CONTINUITY OF INFINITELY DEGENERATE WEAK SOLUTIONS VIA THE TRACE METHOD

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Abstract. In 1971 Fedi˘ı proved in [Fe] the remarkable theorem that the linear second order partial differential operator

\[ L_f u(x,y) \equiv \left\{ \frac{\partial}{\partial x^2} + f(x)^2 \frac{\partial}{\partial y^2} \right\} u(x,y) \]

is hypoelliptic provided that \( f \in C^\infty(\mathbb{R}) \), \( f(0) = 0 \) and \( f \) is positive on \( (-\infty,0) \cup (0,\infty) \). Variants of this result, with hypoellipticity replaced by continuity of weak solutions, were recently given by the authors, together with Cristian Rios and Ruipeng Shen, in [KoRiSaSh] to infinitely degenerate elliptic divergence form equations

\[ \nabla^{tr} A(\mathbf{x},u) \nabla u = \phi(x), \quad x \in \Omega \subset \mathbb{R}^n, \]

where the nonnegative matrix \( A(\mathbf{x},u) \) has bounded measurable coefficients with trace roughly 1 and determinant comparable to \( f^2 \), and where \( F = \ln \frac{1}{f} \) is essentially doubling.

However, in the plane, these variants assumed additional geometric constraints on \( f \), such as \( f(r) \geq e^{-r-\sigma} \) for some \( 0 < \sigma < 1 \), something not required in Fedi˘ı's theorem. In this paper we in particular remove these additional geometric constraints in the plane for homogeneous equations with \( F \) essentially doubling.

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1. Introduction

In 1971 Fedi˘ı proved in [Fe] the remarkable theorem that the linear second order partial differential operator

\[ L_f u(x,y) \equiv \left\{ \frac{\partial}{\partial x^2} + f(x)^2 \frac{\partial}{\partial y^2} \right\} u(x,y) \]

is hypoelliptic, i.e. every distribution solution \( u \in \mathcal{D}'(\mathbb{R}^2) \) to the equation \( L_f u = \phi \in C^\infty(\mathbb{R}^2) \) in \( \mathbb{R}^2 \) is smooth, i.e. \( u \in C^\infty(\mathbb{R}^2) \), provided that \( f \in C^\infty(\mathbb{R}) \), \( f(0) = 0 \) and \( f \) is positive on \( (-\infty,0) \cup (0,\infty) \). In particular \( f \) can vanish to infinite order and \( L_f \) is infinitely degenerate elliptic. See also [KuStr], [Mor], [Chr] and [Koh] for generalizations to smooth equations in higher dimensions, something we do not pursue here.

Variants of this result were then given in [KoRiSaSh] to infinitely degenerate elliptic divergence form equations

\[ \nabla^{tr} A(\mathbf{x},u) \nabla u = \phi(x), \quad x \in \Omega \subset \mathbb{R}^n, \]

where the nonnegative matrix \( A \) has bounded measurable coefficients with trace roughly 1 and determinant comparable to \( f^2 \). The concept of hypoellipticity was interpreted there in terms of local boundedness and continuity of weak solutions. However, additional geometric constraints on the degeneracy \( f \) were needed.
for the methods used there, beyond the minimal restriction that \( F \equiv \ln \frac{1}{r} \) be a ‘structured geometry’, i.e. satisfies the five ‘log doubling’ structure conditions in Definition 1 below (useful for estimating arc length and volume of control balls). While these additional geometric constraints were often necessary in dimension \( n \geq 3 \), as is the case for the smooth equations in higher dimensions in [KuSt] and [Chr], the case of dimension \( n = 2 \) was left open in the rough setting.

The main goal of this paper is to extend this type of result to the plane \( \mathbb{R}^2 \) for all structured geometries without any additional geometric constraints. More precisely, to certain equations in the plane \( \mathbb{R}^2 \) with nonnegative matrices having bounded measurable coefficients with trace roughly 1 and determinant \( f^2 \), and with at least bounded forcing functions \( \phi \). For this purpose we develop a trace method that first constructs a region in \( \mathbb{R}^2 \) on whose boundary a given subsolution \( \subsol \) has a suitable trace, and second applies a maximum principle to derive local boundedness and continuity of weak solutions \( \subsol \) to some infinitely degenerate equations, resulting in our Trace Method Theorem.

We will use both an existing maximum principle for inhomogeneous equations from [KoRiSaSh], that requires a restriction on the geometry \( F \) associated with the operator, as well as a new maximum principle for homogeneous equations, valid for all structured geometries, and all dimensions as well. Table 1 organizes the various conclusions on weak solutions so that they either persist or improve as we move to the right or lower down in the table.

There are three separate notions of admissibility of inhomogeneous data \( \phi \) appearing in this table: the strongest notion is that used in [KoRiSaSh2] which amounts to assuming the data are very close to \( L^\infty \); the weakest notion is that in [KoRiSaSh], denoted \( \phi \in X_f \) here; and an intermediate notion, called \( f \)-admissible, that is used in the current paper. The boxes are color coded as follows: continuity results in red boxes require additional geometric constraints and the strongest notion of admissibility of inhomogeneous data, and were proved in [KoRiSaSh2]; local boundedness results in red boxes require less stringent additional geometric constraints and the weakest notion of admissibility of inhomogeneous data, and were proved in [KoRiSaSh]: results in black boxes are valid for all structured geometries and require the intermediate notion of admissibility of inhomogeneous data, and are proved here; and the single continuity result in the purple box requires a geometric constraint and the intermediate notion of admissibility of inhomogeneous data, and is also proved here. In all cases the inhomogeneous data include bounded measurable functions \( \phi \). The above results are described in more detail below, see e.g. Definition 3 for the meaning of \( f \)-admissible, and Definition 2 for the meaning of \( F_{k,\sigma} \).

| Data | \( \phi(x, y) \) | \( \phi(x, \cdot) \) \( \rho \) cont unif in \( x \) | \( 0 \) |
|------|----------------|----------------|---------|
| \( \mathcal{A}(x, y, u(x, y)) \) | \text{loc bdd } F_{\sigma}, \sigma < 1 | \text{cont } F_{3, \sigma}, \sigma < 1 | \text{loc bdd all } F |
| \( \mathcal{A}(x) \) | \text{loc bdd } F_{\sigma}, \sigma < 1 | \text{cont } F_{\sigma}, \sigma < 1 | \text{cont all } F |

**Table 1. Brief summary of applications**

More specifically, we will consider replacing the Fedi˘ı operator \( L_f \) above with a more general second order divergence form special quasilinear operator \( \mathcal{L} = \nabla^{tr} \mathcal{A}(x, y, u) \nabla \) in \( \mathbb{R}^2 \) with bounded measurable coefficients, and we will consider the special quasilinear equations (special because \( \mathcal{A} \) is independent of \( \nabla u \)) and restricted linear equations (restricted because \( A \) is independent of \( y \)),

\[
\begin{align*}
\text{special quasilinear} & : \quad \mathcal{L}u = \nabla^{tr} \mathcal{A}(x, y, u) \nabla u = \phi, \\
\text{restricted linear} & : \quad Lu = \nabla^{tr} A(x) \nabla u = \phi,
\end{align*}
\]

where \( \phi \) is \( f \)-admissible as in Definition 3. Roughly speaking, we prove the following five new results for such second order divergence form operators in the plane with ‘structured geometry’ \( f \), i.e. \( F = \ln \frac{1}{r} \) satisfies Definition 3 below, which essentially says that \( F \) is a doubling function with some normalizing conditions.

\(^{1}\)The arXiv article [KoRiSaSh2] contains all of the continuity results stated here for inhomogeneous equations, that require the strong constraint \( F_{3,\sigma} \) on the geometry, and also a stronger restriction on the inhomogeneous data \( \phi \). These results were obtained there using an infinitely degenerate Moser scheme that is considerably more complicated than the adaptation of the DeGeorgi / Caffarelli / Vasseur scheme used in [KoRiSaSh].
Notation 1. We will often make mention of the plane ‘geometry’ associated with the functions \( f ( r ) \) or \( F ( r ) = \ln \frac{1}{r} \). By this we mean the geometry of metric balls defined in [KoRiSaSh, Chapter 7] using the degenerate Riemannian metric \( dt^2 = dx^2 + \frac{1}{f ( x )} dy^2 \), with its associated control distance \( d \). Given two structured geometries represented by \( F ( r ) \) and \( G ( r ) \), we say that \( F \) is stronger, or less degenerate, than \( G \), if \( F ( r ) \leq G ( r ) \) for sufficiently small \( r > 0 \). More generally, if the \( 2 \times 2 \) matrix \( A ( x, y, u ( x, y ) ) \) associated with a divergence form operator \( L \) is comparable to a diagonal matrix \( D_f = \begin{bmatrix} 1 & 0 \\ 0 & f ( x )^2 \end{bmatrix} \), then we refer to this geometry as being associated with \( L \) or with \( A ( x, y, u ( x, y ) ) \).

1. (homogeneous maximum principle) For any structured geometry, a maximum principle holds for weak subsolutions to homogeneous equations \( Lu = 0 \). This has an extension to all dimensions \( n \geq 2 \).

2. (special homogeneous quasilinear equation) For any structured geometry, weak solutions \( u \) to a homogeneous special quasilinear equation \( Lu = 0 \) are locally bounded.

3. (special inhomogeneous quasilinear equation) If the geometry \( F \) is stronger than \( F_\sigma \) for some \( \sigma < 1 \), i.e. \( F ( r ) \leq F_\sigma ( r ) \) for \( r > 0 \) sufficiently small, then weak solutions \( u \) to a special quasilinear equation \( Lu = \phi \) are locally bounded provided the forcing function \( \phi \) is \( f \)-admissible.

4. (restricted linear homogeneous equation) For any structured geometry, if \( A ( x, y, u ) = A ( x ) \) depends only on \( x \), then weak solutions \( u \) to the homogeneous restricted linear equation \( Lu = 0 \) are continuous.

5. (restricted linear equation with forcing function continuous in \( y \)) If the geometry \( F \) is stronger than \( F_\sigma \) for some \( \sigma < 1 \), then weak solutions \( u \) to a restricted linear equation \( Lu = \phi \) are continuous provided the \( f \)-admissible forcing function \( \phi ( x, y ) \) is continuous in \( y \) with modulus of continuity uniform in \( x \).

Statement (1) is Theorem 2 and the reader should have no difficulty in deriving statements (2), (3), (4) and (5) from Theorem 3 and the maximum principles in Theorems 1 and 2. Indeed, statements (2) and (4) for homogeneous equations use Theorems 2 and 3; while statements (3) and (5) use Theorems 1 and 3.

The two main new results listed above are statements (2) and (4), which require no additional geometric assumptions on the geometry of the operator \( L \) other than that it is a structured geometry, thus giving results closer in spirit to Fedilii’s theorem, which required no geometric assumptions other than that \( f \) is positive away from 0. After a section on preliminaries, which makes precise the conditions surrounding our equations, the following two sections prove the new homogeneous maximum principle in \( \mathbb{R}^n \), and the Trace Method Theorem in \( \mathbb{R}^2 \) respectively.

1.1. Preliminaries. We begin with the second order special quasilinear equation (where only \( u \), and not \( \nabla u \), appears nonlinearly),

\[
Lu = \nabla^T A ( x, y, u ( x, y ) ) \nabla u = \phi, \quad (x, y) \in \Omega, 
\]

where \( \Omega \) is a bounded domain in the plane \( \mathbb{R}^2 \), and we assume the following quadratic form condition on the ‘quasilinear’ matrix \( A ( x, y, z ) \),

\[
c \xi^T D_f ( x ) \xi \leq \xi^T A ( x, y, z ) \xi \leq C \xi^T D_f ( x ) \xi ,
\]

for a.e. \( (x, y) \in \Omega \) and all \( z \in \mathbb{R}, \xi \in \mathbb{R}^2 \), where \( c, C \) are positive constants. Equivalently, the \( 2 \times 2 \) matrix \( A ( x, y, z ) \) has bounded measurable coefficients and is comparable to the following diagonal matrix \( D_f ( x ) \) depending only on \( x \),

\[
D_f ( x ) = \begin{bmatrix} 1 & 0 \\ 0 & f ( x )^2 \end{bmatrix}.
\]

Define the \( f \)-gradient by

\[
\nabla_f = D_f ( x ) \nabla ,
\]

\[\text{Recall that a vector } \mathbf{v} \text{ is subunit for an invertible symmetric matrix } A, \text{ i.e. } ( \mathbf{v} : \xi )^2 \leq \xi^T A \xi \text{ for all } \xi, \text{ if and only if } \xi^T A \xi \leq 1, \text{ see e.g. } [\text{KoRiSaSh}, \text{Chapter } 7].\]

\[\text{These conclusions were obtained in } [\text{KoRiSaSh}] \text{ under stronger geometric assumptions on the operator } L, \text{ namely } F \geq F_\sigma \text{ for local boundedness, and } F \geq F_{3, \sigma} \text{ for continuity, where } 0 < \sigma < 1.\]
and the associated degenerate Sobolev space $W^{1,2}_f(\Omega)$ to have norm

$$\|v\|_{W^{1,2}_f} \equiv \sqrt{\int_\Omega \left( |v|^2 + \nabla v^\tau D_f \nabla v \right)} = \sqrt{\int_\Omega \left( |v|^2 + |\nabla f|^2 \right)}.$$  

Note that if $A(x)$ is comparable to $D_f(x)$, then

$$\|v\|_{W^{1,2}_f} \approx \sqrt{\int_\Omega \left( |v|^2 + \nabla v^\tau A \nabla v \right)},$$

which shows that $\nabla f$ is an appropriate gradient to use in connection with the operator $\mathcal{L}$. We say $u \in W^{1,2}_f(\Omega)$ is a $W^{1,2}_f(\Omega)$-weak solution to $Lu = \phi$ if

$$-\int (\nabla w^\tau A(x, u(x)) \nabla u) = \int \phi w, \quad \text{for all } w \in W^{1,2}_f(\Omega)_0.$$

We will assume that the degeneracy function $f(r) = e^{-F(r)}$ is even, and that there is $R > 0$ such that $F$ satisfies the following five structure conditions from [KoRiSaSh] for some constants $C \geq 1$ and $\varepsilon > 0$.

**Definition 1** (structure conditions). *A twice continuously differentiable function $F : (0, R) \to \mathbb{R}$ is said to satisfy geometric structure conditions, or to be a structured geometry, if:

1. $\lim_{x \to 0^+} F'(x) = +\infty$;
2. $F'(x) < 0$ and $F''(x) > 0$ for all $x \in (0, R)$;
3. $\frac{1}{x} |F'(r)| \leq |F'(x)| \leq C |F'(r)|$ for $\frac{1}{2} r < x < 2r < R$;
4. $\frac{1}{xF(x)}$ is increasing in the interval $(0, R)$ and satisfies $\frac{1}{xF(x)} \leq \frac{1}{x}$ for $x \in (0, R)$;
5. $\frac{F''(x)}{F'(x)} \approx \frac{1}{r}$ for $x \in (0, R)$.

**Definition 2.** For $0 < r < \infty$ define

$$F_\sigma(r) \equiv \left( \frac{1}{r} \right)^{\frac{1}{\sigma}}, \quad 0 < \sigma < 1,$$

$$F_{k,\sigma}(r) \equiv \left( \ln \frac{1}{r} \right)^{\sigma} \left( \ln^k \frac{1}{r} \right)^{\sigma}, \quad 0 < \sigma < 1 \text{ and } k \in \mathbb{N}.$$  

The functions $F_\sigma$ and $F_{k,\sigma}$ are examples of functions satisfying geometric structure conditions as above. Note that $F_\sigma = e^{-F_{\sigma'}}$ vanishes to infinite order at $r = 0$, and that $F_{\sigma}$ vanishes to a faster order than $F_{\sigma'}$ if $\sigma > \sigma'$. A similar remark applies to $F_{k,\sigma} = e^{-F_{k,\sigma'}}$. The first part of the next definition originates in [KoRiSaSh] see Definition 4).

**Definition 3.** Fix a bounded domain $\Omega \subset \mathbb{R}^2$. Define the space $X_f(\Omega)$ to consist of all functions $\phi$ on $\mathbb{R}^n$ such that

$$\|\phi\|_{X_f(\Omega)} \equiv \sup_{v \in (W^{1,1}_{\text{loc}}(\Omega))_0} \frac{\int_\Omega |v\phi| \, dy}{\int_\Omega \|\nabla f v\| \, dy} < \infty.$$  

We say that $\phi$ is $f$-admissible in $\Omega$ if both $\phi \in X_f(\Omega)$ and $\phi$ satisfies the following $L^q$ growth condition in $\Omega$,

$$\|\phi\|_{L^q_{\text{growth}}(\Omega)} \equiv \sup_{(x, y) \in \Omega \setminus \{y\text{-axis}\}} \|\phi\|_{L^q(B((x, y), \frac{1}{r}))} < \infty, \quad \text{for some } q > \frac{n}{2}.$$  

We norm the $f$-admissible functions with

$$\|\phi\|_{f-\text{adm}}(\Omega) \equiv \|\phi\|_{X_f(\Omega)} + \|\phi\|_{L^q_{\text{growth}}(\Omega)}.$$

The point of including $L^q_{\text{growth}}$ in the definition of $f$-admissible is so that we can apply standard elliptic theory as in [GiTr] away from the $y$-axis with appropriate uniformity. We will apply the definition of $f$-admissible to forcing functions $\phi$ only for structured geometries $f$. In connection with the definition of $\|\phi\|_{X_f(\Omega)}$, note that $\int_\Omega \|\nabla f v\| \, dy \approx \|v\|_{W^{1,1}_f(\Omega)}$ by the $1 - 1$ Poincaré inequality analogous to (25) in Proposition 3 below. Finally note that bounded functions are $f$-admissible; $\|\phi\|_{f-\text{adm}}(\Omega) \lesssim \|\phi\|_{L^\infty(\Omega)}$. 


Definition 4. We say a function \( u \in W^{1,2}_f(\Omega) \) is bounded by a constant \( \ell \in \mathbb{R} \) on the boundary \( \partial \Omega \) if 
\[
(u - \ell)^+ = \max\{u - \ell, 0\} \in \left(W^{1,2}_f\right)_0(\Omega).
\]
We define \( \sup_{x \in \partial \Omega} u(x) \) to be \( \inf \{ \ell \in \mathbb{R} : (u - \ell)^+ \in \left(W^{1,2}_f\right)_0(\Omega) \} \).

Before we start stating our main new results, we recall the geometric maximum principle for weak subsolutions to inhomogeneous equations given in [KoRiSaSh], under additional restrictions on the geometry \( F \) associated with the form \( A \) of the operator. We restrict our attention to the case \( n = 2 \) as that is the only dimension in which we obtain new results for \( W^{1,2}_f(\Omega) \)-weak solutions. Namely, we assume that \( f(x) \neq 0 \) if \( x \neq 0 \), and that \( F \) satisfies the five geometric structure conditions in Definition 4. Note that the admissibility requirement for \( \phi \) in the next theorem is the weakest one, \( \phi \in X_f(\Omega) \), and which is used in [KoRiSaSh].

**Theorem 1** (inhomogeneous maximum principle). Suppose that \( F = \ln \frac{1}{x} \) is a structured geometry, i.e. satisfies the structure conditions in Definition 4 and is stronger than \( F_\sigma \) for some \( 0 < \sigma < 1 \). Assume that \( u \) is a weak subsolution to \( \mathcal{L}u = \phi \) in a domain \( \Omega \subset \mathbb{R}^2 \), where \( \mathcal{L} = \nabla^\sigma \mathcal{A} \nabla \) and \( \mathcal{A} \approx D_f \) has bounded measurable coefficients, and \( \phi \in X_f(\Omega) \). Moreover, suppose that \( u \) is bounded in the weak sense on the boundary \( \partial \Omega \). Then \( u \) is globally bounded in \( \Omega \) and satisfies
\[
\sup_{\Omega} u \leq \sup_{\partial \Omega} u + C \| \phi \|_{X(\Omega)}.
\]

We will use the above maximum principle when dealing with inhomogeneous equations in the plane. On the other hand, when dealing with homogeneous equations, we will use an improved maximum principle, valid for more general geometries, and which is our first main new theorem.

1.2. **Statement of the two main results.** Since our homogeneous maximum principle holds in higher dimensions as well, we will give the statement and proof for domains \( \Omega \subset \mathbb{R}^n \). We refer to [KoRiSaSh] for the straightforward extension of the planar definitions used here to higher dimensions, noting in particular that \( D_f(x_1, \ldots, x_n) \) is the \( n \times n \) diagonal matrix with diagonal entries \( \{1, \ldots, 1, f(x_1)^2\} \). Our first main theorem is a maximum principle for homogeneous equations in \( \mathbb{R}^n \) that holds for all structured geometries.

**Theorem 2** (homogeneous maximum principle). Suppose that \( F = \ln \frac{1}{x} \) is a structured geometry, i.e. satisfies the geometric structure conditions in Definition 4. Assume that \( u \) is a weak subsolution to \( \mathcal{L}u = 0 \) in a domain \( \Omega \subset \mathbb{R}^n \) with \( n \geq 2 \), where \( \mathcal{L} = \nabla^\sigma \mathcal{A} \nabla \) and \( \mathcal{A} \approx D_f \) has bounded measurable coefficients. Moreover, suppose that \( u \) is bounded in the weak sense on the boundary \( \partial \Omega \). Then
\[
\sup_{\Omega} u \leq \sup_{\partial \Omega} u.
\]

Our second main theorem yields a new method for obtaining local boundedness and continuity of weak solutions in the plane, given that we already have an ‘appropriate’ maximum principle. In order to combine statements using either the homogeneous or inhomogeneous maximum principles, it is convenient to define precisely what we mean by an ‘appropriate’ maximum principle in the plane.

**Definition 5.** Let \( \phi \in L^2_{\text{loc}}(\Omega) \) and \( \mathcal{L} = \nabla^\sigma \mathcal{A} \nabla \) where \( \mathcal{A} \approx D_f \) has bounded measurable coefficients and a structured geometry in \( \Omega \). An equation \( \mathcal{L}v = \phi \) satisfies the Maximum Principle Property in \( \Omega \), or \( \text{MPP} \) for short, if for every open rectangle \( R = (a,b) \times (c,d) \) with closure contained in \( \Omega \), there is a constant \( C_R \) such that
\[
\sup_{\Omega} v \leq \sup_{\partial \Omega} v + C_R,
\]
for all weak solutions \( v \) of \( \mathcal{L}v = \phi \) that are bounded in the weak sense on \( \partial \Omega \).

The new method, which we refer to as the trace method, is embodied in the next theorem, for which we need the following somewhat technical definition of a modulus of continuity associated with a structured geometry.

**Definition 6.** For any structured geometry \( F = \ln \frac{1}{x} \), let \( \omega_f \) be the modulus of continuity associated to \( f \) as defined in (3.17) and (3.18) below.

For any modulus of continuity \( \omega \), define a difference operator in the second variable by \( D_{\omega,0}^\omega h(x,y) = \frac{h(x,y+\delta) - h(x,y)}{\omega(\delta)} \).
Theorem 3 (trace method). Let \( \phi \) satisfy the \( L^q \) growth condition \( \| \phi \|_{L^q_{\text{growth}}(\Omega)} < \infty \) for the operator \( L \) in the domain \( \Omega \subset \mathbb{R}^2 \), for some \( q > \frac{2}{3} \). Suppose the equation \( Lv = \nabla A \nabla v = \phi \) satisfies the MPP, i.e. the Maximum Principle Property, in \( \Omega \) with \( A \) as above, and suppose that \( u \in W^{1,2}_f(\Omega) \) is a weak solution to this equation, i.e. \( Lu = \phi \) in \( \Omega \). Then

\( \| \phi \|^2_{X_f(B)} \leq P v(x), \quad \text{a.e.} \ x \in \{ u > 0 \} \cap B. \)

Proof. Since \( u \) is a weak subsolution to \( L u = \phi \), we have using as test function \( w = u_+ \in \left( W^{1,2}_f \right)_0(B) \), that

\[
\int \nabla (u_+) A \nabla u_+ d\mu = \int \nabla u_+ A \nabla u d\mu - \int u_+ \phi d\mu \leq \| \phi \|_{X_f(B)} \int |\nabla f u_+| d\mu \leq C P \int v |\nabla f u_+| d\mu,
\]

where for the last inequality we used conditions \( \text{(L.2)} \) and \( \text{(1.1)} \). Using Hölder’s inequality and \( \text{(L.2)} \) this gives

\[
\int |\nabla f u_+|^2 d\mu \leq \frac{1}{c} \int |\nabla (u_+) A \nabla u_+ d\mu \leq C P \int v^2 d\mu,
\]

which is \( \text{(2.2)} \). Using \( w = u_- \) as a test function we obtain the last statement of the Proposition.

Now we can prove the abstract maximum principle for homogeneous equations, assuming only the ‘straight-across’ Sobolev inequality

\[
\| w \|_{L^2(\Omega)} \leq C(\Omega) \| \nabla f w \|_{L^2(\Omega)}, \quad w \in \left( W^{1,2}_f \right)_0(\Omega)
\]

Theorem 4. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \), and assume the global Sobolev inequality \( \text{(2.3)} \) holds. Assume that \( u \) is a weak subsolution to the homogeneous equation \( Lu = 0 \) in \( \Omega \), and that \( u \) is bounded on the boundary \( \partial \Omega \). Then we have

\[
\sup_{\Omega} u \leq \sup_{\partial \Omega} u.
\]

Proof. If \( u \) is a weak subsolution to \( L u = 0 \), then so is \( u - \sup_{\partial \Omega} u \), therefore we can assume \( u \leq 0 \) on \( \partial \Omega \).

Since \( \phi \equiv 0 \), we can use \( \text{(2.2)} \) with \( v \equiv 0 \) and \( \text{(2.3)} \) applied to \( w = u_+ \) to obtain

\[
\int u_+^2 \leq 0,
\]

which implies \( u \leq 0 \) in \( \Omega \). Applying this to \( u - \sup_{\partial \Omega} u \) gives the result.

To obtain the ‘geometric’ version of this theorem, namely Theorem \text{[2]} we need to show \( \text{(2.3)} \). For this we first recall a proposition from \text{[KoRiSaSh]} Proposition 76].
Proposition 2. Let the balls $B(0,r)$ and the degenerate gradient $\nabla_f$ be as above for a structured geometry. There exists a constant $C$ such that the proportional vanishing $L^1$-Sobolev inequality
\begin{equation}
\int_{B(0,r)} |w| \, dx \leq Cr \int_{B(0,2r)} |\nabla_f w| \, dx,
\end{equation}
holds for any Lipschitz function $w$ that vanishes on a subset $E$ of the ball $B(0, r)$ with $|E| \geq \frac{1}{2} |B(0,r)|$, and all sufficiently small $r > 0$.

Proposition 3. Suppose that $F$ satisfies the geometric structure conditions in Definition 1. Then the following (2.2) global Sobolev inequality holds with geometry $F_n$
\begin{equation}
\|w\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla_f w\|_{L^2(\Omega)}, \quad w \in \left(W^{1.2}_f\right)_0(\Omega)
\end{equation}
Proof. As in the proof of a corresponding proposition in [KoRiSaSh, Proposition 81], it suffices by using a partition of unity to suppose that $\Omega$ is bounded. Then choose a ball $B(0,r_0)$ containing $\Omega$ and extend $w$ to be 0 outside $\Omega$ so that $w \in \left(W^{1.2}_f\right)_0(B(0,r_0))$. Next, choose $R > r_0$ s.t. $E \equiv B(0,R) \setminus B(0,r_0)$ satisfies $|E| \geq |B(0,R)|/2$. Then we can apply Proposition 2 to $w^2 \in \left(W^{1.1}_f\right)_0(B(0,R))$ to obtain
\begin{equation}
\int_{\Omega} w^2 \, dx = \int_{B(0,R)} w^2 \, dx \leq CR \int_{B(0,R)} |\nabla_f w^2| \, dx = 2CR \int_{\Omega} |w||\nabla_f w| \, dx.
\end{equation}
Using Hölder’s inequality we conclude (2.5). \qed

Thus Proposition 3 together with Proposition 4 prove the geometric maximum principle in Theorem 2.

3. Proof of the Trace Method Theorem in $\mathbb{R}^2$

We will prove the Trace Method Theorem in eight steps. Conclusion (1) of Theorem 3 will follow from the first three steps, where the first two will establish ‘smoothness’ properties of functions $u \in W^{1,2}_f(\Omega)$, where it is crucial that $\Omega$ is a planar domain, and the third requires that $u$ be a weak solution. Conclusion (2) will then follow from an additional five steps, two of which are refinements of Steps two and three. It suffices to consider the case $\Omega = (-1,1) \times (-1,1)$, which we assume in all eight steps below. We also use the notation \(\Omega_{a,b}^{c,d} = (a, b) \times (c, d)\) for $-1 \leq a < b \leq 1$ and $-1 \leq c < d \leq 1$.

3.1. Local boundedness of weak solutions. Here we will prove Conclusion (1) of the Trace Method Theorem. We begin with Lebesgue’s differentiation theorem and maximal function for Hilbert space valued functions on an interval $(c,d)$.

Lemma 1. Suppose that $H$ is a separable Hilbert space and that $F \in L^2_H((c,d))$, i.e. $F : (c,d) \rightarrow H$ and
\begin{align*}
\|F\|_{L^2_H((c,d))}^2 &= \sqrt{\int_{c}^{d} |F(y)|_H^2 \, dy} < \infty.
\end{align*}
Then for almost every $x \in (c,d)$ we have both
\begin{align*}
\lim_{I \ni y} \frac{1}{|I|} \int_{I} |F(t) - F(y)|_H^2 \, dt &= 0, \\
M_2 F(y) &= \sup_{I : y \in I} \sqrt{\frac{1}{|I|} \int_{I} |F(t)|_H^2 \, dt} < \infty.
\end{align*}
Moreover we have the weak type estimate
\begin{align*}
|\{y \in (c,d) : M_2 F(y) > \lambda\}| &\leq \frac{5}{\lambda^2} \|F\|_{L^2_H((c,d))}^2, \quad \lambda > 0.
\end{align*}
Proof. This is proved exactly as in the classical case when the Hilbert space $H$ is the scalar field $\mathbb{R}$. \qed
We now apply Lemma \[\text{in the case } \mathcal{H} = L^2 ((a, b))\], so that \( F \in L^2_{\mathcal{H}} ((c, d)) \) can be realized as a real-valued function \( f (x, y) \) defined on \( \Omega_{a,b}^{c,d} \equiv (a, b) \times (c, d) \) with
\[
F (y) = f (x, y), \quad \text{for } (x, y) \in \Omega_{a,b}^{c,d},
\]
and then by (3.1), we have
\[
\| F \|_{L^2_{\mathcal{H}} ((a, b))} = \sqrt{\int_c^d \left( \int_a^b | f (x, y) |^2 \, dx \right) \, dy} = \| f \|_{L^2 (\Omega_{a,b}^{c,d})}.
\]

Conclusion (1) of the Trace Method Theorem can now be completed in three steps, and Conclusion (2) will require an additional five steps.

3.1.1. Step one. Suppose \( u \in W^{1,2}_D (\Omega) \), where \( D \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \), and set
\[
f_1 (x, y) \equiv u (x, y) \quad \text{and} \quad f_2 (x, y) \equiv \frac{\partial u}{\partial x} (x, y).
\]
Suppose \(-1 < a < b < 1 \) and \( z \in (-1, 1) \) satisfies
\[
\lim_{I \searrow \{ z \}} \frac{1}{| I |} \int_I \left( \int_{-1}^1 | f_i (x, y) - f_i (x, z) |^2 \, dx \right) \, dy = 0,
\]
\[
\lim_{j \to \infty} \int_{a}^{b} \left| \varphi_{\varepsilon_j} \ast f_i (x, z) - f_i (x, z) \right|^2 \, dx = 0,
\]
where \( F_i (y) \equiv f_i (\cdot, y) \in L^2_{\mathcal{H}} ((-1, 1)) \) for \( i = 1, 2 \), and \( \{ \varphi_{\varepsilon} \}_{\varepsilon > 0} \) denotes a smooth approximate identity in the plane. Then for \(-1 < a < b < 1 \), we have
\[
\| u (\cdot, z) \|_{L^{p} ((a, b))} \leq C \gamma \Gamma.
\]
To see this define
\[
\Phi_{\varepsilon} (z) (x) \equiv \varphi_{\varepsilon} \ast u (x, z) \quad \text{and} \quad \tilde{\Phi}_{\varepsilon} (z) (x) \equiv \varphi_{\varepsilon} \ast \frac{\partial u}{\partial x} (x, z).
\]
Then both \( \Phi_{\varepsilon} (z) (x) \) and \( \tilde{\Phi}_{\varepsilon} (z) (x) \) are smooth functions of \( x \) satisfying \( \Phi_{\varepsilon} (z)' (x) = \tilde{\Phi}_{\varepsilon} (z) (x) \), and so by the Sobolev embedding theorem in dimension one,
\[
\| \Phi_{\varepsilon} (z) \|_{L^{r}_{\mathcal{H}} ((a, b))} \leq C \left( \| \Phi_{\varepsilon} (z) \|_{L^{2} ((a, b))} + \| \Phi_{\varepsilon} (z)' \|_{L^{2} ((a, b))} \right) = C \left( \| \Phi_{\varepsilon} (z) \|_{\mathcal{H}} + \| \tilde{\Phi}_{\varepsilon} (z) \|_{\mathcal{H}} \right) \leq CM_2 \left( \| \Phi_{\varepsilon} \| + \| \tilde{\Phi}_{\varepsilon} \| \right) (z),
\]
and then by (3.1), we have
\[
\| \Phi_{\varepsilon} (z) \|_{L^{r}_{\mathcal{H}} ((a, b))} \leq c M_2 \left( \| \Phi_{\varepsilon} \| + \| \tilde{\Phi}_{\varepsilon} \| \right) (z) \leq c \Gamma.
\]
Now from the uniform boundedness of the \( L^2 (\Omega) \) norms of \( \Phi_{\varepsilon} (z) \), it follows that for \( 0 < \gamma < \frac{1}{2} \), there is a sequence of functions \( \Phi_{\varepsilon_j} (z) \) that converges in \( L^{p} ((a, b)) \) to \( V (z) \in Lip_{\gamma} ((a, b)) \). Thus \( \Phi_{\varepsilon_j} (z) \) also converges in \( L^{2} ((a, b)) \) to \( V (z) \), which by the second line in (3.1) coincides with the function \( x \to u (x, z) \), i.e. the function \( U (z) = u (\cdot, z) \). This completes the proof of (3.2).

3.1.2. Step two. Fix a smooth approximate identity \( \{ \varphi_{\varepsilon} \}_{\varepsilon > 0} \) in the plane and consider the functions \( f_i \in L^2 (\Omega) \) in Step one. We claim there are points \( c \in (-1, -\frac{1}{2}) \) and \( d \in (\frac{1}{2}, 1) \), and a sequence \( \{ \varepsilon_j \}_{j=1} \), such that for \( z \in \{ c, d \} \),
\[
\lim_{I \searrow \{ z \}} \frac{1}{| I |} \int_I \left( \int_{-1}^1 | f_i (x, y) - f_i (x, z) |^2 \, dx \right) \, dy = 0,
\]
\[
\lim_{j \to \infty} \int_{c}^{d} \left| \varphi_{\varepsilon_j} \ast f_i (x, z) - f_i (x, z) \right|^2 \, dx = 0, \quad -1 < q < r < 1,
\]
\[
M_2 F_i (z) \leq \sqrt{\Pi} \| F_i \|_{L^2 (\mathcal{H}_{((-1,1))})}.
\]
Indeed, the first two lines of (3.4) hold almost everywhere; the first line since the set of Lebesgue points have full measure, and the second line follows from
\[ \lim_{\varepsilon \to 0} \int_s^t \left\{ \int_q^r \| \varphi_\varepsilon * f_i (x, z) - f_i (x, z) \|^2 \, dx \right\} \, dz = 0, \quad -1 < q < r < 1, -1 < s < t < 1, \]
since the square root of the integral in braces is a function of \( z \) that converges to 0 in \( L^2 ((s, t)) \), and hence has an almost everywhere pointwise convergent to 0 sequence \( \{ \varepsilon_j \}_{j=1}^\infty \). The third line follows from the weak type estimate for the maximal operator \( M \),
\[ |\{ y \in (-1, 1) : M_2 F_i (y) > \lambda \}| \leq \frac{5}{\lambda^2} \| F_i \|^2 \mathbb{L}^2((-1,1)) \]
since then
\[ |\{ y \in (-1, 1) : M_2 F_i (y) > \sqrt{11} \| F_i \|^2 \mathbb{L}^2((-1,1)) \}| < \frac{1}{2}, \]
which shows there exist points \( c \in (-1, -\frac{1}{2}) \) and \( d \in (\frac{1}{2}, 1) \) satisfying (3.3).

Before proceeding to Step three, we give a lemma which will play a crucial role in both Steps three and seven.

**Lemma 2.** Given \( v \in W^{1,2}_f (\Omega) \), \( \Omega_{q,r}^{s,t} \subset \Omega \), and a smooth approximate identity \( \{ \varphi_\varepsilon \}_{0 < \varepsilon < 1} \), we have
\[ (\varphi_\varepsilon * v)_+ \to v_+ \]
in the norm of \( W^{1,2}_f (\Omega_{q,r}^{s,t}) \) as \( \varepsilon \to 0 \).

**Proof.** Let \( Y \) be a \( C^1 \) vector field on \( \Omega \). Since \( \Omega_{q,r}^{s,t} \subset \Omega \), we have by a result of Friedrichs [Frie], see also [GaNa] see (A.1) in the Appendix, that the commutator \( \tilde{Y} \) is an integral operator from \( L^2 (\Omega) \) to \( L^2 (\Omega_{q,r}^{s,t}) \) such that \( \| \tilde{Y} \| \mathbb{L}^2(\Omega_{q,r}^{s,t}) \to 0 \) as \( \varepsilon \to 0 \) for all \( w \in L^2 (\Omega) \). Using this with \( Y \) equal to each of the vector fields \( \partial_x \) and \( f (x) \partial_y \), we then obtain
\[ (\varphi_\varepsilon * v)_+ \to v_+ \]
in \( W^{1,2}_f (\Omega_{q,r}^{s,t}) \).

We must show that both
\[ \nabla_f [v_+] \to v_+ \quad \text{in} \quad L^2 (\Omega_{q,r}^{s,t}), \]
\[ \nabla_f [(\varphi_\varepsilon * v)_+] \to v_+ \quad \text{in} \quad L^2 (\Omega_{q,r}^{s,t}). \]

We begin by using the dominated convergence theorem to prove the first line in (3.4). Indeed, pick any decreasing sequence \( \{ \varepsilon_k \}_{k=1}^\infty \subset (0, 1) \) with \( \lim_{k \to \infty} \varepsilon_k = 0 \). From (3.4), we see that there is a subsequence converging pointwise almost everywhere, and we will continue to denote the subsequence by \( \{ \varepsilon_k \}_{k=1}^\infty \). Now let \( \mathcal{L}[v] \) denote the set of Lebesgue points of \( v \), and note that
\[ \left\{ x \in \Omega_{q,r}^{s,t} : \lim_{k \to \infty} \varphi_{\varepsilon_k} * v (x) = v (x) \right\} \subset \mathcal{L}[v], \]
and of course \( \Omega_{q,r}^{s,t} \setminus \mathcal{L}[v] = 0 \). On the set \( \mathcal{L}[v] \) we have
\[ \lim_{k \to \infty} \left[ \varphi_{\varepsilon_k} * v (x) \right]_+ = [v (x)]_+, \quad x \in \mathcal{L}[v], \]
and by [Ste] Theorem 2 on page 62], the supremum over \( k \) satisfies
\[ \sup_{1 \leq k < \infty} \left[ \varphi_{\varepsilon_k} * v (x) \right]_+ \leq CM \bar{v} (x), \quad x \in \Omega_{q,r}^{s,t}, \]
where \( M \) is the maximal function. Since \( \bar{v} \in L^2 (\Omega) \) by the maximal theorem, the dominated convergence theorem now yields the first line in (3.4).

To prove the second line in (3.4), we use identities from [SaW3] see (33) on page 1886, which the reader can easily verify translate into the following in our notation,
\[ \nabla_f (v_+) = 1_{\{ v > 0 \}} \nabla_f v \quad \text{and} \quad \nabla_f [(\varphi_\varepsilon * v)_+] = 1_{\{ \varphi_\varepsilon * v > 0 \}} \nabla_f (\varphi_\varepsilon * v). \]

We claim the same argument as above now yields the limit
\[ 1_{\{ \varphi_\varepsilon * v > 0 \}} \nabla_f (\varphi_\varepsilon * v) \to 1_{\{ v > 0 \}} \nabla_f (v) \quad \text{in} \quad L^2 (\Omega_{q,r}^{s,t}). \]
Indeed, we see from (3.4) that
\[ \varphi_{\varepsilon} \ast \nabla f v \rightarrow \nabla f v \text{ in } L^2(\Omega_{q,r}^{c,d}). \]
Thus every decreasing sequence in $(0, 1)$ with limit 0 at 0, has a subsequence $\{\varepsilon_k\}_{k=1}^\infty$ such that
\[ \nabla f (\varphi_{\varepsilon_k} \ast v) = \varphi_{\varepsilon_k} \ast \nabla f v \rightarrow \nabla f v, \quad \text{pointwise almost everywhere in } \Omega_{q,r}^{c,d}. \]
Then
\[ \left\{ x \in \Omega_{a,b}^{c,d} : \lim_{k \rightarrow \infty} \varphi_{\varepsilon_k} \ast \nabla f v(x) = \nabla f v(x) \right\} \subset \mathcal{L} \left[ \nabla v \right], \]
where $|\Omega_{q,r}^{c,d} \setminus \mathcal{L}[v]| = 0$, and it is easily checked that
\[ 1_{\{\varphi_{\varepsilon} \ast v > 0\}} \nabla f (\varphi_{\varepsilon} \ast v) \rightarrow 1_{\{v > 0\}} \nabla f v, \quad \text{pointwise almost everywhere in } \Omega_{q,r}^{c,d}. \]
Thus from (3.6) we have $\nabla f \left[ (\varphi_{\varepsilon} \ast v)_{+} \right] \rightarrow \nabla f (v_{+})$ pointwise almost everywhere in $\Omega_{q,r}^{c,d}$, and then using $M \nabla f v \in L^2(\Omega_{q,r}^{c,d})$, the dominated convergence theorem yields the second line in (3.5).

3.1.3. Step three. Under the hypotheses of Trace Method Theorem, and with $c, d$ as in Step two, we claim that
\[ u \in L^\infty \left( \Omega_{a,b}^{c,d} \right), \]
which then completes the proof of Conclusion (1) of the Trace Method Theorem.

To prove this we begin with the following lemma.

Lemma 3. Suppose that $u \in W_1^{1,2} \left( (-1, 1)^2 \right) \cap C^\infty \left( (-1, 1)^2 \setminus \text{y-axis} \right)$ satisfies $\nabla A \nabla u = \phi$, where $\phi$ is $f$-admissible, and where $A(x, y) \approx D_f(x)$. Choose $c, d$ as in (3.5) and choose
\[ a = \frac{3}{4} \quad \text{and} \quad b = \frac{3}{4}. \]
Then if $\Omega_{a,b}^{c,d} \equiv (a, b) \times (c, d)$, we have that $u$ is bounded in the weak sense on $\partial \Omega_{a,b}^{c,d}$, i.e. there is a constant $\ell$ such that $(u - \ell)^{+} \in \left( W_1^{1,2} \left( \Omega_{a,b}^{c,d} \right) \right)^{+}$, and in fact one can take
\[ \ell \equiv C' \max \left\{ \|u\|_{W_1^{1,2}(\Omega)}, \|\phi\|_{L^q_{\text{growth}}(\Omega)} \right\}. \]

Proof. By (3.2) and (3.3), we have for $z \in \{c, d\}$ that
\[ \left\| \varphi_{\varepsilon} \ast u(x, z) \right\|_{L^\infty((a, b))} \leq C' \|u\|_{W_1^{1,2}(\Omega)} \leq C' \|u\|_{W_1^{1,2}(\Omega)}. \]
Then from ellipticity away from the $y$-axis, we have for $t \in \{a, b\}$ that
\[ \left\| \varphi_{\varepsilon} \ast u(t, y) \right\|_{L^\infty((c, d))} \leq C' \left( \|u\|_{L^2(\Omega)} + \|\phi\|_{L^q_{\text{growth}}(\Omega)} \right). \]
Define $\ell \equiv C' \max \left\{ \|u\|_{W_1^{1,2}(\Omega)}, \|\phi\|_{L^q_{\text{growth}}(\Omega)} \right\}$. Since $\varphi_{\varepsilon} \ast u(x, y)$ is a smooth function in $\Omega$ provided $2\varepsilon < \min\{a - 1, 1 - b, c - 1, 1 - d\}$, the above inequalities imply
\[ \left| \varphi_{\varepsilon} \ast u(x, y) \right|_{\partial \Omega_{a,b}^{c,d}} \leq \frac{1}{2} \ell. \]
This gives
\[ \left( \varphi_{\varepsilon} \ast u(x, y) - \frac{1}{2} \ell \right)^{+} = 0 \quad \text{on} \quad \partial \Omega_{a,b}^{c,d}, \]
and by continuity,
\[ \supp \left( \varphi_{\varepsilon} \ast u(x, y) - \ell \right)^{+} \subseteq \Omega_{c,d}^{c,d}. \]
Thus we have
\[ \left( \varphi_{\varepsilon} \ast u(x, y) - \ell \right)^{+} \in \left( W_1^{1,2} \right)_{0} \left( \Omega_{a,b}^{c,d} \right), \]
and it remains to show that
\[ (3.8) \quad \left( \varphi_{\varepsilon} \ast u - \ell \right)^{+} \rightarrow (u - \ell)^{+}. \]
in the norm of $W^{1,2}_f(\Omega)$ as $\varepsilon \to 0$. Indeed, since $\left(W^{1,2}_f(\Omega)\right)_0$ is closed in $W^{1,2}_f(\Omega)$, we would then conclude that $(u - l)^+ \in \left(W^{1,2}_f(\Omega)\right)_0$ as required. To prove (3.8), we note that since $\varphi_{x_j} \ast (u(x, y) - \ell) = \varphi_{x_j} \ast (u(x, y) - \ell)$, we may assume without loss of generality that $\ell = 0$, and then Lemma 2 applies to finish the proof of Lemma 3.

We now use Lemma 3 together with Steps one and two, to obtain that $u$ is bounded in the weak sense on $\partial \Omega^{c,d}_\varepsilon$. Finally then, we apply the assumed $\mathcal{MPP}$ for the equation $Lu = \phi$ to conclude that $u$ is bounded in $\Omega^{c,d}_\varepsilon$.

$$\sup_{\Omega^{c,d}_\varepsilon} u \leq C' \left( \|u\|_{W^{1,2}_f(\Omega)} + \|\phi\|_{L^q_{\phi\text{-admissible}}(\Omega)} + CR \right).$$

3.2. Continuity of weak solutions. Here we will prove Conclusion (2) of the Trace Method Theorem in five additional steps. Steps five and seven are refinements of Steps two and three respectively.

3.2.1. Step four. Here we assume that $\phi$ is $f$-admissible in $\Omega$, and in addition satisfies the following property:

$$\phi(x, y) \text{ has modulus of continuity } \rho \text{ in } y, \text{ uniformly in } x.$$  

Let $\omega(\delta)$ be a modulus of continuity with $\omega \geq \rho$, and set $v_\delta(x, y) \equiv u(x, y + \delta)$. Consider the difference operators

$$D_{e_2, \delta} u(x) = \frac{u(x, y + \delta) - u(x, y)}{\omega(\delta)} = \frac{v_\delta(x, y) - u(x, y)}{\omega(\delta)}$$

in the $y$-variable. We claim that if $u$ is a weak solution to $Lu = \phi$, then

$$w = D_{e_2, \delta} u \text{ is a weak solution to } \left\{ \begin{array}{ll}
Lu = 0 & \text{if } \phi(x, y) \text{ is independent of } y \\
Lu = \eta \in L^\infty & \text{if } \phi(x, \cdot) \in \text{Lip}_\rho \text{ uniformly in } x.
\end{array} \right.$$  

To see this, note that

$$L \left(D_{e_2, \delta} u \right)(x) = \frac{Lv_\delta(x, y) - Lu(x, y)}{\omega(\delta)} = \frac{\nabla A(x, y) \nabla u(x, y + \delta) - Lu(x, y)}{\omega(\delta)} = \frac{\nabla A(x, y + \delta) \nabla u(x, y + \delta) - Lu(x, y)}{\omega(\delta)} = \frac{Lu(x, y + \delta) - Lu(x, y)}{\omega(\delta)} \frac{\phi(x, y + \delta) - \phi(x, y)}{\omega(\delta)},$$

since $A(x)$ is independent of $y$. Then if $\phi(x, y)$ is independent of $y$, we have $L \left(D_{e_2, \delta} u \right)(x) \equiv 0$, while if $\phi(x, \cdot) \in \text{Lip}_\rho$ uniformly in $x$, then $L \left(D_{e_2, \delta} u \right)(x)$ is bounded.

3.2.2. Step five (a refinement of Step two). Fix $-1 < a < -\frac{3}{4}$ and $\frac{3}{4} < b < 1$. We claim that for every $0 < \gamma < \frac{1}{4}$, there is a positive constant $C_\gamma$ with the property that for every $0 < \delta < \frac{1}{4a}$, there is a set $\Theta_\delta \equiv \{c_\delta, c_3 + \delta, d_\delta, d_\delta + \delta\}$ of four points with $-1 + \delta < c_\delta < -\frac{3}{2}$ and $\frac{1}{2} < d_\delta < 1 - \delta$, such that $\Omega^{c_\delta, d_\delta}_\delta \subseteq \Omega$ and

$$\|u(\cdot, y)\|_{\text{Lip}_\rho(a, b)} \leq C_\gamma \|u\|_{W^{1,2}_f(\Omega)} \quad \text{for all } u \in W^{1,2}_f(\Omega) \text{ and all } y \in \Theta_\delta.$$  

To prove this, we first note that if $F \in L^2_{\text{loc}}((-1, 1))$, we can realize $F$ as a real-valued function $f(x, y)$ defined on $\Omega$ with

$$F(y) = f(x, y), \quad \text{for } (x, y) \in \Omega,$$

$$\|F\|_{L^2_{\text{loc}}((-1, 1))} = \sqrt{\int_{-1}^{1} \left( \int_{-1}^{1} |f(x, y)|^2 \, dx \right) \, dy} = \|f\|_{L^2(\Omega)}.$$
Next fix $0 < \delta < \frac{1}{10}$, a smooth approximate identity $\{\varphi_\varepsilon\}_{\varepsilon > 0}$ in the plane, and points $c \in (-1, -\frac{1}{2})$ and $d \in (\frac{1}{2}, 1)$. We now use Lemma 1 to choose $c_\delta \in (c, -\frac{1}{2})$ and $d_\delta \in (\frac{1}{2}, d)$ and a sequence $\{\varepsilon_j\}_{j=1}^\infty$ such that for
\[
z \in \Theta_\delta \equiv \{c_\delta, c_\delta + \delta, d_\delta, d_\delta + \delta\},
\]
we have
\[
\lim_{\varepsilon \to 0} \int_{\Delta(z)} \frac{1}{|I|} \int_I \left( \int_{-1}^1 |f(x, y) - f(x, z)|^2 \, dx \right) \, dy = 0, \\
\lim_{j \to \infty} \int_a^b |\varphi_{\varepsilon_j} \ast f(x, z) - f(x, z)|^2 \, dx = 0,
\]
\[
M_2 F(z) \leq \sqrt{101} \|F\|_{L^2_\delta((-1,1))}.
\]

To this end, we first note that the set $E$ of points $z$ for which the first two lines of (3.12) hold is a set of full measure, since the first line holds for Lebesgue points, and since the second line follows from
\[
\frac{\max \{M_2 F(c_\delta), M_2 F(c_\delta + \delta), M_2 F(d_\delta), M_2 F(d_\delta + \delta)\}}{\lambda} = \frac{\sqrt{101} \|F\|_{L^2_\delta((-1,1))}}{\lambda}.
\]
It now follows that for each $j \in \mathbb{N}_{\text{odd}}$ with $j\delta < \frac{1}{10}$, almost every pair of points $(c_\delta, d_\delta)$ in $E \times E$ satisfying
\[
-1 + j\delta \leq c_\delta < -1 + (j + 1)\delta, \\
1 - (j + 1)\delta \leq d_\delta < 1 - j\delta,
\]
has the property that
\[
\max \{M_2 F(c_\delta), M_2 F(c_\delta + \delta), M_2 F(d_\delta), M_2 F(d_\delta + \delta)\} > \lambda \equiv \sqrt{101} \|F\|_{L^2_\delta((-1,1))}.
\]
It now follows that for each $j \in \mathbb{N}_{\text{odd}}$ with $j\delta < \frac{1}{10}$, at least one of the pairwise disjoint sets
\[
\{M_2 F > \lambda\} \cap [-1 + j\delta, -1 + (j + 1)\delta), \\
\{M_2 F > \lambda\} \cap [-1 + (j + 1)\delta, -1 + (j + 2)\delta), \\
\{M_2 F > \lambda\} \cap [1 - (j + 1)\delta, 1 - j\delta), \\
\{M_2 F > \lambda\} \cap [1 - (j + 2)\delta, 1 - (j + 1)\delta),
\]
has measure at least $\frac{1}{4}\delta$. Thus we have
\[
\left|\{M_2 F > \lambda\}\right| \geq \sum_{j \in \mathbb{N}_{\text{odd}}, j\delta < \frac{1}{10}} \frac{1}{4}\delta \geq \frac{1}{20},
\]
by the pairwise disjointedness of these collections of sets in $j$, and together with the weak type estimate for $M_2$ in Lemma 1 we obtain
\[
\frac{1}{20} \leq \left|\{y \in (c, d) : M_2 F(y) > \lambda\}\right| \leq \frac{5}{\lambda^2} \|F\|^2_{L^2_\delta((-1,1))},
\]
which contradicts the choice $\lambda = \sqrt{101} \|F\|_{L^2_\delta((-1,1))}$. This completes the proof of (3.12).

We can also adapt the above argument to show that if, instead of a single function $f$, we have two functions $F_1, F_2 \in \mathcal{H} = L^2((-1, 1))$, then we can choose $c_\delta \in (c, -\frac{1}{2})$ and $d_\delta \in (\frac{1}{2}, d)$ such that
\[
\lim_{\varepsilon \to 0} \int_{\Delta(z)} \frac{1}{|I|} \int_I \left( \int_{-1}^1 |f_i(x, y) - f_i(x, z)|^2 \, dx \right) \, dy = 0, \\
M_2 F_i(z) \leq 2\sqrt{101} \|F_i\|_{L^2_\delta((-1,1))},
\]
for $i = 1, 2$, and $z = c_\delta, c_\delta + \delta, d_\delta, d_\delta + \delta$.
Indeed, if $M_2 F_i (z) \leq \sqrt{401} \| F_i \|_{L^2_k((-1, 1))}$ fails for the four choices of $z$ and each $i = 1, 2$, then there will be eight pairwise disjoint sets instead of four in the above argument, and we obtain

$$\frac{1}{40} \leq |\{y \in (c, d) : \max \{M_2 F_1 (y), M_2 F_2 (y)\} > \lambda\}| \leq 2 \frac{5}{\lambda^2} \| F_i \|_{L^2_k((-1, 1))}.$$  

Now we use (3.13) to apply Step one with

$$\Gamma = \sqrt{401} \left(\|u\|_{L^2(\Omega)} + \| \frac{\partial u}{\partial x}\|_{L^2(\Omega)}\right) = \sqrt{401} \|u\|_{W^{1, 2}(\Omega)},$$

in order to obtain the uniform boundedness of the $Lip_{\kappa} ((a, b))$ norms of $\Phi_{\varepsilon} (z)$. Then it follows that for $0 < \gamma < \frac{1}{4}$, there is a sequence of functions $\Phi_{\varepsilon_j} (z)$ that converges in $Lip_{\gamma} ((a, b))$ to $V (z) \in Lip_{\gamma} ((a, b))$. Thus $\Phi_{\varepsilon_j} (z)$ also converges in $L^\alpha ((a, b))$ to $V (z)$, which by the second line in (3.12) coincides with the function $x \rightarrow u (x, z)$, i.e. $U (z) = u (\cdot, z)$. This completes the proof of (3.11).

3.2.3. Step six. Recall that for a modulus of continuity $\omega$, we defined the corresponding difference operator $D_{\omega, \delta}^z$ in the direction $e_2$ by

$$D_{\omega, \delta}^z u (x) \equiv \frac{u(x+y+\delta)-(x,y)}{\omega(\delta)}.$$  

Then given $L$ as above and $-1 < a < -\frac{1}{4}, \frac{1}{4} < b < 1$, we claim there is a modulus of continuity $\omega_f (\delta)$ defined for $0 < \delta < \frac{1}{4}$ (see (3.13) below for an explicit formula) such that

$$(3.14) \quad \|D_{\omega, \delta}^z u (\cdot, z)\|_{L^\infty((a,b))} \leq C_0 C_1 \left(\|u\|_{W^{1, 2}(\Omega)} \|\phi\|_{L^q(\Theta_{\beta})}, C_R\right),$$

for all weak solutions $u$ to $Lu = \phi$ and points $z \in \Theta_{\beta}$, where the constant $C_R$ is that arising in the MPP.

To prove this we will apply a classical Hölder estimate for elliptic equations in small balls arbitrarily close to the singular $y$-axis, which we now describe. Without loss of generality we may assume $x \geq 0$, and then fix $\beta > 0$. Consider a Euclidean ball $B_{Euc} (x + 2 \beta, y)$ and a Euclidean ball $RB_{Euc} (x + 2 \beta, y), R)$ concentric with $B_{Euc}$ and having radius $0 < R \leq \beta$. Let $\phi$ be $f$-admissible. Then in particular, $\phi \in L^q(\Omega)$ for some $q > \frac{1}{2}$. Fix such a $q$ and set

$$M \equiv \|\phi\|_{L^q(\Theta_{\beta})}. $$

For elliptic operators, we have from Theorem 8.22 in [GT] with notation used there, the following classical Hölder estimate for weak solutions,

$$(3.15) \quad \text{osc}_{RB_{Euc}} u \leq \frac{1}{\gamma} \left(\left(\frac{R}{\beta}\right)^\alpha \sup_{B_{Euc}} |u| + k R^\alpha\right), \quad R < \beta.$$  

We now derive bounds for both $\gamma$ and $\alpha$ from an examination of the proofs in [GT]. Theorems 8.17 and 8.18 in [GT] immediately yield the following Harnack inequality:

$$\sup_{B_{Euc}} u \leq C_H \left(\inf_{B_{Euc}} u + k\beta^{1-\frac{\alpha}{\gamma}}\right),$$

with $C_H \equiv C(n)\beta^\frac{\alpha}{\gamma}$, $k = \frac{C\|\phi\|_{L^q(B_{Euc})}}{\lambda}$,

and so $k$ can be bounded above by $\frac{CM}{\lambda}$. For homogeneous equations, where $k = 0$, this is recorded as Theorem 8.20 in [GT]. An inspection of the inequality at the top of page 202 in [GT] now shows that we can take $\gamma = 1 - \frac{1}{\alpha} \geq \frac{1}{2}$ if $C_H \geq 2$, and

$$\alpha \approx -\ln \left(1 - \frac{1}{2C_H}\right),$$

since $\alpha = (1 - \mu) \frac{\log \gamma}{\log \tau}$ in the notation of [GT].
With the Hölder estimate (3.15) in hand, we now note that from Conclusion (1) of the Trace Method Theorem, already proved in Step three, it follows that for any Euclidean ball $B_{\text{Euc}}$,

$$\sup_{B_{\text{Euc}}} u \leq C \left\{ \| u \|_{W^{1,2}_f(\Omega)} + \| \phi \|_{L^q_{\text{growth}}(\Omega)} + C_R \right\},$$

with $C$ independent of $\lambda$. Combining this with (3.16) and the lower estimate for $\gamma$ gives

$$\sup_{B_{\text{Euc}}(R)} \osc u \leq C \left( \left( \frac{R}{\beta} \right)^{\alpha} \left( \| u \|_{W^{1,2}_f(\Omega)} + \| \phi \|_{L^q_{\text{growth}}(\Omega)} + C_R \right) + kR^\alpha \right),$$

with the constant $C$ independent of $\lambda$.

Our operator is elliptic in $B_{\text{Euc}}$ with the ellipticity constant satisfying

$$\lambda \geq f(\beta)^2.$$

Therefore, for $0 < \delta \leq \beta$, and using $k \leq \frac{CM_f}{f(\beta)^2}$, we have

$$|u(x + 2\beta, y + \delta) - u(x + 2\beta, y)| \leq \sup_{B_{\text{Euc}}(x + 2\beta, y)} \osc u \leq C \left( \frac{\delta}{\beta} \right)^{\alpha(\beta)} \left( \| u \|_{W^{1,2}_f(\Omega)} + \| \phi \|_{L^q_{\text{growth}}(\Omega)} + C_R \right) + C \| \phi \|_{L^q_{\text{growth}}(\Omega)} \frac{\delta^{\alpha(\beta)}}{f(\beta)^2},$$

where

$$\alpha(\beta) = -C' \ln \left( 1 - C^{-\frac{1}{f(\beta)^2}} \right).$$

We now need to choose $\delta = \delta(\beta)$ such that

$$\left( \frac{\delta}{\beta} \right)^{\alpha(\beta)} \to 0 \quad \text{and} \quad \frac{\delta^{\alpha(\beta)}}{f(\beta)^2} \to 0 \quad \text{as} \quad \beta \to 0.$$

To satisfy the first condition it is sufficient to require

$$-\ln \left( 1 - C^{-\frac{1}{f(\beta)^2}} \right) \ln \frac{\delta}{\beta} \to -\infty \quad \text{as} \quad \beta \to 0,$$

$$C^{-\frac{1}{f(\beta)^2}} \ln \frac{\delta}{\beta} \to -\infty \quad \text{as} \quad \beta \to 0,$$

$$\ln \frac{\delta}{\beta} = -C' \frac{1}{f(\beta)^2},$$

which holds for $\delta = \Gamma_f(\beta)$ where

$$\Gamma_f(\beta) \equiv \beta \exp \left( -\exp \left( \frac{C'}{f(\beta)^2} \right) \right).$$

Note that as $\beta \to 0$ we have $\delta \to 0$ very quickly. Thus, as $\delta \to 0$, $\beta \to 0$ very slowly. We now verify that with $\delta$ as above we also have

$$\frac{\delta^{\alpha(\beta)}}{f(\beta)^2} \to 0 \quad \text{as} \quad \beta \to 0.$$

Passing to logarithms again we have

$$\ln \left( \frac{\delta^{\alpha(\beta)}}{f(\beta)^2} \right) = \alpha(\beta) \ln \delta + 2 \ln \frac{1}{f(\beta)} = C' \ln \left( 1 - C^{-\frac{1}{f(\beta)^2}} \right) \left( \exp \left( \frac{C'}{f(\beta)^2} \right) + \ln \frac{1}{\beta} \right) + 2 \ln \frac{1}{f(\beta)} \approx -\exp \left( -\frac{C}{f(\beta)^2} \right) \left( \exp \left( \frac{C'}{f(\beta)^2} \right) + \ln \frac{1}{\beta} \right) + 2 \ln \frac{1}{f(\beta)}$$

and the expression converges to $-\infty$ as $\beta \to 0$ provided we choose $C'$ sufficiently large.
We now calculate a modulus of continuity $\omega(\delta)$ that ensures the function $D_{e_2,\delta}^\omega u(x, y)$ is bounded uniformly for $y \in \{c_3, d_3\}$. Using (3.16) on the second term of line 2, and (3.11) on the first and third terms of line 2, we have for $0 < x < x + 2\beta < b$ and $0 < \delta < \frac{1}{2}\beta$ (and similarly for $x < 0$),

\[
|D_{e_2,\delta}^\omega u(x, y)| = \frac{1}{\omega(\delta)} |u(x, y + \delta) - u(x, y)| \\
\leq \frac{C_0}{\omega(\delta)} \left\{ \beta \gamma + \left( \frac{\delta}{\beta} \right)^{\alpha(\beta)} + \frac{\delta^{\alpha(\beta)}}{(\beta f(\beta)^{2} + \beta \gamma)} \right\} \leq C_0,
\]

where $C_0$ depends on $\|u\|_{W^1,2(\Omega)}$, $\|f\}_{f-\text{adm}(\Omega)}$ and $C_R$, provided we choose $\omega(\delta) = \omega_f(\delta)$ where

\[
\omega_f(\delta) \equiv \Gamma_f^{-1}(\delta)^{\gamma} + \left( \frac{\delta}{\Gamma_f^{-1}((\delta))} \right)^{\alpha(\Gamma_f^{-1}((\delta)))} + \frac{\delta^{\alpha(\Gamma_f^{-1}((\delta)))}}{f(\Gamma_f^{-1}((\delta)))^{2}}.
\]

Note that the modulus of continuity $\omega_f(\delta)$ is increasing on $(0, 1)$ and satisfies $\lim_{\delta \searrow 0} \omega_f(\delta) = 0$.

**Remark 1.** The constant $C_R$ in the Step six arguments above can be replaced

1. by $\|\phi\|_{X_f(\Omega)}$ if we are using the maximum principle in Theorem 7
2. by 0 if we are using the homogeneous maximum principle in Theorem 4

3.2.4. Step seven (a refinement of Step three). We claim that with $\omega(\delta) \equiv \Gamma(\delta, \rho)$, where $\rho$ is the modulus of continuity of $\phi(x, \cdot)$ as in (3.19), we have

\[
\|D_{e_2,\delta}^\omega u\|_{L^\infty(\Omega,\mathbb{R}^n)} \leq C, \quad \text{with a constant } C \text{ independent of } 0 < \delta < \frac{1}{10}.
\]

To prove this we will apply the assumed Maximum Principle Property in $\Omega$ to the function $D_{e_2,\delta}^\omega u(x, y)$, and for this in turn we will need the following lemma.

**Lemma 4.** Suppose that $u \in W^{1,2}_f\left((-1, 1)^2 \cap \overline{C^\infty((-1, 1)^2 \setminus \text{y-axis})}\right)$ satisfies $\nabla A \nabla u = \phi$, where $\phi$ is $f$-admissible and satisfies (3.14), and $A(x) \approx D_f(x)$. Let $0 < \delta < \frac{1}{10}$, and choose $c_3, d_3 + \delta, d_4, d_5 + \delta$ as in (3.17) and choose $a = -\frac{\delta}{2}$ and $b = \frac{\delta}{2}$. Then with $\Omega_{a,b}^\pm = (-\frac{\delta}{2}, \frac{\delta}{2}) \times (a, b)$, we have that $v_\delta = D_{e_2,\delta}^\omega u$ is bounded in the weak sense on $\partial \Omega_{a,b}^\pm$, i.e. there is a constant $\ell$ such that $(v_\delta - \ell)^+ \in \left(W^{1,2}_f\left(\Omega_{a,b}^\pm\right)\right)_0$.

**Proof.** We fix $\delta$ and write $v = v_\delta$. From (3.14), we have for $z \in \{c_3, d_3\}$ that

\[
\|\varphi\ast v(z, \cdot)\|_{L^\infty(\Omega_0,\mathbb{R}^n)} \leq C_0,
\]

and from (3.10) and ellipticity away from the y-axis, we have for $t \in \{-\frac{\delta}{2}, \frac{\delta}{2}\}$ that

\[
\|\varphi\ast v(t, y)\|_{L^\infty((c,d))} \leq C' \left( \|v\|_{L^2(B_{\text{loc}}(\pm \frac{\delta}{2}, \frac{\delta}{2})))} + \|D_{e_2,\delta}^\omega \phi\|_{L^\infty(B_{\text{loc}}(\pm \frac{\delta}{2}))} \right).
\]

so we get

\[
\|v\|_{L^2(B_{\text{loc}}(\pm \frac{\delta}{2}, \frac{\delta}{2})))} \leq \|D_{e_2,\delta}^\omega u\|_{L^\infty(B_{\text{loc}}(\pm \frac{\delta}{2}, \frac{\delta}{2})))} \leq C' \frac{\delta^{\alpha(\frac{\delta}{2})}}{\omega(\frac{\delta}{2})} \left( \|u\|_{W^{1,2}_f(\Omega)} + \|\phi\|_{f-\text{adm}(\Omega)} \right).
\]

Define

\[
\ell \equiv 2 \max \left\{ C_0, \|u\|_{W^{1,2}_f(\Omega)} + \|\phi\|_{f-\text{adm}(\Omega)} \right\}.
\]
Since $\varphi_\varepsilon \ast v(x, y)$ is a smooth function in $\Omega$ provided $2\varepsilon < \min\{a + 1, 1 - b, c + 1, 1 - d\}$, the above inequalities imply

$$|\varphi_\varepsilon \ast v(x, y)|_{\partial \Omega^{\varepsilon, d}} \leq \frac{1}{2}.$$ 

This gives

$$\left(\varphi_\varepsilon \ast v(x, y) - \frac{1}{2}\right)_+ = 0 \quad \text{on} \quad \partial \Omega^{\varepsilon, d},$$

and by continuity,

$$\text{supp} \left(\varphi_\varepsilon \ast v(x, y) - \ell\right)_+ \in \Omega^{\varepsilon, d}_a,b.$$ 

Thus we have

$$\left(\varphi_\varepsilon \ast v(x, y) - \ell\right)_+ \in \left(W^{1,2}_f\right)_0 \left(\Omega^{\varepsilon, d}_a,b\right),$$

and it remains to show that

$$\left(\varphi_\varepsilon \ast v - \ell\right)_+ \rightarrow (v - \ell)_+$$

in the norm of $W^{1,2}_f(\Omega)$ as $\varepsilon \rightarrow 0$. Indeed, since $\left(W^{1,2}_f(\Omega)\right)_0$ is closed in $W^{1,2}_f(\Omega)$, we would then conclude that $u_\varepsilon^+ = (v - \ell)^+ \in \left(W^{1,2}_f(\Omega)\right)_0$ as required. So it remains to prove (3.19), and since $\varphi_\varepsilon \ast v(x, y) - \ell = \varphi_\varepsilon \ast (v(x, y) - \ell)$, we may assume without loss of generality that $\ell = 0$. Lemma 4 now completes the proof of Lemma 4.

With Lemma 4 in hand, we can now apply the assumed MPP for the equation $Lv = D^2_{e_2, \delta} \phi$ in $\Omega$ to conclude that $D^\omega_{e_2, \delta} u \in L^\infty \left(\Omega^{-\frac{3}{4}, \frac{3}{4}}\right)$ uniformly in $0 < \delta < \frac{1}{10}$.

3.2.5. Step eight. In order to complete the proof of Conclusion (2) of the Trace Method Theorem, it remains to show that

$$u \in \text{Lip}_\omega \left(\Omega^{-\frac{3}{4}, \frac{3}{4}}\right)$$

for the modulus of continuity $\omega(\delta)$ in Steps six and seven.

For this, suppose we are given points $P = (x, y)$ and $P + (\delta_1, \delta_2) = (x + \delta_1, y + \delta_2)$, both near the origin, and set

$$\delta \equiv \max \left\{\sqrt{\delta_1}, \delta_2\right\}.$$ 

Then choose a ‘$\delta$-good’ point $z$ near $y$, i.e. such that

$$|z - y| \leq \delta \quad \text{and} \quad \int_0^1 \left(|u(t, z)|^2 + \left|\frac{\partial u}{\partial x}(t, z)\right|^2\right) dt \leq C^2.$$ 

Indeed, this is possible since if we take $\lambda = \frac{C}{\sqrt{\delta}}$ with $C = \frac{1}{\sqrt{10}\|u\|_{W^{1,2}_f}}$ in the weak type estimate in Lemma 11, we obtain

$$\left\{y \in (c, d) : M_2 \sqrt{|u|^2 + \left|\frac{\partial u}{\partial x}\right|^2} (y) > \frac{C}{\sqrt{\delta}}\right\} \leq \frac{5 \left(\|u\|_{L^2_f((c, d))}^2 + \left|\frac{\partial u}{\partial x}\right|^2_{L^2_f((c, d))}\right)}{C^2} \delta = \frac{\delta}{2},$$

$$0 < \delta < \frac{1}{10},$$

and hence conclude that there is $z \in (y, y + \delta)$ with $M_2 \sqrt{|u|^2 + \left|\frac{\partial u}{\partial x}\right|^2} (z) \leq \frac{C}{\sqrt{\delta}}$.

Now we apply (3.2) in Step one with $\frac{1}{4} < \gamma < \frac{1}{2}$ to obtain the inequality

$$|u(x + \delta_1, z) - u(x, z)| \leq C\gamma \delta^2 \frac{C}{\sqrt{\delta}} \leq C\gamma \delta^{2\gamma - \frac{1}{2}}.$$
Altogether then we have using Step seven that
\[
\begin{align*}
|u(x + \delta_1, y + \delta_2) - u(x, y)| & \leq |u(x + \delta_1, y + \delta_2) - u(x + \delta_1, z)| + |u(x + \delta_1, z) - u(x, z)| + |u(x, z) - u(x, y)| \\
& \leq \omega(|y + \delta_2 - z|) + C_\gamma \delta^{2\gamma - \frac{1}{2}} + \omega(|y - z|) \leq 2\omega(\delta) + C_\gamma \delta^{2\gamma - \frac{1}{2}},
\end{align*}
\]
which completes the proof of Step eight since for $\frac{1}{4} < \gamma < \frac{1}{2}$, we have $2\omega(\delta) + C_\gamma \delta^{2\gamma - \frac{1}{2}} \leq C'\omega(\delta)$ for $\delta > 0$ sufficiently small. The proof of the Trace Method Theorem 3 is now complete.

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