Viscous Bianchi type I Universes in Brane Cosmology

T. Harko and M. K. Mak

Department of Physics, The University of Hong Kong, Pokfulam Road, Hong Kong

We consider the dynamics of a viscous cosmological fluid in the generalized Randall-Sundrum model for an anisotropic, Bianchi type I brane. To describe the dissipative effects we use the Israel-Hiscock-Stewart full causal thermodynamic theory. By assuming that the matter on the brane obeys a linear barotropic equation of state, and the bulk viscous pressure has a power-law dependence on the energy density, the general solution of the field equations can be obtained in an exact parametric form. The obtained solutions describe generally a non-inflationary brane world. In the large time limit the brane Universe isotropizes, ending in an isotropic and homogeneous state. The evolution of the temperature and of the comoving entropy of the Universe is also considered, and it is shown that due to the viscous dissipative processes a large amount of entropy is created in the early stages of evolution of the brane world.

I. INTRODUCTION

The idea that our four-dimensional Universe might be a three-brane embedded in a higher dimensional space-time has attracted much attention. According to the brane-world scenario, the physical fields in our four-dimensional space-time, which are assumed to arise as fluctuations of branes in string theories, are confined to the three brane, while gravity can freely propagate in the bulk space-time, with the gravitational self-couplings not significantly modified. The model originated from the study of a single 3-brane embedded in five dimensions, with the 5D metric given by $ds^2 = e^{-f(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2$, which can produce a large hierarchy between the scale of particle physics and gravity due to the appearance of the warp factor. Even if the fifth dimension is uncompactified, standard 4D gravity is reproduced on the brane. Hence this model allows the presence of large or even infinite non-compact extra dimensions and our brane is identified to a domain wall in a 5-dimensional anti-de Sitter space-time.

The Randall-Sundrum model was inspired by superstring theory. The ten-dimensional $E_8 \times E_8$ heterotic string theory, which contains the standard model of elementary particle, could be a promising candidate for the description of the real Universe. This theory is connected with an eleven-dimensional theory, M-theory, compactified on the orbifold $R_{10} \times S^1 / Z_2$. In this model we have two separated ten-dimensional manifolds. For a review of dynamics and geometry of brane Universes see [12].

The static Randall-Sundrum solution has been extended to time-dependent solutions and their cosmological properties have been extensively studied [4,5,6,7,8,9,10,11,20,21,27]. In one of the first cosmological applications of this scenario, it was pointed out that a model with a non-compact fifth dimension is potentially viable, while the scenario which might solve the hierarchy problem predicts a contracting Universe, leading to a variety of cosmological problems [13]. By adding cosmological constants to the brane and bulk, the problem of the correct behavior of the Hubble parameter on the brane has been solved by Cline, Grojean and Servant [14]. As a result one also obtains normal expansion during nucleosynthesis, but faster than normal expansion in the very early Universe. The creation of a spherically symmetric brane-world in AdS bulk has been considered, from a quantum cosmological point of view, with the use of the Wheeler-de Witt equation, by Anchordoqui, Nunez and Olsen [15].

The effective gravitational field equations on the brane world, in which all the matter forces except gravity are confined on the 3-brane in a 5-dimensional space-time with $Z_2$-symmetry have been obtained, by using a geometric approach, by Shiromizu, Maeda and Sasaki [16,17]. The correct signature for gravity is provided by the brane with positive tension. If the bulk space-time is exactly anti-de Sitter, generically the matter on the brane is required to be spatially homogeneous. The electric part of the 5-dimensional Weyl tensor $E_{IJ}$ gives the leading order corrections to the conventional Einstein equations on the brane. The four-dimensional field equations for the induced metric and scalar field on the world-volume of a 3-brane in the five-dimensional bulk with Einstein gravity plus a self-interacting scalar field have been derived by Maeda and Wands [18].

The linearized perturbation equations in the generalized Randall-Sundrum model have been obtained, by using the covariant nonlinear dynamical equations for the gravitational and matter fields on the brane, by Maartens [19].
systematic analysis, using dynamical systems techniques, of the qualitative behavior of the Friedmann-Robertson-Walker (FRW), Bianchi type I and V cosmological models in the Randall-Sundrum brane world scenario, with matter on the brane obeying a barotropic equation of state \( p = (\gamma - 1)\rho \), has been realized by Campos and Sopuerta [22,23]. In particular, they constructed the state spaces for these models and discussed what new critical points appear, the occurrence of bifurcations and the dynamics of the anisotropy for both a vanishing and non-vanishing Weyl tensor in the bulk.

The general exact solution of the field equations for an anisotropic brane with Bianchi type I and V geometry, with perfect fluid and scalar fields as matter sources has been found in [24]. Expanding Bianchi type I and V brane-worlds always isotropize, although there could be intermediate stages in which the anisotropy grows. In spatially homogeneous brane world cosmological models the initial singularity is isotropic and hence the initial conditions problem is solved [25]. Consequently, these models do not exhibit Mixmaster or chaotic-like behavior close to the initial singularity [26]. Other properties of brane world cosmologies have been considered in [27] and [28].

Realistic brane-world cosmological models require the consideration of more general matter sources to describe the evolution and dynamics of the very early Universe. Limits on the initial anisotropy induced by the 5-dimensional Kaluza-Klein graviton stresses by using the CMB anisotropies have been obtained by Barrow and Maartens [29] and Leong, Challinor, Maartens and Lasenby [30]. Anisotropic Bianchi type I brane-worlds with a pure magnetic field and a perfect fluid have also been analyzed [31]. Rotational perturbations of brane world cosmological models have been studied in [32].

Most of the investigations of brane cosmological models have been performed under the simplifying assumption of a perfect cosmological fluid. But in many cosmological situations an idealized fluid model of matter is inappropriate, especially in the case of matter at very high densities and pressures. Such possible situations are the relativistic transport of photons, mixtures of cosmic elementary particles, evolution of cosmic strings due to their interaction with each other and surrounding matter, classical description of the (quantum) particle production phase, interaction between matter and radiation, quark and gluon plasma viscosity etc. From a physical point of view the inclusion of dissipative terms in the energy-momentum tensor of the cosmological fluid seems to be the best motivated generalization of the matter term of the gravitational field equations.

The first attempts at creating a theory of relativistic dissipative fluids were those of Eckart [33] and Landau and Lifshitz [34]. These theories are now known to be pathological in several respects. Regardless of the choice of equation of state, all equilibrium states in these theories are unstable and in addition signals may be propagated through the fluid at velocities exceeding the speed of light. These problems arise due to the first order nature of the theory, that is, it considers only first-order deviations from the equilibrium leading to parabolic differential equations, hence to infinite speeds of propagation for heat flow and viscosity, in contradiction with the principle of causality. Conventional theory is thus applicable only to phenomena which are quasi-stationary, i.e. slowly varying on space and time scales characterized by mean free path and mean collision time.

A relativistic second-order theory was found by Israel [35] and developed by Israel and Stewart [36] into what is called “transient” or “extended” irreversible thermodynamics. In this model deviations from equilibrium (bulk stress, heat flow and shear stress) are treated as independent dynamical variables, leading to a total of 14 dynamical fluid variables to be determined. However, Hiscock and Lindblom [57] and Hiscock and Salmonson [58] have shown that most versions of the causal second order theories omit certain divergence terms. The truncated causal thermodynamics of bulk viscosity leads to pathological behavior in the late Universe, while the solutions of the full causal theory are [38]: a) for stable fluid configurations the dissipative signals propagate causally, b) unlike in Eckart-type’s theories, there is no generic short wave-length secular instability and c) even for rotating fluids, the perturbations have a well-posed initial value problem. For general reviews on causal thermodynamics and its role in relativity see [10] and [11].

Causal bulk viscous thermodynamics has been extensively used for describing the dynamics and evolution of the early Universe or in an astrophysical context. But due to the complicated character of the evolution equations, very few exact cosmological solutions of the gravitational field equations are known in the framework of the full causal theory. For a homogeneous Universe filled with a full causal viscous fluid source obeying the relation \( \xi \sim \rho^{1/2} \), exact general solutions of the field equations have been obtained in [42, 43, 44, 45]. In this case the evolution of the bulk viscous cosmological model can be reduced to a Painleve-Ince type differential equation, whose invariant form can be reduced, by means of non-local transformations, to a linear inhomogeneous ordinary second-order differential equation with constant coefficients [44]. It has also been proposed that causal bulk viscous thermodynamics can model on a phenomenological level matter creation in the early Universe [13, 17]. Exact causal viscous cosmologies with \( \xi \sim \rho^s \) have been obtained in [48].

Because of technical reasons, most investigations of dissipative causal cosmologies have assumed FRW symmetry (i.e. homogeneity and isotropy) or small perturbations around it [19]. The Einstein field equations for homogeneous models with dissipative fluids can be decoupled and therefore are reduced to an autonomous system of first order ordinary differential equations, which can be analyzed qualitatively [50, 51].
The influence of the bulk viscosity of the matter on the brane has been analyzed, for an isotropic flat FRW geometry, in [12]. Dissipative viscous effects in the brane world lead to important differences in the cosmological dynamics, as compared to the standard general relativistic cosmology. Since the effects of the extra-dimensions and also the viscous effects are more important at high matter densities, the most important contribution to the energy density of the matter in this regime comes from the quadratic term in density. Consequently, during the early period of evolution of the brane world the energy density of matter is proportional to the Hubble parameter, in opposition to the standard general relativistic case with energy density proportional to the square of the Hubble parameter.

It is the purpose of the present paper to investigate the effects of the bulk viscosity of the cosmological matter fluid on the dynamics of an anisotropic, Bianchi type I brane world. In order to solve the field equations, we assume a specific equation of state for the bulk viscous pressure. With this choice the general solution of the field equations can be expressed in an exact parametric form. The obtained solution describes a non-inflationary Universe, tending specific equation of state for the bulk viscous pressure. With this choice the general solution of the field equations can be expressed in an exact parametric form. The obtained solution describes a non-inflationary Universe, tending in a large time limit to an isotropic homogeneous geometry. The behavior of the temperature and comoving entropy is also considered.

The present paper is organized as follows. The field equations on the brane describing the evolution of a viscous cosmological fluid are written down in Section II. In Section III we present the general solution of the field equations in the case of a power-law dependence of the bulk viscous pressure on the energy density. In Section IV we discuss and conclude our results.

II. DISSIPATIVE COSMOLOGICAL FLUIDS ON THE ANISOTROPIC BRANE

In the 5D space-time the brane-world is located as $Y(X^I) = 0$, where $X^I$, $I = 0, 1, 2, 3, 4$ are 5-dimensional coordinates. The effective action in five dimensions is [13]

$$ S = \int d^5X \sqrt{-g_5} \left( \frac{1}{2k_5^2} R_5 - \Lambda_5 \right) + \int_{Y=0} d^4x \sqrt{-g} \left( \frac{1}{k_5^2} K^\pm - \lambda + L_{\text{matter}} \right), $$

with $k_5^2 = 8\pi G_5$ the 5-dimensional gravitational coupling constant and where $x^\mu$, $\mu = 0, 1, 2, 3$ are the induced 4-dimensional brane world coordinates. $R_5$ is the 5D intrinsic curvature in the bulk and $K^\pm$ is the intrinsic curvature on either side of the brane.

On the 5-dimensional space-time (the bulk), with the negative vacuum energy $\Lambda_5$ and brane energy-momentum as source of the gravitational field, the Einstein field equations are given by

$$ G_{I\lambda} = k_5^2 T_{I\lambda}, \quad T_{I\lambda} = -\Lambda_5 g_{I\lambda} + \delta(Y) \left[ -\Lambda g_{I\lambda} + T_{I\lambda}^{\text{matter}} \right]. $$

In this space-time a brane is a fixed point of the $Z_2$ symmetry. In the following capital Latin indices run in the range $0, ..., 3$. Assuming a metric of the form $ds^2 = (n_1 n_5 + g_{I\lambda}) dx^I dx^\lambda$, with $n_1 dx^I = dx^I$ the unit normal to the $\chi = \text{const.}$ hypersurfaces and $g_{I\lambda}$ the induced metric on $\chi = \text{const.}$ hypersurfaces, the effective four-dimensional gravitational equations on the brane (which are the consequence of the Gauss-Codazzi equations) take the form [16,17]:

$$ G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu}, $$

where $S_{\mu\nu}$ is the local quadratic energy-momentum correction

$$ S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu\alpha} T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} (3T^{\alpha\beta} T_{\alpha\beta} - T^2), $$

and $E_{\mu\nu}$ is the non-local effect from the bulk free gravitational field, transmitted via the projection of the bulk Weyl tensor $C_{IJAB}$: $E_{IJ} = C_{IJAB} n^A n^B$ and $E_{IJ} \to E_{\mu\nu} \delta_5^\mu \delta_5^\nu$ as $\chi \to 0$. The four-dimensional cosmological constant, $\Lambda$, and the gravitational coupling constant on the brane, $k_4$, are given by $\Lambda = k_5^2 \left( \Lambda_5 + k_5^2 \frac{\chi^2}{n} \right)/2$ and $k_5^2 = k_6^2 \lambda/6$, respectively.

The Einstein equation in the bulk, the Codazzi equation, also implies the conservation of the energy momentum tensor of the matter on the brane,

$$ D_{\nu} T_{\mu}^{\nu} = 0. $$

Moreover, the contracted Bianchi identities imply that the projected Weyl tensor should obey the constraint

\[ \begin{array}{l}
\int d^4x \sqrt{-g} \left( \frac{1}{k_5^2} K^\pm - \lambda + L_{\text{matter}} \right) \nonumber \\
\end{array} \]
Finally, the equations (1) and (4) give the complete set of field equations for the brane gravitational field.

For any matter fields (scalar field, perfect fluids, kinetic gases, dissipative fluids etc.) the general form of the brane energy-momentum tensor can be covariantly given as

\[ T_{\mu\nu} = \rho u_{\mu} u_{\nu} + p_{\text{eff}} h_{\mu\nu} + \pi_{\mu\nu} + 2q(\mu u_{\nu}). \]

The decomposition is irreducible for any chosen 4-velocity \( u^\mu \). Here \( \rho \) and \( p_{\text{eff}} \) are the energy density and the effective isotropic pressure, and \( h_{\mu\nu} = g_{\mu\nu} + u_{\mu} u_{\nu} \) projects orthogonal to \( u^\mu \). The energy flux \( q_{\mu} \) and the anisotropic stress \( \pi_{\mu\nu} \) obey the conditions \( q_{\mu} = q_{<\mu>} \) and \( \pi_{\mu\nu} = \pi_{<\mu\nu>} \), respectively, where angular brackets denote the projected, symmetric and trace-free parts:

\[ V_{<\mu>} = h_{\mu} \nu \nu \nu, \quad W_{<\mu\nu>} = \left[ h_{<\mu} \alpha h_{\nu} \beta \right] - \frac{1}{3} h_{\alpha\beta} h_{\mu\nu} \right] W_{\alpha\beta}. \]

We assume that the equilibrium thermodynamics pressure \( p \) of the cosmological fluid obeys a linear barotropic equation of state \( p = (\gamma - 1) \rho \), \( \gamma \in [1, 2] \) and \( \gamma = \text{constant} \).

The effect of the bulk viscosity of the cosmological fluid on the brane can be considered by adding to the usual thermodynamic pressure \( p \) the bulk viscous pressure \( \Pi \) and formally substituting the pressure terms in the energy-momentum tensor by \( p_{\text{eff}} = p + \Pi \).

The anisotropic stress \( \pi_{\mu\nu} \) of the matter on the brane satisfies the evolution equation (10)

\[ \tau_2 h_{\alpha} \beta h_{\mu} \nu \pi_{\mu\nu} + \pi_{\alpha\beta} = -2\eta \sigma_{\alpha\beta} - \left[ \eta T \left( \frac{\tau_2}{2\eta T} u^\nu \right) \right] \pi_{\alpha\beta}, \]

where \( \eta \) is the shear viscosity coefficient, \( \tau_2 = 2\eta_2 \), with \( \beta_2 \) the thermodynamic coefficient for the tensor dissipative contribution to the entropy density and \( \sigma_{\alpha\beta} \) is the shear tensor. In the following we assume that the main contribution to entropy generation is via scalar dissipation, that is, we consider that the dissipative contribution from the shear viscosity can be neglected, \( \eta \approx 0 \). Consequently, the anisotropic stresses of the matter on the brane also vanish, \( \pi_{\mu\nu} \approx 0 \). We consider that in Eq. (7) the heat transfer is zero, that is, we take \( q_{\mu} = 0 \).

Then the quadratic corrections to the matter energy momentum tensor on the brane are given by

\[ S_{\mu\nu} = \frac{1}{12} \sigma^2 u_{\mu} u_{\nu} + \frac{1}{12} \rho(\rho + 2p_{\text{eff}}) h_{\mu\nu}. \]

The symmetry properties of \( E_{\mu\nu} \) imply that in general we can decompose it irreducibly with respect to a chosen 4-velocity field \( u^\mu \) as

\[ E_{\mu\nu} = -k^4 \left[ \mathcal{U} \left( u_{\mu} u_{\nu} + \frac{1}{3} h_{\mu\nu} \right) + P_{\mu\nu} + 2Q(\mu u_{\nu}) \right], \]

where \( k = k_3/k_4 \). The non-local anisotropic fields \( P_{\mu\nu} \) and \( Q_{\mu} \) include five-dimensional gravitational wave modes. To determine the brane dynamics one must solve the 5D field equations completely, by choosing appropriate boundary conditions in the bulk. For the only known solution of the field equations in the bulk, the Schwarzschild- anti de Sitter bulk that contains a moving brane, the contribution of these terms is zero (3). The effects on the CMB radiation anisotropy of the non-local stresses under various plausible physical assumptions have been discussed in (23) and (30).

One such possibility is to assume that the time evolution of the non-local stress is proportional to the energy density of the anisotropic source \( \rho^{kk} \), with \( \rho^{kk} \) perturbatively small relative to the matter energy density \( \rho \), \( \rho^{kk} \ll \rho \). Since, on the other hand, the viscous dissipative effects are proportional to the matter density, their effects will generally dominate the non-local effects from the bulk. Therefore we shall also assume a vanishing effective non-local energy density, \( P_{\mu\nu} \approx 0 \approx Q_{\mu} \).

The line element of a Bianchi type I space-time, which generalizes the flat Friedmann-Robertson-Walker metric to the anisotropic case, is given by

\[ ds^2 = -dt^2 + a_1^2(t) dx^2 + a_2^2(t) dy^2 + a_3^2(t) dz^2. \]

We define the following variables:
\[ V = \prod_{i=1}^{3} \alpha_i, \quad 3H = \sum_{i=1}^{3} H_i, \quad \dot{H}_i = \frac{\dot{\alpha}_i}{\alpha_i} \Delta H_i = H_i - H, \quad i = 1, 2, 3. \] (13)

In Eqs. (13), \( V \) is the volume scale factor, \( H_i, \quad i = 1, 2, 3 \) are the directional Hubble parameters, and \( H \) is the mean Hubble parameter. From Eqs. (13) we also obtain \( H = \dot{V}/3V \).

The physical quantities of observational interest in cosmology are the expansion scalar \( \theta \), the mean anisotropy parameter \( A \) and the shear scalar \( \sigma^2 \), which are defined according to

\[ \theta = 3H, \quad 3A = \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2, \quad 2\sigma^2 = \sigma_{ik}\sigma^{ik} = \sum_{i=1}^{3} H_i^2 - 3H^2 = 3AH^2. \] (14)

Using the variables (13), the Einstein gravitational field equations, the Bianchi identity and the evolution equation for the non-local dark energy take the form:

\[ 3\dot{H} + \sum_{i=1}^{3} H_i^2 = \Lambda + \frac{k_i^2}{2} [(2 - 3\gamma) \rho - 3\Pi] + \frac{k_i^2 \rho}{2\lambda} [(1 - 3\gamma) \rho - 3\Pi] - \frac{6\dot{\mu}}{k_i^2 \lambda}, \] (15)

\[ \frac{1}{V} \frac{d}{dt}(VH_i) = \Lambda + \frac{k_i^2}{2} [(2 - \gamma) \rho - \Pi] + \frac{k_i^2 \rho}{2\lambda} [(1 - \gamma) \rho - \Pi] + \frac{2\dot{\mu}}{k_i^2 \lambda}, \quad i = 1, 2, 3, \] (16)

\[ \dot{\rho} + 3\gamma H \rho = -3H\Pi, \] (17)

\[ \dot{\mu} + 4H\mu = 0. \] (18)

The causal evolution equation for the bulk viscous pressure \( \Pi \) is given by (11)

\[ \tau \ddot{\Pi} + \Pi = -3\xi H - \frac{1}{2} \tau \Pi \left( 3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{T}{\tau} \right), \] (19)

where \( \tau \) is the relaxation time, \( T \) is the temperature and \( \xi \) is the bulk viscosity coefficient.

Eq. (19) arises as the simplest way (linear in \( \Pi \)) to satisfy the \( H \)-theorem (i.e., for the entropy production to be non-negative, \( S^{\mu}_{;\mu} = \Pi^2/(\xi T) \geq 0 \)). The particle flow vector \( N^\mu \) is given by \( N^\mu = nu^\mu \), where \( n \geq 0 \) is the particle number density. In the framework of causal thermodynamics and limiting ourselves to second-order deviations from equilibrium, the entropy flow vector \( S^\mu \) takes the form \( S^\mu = eN^\mu - \tau \Pi^2 u^\mu /2\xi T \), where \( e \) is the entropy per particle.

An important observational quantity is the deceleration parameter

\[ q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 = \frac{\rho + 3p + 3\Pi + 2\frac{k_i^2}{2\lambda}}{2 (\rho + \frac{k_i^2}{2\lambda})}. \] (20)

The sign of the deceleration parameter indicates whether the model inflates or not. The positive sign of \( q \) corresponds to “standard” decelerating models, whereas the negative sign indicates inflation.

III. GENERAL SOLUTION OF THE FIELD EQUATIONS

Integrating (18) yields for the dark matter term the expression \( \mathcal{U} = \mathcal{U}_0 V^{-4} \), where \( \mathcal{U}_0 \geq 0 \) is an arbitrary constant of integration.

By adding Eqs. (16) we find

\[ \frac{1}{V} \frac{d}{dt}(VH) = \dot{H} + 3H^2 = \Lambda + \frac{k_i^2}{2} [(2 - \gamma) \rho - \Pi] + \frac{k_i^2 \rho}{2\lambda} [(1 - \gamma) \rho - \Pi] + \frac{2\dot{\mu}}{k_i^2 \lambda}. \] (21)

Substracting Eq. (21) from Eqs. (16) we obtain

\[ H_i = H + \frac{K_i}{V}, \quad i = 1, 2, 3, \] (22)

with \( K_i, \quad i = 1, 2, 3 \) constants of integration satisfying the consistency condition \( \sum_{i=1}^{3} K_i = 0 \). By using Eq. (22), it is easy to show that
where we denoted \( K^2 = \sum_{i=1}^{3} K_i^2 \).

Usually, in order to find the general solution of the gravitational field equations, the equations of state of the bulk viscosity coefficient, temperature and relaxation time are given, in terms of the energy density of the cosmological fluid. Then the thermodynamic and geometric parameters are obtained by solving the field equations. But in the present paper we shall use a different approach, namely we shall assume that the functional dependence of the bulk viscous pressure on the energy density is known. Therefore the behavior of the bulk viscosity coefficient, temperature and relaxation time can be obtained from the field equations. Hence for the bulk viscous pressure we assume the following general power-law type density dependence:

\[
\Pi = -\gamma \Pi_0 \rho^m, \quad m = \text{constant} > 0, m \neq 1, \Pi_0 = \text{constant} \geq 0.
\]  

(24)

This form of the bulk viscous pressure is suggested by the form of \( \Pi \) in the non-causal theory, where \( \Pi \sim \xi H \). This relation follows by taking \( \tau = 0 \) in Eq. (19). Assuming a power law density dependence of the bulk viscosity coefficient on the density, \( \xi \sim \rho^s \), \( s = \text{constant} > 0 \), and taking into account that generally the Hubble parameter can also be approximated by some power of the energy density, we obtain Eq. (24) in a natural way. By using such a form for the viscous stress Barrow \[53\] has obtained a class of homogeneous and isotropic cosmological models which begin in a de Sitter state but subsequently deflate towards the flat Friedman model. Barrow \[54\] has also considered the effects of the anisotropies on the early cosmological evolution, within the framework of the non-causal thermodynamical approach, for two distinct classes of models corresponding to the choices \( m > 1 \) and \( m < 1 \) in the expression of the bulk viscous pressure. In both cases the initial anisotropies are smoothed out by the inflationary expansion of the Universe. The range of values of \( m \) can be roughly estimated by considering the isotropic limit of the model. Since for a high density cosmological fluid \( s \leq 1/2 \) \[55\], and for standard cosmological models \( H \sim \rho^{1/2} \), it follows that in conventional general relativity \( m < 1 \). However, in the brane world model, at high densities the quadratic correction term dominates the cosmological evolution and the Hubble parameter is proportional to the energy density, \( H \sim \rho \). Therefore values of \( m \) greater than one, \( m > 1 \), are also allowed to describe high density bulk viscous cosmological fluids. For low density viscous matter \( s \rightarrow 1 \) \[56\] and consequently on the brane \( m \) could take values as high as \( 2, m \rightarrow 2 \).

The Israel-Stewart theory is derived under the assumption that the thermodynamic state of the fluid is close to equilibrium, which means that the non-equilibrium bulk viscous pressure should be small when compared to the local equilibrium pressure, that is \( |\Pi| \ll \rho \). Consequently, the energy density of the cosmological fluid must satisfy for all times the condition \( \rho \ll (\gamma - 1) / 3 \gamma \Pi_0 \). Therefore values of \( m \) greater than one, \( m > 1 \), are also allowed to describe high density bulk viscous cosmological fluids. For low density viscous matter \( s \rightarrow 1 \) \[56\] and consequently on the brane \( m \) could take values as high as \( 2, m \rightarrow 2 \).

In view of the definition of the mean Hubble function \( H \) and Eq. (25), Eq. (26) can be integrated to give

\[
t - t_0 = \sqrt{\frac{2}{3}} \times \int \left[ K^2 + 2 \Lambda V^2 + 2 k_i^2 V^2 \left( \rho_0 V^{\gamma(m-1)} + \Pi_0 \right)^{1-m} + \frac{k_i^2}{\lambda} \rho^2 + \frac{12\lambda}{k_i^2} \rho V^{\frac{1}{2}} \right]^{\frac{1}{2}} dV,
\]  

(27)

with \( t_0 \) an arbitrary constant of integration.
The scale factors $a_i, i = 1, 2, 3$, are obtained from Eq. (22), and are given by

$$a_i(t) = a_{i0}V^\frac{i}{\gamma},$$

where $a_{i0}, i = 1, 2, 3$ are arbitrary constants of integration.

The anisotropy parameter can be expressed in terms of $V$ and $H$ in the form

$$A(t) = \frac{1}{3} \frac{K^2}{V^2 H^2} = 2K^2 \left[ K^2 + 2\Lambda V^2 + 2k_4^2 V^2 \left( \rho_0 V^{\gamma(m-1)} + \Pi_0 \right)^\frac{1}{1-m} + \frac{k_4^2}{\Lambda} V^2 \left( \rho_0 V^{\gamma(m-1)} + \Pi_0 \right)^\frac{1}{1-m} + \frac{12b_0}{k_4^2 \Lambda} V^\frac{1}{2} \right]^{-1}.$$  \tag{29}

The expansion scalar $\theta$ can be represented, in a parametric form, as a function of $V$, as

$$\theta = \sqrt{\frac{3}{2} \left[ K^2 + 2\Lambda V^2 + 2k_4^2 V^2 \left( \rho_0 V^{\gamma(m-1)} + \Pi_0 \right)^{1/(1-m)} + \frac{k_4^2}{\Lambda} V^2 \left( \rho_0 V^{\gamma(m-1)} + \Pi_0 \right)^{2/(1-m)} + \frac{12b_0}{k_4^2 \Lambda} V^{1/2} \right]}.$$  \tag{30}

The shear scalar $\sigma^2$ is given by

$$\sigma^2 = \frac{K^2}{2V^2}.$$  \tag{31}

As a function of $V$ the deceleration parameter $q$ can be expressed as

$$q = \frac{(3\gamma - 2) \left( \rho_0 V^{\gamma(m-1)} + \Pi_0 \right)^{1/(1-m)} - 3\gamma \Pi_0 \rho^m + 2K^2}{2 \left[ \left( \rho_0 V^{\gamma(m-1)} + \Pi_0 \right)^{1/(1-m)} + \frac{k_4^2}{2V^2} \right]}.$$  \tag{32}

Integrating Eq. (19), we obtain the time dependence of the temperature of the matter on the brane in the form

$$T(t) = T_0 \frac{\Pi^2}{\xi} \exp \left[ 2 \int \left( \frac{1}{\tau} + \frac{3\gamma H}{\tau \Pi} + \frac{3H}{2} \right) dt \right],$$

with $T_0$ an arbitrary constant of integration.

Following Belinskii, Nikomarov and Khalatnikov [55], we suppose that, in order to guarantee that the propagation velocity of bulk viscous perturbations, i.e., the non-adiabatic contribution to the speed of sound $c_b$ does not exceed the speed of light, the relation $\tau = \xi/c_b^2(\rho + p) = \xi/c_b^2 \rho$ holds in the dissipative fluid. With this choice for the relaxation time $\tau$ the causal evolution Eq. (13) gives the following evolution law for the temperature:

$$T(t) = T_0 \gamma \frac{\Pi^2}{c_b^2} \rho^{2m-1} \exp \left[ 2 \int \left( \gamma \frac{c_b^2 \rho}{\xi} - 3\gamma \frac{c_b^2}{\Pi_0} \rho^{1-m} H + \frac{3H}{2} \right) dt \right].$$  \tag{34}

An analysis of the relativistic kinetic equation for some simple cases given by Murphy [68], Belinskii and Khalatnikov [67] and Belinskii, Nikomarov and Khalatnikov [55] has shown that in the asymptotic regions of small and large values of the energy density, the viscosity coefficients can be approximated by power functions of the energy density with definite requirements on the exponents of these functions. For small values of the energy density it is reasonable to consider large exponents, equal in the extreme case to one. For large $\rho$ the power of the bulk viscosity coefficient should be considered smaller (or equal) to 1/2. Hence, we shall assume that the bulk viscosity coefficient obeys the simple phenomenological law

$$\xi(t) = \alpha \rho^s,$$  \tag{35}

with $\alpha \geq 0$ and $0 \leq s \leq 1/2$ constants.

Therefore we obtain for the temperature of the dissipative viscous cosmological fluid on the brane the following representation:

$$T = T_0 \gamma \frac{\Pi^2}{c_b^2} \rho^{2m-1} \exp \left[ 2 \int \left( \gamma \frac{c_b^2 \rho}{\xi} - 3\gamma \frac{c_b^2}{\Pi_0} \rho^{1-m} H + \frac{3H}{2} \right) dt \right] = T_0 \gamma \frac{\Pi^2}{c_b^2} \rho^{2m-1} \times$$

$$\exp \left[ \frac{8}{3} \int \frac{\left( \gamma \frac{c_b^2 \rho}{\xi} - 3\gamma \frac{c_b^2}{\Pi_0} \rho^{1-m} H + \frac{3H}{2} \right)}{K^2 + 2\Lambda V^2 + 2k_4^2 V^2 \left( \rho_0 V^{\gamma(m-1)} + \Pi_0 \right)^{1-m} + \frac{k_4^2}{\Lambda} V^2 \left( \rho_0 V^{\gamma(m-1)} + \Pi_0 \right)^{1-m} + \frac{12b_0}{k_4^2 \Lambda} V^{1/2}} dV \right].$$  \tag{36}
The growth of the total comoving entropy $\Sigma(t)$ over a proper time interval $\left( t_0, t \right)$ is given by

$$
\Sigma(t) - \Sigma(t_0) = -\frac{3}{k_B} \int_{t_0}^{t} \frac{\Pi V H}{T} dt = -\frac{1}{k_B} \int_{V_0}^{V} \frac{\Pi}{T} dV,
$$

where $k_B$ is the Boltzmann constant.

**IV. DISCUSSIONS AND FINAL REMARKS**

In the present paper we have considered the evolution of a causal viscous dissipative cosmological fluid in the brane world scenario. As only source of dissipation we have considered the bulk viscosity of the matter on the brane. In this case the standard Einstein equations are modified, due to the presence of the terms from extra dimensions, quadratic in the energy density.

By assuming a power-law dependence of the bulk viscous pressure on the energy density of the cosmological fluid, the general solution of the field equations for an anisotropic Bianchi type I geometry can be obtained in an exact parametric form. By assuming for the bulk viscosity coefficient and for the relaxation time the usual equations of state, the temperature evolution of the cosmological fluid can also be obtained in a closed form.

The cosmological evolution of the bulk viscous brane Universe is expansionary, with all the scale factors monotonically increasing functions of time. For $t \to \infty$ we have $a_i = a = a_0 V^{1/3}$, $i = 1, 2, 3$. Hence the Universe ends in an isotropic state. The expansion parameter $\theta$ is a decreasing function of time, with a singular behavior at $t = 0$.

The time evolution of the energy density of the cosmological fluid essentially depends on the value of the exponent $m$ in the bulk viscous pressure-energy density relation. The variation with respect the cosmological time of $\rho$ is represented, for $m > 1$, in Fig. 1.

For this range of values of $m$ the Universe starts its evolution with finite values of the energy density, $\rho(0) = \Pi_0^{1/(1-m)} \neq 0$. But the scale factors are singular at the initial time, $a_i(t_0) = 0$, $i = 1, 2, 3$. For $m < 1$ the anisotropic brane Universe starts from a singular state, with infinite energy density and zero scale factors. For all values of $m$ the energy density is a monotonically decreasing function of time.

The evolution of the deceleration parameter $q$ is represented in Fig. 2.
FIG. 2. Dynamics of the deceleration parameter $q(t)$ of the Zeldovich fluid ($\gamma = 2$) filled Bianchi type I brane Universe, for different values of the parameter $m$: $m = 1.3$ (solid curve), $m = 1.4$ (dotted curve) and $m = 1.5$ (dashed curve). We have normalized the constant parameters so that $K^2 = 2$, $\Lambda = 0.5$, $\rho_0 = \Pi_0$, $2k_4^2\rho_0^{1/(1-m)} = 1$, $k_4^2\rho_0^{2/(1-m)} = \lambda$ and $12\lambda_0 = k_4^2\lambda$.

Due to the smallness of the bulk viscous pressure, satisfying the condition $|\Pi| < \rho$, the cosmological evolution is non-inflationary, with $q > 0$ for all values of $m$ and for all $t$. In order to have inflationary evolution it is necessary that the negative bulk viscous pressure dominates the thermodynamic pressure.

The dynamics of the mean anisotropy parameter $A$ strongly depends on the value of $m$. The time variation of $A$ for $m < 1$ is represented in Fig. 3. In this case the Bianchi type I Universe starts its evolution from an isotropic (but singular) state, with $A(0) = 0$. For small times the mean anisotropy is increasing, reaching a maximum at a finite time $t = t_c$. For times $t > t_c$ the anisotropy is a monotonically decreasing function of time, and thus in the large time limit the anisotropic brane world Universe will end in an homogeneous and isotropic state. As has already been pointed out in [24], this behavior is specific for brane world cosmological models and cannot be found in conventional general relativistic scenarios. For $m > 1$, the brane Universe starts from a state of maximum anisotropy, also reaching in the long time limit an isotropic phase. Due to the expansion of the Universe, the shear scalar $\sigma^2$ vanishes in the large time limit, $\sigma^2 \to 0$ for $V \to \infty$.

FIG. 3. Evolution of the mean anisotropy parameter $A(t)$ of the Zeldovich fluid ($\gamma = 2$) filled Bianchi type I brane Universe, for different values of the parameter $m$: $m = 0.7$ (solid curve), $m = 0.8$ (dotted curve) and $m = 0.9$ (dashed curve). We have normalized the constants so that $K^2 = 2$, $\Lambda = 0.5$, $\rho_0 = \Pi_0$, $2k_4^2\rho_0^{1/(1-m)} = 1$, $k_4^2\rho_0^{2/(1-m)} = \lambda$ and $12\lambda_0 = k_4^2\lambda$.

According to the formal definition of the isotropization given by Collins and Hawking [58] a cosmological model approaches isotropy if the following four conditions hold as $t \to \infty$: i) the Universe is expanding indefinitely and $H > 0$ ii) $T^{00} > 0$ and $T^{0i}/T^{00} \to 0$, $T^{0i}/T^{00}$ represents an average velocity of the matter relative to the surfaces of homogeneity. If this does not tend to zero, the Universe would not appear homogeneous or isotropic iii) the anisotropy in the locally measured Hubble constant $\sigma/H$ tends to zero, $\sigma/H \to 0$ and iv) the distortion part of the metric tends to a constant. All these conditions are satisfied in the present model, since in the large time limit the bulk viscous
brane Universe is expanding and the energy density of the matter is positive for all times. Moreover, as one can immediately see from Eq. (14) the quantity $\sigma/H$ is proportional to the square root of the anisotropy parameter, $\sigma/H \sim \sqrt{A}$ and in the large time for all values of the parameter $m$ it tends to zero, $\sigma/H \to 0$. For the condition iv), it is also satisfied since in the metric we have considered only the volume part. Spatially homogeneous models can be divided in three classes: those which have less than the escape velocity (i.e., those whose rate of expansion is insufficient to prevent them from recollapsing), those which have just the escape velocity and those which have more than the escape velocity [54]. Models of the third class do not tend, generally, to isotropy. In fact the only types which can tend toward isotropy at arbitrarily large times are types $I$, $V$, $VII_0$ and $VII_b$. For type $VII_b$ there is no nonzero measure set of these models which tends to isotropy [52]. In this sense the Bianchi type I model we have studied in the present paper is a very special case. The Bianchi types that drive flat and open Universes away from isotropy in the Collins-Hawking sense are those of type $VII$.

As a consequence of its energy density dependence, the bulk viscous pressure is a decreasing function of time for all values of the parameters and for large times it tends to zero. The time evolution of $\Pi$ essentially depends on the constant $m$. For $m >> 1$, the time scale in which $\Pi \approx 0$ is very short.

The variation of the temperature, presented in Fig. 4, shows that at the initial times of the cosmological evolution the temperature is an increasing function of time, reaching a maximum value at a finite time $t_c$. For times $t > t_c$, the temperature is a decreasing function of time.

![Temperature Evolution](image)

**FIG. 4.** Time evolution of the temperature $T(t)$ of the Zeldovich fluid ($\gamma = 2$) filled Bianchi type I brane Universe, for $s = 1/4$ and for different values of the parameter $m$: $m = 0.45$ (solid curve), $m = 0.65$ (dotted curve) and $m = 0.9$ (dashed curve). We have normalized the constants so that $K^2 = 2$, $\Lambda = 0.5$, $\rho_0 = \Pi_0$, $2k_0^2\rho_0^{1/(1-m)} = 1$, $k_0^2\rho_0^2/(1-m) = \lambda$, $12\rho_0 = k_0^2\lambda$, $\gamma T_0\Pi_0^2 = \sigma_0^2$, $\gamma c_0^2 = \alpha$ and $3c_0^2 = \Pi_0$.

The comoving entropy, represented in Fig. 5, is an increasing function of time in the early stages of cosmological evolution, but it tends to a constant value in the large time limit.

![Entropy Evolution](image)
FIG. 5. Variation, as a function of time, of the comoving entropy $\Sigma(t)$ of the Zeldovich fluid ($\gamma = 2$) filled Bianchi type I brane Universe, for $s = 1/4$ and for different values of the parameter $m$: $m = 0.45$ (solid curve), $m = 0.65$ (dotted curve) and $m = 0.9$ (dashed curve). We have normalized the constants so that $K^2 = 2$, $\Lambda = 0$, $\rho_0 = \Pi_0$, $2k^2\rho_0^{-1/(1-m)} = 1$, $k^2\rho_0^{-5/(1-m)} = \lambda$, $12\rho_0 = k^2\lambda$, $\gamma T_0 \Pi_0 = \dot{\phi}^2$, $\gamma \dot{\phi}^2 = \alpha$ and $3c_s^2 = \Pi_0$.

The transition from the anisotropic to the isotropic state of the brane world is associated to a large increase in the entropy. The increase of the comoving entropy of the cosmological fluid is independent of the value of the parameter $m$ and is a general feature of the model. In the large time limit $\Pi \to 0$ and, as one can see from Eq. (37), in this limit $\Sigma \to const$.

The equation of state of the Zeldovich fluid with $\gamma = 2$ is equivalent to the equation of state of a massless scalar field $\phi$, with $\rho = p = \dot{\phi}^2/2$ and a dissipative coupling [47], with $\phi$ satisfying the equation

$$\ddot{\phi} + 3H\dot{\phi} = 3 \left(2^{1-m}\right) \Pi_0 \phi^{2m-1} \dot{H}. \quad (38)$$

The evolution of the scalar field can be obtained again in parametric form. Classically, bulk viscous stresses arise because the expansion of the Universe is continually trying to pull the fluid out of thermal equilibrium, and the cosmological fluid is trying to relax back. Hence the viscosity of a classical fluid arises from the differential motion of fluid elements. But in the collisionless periods after the Planck era and during inflation there are no particle interactions responsible for the viscosity. However, the phenomenological description of quantum particle production processes can be consistently modeled by means of bulk viscosities [47, 53]. In particular, the full causal thermodynamics can be reformulated as a theory describing particle production [47]. In this formulation the causal bulk viscous pressure $\Pi$ in the energy conservation equation (37) acts as a creation pressure $p_c$, $p_c = \Pi$, while the particle production rate $\Gamma$ is proportional to the bulk viscous pressure divided by the locally measured Hubble parameter. The particle balance equation, describing the variation of the particle number due to the combined effects of the expansion of the Universe and particle production, takes the form

$$\dot{n} + 3Hn = \Gamma n, \quad (39)$$

where $\Gamma = -\Pi/\gamma H$ [47].

As an example of particle creation processes on the brane we consider particle production from a scalar field obeying an equation of state of the form $p_{\phi} = (\gamma_{\phi} - 1)\rho_{\phi}$, $0 \leq \gamma_{\phi} \leq 1$, where $\rho_{\phi} = \phi^2/2 + U(\phi)$ and $p_{\phi} = \phi^2/2 - U(\phi)$, with $U(\phi)$ the self-interaction potential. The deviation from the conformal invariance of the field or the polarization (trace anomaly) of quantized fields interacting with dynamic spacetimes will lead to particle production [53]. The rate of particle creation in a particular mode depends on its frequency in relation to the expansion rate of the Universe: at any given time $t$ in the quantum regime, new particles with frequency $\omega \leq t^{-1}$ are created from the quantum vacuum, while particles produced earlier interact and have an expansionary dynamics. Whether the newly created particles can reach effective thermal equilibrium depends on the rate of their interaction relative to the rate of expansion.

To obtain a phenomenological classical description of the essentially quantum creation process, we assume again that the negative creation pressure is proportional to the energy density of the scalar field, $p_c = -\rho_{\phi} = \gamma_{\phi} \rho_{\phi}^m$, where, for simplicity, we suppose that all $p_{\phi}$, $\gamma_{\phi}$ and $m$ are constants. Then the time variation of the scalar field is given by

$$\rho_{\phi}(t) = \left[\rho_{0\phi} V^{-\gamma_{\phi}(m-1) + p_{\phi}}\right]^{1/(1-m)}, \quad (40)$$

where $\rho_{0\phi} \geq 0$ is a constant of integration. The rate at which particles are created on the brane is

$$\Gamma(t) = p_c H^{-1} \left[\rho_{0\phi} V^{-\gamma_{\phi}(m-1) + p_{\phi}}\right]^{m/(1-m)}, \quad (41)$$

while the particle number varies as

$$n(t) = \frac{n_0}{V} \exp \left(\int \Gamma(t) dt\right), \quad (42)$$

with $n_0 \geq 0$ a constant of integration. At high temperature $n = g \zeta(3) T^3/\pi^2$, where $\zeta(s)$ is the Riemann function, $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$, with $\zeta(3) = 1.202$ and $g$ is the total spin state of the particles. Therefore the temperature of the very early Universe is given by

$$T(t) = \left(\frac{n_0 \pi^2}{g \zeta(3)}\right)^{1/3} \exp \left(\frac{1}{4} \int \Gamma(t) dt\right)/V^{1/3}. \quad (43)$$
In the particular case $\gamma_\phi = 0$, corresponding to a dissipative scalar field obeying an equation of state of the form $\rho_\phi + p_\phi = 1/0$, the time variation of the energy density of the field is $\rho_\phi \sim t^{1/(1-m)}$, while the creation pressure varies as $p_c \sim t^{m/(1-m)}$. In order to have a decaying field it is necessary that $m > 1$. Even in the presence of particle creation processes, the initial expansion of the brane Universe is non-inflationary. But in the large time limit, for large values of $V$ and for a decaying creation pressure with $p_c \rightarrow 0$, the deceleration parameter becomes $q = [(3\gamma_\phi - 2)]/2$. For $\gamma_\phi < 2/3$, the brane Universe ends in an inflationary epoch.

Since the dynamics on the brane is different from the standard general relativistic one, the particle creation processes are also influenced by the effects of the extra dimensions. Because the rate of expansion is higher, the particle creation processes are also accelerated in the models with large extra dimensions.

In the present paper we have pointed out another important difference between brane world cosmology and standard general relativity predictions about the very early Universe, namely the behavior of the temperature of a realistic cosmological fluid. In brane world cosmological models, the high density Universe starts from a low temperature state. In the initial stages of expansionary evolution the increase of the degree of anisotropy is associated to an increase of the temperature of the Universe, the cosmological fluid experiencing a heating process. An other interesting feature of the model is the presence of a non-singular energy density, associated to a singularity of the scale factors. The present approach can also lead to a better understanding of the entropy generation mechanisms in the very early Universe, where viscous type physical processes have probably played an important role even when the Universe was about 1000s old [1].

Hence dissipative viscous brane world cosmological model with an anisotropic geometry could maybe describe a well-determined period of the very early evolution of our Universe.

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