The Orbifolds of Permutation-Type as Physical String Systems at Multiples of \( c = 26 \)

II. The Twisted BRST Systems of \( \hat{c} = 52 \) Matter

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Abstract

This is the second in a series of papers which consider the orbifolds of permutation-type as candidates for new physical string systems at higher central charge. In the first paper, I worked out the extended actions of the twisted sectors of those orbifolds – which exhibit new permutation-twisted world-sheet gravities and correspondingly extended diffeomorphism groups. In this paper I begin the study of these systems as operator string theories, limiting the discussion for simplicity to the strings with \( \hat{c} = 52 \) matter (which are those governed by \( \mathbb{Z}_2 \)-twisted permutation gravity). In particular, I present here a construction of the twisted reparametrization ghosts and new twisted BRST systems of all \( \hat{c} = 52 \) strings. The twisted BRST systems also imply new extended physical state conditions, whose analysis for individual \( \hat{c} = 52 \) strings is deferred to the next paper of the series.

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1 Introduction

Opening a third, more phenomenonological chapter in the orbifold program [1-11,12-15], I have speculated in Ref. [16] that the orbifolds of permutation-type may describe physical string systems at higher critical central charge. This includes the values

\[ \hat{c} = 26K, \quad K = 2, 3, \ldots \]  \hspace{1cm} (1.1)

for the bosonic cases and

\[ \hat{c} = (10K), \quad (\hat{c}, \hat{\bar{c}}) = (26K, 10K) \]  \hspace{1cm} (1.2)

for superstring orbifolds of permutation-type. There is a surprisingly large variety of orbifolds of permutation-type, including the permutation orbifolds themselves [1-6,11,16], the orientation orbifolds [12,13,15, 16], the open-string permutation orbifolds [14] and their $T$-duals [15,16], the generalized permutation orbifolds [15,16] and others. A short review of these varieties is included in Ref. [16].

The technical observation which underlies this conjecture is as follows: The extended world-sheet actions of the twisted sectors of the bosonic orbifolds exhibit a large class of new extended (twisted) world-sheet gravities – called the permutation gravities [16] – which are accompanied by correspondingly-extended diffeomorphism invariances. Like the sectors of the permutation orbifolds, the permutation gravities are classified by the conjugacy classes of the permutation groups $\hat{H}(\text{perm})$ – where the trivial class of $H(\text{perm})$ corresponds to (decoupled copies of) ordinary Polyakov gravity in the untwisted sectors of the orbifolds. Indeed, the permutation gravities can be understood\(^{1}\) as nothing but the standard orbifold map (the principle of local isomorphisms) of ordinary Polyakov gravity into the twisted sectors. The explicit forms of the corresponding permutation supergravities of the superstring orbifolds have not yet been worked out.

The permutation gravities play the following role in the conjecture: The covariant formulation of the twisted sectors of the orbifolds of permutation-type exhibit an increased number of negative-norm states (time-like currents) in correspondence with their higher central charges. This parallels the situation in the untwisted sectors, where the negative-norm states of the untwisted

\(^{1}\)See in particular Subsec. 3.5 of Ref. [16].
string copies are removed by the ordinary Polyakov gravity of each copy. Since the permutation gravities are the maps of the untwisted gravities into the twisted sectors of the orbifold, one may expect the extended diffeomorphism invariances of the new gravities to similarly remove the negative-norm states of each twisted sector. The size of the new twisted diffeomorphism invariances lends strong support to this speculation.

The classical intuition of Ref. [16] must of course be verified both at the operator level and the interacting level, work which I begin in the present paper. In particular, I construct here the twisted reparametrization ghosts and new twisted BRST systems for all the twisted open- and closed-string sectors which are governed by the simple case of $\mathbb{Z}_2$-twisted permutation gravity, i.e. all twisted strings with matter central charge $\hat{c} = 52$.

The new BRST systems then directly imply new extended physical state conditions for each $\hat{c} = 52$ string, which are in fact the operator realizations of the classical extended Virasoro constraints of Ref. [16]. I emphasize that the number of these conditions at $\hat{c} = 52$ is doubled relative to the ordinary physical state condition of $c = 26$ strings, so the counting at the operator level still appears adequate to eliminate the doubled set of negative-norm states at $\hat{c} = 52$. For brevity however, analysis of the extended physical state conditions and spectrum of particular $\hat{c} = 52$ strings is deferred to the next paper of this series.

I should also emphasize that the construction of this paper is largely self-contained, using only standard operator-product techniques in the orbifold program. Moreover, given the operator-product formulation of untwisted BRST in Sec. 2, the computation given in the succeeding sections can be straightforwardly generalized to obtain the twisted BRST systems of all the orbifolds of permutation-type. Alternately, these systems can be constructed à la Faddeev-Popov from the extended actions of Ref. [16], but I will not follow this line here.

2 Operator-Product Form of Untwisted BRST

In order to apply the techniques of the orbifold program, one first needs the ordinary BRST system [17-19] in operator-product form. The simplest approach to this is the picture of Freericks and Halpern [20], where the (left-
mover) reparametrization ghosts \[3\] of ordinary complex half-integer mode Bardakci-Halpern (BH) fermions \[26\]:

\[
\psi(z) = \sum_{p \in \mathbb{Z}-\frac{1}{2}} \psi(p) z^{-p-\frac{1}{2}}, \quad \bar{\psi}(z) = \sum_{p \in \mathbb{Z}-\frac{1}{2}} \bar{\psi}(p) z^{-p-\frac{1}{2}} \tag{2.1a}
\]

\[
[\psi(p), \bar{\psi}(q)]_+ = \delta_{p+q,0} \tag{2.1b}
\]

\[
\psi(p > 0) |0\rangle = \bar{\psi}(p > 0) |0\rangle = 0 \tag{2.1c}
\]

\[
J(z) \equiv :\bar{\psi}(z)\psi(z): = \sum_{m \in \mathbb{Z}} J(m) z^{-m-1} \tag{2.1d}
\]

\[
T^{BH}(z) \equiv -\frac{1}{2} :\bar{\psi}(z) \partial \psi(z): = \sum_{m} L^{BH}(m) z^{-m-2} \tag{2.1e}
\]

\[
T^{G}(z) \equiv T^{BH}(z) + \frac{3}{2} \partial J(z) \tag{2.1f}
\]

\[
= :\bar{\psi}(z) \partial \psi(z) + 2 \partial \bar{\psi}(z) \psi(z): \tag{2.1g}
\]

\[
= \sum_{m} L^{G}(m) z^{-m-2} \tag{2.1h}
\]

\[
L^{G}(m \geq -1) |0\rangle = L^{BH}(m \geq -1) |0\rangle = 0. \tag{2.1i}
\]

Here \( |0\rangle \) is the BH vacuum and \( T^{BH} \) is the BH stress tensor with central charge 26. The ghost stress tensor \( T^{G} \) is a conformal deformation by \( \partial J \) of the BH stress tensor. In the language of Ref. [20], \( \partial J \) is an example of a c-changing deformation which, according to Eq. (2.1i), is also an sl(2)-preserving deformation. For the later transition to the orbifolds, it is important that all relations be written in terms of operator-product normal ordering \( : \cdot : \) (subtract singular terms) – which in this case is equal to BH-mode normal ordering:

\[
:A(p) B(q):_M \equiv -\theta(p > 0) B(q) A(q) + \theta(p < 0) A(p) B(q) \tag{2.2a}
\]

\[
:A(z) B(\omega): = :A(z) B(\omega):_M \tag{2.2b}
\]
where $A$ and $B$ can be $\psi$ or $\bar{\psi}$.

The operator product description of the ghost system is:

$$\psi(z)\bar{\psi}(\omega) = \frac{1}{z - \omega} + :\psi(z)\bar{\psi}(\omega): \quad (2.3a)$$

$$T^{G}(z)\bar{\psi}(\omega) = \left(-\frac{1}{(z - \omega)^2} + \frac{1}{z - \omega}\partial_{\omega}\right)\bar{\psi}(\omega) + :T^{G}(z)\bar{\psi}(\omega): \quad (2.3b)$$

$$T^{G}(z)\psi(\omega) = \left(\frac{2}{(z - \omega)^2} + \frac{1}{z - \omega}\partial_{\omega}\right)\psi(\omega) + :T^{G}(z)\psi(\omega): \quad (2.3c)$$

$$T^{G}(z)T^{G}(\omega) = \left(-\frac{26}{4(z - \omega)^4} + \left(\frac{2}{(z - \omega)^2} + \frac{1}{z - \omega}\right)T^{G}(\omega) + \right. \left.+ :T^{G}(z)T^{G}(\omega): \quad (2.3d)\right.$$
\[ J^G(z) \equiv :\bar{\psi}(z)\psi(z):_R = J(z) - \frac{3}{z^2} = \sum_m J^G(m)z^{-m-1} \quad (2.5b) \]

\[ J^G(m = 0) = \frac{1}{2}[c(0), b(0)] + \sum_{n=1}^{\infty} (c(-n)b(n) - b(-n)c(n)) \quad (2.5c) \]

which is needed to define the standard ghost current \( J^G \) in Eq. (2.5b). There is no need to pursue this familiar line here however because, as noted above, local operators (not modes) and operator-product normal ordering (not Ramond ordering) are required to apply the principle of local isomorphisms in Section 4.

Returning then to the local formulation, I further introduce the (left-mover) matter stress tensor \( T \) at the central charge \( c = 26 \) of the critical closed bosonic string, and the BRST current \( J_B \):

\[ T(z)T(\omega) = \frac{13}{(z-\omega)^4} + \left( \frac{2}{(z-\omega)^2} + \frac{1}{z-\omega} \right)T(\omega) + :T(z)T(\omega): \quad (2.6a) \]

\[ J_B(z) \equiv :\bar{\psi}(z)T(z) + \frac{1}{2} :\bar{\psi}(z)T^G(z): = \sum_m J^B(m)z^{-m-1}. \quad (2.6b) \]

After some algebra with the Wick expansion for BH fermions, one finds the desired operator products

\[ J^B(z)\psi(\omega) = \frac{J(\omega)}{(z-\omega)^2} + \frac{T^t(\omega)}{z-\omega} + :J^B(z)\psi(\omega): \quad (2.7a) \]

\[ T^t(z) \equiv T(z) + T^G(z) = \sum_m L^t(m)z^{-m-2} \quad (2.7b) \]

\[ J^B(z)J^B(\omega) = \frac{10}{(z-\omega)^3}\partial_\omega \bar{\psi}(\omega)\bar{\psi}(\omega) + \frac{5}{(z-\omega)^2}\partial_\omega^2 \bar{\psi}(\omega)\bar{\psi}(\omega) \]

\[ + \frac{3}{2(z-\omega)}\partial_\omega(\partial_\omega^2 \bar{\psi}(\omega)\bar{\psi}(\omega)) + :J^B(z)J^B(\omega): \quad (2.7c) \]

where \( T^t \) in Eq. (2.7b) is the total stress tensor with zero central charge.

This (and a right-mover copy of the system) completes the operator-product description needed below but, because Eq. (2.7c) is new, I also give selected parts of the mode algebra corresponding to these results:

\[ [J^B(m), b(n)]_+ = mJ(m + n) + L^t(m + n) \quad (2.8a) \]
\[ [J^B(m), J^B(n)]_+ = \frac{1}{2}(10mn - 3(m+n)(m+n-1)) \times \]
\[ \times \sum_{p \in \mathbb{Z}} pc(p)c(m+n+p) \]  \hspace{1cm} (2.8b)

\[ Q \equiv J^B(m=0) \]  \hspace{1cm} (2.8c)

\[ [Q, b(m)]_+ = L^t(m), \quad [Q, L^t(m)] = 0. \]  \hspace{1cm} (2.8d)

\[ Q^2 = 0 \]  \hspace{1cm} (2.8e)

Besides the standard BRST charge \( Q \), I note in passing that other nilpotent operators such as \( (J^B(3))^2 = 0 \) are implied by the anticommutator (2.8b).

### 3 Automorphisms and Eigenfields

In this and the following section, I will apply standard operator techniques in the orbifold program to construct the twisted BRST systems of all \( \hat{c} = 52 \) matter. These systems comprise all the twisted sectors of the orbifolds of permutation-type which are governed by \( \mathbb{Z}_2 \)-twisted permutation gravity [16], or equivalently, those which are governed by an order-two orbifold Virasoro algebra at \( \hat{c} = 52 \) [1,27,9,12,16]. The list of these sectors includes the twisted open-string sectors of the orientation orbifolds and the twisted closed-string sectors of the generalized \( \mathbb{Z}_2 \)-permutation orbifolds

\[ \frac{U(1)^{26}}{H_-}, \quad H_- = \mathbb{Z}_2 \text{(w.s.)} \times H \]  \hspace{1cm} (3.1a)

\[ \frac{U(1)^{26} \times U(1)^{26}}{H_+}, \quad H_+ = \mathbb{Z}_2 \text{(perm)} \times H' \]  \hspace{1cm} (3.1b)

as well as the generalized open-string \( \mathbb{Z}_2 \)-permutation orbifolds and their T-duals. In Eq. (3.1), \( H_\pm \) are automorphism groups generated by \( \tau_\pm \times \omega \), \( \omega \in H \) or \( H' \), where \( \tau_\pm \) is the element of \( \mathbb{Z}_2 \text{(w.s.)} \) which exchanges the left- and right-movers of the untwisted closed string \( U(1)^{26} \) and \( \tau_+ \) is the element of \( \mathbb{Z}_2 \text{(perm)} \) which exchanges the two copies of the closed-string. In both

\[ ^3 \text{The generalized open-string permutation orbifolds and their T-duals [15] are constructed from the left-mover data of the generalized permutation orbifolds } U(1)^{26K}/(H \text{(perm)} \times H')). \text{ The open-string orientation- orbifold sectors are among the } T\text{-duals of the generalized open-string } \mathbb{Z}_2 \text{-permutation orbifolds.} \]
cases, the extra automorphisms \( \omega \) act uniformly on the left- and right-movers of each closed string.

Both orbifold types in Eq. (3.1) can have many twisted sectors, labelled by the conjugacy classes of \( H_\pm \), and half the twisted sectors of the orientation orbifolds (corresponding to the orientation-reversing automorphisms of \( H_- \)) are twisted open string CFT's at \( \hat{c} = 52 \). I will have more to say about the relation of orientation orbifolds to orientifolds [28] in succeeding papers of the series. The extended Polyakov actions (with \( \mathbb{Z}_2 \)-twisted permutation gravity) of all these sectors are given in Eqs. (2.52) and (4.6) of Ref. [16].

The first step in the orbifold program is to specify the action of the automorphisms on the untwisted fields. The action on the matter fields has been studied elsewhere [9,12,16], so I focus here on the fields of the BRST system which (like the extended stress tensors and \( \mathbb{Z}_2 \)-twisted permutation gravity itself) see only the \( \mathbb{Z}_2 \)'s (\( \mathbb{Z}_2 \)(w.s.) or \( \mathbb{Z}_2 \)(perm)) of the examples in Eq. (3.1). Following the development of Refs. [12,13], we may summarize the action of the non-trivial element of each \( \mathbb{Z}_2 \) in the following unified, two-component notation:

\[
T^G_I \equiv :\bar{\psi}_I \partial \psi_I + 2\partial \bar{\psi}_I \psi_I :, \quad I = 0, 1 \tag{3.2a}
\]

\[
J_I = :\bar{\psi}_I \psi_I :, \quad J^B_I = \bar{\psi}_I T_I + \frac{1}{2} :\bar{\psi}_I T^G_I :, \tag{3.2b}
\]

\[
A_I(z)' = (\tau_1)_{IJ} A_J(z), \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.2c}
\]

Here all the operators are functions of complex \( z \) and \( A_I' \) in (3.2c) is the automorphic response of any operator in the system.

I want to emphasize that, with appropriate identifications, the unified action (3.2) describes the BRST systems of all untwisted sectors which can lead to twisted sectors at \( \hat{c} = 52 \). In the case of the generalized \( \mathbb{Z}_2 \)-permutation orbifolds (3.1b), as well as the generalized open-string \( \mathbb{Z}_2 \)-permutation orbifolds [15], the index \( I \)labels the two left-mover copies in \( U(1)^{26} \times U(1)^{26} \) (or the two right-mover copies with \( z \rightarrow \bar{z} \) for the permutation orbifolds). For the orientation orbifolds, 0 and 1 label the left- and right-movers respectively of the single closed string \( U(1)^{26} \), where the right movers are relabelled by the \( \bar{z} \rightarrow z \) trick of Refs. [12,13]

\[
A_0(z) \equiv A(z), \quad A_1(z) \equiv \bar{A}(\bar{z})|_{\bar{z} \rightarrow z} \tag{3.3}
\]

which preserves the desired exchange of the left- and right-mover modes.
Then one easily finds the operator products of the two-component fields, e.g.

\[ \bar{\psi}_I(z) \psi_J(\omega) = \frac{\delta_{IJ}}{z-\omega} + :\bar{\psi}_I(z) \psi_J(\omega): \]  

(3.4a)

\[ T_I^G(z) T_J^G(\omega) = \delta_{IJ} \left( \frac{-13}{(z-\omega)^4} \right) + \left( \frac{2}{(z-\omega)^2} + \frac{1}{z-\omega} \partial_\omega \right) T_I^G(\omega) + \right. 
\[ + :T_I^G(z) T_J^G(\omega): \]  

(3.4b)

where \( \delta_{IJ} \) is Kronecker delta. All the other two-component operator products have the same semi-simple structure, and it is easily checked that (3.2c) is an automorphism of the full two-component system.

The next step in the orbifold program is the definition of eigenfields, which diagonalize the automorphic response:

\[ U = U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]  

(3.5a)

\[ \bar{\Psi}_u \equiv U_u^T \bar{\psi}_I, \quad \Psi_u \equiv U_u^T \psi_I, \quad \bar{u} = 0, 1 \]  

(3.5b)

\[ J_u \equiv \sqrt{2} U_u^T J_I = \sum_{v=0}^1 :\bar{\Psi}_u \bar{\Psi}_v: \]  

(3.5c)

\[ \theta_u^G \equiv \sqrt{2} U_u^T T_I^G = \sum_v :\bar{\Psi}_{u-v} \partial \Psi_v + 2 \partial \bar{\Psi}_{u-v} \Psi_v: \]  

(3.5d)

\[ \theta_u^T \equiv \sqrt{2} U_u^T (T_I + T_I^G) = \theta_u + \theta_u^G \]  

(3.5e)

\[ J_u^B \equiv 2 U_u^T J_P^G = \sum_v (\bar{\Psi}_{u-v} \theta_v + :\bar{\Psi}_{u-v} \theta_v^G:) \]  

(3.5f)

\[ \mathcal{A}_u(z)' = (-1)^u \mathcal{A}_u(z) \]  

(3.5g)

where \( \mathcal{A}_u \) in Eq. (3.5g) can be any eigenfield in the system. It is then straightforward to compute the operator products in the eigenfield basis, for example:

\[ \bar{\Psi}_u(z) \Psi_v(\omega) = \frac{\delta_{u+v,0 \mod 2}}{z-\omega} + :\bar{\Psi}_u(z) \Psi_v(\omega): \]  

(3.6a)

\[ ^4 \text{Here I am using the notation } \bar{u} = 0, 1 \text{ of the orientation orbifolds, while the standard notation for the } \mathbb{Z}_2 \text{-permutation orbifolds is } \bar{u} \rightarrow \bar{j} = 0, 1. \]
\[ \theta^G_u(z)\theta^G_v(\omega) = -\frac{52}{2} \frac{1}{(z - \omega)^4} \delta_{u+v,0 \mod 2} + \left( \frac{2}{(z - \omega)^2} + \frac{1}{z - \omega} \partial_v \right) \theta^G_{u+v}(\omega) + :\theta^G_u(z)\theta^G_v(\omega):. \quad (3.6b) \]

I omit the rest of the eigenfield operator products for brevity, but the full set can easily be read from the result (4.2) of the following section. With Ref. [12], I remark in particular on the central charge \( c = -52 \) of the ghost eigen-stress tensors \( \{\theta^G_u, u = 0, 1\} \) in Eq. (3.6b), which follows because \( \theta^G_0 = T^G_0 + T^G_1 \) and the central charges of the copies \( T^G_{0,1} \) are additive. Similarly, the central charge of the matter eigen-stress tensors \( \{\theta^t_u\} \) is 52, and the anomaly cancels for the total eigen-stress tensors \( \{\theta^t_u\} \).

4 The New Twisted BRST Systems

The next step in the program is the transition to the orbifold, using the principle of local isomorphisms [1,3,5,6,9,11,12]. This principle maps the eigenfields \( \{A\} \) of each conjugacy class of the automorphism group to the corresponding twisted fields \( \{\hat{A}\} \) of each twisted sector, taking the phases of the automorphic response of the eigenfields as the monodromies of the twisted fields \( ^5 \). To complete the transition, the principle also specifies that the operator products of the twisted fields are isomorphic to those of the eigenfield basis, and hence that operator-product normal-ordered products in the eigenfield basis map to operator-product normal-ordered products in the twisted sectors of the orbifold. Correspondingly, central charges do not change under orbifoldization from the eigenfield basis.

In the present application, we then obtain the operators of the new twisted BRST systems

\[ \hat{J}_u = \sum_v :\hat{\psi}_{u-v}^\dagger \hat{\psi}_v:; \quad \hat{\theta}^t_u = \hat{\theta}_u + \hat{\theta}^G_u \quad (4.1a) \]

\[ \hat{\theta}^G_u = -\frac{1}{2} \sum_v :\hat{\psi}_{u-v}^\dagger \partial \hat{\psi}_v: + \frac{3}{2} \hat{J}_u \quad (4.1b) \]

\[ \hat{J}^B_u = \sum_v (\hat{\psi}_{u-v}^\dagger \hat{\theta}_v + :\hat{\psi}_{u-v}^\dagger \hat{\theta}^G_v:) \quad (4.1c) \]

\(^5\)The fields \( A, \hat{A}, \hat{\hat{A}} \) and the familiar fields \( \hat{A} \) with twisted boundary conditions form commuting diagrams (see Refs. [3,6,11]), where the twisted fields \( \hat{A} \) are the monodromy decomposition of \( A \).
\[ \hat{A}_u(z^{2\pi i}) = (-1)^u \hat{A}_u(z), \quad \bar{u} = 0, 1 \]  

where \( \hat{A}_u \) in (4.1d) can be any operator in the system. Moreover, we obtain the twisted operator products of each sector \(^6\)

\[
\hat{\psi}_u(z) \hat{\psi}_v(\omega) = \frac{\delta_{u+v,0} \mod 2}{z - \omega} + : \hat{\psi}_u(z) \hat{\psi}_v(\omega) :) \quad \bar{u}, \bar{v} \in \{0, 1\} \tag{4.2a}
\]

\[
\hat{\theta}_u(z) \hat{\theta}_v(\omega) = \frac{52/2}{(z - \omega)^4} \delta_{u+v,0} \mod 2 + \left( \frac{2}{(z - \omega)^2} + \frac{1}{z - \omega} \partial_\omega \right) \hat{\theta}_{u+v}(\omega) + : \hat{\theta}_u(z) \hat{\theta}_v(\omega) :) \tag{4.2b}
\]

\[
\hat{\theta}_u^G(z) \hat{\theta}_v^G(\omega) = -\frac{52/2}{(z - \omega)^4} \delta_{u+v,0} \mod 2 + \left( \frac{2}{(z - \omega)^2} + \frac{1}{z - \omega} \partial_\omega \right) \hat{\theta}^G_{u+v}(\omega) + : \hat{\theta}_u^G(z) \hat{\theta}_v^G(\omega) :) \tag{4.2c}
\]

\[
\hat{\psi}_u(z) \hat{\psi}_v(\omega) = -\frac{1}{(z - \omega)^2} + \frac{1}{z - \omega} \partial_\omega \hat{\psi}_{u+v}(\omega) + : \hat{\psi}_u(z) \hat{\psi}_v(\omega) :) \tag{4.2d}
\]

\[
\hat{\theta}_u^G(z) \hat{\psi}_v(\omega) = \frac{2}{(z - \omega)^2} + \frac{1}{z - \omega} \partial_\omega \hat{\psi}_{u+v}(\omega) + : \hat{\theta}_u^G(z) \hat{\psi}_v(\omega) :) \tag{4.2e}
\]

\[
\hat{\psi}_u(z) \hat{\psi}_v(\omega) = \left( \frac{2}{(z - \omega)^2} + \frac{1}{z - \omega} \partial_\omega \right) \hat{\psi}_{u+v}(\omega) + : \hat{\psi}_u(z) \hat{\psi}_v(\omega) :) \tag{4.2f}
\]

\[
\hat{J}^B_u(z) \hat{\psi}_v(\omega) = \frac{1}{(z - \omega)^2} \hat{J}_{u+v}(\omega) + \frac{1}{z - \omega} \hat{\theta}_{u+v}(\omega) + : \hat{J}^B_u(z) \hat{\psi}_v(\omega) :) \tag{4.2g}
\]

\[
\hat{J}^B_u(z) \hat{J}^B_{\bar{v}}(\omega) = \frac{20}{(z - \omega)^2} \sum_x \partial_x \hat{\psi}_x(\omega) \hat{\psi}_{u+v-x}(\omega)
+ \frac{10}{(z - \omega)^2} \sum_x \partial^2_x \hat{\psi}_x(\omega) \hat{\psi}_{u+v-x}(\omega)
+ \frac{3}{z - \omega} \partial_\omega \left( \sum_x \partial^2_x \hat{\psi}_x(\omega) \hat{\psi}_{u+v-x}(\omega) \right)
+ : \hat{J}^B_u(z) \hat{J}^B_{\bar{v}}(\omega) :) \tag{4.2h}
\]

\(^6\)Supplementing the examples in Eq. (3.6), the complete set of eigenfield operator products can be read off from the result Eq. (4.2) by the substitution \( A_u \rightarrow \hat{A}_u \).
In Eqs. (4.1) and (4.2), the symbol \( \cdot \) is operator-product normal ordering in the twisted sectors of the orbifolds. The operators \( \{ \hat{\psi}_u, \hat{\psi}_v \} \) and \( \{ \hat{J}_u^B \} \) with \( \hat{u} = 0, 1 \) are respectively the twisted reparametrization ghost fields and the twisted BRST currents. The twisted reparametrization ghosts can also be obtained by the Faddeev-Popov procedure [29] and choice of the conformal gauge from the extended actions with \( \mathbb{Z}_2 \)-twisted permutation gravity of Ref. [16]. The twisted operator products also record that the orbifold central charges \( \hat{c} = -52, 52 \) and 0 of the various extended, twisted stress tensors \( \hat{\theta}_u^G (\text{ghost}), \hat{\theta}_u (\text{matter}) \) and \( \hat{\theta}_u^t (\text{total}) \) have not changed from those of the corresponding eigen-stress tensors. The extended matter stress tensors \( \hat{\theta}_u \) in particular are the operator versions of the classical (conformal-gauge) extended stress tensors of Ref. [16], and the explicit forms of these operators will be given for all \( \hat{c} = 52 \) matter in the following paper of this series.

In the orbifold program the mode formulation of each twisted sector is left to last, where the moding follows from the monodromies of the local fields. In this case, we find that the mode expansions

\[
\hat{\psi}_u(z) = \sum_{m \in \mathbb{Z}} \hat{c}_u(m + \frac{u}{2})z^{-(m + \frac{u}{2})+1}, \quad \hat{u} = 0, 1 \tag{4.3a}
\]

\[
\hat{\psi}_u(z) = \sum_{m \in \mathbb{Z}} \hat{b}_u(m + \frac{v}{2})z^{-(m + \frac{v}{2})-2} \tag{4.3b}
\]

\[
\hat{\theta}_u^t = \sum_m \hat{L}_u^t(m + \frac{u}{2})z^{-(m + \frac{u}{2})-2}, \quad \hat{L}_u^t(m + \frac{u}{2}) = \hat{L}_u(m + \frac{u}{2}) + \hat{L}_u^G(m + \frac{u}{2}) \tag{4.3c}
\]

\[
\hat{J}_u(z) = \sum_m \hat{J}_u(m + \frac{u}{2})z^{-(m + \frac{u}{2})-1}, \quad \hat{J}_u^B(z) = \sum_m \hat{J}_u^B(m + \frac{u}{2})z^{-(m + \frac{u}{2})-1} \tag{4.3d}
\]

follow from the monodromies (4.1d), where \( \{ \hat{c}_u, \hat{b}_u \} \) and \( \{ \hat{J}_u^B \} \) with \( \hat{u} = 0, 1 \) are respectively the twisted reparametrization-ghost modes and the twisted BRST current modes. Then the operator product system (4.2) and standard orbifold contour integrations (see e.g. Ref. [5]) give the mode algebras of the new twisted BRST systems:

\[
[\hat{c}_u(m + \frac{u}{2}), \hat{b}_v(n + \frac{v}{2})]_+ = \delta_{m+n+u+v,0} \quad \hat{u}, \hat{v} \in \{0, 1\} \tag{4.4a}
\]

\[
[\hat{c}_u(m + \frac{u}{2}), \hat{c}_v(n + \frac{v}{2})]_+ = [\hat{b}_u(m + \frac{u}{2}), \hat{b}_v(n + \frac{v}{2})]_+ = 0 \tag{4.4b}
\]

\[
[\hat{L}_u(m + \frac{u}{2}), \hat{L}_v(n + \frac{v}{2})] = (m - n + \frac{u-v}{2})\hat{L}_{u+v}(m + n + \frac{u+v}{2}) + \frac{52}{12}(m + \frac{u}{2})(m + \frac{u}{2})^2 - 1)\delta_{m+n+u+v,0} \tag{4.4c}
\]
[\hat{L}_u^G(m + \frac{u}{2}), \hat{L}_v^G(n + \frac{v}{2})] = (m - n + \frac{u-v}{2})\hat{L}_{u+v}^G(m + n + \frac{u+v}{2}) + 
- \frac{52}{12}(m + \frac{u}{2})(m + \frac{v}{2})((m + \frac{u}{2})^2 - 1)\delta_{m+n+\frac{u+v}{2},0} \quad (4.4d)

[\hat{L}_u^t(m + \frac{u}{2}), \hat{L}_v(n + \frac{v}{2})] = (m - n + \frac{u-v}{2})\hat{L}_{u+v}^t(m + n + \frac{u+v}{2}) \quad (4.4e)

[\hat{L}_u^G(m + \frac{u}{2}), \hat{c}_v(n + \frac{v}{2})] = -(2(m + \frac{u}{2}) + n + \frac{v}{2})\hat{c}_{u+v}(m + n + \frac{u+v}{2}) \quad (4.4f)

[\hat{L}_u^G(m + \frac{u}{2}), \hat{b}_v(n + \frac{v}{2})] = (m - n + \frac{u-v}{2})\hat{b}_{u+v}(m + n + \frac{u+v}{2}) \quad (4.4g)

[\hat{j}_u^B(m + \frac{u}{2}), \hat{b}_v(n + \frac{v}{2})]_+ = (m + \frac{u}{2})\hat{j}_{u+v}(m + n + \frac{u+v}{2}) + \hat{L}_{u+v}^t(m + n + \frac{u+v}{2}) \quad (4.4h)

\begin{align*}
[\hat{j}_u^B(m + \frac{u}{2}), \hat{j}_v^B(n + \frac{v}{2})]_+ & = \\
& = \{10(m + \frac{u}{2})(n + \frac{v}{2}) - 3(m + n + \frac{u-v}{2})(m + n + \frac{u+v}{2} + 1)\} \times \\
\times \sum_x \sum_{p \in \mathbb{Z}} (p + \frac{x}{2})\hat{c}_x(p + \frac{x}{2})\hat{c}_{u+v-x}(m + n - p + \frac{u+v-x}{2}). \quad (4.4i)
\end{align*}

I call attention in particular to the three distinct order-two orbifold Virasoro algebras \cite{1,27,9,12} in Eqs. (4.4c),(4.4d) and (4.4e), whose integral Virasoro subalgebras are generated by \{\hat{L}_0(m)\}, \{\hat{L}_0^G(m)\} and \{\hat{L}_0^t(m)\} with central charges 52, -52 and 0 respectively.

The new twisted BRST systems constructed here are a central result of this paper. The systems are complete for all the twisted open-strings at matter central charge \(\hat{c} = 52\) and, with the addition of a right-mover copy, all the twisted closed strings at \(\hat{c} = 52\) are described as well. The universal form of these systems reflects their common origin in the \(\mathbb{Z}_2\)-twisted permutation gravity \cite{16} which governs all \(\hat{c} = 52\) strings. The extended matter Virasoro generators \{\hat{L}_u(m + \frac{u}{2}), \hat{u} = 0, 1\} of course vary from sector to sector of these orbifolds, but I defer the explicit form of these generators to the next paper of this series.

Using these results, I turn next to some algebraic properties of the new BRST operator \(\hat{Q}\) itself:

\begin{align*}
\hat{Q} & \equiv \hat{j}_0^B(0) \quad (4.5a) \\
\hat{Q}^2 & = 0 \quad (4.5b) \\
[\hat{Q}, \hat{b}_u(m + \frac{u}{2})]_+ & = \hat{L}_u^t(m + \frac{u}{2}) \quad (4.5c) \\
[\hat{Q}, \hat{L}_u^t(m + \frac{u}{2})] & = 0, \quad \hat{u} = 0, 1 \quad (4.5d)
\end{align*}
In particular, the nilpotency (4.5b) follows from Eq. (4.4i), the anticommutator (4.5c) follows from Eq. (4.4h), and the commutator (4.5d) is a consequence of (4.5b) and (4.5c) together. As noted above for the untwisted BRST system, other nilpotent modes of the twisted BRST current $\hat{J}^B$ are implied by the anticommutator (4.4i).

Finally, we may use the new BRST operator and Eq. (4.5c) to define the physical states $\{|\chi\rangle\}$ in each twisted sector as follows:

$$\hat{Q}|\chi\rangle = \hat{b}_u((m + \frac{n}{2}) > 0)|\chi\rangle = 0 \quad (4.6a)$$

$$\rightarrow \hat{L}_u((m + \frac{n}{2}) \geq 0)|\chi\rangle = 0, \quad \bar{u} = 0, 1. \quad (4.6b)$$

The $\{\hat{c}\}$ condition in (4.6a) is not used here, but will play a role in the discussion below.

5 The Extended Physical State Conditions

To go further, we need more explicit forms of various operators in the system. For the twisted reparametrization ghosts, I introduce the following mode normal-ordered product

$$:\hat{A}_u(m + \frac{n}{2})\hat{B}_v(n + \frac{\omega}{2}) :_M \equiv -\theta(m + \frac{n}{2} > 0)\hat{B}_v(n + \frac{\omega}{2})\hat{A}_u(m + \frac{n}{2})$$

$$+ \frac{1}{4}\delta_{m + \frac{n}{2}, 0}[\hat{A}_u(0), \hat{B}_v(n + \frac{\omega}{2})]$$

$$+ \theta(m + \frac{n}{2} < 0)\hat{A}_u(m + \frac{n}{2})\hat{B}_v(n + \frac{\omega}{2}) (5.1)$$

where $\hat{A}$ and $\hat{B}$ can be either $\hat{c}$ or $\hat{b}$.

Then the mode expansions in Eq. (4.3) straightforwardly give (see e.g. Ref. [5]) the alternate form of the $\hat{\psi}, \hat{\bar{\psi}}$ operator product:

$$\hat{\bar{\psi}}_u(z)\hat{\psi}_v(\omega) = \delta_{u+v, 0} \text{mod } 2 \hat{\Delta}_u(z, \omega) + \hat{\bar{\psi}}_u(z)\hat{\psi}_v(\omega) :_M$$

$$\quad \hat{\Delta}_u(z, \omega) = \left(\frac{z}{\omega}\right)^{\frac{1}{2}+1} \frac{1}{z - \omega} + \frac{z}{2\omega^2}(1 - \bar{u}), \quad \bar{u} = 0, 1 \quad (5.2a)$$

$$= \frac{1}{z - \omega} + a_0(\omega) + (z - \omega)a_1(\omega, \bar{u}) + O(z - \omega)^2 \quad (5.2b)$$

$$a_0(\omega) = \frac{3}{2\omega}, \quad a_1(\omega, \bar{u}) = \frac{4 - \bar{u}}{8\omega^2}. \quad (5.2c)$$
Comparing this result with Eq. (4.2a), one obtains the following relations between the two kinds of normal-ordered products

$$
\hat{\psi}_u(z) \hat{\psi}_v(z) = \hat{\bar{\psi}}_u(z) \hat{\psi}_v(z) : M + \frac{3}{2z} \delta_{u+v,0 \mod 2} \tag{5.3a}
$$

$$
\hat{\psi}_u(z) \partial_z \hat{\psi}_v(z) = \hat{\bar{\psi}}_u(z) \partial_z \hat{\psi}_v(z) : M + \frac{u - 16}{8z^2} \delta_{u+v,0 \mod 2} \tag{5.3b}
$$

$$
\partial_z \hat{\psi}_u(z) \hat{\psi}_v(z) = \partial_z \hat{\bar{\psi}}_u(z) \hat{\psi}_v(z) : M + \frac{4 - u}{8z^2} \delta_{u+v,0 \mod 2} \tag{5.3c}
$$

where I remind that $\hat{\bar{\psi}}_u(z)$ is operator-product normal ordering. These relations and Eq. (4.1a,b) give the mode-ordered form of the operators in the twisted ghost system

$$
\hat{J}^G_u(z) \equiv \hat{J}_u(z) - \frac{3}{z} \delta_{u,0 \mod 2} = \sum_{m \in \mathbb{Z}} J^G_u(m + \frac{u}{2}) z^{-(m + \frac{u}{2}) - 1} \tag{5.4a}
$$

$$
\hat{J}^G_u(m + \frac{u}{2}) = \sum_{v} \sum_{p \in \mathbb{Z}} \hat{c}_v(p + \frac{u}{2}) \hat{b}_{u-v}(m - p + \frac{u-v}{2}) : M \tag{5.4b}
$$

$$
\hat{J}^G_0(m = 0) = \frac{1}{2} [\hat{c}_0(0), \hat{b}_0(0)] + \sum_{p=1}^{\infty} (\hat{c}_0(-p) \hat{b}_0(p) - \hat{b}_0(-p) \hat{c}_0(p)) +
\sum_{p=1}^{\infty} (\hat{c}_1(-p + \frac{1}{2}) \hat{b}_{-1}(p - \frac{1}{2}) - \hat{b}_{-1}(-p - \frac{1}{2}) \hat{c}_1(p + \frac{1}{2})) \tag{5.4c}
$$

$$
\hat{\theta}^G_u(z) = \sum_{v} \hat{\psi}_v(z) \partial_z \hat{\psi}_{u-v}(z) + 2 \partial_z \hat{\psi}_v(z) \hat{\psi}_{u-v}(z) : M - \frac{17}{8z^2} \delta_{u,0 \mod 2} \tag{5.4d}
$$

$$
\hat{L}^G_u(m + \frac{u}{2}) = -\sum_{v} \sum_{p \in \mathbb{Z}} (m + p + \frac{u+v}{2}) \hat{c}_v(p + \frac{u}{2}) \hat{b}_{u-v}(m - p + \frac{u-v}{2}) : M
- \frac{17}{8} \delta_{m+v,0} \tag{5.4e}
$$

where $\{\hat{J}^G_u(z)\}$ is the properly-ordered form of the twisted ghost current. The $\{\hat{c}_0, \hat{b}_0\}$ terms of the ghost charge $\hat{J}^G_0(0)$ in Eq. (5.4c) are isomorphic to the untwisted ghost charge in Eq. (2.5c). Using Eqs. (4.1c), (4.2e) (4.3)
and (5.1), mode-ordered expressions can also be written out for the twisted BRST current \( \hat{J}_u^B \) and the BRST charge \( \hat{Q} \), but these will not be needed in the present development.

The mode-ordered forms in Eq. (5.4) and the ghost-mode conditions on the physical states in Eq. (4.6a) then imply the further characterization of the physical states

\[
\langle \hat{J}^G_u((m + \frac{u}{2}) \geq 0) + \frac{1}{2}\delta_{m + \frac{u}{2}, 0})|\chi\rangle = 0 \quad (5.5a)
\]

\[
\langle \hat{L}^G_u((m + \frac{u}{2}) \geq 0) + \frac{17}{8}\delta_{m + \frac{u}{2}, 0})|\chi\rangle = 0, \quad \bar{u} = 0, 1 \quad (5.5b)
\]

in terms of the twisted ghost current and the extended Virasoro generators of the twisted ghost system.

Finally, Eqs. (4.6b) and (5.5b) give the desired description of the physical states of each twisted sector in terms of the extended Virasoro generators of the \( \hat{c} = 52 \) matter:

\[
\langle \hat{L}_u((m + \frac{u}{2}) \geq 0) - \frac{17}{8}\delta_{m + \frac{u}{2}, 0})|\chi\rangle = 0, \quad \bar{u} = 0, 1. \quad (5.6)
\]

Here and in the succeeding papers of this series, I refer to this result as the extended physical-state condition (or extended Virasoro condition) for \( \hat{c} = 52 \) string matter. The two components \( \bar{u} = 0, 1 \) of this condition are the operator analogues of the extended classical Virasoro constraints \( \{ \hat{\theta}_u = 0, \bar{u} = 0, 1 \} \) implied by the corresponding extended actions with \( \mathbb{Z}_2 \)-twisted permutation gravity in Ref. [16].

6 Discussion

This paper begins the discussion of the orbifold-strings of permutation-type at the operator level. In particular, I have constructed here the twisted BRST systems and extended physical-state conditions of all orbifold-string matter at central charge \( \hat{c} = 52 \), which corresponds to the special case of \( \mathbb{Z}_2 \)-twisted permutation gravity in the extended actions of Ref. [16].

I remind the reader that each of the twisted \( \hat{c} = 52 \) strings has twice the conventional number of negative-norm states, and that the two-component form of the extended physical-state condition (5.6) supports our speculation that the orbifold-strings of permutation-type can be free of negative-norm states at higher central charge. The explicit form of the extended matter
Virasoro generators and the physical spectrum of the \( c = 52 \) strings will be discussed in the following paper of the series. In later papers, we will also see that the extended physical-state condition follows as expected from extended Ward identities in the interacting theories.

The derivation given here used the standard operator techniques of the orbifold program [1-15], leaving for another time the more-involved Faddeev-Popov derivation of the twisted BRST systems from the extended actions of Ref. [16]. Using the operator-product form of untwisted BRST in Sec. 2, the standard operator techniques can be straightforwardly applied to obtain the twisted BRST systems of the general orbifold of permutation-type – whose twisted sectors couple to general permutation-twisted world-sheet gravity [16]. I have not yet worked these systems out but, as an educated guess, I would expect the following labelling of the twisted BRST currents \( \{ \hat{J}_j^B \} \) and BRST charges \( \{ \hat{Q}_j \} \) in twisted open-string sector \( \sigma \) of the general orbifold of permutation-type

\[
\hat{J}_j^B (m + \frac{j}{f_j(\sigma)}), \quad \hat{Q}_j \equiv \hat{J}_{0j}^B (0) \quad (6.1a)
\]

\[
[\hat{Q}_i, \hat{Q}_j]_+ = 0 \quad (6.1b)
\]

\[
\hat{c} = 26K, \quad j = 0, 1, \ldots, f_j(\sigma) - 1, \quad \sum_j f_j(\sigma) = K \quad (6.1c)
\]

as well as a right-mover barred copy for the twisted closed-string sectors. In this labelling \( f_j(\sigma) \) is the length of cycle \( j \) in the corresponding element \( h_\sigma \in H(\text{perm}) \), which permutes \( K \) copies of the closed string \( U(1)^{26} \) in the untwisted sector. In particular Eq. (6.1) shows a total of \( K \) twisted BRST currents in each twisted sector \( \sigma \), including a BRST charge for each cycle in \( h_\sigma \). This reduces for \( K = 2 \) to the case studied in this paper because the non-trivial element of \( \mathbb{Z}_2 \) is a single cycle of length 2.

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