MATHEMATICAL REMARKS ON THE
COHOMOLOGY OF GAUGE GROUPS AND ANOMALIES

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ABSTRACT
Anomalies can be viewed as arising from the cohomology of the Lie algebra of the
group of gauge transformations and also from the topological cohomology of the group
of connections modulo gauge transformations. We show how these two approaches are
unified by the transgression map. We discuss the geometry behind the current commu-
tator anomaly and the Faddeev- Mickelsson anomaly using the recent notion of a gerbe.
Some anomalies (notably 3-cocycles) do not have such a geometric origin. We discuss
one example and a conjecture on how these may be related to geometric anomalies

1. Introduction and preliminaries
The transition from classical to quantum physics has lead to a great deal of interest-
ing mathematics. The study of so-called anomalies really begins with Wigner’s theoremsymmetries of a dynamical system manifest themselves as projective representations at the quantum level and hence give rise to 2-cocycles on the symmetry group. In quantum field theory unexpected or anomalous behaviour of gauge symmetry groups (infinite dimensional Lie groups) in the quantised theory has led to an extensive study of Lie group cohomology and to the explicit construction of cocycles (the so-called anomalies). However the infinite dimensional theory exhibits a number of features which have no counterpart in finite dimensions. Necessarily new tools are needed to study the mathematical questions raised by these developments. In this note we will put forward a particular viewpoint about ‘anomalies’. There is nothing in this paper which is particularly deep. Our aim is to expose some mathematical questions, answer some of the easier ones and review or explain related work of other authors in our framework.

In the first section we define the transgression map. This relates the Lie algebra
cohomology of $L\mathcal{G}$, the Lie algebra of the gauge group $\mathcal{G}$, with values in real-valued functions on $\mathcal{A}$, in dimension $p$ to the topological cohomology of $\mathcal{A}/\mathcal{G}$, the space
of connections modulo gauge, in dimension \( p + 1 \). We then consider particular cases of the transgression map and discuss the geometric interpretation of this cohomology and the corresponding anomaly in physics. When \( p = 1 \), \( H^2(A/G, \mathbb{Z}) \) classifies the equivalence classes of complex, one-dimensional vector bundles on \( A/G \). The anomaly of interest is the current commutator anomaly and is associated to the determinant line bundle of the Dirac operator. When \( p = 2 \) the anomaly of interest is the Faddeev-Mickelsson anomaly. Only recently has it become clear the geometric object in this case is what is known as a gerbe. As gerbes are rather recent phenomena we present a short review of their properties. Finally when \( p = 3 \) we give an example of a 3-cocycle which arises as an obstruction to the existence of an extension of one Lie algebra by another. It is an open question whether it has a relationship with \( H^4(A/G, \mathbb{Z}) \).

1.1. Some mathematical preliminaries

Let \( M \) be a compact Riemannian manifold and let \( P \) be a principal bundle over \( M \) with structure group \( G \), a compact Lie group. We let \( A \) denote the affine space of connections on \( P \) with values in the Lie algebra \( LG \) of \( G \). Denote by \( \mathcal{G} \) the group of automorphisms of \( P \) fixing the base. By suitable basepointing we can assume that \( \mathcal{G} \) acts freely on \( A \). Denote by \( LG \) the Lie algebra of \( \mathcal{G} \).

We denote the space of smooth \( p \) forms on a manifold \( X \) by \( \Omega^p(X) \) and de Rham complex by \( \Omega^\ast(X) \). If \( H \) is an \( LG \)-module then we denote Lie algebra cochains on \( LG \) of degree \( p \) with values in \( H \) by \( C^p(LG, H) \). We denote the coboundary operator for this cochain complex by \( \delta \) and the cohomology by \( H^\ast LG(H) \).

If \( X \) is a space on which \( G \) acts freely then there is a map from the de Rham cohomology of \( X \) into the Lie algebra cohomology of \( \text{Map}(X, R) \) defined as follows. If \( \omega \in \Omega^p(X) \) with \( d\omega = 0 \) and \( \pi: A \to A/G \) then \( \pi^*\omega = d\mu \) for some \( \mu \) as \( A \) is contractible. In fact there will be an explicit formula for \( \mu \) as \( A \) has an affine structure. Consider \( c_\mu \) the \( LG \) cocycle obtained by restricting \( \mu \) to the tangent spaces to the fibres of \( A \to A/G \) as in section 1.1. Then \( \delta(c_\mu) = c_{d\mu} = c_\omega \). But \( \omega \) is pulled-back so kills any vectors tangent to the fibres and hence \( c_\omega = 0 \). Hence \( \delta(c_\mu) = 0 \). Moreover if we change \( \mu \) by \( \mu + d\eta \) (the only possible change because \( A \)

2. The Transgression Map

One can think of anomalies as arising from the topology of \( A/G \) or as group cocycles on \( LG \) and both these points of view have been explored in the literature to some extent. We shall show that they are related via the transgression map.

If \( \omega \in \Omega^p(A/G) \), \( d\omega = 0 \) and \( \pi: A \to A/G \) then \( \pi^*\omega = d\mu \) for some \( \mu \) as \( A \) is contractible. In fact there will be an explicit formula for \( \mu \) as \( A \) has an affine structure. Consider \( c_\mu \) the \( LG \) cocycle obtained by restricting \( \mu \) to the tangent spaces to the fibres of \( A \to A/G \) as in section 1.1. Then \( \delta(c_\mu) = c_{d\mu} = c_\omega \). But \( \omega \) is pulled-back so kills any vectors tangent to the fibres and hence \( c_\omega = 0 \). Hence \( \delta(c_\mu) = 0 \). Moreover if we change \( \mu \) by \( \mu + d\eta \) (the only possible change because \( A \)


is contractible) then $\mu$ changes by $\delta\mu$. Lastly if we add to $\omega$ some $d\rho$ then this adds $\pi^*\rho$ to $\mu$ and $\pi^*\rho$ vanishes in the fibre direction unless $\rho$ is a function.

So have we the transgression map on de Rham cohomology:

$$H^p(A/\mathcal{G}) \to H^{p-1}_{L\mathcal{G}}(\text{Map}(A, \mathbb{R})) \quad p > 1 \quad (1.1)$$

Notice this is slightly different to the topologists transgression map, they look at only one fibre and identify it with $\mathcal{G}$ to obtain

$$H^p(A/\mathcal{G}) \to H^{p-1}(\mathcal{G}).$$

The transgression map relates the cohomology of $A/\mathcal{G}$ to the Lie algebra cohomology of the gauge group. In the next two sections we wish to examine two cases of the transgression map which give rise to anomalies in physics and which also have interesting geometric interpretations.

3. The transgression for $p = 1$ and the determinant line bundle.

Consider a $U(1)$ bundle $Q \to A$ on which $\mathcal{G}$ acts covering the action of $\mathcal{G}$ on $A$. This means that we can form the quotient $Q/\mathcal{G}$ which is a $U(1)$ bundle on $A/\mathcal{G}$. As a bundle on $A$, $Q$ is trivial because $A$ is contractible. The question we wish to consider is whether or not $Q/\mathcal{G}$ is trivial. There are two ways of approaching this. We can ask whether $Q$ has a $\mathcal{G}$ equivariant section. If it has this will descend to a section of $Q/\mathcal{G}$ and hence trivialise it. Conversely any section of $Q/\mathcal{G}$ will lift back to a $\mathcal{G}$ equivariant section of $Q$. The second point of view is to consider the Chern class of $Q/\mathcal{G}$, this is a cohomology class in $H^2(A/\mathcal{G}, \mathbb{R})$ and vanishes precisely when $Q/\mathcal{G}$ is trivial. We will show that these two different points of view are related by the transgression map.

Let $s$ be a section of $Q$ over $A$ and define $M(A, g)$ by

$$s(A, g)g^{-1} = M(A, g)s(A).$$

We consider $M$ as a $\mathcal{G}$ cocycle with values in $\text{Map}(A, U(1))$. It measures the failure of $s$ to be a $\mathcal{G}$ equivariant section. Now

$$s(Agh)(gh)^{-1} = M(A, gh)s(A)$$

and

$$s((Ag)h)h^{-1}g^{-1} = M(Ag, h)s(Ag)g^{-1} = M(Ag, h)M(A, g)s(A)$$

so that

$$M(A, gh) = M(Ag, h)M(A, g)$$

or

$$M(Ag, h)M(A, gh)^{-1}M(A, g) = 1$$

and $M$ is a 1-cocycle for $\mathcal{G}$ with values in $\text{Map}(A, U(1))$. 

Of course even if $Q$ admits a $G$ equivariant section $s$ may not be it. However any other section is of the form $s(A)h(A)$ for $h \in \text{Map}(A,U(1))$ and it gives rise to a $M'$ which is related to $M$ by

$$M(A,g) = h(Ag)M'(A,g)h(A).$$

So we see from this that the class of $M$ depends only on the bundle $Q$ and moreover that if $Q$ admits a $G$ equivariant section there is an $h$ such that $M(A,g) = h(Ag)h(A)$. That is, $M$ is the trivial class. Conversely if $M(A,g) = h(Ag)h(A)^{-1}$ for some $h : A \to U(1)$, then $\tilde{s} = s(A)h(A)^{-1}$ satisfies

$$\tilde{s}(Ag)g^{-1} = s(A)h(A)^{-1}g^{-1} = M(A,g)s(A)M(A,g)^{-1}h(A)^{-1} = \tilde{s}(A).$$

So the cohomology class of $M$ is precisely the obstruction to $Q$ having a $G$ invariant trivialization.

To relate this to transgression we consider the Lie algebra version. Let $\eta$ be an element of $LG$, then it gives rise to a vector field $\eta_A$ on $A$. There are two ways of lifting this vector field to the bundle $Q$. The first way is to note that by assumption $G$ acts on $Q$ and hence $\eta$ gives rise to a vector field $\eta_Q$. The second is $s_*(\eta_A)$, the lift of $\eta_A$ using the section $s$. If $s$ is equivariant these coincide so we define

$$m(A,\eta) = s_*(\eta) - \dot{\eta}.$$

Recall that the vertical vectors to the fibering $Q \to A$ can be identified with the Lie algebra of $U(1)$ and that we identify with $\mathbb{R}$ so we will regard, $m$ via these identifications as $\mathbb{R}$ valued. Note that this is related to the group cocycle $M$ by

$$m(A,\eta) = \frac{d}{dt} (M(A,e^{t\eta})) \bigg|_{t=0}.$$

It is straightforward to show that $m$ is a Lie algebra cocycle with values in $\text{Map}(A,\mathbb{R})$. We want to show that $m$ is the transgression of the Chern class of $Q$. To see this pick a connection $\omega$ for $Q/G$, this is a 1-form on $Q/G$, and let $\pi^* (\nabla)$ be its lift to $Q$. Denote by $F$ the curvature of $\nabla$, a two-form on $A/G$. The curvature of $\pi^*(\nabla)$ is $\pi^*(F)$. We will show that the transgression of $F$ is $m$. To see this notice that on the total space of the bundle $Q$ we have

$$p^*\pi^* F = d\pi^*(\omega)$$

where $p : Q \to A$ is the bundle projection. If we pull back with $s$ we have

$$\pi^*(F) = ds^*\pi^*(\omega)$$

so that we can take $\mu = s^*\pi^*(\omega)$ in the definition of the transgression of $F$. Let $\eta_A$ then be the vector field generated by $\eta$ on $A$ and similarly for $\eta_Q$. Notice
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\[ \pi^*(\omega) (\eta_Q) = 0 \] as \( \eta_Q \) is vertical for the projection \( Q \to Q/G \) and \( \pi^*(\omega) \) is pulled-back. Also \( s_* (\eta) - \eta_Q \) is vertical for the fibering of \( Q \to \mathcal{A} \) so that we have

\[ \pi^*(\omega) (s_* (\eta) - \eta_Q) = s_* (\eta) - \eta_Q. \]

Combining these facts gives

\[ \mu (\eta_A) = \pi^*(\omega) (s_* (\eta) = \pi^*(\omega) (s_* (\eta) - \eta_Q) = s_* (\eta) - \eta_Q = m(A, \eta) \]

as required. Notice that if \( G \) is connected \( m \) is precisely the obstruction to \( Q \) having a \( G \) invariant trivialization.

3.1. The determinant line bundle and the current commutator anomaly.

The anomaly that arises in this case is called the current commutator anomaly and is related to the determinant line bundle. Recall that if \( X : V \to W \) is linear map and \( \dim(W) = \dim(V) \) then \( \det(X) \) is not a number but an element of \( \det(V)^* \otimes \det(W) \) where \( \det(V) \) is the highest exterior power of \( V \) and similarly for \( W \). Such a situation arises for the Dirac operator when \( M \) is odd dimensional. Then the Dirac operator maps from plus to minus spinors. If we couple the Dirac operator to a connection in \( \mathcal{A} \) then we have a family of operators \( D_A : H^+ \to H^- \) where \( H^\pm \) is the space of \( \pm \) spinor fields. To define the determinant of the Dirac operator in this case we have the added problem of ‘regularising’ the infinite determinant. This can be done but the regularisation depends on \( A \). The upshot is that \( \det(D_A) \) is a section of a line bundle \( Q \) over \( \mathcal{A} \). If we want \( \det(D_A) \) to be a number we need to be able to trivialise this line bundle. We are now in the situation at the beginning of this section.

4. The transgression for \( p = 2 \) Faddeev’s anomaly and gerbes

When \( p = 2 \) we are interested in the transgression

\[ H^3(\mathcal{A}/\mathcal{G}) \to H^2_{L^2}(\mathcal{G}). \]

We can regard the elements of \( H^3(\mathcal{A}/\mathcal{G}, \mathbb{Z}) \) as characteristic classes of \( PU(H) \) bundles for \( H \) an infinite dimensional separable Hilbert space. They measure the failure of the bundle to extend to a \( U(H) \) bundle and hence to be trivial as \( U(H) \) is contractible.

If \( Q \) is a \( PU(H) \) bundle on \( \mathcal{A} \) then it extends to a \( U(H) \) bundle because \( \mathcal{A} \) is contractible. However if \( \mathcal{G} \) acts on \( Q \) then it may not act on \( PU(H) \) but a central extension will. This central extension is determined by a class in \( H^2_{L^2}(\text{Map}(\mathcal{A}, U(1))) \) and determines a class on \( H^2_{L^2}(\text{Map}(\mathcal{A}, \mathcal{R})). \) We claim that these two classes are related by transgression. To obtain a de Rham version of the three class we need to use the theory of gerbes. In fact we shall use a simplified definition of a gerbe that will be adequate for our purposes. The relationship of this definition to that of Brylinski is similar to the relationship between sheaves and line bundles.
4.1. Gerbes

We want to define a differential geometric object that, for want of a better name, we shall call a gerbe. The reader should beware that these are not the same as the gerbes defined by Brylinski. We shall indicate below the precise relationship. Let \( \pi: Y \to M \) be a fibration. Define the fibre product \( Y \times_M Y \) as the subset of pairs \((y, y')\) in \( Y \times Y \) such that \( \pi(y) = \pi(y') \). Notice that the diagonal is inside \( Y \times_M Y \) and that the map \( \tau \) that transposes two elements fixes \( Y \times_M Y \). Then a gerbe is a principal \( \mathbb{C}^\times \) bundle \( Q \) over \( Y \times_M Y \) with a composition map defined as follows. Inside the product \( Y \times_M Y \times Y \times_M Y \) we can consider all the pairs of the form \((\alpha, \beta, \gamma, \delta)\). We shall denote this subset by \( Y \times_M Y \circ Y \times_M Y \). The bundle \( Q \) tensored with itself defines a line bundle over \( Y \times_M Y \times Y \times_M Y \) which we shall denote by \( Q \otimes Q \). Its restriction to \( Y \times_M Y \circ Y \times_M Y \) we shall denote by \( L \circ L \). The composition is a morphism of bundles \( L \circ L \) which we denote by \( v \otimes w \mapsto v \circ w \). We require that the composition is associative, that is \((v \circ w) \circ u = v \circ (w \circ u)\) whenever the triple product is defined. From the composition we can define an identity and inverse as follows.

The identity is a section denoted \( 1 \) of \( Q \) restricted to the diagonal \( \Delta_Y \) defined by \( v \circ 1 = v \). Notice that this is well-defined as if \( v \in Q_{pq} \) and \( e \in Q_{qq} \) then \( v \circ e \in Q_{pq} \) and hence \( v \circ e = \alpha v \) for some \( \alpha \) in \( U(1) \). Hence we define \( 1 = e/\alpha \). Of course we would also like that \( 1 \circ v = v \). It is clear that \( 1 \circ v = \beta v \) and hence using associativity on the product \( v \otimes v \) we deduce that \( \beta = 1 \).

The inverse is a bundle morphism \( Q \to Q^* \) covering the transposition map \((p, q) \to (q, p)\) on \( Y \times_M Y \). If we denote it by \( v \mapsto v^{-1} \) it is defined by \( v \circ v^{-1} = 1 \).

A simple example is to let \( Q \to Y \) be a principal \( \mathbb{C}^\times \) bundle. Define \( P_{(x,y)} = Aut_{\mathbb{C}^\times}(Q_y, Q_x) \). Then this defines a gerb called the trivial gerb. More generally, associated to every gerbe is a cohomology class in \( H^3(M, \mathbb{Z}) \). This class vanishes if the gerbe is trivial.

The example we are interested in as follows. Let

\[ 0 \to \mathbb{C}^\times \to \hat{G} \to G \to 0 \]

be a central extension of groups. Let \( Y \to M \) be a principal \( G \) bundle. Let \( p: \hat{G} \to G \) be the projection and define a gerbe over \( Y \times_M Y \) by

\[ \hat{Y}_{(x,y)} = \{ g \in \hat{G} \mid x = yp(g) \} \]

Assume that \( Y \) has a lift to a principal \( \hat{G} \) bundle \( Z \) over \( M \) so that there is a projection \( q: Z \to Y \) commuting with \( p \) in the appropriate way. Then \( Z \to Y \) is a \( \mathbb{C}^\times \) bundle over \( Y \). Let \( g \in \hat{Y}_{(x,y)} \) then it defines a map \( Z_y \to Z_x \) which, by centrality, commutes with the \( \mathbb{C}^\times \) action. This defines an isomorphism

\[ \hat{Y}_{(x,y)} \simeq Aut_{\mathbb{C}^\times}(Z_x, Z_y) \]

so that the gerbe \( \hat{Y} \) is trivial. Moreover if the gerbe \( \hat{Y} \) is trivial, say isomorphic to \( Aut(Z, Z) \) for some \( \mathbb{C}^\times \) bundle \( Z \to Y \) then we can define an action of \( \hat{G} \) on \( Z \) and
make it a lift of $Y$. To do this start with $g$ in $\hat{G}$ and the fibre $Z_y$. Then let $x = yp(g)$. Then $g \in \hat{Y}_{(x,y)} = \text{Aut}_{C^*}(Z_y, \hat{Z}_x)$ so apply the corresponding automorphism to any element in $\hat{Z}_y$ to define the action of $g$. It can be checked that this defines a lift of $Y$. In this case the gerbe class is the obstruction to the bundle lifting. The usefulness of gerbes for our purposes is that the give a differential form realisation of this class that can be used to investigate transgression.

To define this differential form we have to introduce the notion of a gerbe connection. This is a connection on $Q \to Y \times_M Y$ which respects the structure of the gerbe. This means that over the diagonal it is the flat connection, that the product map on the gerbe maps the product connection to itself and that the inversion map maps the connection to its dual. The curvature of the gerbe connection is a two form $F$ on $Y \times M$. It can be shown that the curvature can be written as $F = \pi_1^* f - \pi_2^* f$ for some two-form $f$ on $Y$ where $\pi_i$ denotes the projection onto the various factors. It can then be shown that $df$ is the pull-back of a three-form $\omega$ on $M$ which we shall call the gerbe curvature.

The relationship between our line bundles and the gerbes defined by Brylinski is as follows. For any open set $U \subset M$ we can consider the space of all sections $s$ of $Y$. To any two such sections $s$ and $t$ we have section over $U$ of $Y \times M$ defined by $m \mapsto (s(m), t(m))$. We can use this to pullback the line bundle $L$ to a line bundle on $U$. The space of all sections of this we call $\text{Mor}(s, t)$. There is an obvious composition of morphisms in $\text{Mor}(s, t)$ with morphisms in $\text{Mor}(t, r)$ to give elements of $\text{Mor}(s, r)$. We therefore have associated to $U$ a category, in fact a groupoid. This association is a presheaf of groupoids. If we sheafify it we obtain sheaf of groupoids which is a gerbe. More details can be found in the work of Murray.

4.2. Gerbes and co-cycles.

Let $P \to A$ be a fibration and $J \to P$ a gerbe. Assume that $G$ acts on $P$ and $J$ and preserves the gerbe structure. Choose a section $s: A \to P$. This is possible because $A$ is contractible. Let $\pi$ denote the projection $P \to A$ and define a map $\phi: P \to P \times P$ by $\phi(p) = (p, s(\pi(p)))$. Define a $U(1)$ bundle $Q$ on $P$ by $Q = \phi^{-1}(P)$. That is $Q_p = J_{(p, s(\pi(p)))}$. Notice then that

$$Q_p \otimes Q_q^* = P_{p,s(\pi(p))} \otimes P_{s(\pi(p))}^* = P_{p,q}$$

so that $Q$ trivialises the gerbe $P$. Consider now the automorphisms of $Q$ covering the action of $G$ on $P$ and commuting with the identification of $J$ and $Q \otimes Q^*$. That is if $\hat{g}$ is such an automorphism covering $g$ then we have that the diagram

$$
\begin{array}{c}
Q_p \otimes Q_q^* \downarrow & \rightarrow & J_{p,q} \\
\hat{g} \otimes \hat{g} \downarrow & \rightarrow & \hat{g} \\
Q_{gp} \otimes Q_{gq}^* \downarrow & \rightarrow & J_{gp,gq}
\end{array}
$$

commutes.
To see that there are any let $g$ be in $G$ and consider the bundle over $A$ whose fibre at some $A$ is $J_{gs(A),s(A)}$. Because $A$ is contractible we can find a section $\eta$ of this. Then we can define an automorphism of $Q$ covering $g$ by noticing that $g$ maps $Q_p$ to $J_{gp,gs\pi(p)}$ and that $J_{gp,gs\pi(p)} \otimes J_{gs(\pi(p)),gs\pi(p)} = Q_{gp}$. It is straightforward to check that this automorphism preserves the identification $P = Q \otimes Q^*$. Hence if we apply $g$ and then pair with $\eta$ we finish up in $Q_{gp}$. Notice that any two lifts of $g$ defined in this way differ by an element of $\text{Map}(A, R)$. Conversely any two automorphisms of $Q$ covering the action of some element of $G$ must differ by an automorphism fixing $P$. This is a map from $P$ into $U(1)$. Because of the commutativity of the diagram (4.1) we see that this defines a map from $Q_p \otimes Q^*$ to itself which must be the identity. Hence the map into $U(1)$ is constant on the fibres of $P \to A$ and defines an element of $\text{Map}(A, R)$. Let $\hat{G}$ be this group of automorphisms. We have proved that there is a central extension

$$U(1) \to \hat{G} \to G.$$  

We are interested in the co-cycle for the corresponding extension of Lie algebras. The Lie algebra of an automorphism is a vector field. To define explicitly a co-cycle we start we need a method of lifting vector fields from $P$ to $Q$. If this is denoted $X \mapsto \tilde{X}$ then the cocycle is

$$c(X,Y) = [\tilde{X}, \tilde{Y}] - \tilde{([X,Y])}.$$  

A method of defining such a lifting is to pick a connection $\nabla$ for $Q$. This needs to be a connection with the property that the induced connection on $J$ is $G$ invariant. We shall see that this is possible in a moment. Given the connection we can define the lift of a vector field by lifting it horizontally. Standard results then tell us that the cocycle is

$$c(X,Y) = F(X,Y)$$  

where $F$ is the curvature.

Consider now the quotient gerbe over $A/G$. Pick a gerbe connection $\nabla$, two-form $f$ and gerbe curvature $\omega$. Denote the corresponding pulled-back objects by $\hat{\nabla}$, $\hat{f}$ and $\hat{\omega}$. Notice, that by definition they are $G$ invariant. We can then construct a connection on $Q$ of the required type as $\phi^*(\hat{\nabla})$. The curvature of this is $\phi^* (\pi_1^* + \pi_2^*)(\hat{f})$. Hence the cocycle is

$$c(X,Y) = \hat{f}_{\pi_p}(X,Y) - \hat{f}_p(s_*\pi_*(X),s_*\pi_*(Y))$$

$$= \hat{f}_{\pi_p}(X,Y)$$  

as $\hat{f}$ is pulled-back under $G$ and $X$ and $Y$ are tangent to an orbit of $G$. But we also have $p^*(\omega) = ds^*\hat{f}$ so that the class defined by the transgression is

$$s^*\hat{f}(X,Y) = \hat{f}_{\pi_p}(X,Y)$$  

which is the same as the cocycle for the extension.
4.3. The Fock bundle and the Faddeev-Mickelsson anomaly

Let $M$ be odd dimensional so that the Dirac operator maps from spinors to spinors. Following consider the subset $A_0 \subset A \times \mathbb{R}$ defined as all pairs $(A, s)$ where $s$ is not in the spectrum of the Dirac operator coupled to $A$. Given such a pair $(A, s)$ we can decompose the Hilbert space of spinors into the direct sum of $H^+(A,s)$ and $H^-(A,s)$; the sums of the eigenspaces for eigenvalues greater and less than $s$ respectively. We can then form the Fock bundle

$$F(A,s) = \bigwedge (H^+(A,s)) \otimes \left( \bigwedge (H^-(A,s)) \right)^*.$$ 

If leave $A$ fixed and vary $s$ to some $t < s$. Then we have

$$H = (H^-(A,t)) \oplus V_{(A,t,s)} \oplus H^+(A,s)$$

where $V_{(A,t,s)}$ is the sum of all the eigenspaces for eigenvalues between $t$ and $s$. Moreover

$$H^+(A,t) = V_{(A,t,s)} \oplus H^+(A,s) \quad \text{and} \quad H^-(A,s) = H^-(A,t) \oplus V_{(A,t,s)}.$$ 

It follows that

$$F(A,s) = F(A,t) \otimes L_{(A,s,t)}$$

where $L_{(A,s,t)} = (\det V_{(A,t,s)})^2$. Hence the projective spaces $P(F(A,t))$ and $P(F(A,t))$ can be identified and descend to a projective bundle on $A$. Moreover the gauge group acts on the projective bundle and we are therefore in the setting of section 4.

5. 3-cocycles

The main examples of 3-cocycles which have occurred in the physics literature do not fit into transgression map framework that we have presented and they are not given by cocycles on the Lie algebra of the gauge group. Moreover they have been interpreted as the breakdown of the Jacobi identity. However algebras of operators on Hilbert spaces must always satisfy the Jacobi identity and so this forces the conclusion that when a 3-cocycle arises it must signal the absence of a representation by operators of the algebra or group in question. Some time ago one of us showed how one could understand this conclusion in terms of an ‘obstruction’ to representing a symmetry group of a quantum system on a Hilbert space (this was an extension of the traditional mathematical approach to the existence of group extensions). Recently a mathematical framework, originally devised to study group actions on $C^*$-algebras, was adapted to understand the examples of 3-cocycles arising in quantum field theory. This approach was applied in to study 3-cocycles in non-abelian gauge theories. It seems plausible that the geometric framework of this paper can be applied to these examples as well although we have not yet found a way to do so.
5.1. Summary of Jo’s Calculation

We begin by reminding the reader of the standard calculation in the physics literature following Jo. Jo considers chiral fermions in \((3 + 1)\) dimensions coupled to a Yang-Mills gauge field. He constructs an equal-time algebra of quantum fields starting with

\[
A = \sum_{i=1}^{3} \sum_a A^a_i(x)T^a dx^i,
\]

the Yang-Mills field “operator” at fixed time \(t = 0\). Here \(T^a\) are the generators of the Lie algebra \(g\) of the gauge group. Jo finds that the CCR become anomalous. Specifically he calculates commutation relations for \(A\) and its conjugate momentum \(E\) by a perturbation theory method (the BJL method). They are:

\[
\begin{align*}
[A^a_i(x), A^b_j(y)] &= 0 \\
[E^a_i(x), A^b_j(y)] &= -i\delta_{ij}\delta^{ab}\delta^3(x - y) \\
[E^a_i(x), E^b_j(y)] &= i\alpha \epsilon_{ijk} \text{tr}(T^a T^b + T^b T^a)A_k(x)\delta^3(x - y)
\end{align*}
\]

(here \(\epsilon_{ijk}\) is the antisymmetric tensor as usual, \(\text{tr}(\cdot)\) is the trace in \(g\), \(\alpha\) is a constant, \(A_k \equiv \sum_a A^a_k T^a\) and \(g\) is \(n\)-dimensional).

As it stands Jo’s calculation is not in the right form for our purposes. Let \(S\) denote the smooth functions of fast decrease on \(\mathbb{R}^3\) with values in \(\mathbb{R}^3 \times \mathbb{R}^n\), and denote the components of \(f \in S\) by \(f^a_i(x)\). Then

\[
A(f) := \sum_{i=1}^{3} \sum_{a=1}^{n} \int_{\mathbb{R}^3} A^a_i(x) f^a_i(x) d^3x
\]

and similarly we smear \(E(f)\) over \(S\). The commutation relations are:

\[
\begin{align*}
[A(f), A(g)] &= 0 \\
[E(f), A(g)] &= -i \sum_{j=1}^{3} \sum_{a=1}^{n} \int f^a_j(x) g^a_j(x) d^3x =: -i(f, g) \\
[E(f), E(g)] &= i\alpha A(f \star g)
\end{align*}
\]

where

\[
(f \star g)^{ab}_k(x) := \sum_{a,b} \epsilon^{ijk} f^a_i(x) g^b_j(x) \text{tr}[(T^a T^b + T^b T^a)T^c].
\]

The right-hand side of (5.2iii) is not a 2-cocycle, only a 2-cochain with values in the Lie algebra generated by the \(A(f)\). Now proceeding as in Jo with these smeared commutation relations, we find that the triple commutators produce a scalar-valued
three-cocycle:

\[ J(E(f), E(g), E(h)) = \text{Cycl.}[E(f), [E(g), E(h)]] \]
\[ = \text{Cycl.}[E(f), i\alpha A(g \ast h)] \]
\[ = \alpha \text{Cycl.}(f, g \ast h) \]
\[ = \alpha \text{Cycl.} \sum_{a,b,c} \int \epsilon^{ijk} g^a_i(x) h^b_j(x) f^c_k(x) \text{tr}[(T^a + \ldots T^b + T^c) T^d] d^3x \]
\[ = 3\alpha \sum_{a,b,c} \int \epsilon^{ijk} g^a_i(x) h^b_j(x) f^c_k(x) \text{tr}[(T^a T^b + \ldots T^c) T^d] d^3x \]  

where \( \text{Cycl.} \) denotes summation over cyclic permutations of \( f, g, h \).

At this point the view expounded in the literature is to regard the Jacobi identity as failing. So one has to conclude that the \( E(f) \)'s are not operators on a common dense invariant domain. In fact of course it is not clear where the contradiction lies since one assumes the \( E(f) \)'s are such operators in order to define the anomalous commutators (5.2) and so perhaps the BJL method is the real problem. There is a way of making sense of all this which is expounded in following the framework of and which we will now summarise.

### 5.2. 3-cocycles as obstructions

To explain this we need to identify an exact sequence of Lie algebras

\[ 0 \rightarrow \Delta \rightarrow \Gamma \rightarrow \Omega \rightarrow 0 \]  

with \( \Delta \) an Abelian Lie algebra and \( \Gamma \) an Abelian extension of \( \Delta \) by \( \Omega \).

To obtain this sequence, we will think of the “operators” \( E(f), A(g) \) as acting by derivations on the algebra \( \mathcal{A} \) generated by the space time smeared fields, in which case scalars are factored out of the commutation relations (5.2). (One would calculate this action by taking formal commutators of elements of \( \mathcal{A} \) with the \( E(f), A(g) \). This is not inconsistent because even though \( E(f) \) may not exist as an operator one only needs it to specify a quadratic form in order that the formal commutator with elements of \( \mathcal{A} \) should be defined.) Let \( \Delta \) be the Abelian Lie algebra, identical with the test function space \( \mathcal{S} \) as a linear space (its elements are thought of as the \( A(f) \)'s). Let \( \Omega \) be another copy of \( \mathcal{S} \) as an Abelian Lie algebra (but its elements are thought of as the \( E(f) \)'s acting by commutators so the right-hand side of (3.2iii) does not enter).

The action of \( \Omega \) on \( \Delta \) is taken to be trivial, (this is justified by (5.2ii), factoring out the constants). Then the map \( \sigma : \Omega^2 \rightarrow \Delta \) defined by \( \sigma(f, g) := \alpha f \ast g \) (using the identification of \( \Omega \) and \( \Delta \) with \( \mathcal{S} \)) is a two-cycle because

\[ \partial \sigma(f, g, h) = \text{Cycl.}(d_f(\sigma(g, h)) + \sigma(f, [g, h])) = 0 \]

for all \( f, g, h \in \Omega \). So we form the corresponding extension of Lie algebras:

\[ \Gamma = \Omega \oplus \Delta \text{ with bracket } [f \oplus g, h \oplus k] = 0 \oplus \sigma(f, h) \]  

(5.6)
Now that the exact sequence (5.3) is specified, and we identify a section \( \omega : \Omega \rightarrow \Gamma \) by \( \omega_f = f \oplus 0 \), so \([\omega_f, \omega_h] = 0 \oplus [f, g] \), and \( \omega_f \) corresponds to \( E(f) \), now satisfying (5.2ii).

We now aim to construct a second exact sequence

\[
0 \rightarrow \mathcal{W} \rightarrow \mathcal{V} \xrightarrow{ad|A} \Delta \rightarrow 0
\]

by assuming there is a representation \( v : \Delta \rightarrow \mathcal{V} \) by self-adjoint operators on a common dense invariant domain \( D \) in a Hilbert space \( \mathcal{H} \), on which the algebra \( \mathcal{A} \) is also irreducibly represented and such that \( v \) implements \( \Delta \) as derivations on \( \mathcal{A} \) (that is the action of \( d \in \Delta \) on \( \mathcal{A} \) is given by \( ad|\mathcal{A}(d)(A) = [v_d, A] \) for \( A \in \mathcal{A} \)). Let \( \mathcal{V} \) be the abelian Lie algebra given by the linear span of the operators \( \{R^d, v_d \mid d \in \Delta \} \). Then \( \mathcal{W} = \mathcal{R}^1 \) and \( ad|\mathcal{A}(\mathcal{V}) \cong \Delta \).

The framework in requires us to construct an intermediate invariant whose geometric significance is completely opaque. This invariant is constructed from a map \( \lambda : \Gamma \times \Delta \rightarrow \mathcal{R} \) determined by an action \( \delta : \Gamma \rightarrow \text{Der} \mathcal{V} \). The relation (5.2ii) which is suggested by the canonical commutation relations is what we use to define \( \lambda \). Replace \( A(f) \) with \( v_f, f \in \Delta = \mathcal{S} \) and suppose for the moment that there are operators \( u_f, f \in \Omega = \mathcal{S} \) preserving \( D \) (these would be the \( E(f) \)'s if they existed) satisfying the usual commutation relations \([u_f, v_h] = i(f, h)\). Suppose further that these commutation relations provide the action \( \delta : \Omega \rightarrow \text{Der} \mathcal{V} \) by \( d_f(v_h) = (-i)[u_f, v_h] \). So \( \delta : \Gamma \rightarrow \text{Der} \mathcal{V} \) will be

\[
\delta_{f \oplus g}(v_h) = \delta_{f \oplus 0}(v_h) + \delta_{0 \oplus g}(v_h) = \delta_{f \oplus 0}(v_h) = (f, h)
\]

This tells us to define \( \lambda : \Gamma \times \Delta \rightarrow \mathcal{R} \) by using

\[
\lambda(f \oplus g, h) = \delta_{f \oplus g}(v_h) - g|_{f \oplus g, 0 \oplus h}
\]

\[
= \delta_{f \oplus g}(v_h)
\]

\[
= (f, h). \tag{5.7}
\]

Now this method of arriving at (5.7) is not an argument because the \( u_f \)'s need not exist however (5.7) is a good definition provided

\[
\lambda([f \oplus g, h \oplus k], m) = \lambda(f \oplus g, [h \oplus k, 0 \oplus m]) - \lambda(h \oplus k, [f \oplus g, 0 \oplus m])
\]

which is easy to check.

Hence the \( \lambda \) given by (5.7) satisfies the requirements of \( \) and hence defines a 3-cocycle \( K : \Omega^3 \rightarrow \mathcal{R} \):

\[
K(f, g, h) = \lambda(f \oplus 0, \sigma(g, h)) + \lambda(g \oplus 0, \sigma(h, f)) + \lambda(h \oplus 0, \sigma(f, g))
\]

\[
= (f, \sigma(g, h)) + (g, \sigma(h, f)) + (h, \sigma(f, g))
\]

\[
= \alpha \text{Cycl.}(f, g \star h) \tag{5.8}
\]

which is exactly the same 3-cocycle as the one obtained in Jo's calculation. In it is shown that the Lie algebra \( \Gamma \) cannot be represented in such a way that the action \( \delta \) associated to \( \lambda \) can be implemented because \( K \) determines a non-trivial
cohomology class. (The cohomology group with complex coefficients in degree \( n \) of an abelian Lie algebra is given by the space of totally skew \( n \)-multilinear maps).

From the viewpoint of quantum field theory this is further evidence that the canonical equal-time formalism is probably not appropriate in 3+1 dimensions: a fact which is not all that surprising.

The relationship with our earlier discussion arises by letting \( \mathcal{A} \) be the connections on a trivial bundle over \( \mathbb{R}^3 \) (i.e at fixed time \( t = 0 \)) and \( \text{Map}(\mathcal{A}, \mathbb{C}) \) the Abelian Lie algebra of continuously differentiable mappings \( \mathcal{A} \to \mathbb{C} \) (so that \( \delta d/\delta A(f) \) makes sense \( \forall d \in \text{Map}(\mathcal{A}, \mathbb{C}) \)). Then we identify \( \Delta \) as a subgroup of \( \text{Map}(\mathcal{A}, \mathbb{C})/\mathbb{C} \) by regarding each \( f \in \Delta \) as defining a function \( A(f) \) on \( \mathcal{A} \) via (5.1). However it is not clear what the geometric meaning of this 3-cocycle is. There are generalisations of categories to a theory of 2-categories, where one has objects, morphisms between pairs of objects and something else between any triangle of morphisms. Perhaps the correct geometric object is a sheaf of 2-categories?

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References

1. Singer, I.M. Some remarks on the Gribov ambiguity. Commun. Math. Phys. 60, (1978), 7–12.
2. Freed, D.S. On determinant line bundles. In Mathematical Aspects of String Theory, S.T. Yau ed. World Scientific, Singapore, 1987.
3. Brylinski, J-L. Loop spaces, characteristic classes and geometric quantisation., Birkhäuser, Boston, 1993.
4. Murray, M.K. Some differential geometry related to gerbes. In preparation. 1994.
5. Carey, A.L., The origin of 3-cocycles in quantum field theory. Phys. Lett. 194B (1987) 267-272.
6. Maclane, S., Homology. Grundlehren Mathematischen Wissenschaften 114, Springer-Verlag, Berlin, Gottingen, Heidelberg 1963.
7. Carey, A. L. Grundling, H. Raeburn, I. Sutherland, C, Group actions on \( C^* \)-algebras, 3-cocycles and quantum field theory (preprint).
8. Carey, A.L., Grundling, H., Hurst, C. A. and Langmann, E. Realising 3-cocycles as obstructions. In preparation.
9. Jo, S-G, Commutators in an anomolous non-abelian chiral gauge theory, Phys Lett 163B (1985) 353-359.
10. Jackiw, R., Three-cocycle in mathematics and physics, Phys.Rev.Lett. 54, 1985, 159-16.; Jackiw, R. Topological investigations of quantised gauge theories, edited by Trieman, S.B. et al World Scientific, Singapore 1985
11. Segal, G. B., Faddeev’s anomaly in Gauss’ law, unpublished preprint.