A REMARK ON YONEDA’S LEMMA

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ABSTRACT. Yoneda’s Lemma is about the canonical isomorphism of all the natural transformations from a given representable covariant (contravariant, reps.) functor (from a locally small category to the category of sets) to a covariant (contravariant, reps.) functor. In this note we point out that given any representable functor and any functor we have the canonical natural transformation from the given representable functor to the “subset” functor of the given functor, “collecting all the natural transformations”.

1. YONEDA’S LEMMA

The well-known Yoneda’s lemmas about representable functors are the following:

Theorem 1.1. Let C be a locally small category, i.e., \( \text{hom}_C(A, B) \) is a set, and let \( \text{Set} \) be the category of sets.

1. (the covariant case) Let \( F_* : C \to \text{Set} \) be a covariant functor. Let \( h_A := \text{hom}_C(A, -) \) be a covariant hom-set functor \( h_A : C \to \text{Set} \). Then the set of all the natural transformations from the hom-set covariant functor \( h_A = \text{hom}_C(A, -) \) to the covariant functor \( F_* \) is isomorphic to the set \( F_*(A) \):

\[
\text{Natural}(h_A, F_*) \cong F_*(A).
\]

2. (the contravariant case) Let \( F^* : C \to \text{Set} \) be a contravariant functor. Let \( h^A := \text{hom}_C(-, A) \) be a contravariant hom-set functor \( h^A : C \to \text{Set} \). Then the set of all the natural transformations from the hom-set contravariant functor \( h^A = \text{hom}_C(-, A) \) to the contravariant functor \( F^* \) is isomorphic to the set \( F^*(A) \):

\[
\text{Natural}(h^A, F^*) \cong F^*(A).
\]

From now, for the sake of simplicity, we denote \( h^A(X) = \text{hom}_C(X, A) \) simply by \([X, A]\) and similarly \([A, X]\) for \( h_A(X) = \text{hom}_C(A, X)\).

The contravariant case of Yoneda’s Lemma is proved by using the following commutative transformation: Let \( \tau : [-, A] \to F_*(-) \) be a natural transformation:

(1.2)

\[
\begin{array}{ccc}
\text{id}_A \in [A, A] & \xrightarrow{\tau} & F^*(A) \\
\downarrow f^* & & \downarrow f^* \\
f \in [X, A] & \xrightarrow{\tau} & F^*(X) \end{array}
\]

Note that \( f = f^*(\text{id}_A) = f \circ \text{id}_A \). Hence we have

\[
\tau(f) = \tau(f^*(\text{id}_A)) = f^*(\tau(\text{id}_A)) \quad \text{(by the naturality of } \tau\text{)}
\]

Thus the natural transformation \( \tau : [-, A] \to F_*(-) \) is determined by the element \( \tau(\text{id}_A) \in F^*(A) \).

Conversely, given any element \( \alpha \in F^*(A) \) we can define the natural transformation \( \tau_\alpha : [-, A] \to F^*(-) \) by, for each object \( X \in \text{Obj}(C) \),

\[
\tau_\alpha : [X, A] \to F^*(X) \quad \tau_\alpha(f) = f^* \alpha,
\]

in which case \( \tau_\alpha : [A, A] \to F^*(A) \) satisfies that \( \tau_\alpha(\text{id}_A) = \text{id}_A^*(\alpha) = \alpha \). The above isomorphism map is called the Yoneda map:

\[
\mathcal{Y} : \text{Natural}(h^A, F^*) \cong F^*(A) \quad \mathcal{Y}(\tau) := \tau(\text{id}_A), \text{ or}
\]

(*) Partially supported by JSPS KAKENHI Grant Numbers 16H03936.
\[ Y : F^*(A) \cong \text{Natural}(h^A, F^*) \quad Y(\alpha) := \tau_\alpha. \]

2. A Remark

Let \( G \) be a set and let \( \text{Sub}(G) \) be the set of all subsets of \( G \). Let \( h^A(-) = [-, A] : C \to \text{Set} \) and \( F^* : C \to \text{Set} \) be as above. Then for each object \( X \in \text{Obj}(C) \) we have the following canonical map:

\[ \text{Im}_{F^*} : h^A(X) = [X, A] \to \text{Sub}(F^*(X)) \]

defined by

\[ \text{Im}_{F^*}(f) := \text{Image}(f^* : F^*(A) \to F^*(X)) = f^*(F^*(A)) = \{ f^*\alpha \mid \alpha \in F^*(A) \}. \]

The last two parts are written down for an emphasis. As observed in the above, \( f^*\alpha = \tau_\alpha(f) \) which is the image of \( f \) under the natural transformation \( \tau_\alpha \) corresponding to \( \alpha \in F^*(A) \). In other words \( \text{Im}_{F^*}(f) \) is the set consisting of the images of \( f \) by all the natural transformations \( \text{Natural}(h^A, F^*) \).

For a morphism \( g : X \to Y \in C \), we have the following commutative diagram:

\[
\begin{array}{ccc}
  h^A(Y) = [Y, A] & \xrightarrow{\text{Im}_{F^*}} & \text{Sub}(F^*(Y)) \\
  9^* \downarrow & & \downarrow 9^* \\
  h^A(X) = [X, A] & \xrightarrow{\text{Im}_{F^*}} & \text{Sub}(F^*(X))
\end{array}
\]

If we let \( \text{Sub}F^* : C \to \text{Set} \) be the “subset” functor associated to the given functor \( F^* : C \to \text{Set} \), defined by \( \text{Sub}F^*(X) := \text{Sub}(F^*(X)) \), we can consider \( \text{Im}_{F^*}(f) \) as a natural transformation

\[ \text{Im}_{F^*} : h^A(-) \to \text{Sub}F^*(-), \]

which sort of “collects” all the natural transformation images.

The upshot is the following.

Observation 2.1. Let \( C \) be a locally small category and \( \text{Set} \) be the category of sets. Let \( h^A(-) = [-, A] : C \to \text{Set} \) be a representable contravariant functor and \( F^* : C \to \text{Set} \) be another contravariant functor. Then we have the following canonical natural transformation (‘sort of “collecting” or “using” all the natural transformations from \( h^A \) to \( F^* \))

\[ \text{Im}_{F^*} : h^A(-) \to \text{Sub}F^*(-), \]

where for each object \( X \in \text{Obj}(C) \) we have \( \text{Im}_{F^*}(f) = f^*(F^*(A)) \subset F^*(X) \), which is the set consisting of the images of \( f \) by all the natural transformations from \( h^A \) to \( F^* \).

The similar observation for the covariant case is made mutatis mutandis, so omitted.

Remark 2.2. Depending on the situations, the target category of our contravariant functor can have more structures, e.g., groups, abelian groups, rings, commutative rings, etc.

3. Applications/Examples

Example 3.1 (complex vector bundles and characteristic classes (e.g., see [11, 6]).) Let \( \text{Vect}_n(X) \) be the set of isomorphism classes of complex vector bundles of rank \( n \) and for \( f : X \to Y \) the pullback map \( f^* : \text{Vect}_n(Y) \to \text{Vect}_n(X) \) is defined by \( f^*[E] := [f^*E] \). Thus the functor is a contravariant functor from the category of (paracompact) topological spaces to the category \( \text{Set} \) of sets. Then we do know that

\[ \text{Vect}_n(X) \cong [X, G_n(\mathbb{C}^\infty)] \]

where \( G_n(\mathbb{C}^\infty) \) is the infinite Grassmann manifold of complex planes of dimension \( n \), i.e., the classifying space of complex vector bundles of rank \( n \). This isomorphism is by the correspondence \([E] \leftrightarrow [f_E] \), where \( f_E : X \to G_n(\mathbb{C}^\infty) \) is a classifying map of \( E \), i.e., \( E = f_E^*\gamma^n \), where \( \gamma^n \) is the universal complex vector bundle of rank \( n \) over \( G_n(\mathbb{C}^\infty) \). Thus the functor \( \text{Vect}_n(-) \) is a representable contravariant functor. Let \( H^*(-; \mathbb{Z}) \) be the integral cohomology functor. Then by the observation in the previous section we have the following natural transformation:

\[ \text{Im}_{H^*} : \text{Vect}_n(-) \to \text{Sub}H^*(-; \mathbb{Z}), \]
defined by for \( \text{Vect}_n(X) \), \( \text{Im}_{H^n}([E]) := \text{Im} \left( f_E^*: H^*(G_n(\mathbb{C}^\infty); \mathbb{Z}) \to H^*(X; \mathbb{Z}) \right) \). By the definition of characteristic classes, \( \text{Im} \left( f_E^* : H^*(G_n(\mathbb{C}^\infty); \mathbb{Z}) \to H^*(X; \mathbb{Z}) \right) \) is nothing but the subring consisting of all the characteristic classes of \( E \), which is \( \mathbb{Z}[c_1(E), c_2(E), \ldots, c_n(E)] \). Let us denote this subring by \( \text{Char}(E) \). By the definition for isomorphic two vector bundles \( E \) and \( E' \) we do have \( \text{Char}(E) = \text{Char}(E') \). One could define a very “coarse” classification of vector bundles using \( \text{Char}(E) \), i.e.,

\[
E \cong_{\text{coarse}} E' \iff \text{Char}(E) = \text{Char}(E')
\]

For example, a line bundle \( L \) and its inverse \( L^{-1} \) satisfy that \( L \cong_{\text{coarse}} L^{-1} \) because \( c_1(L^{-1}) = -c(L) \), which implies that \( \text{Char}(L) = \text{Char}(L^{-1}) \).

**Remark 3.2.** In the case of real vector bundles, the complex infinite Grassmann \( G_n(\mathbb{C}^\infty) \), the Chern class \( c_i \) and the coefficient ring \( \mathbb{Z} \) are respectively replaced by the real infinite Grassmann \( G_n(\mathbb{R}^\infty) \), the Stiefel-Whitney class \( w_i \), and the coefficient ring \( \mathbb{Z}_2 \).

**Example 3.3** (René Thom’s notion of dependence of cohomology classes). Thom defined the following:

**Definition 3.4** (R. Thom). The cohomology class \( \beta \in H^q(X; B) \) depends on the cohomology class \( \alpha \in H^p(X; A) \), where \( A, B \) are coefficient rings, if, for all (perhaps infinite) polyhedra \( Y \) and all maps \( f : X \to Y \) such that \( \alpha \in f^*(H^p(Y; A)) \), we have \( \beta \in f^*(H^q(Y; B)) \).

Fist we recall that the cohomology theory is a representable contravariant functor, indeed, representable by the Eilenberg-Maclane space, i.e., \( H^j(X, R) \cong [X, K(R, j)] \) where \( K(R, j) \) is the Eilenberg-Maclane space whose homotopy type is completely characterized by the homotopy groups \( \pi_j(K(R, j)) = R \) and \( \pi_i(K(R, i)) = 0, i \neq j \). Then by the Hurewicz Theorem we have \( H_j(K(R, j); \mathbb{Z}) \cong \pi_j(K(R, j)) = R \) and \( H_d(K(R, j)) = 0 \) for \( d < j \). Hence by the universal coefficient theorem we have the isomorphism

\[
\Phi : H^j(K(R, j); R) \cong \text{Hom}(H_j(K(R, j); \mathbb{Z}), R) \cong \text{Hom}(\pi_j(K(R, j)), R) \cong \text{Hom}(\mathbb{R}, R).
\]

Let \( u := \Phi^{-1}(\text{id}_R) \) for the identity map \( \text{id}_R : R \to R \). Then the isomorphism \( \Theta : [X, K(R, j)] \cong H^j(X, R) \) is obtained by \( \Theta([f]) := f^*u \) where \( f^* : H^j(K(R, j); R) \to H^j(X, R) \).

**Proposition 3.5** (R. Thom). Let \( \alpha \in H^p(X; A) \cong [X, K(A, p)] \) and let \( f_\alpha : X \to K(A, p) \) be a map such that the homotopy class \( [f_\alpha] \) corresponds to \( \alpha \). Then \( \beta \in H^q(X, B) \) depends on \( \alpha \) if and only if \( \beta \in f_\alpha^*(H^q(K(A, p); B)) \).

In our set-up, we consider the representable contravariant functor \( h_{K(A, p)}(-) \) and the contravariant cohomology functor \( H^p(-; B) \). Then from the above section we have the following canonical natural transformation:

\[
\text{Im}_{H^p(-; B)} : [-, K(A, p)] \to \text{Sub}H^p(-; B),
\]

where for a topological space \( X \) and for (the homotopy class) of \( f_\alpha : X \to K(A, p) \) corresponding to the cohomology class \( \alpha \in H^p(X; A) \cong [X, K(A, p)] \) we have

\[
\text{Im}_{H^p(-; B)}(f_\alpha) = f_\alpha^*(H^p(K(A, p); B)),
\]

which is, due to the above proposition of Thom, nothing but the subgroup of all the cohomology classes \( \beta \in H^q(X; B) \) depending on the cohomology class \( \alpha \), and also by our observation above it is the subgroup consisting of the image of \( f_\alpha \) by all the natural transformations from the representable functor \( [-, K(A, p)] \), in other words the cohomology functor \( H^p(-; A) \) to the cohomology functor \( H^q(-; B) \).

One can consider some other reasonable or interesting pairs \( (h^A(-), F^*(\cdot)) \) of representable contravariant functors \( h^A(-) \) and contravariant functors \( F^*(\cdot) \). In a different paper we want to study such things, e.g. K-theory and the cohomology theory.

4. **ONE MORE REMARK: POSET-STRATIFIED SPACE STRUCTURES OF REPRESENTABLE FUNCTORS**

The subset functor \( \text{Sub}F^*(\cdot) \) has in fact another simple structure of partial order set due to the set-theoretic inclusion. If we consider the above natural transformation

\[
\text{Im}_{F^*} : h^A(X) = [-, A] \to \text{Sub}F^*(\cdot)
\]

and for an object \( X \in \text{Obj}(\mathcal{C}) \) we have

\[
\text{Im}_{F^*} : h^A(X) = [X, A] \to \text{Sub}(F^*(X))
\]
defined by $\text{Im}_{F^*}(f) = f^*(F^*(A))$. For the following commutative diagram of morphisms in $C$

$$
\begin{array}{c}
X \\
\downarrow g \\
A \\
\end{array}
\xrightarrow{f} 
\begin{array}{c}
A \\
\downarrow t \\
A \\
\end{array}
$$

under the contravariant functor $F^*$, we have that $f^* = g^* \circ t^*$:

$$
\begin{array}{c}
F^*(X) \\
\downarrow g^* \\
F^*(A) \\
\end{array}
\xrightarrow{f^*} 
\begin{array}{c}
F^*(A) \\
\downarrow t^* \\
F^*(A) \\
\end{array}
$$

Hence we have that

$$\text{Im}_{F^*}(f) = f^*(F^*(A)) \subseteq \text{Im}_{F^*}(g) = g^*(F^*(A)).$$

If we define the order $f \leq_L g$ for $f, g \in [X, A]$ by the above commutative diagram $X \xrightarrow{f} A$, $A \xrightarrow{g} A$, $A \xrightarrow{t} A$

i.e., by the condition that $\exists t \in [A, A]$ such that $f = t \circ g$, then this order $\leq_L$ is a preorder, i.e., it is reflexive and transitive, but not necessarily anti-symmetric. With this order we get a preordered set $([X, A], \leq_L)$. A preordered set is called a proset and the category of presets and monotone (order-preserving) maps is denoted by $\mathcal{P}roset$. Namely, by this preorder, the representable contravariant functor $h^A : C \to \mathcal{S}et$ becomes a representable contravariant functor to the category of preorders: $h^A : C \to \mathcal{P}roset$. The contravariant functor $F^*(-)$ also gives rise to the associated contravariant functor $\mathcal{S}ub F^* : C \to \mathcal{P}roset$, to be more precise, the order of inclusion is a partial order, thus in the case of $\mathcal{S}ub F^*$ the target id the category $\mathcal{P}oset$ of posets (partially ordered sets) and monotone (order-preserving) maps. With these preorders the above natural transformation $\text{Im}_{F^*} : h^A(X) = [-, A] \to \mathcal{S}ub F^*(-)$ is a natural transformation $\text{Im}_{F^*} : (h^A(X) = [-, A], \leq_L) \to (\mathcal{S}ub(F^*(-)), \leq))$. Then for each object $X \in \text{Obj}(C)$ we have the map

$$\text{Im}_{F^*} : (h^A(X) = [X, A], \leq_L) \to (\mathcal{S}ub(F^*(X)), \leq))$$

which is a monotone map from a proset to a poset. If we consider the Alexandroff topologies 11(also see [2] [3] [12]) for a proset, this monotone map becomes a continuous map from the proset considered as a topological space with the Alexandroff topology to a poset considered as a topological space with the Alexandroff topology. This is nothing but the so-called a poset-stratified space 10(e.g., also see [3]). From this viewpoint in 15 (cf. 14) we consider poset-stratified space structures of the homotopy set of continuous maps of topological spaces and in 16 for a general locally small category.

**Remark 4.1.** To get a poset-stratified space structure of $\text{Im}_{F^*} : (h^A(X), \leq_L) \to (\mathcal{S}ub(F^*(X)), \leq)$ we appeal to another contravariant functor $F^*(-)$. But, in order to get such a poset-stratified space structure we do not need such a functor. In general, if we have a proset $(P, \leq)$, then we consider the equivalence relation $a \sim b \iff a \leq b, b \leq a$ on $P$ and consider the set $P_\sim$ of the equivalence relations and we define the order $[a] \leq [b] \iff a \sim b$, which is a partial order and the canonical map $\pi : (P, \leq) \to (P_\sim, \subseteq)$ is a monotone map, thus we get a poset-stratified space considering the associated Alexandroff topologies (e.g., [16]). This construction gives a kind of universal poset-stratified space structure $\pi : ([X, A], \leq_L) \to ((X, A), \leq_L)$ and the above $\text{Im}_{F^*} : (h^A(X) = [X, A], \leq_L) \to (\mathcal{S}ub(F^*(X)), \leq))$ involving another contravariant functor $F^*$ give a more geometric one, so to speak.

**Remark 4.2.** For the topological homotopy category $h\mathcal{T}op$ the above relation $f = t \circ g$ becomes $f \sim t \circ g$. This relation was considered in a different context by K. Borsuk [4, 5] and generalized by P. Hilton [7] (cf. [8] [9]).
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