STACKELBERG-NASH NULL CONTROLLABILITY OF HEAT EQUATION WITH GENERAL DYNAMIC BOUNDARY CONDITIONS

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Abstract. This paper deals with the hierarchical control of the anisotropic heat equation with dynamic boundary conditions and drift terms. We use the Stackelberg-Nash strategy with one leader and two followers. To each fixed leader, we find a Nash equilibrium corresponding to a bi-objective optimal control problem for the followers. Then, by some new Carleman estimates, we prove a null controllability result.

1. Introduction. A control problem consists of finding a control that steers the system under consideration from an initial state to a fixed target. In many situations, the resolution process leads to a single-objective optimal control problem, and generally, under some appropriate conditions, the existence and uniqueness of optimal control problems can be proved.

The modeling of many industrial and economic complex problems reveals several criteria to optimize simultaneously. For example, in the heat transfer, in a room, it is meaningful to try to guide the temperature to be close as much as possible to a fixed target at the end of the day and, additionally, keep the temperature not too far from a prescribed value at some regions. This can be done by applying several controls at different locations of the room. This problem can be seen as a game with controls as players.

To solve a multi-objective optimal control problem, there are several strategies to choose the best controls, depending on the character of the problem. Among these strategies, let us mention the cooperative strategy proposed by Pareto [40], the Nash non-cooperative strategy [44], and the Stackelberg hierarchical strategy [41].

In the context of the control of evolution equations, a relevant question is whether one can steer the system to the desired state exactly or approximately by controls

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corresponding to one of these strategies. Up to now, there are several papers related to this topic. In the seminal papers [35] and [36], J. L. Lions introduced and studied the Stackelberg strategy for the hyperbolic and parabolic equations. Later on, the authors in [14] and [15], studied the existence and uniqueness of Stackelberg-Nash equilibrium, as well as its characterization. In these works, the followers and the leader have an approximate controllability objective. Then again, from a theoretical and numerical point of view, the papers [12, 20, 23, 47] dealt with the above questions, for parabolic equations and the Burgers equation, respectively. We refer also to [37], where the authors obtained results of hierarchical control for parabolic equations with moving boundaries. Eventually, let us mention that the Stackelberg-Nash strategy for Stokes systems has been studied in [26]. We emphasize that all the above results deal with the hierarchical control for the evolution equation only in the case of approximate controllability. In [5], the authors developed the first hierarchical results in the context of the controllability to trajectories. These results were recently improved in [4] by imposing some weak conditions on observation domains for the followers. The same idea is also applied for the wave equation and the degenerate parabolic equations in the papers [20] and [6], respectively. More recently, in [7], the authors dealt with the hierarchical exact controllability of parabolic equations with distributed and boundary controls. The same method was also applied in the context of parabolic coupled systems in [27] and [28] and mixed with robust control in [29] and [39]. The previous works have considered Dirichlet and Neumann boundary conditions.

In this paper, we are interested in developing the Stackelberg-Nash strategy for a parabolic equation with dynamic boundary conditions. This type of boundary conditions has been considered in the context of null controllability, when only one control is acting on the system, in [33, 38]. The cost of approximate controllability and an inverse source problem of such equations were also studied in [9] and [2], respectively. For some controllability results of hyperbolic equation with dynamic boundary condition, the reader can see [21]. We refer to [24] for a derivation and physical interpretation of such boundary conditions, and to [13, 25, 32, 42, 43, 48, 49] for the study of the existence, the uniqueness and the regularity of their solutions. Note also that the controllability problems of evolution equations is extensively studied in the case of static boundary conditions in [1, 3, 8, 11, 45] and reference therein. It is worth mentioning that the presence of dynamic boundary conditions, in evolution problems, creates a transmission problem between the dynamics in the domain and on the boundary, also appear in various interesting physical models and motivated by problems occurring diffusion, reaction-diffusion, and phase-transition phenomena, special flows in hydrodynamics, models in climatology, and so on. We refer to [24] for a derivation and physical interpretation of such boundary conditions.

Our main results rely on a new Carleman estimate of a coupled system. In our case, the first equation is a direct heat equation whose second term is in $L^2$ and an adjoint equation with terms in a Sobolev space of negative order $H^{-1}$. To our knowledge, our paper is the first to deal with Carleman estimates for a coupled system with terms in $H^{-1}$, especially in the context of hierarchical control of evolution equations.

This paper is organized as follows. In Section 2, we formulate the problem under consideration and state the main result. In Section 3, we introduce some needed functional spaces and recall some previous results on the well-posedness. Section 4 deals with the proof of existence and characterization of Nash equilibrium.
Section 5, we prove some suitable Carleman estimates and deduce our null controllability result in the linear case. The semilinear case is considered in Section 6.

2. Problem and its formulation. Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\Gamma := \partial \Omega$. Let $\omega$, $\omega_1$ and $\omega_2$ be arbitrary nonempty open subsets strictly contained in $\Omega$. Given $T > 0$, we denote $\Omega_T := \Omega \times (0,T)$ and $\Gamma_T := \Gamma \times (0,T)$. Consider the heat equation with dynamic boundary conditions

$$\begin{cases}
\partial_t y - \text{div}(A \nabla y) + B(x,t) \cdot \nabla y + a(x,t)y = f_1 \omega + v_1 \omega_1 + v_2 \omega_2 & \text{in } \Omega_T, \\
\partial_t y_T - \text{div}(A_T \nabla y_T) + \partial^\omega y + B_T(x,t) \cdot \nabla y_T + b(x,t)y_T = 0 & \text{on } \Gamma_T, \\
(y(0),y_T(0)) = (y_0, y_T,0). & \text{in } \Omega \times \Gamma,
\end{cases}$$

(1)

Here, $y_T$ denotes the trace of $y$ on $\Gamma$, $y_0 \in L^2(\Omega)$ and $y_T,0 \in L^2(\Gamma)$ the initial data in the domain and on the boundary, respectively. We emphasize that $y_T,0$ is not necessarily the trace of $y_0$, since we do not assume that $y_0$ has a trace, but if $y_0$ has a well-defined trace on $\Gamma$, then the trace must coincide with $y_T,0$, and $\nu$ is the outer unit normal field, $\partial^\omega y := (A \nabla y \cdot \nu)_|_\Gamma$ is the co-normal derivative at $\Gamma$. The functions $a$, $b$, $B$ and $B_T$ belong to $L^\infty(\Omega_T)$, $L^\infty(\Gamma_T)$, $L^\infty(\Omega_T)^N$ and $L^\infty(\Gamma_T)^N$, respectively; $\omega$, $\omega_1$ and $\omega_2$ are the characteristic functions of $\omega$, $\omega_1$ and $\omega_2$, respectively, and $f$, $v_1$ and $v_2$ are the control functions which act on the system through the subset $\omega_1$, $\omega_2$, and $\omega_2$, respectively. The boundary $\Gamma$ of $\Omega$ is considered to be a $(N-1)$-dimensional compact Riemannian submanifold equipped by the Riemannian metric $g$ induced by the natural embedding $\Gamma \subset \mathbb{R}^N$. Let $(x_i)$ be the natural coordinate system, we denote $\left(\frac{\partial}{\partial x_j}\right)$ the corresponding tangent vector field and $g_{ij} = \left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)$. We define the inner product and the norm on the tangent space by

$$g(X,Y) = \langle X,Y \rangle_\Gamma = \sum_{i,j=1}^{N-1} g_{ij}(x) \alpha_i \beta_j \quad \text{and} \quad |X|_g = \langle X,X \rangle_\Gamma^{1/2}$$

for all $X = \sum_{i=1}^{N-1} \alpha_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^{N-1} \beta_i \frac{\partial}{\partial x_i}$ in the tangent space and we simply denote $X \cdot Y = \langle X,Y \rangle_\Gamma$. For any smooth function $y$ on $\Gamma$, the tangential gradient of $y$ on $\Gamma$ is defined as $\nabla_\Gamma y = \sum_{i,j=1}^{N-1} g^{ij} \frac{\partial y}{\partial x_j} \frac{\partial}{\partial x_i}$, where the matrix $\tilde{g} = (g^{ij})$ is the inverse of $g = (g_{ij})$, see [34] and [48] for more detail. We also recall, from [33], the following weak definitions of the divergence operator $\text{div}()$ and the tangential divergence operator $\text{div}_T()$. For $F \in L^2(\Omega)$ and $F_T \in L^2(\Gamma)$

$$\text{div}(F) : H^1(\Omega) \rightarrow \mathbb{R}, \quad u \mapsto -\int_\Omega F \cdot \nabla u \, dx + \langle F \cdot \nu, u \rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)},$$

$$\text{div}_T(F_T) : H^1(\Gamma) \rightarrow \mathbb{R}, \quad u_T \mapsto -\int_\Gamma F_T \cdot \nabla_T u_T \, d\sigma.$$

Here, $d\sigma$ is the natural surface measure on $\Gamma$. Viewed as linear forms on $H^1(\Omega)$ and $H^1(\Gamma)$, $\text{div}(F)$ and $\text{div}_T(F_T)$ are continuous. In particular, we have the following estimates.

$$|\langle \text{div}(F), u \rangle| \leq C_1 \|F\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} \quad F \in L^2(\Omega), u \in H^1(\Omega),$$

$$|\langle \text{div}_T(F_T), u_T \rangle| \leq C_2 \|F_T\|_{L^2(\Gamma)} \|u_T\|_{H^1(\Gamma)} \quad F_T \in L^2(\Gamma), u_T \in H^1(\Gamma),$$

where $C_1$ and $C_2$ are positive constants. For $F_T = \nabla_T u_T$, $u_T \in H^1(\Gamma)$, we define the Laplace-Beltrami operator $\Delta_T$ as follows

$$\Delta_T u_T = \text{div}_T(\nabla_T u_T).$$

Throughout this paper, we assume that the matrices $A$ and $A_T$ satisfy the following assumptions.
The followers, and the fixed target functions following main cost functional
\[ \phi_i = 1 \]
for each \( x \in \Omega, x_0 \in \Gamma, \) and \( \zeta \in \mathbb{R}^N. \)

This paper deals with Stackelberg-Nash strategy for the null controllability of the above heat equation with dynamic boundary conditions. To be more specific, we introduce, for \( i = 1, 2, \) the non-empty open sets \( \omega_{i,d} \subset \Omega, \) representing the observation domains of the followers, and the fixed target functions \( y_{i,d} \in L^2(0,T;L^2(\omega_{i,d})). \) Let us define the following main cost functional
\[ J(f) = \frac{1}{2} \int_{\omega \times (0,T)} |f|^2 dx dt, \]
and the secondary cost functional
\[ J_i(f; v_1, v_2) = \frac{\alpha_i}{2} \int_{\omega_{i,d} \times (0,T)} |y - y_{i,d}|^2 dx dt + \frac{\mu_i}{2} \int_{\omega_{i,d} \times (0,T)} |v_i|^2 dx dt, \quad i = 1, 2, \]
where \( \alpha_i, \mu_i \) are positive constants and \( Y(f, v_1, v_2) = (y(f, v_1, v_2), y_1(f, v_1, v_2)) \) is the solution to (1). For a fixed \( f \in L^2(0,T;L^2(\omega)), \) the pair \( (v_1^*, v_2^*) = (v_1^*(f), v_2^*(f)) \) is called a Nash equilibrium for \((J_1, J_2)\) associated to \( f \) if
\[ \begin{align*}
J_1(f; v_1^*, v_2^*) &\leq J_1(f; v, v_1^*) \quad \forall v \in L^2(0,T;L^2(\omega_1)), \\
J_2(f; v_1^*, v_2^*) &\leq J_2(f; v_1^*, v) \quad \forall w \in L^2(0,T;L^2(\omega_2)),
\end{align*} \]
or equivalently
\[ \begin{align*}
J_1(f; v_1^*, v_2^*) &= \min_{v \in L^2(0,T;L^2(\omega_1))} J_1(f; v, v_1^*), \\
J_2(f; v_1^*, v_2^*) &= \min_{v \in L^2(0,T;L^2(\omega_2))} J_2(f; v_1^*, v).
\end{align*} \]
Since the functionals \( J_i (i = 1, 2) \), are differentiable and convex, then, by well-known characterization results, see, e.g. Theorem 3.8 of [30] or [16], the pair \( (v_1^*, v_2^*) = (v_1^*(f), v_2^*(f)) \) is a Nash equilibrium for \((J_1, J_2)\) associated to \( f \) if and only if
\[ \begin{align*}
\frac{\partial J_1}{\partial v_1}(f; v_1^*, v_2^*)(v) &= 0 \quad \forall v \in L^2(0,T;L^2(\omega_1)), \\
\frac{\partial J_2}{\partial v_1}(f; v_1^*, v_2^*)(w) &= 0 \quad \forall w \in L^2(0,T;L^2(\omega_2)).
\end{align*} \]

Our goal is to prove that, for any initial data \( Y_0 = (y_0, y_{0,T}) \in L^2 = L^2(\Omega) \times L^2(\Gamma), \) there exist a control \( f \in L^2(0,T;L^2(\omega)) \) with minimal norm (called leader) and an associated Nash equilibrium \( (v_1^*, v_2^*) = (v_1^*(f), v_2^*(f)) \in L^2(0,T;L^2(\omega_1)) \times L^2(0,T;L^2(\omega_2)) \) (called followers) such that the associated state \( Y_{(f)}(X_0, f, v_1^*, v_2^*) \) of (1) satisfies \( Y(f, v_1^*, v_2^*)(T) = 0. \) To do this, we shall follow the Stackelberg-Nash strategy: for each choice of the leader \( f, \) we look for a Nash equilibrium pair for the cost functionals \( J_i (i = 1, 2); \) which means finding the controls \( v_1^*(f) \in L^2(0,T;L^2(\omega_1)) \) and \( v_2^*(f) \in L^2(0,T;L^2(\omega_2)) \), depending on \( f, \) satisfying (7). Once the Nash equilibrium has been identified and fixed for each \( f, \) we look for a control \( \hat{f} \) such that
\[ J(\hat{f}) = \min_{f \in L^2(0,T;L^2(\omega))} J(f, v_1^*(f), v_2^*(f)), \]
subject to the null controllability constraint
\[ Y(T, \hat{f}, v_1^*(\hat{f}), v_2^*(\hat{f}))(T) = 0. \]
Assume that the control regions satisfy the following assumption
\[ \omega_d = \omega_{1,d} = \omega_{2,d}, \quad \text{and} \quad \omega_d \cap \omega \neq \emptyset. \]

The main result in this paper is the following.
Theorem 2.1. Let us assume that (10) holds and $\mu_i > 0$, $i = 1, 2$, are sufficiently large. There exists a positive weight function $\rho = \rho(t)$ blowing up at $t = T$ such that for every $y_{i,d} \in L^2(0,T;L^2(\omega_i))$ satisfying
\begin{equation}
\int_{\omega_i,d \times (0,T)} \rho^2 y_i^2 dx dt < \infty, \quad i = 1, 2,
\end{equation}
and every $Y_0 = (y_0, y_{1,0}^T) \in L^2$, there exist a control $f \in L^2(0,T;L^2(\omega))$ with minimal norm and an associated Nash equilibrium $(v_1^*, v_2^*) \in L^2(0,T;L^2(\omega_1)) \times L^2(0,T;L^2(\omega_2))$ such that the corresponding solution to (1) satisfies $Y(T, f, v_1^*, v_2^*) = 0$.

Remark 1. (a) As mentioned by several authors, the assumption (11) seems to be natural. It means that the follower targets $y_{i,d}$, $i = 1, 2$, approach to 0 as $T \to 0$, and with this the leader finds no obstruction to control the system. It remains an open problem to verify if this condition is necessary.

(b) In [4], the authors have considered some weak conditions than (10), for instance $\omega_{1,d} \cap \omega \neq \omega_{2,d} \cap \omega$, in the context of the heat equation with Dirichlet conditions. This generalization will be treated in a forthcoming paper for dynamic boundary conditions.

(c) The hierarchical control is motivated by applications where more than one objective is desirable. For example, we can think of $y$ and $y_d$ the concentration of some chemical product in $\Omega$ and on the boundary $\Gamma$, respectively. The process is to guide the system under consideration to 0 by means of an optimal control $f$ acting on the domain $\omega$, and without, in the course of the action, going too far from $y_d$ in a small subdomain $d$.

3. Preliminaries. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary $\Gamma := \partial \Omega$. Following [38], we introduce the product space defined by
$$L^2 = L^2(\Omega) \times L^2(\Gamma).$$
Here, we have considered the Lebesgue measure $dx$ on $\Omega$ and the natural surface measure $d\sigma$ on $\Gamma$. Equipped by the scalar product
$$\langle (u, w), (v, z) \rangle_{L^2} = (u, v)_{L^2(\Omega)} + (w, z)_{L^2(\Gamma)} = \int_\Omega uv dx + \int_\Gamma wz d\sigma.$$  
$L^2$ is a Hilbert space. Recall that $H^1(\Gamma)$ and $H^2(\Gamma)$ are real Hilbert spaces endowed with the respective norms
$$\|u\|_{H^1(\Gamma)} = \langle u, u \rangle_{H^1(\Gamma)}, \quad \text{with } (u, v)_{H^1(\Gamma)} = \int_\Gamma uv d\sigma + \int_\Gamma \nabla u \nabla v d\sigma,$$
and
$$\|u\|_{H^2(\Gamma)} = \langle u, u \rangle_{H^2(\Gamma)}, \quad \text{with } (u, v)_{H^2(\Gamma)} = \int_\Gamma uv d\sigma + \int_\Gamma \Delta u \Delta v d\sigma.$$
We point out that the operator $\Delta$ can be considered as an unbounded linear operator from $L^2(\Gamma)$ in $L^2(\Gamma)$, with domain
$$D(\Delta) = \{u \in L^2(\Gamma) : \Delta u \in L^2(\Gamma)\},$$
and it is known that $-\Delta$ is a self-adjoint and nonnegative operator on $L^2(\Gamma)$. This implies that $-\Delta$ generates an analytic $C_0-$semigroup $(e^{t\Delta})_{t \geq 0}$ on $L^2(\Gamma)$. If $\Gamma$ is smooth, then one can show that $D(\Delta) = H^2(\Gamma)$, and $u \mapsto \|u\|_{L^2(\Gamma)} + \|\Delta u\|_{L^2(\Gamma)}$ defines an equivalent norm on $H^2(\Gamma)$, see [38] and the references therein for more details. As in [38], we denote
$$H^k = \{u, v \} \in H^k(\Omega) \times H^k(\Gamma) : \ u_{\Gamma} = u_{\Gamma}^T, \quad k = 1, 2,$$
viewed as a subspace of $H^k(\Omega) \times H^k(\Gamma)$ with the natural topology inherited by $H^k(\Omega) \times H^k(\Gamma)$ and
$$E(t_0, t_1) = H^2(t_0, t_1; L^2) \cap L^2(t_0, t_1; H^2) \quad \text{for } t_1 > t_0, \text{ and } E_1 = E(0, T).$$
We denote \((H^k(\Omega))', H^{-k}(\Gamma)\) and \(\mathbb{H}^{-k}\) the topological dual of \(H^k(\Omega)\), \(H^k(\Gamma)\) and \(\mathbb{H}^k\), respectively, \(k = 1, 2\), and
\[
\mathbb{W} = \{ U \in L^2(0, T; \mathbb{H}^1) : \ U' \in L^2(0, T; \mathbb{H}^{-1}) \}.
\]

Now we shall recall some results on the well-posedness of the nonhomogeneous forward system
\[
\begin{align*}
\partial_t y - \div (A \nabla y) + B(x, t) \cdot \nabla y + a(x, t)y = f & \quad \text{in } \Omega_T, \\
\partial_t y_T - \div(A_T \nabla y_T) + \partial_t^\delta y + B_T(x, t) \cdot \nabla y_T + b(x, t)y_T = g & \quad \text{on } \Gamma_T, \\
(y(0), y_T(0)) = (y_0, y_{0T}) & \quad \text{in } \Omega_T
\end{align*}
\]
and the nonhomogeneous backward one
\[
\begin{align*}
-\partial_t \varphi - \div(A \nabla \varphi) - \div(B \varphi) + a(x, t)\varphi = f_1 & \quad \text{in } \Omega_T, \\
-\partial_t \varphi_T - \div(A_T \nabla \varphi_T) + \partial_t^\delta \varphi - \div(B_T \varphi_T) + \varphi_T B \cdot \nu + b(x, t)\varphi_T = g_1 & \quad \text{on } \Gamma_T, \\
(\varphi(T), \varphi_T(T)) = (\varphi_T, \varphi_T) & \quad \text{in } \Omega \times \Gamma.
\end{align*}
\]

Remark first that the system (12) can be rewritten as the following abstract Cauchy problem
\[
\begin{align*}
Y'(t) = AY - D(t)Y + F & \quad t > 0, \\
Y(0) = Y_0 & = (y_0, y_{0T}),
\end{align*}
\]
where we denoted \(Y = (y, y_T), F = (f, g)\).
\[
A = \begin{pmatrix} \div(A \nabla) & 0 \\ -\partial_t^\delta & \div(A_T \nabla_T) \end{pmatrix}, \quad D(A) = \mathbb{H}^2
\]
and
\[
D(t) = \begin{pmatrix} (B(t) \cdot \nabla + a(t)) & 0 \\ 0 & B_T(t) \cdot \nabla_T + b(t) \end{pmatrix}.
\]

Following [38], we can show that the operator \(A\) satisfies the following important property.

**Proposition 1** ([38]). The operator \(A\) is densely defined, and generates an analytic C0-semigroup \((e^{tA})_{t \geq 0}\) on \(L^2\). We have also \((L^2, \mathbb{H}^2)_{1,2} = \mathbb{H}^1\).

The following existence and uniqueness results hold, see [9] for the proof.

**Proposition 2** ([9]). For every \(Y_0 = (y_0, y_{0T}) \in \mathbb{L}^2\), \(f \in L^2(\Omega_T)\) and \(g \in L^2(\Gamma_T)\) the system (12) has a unique mild solution given by
\[
Y(t) = e^{tA}Y_0 + \int_0^t e^{(t-s)A} (F(s) - D(s)Y(s))ds
\]
for all \(t \in [0, T]\). Moreover, there exists a constant \(C > 0\) such that
\[
\|Y\|_{C([0,T];L^2)} \leq C(\|Y_0\|_{L^2} + \|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Gamma_T)}).
\]

For the backward system, we have the following well-posedness result, see [32].

**Proposition 3.** For every \(\Phi_T = (\varphi_T, \varphi_{T,T}) \in \mathbb{L}^2\), \(f_1 \in L^2(0, T; (H^1(\Omega))')\) and \(g_1 \in L^2(0, T; H^{-1}(\Gamma))\), the backward system (13) has a unique weak solution \(\Phi = (\varphi, \varphi_T) \in \mathbb{W}\).

That is
\[
\begin{align*}
\int_0^T \langle \partial_t \varphi, v \rangle_{(H^1(\Omega))', H^1(\Omega)} dt + \int_{\Omega_T} A \nabla \varphi \cdot \nabla v dx dt - \int_{\Gamma_T} \varphi B \cdot \nabla v dx dt & \quad t = 0, \\
+ \int_{\Omega_T} a \varphi v dx dt + \int_0^T \langle \partial_t \varphi_T, v_T \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} dt + \int_{\Gamma_T} A_T \nabla \varphi_T \cdot \nabla v_T d\sigma dt \\
- \int_{\Gamma_T} \varphi_T B_T \cdot \nabla v_T d\sigma dt + \int_{\Gamma_T} b \varphi_T v_T d\sigma dt & = \int_0^T \langle f_1, v \rangle_{(H^1(\Omega))', H^1(\Omega)} dt \\
+ \int_0^T \langle g_1, v_T \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} dt
\end{align*}
\]
for each \((v, v_1) \in L^2(0, T; \mathbb{H}^1)\), with \(v(0) = v_1(0) = 0\) and \((\varphi(T), \varphi_T(T)) = (\varphi_T, \varphi_T, T)\). Moreover, we have the estimate

\[
\max_{0 \leq t \leq T} \left\| \Phi(t) \right\|_{L^2}^2 + \left\| \Phi_0 \right\|_{L^2(0, T; \mathbb{H}^1)}^2 + \left\| \Phi' \right\|_{L^2(0, T; \mathbb{H}^{-1})}^2 \leq C \left( \left\| \Phi_T \right\|_{L^2}^2 + \left\| f_1 \right\|_{L^2(0, T; (H(\Omega)))} + \left\| g_1 \right\|_{L^2(0, T; H^{-1}(\Gamma))} \right),
\]

(18)

where \(C\) is a positive constant.

4. **Existence, uniqueness and characterization of Nash-equilibrium.** In this section, using the same ideas as in [5], we shall prove the existence and provide a characterization, in term of an adjoint system, of the Nash-equilibrium in the sense of (7). In the sequel, if \(X\) is a Hilbert space, \((\cdot, \cdot)_X\) stands for the scalar product of \(X\).

For \(i = 1, 2\), consider the functional given by (4) and denote the spaces

\[
\mathcal{H}_i = L^2(0, T; L^2(w_i)), \quad \mathcal{H}_{i,d} = L^2(0, T; L^2(w_{i,d})), \quad \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \quad \text{and} \quad \mathcal{H}_d = \mathcal{H}_{1,d} \times \mathcal{H}_{2,d}.
\]

Define the operators \(L_i \in \mathcal{L}(\mathcal{H}_i, L^2(0, T; L^2(\Omega)))\) and \(\ell_i \in \mathcal{L}(\mathcal{H}_i, L^2(0, T; L^2(\Omega)))\) by

\[
L_i(u_i) = Y_i = (y_i, y_{1,i}), \quad \text{and} \quad \ell_i(v_i) = y_{1,i},
\]

where \(Y_i = (y_i, y_{1,i})\) is the solution to

\[
\begin{cases}
\partial_t y_i - \text{div}(A \nabla y_i) + B(x, t) \cdot \nabla y_i + a(x, t)y_i = v_{1,i} & \text{in} \ \Omega_T, \\
\partial_{\nu_T} y_{1,i} - \text{div}(A_{1,d} \nabla y_{1,i}) + \partial_{\nu_T} y_{1,i} + B_{1}(x, t) \cdot \nabla y_{1,i} + b_{1}(x, t)y_{1,i} = 0 & \text{on} \ \Gamma_T, \\
(y_i(0), y_{1,i}(0)) = 0 & \text{in} \ \Omega \times \Gamma.
\end{cases}
\]

(19)

So, we can write \(Y\) as follows

\[
Y_f(v_1, v_2) = L_1(v_1) + L_2(v_2) + Q(f),
\]

where \(Q(f) = (q_1, q_2)\) solves the system

\[
\begin{cases}
\partial_t q - \text{div}(A \nabla q) + B(x, t) \cdot \nabla q + a(x, t)q = 1_\omega f & \text{in} \ \Omega_T, \\
\partial_{\nu_T} q_{1,i} - \text{div}(A_{1,d} \nabla q_{1,i}) + \partial_{\nu_T} q_{1,i} + B_{1}(x, t) \cdot \nabla q_{1,i} + b_{1}(x, t)q_{1,i} = 0 & \text{on} \ \Gamma_T, \\
(q(0), q_{1,i}(0)) = Y_0 & \text{in} \ \Omega \times \Gamma.
\end{cases}
\]

(20)

Using some ideas of [5], we shall prove the existence and uniqueness of Nash-equilibrium. More precisely, we have the following result.

**Proposition 4.** There exists a constant \(\mu_0 > 0\) such that, if \(\mu_i \geq \mu_0, \ i = 1, 2\), then for each \(f \in L^2(0, T; L^2(\Omega))\) there exists a unique Nash-equilibrium \((v_1^*, v_2^*) \in \mathcal{H}_1 \times \mathcal{H}_2\) for \((J_1, J_2)\) associated to \(f\). Furthermore, there exists a constant \(C > 0\) such that

\[
\| (v_1^*, v_2^*) \|_\mathcal{H} \leq C(1 + \| f \|_{L^2(0, T; L^2(\Omega))}).
\]

(21)

**Proof.** For \(i = 1, 2\) and a fixed \(f\), we have

\[
\partial_t J_i(f; v_1^*, v_2^*)(v_i) = \alpha_i(\ell_1(v_1^*) + \ell_2(v_2^*) + q(f) - y_{i,d}, \ell_i(v_i))_{\mathcal{H}_{1,d}} + \mu_i(v_i, v_i)_{\mathcal{H}_i} \quad \forall v_i \in \mathcal{H}_i,
\]

here, \(\partial_t J_i = \frac{\partial J_i}{\partial v_i}\) stands for the \(i\)-th partial derivative of \(J_i\). Using (7), we deduce that \((v_1^*, v_2^*)\) is a Nash-equilibrium if and only if

\[
\alpha_i(\ell_1(v_1^*) + \ell_2(v_2^*) - y_{i,d}, v_i)_{\mathcal{H}_{1,d}} + \mu_i(v_i, v_i)_{\mathcal{H}_i} = 0 \quad \forall v_i \in \mathcal{H}_i,
\]

where \(\tilde{y}_{i,d} = y_{i,d} - q\). Thus

\[
\alpha_i(\ell_1^*(v_1^*) + \ell_2^*(v_2^*) - \tilde{y}_{i,d}, v_i)_{\mathcal{H}_{1,d}} + \mu_i(v_i, v_i)_{\mathcal{H}_i} = 0 \quad \forall v_i \in \mathcal{H}_i.
\]

Finally, \((v_1^*, v_2^*)\) is a Nash-equilibrium if and only if

\[
\alpha_1 \ell_1^* \ell_1(v_1^*)_{w_{1,d}} + \ell_2 v_2^*_{1}, w_{1,d} + \mu_i v_i^* = \alpha_i \ell_1^* (\tilde{y}_{i,d} 1_{w_{1,d}}).
\]

We define the operator \(R = (R_1, R_2) \in \mathcal{L}(\mathcal{H})\) as follows

\[
R_i(v_1^*, v_2^*) = \alpha_i \ell_1^* \ell_1(v_1^*)_{w_{1,d}} + \ell_2 v_2^*_{1}, w_{1,d} + \mu_i v_i^*.
\]

We have

\[
R(v_1^*, v_2^*) = (\alpha_1 \ell_1^* (\tilde{y}_{1,d} 1_{w_{1,d}}), \alpha_2 \ell_1^* (\tilde{y}_{2,d} 1_{w_{2,d}})).
\]
Let us prove that $R$ is invertible. We have

$$
\langle R(v_1^*, v_1^*), (v_1^*, v_1^*) \rangle_H = \langle R_1(v_1^*, v_1^*), v_1^* \rangle_{H_1} + \langle R_2(v_1^*, v_2^*), v_2^* \rangle_{H_2}
$$

$$
= \alpha_1 \|1_{w_{1,d}} \ell_1(v_1^*) + 1_{w_{1,d}} \ell_2(v_2^*), \ell_1(v_1^*) \|_{H_1} + \mu_1 \|v_1^*\|_{H_1}^2
$$

$$
+ \alpha_2 \|1_{w_{2,d}} \ell_2(v_2^*), \ell_2(v_2^*) \|_{H_2} + \mu_2 \|v_2^*\|_{H_2}^2
$$

$$
= \alpha_1 \|1_{w_{1,d}} \ell_1(v_1^*), \ell_1(v_1^*) \|_{H_1} + \alpha_1 \|1_{w_{1,d}} \ell_1(v_1^*)\|_{H_1}^2 + \mu_1 \|v_1^*\|_{H_1}^2
$$

$$
+ \alpha_2 \|1_{w_{2,d}} \ell_2(v_2^*), \ell_2(v_2^*) \|_{H_2} + \alpha_2 \|1_{w_{2,d}} \ell_2(v_2^*)\|_{H_2}^2 + \mu_2 \|v_2^*\|_{H_2}^2
$$

$$
\geq -\frac{\alpha_1}{4} \|1_{w_{1,d}} \ell_2(\xi_{H_2,H_{1,d}})\|_{H_2}^2 + \mu_1 \|v_1^*\|_{H_1}^2
$$

$$
- \frac{\alpha_2}{4} \|1_{w_{2,d}} \ell_1(\xi_{H_1,H_{2,d}})\|_{H_1}^2 + \mu_2 \|v_2^*\|_{H_2}^2.
$$

Choosing the parameters $\mu_1$ and $\mu_2$ such that

$$
4\mu_1 > \alpha_2 \|1_{w_{2,d}} \ell_1(\xi_{H_1,H_{2,d}})\|_{H_1}^2 \text{ and } 4\mu_2 > \alpha_1 \|1_{w_{1,d}} \ell_2(\xi_{H_2,H_{1,d}})\|_{H_2}^2,
$$

there exists a constant $C_1 > 0$, such that

$$
\langle R(v_1^*, v_1^*), (v_1^*, v_1^*) \rangle_H \geq C_1 \|(v_1^*, v_1^*)\|_{H}^2.
$$

Since $R$ is continuous, we deduce, from Lax-Milgram theorem, that $R$ invertible. Furthermore, we have

$$
\|(v_2^*, v_2^*)\|_H \leq C \|(\alpha_1 \ell_1(y_{1,d}1_{w_{1,d}}), \alpha_2 \ell_2(y_{2,d}1_{w_{2,d}}))\|_H \leq C(1 + \|f\|_{L^2(0,T,L^2(\Omega))}^2).
$$

This achieves the proof. \hfill \square

Now, we shall characterize the Nash-equilibrium in term of the solution to an adjoint system. To this end, let us introduce the following adjoint systems

$$
\begin{aligned}
-\partial_t \phi^i - \text{div}(A \nabla \phi^i) - \text{div}(\phi^i B) + a(x, t) \phi^i &= \alpha_i (y - y_{i,d}) 1_{w_{i,d}} & \text{in } \Omega_T, \\
-\partial_t \phi^i - \text{div}(A_{y_i} \nabla \phi^i) - \text{div}(\phi^i B_{1}) + \phi^i B \cdot \nu + \partial^\alpha \phi^i + b(x, t) \phi^i_1 &= 0 & \text{on } \Gamma_T, \\
(\phi^i(T), \phi^i_1(T)) &= 0 & \text{in } \Omega \times \Gamma, \\
i = 1, 2.
\end{aligned}
$$

(22)

Multiplying (19) by $\Phi^i = (\phi^i, \phi^i_1)$ and integrating by parts, we find

$$
\alpha_i \langle y(f, v_1^i, v_2^i), \ell_i(v_1^i) \rangle_{L^2(0,T,L^2(\Omega))} = \langle \phi^i, v_1^i \rangle_H.
$$

(23)

We deduce that, $(v_1^1, v_2^1)$ is a Nash-equilibrium if and only if

$$
\langle \phi^1, v_1^1 \rangle_{H_1} + \mu_1 (v_1^1, v_1^1)_{H_1} = 0 \quad \text{and} \quad \langle \phi^2, v_2^2 \rangle_{H_2} + \mu_2 (v_2^2, v_2^2)_{H_2} = 0, \quad (v_1, v_2) \in H.
$$

Then

$$
v_i^* = -\frac{1}{\mu_i} \phi^i |_{\omega_i \times (0,T)} \quad \text{for } i = 1, 2.
$$

(24)

Let us collect all the previous results in a same system. We obtain the following forward-backward system, called optimality system

$$
\begin{aligned}
\partial_t y - \text{div}(A \nabla y) + B(x, t) \cdot \nabla y + a(x, t) y &= f_1, & \text{in } \Omega_T, \\
\partial_t y_{1_i} - \text{div}(A_{y_{1_i}} \nabla y_{1_i}) + B_1(x, t) \cdot \nabla y_{1_i} + \partial^\alpha y_{1_i} + b(x, t)y_{1_i} &= 0, & \text{on } \Gamma_T, \\
-\partial_t \phi^i - \text{div}(A \nabla \phi^i) - \text{div}(\phi^i B) + a(x, t) \phi^i &= \alpha_i (y - y_{i,d}) 1_{w_{i,d}} & \text{in } \Omega_T, \\
-\partial_t \phi^i - \text{div}(A_{y_i} \nabla \phi^i) - \text{div}(\phi^i B_{1}) + \phi^i B \cdot \nu + \partial^\alpha \phi^i + b(x, t) \phi^i_1 &= 0 & \text{on } \Gamma_T, \\
(\phi^i(T), \phi^i_1(T)) &= Y_0 & \text{in } \Omega \times \Gamma, \\
i = 1, 2.
\end{aligned}
$$

(25)
In Section 5, we will show that the null controllability for the system (25) is equivalent to a suitable observability inequality for the following adjoint system

\[
\begin{align*}
-\partial_t z - \text{div}(A\nabla z) - \text{div}(Bz) + a(x,t)z &= \alpha_1 \psi_1^1 \omega_1.a + \alpha_2 \psi_1^2 \omega_1.d \quad \text{in } \Omega_T, \\
-\partial_t \psi_i - \text{div}(A_i \nabla \psi_i) - \text{div}(B_i \psi_i) + z(T) \cdot \nu - \partial_t \psi_i(x,t)z &= 0 \quad \text{on } \Gamma_T, \\
\partial_t \psi_i &- \text{div}(A_i \nabla \psi_i) + B_i \nabla \psi_i + a(x,t)\psi_i = -\frac{1}{\alpha_i} \psi_i \quad \text{in } \Omega_T, \\
(\psi^i(0), \psi^i(T)) &= \mathbf{0} \quad \text{in } \Omega \times \Gamma, \\
(\psi^i(0), \psi^i(T)) &= \mathbf{0} \quad \text{in } \Omega \times \Gamma, \\
i &= 1, 2.
\end{align*}
\]

(26)

5. Carleman estimates and null controllability.

5.1. Carleman estimates. In this section, we shall state and show some suitable Carleman estimates needed to prove our main result concerning null controllability. To this end, let us first introduce the following well-known Morse function, see [19] for the existence of such function. Let \( \omega' \) be an open set of \( \Omega \) such that

\[\omega' \subset \omega \cap \omega_d\]

and \( \eta_0 \in C^2(\overline{\Omega}) \) be a function such that

\[
\begin{align*}
\eta_0 &> 0 \quad \text{in } \Omega \quad \text{and} \quad \eta_0 = 0 \quad \text{on } \Gamma, \\
\nabla \eta_0 &\neq 0 \quad \text{in } \overline{\Omega} \setminus \omega', \\
\nabla \eta_0 &\neq 0, \quad \nabla \eta_0 < 0, \quad \nabla \eta_0 = \partial_t \eta_0 \nu \quad \text{on } \Gamma
\end{align*}
\]

for some constant \( c > 0 \).

Introduce the following classical weight functions

\[
\xi(x,t) = e^{\frac{(2\lambda_1 + \eta_0(x))}{t(T - t)}} \quad \text{and} \quad \alpha(x,t) = \frac{e^{2\lambda_1 \|\eta_0\|_{\infty} + s_0(x)}}{t(T - t)},
\]

where \( x \in \overline{\Omega}, \ t \in (0, T) \) and \( \lambda \geq 1 \). The following lemma summarizes some important properties of the above functions. In what follows, \( C \) stands for a generic positive constant only depending on \( \Omega \) and \( \omega' \), whose value can change from line to line.

**Lemma 5.1.** The functions \( \xi \) and \( \alpha \) satisfy the following properties.

1. \( \partial_t \alpha \leq C \xi^2, \ |\partial_t \xi| \leq C \xi^2, \ |\nabla \alpha| \leq C \lambda \xi, \ |\nabla \xi| \leq C \lambda \xi \).
2. \( \partial_t (\xi \lambda^2 \xi e^{-2s\alpha}) \leq C s \lambda^2 \xi^2 e^{-2s\alpha} \).
3. \( \text{div}(\nabla(\xi^2 e^{-2s\alpha})) \leq C s \lambda^2 \xi^2 e^{-2s\alpha} \).
4. For all \( s > 0 \) and \( r \in \mathbb{R} \), the function \( e^{-2s\alpha} \xi^r \) is bounded on \( \Omega_T \).

Now, we recall some Carleman estimates for heat equation with dynamic boundary conditions needed to show our main result. Let us first introduce the following quantity

\[
\mathcal{I}(s, \lambda, \Phi) = s \lambda^2 \int_{\Omega_T} \xi e^{-2s\alpha} |\nabla \varphi|^2 dx \ dt + s \lambda \int_{\Gamma_T} \xi e^{-2s\alpha} |\nabla \varphi|^2 d\sigma \ dt + s \lambda \int_{\Gamma_T} \xi e^{-2s\alpha} |\varphi|^2 d\sigma \ dt + s \lambda \int_{\Gamma_T} \xi e^{-2s\alpha} |\varphi|^2 d\sigma \ dt + s^3 \lambda^3 \int_{\Omega_T} \xi e^{-2s\alpha} |\varphi|^2 d\sigma \ dt,
\]

where \( \lambda \) and \( s \) are positive real numbers and \( \Phi = (\varphi, \varphi_T) \) is a smooth function. Consider the following general form of the adjoint system

\[
\begin{align*}
-\partial_t q - \text{div}(A \nabla q) &= F_0 + \text{div}(F) \quad \text{in } \Omega_T, \\
-\partial_t q_T - \text{div}(A \nabla q_T) + \partial_t^2 q &= -F \cdot \nu + F_{\Gamma,0} + \text{div}(F_{\Gamma}) \quad \text{on } \Gamma_T, \\
(q(T), q_T(T)) &= Q_T \quad \text{in } \Omega \times \Gamma,
\end{align*}
\]

(27)
where $F_0 \in L^2(\Omega_T)$, $F \in (L^2(\Omega_T))^N$, $F_{T,0} \in L^2(\Gamma_T)$, $F_T \in (L^2(\Gamma_T))^N$ and $Q_T \in L^2$. We have the following Carleman estimates, see [2] and [9] for the proof.

**Lemma 5.2.** (i) If $F = F_T = 0$, then there exist $\lambda_1 \geq 1$, $s_1 \geq 1$ and $C_1 = C_1(\omega, \Omega) > 0$ such that the solution $Q = (q, q_T)$ to (27) satisfies

$$
I(s, \lambda; Q) \leq C \left( \int_{\Omega_T} e^{-2s_{\alpha}}|F_0|^2 \, dx \, dt + \int_{\Gamma_T} e^{-2s_{\alpha}}|F_{T,0}|^2 \, d\sigma \, dt 
+ s^3 \lambda^4 \int_{\omega \times (0,T)} \xi^3 e^{-2s_{\alpha}}|q|^2 \, dx \, dt \right)
$$

for all $\lambda \geq \lambda_1$ and $s \geq s_1$.

(ii) If the functions $F$ and $F_T$ are not necessarily zero, then there exist $\lambda_2 \geq 1$, $s_2 \geq 1$ and $C_2 = C_2(\omega, \Omega) > 0$ such that the solution $Q = (q, q_T)$ to (27) satisfies

$$
I(s, \lambda; Q) \leq C \left( \int_{\Omega_T} e^{-2s_{\alpha}}|F_0|^2 \, dx \, dt + \int_{\Gamma_T} e^{-2s_{\alpha}}|F_{T,0}|^2 \, d\sigma \, dt 
+ s^2 \lambda^2 \int_{\Omega_T} e^{-2s_{\alpha}}\xi^2 |F|^2 \, dx \, dt 
+ s^2 \lambda^2 \int_{\Gamma_T} e^{-2s_{\alpha}}\xi^2 |F_T|^2 \, d\sigma \, dt 
+ s^3 \lambda^4 \int_{\omega \times (0,T)} \xi^3 e^{-2s_{\alpha}}|q|^2 \, dx \, dt \right)
$$

for all $\lambda \geq \lambda_2$ and $s \geq s_2$.

Now, we shall prove a Carleman estimate for the coupled system (26).

**Theorem 5.3.** Assume that (10) holds. Then there exist $\lambda_3 \geq 1$, $s_3 \geq 1$ and $C > 0$ such that every solution $(Z, \Psi^1, \Psi^2)$ to (26) satisfies

$$
I(s, \lambda; Z) + I(s, \lambda; H) \leq C s^7 \lambda^8 \int_{\omega \times (0,T)} e^{-2s_{\alpha}}\xi^7 |z|^2 \, dx \, dt 
$$

(28)

for all $s \geq s_3$ and $\lambda \geq \lambda_3$, where $Z = (z, z_T)$ and $H = (h, h_T) = \alpha_1 \Psi^1 + \alpha_2 \Psi^2 = \alpha_1(\psi^1, \psi^1_T) + \alpha_2(\psi^2, \psi^2_T)$.

**Proof.** Let $\omega'$ be open sets such that

$$
\omega' \subset \omega'_1 \subset \subset \omega \cap \omega_d. 
$$

(29)

Let $\theta \in C^\infty(\Omega)$ such that

$$
0 \leq \theta \leq 1, \quad \theta = 1 \text{ in } \omega' \text{ and } \text{supp}(\theta) \subset \omega'_1.
$$

Applying Carleman estimate given in the second point of Lemma 5.2 to $Z$ with the observation region $\omega'$ instead of $\omega$ and $F_0 = h_1 \omega_d - az$, $F = Bz$, $F_{T,0} = -bz_T$, $F_T = B_T z_T$, we find

$$
I(s, \lambda; Z) \leq C \left( \int_{\Omega_T} e^{-2s_{\alpha}}|h|^2 \, dx \, dt + \int_{\Omega_T} e^{-2s_{\alpha}}|az|^2 \, dx \, dt + s^2 \lambda^2 \int_{\Omega_T} \xi^2 e^{-2s_{\alpha}}|Bz|^2 \, dx \, dt + 
\right.
$$

$$
+ s^2 \lambda^2 \int_{\Gamma_T} \xi^2 e^{-2s_{\alpha}}|B_T z_T|^2 \, d\sigma \, dt + \int_{\Gamma_T} e^{-2s_{\alpha}}|b z_T|^2 \, d\sigma \, dt + 
$$

$$
+ s^3 \lambda^4 \int_{\omega' \times (0,T)} \xi^3 e^{-2s_{\alpha}}|z|^2 \, dx \, dt \right) 
$$

$$
\leq C \left( \int_{\Omega_T} e^{-2s_{\alpha}}|h|^2 \, dx \, dt + \|a\|_{L^\infty}^2 \int_{\Omega_T} e^{-2s_{\alpha}}|z|^2 \, dx \, dt + \|B\|_{L^\infty}^2 s^2 \lambda^2 \int_{\Omega_T} \xi^2 e^{-2s_{\alpha}}|z|^2 \, dx \, dt + 
\right.
$$

$$
\left. \|B_T\|_{L^\infty}^2 s^2 \lambda^2 \int_{\Gamma_T} \xi^2 e^{-2s_{\alpha}}|z_T|^2 \, d\sigma \, dt + \|b\|_{L^\infty} \int_{\Gamma_T} e^{-2s_{\alpha}}|z_T|^2 \, d\sigma \, dt + 
\right.
$$

$$
\left. s^3 \lambda^4 \int_{\omega' \times (0,T)} \xi^3 e^{-2s_{\alpha}}|z|^2 \, dx \, dt \right).
$$
Choosing \( s_2 \) large enough, the previous inequality becomes

\[
\mathcal{I}(s, \lambda; Z) \leq C \left( \int_{\Omega_T} e^{-2s\alpha} |h|^2 \, dx \, dt + s^3 \lambda^4 \int_{\omega' \times (0, T)} \xi^3 e^{-2s\alpha} |z|^2 \, dx \, dt \right).
\]

(30)

Using the Carleman estimate given in the first point of Lemma 5.2 for \( H \), and the fact that \( \xi \geq \frac{4}{\alpha} \), we obtain

\[
\mathcal{I}(s, \lambda; H) \leq C \left( \int_{\Omega_T} e^{-2s\alpha} |z|^2 \, dx \, dt + \int_{\Omega_T} e^{-2s\alpha} |ah|^2 \, dx \, dt \\
+ \int_{\Gamma_T} e^{-2s\alpha} |B : \nabla h|^2 \, dx \, dt + \int_{\Gamma_T} e^{-2s\alpha} |bh|^2 \, d\sigma \, dt \\
+ \int_{\Gamma_T} e^{-2s\alpha} |Bh : \nabla h|^2 \, d\sigma \, dt + s^3 \lambda^4 \int_{\omega' \times (0, T)} |\nabla^2z|^2 \, dx \, dt \right).
\]

(31)

By choosing \( s_1 \) large enough, one has

\[
\mathcal{I}(s, \lambda; H) \leq C \left( \int_{\Omega_T} e^{-2s\alpha} |z|^2 \, dx \, dt + s^3 \lambda^4 \int_{\omega' \times (0, T)} \xi^3 e^{-2s\alpha} |h|^2 \, dx \, dt \right).
\]

(32)

From (30) and (31) and choosing \( s_1 \) and \( s_2 \) large enough, we deduce

\[
\mathcal{I}(s, \lambda; Z) + \mathcal{I}(s, \lambda; H) \leq C \left( s^3 \lambda^4 \int_{\omega' \times (0, T)} \xi^3 e^{-2s\alpha} |z|^2 \, dx \, dt + s^3 \lambda^4 \int_{\omega' \times (0, T)} \xi^3 e^{-2s\alpha} |h|^2 \, dx \, dt \right).
\]

Thanks to (10) and (29), we have

\[
h = -\partial_t z - \text{div}(A\nabla z) - \text{div}(Bz) + a(x, t)z \quad \text{in} \quad \omega'.
\]

(33)

Using (33) and the fact that \( \text{supp}(\theta) \subset \omega' \), we find

\[
s^3 \lambda^4 \int_{\omega' \times (0, T)} \theta z^3 e^{-2s\alpha} |h|^2 \, dx \, dt =
\]

\[
s^3 \lambda^4 \int_{\omega' \times (0, T)} \theta z^3 e^{-2s\alpha} h(-\partial_t z - \text{div}(A\nabla z) - \text{div}(Bz) + a(x, t)z) \, dx \, dt
\]

\[
= s^3 \lambda^4 \int_{\omega' \times (0, T)} \theta z^3 e^{-2s\alpha} z \partial_h \, dx \, dt + s^3 \lambda^4 \int_{\omega' \times (0, T)} \partial_h (\theta z^3 e^{-2s\alpha}) \, dx \, dt + s^3 \lambda^4 \int_{\omega' \times (0, T)} \text{div}(A\nabla(\theta z^3 e^{-2s\alpha})) \, dx \, dt
\]

\[
+ s^3 \lambda^4 \int_{\omega' \times (0, T)} \theta z^3 e^{-2s\alpha} ahz \, dx \, dt.
\]

By the symmetry of the matrix \( A \), we obtain

\[
\text{div}(A\nabla(\theta z^3 e^{-2s\alpha})) = \text{div}(A\nabla(\theta z^3 e^{-2s\alpha})) + 2\nabla(\theta z^3 e^{-2s\alpha}) \cdot A\nabla h,
\]

\[
\nabla(\theta z^3 e^{-2s\alpha}) = \theta z^3 e^{-2s\alpha} \nabla h.
\]

Then

\[
s^3 \lambda^4 \int_{\omega' \times (0, T)} \theta z^3 e^{-2s\alpha} |h|^2 \, dx \, dt =
\]
Using the equation satisfied by

\[ s^3 \lambda^4 \int_{\omega'} \partial_t (\xi^3 e^{-2s\alpha} \partial_t h) dx dt + s^3 \lambda^4 \int_{\omega'} \div(\nabla (\xi^3 e^{-2s\alpha} \partial_t h)) dx dt \]

\[ + \frac{\alpha_1}{\mu_1} s^3 \lambda^4 \int_{\omega'} \div(\nabla (\xi^3 e^{-2s\alpha} \partial_t h)) dx dt + \frac{\alpha_2}{\mu_2} s^3 \lambda^4 \int_{\omega'} \div(\nabla (\xi^3 e^{-2s\alpha} \partial_t h)) dx dt \]

\[ + 2s^3 \lambda^4 \int_{\omega'} (\xi^3 e^{-2s\alpha}) \cdot \nabla h dx dt + s^3 \lambda^4 \int_{\omega'} (\xi^3 e^{-2s\alpha}) \cdot \nabla h dx dt \]

\[ + \frac{3s^3 \lambda^4}{\mu_1} \int_{\omega'} \theta \xi^3 e^{-2s\alpha} B \cdot \nabla h dx dt + s^3 \lambda^4 \int_{\omega'} \theta \xi^3 e^{-2s\alpha} h dx dt. \]

Using the equation satisfied by \( h = \alpha_1 \psi^1 + \alpha_2 \psi^2 \), we find

\[ s^3 \lambda^4 \int_{\omega'} \theta \xi^3 e^{-2s\alpha} |h|^2 dx dt \leq \frac{C}{\epsilon} s^3 \lambda^8 \int_{\omega'} \xi^7 e^{-2s\alpha} |z|^2 dx dt \]

\[ + C s^3 \lambda^6 \int_{\omega'} e^{-2s\alpha} |h| |z| dx dt \]

By the properties of the functions \( \alpha \) and \( \xi \) given in Lemma 5.1, we can estimate the the four last terms in the right hand side in the above inequality. To this end, fix \( \epsilon \) small enough and using Young inequality, we deduce for the first term

\[ s^3 \lambda^4 \int_{\omega'} \partial_t (\xi^3 e^{-2s\alpha} \partial_t h) dx dt \leq C s^3 \lambda^4 \int_{\omega'} \theta \xi^3 e^{-2s\alpha} h dx dt \]

\[ \leq \frac{C}{\epsilon} s^3 \lambda^8 \int_{\omega'} \xi^7 e^{-2s\alpha} |z|^2 dx dt + C s^3 \lambda^4 \int_{\omega'} \theta \xi^3 e^{-2s\alpha} |h|^2 dx dt. \]

For the second one, we have

\[ s^3 \lambda^4 \int_{\omega'} \div(\nabla (\xi^3 e^{-2s\alpha} \theta)) h dx dt \leq C s^3 \lambda^5 \int_{\omega'} e^{-2s\alpha} |h| |z| dx dt \]

\[ \leq \frac{C}{\epsilon} s^3 \lambda^8 \int_{\omega'} \xi^7 e^{-2s\alpha} |z|^2 dx dt + C s^3 \lambda^4 \int_{\omega'} \xi^3 e^{-2s\alpha} |h|^2 dx dt. \]

The third one can be estimated as

\[ - s^3 \lambda^4 \int_{\omega'} \nabla h \cdot \nabla (\xi^3 e^{-2s\alpha} \theta) dx dt \leq C s^3 \lambda^5 \int_{\omega'} \xi^4 e^{-2s\alpha} |z| \nabla h dx dt \]

\[ \leq \frac{C}{\epsilon} s^3 \lambda^8 \int_{\omega'} \xi^7 e^{-2s\alpha} |z|^2 dx dt + C s^3 \lambda^4 \int_{\omega'} \xi e^{-2s\alpha} |\nabla h|^2 dx dt. \]

For the last one, we have

\[ s^3 \lambda^4 \int_{\omega'} B \cdot \nabla (\xi^3 e^{-2s\alpha} h) dx dt \leq s^3 \lambda^4 \int_{\omega'} h dx dt \]

\[ \leq \frac{C}{\epsilon} s^3 \lambda^8 \int_{\omega'} \xi^7 e^{-2s\alpha} |z|^2 dx dt + C s^3 \lambda^4 \int_{\omega'} \xi^3 e^{-2s\alpha} |h|^2 dx dt. \]

Using (32), (35)-(38) and choosing \( \epsilon \) small enough, we find

\[ \mathcal{I}(s, \lambda; Z) + \mathcal{I}(s, \lambda; H) \leq \frac{\alpha_1}{\mu_1} s^3 \lambda^4 \int_{\omega'} \xi^3 e^{-2s\alpha} |z|^2 dx dt + \frac{\alpha_2}{\mu_2} s^3 \lambda^4 \int_{\omega'} \theta \xi^3 e^{-2s\alpha} |z|^2 dx dt \]

\[ + C s^3 \lambda^8 \int_{\omega'} \xi^7 e^{-2s\alpha} |z|^2 dx dt. \]
For $s_1$ and $s_2$ large enough, we conclude

$$\mathcal{I}(s, \lambda; Z) + \mathcal{I}(s, \lambda; H) \leq C s^7 \lambda^8 \int_{\omega \times (0, T)} \xi^7 e^{-2s\alpha} |z|^2 \, dx \, dt.$$  

The proof is then finished. \hfill \square

**Remark 2.** As mentioned in [5], the assumption (10) is used to prove the Carleman estimate (28).

To prove the needed observability inequality, we are going to improve the Carleman inequality (28). To this end, following [5], we modify the weight functions $\xi$ and $\alpha$. More precisely, we introduce the functions

$$l(t) = \begin{cases} 
T^2 / 4 & \text{if } t \in (0, T/2), \\
 t(T - t) & \text{if } t \in (T/2, T), 
\end{cases}$$

and

$$\tilde{\xi}(x, t) = \frac{e^{\lambda(2\|\eta_0\|_\infty + \eta_0(x))}}{l(t)}, \quad \tilde{\eta}(x, t) = \frac{e^{4\lambda\|\eta_0\|_\infty - e^{\lambda(2\|\eta_0\|_\infty + \eta_0(x))}}}{l(t)},$$

$$\tilde{\eta}(t) = \min_{x \in \Omega} \tilde{\xi}(x, t) \quad \text{and} \quad \tilde{\xi}(t) = \max_{x \in \Omega} \tilde{\eta}(x, t). \tag{39}$$

Point out that the functions $\tilde{\xi}$ and $\tilde{\eta}$ do not blow up at $t = 0$, this will be important to get an improved Carleman estimate. We also denote

$$\tilde{\mathcal{I}}(s, \lambda; Z) = \int_{\Omega \times (0, T)} e^{-2\tilde{\eta} \tilde{\xi}} |z|^2 \, dx \, dt + \int_{\Gamma \times (0, T)} \xi \xi (z|T|^2 \, d\sigma \, dt,$$

where $s, \lambda > 1$ and $Z = (z, z_1)$. With these definitions and the same notations of Theorem 5.3, we have the following result.

**Lemma 5.4.** Assume that (10) holds, then there exist $\lambda_4 \geq 1$, $s_4 \geq 1$ and $C > 0$ such that every solution $(Z, \Psi_1, \Psi_2)$ to (26) satisfies

$$||Z(0)||^2_2 + \tilde{\mathcal{I}}(s, \lambda; Z) + \mathcal{I}(s, \lambda; H) \leq C \int_{\omega \times (0, T)} \xi^7 e^{-2s\alpha} |z|^2 \, dx \, dt$$

for all $s \geq s_4$ and $\lambda \geq \lambda_4$.

**Proof.** The proof relies on the Carleman estimate (28) and some energy estimates. Following the strategy in [5], let us introduce a function $\eta \in C^1([0, T])$ such that

$$\eta = 1 \quad \text{in } [0, T/2], \quad \eta = 0 \quad \text{in } [3T/4, T] \quad \text{and} \quad \eta' \leq C/T^2. \tag{40}$$

Set $P = \eta Z$. It is clear that $P = (p, \nu)$ satisfies

$$\begin{cases} 
-\partial_t p - \text{div}(A \nabla p) - \text{div}(pB) + a(x, t)p = \eta h_1 \omega + \partial_t \eta z \\
-\partial_t \nu - \text{div}(A \nabla \nu B) - \text{div}(\nu B_T) + p \nu B \cdot \nu + \nu \partial_x^0 p + b(x, t)p = \partial_t \eta z \quad \text{on } \Gamma, \\
(p(T), \nu(T)) = 0, \quad \text{in } \Omega \times \Gamma. \tag{41}
\end{cases}$$

From the energy estimate (18) for $P$, there exists a positive constant $C$ such that

$$||Z(0)||^2_2 + \int_{\Omega \times (0, T/2)} e^{-2\eta \tilde{\xi}} \xi^2 \, dx \, dt + \int_{\Gamma \times (0, T/2)} e^{-2\eta \tilde{\eta}} \xi |z|_T^2 \, d\sigma \, dt \leq C \left( \frac{1}{T^2} \int_{\Omega \times (T/2, 3T/4)} |Z|^2_2 + \int_{\Omega \times (0, T/2)} e^{-2\alpha \eta} \xi^2 |z|^2 \, dx \, dt + \int_{\Gamma \times (T/2, 3T/4)} e^{-2\lambda \eta} \xi |z|_T^2 \, d\sigma \, dt \right).$$

Using the fact that the weight functions $e^{-2\alpha \eta}$ and $\tilde{\xi}$ (resp. $e^{-2\lambda \eta}$ and $\tilde{\xi}$) are bounded in $\Omega \times [0, T/2]$ (resp. $\Omega \times [T/2, 3T/4]$), we deduce

$$\begin{align*}
||Z(0)||^2_2 + \int_{\Omega \times (0, T/2)} e^{-2\eta \tilde{\xi}} \xi^2 \, dx \, dt + \int_{\Gamma \times (0, T/2)} e^{-2\eta \tilde{\eta}} \xi |z|_T^2 \, d\sigma \, dt \\
\leq C \left( \frac{1}{T^2} \int_{\Omega \times (T/2, 3T/4)} e^{-2\alpha \eta} \xi^2 |z|^2 \, dx \, dt + \frac{1}{T^2} \int_{\Omega \times (T/2, 3T/4)} e^{-2\lambda \eta} \xi |z|_T^2 \, d\sigma \, dt \right).
\end{align*}$$
\[
\begin{align*}
&\int_{\Omega \times (0, 3T/4)} |h_2|^2 dx \, dt + \int_{\Gamma \times (0, 3T/4)} |h_1|^2 d\sigma \, dt \\
&\leq C \left( I(s, \lambda, Z) + \|H\|_{L^2(0, 3T/4; L^2)} \right).
\end{align*}
\] (42)

On the other hand, since \( \alpha = \bar{\sigma} \) and \( \xi = \bar{\xi} \) in \((T/2, T)\), we have
\[
\int_{\Omega \times (T/2, T)} e^{-2\bar{\sigma}\bar{\xi}}|z|^2 dx \, dt + \int_{\Gamma \times (T/2, T)} e^{-2\bar{\sigma}\bar{\xi}}|\xi|^2 d\sigma \, dt \leq CI(s, \lambda, Z).
\] (43)

From (42) and (43), we obtain
\[
\|Z(0)\|_{L^2}^2 + \int_{\Omega \times (0, T)} e^{-2\bar{\sigma}\bar{\xi}}|z|^2 dx \, dt + \int_{\Gamma \times (0, T)} e^{-2\bar{\sigma}\bar{\xi}}|\xi|^2 d\sigma \, dt \leq C \left( I(s, \lambda, Z) + \|H\|_{L^2(0, 3T/2; L^2)} \right).
\]

Using again the energy estimate and the fact that the weight functions \( e^{-2\bar{\sigma}\bar{\xi}} \) and \( \bar{\xi} \) are bounded in \([0, 3T/4]\), we see that
\[
\|H\|_{L^2(0, 3T/2; L^2)}^2 \leq C \left( \frac{\alpha_1}{\mu_1} + \frac{\alpha_2}{\mu_2} \right) \left( \int_{\Omega \times (0, T)} e^{-2\bar{\sigma}\bar{\xi}}|z|^2 dx \, dt + \int_{\Gamma \times (0, T)} e^{-2\bar{\sigma}\bar{\xi}}|\xi|^2 d\sigma \, dt \right).
\]

For \( \mu_1 \) and \( \mu_2 \) large enough, we deduce
\[
\|Z(0)\|_{L^2}^2 + \int_{\Omega \times (0, T)} e^{-2\bar{\sigma}\bar{\xi}}|z|^2 dx \, dt + \int_{\Gamma \times (0, T)} e^{-2\bar{\sigma}\bar{\xi}}|\xi|^2 d\sigma \, dt \leq CI(s, \lambda, Z).
\]

Then, we conclude that
\[
\|Z(0)\|_{L^2}^2 + I(s, \lambda; Z) + I(s, \lambda; H) \leq C \left( I(s, \lambda; Z) + I(s, \lambda; H) \right).
\]

By Carleman estimate (28) and this last inequality, we find
\[
\|Z(0)\|_{L^2}^2 + I(s, \lambda; Z) + I(s, \lambda; H) \leq C \int_{\Omega \times (0, T)} e^{-2\alpha s \bar{\xi}}|z|^2 dx \, dt.
\]

This finishes the proof. \( \square \)

5.2. Observability and null controllability. In this section we will prove our main result of controllability given in Theorem 2.1. Let us first recall that the characterization (24) of the follower controls adds two additional equations to the system (1), and our problem is then reduced to look for a control \( f \in L^2(0, T; L^2(\omega)) \) such that the solution \((Y, \Phi^1, \Phi^2)\) to the optimality system (25) satisfies \( Y(T) = 0 \). To this end, we shall prove the following observability inequality.

**Proposition 5.** Assume that (10) holds and \( \mu_i > 0 \), \( i = 1, 2 \), are large enough. Then, there exist a constant \( C > 0 \) and a positive weight function \( \rho = \rho(t) \) blowing up at \( t = T \), such that, for any \( Z_T \in \mathbb{L}^2 \), the solution \((Z, \Psi^1, \Psi^2)\) to (26) satisfies the following inequality
\[
\|Z(0)\|_{L^2}^2 + \sum_{i=1}^{2} \int_{\Omega_T} \rho^{-2} |\psi^i|^2 dx \, dt + \sum_{i=1}^{2} \int_{\Gamma_T} \rho^{-2} |\psi^i|^2 d\sigma \, dt \leq C \int_{\Omega \times (0, T)} |z|^2 dx \, dt.
\] (44)

**Proof.** Fix \( s \) large enough and set \( \rho = e^{|s|} \). Thus \( \rho \) is a positive function blowing up at \( t = T \). For \( i = 1, 2 \), using the equation satisfied by \( \psi^i \) in (25), we readily see that
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|\rho^{-1}\psi^i(t)\|_{L^2}^2) + \int_{\Omega} \rho^{-2} A\nabla \psi^i \cdot \nabla \psi^i dx &- \int_{\Omega} \rho^{-2} \psi^i B \cdot \nabla \psi^i dx + \int_{\Omega} a|\rho^{-1}\psi^i|^2 dx \\
&= \int_{\Omega} \rho^{-2} A_{i1} \nabla \psi^i \cdot \nabla \psi^i dx - \int_{\Gamma} \rho^{-2} \psi^i B_1 \cdot \nabla \psi^i d\sigma + \int_{\Gamma} b|\rho^{-1}\psi^i|^2 d\sigma \\
&= \frac{1}{\mu_i} \|\rho^{-2}\psi^i\|^2_{L^2(0, T; L^2(\omega))} + \rho \rho^{-3} \|\Psi^i\|_{L^2}^2.
\end{align*}
\]
By Young inequality and for a positive $\lambda$, we have
\[
\int_{\Omega} \rho^{-2} |\psi| B \cdot \nabla \psi^i dx \leq \frac{\lambda}{2} \|B\|_\infty^2 \rho^{-1} |\psi|^2 L_2(\Omega) + \frac{1}{2\lambda} \|\nabla \psi^i\|_{L_2(\Omega)}^2,
\]
\[
\int_{\Gamma} \rho^{-2} |\psi| B_T \cdot \nabla \psi^i dx \leq \frac{\lambda}{2} \|B_T\|_\infty^2 \rho^{-1} |\psi|^2 L_2(\Gamma) + \frac{1}{2\lambda} \|\nabla \psi^i\|_{L_2(\Gamma)}^2 \quad \text{in } (0, T).
\]
Using the ellipticity of $A$ and $A_T$ and choosing $\lambda$ large enough, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\rho^{-1} \Psi^i(t)\|_{L_2}^2 \leq C (\|a\|_\infty + \|b\|_\infty + \|B\|_\infty^2 + \|B_T\|_\infty^2) \|\Psi^i(t)\|_{L_2}^2 
+ \frac{1}{\mu_i} \|\rho^{-2} z^i\|_{L_2(0,T;L^2(\omega))}^2, \quad t \in (0, T).
\]
Thus, from Gronwall’s lemma and the fact that $\Psi^i(0) = 0$, it follows that
\[
\|\rho^{-2} (t) \Psi^i(t)\|_{L_2}^2 \leq C \|\rho^{-2} z^i\|_{L_2(0,T;L^2(\omega))}^2, \quad t \in (0, T). \tag{45}
\]
Using Lemma 5.4, we find
\[
\|Z(0)\|_{L_2}^2 + \int_{0}^{T} \rho^{-2} |z|^2 dx dt + I(s, \lambda; H) \leq C \int_{\omega} e^{-2\alpha t} \xi^7 |z|^2 dx dt. \tag{46}
\]
From the inequalities (45) and (46) and the fact that the function $e^{-2\alpha t} \xi^7$ is bounded in $\Omega_T$, we obtain
\[
\|Z(0)\|_{L_2}^2 + \int_{\Omega_T} \rho^{-2} |\psi|^2 dx dt + \int_{\Gamma_T} \rho^{-2} |\psi|^2 d\sigma dt + \int_{\Omega_T} \rho^{-2} |\psi|^2 dx dt + \int_{\Gamma_T} \rho^{-2} |\psi|^2 d\sigma dt \leq C \int_{\omega} |z|^2 dx dt.
\]
This ends the proof. \qed

Now, we prove the controllability result in Theorem 2.1. Since we have proved the existence and uniqueness of Nash equilibrium in Proposition 4, it remains to prove the following result.

**Proposition 6.** Let $\rho = \rho(t)$ the weight function given in Proposition 5. Then, for any $y_i,d \in H_{i,d}, i = 1, 2,$ satisfying (11) and $Y_0 \in L^2$, there exists a control $f \in L^2(0, T; L^2(\omega))$ with minimal norm such that
\[
\|f\|_{L^2(0, T; L^2(\omega))} \leq C \left( \|Y_0\|_{L^2} + \|\rho y_1,d\|_{L^2(\Omega_T)} + \|\rho y_2,d\|_{L^2(\Gamma_T)} \right), \tag{47}
\]
and the associated state satisfies $Y(0) = 0$, where $(Y, \Phi^1, \Phi^2)$ is the solution to (25) and $C$ is a positive constant.

**Proof.** Multiplying the solution $(Y, \Phi^1, \Phi^2)$ to (25) by the solution $(Z, \Psi^1, \Psi^2)$ to (26) and integrating by parts, we find
\[
\langle Y(T), Z(T) \rangle_{L^2} - \langle Y(0), Z(0) \rangle_{L^2} = \int_{\omega \times (0, T)} f \psi dx dt - \sum_{i=1}^{2} \alpha_i \int_{\omega_i,d \times (0, T)} y_i,d \Psi^i dx dt. \tag{48}
\]
Thus, the null controllability property is equivalent to find, for each $Y_0 \in L^2$, a control $f$ such that, for any $Z_T \in L^2$, one has
\[
\int_{\omega \times (0, T)} f \psi dx dt = -\langle Y(0), Z(0) \rangle_{L^2} + \sum_{i=1}^{2} \alpha_i \int_{\omega_i,d \times (0, T)} y_i,d \Psi^i dx dt. \tag{49}
\]
To this end, let $\epsilon > 0$ and $Z_T \in L^2$. Introduce the following functional
\[
J_\epsilon(Z_T) = \frac{1}{2} \int_{\omega \times (0, T)} |\psi|^2 dx dt + \epsilon \|Z_T\|_{L^2} + \langle Y_0, Z(0) \rangle_{L^2} 
- \sum_{i=1}^{2} \alpha_i \int_{\omega_i,d \times (0, T)} y_i,d \Psi^i dx dt.
\]
It is clear that $J_\epsilon : \mathbb{L}^2 \rightarrow \mathbb{R}$ is continuous and convex. Moreover, from Young inequality together with the observability inequality (44), we have, for $\delta > 0$,

$$\langle Y(0), Z(0) \rangle_{\mathbb{L}^2} \geq - \frac{1}{25} \|Y_0\|_{\mathbb{L}^2}^2 - \frac{\delta}{2} \|Z(0)\|_{\mathbb{L}^2}^2 \geq -\delta \left( C \int_{\omega \times (0,T)} |z|^2 \, dx \, dt + \int_{\Omega_T} \rho^{-2} |\psi|^2 \, dx \, dt + \int_{\Omega_T} \rho^{-2} |\psi|^2 \, dx \, dt \right).$$

Choosing $\delta = \frac{1}{2C}$ and using the above inequalities, we get

$$J_\epsilon(Z_T) \geq \frac{1}{4} \int_{\omega \times (0,T)} |z|^2 \, dx \, dt + \epsilon \|Z_T\|_{\mathbb{L}^2} - C \left( \|Y_0\|_{\mathbb{L}^2}^2 + \sum_{i=1}^{2} \alpha_i^2 \int_{\omega_i \times (0,T)} \rho^2 |y_{i,d}|^2 \, dx \, dt \right).$$

(50)

Consequently, $J_\epsilon$ is coercive in $\mathbb{L}^2$, and then $J_\epsilon$ admits a unique minimizer $Z_T^\epsilon$. If $Z_T^\epsilon \neq 0$, we have

$$\langle J'(Z_T^\epsilon), Z_T \rangle_{\mathbb{L}^2} = 0, \quad Z_T \in \mathbb{L}^2.$$  

(51)

Then, for all $Z_T \in \mathbb{L}^2$, we have

$$\int_{\omega \times (0,T)} z^i \, dz \, dt + \epsilon \frac{Z_T^\epsilon}{\|Z_T^\epsilon\|_{\mathbb{L}^2}} , Z_T \rangle_{\mathbb{L}^2} + \langle Y_0, Z(0) \rangle_{\mathbb{L}^2} - \sum_{i=1}^{2} \alpha_i \int_{\omega_i \times (0,T)} y_{i,d} \psi \, dx \, dt = 0,$$

(52)

where we have denoted by $(Z_\epsilon, \Psi_1^\epsilon, \Psi_2^\epsilon)$ the solution to (25) with $Z_T = Z_T^\epsilon$. Take $f_\epsilon = z_\epsilon$ in (48), we find

$$\epsilon \left( \frac{Z_T^\epsilon}{\|Z_T^\epsilon\|_{\mathbb{L}^2}}, Z_T \right)_{\mathbb{L}^2} + \langle Y_\epsilon(T), Z_T \rangle_{\mathbb{L}^2} = 0, \quad Z_T \in \mathbb{L}^2.$$  

(53)

Hence

$$\|Y_\epsilon(T)\|_{\mathbb{L}^2} \leq \epsilon.$$  

(54)

Taking $z = z_\epsilon$ in (52), and using observability inequality (44) together with Young inequality, we deduce

$$\|f_\epsilon\|_{L^2(0,T; L^2(\omega))} \leq C \left( \|Y_0\|^2 + \sum_{i=1}^{2} \alpha_i^2 \int_{\omega_i \times (0,T)} \rho^2 |y_{i,d}|^2 \, dx \, dt \right).$$

(55)

If $Z_T^\epsilon = 0$, arguing as in [17], we deduce that

$$\lim_{t \to 0^+} \frac{J_\epsilon(t Z_T^\epsilon)}{t} \geq 0, \quad Z_T \in \mathbb{L}^2.$$  

(56)

Using (56) and take $f_\epsilon = 0$, we obtain (54) and (55). By (55), we deduce that there exist $f \in L^2(0,T; L^2(\omega))$ and a subsequence, still denoted by $f_\epsilon$, such that

$$f_\epsilon \rightharpoonup f \quad \text{weakly in} \quad L^2(0,T; L^2(\omega)).$$

By energy estimate, we deduce

$$Y_\epsilon(T) \rightharpoonup Y(T) \quad \text{weakly in} \quad \mathbb{L}^2.$$  

(57)

Using (54) and (57), we deduce that $Y(T) = 0$. This concludes the null controllability result. $\square$
As a consequence of the above HUM method, see for instance [46], the control we have constructed is one of the minimal norm, and it is characterized as follows.

**Corollary 1.** Let $Z_T^c$ be the unique minimizer of $J_c$, then the leader control with minimal norm is given by the limit of $f^c = z_1w_1 = z_1\lambda_0$ as $\epsilon$ goes to zero, where $(Y, Z, \Phi^1, \Phi^2, \Psi^1, \Psi^2) = (y, y_T, (z, z_T), (\varphi_1, \psi_1^c), (\varphi_2, \psi_2^c), (\psi_1^1, \psi_1^3), (\psi_2^1, \psi_2^3))$ satisfies

$$
\begin{align*}
&\frac{\partial}{\partial t}y - \text{div}(A\nabla y) + B(x, t) \cdot \nabla y + a(x, t)y = 1_{\omega_1}z - \frac{1}{\rho_1}y_1^1\omega_1 - \frac{1}{\rho_2}y_2^2\omega_2, \quad \text{in } \Omega_T \\
&\frac{\partial}{\partial t}y_T - \text{div}(A\nabla y_T) = 0, \quad \text{on } \Gamma_T \\
&-\partial_z - \text{div}(A\nabla z) - \text{div}(Bz) - a(x, t)z = \alpha_1\omega_1, z = \varphi_1^c + \alpha_2\psi_2^c, \quad \text{in } \Omega_T \\
&-\partial_t\varphi_1^c - \text{div}(A\nabla\varphi_1^c) - \text{div}(B\varphi_1^c) - \varphi_1^c B \cdot \varphi_1^c + b(x, t)\varphi_1^c = 0, \quad \text{in } \Omega_T \\
&\frac{\partial}{\partial t}\psi_1^c - \text{div}(A\nabla\psi_1^c) + B \cdot \nabla\psi_1^c + \alpha_1\omega_1, \quad \text{in } \Omega_T \\
&(y(0), y_T(0)) = Y_0, \quad (z(T), z_T(T)) = Z_T^c, \quad (\varphi_1^1(T), \psi_1^c(T)) = (0, 0), \quad (\psi_1^1(0), \psi_1^c(0)) = (0, 0), \\
&i = 1, 2.
\end{align*}
$$

(58)

**Remark 3.** (a) For a penalized HUM method of constructing controls with minimal norm, we refer to [10] and [22].

(b) We shall give the idea behind the form of the observability inequality (44) and we prove the equivalence between the controllability of (25) and the observability inequality (44). To this end, let us introduce the following functionals

$$
\mathcal{L}_T : L^2(0; T; L^2(\omega)) \rightarrow L^2
$$

and

$$
\mathcal{R}_T : L^2(0; T; L^2(\omega)) \times L^2(0; T; L^2(\omega)) \rightarrow L^2
$$

defined by

$$
\mathcal{L}_T(f) = Y(T, Y_0 = 0, f, y_1, y_2 = 0) = Y_1(T)
$$

and

$$
\mathcal{R}_T(Y_0, y_1, y_2) = Y(T, Y_0 = 0, f = 0, y_1, y_2 = 0) = Y_2(T),
$$

where

$$
L^2(0; T; L^2(\omega)) = \left\{ u \in L^2(0; T; L^2(\omega)) : \int_{0}^{T} \rho^2 u^2 dx dt < \infty \right\}
$$

and $(Y, \Phi^1, \Phi^2)$ is the solution to (25). Multiplying the adjoint system (26) by $(Y, \Phi^1, \Phi^2)$ and integrating by parts, we get

$$
\langle Y(T), Z(T) \rangle_{L^2} \equiv \sum_{i=1}^{2} \int_{0}^{T} \alpha_i y_i \psi_i^c dx dt = \int_{0}^{T} z f dx dt - \sum_{i=1}^{2} \frac{1}{\mu_i} \int_{0}^{T} z_i \psi_i^c dx dt.
$$

(59)

In the same way, multiplying the system (25) by $(Z, \Psi^1, \Psi^2)$ and integrating by parts, we get

$$
\int_{0}^{T} \alpha_i \psi_i^c (y - y_i, dx dt = - \frac{1}{\mu_i} \int_{0}^{T} z_i \psi_i^c dx dt.
$$

(60)

If $Y_0 = 0$ and $y_1 = y_2 = 0$, by using (59) and (60), we find

$$
\langle Y_1(T), Z_T \rangle = \int_{0}^{T} z f dx dt.
$$
This proves that
\[ L^*_r(Z_T) = z_{1}\omega. \]

If now \( f = 0 \), using again (59) and (60), we get
\[ \langle Y_2(T), Z(T) \rangle_{L^2} = \langle Y_0, Z(0) \rangle_{L^2} - \sum_{i=1}^{2} \alpha_i \int_{\omega_{i,d} \times (0, T)} y_i, a\psi^i \, dx \, dt. \]

That is
\[ R^*_r(Z_T) = (Z(0), -\alpha_1 \rho^{-1} \psi^1, -\alpha_2 \rho^{-1} \psi^2). \]

The relation between observability inequality (44) and null controllability of the system (25) follows from Theorem 1.18 of [50].

6. Similar results for semilinear problems. In this section, following [5, 9], and using the results obtained in the linear case with a fixed point argument, we deduce similar results for the following semilinear problem
\[
\begin{cases}
\partial_t y - \text{div}(A \nabla y) + F(y, \nabla y) = f_1 \omega + v_1 \omega_1 + v_2 \omega_2 & \text{in } \Omega_T, \\
\partial_t y_t - \text{div}(A \nabla y_t) + d^2_t y + G_t(y_t, \nabla y_t) = 0 & \text{on } \Gamma_T, \\
(y(0), y_t(0)) \in (y_0, y_t(0), \nu) & \text{on } \Omega \times \Gamma,
\end{cases}
\]
with \( F : (s, \xi_1, ..., \xi_N) \mapsto F(s, \xi_1, ..., \xi_N) \) and \( G : (s, \xi_1, ..., \xi_N) \mapsto G(s, \xi_1, ..., \xi_N) \) are in \( C^1(\mathbb{R} \times \mathbb{R}^N) \) such that \( F(0) = G(0) = 0 \). We assume furthermore that there exist \( L_F, L_G > 0 \) such that
\[
\begin{align*}
|F(s, \xi) - F(r, \xi')| & \leq L_F(|s - r| + \|\xi - \xi'\|), \\
|G(s, \xi) - G(r, \xi')| & \leq L_G(|s - r| + \|\xi - \xi'\|)
\end{align*}
\]
for all \( s, r \in \mathbb{R} \) and \( \xi, \xi' \in \mathbb{R}^N \).

Since the method is standard, we only give the main ideas. In the semilinear framework, the corresponding functionals \( J_1 \) and \( J_2 \) are not convex in general. For this reason, we must consider the following weaker Nash equilibrium.

**Definition 6.1.** Let \( f \in L^2(0, T; L^2(\omega)) \) be given. The pair \((v_1, v_2)\) is called Nash quasi-equilibrium of \((J_1, J_2)\) if
\[
\begin{align*}
\frac{\partial J_1}{\partial v_1}(f; v_1, v_2)(v) & = 0 & \forall v \in L^2(0, T; L^2(\omega_1)) , \\
\frac{\partial J_2}{\partial v_2}(f; v_1, v_2)(w) & = 0 & \forall w \in L^2(0, T; L^2(\omega_2)).
\end{align*}
\]

Introduce the following notations
\[
\begin{align*}
\tilde{F}_1(y) & = \partial_1 F(y, \nabla y), & \tilde{F}_2(y) & = (\partial_1 G(y, \nabla y), ..., \partial_N G(y, \nabla y)), \\
\tilde{G}_1(y) & = \partial_1 G(y, \nabla y), & \tilde{G}_2(y) & = (\partial_1 G(y, \nabla y), ..., \partial_N G(y, \nabla y)).
\end{align*}
\]

Using the same ideas as in [5], the Nash quasi-equilibrium \((v_1^*, v_2^*)\) of \((J_1, J_2)\) is given by
\[
v_i^* = \frac{1}{\mu_i} \varphi_i^* \mid_{\omega_i \times (0, T)} \quad \text{for } i = 1, 2,
\]
where \((\varphi^1, \varphi^2)\) satisfies
\[
\begin{align*}
\begin{cases}
-\partial_t \varphi^1 - \text{div}(A \nabla \varphi^1) - \text{div}(\varphi^1 \tilde{F}_2(y)) + \tilde{F}_1(y) \varphi^1 & = \alpha_i (y - y_{i,d}) 1_{\omega_{i,d}} & \text{in } \Omega_T, \\
-\partial_t \varphi^2 - \text{div}(A \nabla \varphi^2) - \text{div}(\varphi^2 \tilde{G}_2(y_t)) + \varphi^2 \tilde{F}_2^2(y) \cdot \nu + \partial_2 \varphi^2 & = \tilde{G}_1(y_t) \varphi^2 & \text{on } \Gamma_T, \\
(\varphi^1(T), \varphi^2(T)) & = 0 & \text{in } \Omega \times \Gamma,
\end{cases}
\end{align*}
\]
for \( i = 1, 2 \).
Accordingly, we have the next optimality system

\[
\begin{aligned}
\partial_t y - \text{div}(A\nabla y) + F(y, \nabla y) &= f_{1,\omega} + v_1 l_{\omega_1} + v_2 l_{\omega_2} & \text{in } \Omega_T, \\
\partial_t y_i - \text{div}(A_i \nabla y_i) + \partial_i^a y_i + G_i(y_i, \nabla y_i) &= 0 & \text{on } \Gamma_T, \\
-\partial_t \varphi^i - \text{div}(A \nabla \varphi^i) - \text{div}(\varphi^i F_2(y)) + \tilde{F}_1(y) \varphi^i &= \alpha_i(y - y_i, d) l_{\omega_i, d} & \text{in } \Omega_T, \\
-\partial_t \varphi_i - \text{div}(A_i \nabla \varphi_i) - \text{div}(\varphi_i F_2(y)) + \tilde{F}_1(y) \varphi_i &= 0 & \text{on } \Gamma_T, \\
(y(0), y_i(0)) &= (y_0, y_{i,0}) & \text{in } \Omega \times \Gamma, \\
(\varphi^i(T), \varphi_i(T)) &= 0 & \text{in } \Omega \times \Gamma, \\
i &= 1, 2.
\end{aligned}
\]

(65)

Using the same ideas as in the linear case, we can prove the null controllability of the optimality system (65). Then, by some well-known fixed point arguments, we deduce the same results for the semilinear system (61). More precisely, we have the next theorem.

**Theorem 6.2.** *Under the same assumptions of Theorem 2.1, there exists a positive weight function \(p = \rho(t)\) blowing up at \(t = T\) such that for every \(y_{i, d} \in L^2(0, T; L^2(\omega))\) satisfying (11) and every \(y_{0} = (y_0, y_{i,0}) \in L^2\), there exist controls \(f \in L^2(0, T; L^2)\) and associated Nash quasi-equilibrium \((v_1^*, v_2^*) \in L^2(0, T; L^2(\omega_1)) \times L^2(0, T; L^2(\omega_2))\) such that the corresponding solution to (61) satisfies \(Y(T, f, v_1^*, v_2^*) = 0\).

7. **Conclusion.** In this paper we have studied a hierarchical control problem of heat equation with general dynamic boundary conditions. Following the Stackelberg-Nash strategy with one leader and two followers, we have proved, for each fixed leader, the existence and uniqueness of a Nash-equilibrium, and by means of an adjoint system, we have characterized the Nash-equilibrium, and we have deduced an optimality system. By suitable Carleman estimates, we have established an observability inequality which is the key to deduce our controllability result. The similar results are also obtained for the semilinear system.

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