CONFLUENCE ON THE PAINLEVÉ MONODROMY MANIFOLDS, THEIR POISSON STRUCTURE AND QUANTISATION.

MARTA MAZZOCCO, VLADIMIR RUBTSOV

Abstract. In this paper we obtain a system of flat coordinates on the monodromy manifold of each of the Painlevé equations. This allows us to quantise such manifolds. We produce a quantum confluence procedure between cubics in such a way that quantisation and confluence commute. We also investigate the underlying cluster algebra structure and the relation to the versal deformations of singularities of type \(D_4, A_3, A_2,\) and \(A_1.\)

1. Introduction

Following the approach by Sakai [21], there are eight Painlevé equations corresponding to the eight extended Dynkin diagrams \(\tilde{D}_4, \tilde{D}_5, \tilde{D}_6, \tilde{D}_7, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8,\) corresponding respectively to PVI, PV, three different cases of PIII, PIV, PII and PI. Their monodromy manifolds were studied by several authors, but were recently presented in a unified way in [19]:

\[
\begin{align*}
\tilde{D}_4 & : x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0, \\
\tilde{D}_5 & : x_1 x_2 x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0, \\
\tilde{D}_6 & : x_1 x_2 x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_1 - 1 = 0, \\
\tilde{D}_7 & : x_1 x_2 x_3 + x_1^2 + x_2^2 + \omega_1 x_1 = 0, \\
\tilde{D}_8 & : x_1 x_2 x_3 + x_1^2 + x_2^2 + 1 = 0, \\
\tilde{E}_6 & : x_1 x_2 x_3 + x_1^2 + \omega_1 x_1 + \omega_2 x_2 + x_3 + 1 + \omega_4 = 0, \\
\tilde{E}_7^* & : x_1 x_2 x_3 + x_1 + x_2 + x_3 + \omega_4 = 0, \\
\tilde{E}_7^{**} & : x_1 x_2 x_3 + x_1 + \omega_2 x_2 + x_3 - \omega_2 + 1 = 0, \\
\tilde{E}_8 & : x_1 x_2 x_3 + x_1 + x_2 + 1 = 0,
\end{align*}
\]

where \(\omega_1, \ldots, \omega_4\) are some constants related to the parameters appearing in the Painlevé equations as described in Section 2 here below and the two cubics \(\tilde{E}_7^*\)
and $\tilde{E}_7^*$ correspond to the two different isomonodromy problems for PII found by Flaschka–Newell [8] and Jimbo–Miwa [16] respectively.

By looking at the above list of cubics it is immediately evident that one can follow the famous Painlevé confluence scheme (indeed the generalised one appearing in Sakai’s paper [21]) on the cubics by simple operations. For example, we can produce the PV $\tilde{D}_5$ cubic by considering the PVI $\tilde{D}_4$ one and rescaling $x_1 \to \frac{x_1}{\epsilon}$, $x_2 \to \frac{x_2}{\epsilon}$, $\omega_1 \to \frac{\omega_1}{\epsilon}$, $\omega_2 \to \frac{\omega_2}{\epsilon}$, $\omega_3 \to \frac{\omega_3}{\epsilon}$ and $\omega_4 \to \frac{\omega_4}{\epsilon}$ and then keeping the dominant term as $\epsilon \to 0$. This simple idea allows to us to extend the parameterisation of the PVI cubic in terms of shear coordinates obtained in [4] to all other Painlevé equations.

In particular, in this paper we study the natural Poisson bracket defined on these cubics, its relation with the log-canonical Poisson bracket, provide flat coordinates in terms of shear coordinates obtained in [4] to all other Painlevé equations.

For the $\tilde{D}_5, \tilde{D}_6, \tilde{E}_6, \tilde{E}_7$ cubics we also associate a Riemann surface and its fatgraph to each cubic, so that our flat coordinates are indeed the Thurston shear coordinates on the fat–graph. Following the Fock–Goncharov philosophy, we also address the problem of whether there is some cluster algebra structure hidden in each cubic. We prove that indeed for $\tilde{D}_4, \tilde{D}_5, \tilde{D}_6$ and $\tilde{E}_6$ there is a tagged cluster algebra structure [5]. In particular this implies that the procedure of analytic continuation of the solutions to the sixth Painlevé equation satisfies the Laurent phenomenon as explained in Section 4.

Last but not least, we interpret each Painlevé cubic as versal deformation of an Arnold singularity according to Sakai’s table.

This paper is organised as follows: in Section 2 we recall the link between the parameters $\omega_1, \ldots, \omega_4$ and the Painlevé parameters $\alpha, \beta, \gamma$ and $\delta$. In Section 3 we discuss the Cayley cubic and its relation to the log-canonical Poisson bracket and we interpret each Painlevé cubic as versal deformation of an Arnold singularity according to Sakai’s table. In Section 4 we explain the tagged cluster algebra structure appearing in the case of PVI, PV, PIII and PIV. In Section 5 we present the flat coordinates for each cubic. In Section 6 we present the quantisation and the quantum confluence.

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2. Unified approach to the monodromy manifolds

According to [19], the monodromy manifolds $\mathcal{M}^{(d)}$ have all the form

$$x_1^2 x_3 + \epsilon_1^{(d)} x_1^2 + \epsilon_2^{(d)} x_2^2 + \epsilon_3^{(d)} x_3 + \omega^{(d)}_1 x_1 + \omega^{(d)}_2 x_2 + \omega^{(d)}_3 x_3 + \omega^{(d)}_4 = 0,$$

where $d$ is an index running on the list of the extended Dynkin diagrams $\tilde{D}_4, \tilde{D}_5, \tilde{D}_6, \tilde{D}_7, \tilde{D}_8, \tilde{E}_6, \tilde{E}_7^*, \tilde{E}_7^*, \tilde{E}_8$ and the parameters $\epsilon^{(d)}_i, \omega^{(d)}_i, i = 1, 2, 3$ are given by:

$$\epsilon^{(d)}_1 = \begin{cases} 1 & \text{for } d = \tilde{D}_4, \tilde{D}_5, \tilde{D}_6, \tilde{D}_7, \tilde{D}_8, \tilde{E}_6, \\ 0 & \text{for } d = \tilde{E}_7^*, \tilde{E}_7^*, \tilde{E}_8, \end{cases}$$
where the parameters \( \theta \) obtained from our cubic (2.1) by a simple sign change:

\[
\epsilon_2^{(d)} = \begin{cases} 
1 & \text{for } d = \bar{D}_4, \bar{D}_5, \bar{D}_6, \bar{D}_7, \bar{D}_8 \\
0 & \text{for } d = \bar{E}_6, \bar{E}_7^*, \bar{E}_7^{**}, \bar{E}_8,
\end{cases}
\]

\[
\epsilon_3^{(d)} = \begin{cases} 
1 & \text{for } d = \bar{D}_4, \\
0 & \text{for } d = \bar{D}_5, \bar{D}_6, \bar{D}_7, \bar{D}_8, \bar{E}_6, \bar{E}_7^*, \bar{E}_8.
\end{cases}
\]

while

\[
\begin{align*}
\omega_1^{(d)} &= -G_1^{(d)} G_\infty^{(d)} - \epsilon_1^{(d)} G_2^{(d)} G_3^{(d)}, \\
\omega_2^{(d)} &= -G_2^{(d)} G_\infty^{(d)} - \epsilon_2^{(d)} G_1^{(d)} G_3^{(d)}, \\
\omega_3^{(d)} &= -G_3^{(d)} G_\infty^{(d)} - \epsilon_3^{(d)} G_1^{(d)} G_2^{(d)}, \\
\omega_4^{(d)} &= \epsilon_2^{(d)} \epsilon_3^{(d)} G_1^{(d)} G_2^{(d)} G_3^{(d)} G_\infty^{(d)} + 4 \epsilon_1^{(d)} \epsilon_2^{(d)} \epsilon_3^{(d)} G_\infty^{(d)}
\end{align*}
\]

where \( G_1^{(d)}, G_2^{(d)}, G_3^{(d)}, G_\infty^{(d)} \) are some constants related to the parameters appearing in the Painlevé equations as follows:

\[
G_1^{(d)} = \begin{cases} 
2 \cos \pi \theta_0 & d = \bar{D}_4, \bar{D}_5, \bar{E}_6 \\
e^{-\pi i \theta_0} & d = \bar{E}_7^* \\
1 & d = \bar{D}_6
\end{cases}
\]

\[
G_2^{(d)} = \begin{cases} 
2 \cos \pi \theta_1 & d = \bar{D}_4, \bar{D}_5, \\
2 \cos \pi \theta_\infty & d = \bar{E}_6 \\
e^{-\pi i \theta_0} & d = \bar{E}_7^* \\
1 & d = \bar{D}_6
\end{cases}
\]

\[
G_3^{(d)} = \begin{cases} 
2 \cos \pi \theta_1 & d = \bar{D}_4, \bar{D}_5, \\
1 & d = \bar{D}_6, \bar{D}_7 \\
2 \cos \pi \theta_\infty & d = \bar{E}_6 \\
e^{-\pi i \theta_0} & d = \bar{E}_7^* \\
0 & d = \bar{E}_7^{**}
\end{cases}
\]

\[
G_\infty^{(d)} = \begin{cases} 
2 \cos \pi \theta_\infty & d = \bar{D}_4, \bar{D}_5, \bar{E}_6 \\
e^{-\pi i \theta_0} & d = \bar{E}_7^* \\
e^{i \pi \theta_0} & d = \bar{E}_7^{**} \\
1 & d = \bar{D}_6 \\
e^{\pi i \theta_\infty} & d = \bar{D}_7
\end{cases}
\]

where the parameters \( \theta_0, \theta_1, \theta_\infty \) are related to the Painlevé equations parameters in the usual way [16].

**Remark 2.1.** Observe that in the original article [19] the \( \bar{E}_7^* \) cubic corresponding to the Flaschka–Newell isomonodromic problem [8] has different signs. This can be obtained from our cubic (2.1) by a simple sign change: \( x_2 \to -x_2 \) Analogously the
\( \tilde{E}_8 \) and \( D_6 \) cubics have different signs in \([19]\), which can both be obtained by the same sign change \( x_i \to -x_i \) for \( i = 1, 2 \). Finally to obtain the \( D_8 \) cubic as in \([19]\) we just need to rescale \( x_1 \to ix_1 \) and \( x_3 \to ix_3 \).

**Remark 2.2.** The cubic family of the monodromy manifolds \( M^{(d)} \) \( \tilde{A}_4 \) type appears in many different contexts. The \( \tilde{A}_4 \)-case was studied in Oblomkov’s work (see \([18]\)). W. Goldman and D. Toledo (\([11]\)) had proved that every cubic surface with \( \epsilon_i^d = 1 \) for all \( i = 1, 2, 3 \) (and at least one \( \omega_i^d \neq 0 \) for \( i = 1, 2, 3 \)) arises from a representation of the fundamental group of the 4-holed sphere in \( SL(2, \mathbb{C}) \). They also have shown that if all \( \omega_i^d = 0 \) for \( i = 1, 2, 3 \) and \( \omega_4 \neq 0 \) then the 4-holed sphere should be replaced by 1-hole torus. In the Painlevé context the family of surfaces were considered by S. Cantat et F. Loray (\([2]\)) and M. Inaba, K. Iwasaki and M. Saito in \([13]\). The first author (together with L. Chekhov) has studied the shear coordinates on \( D_4 \)-type family in the paper \([4]\). We want to mention also that M. Gross, P. Hacking and S. Keel (see Example 5.12 of \([12]\)) claim that the family \([24]\) can be “uniformize” by some analogues of theta-functions related to toric mirror data on log-Calabi-Yau surfaces.

3. A digression on volume forms, singularities and different Dynkin diagrams

We would like to address here some natural facts that arise when comparing the various descriptions of family of affine cubics surfaces with 3 lines at infinity \((2.1)\).

First of all, the projective completion of the family of cubics \((2.1)\) with \( \epsilon_i^{(d)} \neq 0 \) for all \( i = 1, 2, 3 \) has singular points only in the finite part of the surface and if any of \( \epsilon_i^{(d)} , i = 1, 2, 3 \) vanish, then \( M^{(d)} \) is singular at infinity with singular points in homogenous coordinates \( X_i = 1 \) and \( X_j = 0, j \neq i \) \((18)\). Here \( x_i = \frac{\bar{x}_i}{\bar{x}_4} \).

One can consider this family of cubics as a variety \( S = \{ (\bar{x}, \bar{\omega}) \in \mathbb{C}^3 \times \Omega : S(\bar{x}, \bar{\omega}) = 0 \} \) where \( \bar{x} = (x_1, x_2, x_3) \), \( \bar{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4) \) and the “\( \bar{x} \)-forgetful” projection \( \pi : S \to \Omega : \pi(\bar{x}, \bar{\omega}) = \bar{\omega} \). This projection defines a family of affine cubics with generically non-singular fibres \( \pi^{-1}(\bar{\omega}) \) (we will discuss the nature of these singularities in Subsection \((3.1)\).

The cubic surface \( S_{\bar{\omega}} \) has a volume form \( \vartheta_{\bar{\omega}} \) given by the Poincaré residue formulae:

\[
\vartheta_{\bar{\omega}} = \frac{dx_1 \wedge dx_2}{(\partial S_{\bar{\omega}})/(\partial x_3)} = \frac{dx_2 \wedge dx_3}{(\partial S_{\bar{\omega}})/(\partial x_1)} = \frac{dx_3 \wedge dx_1}{(\partial S_{\bar{\omega}})/(\partial x_2)}.
\]

The volume form is a holomorphic 2-form on the non-singular part of \( S_{\bar{\omega}} \) and it has singularities along the singular locus. This form defines the Poisson brackets on the surface in the usual way as

\[
\{x_1, x_2\}_{\bar{\omega}} = \frac{\partial S_{\bar{\omega}}}{\partial x_3}
\]

and the other brackets are defined by circular transposition of \( x_1, x_2, x_3 \). It is a straightforward computation to show that for \( (i, j, k) = (1, 2, 3) \):

\[
\{x_i, x_j\}_{\bar{\omega}} = \frac{\partial S_{\bar{\omega}}}{\partial x_k} = x_i x_j + 2\epsilon_i^d x_k + \omega_i^d
\]
and the volume form reads as

\[
\vartheta = \frac{dx_i \wedge dx_j}{(\partial S_\bar{\omega})/(\partial x_k)} = \frac{dx_i \wedge dx_j}{(x_i x_j + 2 \xi^i \xi^j + \omega_i^j)}.
\]

In a special case of PVI, i.e. the \( \tilde{D}_4 \) cubic with parameters \( \omega_i = 0 \) for \( i = 1, 2, 3 \) and \( \omega_4 = -4 \), there is an isomorphism \( \pi : \mathbb{C}^* \times \mathbb{C}^*/\eta \rightarrow S_\bar{\omega} : \eta \rightarrow u \rightarrow \frac{1}{u}, \quad \xi \rightarrow \frac{1}{\xi}. \)

The log-canonical 2-form \( \bar{\vartheta} = \bar{\vartheta} = \bar{\vartheta} = \bar{\vartheta} \) defines a symplectic structure on \( \mathbb{C}^* \times \mathbb{C}^* \) which is invariant with respect the involution \( \eta \) and therefore defines a symplectic structure on the non-singular part of the cubic surface \( S_\bar{\omega} \) for \( \omega_i = 0 \) for \( i = 1, 2, 3 \) and \( \omega_4 = -4 \).

The relation between the log-canonical 2-form \( \bar{\vartheta} = \bar{\vartheta} = \bar{\vartheta} = \bar{\vartheta} \) and the Poisson brackets on the surface \( S_\bar{\omega} \) can be extended to all values of the parameters \( \bar{\omega} \) and for all the Painlevé cubics as we shall show in this paper. In fact the flat coordinates that we will introduce in Section 5 are such that their exponentials satisfy the log-canonical Poisson bracket. Before doing so, we clarify the relation between the Painlevé cubics and singularity theory.

3.1. Singularity theory approach to the Painlevé cubics. As mentioned above, for special values of \( \omega_1, \ldots, \omega_4 \) the fibre may have a singularity. Such singularities were classified in [13] for PVI and in [19] for all other Painlevé equations. These results can be summarised in the following table:

| Dynkin | Painlevé equations | Surface singularity type |
|--------|--------------------|--------------------------|
| \( D_4 \) | \( P_{VI} \) | \( D_4 \) |
| \( D_5 \) | \( P_{V} \) | \( A_3 \) |
| \( D_6 \) | \( \deg P_{V} = P_{II} (D_6) \) | \( A_1 \) |
| \( D_7 \) | \( P_{II} (D_7) \) | \( A_1 \) |
| \( D_8 \) | \( P_{II} (D_8) \) | non-singular |
| \( E_6 \) | \( P_{IV} \) | \( A_2 \) |
| \( E_7 \) | \( P_{II} (FN) \) | \( A_1 \) |
| \( E_8 \) | \( P_{II} (MJ) \) | \( A_1 \) |
| \( E_8 \) | \( P_{I} \) | non-singular |

Table 1.

The meaning of the table is the following: for each Painlevé equation of type specified by the first column in the table, there is at least one singular fibre with singularity of the type given in the second column of the table, and at least one singular fibre with singularity of type specified by any Dynkin sub-diagram of the Dynkin diagram given in the second column of the table.

For example PIV is the equation corresponding to \( E_6 \) and it has a two singular fibres with singularity of type \( A_2 \) and at three singular fibres with singularity of type \( A_1 \).
The scope of this section is to show that the non singular fibres of each family of affine cubics are locally diffeomorphic to the versal unfolding of the singularity of the type given in the second column of the table.

3.1.1. $D_4$. This case corresponds to the sixth Painlevé equation. The cubic in this case is (we drop the indices $(\bar{D}_4)$ for convenience):

$$x_1x_2x_3 + x_1^3 + x_2^3 + x_3^3 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0. \tag{3.9}$$

To show that this is diffeomorphic to the versal unfolding of $D_4$ we need to map this cubic to Arnol’d form. To this aim we first shift all variables by $x_i \rightarrow x_i + 2$, $i = 1, 2, 3$ to obtain

$$x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3 + x_1x_2x_3 + \bar{\omega}_1 x_1 + \bar{\omega}_2 x_2 + \bar{\omega}_3 x_3 + \bar{\omega}_4 = 0, \tag{3.10}$$

where

$$\bar{\omega}_i = \omega_i + 8, \quad \text{for } i = 1, 2, 3, \quad \bar{\omega}_4 = \omega_4 + 2(\omega_1 + \omega_2 + \omega_3) + 20.$$

As a second step we use the following diffeomorphism around the origin:

$$x \rightarrow x - \frac{1}{2} y, \quad y \rightarrow x + \frac{1}{2} y, \quad z \rightarrow z + \frac{y^2}{8} - 2x - \frac{x^2}{2} - \frac{\bar{\omega}_3}{2}$$

so that the new cubic (up to a Morse singularity that we throw away and after a shift $x \rightarrow x - \frac{\bar{\omega}_4}{4}$) becomes indeed the versal unfolding of a $D_4$ singularity in Arnol’d form:

$$-2x_1^3 + \frac{x_1x_2^2}{2} + \bar{\omega}_1 x_1 + \bar{\omega}_2 x_2 + \bar{\omega}_3 x_1^2 + \bar{\omega}_4,$$

where

$$\bar{\omega}_1 = \omega_1 + \omega_2 - 8 - 4\omega_3 - \frac{\omega_3^2}{8}, \quad \bar{\omega}_2 = \frac{\omega_2 - \omega_1}{2},$$

$$\bar{\omega}_3 = 8 + \omega_3, \quad \bar{\omega}_4 = \omega_4 + 2\omega_3 - \frac{\omega_3(\omega_1 + \omega_2 - \omega_3)}{4} + 4.$$
It is easy to prove that any solution \( u(x_2) \) of the equation

\[
\frac{G^2_{\infty}}{u^2} - \frac{G_{\infty}(G_2 + G_1 G_{\infty})}{u} - (G_1 + G_2 G_{\infty}) u + u^2 = x_1^2 + (G_2 + G_1 G_{\infty}) x_2^2 + (G_1 + G_2 G_{\infty}) x_2
\]

will define a diffeomorphism by mapping (3.13) to the versal unfolding of \( A_3 \).

3.3. \( \overline{D}_6 \). This case corresponds to the third Painlevé equation. The cubic in this case is (we drop the indices \((\overline{D}_3)\) for convenience):

\[
(3.13) \quad x_1 x_2 x_3 + x_1^3 + x_2^3 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 = 0,
\]

where only two parameters are free:

\[
\omega_1 = -1 - G^2_{\infty}, \quad \omega_2 = -G_2 G_{\infty}, \quad \omega_3 = G^2_{\infty}.
\]

The most singular fibre is given by \( G_{\infty} = 1 \) and \( G_2 = 2 \) and has two singular points at \((1, 0, 2)\) and \((0, 1, 2)\) respectively. We can define two local diffeomorphisms, one around \((1, 0, 2)\), the other around \((0, 1, 2)\), which map our cubic to the versal unfolding of a \( A_1 \) singularity.

The first diffeomorphism is given by:

\[
x_1 \rightarrow \frac{1 + G^2_{\infty}}{2} + x_1, \quad x_2 \rightarrow -x_2 + x_3, \quad x_3 \rightarrow \frac{2(G_2 G_{\infty} - 2x_3)}{1 + G^2_{\infty} + 2x_1}
\]

The second diffeomorphism is:

\[
x_1 \rightarrow -x_1 + x_3, \quad x_2 \rightarrow \frac{G_2 G_{\infty}}{2} - x_2, \quad x_3 \rightarrow \frac{2(1 + G^2_{\infty} - 2x_3)}{G_2 G^2_{\infty} - 2x_2}.
\]

3.3.1. \( \overline{E}_6 \). This case corresponds to the fourth Painlevé equation. The cubic in this case is (we drop the indices \((\overline{E}_6)\) for convenience):

\[
(3.14) \quad x_1 x_2 x_3 + x_1^3 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0,
\]

where only two parameters are free:

\[
\omega_1 = -G_1 G_{\infty} - G^2_{\infty}, \quad \omega_2 = -G^2_{\infty}, \quad \omega_3 = -G^2_{\infty}, \quad \omega_4 = G^2_{\infty} + G_1 G^3_{\infty}.
\]

Again we want to show that this is diffeomorphic to the versal unfolding of \( A_2 \). To this aim we impose the following change of variables:

\[
(3.15) \quad x_1 \rightarrow x_1 - x_3 + \frac{G^2_{\infty}}{u}, \quad x_2 \rightarrow u, \quad x_3 \rightarrow \frac{2x_3}{u} + \frac{G_{\infty}}{u} (G_1 + G_{\infty}) - \frac{2G^2_{\infty}}{u^2}
\]

where \( u \) is function of \( x_3 \) satisfying the following

\[
\frac{G^4_{\infty}}{u^2} - \frac{G^3_{\infty}(G_{\infty} + G_1)}{u} - G^2_{\infty} u = x_3^2 + G_{\infty} x_2.
\]

It is easy to prove that this transformation is a local diffeomorphism mapping our cubic to

\[
x_1^2 - x_2^3 + x_2^3 + G_{\infty} x_2 + G_{\infty} + G_1 G^3_{\infty},
\]

the versal unfolding of the \( A_2 \) singularity.
This case corresponds to the second Painlevé equation. Since the treatment of the two cubics $\tilde{E}_7^*$ and $\tilde{E}_7^{**}$ is completely equivalent, we choose to work with the former:

\begin{equation}
\lambda_1\lambda_2\lambda_3 - \lambda_1 - \lambda_2 - \lambda_3 + \omega_4 = 0,
\end{equation}

where:

$$
\omega_4 = G_\infty^2 + G_\infty^{-2}
$$

The following change of variables:

$$
x_1 \to x_1 - x_3 + \frac{1}{u}, \quad x_2 \to u, \quad x_3 \to \frac{x_1 + x_3 + 1}{u},
$$

where $u$ is a function of $x_2$ satisfying

$$
-\frac{1}{u} - u = x_2^2,
$$

is a local diffeomorphism mapping our cubic to the versal unfolding of the $A_1$ singularity:

$$
x_1^2 - x_3^3 + x_2^2 + \omega_4.
$$

\section*{4. Painlevé VI: analytic continuation and cluster mutations}

In \cite{Okamoto} it was proved that the procedure of analytic continuation of a local solution to the sixth Painlevé equation corresponds to the following action of the braid group on the monodromy manifold:

\begin{align}
\beta_1 : \quad & x_1 \to -x_1 - x_2 x_3 - \omega_1, \\
& x_2 \to x_3, \\
& x_3 \to x_2,
\end{align}

\begin{align}
\beta_2 : \quad & x_1 \to x_3, \\
& x_2 \to -x_2 - x_1 x_2 - \omega_2, \\
& x_3 \to x_1,
\end{align}

\begin{align}
\beta_3 : \quad & x_1 \to x_2, \\
& x_2 \to x_1, \\
& x_3 \to -x_3 - x_1 x_2 - \omega_3.
\end{align}

Note that two of these are enough to generate the whole braid group.

We are now going to show that when $G_\infty = 2$ (geometrically this means that we have a puncture at infinity), the action of the braid group coincides with a tagged cluster algebra structure \cite{FominZelevinsky}.

In order to see this let us compose each braid with a Okamoto symmetry in order to obtain the following

\begin{align}
\beta_i : \quad & x_i \to -x_i - x_j x_k - \omega_i, \quad j, k \neq i, \\
& x_j \to x_j, \quad \text{for } j \neq i
\end{align}

By using \cite{Okamoto} this transformation acquires a cluster flavour:

\begin{align}
\tilde{\beta}_i : \quad & x_i x_i' = x_j^2 + x_k^2 + \omega_j x_j + \omega_k x_k + \omega_4 \quad j, k \neq i.
\end{align}

Indeed let us introduce the shifted variables:

$$
y_i := x_i - G_i, \quad i = 1, 2, 3,
$$
they satisfy the tagged cluster algebra relation:

\[
\mu_i : y_i y'_i = y_j^2 + y_k^2 + G_i y_j y_k \quad j, k \neq i.
\]

Note that tagged cluster algebras satisfy the Laurent phenomenon. In particular this result implies that procedure of analytic continuation of the solutions to the sixth Painlevé equation satisfies the Laurent phenomenon: if we start from a local solution corresponding to the point \((y_1^0, y_2^0, y_3^0)\) on the shifted Painlevé cubic

\[
y_1 y_2 y_3 + y_1^2 + y_2^2 + y_3^2 + G_1 y_1 y_2 y_3 + G_2 y_1 y_3 + G_3 y_1 y_2 = 0
\]

any other branch of that solution will corresponds to points \((y_1, y_2, y_3)\) on the same cubic such that each \(y_i\) is a Laurent polynomial of the initial \((y_1^0, y_2^0, y_3^0)\).

**Remark 4.1.** A similar tagged cluster algebra structure can be found also for the fifth and the third Painlevé equation (see Subsections 4.2 and 4.2 below). However the meaning of this tagged cluster algebra structure in terms of analytic continuation of the solutions is still to be clarified and is postponed to subsequent publications.

### 4.1 Tagged cluster algebra structure for PV and PIII.

In this case, only \(x_1\) and \(x_2\) can be mutated. In the case of the \(\tilde{D}_5\) cubic, the formula for these mutation is the same as before:

\[
\tilde{\beta}_i : \begin{align*}
x_i &\to -x_i - x_j x_k - \omega_i, \\
x_j &\to x_j, \quad \text{for} \quad j \neq i
\end{align*}
\]

where \(\omega_i = \omega_i^{(\tilde{D}_5)}\). The the shifted variables:

\[
y_i := x_i + t_i, \quad i = 1, 2, 3,
\]

where

\[
t_1 = -\frac{G_\infty (G_2 G_\infty - G_1)}{G_\infty^2 - 1}, \quad t_2 = -\frac{G_\infty (G_1 G_\infty - G_2)}{G_\infty^2 - 1}, \quad t_3 = -\frac{1 + G_\infty^2}{G_\infty},
\]

satisfy the tagged cluster algebra:

\[
\mu_1 : y_1 y'_1 = y_2^2 - t_1 y_2 y_3 + \nu y_3,
\]

\[
\mu_2 : y_2 y'_2 = y_1^2 - t_2 y_1 y_3 + \nu y_3,
\]

where

\[
\nu = -\frac{G_\infty (1 + G_1^2 G_\infty^2 - (2 - G_1^2) G_\infty^2 + G_1^4 - G_1 G_2 G_\infty (1 + G_\infty^2))}{(G_\infty^2 - 1)^2}.
\]

Analogous computations can be repeated for PIII.

### 4.2 Tagged cluster algebra structure for PIV.

In the case of the \(\tilde{E}_6\) cubic, only \(x_1\) can be mutated. The formula for this mutation is the same as usual:

\[
\tilde{\beta} : \begin{align*}
x_1 &\to -x_1 - x_2 x_3 - \omega_1, \\
x_j &\to x_j, \quad \text{for} \quad j = 2, 3
\end{align*}
\]

where \(\omega_i = \omega_i^{(\tilde{E}_6)}\). We present here the shifted variables for the case \(G_\infty = 1\):

\[
y_i := x_i + t_i, \quad i = 1, 2, 3,
\]

where

\[
t_1 = -1 + \frac{5}{4G_1}, \quad t_2 = \frac{1}{2} - \frac{5}{4G_1}, \quad t_4 = -2,
\]
satisfy the tagged cluster algebra:
\[ (4.25) \quad \mu_1 : y_1y'_1 = y_2^2 - t_1y_2y_3 + \nu y_3, \]
where
\[ \nu = -\frac{25 - 30G_1 + 24G_1^2}{32G_1^2}. \]

5. Shear coordinates for the Painlevé monodromy manifolds

In the \( D_4 \) case the following parameterisation of the cubic in shear coordinates
on the fat-graph of a 4–holed sphere was found in \([4]\):
\[ (5.26) \quad x_1 = -e^{\tilde{s}_2 + \tilde{s}_3} - e^{-\tilde{s}_2 - \tilde{s}_3} - e^{-\tilde{s}_2 + \tilde{s}_3} - G_2 e^{\tilde{s}_3} - G_3 e^{-\tilde{s}_2} \\
 x_2 = -e^{\tilde{s}_3 + \tilde{s}_1} - e^{-\tilde{s}_3 - \tilde{s}_1} - e^{-\tilde{s}_3 + \tilde{s}_1} - G_3 e^{\tilde{s}_1} - G_1 e^{-\tilde{s}_3}, \\
 x_3 = -e^{\tilde{s}_1 + \tilde{s}_2} - e^{-\tilde{s}_1 - \tilde{s}_2} - e^{-\tilde{s}_1 + \tilde{s}_2} - G_1 e^{\tilde{s}_2} - G_2 e^{-\tilde{s}_1}, \]
where
\[ G_i = e^{\frac{\nu_i}{2}} + e^{-\frac{\nu_i}{2}}, \quad i = 1, 2, 3, \]
and
\[ G_\infty = e^{\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3} + e^{-\tilde{s}_1 - \tilde{s}_2 - \tilde{s}_3}, \]
and \( \tilde{s}_i \) are actually the shifted shear coordinates \( \tilde{s}_i = s_i + \frac{\nu_i}{2}, \quad i = 1, 2, 3. \)

We recall that according to Fock \([9,10]\), the fat graph associated to a Riemann
surface \( \Sigma_{g,n} \) of genus \( g \) and with \( n \) holes is a connected three–valent graph drawn
without self-intersections on \( \Sigma_{g,n} \) with a prescribed cyclic ordering of labelled edges
entering each vertex; it must be a maximal graph in the sense that its complement
on the Riemann surface is a set of disjoint polygons (faces), each polygon containing
exactly one hole (and becoming simply connected after gluing this hole). In the case
of a Riemann sphere \( \Sigma_{0,4} \) with 4 holes, the fat–graph is represented in Fig.1.

The geodesic length functions, which are traces of hyperbolic elements in the
Fuchsian group \( \Delta_{g,s} \) such that
\[ \Sigma_{g,s} \sim \mathbb{H}/\Delta_{g,s} \]
are obtained by decomposing each hyperbolic matrix \( \gamma \in \Delta_{g,s} \) into a product of
the so–called right, left and edge matrices:
\[ R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_{s_i} := \begin{pmatrix} 0 & -\exp\left(\frac{\nu_i}{2}\right) \\ \exp\left(-\frac{\nu_i}{2}\right) & 0 \end{pmatrix}. \]
In this setting our \( x_1, x_2x_3 \) are the geodesic lengths of the geodesics which go
around two holes without self–intersections, for example \( x_3 \) corresponds to the
dashed geodesic in Fig.1.

In \([3]\) it was shown that flips on the shear coordinates correspond to the action
of the braid group on the cubic. The flips of the shear coordinates which give rise
to the braid transformations \( \tilde{\beta}_1, \tilde{\beta}_2 \) and \( \tilde{\beta}_3 \) have the following form
\[ (5.27) \quad f_1 : \begin{align*}
\tilde{s}_1 & \rightarrow \tilde{s}_1, \\
\tilde{s}_2 & \rightarrow -\tilde{s}_2 - \log \left[ 1 + G_1 e^{\tilde{s}_1} + e^{2\tilde{s}_1} \right], \\
\tilde{s}_3 & \rightarrow -\tilde{s}_3 + \log \left[ 1 + G_1 e^{-\tilde{s}_1} + e^{-2\tilde{s}_1} \right],
\end{align*} \]
\[ (5.28) \quad f_2 : \begin{align*}
\tilde{s}_2 & \rightarrow -\tilde{s}_2, \\
\tilde{s}_3 & \rightarrow -\tilde{s}_3 - \log \left[ 1 + G_2 e^{-\tilde{s}_2} + e^{-2\tilde{s}_2} \right],
\end{align*} \]
Figure 1. The fat graph of the 4 holed Riemann sphere. The dashed geodesic corresponds to $x_3$.

\begin{align}
\hat{s}_1 &\to -\hat{s}_1 - \log\left[1 + G_3 e^{\hat{s}_3} + e^{2\hat{s}_3}\right], \\
\hat{s}_2 &\to -\hat{s}_2 + \log\left[1 + G_3 e^{-\hat{s}_3} + e^{-2\hat{s}_3}\right], \\
\hat{s}_3 &\to \hat{s}_3.
\end{align}

Remark 5.1. Observe that in [5] it was proved that shear coordinate flips (5.27), (5.28), (5.29) are indeed dual to the tagged cluster mutations (4.22) for the corresponding $\lambda$-lengths.

We are now going to produce a similar shear coordinate description of each of the other Painlevé cubics. For $\tilde{D}_5$, $\tilde{D}_6$, $\tilde{E}_6$, $\tilde{E}_7$ we will provide a geometric description of the corresponding Riemann surface and its fat-graph. Our geometric description agrees with the one obtained in [22], which was obtained by building a Strebel differential from the isomonodromic problems associated to each of the Painlevé equations.

5.1. Shear coordinates for $\tilde{D}_5$. The confluence from the cubic associated to PVI to the one associated to PV is realised by

$\hat{s}_3 \to \hat{s}_3 - \log[\epsilon], \quad p_3 \to p_3 - 2\log[\epsilon],$

in the limit $\epsilon \to 0$. We obtain the following shear coordinate description for the $\tilde{D}_5$ cubic:

\begin{align}
x_1 &= -e^{\hat{s}_2 + \hat{s}_3} - e^{-\hat{s}_2 + \hat{s}_3} - G_2 e^{\hat{s}_3} - G_3 e^{-\hat{s}_2} \\
x_2 &= -e^{\hat{s}_3 + \hat{s}_1} - G_3 e^{\hat{s}_1}, \\
x_3 &= -e^{\hat{s}_1 + \hat{s}_2} - e^{-\hat{s}_1 - \hat{s}_2} - e^{-\hat{s}_1 + \hat{s}_2} - G_1 e^{\hat{s}_2} - G_2 e^{-\hat{s}_1}
\end{align}

where

$G_i = e^{\frac{p_i}{2}} + e^{-\frac{p_i}{2}}, \quad i = 1, 2, \quad G_3 = e^{\frac{p_3}{2}}, \quad G_\infty = e^{\hat{s}_1 + \hat{s}_2 + \hat{s}_3}.$

To obtain the cubic in our form we need to specialise $p_3 = 0$.

To understand the geometry of this confluence we need to revert to the non-shifted shear coordinates, in which the confluence is realised by

$s_3 \to s_3, \quad p_3 \to p_3 - 2\log[\epsilon].$

This means that we send the perimeter $p_3$ to infinity, which is the same as opening one of the faces in two infinite directions as in Fig. 2. Geometrically speaking this
correspond to a Riemann sphere with three holes and two marked points on one of them.

5.2. **Shear coordinates for \( \tilde{E}_6 \).** The confluence from PV to PIV is realised by the substitution

\[ \bar{s}_2 \to \bar{s}_2 - \log|\epsilon|, \quad p_2 \to p_2 - 2\log|\epsilon|, \]

in formulae (5.30). In the limit \( \epsilon \to 0 \) we obtain:

\[
\begin{align*}
  x_1 &= -e^{\bar{s}_2 + \bar{s}_3} - G_2 e^{\bar{s}_3} \\
  x_2 &= -e^{\bar{s}_3 + \bar{s}_1} - G_3 e^{\bar{s}_1} \\
  x_3 &= -e^{\bar{s}_1 + \bar{s}_2} - e^{\bar{s}_1 + \bar{s}_2} - G_1 e^{\bar{s}_2} - G_2 e^{-\bar{s}_1}
\end{align*}
\]

(5.31)

where

\[
G_1 = e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}}, \quad G_i = e^{\frac{p_i}{2}}, \quad i = 2, 3, \quad G_\infty = e^{\bar{s}_1 + \bar{s}_2 + \bar{s}_3}.
\]

To obtain the cubic in our form we need to specialise \( p_3 = p_2 = 2\bar{s}_1 + 2\bar{s}_2 + 2\bar{s}_3 \).

Again, to understand the geometry of this confluence we need to revert to the non–shifted shear coordinates, in which the confluence is realised by

\[ s_2 \to s_2, \quad p_2 \to p_2 - 2\log|\epsilon|. \]

Similarly to the previous case, this means that we send the perimeter \( p_2 \) to infinity, which is the same as opening one of the two faces without marked points in two infinite directions. Geometrically speaking this correspond to a Riemann sphere with two holes, one of which has 4 marked points.

5.3. **Shear coordinates for \( \tilde{E}_7 \).** The confluence from PIV to PII is realised by the substitution

\[ \bar{s}_1 \to \bar{s}_1 - \log|\epsilon|, \quad p_1 \to p_1 - 2\log|\epsilon|, \]

in formulae (5.31). In the limit \( \epsilon \to 0 \) we obtain:

\[
\begin{align*}
  x_1 &= -e^{\bar{s}_2 + \bar{s}_3} - G_2 e^{\bar{s}_3} \\
  x_2 &= -e^{\bar{s}_3 + \bar{s}_1} - G_3 e^{\bar{s}_1} \\
  x_3 &= -e^{\bar{s}_1 + \bar{s}_2} - G_1 e^{\bar{s}_2}
\end{align*}
\]

(5.32)

where

\[
G_i = e^{\frac{p_i}{2}}, \quad i = 1, 2, 3, \quad G_\infty = e^{\bar{s}_1 + \bar{s}_2 + \bar{s}_3}.
\]
To obtain the $\tilde{E}_7$ cubic we need to specialise $p_3 = p_2 = p_1 = -2(\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3)$, while to get the $\tilde{E}_7^*$ one we need $p_3 = -p_2 = p_1 = -2(\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3)$.

Again geometrically this confluence gives rise to a Riemann sphere with one hole and 6 marked points.

### 5.4. Shear coordinates for $\tilde{E}_7$

The confluence from PII to PI is realised by

\[
\tilde{s}_3 \to \tilde{s}_3 - \log[\epsilon], \quad p_3 \to p_3 + 2 \log[\epsilon],
\]

in formulae (5.32). In the limit $\epsilon \to 0$ we obtain:

\[
\begin{align*}
x_1 &= -e^{\tilde{s}_2 + \tilde{s}_3} - G_2 e^{\tilde{s}_3} \\
x_2 &= -e^{\tilde{s}_3 + \tilde{s}_1} \\
x_3 &= -e^{\tilde{s}_1 + \tilde{s}_2} - G_1 e^{\tilde{s}_2} - G_2 e^{\tilde{s}_1}
\end{align*}
\]

where

\[
G_i = e^{\frac{\pi_i}{2}}, \quad i = 1, 2, \quad G_3 = 0, \quad G_\infty = e^{\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3}.
\]

To obtain the cubic in our form we need to specialise $p_2 = p_1 = 0$ and $\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3 = 0$.

### 5.5. Shear coordinates for $\tilde{D}_6$

The confluence from PV to PIII $\tilde{D}_6$ is realised by the substitution

\[
\tilde{s}_3 \to \tilde{s}_3 - \log[\epsilon], \quad p_3 \to p_3 + 2 \log[\epsilon],
\]

in formulae (5.30). In the limit $\epsilon \to 0$ we obtain:

\[
\begin{align*}
x_1 &= -e^{\tilde{s}_2 + \tilde{s}_3} - e^{-\tilde{s}_2 + \tilde{s}_3} - G_2 e^{\tilde{s}_3} \\
x_2 &= -e^{\tilde{s}_3 + \tilde{s}_1} \\
x_3 &= -e^{\tilde{s}_1 + \tilde{s}_2} - e^{-\tilde{s}_1 + \tilde{s}_2} - e^{-\tilde{s}_1 - \tilde{s}_2} - G_1 e^{\tilde{s}_2} - G_2 e^{-\tilde{s}_1}
\end{align*}
\]

where

\[
G_i = e^{\frac{\pi_i}{2}} + e^{\frac{-\pi_i}{2}}, \quad i = 1, 2, \quad G_3 = 0, \quad G_\infty = e^{\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3}.
\]

To obtain the cubic in our form we need to specialise $p_1 = 2\tilde{s}_1 + 2\tilde{s}_2 + 2\tilde{s}_3$.

To understand this confluence from a geometric point of view we first revert to the unshifted shear coordinates, which are transformed as follows:

\[
s_3 \to s_3 - \log[\epsilon^2], \quad p_3 \to p_3 + 2 \log[\epsilon],
\]

the second part of this scaling corresponds to dragging back from $\infty$ the two infinite directions produced by the PVI to PV confluence, and clashing them into a point with $G_3 = 0$. The first part of the scaling sends this point to infinity. This corresponds to a Riemann sphere with three holes and one marked point on one of them. The best way to understand this is to consider a Riemann sphere with 5 holes and an involution which identifies two couples of opposite holes and rotates the fifth (see figure 3). This involution admits an invariant curve with a stable point on it. We select a geodesic homothopic to this curve and cut along it, obtaining then a Riemann sphere with three holes and a marked point (the projection of the stable one).

---

1The authors are grateful to L. Chekhov for his insights on the geometric interpretation of this confluence.
Figure 3. Geodesics of the same color are identified. The dashed line corresponds to the invariant curve and its stable point. The green geodesic is the one homothopic to it.

5.6. **Shear coordinates for** $\tilde{D}_7$. The confluence from PV to PIII $\tilde{D}_7$ is realised by the substitution

$$
\begin{align*}
\tilde{s}_1 &\to \tilde{s}_1 - \log[\epsilon], \\
\tilde{s}_2 &\to \tilde{s}_2 + \log[\epsilon], \\
\tilde{s}_3 &\to \tilde{s}_3 - \log[\epsilon], \\
p_2 &\to p_2 - 2 \log[\epsilon], \\
p_3 &\to p_3 - 2 \log[\epsilon],
\end{align*}
$$

in formulae (5.30). In the limit $\epsilon \to 0$ we obtain:

$$
\begin{align*}
x_1 &= -e^{-\tilde{s}_2 + \tilde{s}_3} - G_2 e^{\tilde{s}_3} - G_3 e^{-\tilde{s}_2} \\
x_2 &= -e^{\tilde{s}_3 + \tilde{s}_1} - G_3 e^{\tilde{s}_1} \\
x_3 &= -e^{\tilde{s}_1 + \tilde{s}_2} - e^{-\tilde{s}_1 - \tilde{s}_2} - G_2 e^{-\tilde{s}_1}
\end{align*}
$$

where

$$
G_i = e^{\frac{p_i}{2}}, \quad i = 1, 2, 3, \quad G_\infty = e^{\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3}.
$$

To obtain the cubic in our form we need to specialise $p_1 = -2(\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3)$.

5.7. **Shear coordinates for** $\tilde{D}_8$. The confluence from PV to PIII $\tilde{D}_8$ is realised by the substitution

$$
\begin{align*}
s_1 &\to s_1 - \log[\epsilon], \\
s_2 &\to s_2 + \log[\epsilon], \\
p_2 &\to p_2 - 2 \log[\epsilon], \\
p_3 &\to p_3 - 2 \log[\epsilon],
\end{align*}
$$

in formulae (5.34). In the limit $\epsilon \to 0$ we obtain:

$$
\begin{align*}
x_1 &= -e^{-\tilde{s}_2 + \tilde{s}_3} - G_2 e^{\tilde{s}_3} \\
x_2 &= -e^{\tilde{s}_3 + \tilde{s}_1} - G_3 e^{\tilde{s}_1}, \\
x_3 &= -e^{\tilde{s}_1 + \tilde{s}_2} - e^{-\tilde{s}_1 - \tilde{s}_2} - G_2 e^{-\tilde{s}_1}
\end{align*}
$$

where

$$
G_1 = G_3 = 0, \quad G_2 = e^{\frac{p_2}{2}}, \quad G_\infty = e^{\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3}.
$$

To obtain the cubic in our form we need to specialise $G_2 = 0$. 
6. Quantisation

In this section we provide the quantisation of all the Painlevé cubics and produce the corresponding quantum confluence in such a way that quantisation and confluence commute.

As discussed in Section 3 on each Painlevé cubic surface denoted by an index $d$ running on the list of the extended Dynkin diagrams $\tilde{D}_5, \tilde{D}_6, \tilde{D}_7, \tilde{D}_8, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, we have the following Poisson bracket:

\[ \{ x_i, x_{i+1} \} = x_i x_{i+1} + 2 \epsilon^{(d)}_k x_k + \omega^{(d)}_k, \quad k \neq i, i+1, \]

where we use the cyclic notation $x_{i+3} = x_i$, $i = 1, 2, 3$ and the parameters $\epsilon^{(d)}_i$, and $\omega^{(d)}_{1,2,3}$ are given by the formulae (2.2) and (2.3) respectively.

This Poisson algebra is induced by the Poisson algebras of geodesic length functions constructed in [3] by postulating the Poisson relations on the level of the shear coordinates $s_a$ of the Teichmüller space. In our case these are:

\[ \{ s_1, s_2 \} = \{ s_2, s_3 \} = \{ s_3, s_1 \} = 1, \]

while the perimeters $p_1, p_2, p_3$ are assumed to be Casimirs so that the shifted shear coordinates $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3$ satisfy the same Poisson relations:

\[ \{ \tilde{s}_1, \tilde{s}_2 \} = \{ \tilde{s}_2, \tilde{s}_3 \} = \{ \tilde{s}_3, \tilde{s}_1 \} = 1. \]

It is worth reminding that the exponentials of the shear coordinates satisfy the log-canonical Poisson bracket. To produce the quantum Painlevé cubics, we introduce the Hermitian operators $S_1, S_2, S_3$ subject to the commutation inherited from the Poisson bracket of $s_i$:

\[ [S_i, S_{i+1}] = i\hbar \{ s_i, s_{i+1} \} = i\pi \hbar, \quad i = 1, 2, 3, \quad i + 3 \equiv i. \]

Observe that thanks to this fact, the commutators $[S_i, S_j]$ are always numbers and therefore we have

\[ \exp(aS_i) \exp(bS_j) = \exp \left( aS_i + bS_j + \frac{ab}{2} [S_i, S_j] \right), \]

for any two constants $a, b$. Therefore we have the Weyl ordering:

\[ e^{S_1+S_2} = q^{\frac{1}{2}} e^{S_1} e^{S_2} = q^{-\frac{1}{2}} e^{S_2} e^{S_1}, \quad q \equiv e^{-i\pi \hbar}. \]

After quantisation, the perimeters $p_1, p_2, p_3$ and $\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3$ remain non–deformed, so we preserve the previous notation for them. This is equivalent to say that the constants $\omega^{(d)}_i$ remain non-deformed.

We introduce the Hermitian operators $X_1, X_2, X_3$ as follows: consider the classical expressions for $x_1, x_2, x_3$ in terms of $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3$ and $p_1, p_2, p_3$. Write each product of exponential terms as the exponential of the sum of the exponents and replace those exponents by their quantum version. For example, in the case of $\tilde{D}_5$ we have:

\[ x_1 = -e^{-S_2 + \tilde{s}_3} - e^{-\tilde{s}_2 + \tilde{s}_3} - G_2 e^{\tilde{s}_3} - G_3 e^{-\tilde{s}_2}, \]

and its quantum version is defined as

\[ X_1 = -e^{-S_2} - (e^{\frac{\pi i}{2}} + e^{-\frac{\pi i}{2}}) e^{S_3} - e^{-S_2 + S_3} - e^{S_2 + S_3} = \]

\[ = e^{-S_2} - (e^{\frac{\pi i}{2}} + e^{-\frac{\pi i}{2}}) e^{S_3} - q^{-\frac{1}{2}} e^{-S_2} e^{S_3} - q^{\frac{1}{2}} e^{S_2} e^{S_3}. \]
Theorem 6.1. Denote by $X_1, X_2, X_3$ the quantum Hermitian operators corresponding to $x_1, x_2, x_3$ as above. The quantum commutation relations are:

$$q^\frac{i}{2} X_i X_{i+1} - q^{-\frac{i}{2}} X_{i+1} X_i = \left( \frac{1}{q} - q \right) \epsilon^{(d)}_k X_k + (q^{-\frac{i}{2}} - q^\frac{i}{2}) \omega^{(d)}_i$$

where $\epsilon^{(d)}_i$ and $\omega^{(d)}_i$ are the same as in the classical case. The quantum operators satisfy the following quantum cubic relations:

$$q^\frac{i}{2} X_1 X_2 X_3 + q^{-\frac{i}{2}} \epsilon^{(d)}_1 X_1^2 + q^\frac{i}{2} \epsilon^{(d)}_2 X_2^2 + q^{-\frac{i}{2}} \epsilon^{(d)}_3 X_3^2 + q^\frac{i}{2} \omega^{(d)}_1 X_1^2 + q^{-\frac{i}{2}} \omega^{(d)}_2 X_2^2 + q^\frac{i}{2} \omega^{(d)}_3 X_3^2 = 0.$$

Remark 6.2. Observe that in the case of PVI the above quantum commutation relations (6.38) and quantum cubic (6.39) appeared already in the paper by Oblomkov [18] in which the classical cubic (2.1) for the generalised rank 1 double affine Hecke (DAHA) algebra studied in [20]. Their work followed a theorem by Oblomkov [18] in which the classical cubic (2.1) for the $D_4$ case appeared as the spectrum of the centre of the same generalised DAHA.

Remark 6.3. In the case of PII the quantum commutation relations (6.38) (up to re-scaling) coincide with the equitable presentation of $U_q(\mathfrak{sl}_2)$, due to Ito, Terwilliger and Weng [15]. This algebra is generated by $x, y$ and $z^{\pm 1}$ subject to the relations $q^i xy - yx = q^i - 1, q^i yz - zy = q^i - 1$ and $q^i zx - xz = q^i - 1$.

Remark 6.4. It is clear that if we define the quantum confluence as the obvious analogue of the classical one, i.e. we rescale the quantum Hermitian operators by a constant $\epsilon$ and take the limit as $\epsilon \to 0$, then quantisation and confluence commute.

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