A sharp Bernstein–type inequality and application to the Carleson embedding theorem with matrix weights

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Abstract
We prove a sharp Bernstein-type inequality for complex polynomials which are positive and satisfy a polynomial growth condition on the positive real axis. This leads to an improved upper estimate in the recent work of Culiuc and Treil (Int. Math. Res. Not. 2019: 3301–3312, 2019) on the weighted martingale Carleson embedding theorem with matrix weights. In the scalar case this new upper bound is optimal.

Keywords Carleson embedding theorem · Bernstein-type inequality

Mathematics Subject Classification Primary 42B35 · Secondary 30C10

1 Result

Lemma 1.1 Let $n$ be a positive integer and $p : \mathbb{C} \to \mathbb{C}$ a polynomial such that $p(s) \geq 0$ for all $s \geq 0$ and

$$|p(s)| \leq s^{-1} (1 + s)^n \quad \text{for all} \ s > 0.$$  \hfill (1.1)

Then

$$|p(0)| \leq n^2,$$  \hfill (1.2)

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with equality if

\[ p(s) = p_n(s) := \frac{1}{2} \frac{(s + 1)^n}{s} \left( 1 - T_n \left( \frac{1 - s}{1 + s} \right) \right). \]  \tag{1.3}

Here, \( T_n(x) = \cos(n \arccos x) \) is the \( n \)-th Chebyshev polynomial of the first kind.

The source of motivation for Lemma 1.1 has been the recent work of Culiuc and Treil [1] on the Carleson embedding theorem with matrix weights. In fact, Lemma 2.2 in [1], which they attribute to F. Nazarov and M. Sodin, provides the (weaker) estimate

\[ |p(0)| \leq e^2 n^2 \]  \tag{1.4}

for any polynomial \( p : \mathbb{C} \to \mathbb{C} \) satisfying (1.1). Developing a sophisticated Bellman function technique and making use of estimate (1.4), Culiuc and Treil [1] proved the following result ([1, Theorem 1.2]). We refer to [1] for the relevant terminology and notation.

**Theorem A** (Carleson embedding theorem for matrix weights) Let \( W \) be a \( d \times d \) matrix–valued measure and let \( A_I, I \in \mathcal{D} \) be a sequence of positive semidefinite \( d \times d \) matrices. Then the following are equivalent:

(i) \[ \sum_{I \in \mathcal{D}} \left\| A_I^{1/2} (W^{1/2} f)_I \right\|^2 |I| \leq A \| f \|^2_{L^2}. \]

(ii) \[ \sum_{I \in \mathcal{D}} \left\| A_I^{1/2} \langle W f \rangle_I \right\|^2 |I| \leq A \| f \|^2_{L^2}. \]

(iii) \[ \frac{1}{|I_0|} \sum_{I \in \mathcal{D}, I \subset I_0} \langle W \rangle_I A_I \langle W \rangle_I |I| \leq B \langle W \rangle_{I_0} \text{ for all } I_0 \in \mathcal{D}. \]

Moreover, the best constants \( A \) and \( B \) satisfy \( B \leq A \leq CB \), where \( C = C(d) = 4e^2 d^2 \).

In fact, the proof of Theorem A in [1] requires the estimate (1.4) only for polynomials \( p : \mathbb{C} \to \mathbb{C} \) with degree \( n = 2d \), which satisfy (1.1) and are real and positive on the positive real axis. Therefore Lemma 1.1 implies that one can take

\[ C(d) = 4d^2 \]

instead of \( C(d) = 4e^2 d^2 \) in Theorem A. In the scalar case \( (d = 1) \) this new upper bound produces the upper estimate \( A \leq 4B \), which is known to be optimal [4, Theorem 3.3].

**Remark 1** The method we use for the proof of Lemma 1.1 can also be used to improve the bound (1.4) given by [1, Lemma 2.2], which holds for any polynomial \( p : \mathbb{C} \to \mathbb{C} \) satisfying (1.1). This leads to

\[ |p(0)| \leq 2n^2 - n, \]  \tag{1.5}

see the next section for the proof. The estimate (1.5) is presumably not best possible.
2 Proofs

The idea is to view both estimates, (1.2) and (1.5), as Bernstein–type estimates. Recall that for a polynomial $h$ of degree $N$ the classical Bernstein inequality says that

$$\max_{|z|=1} |h'(z)| \leq N \cdot \max_{|z|=1} |h(z)|.$$ 

Proof of Lemma 1.1 By assumption, $p : \mathbb{C} \rightarrow \mathbb{C}$ is a polynomial satisfying (1.1) and $p(s) \geq 0$ for all $s \geq 0$. Then $q(z) := zp(z)$ is polynomial of degree at most $n$ with $q(0) = 0$, $p(0) = q'(0)$, and $q(s) \geq 0$ for all $s \geq 0$. We define the auxiliary function

$$f(z) := \frac{(1+z)^{2n}}{(4z)^n} q \left( -\left( \frac{1-z}{1+z} \right)^2 \right) = \sum_{k=-n}^{n} a_k z^k,$$

a Laurent polynomial of degree $\leq n$. It is not difficult to see that the growth condition (1.1) for $p$ implies the uniform bound

$$|f(z)| \leq 1 \quad \text{for all } |z| = 1.$$

We also note that

$$p(0) = q'(0) = -2f''(1),$$

so our task is to find the best upper bound for $|f''(1)|$.

In order to find such an estimate, it turns out to be essential that the auxiliary function $f$ is real and positive (i.e., $\geq 0$) on $|z| = 1$. To see this just note that

$$k(z) = \frac{z}{(1+z)^2} = \frac{1}{4} \left( 1 - \left( \frac{1-z}{1+z} \right)^2 \right)$$

is the Koebe function, familiar from the classical theory of univalent functions, which maps the unit circle $|z| = 1$ onto the half–line $[1/4, +\infty)$. Hence, on $|z| = 1$, $f(z)$ is the product of two real and positive functions.

We are thus in a position to apply the Fejér–Riesz theorem [2] for the Laurent polynomial $f$. This gives us a complex polynomial $P$ of degree $\leq n$ with no zeros in $|z| < 1$ such that

$$f(z) = P(z) \overline{P(1/z)}, \quad z \in \mathbb{C} \setminus \{0\}.$$

Clearly, $|P(z)| \leq 1$ for all $|z| = 1$. We can therefore apply a sharpening of Bernstein’s inequality due to P. Lax [3] (confirming an earlier conjecture of Erdös) which asserts that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \cdot \max_{|z|=1} |P(z)| \leq \frac{n}{2}.$$
In particular,

\[|p(0)| = |q'(0)| = 2|f''(1)| = 4|P'(1)|^2 \leq n^2,\]

proving (1.2). Clearly, the polynomial \( P_n(z) = (z^n - 1)/2 \) has the property \(|P'_n(1)| = n/2\), so \(|f''_n(1)| = n^2/2\) for \( f_n(z) := P_n(z)P_n(1/z) \). It is easy to see that

\[f_n(z) = \left(1 + z\right)^{2n} - q_n \left(-\left(\frac{1 - z}{1 + z}\right)^2\right)\]

for a polynomial \( q_n \) of degree at most \( n \) with \( q_n(0) = 0 \), and it is straightforward to check that \( p_n(z) := q_n(z)/z \) has the form (1.3).

\[\square\]

**Proof of (1.5)** By assumption, \( p : \mathbb{C} \to \mathbb{C} \) is a polynomial satisfying (1.1). Then \( q(z) := zp(z) \) is polynomial of degree at most \( n \) with \( q(0) = 0 \) and \( p(0) = q'(0) \). We define, closely following the proof of [1, Lemma 2.2], the auxiliary function

\[g(z) := \frac{(1 + z)^{2n}}{4^n} - q \left(-\left(\frac{1 - z}{1 + z}\right)^2\right),\]

a polynomial of degree \( N \leq 2n \). As before, the polynomial \( g \) has the property that

\[|g(z)| \leq 1 \quad \text{for all } |z| = 1.\]

Now note that

\[p(0) = -2g''(1).\]

Hence, we could apply the classical Bernstein inequality twice, first for \( g' \) and then for \( g'' \), but this would result in

\[|p(0)| = 2|g''(1)| \leq 2N(N - 1) \leq 4n(2n - 1),\]

which is not particularly good. However, as observed in [1, Proof of Lemma 2.2] we can assume without loss of generality that \( g \) has no zeros in \(|z| < 1\). We can therefore apply as above the inequality of Lax which leads to

\[\max_{|z|=1} |g'(z)| \leq \frac{N}{2} \cdot \max_{|z|=1} |g(z)| \leq n.\]

This brings us in a position to apply Corollary 14.2.8 in [5] for the polynomial \( g' \) which has degree \( \leq 2n - 1 \). Hence

\[|g''(z)| + |(2n - 1)g'(z) - zg''(z)| \leq n(2n - 1), \quad |z| \leq 1.\]

Taking \( z = 1 \) and noting that \( g'(1) = nq(0) = 0 \), gives \( 2|g''(1)| \leq n(2n - 1) \), as required. \(\square\)
3 Remarks

The polynomials $p$ which occur in the proof of Theorem A in [1] are of the form

$$p(s) = \sum_{I \in D} p_I(s)|I|,$$

with $p_I(s) \geq 0$ for all $s \geq 0$ and each $p_I$ a polynomial of degree at most $2(d - 1)$. The extremal polynomial $p_{2d}$ in Lemma 1.1 has degree $2(d - 1)$ and all its $2(d - 1)$ zeros are on the positive real axis and are double zeros. This implies that

$$p(s) = p_{2d}(s) \iff \forall I \in D \exists c(I) \geq 0 p_I|I| = c(I)p_{2d}.$$

Hence the extremal polynomial $p_{2d}$ of Lemma 1.1 shows up in the proof of Theorem A only if each $p_I$ is a multiple of $p_{2d}$.

After acceptance of the paper the authors found another short proof of Lemma 1.1 based on Markov’s inequality [5, Theorem 15.1.4] which allows to identify all extremal polynomials. In fact, using the change of variables $s = (1 - x) / (1 + x)$ we have

$$q(x) := 1 - 2^{1-n}(1 + x)^{n-1}(1 - x)p\left(\frac{1-x}{1+x}\right) = 1 - 2\frac{sp(s)}{(1+s)^n}, \quad x \in (-1, 1).$$

By assumptions, $q$ is a polynomial of degree at most $n$ such that $q(1) = 1, q'(1) = p(0)$ and $|q(x)| \leq 1$ for all $x \in [-1, 1]$. By Markov’s inequality, $|p(0)| = |q'(1)| \leq n^2$ with equality if and only if $q(x) = T_n(x)$. This proves (1.2) with equality if and only if $p = p_n$ as in (1.3).

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