Assigning tolerances to J-values used in safety analysis

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Abstract. This paper describes the methodology employed in the estimation of the input parameters required for J-value analysis. The conceptual foundations and theory behind J-value analysis are first presented, and the relevant parameters are derived. Evaluations of the parameters are then shown and their implications discussed, including an estimate of the coefficient of relative risk aversion, risk-aversion for short. Uncertainties of the parameters are calculated. It is shown that the internal accuracy of the J-value is +/-4%, but that other external, case-dependant effects may reduce this accuracy. Some of these case-dependant sources of uncertainty are discussed and quantified.

1. Introduction
J-value analysis is a fully objective method of assessing whether or not a given safety system represents good value for money. Given that the safety scheme will prolong human life expectancy for a given cost, the J-value method imputes a maximum reasonable spend for the safety system. This maximum spend also takes into account society’s preferences for risk, in that the majority of people are held to be risk averse in safety matters. The method then computes the ratio of the actual cost of the safety system to the theoretical maximum spend, and returns the J-value, denoted $J$. When $J$ is less than unity then the safety system represents good value for money. When $J$ is greater than unity then the safety system is too expensive. The J-value technique is a general method of assessing any safety system which affects human life span. It has been used to analyse a diverse range of health schemes such as the cost effectiveness of drugs for the treatment of cancer, whether it was sensible to install warning systems on train lines throughout the UK, and to decide the best way of reducing radiation exposure to nuclear power plant workers [1].

The J-value method presents a novel way to measure the cost of physical risk and, as with all measurements, it is important to quantify the level of accuracy and precision that goes with the measurement. When we calculate the theoretical maximum spend, how close to the true value is this measurement? How large are the variations? What kind of errors and biases are inherent in the J-value framework? Here, we aim to address these issues and show how the uncertainties on the input parameters propagate through onto the sole output parameter, the J-value, which then enables us to present $J$ with its tolerance limits. Clearly, the wider the tolerance limits, the less meaningful it is to state whether or not a safety system represents value for money.
2. The J-value framework

The first building block in the J-value framework is the Life Quality Index (LQI), developed by Pandey and Nathwani [2]. This quantifies the overall level of wellbeing of either an individual or a whole society in terms of two relevant predictors of wellbeing. It is postulated that the two most relevant factors influencing the quality of life of a society are its average life expectancy and its GDP per person. At an individual level, this corresponds to (i) how long an individual can expect to live for, and (ii) how much money he will have available to spend, on life’s necessities and on its luxuries. The further stipulation is made that quality of life is more accurately determined from expected free time, rather than expected total time. This is because people value their free time greater than their time spent working. The expected free time, \( T \), can be written in terms of life expectancy \( X \) and the fraction of time spent in work from now on, \( w \):

\[
F = (1 - w)X
\]  

(1)

The income of the individual is denoted as \( G \). These factors are then combined into a Cobb-Douglas utility function [3], to give the LQI as:

\[
Q_1 = G^\alpha F^\beta
\]  

(2)

Where \( \alpha \) and \( \beta \) are positive constants. A property of the Cobb-Douglas function is that any monotonic increasing function of \( Q \) will also suffice as a LQI. Therefore, a new LQI may be defined as:

\[
Q_2 = Q_1^{1/\beta} = G^{\alpha/\beta}T = (1 - w)G^{1 - \varepsilon}X
\]  

(3)

Where \( 1 - \varepsilon = \alpha/\beta \). Here \( \varepsilon \) is the risk-aversion, which is an important economic parameter that characterises an individual’s preferences for risk. A positive value indicates that the individual is risk averse. The individual’s income is also dependent on the fraction of time spent working. It may be shown that [4]:

\[
G = \mathcal{A}w^\theta
\]  

(4)

Where \( \mathcal{A} \) is a constant and \( \theta \) is the fraction of wages in the GDP, and is about 0.6. It is now assumed that members of society have, on average, chosen their fraction of time spent working so that their LQI is optimised. The value of \( w \) which optimises the LQI is denoted as \( w_0 \), and can be derived by substituting (4) into (3), differentiating (3) and setting the result equal to zero (although it will not be proved here one can verify that the result is indeed a maximum). This method gives \( w_0 \) in terms of \( \varepsilon \). However, \( w_0 \) is assumed to be equal to the population-averaged work-time fraction, which can be observed, and so the resulting equation is inverted to estimate \( \varepsilon \) in terms of \( w_0 \):

\[
\varepsilon = 1 - \left( \frac{1}{\theta} \frac{w_0}{1 - w_0} \right)
\]  

(5)
To derive the J-value, it is assumed that, if an individual purchases a safety system, so that he exchanges some of his income for an increase in life expectancy, then his new LQI should be equal to or greater than his initial LQI, i.e.: 

\[ Q_x(G - \delta G, X + \delta X) \geq Q_x(G, X) \quad (6) \]

One can apply condition (6) to (3) to derive a maximum annual individual spend, which is:

\[ -\delta G = \frac{G}{1 - \epsilon} \frac{\delta X}{X} \quad (7) \]

The total spend is this value multiplied by the individual’s life expectancy, X. Furthermore, if a population of N individual’s are affected by the hazard which is being mitigated by the safety system, then each of these individuals will also pay this amount. The total amount paid is denoted as \( \delta V_N \) and is given as:

\[ \delta V_N = -NX\delta G = N \frac{G}{1 - \epsilon} \delta X \quad (8) \]

Where \( X \) and \( G \) are now the average life expectancy and income of the population affected by the hazard. This is the theoretical maximum reasonable a group should be willing to pay to mitigate a risk in order to achieve the given change in life expectancy. The group affected is assumed to be representative of the general UK population, so that \( X \) is the national average life expectancy and \( G \) is the GDP per person. If the actual cost of the system is known, and is denoted as \( \hat{\delta V}_N \), then the J-value is:

\[ J = \frac{\delta V_N}{\hat{\delta V}_N} = \frac{(1 - \epsilon)\delta V_N}{NG\delta X} \quad (9) \]

If the system is to represent good value for money, then the actual cost should be less than the theoretical maximum, so that \( J \leq 1 \). If \( J > 1 \) then the safety system is too expensive.

3. Measurement of the input parameters and their uncertainties

There are five parameters which are required to calculate the J-value. Some of these are in turn estimated from other observable parameters. In order to calculate the tolerance limits of the J-value, the uncertainties for each parameter must be measured, and the way in which they propagate through to the J-value must also be ascertained. The main source of uncertainty in the measurement process presented here is uncertainty due to random error. Random errors are due to imperfect measuring apparatus or imperfect knowledge of the quantity being measured and are observed when repeated measurements of the same parameter shows some dispersion. A wide dispersion would mean there is a large amount of uncertainty. There is also another source of uncertainty, which is due to systematic errors. Systematic errors occur when there is an inherent bias in the measurement process which shifts the result away from its true value. The uncertainty is quantified by the standard deviation. For a
variable $a$, the standard deviation is denoted by $\sigma_a$. The relative standard deviation, or coefficient of variation, $\sigma_a/a$, is also be used.

If a parameter is a function of other variables, then the uncertainty of the variables will affect the uncertainty of the parameter, depending on the function. This is known as ‘propagation of uncertainty’, and can be quantified as such:

If a parameter $a$ is a function of other variables $b, c, d$, etc., i.e.:

$$a = a(b, c, \ldots, z)$$

(10)

Then the variance on $a$, $\sigma_a^2$ is given by:

$$\sigma_a^2 = \left(\frac{\partial a}{\partial b} \sigma_b\right)^2 + \left(\frac{\partial a}{\partial c} \sigma_c\right)^2 + \ldots + \left(\frac{\partial a}{\partial z} \sigma_z\right)^2 + \text{correlations}$$

(11)

And the uncertainty is given by the standard deviation, $\sigma_a$, the square root of equation (11). The ‘correlations’ term represents the contribution to the variance when two or more of the variables are correlated with each other. The propagation of uncertainty is important for all $J$-value parameters and variables, including the output variable, $J$. The methodology used in estimating each parameter is described below.

3.1 The Gross Domestic Product per Person, $G$

The Gross Domestic Product (GDP) of a country is a measure of economic activity. It is the value of all goods and services produced within the country over the year. The GDP per person is the GDP divided by the total population of the country. In the UK, these figures are published annually by the Office for National Statistics (ONS), in a publication entitled “United Kingdom National Accounts: The Blue Book” [5]. The value of $G$, as taken from the Blue Book 2009 is £23,571. To assess the uncertainty of $G$, we first note that it is given by:

$$G = \frac{GDP}{n_{\text{Pop}}}$$

(12)

Both the measurement of the GDP and the population are subject to systematic uncertainty. Furthermore, the GDP and population are also highly correlated – a rising population will tend to be associated with a rising GDP. By applying the propagation of uncertainty conditions, the coefficient of variation on the GDP per person can be calculated as:

$$\frac{\sigma_{\text{GDP}}}{G} = \sqrt{\left(\frac{\sigma_{\text{GDP}}}{n_{\text{Pop}}}\right)^2 + \left(\frac{\sigma_{\text{Pop}}}{GDP \times n_{\text{Pop}}}\right)^2 - 2 \frac{\sigma_{\text{GDP}} \sigma_{\text{Pop}}}{GDP \times n_{\text{Pop}}} \rho_{\text{GDP,Pop}}}$$

(13)

The uncertainty on the GDP measurement is estimated from [6], which gives data on the subsequent revisions in the estimates of the GDP in a previous publication of the Blue Book. It is assumed that the most up to date value of the GDP will be subject to similar revisions, and that this is the major source of uncertainty on the GDP estimate. The total revisions after the initial Blue Book publication are 0.1% of the initial estimate, and so this value will be taken for the coefficient of variation of the GDP.
The uncertainty on the population can also be estimated from data published by the ONS. An analysis performed by the ONS of 2001 Census data showed that there was a 1.5% difference between the population estimates made for that year and the population data received from the ONS [7]. Since the data from the Census is also likely to carry uncertainties, it will be assumed that the true population number is likely to be somewhere between the Census estimate and the ONS estimate. Hence, if the true value lies somewhere between the Census estimate and the ONS estimate, which is 1.5% higher, then it seems reasonable to suppose that the uncertainty on the true population value, wherever it lies, will be the average distance between the Census estimate and the true value, and the ONS estimate and the true value. This means that the coefficient of variation of the population will be half of the total distance between the Census and the ONS estimates, i.e. 0.75%.

The final estimate required to calculate (13) is the correlation coefficient between the GDP and the Population, \( \rho_{\text{GDP},\text{Pop}} \). This can be estimated from the Blue Book time series data [8], which provides the historical values of the GDP and the national population from 1948 to 2008. One may then determine how the two vary together, and hence obtain \( \rho_{\text{GDP},\text{Pop}} \). Performing this calculation gives \( \rho_{\text{GDP},\text{Pop}} = 0.94 \).

Thus using the above the coefficient of variation on the GDP per person is evaluated as 0.7%.

3.2 The Risk-aversion, \( \varepsilon \)

The risk-aversion is given by equation (5), which shows that it is dependant upon two parameters: \( \theta \), the fraction of wages in the GDP and \( w_0 \), the average work-time fraction. Economic theory predicts that \( \theta \) should be constant over time. Data from the UK [9] show reasonable agreement with this prediction. Fig 1. shows the value of \( \theta \) from 1955 to present. Because of the relative stability of \( \theta \), a time averaged value will be used. The standard deviation of this averaged value will then be used for the uncertainty of \( \theta \). The average value over the period 1955 to 2010 is 0.61. However, there is a large peak at 1975, which began in the early 70’s and returned to normal levels during the 80’s. This period corresponds to a period of great industrial unrest in the UK. The period from 1984 to present is more stable, and a better indicator of the future, and so it will be this period used to calculate \( \theta \). The average value for this period is 0.58 and the standard deviation is 0.01. This gives the coefficient of variation in for \( \theta \) as 2%.

The average work-time fraction \( w_0 \) can be shown to be equal to the population averaged future amount of time the population will spend working divided by the population averaged life expectancy [4]. Both of these figures can be estimated from ONS data, and calculations show they are equal to 3.7 y and 41.3 y respectively, giving \( w_0 \) as 0.09, which means that the current population will, on average, spend less than 10% of it’s remaining lifespan in work. The estimation of the uncertainty on \( w_0 \) is very complex, as it requires a detailed analysis of work patterns and death rates throughout the population. The methodology will not be presented here, but may be found in [4]. The coefficient of variation for \( w_0 \) is calculated as 0.004%, which is very small. This is due to the high quality of data from the ONS and large sample population sizes. Applying the propagation of uncertainty condition given by equation (11) allows the uncertainty on the risk-aversion to be calculated as 0.4%
3.3 The Change in Average Discounted Life Expectancy, $\delta X_d$

The change in average life expectancy is dependant upon the specifics of the protection system, and consequently cannot be determined \textit{a priori}. The most general way of estimating the change in life expectancy is by calculating the impact of a hazard reduction on an individual’s life expectancy, which can be done by using the life tables. The life tables give a breakdown of life expectancy, survival probability and the average hazard rate for individuals of a certain age and gender [10]. If a safety scheme affects a large number of people, then the population averaged life-expectancy should be used. If a safety scheme affects a smaller group, for example a group of workers, then the life expectancy averaged over those of working ages should be used.

A typical safety scheme will be applied to a group of people who are exposed to a hazard rate significantly greater than what the average individual is exposed to, and will propose to mitigate the hazard. Typically, a hazard reduction is presented as a reduction in probability of death of “$x$ in 1,000,000 per annum”, for example. This means that if the hazard affects one million people, then the reduction would be expected to save $x$ lives per year, although this statement is misleading as the effects may not be observed as lives saved directly, but rather as a collective increase in life expectancy. If the hazard reduction is denoted as $\delta h$, and length of time which the hazard reduction lasts for is $T$, then the change in life expectancy is approximately given as:

$$\delta X \approx \delta h TX$$

Equation (14) can also be used to calculate the uncertainty on $\delta X$. For most protection systems the main source of uncertainty will be from the hazard reduction as the length of time the reduction lasts for is usually easy to determine and the contribution to uncertainty from the life expectancy $X$ is negligible.
3.4 The Cost of the Safety Scheme, $\delta \hat{\mathbf{V}}_N$, and the Number of People Affected, $N$

Both of these parameters are dependant upon the specifics of the case under analysis, and are not known \textit{a priori}. However, when a protection system is being considered, these parameters are usually known. If the $N$ represents a group of workers, then the company will know how many people it employs. If $N$ is the public population, then the value presented in section 3.1 can be used for the uncertainty.

3.5 The J-Value, $J$

The uncertainty on the J-value will now be presented. Here it will be assumed that there is no uncertainty on the case-dependent parameters $\delta \mathbf{X}$, $N$ and $\delta \mathbf{V}_N$. This calculation may be thought of as representing the internal accuracy of the J-value – that is, the uncertainty contribution from parameters which have we have prior knowledge of. Applying the propagation of uncertainty condition to equation (9) gives the uncertainty of $J$ as:

$$
\frac{\sigma_J}{J} = \sqrt{\left(\frac{\sigma_G}{G}\right)^2 + \left(\frac{\sigma_\varepsilon}{1-\varepsilon}\right)^2}
$$

(15)

Using the numbers shown above, the coefficient of variation on $J$ is calculated as 2%, implying a tolerance of 4% at 2 s.d.

4. Conclusions

The J-value technique for assessing the value for money of protection systems has been shown to have an internal accuracy of about 4%, which, given that the technique is a measurement of the cost of risk to human life, represents good accuracy. However, this accuracy will generally be reduced when specific systems are being considered, due to the case-dependent uncertainties in the accident frequency before and after the protection system has been installed, and also in the system's cost.

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