Fundamental Diagram of Traffic Flow from Prigogine-Herman-Enskog Equation

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Recent applications of a new methodology to measure fundamental traffic relations on freeways shows that many of the critical parameters of the flow-density and speed-spacing diagrams depend on vehicle length. In response to this fact, we present in this work a generalization of the Prigogine-Herman traffic equation for aggressive drivers which takes into account the fact that vehicles are not point-like objects but have an effective length. Our approach is similar to that introduced by Enskog for dense gases and provides the construction of fundamental diagrams which are in excellent agreement with empirical traffic data.

Recently, Coifman5,6 introduces a new methodology to measure fundamental diagrams on freeways which eliminates the search for stationary conditions and provides a new level of accuracy to calibrate macroscopic and microscopic traffic models. After applying his method to three independent data set, Coifman found out that many of the critical parameters (e.g., the critical density and the slope of the congested regime \( w = (dQ/d\rho)_{\rho=\rho_c} \)) appearing on the flow-density fundamental diagram depend on vehicle length. Thus, Coifman’s results indicate that traffic flow models must be updated in order to take into account the length of the vehicles.

In this letter, we address this problem by considering the fundamental diagram from a mesoscopic point of view, i.e., we start from the gas-kinetic-like traffic equation5,8

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = -\frac{f-f_0}{\tau} + I(f, f),
\]

where \( f = f(x, v, t) \) denotes the distribution function. The distribution function is defined in such a way that \( f(x, v, t)\,dx\,dv \) gives the number of vehicles at time \( t \) with position around \( x \) and velocity around \( v \). The first term on the right-hand side of the kinetic traffic equation\(^5\) describes the relaxation of the distribution function to a desired distribution \( f_0 = f_0(x, v, t) \) on a time scale \( \tau \), while the second one is the so-called interaction term. By assuming that vehicles are not point-like objects but have a length \( \ell \) and also require a safety distance \( v\tau_c \), where \( \tau_c \) denotes drivers reaction time, it is possible to write the interaction term as

\[
I(f, f) = \int_v^\infty \chi(1-p)(v' - v)f(x + s(v'), v, t)f(x, v', t)dv' - \int_v^\infty \chi(1-p)(v - v')f(x, v, t)f(x + s(v), v', t)dv',
\]

where \( s(v) = \ell + v\tau_c \) represents the effective length of a vehicle moving with velocity \( v \), \( p \) denotes the overtaking
probability and \( \chi \) is a factor describing the increase in the interaction rate due to the spatial extension of the vehicles. Following Prigogine and Herman \[2\], we assume that the overtaking probability \( p \) and the relaxation time \( \tau \) are given by \( p = 1 - z \) and \( \tau = \tau_0 (1 - p) / \rho \), where \( z = \rho / \rho_0 = \rho \ell \) is the so-called reduced density and \( \tau_0 \) is a proportionality constant. The factor \( \chi \) is given by the expression \( \chi = (1 - (z + Q \tau_c))^{-1} \) which follows by taking into account the mean effective length \( s(v) = \ell + v \tau_0 \) common to two vehicles at the interaction process. Note that the factor \( \chi \) is equal to unity for dilute traffic and may becomes negative at high density values. Since \( \tau_0 = 1 \) where \( \alpha > 0 \)

\[
\frac{f}{1 - \gamma \bar{v} + \gamma v},
\]

(3)

where \( \gamma = \rho \chi (1 - p) \tau \). The above non-linear equation has two types of solutions corresponding to individual and collective flow patterns. Conditions determining the occurrence of both patterns were discussed in detail by Prigogine and Herman. To go further, let us consider the gamma distribution

\[
f_0 = \frac{\alpha}{\Gamma(\alpha)} \rho (\frac{\alpha \bar{v}}{\bar{v}_0})^{\alpha-1} \exp \left( -\frac{\alpha \bar{v}}{\bar{v}_0} \right),
\]

(4)

where \( \alpha > 1 \) is the shape parameter and \( \bar{v}_0 \) denotes the mean desired velocity. According to Velasco and Marques Jr. \[10\], the behavior of aggressive drivers can be described by a gamma distribution with a shape parameter which is directly connected with driver’s aggressiveness. Note that the gamma distribution reduces to the exponential distribution when we set the shape parameter equal to one. The normalization condition for the homogeneous and stationary solution (3) can be written as

\[
\rho = \int_0^{\infty} \frac{f_0}{\sigma + \gamma v} dv,
\]

(5)

where \( \sigma = 1 - \gamma \bar{v} \). Based on the expressions for \( \chi \), \( p \) and \( \tau \), we verify that \( \sigma = 1 \) in the limit of vanishing density and may becomes negative at high density values. Since \( \sigma \) cannot be negative, one can define a critical density \( \rho_c \) which determines the transition between individual and collective regimes. In the case of a gamma distribution, the critical density satisfies the condition

\[
\beta z_c^3 - \frac{\alpha}{\alpha - 1} (1 - z_c) (1 - z_c - Q \tau_c) = 0,
\]

(6)

where \( \beta = \bar{v}_0 \tau_0 / \ell > 0 \) is a dimensionless parameter.

In the collective regime, where the relation \( \bar{v} = \gamma^{-1} \) holds, traffic flow follows the collective flow curve

\[
Q = Q_0 \frac{(1 - z)^2}{(1 - z) Q_0 \tau_c + (z / z_c)^2 (1 - z_c) (1 - z_c - Q \tau_c)}
\]

(7)

which is independent of the shape parameter. This result reflects the fact that, at high densities, drivers abandon their individual characteristics and begin to follow the collective behavior of the system. Once critical traffic conditions are known, the slope \( w = (dQ/d\rho)_{\rho = \rho_c} \) of the fundamental diagram can be used to determine drivers reaction time \( \tau_c \) in the collective regime.

In the individual (or low-density) regime, we find that driver’s aggressive behavior reaches its magnitude since traffic flow depends on the shape parameter. In this case, normalization condition (5) defines a function \( \gamma(\sigma) \) which can be used to obtain the individual flow curve. Fig. 1 shows the ratio \( \gamma / \gamma_c \) (where \( \gamma_c = \bar{v}_0^{-1} \alpha / (\alpha - 1) \) denotes the value of \( \gamma \) at the critical point) as a function of \( \sigma \) for the gamma distribution. The curves displayed in this figure are for the cases: \( \alpha = 120 \) (solid line) and \( \alpha = 3 \) (dashed line). The linear behavior of the solid curve is quite evident and it allows us to write \( \gamma = (1 - \sigma) / \bar{v}_0 \) for the case \( \alpha \gg 1 \). Here, it is important to mention that the application of well-known solution methods of gas-kinetic theory to the Boltzmann-like traffic equation for aggressive drivers \[11\] allows to identify the inverse of the shape parameter as the prefactor of the velocity variance. Experimental data reported in the literature \[8\] show that the prefactor of the velocity variance is almost constant at the low-density region and they can be used to set values for the shape parameter. Thus, we can see that the case \( \alpha = 120 \) is in agreement with empirical traffic data and traffic flow, in the individual regime, is given by the linear relation \( Q = \rho \bar{v} \).

Figures 2 and 3 compare the theoretical flow-density and speed-spacing curves derived from our non-local Prigogine-Herman traffic equation with the empirical traffic data of Coifman for different vehicle length bins. Coifman’s empirical traffic data \[5, 6\] follows by applying the single vehicle passage (svp) method to a primary data set which were collected from a two miles section of the I-80 (in the East Bay portion of the San Francisco Bay area) by using Berkeley Highway Laboratory detector system. Regarding the theoretical results, they follow from our individual and collective flow curves by using traffic and model parameters given in Table I. In this table, the

| Table I. Traffic and model parameters |
|---------------------------------------|
| 18-22 ft | 22-28 ft | 28-38 ft | 58-68 ft |
| \ell (ft) | 20 | 25 | 33 | 63 |
| \( Q_0 \) (veh/h) | 2350 | 2120 | 1525 | 1170 |
| \( \rho_c \) (veh/mi) | 48.4 | 39.5 | 35.1 | 25.6 |
| \( w \) (mi/h) | -14.9 | -16.6 | -17.4 | -26.9 |
| \( \tau_c \) (s) | 1.21 | 1.34 | 1.73 | 1.99 |
version of the Prigogine-Herman kinetic traffic equation for aggressive drivers can be used to describe properly the dependence of the fundamental diagrams on the length of the vehicles. Since such diagrams are central to most traffic flow models, we are quite sure that our mesoscopic description represents an important contribution to the understanding of some interesting phenomena as traffic breakdown and jam propagation. The present successful comparison between theory and experiments marks a step towards a construction of a fundamental diagram for a multi-class traffic flow. In this case, modifications must be introduced in order to include different types of vehicles having different lengths and safety distances. For each vehicle class, one can assign a desired distribution function which, together with the fractions of vehicles of each class, will govern the transition from individual to collective flow.

values of the critical density, flow capacity and slope of the flow-density curve in the collective regime were taken from Coifman’s traffic data. Besides, we take the length of vehicles as mean length value in each length bin, while drivers reaction time in collective regime were obtained, as pointed out previously, from the slope of the flow-density diagram. One can verify that the agreement between experimental data and theoretical predictions is really impressive, even for long vehicles where curves show more scatter than for shorter vehicles. Such scatter event in empirical traffic data occur because there are fewer long vehicles on the highway than shorter ones.

In conclusion, we show in this letter that a non-local version of the Prigogine-Herman kinetic traffic equation for aggressive drivers can be used to describe properly the dependence of the fundamental diagrams on the length of the vehicles. Since such diagrams are central to most traffic flow models, we are quite sure that our mesoscopic description represents an important contribution to the understanding of some interesting phenomena as traffic breakdown and jam propagation. The present successful comparison between theory and experiments marks a step towards a construction of a fundamental diagram for a multi-class traffic flow. In this case, modifications must be introduced in order to include different types of vehicles having different lengths and safety distances. For each vehicle class, one can assign a desired distribution function which, together with the fractions of vehicles of each class, will govern the transition from individual to collective flow.

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