ON THE STABILITY OF THE $L_p$-CURVATURE

MOHAMMAD N. IVAKI

Abstract. It is known that the $L_p$-curvature of a smooth, strictly convex body in $\mathbb{R}^n$ is constant only for origin-centred balls when $1 \neq p > -n$, and only for balls when $p = 1$. If $p = -n$, then the $L_{-n}$-curvature is constant only for origin-symmetric ellipsoids. We prove ‘local’ and ‘global’ stability versions of these results. For $p \geq 1$, we prove a global stability result: if the $L_p$-curvature is almost a constant, then the volume symmetric difference of $\tilde{K}$ and a translate of the unit ball $B$ is almost zero. Here $\tilde{K}$ is the dilation of $K$ with the same volume as the unit ball. For $0 \leq p < 1$, we prove a similar result in the class of origin-symmetric bodies in the $L^2$-distance. In addition, for $-n < p < 0$, we prove a local stability result: There is a neighborhood of the unit ball that any smooth, strictly convex body in this neighborhood with ‘almost’ constant $L_p$-curvature is ‘almost’ the unit ball. For $p = -n$, we prove a global stability result in $\mathbb{R}^2$ and a local stability result for $n > 2$ in the Banach-Mazur distance.

1. Introduction

A compact convex subset of $\mathbb{R}^n$, $n$-dimensional Euclidean space, with non-empty interior is called a convex body. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$ and those with the origin contained in the interior are denoted by $\mathcal{K}^n_0$. We write $\mathcal{F}^n$, and $\mathcal{F}^n_0$ for the set of smooth, strictly convex bodies in $\mathcal{K}^n$ and $\mathcal{K}^n_0$ respectively. Moreover, $B$ and $S^{n-1}$ denote the unit ball and the unit sphere of $\mathbb{R}^n$ respectively. $\tilde{K}$ denotes the dilation of $K$ whose $n$-dimensional Lebesgue measure, $V(\tilde{K})$, equals to that of the unit ball; $V(\tilde{K}) = V(B) = \kappa_n$.

The support function of a convex body $K$ is defined by

$$h_K(u) := \max_{x \in K} x \cdot u, \quad \forall u \in S^{n-1}.$$ 

Let $K \in \mathcal{F}^n_0$ and $\nu_K : \partial K \to S^{n-1}$ be the Gauss map which takes $x$ on the boundary of $K$ to its unique outer unit normal vector, and let $\nu_{\tilde{K}}^{-1} : S^{n-1} \to \mathbb{R}^n$ be the Gauss parameterization of $\partial \tilde{K}$. In this case, we have

$$h_K(u) = u \cdot \nu_{\tilde{K}}^{-1}(u).$$
We write $g, \nabla$ for the standard round metric and the corresponding Levi-Civita connection of the unit sphere. The Gauss curvature of $\partial K$, $K$, and the curvature function of $\partial K$, $f_K$ (as a function on the unit sphere), are related to the support function of the convex body by

$$f_K = \frac{1}{K_K \circ \nu_K^{-1}} = \frac{\det(\nabla^2_{i,j} h_K + g_{ij} h_K)}{\det(g_{ij})}.$$  

The function $h_K^{1-p} f_K$ is called the $L_p$-curvature function of $K$.

For $K \in \mathcal{F}_0^n$ we define the scale invariant quantity

$$R_p(K) = \max_{S^{n-1}} (h_K^{1-p} f_K) / \min_{S^{n-1}} (h_K^{1-p} f_K).$$

We prove stability versions of the following theorem, which is due to a collective work of Firey, Lutwak, Andrews, Brendle, Choi, and Daskalopoulos [Fir74, Lut93, And99a, BCD17]:

**Theorem.** Let $p \in (-n, \infty)$, $p \neq 1$. If $K \in \mathcal{F}_0^n$ satisfies

$$h_K^{1-p} f_K \equiv 1,$$

then $K$ is the unit ball.

**Question 1.** Is there an increasing function $f$ with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ with the following property? If $K \in \mathcal{F}_0^n$ satisfies $R_p(K) \leq 1 + \varepsilon$, then $K$ is $f(\varepsilon)$-close to a ball in a suitable sense?

For $p \geq 1$ due to the $L_p$-Minkowski inequality and a refinement of Urysohn’s inequality from [Seg12] we can obtain a global stability result in a very strong sense. The relative asymmetry of two convex bodies $K, L$ is defined as

$$\mathcal{A}(K, L) := \inf_{x \in \mathbb{R}^n} \frac{V(K \Delta (\lambda L + x))}{V(K)}, \quad \text{where } \lambda^n = \frac{V(K)}{V(L)}$$

and $K \Delta L = (K \setminus L) \cup (L \setminus K)$.

**Theorem 1.1.** Let $p \geq 1$. There exists a constant $C$ independent of dimension with the following property. Any $K \in \mathcal{F}_0^n$ satisfies

$$\mathcal{A}(K, B) \leq C n^{2.5} \left( R_p(K)^{\frac{1}{p}} - 1 \right)^{\frac{1}{2}}.$$

Interestingly, the $L_{p+1}$-Minkowski inequality also allows us to prove the global stability for $0 \leq p < 1$ in the class of origin-symmetric bodies in the $L^2$-distance. The $L^2$-distance of $K, L$ is defined by

$$\delta_2(K, L) = \left( \frac{1}{\omega_n} \int |h_K - h_L|^2 d\sigma \right)^{\frac{1}{2}}.$$
Here $\sigma$ is the spherical Lebesgue measure on $S^{n-1}$, and $\omega_i$ is the surface area of the $i$-dimensional ball.

**Theorem 1.2.** Let $0 \leq p < 1$ and $K \in \mathcal{F}^n$ be origin-symmetric. There exists an origin-centred ball $B_r$ with radius $1 \leq r \leq R_p(K)$, such that

$$
\delta_2(\tilde{K}, B_r) \leq D(\tilde{K}) \left(1 - R_p(K)^{-1}\right)^{\frac{1}{2}}.
$$

Here the diameter of $\tilde{K}$, $D(\tilde{K})$, satisfies the inequality

$$
D(\tilde{K}) \leq 2 \left(\left(1 + \left(\frac{4\omega_{n-1}}{\omega_n}\right)^{\frac{1}{2}}\right) R_p(K)\right)^{\frac{3}{2}}.
$$

For $p \in (-n, 0)$, we also establish a local stability result. The points $e_p$ will be defined in Definition 2.1.

**Theorem 1.3.** Let $p \in (-n, 0)$. There exist positive constants $\gamma$, $\delta$, depending only on $n$, $p$ with the following property. If $K \in \mathcal{F}^n_0$ with $e_p(K) = 0$ satisfies $|h_{\lambda K} - 1|_{C^3} \leq \delta$ for some $\lambda > 0$, then

$$
\delta_2(\tilde{K}, B) \leq \gamma (R_p(K) - 1).
$$

**Remark 1.4.** For the case $p = 0$, some progress recently has been made on the stability of the cone-volume measure by Böröczky and De in [BD21] based on the logarithmic Minkowski inequality in the class of convex bodies with many symmetries proved by Böröczky and Kalantzopoulos in [BK20]. Although our proof of Theorem 1.2 is independent of the existence of $L_p$-Minkowski inequality for $0 \leq p < 1$, it is worth pointing out that such an inequality exists in some particular cases:

- $p \in [0, 1)$ and in the class of origin-symmetric convex bodies in the plane, or in any dimension and in the class of origin-symmetric bodies for $p \in (p_0, 1)$ where $p_0 > 0$ is some constant depending on $n$; see [CYLL20, KM17, Mil21a, Mil21b].

Let $K \in \mathcal{F}^n_0$. The centro-affine curvature of $K$, $H_K$, is defined by

$$
H_K := (h_K^{n+1} f_K)^{-1}.
$$

It is known that $H_K(u)$ is (up to a constant) just a power of the volume of the origin-centred ellipsoid touching $K$ at $\nu_K^{-1}(u)$ of second-order (osculating ellipsoid), and thus is an $SL(n)$ covariant notion. In particular, one of the key properties of the centro-affine curvature is that $\min H_K$ and $\max H_K$ are invariant under special linear transformation $SL(n)$. That is,

$$
\min_{S^{n-1}} H_K = \min_{S^{n-1}} H_{\ell K}, \quad \max_{S^{n-1}} H_K = \max_{S^{n-1}} H_{\ell K}, \quad \forall \ell \in SL(n).
$$

(1.1)
A remarkable theorem of Pogorelov states that if the centro-affine curvature of a smooth, strictly convex body is constant, then the body is an origin-centred ellipsoid; cf. [Gut12, Thm. 10.5.1], [Cal58, CY86, MdP14]. It is of great interest to find a stability version of this statement, for example, in the Banach-Mazur distance $d_{BM}$. For two convex bodies $K, L$, $d_{BM}(K, L)$ is defined by

$$\min\{\lambda \geq 1 : (K-x) \subseteq \ell(L-y) \subseteq \lambda(K-x), \ell \in GL(n), x, y \in \mathbb{R}^n\}.$$  

**Question 2.** Is there an increasing function $f$ with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ with the following property? If $K \in \mathcal{F}_0$ satisfies

$$R_{-n}(K) = \frac{\max H_K}{\min H_K} \leq 1 + \varepsilon,$$

then $K$ is $f(\varepsilon)$-close to an ellipsoid in the Banach-Mazur distance.

The following theorem gives a positive answer to this question in the plane under no additional assumption.  

**Theorem 1.5.** There exist $\gamma, \delta > 0$ with the following property. If $K \in \mathcal{F}_0$ satisfies $R_{-2}(K) \leq 1 + \delta$, then we have

$$(d_{BM}(K, B) - 1)^4 \leq \gamma(R_{-2}(K) - 1).$$

If $K$ has its Santaló point at the origin, then

$$(d_{BM}(K, B) - 1)^4 \leq \gamma(\sqrt{R_{-2}(K)} - 1).$$

Moreover, if $K$ is origin-symmetric, then

$$d_{BM}(K, B) \leq \sqrt{R_{-2}(K)}.$$  

In this case, we may allow $\delta = \infty$.

In higher dimensions, we have the following 'local' stability result.

**Theorem 1.6.** There exist positive numbers $\gamma, \delta$, depending only on $n$ with the following property. Suppose $K \in \mathcal{F}_0$ has its Santaló point at the origin, and for some $\ell \in GL(n)$ we have $|h_{\ell K} - 1|_{C^3} \leq \delta$. Then

$$d_{BM}(K, B) \leq \gamma(R_{-n}(K) - 1)^{\frac{4}{n(n+1)}} + 1.$$  

2. BACKGROUND

A convex body $K$ is said to be of class $C^2_+$, if its boundary hypersurface is two-times continuously differentiable and the support function is differentiable.

Let $K, L$ be two convex bodies with the origin of $\mathbb{R}^n$ in their interiors. In the following, we put $a \cdot K := a^2 K$ and $b \cdot L := b^2 L$ where $a, b > 0$.  

For $p \geq 1$, the $L_p$-linear combination $a \cdot K + p b \cdot L$ is defined as the convex body whose support function is given by $(ah_K^p + bh_L^p)^{1/p}$.

For $K, L \in K^n_0$, the mixed $L_p$-volume $V_p(K, L)$ is defined as the first variation of the usual volume with respect to the $L_p$-sum:

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon \cdot L) - V(K)}{\varepsilon}.$$ 

Aleksandrov, Fenchel and Jessen for $p = 1$ and Lutwak [Lut93] for $p > 1$ have shown that there exists a unique Borel measure $S_p(K, \cdot)$ on $S^{n-1}$, $L_p$-surface area measure, such that

$$V_p(K, L) = \frac{1}{n} \int h_K^p(u) dS_p(K, u).$$

Moreover, $S_p(K, \cdot)$ is absolutely continuous with respect to the surface area measure of $K$, $S(K, \cdot)$, and has the Radon–Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h_K^{1-p}(\cdot).$$

The measure $dS_{p,K} = h_K^{1-p} dS_K$ is known as the $L_p$-surface area measure. If the boundary of $K$ is $C^2$, then

$$\frac{dS_K}{d\sigma} = \frac{1}{\mathcal{K}_K \circ \nu_K^{-1}} = f_K.$$

For $p > 1$, the $L_p$-Minkowski inequality states that for convex bodies $K, L$ with the origin in their interiors we have

$$\frac{1}{n} \int h_K^p dS_p(K) \geq V(K)^{1-\frac{2}{p}} V(L)^{\frac{2}{p}},$$

with equality holds if and only if $K$ and $L$ are dilates (i.e. for some $\lambda > 0$, $K = \lambda L$); see [Lut93]. For $p = 1$, the same inequality holds for all $K, L \in \mathcal{K}^n$, and equality holds if and only if $K$ is homothetic to $L$.

The polar body, $K^*$, of $K \in \mathcal{K}^n_0$ is the convex body defined by

$$K^* = \{ y \in \mathbb{R}^n : x \cdot y \leq 1, \ \forall x \in K \}.$$ 

All geometric quantities associated with the polar body are furnished by $\ast$. For $x \in \text{int } K$, let $K^x := (K - x)^\ast$. The Santaló point of $K$, denoted by $s = s(K)$, is the unique point in $\text{int } K$ such that

$$V(K^s) \leq V(K^x) \ \forall x \in \text{int } K.$$ 

If $K = -K$, then $s(K) = 0$ and $K^* = K^s$.

The Blaschke-Santaló inequality states that

$$V(K^s) V(K) \leq V(B)^2,$$

and equality holds if and only if $K$ is an ellipsoid.
Definition 2.1. The $L_p$-widths of $K \in \mathcal{K}^n$ are defined as follows.

(1) For $p > 1$ : $E_p(K) = \frac{1}{\omega_n} \inf_{x \in \text{int}K} \int h^p_{K-x} d\sigma$.

(2) For $p = 0$ : $E_0(K) = \frac{1}{\omega_n} \sup_{x \in \text{int}K} \int \log h^0_{K-x} d\sigma$.

(3) For $0 < p < 1$ : $E_p(K) = \frac{1}{\omega_n} \sup_{x \in \text{int}K} \int h^p_{K-x} d\sigma$.

(4) For $-n \leq p < 0$ : $E_p(K) = \frac{1}{\omega_n} \inf_{x \in \text{int}K} \int h^p_{K-x} d\sigma$.

Here $\omega_n = n \kappa_n = \int d\sigma$.

Here and in the sequel, $e_p$ denotes the unique point at which the corresponding sup or inf is attained. The points $e_p$ are always in the interior of the convex body; see e.g. [Iva16, Lem. 3.1]. If $K$ is origin-symmetric, then $e_p(K)$ lies at the origin.

For $p \geq 1$ by the $L_p$-Minkowski inequality we have

\[ E_p(\tilde{K}) \geq 1. \]  

(2.1)

For $p \in (-n, 0]$ by the Blaschke-Santaló inequality,

\[ E_0(K) \geq 0, \quad E_p(K) \leq 1, \]

(2.2)

and equality holds when $K$ is a ball. Moreover, for $p \in (0, 1)$ we have

\[ E_p(\tilde{K})E_{-p}(\tilde{K}) = \frac{1}{\omega_n^2} \int h^p_{K-e_p(\tilde{K})} d\sigma \int h^{-p}_{K-e_{-p}(\tilde{K})} d\sigma \geq \frac{1}{\omega_n^2} \int h^p_{K-e_p(\tilde{K})} d\sigma \int h^{-p}_{K-e_{-p}(\tilde{K})} d\sigma \geq 1, \]

where we used the definition of $e_p$ in the last line. Therefore we obtain

\[ E_p(\tilde{K}) \geq 1, \]

(2.3)

and the equality holds only for balls.

We conclude this section by remarking that $E_p$ enjoys the second Łojasiewicz-Simon gradient inequality; see [Sim83, Sim96] for further details.

3. Stability of the width functionals

In this section we prove the stability of the inequalities (2.1) and (2.2) ($p \neq 0$).

Lemma 3.1. Suppose $p \in [-n, 0)$. Let $K \in \mathcal{K}^n$ with $V(K) = V(B)$. Then

\[ |e_p(K) - s(K)|^2 \leq c_0 (1 - E_p(K)) D(K)^{2-p}, \]

where $c_0^{-1} := \frac{v(p-1)}{2\omega_n} \int (u \cdot v)^2 d\sigma(u) = \frac{v(p-1)}{2n} \Omega^{n-1}$ for any vector $v$, and $D(K)$ denotes the diameter of $K$. 


Proof. We may suppose \( e_p(K) \neq s(K) \). Define \( v = -\frac{e_p(K) - s(K)}{|e_p(K) - s(K)|} \) and 
\[ e(t) = e_p(K) + tv, \quad t \in [0, |e_p(K) - s(K)|]. \]
Let us denote the support function of \( K - e(t) \) by \( h_t \) and 
\[ E(t) := \frac{1}{\omega_n} \int h_t^p d\sigma. \]
Note that \( E(0) = E_p(K) \) and \( E'(0) = 0 \). Moreover, the second 
derivative of \( E \) is given by 
\[ E''(t) = \frac{p(p-1)}{\omega_n} \int h_t^{p-2} (u \cdot v)^2 d\sigma(u). \]
Due to \( h_t \leq D(K) \) we obtain 
\[ D(K)^{p-2} |e_p(K) - s(K)|^2 \leq c_0 \left( \frac{1}{\omega_n} \int h_t^{p} d\sigma - E_p(K) \right). \]
Now the claim follows from the Blaschke-Santaló inequality. \( \square \)

**Theorem 3.2.** The following statements hold.

1. Let \( p \geq 1 \). If \( E_p(K) \leq 1 + \varepsilon \), then 
\[ \mathcal{A}(\tilde{K}, B)^2 \leq C n^5 \left( (1 + \varepsilon)^{\frac{1}{p}} - 1 \right). \]
Here \( C \) is a universal constant that does not depend on \( n \).

2. Let \( p \in (-n, 0) \). If \( E_p(K) \geq 1 - \varepsilon \), then there exists an origin-
centred ball of radius \( r \), \( B_r \), such that 
\[ \delta_2(\tilde{K} - e_p(K), B_r) \leq 2c_1 \left( D(\tilde{K}) + r \right)^{n+1} \varepsilon^{\frac{1}{p}} + \left( c_0 D(\tilde{K})^{2-p} \varepsilon \right)^{\frac{1}{2}}. \]
Moreover, if \( \tilde{K} \) is origin-symmetric, then the last term on the right-
hand-side can be dropped and \( D(\tilde{K}) \) can be replaced by \( \frac{1}{2} D(\tilde{K}) \). Here 
\[ 1 \leq r \leq (1 - \varepsilon)^{\frac{1}{p}}, \quad c_1 := \max \left\{ \frac{n}{p + n}, -\frac{n}{p} \right\}, \]
and \( c_0 \) is the constant from Lemma 3.1.

Proof. Case \( p \geq 1 \): Since \( E_p(\tilde{K}) \leq 1 + \varepsilon \), we have 
\[ \frac{1}{\omega_n} \int h_{\tilde{K}} d\sigma \leq E_p(\tilde{K})^{\frac{1}{p}} \leq (1 + \varepsilon)^{\frac{1}{p}}. \]
The refinement of Urysohn’s inequality in [Seg12] completes the proof.

Case \(-n < p < 0\) : Assume \( V(\tilde{K}) = V(B) \). We denote the support 
function of \( \tilde{K} - e_p(K) \) by \( h_p \) and the support function of \( \tilde{K} - s(K) \) 
by \( h_s \). Since \( s(K) \), \( e_p(K) \) are in the interior of \( K \), both \( h_s \) and \( h_p \) are 
positive functions.
Let us put
\[ f = h^p, \quad g = 1, \quad a = -\frac{n}{p}, \quad b = \frac{n}{n+p}, \quad c_1 = \max\{a, b\}. \]

By [Ald08, Thm. 2.2], we have
\[
\left( \int \frac{h^p d\sigma}{(\int h^n d\sigma)^{-\frac{n}{n+p}} \omega_n^\frac{n}{n+p}} \right)^{\frac{p}{n}} \leq 1 - \frac{1}{c_1} \left| \frac{h_s^{\frac{p}{2}}}{(\int \frac{1}{h_s^n} d\sigma)^{\frac{1}{p}}} - \frac{1}{\omega_n^\frac{n}{p}} \right|_{L^2}^2.
\]

Due to our assumption,
\[
\int h^p d\sigma \geq \int h^n d\sigma \geq \omega_n (1 - \varepsilon).
\]

By the Blaschke-Santaló inequality, we have
\[
\int \frac{1}{h^n_s} d\sigma \leq \omega_n.
\]

From (3.2), (3.3), it follows that
\[
1 - \varepsilon \leq \frac{\int h^n d\sigma}{(\int h^n d\sigma)^{-\frac{n}{n+p}} \omega_n^\frac{n}{n+p}},
\]
\[
(1 - \varepsilon) \omega_n \leq \int h^p d\sigma \leq \left( \int \frac{1}{h_s^n} d\sigma \right)^{-\frac{p}{n}} \omega_n^{\frac{n+p}{n}}.
\]

Combining (3.1) and (3.4) we obtain
\[
\left| h_s^{\frac{n}{2}} - r^{\frac{n}{2}} \right|_{L^2}^2 \leq c_1 \omega_n D(K)^n \varepsilon,
\]

where
\[
r^n := \omega_n \left( \int \frac{1}{h_s^n} d\sigma \right)^{-1}, \quad 1 \leq r \leq (1 - \varepsilon)^{\frac{1}{n}}.
\]

In view of (3.5) and (3.6) we have
\[
\left| h_s - r \right|_{L^2}^2 \leq c_1 \omega_n (D(K)^{\frac{1}{n}} + r^{\frac{1}{n}})^2 D(K)^n \varepsilon.
\]

If $K$ is origin-symmetric, then $s(K) = e_p(K)$ and the proof is complete. Moreover, in this case we could have replaced $D(K)$ by $\frac{1}{2} D(K)$. Otherwise, to bound $|h_p - r|_{L^2}$, note that by Lemma 3.1 we have
\[
|e_p(K) - s(K)|^2 \leq c_0 D(K)^{2-p} \varepsilon.
\]
Therefore,

$$|h_p - r|_{L^2} \leq |h_s - r|_{L^2} + \omega_n^\frac{1}{2}|c_p(K) - s(K)|$$

$$\leq (2c_1\omega_n (D(K) + r)^{n+1}\varepsilon)^\frac{1}{2} + (c_0\omega_n D(K)^{2-p}\varepsilon)^\frac{1}{2}.$$ 

□

Remark 3.3. The exponent 1/2 in (1) is sharp; cf. [FMP10]. Moreover, using [Sch14, Thm. 7.2.2] it is also possible to give a stability result of order 1/(n+1) in (1) for the Hausdorff distance $$d_H(\tilde{K} - \text{cent}(\tilde{K}), B)$$; we leave out the details to the interested reader. By cutting off opposite caps of height $$\varepsilon$$ of the unit ball, one can see that the optimal order cannot be better than 1 in (2).

For proving Theorem 1.2, we only need the stability of the $$L_p$$-width functional for $$p = -1$$. In this case, we give a slightly better stability result along with a diameter bound.

**Theorem 3.4.** Suppose $$K$$ is an origin-symmetric convex body with

$$\mathcal{E}_{-1}(\tilde{K}) \geq 1 - \varepsilon$$ for some $$\varepsilon \in (0, 1).$$

Then there exists an origin-centred ball $$B_r$$ of radius $$1 \leq r \leq (1 - \varepsilon)^{-1}$$ such that

$$\delta_2(\tilde{K}, B_r) \leq D(\tilde{K})\sqrt{\varepsilon}.$$

Moreover, we have

$$\left(\frac{1}{2}D(\tilde{K})\right)^{\frac{1}{2}} \leq \left(1 + \left(\frac{4\omega_{n-1}}{\omega_n}\right)^\frac{1}{2}\right) \frac{1}{1 - \varepsilon}.$$

**Proof.** Set $$h = h_{\tilde{K}}.$$ We have

$$\frac{\int \frac{1}{h}d\sigma}{(\int \frac{1}{h^2}d\sigma)^\frac{1}{2}} \omega_n^\frac{1}{2} = 1 - \frac{1}{2} \left(\frac{1}{h} - \frac{1}{\omega_n^\frac{1}{2}}\right)^2.$$

By our assumption and the Blaschke-Santaló inequality,

$$\int \frac{1}{h}d\sigma \geq \omega_n(1 - \varepsilon), \quad \int \frac{1}{h^2}d\sigma \leq \omega_n.$$

Therefore,

$$1 - \varepsilon \leq \frac{\int \frac{1}{h}d\sigma}{(\int \frac{1}{h^2}d\sigma)^\frac{1}{2}} \omega_n^\frac{1}{2}, \quad (1 - \varepsilon)\omega_n \leq \int \frac{1}{h}d\sigma \leq \left(\int \frac{1}{h^2}d\sigma\right)^\frac{1}{2} \omega_n^\frac{1}{2}.$$
Combining these inequalities we obtain
\[ |h - r|^2_{L^2} \leq \omega_n D(\bar{K})^2 \varepsilon, \]
where \( r^2 := \omega_n \left( \int \frac{1}{h^2} d\sigma \right)^{-1} \) and \( 1 \leq r \leq (1 - \varepsilon)^{-1} \).

Next we estimate the diameter from above. Define
\[ S = \{ v \in S^{n-1} : h(\bar{K})(v) \leq R^{\frac{1}{n}} \}, \]
where \( R := \max h_{\bar{K}} = h_{\bar{K}}(u) \) for some vector \( u \in S^{n-1} \). We may assume \( R > 1 \). Then by the Blaschke-Santaló inequality we have
\[ (1 - \varepsilon) \omega_n \leq \int_{S} h_{\bar{K}}^2 d\sigma + \int_{S^c} h_{\bar{K}} d\sigma \leq \left( \int_{S} h_{\bar{K}}^2 d\sigma \right)^{\frac{1}{2}} |S|^{\frac{1}{2}} + |S^c|^{\frac{1}{2}} R^{\frac{1}{n}} \omega_n. \]
Moreover, by convexity we have \( h_{\bar{K}}(v) \geq R |u \cdot v| \) for all \( v \in S^{n-1} \). Hence if \( v \in S \), then \( |u \cdot v| \leq R^{-\frac{2}{n}} \). Now using
\[ \frac{\pi}{2} - \arccos x \leq 2x, \quad \forall x \in [0, 1], \]
we obtain
\[ \frac{1}{2} |S| \leq \omega_{n-1} \int_{\arccos R^{-\frac{2}{n}}}^{\frac{\pi}{2}} \sin^{n-2} \theta d\theta \leq \frac{2 \omega_{n-1}}{R^{\frac{1}{n}}}, \]
Therefore,
\[ 1 - \varepsilon \leq \left( 1 + \left( \frac{4 \omega_{n-1}}{\omega_n} \right)^{\frac{1}{2}} \right) \frac{1}{R^{\frac{1}{n}}}. \]
\[ \square \]

We are ready to give the proofs of our main theorems.

**Proof of Theorem 1.1.** Suppose \( m \leq h_K^{1-p} dS_K/d\sigma \leq M \). Therefore by the \( L_p \)-Minkowski inequality,
\[
\frac{m}{n} \frac{\int h_K^p d\sigma}{V(B)^{1-\frac{p}{n}} V(K)^{\frac{n}{p}}} \leq \frac{1}{n} \frac{\int h_K^p h_K^{1-p} dS_K}{V(K)^{1-\frac{p}{n}}} \leq \frac{V(K)^{1-\frac{p}{n}}}{V(B)^{1-\frac{p}{n}}} \leq \frac{V(B)^{-\frac{p}{n}}}{n} \frac{\int h_K^{1-p} dS_K}{V(B)^{1-\frac{p}{n}}} \leq M.
\]
Hence $\mathcal{E}_p(\tilde{K}) \leq \mathcal{R}_p(\tilde{K})$, and by Theorem 3.2 the proof is complete. □

**Proof of Theorem 1.2.** Assume $m \leq h_K^{1-p}dS_K/d\sigma \leq M$. Then by the $L_q$-Minkowski inequality for $q = p + 1$ we have

$$\frac{1}{n} \int h_K^{q-p} h_K^{1-p} dS_K \geq V(K)^{1-\frac{n}{q}} V(B)^{\frac{n}{q}}.$$  

Therefore,

$$\frac{M}{n} V(K)^{\frac{q-n}{n}} \int h_K^{q-p} d\sigma \geq V(K)^{1-\frac{n}{q}} V(B)^{\frac{n}{q}}. \quad (3.8)$$

Owing to (2.4) for $p > 0$ we have

$$V(K) \geq \frac{m}{n} \int h_K^p d\sigma \geq m V(K)^{\frac{n}{p}} V(B)^{1-\frac{n}{p}}$$

and hence for $p \geq 0$,

$$V(K)^{1-\frac{n}{p}} \geq m V(B)^{1-\frac{n}{p}}. \quad (3.9)$$

Since $e_{p-q}(K) = 0$, in view of (3.8) we obtain $\mathcal{E}_{p-q}(\tilde{K}) \geq \mathcal{R}_p(K)^{-1}$. The claim follows from Theorem 3.4. □

**Remark 3.5.** It is clear from the proofs of Theorem 1.1 and Theorem 1.2 that if $K$ has only a positive continuous curvature function, then the same conclusions hold.

**Remark 3.6.** Applying the Blaschke-Santaló inequality to the left-hand side of (3.8), we obtain

$$\left( \frac{V(K)}{V(B)} \right)^{\frac{n-p}{n}} \leq M.$$ 

This combined with (3.9) yields

$$m \leq \left( \frac{V(K)}{V(B)} \right)^{\frac{n-p}{n}} \leq M.$$

Hence in the class of origin-symmetric bodies if $V(K) = V(B)$, then for any $p \geq 0$ the $L_p$-curvature function attains the value 1 at some point; see also Question 3.

**Proof of Theorem 1.3.** Define $\mathcal{E}_p : \mathcal{F}_0^n \to (0, \infty)$ by

$$\mathcal{E}_p(h_L) = \left( \int h_L^p d\sigma \right)^{\frac{1}{p}} / V(L).$$

By the divergence theorem we have
\[ (\nabla \tilde{E}_p)(h_K) = \frac{h_K^{p-1} \left( \int h_K^p d\sigma \right)^{\frac{p}{n}}} {V(K)^2} \left( \frac{nV(K)}{\int h_K^p d\sigma} - h_K^{1-p} f_K \right). \]

By [Sim96, Sec. 3.13 (ii)] and [Sim96, p. 80], there exist \( c_2, \delta > 0 \), such that for any \( K \) with \( |h_K - 1| C^3 \leq \delta \), there holds
\[ \left| \tilde{E}_p(K) - \tilde{E}_p(B) \right|^{\frac{1}{2}} \leq c_2 \left| (\nabla \tilde{E}_p)(h_K) \right|_{L^2}. \]

Assuming \( m \leq h_K^{1-p} f_K \leq M \) gives
\[ m \leq \frac{nV(K)}{\int h_K^p d\sigma} \leq M. \]

This in turn implies
\[ \left| \tilde{E}_p(K) - \tilde{E}_p(B) \right|^{\frac{1}{2}} \leq c_2 \left| (\nabla \tilde{E}_p)(h_K) \right|_{L^2}, \]

as well as
\[ \mathcal{E}_p(K) \geq \left( 1 + c_3 \left( \mathcal{R}_p(K) - 1 \right)^2 \right)^{\frac{2}{n}}. \]

Due to Theorem 3.2, the proof is complete. \( \square \)

**Proof of Theorem 1.5.** Suppose \( m \leq H_K \leq M \). By [And99b, Lem. 10],
\[ V(K) \geq \frac{\pi}{\sqrt{M}}. \]

In fact, the lemma states that if \( V(K) = \pi \), then centro-affine curvature at some point attains 1. Therefore, since \( V(\sqrt{\pi/V(K)}K) = \pi \), the function \((V(K)/\pi)^2 H_K\) takes the value 1 at some point. Hence using (3.10) and the Hölder inequality we obtain
\[ V(K)V(K^s) \geq \left( \frac{\int h_K f_K H_K^2 d\sigma}{4 \int h_K^p d\sigma} \right)^3 \geq mV(K)^2 \geq \pi^2 \frac{m}{M}. \]

If the Santaló point is at the origin, then we can obtain a slightly better lower bound for the volume product. By [Hug96], we have
\[ H_K(u)H_{K^*}(u^*) = 1, \]
where \( u \) and \( u^* \) are related by \( \langle \nu_K^{-1}(u), \nu_{K^*}^{-1}(u^*) \rangle = 1. \) Since \( K^s = K^* \), this yields
\[ \frac{1}{M} \leq H_{K^s} \leq \frac{1}{m}, \quad V(K^s) \geq \pi \sqrt{m}. \]

Therefore, \( V(K)V(K^s) \geq \pi^2 \sqrt{\frac{M}{m}}. \) Now in both cases, the result follows from [Iva15]. The third claim is exactly [Iva15, Cor. 4]. \( \square \)
Question 3. Given the previous argument, we would like to raise a question. Suppose $K \in \mathcal{F}_0^n$, $n \geq 3$, and $V(K) = V(B)$. Is it true that the centro-affine curvature of $K$ attains the value 1 at some point?

Proof of Theorem 1.6. For all $\ell \in GL(n)$, we have
\[
s(\ell K) = \ell s(K) = 0, \quad d_{BM}(\ell K, B) = d_{BM}(K, B).
\]
Thus we may assume without loss of generality that
\[
|h_K - 1|_{C^3} \leq \delta,
\]
for some $\delta > 0$ to be determined.

Define the functional $\mathcal{P} : \mathcal{F}_0^n \to (0, \infty)$ by
\[
\mathcal{P}(L) = \mathcal{P}(h_K) = \frac{1}{V(L)V(L^*)}.
\]
We have
\[
(\text{grad } \mathcal{P})(h_K) = \mathcal{P}^2(K) \left( \frac{V(K)}{h_K^{n+1}} - \frac{V(K^*)}{f_K} \right)
\]
\[
= \frac{V(K^*)\mathcal{P}^2(K)}{h_K^{n+1}} \left( \frac{V(K)}{V(K^*)} - \frac{1}{H_K} \right).
\]

By [Sim96, Sec. 3.13 (ii)], there exist $\delta, c_2 > 0$ and $\alpha \in (0, 1/2]$, such that for any $K$ with $|h_K - 1|_{C^3} \leq \delta$, we have
\[
\left| \frac{1}{V(K)V(K^*)} - \frac{1}{V(B)^2} \right|^{1-\alpha} \leq c_2 \|(\text{grad } \mathcal{P})(h_K)|_L^2.
\]

By [Sim96, p. 80] and [Iva18, Lem. 4.1, 4.2] we can choose $\alpha = 1/2$.

We estimate the right-hand side of (3.12) in terms of $R_{-n}(K) - 1$. Note that $m \leq H_K \leq M$ implies that
\[
\frac{1}{M} \leq \frac{V(K)}{V(K^*)} = \frac{\int h_K f_K d\sigma}{\int h_K f_K H_K d\sigma} \leq \frac{1}{m}.
\]
Therefore we obtain
\[
\frac{1}{M} \leq \frac{V(K)}{V(K^*)} \quad \text{and} \quad \left| \frac{V(K)}{V(K^*)} - \frac{1}{H_K} \right| \leq \frac{M - m}{Mm}.
\]

On the other hand, by (3.13) and the Blaschke-Santaló inequality,
\[
V(K^*)^2 \leq MV(B)^2.
\]
Putting (3.11), (3.12), (3.13), and (3.14) all together we arrive at
\[
\left| \frac{1}{V(K)V(K^*)} - \frac{1}{V(B)^2} \right|^{\frac{1}{2}} \leq c_3 (R_{-n}(K) - 1) \mathcal{P}^2(K) \frac{|h_K^{-n-1}|_{L^2}}{V(K^*)^2}.
\]
Since we are in a small neighborhood of the unit ball, the term
\[ \frac{\mathcal{P}^2(K) |h_k - n|}{V(K^*)} \]
is bounded. Using again the Blaschke-Santaló inequality we obtain
\[ 1 - c_4 (R_n(K) - 1)^2 \leq \frac{V(K)V(K^*)}{V(B)^2}. \]
In view of [Bör10, Thm. 1.1], the proof is complete. \(\square\)

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**Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstr 8-10, 1040 Wien, Austria, mohammad.ivaki@tuwien.ac.at**