CONCENTRATION OF LAPLACE EIGENFUNCTIONS AND STABILIZATION OF WEAKLY DAMPED WAVE EQUATION

N Burq, Claude Zuily

To cite this version:
N Burq, Claude Zuily. CONCENTRATION OF LAPLACE EIGENFUNCTIONS AND STABILIZATION OF WEAKLY DAMPED WAVE EQUATION. 2015. <hal-01126742>

HAL Id: hal-01126742
https://hal.archives-ouvertes.fr/hal-01126742
Submitted on 6 Mar 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
CONCENTRATION OF LAPLACE EIGENFUNCTIONS AND STABILIZATION OF WEAKLY DAMPED WAVE EQUATION

by

N.Burq & C.Zuily

Abstract. — In this article, we prove some universal bounds on the speed of concentration on small (frequency-dependent) neighborhoods of submanifolds of $L^2$-norms of quasi modes for Laplace operators on compact manifolds. We deduce new results on the rate of decay of weakly damped wave equations.

Résumé. — On démontre dans cet article des bornes universelles sur la vitesse de concentration dans de petits voisinages (dépendant de la fréquence) de sous variétés pour les normes $L^2$ de quasimodes du Laplacien sur une variété compacte. On en déduit de nouveaux résultats sur la décroissance des équations des ondes faiblement amorties.

1. Notations and main results

Let $(M, g)$ be a smooth compact Riemannian manifold without boundary of dimension $n$, $\Delta_g$ the Laplace-Beltrami operator on $M$ and $d(\cdot, \cdot)$ the geodesic distance on $M$.

The purpose of this work is to investigate the concentration properties of eigenfunctions of the operator $\Delta_g$ (or more generally quasimodes). There are many ways of measuring such possible concentrations. The most classical is by describing semi-classical (Wigner) measures (see the works by Shnirelman [22], Zelditch [28], Colin de Verdière [14], Gérard-Leichtnam [15], Zelditch-Zworski [29], Helffer-Martinez-Robert [16]). Another approach was initiated by Sogge and consists in the studying the potential growth of $\|\varphi_\lambda\|_{L^p(M)}$, see the works by Sogge [23, 24], Sogge-Zelditch [25], Burq-Gérard-Tzvetkov [8, 7, 9]. Finally in [10, 4, 27] the concentration of restrictions on submanifolds was considered. Here, we focus on a situation intermediate between the latter (concentration on submanifolds) and the standard $L^2$-concentration (Wigner measures). Indeed, we study the concentration (in $L^2$ norms) on small (frequency dependent) neighborhoods of submanifolds. Our first result is the following

**Theorem 1.1.** — Let $k \in \{1, \ldots, n - 1\}$ and $\Sigma^k$ be a submanifold of dimension $k$ of $M$. Let us introduce for $\beta > 0$,

\[
N_\beta = \{p \in M : d(p, \Sigma^k) < \beta\}.
\]

N.B. was supported in part by Agence Nationale de la Recherche project NOSEVOL, 2011 BS01019 01. N.B. and C. Z. were supported in part by Agence Nationale de la Recherche project ANAÉ ANR-13-BS01-0010-03.
There exists $C > 0$, $h_0 > 0$ such that for every $0 < h \leq h_0$, every $\alpha \in (0, 1)$ and every solution $\psi \in H^2(M)$ of the equation on $M$

$$(h^2 \Delta_g + 1)\psi = g$$

we have the estimate

$$(1.2) \quad \|\psi\|_{L^2(N, \alpha h^{1/2})} \leq C\alpha \sigma \left(\|\psi\|_{L^2(M)} + \frac{1}{h}\|g\|_{L^2(M)}\right)$$

where $\sigma = 1$ if $k \leq n - 3$, $\sigma = 1^{-}$ if $k = n - 2$, $\sigma = \frac{1}{2}$ if $k = n - 1$.

Here $1^{-}$ means that we have a logarithm loss i.e. a bound by $C\alpha|\log(\alpha)|$.

**Remark 1.2.** — As pointed to us by M. Zworski, the result above is not invariant by conjugation by Fourier integral operators. Indeed, it is well known that micro locally, $-h^2\Delta - 1$ is conjugated by a (micro locally unitary) FIO to the operator $hD_{x_1}$. However the result above is clearly false is one replaces the operator $-h^2\Delta - 1$ by $hD_{x_1}$.

Another motivation for our study was the question of stabilization for weakly damped wave equations.

$$(1.3) \quad (\partial^2_t - \Delta_g + b(x)\partial_t)u = 0, (u, \partial_t u) |_{t=0} (u_0, u_1) \in H^{s+1}(M) \times H^{s}(M),$$

where $0 \leq b \in L^\infty(M)$. Let

$$E(u)(t) = \int_M \left\{ g_p(\nabla_g u(p), \nabla_g u(p)) + |\partial_t u(p)|^2 \right\} dv_g(p)$$

where $\nabla_g$ denotes the gradient with respect to the metric $g$.

It is known that as soon as the damping $b \geq 0$ is non trivial, the energy of every solution converge to 0 as $t$ tends to infinity. On the other hand the rate of decay is *uniform* (and hence exponential) in energy space if and only if the *geometric control condition* [2, 5] is satisfied. Here we want to explore the question when some trajectories are trapped and exhibit decay rates (assuming more regularity on the initial data). This latter question was previously studied in a general setting in [19] and on tori in [11, 21, 1] (see also [12, 13]) and more recently by Leautaud-Lerner [18]. According to the works by Borichev-Tomilov [3], stabilization results for the wave equation are equivalent to resolvent estimates. On the other hand, Theorem 1.1 implies easily (see Section 2.2) the following resolvent estimate

**Corollary 1.3.** — Consider for $h > 0$ the following operator

$$(1.4) \quad L_h = -h^2 \Delta_g - 1 + i\alpha b, \quad b \in L^\infty(M).$$

Assume that there exists a global compact submanifold $\Sigma^k \subset M$ of dimension $k$ such that

$$(1.5) \quad b(p) \geq C d(p, \Sigma^k)^{2\kappa}, \quad p \in M$$

for some $\kappa > 0$. Then there exist $C > 0$, $h_0 > 0$ such that for all $0 < h \leq h_0$

$$\|\varphi\|_{L^2(M)} \leq Ch^{-(1+\kappa)}\|L_h\varphi\|_{L^2(M)},$$

for all $\varphi \in H^2(M)$.

This result will imply the following one.
Theorem 1.4. — Under the geometric assumptions of Corollary 1.3, there exists $C > 0$ such that for any $(u_0, u_1) \in H^2(M) \times H^1(M)$, the solution $u$ of (1.3) satisfies

$$E(u(t))^{\frac{1}{2}} \leq \frac{C}{t^\kappa} (\|u_0\|_{H^2} + \|u_1\|_{H^1}).$$

Remark 1.5. — Notice that in Theorem 1.4 the decay rate is worst than the rates exhibited by Leautaud-Lerner [18] in the particular case when the submanifold $\Sigma$ is a torus (and the metric of $M$ is flat near $\Sigma$). We shall exhibit below examples showing that the rate (1.6) is optimal in general.

A main drawback of the result above (and Leautaud-Lerner’s results) is that we were led to global assumptions on the geometry of the manifold $M$ and the trapped region $\Sigma^k$. However, the flexibility of Theorem 1.1 is such that we can actually dropp all global assumptions and keep only a local weak controlability assumption.

Theorem 1.6. — Let us assume the following weak geometric control property: for any $\rho_0 = (p_0, \xi_0) \in S^* M$, there exists $s \in \mathbb{R}$ such that the point $(p_1, \xi_1) = \Phi(s)(\rho_0)$ on the bicharacteristic issued from $\rho_0$ satisfies

- either $p_1 \in \omega = \bigcup \{U \text{ open} : \text{essinf}_U b > 0\}$
- or there exists $\kappa > 0, C > 0$ and a local submanifold $\Sigma^k$ of dimension $k \geq 1$ such that $p_1 \in \Sigma^k$ and near $p_1$,

$$b(p) \geq Cd(p, \Sigma^k)^{2\kappa}.$$

Notice that since $S^* M$ is compact, we can assume in the assumption above that $s \in [-T, T]$ is bounded and that a finite number of submanifolds (and kappa’s) are sufficient. Let $\kappa_0$ be the largest. Then there exists $C > 0$ such that for any $(u_0, u_1) \in H^2(M) \times H^1(M)$, the solution $u$ of (1.3) satisfies

$$E(u(t))^{\frac{1}{2}} \leq \frac{C}{t^{\kappa_0}} (\|u_0\|_{H^2} + \|u_1\|_{H^1}).$$

The results in Theorem 1.1 are in general optimal. On spheres $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, an explicit family of eigenfunctions $e_j(x_1, \ldots, x_{n+1}) = (x_1 + ix_2)^j$ (eigenvalues $\lambda_j^2 = j(j + n - 1)$) is known. We have

$$|e_j(x)|^2 = (1 - |x'|^2)^j = e^{j \log(1 - |x'|^2)}, \quad x' = (x_3, \ldots, x_{n+1}),$$

and consequently, these eigenfunctions concentrate exponentially on $j^{-1/2}$ neighborhoods of the geodesic curve given by $\{x \in S^n : x' = 0\}$ (the equator). As a consequence, the sequence $j^{\frac{2-n}{2}} e_j$ is (asymptotically) normalized by a constant in $L^2(S^n)$, and if $\Sigma^k$ contains the equator, we can see optimality. Indeed, we work in local coordinates $(y, x')$ where $y \in T$ and $x' \in V$ close to 0 in $\mathbb{R}^{n-1}$. This localization being licit since according to (1.7), the function is $O(e^{-\delta j})$ outside of a fixed neighborhood of the equator. Let $h = j^{-1}$. Let us decompose $x' = (y', z') \in \mathbb{R}^{k-1} \times \mathbb{R}^{n-k}$ and consider the submanifold defined by $z' = 0$. Then

$$\|e_j\|_{L^2(S_{\alpha h^{1/2}})} \sim \int_{y=0}^1 \int_{|y'| \leq 1} \int_{|z'| \leq \alpha h^{1/2}} j^{\frac{n-2}{2}} e^{-j|y'|^2 + |z'|^2} dy' dz' \sim \alpha^{-n-k}.$$
are known (the so called zonal spherical harmonics). These are known to have size of order \( \lambda_j^{(n-1)/2} \) in a neighborhood of size \( \lambda_j^{-1} \) of two antipodal points (north and south poles). As a consequence, a simple calculation shows that if the submanifold contains such a point (which if always achievable by rotation invariance), we have, for \( \alpha = \epsilon h^{1/2} \)
\[
\|f_j\|_{L^2(N, h^{1/2})}^2 \geq c h \sim \alpha^2,
\]
which shows that (1.2) is optimal on spheres (at least in the regime \( \alpha \sim h^{1/2} \)). To get the full optimality might be possible by studying other families of spherical harmonics. For general manifolds, following the analysis in [10, Section 5]) should give the optimality of our results for quasi-modes on any manifold.

The paper is organized as follows. We first show how the non concentration result (Theorem 1.1) imply resolvent estimates for the damped Helmholtz equation, which in turn imply very classically the stabilization results for the damped wave equation. We then focus on the core of the article and prove Theorem 1.1. We start with the case of curves for which we have an alternative proof. Then we focus on the general case. We first show that the resolvent estimate is implied by a similar estimate for the spectral projector. To prove this latter estimate, we rely harmonic analysis and the precise description of the spectral projector given in [10]. Finally, we gathered in an appendix several technical results.

Acknowledgments We’d like to thank M. Zworski for fruitful discussions about these results.

2. From concentration estimates to stabilization results

2.1. A priori estimates. — Recall that \((M, g)\) is a compact connected Riemannian manifold. We shall denote by \(\nabla_g\) the gradient operator with respect to the metric \(g\) and by \(dv_g\) the canonical volume form on \(M\). In all this section we set
\[
L_h = -h^2 \Delta_g - 1 + i hb.
\]
We shall first derive some a-priori estimates on \(L_h\).

Lemma 2.1. — Let \(L_h = -h^2 \Delta_g - 1 + i hb\). Assume \(b \geq 0\) and set \(f = L_h \varphi\). Then
\[
\begin{align*}
(i) \quad & h \int_M b|\varphi(p)|^2 \, dv_g(p) \leq \|\varphi\|_{L^2(M)} \|f\|_{L^2(M)}, \\
(ii) \quad & h^2 \int_M g_p(\nabla_g \varphi(p), \nabla_g \overline{\varphi(p)}) \, dv_g(p) \leq \|\varphi\|_{L^2(M)}^2 + \|\varphi\|_{L^2(M)} \|f\|_{L^2(M)}.
\end{align*}
\]
Proof. — We know that \(\Delta_g = \text{div}\nabla_g\) and by the definition of these objects we have
\[
A =: \int_M g_p(\nabla_g \varphi(p), \nabla_g \overline{\varphi(p)}) \, dv_g(p) = - \int_M \Delta_g \varphi(p) \overline{\varphi(p)} \, dv_g(p).
\]
Multiplying both sides by \(h^2\) and since \(-h^2 \Delta_g \varphi = f + \varphi - i hb \varphi\) we obtain
\[
h^2 A = \int_M |\varphi(p)|^2 \, dv_g(p) - i h \int_M b(p)|\varphi(p)|^2 \, dv_g(p) + \int_M f(p) \overline{\varphi(p)} \, dv_g(p).
\]
Taking the real and the imaginary parts of this equality we obtain the desired estimates. \(\square\)
2.2. Proof of Corollary 1.3 assuming Theorem 1.1. — According to condition (1.5) we have on $N_{\alpha h}^{2}$ \[ b(p) \geq C d(p, \Sigma^k)_{2^\kappa} \geq C \alpha^2 \kappa h^\kappa. \] Writing $\int_{N_{\alpha h}^{2}} |\varphi(p)|^2 dv_g(p) = \int_{N_{\alpha h}^{2}} \frac{1}{b(p)} |\varphi(p)|^2 dv_g(p)$, we deduce from Lemma 2.1 that \[ \int_{N_{\alpha h}^{2}} |\varphi(p)|^2 dv_g(p) \leq \frac{1}{C \alpha^2 \kappa h^{-(1+\kappa)}} \| \varphi \|_{L^2(M)} \| f \|_{L^2(M)}. \] Therefore we are left with the estimate of the $L^2(N_{\alpha h}^{2})$ norm of $\varphi$. According to (2.1) we see that $\varphi$ is a solution of \[ (h^2 \Delta_g + 1) \varphi = -f + i h b \varphi =: g_h \] where $g_h$ satisfies \[ \|g_h\|_{L^2(M)} \leq \|f\|_{L^2(M)} + Ch \|\varphi\|_{L^2(M)}. \] It follows from (2.3) and Theorem 1.1 that \[ \|\varphi\|_{L^2(M)} \leq \frac{1}{C \alpha^2 \kappa} h^{-\frac{1+\kappa}{2}} \|f\|_{L^2(M)} + \frac{1}{C \alpha^2 \kappa} \|f\|_{L^2(M)} + \frac{1}{h^\kappa} \|f\|_{L^2(M)}. \] Now we fix $\alpha$ so small that $C \alpha^2 \leq \frac{1}{2}$ and we use the inequality $a^\frac{1}{2} b^{\frac{1}{2}} \leq a \epsilon + \frac{1}{4\epsilon} b$ to obtain eventually \[ \|\varphi\|_{L^2(M)} \leq C' h^{-(1+\kappa)} \|f\|_{L^2(M)} \] which completes the proof of Corollary 1.3.

2.3. Proof of Theorem 1.4 assuming Corollary 1.3. — The proof is an immediate consequence of a work by Borichev-Tomilov [3] and Corollary 1.3. We quote the following proposition from [18, Proposition 1.5].

Proposition 2.2. — Let $\kappa > 0$. Then the estimate (1.6) holds if and only if there exist positive constants $C, \lambda_0$ such that for all $u \in H^2(M)$, for all $\lambda \geq \lambda_0$ we have \[ C \|(-\Delta_g - \lambda^2 + i \lambda b)u\|_{L^2(M)} \geq \lambda^{1-\kappa} \|u\|_{L^2(M)}. \]

2.4. Proof of Theorem 1.6 assuming Theorem 1.1. — As before Theorem 1.6 will follow from the resolvent estimate \[ \exists C > 0, h_0 > 0 : \forall h \leq h_0 \quad \|\varphi\|_{L^2(M)} \leq Ch^{-(1+\kappa)} \|f\|_{L^2(M)}. \] for every $\varphi \in C^\infty(M)$. We prove (2.4) by contradiction. If it is false one can find sequences $(\varphi_j), (h_j), (f_j)$ such that \[ (-h_j^2 \Delta_g - 1 + i h_j b) \varphi_j = f_j \quad \text{and} \quad \|\varphi_j\|_{L^2(M)} > \frac{1}{h_j^{1+\kappa}} \|f_j\|_{L^2(M)}. \] Then $\|\varphi_j\|_{L^2(M)} > 0$ and we may therefore assume that $\|\varphi_j\|_{L^2(M)} = 1$. It follows that \[ \|f_j\|_{L^2(M)} = o(h_j^{1+\kappa}), \quad j \to +\infty. \] Let $\mu$ be a semiclassical measure for $(\varphi_j)$. By Lemma 2.1 we have \[ \left| \int_M \left\{ \|h_j \nabla_g \varphi_j(p)\|^2 - |\varphi_j(p)|^2 \right\} dv_g(p) \right| \leq \|f_j\|_{L^2(M)}. \]
It follows that $(\varphi_j)$ is $h_j$-oscillating which implies that $\mu(S^*(M)) = 1$. We therefore shall reach a contradiction if we can show that $\text{supp} \mu = \emptyset$ and (2.4) will be proved. First of all by elliptic regularity we have

\begin{equation}
\text{supp} \mu \subset \{(p, \xi) \in S^*(M) : g_p(\xi, \xi) = 1\}.
\end{equation}

On the other hand using Lemma 2.1 we have

\begin{equation}
\int b(p)|\varphi_j(p)|^2 \, dv_g(p) \leq \frac{1}{h_j} \|f_j\|_{L^2(M)}
\end{equation}

since $\|\varphi_j\|_{L^2(M)} = 1$. We deduce from (2.5), (2.8) and (2.6) that

\begin{equation}
(h_j^2 \Delta_g + 1)\varphi_j = G_j, \quad \text{where} \quad \|G_j\|_{L^2(M)} = o(h_j^{1+\frac{2}{p}}) \to 0, j \to +\infty.
\end{equation}

This shows that the support of $\mu$ is invariant by the geodesic flow. Let $\rho_0 \in S^*(M)$ and $\rho_1 = (p_1, \xi_1) \in S^*(M))$ belonging to the geodesic issued from $\rho_0$. Then

$$\rho_0 \notin \text{supp} \mu \iff \rho_1 \notin \text{supp} \mu.$$ 

But according to our assumption of weak geometric control, either a neighborhood of $p_1$ belongs to the set \{b(p) \geq c > 0\} or $p_1 \in \Sigma_k$ and $b(p) \geq Cd(p, \Sigma_k)^{2\varepsilon}$ near $p_1$. In the first case in a neighborhood of $p_1$ the essential inf of $b$ is positive and hence by (2.8) $\rho_1 \notin \text{supp} \mu$. In the second case taking a small neighborhood $\omega$ of $p_1$ we write

$$\int_\omega |\varphi_j(p)|^2 \, dv_g(p) = \int_{\omega \cap N_{\alpha h_j^{1/2}}} |\varphi_j(p)|^2 \, dv_g(p) + \int_{\omega \cap N_{\alpha h_j^{1/2}}} |\varphi_j(p)|^2 \, dv_g(p) = (1) + (2).$$

By Theorem 1.1 and (2.9) we have

\begin{align*}
(1) & \leq C\alpha^\varepsilon(1 + \frac{1}{h_j} \|g_j\|_{L^2(M)}) \leq C\alpha^\varepsilon(1 + o(h_j^{\frac{2}{p}})) \\
(2) & \leq \frac{C}{\alpha^{2\varepsilon} h_j^2} \int_M b(p)|\varphi_j(p)|^2 \, dv_g(p) \leq \frac{C'}{\alpha^{2\varepsilon} h_j^{1+\varepsilon}} = \frac{o(1)}{\alpha^{2\varepsilon}}.
\end{align*}

It follows that

$$\int_\omega |\varphi_j(p)|^2 \, dv_g(p) \leq C\alpha^\varepsilon + \frac{o(1)}{\alpha^{2\varepsilon}}.$$ 

Let $\varepsilon > 0$. We first fix $\alpha(\varepsilon) > 0$ such that $C\alpha(\varepsilon)^\varepsilon \leq \frac{1}{\varepsilon}$ then we take $j_0$ large enough such that for $j \geq j_0, o(1) \leq \alpha(\varepsilon)^{2\varepsilon} \frac{1}{\varepsilon}$. It follows that for $j \geq j_0$ we have $\int_\omega |\varphi_j|^2 \, dv_g \leq \varepsilon$. This shows that $\lim_{j \to +\infty} \int_\omega |\varphi_j|^2 \, dv_g = 0$ which implies that $\rho_1 \notin \text{supp} \mu$ thus $\rho_0 \notin \text{supp} \mu$. Since $\rho_0$ is arbitrary we deduce that $\text{supp} \mu = \emptyset$ which the desired contradiction.

3. Concentration estimates (Proof of Theorem 1.1)

The rest of the paper will be devoted to the proof of Theorem 1.1. The case $k = 1$ i.e. the case of curves, is easier, so we shall start by this case before dealing with the general case.
3.1. The case of curves. — In this case we follow the strategy in [6, Section 2.4], [17] and see the equation satisfied by quasi modes as an evolution equation with respect to a well chosen variable. One can find an open neighborhood $U_p$ of $p$ in $M$, a neighborhood $B_0$ of the origin in $\mathbb{R}^n$ a diffeomorphism $\theta$ from $U_p$ to $B_0$ such that

$$
\begin{align}
(i) & \quad \theta(U_p \cap \Sigma^1) = \{x = (x', x_n) \in (\mathbb{R}^{n-1} \times \mathbb{R}) \cap B_0 : x' = 0\} \\
(ii) & \quad \theta(N) \subset \{x \in B_0 : |x'| \leq \alpha h^{\frac{1}{2}}\}.
\end{align}
$$

Now $\Sigma^1$ is covered by a finite number of such open neighborhoods i.e. $\Sigma^1 \subset \bigcup_{j=1}^{n_0} U_{p_j}$. We take a partition of unity relative to this covering i.e. $(\chi_j) \in C^\infty(M)$ with $\text{supp} \chi_j \subset U_{p_j}$ and $\sum_{j=1}^{n_0} \chi_j = 1$ in a fixed neighborhood of $\Sigma^1$. Taking $h$ small enough we can write

$$
\psi_h = \sum_{j=1}^{n_0} \chi_j \psi_j, \quad (h^2 \Delta g + 1)\psi_h = \sum_{j=1}^{n_0} (h^2 \Delta g + 1)(\chi_j \psi_j) \quad \text{on } N.
$$

Now for $j = 1, \ldots, n_0$ set

$$
(3.1) \quad F_{j,h} = (h^2 \Delta g + 1)(\chi_j \psi_j).
$$

Then

$$
F_{j,h} \equiv \chi_j g_h - h^2(\Delta g \chi_j) \psi_h - 2h^2 g_p(\nabla g \chi_j, \nabla g \psi) =: 1 - 2 - (3).
$$

We have $\|(1)\|_{L^2(M)} \leq C\|g_h\|_{L^2(M)}$ and $\|(2)\|_{L^2(M)} \leq Ch^2\|\psi_h\|_{L^2(M)}$. By the Cauchy Schwarz inequality we can write

$$
h^2 g_p(\nabla g \chi_j, \nabla g \psi) \leq h^2 g_p(\nabla g \chi_j, \nabla g \psi)^2 \leq h^2 g_p(\nabla g \chi_j, \nabla g \psi)^{\frac{1}{2}}
$$

which implies that $|\langle 3 \rangle|^2 \leq Ch^4 g_p(\nabla g \chi_j, \nabla g \psi)$. It follows from Lemma 2.1 with $b \equiv 0$ that

$$
\|(3)\|_{L^2(M)} \leq Ch(\|\psi_h\|_{L^2(M)} + \|\psi_h\|_{L^2(M)}^2)\|g_h\|_{L^2(M)}^2.
$$

Summing up we have proved that for $j = 1, \ldots, n_0$

$$
(3.2) \quad \|F_{j,h}\|_{L^2(M)} \leq C(\|\psi_h\|_{L^2(M)} + \|g_h\|_{L^2(M)}).
$$

Setting $u_{j,h}(x) = (\chi_j \psi_j) \circ \theta_{j}^{-1}(x)$ we see that we have

$$
(3.3) \quad \|\psi_h\|_{L^2(N_{a,h}^{1/2})} \leq \sum_{j=1}^{n_0} \|\chi_j \psi_j\|_{L^2(N_{a,h}^{1/2})} \leq C \sum_{j=1}^{n_0} \|u_{j,h}\|_{L^2(B_{a,h})}
$$

where $B_{a,h} = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| \leq \alpha h^{\frac{1}{2}}, |x_n| \leq c_0\}$.

Our aim is to bound $\|u_{j,h}\|_{L^2(B_{a,h})}$ for $j = 1, \ldots, n_0$. Therefore we can fix $j$ and omit it in what follows. Without loss of generality we can assume that $\text{supp} u_h \subset K$ where $K$ is a fixed compact independent of $h$.

Notice that Lemma 2.1 with $b \equiv 0$ implies that

$$
(3.4) \quad h^2 \|\nabla u_h\|_{L^2(\mathbb{R}^n)}^2 \leq C(\|\psi_h\|_{L^2(M)}^2 + \|\psi_h\|_{L^2(M)}^2 \|f_h\|_{L^2(M)}).
$$

From (3.2) we see that

$$
(3.5) \quad (h^2 P + 1)u_h = G_h,
$$

where $P = \frac{1}{g(x)^{\frac{1}{2}}} \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k} (g(x)^{\frac{1}{2}} g^{kl} (x) \frac{\partial}{\partial x_l})$ is the image of the Laplace Beltrami operator under the diffeomorphism and

$$
(3.6) \quad \|G_h\|_{L^2(\mathbb{R}^n)} \leq C(\|\psi_h\|_{L^2(M)} + \|g_h\|_{L^2(M)}).
$$
Now let \( \Psi_1 \in C^\infty(\mathbb{R}^n) \), \( \Psi_1(\xi) = \frac{1}{2} \) if \( |\xi| \leq 1 \), \( \Psi_1(\xi) = 0 \) if \( |\xi| \geq 1 \) and \( \Psi \in C^\infty_0(\mathbb{R}^n) \), \( \Psi = 1 \) on the support of \( \Psi_1 \). Then \((1 - \Psi_1)(1 - \Psi) = 1 - \psi \). Write

\[
(3.7) \quad u_h = (I - \Psi(hD))u_h + \Psi(hD)u_h =: v_h + w_h.
\]

We have

\[
(h^2P + 1)v_h = (h^2P + 1)(I - \Psi_1(hD))v_h = (I - \Psi_1(hD))\tilde{f}_h - [h^2P, \Psi_1(hD)]u_h =: G_{1h}
\]

By (3.6), (3.4) and the semi classical symbolic calculus we have

\[
\|G_{1h}\|_{L^2(\mathbb{R}^n)} \leq C(h\|\psi_h\|_{L^2(M)} + \|g_h\|_{L^2(M)}).
\]

Now on the support of \( 1 - \Psi_1(\xi) \), the principal symbol of the semi classical pdo, \( Q = (h^2P + 1) \) does not vanish. By the elliptic regularity we have therefore

\[
(3.8) \quad \sum_{k=0}^2 \|(h\nabla)^kv_h\|_{L^2(\mathbb{R}^n)} \leq C\|G_{1h}\|_{L^2(\mathbb{R}^n)} \leq C(h\|\psi_h\|_{L^2(M)} + \|g_h\|_{L^2(M)}).
\]

It follows that for \( \varepsilon > 0 \) small we have

\[
(3.9) \quad h^{1+\varepsilon}\|v_h\|_{H^{1+\varepsilon}(\mathbb{R}^n)} \leq C(h\|\psi_h\|_{L^2(M)} + \|g_h\|_{L^2(M)}).
\]

Now recall that \( x = (x', x_n) \) where \( x' \in \mathbb{R}^{n-1} \). Let \( r = 1 \) if \( n = 2 \), \( r = 2 \) if \( n \geq 3 \). Then \( H^{1+\varepsilon}(\mathbb{R}^r) \subset L^\infty(\mathbb{R}^r) \). Set \( x' = (y, z) \in \mathbb{R}^r \times \mathbb{R}^{n-1-r} \). We can write

\[
\|v_h\|_{L^2(B)} \leq \left( (\alpha h^{\frac{1}{4}})^r \sup_{y \in \mathbb{R}^r} |v_h(y, z, x_n)|^2 dzdx_n \right)^{\frac{1}{2}} \leq (\alpha h^{\frac{1}{4}})^r \left( \int \|v_h(\cdot, z, x_n)|^2_{H^{1+\varepsilon}(\mathbb{R}^r)} dzdx_n \right)^{\frac{1}{2}}
\]

\[
\leq C(\alpha h^{\frac{1}{4}})^r \|v_h\|_{H^{1+\varepsilon}(\mathbb{R}^r)} \leq C\alpha h^{\frac{1}{4} - \varepsilon} \frac{1}{h}\|\psi_h\|_{L^2(M)} + \|g_h\|_{L^2(M)}.
\]

and since \( \frac{1}{4} - \varepsilon > 0 \) we obtain eventually

\[
(3.10) \quad \|v_h\|_{L^2(B)} \leq C\alpha^\sigma (\|\psi_h\|_{L^2(M)} + \frac{1}{h}\|g_h\|_{L^2(M)})
\]

where \( \sigma = \frac{1}{4} \) if \( n = 2 \), \( \sigma = 1 \) if \( n \geq 3 \).

Now let us consider \( w_h \). First of all we have

\[
(3.11) \quad (h^2P + 1)w_h = \Psi(hD)\tilde{f}_h + [h^2P, \Psi(hD)]u_h =: G_{2h}
\]

with, as above

\[
(3.12) \quad \|G_{2h}\|_{L^2(\mathbb{R}^n)} \leq C(h\|\psi_h\|_{L^2(M)} + \|g_h\|_{L^2(M)}).
\]

We notice that the semi classical principal symbol \( q \) of the operator \( Q =: h^2P + 1 \) satisfies the following property

\[
(3.13) \quad \text{on the set } \{(x, \xi) \in T^*(\mathbb{R}^n) : q(x, \xi) = 0 \} \text{ we have } \frac{\partial q}{\partial \xi} \neq 0.
\]

Since \( \mathcal{K} := \mathcal{K} \times \text{supp} \Psi \) is a compact subset of \( T^*(\mathbb{R}^n) \) we can find a finite number of subsets of \( T^*(\mathbb{R}^n), \mathcal{V}_1, \ldots, \mathcal{V}_N \) such that \( \mathcal{K} \subset \bigcup_{j=1}^N \mathcal{V}_j \) and in which

\[
(i) \quad \text{either } |q(x, \xi)| \geq c_0 > 0
\]

\[
(ii) \quad \text{or } q(x, \xi) = e(x, \xi)(\xi + a(x, \xi')), \text{ a real, } e(x, \xi) \neq 0.
\]
Then we can find \((\zeta_j)_{j=1,\ldots,N}\) such that
\[
\zeta_j \in C_0^\infty(\mathcal{V}_j), \quad \text{and} \quad \sum_{j=1}^N \zeta_j = 1 \quad \text{in a neighborhood of } K.
\]

Therefore we can write
\[
(3.15) \quad \Psi(hD)u_h = w_h = \sum_{j=1}^N \zeta_j(x,hD)w_h.
\]

It is sufficient to bound each term so we shall skip the index \(j\).

**case 1.** In \(\mathcal{V}\) we have \(|q(x,\xi)| \geq c_0 > 0\).

In that case the symbol \(a = \frac{\xi}{q}\) belongs to \(S^0(\mathbb{R}^n \times \mathbb{R}^n)\). By the semi classical symbolic calculus and (3.11) we can write
\[
\zeta(x,hD)w_h = \zeta(x,hD)\Psi(hD)u_h = a(x,hD)Q(x,hD)\Psi(hD)u_h + R_hu_h
\]
where
\[
\|R_hu_h\|_{L^2(\mathbb{R}^n)} \leq Ch\|u_h\|_{L^2(\mathbb{R}^n)}.
\]

It follows from (3.12) that
\[
(3.16) \quad \sum_{k=0}^2 \| (h\nabla)^k \zeta(x,hD)w_h \|_{L^2(\mathbb{R}^n)} \leq C(h\|\psi_h\|_{L^2(M)} + \|g_h\|_{L^2(M)})
\]
so we see that \(\zeta(x,hD)w_h\) satisfies the same estimate (3.8) as \(v_h\). Therefore the same argument as before leads to
\[
(3.17) \quad \|\zeta(x,hD)w_h\|_{L^2(B)} \leq C\alpha^\sigma(\|\psi_h\|_{L^2(M)} + \frac{1}{h}\|g_h\|_{L^2(M)}),
\]
where \(\sigma = \frac{1}{2} \) if \(n = 2\), \(\sigma = 1\) if \(n \geq 3\).

**case 2.** In \(\mathcal{V}\) we have \(q(x,\xi) = e(x,\xi)(\xi_1 + a(x,\eta)), a\) real, \(|e(x,\xi)| \geq c_0 > 0\).

\(l \in \{1,\ldots,n-1\}, \eta = (\xi_1,\ldots,\xi_{l-1},\xi_{l+1},\ldots,\xi_n)\), \(e \in S^0, \ |e(x,\xi)| \geq c_0 > 0\).

Let us set \(x_l = t, x = (x_1,\ldots,x_{l-1},x_{l+1},\ldots,x_n)\). Recall (see (3.3)) that \(B_{0,h} \subset \{(t,x) : |t| \leq \alpha h^{\frac{1}{2}}\}\).

Using the symbolic calculus and (3.12) we see easily that
\[
(3h \frac{\partial}{\partial t} + a(t,x,hD_x))\zeta(x,hD)w_h = G_{3h},
\]
where
\[
(3.18) \quad \|G_{3h}\|_{L^2(\mathbb{R}^n)} \leq C(h\|\psi_h\|_{L^2(M)} + \|g_h\|_{L^2(M)}).
\]

Since the symbol \(a\) is real, computing \(\frac{d}{dt}\|w(t,\cdot)\|_{L^2(\mathbb{R}^{n-1})}^2\) we see easily that
\[
\|\zeta(x,hD)w_h(t,\cdot)\|_{L^2(\mathbb{R}^{n-1})} \leq C \int_{t_0}^t \|\zeta(x,hD)w_h(s,\cdot)\|_{L^2(\mathbb{R}^{n-1})} ds + \frac{1}{h} \int_{t_0}^t \|G_{3h}(s,\cdot)\|_{L^2(\mathbb{R}^{n-1})} ds.
\]

9
Now since $|t| \leq \alpha h^{\frac{1}{2}}$, $|t_0| \leq \alpha h^{\frac{1}{2}}$ using the Cauchy Schwarz inequality, (3.18) and the Gronwall inequality we obtain

$$
\|\zeta(x, hD)w_h(t, \cdot)\|_{L^2(\mathbb{R}^{n-1})} \leq C\alpha^{\frac{1}{2}} h^{\frac{1}{2}} (\|\psi_h\|_{L^2(M)} + \frac{1}{h} \|g_h\|_{L^2(M)}).
$$

It follows that

(3.19) \hspace{1cm} \|\zeta(x, hD)w_h\|_{L^2(B_{a,h})} \leq C\alpha h^{\frac{1}{2}} (\|\psi_h\|_{L^2(M)} + \frac{1}{h} \|g_h\|_{L^2(M)}).

**case 3.** In $\mathcal{V}$ we have $q(x, \xi) = e(x, \xi)(\xi_n + a(x, \xi'))$, a real, $|e(x, \xi)| \geq c_0 > 0$.

Since $B_{a,h} = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| \leq \alpha h^{\frac{1}{2}}, |x_n| \leq c_0\}$ we cannot use the same argument as in case 2. Instead we shall use Strichartz estimates proved in [6, Section 2.4] and [17] (see also [30]). First of all, as before we see that

$$
(\partial_t - a(x, hD_{x'}))\zeta(x, hD)w_h = G_{4h}
$$

where $t = x_n$ and $G_{4h}$ satisfies (3.18).

Assume first $n \geq 4$. It is proved in the above works that with $I = \{t| \leq c_0\}$ one has

(3.20) \hspace{1cm} \|\zeta(x, hD)w_h\|_{L^2(I, L^r_{x'}(\mathbb{R}^{n-1}))} \leq Ch^{-\frac{1}{2}} \frac{1}{h} \|G_{4h}\|_{L^1(I, L^2_{x'}(\mathbb{R}^{n-1}))}, \ r = \frac{2(n-1)}{n-3}.

Now set $B' = \{x' \in \mathbb{R}^{n-1} : |x'| \leq \alpha h^{\frac{1}{2}}\}$. Using the Hölder inequality we obtain

$$
\|\zeta(x, hD)w_h(t, \cdot)\|_{L^2(B')} \leq C\alpha h^{\frac{1}{2}} \|\zeta(x, hD)w_h(t, \cdot)\|_{L^r(\mathbb{R}^{n-1})}
$$

which implies, using (3.20) and (3.18) that

(3.21) \hspace{1cm} \|\zeta(x, hD)w_h(t, \cdot)\|_{L^2(B_{a,h})} \leq C\alpha (\|\psi_h\|_{L^2(M)} + \frac{1}{h} \|g_h\|_{L^2(M)}).

When $n = 3$ the Strichartz estimate (3.20) does not hold but we have the weaker ones, with $\frac{1}{q} + \frac{2}{r} = 1$, $r < +\infty$

(3.22) \hspace{1cm} \|\zeta(x, hD)w_h\|_{L^2(I, L^r_{x'}(\mathbb{R}^2))} \leq C_r h^{-\frac{1}{2} - \frac{1}{r}} \frac{1}{h} \|G_{4h}\|_{L^1(I, L^2_{x'}(\mathbb{R}^2))}

where (see [20])

$$
C_r \leq C \alpha^{1/2}.
$$

Then the Hölder inequality gives

$$
\|\zeta(x, hD)w_h(t, \cdot)\|_{L^2(B')} \leq C_r (\alpha h^{\frac{1}{2}})^{2(\frac{1}{2} - \frac{1}{r})} \|\zeta(x, hD)w_h(t, \cdot)\|_{L^r}
$$

and therefore

(3.23) \hspace{1cm} \|\zeta(x, hD)w_h(t, \cdot)\|_{L^2(B_{a,h})} \leq C_r^{1/2} \alpha^{1/2} (\|\psi_h\|_{L^2(M)} + \frac{1}{h} \|g_h\|_{L^2(M)}).

Optimizing with respect to $r < +\infty$ leads to the choice $r = 4\log(\alpha^{-1})$, which gives a

$$
\sqrt{\log(\alpha^{-1})}
$$

loss in the final estimate. In the case $n = 2$ we have instead the estimate

$$
\|\zeta(x, hD)w_h\|_{L^2(I, L^\infty_{x'}(\mathbb{R}^2))} \leq Ch^{-\frac{1}{2}} \|G_{4h}\|_{L^1(I, L^2_{x'}(\mathbb{R}^2))},
$$

which gives eventually

(3.24) \hspace{1cm} \|\zeta(x, hD)w_h(t, \cdot)\|_{L^2(B_{a,h})} \leq C\alpha^{\frac{1}{2}} (\|\psi_h\|_{L^2(M)} + \frac{1}{h} \|g_h\|_{L^2(M)}).
Then the conclusion in Proposition 1.1 follows from (3.3), (3.7), (3.10), (3.17), (3.19), (3.21), (3.23), (3.24).

3.2. The general case: submanifolds of dimension $k$, $1 \leq k \leq n - 1$. — The Laplace-Beltrami operator $-\Delta_g$ with domain $D = \{u \in L^2(M) : \Delta_g u \in L^2(M)\}$ has a discrete spectrum which can be written

$$0 = \lambda_0^2 < \lambda_1^2 < \cdots < \lambda_j^2 \cdots \to +\infty$$

where $\lambda_j > 0, j \geq 1$ and $-\Delta_g \phi = \lambda_j^2 \phi$.

Moreover we can write $L^2(M) = \bigoplus_{j=0}^{+\infty} H_j$, where $H_j$ is the subspace of eigenvectors associated to the eigenvalue $\lambda_j^2$ and $H_j \perp H_k$ if $j \neq k$.

For $\lambda \geq 0$ we define the spectral projector $\Pi_\lambda : L^2(M) \to L^2(M)$ by

$$(3.25)\quad L^2(M) \ni f = \sum_{j \in \mathbb{N}} \varphi_j, \mapsto \Pi_\lambda f = \sum_{j \in \Lambda_\lambda} \varphi_j, \quad \Lambda_\lambda = \{j \in \mathbb{N} : \lambda_j \in [\lambda, \lambda + 1]\}.$$  

Then $\Pi_\lambda$ is self adjoint and $\Pi_\lambda^2 = \Pi_\lambda$.

Theorem 1.1 will be a consequence of the following one. Recall $N_{\alpha h^{1/2}}$ has been defined in (1.1).

**Proposition 3.1.** — There exist $C > 0, h_0 > 0$ such that for every $h \leq h_0$ and every $\alpha \in (0, 1)$

$$(3.26)\quad \|\Pi_\lambda u\|_{L^2(N_{\alpha h^{1/2}})} \leq C \alpha^\sigma\|u\|_{L^2(M)}, \quad \lambda = \frac{1}{h},$$

for every $u \in L^2(M)$, Here $\sigma = 1$ if $k \leq n - 3$, $\sigma = 1^-$ if $k = n - 2$, $\sigma = \frac{1}{2}$ if $k = n - 1$.

Here, as before, $1^-$ means that we have an estimate by $C \alpha|\log(\alpha)|$.

3.2.1. Proof of Theorem 1.1 assuming Proposition 3.1. — If $\psi = \sum_{j \geq 0} \varphi_j$ we have $g = (h^2\Delta_g + 1)\psi = \sum_{j \geq 0} (h^2\Delta_g + 1)\varphi_j$. Therefore by orthogonality

$$(3.27)\quad \|g\|^2_{L^2(M)} = \sum_{j \geq 0} |1 - h^2\lambda_j^2|^2\|\varphi_j\|^2_{L^2(M)}.$$  

Let $\varepsilon_0$ be a fixed number in $]0, 1[$. With $N = [\varepsilon_0 \lambda]$ we write

$$\psi = \sum_{k = -N}^N \Pi_{\lambda + k}\psi + R_N.$$  

Recall that $\Pi_{\lambda + k}\psi = \sum_{j \in E_k} \varphi_j$, where $E_k = \{j \geq 0 : \lambda_j \in [\lambda + k, \lambda + k + 1]\}$.

Assume $|k| \geq 2$. Since $\lambda + k \leq \lambda, 1$ we have $|\lambda_j - \lambda| \geq \frac{1}{2}|k|$ which implies that $|\lambda_j^2 - \lambda^2| \geq \frac{1}{2}k|\lambda|$. By orthogonality we have

$$\|\Pi_{\lambda + k}\psi\|^2_{L^2(M)} = \sum_{j \in E_k} \|\varphi_j\|^2_{L^2(M)} = \sum_{j \in E_k} \frac{1}{|\lambda_j^2 - \lambda^2|^2}|\lambda_j^2 - \lambda^2|^2\|\varphi_j\|^2_{L^2(M)} \leq \frac{4\lambda^2}{|k|^2} \sum_{j \in E_k} |\lambda_j^2 - \lambda^2|^2\|\varphi_j\|^2_{L^2(M)} \leq \frac{4\lambda^2}{|k|^2} \sum_{j \in E_k} |h^2\lambda_j^2 - 1|^2\|\varphi_j\|^2_{L^2(M)}.$$  

11
Since $\Pi_{\lambda+k}^2 = \Pi_{\lambda+k}$, using Proposition 3.1 and the above estimate we obtain

$$
\| \sum_{2 \leq |k| \leq N} \Pi_{\lambda+k} \psi \|_{L^2(N_\alpha^{1/2})} \leq \sum_{2 \leq |k| \leq N} \| \Pi_{\lambda+k} \psi \|_{L^2(N_\alpha^{1/2})} \leq C \alpha^\sigma \sum_{2 \leq |k| \leq N} \| \Pi_{\lambda+k} \psi \|_{L^2(M)} 
\leq 2 C \alpha^\sigma \lambda \sum_{2 \leq |k| \leq N} \frac{1}{|k|} \left( \sum_{j \in E_k} |h^2 \lambda_j^2 - 1| \right) \| \varphi_j \|_{L^2(M)}^2 \right)^{1/2}.
$$

Using Cauchy-Schwarz inequality, (3.27) and the fact that the $E_k$ are pairwise disjoints we obtain eventually

$$
\| \sum_{2 \leq |k| \leq N} \Pi_{\lambda+k} \psi \|_{L^2(N_\alpha^{1/2})} \leq C \alpha^\sigma \frac{1}{h} \| g \|_{L^2(M)}.
$$

Now a direct application of Proposition 3.1 shows that

$$
\| \sum_{|k| \leq 1} \Pi_{\lambda+k} \psi \|_{L^2(N_\alpha^{1/2})} \leq C \alpha^\sigma \| \psi \|_{L^2(M)}.
$$

Eventually let us consider the remainder $R_N$. We have

$$
R_N = \sum_{j \in A} \varphi_j + \sum_{j \in B} \varphi_j, \quad A = \{ j : \lambda_j \leq \lambda - N \}, \quad B = \{ j : \lambda_j \geq \lambda + N + 1 \}.
$$

The two sums are estimated by the same way since in both cases we have $|\lambda_j - \lambda| \geq c \lambda$ thus $|\lambda_j^2 - \lambda^2| \geq c \lambda^2$. Then by orthogonality we write

$$
\| \sum_{j \in A} \varphi_j \|_{L^2(M)}^2 = \sum_{j \in A} \| \varphi_j \|_{L^2(M)}^2 = \sum_{j \in A} \frac{1}{| \lambda_j^2 - \lambda^2 |} | \lambda_j^2 - \lambda^2 | \| \varphi_j \|_{L^2(M)}^2 
\leq \frac{C}{\lambda^2} \sum_{j \in A} | \lambda_j^2 - \lambda^2 | \| \varphi_j \|_{L^2(M)}^2 \leq \sum_{j \in \mathbb{N}} | h^2 \lambda_j^2 - 1 | \| \varphi_j \|_{L^2(M)}^2 = \| g \|_{L^2(M)}^2.
$$

It follows that $\| R_N \|_{L^2(M)} \leq C \| g \|_{L^2(M)}$. Now $(h^2 \Delta_g + 1)R_N = \sum_{j \in A \cup B} (1 - h^2 \lambda_j^2) \varphi_j =: g_N$ and $\| g_N \|_{L^2(M)} \leq \| g \|_{L^2(M)}$. So using Lemma A.1 we obtain

$$
\| R_N \|_{L^2(N_\alpha^{1/2})} \leq C \alpha^\sigma \frac{1}{h} \| g \|_{L^2(M)}
$$

where $\sigma = \frac{1}{2}$ if $k = n - 1$, $\sigma = 1$ if $1 \leq k \leq n - 2$.

Then Theorem 1.1 follows from (3.28), (3.29) and (3.30).

3.2.2. Proof of Proposition 3.1. — This proposition will be a consequence of the following one.

**Proposition 3.2.** Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(0) \neq 0$. There exist $C > 0, h_0 > 0$ such that for every $h \leq h_0$, every $\alpha \in (0,1)$, and every $u \in L^2(M)$ we have

$$
\| \chi(\sqrt{-\Delta_g} - \lambda) u \|_{L^2(N_\alpha^{1/2})} \leq C \alpha^\sigma \| u \|_{L^2(M)}, \quad \lambda = \frac{1}{h}
$$

where $\chi(\sqrt{-\Delta_g} - \lambda) u = \sum_{j \in \mathbb{N}} \chi(\lambda_j - \lambda) \varphi_j$ if $u = \sum_{j \in \mathbb{N}} \varphi_j$. 

12
Proof of Proposition 3.1 assuming Proposition 3.2. — There exists $\delta = \frac{1}{N} > 0$ and $c > 0$ such that $\chi(t) \geq c$ for every $t \in [-\delta, \delta]$. Now let $E = \{ j \in \mathbb{N} : \lambda_j \in [\mu, \mu + \delta] \}$ and set $\Pi^\delta_\mu u = \sum_{j \in E} \varphi_j$. On $E$ we have $\chi(\lambda_j - \mu) \geq c > 0$ therefore we can write

$$1_E(j) = \chi(\lambda_j - \mu) \frac{1_E(j)}{\chi(\lambda_j - \mu)}.$$

It follows that

$$\Pi^\delta_\mu u = \chi(\sqrt{-\Delta g} - \lambda) \circ Ru$$

where $R$ is continuous from $L^2(M)$ to itself with norm bounded by $\frac{1}{2}$. Therefore assuming Proposition 3.2 we can write

$$(3.32) \quad \|\Pi^\delta_\mu u\|_{L^2(N_{ah^{1/2}})} \leq C\alpha^\sigma \|Ru\|_{L^2(M)} \leq \frac{C}{c} \alpha^\sigma \|u\|_{L^2(M)}.$$

where the constants in the right are independent of $\mu$. Now since

$$\{ j : \lambda_j \in [\lambda, \lambda + 1) \} = \bigcup_{k=0}^{N-1} \{ j : \lambda_j \in [\lambda + k\delta, \lambda + (k+1)\delta) \}$$

where the union is disjoint, one can write $\Pi\lambda u = \sum_{k=0}^{N-1} \Pi^\delta_{\lambda + k\delta}$. It follows from (3.32) that

$$\|\Pi\lambda u\|_{L^2(N_{ah^{1/2}})} \leq C' \alpha^\sigma \|u\|_{L^2(M)}$$

which proves Proposition 3.1. \hfill \Box

It remains to prove Proposition 3.2. Until the end of this section $\sigma$ will be a real number such that

$$\sigma = 1 \quad \text{if} \quad k \leq n - 3, \quad \sigma = 1 - \varepsilon (\varepsilon > 0) \quad \text{if} \quad k = n - 2, \quad \sigma = \frac{1}{2} \quad \text{if} \quad k = n - 1.$$

As before for every $p \in \Sigma^k$ one can find an open neighborhood $U_p$ of $p$ in $M$, a neighborhood $B_0$ of the origin in $\mathbb{R}^n$ a diffeomorphism $\theta$ from $U_p$ to $B_0$ such that

$$\theta(U_p \cap \Sigma^k) = \{ x = (x_a, x_b) \in (\mathbb{R}^k \times \mathbb{R}^{n-k}) \cap B_0 : x_b = 0 \}$$

$$\theta(U_p \cap N_{ah^{1/2}}) \subset B_{a,h} = \{ x \in B_0 : |x_b| \leq ah^{1/2} \}.$$

Now $\Sigma^k$ and $N_{ah^{1/2}}$ for $h$ small, are covered by a finite number of such open neighborhoods i.e. $N_{ah^{1/2}} \subset \bigcup_{j=1}^{n_0} U_{p_j}$. We take a partition of unity relative to this covering i.e. $(\zeta_j) \in C^\infty(M)$ with $\text{supp} \zeta_j \subset U_{p_j}$ and $\sum_{j=1}^{n_0} \zeta_j = 1$ in a fixed neighborhood $O$ of $\Sigma^k$ containing $N_{ah^{1/2}}$. For $p \in O$ we can therefore write

$$\chi(\sqrt{-\Delta g} - \lambda)u(p) = \sum_{j=1}^{n_0} \chi(\sqrt{-\Delta g} - \lambda)(\zeta_ju)(p).$$

Our aim being to bound each term of the right hand side, we shall skip the index $j$ in what follows. Moreover we shall set for convenience

$$\chi_\lambda =: \chi(\sqrt{-\Delta g} - \lambda)$$

We shall use some results in [BGT] from which we quote the following ones.

Theorem 3.3 (10) Theorem 4. — There exists $\chi \in \mathcal{S}(\mathbb{R})$ such that $\chi(0) = 1$ and for any $p_0 \in \Sigma^k$ there a diffeomorphism $\theta$ as above, open sets $W \subset V = \{ x \in \mathbb{R}^n : |x| \leq \varepsilon_0 \}$, a smooth function $a : W_x \times V_y \times R^+_\lambda \to \mathbb{C}$ supported in the set

$$\{(x, y) \in W \times V : |x| \leq c_0 \varepsilon \leq c_1 \varepsilon \leq |y| \leq c_2 \varepsilon \ll 1\}$$

13
satisfying
\[ \forall \alpha \in \mathbb{N}^{2n}, \exists C_\alpha > 0 : \forall \lambda \geq 0, |\partial^\alpha_{x,y} a(x, y, \lambda)| \leq C_\alpha, \]

an operator \( R_\lambda : L^2(M) \to L^\infty(M) \) satisfying
\[ \|R_\lambda u\|_{L^\infty(M)} \leq C\|u\|_{L^2(M)}, \]
such that for every \( x \in U =: W \cap \{ x : |x| \leq c\varepsilon \} \), setting \( \tilde{u} = \zeta u \circ \theta^{-1} \) we have
\[ \chi_\lambda(\zeta u)(\theta^{-1}(x)) = \lambda^{\frac{n-1}{2}} \int_{y \in V} e^{i\lambda \psi(x,y)} a(x, y, \lambda)\tilde{u}(y) dy + (R_\lambda(\zeta u))(\theta^{-1}(x)) \]
where \( \psi(x, y) = -d_g((\theta^{-1}(x)), (\theta^{-1}(y))) \) is the geodesic distance on \( M \) between \( \theta^{-1}(x) \) and \( \theta^{-1}(y) \). Furthermore the symbol \( a \) is real non negative, does not vanish for \( |x| \leq c\varepsilon \) and \( d_g((\theta^{-1}(x)), (\theta^{-1}(y))) \in [c_3\varepsilon, c_4\varepsilon] \).

Let us set
\[ T_\lambda \tilde{u}(x) = \int_{y \in V} e^{i\lambda \psi(x,y)} a(x, y, \lambda)\tilde{u}(y) dy. \]
It follow from (3.34) that
\[ \|\chi_\lambda(\zeta u)\|_{L^2(N_{a,\lambda})} \leq \lambda^{\frac{n-1}{2}} \|T_\lambda \tilde{u}\|_{L^2(B_{a,\lambda})} + \|R_\lambda(\zeta u)\|_{L^2(N_{a,\lambda})} \]
Let us look to the contribution of \( R_\lambda \). Since (see (3.33)) the volume of \( N_{a\lambda^{1/2}} \) is bounded by \( C(\alpha h^2)^{n-k} \) we can write
\[ \|R_\lambda(\zeta u)\|_{L^2(N_{a\lambda^{1/2}})} \leq C(\alpha h^2)^{\frac{n-k}{2}} \|\chi_\lambda(\zeta u)\|_{L^\infty(M)} \leq C(\alpha h^2)^{\frac{n-k}{2}} \|u\|_{L^2(M)}. \]
If \( k = n - 1 \) we have \( \alpha^{\frac{n-k}{2}} = \alpha^{\frac{1}{2}} \) and if \( 1 \leq k \leq n - 2 \) we have \( \alpha^{\frac{n-k}{2}} \leq \alpha \). Therefore we get
\[ \|R_\lambda(\zeta u)\|_{L^2(N_{a\lambda^{1/2}})} \leq C\alpha^{\sigma} \|u\|_{L^2(M)}. \]
According to (3.36) Proposition 3.2 will be a consequence of the following result.

**Proposition 3.4.** — There exists positive constants \( C, \lambda_0 \) such that
\[ \lambda^{\frac{n-1}{2}} \|T_\lambda \tilde{u}\|_{L^2(B_{a,\lambda})} \leq C\alpha^{\sigma} \|u\|_{L^2(M)} \]
for every \( \lambda \geq \lambda_0 \) and every \( u \in L^2(M) \).

**Proof of Proposition 3.4.** — Set \( S_\lambda = T_\lambda T_\lambda^* \) and denote by \( 1_B \) the indicator function of the set \( B_{a,\lambda} \). By the usual trick (3.38) will be a consequence of the following estimate.
\[ \|1_B S_\lambda 1_B v\|_{L^2(\mathbb{R}^n)} \leq C h^{n-1} \alpha^{2\sigma} \|v\|_{L^2(\mathbb{R}^n)}, \quad h = \frac{1}{\lambda}. \]
Let \( K_\lambda(x, x') \) be the kernel of \( S_\lambda \). By (3.35) it is given by
\[ K_\lambda(x, x') = \int e^{i\lambda [\psi(x,y) - \psi(x',y)]} a(x, y, \lambda)\overline{u}(x', y, \lambda) dy. \]
We shall decompose
\[
\begin{cases}
K_\lambda = K_\lambda^1 + K_\lambda^2, \\
K_\lambda^1 = 1_{\{|x-x'| \leq \frac{1}{\lambda}\}}K_\lambda, \\
K_\lambda^2 = 1_{\{|x-x'| > \frac{1}{\lambda}\}}K_\lambda,
\end{cases}
\]
(3.41)
\[
S_\lambda = \sum_{j=1}^{2} S_{\lambda j}\hat{u}(x) = \int K_{\lambda j}(x, x')\hat{u}(x') \, dx'
\]
and treat separately each piece.

3.2.3. Estimate of $S_\lambda^1$. — When $|x-x'| \leq \frac{1}{\lambda}$ the kernel $K_\lambda$ is uniformly bounded. Therefore $|K_\lambda^1| \leq C1_{\{|x-x'| \leq \frac{1}{\lambda}\}}$, so by Schur lemma we have
\[
\|S_\lambda^1 v\|_{L^2(\mathbb{R}^n)} \leq C h^n \|v\|_{L^2(\mathbb{R}^n)}.
\]
Therefore
\[
\|1_B S_\lambda^1 1_B v\|_{L^2(\mathbb{R}^n)} \leq C h^n \|v\|_{L^2(\mathbb{R}^n)}.
\]
(3.42)
On the other hand writing $x = (x_a, x_b), x' = (x'_a, x'_b)$ we have
\[
\|S_\lambda^1 v(\cdot, x_b)\|_{L^2(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} 1_{|x_a-x'_a| \leq h} \|v(x'_a, x'_b)\|_{L^2(\mathbb{R}^k)} \, dx'_a \, dx'.
\]
Again by Schur lemma we get
\[
\|S_\lambda^1 v\|_{L^\infty(\mathbb{R}^{n-k}, L^2(\mathbb{R}^k))} \leq C h^k \|v\|_{L^1(\mathbb{R}^{n-k}, L^2(\mathbb{R}^k))}.
\]
We deduce that
\[
\|1_B S_\lambda^1 1_B v\|_{L^2(\mathbb{R}^n)} \leq C (\alpha h)^{n-k} h^k \|v\|_{L^2(\mathbb{R}^n)}.
\]
This estimate can be rewritten as
\[
\|1_B S_\lambda^1 1_B v\|_{L^2(\mathbb{R}^n)} \leq C \alpha^{n-k-2\sigma} h^{\frac{n-2\sigma}{2} + k} \|v\|_{L^2(\mathbb{R}^n)}.
\]
(3.43)
Now if $h^{\frac{n}{2}} \leq \alpha$ we use (3.42) and we obtain
\[
\|1_B S_\lambda^1 1_B v\|_{L^2(\mathbb{R}^n)} \leq C h^n \|v\|_{L^2(\mathbb{R}^n)}.
\]
If $\alpha \leq h^{\frac{n}{2}}$ we use instead (3.43). Since $n-k-2\sigma \geq 0$ we can write
\[
\|1_B S_\lambda^1 1_B v\|_{L^2(\mathbb{R}^n)} \leq C \alpha^{2\sigma} h^{\sigma(n-k-2\sigma)+\frac{1}{2}(n-k)+2k} \|v\|_{L^2(\mathbb{R}^n)} = C \alpha^{2\sigma} h^{n-\sigma} \|v\|_{L^2(\mathbb{R}^n)} \leq C \alpha^{2\sigma} h^{n-1} \|v\|_{L^2(\mathbb{R}^n)}.
\]
Therefore in all cases we have
\[
\|1_B S_\lambda^1 1_B v\|_{L^2(\mathbb{R}^n)} \leq C \alpha^{2\sigma} h^{n-1} \|v\|_{L^2(\mathbb{R}^n)}.
\]
(3.44)
To deal with the other regime we need the description of the kernel $K$ given in [10].

**Lemma 3.5** ([10] Lemma 6.1). — There exists $\varepsilon \ll 1, (a_p^+, b_p)_{p \in \mathbb{N}} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ such that $|x-x'| \geq \lambda^{-1}$ and any $N \in \mathbb{N}^*$ we have
\[
K_\lambda(x, x') = \sum_{p=0}^{N-1} \sum_{\pm} \frac{C_{\pm} \lambda \psi(x, x')}{\lambda |x-x'|^{n-1}} a_{\pm}^p(x, x', \lambda) + b_N(x, x', \lambda)
\]
where $\tilde{\psi}(x,x')$ is the geodesic distance between the points $\theta^{-1}(x)$ and $\theta^{-1}(x')$. Moreover $a^\pm_p$ are real, have supports of size $O(\varepsilon)$ with respect to the two first variables and are uniformly bounded with respect to $\lambda$. Finally

$$|b_N(x,x',\lambda)| \leq C_N(\lambda|x-x'|)^{-(\frac{d+1}{2}+N)}.$$  

3.2.4. Estimate of $S^2_\chi$. — We cut the set $\frac{1}{\lambda} \leq |x-x'| \leq \varepsilon$ into pieces

$$|x-x'| \sim 2^{-j}, \quad \frac{1}{\lambda} \leq 2^{-j} \leq \varepsilon$$

and we estimate the contribution of each term. According to Lemma (3.46) we are lead to work with the operator

$$A_j \psi(x) = \int k_j(x,x',\lambda) \psi(x') \, dx'$$

where

$$(3.45) \quad k_j(x,x',\lambda) = (\lambda 2^{-j})^{-\frac{n+1}{2}} \chi_0(2^j(x-x')) e^{i\lambda \tilde{\psi}(x,x')} \sum_{p=0}^{N-1} \lambda^{-p} a_p(x,x',\lambda).$$

Now there exists $\chi \in C^\infty(\mathbb{R}^n)$ such that supp $\chi \subset \{x : |x| \leq 1\}$, $\chi(x) = 1$ if $|x| \leq \frac{1}{4}$ and

$$\sum_{p \in \mathbb{Z}^n} \chi(x-p) = 1, \forall x \in \mathbb{R}^n.$$

Following [10] we write

$$(3.46) \quad k_{jpq}(x,x',\lambda) = \chi(2^j x - p) k_j(x,x',\lambda) \chi(2^j x' - q)$$

and we denote by $A_{jpq}$ the operator with kernel $k_{jpq}$.

Notice that the sum appearing in (3.46) is to be taken only for $|p - q| \leq 2$.

We claim that by quasi orthogonality in $L^2$ we have

$$(3.47) \quad \|B A_j B\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C \sup_{|p-q| \leq 2} \|B A_{jpq} B\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}.$$

Indeed let us forget $B$ which plays any role. We have

$$\|A_j \psi\|_{L^2(\mathbb{R}^n)} = \sum_{|p-q| \leq 2} \sum_{|p'-q'| \leq 2} \int A_{jpq} [\tilde{\chi}(2^j - q) \psi(x)] A_{jp'q'} [\tilde{\chi}(2^j - q') \psi(x)] \, dx$$

where $\tilde{\chi} \in C^\infty(\mathbb{R}^n)$, $\tilde{\chi} = 1$ on the support of $\chi$ and $\sum_{p \in \mathbb{Z}^n} |\tilde{\chi}(x-p)|^2 \leq M, \forall x \in \mathbb{R}^n$. Due to the presence of $\chi(2^j x - p)$, $\chi(2^j x - p')$ ans $\chi(2^j (x - x'))$ inside the above integral one must also have $|p - p'| \leq 2$ in the sum. Therefore we are summing on the set $E = \{(p,q,p',q') : |p - q| \leq 2, |p' - q| \leq 4, |q' - q| \leq 6\}$. We have

$$E \subset E_1 = \{(p,q,p',q') : |p - q| \leq 2, |p' - q| \leq 4, |q' - q| \leq 6\};$$

$$E \subset E_2 = \{(p,q,p',q') : |p' - q'| \leq 2, |p - q'| \leq 4, |q' - q| \leq 6\}.$$

It follows from the Cauchy-Schwarz inequality that $\|A_j \psi\|_{L^2(\mathbb{R}^n)}$ can be bounded by

$$\left( \sum_{E_1} \|A_{jpq}\|^2_{L^2 \to L^2} \|\tilde{\chi}(2^j - q) \psi\|^2_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{2}} \left( \sum_{E_2} \|A_{jp'q'}\|^2_{L^2 \to L^2} \|\tilde{\chi}(2^j - q') \psi\|^2_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{2}}$$

16
and therefore by the choice of $\tilde{\chi}$ by $C \sup_{|p-q| \leq 2} \| A_{jpq} \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}^2 \| v \|_{L^2(\mathbb{R}^n)}^2$ which proves our claim.

Now let us consider the operator $Q_{jpq}$ defined by

$$Q_{jpq}v(X) = \int_{\mathbb{R}^n} \sigma_{jpq}(X, X', \lambda) v(X') \, dX' \tag{3.48}$$

$$\sigma_{jpq}(X, X', \lambda) = \chi(X - p)k_j(2^{-j}X, 2^{-j}X', \lambda) \chi(X' - q).$$

Then by the change of variables $(x = 2^{-j}X, x' = 2^{-j}X')$ we can see easily that

$$\| 1_{2^j B} Q_{jpq}1_{2^j B} v \|_{L^2(\mathbb{R}^n)} \leq K_j \| v \|_{L^2(\mathbb{R}^n)} \quad \text{implies} \quad \| 1_B A_{jpq}1_B v \|_{L^2(\mathbb{R}^n)} \leq 2^{-jn}K_j \| v \|_{L^2(\mathbb{R}^n)}. \tag{3.50}$$

Setting

$$\mu_j = \lambda 2^{-j}, \quad \tilde{\psi}_j(X, X') = 2^j \tilde{\psi}(2^{-j}X, 2^{-j}X'),$$

we deduce from (3.45) and (3.48) we have

$$\sigma_{jpq}(X, X', \lambda) = \mu_j^{-\frac{na+1}{2}} e^{i\mu_j \tilde{\psi}_j(X, X')} \chi(X - p)\chi(X - q)\chi_0(X - X'). \tag{3.52}$$

We shall derive two estimates of the left hand side of (3.49). On one hand using Theorem A.4 with $p = k - 1$ we can write,

$$\| 1_{2^j B} Q_{jpq}1_{2^j B} v \|_{L^2(\mathbb{R}^n)} \leq C(\alpha h^\frac{1}{2}2^j)^{\frac{n-k}{2}} \| Q_{jpq}1_{2^j B} v \|_{L^\infty(\mathbb{R}_b^{n-k} \times \mathbb{R}^{a_1}, L^2(\mathbb{R}_b^{n-k}))},$$

$$\leq C\mu_j^{-\frac{na+1}{2}} (\alpha h^\frac{1}{2}2^j)^{\frac{n-k}{2}} \mu_j^{\frac{k+1}{2}} \| 1_{2^j B} v \|_{L^1(\mathbb{R}_b^{n-k} \times \mathbb{R}^{a_1}, L^2(\mathbb{R}_b^{n-k}))},$$

$$\leq C\mu_j^{-\frac{na+1}{2}} (\alpha h^\frac{1}{2}2^j)^{n-k} \mu_j^{\frac{k+1}{2}} \| v \|_{L^2(\mathbb{R}^n)}. \tag{3.53}$$

We deduce from (3.50) and (3.47) that

$$\| 1_B A_1 1_B v \|_{L^2(\mathbb{R}^n)} \leq CH^{n-1} \alpha^{n-k}2^{j (\frac{na+1}{2}-1)} \| v \|_{L^2(\mathbb{R}^n)}. \tag{3.54}$$

On the other hand using Theorem A.2 with $p = n - 1$ we can write

$$\| 1_{2^j B} Q_{jpq}1_{2^j B} v \|_{L^2(\mathbb{R}^n)} \leq \| Q_{jpq}1_{2^j B} v \|_{L^2(\mathbb{R}^n)} \leq C\mu_j^{-\frac{na+1}{2}} \mu_j^{-\frac{n-k}{2}} \| v \|_{L^2(\mathbb{R}^n)},$$

from which we deduce using (3.50) and (3.47) that

$$\| 1_B A_1 1_B v \|_{L^2(\mathbb{R}^n)} \leq C2^{-jn}(2^j h)^{n-1} \leq CH^{n-1}2^{-j}. \tag{3.54}$$

Recall that we have $S^2_\lambda = \sum_{j \in E} A_j$ where $E = \{ j : \frac{1}{\varepsilon} \leq 2^j \leq \lambda \}$. Then we write

$$1_B S^2_\lambda 1_B v = \sum_{j \in E_1} 1_B A_j 1_B v + \sum_{j \in E_2} 1_B A_j 1_B v = (1) + (2), \quad \text{where} \quad E_1 = \{ j : \frac{1}{\varepsilon} \leq 2^j \leq \alpha^{-2} \}, \quad E_2 = \{ j : \alpha^{-2} \leq 2^j \leq \lambda \}. \tag{3.55}$$
To estimate the term (1) we use (3.53). We obtain
\[ \| (1) \|_{L^2(\mathbb{R}^n)} \leq C h^{n-1} \alpha^{n-k} \sum_{j \in B_k} 2^{j(\frac{n-k}{2}-1)} \| v \|_{L^2(\mathbb{R}^n)}. \]
Then we have three cases.
If \( \frac{n-k}{2} - 1 > 0 \) that is if \( k \leq n-3 \) then
\[ \| (1) \|_{L^2(\mathbb{R}^n)} \leq C h^{n-1} \alpha^{n-k} \left( \frac{1}{\alpha^2} \right)^{\frac{n-k}{2}-1} \| v \|_{L^2(\mathbb{R}^n)} \leq C h^{n-1} \alpha^2 \| v \|_{L^2(\mathbb{R}^n)}. \]
If \( \frac{n-k}{2} - 1 = 0 \) that is if \( k = n-2 \) then
\[ \| (1) \|_{L^2(\mathbb{R}^n)} \leq C h^{n-1} \alpha^2 \log(\alpha^{-1}) \| v \|_{L^2(\mathbb{R}^n)}. \]
If \( k = n-1 \) then
\[ \| (1) \|_{L^2(\mathbb{R}^n)} \leq C h^{n-1} \alpha \sum_{j=0}^{\infty} 2^{-j} \| v \|_{L^2(\mathbb{R}^n)} \leq C h^{n-1} \alpha \| v \|_{L^2(\mathbb{R}^n)}. \]
To estimate the term (2) we use (3.54). We obtain
\[ \| (2) \|_{L^2(\mathbb{R}^n)} \leq C h^{n-1} \alpha^2 \| v \|_{L^2(\mathbb{R}^n)}. \]
Using these estimates and (3.55) we deduce
\[ \| 1_B S_{\lambda}^2 1_B v \|_{L^2(\mathbb{R}^n)} \leq C \alpha^2 \| v \|_{L^2(\mathbb{R}^n)}. \]
where \( \sigma = 1 \) if \( k \leq n-3 \), \( \sigma = 1 - \varepsilon \) if \( k = n-2 \), \( \sigma = \frac{1}{2} \) if \( k = n-1 \).

Gathering the estimates proved in (3.44) and (3.56) we obtain (3.39) which proves Proposition 3.4 and therefore Proposition 3.1. The proof of Theorem 1.1 is complete.

A. Some technical results

Lemma A.1. — Let \( w \in C^\infty(M) \) be a solution of the equation \( (h^2 \Delta_g + 1)w = F \) Then
\[ \| w \|_{L^2(N_{\alpha h^{1/2}})} \leq C \frac{\alpha^\gamma}{h} (\| F \|_{L^2(M)} + \| w \|_{L^2(M)}) \]
where \( \gamma = \frac{1}{2} \) if \( k = n-1 \), \( \gamma = 1 \) if \( 1 \leq k \leq n-2 \).

Proof. — Setting \( \| \nabla_g w \|_{L^2(M)} = \left( \int_M g_p (\nabla_g w(p), \nabla_g w(p)) \, dv_g(p) \right)^{\frac{1}{2}} \) we deduce from Lemma 2.1 and from the equation that
\begin{align*}
&h \| \nabla_g w \|_{L^2(M)} \leq C (\| F \|_{L^2(M)} + \| w \|_{L^2(M)}), \\
&h^2 \| \Delta_g w \|_{L^2(M)} \leq C (\| F \|_{L^2(M)} + \| w \|_{L^2(M)}).
\end{align*}
Now setting \( \tilde{w}_j = (\chi_j w) \circ \theta^{-1} \) we have
\[ \| w \|_{L^2(N_{\alpha h^{1/2}})} \leq \sum_{j=1}^{n_0} \| \chi_j w \|_{L^2(N_{\alpha h^{1/2}})} \leq C \sum_{j=1}^{n_0} \| \tilde{w}_j \|_{L^2(B_{a,h})}. \]
For fixed \( j \in \{1, \ldots, n_0 \} \) we deduce from (A.1) that
\[
(A.3) \quad h \| \tilde{w}_j \|_{H^1(B_{\alpha,h}, \mathbb{R})} + h^2 \| \tilde{w}_j \|_{H^2(B_{\alpha,h}, \mathbb{R})} \leq C \left( \| F \|_{L^2(M)} + \| w \|_{L^2(M)} \right),
\]
from which we deduce that for \( \varepsilon > 0 \) small
\[
(A.4) \quad h^{1+\varepsilon} \| \tilde{w}_j \|_{H^{1+\varepsilon}(B_{\alpha,h}, \mathbb{R})} \leq C \left( \| F \|_{L^2(M)} + \| w \|_{L^2(M)} \right).
\]
Using the Sobolev embeddings \( H^1(\mathbb{R}) \subset L^\infty(\mathbb{R}) \) and \( H^{1+\varepsilon}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2) \), the fact that \( B_{\alpha,h} \subset \{ x = (x_a, x_b) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |x_b| \leq \alpha h^{\frac{1}{2}} \} \) and (A.3), (A.4) we obtain
\[
\| \tilde{w}_j \|_{L^2(B_{\alpha,h}, \mathbb{R})} \leq (\alpha h^{\frac{1}{2}})^{\frac{1}{2}} \| \tilde{w}_j \|_{H^1(B_{\alpha,h}, \mathbb{R})} \leq C \frac{1}{h} \left( \| F \|_{L^2(M)} + \| w \|_{L^2(M)} \right), \quad \text{if } k = n - 1,
\]
\[
\| \tilde{w}_j \|_{L^2(B_{\alpha,h}, \mathbb{R})} \leq \alpha h^{\frac{1}{2}} \| \tilde{w}_j \|_{H^{1+\varepsilon}(B_{\alpha,h}, \mathbb{R})} \leq C \frac{\alpha}{h} \left( \| F \|_{L^2(M)} + \| w \|_{L^2(M)} \right), \quad \text{if } k \leq n - 2.
\]
Lemma A.1 follows then from (A.2). \( \square \)

**A.2. Stein's lemma.** — In this section we prove a version of Stein Lemma [26, Chap 9, Proposition 1.1]. For \( \lambda > 0 \) we consider the operator
\[
(A.5) \quad T^\lambda u(\Xi) = \int_{\mathbb{R}^n} e^{i\lambda \phi(X, \Xi)} a(X, \Xi, \lambda) u(X) \, dX
\]
where \( \phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is a smooth real valued phase and \( a \) a smooth symbol.

We shall make the following assumptions.

1. (H1) there exists a compact \( K \subset \mathbb{R}^n \times \mathbb{R}^n \) such that \( \text{supp}_{X,\Xi} a \subset K \), \( \forall \lambda > 0 \),
2. (H2) \( \text{rank} \left( \frac{\partial^2 \phi}{\partial X_i \partial \Xi_j}(X, \Xi) \right)_{1 \leq i, j \leq n} \geq p \in \{1, \ldots, n\}, \forall (X, \Xi) \in K. \)

Our purpose is to prove the following result.

**Theorem A.2.** — Under the hypotheses (H1) and (H2) there exists \( C > 0 \) such that
\[
\| T^\lambda u \|_{L^2(\mathbb{R}^n)} \leq C \lambda^{-\frac{n}{2}} \| u \|_{L^2(\mathbb{R}^n)}
\]
for every \( \lambda > 0 \) and all \( u \in L^2(\mathbb{R}^n) \).

**Remark A.3.** — We shall actually apply Theorem A.2 for a family of phases \( \phi_j \) and symbols \( a_j \) converging in \( C^\infty \) topology to a fixed phase \( \phi \) and symbol \( a \) and use that in such case the estimates are uniform with respect to the parameter \( j \), which will be a consequence of the proof given below.

Below we shall prove a slightly stronger result.

First of all by the hypothesis (H1), using partitions of unity, we may assume without loss of generality that with a small \( \varepsilon > 0 \)
\[
\text{supp}_{X,\Xi} a \subset V_{\rho_0} = \{(X, \Xi) \in \mathbb{R}^n \times \mathbb{R}^n : |X - X_0| + |\Xi - \Xi_0| \leq \varepsilon \}, \quad \rho_0 = (X_0, \Xi_0).
\]
Moreover changing if necessary the orders of the variables we may assume that near \( \rho_0 \)
\[
X = (x, y) \in \mathbb{R}^p \times \mathbb{R}^{n-p}, \quad \Xi = (\xi, \eta) \in \mathbb{R}^p \times \mathbb{R}^{n-p}
\]
and for all \( (X, \Xi) \in V_{\rho_0} \) the \( p \times p \)-matrix
\[
(A.6) \quad M_p(X, \Xi) = \left( \frac{\partial^2 \phi}{\partial X_i \partial \Xi_j}(X, \Xi) \right)_{1 \leq i, j \leq p}
\]
is invertible with \( \|M_p(X, \Xi)^{-1}\| \leq c_0 \).

Then we have

**Theorem A.4.** — There exists a positive constant \( C \) such that for every \( \lambda > 0 \) we have

\[
\|T^\lambda u\|_{L^\infty(R^{n-p}_y, L^2(R^n_\xi))} \leq C\lambda^{-\frac{p}{p-2}}\|u\|_{L^1(R^{n-p}_y, L^2(R^n_\xi))}
\]

for all \( u \in L^1(R^{n-p}_y, L^2(R^n_\xi)) \).

Theorem A.2 follows from Theorem A.4 using (H1).

**Proof of Theorem A.4.** — It is an easy consequence of the proof of a proposition in section 1.1 Chapter IX in [26]. Indeed let us set for \( (y, \eta) \in R^{n-p} \times R^{n-p} \)

(A.7) \( \phi_{(y, \eta)}(x, \xi) = \phi(x, y, \xi, \eta) \), \( a_{(y, \eta)}(x, \xi) = a(x, y, \xi, \eta) \), \( u_y(x) = u(x, y) \),

(A.8) \( T^\lambda_{(y, \eta)}u_y(\xi) = \int_{R^p} e^{i\lambda\phi_{(y, \eta)}(x, \xi)}a_{(y, \eta)}(x, \xi)u_y(x)\,dx. \)

Then we have

(A.9) \( T^\lambda u(\Xi) = \int_{R^{n-p}} T^\lambda_{(y, \eta)}u_y(\xi)\,dy. \)

We claim that there exists \( C > 0 \) such that for every \( (y, \eta) \in V_{(y_0, \eta_0)} \) we have

(A.10) \( \|T^\lambda_{(y, \eta)}u_y\|_{L^2(R^n_\xi)} \leq C\lambda^{-\frac{p}{p-2}}\|u_y\|_{L^2(R^n_\xi)} \quad \forall \lambda > 0. \)

Assuming for a moment that (A.10) is proved we obtain

\[
\|T^\lambda u(\cdot, \eta)\|_{L^2(R^n_\xi)} \leq \int_{R^{n-p}} \|T^\lambda_{(y, \eta)}u_y\|_{L^2(R^n_\xi)}\,dy \leq C\lambda^{-\frac{p}{p-2}}\int_{R^{n-p}} \|u(\cdot, \eta)\|_{L^2(R^n_\xi)}\,dy
\]

which implies immediately the conclusion of Theorem A.2.

The claim (A.10) follows immediately from the proof of proposition in 1.1 Chapter IX in [Stein]. However, for the convenience of the reader, we shall give it here.

For simplicity we shall skip the subscript \( (y, \eta) \), keeping in mind the uniformity, with respect to \( (y, \eta) \in V_{(y_0, \eta_0)} \), of the constants in the estimates. Therefore we set

\[
S_{\lambda} = T^\lambda_{(y, \eta)}, \quad \phi_{(y, \eta)} = \psi, \quad b = a_{(y, \eta)}. \]

It follows from (A.6) that the matrix

\[
N(x, \xi) = \left( \frac{\partial^2 \psi}{\partial x_i \partial \xi_j}(x, \xi) \right)_{1 \leq i, j \leq p}
\]

is invertible and \( \|N(x, \xi)^{-1}\| \leq c_0 \) where \( c_0 \) is independent of \( (y, \eta) \). Now by the usual trick the estimate (A.10) is satisfied if and only if we have

(A.11) \( \|S_{\lambda}S_{\lambda}^{-1}f\|_{L^2(R^p)} \leq C\lambda^{-p}\|f\|_{L^2(R^p)} \)

with \( C \) independent of \( (y, \eta) \). It is easy to see that

(A.12) \( S_{\lambda}S_{\lambda}^{-1}f(\xi) = \int_{R^p} K(\xi, \xi')f(\xi')\,d\xi' \)

with

\[
K(\xi, \xi') = \int_{R^k} e^{i\lambda(\psi(x, \xi) - \psi(x, \xi'))}b(x, \xi)\overline{b}(x, \xi')\,dx.
\]

20
Let us set
\[ c(x, \xi, \xi') = N(x, \xi)^{-1} \frac{\xi - \xi'}{|\xi - \xi'|}. \]
Then we can write
\[ (A.13) \quad c(x, \xi, \xi') \cdot \nabla_x e^{i\lambda(\psi(x,\xi)-\psi(x,\xi'))} = e^{i\lambda(\psi(x,\xi)-\psi(x,\xi'))} i\lambda \Delta(x, \xi, \xi') \]
where
\[ \Delta(x, \xi, \xi') = \sum_{j=1}^{k} c_j(x, \xi, \xi') \left( \frac{\partial \phi}{\partial x_j}(x, \xi) - \frac{\partial \phi}{\partial x_j}(x, \xi') \right), \]
\[ = \sum_{j,l=1}^{k} c_j(x, \xi, \xi') \left( \frac{\partial^2 \phi}{\partial x_j \partial \xi_l}(x, \xi)(\xi_l - \xi'_l) + O(|\xi - \xi'|^2), \right) \]
\[ = \langle N(x, \xi)c(x, \xi, \xi'), \xi - \xi' \rangle + O(|\xi - \xi'|^2) = |\xi - \xi'| + O(|\xi - \xi'|^2), \]
where \( O(|\xi - \xi'|^2) \) is independent of \((y, \eta)\). Since \( b \) has small support in \( \xi \) we deduce that
\[ (A.14) \quad \Delta(x, \xi, \xi') \geq C|\xi - \xi'|. \]
Moreover since the derivatives with respect to \( x \) of \( N(x, \xi)^{-1} \) are products of \( N(x, \xi)^{-1} \) and derivatives of \( N(x, \xi) \), we see that all the derivatives with respect to \( x \) of \( \Delta(x, \xi, \xi') \) are uniformly bounded in \((y, \eta)\) near \((y_0, \eta_0)\). Let us set
\[ L = \frac{1}{i\lambda \Delta(x, \xi, \xi')}c(x, \xi, \xi') \cdot \nabla_x. \]
It follows from (1.4) and the fact that \( b \) has compact support in \( x \) that for every \( N \in \mathbb{N} \) we can write
\[ K(\xi, \xi') = \int_{\mathbb{R}^p} e^{i\lambda(\psi(x,\xi)-\psi(x,\xi'))} (L)^N[b(x, \xi) \overline{b}(x, \xi')] dx. \]
We deduce from (A.14) that for every \( N \in \mathbb{N} \) there exists \( C_N > 0 \) independent of \((y, \eta)\) such that
\[ |K(\xi, \xi')| \leq \frac{C_N}{(1 + \lambda |\xi - \xi'|)^N}. \]
Taking \( N > p \) we deduce from (A.12) and Schur lemma that (A.11) holds with a constant \( C \) independent of \((y, \eta)\). This completes the proof. \( \Box \)

**Lemma A.5.** — Let \( d \geq 1 \), \( \delta \in \mathbb{R} \) and \( \varphi_0(x, x') = (\sum_{j=1}^{d}(x_j - x'_j)^2 + \delta^2)^{\frac{1}{2}} \). Let \( M = \left( \frac{\partial^2 \varphi_0}{\partial x_j \partial x_k}(x, x') \right)_{1 \leq j, k \leq d} \). Then
\[ (i) \quad \text{if } \delta \neq 0 \quad M \text{ has rank } d \text{ for all } x, x' \in \mathbb{R}^d, \]
\[ (ii) \quad \text{if } \delta = 0 \quad M \text{ has rank } d - 1 \text{ for } x \neq x'. \]

**Proof.** — (i) A simple computation shows that
\[ M = \varphi_0(x, x')^{-1}(-\delta_{jk} + \omega_j \omega_k), \quad \omega_j = \frac{x_j - x'_j}{\varphi_0(x, x')} \]
where \( \delta_{jk} \) is the Kronecker symbol. For \( \lambda \in \mathbb{R} \) consider the polynomial in \( \lambda \)
\[ F(\lambda) = \det \left( -\delta_{jk} + \lambda \omega_j \omega_k \right)_{1 \leq j, k \leq d} \]
We have obviously $F(0) = (-1)^d$. Now denote by $C_j(\lambda)$ the $j^{th}$ column of this determinant. Then

$$F'(\lambda) = \sum_{k=1}^d \det (C_1(\lambda), \ldots, C_k'(\lambda), \ldots C_d(\lambda)).$$

Since $\det (C_1(\lambda), \ldots, C_k'(\lambda), \ldots C_d(\lambda)) = (-1)^{d-1} \omega^2$ we obtain $F'(0) = (-1)^{d-1} \sum_{j=1}^d \omega^2 j$. Now $C_j(\lambda)$ being linear with respect to $\lambda$ we have $C_j'(\lambda) = 0$. Therefore

$$F''(\lambda) = \sum_{j=1}^d \sum_{k=1, k \neq j}^d \det (C_1(\lambda), \ldots, C_j'(\lambda), \ldots C_k'(\lambda), \ldots C_d(\lambda)).$$

Since $C_j'(\lambda) = \omega_j(\omega_1, \ldots, \omega_d)$ and $C_k'(\lambda) = \omega_k(\omega_1, \ldots, \omega_d)$ we have $F''(\lambda) = 0$ for all $\lambda \in \mathbb{R}$. It follows that $F(\lambda) = (-1)^d(1 - \sum_{j=1}^d \omega^2 j)$. Therefore

$$\det M = (-1)^d(1 - \sum_{j=1}^d \omega^2 j) = (-1)^d \frac{\delta^2}{\varphi_0(x, x')^2} \neq 0.$$

(ii) Since $x - x' \neq 0$ we may assume without loss of generality that $\omega_d \neq 0$. Set

$$A = (-\delta_{jk} + \omega_j \omega_k)_{1 \leq j,k \leq d-1}.$$

Introducing $G(\lambda) = \det (-\delta_{jk} + \lambda \omega_j \omega_k)_{1 \leq j,k \leq d-1}$ the same computation as above shows that

$$\det A = (-1)^{d-1}(1 - \sum_{j=1}^{d-1} \omega^2 j) = (-1)^{d-1} \omega_d^2 \neq 0.$$

\[\square\]

References

[1] N. Anantharaman and M. Léautaud. Sharp polynomial decay rates for the damped wave equation on the torus. \textit{Anal. PDE}, 7(1):159–214, 2014. With an appendix by Stéphane Nonnenmacher.

[2] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. \textit{S.I.A.M. Journal of Control and Optimization}, 305:1024–1065, 1992.

[3] A. Borichev and Y. Tomilov. Optimal polynomial decay of functions and operator semigroups. \textit{Math. Ann.}, 347(2):455–478, 2010.

[4] J. Bourgain and Z. Rudnick. Restriction of toral eigenfunctions to hypersurfaces and nodal sets. \textit{Geom. Funct. Anal.}, 22(4):878–937, 2012.

[5] N. Burq and P. Gérard. Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. \textit{Comptes Rendus de L’Académie des Sciences}, pages 749–752, 1997. t.325, Série I.

[6] N. Burq, P. Gérard, and N. Tzvetkov. The Cauchy problem for the nonlinear Schrödinger equation on compact manifolds. In \textit{Phase space analysis of partial differential equations. Vol. I}, Pubbl. Cent. Ric. Mat. Ennio Giorgi, pages 21–52. Scuola Norm. Sup., Pisa, 2004.

[7] N. Burq, P. Gérard, and N. Tzvetkov. Multilinear estimates for the Laplace spectral projector on compact manifolds. \textit{Comptes rendus de l’académie des sciences}, 338(Sér. I):359–364, 2004.

[8] N. Burq, P. Gérard, and N. Tzvetkov. Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces. \textit{Inventiones Mathematicae}, 159(1):187 – 223, 2005.
[9] N. Burq, P. Gérard, and N. Tzvetkov. Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations. *Ann. Sci. École Norm. Sup. (4)*, 38(2):255–301, 2005.

[10] N. Burq, P. Gérard, and N. Tzvetkov. Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds. *Duke Math. J.*, 138(3):445–486, 2007.

[11] N. Burq and M. Hitrik. Energy decay for damped wave equations on partially rectangular domains. *Math. Res. Lett.*, 14(1):35–47, 2007.

[12] N. Burq and M. Zworski. Geometric control in the presence of a black box. *Jour. of the American Math. Society*, 17(2):443–471, 2004.

[13] N. Burq and M. Zworski. Bouncing ball modes and quantum chaos. *S.I.A.M. Review*, 47(1):43–49, 2005.

[14] Y. Colin de Verdière. Ergodicité et foctions propres du laplacien. *Comm. Math. Phys.*, 102:187–214, 1985.

[15] P. Gérard and E. Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Mathematical Journal*, 71:559–607, 1993.

[16] B. Helffer, A. Martinez, and D. Robert. Ergodicité et limite semi-classique. *Communications in Mathematical Physics*, 109:313–326, 1987.

[17] H. Koch, D. Tataru, and M. Zworski. Semiclassical $L^p$ estimates. *Ann. Henri Poincaré*, 8(5):885–916, 2007.

[18] M. Leautaud and N. Lerner. Energy decay for a locally undamped wave equation. *arxiv: http://arxiv.org/abs/1411.7271*, 2014.

[19] C. Sogge. Concerning the $L^p$ norm of spectral clusters for second order elliptic operators on compact manifolds. *Jour. of Funct. Anal.*, 77:123–138, 1988.

[20] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[21] K.D. Phung. Polynomial decay rate for the dissipative wave equation. *J. Differential Equations*, 240(1):92–124, 2007.

[22] A.I. Shnirelman. Ergodic properties of eigenfunctions. *Uspekhi Mat. Nauk*, 29:181–182, 1974.

[23] C. Sogge. Concerning the $L^p$ norm of spectral clusters for second order elliptic operators on compact manifolds. *J. Fourier Anal. Appl.*, 77:123–138, 1988.

[24] C. Sogge. Fourier integrals in classical analysis. *Cambridge tracts in Mathematics*, 1993.

[25] C. Sogge and S. Zelditch. Riemannian manifolds with maximal eigenfunction growth. *Duke Math. J.*, 114(3):387–437, 2002.

[26] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[27] M. Tacy. Semiclassical $L^p$ estimates of quasimodes on submanifolds. *Comm. Partial Differential Equations*, 35(8):1538–1562, 2010.

[28] S. Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. Jour.*, 55:919–941, 1987.

[29] S. Zelditch and M. Zworski. Ergodicity of eigenfunctions for ergodic billiards. *Communications in Mathematical Physics*, 175:673–682, 1996.

[30] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.
