Generic instabilities in a thin polar ordered active fluid layer

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We set up and study the generic coarse-grained dynamics of a thin inflexible layer of polar-ordered suspension of active particles embedded in an isotropic passive fluid medium. Investigations of the linear instabilities around a chosen orientationally ordered uniform reference state reveal generic moving and static instabilities in the system. We illustrate how the interfacial friction between the active fluid layer and the embedding fluid medium significantly affects the instabilities.

I. INTRODUCTION

The emergence of large-scale collective dynamics is one of the most intriguing and fascinating features of a large variety of driven, active systems made of active particles [1]. These are generally elongated and their direction of self-propulsion is set by their own anisotropy (i.e., the two ends are distinguishable, hence polar), instead of being determined by an externally imposed field. In contrast, active nematics [2], made of active particles which are head-tail symmetric, do not show any self-propulsion. These active systems, polar or nematic, are generically characterised by the existence of orientationally ordered states. These are nonequilibrium analogues of the equilibrium nematic liquid crystals. There are numerous examples, which include both living systems (living matter) as well as their artificially prepared non-living analogues. Biological examples of active systems include both small and large in-vitro and in-vivo systems, e.g., reconstituted bio-filaments and the associated motor proteins [3], the cytoskeleton of living cells and bacterial suspensions [4], cell layers [5], and also larger-size objects, e.g., flock of birds or school of fishes [6]. Analogous non-living examples of active matter systems also arise in various contexts, e.g., layers of vibrated granular rods [7] and colloidal or nanoscale particles propelled through a fluid by catalytic activity at their surface [8]. All these examples of active systems are distinguished by a local energy supply in the bulk that drives the systems away from equilibrium. This is in contrast to other well-known examples of driven systems, e.g., in sheared systems, where the external drives act at the boundaries. For instance, in cell biology contexts, this supply of energy takes place due to the hydrolysis of adenosine triphosphate (ATP) to adenosine diphosphate (ADP) and other phosphates (Ph) by the molecular motors, thus converting the chemical fuel into mechanical motion.

Despite hugely varying microscopic details, different active matter systems display a host of intriguing nonequilibrium phenomena with generic features independent of system details, e.g., pattern formations, wave propagations, oscillations and unusually strong fluctuations etc [15–17] etc. Due to the large number of diverse microscopic variables present (especially in the cell biology context), the level of complexities in active matter systems at microscopic levels is very high. Instead, it is convenient to formulate the coarse-grained dynamics of the active systems based on identifying global features, e.g., the presence or absence of conservation laws, symmetries, the presence of appropriate broken symmetry variables and the nature of the underlying momentum damping. These are similar in spirit and nonequilibrium generalisation of the general principles and laws developed to describe the statistical mechanics and dynamics of the ordered phases in equilibrium systems [9]. These active fluid theories, parametrised by a set of phenomenological constants [2, 10–14], serve as as generic coarse-grained descriptions for a driven orientable fluid with nematic or polar symmetries and are particularly useful to uncover and elucidate the long wavelength behaviour observed in very different physical systems and at very different length scales [6, 15–18].

In a bulk fluid (both active and passive) the viscosity damps out any local momentum gradient and thus reduces any relative velocities between neighbouring regions. The total momentum of the system is however kept conserved; such systems are known as wet active matters in the language of Ref. [13]; see, e.g., Refs. [15–19]. In contrast, for systems in contact with a rigid substrate (e.g., a layer of active fluid on a solid substrate) there is a drag on the system acted typically through a no-slip boundary condition on the active matter velocity at the active matter-rigid substrate interface. This drag leads to nonconservation of the momentum of the active system and cuts off any long-ranged hydrodynamic interactions. These are known as dry active matter in the classification used in Ref. [13] and have been studied extensively, see, e.g., [6, 15–20]. The properties of active matter systems are often considered
in the form of thin, quasi two-dimensional (2D) layer. Such quasi-2D active matter systems exist both *in-vitro* and *in-vivo* cell cortex [21] or the cortical actin layers and cell ruffles, e.g., lamellipodia [22] are examples belonging to the former category, where as reconstituted actin layers on liposomes [23] are examples of 2D *in-vitro* active fluid systems. In the present work we are interested in a class of 2D wet polar active matter layer, where the active particle system is embedded inside a three-dimensional (3D) bulk isotropic passive fluid. The active fluid and the embedding passive fluid interact via a mutual friction at the interfaces of the active fluid-bulk fluid interface, leading to momentum damping of the active particles. Inspired by the current studies on both wet and dry active matters and their significant differences in terms of their long wavelength properties, we analyse the dependences of the long wavelength properties of a 2D active orientationally ordered system inside a 3D fluid medium on the mutual friction.

In this work, we construct a generic coarse-grained dynamical description of a thin inflexible flat layer of a suspension of active particles, embedded in passive isotropic fluid. To this end, we construct a set of 2D continuum equations of motion for the local orientation and number density of the polar active species. In a linearised treatment about a chosen orientationally ordered state, we analyse the linear instabilities in the system. We show how the nature of the instabilities depend on the various active contributions to the dynamics. We illustrate the role of the mutual friction between the active fluid layer and the bulk embedding fluid in controlling the instabilities. We also study the nematic limit of the dynamics. We highlight how a change in the magnitude of the interfacial friction relative to the viscous dampings may drastically alter the long wavelength dynamics. Our results may be used to study the nature of friction, say, e.g., in reconstituted actin filaments deposited on a liposome embedded in a fluid. In addition, in an *in-vitro* system of two eukaryotic cells with a substantial area of contact, then the dynamics of the cortical actin layers of the two cells on both sides of the contact plane should be describable by our dynamical equations at a coarse-grained level. Nonetheless, our formulation is sufficiently general and does not specifically relate to any cell biological example. Our results show how experimental studies may be used to obtain information about the mutual friction at the interface. The rest of the paper is organised as follows. Equivalently, a large enough system should display very different linear instabilities at various length scales, controlled by the interfacial friction. The rest of the article is organised as follows: In Sec. II we define our model and set up the basic equations of motion. Then in Secs. IIIA, IIIB, and IIIC we analyse the instabilities for high, intermediate and low values of the mutual friction. Then in Sec. IV we briefly compare the linear instabilities in the different regimes of the model, delineated by the magnitude of the mutual friction. In the next Sec. V we analyse the nematic limit of our model dynamics. We discuss and summarise in Sec. VI. Finally, we provide some calculational details and the obtain the ambient velocity profiles in Appendix.

II. MODEL EQUATIONS

We consider an inflexible thin planar layer of a viscous active fluid with a vanishingly small thickness, located at the xy-plane, i.e., at $z = 0$. We treat it as a quasi 2D system, for which a 2D description should be appropriate. The local number densities of the active species and the solvent are $\rho(x)$ and $\phi(x)$, where $x = (x, y)$, respectively. The active fluid layer, with a 2D viscosity $\eta$, is embedded in a 3D passive incompressible ambient fluid with a 3D viscosity $\eta'$, both above ($z > 0$) and below ($z < 0$); see Fig. I for a schematic diagram of our model system.

The centre of mass velocity of the active particles and the solvent combined is given by $v$. The total number of both the active and solvent particles are separately conserved: The continuity equations for $\rho$ and $\phi$ are written as

$$\partial_t \rho + \nabla \cdot J_\rho = 0, \tag{1}$$

$$\partial_t \phi + \nabla \cdot J_\phi = 0. \tag{2}$$

Here, $\nabla \equiv \hat{\text{x}} \partial/\partial x + \hat{\text{y}} \partial/\partial y$ is the 2D gradient operator, $\hat{\text{x}}, \hat{\text{y}}$ are the unit vectors along $x$- and $y$-directions. The particle currents $J_\rho$ and $J_\phi$ can be expressed in terms of the 2D centre-of-mass velocity $v$ and the diffusion current $j$.

$$J_\rho = \rho v + j, \tag{3}$$

$$J_\phi = \phi v - j. \tag{4}$$

Here, the molecular masses of both the active and solvent particles are assumed to be equal to unity for calculational convenience. We are interested in an orientationally ordered state of the model system. To this end, we introduce a 2D local polarisation vector $p = (p_x, p_y)$, with a fixed magnitude, $p^2 = 1$, as appropriate for an orientationally ordered state. Microscopically, it describes the local orientations of the actin filaments or bacteria. We consider the active fluid to be incompressible, i.e., $\nabla \cdot v = 0$. Our chosen reference state is defined by $p_x = 1$ with no macroscopic flow, i.e., $\langle v_\alpha \rangle = 0$. In the Stokesian limit of the flow dynamics, the force balance equation

$$\nabla_\alpha q_{\alpha\beta} - \partial_\beta \Pi + F_\beta = 0, \text{ with } \alpha, \beta = x, y \tag{5}$$

in Appendix.
yields the generalised Stokes equation for \( \mathbf{v} \). The 2D pressure \( \Pi \) may be eliminated by using the incompressibility condition \( (\nabla \cdot \mathbf{v} = 0) \). Here, \( \sigma_{\alpha \beta} \) is the total stress tensor and external forces \( f_\beta \) are the tangential stresses of the embedding fluid on the two sides (top and bottom) of the active fluid layer.

\[
F_\beta = \eta' \left( \partial_z v'_\beta + \partial_\beta v'_z \right)|_{z=\epsilon} - \eta' \left( \partial_z v'_\beta + \partial_\beta v'_z \right)|_{z=-\epsilon}
\]

where \( \epsilon \to 0; \mathbf{v}'(\mathbf{r}) \) (with \( \mathbf{r} = (x, y, z) \)) is the 3D ambient fluid velocity.

In the spirit of linear response theories \([9]\), the dynamics of the active fluid layer is described in terms of linear relations between the thermodynamic fluxes \( (\sigma^s_{\alpha \beta}, j_\alpha, P_\alpha) \) and the corresponding generalised forces \( (v_{\alpha \beta}, \partial_\alpha \overline{\mu}, h_\alpha) \) \([24]\).
Here, \( \sigma_{\alpha\beta}^{a} \) is the symmetric part of the deviatoric stress

\[
\sigma_{\alpha\beta}^{a} = \sigma_{\alpha\beta} + \rho_{i} v_{i\alpha} v_{\beta} - \sigma_{\alpha\beta}^{a} + \Pi \delta_{\alpha\beta},
\]

with \( \sigma_{\alpha\beta}^{a} = (p_{\alpha} h_{\beta} - p_{\beta} h_{\alpha})/2 \) is the antisymmetric part of the stress tensor, \( h_{\alpha} \) being the thermodynamic force conjugate to polarisation \( p_{\alpha} \). Further, \( \rho_{i} = \rho + \phi \) is the total density of the two species combined and \( v_{i\alpha} = (\partial_{\alpha} v_{\beta} + \partial_{\beta} v_{\alpha})/2 \) is local strain rate tensor. In addition, \( P \) is the convected co-rotational derivative of the polarisation vector given by

\[
P_{\alpha} = \frac{D}{Dt} p_{\alpha} = \partial_{t} p_{\alpha} + v_{\beta} \partial_{\beta} p_{\alpha} + \omega_{\alpha\beta} p_{\beta},
\]

with \( \omega_{\alpha\beta} = \frac{1}{2} (\partial_{\alpha} v_{\beta} - \partial_{\beta} v_{\alpha}) \) is the vorticity tensor. Furthermore, \( \bar{\mu} = \mu_{\rho} - \mu_{\phi} \) is the effective chemical potential; \( \mu_{\rho} \) and \( \mu_{\phi} \) are individual chemical potentials of the active particles and the solvent molecules, respectively. For simplicity we consider the dilute limit of the active particles, i.e., \( J_{\phi} \approx 0 \). In this limit, the overall incompressibility is equivalent to considering \( \phi = const. \), so that the dynamics of \( \phi \) can be neglected and we consider the dynamics of \( \rho \) alone. In this dilute limit, \( \bar{\Pi} \) may be replaced by the chemical potential \( \mu_{\rho} \) for the active particles.

The stress field is assumed to contain a nonequilibrium active stress of the form

\[
\sigma_{\alpha\beta}^{act} = \zeta'(\rho) \Delta \mu_{\rho} p_{\alpha} p_{\beta}.
\]

Microscopically, \( \sigma_{\alpha\beta}^{act} \) is due to the local nonequilibrium dynamics of the active particles. Therefore, the magnitude of \( \sigma_{\alpha\beta}^{act} \) should depend on the local density \( \rho \) of the active particles; hence the form \( \zeta' = \zeta'(\rho) \). We write \( \rho = \rho_{0} + \delta \rho \), where \( \rho_{0} \) is the mean active particle density and \( \delta \rho \) are fluctuations (assumed small) about \( \rho_{0} \). Expanding for small \( \delta \rho \), we write \( \zeta'(\rho) = \zeta + \zeta'_{0} \delta \rho \), where \( \zeta \approx \zeta'(\rho_{0}) \) and \( \zeta' = \zeta'(\rho_{0}) \). Parameter \( \Delta \mu \) represents the strength of \( \sigma_{\alpha\beta}^{act} \); the latter is said to be contractile or extensile depending on whether \( \Delta \mu \) is negative or positive, respectively; 

\( \Delta \mu \) is a measure of the rate of supply of (free) energy that pushes the system out of equilibrium; in the context of the cortical actins in a cell, it is the hydrolysis of the ATP molecules to ADP and phosphates that supplies this energy; \( \Delta \mu = \mu_{ATP} - \mu_{ADP} - \mu_{pH} \) where \( \mu_{ATP}, \mu_{ADP}, \mu_{pH} \) are the chemical potentials of ATP, ADP and phosphate molecules. Parameter \( \Delta \mu \) has the dimension of energy/(mass.mole). Numerical estimation of \( \Delta \mu \) is not easy: In the particular context of cell biology, we use the fact that approximately 7 kCal energy released per mole of ATP due to its hydrolysis. Since 1 molar mass of ATP ~ 500, we obtain from its definition \( \Delta \mu \approx 7kCal/(500gm/10^{23}) \), the free energy release per unit mass per molecule.

The relevant linear flux-force relations \[24\], that include the active stress contribution to the stress and allow for polar terms, i.e., not invariant under \( p \rightarrow -p \), are

\[
\sigma_{\alpha\beta}^{a} = 2\eta \tilde{v}_{\alpha\beta} + \zeta'(\rho) \Delta \mu_{\rho} p_{\alpha} p_{\beta} + \frac{\nu_{1}}{2} (p_{\alpha} h_{\beta} + p_{\beta} h_{\alpha}) - \frac{\epsilon_{0}}{2} (p_{\alpha} \partial_{\beta} \mu_{p} + p_{\beta} \partial_{\alpha} \mu_{p}), \tag{10}
\]

\[
j_{\alpha} = -\gamma_{pp} \partial_{\alpha} \tilde{p} + \tilde{\lambda} h_{\alpha} + \kappa_{p} \Delta \mu \mu_{w} \nabla_{\beta} (pp_{\beta}p_{\beta}) - \frac{\epsilon_{0}}{2} \mu_{p} (p_{\alpha} \partial_{\beta} v_{\alpha} - \partial_{\beta} v_{\alpha}), \tag{11}
\]

\[
P_{\alpha} = \frac{h_{\alpha}}{\gamma_{v}} + \lambda_{1} p_{\alpha} \Delta \mu + \nu_{1} p_{\beta} v_{\alpha\beta} - \nu \delta_{\alpha} \bar{\Pi} + \lambda_{2} (p \cdot \nabla) p_{\alpha} \Delta \mu + \lambda_{p} \partial_{\alpha} \rho \Delta \mu. \tag{12}
\]

Coupling constant \( \nu_{1} \) denotes the equilibrium flow-orientation coupling \[25\] and should have magnitude larger than unity for stability reasons \[25\]; similarly, \( \epsilon_{0} \) denotes the equilibrium coupling between the flow and the particle current \[24\]. Parameter \( \gamma_{pp} > 0 \) is a mobility coefficient (an equilibrium coupling constant) and related to the diffusion coefficient. In addition, particle current \( j_{\beta} \) should have active contributions \( \kappa_{p} \Delta \mu p_{\beta} \) and \( w \Delta \mu \nabla_{\beta} (pp_{\beta}p_{\beta}) \), such that there should be an active flow of the particles in the direction of \( p \), with amplitudes proportional to \( \Delta \mu \). In addition, \( \tilde{\lambda} \) is a cross-coupling equilibrium coupling constant. (In general, \( \tilde{\alpha} \) may be a tensor reflecting the anisotropy of the polar ordered state; we neglect this here.) Notice that in \[12\] we include two symmetry-permitted active terms with coefficients \( \lambda_{2} \) and \( \lambda_{p} \), respectively: the \( \lambda_{2} \)-term is a self-advection term and the \( \lambda_{p} \)-term is a nonequilibrium osmotic pressure term, modeling motion of the active particles along or opposite to the concentration gradient (depending upon the sign). Coefficients \( \zeta', \kappa_{p}, \lambda_{1}, \lambda_{2}, w, \mu_{p} \) are "active coefficients", i.e., coefficients of different active terms in Eqs. \[10\,12\]. Out of these, \( \kappa_{p}, \lambda_{2} \) and \( \lambda_{p} \) are coefficients of the different polar terms, which break the symmetry under \( p \rightarrow -p \), whereas \( w, \zeta \) and \( \zeta' \) are coefficients of the nematic active terms in the dynamics. Thus, in the nematic limit of the model, \( \kappa_{p}, \lambda_{2} \) and \( \lambda_{p} \) are all zero, and the only source of nonequilibrium drive is the active stress \[10\] and the active particle current represented by the \( w \)-term. For reasons similar to the \( \rho \)-dependence of \( \zeta' \), active coefficient \( \kappa_{p} \) should depend on \( \rho \). We write for small density fluctuations \[13\]

\[
\kappa_{p}(\rho) = \kappa_{0} + \kappa_{pp} \delta \rho, \quad \kappa_{pp} = \frac{\kappa_{p}}{\partial \rho} \rho = \rho_{0}, \tag{13}
\]
where \( \kappa_0 = \kappa_\rho(\rho_0) \) depends on the mean density and \( \kappa_\rho \) incorporates the effects of the fluctuations of \( \rho \) about \( \rho_0 \). We ignore any \( \rho \)-dependence of \( \lambda_\rho \) and \( \lambda_\rho \) and treat them as a constant, since we are interested in a linearised treatment.

Thermodynamic forces \( h \) and \( \mu_\rho \) are defined as follows

\[
h_\alpha = -\frac{\delta F_0}{\delta \rho_\alpha}, \quad \mu_\rho = \frac{\delta F_0}{\delta \rho},
\]

where \( F_0 \) is a free energy functional that controls the relaxation of the system to its thermal equilibrium state in the absence of any activity. At the bilinear order in fields

\[
F_0 = \int d^2x \frac{1}{2}[D(\nabla_\alpha p_\beta)^2 + A(\delta \rho)^2 + 2\chi \rho \nabla \cdot p],
\]

where \( D \) is a 2D Frank elastic constant (we have assumed equal Frank’s constants for simplicity), \( A \sim T \rho_0 \) is an Osmotic modulus with \( T \) being the temperature when the system is in thermal equilibrium, \( \chi \) provides a symmetry-allowed coupling between the density fluctuations and splay. Assuming the minimum free energy configuration to be

\[
\kappa = \frac{1}{2} \delta \rho \chi, \quad \rho \big|_{\min} = \rho_0
\]

we obtain to the lowest order in spatial gradients (see Appendix.)

\[
\eta \nabla^2 v_\beta + \frac{\zeta}{\rho} \nabla \rho \nabla p_\beta = 0 = \eta \nabla^2 v_\beta + \frac{\zeta}{\rho} \nabla \rho \nabla p_\beta + \frac{\zeta}{\rho} \nabla \nabla^2 p_\beta \rho = -F_\beta,
\]

valid for both the super- \( (z > 0) \) and sub- \( (z < 0) \) phases, \( \Pi' \) is the ambient fluid pressure and \( \nabla_3 \) is the 3D gradient operator. The boundary conditions on \( v'_i \) are as follows:

- No flow at infinity: At both \( z \rightarrow \pm \infty \), \( v'_i \) should vanish,
- Balance of the normal stresses of the ambient fluid at the 2D active fluid layer,
- Due to the assumed inflexibility of the active fluid layer, the normal velocity of the ambient fluid at the active fluid layer should be zero: \( v'_y(z = \epsilon) = 0 = v'_y(z = -\epsilon) \),
- Boundary conditions at the active fluid-bulk fluid interfaces requires careful consideration; see Fig. 2. The most common boundary condition used in this context is the "no-slip" condition, i.e., equality of the active fluid velocity and the in-plane component of the 3D ambient fluid velocity at the top and bottom interfaces between the ambient fluid and the active fluid layer. We generalise this by allowing a slip. We implement this by introducing a friction coefficient or a slip coefficient, such that the shear stresses are balanced by the friction forces at the interfaces. This implies (using \( v'_z = 0 \) at \( z = \pm \epsilon \))

\[
\eta \partial_z v'_\beta|_{z=\epsilon} = \Gamma(v'_\beta|_{z=\epsilon} - v'_\beta), \quad \eta \partial_z v'_\beta|_{z=-\epsilon} = -\Gamma(v'_\beta|_{z=-\epsilon} - v'_\beta), \quad \beta = x, y
\]

where \( \Gamma \) is the coefficient of friction (or slip coefficient) at the upper and lower interfaces (we assume equal friction coefficients at the upper and lower interfaces for simplicity); this corresponds to a slip length \( l_s \sim \eta'/\Gamma \).

Notice that for a finite \( \Gamma \), boundary conditions (21) implies partial slip between \( v_\beta(x, y) \) and \( v'_\beta(x, y, z = \pm \epsilon) \), \( \beta = x, y \). While the no-slip boundary condition is more conventionally used, on mesoscopic scales, however, instances of violation of the no-slip boundary conditions are known. For instance, Ref. [20] has shown that beyond a critical shear stress that depends strongly on the surface roughness, departure from the no-slip conditions may be observed. It has also been found that upon addition of surfactant in the fluid, the boundary condition changes from no-slip to partial slip [21]. In addition, there are now strong evidences in favour of slip in polymer melts; see, e.g., Refs. [22, 23]. Furthermore, it has been demonstrated in Ref. [24] how a large slip at a liquid-liquid interface may be introduced experimentally. While no systematic measurements of slip at interfaces involving active fluids are known, the above existing results suggest that considering the complex internal structure of the active fluid (e.g., the presence of actin filaments), a partial slip at the interfaces between the active fluid layer and the 3D embedding fluid cannot be ruled out. Thus it
is important to study implications of finite slips in an active fluid problem, which we set out to do below by using our model system. Notice that vanishing $v'_z$ at $z = \pm \epsilon$ implies that the shear forces $F_\beta$ on the active fluid layer as given in (16) take the simpler form

$$F_\beta = \eta'(\partial_z v'_\beta|_{z=\epsilon} - \partial_z v'_\beta|_{z=-\epsilon}).$$

(20)

By using the boundary conditions prescribed above, together with the incompressibility of the ambient fluid $\nabla \cdot \mathbf{v}' = 0$, Stokes’ Eq. (17) may be solved to yield $v'_z$ (see Appendix) and obtain $F_\beta$.

Equation of motion for the orientational field $p_\alpha$ may be written combining equations (8) and (12).

$$\partial_t p_\alpha + v_\beta \partial_\beta p_\alpha + \omega_{\alpha \beta} p_\beta = \frac{h_\alpha}{\gamma_0} + \lambda_1 p_\alpha \Delta \mu + \nu_1 p_\beta v_{\alpha \beta} - \lambda \partial_\alpha \rho - \nu_2 (p \cdot \nabla) p_{\alpha} \Delta \mu + \nu_1 p_\alpha \partial_\alpha \rho \Delta \mu.\tag{21}$$

With $p_x = 1$ defining the reference state, $p_y$ is a broken symmetry (slow) mode. We linearise (21) above about $p_x = 1$ for small $p_y$. This yields

$$\partial_t p_y = \frac{(D\nabla^2 p_y + \chi \partial_\rho \rho)}{\gamma_0} + \lambda_2 \Delta \mu \partial_\alpha p_y + \lambda_3 \rho \Delta \rho \partial_\rho p_y + \left(\frac{\nu_1 - 1}{2}\right) \partial_\rho v_x + \left(\frac{\nu_1 + 1}{2}\right) \partial_\rho v_y - \lambda A \partial_\rho - \lambda \chi \partial_\rho^2 p_y.\tag{22}$$

Note that in Eq. (22), $\rho$ enters into the dynamics of $p_y$ through both equilibrium and nonequilibrium contributions. Both are equally relevant being the lowest order terms in gradient expansions.

The equation of motion for $p$ is obtained by using Eqs. (11), (3) and (11). Up to the order $q^2$ the equation of motion for $\rho$ in the Fourier space, linearised about $p_x = 1$, is obtained as (set $A = 1$)

$$\partial_t \rho = -\gamma_\rho q_y^2 \rho + \lambda_2 \rho q_y^2 \rho + \Delta \mu q_y^2 \rho + i \mu q_y \rho q_z \rho - i \lambda \rho q_y \rho q_z \rho + D \lambda_q q_y^2 \rho + \lambda \chi q_y^2 \rho,\tag{23}$$

where $q = (q_x, q_y)$ is the in-plane Fourier wavevector, conjugate to $x = (x, y)$.

Notice that in the linear equations (16), (22) and (23) there are seven active coefficients (excluding $\Delta \mu$). In terms of an underlying equivalent agent-based microscopic dynamics all these coefficients should depend on the local density of the active particles and the alignment rules (favouring nematic or polar alignment). Thus, it is possible that all the seven active coefficients are not independent parameters. On dimensional ground we argue that the two nematic active coefficients $\zeta$ and $\bar{\zeta}$ should be related as $\zeta \sim \rho_0 \zeta$ and the pairs of polar active coefficients in the active particle current $(\kappa_0, \kappa_\rho)$ and in the active alignment $(\lambda_2, \lambda_\rho)$ are related as $\kappa_\rho \rho_0^3 \sim \kappa_\rho$ and $\lambda_\rho \rho_0 \sim \lambda_2$, respectively. With the expectation that the alignment polar and active current terms originate from same underlying (polar) microscopic rules, we expect them to be mutually simply related. Again on dimensional ground we expect $\lambda_2 \sim \kappa_\rho$. Note however that in the all the above heuristic relations, there are dimensionless proportionality constants which we cannot obtain on simple physical ground. In what follows below, we ignore this issue and treat all the seven coefficients as independent model parameters.

### A. High friction limit

We consider a “large” $\Gamma$. Formally in the limit $\Gamma \to \infty$ (equivalently, $l_s \to 0$), the stress balance equations (18) and (19) yield

$$v'|_{z=\epsilon} = v'|_{z=-\epsilon} = v_\alpha, \quad \alpha = x, y.$$

(24)

Thus, there is no slip between the ambient fluid velocity at the active fluid layer $v'_\alpha|_{z=\pm \epsilon}$ and the active fluid velocity $v_\alpha$. Equation (24) forms one of the boundary conditions on the ambient fluid velocity $v'_\alpha$.

Forces $F_\alpha$ then may be expressed as (see Appendix B),

$$F_x = -2qv_y q_\eta', \tag{25}$$

$$F_y = -2qv_x q_\eta', \tag{26}$$

where $q = (q_x, q_y)$ is the in-plane Fourier wavevector. Putting the values of (25) and (26) in the Stokes equation (16), the expressions for $v_x$ and $v_y$ can be derived up to the lowest order in $q$ linearising about $p_x = 1$ and $\rho = \rho_0$.

$$v_x = -i \frac{\zeta q^2 q_y}{2q' q^3} \Delta \mu p_y + i \frac{\zeta q^2 q_x}{2q' q^3} \Delta \mu p_y + i \frac{\bar{\zeta} q^2 q_y}{2q' q^3} \Delta \mu \rho, \tag{27}$$

$$v_y = -i \frac{\zeta q^2 q_x}{2q' q^3} \Delta \mu p_y + i \frac{\zeta q^2 q_y}{2q' q^3} \Delta \mu p_y - i \frac{\bar{\zeta} q^2 q_y}{2q' q^3} \Delta \mu \rho. \tag{28}$$
Thus, $v_a$ at $O(q^0)$ has only active contributions. Equation (22) may be written by substituting for $v_x$ and $v_y$ from Eqs. (27) and (28). We thus obtain

$$
\partial_t p_y = \frac{-Dq_y^2 p_y + i\lambda q_y \rho}{\gamma_0} + i\lambda_x \Delta \mu q_y \rho + i\lambda_x^2 \Delta \mu q_x p_y - i\lambda q_y \rho - i\lambda \chi q_y^2 p_y - \frac{1}{4\eta' q}[(v_1 - 1)q_y^2 - (v_1 + 1)q_x^2] \left[\zeta \Delta \mu \left(1 - \frac{2q_y^2}{q^2}\right)p_y + \frac{\bar{\zeta} \Delta \mu q_y}{q^2} \rho\right],
$$

in the Fourier space.

### B. Intermediate friction

For intermediate values of $\Gamma$, there are a considerable slip between the ambient fluid velocity $v_{a1} = \pm \epsilon$ and the active fluid velocity $v_a$, $\alpha = x, y$. In the limit $q \gg \Gamma/\eta'$ (equivalently $q \gg l_s^{-1}$), (18) and (19) reduce to (see Appendix C)

$$
\eta' \partial_x v'_y = -\Gamma v_y,
$$

\(\eta' \partial_x v'_y = \Gamma v_y\).

Clearly, these would be valid for a system with system size $L \ll l_s$.

Substituting (30) and (31) in (18) and linearising about $p_x = 1$ and $\rho = \rho_0$, the generalised Stokes equations for $v_x$ and $v_y$ are obtained as (see Appendix C)

$$
v_x = \frac{i \zeta \Delta \mu q_y}{2\Gamma} \left(1 - \frac{2q_y^2}{q^2}\right)p_y + \frac{\bar{\zeta} \Delta \mu q_x q_y^2}{2\Gamma q^2} \rho,
$$

$$
v_y = \frac{i \zeta \Delta \mu q_x}{2\Gamma} \left(1 - \frac{2q_y^2}{q^2}\right)p_y - \frac{\bar{\zeta} \Delta \mu q_y^2 q_x}{2\Gamma q^2} \rho,
$$

where we have neglected $\eta q^2 v_a$ in the limit $\eta q^2 \ll 2\Gamma$. This should be valid in the wavevector range satisfying $\eta q^2 \ll \Gamma \ll \eta'/q$. We have $\eta \sim \eta' d$, where $d$ is the thickness of the active fluid layer. Taking $d \sim 10^{-7} m$ for a cortical actin layer and $\eta' \sim 10^{-3} N \cdot sec/m^2$ for water, the above inequality should hold over a wide range of $q$. As before, $v_a$ has only active contributions at the lowest order in $q$. Similar to an ordered active polar fluid layer on a solid substrate, the hydrodynamic interactions here are completely cut off by the friction $\Gamma$ and consequently $v \sim O(q)$ to the lowest order in the wavevector. Not surprisingly, Eqs. (32) and (33) are identical in structure with the form of the velocities of an active polar fluid layer resting on a solid surface. This is due to the fact that for $q l_s \gg 1$, $v'_i, i = x, y$ are effectively very small and hence ignored. This background fluid thus effectively behaves as a fixed background (similar to a rigid substrate). Thus, with an intermediate value for $\Gamma$, our model active system corresponds surprisingly to a dry active matter, despite being in contact with an embedding bulk fluid.

Using the above Eqs. (32) and (33) in eq. (22), the equation for $p_y$ can be written as

$$
\partial_t p_y = i\left(\frac{X}{\gamma_0} + \lambda_p \Delta \mu \rho\right)q_y \rho + \frac{i}{4\Gamma}[(v_1 - 1)q_y^2 - (v_1 + 1)q_x^2] \left[\zeta \Delta \mu \left(1 - \frac{2q_y^2}{q^2}\right)p_y + \frac{\bar{\zeta} \Delta \mu q_y}{q^2} \rho\right]
$$

Density $\rho$ of the active particles still follows Eq. (29). Notice that Eqs. (34) and (35) are the linearised version of the model Eqs. for a polar flock in Ref. [6], which is a coarse-grained model for an active polar flock in a frictional medium. Thus with intermediate $\Gamma$, our model is a representation of the model in Ref. [6].

### C. Weak friction limit

In this case, $\Gamma$ is so small that $\eta q^2 \gg \Gamma$. From the generalised Stokes Eq. for $v$ (valid now for system size $L < (\eta/\Gamma)^{1/2}$) we find

$$
v_x = -i\frac{\zeta q_x q_y}{\eta q^2} \Delta \mu p_y + i\frac{\zeta q_y^3}{\eta q^2} \Delta \mu p_y + i\frac{\bar{\zeta} q_x q_y}{\eta q^2} \Delta \mu p_y,
$$

$$
v_y = -i\frac{\zeta q_x q_y}{\eta q^2} \Delta \mu p_y + i\frac{\zeta q_y^3}{\eta q^2} \Delta \mu p_y - i\frac{\bar{\zeta} q_x q_y}{\eta q^2} \Delta \mu p_y.
$$
Thus, \( v \sim O(1/q) \) at the lowest order, in contrast to the \( q \)-dependences of the velocities for large or intermediate \( \Gamma \) above. The differences are due to the lack of any screening of the hydrodynamic interactions in the present case. Effectively, in this limit, the active fluid layer is a free standing system being completely decoupled dynamically from the ambient fluid. The dynamical equation for \( p_y \) takes the form

\[
\partial_t p_y = i \left( \frac{\sqrt{\eta}}{\eta_0} + \lambda_2 \Delta \mu \right) q_x p_y + \frac{i}{\eta_0} \left[ (\nu_1 - 1) q_x^2 - (\nu_1 + 1) q_z^2 \right] \left[ i \zeta \Delta \mu \left( 1 - \frac{2q_y^2}{q^2} \right) q_y p_y + i \frac{\xi q_z q_y^2}{q^2} p_y \right] - \frac{D q^2 p_y}{\gamma_0} + i \lambda_2 \Delta \mu q_x p_y - i \lambda q_y p_y - \lambda \phi q_y^2 p_y.
\]

Equation of motion of \( \rho \) is still given by Eq. (23).

### III. LINEAR INSTABILITIES

We now analyse the linear stability of the system from the dynamical equations obtained above by assuming a time-dependence for \( p_y \) and \( \rho \) of the general form \( \exp(\Lambda t) \). There are two independent modes, which may be static or moving, stable or unstable, given by two values of \( \Lambda \). We calculate \( \Lambda \) up to the lowest order in wavevector \( q \) for the different cases elucidated above.

#### A. High friction limit

Consider first strong nonequilibrium osmotic pressure, i.e., \( \chi/\gamma_0 - \tilde{\lambda} \ll \lambda_p \Delta \mu \). The eigenvalues \( \Lambda \) of the stability matrix corresponding to Eqs. (29) and (23) in polar coordinates \( q = (q \cos \theta, q \sin \theta) \), where \( \theta \) is the angle between the wavevector \( q \) and the ordering direction (x-axis), up to the linear order in \( q \) are

\[
\Lambda = \left( \frac{\lambda_2 - \kappa_{pp}}{2} \right) q \cos \theta + \frac{B \zeta \Delta \mu}{4\eta'} q \cos 2\theta \pm \frac{q \Delta \mu}{2 \eta'} \left[ \left( i \left( \lambda_2 - \kappa_{pp} \right) \cos \theta + \frac{B \zeta}{2\eta'} \cos 2\theta \right)^2 \right.
\]

\[
+ 4\lambda_p \kappa_0 \sin^2 \theta + 4i \frac{B \zeta \kappa_0}{4\eta'} \sin \theta \sin 2\theta + 4i \frac{B \kappa \zeta}{2\eta'} \cos \theta \cos 2\theta
\]

\[
- 4\lambda_2 \kappa_{pp} \cos^2 \theta |^{1/2} = \Lambda^h_\pm, \Lambda^l_\pm.
\]

where, \( B = (\nu_1 + 1) \cos^2 \theta - (\nu_1 - 1) \sin^2 \theta \). Clearly, \( \Lambda \) scales with \( q \) and \( \Delta \mu \). Thus, \( \Lambda \sim q \), the coefficient of proportionality should be a function of \( \theta \) (hence anisotropic) and other model parameters, and may be real or imaginary. This linear \( q \)-dependence is different from \( q \)-independent eigenmodes in bulk polar active fluids (see, e.g., Ref. [15]) and is a consequence of the hydrodynamic interactions mediated by the ambient fluid. It is evident from [35] that the real parts of \( \Lambda^h_+, \Lambda^l_+ \) vanish when \( B \) vanishes, i.e., for particular values of \( \theta \). There may be additional zeros for them for other isolated values of \( \theta \). In Fig. 3, representative plots of \( \Lambda^h_+ \), \( \Lambda^l_+ \) as functions of \( \theta \) are shown for both signs of \( \kappa_0 \lambda_p \) and \( \zeta \), for fixed values of other parameters and \( q = 1, \Delta \mu > 0 \). There are few notable features clearly visible in Fig. 3. (i) the profiles of \( \Lambda^h_+ \), \( \Lambda^l_+ \) depend strongly on the signs of \( \zeta \) and \( \kappa_0 \lambda_p \), (ii) the zeros of the the real parts of \( \Lambda^h_+ \) and \( \Lambda^l_+ \) are not necessarily overlapping. The dependences of \( \Lambda^h_+ \), \( \Lambda^l_+ \) on \( \zeta \) turn out to be weak (not shown).

It is also instructive to explore the limit \( B \to 0 \), or,

\[
\tan^2 \theta = \frac{\nu_1 + 1}{\nu_1 - 1} = \tan^2 \theta_0,
\]

yielding \( \theta = \pm \theta_0, \pm (\theta_0 + \pi) \), such that \( B(\theta_0) = 0 \). The eigenvalues in this case are

\[
\Lambda(\theta_0) = \left( \frac{\lambda_2 - \kappa_{pp}}{2} \right) q \cos \theta \pm \frac{\Delta \mu q}{2} \left[ \left( \lambda_2 + \kappa_{pp} \right)^2 \cos^2 \theta_0 + 4\lambda_p \kappa_0 \sin^2 \theta_0 \right]^{1/2}.
\]

It is evident from Eq. (40) that for \( \lambda_p \kappa_0 < 0 \), \( \Lambda(\theta_0) \) is fully imaginary i.e., two propagating modes, which are oppositely moving, are present in the system with an anisotropic \( q \)-independent wave speed. For \( \lambda_p \kappa_0 > 0 \) and
FIG. 3: Representative plots of the real (black thin line) and imaginary (purple thick) parts of (a) eigenmode $\Lambda^h_+$ vs $\theta$ for $\kappa_0\lambda_\rho > 0$ and $\zeta > 0$, (b) eigenmode $\Lambda^h_- vs \theta$ for $\kappa_0\lambda_\rho > 0$ and $\zeta > 0$, (c) eigenmode $\Lambda^h_+ vs \theta$ for $\kappa_0\lambda_\rho < 0$ and $\zeta > 0$, (d) eigenmode $\Lambda^h_- vs \theta$ for $\kappa_0\lambda_\rho < 0$ and $\zeta > 0$, (e) eigenmode $\Lambda^h_+ vs \theta$ for $\kappa_0\lambda_\rho > 0$ and $\zeta < 0$, (f) eigenmode $\Lambda^h_- vs \theta$ for $\kappa_0\lambda_\rho > 0$ and $\zeta < 0$, (g) eigenmode $\Lambda^h_+ vs \theta$ for $\kappa_0\lambda_\rho < 0$ and $\zeta < 0$, and (h) eigenmode $\Lambda^h_- vs \theta$ for $\kappa_0\lambda_\rho < 0$ and $\zeta < 0$, with fixed values of the other parameters; $q = 1$. Here $\Delta\mu > 0$ for all the plots. Notice that the real part vanishes only at isolated, discrete values of $\theta$. Nonzero imaginary part implies propagating modes (see text).
$$|4\lambda_\rho\kappa_0\sin^2 \theta_0| > |(\lambda_2 + \kappa_{\rho\rho})^2 \cos^2 \theta_0|,$$ \(\Lambda(\theta_0)\) has a real part in addition to propagating modes. The real part comes from

$$\pm \left[ - (\lambda_2 + \kappa_{\rho\rho})^2 \cos^2 \theta_0 + 4\lambda_\rho\kappa_0\sin^2 \theta_0 \right]^{1/2}.$$  \hfill (41)

Evidently, the real part displays instability for both signs of \(\Delta \mu\) in this case, with anisotropic decay/growth rates which scale with \(q\). A schematic diagram showing static and moving instabilities in the \(\kappa_0\lambda_\rho - (\lambda_2 + \kappa_{\rho\rho})^2\) plane is shown in Fig. 4 which clearly indicates the region in the \(\kappa_0\lambda_\rho - (\lambda_2 + \kappa_{\rho\rho})^2\) plane.

![Schematic diagram](image)

**FIG. 4:** A schematic diagram displaying static and moving instabilities in the \(\kappa_0\lambda_\rho - (\lambda_2 + \kappa_{\rho\rho})^2\) plane at a given value of \(\theta = \theta_0\) (see text).

A schematic plot of \(\kappa_0\lambda_\rho\) vs \(\theta\) for chosen values of \(\kappa_{\rho\rho}\) and \(\lambda_2\) is shown in Fig. 5, clearly indicating the unstable regions and propagating modes.

![Schematic plot](image)

**FIG. 5:** A plot of \(\kappa_0\lambda_\rho\) vs \(\theta\) for some chosen values of \(\kappa_{\rho\rho}\) and \(\lambda_2\) is shown. The regions inside the upward parabolas indicate the presence of moving instabilities and all other regions outside have propagating modes without damping or growth at \(O(q)\) (see text).

Consider the case when there are only propagating modes at \(O(q)\) at, say, \(\theta = \theta_0\). Now assume \(\theta\) very close to \(\theta_0\); we write \(\theta = \theta_0 + \delta \theta\), where \(\delta \theta\) is a very small. In that case \(B \approx -\nu_1 \sin 2\theta_0 \delta \theta\), up to order \(O(\delta \theta)\). The eigenvalues corresponding to \(\theta = \theta_0 + \delta \theta\) are given by

$$\Lambda(\theta_0 + \delta \theta) = \Lambda(\theta_0) - \frac{\zeta \Delta \mu \nu_1}{4\gamma q} q \sin 2\theta_0 \cos 2\theta_0 \delta \theta + iO(\delta \theta).$$  \hfill (42)

Noting that \(\Lambda(\theta_0)\) is fully imaginary, (42) shows that \(\Lambda(\theta_0 + \delta \theta)\) has real parts, whose signs depend on \(\Delta \mu\) for a given \(\delta \theta\). Thus, we conclude that the system shows instability for either sign of \(\Delta \mu\) along with generic propagating modes.
with an anisotropic wave speed proportional to $\Delta \mu$. Considering $\Lambda$ in the $(q, \theta)$ plane, we thus notice that there are special directions given by $\theta = \pm \theta_0, \pm (\theta_0 + \pi)$ along which (small) perturbations move as waves without any growth or damping (to the linear order in $q$), provided $\Delta = 4 \lambda_\rho \kappa_0 \sin^2 \theta_0 - (\lambda_2 + \kappa_\rho \eta)^2 \cos^2 \theta_0 < 0$ is satisfied; see Fig. 6. Additional values of $\theta$ for which for which the real parts of $\Lambda^h_+$ or $\Lambda^h_-$ vanish may be found from Eqs. (38). However, both the real parts will not vanish simultaneously at these angles; see Fig. 6. Along all other directions, at least one of $\Lambda^h_+$ should have a real part, and hence perturbations will grow/decay and move. If $\Delta > 0$, there are no special directions with only propagating modes. Since, $p_y(x,t)$ and $\rho(x,t)$ depend on $p_y(q,t)$ and $\rho(q,t)$ for all $q$, hence,

\[\begin{align*}
\pi/2 - \theta_0 & \quad \text{propagating modes} \\
\theta_0 & \quad q_x \\
q_y
\end{align*}\]

FIG. 6: Schematic diagram ($\kappa_0 \lambda_\rho < 0$ or $\Delta < 0$) displaying the special angular directions given by $\theta = \theta_0$ in the plane along which there are only propagating modes up to $O(q)$ (see text).

the two eigenmodes for all $q$, $p_y(x,t)$ and $\rho(x,t)$ show generic moving instabilities at $O(q)$ for both signs of $\Delta \mu$ for arbitrary choice for the active coefficients. It is also clear that at $O(q)$, the system can be stable only if $\zeta = 0 = \bar{\zeta}$ and $\lambda_\rho \kappa_0 < 0$. Thus, the active stresses clearly destabilise the system. Of course, at higher order in $q$, the system will be stabilised by large enough $D$ or $\gamma_{\rho\rho}$.

It is useful to analyse the stability of the system for some particular values of $\theta$. First we start with $\theta = 0$. In this limit the eigenvalues are given by

$$
\Lambda(\theta = 0) = i \lambda_2 \Delta \mu q + \frac{(\nu_1 + 1) \zeta \Delta \mu}{4 \eta'} q, -i \Delta \mu \kappa_{\rho\rho} q.
$$

(43)

Thus there are two modes; one is purely imaginary and hence just a propagating mode, the other has both real and imaginary parts. The sign of the real part is determined by $\Delta \mu$. Thus this eigenvalue is moving and either growing (unstable) or decaying (stable) in time, respectively, when $(\nu_1 + 1) \zeta \Delta \mu > 0$, or, $< 0$.

For $\theta = \frac{\pi}{2}$, the stability eigenvalues are given by

$$
\Lambda(\theta = \frac{\pi}{2}) = \frac{(\nu_1 - 1) \zeta \Delta \mu}{8 \eta'} q \pm \frac{\Delta \mu q}{2} \left[ \frac{(\nu_1 - 1) \zeta}{4 \eta'} \right]^2 + 4 \lambda_\rho \kappa_0
$$

(44)

From Eq. (44), we note that for $\lambda_\rho \kappa_0 > 0$, the system is unstable for both $\Delta \mu > 0$ and $\Delta \mu < 0$. Next, for $\lambda_\rho \kappa_0 < 0$ and $(\nu_1 - 1) \zeta > 0$,

- If $|4 \lambda_\rho \kappa_0| > \frac{(\nu_1 - 1) \zeta^2}{4 \eta'}$ and $\Delta \mu > 0$, the modes are unstable and oppositely moving.
- However, when $|4 \lambda_\rho \kappa_0| < \frac{(\nu_1 - 1) \zeta^2}{4 \eta'}$ with $\Delta \mu > 0$, both the modes are unstable. There are no propagating waves.

In the special case with $\bar{\zeta} = 0 = \lambda_\rho$, i.e., if we ignore the density dependences of the active coefficients, the eigenvalues of the stability matrix take a simpler form

$$
\Lambda(\bar{\zeta} = 0 = \lambda_\rho) = i \lambda_2 \Delta \mu q \cos \theta + \frac{B \zeta \Delta \mu}{2 \eta'} q \cos 2\theta, -i \kappa_{\rho\rho} \Delta \mu q \cos \theta,
$$

(45)

which indicates the presence of propagating modes and instability for both signs of $\Delta \mu$ above or below $\theta = \frac{\pi}{4}$.
Now briefly consider the instabilities with $|\chi/\gamma_0 - \bar{\lambda}| \gg |\lambda_0 \Delta \mu|$ (weak nonequilibrium osmotic pressure): Neglecting $\lambda_0 \Delta \mu$ in comparison with $\chi/\gamma_0 - \bar{\lambda}$, the eigenvalues $\Lambda$ are given by
\[
\Lambda = \frac{\Delta \mu q}{2} [-i \kappa_{pp} \cos \theta - \frac{\zeta B}{4\eta}(1 - 2 \cos^2 \theta) + i \lambda_2 q \cos \theta] \pm \frac{q}{2} \sqrt{\left[\frac{\zeta B(1 - 2 \cos^2 \theta)}{4\eta} - i \lambda_2 \cos \theta + i \kappa_{pp} \cos \theta\right]^2 \Delta \mu^2 + 4i \frac{\Delta \mu}{4\eta} \zeta B \sin^2 \theta \cos \theta + \left(\frac{\chi}{\gamma_0} - \bar{\lambda}\right) \sin^2 \theta |\kappa_0 \Delta \mu - 4i \kappa_{pp} \Delta \mu^2 \cos \theta|-i \lambda_2 \cos \theta + \frac{\zeta B}{4\eta}(1 - 2 \cos^2 \theta)\}^{1/2}. \tag{46}
\]
Thus, $\Lambda$ are no longer homogeneous functions of $\Delta \mu$. In order to progress further, assume a "small" $\Delta \mu$. Then, in an expansion in powers of $\Delta \mu$, we obtain to the lowest order in $q$ and $\Delta \mu$
\[
\Lambda = \pm \sqrt{\left(\frac{\chi}{\gamma_0} - \bar{\lambda}\right) \kappa_0 \Delta \mu q \sin \theta, \quad \left(\frac{\chi}{\gamma_0} - \bar{\lambda}\right) \kappa_0 \Delta \mu > 0,} \tag{47}
\]
\[
\Lambda = \pm iq \sin \theta \sqrt{\left(\frac{\chi}{\gamma_0} - \bar{\lambda}\right) \kappa_0 \Delta \mu |, \quad \left(\frac{\chi}{\gamma_0} - \bar{\lambda}\right) \kappa_0 \Delta \mu < 0.} \tag{48}
\]
Thus, in the former case, we find instabilities for either sign of $\Delta \mu$, where as in the second case, we find oppositely moving propagating waves.

### B. Intermediate friction

We again consider $|\chi/\gamma_0 - \bar{\lambda}| \ll |\lambda_0 \Delta \mu|$ first. To the lowest order (linear order) in $q$, the eigenvalues of the linear stability matrix are
\[
\Lambda = -i \frac{\Delta \mu q}{2} (\kappa_{pp} - \lambda_2) \cos \theta \pm \frac{i q \Delta \mu}{2} \left[\cos^2 \theta (\kappa_{pp} + \lambda_2)^2 - 4 \kappa_0 \lambda_2 \sin^2 \theta\right]^{1/2} = \Lambda_+, \Lambda_- . \tag{49}
\]
We can make the following general conclusions about the mode structures from (49). First of all, none of the active stress coefficients $\zeta$ and $\bar{\zeta}$ appear in (49). Thus the active stress is irrelevant in the dynamics to the linear order in $q$. The dynamics at this order in $q$ is controlled by the remaining active coefficients, viz, $\kappa_0, \kappa_{pp}, \lambda_2$ and $\lambda_0$. This is clearly in contrast to the situation with large $\Gamma$. Secondly, if $\kappa_0 \lambda_2 < 0$, then the discriminant is positive for all values of $\theta$ to the linear order in $q$. Then only propagating modes will be present for all values of $\theta$. As before, there should two oppositely moving propagating modes with the speed of wave being anisotropic and proportional to $\Delta \mu$. If on the other hand $\kappa_0 \lambda_2 > 0$, the discriminant in (49) is negative for all magnitudes of $\kappa_0 \lambda_2 \neq 0$ at least at $\theta = \pi/2$, giving rise to instability in the system for both signs of $\Delta \mu$. These instabilities are moving in the opposite directions. In general, for any value of $\theta$ satisfying $\cos^2 \theta (\kappa_{pp} + \lambda_2)^2 - 4 \kappa_0 \lambda_2 \sin^2 \theta > 0$ and for both signs of $\Delta \mu$, both the modes are propagating without damping (or growth). Thus, any perturbation in a region of the polar plane satisfying the above condition moves without any growth or decay in the amplitude (up to $O(q)$). Else, in the remaining region of the polar plane, one of the modes is stable and the other stable. The speed of the moving stabilities are unsurprisingly anisotropic and proportional to $\Delta \mu$. The above consideration for $\kappa_0 \lambda_2 > 0$ allows us to define an angle $\hat{\theta}$ such that
\[
\cos^2 \hat{\theta} (\kappa_{pp} + \lambda_2)^2 - 4 \kappa_0 \lambda_2 \sin^2 \hat{\theta} = 0 . \tag{50}
\]
Then, for $\kappa_0 \lambda_2 > 0$ in the shaded region in Fig. 7 characterised by $\hat{\theta}$ there are only propagating waves at $O(q)$, outside of this region, the system is linearly unstable at $O(q)$.

The growth rate or relaxation rates of the unstable and stable modes are also anisotropic and scale with $\Delta \mu$. Representative plots of the real and imaginary parts of $\Lambda_+, \Lambda_-$ as functions of $\theta$ for some chosen parameter values are shown in Fig. 8 showing the presence of propagating modes. The regions of instabilities and propagating modes are clearly indicated. In particular, there are a few notable features as displayed by Fig. 8 consistent with the forms of the eigenvalues (49). For instance, for $\kappa_0 \lambda_2 < 0$, $\Lambda_+, \Lambda_-$ are wholly imaginary for all $\theta$ and unequal, i.e., the speed of the two modes are different in magnitude. In contrast, for $\kappa_0 \lambda_2 > 0$, the real parts vanish over an identical range of $\theta$ for both the modes, that belongs to the shaded region in Fig. 7 with unequal imaginary parts, i.e., different speeds for the two modes. For the other values of $\theta$, the real parts are nonzero and opposite of each other, representing stable and unstable modes, with same speeds of propagation. The overall differences with the eigenvalues for large (diverging) $\Gamma$ are clearly visible.

Evidently, the model is overall stable at the linear order in $q$, provided, $\lambda_0 \kappa_0 \leq 0$. In this stable sector of the parameter space, the results of Ref. 6 that includes the effects of the nonlinearities and noises should directly apply here.
We now consider briefly the case with $|\chi/\gamma_0 - \bar{\lambda}| \gg |\lambda_\mu \Delta \mu|$. Proceeding as in Sec. III A above, the eigenvalues to the lowest order in $q$ and $\Delta \mu$ are given by

$$\Lambda = \pm \sqrt{\left(\frac{\chi}{\gamma_0} - \bar{\lambda}\right)\kappa_0 \Delta \mu q \sin \theta},$$

yielding instability for $(\chi/\gamma_0 - \bar{\lambda})\kappa_0 \Delta \mu > 0$ and oppositely moving propagating modes for $(\chi/\gamma_0 - \bar{\lambda})\kappa_0 \Delta \mu < 0$. These results are identical to the corresponding results in Sec. III A.

C. Weak friction limit

We now analyse the linear instabilities for $\eta q^2 \gg \Gamma$. One of the eigenvalues $\Lambda$ of the linear stability matrix is non-zero at $O(q^0)$. We find

$$\Lambda = \frac{\zeta \Delta \mu}{2\eta} \left(\cos^2 \theta - \sin^2 \theta\right) \left[(\nu_1 - 1) \sin^2 \theta - (\nu_1 + 1) \cos^2 \theta\right].$$

With a given choice for the sign of $\zeta \Delta \mu$ (say positive), $\Lambda > (\leqslant)0$ for $\cos^2 \theta - \sin^2 \theta$ and $\cos^2 \theta$ have the same (opposite) signs and vice versa for $\sin^2 \theta$. These results are identical to those in Ref. [15] for a bulk polar ordered active fluid. It is not a surprise that our results are same as those in Ref. [15], for in the weak $\Gamma$ limit, the active fluid layer in our model is effectively dynamically decoupled from the ambient fluid and hence acts as a free standing system, and hence, identical to the system considered in Ref. [15].

IV. INFLUENCE OF $\Gamma$ ON THE LINEAR INSTABILITIES

As our results above reveal, the magnitude of $\Gamma$ delineates different regimes of the model. While all these regimes display generic long wavelength instabilities in the different regions of the parameter space, the detailed nature of the instabilities and the regions in the parameter space where they are present vary. For both large or moderate $\Gamma$, we generically find moving and static instabilities. With $|\chi/\gamma_0 - \bar{\lambda}| \ll |\lambda_\mu \Delta \mu|$, both predict propagating modes with anisotropic $q$-independent speed proportional to $\Delta \mu$ and growth or decay rate, again anisotropic and linear in $q$. On the other hand, in the weak friction limit, the eigenmodes are $q$-independent and has no propagating modes at this order. Despite the loose similarities between the nature of the long wavelength instabilities for large and moderate $\Gamma$, closer inspection reveals significant differences between the two cases. With a large (formally diverging) $\Gamma$, the active fluid velocities $v_\alpha \sim O(q^0)$, where as, for moderate $\Gamma$, $v_\alpha \sim O(q)$, $\alpha = x, y$. Furthermore, with a diverging $\Gamma$, the system is unstable in the full parameter space to the lowest order in $q$ along all angles in the polar plane, except for along the lines $\theta = \pm \theta_0, \pm (\theta_0 + \pi)$. Along these special directions, there are only propagating modes without any
damping or growth (to the linear order in \( q \)). At every other value of \( \theta \), one mode is unstable. Thus, for all (finite) values of the parameters and both signs of \( \Delta \mu \), there are moving instabilities with anisotropic speeds. In contrast, with intermediate interfacial friction, there are regions in the parameter space where there are only propagating modes with no instabilities at the lowest order in \( q \) for any \( \theta \); only in a subspace of the parameter space, one encounters moving instabilities for either sign of \( \Delta \mu \). In addition, for an intermediate \( \Gamma \), the active stress has no role to play in determining the long wavelength instabilities; these are fully controlled by the active diffusive current and the active self-advection of the active species. In contrast, the strength of the active stress enters into the eigenmodes (unstable) in the large \( \Gamma \) case. However, if there are no active stresses present (but with finite active currents, active self-convection and active osmotic pressure), instabilities for large \( \Gamma \) is identical to those for intermediate \( \Gamma \) to the linear order in \( q \).

V. NEMATIC LIMIT OF THE DYNAMICS

Until now we have considered polar active particles, so that the corresponding dynamics is not invariant under \( p \to -p \). In the nematic limit, the dynamics is invariant under \( p \to -p \). Hence, active coefficients \( \lambda_2, \lambda_\rho, \kappa_0 \) and \( \kappa_{\rho\rho} \) and equilibrium couplings \( \chi \) and \( \lambda \) are zero. Thus, to the lowest order, the dynamical equations in the strong friction case are

\[
\frac{\partial \rho_y}{\partial t} = -\frac{1}{4q_y^2 \eta} [(\nu_1 - 1)q_y^2 - (\nu_1 + 1)q_x^2] \left[ \zeta \Delta \mu q_y \left( 1 - \frac{2q_x^2}{q_y^2} \right) p_y + \bar{\zeta} \Delta \mu q_x q_y \right],
\]

\[
\frac{\partial \rho}{\partial t} = -\gamma_{\rho\rho} q_y^2 \rho + w \Delta \mu \rho + w \rho_0 \Delta \mu q_x q_y p_y.
\]
As before, assume a time dependence of the form \(\exp(\Lambda t)\) for the fluctuations. Then, to the lowest order in \(q\)

\[
\Lambda = 0, \quad \frac{\zeta \Delta \mu}{4\eta'} \left(\cos^2 \theta - \sin^2 \theta\right) \left[(\nu_1 - 1)\sin^2 \theta - (\nu_1 + 1)\cos^2 \theta\right].
\]

(55)

Thus, with positive \(\zeta \Delta \mu\), \(\Lambda\) is positive (negative) for \(\cos^2 \theta - \sin^2 \theta\) and \((\nu_1 - 1)\sin^2 \theta - (\nu_1 + 1)\cos^2 \theta\) having the same (opposite) signs. Similarly for \(\zeta \Delta \mu < 0\). Thus, the system is unstable for both signs of \(\Delta \mu\).

For a finite \(\Gamma\), to the lowest order in \(q\), the corresponding dynamical equations with nematic symmetry are

\[
\frac{\partial p_y}{\partial t} = -\frac{Dq^2}{\gamma_0} p_y - \frac{\zeta \Delta \mu}{4\Gamma} \left[(\nu_1 - 1)q_y^2 - (\nu_1 + 1)q_x^2\right] \left(1 - \frac{2q_y^2}{q^2}\right) p_y,
\]

(56)

\[
\frac{\partial \rho}{\partial t} = -\gamma \rho q^2 \rho + w\Delta \mu q_x^2 \rho + w\rho_0 \Delta \mu q_y q_y p_y.
\]

(57)

Interestingly, Eqs. (56) and (57) are identical to those in Ref. [16] for active nematics on a substrate. Thus, the results of Ref. [16] are to hold here. We do not discuss these here in details. Regardless of the details, in the nematic limit there are no propagating waves and the instabilities are always static or localised. In contrast, active polar ordered systems are characterised by the presence of generic propagating modes and moving instabilities. Finally, the eigenmodes in both the nematic and polar ordered systems with strong interfacial friction with the embedding fluid scale with \(q\). However, for intermediate friction, the eigenmodes for the nematic system scale as \(q^2\), where as for the corresponding polar ordered system, they scale as \(q\).

**VI. SUMMARY**

In this work, we have set up the coarse-grained dynamics of a thin layer of polar ordered active particle suspensions embedded in bulk isotropic passive fluid. We have obtained coupled coarse-grained equations of motion for the local density and orientation of the polar active particles. Our framework is sufficiently general and holds for any flat 2D polar-ordered system embedded inside a 3D fluid medium. In a linearised treatment, we have studied the instabilities of small fluctuations around a reference polar ordered state. The mutual friction at the interfaces between the active fluid layer and the embedding bulk fluid is shown to affect the detailed nature of the instabilities significantly. For instance, in the limit of large friction or for no-slip between the active fluid velocity and the embedding fluid velocity at the interface, there are generic instabilities at \(O(q)\) for both contractile and extensile activities along all directions in the plane (except for four special angles), regardless of the values and signs of the different active coefficients in the model, except for four special angles along which there are only propagating modes at \(O(q)\). In contrast for moderate friction in a system with \(L < l_s\), i.e., with a finite slip at the interfaces, instabilities occur at \(O(q)\) only in range of directions when \(\kappa_0 \lambda_0 > 0\), i.e., for specific relations between the coefficients of the active current and active osmotic pressure. Else there are only propagating modes at the linear order in \(q\). Our model in this regime actually behaves like a dry active matter and is identical to the linearised version of the model of active flocks in Ref. [6] up to \(O(q)\). Lastly, in the limit of very weak friction, the active fluid layer gets dynamically decoupled from the embedding fluid and behaves as a free standing system. There are then instabilities at \(O(q^3)\) for both extensile and contractile activities, which are same as those discussed in Ref. [13]. It is interesting to understand the implications of our results from the perspectives of the system size \(L\). For a sufficiently large \(L\) with a given \(\Gamma\), there is a wavevector regime of sufficiently small magnitude, such that \(q \ll \Gamma/\eta'\) is realised. In that wavevector regime, the mode fluctuations will be described by our results in Sec. [IIIB]. Then there is an intermediate wavevector regime with \(q \gg \Gamma/\eta'\) with our results in Sec. [III C] should be observed. Finally, for sufficiently high wavevectors, \(q^2 \gg \Gamma/\eta\), the mode fluctuations are to follow our results in Sec. [III D]. We also discuss the nematic limit of the dynamics and compare it with their polar analogues. Our results evidently highlight the crucial role played by the interfacial friction and demonstrate how experimental knowledge about the linear instabilities may be used to extract information about the friction coefficient. Actual biological realisations of quasi-2D active fluids have more complicated structures. Our work should be considered only as a first step towards a more complete physical understanding of such systems. We expect our results to be useful in understanding in-vitro experiments on reconstituted layers of ordered actin filaments with molecular motors in an embedding fluid (e.g., water). In order to distinguish the different cases of high friction and moderate friction, one would require to experimentally find the dependences of the linear instabilities on the various active coefficients in the model used above. As our results illustrate, systems with different sizes for a given set of parameters are to have strikingly varying long wavelength dynamics as the size varies. Thus, our results may alternatively be validated by experimenting on various sizes of the chosen system, e.g., by taking reconstituted actin layers in a passive solvent of various sizes. These are expected to be experimentally challenging tasks. Nevertheless, we look forward to possible experimental attempts to study the issues highlighted here. Lastly the formal similarities between the dynamical
equations with moderate interfacial friction and those for a polar ordered system resting on a solid substrate open up the possibilities of studying the physics of moderate friction by performing experiments on an analogous system resting on a solid substrate.

Our analyses are valid for small fluctuations around an ordered state. Thus no conclusions may be drawn from our studies about the eventual steady states in the event of the linearly unstable uniform initial states. Numerical solutions of the full model equations should yield valuable information in this regard. In the present work, we have assumed the system to be inextensible and hence the out-of-plane fluctuations are prohibited. However, this condition may be violated for reconstituted actin filaments on a liposome. Thus for better quantitative understanding of the experimental results, a thin layer of active fluid with finite flexibility (i.e., with a finite surface tension or bending modulus) should be studied. It will also be interesting to study the diffusivity of a test particle inside the 2D active polar system. It is well-known that the diffusivity of a test particle in a 2D passive fluid is rendered finite by the hydrodynamic interactions mediated by the embedding medium \[^{31}\]. It is expected that the diffusivity of a test particle in a 2D polar ordered medium is affected by the interplay of hydrodynamic interactions of by the embedding medium and the strength of the interfacial friction. It would be interesting to study this theoretically or experimentally.

VII. ACKNOWLEDGEMENT

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Appendix A: Derivation of the full 2D generalised Stokes equation for \( v_i \)

Using Eqs. (7), (10) in Eq. (5), the generalised Stokes Eq. may be written as

\[
\eta \nabla^2 v_\beta + \Delta \mu \partial_\alpha (\zeta' (\rho) p_\alpha p_\beta) + \nu_1 \partial_\alpha (p_\alpha h_\beta + p_\beta h_\alpha) - \frac{\epsilon_0}{2} \partial_\alpha (p_\alpha \partial_\beta \mu + p_\beta \partial_\alpha \mu) + \frac{1}{2} \partial_\alpha (p_\alpha h_\beta - p_\beta h_\alpha) = \partial_\beta \Pi - F_\beta. \tag{A1}
\]

Using incompressibility \( \Pi \) can be derived from (A1) as

\[
\Pi = \frac{\Delta \mu}{\nu_2} \partial_\alpha \partial_\beta (\zeta' (\rho) p_\alpha p_\beta) + \frac{\nu_1}{2} \partial_\alpha \partial_\beta (p_\alpha h_\beta + p_\beta h_\alpha) - \frac{\epsilon_0}{2} \partial_\alpha \partial_\beta (A p_\alpha \partial_\beta \rho + A p_\beta \partial_\alpha \rho + 2 \chi p_\alpha \partial_\beta \nabla \cdot \mathbf{p} + 2 \chi p_\beta \partial_\alpha \nabla \cdot \mathbf{p}). \tag{A2}
\]

Using this value of \( \Pi \) in (A1), the Stoke’s equation is derived as

\[
\eta \nabla^2 v_\beta + \Delta \mu P_{\beta \gamma} \partial_\alpha (\zeta' (\rho) p_\alpha) + \frac{\nu_1}{2} P_{\beta \gamma} \partial_\alpha (p_\alpha h_\gamma + p_\gamma h_\alpha) - \epsilon_0 P_{\beta \gamma} \partial_\alpha (A p_\alpha \partial_\gamma \rho + A p_\gamma \partial_\alpha \rho + 2 \chi p_\alpha \partial_\gamma \nabla \cdot \mathbf{p} + 2 \chi p_\gamma \partial_\alpha \nabla \cdot \mathbf{p} ) + \frac{1}{2} P_{\beta \gamma} \partial_\alpha (p_\alpha h_\gamma - p_\gamma h_\alpha) = - F_\beta, \tag{A3}
\]

where \( F_\beta \) are given by (10) and \( P_{\alpha \beta} \) is the transverse projection operator written as \( P_{\alpha \beta} = \delta_{\alpha \beta} - \frac{\partial_\alpha \partial_\beta}{\nabla^2} \). Linearising about \( p_x = 1 \), the Stokes equation (A3) is simplified to

\[
\eta \nabla^2 v_\beta + \Delta \mu P_{\beta x} \partial_\alpha (\zeta' (\rho) p_\alpha) + \Delta \mu P_{\beta y} \partial_\alpha (\zeta' (\rho) p_\gamma) + \frac{\nu_1}{2} P_{\beta x} (\partial_\alpha h_\alpha - 2 \partial_\alpha h_x) - \frac{\nu_1}{2} P_{\beta y} (\partial_\alpha h_\alpha - 2 \partial_\alpha h_y) - \frac{1}{2} P_{\beta \gamma} \partial_\alpha h_\gamma + \frac{1}{2} P_{\beta \gamma} \partial_\alpha h_x
\]

\[
= - \epsilon_0 P_{\beta x} (\partial_\alpha \rho - 2 \partial_\alpha \rho) - \epsilon_0 P_{\beta y} (\partial_\alpha \rho - 2 \partial_\alpha \rho) = - F_\beta, \tag{A4}
\]

Now using Eq. (10), we find \( h_y = - \frac{\partial F}{\partial p_y} = D \nabla^2 p_y + \chi \partial_y \rho \). In addition, \( h_x \) acts as a Lagrange multiplier to enforce the constraint \( \rho^2 = 1 \). Notice that \( h_x \) contributes terms which are higher order in gradients in Eq. (A4). Thus neglecting all the higher order terms, the generalised Stokes equation up to the lowest order in gradients is given by Eq. (10).

Appendix B: Form of \( F_\beta \): strong friction

The velocity and hydrodynamic pressure for the subphase and superphase are given by Eqs. (B1), (B2). We impose incompressibility on the 3D ambient fluid:

\[
\partial_z v'_i = - \nabla_i v'_i \text{ with } i=x,y, \tag{B1}
\]
for both $z > 0$ and $z < 0$. Fourier transforming the in-plane coordinates $x = (x, y)$,

$$\eta'(-q^2 + \partial_x^2)v'_x = \partial_z \Pi', \quad \eta'(-q^2 + \partial_y^2)v'_y = iq \Pi', \quad \eta'(-q^2 + \partial_z^2)v'_z = 0,$$

(E2)

(B3)

(B4)

where $i = x, y$; $\mathbf{q} = (q_x, q_y)$ is the in-plane Fourier wavevector. The above equations can be solved together to obtain the solutions for $v'_x, v'_y, v'_z$ and $\Pi'$. We write

$$v'_x = (A_1 + B_1z)e^{-qz} \quad \text{for} \quad z > 0,$$

(B5)

$$v'_x = (A_2 + B_2z)e^{qz} \quad \text{for} \quad z < 0,$$

(B6)

$$v'_y = (A_3 + B_3z)e^{-qz} \quad \text{for} \quad z > 0,$$

(B7)

$$v'_y = (A_4 + B_4z)e^{qz} \quad \text{for} \quad z < 0,$$

(B8)

$$v'_z = (C_1 + D_1z)e^{-qz} \quad \text{for} \quad z > 0,$$

(B9)

$$v'_z = (C_2 + D_2z)e^{qz} \quad \text{for} \quad z < 0,$$

(B10)

$$\Pi' = E_1e^{-qz} \quad \text{for} \quad z > 0,$$

(B11)

$$\Pi' = E_2e^{qz} \quad \text{for} \quad z < 0,$$

(B12)

where coefficients $A_1, A_2, ..., E_2$ are real or imaginary functions of $\mathbf{q}$.

The incompressibility condition (B1), evaluated at $z = 0$ yield

$$D_1 = -iq_xA_1 - iq_yA_3 + qC_1 = \frac{iq_x}{q}B_1 + \frac{iq_y}{q}B_3,$$

(B13)

$$D_2 = -iq_xA_2 - iq_yA_4 - qC_2 = -\frac{iq_x}{q}B_2 + \frac{iq_y}{q}B_4.$$

(B14)

The continuity of velocity or Eq. (24) gives

$$A_1 = A_2 = v_x,$$

(B15)

$$A_3 = A_4 = v_y,$$

(B16)

$$C_1 = C_2,$$

(B17)

As the active fluid film is two dimensional, there is no discontinuity over the vertical gradient of $v'_z$ (since $v_z = 0$). This allows us to write

$$\partial_z v'_z|_{z=\varepsilon} = \partial_z v'_z|_{z=-\varepsilon},$$

(B18)

which yields using Eqs. (B9) and (B10)

$$D_1 = 2qC_1 + D_2.$$

(B19)

The tangential stress $F_t$ may be evaluated using the Stokes equation (17). The Stokes equation for $v'_i$ yields

$$\eta'\nabla^2 v'_i = 0.$$  

(B20)

Eq. (B20) gives us further relations

$$B_3 = -\frac{iq_x}{q}D_1 = \frac{iq_y}{q}B_1 \quad \text{and}$$

(B21)

$$B_4 = \frac{iq_y}{q}D_2 = \frac{iq_y}{q}B_2.$$  

(B22)

Using Eqs. (B15), (B17), (B19), (B21) and (B22), the $x$-component of 3D force $F$ is obtained as

$$F_x = \eta' \left( \partial_z v'_x + \partial_x v'_x \right)\varepsilon - \eta' \left( \partial_z v'_x + \partial_x v'_x \right)|_{-\varepsilon}$$

$$= \eta' \left[ -2\frac{q_x}{q}v_x - 2qv_x - 2\frac{q_y}{q}v_y - i\frac{q_x}{q}(D_1 + D_2) + B_1 - B_2 \right]$$

$$= -2\eta'qv_x,$$

(B23)

where we have used incompressibility of the 3D velocity in the last line. Similarly we get the $y$-component of $F$ as

$$F_y = -2\eta'qv_y.$$  

(B24)
Appendix C: Form of $F_\beta$: intermediate friction

We start with

$$\eta' \frac{\partial v'_\alpha}{\partial z}|_{z=\epsilon} = \Gamma(v'_\alpha|_{z=\epsilon} - v_\alpha), \alpha = x, y. \quad \text{(C1)}$$

A similar condition exists at $z = -\epsilon$. Now using the forms of $v'_x, v'_y$ and $v'_z$ as given by (B5), (B7) and (B9) we obtain

$$\eta'(-A_1 q + B_1) = \Gamma(A_1 - v_x), \quad \text{(C2)}$$
$$\eta'(-A_3 q + B_3) = \Gamma(A_3 - v_y). \quad \text{(C3)}$$

In the weak friction limit, $\Gamma \ll O(\eta' q)$ in the wavevector range of interest. Thus,

$$\eta'(-A_1 q + B_1) = \Gamma(-v_x), \quad \text{(C4)}$$
$$\eta'(-A_3 q + B_3) = \Gamma(-v_y). \quad \text{(C5)}$$

or, equivalently,

$$\eta' \frac{\partial v'_i}{\partial z} = -\Gamma v_x, \quad \text{(C6)}$$
$$\eta' \frac{\partial v'_i}{\partial z} = -\Gamma v_y \quad \text{(C7)}$$

at $z = \epsilon$. Similar considerations at $z = -\epsilon$ finally yields

$$\eta' \left[ \frac{\partial v'_\alpha}{\partial z} |_{z=\epsilon} - \frac{\partial v'_\alpha}{\partial z} |_{z=-\epsilon} \right] = -2\Gamma v_\alpha. \quad \text{(C8)}$$

This yields for the 2D generalised Stokes equation which $v_\alpha$ satisfy

$$\eta \nabla^2 v_\alpha + \zeta \Delta_\mu P_{\alpha x}\partial_x p + \zeta \Delta_\mu P_{\alpha y}\partial_y p + \zeta \Delta_\mu P_{\alpha z}\partial_z \rho - 2\Gamma v_\alpha = 0. \quad \text{(C9)}$$

Now write Eq. (C9) in the Fourier space and neglect $\eta q^2 v_\alpha$, assuming $\eta q^2 \ll 2\Gamma$. This yields Eqs. (32) and (33).

Appendix D: Velocity profiles of the ambient fluid

It is instructive to obtain the flow profiles of three-dimensional velocity fields, that are created by the (small) fluctuations in $\rho$ and $p_y$, in the three different regimes of our model as delineated by the values of $\Gamma$.

1. Large $\Gamma$

In this case $v'_i(x, y, z = \pm \epsilon) = v_i(x, y), i = x, y$. Since $v'_z(z = \pm \epsilon) = 0$, from (B9) and (B10), $C_1 = 0 = C_2$. Using the no-slip condition on $v'_i(z = \pm)$ and the 3D incompressibility of $v'_i, \alpha = x, y, z, (iq_x v'_x + iq_y)|_{z=\pm} = 0 = \frac{\partial v'_z}{\partial z}|_{z=\pm}$ in the Fourier space. This yields $D_1 = 0 = D_2$. Thus, $v'_z = 0$ everywhere above and below the active fluid layer. Hence, the flow in the surrounding fluid is actually 2D, parallel to the active fluid layer. We further find $B_1 = 0 = B_2$ and $B_3 = 0 = B_4$. Thus in the Fourier space,

$$v'_i(q_x, q_y, z) = v_i(q_x, q_y) \exp(-qz), z > 0, \quad \text{and} \quad v'_i(q_x, q_y, z) = v_i(q_x, q_y) \exp(qz), z < 0. \quad \text{(D1)}$$

Therefore, $v'_i$ has the same form as $v_i$ with an exponentially damped amplitude by a factor $\exp(-q|z|)$ and hence shows the same instabilities at $O(q)$. 
2. Intermediate Γ

In the intermediate friction case, the 3D shear stress balance is given by

\[ \eta' \frac{\partial v'_i}{\partial z} |_{z = \pm} = -\Gamma v_i |_{z = \pm} \]  
\[ \eta' \frac{\partial v'_i}{\partial z} |_{-\epsilon} = \Gamma v_i |_{-\epsilon} \]  
\[ (D2) \]
\[ (D3) \]

Using the above equations and (B5), (B6), (B7) and (B8), we get a set of relations between the couplings given by

\[ -qA_1 + B_1 = -\frac{\Gamma}{\eta'} v_x, \]  
\[ (D4) \]
\[ qA_2 + B_2 = \frac{\Gamma}{\eta'} v_x, \]  
\[ (D5) \]
\[ -qA_3 + B_3 = -\frac{\Gamma}{\eta'} v_y, \]  
\[ (D6) \]
\[ qA_4 + B_4 = \frac{\Gamma}{\eta'} v_y. \]  
\[ (D7) \]

Now using (D4), (D5), (D6), (D7), 3D incompressibility \((\nabla \cdot \mathbf{v} = 0)\) of ambient fluid, 2D incompressibility \((\nabla \perp \cdot \mathbf{v} = 0)\) of active fluid layer and equations (B21) and (B22), we can show that \(B_1 = B_2 = B_3 = B_4 = 0\) and, \(D_1 = D_2 = 0\).

3. Small Γ

In the limit of very small Γ, we have \(\eta' \frac{\partial v'_i}{\partial z} |_{z = \pm} \approx 0\). i.e., we effectively have the stress-free boundary condition on \(v'_i\) at \(z = 0\). In addition, \(v'_z = 0\) at \(z = 0\). Now using the results in Sec. [C] we find \(A_1 = A_2 = B_1 = B_2 = A_3 = A_4 = B_3 = B_4 = 0\), i.e., \(v'_i = 0\) at all \(z > 0\) and \(z < 0\) identically. In addition, \(v'_z = 0\) everywhere. Thus, the 3D velocity field vanishes.

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The thin active fluid layer in our model corresponds to the cortical actin layer in a cell, where as the embedding fluid in our model represents the bulk cytoplasm. While these straightforward relations between our model and the cell cortex seem self-evident and simple, it should be remembered that in an eukaryotic cell, there are no sharp boundaries between the cortical actin layer and the bulk cytoplasm. In addition, the bulk cytoplasm is not a passive fluid; it has active processes going on in it.