Euclidean Solutions in Broken Phase and Electro-Weak Dynamics

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Abstract

A Higgs-Yukawa system in a broken phase and Euclidean solutions are investigated. Although it has been believed that there are no Euclidean solutions in the broken phase in 4-dimension, we find numerically ones in the phase due to the effect of a strong Yukawa coupling. The complex Yukawa coupling is necessary for the stability of the solution. The extension to a complex Higgs-Yukawa system is also investigated.
1. Introduction In the electro-weak (EW) theory with rich structure of the dynamics, it is an essential point that the Higgs field has a non-zero vacuum expectation value (VEV). Its typical example is the existence of static solutions as sphaleron which is unstable and combine distinct vacua from each other in the EW theory. It is important that there is an instanton solution in the system to estimate the tunneling effect between distinct vacua semi-classically. On the other hand, by the famous scaling argument \cite{1,2}, it is pointed out that the EW theory cannot have instanton solutions because of collapsing phenomena.

To solve the above problem we know a constrained instanton, a streamline method and new valley one \cite{2-4}. They are not exact solutions in the theory but an effective collection of configurations in the path integral. Another approach is to introduce some external forces against the collapse. One possibility is given by taking account of the effect of heavy fermions. One of the evidences is the fermion effect on sphaleron \cite{5-9}. The sphaleron barrier between two vacua is more lowered by heavier fermion effects. This means the transition between distinct vacua becomes to be easier.

Now, we can see the naive scaling argument as the following: For the Higgs field, we can give a scale transformation with fixing its VEV as

$$\phi(x, a) \rightarrow \phi(ax).$$

In addition to spatial coordinates, other fields are transformed like as canonical scaling. Then the scaling of the present action is given as follows;

$$S(a) = S_0 + S_{-4}a^{-4} + S_{-2}a^{-2} + S_{-1}a^{-1},$$

where $S_0$ is a scale invariant term such as gauge and matter kinetic ones without VEV. The coefficient $S_{-4}$ is a positive number due to giving a stability of the vacuum and the coefficient $S_{-2}$ is a positive number which comes from the kinetic term of the scalar field with VEV in the Lagrangian. On the
other hand, the sign of the coefficient $S_{-1}$ can not be definitely determined because of the Yukawa coupling effect.

From the above scaling argument, one can understand that the Yukawa term $(S_{-1})$ plays an important role in the short distance phenomena. If the term is negative and has enough magnitude, the collapsing of the solution becomes to be impossible by the repulsing effect.

In this article, we investigate a Higgs-Yukawa system in a broken phase and construct Euclidean solutions (instantons) numerically for a strong Yukawa coupling.

2. *simple Higgs-Yukawa System* We shall derive equations of motion for a simple Higgs-Yukawa system (the Higgs filed is real) in a Euclidean space. Since fermi fields $\Psi$ and $\bar{\Psi}$ are independent of each other in the Euclidean space, the real positiveness of $\bar{\Psi}\Psi$ is not guaranteed. So, we can introduce a complex Yukawa coupling $g(=|g|e^{i\theta})$, keeping a Lagrangian hermite. Here, the Lagrangian is given as

$$\mathcal{L} = \frac{1}{2}(\partial_{\mu}\Phi)^2 + 2\lambda^2(\Phi^2 - v^2)^2 + \bar{\Psi}\gamma_{\mu}\partial_{\mu}\Psi + \Phi\text{Re}(g\bar{\Psi}\Psi),$$

where $v$ is the VEV of the Higgs field $\Phi$ and is taken as $v = 246$ GeV which correspond to a standard Higgs in the EW theory. By introducing the spherical ansatz for the scalar field, one can be applicable for simplicity as

$$\Phi(x) = \Phi(r),$$

where $r = \sqrt{x^2}$.

The ansatz for the fermi fields is non trivial because of taking account of freedom of spinor. Here, we adopt the following ones;

$$\Psi(x) = (f_1(r) + e^{i\theta}f_2(r)\gamma_{\mu}\frac{x_{\mu}}{r})\xi,$$

$$\bar{\Psi}(x) = \xi(f_1(r) - e^{i\theta}f_2(r)\gamma_{\mu}\frac{x_{\mu}}{r}),$$

...
where $\xi$ and $\bar{\xi}$ denote constant spinors. The normalization is given as

$$\bar{\xi}\xi = \pm e^{-i\theta},$$
$$\bar{\xi}\gamma_\mu\xi = 0.$$

Unfortunately, there are no explicit forms of the equations of motion for $\Psi$ and $\bar{\Psi}$ except for the $\theta = 0$ or $\frac{\pi}{2}$ case. Thus, by substituting the above ansatz into the Lagrangian (1), one can derive the fermion equations of motion in term of $f_1$ and $f_2$. So, we can write down equations of motion by $\Phi$, $f_1$ and $f_2$ as

$$\Phi'' + \frac{3}{r}\Phi' = 8\lambda^2\Phi(\Phi^2 - v^2) \pm |g|(f_1^2 - f_2^2 \cos 2\theta),$$  \hspace{1cm} (2)

$$f_1' = |g|\Phi f_2 \cos 2\theta, \hspace{1cm} (3)$$

$$f_2' + \frac{3}{r}f_2 = |g|\Phi f_1. \hspace{1cm} (4)$$

Here, one can see that in the $\theta = 0$ or $\frac{\pi}{2}$ case, above equations are reduced ones which are derived from equations of motion of $\Psi$ and $\bar{\Psi}$. Particularly, the Yukawa coupling with $\theta = 0$ becomes to be real. Thus both $f_1$ and $f_2$ have real masses and then have normal and exponential dumping behaviors. For the exponential behavior of $\Phi$, the Eq.(2) indicates that the fermion mass must be heavier than a half of the Higgs mass, thus the Yukawa coupling must be sufficiently strong. Among present known fermions, only top quark may be satisfied with this condition, so we expect that top quark plays a main role for an instanton of the Higgs-Yukawa system.

We examine Eq.(2) - Eq.(4) numerically to find a solution that obeys the following boundary conditions;

$$\Phi'(0) = f_2(0) = 0,$$
$$\Phi(\infty) = v,$$
$$f_1(\infty) = f_2(\infty) = 0.$$
The first condition is given for the smoothness at the origin, the second one
means the vacuum condition at \( r = \infty \). The third one gives the normaliz-
ability condition of fermionic wave function. In the calculation, both values
of \( f_1(0) \) and \( \Phi(0) \) are free parameters. So, we treat the \( f_1(0) \) as a variable pa-
rameter with a fixed \( \Phi(0) \) to find a solution that satisfies the above boundary
conditions.

Before numerical investigation of the Higgs-Yukawa system, we examine
a pure Higgs system analytically. In the pure Higgs system, there are two
kinds of “solutions” that do not obey boundary conditions. Eq.(2) without
Yukawa term is interpreted as the point particle’s equation of motions with
a friction in a potential of \( -(\Phi^2 - v^2)^2 \) \[10\]. In a first type solution with
\( |\Phi(0)| < v \), the particle falls into the origin and oscillates dumpingly around
the origin. In a second type one with \( |\Phi(0)| > v \), the particle departs from
the origin and the “solution” behaves asymptotically as \( \Phi(r) \propto \frac{1}{a+r} \) where \( a \)
is a constant. Then, we can determine a sign of Yukawa coupling to improve
the above behavior.

Now, we try to search the numerical solution in the special three cases as
\( \theta = 0 \) (a real Yukawa coupling), \( \theta = \frac{\pi}{2} \) (a imaginary Yukawa coupling) and
\( \theta = \frac{\pi}{4} \).

In the case of \( \theta = 0 \), we can not find any solutions which obey the bound-
ary conditions for any \( \Phi(0) \). However, we can find “solutions” very similar to
one in the pure Higgs system. It is pointed out that when \( \Phi(r), f_1(r), f_2(r) >> 0 \),
Eq.(2) - Eq.(4) have simple pole solutions as
\[
\Phi(r) \propto f_1(r) \propto f_2(r) \sim \frac{1}{a + r}.
\]
The result for \( \Phi(r) \) obtained for many values of \( f_1(0) \) are given in Fig.1. One
can see in this figure that all calculated \( \Phi(r) \) show a simple pole behavior in
all values of \( f_1(0) \).

Next, we try to search for a solution in the case of \( \theta = \frac{\pi}{2} \). We can get
many numerical solutions which obey the boundary conditions. The typical results for $\Phi(r)$, $f_1(r)$ and $f_2(r)$ are shown in Fig.2 - Fig.4, respectively. These figures show that these solutions localizes around the origin with nonzero finite size. It means that the contribution from fermion supports for the Higgs field not to collapse to the trivial solution. It has already be pointed out that there is a analytic solution in this theory with $v = 0$ by Inagaki [11]. The solution is shown with a dashed line in Fig.2 - Fig.4, too.

In the case of $\theta = \frac{\pi}{4}$, only $f_1$ contributes to Higgs field, and $f_1$ is linear in $r$. Thus, one can easily know that there are no solutions. This fact means that there exists a gap between two regions connected to $\theta = 0$ and to $\theta = \frac{\pi}{2}$.

We investigate numerically whether there are solutions in the present system or not, in the two cases of $0 \leq \theta < \frac{\pi}{4}$ and $\frac{\pi}{4} < \theta \leq \frac{\pi}{2}$. We find that there is not any solution in the case of $0 \leq \theta < \frac{\pi}{4}$. On the other hand, there are solutions for $\frac{\pi}{4} < \theta \leq \frac{\pi}{2}$. To show these facts, the actions calculated with the obtained solution is given as a function of $\theta$ in Fig.5.

We can show also the relation of imaginary fermion masses to the calculated action in Fig.6. One can see that the mass dependency of the action is consistent with the result given by Nolte and Kunz [6].

Finally, we can point out that important one of present findings is that the solutions localize around the origin with a nonzero finite size, and the contribution from fermions plays an important role for the Higgs field not to collapse to the trivial solution.

3. Complex Higgs-Yukawa System Next, we try the extension to complex Higgs-Yukawa system. We introduce two same fermions except for constant spinors normalization to avoid a linear term of $x$ in Lagrangian with a suitable ansatz for fields. Thus, our Lagrangian is,

$$
\mathcal{L} = |\partial \Phi|^2 + 2\lambda(|\Phi|^2 - v^2)^2 + \bar{\Psi} \bar{\partial} \Psi + \bar{\Psi}_L \Phi \Psi_R + \bar{\Psi}_R \Phi^\dagger \Psi_L,
$$

where $\Psi = [\psi_1, \psi_2]$ and $\Phi = \Phi_R + i\Phi_I$. 

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In the present investigation, the ansatz for each field is given the following,
\[ \Phi = \Phi(r), \]
\[ \psi_i = (f_1 + f_2(\gamma \cdot \hat{x}) + if_3\gamma_5 + if_4\gamma_5(\gamma \cdot \hat{x}))\omega_i, \]
\[ \bar{\psi}_i = \bar{\omega}_i(f_1 - f_2(\gamma \cdot \hat{x}) + if_3\gamma_5 - if_4(\gamma \cdot \hat{x})\gamma_5), \]
\[ \bar{\omega}_i\omega_i = A \text{ (real number)}, \]
\[ \bar{\omega}_i\gamma_5\omega_i = B \text{ (imaginary number)}, \]
\[ \bar{\omega}_i\gamma_5\gamma_\mu\omega_i = \begin{cases} C & \text{ (imaginary number) for } i = 1, \\ -C & \text{ (imaginary number) for } i = 2. \end{cases} \]

Finally, we can obtain following equations of motion, using the above ansatz.
\[ \Phi''_R(r) + \frac{3}{r}\Phi'_R(r) = 4\lambda\Phi_R(|\Phi|^2 - v^2) \]
\[ + gA(f_1^2 - f_2^2 - f_3^2 + f_4^2) + 2igB(f_1f_3 + f_2f_4), \]  
\[ \Phi''_I(r) + \frac{3}{r}\Phi'_I(r) = 4\lambda\Phi_I(|\Phi|^2 - v^2) \]
\[ + igB(f_1^2 + f_2^2 - f_3^2 - f_4^2) - 2gA(f_1f_3 - f_2f_4), \]
\[ f_1' = g(\Phi_Rf_2 - \Phi_If_4), \]   
\[ f_2' + \frac{3}{r}f_2 = g(\Phi_Rf_1 - \Phi_If_3), \]   
\[ f_3' = -g(\Phi_Rf_4 + \Phi_If_2), \]   
\[ f_4' + \frac{3}{r}f_4 = -g(\Phi_Rf_3 + \Phi_If_1). \]

Here, boundary conditions are as follows,
\[ \Phi'_R(0) = \Phi'_I(0) = f_2(0) = f_4(0) = 0, \]
\[ \Phi_R(\infty)^2 + \Phi_I(\infty)^2 = v^2, \]
\[ f_1(\infty) = f_2(\infty) = f_3(\infty) = f_4(\infty) = 0. \]

The first condition is given by the smoothness at the origin as same as the one in simple Higgs case. the second one means vacuum at infinity and the third one denotes fermion normalizability.
We shall examine equations of motion in this system in comparison with those in the previous simple Higgs-Yukawa system. Putting $\Phi = \Phi_R$, $f_3 = f_4 = 0$ and $B = 0$ in Eq.(5) - Eq.(10), we can obtain the same equations in the simple Higgs-Yukawa system with $\theta = 0$. Then one can know that there are simple pole solutions in the present system. If we take as $f_2 \to \mathrm{i}f_2$, $g \to \mathrm{i}g$ and $f_3 = f_4 = 0 = B = 0$ in Eq.(5) - Eq.(10), we can get same equations in the previous system with $\theta = \frac{\pi}{2}$. In this case, we can say that there is a solution with a nonzero finite size.

Now, we are going to search for a solution with a nonzero finite size in the present system. It is important for constructing instanton solution in the EW theory to get a solution in this system.

4. Summary We shall summarize this article. Due to the strong Yukawa coupling, the Euclidean solution in a broken phase is stabilized. Explicit solutions are found numerically. Fermi fields can give the effect as external sources against the collapsing instanton. Generally, external sources enough to support instantons must have suitable values or sign in the following three points; (1)magnitude, (2)spatial size and (3)sign.

It is so difficult to change the short distance behavior in the standard dynamics except Yukawa couplings, and then we can not find the other possibility for external sources against collapse.

Changing of the Yukawa coupling from real to imaginary corresponds to the above third point and prefer repulsive effect a to attractive ones owing to giving against collapsing forces.

Our final aim is to construct instanton solutions in the EW theory. In the theory, there are other problems gauge fields, index theorem and fermion zero modes. They shall be studied in the succeeding paper.
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Figure Captions

• Fig.1  Numerical “Solutions” for the Higgs-field in the real Yukawa system. Each line denotes one for different values of $f_1(0)$.

• Fig.2  Numerical Solutions for the Higgs-field in the imaginary Yukawa system. A solid line and dash-dotted line correspond to solutions with the Yukawa coupling $|g| = 175 \text{ GeV}/v$ for $M_h = 60 \text{ GeV}$ and $120 \text{ GeV}$ respectively. A dashed line denotes Inagaki’s analytic solution with $v = 0$.

• Fig.3  Numerical Solutions for the $f_1$ in the imaginary Yukawa system. Each line corresponds to the same one in Fig.2.

• Fig.4  Numerical Solutions for the $f_2$ in the imaginary Yukawa system. Each line corresponds to the same one in Fig.2.

• Fig.5  Action ($S$) versus Yukawa coupling phase ($\theta$) in the imaginary Yukawa system.

• Fig.6  Action ($S$) versus Yukawa coupling strength ($M_f = |g|v$) in the imaginary Yukawa system.
\[ \frac{\Phi}{\nu} \]

Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6