
0, & k < 0
\end{array} \right.
\] \\
H^1(\mathbb{P}^1, \mathcal{O}(k)) &= \left\{ \begin{array}{ll}
0, & k \geq -1 \\
\oplus_{i=1}^{-k-1} \mathbb{C}_{a+i}, & k < -1
\end{array} \right.
\]

**Example 12.** This example will arise in the localization calculations in Section 4.4.

Let \( T = \mathbb{C}^* \) act on \( \mathbb{P}^1 \) and on \( \mathcal{O}(k) \) as in the previous example. Let \( f : C \cong \mathbb{P}^1 \to \mathbb{P}^1 \) be the degree \( d \) map given by \([u, v] \mapsto [u^d, v^d] \). Let \( p^0 = [0, 1], p^1 = [1, 0] \in C \). Then \( f(p^i) = q^i \) for \( i = 0, 1 \), and \( f^*T\mathbb{P}^1 \cong (TC)^{\otimes d} \). We have

\[
\begin{align*}
e_T(T_{p^0}C) &= \frac{1}{d} e_T(T_{q^0}\mathbb{P}^1) = \frac{u}{d}, \\
e_T(T_{p^1}C) &= \frac{1}{d} e_T(T_{q^1}\mathbb{P}^1) = \frac{-u}{d}, \\
e_T((f^*\mathcal{O}(k))|_{p^0}) &= e_T(\mathcal{O}(k)|_{q^0}) = au, \\
e_T((f^*\mathcal{O}(k))|_{p^1}) &= e_T(\mathcal{O}(k)|_{q^1}) = (a - k)u.
\]

By (11),
\[
\operatorname{ch}_TH^0(C, f^*O(k)) - \operatorname{ch}_TH^1(C, f^*O(k)) = \frac{e^{au} + e^{(a-k)u}}{1 - e^{-u/d}} - \frac{e^{au} - (kd+1)u/d}{1 - e^{-u/d}} \\
= \begin{cases} 
\sum_{i=0}^{kd} e^{au - \frac{iud}{d}}, & k \geq 0 \\
- \sum_{i=1}^{-kd-1} e^{au + \frac{iud}{d}}, & k < 0
\end{cases}
\]
Indeed we have
\[
\operatorname{ch}_TH^0(C, f^*O(k)) = \begin{cases} 
\sum_{i=0}^{kd} e^{au - \frac{iud}{d}}, & k \geq 0 \\
0, & k < 0
\end{cases}
\]
\[
\operatorname{ch}_TH^1(C, f^*O(k)) = \begin{cases} 
0, & k \geq 0 \\
\sum_{i=1}^{-kd-1} e^{au + \frac{iud}{d}}, & k < 0
\end{cases}
\]
Given \(a \in \mathbb{Z}\), let \(\mathcal{C}_{\frac{a}{d}}\) denote the orbibundle on \(\text{pt}/\mathbb{Z}_d\) characterized by \(e_T(\mathcal{C}_{\frac{a}{d}}) = \frac{au}{d}\). Then
\[
H^0(C, f^*O(k)) = \begin{cases} 
\bigoplus_{i=0}^{kd} \mathcal{C}_{\frac{ai}{d}}, & k \geq 0 \\
0, & k < 0
\end{cases}
\]
\[
H^1(C, f^*O(k)) = \begin{cases} 
0, & k \geq 0 \\
\bigoplus_{i=1}^{-kd-1} \mathcal{C}_{\frac{ai}{d}}, & k < 0
\end{cases}
\]

4. Proof of the ELSV Formula by Virtual Localization

In this section, we present the proof of the ELSV formula by virtual localization on moduli spaces of relative stable maps, following Graber-Vakil [14]. In Section 4.1 we define moduli spaces of relative stable maps \(\overline{M}_{g,0}(\mathbb{P}^1, \mu)\). In Section 4.2 we identify the Hurwitz number \(H_{g,\mu}\) with a top intersection on \(\overline{M}_{g,0}(\mathbb{P}^1, \mu)\), so that it can be viewed as a relative Gromov-Witten invariant for the pair \((\mathbb{P}^1, \infty)\). Section 4.3 contains the localization calculations (using the torus action introduced in Section 4.3) which yield the ELSV formula.

4.1. Moduli spaces. We fix a pair \((g, \mu)\), where \(g\) is a nonnegative integer (which will be the genus) and \(\mu = (\mu_1 \geq \cdots \geq \mu_h > 0)\) is a partition. Let \(\mathcal{M}_{g,0}(\mathbb{P}^1, \mu)\) be the moduli space of ramified covers
\[
f : (C, x_1, \ldots, x_h) \rightarrow (\mathbb{P}^1, q^1)
\]
of degree $d$ from a nonsingular complex algebraic curve (Riemann surface) $C$ of genus $g$ to $\mathbb{P}^1$ such that the ramification type over a distinguished point $q^1 = \infty \in \mathbb{P}^1$ is specified by the partition $\mu$, i.e.,

$$f^{-1}(q^1) = \mu_1 x_1 + \cdots + \mu_h x_h$$

as Cartier divisors. The moduli space $\mathcal{M}_{g,0}(\mathbb{P}^1, \mu)$ is not compact. To compactify it, we consider the moduli space

$$\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$$

of relative stable maps to $(\mathbb{P}^1, q^1)$. Moduli spaces of relative stable maps were first constructed in the symplectic category by A.Li-Ruan [29] and by Ionel-Parker [16, 17]; later J. Li constructed such moduli spaces in the algebraic category [26, 27]. We need to use the algebraic version constructed by J. Li because virtual localization on moduli spaces of relative stable maps has only been proved in the algebraic setting [13, 15]. Moduli spaces of relative stable maps are defined for a general pair $(X, D)$ where $X$ is a smooth projective variety, and $D$ is a smooth divisor in $X$. Here we content ourselves with the definition for the special case we need for the ELSV formula: $X = \mathbb{P}^1$ and $D$ is a point.

We use notation similar to that in [31] and [33]. Let $\mathbb{P}^1(m) = \mathbb{P}^1_1 \cup \cdots \cup \mathbb{P}^1_m$ be a chain of $m$ copies of $\mathbb{P}^1$. For $l = 1, \ldots, m - 1$, let $q^1_l$ be the node at which $\mathbb{P}^1_l$ and $\mathbb{P}^1_{l+1}$ intersect. Let $q^1_0 \in \mathbb{P}^1_l$ and $q^1_m \in \mathbb{P}^1_m$ be smooth points.

A point of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ is a morphism

$$f : (C, x_1, \ldots, x_h) \to (\mathbb{P}^1[m], q^1_m)$$

where $C$ is complex algebraic curve of (arithmetic) genus $g$, with at most nodal singularities, and $\mathbb{P}^1[m]$ is obtained by identifying $q^1 \in \mathbb{P}^1$ with $q^1_0 \in \mathbb{P}^1(m)$. In particular, $\mathbb{P}^1[0] = \mathbb{P}^1$. We call the original $\mathbb{P}^1 = \mathbb{P}^1_0$ the root component and $\mathbb{P}^1_1, \ldots, \mathbb{P}^1_m$ the bubble components. For $l = 0, \ldots, m$, let $C_l = f^{-1}(\mathbb{P}^1_l)$, so that $C = C_0 \cup C_1 \cup \cdots \cup C_m$, and let $f_l : C_l \to \mathbb{P}^1_l$ be the restriction of $f$. Then $f$ satisfies the following properties:

1. (degree) $\deg f_l = d$, for $l = 0, \ldots, m$.
2. (ramification) $f^{-1}(q^1_m) = \sum_{i=1}^h \mu_i x_i$ as Cartier divisors.
3. (predeformability) The preimage of each node of the target consists of nodes, at each of which two branches have the same contact order; distinct $C_i$ share no common irreducible components. This is the predeformable condition: so that one can smooth both the target and the domain to obtain a morphism to $\mathbb{P}^1$. 
(4) **(stability)** The automorphism group of $f$ is finite.

Two morphisms satisfying (1)–(3) are equivalent if (a) they have the same target $\mathbb{P}^1[m]$ for some nonnegative integer $m$, and (b) they differ by an isomorphism of the domain and an element of $\text{Aut}(\mathbb{P}^1(m), q^1_0, q^1_m) \cong (\mathbb{C}^*)^m$. In particular, this defines the automorphism group in (4). For fixed $g, \mu$, the stability condition (4) gives an upper bound of the number $m$ of bubble components of the target.

By the results in [26], $\overline{M}_{g,0}(\mathbb{P}^1, \mu)$ is a proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension $r := 2g - 2 + d + h$. Roughly speaking, this means that it is a compact, Hausdorff, singular orbifold, with a “virtual tangent bundle” of rank $r$ (over $\mathbb{C}$). It has a virtual fundamental class $[\overline{M}_{g,0}(\mathbb{P}^1, \mu)]_{\text{vir}} \in H_{2r}(\overline{M}_{g,0}(\mathbb{P}^1, \mu); \mathbb{Q})$.

4.2. **Branch morphism.** Given a point $[\tilde{f} : (C, x_1, \ldots, x_h) \to \mathbb{P}^1[m]] \in \overline{M}_{g,0}(\mathbb{P}^1, \mu)$, let $\tilde{f} = \pi_m \circ f : (C_1, x_1, \ldots, x_h) \to \mathbb{P}^1$, where $\pi_m : \mathbb{P}^1[m] \to \mathbb{P}^1$ is the map that contracts the bubble components. When $C$ is nonsingular (in this case we must have $m = 0$ and $\tilde{f} = f$), let $\text{Br}(\tilde{f})$ denote the branch divisor of $\tilde{f}$, namely,

$$\text{Br}(\tilde{f}) = \sum_{q \in \mathbb{P}^1} (d - \# \tilde{f}^{-1}(q))q$$

where $\# \tilde{f}^{-1}(q) \leq d$, and $\# \tilde{f}^{-1}(q) < d$ if and only if $q$ is a critical value of $\tilde{f}$. Then

$$\text{Br}(\tilde{f}) = b_1 + \cdots + b_r + (d - h)q^1$$

where $b_1, \ldots, b_r \in \mathbb{P}^1 - \{q^1\}$ (not necessarily distinct). In general $\text{Br}(\tilde{f}) = \text{Br}'(\tilde{f}) + (d - h)q^1$, where $\text{Br}'(\tilde{f})$ is an effective divisor on $\mathbb{P}^1$ of degree $r$ and can be viewed as a point in $\text{Sym}^r \mathbb{P}^1$, the $r$-th symmetric produce of $\mathbb{P}^1$; the map

$$\text{Br} : \overline{M}_{g,0}(\mathbb{P}^1, \mu) \to \text{Sym}^r \mathbb{P}^1 \cong \mathbb{P}^r$$

$$[f : (C, x_1, \ldots, x_h) \to \mathbb{P}^1[m]] \mapsto \text{Br}'(\tilde{f})$$

is a morphism, known as the branch morphism (c.f. Fantechi-Pandharipande [10]). The Hurwitz number $H_{g,\mu}$ is essentially the degree of the branch morphism $\text{Br} : \overline{M}_{g,0}(\mathbb{P}^1, \mu) \to \mathbb{P}^r$. More precisely, let $H \in H^2(\mathbb{P}^r; \mathbb{Z})$ be the hyperplane class, so that $H^*(\mathbb{P}^r; \mathbb{Z}) = \mathbb{Z}[H]/(H^{r+1})$. Then
$H^* \in H^{2r}(\mathbb{P}^r; \mathbb{Z})$ is the Poincaré dual of the point class $[pt] \in H_0(\mathbb{P}^r; \mathbb{Z})$, and $H^{2i}(\mathbb{P}^r; \mathbb{Q}) = \mathbb{Q}H^i$, $i = 1, \ldots, r$. We have

$$H_{g,\mu} = \frac{1}{\#\text{Aut}(\mu)} \deg \text{Br} = \frac{1}{\#\text{Aut}(\mu)} \int_{[\mathcal{M}_{g,0}(\mathbb{P}^1,\mu)]^\text{vir}} \text{Br}^* H^r$$

where

$$\left[\mathcal{M}_{g,0}(\mathbb{P}^1; \mu)\right]^\text{vir} \in H_{2r}(\mathcal{M}_{g,0}(\mathbb{P}^1,\mu); \mathbb{Q}), \quad \text{Br}^* H^r \in H^{2r}(\mathcal{M}_{g,0}(\mathbb{P}^1,\mu); \mathbb{Q}).$$

We will use virtual localization to compute the right hand side of (12) and obtain the right hand side of the ELSV formula (10).

4.3. Torus action. Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by $t \cdot [x, y] = [tx, ty]$ where $t \in \mathbb{C}^*$ and $[x, y] \in \mathbb{P}^1$. Then $\mathbb{C}^*$ acts on $\mathcal{M}_{g,0}(\mathbb{P}^1, \mu)$ and on $\text{Sym}^r \mathbb{P}^1 \cong \mathbb{P}^r$, and the branch morphism $\text{Br} : \mathcal{M}_{g,0}(\mathbb{P}^1, \mu) \to \mathbb{P}^r$ is $\mathbb{C}^*$-equivariant. The isomorphism $\mathbb{P}^r \xrightarrow{\cong} \text{Sym}^r(\mathbb{P}^1)$ is given by

$$[a_0, a_1, \ldots, a_r] \mapsto \text{div}(\sum_{i=0}^r a_i x^i y^{r-i})$$

where $[x, y]$ are homogeneous coordinates of $\mathbb{P}^1$, and $\text{div}(p(x, y))$ is the divisor defined by the equation $p(x, y) = 0$. The $\mathbb{C}^*$-action on $\mathbb{P}^r$ is given by $t \cdot [a_0, \ldots, a_r] = [a_0, t^{-1}a_1, \ldots, t^{-r}a_r]$ for $t \in \mathbb{C}^*$, $[a_0, \ldots, a_r] \in \mathbb{P}^r$. By Example 7, the $\mathbb{C}^*$-equivariant cohomology of $\mathbb{P}^r$ is given by

$$H^*(\mathbb{P}^r; \mathbb{Q}) = \mathbb{Q}[H, u]/(H(H - u) \cdots (H - ru)).$$

The $\mathbb{C}^*$-fixed points on $\mathbb{P}^r$ are

$$p_i = \text{div}(x^i y^{r-i}) = iq^0 + (r - i)q^1, \quad i = 0, \ldots, r.$$

4.4. Localization. We lift $H^* \in H^{2r}(\mathbb{P}^r; \mathbb{Q})$, the Poincaré dual of the point class $[pt] \in H_0(\mathbb{P}^r; \mathbb{Q})$, to $\prod_{i=0}^{r-1} (H - iu) \in H^*_{\mathbb{C}^*}(\mathbb{P}^r; \mathbb{Q})$, the $\mathbb{C}^*$-equivariant Poincaré dual of the fixed point $p_r \in \mathbb{P}^r$. Then

$$H_{g,\mu} = \frac{1}{|\text{Aut}(\mu)|} \int_{[\mathcal{M}_{g,0}(\mathbb{P}^1,\mu)]^\text{vir}} \text{Br}^* \prod_{i=0}^{r-1} (H - iu)$$

where the sum is over all connected components of the $\mathbb{C}^*$-fixed point set $\mathcal{M}_{g,0}(\mathbb{P}^1, \mu)^{\mathbb{C}^*}$, and $e_{\mathbb{C}^*}(N^\text{vir}_F)$ is the $\mathbb{C}^*$-equivariant Euler class of the virtual normal bundle $N^\text{vir}_F$ of the fixed locus $F$. If $\mathcal{M}_{g,0}(\mathbb{P}^1, \mu)$ were a compact complex manifold, and each $F$ were a compact complex submanifold of dimension $d_F$ then $[F]^\text{vir}$ would be the usual fundamental class $[F] \in H_{2d_F}(F; \mathbb{Q})$, $N^\text{vir}_F$ would be the usual normal bundle $N_F$ in
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\(\overline{M}_{g,0}(\mathbb{P}^1, \mu)\), and the second equality would be the Atiyah-Bott localization formula. Here we need to use the virtual localization formula proved by Graber-Pandharipande \cite{Graber}. By \cite{Kaid}, we may apply virtual localization to moduli space of relative maps \(\overline{M}_{g,0}(\mathbb{P}^1, \mu)\).

For each \(F\), we have \(\text{Br}(F) = p_i\) for some \(i \in \{0, 1, \ldots, r\}\). Note that \(H|_{p_i} = iu\). Let

\[F_i = \overline{M}_{g,0}(\mathbb{P}^1, \mu)^{C_i} \cap \text{Br}^{-1}(p_i), \quad i = 0, \ldots, r.\]

Then \(\overline{M}_{g,0}(\mathbb{P}^1, \mu)^{C_i} = F_0 \cup \cdots \cup F_r\), and

\[
\text{Br}^*(\prod_{j=0}^{r-1} (H - ju))|_{F_i} = \prod_{j=0}^{r-1} (H - ju)|_{p_i} = \begin{cases} 0, & 0 \leq i \leq r - 1, \\ r!u^r, & i = r. \end{cases}
\]

Therefore,

\[H_{g, \mu} = \frac{1}{|\text{Aut}(\mu)|} \int_{[F_r]_{\text{vir}}} \frac{r!u^r}{C^*_{\text{vir}}(N_{F_r}^{\text{vir}})}.\]

We need to identify the substack \(F_r\), and its virtual tangent bundle \(T_{F_r}^{\text{vir}}\) and virtual normal bundle \(N_{F_r}^{\text{vir}}\).

Let \(f : (C, x_1, \ldots, x_h) \to (\mathbb{P}^1[1], q_0^1)\) be a relative stable map which represents a point in \(F_r\). One can show that \(f\) must be of the following form.

(1) \(m = 0\), so \(C = C_0\).
(2) \(D_0 := f^{-1}(q^0)\) is a curve of arithmetic genus \(g\).
(3) \(f^{-1}(q^1) = \{x_1, \ldots, x_h\}\).
(4) \(f^{-1}(\mathbb{P}^1 \setminus \{q^0, q^1\})\) is a disjoint union of of twice punctured spheres \(L_1, \ldots, L_h\), and \(f|_{L_i} : L_i \to \mathbb{P}^1 \setminus \{q^0, q^1\}\) is an honest covering map of degree \(\mu_i\).
(5) For \(i = 1, \ldots, h\), let \(D_i\) be the closure of \(L_i\) in \(C\). Then \(D_i = L_i \cup \{y_i, x_i\} \cong \mathbb{P}^1\), where \(x_i\) is the \(i\)-th marked point, and \(y_i\) is a node at which \(D_i\) intersects \(D_0\). \(\hat{f}_i := f|_{D_i} : D_i \to \mathbb{P}^1\) is fully ramified over \(q^0\) and \(q^1\).
(6) \((D_0, y_1, \ldots, y_h)\) is an \(h\) pointed, genus \(g\) stable curve, so it represents a point in \(\overline{M}_{g,h}\).

We have

\[
\text{Aut}(f) = \prod_{i=1}^{h} \text{Aut} \left(\hat{f}_i : (D_i, y_i, x_i) \to (\mathbb{P}^1, q^0, q^1)\right) = \prod_{i=1}^{h} \mathbb{Z}_{\mu_i}
\]

where \(\mathbb{Z}_{\mu_i}\) is the cyclic group of order \(\mu_i\). Note that \(\hat{f}_i : (D_i, y_i, x_i) \to (\mathbb{P}^1, q^0, q^1)\) are the same over \(F_r\) while \((D_0, y_1, \ldots, y_h)\) can be any point
in $\overline{M}_{g,h}$. We have a surjective morphism

$$\iota : \overline{M}_{g,h} \to F_r = \overline{M}_{g,h}/\prod_{i=1}^{h} \mathbb{Z}_{\mu_i} \cong \overline{M}_{g,h} \times \prod_{i=1}^{h} B\mathbb{Z}_{\mu_i}.$$  

To simplify the notation, we write $(C, x)$ instead of $(C, x_1, \ldots, x_h)$, and write $(C, x, f)$ instead of $f : (C, x_1, \ldots, x_n) \to \mathbb{P}^1$. Let $D_0, D_1, \ldots, D_h$ and $\hat{f}_i := f|_{D_i}$ be defined as above, and write $(D_0, y)$ instead of $(D_0, y_1, \ldots, y_h)$.

The tangent space $T^1_{(C, x, f)}$ and the obstruction space $T^2_{(C, x, f)}$ at a moduli point $[(C, x, f)] \in F_r$ fit in the following long exact sequence of $\mathbb{C}^*$-representations:

$$0 \to \text{Aut}(C, x) \to \text{Def}(f) \to T^1_{(C, x, f)} \to \text{Def}(C, x) \to \text{Obs}(f) \to T^2_{(C, x, f)} \to 0.  \tag{13}$$

- $\text{Aut}(C, x) = \text{Ext}^0(\Omega_C(\sum_{i=1}^{h} x_i), \mathcal{O}_C)$ is the space of infinitesimal automorphism of the domain $(C, x)$. We have

$$\text{Aut}(C, x) = \text{Aut}(D_0, y) \oplus \bigoplus_{i=1}^{h} \text{Aut}(D_i, y_i, x_i),$$

where $\text{Aut}(D_0, y) = 0$ since $(D_0, y)$ is stable, and $\text{Aut}(D_i, y_i, x_i) = H^0(\mathbb{P}^1, T_{\mathbb{P}^1}(-0-\infty)) \cong \mathbb{C}_0$ (trivial 1-dimensional representation of $\mathbb{C}^*$).

- $\text{Def}(C, x) = \text{Ext}^1(\Omega_C(-\sum_{i=1}^{h} x_i), \mathcal{O}_C)$ is the space of infinitesimal deformation of the domain $(C, x)$. We have a short exact sequence of $\mathbb{C}^*$-representations:

$$0 \to \text{Def}(D_0, y) \to \text{Def}(C, x) \to \bigoplus_{i=1}^{h} T_{y_i} D_0 \otimes T_{y_i} D_i \to 0$$

where $\text{Def}(D_0, y) = T_{(D_0, y)} \overline{M}_{g,h}$.

- $\text{Def}(f) = H^0(C, f^*(T_{\mathbb{P}^1}(-q^1)))$ is the space of infinitesimal deformation of the map $f$, and

- $\text{Obs}(f) = H^1(C, f^*(T_{\mathbb{P}^1}(-q^1)))$ is the space of obstruction to deforming $f$.

For $i = 1, 2$, let $T^i_{\text{v}}$ and $T^i_{\text{m}}$ be the fixed and moving parts of $T^i|_{F_r}$. Then

$$T^1 = T^1_{\text{v}} + T^2_{\text{m}}, \quad T^2 = T^2_{\text{v}} + T^2_{\text{m}}.$$  

The virtual tangent bundle of $F_r$ is $T^\text{vir}_{F_r} = T^1_{\text{v}} - T^2_{\text{v}}$ and the virtual normal bundle of $F$ is $N^\text{vir}_{F_r} = T^1_{\text{m}} - T^2_{\text{m}}$. Let

$$B_1 = \text{Aut}(C, x), \quad B_2 = \text{Def}(f), \quad B_4 = \text{Def}(C, x), \quad B_5 = \text{Obs}(f),$$
and let $B^f_i$ and $B^m_i$ be the fixed and moving parts of $B_i$. Then

\[ T_{F_r}^{vir} = T_1^f - T_2^f, \]

\[
\frac{1}{e_{C^*}(N_{F_r}^{vir})} = \frac{e_{C^*}(B^m_2^i) e_{C^*}(B^m_4^i)}{e_{C^*}(B^m_2^i) e_{C^*}(B^m_4^i)}
\]

We have

\[
B^f_1 = \mathbb{C}_{\oplus h}, \quad B^m_1 = 0,
\]

\[
B^f_4 = T_{(D_0, y)} \overline{M}_{g, h}, \quad B^m_4 = \bigoplus_{i=1}^h T_y D_0 \otimes T_y D_i = \bigoplus_{i=1}^h (L^y_i)(c_{0, y}) \otimes \mathbb{C}_{\mu_i}.
\]

Let \( \iota^* : H^*(F_r) \otimes \mathbb{Q}(u) \rightarrow H^*(\overline{M}_{g, h}) \otimes \mathbb{Q}(u) \) be induced by the surjective morphism \( i : \overline{M}_{g, h} \rightarrow F_r \). Then

\[
\iota^* \left( \frac{e_{C^*}(B^m_1^i)}{e_{C^*}(B^m_4^i)} \right) = \frac{1}{\prod_{i=1}^h \left( \frac{u_{\mu_i} - \psi_i}{\mu_i} \right)} = \prod_{i=1}^h (u - \mu_i \psi_i).
\]

For \( k = 0, 1 \) and \( i = 0, 1, \ldots, h \), let

\[
H^k(D_i) = H^k \left( D_i, \hat{f}_i^* (T_{\mathbb{P}^1 1} (-q^1)) \right).
\]

Then we have a long exact sequence of \( \mathbb{C}^* \)-representations:

\[
\begin{array}{c}
0 \\
B_2 \rightarrow H^0(D_0) \oplus \bigoplus_{i=1}^h H^0(D_i) \rightarrow (T_{\mathbb{P}^1 1})^{\oplus h} \rightarrow B_5 \rightarrow H^1(D_0) \oplus \bigoplus_{i=1}^h H^1(D_i) \rightarrow 0.
\end{array}
\]

We have

\[
H^k(D_0) \cong H^k(D_0) \otimes T_{\mathbb{P}^1 1} = \begin{cases} 
\mathbb{C}_1, & k = 0; \\
\mathbb{C}_{(E^y)(D_0, y)} \otimes \mathbb{C}_1, & k = 1.
\end{cases}
\]

Note that

\[
T_{\mathbb{P}^1 1} (-q^1) \cong O_{\mathbb{P}^1 1}, \quad e_{C^*} ((T_{\mathbb{P}^1 1} (-q^1))_{\phi}) = u.
\]

By Example 12, for \( i = 1, \ldots, h \), we have

\[
H^k(D_i) \cong H^k(D_i, \hat{f}_i^* O(1)) = \begin{cases} 
\mathbb{C} \oplus_{a=0}^h \mathbb{C} \mu_i, & k = 0; \\
0, & k = 1.
\end{cases}
\]

Therefore,

\[
B^f_2 = \mathbb{C}^{\oplus h}, \quad B^f_5 = 0, \quad B^m_5 - B^m_2 = \mathbb{C}_{(D_0, y)} \otimes \mathbb{C}_1 \oplus \mathbb{C}_1^{\oplus (h-1)} \bigoplus_{i=1}^h \bigoplus_{a=1}^h \mathbb{C} \mu_i.
\]
So
\[ \iota^* \left( \frac{e_{\mathbb{C}^*}(B_{5}^{\mu})}{e_{\mathbb{C}^*}(B_{2}^{\mu})} \right) = \prod_{i=1}^{h} \frac{\mu_i^{\mu_i}}{\mu_1!} u^{h-d-1} \cdot \Lambda_g^+(u), \]
where
\[ \Lambda_g^+(u) = \sum_{i=0}^{g} (-1)^i \lambda_i u^{g-i}. \]

We conclude that
\[ T_{(C,x,f)}^{1,f} = T_{(D_0,y)}(\overline{\mathcal{M}}_{g,h}, T^{2,f} = 0, \]
so
\[ [F_r]^\text{vir} = \frac{1}{\mu_1 \cdots \mu_h} \iota_* [\overline{\mathcal{M}}_{g,h}] \]
where \( \iota_* : H_{2r}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}) \to H_{2r}(F_r; \mathbb{Q}) \). We also have
\[ \frac{1}{\iota^* e_{\mathbb{C}^*}(N^\text{vir}_{F_r})} = \prod_{i=1}^{h} \frac{\mu_i^{\mu_i}}{\mu_i!} \cdot \frac{\mu_1 \cdots \mu_h \Lambda_g^+(u) u^{h-d-1}}{\prod_{i=1}^{h} (u - \mu_i \psi_i)}. \]

\[ H_{g,\mu} = \frac{1}{\# \text{Aut}(\mu)} \int_{[F_r]^\text{vir}} e_{\mathbb{C}^*}(N^\text{vir}_{F_r}) \frac{r! u^r}{\mu_1 \cdots \mu_h \cdot \# \text{Aut}(\mu) \int_{\overline{\mathcal{M}}_{g,h}} e_{\mathbb{C}^*}(N^\text{vir}_{F_r}) u^r} \]
\[ = \frac{r!}{\# \text{Aut}(\mu)} \prod_{i=1}^{h} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,h}} \Lambda_g^+(u) u^{2g-3+2h} \cdot \prod_{i=1}^{h} (u - \mu_i \psi_i) \]
\[ = \frac{r!}{\# \text{Aut}(\mu)} \prod_{i=1}^{h} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,h}} \Lambda_g^+(1) \prod_{i=1}^{h} (1 - \mu_i \psi_i). \]

In the last equality, we may let \( u = a \), where \( a \) is any nonzero constant rational number; the answer is independent of \( a \). This completes the proof of the ELSV formula (10).

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