Transition Probabilities for Flavor Eigenstates of Non-Hermitian Hamiltonians in the PT-Broken Phase

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Abstract

We investigate the transition probabilities for the “flavor” eigenstates in the two-level quantum system, which is described by a non-Hermitian Hamiltonian with the parity and time-reversal (PT) symmetry. Particularly, we concentrate on the so-called PT-broken phase, where two eigenvalues of the non-Hermitian Hamiltonian turn out to be a complex conjugate pair. In this case, we find that the transition probabilities will be unbounded in the limit of infinite time $t \to +\infty$. After making a connection between the PT-broken phase and the neutral-meson system in particle physics, we observe that the infinite-time behavior of the transition probabilities can be attributed to the negative decay width of one eigenstate of the non-Hermitian Hamiltonian. We also present some brief remarks on the situation at the so-called exceptional point, where both the eigenvalues and eigenvectors of the Hamiltonian coalesce.
1 Introduction

Non-Hermitian Hamiltonians with the joint parity and time-reversal (PT) symmetry have recently attracted a lot of attention [1] and very interesting applications have been found for a number of physical systems in particle physics, nuclear physics, optics, electronics, and many others [2]. In the existing literature, the two-level system with PT-symmetric non-Hermitian Hamiltonians [3,4] has been extensively investigated, as a simple but instructive example, to explore and clarify all related conceptual issues [5–7]. However, it is worthwhile to mention that the transition amplitudes and probabilities between the “flavor” eigenstates have rarely been studied, except for some general discussion in Refs. [8,9].

Since the phenomenon of “flavor” mixing is quite common in particle physics, such as flavor oscillations of massive neutrinos [10] and the neutral-meson system $P^0 - P^0$ [11] (e.g., $K^0 - K^0$, $D^0 - D^0$, and $B^0 - \bar{B}^0$), it is intriguing to consider the transitions among “flavor” eigenstates in the system with PT-symmetric non-Hermitian Hamiltonians [12,13]. In our previous work [13], we calculated the transition probabilities for the “flavor” eigenstates in the scenario, where the PT symmetry is always preserved and two eigenvalues of the non-Hermitian Hamiltonian are real, which is known as the PT-symmetric phase. In the present work, we aim to extend the previous study in the PT-symmetric phase to the PT-broken phase. The primary motivation for such an extension is at least two-fold.

First, by “flavor” eigenstates of a two-level quantum system with a non-Hermitian Hamiltonian $\mathcal{H}$, we mean the complete set of basis vectors $\{\left|u_\beta\right>\}$ (for $\beta = a, b$), in which the matrix representation of the Hamiltonian is given by $\mathcal{H}_{\alpha\beta} \equiv \left< u_\alpha \right| \mathcal{H} \left| u_\beta \right>$ and the left vectors $\left< u_\alpha \right> \equiv \left| u_\alpha \right>^\dagger$ and $\left< u_\beta \right> \equiv \left| u_\beta \right>^\dagger$ (for $\alpha, \beta = a, b$) have been defined as in conventional quantum mechanics. Therefore, the transition amplitudes in the following discussion will be referred to the projection of the time-evolved flavor eigenstates $\{\left|u_\alpha(t)\right>\}$ into their initial states $\{\left|u_\beta\right>\}$. However, whenever the transition amplitudes $A_{\alpha\beta} \equiv \left< u_\beta \right| u_\alpha(t) \left> \right>$ are calculated, we will clearly indicate the exact definition of the involved inner product as well as that of the left state vectors. As mentioned, for PT-symmetric non-Hermitian Hamiltonians, the transition amplitudes $A_{\alpha\beta}$ and the corresponding probabilities $P_{\alpha\beta} \equiv |A_{\alpha\beta}|^2$ have been explicitly computed and extensively studied in Ref. [13]. Hence, it is a natural continuation to extend the investigation to the PT-broken phase.

Second, in contrast to the PT-symmetric phase, where two eigenvalues of the non-Hermitian Hamiltonian are real, the PT-broken phase will be complicated by a complex-conjugate pair of eigenvalues. As is well known, if PT symmetry is maintained, it is always possible to find a similarity transformation that converts a non-Hermitian Hamiltonian into its Hermitian counterpart [5,7,13]. However, this is impossible for the PT-broken phase, rendering it rather different. Hence, in this particular case, the transition probabilities deserve a dedicated study.

The remaining part of this work is organized as follows. In Sec. 2, we present the general formalism for the investigation of PT-symmetric non-Hermitian Hamiltonians and summarize the main features of the PT-symmetric phase, the PT-broken phase, and the
exceptional point, where the transition between these two phases occurs. Then, in Sec. 3, the transition amplitudes and probabilities in the PT-broken phase will be introduced and studied, where the connection between the PT-broken phase and the neutral-meson system is also performed. Finally, in Sec. 4, we summarize the main results and draw our conclusions.

2 General Formalism

For a general discussion about the properties of PT-symmetric non-Hermitian Hamiltonians and their applications, one should be referred to the excellent review by Bender [1] and references therein. Particularly, in this work, we focus on the simple two-level system, for which the Hamiltonian is diagonalizable and space-time independent. The space-reflection operator $\mathcal{P}$ is defined as [13]

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the time-reversal operator $\mathcal{T}$ is taken to be just the complex conjugation $\mathcal{K}$, namely, $\mathcal{T}\mathcal{O}\mathcal{T}^{-1} = \mathcal{O}^*$ for any operators $\mathcal{O}$ in the Hilbert space. The most general form of the Hamiltonian for the two-level system is given by

$$\mathcal{H} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\{a, b, c, d\}$ are arbitrary complex constants. The PT symmetry of the Hamiltonian system requires that $[\mathcal{P}\mathcal{T}, \mathcal{H}] = 0$, so we have

$$\langle \mathcal{P}\mathcal{T}\mathcal{H} \rangle \Psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \Psi^* = \begin{pmatrix} c^* & d^* \\ a^* & b^* \end{pmatrix} \Psi^*,$$

$$\langle \mathcal{H}\mathcal{P}\mathcal{T} \rangle \Psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi^* = \begin{pmatrix} b & a \\ d & c \end{pmatrix} \Psi^*,$$

where $\Psi$ stands for any vectors in the Hilbert space that the operators are acting on. From Eqs. (3) and (4), one can recognize that the PT symmetry of the system implies that $a = d^*$ and $b = c^*$. Therefore, the most general non-Hermitian Hamiltonian $\mathcal{H}$ actually contains only four degrees of freedom (in terms of the number of real parameters), when it respects the PT symmetry. This is equal to the number of free parameters in the two-level system with the Hermitian Hamiltonian, where $a$ and $d$ are real while $b = c^*$.

For later convenience, we adopt the following parametrization of the most general PT-symmetric non-Hermitian Hamiltonian, viz.,

$$\mathcal{H} = \begin{pmatrix} \rho e^{i\varphi} & \sigma e^{i\phi} \\ \sigma e^{-i\phi} & \rho e^{-i\varphi} \end{pmatrix},$$

where all parameters $\{\rho, \varphi\}$ and $\{\sigma, \phi\}$ are real and time-independent. For a recent study on time-dependent parameters in $\mathcal{H}$, see Refs. [14,15]. With the Hamiltonian in Eq. (5), one can
immediately figure out the eigenvalues of the system and their corresponding eigenvectors. More explicitly, the characteristic equation for this system is given by

\[ \det (\lambda \mathbb{1}_2 - \mathcal{H}) = 0 , \]  

where \( \lambda \) denotes the eigenvalues and \( \mathbb{1}_2 \) is the 2 \times 2 identity matrix. From Eq. \( (6) \), it is straightforward to find out two eigenvalues \( \lambda_\pm \) as

\[ \lambda_\pm = \rho \cos \varphi \pm \sqrt{\sigma^2 - \rho^2 \sin^2 \varphi} . \]  

(7)

Under the condition that \( \rho^2 \sin^2 \varphi < \sigma^2 \) is satisfied, the two eigenvalues are real. If this condition is not satisfied, namely, \( \rho^2 \sin^2 \varphi \geq \sigma^2 \), we obtain either (i) two complex eigenvalues (if \( \rho^2 \sin^2 \varphi > \sigma^2 \) holds)

\[ \lambda_\pm = \rho \cos \varphi \pm i \sqrt{\rho^2 \sin^2 \varphi - \sigma^2} , \]  

(8)

which are complex conjugates to each other, or (ii) a degenerate real eigenvalue \( \lambda_\pm = \lambda_0 = \rho \cos \varphi \) with multiplicity 2, since \( \rho^2 \sin^2 \varphi = \sigma^2 \) holds. In Fig. 1 we present the eigenvalues as a function of \( \sin \varphi \) for different choices of the ratio of the parameters \( \rho \) and \( \sigma \). The eigenvalues are displayed in the PT-symmetric phase (\( \rho^2 \sin^2 \varphi < \sigma^2 \): two real eigenvalues, see Subsec. 2.1) and the PT-broken phase (\( \rho^2 \sin^2 \varphi > \sigma^2 \): two complex-conjugate eigenvalues, see Subsec. 2.2) as well as the exceptional points are indicated (\( \rho^2 \sin^2 \varphi = \sigma^2 \): one degenerate real eigenvalue, see Subsec. 2.3). Some helpful comments on the eigenvalues and their corresponding eigenvectors of the non-Hermitian Hamiltonian \( \mathcal{H} \) in Eq. \( (5) \) are in order.

2.1 PT-Symmetric Phase

As mentioned before, the two eigenvalues in Eq. \( (7) \) are real if the condition \( \rho^2 \sin^2 \varphi < \sigma^2 \) is fulfilled. This is usually called the PT-symmetric phase of the system. In this case, we write the two eigenvectors corresponding to \( E_\pm = \lambda_\pm = \lambda_0 = \rho \cos \varphi \) as \( |u_\pm\rangle = (a_+, b_+)^T \) and \( |u_-\rangle = (a_-, b_-)^T \), where \( a_\pm \) and \( b_\pm \) are all complex numbers, and solve the equations

\[ \mathcal{H} \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} = E_+ \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} , \]  

(9)

Using Eqs. \( (5) \), \( (7) \), and \( (9) \), we obtain the following solutions

\[ \frac{a_+}{b_+} = i e^{-i(\alpha - \phi)} , \quad \frac{a_-}{b_-} = i e^{+i(\alpha + \phi)} , \]  

(10)

where \( \cos \alpha \equiv (\rho \sin \varphi)/\sigma \) and \( \sin \alpha = (\sigma^2 - \rho^2 \sin^2 \varphi)^{1/2}/\sigma \) have been defined\(^1\). For clarity, the parameter space of \( \{ \rho, \varphi \} \) and \( \{ \sigma, \phi \} \) will be constrained as follows. First, \( \rho \) and \( \sigma \) are moduli of the matrix elements of the Hamiltonian in Eq. \( (5) \) such that \( \rho \geq 0 \) and \( \sigma \geq 0 \) hold.

\(^1\)Note that the definition of the parameter \( \alpha \) differs from that in Ref. \( 13 \), where \( \sin \alpha \equiv (\rho \sin \varphi)/\sigma \). The reason for such a change is to make a coherent presentation in both the PT-symmetric and -broken phases.
Figure 1: Illustration for the real (solid curves) and imaginary (dotted curves) parts of the normalized eigenvalues \( \lambda_{\pm}/\sigma = \xi \sqrt{1 - \sin^2 \varphi} \pm \sqrt{1 - \xi^2 \sin^2 \varphi} \) as functions of \( \sin \varphi \) for three different choices of the ratio of the parameters \( \rho \) and \( \sigma \), i.e., \( \xi \equiv \rho/\sigma = 2 \) (red curves), 3 (orange curves), 4 (yellow curves), in the \( \text{PT-symmetric phase} \) (none or less shaded areas) and the \( \text{PT-broken phase} \) (shaded areas). The corresponding exceptional points are marked by black points (‘\( \bullet \)’) and the black thin dashed curve shows the trajectory of the exceptional points.

Second, \( E_{\pm} \) are conventionally identified with the energy eigenvalues, so we assume both of them to be non-negative. Such a requirement implies \( E_- = \rho \cos \varphi \geq 0 \) for \( \sigma^2 = \rho^2 \sin^2 \varphi \), i.e., \( \varphi \in [-\pi/2, \pi/2] \). Furthermore, we must have \( \rho^2 \geq \sigma^2 \geq \rho^2 \sin^2 \varphi \). Otherwise, for \( \sigma > \rho \), we get \( E_- = \rho \cos \varphi - \sqrt{\sigma^2 - \rho^2 \sin^2 \varphi} < \rho \cos \varphi - \sqrt{\rho^2 - \rho^2 \sin^2 \varphi} = 0 \). In summary, we assume \( \rho \geq \sigma > \rho \sin \varphi > 0 \) with \( \varphi \in [0, \pi/2] \) and \( \phi \in [0, 2\pi] \). In this parameter space, we have \( \alpha \in [0, \pi/2] \). Certainly, one can also choose a different range of \( \varphi \) such that the allowed region of \( \alpha \) will be different. However, a different choice of the parameter space will essentially not affect our discussion.

In accordance with the general solutions in Eq. (10), we choose the following forms of the corresponding eigenvectors

\[
|u_+\rangle = N_+ \left( e^{i\pi/4} \cdot e^{-i\alpha_-/2} e^{-i\pi/4} \cdot e^{+i\alpha_+/-2} \right), \quad |u_-\rangle = N_- \left( e^{i\pi/4} \cdot e^{+i\alpha_-/2} e^{-i\pi/4} \cdot e^{-i\alpha_+/-2} \right),
\]

where \( N_\pm \) are two arbitrary normalization constants and \( \alpha_\pm \equiv \alpha \pm \phi \) have been introduced. In order to fix the normalization constants, we compute the explicit \( \mathcal{PT} \)-inner products of
In our choice of the phase convention for $N$ the same real eigenvalues $E$ Hamiltonian in Eq. (17). First, we have to find out the eigenvectors of these two eigenvectors, i.e.,

$$\langle u_+|u_+\rangle_{\mathcal{P}\mathcal{T}} \equiv (\mathcal{P}\mathcal{T}|u_+\rangle)^T \cdot |u_+\rangle = +2|N_\pm|^2 \sin \alpha_\pm,$$

(12)

$$\langle u_-|u_-\rangle_{\mathcal{P}\mathcal{T}} \equiv (\mathcal{P}\mathcal{T}|u_-\rangle)^T \cdot |u_-\rangle = -2|N_\pm|^2 \sin \alpha_+,$$

(13)

$$\langle u_+|u_+\rangle_{\mathcal{P}\mathcal{T}} \equiv (\mathcal{P}\mathcal{T}|u_+\rangle)^T \cdot |u_-\rangle = -2N_+^* N_- \sin \phi,$$

(14)

$$\langle u_-|u_+\rangle_{\mathcal{P}\mathcal{T}} \equiv (\mathcal{P}\mathcal{T}|u_-\rangle)^T \cdot |u_+\rangle = -2N_-^* N_+ \sin \phi,$$

(15)

where the superscript “$T$” denotes matrix transpose. Demanding the conventional normalization conditions [1], namely,

$$\langle u_\pm|u_\pm\rangle_{\mathcal{P}\mathcal{T}} \equiv (\mathcal{P}\mathcal{T}|u_\pm\rangle)^T \cdot |u_\pm\rangle = \pm 1,$$

(16)

$$\langle u_\pm|u_\pm\rangle_{\mathcal{P}\mathcal{T}} \equiv (\mathcal{P}\mathcal{T}|u_\pm\rangle)^T \cdot |u_\pm\rangle = 0,$$

one can determine the constants $N_\pm$ up to an overall phase. The normalization conditions in the first identity in Eq. (16) lead to $|N_\pm|^2 = 1/(2 \sin \alpha_\pm)$, while the orthogonality conditions in the second identity in Eq. (16) give rise to $\phi = 0$ or $\phi = \pi$. Therefore, we conclude that the PT symmetry together with the orthogonality of the two eigenvectors under the $\mathcal{P}\mathcal{T}$-inner product justifies the particular form of the non-Hermitian Hamiltonian in Eq. (5) with $\phi = 0$ or $\phi = \pi$. Thus, in the subsequent discussion about the PT-symmetric phase, we will concentrate on this particular form for a PT-symmetric non-Hermitian Hamiltonian, i.e.,

$$\mathcal{H} = \begin{pmatrix} \rho e^{+i\varphi} & \sigma \\ \sigma & \rho e^{-i\varphi} \end{pmatrix},$$

(17)

and fix the normalization constants as $N_+ = 1/\sqrt{2 \sin \alpha}$ and $N_- = i/\sqrt{2 \sin \alpha}$. The eigenvectors are given in Eq. (11), but now with $\alpha_\pm = \alpha$ for $\phi = 0$, and can be rewritten as

$$|u_+\rangle = \frac{1}{\sqrt{2 \sin \alpha}} \begin{pmatrix} e^{+i\pi/4} \cdot e^{-i\alpha/2} \\ e^{-i\pi/4} \cdot e^{+i\alpha/2} \end{pmatrix}, \quad |u_-\rangle = \frac{i}{\sqrt{2 \sin \alpha}} \begin{pmatrix} e^{+i\pi/4} \cdot e^{+i\alpha/2} \\ e^{-i\pi/4} \cdot e^{-i\alpha/2} \end{pmatrix}.$$
where $\langle v_\pm | \equiv |v_\pm \rangle^\dagger$ and $\langle u_\pm | \equiv |u_\pm \rangle^\dagger$. Then, one can verify that $\det \eta = \csc^2 \alpha - \cot^2 \alpha = 1 > 0$ and the inverse of $\eta$ is given by

$$\eta^{-1} = \sum_{s=\pm} |u_s \rangle \langle u_s| = \begin{pmatrix} \csc \alpha & +i \cot \alpha \\ -i \cot \alpha & \csc \alpha \end{pmatrix} .$$

(22)

By construction, the relation $\eta H \eta^{-1} = H^\dagger$ holds, so one can easily prove that there exists a charge-conjugation operator $C$ defined as

$$C \equiv P^{-1} \eta = \eta^{-1} P = \begin{pmatrix} +i \cot \alpha & \csc \alpha \\ \csc \alpha & -i \cot \alpha \end{pmatrix} ,$$

(23)

satisfying the commutation relation

$$[C, H] = (P^{-1} \eta) H - H (P^{-1} \eta) = P^{-1} (\eta H \eta^{-1} - PHP^{-1}) \eta = 0 .$$

(24)

This non-Hermitian Hamiltonian system respects both the $\mathcal{C}$ and $\mathcal{PT}$ symmetries, and thus, the $\mathcal{CPT}$ symmetry. Since the $\mathcal{PT}$-inner product is actually not positive-definite (due to $\det \mathcal{P} = -1 < 0$), it is necessary to introduce the $\eta$- and $\mathcal{CPT}$-inner products for the definitions of the transition amplitudes and probabilities [18]. More explicitly,

- **The $\eta$-inner product** for any two state vectors $|\psi\rangle$ and $|\chi\rangle$ reads

$$\langle \psi | \chi \rangle_\eta \equiv \langle \psi | \eta | \chi \rangle = |\psi\rangle^\dagger \cdot \eta \cdot |\chi\rangle .$$

(25)

- **The $\mathcal{CPT}$-inner product** for any two state vectors $|\psi\rangle$ and $|\chi\rangle$ reads

$$\langle \psi | \chi \rangle_{\mathcal{CPT}} \equiv (\mathcal{CPT} |\psi\rangle)^T \cdot |\chi\rangle = |\psi\rangle^\dagger \cdot \mathcal{P} \mathcal{C} \cdot |\chi\rangle = \langle \psi | \chi \rangle_\eta ,$$

(26)

where $\mathcal{PC} = \eta$ from the definition of the $\mathcal{C}$ operator in Eq. (23) has been used in the last step in Eq. (26).

Therefore, the $\eta$- and $\mathcal{CPT}$-inner products are equivalent and we can use either of them to calculate the transition amplitudes and probabilities between two quantum states. These calculations have been performed in Ref. [13].

### 2.2 PT-Broken Phase

Under the condition $\sigma^2 < \rho^2 \sin^2 \varphi$, one can check that $[\mathcal{PT}, H] = 0$ remains to be valid for the most general form of the Hamiltonian $H$ in Eq. (5). This should be the case as we have derived the general form of the PT-symmetric Hamiltonian for arbitrary values of the parameters. However, as we have shown in Eq. (8), the Hamiltonian (5) has two complex eigenvalues $E_{\pm}^\prime = \lambda_{\pm}$, which in this PT-broken phase are labeled by primes in order to avoid any confusion with the ones in the PT-symmetric phase.
Following the same procedure as in the PT-symmetric phase to calculate the eigenvectors $|u'_\pm\rangle \equiv (a'_\pm, b'_\pm)^T$ corresponding to the eigenvalues $E'_\pm$, we need to solve the equations

$$H \begin{pmatrix} a'_\pm \\ b'_\pm \end{pmatrix} = E'_\pm \begin{pmatrix} a'_\pm \\ b'_\pm \end{pmatrix} .$$

Introducing $\cosh \alpha' \equiv (\rho \sin \varphi)/\sigma$ and $\sinh \alpha' = (\rho^2 \sin^2 \varphi - \sigma^2)^{1/2}/\sigma$, where the identity $\cosh^2 \alpha' - \sinh^2 \alpha' = 1$ can be easily verified, we obtain the solutions to Eq. (27) as

$$a'_+ / b'_+ = i e^{+(\alpha'+i\phi)} , \quad a'_- / b'_- = i e^{-(\alpha'-i\phi)} .$$

Making a comparison between Eq. (10) in the PT-symmetric phase and Eq. (28) in the PT-broken phase, one observes the connection between these two cases by simply identifying $\alpha' = -i\alpha$. Using Eq. (28), we explicitly rewrite the eigenvectors as

$$|u'_+\rangle = N'_+ \begin{pmatrix} e^{+i\pi/4} \cdot e^{+(\alpha'+i\phi)/2} \\ e^{-i\pi/4} \cdot e^{-(\alpha'+i\phi)/2} \end{pmatrix} , \quad |u'_-\rangle = N'_- \begin{pmatrix} e^{-i\pi/4} \cdot e^{-\alpha'/2} \\ e^{+i\pi/4} \cdot e^{-(\alpha'+i\phi)/2} \end{pmatrix}$$

and try to fix the two normalization constants $N'_{\pm}$ by examining the $\mathcal{PT}$-inner products of these two eigenvectors. As in Eqs. (12)-(15), we compute the $\mathcal{PT}$-inner products, namely,

$$\langle u'_+|u'_+\rangle_{\mathcal{PT}} = -2|N'_+|^2 \sin \phi , \quad \langle u'_-|u'_-\rangle_{\mathcal{PT}} = -2|N'_-|^2 \sin \phi ,$$

$$\langle u'_+|u'_-\rangle_{\mathcal{PT}} = -2iN'_+N'_- \sinh(\alpha' - i\phi) , \quad \langle u'_-|u'_+\rangle_{\mathcal{PT}} = +2iN'_+N'_- \sinh(\alpha' + i\phi) .$$

At first sight, it seems that one can choose proper values of $N'_\pm$ to guarantee the orthogonality conditions $\langle u'_\pm|u'_\pm\rangle_{\mathcal{PT}} = 0$. However, as one can see from Eq. (31), this is only possible if both $\alpha' = 0$ and $\sin \phi = 0$ hold, or equivalently, at the so-called exceptional point with $\rho^2 \sin^2 \varphi = \sigma^2$.

Therefore, for $\rho^2 \sin^2 \varphi > \sigma^2$ under consideration, we have to determine the normalization constants $N'_\pm$ by requiring $\langle u'_\pm|u'_\pm\rangle_{\mathcal{PT}} = 0$ and $\langle u'_\pm|u'_\pm\rangle_{\mathcal{PT}} = +1$. These requirements differ significantly from those in the PT-symmetric phase. From Eq. (30) with $\langle u'_\pm|u'_\pm\rangle_{\mathcal{PT}} = 0$, we immediately get $\phi = 0$ (or $\phi = \pi$), which is also consistent with our previous convention in the PT-symmetric phase. In addition, from Eq. (31) with $\phi = 0$ and $\langle u'_\pm|u'_\pm\rangle_{\mathcal{PT}} = +1$, we obtain $N'_+ = e^{-i\pi/4}/\sqrt{2\sinh \alpha'}$ and $N'_- = e^{+i\pi/4}/\sqrt{2\sinh \alpha'}$, and thus, using Eq. (29), we find the two eigenvectors as

$$|u'_+\rangle = \frac{e^{-i\pi/4}}{\sqrt{2\sinh \alpha'}} \begin{pmatrix} e^{+i\pi/4} \cdot e^{+\alpha'/2} \\ e^{-i\pi/4} \cdot e^{-\alpha'/2} \end{pmatrix} = \frac{1}{\sqrt{2\sinh \alpha'}} \begin{pmatrix} e^{+\alpha'/2} \\ -ie^{-\alpha'/2} \end{pmatrix} ,$$

$$|u'_-\rangle = \frac{e^{+i\pi/4}}{\sqrt{2\sinh \alpha'}} \begin{pmatrix} e^{+i\pi/4} \cdot e^{-\alpha'/2} \\ e^{-i\pi/4} \cdot e^{+\alpha'/2} \end{pmatrix} = \frac{1}{\sqrt{2\sinh \alpha'}} \begin{pmatrix} e^{-\alpha'/2} \\ +ie^{+\alpha'/2} \end{pmatrix} .$$

It is helpful to make some comments on the further connection between the PT-symmetric and PT-broken phases. In the latter case, we have two eigenvalues $E'_\pm = \rho \cos \varphi \pm i \sigma \sin \alpha'$,
in which the replacement of \( \alpha' = -i\alpha \) leads to the two eigenvalues \( E_{\pm} \) in the former case. At the same time, if we replace \( \alpha' \) by \(-i\alpha\) everywhere in Eqs. (32) and (33), the eigenvectors \( |u'_{\pm}\rangle \) will reduce to \( |u_{\pm}\rangle \) in Eq. (18).

Nevertheless, given the eigenvectors \( |u'_{\pm}\rangle \) in Eqs. (32) and (33), one can check that \( \mathcal{PT}|u'_{\pm}\rangle = |u_{\pm}\rangle \) and \( \mathcal{H}|u'_{\pm}\rangle = E'_{\pm}|u_{\pm}\rangle \), indicating that the energy eigenstates \( |u'_{\pm}\rangle \) are not eigenstates of the \( \mathcal{PT} \) operator. This is the reason why this scenario is called the PT-broken phase. However, this is not in contradiction with the fact that \( [\mathcal{PT}, \mathcal{H}] = 0 \). Since \( \mathcal{PT} \) is an anti-linear operator and \( E'_{\pm} = E''_{\pm} \), one should note that \( \mathcal{PT} E'_{\pm}|u'_{\pm}\rangle = E''_{\pm} \mathcal{PT}|u'_{\pm}\rangle \). More explicitly, we have

\[
\mathcal{PT}\mathcal{H}|u'_{\pm}\rangle = \mathcal{PT} E'_{\pm}|u'_{\pm}\rangle = E''_{\pm}|u'_{\pm}\rangle, \quad \mathcal{H}\mathcal{PT}|u'_{\pm}\rangle = \mathcal{H}|u'_{\pm}\rangle = E'_{\pm}|u'_{\pm}\rangle,
\]

implying \( [\mathcal{PT}, \mathcal{H}] = 0 \). This is quite different from the PT-symmetric phase, in which the two eigenvalues \( E_{\pm} \) are real.

Now, we apply the bi-orthogonal formalism to the non-Hermitian Hamiltonian system in the PT-broken phase. As before, we have to find out the eigenvectors of \( \mathcal{H}' \), namely,

\[
\mathcal{H}'|v'_{\pm}\rangle = E'_{\pm}|v'_{\pm}\rangle.
\]

The identity \( \mathcal{P}\mathcal{H}'\mathcal{P}^{-1} = \mathcal{H} \) is still applicable, so we multiply Eq. (35) on both sides from the left by the \( \mathcal{P} \) operator and then obtain

\[
(\mathcal{P}\mathcal{H}'\mathcal{P}^{-1}) \mathcal{P}|v'_{\pm}\rangle = E'_{\pm}\mathcal{P}|v'_{\pm}\rangle,
\]

indicating \( \mathcal{P}|v'_{\pm}\rangle \propto |u'_{\pm}\rangle \). Identifying \( |v'_{\pm}\rangle = \mathcal{P}|u'_{\pm}\rangle \), we can immediately compute the metric operator \( \eta' \), i.e.,

\[
\eta' \equiv \sum_{s=\pm} |v'_{+s}\rangle\langle v'_{-s}| = \mathcal{P} (|u'_{+}\rangle\langle u'_{-}| + |u'_{-}\rangle\langle u'_{+}|) \mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

and its inverse

\[
\eta'^{-1} = \sum_{s=\pm} |u'_{+s}\rangle\langle u'_{-s}| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Note that \( \eta' = \eta'^{-1} = \mathcal{P} \) with \( \det \eta' = -1 < 0 \), so it is not positive-definite. In this case, it is impossible to find a Hermitian matrix to convert the non-Hermitian Hamiltonian into a Hermitian one via a similarity transformation. Furthermore, the \( \mathcal{C} \) operator is given by \( \mathcal{C} = \mathcal{P}^{-1}\eta' = \eta'^{-1}\mathcal{P} = 1_2 \), which turns out to be the trivial \( 2 \times 2 \) identity matrix.

Similar to the PT-symmetric phase, we can define the \( \eta' \)-inner product as well as the \( \mathcal{CPT} \)-inner product as follows

- The \( \eta' \)-inner product for any two state vectors \( |\psi\rangle \) and \( |\chi\rangle \) reads

\[
\langle \psi|\chi \rangle_{\eta'} \equiv \langle \psi|\eta'|\chi \rangle = |\psi\rangle^\dagger \cdot \eta' \cdot |\chi\rangle.
\]
• The $\mathcal{CPT}$-inner product for any two state vectors $|\psi\rangle$ and $|\chi\rangle$ reads
\[
\langle \psi|\chi \rangle_{\mathcal{CPT}} \equiv (\mathcal{CPT}|\psi\rangle)^T \cdot |\chi\rangle = |\psi\rangle^\dagger \cdot \mathcal{P} \cdot |\chi\rangle = \langle \psi|\chi \rangle_{\eta'},
\]
where $\mathcal{P}C = \eta' = \mathcal{P}$ has been used in the last step.

Therefore, the two inner products are equivalent and we will not distinguish between them. In addition, since the $\mathcal{C}$ operator is trivial, these two inner products are also identical with the $\mathcal{PT}$-inner product. However, as we have mentioned, the metric operator $\eta' = \mathcal{P}$ is no longer positive-definite, and thus, the norm $\langle \psi|\psi \rangle_{\eta'}$ cannot be guaranteed to be positive. In fact, for the energy eigenstates $|u'_\pm\rangle$ and $|u'_-\rangle$, we have $\langle u'_\pm|u'_\pm \rangle_{\mathcal{PT}} = \langle u'_\pm|u'_\pm \rangle_{\eta'} = 0$ and $\langle u'_\pm|u'_\mp \rangle_{\mathcal{PT}} = \langle u'_\pm|u'_\mp \rangle_{\eta'} = +1$. The identity $\eta'\mathcal{H}\eta'^{-1} = \mathcal{H}^\dagger$, which now coincides with $\mathcal{P}\mathcal{H}\mathcal{P}^{-1} = \mathcal{H}^\dagger$, indeed leads to a unitary time evolution of the energy eigenstates.

### 2.3 Exceptional Point

Finally, let us give a brief discussion about the exceptional point (EP) at $\rho^2 \sin^2 \varphi = \sigma^2$. The EP can be identified as either the limiting case of $\alpha \to 0$ in the PT-symmetric phase or that of $\alpha' \to 0$ in the PT-broken phase. In either limit, the energy eigenvalues become degenerate $E_{\pm} \to E_0 = \rho \cos \varphi$. Moreover, for the eigenvectors $|u'_\pm\rangle$ in Eq. (18) and $|u'_\pm\rangle$ in Eqs. (32) and (33), the normalization constants $N_{\pm} \propto 1/\sqrt{\sin \alpha}$ and $N'_{\pm} \propto 1/\sqrt{\sinh \alpha'}$ are divergent in the respective limits of $\alpha \to 0$ and $\alpha' \to 0$. However, this is an artificial divergence, since $N_{\pm}$ (or $N'_{\pm}$) in the limit of $\alpha \to 0$ (or $\alpha' \to 0$) cannot be determined from the $\mathcal{PT}$-inner products of the relevant eigenvectors. The proper normalization can be taken as $\langle u_0|u_0 \rangle = 1$, with $\langle u_0 | \equiv |u_0\rangle^\dagger$, so we have

\[
|u_\pm\rangle (\text{or } |u'_\pm\rangle) \to |u_0\rangle = \frac{1}{\sqrt{2}} \left( e^{+i\pi/4} e^{-i\pi/4} \right),
\]
corresponding to the degenerate eigenvalue $E_0$ at the EP. The rich physics at the EPs and their practical applications have been briefly summarized in Refs. [19,20].

Since the time evolution of $|u_0\rangle$ is governed by the Schrödinger equation, we have $|u_0(t)\rangle = e^{-iE_0 t}|u_0\rangle$, implying that only an overall phase factor will develop and no transitions between any two quantum states are expected. This is also true for the flavor eigenstates $|u_a\rangle = (1, 0)^T$ and $|u_b\rangle = (0, 1)^T$, which are linear superpositions of the energy eigenstates.

### 3 Transitions in the PT-Broken Phase

#### 3.1 PT-Inner Product

Since the transition amplitudes and probabilities between two flavor eigenstates in the PT-symmetric phase have been examined in detail in Ref. [13], we now consider the transitions
between two flavor eigenstates in the PT-broken phase in this section. In this scenario, the Schrödinger equation for the time evolution of the energy eigenstates is

\[
\frac{1}{i} \frac{d}{dt} |u_\pm'(t)\rangle = \mathcal{H}|u_\pm'(t)\rangle = E_\pm'|u_\pm'\rangle ,
\]

and thus, we have

\[
|u_+(t)\rangle = e^{-iE_+ t}|u_+(0)\rangle = \frac{e^{-i\omega t + \gamma t}}{\sqrt{2 \sinh \alpha'}} \begin{pmatrix} e^{+\alpha'/2} \\ -ie^{-\alpha'/2} \end{pmatrix},
\]

\[
|u_-(t)\rangle = e^{-iE_- t}|u_-(0)\rangle = \frac{e^{-i\omega t - \gamma t}}{\sqrt{2 \sinh \alpha'}} \begin{pmatrix} +ie^{-\alpha'/2} \\ e^{+\alpha'/2} \end{pmatrix},
\]

where the auxiliary parameters \(\omega \equiv \rho \cos \varphi\) and \(\gamma \equiv \sqrt{\rho^2 \sin^2 \varphi - \sigma^2}\) have been defined. For reference, we list below the correspondences between the three new parameters \(\{\omega, \gamma, \alpha'\}\) and the three original ones \(\{\rho, \sigma, \varphi\}\), where \(\phi = 0\) has been assumed as in the previous section

\[
\omega = \rho \cos \varphi, \quad \gamma = \sqrt{\rho^2 \sin^2 \varphi - \sigma^2}, \quad \alpha' = \text{arccosh}\left(\frac{\rho \sin \varphi}{\sigma}\right)
\]
or

\[
\rho = \sqrt{\omega^2 + \gamma^2 \coth^2 \alpha'}, \quad \sigma = \frac{\gamma}{\sinh \alpha'}, \quad \varphi = \text{arccos}\left(\frac{\omega}{\sqrt{\omega^2 + \gamma^2 \coth^2 \alpha'}}\right).
\]

In the following, we adopt the new set of parameters \(\{\omega, \gamma, \alpha'\}\), which can be converted back to the original one by using Eq. \(46\).

To demonstrate the unitary time evolution, we calculate the norms of the time-evolved energy eigenstates to find

\[
\langle u_+'(t)|u_+'(t)\rangle_{PT} = |u_+'(t)\rangle^\dagger \cdot \mathcal{P} \cdot |u_+'(t)\rangle = 0 ,
\]

\[
\langle u_-'(t)|u_-'(t)\rangle_{PT} = |u_-'(t)\rangle^\dagger \cdot \mathcal{P} \cdot |u_-'(t)\rangle = 0 .
\]

Similarly, one can also verify that \(\langle u_+'(t)|u_+'(t)\rangle_{PT} = +1\), which is time-independent as it should be.

Next, we introduce the flavor eigenstates in which basis the explicit form of the non-Hermitian Hamiltonian is specified. Recall the diagonalization of the Hamiltonian, i.e.,

\[
A'\mathcal{H}A'^{-1} = \tilde{\mathcal{H}} \equiv \begin{pmatrix} E_+'' & 0 \\ 0 & E_-'' \end{pmatrix} \quad \Rightarrow \quad (\mathcal{H}|w_+'\rangle, \mathcal{H}|w_-\rangle) = (|w_+'\rangle E_+''|w_-'\rangle, |w_-\rangle E_-') ,
\]

where we have written \(A'^{-1} = (|w_+'\rangle, |w_-\rangle)\) with \(|w_\pm\rangle\) being two column vectors. Obviously, we can identify \(|w_\pm\rangle\) with \(|u_\pm'\rangle\) in Eqs. \(32\) and \(33\), since \(\mathcal{H}|u_\pm'\rangle = E_\pm'|u_\pm'\rangle\). Hence, it is easy to derive

\[
A'^{-1} = \frac{1}{\sqrt{2 \sinh \alpha'}} \begin{pmatrix} e^{+\alpha'/2} & +ie^{-\alpha'/2} \\ -ie^{-\alpha'/2} & e^{+\alpha'/2} \end{pmatrix} , \quad A' = \frac{1}{\sqrt{2 \sinh \alpha'}} \begin{pmatrix} e^{+\alpha'/2} & -ie^{-\alpha'/2} \\ +ie^{-\alpha'/2} & e^{+\alpha'/2} \end{pmatrix} ,
\]
where one can note that $A^{-1} = A^T$ and $A^\dagger = A'$. Furthermore, it is straightforward to verify that the flavor eigenstates are given by

$$|u'_a\rangle = (A^{-1})_{a+} |u'_+\rangle + (A^{-1})_{a-} |u'_-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ \hfill (51)

$$|u'_b\rangle = (A^{-1})_{b+} |u'_+\rangle + (A^{-1})_{b-} |u'_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ \hfill (52)

which resemble the forms in the PT-symmetric phase. This should be the case as the explicit form of the Hamiltonian in Eq. (17) remains the same in the PT-broken phase. In Eqs. (51) and (52), $(A^{-1})_{\beta s}$ for $\beta = a, b$ and $s = +, -$ denote the matrix elements of $A^{-1}$. One can also prove that the norms $\langle u'_a(t)|u'_a(t)\rangle_{PT} = \langle u'_b(t)|u'_b(t)\rangle_{PT} = 0$ and $\langle u'_a(t)|u'_b(t)\rangle_{PT} = \langle u'_b(t)|u'_a(t)\rangle_{PT} = +1$ are time-independent.

Then, we proceed to compute the amplitudes and probabilities for the transitions between two flavor eigenstates. After some calculations, the transition amplitudes are found to be

$$\mathcal{A}'_{aa} \equiv \langle u'_a|u'_a(t)\rangle_{PT} = -i e^{-i\omega t} \frac{\sinh(\gamma t)}{\sinh \alpha'},$$ \hfill (53)

$$\mathcal{A}'_{ab} \equiv \langle u'_a|u'_b(t)\rangle_{PT} = e^{-i\omega t} \frac{\sinh(\alpha' + \gamma t)}{\sinh \alpha'},$$ \hfill (54)

$$\mathcal{A}'_{ba} \equiv \langle u'_b|u'_a(t)\rangle_{PT} = e^{-i\omega t} \frac{\sinh(\alpha' - \gamma t)}{\sinh \alpha'},$$ \hfill (55)

$$\mathcal{A}'_{bb} \equiv \langle u'_b|u'_b(t)\rangle_{PT} = -i e^{-i\omega t} \frac{\sinh(\gamma t)}{\sinh \alpha'},$$ \hfill (56)

while the corresponding transition probabilities are defined as $\mathcal{P}'_{\alpha\beta} \equiv |\mathcal{A}'_{\alpha\beta}|^2$ (for $\alpha, \beta$ running over $a, b$) and explicitly calculated as

$$\mathcal{P}'_{aa} = \sinh^2(\gamma t)/\sinh^2 \alpha',$$ \hfill (57)

$$\mathcal{P}'_{ab} = \sinh^2(\alpha' + \gamma t)/\sinh^2 \alpha',$$ \hfill (58)

$$\mathcal{P}'_{ba} = \sinh^2(\alpha' - \gamma t)/\sinh^2 \alpha',$$ \hfill (59)

$$\mathcal{P}'_{bb} = \sinh^2(\gamma t)/\sinh^2 \alpha'.$$ \hfill (60)

One can observe non-conservation of the total probability, i.e., $\mathcal{P}'_{aa} + \mathcal{P}'_{bb} \neq 1$ or $\mathcal{P}'_{ba} + \mathcal{P}'_{ab} \neq 1$. Moreover, all probabilities in Eqs. (57)–(60) go to infinity for $t \to +\infty$, rendering them to be physically meaningless. However, the metric operator $\eta' = \mathcal{P}$ is not positive-definite, so we should not expect the sum of transition probabilities to be conserved. One may instead compute the differences between the probabilities, i.e.,

$$\mathcal{P}'_{aa} - \mathcal{P}'_{ab} = -\left[ \sinh^2(\alpha' + \gamma t) - \sinh^2(\gamma t) \right] / \sinh^2 \alpha' = -\sinh(\alpha' + 2\gamma t)/\sinh \alpha',$$ \hfill (61)

$$\mathcal{P}'_{ba} - \mathcal{P}'_{bb} = +\left[ \sinh^2(\alpha' - \gamma t) - \sinh^2(\gamma t) \right] / \sinh^2 \alpha' = +\sinh(\alpha' - 2\gamma t)/\sinh \alpha',$$ \hfill (62)

which are unfortunately time-dependent. As a remedy for this problem, following the same
Therefore, it seems more reasonable to define the flavor eigenstates \(|\tilde{u}_a\rangle\) and \(|\tilde{u}_b\rangle\) as follows

\[
|\tilde{u}_a\rangle = \frac{1}{\sqrt{2}} (|u_a\rangle + |u_b\rangle) = \frac{1}{\sqrt{2}} \left( +1 \right),
\]

\[
|\tilde{u}_b\rangle = \frac{1}{\sqrt{2}} (|u_a\rangle - |u_b\rangle) = \frac{1}{\sqrt{2}} \left( +1 \right),
\]

(63)

where the expressions for the two original flavor eigenstates \(|u_a\rangle\) and \(|u_b\rangle\) in Eqs. (51) and (52) have been used. Since the C operator is trivial in the PT-broken phase, we can easily prove that \(\mathcal{CPT}|\tilde{u}_a\rangle = \mathcal{PT}|\tilde{u}_a\rangle = +|\tilde{u}_a\rangle\) and \(\mathcal{CPT}|\tilde{u}_b\rangle = \mathcal{PT}|\tilde{u}_b\rangle = -|\tilde{u}_b\rangle\). Therefore, the newly-constructed flavor eigenstates are eigenstates of both the \(\mathcal{CPT}\) and \(\mathcal{PT}\) operators. With these \(\mathcal{CPT}\) flavor eigenstates, we repeat the calculations of the transition amplitudes and probabilities, and then obtain the amplitudes \(\tilde{\mathcal{A}}_{\alpha\beta} \equiv \langle \tilde{u}_\beta'|u_a'(t)\rangle\) as

\[
\tilde{\mathcal{A}}'_{aa} \equiv \langle \tilde{u}_a'|u_a'(t)\rangle_{\mathcal{PT}} = \frac{1}{\sqrt{2}} (\mathcal{A}'_{aa} + \mathcal{A}'_{ab}),
\]

(65)

\[
\tilde{\mathcal{A}}'_{ab} \equiv \langle \tilde{u}_b'|u_a'(t)\rangle_{\mathcal{PT}} = \frac{1}{\sqrt{2}} (\mathcal{A}'_{aa} - \mathcal{A}'_{ab}),
\]

(66)

\[
\tilde{\mathcal{A}}'_{ba} \equiv \langle \tilde{u}_a'|u_b'(t)\rangle_{\mathcal{PT}} = \frac{1}{\sqrt{2}} (\mathcal{A}'_{ba} + \mathcal{A}'_{bb}),
\]

(67)

\[
\tilde{\mathcal{A}}'_{bb} \equiv \langle \tilde{u}_b'|u_b'(t)\rangle_{\mathcal{PT}} = \frac{1}{\sqrt{2}} (\mathcal{A}'_{ba} - \mathcal{A}'_{bb}),
\]

(68)

and the probabilities \(\tilde{\mathcal{P}}_{\alpha\beta} \equiv |\tilde{\mathcal{A}}'_{\alpha\beta}|^2\) as

\[
\tilde{\mathcal{P}}'_{aa} = \tilde{\mathcal{P}}'_{ab} = \frac{1}{2} (\mathcal{P}'_{aa} + \mathcal{P}'_{ab}) = \frac{1}{2 \sinh^2 \alpha'} \left[ \sinh^2(\gamma t) + \sinh^2(\alpha' + \gamma t) \right],
\]

(69)

\[
\tilde{\mathcal{P}}'_{ba} = \tilde{\mathcal{P}}'_{bb} = \frac{1}{2} (\mathcal{P}'_{ba} + \mathcal{P}'_{bb}) = \frac{1}{2 \sinh^2 \alpha'} \left[ \sinh^2(\gamma t) + \sinh^2(\alpha' - \gamma t) \right].
\]

(70)

Although these probabilities still become infinite in the limit of \(t \to +\infty\), one can check that \(\tilde{\mathcal{P}}'_{aa} - \tilde{\mathcal{P}}'_{ab} = 0\) and \(\tilde{\mathcal{P}}'_{ba} - \tilde{\mathcal{P}}'_{bb} = 0\), where the time dependence is completely canceled out.

It is interesting to note that there is no interference between the two amplitudes \(\mathcal{A}'_{aa}\) and \(\mathcal{A}'_{ab}\) when squaring the modified amplitudes \(\tilde{\mathcal{A}}'_{aa}\) and \(\tilde{\mathcal{A}}'_{ab}\) to calculate \(\tilde{\mathcal{P}}'_{aa}\) and \(\tilde{\mathcal{P}}'_{ab}\), leading to a simple average of the probabilities in Eq. (69). The main reason can be traced back to the amplitudes in Eqs. (53) and (54), where \(\mathcal{A}'_{aa}\) is purely imaginary, whereas \(\mathcal{A}'_{ab}\) is real up to the same phase factor \(e^{-i\omega t}\). Similar observations can be made for \(\tilde{\mathcal{P}}'_{ba}\) and \(\tilde{\mathcal{P}}'_{bb}\) in Eq. (70). Therefore, it seems more reasonable to define the \(\mathcal{CPT}\) flavor eigenstates as the final states in the sense of the time-independence of the probability differences.

### 3.2 Connection between the PT-Broken Phase and the Neutral-Meson System

The non-Hermitian Hamiltonian with complex eigenvalues has been known in particle physics for a long time. As a concrete example, the mixing and oscillation of the neutral-meson
such as $K^0\overline{K}^0$, $D^0\overline{D}^0$, and $B^0\overline{B}^0$, can be described by an effective non-Hermitian Hamiltonian \cite{21,23}

$$H = M - \frac{i}{2} \Gamma \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_{22} \end{pmatrix},$$

(71)

where both $M$ and $\Gamma$ are $2 \times 2$ Hermitian matrices. In order to make a distinction between the neutral-meson system and the PT-broken phase under consideration, we have set all $2 \times 2$ matrices in the former case in a sans-serif typeface. Without imposing either CPT or CP invariance\footnote{Note that the C, P, and T transformations, as well as their combinations CPT and CP, for the neutral-meson system should be understood in the same way as in particle physics or relativistic quantum field theories in general.}, the time-evolved neutral-meson states can be written as \cite{11}

$$|P_0(t)\rangle = \left[ g_+(t) + zg_-(t) \right] |P_0\rangle - \frac{q}{p} \sqrt{1 - z^2} g_-(t) |P_0\rangle,$$

(72)

$$|\overline{P}(t)\rangle = \left[ g_+(t) - zg_-(t) \right] |P_0\rangle - \frac{p}{q} \sqrt{1 - z^2} g_-(t) |P_0\rangle,$$

(73)

where $z = 0$ corresponds to the case of either CPT or CP invariance and the relevant time-evolution functions are given by

$$g_\pm(t) \equiv \frac{1}{2} \left[ \exp \left( -iM_\pm t - \frac{1}{2} \Gamma_\pm t \right) \pm \exp \left( -iM_\mp t - \frac{1}{2} \Gamma_\mp t \right) \right].$$

(74)

Note that for $i = 1, 2$, $M_i$ stand for the masses of the energy eigenstates $|P_i\rangle$, while $\Gamma_i$ for the corresponding total decay widths. The masses and decay widths are related to the matrix elements of the effective Hamiltonian with the eigenvalues $\{E_1, E_2\}$ via

$$E_1 \equiv M_1 - \frac{i}{2} \Gamma_1 = M_{11} - \frac{i}{2} \Gamma_{11} + pq \left[ \kappa + \sqrt{1 + \kappa^2} \right],$$

(75)

$$E_2 \equiv M_2 - \frac{i}{2} \Gamma_2 = M_{22} - \frac{i}{2} \Gamma_{22} - pq \left[ \kappa + \sqrt{1 + \kappa^2} \right],$$

(76)

where $\kappa \equiv [(M_{22} - i\Gamma_{22}/2) - (M_{11} - i\Gamma_{11}/2)]/(2pq)$ and

$$p^2 \equiv M_{12} - \frac{i}{2} \Gamma_{12}, \quad q^2 \equiv M_{12}^* - \frac{i}{2} \Gamma_{12}^*.$$  

(77)

The complex parameter $z$ can be expressed as follows

$$z \equiv \frac{\kappa}{\sqrt{1 + \kappa^2}} = \frac{\delta m - \frac{i}{2} \delta \Gamma}{\Delta m - \frac{i}{2} \Delta \Gamma}$$

(78)

with $\delta m \equiv M_{11} - M_{22}$, $\Delta m \equiv M_2 - M_1$, $\delta \Gamma \equiv \Gamma_{11} - \Gamma_{22}$, and $\Delta \Gamma \equiv \Gamma_2 - \Gamma_1$. Now, it is evident that $z = 0$ corresponds to $M_{11} = M_{22}$ and $\Gamma_{11} = \Gamma_{22}$, as implied by the CPT theorem for local quantum field theories.
It is straightforward to calculate the transition amplitudes for \(|P^0\rangle \rightarrow |P^0\rangle\) and \(|P^0\rangle \rightarrow |P^0\rangle\), namely,

\[
A_{p_0 p_0}(t) \equiv \langle P^0| P^0(t) \rangle = g_+(t) + zg_-(t) , \tag{79}
\]

\[
A_{p_0 \bar{P}^0}(t) \equiv \langle \bar{P}^0| P^0(t) \rangle = -\frac{q}{p} \sqrt{1 - z^2} g_-(t) , \tag{80}
\]

where \(\langle P^0\rangle \equiv |P^0\rangle^\dagger\) and \(\langle \bar{P}^0\rangle \equiv |\bar{P}^0\rangle^\dagger\) have been defined. Accordingly, the corresponding transition probabilities turn out to be

\[
P_{p_0 p_0}(t) \equiv |A_{p_0 p_0}(t)|^2 = \frac{1}{4} \left[ e^{-\Gamma_1 t} + e^{-\Gamma_2 t} + 2e^{-\Gamma_1 t} \cos(\Delta mt) \right] + \frac{1}{4} \left[ e^{-\Gamma_1 t} + e^{-\Gamma_2 t} - 2e^{-\Gamma_1 t} \cos(\Delta mt) \right] |z|^2 + \frac{1}{2} \left( e^{-\Gamma_2 t} - e^{-\Gamma_1 t} \right) \Re(z) + e^{-\Gamma_1 t} \sin(\Delta mt) \Im(z) , \tag{81}
\]

\[
P_{p_0 \bar{P}^0}(t) \equiv |A_{p_0 \bar{P}^0}(t)|^2 = \frac{|q|^2}{|p|^2} \left[ e^{-\Gamma_1 t} + e^{-\Gamma_2 t} - 2e^{-\Gamma_1 t} \cos(\Delta mt) \right] \sqrt{1 - 2\Re(z^2) + |z|^4} , \tag{82}
\]

with \(\Gamma \equiv (\Gamma_1 + \Gamma_2)/2\). Since the decay widths \(\Gamma_1\) and \(\Gamma_2\) are positive, the transition probabilities \(P_{p_0 p_0}(t)\) and \(P_{p_0 \bar{P}^0}(t)\) will vanish in the limit of \(t \rightarrow +\infty\).

Since the effective Hamiltonian in Eq. (71) takes the most general form, it can also be applied to the PT-symmetric non-Hermitian Hamiltonian in the PT-broken phase. Comparing the Hamiltonians in these two cases, i.e., Eqs. (17) and (71), we identify the following relations

\[
M_{11} - \frac{i}{2} \Gamma_{11} = \rho \cos \varphi + i\rho \sin \varphi , \tag{83}
\]

\[
M_{22} - \frac{i}{2} \Gamma_{22} = \rho \cos \varphi - i\rho \sin \varphi , \tag{84}
\]

\[
M_{12} - \frac{i}{2} \Gamma_{12} = \sigma , \tag{85}
\]

\[
M_{12}^* - \frac{i}{2} \Gamma_{12}^* = \sigma , \tag{86}
\]

implying that \(M_{11} = M_{22} = \rho \cos \varphi, \Gamma_{11} = -\Gamma_{22} = -2\rho \sin \varphi, M_{12} = \sigma, \) and \(\Gamma_{12} = 0\), together with \(p = q = \sqrt{M_{12}} = \sqrt{\sigma}\) and \(\kappa = -i(\rho \sin \varphi)/\sigma\). Using Eqs. (75) and (76), we obtain

\[
M_1 = \rho \cos \varphi , \quad \Gamma_1 = -2\sqrt{\rho^2 \sin^2 \varphi - \sigma^2} , \tag{87}
\]

\[
M_2 = \rho \cos \varphi , \quad \Gamma_2 = +2\sqrt{\rho^2 \sin^2 \varphi - \sigma^2} , \tag{88}
\]

where it should be noted that \(\rho^2 \sin^2 \varphi > \sigma^2\) and \((1 - \rho^2 \sin^2 \varphi/\sigma^2)^{1/2} = +i\sqrt{\rho^2 \sin^2 \varphi - \sigma^2/\sigma}\) has been utilized. The parameter \(z\) in Eq. (78) is determined by \(\delta m \equiv M_{11} - M_{22} = 0,\)
\( \Delta m \equiv M_2 - M_1 = 0, \delta \Gamma \equiv \Gamma_{11} - \Gamma_{22} = -4\rho \sin \varphi, \) and \( \Delta \Gamma \equiv \Gamma_2 - \Gamma_1 = 4\sqrt{\rho^2 \sin^2 \varphi - \sigma^2}, \) namely,
\[
z = \frac{\delta \Gamma}{\Delta \Gamma} = -\frac{\rho \sin \varphi}{\sqrt{\rho^2 \sin^2 \varphi - \sigma^2}} = -\coth \alpha'.
\] From the previous discussion, one can recognize that \( E_1 = E'_+ = \omega + i\gamma \) and \( E_2 = E'_- = \omega - i\gamma \) with \( \omega = \rho \cos \varphi \) and \( \gamma = \sqrt{\rho^2 \sin^2 \varphi - \sigma^2}, \) and thus, \( M_1 = M_2 = \omega \) and \( \Gamma_1 = -\Gamma_2 = -2\gamma. \) It is interesting to observe that \( z \neq 0 \) and \( q/p = 1 \) are valid in the PT-broken phase, which cannot be simultaneously true for the neutral-meson system.

At this point, it is helpful to give some remarks on the CPT and CP symmetries in the neutral-meson system, and the C\( \mathcal{PT} \) and PT symmetries in the PT-broken phase. Following the convention in Ref. [23], one can write down the discrete space-time symmetry transformations for the neutral-meson system as
\[
C|P^0\rangle = -|P^0\rangle, \quad P|P^0\rangle = -|P^0\rangle, \quad T|P^0\rangle = |P^0\rangle,
\] implying that \( CP|P^0\rangle = |P^0\rangle \) and \( CP|P^0\rangle = |P^0\rangle. \) Observe that the time-reversal transformation will interchange the initial and final states, which form separately complete bases, and it is same in both the neutral-meson system and the PT-broken phase, i.e., \( T = \mathcal{T}. \) In the matrix representation, we consider the two flavor eigenstates as \( |P^0\rangle = (1, 0)^T \) and \( |P^0\rangle = (0, 1)^T \) and their Hermitian conjugated states \( \langle P^0| = |P^0\rangle^\dagger = (1, 0) \) and \( \langle P^0| = |P^0\rangle^\dagger = (0, 1). \) It is then straightforward to obtain
\[
C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad CP = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\] where one can observe that the matrix forms of the CP and \( \mathcal{P} \) operators are exactly the same. It is not difficult to verify that the CPT or CP invariance in the neutral-meson system guarantees \( M_{11} = M_{22} \) and \( \Gamma_{11} = \Gamma_{22}, \) while CP or T invariance leads to \( \Im(M_{12}) = \Im(\Gamma_{12}) = 0. \) However, as we have seen, the relations \( M_{11} = M_{22}, \Gamma_{11} \neq \Gamma_{22}, \) and \( \Im(M_{12}) = \Im(\Gamma_{12}) = 0 \) hold in the PT-broken phase.

Since the transition probabilities for the flavor eigenstates in the neutral-meson system have been calculated, we can apply them directly to the PT-broken phase. Using Eq. (81) as well as \( \Delta m = 0 \) and \( \Gamma = (\Gamma_1 + \Gamma_2)/2 = 0, \) we find that
\[
\mathcal{P}_{aa}'(t) = \frac{1}{4} \left( e^{2\gamma t} + e^{-2\gamma t} + 2 \right) + \frac{1}{4} \left( e^{2\gamma t} + e^{-2\gamma t} - 2 \right) \cosh^2 \alpha' + \frac{1}{2} \left( e^{2\gamma t} - e^{-2\gamma t} \right) \coth \alpha' \\
= \cosh^2(\gamma t) + \sinh^2(\gamma t) \cosh^2 \alpha' \sinh^2 \alpha' + \sinh(2\gamma t) \coth \alpha' \cosh \alpha' = \frac{\sinh^2(\alpha' + \gamma t)}{\sinh^2 \alpha'},
\] and similarly using Eq. (82), we obtain
\[
\mathcal{P}_{ab}'(t) = \frac{1}{4} \left( e^{2\gamma t} + e^{-2\gamma t} - 2 \right) \sqrt{(1 - \cosh^2 \alpha')^2} = \frac{\sinh^2(\gamma t)}{\sinh^2 \alpha'},
\] Comparing the above results with the ones in Eqs. (57) and (58), we realize the exchange between the expressions of \( \mathcal{P}_{aa}' \) and \( \mathcal{P}_{ab}'. \) Such an observation can be understood by noticing
the fact that the $PT$-inner product and the ordinary inner product (i.e., the $T$-inner product) differ by the parity operator $P$ that causes the exchange of the final flavor eigenstates.

With the above comparative study, we can observe that the transition probabilities $P_{aa}'$ and $P_{ab}'$ calculated using the ordinary inner product become infinite in the limit of $t \to +\infty$ as well, as in the case of the $PT$-inner product. In analogy to the neutral-meson system, this observation can be attributed to the fact that the total decay width $\Gamma_1 = -2\gamma < 0$ of one energy eigenstate is negative. Therefore, it remains to be explored whether the PT-broken phase can be practically applied to a realistic dynamical system beyond particle physics or not.

4 Summary and Conclusions

The basic properties of non-Hermitian Hamiltonians in both the PT-symmetric and -broken phases are interesting and their practical applications have recently received a lot of attention. In this work, we have focused on the flavor transitions in the two-level quantum system with PT-symmetric non-Hermitian Hamiltonians. Extending our previous investigation on the PT-symmetric phase with two real eigenvalues, we have considered the PT-broken phase, in which the two eigenvalues are complex conjugates to each other.

First, after solving the eigenvalues and eigenvectors of the non-Hermitian Hamiltonian in the PT-broken phase, we have explicitly constructed the charge-conjugation operator $C$ and the metric operator $\eta'$, for which the identities $C = \frac{1}{2}$ and $\eta' = P$ are valid. Second, using the $PT$-inner product, we have calculated the transition amplitudes and probabilities for the flavor eigenstates, i.e., $|u'_\alpha\rangle \to |u'_\beta\rangle$ for $\alpha, \beta = a, b$. After introducing the $CPT$ flavor eigenstates $CPT|\tilde{u}'_a\rangle = +|\tilde{u}'_a\rangle$ and $CPT|\tilde{u}'_b\rangle = -|\tilde{u}'_b\rangle$ as the final states, we have found that the difference $\tilde{P}'_{aa} - \tilde{P}'_{ab}$ between, instead of the sum $\tilde{P}'_{aa} + \tilde{P}'_{ab}$ of, the corresponding transition probabilities, vanishes and is time-independent. However, the probabilities themselves in the PT-broken phase have been found to be infinite in the limit of $t \to +\infty$, which is totally different from the corresponding result in the PT-symmetric phase. Third, in analogy to the neutral-meson system, we have also calculated the transition probabilities using the ordinary inner product, which is equivalent to the $T$-inner product, and observed that the infinite-time behavior of the probabilities originates from the negative decay width $\Gamma_1 = -2\gamma < 0$ of one energy eigenstate. For this reason, one might have to find practical applications of the PT-broken phase in dynamical systems beyond particle physics.

The results presented in this work indicate that the PT-broken phase has very different properties compared with the PT-symmetric phase and this deserves further exploration. As shown in Refs. [24][26], the microcavity sensors prepared at the exceptional point will be much more sensitive to small perturbations, which can be implemented to realize a one-particle detection. In a similar way, the practical applications of the non-Hermitian Hamiltonian in the PT-broken phase may be accomplished only after coupling it to another system. This is the case for the neutral-meson system, where the weak interaction is switched on in order
for the neutral mesons to decay. As we have mentioned, realistic applications may be lying beyond particle physics due to the negative decay width. We leave all these important points for future works.

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