ALMOST PERIODICITY ANALYSIS FOR A DELAYED NICHOLSON’S BLOWFLIES MODEL WITH NONLINEAR DENSITY-DEPENDENT MORTALITY TERM

CHUANGXIA HUANG*, HUA ZHANG AND LIHONG HUANG

School of Mathematics and Statistics
Changsha University of Science and Technology
Changsha 410114, Hunan, China
Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha 410114, Hunan, China

(Communicated by Jianshe Yu)

Abstract. This paper mainly investigates a class of almost periodic Nicholson’s blowflies model involving a nonlinear density-dependent mortality term and time-varying delays. Combining Lyapunov function method and differential inequality approach, some novel assertions are established to guarantee the existence and exponential stability of positive almost periodic solutions for the addressed model, which generalize and refine the corresponding results in some recent published literatures. Particularly, an example and its numerical simulations are given to support the proposed approach.

1. Introduction. Many problems in the fields of physics [1, 2], mathematical biology [3, 4, 5, 6, 7, 8, 9] and control theory can be attributed to study of the nonlinear differential equations, especially it is almost periodicity because there is almost no phenomenon that is purely periodic[10, 11, 12]. Consequently, the qualitative theory of differential equations involving almost periodicity has been the new world-wide focus. In particular, more attention has been paid to the existence and global stability of almost periodic solutions for delayed Nicholson’s blowflies equation and its variants [9, 13, 14, 15, 16, 17, 18, 19, 24, 25]. For example, some sufficient conditions ensuring the global exponential stability of positive periodic solutions and almost periodic solutions of classical Nicholson’s blowflies equation with time-varying delays have been established in [10] and [11, 12] respectively. Furthermore, the global exponential stability of positive almost periodic solutions for Nicholson’s blowflies models involving nonlinear density-dependent mortality terms has been obtained in [14, 15, 16, 17, 18, 19, 20]. The results included in each paper of [10, 11, 12, 14, 15, 16, 17, 18, 19, 20] gave an answer to the open problem:

2000 Mathematics Subject Classification. 34C25; 34K13.

Key words and phrases. Nicholson’s blowflies model, density-dependent mortality term, almost periodic solution, stability.

The work is partially supported by the National Natural Science Foundation of China (Nos.11771059, 51839002); Hunan Provincial Natural Science Foundation of China (No. 2016JJ1001); Scientific Research Fund of Hunan Provincial Education Department (Nos. 15A003, 16C0036).

* Corresponding author.

3337
Find global stability conditions for the positive periodic solution of delayed non-autonomous Nicholson’s blowflies equation, which was proposed by Berezansky et al. [21].

It should be pointed out that, to a large extent, all results involving the global exponential stability of periodic solutions and almost periodic solutions for delayed non-autonomous Nicholson-type equations established in [10, 11, 12, 14, 15, 16, 17, 18, 19, 20] are based on that the solutions exist in a smaller interval \([\kappa, \tilde{\kappa}]\approx [0.7215355, 1.342276]\), where

\[
\kappa \in (0, 1), \quad \tilde{\kappa} \in (1, +\infty), \quad \frac{1 - \kappa}{e^\kappa} = \frac{1}{e^\tilde{\kappa}}, \quad \sup_{x \geq \kappa} \frac{1 - x}{e^x} = \frac{1}{e^\kappa}, \quad \kappa e^{-\kappa} = \tilde{\kappa} e^{-\tilde{\kappa}}.
\]

Obviously, the existence of periodic solutions and almost periodic solutions is restricted to \([\kappa, \tilde{\kappa}]\) will inevitably impose many constraints on mathematical modeling. To the authors’ best knowledge, no existing work has discussed the global stability of periodic solutions for Nicholson’s blowflies equation when the existence interval of periodic solution exceeding \([\kappa, \tilde{\kappa}]\), such studies are however important for us to understand the dynamical characteristics of population models.

Motivated by the above discussions, in this paper, our purpose is to study the existence and global exponential stability of almost periodic solutions of the following Nicholson’s blowflies model with a nonlinear density-dependent mortality term:

\[
x'(t) = -a(t) + b(t)e^{-x(t)} + \sum_{j=1}^{m} \beta_j(t)x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))}, \quad (1.1)
\]

where \(x(t)\) denotes the population of sexually mature adults at time \(t\), \(a, b, \beta_j, \gamma_j : \mathbb{R} \to (0, +\infty)\) and \(\tau_j : \mathbb{R} \to [0, +\infty)\) are almost periodic functions, and \(j \in S = \{1, 2, \ldots, m\}\). The definition of almost periodic function can be found in [1, 3]. For more details on biological explanations of the coefficients of (1.1), we refer the reader to [12, 14, 15, 16, 17, 18, 19, 20, 21]. The significance of this paper is as follows. Firstly, based on differential inequality techniques, a novel proof of the positivity of solutions of delayed non-autonomous Nicholson’s blowflies model with a nonlinear density-dependent mortality term is given. Secondly, by using the fluctuation lemma, we establish the boundedness interval on all solutions of (1.1) without involving \([\kappa, \tilde{\kappa}]\). Thirdly, a almost periodic solution for system (1.1) is given in bounded interval without adopting \([\kappa, \tilde{\kappa}]\), which is global exponential stable.

For simplicity, we introduce the following notations:

\[
w^s = \sup_{t \in \mathbb{R}} w(t), \quad w^i = \inf_{t \in \mathbb{R}} w(t), \quad \sigma = \max_{1 \leq j \leq m} \tau^* j , \quad C = C([\sigma, 0], \mathbb{R}), \quad C_+ = C([-\sigma, 0], \mathbb{R}^+).
\]

Then, we introduce the initial value conditions of (1.1) as follows:

\[
x(t_0 + \theta) = \varphi(\theta), \quad \theta \in [-\sigma, 0], \quad \varphi \in C_+ \quad \text{and} \quad \varphi(0) > 0. \quad (1.2)
\]

Let \(x(t; t_0, \varphi)\) be a solution of the initial value problem (1.1) and (1.2), and \([t_0, \eta(\varphi))\) be the maximal right-interval of existence of \(x_t(t_0, \varphi)\). Particularly, the existence and uniqueness of \(x(t; t_0, \varphi)\) is straightforward in [22].

2. Main results. Before proceeding, the following two lemmas will be introduced.
Lemma 2.1. Let
\[ b(t) > a(t) > \sum_{j=1}^{m} \frac{1}{e^{\gamma_j(t)}} \beta_j(t) \text{ for all } t \geq t_0. \]  \hspace{1cm} (2.1)

Then, \( x(t) = x(t; t_0, \varphi) \) is positive and bounded on \([t_0, +\infty), \) and
\[ S_- := \lim \inf_{t \to +\infty} \frac{b(t)}{a(t)} \leq \lim \inf_{t \to +\infty} x(t) \leq \lim \sup_{t \to +\infty} x(t) \]
\[ \leq S^+ := \lim \sup_{t \to +\infty} \frac{b(t)}{a(t)} - \sum_{j=1}^{m} \frac{1}{e^{\gamma_j(t)}} \beta_j(t). \] \hspace{1cm} (2.2)

Proof. We must show the positivity of \( x(t) \) on \([t_0, \eta(\varphi)), \) for otherwise we can take \( \bar{t} \in (t_0, \eta(\varphi)) \) to obey that
\[ x(\bar{t}) = 0, \quad x(t) > 0 \quad \text{for all } t \in [t_0, \bar{t}). \]

Apparently,
\[ x'(t) \geq -a(t) + b(t)e^{-x(t)} \]
and
\[ (e^{x(t)})' = x'(t)e^{x(t)} \geq -a(t)e^{x(t)} + b(t) \text{ for all } t \in [t_0, \bar{t}]. \]

Letting \( N(t) = e^{x(t)} \) leads to
\[ N'(t) \geq -a(t)N(t) + b(t)(t \in [t_0, \bar{t}]), \]
and
\[ N(\bar{t}) \geq e^{-\int_{t_0}^{\bar{t}} a(u)du} [N(t_0) + \int_{t_0}^{\bar{t}} e^{\int_{t_0}^{u} a(\nu)d\nu} b(u)du], \]
which, together with (2.1), entails that
\[ 0 = x(\bar{t}) \]
\[ \geq -\int_{t_0}^{\bar{t}} a(u)du + \ln[e^{x(0)}] + \int_{t_0}^{\bar{t}} e^{\int_{t_0}^{u} a(\nu)d\nu} b(u)du \]
\[ \geq -\int_{t_0}^{\bar{t}} a(u)du + \ln[e^{x(0)}] + \int_{t_0}^{\bar{t}} e^{\int_{t_0}^{u} a(\nu)d\nu} a(u)du \]
\[ = \ln[1 + (e^{x(0)} - 1)e^{-\int_{t_0}^{\bar{t}} a(u)du}] > 0. \]

This contradiction suggests that \( x(t) > 0 \) for all \( t \in [t_0, \eta(\varphi)) \). In order to demonstrate the boundedness of \( x(t) \), define
\[ \omega(t) = \max\{\xi : \xi \leq t, x(\xi) = \max_{t_0 - \sigma \leq s \leq t} x(s)\}, \quad t \in [t_0 - \sigma, \eta(\varphi)). \]

Suppose that \( x(t) \) is unbounded on \([t_0, \eta(\varphi)). \) Then
\[ \lim_{t \to \eta(\varphi)^-} \omega(t) = \eta(\varphi), \quad \text{and} \quad \lim_{t \to \eta(\varphi)^-} x(\omega(t)) = +\infty. \] \hspace{1cm} (2.3)

Evidently,
\[ x(\omega(t)) = \max_{t_0 - \tau \leq s \leq t} x(s), \text{ and so } x'(\omega(t)) \geq 0, \text{ where } \omega(t) > t_0, \]
which, together with the fact that $\sup_{u \geq 0} ue^{-u} = \frac{1}{e}$, suggests that
\[
0 \leq x'(\omega(t)) = -a(\omega(t)) + b(\omega(t))e^{-x(\omega(t))} + \sum_{j=1}^{m} \frac{\beta_j(\omega(t))}{\gamma_j(\omega(t))} \gamma_j(\omega(t))x(\omega(t) - \tau_j(\omega(t)))e^{-\gamma_j(\omega(t))x(\omega(t) - \tau_j(\omega(t)))}
\]
\[
\leq -a(\omega(t)) + b(\omega(t))e^{-x(\omega(t))} + \sum_{j=1}^{m} \frac{\beta_j(\omega(t))}{\gamma_j(\omega(t))} \frac{1}{e}, \quad \text{where } \omega(t) > t_0.
\]
With the help of (2.1) and (2.3), letting $t \to \eta(\varphi)$ yields
\[
0 \leq \limsup_{t \to \eta(\varphi)} [-a(t) + \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)} \frac{1}{e}] < 0,
\]
which is a contradiction and reveals the boundedness of $x(t)$. In view of Theorem 2.3.1 in [22], on can see that $\eta(\varphi) = +\infty$.

Next, we demonstrate that (2.2) holds. In fact, according to the fluctuation lemma [[23], Lemma A.1.], we can select two sequences $\{t_k^1\}_{k=1}^{+\infty}$ and $\{t_k^2\}_{k=1}^{+\infty}$ satisfying
\[
\lim_{k \to +\infty} t_k^1 \to +\infty, \quad \lim_{k \to +\infty} x(t_k^1) \to l = \liminf_{k \to +\infty} x(t), \quad \lim_{k \to +\infty} x'(t_k^1) \to 0, \quad (2.4)
\]
and
\[
\lim_{k \to +\infty} t_k^2 \to +\infty, \quad \lim_{k \to +\infty} x(t_k^2) \to L = \limsup_{k \to +\infty} x(t), \quad \lim_{k \to +\infty} x'(t_k^2) \to 0, \quad (2.5)
\]
respectively. Regard to the boundedness of $a(t), b(t), \beta_j(t), \gamma_j(t)$ and $x(t - \tau_j(t))$, we can select a subsequence of $\{k\}_{k=1}^{+\infty}$, still denoted by $\{k\}_{k=1}^{+\infty}$, such that
\[
\lim_{k \to +\infty} a(t_k^1), \quad \lim_{k \to +\infty} b(t_k^1), \quad \lim_{k \to +\infty} \beta_j(t_k^1), \quad \lim_{k \to +\infty} \gamma_j(t_k^1), \quad \lim_{k \to +\infty} x(t_k^1 - \tau_j(t_k^1))
\]
and
\[
\lim_{k \to +\infty} a(t_k^2), \quad \lim_{k \to +\infty} b(t_k^2), \quad \lim_{k \to +\infty} \beta_j(t_k^2), \quad \lim_{k \to +\infty} \gamma_j(t_k^2), \quad \lim_{k \to +\infty} x(t_k^2 - \tau_j(t_k^2))
\]
each exist for all $j \in S$. Consequently, it follows from (2.4) and (2.5) that
\[
0 = \lim_{k \to +\infty} x'(t_k^1) \geq -\lim_{k \to +\infty} a(t_k^1) + \lim_{k \to +\infty} b(t_k^1)e^{-l},
\]
and
\[
0 = \lim_{k \to +\infty} x'(t_k^2) = -\lim_{k \to +\infty} a(t_k^2) + \lim_{k \to +\infty} b(t_k^2)e^{-L} + \sum_{j=1}^{m} \lim_{k \to +\infty} \beta_j(t_k^2) \lim_{k \to +\infty} x(t_k^2 - \tau_j(t_k^2))e^{-\gamma_j(t_k^2)}x(t_k^2 - \tau_j(t_k^2))
\]
\[
\leq -\lim_{k \to +\infty} a(t_k^2) + \lim_{k \to +\infty} b(t_k^2)e^{-L} + \sum_{j=1}^{m} \frac{\beta_j(t_k^2)}{\gamma_j(t_k^2)}.
which yield
\[
S_- := \lim \inf_{t \to +\infty} \frac{b(t)}{a(t)} \leq \lim \inf_{t \to +\infty} x(t) \leq \lim \sup_{t \to +\infty} x(t) \\
\leq S^+ := \lim \sup_{t \to +\infty} \frac{b(t)}{a(t) - \sum_{j=1}^{m} \frac{1}{\gamma_j(t)}}.
\]

This finishes the proof of Lemma 2.1.

Lemma 2.2. Let (2.1) and
\[
S_+ > 0, \quad \lim_{t \to +\infty} \sup \left\{ -b(t)e^{-S^+} + \sum_{j=1}^{m} \max \left\{ \frac{1}{e^{2}}, \frac{1 - \gamma_j S_-}{e^{\gamma_j S_-}} \right\} \beta_j(t) \right\} < 0.
\]

Moreover, suppose that \( x(t) = x(t; t_0, \varphi) \) satisfies that \( \varphi' \) is bounded continuous on \([-\sigma, 0]\). Then, for any \( \epsilon > 0 \), there exists \( l = l(\epsilon) > 0 \), such that each interval \([\alpha, \alpha + l]\) includes at least one number \( \delta \) for which there exists \( \bar{Q} > 0 \) satisfies
\[
|x(t + \delta) - x(t)| \leq \epsilon, \quad \text{for all} \quad t > \bar{Q}.
\]

Proof. According to (2.6), it is easy to see that there exists \( t_0^* \geq t_0 \) such that
\[
\sup_{t \geq t_0^*} \left\{ -b(t)e^{-S^+} + \sum_{j=1}^{m} \max \left\{ \frac{1}{e^{2}}, \frac{1 - \gamma_j S_-}{e^{\gamma_j S_-}} \right\} \beta_j(t) \right\} < 0.
\]

Set
\[
H(u, v) = \sup_{t \geq t_0^*} \left\{ -b(t)e^{-S^+ - v} - u \right\} \\
+ \sum_{j=1}^{m} \max \left\{ \frac{1}{e^{2}}, \frac{1 - \gamma_j (S_- - v)}{e^{\gamma_j (S_- - v)}} \right\} \beta_j(t)e^{u*}, \quad u, v \in [0, 1].
\]

Then, \( H(u, v) \) is a continuous function, and
\[
H(0, 0) = \sup_{t \geq t_0^*} \left\{ -b(t)e^{-S^+} + \sum_{j=1}^{m} \max \left\{ \frac{1}{e^{2}}, \frac{1 - \gamma_j S_-}{e^{\gamma_j S_-}} \right\} \beta_j(t) \right\} < 0,
\]
which implies that there exist two constants \( \eta, \lambda \in (0, 1] \) such that
\[
H(\lambda, \varepsilon) = \sup_{t \geq t_0^*} \left\{ -b(t)e^{-S^+ - \varepsilon - \lambda} \right\} \\
+ \sum_{j=1}^{m} \max \left\{ \frac{1}{e^{2}}, \frac{1 - \gamma_j (S_- - \varepsilon)}{e^{\gamma_j (S_- - \varepsilon)}} \right\} \beta_j(t)e^{\lambda*} \right\} < 0,
\]
for all \( \varepsilon \in [0, \eta] \). Consequently, we can pick a positive constant \( B \) such that
\[
\sup_{\varepsilon \in [0, \eta]} H(\lambda, \varepsilon) = -B < 0.
\]
For \( t \in (-\infty, t_0 - \sigma] \), we add the definition of \( x(t) \) with \( x(t) \equiv x(t_0 - \sigma) \). Set
\[
A(\delta, t) = [b(t + \delta) - b(t)]e^{-x(t + \delta)}
\]
\[
+ \sum_{j=1}^{m} [\beta_j(t + \delta) - \beta_j(t)]x(t + \delta - \tau_j(t + \delta))e^{-\gamma_j(t + \delta)x(t + \delta - \tau_j(t + \delta))}
\]
\[
+ \sum_{j=1}^{m} \beta_j(t)x(t + \delta - \tau_j(t + \delta))e^{-\gamma_j(t + \delta)x(t + \delta - \tau_j(t + \delta))}
\]
\[
-x(t - \tau_j(t) + \delta)e^{-\gamma_j(t + \delta)x(t - \tau_j(t) + \delta)}
\]
\[
+ \sum_{j=1}^{m} \beta_j(t) \left[ x(t - \tau_j(t) + \delta)e^{-\gamma_j(t + \delta)x(t - \tau_j(t) + \delta)} + \right.
\]
\[
- x(t - \tau_j(t) + \delta)e^{-\gamma_j(t)x(t - \tau_j(t) + \delta)} \right] - [a(t + \delta) - a(t)], \quad t \in \mathbb{R}.
\] (2.9)

For any \( \varepsilon \in (0, \min\{\eta, S_{-}\}) \), it follows from Lemma 2.1 that there exists \( T_{\varphi} > t_0^* \) such that
\[
S_{-} - \varepsilon < x(t) < S^* + \varepsilon, \quad \text{for all } t \in [T_{\varphi} - \sigma, +\infty),
\] (2.10)

which implies that the right side of (1.1) is also bounded, and \( x'(t) \) is a bounded function on \([t_0 - \sigma, +\infty)\). Thus, with the help of the fact that \( x(t) \equiv x(t_0 - \sigma) \) for \( t \in (-\infty, t_0 - \sigma] \), we gain that \( x(t) \) is uniformly continuous on \( \mathbb{R} \). From uniformly almost periodic family theory in [1], Corollary 2.3, p. 19], for each \( \varepsilon \in (0, \min\{\eta, S_{-}\}) \), there exists \( l = l(\varepsilon) > 0 \), such that every interval \([\alpha, \alpha + l] \subseteq \mathbb{R} \), includes a \( \delta \) for which
\[
|A(\delta, t)| \leq \frac{1}{2} B\varepsilon, \quad \text{for all } t \in \mathbb{R}.
\] (2.11)

Let \( Q_0 \geq \max\{t_0, t_0 - \delta, T_{\varphi} + \sigma, T_{\varphi} + \sigma - \delta\} \). For \( t \in \mathbb{R} \), denote
\[
u(t) = x(t + \delta) - x(t).
\]

Then, for all \( t \geq Q_0 \), we get
\[
\frac{du(t)}{dt} = b(t)[e^{-x(t + \delta)} - e^{-x(t)}] + \sum_{j=1}^{m} \beta_j(t) \left[ x(t - \tau_j(t) + \delta)e^{-\gamma_j(t)x(t - \tau_j(t) + \delta)}
\]
\[
- x(t - \tau_j(t))e^{-\gamma_j(t)x(t - \tau_j(t))} \right] + A(\delta, t).
\] (2.12)

Note the following inequalities
\[
\gamma_j^1(S_{-} - \varepsilon) \leq \gamma_j(t)x(t - \tau_j(t)), \quad \text{for all } t \geq Q_0, \; j \in S,
\] (2.13)
\[
(e^{-s} - e^{-t})\text{sgn}(s - t) \leq -e^{-(S^* + \varepsilon)}|s - t| \quad \text{where } s, t \in [S_{-} - \varepsilon, S^* + \varepsilon],
\] (2.14)

and
\[
|se^{-s} - te^{-t}| \leq \max\left\{ \frac{1}{e^2}, \frac{1 - \gamma_j^1(S_{-} - \varepsilon)}{e^1(S_{-} - \varepsilon)} \right\}|s - t|.
\] (2.15)
where \( s, t \in [\gamma_j^*(S_-, \varepsilon), +\infty), j \in S \), we obtain

\[
D^-\left(e^{\lambda t}|u(s)|\right)|_{s=t} \leq \lambda e^{\lambda t}|u(t)| + e^{\lambda t}\left\{ b(t)\left[ e^{-x(t+\delta)} - e^{-x(t)} \right] \sgn(x(t+\delta) - x(t)) + \left| \sum_{j=1}^{\infty} \beta_j(t) [x(t-\tau_j(t) + \delta) - \tau_j(t) + \delta] \right| + \sum_{j=1}^{\infty} \beta_j(t) \right\}
\]

Step one. If \( 0 < \lambda \leq 2 \), then \( E(t) \) is non-decreasing. Now, the remaining proof will be divided into two steps.

**Step one.** If \( E(t) > \lambda e^{\lambda t}|u(t)| \) for all \( t \geq Q_0 \), we assert that

\[
E(t) \equiv E(Q_0), \quad \text{for all } t \geq Q_0.
\]

In the contrary case, one can pick \( Q_1 > Q_0 \) such that \( E(Q_1) > E(Q_0) \). Because

\[
e^{\lambda t}|u(t)| \leq E(Q_0) \quad \text{for all } t \leq Q_0,
\]

there must exist \( \beta^* \in (Q_0, Q_1) \) such that

\[
e^{\lambda \beta^*}|u(\beta^*)| = E(Q_1) \geq E(\beta^*),
\]

which contradicts the fact that \( E(\beta^*) > e^{\lambda \beta^*}|u(\beta^*)| \) and proves the above assertion. Then, we can select \( Q_2 > Q_0 \) satisfying

\[
|u(t)| \leq e^{-\lambda t}E(t) = e^{-\lambda t}E(Q_0) < \varepsilon \quad \text{for all } t \geq Q_2.
\]
Step two. If there exists $Q^* \geq Q_0$ such that $E(Q^*) = e^{\lambda Q^*} |u(Q^*)|$, we can have from (2.13) and (2.20) that
\[
0 \leq D^-(e^{\lambda s}|u(s)|)|_{s=Q^*} \\
\leq -[b(Q^*)e^{-(S^t+\epsilon)} - \lambda]e^{\lambda Q^*}|u(Q^*)| \\
+ \sum_{j=1}^m \beta_j(Q^*) \max\left\{ \frac{1}{e^2}, \frac{1 - \gamma_j(S-\epsilon)}{e^{\gamma_j(S-\epsilon)}} \right\} e^{\lambda \tau_j(Q^*)} \\
\times e^{\lambda(Q^*-\tau_j(Q^*))}|u(Q^*-\tau_j(Q^*))| + e^{\lambda Q^*}|A(\delta, Q^*)| \\
\leq \left\{-[b(Q^*)e^{-(S^t+\epsilon)} - \lambda] \\
+ \sum_{j=1}^m \beta_j(Q^*) \max\left\{ \frac{1}{e^2}, \frac{1 - \gamma_j(S-\epsilon)}{e^{\gamma_j(S-\epsilon)}} \right\} e^{\lambda \tau_j(Q^*)} \right\} E(Q^*) + \frac{1}{2} B \varepsilon e^{\lambda Q^*} \\
< -BE(Q^*) + B \varepsilon e^{\lambda Q^*},
\] (2.19)
which leads to
\[
e^{\lambda Q^*}|u(Q^*)| = E(Q^*) < \varepsilon e^{\lambda Q^*}, \quad \text{and} \quad |u(Q^*)| < \varepsilon. \quad \text{(2.20)}
\]

For any $t > Q^*$ satisfying $E(t) = e^{\lambda t}|u(t)|$, by the same method as that in the derivation of (2.20), we can show
\[
e^{\lambda t}|u(t)| < \varepsilon e^{\lambda t}, \quad \text{and} \quad |u(t)| < \varepsilon. \quad \text{(2.21)}
\]

Furthermore, if $E(t) > e^{\lambda t}|u(t)|$ and $t > Q^*$, one can pick $Q_3 \in [Q^*, t)$ such that
\[
E(Q_3) = e^{\lambda Q_3}|u(Q_3)| \quad \text{and} \quad E(s) > e^{\lambda s}|u(s)| \quad \text{for all} \quad s \in (Q_3, t],
\]
which, together with (2.20) and (2.21), suggests that
\[
|u(Q_3)| < \varepsilon. \quad \text{(2.22)}
\]

With a similar reasoning as that in the proof of Step one, we can entail that
\[
E(s) \equiv E(Q_3) \quad \text{is a constant for all} \quad s \in (Q_3, t],
\]
which, together with (2.22), follows that
\[
|u(t)| < e^{-\lambda t} E(t) = e^{-\lambda t} E(Q_3) = |u(Q_3)|e^{-\lambda(t-Q_3)} < \varepsilon.
\]

Finally, the above discussion infers that there exists $\hat{Q} > \max\{Q^*, Q_0, t_2\}$ obeying that
\[
|u(t)| \leq \varepsilon \quad \text{for all} \quad t > \hat{Q},
\]
which finishes the proof of Lemma 2.2. \hfill \Box

3. Global exponential stability of almost periodic solutions. Combined with Lemmas 2.1 and 2.2, we can have the following theorem:

Theorem 3.1. Assume that all assumptions of Lemmas 2.1 and 2.2 are satisfied. Then, (1.1) has a globally exponentially stable positive almost periodic solution $x^*(t)$. Moreover, there exist constants $K_{\varphi, x^*}$ and $t_{\varphi, x^*}$ such that
\[
|x(t; t_0, \varphi) - x^*(t)| < K_{\varphi, x^*} e^{-\lambda t} \quad \text{for all} \quad t > t_{\varphi, x^*}.
\]
Proof. Let \( v(t) = v(t; t_0, \varphi^v) \) be a solution of equation (1.1) with initial conditions satisfying the assumptions in Lemma 2.2. We also add the definition of \( v(t) \) with \( v(t) \equiv v(t_0 - \sigma) \) for all \( t \in (-\infty, t_0 - \sigma] \). Set

\[
\epsilon(k, t) = |b(t + t_k) - b(t)|e^{-\gamma_j(t + t_k)} + \sum_{j=1}^{m} |\beta_j(t + t_k) - \beta_j(t)|
\]

\[
\times v(t + t_k - \tau_j(t + t_k))e^{-\gamma_j(t + t_k)v(t + t_k - \tau_j(t + t_k))} + \sum_{j=1}^{m} \beta_j(t)[v(t + t_k - \tau_j(t + t_k))e^{-\gamma_j(t + t_k)v(t + t_k - \tau_j(t + t_k))}
\]

\[-v(t - \tau_j(t) + t_k)e^{-\gamma_j(t + t_k)v(t - \tau_j(t) + t_k)}] + \sum_{j=1}^{m} \beta_j(t)[v(t - \tau_j(t) + t_k)e^{-\gamma_j(t + t_k)v(t - \tau_j(t) + t_k)} - v(t - \tau_j(t) + t_k)e^{-\gamma_j(t)v(t - \tau_j(t) + t_k)} - [a(t + t_k) - a(t)], \quad t \in \mathbb{R},
\]

(3.1)

where \( \{t_k\} \) is any sequence of real numbers. For any \( \varepsilon \in (0, \min\{\eta, S_-\}) \), by Lemma 2.1, we can choose \( t_{\varphi^v} > t_0 \) such that

\[
S_- - \varepsilon < v(t) < S^+ + \varepsilon, \quad \text{for all} \quad t \geq t_{\varphi^v},
\]

(3.2)

which together with the boundedness of \( v'(t) \) and the fact that \( v(t) \equiv v(t_0 - \sigma) \) for \( t \in (-\infty, t_0 - \sigma] \), entails that \( v(t) \) is uniformly continuous on \( \mathbb{R} \). Then, from the almost periodicity of \( a, b, \tau_j, \gamma_j \) and \( \beta_j \), we can select a sequence \( \{t_k\} \to +\infty \) such that

\[
|a(t + t_k) - a(t)| \leq \frac{1}{K}, \quad |b(t + t_k) - b(t)| \leq \frac{1}{K}, \quad |\tau_j(t + t_k) - \tau_j(t)| \leq \frac{1}{K}, \quad |\gamma_j(t + t_k) - \gamma_j(t)| \leq \frac{1}{K}, \quad |\epsilon(k, t)| \leq \frac{1}{K}
\]

(3.3)

for all \( j, t \).

Since \( \{v(t + t_k)\}_{k=1}^{\infty} \) is uniformly bounded and uniformly continuous, by Arzala-Ascoli Lemma and diagonal selection principle, we can choose a subsequence \( \{t_{k_j}\} \) of \( \{t_k\} \), such that \( v(t + t_{k_j}) \) [for convenience, we still denote by \( v(t + t_k) \)] uniformly converges to a continuous function \( x^*(t) \) on any compact set of \( \mathbb{R} \), and

\[
S_- - \varepsilon \leq x^*(t) \leq S^+ + \varepsilon, \quad \text{for all} \quad t \in \mathbb{R}.
\]

(3.4)

Now, by similar lines used in the proof of Theorem 3.1 in [21], we can prove that \( x^*(t) \) is a positive almost periodic solution of (1.1), which, together with the arbitrariness of \( \varepsilon \), implies that

\[
S_- \leq x^*(t) \leq S^+, \quad \text{for all} \quad t \in \mathbb{R}.
\]

(3.5)

Finally, we prove that \( x^*(t) \) is globally exponentially stable.

Let \( x(t) = x(t; t_0, \varphi) \) and \( y(t) = x(t) - x^*(t) \), where \( t \in [t_0 - \sigma, +\infty) \). Define

\[
z(t) = x(t) - x^*(t), \quad \text{and} \quad U(t) = |z(t)|e^{\lambda t} \quad \text{for all} \quad t \in [t_0 - \sigma, +\infty).
\]

Obviously,

\[
z'(t) = b(t)[e^{-\gamma_j(t)v(t - \tau_j(t))} - e^{-\gamma_j(t)v(t - \tau_j(t))}] + \sum_{j=1}^{m} \beta_j(t)x(t - \tau_j(t))e^{-\gamma_j(t)v(t - \tau_j(t))} - x^*(t - \tau_j(t))e^{-\gamma_j(t)v(t - \tau_j(t))}. \quad \text{(3.6)}
\]
which, together with (2.8), derives

\[ S_--\varepsilon \leq x(t), \quad x^*(t) \leq S^+ + \varepsilon, \quad \text{for all} \quad t \in [T_{\varphi,x^*}, \sigma, +\infty). \] (3.7)

By (3.6) and calculating the upper-right Dini derivative of \( U(t) \), for all \( t > T_{\varphi,x^*} \), we obtain

\[
D^- (U(t)) \leq b(t)[e^{-x(t)} - e^{-x^*(t)}] \text{sgn}(x(t) - x^*(t))e^{\lambda t} \\
+ \sum_{j=1}^{m} \beta_j(t)|x(t - \tau_j(t))e^{-\gamma_j(t)}x(t - \tau_j(t)) - x^*(t - \tau_j(t))|e^{\lambda t} + \lambda|z(t)|e^{\lambda t}.
\] (3.8)

We state that

\[ U(t) < e^{\lambda T_{\varphi,x^*}} \left( \max_{t \in [t_0 - \sigma, T_{\varphi,x^*}]} |x(t) - x^*(t)| + 1 \right) =: M_{\varphi,x^*} \quad \text{for all} \quad t > T_{\varphi,x^*}. \] (3.9)

Suppose the contrary and choose \( T_* > T_{\varphi,x^*} \) such that

\[ U(T_*) = M_{\varphi,x^*} \quad \text{and} \quad U(t) < M_{\varphi,x^*} \quad \text{for all} \quad t \in [t_0 - \sigma, T_*). \] (3.10)

From (2.14), (2.15), (3.7), (3.8), (3.10) and the following inequality:

\[ \gamma_j^s(S_--\varepsilon) \leq \gamma_j(T_*)x(T_* - \tau_j(T_*)), \quad \gamma_j(T_*)x^*(T_* - \tau_j(T_*)), \quad j \in S, \]

we gain

\[
0 \leq D^- (U(T_*)) \\
\leq b(T_*)[e^{-x(T_*)} - e^{-x^*(T_*)}] \text{sgn}(x(T_*) - x^*(T_*))e^{\lambda T_*} \\
+ \sum_{j=1}^{m} \beta_j(T_*)|x(T_* - \tau_j(T_*))e^{-\gamma_j(T_*)}x(T_* - \tau_j(T_*)) - x^*(T_* - \tau_j(T_*))|e^{\lambda T_*} + \lambda|z(T_*)|e^{\lambda T_*} \\
\leq -[b(T_*)e^{-\lambda(T_* + \varepsilon)} - \lambda]|z(T_*)|e^{\lambda T_*} \\
+ \sum_{j=1}^{m} \beta_j(T_*) \gamma_j(T_*)x(T_* - \tau_j(T_*))e^{-\gamma_j(T_*)}x(T_* - \tau_j(T_*)) - \gamma_j(T_*)x^*(T_* - \tau_j(T_*))e^{\lambda T_*} \\
\leq -[b(T_*)e^{-\lambda(T_* + \varepsilon)} - \lambda]|z(T_*)|e^{\lambda T_*} \\
+ \sum_{j=1}^{m} \beta_j(T_*) \max \left\{ \frac{1}{\varepsilon^2}, \frac{1 - \gamma_j^s(S_--\varepsilon)}{e^{\gamma_j^s(S_--\varepsilon)}} \right\} |z(T_* - \tau_j(T_*))|e^{\lambda(T_* - \tau_j(T_*))}e^{\lambda \tau_j(T_*)} \\
\leq \left\{ -[b(T_*)e^{-\lambda(T_* + \varepsilon)} - \lambda] + \sum_{j=1}^{m} \beta_j(T_*) \max \left\{ \frac{1}{\varepsilon^2}, \frac{1 - \gamma_j^s(S_--\varepsilon)}{e^{\gamma_j^s(S_--\varepsilon)}} \right\} e^{\lambda \tau_j(T_*)} \right\} M_{\varphi,x^*},
\]

which, together with (2.8), derives

\[ 0 \leq -[b(T_*)e^{-\lambda(T_* + \varepsilon)} - \lambda] + \sum_{j=1}^{m} \beta_j(T_*) \max \left\{ \frac{1}{\varepsilon^2}, \frac{1 - \gamma_j^s(S_--\varepsilon)}{e^{\gamma_j^s(S_--\varepsilon)}} \right\} e^{\lambda \sigma} < 0. \]

This is a clear contradiction and proves (3.9). Hence,

\[ |z(t)| < M_{\varphi,x^*}e^{-\lambda t} \quad \text{for all} \quad t > T_{\varphi,x^*}. \]
4. A numerical example. Example 4.1. Regard the following almost periodic Nicholson’s blowflies model involving a nonlinear density-dependent mortality term and time-varying delays:

\[
x'(t) = -e^{-(2+\cos t)} + (10 + 10|\cos t + \cos \sqrt{2}t|)e^{-x(t)} \\
+ \frac{1 + |\cos t + \cos \sqrt{2}t|}{2000} x(t - 2e^{\sin t})e^{-\frac{1}{2}x(t-2e^{\sin t})} \\
+ \frac{1 + |\cos t + \cos \sqrt{2}t|}{2000} x(t - 2e^{\cos t})e^{-\frac{1}{2}x(t-2e^{\cos t})}.
\]

Clearly,

\[
3 < S_- := \liminf_{t \to +\infty} \frac{b(t)}{a(t)} < S^+ := \limsup_{t \to +\infty} \frac{b(t)}{a(t)} - \sum_{j=1}^{m} \frac{1}{e^{\frac{1}{2}S^-}} \beta_j(t)
\]

and

\[
\limsup_{t \to +\infty} \left\{ -b(t)e^{-S^+} + \sum_{j=1}^{m} \max\left\{ \frac{1}{e^{\frac{1}{2}S^-}} \frac{1 - \gamma_j S^-}{e^{\gamma_j S_-}} \right\} \beta_j(t) \right\} < -0.3.
\]

Thus, (4.1) satisfies the assumptions adopted in Theorem 3.1, which has a globally exponentially stable positive almost periodic solution \(x^*(t)\), which belongs to \((3, +\infty)\). The simulation results given in Figure 1 strongly validate the theoretical analysis.
Remark 1. Note that $x^*(t) \geq 3$ does not belong to $[\kappa, \tilde{\kappa}] \approx [0.7215355, 1.342276]$, and $\gamma_1(t) = \gamma_2(t) = \frac{1}{2}$ does not satisfy the following condition:

$$
\gamma_j(t) \geq 1, \text{ for all } t \in \mathbb{R}, j \in S,
$$

which has been considered as fundamental for the considered periodicity and almost periodicity of delayed non-autonomous Nicholson’s blowflies models in [10, 11, 12, 14, 15, 16, 17, 18, 19, 20]. Hence, all the results in these above mentioned references cannot be applicable to show the global exponential stability on the positive almost periodic solution of (4.1). Whether or not our approach and method in this paper are available to study the global stability of almost periodic solutions for other delayed Nicholson’s blowflies models (1.1) in the case that the almost periodic solution does not belong to $[\kappa, \tilde{\kappa}] \approx [0.7215355, 1.342276]$, it is an interesting topic and we leave this as our future research.

Acknowledgments. We are extremely grateful to the editor, Professor Shouchuan Hu and the anonymous reviewers for their valuable comments and suggestions which have contributed to the improved presentation of the manuscript.

REFERENCES

[1] A. M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Mathematics, Vol. 377, Springer-Verlag, Berlin, 1974.

[2] C. Huang and H. Zhang, *Periodicity of non-autonomous inertial neural networks involving proportional delays and non-reduced order method*, International Journal of Biomathematics, 12 (2019), 1950016.

[3] C. Zhang, *Almost Periodic Type Functions and Ergodicity*, Kluwer Academic/Science Press, Beijing, 2003.

[4] T. M. Touaoula, *Global stability for a class of functional differential equations (Application to Nicholson’s blowflies and Mackey-Glass models)*, Discrete & Continuous Dynamical Systems - A, 38 (2018), 4391–4419.

[5] S. Chen and J. Yu, *Stability and bifurcation on predator-prey systems with nonlocal prey competition*, Discrete & Continuous Dynamical Systems - A, 38 (2018), 43–62.

[6] T. Shibata, *Global behavior of bifurcation curves for the nonlinear eigenvalue problems with periodic nonlinear terms*, Communications on Pure & Applied Analysis, 17 (2018), 2139–2147.

[7] C. Huang, Y. Qiao, L. Huang and R. Agarwal, *Dynamical behaviors of a food-chain model with stage structure and time delays*, Discrete & Continuous Dynamical Systems - A, 38 (2018), 1387–1419.

[8] Z. Yang, C. Huang and X. Zou, *Effect of impulsive controls in a model system for age-structured population over a patchy environment*, Journal of Mathematical Biology, 76 (2018), 185–195.

[9] C. Huang, Z. Yang, T. Yi and X. Zou, *On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities*, Journal of Differential Equations, 256 (2014), 2101–2114.

[10] B. Liu, *Global exponential stability of positive periodic solutions for a delayed Nicholson’s blowflies model*, Journal of Mathematical Analysis and Applications, 412 (2014), 212–221.

[11] B. Liu, *New results on global exponential stability of almost periodic solutions for a delayed Nicholson blowflies model*, Annales Polonici Mathematici, 113 (2015), 191–208.

[12] W. Xiong, *New results on positive pseudo-almost periodic solutions for a delayed Nicholson’s blowflies model*, Nonlinear Dynamics, 85 (2016), 563–571.

[13] Y. Xu, *New stability theorem for periodic Nicholson’s model with mortality term*, Applied Mathematics Letters, (2019).

[14] B. Liu, *Almost periodic solutions for a delayed Nicholson’s blowflies model with a nonlinear density-dependent mortality term*, Advances in Difference Equations, 72 (2014), 1–16.

[15] B. Liu, *Positive periodic solutions for a nonlinear density-dependent mortality Nicholson’s blowflies model*, Kodai Mathematical Journal, 37 (2014), 157–173.

[16] L. Yao, *Dynamics of Nicholson’s blowflies models with a nonlinear density-dependent mortality*, Applied Mathematical Modelling, 64 (2018), 185–195.
[17] Y. Tang, Global attractivity of asymptotically almost periodic Nicholson’s blowflies models with a nonlinear density-dependent mortality term, *International Journal of Biomathematics*, 11 (2018), 1850079.

[18] Y. Xu, Existence and global exponential stability of positive almost periodic solutions for a delayed Nicholson’s blowflies model, *Journal of the Korean Mathematical Society*, 51 (2014), 473–493.

[19] W. Chen and W. Wang, Almost periodic solutions for a delayed Nicholson’s blowflies system with nonlinear density-dependent mortality terms and patch structure, *Advances in Difference Equations*, 205 (2014), 1–19.

[20] P. Liu and L. Zhang, et al, Global exponential stability of almost periodic solutions for Nicholson’s Blowflies system with nonlinear density-dependent mortality terms and patch structure, *Mathematical Modelling and Analysis*, 22 (2017), 484–502.

[21] L. Berezansky, E. Braverman and L. Idels, Nicholson’s blowflies differential equations revisited: Main results and open problems, *Applied Mathematical Modelling*, 34 (2010), 1405–1417.

[22] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[23] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer-Verlag, New York, 2011.

[24] L. Duan, X. Fang and C. Huang, Global exponential convergence in a delayed almost periodic Nicholson’s blowflies model with discontinuous harvesting, *Mathematical Methods in the Applied Sciences*, 41 (2018), 1954–1965.

[25] L. Duan and C. Huang, Existence and global attractivity of almost periodic solutions for a delayed differential neoclassical growth model, *Mathematical Methods in the Applied Sciences*, 40 (2017), 814–822.

Received October 2018; revised February 2019.

E-mail address: cxiahuang@amss.ac.cn
E-mail address: ZH18774856802@163.com
E-mail address: lhhuang@csust.edu.cn