CENTRAL LIMIT THEOREM AND COHOMOLOGICAL EQUATION ON HOMOGENEOUS SPACES

BY

RONGGANG SHI*

Shanghai Center for Mathematical Sciences, Jiangwan Campus, Fudan University
No. 2005 Songhu Road, Shanghai 200433, China
e-mail: ronggang@fudan.edu.cn

ABSTRACT

The dynamics of one-parameter diagonal group actions on finite volume homogeneous spaces has a partially hyperbolic feature. In this paper we extend the Livšic-type result to these possibly noncompact and nonaccessible systems. We also prove a central limit theorem for the Birkhoff averages of points on a horospherical orbit. The Livšic-type result allows us to show that the variance of the central limit theorem is nonzero provided that the test function has nonzero mean with respect to an invariant probability measure.

1. Introduction

Let \( X = G/\Gamma \) be a finite volume homogeneous space, where \( G \) is a connected noncompact semisimple Lie group with finite center and \( \Gamma \) is an irreducible lattice of \( G \). Recall that a lattice \( \Gamma \) is said to be irreducible if for any noncompact simple factor \( N \) of \( G \) the group \( NT\Gamma \) is dense in \( G \). The left translation action of \( G \) on \( X \) is (strongly) mixing with respect to the probability Haar measure \( \mu \). The mixing is exponential when the action of \( G \) on \( X \) has a strong spectral gap, i.e., the action of each noncompact simple factor of \( G \) on \( X \) has a spectral

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gap. Recall that the action of a closed subgroup $H$ of $G$ on $X$ is said to have a spectral gap, if there exists $\delta > 0$ and a compactly supported probability measure $\nu$ on $H$ such that for all

$$\varphi \in L^2_{\mu,0} := \{ \varphi \in L^2_\mu : \mu(\varphi) = 0 \}$$

one has

$$\int_X \left| \int_H \varphi(h^{-1}x) \, d\nu(h) \right|^2 \, d\mu(x) \leq (1 - \delta) \int_X |\varphi(x)|^2 \, d\mu(x).$$

It was proved by Mozes [18] that the system $(X, \mu, G)$ is mixing of all orders. The effective version of multiple mixing was proved by Björklund–Einsiedler–Gorodnik [3] under the assumption of the existence of the strong spectral gap. As an application, they obtained a central limit theorem in [5]. The aim of this paper is to give a sufficient condition of nonzero variance and prove new central limit theorems based on [21] where the author proved effective multiple correlations for the trajectory of a certain measure $\nu$ singular to $\mu$.

The irreducibility assumption for $\Gamma$ is unnecessary, so we assume from now on that $\Gamma$ is just a lattice of $G$. We consider the action of a one-parameter Ad-diagonalizable subgroup $F = \{ a_t : t \in \mathbb{R} \}$ on $X$. Here Ad-diagonalizable means the image of $F$ through the adjoint representation of $G$ is diagonal with respect to some basis of the Lie algebra.

Let $\phi : X \to \mathbb{R}$ be a continuous function integrable with respect to $\mu$, and let $\nu$ be a probability measure on $X$. It is said that $(\phi, \nu, F)$ obeys the central limit theorem if the random variables

$$\frac{1}{\sqrt{T}} \int_0^T \phi(a_t x) - \mu(\phi) \, dt \quad \text{given by } (X, \nu)$$

converge as $T \to \infty$ to the normal distribution with mean zero and variance $\sigma = \sigma(\phi, \nu, F) \geq 0$, i.e., for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{T \to \infty} \int_X f \left( \frac{1}{\sqrt{T}} \int_0^T \phi(a_t x) - \mu(\phi) \, dt \right) \, d\nu(x) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} f(s) e^{-\frac{s^2}{2\sigma}} \, ds.$$

If $\sigma = 0$, then the right-hand side of (1.2) is interpreted to be $f(0)$ and we say the central limit theorem is degenerate.

Let $C^\infty_c(X)$ be the space of compactly supported smooth and real valued functions on $X$. It was proved in [5] that $(\phi, \mu, F)$ obeys the central limit theorem if $\phi \in C^\infty_c(X)$ and the action of $F$ on $X$ is exponential mixing of
all orders, e.g., the action of $G$ on $X$ has a strong spectral gap [3]. Some special cases were known earlier in [22, 19, 15] when $G$ is the isometric group of a hyperbolic manifold with constant negative curvature. If in addition $X$ is compact, then the characterization of nonzero variance is well understood. It was proved by Ratner [19] that $\sigma(\phi, \mu, F) = 0$ if and only if the system of cohomological equations parameterized by $s > 0$

\[ \int_0^s \phi(a_t x) - \mu(\phi) \, dt = \varphi(a_s x) - \varphi(x) \quad (1.3) \]

has a measurable solution $\varphi \in L^2_\mu$. Here a measurable solution means (1.3) holds for $\mu$ almost every $x \in X$. Using Livšic’s theorem [16] which says that a measurable solution to (1.3) is equal to a continuous function almost everywhere, Melbourne and Török [17] showed that $\sigma(\phi, \mu, F) = 0$ if and only if

\[ \int_0^s \phi(a_t x) - \mu(\phi) \, dx = 0 \quad \text{for all } x \in X \text{ and } s > 0 \text{ with } a_s x = x. \quad (1.4) \]

Our first main result is to extend Livšic’s theorem to the homogeneous space $X$. This will allow us to give a sufficient condition of nonzero variance similar to (1.4). For $a \in G$, we use $G'_a$ to denote the group generated by the unstable horospherical subgroup $G^+_a$ and the stable horospherical subgroup $G^-_a$, where

\[ G^+_a = \{ g \in G : \lim_{n \to \infty} a^{-n} g a^n \to 1_G \} \quad \text{and} \quad G^-_a = \{ g \in G : \lim_{n \to \infty} a^n g a^{-n} \to 1_G \}. \]

Here and hereafter $1_G$ denotes the identity element of a group $G$. The group $G'_a$ is a connected semisimple Lie group without compact factors and it is normal in $G$. Let $\widehat{C}_c^\infty(X) = C_c^\infty(X) + \mathbb{R}$ be the set of functions which can be written as a sum of a function in $C_c^\infty(X)$ and a constant.

**Theorem 1.1:** Let $X = G/\Gamma$ where $G$ is a connected semisimple Lie group with finite center and $\Gamma$ is a lattice. Let $\{a_t : t \in \mathbb{R}\}$ be a one-parameter Ad-diagonalizable subgroup of $G$ and let $\mu$ be the probability Haar measure on $X$. Suppose $a = a_1$ and the action of $G'_a$ on $X$ has a spectral gap. If $\varphi$ is a measurable solution to the cohomological equation

\[ \psi(x) = \varphi(ax) - \varphi(x) \]

where $\psi \in \widehat{C}_c^\infty(X)$, then $\varphi \in L^2_\mu$ and there is a smooth function $\tilde{\varphi} : X \to \mathbb{R}$ such that $\varphi = \tilde{\varphi}$ almost everywhere with respect to $\mu$. 
In the case where $X$ is compact and accessible, Theorem 1.1 is a special case of Wilkinson [23, Thm. A]. The accessibility assumption in our setting is the same as $G'_a = G$. Our result is new in the case where $X$ is nonaccessible or non-compact. The spectral gap assumption is always satisfied if $G'_a$ is nontrivial, $G$ has no compact factors and $\Gamma$ is irreducible; see Kelmer–Sarnak [13]. In §5, we prove the smoothness of $\varphi$ along $G'_a \pm$ orbits using the method of [23] and Avila–Santamaria–Viana [1]. The method allows us to prove the Hölder continuity of $\varphi$ along $G'_a$ orbits if we only assume $\psi$ is Hölder. The Hölder continuity in the case where $X = \text{SL}_d(\mathbb{R})/\text{SL}_d(\mathbb{Z})$ is due to Le Borgne [14]. This part doesn’t use the spectral gap assumption. In §6, we prove the smoothness of $\varphi$ along the central foliations of $a$ using the ideas from Gorodnik–Spatzier [11] and Fisher–Kalinin–Spatzier [8]. Here we need to use the spectral gap assumption which implies that the dynamical system $(X, \mu, a)$ is exponential mixing. The smoothness of $\varphi$ then follows from a result of Journé [12] which says that a function uniformly smooth on a transverse family of foliations is smooth.

Now we state our result on the central limit theorem and give sufficient conditions for the nonzero variance. In order to define the measure $\nu$ singular to $\mu$ we need to review some concepts from [21]. We first set up the notation.

1. $H$ is a connected normal subgroup of $G$ without compact factors.
2. $P$ is an absolutely proper parabolic subgroup of $H$, i.e., $P$ contains none of the simple factors of $H$.
3. $U$ is the unipotent radical of $P$.
4. $A$ is a maximal Ad-diagonalizable subgroup of $H$ contained in $P$.

The group $U$ is said to be $a$-expanding for some $a \in A$ if, for any nontrivial irreducible representation $\rho : H \to \text{GL}(V)$ on a finite-dimensional real vector space $V$, one has

$$\lim_{n \to \infty} \rho(a^{-n})v = 0 \quad \forall U\text{-fixed } v \in V.$$ 

The expanding cone of $U$ in $A$ is

$$A_U^+ = \{a \in A : U \text{ is } a\text{-expanding}\}$$

and it has nice dynamical properties while translating $U$-slices on $X$. One of the main results of [21] is to explicitly describe the expanding cone $A_U^+$ of $U$ in $A$. We postpone the explicit description of $A_U^+$ to §2 where we also review effective multiple correlations. Here we only give an example which is the motivation of this concept.
Example 1.2: Let $m, n$ be positive integers; 

$$H = G = SL_{m+n}(\mathbb{R}), U = \left\{ \begin{pmatrix} 1_m & h \\ 0_{nm} & 1_n \end{pmatrix} \in H : h \in M_{mn}(\mathbb{R}) \right\}$$

where $1_m$ and $1_n$ are the identity matrices of rank $m$ and $n$, respectively, $A$ is the identity component of diagonal matrices in $G$, and 

$$(1.5) \quad A^+_U = \{ \text{diag}(e^{r_1}, \ldots, e^{r_m}, e^{-t_1}, \ldots, e^{-t_n}) \in G : r_i, t_j > 0 \}.$$ 

Suppose $p \in A$ is a regular element, where regular refers to the projection of $p$ to each simple factor of $H$ is not the identity element. Let $H^+_p$ be the unstable horospherical subgroup of $p$ in $H$. We assume that $H^+_p \leq U$, $a_1 \in A^+_U$ and the conjugation of $a_1$ expands $H^+_p$, i.e., 

$$(\ast) \quad \{ a_t : t > 0 \} \subset A^+_U \overset{\text{def}}{=} \{ a \in A^+_U : H^+_p \leq H^+_a \cap U \}.$$ 

The assumption $(\ast)$ is always checkable due to the explicit description of $A^+_U$; see §2 for more details.

Let $\nu = \nu_{q, z}$ be a probability measure on $X$ given by a compactly supported nonnegative smooth function $q$ on $U$ and $z \in X$ in the following way: 

$$(1.6) \quad \nu_{q, z}(\varphi) = \int_U \varphi(uz) q(u) \, d\mu_U(u) \quad \forall \varphi \in C_c(X),$$

where $\mu_U$ is a fixed Haar measure on $U$ with $\mu_U(q) = 1$. Now we are ready to state the central limit theorem.

Theorem 1.3: Let $X = G/\Gamma$ and $\mu$ be as in Theorem 1.1. Let $H, P, U$ and $A$ be as in (n.1)–(n.4). Suppose the action of $H$ on $X$ has a spectral gap, $F = \{ a_t : t \in \mathbb{R} \}$ is a one-parameter subgroup of $H$ satisfying $(\ast)$ for a regular element $p \in A$ and $\nu = \nu_{q, z}$ is a probability measure on $X$ given by (1.6) for $z \in X$ and $q \in C^\infty_c(U)$. Then for all $\phi \in \widehat{C}^\infty_c(X)$ with $\int_X \phi \, d\mu = 0$, the system $(\phi, \nu, F)$ obeys the central limit theorem with variance

$$(1.7) \quad \sigma(\phi, F) = \lim_{l \to \infty} 2 \int_0^l \int_X \phi(a_t x) \phi(x) \, d\mu(x) \, dt \in [0, \infty).$$

It can be seen from (1.7) that the variance $\sigma(\phi, F)$ does not depend on $\nu$ and is the same as $\sigma(\phi, \mu, F)$. So the sufficient condition for the nonzero $\sigma(\phi, F)$ below can also be applied to the central limit theorem proved in [5]. In the setting of Example 1.2 with

$$\Gamma = SL_{m+n}(\mathbb{Z}),$$
Theorem 1.3 is due to Björklund–Gorodnik [4]. The difference between our
proof and that in [4] is that instead of using commulants, we prove directly that
the sequence of moments of each fixed order converges to the corresponding
moment of the normal distribution.

**Theorem 1.4:** Let the notation and the assumptions be as in Theorem 1.3.
Then the variance \(\sigma(F,\phi)\) in (1.7) is equal to zero if and only if the system of
cohomological equations

\[
\int_0^s \phi(a_t x) \, dt = \varphi(a_s x) - \varphi(x) \quad (s > 0)
\]

has a solution \(\varphi \in L^2_\mu\).

This theorem together with Theorem 1.1 allow us to give a sufficient condition
for the nonzero variance.

**Definition 1.5:** A function \(\phi \in \mathcal{O}_c^\infty(X)\) is said to be **dynamically null** with
respect to \((X,F)\) if for any \(F\)-invariant probability measure \(\tilde{\mu}\) on \(X\) one has
\[
\tilde{\mu}(\phi) = 0.
\]

**Theorem 1.6:** Let the notation and assumptions be as in Theorem 1.3. If the
variance \(\sigma(F,\phi)\) in (1.7) is zero, then the function \(\phi\) is dynamically null with
respect to \((X,F)\).

The above theorem says that the central limit theorem is nondegenerate if
the integral of \(\phi\) with respect to an \(F\)-invariant probability measure is nonzero.

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2. Preliminaries

In this section we review some facts and prove a couple of auxiliary results.
Let the notation and assumptions be as in Theorem 1.3. Some results in this
section are also used in the proof of Theorem 1.1 where we take \(H = G'_a\).

We first state the explicit description of the expanding cone and the result of
effective multiple correlations in [21]. Let \(\mathfrak{h}, \mathfrak{u}\) and \(\mathfrak{a}\) be the Lie algebras of \(H, U\)
and \(A\), respectively. Let \(\text{Ad} : H \to \text{GL}(\mathfrak{h})\) and \(\text{ad} : \mathfrak{h} \to \text{End}(\mathfrak{h})\) be the adjoint
representations of the Lie group and the Lie algebra, respectively.
Recall that we assume (⋆) holds, namely, \(a_1\) belongs to the expanding cone \(A_U^+\) and the conjugation of \(a_1\) expands a horospherical subgroup \(H_p^+\) contained in \(U\). The latter means that the eigenvalues of \(\text{Ad}(a_1)\) on the Lie algebra of \(H_p^+\) are bigger than one. The assumption \(a_1 \in A_U^+\) is also easy to check and the details are given below.

Let \(\Phi(u)\) be the set of nonzero linear forms \(\beta\) on \(a\) such that there exists a nonzero \(v \in u\) with
\[
\text{ad}(s)v = \beta(s)v \quad \forall s \in a.
\]
Let \(B(\cdot, \cdot)\) be the Killing form on \(h\). Then for each \(\beta \in \Phi(u)\), there exists a unique \(s_\beta \in a\) such that
\[
B(s_\beta, s) = \beta(s) \quad \forall s \in a.
\]

**Theorem 2.1 ([21]):** The expanding cone \(A_U^+ = \exp \mathfrak{a}_u^+\) where
\[
\mathfrak{a}_u^+ = \left\{ \sum_{\beta \in \Phi(u)} t_\beta h_\beta : t_\beta > 0 \right\}.
\]

In the statement of the effective multiple correlations we need to use the \((2, \ell)\)-Sobolev norm for a positive integer \(\ell\). Recall that a vector field on a smooth manifold \(Y\) is a smooth section of the tangent bundle of \(Y\). A vector field \(v\) on \(Y\) defines a partial differential operator \(\partial^v\) on \(C^\infty(Y)\). Given \(\alpha = (v_1, \ldots, v_k)\) where \(v_i\) \((1 \leq i \leq k)\) are vector fields on \(Y\) we take \(\partial^\alpha = \partial^{v_1} \cdots \partial^{v_k}\) and \(|\alpha| = k\).

We also allow \(\alpha\) to be the null set where \(\partial^\alpha\) is the identity operator and \(|\alpha| = 0\).

Elements of the Lie algebra \(\mathfrak{g}\) are naturally identified with the right invariant vector fields on \(G\) which descend naturally to vector fields on \(X\). For every \(v \in \mathfrak{g}\) we use the same notation \(v\) for the induced vector field on \(X\). We fix a basis \(\mathfrak{b}\) of \(\mathfrak{g}\) consisting of eigenvectors of \(\text{Ad}(a_1)\) and for \(\phi \in \hat{C}_c^\infty(X)\) denote
\[
\|\phi\|_\ell = \max_{|\alpha| \leq \ell} \|\partial^\alpha \phi\|_{L^2_\mu},
\]
where the maximum is taken over all the \(k\)-tuples \(\alpha\) \((0 \leq k \leq \ell)\) with alphabet \(\mathfrak{b}\).

We fix an inner product on \(\mathfrak{g}\) such that elements of \(\mathfrak{b}\) are orthogonal to each other. Let \(d_G\) be the Riemannian distance on \(G\) given by a right invariant Riemannian manifold structure induced from the inner product on \(\mathfrak{g}\). The advantage of \(d_G\) is that there exists \(\kappa > 0\) such that
\[
d_G(1_G, a_t g a_{-t}) \leq e^{\kappa |t|} d_G(1_G, g) \quad (2.1)
\]
for any $t \in \mathbb{R}$ and $g \in G$. The Lipschitz norm on $\hat{C}^\infty_c(X)$ is defined by
\[
\|\phi\|_{\text{Lip}} = \sup_{g,h \in G, g \Gamma \neq h \Gamma} \frac{|\phi(g \Gamma) - \phi(h \Gamma)|}{d(g \Gamma, h \Gamma)} \quad \text{where} \quad d(g \Gamma, h \Gamma) = \inf_{\gamma \in \Gamma} d_G(g \gamma, h).
\]
The sup norm on $\hat{C}^\infty_c(X)$ is defined by
\[
\|\phi\|_{\text{sup}} = \sup_{x \in X} |\phi(x)|.
\]
Recall that $\nu = \nu_{q,z}$ is a fixed probability measure on $X$ given by $z \in X$ and $q \in C^\infty_c(U)$ through the formula (1.6).

**Theorem 2.2** ([21, Thm. 4.5]): There exist absolute constants $\delta > 0$, $\ell \in \mathbb{N}$ and $M \geq 1$ with the following properties: for any positive integer $k$, any $\phi_i \in \hat{C}^\infty_c(X)$ $(1 \leq i \leq k)$ and any real numbers $t_1, t_2, \ldots, t_k$ one has
\[
(2.2) \quad \left| \int_X \prod_{i=1}^k \phi_i(a_{t_i}x) \, d\nu(x) - \int_X \phi_k \, d\mu \cdot \int_X \prod_{i=1}^{k-1} \phi_i(a_{t_i}x) \, d\nu(x) \right| \\
\leq Mk \cdot \max_{1 \leq i \leq k} \|\phi_i\| \left( \max_{1 \leq i \leq k} \|\phi_i\|_{\text{sup}} \right)^{k-1} \cdot e^{-\delta \min\{t_k, t_k-t_1, t_k-t_2, \ldots, t_k-t_{k-1}\}},
\]
where $\| \cdot \|$ is the norm on $\hat{C}^\infty_c(X)$ given by
\[
\|\phi\| = \max\{\|\phi\|_{\text{sup}}, \|\phi\|_\ell, \|\phi\|_{\text{Lip}}\} \quad \forall \phi \in \hat{C}^\infty_c(X).
\]
We make a convention in this paper that the product indexed by the null set is 1 and the sum indexed by the null set is 0. So in (2.2), if $k = 1$ then
\[
\prod_{i=1}^{k-1} \phi_i(a_{t_i}x) = 1.
\]

**Lemma 2.3:** There exists $\delta' > 0$, $E_0 > 0$ and $\ell_0 \in \mathbb{N}$ such that for any functions $\phi, \psi \in \hat{C}^\infty_c(X)$ and any $t \in \mathbb{R}$ one has
\[
\left| \int_X \phi(a_t x) \psi(x) \, d\mu(x) - \int_X \phi \, d\mu \int_X \psi \, d\mu \right| \leq E_0 \|\phi\|_{\ell_0} \|\psi\|_{\ell_0} e^{-\delta'|t|}.
\]

**Proof.** Recall that we assume the action of $H$ on $X$ has a spectral gap and $F$ has nontrivial projection to each simple factor of $H$. So the conclusion follows from [7, §6.2.2].

Actually the dynamical system $(X, \mu, a_t)$ is mixing of all orders. This fact is proved in [3, Thm. 1.1] under the assumption of strong spectral gap. It might be possible to get a proof from [3] under our weaker assumption, but
we give a simpler proof here using the method of [21] for completeness. The exponential mixing of all orders will follow from an estimate similar to (2.2) with $\nu$ replaced by $\mu$. During the proof we will use the notation $f_1 \lesssim_* f_2$ for two nonnegative functions which means $f_1 \leq C f_2$ for some positive constant $C$ possibly depending on $\ast$.

**Lemma 2.4:** The conclusion of Theorem 2.2 holds with $\nu$ replaced by $\mu$.

**Sketch of Proof.** The proof is the same as that of [21, Thm. 4.5], so we only give a sketch of the key steps. The main difference is that instead of using quantitative nonescape of mass [21, Thm. 1.3] we need to use the effective estimate of the volume in the cusp.

We use $B^G_r$ to denote the open ball of radius $r$ in $G$ centered at the identity element. The injectivity radius at $x \in X$ is defined by

$$I(x, X) = \sup \{ r > 0 : g \mapsto gx \text{ is injective on } B^G_r \}.$$  

For $\varepsilon > 0$, let

$$\text{Inj}_\varepsilon = \{ x \in X : I(x, X) \geq \varepsilon \}.$$

We claim that there exist $C_1, \delta_1 > 0$ such that

$$\mu(X \setminus \text{Inj}_\varepsilon) \leq C_1 \varepsilon^{\delta_1} \quad \text{for all } \varepsilon > 0. \tag{2.3}$$

We don’t have a direct proof of this fact, but it is a corollary of the equidistribution of measures in [21, Thm. 1.4] and the quantitative nonescape of mass in [21, Thm. 1.3]. More precisely, we fix a Haar measure $\mu_1$ on $G^+$ such that the open ball of radius 1 in $G^+$ (denoted by $B^+_1$) has measure 1. Since the action of $H$ on $X$ has a spectral gap, there exists $x \in X$ such that $Hx$ is dense in $X$. Therefore, by [21, Thm. 1.4] we have for any $\varphi \in C_c(X)$

$$\lim_{t \to \infty} \int_{B^+_1} \varphi(a_t hx) \, d\mu_1(h) = \mu(\varphi). \tag{2.4}$$

On the other hand, by [21, Thm. 1.3], there exists $C_1 > 0$ and $\delta_1 > 0$ such that

$$\mu_1(\{ h \in B^+_1 : a_t hx \in X \setminus \text{Inj}_\varepsilon \}) \leq C_1 \varepsilon^{\delta_1} \quad \text{for all } t \geq 0 \text{ and } \varepsilon > 0. \tag{2.5}$$

In view of (2.4) and (2.5), if $\varphi : X \to [0, 1]$ and $\text{supp}(\varphi) \subset X \setminus \text{Inj}_\varepsilon$, then

$$\mu(\varphi) \leq C_1 \varepsilon^{\delta_1}.$$ 

Now (2.3) follows from taking a sequence of increasing functions $\varphi$ which converges to the characteristic function of $X \setminus \text{Inj}_\varepsilon$. 


To prove the lemma, we assume without loss of generality that
\[ k \geq 2 \quad \text{and} \quad t = \min\{t_k, t_k - t_1, \ldots, t_k - t_{k-1}\} > 0, \]
since otherwise the conclusion is trivial. Let \( \ell = \ell_0 + \dim G \) where \( \ell_0 \) is as in Lemma 2.3 and \( r = e^{\frac{-t_0}{3\ell+1}} \). There is a smooth function \( \theta : G^+ \to [0, \infty) \) such that \( \text{supp} (\theta) \) is contained in the ball of radius \( r \) in \( G^+ \), \( \mu_1(\theta) = 1 \) and
\[ \|\theta\|_{\ell} \lesssim r^{-2\ell}. \]

Let \( G^+ \) be the unstable horospherical subgroup of \( a_1 \) in \( G \). Since \( H \) is a normal subgroup of \( G \), one has \( G^+ \) is a subgroup of \( H \). Then
\[
\int_X \prod_{i=1}^k \phi_i(a_{t_i}x) \, d\mu(x) = \int_{G^+} \theta(g) \int_X \prod_{i=1}^k \phi_i(a_{t_i}x) \, d\mu(x) \, dg
\]
\[ = \int_{G^+} \int_X \theta(g) \prod_{i=1}^k \phi_i(a_{t_i}a_{t_k-t_i}ga_{t_k-t_{i-1}}) \, d\mu(x) \, dg \]
\[ = \int_{G^+} \int_X \theta(g) \phi_k(a_{t_k}gx) \prod_{i=1}^{k-1} \phi_i(g_i a_{t_i}x) \, d\mu(x) \, dg, \]
where \( g_i = a_{t_i-t_{k+1}} + t_i ga_{t_k-t_{i-1}} \) satisfies
\[ d_G(1_G, g_i) \lesssim r. \]
So
\[ |\phi_i(g_i a_{t_i}x) - \phi_i(a_{t_i}x)| \leq \|\phi_i\|_{\text{Lip}} r. \] (2.7)

Using (2.7), we can effectively replace \( \phi_i(g_i a_{t_i}x) \) on the right-hand side of (2.6) by \( \phi_i(a_{t_i}x) \), and reduce the estimate of (2.6) to the estimate of
\[ \int_X \int_{G^+} \theta(g) \phi_k(a_{t_k}gx) \, dg \prod_{i=1}^{k-1} \phi_i(a_{t_i}x) \, d\mu(x). \] (2.8)

We have an effective estimate of
\[ \int_{G^+} \theta(g) \phi_k(a_{t_k}gx) \, dg \]
for \( x \in \text{Inj}_\varepsilon \) using [21, Lemma 4.3]. We have an effective estimate of the volume of \( X \setminus \text{Inj}_\varepsilon \) using (2.3). These two estimates together allow us to give an effective estimate of (2.8).
In the proof of the central limit theorem, we need to know the growth of the norm $\| \cdot \|$ of the functions

$$\phi \circ a_t(x) = \phi(a_t x) \quad \text{for } t \geq 0.$$  

Clearly, for all $\phi \in \hat{C}_c^\infty(X)$ we have $\| \phi \|_{\sup} = \| \phi \circ a_t \|$. For the other two norms we have the following lemma.

**Lemma 2.5:** There exists $\kappa > 0$ such that for all $\phi \in \hat{C}_c^\infty(X)$ and $t \geq 0$ one has

$$\| \phi \circ a_t \|_\ell \leq e^{\kappa t} \| \phi \|_\ell \quad \text{and} \quad \| \phi \circ a_t \|_{\Lip} \leq e^{\kappa t} \| \phi \|_{\Lip}.$$  

Therefore,

$$\| \phi \circ a_t \| \leq e^{\kappa t} \| \phi \|.$$  

**Proof.** Let $v \in \mathfrak{b}$ which is considered as a right invariant vector field and let $v_1$ be the value of $v$ at the identity element. Let $\tilde{\phi}$ be the lift of $\phi$ to $G$. For $g \in G$,

$$\partial^v(\phi \circ a_t)(g\Gamma) = v_1(\tilde{\phi}(a_t x g)) = v_1(\tilde{\phi}(a_t x a_{-t} \cdot a_t g)) = \partial^{\Ad(a_t)}v \tilde{\phi}(a_t g\Gamma) = e^{\kappa_1 t} \partial^v \tilde{\phi}(a_t g\Gamma),$$

where we use the assumption that $\mathfrak{b}$ consists of eigenvectors of $\Ad(a_1)$ so that there exists $\kappa_1 \in \mathbb{R}$ with $\Ad(a_t)v = e^{\kappa_1 t}v$. In general for any $\alpha \in B_k$ there exists $\kappa_\alpha \in \mathbb{R}$ such that

$$\partial^\alpha(\phi \circ a_t) = e^{\kappa_\alpha} \cdot (\partial^\alpha \phi) \circ a_t.$$  

Therefore the first inequality of (2.9) holds for any $\kappa \geq \max_{|\alpha| \leq \ell} \kappa_\alpha$.

For $g, h \in G$ with $g\Gamma \neq h\Gamma$, we have

$$\frac{|\phi(a_t g\Gamma) - \phi(a_t h\Gamma)|}{d(g\Gamma, h\Gamma)} \leq \|\phi\|_{\Lip} \frac{d(a_t g\Gamma, a_t h\Gamma)}{d(g\Gamma, h\Gamma)},$$

Suppose $d(g\Gamma, h\Gamma) = d_G(g, h\gamma)$ for some $\gamma \in \Gamma$. Then by the right invariance of $d_G$, the definition of $d$ and (2.1), we have

$$d(a_t g\Gamma, a_t h\Gamma) \leq d_G(1_G, a_t h\gamma g^{-1}a_{-t}) \leq e^{\kappa t}d_G(1_G, h\gamma g^{-1}) = e^{\kappa t}d(g\Gamma, h\Gamma).$$

Therefore the second inequality of (2.9) holds. \qed
Lemma 2.6: For any sequence \( \{ \phi_n \} \) of functions in \( \hat{C}_c^\infty(X) \) one has

\[
\| \phi_1 \cdots \phi_n \| \leq n^\ell \cdot \left( \max_{1 \leq i \leq n} \| \phi_i \| \right)^\ell \cdot \left( \max_{1 \leq i \leq n} \| \phi_i \|_{\text{sup}} \right)^{n-\ell}.
\] (2.10)

Proof. It suffices to show that (2.10) holds if we replace the norm in the left-hand side by any of the three norms used to define \( \| \cdot \| \). It is clear that this is true for \( \| \cdot \|_{\text{sup}} \).

For different \( x, y \in X \), we have

\[
\left| \prod_{i=1}^n \phi_i(x) - \prod_{i=1}^n \phi_i(y) \right| \leq \sum_{k=1}^n \left| \prod_{i=1}^k \phi_i(x) \prod_{j=k+1}^n \phi_j(y) - \prod_{i=1}^{k-1} \phi_i(x) \prod_{i=k}^n \phi_j(y) \right| \leq n \max_{1 \leq i \leq n} \left( \| \phi_i \|_{\text{Lip}} \prod_{1 \leq j \leq n, j \neq i} \| \phi_j \|_{\text{sup}} \right).
\]

Therefore

\[
\| \phi_1 \cdots \phi_n \|_{\text{Lip}} \leq \text{RHS of (2.10)}.
\]

Let \( \alpha = (v_1, \ldots, v_k) \in b^k \) where \( 1 \leq k \leq \ell \). The product rule of the differential operators implies

\[
\partial^\alpha \prod_{1 \leq i \leq n} \phi_i = \sum \prod_{i=1}^n \partial^{\alpha_i} \phi_i,
\]

where there are \( n^k \) terms in the summation and at most \( k \leq \ell \) of the \( \partial^{\alpha_i} \) are not the identity operator. Therefore,

\[
\left\| \partial^\alpha \prod_{1 \leq i \leq n} \phi_i \right\|_{L^2_{\mu}} \leq n^\ell \left( \max_{1 \leq i \leq n} \| \phi_i \| \right)^\ell \left( \max_{1 \leq m \leq n} \| \phi_m \|_{\text{sup}} \right)^{n-\ell} \leq \text{RHS of (2.10)},
\]

which completes the proof.

3. Variance

In this section we prove the finiteness of the variance in (1.7) and Theorem 1.4. All of them are contained in the following lemma.
Lemma 3.1: Let the notation and the assumptions be as in Theorem 1.3. In particular, \( \phi \in \hat{C}_c^\infty(X) \) and \( \mu(\phi) = 0 \). For a real number \( t \in \mathbb{R} \), let \( \phi_t \) be the function \( \phi(atx) \). Then:

(i) \( \lim_{l \to \infty} 2 \int_0^l \mu(\phi_t \phi) \, dt \) converges to a nonnegative real number, i.e., \( \sigma(\phi, F) \) of (1.7) is well-defined and nonnegative.

(ii) \( \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^T \int_X \phi_t \phi_s \, d\nu \, dt \, ds = \sigma(\phi, F) \).

(iii) \( \sigma(\phi, F) = 0 \) if and only if there exists \( \varphi \in L^2_\mu \) such that for all \( s > 0 \)

\[
\int_0^s \phi_t(x) \, dt = \varphi(asx) - \varphi(x) \quad \text{for } \mu\text{-a.e. } x \in X.
\]

Proof. (i) In view of Lemma 2.3, there exists \( \delta' > 0 \) such that \( |\mu(\phi_t \phi)| \lesssim \phi e^{-\delta't} \) for all \( t \geq 0 \). Therefore, \( 2 \int_0^l \mu(\phi_t \phi) \, dt \) converges as \( l \to \infty \). This proves that \( \sigma(\phi, F) \) is well-defined.

To prove \( \sigma(\phi, F) \) is nonnegative, we show that it is equal to

\[
\sigma' \overset{\text{def}}{=} \lim_{T \to \infty} \frac{1}{T} \int_X \left( \int_0^T \phi_t \, dt \right)^2 \, d\mu = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^T \mu(\phi_t \phi_s) \, dt \, ds,
\]

which is obviously nonnegative. By the symmetry of \( t \) and \( s \), and the invariance of \( \mu \) under \( F \), we have

\[
\sigma' = \lim_{T \to \infty} \frac{2}{T} \int_0^T \int_s^T \mu(\phi_t \phi_s) \, dt \, ds = \lim_{T \to \infty} \frac{2}{T} \int_0^T \int_s^T \mu(\phi_{t-s} \phi) \, dt \, ds.
\]

We make a change of variables \( (s, t) \to (s, r) = (s, t-s) \), then

\[
\sigma' = \lim_{T \to \infty} \frac{2}{T} \int_0^T \int_0^{T-r} \mu(\phi_r \phi) \, ds \, dr = \lim_{T \to \infty} \frac{2}{T} \int_0^T (T-r) \mu(\phi_r \phi) \, dr
\]

\[
= \sigma(\phi, F) - \lim_{T \to \infty} \frac{2}{T} \int_0^T r \mu(\phi_r \phi) \, dr.
\]

Since \( |\mu(\phi_r \phi)| \lesssim \phi e^{-\delta' r} \) for \( r \geq 0 \), the value \( |\int_0^T r \mu(\phi_r \phi) \, dr| \) is uniformly bounded for all \( T > 0 \). So

\[
\lim_{T \to \infty} \frac{2}{T} \int_0^T r \mu(\phi_r \phi) \, dr = 0,
\]

and hence \( \sigma(\phi, F) = \sigma' \).
(ii) Suppose \( l \geq 1 \) and \( T \geq 10l \). For every \( r \) with \( 0 \leq r \leq l \), we apply (2.2) for \( k = 1 \), the function \( \phi \phi_r \) and \( s \geq 0 \); then we have

\[
\int_X \phi_s \phi_{r+s} \, d\nu = \mu(\phi \phi_r) + O_{\phi,l}(e^{-\delta s}).
\]  

(3.3)

We decompose the function on \( X \) defined by

\[
\frac{1}{2} \int_0^T \int_0^T \phi_s(x) \phi_t(x) \, dt \, ds = \int_0^T \int_s^T \phi_s(x) \phi_t(x) \, dt \, ds
\]

(3.4)

into \( \xi(x) + \eta(x) \), where

\[
\xi(x) = \int_0^T \int_s^{\min\{T,s+l\}} \phi_s(x) \phi_t(x) \, dt \, ds,
\]

\[
\eta(x) = \int_0^{T-l} \int_s^{T+l} \phi_s(x) \phi_t(x) \, dt \, ds.
\]

To calculate \( \nu(\xi) \) we make a change of variables \((s,t) \to (s,r) = (s,t-s)\) and apply Fubini's theorem:

\[
\nu(\xi) = \int_0^l \int_0^{T-r} \int_X \phi_s \phi_{r+s} \, d\nu \, ds \, dr
\]

(3.5)

\[
= \int_0^l \int_0^{T-r} \mu(\phi_r \phi) + O_{\phi,l}(e^{-\delta s}) \, ds \, dr \quad \text{(by (3.3))}
\]

\[
= T \int_0^l \mu(\phi_r \phi) \, dr + O_{\phi,l,\delta}(1).
\]

On the other hand, by (2.2) with \( k = 2 \), \( t_2 = t \) and \( t_1 = s \),

\[
\nu(\eta) = \int_0^{T-l} \int_s^{T+l} \phi_t \phi_s \, d\nu \, dt \, ds
\]

(3.6)

\[
= \int_0^{T-l} \int_s^{T+l} O_\phi(e^{-\delta(t-s)}) \, dt \, ds
\]

\[
= \int_0^{T-l} O_{\phi,\delta}(e^{-\delta l}) \, ds
\]

\[
= (T - l)O_{\phi,\delta}(e^{-\delta l}).
\]
By (3.4), (3.5) and (3.6), we have

$$\frac{1}{T} \int_0^T \int_0^T \int_X \phi_t(x) \phi_s(x) \, d\nu \, dt \, ds$$

(3.7)

$$= \frac{2\nu(\xi) + 2\nu(\eta)}{T}$$

$$= 2 \int_0^l \mu(\phi_r \phi) \, dr + \frac{1}{T} O_{\phi,l,\delta} (1) + O_{\phi,\delta} (e^{-\delta l}).$$

We show that the left-hand side of (3.7) is arbitrarily close to $\sigma(\phi, F)$ provided that $T$ is sufficiently large. Given $\varepsilon > 0$, in view of (i), there exists $l \geq 1$ such that

$$\left| 2 \int_0^l \mu(\phi_r \phi) \, dr - \sigma(\phi, F) \right| < \varepsilon \quad \text{and} \quad \left| O_{\phi,\delta} (e^{-\delta l}) \right| < \varepsilon.$$

For this fixed $l$ we have

$$\left| \frac{1}{T} O_{\phi,l,\delta} (1) \right| < \varepsilon$$

provided that $T$ is sufficiently large. Therefore (3.1) holds.

(iii) Suppose (3.2) holds for all $s > 0$. Recall that we have proved in (i) that

$$\sigma(\phi, F) = \lim_{T \to \infty} \int_X \frac{1}{T} \left( \int_0^T \phi_t \, dt \right)^2 \, d\mu.$$

(3.8)

By (3.2),

$$\int_0^T \phi_t(x) \, dt = \varphi(a_T x) - \varphi(x) \quad \text{for } \mu\text{-a.e. } x.$$

This together with (3.8) and $\varphi \in L^2_\mu$ imply $\sigma(\phi, F) = 0$.

Now we assume $\sigma(\phi, F) = 0$. We claim that

the $L^2$-norm of $\xi_T(x) \overset{\text{def}}{=} \int_0^T \phi_t(x) \, dt$ is uniformly bounded for all $T \geq 0$.

Let $\eta(x) = \xi_1(x)$ and $\eta_i(x) = \eta(a_i x)$. Then the claim is equivalent to stating that the $L^2$-norm of

$$\xi_n(x) = \sum_{i=0}^{n-1} \eta_i$$

is uniformly bounded for all $n \geq 2$. 

Since $\mu$ is $F$-invariant,

$$\int_X \left( \sum_{i=0}^{n-1} \eta_i \right)^2 d\mu = n\mu(\eta)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} \mu(\eta_i - j\eta)$$

(3.9)

$$= n\mu(\eta)^2 + 2 \sum_{i=1}^{n-1} (n - i)\mu(\eta_i\eta).$$

By (3.8) and (3.9)

$$0 = \sigma(\phi, F) = \lim_{m \to \infty} \int_X \frac{1}{m} \left( \sum_{i=0}^{m-1} \eta_i \right)^2 d\mu$$

$$= \mu(\eta^2) + \lim_{m \to \infty} 2 \sum_{i=1}^{m-1} \left( 1 - \frac{i}{m} \right) \mu(\eta_i\eta).$$

We solve $\mu(\eta^2)$ from the above equation and plug it in (3.9); then

$$\int_X \left( \sum_{i=0}^{n-1} \eta_i \right)^2 d\mu = -2n \left( \lim_{m \to \infty} \sum_{i=1}^{n-1} \left( 1 - \frac{i}{m} \right) \mu(\eta_i\eta) \right) + 2 \sum_{i=1}^{n-1} (n - i)\mu(\eta_i\eta)$$

(3.10)

$$+ \sum_{i=1}^{n-1} (n - i)\mu(\eta_i\eta)$$

$$= -2n \lim_{m \to \infty} \sum_{i=n}^{m-1} \left( 1 - \frac{i}{m} \right) \mu(\eta_i\eta) - 2 \sum_{i=1}^{n-1} i\mu(\eta_i\eta).$$

Note that the function $\eta \in \hat{C}_c^\infty(X)$. So Lemma 2.3 implies

$$\left| 2n \lim_{m \to \infty} \sum_{i=n}^{m-1} \left( 1 - \frac{i}{m} \right) \mu(\eta_i\eta) \right| \lesssim n \sum_{i=n}^{m-1} e^{-\delta' i} \leq \frac{ne^{-\delta' n}}{1 - e^{-\delta'}},$$

which converges to zero as $n \to \infty$. So the absolute value of the first term of (3.10) is uniformly bounded for all $n \in \mathbb{N}$. Similarly, we can uniformly bound the second term of (3.10) as

$$\left| 2 \sum_{i=1}^{n-1} i\mu(\eta_i\eta) \right| \lesssim n \sum_{i=1}^{n-1} ie^{-\delta' i} \lesssim \int_1^\infty te^{-\delta' t} dt < \infty.$$

Therefore, the $L^2$-norm of $\xi_n$ is uniformly bounded for all $n \in \mathbb{N}$ and the proof of the claim is complete.
Note that the Hilbert space $L^2_\mu$ is self-dual. So the claim and Alaoglu's theorem imply that there exists a subsequence $\{n_i\}$ of natural numbers and $\varphi \in L^2_\mu$ such that
\[
\lim_{i \to \infty} \xi_{n_i} = -\varphi
\]
in the weak* topology. We show that (3.2) holds for this $\varphi$. It is not hard to see that the function $\varphi(a_s x)$ is a weak* limit of the sequence $\{-\xi_{n_i}(a_s x)\}_i$ in $L^2_\mu$. Therefore, in the weak* topology we have
\[
\varphi \circ a_s - \varphi = \lim_{i \to \infty} (\xi_{n_i} - \xi_{n_i} \circ a_s) = \int_0^s \phi_t \, dt - \lim_{i \to \infty} \int_0^{s} \phi_{t+n_i} \, dt.
\]
On the other hand, given any $\psi \in C_c^\infty(X)$, by Lemma 2.3
\[
\left| \lim_{i \to \infty} \int_0^s \int_X \phi_{t+n_i}(x) \psi(x) \, d\mu(x) \, dt \right| \leq \lim_{i \to \infty} \int_0^s \left| \int_X \phi_{t+n_i}(x) \psi(x) \, d\mu(x) \right| \, dt \\
\lesssim_{\phi, \psi} \lim_{i \to \infty} \int_0^s e^{-\delta'(t+n_i)} \, dt = 0.
\]
Therefore
\[
\lim_{i \to \infty} \int_0^s \phi_{t+n_i}(x) \, dt = 0
\]
in the weak* topology. This observation together with (3.11) imply (3.2).

4. Proof of the central limit theorem

Let the notation and the assumptions be as in Theorem 1.3. In particular, we fix a function $\phi \in \hat{C}_c(X)$ and take $\phi_t = \phi \circ a_t$. In this section the dependence of constants on $\phi$ and the probability measure $\nu$ will not be specified. By possibly replacing $\phi$ by $\frac{\phi}{\|\phi\|}$ we assume without loss of generality that $\|\phi\|_{\text{sup}} \leq \|\phi\| \leq 1$. We assume that $\phi$ is not identically zero, since otherwise the conclusion holds trivially. Let $M, \ell, \kappa \geq 1$ and $\delta > 0$ be such that Theorem 2.2 and Lemma 2.5 hold.

We need to show that the random variables
\[
S_T : (X, \nu) \to \mathbb{R} \quad \text{where} \quad S_T(x) = \frac{1}{\sqrt{T}} \int_0^T \phi_t(x) \, dt
\]
converges as $T \to \infty$ to the normal distribution with mean zero and variance $\sigma = \sigma(\phi, F)$. Our main tool is the following so-called second limit theorem in the theory of probability.
THEOREM 4.1 ([10]): If for every \( n \geq 0 \)

\[
\lim_{T \to \infty} \frac{1}{n!} \int_X S^n_T(x) \, d\nu(x) = \begin{cases} 
\frac{(\sigma/2)^{n/2}}{(n/2)!} & \text{for } n \text{ even}, \\
0 & \text{for } n \text{ odd}, 
\end{cases}
\]

then as \( T \to \infty \) the distribution of random variables \( S_T \) on \((X,\nu)\) converges to the normal distribution with mean zero and variance \( \sigma \).

Note that (4.1) obviously holds for \( n = 0 \) and \( n = 1 \). The case of \( n = 2 \) is proved in Lemma 3.1(ii). So we assume \( n \geq 3 \) in the rest of this section unless otherwise stated. We will estimate

\[
b_{n,T} = \frac{1}{n!} \int_X \left( \int_0^T \phi_t(x) \, dt \right)^n \, d\nu
\]

for each fixed \( n \geq 3 \). Write \( t = (t_1, \ldots, t_n) \), \( dt = dt_1 \cdots dt_n \), \( \phi_t = \prod_{i=1}^n \phi_{t_i} \) and

\[
[0, T]_\leq^n = \{ t \in [0, T]^n : t_1 \leq t_2 \leq \cdots \leq t_n \}.
\]

By the symmetry of the variables of \( t \) and the Fubini’s theorem,

\[
b_{n,T} = \int_X \int_{[0, T]_\leq^n} \phi_t(x) \, dt \, d\nu = \int_{[0, T]_\leq^n} \int_X \phi_t(x) \, d\nu \, dt.
\]

An ordered partition

\[
P = \{ \{1, 2, \ldots, m_1\}, \{m_1 + 1, \ldots, m_2\}, \ldots, \{m_{|P|-1} + 1, \ldots, n - 1, n\} \}
\]

of \{1, \ldots, n\} determines \( |P| - 1 \) positive integers \( 1 \leq m_1 < \cdots < m_{|P|-1} < n \) and vice versa. We use \( P \prec n \) to denote \( P \) is an ordered partition of \{1, \ldots, n\}. Let \( |P| = k \) be the cardinality of \( P \) and let \( m_0 = 0, m_k = n \). Although \( k \) and \( m_i \) depend on \( P \), we will not specify it for simplicity. We fix a positive real number

\[
b = n + \frac{4n\ell + 2}{\delta}
\]

and set

\[
I_P = \{ t \in [0, T]_\leq^n : t_{m_{i+1}} - t_{m_i} > b^{m_{i+1} - m_i} n \log T \text{ for } 1 \leq i < k \text{ and } t_{j+1} - t_j \leq b^{m_{i+1} - m_i} n \log T \text{ for } m_{i-1} < j < m_i, 1 \leq i \leq k \}.
\]

We will always assume

\[
T > Mn^2 b^n \log T
\]

so that \( I_P \neq \emptyset \) for any \( P \prec n \). This will allow us to avoid ambiguity in the discussions below.
Lemma 4.2: The set $[0, T]_n^{\leq}$ is a disjoint union of $I_P$ where $P$ is taken over all the ordered partitions of $\{1, \ldots, n\}$.

Proof. Given $t \in [0, T]_n^{\leq}$, we show that there exists a $P < n$ such that $t \in I_P$. We find the blocks of $P$ from the top to the bottom. Let

$$m = \max\{1 \leq j < n : t_{j+1} - t_j > b^{n-j}n \log T\},$$

where we interpret $m = 0$ if the set before taking the maximum is empty. This $m$ is $m_{k-1}$. To find other $m_i$ we do the same calculation for the set $\{1, 2, \ldots, m\}$. The process will terminate with $m = 0$ in finite steps and it gives a partition $P$ such that $t \in I_P$. It is not hard to see from the construction that $P$ is uniquely determined by $t$. So $[0, T]_n^{\leq}$ is a disjoint union of $I_P$. ■

In view of (4.2) and Lemma 4.2, we have

$$b_{n,T} = \sum_{P < n} b_{n,T,P} \quad \text{where} \quad b_{n,T,P} = \int_{I_P} \int_X \phi_t(x) \, d\nu \, dt.$$ 

Now we estimate $b_{n,T,P}$ for each fixed partition $P$. We write $t = (t_1, \ldots, t_k)$ according to the partition $P$, i.e., $t_i = (t_{m_{i-1}+1}, \ldots, t_{m_i})$. For $1 \leq i \leq k$ and $t \in [0, T]^n$ we let

$$r_{i,t} = t_{m_{i-1}+1}, \quad s_{i,t} = t_{m_i}, \quad t_i - r_{i,t} = (t_{m_{i-1}+1} - r_{i,t}, \ldots, t_{m_i} - r_{i,t})$$

and

$$\phi_{t_i} = \prod_{m_{i-1} < j \leq m_i} \phi_{t_j}.$$ 

Let

$$r = (r_1, \ldots, r_k) \in \mathbb{R}^k \quad \text{and} \quad I_P(r) = \{t \in I_P : r_{i,t} = r_i\}.$$ 

Let

$$R_P = \{r \in [0, T]^k : I_P(r) \neq \emptyset\}, \quad dr = \prod_{i=1}^k dr_i \quad \text{and} \quad dt_P = \prod_{i \in E_P} dt_i,$$

where

$$E_P = \{1, \ldots, n\} \setminus \{m_{i-1} + 1 : 1 \leq i \leq k\}.$$ 

By slight abuse of notation we write $dt = dr \, dt_P$ by identifying $t_{m_{i-1}+1}$ with $r_i$ for $1 \leq i \leq k$. 

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Lemma 4.3: The volume of $I_P(r)$ with respect to $dt_P$ is at most $b^n n^n \log^n T$.

Proof. This is almost a trivial estimate by noting that if $m_{i-1} < j < m_i$ for some $1 \leq i \leq k$, then $t_{j+1} - t_j \leq b^n n \log T$. 

Lemma 4.4: One has

$$0 \leq \frac{T^k}{k!} - \int_{R_P} 1 \, dr \leq n^3 b^n T^{k-1} \log T.$$  

(4.6)

Proof. Since $R_P$ is a subset of $[0,T]^k_\leq$, one has

$$0 \leq \int_{[0,T]^k_\leq} 1 \, dr - \int_{R_P} 1 \, dr = \frac{T^k}{k!} - \int_{R_P} 1 \, dr,$$

which gives the lower bound in (4.6).

To prove the upper bound we need more precise information about $R_P$. Let

$$b_{k-1} = b^{m_{k-1}-m_{k-1}} n \log T,$$
$$b_{k-2} = (b^{m_{k-1}-m_{k-1}} + b^{m_{k-2}-m_{k-2}}) n \log T,$$
$$\vdots$$
$$b_1 = (b^{m_{k-1}-m_{k-1}} + \cdots + b^{m_2-m_1}) n \log T.$$

We set $r_0 = 0$, $b_0 = b_1$ and $b_k = 0$. We claim that

$$\int_{R_P} 1 \, dr = \prod_{i=1}^k \int_{r_{i-1} + b_{i-1} - b_i}^{T-b_i} 1 \, dr_i.$$  

(4.7)

If $k = 1$, then $R_P = [0,T]$ and (4.7) holds. Suppose $k \geq 2$. If $r \in R_P$, then by definition $I_P(r) \neq \emptyset$. So there exists $t \in I_P$ such that $r_{i,t} = r_i$. In view of the definition of $I_P$ in (4.4), one has

$$\forall 1 < i \leq k.$$

(4.8)

On the other hand, if (4.8) holds for some $r$, then $I_P(r)$ is nonempty and hence $r \in R_P$. The formula (4.7) follows from (4.8).
Now we prove the upper bound in (4.6). Note that the length of
\[ L_i \overset{\text{def}}{=} [r_{i-1}, T] \setminus [r_{i-1} + b_{i-1} - b_i, T - b_i] \]
is at most \( b_{i-1} \leq b n^2 \log T \). So
\[
\prod_{i=1}^{k} \int_{r_{i-1}}^{T} 1 \, dr_i - \prod_{i=1}^{k} \int_{r_{i-1} + b_{i-1} - b_i}^{T - b_i} 1 \, dr_i
\]
\[
= \sum_{j=1}^{k} \left( \prod_{i=1}^{j-1} \int_{r_{i-1} + b_{i-1} - b_i}^{T - b_i} 1 \, dr_i \int_{L_j}^{T} 1 \, dr_j \prod_{s=j+1}^{k} \int_{r_{s-1}}^{T} 1 \, dr_s \right)
\]
\[
\leq \sum_{j=1}^{k} \left( \prod_{i=1}^{j-1} \int_{0}^{T} 1 \, dr_i \int_{L_j}^{T} 1 \, dr_j \prod_{s=j+1}^{k} \int_{0}^{T} 1 \, dr_s \right)
\]
\[
\leq k \cdot b n^2 \log T \cdot T^{k-1} \leq b^n n^3 T^{k-1} \log T. \quad \blacksquare
\]

**Corollary 4.5:** If \( |P| = k < \frac{n}{2} \), then \( \lim_{T \to \infty} \frac{b_{n,T,P}}{T^2} = 0. \)

**Proof.** Recall that we assume \( \| \phi \| \leq 1 \). So by Lemmas 4.3 and 4.4
\[
|b_{n,T,P}| \leq \int_{R_P} \int_{I_P(r)} 1 \, dt_P \, dr \leq \frac{T^k}{k!} \cdot b^n n^2 \log^n T.
\]
The conclusion follows from the above estimate and the assumption that \( k < \frac{n}{2} \). \quad \blacksquare

Now we estimate \( b_{n,T,P} \) for \( k = |P| \geq \frac{n}{2} > 1 \) using Theorem 2.2. By Lemma 2.5, \( \| \phi_t \| \leq e^{\kappa t} \| \phi \| \) for \( t \geq 0 \). So in view of the definition of \( I_P \) in (4.4) and the assumption \( \| \phi \| \leq 1 \)
\[
\max_{m_{i-1} < j \leq m_i} \| \phi_{t_j - r_{i,t}} \| \leq \exp(\kappa(t_{m_{i}} - r_{i,t}))
\]
\[
= \exp \left( \kappa \sum_{m_{i-1} < j < m_i} (t_{j+1} - t_j) \right)
\]
\[
\leq \exp \left( (\kappa n \log T) \sum_{m_{i-1} < j < m_i} b^{m_{i} - j} \right)
\]
\[
\leq \exp(2\kappa n b^{m_{i} - m_{i-1} - 1} \log T),
\]
where in the last estimate we use \( b \geq n \). In (4.10), we interpret
\[
\sum_{m_{i-1} < j < m_i} (t_{j+1} - t_j) = 0
\]
if there is no integer \( j \) satisfying \( m_{i-1} < j < m_i \). By Lemma 2.6, the assumption \( \| \phi \|_{\text{sup}} \leq \| \phi \| \leq 1 \) and (4.10),

\[
\| \phi_{t_i-r_i,t} \| \leq (m_i - m_{i-1})^\ell \cdot (\max_{m_{i-1} < j \leq m_i} \| \phi_{t_j-r_i,t} \|)^\ell 
\leq (m_i - m_{i-1})^\ell \cdot \exp(2\kappa n \ell b^{m_i-m_{i-1}-1} \log T) 
\leq n^\ell \exp(2\kappa n \ell b^{m_i-m_{i-1}-1} \log T) 
\leq \exp(4\kappa n \ell b^{m_i-m_{i-1}-1} \log T),
\]

where in the last estimate we use \( n \leq e^n \).

For \( 1 \leq i \leq k \), using Theorem 2.2 with the product of \( m_{i-1} + 1 \) functions \( \phi_s \) \((1 \leq s \leq m_{i-1})\) and \( \phi_{t_i} \), we have

\[
\left| \int_X \prod_{j=1}^i \phi_{t_j} \, d\nu - \int_X \prod_{j=1}^{i-1} \phi_{t_j} \, d\nu \int_X \phi_{t_i-r_i,t} \, d\mu \right|
\leq \int_X \phi_{t_1} \phi_{t_2} \cdots \phi_{t_{m_{i-1}}-1} \phi_{(t_i-r_i,t)+r_i,t} \, d\nu 
- \int_X \phi_{t_1} \phi_{t_2} \cdots \phi_{t_{m_{i-1}}} \, d\nu \int_X \phi_{t_i-r_i,t} \, d\mu
\leq Mn \exp(4\kappa b^{(m_i-m_{i-1})}n \log T) \cdot \exp(-\delta b^{(m_i-m_{i-1})}n \log T)
\leq e^{-n \log T} \leq T^{-n},
\]

where we use \( \| \phi \|_{\text{sup}} \leq 1 \) and set \( s_{0,t} = 0 \). The definition of \( I_P \) in (4.4) implies

\[
(4.13) \quad \quad r_i,t - s_{i-1,t} > b^{(m_i-m_{i-1})}n \log T \quad \text{for } i \geq 2.
\]

By (4.11), (4.12) and (4.13) we have for \( 2 \leq i \leq k \)

\[
(4.14) \quad \quad \left| \int_X \prod_{j=1}^i \phi_{t_j} \, d\nu - \int_X \prod_{j=1}^{i-1} \phi_{t_j} \, d\nu \int_X \phi_{t_i-r_i,t} \, d\mu \right|
\leq Mn \exp(4\kappa b^{(m_i-m_{i-1})}n \log T) \cdot \exp(-\delta b^{(m_i-m_{i-1})}n \log T)
\leq e^{-n \log T} \leq T^{-n},
\]

where in the last line we use \( b \geq \frac{4\kappa \ell + 2}{\delta} \) in (4.3) and \( T \geq Mn \) in (4.5).

Remark 4.6: The estimate (4.14) also holds for \( i = 1 \) provided that \( t_1 \geq b^{m_1}n \log T \), which will be used later.
By (4.14) and the assumption \( \| \phi \|_{\sup} \leq 1 \) we have

\[
\left| b_{n,T,P} - \int_{I_P} \left( \int_X \phi_{t_1} \, d\nu \right) \left( \prod_{i=2}^k \int_X \phi_{t_{i-r_{i-1}}} \, d\mu \right) \, dt \right|
\leq \int_{I_P} \left| \int_X \prod_{i=1}^k \phi_{t_i} \, d\nu - \int_X \phi_{t_1} \, d\nu \prod_{i=2}^k \int_X \phi_{t_{i-r_{i-1}}} \, d\mu \right| \, dt
\leq \int_{I_P} \sum_{i=2}^k \left| \int_X \prod_{j=1}^i \phi_{t_j} \, d\nu - \int_X \prod_{j=1}^{i-1} \phi_{t_j} \, d\nu \int_X \phi_{t_{i-r_{i-1}}} \, d\mu \right| \, dt
\leq \int_{I_P} n T^{-n} \, dt \leq 1.
\]

\[ (4.15) \]

**Lemma 4.7**: Suppose \( n \geq 3 \) and \( k = |P| \geq \frac{n+2}{2} \); then

\[
\lim_{T \to \infty} \frac{b_{n,T,P}}{T^k} = 0.
\]

**Proof.** If \( k > \frac{n}{2} \), then the partition \( P \) must contain a single number. Since \( k \geq \frac{n+2}{2} \), there exists \( i > 1 \) such that \( \{m_i\} \in P \). Therefore,

\[
\prod_{i=2}^k \int_X \phi_{t_{i-r_{i-1}}} \, d\mu = 0.
\]

This equality and (4.15) imply \( \lim_{T \to \infty} \frac{b_{n,T,P}}{T^k} = 0 \). \( \square \)

Now there are two cases left, namely,

\[
(4.16) \quad n \text{ is odd and } P = P_1 = \{\{1\}, \{2, 3\}, \ldots, \{n-1, n\}\},
\]

\[
(4.17) \quad n \text{ is even and } P = P_2 = \{\{1, 2\}, \{3, 4\}, \ldots, \{n-1, n\}\}.
\]

**Lemma 4.8**: If \( n \geq 3 \) is even, and \( P = P_2 \) is given by (4.17), then for \( k = |P| = \frac{n}{2} \)

\[
\lim_{T \to \infty} \frac{b_{n,T,P}}{T^k} = \frac{(\sigma^2)^k}{k!}.
\]

\[ (4.18) \]

**Proof.** Let

\[
R'_P = \{r \in R_P : r_i - r_{i-1} \geq 2b^2 n \log T \text{ for } 1 \leq i \leq k \text{ and } r_k \leq T - bn \log T\},
\]

where \( r_0 = 0 \). It can be checked directly from the definition that for every \( r \in R'_P \),

\[
I_P(r) = \{t \in \mathbb{R}^n : t_{2i-1} = r_i \text{ and } r_i \leq t_{2i} \leq r_i + bn \log T\}.
\]

\[ (4.19) \]
By Fubini’s theorem and (4.19)

\[(4.20) \quad \int_{R'} \int_{P(r)} \prod_{i=1}^{k} \mu(\phi_{t_i-r_i,t}) \, dt \, dr = \int_{R'} \left( \int_{0}^{b_n \log T} \mu(\phi_s \phi) \, ds \right)^k \, dr. \]

Similar to (4.15) and using Remark 4.6 we have

\[(4.21) \quad \left| \int_{R'} \int_{I_P(r)} \sum_{X_{i=1}}^{k} \phi_{t_i} \, d\nu \, dt - \int_{R'} \int_{I_P(r)} \prod_{i=1}^{n} \mu(\phi_{t_i-r_i,t}) \, dt \, dr \right| \leq 1. \]

On the other hand, similar to (4.9), we have

\[(4.22) \quad \int_{R \setminus R'} 1 \, dr \leq 2 b_n^2 n^2 T^{k-1} \log T. \]

By Lemma 4.3, (4.22) and the assumption \(\|\phi\|_{\text{sup}} \leq 1\)

\[(4.23) \quad \left| b_{n,T,P} - \int_{R'} \int_{I_P(r)} \sum_{X_{i=1}}^{k} \phi_{t_i} \, d\nu \, dt \right| \leq \int_{R \setminus R'} 1 \, dt \leq 2 n^{n+2} T^{k-1} \log^{n+1} T. \]

By Lemma 4.4 and (4.22)

\[(4.24) \quad \frac{T^k}{k!} - \int_{R'} 1 \, dr = \frac{T^k}{k!} - \int_{R_P} 1 \, dr + \int_{R \setminus R'} 1 \, dr \leq 3 b_n^3 n^3 T^{k-1} \log T. \]

To sum up, by (4.23)

\[
\lim_{T \to \infty} \frac{b_{n,T,P}}{T^k} = \lim_{T \to \infty} \frac{k! T^{-k}}{T^k} \int_{R'} \left( \int_{0}^{b_n \log T} \mu(\phi_s \phi) \, ds \right)^k \, dr \quad \text{(by 4.21)} \\
= \lim_{T \to \infty} \frac{k! T^{-k}}{T^k} \int_{R'} \left( \int_{0}^{b_n \log T} \mu(\phi_s \phi) \, ds \right)^k \, dr \quad \text{(by 4.20)} \\
= \lim_{T \to \infty} \left( \int_{0}^{b_n \log T} \mu(\phi_s \phi) \, ds \right)^k \quad \text{(by Lemma 3.1(i))} \\
= \left( \frac{\sigma^2}{2} \right)^k. \]

**Lemma 4.9:** If \(n \geq 3\) is odd, and \(P = P_1\) is given by (4.16), then

\[(4.25) \quad \lim_{T \to \infty} \frac{b_{n,T,P}}{T^k} = 0. \]
Proof. Recall that \( k = |P| = \frac{n}{2} + \frac{1}{2} \) and \( \|\phi\| \leq 1 \). By (4.15)
\[
|b_{n,T,P}| \leq 1 + \int_{R_P} \left| \int_X \phi_{r_1} \, d\nu \right| \left( \int_{I_P(x)} \prod_{i=2}^{k} \int_X |\phi_{t_i-r_{i-1},t}| \, d\mu \, dt_P \right) \, dr
\]
\[
\leq 1 + b^2 n^2 (\log T) \int_{R_P} \left| \int_X \phi_{r_1} \, d\nu \right| \, dr \quad \text{(by Lemma 4.3)}
\]
\[
\leq 1 + b^2 n^2 (\log T) T^{k-1} \int_{0}^{\infty} \left| \int_X \phi_{r_1} \, d\nu \right| \, dr_1
\]
\[
\leq 1 + b^2 n^2 (\log T) T^{k-1} M \int_{0}^{\infty} e^{-\delta t} \, dr_1. \quad \text{(by (2.2))}
\]
So (4.25) follows from the above estimate and the observation
\[
k - 1 = \frac{n - 1}{2} < \frac{n}{2}.
\]

Proof of Theorem 1.3. By Theorem 4.1 it suffices to prove (4.1). As noted after Theorem 4.1, (4.1) holds for \( n = 0, 1 \) and 2. For an odd integer \( n \geq 3 \), one has
\[
\lim_{T \to \infty} \frac{b_{n,T}}{T^{\frac{n}{2}}} = \left( \sum_{|P| < \frac{n}{2}} + \sum_{|P| > \frac{n+1}{2}} + \sum_{|P| = \frac{n+1}{2}} \right) \lim_{T \to \infty} \frac{b_{n,T,P}}{T^{\frac{2}{2}}}. \]
Then it follows from Corollary 4.5, Lemma 4.7 and Lemma 4.9 that (4.1) holds for odd \( n \geq 3 \). For an even integer \( n \geq 3 \), one has
\[
\lim_{T \to \infty} \frac{b_{n,T}}{T^{\frac{n}{2}}} = \left( \sum_{|P| < \frac{n}{2}} + \sum_{|P| > \frac{n}{2}} + \sum_{|P| = \frac{n}{2}} \right) \lim_{T \to \infty} \frac{b_{n,T,P}}{T^{\frac{2}{2}}}. \]
So it follows from Corollary 4.5, Lemma 4.7 and Lemma 4.8 that (4.1) holds for even \( n \geq 3 \). 

5. Regularity along stable and unstable leaves

Let \( \Gamma \) be a lattice of a connected semisimple Lie group \( G \) with finite center and \( a \in G \) be Ad-diagonalizable. We assume in this section that the action of \( a \) on \( X = G/\Gamma \) is ergodic with respect to the probability Haar measure \( \mu \). In view of Mautner’s phenomenon [2, Thm. III.1.4] and Ratner’s measure classification theorem [20, Thm. 3], the ergodicity assumption is equivalent to stating that \( G_a' \Gamma \) is dense is \( G \).
A function $\psi : Y \rightarrow \mathbb{R}$ on a metric space $(Y, \text{dist})$ is said to be $\theta$-Hölder ($0 < \theta \leq 1$) if
\[ \|\psi\|_\theta' := \sup_{x, y \in Y, 0 < \text{dist}(x, y) < 1} \frac{|\psi(x) - \psi(y)|}{\text{dist}(x, y)^\theta} < \infty. \]

Here the upper bound of the distances between $x$ and $y$ is only needed in the case where $Y$ is noncompact. Recall that we have fixed a right invariant metric on $G$ which induces a metric on $X$. Any closed subgroup of $G$ is considered as a metric space with the metric inherited from that of $G$.

Let $H$ be a closed subgroup of $G$ with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Recall from §2 that each $\alpha \in \mathfrak{h}$ defines a differential operator $\partial^\alpha$ on $X$. A function $\varphi : X \rightarrow \mathbb{R}$ is said to be uniformly smooth along $H$ orbits of a subset $X' \subset X$ if for any bounded open subset $U$ of $H$, any $k \geq 0$ and any basis $\mathfrak{b}_k$ of $\mathfrak{h}$, there exists $M \geq 1$ such that
\[ |\partial^\alpha \varphi(gx)| \leq M \quad \text{for all } x \in X', g \in U \text{ and } \alpha \in \mathfrak{b}_k^k. \]

It is not hard to see that to show (5.1) holds for any $\mathfrak{b}_k$ it suffices to prove it for a fixed $\mathfrak{b}_k$.

A useful tool in finding a continuous function which is equal to a given measurable function $\varphi : X \rightarrow \mathbb{R}$ is to assign the density value at every point of measurable continuity. Recall that a point $x \in X$ is a point of measurable continuity of $\varphi$ if there is $s \in \mathbb{R}$ such that $x$ is a Lebesgue density point of $\varphi^{-1}(U)$ for every neighborhood $U$ of $s$ in $\mathbb{R}$. The value $s$ is called the density value of $\varphi$ at $x$. Let $\text{MC}(\varphi)$ be the set of measurable continuity points of $\varphi$, and let $\tilde{\varphi} : \text{MC}(\varphi) \rightarrow \mathbb{R}$ be the map which sends $x$ to the density value of $\varphi$ at $x$.

**Theorem 5.1:** Suppose $(X, \mu, a)$ is ergodic and $\psi : X \rightarrow \mathbb{R}$ is $\theta$-Hölder. Let $\varphi : X \rightarrow \mathbb{R}$ be a measurable solution to the cohomological equation
\[ \psi(x) = \varphi(ax) - \varphi(x). \]

Let $\tilde{\varphi} : \text{MC}(\varphi) \rightarrow \mathbb{R}$ be the map which sends $x$ to the density value of $\varphi$ at $x$. Then the following hold:

(i) $\text{MC}(\varphi)$ is a $G'_a$-invariant full measure subset of $X$ and $\varphi = \tilde{\varphi}$ almost everywhere;
(ii) $\tilde{\varphi}(gx) - \tilde{\varphi}(x)$ is continuous on $G'_a \times \text{MC}(\varphi)$;
(iii) the Hölder norms $\|\tilde{\varphi}(gx)\|_\theta (x \in \text{MC}(\varphi))$ of functions on $G'_a$ are uniformly bounded;
(iv) if $\psi \in \hat{C}_c^\infty(X)$, then $\tilde{\varphi}$ is uniformly smooth along $G_a^-$ and $G_a^+$ orbits of $\text{MC}(\varphi)$.

For every $x \in X$, $u \in G_a^-$ and $s \in \mathbb{R}$ we define the stable holonomy map by

$$H_{x,u}^-(s) = s + \sum_{n=0}^{\infty} \psi(a^n x) - \psi(a^n ux).$$

(5.3)

In the context of [1], $H_{x,u}^-$ is a map from the fiber of $x$ to the fiber of $ux$ in the fiber bundle $X \times \mathbb{R}$. We do not use the fiber bundle language but adopt the name holonomy map. Similarly, for $u \in G_a^+$, the unstable holonomy map is defined by

$$H_{x,u}^+(s) = s + \sum_{n=-\infty}^{-1} \psi(a^n ux) - \psi(a^n x).$$

(5.4)

Since $\psi$ is $\theta$-Hölder, both of the holonomy maps are well-defined and continuous for $(u, x, s) \in G_a^\pm \times X \times \mathbb{R}$.

A measurable function $\varphi : X \to \mathbb{R}$ is said to be essentially $\mathcal{H}^-$ invariant if there is a full $\mu$ measure subset $X' \subset X$ such that for all $x \in X, u \in G_a^-$ with $x, ux \in X'$, one has

$$H_{x,u}^-(\varphi(x)) = \varphi(ux).$$

(5.5)

We define essentially $\mathcal{H}^+$ invariant in a similar way.

**Lemma 5.2:** If $\varphi$ is a measurable solution to the cohomological equation (5.2), then $\varphi$ is essentially $\mathcal{H}^+$ and $\mathcal{H}^-$ invariant.

**Proof.** By Lusin’s theorem, there is a compact subset $K$ of $X$ such that $\varphi$ is uniformly continuous on $K$ and $\mu(K) > 0.9$. Since $(X, \mu, a)$ is ergodic, there exists an $a$-invariant full measure subset $X_1$ such that the Birkhoff average of the characteristic function of $K$ at any $x \in X_1$ converges to $\mu(K)$. By assumption, there is an $a$-invariant full measure subset $X_2$ of $X$ such that (5.2) holds for all $x \in X_2$. We will show that $\varphi$ is essentially $\mathcal{H}^-$ invariant by taking

$$X' = X_1 \cap X_2.$$

Suppose $x \in X', u \in G_a^-$ and $ux \in X'$. Since $X_2$ is $a$-invariant, we have $a^n x, a^n ux \in X_2$ for all $n \in \mathbb{N} \cup \{0\}$. Using (5.2) for all $a^n x$ and $a^n ux$, we have

$$\sum_{n=0}^{\infty} \psi(a^n x) - \psi(a^n ux) = \varphi(ux) - \varphi(x) + \lim_{n \to \infty} \varphi(a^n x) - \varphi(a^n ux),$$

(5.6)
where the existence of an infinite sum on the left-hand side of (5.6) follows from the Hölder property of \( \psi \). So \( \lim_{n \to \infty} \varphi(a^nux) - \varphi(a^nx) \) converges. Since \( x, ux \in X_1 \), there are infinitely many \( n \) with \( a^nx \) and \( a^nux \) belonging to \( K \). Since \( \varphi \) is uniformly continuous on \( K \) and 
\[
d(a^nux, a^nx) \leq d_G(a^nua^{-n}, 1_G) \to 0 \quad \text{as } n \to \infty,
\]
we know that \( \lim_{n \to \infty} \varphi(a^nux) - \varphi(a^nx) = 0 \). Therefore (5.5) holds. This proves that \( \varphi \) is essentially \( H^- \) invariant. The proof of the essential \( H^+ \) invariance is similar.

**Lemma 5.3:** If \( \varphi \) is a measurable solution to the cohomological equation (5.2), then the set \( \text{MC}(\varphi) \) is \( G'_a \)-invariant and
\[
(5.7) \quad \tilde{\varphi}(ux) = H^*_x,u(\tilde{\varphi}(ux)) \quad \text{for all } x \in \text{MC}(\varphi) \text{ and } u \in G_a^*
\]
where \( * \in \{+,-\} \).

**Proof.** We show that \( \text{MC}(\varphi) \) is \( G_a^- \)-invariant and (5.7) holds for \( * = - \). One can prove the \( G_a^+ \) invariance of \( \text{MC}(\varphi) \) and (5.7) for \( * = + \) similarly. Since \( G_a' \) is generated by \( G_a^- \) and \( G_a^+ \), the \( G_a' \) invariance of \( \text{MC}(\varphi) \) is a direct corollary. The argument here is essentially the same as that in [1] but much simpler. We provide details here for completeness.

By Lemma 5.2, the function \( \varphi \) is essentially \( H^- \) invariant. So there exists a full measure subset \( X' \) such that (5.5) holds for all \( x \in X' \) and \( ux \in X' \) where \( u \in G_a^- \). Let \( x \in \text{MC}(\varphi) \) and \( u \in G_a^- \). Suppose \( s \) is the density value of \( \varphi \) at \( x \). We will show that \( H^-_{x,u}(s) \) is the density value of \( \varphi \) at \( ux \). Let \( U \) be a neighborhood of \( H^-_{x,u}(s) \). In view of the continuity of \( H^-_{*,*}(*) \), there exist open neighborhoods \( N \) and \( V \) of \( x \) and \( s \), respectively, such that
\[
(5.8) \quad H^-_{y,u}(r) \in U \quad \text{for all } (y, r) \in N \times V.
\]

As \( u : X \to X \) is a diffeomorphism preserving \( \mu \), both \( X' \) and \( u^{-1}X' \) are full measure subsets. Since \( x \in \text{MC}(\varphi) \), it is a Lebesgue density point of
\[
N \cap \varphi^{-1}(V) \cap X' \cap (u^{-1}X').
\]
It follows that \( ux \) is a Lebesgue density point of
\[
N' = u(N \cap \varphi^{-1}(V) \cap X') \cap X'.
\]
We claim that \( N' \subset \varphi^{-1}(U) \). The claim will imply that \( ux \) is a point of measurable continuity of \( \varphi \) with density value \( H^-_{x,u}(s) \).
To prove the claim let \(uy\) be an arbitrary point of \(N'.\) So \(uy \in X'\) and \(y \in N \cap \varphi^{-1}(V) \cap X'.\) Since \(y, uy \in X',\) one has \(\mathcal{H}_{y,u}^{-}(\varphi(y)) = \varphi(uy).\) On the other hand, since \(y \in N\) and \(\varphi(y) \in V,\) it follows from (5.8) that \(\mathcal{H}_{y,u}^{-}(\varphi(y)) \in U.\) So \(\varphi(uy) \in U.\) This completes the proof of the claim and hence the lemma. □

**Proof of Theorem 5.1.** (i) By [1, Lem. 7.10], the set \(MC(\varphi)\) has full measure with respect to \(\mu\) and \(\varphi = \tilde{\varphi}\) almost everywhere. It follows from Lemma 5.3 that \(MC(\varphi)\) is \(G'_a\)-invariant.

(ii) To simplify the notation, we set \(X' = MC(\varphi).\) In view of (5.5) and (5.7), the map on \(G^{-} \times X'\) which sends \((u, x)\) to

\[
\tilde{\varphi}(ux) - \tilde{\varphi}(x) = \mathcal{H}^{-}_{x,u}(\tilde{\varphi}(x)) - \tilde{\varphi}(x) = \sum_{n=0}^{\infty} \psi(a^n x) - \psi(a^n ux)
\]

is continuous on \(G^{-} \times X'.\) Similarly

\[
\tilde{\varphi}(ux) - \tilde{\varphi}(x)
\]

is continuous for \((u, x) \in G^+_{a} \times X'.\)

In general, given \(h \in G'_a,\) there exists \(m \in \mathbb{N}\) and \(\kappa_1, \ldots, \kappa_m \in \{+, -\}\) such that the map

\[
p : G_{a}^{\kappa_1} \times \cdots \times G_{a}^{\kappa_m} \to G'_a\]

given by \(p(g_1, \ldots, g_m) = g_m \cdots g_1\)

contains an open neighborhood of \(h;\) see [6, §2.2]. Moreover, we can write \(h = h_m \cdots h_1\) for \(h_i \in G_{a}^{\kappa_i}\) such that \(p\) is an open map in a neighborhood of \((h_1, \ldots, h_m).\) Let

\[
q : G_{a}^{\kappa_1} \times \cdots \times G_{a}^{\kappa_m} \times X' \to \mathbb{R}
\]

be the map given by

\[
q(g_1, \ldots, g_m, x) = \tilde{\varphi}(g_m \cdots g_1 x) - \tilde{\varphi}(x).
\]

For \(g_0 = 1_G\) one has

\[
q(g_1, \ldots, g_m, x) = \sum_{n=1}^{m} \tilde{\varphi}(g_n \cdots g_1 g_0 x) - \tilde{\varphi}(g_{n-1} \cdots g_1 g_0 x).
\]

So \(q\) is continuous on \(G_{a}^{\kappa_1} \times \cdots \times G_{a}^{\kappa_m} \times X'.\) Since \(p\) is an open map in a neighborhood of \((h_1, \ldots, h_m),\) the map

\[
\tilde{\varphi}(gx) - \tilde{\varphi}(x)
\]

is continuous at \((h, x) \in G'_a \times X'.\) Therefore, \(\tilde{\varphi}(gx) - \tilde{\varphi}(x)\) is a continuous function on \(G'_a \times X'.\)
(iii) Next we prove the uniform boundedness of the H"older norms. Recall that the metric on $G'_a$ is induced from the right invariant metric on $G$. So it suffices to prove that there is an open neighborhood $U$ of the identity in $G'_a$ and $M \geq 1$ such that for any $x \in X'$ and $g \in U$

\begin{equation}
|\tilde{\varphi}(gx) - \tilde{\varphi}(x)| \leq Md_G(g, 1_G)^\theta. \tag{5.12}
\end{equation}

There is a map $p$ as in (5.10) such that the $p(1_G, \ldots, 1_G) = 1_G$ and the differential of $p$ at $(1_G, \ldots, 1_G)$ is surjective. Therefore, there is a submanifold $V$ passing through $(1_G, \ldots, 1_G)$ such that $p|_V$ is a diffeomorphism onto its image. In particular, $p|_V$ is a bi-Lipschitz map onto $U = p(V)$. So for all $(g_1, \ldots, g_m) \in V$ and $g = p(g_1, \ldots, g_m)$ we have

\begin{equation}
\max_{i=1}^m d_G(g_i, 1_G) \lesssim d_G(g, 1_G). \tag{5.13}
\end{equation}

So for $g_i \in G^-_a$ and $y \in X'$, by (5.9) and (5.13), we have

\begin{equation}
|\tilde{\varphi}(g_iy) - \tilde{\varphi}(y)| \lesssim \|\psi\|_\theta d_G(g, 1_G)^\theta. \tag{5.14}
\end{equation}

The same estimate holds for $g_i \in G^+_a$. So for any $x \in X'$ and $g \in U$, by (5.11) and (5.14), we have

\begin{equation}
|\tilde{\varphi}(gx) - \tilde{\varphi}(x)| \leq m \max_n |\tilde{\varphi}(g_n \cdots g_1 g_0 x) - \tilde{\varphi}(g_n-1 \cdots g_1 g_0 x)| \lesssim m\|\psi\|_\theta d_G(g, 1_G)^\theta,
\end{equation}

from which (5.12) follows.

(iv) Let $\mathfrak{b}_-$ be a basis of the Lie algebra of $G^-_a$ consisting of eigenvectors of $\text{Ad}(a)$. We assume without loss of generality that all the eigenvalues of $\text{Ad}(a)$ are positive. Suppose $\alpha = (v_1, \ldots, v_r) \in \mathfrak{b}^-_a$ and $t_\alpha = \sum_{i=1}^r t_i$ where $t_i > 0$ satisfies $\text{Ad}(a)v_i = e^{-t_i}v_i$. We use $\partial^\alpha_u$ to denote the differentiable operator $\partial^{\alpha}$ on $G^-_a$ with respect to the variable $u \in G^-_a$. Then

\begin{equation}
\partial^\alpha_u \psi(a^nux) = \partial^{\text{Ad}(a^n)\alpha} \psi(a^nux) = e^{-t_\alpha n} \partial^\alpha \psi(a^nux), \tag{5.15}
\end{equation}

whose sum over $n \geq 0$ converges uniformly for $u$ in a fixed compact subset and $x \in X$. So by (5.9) and (5.15), for any $x \in \text{MC}(\varphi)$ and $u \in G^-_a$, we have

\begin{equation}
\partial^\alpha \tilde{\varphi}(ux) = -\sum_{n=0}^{\infty} e^{-t_\alpha n} \partial^\alpha \psi(a^nux). \nonumber
\end{equation}

Therefore $\tilde{\varphi}$ is uniformly smooth along $G^-_a$ orbits of $\text{MC}(\varphi)$. By similar arguments one can prove that $\tilde{\varphi}$ is uniformly smooth along $G^+_a$ orbits of $\text{MC}(\varphi)$. 

\quad \blacksquare
6. Livšic-type theorem

Let the notation and assumptions be as in Theorem 1.1. In particular, the action of $G'_a$ on $X = G/\Gamma$ has a spectral gap. This implies that the action of $G'_a$ on $(X, \mu)$, where $\mu$ is the probability Haar measure, is mixing. As a consequence, the dynamical system $(X, \mu, a)$ is ergodic, hence Theorem 5.1 holds.

**Lemma 6.1:** Suppose $\varphi$ is a measurable solution to the cohomological equation $\psi(x) = \varphi(ax) - \varphi(x)$ where $\psi \in \hat{C}_c^\infty(X)$. Then $\varphi \in L^2_\mu$ and $\mu(\psi) = 0$.

**Proof.** We first prove $\mu(\psi) = 0$. By the Birkhoff ergodic theorem, for $\mu$ almost every $x \in X$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \psi(a^k x) = \frac{1}{n} (\varphi(a^n x) - \varphi(x))$$

converges to $\varphi^*(x)$, and moreover, $\mu(\varphi) = \mu(\varphi^*)$. Since the dynamical system $(X, \mu, a)$ is ergodic, there exist a measurable subset $E$ of $X$ and $\ell > 0$ such that $\mu(E) > 0$ and $|\varphi(x)| < \ell$ for any $x \in E$. The Birkhoff ergodic theorem implies that for almost every $x \in X$, there are infinitely many $n \in \mathbb{N}$ such that $a^n x \in E$. So $\varphi^*(x) = 0$ almost surely, and hence $\mu(\psi) = 0$.

Now we prove $\varphi \in L^2_\mu$. Since $(X, \mu, a)$ is ergodic, the measurable solution $\varphi$ to the cohomological equation $\psi(x) = \varphi(ax) - \varphi(x)$ is unique up to constants. Therefore, it suffices to prove that it has a solution in $L^2_\mu$, i.e., $\psi$ is cohomologous to 0 in $L^2_\mu$. Since the action of $G'_a$ on $X$ has a spectral gap, the dynamical system $(X, \mu, a)$ is exponential mixing of all orders by Lemma 2.4. Therefore, by [5, Thm. 1.1] the random variables

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \psi(a^k x) = \frac{1}{\sqrt{n}} (\varphi(a^{n-1} x) - \varphi(x))$$

given by $(X, \mu)$ converges as $n \to \infty$ to the normal distribution with mean zero and variance

$$\sigma = \int_X \psi(x)^2 \, d\mu(x) + 2 \sum_{n=1}^{\infty} \int_X \psi(a^n x) \psi(x) \, d\mu(x).$$

(6.1)

We claim that the random variable $\frac{1}{\sqrt{n}} (\varphi(a^{n-1} x) - \varphi(x))$ converges to zero in distribution as $n \to \infty$. To see this, given $\varepsilon > 0$, there is $M \geq 1$ such that

$$\mu(\{ x \in X : \varphi(x) \leq M \}) > 1 - \varepsilon.$$
Since the measure $\mu$ is $\alpha$-invariant, for any $n > 0$
\[ \mu(\{ x \in X : \varphi(a^{n-1}x) \leq M \text{ and } \varphi(x) \leq M \}) > 1 - 2\varepsilon, \]
from which the claim follows. Therefore, the central limit theorem has to be degenerate and $\sigma = 0$. Then the proof of Lemma 3.1(iii) implies that $\psi$ is cohomologous to 0 in $L^2_{\mu}$. In view of the uniqueness of the measurable solution to the cohomological equation, we have $\varphi \in L^2_{\mu}$. \[ \square \]

Recall that an element $v \in g$ is identified with the right invariant vector field on $G$ and it descends to a vector field on $X$ with the same notation $v$.

**Lemma 6.2:** Suppose $v \in g$ and $\psi, \phi \in \hat{C}_c^\infty(X)$. Then
\[ (6.2) \quad \int_X \partial^v \psi \phi \, d\mu = \int_X \psi \partial^v \phi \, d\mu. \]

**Proof.** For any nonzero real number $t$ and $x \in X$, we let
\[ f_t(x) = \frac{\psi(\exp(tv)x)\phi(\exp(tv)x) - \phi(x)\psi(x)}{t}. \]
The functions $f_t$ are uniformly bounded by the mean value theorem. So the dominated convergence theorem implies
\[ \lim_{t \to 0} \int_X f_t \, d\mu = \int_X \lim_{t \to 0} f_t \, d\mu = \int_X \partial^v \psi \phi - \psi \partial^v \phi \, d\mu. \]
On the other hand, since $G$ preserves $\mu$, we have $\int_X f_t \, d\mu = 0$ for all $t$. Therefore (6.2) holds. \[ \square \]

Lemma 6.2 allows us to define the distribution derivative of $\varphi \in L^2_{\mu}$. For each $\alpha \in g^k$, we define $\partial^\alpha \varphi$ as a linear functional on $\hat{C}_c^\infty(X)$ such that
\[ \langle \partial^\alpha \varphi, \phi \rangle = (-1)^{|\alpha|} \langle \varphi, \partial^\alpha \phi \rangle \quad \text{where } |\alpha| = k. \]

In view of Lemma 6.2, this definition is consistent with the usual definition in the sense that the distribution derivative of a function $\psi \in \hat{C}_c^\infty(X)$ is represented by the function $\partial^\alpha \psi$. This property implies that our definition of distribution derivative is independent of the choice of coordinates and coincides with the definition using local charts. Let $g^{-}$ and $g^{+}$ be the Lie algebras of $G^{-}_{a}$ and $G^{+}_{a}$, respectively. Let $l$ be the Lie algebra of $L = \{ g \in G : ga = ag \}$. For a square integrable function $\xi$ on $L$ and $\alpha \in l^k$, one can define the distribution derivative $\partial^\alpha \xi$ in a similar way. Moreover, this definition coincides with the definition using local charts.
Lemma 6.3: Suppose $\psi \in \hat{C}_c^\infty(X)$ and $\varphi$ is an $L^2_{\mu,0}$ solution to the cohomological equation $\psi(x) = \varphi(ax) - \varphi(x)$. Then for all $\alpha \in l^k$ and $\phi \in \hat{C}_c^\infty(X)$, one has

$$\langle \partial^\alpha \varphi, \phi \rangle = -\sum_{n=0}^{\infty} \int_X \partial^\alpha \psi(a^n x) \phi(x) \, d\mu(x).$$

Proof. According to the definition and the assumption $\psi(x) = \varphi(ax) - \varphi(x)$,

$$\langle \partial^\alpha \varphi, \phi \rangle = (-1)^{|\alpha|} \int_X \varphi \partial^\alpha \phi \, d\mu$$

(6.3) $$= (-1)^{|\alpha|} \int_X \left( \varphi(a^k x) - \sum_{n=0}^{k-1} \psi(a^n x) \right) \partial^\alpha \phi(x) \, d\mu(x)$$

$$= (-1)^{|\alpha|} \left( \int_X \varphi(a^k x) \partial^\alpha \phi(x) \, d\mu - \sum_{n=0}^{k-1} \int_X \psi(a^n x) \partial^\alpha \phi(x) \, d\mu \right).$$

As the action of $a$ on $(X, \mu)$ is mixing and $\mu(\varphi) = 0$, we have

$$\lim_{k \to \infty} \int_X \varphi(a^k x) \partial^\alpha \phi(x) \, d\mu = 0.$$

On the other hand, by Lemma 2.3 the mixing is exponential for functions in $\hat{C}_c^\infty(X)$. So by (6.3) and Lemma 6.2,

$$\langle \partial^\alpha \varphi, \phi \rangle = (-1)^{|\alpha|+1} \sum_{n=0}^{\infty} \int_X \psi(a^n x) \partial^\alpha \phi(x) \, d\mu$$

$$= -\sum_{n=0}^{\infty} \int_X \partial^\alpha (\psi \circ a^n)(x) \phi(x) \, d\mu(x)$$

$$= -\sum_{n=0}^{\infty} \int_X \partial^\alpha \psi(a^n x) \phi(x) \, d\mu(x). \quad \blacksquare$$

A vector field $v$ on $X$ is said to be tangent to $L$ orbits if its value $v_x$ at each point $x \in X$ is tangent to the submanifold $Lx$. The space of all such vector fields is denoted by $\mathcal{F}_L(X)$. We fix a basis $b$ of $\mathfrak{g}$ consisting of eigenvectors of $\text{Ad}(a)$ and let $b_{l} = b \cap l$.

Lemma 6.4: Suppose $\beta \in \mathcal{F}_L(X)^k$. Then there exists a family of smooth functions $f_\alpha$ ($\alpha \in b_{l}^r, r \leq k$) such that

$$\partial^\beta = \sum_{\alpha \in b_{l}^r, r \leq k} f_\alpha \partial^\alpha.$$
Proof. Since the values of $b_1$ at any point form a basis of the tangent space of the point, the conclusion is clear for $k = 1$. We prove the general case by induction. Suppose the conclusion holds for $|\beta| \leq k$ and $\partial^\beta = \partial^{v_1} \cdots \partial^{v_{k+1}}$ where $v_i \in \mathcal{F}_L(X)$. By the induction hypothesis

$$\partial^{v_1} = \sum_{v \in b_1} f_v \partial^v \text{ and } \partial^{v_2} \cdots \partial^{v_{k+1}} = \sum_{\alpha \in b_1^r, r \leq k} f_\alpha \partial^\alpha,$$

where $f_v$ and $f_\alpha$ are smooth functions on $X$. Let $\phi \in C_c^\infty(X)$ be a test function. Then

$$\partial^\beta \phi = \sum_{v \in b_1} f_v \partial^v \left( \sum_{|\alpha| \leq k} f_\alpha \partial^\alpha \phi \right) = \sum_{|\alpha| \leq k} \left( \sum_{v \in b_1} f_v \partial^v f_\alpha \right) \partial^\alpha \phi + \sum_{v \in b_1} f_v f_\alpha \partial^{(v,\alpha)} \phi.$$

So the conclusion holds for $|\beta| = k + 1$. ■

Proof of Theorem 1.1. We have proved $\varphi \in L^2_\mu$ in Lemma 6.1. Next we show that $\varphi$ is equal to a smooth function almost surely. By possibly replacing $\varphi$ by $\varphi - \mu(\varphi)$, we assume that $\mu(\varphi) = 0$. In view of Theorem 5.1, we assume that $\varphi = \tilde{\varphi}$ is defined on the $G'_\mu$-invariant full measure subset $X' = \text{MC}(\varphi)$ so that the conclusions of Theorem 5.1 hold. We will show that $\text{MC}(\varphi) = X$ and $\varphi$ is smooth.

The question is local, so we fix $x \in X$. We choose open neighborhoods $U$, $U^-$ and $U^+$ of the identity in $L$, $G_a^-$ and $G_a^+$, respectively, such that $U \times U^- \times U^+$ is diffeomorphic to its image via the map $(g, h_0, h_1) \rightarrow gh_0h_1x$. Let $\xi : U \rightarrow X$ be defined by $g \in U \rightarrow \varphi(gx)$. We fix Haar measures on $G, G_a^-, G_a^+$ and $L$ so that a fundamental domain of $\Gamma$ has measure 1 and

$$(6.4) \quad d(gh_0h_1) = dg \; dh_0 \; dh_1 \quad \text{where } g \in L, h_0 \in G_a^- \text{ and } h_1 \in G_a^+.$$

Since $\varphi \in L^2_\mu$, the function $\xi$ is square integrable.

Suppose $g \in L$ and $gx \in X'$. Note that $L$ normalizes $G_a^+$ and $G_a^-$. So by (5.7), (5.3) and (5.4)

$$\varphi(gh_0h_1x) = \varphi(gh_0g^{-1}, gh_1x)$$

$$= \varphi(gh_1x) + \sum_{n=0}^{\infty} \psi(a^ngh_1x) - \psi(a^ngh_0h_1x)$$

$$= \varphi(gh_1g^{-1}, gx) + \sum_{n=0}^{\infty} \psi(a^ngh_1x) - \psi(a^ngh_0h_1x)$$

$$= \varphi(gx) + \lambda_\psi(g, h_0, h_1),$$

where $\psi$ is a smooth function on $X$. Let $\varphi \in C_c^\infty(X)$ be a test function. Then

$$\varphi(gh_0h_1x) = \varphi(gh_0g^{-1}, gh_1x)$$

$$= \varphi(gh_1x) + \sum_{n=0}^{\infty} \psi(a^ngh_1x) - \psi(a^ngh_0h_1x)$$

$$= \varphi(gh_1g^{-1}, gx) + \sum_{n=0}^{\infty} \psi(a^ngh_1x) - \psi(a^ngh_0h_1x)$$

$$= \varphi(gx) + \lambda_\psi(g, h_0, h_1),$$

where $\psi$ is a smooth function on $X$. Let $\phi \in C_c^\infty(X)$ be a test function. Then

$$\partial^\beta \phi = \sum_{v \in b_1} f_v \partial^v \left( \sum_{|\alpha| \leq k} f_\alpha \partial^\alpha \phi \right) = \sum_{|\alpha| \leq k} \left( \sum_{v \in b_1} f_v \partial^v f_\alpha \right) \partial^\alpha \phi + \sum_{v \in b_1} f_v f_\alpha \partial^{(v,\alpha)} \phi.$$
where

\[
\lambda_{\psi}(g, h_0, h_1) = \sum_{n=-\infty}^{-1} \left[ \psi(a^n gh_1 x) - \psi(a^n gx) \right] \\
+ \sum_{n=0}^{\infty} \left[ \psi(a^n gh_1 x) - \psi(a^n gh_0 h_1 x) \right]
\]

\[
= \sum_{n=-\infty}^{-1} \left[ \psi(a^n h_1 x) - \psi(ga^n x) \right] \\
+ \sum_{n=0}^{\infty} \left[ \psi(a^n h_1 x) - \psi(ga^n h_0 h_1 x) \right].
\]

(6.6)

For \( \alpha \in \mathbb{I}^k \), let \( \partial^\alpha_g \) be the differential operator \( \partial^\alpha \) with respect to the variable \( g \in L \). Then

\[
\partial^\alpha_g [\psi(ga^n h_1 x) - \psi(ga^n x)] = \partial^\alpha\psi(ga^n h_1 x) - \partial^\alpha\psi(ga^n x) \\
= \partial^\alpha\psi(a^n gh_1 g^{-1} \cdot gx) - \partial^\alpha\psi(a^n \cdot gx),
\]

whose sum over negative integers \( n \) converges uniformly with respect to \( h_1 \in U^+ \) and \( g \in U \). A similar statement holds for

\[
\partial^\alpha_g [\psi(ga^n h_1 x) - \psi(ga^n h_0 h_1 x)].
\]

Therefore, (6.6) converges uniformly on \( U \times U^- \times U^+ \) and

\[
\partial^\alpha_g \lambda_{\psi}(g, h_0, h_1) = \lambda_{\partial^\alpha\psi}(g, h_0, h_1).
\]

(6.8)

The uniform convergence of (6.6) implies that \( \lambda(g, h_0, h_1) \) is continuous on \( U \times U^- \times U^+ \). In view of (6.5), if \( \xi \) is equal to a continuous function almost everywhere, then \( \varphi \) is equal to a continuous function almost everywhere. This implies that \( MC(\varphi) \cap UU^-U^+ x = UU^-U^+ x \) and \( \varphi \) is continuous on \( UU^-U^+ x \). It follows from (6.5), (6.6), (6.7) and (6.8) that if \( \xi \) is smooth on \( U \), then for any \( \alpha \in \mathbb{I}^k \) we have \( \partial^\alpha\varphi(y) \) exists for all \( y \in UU^-U^+ x \) and these values are uniformly bounded on \( UU^-U^+ x \). On the other hand, by Theorem 5.1(iv), the function \( \varphi \) is uniformly smooth along \( G^-_a \) and \( G^+_a \) orbits of \( MC(\varphi) \). Therefore, if \( \xi \) is equal to a smooth function almost everywhere, then \( \varphi \) is defined everywhere on \( UU^-U^+ x \) and all the partial derivatives of \( \varphi \) along foliations are uniformly bounded on \( UU^-U^+ x \). So a theorem of Journé [12] implies that \( \varphi \) is smooth on \( UU^-U^+ x \).
Therefore, it suffices to show that \( \xi \) is equal to a smooth function almost everywhere. Let \( m \) be the dimension of \( L \). We assume that there is a coordinate map \( b : U \to \Omega \) where \( \Omega \) is an open ball in \( \mathbb{R}^m \). Moreover, we assume there is a smooth function \( \rho : U \to \mathbb{R} \) that is bounded from above and below by some positive constants so that \( \rho(g) \, dg \) is mapped by \( b \) to the Lebesgue measure on \( \Omega \). For a compactly supported smooth function \( \zeta_0 : U \to \mathbb{R} \), we consider the Fourier transform of

\[
f(y) = \xi(b^{-1}y)\zeta_0(b^{-1}y)
\]
defined by

\[
\hat{f}(z) = \int_{\Omega} \xi(b^{-1}y)\zeta_0(b^{-1}y)e^{-2\pi iy \cdot z} \, dy,
\]
which is a continuous function on \( \mathbb{R}^m \). We claim that for any positive integer \( n \) and \( z = (z_1, \ldots, z_m) \in \mathbb{R}^m \), we have

\[
|z|^n \sup_{\mathbb{R}^m} \hat{f}(z) \in L^1(\mathbb{R}^m) \quad \text{where } |z|_{\text{sup}} = \max\{|z_1|, \ldots, |z_m|\}.
\]

Assume the claim, so it follows from the Fourier inversion formula that \( f \) equals a smooth function almost everywhere; see, for example, Folland [9, Thm. 8.22.d, 8.26]. By choosing \( \zeta_0 \) with arbitrarily large support, we have \( \xi \) is equal to a smooth function almost everywhere.

Now we prove the claim. We will use the usual multiple index notation for the partial derivatives on \( \mathbb{R}^m \) (see [9, §8.1]). For example, if \( \gamma = (k_1, \ldots, k_m) \in \mathbb{Z}^m_{\geq 0} \) is a multiple index, then

\[
z^\gamma = z_1^{k_1} \cdots z_m^{k_m}, \quad \partial^\gamma = \left( \frac{\partial}{\partial y_1} \right)^{k_1} \cdots \left( \frac{\partial}{\partial y_m} \right)^{k_m}
\]
and \( |\gamma| = k_1 + \cdots + k_m \).

We have

\[
(2\pi iz)^\gamma \hat{f}(z) = (-1)^{|\gamma|} \int_{\Omega} \xi(b^{-1}y)\zeta_0(b^{-1}y)\partial^\gamma e^{-2\pi iy \cdot z} \, dy
\]

(6.10)

\[
= \int_{\Omega} \partial^\gamma(\xi(b^{-1}y)\zeta_0(b^{-1}y))e^{-2\pi iy \cdot z} \, dy,
\]
where \( \partial^\gamma \) refers to the distribution derivative. By the product rule

\[
\partial^\gamma(\xi(b^{-1}y)\zeta_0(b^{-1}y)) = \sum_{\tau \leq \gamma} \partial^\tau \xi(b^{-1}y)\partial^{\gamma-\tau} \zeta_0(b^{-1}y),
\]

(6.11)
where the sum is taken over all the multiple indices \( \tau \) such that \( \gamma - \tau \) has nonnegative entries and each \( \tau \) appears

\[
\frac{\gamma!}{\tau!(\gamma - \tau)!}
\]
times. Since \( \zeta_0 \) is a compactly supported function on \( U \), there are differential operators \( \tilde{\tau} \in \mathcal{F}_L^{[\tau]} \) and \( \tilde{\gamma} - \tau \in \mathcal{F}_L^{[\gamma - \tau]} \) such that

\[
(6.12) \quad \partial^{\tilde{\tau}} \xi(b^{-1}y) \partial^{\tilde{\gamma} - \tau} \zeta_0(b^{-1}y) = \partial^{\tilde{\tau}} \xi(g) \partial^{\tilde{\gamma} - \tau} \zeta_0(g)
\]
where \( b(g) = y \).

By (6.10), (6.11) and (6.12)

\[
(6.13) \quad (2\pi iz)^\gamma \hat{f}(z) = \int_U \sum_{\tau \leq \gamma} \partial^{\tilde{\tau}} \xi(g) \partial^{\tilde{\gamma} - \tau} \zeta_0(g) e^{-2\pi ib(g) \cdot z} \rho(g) \, dg.
\]

By Lemma 6.4, for each \( \tilde{\tau} \) one has

\[
(6.14) \quad \partial^{\tilde{\tau}} = \sum_{\alpha \in b^r, r \leq |\tau|} f_\alpha \partial^\alpha,
\]
where \( f_\alpha \) are smooth functions. In view of (6.13) and (6.14)

\[
(6.15) \quad (2\pi iz)^\gamma \hat{f}(z) = \sum_{\alpha \in b^r, r \leq |\gamma|} \int_U \partial^\alpha \xi(g) \eta_\alpha(g) e^{-2\pi ib(g) \cdot z} \, dg,
\]
where \( \eta_\alpha(g) \) are compactly supported smooth functions on \( U \).

Now we lift all the \( \eta_\alpha \) to compactly supported smooth functions \( \tilde{\eta}_\alpha \) on \( X \). Let \( \zeta : U^- \times U^+ \to [0, \infty) \) be a compactly supported smooth function such that

\[
\int_{U^+} \int_{U^-} \zeta(h_0, h_1) \, dh_0 \, dh_1 = 1.
\]

Each function \( \eta \in C_c^\infty(U) \) is lifted to \( \tilde{\eta} \in C_c^\infty(X) \) as follows:

\[
\eta(y) = \begin{cases} 
\eta(g) \zeta(h_0, h_1) & \text{if } y = gh_0h_1x \text{ for } g \in U, h_0 \in U^-, h_1 \in U^+, \\
0 & \text{otherwise}.
\end{cases}
\]
By (6.4), (6.5) and Fubini’s theorem we have

\[ \int_X \varphi(x) \tilde{\eta}(x) \, d\mu(x) = \int_{U^+} \int_{U^-} \int_U \varphi(gh_0h_1x) \tilde{\eta}(gh_0h_1x) \, dg \, dh_0 \, dh_1 \]

(6.16)

\[ = \int_U \int_{U^+} \int_{U^-} (\varphi(gx) + \lambda_\psi(g, h_0, h_1)) \eta(g) \zeta(h_0, h_1) \, dh_0 \, dh_1 \, dg \]

\[ = \int_U \lambda_\psi(g) \eta(g) \, dg + \int_U \xi(g) \eta(g) \, dg, \]

where

\[ \lambda_\psi(g) = \int_{U^+} \int_{U^-} \lambda_\psi(g, h_0, h_1) \zeta(h_0, h_1) \, dh_0 \, dh_1. \]

By (6.8) and the uniform convergence of (6.6), the function \( \lambda_\psi(g) \) is smooth on \( U \) and for any \( \alpha \in b^r \)

(6.17)

\[ \partial^\alpha \lambda_\psi(g) = \lambda_{\partial^\alpha \psi}(g). \]

For any \( \eta \in C_c^\infty(U) \), by (6.16) and (6.17), we have

\[ \int_U (\partial^\alpha \xi) \eta \, dg = (-1)^r \int_U \xi(\partial^\alpha \eta) \, dg \]

(6.18)

\[ = (-1)^r \int_X \varphi(x) \tilde{\partial^\alpha \eta}(x) \, d\mu(x) + (-1)^{r+1} \int_U \lambda_\psi(g)(\partial^\alpha \eta)(g) \, dg \]

\[ = \int_X \partial^\alpha \varphi(x) \tilde{\eta}(x) \, d\mu(x) - \int_U \partial^\alpha \lambda_\psi(g) \eta(g) \, dg \]

\[ = \int_X \partial^\alpha \varphi(x) \tilde{\eta}(x) \, d\mu(x) - \int_U \lambda_{\partial^\alpha \psi}(g) \eta(g) \, dg. \]

Write \( \eta_{\alpha, z}(g) = \eta_{\alpha}(g)e^{-2\pi ib(g)z} \); then by (6.18)

(6.19) \[ \int_U \partial^\alpha \xi(g) \eta_{\alpha, z}(g) \, dg = \int_X \partial^\alpha \varphi(x) \tilde{\eta}_{\alpha, z}(x) \, d\mu(x) - \int_U \lambda_{\partial^\alpha \psi}(g) \eta_{\alpha, z}(g) \, dg. \]

By Lemma 6.3,

\[ \left| \int_X \partial^\alpha \varphi(x) \tilde{\eta}_{\alpha, z}(x) \, d\mu(x) \right| = \left| \sum_{n=0}^{\infty} \int_X \partial^\alpha \varphi(a^n x) \tilde{\eta}_{\alpha, z}(x) \, d\mu(x) \right| \]

(6.20)

\[ \leq \sum_{n=0}^{\infty} \left| \int_X \partial^\alpha \varphi(a^n x) \tilde{\eta}_{\alpha, z}(x) \, d\mu(x) \right|. \]
By Lemma 2.3,
\begin{equation}
\left| \int_X \partial^\alpha \psi(a^n x) \tilde{\eta}_{\alpha,z}(x) \, d\mu(x) \right| \lesssim_{\alpha} e^{-n\delta'} \| \eta_{\alpha,z} \|_{\ell_0} \lesssim_{\alpha} e^{-n\delta'} |z|_{\text{sup}}^\ell_0.
\end{equation}
So (6.20) and (6.21) imply
\begin{equation}
\left| \int_X \partial^\alpha \varphi(x) \tilde{\eta}_{\alpha,z}(x) \, d\mu(x) \right| \lesssim_{\alpha} |z|_{\text{sup}}^\ell_0.
\end{equation}
On the other hand,
\begin{equation}
\left| \int_U \lambda_{\partial^\alpha \psi}(g) \eta_{\alpha,z}(g) \, dg \right| \leq \int_U |\lambda_{\partial^\alpha \psi}(g)\eta_{\alpha}(g)| \, dg \lesssim_\alpha 1.
\end{equation}
By (6.19), (6.22) and (6.23), we have
\begin{equation}
\left| \int_U \partial^\alpha \xi(g) \eta_{\alpha,z}(g) \, dg \right| \lesssim_\alpha |z|_{\text{sup}}^\ell_0 + 1.
\end{equation}
Using (6.24) to estimate each term on the right-hand side of (6.15), we get
\begin{equation}
|(2\pi iz)^\gamma \hat{f}(z)| \lesssim_\gamma |z|_{\text{sup}}^\ell_0 + 1.
\end{equation}
For any positive integer \( k \), we sum the left-hand side of (6.25) for all \( \gamma = 2k \gamma_i \) where \( \{ \gamma_i : 1 \leq i \leq m \} \) is the standard basis of \( \mathbb{R}^m \). Then
\begin{equation}
|z|_{\text{sup}}^{2k} |\hat{f}(z)| \lesssim_k |z|_{\text{sup}}^\ell_0 + 1,
\end{equation}
from which the claim (6.9) follows. \( \blacksquare \)

**Proof of Theorem 1.6.** Suppose the variance \( \sigma(F, \phi) \) is zero; then Theorem 1.4 implies that there is a measurable solution \( \varphi \) to the system of cohomological equations (1.8). Note that \( \psi(x) = \int_0^1 \phi(a_t x) \, dt \) belongs to \( \hat{C}_c^\infty(X) \) and \( \varphi \) is a measurable solution to the cohomological equation \( \psi(x) = \varphi(a_1 x) - \varphi(x) \). By assumption, the projection of \( F \) to each simple factor of \( H \) is nontrivial, so the action of \( G'_{a_1} = H \) on \( X \) has a spectral gap. Theorem 1.1 implies that there is a smooth function \( \tilde{\varphi} \) on \( X \) such that \( \tilde{\varphi} = \varphi \) almost everywhere. This means that there is a continuous function \( \varphi \) such that (1.8) holds for all \( s > 0 \) and \( x \in X \).

To prove \( \phi \) is dynamically null with respect to \( (X, F) \), it suffices to show that for any \( F \)-invariant and ergodic probability measure \( \tilde{\mu} \) on \( X \) the integral \( \tilde{\mu}(\phi) = 0 \). By the Birkhoff ergodic theorem and Poincaré recurrence theorem, there is \( x \in X \) such that \( x \) is in the closure of \( \{ a_t x : t \geq T_0 \} \) for any \( T_0 \geq 0 \) and
\[ \tilde{\mu}(\phi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(a_t x) \, dt = \lim_{T \to \infty} \frac{1}{T} (\varphi(a_T x) - \varphi(x)). \]
The above two properties together with the continuity of \( \varphi \) imply \( \tilde{\mu}(\phi) = 0 \). \( \blacksquare \)
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