Some algebraically solvable two-dimensional dynamical systems with polynomial interactions

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Abstract. We tersely review a recently introduced technique to identify systems of two nonlinearly-coupled Ordinary Differential Equations (ODEs) solvable by algebraic operations; and we report some specific examples of this kind, namely systems of 2 first-order ODEs with polynomial right-hand sides,

\[ \dot{x}_n = P^{(n)}(x_1, x_2), \quad n = 1, 2, \]

satisfied by the 2 (possibly complex) dependent variables \( x_n \equiv x_n(t) \). Here \( P^{(n)}(x_1, x_2) \) indicates some specific polynomial. These examples are analogous, but different, from those previously reported.

1. Introduction

A technique to identify dynamical systems characterized by systems of nonlinearly-coupled Ordinary Differential Equations (ODEs) solvable by algebraic operations is based on the relations among the \( N \) coefficients \( y_m(t) \) and the \( N \) zeros \( x_n(t) \) of a time-dependent (monic) polynomial of degree \( N \) in the (complex) variable \( z \):

\[ p_N(z; t) = z^N + \sum_{m=1}^{N} y_m(t) z^{N-m} = \prod_{n=1}^{N} [z - x_n(t)]. \]  

(1)

The basic idea is to consider "simple" time evolutions of the \( N \) coefficients \( y_m(t) \)—possibly explicitly solvable evolutions, maybe featuring remarkable properties such as isochrony (implying that all the \( N \) coefficients \( y_m(t) \) are periodic in \( t \) with the same fixed period independent of the initial data); and to then consider the corresponding time evolutions of the \( N \) zeros \( x_n(t) \), which are generally characterized by "more nonlinear" equations of motions, the solutions of which are then obtainable via the algebraic operation of computing the \( N \) zeros \( x_n(t) \) of the polynomial \( p_N(z; t) \) defined in terms of its \( N \) coefficients \( y_m(t) \). This generally entails that the time evolution of the zeros \( x_n(t) \) inherits properties (such as isochrony) of the evolution of the coefficients \( y_m(t) \). This idea has a long history [1], and it has produced many developments, see for instance [2] [3] [4] [5] [6] and references therein.
An additional recent development has explored the modifications of the approach outlined above which emerge if, rather than considering generic time-dependent polynomials, one focusses on special time-dependent polynomials featuring (for all time) multiple zeros: in particular, as a first step in this direction, results have been reported for the case of a polynomial featuring one double zero [7], then some significant progress has been made on the case of a polynomial featuring one zero of arbitrary multiplicity [8], and finally the most general case has been treated of a polynomial $p_{M}(z;t)$ of degree $M$ featuring an arbitrary number $N$ of zeros each of them of arbitrary multiplicity $\mu_n$ [9]:

$$p_{M}(z;t) = z^N + \sum_{m=1}^{M} y_m(t) z^{M-m} = \prod_{n=1}^{N} \{[z - x_n(t)]^{\mu_n}\} ,$$

(2)

of course with the $N$ positive integers $\mu_n$ related to the positive integer parameters $M$ and $N$ by the relation

$$M = \sum_{n=1}^{N} (\mu_n) .$$

(3)

A second twist of this development restricted attention to the $N = 2$ case—i.e., polynomials (see (2)) featuring only 2 zeros [10] [9]. This stringent limitation has opened the way to the identification of algebraically solvable two-dimensional dynamical systems with polynomial interactions, namely systems characterized by the following systems of 2 nonlinearly coupled ODEs:

$$\dot{x}_n = p^{(n)}(x_1, x_2) , \quad n = 1, 2 .$$

(4)

**Notation 1.1.** Here and hereafter superimposed dots denote differentiations with respect to the dependent variable $t$ ("time"), $x_n \equiv x_n(t)$ are the 2 (generally complex) dependent variables, and $p^{(n)}(x_1, x_2)$ are generally 2 polynomials (or "almost polynomials": see below) in these 2 dependent variables $x_1, x_2$. Parameters such $a, b, c, \alpha, \beta, \gamma$ (possibly equipped with subscripts) are time-independent; they can be arbitrarily assigned (up to explicitly indicated relations among them). Note that often, above and hereafter, the $t$-dependence of variables is not explicitly indicated (when this is unlikely to generate any misunderstanding).

A few such systems have been identified and tersely discussed in [10] by taking as point of departure the case of a polynomial such as (2) with $M = 3$, and several such cases have been treated in [9] by focussing on the case with $M = 4$. In the present paper—after a terse reminder of the methodology to obtain these results—we focus again on the $M = 3$ case, reporting several solvable models of type (4) not previously identified. Analogous treatments of cases with $M > 4$ shall be performed by ourselves or by others in the future.

In the following Section 2 we tersely review our notation and some results which are basic for the following treatment. In Section 3 and its subsections several specific examples are reported (with some related results confined to Appendix A). The last Section 4 outlines possible future developments.

2. Preliminaries

A basic tool of our treatment are the following definitions of the 3 variables $y_m \equiv y_m(t)$, $m = 1, 2, 3$ in terms of the 2 variables $x_n \equiv x_n(t)$, $n = 1, 2$:

$$y_1 = -(2x_1 + x_2) , \quad y_2 = x_1 (x_1 + x_2) , \quad y_3 = -(x_1)^2 x_2 ;$$

(5)

they of course imply (see (2)) that $x_1$ and $x_2$ are the 2 zeros—with respective multiplicities 2 and 1—of the (monic) third-degree polynomial $p_3(z;t)$ with coefficients $y_1, y_2, y_3$:

$$p_3(z;t) = z^3 + \sum_{m=1}^{3} y_m(t) z^{3-m} = [z - x_1(t)]^2 [z - x_2(t)] .$$

(6)
Note that this entails that $x_1$ and $x_2$ can be obtained—in terms of $y_1$ and $y_2$, or $y_1$ and $y_3$, or $y_2$ and $y_3$—by solving the following algebraic equations:

$$3(x_1)^2 + 2y_1x_1 + y_2 = 0, \quad x_2 = -(2x_1 + y_1), \tag{7}$$

or

$$2(x_1)^3 + y_1(x_1)^2 + y_3 = 0, \quad x_2 = -(2x_1 + y_1), \tag{8}$$

or

$$(x_1)^3 - y_2x_1 - 2y_3 = 0, \quad x_2 = \frac{-(x_1)^2 + y_2}{2x_1}. \tag{9}$$

It can moreover be easily shown—or see [7] or [9] or [10]—that these formulas imply the following differential relations:

$$\dot{x}_1 = -\frac{2x_1\dot{y}_1 + y_2}{2(x_1 - x_2)}, \quad \dot{x}_2 = \frac{(x_1 + x_2)\dot{y}_1 + y_2}{x_1 - x_2}, \tag{10}$$

$$\dot{x}_1 = \frac{-(x_1)^2 \dot{y}_1 + y_3}{2x_1(x_1 - x_2)}, \quad \dot{x}_2 = \frac{x_1x_2\dot{y}_1 - y_3}{x_1(x_1 - x_2)}, \tag{11}$$

$$\dot{x}_1 = \frac{x_1y_2 + 2y_3}{2x_1(x_1 - x_2)}, \quad \dot{x}_2 = -\frac{x_1x_2\dot{y}_2 + (x_1 + x_2)\dot{y}_3}{(x_1)^2(x_1 - x_2)}. \tag{12}$$

It is plain from these formulas that if the two variables $y_1 \equiv y_1(t)$ and $y_2 \equiv y_2(t)$, or $y_1 \equiv y_1(t)$ and $y_3 \equiv y_3(t)$, or $y_2 \equiv y_2(t)$ and $y_3 \equiv y_3(t)$ satisfy an algebraically solvable system of 2 first-order ODEs, say

$$\dot{y}_m = f_m(y_m, y_m), \quad \dot{y}_m = f_m(y_m, y_m) \tag{13}$$

with $m_1 = 1$, $m_2 = 2$ or $m_1 = 1$, $m_2 = 3$ or $m_1 = 2$, $m_2 = 3$, then the corresponding system of 2—generally nonlinearly-coupled, first-order—ODEs satisfied by $x_1 \equiv x_1(t)$ and $x_2 \equiv x_2(t)$ is as well algebraically solvable. Moreover, if the functions $f_{m_1}(y_m_1, y_m_2)$ and $f_{m_2}(y_m_1, y_m_2)$ are appropriately chosen, then the right-hand sides of the ODEs (10), (11), (12) become polynomial (or "almost polynomial": see below). In the following Section 3 we report several algebraically solvable systems satisfied by the 2 dependent variables $x_1 \equiv x_1(t)$ and $x_2 \equiv x_2(t)$ obtained in this manner, i.e. by first replacing in the relevant equations (10), (11) and (12), the expressions of the time-derivatives of the relevant 2 variables $y_m$ via (13) and subsequently replacing the expressions of these 2 variables $y_m$ in terms of the 2 variables $x_n$ via the relevant equations (5); of course with the functions $f_{m_1}(y_m_1, y_m_2)$ and $f_{m_2}(y_m_1, y_m_2)$ appropriately chosen polynomials (see below).

3. Results

Our first step is to identify 3 solvable systems of 2 evolution equations satisfied by the dependent variables $y_{m_1} \equiv y_{m_1}(t)$ and $y_{m_2} \equiv y_{m_2}(t)$. Their solvability is discussed in Appendix A of Ref. [10] and tersely reviewed below in Appendix A.

The first of these 3 systems—hereafter identified as A1 (see Appendix A)—is characterized by the following 2 uncoupled ODEs:

$$\dot{y}_{m_1} = \sum_{\ell=0}^L \alpha_{\ell}(y_{m_1})^{m_2+1}, \quad \dot{y}_{m_2} = \sum_{\ell=0}^L [\beta_{\ell}(y_{m_2})^{m_1+1}]. \tag{14}$$
The second of these 3 systems—hereafter identified as A2 (see Appendix A)—is characterized by the following 2 coupled ODEs:

\[
\dot{y}_m = \sum_{\ell=0}^{L} \left[ \alpha_\ell \left( \dot{y}_m \right)^{\ell-1} \right], \quad \dot{y}_m = \sum_{\ell=1}^{L} \left[ \beta_\ell \left( y_m \right)^{\ell-1} \right] + \sum_{\ell=0}^{L} \left[ \gamma_\ell \left( \dot{y}_m \right)^{\ell-1+\left(m_2/m_1\right)} \right]. \quad (15)
\]

The third of these 3 systems—hereafter identified as A3 (see Appendix A)—is characterized by the following 2 coupled ODEs:

\[
\begin{align*}
\dot{y}_m &= \alpha_0 + \alpha_1 y_m, \quad \dot{y}_m = \beta_0 \left( y_m \right)^{-1+m} + \beta_1 \left( y_m \right)^{-1+2m}, \\
m &= m_2/m_1, \quad m_1 = 1, \quad m_2 = 2, 3. \quad (16)
\end{align*}
\]

Our next step is to list in the following 11 subsections 11 algebraically solvable systems of 2 nonlinearly-coupled ODEs (4) with polynomial (or "almost polynomial": see below) right-hand sides; in each case we identify the corresponding appropriately chosen algebraically solvable system of ODEs—see above and Appendix A—satisfied by the corresponding functions \( y_m(t) \) and \( y_m(t) \). But note that the algebraically solvable systems (4) thus identified are only 9, because 2 pairs of the systems identified below—although arrived at differently—are in fact identical (a phenomenon already noted in [9]).

3.1. Models of type A1

In the 3 cases listed in this Subsection 3.1 the variables \( y_m(t) \) and \( y_m(t) \) are supposed to satisfy the system A1 of 2 uncoupled ODEs (see (14) and Appendix A), with the indicated assignments of the various parameters.

Model A1.1: \( m_1 = 1, \quad m_2 = 2; \quad L = 1; \quad \alpha_0 = a, \quad \alpha_1 = b; \quad \beta_0 = 2a, \quad \beta_1 = 6b; \)

\[
\begin{align*}
\dot{x}_1 &= ax_1 + bx_1 \left[ 5 \left( x_1 \right)^2 + 5x_1 x_2 - \left( x_2 \right)^2 \right], \\
\dot{x}_2 &= ax_2 - b \left[ 2 \left( x_1 \right)^3 - 2 \left( x_1 \right)^2 x_2 - 8x_1 \left( x_2 \right)^2 - \left( x_2 \right)^3 \right]. \quad (17)
\end{align*}
\]

Model A1.2: \( m_1 = 1, \quad m_2 = 3; \quad L = 1; \quad \alpha_0 = a, \quad \alpha_1 = -2b; \quad \beta_0 = 3a, \quad \beta_1 = -162b; \)

\[
\begin{align*}
\dot{x}_1 &= x_1 \left\{ a + b \left[ 16 \left( x_1 \right)^3 + 48 \left( x_1 \right)^2 x_2 - 9x_1 \left( x_2 \right)^2 - \left( x_2 \right)^3 \right] \right\}, \\
\dot{x}_2 &= x_2 \left\{ a - 2b \left[ 16 \left( x_1 \right)^3 - 33 \left( x_1 \right)^2 x_2 - 9x_1 \left( x_2 \right)^2 - \left( x_2 \right)^3 \right] \right\}. \quad (18)
\end{align*}
\]

Model A1.3: \( m_1 = 2, \quad m_2 = 3; \quad L = 1; \quad \alpha_0 = 2a, \quad \alpha_1 = 2b; \quad \beta_0 = 3a, \quad \beta_1 = 81b; \)

\[
\begin{align*}
\dot{x}_1 &= x_1 \left\{ a + b \left( x_1 \right)^3 \left[ \left( x_1 \right)^3 + 9 \left( x_1 \right)^2 x_2 + 33x_1 \left( x_2 \right)^2 - 16 \left( x_2 \right)^3 \right] \right\}, \\
\dot{x}_2 &= x_2 \left\{ a - b \left( x_1 \right)^3 \left[ 2 \left( x_1 \right)^3 + 18 \left( x_1 \right)^2 x_2 - 15x_1 \left( x_2 \right)^2 - 32 \left( x_2 \right)^3 \right] \right\}. \quad (19)
\end{align*}
\]

3.2. Models of type A2

In the 6 cases listed in this Subsection 3.2 the variables \( y_m(t) \) and \( y_m(t) \) are supposed to satisfy the system A2 of 2 coupled ODEs (see (15) and Appendix A), with the indicated assignments of the various parameters.

Model A2.1: \( m_1 = 1, \quad m_2 = 2; \quad L = 3; \quad \alpha_0 = 3b_0, \quad \alpha_1 = a_0 - 3b_1, \quad \alpha_2 = -a_1 + 3b_2, \quad \alpha_3 = a_2 - 3b_3; \quad \beta_\ell = (-1)^{\ell+1} 2a_{\ell-1}, \quad \ell = 1, 2, 3; \quad \gamma_\ell = (-1)^{\ell+1} 2b_\ell, \quad \ell = 0, 1, 2, 3; \)

\[
\begin{align*}
\dot{x}_n &= \left( a_0 + a_1 X + a_2 X^2 \right) - \left[ b_0 + b_1 X + b_2 X^2 + b_3 X^3 \right], \\
n &= 1, 2; \quad X = 2x_1 + x_2. \quad (20)
\end{align*}
\]
Model A2.2: \( m_1 = 1, \ m_2 = 2; \ L = 3; \ \alpha_0 = 0, \ \alpha_\ell = c_\ell; \ \beta_\ell = 2c_\ell; \ \ell = 1, 2, 3 \)
\[
\dot{x}_n = x_n \left( c_1 + c_2 X + c_3 X^2 \right), \quad n = 1, 2, \quad X = x_1 (x_1 + 2x_2).
\] (21)

Model A2.3: \( m_1 = 1, \ m_2 = 3; \ L = 3; \ \alpha_0 = 9b_0, \ \alpha_\ell = (-1)^{\ell-1} (a_0 - 9b_\ell), \ \beta_\ell = (-1)^{\ell-1} 3a_\ell - 1, \ \ell = 1, 2, 3; \ \gamma_\ell = (-1)^{\ell} b_\ell, \ \ell = 0, 1, 2, 3. \)
\[
\dot{x}_1 = x_1 \left( a_0 + a_1 X + a_2 X^2 \right) - \frac{5x_1 + x_2}{2x_1} \left( b_0 + b_1 X + b_2 X^2 + b_3 X^3 \right),
\]
\[
\dot{x}_2 = x_2 \left( a_0 + a_1 X + a_2 X^2 \right) - \frac{4x_1 - x_2}{x_1} \left( b_0 + b_1 X + b_2 X^2 + b_3 X^3 \right),
\]
\[X = 2x_1 + x_2.\] (22)

Note that the right hand sides of these ODEs (22) are polynomial only if all the coefficients \( b_\ell \) vanish.

Model A2.4: \( m_1 = 3, \ m_2 = 1; \ L = 3; \ \alpha_0 = 0, \ \alpha_\ell = (-1)^{\ell-1} 3c_\ell, \ \beta_\ell = (-1)^{\ell-1} c_\ell - 1, \ \ell = 1, 2, 3; \ \gamma_\ell = 0, \ \ell = 0, 1, 2, 3. \)
\[
\dot{x}_n = x_n \left( c_0 + c_1 X + c_2 X^2 \right), \quad n = 1, 2, \quad X = (x_1)^2 x_2.
\] (23)

Model A2.5: \( m_1 = 2, \ m_2 = 3; \ L = 3; \ \alpha_0 = 0, \ \alpha_\ell = 2c_\ell - 1, \ \beta_\ell = 3c_\ell - 1, \ \ell = 1, 2, 3; \ \gamma_\ell = 0, \ \ell = 0, 1, 2, 3. \)
\[
\dot{x}_n = x_n \left( c_0 + c_1 X + c_2 X^2 \right), \quad n = 1, 2, \quad X = x_1 (x_1 + 2x_2).
\] (24)

Model A2.6: \( m_1 = 3, \ m_2 = 2; \ L = 3; \ \alpha_0 = 0, \ \alpha_\ell = (-1)^{\ell-1} 3c_\ell - 1, \ \beta_\ell = (-1)^{\ell-1} 2c_\ell - 1, \ \ell = 1, 2, 3; \ \gamma_\ell = 0, \ \ell = 0, 1, 2, 3. \)
\[
\dot{x}_n = x_n \left( c_0 + c_1 X + c_2 X^2 \right), \quad n = 1, 2, \quad X = (x_1)^2 x_2.
\] (25)

Remark 3.2-1. Note that the systems of ODEs (21) and (24) are identical, and likewise the systems of ODEs (23) and (25) are identical. ☐

3.3. Models of type A3

In the 2 cases listed in this Subsection the variables \( y_{m_1} (t) \) and \( y_{m_2} (t) \) are supposed to satisfy the system A3 of 2 coupled ODEs (see (16) and Appendix A), with the indicated assignments of the various parameters.

Model A3.1: \( m_1 = 1, \ m_2 = 2; \ \alpha_0 = -3a, \ \alpha_1 = -9b; \ \beta_0 = -2a, \ \beta_1 = -2b; \)
\[
\dot{x}_1 = a + b \left[ (x_1)^2 + 7x_1 x_2 + (x_2)^2 \right],
\]
\[
\dot{x}_2 = a + b \left[ 7(x_1)^2 + 4x_1 x_2 - 2(x_2)^2 \right].
\] (26)

Model A3.2: \( m_1 = 1, \ m_2 = 3; \ \alpha_0 = -18a, \ \alpha_1 = -486b; \ \beta_0 = -2a, \ \beta_1 = -2b; \)
\[
\dot{x}_1 = (x_1)^{-1} \left\{ a (5x_1 + x_2)
\right. \\
+ b \left[ 32 (x_1)^4 - 131 (x_1)^3 x_2 - 51 (x_1 x_2)^2 - 11x_1 (x_2)^3 - (x_2)^4 \right] \right\},
\]
\[
\dot{x}_2 = (x_1)^{-1} \left\{ 2a (4x_1 - x_2)
\right. \\
- 2b \left[ 32 (x_1)^4 + 112 (x_1)^3 x_2 - 51 (x_1 x_2)^2 - 11x_1 (x_2)^3 - (x_2)^4 \right] \right\}.
\] (27)
4. Outlook
Space limitations do not allow to report here some natural developments of the results reported above, such as the identification of the more general solvable systems which obtain from those reported in the preceding Section 3 via linear transformations of the dependent variables $x_n(t)$, say
\[ x_1(t) = A_1 + A_{11}z_1(t) + A_{12}z_2(t) , \quad x_2(t) = A_2 + A_{21}z_1(t) + A_{22}z_2(t) , \]
with the 6 parameters $A_1, A_2, A_{11}, A_{12}, A_{21}, A_{22}$ arbitrarily assigned: the interested reader is referred to the analogous developments discussed in Section 5 of Ref. [9].

And of course the investigation in various fields of applied mathematics of the algebraically solvable systems of 2 nonlinearly-coupled ODEs reported above is an interesting open task.

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Appendix A.
The 2 ODEs of the system A1 (see (14)) are uncoupled, and each of them is clearly solvable by quadratures. Note in particular that the initial-value problem of the ODE
\[ \dot{y} = ay + by^{m+1} \]
is given by the explicit formula
\[ y(t) = y(0) \exp \left( a t \right) \left\{ 1 + \left[ b \left[ y(0) \right] \right] m \left[ 1 - \exp \left( mat \right) \right] \right\} . \]

To solve the system A2 one treats firstly the first of the 2 ODEs (15)—essentially as just above. Then one notes—see Appendix A of Ref. [10]—that the solution of the second ODE of the system (15) reads
\[ y_m^2(t) = F(t) \left[ y_m^2(0) + \int_0^t dt' \left\{ \gamma \left[ y_m^1(t') \right] \right\} \right] \]
with
\[ F(t) = \exp \left\{ \int_0^t \left[ \beta \left[ y_m^1(t') \right] \right] \right\} . \]

Explicit solutions can be easily obtained in the following cases:
\[ m_1 = 1 , \quad m_2 = 2, 3 , \quad m = m_2/m_1 \] \]
\[ \dot{y}_m^1 = \alpha_0 + \alpha_1 y_m^1 + \alpha_2 \left( y_m^1 \right)^2 \] \]
\[ \dot{y}_m^2 = y_m^2 \left( \beta_0 + \beta_1 y_m^1 \right) + \gamma_0 \left( y_m^1 \right)^{1+m} + \gamma_1 \left( y_m^1 \right)^m + \gamma_2 \left( y_m^1 \right)^{1+m} ; \]
\[ y_m^1(t) = \frac{ y_m^1(0) \left[ 1 + \left( \Delta/\alpha_1 \right) \tanh \left( \Delta t \right) \right] - 2 \left( \alpha_0/\alpha_1 \right) \tanh \left( \Delta t \right) }{ 1 - \left\{ 2\alpha_2 y_m^1(0) + \Delta \right\} /\alpha_1 \} \tanh \left( \Delta t \right) , \]
\[ \Delta^2 = (\alpha_1)^2 - 4\alpha_0\alpha_2 . \]
\[ y_{m_2}(t) = f(t) \left[ y_{m_2}(0) + \int_0^t dt' \left[ f(t') \right]^{-1} \sum_{\ell=0}^L \left\{ \gamma_\ell \left[ y_{m_1}(t') \right]^{\ell-1+m} \right\} \right], \quad (A.9) \]

\[ f(t) = \exp \left\{ \int_0^t dt' \sum_{\ell=1}^L \left\{ \beta_\ell \left[ y_{\tilde{m}_1}(t') \right]^{\ell-1} \right\} \right\}. \quad (A.10) \]

Finally, to identify the solution of the system \textbf{A3} (see (16)) we note that time-differentiation of the first of its 2 ODEs entails that \( y_{m_1}(t) \) satisfies the decoupled second-order ODE

\[ \ddot{y}_{m_1} = \alpha_1 \left[ \beta_0 (y_{m_1})^{-1+m} + \beta_1 (y_{m_1})^{-1+2m} \right], \quad m = \frac{m_2}{m_1}; \quad (A.11) \]

hence (see (16)) \( y_1(t) \) is an elliptic (for \( m = 2 \)) or hyperelliptic (for \( m = 3 \)) function; and likewise for \( y_2(t) \) (see the first of the 2 ODEs (16)).

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