Classifying semi-free Hamiltonian $S^1$-manifolds

Eduardo González
Mathematics Department, UMASS Boston.

Correspondence to be sent to: eduardo@math.umb.edu

In this paper we describe a method to establish when a symplectic manifold $M$ with semi-free Hamiltonian $S^1$-action is unique up to isomorphism (equivariant symplectomorphism). This will rely on a study of the symplectic topology of the reduced spaces. We prove that if the reduced spaces satisfy a rigidity condition, then the manifold $M$ is uniquely determined by fixed point data. In particular we can prove that there is a unique family up to isomorphism of 6-dimensional symplectic manifolds with semi-free $S^1$-action and isolated fixed points.

1 Introduction.

Let $(M, \omega)$ be a $2n$ dimensional compact, connected, symplectic manifold, and let $\{\Lambda_{i}\} = \Lambda : S^1 \rightarrow \text{Symp}(M, \omega)$ denote a symplectic circle action on $M$, that is, if $X$ is the vector field generating the action, then $L_X \omega = \omega$ and $\lambda^X = 0$. Recall that the action is semi-free if it is free on $M \setminus M^{S^1}$, i.e. the only weights at every fixed point are $\pm 1, 0$. A circle action is said to be Hamiltonian if there is a $C^\infty$ function $H : M \rightarrow \mathbb{R}$ such that $i_X \omega = -dH$. Such a function is called a Hamiltonian for the action. $H$ is unique if we normalize it by requiring its minimum to be 0. $H$ is a perfect Morse-Bott function and its critical points are the fixed points of $\Lambda$. Let $C(M)$ denote the critical values of $H$. This set is determined by the cohomology class of $\omega$ (cf. Lemma 2.1). We will denote by $\text{HSymp}_{2n}$ the class of manifolds $(M, H, \omega)$ with semi-free Hamiltonian circle actions with normalized Hamiltonians.

In this paper we will provide a mechanism that classifies, under restricted conditions, Hamiltonian $S^1$-manifolds $(M, H, \omega)$ up to isomorphism, i.e. equivariant symplectomorphism. The idea is to investigate how to reconstruct $M$ from its local data $\mathcal{L}(M)$ and what type of information determines the local data. $\mathcal{L}(M)$ can be thought as an atlas for Hamiltonian $S^1$-actions. It will be, roughly speaking, given by neighborhoods of the critical levels of $H$ called critical germs and open submanifolds of regular points called slices. Precise definitions can be found in section 3.2. Our first result can be stated roughly as follows (see Theorem 2.6).

Theorem 1.1. The local data $\mathcal{L}(M)$ determines the Hamiltonian $S^1$-manifold $(M, H, \omega)$ up to isomorphism.

Although we carry out the proof of this theorem in the semi-free case, it holds without this assumption. Our second main objective is to describe the local data in terms of more intrinsic information from the critical levels. To accomplish that, we have to make certain concessions. First, to describe the germs we restrict to the case when all non-extremal critical levels contain fixed-point components of Morse (co)index at most 2. In this case, as shown in section 3.3.1, all the reduced spaces $\overline{M_\lambda}$ at critical values $\lambda$ have a natural structure of a smooth symplectic manifold $(\overline{M_\lambda}, \overline{\omega})$. For clarity we will further assume that the critical levels are simple (see Section 3.3.1 for the general case), this means that all the fixed point components in the same level have a common index $i_\lambda$. Under this assumption, the diffeomorphism type of the reduced spaces $\overline{M_t}$ is constant on the semi-closed interval $(\lambda - \epsilon, \lambda]$. In general, even in the non-simple case, the diffeomorphism type of the reduction bundle $P_t := H^{-1}(t) \rightarrow H^{-1}(t)/S^1 = \overline{M_t}$ is constant for $t$ in any interval of regular values of $H$. We denote by $P^+_{t, \lambda}$ and $P^-_{t, \lambda}$ the diffeomorphism types for regular values immediately below and above $\lambda$, respectively. We let $e(P^\pm_{t, \lambda}) \in H^2(\overline{M_\lambda})$ denote the Euler class of $P^\pm_{t, \lambda}$.

Before we describe the isomorphism type of the germs we need the following definition.

Definition 1.2. Let $(M, H, \omega)$ denote a Hamiltonian $S^1$-manifold. Suppose that all the non-extremal critical levels are simple and of index 2. If $\lambda$ denote a non-extremal critical value of $H$, the fixed point data of $M$ at $\lambda$ consists of the following.

Received 1 January 2009; Revised 11 January 2009; Accepted 21 January 2009

© The Author 2009. Published by Oxford University Press. All rights reserved. For permissions, please e-mail: journals.permissions@oupjournals.org.
• The critical value $\lambda$ and its index $i_{\lambda}$.
• For each fixed component $F$ at $\lambda$, the tuple $(\overline{M}_{\lambda}, \overline{\mathcal{F}}_{\lambda})$ of the reduced space and the image of the fixed point component embedded as a symplectic submanifold. (This information contains the normal bundle $(N_{\overline{\mathcal{F}}_{\lambda}})$.)
• The bundle $P_{\lambda}$.

A similar definition applies if $\lambda$ has coindex 2. For extremal fixed point components, let $e_{F}$ denote either max or min. One has that $\overline{M}_{\lambda = \lambda_{\text{ext}}} = \overline{\mathcal{F}}_{\lambda_{\text{ext}}}$. Then, we take the fixed point data as

• The critical value $\lambda_{\text{ext}}$.
• The symplectic manifold $(\overline{\mathcal{F}}_{\lambda_{\text{ext}}}, \overline{\omega}_{\lambda_{\text{ext}}})$.
• The symplectic normal bundle $N_{F_{\lambda_{\text{ext}}}}/M$ of $F_{\lambda_{\text{ext}}}$ in $M$.

In this situation the index $i_{\lambda_{\text{ext}}}$ can be recovered by the normal bundle $N_{F_{\lambda_{\text{ext}}}}/M$, and thus it is not necessary. The **fixed point data** of $M$ consists of the collection of all of the above tuples for $\lambda \in \mathcal{C}(M)$.

It is important to recall that in general the fixed point set at non-extremal critical levels might be disconnected and in the non-simple case they might have different Morse indexes. The extremal fixed point sets are always connected.

It is already stated in Guillemin-Sternberg’s [9] that the isomorphism type of the critical germs is determined by the fixed point data. As we will see, under further assumptions, we can classify manifolds knowing less information in the fixed point data. As follows.

**Definition 1.3.** The **small fixed point data** of $M$ consist of the following information at each $\lambda$.

• If $\lambda$ is the minimum, the same information as above.
• If $\lambda$ is non-extremal, then for each fixed point component $F$ at $\lambda$ we consider the following data.
  - If $\dim \overline{\mathcal{F}} > 0$, then we keep the same information above but excluding $e(P_{\lambda})$.
  - If $\dim \overline{\mathcal{F}} = 0$ then we only consider $\lambda$ and its index $i_{\lambda}$.
• If $\lambda$ is the maximum, then we only take $\lambda$ and the symplectic manifold $(\overline{\mathcal{F}}_{\lambda_{\text{max}}}, \overline{\omega}_{\lambda_{\text{max}}})$.

To understand the regular slices, we first recall the content of Duistermaat-Heckman theorem. On an interval $I = [t_0, t_1]$ of regular values of $H$, all the reduced bundles $P_t \to \overline{M}_t, t \in I$ are diffeomorphic, say to a fixed one $P \to B$. Moreover the symplectic structure $\omega$ on $M$ yields a family $\{\overline{\omega}_t\}_{t \in I}$ of reduced symplectic forms on $B$. The pair $(B, \{\overline{\omega}_t\})$ and the bundle $P \to B$ determines the isomorphism class of the regular slice $H^{-1}(I)$. Theorem 3.5 extends this result, showing which families of symplectic forms give isomorphic regular slices in the case when one has the following property.

**Definition 1.4.** Let $B$ be a manifold and $\{\overline{\omega}_t\}$ be a family of symplectic structures on $B$ smoothly parametrised by $t \in I$. The pair $(B, \{\overline{\omega}_t\})$ is said to be *rigid* if

1. $\text{Symp}(B, \overline{\omega}_t) \cap \text{Diff}(B)$ is path connected for all $t \in I$.
2. Any deformation between any two cohomologous symplectic forms which are deformation equivalent to $\omega_{t_0}$ on $B$ may be homotoped through deformations with fixed endpoints into an isotopy.

Rigidity can be understood as the symplectic analogue of complex rigidity, in the sense that even if we deform the symplectic forms, we would obtain isotopic forms. Property 1 is in fact non-generic, and Seidel [23] has shown it is not satisfied for the monotone blow up $\mathbb{CP}^2 \# 5 \mathbb{CP}^2$.

Once we have understood the regular slices and the germs, we then glue them near the critical levels, a construction that is already used by Hui Li [14]. Assuming all the reduced spaces are rigid, the resulting manifold is independent of the gluing information and we prove the following theorem.

**Theorem 1.5** (Weak classification Theorem). Let $(M, H, \omega) \in \text{HSymp}_{2n}$. Suppose that all critical levels at $\lambda \in \mathcal{C}(M)$ are simple. Suppose further that for any two consecutive $\lambda', \lambda \in \mathcal{C}(M), \lambda < \lambda'$, the pair $(\overline{M}_{\lambda}, \{\overline{\omega}_t\}_{t \in (\lambda', \lambda)})$ is rigid. Then $(M, H, \omega)$ is determined by its fixed point data up to equivariant symplectomorphism.

In the case when $\dim M = 6$ all the non-extremal fixed-point components have (co)index 2. Following the notation of Karshon-Tolman [11], this corresponds to complexity two (half-dimension of the reduced spaces) Hamiltonian symplectic manifolds. Therefore we can use established results on the symplectic structures and on the topology of the group of symplectomorphisms. When the fixed point components are simple enough it is possible to describe some of the fixed point data at each level in terms of fixed point data associated to the previous critical level. Therefore the isomorphism type only depends on the small fixed point data of Definition 1.3. We have got the following result.
Theorem 1.6. Let \((M, H, \omega) \in \text{HSymp}_n\). Suppose that all critical levels at \(\lambda \in C(M)\) are simple and that for any two consecutive \(\lambda', \lambda \in C(M)\), \(\lambda' < \lambda\), the pair \((M, \{\lambda_i\}_{i \in (\lambda, \lambda')}\) is rigid. If the fixed point sets \((\overline{F}_\lambda, \overline{\omega})\) are either surfaces or isolated fixed points, then the isomorphism class of \((M, H, \omega)\) is uniquely determined by the small fixed point data.

In [13] Hui Li analysed manifolds using even weaker fixed point data, for instance, the normal bundle of all the fixed point sets is not considered. Her information is in spirit purely topological and in that case she has constructed manifolds with the same fixed point data (in her sense) but not diffeomorphic, she calls this phenomena a twist. Theorem 1.6 shows that provided all reduced spaces are rigid and we know the fixed point data as in Definition 1.2, such twist cannot exist.

We will push our method even further by classifying a family of manifolds with minimal fixed point information. Let \(Y^n = S^2 \times \cdots \times S^2\) be the \(n\)-fold product of spheres and let \(\sigma\) be the canonical area form on \(S^2\). Provide \(Y^n\) with the product symplectic form \(\lambda_1 \sigma \times \cdots \times \lambda_n \sigma\) that takes the value \(\lambda_i > 0\) on each of the spheres of \(Y^n\). Let the circle act diagonally on \(Y^n\) in the standard semi-free Hamiltonian fashion. \(Y^n\) is the only known example of a 2\(n\)-dimensional symplectic manifold that admits a semi-free circle action with isolated fixed points. Thus, it is natural to ask if this is the only manifold up to equivariant symplectomorphism that has this property. In the case \(n = 2\) the methods of Karshon [10] answer this question positively. In the present paper we establish the result for \(n = 3\).

Theorem 1.7. Let \(M\) be a 6-dimensional symplectic manifold with a semi-free circle action that has isolated fixed points. Then, \(M\) is equivariantly symplectomorphic to \(Y^3\) with the canonical product form for some \(\lambda_i\).

Plenty of information was known about these manifolds. In [24] it is proved that under the hypothesis of Theorem 1.7 the action must be Hamiltonian. The \(\lambda_i\) are in fact the critical values of the Hamiltonian function of the (only) three fixed points of index 2, assuming the minimum to be zero. Moreover, there is an isomorphism of equivariant cohomology rings \(H^2_S(M) \cong H^2_S(Y^3)\). The isomorphism is such that takes Chern classes into Chern classes, thus by Wall [25] is diffeomorphic to \(Y^3\). More recently and in higher dimensions, a result of Ilinski reproved by Matsuda and Panov [13] Corollary 4.9 shows that if \(M\) is known to be a 2\(n\) dimensional toric space with a circle subgroup acting semi-freely and with isolated fixed points, then \(M\) must be diffeomorphic to \(Y^n\). Nothing was known about the symplectomorphic type of \(M\). An approach to understand its symplectomorphic type was given in [7], where we proved that the quantum cohomology ring of \(M\) and \(Y^3\) are isomorphic.

The proof of Theorems 1.5 and 1.7 will be done in stages. We will show that one can construct the desired isomorphism starting from the minimum and extending it by gluing regular slices and germs. As we mention above, this local information only depends on the fixed point data and the symplectomorphic type of the reduced manifolds, which in this very particular example, are \(\mathbb{C}P^2\) and its blow-ups, whose symplectic topology is well known. As we will see, since the fixed components are isolated points, the local data at a critical level will depend only on the data of the previous one. Thus we can "bootstrap" the construction to extend the isomorphism by attaching a symplectic manifold whose symplectomorphic type is determined by previous information. For instance, one knows that after passing a critical level, the reduced manifolds are related by a blow-up and blow-down, and thus the resulting manifold after passing a critical level is determined by the previous data.

The results in [24, 7] work in all dimensions. However, we do not expect the techniques presented in the current paper to extend easily to higher dimensions. This is because our rigidity arguments strongly rely on results in four dimensional symplectic geometry, which as of now, do not have higher dimensional versions. We finally note that the proof of Theorem 1.7 does not use Wall’s theorem, in contrast to previous methods.

As a last remark, we point out that it is possible to remove the semi-free assumption from our results. In this case one has to deal with generalizations of our tools to the orbifold category, as the work of L. Godinho [2, 6] and W. Chen [3]. This has already been explored by the recent work of McDuff [20] in some particular case.

Acknowledgments: I thank Dusa McDuff and Chris Woodward for encouragement and support during the preparation and revision of this paper. I also owe a special debt of gratitude to Sue Tolman for her illuminating objections and in-depth revision of the first version of this paper. Finally, I thank Hui Li and Martin Pinsonnault for discussions on an early draft and to the anonymous Referee for pointing out many inconsistencies in the manuscript.

2 General Setting.

In this section we will be using the term Hamiltonian \(S^1\)-manifold for triples \((M, H, \omega)\), where \(M\) is a closed, smooth, connected 2\(n\)-manifold, \(\omega\) a symplectic form on \(M\) and \(H\) is a normalized Hamiltonian function on \(M\).
that generates a circle action compatible with \( \omega \) on \( M \). We will assume the action to be semi-free, just to be consistent with the main objective of the paper. For this section this hypothesis can be removed. Although we will be working with general, not necessary closed manifolds, we will always think of them as (isomorphic to) submanifolds of a closed one, say \( M \). Therefore, we sometimes will not make explicit the presence of the symplectic form. We will use the notation \( \text{HamSym}_{2n} \) referring to the class of Hamiltonian \( S^1 \)-manifolds, closed or not, up to isomorphism. Here \((M, H, \omega)\) and \((M', H', \omega')\) are isomorphic if there exist an equivariant diffeomorphism \( f : M \to M' \) such that \( f^*(\omega') = \omega \). Note that \( f \) is equivariant if and only if \( H \circ f = H' \).

Let \((M, L, \omega)\) be a closed Hamiltonian \( S^1 \)-manifold and denote by

\[ C(M) = \{0 = \lambda_0 < \cdots < \lambda_n\} \]

the collection of critical values of \( L \). Thus \( L(M) = [\lambda_0, \lambda_n] \).

**Lemma 2.1.** The set \( C(M) \) is invariant under isomorphism and is determined by the cohomology class of the symplectic structure.

**Proof.** The first statement is obvious. For the second, choose any invariant metric in \( M \). Let \( p \in M^\ast \) a fixed point and take any gradient line of \( L \) joining \( p \) to a point \( q \) in the minimum. Let \( S \) denote the 2-cycle (sphere) obtained by rotating the gradient line by the circle action. By the Hamiltonian assumption its symplectic area \( \int_S \omega = L(p) - L(q) = L(p) \).

This determines \( L(p) \) purely on cohomological data. For more details the reader can consult [2].

We now describe how to get pieces of \( M \) by localising to critical levels using the Hamiltonian. This is, for each \( \varepsilon > 0 \) consider the neighborhood \( L^{-1}(\lambda - \varepsilon, \lambda + \varepsilon) \) of the level set \( L^{-1}(\lambda) \), and for two consecutive values \( \lambda, \lambda' \in C(M) \) take the open submanifold \( L^{-1}(\lambda, \lambda') \). In this paper we are interested on knowing what is needed to reconstruct \( M \) if one just knows the isomorphism type of the pieces \( L^{-1}(\lambda - \varepsilon, \lambda + \varepsilon) \) and \( L^{-1}(\lambda, \lambda') \). In general one would need to specify how to glue these pieces to get back to \( M \). To see what type of gluing maps are allowed, suppose that \((Y, H), (Z, K)\) are manifolds isomorphic to the pieces \( L^{-1}(\lambda - \varepsilon, \lambda + \varepsilon) \) and \( L^{-1}(\lambda, \lambda') \) respectively. Note that one is tempted to glue \( Y \) and \( Z \) along the overlap, that is we want to identify \( H^{-1}(\lambda, \lambda + \varepsilon) \) and \( K^{-1}(\lambda, \lambda + \varepsilon) \) through isomorphisms, to obtain back a manifold isomorphic to \( L^{-1}(\lambda - \varepsilon, \lambda') \) such that \( Y \) and \( K \) are symplectic submanifolds. Therefore one needs to consider a maximal set of gluing maps with this property. One may think of these data as a symplectic atlas of compatible Hamiltonian charts for \( M \). We now provide the precise formalism that allows this to work.

**Definition 2.2.** Let \( \lambda \in \mathbb{R} \) and let \( \varepsilon_0 > 0 \).

(i) A **cobordism** at \( \lambda \) is a tuple \((Y, H, \epsilon)\) such that \( 0 < \epsilon < \varepsilon_0 \) and \( Y \) is a Hamiltonian \( S^1 \)-manifold whose Hamiltonian function \( H \) takes \( Y \) onto \( I_\epsilon = (\lambda - \varepsilon, \lambda + \varepsilon) \) and \( \lambda \) is the only critical value of \( H \). Moreover we require that if \( \epsilon' < \epsilon \) the restriction \((H^{-1}(I_{\epsilon'}), H, \epsilon')\) is identified with \((Y, H, \epsilon')\). Two cobordisms are equivalent, \((Y, H, \epsilon) \sim (Y', H', \epsilon')\), if and only if there is \( \epsilon' < \min(\epsilon, \epsilon') \) such that \((Y, H, \epsilon'')\) and \((Y', H', \epsilon'')\) are isomorphic. That is, there is an isomorphism \( f : H^{-1}(I_{\epsilon''}) \to H'^{-1}(I_{\epsilon''}) \).

A **critical germ** \( G(\lambda, \varepsilon_0) \) at \( \lambda \) is an equivalence class in \( \text{HamSym}_{2n} \) of such tuples.

(ii) Similarly, consider tuples \((Y, H, \epsilon)\) as above where the only critical value of \( H \) is its minimum (maximum) value \( \lambda \). Thus \( H(Y) = (\lambda, \lambda + \varepsilon) \) \((H(Y) = (\lambda - \epsilon, \lambda))\); we have got a similar equivalence relation between them as before. An equivalence class \( m(\lambda, \varepsilon)(M(\lambda, \varepsilon)) \) is called a **minimal (maximal) germ** at \( \lambda \).

We will often refer to critical, maximal or minimal germs just as germs. Note that if \( \delta < \varepsilon \), there is a natural restriction map \( G(\lambda, \varepsilon) \to G(\lambda, \delta) \). The triples \((Y, H, \epsilon)\) as in Definition 2.2(ii) are neighborhoods of the maximal and minimal sets. To see this, first note that there is a unique maximum component \( F_{\text{max}} \) of the fixed point set \( \mathbb{R} \). Then by using an equivariant version of the Darboux Theorem applied to points in \( F_{\text{max}} \), there is a triple of the form \((Y, H, \epsilon)\) for \( \epsilon \) small enough. Moreover, its maximal germ is determined uniquely by the symplectomorphism type of \( F_{\text{max}} \) and its normal bundle. A similar remark applies to the minimum.
**Definition 2.3.** Let $I$ be an open interval. A regular slice is a tuple $(Z, K, I, \omega)$ where $K : Z \rightarrow I$ is a surjective moment map for a free $S^1$-action on the symplectic manifold $(Z, \omega)$. We say that two slices $(Z, K, I, \omega) \sim (Z', K', I, \omega')$ are equivalent if they are isomorphic. We denote by $F(I)$ an equivalence class of such slices.

**Definition 2.4.** Let $G(\lambda, \varepsilon)$ be a germ at $\lambda$ and let $F(I)$ be a class of regular slices for $I = (\lambda', \lambda)$. Let $(Y, H, \varepsilon) \in G(\lambda, \varepsilon)$ and $(Z, K, I, \omega) \in F(I)$. A gluing map $(\phi, \varepsilon) : (Y, H, \varepsilon) \rightarrow (Z, K, I, \omega)$ is given by a pair $(\phi, \varepsilon)$ where $0 < \varepsilon < \varepsilon'$ and $\phi$ is an isomorphism
\[ H^{-1}(\lambda - \varepsilon, \lambda) \xrightarrow{\phi} K^{-1}(\lambda - \varepsilon, \lambda). \]

Two gluing maps $(\phi, \varepsilon), (\phi', \varepsilon')$ are equivalent if there are $\varepsilon'' < \min(\varepsilon, \varepsilon')$ and isomorphisms $f : Y \rightarrow Y', g : Z \rightarrow Z'$ such that the following diagram is commutative.
\[
\begin{array}{ccc}
H^{-1}(\lambda - \varepsilon', \lambda) & \xrightarrow{\phi} & K^{-1}(\lambda - \varepsilon', \lambda) \\
\downarrow f & & \downarrow g \\
H^{-1}(\lambda - \varepsilon'', \lambda) & \xrightarrow{\phi'} & K^{-1}(\lambda - \varepsilon'', \lambda)
\end{array}
\]

A gluing class $\Phi : G(\lambda, \varepsilon) \rightarrow F(I)$ is an equivalence class of pairs $(\phi, \varepsilon)$. This maps are well defined by Diagram (I). Analogously one can define gluing maps for $F(I)$ and the germ $G(\lambda, \varepsilon)$ when $I = (\lambda, \lambda')$. Note that this definition also applies if we substitute the germ $G(\lambda, \varepsilon)$ by a maximal or a minimal one.

Recall that we are interested in building symplectic manifolds out of germs and regular slices. We now describe the simplest case where we can do that. Suppose we have got $G(\lambda, \varepsilon)$ and $F(\lambda, \lambda')$ a germ and a class of regular slices, we want to see how to obtain a new manifold via a gluing class $\Phi : G(\lambda, \varepsilon) \rightarrow F(\lambda, \lambda')$. Choose representatives $(Y, H, \varepsilon) \in G(\lambda, \varepsilon), (Z, K, I, \omega) \in F(I)$ and $(\phi, \varepsilon) \in \Phi$. Then, consider the manifold
\[ Y \cup_{(\phi, \varepsilon)} Z \]
obtained by gluing $Y$ and $Z$ along the overlap $(\lambda, \lambda + \varepsilon)$, that is
\[ Y \sqcup Z/ \sim \text{ where } x \sim y \iff \phi(x) = y \]
where
\[ \phi : H^{-1}(\lambda, \lambda + \varepsilon) \rightarrow K^{-1}(\lambda, \lambda + \varepsilon) \]
is the restricted isomorphism on the interval $(\lambda, \lambda + \varepsilon)$.

If $(\phi', \varepsilon'), (Y', H', \varepsilon'), (Z', K', I, \omega)$ is another set of choices, there exist $0 < \varepsilon'' < \min(\varepsilon, \varepsilon') < \varepsilon$ and isomorphisms $f, g$ as in the commutative Diagram (I). Therefore, by restricting the gluing maps and $f, g$ to the open interval $(\lambda, \lambda + \varepsilon'')$ one gets that $\phi(x) = y$ if and only if $\phi'(f(x)) = g(y)$. Thus $(f, g)$ induces an isomorphism
\[ Y \cup_{(\phi, \varepsilon)} Z \xrightarrow{\cong} Y' \cup_{(\phi', \varepsilon)} Z'. \]

Denote by
\[ G(\lambda, \varepsilon) \cup_{\Phi} F(I) \]
the isomorphism class produced by this gluing. Note that the Hamiltonian function on this new manifold is the one defined by $(H, K) : Y \sqcup Z \rightarrow (\lambda - \varepsilon, \lambda')$ after passing to the quotient. Therefore this is a well defined operation in $\mathbb{H}\text{Symp}_{2n}$.

Conversely, the neighborhoods $(H, K)^{-1}(\lambda - \varepsilon, \lambda + \varepsilon) \subseteq Y \cup_{(\phi, \varepsilon)} Z$ are isomorphic to $H^{-1}(\lambda - \varepsilon, \lambda + \varepsilon)$ for all $\varepsilon$. Similarly $(H, K)^{-1}(\lambda, \lambda') \cong K^{-1}(\lambda, \lambda')$. Then we have got the following lemma.

**Lemma 2.5.** Suppose we have given a germ of cobordism $G(\lambda, \varepsilon), a regular slice $F(\lambda, \lambda')$ and a gluing class $\Psi : G(\lambda, \varepsilon) \rightarrow F(\lambda, \lambda')$. Then we can associate a unique isomorphism class
\[ G(\lambda, \varepsilon) \cup_{\Psi} F(\lambda, \lambda') \] with Hamiltonian $(H, K)$ in $\mathbb{H}\text{Symp}_{2n}$. Moreover, the manifolds $(H, K)^{-1}(I)$ and $(H, K)^{-1}(\lambda, \lambda')$ represent the classes $G(\lambda, \varepsilon)$ and $F(\lambda, \lambda')$.\[ \square \]
It is clear that we can apply the same idea to define a gluing of the form \( F(\lambda', \lambda) \cup \Phi G(\lambda, \varepsilon) \). Similarly, when we have got maximal and minimal germs \( M(\lambda, \varepsilon), m(\lambda', \varepsilon) \) one can construct manifolds

\[
m(\lambda', \varepsilon') \cup \Phi F(\lambda', \lambda) \text{ and } F(\lambda', \lambda) \cup \Phi M(\lambda, \varepsilon).
\]

It is important to note that order of the gluing does not matter provided \( \varepsilon \) is small enough.

To reconstruct \( M \), we would like to define this process more generally. We want to glue more general data, as we now explain. **A set of local data** \( \mathcal{L} \) consists of:

- A collection \( \mathcal{C} = \{0 = \lambda_0 < \cdots < \lambda_{s+1}\} \) of critical levels.
- Germs \( G(\lambda_i, \varepsilon_i) \) at \( \lambda_i \) for all \( i = 1, \ldots, s \), minimal and maximal germs \( m(\lambda_0, \varepsilon_0), M(\lambda_{s+1}, \varepsilon_{s+1}) \) respectively.
- For all \( j = 0, \ldots, s \), equivalence classes of regular slices \( F(I_j) \) where \( I_j = (\lambda_j, \lambda_{j+1}) \) is a maximal open interval of regular values.
- Gluing classes \( \Phi_i^+, \Phi_i^- \) from \( G(\lambda_i, \varepsilon_i) \) to \( F(I_i) \) and \( F(I_{i-1}) \) respectively for all \( i = 1, \ldots, s \).

As an example consider \( (M, \omega, H) \in \text{H\text{Symp}}_{2n} \). Its localisations

\[
H^{-1}(\lambda - \varepsilon, \lambda + \varepsilon), \text{ and the free sets } H^{-1}(\lambda, \lambda')
\]

for \( \varepsilon > 0 \) and \( \lambda, \lambda' \in \mathcal{C}(M) \) define the germs and regular slices. Then the **local data associated to** \( M \), denoted by \( \mathcal{L}_M \), is the collection of all the isomorphism classes of these sets and the gluing classes on the overlaps. Now we have got the following theorem.

**Theorem 2.6.** Given a set of local data \( \mathcal{L} \), there exists a closed Hamiltonian \( S^1 \)-manifold \( M_{\mathcal{L}} \) such that its associated set of local data is \( \mathcal{L} \). Moreover, this manifold is unique up to isomorphism, this is we have got a one-to-one association

\[
\{ \text{Sets of local data} \} \longrightarrow \text{H\text{Symp}}_{2n}.
\]

**Proof.** Consider representatives

\[
(Y_{s+1}, H_{s+1}, \varepsilon_{s+1}) \in M(\lambda_{s+1}, \varepsilon_{s+1}), \quad (Y_0, H_0, \varepsilon_0) \in m(\lambda_0, \varepsilon_0)
\]

\[
((Y_i, H_i, \varepsilon_i) \in G(\lambda_i, \varepsilon_i), \varepsilon_{s+1}), \quad (Z_i, K_i, I_i) \in F(I_i)
\]

and

\[
(\phi_i^+, \varepsilon) \in \Phi^+.
\]

On the disjoint union

\[
M_{\mathcal{L}} = Y_0 \cup Z_1 \cup Y_1 \cdots \cup Z_{s+1} \cup Y_{s+1}
\]

define the equivalence relation by

\[
y \sim_{\mathcal{L}} z \iff \text{for some } j, (y \in Y_j, z \in Z_{j+1}, \phi_j^-(y) = z) \text{ or } (y \in Y_j, z \in Z_j, \phi_j^+(y) = z)
\]

Define \( M_{\mathcal{L}} := M_{\mathcal{L}}/\sim_{\mathcal{L}} \) with Hamiltonian \( H_{\mathcal{L}} = [H_0, K_1, \ldots, K_{s+1}, H_{s+1}] \). Note that \( H_{\mathcal{L}} : M_{\mathcal{L}} \longrightarrow [\lambda_0, \lambda_{s+1}] \)
and then \( M_{\mathcal{L}} \) is a closed symplectic manifold whose isomorphism class is completely determined by the local data \( \mathcal{L} \), since another set of choices would give an isomorphism as in Equation (2). Finally, to see that the association \( \mathcal{L} \mapsto M_{\mathcal{L}} \) is one-to-one, we proceed as before, by considering the localisations of \( M_{\mathcal{L}} \) at the critical levels with the Hamiltonian \( H_{\mathcal{L}} \). One can see that this recovers \( \mathcal{L} \).

As a last note, we clarify that for the rest of this paper we will treat isomorphic manifolds as equal, unless we specify the contrary.
3 Determining Local Data.

3.1 Basic Notations.

The background material for this section can be found in [21] and [9]. Let \((M, H, \omega) \in \text{HSymp}_{2n}\). Let \(t \in \mathbb{R}\) be a regular value of \(H\). \(S^1\) acts freely on the level set \(H^{-1}(t)\). The orbit space \(\overline{M}_t := H^{-1}(t)/S^1\) is the reduced space of \(M\) at the level \(t\). This space is symplectic with the reduced symplectic form \(\overline{\omega}\). The fibration \(\pi : H^{-1}(t) \to \overline{M}_t\) is a principal \(S^1\) bundle over \(\overline{M}_t\). Denote its total space by \(P_t\) or just by \(P\), whenever there is no risk of confusion. Denote its Euler class by \(e(P) \in H^2(\overline{M}_t)\). The reduced symplectic form \(\overline{\omega}\) on \(\overline{M}_t\) satisfies

\[
\pi^*\overline{\omega} = i_t^*\omega.
\]

We now introduce the precise definitions of common relations in symplectic geometry that we use in this paper. Consider two symplectic forms \(\omega_0, \omega_1\) on a manifold \(X\). These forms are said to be symplectomorphic if there is a diffeomorphism \(f : X \to X\) such that \(f^*\omega_1 = \omega_0\). A deformation between \(\omega_0, \omega_1\) is a (smooth) family \(\{\omega_s\}\) of symplectic forms that join them. A deformation is an isotopy if the elements in the family \(\{\omega_s\}\) all lie in the same cohomology class. It is well known (Moser’s lemma) that two symplectic forms are isotopic if and only if there is a family of diffeomorphisms \(\{h_s\}\) on \(X\) such that \(h_s^*\omega_s = \omega_0\) and \(h_0 = \text{id}\). The concepts of isotopy and deformation are in general not equivalent (cf. Example 13.20 in [21]), but for some special cases as we will see in Theorem 4.1, they agree. For the objectives of the present work, manifolds where these two properties agree will be a key ingredient as we will see in Lemma 3.3.

When the manifolds are equipped with circle actions, we will make the natural assumption that all the \(P\)-families satisfy

\[
\{\omega_t\} = \{\omega_0\} + (t - t_0)e(P), t \in I.
\]

The symplectic structure of \(H^{-1}(I)\) is completely characterized as follows.

**Lemma 3.1** (Proposition 5.8 in [21]). The symplectic manifold \((H^{-1}(I), \omega)\) is determined by the bundle \(P \to B\) and the family \(\{\omega_t\}_{t \in I}\) up to an equivariant symplectomorphism.

Our aim in this section is to understand the isomorphism class of \(H^{-1}(I)\) in terms of less information. Suppose that we have got two invariant symplectic forms \(\omega, \omega'\) on \(H^{-1}(I)\). Equation (4) shows that the paths defined by the cohomology classes of the reduced forms satisfy \(\overline{\omega_t} = \overline{\omega_t'}\) in \(B\) provided \(\overline{\omega_{t_0}} = \overline{\omega_{t_0}'}\) for some class \(e(P)\). We denote by \(\overline{\omega}_t\) the weak equivalence class of the path \(\{\overline{\omega_t}\}_{t \in I}\).

**Definition 3.2.** Let \(B\) be a manifold. We say that two paths \(\overline{\omega}_t, \overline{\omega}_t', t \in I\) of symplectic forms are weakly-equivalent if \(\overline{\omega_t} = \overline{\omega_t'}\) for all \(t \in I\). This is equivalent to require \(\overline{\omega_{t_0}} = \overline{\omega_{t_0}'}\) and impose Equation (4) for some class \(e(P)\). We denote by \(\overline{\omega}_t\) the weak equivalence class of the path \(\{\overline{\omega_t}\}_{t \in I}\).

In particular, any two invariant symplectic forms \(\omega, \omega'\) in \(H^{-1}(I)\) yield weakly-equivalent paths of reduced forms if their symplectic reductions at \(t_0\) agree. What we want to know is when any two families of forms \(\{\overline{\omega_t}\}, \{\overline{\omega_t}'\}\) in \(B\) satisfying Equation (4) lift to isomorphic forms on \(H^{-1}(I)\). Equivalently we want to determine the isomorphism type of \(X \times I\) in terms of these paths of forms. Suppose that there is an equivariant diffeomorphism \(\phi : P \times I \to P \times I\) such that \(\phi^*(\omega) = \omega'\). \(\phi\) defines a family \(\phi_t : P_t \to P_t\) by restricting to the regular level at \(t\). These maps descend to the quotient \(\overline{\phi}_t : B \to B\), satisfying the following.

\[
\begin{array}{ccc}
P_t & \xrightarrow{\phi_t} & P_t \\
\downarrow{\pi} & \cong & \downarrow{\pi} \\
B & \xrightarrow{\overline{\phi}_t} & B
\end{array}
\]

The maps \(\overline{\phi}_t\) satisfy \((\overline{\phi}_t)^*\overline{\omega}_t = \overline{\omega}_t\) and preserve the Euler class \(e(P)\) for all \(t \in I\).

Conversely, given any smooth family of maps \(\overline{\phi}_t : B \to B\) such that \((\overline{\phi}_t)^*\overline{\omega}_t = \overline{\omega}_t\) and \((\overline{\phi}_t)^*e(P) = e(P)\), it lifts to a family of isomorphisms \(f_t : P \to P\) that make Diagram (4) commute. Therefore, these maps bundle together to a diffeomorphism \(f : P \times I \to P \times I\) such that \(f^*(\omega) = \omega'\).

For simplicity of notation, assume that \(I = [0, 1]\).
Definition 3.3. Two weakly-equivalent families \( \{\omega_t\}_{t \in T} \) and \( \{\omega'_t\}_{t \in T} \) of symplectic forms on \( B \) are said to be equivalent if there is a smooth family \( \{\omega_{s,t}\} \) of symplectic forms such that

\[
\frac{d}{ds} [\omega_{s,t}] = 0 \quad \text{and} \quad \omega_{0,t} = \omega_t, \quad \omega_{1,t} = \omega'_t, \tag{6}
\]

for \( 0 \leq t, s \leq 1 \).

If \( \{\omega_t\} \) is equivalent to \( \{\omega'_t\} \), then there is an isotopy \( \Omega_t \) of forms on \( P \times I \) that lifts \( \omega_{s,t} \) and such that \( \Omega_0 = \omega \), \( \Omega_1 = \omega' \). By Moser’s lemma, we have got a family of maps \( f_s : P \times I \longrightarrow P \times I \) such that \( f_s^* \omega_s = \omega \). Therefore the isomorphism type of \( P \times I \) depends exclusively on the equivalence class of \( \{\omega_t\} \). Using rigidity (Definition [12]), we now understand the relation amongst weakly-equivalent and equivalent.

Lemma 3.4. Let \( (B, \{\omega_t\}_{t \in T}) \) be a rigid pair. Then, any family \( \{\omega_t\} \) weakly-equivalent to \( \{\omega_t\} \) and with \( \omega_0 = \omega'_0 \), is also equivalent to \( \{\omega_t\} \).

Proof. We want to see that there is a family of symplectic forms \( \{\omega_{s,t}\}_{0 \leq s, t \leq 1} \) such that for each fixed \( t \) the path \( \omega_{s,t} \) is an isotopy from \( \omega_t \) to \( \omega'_t \). First, since the families are weakly-equivalent, \( \omega_t \) is cohomologous to \( \omega'_t \) for all \( t \), then for \( s \in [0, 1] \) the cohomology class of the form \( s \omega_t + (1 - s) \omega'_t \) is constant with respect to \( s \). If \( \omega_0 = \omega'_0 \), Moser’s argument shows that there is an \( \epsilon > 0 \) and a family \( \omega_{s,t} \) satisfying Equation (6) for \( 0 \leq t \leq \epsilon \). (Compare with Example 3.20 in [21]). We want to see that we can take \( \epsilon = 1 \). To see this, define

\[
D = \{ T : \exists \{\omega_{s,t}\}_{0 \leq s, t \leq 1, 0 \leq \epsilon \leq T} \text{ satisfying Equation (6)} \} \subset [0, 1].
\]

We claim that \( D \) is \([0, 1]\). We will do this by proving that \( D \) is open and closed. Let \( T \in D \). Since \( \omega_T \) is isotopic to \( \omega'_T \), we can assume that \( \omega_T = \omega'_T \). Then, by the same argument as when \( T = 0 \), there is an \( \epsilon > 0 \) such that \( T + \epsilon \in D \). Thus \( D \) is open.

To see that \( D \) is closed, take \( T \) such that \( 0 < T \leq 1 \), and \( T - \epsilon \in D \) for every \( \epsilon > 0 \) small. The path

\[
\alpha_s = \begin{cases} \omega,(s) & v(s) = (1 - 2s)T, 0 \leq s \leq \frac{1}{2} \\ \omega,(s) & u(s) = (2s - 1)T, \frac{1}{2} \leq s \leq 1 \end{cases}
\]

is a deformation between \( \omega_T \) and \( \omega'_T \). Since \( (B, \{\omega_t\}) \) is rigid and \( \omega_T = \omega'_T \), \( \alpha_s \) can be homotoped through deformations with fixed endpoints to an isotopy. Let \( \beta_s \) such isotopy with \( \beta_0 = \omega_T \) and \( \beta_1 = \omega'_T \). Again, by Moser’s argument, we can extend \( \beta_s \) in a small neighborhood of \( T \). That is, for an \( \epsilon > 0 \) there is a family \( \beta_{s,T-\epsilon} \) of symplectic forms which is homotopic to \( \beta_s \).

On the other hand, for the hypothesis on \( T \), we have \( T - \epsilon \in D \). Thus there is a family \( \omega_{s,t} \) that satisfies Equation (6), for all \( t \in [0, T - \epsilon] \). The concatenation of the two isotopies \( \omega_{s,T-\epsilon}, -\beta_{s,T-\epsilon} \) defined by

\[
\gamma_s := \begin{cases} \omega_{s,T-\epsilon} & 0 \leq s \leq 1 \\ \beta_{2-s,T-\epsilon} & 1 \leq s \leq 2 \end{cases}
\]

(after smoothing) defines a loop at \( \omega_{T-\epsilon} \) in the space of symplectic structures \( S(a) \) with fixed cohomology class \( a = [\omega_{T-\epsilon}] \).

The fibration

\[
\text{Symp}(B, \omega_{T-\epsilon}) \cap \text{Diff}_0(B) \longrightarrow \text{Diff}_0(B) \xrightarrow{\pi} S(a), \quad \pi : f \mapsto f^*(\omega_{T-\epsilon}),
\]

gives a lift \( \{f_s\}_{s \in [0, 2]} \) in \( \text{Diff}_0(B) \) such that \( f_0 = id \) and \( f_s^*(\gamma_s) = \omega_{T-\epsilon} \) for all \( s \). By rigidity hypothesis, the fibre \( \text{Symp}(B, \omega_{T-\epsilon}) \cap \text{Diff}_0(B) \) is path connected, we can assume that \( f_2 = id \) as well.

The map

\[
h_s := f_s \circ (f_{2-s})^{-1}, \quad 0 \leq s \leq 1
\]

is such that

\[
h_0 = h_1 = id, \quad h_s^*(\gamma_{2-s}) = \gamma_s
\]

that is,

\[
h_s^*(\beta_{2-s,T-\epsilon}) = \omega_{s,T-\epsilon}.
\]

Therefore, the new family

\[
\hat{\beta}_{s,t} := h_s^*(\beta_{s,t})
\]

satisfies Equation (6) for \( 0 \leq s \leq 1, T - \epsilon \leq t \leq T \) and agrees with \( \omega_{s,t} \) at \( t = T - \epsilon \). After smoothing, we see that \( \omega_{s,t} \) can be extended to all \( t \leq T \) via \( \hat{\beta}_{s,t} \). Then \( T \in D \). This proves that \( D \) is closed.

We would like to emphasise where our argument fails if \( \text{Symp}(B, \omega_{T-\epsilon}) \cap \text{Diff}_0(B) \) is not path connected. In this case, we cannot consider \( f_2 \) to be the identity and then \( h_0 \neq id \). For our boundary conditions, we need the extension \( \hat{\beta}_{0,t} \) to agree with \( \omega_t \) for \( T - \epsilon \leq t \leq T \). We cannot conclude that if \( h_0 \) is not the identity. ■
Combining all the results above, yield the most important result of this section.

**Theorem 3.5.** Let \((M, H, \omega)\) be a Hamiltonian \(S^1\)-manifold with proper Hamiltonian and let \(\lambda' ,\lambda\) be any two consecutive critical values. Let \(I = [t_0, t_1] \subset (\lambda', \lambda)\), and suppose that under the identification \(\overline{M}_t = \overline{M}_{t_0}\), \((\overline{M}_t, \{\overline{\omega}_t\})\) is rigid. Then any other invariant form \(\omega'\) on \(H^{-1}(I)\) is isomorphic to \(\omega\) provided its reduction at \(t_0\) is (diffeomorphic to) \(\overline{\omega}_{t_0}\). In other words, the isomorphism class of the regular slice \(H^{-1}(I)\) is determined up to isomorphism by the bundle \(P \to \overline{M}_{t_0}\), the family of cohomology classes \(\overline{\omega}_t\) and the initial form \(\overline{\omega}_{t_0}\).

**Proof.** Since any two invariant forms on \(H^{-1}(I)\) with \(\overline{\omega}_{t_0} = \overline{\omega}_{t_0}\) yield weakly-equivalent paths of forms the family of reduced forms \(\{\overline{\omega}_t\}\) is weakly-equivalent to \(\{\overline{\omega}_t\}\), Lemma 3.4 shows they are also equivalent, therefore \(\omega, \omega'\) are isomorphic.

3.3 Germs at critical levels.

We now describe the germ \(G(\lambda, \varepsilon)\) when \(H^{-1}(\lambda)\) contains only fixed points of index 2 and the action is semi-free. We will also discuss the change in the fixed point data.

3.3.1 Smooth and symplectic structure on the critical reduced space.

Let \(H^{-1}(\lambda)\) be a level with only index 2 fixed-point components for \((M, H, \omega)\). In general \(H^{-1}(\lambda)\) is a singular space and thus \(\overline{M}_\lambda\) is not naturally smooth. Although it is well known that we can provide this space with a smooth symplectic manifold structure [9], one can put a canonical, independent of choices, smooth structure on \(\overline{M}_\lambda\) using grommets near the fixed point set, as in Tolman-Karshon [11]. In what follows we use a reinterpretation due to McDuff [21, §3.2]. Let \(F \subset H^{-1}(\lambda)\) denote the fixed point set, and for simplicity assume it is connected of codimension 2k. The equivariant Darboux-Weinstein theorem states that germs of neighborhoods \(U \subset M\) containing \(F\) are isomorphic. One can identify any such \(U\) with the normal bundle \(N_{F/M}\) of \(F\) in \(M\). The isomorphism class of \(U\) is determined by the restriction of the symplectic structure to \(F\) and the isomorphism class of the normal bundle \((\overline{M}_\lambda, \overline{\omega}_\lambda, F)\) when \(\lambda\) is a singular point of \(H\).

To be precise, an embedding of a ball bundle and a connection on a principal \(U(n)\)-bundle.

3.3.2 Cobordism and the change of fixed point data.

Let \((M, \omega)\) be a symplectic manifold and let \(S \subset M\) be a closed submanifold. For \(\varepsilon > 0\) small enough, we denote by \(B\) the \(\varepsilon\)-symplectic blow up of \(M\) along \(S\) and the blow-down map as defined by Guillemin and Sternberg [8]. The construction of the manifold \(B\) depends on several choices, but its diffeomorphism type is independent of them. \(B\) admits a blow-up symplectic structure denoted by \(\overline{\omega}(\varepsilon)\). This form is not independent of the choice, but its germ of isotopy classes is. That is, if \(\overline{\omega}(\varepsilon)\) is a form obtained by making different choices, then for some \(\varepsilon_0\) small enough there exist a smooth family \(f_\varepsilon \in \text{Diff}(M)\) such that \(f_\varepsilon^* \overline{\omega}(\varepsilon) = \overline{\omega}(\varepsilon)\) for all \(0 < \varepsilon < \varepsilon_0\). With this in mind, assume we have got the following data.

1. A compact symplectic manifold \((\overline{M}, \overline{\omega})\).
2. A symplectic submanifold \(\overline{F} \subset \overline{M}\).
3. A principal \(S^1\)-bundle \(\pi : \overline{P} \to \overline{M}\).
4. A connection 1-form \(\alpha\) on \(\overline{P}\).

Then it is possible to create a cobordism \((Y(\lambda), H, \varepsilon) \in \text{HSymp}_2\) having the following properties.

Not even when one blows-up a point. This is because there is a non-compact family of choices.
1. \( H \) maps \( Y(\lambda) \) onto \( I = (\lambda - \epsilon, \lambda + \epsilon) \).
2. For all \( t > 0 \), \( H^{-1}(\lambda - t) \) is equivariantly diffeomorphic to \( P \).
3. For all \( t > 0 \) the symplectic reduction \( Y(\lambda) \) at the level \( \lambda - t \) is symplectomorphic to \( \overline{M} \) with symplectic form \( \omega - tda \).
4. \( \lambda \) is a critical level of \( H \) of index 2.
5. For all \( t > 0 \) the reduction of \( Y(\lambda) \) at \( \lambda + t \) is the blow up \( Bl_F(\overline{M}) \) of \( \overline{M} \) along \( F \) with symplectic structure \( \tilde{\omega}(t) + \beta(tda) \).

Here \( \tilde{\omega}(t) \) is the \( t \)-blow up form and \( \beta : Bl_F(\overline{M}) \to \overline{M} \) is the blow down map.

6. The fixed point set at \( \lambda \) is \( F \).

**Theorem 3.6** (Guillemin-Sternberg). Let \((M,K,\omega) \in \mathbf{HSymp}_{2n} \) and \( \lambda \) be an index 2 critical value. Let \( I = (\lambda - \epsilon, \lambda + \epsilon) \) be a sufficiently small interval in \( \mathbb{R} \). Then, the open submanifold \( K^{-1}(I) \) is equivariantly symplectomorphic to the manifold \((Y(\lambda),H,\epsilon)\). Moreover, the germ of diffeomorphism of \( M \) near \( \lambda \) only depends on the fixed point data \((\overline{M}_\lambda,F,\overline{\omega}_\lambda),\epsilon(\lambda) \in H^2(\overline{M}_\lambda)\).

Theorem 3.6 describes the change in the fixed point data after crossing a critical level \( \lambda \) with index 2 fixed point sets. A similar analysis applies for coindex 2. We will include in Lemma 3.10 details of the proof of the uniqueness statement, since we will need it later.

Let \( N_{F/M} \) be the normal bundle of the fixed point submanifold \( F \) in \( M \). This bundle decomposes as

\[
N_{F/M} = N^+_{F/M} \oplus N^-_{F/M}
\]

(10)

taking positive and negative directions. A small neighborhood of the zero section of \( N_{F/M} \) is \( S^1 \)-isomorphic to a neighborhood \( U \cong U^+ \oplus U^- \) of \( F \) in \( M \). Denote by \( \overline{X}_t \) the symplectic reduction of \( U^- \) at \( t \) if \( t \leq \lambda \) or the reduction of \( U^+ \) at \( t \) if \( t > 0 \). The diffeotype of the triple \((P_\lambda,\overline{M}_t,\overline{X}_t)\) depends smoothly on \( t \) for \( t \in (\lambda - \epsilon,\lambda] \) and it is constant. Denote it by \((P_-,\overline{M}_-,\overline{X}_-)(t)\) where \( \overline{X}_- = \overline{F}_\lambda \). Similarly, we denote the common diffeotype for \( t \in (\lambda, \lambda + \epsilon) \) by \((P_+,\overline{M}_+,\overline{X}_+)(t)\).

We now remark an important relation of the bundles \[ \text{Frame}(N_{F/M}) \] (pp. 516). By analysing the negative directions, there is an isomorphisms of \( S^1 \)-bundles

\[
P_\lambda |_\overline{F} \cong \text{Frame}(N^-_{F/M})
\]

(11)

where \( P_\lambda |_\overline{F} \) is the restriction of \( P_\lambda \) to the fixed point set in the reduced space, and \( \text{Frame}(N^-_{F/M}) \) is the frame bundle of \( N^-_{F/M} \). Similarly there is an isomorphism of bundles

\[
N^-_{\overline{F}/\overline{M}_\lambda} \cong N^+_{F/M}.
\]

(12)

Equations (10), (11) and (12) show that the normal bundle \( N_{F/M} \) is determined by \( N^-_{\overline{F}/\overline{M}_\lambda}, \overline{F} \) and \( P_\lambda \).

Now we address the relation amongst the principal bundles \( P_\lambda \) and \( P_- \). This is quite subtle. By Theorem 3.6, \( \overline{M}_+ \) is the blow up of \( \overline{M}_- \) along \( \overline{X}_- \). Let \( \beta : \overline{M}_+ \to \overline{M}_- \) be the blow down map. \( \beta \) restricts to a diffeomorphism \( \overline{M}_+ - \overline{X}_+ \to \overline{M}_- - \overline{X}_- \), and when restricted to \( \overline{X}_+ \) it is a fibration

\[
\beta : \overline{X}_+ \to \overline{X}_-
\]

whose fibres are all diffeomorphic to \( \mathbb{CP}^{k-1} \). Here \( 2k \) is the codimension of \( \overline{X}_- \) in \( \overline{M}_- \). Denote by \( L' \) the line bundle on \( \overline{M}_+ \) whose Chern class is dual to its codimension 2 submanifold \( \overline{X}_+ \). Let \( L \) be the circle bundle associated to \( L' \). Then we have [3, Formula 13.3]

\[
P_+ = \beta^*(P_-) \otimes L
\]

(13)

as circle bundles over \( \overline{M}_+ \). Since \( L \) is trivial on \( \overline{M}_+ - \overline{X}_+ \), then \( P_- \cong P_+ \) on \( \overline{M}_+ - \overline{X}_+ \). The construction of \( L \) depends on the normal bundle of \( \overline{X}_+ \) in \( \overline{M}_+ \), and hence on the pair \((\overline{M}_-,\overline{X}_-)(\overline{M}_\lambda,\overline{F}_\lambda)\). Then one is tempted to describe \( P_\lambda \) in terms of \( P_-(\overline{M}_-,\overline{F}_\lambda) \). We now see examples on which the relation between \( P_- \) and \( P_+ \) is easy to depict.

**Example 3.8.**

\(^1\) a 2-form on \( F \), but it descends to \( \overline{M} \). This is the form that we consider here.
11

**Fig. 1.** The cobordism around an isolated fixed point \( p \) of index 2. Here we assume that the minimum is also isolated. The base of the figure represents \( \mathbb{C}P^2 \) and the top the blow-up \( \mathbb{C}P^2\#\mathbb{C}P^2 \). If the fixed point is of index 4 this cobordism is up-side down.

**Example 3.7.** Suppose \( F \) is an isolated fixed point \( p \in M \) of index 2. The relation between the Euler classes \( e(P_-) \) and \( e(P_+) \) is clear. Let \( \beta : \overline{M}_+ \to \overline{M}_- \) denote the blow down map that collapses the exceptional divisor \( E \) to \( p \in \overline{M}_- \). Then we have

\[
e(P_+) = \beta^* e(P_-) + \text{PD}(E).
\]

In the case \( p \) has coindex 2, the relation is inverted

\[
\beta^* e(P_+) = e(P_-) + \text{PD}(E),
\]

since now \( \overline{M}_- \) blows down to \( \overline{M}_+ \).

Suppose \( \text{codim} \ F = 4 \) and that the index of \( F = 2 \). By Lemma 5 in McDuff [16] all the reduced spaces \( \overline{M}_t \) for \( t \in I, t \neq \lambda_1 \) are diffeomorphic, to \( B_- \) say. In particular \( B_+ \) is diffeomorphic to \( B_- \). Moreover

\[
e(P_+) = e(P_-) + \text{PD}(F)
\]

where \( F \) is embedded in \( B_- \) as before. If \( \dim M = 6 \) this case applies when \( F \) is a surface.

These two examples show, that in special circumstances the bundles \( P \) are determined by the rest of the fixed point data at and below the critical level \( \lambda \) and thus they might be discarded as input information for the classification problem.

### 3.3.3 The critical germ at non-extremal critical levels.

It is pointed out in [3, pp. 514] that the symplectomorphism type of the germs is independent of choices and it depends exclusively in the fixed point data. We include the argument in [20] to show that any two germs with the same fixed point data at a critical value are isomorphic. We then adapt all these ideas to show, that using rigidity we can remove more information from the fixed point data without altering the isomorphism type of the germ. Before we state the results, we recall that we are using the smooth structure for reduced spaces as in (3.3.1) and that to avoid complications we are assuming each critical level to be simple. Please refer to (3.3.1) for the non-simple case.

**Definition 3.9.** Let \( (M_i, H_i, \omega_i), i = 1, 2 \) denote two Hamiltonian \( S^1 \)-manifolds. We say that they have the same fixed point data at a non-extremal critical value \( \lambda \) if

1. \( \lambda \) is a non-extremal critical value for both \( H_i \).
2. There is a symplectomorphism of reduced spaces \( \phi : (\overline{M}_1,\lambda, \overline{F}_1,\lambda, \overline{\omega}_1,\lambda) \to (\overline{M}_2,\lambda, \overline{F}_2,\lambda, \overline{\omega}_2,\lambda) \) such that \( \phi^* e(\overline{F}_2,\lambda) = e(\overline{F}_1,\lambda) \).
3. The index functions are related by \( i_{2,\lambda} \circ \phi = i_{1,\lambda} \).

We say that they have the same fixed point data at the minimum if \( \lambda \) is the minimum value for both \( H_i \) and there is a symplectomorphism \( \phi : (\overline{F}_{1,\min}, \omega_{F_{1,\min}}) \to (\overline{F}_{2,\min}, \omega_{F_{2,\min}}) \) such that \( \phi^* (\overline{N}_{F_{2,\min}/M_2}) = N_{F_{1,\min}/M_1} \).

Similarly for the maximum.
The following two lemmas deal with the isomorphism type of the germs. The first one is a minor generalisation of the result in [20]. The second one is just a small adaptation of the proof of Theorem 13.1.

Lemma 3.10. Let \((M_i, H_i, \omega_i), i = 1, 2\) denote two (possibly open) Hamiltonian \(S^1\)-manifolds with proper Hamiltonians \(H_i\). Suppose that \(\lambda\) is a common critical value of \((M_i, H_i, \omega_i), i = 1, 2\) with only fixed point components of index \(2\), and they have the same fixed point data at \(\lambda\). Then, there is an \(\varepsilon_0 > 0\) such that \(H^{-1}_1(\lambda - \varepsilon, \lambda + \varepsilon)\) is isomorphic to \(H^{-1}_2(\lambda - \varepsilon, \lambda + \varepsilon)\), for all \(\varepsilon \leq \varepsilon_0\).

Proof. Assume for now that the fixed point set is a single isolated fixed point \(p_i\). Let \(\chi : U_i \rightarrow U \subset \mathbb{C}^n\) denote a Darboux chart near \(p_i\) so that \(H_i \circ \chi^{-1}\) are in standard form \(\lambda - |z_i|^2 + \sum_{1 \leq j \leq n} |z_j|^2\). Since the fixed point data is the same near \(\lambda\) there is a symplectomorphism \(\phi : (\overline{M}_{1, \lambda}, \overline{\omega}_{1, \lambda}) \rightarrow (\overline{M}_{2, \lambda}, \overline{\omega}_{2, \lambda})\). In the local model we can assume that this map can be isotoped to the identity, by shrinking the neighborhoods \(U_i\), it is necessary. Thus using the charts in the reduced spaces given by \([S]\) we can assume that \(\ell^{-1} \circ \chi_2 \circ \phi \circ \chi_1^{-1} \circ \ell\) is the identity near the fixed point set.

Claim 1. For \(\varepsilon > 0\) small enough, the two manifolds \(H^{-1}_i(\lambda - \varepsilon, \lambda + \varepsilon)\backslash U_i\) are equivariantly diffeomorphic.

This claim follows essentially from the fact that on the complement \(H^{-1}_i(\lambda - \varepsilon, \lambda + \varepsilon)\backslash U_i\), the action is free, and since both manifolds have the same symplectic reduction at zero, it follows from Duistermaat-Heckman theorem. To make this explicit, choose an \(S^1\)-invariant and compatible almost complex structure on \(M_i\), so that near \(p_i\) they agree with \(\chi^{-1}(J_0)\), where \(J_0\) is the standard complex structure in \(\mathbb{C}^n\). Let \(g_i\) denote the metric induced by \(\omega_i, J_i\) and let \(\tau_i, t\) be the downwards gradient flow with respect to the metrics \(g_i\). By taking \(\varepsilon > 0\) small enough we can assume that the neighborhoods \(U_i\) are compatible with the gradient flow. This is we can assume that the set \(U_i \cap H^{-1}_i(\lambda - \varepsilon, \lambda + \varepsilon)\) is not empty and that it contains all the orbits in \(H^{-1}_i(\lambda, \lambda + \varepsilon)\) whose downward gradient flow converges to the fixed points. By parametrisering if needed one can assume that \(\tau_{i, t}\) takes any level set \(H^{-1}_i(c)\) to \(H^{-1}_i(c - t)\). These flows are non-singular away from \(p_i\), so that outside \(U_i\) one can define the equivariant diffeomorphism \(\tau(x, t) = \tau_{2, \lambda} \circ \phi \circ \tau_{1, t}^{-1}(x)\), where \(\phi\) is the symplectomorphism of \(H^{-1}_1(\lambda)\backslash U_1\) to \(H^{-1}_2(\lambda)\backslash U_2\) lifting \(\phi\). This proves the claim.

Now define \(\tau\) on \(U_1\) to be the isomorphism \(\chi_2^{-1} \circ \chi_1\). Then \(\tau \circ \omega_2 = \omega_1\) on \(U_1\) by construction on \(U_1\), but it might not be outside \(U_1\). The family of forms \((\tau^{-1} \omega_i)_{t, \tau}((\omega_1)_{t, \tau})\) induced in the reduced spaces \((H^{-1}_i(t)\backslash U_1)/S^1\) define a path of non-degenerate forms \(s(\tau^{-1} \omega_2)_t + (1 - s)(\omega_1)_t, s \in [0, 1]\) provided \(t \in (\lambda - \varepsilon, \lambda + \varepsilon)\) and \(\varepsilon < \varepsilon_0\) for a sufficiently small \(\varepsilon_0\). Moser’s lemma now provides a family of isotopies from \((\omega_1)_{t, \tau}\) to \((\tau^{-1} \omega_2)_t\), which can be used to correct \(\tau\) to the desired equivariant symplectomorphism. This shows the theorem in the case that the fixed point components are isolated points.

The general case reduces to the case above as in [3.3.1] using normal directions. We have the following claim.

Claim 2. The manifolds \(M_i\) are isomorphic near the fixed point components.

Suppose \(F_1 \subset \overline{M}_{1, \lambda}\) is a fixed point component. The symplectomorphism \(\phi\) of Definition 3.9 maps \(F_1\) to a unique component \(F_2 \subset \overline{M}_{2, \lambda}\). Moreover \(\phi^*(N_{F_1/\overline{M}_{1, \lambda}}) = N_{F_2/\overline{M}_{2, \lambda}}\) and \(\phi^*(P_{1, \lambda}) = P_{1, \lambda}\). Using Equations (10), (11) and (12) the normal bundles \(N_{F_i/\overline{M}_i}\) are also identified through \(\phi\). Weinstein’s symplectic neighborhood theorem shows that a neighborhood of \(F_1\) is symplectomorphic to one of \(F_2\). This proves Claim 2 and the Theorem.

In [20] McDuff uses the previous argument to show that in the case when the fixed point components are isolated points and the reduced spaces are rigid (as in Definition 1.4) and unique in the sense that any two cohomologous symplectic forms are diffeomorphic, then the germ is uniquely determined by the fixed point data. We want to remark that it is not known whether rigidity implies uniqueness.

Before we prove our next result, we need two more definitions.

Definition 3.11. Let \((M_i, H_i, \omega_i), i = 1, 2\) denote two Hamiltonian \(S^1\)-manifolds. We say that they have the same small fixed point data at a non-extremal critical value \(\lambda\) if

1. \(\lambda\) is a non-extremal critical value for both \(H_i\).
2. There is a diffeomorphism \(\phi : (\overline{M}_{1, \lambda}, F_{1, \lambda}) \rightarrow (\overline{M}_{2, \lambda}, F_{2, \lambda})\), such that the restriction \(\phi|_{F_1}\) is a symplectomorphism on each component.
3. The index functions are related by \(i_{2, \lambda} \circ \phi = i_{1, \lambda}\).

\[\text{Compare this and what follows with proof of Theorem 13.1 in [9], they denote this piece as } Q, \text{ note that their hypothesis and statement is slightly different.}\]
Here we are only assuming the case when the fixed point components have positive dimension in Definition 13. The other cases can be easily adapted. For the minimum, we assume the same as in Definition 5.5. For the maximum we only require the existence of a symplectomorphism $\phi: (F_{1,\text{max}}, \omega_{F_{1,\text{max}}}) \rightarrow (F_{2,\text{max}}, \omega_{F_{2,\text{max}}})$ without assuming anything about its normal bundle.

**Definition 3.12.** Let $(M_i, H_i, \omega), i = 1, 2$ denote two Hamiltonian $S^1$-manifolds. We say that the spaces $M_i$ are identified at a regular value $t$ if there is an $S^1$-bundle isomorphism $P_{1,t} \rightarrow P_{2,t}$ inducing a symplectomorphism of reduced spaces $\phi: (\overline{M}_{1,t}, \overline{\omega}_{1,t}) \rightarrow (\overline{M}_{2,t}, \overline{\omega}_{2,t})$.

What the next lemma says is that in the case that we have got two manifolds with the same small fixed point data, then any identification of the manifolds at a regular value $t$ right before the critical level extends to an isomorphism over the critical level, provided the family of reduced spaces is rigid. Note that this identification plays the role of the symplectomorphism in Lemma 5.10.

**Lemma 3.13.** Let $(M_i, H_i, \omega), i = 1, 2$ denote two (possibly open) Hamiltonian $S^1$-manifolds with proper Hamiltonians $H_i$. Assume that $\lambda$ is a common critical value of $(M_i, H_i, \omega), i = 1, 2$ with only fixed point components of index 2. Suppose further that:

1. The manifolds have the same small fixed point data.
2. $(M_i, H_i, \omega)$ are identified at a regular value $t_0 < \lambda$.
3. The pair $(\overline{M}_{1,t_0}, \overline{\omega}_{1,t_0})$ is rigid.

Then, there is an $\varepsilon > 0$ such that $H_1^{-1}(\lambda - \varepsilon, \lambda + \varepsilon)$ is isomorphic to $H_2^{-1}(\lambda - \varepsilon, \lambda + \varepsilon)$, for all $\varepsilon \leq \varepsilon_0$.

**Proof.** The proof is similar to that of Lemma 5.10. We only need two modifications. First, the equivalent of Claim 2 follows since the manifolds are identified at the regular value $t_0$, thus they have isomorphic bundles $P_{t_0}$. If they have the same small fixed point data, the normal bundles $N_{F_{i,M}}$ are also isomorphic. The same argument as in Claim 2 of Lemma 5.10 shows that there are neighborhoods $U_i$ of $F_i$ and a symplectomorphism $\phi: (U_1, \omega_1) \rightarrow (U_2, \omega_2)$.

We only need to show the equivalent of Claim 1, that is the complements $H^{-1}_i(\lambda - \varepsilon, \lambda + \varepsilon) \setminus U_i$ are also symplectomorphic for $\varepsilon$ small enough. Just as before, they are clearly diffeomorphic, since the action is free on this piece and by hypothesis there is an identification $\overline{M}_{1,t_0} \cong \overline{M}_{2,t_0}$ of the reduced spaces at regular level $t_0$. Also, the family of reduced symplectic forms $\overline{\omega}_{1,t}$ is weakly equivalent to $\overline{\omega}_{2,t}$ for $t \in [t_0, t_1]$ for any $t_1 < \lambda$. By hypothesis $(\overline{M}_{1,t}, \overline{\omega}_{1,t})$ spaces is rigid for $t \in [t_0, t_1]$. By taking $t_1$ sufficiently close to $\lambda$ and using the same arguments as in Lemma 5.10 along with Theorem 5.5, we conclude that $H^{-1}_i(\lambda - \varepsilon, \lambda + \varepsilon) \setminus U_i \cong H^{-1}_i(\lambda - \varepsilon, \lambda + \varepsilon) \setminus U_2$. Now, by taking $\varepsilon$ small enough the symplectic structures on the two pieces $U_i$ and $H^{-1}_i(\lambda - \varepsilon, \lambda + \varepsilon) \setminus U_i$ are close enough. By Moser’s Lemma we can assume they agree. This finishes the proof.

### 3.4 Non-simple levels

We now describe how to deal with non-simple levels. Suppose $(M, H, \omega)$ has a critical level at $\lambda$, with fixed point components of index and coindex 2. In Definition 12 the index attached to $\lambda$ is not well defined, rather it is an integer valued function $i_\lambda$ on the set of connected components of the fixed point set. Therefore this definition needs to be modified to accommodate this. Let $F_1$ denote the union of index 2 components, i.e. $F_1 = i_\lambda^{-1}(2)$ and let $F_2$ denote the coindex 2 part. The modification necessary to the make our previous results work in this setting are as follows.

1. As before, we can decompose the normal bundles as positive and negative directions $N_{F_{i/M}} = N_+ \oplus N_-^{\pm}$, $i = 1, 2$. A small neighborhood of the zero section in $N_{F_{i/M}}$ is isomorphic to a neighborhood $U_i = N_+^{\pm} \cup U_-^{\pm}$ of $F_i$ in $M$.

2. The symplectic reduction $\overline{M}_\lambda$ at $\lambda$ also has a smooth structure. This is because the construction of the charts in 3.3.1 is done locally near $F_1$. We denote $F_2$ the images of the fixed point set.

3. For $\varepsilon > 0$ the space $\overline{M}_\lambda^{\varepsilon}$ is diffeomorphic to the blow-up of $\overline{M}_\lambda$ along $F_1$ only. $\overline{M}_{\lambda - \varepsilon}$ is the blow up along $F_2$. Let $\beta_1: \overline{M}_{\lambda + \varepsilon} \rightarrow \overline{M}_\lambda$ and $\beta_2: \overline{M}_{\lambda - \varepsilon} \rightarrow \overline{M}_\lambda$ denote the respective blow-down maps. The tuple $(\beta_1, \overline{M}_{\lambda}, \overline{M}_{\lambda - \varepsilon})$ depends smoothly on $t$, and its diffeotype is constant for $t$ close enough to $\lambda$. The main difference to the simple case is that none of reduced spaces for $\varepsilon > 0$ is diffeomorphic to the one at $\lambda$. Denote by $(P_{\pm}, \overline{M}_{\pm}, X_{1,\pm}, X_{2,\pm})$, the diffeotypes for $t = \lambda \pm \varepsilon$ respectively and $\varepsilon > 0$ small.
4. There exist a smooth $S^1$ principal bundle $P$ over $\overline{M}_λ$ such that the principal bundles $P_±$ are related by
\[ β_1^*P = P_+ ⊗ L_1, \quad β_2^*P = P_- ⊗ L_2, \]  
(14)
for some circle bundles $L_i$ depending completely on the fixed point data at $λ$, as in Equation (13). More precisely, Denote by $L'_1$ (resp. $L'_2$) the line bundle on $\overline{M}_+$ (resp. $\overline{M}_-$) whose Chern class is dual to its codimension 2 submanifold $X_{1,+}$ (resp. $X_{2,-}$). Let $L_1$ (resp. $L_2$) be the circle bundle associated to $L'_1$ (resp. $L'_2$).
5. The family of $ε$-blow up symplectic forms has a well defined germ, just as in §3.3.2.
6. Definitions 3.9 and 3.11 are perfectly valid for the case at hand, since we require the morphism $φ$ to preserves the index.
7. Claim 2 in the proofs of Lemma 3.10 and Lemma 3.13 also holds in this case, since it is local in nature. Claim 1 also follows. Details are left to the reader.

4 Six dimensional case.

In this section we will apply our previous analysis in dimension six.

4.1 Some results in 4-dimensional symplectic topology.

For our analysis, we will need the following results. The techniques in our paper rely on them, and the possibility of extending our results beyond six dimensions depends on the validity of analogous theorems in higher dimensions.

**Theorem 4.1 (L).** Let $X$ be $CP^2$, a blow-up of $CP^2$ or any rational surface. Then
1. Any deformation of two cohomologous symplectic forms on $X$ may be homotoped through deformations with fixed endpoints to an isotopy. ❑
2. Any two cohomologous symplectic forms on $X$ are symplectomorphic.

**Lemma 4.2** (Abreu, Gromov, McDuff, Lalonde, Pinsonnault, Evans). Let $(X, ω)$ be $CP^2$ with the form $ω_t(L) = t$ or its one-point blow-up $CP^2 # CP^2$ such that on the exceptional divisor $ω_t(E) = t - λ_1$. Then the group $\text{Symp}(X, ω_t)$ is connected for all $t$. Similarly if $(X, ω)$ denotes any of the blow ups $CP^2 # kCP^2$ for $k ≤ 3$, $ω_t(E_i) = t - λ_i$ for the exceptional curves $E_i$, then the group $\text{Symp}^H(X, ω)$ of symplectomorphisms that induce the identity on $H_*(X)$ is path connected. ❑

The reader can consult the original articles [3, 1], [12], [22], [4] the survey [19] for these results.

4.2 Proofs of the main theorems.

Assume that $(M, ω)$ is a symplectic 6-manifold with a semi-free $S^1$-action with isolated fixed points. Since the fixed points are isolated it is not necessary to assume that the action is Hamiltonian, it would follow from [24]. The fixed points are given as follows. The minimum, three critical points $p_1, p_2, p_3$ of index two, three $p_{12}, p_{23}, p_{13}$ of index four and a maximum. We will denote by $λ_i$ the critical values $H(p_i)$. Without lost of generality we may assume the minimum value of $H$ is zero, and that $λ_1 ≤ λ_2 ≤ λ_3$. It is not hard to see [24] that the fixed points of index 4 are in the level sets $H^{-1}(λ_i + λ_j), j ≠ i$ and the maximum is the unique point in $H^{-1}(λ_1 + λ_2 + λ_3)$ (See Figure 2). Before going any further recall that we want to prove that $M$ is isomorphic to $Y^3 = S^2 × S^2 × S^2$ with the product symplectic form $ω = λ_1σ × λ_2σ × λ_3σ$ here $σ$ is the canonical area form on $S^2$. We are assuming that the circle acts on $Y^3$ by
\[ e^{2πit}(x, y, z) \mapsto (e^{2πit}x, e^{2πit}y, e^{2πit}z). \]
and denote by $K : Y \to \mathbb{R}$ its Hamiltonian. For simplicity of the notation we write $Y^t := K^{-1}[0, t)$ and similarly $M^t := H^{-1}[0, t)$.

*This is actually true for more general 4-manifolds, namely manifolds of non-simple SW-type.
Proof of Theorem 1.7. We start by noticing that there are two main cases to analyse (but not only), when \( \lambda_1 + \lambda_2 \leq \lambda_3 \) and when \( \lambda_1 + \lambda_2 > \lambda_3 \). The difference of this two cases is the order in which we reach fixed points. For simplicity, one can treat the first case since the second one is analogous. We start by assuming that none of the \( \lambda_i \) agree, since this case is slightly easier than the case where some of them are equal.

Our aim is to give a construction of the isomorphism by showing that the manifolds \( M' \) and \( Y' \) are isomorphic as \( t \) increases. We will start from the minimum level. We have already explained that the minimal germs are determined by the fixed point data at the minimum (see comment after Definition 2.2). We now explain this for this particular example, since we want to be as explicit as possible. From the equivariant version of the Darboux theorem at the minimum, one gets a neighborhood of the minimum isomorphic to \( C^5 \) with the diagonal circle action. The Hamiltonian function in these coordinates is given by

\[
(z_1, z_2, z_3) \mapsto |z_1|^2 + |z_2|^2 + |z_3|^2.
\]

If \( 0 < t < \epsilon \) for \( \epsilon > 0 \) small enough, we have that the level set \( H = t \) is the 5-sphere,

\[
S^5 : |z_1|^2 + |z_2|^2 + |z_3|^2 = t.
\]

Therefore, the symplectic reduction \( M_t = S^5/S^1 \) is the projective space \( (\mathbb{C}P^2, \varpi_t) \) with the symplectic form \( \varpi_t \), that takes the value \( t \) on the line \( L \). The bundle \( P = H^{-1}(t) \hookrightarrow M_t \) is now just the Hopf fibration \( S^5 \to \mathbb{C}P^2 \), whose Euler class \( e(P) \in H^2(\mathbb{C}P^2) \) is the negative generator of \( H^2(\mathbb{C}P^2) \). The diffeomorphism type of the reduced space \( Y_t \) for all \( 0 < t < \lambda_1 \) is \( \mathbb{C}P^2 \). Theorem 3.10 and Lemma 3.12 show that the pair \( (\mathbb{C}P^2, \{\varpi_t\}_{t \in I}) \) is rigid for all subinterval \( I \subset (0, \lambda_1) \), and thus by Theorem 4.3 the symplectomorphic type of the slice corresponding to the interval \( I \) is determined. The diffeotype of the reduced spaces and the respective bundles is constant for all \( t \leq \lambda_1 \), in particular for \( t = \lambda_1 \). Thus (Definition 2.9) both manifolds have the same fixed point data at \( \lambda_1 \) and they are identified at a regular level immediately before \( \lambda_1 \). Lemma 3.13 shows we can extend the isomorphism pass the critical level. One then obtains an isomorphism \( M^{\lambda_1 + \epsilon} \cong Y^{\lambda_1 + \epsilon} \) by gluing along any regular level \( t < \lambda_1 \).

It is important to note that any gluing map \( \Phi : H^{-1}(t) \to K^{-1}(t) \) would suffice, since we just want to show the existence of the isomorphism.

For \( t - \lambda_1 > 0 \) small enough, Theorem 3.10 asserts that the reduction at \( t \) is the blowup \( \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) with the symplectic form \( \varpi_t \) such that the line class \( L \) has symplectic area \( t \) and the exceptional class \( E_1 \) has symplectic area \( t - \lambda_1 \). The principal bundle \( P_t \) corresponding to the level \( t \) is explicitly given in terms of the previous fixed point data Example 3.7. The germ near \( \lambda_1 \) is determined by the fixed point data at \( t = \lambda_1 \), so we have \( M^{t \cong Y^t} \) for \( t - \lambda_1 > 0 \). Since \( (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \{\varpi_t\}_{t \in (\lambda_1, \lambda_2)}) \) is rigid, the same argument above shows, that \( M^{\lambda_2} \cong Y^{\lambda_2} \).

The symplectic area of the exceptional class \( E_1 \) in \( \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) depends linearly in \( t \), and thus cannot blow down as \( t \) reaches \( \lambda_2 \). For \( t - \lambda_2 > 0 \) small enough the reduced spaces blow up and thus it is \( \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2} \) equipped with the family of blow up symplectic forms \( \{\varpi_t\} \) taking values \( t, t - \lambda_1, t - \lambda_2 \) on \( L, E_1, E_2 \) respectively. The pair \( (\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}, \{\varpi_t\}_{t \in (\lambda_1, \lambda_2)}) \) is again by Theorem 3.10 and Lemma 3.12 rigid, and then we have got an isomorphism \( Y^t \cong M^t \) for \( t < \lambda_1 + \lambda_2 \). To see what happens after we pass the critical level \( \lambda_1 + \lambda_2 \), we note that there are only three exceptional classes: \( E_1, E_2 \) and \( E_{12} := L - E_1 - E_2 \) with respective areas given by \( t - \lambda_1, t - \lambda_2 \) and \( t - \lambda_1 - \lambda_2 \). The areas of the exceptional curves \( E_1, E_2 \) are growing, thus do not blow down as \( t \) reaches \( \lambda_1 + \lambda_2 \), then only a curve in class \( E_{12} \) blows down when crossing \( \lambda_1 + \lambda_2 \). One important issue is to note that there is only one way of blowing down. This is, regardless of the actual curve that blows down, the symplectomorphic type of the blow-down manifold only depends on the homology class of the curves [13] proof of Theorem 1.1. In our case the class \( E_{12} \) is the one that blows down, and thus we have got an isomorphism \( Y^t \cong M^t \) for \( t = \lambda_1 + \lambda_2 + \epsilon \) and \( \epsilon > 0 \) small enough. Following this process, one notices that the only possible reduced spaces that can appear are \( \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2} \) for \( k \leq 3 \), which have associated rigid families of symplectic structures, by Theorem 4.1 and Lemma 4.2. The classes that blow down are determined by the numbers \( \lambda_i \) and thus the same arguments we have used before give the desired isomorphism \( M^{t \cong Y^t} \). The process continues until \( t \) reaches the maximum, obtaining the desired isomorphism, in the case when all the critical values \( \lambda_i \) are different.

The case when two or three of the values \( \lambda_i \) agree, one has to pay attention to the validity of condition 1 in Definition 1.4 there might be several components in the group of symplectomorphisms induced by symplectomorphisms that permute exceptional classes of equal size in the blow up. Nevertheless, the reduced spaces are still rigid, since the symplectomorphisms in the base that we are considering are those which also are in the identity component of the diffeomorphisms which is connected by 22 [4].

The more general results stated in the introduction now follow similarly.

Proof of Theorem 1.1. Recall that \((M, H, \omega) \in \mathbf{HSymp}_{2n}, \text{ and } \mathcal{C}(M) \) is the set of critical values. We are assuming that each non-extremal critical value \( \lambda \in \mathcal{C}(M) = \{\lambda_0, \ldots, \lambda_s\} \) contains only fixed points of (co)index 2 and that
all the families of reduced spaces \((\mathcal{M}_i, \mathcal{m}_i)_{t \in (\lambda_i, \lambda_{i+1})}\) for consecutive critical values are rigid. Therefore, by Lemma 3.10 and Theorem 3.5 the regular slices and germs are completely determined by the fixed point data. Given any other manifold with the same fixed point data. We use the same procedure as in the proof of Theorem 1.7 to build the isomorphism starting from the minimum and gluing slices and germs.

**Fig. 2.** The fixed points, their critical values and some reduced spaces of a manifold with isolated fixed points (Example A). Figures a), b), c) and d) represent the reduced spaces for \(t\) in the intervals \((0, \lambda_1), (\lambda_3, \lambda_1 + \lambda_2), (\lambda_1 + \lambda_2, \lambda_1 + \lambda_3), (\lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_3)\) respectively. Note that as \(t \to \lambda_1 + \lambda_2\) the exceptional sphere \(L - E_1 - E_2\) blows down. d) is the manifold obtained after blowing down the exceptional spheres \(L - E_1 - E_j\). a) and d) are diffeomorphic (via Cremona transformation) but the Euler class of the principal bundles associated to these reduced spaces differ by a sign.

**Proof of Theorem 1.7** Recall that we are assuming that for each \(\lambda \in C(M)\) the fixed point submanifolds \((\mathcal{T}_{\lambda_0}, \mathcal{m}_{\lambda_0})\) are isolated points or surfaces of index 2.

The proof is again by exhaustion on the sets \(H^{-1}(-\infty, t)\). Following the same notations as before, suppose \(C(M) = \{\lambda_0 = 0, \lambda_1, \ldots, \lambda_s\}\). Starting at the minimum, the normal form given by the Darboux-Weinstein Theorem applied to the fixed point set at \(\lambda_0\), shows that the minimal germ is determined by the index and the submanifold \((\mathcal{T}_{\lambda_0}, \mathcal{m}_{\lambda_0})\) (with its normal bundle). By taking the reduction at any level \(t\) close enough to \(\lambda_0\) one obtains the diffeomorphism type of the reduced bundle \(P_t \to \mathcal{M}_t\). The rigidity of the reduced spaces and Theorem 3.5 ensures that the isomorphism type of the regular slice for the interval \((\lambda_0, \lambda_1)\) is determined by the small fixed point data at \(\lambda_0, \lambda_1\).

Let \(t = \lambda_1 - \varepsilon\), for \(\varepsilon > 0\) small enough, thus any other manifold with the same fixed point data is isomorphic to \(M\) up to level \(t\) and thus they are identified at level \(t\) (cf. Definition 3.11), satisfying the hypotheses of Lemma 3.13. This shows that the critical germ at \(\lambda_1\) is determined, and therefore any other space with the same fixed point data is identified with \(M\) up to level \(t = \lambda_1 + \varepsilon\), for some \(\varepsilon > 0\) small enough. Examples 3.7 and 3.8 show that the principal bundle \(P_{\lambda_2} \to \mathcal{M}_{\lambda_2}\) is also determined by the known data at \(\lambda_1\). We continue this process, until \(t\) reaches \(\lambda_s\).

Finally, using the analysis of Section 3.4 one can show that this theorems hold even in the non-simple case. The details are left to the reader.

**References**

[1] Miguel Abreu and Dusa McDuff. Topology of symplectomorphism groups of rational ruled surfaces. *J. Amer. Math. Soc.*, 13(4):971–1009, 2000. ISSN 0894-0347.

[2] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984. ISSN 0040-9383.

[3] Weimin Chen. Pseuoholomorphic curves in four-orbifolds and some applications. In *Proceedings of Conference on Geometry and Topology of Manifolds*, McMaster University. May 14-18. Fields Institute, May 2004.

[4] Jonathan David Evans. Symplectic mapping class groups of some Stein and rational surfaces, 2009. URL [http://www.citebase.org/abstract?id=oai:arXiv.org:0909.5622](http://www.citebase.org/abstract?id=oai:arXiv.org:0909.5622)
[5] Leonor Godinho. Blowing up symplectic orbifolds. *Annals of Global Analysis and Geometry*, (20):117–162, 2001.

[6] Leonor Godinho. On certain symplectic circle actions. *J. Symplectic Geom.*, 3(3):357–383, 2005. ISSN 1527-5256.

[7] Eduardo Gonzalez. Quantum cohomology and $S^1$-actions with isolated fixed points. *Trans. Amer. Math. Soc.*, 358(7):2927–2948, 2006. ISSN 0002-9947.

[8] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985. ISSN 0020-9910.

[9] V. Guillemin and S. Sternberg. Birational equivalence in the symplectic category. *Invent. Math.*, 97(3):485–522, 1989. ISSN 0020-9910.

[10] Yael Karshon. Periodic Hamiltonian flows on four-dimensional manifolds. *Mem. Amer. Math. Soc.*, 141 (672):viii+71, 1999. ISSN 0065-9266.

[11] Yael Karshon and Susan Tolman. Centered complexity one Hamiltonian torus actions. *Trans. Amer. Math. Soc.*, 353(12):4831–4861 (electronic), 2001. ISSN 0002-9947.

[12] François Lalonde and Martin Pinsonnault. The topology of the space of symplectic balls in rational 4-manifolds. *Duke Math. J.*, 122(2):347–397, 2004. ISSN 0012-7094.

[13] Hui Li. Semi-free Hamiltonian circle actions on 6-dimensional symplectic manifolds. *Trans. Amer. Math. Soc.*, 355(11):4543–4568 (electronic), 2003. ISSN 0002-9947.

[14] Hui Li. On the construction of certain 6-dimensional symplectic manifolds with Hamiltonian circle actions. *Trans. Amer. Math. Soc.*, 357(3):983–998 (electronic), 2005.

[15] Mikiya Masuda and Taras Panov. Semifree circle actions, Bott towers, and quasitoric manifolds, 2006. URL http://arXiv.org:math/0607094v2.

[16] Dusa McDuff. The moment map for circle actions on symplectic manifolds. *J. Geom. Phys.*, 5(2):149–160, 1988. ISSN 0393-0440.

[17] Dusa McDuff. The structure of rational and ruled symplectic 4-manifolds. *J. Amer. Math. Soc.*, 3(3):679–712, 1990. ISSN 0894-0347.

[18] Dusa McDuff. From symplectic deformation to isotopy. In *Topics in symplectic 4-manifolds (Irvine, CA, 1996)*, First Int. Press Lect. Ser., I, pages 85–99. Internat. Press, Cambridge, MA, 1998.

[19] Dusa McDuff. Lectures on groups of symplectomorphisms. *Rend. Circ. Mat. Palermo (2) Suppl.*, (72):43–78, 2004.

[20] Dusa McDuff. Some 6-dimensional Hamiltonian $S^1$-manifolds. *J Topology*, 2(3):589–623, 2009. doi: 10.1112/jtopol/jtp023. URL http://jtopol.oxfordjournals.org/cgi/content/abstract/2/3/589.

[21] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998. ISBN 0-19-850451-9.

[22] Martin Pinsonnault. In preparation.

[23] Paul Seidel. Lectures on four-dimensional Dehn twists. In *Symplectic 4-manifolds and algebraic surfaces*, volume 1938 of *Lecture Notes in Math.*, pages 231–267. Springer, Berlin, 2008.

[24] Susan Tolman and Jonathan Weitsman. On semifree symplectic circle actions with isolated fixed points. *Topology*, 39(2):299–309, 2000. ISSN 0040-9383.

[25] C. T. C. Wall. Classification problems in differential topology. V. On certain 6-manifolds. *Invent. Math. 1 (1966), 355-374; corrigendum, ibid, 2:306, 1966.*