Weak limits of entropy regularized Optimal Transport: potentials, plans and divergences

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Abstract: This work deals with the asymptotic distribution of both potentials and couplings of entropic regularized optimal transport for compactly supported probabilities in $\mathbb{R}^d$. We first provide the central limit theorem of the Sinkhorn potentials—the solutions of the dual problem—as a Gaussian process in $C^s(\Omega)$. Then we obtain the weak limits of the couplings—the solutions of the primal problem—evaluated on integrable functions, proving a conjecture of Harchaoui, Liu, and Pal (2020). Finally we consider the weak limit of the entropic Sinkhorn divergence between $P$ and $Q$ under both assumptions $H_0 : P = Q$ or $H_1 : P \neq Q$. Under $H_0$ the limit is a quadratic form applied to a Gaussian process in a Sobolev space, while under $H_1$, the limit is Gaussian. We provide also a different characterisation of the limit under $H_0$ in terms of an i.i.d. sequence of standard Gaussian random variables. Such results enable statistical inference based on entropic regularized optimal transport.

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1. Introduction

Optimal transport has proven its effectiveness as a powerful tool in statistical data analysis. Formulated as a minimization problem, it reads

$$T_\mathcal{E}(P, Q) = \min_{\pi \in \Pi(P, Q)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2} \|x - y\|^2 d\pi(x, y),$$

where $\Pi(P, Q)$ denotes the set of couplings between the probabilities $P$ and $Q$. Optimal transport provides a notion of discrepancy between distributions [37, Chapter 7] useful for testing similarity between probabilities [8, 9, 24, 13] and making inference. For this, it is necessary to know the weak limit of $T_\mathcal{E}(P_n, Q_m)$ when the empirical measures $P_n$ and $Q_m$ are used in the place of the population ones, $P$ and $Q$. Unfortunately, in general dimension, with the exception of perhaps a few simplified cases [10, 30]—the limit is unknown. Moreover, the rate is slower than the usual parametric rate $n^{-\frac{1}{2}}$ [38, 18]. This motivates the use of regularization methods for optimal transport, since they are not affected by the curse of dimension, such as entropic regularization [5], or simplified versions, such as sliced optimal transport [33]. Regularized optimal transport is now used for many practical applications such as domain adaptation [4], counterfactual explanations [6], music transcription [17], diffeomorphic registration [15, 7] and measure colocalization in super-resolution images [30].

Thought weak limits are known for the optimal value in the regularized optimal transport problem, less is known about the distributional limits of the optimizers themselves. In this paper we provide the limits of the empirical solutions of the primal problem (plans/couplings), the dual problem (potentials), and the celebrated Sinkhorn divergence [20].

Let $\Omega \subset \mathbb{R}^d$ be a compact set. The entropic regularized optimal transport cost between two probability measures $P, Q \in \mathcal{P}(\Omega)$ is defined as the solution to the optimization problem

$$S_\epsilon(P, Q) = \min_{\pi \in \Pi(P, Q)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2} \|x - y\|^2 d\pi(x, y) + \epsilon H(\pi | P \otimes Q),$$

(1)
where the relative entropy between two probability measures $\alpha$ and $\beta$ is written as $H(\alpha \mid \beta) = \int \log(\frac{d\alpha}{d\beta})(x)\,d\alpha(x)$ if $\alpha$ is absolutely continuous with respect to $\beta$, $\alpha \ll \beta$, and $+\infty$ otherwise. We denote the solution of (1) by $\pi_{P,Q}$. This problem can also be written in its dual formulation

$$S_\epsilon(P,Q) = \sup_{f \in L_1(P)} \int f(x)dP(x) + \int g(y)dQ(y) - \epsilon \int e^{\frac{|f(x)+g(y)-\frac{1}{2}|x-y|^2}{\epsilon}}dP(x)dQ(y) + \epsilon. \tag{2}$$

There exists a unique pair $(f_{P,Q}, g_{P,Q})$ of solutions of the above optimization problem such that

$$\int g_{P,Q}(x)dQ(y) = 0. \tag{3}$$

Moreover, it holds that $\pi_{P,Q} = \xi_{P,Q}d(P \otimes Q)$ with

$$\xi_{P,Q}(x,y) = e^{\frac{f_{P,Q}(x)+g_{P,Q}(y)-\frac{1}{2}|x-y|^2}{\epsilon}}. \tag{4}$$

It can be shown that this optimality condition implies the unique extension of the solutions of (4) to the space $C^\alpha(\Omega) \times C^\alpha(\Omega)$, given by the relations

$$f_{P,Q} = \epsilon \log \left( \int e^{\frac{g_{P,Q}(y)-\frac{1}{2}|y|^2}{\epsilon}}dQ(y) \right) \text{ and } g_{P,Q} = \epsilon \log \left( \int e^{\frac{f_{P,Q}(x)-\frac{1}{2}|x|^2}{\epsilon}}dP(x) \right). \tag{5}$$

These relations have an important consequence—the optimization class can be reduced from $L_1(\Omega)$ to $C^\alpha(\Omega)$, with, moreover, uniformly (for all $P, Q \in \mathcal{P}(\Omega)$) bounded derivatives, see [19, 31]. Since the class $C^\alpha(\Omega)$, with an appropriate choice of $s$, is uniformly Donsker (eg. Section 2.7.1. in [36]), one can obtain the bound $\sqrt{n}E[\|S_\epsilon(P_n,Q) - S_\epsilon(P,Q)\|] \leq C_0$, where the constant $C_0$ depends polynomially on $\text{diam}(\Omega)$, see [31]. Moreover, since the convergence of $\mathbb{E}[S_\epsilon(P_n,Q)]$ towards its population counterpart occurs at a faster rate (see, e.g., [12]) and the fluctuations are asymptotically Gaussian ([14, 31, 11]), the weak limit

$$\sqrt{n}(S_\epsilon(P_n,Q) - S_\epsilon(P,Q)) \xrightarrow{w} \mathcal{N}(0, \text{Var}_{\mathbf{X} \sim P}(f_{P,Q}(\mathbf{X}))) \tag{6}$$

holds. Though (6) gives a weak limit for $S_\epsilon(P_n,Q)$, it does not give information about the coupling $\pi_{P,Q}$ or the optimal dual variables $(f_{P,Q}, g_{P,Q})$.

For statistical inference, obtaining the asymptotic behaviour of optimal regularized couplings themselves, i.e., the limits of

$$\sqrt{n}\left( \int \frac{m}{m+n} \eta(d\pi_{P_n,Q_m} - d\pi_{P,Q}) \right), \quad \eta \in L^\infty(P \otimes Q), \tag{7}$$

is highly desirable. Indeed, in many of the previously cited applications, the optimal coupling itself is the object of interest. Obtaining the limit of (7) would allow the statistician to obtain consistent confidence intervals for inference on the coupling. With regards to the limit of (7), the recent work [28] studied a modified regularization procedure inspired by Schrodinger’s lazy gas experiment giving rise to a different empirical estimator $\pi_{P_n,Q_m}$. That work showed that if $\frac{m}{n+m} \to \lambda \in (0,1)$, then $\pi_{P_n,Q_m}$ satisfies

$$\sqrt{n}\left( \int \frac{m}{m+n} \eta(d\pi_{P_n,Q_m} - d\pi_{P,Q}) \right) \xrightarrow{w} \mathcal{N}(0, \sigma_{\lambda,\epsilon}^2(\eta)), \quad \eta \in L^2(P \otimes Q), \tag{8}$$

where the variance $\sigma_{\lambda,\epsilon}^2(\eta)$ is

$$\lambda \text{Var}_{\mathbf{X} \sim P}((1 - A_{\mathbf{Q}}A_{\mathbf{P}})^{-1}(\eta_x - A_{\mathbf{Q}}\eta_y)(\mathbf{X})) + (1-\lambda) \text{Var}_{\mathbf{Y} \sim Q}((1 - A_{\mathbf{P}}A_{\mathbf{Q}})^{-1}(\eta_y - A_{\mathbf{P}}\eta_x)(\mathbf{Y})),$$
see section 2 for the precise definitions of the operators \( A_P, A_Q \) and section 3 for the ones of \( \eta_x, \eta_y \). Moreover, the authors of [28] conjectured that the distributional limit (8) holds also for the classic Sinkhorn regularization. [27] proved that (7) is tight and the limit is centered, however the conjecture remained opened. In Theorem 3.1 we prove that (8) holds for compactly supported measures, and therefore the conjecture of [28] is true.

Theorem 3.1 is derived as a consequence of the first-order linearization of the potentials, described in Theorem 2.2, whose proof is based on a reformulation of the optimality conditions (5) as a Z-estimation problem (see for instance [36]). Differentiating in the Fréchet sense the objective function and using the uniform bounds provided by [12], the problem is reduced to the continuity and existence of the following operator in \( C^0(\Omega) \times C^0(\Omega) \)

\[
\begin{pmatrix}
(1 - A_Q \hat{A}_P)^{-1} & -(1 - A_Q \hat{A}_P)^{-1} A_Q \\
-\hat{A}_P(1 - A_Q \hat{A}_P)^{-1} & (1 - \hat{A}_P A_Q)^{-1}
\end{pmatrix},
\]

which follows from Fredholm alternative [2, Theorem 6.6] (note that a similar result of invertibility of these operators between different Banach spaces has been obtained by [3]). As a consequence, Theorem 2.2 yields the limits, if \( m = m(n) \to \infty \) and \( \frac{m}{n+m} \to \lambda \in (0,1) \),

\[
\sqrt{\frac{n}{n+m}} \left( \frac{f_{P_n, Q_m} - f_{P, Q}}{g_{P_n, Q_m} - g_{P, Q}} \right) \quad \overset{w}{\to} \quad \left( \sqrt{1 - A_Q A_P^{-1}} A_Q k_{P, Q} G_P - \sqrt{1 - \lambda(1 - A_Q A_P^{-1}) k_{P, Q}} G_Q \right) / \left( \sqrt{1 - \lambda A_P(1 - A_Q A_P^{-1}) k_{P, Q}} G_P - \sqrt{1 - \lambda(1 - A_P A_Q^{-1}) k_{P, Q}} G_Q \right),
\]

weakly in \( C^0(\Omega) \times C^0(\Omega) \), where \( G_P \) and \( G_Q \) are independent P and Q-Brownian bridges. Moreover, in the one-sample case,

\[
\sqrt{n} \left( \frac{f_{P_n, Q} - f_{P, Q}}{g_{P_n, Q} - g_{P, Q}} \right) \quad \overset{w}{\to} \quad \left( (1 - A_Q A_P^{-1}) A_Q G_{P, s}^{-1} - (1 - A_P A_Q^{-1}) G_{P, s}^{-1} \right),
\]

weakly in \( C^0(\Omega) \times C^0(\Omega) \). Theorem 2.2, apart from being interesting in itself, has many applications since, among other things, the derivative of \( f_{P, Q} \) may be used to estimate the transport map from \( P \) to \( Q \) (eg. [32, 34]), which is also a useful tool for inference.

The regularized transport cost is easier to compute than the usual optimal transport cost but is unsuitable for two-sample testing, since \( S_r(P, P) \neq 0 \). In [20], the authors propose to remedy this deficiency by defining the quadratic Sinkhorn divergence:

\[
D_r(P, Q) = \frac{1}{2} \left( \frac{1}{n} \sum_{i=1}^n (f_{P_i, Q_i} - f_{P, Q})^2 \right) \quad \text{where} \quad D_r(P, Q) = 0 \quad \text{if and only if} \quad P = Q.
\]

Under \( H_1 \) the limit can be derived by means of Efron-Stein linearization [14, 31, 11, 24], giving

\[
\sqrt{n} \left( D_r(P_n, Q_m) - D_r(P, Q) \right) \quad \overset{w}{\to} \quad \mathcal{N}(0, \lambda \text{Var}_P(\psi_{P, Q}) + (1 - \lambda) \text{Var}_Q(\psi_{Q, P})),
\]

and

\[
\sqrt{m} \left( D_r(P_n, Q) - D_r(P, Q) \right) \quad \overset{w}{\to} \quad \mathcal{N}(0, \text{Var}_P(\psi_{P, Q})),
\]

where \( \psi_{P, Q} = f_{P, Q} - \frac{1}{2} (f_{P^1, P} + g_{P, P}) \) and \( \psi_{Q, P} = f_{Q, P} - \frac{1}{2} (f_{Q, Q^1} + g_{Q, Q}) \). Under \( H_0 \), however, \( \psi_{P, Q} = \psi_{Q, P} = 0 \), so the limit is degenerate, and in fact \( D_r(P_n, Q) = \mathcal{O}(\frac{1}{n}) \). To obtain a non-trivial limit, we therefore conduct a second order analysis. The limit, in this case, behaves as \( n D_1(P_n, P) \approx \frac{1}{2} \lambda \sum_{j=1}^N 1_j N_j^2 \) where \( \{N_i\}_{i \in \mathbb{N}} \) is a sequence of i.i.d. random variables with \( N_i \sim \mathcal{N}(0,1) \) and \( \{ \lambda_j \}_{j \in \mathbb{N}} \subset [0, \infty) \) is a square-summable sequence depending on \( P \) and \( \epsilon \). A similar expression was previously known to hold only in the case where \( P \) is discrete [1]. Extending those results to the general case is not a trivial process (cf. [22, Remark 8]), and verifying the necessary conditions to obtain this limit is the most technical part of our argument.
1.1. State-of-art

In recent years there has been a substantial body of work studying the weak limits of the optimal transport problem. Since, for obvious reasons we cannot cite every one of them, we refer to [29] for a comprehensive survey. Focusing on entropic regularized optimal transport, similar results for finitely supported measures are obtained by [30] and [1]. The limits of the regularized cost (6) has been proven first in [12] and [31]—using the Efron-Stein linearization—and then in [23]—using Hadamard linearization.

The conjecture of [28] has been also investigated in [27], under slightly different assumptions. That work shows the existence of a weak limit for (7), which they conjecture is Gaussian. We prove their conjecture. They also derive a similar result to Theorem 2.2, but in a weaker norm. We prove convergence in the space $C^s(\Omega) \times C^s_0(\Omega)$, which allows us to derive the limit of the Sinkhorn divergence.

Concurrently and independently of our work, [22] derive also the limits of the Sinkhorn optimal transport potentials potentials and divergences. In that work the first order development is done by means of the functional Hadamard differentiability of the potentials. It is worth mentioning that since a preprint of this work first appeared online, [25] has generalized our results for non-smooth bounded costs. To obtain this generalization [25] linearizes the potentials in the space $L^2(P_n)$ instead of in $C^s(\Omega)$. This allows them to avoid suprema over function classes, which, for non-smooth costs, are not Donkser.

1.2. Outline of the paper

The paper is organized as follows. The notation is given in Section 1.3. The central limit theorem for the regularized optimal transport potentials and its proof can be found in Section 2; however, auxiliary Lemmas are proven in Section C. In Section 3 we enunciate and prove the central limit theorem for the couplings, Theorem 3.1, and its immediate consequence, Corollary 3.2. Section 4 deals with the weak limits of the Sinkhorn divergence, formally stated in Theorems 4.1 and 4.4. Also in this section the reader can find a discussion about the simplification of the limit by embedding it into a Hilbert space and the proof of the main result. Auxiliary results and their proofs are postponed to Section B.

1.3. Notations

For the reader's convenience, this section sets the notation used throughout this work. Unless otherwise specified, the probabilities $P$ and $Q$ are supported in the compact set $\Omega \subset \mathbb{R}^d$, meaning that $P, Q \in \mathcal{P}(\Omega)$. In the proofs we will use the notations

$$
\begin{align*}
 f_{n,m}(x) &= \frac{f_{P_n,Q_m}(x)}{\epsilon}, & g_{n,m}(y) &= \frac{g_{P_n,Q_m}(y)}{\epsilon}, & C(x,y) &= e^{-\frac{1}{2\epsilon}||x-y||^2}, \\
 f_*(x) &= \frac{f_{P,Q}(x)}{\epsilon}, & g_*(y) &= \frac{g_{P,Q}(y)}{\epsilon},
\end{align*}
$$

(9)

and

$$
h_{n,m}(x,y) = f_{n,m}(x) + g_{n,m}(y), & h_*(x,y) = f_*(x) + g_*(y)
$$

to reduce the size of the displays. For two probabilities $\mu, \nu \in \mathcal{P}(\Omega)$ the pair $(f_{\mu,\nu}, g_{\mu,\nu})$ is one solving (2) for $\mu, \nu$, its direct sum $(x,y) \mapsto f_{\mu,\nu}(x) + g_{\mu,\nu}(y)$ is denoted by $h_{\mu,\nu}$. The solution of (1) is denoted by $h_{\mu,\nu}$ and its density with respect to the direct product measure $\mu \otimes \nu$ by $e_{\mu,\nu}^0$.

We set $s = \left\lceil \frac{d}{2} \right\rceil + 1$, and denote, for any $\alpha \in \mathbb{N}$, the space of all functions on $\Omega$ that possess uniformly bounded partial derivatives up to order $\alpha$ as $\mathcal{C}^\alpha(\Omega)$, in which we consider the norm

$$
\|f\|_{\mathcal{C}^\alpha(\Omega)} = \|f\|_{\alpha} = \sum_{i=0}^{\alpha} \sum_{|\beta|=i} \|D^\beta f\|_\infty.
$$

In the product space $\mathcal{C}^\alpha(\Omega) \times \mathcal{C}^\alpha(\Omega)$ we consider the norm

$$
\|(f,g)\|_{\mathcal{C}^\alpha(\Omega) \times \mathcal{C}^\alpha(\Omega)} = \|(f,g)\|_{\alpha \times \alpha} = \|f\|_{\alpha} + \|g\|_{\alpha}.
$$
We define the spaces
\[ C_0^\alpha(\Omega) = \{ f \in C^\alpha(\Omega) : \text{Q}(f) = 0 \} \quad \text{and} \quad L_0^2(\Omega) = \left\{ f \in L^2(\Omega) : \int f dQ = 0 \right\}. \]

Moreover, for a general Banach space \( H \), the norm is denoted as \( \| \cdot \|_H \). The operator norm of an operator \( F : H_1 \to H_2 \) is denoted as \( \| F \|_{H_1 \to H_2} \). Unless otherwise stated, the random vectors \( \mathbf{X} \) and \( \mathbf{Y} \) are independent and follow respectively the distributions \( P \) and \( Q \). For a measurable function \( f : \Omega \to \mathbb{R} \), the following expressions are equivalent:
\[
\mathbb{E}[f(\mathbf{X})] = \mathbb{E}_{\mathbf{X} \sim P}[f(\mathbf{X})] = \int f(x) dP(x) = \int f dP = P(f).
\]

The function \( y \mapsto \int f(x, y) dP(x) \) is denoted by \( \int f(x, \cdot) dP(x) \). Given a random sequence \( \{w_n\}_{n \in \mathbb{N}} \subset \mathcal{H} \) and a real random sequence \( \{a_n\}_{n \in \mathbb{N}} \), the notation \( w_n = o_P(a_n) \) means that the sequence \( w_n/a_n \) tends to 0 in probability, and \( w_n = O_P(a_n) \) that \( w_n/a_n \) is stochastically bounded, for further details see Section 2.2 in [35]. Finally, for a Borel measure \( \mu \) in \( \mathbb{R}^d \), \( L^2(\mu) \) denotes the space of square integrable functions. The special case of the Lebesgue measure \( \ell_2 \) in \( \Omega \) is denoted by \( L^2(\Omega) \). Probability measures are sometimes viewed as elements of the dual space of \( C^\alpha(\Omega) \), namely \( (C^\alpha(\Omega))' \), with the standard dual norm \( \|P\|_\alpha' = \sup_{\|f\|_\alpha \leq 1} |P(f)| \).

2. Central Limit Theorem of Sinkhorn potentials

The main result of this section is Theorem 2.2, which gives the first-order linearization of the difference between the empirical and population Sinkhorn potentials. As a consequence, Corollary 2.4 gives the central limit theorem of that difference with rate \( \sqrt{\frac{n}{n+m}} \). This section contains also the proof of Theorem 2.2 and Corollary 2.4. The proofs of auxiliary results are postponed to the Appendix.

Set two probabilities \( P, Q \in \mathcal{P}(\Omega) \), recall that \( \xi_{P, Q} \) is the density of \( \pi_{P, Q} \) w.r.t to \( P \otimes Q \) and define the operators
\[
\mathcal{A}_P : L^2(P) \ni f \mapsto \int \xi_{P, Q}(x, \cdot) f(x) dP(x) \in C^\alpha(\Omega),
\]
\[
\mathcal{A}_Q : L_0^2(Q) \ni g \mapsto \int \xi_{P, Q}(\cdot, y) g(y) dQ(y) \in C^\alpha(\Omega),
\]
\[
\tilde{\mathcal{A}}_P : L_0^2(P) \ni f \mapsto \mathcal{A}_P f - \int \mathcal{A}_P f(x) dQ(y) \in C^\alpha(\Omega)
\]
and, for a function \( h \in C^\infty(\Omega^2) \),
\[
i_h : (C^\alpha(\Omega))' \to C^\infty(\Omega)
\]
\[
\nu \mapsto (y \mapsto \nu(h(\cdot, y)))
\]
which for Radon measures \( \nu \in (C^\alpha(\Omega))' \) takes the form \( i_h(\nu) = \int h(x, \cdot) d\nu(x) \). Note that by convention, \( i_h \) always corresponds to the application of \( \nu \) to the first coordinate of \( h \). At times during our argument, we will wish to apply this operation to either the first or second coordinate of \( \xi_{P, Q} \) (or to its empirical counterpart), which we will indicate by swapping the order of the subscripts, e.g.,
\[
i_{\xi_{Q, P}}(\nu) := (x \mapsto \nu(\xi_{P, Q}(x, \cdot))).
\]

The following result shows this operator is bounded. For a multiindex \( b = (b_1, \ldots, b_d) \in \mathbb{N}^d \) We use the notation \( \partial_{b,y} h = \partial_{(b_1, \ldots, b_d, 0, \ldots, 0)} h \).

**Lemma 2.1.** For all \( b \in \mathbb{N}^d \) and \( \nu \in C^\alpha(\Omega) \), it holds that \( \partial_{b,y} i_h(\nu) = i_{\partial_{b,y} h}(\nu) \), where \( \partial_{b,y} h(y, x) \) denotes the partial derivative (in the standard multi-index notation) with respect to the \( y \)'s component. Moreover, for all \( \beta \in \mathbb{N} \) there exists a constant \( C(\beta) \) such that
\[
\|i_h(\nu)\|_{C^\beta(\Omega)} \leq C(\beta) \|\nu\|_{(C^\alpha(\Omega))'},
\]
so that \( i_h : (C^\alpha(\Omega))' \to C^\beta(\Omega) \) is a well-defined bounded operator for any \( \beta \in \mathbb{N} \).
Remark first that, given the optimal entropic coupling $\pi_{P,Q}$, the operators $A_P$ and $A_Q$ correspond to the barycentric projection of a generic function $f$ in $L^2(P)$. Our definition if therefore a natural generalization of the one proposed in [32], which studies the projection $f \xi_{P,Q}(\cdot, y)dQ(y)$. $A_C$ is nothing more than the centered version of the operator $A_P$. Note that in our work we only need to consider this operator but we could have defined it in the same way $A_Q$. In the following result such operators appear naturally when describing the limit of the Sinkhorn potentials.

**Theorem 2.2.** Let $\Omega \subset \mathbb{R}^d$ be a compact set, $P, Q \in \mathcal{P}(\Omega)$ and $P_n$ (resp. $Q_m$) be the empirical measure of the i.i.d. sample $X_1, \ldots, X_n$ (resp. $Y_1, \ldots, Y_m$) distributed as $P$ (resp. $Q$). If $m = m(n) \to \infty$ and $\frac{m}{m+n} \to \lambda \in (0,1)$,

$$
\left( f_{P_n,Q_m} - f_{P,Q} \right) = \epsilon \left( \frac{1}{n} \sum_{k=1}^{n} \xi_{P,Q}(X_k, \cdot) - \mathbb{E} \left[ \xi_{P,Q}(X, \cdot) \right] \right)
$$

in $C^0(\Omega) \times C^0(\Omega)$, for any $\alpha \in \mathbb{N}$.

**Remark 2.3.** Recall that our notational convention implies that $i_{\xi_{P,Q}}(P_n - P)$ and $i_{\xi_{Q,P}}(Q_m - Q)$ implicitly refer to integration of $\xi_{P,Q}$ with respect to different coordinates.

By proving that

$$
\left( i_{\xi_{P,Q}}(P_n - P) \right) = \left( \frac{1}{n} \sum_{k=1}^{n} \xi_{P,Q}(X_k, \cdot) - \mathbb{E} \left[ \xi_{P,Q}(X, \cdot) \right] \right)
$$

satisfies the central limit theorem in $C^0(\Omega) \times C^0(\Omega)$ and the linearization given in Theorem 2.2 we obtain the limit behavior. There are several ways to prove (11): the first involves utilizing the fact that the empirical process satisfies the central limit theorem in $(C^0(\Omega))^\prime$ ($\alpha \geq s$), followed by the application of Lemma 2.5. The second method involves using embeddings of Sobolev spaces into $(C^0(\Omega))^\prime$ that are of type II and thus satisfy the central limit theorem. The last approach is to demonstrate that $\|i_{\xi_{P,Q}}(P_n - P)\|_{C^{\alpha+1}(\Omega)} = O_P(1/n)$ thereby establishing that $i_{\xi_{P,Q}}(P_n - P)$ is tight in $C^0(\Omega)$. We leave the details for the reader.

**Corollary 2.4.** Let $\Omega \subset \mathbb{R}^d$ be a compact set, $P, Q \in \mathcal{P}(\Omega)$ and $P_n$ (resp. $Q_m$) be the empirical measure of the i.i.d. sample $X_1, \ldots, X_n$ (resp. $Y_1, \ldots, Y_m$) distributed as $P$ (resp. $Q$). Set $\alpha \in \mathbb{N}$ and suppose that both samples are mutually independent. Then, if $m = m(n) \to \infty$ and $\frac{m}{m+n} \to \lambda \in (0,1)$,

$$
\sqrt{n} \left( f_{P_n,Q_m} - f_{P,Q} \right) \to \epsilon \left( \frac{1}{n} \sum_{k=1}^{n} \xi_{P,Q}(X_k, \cdot) - \mathbb{E} \left[ \xi_{P,Q}(X, \cdot) \right] \right)
$$

weakly in $C^0(\Omega) \times C^0(\Omega)$, where $G_P$ and $G_Q$ are independent $P$ and $Q$-Brownian bridges. Moreover, in the one-sample case;

$$
\frac{1}{n} \left( f_{P_n,Q} - f_{P,Q} \right) \to \epsilon \left( \frac{1}{n} \sum_{k=1}^{n} \xi_{P,Q}(X_k, \cdot) - \mathbb{E} \left[ \xi_{P,Q}(X, \cdot) \right] \right)
$$

weakly in $C^0(\Omega) \times C^0(\Omega)$.

**Proof of Theorem 2.2.** We can assume that $\alpha \geq s$ without loss of generality. In order to simplify the expressions we define the bilinear operator $\tilde{A}_C$ as

$$
\tilde{A}_C : (C^0(\Omega))^\prime \times C^0(\Omega) \to C^0(\Omega)
$$

$$(f, \nu) \mapsto (y \mapsto \nu(f(\cdot) C(\cdot, y)))
$$

which for measures $\nu \in (C(\Omega))^\prime$ takes the form

$$
\tilde{A}_C(f, \nu) = \int f(x) C(x, \cdot) d\nu(x).
$$
The relation (5) means that \((f_x, g^*_x), (f_{n,m}, g_{n,m}) \in C^\alpha(\Omega) \times C^\alpha_0(\Omega)\) are just the solutions of \(\Psi(f_x, g^*_x) = \Psi_{n,m}(f_{n,m}, g_{n,m}) = 0\), where

\[
\Psi, \Psi_{n,m} : C^\alpha(\Omega) \times C^\alpha_0(\Omega) \to C^\alpha(\Omega) \times C^\alpha_0(\Omega)
\]

are respectively defined as

\[
\Psi \left( \begin{array}{c} f \\ g \end{array} \right) = \left( \begin{array}{c} f + \log (\mathcal{A}_c (e^f, Q)) \\ g + \log (\mathcal{A}_c (e^g, P)) \end{array} \right) - Q \left( \log (\mathcal{A}_c (e^f, P)) \right)
\]

and

\[
\Psi_{n,m} \left( \begin{array}{c} f \\ g \end{array} \right) = \left( \begin{array}{c} f + \log (\mathcal{A}_c (e^{f_{n,m}}, P)) \\ g + \log (\mathcal{A}_c (e^{g_{n,m}}, P)) \end{array} \right).
\]

Note that we are subtracting the expectation w.r.t. \(Q\) in the second component so that the image of \(\Psi\) and \(\Psi_{n,m}\) lies in \(C^\alpha(\Omega) \times C^\alpha_0(\Omega)\). The following estimates, derived from Lemmas B.3 and B.4, are fundamental:

\[
\|\mathcal{A}_c(e^{f_x}, P) - \mathcal{A}_c(e^{f_{n,m}}, P)\|_\alpha = O_P \left( \sqrt{\frac{n + m}{nm}} \right), \tag{12}
\]

\[
\|\mathcal{A}_c(e^{f_x}, P) - \mathcal{A}_c(e^{f_{n,m}}, P_n)\|_\alpha = O_P \left( \sqrt{\frac{n + m}{nm}} \right), \tag{13}
\]

\[
\|\mathcal{A}_c(e^{f_x}, P) - \mathcal{A}_c(e^{f_x}, P_n)\|_\alpha = O_P \left( \sqrt{\frac{n + m}{nm}} \right). \tag{14}
\]

The same proof of Lemma 2.1 gives the following result.

**Lemma 2.5.** There exists a constant \(C\) such that for each \(\nu \in (C^\alpha(\Omega))'\) and \(f \in C^\alpha(\Omega)\)

\[
\|\mathcal{A}_c(f, \nu)\|_{C^\alpha(\Omega)} \leq \|f\|_{C^\alpha(\Omega)} \|\nu\|_{(C^\alpha(\Omega))'}.
\]

As a consequence, \(f \mapsto \mathcal{A}_c(f, \nu)\) and \(\nu \mapsto \mathcal{A}_c(f, \nu)\) are bounded operators.

*Step 1: Fréchet differentiability of \(\Psi\).* The following result yields the Fréchet differentiability of \(\Psi\) at \((f_x, g^*_x)\), which implies that

\[
\frac{\|\Psi(f_{n,m}, g_{n,m}) - \Psi(f_x, g^*_x) - D(f_x, g^*_x)\Psi(f_x, g^*_x)(\delta_{n,m})\|_{C^\alpha(\Omega) \times C^\alpha_0(\Omega)}}{\|\delta_{n,m}\|_{C^\alpha(\Omega) \times C^\alpha_0(\Omega)}} \to 0. \tag{15}
\]

for \(\delta_{n,m} = (f_{n,m} - f_x, g_{n,m} - g^*_x)\).

**Lemma 2.6.** The functional \(\Psi : C^\alpha(\Omega) \times C^\alpha_0(\Omega) \to C^\alpha(\Omega) \times C^\alpha_0(\Omega)\) is Fréchet differentiable at \((f_x, g^*_x)\) with derivative

\[
\left( \begin{array}{c} h_1 \\ h_2 \end{array} \right) \mapsto \left( \begin{array}{c} h_1 \\ h_2 \end{array} \right) + \left( \begin{array}{c} \mathcal{A}_Q(h_1) \\ \mathcal{A}_P(h_2) \end{array} \right).
\]

**Proof.** Note that it is enough to check out that the limits

\[
\limsup_{\|h_1\|_\alpha \to 0} \frac{\|\log (\mathcal{A}_c(e^{f_x + h_1}, P)) - \log (\mathcal{A}_c(e^{f_x}, P)) - \mathcal{A}_P(h_1)\|_\alpha}{\|h_1\|_\alpha}
\]

and

\[
\limsup_{\|h_2\|_\alpha \to 0} \frac{\|\log (\mathcal{A}_c(e^{g^*_x + h_2}, Q)) - \log (\mathcal{A}_c(e^{g^*_x}, Q)) - \mathcal{A}_Q(h_2)\|_\alpha}{\|h_1\|_\alpha}
\]

are 0. We focus only on the first one, as the second follows by the same arguments. Lemma A.1 yields \(\|e^{f_x + h_1} - e^{f_x} - h_1 e^{f_x}\|_\alpha = o(\|h_1\|_\alpha)\). The linearity and continuity of the operator \(f \mapsto \mathcal{A}_c(f, P)\) implies its Fréchet differentiability. By composition, the operator \(f \mapsto \mathcal{A}_c(e^f, P)\) is also Fréchet differentiable. Since \(\mathcal{A}_c(e^f, P) > 0\), Lemma A.1 applied to the logarithm concludes the proof. \(\square\)
Step 2: Invertibility of $C^\alpha(\Omega) \times C^0_0(\Omega)$. We adopt matrix notation, i.e.,

\[
\begin{pmatrix}
(1 - A_Q A_P)^{-1} & -(1 - A_Q A_P)^{-1} A_Q \\
-A_P (1 - A_Q A_P)^{-1} & (1 - A_P A_Q)^{-1}
\end{pmatrix}
\begin{pmatrix}
f \\
g
\end{pmatrix}
= \begin{pmatrix}
(1 - A_P A_P)^{-1} f - (1 - A_P A_P)^{-1} A_P g \\
-A_P (1 - A_P A_P)^{-1} f + (1 - A_P A_Q)^{-1} g
\end{pmatrix}.
\]

Lemma 2.7. Let $\Omega \subset \mathbb{R}^d$ be a compact set, $P, Q \in \mathcal{P}(\Omega)$, then

(i) $(1 - A_Q A_P)$ and $(1 - A_P A_Q)$ are continuously invertible operators in $C^\alpha(\Omega)$ and $C^0_0(\Omega)$ respectively.
(ii) $A_P = A_P$ in the space $\{f \in C^\alpha(\Omega) : \int f(x)dP(x) = 0\}$.
(iii) the relation

\[
\begin{pmatrix}
(1 - A_Q A_P)^{-1} & -(1 - A_Q A_P)^{-1} A_Q \\
-A_P (1 - A_Q A_P)^{-1} & (1 - A_P A_Q)^{-1}
\end{pmatrix} = (D(f, s))^\dagger
\]

holds in $C^\alpha(\Omega) \times C^0_0(\Omega)$.

Moreover, (i), (ii) and (iii) also hold in $L^2(P) \times L^2_0(Q)$ instead of $C^\alpha(\Omega) \times C^0_0(\Omega)$.

Proof. If we show that the operators $A_Q, A_P$ are compact and that $(1 - A_P A_Q)$ and $(1 - A_Q A_P)$ are injective, the Fredholm alternative [2, Theorem 6.6] and continuous inverse theorem [2, Corollary 2.7] would prove (i). Claim (ii) follows by (17) below and the last claim by basic algebra. The proof in $L^2(P) \times L^2_0(Q)$ is the same; to avoid repeating arguments, we leave the details to the reader.

Compactness of $A_P$ and $A_Q$: Let $\{f_k\}_{k \in \mathbb{N}}$ be a bounded sequence in $C^\alpha(\Omega)$. Since the sequence

\[
A_P f_k = \int \xi_{P,Q}(x, \cdot) f_k(x) dP(x) \in C^{\alpha+1}(\Omega), \quad k \in \mathbb{N}
\]

has its derivatives up to order $\alpha + 1$ uniformly bounded by some $C(\Omega, \alpha, d)$ [12, Lemma 4.3], the Arzelà–Ascoli theorem yields the relatively compactness of $\{A_P f_k\}_{k \in \mathbb{N}}$ in $C^\alpha(\Omega)$. The same argument applies to $A_Q$.

Injectivity of Fredholm operators: We prove it by reductio ad absurdum. Suppose that $A_Q A_P f = A_P A_Q f = \lambda f$, for some nonzero $f \in C^\alpha(\Omega)$ and $\lambda \in \{-1, 1\}$. In this case, since the optimality condition (4) implies

\[
P A_Q A_P f = Q A_P f = \int \int \xi_{P,Q}(x,y) f(x) dP(x) dQ(y) = \lambda \int f dP,
\]

we have

\[
P A_Q A_P f = \int \int \xi_{P,Q}(x,y) ((A_P - Q A_P) f) dP(x) dQ(y) = \lambda Q (A_P - Q A_P) f = 0,
\]

and hence by assumption $0 = \int f dP = Q A_P f$. We therefore obtain

\[
f(x')^2 = (A_Q A_P f(x'))^2 = (A_Q A_P f(x'))^2 = \left( \int \xi_{P,Q}(x', y) dP(x) dQ(y) \right)^2
\]

for all $x' \in \Omega$. On the other hand, Jensen’s inequality gives

\[
f(x')^2 \leq \int \xi_{P,Q}(x', y) \left( \int \xi_{P,Q}(x, y) f(x) dP(x) \right)^2 dQ(y) \leq \|f\|^2_{\infty}, \quad \text{for all } x' \in \Omega,
\]

and the first inequality is strict unless

\[
\int \xi_{P,Q}(x, y) f(x) P(x) = c \in \mathbb{R} \text{ for Q-a.e. } y.
\]
Since there exists \( x' \in \Omega \) such that \( |f(x')| = \|f\|_\infty \), (18) holds. Therefore,

\[
\hat{f}_{P,Q} : x \mapsto \log \left( \frac{e^{f_{P,Q}(x)}(f(x) + 1 + \|f\|_\infty)}{c + 1 + \|f\|_\infty} \right)
\]

is a solution of the dual problem (2). Since it is unique, we obtain \( e^{f_{P,Q}(x)}(f(x) + 1 + \|f\|_\infty) = e^{f_{P,Q}(x)} \) and \( f = c \in \Omega \). Since \( f \) is centered, we conclude \( f = 0 \). \( \square \)

**Step 3: Linearization of \( \Psi_{n,m} - \Psi \).** The optimality condition \( \Psi_{n,m}(f_{n,m}, g_{n,m}) = \Psi(f_*, g_*) = 0 \) and (15) yields

\[
\Psi \left( f_{n,m}, g_{n,m} \right) - \Psi_{n,m} \left( f_{n,m}, g_{n,m} \right) = D_{(f_*, g_*)} \Psi_{P,Q}(\delta_{n,m}) + o_p \left( \sqrt{\frac{n + m}{nm}} \right) \tag{19}
\]

The following result allows to exchange, up to additive \( o_p \left( \sqrt{\frac{n + m}{nm}} \right) \) terms, \( (f_{n,m}, g_{n,m}) \) by \( (f_*, g_*) \) in (19).

**Lemma 2.8.** The asymptotic equality

\[
\Psi \left( f_{n,m}, g_{n,m} \right) - \Psi_{n,m} \left( f_{n,m}, g_{n,m} \right) = \Psi \left( f_*, g_* \right) - \Psi_{n,m} \left( f_*, g_* \right) + o_p \left( \sqrt{\frac{n + m}{nm}} \right)
\]

holds in \( C^\alpha(\Omega) \times C^\alpha_0(\Omega) \). As a consequence,

\[
\Psi \left( f_*, g_* \right) - \Psi_{n,m} \left( f_*, g_* \right) = D_{(f_*, g_*)} \Psi_{P,Q}(\delta_{n,m}) + o_p \left( \sqrt{\frac{n + m}{nm}} \right)
\]

holds in \( C^\alpha(\Omega) \times C^\alpha_0(\Omega) \).

**Proof.** It is enough to show that

\[
\log \left( \mathcal{A}_C(e^{f_{n,m}}, P) \right) - \log \left( \mathcal{A}_C(e^{f_{n,m}}, P_n) \right) - \log \left( \mathcal{A}_C(e^{f_*}, P) \right) + \log \left( \mathcal{A}_C(e^{f_*}, P_n) \right) \tag{20}
\]

and

\[
\log \left( \mathcal{A}_C(e^{g_{n,m}}, Q) \right) - \log \left( \mathcal{A}_C(e^{g_{n,m}}, Q_m) \right) - \log \left( \mathcal{A}_C(e^{g_*}, Q) \right) + \log \left( \mathcal{A}_C(e^{g_*}, Q_m) \right)
\]

are both \( o_p \left( \sqrt{\frac{n + m}{nm}} \right) \) in \( C^\alpha(\Omega) \). To avoid repeated arguments we only prove this claim for (20). In accordance with Lemma A.1—applied to the natural logarithm—and equations (12), (13) and (14) we obtain the estimates

\[
\log \left( \mathcal{A}_C(e^{f_{n,m}}, P) \right) = \log \left( \mathcal{A}_C(e^{f_*}, P) \right) + \frac{\mathcal{A}_C(e^{f_{n,m}}, P)}{\mathcal{A}_C(e^{f_*}, P)} - 1 + o_p \left( \sqrt{\frac{n + m}{nm}} \right),
\]

\[
\log \left( \mathcal{A}_C(e^{f_{n,m}}, P_n) \right) = \log \left( \mathcal{A}_C(e^{f_*}, P_n) \right) + \frac{\mathcal{A}_C(e^{f_{n,m}}, P_n)}{\mathcal{A}_C(e^{f_*}, P_n)} - 1 + o_p \left( \sqrt{\frac{n + m}{nm}} \right),
\]

\[
\log \left( \mathcal{A}_C(e^{f_*}, P) \right) = \log \left( \mathcal{A}_C(e^{f_*}, P) \right) + \frac{\mathcal{A}_C(e^{f_*}, P_n)}{\mathcal{A}_C(e^{f_*}, P)} - 1 + o_p \left( \sqrt{\frac{n + m}{nm}} \right).
\]

From here we derive the the following rewriting of (20);

\[
\frac{\mathcal{A}_C(e^{f_{n,m}} - e^{f_*}, P_n - P)}{\mathcal{A}_C(e^{f_*}, P)} + o_p \left( \sqrt{\frac{n + m}{nm}} \right). \tag{21}
\]

Then, since \( \mathcal{A}_C(e^{f_*}, P) \) is bounded away from 0 and infinity, by Lemma A.2 the proof is concluded by showing \( \mathcal{A}_C(e^{f_{n,m}} - e^{f_*}, P_n - P) = o_p \left( \sqrt{\frac{n + m}{nm}} \right) \). That holds due to Lemma 2.5, (42) and the fact that the unit ball of \( C^\alpha(\Omega) \) is uniformly Donsker. \( \square \)
With the results we have obtained, we are now in a position to complete the proof of the theorem. For \(\alpha \geq s\), it holds that \(\sqrt{\frac{n}{n+m}} \| P_n - P\|_{C^s(\Omega)}\) and \(\sqrt{\frac{m}{n+m}} \| Q_m - Q\|_{C^s(\Omega)}\) are \(O_P(1)\). Hence Lemma 2.5 and Lemma A.1 (applied to the natural logarithm) yield

\[
\Psi \left( \frac{f_*}{g_*} \right) - \Psi_{n,m} \left( \frac{f_*}{g_*} \right) = \left( \log(\mathcal{A}_C(e^{\mathcal{g}_r}, Q)) - \log(\mathcal{A}_C(e^{\mathcal{g}_r}, Q_m)) \right) - \left( \log(\mathcal{A}_C(e^{\mathcal{f}_r}, P)) - \log(\mathcal{A}_C(e^{\mathcal{f}_r}, P_n)) \right)
\]

\[
= \left( e^{\mathcal{f}_r} \mathcal{A}_C(e^{\mathcal{g}_r}, Q - Q_m) \right) + o_P \left( \sqrt{\frac{n+m}{n}} \right) \text{ in } C^s(\Omega) \times C^s_0(\Omega).
\]

Lemmas 2.7 and 2.8 give

\[
\left( \frac{f_{n,m} - f_*}{g_{n,m} - g_*} \right) = (D(f_* , g_* , \Psi))^{-1} \left( e^{\mathcal{f}_r} \mathcal{A}_C(e^{\mathcal{g}_r}, Q - Q_m) \right) + o_P \left( \sqrt{\frac{n+m}{n}} \right) \text{ in } C^s(\Omega) \times C^s_0(\Omega).
\]

As a consequence,

\[
\left( \frac{f_{P_n,Q_m} - f_{P,Q}}{g_{P_n,Q_m} - g_{P,Q}} \right) = e^{\epsilon} \left( 1 - \mathcal{A}_Q \tilde{A}_P \right)^{-1} \epsilon \mathcal{A}_Q k_{P,Q}(P_n - P) - \left( 1 - \mathcal{A}_Q \tilde{A}_P \right)^{-1} \epsilon \mathcal{A}_Q k_{P,Q}(Q_m - Q) \right)
\]

\[
+ o_P \left( \sqrt{\frac{n+m}{n}} \right). \quad (22)
\]

Since \(\int k_{P,Q}(P_n - P)dQ = \frac{1}{n} \sum_{k=1}^n \mathcal{R}_i(P_n, \mathcal{X}_k, \mathcal{Y}_k) - \mathbb{E}(\mathcal{R}_i(P_n, \mathcal{X}, \mathcal{Y})) = 0\), in view of Lemma 2.7 (ii), we can thus exchange \(\tilde{A}_P\) by \(A_P\) in (22) and the proof of the theorem is completed.

\[\square\]

3. Central limit theorem for the solution of the primal problem and Sinkhorn distances

This section covers the weak limit of the quantity

\[
\sqrt{\frac{n}{n+m}} \int \eta(d\pi_{P_n,Q_m} - d\pi_{P,Q}), \quad \text{where } \eta \in L^\infty(P \otimes Q). \quad (23)
\]

Before stating this result, for a fixed function \(\eta \in L^\infty(P \otimes Q)\) we introduce the notation:

\[
\eta_x : x \rightarrow \int \eta(x,y)\xi_{P,Q}(x,y)dQ(y) \quad \text{and} \quad \eta_y : y \rightarrow \int \eta(x,y)\xi_{P,Q}(x,y)dP(x),
\]

which substantially simplifies the description of the first-order decomposition of (23), described in the following theorem.

**Theorem 3.1.** Let \(\Omega \subset \mathbb{R}^d\) be a compact set, \(P, Q \in P(\Omega)\) and \(P_n\) (resp. \(Q_m\)) be the empirical measure of the i.i.d. sample \(X_1, \ldots, X_n\) (resp. \(Y_1, \ldots, Y_m\)) distributed as \(P\) (resp. \(Q\)). Then, if \(m = m(n) \rightarrow \infty, \frac{m}{n+m} \rightarrow \lambda \in (0,1)\) and \(\eta \in L^\infty(P \otimes Q),\)

\[
\int \eta(d\pi_{P_n,Q_m} - d\pi_{P,Q}) = \frac{1}{n} \sum_{k=1}^n (1 - \mathcal{A}_Q \mathcal{A}_P)^{-1}(\eta_x - \mathcal{A}_Q \eta_y)(X_k) + \frac{1}{m} \sum_{j=1}^m (1 - \mathcal{A}_P \mathcal{A}_Q)^{-1}(\eta_y - \mathcal{A}_P \eta_x)(Y_j)
\]

\[
+ o_P \left( \sqrt{\frac{n+m}{n}} \right).
\]
Theorem 3.1 gives the first-order decomposition of the solutions of the regularized optimal transport problem \(1\). We recall that, for a regularization based on the Schrödinger bridge, \([28]\) arrived at exactly the same thing, conjecturing the truth of Theorem 3.1. The techniques of the proofs are completely different; ours is based on the theory of empirical processes, while that of \([28]\) on a change of measure and projections in \(L^\infty(P \otimes Q)\).

**Corollary 3.2.** Let \(\Omega \subset \mathbb{R}^d\) be a compact set, \(P, Q \in \mathcal{P}(\Omega)\) and \(P_n \) (resp. \(Q_m\)) be the empirical measure of the i.i.d. sample \(X_1, \ldots, X_n\) (resp. \(Y_1, \ldots, Y_m\)) distributed as \(P\) (resp. \(Q\)). Then, if \(m = m(n) \to \infty\), \(\frac{m}{n + m} \to \lambda \in (0, 1)\) and \(\eta \in L^\infty(P \otimes Q)\),

\[
\sqrt{\frac{nm}{n + m}} \left( \int \eta d\pi_{P_n,Q_m} - \int \eta d\pi_{P,Q} \right) \xrightarrow{w} N(0, \sigma^2_{\lambda, \epsilon}(\eta)),
\]

where the variance \(\sigma^2_{\lambda, \epsilon}(\eta)\) is

\[
\lambda \text{Var}_{X \sim P} [(1 - A_{Q}A_{P})^{-1}(\eta_{x} - A_{Q}\eta_{y})(X)]
\]

\[
+ (1 - \lambda) \text{Var}_{Y \sim Q} [(1 - A_{P}A_{Q})^{-1}(\eta_{y} - A_{P}\eta_{x})(Y)].
\]

Moreover, in the one-sample case we have

\[
\sqrt{n} \left( \int \eta d\pi_{P_n,Q} - \int \eta d\pi_{P,Q} \right) \xrightarrow{w} N(0, \sigma^2_{P, \epsilon}(\eta)),
\]

with \(\sigma^2_{P, \epsilon}(\eta) = \text{Var}_{X \sim P} [(1 - A_{Q}A_{P})^{-1}(\eta_{x} - A_{Q}\eta_{y})(X)]\).

### 3.1. Applications

An immediate corollary of Theorem 3.1 is its application to the square norm function, where we obtain the weak limit of the Sinkhorn cost introduced in \([5]\). Formally it is defined as

\[
d_S(P, Q) = \mathbb{E}_{(X,Y) \sim \pi_{P,Q}} \left[ \frac{\|X - Y\|^2}{2} \right],
\]

and represents the cost of ‘transporting mass’ from \(P\) to \(Q\) when using the coupling given by the entropic regularization.

**Corollary 3.3.** Let \(\Omega \subset \mathbb{R}^d\) be a compact set, \(P, Q \in \mathcal{P}(\Omega)\) and \(P_n \) (resp. \(Q_m\)) be the empirical measure of the i.i.d. sample \(X_1, \ldots, X_n\) (resp. \(Y_1, \ldots, Y_m\)) distributed as \(P\) (resp. \(Q\)). Then, if \(m = m(n) \to \infty\), \(\frac{m}{n + m} \to \lambda \in (0, 1)\),

\[
\sqrt{\frac{nm}{n + m}} (d_S(P_n,Q_m) - d_S(P,Q)) \to N \left( 0, \sigma^2_{\lambda, \epsilon} \left( \frac{\| \cdot - \| ^2}{2} \right) \right), \quad \text{weakly},
\]

where the variance \(\sigma^2_{\lambda, \epsilon} \left( \frac{\| \cdot - \| ^2}{2} \right)\) is defined in Corollary 3.2 for the function \((x, y) \to \frac{\| x - y \|^2}{2}\). Moreover, in the one-sample case we have

\[
\sqrt{n} (d_S(P_n,Q) - d_S(P,Q)) \to N \left( 0, \sigma^2_{P, \epsilon} \left( \frac{\| \cdot - \| ^2}{2} \right) \right), \quad \text{weakly},
\]

with \(\sigma^2_{P, \epsilon} \left( \frac{\| \cdot - \| ^2}{2} \right)\) as in Corollary 3.2.

Another interesting application of Corollary 3.2 is to the function \((x, y) \to 1_{\| x - y \|^2 \leq t}\), for \(t \geq 0\). \([30]\) computed the regularized optimal transport problem to match two protein intensity distributions and defined the regularized colocalization measure \(\text{RCol}\)

\[
\text{RCol}(\pi_{P,Q}, t) = \int 1_{\| x - y \|^2 \leq t} d\pi_{P,Q}(x, y) = \pi_{P,Q}(\| \cdot - \| ^2 \leq t),
\]

which represents the mass of the pixel intensity transported on scales smaller or equal to \(t\). Theorem 7.1. in \([30]\) gives confidence intervals for the discretized images (finite number of pixels). The following result extend it to general probability distributions representing the pixels.
Corollary 3.4. Let $\Omega \subset \mathbb{R}^d$ be a compact set, $P, Q \in \mathcal{P}(\Omega)$ and $P_n$ (resp. $Q_m$) be the empirical measure of the i.i.d. sample $X_1, \ldots, X_n$ (resp. $Y_1, \ldots, Y_m$) distributed as $P$ (resp. $Q$). Then, if $m = m(n) \to \infty$, \[
 \frac{m}{n+m} \to \lambda \in (0,1),
\]
where the variance $\sigma_{\lambda,\epsilon}^2(1_{\|\cdot\|_2 \leq \epsilon})$ is defined in Corollary 3.2 for the function $(x, y) \mapsto 1_{\|x-y\|_2 \leq \epsilon}$. Moreover, in the one-sample case we have \[
 \sqrt{n}(RCol(\pi_{P_n}, Q, t) - RCol(\mu, \epsilon)) \to N\left(0, \sigma_{\lambda,\epsilon}^2(1_{\|\cdot\|_2 \leq \epsilon})\right), \quad \text{weakly,}
\]
with $\sigma_{\lambda,\epsilon}^2(1_{\|\cdot\|_2 \leq \epsilon})$ as in Corollary 3.2

3.2. Proof of Theorem 3.1

Recall that $1 - A_Q A_P$ and $1 - A_P A_Q$ are self adjoint due to the following relation, which is consequence of Fubini’s theorem,

\[
\langle f, A_Q g \rangle_{L^2(P)} = \langle A_P f, g \rangle_{L^2(Q)}, \quad \text{for all } f \in L^2(P), g \in L^2(Q).
\]

For the sake of readability, we adopt the notation (9). We define first the bilinear form $B_C$ as

\[
B_C : C^1(\Omega^2) \times (C^1(\Omega^2))' \to \mathbb{R}
\]

\[
(h, \nu) \mapsto \nu(hC),
\]

which for measures $\nu \in (C(\Omega))'$ takes the form $B_C(h, \nu) = \int h(x, y) C(x, y) \nu(x, y) \, dx \, dy$. The following are the main properties of the form $B_C$. The proofs of (25) and (27) are straightforward, while the proof of (26) is direct consequence of [36, Theorems 2.7.11 and 2.7.1].

Lemma 3.5. The following holds:

- (Continuity) For all $(h, \nu) \in C^1(\Omega^2) \times (C^1(\Omega^2))'$

\[
|B_C(h, \nu)| \leq \|h\|_{C^1(\Omega^2)} \|\nu\|_{(C^1(\Omega^2))'} \leq C \|h\|_{C^1(\Omega^2)} \|\nu\|_{(C^1(\Omega^2))'}.
\]

(25)

- (Consistent empirical process)

\[
\|\eta(Q_m \otimes (P_n - P))\|_{(C^1(\Omega^2))'} \leq \|\eta(Q_m - Q) \otimes P_n\|_{(C^1(\Omega))'} = o_P(1).
\]

(26)

- (Rate for fixed $h \in C^1(\Omega^2)$)

\[
|B_C(h, \eta(Q_m - Q) \otimes (P_n - P))| = O_P \left(\frac{n+m}{nm}\right).
\]

(27)

We also have the following “rules of calculus” of the operators $A_P$, $A_Q$ and $\xi_{\epsilon_{\epsilon}, Q}$ in the space

\[
S = \{\nu \in (C(\Omega))' : \nu(\Omega) = 0\}^{C^*(\Omega)'} \subset (C^*(\Omega))'.
\]

Such a Banach space is convenient due to two facts; (i) the equality

\[
\nu A_P^2 \xi_{\epsilon_{\epsilon}, P}(\mu) = \langle A_P \xi_{\epsilon_{\epsilon}, P}(\mu), \xi_{\epsilon_{\epsilon}, P}(\nu)\rangle_{L^2(P)}
\]

holds for $\mu, \nu \in S$ due to Fubini’s theorem; (ii) \[
Q \xi_{\epsilon_{\epsilon}, Q}(\mu) = \int \xi_{\epsilon_{\epsilon}, Q}(x, y) dQ(y) d\mu(x) = 0,
\]

so that $(1 - A_P A_Q)^{-1}$ and $(1 - A_Q A_P)^{-1}$ are well-defined in $S$, and the P-Gaussian bridge belongs to $S$. 

Lemma 3.6 (Rules of Calculus). Set \( \nu \in \mathcal{S} \) and \( f \in \mathcal{C}^*(\Omega) \). It holds that

\[
A_P (1 - A_Q A_P)^{-1} = (1 - A_P A_Q)^{-1} A_P, \quad A_Q (1 - A_P A_Q)^{-1} = (1 - A_Q A_P)^{-1} A_Q.
\]

\[
\langle f, i_{\xi_{\mathcal{Q}, \nu}} \nu \rangle_{L^2(\mathcal{P})} = \nu(A_P f), \quad \text{and} \quad \langle f, i_{\xi_{\mathcal{Q}, \nu}} \nu \rangle_{L^2(\mathcal{Q})} = \nu(A_Q f).
\]

Proof. (29) holds due to

\[
A_P = (1 - A_P A_Q)^{-1} (1 - A_P A_Q) A_P = (1 - A_P A_Q)^{-1} (A_P - A_P A_Q A_P) = (1 - A_P A_Q)^{-1} A_P (1 - A_Q A_P).
\]

We prove (30) for \( \nu \in \mathcal{C}(\Omega)' \). Fubini's theorem implies that

\[
\langle f, i_{\xi_{\mathcal{Q}, \nu}} \nu \rangle_{L^2(\mathcal{P})} = \int f(x) \xi_{\mathcal{P}, Q}(x, y) d(\mathcal{P} \otimes \nu)(x, y)
\]

\[
= \int \left( \int f(x) \xi_{\mathcal{P}, Q}(x, y) d\mathcal{P}(x) \right) d\nu(y),
\]

yielding the result. In the general case \( \nu \in \mathcal{S} \) the same result holds due to the continuity of the operators and a standard density argument. \( \square \)

We call \( v_{n,m}^Q = \sqrt{\frac{n}{n+m}} (Q_m - Q) \) and \( v_{n,m}^P = \sqrt{\frac{n}{n+m}} (P_n - P) \). With this estimates in mind, we split (23) in three terms: \( \int \eta (d\pi_{P_n, Q_m} - d\pi_{P, Q}) = A + B + C \), where each of them can be easily estimated via (27), (26), (25) and Lemma B.1 as follows. First the term capturing the difference between the densities

\[
A = \sqrt{\frac{n}{n+m}} \int \eta(x, y) C(x, y) \left( e^{h_{n,m}(x,y)} - e^{h_{n,m}(x,y)} \right) dQ_m(y) dP_n(x)
\]

\[
= \sqrt{\frac{n}{n+m}} C(e^{h_{n,m} - e^{h_{n,m}}, \eta(Q_m \otimes P_n))}
\]

\[
= \sqrt{\frac{n}{n+m}} \left( C(e^{h_{n,m} - h_{n,m}}, \eta(Q \otimes P)) + C(e^{h_{n,m} - h_{n,m}}, \eta(Q_m \otimes P_n - Q \otimes P)) \right)
\]

\[
= \sqrt{\frac{n}{n+m}} C(e^{h_{n,m} - h_{n,m}}, \eta(Q \otimes P)) + o_P(1).
\]

Applying Lemma A.1 and Lemma B.1 yields

\[
A = \sqrt{\frac{n}{n+m}} C(h_{n,m} - h_{n,m}, e^{h_{n,m}}, \eta(Q \otimes P)) + o_P(1).
\]

Then the one dealing with the empirical process of \( P \)

\[
B = C(e^{h_{n,m}}, \eta(Q_m \otimes v_{n,m}^P))
\]

(by (27))

\[
= C(e^{h_{n,m}}, \eta(Q \otimes v_{n,m}^P)) + o_P(1)
\]

\[
= (v_{n,m}^P \otimes \eta) (\eta \xi_{\mathcal{P}, Q}) + o_P(1)
\]

\[
= v_{n,m}^P (\eta_x) + o_P(1)
\]

and the last one with that of \( Q \), namely \( C_m = v_{n,m}^Q (\eta_y) \). In view of Theorem 2.2, we have

\[
A = \langle \eta, (1 - A_Q A_P)^{-1} A_Q i_{\xi_{\mathcal{P}, \nu}} v_{n,m}^P - (1 - A_Q A_P)^{-1} i_{\xi_{\mathcal{Q}, \nu}} v_{n,m}^Q \rangle_{L^2(\pi_{P, Q})}
\]

\[
+ \langle \eta, A_P (1 - A_Q A_P)^{-1} i_{\xi_{\mathcal{P}, \nu}} v_{n,m}^Q - (1 - A_P A_Q)^{-1} i_{\xi_{\mathcal{Q}, \nu}} v_{n,m}^P \rangle_{L^2(\pi_{P, Q})} + o_P(1). \quad (31)
\]
In order to introduce the term \( \eta \) inside the operator, we observe that
\[
\langle \eta, (1 - A_Q A_P)^{-1} i_{\xi_{Q,P}} v_{n,m}^Q \rangle_{L^2(\pi_{P,Q})} = \langle \eta, (1 - A_Q A_P)^{-1} i_{\xi_{Q,P}} v_{n,m}^Q \rangle_{L^2(P)}
\]
(by (24)) = \( \langle (1 - A_Q A_P)^{-1} \eta, i_{\xi_{Q,P}} v_{n,m}^Q \rangle_{L^2(P)} \).
Each term of (31) can be treated in the same way, which gives the relations:
\[
\langle \eta, A_P (1 - A_Q A_P)^{-1} i_{\xi_{Q,P}} v_{n,m}^Q \rangle_{L^2(\pi_{P,Q})} = \langle (1 - A_Q A_P)^{-1} A_Q \eta, i_{\xi_{Q,P}} v_{n,m}^Q \rangle_{L^2(P)}
\]
\[
\langle \eta, (1 - A_P A_Q)^{-1} i_{\xi_{P,Q}} v_{n,m}^P \rangle_{L^2(\pi_{P,Q})} = \langle (1 - A_P A_Q)^{-1} A_P \eta, i_{\xi_{P,Q}} v_{n,m}^P \rangle_{L^2(Q)}
\]
and
\[
\langle \eta, (1 - A_Q A_P)^{-1} A_Q i_{\xi_{P,Q}} v_{n,m}^P \rangle_{L^2(\pi_{P,Q})} = \langle (1 - A_P A_Q)^{-1} A_P \eta, i_{\xi_{P,Q}} v_{n,m}^P \rangle_{L^2(Q)}.
\]
Therefore, we obtain
\[
A = \langle (1 - A_Q A_P)^{-1} (A_Q \eta - \eta), i_{\xi_{Q,P}} v_{n,m}^Q \rangle_{L^2(P)}
\]
\[
+ \langle (1 - A_P A_Q)^{-1} (A_P \eta - \eta), i_{\xi_{P,Q}} v_{n,m}^P \rangle_{L^2(Q)}
\]
(by (30)) = \( v_{n,m}^Q A_P (1 - A_Q A_P)^{-1} (A_Q \eta - \eta) + v_{n,m}^P A_Q (1 - A_P A_Q)^{-1} (A_P \eta - \eta) \).
Adding the missing \( B \) and \( C \) terms and omitting \( o_P(1) \) errors, we obtain that (23) is equal to
\[
v_{n,m}^Q (\eta + A_P (1 - A_Q A_P)^{-1} (A_Q \eta - \eta)) + v_{n,m}^P (\eta + A_Q (1 - A_P A_Q)^{-1} (A_P \eta - \eta)).
\]
(32)
Focusing on the first term of (32), we compute
\[
v_{n,m}^Q (\eta + A_P (1 - A_Q A_P)^{-1} (A_Q \eta - \eta))
\]
(by (30)) = \( v_{n,m}^Q (\eta + (1 - A_P A_Q)^{-1} A_P (A_Q \eta - \eta)) \)
\[
= v_{n,m}^Q (\eta + (1 - A_P A_Q)^{-1} A_P (A_Q \eta - A_P \eta)) \)
\[
= v_{n,m}^Q (\eta + (1 - A_P A_Q)^{-1} (A_P \eta - \eta)) \)
\[
= v_{n,m}^Q ((1 - A_P A_Q)^{-1} (\eta - A_P \eta)).
\]
Simplifying the second term of (32) in an identical fashion yields the result.

4. Weak limit of the Divergences
Recall that, for probabilities \( P, Q \in \mathcal{P}(\Omega) \), the quadratic Sinkhorn divergence [20] is defined as
\[
D_\varepsilon(P, Q) = S_\varepsilon(P, Q) - \frac{1}{2} \left( S_\varepsilon(P, P) + S_\varepsilon(Q, Q) \right).
\]
The correction terms in the definition of the Sinkhorn divergence are designed to debias the entropic optimal transport distance so that it becomes a bona fide discrepancy measure (i.e., \( D_\varepsilon(P, Q) = 0 \) if and only if \( P = Q \)). This distance is of great interest for statistical applications. In this section, we study its limit distribution. Theorem 4.4 provides limits of the quantity \( a_n (D_\varepsilon(P_n, Q) - D_\varepsilon(P_n, Q)) \), where the sequence \( \{a_n\}_{n \in \mathbb{N}} \) depends on the hypothesis \( H_0 : P = Q \) or \( H_1 : P \neq Q \). In the latter case, the limit can be established by means of the [14]'s technique based on Efron-Stein inequality—see also [31, 11] for reproduction and improvement of this argument—with common rate \( a_n = \sqrt{n} \). Since the proof under \( H_1 \) does not have any technical inovation, we omit the details. The case \( P = Q \), however, has the faster rate \( a_n = n \) and the limit depends on the P-Brownian bridge \( G_P \) in \( (C^\infty(\Omega))' \). More precisely, the limit in the one sample case will be the action of the random operator \( G_P \) on the random function \( (1 - A_P^2)^{-1} i_{\xi_{P,P}} (G_P) \). To state our result, we define the bilinear form
\[
\mathbb{M} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}
\]
\[
(\nu, \mu) \mapsto \nu ((1 - A_P^2)^{-1} i_{\xi_{P,P}} (\mu)),
\]
where \( \mathcal{S} \) is defined in (28). \( \mathbb{M} \) can be seen as the quadratic approximation of the Sinkhorn divergence under \( H_0 \).
Theorem 4.1. Let $\Omega \subset \mathbb{R}^d$ be a compact set, $P, Q \in \mathcal{P}(\Omega)$ and $P_n$ (resp. $Q_n$) be the empirical measure of the i.i.d. sample $X_1, \ldots, X_n$ (resp. $Y_1, \ldots, Y_m$) distributed as $P$ (resp. $Q$). Then, if $m = m(n) \to \infty$ and $\frac{m}{n+m} \to \lambda \in (0,1)$, we have the following limits.

- Under $H_0$: $P = Q$,
  \[
  D_1(P_n, Q_n) = \frac{1}{2} \text{KL}(P_n \mid Q_n) + o_P \left( \frac{n+m}{nm} \right)
  \]
  and $D_1(P_n, P) = \frac{1}{2} \text{KL}(P_n \mid P) + o_P \left( \frac{1}{n} \right)$.

- Under $H_1$: $P \neq Q$,
  \[
  \sqrt{\frac{n+m}{n}} \left( D_1(P_n, Q_n) - D_1(P, Q) \right) = \int \psi_{P, Q} d(P_n - P) + \int \psi_{Q, P} d(Q_m - Q) + o_P \left( \sqrt{\frac{n+m}{nm}} \right),
  \]
  \[
  \text{and } \sqrt{n} \left( D_1(P_n, Q) - D_1(P, Q) \right) = \int \psi_{P, Q} d(P_n - P) + o_P \left( n^{\frac{3}{2}} \right), \text{ where } \psi_{P, Q} = f_P - \frac{1}{2} (f_P + g_{P, P}) \text{ and } \psi_{Q, P} = f_Q - \frac{1}{2} (f_{Q, Q} + g_{Q, Q}).
  \]

Using Theorem 4.1 to prove that $n D_1(P_n, P'_m) \overset{a.s.}{\to} \mathbb{M}(G, G_P)$ is trivial due to the continuity of all the operators involving $\mathbb{M}$. However, characterizing in terms of known random variables is not immediate. We would like to use the following result.

Lemma 4.2 (26, Theorem 6.32). A Gaussian random variable $G$ in a separable Hilbert space $\mathcal{H}$ can be represented as the a.s. limit as $k \to \infty$ in $\mathcal{H}$ of the sum $\sum_{i=1}^{k} \lambda_i e_i$, where $\{\lambda_i\}_{i=1}^{k}$ is an i.i.d. sequence of $N(0,1)$ and $\{e_i\}_{i=1}^{k}$ an orthonormal basis of $\mathcal{H}$. Therefore, $\|G\|_{\mathcal{H}}^2 = \sum_{i=1}^{k} \lambda_i^2 \chi(1)_i$ where $\{\chi(1)_i\}_{i \in \mathbb{N}}$ is an i.i.d sequence of Chi-squares with one degree of freedom.

However, $(\mathcal{C}^*(\Omega))'$ is not Hilbertian or separable, and this is not immediately deduced. To write everything in terms of a inner product in a Hilbert space we use the following result.

Lemma 4.3. The operator $\mathbb{M}$ is bilinear, definite positive and bounded. As a consequence, the space $(\mathcal{H}_M, \mathbb{M})$ is a separable Hilbert space, where $\mathcal{H}_M = \mathcal{S}^M$.

Proof. The fact that $\mathbb{M}$ is bilinear is straightforward. It is also bounded due to the following:

\[
\nu((1 - A_P^2)^{-1} \iota_{P,P}(\mu)) \leq \nu\|\cdot\|_{(\mathcal{C}^*(\Omega))'} \|\cdot\|_{(\mathcal{C}^*(\Omega))} \|(1 - A_P^2)^{-1}\|_{(\mathcal{C}^*(\Omega))' \to (\mathcal{C}^*(\Omega))} \|\iota_{P,P}\|_{(\mathcal{C}^*(\Omega))' \to (\mathcal{C}^*(\Omega))},
\]

(33)

where $\| \cdot \|_{\mathcal{B} \to \mathcal{G}}$ denotes the operator norm between the Banach spaces $\mathcal{B}$ and $\mathcal{G}$. Note that $\|(1 - A_P^2)^{-1}\|_{(\mathcal{C}^*(\Omega))' \to (\mathcal{C}^*(\Omega))} \|\iota_{P,P}\|_{(\mathcal{C}^*(\Omega))' \to (\mathcal{C}^*(\Omega))}$ is bounded due to Lemmas 2.7 and 2.1. We need to prove now that $\mathbb{M}$ is symmetric and positive for measures $\mu$ and $\nu$. The complete statement holds by a density argument. First note that

\[
(1 - A_P^2)^{-1} = 1 + A_P^2 (1 - A_P^2)^{-1}.
\]

(34)

The spectrum of the self-adjoint compact operator $A_P$ in the Hilbert space $L^2_0(P)$ is contained in $[0, 1 - \delta]$ for some $\delta > 0$ (see Lemma 2.7). Since, self-adjoint compact operators in Hilbert spaces attain its norm [2, Proposition 6.9], we have $\|A_P\|_{L^2_0(P) \to L^2_0(P)} < 1$. Hence,

\[
\lim_{k \to \infty} \left\| (1 - A_P^2)^{-1} - \sum_{i=0}^{k} A_P^{2k} \right\|_{L^2_0(P) \to L^2_0(P)} = 0.
\]

(35)

Set $f \in \mathcal{C}(\Omega)$. By Fubini’s theorem, it holds

\[
\nu A_P^2(f) = \int \xi_{P,P}(u, z) \iota_{P,P}(z, y) f(y) dP(y) P(z) d\nu(u)
\]

\[
= \langle A_P f, \iota_{P,P}(\nu) \rangle_{L^2_0(P)}.
\]
By using (34), (35) and (24) we get

\[ M(\mu, \nu) = \int \xi_{\mathcal{P}, \mathcal{P}} d(\mu \otimes \nu) + \sum_{i=0}^{\infty} (A^{2k+1}_{\mathcal{P}} i_{\mathcal{P}, \mathcal{P}}(\mu), i_{\mathcal{P}, \mathcal{P}}(\nu))_{L^2(\mathcal{P})} \]

\[ = \int \xi_{\mathcal{P}, \mathcal{P}} d(\nu \otimes \mu) + \sum_{i=0}^{\infty} (A^{2k+1}_{\mathcal{P}} i_{\mathcal{P}, \mathcal{P}}(\nu), i_{\mathcal{P}, \mathcal{P}}(\mu))_{L^2(\mathcal{P})}, \]

from where we deduce the symmetry. Since, \( \xi_{\mathcal{P}, \mathcal{P}}(z, y) = \xi_{\mathcal{P}, \mathcal{P}}(y, z) = e^{f(x)} C(z, y) e^{f(x)} \) and \( C \) is the Gaussian kernel with bandwidth \( \epsilon \), the bilinear form \( M \) is positive.

To prove the last claim we only need to show that \( \mathcal{H}_{\mathcal{M}} \) is separable. First we notice that (33) implies

\[ \mathcal{H}_{\mathcal{M}} = \{ \nu \in (\mathcal{C}(\Omega))' : \nu(\Omega) = 0 \} . \]

Recall that, by the uniform boundedness principle, the sequence \( \{d_n\}_{n \in \mathbb{N}} \) is bounded in \((\mathcal{C}(\Omega))'\). Since the weak* topology of \((\mathcal{C}(\Omega))'\) is separable, we can find a countable dense \( D \subset \{ \nu \in (\mathcal{C}(\Omega))' : \nu(\Omega) = 0 \} \) such that for any \( \mu \in (\mathcal{C}(\Omega))' \) with \( \mu(\Omega) = 0 \) there exists \( d_n \xrightarrow{\nu} \mu \), with \( d_n \in D \) for all \( n \). Here the convergence \( \xrightarrow{\nu} \) is the weak* in \((\mathcal{C}(\Omega))'\). Since the operator \( i_{\mathcal{P}, \mathcal{P}} \) is compact (apply the Arzelà–Ascoli theorem to Lemma 2.1) and \((1 - A^2_{\mathcal{P}})^{-1} \) is bounded, then \((1 - A^2_{\mathcal{P}})^{-1} i_{\mathcal{P}, \mathcal{P}}(d_n - \mu) \xrightarrow{\mathcal{C}(\Omega)} 0 \). Therefore, \( M(d_n - \mu, d_n - \mu) \to 0 \), which shows that \( D \) is also dense in \( \mathcal{H}_{\mathcal{M}} \). \( \square \)

We can now use Lemma 4.2 to get the following representation.

**Theorem 4.4.** Let \( \Omega \subset \mathbb{R}^d \) be a compact set, \( P \in \mathcal{P}(\Omega) \), \( P_n \) and \( P'_m \) be independent empirical measures of \( P \). Then, if \( m = m(n) \to \infty \) and \( \frac{m}{n+m} \to \lambda \in (0, 1) \), we have the following limits:

\[ n D_1(P_n, P) \xrightarrow{w} \frac{\epsilon}{2} \sum_{i=1}^{\infty} \lambda_i^2 N_i^2 \]

and

\[ \frac{n+m}{n+m} D_e(P_n, P'_m) \xrightarrow{w} \frac{\epsilon}{2} \sum_{i=1}^{\infty} \lambda_i^2 N_i^2, \]

where \( \{N_i\}_{i \in \mathbb{N}} \) is a sequence of i.i.d. random variables with \( N_i \sim N(0, 1) \) and \( \{\lambda_i\}_{i \in \mathbb{N}} \subset [0, \infty) \) is such that \( \sum_{i=1}^{\infty} \lambda_i^2 < \infty \). More precisely, \( \{\lambda_i\}_{i \in \mathbb{N}} \subset [0, \infty) \) are the eigenvalues of the covariance operator of \( \mathcal{G}_P \) in the separable Hilbert space \( \mathcal{H}_{\mathcal{M}} \).

**Proof of Theorem 4.1** if \( P = Q \). As usually, we prove it in the two-sample case. We denote \( P'_m = Q_m = \frac{1}{m} \sum_{k=1}^{m} \delta_{X_k^m} \). We want to derive the limit of

\[ D_e(P_n, P'_m) = S_e(P_n, P'_m) - \frac{1}{2} (S_e(P_n, P_n) + S_e(P'_m, P'_m)) \]

\[ = \frac{1}{2} (S_e(P_n, P'_m) - S_e(P_n, P_n)) + \frac{1}{2} (S_e(P'_m, P'_m) - S_e(P_n, P'_m)). \]

The following result writes the previous display in a way that we can apply Theorem 2.2.

**Lemma 4.5.** It holds that

\[ D_e(P_n, P'_m) = \frac{1}{2} \int (g_{P_n, P'_m} - g_{P_n, P'_m})(dP'_m - dP'_m) \]

\[ + \frac{1}{4e} \int ((h_{P_n, P'_m} - h_{P_n, P'_m})^2 + (h_{P_n, P'_m} - h_{P_n, P'_m})^2)d\pi_{P, P} + o_P \left( \frac{n+m}{nm} \right). \]  

(36)

There are two important terms in (36); we linearize both of them of separately. 

**First term of (36):** In view of Theorem 2.2, \( (g_{P_n, P'_m} - g_{P_n, P'_m}) \) can be expressed, up to additive \( o_P \left( \sqrt{\frac{n+m}{nm}} \right) \) terms, in \( \mathcal{C}^2(\Omega) \) as

\[ ((1 - A^2_{\mathcal{P}})^{-1} i_{\mathcal{P}, \mathcal{P}}(P_n - P'_m) + (1 - A^2_{\mathcal{P}})^{-1} A_P i_{\mathcal{P}, \mathcal{P}}(P_n - P'_m)) \]

\[ = ((1 - A^2_{\mathcal{P}})^{-1} - (1 - A^2_{\mathcal{P}})^{-1} A_P) i_{\mathcal{P}, \mathcal{P}}(P_n - P'_m) \]

\[ = ((1 - A^2_{\mathcal{P}})^{-1} - (1 - A_{\mathcal{P}})) i_{\mathcal{P}, \mathcal{P}}(P_n - P'_m). \]
These $o_p \left( \sqrt{\frac{n+m}{n}} \right)$ error terms in $C^*(\Omega)$ become $o_p \left( \frac{n+m}{n} \right)$ when $(P_n - P'_n)$ acts over them. Hence,

$$(P_n - P'_n) (g_{P_n, P'_n} - g_{P_n, P_n}) = \epsilon (P_n - P'_n) (1 - A_P^2)^{-1} (1 - A_P) (i_{\xi P, P} (P_n - P'_n)) + o_p \left( \frac{n+m}{n} \right). \quad (37)$$

**Second term of (36):** Since, up to $o_p \left( \sqrt{\frac{n+m}{n}} \right)$ terms, we have

$$h_{P_n, P'_n} (x, y) - h_{P'_n, P_m} (x, y) = -((1 - A_P^2)^{-1} A_P i_{\xi P, P} (P_n - P'_n)) (x) + ((1 - A_P^2)^{-1} i_{\xi P, P} (P_n - P'_n)) (y),$$

so that

$$(h_{P_n, P'_n} (x, y) - h_{P'_n, P_m} (x, y))^2 = \epsilon^2 \left( ((1 - A_P^2)^{-1} A_P i_{\xi P, P} (P_n - P'_n)) (x) \right)^2 + \epsilon^2 \left( ((1 - A_P^2)^{-1} i_{\xi P, P} (P_n - P'_n)) (y) \right)^2 - 2 \epsilon^2 \left( (1 - A_P^2)^{-1} A_P i_{\xi P, P} (P_n - P'_n) \right) (x) \left( (1 - A_P^2)^{-1} i_{\xi P, P} (P_n - P'_n) \right) (y). \quad (38)$$

Integrating (38) with respect to $\pi_{P, P}$ yields

$$\epsilon^2 \|(1 - A_P^2)^{-1} A_P i_{\xi P, P} (P_n - P'_n)\|_{L^2(P)}^2 + \epsilon^2 \|(1 - A_P^2)^{-1} i_{\xi P, P} (P_n - P'_n)\|_{L^2(P)}^2$$

$$- 2 \epsilon^2 \|(1 - A_P^2)^{-1} A_P i_{\xi P, P} (P_n - P'_n), A_P (1 - A_P^2)^{-1} i_{\xi P, P} (P_n - P'_n)\|_{L^2(P)},$$

which is equal to

$$\epsilon^2 \|(1 - A_P^2)^{-1} i_{\xi P, P} (P_n - P'_n)\|_{L^2(P)}^2 - \epsilon^2 \|(1 - A_P^2)^{-1} A_P i_{\xi P, P} (P_n - P'_n)\|_{L^2(P)}^2; \quad (39)$$

where we use that all the operators commute. Expanding the squares and using commutativity again, we obtain

$$\|(1 - A_P^2)^{-1} A_P i_{\xi P, P} (P_n - P'_n)\|_{L^2(P)}^2 = \langle (1 - A_P^2)^{-2} A_P^2 i_{\xi P, P} (P_n - P'_n), i_{\xi P, P} (P_n - P'_n) \rangle_{L^2(P)}$$

and

$$\|(1 - A_P^2)^{-1} i_{\xi P, P} (P_n - P'_n)\|_{L^2(P)}^2 = \langle (1 - A_P^2)^{-2} i_{\xi P, P} (P_n - P'_n), i_{\xi P, P} (P_n - P'_n) \rangle_{L^2(P)}.$$ 

As a consequence, (39) can be rewritten as

$$\epsilon^2 \|(1 - A_P^2)^{-2} A_P^2 i_{\xi P, P} (P_n - P'_n), i_{\xi P, P} (P_n - P'_n)\|_{L^2(P)}$$

$$= \epsilon^2 \langle (1 - A_P^2)^{-1} i_{\xi P, P} (P_n - P'_n), i_{\xi P, P} (P_n - P'_n) \rangle_{L^2(P)},$$

(by (30)) = $\epsilon^2 (P_n - P'_n) [A_P (1 - A_P^2)^{-1} i_{\xi P, P} (P_n - P'_n)], \quad (40)$

which turns out to be more convenient for the sequel.

**Derivation of the limit.** From Lemma 4.5 and equations (37) and (40), we get (up to $o_p \left( \frac{n+m}{n} \right)$ additive terms)

$$D_c (P_n, P'_m) = \frac{\epsilon}{2} (P_n - P'_m) [(1 - A_P^2)^{-1} i_{\xi P, P} (P_n - P'_n)],$$

which proves Theorem 4.1 under $H_0$. \qed
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Appendix A: Auxiliary results of Hölder spaces

Lemma A.1. Let $\alpha \in \mathbb{N}$, $I \subset \mathbb{R}$ be a compact interval, $F \in C^\infty(I)$ and $g \in C^\alpha(\Omega)$, with $g(\Omega) \subset I$. Then the operator

$$\delta_F : C^\alpha(\Omega) \rightarrow C^\alpha(\Omega)$$

$$g \mapsto F(g),$$

is Fréchet differentiable in $g$ with derivative $D\delta_F(g)h = F'(g)h$. Moreover, setting $f \in C^\alpha(\Omega)$, the operator

$$C^\alpha(\Omega) \rightarrow C^\alpha(\Omega)$$

$$g \mapsto fg,$$

is Fréchet differentiable in $g$ with derivative $h \rightarrow fh$.

Proof. To verify the first claim, i.e.

$$\|F(g + h) - F(g) - F'(g)h\|_{C^\alpha} = o(\|h\|_{C^\alpha}),$$

we prove the bound for each derivative in a direction $v$ by a recursive argument on $a = |v|$. Note that the first case, the uniform norm, is trivial; set $x \in \Omega$ and apply Taylor’s theorem to obtain

$$|F(g(x) + h(x)) - F(g(x)) - F'(g(x))h(x)| \leq |F''(g(x))h(x)|^2 \leq \sup_I |F''(x)| \cdot \|h\|_s.$$ 

Suppose that the result holds for $a \in \mathbb{N}$. Given a derivative in a direction $b$ with $|b| = a + 1$, there exists a decomposition $D_b = D_c D_d$ with $|d| = 1$ and $|c| = a$. Therefore, the chain rule yields

$$D_{b}(F(g + h) - F(g) - F'(g)h)$$

$$= D_c (F'(g + h)(D_{d} g + D_{d} h) - F'(g))D_{d} g - F''(g)D_{d} g h$$

$$= D_c (F'(g + h) - F'(g) - F'(g)h)D_{d} g + (F'(g + h) - F'(g))D_{d} h h.$$ 

Since the function $F'$ still satisfies the assumptions of the theorem we obtain, by induction hypothesis and Lemma A.2, the limits

$$\|D_c ((F'(g + h) - F'(g) - F''(g)h)D_{d} g)\|_{C^\infty}$$

$$\leq C \|D_c (F'(g + h)(D_{d} g + D_{d} h) - F'(g))\|_{C^\infty} = o(\|h\|_s)$$

and

$$\|D_c ((F'(g + h) - F'(g))D_{d} h)\|_{C^\infty} \leq C \|D_c (F'(g + h) - F'(g))\|_{C^\infty} = o(\|h\|_s),$$

which finish the proof of the first claim. For the second one the relation $f(g + h) - fg - fh = 0$ and Lemma A.2 conclude.
Lemma A.2. Let \( f, g \in \mathcal{C}^s(\Omega) \) then there exists a constant \( C \) depending on \( \Omega, s \) and \( d \) such that
\[
\|fg\|_s \leq C\|f\|_s\|g\|_s.
\]

**Proof.** The proof is direct consequence of the multivariate Leibniz rule, i.e.
\[
\|D_a(fg)\|_\infty \leq \sum_{|\alpha| \leq a} \binom{a}{\alpha} \|D_\alpha f\|_\infty \|D_{\alpha^c}g\|_\infty \leq C\|f\|_s\|g\|_s.
\]

\[\square\]

Appendix B: Refined convergence rates of regularized potentials

The aim of this section is to refine the results of [12] so that any errors that arise in the linearizations vanish rapidly enough. The proofs lack interest; they are merely calculations and Taylor expansions of the exponential function.

**Lemma B.1.** Let \( \Omega \subset \mathbb{R}^d \) be a compact set, \( P, Q \in \mathcal{P}(\Omega) \), and its empirical versions \( P_n, Q_m \), with \( \frac{n}{n+m} \to \lambda \in (0,1) \). Then
\[
\|e^{h_{n,m}} - e^{h_*}(h_{n,m} - h_*)\|_{\mathcal{C}^r(\Omega \times \Omega)} = o_p\left( \sqrt{\frac{n+m}{nm}} \right), \quad \text{and} \tag{41}
\]
\[
\|e^{h_{n,m}} - e^{h_*}\|_{\mathcal{C}^r(\Omega \times \Omega)} = O_p\left( \sqrt{\frac{n+m}{nm}} \right). \tag{42}
\]

**Proof.** Since the functions \( h_{n,m} \) and \( h_{n,m} \) are uniformly bounded, for all \( n, m \in \mathbb{N} \), Lemma A.1 applied to the exponential gives
\[
\|e^{h_{n,m}} - e^{h_*} - e^{h_*}(h_{n,m} - h_*)\|_s = o_p(\|h_* - h_{n,m}\|_s)
\]
and, using Theorem 4.5 in [12], we obtain (41). To prove (42) we apply the inverse triangle inequality to (41);
\[
\|e^{h_{n,m}} - e^{h_*} - e^{h_*}(h_{n,m} - h_*)\|_{\mathcal{C}^r(\Omega \times \Omega)} \geq \|e^{h_{n,m}} - e^{h_*}\|_{\mathcal{C}^r(\Omega \times \Omega)} - \|e^{h_*}(h_{n,m} - h_*)\|_{\mathcal{C}^r(\Omega \times \Omega)}.
\]
Then we apply Lemma A.2 and (41) to obtain
\[
\|e^{h_{n,m}} - e^{h_*}\|_{\mathcal{C}^r(\Omega \times \Omega)} \leq C\|(h_{n,m} - h_*)\|_{\mathcal{C}^r(\Omega \times \Omega)} + o_p\left( \sqrt{\frac{n+m}{nm}} \right).
\]
Theorem 4.5 in [12] concludes. \[\square\]

**Lemma B.2.** Let \( \Omega \subset \mathbb{R}^d \) be a compact set and \( P, Q \in \mathcal{P}(\Omega) \), then
\[
\left| \int \frac{1}{e} S_e(P, Q) - \frac{1}{e} \int (h - h_*)^2 d\pi_{P,Q} \right| \leq \frac{1}{6} \|h - h_*\|_\infty^3 e\|h - h_*\|_\infty,
\]
for all \( h(x, y) = f(x) + g(y) \), with \( f, g \in \mathcal{C}(\Omega) \) and \( \int e^h d \mathcal{P}(P \otimes Q) = 1 \).

**Proof.** The application of Taylor’s theorem to the exponential gives
\[
|e^x - 1 - x - \frac{1}{2}x^2| \leq \frac{1}{6} |x^3| e^{|x|}.
\]
(43)
Since also \( \int e^h \cdot C \cdot d(P \otimes Q) = 1 \), (5) yields
\[
\frac{1}{\varepsilon} S_\varepsilon(P, Q) = \int h_s dPQ - \int (e^{h_s} - e^h) \cdot C \cdot d(P \otimes Q).
\]
We can decompose
\[
- \int (e^{h_s} - e^h) \cdot C \cdot d(P \otimes Q) = \int (e^{h_s - h} - 1) e^{h_s} \cdot C \cdot d(P \otimes Q)
\]
and, finally, (43) implies
\[
\left| \frac{1}{\varepsilon} S_1(P, Q) - \int h_s d(P \otimes Q) - \int (h - h_s + \frac{1}{2} (h - h_s)^2) \right| d\pi_{P, Q} \leq \frac{1}{6} \| h - h_s \|_\infty^3 e^{\|h_s\|_\infty}.
\]
Since (5) cancels the linear terms, the proof is completed.

**Lemma B.3.** Let \( \Omega \subset \mathbb{R}^d \) be a compact set, \( P, Q \in \mathcal{P}(\Omega) \) and the associate empirical measures \( P_n \) and \( Q_m \). Then there exists a constant \( C(\Omega, d, s) \) such that
\[
\left\| (e^{f_{n,m}(x)} - e^{f_s(x)}) \cdot C(x, \cdot) \right\|_s \leq C(\Omega, d, s) \| f_{n,m} - f_s \|_\infty,
\]
for all \( x \in \Omega \) and
\[
\mathbb{E} \left[ \left\| (e^{f_{n,m}(x)} - e^{f_s(x)}) \cdot C(x, \cdot) \right\|_s^2 \right] \leq C(\Omega, d, s) \left( \frac{1}{n} + \frac{1}{m} \right).
\]
**Proof.** The first claim is straightforward;
\[
\left\| (e^{f_{n,m}(x)} - e^{f_s(x)}) \cdot C(x, \cdot) \right\|_s \leq \| f_{n,m} - f_s \|_\infty \| C(x, \cdot) \|_s \leq C \| f_{n,m} - f_s \|_\infty.
\]
The second claim is thus completed by applying Theorem 4.5 in [12].

**Lemma B.4.** Let \( \Omega \subset \mathbb{R}^d \) be a compact set, \( P, \in \mathcal{P}(\Omega) \) and \( P_n \) be the empirical measure of the i.i.d. sample \( X_1, \ldots, X_n \) distributed as \( P \). Then
\[
n \mathbb{E} \left[ \sup_{y \in \Omega, f \in C^\ast(\Omega), \| f \|_s \leq 1} \left| \int f(x) \cdot C(x, y) d(P_n - P)(x) \right| \right] = O(1).
\]
Moreover, the class \( \{ x \mapsto g(y) \cdot C(x, y) f(x), \ y \in \Omega, \| f \|_s \leq 1, \| g \|_s \leq 1 \} \) is \( P \)-Donsker.

**Proof.** The inclusion
\[
\{ x \mapsto g(y) \cdot C(x, y) f(x), \ y \in \Omega, \| f \|_s \leq 1, \| g \|_s \leq 1 \} \subset \{ f \in C^\ast(\Omega), \| f \|_s \leq C \}
\]
holds for certain constant \( C > 0 \) and [36, 2.7.2 Corollary] and [21, Exercise 2.3.1] give
\[
\mathbb{E} \left[ \left( \sup_{\| f \|_s \leq 1} (P_n - P)(f) \right)^2 \right] \leq \frac{C}{n}.
\]
Therefore the first statement holds. The last one is consequence of (44), Theorems 2.5.2 and 2.7.1 in [36].
Appendix C: Proofs of the Lemmas

Proof of Lemma 2.1. We prove $\partial_h i_b(\nu) = i_{\partial_h \beta} h(\nu) \in C(\Omega)$ inductively in $\beta = |b| \in \mathbb{N}$ the result. For $\beta = 0$ we only need to check the continuity, which holds by the same steps of (45) below. We avoid repeated arguments and skip the proof. Assume that $\partial_h i_b(\nu) = i_{\partial_h \beta} h(\nu)$ for all $|b'| \leq \beta$. Let $b = (b_1, \ldots, b_d)$ be a multi-index with $|b| \leq \beta + 1$. Assume without loss of generality that $b_1 \geq 1$ and consider $b' = (b_1 - 1, b_2, \ldots, b_d)$ which obviously satisfies $|b'| \leq \beta$. Therefore,

$$i_{b', \beta} h(\nu(y + t e_1) - i_{b', \beta} h(\nu)(y) = \frac{1}{t} \nu \left( \partial_{b', \beta} h(\nu(y + t e_1) - \partial_{b', \beta} h(\nu(y)) \right)$$

Due to the limit

$$\lim_{t \to 0} \left\| \frac{\partial_{b', \beta} h(\nu(y + t e_1) - \partial_{b', \beta} h(\nu(y))}{t} - \partial_b h(\nu(y)) \right\|_{C^\infty(\Omega)} = 0$$

we get

$$\lim_{t \to 0} \frac{i_{b', \beta} h(\nu(y + t e_1) - i_{b', \beta} h(\nu)(y)}{t} = i_{b', \beta} h(\nu(y), \forall y \in \Omega.$$  

Since

$$|i_{b', \beta} h(\nu)(y) - i_{b', \beta} h(\nu)(y)| = \nu (\partial_{b', \beta} h(\nu(y) - \partial_b h(\nu(y))$$

$$\leq \|\nu\|_{C^\infty(\Omega)} \|\partial_{b', \beta} h(\nu(y) - \partial_b h(\nu(y)\|_{C^\infty(\Omega)} (45)$$

and $h \in C^\infty(\Omega^2)$, we get $i_{b', \beta} h(\nu) \in C(\Omega)$. As a consequence, $\partial_h i_b(\nu) = i_{\partial_h \beta} h(\nu) \in C(\Omega)$ for all $b \in \mathbb{N}^d$ and, of course, $i_{b'}(\nu) \in C^\infty(\Omega)$. 

Proof of Lemma 4.5. First note that, since $\int h_{P_n, P_m} \mathbb{E} d(P_n \otimes P_m) = 1$, Lemma B.2 yields

$$S_{\epsilon}(P_n, P_m) = \int h_{P_n, P_m} d(P_n \otimes P_m) + \frac{1}{2\epsilon} \int (h_{P_n, P_m} - h_{P_n, P_m})^2 d\pi_{P_n, P_m} + \frac{1}{2\epsilon} \int \left( h_{P_n, P_m} - h_{P_n, P_m} \right) d\pi_{P_n, P_m} + \frac{1}{2\epsilon} \int h_{P_n, P_m} \mathbb{E} d(P_n \otimes P_m),$$

where the last term is consequence of Lemma A.2 and Theorem 4.5 in [12]. Therefore

$$S_{\epsilon}(P_n, P_m) = S_{\epsilon}(P_n, P_m) + \int h_{P_n, P_m} d((P_n - P_m) \otimes P_m)$$

$$+ \frac{1}{2\epsilon} \int (h_{P_n, P_m} - h_{P_n, P_m})^2 d\pi_{P_n, P_m} + \frac{1}{2\epsilon} \int h_{P_n, P_m} \mathbb{E} d(P_n \otimes P_m).$$

By the same argument we also have

$$S_{\epsilon}(P_n, P_m) = S_{\epsilon}(P_n, P_m) + \int h_{P_n, P_m} d(P_n \otimes (P_m - P_n))$$

$$+ \frac{1}{2\epsilon} \int (h_{P_n, P_m} - h_{P_n, P_m})^2 d\pi_{P_n, P_m} + \frac{1}{2\epsilon} \int h_{P_n, P_m} \mathbb{E} d(P_n \otimes (P_m - P_n)).$$

and, as a consequence,

$$D_{\epsilon}(P_n, P_m) = \frac{1}{2} \left( \int h_{P_n, P_m} d((P_n - P_m) \otimes P_m) + \int h_{P_n, P_m} d(P_n \otimes (P_m - P_m)) \right)$$

$$+ \frac{1}{4\epsilon} \int (h_{P_n, P_m} - h_{P_n, P_m})^2 + (h_{P_n, P_m} - h_{P_n, P_m})^2 d\pi_{P_n, P_m} + \frac{1}{2\epsilon} \int h_{P_n, P_m} \mathbb{E} d(P_n \otimes (P_m - P_m)).$$
The relations
\[
\int h_{P_n, P'_m} d((P_n - P'_m) \otimes P'_m) = \int f_{P_n, P'_m} (dP_n - dP'_m),
\]
\[
\int h_{P_n, P'_m} d(P_n \otimes (P'_m - P_n)) = \int g_{P'_m, P'_m} (dP'_m - dP_n)
\]
and the fact that, in this symmetric case, \( f_{P_n, P'_m} = g_{P'_m, P'_m} + c \), give
\[
D_\varepsilon(P_n, P'_m) = \frac{1}{2} \int (g_{P_n, P'_m} - g_{P'_m, P'_m})(dP'_m - dP'_m)
+ \frac{1}{4\varepsilon} \int ((h_{P_n, P'_m} - h_{P'_m, P'_m})^2 + (h_{P'_m, P'_m} - h_{P_n, P'_m})^2) d\pi_{P_n, P'_m} + \mathcal{O}_\varepsilon \left( \frac{n + m}{nm} \right).
\]
Since \( \|\pi_{P_n, P'_m} - \pi_{P, P}\|_{\mathcal{C}([0,\varepsilon])} \to 0 \) a.s. and \( \|h_{P_n, P'_m} - h_{P'_m, P'_m}\|_s = \mathcal{O}_\varepsilon \left( \frac{n + m}{nm} \right) \), the inequality
\[
\left| \int (h_{P_n, P'_m} - h_{P'_m, P'_m})^2 (d\pi_{P_n, P'_m} - d\pi_{P, P}) \right| \leq \|h_{P_n, P'_m} - h_{P'_m, P'_m}\|_s \|\pi_{P_n, P'_m} - \pi_{P, P}\|_{\mathcal{C}([0,\varepsilon])}
\]
yields the result. \( \square \)

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