On Measure Theoretic definitions of Generalized Information Measures and Maximum Entropy Prescriptions

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Abstract. Though Shannon entropy of a probability measure $P$, defined as $-\int_X \frac{dP}{d\mu} \ln \frac{dP}{d\mu} \, d\mu$ on a measure space $(X, \mathcal{M}, \mu)$, does not qualify itself as an information measure (it is not a natural extension of the discrete case), maximum entropy (ME) prescriptions in the measure-theoretic case are consistent with that of discrete case. In this paper, we study the measure-theoretic definitions of generalized information measures and discuss the ME prescriptions. We present two results in this regard: (i) we prove that, as in the case of classical relative-entropy, the measure-theoretic definitions of generalized relative-entropies, Rényi and Tsallis, are natural extensions of their respective discrete cases, (ii) we show that, ME prescriptions of measure-theoretic Tsallis entropy are consistent with the discrete case.

PACS numbers:
1. Introduction

Shannon measure of information was developed essentially for the case when the random variable takes a finite number of values. However, in the literature, one often encounters an extension of Shannon entropy in the discrete case to the case of a one-dimensional random variable with density function $p$ in the form (e.g. [2])

$$S(p) = - \int_{-\infty}^{+\infty} p(x) \ln p(x) \, dx.$$  

This entropy in the continuous case as a pure-mathematical formula (assuming convergence of the integral and absolute continuity of the density $p$ with respect to Lebesgue measure) resembles Shannon entropy in the discrete case, but can not be used as a measure of information. First, it is not a natural extension of Shannon entropy in the discrete case, since it is not the limit of the sequence finite discrete entropies corresponding to pmf which approximate the pdf $p$. Second, it is not strictly positive. 

Inspite of these shortcomings, one can still use the continuous entropy functional in conjunction with the principle of maximum entropy where one wants to find a probability density function that has greater uncertainty than any other distribution satisfying a set of given constraints. Thus, in this use of continuous measure one is interested in it as a measure of relative uncertainty, and not of absolute uncertainty. This is where one can relate maximization of Shannon entropy to the minimization of Kullback-Leibler relative-entropy (see [3, pp. 55]).

Indeed, during the early stages of development of information theory, the important paper by Gelfand, Kolmogorov and Yaglom [4] called attention to the case of defining entropy functional on an arbitrary measure space $(X, \mathcal{M}, \mu)$. In this respect, Shannon entropy of a probability density function $p : X \to \mathbb{R}^+$ can be written as,

$$S(p) = - \int_X p(x) \ln p(x) \, d\mu.$$ 

One can see from the above definition that the concept of “the entropy of a pdf” is a misnomer: there is always another measure $\mu$ in the background. In the discrete case considered by Shannon, $\mu$ is the cardinality measure§ [1, pp. 19]; in the continuous case considered by both Shannon and Wiener, $\mu$ is the Lebesgue measure cf. [2, pp. 54] and [5, pp. 61, 62]. All entropies are defined with respect to some measure $\mu$, as Shannon and Wiener both emphasized in [1, pp.57, 58] and [5, pp.61, 62] respectively.

This case was studied independently by Kallianpur [6] and Pinsker [7], and perhaps others were guided by the earlier work of Kullback [8], where one would define entropy in terms of Kullback-Leibler relative entropy. Unlike Shannon entropy, measure-theoretic definition of KL-entropy is a natural extension of definition in the discrete case.

In this paper we present the measure-theoretic definitions of generalized information measures and show that as in the case of KL-entropy, the measure-theoretic definitions

§ Counting or cardinality measure $\mu$ on a measurable space $(X, \mathcal{M})$, when is $X$ is a finite set and $\mathcal{M} = 2^X$, is defined as $\mu(E) = \#E, \forall E \in \mathcal{M}$.
of generalized relative-entropies, Rényi and Tsallis, are natural extensions of their respective discrete cases. We discuss the ME prescriptions for generalized entropies and show that ME prescriptions of measure-theoretic Tsallis entropy are consistent with the discrete case, which is true for measure-theoretic Shannon-entropy.

Rigorous studies of the Shannon and KL entropy functionals in measure spaces can be found in the papers by Ochs [9] and by Masani [10, 11]. Basic measure-theoretic aspects of classical information measures can be found in [7, 12, 13].

We review the measure-theoretic formalisms for classical information measures in §2 and extend these definitions to generalized information measures in §3. In §4 we present the ME prescription for Shannon entropy followed by prescriptions for Tsallis entropy in §5. We revisit measure-theoretic definitions of generalized entropic functionals in §6 and present some results.

2. Measure-Theoretic definitions of Classical Information Measures

2.1. Discrete to Continuous

Let \( p : [a, b] \to \mathbb{R}^+ \) be a probability density function, where \([a, b] \subset \mathbb{R}\). That is, \( p \) satisfies
\[
p(x) \geq 0, \ \forall x \in [a, b] \text{ and } \int_a^b p(x) \, dx = 1.
\]

In trying to define entropy in the continuous case, the expression of Shannon entropy was automatically extended by replacing the sum in the Shannon entropy discrete case by the corresponding integral. We obtain, in this way, Boltzmann’s H-function (also known as differential entropy in information theory),
\[
S(p) = -\int_a^b p(x) \ln p(x) \, dx. \tag{1}
\]

But the “continuous entropy” given by (1) is not a natural extension of definition in discrete case in the sense that, it is not the limit of the finite discrete entropies corresponding to a sequence of finer partitions of the interval \([a, b]\) whose norms tend to zero. We can show this by a counter example. Consider a uniform probability distribution on the interval \([a, b]\), having the probability density function
\[
p(x) = \frac{1}{b - a}, \quad x \in [a, b].
\]

The continuous entropy (1), in this case will be
\[
S(p) = \ln(b - a).
\]

On the other hand, let us consider a finite partition of the the interval \([a, b]\) which is composed of \(n\) equal subintervals, and let us attach to this partition the finite discrete uniform probability distribution whose corresponding entropy will be, of course,
\[
S_n(p) = \ln n.
\]
Obviously, if \( n \) tends to infinity, the discrete entropy \( S_n(p) \) will tend to infinity too, and not to \( \ln(b - a) \); therefore \( S(p) \) is not the limit of \( S_n(p) \), when \( n \) tends to infinity. Further, one can observe that \( \ln(b - a) \) is negative when \( b - a < 1 \).

Thus, strictly speaking continuous entropy \( S \) cannot represent a measure of uncertainty since uncertainty should in general be positive. We are able to prove the “nice” properties only for the discrete entropy, therefore, it qualifies as a “good” measure of information (or uncertainty) supplied by a random experiment. The “continuous entropy” not being the limit of the discrete entropies, we cannot extend the so-called nice properties to it.

Also, in physical applications, the coordinate \( x \) in (1) represents an abscissa, a distance from a fixed reference point. This distance \( x \) has the dimensions of length. Now, with the density function \( p(x) \), one can specify the probabilities of an event \([c, d] \subset [a, b]\) as \( \int_c^d p(x) \, dx \), one has to assign the dimensions (length)\(^{-1} \), since probabilities are dimensionless. Now for \( 0 \leq z < 1 \), one has the series expansion

\[
-\ln(1 - z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \ldots ,
\]

(2)

it is necessary that the argument of the logarithm function in (1) be dimensionless. Hence the formula (1) is then seen to be dimensionally incorrect, since the argument of the logarithm on its right hand side has the dimensions of a probability density \( [14] \). Although Shannon \([15]\) used the formula (1), he does note its lack of invariance with respect to changes in the coordinate system.

In the context of maximum entropy principle Jaynes \([16]\) addressed this problem and suggested the formula,

\[
S'(p) = -\int_a^b p(x) \ln \frac{p(x)}{m(x)} \, dx ,
\]

(3)

in the place of (1), where \( m(x) \) is a prior function. Note that when \( m(x) \) is probability density function, (3) is nothing but the relative-entropy. However, if we choose \( m(x) = c \), a constant (e.g \([17]\)), we get

\[
S'(p) = S(p) - \ln c ,
\]

where \( S(p) \) refers to the continuous entropy (1). Thus, maximization of \( S'(p) \) is equivalent to maximization of \( S(p) \). Further discussion on estimation of probability density functions by ME-principle in the continuous case can be found in \([18, 17, 19]\).

Prior to that, Kullback \([8]\) too suggested that in the measure-theoretic definition of entropy, instead of examining the entropy corresponding to only on given measure, we have to compare the entropy inside a whole class of measures.

### 2.2. Classical information measures

Let \((X, \mathcal{M}, \mu)\) be a measure space. \( \mu \) need not be a probability measure unless otherwise specified. Symbols \( P, R \) will denote probability measures on measurable space \((X, \mathcal{M})\) and \( p, r \) denote \( \mathcal{M} \)-measurable functions on \( X \). An \( \mathcal{M} \)-measurable function \( p : X \to \mathbb{R}^+ \) is said to be a probability density function (pdf) if \( \int_X p \, d\mu = 1 \).
In this general setting, Shannon entropy \( S(p) \) of pdf \( p \) is defined as follows \[20\].

**Definition 2.1.** Let \((X, \mathcal{M}, \mu)\) be a measure space and \(\mathcal{M}\)-measurable function \( p : X \to \mathbb{R}^+ \) be pdf. Shannon entropy of \( p \) is defined as
\[
S(p) = -\int_X p \ln p \, d\mu ,
\]
provided the integral on right exists.

Entropy functional \( S(p) \) defined in (4) can be referred to as entropy of the probability measure \( P \), in the sense that the measure \( P \) is induced by \( p \), i.e.,
\[
P(E) = \int_E p(x) \, d\mu(x) , \quad \forall E \in \mathcal{M} .
\]
This reference is consistent because the probability measure \( P \) can be identified \( a.e \) by the pdf \( p \).

Further, the definition of the probability measure \( P \) in (5), allows us to write entropy functional (4) as,
\[
S(p) = -\int_X \frac{dP}{d\mu} \ln \frac{dP}{d\mu} \, d\mu ,
\]
since (5) implies \( P \ll \mu \), and pdf \( p \) is the Radon-Nikodym derivative of \( P \) w.r.t \( \mu \).

Now we proceed to the definition of Kullback-Leibler relative-entropy or KL-entropy for probability measures.

**Definition 2.2.** Let \((X, \mathcal{M})\) be a measurable space. Let \( P \) and \( R \) be two probability measures on \((X, \mathcal{M})\). Kullback-Leibler relative-entropy KL-entropy of \( P \) relative to \( R \) is defined as
\[
I(P||R) = \begin{cases} 
\int_X \ln \frac{dP}{dR} \, dP & \text{if } P \ll R , \\
+\infty & \text{otherwise.}
\end{cases}
\]

The divergence inequality \( I(P||R) \geq 0 \) and \( I(P||R) = 0 \) if and only if \( P = R \) can be shown in this case too. KL-entropy (7) also can be written as
\[
I(P||R) = \int_X \frac{dP}{dR} \ln \frac{dP}{dR} \, dR .
\]

Let the \( \sigma \)-finite measure \( \mu \) on \((X, \mathcal{M})\) such that \( P \ll R \ll \mu \). Since \( \mu \) is \( \sigma \)-finite, from Radon-Nikodym theorem, there exists a non-negative \( \mathcal{M} \)-measurable functions \( p : X \to \mathbb{R}^+ \) and \( r : X \to \mathbb{R}^+ \) unique \( \mu \)-a.e, such that
\[
P(E) = \int_E p \, d\mu , \quad \forall E \in \mathcal{M} ,
\]
\(\|\) Say \( p \) and \( r \) are two pdfs and \( P \) and \( R \) are corresponding induced measures on measurable space \((X, \mathcal{M})\) such that \( P \) and \( R \) are identical, i.e., \( \int_E p \, d\mu = \int_E r \, d\mu, \forall E \in \mathcal{M} \). Then we have \( p \equiv r \) and hence \( -\int_X p \ln p \, d\mu = -\int_X r \ln r \, d\mu \).

\(\|\) If a nonnegative measurable function \( f \) induces a measure \( \nu \) on measurable space \((X, \mathcal{M})\) with respect to a measure \( \mu \), defined as \( \nu(E) = \int_E f \, d\mu, \forall E \in \mathcal{M} \) then \( \nu \ll \mu \). Converse is given by Radon-Nikodym theorem \[21 \] pp.36, Theorem 1.40(b)].
and

\[ R(E) = \int_E r \, d\mu \ , \ \forall E \in \mathcal{M} . \]  

(10)

The pdfs \( p \) and \( r \) in (9) and (10) (they are indeed pdfs) are Radon-Nikodym derivatives of probability measures \( P \) and \( R \) with respect to \( \mu \), respectively, i.e., \( p = \frac{dP}{d\mu} \) and \( r = \frac{dR}{d\mu} \).

Now one can define relative-entropy of pdf \( p \) w.r.t \( r \) as follows:

**Definition 2.3.** Let \((X, \mathcal{M}, \mu)\) be a measure space. Let \( \mathcal{M} \)-measurable functions \( p, r : X \to \mathbb{R}^+ \) be two pdfs. The KL-entropy of \( p \) relative to \( r \) is defined as

\[ I(p \| r) = \int_X p(x) \ln \frac{p(x)}{r(x)} \, d\mu(x) , \]  

(11)

provided the integral on right exists.

As we have mentioned earlier, KL-entropy (11) exist if the two densities are absolutely continuous with respect to one another. On the real line the same definition can be written as

\[ I(p \| r) = \int_{\mathbb{R}} p(x) \ln \frac{p(x)}{r(x)} \, dx , \]

which exist if the densities \( p(x) \) and \( r(x) \) share the same support. Here, in the sequel we use the convention

\[ \ln 0 = -\infty, \quad \ln \frac{a}{0} = +\infty \text{ for any } a \in \mathbb{R}, \quad 0.(\pm \infty) = 0. \]  

(12)

Now we turn to the definition of entropy functional on a measure space. Entropy functional in (6) is defined for a probability measure that is induced by a pdf. By the Radon-Nikodym theorem, one can define Shannon entropy for any arbitrary \( \mu \)-continuous probability measure as follows.

**Definition 2.4.** Let \((X, \mathcal{M}, \mu)\) be a \( \sigma \)-finite measure space. Entropy of any \( \mu \)-continuous probability measure \( P \) \((P \ll \mu)\) is defined as

\[ S(P) = -\int_X \ln \frac{dP}{d\mu} \, dP . \]  

(13)

Properties of entropy of a probability measure in the Definition 2.4 are studied in detail by Ochs [9] under the name generalized Boltzmann-Gibbs-Shannon Entropy. In the literature, one can find notation of the form \( S(P|\mu) \) to represent the entropy functional in (13), viz., the entropy of a probability measure, to stress the role of the measure \( \mu \) (e.g. [9, 20]). Since all the information measures we define are with respect to the measure \( \mu \) on \((X, \mathcal{M})\), we omit \( \mu \) in the entropy functional notation.

By assuming \( \mu \) as a probability measure in the Definition 2.4 one can relate Shannon entropy with Kullback-Leibler entropy as,

\[ S(P) = -I(P \| \mu) . \]  

(14)

\( + \) This follows from the chain rule for Radon-Nikodym derivative:

\[ \frac{dP}{dR} \bigg{\mid} \frac{\partial}{\partial R} \frac{P}{\partial \mu} \bigg{\mid} \frac{\partial}{\partial \mu} \left( \frac{dR}{d\mu} \right)^{-1} . \]
Note that when \( \mu \) is not a probability measure, the divergence inequality \( I(P\|\mu) \geq 0 \) need not be satisfied.

A note on the \( \sigma \)-finiteness of measure \( \mu \). In the definition of entropy functional we assumed that \( \mu \) is a \( \sigma \)-finite measure. This condition was used by Ochs [9], Csiszár [22] and Rosenblatt-Roth [23] to tailor the measure-theoretic definitions. For all practical purposes and for most applications, this assumption is satisfied. (See [9] for a discussion on the physical interpretation of measurable space \((X, \mathcal{M})\) with \( \sigma \)-finite measure \( \mu \) for entropy measure of the form (13), and of the relaxation \( \sigma \)-finiteness condition.) By relaxing this condition, more universal definitions of entropy functionals are studied by Masani [10, 11].

2.3. Interpretation of Discrete and Continuous Entropies in terms of KL-entropy

First, let us consider discrete case of \((X, \mathcal{M}, \mu)\), where \( X = \{x_1, \ldots, x_n\}, \mathcal{M} = 2^X \) and \( \mu \) is a cardinality probability measure. Let \( P \) be any probability measure on \((X, \mathcal{M})\). Then \( \mu \) and \( P \) can be specified as follows.

\[
\mu: \mu_k = \mu(\{x_k\}) \geq 0, \quad k = 1, \ldots, n, \quad \sum_{k=1}^{n} \mu_k = 1 ,
\]

and

\[
P: \quad P_k = P(\{x_k\}) \geq 0, \quad k = 1, \ldots, n, \quad \sum_{k=1}^{n} P_k = 1 .
\]

The probability measure \( P \) is absolutely continuous with respect to the probability measure \( \mu \) if \( \mu_k = 0 \) implies \( P_k = 0 \) for any \( k = 1, \ldots, n \). The corresponding Radon-Nikodym derivative of \( P \) with respect to \( \mu \) is given by

\[
\frac{dP}{d\mu}(x_k) = \frac{P_k}{\mu_k}, \quad k = 1, \ldots, n .
\]

The measure-theoretic entropy \( S(P) \) [13], in this case, can be written as

\[
S(P) = - \sum_{k=1}^{n} P_k \ln \frac{P_k}{\mu_k} = \sum_{k=1}^{n} P_k \ln \frac{1}{P_k} = \sum_{k=1}^{n} P_k \ln P_k - \sum_{k=1}^{n} P_k \ln P_k .
\]

If we take referential probability measure \( \mu \) as a uniform probability distribution on the set \( X \), i.e. \( \mu_k = \frac{1}{n} \), we obtain

\[
S(P) = S_n(P) - \ln n , \tag{15}
\]

where \( S_n(P) \) denotes the Shannon entropy of pmf \( P = (P_1, \ldots, P_n) \) and \( S(P) \) denotes the measure-theoretic entropy in the discrete case.

Now, let us consider the continuous case of \((X, \mathcal{M}, \mu)\), where \( X = [a, b] \subset \mathbb{R}, \mathcal{M} \) is set of Lebesgue measurable sets of \([a, b]\), and \( \mu \) is the Lebesgue probability measure. In this case \( \mu \) and \( P \) can be specified as follows.

\[
\mu: \mu(x) \geq 0, \quad x \in [a, b], \quad \exists \mu(E) = \int_{E} \mu(x) \, dx, \quad \forall E \in \mathcal{M}, \quad \int_{a}^{b} \mu(x) \, dx = 1 ,
\]
and
\[ P: \quad P(x) \geq 0, x \in [a, b], P(E) = \int_{E} P(x) \, dx, \forall E \in \mathcal{M}, \int_{a}^{b} P(x) \, dx = 1. \]

Note the abuse of notation in the above specification of probability measures \( \mu \) and \( P \), where we have used the same symbols for both measures and pdfs.

The probability measure \( P \) is absolutely continuous with respect to the probability measure \( \mu \), if \( \mu(x) = 0 \) on a set of a positive Lebesgue measure implies that \( P(x) = 0 \) on the same set. The Radon-Nikodym derivative of the probability measure \( P \) with respect to the probability measure \( \mu \) will be
\[ \frac{dP}{d\mu}(x) = \frac{P(x)}{\mu(x)}. \]

Then the measure-theoretic entropy \( S(P) \) in this case can be written as
\[ S(P) = -\int_{a}^{b} P(x) \ln \frac{P(x)}{\mu(x)} \, dx. \]

If we take referential probability measure \( \mu \) as a uniform distribution, i.e. \( \mu(x) = \frac{1}{b-a}, x \in [a, b] \), then we obtain
\[ S(P) = S_{[a,b]}(P) - \ln(b-a), \]
where \( S_{[a,b]}(P) \) denotes the Shannon entropy of pdf \( P(x), x \in [a, b] \) \(^{(1)}\) and \( S(P) \) denotes the measure-theoretic entropy in the continuous case.

Hence, one can conclude that measure theoretic entropy \( S(P) \) defined for a probability measure \( P \) on the measure space \((X, \mathcal{M}, \mu)\), is equal to both Shannon entropy in the discrete and continuous case up to an additive constant, when the reference measure \( \mu \) is chosen as a uniform probability distribution. On the other hand, one can see that measure-theoretic KL-entropy, in discrete and continuous cases are equal to its discrete and continuous definitions.

Further, from \((14)\) and \((15)\), we can write Shannon Entropy in terms Kullback-Leibler relative entropy
\[ S_{\mu}(P) = \ln n - I(P||\mu). \quad (16) \]

Thus, Shannon entropy appears as being (up to an additive constant) the variation of information when we pass from the initial uniform probability distribution to new probability distribution given by \( P_{k} \geq 0, \sum_{k=1}^{n} P_{k} = 1 \), as any such probability distribution is obviously absolutely continuous with respect to the uniform discrete probability distribution. Similarly, by \((14)\) and \((2.3)\) the relation between Shannon entropy and Relative entropy in discrete case we can write Boltzmann H-function in terms of Relative entropy as
\[ S_{[a,b]}(p) = \ln(b-a) - I(P||\mu). \quad (17) \]

Therefore, the continuous entropy or Boltzmann H-function \( S(p) \) may be interpreted as being (up to an additive constant) the variation of information when we pass from the initial uniform probability distribution on the interval \([a, b] \) to the new probability
measure defined by the probability distribution function \( p(x) \) (any such probability measure is absolutely continuous with respect to the uniform probability distribution on the interval \([a, b]\)).

Thus, KL-entropy equips one with unitary interpretation of both discrete entropy and continuous entropy. One can utilize Shannon entropy in the continuous case, as well as Shannon entropy in the discrete case, both being interpreted as the variation of information when we pass from the initial uniform distribution to the corresponding probability measure.

Also, since measure theoretic entropy is equal to the discrete and continuous entropy up to an additive constant, ME prescriptions of measure-theoretic Shannon entropy are consistent with discrete case and the continuous case.

3. Measure-Theoretic Definitions of Generalized Information Measures

We begin with a brief note on the notation and assumptions used. We define all the information measures on the measurable space \((X, \mathcal{M})\), and default reference measure is \( \mu \) unless otherwise stated. To avoid clumsy formulations, we will not distinguish between functions differing on a \( \mu \)-null set only; nevertheless, we can work with equations between \( \mathcal{M} \)-measurable functions on \( X \) if they are stated as valid as being only \( \mu \)-almost everywhere (\( \mu \)-a.e or a.e). Further we assume that all the quantities of interest exist and assume, implicitly, the \( \sigma \)-finiteness of \( \mu \) and \( \mu \)-continuity of probability measures whenever required. Since these assumptions repeatedly occur in various definitions and formulations, these will not be mentioned in the sequel. With these assumptions we do not distinguish between an information measure of pdf \( p \) and of corresponding probability measure \( P \) – hence we give definitions of information measures for pdfs, we use corresponding definitions of probability measures as well, whenever it is convenient or required – with the understanding that \( P(E) = \int_E p \, d\mu \), the converse being due to the Radon-Nikodym theorem, where \( p = \frac{dP}{d\mu} \). In both the cases we have \( P \ll \mu \).

First we consider the Rényi generalizations. Measure-theoretic definition of Rényi entropy can be given as follows.

**Definition 3.1.** Rényi entropy of a pdf \( p : X \to \mathbb{R}^+ \) on a measure space \((X, \mathcal{M}, \mu)\) is defined as

\[
S_\alpha(p) = \frac{1}{1 - \alpha} \ln \int_X p(x)^\alpha \, d\mu(x) ,
\]

provided the integral on the right exists and \( \alpha \in \mathbb{R}, \alpha > 0 \).

The same can be defined for any \( \mu \)-continuous probability measure \( P \) as

\[
S_\alpha(P) = \frac{1}{1 - \alpha} \ln \int_X \left( \frac{dP}{d\mu} \right)^{\alpha-1} \, dP .
\]

On the other hand, Rényi relative-entropy can be defined as follows.
Definition 3.2. Let \( p, r : X \to \mathbb{R}^+ \) be two pdfs on measure space \((X, \mathcal{M}, \mu)\). The Rényi relative-entropy of \( p \) relative to \( r \) is defined as

\[
I_\alpha(p\|r) = \frac{1}{\alpha - 1} \ln \int_X \frac{p(x)^\alpha}{r(x)^{\alpha - 1}} d\mu(x) ,
\]

provided the integral on the right exists and \( \alpha \in \mathbb{R}, \alpha > 0 \).

The same can be written in terms of probability measures as,

\[
I_\alpha(P\|R) = \frac{1}{\alpha - 1} \ln \int_X \left( \frac{dP}{dR} \right)^{\alpha - 1} dP
\]

\[
= \frac{1}{\alpha - 1} \ln \int_X \left( \frac{dP}{dR} \right)^\alpha dR ,
\]

whenever \( P \ll R; I_\alpha(P\|R) = +\infty \), otherwise. Further if we assume \( \mu \) in (19) is a probability measure then

\[
S_\alpha(P) = I_\alpha(P\|\mu) .
\]

Tsallis entropy in the measure theoretic setting can be defined as follows.

Definition 3.3. Tsallis entropy of a pdf \( p \) on \((X, \mathcal{M}, \mu)\) is defined as

\[
S_q(p) = \int_X p(x) \ln_q \frac{1}{p(x)} d\mu(x) = \frac{1 - \int_X p(x)^q d\mu(x)}{q - 1} ,
\]

provided the integral on the right exists and \( q \in \mathbb{R} \) and \( q > 0 \).

\( \ln_q \) in (23) is referred to as \( q \)-logarithm and is defined as \( \ln_q x = \frac{x^{1-q} - 1}{1-q} \) \((x > 0, q \in \mathbb{R})\). The same can be defined for \( \mu \)-continuous probability measure \( P \), and can be written as

\[
S_q(P) = \int_X \ln_q \left( \frac{dP}{d\mu} \right)^{-1} dP .
\]

The definition of Tsallis relative-entropy is given below.

Definition 3.4. Let \((X, \mathcal{M}, \mu)\) be a measure space. Let \( p, r : X \to \mathbb{R}^+ \) be two probability density functions. The Tsallis relative-entropy of \( p \) relative to \( r \) is defined as

\[
I_q(p\|r) = -\int_X p(x) \ln_q \frac{r(x)}{p(x)} d\mu(x) = \frac{\int_X p(x)^q d\mu(x) - 1}{q - 1}
\]

provided the integral on right exists and \( q \in \mathbb{R} \) and \( q > 0 \).

The same can be written for two probability measures \( P \) and \( R \), as

\[
I_q(P\|R) = -\int_X \ln_q \left( \frac{dP}{dR} \right)^{-1} dP ,
\]

whenever \( P \ll R; I_q(P\|R) = +\infty \), otherwise. If \( \mu \) in (24) is a probability measure then

\[
S_q(P) = I_q(P\|\mu) .
\]
4. Maximum Entropy and Canonical Distributions

For all the ME prescriptions of classical information measures we consider set of constrains of the form
\[ \int_X u_m \, dP = \int_X u_m(x) p(x) \, d\mu(x) = \langle u_m \rangle , \quad m = 1, \ldots, M , \]  
(28)
with respect to \( \mathcal{M} \)-measurable functions \( u_m : X \to \mathbb{R} , \quad m = 1, \ldots, M \), whose expectation values \( \langle u_m \rangle, m = 1, \ldots, M \) are (assumed to be) a priori known, along with the normalizing constraint \( \int_X dP = 1 \). (From now on we assume that any set of constraints on probability distributions implicitly includes this constraint, which will not be mentioned in the sequel.)

To maximize the entropy \( H \) with respect to the constraints \( 28 \), the solution is calculated via the Lagrangian:
\[ L(x, \lambda, \beta) = -\int_X \ln dP d\mu(x) - \lambda \left( \int_X dP(x) - 1 \right) - \sum_{m=1}^M \beta_m \left( \int_X u_m(x) dP(x) - \langle u_m \rangle \right) , \]
(29)
where \( \lambda \) and \( \beta_m, m = 1, \ldots, M \) are Lagrange parameters (we use the notation \( \beta = (\beta_1, \ldots, \beta_M) \)). The solution is given by
\[ \ln \frac{dP}{d\mu}(x) + \lambda + \sum_{m=1}^M \beta_m u_m(x) = 0 . \]
The solution can be calculated as
\[ dP(x, \beta) = \exp \left( -\ln Z(\beta) - \sum_{m=1}^M \beta_m u_m(x) \right) d\mu(x) \]
(30)
or
\[ p(x) = \frac{dP}{d\mu}(x) = \frac{e^{-\sum_{m=1}^M \beta_m u_m(x)}}{Z(\beta)} , \]
(31)
where the partition function \( Z(\beta) \) is written as
\[ Z(\beta) = \int_X \exp \left( -\sum_{m=1}^M \beta_m u_m(x) \right) d\mu(x) . \]
(32)
The Lagrange parameters \( \beta_m, m = 1, \ldots, M \) are specified by the set of constraints \( 28 \).

The maximum entropy, denoted by \( S \), can be calculated as
\[ S = \ln Z + \sum_{m=1}^M \beta_m \langle u_m \rangle . \]
(33)
The Lagrange parameters \( \beta_m, m = 1, \ldots, M \), are calculated by searching the unique solution (if it exists) of the following system of nonlinear equations:
\[ \frac{\partial}{\partial \beta_m} \ln Z(\beta) = -\langle u_m \rangle , \quad m = 1, \ldots, M . \]
(34)
We also have
\[
\frac{\partial S}{\partial \langle u_m \rangle} = -\beta_m , \quad m = 1, \ldots M .
\] (35)

Equations (34) and (34) are referred to as the thermodynamic equations.

5. ME prescription for Tsallis Entropy

The great success of Tsallis entropy is attributed to the power-law distributions one can derive as maximum entropy distributions by maximizing Tsallis entropy with respect to the moment constraints. But there are subtleties involved in the choice of constraints one would choose for ME prescriptions of these entropy functionals. These subtleties are still part of the major discussion in the nonextensive formalism [24, 25, 26].

In the nonextensive formalism maximum entropy distributions are derived with respect to the constraints which are different from (28), which are used for classical information measures. The constraints of the form (28) are inadequate for handling the serious mathematical difficulties (see [27]). To handle these difficulties constraints of the form
\[
\frac{\int_X u_m(x) p(x)^q \, d\mu(x)}{\int_X p(x)^q \, d\mu(x)} = \langle \langle u_m \rangle \rangle_q , \quad m = 1, \ldots, M
\] (36)
are proposed. (36) can be considered as the expectation with respect to the modified probability measure \( P_{(q)} \) (it is indeed a probability measure) defined as
\[
P_{(q)}(E) = \left( \int_X p(x)^q \, d\mu \right)^{-1} \int_E p(x)^q \, d\mu .
\] (37)
The measure \( P_{(q)} \) is known as escort probability measure.

The variational principle for Tsallis entropy maximization with respect to constraints (36) can be written as
\[
\mathcal{L}(x, \lambda, \beta) = \int_X \ln_q \frac{1}{p(x)} \, dP(x) - \lambda \left( \int_X dP(x) - 1 \right) - \sum_{m=1}^M \beta_m^{(q)} \left( \int_X p(x)^{q-1} \left( u_m(x) - \langle \langle u_m \rangle \rangle_q \right) \, dP(x) \right),
\] (38)
where the parameters \( \beta_m^{(q)} \) can be defined in terms of true Lagrange parameters \( \beta_m \) as
\[
\beta_m^{(q)} = \left( \int_X p(x)^q \, d\mu \right)^{-1} \beta_m , \quad m = 1, \ldots, M.
\] (39)
The maximum entropy distribution in this case can be written as
\[
p(x) = \left[ 1 - (1 - q) \left( \int dx \, p(x)^q \right)^{-1} \sum_{m=1}^M \beta_m \left( u_m(x) - \langle \langle u_m \rangle \rangle_q \right) \right]^{1/q}
\] (40)
and
\[
p(x) = \frac{e^{-\left( \int_X p(x)^q \, d\mu \right)^{-1} \sum_{m=1}^M \beta_m (u_m(x) - \langle \langle u_m \rangle \rangle_q)}}{Z_q},
\] (41)
where
\[
Z_q = \int_X e^{-\left(\int_X p(x)^q \, d\mu\right)^{-1} \sum_{m=1}^{M} \beta_m (u_m(x) - \langle\langle u_m\rangle\rangle_q) \, d\mu(x)}.
\] (42)

Maximum Tsallis entropy in this case satisfies
\[
S_q = \ln_q Z_q,
\] (43)
while corresponding thermodynamic equations can be written as
\[
\frac{\partial}{\partial \beta_m} \ln_q Z_q = -\langle\langle u_m\rangle\rangle_q, \quad m = 1, \ldots, M,
\] (44)
\[
\frac{\partial S_q}{\partial \langle\langle u_m\rangle\rangle_q} = -\beta_m, \quad m = 1, \ldots, M,
\] (45)
where
\[
\ln_q Z_q = \ln_q Z_q - \sum_{m=1}^{M} \beta_m \langle\langle u_m\rangle\rangle_q.
\] (46)

6. Measure-Theoretic Definitions: Revisited

It is well known that unlike Shannon entropy, Kullback-Leibler relative-entropy in the discrete case can be extended naturally to the measure-theoretic case. In this section, we show that this fact is true for generalized relative-entropies too. Rényi relative-entropy on continuous valued space \( \mathbb{R} \) and its equivalence with the discrete case is studied by Rényi [28]. Here, we present the result in the measure-theoretic case and conclude that both measure-theoretic definitions of Tsallis and Rényi relative-entropies are equivalent to its discrete case.

We also present a result pertaining to ME of measure-theoretic Tsallis entropy. We show that ME of Tsallis entropy in the measure-theoretic case is consistent with the discrete case.

6.1. On Measure-Theoretic Definitions of Generalized Relative-Entropies

Here we show that generalized relative-entropies in the discrete case can be naturally extended to measure-theoretic case, in the sense that measure-theoretic definitions can be defined as a limit of a sequence of finite discrete entropies of pmfs which approximate the pdfs involved. We call this sequence of pmfs as “approximating sequence of pmfs of a pdf”. To formalize these aspects we need the following lemma.

Lemma 6.1. Let \( p \) be a pdf defined on measure space \((X, \mathcal{M}, \mu)\). Then there exists a sequence of simple functions \( \{f_n\} \) (we refer to them as approximating sequence of simple functions of \( p \)) such that \( \lim_{n \to \infty} f_n = p \) and each \( f_n \) can be written as
\[
f_n(x) = \frac{1}{\mu(E_{n,k})} \int_{E_{n,k}} p \, d\mu, \quad \forall x \in E_{n,k}, \quad k = 1, \ldots, m(n),
\] (47)
where \((E_{n,1}, \ldots, E_{n,m(n)})\) is the measurable partition corresponding to \(f_n\) (the notation \(m(n)\) indicates that \(m\) varies with \(n\)). Further each \(f_n\) satisfies
\[
\int_X f_n \, d\mu = 1.
\]

**Proof.** Define a sequence of simple functions \(\{f_n\}\) as
\[
f_n(x) = \begin{cases} 
\frac{1}{\mu E_{n,k}} \int_{E_{n,k}} p \, d\mu, & \text{if } k = 0, 1, \ldots, n2^n - 1, \\
\frac{1}{\mu E_{n,\infty}} \int_{E_{n,\infty}} p \, d\mu, & \text{if } n \leq p(x),
\end{cases}
\]
(49)
Each \(f_n\) is indeed a simple function and can be written as
\[
f_n = \sum_{k=0}^{n2^n-1} \left( \frac{1}{\mu E_{n,k}} \int_{E_{n,k}} p \, d\mu \right) \chi_{E_{n,k}} + \left( \frac{1}{\mu F_n} \int_{F_n} p \, d\mu \right) \chi_{F_n},
\]
(50)
where \(E_{n,k} = p^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}))\), \(k = 0, \ldots, n2^n - 1\) and \(F_n = p^{-1}([n, \infty))\). Since \(\int_E p \, d\mu < \infty\) for any \(E \in \mathcal{M}\), we have \(\int_{E_{n,k}} p \, d\mu = 0\) whenever \(\mu E_{n,k} = 0\), for \(k = 0, \ldots, n2^n - 1\). Similarly \(\int_{F_n} p \, d\mu = 0\) whenever \(\mu F_n = 0\). Now we show that \(\lim_{n \to \infty} f_n = p\), point-wise.

First assume that \(p(x) < \infty\). Then \(\exists n \in \mathbb{Z}^+ \ni p(x) \leq n\). Also \(\exists k \in \mathbb{Z}^+, 0 \leq k \leq n2^n - 1 \ni \frac{k}{2^n} \leq p(x) < \frac{k+1}{2^n}\) and \(\frac{k}{2^n} \leq f_n(x) < \frac{k+1}{2^n}\). This implies \(0 \leq |p - f_n| < \frac{1}{2^n}\) as required.

If \(p(x) = \infty\), for some \(x \in X\), then \(x \in F_n\) for all \(n\), and therefore \(f_n(x) \geq n\) for all \(n\); hence \(\lim_{n \to \infty} f_n(x) = \infty = p(x)\).

Finally we have
\[
\int_X f_n \, d\mu = \sum_{k=1}^{n(m)} \left[ \frac{1}{\mu(E_{n,k})} \int_{E_{n,k}} p \, d\mu \right] \mu(E_{n,k})
= \sum_{k=1}^{n(m)} \int_{E_{n,k}} p \, d\mu
= \int_X p \, d\mu = 1
\]
\[
\square
\]

The above construction of a sequence of simple functions which approximate a measurable function is similar to the approximation theorem [21, pp.6, Theorem 1.8(b)] in the theory of integration. But, approximation in Lemma [6.1] can be seen as a mean-value approximation where as in the later case it is the lower approximation. Further, unlike in the case of lower approximation, the sequence of simple functions which approximate \(p\) in Lemma [6.1] are neither monotone nor satisfy \(f_n \leq p\).
Now one can define a sequence of pmfs \( \tilde{p}_n \) corresponding to the sequence of simple functions constructed in Lemma 6.1, denoted by \( \tilde{p}_n = (\tilde{p}_{n,1}, \ldots, \tilde{p}_{n,m(n)}) \), as

\[
\tilde{p}_{n,k} = \mu(E_{n,k}) f_n \chi_{E_{n,k}} = \int_{E_{n,k}} p \, d\mu, \quad k = 1, \ldots, m(n),
\]

for any \( n \). We have

\[
\sum_{k=1}^{m(n)} \tilde{p}_{n,k} = \sum_{k=1}^{m(n)} \int_{E_{n,k}} p \, d\mu = \int_X p \, d\mu = 1,
\]

and hence \( \tilde{p}_n \) is indeed a pmf. We call \( \{\tilde{p}_n\} \) as the approximating sequence of pmfs of pdf \( p \).

Now we present our main theorem, where we assume that \( p \) and \( r \) are bounded. The assumption of boundedness of \( p \) and \( r \) simplifies the proof. However, the result can be extended to an unbounded case. See [29] analysis of Shannon entropy and relative entropy on \( \mathbb{R} \).

**Theorem 6.2.** Let \( p \) and \( r \) be pdf, which are bounded, defined on a measure space \( (X, \mathcal{M}, \mu) \). Let \( \tilde{p}_n \) and \( \tilde{r}_n \) be the approximating sequence of pmfs of \( p \) and \( r \) respectively. Let \( I_\alpha \) denote the Rényi relative-entropy as in (20) and \( I_q \) denote the Tsallis relative-entropy as in (25) then

\[
\lim_{n \to \infty} I_\alpha(\tilde{p}_n \| \tilde{r}_n) = I_\alpha(p \| r) \quad (53)
\]

and

\[
\lim_{n \to \infty} I_q(\tilde{p}_n \| \tilde{r}_n) = I_q(p \| r) \quad (54)
\]

**Proof.** It is enough to prove the result for either Tsallis or Rényi since each are monotone and continuous functions of each other. Hence we write down the proof for the case of Rényi and we use the entropic index \( \alpha \) in the proof.

Corresponding to pdf \( p \), let \( \{f_n\} \) be the approximating sequence of simple functions such that \( \lim_{n \to \infty} f_n = p \) as in Lemma 6.1. Let \( \{g_n\} \) be the approximating sequence of simple functions for \( r \) such that \( \lim_{n \to \infty} g_n = r \). Corresponding to simple functions \( f_n \) and \( g_n \) there exists a common measurable partition* \( \{E_{n,1}, \ldots, E_{n,m(n)}\} \) such that \( f_n \) and \( g_n \) can be written as

\[
f_n(x) = \sum_{k=1}^{m(n)} (a_{n,k}) \chi_{E_{n,k}}(x), \quad a_{n,k} \in \mathbb{R}^+, \quad \forall k = 1, \ldots, m(n),
\]

and

\[
g_n(x) = \sum_{k=1}^{m(n)} (b_{n,k}) \chi_{E_{n,k}}(x), \quad b_{n,k} \in \mathbb{R}^+, \quad \forall k = 1, \ldots, m(n),
\]

* Let \( \varphi \) and \( \phi \) are two simple functions defined on \( (X, \mathcal{M}) \). Let \( \{E_1, \ldots, E_n\} \) and \( \{F_1, \ldots, F_m\} \) be the measurable partitions corresponding to \( \varphi \) and \( \phi \) respectively. Then partition defined as \( \{E_i \cap E_j | i = 1, \ldots, n, \ j = 1, \ldots, m\} \) is a common measurable partition for both \( \varphi \) and \( \phi \).
where \( \chi_{E_{n,k}} \) is the characteristic function of \( E_{n,k} \), for \( k = 1, \ldots, m(n) \). By (55) and (56) the approximating sequences of pmfs \( \{ \tilde{p}_n = (\tilde{p}_{n,1}, \ldots, \tilde{p}_{n,m(n)}) \} \) and \( \{ \tilde{r}_n = (\tilde{r}_{n,1}, \ldots, \tilde{r}_{n,m(n)}) \} \) can be written as (see (51))

\[
\tilde{p}_{n,k} = a_{n,k} \mu(E_{n,k}) \quad k = 1, \ldots, m(n) ,
\]

\[
\tilde{r}_{n,k} = b_{n,k} \mu(E_{n,k}) \quad k = 1, \ldots, m(n) .
\]

Now Rényi relative entropy for \( \tilde{p}_n \) and \( \tilde{r}_n \) can be written as

\[
S_\alpha(\tilde{p}_n \| \tilde{r}_n) = \frac{1}{\alpha - 1} \ln \sum_{k=1}^{m(n)} \frac{a_{n,k}^\alpha}{b_{n,k}^{\alpha-1}} \mu(E_{n,k}) .
\]

To prove \( \lim_{n \to \infty} S_\alpha(\tilde{p}_n \| \tilde{r}_n) = S_\alpha(p \| r) \) it is enough to prove that

\[
\lim_{n \to \infty} \frac{1}{\alpha - 1} \ln \int_X \frac{f_n(x)^\alpha}{g_n(x)^{\alpha-1}} \, d\mu(x) = \frac{1}{\alpha - 1} \ln \int_X \frac{p(x)^\alpha}{r(x)^{\alpha-1}} \, d\mu(x) ,
\]

since we have

\[
\int_X \frac{f_n(x)^\alpha}{g_n(x)^{\alpha-1}} \, d\mu(x) = \sum_{k=1}^{m(n)} \frac{a_{n,k}^\alpha}{b_{n,k}^{\alpha-1}} \mu(E_{n,k}) .
\]

Further it is enough to prove that

\[
\lim_{n \to \infty} \int_X h_n(x)^\alpha g_n(x) \, d\mu(x) = \int_X \frac{p(x)^\alpha}{r(x)^{\alpha-1}} \, d\mu(x) ,
\]

where \( h_n \) is defined as \( h_n(x) = \frac{f_n(x)}{g_n(x)} \).

**Case 1: \( 0 < \alpha < 1 \)**

In this case the *Lebesgue dominated convergence theorem* [30, pp.26] gives that,

\[
\lim_{n \to \infty} \int_X f_n^\alpha \, d\mu = \int_X p^\alpha \, d\mu .
\]

and hence (54)

**Case 2: \( \alpha > 1 \)**

\[\text{‡} \] Since simple functions \( (f_n)^\alpha \) and \( (g_n)^{\alpha-1} \) can be written as

\[
(f_n)^\alpha(x) = \sum_{k=1}^{m(n)} \left( a_{n,k}^\alpha \right) \chi_{E_{n,k}}(x) , \quad \text{and}
\]

\[
(g_n)^{\alpha-1}(x) = \sum_{k=1}^{m(n)} \left( b_{n,k}^{\alpha-1} \right) \chi_{E_{n,k}}(x) .
\]

Further,

\[
\frac{f_n^\alpha}{g_n^{\alpha-1}}(x) = \sum_{k=1}^{m(n)} \left( \frac{a_{n,k}^\alpha}{b_{n,k}^{\alpha-1}} \right) \chi_{E_{n,k}}(x) .
\]
We have $h_n^\alpha f_n \to \frac{f(x)^\alpha}{g(x)^{\alpha-1}}$ a.e. By Fatou’s Lemma [30, pp.23] we obtain that,

$$
\lim_{n \to \infty} \inf \int_X h_n(x)^\alpha g_n(x) \, d\mu(x) \geq \int_X \frac{p(x)^\alpha}{r(x)^{\alpha-1}} \, d\mu(x) .
$$

(64)

From the construction of $f_n$ and $g_n$ (Lemma 6.1) we have

$$
h_n(x)f_n(x) = \frac{1}{\mu(E_{n,i})} \int_{E_{n,i}} \frac{p(x)}{r(x)} \, d\mu , \quad \forall x \in E_{n,i} .
$$

(65)

By Jensen’s inequality we get

$$
h_n(x)^\alpha f_n(x) \leq \frac{1}{\mu(E_{n,i})} \int_{E_{n,i}} \frac{p(x)^\alpha}{r(x)^{\alpha-1}} \, d\mu , \quad \forall x \in E_{n,i} .
$$

(66)

By (55) and (56) we can write (66) as

$$
\frac{a_{n,i}}{b_{n,i}} \mu(E_{n,i}) \leq \int_{E_{n,i}} \frac{p(x)^\alpha}{r(x)^{\alpha-1}} \, d\mu , \quad \forall i = 1, \ldots m(n) .
$$

(67)

By taking summations both sides of (67) we get

$$
\sum_{i=1}^{m(n)} \frac{a_{n,i}}{b_{n,i}} \mu(E_{n,i}) \leq \sum_{i=1}^{m(n)} \int_{E_{n,i}} \frac{p(x)^\alpha}{r(x)^{\alpha-1}} \, d\mu , \quad \forall i = 1, \ldots m(n) .
$$

(68)

The above equation (68) nothing but

$$
\int_X h_n^\alpha(x)f_n(x) \, d\mu(x) \leq \int_X \frac{p(x)^\alpha}{r(x)^{\alpha-1}} \, d\mu , \quad \forall n ,
$$

and hence

$$
\sup_{i>n} \int_X h_i^\alpha(x)f_i(x) \, d\mu(x) \leq \int_X \frac{p(x)^\alpha}{r(x)^{\alpha-1}} \, d\mu , \quad \forall n .
$$

Finally we have

$$
\lim_{n \to \infty} \sup \int_X h_n^\alpha(x)f_n(x) \, d\mu(x) \leq \int_X \frac{p(x)^\alpha}{r(x)^{\alpha-1}} \, d\mu .
$$

(69)

From (64) and (69) we have

$$
\lim_{n \to \infty} \int_X \frac{f_n(x)^\alpha}{g_n(x)^{\alpha-1}} \, d\mu(x) = \int_X \frac{p(x)^\alpha}{r(x)^{\alpha-1}} \, d\mu ,
$$

(70)

and hence (54).
6.2. On ME of Measure-Theoretic definition of Tsallis entropy

With the shortcomings of Shannon entropy that it cannot be naturally extended to the non-discrete case, we have observed that Shannon entropy in its general case on measure space can be used consistently for the ME-prescriptions. One can easily see that generalized information measures of Rényi and Tsallis too cannot be extended naturally to measure-theoretic case, i.e., measure-theoretic definitions are not equivalent to the discrete case in the sense that they can not be defined as a limit of sequence of finite discrete entropies corresponding to pmfs defined on measurable partitions which approximates the pdf. One can use the same counter example we discussed in §2.1.

We have already given the ME-prescriptions of Tsallis entropy in the measure-theoretic case. In this section, we show that the ME-prescriptions in the measure-theoretic case are consistent with the discrete case.

Proceeding as in the case of measure-theoretic entropy in §2.3, measure-theoretic Tsallis entropy $S_q(P)$ in the discrete case can be written as

$$S_q(P) = \sum_{k=1}^{n} P_k \ln_q \frac{\mu_k}{P_k}.$$  (71)

By (72) we get

$$S_q(P) = \sum_{k=1}^{n} P_k^q \left[ \ln_q \mu_k - \ln_q P_k \right] = S_q^n(P) + \sum_{k=1}^{n} P_k^q \ln_q \mu_k ,$$  (72)

where $S_q^n(P)$ is the Tsallis entropy in discrete case. When $\mu$ is a uniform distribution i.e., $\mu_k = \frac{1}{n} \forall n = 1, \ldots n$ we get

$$S_q(P) = S_q^n(P) - n^{q-1} \ln_q n \sum_{k=1}^{n} P_k^q .$$  (73)

Now we show that the quantity $\sum_{k=1}^{n} P_k^q$ is constant in maximization of $S_q(P)$ with respect to the set of constraints (30).

The claim is that

$$\int p(x)^q \, d\mu(x) = (Z_q)^{1-q} ,$$  (74)

which holds for Tsallis maximum entropy distribution in general. This can be shown as follows. From the maximum entropy distribution, we have

$$p(x)^{1-q} = \frac{1 - (1 - q) \left( \int_X p(x)^q \, d\mu(x) \right)^{-1} \sum_{m=1}^{M} \beta_m \left( u_m(x) - \langle \langle u_m \rangle \rangle_q \right)}{(Z_q)^{1-q}} ,$$

which can be rearranged as

$$\left( Z_q \right)^{1-q} p(x) = \left[ 1 - (1 - q) \frac{\sum_{m=1}^{M} \beta_m \left( u_m(x) - \langle \langle u_m \rangle \rangle_q \right)}{\int p(x)^q \, d\mu(x)} \right] p(x)^q .$$
By integrating both sides in the above equation, and by using (36) we get (74).

\[
\sum_{k=1}^{n} \frac{P_{k}^{q}}{\mu_{k}^{q-1}} = (Z_{q})^{1-q}.
\]

(75)

When \(\mu\) is uniform distribution we get

\[
\sum_{k=1}^{n} P_{k}^{q} = n^{1-q} (Z_{q})^{1-q}
\]

which is a constant.

Hence by (73) and (76), one can conclude that with respect to a particular instance of ME, measure-theoretic Tsallis entropy \(S(P)\) defined for a probability measure \(P\) on the measure space \((X, \mathcal{M}, \mu)\), is equal to discrete Tsallis entropy up to an additive constant, when the reference measure \(\mu\) is chosen as a uniform probability distribution. There by, one can further conclude that with respect to a particular instance of ME of measure-theoretic Tsallis entropy is consistent with its discrete definition.

7. Conclusions

In this paper we presented measure-theoretic definitions of generalized information measures. We proved that the measure-theoretic definitions of generalized relative-entropies, Rényi and Tsallis, are natural extensions of their respective discrete cases. We also showed that, ME prescriptions of measure-theoretic Tsallis entropy are consistent with the discrete case.

References

[1] C. E. Shannon and W. Weaver. *The Mathematical Theory of Communication*. University of Illinois Press, Urbana, Illinois, 1949.
[2] R. B. Ash. *Information Theory*. Interscience, New York, 1965.
[3] J. N. Kapur and H. K. Kesavan. *Entropy Optimization Principles with Applications*. Academic Press, 1997.
[4] I. M. Gelfand, N. A. Kolmogorov, and A. M. Yaglom. On the general definition of the amount of information. *Dokl. Akad. Nauk USSR*, 111(4):745–748, 1956. (In Russian).
[5] N. Wiener. *Cybernetics*. Wiley, New York, 1948.
[6] G. Kallianpur. On the amount of information contained in a \(\sigma\)-field. In I. Olkin and S. G. Ghurye, editors, *Essays in Honor of Harold Hotelling*, pages 265–273. Stanford Univ. Press, Stanford, 1960.
[7] M. S. Pinsker. *Information and Information Stability of Random Variables and Process*. Holden-Day, San Francisco, CA, 1960. (English ed., 1964, translated and edited by Amiel Feinstein).
[8] S. Kullback and R. A. Leibler. On information and sufficiency. *Ann. Math. Stat.*, 22:79–86, 1951.
[9] W. Ochs. Basic properties of the generalized Boltzmann-Gibbs-Shannon entropy. *Reports on Mathematical Physics*, 9:135–155, 1976.
[10] P. R. Masani. The measure-theoretic aspects of entropy, Part 1. *Journal of Computational and Applied Mathematics*, 40:215–232, 1992.
[11] P. R. Masani. The measure-theoretic aspects of entropy, Part 2. *Journal of Computational and Applied Mathematics*, 44:245–260, 1992.
[12] Silviu Guiaşu. *Information Theory with Applications*. McGraw-Hill, Great Britain, 1977.
[13] Robert M. Gray. *Entropy and Information Theory*. Springer-Verlag, New York, 1990.
[14] Jonathan D. H. Smith. Some observations on the concepts of information theoretic entropy and randomness. *Entropy*, 3:1–11, 2001.
[15] C. E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379, 1948.
[16] E. T. Jaynes. Prior probabilities. *IEEE Transactions on Systems Science and Cybernetics*, 4(3):227–241, 1968.
[17] Arnold Zellner and Richard A. Highfield. Calculation of maximum entropy distributions and approximation of marginal posterior distributions. *Journal of Econometrics*, 37:195–209, 1988.
[18] Aida C. G. Verdugo Lazo and Pushpa N. Rathie. On the entropy of continuous probability distributions. *IEEE Transactions on Information Theory*, IT-24(1):120–122, 1978.
[19] Hang K. Ryu. Maximum entropy estimation of density and regression functions. *Journal of Econometrics*, 56:397–440, 1993.
[20] K. B. Athreya. Entropy maximization. IMA Preprint Series 1231, Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, 1994.
[21] Shmuel Kantorovitz. *Introduction to Modern Analysis*. Oxford, New York, 2003.
[22] Imre Csiszár. On generalized entropy. *Studia Sci. Math. Hung.*, 4:401–419, 1969.
[23] M. Rosenblatt-Roth. The concept of entropy in probability theory and its applications in the theory of information transmission through communication channels. *Theory Probab. Appl.*, 9(2):212–235, 1964.
[24] G. L. Ferri, S. Martínez, and A. Plastino. The role of constraints in tsallis’ nonextensive treatment revisited. *Physica A*, 347:205–220, 2005.
[25] Sumiyoshi Abe and G. B. Bagci. Necessity of $q$-expectation value in nonextensive statistical mechanics. *Physical Review E*, 71:016139, 2005.
[26] T. Wada and A. M. Scarfone. Connections between Tsallis’ formalism employing the standard linear average energy and ones employing the normalized $q$-average energy. *Physics Letters A*, 335:351–362, 2005.
[27] Constantino Tsallis, Renio S. Mendes, and A. R. Plastino. The role of constraints within generalized nonextensive statistics. *Physica A*, 261:534–554, 1998.
[28] Alfred Rényi. Some fundamental questions of information theory. *MTA III. Oszt. Közl.*, 10:251–282, 1960. (reprinted in [31], pp. 526–552).
[29] Alfred Rényi. On the dimension and entropy of probability distributions. *Acta Math. Acad. Sci. Hung.*, 10:193–215, 1959. (reprinted in [31], pp. 320–342).
[30] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill, 1964. (International edition, 1987).
[31] Pál Turán, editor. *Selected Papers of Alfréd Rényi*. Akademia Kiado, Budapest, 1976.