INFINITE-DIMENSIONAL QUADRATURE AND QUANTIZATION

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ABSTRACT. We study numerical integration of Lipschitz functionals on a Banach space by means of deterministic and randomized (Monte Carlo) algorithms. This quadrature problem is shown to be closely related to the problem of quantization of the underlying probability measure. In addition to the general setting we analyze in particular integration w.r.t. Gaussian measures and distributions of diffusion processes. We derive lower bounds for the worst case error of every algorithm in terms of its computational cost, and we present matching upper bounds, up to logarithms, and corresponding almost optimal algorithms. As auxiliary results we determine the asymptotic behaviour of quantization numbers and Kolmogorov widths for diffusion processes.

1. INTRODUCTION

Let \( \mu \) be a Borel probability measure on a Banach space \((\mathcal{X}, \| \cdot \|)\) such that
\[
\int_{\mathcal{X}} \| x \| \, \mu(dx) < \infty.
\]
Moreover, let \( F \) denote the class of all Lipschitz continuous functionals \( f : \mathcal{X} \to \mathbb{R} \) with Lipschitz constant at most one, i.e.,
\[
|f(x) - f(y)| \leq \| x - y \|, \quad x, y \in \mathcal{X}.
\]
We wish to compute
\[
S(f) = \int_{\mathcal{X}} f(x) \, \mu(dx)
\]
for \( f \in F \) by means of deterministic or randomized (Monte Carlo) algorithms that use the values \( f(x) \) of the functional \( f \) at a finite number of sequentially (adaptively) chosen points \( x \in \mathcal{X} \). We present a worst case analysis, and we optimally relate the error and the cost of algorithms.

The classical instance of this quadrature problem is given by \( \mathcal{X} = \mathbb{R}^d \) and \( \mu \) being the uniform distribution on \([0, 1]^d\), say, or the \( d \)-dimensional standard normal distribution. See, e.g., Novak (1988) and Woźniakowski, Woźniakowski (2001) for results and references. We are mainly interested in infinite-dimensional spaces \( \mathcal{X} \), and in particular we study Gaussian measures \( \mu \) and distributions \( \mu \) of diffusion processes, see also Wasilkowski, Woźniakowski (1996) and Pages, Printems (2004). Infinite-dimensional quadrature is applied, e.g., in mathematical finance and quantum physics, and moreover it is used as a computational tool to solve parabolic or elliptic partial differential equations.

The appropriate framework for the analysis of finite- and infinite-dimensional quadrature problems is provided by the real-number model of computation. Informally, a real-number algorithm is like a C-program that carries out exact computations with real
numbers. Furthermore, a perfect generator for random numbers from $[0, 1]$ is available, and algorithms have access to the functionals $f \in F$ via an oracle (subroutine) that provides values $f(x)$ for points $x$ from a finite-dimensional subspace $\mathfrak{X}_0 \subset \mathfrak{X}$. The subspace may be chosen arbitrarily, but it is fixed for a specific algorithm. If, for instance, $\mu$ is the Wiener measure on $\mathfrak{X} = C([0, 1])$ or, more generally, the distribution of a diffusion process, then spaces $\mathfrak{X}_0$ of piecewise linear functions are frequently used in computational practice. The cost of an oracle call for $f(x)$ is given by the dimension of the corresponding subspace $\mathfrak{X}_0$, while real number operations, evaluations of elementary functions, and calls of the random number generator are performed at cost one. Furthermore, in the case of a diffusion process, function values of its drift and diffusion coefficients are provided at cost one, too.

By $\varepsilon^\text{det}_N$ and $\varepsilon^\text{ran}_N$, we denote the smallest worst case error that can be achieved by any deterministic or randomized algorithm, resp., whose computational cost is bounded by $N$. We wish to determine the asymptotic behaviour of the minimal errors $\varepsilon^\text{det}_N$ and $\varepsilon^\text{ran}_N$ and to find algorithms with cost close to $N$ and error close to the corresponding minimal error. We write $a_N \preceq b_N$ for sequences of positive real numbers $a_N$ and $b_N$ if $\sup_{N \in \mathbb{N}} a_N/b_N < \infty$. Moreover, $a_N \asymp b_N$ means $a_N \preceq b_N$ and $b_N \preceq a_N$.

Our main results are the following upper and lower bounds for the minimal errors in the diffusion case and in the Gaussian case.

Suppose that $\mu$ is the distribution of an $m$-dimensional diffusion process on the space $\mathfrak{X} = C([0, 1], \mathbb{R}^m)$, equipped with the supremum norm. Under mild assumptions on the drift and diffusion coefficients the minimal errors $\varepsilon^\text{det}_N$ and $\varepsilon^\text{ran}_N$ satisfy

$$\varepsilon^\text{det}_N \gtrsim (\ln N)^{-1/2}$$

see Theorem 1 and Proposition 3 and

$$N^{-1/4} \cdot (\ln N)^{-3/4} \leq \varepsilon^\text{ran}_N \leq N^{-1/4} \cdot (\ln N)^{1/4},$$

see Theorems 4 and 5.

We conclude that the quadrature problem for diffusion processes is intractable by means of deterministic algorithms, since $(\ln N)^{-1/2}$ tends to zero too slowly, but randomization helps substantially. The upper bound for $\varepsilon^\text{ran}_N$ is achieved by a suitably adjusted weak Euler scheme. In view of the lower bound, this algorithm is optimal, up to a multiple of at most $\ln N$, in the class of all randomized algorithms.

Suppose that $\mu$ is a zero mean Gaussian measure on a separable Banach space $\mathfrak{X}$, whose small ball function

$$\varphi(\varepsilon) = -\ln \mu(\{x \in \mathfrak{X} : \|x\| \leq \varepsilon\})$$

satisfies

$$\varphi(\varepsilon) \asymp \varepsilon^{-\alpha} \cdot (\ln \varepsilon^{-1})^\beta$$

for some constants $\alpha > 0$ and $\beta \in \mathbb{R}$ as $\varepsilon$ tends to zero. This asymptotic behaviour typically holds for Gaussian measures on infinite-dimensional spaces, see, e.g., the review article by Li, Shao (2001). Consider, for instance, the distribution $\mu$ of the fractional Brownian motion with Hurst parameter $H \in ]0, 1]$ on the space $\mathfrak{X} = C([0, 1])$ or $\mathfrak{X} = L_p([0, 1])$ with $p \in [1, \infty]$. Then $\alpha = 1/H$ and $\beta = 0$. A non-zero constant $\beta$ appears, for example, in case of $\mu$ being the distribution of the $d$-dimensional Brownian sheet on the space $\mathfrak{X} = L_2([0, 1]^d)$. Then $\alpha = 2$ and $\beta = 2(d - 1)$, see Csáki (1984) and Fill, Torcaso (2004).
Given the above small ball asymptotics, the minimal error $\varepsilon^\text{det}_N$ satisfies

$$\varepsilon^\text{det}_N \geq (\ln N)^{-1/\alpha} \cdot (\ln \ln N)^{\beta/\alpha},$$

see Theorem 1 and Proposition 3. Furthermore, for the minimal error $\varepsilon^\text{ran}_N$ we have

$$\varepsilon^\text{ran}_N \leq N^{-1/(2+\alpha)} \cdot (\ln N)^{(\alpha+\beta)/(2+\alpha)},$$

see Theorem 4 as well as

$$\limsup_{N \to \infty} \varepsilon^\text{ran}_N \cdot N^{1/(2+\alpha)} \cdot (\ln N)^{(2+2\alpha-\alpha\beta)/(\alpha(2+\alpha))} \cdot (\ln \ln N)^{-2\beta/(\alpha(2+\alpha))} > 0,$$

see Theorem 6. Note that the bounds for $\varepsilon^\text{ran}_N$ only differ by powers of $\ln N$ and $\ln \ln N$ for an infinite sequence of integers $N$.

We observe intractability of the quadrature problem for Gaussian measures by means of deterministic methods, and randomization helps substantially. The upper bound for $\varepsilon^\text{ran}_N$ is achieved by the classical Monte Carlo algorithm based on a normal distribution on a properly chosen subspace $X_0 \subset X$.

We briefly outline the content of the paper. For the analysis of the quadrature problem we establish general relations of the minimal errors $\varepsilon^\text{det}_N$ and $\varepsilon^\text{ran}_N$ to quantization numbers and average Kolmogorov and linear widths of probability measures on Banach spaces. See, e.g., Creutzig (2002), Dereich (2003), Dereich et al. (2003), Graf, Luschgy (2000), Luschgy, Pagès (2004), and Ritter (2000) for results and references concerning the latter quantities.

In Section 2 we only use the fact that algorithms evaluate the functionals $f \in F$ at a finite number of points $x \in X$. The minimal errors $\varepsilon^\text{det}_n$ and $\varepsilon^\text{ran}_n$ are defined as the smallest worst case error that can be achieved by any algorithm that uses $n$ functional evaluations (on average). These minimal errors turn out to be closely related to the quantization numbers $q^{(r)}_n$, which are defined as a distance of the measure $\mu$ to the class of all discrete probability measure on $X$ with support of size $n$. More precisely, we have

$$\varepsilon^\text{det}_n = q^{(1)}_n,$$

see Theorem 1 and

$$n^{1/2} \cdot \sup_{m \geq 4n} (q^{(1)}_{m-1} - q^{(1)}_m) \preceq \varepsilon^\text{ran}_n \preceq n^{-1/2} \cdot q^{(2)}_{[n/2]},$$

see Theorems 2 and 3. The latter estimate yields the well-known result $\varepsilon^\text{ran}_n \asymp n^{-1/2-1/d}$ in the finite-dimensional case $X = \mathbb{R}^d$.

In Section 3 we examine the computational cost of algorithms more closely, and we take into account that the functionals $f \in F$ may only be evaluated at points $x$ from finite-dimensional subspaces $X_0 \subset X$. The latter restriction leads to the consideration of average Kolmogorov widths $d^{(p)}_k$ of the measure $\mu$, which are defined as average errors of best approximation by means of optimally chosen $k$-dimensional subspaces $X_0 \subset X$. It turns out that

$$\varepsilon^\text{ran}_N \geq \inf_{k, n \leq N} \max(\varepsilon^\text{ran}_n, d^{(1)}_k)$$

for every measure $\mu$, see Proposition 2 which is the key tool to derive the lower bounds for randomized algorithms.

In Sections 4 and 5 we study diffusion processes and Gaussian measures, resp., and we apply the general results from Sections 2 and 3. As auxiliary results we determine the
asymptotic behaviour of the quantization numbers and the Kolmogorov widths in the diffusion case, see Proposition 3 and Remark 7.

2. Quadrature of Lipschitz Functionals and Quantization

At first we disregard the details of the real number model. We only take into account that algorithms may only evaluate the functionals \( f \in F \) at a finite number of sequentially chosen points in the Banach space \( \mathcal{X} \) in order to approximate the integrals \( S(f) \).

2.1. Basic Definitions. Any deterministic sequential evaluation is formally defined by a point

\[ x_1 \in \mathcal{X} \]

and a sequence of mappings

\[ \psi_\ell : \mathbb{R}^{\ell-1} \to \mathcal{X}, \quad \ell \geq 2. \]

For every \( f \in F \) the evaluation starts at the point \( x_1 \), and the mappings \( \psi_\ell \) determine the subsequent evaluation points. More precisely, after \( n \) steps the functional values

\[ y_1 = f(x_1) \]

and

\[ y_\ell = f(\psi_\ell(y_1, \ldots, y_{\ell-1})), \quad \ell = 2, \ldots, n, \]

are known. A decision to stop or to further evaluate \( f \) is made after each step. This is formally described by a sequence of mappings

\[ \tau_\ell : \mathbb{R}^\ell \to \{0, 1\}, \quad \ell \geq 1, \]

and the total number \( n(f) \) of evaluations is given by

\[ n(f) = \min\{\ell \geq 1 : \tau_\ell(y_1, \ldots, y_\ell) = 1\}, \]

which is finite for every \( f \in F \) by assumption. Finally, an approximation

\[ \hat{S}(f) = \phi_{n(f)}(y_1, \ldots, y_{n(f)}) \]

to \( S(f) \) is defined by a sequence of mappings

\[ \phi_\ell : \mathbb{R}^\ell \to \mathbb{R}, \quad \ell \geq 1. \]

Any such mapping \( \hat{S} : F \to \mathbb{R} \) could be considered as a deterministic algorithm, with algorithm being understood in a broad sense, and the corresponding class of mappings is denoted by \( S^{\text{det}} \). For convenience, we identify \( \hat{S} \) with the point \( x_1 \) and the sequences of mappings \( \psi_\ell, \tau_\ell, \) and \( \phi_\ell \). Moreover, we write \( \text{card}(\hat{S}, f) \) instead of \( n(f) \), and this quantity is called the cardinality of \( \hat{S} \) applied to \( f \). Note that \( S^{\text{det}} \) contains in particular all quadrature formulas

\[ \hat{S}(f) = \sum_{i=1}^{n} a_i \cdot f(x_i) \]

with \( a_i \in \mathbb{R} \) and \( x_i \in \mathcal{X} \). Here all mappings \( \psi_\ell \) and \( \tau_\ell \) are constant with \( \psi_2 = x_2, \ldots, \psi_n = x_n \) and \( \tau_1 = \cdots = \tau_{n-1} = 0 \) while \( \tau_n = 1 \), i.e., all functionals \( f \in F \) are evaluated non-sequentially at the same set of \( n \) points, and \( \phi_n \) is linear.

A randomized (or Monte Carlo) broad sense algorithm based on sequential evaluation is formally defined by a probability space \( (\Omega, \mathcal{A}, P) \) and a mapping

\[ \hat{S} : \Omega \times F \to \mathbb{R} \]
such that
\( (i) \hat{S}(\omega, \cdot) \in \mathbb{S}^{\text{det}} \) for every \( \omega \in \Omega \),
\( (ii) \hat{S}(\cdot, f) \) is measurable for every \( f \in F \),
\( (iii) \omega \mapsto \text{card}(\hat{S}(\omega, \cdot), f) \) is measurable for every \( f \in F \).

We refer to Nemirovsky, Yudin (1983) and Wasilkowski (1989) for this and an equivalent
definition of randomized algorithms. In the sequel the random variables from (ii) and (iii)
are denoted by \( \hat{S}(f) \) and \( \text{card}(\hat{S}, f) \), respectively.

By \( \mathbb{S}^{\text{ran}} \) we denote the class of all mappings \( \hat{S} \) with properties (i)–(iii) on any probability
space. Clearly, \( \mathbb{S}^{\text{det}} \subset \mathbb{S}^{\text{ran}} \). Note that \( \mathbb{S}^{\text{ran}} \) contains in particular the classical (abstract)
Monte Carlo method
\[
(1) \quad \hat{S}(f) = \frac{1}{n} \sum_{i=1}^{n} f(X_i)
\]
with \( X_1, \ldots, X_n \) being independent and distributed according to \( \mu \).

The worst case error of \( \hat{S} \in \mathbb{S}^{\text{ran}} \) is defined by
\[
e(\hat{S}) = \sup_{f \in F} \left( \mathbb{E}|S(f) - \hat{S}(f)|^2 \right)^{1/2},
\]
which in particular for \( \hat{S} \in \mathbb{S}^{\text{det}} \) reads
\[
e(\hat{S}) = \sup_{f \in F} |S(f) - \hat{S}(f)|.
\]
The worst case cardinality of \( \hat{S} \in \mathbb{S}^{\text{ran}} \) is defined by
\[
\text{card}(\hat{S}) = \sup_{f \in F} \mathbb{E}(\text{card}(\hat{S}, f)),
\]
which in particular for \( \hat{S} \in \mathbb{S}^{\text{det}} \) reads
\[
\text{card}(\hat{S}) = \sup_{f \in F} \text{card}(\hat{S}, f).
\]

For simplicity we assume that \( \text{card}(\hat{S}) \in \mathbb{N} \) for randomized algorithms, too.

Minimization of the worst case error among those broad sense algorithms that use at
most \( n \) evaluations (on average) leads to the definition of the \( n \)-th minimal errors
\[
\epsilon_n^{\text{ran}} = \inf \{ e(\hat{S}) : \hat{S} \in \mathbb{S}^{\text{ran}}, \text{card}(\hat{S}) \leq n \}
\]
and
\[
\epsilon_n^{\text{det}} = \inf \{ e(\hat{S}) : \hat{S} \in \mathbb{S}^{\text{det}}, \text{card}(\hat{S}) \leq n \}.
\]

We add that minimal errors are key quantities in information-based complexity, see, e.g.,
Traub, Wasilkowski, Woźniakowski (1988), Novak (1988), and Ritter (2000).

In Sections 2.2 and 2.3 we relate the minimal errors to quantization numbers. The \( n \)-th quantization number \( q_n^{(r)} \) of order \( r > 0 \) is defined as
\[
q_n^{(r)} = \inf_{x_1, \ldots, x_n \in \mathbb{X}} q^{(r)}(x_1, \ldots, x_n),
\]
where
\[
q^{(r)}(x_1, \ldots, x_n) = \left( \int_{\mathbb{X}} \min_{i=1, \ldots, n} \| x - x_i \|^{r} \mu(dx) \right)^{1/r},
\]
see, e.g., Graf, Luschgy (2000). In this context a collection of points \( x_1, \ldots, x_n \in \mathbb{X} \) is called
a codebook for quantization of the probability measure \( \mu \). For notational convenience we
let $q_n = q_n^{(1)}$ and $q = q^{(1)}$. Note that $q_n < \infty$, and furthermore $\lim_{n \to \infty} q_n = 0$ if $\mathfrak{X}$ is separable.

2.2. Deterministic Algorithms. The quantization problem and the quadrature problem by means of broad sense deterministic algorithms are equivalent in the following sense. Since $S$ is a real-valued linear mapping on a convex and symmetric set $F$, it follows that

$$e_{\text{det}}^n = \inf \{ e(\widehat{S}) : \widehat{S} \in S^{\text{det}} \text{ is a quadrature formula, } \text{card}(\widehat{S}) \leq n \},$$

see Smolyak (1965), Bakhvalov (1971), and also Traub, Wasilkowski, Woźniakowski (1988, Chap. 4.5). Furthermore, for $F$ and $S$ as studied in this paper we have

$$\inf \{ e(\widehat{S}) : \widehat{S} \in S^{\text{det}} \text{ is a quadrature formula based on } x_1, \ldots, x_n \} = q(x_1, \ldots, x_n)$$

for every codebook $x_1, \ldots, x_n \in \mathfrak{X}$, see Kantorovich, Rubinstein (1958) and Gray, Neuhoff, Shields (1975). The latter infimum is attained by the quadrature formula

$$\widehat{S}(f) = \sum_{i=1}^{n} \mu(V_i) \cdot f(x_i),$$

if $V_1, \ldots, V_n$ is a corresponding Voronoi partition of $\mathfrak{X}$. An (almost) optimal codebook therefore yields an (almost) optimal quadrature formula (3), and the $n$-th minimal error $e_{\text{det}}^n$ coincides with the $n$-th quantization number of order one.

**Theorem 1.** For every $n \in \mathbb{N}$

$$e_{\text{det}}^n = q_n.$$

**Remark 1.** There are numerous results on $e_{\text{det}}^n$ or $q_n$ for finite-dimensional spaces $\mathfrak{X} = \mathbb{R}^d$, see, e.g., Novak (1988), Graf, Luschgy (2000), Wasilkowski, Woźniakowski (1996, 2001).

Assume $r \geq 1$. Then, under rather mild assumptions on $\mu$, and in particular for the uniform distribution on $[0, 1]^d$, the quantization numbers satisfy

$$\lim_{n \to \infty} q^{(r)}_n \cdot n^{1/d} = c^{(r)}$$

with some constant $c^{(r)} = c^{(r)}(\mu, d, \| \cdot \|) > 0$, see Graf, Luschgy (2000, Thm. 6.2).

**Remark 2.** Much less is known about $e_{\text{det}}^n$ or $q_n$ for infinite-dimensional spaces $\mathfrak{X}$, see Wasilkowski, Woźniakowski (1996), Dereich et al. (2003), Dereich (2003, 2004), and Luschgy, Pagès (2003, 2004) for results and references. If $\mu$ is the distribution of a diffusion process or a Gaussian process then, typically, the quantization numbers $q^{(r)}_n$ tend to zero only with logarithmic order, see Sections 4 and 5. For such processes we conclude from Theorem 4 that quadrature of arbitrary Lipschitz functionals by means of (broad sense) deterministic algorithms is intractable.

As an example consider the Wiener measure $\mu$ on $\mathfrak{X} = C([0, 1])$ endowed with the supremum norm. In this case

$$\lim_{n \to \infty} q^{(r)}_n \cdot (\ln n)^{1/2} = c$$

with some constant $c > 0$, see Dereich, Scheutzow (2005).
2.3. Randomized Algorithms. We first state an upper bound for the minimal error $e_{2n}^{ran}$ in terms of the quantization number $q_n^{(2)}$, which is a consequence of a well-known variance reduction technique based on quantization, see Pagès, Printems (2004). Note that $\lim_{n \to \infty} q_n^{(2)} = 0$ if $\mathcal{X}$ is separable and $\int_{\mathcal{X}} \|x\|^2 \mu(dx) < \infty$. Under the latter assumption the classical Monte Carlo method (I) without variance reduction only yields errors of order $n^{-1/2}$ in all non-trivial cases.

**Theorem 2.** For every $n \in \mathbb{N}$

$$
e_{2n}^{ran} \leq 2 \cdot n^{-1/2} \cdot q_n^{(2)}.
$$

**Proof.** Consider a codebook $x_1, \ldots, x_n \in \mathcal{X}$ as well as a corresponding Voronoi partition $V_1, \ldots, V_n$ of $\mathcal{X}$. For $f \in F$ let $J(f)$ denote the interpolation of $f$ at the points $x_i$ that is constant on the corresponding cells $V_i$, i.e.,

$$J(f) = \sum_{i=1}^{n} f(x_i) \cdot 1_{V_i}.
$$

The deterministic broad sense algorithm (3) approximates $S(f)$ by $S(J(f))$. Define a broad sense randomized algorithm $\hat{S} \in S^{ran}$ with $\text{card}(\hat{S}) \leq 2n$ by

$$
\hat{S}(f) = S(J(f)) + \frac{1}{n} \sum_{i=1}^{n} (f - J(f))(X_i)
$$

with $X_1, \ldots, X_n$ being independent and distributed according to $\mu$. Hence the non-deterministic part of $\hat{S}$ consists of applying the classical Monte Carlo method (I) to $\bar{f} = f - J(f)$. It follows that

$$e(\hat{S}) = n^{-1/2} \cdot \sup_{f \in F} \left( \int_{\mathcal{X}} \left( \bar{f}(x) - S(\bar{f}) \right)^2 \mu(dx) \right)^{1/2}.
$$

Since $|\bar{f}(x)| \leq \min_{i=1,\ldots,n} \|x - x_i\|$, we obtain

$$
\left( \int_{\mathcal{X}} \left( \bar{f}(x) - S(\bar{f}) \right)^2 \mu(dx) \right)^{1/2} \leq \left( \int_{\mathcal{X}} \bar{f}^2(x) \mu(dx) \right)^{1/2} + |S(\bar{f})|
$$

$$
\leq q^{(2)}(x_1, \ldots, x_n) + q^{(1)}(x_1, \ldots, x_n)
\leq 2 \cdot q^{(2)}(x_1, \ldots, x_n),
$$

which completes the proof. \qed

We now turn to lower bounds for (broad sense) randomized algorithms. In this setting a result analogous to (2) is not available in general, and therefore considerations cannot a priori be restricted to randomized quadrature formulas. We use the following tool, which is due to Bakhvalov (1959) and Novak (1988) and which holds for integration problems in general, see Novak (1988, Sec. 2.2.10).

**Proposition 1.** Let $m \geq 4n$, and suppose there are functionals $f_1, \ldots, f_m : \mathcal{X} \to \mathbb{R}$ such that

$$
\{x \in \mathcal{X} : f_i(x) \neq 0\} \cap \{x \in \mathcal{X} : f_j(x) \neq 0\} = \emptyset
$$

(7)
for all \( i \neq j \) and

\[
\sum_{i=1}^{m} \delta_i \cdot f_i \in F
\]

for all \( \delta_1, \ldots, \delta_m \in \{\pm 1\} \). Then

\[
e_{\text{ran}}^n \geq \frac{1}{4} \cdot n^{1/2} \cdot \min_{i=1,\ldots,m} S(f_i).
\]

A proper choice of the functionals \( f_i \) in Proposition 1 yields a lower bound for the minimal error \( e_{\text{ran}}^n \) in terms of consecutive differences of quantization numbers.

**Theorem 3.** For every \( n \in \mathbb{N} \)

\[
e_{\text{ran}}^n \geq \frac{1}{8} \cdot n^{1/2} \cdot \sup_{m \geq 4n} (q_{m-1} - q_m).
\]

**Proof.** For \( \varepsilon \in ]0,1[ \) and \( m \geq 4n \) choose \( x_1, \ldots, x_m \in \mathcal{X} \) with

\[
qu(x_1, \ldots, x_m) \leq \varepsilon \cdot q_{m-1} + (1 - \varepsilon) \cdot q_m + \varepsilon,
\]

and consider the functionals

\[
f_i(x) = \frac{1}{2} \cdot \max(0, \min_{j \neq i} \|x - x_j\| - \|x - x_i\|), \quad i = 1, \ldots, m.
\]

Clearly (7) is satisfied and \( f_1, \ldots, f_m \in F \). Consequently (8) holds, too.

We claim that

\[
S(f_i) \geq \frac{1}{2} \cdot \min_{k=1,\ldots,m-1} \|x - x_k\| = \|x - x_i\|.
\]

It suffices to prove the statement for \( i = m \). To this end consider a Voronoi partition \( V_1, \ldots, V_m \) corresponding to \( x_1, \ldots, x_m \), and let \( U_1, \ldots, U_{m-1} \) be a Voronoi partition corresponding to \( x_1, \ldots, x_{m-1} \). If \( j \leq m-1 \) and \( x \in V_m \cap U_j \) then

\[
f_m(x) = \frac{1}{2} \cdot (\|x - x_j\| - \|x - x_m\|).
\]

Hence

\[
\int_{\mathcal{X}} f_m(x) \mu(dx) = \int_{V_m} f_m(x) \mu(dx)
\]

\[
= \frac{1}{2} \cdot \sum_{j=1}^{m-1} \int_{V_m \cap U_j} \|x - x_j\| \mu(dx) - \frac{1}{2} \cdot \int_{V_m} \|x - x_m\| \mu(dx)
\]

\[
= \frac{1}{2} \cdot \sum_{j=1}^{m-1} \int_{(V_m \cap U_j) \cup V_j} \|x - x_j\| \mu(dx) - \frac{1}{2} \cdot \sum_{j=1}^{m} \int_{V_j} \|x - x_j\| \mu(dx).
\]

Note that the sets \((V_m \cap U_j) \cup V_j\) with \( j \leq m-1 \) form a partition of \( \mathcal{X} \) as well, and every \( x \in (V_m \cap U_j) \cup V_j \) satisfies

\[
\min_{k=1,\ldots,m-1} \|x - x_k\| = \|x - x_j\|.
\]

Thus

\[
S(f_m) = \frac{1}{2} \cdot (q(x_1, \ldots, x_{m-1}) - q(x_1, \ldots, x_m))
\]

and (10) follows from (9).

It remains to apply Proposition 1 and to let \( \varepsilon \) tend to zero. \( \square \)
The following consequence of Theorem 3 is useful, in particular, for finite-dimensional spaces $X$.

**Corollary 1.** If the sequence $(q_n)_{n \in \mathbb{N}}$ is regularly varying with index $-\alpha < 0$ then

$$
\lim \inf_{n \to \infty} \epsilon_{\text{ran}}^n \cdot n^{1/2}/q_n \geq \frac{\alpha}{25 + 2\alpha}.
$$

*Proof.* Put

$$
g(m) = \sup_{\ell \geq m} (q_{\ell-1} - q_{\ell})
$$

for $m \in \mathbb{N} \setminus \{1\}$ and let $\kappa > 1$. Clearly,

$$
g(m) \geq \frac{q_m - q_{[km]}}{[km] - m} = q_m \cdot \frac{1 - q_{[km]}/q_m}{[km] - m}.
$$

Since $\lim_{m \to \infty} q_{[km]}/q_m = \kappa^{-\alpha}$ it follows that

$$
\lim \inf_{m \to \infty} g(m) \cdot m/q_m \geq \frac{1 - \kappa^{-\alpha}}{\kappa - 1}.
$$

Letting $\kappa$ tend to one yields

$$
\lim \inf_{m \to \infty} g(m) \cdot m/q_m \geq \alpha.
$$

Combining the latter estimate and Theorem 3 completes the proof. □

**Remark 3.** Suppose that the quantization numbers satisfy (4), which typically holds in the finite-dimensional case $X = \mathbb{R}^d$, see Remark 1. Then Corollary 1 is applicable with $\alpha = 1/d$, and we obtain

$$
\lim \inf_{n \to \infty} \epsilon_{\text{ran}}^n \cdot n^{1/2 + 1/d} \geq \frac{c^{(1)}}{d \cdot 25 + 2/d}.
$$

A matching upper bound is provided by Theorem 2 so that we end up with the well-known fact

$$
\epsilon_{\text{ran}}^n \asymp n^{-1/2 - 1/d},
$$

see Novak (1988, Sec. 2.2.6) for the case of the uniform distribution $\mu$ on $[0, 1]^d$.

From the previous remark we conclude that, up to multiplicative constants, neither the upper bound in Theorem 2 nor the lower bound in Theorem 3 can be improved in general.

Corollary 1 is not applicable, if the quantization numbers are slowly varying, cf. Remark 2. Instead, one may use the following result.

**Corollary 2.** Let $f : [0, \infty[ \to ]0, \infty[$ be a convex and differentiable function. If

$$
\lim \sup_{n \to \infty} q_n/f(n) \geq 1
$$

and

$$
\lim_{n \to \infty} q_n = 0,
$$

then

$$
\lim \sup_{n \to \infty} \epsilon_{\text{ran}}^n / (n^{1/2} \cdot |f'(4n + 3)|) \geq 1/8.
$$
Proof. Fix $\varepsilon \in [0, 1]$. By assumption
\[ q_{m-1} \geq (1 - \varepsilon) \cdot f(m - 1) = (1 - \varepsilon) \cdot \int_{m-1}^{\infty} -f'(s) \, ds \]
holds for infinitely many integers $m$. Since $q_{m-1} = \sum_{k=m}^{\infty} (q_{k-1} - q_k)$, we also have
\[ q_{m-1} - q_m \geq (1 - \varepsilon) \cdot \int_{m-1}^{m} -f'(s) \, ds \geq -(1 - \varepsilon) \cdot f'(m) \]
infinity often. To every such $m$ we associate $n = \lfloor m/4 \rfloor$. Then $m \in [4n, 4n + 3]$ and Theorem 3 implies
\[ e_{\text{ran}}^n \geq -(1 - \varepsilon)/8 \cdot n^{1/2} \cdot f'(4n + 3). \]
Letting $\varepsilon$ tend to zero finishes the proof. \qed

Remark 4. Suppose that the quantization numbers satisfy
\[ q_n^{(r)} \approx (\ln n)^{-1/2}, \]
which typically holds for diffusion processes, see Proposition 3 and in particular for the Wiener measure, see Remark 2. Then Corollary 2 is applicable with $f(t) = c \cdot (\ln t)^{-1/2}$ for some constant $c > 0$, and we obtain
\begin{equation}
\limsup_{n \to \infty} e_{\text{ran}}^n \cdot n^{1/2} \cdot (\ln n)^{3/2} > 0. \tag{11}
\end{equation}
On the other hand,
\begin{equation}
\limsup_{n \to \infty} e_{\text{ran}}^n \cdot n^{1/2} \cdot (\ln n)^{1/2} < \infty \tag{12}
\end{equation}
by Theorem 2. This upper bound is achieved by a sequence of comparatively simple broad sense randomized algorithms, see (6), which are far superior to any sequence of (broad sense) deterministic algorithms, see Theorem 1. Moreover, upper and lower bounds do not differ much for an infinite sequence of integers $n$. We add that, for a large class of diffusion processes, inequality (11) holds true with limes superior replaced by limes inferior, see Proposition 4.

3. Finite-dimensional Sampling and Kolmogorov Widths

So far we have studied broad sense algorithms $\hat{S} \in \mathbb{S}^\text{ran}$, and we have expressed the quality of such an algorithm in terms of its error $e(\hat{S})$ and its cardinality $\text{card}(\hat{S})$. The cardinality serves as a crude measure of the cost of $\hat{S}$, if one assumes that evaluation of functionals $f \in F$ is possible at any point $x \in \mathbb{X}$ at cost one and if all further operations are not taken into account. Moreover, by definition of $\mathbb{S}^\text{ran}$, a broad sense randomized algorithm may use perfect generators for random elements according to any Borel probability measure on $\mathbb{X}$, in particular according to $\mu$. These assumptions are rather unrealistic and do not correspond to a reasonable model of computation, and the practical relevance of algorithms like (6) and upper bounds like (12) seems to be doubtful. We stress that this point of view concerns lower bounds like (11) only in the sense that they are ‘too weak’.

It is more appropriate to take the real number model of computation as a basis for quadrature problems. See Traub, Wasilkowski, Woźniakowski (1988) and Novak (1995) for the definition of this model. Informally, a real number algorithm is like a C-program that carries out exact computations with real numbers. Furthermore, a perfect generator for random numbers from $[0, 1]$ as well as elementary functions like exp, ln, etc. are
available. We think that these assumptions are present at least implicitly in most of the work dealing with quadrature problems. Algorithms have access to the functionals \( f \in F \) via an oracle (subroutine) that provides values \( f(x) \) for points \( x \) from a finite-dimensional subspace \( X_0 \subset X \). The subspace may be chosen arbitrarily but it is fixed for a specific algorithm, and the cost for each oracle call is proportional to the dimension of \( X_0 \).

**Example 1.** Consider the distribution \( \mu \) of a diffusion process \( X \) with values in \( X = C([0,1], \mathbb{R}^m) \). Let \( \hat{X}^{(k)} \) denote the Euler scheme with uniform step-size \( 1/(k-1) \) and piecewise linear interpolation, and define the classical Euler Monte Carlo algorithm \( \hat{S}_n^{(k)} \) by

\[
\hat{S}_n^{(k)}(f) = 1/n \cdot \sum_{i=1}^{n} f(\hat{X}_i^{(k)})
\]

with independent copies \( \hat{X}_1^{(k)}, \ldots, \hat{X}_n^{(k)} \) of \( \hat{X}^{(k)} \). This algorithm uses an oracle for the \( k \)-dimensional subspace of piecewise linear functions with breakpoints at \( \ell/(k-1) \). Moreover, only random numbers from \([0,1]\) are needed for the computation of \( \hat{S}_n^{(k)}(f) \).

For simplicity we assume that the cost of an oracle call for functional evaluation coincides with the dimension \( k \) of the corresponding subspace \( X_0 \) and that real number operations as well as calls of the random number generator and evaluations of elementary functions are performed at cost one. Furthermore, in case of \( \mu \) being the distribution of a diffusion process, function values of its drift and diffusion coefficients are provided at cost one, too. Then the total cost of a computation is given by \( k \times \text{number of oracle calls for functional evaluation} + \text{total number of real number operations, calls of the random number generator, evaluations of elementary functions, and, eventually, function evaluations of drift and diffusion coefficients} \).

For randomized algorithms \( \hat{S} \) the computational cost is a random variable, which also may depend on the integrand \( f \in F \). Analogously to \( \text{card}(\hat{S}) \) we therefore define \( \text{cost}(\hat{S}) \), the worst case cost of \( \hat{S} \), by its maximal expected cost over the class \( F \).

**Remark 5.** For the classical Euler Monte Carlo algorithm we have

\[
\text{cost}(\hat{S}_n^{(k)}) \approx k \cdot n,
\]

i.e., the cost is proportional to the product of the dimension of the subspace and the number of oracle calls for functional evaluation. Equivalently, the cost is proportional to the product of the number of time steps and the number of repetitions.

Analogously to \( \varepsilon_n^{\text{ran}} \) we introduce the \( N \)-th minimal error

\[
\varepsilon_N^{\text{ran}} = \inf \{ e(\hat{S}) : \hat{S} \text{ randomized algorithm with cost}(\hat{S}) \leq N \}
\]

in the real number model. By just counting the number of oracle calls we get \( \varepsilon_N^{\text{ran}} \geq \varepsilon_N^{\text{ran}} \). To derive a lower bound for \( \varepsilon_N^{\text{ran}} \) that also takes into account the dimension of the subspaces \( X_0 \) we study the \( k \)-th average Kolmogorov width of order \( p > 0 \)

\[
d_k^{(p)} = \inf \left\{ \left( \int_X \text{dist}^p(x, X_0) \mu(dx) \right)^{1/p} : \dim(X_0) = k \right\}
\]

for the measure \( \mu \). For notational convenience we let \( d_k = d_k^{(1)} \). See, e.g., Ritter (2000, Sec. VII.2.5) and Creutzig (2002) for results and references.
The following lower bound corresponds to the extremal cases, where either the dimension $k$ of the subspace or the number $n$ of evaluations may be arbitrarily large.

**Proposition 2.** For every $N \in \mathbb{N}$

$$\varepsilon_{\text{ran}} \geq \inf_{k,n \leq N} \max(e_{\text{ran}}^n, d_k).$$

**Proof.** Consider any randomized algorithm $\hat{S}$ with cost($\hat{S}$) $\leq N$, and assume that its oracle is based on a $k$-dimensional subspace $\mathcal{X}_0 \subset \mathcal{X}$. Define a functional $f_0 \in F$ by $f_0 = \text{dist}(\cdot, \mathcal{X}_0)$. Since $\hat{S}$ evaluates $f_0$ only at points from $\mathcal{X}_0$ we have

$$\hat{S}(f_0) = \hat{S}(-f_0),$$

and consequently

$$e(\hat{S}) \geq \frac{1}{2} \cdot \left( \left( \mathbb{E}|S(f_0) - \hat{S}(f_0)|^2 \right)^{1/2} + \left( \mathbb{E}|S(-f_0) - \hat{S}(-f_0)|^2 \right)^{1/2} \right) \geq S(f_0).$$

Hence

$$e(\hat{S}) \geq \int_{\mathcal{X}} \text{dist}(x, \mathcal{X}_0) \mu(dx) \geq d_k.$$ 

On the other hand, put $n = \text{card}(\hat{S})$ to obtain

$$e(\hat{S}) \geq e_{\text{ran}}^n.$$ 

We conclude that $e(\hat{S}) \geq \max(e_{\text{ran}}^n, d_k)$ for some $k, n \in \mathbb{N}$ such that $k \cdot n \leq N$. \hfill $\square$

4. **Randomized Algorithms for Diffusion Processes**

In this section we consider the distribution $\mu$ of an $m$-dimensional diffusion process $X$ on the space $C([0,1], \mathbb{R}^m)$, equipped with the supremum norm. More precisely, $X$ is given by

(14)

$$dX_t = a(X_t) \, dt + b(X_t) \, dW_t,$$

$$X_0 = u_0 \in \mathbb{R}^m$$

for $t \in [0,1]$ with an $m$-dimensional Brownian motion $W$, and we assume that the following conditions are satisfied:

(i) $a : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is Lipschitz continuous

(ii) $b : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$ has bounded first and second order partial derivatives and is of class $C^\infty$ in some neighborhood of $u_0$

(iii) $\det b(u_0) \neq 0$

We first present bounds for the quantization numbers and the Kolmogorov widths, see also Remark 7. The corresponding proofs are postponed to Section 4.2.

**Proposition 3.** The quantization numbers $q_n^{(r)}$ satisfy

$$q_n^{(r)} \approx (\ln n)^{-1/2}$$

for every $r > 0$. The average Kolmogorov widths $d_k^{(p)}$ satisfy

$$d_k^{(p)} \approx k^{-1/2}$$

for every $p > 0$. 
The asymptotic behavior of the quantization numbers stated in Proposition 3 is partially known. Luschgy and Pages (2003) study scalar stochastic differential equations under suitable growth and smoothness conditions. In this work the upper bound is established for equations with a strictly positive diffusion coefficient \( b : [0, 1] \times \mathbb{R} \to \mathbb{R} \), and a matching lower bound is derived if \( \inf_{(t,x) \in [0,1] \times \mathbb{R}} b(t, x) > 0 \) and \( r \geq 1 \). More generally, \( m \)-dimensional diffusions with a scalar diffusion coefficient \( b : [0, 1] \times \mathbb{R}^m \to \mathbb{R} \) are analyzed by Dereich (2004), who determines the exact asymptotic behavior of the quantization numbers for \( r \geq 1 \) under rather mild smoothness assumptions. The asymptotic behavior of the Kolmogorov widths is determined by Maiorov (1993) for the Brownian motion.

Observing Theorem 4, we conclude that quadrature of arbitrary Lipschitz functionals is intractable by means of deterministic algorithms.

We next present a lower bound for the minimal error \( \varepsilon_n^{\text{ran}} \), which improves the estimate (11). See Section 4.3 for the corresponding proof.

**Proposition 4.** The minimal errors \( \varepsilon_n^{\text{ran}} \) satisfy
\[
\varepsilon_n^{\text{ran}} \geq n^{-1/2} \cdot (\ln n)^{-3/2}.
\]

Propositions 2, 3, and 4 immediately yield the following lower bound.

**Theorem 4.** The minimal errors \( \varepsilon_N^{\text{ran}} \) satisfy
\[
\varepsilon_N^{\text{ran}} \geq N^{-1/4} \cdot (\ln N)^{-3/4}.
\]

Consider the Euler Monte Carlo algorithm \( \hat{S}_n^{(k)} \) for equation (14) with normally distributed increments. More precisely, put \( \hat{X}_{i,0}^{(k)} = u_0 \) and define
\[
\hat{X}_{i,\ell+1}^{(k)} = \hat{X}_{i,\ell}^{(k)} + 1/(k - 1) \cdot a(\hat{X}_{i,\ell}^{(k)}) + 1/\sqrt{k - 1} \cdot b(\hat{X}_{i,\ell}^{(k)}) \cdot Z_{i,\ell}
\]
for \( i = 1, \ldots, n \) and \( \ell = 0, \ldots, k - 2 \). Here \((Z_{i,\ell})_{i,\ell}\) is an independent family of \( m \)-dimensional standard normally distributed random vectors. Finally, let \( \hat{X}_{i}^{(k)} \) denote the piecewise linear interpolation of \( \hat{X}_{i,0}, \ldots, \hat{X}_{i,k-1} \) at the breakpoints \( \ell/(k - 1) \). Then \( S_n^{(k)} \) is given by (13).

**Theorem 5.** The Euler Monte Carlo algorithm \( \hat{S}_N = \hat{S}_n^{(k)} \) with \( n = \lfloor N^{1/2} \cdot (\ln N)^{-1/2} \rfloor \) and \( k = \lfloor N^{1/2} \cdot (\ln N)^{1/2} \rfloor \) satisfies
\[
e(\hat{S}_N) \leq N^{-1/4} \cdot (\ln N)^{1/4}
\]
and
\[
\text{cost}(\hat{S}_N) \leq N.
\]

**Proof.** Consider the strong Euler scheme \( \hat{X}^{(k)} \) with step-size \( 1/(k - 1) \) and piecewise linear interpolation for equation (13). Then
\[
E\|X - \hat{X}^{(k)}\|_{\infty} \leq c_1 \cdot k^{-1/2} \cdot (\ln k)^{1/2}
\]
with some constant \( c_1 > 0 \) that does not depend on \( k \), see Faure (1992). Let \( f \in F \). Since \( S(f) = E(f(X)) \) and \( E(\hat{S}_N(f)) = E(f(\hat{X}^{(k)})) \), we get
\[
|S(f) - E(\hat{S}_N(f))| \leq c_1 \cdot (\ln k/k)^{1/2}
\]
for the bias of \( \hat{S}_N(f) \) by means of (14). Put \( g = f - f(0) \) to obtain
\[
\nabla(\hat{S}_N(f)) = \nabla(\hat{S}_N(g)) \leq 1/n \cdot E(g^2(\hat{X}^{(k)})) \leq 1/n \cdot E(\|\hat{X}^{(k)}\|_{\infty}) \leq c_2 \cdot 1/n
\]
for the variance of $\widehat{S}_N(f)$, where the constant $c_2 > 0$ depends neither on $k$ nor on $f$. We conclude that

$$
\mathbb{E}(S(f) - \widehat{S}_N(f))^2 \leq \max(c_1^2, c_2) \cdot (1/n + \ln k/k),
$$

and with the particular choice of $n$ and $k$ the asymptotic upper bound for the error of $\widehat{S}_N = \widehat{S}_n^{(k)}$ follows. The cost of $\widehat{S}_N$ is determined in Remark \(\square\)

Combine Theorems 4 and 5 to conclude that the Euler Monte Carlo algorithm $\widehat{S}_N$ is almost optimal.

Corollary 3.

$$
e(\widehat{S}_N) \preceq e_{ran}^N \cdot \ln N.
$$

4.1. Preliminaries. A basic idea in the proofs of Propositions 3 and 4 is to reduce the case of an $m$-dimensional diffusion process with properties (i)–(iii) to the particular case of a one-dimensional Brownian motion by means of Lipschitz transformations and stopping.

Let $X$ denote any random element with values in some Banach space $\mathfrak{X}$ and consider its distribution $\mu$ on this space. We use the notation

$$
e_n^{ran}(X, \mathfrak{X}) = e^{ran}_n
$$

for the $n$-th minimal error of randomized algorithms,

$$
d_k^{(p)}(X, \mathfrak{X}) = d_k^{(p)}
$$

for the $k$-th average Kolmogorov width of order $p$, and

$$
q_n^{(r)}(X, \mathfrak{X}) = q_n^{(r)}
$$

for the $n$-th quantization number of order $r$.

Consider a measurable mapping $T : \mathfrak{X} \to \mathfrak{Y}$, where $\mathfrak{Y}$ is a Banach space, too. The following observation is straightforward to verify. We add that an analogous result for Kolmogorov widths is not available.

Lemma 1. Suppose that $T$ is Lipschitz continuous with a Lipschitz constant $L > 0$. Then

$$
e_n^{(ran)}(TX, \mathfrak{Y}) \leq L \cdot e_n^{ran}(X, \mathfrak{X})
$$

and

$$
q_n^{(r)}(TX, \mathfrak{Y}) \leq L \cdot q_n^{(r)}(X, \mathfrak{X}).
$$

We formulate a simplified version of a general relation between quantization numbers and average Kolmogorov widths, which is due to Creutzig (2002, Thm. 4.6.1).

Lemma 2. For $0 < r < p$

$$
\sup_{n \leq 2^\ell} \ln n \cdot q_n^{(r)}(X, \mathfrak{X}) \preceq \sup_{k \leq \ell} k \cdot d_k^{(p)}(X, \mathfrak{X}).
$$

The following contraction principle holds for best approximation of sums of independent and symmetric random elements.

Lemma 3. Let $X_1, \ldots, X_k$ denote a sequence of independent and symmetric random elements with values in $\mathfrak{X}$ and let $p \geq 1$. Then

$$
\mathbb{E} \left( \text{dist}^p \left( \sum_{\ell=1}^k \lambda_\ell X_\ell, \mathfrak{X}_0 \right) \right) \leq \max_{\ell=1, \ldots, k} |\lambda_\ell|^p \cdot \mathbb{E} \left( \text{dist}^p \left( \sum_{\ell=1}^k X_\ell, \mathfrak{X}_0 \right) \right)
$$

for all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and every closed linear subspace $\mathcal{X}_0 \subset \mathcal{X}$.

**Proof.** Take Rademacher variables $\varepsilon_1, \ldots, \varepsilon_k$ such that $\varepsilon_1, \ldots, \varepsilon_k, X_1, \ldots, X_k$ are independent, and consider the quotient mapping $Q : \mathcal{X} \to \mathcal{X}/\mathcal{X}_0$. Since $(X_1, \ldots, X_k)$ and $(\varepsilon_1 X_1, \ldots, \varepsilon_k X_k)$ coincide in distribution, the same holds true for $(QX_1, \ldots, QX_k)$ and $(\varepsilon_1 QX_1, \ldots, \varepsilon_k QX_k)$. Hence

$$
\mathbb{E}(\text{dist}^p\left(\sum_{\ell=1}^k \lambda_\ell X_\ell, \mathcal{X}_0\right)) = \mathbb{E}\left(\sum_{\ell=1}^k \lambda_\ell \cdot QX_\ell\right)^p_{\mathcal{X}/\mathcal{X}_0} = \mathbb{E}\left(\sum_{\ell=1}^k \lambda_\ell \varepsilon_\ell \cdot QX_\ell\right)^p_{\mathcal{X}/\mathcal{X}_0}.
$$

For any choice of elements $y_\ell \in \mathcal{X}/\mathcal{X}_0$

$$
\mathbb{E}\left(\sum_{\ell=1}^k \lambda_\ell \varepsilon_\ell \cdot y_\ell\right)^p_{\mathcal{X}/\mathcal{X}_0} \leq \max_{\ell=1, \ldots, k} |\lambda_\ell|^p \cdot \mathbb{E}\left(\sum_{\ell=1}^k \varepsilon_\ell \cdot y_\ell\right)^p_{\mathcal{X}/\mathcal{X}_0},
$$
due to Kahane’s contraction principle, see Kahane (1993, p. 21). Thus

$$
\mathbb{E}\left(\sum_{\ell=1}^k \lambda_\ell \varepsilon_\ell \cdot QX_\ell\right)^p_{\mathcal{X}/\mathcal{X}_0} \leq \max_{\ell=1, \ldots, k} |\lambda_\ell|^p \cdot \mathbb{E}\left(\sum_{\ell=1}^k \varepsilon_\ell \cdot QX_\ell\right)^p_{\mathcal{X}/\mathcal{X}_0},
$$

which completes the proof. \(\square\)

Now we turn to the diffusion process $X$ given by (14).

**Lemma 4.** There exists a neighborhood $U$ of $u_0$ and a function $f \in C^\infty(U)$ such that

$$(\nabla f)^{bb^*}\nabla f = 1.$$  

**Proof.** Choose a radius $r > 0$ such that $\det bb^*(u) \neq 0$ if $|u - u_0| < r$. Furthermore, take $g \in C^\infty(\mathbb{R}^m, \mathbb{R}^{m \times m})$ with symmetric and positive definite values such that

$$g(u) = (bb^*)^{-1}(u)$$

if $|u - u_0| < r/2$ and $g(u)$ is the identity matrix if $|u - u_0| > r$. Then $M = \mathbb{R}^m$ endowed with the metric tensor $\sum_{i,j=1}^m g_{ij}(u) \cdot du^i \otimes du^j$ is a complete $C^\infty$-Riemannian manifold. Here $u^1, \ldots, u^m$ are the local coordinates obtained when taking the identity as chart. Moreover, let $d_M$ denote the corresponding Riemannian distance.

Choose $v_0 \in M$ such that $0 < |v_0 - u_0| < r/2$ and $0 < d_M(v_0, u_0) < i_{v_0}(M)$, where $i_{v_0}(M)$ denotes the injectivity radius at $v_0$, see Sakai (1996, Prop. III.4.13). Define

$$U = \{u \in M : 0 < |v_0 - u| < r/2, \ 0 < d_M(v_0, u) < i_{v_0}(M)\}$$

as well as

$$f(u) = d_M(v_0, u)$$

for $u \in U$. Then $f \in C^\infty(U)$ and $(\nabla f)^{bb^*}\nabla f = 1$, see Sakai (1996, Prop. III.4.8). \(\square\)

In addition to $C = C([0,1], \mathbb{R}^m)$ we also consider the Banach space $L_1 = L_1([0,1], \mathbb{R}^m)$.

**Lemma 5.** Either let $\mathcal{X} = C$ and $\mathcal{Y} = C([0,1], \mathbb{R})$ or let $\mathcal{X} = L_1$ and $\mathcal{Y} = L_1([0,1], \mathbb{R})$. There exists a Lipschitz continuous mapping $T : \mathcal{X} \to \mathcal{Y}$ and a stopping time $\tau$ with $\mathbb{P}(\tau > 0) = 1$ such that the stopped process

$$\left( TX \right)_t^\tau = \left( TX \right)_{t \wedge \tau}, \quad t \in [0,1],$$

is a Brownian motion stopped at time $\tau$. 

**QUADRATURE AND QUANTIZATION 15**
Proof. Due to Lemma 4 there exists a function $h \in C^\infty(\mathbb{R}^m)$ with bounded derivatives that satisfies
\[(\nabla h)^*bb^*\nabla h = 1\]
on a closed ball with radius $r > 0$ around $u_0$. Define the stopping time
\[\tau = \inf\{t \in [0, 1] : |X_t - u_0| = r\}.
Clearly, $\mathbb{P}(\tau > 0) = 1$.

In both cases, $X = C$ and $X = L_1$ we define a Lipschitz continuous mapping $T : \mathfrak{X} \to \mathfrak{Y}$ by
\[(Tx)(t) = h(x(t)) - h(u_0) - \int_0^t \left((\nabla h)^*a + \frac{1}{2} \sum_{i,j=1}^m (bb^*)_{i,j} \frac{\partial^2}{\partial u_i \partial u_j} h\right)(x(s)) \, ds.
Itô’s formula implies
\[(TX)_t = \int_0^t ((\nabla h)^*b)(X_t) \, dW_t.
Observing (16) we conclude that the stopped process $(TX)^\tau$ is a continuous martingale with quadratic variation
\[\langle (TX)^\tau \rangle_t = \int_0^{t \land \tau} ((\nabla h)^*bb^*\nabla h)(X_s) \, ds = t \land \tau,
which completes the proof. □

Remark 6. The assumption that the diffusion coefficient $b$ is of class $C^\infty$ in a neighborhood of the initial value $u_0$ can be relaxed. For instance, in the one-dimensional case it suffices to assume $b \in C^1([0, 1])$ with Lipschitz continuous first derivative. Then
\[f(u) = \int_{u_0}^u |1/b(v)| \, dv\]
is well defined in a neighborhood of $u_0$, and the statement of Lemma 5 follows with the same proof.

4.2. Proof of Proposition 3. We use the contraction principle from Lemma 3 to establish the upper bound for the Kolmogorov widths.

Lemma 6. For every $p > 0$
\[d^{(p)}_k(X, C) \leq k^{-1/2}.
Proof. Assume that $p \geq 1$ without loss of generality. Fix $k \in \mathbb{N}$, put $t_\ell = \ell/k$ for $\ell = 0, \ldots, k$, and consider the corresponding Euler process $\overline{X}^{(k)}_t$ defined by $\overline{X}^{(k)}_0 = u_0$ and
\[\overline{X}^{(k)}_t = \overline{X}^{(k)}_{t_\ell} + a(\overline{X}^{(k)}_{t_\ell}) \cdot (t - t_\ell) + b(\overline{X}^{(k)}_{t_\ell}) \cdot (W_t - W_{t_\ell})\]
for $t \in [t_\ell, t_{\ell+1}]$. We have
\[\mathbb{E}||X - \overline{X}^{(k)}||_\infty^p \leq k^{-p/2},
see Bouleau, Lépingle (1994, p. 276), and therefore
\[d^{(p)}_k(X, C) \leq k^{-1/2} + d^{(p)}_k(\overline{X}^{(k)}, C),\]
Let \( \tilde{W}^{(k)} \) denote the piecewise linear interpolation of the Brownian motion \( W \) at the breakpoints \( t_\ell \) and define the continuous process \( V^{(k)} \) by
\[
V^{(k)}_t = b(\overline{X}_t^{(k)}) \cdot (W_t - \tilde{W}^{(k)}_t)
\]
for \( t \in [t_\ell, t_{\ell+1}] \). Note that \( \overline{X}^{(k)} - V^{(k)} \) takes values in the \((k + 1)\)-dimensional subspace of piecewise linear functions with breakpoints \( t_\ell \). Hence
\[
d_{2k+1}^{(p)}(\overline{X}^{(k)}, C) \leq d_k^{(p)}(V^{(k)}, C).
\]
Let \( \mathfrak{A} \) denote the \( \sigma \)-algebra generated by \( W(t_1), \ldots, W(t_k) \). The random variables \( b(\overline{X}_t^{(k)}) \) are measurable with respect to \( \mathfrak{A} \), and conditioned on \( \mathfrak{A} \) the process \( W - \tilde{W}^{(k)} \) consists of independent Brownian bridges on the subintervals \([t_\ell, t_{\ell+1}]\). We apply Lemma 3 with \( X_\ell = 1_{[t_{\ell-1}, t_\ell]} \cdot (W - \tilde{W}^{(k)}) \) to obtain
\[
d_{2k}^{(p)}(V^{(k)}, C) \leq (\mathbb{E}\|b(\overline{X}^{(k)})\|_p^p)^{1/p} \cdot d_{2k}^{(p)}(W - \tilde{W}^{(k)}, C) \leq d_k^{(p)}(W, C).
\]
From Maiorov (1993) we get \( d_k^{(p)}(W, C) \asymp k^{-1/2} \). \( \Box \)

The lower bound for the quantization numbers even holds for the space \( X = L_1 \).

**Lemma 7.** For every \( r > 0 \)
\[
q_n^{(r)}(X, L_1) \geq (\ln n)^{-1/2}.
\]

**Proof.** Observe that, due to Lemma 11 and Lemma 5, it suffices to show that
\[
q_n^{(r)}(Y, L_1) \geq (\ln n)^{-1/2}
\]
for every one-dimensional process \( Y \) such that
\[
Y_{t \wedge \tau} = W_{t \wedge \tau}, \quad t \in [0, 1],
\]
with a stopping time \( \tau \) that satisfies \( P(\tau = 0) = 0 \).

To this end fix \( \varepsilon \in ]0, 1] \) with \( \mathbb{P}(\tau \geq \varepsilon) > 0 \) and define a bounded linear operator \( T : L_1 \to L_1 \) by
\[
(Tx)(t) = \varepsilon^{-1/2} \cdot x(\varepsilon \cdot t).
\]
Clearly \( TW \) is a Brownian motion, too. The quantization problem for Gaussian processes in the space \( L^1 \) is analyzed in Dereich, Scheutzow (2005). In particular there exists a constant \( \kappa > 0 \) such that
\[
\lim_{n \to \infty} (\ln n)^{1/2} \cdot q_n^{(r)}(TW, L_1) = \kappa
\]
for every \( r > 0 \), see Dereich, Scheutzow (2005, Thm. 6.1).

For \( n \in \mathbb{N} \) let \( M_n \subset L_1 \) denote any set of cardinality \( n \), fix \( \delta \in ]0, 1[ \), and put
\[
A_n = \{ \text{dist}(TW, M_n) \geq (1 - \delta) \cdot q_n^{(r)}(TW, L_1) \}.
\]
Due to 118 we can complement the sets \( M_n \) to sets \( \tilde{M}_n \) of cardinality \( 2n \) such that
\[
\lim_{n \to \infty} (\ln n)^{1/2} \cdot (\mathbb{E}(\text{dist}^r(TW, \tilde{M}_n)))^{1/2r} = \kappa.
\]
as well as
\[
\lim_{n \to \infty} (\ln n)^{1/2} \cdot (\mathbb{E}(\text{dist}^r(TW, \tilde{M}_n)))^{1/r} = \kappa.
\]
Employing Lemma A.1 in Dereich, Scheutzow (2005) we conclude that
\[ \lim_{n \to \infty} \mathbb{P}(A_n) = 1. \]
Consequently
\[
\mathbb{E}(\text{dist}'(TY, M_n)) \geq \mathbb{E}(1_{\{\tau \geq \varepsilon\}} \cdot \text{dist}'(TW, M_n)) \\
\geq (1 - \delta)^r \cdot \mathbb{P}(\{\tau \geq \varepsilon\} \cap A_n) \cdot (q_n^{(r)}(TW, L_1))^r \\
\geq (\ln n)^{-r/2},
\]
which yields
\[ q_n^{(r)}(TY, L_1) \geq (\ln n)^{-1/2}. \]
The latter bound implies (17) by Lemma 1.

\[ \square \]

Proof of Proposition 3. In view of Lemma 6 and Lemma 7 it suffices to show that
\[ q_n^{(r)}(X, C) \leq (\ln n)^{-1/2} \]
and
\[ d_k^{(p)}(X, L_1) \geq k^{-1/2}. \]
By Lemma 2 and Lemma 5 we have
\[ \ln n \cdot q_n^{(r)}(X, C) \geq \sup_{k \leq 2\ln n} k \cdot d_k^{(2r)}(X, C) \geq (\ln n)^{1/2}, \]
which yields (19). From Lemma 6 we also get
\[ d_k^{(p)}(X, L_1) \leq c \cdot k^{-1/2} \]
with some constant \( c > 0 \). Moreover, by Lemma 7
\[ \sup_{n \leq 2^\ell} \ln n \cdot q_n^{(p/2)}(X, L_1) \geq \sup_{n \leq 2^\ell} (\ln n)^{1/2} \geq \ell^{1/2}. \]
Consequently, by Lemma 2
\[ \sup_{k \leq \ell} k \cdot d_k^{(p)}(X, L_1) \geq \tilde{c} \cdot \ell^{1/2} \]
with some constant \( \tilde{c} \in [0, c] \). Put \( c = (\tilde{c}/c)^2 \). Since
\[ \sup_{k < c \cdot \ell} k \cdot d_k^{(p)}(X, L_1) < \tilde{c} \cdot \ell^{1/2} \]
by (21), we conclude that
\[ \ell \cdot d_{[c \cdot \ell]}^{(p)}(X, L_1) \geq \sup_{c \cdot \ell \leq k \leq \ell} k \cdot d_k^{(p)}(X, L_1) \geq \tilde{c} \cdot \ell^{1/2}, \]
which yields (20).

\[ \square \]

Remark 7. According to Lemma 4 and (20), Proposition 3 is valid, too, for \( X = L_1 \) instead of \( X = C \).
4.3. **Proof of Proposition 4** Consider a one-dimensional Brownian motion $W$. Given $\ell \in \mathbb{N}$ and $\varepsilon \in [0, 1]$ let $s_i = i/\ell \cdot \varepsilon$ and put

$$B_{i,0}^{\ell,\varepsilon} = \{ x \in C([0, 1]) : x(s_i) - x(s_{i-1}) \geq \varepsilon^{1/2}/\ell^{3/2} \}$$

as well as

$$B_{i,1}^{\ell,\varepsilon} = \{ x \in C([0, 1]) : x(s_i) - x(s_{i-1}) < -\varepsilon^{1/2}/\ell^{3/2} \}$$

for $i = 1, \ldots, \ell$. Moreover, define

$$A_{\alpha}^{\ell,\varepsilon} = \bigcap_{i=1}^{\ell} \{ W \in B_{i,\alpha_i}^{\ell,\varepsilon} \}$$

for any multi-index $\alpha \in \{0, 1\}^\ell$.

**Lemma 8.** There exists a constant $c_0 \in ]0, 1[$ such that

$$c_0 \cdot 2^{-\ell} \leq \mathbb{P}(A_{\alpha}^{\ell,\varepsilon}) \leq 2^{-\ell}$$

for all $\ell \in \mathbb{N}$, $\varepsilon \in [0, 1]$, and $\alpha \in \{0, 1\}^\ell$.

**Proof.** Obviously, the probability $\mathbb{P}(A_{\alpha}^{\ell,\varepsilon})$ does not depend on $\alpha$. Hence

$$\mathbb{P}(W \in B_{i,0}^{\ell,\varepsilon}) = \frac{1}{2} - \mathbb{P}(0 \leq W_{s_i} - W_{s_{i-1}} \leq \varepsilon^{1/2}/\ell^{3/2})$$

$$= \frac{1}{2} - \int_0^{1/\ell} (2\pi)^{-1/2} \exp(-x^2/2) \, dx$$

$$\geq \frac{1}{2} \cdot (1 - \sqrt{2/\pi} \cdot \ell^{-1})$$

implies

$$2^\ell \cdot \mathbb{P}(A_{\alpha}^{\ell,\varepsilon}) \geq (1 - \sqrt{2/\pi} \cdot \ell^{-1})^\ell.$$  

The latter bound tends to $\exp(-\sqrt{2/\pi})$ as $\ell$ tends to infinity, which completes the proof. □

Let $A$ be any event with $\mathbb{P}(A) \geq 1 - c_0/2$ and put

$$N(\varepsilon, \ell) = \# \{ \alpha \in \{0, 1\}^\ell : \mathbb{P}(A_{\alpha}^{\ell,\varepsilon} \cap A) > c_0 \cdot 2^{-\ell-2} \}.$$  

**Lemma 9.** For all $\varepsilon > 0$ and $\ell \in \mathbb{N}$

$$N(\varepsilon, \ell) \geq c_1 \cdot 2^\ell,$$

where $c_1 = c_0/(4 - c_0)$.

**Proof.** Due to Lemma \[\]  

$$\mathbb{P}\left( \bigcup_{\alpha \in \{0,1\}^\ell} A_{\alpha}^{\ell,\varepsilon} \cap A \right) \geq \mathbb{P}(A) + \mathbb{P}\left( \bigcup_{\alpha \in \{0,1\}^\ell} A_{\alpha}^{\ell,\varepsilon} \right) - 1$$

$$\geq \mathbb{P}(A) + c_0 - 1 \geq c_0/2.$$  

On the other hand, by the definition of $N(\varepsilon, \ell)$ and Lemma \[\]

$$\mathbb{P}\left( \bigcup_{\alpha \in \{0,1\}^\ell} A_{\alpha}^{\ell,\varepsilon} \cap A \right) \leq (2^\ell - N(\varepsilon, \ell)) \cdot c_0 \cdot 2^{-\ell-2} + N(\varepsilon, \ell) \cdot 2^{-\ell}.$$  

It remains to combine both estimates. □
Proof of Proposition 4. Because of Lemma 1 and Lemma 5, it suffices to prove that

\[ e_n^{\text{ran}}(Y,C) \geq n^{-1/2} \cdot (\ln n)^{-3/2} \]

for every one-dimensional process \( Y \) such that

\[ Y_{t \wedge \tau} = W_{t \wedge \tau}, \quad t \in [0,1], \]

with a stopping time \( \tau \) that satisfies \( P(\tau = 0) = 0 \). To this end we use Proposition 1.

Put

\[ B_{\alpha}^\ell = \bigcap_{i=1}^\ell B_{i,\alpha_i} \]

and define \( f_\alpha^\ell \in F \) by

\[ f_\alpha^\ell(x) = \text{dist} \left( x, \left( B_{\alpha}^\ell \right)^c \right) \]

for \( \alpha \in \{0,1\}^\ell \). Note that

\[ f_\alpha^\ell(x) \geq \frac{1}{2} \cdot \min_{i=1,...,\ell} |x(s_i) - x(s_{i-1})| \]

for \( x \in B_{\alpha}^\ell \). Choose \( \varepsilon \in [0,1] \) with \( P(\tau \geq \varepsilon) \geq 1 - c_0/2 \), and let \( A = \{ \tau \geq \varepsilon \} \). Then

\[ S(f_\alpha^\ell) \geq \frac{1}{2} \cdot \mathbb{E} \left( 1_A \cdot 1_{B_{\alpha}^\ell} W \cdot \min_{i=1,...,\ell} |W_{s_i} - W_{s_{i-1}}| \right) \]

\[ \geq \frac{1}{2} \cdot \mathbb{E} \left( 1_{A_{\alpha}^\ell \cap A} \cdot \min_{i=1,...,\ell} |W_{s_i} - W_{s_{i-1}}| \right) \]

\[ \geq \frac{1}{2} \cdot \varepsilon^{1/2} / \ell^{3/2} \cdot \mathbb{P}(A_{\alpha}^\ell \cap A). \]

Take \( n = \lfloor c_1 \cdot 2^{\ell-1} \rfloor \) and use Lemma 9 to conclude that

\[ S(f_\alpha^\ell) \geq n^{-1} \cdot (\ln n)^{-3/2} \]

holds uniformly for at least \( 2n \) multi-indices \( \alpha \in \{0,1\}^\ell \). Finally, apply Proposition 1 to complete the proof of (22). \( \square \)

5. Randomized Algorithms for Gaussian Measures

In this section we consider zero mean Gaussian measures \( \mu \) on separable Banach spaces \( X \), and throughout we assume that the corresponding small ball function

\[ \varphi(\varepsilon) = -\ln \mu(\{ x \in X : \|x\| \leq \varepsilon \}) \]

satisfies

\[ \varphi(\varepsilon) \asymp \varepsilon^{-\alpha} \cdot (\ln \varepsilon^{-1})^\beta \]

for some constants \( \alpha > 0 \) and \( \beta \in \mathbb{R} \) as \( \varepsilon \) tends to zero.

Remark 8. Typically, (23) holds for infinite-dimensional spaces \( X \), see Li, Shao (2001).

For example, if \( \mu \) is the distribution of a fractional Brownian motion with Hurst parameter \( H \in ]0,1] \) on \( X = C([0,1]) \) or \( X = L_p([0,1]) \) for some \( p \in [1, \infty[ \), then \( \alpha = 1/H \) and \( \beta = 0 \). Moreover, \( \alpha = 1/(H - \gamma) \) and \( \beta = 0 \) when \( \| \cdot \| \) denotes the \( \gamma \)-Hölder norm. Similar results are known for Sobolev norms, see Kuelbs, Li, Shao (1995) and Li, Shao (1999).

If \( X = C([0,1]^2) \) and \( \mu \) is the distribution of the two-dimensional fractional Brownian sheet, then \( \alpha = 1/H \) and \( \beta = 1 + 1/H \) due to Belinsky, Linde (2002). Moreover, for a \( d \)-dimensional Brownian sheet considered in \( X = L_2([0,1]^d) \) one has \( \alpha = 2 \) and \( \beta = 2(d-1) \), see Csáki (1984) and Fill, Torcaso (2004).
Assumption 23 determines the asymptotic behavior of the quantization numbers and the Kolmogorov widths, see Dereich (2003, Thm. 3.1.2) and Creutzig (2002, Cor. 4.7.2).

**Proposition 5.** The quantization numbers \( q_n^{(r)} \) satisfy
\[
q_n^{(r)} \asymp (\ln n)^{-1/\alpha} \cdot (\ln \ln n)^{\beta/\alpha}
\]
for every \( r > 0 \). The average Kolmogorov widths \( d_k^{(p)} \) satisfy
\[
d_k^{(p)} \asymp k^{-1/\alpha} \cdot (\ln k)^{\beta/\alpha}
\]
for every \( p > 0 \).

Hence, by Theorem 1, quadrature of arbitrary Lipschitz functionals by means of deterministic algorithms is intractable. Now we turn to the analysis of randomized algorithms.

**Proposition 6.** The minimal errors \( \varepsilon_n^{\text{ran}} \) satisfy
\[
\limsup_{n \to \infty} \varepsilon_n^{\text{ran}} \cdot n^{1/2} \cdot (\ln n)^{1+1/\alpha} \cdot (\ln \ln n)^{-\beta/\alpha} > 0.
\]

*Proof.* Apply Corollary 2 with \( f \) given by \( f(t) = c \cdot (\ln t)^{-1/\alpha} \cdot (\ln \ln t)^{\beta/\alpha} \) for \( t \) sufficiently large and a suitable constant \( c > 0 \).

Proposition 6 provides a lower bound for the error of broad sense randomized algorithms in terms of the number of functional evaluations. The lower bound depends on the specific properties of the Gaussian measure only via logarithmic terms. This is no longer the case if we relate the error of randomized algorithms to their computational cost.

**Theorem 6.** The minimal errors \( \varepsilon_N^{\text{ran}} \) satisfy
\[
\limsup_{N \to \infty} \varepsilon_N^{\text{ran}} \cdot N^{1/(2+\alpha)} \cdot (\ln N)^{(2+2\alpha-\alpha\beta)/(\alpha(2+\alpha))} \cdot (\ln \ln N)^{-2\beta/(\alpha(2+\alpha))} > 0.
\]

*Proof.* We combine Propositions 2, 5, and 6. Due to Proposition 6 there exists a constant \( c > 0 \) and an increasing sequence of integers \( n_\ell \in \mathbb{N} \) such that
\[
\varepsilon_n^{\text{ran}} \geq c \cdot n_\ell^{-1/2} \cdot (\ln n_\ell)^{-1-1/\alpha} \cdot (\ln \ln n_\ell)^{\beta/\alpha}
\]
for every \( \ell \in \mathbb{N} \). Put
\[
N_\ell = \left[ n_\ell^{(2+\alpha)/2} \cdot (\ln n_\ell)^{(\alpha+\beta+1)} \cdot (\ln \ln n_\ell)^{-\beta} \right],
\]
and let \( n, k \in \mathbb{N} \) with \( n \cdot k \leq N_\ell \). If \( n > n_\ell \) then \( k < N_\ell/n_\ell \), and Proposition 5 implies
\[
d_k \geq d_{[N_\ell/n_\ell]} \geq (N_\ell/n_\ell)^{-1/\alpha} \cdot (\ln(N_\ell/n_\ell))^{\beta/\alpha} \asymp n_\ell^{-1/2} \cdot (\ln n_\ell)^{-1-1/\alpha} \cdot (\ln \ln n_\ell)^{\beta/\alpha}.
\]
On the other hand, if \( n \leq n_\ell \) then \( \varepsilon_n^{\text{ran}} \geq \varepsilon_n^{\text{ran}} \). Consequently, by Proposition 2 and (24)
\[
\varepsilon_N^{\text{ran}} \geq n_\ell^{-1/2} \cdot (\ln n_\ell)^{-1-1/\alpha} \cdot (\ln \ln n_\ell)^{\beta/\alpha}.
\]
Straightforward computations show
\[
n_\ell^{-1/2} \cdot (\ln n_\ell)^{-1-1/\alpha} \cdot (\ln \ln n_\ell)^{\beta/\alpha}
\asymp N_\ell^{-1/(2+\alpha)} \cdot (\ln N_\ell)^{-(2+2\alpha-\alpha\beta)/(\alpha(2+\alpha))} \cdot (\ln \ln N_\ell)^{2\beta/(\alpha(2+\alpha))},
\]
which completes the proof. \( \square \)
It is quite common to approximately compute the integrals $S(f)$ with respect to Gaussian measures by sampling from a standard normal distribution on a suitable finite-dimensional subspace of $\mathcal{X}$. A proper choice of the subspace is suggested by the following general result on average linear widths, which is due to Creutzig (2002, Thm. 4.4.1). There exist points $x_{\ell}^{(k)} \in \mathcal{X}$ and bounded linear functionals $\xi_{\ell}^{(k)} \in \mathcal{X}^*$ such that

\begin{equation}
\left( \int_{\mathcal{X}} \| x - \hat{X}^{(k)}(x) \|^2 \, \mu(dx) \right)^{1/2} \leq \ln k \cdot d_k
\end{equation}

for

$$\hat{X}^{(k)}(x) = \sum_{\ell=1}^{k} \xi_{\ell}^{(k)}(x) \cdot x_{\ell}^{(k)}.$$

Clearly we may assume that $\xi_{1}^{(k)}, \ldots, \xi_{k}^{(k)}$ are independent with respect to $\mu$. Take independent copies $\hat{X}_1^{(k)}, \ldots, \hat{X}_n^{(k)}$ of $\hat{X}^{(k)}$ and define the randomized algorithm $\hat{S}_n^{(k)}$ by (13).

**Theorem 7.** The algorithm $\hat{S}_n^{(k)} = \hat{S}_n^{(k)}(\cdot)$ with $n = [N^{2/(2+\alpha)} \cdot (\ln N)^{-2(\alpha+\beta)/(2+\alpha)}]$ and $k = [N^{\alpha/(2+\alpha)} \cdot (\ln N)^{(\alpha+\beta)/(2+\alpha)}]$ satisfies

$$e(\hat{S}_n^{(k)}) \leq N^{-1/(2+\alpha)} \cdot (\ln N)^{\alpha+\beta)/(2+\alpha)}$$

and

$$\text{cost}(\hat{S}_n^{(k)}) \preceq N.$$

**Proof.** Proceed as in the proof of Theorem 5 to obtain

$$e^2(\hat{S}_n^{(k)}) \leq 1/n + (\ln k)^2 \cdot d_k^2 \leq 1/n + k^{-2/\alpha} \cdot (\ln k)^{2(\alpha+\beta)/\alpha}$$

by means of (25) and Proposition 5. The asymptotic upper bound for the error of $\hat{S}_n = \hat{S}_n^{(k)}$ now follows from the particular choice of $n$ and $k$. Clearly, $\text{cost}(\hat{S}_n^{(k)}) \asymp k \cdot n$. \hfill \Box

Combine Theorems 5 and 7 to conclude that the algorithm $\hat{S}_n$ is almost optimal in the following sense.

**Corollary 4.** There exists a constant $c > 0$ such that

$$e(\hat{S}_n) \leq c \cdot \varepsilon_{n}^{\text{ran}} \cdot (\ln N)^{1+2/(\alpha+2+\alpha)} \cdot (\ln \ln N)^{-2\beta/(\alpha(2+\alpha))}$$

holds for infinitely many integers $N$.

**Remark 9.** A slightly better upper bound is available if the Banach space $\mathcal{X}$ is $B$-convex, e.g., if $\mathcal{X}$ is an $L_p$-space with $p \in [1, \infty]$. Instead of (25) we then have

\begin{equation}
\left( \int_{\mathcal{X}} \| x - \hat{X}^{(k)}(x) \|^2 \, \mu(dx) \right)^{1/2} \leq d_k,
\end{equation}

see Creutzig (2002, Cor. 3.4.2), which yields

$$e(\hat{S}_n) \preceq N^{-1/(2+\alpha)} \cdot (\ln N)^{\beta/(2+\alpha)}$$

in Theorem 7. Both of the estimates (25) and (26) are proven non-constuctively.

For a number of Gaussian measures on function spaces the Karhunen-Loève expansion is explicitly known, and hereby we get an approximation $\hat{X}^{(k)}$ that satisfies (26), if $\mathcal{X}$ is any $L_p$-space with $p \in [1, \infty]$. In particular for an $L_2$-space $\mathcal{X}$ and $\beta = 0$ the upper bound (26) is due to Wasilkowski, Woźniakowski (1996, p. 2076).
Consider the distribution $\mu$ of the $d$-dimensional fractional Brownian sheet with Hurst parameter $H \in ]0, 1[$ on the space $X = C([0, 1]^d)$. In this case a direct approach yields
\[
(\int_X \| x - \hat{X}^{(k)}(x) \|^2 \mu(dx))^{1/2} \preceq k^{-H} \cdot (\ln k)^{H(d-1)+d/2},
\]
see Kühn, Linde (2002). See also Ayache, Taqqu (2003) for a wavelet approximation $\hat{X}^{(k)}$ in the case $d = 1$ and Dzhaparidze, van Zanten (2005) for a trigonometric approximation $\hat{X}^{(k)}$ in the case $d \geq 1$, which both satisfy this estimate. From (27) we get
\[
e(\hat{S}_N) \preceq N^{-1/(2+1/H)} \cdot (\ln N)^{d/2-1/(2+1/H)}
\]
in Theorem 7.

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References
Ayache, A., Taqqu, M. S. (2003), Rate optimality of wavelet series approximations of fractional Brownian motion, J. Fourier Anal. Appl. 9, 451–471.
Bakhvalov, N. S. (1959), On approximate computation of integrals (in Russian), Vestnik MGV, Ser. Math. Mech. Astron. Phys. Chem. 4, 3–18.
Bakhvalov, N. S. (1971), On the optimality of linear methods for operator approximation in convex classes of functions, USSR Comput. Math. Math. Phys. 11, 244–249.
Belinsky, E., Linde, W. (2002), Small ball probabilities of fractional Brownian sheets via fractional integration operators, J. Theoret. Probab. 15, 589–612.
Bouleau, N., Lépingle, D. (1994), Numerical Methods for Stochastic Processes, Wiley, New York.
Csáki, E. (1984), On small values of the square integral of a multiparameter Wiener process, in: Statistics and Probability, J. Mogyorodi, I. Vincze, W. Wertz, eds., pp. 19–26, Reidel, Dordrecht.
Creutzig, J. (2002), Approximation of Gaussian random vectors in Banach spaces, Ph.D. Dissertation, Universität Jena.
Dereich, S. (2003), High resolution coding of stochastic processes and small ball probabilities. Ph.D. Dissertation, TU Berlin.
Dereich, S. (2004), The quantization complexity of diffusion processes, Preprint, arXiv:math.PR/0411597
Dereich, S., Fehringer, F., Matoussi, A., and Scheutzow, M. (2003), On the link between small ball probabilities and the quantization problem, J. Theoret. Probab. 16, 249–265.
Dereich, S., Scheutzow, M. (2005), High-resolution quantization and entropy coding for fractional Brownian motion, Preprint, arXiv:math.PR/0504480
Dzhaparidze, K., van Zanten, H. (2005), Optimality of an explicit series expansion of the fractional Brownian sheet, Stat. Prob. Letters 71, 295–301.
Faure, O. (1992), Simulation du mouvement brownien et des diffusions, Thèse, ENPC Paris.
Fill, J. A., Torcaso, F. (2004), Asymptotic analysis via Mellin transforms for small deviations in $L^2$-norm of integrated Brownian sheets, Probab. Theory Relat. Fields 130, 259–288.
Graf, S., Luschgy, H. (2000), Foundations of Quantization for Probability Distributions, Lect. Notes in Math. 1730, Springer-Verlag, Berlin.

Gray, R. M., Neuhoff, D. L., Shields, P. C. (1975), A generalization of Ornstein’s $\bar{d}$ distance with applications to information theory, Ann. Appl. Prob. 3, 315–328.

Kahane, J.-P. (1993), Some Random Series of Functions, Cambridge Univ. Press, Cambridge.

Kantorovich, L. V., Rubinstein, G. S. (1958), On a space of completely additive functions (in Russian), Vestnik Leningrad Univ. 13, no. 7, Ser. Mat. Astron. Phys. 2, 52–59.

Kühn, T., Linde, W. (2002), Optimal series representation of fractional Brownian sheets, Bernoulli 8, 669–696.

Kuelbs, J., Li, W. V., Shao Q. M. (1995), Small ball estimates for fractional Brownian motion under Hölder norm and Chung’s functional LIL, J. Theoret. Probab. 8, 361–386.

Li, W.V., Shao, Q.-M. (1999), Small ball estimates for Gaussian processes under the Sobolev norm, J. Theoret. Probab. 12, 699–720.

Li, W. V., Shao, Q.-M. (2001), Gaussian processes: inequalities, small ball probabilities and applications, in: Stochastic Processes: Theory and Methods, Handbook of Statist., Vol. 19, D. N. Shanbhag, C. R. Rao, eds., pp. 533–597, North-Holland, Amsterdam.

Luschgy, H., Pagès, G. (2003), Functional quantization of 1-dimensional Brownian diffusion processes, Preprint, Université de Paris VI, LPMA no. 853.

Luschgy, H., Pagès, G. (2004), Sharp asymptotics of the functional quantization problem for Gaussian processes, Ann. Appl. Prob. 32, 1574–1599.

Maiorov, V. (1993), Average $n$-widths of the Wiener space in the $L_\infty$-norm, J. Complexity 9, 222–230.

Nemirovsky, A. S., Yudin, D. B. (1983), Problem Complexity and Method Efficiency in Optimization, Wiley, New York.

Novak, E. (1988), Deterministic and Stochastic Error Bounds in Numerical Analysis, Lect. Notes in Math. 1349, Springer-Verlag, Berlin.

Novak, E. (1995), The real number model in numerical analysis, J. Complexity 11, 57–73.

Pagès, G., Printems, J. (2004), Functional quantization for pricing derivatives, Preprint, Université de Paris VI, LPMA no. 930.

Ritter, K. (2000), Average-Case Analysis of Numerical Problems, Lect. Notes in Math. 1733, Springer-Verlag, Berlin.

Sakai, T. (1996), Riemannian Geometry, Transl. Math. Monogr. 149, AMS, Rhode Island.

Smolyak, S. A. (1965), On optimal restoration of functions and functionals of them (in Russian), Candidate Dissertation, Moscow State University.

Traub, J. F., Wasilkowski, G. W., Woźniakowski, H. (1988), Information-Based Complexity, Academic Press, New York.

Wasilkowski, G. W. (1989), Randomization for continuous problems, J. Complexity 5, 195–218.

Wasilkowski, G. W., Woźniakowski, H. (1996), On tractability of path integration, J. Math. Phys. 37, 2071–2088.

Wasilkowski, G. W., Woźniakowski, H. (2001), Complexity of weighted approximation over $\mathbb{R}^d$, J. Complexity 17, 722–740.
