OPTIMAL CONTROL OF STURM-LIOUVILLE TYPE EVOLUTION DIFFERENTIAL INCLUSIONS WITH ENDPOINT CONSTRAINTS

ELIMHAN N. MAHMUDOV*
Department of Mathematics, Istanbul Technical University
34469, Maslak, Istanbul, Turkey
Azerbaijan National Academy of Sciences Institute of Control Systems
Baku, Azerbaijan

(Communicated by N. U. Ahmed)

ABSTRACT. The present paper studies a new class of problems of optimal control theory with linear second order self-adjoint Sturm-Liouville type differential operators and with functional and non-functional endpoint constraints. Sufficient conditions of optimality, containing both the second order Euler-Lagrange and Hamiltonian type inclusions are derived. The presence of functional constraints generates a special second order transversality inclusions and complementary slackness conditions peculiar to inequality constraints; this approach and results make a bridge between optimal control problem with Sturm-Liouville type differential differential inclusions and constrained mathematical programming problems in finite-dimensional spaces. The idea for obtaining optimality conditions is based on applying locally-adjoint mappings to Sturm-Liouville type set-valued mappings. The result generalizes to the problem with a second order non-self-adjoint differential operator. Furthermore, practical applications of these results are demonstrated by optimization of some semilinear optimal control problems for which the Pontryagin maximum condition is obtained. A numerical example is given to illustrate the feasibility and effectiveness of the theoretic results obtained.

1. Introduction. It is well known that, optimization of second order differential inclusions arise from a wide variety of problems in science, economy, engineering design, industrial optimization, game theory and so on. Optimal control of discrete-differential inclusions with lumped and distributed parameters has been expanding in all directions at an astonishing rate during the last few decades. Note that the differential inclusions are not only models for many dynamical processes but they also provide a powerful tool for various branches of mathematical analysis; see more discussions and comments in the relatively recent publications [9, 20, 7] and the references therein. First order differential inclusions, set-valued maps play a crucial role in the mathematical theory of optimal processes given in the next papers [2, 4, 6, 12]. The objective of the paper [2] is to briefly summarize some recent results on Differential Inclusions and their optimal control. Then, using vector measures as controls, are presented some new results on the necessary and sufficient conditions of...
optimality. Further, are considered systems having structural perturbation modeled by operator valued measures. In the work [8], for set-valued maps are introduced the concept of a generalized second-order composed contingent epiderivative. Then, by virtue of the generalized second-order composed contingent epiderivative, are established a unified second-order necessary and sufficient condition of optimality for set-valued optimization problems. The new contribution [1] focuses on the case when the interior of the domain of the maximally monotone operator governing the given differential inclusion is nonempty; this includes in a natural way the finite-dimensional case. Some consequences of the viability of closed sets are given. This analysis makes use of standard tools from convex and variational analysis.

Closely related optimality problems for a first order differential inclusions were considered by Rockafellar [11] and Mordukhovich [19] and the present study is an important generalization of their works to the problem with second order Sturm-Liouville type differential inclusions.

In the paper [11] are considered a Mayer problem of optimal control, whose dynamic constraint is given by a convex-valued first order differential inclusion. Both state and endpoint constraints are involved. Are proved necessary conditions incorporating the Hamiltonian inclusion, the Euler-Lagrange inclusion, and the Pontryagin maximum condition. These results weaken the hypotheses and strengthen the conclusions of earlier works. The paper [19] is devoted to the study of a Mayer-type optimal control problem for semilinear unbounded evolution inclusions in reflexive and separable Banach spaces subject to endpoint constraints described by finitely many Lipschitzian equalities and inequalities. First are constructed a sequence of discrete approximations to the optimal control problem for evolution inclusions and proved that optimal solutions to discrete approximation problems uniformly converge to a given optimal solution for the original continuous-time problem.

In fact, the difficulty in the problems with second order ordinary differential inclusions is rather to construct the Euler-Lagrange type second order adjoint inclusions and the suitable boundary of so-called transversality conditions. That is why on the whole in literature only the qualitative properties of second order differential inclusions are investigated (see [3, 5, 10] and references therein). The paper [3] studies, existence solutions for second order differential inclusions with mixed semicontinuous maps, which are upper semicontinuous in some points and lower semicontinuous in remaining points. It should be noted that there are papers [5, 10] in the literature devoted to the study of some qualitative properties of the second order (S-L)-type differential inclusions. The work [5] studies second-order Sturm-Liouville-type differential inclusions. Using Bressan-Colombo results concerning the existence of continuous selections of lower semi-continuous set-valued mappings with decomposable values, a continuous version of Filippovs theorem for a Sturm-Liouville differential inclusion is proved. In the paper [10], the authors investigate the existence of solutions of impulsive boundary value problems for Sturm-Liouville type differential inclusions which admit nonconvex set-valued mappings on the right-hand side. Two results under weaker conditions are presented. The methods rely on a fixed-point theorem for the contraction of set-valued maps due to Covitz and Nadler and on Schaefers fixed-point theorem combined with lower semi-continuous set-valued operators with decomposable values. Besides, since $p(t) > 0,$ taking $p(t) \equiv 1$ identically in Sturm-Liouville type differential inclusion, it can be easily seen that in fact, we have “usual” second order differential inclusion for which there are different existence theorems (see [3] and references therein).
In spite of the presence of the above mentioned qualitative studies, optimal control of problems with the higher order DFIs has not been studied in the literature; up to my best knowledge, there a few papers of Mahmudov [12]-[18] devoted to such studies. The paper [16] studies the Bolza problem of optimal control theory with a fixed time interval given by convex and nonconvex second order differential inclusions. Our main goal is to derive sufficient optimal conditions for a Cauchy problem of second order differential inclusions. As supplementary problems, discrete and discrete approximation inclusions are considered, necessary and sufficient conditions, including distinctive transversality ones, are proved by incorporating the Euler-Lagrange and Hamiltonian type of inclusions. In the paper [16] are studied the sufficient conditions of optimality for Cauchy problem of fourth-order differential inclusions. The main purpose is to derive sufficient optimality conditions for mentioned problems with fourth-order differential inclusions and transversality conditions. The paper [17] studies a new class of problems of optimal control theory with Sturm-Liouville type differential inclusions involving second order linear self-adjoint differential operators. Necessary and sufficient conditions, containing both the second order Euler-Lagrange and Hamiltonian type inclusions and “transversality” conditions are derived. The result strengthens and generalizes to the problem with a second order non-self-adjoint differential operator; a suitable choice of coefficients then transforms this operator to the desired Sturm-Liouville type problem.

The present paper is dedicated to one of the most difficult and interesting fields optimization of the second order Sturm-Liouville (S-L)-type differential inclusions with functional and non-functional endpoint constraints. To the best of our knowledge, there is no paper which considers optimality conditions for these problems in the literature and we aim to fill this gap.

The problems posed in the present paper and the corresponding optimality conditions are new and organized in the following order; in Section 2, the needed facts and supplementary results from the book of Mahmudov [12] are summarized for the readers convenience. In particular, Hamiltonian function, argmaximum sets of a set-valued mapping, the locally-adjoint mapping (LAM) are introduced and the (S-L) problem for differential inclusions is formulated. In Section 3, sufficient conditions of optimality for (S-L) problem are derived. The right-hand side of the transversality inclusion and complementary slackness condition will surprise any one working in the field because this approach and results make a bridge between optimal control problem with higher order differential inclusion (S-L) and constrained mathematical programming problems in finite-dimensional spaces. Thus, using LAM we deduce the sufficient conditions of optimality for (S-L) problem in second order Euler-Lagrange forms. In addition, under the closedness condition of set-valued mapping, the optimality conditions for the (S-L) problem are derived in terms of Hamiltonian function in the much more pleasant form. Note that the basic idea in the paper for the continuous problem (S-L) was to replace it by the discrete-approximate problem and then by passing to the limit to formulate sufficient conditions optimality for the original problem. In this paper establishment of these conditions is omitted and we start our discussion with sufficient optimality conditions for problem (S-L). And in this sense the obtained results of Section 3 as well as of the next sections are only the visible part of the “icebergs”. In Section 4 is investigated a Bolza problem with non-self-adjoint second order linear differential operator and with functional and non-functional endpoint constraints. A sufficient condition of optimality is formulated in term of second order an adjoint linear differential operator with variable
coefficients. A suitable choice of coefficients then transforms this operator to the desired Sturm-Liouville type operator. In particular, if a positive-valued, scalar function specific to Sturm-Liouville differential inclusions is identically equal to one, we have immediately the optimality conditions for the second order differential inclusions. In Section 5 the optimality conditions are given for a Bolza problem with a more general second order differential inclusions. The basic difference of this problem is that here set-valued mapping depend not only on sought function, but on its derivative, too. This fact seriously complicates the construction of proof of optimality conditions for higher order differential inclusions. These obtained conditions involve useful forms of the Pontryagin maximum condition and second order optimality conditions for higher order differential inclusions. These obtained conditions involve useful forms of the Pontryagin maximum condition and second order Euler-Lagrange type adjoint inclusions. As example, we consider a Mayer problem with a semilinear second order continuous-time evolution inclusions and missing non-functional endpoint constraint. In addition, numerical computation developed for a Mayer problem with particular case of Newtons second law of motion.

2. Needed facts and problem statement. The necessary notions can be found in the monograph of Mahmudov [12]. Let $\mathbb{R}^n$ be $n$ dimensional Euclidean space, $\langle x, y \rangle$ be an inner product of elements $x, y \in \mathbb{R}^n$, $(x, y)$ be a pair of $x, y$. Assume we are given the evolution set-valued mapping $F(\cdot, t) : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n, t \in [t_0, t_1]$. Then $F(\cdot, t)$ is convex, if its graph $gph F(\cdot, t) := \{(x, y, z) : z \in F(x, y, t)\}$ is a convex subset of $\mathbb{R}^{3n}$. $F(\cdot, t)$ is convex and closed, if its graph is a convex and closed set in $\mathbb{R}^{3n}$. The domain of $F(\cdot, t)$ is denoted by $dom F(\cdot, t) = \{(x, y) : F(x, y, t) \neq \emptyset\}$. $F(\cdot, t)$ is convex-valued, if $F(x, y, t)$ is a convex set for each $(x, y) \in dom F(\cdot, t)$.

Let us introduce the Hamiltonian function and argmaximum set for a set-valued mapping $F(\cdot, t)$

$$H_{F(\cdot), t}(x, y, \eta) = \sup_{z} \{\langle z, \eta \rangle : z \in F(x, y, t)\}, \eta \in \mathbb{R}^n,$$

$$F_{A}(x, y; \eta, t) = \{ z \in F(x, y, t) : \langle z, \eta \rangle = H_{F(\cdot), t}(x, y, \eta) \},$$

respectively. For convex $F(\cdot, t)$ we set $H_{F(\cdot), t}(x, y, \eta) = -\infty$, if $F(x, y, t) = \emptyset$.

The convex cone $K_{A}(x, y, z), (x, y, z) \in A$ is called the cone of tangent directions at a point $(x, y, z) \in A$ to the set $A$, if from $(\bar{x}, \bar{y}, \bar{z}) \in K_{A}(x, y, z)$ it follows that $(\bar{x}, \bar{y}, \bar{z})$ is a tangent vector to the set $A$ at point $(x, y, z) \in A$, i.e., there exists such function $\xi : \mathbb{R}^1 \to \mathbb{R}^{3n}$ that $(x, y, z) + \lambda(\bar{x}, \bar{y}, \bar{z}) + \xi(\lambda) \in A$ for sufficiently small $\lambda > 0$ and $\lambda^{-1}\xi(\lambda) \to 0$, as $\lambda \downarrow 0$.

Obviously, on the definition the cone of tangent directions cannot be nonconvex; it should be pointed out that the cone of tangent directions is not uniquely defined. In any case the wider a cone of tangent directions the essentially optimality conditions.

In general, for a set-valued mapping $F(\cdot, t)$ a set-valued mapping $F^* (\cdot; x, y, z, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^{2n}$ defined by

$$F^*(\eta; (x, y, z), t) := \{(\hat{x}, \hat{y}) : (\hat{x}, \hat{y}, -\eta) \in K_{gph F(\cdot), t}^*(x, y, z)\},$$

is called the LAM to the set-valued mapping $F(\cdot, t)$ at a point $(x, y, z) \in gph F(\cdot, t)$, where $K_{gph F(\cdot), t}^*(x, y, z)$ is the dual to the cone of tangent directions $K_{gph F(\cdot), t}(x, y, z)$. We provide another definition of LAM to nonconvex mapping $F(\cdot, t)$ which is more relevant for further development

$$F^*(\eta; (x, y, z), t) := \{(\hat{x}, \hat{y}) : H_{F(\cdot), t}(x_1, y_1, \eta) - H_{F(\cdot), t}(x, y, \eta) \leq \langle \hat{x}, x_1 - x \rangle$$

$$+ \langle \hat{y}, y_1 - y \rangle, \forall (x_1, y_1) \in \mathbb{R}^{2n}\}, (x, y, z) \in gph F(\cdot, t), z \in F_{A}(x, y, \eta, t).$$
Clearly, for the convex mapping the Hamiltonian $H_{F(\cdot,t)}(\cdot,\eta)$ is concave and the latter and previous definitions of LAMs coincide. Moreover, by Lemma 5.2 [17] if $F(\cdot,t)$ is a convex-valued continuous mapping, then Hamiltonian function $H_{F(\cdot,t)}$ is continuous. In addition, if $F(\cdot,t)$ in the sense of Hausdorff metric is a Lipschitz set-valued map, then $H_{F(\cdot,t)}$ is also Lipschitz function.

In Section 3 is considered a Bolza problem with Sturm-Liouville type evolution differential Inclusions and with functional and non-functional constraints:

$$\minimize \ J[x(\cdot)] = \int_{t_0}^{t_1} g(x(t),t)dt + \varphi_0(x(t_1),x'(t_1)), \quad (1)$$

$$(S-L) \quad (p(t)x'(t))' \in F(x(t),t), \ \text{a.e.} \ t \in [t_0,t_1], \quad (2)$$

$$\varphi_k(x(t_1),x'(t_1)) \leq 0, \ k = 1, \ldots, r; \quad (3)$$

$$x(t_0) = \alpha_0, \ x'(t_0) = \alpha_1, \ (x(t_1),x'(t_1)) \in Q. \quad (4)$$

Here $F(\cdot,t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a time dependent set-valued mapping, $g(\cdot,t) : \mathbb{R}^n \rightarrow \mathbb{R}^1$ and $\varphi_k : \mathbb{R}^{2n} \rightarrow \mathbb{R}^1 (k = 0, \ldots, r)$ are continuous functions, $p : [t_0, t_1] \rightarrow \mathbb{R}$ is continuously differentiable function, $Q \subseteq \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ is nonempty subset, $(Ax)(t) = \frac{d}{dt}(p(t)x') \equiv (p(t)x')'$ is a second order Sturm-Liouville operator. The problem is to find an arc $\tilde{x}(\cdot)$ of the problem (1)-(4) satisfying (2) almost everywhere (a.e.) on $[t_0, t_1]$ and the functional (3) and non-functional (4) constraints that minimizes the Bolza functional $J[x(\cdot)]$. We label this problem as (S-L). First we derive sufficient optimality conditions in the convex problem and then generalize the obtained results to the nonconvex case. Here, a feasible trajectory $x(\cdot)$ is understood to be an absolutely continuous function on a time interval $[t_0, t_1]$ together with the first order derivatives for which $x''(\cdot) \in L^2_t([t_0, t_1])$. Obviously, such class of functions is a Banach space, endowed with the different equivalent norms. Note that the second order Sturm-Liouville operator $(Ax)(t)$ is a well-known example of a self-adjoint operator, i.e., $(A^*\eta)(t) = p(t)\eta'' + p'(t)\eta'$. As is known, in mathematics, this operator is central to Sturm-Liouville where the eigenfunctions (analogues to eigenvectors) of this operator are considered.

3. Sufficient conditions of optimality for (S-L) problem with endpoint constraints. Let us explain the principal method that we use to obtain the sufficient condition of optimality for the above posed problem (S-L); the basic idea is to replace the continuous problem (S-L) by the discrete-approximate problem that can be studied effectively, and then by passing to the limit to formulate sufficient conditions optimality for the original problem. In this paper establishment of these conditions is omitted and we start our discussion with a presentation of sufficient optimality conditions for problem (S-L). And in this sense the obtained results of paper are only the visible parts of the "icebergs". In this way we establish so-called the second order (S-L) type Euler-Lagrange and transversality inclusions and the complementary slackness conditions.

The second order Euler-Lagrange differential inclusion

$$(i) \quad (A\eta)(t) \in F^*[\eta(t);(\tilde{x}(t),A\tilde{x}(t)), \tilde{t}] - \partial g(\tilde{x}(t),t), \text{a.e.} \ t \in [t_0, t_1].$$

The second order transversality condition for endpoint constraints at $t = t_1$

$$(ii) \quad p(t_1)(\eta'(t_1),-\eta(t_1)) \in \partial \varphi_0(\tilde{x}(t_1),\tilde{x}'(t_1)) + \sum_{k=1}^{k=r} \lambda_k \partial \varphi_k(\tilde{x}(t_1),\tilde{x}'(t_1))$$
The complementary slackness conditions at \( t = t_1 \)

\[
(iii) \quad \lambda_k \varphi_k(\tilde{x}(t_1), \tilde{x}'(t_1)) = 0, \quad \lambda_k \geq 0, \quad k = 1, \ldots, r.
\]

In what follows, we refer to \( \lambda_k \) as the Lagrange multiplier associated with the \( k \)-th inequality constraint \( \varphi_k(x(t_1), x'(t_1)) \leq 0 \) and usually \( \lambda_{k_0} > 0 \), if \( \varphi_{k_0}(x(t_1), x'(t_1)) = 0 \) and \( \lambda_{k_0} = 0 \) if \( \varphi_{k_0}(x(t_1), x'(t_1)) < 0 \) for some \( k_0 \). Moreover, \( K_Q^*(\tilde{x}(t_1), \tilde{x}'(t_1)) \) is the dual to the cone of tangent directions \( K_Q(\tilde{x}(t_1), \tilde{x}'(t_1)), (\tilde{x}(t_1), \tilde{x}'(t_1)) \in Q \subseteq \mathbb{R}^{2n} \).

We assume that \( \eta(t), t \in [t_0, t_1] \) is absolutely continuous function together with the first order derivative and \( \eta''(\cdot) \in \mathbb{L}^0_n \). It should be noted that the LAM \( F^* \) is nonempty at a given point provided that

\[
(iv) \quad (p(t)\tilde{x}'(t))' \in F_A(\tilde{x}(t), \eta(t), t), \text{ a.e. } t \in [t_0, t_1]
\]

It turns out that the following assertion is true.

**Theorem 3.1.** *(Sufficient conditions of optimality for second order evolution Sturm-Liouville differential inclusions with endpoint constraints)* Suppose that \( \varphi_k : \mathbb{R}^{2n} \to \mathbb{R}^1, k = 0, 1, \ldots, r \) are convex continuous functions, \( F(\cdot, t) \) is an evolution convex set-valued mapping and \( Q \subseteq \mathbb{R}^{2n} \) is a convex set. Then for optimality of the trajectory \( \tilde{x}(\cdot) \) in the \((S-L)\) problem it is sufficient that there exists an absolutely continuous function \( \eta(t), t \in [t_0, t_1] \) together with the first order derivative and not all zero Lagrange multipliers \( \lambda_k \geq 0, k = 0, \ldots, r \), satisfying a.e. the second order \((S-L)\) type Euler-Lagrange inclusion (i), the transversality condition (ii) and the complementary slackness conditions (iii) at \( t = t_1 \).

**Proof.** By Theorem 2.1 [12, p.62] we have \( F^*(\eta; (x, z), t) = \partial_z H_{F(t)}(x, \eta), z \in F_A(x, \eta, t) \). Then from the Euler-Lagrange inclusion (i) we obtain the second order adjoint differential inclusion

\[
(A\eta)(t) \in \partial_z[H_{F(\cdot, t)}(\tilde{x}(t), \eta(t)) - g(\tilde{x}(t), t)], \quad t \in [t_0, t_1],
\]

which means that

\[
H_{F(\cdot, t)}(x(t), \eta(t)) - g(x(t), t) - [H_{F(\cdot, t)}(\tilde{x}(t), \eta(t)) - g(\tilde{x}(t), t)]
\leq \langle (A\eta)(t), x(t) - \tilde{x}(t) \rangle.
\]

Then by using the definition of Hamiltonian function and the condition (iv) we have

\[
\langle Ax(t) - A\tilde{x}(t), \eta(t) \rangle - g(x(t), t) + g(\tilde{x}(t), t) \leq \langle (A\eta)(t), x(t) - \tilde{x}(t) \rangle.
\] (5)

Let us integrate (5) over the interval \([t_0, t_1]\)

\[
\int_{t_0}^{t_1} [\langle Ax(t) - A\tilde{x}(t), \eta(t) \rangle - \langle (A\eta)(t), x(t) - \tilde{x}(t) \rangle] dt \leq \int_{t_0}^{t_1} [g(x(t), t) - g(\tilde{x}(t), t)] dt.
\] (6)
It can be easily seen that integration by parts on the left-hand side of (6) (here
is taken into account that $x(t)$ and $\tilde{x}(t), t \in [t_0, t_1]$ are feasible trajectories) give us

$$
\int_{t_0}^{t_1} \left[ \langle Ax(t) - A\tilde{x}(t), \eta(t) \rangle - \langle (A\eta)(t), x(t) - \tilde{x}(t) \rangle \right] dt
= \int_{t_0}^{t_1} p(t) \left[ \langle x'(t) - \tilde{x}'(t), \eta(t) \rangle - \langle \eta'(t), x(t) - \tilde{x}(t) \rangle \right] dt
+ \int_{t_0}^{t_1} p'(t) \left[ \langle x'(t) - \tilde{x}'(t), \eta(t) \rangle - \langle \eta'(t), x(t) - \tilde{x}(t) \rangle \right] dt
= p(t) \left[ (x'(t_1) - \tilde{x}'(t_1), \eta(t_1)) - \langle \eta'(t_1), x(t_1) - \tilde{x}(t_1) \rangle \right].
$$

Then by virtue of (7) the inequality (6) can be transformed into inequality

$$
\int_{t_0}^{t_1} \left[ g(x(t), t) - g(\tilde{x}(t), t) \right] dt \geq p(t) \left[ (x'(t_1) - \tilde{x}'(t_1), \eta(t_1)) - \langle \eta'(t_1), x(t_1) - \tilde{x}(t_1) \rangle \right].
$$

(8)

Now recall that on the definition of dual cone for all pair $(\eta(t_1), \eta'(t_1))$
$\in K_0^* (\tilde{x}(t_1), \tilde{x}'(t_1))$ the functional transversality condition (ii) of theorem implies

$$
\varphi_0(x(t_1), x'(t_1)) + \sum_{k=1}^r \lambda_k \varphi_k(x(t_1), x'(t_1)) - \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1))
- \sum_{k=1}^r \lambda_k \varphi_k(\tilde{x}(t_1), \tilde{x}'(t_1)) \geq p(t_1) \left[ (\eta'(t_1), x(t_1) - \tilde{x}(t_1))
- \langle \eta(t_1), x'(t_1) - \tilde{x}'(t_1) \rangle \right] + \langle \eta(t_1), x(t_1) - \tilde{x}(t_1) \rangle \langle \eta'(t_1), x'(t_1) - \tilde{x}'(t_1) \rangle
\geq p(t_1) \left[ (\eta'(t_1), x(t_1) - \tilde{x}(t_1)) - \langle \eta(t_1), x'(t_1) - \tilde{x}'(t_1) \rangle \right].
$$

(9)

Thus from the inequalities (8) and (9) thereby we get

$$
\int_{t_0}^{t_1} \left[ g(x(t), t) - g(\tilde{x}(t), t) \right] dt + \varphi_0(x(t_1), x'(t_1))
+ \sum_{k=1}^r \lambda_k \varphi_k(x(t_1), x'(t_1)) - \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1))
- \sum_{k=1}^r \lambda_k \varphi_k(\tilde{x}(t_1), \tilde{x}'(t_1)) \geq 0.
$$

(10)

Recall that $x(\cdot)$ is a feasible solution and $\lambda_k \geq 0, k = 1, \ldots, r$, that is,
$\sum_{k=1}^r \lambda_k \varphi_k(x(t_1), x'(t_1)) \leq 0$. Hence, using the complementary slackness conditions (iii) at $t = t_1$ for an arbitrary feasible solution $x(\cdot)$ from (10) we have

$$
\int_{t_0}^{t_1} \left[ g(x(t), t) - g(\tilde{x}(t), t) \right] dt + \varphi_0(x(t_1), x'(t_1)) - \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1)) \geq 0,
$$

that is, $J[x(t)] \geq J[\tilde{x}(t)], \forall x(t), t \in [t_0, t_1]$ i.e. $\tilde{x}(\cdot)$ is optimal.

\[\square\]

**Remark 1.** We note that if non-functional constraint is inactive, i.e., $Q = \mathbb{R}^{2n}$,
then $K_0^* (\tilde{x}(t_1), \tilde{x}'(t_1)) = \{(0, 0)\}$. Therefore, the transversality inclusion (ii) for functional constraints is $p(t_1)(\eta(t_1), -\eta(t_1)) \in \partial \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1)) + \sum_{k=1}^r \lambda_k \partial \varphi_k(\tilde{x}(t_1), \tilde{x}'(t_1))$, and
\[ \ddot{x}(t_1) \] and the inequality (9) is simplified as follows
\[ \varphi_0(x(t_1), x'(t_1)) - \varphi_0(\ddot{x}(t_1), \dddot{x}(t_1)) \geq p(t_1) \left[ \langle \eta'(t_1), x(t_1) - \ddot{x}(t_1) \rangle - \langle \eta(t_1), x'(t_1) - \dddot{x}(t_1) \rangle \right], \]
which implies again that \( J[x(t)] \geq J[\ddot{x}(t)], \forall x(t), t \in [t_0, t_1] \).

**Remark 2.** Suppose now that the functional conditions \( \varphi_k(x(t_1), x'(t_1)) \leq 0, k = 1, \ldots, r \) of problem (S-L) are missing, for example \( \varphi_k(x(t_1), x'(t_1)) \equiv 0 \) and the conditions (3) always hold. Then instead of transversality inclusion (ii) we have
\[ p(t_1)(\eta'(t_1), -\eta(t_1)) \in \partial \varphi_0(\ddot{x}(t_1), \dddot{x}(t_1)) - K^*_0(\ddot{x}(t_1), \dddot{x}(t_1)). \]

Then the inequality (10) is replaced by \( J[x(t)] - J[\ddot{x}(t)] \geq 0, \forall x(\cdot) \), as was to be proved. And obviously, the complementary slackness conditions (iii) at \( t = t_1 \) is not required.

Remember now that in the problem (S-L) a function \( p(\cdot) \) is a positive-valued scalar function, i.e. \( p(\cdot) : [t_0, t_1] \to [0; \infty] \). Therefore, an important consequence of Theorem 3.1 is the following particular case.

**Corollary 1.** Suppose that the conditions of Theorem 3.1 are satisfied. Then by a particular choice of function \( p(t) \equiv 1, t \in [t_0, t_1] \) for the problem (S-L) the second order Euler-Lagrange inclusion and the transversality condition of Theorem 3.1 for functional constraints at \( t = t_1 \) consist of the following
\begin{enumerate}
    \item \( \eta''(t) \in F^*[\eta(t); (\ddot{x}(t), \dddot{x}(t)), t] - \partial g(\ddot{x}(t), t), \quad a.e. \ t \in [t_0, t_1], \)
    \item \( \left( \frac{d\eta(t_1)}{dt}, -\eta(t_1) \right) \in \partial \varphi_0(\ddot{x}(t_1), \dddot{x}(t_1)) + \sum_{k=1}^r \lambda_k \partial \varphi_k(\ddot{x}(t_1), \dddot{x}(t_1)). \)
\end{enumerate}

**Proof.** Since \( p(t) \equiv 1, t \in [t_0, t_1] \) and \( p(t_1) = 0 \) then \( A\ddot{x}(t) = \dddot{x}(t), (A\eta)(t) = \eta''(t), \)
we have immediately the needed result.

Under the usual closedness of \( F(\cdot, t) \) (\( \text{gph} F(\cdot, t) \) is closed) the conditions of the Theorem 3.1 can be rewritten in a more symmetrical form. Obviously, the “pointwise” closedness of \( F(\cdot, t) \) (\( F(x, t) \) is closed set for each \( x \)) is weaker than this assumption.

**Corollary 2.** In addition to assumptions of Theorem 3.1 let \( F(\cdot, t) \) be a closed set-valued evolution mapping. Then, the conditions (i), (iii) of Theorem 3.1 can be rewritten in term of Hamiltonian function as follows
\[ (A\eta)(t) \in \partial_z \left[ H_{F(\cdot,t)}(\ddot{x}(t), \eta(t)) - g(\ddot{x}(t), t) \right], \]
\[ A\ddot{x}(t) \in \partial_y H_{F(\cdot,t)}(\ddot{x}(t), \eta(t)), \ a.e. \ t \in [t_0, t_1]. \]

**Proof.** Indeed by Theorem 2.1[12, p.62] and Lemma 5.1[17] above the LAM at a given point and argmaximum set are the subdifferentials of the Hamiltonian function on \( x \) and \( \eta \) \( F^*(\eta; (x, z), t) = \partial z H_{F(\cdot,t)}(x, \eta), F_A(x, \eta, t) = \partial_y H_{F(\cdot,t)}(x, \eta), \) respectively. Then the assertions of corollary are equivalent with the conditions (i), (iii) of Theorem 3.1.

**Theorem 3.2.** *(Sufficient conditions of optimality for second order Sturm-Liouville type evolution differential inclusions with endpoint constraints)* Suppose that \( \varphi_k : \mathbb{R}^{2n} \to \mathbb{R}, k_0, 1, \ldots, r \) and a function \( g(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^1 \) be nonconvex functions and
Now we consider optimization of Bolza problem with differential inclusions
\[ \eta \text{ absolutely continuous functions} \]
\[ Q \text{ being a nonconvex subset of } \mathbb{R}^{2n} \text{ having at every point a cone of tangent directions,} \]
\[ F(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n \text{ be an evolution convex set-valued mapping. Then for optimality} \]
\[ \text{of the feasible trajectory } \ddot{x}(\cdot) \text{ in the (S-L) problem it is sufficient that there exists an} \]
\[ \text{absolutely continuous functions } \eta(t), t \in [t_0, t_1], \text{ together with the first order derivative} \]
\[ \text{and not all zero Lagrange multipliers } \lambda_k \geq 0, k = 0, \ldots, r(\lambda_0 = 1), \text{ satisfying} \]
\[ \text{a.e. the second order (S-L) type Euler-Lagrange inclusion (a), the transversality} \]
\[ \text{condition (c), the complementary slackness condition (iii) of Theorem 3.1 and conditions (b), (d):} \]
\[ (a) \ (A\eta)(t) + \eta(t) \in F^* [\eta(t); (\ddot{x}(t), A\ddot{x}(t), t)], \text{ a.e. } t \in [t_0, t_1], \]
\[ (b) \ g(x, t) - g(\ddot{x}(t), t) \geq \langle \eta(t), x - \ddot{x}(t) \rangle, \forall x \in \mathbb{R}^n, \]
\[ (c) \ \varphi_0(x, y) + \sum_{k=1}^{\infty} \lambda_k \varphi_k(x, y) - \varphi_0(\ddot{x}(t_1), \dddot{x}'(t_1)) - \sum_{k=1}^{\infty} \lambda_k \varphi_k(\ddot{x}(t_1), \dddot{x}'(t_1)) \]
\[ \geq p(t_1)[\langle \eta(t_1), x - \ddot{x}(t_1) \rangle - \langle \eta(t_1), y - \dddot{x}(t_1) \rangle] + \langle \omega_0(t_1), x - \ddot{x}(t_1) \rangle \]
\[ + \langle \omega_1(t_1), y - \dddot{x}(t_1) \rangle, \ (\omega_0(t_1), \omega_1(t_1)) \in K_Q(\ddot{x}(t_1), \dddot{x}'(t_1)), \ (x, y) \in \mathbb{R}^{2n}, \]
\[ (d) \ \langle Ax(t), \eta(t) \rangle = H_F(\ddot{x}(t), \eta(t)), \text{ a.e. } t \in [t_0, t_1]. \]

Proof. Recall that in the left hand side of the second order Euler-Lagrange inclusion (a) of theorem we have the sum \((A\eta)(t) + \eta(t)\) instead of \((A\eta)(t)\) in condition (i) of Theorem 3.1. Therefore, by definition of LAM in the nonconvex case (see, Section 2).
\[
H_F(x(t), \eta(t)) - H_F(\ddot{x}(t), \eta(t)) \leq \langle (A\eta)(t) + \eta(t), x(t) - \ddot{x}(t) \rangle,
\]
whereas
\[
\langle Ax(t), \eta(t) \rangle - \langle A\ddot{x}(t), \eta(t) \rangle \leq \langle (A\eta)(t) + \eta(t), x(t) - \ddot{x}(t) \rangle. \quad (11)
\]

Hence from (11) and condition (b) of theorem is justified the inequality (5) of Theorem 3.1 Then in the case \(x = x(1), y = x'(1)\) the furthest proof of theorem is similar to the one for Theorem 3.1.

We note that in the convex case, the conditions (a), (b) and (c) of Theorem 3.2 are equivalent to the second Euler-Lagrange and transversality inclusions (i), (ii) of Theorem 3.1, respectively. It means that the conditions of Theorem 3.1 and Theorems 3.2 in the convex case coincide. Therefore, the objective of this section is accomplished.

4. Optimization of Bolza problem with non-self-adjoint second order operators. Now we consider optimization of Bolza problem with differential inclusions described by non-self-adjoint second order linear differential operators:
\[
\text{minimize } J[x(\cdot)] = \int_{t_0}^{t_1} g(x(t), t) dt + \varphi_0(x(t_1), x'(t_1)),
\]
\[
(PSL) \quad Bx(t) \in F(x(t), t), \text{ a.e. } t \in [t_0, t_1], \ x(t_0) = x_0, \ x'(t_0) = \alpha_1, \]
\[
\varphi_k(x(t_1), x'(t_1)) \leq 0, \ k = 1, \ldots, r; \ (x(t_1), x'(t_1)) \in Q.
\]
where \(Bx = s(t)x'' + q(t)x' = s(t)D^2x + q(t)Dx \) \((D^k, k = 1, 2 \text{ is } k\text{-th order derivatives}) \) in general, is a non-self-adjoint second order linear differential operator with variable continuous \(k\)-th order differentiable coefficients \(s(\cdot), q(\cdot) : [t_0, t_1] \to \mathbb{R}^1\). Here \(F(\cdot, t) \) is an evolution set-valued mapping, \(\varphi_k, k = 0, \ldots, r \text{ are continuous} \)
functions, and $Q$ is nonempty subset of $\mathbb{R}^{2n}$. It can be noted that in the particular case, if $s(t) = p(t), q(t) = p'(t)$, where $p(\cdot) : [t_0, t_1] \rightarrow [0; \infty]$ the second order linear differential operator, $B$ is self-adjoint and the (S-L) and (PSL) problems coincide. This property can be proven using the formal adjoint definition:

$$
(B^* \eta)(t) = D^2(s(t) \eta) - D(q(t) \eta) = (p(t) \eta)' - (p'(t) \eta)'
$$

$$
= p(t) \eta'' + p'(t) \eta' = s(t) \eta'' + q(t) \eta'.
$$

For simplicity we consider a convex problem (PSL).

**Theorem 4.1.** Suppose that $\varphi_k : \mathbb{R}^{2n} \rightarrow \mathbb{R}, k = 0, \ldots, r$ are continuous convex functions, $F(\cdot, t)$ is a convex evolution mapping and $Q$ is convex subset of $\mathbb{R}^{2n}$. Then for optimality of the arc $\tilde{x}(\cdot)$ in the problem (PSL) it is sufficient that there exist Lagrange multipliers not all zero and an absolutely continuous function $\eta(\cdot)$, together with the first order derivative, satisfying a.e. the second order Euler-Lagrange inclusion

$$(a_1) \quad (B^* \eta)(t) \in F^*[\eta(t); (\tilde{x}(t), B\tilde{x}(t), t)] - \partial g(\tilde{x}(t), t);$$

$$B\tilde{x}(t) \in F_A(\tilde{x}(t), \eta(t), t), \text{ a.e. } t \in [t_0, t_1],$$

the second order transversality condition:

$$(b_1) \quad \left[(s(t_1) \eta(t_1))' - q(t_1) \eta(t_1), -s(t_1) \eta(t_1)\right] \in \partial \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1))$$

$$+ \sum_{k=1}^{r} \lambda_k \partial \varphi_k(\tilde{x}(t_1), \tilde{x}'(t_1)) - K^*_Q(\tilde{x}(t_1), \tilde{x}'(t_1))$$

and the complementary slackness condition at $t = t_1$.

**Proof.** As in the proof of Theorem 3.1 the second order Euler-Lagrange inclusion $(a_1)$ of theorem implies that

$$H_F(x(t) \eta(t)) - g(x(t), t) - \left[H_F(\tilde{x}(t) \eta(t)) - g(\tilde{x}(t), t)\right] \leq \left< (B^* \eta)(t), x(t) - \tilde{x}(t) \right>,$$

whereas $\left< Bx(t) - B\tilde{x}(t), \eta(t) \right> \leq \left< (B^* \eta)(t), x(t) - \tilde{x}(t) \right> + g(x(t), t) - g(\tilde{x}(t), t)$.

Therefore, we have

$$\left< s(t)x''(t) + q(t)x'(t), \eta(t) \right> - \left< s(t)x'(t) + q(t)x'(t), \eta(t) \right>$$

$$\leq \left< (s(t) \eta(t))'', x(t) - \tilde{x}(t) \right> + g(x(t), t) - g(\tilde{x}(t), t).$$

By integration of this inequality over the interval $[t_0, t_1]$ we obtain

$$\int_{t_0}^{t_1} \left[ \left< s(t)x''(t) + q(t)x'(t), \eta(t) \right> - \left< s(t) \eta(t) \right>'' \right] dt \leq \int_{t_0}^{t_1} [g(x(t), t) - g(\tilde{x}(t), t)] dt. \tag{12}$$

Transforming the expression in the square parentheses in the left hand side of $(12)$ as in the proof of Theorem 3.1 we have

$$\int_{t_0}^{t_1} [g(x(t), t) - g(\tilde{x}(t), t)] dt \geq \left< \left( x(t_1) - \tilde{x}(t_1) \right)' - q(t_1) \eta(t_1) \right> s(t_1) \eta(t_1)$$

$$- \left< \left( s(t_1) \eta(t_1) \right)' - q(t_1) \eta(t_1) \right> x(t_1) - \tilde{x}(t_1). \tag{13}$$
Now, by second order functional transversality inclusion we have
\[
\varphi_0(x(t_1), x'(t_1)) - \varphi_0(\bar{x}(t_1), \bar{x}'(t_1)) + \sum_{k=1}^{r} \lambda_k \left[ \varphi_k(x(t_1), x'(t_1)) - \varphi_k(\bar{x}(t_1), \bar{x}'(t_1)) \right]
\geq -\left( \langle s(t_1)\eta(t_1) \rangle - q(t_1)\eta(t_1), x(t_1) - \bar{x}(t_1) \right) - \langle s(t_1)\eta(t_1), x'(t_1) - \bar{x}'(t_1) \rangle
+ \langle \eta(t_1), x(t_1) - \bar{x}(t_1) \rangle + \langle \eta'(t_1), \bar{x}'(t_1) \rangle, (\eta(t_1), \eta'(t_1)) \in K^*_Q(\bar{x}(t_1), \bar{x}'(t_1)).
\]

Note that for all pair satisfying \((\eta(t_1), \eta'(t_1)) \in K^*_Q(\bar{x}(t_1), \bar{x}'(t_1))\) it follows that \((\bar{x}(t_1), \bar{x}'(t_1)) \in Q\), i.e., \(x(\cdot)\) is feasible and by virtue of the inequality \(\sum_{k=1}^{r} \lambda_k \varphi_k(x(t_1), x'(t_1)) \leq 0\) and by the complementary slackness condition
\[
\varphi_0(x(t_1), x'(t_1)) - \varphi_0(\bar{x}(t_1), \bar{x}'(t_1)) \geq \left( \langle s(t_1)\eta(t_1) \rangle - q(t_1)\eta(t_1), x(t_1) - \bar{x}(t_1) \right),\]
\[
\forall (x(t_1), x'(t_1)) \in Q. \quad (14)
\]

Finally, summing the inequalities (13) and (14) we have the desired result \(J[x(t)] \geq J[\bar{x}(t)], \forall x(t), t \in [t_0, t_1]\) i.e., \(\bar{x}(\cdot)\) is optimal.

\section*{Corollary 3}
\textbf{Suppose now that in the problem \((PSL)\) \(s(t) \equiv p(t), q(t) \equiv p'(t)\), where \(p(t) > 0\). Then the problem \((PSL)\) is converted to the problem \((S-L)\) in the convex case and the conditions of Theorem 3.1 and Theorem 4.1 coincide.}

\textbf{Proof.} Since \((Bx)(t) = s(t)x'' + q(t)x'\) and \(s(t) \equiv p(t), q(t) \equiv p'(t), t \in [t_0, t_1]\) we have \((Bx)(t) = p(t)x'' + p'(t)x' = (Ax)(t)\), that is, the second order linear differential operator \(B\) is self-adjoint and \((B^*\eta)(t) = (A^*\eta)(t)\). Thus \((a1)\) of Theorem 4.1 implies (i) of Theorem 3.1. Besides, \(p(t) > 0\) and implication \((b1) \Rightarrow (ii)\) is true
\[
\left[ \langle s(t_1)\eta(t_1) \rangle - p(t_1)\eta(t_1), -s(t_1)\eta(t_1) \right] = \left[ (p(t_1)\eta(t_1))' - p'(t_1)\eta(t_1), -p(t_1)\eta(t_1) \right]
\]

The converse implications can be revised by analogy.

\section*{5. Sufficient conditions of optimality for generalization of second order differential inclusions.}
In this section the optimality conditions are given for a Bolza problem with a more general second order differential inclusions. These conditions involve useful forms of the Pontryagin condition and second order Euler-Lagrange type adjoint inclusions. Thus, let us consider the following optimal control problem, labelled as \((PS)\),
\[
\text{minimize } J[x(\cdot)] = \int_{t_0}^{t_1} g(x(t), t) dt + \varphi_0(x(t_1), x'(t_1)),
\]
\[(PS) \quad x''(t) \in F(x(t), x'(t), t), \text{ a.e. } t \in [t_0, t_1], x(t_0) = \alpha_0, x'(t_0) = \beta_1,
\varphi_k(x(t_1), x'(t_1)) \leq 0, k = 1, \ldots, r; (x(t_1), x'(t_1)) \in Q.
\]
where for functions, set-valued mapping and set are satisfied the usually conditions of Theorem 3.1. The basic difference of this problem is that here \(F(x, x', t)\) depend not only on \(x\), but on \(x'\), too. This fact seriously complicates the construction of proof of optimality conditions for higher order differential inclusions. It occurs because on the construction of LAM necessarily appears here an additional conjugate
variable. Let us formulate the second order Euler-Lagrange inclusion (a), (b), and the second order transversality condition (c) at $t = t_1$.

\[
(a) \quad \left( \frac{d^2 \eta(t)}{dt^2} + \frac{dv(t)}{dt}, v(t) \right) \in F^* (\eta(t); (\bar{x}(t), \bar{x}'(t), \bar{x}''(t)), t)
\]
\[
- \partial g(\bar{x}(t), t) \times \{0\}, \text{ a.e. } t \in [t_0, t_1],
\]

where

\[
(b) \quad \bar{x}'' \in F_A(\bar{x}(t), \bar{x}'(t); \eta(t), t) \text{ a.e. } t \in [t_0, t_1].
\]

In what follows we assume that $\eta(t), t \in [t_0, t_1]$ is absolutely continuous function together with the first order derivatives for which $\eta''(\cdot) \in L^1([t_0, t_1])$. Besides the auxiliary function $v(t), t \in [t_0, t_1]$ is absolutely continuous and $v'(\cdot) \in L^1([t_0, t_1])$.

The second order transversality conditions at the endpoint $t = t_1$ consist of the following

\[
(c) \quad \left( \eta''(t) + v'(t) + u(t), v(t) \right) \in F^* (\eta(t); (\bar{x}(t), \bar{x}'(t), \bar{x}''(t)), t) \text{ } t \in [t_0, t_1],
\]

In what follows we assume that $\eta(t), t \in [t_0, t_1],$ is absolutely continuous function together with the first order derivatives and $\eta''(\cdot) \in L^1([t_0, t_1])$. Besides $v(\cdot)$ is absolutely continuous and $v'(\cdot) \in L^1([t_0, t_1])$.

In what follows, Theorem 5.1 is very important.

**Theorem 5.1. (Sufficient conditions of optimality for second order evolution differential inclusions with functional and non-functional constraints)** Suppose that $\varphi_k(\cdot), k = 0, 1, \ldots, r$ are convex continuous functions, $F(\cdot, t) : \mathbb{R}^{2n} \rightharpoonup \mathbb{R}^n$ is an evolution mapping, $Q$ is convex subset of $\mathbb{R}^{2n}$. Then for optimality of the trajectory $\bar{x}(\cdot)$ in the problem (PS) it is sufficient that there exists a pair of absolutely continuous functions $\{\eta(t), v(t)\}, t \in [t_0, t_1]$, and Lagrange multipliers $\lambda_k \geq 0, k = 0, \ldots, r(\lambda_0 = 1)$, not all zero, satisfying a.e. the second order Euler-Lagrange inclusion (a), (b), the second order transversality condition (c) and the complementary slackness conditions (iii) of Theorem 3.1.

**Proof.** In the present case by Theorem 2.1[12, p.62] we can write $F^* (\eta; (x, y, z), t) = \partial_{(x,y)}F_{F(\cdot, t)}(x, y, \eta), z \in F_A(x, y; \eta, t)$. Then using the Moreau-Rockafellar Theorem 1.29 [12, p.6] from condition (a) we obtain

\[
\left( \eta''(t) + v'(t), v(t) \right) \in \partial_{(x,y)} \left[ H_{F(\cdot, t)}(\bar{x}(t), \bar{x}'(t), \eta(t)) - g(\bar{x}(t), t) \right], \text{ } t \in [t_0, t_1]
\]

or equivalently,

\[
H_{F(\cdot, t)}(x(t), x'(t), \eta(t)) - H_{F(\cdot, t)}(\bar{x}(t), \bar{x}'(t), \eta(t))
\]
\[
\leq \langle \eta''(t) + v'(t), x(t) - \bar{x}(t) \rangle + \langle v(t), x'(t) - \bar{x}'(t) \rangle + g(x(t), t) - g(\bar{x}(t), t), \tag{15}\]
According to the definition of Hamiltonian function, the inequality (15) can be reduced to the inequality

\[ \langle x''(t), \eta(t) \rangle - \langle \ddot{x}(t), \eta(t) \rangle \leq \langle \eta''(t), x(t) - \ddot{x}(t) \rangle \]

\[ + \frac{d}{dt} \langle v(t), x(t) - \ddot{x}(t) \rangle + g(x(t), t) - g(\ddot{x}(t), t) \]

which implies that

\[ g(x(t), t) - g(\ddot{x}(t), t) \geq \langle (x(t) - \ddot{x}(t))'', \eta(t) \rangle - \langle \eta''(t), x(t) - \ddot{x}(t) \rangle - \frac{d}{dt} \langle v(t), x(t) - \ddot{x}(t) \rangle. \] (16)

Then integrating of the inequality (16) over the interval \([t_0, t_1]\) we have

\[ \int_{t_0}^{t_1} [g(x(t), t) - g(\ddot{x}(t), t)] dt \geq \int_{t_0}^{t_1} \left[ \langle (x(t) - \ddot{x}(t))'', \eta(t) \rangle - \langle \eta''(t), x(t) - \ddot{x}(t) \rangle - \frac{d}{dt} \langle v(t), x(t) - \ddot{x}(t) \rangle \right] dt + \langle v(t_0), x(t_0) - \ddot{x}(t_0) \rangle - \langle v(t_1), x(t_1) - \ddot{x}(t_1) \rangle. \] (17)

Now transform the expression in the square parentheses on the right hand side of (17) as follows

\[ \langle (x(t) - \ddot{x}(t))'', \eta(t) \rangle - \langle \eta''(t), x(t) - \ddot{x}(t) \rangle = \frac{d}{dt} \langle (x(t) - \ddot{x}(t))', \eta(t) \rangle - \frac{d}{dt} \langle \eta'(t), x(t) - \ddot{x}(t) \rangle. \]

Then substituting \( p(t) \equiv 1 \) into the equality (7) we can compute the integral on the right hand side of (17) as follows

\[ \int_{t_0}^{t_1} \left[ \langle x''(t) - \ddot{x}'(t), \eta(t) \rangle - \langle \eta''(t), x(t) - \ddot{x}(t) \rangle \right] dt = \langle x'(t_1) - \ddot{x}(t_1), \eta(t_1) \rangle - \langle \eta'(t_1), x(t_1) - \ddot{x}(t_1) \rangle. \] (18)

Then taking into account (18) in the inequality (17), in view of the initial condition \( x(t_0) = \ddot{x}(t_0) = \alpha_0 \) we have

\[ \int_{t_0}^{t_1} [g(x(t), t) - g(\ddot{x}(t), t)] dt \geq \langle x'(t_1) - \ddot{x}(t_1), \eta(t_1) \rangle - \langle \eta'(t_1) + v(t_1), x(t_1) - \ddot{x}(t_1) \rangle. \] (19)

On the other hand substituting \( p(t_1) \equiv 1 \) into the inequality (9) for all feasible trajectories \( x(t), t \in [t_0, t_1] \) we can write

\[ \varphi_0 \left( x(t_1), x'(t_1) \right) - \varphi_0 \left( \ddot{x}(t_1), \ddot{x}'(t_1) \right) + \sum_{k=1}^{r} \lambda_k \left[ \varphi_k \left( x(t_1), x'(t_1) \right) - \varphi_k \left( \ddot{x}(t_1), \ddot{x}'(t_1) \right) \right] \]

\[ \geq \langle v(t_1) + \eta'(t_1), x(t_1) - \ddot{x}(t_1) \rangle - \langle \eta(t_1), x'(t_1) - \ddot{x}'(t_1) \rangle. \] (20)

At the same time it is shown that \( \ddot{x} \cdot \) is feasible, i.e. \((x(t_1), x'(t_1)) \in Q \). Then by virtue of (19), (20) and second order functional transversality inclusion (c) at \( t = t_1 \) for an arbitrary feasible solution \( x(\cdot) \) by analogy with the proof of Theorem 3.1 we deduce that \( J[x(t)] \geq J[\ddot{x}(t)], \forall x(\cdot) \), that is, \( \ddot{x}(\cdot) \) is optimal. \( \square \)
Corollary 4. In addition to assumptions of Theorem 5.1 let $F(\cdot, t)$ be a closed set-valued mapping. Then the conditions (a), (d) of Theorem 5.1 can be rewritten in term of Hamiltonian function as follows

$$
\left( \eta''(t) + v'(t), v(t) \right) \in \partial_{(x,y)} \left[ H_{F(\cdot,t)}(\tilde{x}(t), \tilde{x}'(t), \eta(t)) - g(\tilde{x}(t), t) \right],
$$

$$
\tilde{x}''(t) \in \partial_{\eta} H(\tilde{x}(t), \tilde{x}'(t), \eta(t)).
$$

Proof. In fact, the validity of the first relation we have seen in the proof of Theorem 3.1, where $F(x, x', t)$ does not depend on $x'$. On the other hand by Lemma 5.1 [17] $\partial_{\eta} H_{F(\cdot,t)}(x, y, \eta) = F_A(x, y, \eta, t)$ and the assertions of corollary are equivalent with the conditions (a), (b) of Theorem 5.1.

Below we prove that if a multivalued mapping $F(\cdot, t)$ depends only on $x$, then the adjoint inclusion involves only one conjugate variable.

Corollary 5. Suppose that for the convex problem with second order continuous-time evolution inclusions (PS) $F(x, x', t) \equiv G(x, t)$, and that the conditions of Theorem 5.1 are satisfied. Then the second order Euler-Lagrange differential inclusion (a), transversality conditions (c) and the complementary slackness conditions at $t = t_1$ of Theorem 5.1 consist of the following

(i) $\eta''(t) \in G^* (\eta(t); (\tilde{x}(t), \tilde{x}'(t), t)) - \partial g(\tilde{x}(t), t)$;

(ii) $\left( \eta(t_1), -\eta(t_1) \right) \in \partial \varphi_0 (\tilde{x}(t_1), \tilde{x}'(t_1)) + \sum_{k=1}^r \lambda_k \partial \varphi_k (\tilde{x}(t_1), \tilde{x}'(t_1))$

$$
- K_Q^* (\tilde{x}(t_1), \tilde{x}'(t_1)), \lambda_k \varphi_k (\tilde{x}(t_1), \tilde{x}'(t_1)) = 0, \lambda_k \geq 0, k = 1, \ldots, r.
$$

Proof. In the present case $F(x, y, t) \equiv G(x, t), dom F(\cdot, t) = dom G(\cdot, t) \times \mathbb{R}^n$ and

$$
F^*(\eta; (x, y, z), t) := \{ (\hat{x}, 0) : HG_G(x_1, \eta) - HG_G(x, \eta) \leq \langle \hat{x}, x_1 - x \rangle, \forall x_1 \in \mathbb{R}^n \} = G^* (\eta; (x, y, z), t) \times \{ 0 \}, (x, z) \in gph G(\cdot, t), v \in G_A(x, y, t).
$$

It means that in the second order Euler-Lagrange inclusion (a) of Theorem 5.1 $v(t) \equiv 0$, $t \in [t_0, t_1]$.

To illustrate how to apply the results of Theorem 5.1 consider the example; suppose now we have a Mayer problem (PS) with a semilinear second order continuous-time evolution inclusions:

minimize $J(x(\cdot)) = \varphi_0 (x(t_1), x'(t_1))$,

$$
x''(t) \in F(x(t), x'(t), t), F(x, y, t) = A_1(t)x + A_2(t)y + B(t)U,
$$

$$
\varphi_k (x(t_1), x'(t_1)) \leq 0, \text{ a.e. } t \in [t_0, t_1], k = 1, \ldots, r; \ x(t_0) = \alpha_0, x'(t_0) = \beta_1,
$$

where $\varphi_k, k = 0, \ldots, r$ are convex continuous functions, $A_i(t)$ ($i = 1, 2), B(t)$ are $n \times n$ and $n \times r$ continuous matrices, respectively, $U$ is a convex compact, i.e., convex closed and bounded subset of $\mathbb{R}^r$. It is required to find a controlling parameter $\hat{w}(t) \in U$ such that the trajectory $\hat{x}(\cdot)$ corresponding to it minimizes $J[x(\cdot)]$.

Theorem 5.2. The feasible trajectory $\hat{x}(\cdot)$ of problem (21) corresponding to the control function $\hat{w}(\cdot)$ minimizes $J[x(\cdot)]$ in the second order semilinear convex differential inclusions with functional constraints if there exists an absolutely continuous function $\eta(\cdot)$ satisfying the complementary slackness conditions at $t = t_1$, the
following second order adjoint differential equation, the transversality inclusion and Pontryagin condition:

\[ \eta''(t) = -A_2(t)\eta'(t) + A_1(t)\eta(t), \text{ a.e. } t \in [t_0, t_3], \]

\[ \langle B(t)\dot{\omega}(t), \eta(t) \rangle = \sup_{w \in U} \{ B(t)w, \eta(t) \}; \quad (A_2^*(t)\eta(t_1) + \eta'(t_1), -\eta(t_1) \} \]

\[ \in \sum_{k=0}^{r} \lambda_k \partial \varphi_k(\hat{x}(t_1), \hat{x}'(t_1)), \lambda_k \geq 0(\lambda_0 = 1), k = 1, \ldots, r. \]

**Proof.** Obviously, if for a convex map \( F(x, y, t) = A_1(t)x + A_2(t)y + B(t)U \) the LAM is nonempty, then \( F^*(\eta; (x, y, z), t) = (A_1^*(t)\eta, A_2^*(t)\eta) \) as \( -B^*(t)\eta \in K_U^*(w) \). Then according to Theorem 3.1 the Euler-Lagrange inclusion is

\[ \eta''(t) + v'(t) = A_1^*(t)\eta(t), \quad v(t) = A_2^*(t)\eta(t). \quad (22) \]

Differentiating second equation of (22) and substituting into first equation we immediately have the needed second order adjoint equation. The other conditions of theorem is an immediate consequence of the conditions of Theorem 3.1. The proof is completed. \( \square \)

**Example 1.** Let us consider the following second order Mayer problem with one functional constraint:

\[
\begin{align*}
\text{minimize } & \varphi_0(x(t_1), x'(t_1)) \text{ subject to } x'' = u, \ |u| \leq 1, \\
\varphi_1(x(t_1), x'(t_1)) < 0; \ x(t_0) = 0, \ x(t_1) = 1,
\end{align*}
\]

(23)

where \( \varphi_0(x(t_1), x'(t_1)) = x'^2(t_1) - x(t_1) \) and \( \varphi_1(x(t_1), x'(t_1)) = x(t_1) - x'(t_1) \).

It is well known that, \( x'' = u \) is a particular case of Newton’s second law of motion described by equation \( F(t) = ma(t) \), where \( F(t) \) acts on the particle and \( a(t) \) is acceleration (in our case \( F(t) = u(t), x''(t) = a(t), m = 1 \)). It can be easily checked that \( \varphi_0(x, y) = y^2 - x \) is a convex function; indeed it is sufficient to show that the \( 2 \times 2 \) Hessian matrix

\[
\varphi''(x, y) = \begin{bmatrix}
\varphi''_{xx}(x, y) & \varphi''_{xy}(x, y) \\
\varphi''_{yx}(x, y) & \varphi''_{yy}(x, y)
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 2
\end{bmatrix} = A
\]

is a positive semidefinite matrix. Consequently, \( \varphi(x, y) \) is convex and \( \partial \varphi(x, y) = (-1, 2y) \). It is required to find the optimal control \( \hat{u}(\cdot) \) such that the corresponding arc \( \hat{x}(\cdot) \) minimizes \( \varphi(x(t_1), x'(t_1)) \). In the present case \( F(x, t) \equiv F(x) = \{ u : |u| \leq 1 \} \) for all \( t \in [t_0, t_1] \).

So the Mayer optimal control problem (23) with second order differential inclusion can be written as follows:

\[
\begin{align*}
\text{minimize } & \varphi_0(x(t_1), x'(t_1)), \\
x'' \in F(x), \ t \in [t_0, t_1], \ x(t_0) = 0, \ x'(t_0) = 1, \\
\varphi_1(x(t_1), x'(t_1)) < 0; \ Q = \mathbb{R}^2.
\end{align*}
\]

Obviously, in the adjoint Euler-Lagrange inclusion (i) of Corollary 5 \( K_Q^*(\hat{x}(t_1), \hat{x}'(t_1)) = \{ 0 \} \) and \( \partial \varphi(\hat{x}(t), t) = \{ 0 \} g(x(t), t) \equiv 0 \) i.e., we have the inclusion \( \eta''(t) \in F^*(\eta(t); (\hat{x}(t), \hat{x}'(t))) \). Then by Theorem 2.1 on LAM [12, p.62] in the convex case \( F^*(\eta; (x, z)) = \partial \varphi H_F(x, \eta) \).

Clearly

\[ H_F(x, \eta) = \max_u \{ u\eta : |u| \leq 1 \} = |\eta| \]

(24)
and so
\[ F^*(\eta; (x, z)) = \partial_x H_F(x, \eta) \equiv 0, \ z \in F_A(x, \eta) = \{-1, +1\}. \]  (25)

Then as a result of Corollary 5 from (25) we deduce that
\[ \frac{d^2 \eta}{dt^2} = 0, \ t \in [t_0, t_1], \]
for which the solution is a linear function of the form \( \eta(t) = C_1 t + C_2 \), where \( C_1, C_2 \) are arbitrary constants. Then (24) implies that \( \tilde{u}(t) \eta(t) = |\eta(t)| \) or
\[ \tilde{u}(t) = \begin{cases} \text{sgn} \eta(t), & \text{if } \eta(t) \neq 0, \\ \forall u_0 \in [-1, 1], & \text{if } \eta(t) = 0 \end{cases} \]  (26)

Furthermore, since \( \eta(t), t \in [t_0, t_1] \) as a linear function, does not change sign more than once in the time interval \([t_0, t_1]\), it follows from (26) that every optimal control \( \tilde{u}(t), t \in [t_0, t_1] \) is a piecewise-constant function, having the values \pm 1 and having not more than two interval of constancy. Let us return to the transversality condition of Corollary 5. By this condition at a point \( t = t_1 \) we have
\[ (\eta'(t_1), -\eta'(t_1)) \in \partial \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1)) + \lambda_1 \partial \varphi_1(\tilde{x}(t_1), \tilde{x}'(t_1)), \]
(27)
where \( \partial \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1)) = (-1, 2\tilde{x}'(t_1)) \) and \( \partial \varphi_1(\tilde{x}(t_1), \tilde{x}'(t_1)) = (1, -1) \). Comparing this relation with (27) we have \( (\eta'(t_1), -\eta'(t_1)) = (-1, 2\tilde{x}'(t_1)) + \lambda_1 (1, -1), \)
which implies \( \eta'(t_1) = \lambda_1 - 1, \eta(t_1) = 1 - 2\tilde{x}'(t_1) \). Then from the general solution of the adjoint Euler-Lagrange equation \( \eta(t) = C_1 t + C_2 \) we have \( 1 - 2\tilde{x}'(t_1) = \eta(t_1) = C_1 + C_2, \lambda_1 - 1 = \eta'(t_1) = C_1 \), where \( C_1, C_2 \) are arbitrary constants (for further convenience we take \( t_1 = 1 \)). Then \( \eta(t) = (\lambda_1 - 1) t + 2 - \lambda_1 - 2\tilde{x}'(1) \), whence \( \eta(t) \neq 0, \) if \( t \neq \tau = 1 + (1 - 2\tilde{x}'(1))/(1 - \lambda_1) \). Therefore, (26) implies that in general, for optimal control \( \tilde{u}(\cdot) \) there are four possibilities
\[ \tilde{u}(t) = 1, \ \eta(t) > 0, \ t \in [t_0, 1]. \]  (28)
\[ \tilde{u}(t) = -1, \ \eta(t) < 0, \ t \in [t_0, 1]. \]  (29)
\[ \tilde{u}(t) = \begin{cases} 1, & \text{if } t_0 \leq t < \tau, \\ -1, & \text{if } \tau < t \leq 1. \end{cases} \]  (30)
\[ \tilde{u}(t) = \begin{cases} -1, & \text{if } t_0 \leq t < \tau, \\ 1, & \text{if } \tau < t \leq 1. \end{cases} \]  (31)

where \( \tau \) is a point of discontinuity of \( \tilde{u}(\cdot) \). Moreover, by the complementary slackness condition of Corollary 5 \( \lambda_1 \varphi_1(\tilde{x}(1), \tilde{x}'(1)) = \lambda_1 (\tilde{x}(1) - \tilde{x}'(1)) = 0. \) Here \( \varphi_1(\tilde{x}(1), \tilde{x}'(1)) < 0 \) and so \( \lambda_1 = 0. \) As a consequence, for \( \lambda_1 = 0, \eta'(1) = -1 \) and \( \eta(t) = -t + 2(1 - \tilde{x}'(1)) \) whereas \( \eta'(t) = -1, t \in [t_0, 1]. \) It follows that either the sign of the linear function \( \eta[\cdot] \) doesn’t changes for the whole interval \([t_0, 1]\) or \( \eta(t) > 0, t_0 \leq t < \tau: \eta(t) < 0, \tau < t \leq 1 \) for a some \( \tau \) in the interval \( t_0 < \tau < 1 \) (the case (31) is excluded).

Therefore, since \( \tilde{u}(t) \) is a piecewise-constant function, having not more than two interval of constancy we have either the cases (28), (29) or the case (30). In general, using (28)-(30), by solving the Cauchy problem
\[ \frac{d^2 x(t)}{dt^2} = u(t), \ x(t_0) = 0, \ x'(t_0) = 1 \]  (32)
we have a unique solution of the initial value problem (32). Thus for the time
interval on which \( u = 1 \) we have \( x'(t) = t + c_1; x(t) = t^2/2 + c_1t + c_2 \) (\( c_1, c_2 \) are
arbitrary constants). Taking into account the initial values in (32) we derive that
\[
x'(t) = t + 1; \quad x(t) = (t^2/2) + t.
\]

Similarly, we have for the time interval on which \( u = -1 \)
\[
x'(t) = 1 - t; \quad x(t) = -(t^2/2) + t.
\]

Let us denote the parabolas (33) and (34) by \( x_1(t), x_2(t) \), correspondingly. Obvi-
osely, in the case (28) \( \tilde{u}(t) = 1, t \in [t_0, 1] \) and from (33) we have
\( \tilde{x}_1(1) = (1/2) + 1 = 1.5 \) and \( \tilde{x}'_1(1) = 2 \). Note that \( \tilde{x}_1(1) = 1.5, \tilde{x}'_1(1) = 2 \) satisfy the
functional condition \( \varphi_1(x(1), x'(1)) < 0 \). As a result, the value of our Mayer
problem (23) would be \( \varphi_0(\tilde{x}_1(1), \tilde{x}'_1(1)) = \tilde{x}'_2(1) - \tilde{x}_1(1) = 2^2 - (3/2) = 2.5 \), if
\( \tilde{u}(t) = 1, t \in [t_0, 1] \). By analogy, if now \( \tilde{u}(t) = -1, t \in [t_0, 1] \) from (34) we deduce
that \( \tilde{x}_2(1) = -(t^2/2) + 1 + 0.5, \tilde{x}'_2(1) = 0 \) and \( \varphi_1(\tilde{x}_2(1), \tilde{x}'_2(1)) = 0.5 \). Finally,
\( \varphi_1(\tilde{x}_2(1), \tilde{x}'_2(1)) < 0 \) is not satisfied and this case is excluded.

On the other hand, in the case (30) the control function \( \tilde{u}(t) \) first is equal to
+1, then equal to -1 and the trajectory \( \tilde{x}(t) \) consists of two pieces of parabolas
\( \tilde{x}_1(t), t_0 \leq t \leq \tau \); \( \tilde{x}_2(t), \tau < t \leq 1 \) \((t) \) is continuous and piecewise smooth on the
interval \( t_0 \leq t \leq 1 \). In this case the solution of the equation (32) on the interval
\( t_0 \leq t \leq \tau \) is given by (33); at a point \( \tau \) are satisfied \( x_1(\tau) = (\tau^2/2) + \tau, x'_1(\tau) = 1 + \tau \).
Consider now the initial value problem
\[
x''(t) = -1, \quad x_2(\tau) = (\tau^2/2) + \tau, \quad x'_2(\tau) = 1 + \tau, \quad t \in [\tau, 1].
\]

It is clear that \( (\tau^2/2) + \tau = x_2(\tau) = -(\tau^2/2) + c_1\tau + c_2 \) and \( 1 + \tau = x'_2(\tau) =
-\tau + c_1 \) from which we obtain that the solution of the initial value problem (35)
is \( \tilde{x}_2(t) = -(t^2/2) + (1 + 2\tau)t - \tau^2 \). Substituting the value \( \tau = 2(1 - \tilde{x}'(1)) \) into
equation \( \tilde{x}'_2(t) = 1 - t + 2\tau \) we have 5\( \tilde{x}'_2(1) = 4 \), \( \tilde{x}'_2(1) = \tilde{x}'(1) = 0.8 \). Moreover,
\( \tilde{x}_2(1) = 2\tau - \tau^2 + (0.5) \) and \( \tilde{x}_2(1) = \tilde{x}(1) = 4(\tilde{x}'(1)) - \tilde{x}'(1) = 0.5 = 1.14 \). Thus,
\( \varphi_1(\tilde{x}_2(1), \tilde{x}'_2(1)) = 1.14 - 0.8 = 0.34 > 0 \) and the functional constraint is not
satisfied, i.e., we believe that the value of our problem is \( \varphi_0(\tilde{x}_1(1), \tilde{x}'_1(1)) = 2.5 \).

Acknowledgments. The author wishes to express his sincere thanks to the Prof.
Kok Lay Teo, Editor-in-Chief of JIMO and to the anonymous reviewers for their
careful reading of my manuscript and their many insightful comments and sugges-
tions.

REFERENCES

[1] S. Adly, A. Hantoute and M. Th’era, Nonsmooth Lyapunov pairs for differential inclusions
governed by operators with nonempty interior domain, Mathem. Program., 157 (2016), 349–374.
[2] N. U. Ahmed, Differential inclusions operator valued measures and optimal control, Dynamic
Syst. Appl., 16 (2007), 13–35.
[3] D. Azzam-Laouir and L. Sabrina, Existence solutions for a class of second order differential inclusions,
Pacific Journ. of Optim., 6 (2005), 339–346.
[4] A. Bagirov, N. Karmitsa and M. Makela, Introduction to Nonsmooth Optimization, Springer,
2014.
[5] A. Cernea, Continuous version of Filippov’s theorem for a Sturm-Liouville type differential
inclusion, E.J. Differ. Equat., 2008 (2008), 1–7.
[6] F. H. Clarke, Functional Analysis, Calculus of Variations and Optimal Control, Graduate
Texts in Mathematics, 264, Springer, 2013.
[7] Y. Gao, X. Yang, J. Yang and H. Yan, Scalarizations and Lagrange multipliers for approximat
solutions in the vector optimization problems with set-valued maps, J. Industrial Manag.
Optim., 11 (2014), 673–683.
[8] S. J. Li, S. K. Zhu and K. Lay Teo, New generalized second-order contingent epiderivatives
and set-valued optimization problems, J. Optim. Theory Appl., 152 (2012), 587–604.
[9] Q. Liqun, K. Lay Teo and X. Yang, Optimization and Control with Applications, Springer,
2005.
[10] Y. Liu, J. Wu and Z. Li, Impulsive boundary value problems for Sturm-Liouville type differ-
ential inclusions, J. Syst. Sci. Complexity, 20 (2007), 370–380.
[11] P. D. Loewen and R. T. Rockafellar, Optimal control of unbounded differential inclusions,
SIAM J Contr Optim., 32 (1994), 442–470.
[12] E. N. Mahmudov, Approximation and Optimization of Discrete and Differential Inclusions,
Elsevier: Boston, USA, 2011.
[13] E. N. Mahmudov, Approximation and Optimization of Higher order discrete and differential
inclusions, Nonlin. Diff. Equat. Appl. (NoDEA), 21 (2014), 1–26.
[14] E. N. Mahmudov, Optimal control of second order delay-discrete and delay differential inclusions
with state constraints, Evol. Equat. Cont. Theory (EECT), 7 (2018), 501–529.
[15] E. N. Mahmudov, Optimization of Fourth-Order Differential Inclusions, Proceed. Institute
Mathem. Mechanics, 44 (2018), 90–106.
[16] E. N. Mahmudov, Optimization of second-order discrete approximation inclusions, Numeric.
Funct. Anal. Optim., 36 (2015), 624–643.
[17] E. N. Mahmudov, Optimization of Mayer problem with Sturm-Liouville-type differential inclusions, J. Optim. Theory Appl., 177 (2018), 345–375.
[18] E. N. Mahmudov, Optimization of fourth order Sturm-Liouville type differential inclusions
with initial point constraints, J. Industrial Manag. Optim., (2018).
[19] B. S. Mordukhovich, Optimal control of semilinear unbounded evolution inclusions with functional constraints, J. Optim. Theory Appl., 167 (2015), 821–841.
[20] Y. Xu and Z. Peng, Higher-order sensitivity analysis in set-valued optimization under Henig
efficiency, J. Industrial Manag. Optim., 13 (2017), 313–327.

Received November 2018; revised February 2019.
E-mail address: elimhan22@yahoo.com,mahmudov@itu.edu.tr