Five-Loop Vacuum Energy $\beta$ Function in $\phi^4$ Theory with $O(N)$-Symmetric and Cubic Interactions

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Abstract
The beta function of the vacuum energy density is analytically computed at the five-loop level in $O(N)$-symmetric $\phi^4$ theory, using dimensional regularization in conjunction with the $\overline{\text{MS}}$ scheme. The result for the case of cubic anisotropy is also given. It is pointed out how to also obtain the beta function of the coupling and the gamma function of the mass from vacuum graphs. This method may be easier than traditional approaches.

I Introduction
In this article, we extend earlier work [1], where the beta function $\beta_v$ of the vacuum energy density in $O(N)$-symmetric $\phi^4$ theory was computed at the four-loop level, to five loops. Integrals are dimensionally regulated and divergences removed by modified minimal subtraction ($\overline{\text{MS}}$). For motivations for this work, see [1]. We employ two different methods to arrive at our result for the $O(N)$-symmetric case. In section [II], we use the scheme from [1] to determine the five-loop contributions to the vacuum energy renormalization constant $Z_v$ and to $\beta_v$. All necessary subtractions result from consistency requirements while renormalizing the vacuum energy. As explained in [II], this yields as a by-product $\gamma_m$ through four loops and $\beta_g$ through three loops. In section [III], we check the relevant recursion relations for the contributions to $Z_v$. In section [IV], we give $\beta_v$ through five loops for the $O(N)$-symmetric case. In section [V], we use a more traditional approach to check our result for $\beta_v$. In section [VI], we extend it for the case of an additional cubic interaction. Our
work brings the evaluation of $\beta_v$ on a par with that of the other beta and gamma functions in $\phi^4$ theory (see [2], [3] and references therein).

For definitions and conventions, the reader is referred to the detailed article in [1]. The only exception is $\beta_v$ itself, which we define here as

$$\beta_{v,\epsilon}(g, \epsilon) = \frac{m^{2+\epsilon}}{m^4} \left[ \frac{\partial}{\partial \mu^2} \left( \frac{m^4 h}{\mu^4 g} \right) \right]_B,$$

where the subscript $B$ indicates that bare quantities are kept fixed. I.e. $\beta_v$ and $\beta_{v,\epsilon}$ in this article are just $\beta_v$ and $\beta_{v,\epsilon}$ from [1] divided by the renormalized coupling $g$. The connection to the constant $Z_v = 1 + h^{-1} \sum_{k=1}^{\infty} Z_{v,k}(g)/\epsilon^k$, which renormalizes the vacuum energy, is now

$$\beta_{v,\epsilon}(g, \epsilon) = \beta_v(g) = \frac{1}{2} Z'_{v,1}.$$  

(2)

Our revised definition is more natural since even in the non-interacting theory, i.e. for $g = 0$, the vacuum energy density acquires a divergent part from the zero-point fluctuations of the uncoupled harmonic oscillators. It has to be renormalized and therefore runs with $\mu$, such that

$$\beta_v = \frac{N}{4}$$

(3)

for the free theory. E.g., the renormalization group equation for the effective potential in $d = 4 - \epsilon$ dimensions for $h = 0$ reads now [1]

$$\left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta_{g,\epsilon} \frac{\partial}{\partial g} + \gamma_{m,\epsilon} m^2 \frac{\partial}{\partial m^2} - \gamma_{\phi,\epsilon} \phi^2 \frac{\partial}{\partial \phi^2} \right] V_\epsilon(g, m^2, \phi^2, h = 0, \mu^2) = -\beta_{v,\epsilon} m^4 \frac{(4\pi)^2}{\mu^4}.$$

(4)

For the free theory, we have $\beta_{g,\epsilon} = \gamma_{m,\epsilon} = \gamma_{\phi,\epsilon} = 0$ and, going to four dimensions, the $\overline{\text{MS}}$ vacuum energy

$$V(g = 0, m^2, \phi^2 = 0, h = 0, \mu^2) = \frac{N m^4}{4(4\pi)^2} \left( \ln \frac{m^2}{\bar{\mu}^2} - \frac{3}{2} \right),$$

(5)

with the $\overline{\text{MS}}$ renormalization scale

$$\bar{\mu}^2 \equiv 4\pi \mu^2 e^{-\gamma_E}$$

(6)

is immediately seen to satisfy (4) with $\beta_v$ given by (3) with no spurious appearances of a coupling.

**II Five-Loop Contribution to $Z_v$ and $\beta_v$**

In this section, we determine the five-loop contributions to $Z_v$ and to $\beta_v$ in the $O(N)$-symmetric model. We proceed in the same way as in [1]: We employ modified Feynman rules, where the one-loop mass correction is absorbed into a modified mass in the propagator. This reduces the full set of diagrams in table 1 to the reduced set in table 2, where we have also included the six-loop diagrams, whose evaluation we save for a later day. We arrange that the appearances of the wave function renormalization constant $Z_{\phi}$ in the propagator and coupling cancel from the outset.
| number of loops | order in $g$ | diagrams and symmetry factors |
|-----------------|--------------|-------------------------------|
| 0               | $g^{-1}$     | $1 \bullet$                  |
| 1               | $g^0$        | $\frac{1}{2} \bigcirc$      |
| 2               | $g^1$        | $\frac{1}{8} \bigcirc \bigcirc$ |
| 3               | $g^2$        | $\frac{1}{48} \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$ |
| 4               | $g^3$        | $\frac{1}{48} \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$ |
| 5               | $g^4$        | $\frac{1}{256} \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$ |

Table 1: Vacuum diagrams through five loops and their symmetry factors. In the equations in the text the symmetry factor is considered part of each respective diagram.

There are no extra counterterm rules, since all counterterms are already contained in the Feynman rules for vacuum energy, propagator and coupling; i.e. we use bare values for vacuum energy, mass and coupling in the Feynman rules and expand our results for diagrams to the necessary order in the renormalized coupling $g$. For this purpose, we have to formally construct the renormalization constants $Z_m$, $\bar{Z}_m$ and $Z_g$, for mass, modified mass and coupling, through four, four and three loops, respectively, from $\gamma_m$ and $\beta_g$.

The calculation through four loops is detailed in [1]. Continuing to five loops, we keep all divergent terms in zero through five loops through order $g^4$. The relevant five-loop diagrams are

\[
\begin{align*}
\text{Diagram (7)} & \quad = \frac{N(N+2)(N^2+6N+20)}{10368} (4\pi)^8 g^4 Z_m^4 Z_\bar{m} 2^{-\frac{5}{2}} \epsilon I_{5a}, \\
\text{Diagram (8)} & \quad = \frac{N(N+2)^2}{1296} (4\pi)^8 g^4 Z_m^4 Z_\bar{m} 2^{-\frac{5}{2}} \epsilon I_{5b},
\end{align*}
\]
| number of loops | order in $g$ | remaining diagrams and revised symmetry factors |
|-----------------|-------------|-------------------------------------------------|
| 0               | $g^{-1}$    | $1$ •                                           |
| 1               | $g^0$       | $\frac{1}{2}$ ●                               |
| 2               | $g^1$       | $-\frac{1}{8}$ ○                              |
| 3               | $g^2$       | $\frac{1}{48}$ ○                              |
| 4               | $g^3$       | $\frac{1}{48}$ ○                              |
| 5               | $g^4$       | $\frac{1}{128}$ ○ $\frac{1}{144}$ ○ $\frac{1}{32}$ ○ |
| 6               | $g^5$       | $\frac{1}{320}$ ○ $\frac{1}{288}$ ○ $\frac{1}{144}$ ○ $\frac{1}{16}$ ○ $\frac{1}{48}$ ○ $\frac{1}{120}$ ○ |

Table 2: Remaining diagrams through six-loop order after absorption of the one-loop mass correction into a modified mass in the propagator, i.e. after a careful resummation of the quadratic part of the Lagrange density.

\[
\underbrace{\text{Diagram}} = \frac{N(N + 2)(5N + 22)}{2592} \frac{(4\pi)^8 g^4 Z_g^4 Z^{2-\frac{2}{g}}}{\bar{m}^2} I_{5c},
\]

where $I_{5a}$, $I_{5b}$ and $I_{5c}$ are defined, their evaluation sketched and their results given in appendix A. Using these, the formally reconstructed $Z_m$, $Z_{\bar{m}}$ and $Z_g$, as well as the results of [4], one gets

\[
\begin{align*}
\cdot + & \quad \circ + \quad \circ \circ + \quad \circ \circ \circ + \quad \circ \circ \circ \circ + \quad \circ \circ \circ \circ \circ + \quad \circ \circ \circ \circ \circ \circ + \\
& \quad \cdot + \quad \circ + \quad \circ \circ + \quad \circ \circ \circ + \quad \circ \circ \circ \circ + \quad \circ \circ \circ \circ \circ + \quad \circ \circ \circ \circ \circ \circ \\
& - \frac{m^4}{(4\pi)^2 g} \left\{ h \ight. \\
& \quad + \left[ Z^{\nu}_{15} + \frac{5N(N+2)}{144} \left( \beta_3 - \frac{33N^2 + 922N + 2960 + 96(5N+22)\zeta(3)}{432} \right) \left( \ln \frac{m^2}{\mu^2} - 1 \right)^2 \ight.
\end{align*}
\]
Using the results from [1], it is straightforward to check that all of the above relations hold.

Demanding the cancellation of logarithmic terms gives the three-loop coefficient of \( \beta_2 \) and the four-loop coefficient of \( \gamma_m \),

\[ \begin{align*}
\beta_3 &= \frac{33N^2+922N+2960+96(5N+22)\zeta(3)}{432}, \\
\alpha_4 &= \frac{(N+2)(N^2-7578N-31060-48(3N^2+10N+68)\zeta(3)-288(5N+22)\zeta(4))}{15352},
\end{align*} \]

which coincide with known results (see, e.g., [2, 3]). Demanding subsequently (10) to be finite as \( \epsilon \to 0 \) gives

\[ \begin{align*}
Z_{15}^v &= \frac{N(N+2)[-319N^2+13968N+64804+16(3N^2-382N-1700)\zeta(3)+96(4N^2+39N+146)\zeta(4)-1024(5N+22)\zeta(5)]}{207360}, \\
Z_{25}^v &= -\frac{N(N+2)[31N^2+2354N+9306+3(7N^2-28N-48)\zeta(3)+72(5N+22)\zeta(4)]}{9720}, \\
Z_{35}^v &= \frac{N(N+2)[51N^2+8462N+25048+288(5N+22)\zeta(3)]}{19440}, \\
Z_{45}^v &= -\frac{N(N+2)[293N^2+2624N+5840]}{4860}, \\
Z_{55}^v &= \frac{N(N+2)(N+4)(5N+28)}{810}.
\end{align*} \]

The relation (2) then gives us the five-loop coefficient of \( \beta_v = \sum_{k=1}^{\infty} \delta_k g^{k-1} \) through \( \delta_5 = 5Z_{15}^v/2 \).

### III Recursion Relations for the \( Z_{kl}^v \)

The relevant recursion relations among the renormalization constants are [1]

\[ \begin{align*}
Z_{25}^v &= \frac{1}{5} \left[ (2Z_{11}^m + 3Z_{11}^g) Z_{14}^v + 2(2Z_{12}^m + 2Z_{12}^g) Z_{13}^v + 3(2Z_{13}^m + Z_{13}^g) Z_{12}^v + 4(2Z_{14}^m) Z_{11}^v \right], \\
Z_{35}^v &= \frac{1}{5} \left[ (2Z_{11}^m + 3Z_{11}^g) Z_{24}^v + 2(2Z_{12}^m + 2Z_{12}^g) Z_{23}^v + 3(2Z_{13}^m + Z_{13}^g) Z_{22}^v \right], \\
Z_{45}^v &= \frac{1}{5} \left[ (2Z_{11}^m + 3Z_{11}^g) Z_{34}^v + 2(2Z_{12}^m + 2Z_{12}^g) Z_{33}^v \right], \\
Z_{55}^v &= \frac{1}{5} \left[ (2Z_{11}^m + 3Z_{11}^g) Z_{44}^v \right].
\end{align*} \]

Using the results from [1], it is straightforward to check that all of the above relations hold.
IV $\beta_v$ Through Five Loops

Combining our result for $\delta_5$ with the four-loop result from [1], we arrive at the five-loop result for the $O(N)$-symmetric case,

$$
\beta_v(g) = \frac{N}{4} + \frac{N(N+2)}{96}g^2 + \frac{N(N+2)(N+8)[12\zeta(3)-25]}{1296}g^3 + \frac{N(N+2)[-319N^2+13968N+64864+16(3N^2-382N-1700)\zeta(3)+96(4N^2+39N+146)\zeta(4)-1024(5N+22)\zeta(5)]}{82944}g^4 + O(g^5).
$$

(22)

We note that among the beta and gamma functions, $\beta_v$ is the easiest to compute. Therefore, it would be the prime candidate for the first complete non-trivial six-loop calculation, since only the last six diagrams in table [2] have to be computed, of which the first three are essentially trivial. After converting the quartically divergent integrals into logarithmically divergent ones by twice differentiating with respect to $m^2$ (see appendix A), a subset of the diagrams necessary for the six-loop renormalization of the coupling has to be evaluated. Since this subset consists of the diagrams where the four external lines are attached to only two vertices, the most difficult topologies are absent and the computation should be considerably more easy than the full coupling renormalization at this level. A similar statement is true for the comparison with mass and wave function renormalization.

We further note that the $k$-loop coefficients $\alpha_k$ of $\gamma_m$ we get as by-products here and in [1] come from $\epsilon^{-1}\ln(m^2/\bar{\mu}^2)$ terms—and therefore originally from $\epsilon^{-2}$ terms—of $(k+1)$-loop vacuum diagrams. Similarly, the $k$-loop coefficients $\beta_k$ of $\beta_g$ we get from $\epsilon^{-1}\ln^2(m^2/\bar{\mu}^2)$ terms—and therefore originally from $\epsilon^{-3}$ terms—of $(k+2)$-loop vacuum diagrams. These may be easier to compute than the $\epsilon^{-1}$ terms of generic $k$-loop diagrams contributing to $\beta_k$ and $\alpha_k$. In particular, and in contrast to [9], we do not have to consider the topology of eight-loop vacuum diagrams to obtain the five-loop contribution to $\beta_g$, but only to compute the $1/\epsilon^3$ terms of seven-loop vacuum diagrams. To obtain the five-loop contribution to $\gamma_m$, we have to compute the $1/\epsilon^2$ terms of six-loop vacuum diagrams.

However, in the current approach of evaluating loop integrals (to be described in appendix A), the divergences related to the two-point function at one lower loop (and therefore contributing to $\gamma_m$ at one lower loop) and to the four-point function at two lower loops (and therefore contributing to $\beta_g$ at two lower loops) than the considered vacuum graph integrals need to be evaluated as subdivergences of these vacuum graphs anyway. Therefore, when evaluating the integrals as described in appendix A, no advantage has been gained for computing $\gamma_m$ and $\beta_g$, at least as far as the necessary computational techniques are concerned.

V $\beta_v$ and $Z_v$ from the Standard Method

Along more traditional lines, we have

$$
Z_v = 1 + \frac{g}{hm^2}K\hat{R} \sum \text{vacuum graphs},
$$

(23)

where $\hat{R}$ removes subdivergences of graphs and $K$ isolates the negative powers in $\epsilon$ (see [11] for a clear definition of $K$ and $\hat{R}$). The sum in (23) goes over the graphs in table [1] and the Feynman rules.
to be used are
\[ a \rightarrow b = \frac{\delta_{ab}}{p^2 + m^2}, \]  
\[ a \times b \rightarrow c \times d = -[\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}] \frac{(4\pi)^2 g}{3} \]  
with renormalized quantities \( m^2 \) and \( g \), since all subdivergences are removed by the \( \bar{R} \) operation. We have carried out this program from one through five loops in order to also check the results in [1]. Using the results of appendix B, it is easy to see that we get the same \( Z_v \) and therefore also \( \beta_v \) as in the approach above.

If a diagram \( D \) separates into \( n \) diagrams \( D_k \) with independent integrations, then the standard definition of \( K\bar{R} \) yields
\[ K\bar{R}D = K\bar{R} \prod_{k=1}^{n} D_k = (-1)^{n+1} \prod_{k=1}^{n} K\bar{R}D_k. \]  
Since \( K \) picks out only the pole terms in \( \epsilon \), the \( 1/\epsilon \) terms of \( Z_v \) and therefore also \( \beta_v \) receive only contributions from diagrams whose integrations are not separable. Thus, for the calculation of \( \beta_v \), it is again sufficient to consider the diagrams of table II. The two-loop diagram is no longer needed, though. However, since, within this program, separable diagrams can be simply put algebraically together from lower-loop diagrams, including them in the calculation provides together with the recursion relations between the \( Z_x \) [1] a convenient cross-check in the determination of \( Z_v \) and \( \beta_v \).

VI Cubic Anisotropy

Here we give \( \beta_v \) for the case of a cubic anisotropy [3, 9]. We rename \( g \rightarrow g_1 \) and introduce a second coupling \( g_2 \) through
\[ L = \frac{1}{2} \partial_{\mu} \phi_B \partial_{\mu} \phi_B + \frac{1}{2} m_B^2 \phi_B \phi_B + \frac{(4\pi)^2}{4!} \left( g_{1B} T^{(1)}_{ijkl} + g_{2B} T^{(2)}_{ijkl} \right) \phi_i \phi_j \phi_k \phi_l + \frac{m_B^4 h_B}{(4\pi)^2 g_{1B}}, \]  
where repeated indices are summed over (\( \mu = 1, \ldots, d \) and \( i, j, k, l = 1, \ldots, N \)), the subscript \( B \) refers to bare quantities and where
\[ T^{(1)}_{ijkl} = \frac{1}{3} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \]  
\[ T^{(2)}_{ijkl} = \delta_{ijkl} \equiv \begin{cases} 1, & i = j = k = l, \\ 0, & \text{else.} \end{cases} \]  
Using standard methods [4], one arrives at
\[ \beta_v = \frac{1}{2g_1} \left( g_{1B} \frac{\partial Z_v}{\partial g_1} + g_{2B} \frac{\partial Z_v}{\partial g_2} \right). \]  
The seemingly asymmetric treatment of \( g_1 \) and \( g_2 \) is entirely due to our definition of the constant term in (27), since the definitions (28) and (29) are not used for the derivation of (31). However,
it is easy to see that e.g. replacing the constant term in (27) by $m_B^4 h_B / [(4\pi)^2 g_{2b}]$ and redefining $Z_v$ accordingly exchanges $g_1$ and $g_2$ in (30), but leaves $\beta_v$ invariant.

Through five loops, we get

$$
\beta_v(g_1, g_2) = \frac{N}{4} + \frac{N(N+2)}{96} g_1^2 + \frac{N}{16} g_1 g_2 + \frac{N}{32} g_2^2 \\
+ \frac{N(N+2)(N+8)[12\zeta(3)-25]}{1296} g_1^3 + \frac{N(N+8)[12\zeta(3)-25]}{144} g_1^2 g_2 \\
+ \frac{N[12\zeta(3)-25]}{16} g_1 g_2^2 + \frac{N[12\zeta(3)-25]}{48} g_2^3
$$

which reduces to (22) for $(g_1, g_2) \rightarrow (g, 0)$.

**Note Added**

Recently, the result (22) has been confirmed in an independent calculation by S.A. Larin, M. Mönnigmann, M. Strösser and V. Dohm, [cond-mat/9711069](cond-mat/9711069), where also its application to three-dimensional systems is considered.

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**Appendix**

A Five-Loop Integrals

Our strategy for computing the five-loop integrals $I_{5a}$, $I_{5b}$, $I_{5c}$ is different from the way of computing integrals in [1]. Here we take two derivatives of each integral with respect to $m^2$ to convert it into a sum of logarithmically divergent integrals. Then we subtract subdivergences using the $\mathcal{K}\mathcal{R}$ operation, so that we can set most masses in propagators in the resulting expression to zero without changing its divergent part. This allows us to do the relevant integrals using the methods of infrared rearrangement [3], (our own modified version of) the $R^*$ operation [4] and the results of [4], being in turn partially based also on the integration-by-parts algorithm [8]. For an introduction to these techniques and the multi-loop renormalization of $\phi^4$ theory in general, see [5]. At last, we evaluate the terms containing subdivergences that we have subtracted above and add them again. These terms have less than five loops and can be computed by either recursively continuing this procedure or by using the results from [1].
One might argue that this method of computing integrals is not independent of the standard method referred to in section II. Nevertheless, it provides us with many more cross-checks as described in sections III and IV.

All diagrams in this section only refer to momentum space integrals without symmetry and group factors. To get the notation in line with appendix B, we rename the integrals \( I_2^c \) and \( I_3^c \) from II, \( I_{3a} \) and \( I_{4a} \), respectively, so that their finite parts defined in III are now \( I_{3a,f} = I_{2,f}^c \) and \( I_{4a,f} = I_{3,f}^c \).

### A.1 \( I_{5a} \)

Define \( I_{5a} \) by

\[
I_{5a} = \begin{pmatrix}
\begin{array}{c}
\bigcirc
\end{array}
\end{pmatrix} = \frac{1}{\kappa p q r s (p^2+m^2)(k^2+p^2+m^2)(q^2+m^2)(r^2+m^2)(s^2+m^2)}. \tag{32}
\]

Since \( I_{5a} \propto (m^2)^{d-8} = (m^2)^{2-2\epsilon} \) we can write

\[
I_{5a} = \frac{m^4}{(2-\frac{d}{2})(2-\frac{d}{2})} \left( \frac{\partial}{\partial m^2} \right)^2 I_{5a} = \frac{8m^4}{(2-\frac{d}{2})(2-\frac{d}{2})} \left[ 2 \begin{pmatrix}
\begin{array}{c}
\bigcirc
\end{array}
\end{pmatrix} + \begin{pmatrix}
\begin{array}{c}
\bigcirc
\end{array}
\end{pmatrix} + 6 \begin{pmatrix}
\begin{array}{c}
\bigcirc
\end{array}
\end{pmatrix} \right]. \tag{33}
\]

The three integrals on the right hand side can now be evaluated as described above. The result is

\[
I_{5a} = \frac{m^4}{(4\pi)^{10}} \left\{ \frac{192}{5\epsilon^6} + \frac{1}{\epsilon^2} \left[ -96 \ln \frac{m^2}{\mu^2} + \frac{848}{5} \right] + \frac{1}{\epsilon^3} \left[ 120 \ln^2 \frac{m^2}{\mu^2} - 424 \ln \frac{m^2}{\mu^2} + \left( \frac{2004}{5} + 24\zeta(2) \right) \right] \\
+ \frac{1}{\epsilon^4} \left[ 116 \ln^3 \frac{m^2}{\mu^2} - 298 \ln^2 \frac{m^2}{\mu^2} + (258 + 156\zeta(2)) \ln \frac{m^2}{\mu^2} + \left( -\frac{317}{5} - 170\zeta(2) + \frac{88}{5}\zeta(3) \right) \right] \\
+ \frac{1}{\epsilon^5} \left[ \frac{31}{2} \ln^4 \frac{m^2}{\mu^2} - 45 \ln^3 \frac{m^2}{\mu^2} + \left( \frac{107}{2} + 45\zeta(2) \right) \ln \frac{m^2}{\mu^2} - \left( \frac{19}{2} + 71\zeta(2) + 12\zeta(3) \right) \ln \frac{m^2}{\mu^2} \\
+ \left( -\frac{609}{20} + \frac{59}{2}\zeta(2) + \frac{82}{5}\zeta(3) + \frac{141}{5}\zeta(4) + \frac{211}{2}\zeta(2)^2 \right) \right] \right\} \\
+ \frac{24}{(4\pi)^{10} \epsilon^2} I_{3a,f} + \frac{8}{(4\pi)^{10} \epsilon} I_{4a,f} + I_{5a,f}, \tag{34}
\]

where \( I_{5a,f} = \mathcal{O}(\epsilon^0) \).

### A.2 \( I_{5b} \)

Define \( I_{5b} \) by

\[
I_{5b} = \begin{pmatrix}
\begin{array}{c}
\bigcirc
\end{array}
\end{pmatrix} = \frac{1}{\kappa p q r s (k^2+m^2)(k+p+q)^2+m^2)(p^2+m^2)(q^2+m^2)(r^2+m^2)(s^2+m^2)}. \tag{35}
\]
We can write

\[ I_{5b} = \frac{m^4}{(2-\frac{2}{3})^4(1-\frac{2}{3})} \left( \frac{\partial}{\partial m^2} \right)^2 I_{5b} \]

\[ = \frac{6m^4}{(2-\frac{2}{3})^4(1-\frac{2}{3})} \left[ 2 \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) + 2 \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) + 3 \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) + 4 \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) + \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) \right] \]  

(36)

and evaluate this to give

\[ I_{5b} = \frac{m^4}{(4\pi)^m} \left\{ \frac{109}{5\pi^5} + \frac{1}{\pi^4} \left\{ -96 \ln \frac{m^2}{\mu^2} + \frac{452}{5} \right\} + \frac{1}{\pi^4} \left[ 120 \ln^2 \frac{m^2}{\mu^2} - 226 \ln \frac{m^2}{\mu^2} + \left( \frac{2241}{5} + 24 \zeta(2) \right) \right] \right\} \]

\[ + \frac{1}{\pi^4} \left\{ -46 \ln^3 \frac{m^2}{\mu^2} + \frac{191}{2} \ln^2 \frac{m^2}{\mu^2} + \left( \frac{109}{2} - 6 \zeta(2) \right) \ln \frac{m^2}{\mu^2} + \left( -\frac{2463}{20} - \frac{25}{2} \zeta(2) + 4 \zeta(3) \right) \right\} \]

\[ + \frac{1}{\pi^4} \left\{ -47 \ln^4 \frac{m^2}{\mu^2} + \frac{1225}{12} \ln^3 \frac{m^2}{\mu^2} - \left( \frac{2633}{8} + \frac{39}{2} \zeta(2) \right) \ln^2 \frac{m^2}{\mu^2} + \left( \frac{291}{2} + \frac{233}{4} \zeta(2) - 10 \zeta(3) \right) \ln \frac{m^2}{\mu^2} \right\} \]

\[ + \left( -\frac{1214}{5} - \frac{385}{8} \zeta(2) + \frac{61}{6} \zeta(3) - \frac{3}{4} \zeta(4) + \frac{3}{4} \zeta(2) \right) \right\} \]

\[ + \frac{6}{(4\pi)^m} \left\{ \frac{1}{\pi^4} \left[ -8 \right] \right\} I_{3a,f} + I_{5b,f} \]

(37)

where \( I_{5b,f} = O(e^0) \).

**A.3 \( I_{5c} \)**

Define \( I_{5c} \) by

\[ I_{5c} \equiv \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) = \int_{kpqr} \left( p^2 + m^2 \right)^{1/2} \left( q^2 + m^2 \right)^{1/2} \left( r^2 + m^2 \right)^{1/2} \left( s^2 + m^2 \right)^{1/2} \]

(38)

We can write

\[ I_{5c} = \frac{m^4}{(2-\frac{2}{3})^4(1-\frac{2}{3})} \left( \frac{\partial}{\partial m^2} \right)^2 I_{5c} \]

\[ = \frac{4m^4}{(2-\frac{2}{3})^4(1-\frac{2}{3})} \left[ 2 \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) + \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) + 8 \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) + 2 \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) + 2 \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) \right. \]

\[ + \left. \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) + 2 \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) \right] \]  

(39)

and evaluate this to give

\[ I_{5c} = \frac{m^4}{(4\pi)^m} \left\{ \frac{96}{5\pi^5} + \frac{1}{\pi^4} \left\{ -48 \ln \frac{m^2}{\mu^2} + \frac{456}{5} \right\} + \frac{1}{\pi^4} \left[ 60 \ln^2 \frac{m^2}{\mu^2} - 228 \ln \frac{m^2}{\mu^2} + \frac{3541}{15} + 12 \zeta(2) + \frac{32}{3} \zeta(3) \right] \right\} \]

\[ + \frac{1}{\pi^4} \left\{ 58 \ln^3 \frac{m^2}{\mu^2} - 129 \ln^2 \frac{m^2}{\mu^2} + \left( \frac{119}{3} + 78 \zeta(2) - 16 \zeta(3) \right) \right\} \ln \frac{m^2}{\mu^2} \]
\[
\begin{align*}
&+ \left( \frac{457}{6} - 81\zeta(2) + \frac{84}{5}\zeta(3) + \frac{24}{5}\zeta(4) \right) \\
&\frac{1}{\varepsilon} \left[ \frac{31}{4} \ln^4 \frac{m^2}{\mu^2} - \frac{19}{6} \ln^3 \frac{m^2}{\mu^2} \left( \frac{5}{12} + \frac{45}{2}\zeta(2) + 20\zeta(3) \right) \ln \frac{m^2}{\mu^2} \\
&+ \left( -\frac{773}{12} - \frac{10}{2}\zeta(2) - 26\zeta(3) - 12\zeta(4) \right) \ln \frac{m^2}{\mu^2} \\
&+ \left( 2\frac{299}{24} - \frac{107}{12}\zeta(2) + \frac{5}{2}\zeta(3) + \frac{201}{10}\zeta(4) - \frac{64}{5}\zeta(5) + \frac{24}{5}\zeta(2)^2 + 4\zeta(2)\zeta(3) \right) \right] \\
&+ \frac{1}{(4\pi)^2} \left( \frac{12}{\varepsilon^2} + \frac{4}{\varepsilon} \right) I_{3\alpha,f} + \frac{4}{(4\pi)^2 \varepsilon} I_{4\alpha,f} + I_{5c,f},
\end{align*}
\]

where \( I_{5c,f} = \mathcal{O}(\varepsilon^0) \).

\section{\( \mathcal{K}\bar{R} \) on Quartically Divergent Diagrams}

Using standard methods, we have evaluated the quartically divergent diagrams from table II with subdivergences removed, as needed for the determination of \( Z_v \) in section V. As in the last section, the diagrams in the following refer to momentum space integrals, while symmetry factors \( S_x \) and group factors \( G_x(g_1, g_2) \) are written out separately. As indicated, we define the group factors to contain not only the contributions from the contraction of the \( \delta_{ij} \) and \( \delta_{ijkl} \) tensors, but also their accompanying factors \( -g_1/3 \) and \( -g_2 \), respectively. For diagrams with separable integrations, we only give the \( \mathcal{O}(N) \)-symmetric result \( G_x(g, 0) \), since these diagrams are exclusively used for cross-checks, which we have performed only for the \( \mathcal{O}(N) \)-symmetric case. For diagrams whose integrations do not separate, we give \( G_x(g_1, g_2) \).

Our strategy to compute the quartically divergent integrals with subdivergences removed is to convert quartically divergent integrals by twice differentiating with respect to \( m^2 \) to logarithmically divergent ones and subsequently evaluate these by the methods of [5]-[8]. We exploit the fact that the operator \( \partial/\partial m^2 \) commutes with \( \mathcal{K}\bar{R} \) (in the same way as differentiation with respect to an external momentum commutes with \( \mathcal{K}\bar{R} \), see [10]) and that the result of \( \mathcal{K}\bar{R} \) acting on a quartically divergent diagram is proportional to \( m^4 \),

\[
\mathcal{K}\bar{R} \begin{array}{c} \hline \hline \end{array} = \frac{1}{2} m^4 \mathcal{K}\bar{R} \left( \frac{\partial}{\partial m^2} \right)^2 \begin{array}{c} \hline \hline \end{array} .
\]  

\section*{B.1 One Loop}

\[
I_{1\alpha} \equiv \mathcal{K}\bar{R} \begin{array}{c} \hline \hline \end{array} = \frac{1}{2} m^4 \mathcal{K}\bar{R} \begin{array}{c} \hline \hline \end{array} = \frac{m^4}{(4\pi)^2 \varepsilon},
\]
\[ S_{1a} = \frac{1}{2}, \quad G_{1a}(g_1, g_2) = N. \quad (43) \]

### B.2 Two Loops

\[ I_{2a} \equiv K \bar{R} \begin{array}{c} \bigcirc \end{array} \begin{array}{c} \bigcirc \end{array} = m^4 K \bar{R} \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} = -\frac{4m^4}{(4\pi)^4 e^2}, \quad (45) \]

\[ S_{2a} = \frac{1}{8}, \quad G_{2a}(g, 0) = -\frac{1}{3} N(N + 2)g. \quad (47) \]

### B.3 Three Loops

\[ I_{3a} \equiv K \bar{R} \begin{array}{c} \bigcirc \end{array} \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} = \frac{1}{2} m^4 K \bar{R} \left[ 8 \begin{array}{c} \bigcirc \end{array} \begin{array}{c} \bullet \end{array} + 12 \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} \right] = \frac{m^4}{(4\pi)^6} \left( \frac{16}{e^4} - \frac{40}{3e^2} + \frac{1}{e^2} \right), \quad (48) \]

\[ S_{3a} = \frac{1}{48}, \quad G_{3a}(g_1, g_2) = \frac{1}{2} N(N + 2)g_1^2 + 2N g_1 g_2 + Ng_2^2. \quad (50) \]

\[ I_{3b} \equiv K \bar{R} \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} = m^4 K \bar{R} \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} = \frac{8m^4}{(4\pi)^6 e^3}, \quad (51) \]

\[ S_{3b} = \frac{1}{16}, \quad G_{3b}(g, 0) = \frac{1}{9} N(N + 2)^2 g^2. \quad (53) \]

### B.4 Four Loops

\[ I_{4a} \equiv K \bar{R} \begin{array}{c} \bigcirc \end{array} \begin{array}{c} \bigcirc \end{array} \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} = \frac{1}{2} m^4 K \bar{R} \left[ 12 \begin{array}{c} \bigcirc \end{array} \begin{array}{c} \bigcirc \end{array} + 6 \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} + 24 \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} \right] = \frac{m^4}{(4\pi)^6} \left[ -\frac{24}{e^4} + \frac{44}{e^2} - \frac{42}{e^2} + \frac{1}{e^2} \left( \frac{25}{2} - 6\zeta(3) \right) \right], \quad (54) \]

\[ S_{4a} = \frac{1}{48}, \quad G_{4a}(g_1, g_2) = -\frac{1}{27} N(N + 2)(N + 8)g_1^3 - \frac{1}{3} N(N + 8)g_1^2 g_2 - 3N g_1 g_2^2 - Ng_2^3. \quad (56) \]

\[ I_{4b} \equiv K \bar{R} \begin{array}{c} \bigcirc \end{array} \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} = \frac{1}{2} m^4 K \bar{R} \left[ 4 \begin{array}{c} \bigcirc \end{array} \begin{array}{c} \bullet \end{array} + 6 \begin{array}{c} \bullet \end{array} \begin{array}{c} \bullet \end{array} \right] = \frac{m^4}{(4\pi)^6} \left( -\frac{16}{e^4} + \frac{40}{3e^2} - e^2 \right), \quad (57) \]
\[ S_{4b} = \frac{1}{32}, \]
\[ G_{4b}(g, 0) = -\frac{1}{9} N(N + 2)^2 g^3. \]  

\[ \bar{I}_{4c} \equiv \mathcal{K}\bar{R} \quad \text{= m}^4 \mathcal{K}\bar{R} = -\frac{16 m^4}{(4\pi)^3}, \] 
\[ S_{4c} = \frac{1}{32}, \]
\[ G_{4c}(g, 0) = -\frac{1}{27} N(N + 2)^3 g^3. \]  

\[ I_{4d} \equiv \mathcal{K}\bar{R} = 0, \] 
\[ S_{4d} = \frac{1}{48}, \]
\[ G_{4d}(g, 0) = -\frac{1}{27} N(N + 2)^3 g^3. \]  

### B.5 Five Loops

\[ \bar{I}_{5a} \equiv \mathcal{K}\bar{R} \quad \text{= m}^4 \mathcal{K}\bar{R} \left[ 16 \left( \begin{array}{c} 16 \\ \end{array} \right) + 8 \left( \begin{array}{c} 8 \\ \end{array} \right) - 48 \left( \begin{array}{c} 1 \end{array} \right) \right] \]
\[ = -\frac{m^4}{(4\pi)^3} \left[ \frac{192}{5e^2} - \frac{352}{5e^4} + \frac{144}{5e^3} + \frac{1}{2} \left( \frac{168}{5} - \frac{272}{5} \zeta(3) \right) + \frac{1}{e^2} \left( -\frac{319}{20} + \frac{12}{5} \zeta(3) + \frac{96}{5} \zeta(4) \right) \right], \] 
\[ S_{5a} = \frac{1}{128}, \]
\[ G_{5a}(g_1, g_2) = \frac{1}{81} N(N + 2)(N^2 + 6N + 20) g_1^4 + \frac{4}{27} N(N^2 + 6N + 20) g_1^3 g_2 + \frac{2}{3} N(N + 8) g_2^2 + 4N g_1 g_2 + N g_2^2. \]  

\[ \bar{I}_{5b} \equiv \mathcal{K}\bar{R} \quad \text{= m}^4 \mathcal{K}\bar{R} \left[ 12 \left( \begin{array}{c} 12 \\ \end{array} \right) + 12 \left( \begin{array}{c} 12 \\ \end{array} \right) + 18 \left( \begin{array}{c} 18 \\ \end{array} \right) + 24 \left( \begin{array}{c} 24 \\ \end{array} \right) + 6 \left( \begin{array}{c} 6 \\ \end{array} \right) \right] \]
\[ = -\frac{m^4}{(4\pi)^3} \left( \frac{192}{5e^2} - \frac{208}{5e^4} - \frac{74}{5e^3} + \frac{138}{5e^2} - \frac{71}{20e} \right), \] 
\[ S_{5b} = \frac{1}{144}, \]
\[ G_{5b}(g_1, g_2) = \frac{1}{9} N(N + 2)^2 g_1^4 + \frac{4}{3} N(N + 2) g_1^3 g_2 + \frac{2}{3} N(N + 8) g_1^2 g_2^2 + 4N g_1 g_2^3 + N g_2^4. \]
\[
I_{5c} \equiv \mathcal{K}R \quad \equiv \quad \frac{1}{2}m^4\mathcal{K}\bar{R} \left[ 8 \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1}
\end{array} + 4 \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram2}
\end{array} + 32 \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram3}
\end{array} + 8 \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram4}
\end{array} \\
+ 8 \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram5}
\end{array} + 4 \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram6}
\end{array} + 8 \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram7}
\end{array} \right] \\
= \frac{m^4}{(4\pi)^4} \left[ \frac{96}{5\mathcal{K}e^2} - \frac{304}{5e^4} + \frac{1}{e^2} \left( \frac{1592}{15} + \frac{192}{5} \zeta(3) \right) + \frac{1}{e^4} \left( \frac{340}{3} + \frac{24}{5} \zeta(3) - \frac{96}{5} \zeta(4) \right) + \frac{1}{e^6} \left( \frac{320}{3} - 16\zeta(3) + \frac{18}{5} \zeta(4) - \frac{64}{5} \zeta(5) \right) \right], \quad \text{(72)}
\]

\[
S_{5c} = \frac{1}{32},
\]

\[
G_{5c}(g_1, g_2) = \frac{1}{81}N(N + 2)(5N + 22)g_1^4 + \frac{4}{27}N(5N + 22)g_1^2g_2^2 + \frac{2}{9}N(N + 26)g_1^2g_2^2 + 4Ng_1g_3 + Ng_2^4. \quad \text{(74)}
\]

\[
\bar{I}_{5d} \equiv \mathcal{K}\bar{R} \quad \equiv \quad \frac{1}{2}m^4\mathcal{K}\bar{R} \left[ 4 \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram9}
\end{array} + 2 \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram10}
\end{array} + 8 \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram11}
\end{array} \right] \\
= \frac{m^4}{(4\pi)^4} \left[ \frac{16}{e^4} - \frac{88}{3e^4} + \frac{28}{5e^4} + \frac{1}{e^2} \left( -\frac{25}{7} + 4\zeta(3) \right) \right], \quad \text{(75)}
\]

\[
S_{5d} = \frac{1}{16},
\]

\[
G_{5d}(g, 0) = \frac{1}{81}N(N + 2)^2(N + 8)g^4. \quad \text{(77)}
\]

\[
\bar{I}_{5e} \equiv \mathcal{K}\bar{R} \quad \equiv \quad m^4\mathcal{K}\bar{R} \left[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram12}
\end{array} \right] = \frac{m^4}{(4\pi)^4} \left( \frac{8}{3e^4} - \frac{3}{e^2} \right), \quad \text{(78)}
\]

\[
S_{5e} = \frac{1}{48},
\]

\[
G_{5e}(g, 0) = \frac{1}{27}N(N + 2)^3g^4. \quad \text{(79)}
\]

\[
\bar{I}_{5f} \equiv \mathcal{K}\bar{R} \quad \equiv \quad m^4\mathcal{K}\bar{R} \left[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram13}
\end{array} \right] = \frac{m^4}{(4\pi)^4} \left( \frac{32}{3e^4} - \frac{32}{3e^4} + \frac{8}{3e^4} \right), \quad \text{(81)}
\]

14
\[
S_{5f} = \frac{1}{32},
\]
\[
G_{5f}(g, 0) = \frac{1}{27} N(N + 2)^3 g^4.
\]
\[
\bar{I}_{5g} \equiv \mathcal{K}\bar{R} = \frac{1}{2} m^4 \mathcal{K}\bar{R}

\[
= 4 \left[ \begin{array}{c}
\text{diagram}
\end{array} \right] + 6 \left[ \begin{array}{c}
\text{diagram}
\end{array} \right]
\]
\[
= \frac{m^4}{(4\pi)^{10}} \left( \frac{32}{e^4} - \frac{80}{3e^4} + \frac{2}{e^4} \right),
\]
\[
S_{5g} = \frac{1}{48},
\]
\[
G_{5g}(g, 0) = \frac{1}{27} N(N + 2)^3 g^4.
\]
\[
\bar{I}_{5h} \equiv \mathcal{K}\bar{R}

\[
= m^4 \mathcal{K}\bar{R}
\]
\[
= \frac{32m^4}{(4\pi)^{10} e^4},
\]
\[
S_{5h} = \frac{1}{64},
\]
\[
G_{5h}(g, 0) = \frac{1}{81} N(N + 2)^4 g^4.
\]
\[
\bar{I}_{5i} \equiv \mathcal{K}\bar{R}

\[
= 0,
\]
\[
S_{5i} = \frac{1}{32},
\]
\[
G_{5i}(g, 0) = \frac{1}{81} N(N + 2)^4 g^4.
\]
\[
\bar{I}_{5j} \equiv \mathcal{K}\bar{R}

\[
= 0,
\]
\[
S_{5j} = \frac{1}{128},
\]
\[
G_{5j}(g, 0) = \frac{1}{81} N(N + 2)^4 g^4.
\]
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