Functions with no unbounded Fatou components

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ABSTRACT
For a transcendental entire function with sufficiently small growth, Baker raised the question whether it has no unbounded Fatou components. We have shown that if the function is of order strictly less than half, minimal type, then it has no unbounded Fatou components. This, in particular gives a partial answer to Baker’s question. In addition, we have addressed Wang’s question on Fejér gaps. Certain results about functions with Fabry gaps and of infinite order have also been generalized.

1. Introduction and preliminaries
This article investigates the question raised by Baker for transcendental entire functions, which says, whether every Fatou component of a transcendental entire function is bounded when the function is of sufficiently small growth. Let us recall some basic definitions that we will be using throughout this article.

Let \( f \) be a transcendental entire function and let \( f^n \) denote the \( n \) – th iterate of \( f \). The Fatou set \( F(f) \) is defined to be the set of all \( z \in \mathbb{C} \) such that \( \{f^n\}_{n \in \mathbb{N}} \) forms a normal family in some neighbourhood of \( z \). The complement \( J(f) \) of \( F(f) \) is called as the Julia set of \( f \). The basic properties of these sets can be found in Refs [1–3].

The order of growth \( \rho \), lower order of growth \( \lambda \), and type \( \sigma \) of a transcendental entire function \( f \) are defined as follows:

\[
\rho = \lim \sup_{r \to \infty} \frac{\log \log M(r,f)}{\log r},
\]

\[
\lambda = \lim \inf_{r \to \infty} \frac{\log \log M(r,f)}{\log r}, \quad \text{and}
\]

\[
\sigma = \lim \sup_{r \to \infty} \frac{\log M(r,f)}{r^\rho},
\]

where \( M(r,f) = \max\{|f(z)| : |z| = r\} \).

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The growth of $f$ is said to be of minimal type if $\sigma = 0$, mean type if $\sigma \in (0, \infty)$, and maximal type if $\sigma = \infty$. Also, the minimum modulus of $f$ on $|z| = r$ is denoted by $L(r, f)$, i.e.

$$L(r, f) = \min\{|f(z)| : |z| = r\}.$$  

For each $r \geq 0$, the maximum term $\mu(r, f)$ is defined as

$$\mu(r, f) = \max\{|a_k|r^k : k = 0, 1, 2, \ldots\},$$

where $a_k$'s are the coefficients of $f$ in its power series expansion around zero. For $r > 0$, the central index $\nu(r, f)$ is defined as the largest $k$ for which $|a_k|r^k = \mu(r, f)$.

In Ref. [4], Baker observed that for the function

$$f(z) = \frac{\sin z^{\frac{1}{2}}}{z^{\frac{1}{2}}} + z + a,$$

where $a$ is sufficiently large, $F(f)$ contains a segment $[x_0, \infty)$ of the positive real axis. This function is of order $\frac{1}{2}$, mean type. Hence, the sufficiently small growth condition appears to be of order $\frac{1}{2}$, minimal type at most.

Along with Baker, several researchers gave an affirmative answer to this question by assuming certain regularity conditions on the function $f$. We list below some of these results.

Baker proved that if $f$ satisfies the condition that

$$\log M(r, f) = O((\log r)^p), \quad \text{for some } p \in (1, 3),$$

then every component of $F(f)$ is bounded.

Stallard improved the growth condition to

$$\log \log M(r, f) < \frac{(\log r)^{\frac{1}{2}}}{(\log \log r)^c}, \quad \text{for some } c \in (0, 1)$$

for sufficiently large $r$ [5]. In the same paper, it is also proved that if $\rho < \frac{1}{2}$ and

$$\frac{\log \log M(2r, f)}{\log M(r, f)} \to c \neq \infty \quad \text{as } r \to \infty,$$

then every component of $F(f)$ is bounded.

In Ref. [6], Zheng proved that there are no unbounded periodic Fatou components if the growth of $f$ is at most of order $\frac{1}{2}$, minimal type. Anderson and Hinkanen used the notion of self sustaining spread to prove some results in this direction [7]. Rippon and Stallard also gave some sufficient conditions which imply that $F(f)$ has no unbounded Fatou component [8]. They also improved some results of Hinkanen, Wang [9] and of several other authors. In addition, they proved that Eremenko’s conjecture holds for functions with no unbounded Fatou components. Thus, Baker’s question is still open for functions with wandering domains.
2. Main result

In this article, we prove the following:

**Theorem 2.1:** Let $f$ be a transcendental entire function. If for given $\epsilon > 0$, the condition

$$\log L(r, f) > (1 - \epsilon) \log M(r, f)$$

is satisfied for every $r$ outside a set of logarithmic density zero, then $F(f)$ has no unbounded components.

The above theorem generalizes [10, Theorem 1] and [11, Theorem 1]. In Ref. [10], Singh considered $f$ as a composition of two transcendental entire functions of positive order with certain conditions on them; meanwhile Wang’s paper [9] deals with functions of finite order possessing Fabry gaps with positive lower order. Note that functions of finite order with Fabry gaps certainly satisfy the hypothesis of Theorem 2.1 [12]. In the same paper, Wang also proposed the question whether a function with Fejér gaps has no unbounded Fatou components. Theorem 2.1 gives an affirmative answer to this question also, as a function with Fejér gaps satisfies the condition of Theorem 2.1 [13].

In the proof of Theorem 2.1, we need two real sequences $\{R_n\}$ and $\{S_n\}$ satisfying the following properties:

1. For any $\alpha > 1$, we have $R_{n+1} = M(R_n^{\frac{1}{2\alpha}}, f)$, $S_{n+1} = M(S_n, f)$, and
2. $S_n \leq R_n^{\frac{1}{2\alpha}}$, for every sufficiently large $n$.

Before considering the construction of the sequences in the most general case of $f$ being any transcendental entire function, we shall illustrate a construction for a special case as given below.

Suppose that $f$ is of positive lower order and having finite order. As $\frac{\rho}{\lambda} > 0$, there exists $n_{\lambda, \rho} \in \mathbb{N}$ such that $n_{\lambda, \rho} > \frac{\rho}{\lambda}$. Now, consider

$$\liminf_{r \to \infty} \log \frac{\log M\left(\frac{r^{\frac{1}{2\alpha}}}{16\alpha^4 n_{\lambda, \rho}}, f\right)}{\log r^{2\alpha}} = \liminf_{r \to \infty} \left(\frac{\log \frac{1}{16\alpha^4 n_{\lambda, \rho}}}{\log r^{2\alpha}} + \frac{\log \log M\left(\frac{r^{\frac{1}{2\alpha}}, f\right)}{\log r^{2\alpha}}\right)$$

$$= \liminf_{r \to \infty} \frac{\log \log M\left(\frac{r^{\frac{1}{2\alpha}}, f\right)}{\log r^{2\alpha}} = \frac{\lambda}{4\alpha^2} \quad \text{(by the definition of lower order).}$$

Hence, there exists $r_3 > 0$ such that

$$\frac{\log \log M\left(\frac{r^{\frac{1}{2\alpha}}, f\right)}{\log r^{2\alpha}} \geq \frac{\lambda}{8\alpha^2} \quad \text{for every } r \geq r_3.$$

On similar lines, using the definition of order of $f$, there exists $r_4 > 0$ such that

$$\log \log M(S_n, f)^{\frac{1}{2\alpha}} \leq 2\rho \log S_n \quad \text{for every } r \geq r_4.$$
Now, take $R_1, S_1 > \max\{r_0, r_1, r_3, r_4\}$ such that $R_1^{\frac{1}{16\alpha^4n_{\lambda, \rho}}} \geq S_1^{\frac{1}{\alpha}}$, where $r_0, r_1$ are chosen from the steps 1 and 2 of the proof of Theorem 2.1 (to follow). Further, for any $\alpha > 1$, set $R_{n+1} = M(R_n^{\frac{1}{16\alpha^4n_{\lambda, \rho}}}, f)$ and $S_{n+1} = M(S_n, f)$. To prove (2), it is sufficient to show that $S_n^{\frac{1}{\alpha}} \leq R_n^{\frac{1}{16\alpha^4n_{\lambda, \rho}}}$ for every $n \in \mathbb{N}$, which we shall prove by induction on $n$.

Consider

$$
\frac{\log \log M(R_n^{\frac{1}{16\alpha^4n_{\lambda, \rho}}}, f)^{\frac{1}{\alpha}}}{\log \log M(S_n, f)^{\frac{1}{\alpha}}} \geq \frac{\lambda}{8\alpha^2} \log R_n^{2\alpha} \frac{2\rho \log S_n}{\lambda \log S_n^{\frac{1}{\alpha}}} \geq \frac{\lambda \log R_n^{2\alpha}}{16\alpha^2 \rho \log S_n^{\frac{1}{\alpha}}} \geq \frac{\log R_n^{\frac{1}{16\alpha^4n_{\lambda, \rho}}}}{\log S_n^{\frac{1}{\alpha}}} \geq 1.
$$

Consequently,

$$
S_{n+1}^{\frac{1}{\alpha}} = M(S_n, f)^{\frac{1}{\alpha}} \leq M(R_n^{\frac{1}{16\alpha^4n_{\lambda, \rho}}}, f)^{\frac{1}{\alpha}} = R_{n+1}^{\frac{1}{16\alpha^4n_{\lambda, \rho}}},
$$

which completes the induction process.

General Case: Here, lower order of $f$ can be zero or order of $f$ can be infinity, and hence we can not apply the method as done above. To this end, we first make the following observations.

Using [14, Lemma 2.2.7], we may choose a constant $K = \log \mu(s, f) \geq 2$ for sufficiently large $s \geq 1$ such that

$$
\log \mu(r, f) \leq v(r, f) \log r + K \quad \text{for sufficiently large } r.
$$

Replacing $r$ with $2r$, we get

$$
\log \mu(2r, f) \leq v(2r, f) \log 2r + K \quad \text{for sufficiently large } r. \quad (1)
$$

Now, for $r > 0$, by [14, Lemma 2.2.2], we have $M(r, f) \leq 2\mu(2r, f)$. Taking log on both sides, we get

$$
\log M(r, f) \leq \log 2 + \log \mu(2r, f) \quad \text{for sufficiently large } r.
$$

This further implies that

$$
\log 2M(r, f) \leq 2\log 2 + \log \mu(2r, f) \quad \text{for sufficiently large } r.
$$

Hence,

$$
\log 2M(r, f) \leq 2\log 2 + v(2r, f) \log 2r + K
$$
\[ \leq K' + \nu(2r, f) \log 2r \]
\[ = \log K'' + \nu(2r, f) \log 2r \]
\[ = \log (K'' 2r^{\nu(2r, f)}) \]

for sufficiently large \( r \). Therefore, there exists \( s_0 > 0 \) such that
\[ \log 2M(r, f) \leq \log (K'' (2r)^{\nu(2r, f)}) \text{ for every } r \geq s_0. \]

Again, using [14, Lemma 2.2.7], we have
\[ \log \mu(r, f) + \nu(r, f) \log r \leq \log \mu(r^2, f), \text{ for sufficiently large } r. \]

Using [14, Lemma 2.2.2], we obtain
\[ \log \mu(r, f) + \nu(r, f) \log r \leq \log M(r^2, f) \text{ for sufficiently large } r. \quad (2) \]

This gives us the existence of \( s_1 > 0 \) such that
\[ \log \mu(r, f) + \nu(r, f) \log r \leq \log M(r^2, f) \text{ for every } r \geq s_1. \]

Choose \( R_1, S_1 \) such that \( R_1^{1/2} \geq 2S_1 \geq \max\{s_0, s_1, r_1, r_2\} \) and \( \mu(R_1^{1/2}, f) \geq K'' \). For \( n \in \mathbb{N} \), define \( R_{n+1} = M(R_n^{1/2}, f) \), and \( S_{n+1} = M(S_n, f) \).

**Lemma 2.1:** There exists a sequence \( \{k_n\} \) in \((0, 1)\) such that
\[ a_n := \frac{\nu(R_n^{1/2}, f)}{\nu((8S_n)^{2k_n}, f)}(\log M(4S_n^2, f))^2 \]

satisfies \( k_n \leq a_n \) for every \( n \in \mathbb{N} \), and \( (8S_n)^{2k_n} \) approaches to some finite number as \( n \) tends to infinity.

**Proof:** First, we choose a sequence \( \{l_n\} \) of real numbers such that \( (8S_n)^{2l_n} \) approaches to some \( b \) as \( n \) tends to infinity. On using right continuity of \( \nu(r, f) \) at \( b \), we have the existence of \( \delta > 0 \) such that \( \nu((8S_n)^{2l_n}, f) = \nu(b, f) \) for every \( n \) satisfying \( b < (8S_n)^{2l_n} < b + \delta \).

Now, consider \( \frac{\nu(R_n^{1/2}, f)}{(\log M(4S_n^2, f))^2} = b_n \) (say). The above observations gives us that
\[ a_n = \frac{b_n}{\nu(b, f)} \text{ for sufficiently large } n. \]

Now, for every \( n \in \mathbb{N} \), define \( k_n = \min\{a_n, l_n\} \). From the definition, it is clear that \( \{k_n\} \) satisfies the required properties, i.e. \( k_n \leq a_n \) and \( (8S_n)^{2k_n} \) approaches to some finite number, say \( a \) as \( n \) tends to infinity. ■
Now, using the above observations and Lemma 2.1, we shall prove that

\[ \nu(2S_n, f) \leq \frac{a_n \nu((8S_n)^{2k_n}, f) (\log M(4S_n^2, f))^2}{4\alpha} \quad \text{for sufficiently large } n. \]

For this, consider

\[ \frac{\nu(2S_n, f)}{\nu((8S_n)^{2k_n}, f)(\log M(4S_n^2, f))^2} \leq \frac{K(\log M(4S_n^2, f) - \log \mu(2S_n, f)) \log (8S_n)^{2k_n}}{\log 2S_n \log M\left(\frac{(8S_n)^{2k_n}}{2}, f\right) (\log M(4S_n^2, f))^2} \]

\[ = \frac{K \log M(4S_n^2, f) \log (8S_n)^{2k_n}}{\log 2S_n \log M\left(\frac{(8S_n)^{2k_n}}{2}, f\right) (\log M(4S_n^2, f))^2} - \frac{K \log \mu(2S_n, f)}{\log 2S_n \log M\left(\frac{(8S_n)^{2k_n}}{2}, f\right) (\log M(4S_n^2, f))^2} \]

\[ \leq \frac{K \log M(4S_n^2, f) \log (2S_n)^{6k_n}}{\log 2S_n \log M\left(\frac{(8S_n)^{2k_n}}{2}, f\right) (\log M(4S_n^2, f))^2} \]

\[ = \frac{6Kk_n \log 2S_n}{m_0 \log M(4S_n^2, f)} \]

\[ \leq \frac{a_n}{4\alpha}, \quad \text{(5)} \]

for sufficiently large \( n \).

This means that there exists \( n_1 \in \mathbb{N} \) such that

\[ \nu(2S_n, f) \leq \frac{a_n \nu((8S_n)^{2k_n}, f) \log (8S_n)^{2k_n}}{4\alpha} \quad \text{for every } n \geq n_1. \]

Justification for the deduction on each line of the above multilime equation:

- On the first line Equation (3), we have used Equation (1), Equation (2) and the following inequality: for \( \beta > 1 \), we have \( x - \beta \geq \frac{x}{\beta} \), for sufficiently large \( x \).
- For the fifth inequality Equation (4), using Lemma 2.1, we get \( k_n \leq a_n \) and \( \log M\left(\frac{(8S_n)^{2k_n}}{2}, f\right) \) approaches to \( \log M\left(\frac{a}{2}, f\right) = m_0 \) (say).
- For the sixth inequality Equation (5), use the fact that \( \frac{1}{\log M(4S_n^2, f)} \) approaches to zero as \( n \) tends to \( \infty \).

Now, assume that \( R_{n_1}^{\frac{1}{2k_n}} \geq 2S_{n_1} \). Then, for \( n > n_1 \), consider

\[ \log M(R_{n_1}^{\frac{1}{2k_n}}, f) \leq \frac{1}{4\alpha} \log \mu(R_{n_1}^{\frac{1}{2k_n}}, f) \leq \frac{1}{4\alpha} \frac{R_{n_1}^{16\alpha^2}}{\log \left(\frac{1}{\nu(R_{n_1}^{\frac{1}{2k_n}}, f)}\right)^{v(2S_{n_1}, f)}} \]
\[ \geq \frac{\log \mu(R_{n1}^{\frac{1}{2\alpha}}, f) \frac{1}{16\alpha^2} \nu(R_{n1}^{\frac{1}{2\alpha}}, f)}{\log \left(K''(2^{\frac{\alpha}{4}} n_1) \frac{\nu n_1 (8n_1)^{2k_1} f \log (8n_1)^{2k_1}}{4\alpha} \right)} \geq 1. \]

This gives us that \( R_{n1+1}^{\frac{1}{2\alpha}} \geq 2S_{n1+1} \). On applying the same process inductively, we get
\[ R_{n}^{\frac{1}{2\alpha}} \geq 2S_n \text{ for every } n \geq n_1. \]

We now prove the following result which will also be used in the proof of Theorem 2.1.

**Lemma 2.2:** Let \( f \) be a transcendental entire function and let \( m > 1 \). Then,
\[ M(\log r^{\frac{1}{2m}} r^{\frac{1}{2m}}, f) \geq \left( \log M(r^{\frac{1}{2m}}, f) M(r^{\frac{1}{2m}}, f)^{\frac{1}{2m}} \right)^m \]
for sufficiently large \( r \).

**Proof:** Consider,
\[ \log M(r^{\frac{1}{2m}}, f) M(r^{\frac{1}{2m}}, f)^{\frac{1}{2m}} = M(r^{\frac{1}{2m}}, f) \frac{1}{2m} \log M(r^{\frac{1}{2m}}, f)^{\frac{1}{2m}} \leq M(r^{\frac{1}{2m}}, f)^{\frac{1}{2m}} \]
for \( m \) large enough. As \( \log r^{\frac{1}{2m}} r^{\frac{1}{2m}} \geq r^{\frac{1}{2m}} \) for sufficiently large \( r \), we have \( M(\log r^{\frac{1}{2m}} r^{\frac{1}{2m}}, f) \geq M(r^{\frac{1}{2m}}, f) \) for sufficiently large \( r \). This further implies that
\[ \frac{\log M(r^{\frac{1}{2m}}, f) M(r^{\frac{1}{2m}}, f)^{\frac{1}{2m}}}{M(\log r^{\frac{1}{2m}} r^{\frac{1}{2m}}, f)^{\frac{1}{2m}}} \leq 1 \]
for sufficiently large \( r \). As a result, we obtain
\[ \left( \log M(r^{\frac{1}{2m}}, f)^{\frac{1}{2m}} M(r^{\frac{1}{2m}}, f) \right)^m \leq M(\log r^{\frac{1}{2m}} r^{\frac{1}{2m}}, f) \]
for sufficiently large \( r \).

**Proof of the Theorem 2.1:** We will prove the result through the following steps:

Step 1: Using the given hypothesis, there exist real numbers \( \alpha > 1 \) and \( r_0 \) such that for each \( r \geq r_0 \), there exists \( \sigma \) satisfying \( r \leq \sigma \leq r^\alpha \) and \( L(\sigma, f) = M(r, f) \) [10].
Step 2: In this step, we will observe some inequalities:

(i) Using Lemma 2.2, for $m = \alpha$ there exists $r_1$ such that

\[
\left( \log M(r^{\frac{1}{2\alpha}}, f) \left( r^{\frac{1}{2\alpha}} \right)^{\frac{1}{2\alpha}} \right)^{\alpha} \leq M\left( \log r^{\frac{1}{2\alpha}}, f \right) \text{ for } r \geq r_1.
\]

(ii) From [8, Lemma 2.2], for $c = 2\alpha$, there exists $r_2$ such that

\[M(r^{2\alpha}, f) \geq M(r, f)^{2\alpha} \text{ for } r \geq r_2.\]

Step 3: Observe that the sequence \( \left\{ \log R_n^{\frac{1}{2\alpha}} \right\} \) tends to infinity as \( n \) tends to infinity. Using Step 1, for each \( n \) there exists \( \sigma_n \) satisfying

\[\log R_n^{\frac{1}{2\alpha}} \leq \sigma_n \leq \left( \log R_n^{\frac{1}{2\alpha}} \right)^{\alpha}\]

such that \( L(\sigma_n, f) = M(\log R_n^{\frac{1}{2\alpha}}, f) \). Also, by Step 2, we have

\[L(\sigma_n, f) = M(\log R_n^{\frac{1}{2\alpha}}, f)\]

\[\geq \left( \log M(R_n^{\frac{1}{2\alpha}}, f) \left( R_n^{\frac{1}{2\alpha}} \right)^{\frac{1}{2\alpha}} \right)^{\alpha}\]

\[= \left( \log R_n^{\frac{1}{2\alpha}} \right)^{\alpha}.\]

Hence \( \liminf_{n \to \infty} L(\sigma_n, f) = \infty \). This gives us that the image of unbounded Fatou component is unbounded [15].

Step 4: Suppose that \( F(f) \) has an unbounded component, say \( U \). Without loss of generality, we can assume that \( 0, 1 \in J(f) \) such that \( f(0) = 1 \). As \( U \) is an unbounded component, for sufficiently large \( n \geq n_2 \geq n_1 \), the Fatou component \( U \) intersects the following three circles:

\[T_n = \{ z : |z| = S_n \},\]

\[T_n^1 = \{ z : |z| = \sigma_n \}, \text{ and}\]

\[T_n^2 = \left\{ z : |z| = \left( \log R_n^{\frac{1}{2\alpha}} \right)^{\alpha} \right\}.\]
As $U$ is a connected Fatou component, we can join any two points by a path. Let $\gamma : [0, 1] \to U$ be a path joining the points $z_n \in T_n$ in $U$ and $z_{n+1}^2 \in T_{n+1}^2$ in $U$. Then $\gamma$ intersects the circle $T_{n+1}^1$ at some point, say $z_{n+1}^1$ (Figure 1). By Step 3, $f(U)$ is an unbounded Fatou component containing the path $f \circ \gamma$. Also, we have $|f(z_n)| \leq S_{n+1}$ and

$$|f(z_{n+1}^1)| \geq \left( \log \frac{1}{R_{n+2}^\alpha} \right)^\alpha R_{n+2}^\alpha.$$ 

These observations imply the existence of two points $z_{n+1} \in T_{n+1}$ and $z_{n+2}^2 \in T_{n+2}^2$ which lie on the path $f \circ \gamma$.

By continuing this process inductively, we get that $f^k(U)$ will contain the path $f^k \circ \gamma$ which intersects the circles $T_{n+k}$ at $z_{n+k}$ and $T_{n+k+1}^2$ at $z_{n+k+1}^2$. Therefore, $f^k$ takes a value of modulus at least $S_{n+k}$ on $\gamma$. This shows that $\{f^k\}$ goes to infinity locally uniformly on $U$. This, in particular, implies that there exists $N_0 \in \mathbb{N}$ such that for any $k > N_0$ and for every $z \in \gamma([0, 1])$, we have $|f^k(z)| > 1$. Applying [4, Lemma 5] to the compact set $\gamma([0, 1])$, for all $z, w \in \gamma([0, 1])$, we have

$$|f^k(z)| < B|f^k(w)|^C \quad \text{for every } k > N_0.$$
Now, for any $k > N_0$, we can choose $u_k, u_k^2 \in \gamma([0, 1])$ such that $f^k(u_k) = z_{n+k}$ and $f^k(u_k^2) = z_{n+k+1}^2$. Hence, we have $|z_{n+k+1}^2| < |z_{n+k}|^s$ for any $k > N_0$, i.e.

$$\left( \log R\frac{1}{n+k+1} \right)^\alpha < B\frac{C}{n+k}$$

for any $k > N_0$.

Now, using the relation between the sequences $R_n$ and $S_n$, we have

$$M(S_{n+k}, f) = S_{n+k+1} < \left( \log R\frac{1}{n+k+1} \right)^\alpha < B\frac{C}{n+k}$$

for every $k > N_0$, which is a contradiction as $f$ is a transcendental entire function.

As a consequence, we have the following corollary giving a partial answer to Baker’s question.

**Corollary 2.1:** Suppose $f$ is a transcendental entire function of order $\rho < \frac{1}{2}$, minimal type. Then $F(f)$ has no unbounded component.

**Proof:** As $\rho < \frac{1}{2}$, then by Ref. [16], the conclusion of the Step 1 of the proof of Theorem 2.1 is already satisfied. This gives us that $F(f)$ has no unbounded component.

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