Uniform Mixing on Cayley Graphs

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Abstract

We provide new examples of Cayley graphs on which the quantum walks reach uniform mixing. Our first result is a complete characterization of all $2(d+2)$-regular Cayley graphs over $\mathbb{Z}_d^3$ that admit uniform mixing at time $2\pi/9$. Our second result shows that for every integer $k \geq 3$, we can construct Cayley graphs over $\mathbb{Z}_q^d$ that admit uniform mixing at time $2\pi/q^k$, where $q = 3, 4$.

We also find the first family of irregular graphs, the Cartesian powers of the star $K_{1,3}$, that admit uniform mixing.

Keywords: quantum walk, uniform mixing, Cayley graph

1 Introduction

A continuous-time quantum walk on a graph $X$ is defined by the transition matrix

$$U(t) := \exp(itA) = \sum_{k \geq 0} \frac{(itA)^k}{k!}$$

where $A$ is the adjacency matrix of $X$. The probability that at time $t$, the quantum walk with initial state represented by $u$ is in the state represented by $v$ is

$$|U(t)_{uv}|^2.$$ 

We say $X$ admits uniform mixing at time $t$ if the above probability is the same for all vertices $u$ and $v$. A weaker version of uniform mixing, called local uniform mixing, occurs if the probability distribution given by a column of $U(t)$ is uniform.
Uniform mixing on graphs is rare, and almost all the known examples are Cayley graphs. The current list of Cayley graphs contains the complete graphs \( K_2 \), \( K_3 \) and \( K_4 \), the Hamming graphs \( H(d, 2) \), \( H(d, 3) \) and \( H(d, 4) \), the Paley graph of order nine, some strongly regular graphs from regular symmetric Hadamard matrices, some linear Cayley graphs over \( \mathbb{Z}_2 \), \( \mathbb{Z}_3 \) and \( \mathbb{Z}_4 \), and the Cartesian product of graphs which admit uniform mixing at the same time. These graphs share the following features.

(a) They are Cayley graphs over abelian groups.

(b) They have integer eigenvalues.

(c) Their mixing times are rational multiples of \( \pi \).

That being said, a graph that admits uniform mixing is not necessarily vertex-transitive or even regular. As we will see in Section 9, the star \( K_{1,3} \) admits uniform mixing at time \( 2\pi/\sqrt{27} \). The Cartesian powers of \( K_{1,3} \) are so far the only family of irregular graphs found to admit uniform mixing, and the only family to which none of the above features applies.

Although characterizing uniform mixing in general seems daunting, the problem can be reduced if we require some regularity on the graphs. The vertex transitive graphs, which provide most of the known examples, satisfy the property that for any two vertices \( u \) and \( v \), there is a graph automorphism that maps \( u \) to \( v \). The transition matrix of a vertex-transitive graph is entirely determined by one of its rows. Therefore for all Cayley graphs, uniform mixing is equivalent to local uniform mixing.

In this paper, we provide new examples of Cayley graphs over \( \mathbb{Z}_q^d \) that admit uniform mixing. Our examples contain both an infinite family of Cayley graphs on which mixing occurs at time \( 2\pi/9 \), and infinite families of Cayley graphs on which mixing occurs arbitrarily faster. The first part of this paper is devoted to giving a complete characterization of \( 2(d + 2) \)-regular Cayley graphs over \( \mathbb{Z}_q^d \) that admit uniform mixing at time \( 2\pi/9 \). In the second part, we show that for an arbitrarily large integer \( k \), there are families of Cayley graphs over \( \mathbb{Z}_q^d \) that admit uniform mixing at time \( 2\pi/q^k \), where \( q = 3, 4 \). These examples extend the results of Mullin and Chan.
2 Quotients of Hamming Graphs

A Cayley graph over the additive group \( \mathbb{Z}_q^d \) is a graph with vertex set \( \mathbb{Z}_q^d \) and edge set 
\[
\{(g, h) : h - g \in C\}
\]
for some subset \( C \) of \( \mathbb{Z}_q^d \). This graph is denoted by \( X(\mathbb{Z}_q^d, C) \), and the subset \( C \) is called the connection set. We assume \( C \) is inverse-closed and does not contain the identity element, so that \( X(\mathbb{Z}_q^d, C) \) is a simple graph. Note here \( q \) is not necessarily a prime power – we will think of the vertex set \( \mathbb{Z}_q^d \) as a module in this paper.

While no extra algebraic structure of \( \mathbb{Z}_q^d \) is needed to define a Cayley graph, it is often convenient to view the underlying group \( \mathbb{Z}_q^d \) as a \( \mathbb{Z}_q \)-module.

Two easy observations follow. First, if \( \phi \) is a module automorphism of \( \mathbb{Z}_q^d \), then the two Cayley graphs \( X(\mathbb{Z}_q^d, C) \) and \( X(\mathbb{Z}_q^d, \phi(C)) \) are isomorphic. Second, \( X(\mathbb{Z}_q^d, C) \) is connected if and only if its connection set contains a basis of \( \mathbb{Z}_q^d \). An example of connected Cayley graphs over \( \mathbb{Z}_q^d \) is the Hamming graph \( H(d, q) \), whose connection set consists of non-zero multiples of the standard basis \( \{e_1, e_2, \ldots, e_d\} \) of \( \mathbb{Z}_q^d \).

We will pay special attention to the quotient graphs of Hamming graphs, as they form an important family of Cayley graphs over \( \mathbb{Z}_q^d \). Consider a submodule \( \Gamma \) of \( \mathbb{Z}_q^d \) with Hamming distance at least three. The partition of \( \mathbb{Z}_q^d \) by the cosets of \( \Gamma \) satisfies three properties:

(i) every coset is a coclique;

(ii) every vertex is adjacent to at most one vertex in a coset;

(iii) if some vertex in the coset \( g + \Gamma \) is adjacent to a vertex in another coset \( h + \Gamma \), then there is a matching between \( g + \Gamma \) and \( h + \Gamma \).

The quotient graph of \( H(d, q) \) induced by \( \Gamma \), denoted \( H(d, q)/\Gamma \), is a graph with a vertex for each coset of \( \Gamma \), such that two vertices are adjacent if there is a matching between the two associated cosets. We note that every quotient graph of \( H(d, q) \) is a Cayley graph for a quotient module of \( \mathbb{Z}_q^d \).

2.1 Lemma. Let \( C \) be the connection set of the Hamming graph \( H(d, q) \). For a submodule \( \Gamma \) of \( \mathbb{Z}_q^d \) with Hamming distance at least three, let 
\[
C/\Gamma = \{c + \Gamma : c \in C\}.
\]
Then

\[ H(d, q)/\Gamma \cong X(\mathbb{Z}_q^d/\Gamma, C/\Gamma). \]

Proof. Let \( g + \Gamma \) and \( h + \Gamma \) be two vertices of \( X(\mathbb{Z}_q^d, C)/\Gamma \). They are adjacent if and only if there exist \( x, y \in \Gamma \) and \( c \in C \) such that \( (g + x) - (h + y) = c \), or equivalently,

\[(g + \Gamma) - (h + \Gamma) = c + \Gamma \in C + \Gamma. \]

\[\Box\]

3 Linear Cayley Graphs

A Hamming graph \( H(d, q) \) is known to admit uniform mixing if and only if \( q \in \{2, 3, 4\} \). Since uniform mixing on \( H(d, q) \) implies uniform mixing on some of its quotient graphs, for our interest it is important to understand which Cayley graphs are quotients of \( H(d, q) \). In this section, we show that quotient graphs of \( H(d, q) \) are exactly the linear Cayley graphs over \( \mathbb{Z}_q^d \), that is, graphs \( X(\mathbb{Z}_q^d, C) \) for which \( C \cup \{0\} \) is closed under multiplication by \( \mathbb{Z}_q \). Notice that all Cayley graphs over \( \mathbb{Z}_2^d \) or \( \mathbb{Z}_3^d \) are linear, so we can characterize uniform mixing on them in terms of the submodules that induce the quotients.

3.1 Theorem. Let \( X(\mathbb{Z}_q^d, C) \) be a connected linear Cayley graph with valency \( d(q - 1) \). Then \( X(\mathbb{Z}_q^d, C) \) is isomorphic to a quotient graph \( H(d, q)/\Gamma \) for some submodule \( \Gamma \) of \( \mathbb{Z}_q^d \), where \( |\Gamma| = q^{d-r} \) and \( \Gamma \) has Hamming distance at least three.

Proof. Let \( C \) be the connection set of \( X \). We can partition \( C \) into cells \( C_1, C_2, \ldots, C_d \) such that two elements lie in the same cell if and only if they are multiples of each other, and the first \( r \) cells contains a basis of \( \mathbb{Z}_q^d \). Since \( C_j \) is a cyclic group, we may assume \( C_j = \langle v_j \rangle \). Define a module homomorphism from \( C \) to \( \mathbb{Z}_q^r \times \mathbb{Z}_q^{d-r} \) by

\[ f(v_j) = \begin{cases} (v_j, 0), & \text{if } v \in C_j \text{ for } j \in \{1, 2, \ldots, r\}, \\ (v_j, e_j), & \text{if } v \in C_j \text{ for } j \in \{r+1, r+2, \ldots, d\} \end{cases} \]

where \( \{e_1, e_2, \ldots, e_d\} \) is the standard basis of \( \mathbb{Z}_q^d \). Let \( C' = \{f(v) : v \in C\} \)
Then $C'$ consists of all non-zero multiples of a basis of $\mathbb{Z}_q^r \times \mathbb{Z}_q^{d-r}$. Thus, $X(\mathbb{Z}_q^d, C')$ is isomorphic to the Hamming graph $H(d, q)$. Let $\phi$ be a module automorphism of $\mathbb{Z}_q^d$ that maps a basis in $C'$ to the standard basis. Let $\Gamma'$ be the submodule of $\mathbb{Z}_q^d$ generated by $\{e_{r+1}, e_{r+2}, \ldots, e_d\}$, and $\Gamma$ the image of $\Gamma'$ under $\phi$. Clearly $|\Gamma| = q^{d-r}$. By Lemma 2.1

$$H(d, q)/\Gamma \cong X(\mathbb{Z}_q^d, C')/\Gamma' \cong X(\mathbb{Z}_q^r, C).$$

Since we started with a simple Cayley graph, $\Gamma$ must have Hamming distance at least three.

Conversely, for any quotient graph $H(d, q)/\Gamma$ where $\Gamma$ has minimum distance at least three, we can find a connection set of the linear Cayley graph isomorphic to $H(d, q)/\Gamma$.

3.2 Theorem. Let $\Gamma$ be a submodule of $\mathbb{Z}_q^d$ with size $q^{d-r}$ and Hamming distance at least three. Then $H(d, q)/\Gamma$ is isomorphic to a linear Cayley graph over $\mathbb{Z}_q^r$ with valency $d(q-1)$.

Proof. Without loss of generality we may assume $\Gamma$ is generated by the columns of

$$\begin{pmatrix} R \\ S \end{pmatrix}$$

where $S$ is square and invertible over $\mathbb{Z}_q$. Let

$$P = \begin{pmatrix} I & -RS^{-1} \\ 0 & S^{-1} \end{pmatrix}.$$ 

We have

$$P \begin{pmatrix} R \\ S \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

that is, $P$ defines a module automorphism $\phi$ of $\mathbb{Z}_q^d$ that maps $\Gamma$ to the submodule generated by $\{e_{r+1}, e_{r+2}, \ldots, e_d\}$. Let $C$ be the connection set of $H(d, q)$, and let $C'$ be the set of non-zero multiples of the columns of $Q = (I \ -RS^{-1})$.

By Lemma 2.1 we have

$$H(d, q)/\Gamma \cong X(\mathbb{Z}_q^r, \phi(C)/\phi(\Gamma)) \cong X(\mathbb{Z}_q^r, C').$$

Since $\Gamma$ has Hamming distance at least three, no two columns of $R$ are multiples of each other. Thus no two columns of $Q$ are multiples of each other, and it follows that $X(\mathbb{Z}_q^r, C')$ has valency $d(q-1)$.

\[\square\]
4 Quotient Graphs with One Generator

For a quotient graph $H(d,q)/\Gamma$, the entries of its transition matrix are block sums of the transition matrix of $H(d,q)$. As functions of the time $t$, these block sums can be greatly simplified if we plug in the time $\tau_q$ when $H(d,q)$ admits uniform mixing. Thus, at the specific time $\tau_q$, whether uniform mixing occurs on $H(d,q)/\Gamma$ is fully determined by the weight distributions of the cosets of $\Gamma$. For details see Mullin’s Ph.D. thesis [8, Ch 8].

In this section we characterize quotients $H(d,q)/\langle a \rangle$ that admit uniform mixing at time $\tau_q$, where the submodule is generated by one element $a$ with Hamming weight $\text{wt}(a)$ at least three. As a special case, the “folded” Hamming graphs $H(d,q)/\langle 1 \rangle$ have been studied in [8]. In general, Theorem 3.2 gives the matrix $Q$ representing the connection set of $H(d,q)/\langle a \rangle$. By row reduction, column permutation, and column scaling of $Q$, it is not hard to see that $H(d,q)/\langle a \rangle$ is a Cartesian product of a Hamming graph and a folded Hamming graph. Combining this observation and the results on the folded Hamming graphs, we give the following characterization.

4.1 Theorem. (a) Uniform mixing occurs on $H(d,2)/\langle a \rangle$ at time $\tau_2 = \pi/4$ if and only if $\text{wt}(a)$ is odd.

(b) Uniform mixing occurs on $H(d,3)/\langle a \rangle$ at time $\tau_3 = 2\pi/9$ if and only if $\text{wt}(a)$ is not divisible by three.

(c) Uniform mixing occurs on $H(d,4)/\langle a \rangle$ at time $\tau_4 = \pi/4$ if and only if $\text{wt}(a)$ is odd. \hfill $\Box$

This Theorem takes care of all the connected $(d+1)(q-1)$-regular Cayley graphs over $Z_q^d$, which admit uniform mixing at $\tau_q$, for $q \in \{2,3,4\}$.

5 Quotient Graphs with Two Generators

We move on to the quotients of Hamming graphs where the submodules are generated by two elements. In this section, we characterize the $2(d+2)$-regular Cayley graphs over $Z_q^d$ that admit uniform mixing at time $2\pi/9$. To begin, we need the weight distribution conditions over $Z_3$ from [8, Ch 8].

5.1 Theorem (Mullin). Let $\Gamma$ be a submodule of $Z_3^d$ with Hamming distance at least three such that $|\Gamma| = 3^s$. For any coset of $\Gamma$, let $n_j$ be the number
of elements in it with weight $j$ modulo three. Uniform mixing occurs on $H(d, q)/\Gamma$ at time $2\pi/9$ if and only if the weight distribution of every coset of $\Gamma$ satisfies

$$n_0n_1 + n_0n_2 + n_1n_2 = 3^{2s-1} - 3^{s-1}.$$  

Since we have to examine the weight distribution of every coset of $\Gamma$, it helps to understand the relation between the weights of $\Gamma$ and the weights of $\Gamma + c$ for each vector $c$. Suppose $\Gamma$ is generated by $s$ elements, and let $M$ be the matrix with the generators of $\Gamma$ as its columns. Then each element in $\Gamma$ can be written as $My$ for some vector $y \in \mathbb{Z}_3^s$, and is uniquely associated to an element $My + c$ in the coset $\Gamma + c$. We will refer to

$$\text{wt}(My + c) - \text{wt}(My)$$

as the weight change of the element $My$ with respect to $c$. The following gives the necessary condition on the weight changes for both $\Gamma$ and $\Gamma + c$ to satisfy the weight distribution condition in Theorem 5.1.

**5.2 Lemma.** Let $\Gamma$ be a submodule of $\mathbb{Z}_3^d$ with size $3^s$ and minimum distance three, and let $\Gamma + c$ be a coset of $\Gamma$. For $j = 0, 1, 2$, let $\Gamma_j$ denote the set of elements in $\Gamma$ with weight congruent to $j$ modulo three. Let $m_j$ be the number of elements in $\Gamma_j$ whose weight change with respect to $c$ is one modulo three. If both $\Gamma$ and $\Gamma + c$ satisfy the weight distribution condition in Theorem 5.1, then either $m_j = 0$ for all $j$, or $m_j$ satisfies the following:

$$m_0 + m_1 + m_2 = 3^{s-1}$$

$$(m_0n_0 + m_1n_1 + m_2n_2) + 3(m_0m_1 + m_1m_2 + m_0m_2) = 3^{2s-1} - 3^{2s-2}.$$  

**Proof.** We will calculate the weights over $\mathbb{Z}_3$. Let $My$ be an element in $\Gamma$. Notice that

$$\text{wt}(My + c) = (My + c)^T(My + c) = \text{wt}(My) + \text{wt}(c) + 2c^TMy.$$  

Since $\text{wt}(c)$ depends only on $c$, and the condition in Theorem 5.1 is symmetric on $n_0, n_1, n_2$, we may assume without loss of generality that $\text{wt}(c) = 0$. If
$c^TM = 0$, then the weight change of each element in $\Gamma$ is zero. Otherwise, there are exactly $3^{s-1}$ vectors $y$ such that

$$2c^TMy = 1.$$ 

Therefore

$$m_0 + m_1 + m_2 = 3^{s-1}.$$ 

Notice that for each solution $y$ to $2c^TMy = 1$, the vector $2y$ is a solution to $2c^TMy = 2$. Thus there are equal number of elements in $\Gamma_j$ with weight change one and weight change two. It follows that in the coset $\Gamma + c$, the number of elements with weight $j$ is

$$n'_j = n_j - 2m_j + m_{j-1} + m_{j+1}$$ 

where the subscripts are calculated modulo three. Since $\Gamma + c$ satisfies the weight distribution condition in Theorem 5.1

$$n_0'n_1 + n_0'n_2 + n_1'n_2 = 3^{2s-1} - 3^{s-1}.$$ 

This together with the fact that

$$n_0n_1 + n_0n_2 + n_1n_2 = 3^{2s-1} - 3^{s-1}$$

yields

$$(m_0n_0 + m_1n_1 + m_2n_2) + 3(m_0m_1 + m_1m_2 + m_0m_2) = 3^{2s-1} - 3^{2s-2}. \Box$$

With the above observation, we characterize the quotient graphs $H(d, q)/\langle a, b \rangle$ that admit uniform mixing at time $2\pi/9$ in terms of the generators $a$ and $b$.

5.3 Theorem. Uniform mixing occurs on $H(d, 3)/\langle a, b \rangle$ at time $2\pi/9$ if and only if one of the following holds:

(i) $a^Tb \equiv 0 \pmod{3}$, wt$(a) \not\equiv 0 \pmod{3}$, and wt$(b) \not\equiv 0 \pmod{3}$.

(ii) $a^Tb \not\equiv 0 \pmod{3}$, and wt$(a) \not\equiv$ wt$(b) \pmod{3}$ unless wt$(a) \equiv$ wt$(b) \equiv 0 \pmod{3}$. 

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Proof. For notational convenience, we define the weight structure of the coset \( \Gamma + c \) to be the tuple with coordinates \( n_0, n_1, n_2 \) in non-descending order, denoted by \( W(\Gamma + c) \). By Theorem 5.1, the quotient graph \( H(d, 3)/\langle a, b \rangle \) admits uniform mixing at time \( 2\pi/9 \) if and only if the weight distribution of every coset \( \Gamma + c \) satisfies

\[
\begin{align*}
n_0n_1 + n_0n_2 + n_1n_2 &= 24 \\
n_0 + n_1 + n_2 &= 9
\end{align*}
\]

which holds if and only if for all \( c \),

\[W(\Gamma + c) \in \{(1, 4, 4), (2, 2, 5)\} \]

We first show that \( W(\Gamma) \) lies in the above set if and only if one of the conditions (i) and (ii) holds. Let

\[M = \begin{pmatrix} a & b \end{pmatrix}\]

be a matrix and let

\[y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\]

be a vector in \( \mathbb{Z}_3^2 \). The weight of \( My \) is

\[
\text{wt}(My) = y^T M^T My = \text{wt}(y_1) \text{wt}(a) + \text{wt}(y_2) \text{wt}(b) + 2y_1y_2a^T b.
\]

Thus the weights of the elements in \( \Gamma \) are

| label | weight | multiplicity |
|-------|--------|--------------|
| \( w_0 \) | 0 | 1 |
| \( w_1 \) | \text{wt}(a) | 2 |
| \( w_2 \) | \text{wt}(b) | 2 |
| \( w_3 \) | \text{wt}(a) + \text{wt}(b) + a^T b | 2 |
| \( w_4 \) | \text{wt}(a) + \text{wt}(b) + 2a^T b | 2 |

Since \( \Gamma \) is a group of order nine, \( n_0 \) is odd and \( n_1, n_2 \) are even. We consider two cases.

(a) Suppose

\[W(\Gamma) = (1, 4, 4).\]

Then half of \( \{w_1, w_2, w_3, w_4\} \) are one, and the rest are two.
• \( \text{wt}(a) = \text{wt}(b) \neq 0 \). Then \( w_3 = w_4 \notin \{0, w_1\} \) if and only if \( a^Tb = 0 \).

• \( \text{wt}(a) = 2\text{wt}(b) \neq 0 \). Then \( w_3 = a^Tb = 2w_4 \). It follows that two of \( \{w_1, w_2, w_3, w_4\} \) are one and the others are two if and only if \( a^Tb \neq 0 \).

\( \text{wt}(a) = \text{wt}(b) = 0 \). Then \( w_3 = a^Tb = 2w_4 \). Thus \( \{w_3, w_4\} = \{1, 2\} \) if and only if \( a^Tb \neq 0 \).

\( \text{wt}(a) = 2\text{wt}(b) \neq 0 \). Then \( w_3 = a^Tb = 2w_4 \). It follows that \( w_3 = w_4 = 0 \) if and only if \( a^Tb = 0 \).

(b) Suppose \( W(\Gamma) = (2, 2, 5) \).

Then half of \( \{w_1, w_2, w_3, w_4\} \) are zero, and the rest are one and two respectively.

\( \text{wt}(a) = \text{wt}(b) = 0 \). Then \( w_3 = a^Tb = 2w_4 \). Thus \( \{w_3, w_4\} = \{1, 2\} \) if and only if \( a^Tb \neq 0 \).

\( \text{wt}(a) = 2\text{wt}(b) \neq 0 \). Then \( w_3 = a^Tb = 2w_4 \). It follows that \( w_3 = w_4 = 0 \) if and only if \( a^Tb = 0 \).

Summarizing the above yields the conditions (i) and (ii).

Next we show that if
\[
W(\Gamma) \in \{(1, 4, 4), (2, 2, 5)\}
\]
then
\[
W(\Gamma + c) = W(\Gamma)
\]
for all \( c \in \mathbb{Z}^d \). By Lemma 5.2 the weight changes \( m_0, m_1, m_2 \) are either zero, or satisfy
\[
 m_0 + m_1 + m_2 = 3 \\
 (m_0n_0 + m_1n_1 + m_2n_2) + 3(m_0m_1 + m_0m_2 + m_0m_3) = 18 \tag{1}
\]
In the latter case, the weight changes are either
\[
 m_0 = m_1 = m_2 = 1
\]
or
\[
 \{m_0, m_1, m_2\} = \{0, 1, 2\}.
\]
It is easy to see that if \( m_0 = m_1 = m_2 = 1 \), then \( W(\Gamma + c) = W(\Gamma) \). Suppose
\[
 \{m_0, m_1, m_2\} = \{0, 1, 2\}.
\]
Then by Equation (I), when $W(\Gamma) = (1, 4, 4)$, we have

$$
n_0 = 1, \quad n_1 = n_2 = 4
m_0 = 0, \quad \{m_1, m_2\} = \{1, 2\}
$$

and when $W(\Gamma) = (2, 2, 5)$, we have

$$
n_0 = 5, \quad n_1 = n_2 = 2
m_0 = 2, \quad \{m_1, m_2\} = \{0, 1\}.
$$

Since the weight distribution of $\Gamma + c$ is

$$n'_j = n_j - 2m_j + m_{j-1} + m_{j+1}
$$

again we have $W(\Gamma + c) = W(\Gamma)$.

It is perhaps surprising that in all the characterizations discussed so far, the condition for a quotient graph $H(d, q)/\Gamma$ to admit uniform mixing only relies on the group generators, although Theorem 5.1 suggests we check the weight distribution of every coset of this group. It would reduce the problem of checking uniform mixing on $H(d, q)/\Gamma$ at time $\tau_q$ considerably if this were true in general.

6 Hamming Schemes

In this section, we construct Cayley graphs over $\mathbb{Z}_q^d$ that admit uniform mixing earlier than the complete graph $K_q$. The construction is based on the association scheme which spans the adjacency algebra of a Hamming graph, called the Hamming scheme. We introduce the basic concepts of association schemes and some useful results on the eigenvalues of Hamming schemes.

An association scheme with $d$ classes is a set $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ of 01-matrices that satisfies the following conditions:

- $A_0 = I$.
- $\sum_{r=0}^{d} A_j = J$.
- $A_r^T \in \mathcal{A}$ for $r = 0, 1, \ldots, n$.
- $A_rA_s = A_rA_s \in \text{span}(\mathcal{A})$. 

The association scheme $\mathcal{A}$ generates an algebra over $\mathbb{C}$, which is referred to as the Bose-Mesner algebra of $\mathcal{A}$. This algebra has an orthogonal basis of idempotents $E_0, E_1, \ldots, E_d$. Thus for each matrix $A_r$ in the scheme, there are scalars $p_r^{(d)}(s)$ such that

$$A_r = \sum_{s=0}^{d} p_r^{(d)}(s) E_s.$$  

These scalars are called the eigenvalues of the scheme $\mathcal{A}$. When $d$ is clear from the context, we drop the superscript and write $p_r(s)$. The following theorem due to Chan [3] shows that whether a graph admits uniform mixing depends only on its spectrum and the eigenvalues of the Bose-Mesner algebra containing its adjacency matrix.

6.1 Theorem (Chan). Let $X$ be a graph on $v$ vertices whose adjacency matrix belongs to the Bose-Mesner algebra of $\mathcal{A}$. Let $p_j(s)$ be the eigenvalues of $\mathcal{A}$. Suppose the spectral decomposition of $A(X)$ is

$$A(X) = \sum_{s=0}^{d} \theta_s E_s$$

The continuous quantum walk of $X$ is uniform mixing at time $\tau_q$ if and only if there exist scalars $t_0, t_1, \ldots, t_d$ such that

- $|t_0| = |t_1| = \cdots = |t_d| = 1$
- $\sqrt{\tau_q} e^{i \tau_q \theta_s} = \sum_{j=0}^{d} p_j(s)t_j$ for $s = 0, 1, \ldots, d$. \hfill $\square$

A Hamming scheme $\mathcal{H}(d, q)$ is an association scheme constructed from the Hamming graph $H(d, q)$. The matrix $A_j$ is the adjacency matrix of the $r$-th distance graph of $H(d, q)$, which has the same vertex set as $H(d, q)$ such that two vertices are adjacent if they are at distance $r$ in $H(d, q)$. The eigenvalues of the scheme $\mathcal{H}(d, q)$ satisfy

$$p_r^{(d)}(s) = [x^r](1 + (q - 1)x)^{d-s}(1 - x)^s$$

for $s = 0, 1, \cdots, d$. These are called the Krawtchouk polynomials. They satisfy the following properties.
6.2 Lemma. Let \( p_r(s) \) be the eigenvalues of the Hamming scheme \( H(d, q) \). Then

\[
\begin{align*}
(i) & \quad p_r(s) = \sum_h (-q)^h (q - 1)^{r-h} \binom{d-h}{r-h} \binom{s}{h}. \\
(ii) & \quad p_r(s) - p_r(s - 1) + (q - 1)p_{r-1}(s) + p_{r-1}(s - 1) = 0. \\
(iii) & \quad p^{(d+1)}_r(s) - p^{(d+1)}_r(s + 1) = q p^{(d)}_{r-1}(s). \\
(iv) & \quad \text{If } q = 2, \text{ then } \\
& \quad p_{r-1}(s) - p_{r-1}(s + 2) = 4 \sum_h (-2)^h \binom{d-2-h}{r-2-h} \binom{s}{h}.
\end{align*}
\]

Proof. By Equation (2),

\[
\begin{align*}
p_r(s) &= [x^r](1 + (q - 1)x)^{d-s}(1 + (q - 1)x - qx)^s \\
&= [x^r] \sum_h \binom{s}{h} (1 + (q - 1)x)^{d-h} (-qx)^h \\
&= \sum_h \binom{[x^h]}{s} (-qx)^h \binom{[x^{r-h}]}{(d-h)(1 + (q - 1)x)^{d-h}} \\
&= \sum_h (-q)^h (q - 1)^{r-h} \binom{d-h}{r-h} \binom{s}{h}.
\end{align*}
\]

Properties (ii) and (iii) follow from Equation (2), and Property (iv) follows from the above three properties for \( q = 2 \).

7 Sufficient Conditions for Uniform Mixing in Hamming Schemes

A graph on \( n \) vertices admits uniform mixing at time \( t \) if and only if \( \sqrt{n}U(t) \) is equal to a complex Hadamard matrix. For notational convenience, let \( \mathcal{B}(q) \) denote the Bose-Mesner algebra of the Hamming scheme \( H(d, q) \). For \( q \in \{2, 3, 4\} \), we can construct complex Hadamard matrices in \( \mathcal{B}(q) \) from a primitive \( q \)-th roots of unity.
7.1 Lemma. Let $\zeta_q$ be a primitive $q$-th root of unity. For $q \in \{2, 3, 4\}$, the matrix

$$e^{i\beta} (I_q + \zeta_{6-q}(J_q - I_q)) \otimes d$$

is a complex Hadamard matrix in $B(q)$.

Proof. First note that $I_q + \zeta_{6-q}(J_q - I_q)$ is a complex Hadamard matrix of order $q$ for $q \in \{2, 3, 4\}$. In fact, it is a scalar multiple of $U_{K_q}(\tau_q)$ for some mixing time $\tau_q$ of $K_q$. Now if $H_1$ and $H_2$ are both complex Hadamard matrices of order $q$, then $H_1 \otimes H_2$ is flat and thus a complex Hadamard matrix of order $q^2$. Lastly, a unimodular scalar multiple of a complex Hadamard matrix is again a complex Hadamard matrix.

The following generalizes Lemma 3.2 in [3].

7.2 Lemma. Let $X$ be a graph in $B(q)$ with eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$, and let $\epsilon \in \{1, -1\}$. Suppose $k \geq 2$.

(i) If $q = 2$, and

$$\theta_s - \theta_0 \equiv \epsilon 2^{k-1} s \pmod{2^{k+1}}$$

for $s = 0, 1, \ldots, d$, then $X$ admits uniform mixing at time $\pi/2^k$.

(ii) If $q = 3$, and

$$\theta_s - \theta_0 \equiv \epsilon 3^{k-1} s \pmod{3^k}$$

for $s = 0, 1, \ldots, d$, then $X$ admits uniform mixing at time $2\pi/3^k$.

(iii) If $q = 4$, and

$$\theta_s - \theta_0 \equiv 2^k s \pmod{2^{k+1}}$$

for $s = 0, 1, \ldots, d$, then $X$ admits uniform mixing at time $\pi/2^k$.

Proof. Suppose $q \in \{2, 3, 4\}$. Let

$$H_q = e^{i\beta} (I_q + \zeta_{6-q}(J_q - I_q)) \otimes d$$

$$= e^{i\beta} \left( (1 + (q-1)\zeta_{6-q}) \left( \frac{1}{q} J_q \right) + (1 - \zeta_{6-q}) \left( I_q - \frac{1}{q} J_q \right) \right) \otimes d.$$
By Equation (4.2) in [5, Sec 4],

\[ H_q = e^{i\beta} \sum_{r=0}^{d} (\zeta_{6-q})^r A_r \]

where \( A_r \) is the adjacency matrix of the \( r \)-distance graph of \( H(d, q) \). Hence the condition

\[ \sqrt{q^d} e^{it A} = H_q \]

is equivalent to

\[ \sqrt{q^d} e^{i\theta_s t} = e^{i\beta} (1 + (q - 1)\zeta_{6-q})^{d-s} (1 - \zeta_{6-q})^s \]

for \( s = 0, 1, \ldots, d \). It follows that

\[ \sqrt{q^d} e^{i\theta_0 t} = e^{i\beta} (1 + (q - 1)\zeta_{6-q})^d \]

(3)

and

\[ e^{i(\theta_s - \theta_0)t} = \left( \frac{1 - \zeta_{6-q}}{1 + (q - 1)\zeta_{6-q}} \right)^s \]

(4)

for \( s = 0, 1, \ldots, d \).

(i) For \( q = 2 \), Equation (4) reduces to

\[ \frac{2^k}{\pi} (\theta_s - \theta_0) t \equiv \epsilon 2^{k-1} s \pmod{2^{k+1}}. \]

(ii) For \( q = 3 \), Equation (4) reduces to

\[ \frac{3^k}{2\pi} (\theta_s - \theta_0) t \equiv \epsilon 3^{k-1} s \pmod{3^k}. \]

(iii) For \( q = 4 \), Equation (4) reduces to

\[ \frac{2^k}{\pi} (\theta_s - \theta_0) t \equiv 2^k s \pmod{2^{k+1}}. \]

Thus, for \( q \in \{2, 3, 4\} \), if \( \theta_s - \theta_0 \) satisfies the corresponding condition in the lemma, then there exists \( t, \beta \in \mathbb{R} \) that satisfy Equation (3) and (4). That is, \( X \) admits uniform mixing at time \( t \). \( \square \)
8 Faster Uniform Mixing on Distance Graphs

We apply the sufficient conditions developed in the last section to the distance graphs of Hamming graphs, as their eigenvalues are known. In Lemma 3.3 of \[3\], Chan derived a more accessible condition for uniform mixing in $H(d, 2)$ using Property (iv) in Lemma 6.2. However, this property holds only for $q = 2$. To extend her result, we need more general properties for $q \in \{2, 3, 4\}$. The first one is a corollary to Lemma 7.2.

8.1 Corollary. Suppose $d \geq 1$, $r \geq 1$ and $k \geq 2$. Let $X_r$ be the $r$-distance graph of the Hamming graph $H(d, q)$, and let $\epsilon \in \{1, -1\}$.

(i) If $q = 2$, and

$$p_{r-1}^{(d-1)}(s) \equiv \epsilon 2^{k-2} \pmod{2^k}$$

for $s = 0, 1, \ldots, d - 1$, then $X_r$ admits uniform mixing at time $\pi/2^k$.

(ii) If $q = 3$, and

$$p_{r-1}^{(d-1)}(s) \equiv \epsilon 3^{k-2} \pmod{3^{k-1}}$$

for $s = 0, 1, \ldots, d - 1$, then $X_r$ admits uniform mixing at time $2\pi/3^k$.

(iii) If $q = 4$, and

$$p_{r-1}^{(d-1)}(s) \equiv 2^{k-2} \pmod{2^{k-1}}$$

for $s = 0, 1, \ldots, d - 1$, then $X_r$ admits uniform mixing at time $\pi/2^k$.

Proof. We prove this for $q = 2$; the other two cases are similar.

Suppose

$$p_{r-1}^{(d-1)}(s) \equiv \epsilon 2^{k-2} \pmod{2^k}$$

for $s = 0, 1, \ldots, d - 1$. By Property (iii) in Lemma 6.2 this implies

$$p_r(s + 1) - p_r(s) \equiv -\epsilon 2^{k-1} \pmod{2^{k+1}}.$$ 

It follows that

$$p_r(s) - p_r(0) = p_r(s) - p_r(s - 1) + \cdots + p_r(1) - p_r(0) = -\epsilon s 2^{k-1} \pmod{2^{k+1}}.$$ 

By Lemma 7.2, $X_r$ admits uniform mixing at time $\pi/2^k$. □
With the help from the smaller scheme $H(d-1,q)$, we are able to construct examples in $H(d,q)$ that admit faster uniform mixing. It turns out that the conditions on the eigenvalues can be further simplified using Lemma 6.2. From now on, we focus on the Hamming schemes $H(d,3)$ and $H(d,4)$, as the examples in $H(d,2)$ are already given in [3].

8.2 Lemma. For $d \geq 1$, $r \geq 1$ and $k \geq 2$, if there exists $\epsilon \in \{-1, 1\}$ such that the following holds

(i) $2^{r-1} \binom{d-1}{r-1} \equiv \epsilon 3^{k-2} \pmod{3^{k-1}}$,

(ii) $3^{k-h-1}$ divides $\binom{d-h-1}{r-h-1}$ for $h = 1, 2, \cdots, k-2$,

then the distance graphs $X_r$ and $X_{d-r+1}$ in the Hamming scheme $H(d,3)$ admit uniform mixing at time $2\pi/3^k$.

Proof. From Lemma 6.2 we have

$$p_{r-1}^{(d-1)}(s) \equiv \sum_{h=0}^{d-1} (-3)^h 2^{r-h-1} \binom{d-h-1}{r-h-1} \binom{s}{h} \pmod{3^{k-1}}$$

$$\equiv \sum_{h=0}^{k-2} (-3)^h 2^{r-h-1} \binom{d-h-1}{r-h-1} \binom{s}{h} \pmod{3^{k-1}}.$$

By condition (i), when $s = 0$,

$$p_{r-1}^{(d-1)}(0) = 2^{r-1} \binom{d-1}{r-1} \equiv \epsilon 3^{k-2} \pmod{3^{k-1}}$$

and condition (ii), when $s \geq 1$,

$$p_{r-1}^{(d-1)}(s) = p_{r-1}^{(d-1)}(0) + \sum_{h=1}^{k-2} (-3)^h 2^{r-h-1} \binom{d-h-1}{r-h-1} \binom{s}{h} \pmod{3^{k-1}}$$

$$\equiv \epsilon 3^{k-2} \pmod{3^{k-1}}.$$

It follows from Corollary 8.1 that $X_r$ in $H(d,3)$ admit uniform mixing at time $2\pi/3^k$. For $X_{d-r+1}$, first note that condition (i) is symmetric on $r$ and $d-r+1$. By condition (ii), $3^{k-h-1}$ divides

$$\binom{d-h}{r-h} - \binom{d-h-1}{r-h-1} = \binom{d-h-1}{r-h}$$
for $h = 1, 2, \cdots, k - 2$. It follows that $3^{k-h-2}$ divides
\[
\binom{d - h - 1}{r - h} - \binom{d - h - 2}{r - h - 1} = \binom{d - h - 2}{r - h}
\]
for $h = 1, 2, \cdots, k - 2$. Continuing this procedure, we see that $3^{k-h-l}$ divides
\[
\binom{d - h - l}{r - h}
\]
for $h = 1, 2, \cdots, k - 2$ and $l = 1, 2, \cdots, k - 2$. Taking $h = 1$ for all $l$ shows that $3^{k-l-1}$ divides
\[
\binom{d - l - 1}{r - 1} = \binom{d - l - 1}{(d - r + 1) - l - 1}
\]
for $l = 1, 2, \cdots, k - 2$, which is exactly condition (ii) with $r$ replaced by $d - r + 1$. Hence, $X_{d-r+1}$ admits uniform mixing at time $2\pi/3^k$ as well.

With a similar argument, we can reduce the conditions for faster uniform mixing in $\mathcal{H}(d, 4)$ to the following.

**8.3 Lemma.** For $d \geq 1$, $r \geq 1$ and $k \geq 2$, if the following two conditions hold

(i) $3^{r-1}\binom{d-1}{r-1} \equiv 2^{k-2} \pmod{2^{k-1}},$

(ii) $2^{k-2h-1}$ divides $\binom{d-h-1}{r-h-1}$ for $h = 1, 2, \cdots, \lfloor k/2 \rfloor - 1,$

then the distance graphs $X_r$ and $X_{d-r+1}$ in the Hamming scheme $\mathcal{H}(d, 4)$ admit uniform mixing at time $\pi/2^k$.

The above observations imply that our potential examples rely heavily on the divisibility of a binomial coefficient by some prime power. In fact, this is closely related to the base $p$ representation of the binomial coefficients, where $p$ is a prime. To find the pairs $(d, r)$ that satisfy the divisibility conditions, we need the following number theory result due to Kummer [4, Ch 9].

**8.4 Theorem** (Kummer). Let $p$ be a prime. The largest integer $k$ such that $p^k$ divides $\binom{N}{M}$ is the number of carries in the addition of $N - M$ and $M$ in base $p$ representation.
For our purposes, we look at the ternary representations of \(d - r\) and \(r - h - 1\) for \(\mathcal{H}(d, 3)\), and their binary representations for \(\mathcal{H}(d, 4)\). The following are our new examples of distance graphs that admit uniform mixing at times earlier than the Hamming graphs.

**8.5 Theorem.** For \(k \geq 2\) and \(r \in \{3^k - 1, 3^k - 4, 3^k - 7\}\), the \(r\)-distance graphs \(X_r\) of the Hamming graph \(H(2 \cdot 3^k - 9, 3)\) admit uniform mixing at time \(2\pi/3^k\).

**Proof.** Let \(d = 2 \cdot 3^k - 9\) and \(r = 3^k - 1\). Then
\[
\begin{align*}
d - r &= 2 \cdot 3^{k-1} + 2 \cdot 3^{k-2} + \cdots + 2 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 \\
r - 1 - h &= 2 \cdot 3^{k-1} + 2 \cdot 3^{k-2} + \cdots + 2 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0 - h.
\end{align*}
\]
When \(h = 0\), since \((d - r) + (r - 1)\) has exactly \(k - 2\) carries, \(\binom{d-1}{r-1}\) is divisible by \(3^{k-2}\) but not divisible by \(3^{k-1}\). Then there exists \(\epsilon \in \{-1, 1\}\) such that
\[
2^{r-1} \binom{d-1}{r-1} \equiv \epsilon 3^{k-2} \pmod{3^{k-1}}.
\]
For \(h = 1\), the number of carries in \((d - r) + (r - 2)\) is still \(2^{k-2}\). When \(h = 2, \cdots, k - 2\), the number of carries in \((d - r) + (r - h + 1)\) drops by at most one as \(h\) increases by one. Therefore \((d - r) + (r - h + 1)\) has at least \(k - h - 1\) carries, and so \(3^{k-h-1}\) divides \(\binom{n-h}{r-h-1}\). By Theorem 8.2, \(X_{3^k-1}\) and \(X_{3^k-7}\) in \(\mathcal{H}(2 \cdot 3^k - 9, 3)\) admit uniform mixing at time \(2\pi/3^k\). Similar argument applies to \(X_{3^k-4}\).

Some new examples in \(\mathcal{H}(d, 4)\) can be obtained in a similar way.

**8.6 Theorem.** For \(k \geq 2\), the distance graph \(X_{2^k-2}\) of the Hamming graph \(H(2^{k-1} - 1, 4)\), and the distance graphs \(X_{2^{k-1}-1}, X_{2^k-1}\) of the Hamming graph \(H(2^k - 2, 4)\) admit uniform mixing at time \(\pi/2^k\).

9 Local and Global Uniform Mixing on Stars

For irregular graphs, local uniform mixing may be a better choice to start with. We follow Carlson et al [2] and show that the star \(K_{1,n}\) admits local
uniform mixing. In particular, uniform mixing in the global sense occurs on the claw $K_{1,3}$.

We apply spectral decomposition to the adjacency matrix $A$ of the star $K_{1,n}$. The eigenvalues of $K_{1,n}$ are $\theta_0 = 0$, $\theta_1 = \sqrt{n}$ and $\theta_2 = -\sqrt{n}$. Denote the projections onto these eigenspaces by $E_0$, $E_1$ and $E_2$. We have

$$E_0 = \begin{pmatrix} 0 & 0^T \\ 0 & I - \frac{1}{n} J \end{pmatrix}$$

$$E_1 = \frac{1}{2n} \begin{pmatrix} n & \sqrt{n}1^T \\ \sqrt{n}1 & J \end{pmatrix}$$

$$E_2 = \frac{1}{2n} \begin{pmatrix} n & -\sqrt{n}1^T \\ -\sqrt{n}1 & J \end{pmatrix}$$

where $J$ denotes the all-ones matrix. It follows that the transition matrix of $K_{1,n}$ is

$$U(t) = e^{0-it}E_0 + e^{\sqrt{n}it}E_1 + e^{-\sqrt{n}it}E_2$$

$$= \begin{pmatrix} \cos(\sqrt{n}t) & \frac{1}{\sqrt{n}} \sin(\sqrt{n}t)1 & \frac{1}{\sqrt{n}} \sin(\sqrt{n}t)1 & I + \frac{1}{n}(\cos(\sqrt{n}t) - 1)J \end{pmatrix}.$$

The quantum walk starting with the central vertex is uniform mixing at time $t$ if and only if

$$\left| \cos(\sqrt{n}t) \right| = \left| \frac{\sin(\sqrt{n}t)}{\sqrt{n}} \right|$$

or equivalently,

$$\tan(\sqrt{n}t) = \pm \sqrt{n}. \quad (5)$$

Thus, the star $K_{1,n}$ admits local uniform mixing at time

$$\pm \frac{\arctan(\sqrt{n})}{\sqrt{n}} + k\pi$$

for all integers $k$.

For uniform mixing, one additional condition from the lower right block of $U(t)$ is

$$\left| 1 + \frac{1}{n} (\cos(\sqrt{n}t) - 1) \right| = \left| \frac{1}{n} (\cos(\sqrt{n}t) - 1) \right|$$

or equivalently,

$$\cos(\sqrt{n}t) = 1 - \frac{n}{2}. \quad (6)$$
Combining Equation (5) and Equation (6), we see that the only solution is

\[ n = 3, \quad t = \pm \frac{2\pi}{\sqrt{27}} + 2k\pi \]

for all integers \( k \). Plugging this into \( U(t) \) yields a flat matrix. We conclude that the only star that admits uniform mixing is the claw \( K_{1,3} \), with earliest mixing time \( 2\pi/\sqrt{27} \). The Cartesian powers of \( K_{1,3} \) then form an infinite family of irregular graphs that admit uniform mixing.

### 10 Open Problems

There are a number of open problems on uniform mixing, ranging across characterizing graphs that admit uniform mixing in some common family, determining the mixing times of a given graph, and constructing new examples. Following our notation in Section 4, we let \( \tau_q \) denote the earliest time at which the complete graph \( K_q \) admits uniform mixing.

1. **Question:** To determine whether uniform mixing occurs on the quotient graph \( H(d, q)/\Gamma \) at time \( \tau_q \), we have to check the weight distribution of every coset of \( \Gamma \). Is it sufficient to just check the weight distribution of \( \Gamma \)?

   For groups with one or two generators, the weight distribution of any coset \( \Gamma + c \) is merely a permutation of the weight distribution of \( \Gamma \). For an example see Theorem 5.3. If this were true in general, it would be helpful in characterizing linear Cayley graphs with higher degrees.

2. **Question:** Is there a characterization of uniform mixing on non-linear Cayley graphs over \( \mathbb{Z}_d \)?

   This is one thing that the weight distribution condition does not tell. For \( q \geq 4 \), a Cayley graph over \( \mathbb{Z}_q^d \) may not be linear, and thus may not be a quotient graph of \( H(d, q) \). It is desirable to find another approach for these non-linear Cayley graphs.

3. **Question:** If a Cayley graph over \( \mathbb{Z}_q^d \) admits uniform mixing, must its eigenvalues be integral?

   As we mentioned in Section 11 all the known Cayley graphs that admit uniform mixing have integer eigenvalues. It is unclear if this is a necessary condition. The first place to find a counterexample might be the
Cayley graphs over $\mathbb{Z}_q^d$, as the eigenvalues are no longer guaranteed to be integral.

4. Question: If a Cayley graph over $\mathbb{Z}_q^d$ admits uniform mixing at time $t$, must $t$ be a rational multiple of $\pi$?

Again this is true for all the known examples. However, it may only apply to graphs with integer eigenvalues. Even for this smaller class of graphs, it would be interesting to confirm such an algebraic property of the mixing times.

5. Question: How fast can a Cayley graph over $\mathbb{Z}_q^d$ admit uniform mixing?

So far, the best examples that admit uniform mixing earlier than $\tau_q$ are the distance graphs of $H(d, 2)$ found in [3], and the distance graphs of $H(d, 3)$ and $H(d, 4)$ found in Section 8 of this paper. These families provide arbitrarily faster uniform mixing, although at the cost of larger vertex sets. In an effort to construct new examples with faster uniform mixing, a question arises as to whether there is a lower bound on the mixing time of a given graph.

6. Question: Are there more irregular graphs that admit uniform mixing?

The star $K_{1,3}$ and its Cartesian powers suggest that there could be other irregular graphs that admit uniform mixing. As we did in Section 9, one may look at local uniform mixing on some common familes of irregular graphs, and then impose more conditions for global uniform mixing.

References

[1] Amir Ahmadi, Ryan Belk, Christino Tamon, and Carolyn Wendler, *Mixing in Continuous Quantum Walks on Graphs*, arXiv:quant-ph/0209106 (2002).

[2] W. Carlson, A. Ford, E. Harris, J. Rosen, C. Tamon, and K. Wrobel, *Universal Mixing of Quantum Walk on Graphs*, arXiv:quant-ph/0608044 (2006).

[3] Ada Chan, *Complex Hadamard Matrices, Instantaneous Uniform Mixing and Cubes*, arXiv:1305.5811 (2013).
[4] Leonard Eugene Dickson, *History of the Theory of Numbers. Vol. I: Divisibility and Primality*, Chelsea Publishing Co., New York, 1966.

[5] Chris Godsil, *Generalized Hamming schemes*, arXiv:1011.1044 (2010).

[6] Chris Godsil, Natalie Mullin, and Aidan Roy, *Uniform Mixing and Association Schemes*, arXiv:1301.5889 (2013).

[7] Cristopher Moore and Alexander Russell, *Quantum Walks on the Hypercube*, arXiv:quant-ph/0104137 (2001).

[8] Natalie Ellen Mullin, *Uniform Mixing of Quantum Walks and Association Schemes*, Ph.D. thesis, University of Waterloo, September 2013.