Manipulation of non-linear heat currents in the dissipative Anderson–Holstein model

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Abstract

The precise control of phonon heat currents will be of primary importance in emerging phononic devices. In this paper, a detailed analysis of electronically controled phonon transport is carried out using an Anderson–Holstein based dissipative quantum dot setup. We consider two relevant electronic bias situations: (a) a voltage bias in the absence of an electronic temperature gradient and (b) an electronic temperature gradient at zero voltage. It is shown that the direction of phonon transport in the non-linear regime is different in the two cases since the first case facilitates the accumulation of phonons in the dot and the second case leads to the absorption of phonons in the dot. In the linear regime, both the phonon and electronic transport get decoupled and Onsager’s symmetry is verified. We explain the observed cumulative effects of voltage and electronic temperature gradients on the non-linear phonon currents by introducing a new transport coefficient that we term as the electron induced phonon thermal conductivity. It is demonstrated that under suitable operating conditions in Case (a) the dot can pump in phonons into the hotter phonon reservoirs and in Case (b) the dot can extract phonons out of the colder phonon reservoirs. Finally, as a corollary, we elaborate on how the non-linear electronic heat current can be stimulated and controlled by manipulating the temperature of the phonon reservoirs even under vanishing effective electronic charge flow.

Keywords: quantum dots, phononics, thermoelectric cooling

(Some figures may appear in colour only in the online journal)
phonon temperature. Hence we additionally augment our set up with phonon reservoirs \( H \) and \( C \) in order to make a distinction between electronic temperature and phonon temperature. Additionally the extent of non-equilibrium phonon population in the dot can be evaluated via a phonon thermometer \([30, 31]\) which serves to estimate the dot temperature. Hence this set up can be used to excite electronic and phonon currents driven by applying a bias voltage and/or a thermal bias across the contacts/reservoirs.

Our objective in this work is to comprehensively study the electronic control of phonon transport in two cases via: Case (a): a voltage bias in the absence of an electronic temperature bias and Case (b): an electronic temperature bias at zero voltage. It is shown by the simulation framework that the polarity of phonon currents in the non-linear regime is opposite in the two cases since the first case facilitates the accumulation of phonons in the dot and the second case leads to the absorption of phonons in the dot. In Case (a), excess phonons heat up the dot and are suitably extracted by the phonon reservoirs kept in equilibrium. In Case (b), the dot cools down and the reservoirs pump phonons into it. However in the linear regime, phonon and electron transport get decoupled thus verifying the Onsager’s reciprocity. In a previous work \([31]\), we explored the non-linear phonon Peltier effect by noticing the voltage dependence of the non-linear phonon current. In the current work, we propose a new parameter called, electron-assisted phonon thermal conductivity to signify the mutual dependence of phonon currents on the voltage and electronic temperature gradient.

Additionally, we notice that the polarity of phonon transport can be controlled by differing the temperature of phonon reservoirs from the temperature of electronic contacts. In Case (a), phonons flow from the reservoirs into the dot at low voltage when the reservoirs are hotter than the contacts. At large voltage, the polarity of phonon flow is reversed and the phonons are pumped into the hotter reservoirs. An opposite situation is observed in Case (b) given the reservoirs are colder than the contacts. In this case, phonons relax from the dot to the reservoirs at small electronic temperature gradients. When the temperature gradient is large, the direction of phonon flow is reversed and phonons are extracted from the colder reservoirs. In both cases, the direction of phonon current is verified by estimating the dot temperature with a phonon thermometer bath weakly coupled to the dot. Finally, we elaborate on how the non-linear electronic heat current can be stimulated and controlled by manipulating the temperature of the phonon reservoirs even under vanishing effective electronic current flow.

This paper is organized as follows: section 2 introduces the device model and formulates the transport equations. In section 3.1, we focus on the comparative study of the characteristics of the phonon currents in the two cases under consideration. Section 3.2 introduces the electron assisted phonon thermal conductivity coefficient and compares it with the conventional electronic thermal conductivity. In section 3.3, we describe the manipulation of phonon bath temperatures to
modulate the flow of the phonon and electronic heat currents. In section 4, we summarize our results and conclude.

2. Physics and formulation

The schematics of the set ups to be studied here are depicted in figures 1(a) and (b). Each of them typically comprises a single quantum dot weakly coupled to the electronic contacts \( \alpha_1 (\alpha_1 \in L, R) \), and the phonon reservoirs \( \alpha_2 (\alpha_2 \in H, C) \). The dot is described by the dissipative Anderson–Holstein model which has both electronic and phonon degrees of freedom interacting with each other. The electronic current is driven through the device in two ways: Case (a): by applying a voltage bias (shown in figure 1(a)) across the contacts \( L \) and \( R \) and Case (b) by applying an electronic temperature gradient (shown in figure 1(b)) across the contacts \( L \) and \( R \). The phonon current is set up by applying a thermal bias across the phonon reservoirs \( H \) and \( C \). In addition, the electron current gives rise to a phonon current even in the absence of a temperature bias across the reservoirs \( H \) and \( C \), provided the electron–phonon coupling is finite. The direction of phonon flow depends on the dot temperature \( T_D \), which is estimated by a phonon thermometer bath weakly coupled to the dot.

2.1. Model Hamiltonian

The composite system Hamiltonian \( H \) can be written as

\[
H = H_D + H_{\alpha_1} + H_{\alpha_2} + H_{\alpha_1D} + H_{\alpha_2D} \quad \text{Here } \quad H_D, \ H_{\alpha_1}, \ H_{\alpha_2} \ \text{denote the respective Hamiltonians of the dot, the contacts and the reservoirs, while } \ H_{\alpha_1D} \text{ and } H_{\alpha_2D} \text{ represent the dot-to-contact electronic tunneling processes and the dot-to-reservoir phonon relaxation processes respectively. The dot Hamiltonian is further divided as}
\]

\[
H_D = H_{d} + H_{ph} + H_{el-ph}.
\]  

(1)

The electronic part \( (H_d) \) consists of a single spin degenerate energy level with an on-site energy \( \varepsilon \) and a Coulomb interaction energy \( U \) for double occupancy. The phonon part \( (H_{ph}) \) comprises of a single phonon mode \( \nu \) with angular frequency \( \omega_\nu \). Inside the dot, the electrons and phonons interact through a dimensionless parameter \( \lambda_\nu \). Hence, \( H_d, \ H_{ph}, \ H_{el-ph} \) are given by

\[
H_d = \left( \sum_\sigma c_\sigma^\dagger c_\sigma + Ud_{\downarrow}^\dagger d_{\downarrow} + Ud_{\uparrow}^\dagger d_{\uparrow} \right),
\]

\[
H_{ph} = \hbar \omega_\nu b_{\nu}^\dagger b_{\nu},
\]

\[
H_{el-ph} = \sum_\nu \lambda_\nu \hbar \omega_\nu d_{\downarrow}^\dagger d_{\downarrow} (b_{\nu}^\dagger + b_{\nu}).
\]

In the above expressions, \( d_{\sigma}^\dagger (d_{\sigma}) \) and \( b_{\nu}^\dagger (b_{\nu}) \) are the respective creation (annihilation) operators of the electrons and phonons. The electronic contacts \( \alpha_1 \) characterize a macroscopic body of non-interacting electrons in the momentum eigen state \( \alpha_{1k} \) and spin \( \sigma \) with energies \( \epsilon_{\alpha_{1k}\nu} \). Similarly, the phonon reservoirs \( \alpha_2 \) comprise of numerous phonon modes \( \alpha_2 \nu \) with an angular frequency \( \omega_{\alpha_2\nu} \). They are defined by the following Hamiltonians,

\[
\begin{align*}
\hat{H}_{\alpha_1} &= \sum_{\alpha_1 \in L,R} \sum_{\nu,\sigma} \hat{c}_{\alpha_1 \nu \sigma k}^\dagger \hat{c}_{\alpha_1 \nu \sigma k}, \\
\hat{H}_{\alpha_2} &= \sum_{\alpha_2 \in H,C} \sum_{\nu,\sigma} \hat{B}_{\alpha_2 \nu \sigma}^\dagger \hat{B}_{\alpha_2 \nu \sigma},
\end{align*}
\]  

(3)

where \( \hat{c}_{\alpha_1 \nu \sigma k}(\hat{c}_{\alpha_1 \nu \sigma k}) \) creates (annihilates) an electron with momentum \( k \) and spin \( \sigma \) in the contact \( \alpha_1 \), and \( \hat{B}_{\alpha_2 \nu \sigma}(\hat{B}_{\alpha_2 \nu \sigma}) \) creates (annihilates) a phonon of angular frequency \( \omega_{\alpha_2 \nu} \) in the reservoir \( \alpha_2 \). If the electrons in the dot are coupled to the electrons in the contacts \( \alpha_1 \) through an energy \( \tau_{el-ph} \) and the phonons in the dot are coupled to the phonons in the reservoirs \( \alpha_2 \) through an energy \( \tau_{ph} \) then the coupling Hamiltonians \( H_{\alpha_1D} \) and \( H_{\alpha_2D} \) are given as follows

\[
\begin{align*}
\hat{H}_{\alpha_1D} &= \sum_{\nu,\sigma,\tau} \tau_{el-ph} \hat{c}_{\alpha_1 \nu \sigma k}^\dagger \hat{c}_{\alpha_2 \nu \sigma k}^\dagger + \hat{c}_{\alpha_1 \nu \sigma k} \hat{c}_{\alpha_2 \nu \sigma k} + \hat{c}_{\alpha_1 \nu \sigma k}^\dagger \hat{c}_{\alpha_2 \nu \sigma k}^\dagger, \\
\hat{H}_{\alpha_2D} &= \sum_{\nu,\sigma,\tau} \tau_{ph} \hat{B}_{\alpha_2 \nu \sigma}^\dagger \hat{B}_{\alpha_1 \nu \sigma}^\dagger + \hat{B}_{\alpha_2 \nu \sigma} \hat{B}_{\alpha_1 \nu \sigma} + \hat{B}_{\alpha_2 \nu \sigma}^\dagger \hat{B}_{\alpha_1 \nu \sigma}^\dagger.
\end{align*}
\]  

(4)

In this work, we assume that the phonon transition processes are mode independent and the electron tunneling processes are momentum conserved and spin independent. With this approximation we can rewrite \( \tau_{el-ph} \) and \( \tau_{ph} \) as \( \tilde{\tau}_{el-ph} \) and \( \tilde{\tau}_{ph} \) respectively.

The dot Hamiltonian is diagonalized by the polaron transformation (such that \( \hat{H}_D \rightarrow \tilde{H}_D = e^{\tilde{\tau}_{el-ph} \Gamma} H_D e^{-\tilde{\tau}_{el-ph} \Gamma} \), where \( S = \sum_{\nu,\sigma} \lambda_\nu [\hat{b}_{\nu} - \hat{b}_{\nu}^\dagger] \)). It transforms the dot fermionic operators \( d_{\nu}^\dagger (d_{\nu}) \) and leads to the renormalization of \( \epsilon \) and \( U \), such that \( \tilde{\epsilon} = \epsilon - \lambda_\nu \hbar \omega_\nu \), and \( \tilde{U} = U - 2 \lambda_\nu^2 \hbar \omega_\nu \). The eigen states of \( \tilde{H}_D \) are represented by \( |n,q\rangle \), where \( n \) and \( q \) are the electron and phonon number of the eigen states. The energy eigen values of those states become \( E_{n,q} = \tilde{E}_n + q \hbar \omega_\nu \), where \( \tilde{E}_0 = 0, \tilde{E}_1 = \tilde{\epsilon}, \tilde{E}_2 = 2\tilde{\epsilon} + \tilde{U}, \) corresponding to the \( n = 0, 1 \) and 2 electron space respectively. This transformation also renormalizes the electronic tunneling energy \( \tilde{\tau}_{el-ph} \), such that \( \tilde{\tau}_{el-ph} = \tau_{el-ph} \exp [-\lambda_\nu (\hat{b}_{\nu} - \hat{b}_{\nu}^\dagger)] \). However, the phonon coupling energy \( \tilde{\tau}_{ph} \) is left unaltered due to the polaron transformation since the operator \( S \) commutes with the phonon operators \( \hat{b}_{\nu}^\dagger (\hat{b}_{\nu}) \) of the dot.

With the derived expressions of \( \tilde{\tau}_{el-ph} \) and \( \tilde{\tau}_{ph} \), we can evaluate the dot-to-contact electron tunneling rate \( \tilde{\gamma}_{\alpha_1} \) and the dot-to-phonon relaxation rate \( \tilde{\beta}_{\alpha_2} \) using the Fermi’s golden rule. They are represented as: \( \tilde{\gamma}_{\alpha_1} = \frac{2\pi}{\hbar} \sum_{\nu,\sigma} |\tilde{\tau}_{el-ph}|^2 \rho_{\alpha_1\sigma} \) and \( \tilde{\beta}_{\alpha_2} = \frac{2\pi}{\hbar} \sum_{\nu,\sigma} |\tilde{\tau}_{ph}|^2 D_{\alpha_2} \), where \( \rho_{\alpha_1\sigma} \) and \( D_{\alpha_2} \) are the constant electron and phonon density of states associated with the contacts \( \alpha_1 \) and the reservoirs \( \alpha_2 \) respectively. In the Anderson–Holstein model, the rate of phonon transition becomes mode independent since a single mode is involved. The study of the model Hamiltonian enables us to formulate the transport methodology, which we will analyze in the next subsection.
2.2. Transport methodology

Elaborating on the assumptions firstly, we set the rate of dot-to-reservoir phonon relaxation processes much lower than the rate of dot-to-contact electron tunneling processes \( \hbar \gamma_{\alpha 1} \gg \hbar \beta_{\alpha 2} \) to rule out the system damping \cite{32}. Second, we set \( \hbar \gamma_{\alpha 1}, \hbar \beta_{\alpha 2} \ll k_B T \), so that the transport of the electrons and phonons through a single quantum dot weakly coupled to the electronic contacts and the phonon reservoirs can be described in the sequential tunneling limit in which charge and phonon currents are calculated via the rate equation approach \cite{35–41}, unlike the non-equilibrium Green’s function approach used in the heterostructure based thermoelectric devices \cite{42–45}. Lastly, we neglect the overlap of the adjacent phonon sidebands, by setting the energy gap between the sidebands larger than the tunnel induced broadening of energy levels \( \hbar \omega_p \gg \hbar \gamma_{\alpha 1} \) \cite{46}. With this assumption, the Markov approximation is justified and two consecutive electron tunneling processes are completely uncorrelated \cite{47}, leading to the justification of using only the diagonal elements via the rate equation \cite{48–50}.

The tunneling rate between two eigen states \( |n, q \rangle \) and \( |n \pm 1, q' \rangle \) is determined by the contact Fermi function of the energy difference of the two states and it is given by:

\[
R_{(n,q)\rightarrow(n\pm1,q')} = \sum_{\alpha \in \{LR\}} \gamma_{\alpha}| \langle n, q | \tilde{d}_{\alpha} | n \pm 1, q' \rangle |^2 f_{\alpha}(E_{n+1,q'} - E_{n,q}), \tag{5}
\]

\[
R_{(n,q)\rightarrow(n-1,q')} = \sum_{\alpha \in \{LR\}} \gamma_{\alpha}| \langle n, q | \tilde{d}_{\alpha} | n - 1, q' \rangle |^2 f_{\alpha}(E_{n,q} - E_{n-1,q'}), \tag{6}
\]

where \( f_{\alpha} (\zeta) = 1/(1 + \exp(-\zeta/\mu_{\alpha})) \) is the Fermi–Dirac distribution function of the contact \( \alpha \) with chemical potential \( \mu_{\alpha} \) and temperature \( T_{\alpha} \). The phonon relaxation process between the reservoirs and the dot lead to the transition between two eigen states \( |n, q \rangle \) and \( |n, q' \pm 1 \rangle \) and the relaxation rate follows the Boltzmann ratio:

\[
R_{(n,q)\rightarrow(n,q+1)} = \sum_{\alpha \in \{RL\}} \beta_{\alpha}(q + 1)f(n, q', \zeta) \exp \left( -\frac{\hbar \omega_p}{k_B T_{s2}} \right), \tag{7}
\]

\[
R_{(n,q)\rightarrow(n,q-1)} = \sum_{\alpha \in \{RL\}} \beta_{\alpha}(q + 1). \tag{8}
\]

Using the rate equations, the master equation for the probabilities \( P_{n,q} \) of the many-body electron–phonon states \( |n, q \rangle \) take the following form:

\[
\frac{dP_{n,q}}{dt} = \frac{N_s}{q'} \sum_{q' \neq q} \left[ R_{(n,q')\rightarrow(n,q)}^R P_{n,q'} - R_{(n,q)\rightarrow(n,q')}^R P_{n,q} \right] + \left[ R_{(n,q')\rightarrow(n,q)}^{ph} P_{n,q'} - R_{(n,q)\rightarrow(n,q')}^{ph} P_{n,q} \right] \delta(n \pm 1, q') \delta(q \pm 1, q'). \tag{9}
\]

In the steady state, the derivative in the left hand side of (9) vanishes and the steady state probabilities \( P_{n,q} \) can be determined by solving the algebraic equations. We use the probabilities to compute the charge currents, the electronic heat currents associated with the contacts \( \alpha_1 \) and the phonon heat currents associated with reservoirs \( \alpha_2 \). They are given as

\[
I_{\alpha_1} = \sum_{q=0}^{N_q} \sum_{q'=0}^{N_q'} -q \left[ R_{(n+1,q')\rightarrow(n,q)}^{\alpha_1} P_{n+1,q'} - R_{(n,q')\rightarrow(n,q)}^{\alpha_1} P_{n,q} \right], \tag{10}
\]

\[
F_{Q_\alpha_1}^{\alpha_1} = \sum_{q=0}^{N_q} \sum_{q'=0}^{N_q'} (E_{n+1,q'} - E_{n,q} - \mu_{\alpha_1}) \left[ R_{(n+1,q')\rightarrow(n,q)}^{\alpha_1} P_{n+1,q'} - R_{(n,q')\rightarrow(n,q)}^{\alpha_1} P_{n,q} \right], \tag{11}
\]

\[
F_{\phi}^{\alpha_1} = \sum_{q=0}^{N_q} \sum_{q'=0}^{N_q'} \hbar \omega_p \left[ R_{(n,q')\rightarrow(n,q)}^{\phi} P_{n,q'} - R_{(n,q)\rightarrow(n,q')}^{\phi} P_{n,q} \right] \delta(q, q') \delta(n, n'). \tag{12}
\]

Now it is evident from (5) and (6) that the transition between the states \( |n, q \rangle \) and \( |n \pm 1, q' \rangle \) leads to the net phonon generation (or absorption) in the dot. In the next subsection we will focus on the role of \( \lambda_p \) in controlling the phonon generation (or absorption). Considering the law of charge conservation, in the rest of the paper we will denote \( I_L = -I_R = I \).

2.3. Effect of interaction on phonon transport

The effective electron tunneling rate \( \gamma_{\alpha \alpha}^{\text{eff}} \) between the states \( |n, q \rangle \) and \( |n \pm 1, q' \rangle \) is modified by the \textit{Franck–Condon} overlapping factor between the two states \cite{51–53}. The effective electron tunneling rate is given as

\[
\gamma_{\alpha \alpha}^{\text{eff}} = \gamma_{\alpha \alpha} |FC_{q,q'}|^2 = \gamma_{\alpha \alpha} \left[ |\langle n, q | \tilde{d}_{\alpha} | n', q' \rangle |^2 \delta(n', n - 1) + |\langle n, q | \tilde{d}_{\alpha} | n', q' \rangle |^2 \delta(n', n + 1) \right]. \tag{13}
\]

where \( |FC_{q,q'}|^2 = \exp(-\lambda_{p^2} K_{\alpha}^4 K_{\alpha}^{2-k} |\lambda_{p}^{k-2} L_{\alpha}^{2-k}|^4) \) is the \textit{Franck–Condon} factor between the two states with phonon number \( q \) and \( q' \) and \( L_{\alpha}^{2-k} \) is the associated Laguerre polynomial with \( k = \min(q, q') \) and \( K = \max(q, q') \). The net phonon generation (or absorption) in the dot due to the transition between the states \( |n, q \rangle \) and \( |n \pm 1, q \rangle \), takes place at a rate

\[
G_{\phi}^{\alpha_1} = \sum_{n,q \neq n' \pm 1} (q' - q) P_{n,q} R_{(n,q)\rightarrow(n,\pm1,q')}^{\alpha_1}. \tag{14}
\]

It is evident that in the limit \( \hbar \gamma_{\alpha_1} \gg \hbar \beta_{\alpha_2} \), the phonon distribution in the dot is primarily determined by the
electron transport. When \( q \neq q' \), the phonons generate (or get absorbed) in the dot and the average phonon number in the dot 

\[
\langle N_{ph} \rangle = \sum_{q} q \mathbb{P}_{n,q} \text{deviates from the equilibrium phonon distribution } \langle N_{ph}^{eq} \rangle \text{ of the phonon reservoirs. The excess phonons are extracted by the reservoirs at a rate [54]}
\]

\[
RE_{\alpha\gamma}^{ph} = \sum_{q=0}^{\alpha} \sum_{q'=0}^{\gamma} \hbar \omega_{q} \left[ R^{ph}_{\alpha\gamma} \left( (n,q) \rightarrow (n',q') \right) \mathbb{P}_{n,q} \right] - \left[ R^{ph}_{\alpha\gamma} \left( (n',q') \rightarrow (n,q) \right) \mathbb{P}_{n',q'} \right] \delta(q+1,q')
\]

(15)

It is evident from (12) and (15) that the phonon current \( J_{Q_{\alpha\gamma}}^{ph} \) vanishes until \( q = q' \). In this case, \( \langle N_{ph} \rangle \) equals with \( \langle N_{ph}^{eq} \rangle \) and \( J_{Q_{\alpha\gamma}}^{ph} \) vanishes. On the other way, when \( \lambda_{\alpha} \) is non-zero, \( \langle N_{ph} \rangle \) deviates from \( \langle N_{ph}^{eq} \rangle \) and \( J_{Q_{\alpha\gamma}}^{ph} \) becomes finite. The deviation of \( \langle N_{ph} \rangle \) from \( \langle N_{ph}^{eq} \rangle \) causes the variation of dot temperature \( T_{D} \) from the equilibrium reservoir temperature \( T_{\alpha} \). The dot temperature can be computed from the Boltzmann ratio with a quasi-equilibrium approximation [55] as:

\[
T_{M} = \frac{h \omega_{\gamma}}{k_{B}} \left[ \ln \left( \frac{\mathbb{P}_{n,q}}{\mathbb{P}_{n,q+1}} \right) \right]^{-1}.
\]

We will notice in the next section that the electron induced phonon current shows different characteristics depending on whether the electron flow is stimulated by a voltage bias or by an electronic temperature bias. From now on, we will denote \( \lambda_{\gamma}, \omega_{\gamma}, G_{\alpha\gamma}^{ph} \) and \( RE_{\alpha\gamma}^{ph} \) simply as \( \lambda, \omega, G_{ph} \) and \( R_{ph} \). Also, unless otherwise mentioned, the electron contact coupling and reservoir phonon couplings are assumed to be symmetric (i.e. \( \gamma_{\alpha 1} = \gamma, \gamma_{\alpha 2}^{eff} = \gamma_{\alpha 2}^{eq} \) and \( \beta_{\alpha 1} = \beta, \beta_{\alpha 2} = \beta \), where \( \alpha_1 \in L, R \), \( \alpha_2 \in H, C \).

3. Results

3.1. Electron–phonon coupled transport

In this section we focus on the electron–phonon coupled transport in our setup. Each operating point is signified by a voltage bias, an electronic temperature gradient \( \Delta T_{el} = T_{L} - T_{R} \) applied across the contacts and a phonon temperature gradient \( \Delta T_{ph} = T_{H} - T_{C} \) applied between the reservoirs. Additionally we assume the dot functions as n-type, i.e. \( \epsilon > \mu \). In our simulation framework we set \( \epsilon = 1 \) meV, \( U = 100 \) meV, \( \gamma = 10^{11} s^{-1}, \beta = 10^{9} s^{-1} \) and \( \omega = 5 \times 10^{11} s^{-1} \). In the limit \( (\hbar \gamma > h\beta) \), the dot phonon distribution is controlled by the electron current rather than the phonon reservoirs. We also set the Fermi energy of the leads at zero and voltage is dropped around the Fermi level in a symmetric manner, i.e. \( \mu_{L} = -\mu_{R} = V_{LR}/2 \).

First we demonstrate the behavior of electronic induced \( I_{Q}^{ph} \) in two regimes: Case (a) charge currents are driven by a voltage bias when \( \Delta T_{el} (\Delta T_{el} = T_{L} - T_{R}) \) is zero and Case (b) charge currents are set up by an electronic temperature gradient \( \Delta T_{el} = T_{L} - T_{R} \neq 0 \) at zero voltage. Each case operates at zero \( \Delta T_{ph} \) to ensure that only electronic current influences the phonon distribution and considers that the electronic contacts and phonon reservoirs are in thermal equilibrium \( (T_{L} = T_{R} = T_{H} = T_{C}) \) at zero applied electronic bias (when \( \Delta V = 0 \) for Case (a) and \( \Delta T_{el} = 0 \) for Case (b)). To put things simply, reservoirs \( H \) and \( C \) are merged to a single reservoir \( B \) with temperature \( T_{B} \) and relaxation rate \( \beta \). The modified setups in Case (a) and (b) are depicted in figures 2(a) and (b) respectively. In each operating point, the direction of phonon currents is decided by the gradient between \( T_{B} \) and the dot temperature \( T_{M} \).

The preceding section established that \( I_{Q}^{ph} \) vanishes at \( \lambda = 0 \). As we turn on \( \lambda \), the magnitude of \( I_{Q}^{ph} \) increases with voltage and \( \Delta T_{el} \). This is observed in both Cases (a) and (b) as shown in figures 2(c) and (d). However, we find that the direction of \( I_{Q}^{ph} \) is opposite in these cases. Figure 2(c) notes that in Case (a), the dot phonons relax to the reservoir and the magnitude of \( I_{Q}^{ph} \) increases in the negative direction. On the other hand, figure 2(d) shows that in Case (b), the dot becomes phonon deficient and the reservoir pumps phonons into the dot. This clearly indicates that modulating \( \lambda \) makes our setup behave like a phonon switch whereas voltage and \( \Delta T_{el} \) control the direction of \( I_{Q}^{ph} \). These properties can be utilized in the design of phonon switches and rectifiers which aim to implement digital logic by modulating the phonon current.

The difference in the direction of \( I_{Q}^{ph} \) can be explained by studying the phonon generation rate \( (G_{ph}) \) in both cases. Figure 3(a) notes that in Case (a), \( G_{ph} \) is positive and rises with voltage when \( \lambda \) is finite. As voltage increases, phonons accumulate in the dot as \( \langle N_{ph} \rangle \) exceeds \( \langle N_{ph}^{eq} \rangle \). The voltage response of \( \langle N_{ph} \rangle \) is presented in figure 3(b). In contrast, figure 3(c) notes that in Case (b), \( G_{ph} \) becomes negative and falls with \( \Delta T_{el} \) when \( \lambda \) is non-zero. It implies that the rise of \( \Delta T_{el} \) facilitates phonon absorption and \( \langle N_{ph} \rangle \) falls below the level of \( \langle N_{ph}^{eq} \rangle \) as depicted in figure 3(d). As a result in Case (b), reservoirs pump phonons into the dot.

One should note that the variation of \( I_{Q}^{ph} \) with voltage and \( \Delta T_{el} \) takes place in a non-linear fashion. This non-linearity \( (qV \gg k_{B}T \text{ or } \Delta T_{el} \gg T) \) is related to the non-linearity of the Fermi–Dirac distribution function and it has nothing to do with the higher order terms in the rate equation. Now it is essential to test the features of \( I_{Q}^{ph} \) in the linear regime. From a fundamental standpoint the Onsager’s reciprocity should be validated in the linear regime. Our simulation framework detects that the phonon heat current does not vary with voltage in the linear limit \( (qV \ll k_{B}T \text{ or } \Delta T_{el} \ll T) \). To explore the linear response of \( I_{Q}^{ph} \), we test the nature of many body electron–phonon probabilities \( \mathbb{P}_{n,q} \) of the states \( [n, q] \) in both cases, when \( \lambda \) is non-zero. Figure 4(a) concludes that in Case (a), \( \mathbb{P}_{n,q} \) does not vary with voltage. Similarly, figure 4(b) depicts that in Case (b), \( \mathbb{P}_{n,q} \) remains constant with \( \Delta T_{el} \). Hence, in
In the linear regime, the average population of phonons in the dot ($\langle N_{\text{ph}} \rangle = \sum_n q_n P_{n,q}$) remains unaltered and $I_{\text{ph}}^Q$ remains constant as dictated in (15). This implies that the electron and phonon transport are uncoupled in the linear regime.

After the discussion of linear response of $I_{\text{ph}}^Q$, it is essential to examine the inter-relation of the electronic charge current $I$ and the phonon temperature gradient ($\Delta T_{\text{ph}} = T_H - T_C$). In this course, first we set $\Delta T_{\text{el}} = 0$ and test the variation of $I$ as a function of voltage and $\Delta T_{\text{ph}}$. Figures 5(a) and (b) depict the color variation of $I_{\text{ph}}^Q$ as a function of $V$ and $\Delta T_{\text{ph}}$ for $\lambda = 0$ and $\lambda = 0.5$. The direction of phonon currents is opposite in two cases. This figure establishes that $V$ and $\Delta T_{\text{el}}$ decide the direction of $I_{\text{ph}}^Q$ and $\lambda$ works like a phonon switch.

The magnitude of phonon current increases with voltage when $\lambda$ is non-zero (c) Variation of $G_{\text{ph}}$ as a function of $\Delta T_{\text{el}}$ for different $\lambda$ in Case (b). (d) Negative deviation of $\langle N_{\text{eq}}^{\text{ph}} \rangle$ from the level of $\langle N_{\text{eq}}^{\text{ph}} \rangle$ as $\Delta T_{\text{el}}$ increases for non-zero $\lambda$ in Case (b). We infer that in Case (a) phonons accumulate in the dot and in Case (b) phonons get absorbed and explain opposite polarity of phonon current in the two cases.

the linear regime, the average population of phonons in the dot ($\langle N_{\text{ph}} \rangle = \sum_n q_n P_{n,q}$) remains unaltered and $I_{\text{ph}}^Q$ remains constant as dictated in (15). This implies that the electron and phonon transport are uncoupled in the linear regime.

After the discussion of linear response of $I_{\text{ph}}^Q$, it is essential to examine the inter-relation of the electronic charge current $I$ and the phonon temperature gradient ($\Delta T_{\text{ph}} = T_H - T_C$). In this course, first we set $\Delta T_{\text{el}} = 0$ and test the variation of $I$ as a function of voltage and $\Delta T_{\text{ph}}$. Figures 5(a) and (b) depict the color variation of $I$ as a function of $V$ and $\Delta T_{\text{ph}}$ for $\lambda = 0$ and
are related to the variation of $I$ as a function of $\Delta T_{ph}$ and $V$ for (a) $\lambda = 0$ and (b) $\lambda = 0.5$ when $\Delta T_e = 0$. Color variation of $I$ as a function of $\Delta T_{ph}$ and $\Delta T_e$ for (c) $\lambda = 0$ and (d) $\lambda = 0.5$ at zero voltage. It implies that in our setup $I$ can not be stimulated by $\Delta T_{ph}$ even in the non-linear regime.

$\lambda = 0.5$ respectively. It is observed that even in the non-linear regime, $I$ shows no dependence on $\Delta T_{ph}$. Next, we study the variation of $I$ with $\Delta T_{el}$ and $\Delta T_{ph}$ at zero voltage. The color variation of $I$ as a function of $\Delta T_{el}$ and $\Delta T_{ph}$ is depicted in figures 5(c) and (d) for $\lambda = 0$ and $\lambda = 0.5$ respectively. Here also, we notice that $I$ shows no dependence on $\Delta T_{ph}$. This result is consistent with the previous report [56] which demonstrated why $I$ can not be stimulated by $\Delta T_{ph}$ in quantum dot based devices with a single vibronic mode. In the linear regime, charge and heat currents $I_{Qel}$ and $I_{Qph}$ are related to the voltage and temperature gradients $V$, $\Delta T_{el}$ and $\Delta T_{ph}$ by the Onsager’s matrix, such that

$$
\begin{pmatrix}
\sigma_{el} \\
\kappa_{el} \\
\kappa_{ph}
\end{pmatrix}
= 

\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
\sigma \\
\kappa_{el} \\
\kappa_{ph}
\end{bmatrix}.
$$

In the present scenario, we find that $L_{13} = L_{31} = 0$ and $L_{23} = L_{32} = 0$. It confirms that Onsager’s reciprocity is obeyed. Now the dependence of $I_{Q}$ on the voltage and $\Delta T_{el}$, immediately hints at the concept of a new thermal conductivity coefficient which is different from the conventional electronic thermal conductivity. We will elaborate on it in the next subsection.

### 3.2. Non-linear phonon Peltier and electron-induced phonon thermal conductivity

Beyond the linear regime, the conventional electronic Peltier coefficient is defined as $\pi_{el} = \frac{\partial I}{\partial \Delta T_{el}}$, when there is no electronic temperature gradient ($\Delta T_{el} = 0$). In analogy with the electronic Peltier coefficient, our previous work [31] introduced the phonon Peltier coefficient as $\pi_{ph} = \frac{\partial I}{\partial \Delta T_{ph}}$. In this subsection, we draw a comparative study between the two Peltier coefficients.

Both electronic and phonon Peltier coefficients are computed in the differential form to avoid the singularity at the short-circuit point ($V = 0$). First, we plot the color map of $\pi_{el}$ and $\pi_{ph}$ for different $\lambda$.
as a function of voltage and $\lambda$ in figure 6(a). It shows that $\pi_{el}$ varies with voltage in an asymmetric manner but remains completely independent of $\lambda$. We note from (10) and (11) that $I_{Q}^{\text{ph}}$ differs from $I$ by a prefactor $\frac{\kappa_{el-\text{I}-\text{ph}}}{\Delta T_{el}}$, which varies with voltage (or with the position of electrochemical potential $\mu$) but does not change with $\lambda$. As a consequence, $\pi_{el}$ remains independent of $\lambda$. Figure 6(b) notes that $I$ is a perfectly anti-symmetric function of voltage although $I_{Q}^{\text{ph}}$ changes with voltage in a symmetric manner. Hence, the color map in figure 6(a) shows that $\pi_{el} = \frac{dQ}{dT}$ is a symmetric function of voltage.

Now we analyze the effect of voltage and $\lambda$ on $\pi_{ph}$. Figure 6(c) notes that $\pi_{ph}$ is a joint function of voltage and $\lambda$. It vanishes when $\lambda$ is zero and varies linearly with the voltage as we switch on a finite $\lambda$. Earlier, figure 2 showed that the magnitude of $I_{Q}^{\text{ph}}$ increases with $\lambda$ and figure 5 showed that the magnitude of $I$ falls with $\lambda$. Therefore, the absolute magnitude of $\pi_{ph}$ increases with $\lambda$ as shown in figure 6(c). On the other way, figure 6(d) depicts that $I_{Q}^{\text{ph}}$ and $I$ maintain perfect symmetry and antisymmetry with voltage respectively. Hence, unlike $\pi_{el}$, $\pi_{ph}$ are voltage anti-symmetric with respect to $\lambda$ (as shown in figure 6(c)). However one must appreciate that $I_{Q}^{\text{ph}}$ varies with voltage only in the non-linear regime. Hence we term the phonon Peltier co-efficient ($\pi_{ph}$) as the non-linear phonon Peltier coefficient.

The concept of $\pi_{ph}$ was manifested since a voltage bias stimulates and modulates $I_{Q}^{\text{ph}}$ when $\Delta T_{el}$ is zero. We have already observed that, $I_{Q}^{\text{ph}}$ is a joint function of $V$ and $\Delta T_{el}$. This motivates us to propose a new thermal conductivity coefficient $\kappa_{el-\text{I}-\text{ph}}$, which is different from the conventional electronic thermal conductivity $\kappa_{el}$. Both thermal conductivity coefficients $\kappa_{el}$ and $\kappa_{el-\text{I}-\text{ph}}$ are defined as

$$\kappa_{el} = -\frac{\partial I_{Q}^{\text{ph}}}{\partial \Delta T_{el}} \bigg|_{I_{Q}^{el}=0},$$

$$\kappa_{el-\text{I}-\text{ph}} = -\frac{\partial I_{Q}^{\text{ph}}}{\partial \Delta T_{el}} \bigg|_{I_{Q}^{el}=0}.$$
We present a schematic of the setups in Case (HR) and (CR) in figure 8(a). Figure 8(b) plots the voltage response of $I_{ph}^{th}$ in Case (HR). In this case, $I_{ph}^{th}$ flows from the dot to the reservoir in the low voltage range but reverses its polarity at $|V| = V_0$, when $\lambda$ is non-zero. This non-trivial phenomenon can be explained by studying the voltage variation of $\langle N_{ph}^{eq} \rangle$ depicted in figure 8(c). As we increase the voltage, phonons accumulate in the dot and at $|V| = V_0$, the average phonon number in the dot exceeds the equilibrium phonon number $\langle N_{ph}^{eq} \rangle$ of the reservoir. Hence, the polarity of $I_{ph}^{th}$ is flipped. However, figure 8(d) captures no such non-trivial polarity reversal of $I_{ph}^{th}$ in Case (CR). In this case, figure 8(e) notes that $\langle N_{ph} \rangle$ is always higher than the mark of $\langle N_{ph}^{eq} \rangle$ and $I_{ph}^{th}$ is directed from the dot to the cold reservoir. The voltage variation of $\langle N_{ph} \rangle$ presented in figures 8(c) and (e) is consistent with the voltage variation of the dot temperature $T_M$ shown in their respective insets. It is interesting to observe that the electron current emerging from electronic contacts can pump phonons into the hotter reservoir at large voltage. This is a type of counter-intuitive heating of a hot reservoir. Figure 8(f) depicts the schematic of counter-intuitive heating in Case (HR).

An opposite response of $I_{ph}^{th}$ is observed when we analyze Case (HR) and (CR) in the limit of Case (b) where we drive the electronic temperature gradient at zero voltage. Figure 9(a) depicts the modified device schematics in Case (HR) and (CR). Figure 9(b) plots the variation of $I_{ph}^{th}$ with $\Delta T_{el}$ for $\lambda = 0$ and $\lambda \neq 0$. In this case, we note no flip in the direction of $I_{ph}^{th}$ and the dot pumps phonons into the reservoir for all values of $\Delta T_{el}$. Figure 9(c) shows that in this case, $\langle N_{ph} \rangle$ is always lower than the level of $\langle N_{ph}^{eq} \rangle$ and
Finally, we elaborate on the manipulation of the electronic heat currents ($I_{el}^Q$) in Case (HR) and (CR). In this context we rename $I_{el}^Q$ associated with the contacts $L$ and $R$ as $I_{el}^{QR}$ and $I_{el}^{QR}$ respectively. Figures 10(a) and (b) depict the voltage variation of $I_{el}^{QR}$ and $I_{el}^{QR}$ for different values of $\lambda$ in Case (HR). We observe that both $I_{el}^{QR}$ and $I_{el}^{QR}$ vanish when $\lambda = 0$. When $\lambda$ is non-zero, both $I_{el}^{QR}$ and $I_{el}^{QR}$ become finite and they diverge uniformly into the contacts even in the short-circuit point ($V = 0$).

When $\lambda \neq 0$, the Hot Bath pushes phonons into the dot and the excess heat flows away into the contacts in the form of $I_{el}^Q$.

Similarly, figures 10(c) and (d) explain the same phenomenon in Case (CR). In this case, the Cold Bath extracts phonons out of the dot and $I_{el}^Q$ flows from the contacts into to the dot. The whole scenario is pictorially presented in figure 10(e).

It is intriguing to notice that $I_{el}^Q$ can be stimulated by varying the temperature of phonon reservoir even at the short circuit point $V = 0$, where the effective electronic charge flow vanishes (i.e $I = 0$). Now if the contact $L$ is strongly coupled to the dot as compared to the contact $R$, (i.e $\gamma_L \gg \gamma_R$), $I_{el}^Q$ is dragged out from (or pushed into) the dot by the contact $L$. Figures 10(f) and (g) plot the small voltage response of $I_{el}^{QR}$ and $I_{el}^{QR}$ for non-zero $\lambda$ in Case (HR) and (CR) respectively. They show that the hotter (colder) reservoir can selectively heat up (or cool down) $L$ even when $V = 0$ as shown in the schematic presented in figure 10(h). Thus we can selectively channel $I_{el}^Q$ by varying the temperature of the reservoir using asymmetric dot-to-contact coupling. Therefore, the temperature of the phonon reservoir plays a major role in the stimulation and control of the electronic and phonon heat currents in the dot. We believe that this intriguing physics can be implemented in the design of thermal transistors or thermal logic devices.

4. Conclusion

In this paper, a detailed analysis of electronically controlled phonon transport was carried out using an Anderson–Holstein based dissipative quantum dot setup. We considered two relevant electronic bias situations: (a) a voltage bias in the absence of an electronic temperature gradient and (b) an electronic temperature gradient at zero voltage. It was shown that the direction of phonon transport in the non-linear regime is different in the two cases since the first case facilitates the accumulation of phonons in the dot and the second case leads to the absorption of phonons in the dot. We explained the observed cumulative effects of voltage and electronic temperature gradients on the non-linear phonon currents by introducing a new transport coefficient that we termed as the electron induced phonon thermal conductivity. It was demonstrated that under suitable operating conditions in Case (a) the dot can pump in phonons into the hotter phonon reservoirs and in Case (b) the dot can extract phonons out of the colder phonon reservoirs. Finally, as a corollary, we elaborated on how the non-linear electronic heat current can be stimulated and controlled by manipulating the temperature of the phonon reservoirs even according to the thermal gradient and the setup should not be interpreted as heat pump or refrigerator.
under vanishing effective electronic charge flow. The results presented here may be used to design modern phononic devices aimed at thermal rectification or phonon computation.

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