Dissipative dynamics of the Josephson effect in the binary Bose-condensed mixtures

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The dissipative dynamics of a pointlike Josephson junction in binary Bose-condensed mixtures is analyzed within the framework of the model of a tunneling Hamiltonian. The transmission of unlike particles across a junction is described by the different transmission amplitudes. The effective action that describes the dynamics of the phase differences across the junction for each of two condensed components is derived employing the functional integration method. In the low-frequency limit the dynamics of a Josephson junction can be described by two coupled equations in terms of the potential energy and dissipative Rayleigh function using a mechanical analogy. The interplay between mass currents of each mixture component appears in the second-order term in the tunneling amplitudes due to inter-species hybridizing interaction. The asymmetric case of the binary mixtures with the different concentration and order parameters is considered as well.

PACS numbers: 03.75.Lm, 67.85.Fg, 74.50+r

INTRODUCTION

Recently, experimental study of multicomponent Bose-Einstein condensates has made a substantial progress. The study of multiple atomic condensates is intriguing since they can produce a laboratory mixture of distinguishable boson superfluids at sufficiently low temperatures. A considerable amount of theoretical work has been devoted to binary Bose-condensed mixtures, focusing, mainly, on the mean-field description of trapped binary mixtures [1], stability and phase separation [2–4], collective excitations [5–7], condensate depletions [8], and quantum merging of two different condensates [9–11].

In this connection we will consider here the dissipative and interference aspects of the Josephson effect in the binary Bose-condensed mixtures. The Josephson effect, first predicted and discovered for two superconductors separated with a thin insulator layer, is a macroscopic quantum phenomenon in a condensed medium. The dynamics of the effect is described in terms of the difference between the phases of the superconductors, playing the role of a macroscopic quantum variable. In spite of wide application of the effect in devices for extremely high-sensitivity measurements of currents, voltages, and magnetic fields the Josephson effect is still of interest in the fundamental modern physics. Like superconductors and Fermi superfluids, the Josephson effect is also inherent in Bose superfluids [12–14]. The effect has been observed by the mass flow of superfluid $^4$He through nanoscale apertures coupling two bulk superfluid reservoirs. The Josephson tunneling junction in ultracold dilute atomic gases is formed with a laser separating two Bose-Einstein condensates [15–16].

In general, the Josephson effect can include both the so-called internal effect for the atoms in different hyperfine states much as NMR phenomena in $^3$He [17] and the conventional case of two Bose condensates separated with a potential barrier which acts as a tunneling junction. A lot of work [18–32] has been done in the latter case as a direct analogy with conventional superconductors. Those studies dealt with the one-component Bose-Einstein condensates alone.

Multicomponent Bose-Einstein condensates are also a very interesting subject for studying various macroscopic tunneling phenomena [33–37]. One may expect novel and richer manifestations of the Josephson effect. For the system of two Bose-condensed mixtures connected with a weakly coupled junction, the dynamics of the Josephson effect should be governed by the difference between the phases for each Bose-condensed component of a mixture. In other words, two relative phases $\phi_1$ and $\phi_2$ must be involved into consideration. In addition, we must take into account the different tunneling transition amplitudes $I_1$ and $I_2$ across the junction for various bosonic atoms of masses $m_1$ and $m_2$ composing the mixture. This results in two Josephson currents associated with the mass flow of each component of a mixture across the junction. From the general point of view one may expect interference and coupling between the Josephson currents. The dynamics of the Josephson effect in the vicinity of the phase separation of a mixture is an additional motivation for studying binary condensates. Furthermore, the dissipative aspects of the Josephson dynamics in binary condensed mixtures have not yet received a proper and wide investigation.

The dissipative effects and dephasing of the Josephson oscillations come from the coupling between the macroscopic relative phase variable and the infinite number of the microscopic degrees of freedom. The successive method of eliminating microscopic degrees of freedom from the Hamiltonian was developed first for the superconducting Josephson systems [38–40]. Later that functional integration approach [27, 28] and the Keldysh Green function method [30] were extended and applied to studying dissipative and nonequilibrium Josephson dynamics in the one-component Bose-condensed systems.

In this paper, we will generalize the energy dissipation effects in the Josephson dynamics to the case of the...
binary Bose-condensed systems, employing functional integration approach of Ref. [28]. We will derive the expression for the effective action depending on two relative phases $\varphi_1$ and $\varphi_2$ between two condensed mixtures connected by a pointlike tunneling junction. The response functions in the effective action give the full information on the dynamics of the junction. The low-frequency expansion of the response functions allows us to determine two coupled Josephson equations for the relative phases $\varphi_1$ and $\varphi_2$, Josephson energy $U(\varphi_1, \varphi_2)$ and dissipative Rayleigh function $R(\varphi_1, \varphi_2)$. Of course, we consider the region of the parameters in which the homogeneous state of the both mixtures in the left-hand and right-hand regions of the parameters in which the homogeneous state of the both mixtures in the left-hand and right-hand bulks is stable and the mixtures are not phase-separated.

**EFFECTIVE ACTION**

We keep in mind the case of a pointlike and weakly coupled junction between two macroscopic infinite reservoirs containing binary condensed mixtures. In addition, we neglect the feedback effect of the junction on the mixtures and assume that both the mixtures are always in the thermal equilibrium state. The image of the system is two bulks with one common point through which the transmission of particles is possible with the different tunneling amplitudes $I_1$ and $I_2$ depending on the type of particles.

So, our starting point is the so-called tunneling Hamiltonian ($\hbar = 1$, volume $V = 1$)

$$H = H_l + H_r + H_U + H_t,$$

where $H_{l,r}$ describes the bulk binary Bose-condensed mixture on the left-hand and right-hand sides, respectively,

$$H_l = \sum_{i=1,2} \int d^3r \, \Psi_{k,i,l}^\dagger \left( -\frac{\nabla^2}{2m_i} - \mu_i \right) \Psi_{i,l} + \frac{1}{2} \sum_{i,k=1,2} u_{ik,l} \int d^3r \, \Psi_{k,i,l}^\dagger \, \Psi_{k,i,l} \, \Psi_{k,i,l}.$$

Here $i$ and $k$ take 1 or 2 and denote one of the components of a mixture composed with particles of mass $m_1$ and $m_2$. The coupling between particles is specified by the constants $u_{ik,l}$ which can be expressed by means of the $s$-scattering length $a_{ik,l}$ according to $u_{ik,l} = 2\pi a_{ik,l} (m_i^{-1} + m_k^{-1})$. The same expressions with the substitution $l \rightarrow r$ refer to the mixture on the right-hand side.

The energy, associated with varying the number of particles on the left-hand and right-hand sides,

$$H_U = \frac{1}{2} \sum_{i,k=1,2} \frac{N_{i,l} - N_{i,r}}{2} U_{ik} \frac{N_{k,l} - N_{k,r}}{2},$$

is analogous to the capacity energy of a junction in the case of superconductors. The constants $U_{ik}$ can be connected with the second derivatives of the total energy

$$E = E([N_{1,l}, N_{2,l}, (N_{1,r}, N_{2,r}]) = E_l(N_{1,l}, N_{2,l}) + E_r(N_{1,r}, N_{2,r})$$

with respect to the relative change in the number of the particles across the junction,

$$U_{ik} = \frac{\partial^2 E}{\partial N_{i,l} \partial N_{k,l}} + \frac{\partial^2 E}{\partial N_{i,r} \partial N_{k,r}} = \frac{\partial^2 E}{\partial N_{i,l} \partial N_{k,l}} + \frac{\partial^2 E}{\partial N_{i,r} \partial N_{k,r}},$$

with the obvious symmetrical relation $U_{ik} = U_{ki}$. The constants $U_{ik}$ can usually be estimated also as

$$U_{ik} = \frac{\partial \mu_{i,l}}{\partial N_{k,l}} + \frac{\partial \mu_{i,r}}{\partial N_{k,r}}.$$

The term $H_U$ describes the point that the energy of the system on the whole may depend on the relative numbers of particles from the left-hand and right-hand bulks. In order to avoid an instability of the total system against an infinite growth of the number of particles on the left-hand or right-hand sides, it is necessary to suppose that $H_U > 0$ for any variations $N_{1,l} - N_{1,r}$ and $N_{2,l} - N_{2,r}$. In other words, the matrix of coefficients $U_{ik}$ should be positively determined, i.e., $\|U_{ik}\| > 0$. The total number of the particles of the type labelled by $i$ in each bulk $l$ or $r$ is given by

$$N_{i,l} = \int d^3r \, \Psi_{i,l}^\dagger \Psi_{i,l} \text{ and } (l \rightarrow r).$$

The last term,

$$H_t = -\int_{r \in l, r' \in r'} d^3r \, d^3r' \left[ \Psi_{1,l}^\dagger(r) I_1(r, r') \Psi_{1,r}(r') + \Psi_{2,l}^\dagger(r) I_2(r, r') \Psi_{2,r}(r') \right] + \text{H.c.},$$

is responsible for the transitions of particles from the right-hand to the left-hand bulk and *vice versa*. In general, the transition amplitudes $I_1$ and $I_2$ are different for the various species of the particles composing the mixture. We consider here the simplest case of a pointlike junction, i.e.,

$$I_i(r, r') = I_i \delta(r) \delta(r'), \quad i = 1, 2.$$

To study the properties of the system described by Eq. (1), we calculate the partition function $Z$ using the analogy of the superconducting junction [33, 40] and the approaches employed for the Bose junction [27, 28],

$$Z = \int D^2 \Psi_{1,l} D^2 \Psi_{2,l} D^2 \Psi_{1,r} D^2 \Psi_{2,r} \exp[-SE],$$
where the action $S_E$, defined on the imaginary (Matsubara) time $\tau$, reads

$$S_E = \int_{-\beta/2}^{\beta/2} d\tau L_E,$$

$$L_E = \int d^3r \sum_{i,k=1,2} \left( \Psi_{i,l}^{\dagger} \frac{\partial}{\partial \tau} \Psi_{i,l} + \Psi_{i,r}^{\dagger} \frac{\partial}{\partial \tau} \Psi_{i,r} \right) + H$$

and, as usual, $\beta = 1/T$ is an inverse temperature.

To eliminate the quartic terms in the action, which come from the energy $H_U$, we employ the Hubbard-Stratonovich procedure by introducing additional gauge fields $V_1(\tau)$ and $V_2(\tau)$ in analogy with the so-called plasmon gauge field in metals,

$$\exp \left( -\int d\tau H_U \right) = \int DV_1(\tau) DV_2(\tau) \exp \left( -\int d\tau \left( \sum_{i,k=1,2} \frac{V_{U_{ik}}}{2} + i \sum_{i=1,2} N_{i,1} - N_{i,2} \right) \right);$$

$$\int DV_1(\tau) DV_2(\tau) \exp \left[ -\int d\tau \sum_{i,k=1,2} \frac{V_{U_{ik}}}{2} \right] = 1.$$ 

Here $\hat{U}_{ik}$ is an inverse matrix for the $2 \times 2$ matrix $U_{ik}$

$$\hat{U}_{ik}^{-1} = \frac{1}{U_{11}U_{22} - U_{12}U_{21}} \begin{pmatrix} U_{22} & -U_{12} \\ -U_{21} & U_{11} \end{pmatrix}.$$ 

After the introduction of the fields $V_1(\tau)$ and $V_2(\tau)$ the partition function $Z$ takes the form

$$Z = \int DV_1 DV_2 \prod_{i=1,2} D^2 \Psi_{i,l} D^2 \Psi_{i,r} \exp \left[ -\hat{S}_E \right],$$

where

$$\hat{S}_E = \hat{S}_l + \hat{S}_r + \int d\tau H_t + \frac{1}{2} \sum_{i,k=1,2} \int d\tau V_i \hat{U}_{ik}^{-1} V_k.$$ 

Here $\hat{S}_l$ and $\hat{S}_r$ denote

$$\hat{S}_l = S_l + i \sum_{i=1,2} \int d\tau d^3r \Psi_{i,l}^{\dagger} V_1(\tau) \Psi_{i,l},$$

$$\hat{S}_r = S_r - i \sum_{i=1,2} \int d\tau d^3r \Psi_{i,r}^{\dagger} V_1(\tau) \Psi_{i,r}.$$ 

In essence, this replacement looks like the renormalization of the chemical potentials for each component of a mixture on the left and right-hand sides of the junction,

$$\mu_{i,l} \rightarrow \mu_{i,l} - iV_1(\tau)/2,$$

$$\mu_{i,r} \rightarrow \mu_{i,r} + iV_1(\tau)/2.$$ 

At this point it is advantageous to perform a gauge transformation of the field operators $\Psi^{\dagger}$ and $\Psi$, which makes the future Green functions real. This is achieved by introducing the phases $\varphi_{i,l}$ and $\varphi_{i,r}$ according to

$$\Psi_{i,l}(i,r) \rightarrow \exp[i\varphi_{i,l}(i,r)] \Psi_{i,l}(i,r),$$

$$\Psi_{i,r}(i,l) \rightarrow \exp[-i\varphi_{i,r}(i,r)] \Psi_{i,r}(i,l),$$

and by imposing the conditions $\varphi_{i,l} = -V_1(\tau)/2$ and $\varphi_{i,r} = V_1(\tau)/2$. Thus, we arrive at the first Josephson relations for the relative phases $\varphi_1(\tau)$ and $\varphi_2(\tau)$:

$$\dot{\varphi}_i(\tau) = V_1(\tau) \quad \text{and} \quad \varphi_i = \varphi_{i,r} - \varphi_{i,l} \quad (i = 1,2). \quad (2)$$

That we have achieved is only a formal elimination of the explicit dependence of the chemical potentials $\mu_{i,l}$ and $\mu_{i,r}$ upon the time $\tau$. On the other hand, the tunneling amplitudes $I_1$ and $I_2$ acquire additional factors depending on the phase differences $\varphi_1(\tau)$ and $\varphi_2(\tau)$ across the junction,

$$I_1 \rightarrow \tilde{I}_1 = I_1 e^{i\varphi_1(\tau)} \quad \text{and} \quad I_2 \rightarrow \tilde{I}_2 = I_2 e^{i\varphi_2(\tau)}.$$ 

Hence we arrive at

$$Z = \int DV_1 DV_2 \prod_{i=1,2} D^2 \Psi_{i,l} D^2 \Psi_{i,r} e^{-S},$$

$$S = S_0 + \int d\tau \tilde{H}_t + \int d\tau \sum_{i,k=1,2} \frac{V_i \hat{U}_{ik}^{-1}}{2} V_k,$$

where $S_0 = S_l + S_r$. The tunneling term $\tilde{H}_t$ is given by

$$\tilde{H}_t = -\int d^3r d^3r' \sum_{i=1,2} \left[ \Psi_{i,l}^{\dagger}(i,r',\tau) \Psi_{i,r} + \text{H.c.} \right].$$ 

Next, one must integrate over fields $\Psi^{\dagger}$ and $\Psi$ in order to obtain the effective action $S_{\text{eff}}$ depending on $V_1(\tau)$ and $V_2(\tau)$ alone. In calculations we treat $\tilde{H}_t$ as a perturbation and restrict ourselves by second-order perturbation in the tunneling amplitudes $I_1$ and $I_2$. Omitting the term independent of $V_1(\tau)$ and employing the Josephson relations $V_i(\tau) = \varphi_i(\tau)$, we find the effective action as

$$S_{\text{eff}}[\varphi_1,\varphi_2] = \int d\tau \left[ \sum_{i,k=1,2} \varphi_i \hat{U}_{ik}^{-1} \frac{\dot{\varphi}_k}{2} + \langle \tilde{H}_t \rangle - \frac{\langle \tilde{H}_t^2 \rangle}{2} \right].$$ 

Here $\langle A \rangle_0$ means the averaging over decoupled action $S_0 = S_l + S_r$ corresponding to $H_0 = H_l + H_r$, i.e.,

$$\langle \tilde{H}_t \rangle = \frac{\langle \tilde{H}_t e^{-S_0} \rangle}{e^{-S_0}} \quad \text{and} \quad \langle \tilde{H}_t^2 \rangle_0 = \langle \tilde{H}_t^2 \rangle - \langle \tilde{H}_t \rangle^2.$$ 

It is obvious that

$$\langle \tilde{H}_t^2 \rangle_0 = \langle \tilde{H}_t^2 \rangle_0 + 2\langle \tilde{H}_t \tilde{H}_2 \rangle_0 + \langle \tilde{H}_2 \rangle^2.$$ 

The first-order terms in the tunneling transparency are obviously decoupled. Since $H_1H_2 \sim I_1I_2$, second-order
two mixtures, using the Bogoliubov nomenclature

\[ \Psi_{i,(i,r)} = C_{i,(i,r)} + \Phi_{i,(i,r)}, \]

with the conventional relation \( C_{i,(i,r)} = \sqrt{n_{i,(i,r)}} \)

where \( n_{i,(i,r)} \) is the density of particles labelled with \( i = 1, 2 \) in the condensate fraction in the left-hand and right-hand bulks, respectively.

In a binary Bose-condensed mixture the Green function \( \hat{G}(\omega_n, \mathbf{p}) \) represents a block \( 4 \times 4 \) matrix. The Green function can readily be found from the inverse matrix whose Fourier representation in the approximation of a weakly interacting two-component Bose-condensed gas mixture is given by

\[
\hat{G}^{-1}(\omega_n, \mathbf{p}) = \begin{pmatrix}
\hat{G}_{11}^{-1} & \hat{G}_{12}^{-1} \\
\hat{G}_{12}^{-1} & \hat{G}_{22}^{-1}
\end{pmatrix} = \begin{pmatrix}
-i\omega_n + \eta_1 + \Delta_{11} & \Delta_{12} \\
\Delta_{11} & -i\omega_n + \eta_1 + \Delta_{11}
\end{pmatrix}^{-1} \begin{pmatrix}
\Delta_{11} & \Delta_{12} \\
\Delta_{12} & \Delta_{22}
\end{pmatrix}.
\]

Here \( \omega_n = 2\pi nT \) is the Matsubara frequency and \( \mathbf{p} \) is the momentum. Also we have introduced the following notations for the free-particle energies

\[ \eta_1 = \eta_1(\mathbf{p}) = \mathbf{p}^2 / 2m_1, \quad \eta_2 = \eta_2(\mathbf{p}) = \mathbf{p}^2 / 2m_2; \]

and for the order parameters

\[ \Delta_{11} = u_{11}n_1, \quad \Delta_{22} = u_{22}n_2, \quad \Delta_{12} = u_{12}\sqrt{n_1n_2}. \]

Accordingly, for the direct matrix Green function

\[
\hat{G}(\omega_n, \mathbf{p}) = \begin{pmatrix}
\hat{G}_{11} & \hat{G}_{12} \\
\hat{G}_{21} & \hat{G}_{22}
\end{pmatrix} = \begin{pmatrix}
G_{11} & F_{11} & G_{12} & F_{12} \\
G_{21} & F_{21} & G_{22} & F_{22}
\end{pmatrix},
\]

we arrive at the following components of the matrix:

\[
G_{11}(\omega_n, \mathbf{p}) = G_{11}(\omega_n, \mathbf{p}) = \frac{(i\omega_n + \eta_1 + \Delta_{11})(\omega_1^2 + \epsilon_1^2) - 2\Delta_{12}^2\eta_2}{(\omega_n^2 + \omega_1^2)(\omega_n^2 + \omega_2^2)}
\]

\[
F_{11}(\omega_n, \mathbf{p}) = F_{11}(\omega_n, \mathbf{p}) = -\frac{\Delta_{11}(\omega_n^2 + \epsilon_2^2) + 2\Delta_{12}^2\eta_2}{(\omega_n^2 + \omega_1^2)(\omega_n^2 + \omega_2^2)}
\]

\[
G_{12}(\omega_n, \mathbf{p}) = G_{12}(\omega_n, \mathbf{p}) = -\frac{\Delta_{12}(i\omega_n + \eta_1)(i\omega_n + \eta_2)}{(\omega_n^2 + \omega_1^2)(\omega_n^2 + \omega_2^2)}
\]

\[
F_{12}(\omega_n, \mathbf{p}) = F_{12}(\omega_n, \mathbf{p}) = -\frac{\Delta_{12}(i\omega_n + \eta_1)(-i\omega_n + \eta_2)}{(\omega_n^2 + \omega_1^2)(\omega_n^2 + \omega_2^2)}
\]

The lower part of the Green function matrix is determined by permutation \( 1 \leftrightarrow 2 \) and \( \Delta_{21} = \Delta_{12} \). We have introduced above the similar abbreviations for either of two mixtures, using the Bogoliubov nomenclature

\[
\epsilon_1^2 = \eta_1^2 + 2\Delta_{11}\eta_1 \quad \text{and} \quad \epsilon_2^2 = \eta_2^2 + 2\Delta_{22}\eta_2
\]

where \( \eta_{1,2} = \mathbf{p}^2 / 2m_{1,2} \) is the free-particle energy. The energies \( \epsilon_1 \) and \( \epsilon_2 \) are the Bogoliubov energies of each component of a mixture taken separately. In the mixture the interspecies interaction \( n_{12} \) hybridizes these two modes, resulting in two familiar branches of elementary excitations, e.g., \( \omega_{1,2}^2 \)

\[
\omega_{1,2}^2(p) = \frac{1}{2} \left( \epsilon_1^2 + \epsilon_2^2 \pm \sqrt{(\epsilon_1^2 - \epsilon_2^2)^2 + 16\Delta_{12}^2\eta_1\eta_2} \right)
\]

with the crossover to the sound-like dispersion \( \omega_{1,2} = p c_{1,2} \) at small \( p \to 0 \) momentum. The sound velocities \( c_{1,2} \) in a mixture are determined by the well-known relations as well \( \frac{3}{8} \)

\[
c_{1,2}^2 = \frac{1}{2} \left( \frac{\Delta_{11}}{m_1} + \frac{\Delta_{22}}{m_2} \pm \sqrt{\left( \frac{\Delta_{11}}{m_1} - \frac{\Delta_{22}}{m_2} \right)^2 + 4\Delta_{12}^2 \frac{m_1 m_2}{m_1 m_2}} \right).
\]
The inequalities \( u_{11}u_{22} > u_{12}^2 \) and \( c_{1,2}^2 > 0 \) are certainly supposed to guarantee the stability of a mixture against its demixing.

Taking into account (2), we arrive finally at the following generalization of the effective action compared with that in the one-component condensed system [28]:

\[
\mathcal{S}_{\text{eff}}[\varphi_1(\tau), \varphi_2(\tau)] = \int d\tau \left[ \sum_{i,k=1,2} \frac{\delta_{ik}}{2} \dot{\varphi}_k(\tau) - 2\dot{I}_1 \sqrt{n_{11}n_{r1}} \cos \varphi_1(\tau) - 2\dot{I}_2 \sqrt{n_{22}n_{r2}} \cos \varphi_2(\tau) \right] - \sum_{i,k=1,2} I_i I_k \int d\tau d\tau' \left[ \alpha_{ik}(\tau - \tau') \cos \left( \varphi_i(\tau) - \varphi_k(\tau') \right) + \beta_{ik}(\tau - \tau') \cos \left( \varphi_i(\tau) + \varphi_k(\tau') \right) \right].
\]

(3)

Here \( \alpha_{ik} \) and \( \beta_{ik} \) are the so-called response functions which can be written using the Green functions:

\[
\alpha_{ik}(\tau) = \sqrt{n_{11}n_{r1}} g_{ik, i}(\tau) + \sqrt{n_{11}n_{r2}} g_{ik, r}(\tau) + G_{ik}(\tau), \quad \beta_{ik}(\tau) = \sqrt{n_{11}n_{r1}} f_{ik, i}(\tau) + \sqrt{n_{11}n_{r2}} f_{ik, r}(\tau) + F_{ik}(\tau).
\]

The Green functions are calculated at the junction point, i.e., at \( \tau = 0 \) and \( \tau' = 0 \):

\[
g_{ik, i}(\tau) = \frac{1}{2} \left[ 2 \pi \sin(\pi \tau) \right]_\tau \left[ G_{ik, i}(\tau, \tau) + G_{ik, r}(\tau, \tau) \right] + \int \frac{d^3p}{2(2\pi)^3} \left[ G_{ik, i}(\tau, \tau) + G_{ik, r}(\tau, \tau) \right],
\]

\[
f_{ik, i}(\tau) = \frac{1}{2} \left[ 2 \pi \sin(\pi \tau) \right]_\tau \left[ F_{ik, i}(\tau, \tau) + F_{ik, r}(\tau, \tau) \right] + \int \frac{d^3p}{2(2\pi)^3} \left[ F_{ik, i}(\tau, \tau) + F_{ik, r}(\tau, \tau) \right].
\]

In order to comprehend the dynamics of the relative phase differences \( \varphi_1 \) and \( \varphi_2 \) across the junction, we should analyze the behavior of the response functions \( \alpha_{ik}(\tau) \) and \( \beta_{ik}(\tau) \) as a function of time. Note that the contribution of the terms \( G_{ik} \) and \( F_{ik} \) to the response functions \( \alpha_{ik} \) and \( \beta_{ik} \) is much smaller than that of the first two others [28]. The order-of-magnitude smallness is about a ratio of the noncondensate density to the condensate density or about gas parameter \((na)^{3/2} \ll 1\). Below, analyzing \( \alpha_{ik} \) and \( \beta_{ik} \), we will concentrate our attention on the first two terms which can be attributed to the condensate-noncondensate tunneling processes.

THE RESPONSE FUNCTIONS

The calculation of the response functions in the general form in a mixture is a complicated problem. Keeping in mind the study of the low-frequency dynamics of a junction, we will restrict our calculation by analyzing the behavior of the response functions on the long-time scale. This means that we should find the low-frequency decomposition of the response functions in the Matsubara frequencies \( \omega_n \). In fact, we imply the inequality \( |\omega_n| \ll \omega_1, \omega_2 \). Next, we will employ the procedure of analytical continuation in order to derive the dynamic Josephson equations which the relative phases \( \varphi_1 \) and \( \varphi_2 \) obey. To obtain the dissipative terms, it is sufficient to expand the response functions \( \alpha_{ik} \) and \( \beta_{ik} \) up to linear terms in frequency \( \omega_n \).

So, we look for the following first coefficients in the low-frequency decomposition

\[
\alpha_{ik}(\omega_n) = -\alpha_{ik}^{(0)}(\omega_n) + \cdots \quad \beta_{ik}(\omega_n) = -\beta_{ik}^{(0)}(\omega_n) + \cdots
\]

Accordingly, the expressions for the response functions in the imaginary-time representation read as

\[
\alpha_{ik}(\tau) = -\alpha_{ik}^{(0)}(\delta(\tau) + \alpha_{ik}^{(1)}(1) \frac{\pi T}{\sin(\pi T \tau)} + \cdots \quad \beta_{ik}(\tau) = -\beta_{ik}^{(0)}(\delta(\tau) + \beta_{ik}^{(1)}(1) \frac{\pi T}{\sin(\pi T \tau)} + \cdots
\]

First of all, we should note that zero harmonic \( \alpha_{i0} = 0 \) in the \( \alpha_{ik} \) response is unimportant if \( i = k \), and thus we can deal with the difference \( \hat{\alpha}_{ii}(\omega_n) = \alpha_{ii}(\omega_n) - \alpha_{i0}(\omega_n) \). In fact, this corresponds to the substitution \( \alpha_{ii}(\tau) = \hat{\alpha}_{ii}(\tau) + \alpha_{i0}(0) \delta(\tau) \) into effective action [38]. The second term \( \alpha_{i0}(0) \delta(\tau) \) yields a physically unimportant time- and phase-independent contribution to the action, meaning a shift of the ground-state energy of a junction. For \( i \neq k \), this does not hold for. As we will see below, \( \alpha_{ij}^{(0)} \) and \( \beta_{ij}^{(0)} \) are connected with the Josephson potential energy and \( \alpha_{ij}^{(1)} \), \( \beta_{ij}^{(1)} \) determine the dissipative properties of the junction.

Let us start from \( \beta_{ij}^{(0)} \). For the sake of brevity, we present here the expressions for \( \alpha_{ik} \) and \( \beta_{ik} \) in the case of a symmetric junction with the identical mixtures on the both left-hand and right-hand sides of the junction when \( \Delta_{ik} = \Delta_{ik} = \Delta_1 \) and \( n_{ik} = n_{ik} = n_{1} \). The general case \( l \neq r \) will be given in the Appendix. The simple calculation yields

\[
\beta_{11}^{(0)} = -2n_1 f_{11}^+(\omega_n = 0) = \frac{n_1}{\pi} \left( m_1^2 (\Delta_{11} + m_2 c_1 c_2)^{1/2} \right)
\]

\[
\beta_{12}^{(0)} = \alpha_{12}^{(0)} = -2n_1 f_{12}^+(\omega = 0) = \frac{\sqrt{m_1 m_2}}{\pi} \left( m_1 (\Delta_{11} + 2m_2 c_2 (\Delta_{12} + m_2 c_2))^{1/2} \right)
\]

and the other quantities can be obtained with \( 1 \equiv 2 \).

The calculation of \( \beta_{ik}^{(1)} \) and \( \alpha_{ik}^{(1)} \) is more complicated:

\[
\alpha_{11}^{(1)} = \beta_{11}^{(1)} = \frac{m_1}{4\pi c_1 c_2} \frac{c_1 c_2 + \Delta_{12} m_2}{c_1 + c_2}
\]

\[
\alpha_{12}^{(1)} = \beta_{12}^{(1)} = 2\sqrt{m_1} \frac{1}{4\pi c_1 c_2} \frac{\Delta_{12}}{c_1 + c_2}
\]
Again all the remaining quantities are given by $1 \Rightarrow 2$. Note only that all $\beta^{(0)}_{ik}$ and $\alpha^{(0)}_{\neq ik}$ remain finite and non-singular at the demixing point $\Delta_{11} \Delta_{22} = \Delta_{12}^2$ or when one of the sound velocities vanishes $c_2 = 0$. On the contrary, both $\alpha^{(1)}_{ik}$ and $\beta^{(1)}_{ik}$ diverge with approaching at demixing point as $c_2 \to 0$.

**JOSEPHSON EQUATIONS**

To obtain the dynamics of the relative phases $\varphi_1$ and $\varphi_2$ in real time, we now follow the standard procedure of analytical continuation. Accordingly, the substitution of Matsubara frequencies $\omega_n \to -i\omega$ in the Fourier transform of the Euler-Lagrange equations $\partial S_{eff}/\partial \varphi_i(\tau) = 0$, we finally arrive at the form of the Euler-Lagrange equations

\[
- \sum_{k=1,2} \tilde{U}^{-1}_{ik} \hat{\varphi}_k(t) + 2I_i n_i \sin \varphi_i(t) + 2I_i \sum_{k=1,2} \sum_{k=1,2} 2I_i I_k \times \\
\left[ \alpha^{(0)}_{ik} \sin \left( \varphi_i(t) - \varphi_k(t) \right) + \beta^{(0)}_{ik} \sin \left( \varphi_i(t) + \varphi_k(t) \right) \right] \\
+ \sum_{k=1,2} 2I_i I_k \hat{\varphi}_k(t) \times \\
\left[ \beta^{(1)}_{ik} \cos \left( \varphi_i(t) - \varphi_k(t) \right) + \alpha^{(1)}_{ik} \cos \left( \varphi_i(t) + \varphi_k(t) \right) \right] = 0.
\]

Using relations from (4) and (5), we finally arrive at the following decomposition

\[
\varphi_i(t) = -\delta \varphi_i(t) + \sum_{k=1,2} 2I_i I_k \times \\
\left[ \alpha^{(0)}_{ik} \sin \left( \varphi_i(t) - \varphi_k(t) \right) + \beta^{(0)}_{ik} \sin \left( \varphi_i(t) + \varphi_k(t) \right) \right] \\
+ \sum_{k=1,2} 2I_i I_k \hat{\varphi}_k(t) \times \\
\left[ \beta^{(1)}_{ik} \cos \left( \varphi_i(t) - \varphi_k(t) \right) + \alpha^{(1)}_{ik} \cos \left( \varphi_i(t) + \varphi_k(t) \right) \right] = 0.
\]

The dissipative energy effect can be described by introducing the dissipative Rayleigh function according to

\[
R(\dot{\varphi}_1, \dot{\varphi}_2) = \frac{1}{2} \sum_{i,k=1,2} r_{ik} (\dot{\varphi}_1, \dot{\varphi}_2) \dot{\varphi}_i \dot{\varphi}_k = \frac{1}{2} \sum_{i,k=1,2} r_{ik} \cos \varphi_i \cos \varphi_k \dot{\varphi}_i \dot{\varphi}_k, \text{ and } r_{ik} = 4I_i I_k \alpha^{(1)}_{ik}.
\]

Finally, the Josephson equations can be written in the general form as

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_i} - \frac{\partial L}{\partial \varphi_i} = -\frac{\partial R}{\partial \dot{\varphi}_i}, \quad i = 1, 2,
\]

where in accordance with Eq. (2),

\[
\dot{\varphi}_i(t) = -\delta \mu_i(t) = \mu_{i,t} - \mu_{i,r}.
\]

The dissipative function $R$ has a sense of the energy dissipation power in the system. This is obvious from the following equation,

\[
\frac{dH}{dt} = \frac{d}{dt} \left( \sum_i \varphi_i \frac{\partial L}{\partial \dot{\varphi}_i} - L \right) = -\sum_i \dot{\varphi}_i \frac{\partial R}{\partial \dot{\varphi}_i} = -2R,
\]

which means that the energy dissipation power equals the double dissipative function. Since the energy dissipation must result in decreasing the total energy $H = K + U$ of the system, the dissipative function $R$ must be a positively determined matrix $R > 0$, i.e.,

\[
r_{11} > 0 \quad \text{and} \quad r_{11} r_{22} > r_{12} r_{21}.
\]

We are persuaded that this is true from

\[
\frac{r_{11} r_{22}}{r_{12} r_{21}} - 1 = \frac{\alpha_{11}^{(1)} \alpha_{22}^{(1)}}{\alpha_{21}^{(1)} \alpha_{12}^{(1)}} - 1 = \frac{m_1 m_2 c_1 c_2 (c_1 + c_2)^2}{\Delta_{12}^2} > 0.
\]

Note that the condition $R > 0$ gets broken simultaneously with the condition $c_{1,2} > 0$ necessary for the stability of a mixture against its demixing. In addition, we also disclose a symmetry of kinetic dissipative coefficients $r_{ik} = r_{ki}$ in accordance with the Onsager principle.
It is interesting that the point of demixing instability \( c_1 c_2 = 0 \) is not singular for the potential energy coefficients \( \varepsilon_{ik} \). On the contrary, the dissipative coefficients \( r_{ik} \) become infinite. The latter means that the Josephson dynamics should slow down and demonstrate an enhancement of decoherence and damping of the Josephson oscillations in the vicinity of the phase demixing. From the mechanical point of view the low-frequency dynamics of a Josephson junction in a Bose-condensed mixture can be described as a system of two coupled particles or pendula moving or oscillating in a viscous medium in a periodic potential relief.

In the lack of hybridization between the different atom species, i.e., when \( \Delta_{12} = 0 \), the crossed terms in the response functions \( \alpha_{i\neq k} \) and \( \beta_{i\neq k} \) vanish as well. As both \( \alpha_{i\neq k} = 0 \) and \( \beta_{i\neq k} = 0 \), the Josephson equations split into two decoupled equations for each component of a mixture. In this case no interference in the mass currents appears and the response functions together with nonzero diagonal coefficients \( \alpha_{ii}, \beta_{ii} \) go over to the quantities corresponding to the case of a single-component Bose-condensed gas \(^{27, 28}\).

CONCLUSION

To summarize, in this paper we have used a functional integration method for the model of a tunneling Hamiltonian in order to analyze the energy dissipation effects in the dynamics of a pointlike Josephson junction between two weakly nonideal Bose-condensed gas mixtures in the thermal equilibrium. The transmission of particles of each component of a mixture across the junction is described by two different tunneling amplitudes. The effective action and response functions that describe the dynamics of two relative phases \( \varphi_1 \) and \( \varphi_2 \) corresponding to each condensed component of a mixture are found. The quasiclassical Josephson equations for the relative phases are derived from the low-frequency decomposition of the response functions.

The dynamics of a pointlike junction in a mixture displays a dissipative Ohmic nature. The energy dissipation effects result from the noncondensate excitations and appear in second order in the tunneling amplitudes. The latter fact favors low damping rates of the Josephson oscillations in the pointlike junctions. A growth of the temperature leads to decreasing the Josephson energy and to increasing the energy dissipation power. The closeness to the phase separation of a mixture enhances the Ohmic character of the phase dynamics. The dissipative Rayleigh function is determined.

On the whole, the Josephson phase dynamics in binary mixtures is described by two coupled equations. This means, in particular, an existence of two Josephson frequencies for small oscillations of the phases around \( \varphi_1 = 0 \) and \( \varphi_2 = 0 \). Since \( r_{ik} \neq 0 \), the oscillations are weakly damped. Emphasize that the interference between the Josephson and dissipative Ohmic components of a mass current for each type of particles starts only from the second-order terms in the tunneling amplitudes and, eventually, due to the presence of the noncondensate fractions. The interference entails, in particular, that the maximum amplitude of the Josephson current of one species atoms depends on the relative phase difference of the second component of a mixture. In addition, it becomes possible that the imbalance in the chemical potential of one component of a mixture can induce also the Ohmic contribution into the mass current of the other component. We believe these aspects deserve a further study.

ACKNOWLEDGMENTS

This study is supported in part by the RFBR grant No. 10-02.00047a.

APPENDIX

Here we present the general asymmetric case of a junction if \( n_{i l} \neq n_{i r} \) and \( \Delta_{ik,l} \neq \Delta_{ik,r} \). For the Josephson potential energy

\[
U(\varphi_1, \varphi_2) = - \sum_{i=1,2} E_i \cos \varphi_i + \frac{1}{2} \sum_{i=1,2} \varepsilon_{ik} \cos \varphi_i \cos \varphi_k,
\]

we find

\[
E_i = 2I_i \sqrt{n_{i l} n_{i r}} \quad (i = 1, 2)
\]

and the next terms

\[
\varepsilon_{11} = 2I_1^2 m_1^2 \pi \left( n_{1l}(\Delta_{11,l} + 2m_1 c_{1r} c_{2r}) \right) \left( n_{1r}(\Delta_{11,l} + 2m_1 c_{1l} c_{2r}) \right) \left( n_{2r}(\Delta_{22,r} + m_1 c_{1l} c_{2r}) \right)
\]

\[
\varepsilon_{22} = 2I_2^2 m_2^2 \pi \left( n_{2l}(\Delta_{22,l} + 2m_1 c_{1l} c_{2r}) \right) \left( n_{2r}(\Delta_{22,l} + 2m_1 c_{1r} c_{2r}) \right) \left( n_{2l}(\Delta_{22,l} + m_1 c_{1l} c_{2r}) \right)
\]

The other two nondiagonal coefficients \( \varepsilon_{12} = \varepsilon_{21} \) are given by the expression

\[
\varepsilon_{12} = 2I_1 I_2 \pi \left[ \sqrt{n_{1l} n_{2l}} \Delta_{12,r} c_{1r} c_{2r} + \sqrt{n_{1r} n_{2r}} \Delta_{12,l} c_{1l} c_{2l} \right].
\]

For the kinetic coefficients in the dissipative Rayleigh function

\[
R(\dot{\varphi}_1, \dot{\varphi}_2) = \frac{1}{2} \sum_{i=1,2} r_{ik} \cos \varphi_i \cos \varphi_k \dot{\varphi}_1 \dot{\varphi}_2,
\]
we have

\[
  r_{11} = \frac{I_1^2 m_1}{\pi} \left[ \frac{n_{1l} c_{1r} c_{2r} + \Delta_{22,r}/m_2}{c_{1r} + c_{2r}} + \frac{n_{1r}}{c_{1l}} c_{1l} c_{2l} + \frac{\Delta_{22,l}/m_2}{c_{1l} + c_{2l}} \right],
\]

\[
  r_{22} = \frac{I_2^2 m_2}{\pi} \left[ \frac{n_{2l} c_{1r} c_{2r} + \Delta_{11,r}/m_1}{c_{1r} + c_{2r}} + \frac{n_{2r}}{c_{1l}} c_{1l} c_{2l} + \frac{\Delta_{11,l}/m_1}{c_{1l} + c_{2l}} \right].
\]

The other two nondiagonal coefficients \( r_{12} = r_{21} \) can be found from the expression

\[
  r_{12} = \frac{I_1 I_2}{\pi} \left[ \frac{n_{1l} n_{2l}}{c_{1r} c_{2r}} \frac{\Delta_{12,r}}{c_{1l} + c_{2l}} + \frac{n_{1r} n_{2r}}{c_{1l} c_{2l}} \frac{\Delta_{12,l}}{c_{1l} + c_{2l}} \right].
\]

The expressions derived above for the dynamical coefficients in the Josephson equations governing the phase differences \( \varphi_1(t) \) and \( \varphi_2(t) \) across a pointlike junction allow us to describe the low-frequency dynamics in the asymmetric case of Bose-condensed gas mixtures with the different densities of the atom species and with the different order parameters. The stability of mixtures against phase separation and the positive definiteness of the dissipative Rayleigh function imply the fulfillment of conditions \( c_{1l}, c_{2l} > 0 \) and \( c_{1r}, c_{2r} > 0 \).

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