PARAMETRIZATION OF U(N)-MONOPOLES ON BLACK HOLES
BY THE MODULI SPACE OF HOLOMORPHIC VECTOR BUNDLES
OVER TWO-SPHERE AND BLACK HOLE ENTROPY

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Abstract
We discuss how to describe $U(N)$-monopoles on the Schwarzschild and Reissner-Nordström black holes by the parameters of the moduli space of holomorphic vector bundles over $S^2$. For $N = 2, 3$ we obtain such a description in an explicit form as well as the expressions for the corresponding monopole masses. This gives a possibility to adduce some reasonings in favour of existence of both a fine structure for black holes and the statistical ensemble tied with it which might generate the black hole entropy. Also there arises some analogy with the famous $K$-theory in topology.

1 Introductory Remarks
The present paper is a natural continuation of our previous work of Ref. [1], so we shall not dwell upon the motivation of studying the topics being considered here so long as it has been done in Ref. [1]. It should be here only recalled that one of the motivations of writing Ref. [1] was the search for the additional quantum numbers (nonclassical hair) characterizing black holes that might help in building a statistical ensemble necessary to generate the black hole entropy.

For this purpose in Ref. [1] with the help of the classification of complex vector bundles over $S^2$ and the Grothendieck splitting theorem a number of infinite series of $U(N)$-magnetic monopoles at $N \geq 1$ were constructed in an explicit form on the Schwarzschild and Reissner-Nordström black holes. Also
the masses of the given monopoles were estimated to show that they might reside in black holes as quantum objects. Any applications to the problem of statistical substantiation of the black hole entropy have not, however, been given in Ref. [1].

It should be noted that the considerations of Ref. [1] employed only the differentiable structure of complex vector $U(N)$-bundles over the chosen class of black holes. Therefore, the additional quantum numbers (topological charges) parametrizing $U(N)$-monopoles were obtained as the Chern numbers of the bundles under discussion. It is known, however, (see, e. g., Ref. [2]) that the given bundles admit holomorphic structures whose moduli space can be actually obtained from the Grothendieck splitting theorem [3]. Accordingly, it is of significant interest to get some description of the mentioned $U(N)$-monopoles in dependence of the parameters of the moduli space in question, so long as, in this way, we obtain a marked increase of the additional quantum numbers characterizing black holes.

The given paper will be devoted to developing all the mentioned above questions. After general considerations in Sec. 2 for arbitrary $N \geq 1$, the generic scheme is concretized in Sec. 3 for $U(2)$-monopoles, while in Sec. 4 for $U(3)$-monopoles. Sec. 5 contains the estimations of the corresponding monopole masses. In Sec. 6 we adduce some reasonings in favour of existence of both a fine structure for black holes which is related with $U(N)$-monopoles and a statistical ensemble for generating black hole entropy. Also we discuss an analogy arising in the 4D black hole physics and reminding us the famous K-theory in topology. Finally, Sec. 7 contains concluding remarks, in particular, concerning the higher-dimensional black holes, while Appendix for inquiry adduces some facts about functions employed in Sec. 2.

We write down the black hole metrics under discussion (using the ordinary set of local coordinates $t, r, \vartheta, \varphi$, covering all the spacetime background manifold of the 4D black hole physics $\mathbb{R}^2 \times S^2$ except for a set of the zero measure) in the form

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu \equiv C dt^2 - C^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \quad (1.1)$$

with $C = 1-2M/r$ for the Schwarzschild case and $C = 1-2M/r+Q^2/r^2$ for the Reissner-Nordström case, where $M, Q$ are, respectively, a black hole mass and an electric charge. Besides in generally $r_+ \leq r < \infty$ with $r_+ = M+\sqrt{M^2-Q^2}$.

Under the circumstances we have the spatial part of the metric (1.1) defined on $\mathbb{R} \times S^2$-topology as

$$d\sigma^2 = C^{-1} dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \equiv \gamma_{ij} dx^i \otimes dx^j \quad (1.2)$$

with $\sqrt{\gamma} = r^2 \sin \vartheta C^{-1/2} = \sqrt{\det(\gamma_{ij})}$.

Throughout the paper we employ the system of units with $\hbar = c = G = 1$, unless explicitly stated. Finally, we shall denote $L_2(B)$ the set of the modulo square integrable complex functions on $B$ for any manifold $B$ furnished with an integration measure.
2 General Considerations

According to Ref. [1] in order to obtain the infinite families of $U(N)$-monopoles for $N \geq 1$, we should use the Grothendieck splitting theorem [2, 3] which asserts that any complex vector bundle over $S^2$ (and, as a consequence, over $\mathbb{R}^2 \times S^2$) of rank $N \geq 1$ [i.e., with the structural group $U(N)$] is a direct sum of $N$ suitable complex line bundles over $S^2$. The standard results of algebraic topology (see, e.g., Ref. [4]) say that $U(N)$-bundles over $S^2$ are in one-to-one correspondence with elements of the fundamental group of $U(N)$, $\pi_1[U(N)]$. On the other hand, in virtue of the Bott periodicity $\pi_1[U(N)] = \mathbb{Z}$ at $N \geq 1$ and, as a result, there exists the countable number of nontrivial complex vector bundles of any rank $N > 1$ over $\mathbb{R}^2 \times S^2$. The sections of such bundles can be qualified as topologically inequivalent configurations (TICs) of $N$-dimensional complex scalar field. The above classification confronts some $n \in \mathbb{Z}$ with each $U(N)$-bundle over $\mathbb{R}^2 \times S^2$-topology. In what follows we shall call it the Chern number of the corresponding bundle. TIC with $n = 0$ can be called untwisted one while the rest of the TICs with $n \neq 0$ should be referred to as twisted.

So far we tacitly implied that the $U(N)$-bundles were supposed to be differentiable. Really, they admit holomorphic structures and since each differentiable complex line bundle over $S^2$ admits only one holomorphic structure (i.e., the holomorphic and differentiable classifications of complex line bundles over $S^2$ coincide [2]) then the Grothendieck splitting theorem in fact gives a description of the moduli space of $N$-dimensional holomorphic complex vector bundles over $S^2$. Namely, each $N$-dimensional holomorphic complex vector bundle over $S^2$ is defined by the only $N$-plet of integers $(r_1, r_2, \ldots, r_N) \in \mathbb{Z}^N$, $r_1 \geq r_2 \geq \ldots \geq r_N$. Two of such $N$-plets $(r_i)$ and $(r'_i)$ define the same differentiable $N$-dimensional bundle if and only if $\sum_i r_i = \sum_i r'_i$. In Ref. [1] we neglected the above holomorphic structures and, in consequence, we chose the $N$-plet characterizing a $N$-dimensional complex bundle with the Chern number $n \in \mathbb{Z}$ in the form $(n, 0, \ldots, 0)$. Let us now take into account the existence of the given holomorphic structures, so we should consider all the above $N$-plets $(r_i)$, each $N$-plet representing the point of the moduli space of $N$-dimensional complex vector bundles over $S^2$.

As was shown in Ref. [3], each complex line bundle (with the Chern number $n$) over $\mathbb{R}^2 \times S^2$ with the metric (1.1) has a complete set of sections in $L_2(\mathbb{R}^2 \times S^2)$, so using the fact that all the $U(N)$-bundles over $\mathbb{R}^2 \times S^2$ can be trivialized over the bundle chart of local coordinates $(t, r, \vartheta, \varphi)$ covering almost the whole manifold $\mathbb{R}^2 \times S^2$, the mentioned set can be written on the given chart in the form

$$f_{\omega lm} = \frac{1}{r} e^{i\omega t} R_{\omega l}(r) e^{im\varphi} P_{mn}^l(\vartheta), l = |n|, |n| + 1, \ldots, |m| \leq l,$$

(2.1)

where the explicit form and some properties of the functions $P_{mn}^l(\vartheta)$ can be found in Appendix (see also Ref. [5]), but we shall not need them further. It
should be noted that in physical literature devoted to the Dirac monopoles (see, e. g., Refs.[6]), the combinations \( e^{im\varphi} P_{ml}^n(\vartheta) = Y_{nlm}(\vartheta, \varphi) \) are called the monopole (spherical) harmonics [that coincide with the ordinary ones at \( n = 0 \), i. e., \( Y_{0lm}(\vartheta, \varphi) = Y_{lm}(\vartheta, \varphi) \)]. As to the functions \( R_{\omega l}(r) \) then they obey the equation

\[
\frac{d}{dr} \left( C r^2 \frac{d}{dr} R_{\omega l} \right) + (k^2 + \omega^2 C^{-1}) \frac{R_{\omega l}}{r} = 0 \quad (2.2)
\]

with \( k^2 = -l(l + 1) \), \( l = |n|, |n| + 1, \ldots \).

Now, in accordance with the Grothendieck splitting theorem, any section of \( N \)-dimensional complex bundle \( \xi_n \) over \( \mathbb{R}^2 \times S^2 \) with the Chern number \( n \in \mathbb{Z} \) can be represented by a \( N \)-plet \((\phi_1, \ldots, \phi_N)\) of complex scalar fields \( \phi_i \), where each \( \phi_i \) is a section of a complex line bundle over \( \mathbb{R}^2 \times S^2 \). According to the above, we can consider \( \phi_i \) the section of complex line bundle with the Chern number \( r_i \in \mathbb{Z} \), where the numbers \( r_i \) are subject to the conditions

\[
r_1 \geq r_2 \geq \ldots \geq r_N ,
\]

\[
r_1 + r_2 + \ldots + r_N = n . \quad (2.3)
\]

As a consequence, we can require the \( N \)-plets

\[
(\frac{1}{r} e^{i\omega t} R_{\omega t1}(r) Y_{rl1m1}, \frac{1}{r} e^{i\omega t} R_{\omega t2}(r) Y_{rl2m2}, \ldots, \frac{1}{r} e^{i\omega t} R_{\omega tN}(r) Y_{rlNmN})
\]

to form the basis in \( \left[ L_2(\mathbb{R}^2 \times S^2) \right]^N \) for the sections of \( \xi_n \), \( l_i = |r_i|, |r_i| + 1, \ldots, |m_i| \leq l_i \), and this will define the wave equation for a section \( \phi = (\phi_1, \ldots, \phi_N) \) of \( \xi_n \) with respect to the metric (1.1)

\[
\begin{pmatrix}
I_N \Box - \frac{1}{r^2 \sin^2 \vartheta} \times \\
2ir_1 \cos \vartheta \partial \varphi - r_1^2 & 0 & \ldots & 0 \\
0 & 2ir_2 \cos \vartheta \partial \varphi - r_2^2 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 2ir_N \cos \vartheta \partial \varphi - r_N^2
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_N
\end{pmatrix} = 0 , \quad (2.4)
\]

where \( I_N \) is the unit matrix \( N \times N \), \( \Box = (\delta)^{-1/2} \partial_{\mu}(g^{\mu\nu}(\delta)^{1/2} \partial_{\nu}) \) — the conventional wave operator conforming to metric (1.1) with the module of its determinant \( \delta = r^4 \sin^2 \vartheta \).

The Eq. (2.4) will, in turn, correspond to the lagrangian

\[
\mathcal{L} = \delta^{1/2} g^{\mu\nu} \overline{\phi} D_{\mu} \phi D_{\nu} \phi , \quad (2.5)
\]
with \( \phi = (\phi_i) \) and a covariant derivative \( D_\mu = \partial_\mu - igA^a_\mu T_a \) on sections of the bundle \( \xi_n \), while the line in (2.5) signifies hermitean conjugation and the matrices \( T_a \) will form a basis of the Lie algebra of \( U(N) \) in \( N \)-dimensional space, \( a = 1, \ldots, N^2 \), \( g \) is a gauge coupling constant, i. e., we come to a theory describing the interaction of a \( N \)-dimensional twisted complex scalar field with the gravitational field described by metric (1.1). The coefficients \( A^a_\mu \) will represent a connection in the given bundle \( \xi_n \) and will describe some nonabelian \( U(N) \)-monopole.

As can be seen, the Eq.(2.4) has the form \( D^\mu D_\mu \phi = 0 \), where \( D_\mu \) is a formal adjoint to \( D_\mu \) with regards to the scalar product induced by metric (1.1) in \([L_2(\mathbb{R}^2 \times S^2)]^N\). That is, the operator \( D_\mu \) acts on the differential forms \( a_\mu dx^\mu \) with coefficients in the bundle \( \xi_n \) in accordance with the rule

\[
D_\mu(a_\mu) = -\frac{1}{\sqrt{\delta}} \partial_\mu (g^{\mu\nu} \sqrt{\delta} a_\nu) + ig A_\mu g^{\mu\nu} a_\nu \tag{2.6}
\]

with \( A_\mu = A^a_\mu T_a \).

As a result, the equation \( D^\mu D_\mu \phi = 0 \) takes the form

\[
I_N \Box \phi - \frac{1}{\sqrt{\delta}} ig \partial_\mu (g^{\mu\nu} \sqrt{\delta} A_\nu \phi) - (ig A_\mu g^{\mu\nu} \partial_\nu + g^2 g^{\mu\nu} A_\mu A_\nu) \phi = 0 \tag{2.7}
\]

Comparing (2.4) with (2.7) gives a row of the (gauge) conditions:

2.1

\( A_t = A_r = 0 \)

2.2

\( A^a_\vartheta = -\overline{A^a_\varphi} \) is pure imaginary, since we, as is accepted in physics, consider the matrices \( \overline{T_a} \) hermitean.

2.3

\( A^a_\varphi = \overline{A^a_\varphi} \) and

\[
g A^a_\varphi T_a = -\begin{pmatrix} r_1 \cos \vartheta & 0 & \cdots & 0 \\ 0 & r_2 \cos \vartheta & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & r_N \cos \vartheta \end{pmatrix}, \tag{2.8}
\]

so, accordingly, \( A^a_\varphi = A^a_\varphi (\vartheta) \).
\[ i \frac{g}{2} \left[ \cot \vartheta A_{\vartheta}^a T_a + (\partial_\vartheta A_{\vartheta}^a) T_a \right] + (A_{\vartheta}^a T_a) (A_{\vartheta}^b T_b) = \]

\[ = \left( \frac{1}{g} \right) \left( \begin{array}{cccc}
\frac{1}{2} & 0 & \ldots & 0 \\
0 & \frac{1}{2} & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & \frac{1}{2} \end{array} \right), \tag{2.9} \]

where \( A_{\vartheta}^a = A_{\vartheta}^a (\vartheta) \) depends only on \( \vartheta \) and, as a consequence, the connection matrix \( A \) for \( \xi_n \)-bundle is equal to \( A = A_{\vartheta}^a T_a d\vartheta + A_{\varphi}^a (\varphi) T_a d\varphi \) with the \( A_{\vartheta, \varphi}^a \) subject to the above conditions.

This yields the curvature matrix \( F = dA + A \wedge A \) for \( \xi_n \)-bundle in the form

\[ F = F_{a \mu \nu} T_a dx^\mu \wedge dx^\nu = (\partial_\vartheta A_{\varphi}^a) T_a d\vartheta \wedge d\varphi = \]

\[ = \frac{1}{g} \sin \vartheta d\vartheta \wedge d\varphi \left( \begin{array}{cccc}
1 & 0 & \ldots & 0 \\
0 & 2 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & N \end{array} \right), \tag{2.10} \]

because the exterior differential \( d = \partial_\vartheta dt + \partial_r dr + \partial_\vartheta d\vartheta + \partial_\varphi d\varphi \) in coordinates \( t, r, \vartheta, \varphi \).

From here it follows that the first Chern class \( c_1(\xi_n) \) of the bundle \( \xi_n \) can be chosen in the form

\[ c_1(\xi_n) = \frac{g}{4\pi} \text{Tr}(F) = \frac{n}{4\pi} \sin \vartheta d\vartheta \wedge d\varphi, \tag{2.11} \]

where we employed (2.3) and (2.10), so that if integrating \( c_1(\xi_n) \) over topological two sphere \( S^2 \) (which can be described by the relations \( 0 \leq \vartheta < \pi, 0 \leq \varphi < 2\pi \) in the manifold in question) we have

\[ \int_{S^2} c_1(\xi_n) = n, \tag{2.12} \]

which is equivalent to the conventional Dirac charge quantization condition \( qg = 4\pi n \) with (nonabelian) magnetic charge

\[ q = \int_{S^2} \text{Tr}(F). \tag{2.13} \]

Introducing the Hodge star operator \( * \) conforming metric (1.1) on 2-forms \( F = F_{\mu \nu} T_a dx^\mu \wedge dx^\nu \) with the values in the Lie algebra of \( U(N) \) by the relation (see, e. g., Refs. [7])
\( (F_{\mu}^{a} dx^{\mu} \wedge dx^{\nu}) \wedge (* F_{\alpha}^{a} dx^{\alpha} \wedge dx^{\beta}) = g^{\mu \alpha} g^{\nu \beta} F_{\mu \nu}^{a} F_{\alpha \beta}^{a} \sqrt{\delta} dx^{0} \wedge \cdots \wedge dx^{3} \), \hspace{1em} (2.14)

written in local coordinates \( x^{\mu} \) (there is no summation over \( a \) in (2.14)), in coordinates \( t, r, \vartheta, \varphi \) we have for \( F \) of (2.10)

\[ *F = *F_{\mu \nu}^{a} T_{a} dx^{\mu} \wedge dx^{\nu} = \frac{1}{g} r^{-2} dt \wedge dr \begin{pmatrix} r_{1} & 0 & \ldots & 0 \\ 0 & r_{2} & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & r_{N} \end{pmatrix} \] \hspace{1em} (2.15)

We can now consider the Yang-Mills (Maxwell at \( N = 1 \)) equations

\[ dF = F \wedge A - A \wedge F, \] \hspace{1em} (2.16)

\[ d*F = *F \wedge A - A \wedge *F. \] \hspace{1em} (2.17)

It is clear that (2.16) is identically satisfied by the above \( A, F \). As for the Eq. (2.17) that it reduces to the condition

\[ A_{\vartheta}^{a}(\vartheta) T_{a} \begin{pmatrix} r_{1} & 0 & \ldots & 0 \\ 0 & r_{2} & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & r_{N} \end{pmatrix} A_{\vartheta}^{a}(\vartheta) T_{a} \] \hspace{1em} (2.18)

which can be fulfilled (at \( N = 1 \) always), for example, if the matrix \( A_{\vartheta}^{a} T_{a} \) is diagonal, so there exist nontrivial solutions of (2.18).

To evaluate the monopole masses we should use the \( T_{00} \)-component of the energy-momentum tensor

\[ T_{\mu \nu} = \frac{1}{4\pi} (- F_{\mu \alpha}^{a} F_{\nu \beta}^{a} g^{\alpha \beta} + \frac{1}{4} F_{\beta \gamma}^{a} F_{\alpha \delta}^{a} g^{\alpha \beta} g^{\gamma \delta} g_{\mu \nu} ) \] \hspace{1em} (2.19)

which does obviously not depend on \( A_{\vartheta}^{a}(\vartheta) \), hence the main thing is the solutions of (2.8). Having obtained them we can find the monopole masses according to

\[ m_{\text{mon}}(n) = \int_{\mathbb{R} \times S^{2}} T_{00} \sqrt{\gamma} d^{3}x = \int_{\mathbb{R} \times S^{2}} T_{00} r^{2} \sin \vartheta C^{-1/2} d^{3}x. \] \hspace{1em} (2.20)

with \( C \) of (1.1) and \( \gamma \) of (1.2) while \( T_{00} \)-component is evaluated on the above solutions. Therefore, let us concretize the above construction for the cases \( N = 2, 3 \) since the \( N = 1 \) case does not differ from the one of Ref.[1]. But before one should do a remark.
Generally speaking for arbitrary spacetimes the formula (2.20) is badly defined since it is impossible in general case to separate the full energy of a gravitating system into the energy of matter (including the electromagnetic field) and the energy of gravitational field itself. However, as was discussed many years ago [8], there exists a class of spacetimes where such a separation is possible. These are the so-called asymptotically flat spacetimes (AFS). Moreover, for AFS one can introduce the so-called the pseudotensor of Landau-Lifshits $t^{L-L}_{\mu\nu}$ of gravitational field [8, 9], so that the full effective energy-momentum tensor of the gravitating system will be $T^{eff}_{\mu\nu} = T_{\mu\nu} + t^{L-L}_{\mu\nu}$ with the energy-momentum tensor of matter (including electromagnetic field) $T_{\mu\nu}$.

If $V$ is a spatial part of the AFS then the quantity $E = \int_{V} T^{eff}_{00} \sqrt{\gamma} d^3x$ is interpreted as the full energy of the gravitating system and, clearly, $E$ can be considered as the sum of the energy of matter and the energy of gravitational field itself. As is well known [5], the black hole spacetimes are just of the AFS and, for example, in Ref.[9] the contribution $\int_{V} t^{L-L}_{00} \sqrt{\gamma} d^3x$ was evaluated for the Schwarzschild spacetime and proved to be equal to $M$, black hole mass. Therefore, the rest of the full energy in AFS equal to $\int_{V} T_{00} \sqrt{\gamma} d^3x$ can be interpreted as the energy (or, in units used by us, as mass) of matter (including electromagnetic field) and, as a result, the formula (2.20) makes sense in black hole spacetimes with the interpretation used further in our paper.

3 U(2)-monopoles

In this case we can take $T_1 = I_2$, $T_a = \sigma_{a-1}$ at $a = 2, 3, 4$, where $\sigma_{a-1}$ are the ordinary Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}
\]

Then the Eq. (2.8) will be consistent with such a choice, if we put the connection matrix $A = A^a_\mu T_a dx^\mu$ to be equal to

\[
A = -\frac{1}{2g} \cos \vartheta d\varphi [(r_1 + r_2)I_2 + (r_1 - r_2)\sigma_3] + A^\vartheta_\vartheta T_\vartheta d\vartheta, \tag{3.2}
\]

where $A^\vartheta_\vartheta(\vartheta)$ possesses the properties described in Sec. 2.

This yields the curvature matrix $F = dA + A \wedge A$ in the form

\[
F = F^a_{\mu\nu} T_a dx^\mu \wedge dx^\nu = \frac{1}{2g} \sin \vartheta d\vartheta \wedge d\varphi [(r_1 + r_2)I_2 + (r_1 - r_2)\sigma_3] = \\
\frac{1}{2g} \sin \vartheta d\vartheta \wedge d\varphi [nI_2 + (r_1 - r_2)\sigma_3], \tag{3.3}
\]
where we used the relation (2.3).

In accordance with (2.20) we find

\[ T_{00} = \frac{1}{16\pi} \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \frac{1}{r^4} \left( \frac{1}{2g} \right)^2 [n^2 + (r_1 - r_2)^2] . \] (3.4)

In what follows we denote \( A_2(n) = n^2 + (r_1 - r_2)^2 \).

4 U(3)-monopoles

In the given situation we can take \( T_1 = I_3, \ T_a = \lambda_{a-1} \) at \( a = 2, \ldots, 9 \), where \( \lambda_{a-1} \) are the Gell-Mann matrices

\[
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \] (4.1)

For the Eq. (2.8) to be consistent with such a choice, it should be put

\[
A = A^a_T \ dx^a = -\frac{1}{3g} \cos \vartheta \, d\varphi [(r_1 + r_2 + r_3)I_3 + \frac{3}{2}(r_1 - r_2)\lambda_3 + \frac{\sqrt{3}}{2}(2r_3 - r_1 - r_2)\lambda_8] + A^a_\vartheta \, T_a \, d\vartheta . \] (4.2)

This yields

\[
F = F^a_{\mu \nu} \ dx^a \wedge \ dx^\nu = \frac{1}{3g} \sin \vartheta \, d\vartheta \wedge d\varphi [(r_1 + r_2 + r_3)I_3 + \frac{3}{2}(r_1 - r_2)\lambda_3 + \frac{\sqrt{3}}{2}(2r_3 - r_1 - r_2)\lambda_8] . \] (4.3)

Finally, taking into account (2.3)

\[
T_{00} = \frac{1}{16\pi} \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \frac{1}{r^4} \left( \frac{1}{2g} \right)^2 \times
\]
\[ n^2 + \frac{9}{4}(r_1 - r_2)^2 + \frac{3}{4}(2r_3 - r_1 - r_2)^2 \]. \quad (4.4)

In what follows we denote \( A_3(n) = n^2 + \frac{9}{4}(r_1 - r_2)^2 + \frac{3}{4}(2r_3 - r_1 - r_2)^2 \).

It is clear that the case of arbitrary \( N \) can be treated analogously but we shall not dwell upon it.

## 5 Monopole Masses

In accordance with (2.20) let us consider miscellaneous cases and for inquiry we shall also adduce the results for \( N = 1 \) case of Ref.\[1\].

### 5.1 Schwarzschild black hole

We have \( Q = 0, r_+ = 2M \).

#### 5.1.1 \( U(1) \)-monopoles

\[ m_{\text{mon}}(n) = \frac{1}{12M}\left(\frac{n}{e}\right)^2 \] \quad (5.1)

with \( g = e = 4.8 \cdot 10^{-10} \text{ cm}^{3/2} \text{ g}^{1/2} \text{ s}^{-1} \) in usual units.

#### 5.1.2 \( U(2) \)-monopoles

\[ m_{\text{mon}}(n, r_1, r_2) = \left(\frac{1}{2g}\right)^2 \frac{A_2(n)}{4} \int_{2M}^{\infty} \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} dr = \left(\frac{1}{2g}\right)^2 \frac{A_2(n)}{12M}. \] \quad (5.2)

#### 5.1.3 \( U(3) \)-monopoles

\[ m_{\text{mon}}(n, r_1, r_2, r_3) = \left(\frac{1}{3g}\right)^2 \frac{A_3(n)}{12M}. \] \quad (5.3)

### 5.2 Reissner-Nordström black hole with \( Q \neq M \)

We have here \( r_+ = M + \sqrt{M^2 - Q^2} \). It is easy to notice that at \( r_+ \leq r < \infty \) the quantity \( (1 - 2M/r + Q^2/r^2) \leq 1 \).
5.2.1 $U(1)$-monopoles

\[ m_{\text{mon}}(n) = \frac{n^2}{4e^2} \int_{r_+}^{\infty} \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} \, \frac{dr}{r^2} \approx \frac{n^2}{4e^2} \frac{1}{r_+} = \frac{n^2}{4e^2 r_+}. \] (5.4)

5.2.2 $U(2)$-monopoles

\[ m_{\text{mon}}(n, r_1, r_2) = \left( \frac{1}{2g} \right)^2 \frac{A_2(n)}{4} \int_{r_+}^{\infty} \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} \, \frac{dr}{r^2} \approx \left( \frac{1}{2g} \right)^2 \frac{A_2(n)}{4} \int_{r_+}^{\infty} \frac{dr}{r^2} = \left( \frac{1}{2g} \right)^2 \frac{A_2(n)}{4r_+}. \] (5.5)

5.2.3 $U(3)$-monopoles

\[ m_{\text{mon}}(n, r_1, r_2, r_3) \approx \left( \frac{1}{3g} \right)^2 \frac{A_3(n)}{4r_+}. \] (5.6)

5.3 Extremal Reissner-Nordström black hole

We have $Q = M$ and $r_+ = M$. Under these conditions we can make replacement $1/r = x$, $x_+ = 1/r_+ = 1/M$ and take into account that $\sqrt{Q^2 x^2 - 2M x + 1} = M|1/M - x|$ which gives

\[ \int_0^{x_+} \sqrt{Q^2 x^2 - 2M x + 1} \, dx = \frac{1}{8M}. \] (5.7)

This yields

5.3.1 $U(1)$-monopoles

\[ m_{\text{mon}}(n) = \left( \frac{n}{e} \right)^2 \frac{1}{8M}. \] (5.8)

5.3.2 $U(2)$-monopoles

\[ m_{\text{mon}}(n, r_1, r_2) = \left( \frac{1}{2g} \right)^2 \frac{A_2(n)}{8M}. \] (5.9)
5.3.3 \( U(3) \)-monopoles

\[
m_{\text{mon}}(n, r_1, r_2, r_3) = \left( \frac{1}{3g} \right)^2 \frac{A_3(n)}{8M}.
\] (5.10)

It should be noted that to obtain the above monopole masses in usual units it is enough to multiply the values (5.1)–(5.6) and (5.8)–(5.10) by \( \bar{\hbar}^2 c^2 / G \).

Introducing parametrization \( Q = \alpha M \) with \( 0 \leq \alpha \leq 1 \), we find \( r_+ = M f(\alpha) \) with \( f(\alpha) = 1 + \sqrt{1 - \alpha^2} \), \( 1 \leq f(\alpha) \leq 2 \). Under the circumstances, evaluating the corresponding Compton wavelength \( \lambda_{\text{mon}}(n, r_i) = \hbar / m_{\text{mon}}(n, r_i) c \), we can see that at any \( n \neq 0, N \geq 1 \), \( \lambda_{\text{mon}}(n, r_i) \ll r_g \), where \( r_g = r_+ G / c^2 \) is a gravitational radius of black hole, if \( g^2 / \bar{h} c \ll 1 \). The latter always holds true for \( U(1) \)-monopoles so long as \( e^2 / \bar{h} c \sim 1/137 \). As a consequence, we come to the conclusion that under certain conditions \( U(N) \)-monopoles might reside in black holes as quantum objects.

So, we can see that the masses of \( U(N) \)-monopoles really depend on the parameters of the moduli space of holomorphic vector bundles over \( S^2 \). Let us consider some possible issues for the 4D black hole physics from this fact, having denoted this moduli space (whose description has been adduced in Sec. 2) as \( \mathfrak{M}_N \) for some given \( N \geq 1 \).

6 Fine Structure of Black Holes, a Statistical Ensemble for Generating Black Hole Entropy, Quantum \( K \)-theory

Among the unsolved questions of modern 4D black hole physics the so-called black hole information problem admittedly ranks high. Referring for more details, e.g., to Refs.\[10\] (and references quoted therein), it should be noted here that one aspect of the problem consists in that for an external observer any black hole looks like an object having in general only a finite number of parameters (classical hair — mass \( M \), charge \( Q \), angular momentum \( J \)) and it is, therefore, unclear how these parameters can encode all the information about quantum particles of matter (which has been collapsed to the black hole), particles that are being radiated à la Hawking. As a consequence, it is impossible to distinguish all the black hole (pure) states, so a black hole should, therefore, be described by a mixed state. In other words, the system (black hole) has an entropy \( S \) while the latter does not correspond to any statistical ensemble, so long as there is no infinite number of quantum (discrete) numbers connected with this system to build an appropriate statistical ensemble.

As the results of both Ref.\[1\] and the present paper, however, show, the natural candidates for additional quantum numbers (nonclassical hair) for
black holes might be the topological quantum numbers parametrizing $U(N)$-monopoles on black holes, so these numbers could be identified with $\mathfrak{m}_N$. Really, as has been demonstrated recently in Ref.\cite{11}, black holes can radiate à la Hawking for any TICs, for instance, of complex scalar field with the Chern number $n \in \mathbb{Z} = \mathfrak{m}_1$ and this occurs independently of other field configurations. More exact analytical and numerical considerations \cite{12} show that, for instance, in the Schwarzschild black hole case, twisted TICs can give the marked additional contribution of order 17% to the total luminosity (summed up over all the TICs). This tells us that there exists some fine structure in black hole physics which is conditioned by nontrivial topological properties of black holes and the given fine structure is able to markedly modify the black hole characteristics, so long as, for example, the words ”Hawking radiation for complex scalar field ” should be now understood as the radiation summed up over all the TICs of complex scalar field on black hole. In a sense, the black hole fine structure is quite analogous to the one of atomic spectra in atomic physics where its existence enables us to achieve an essentially better understanding of the whole structure of atoms.

Let us consider, therefore, more in detail in which way the above fine structure might help to black holes to form a statistical ensemble necessary to generate the black hole entropy.

6.1 Schwarzschild black hole

As is known (see, e. g., Ref.\cite{13}), the entropy $S$ of black hole can be introduced from purely thermodynamical considerations and, for example, $S = 4\pi M^2$ in the Schwarzschild black hole case, so when putting the internal energy of black hole $U = M$, we obtain the temperature of black hole $T = \frac{\partial U}{\partial S} = \frac{1}{8\pi M}$ through the standard thermodynamical relation. It is obvious that $S$ corresponds to a formal partition function $Z = e^{-\frac{M}{2T}}$ for the given $M$ and $T$ (we took the Boltzmann constant $k_B = 1$). The quantity $Z$ is formal because we cannot point out any infinite statistical ensemble conforming to it, so that one could obtain $Z$ by the usual Gibbs procedure, i. e., by averaging over this ensemble. The results of Refs.\cite{11, 12} show that black hole can radiate à la Hawking for any TIC of complex scalar field with the Chern number $n \in \mathfrak{m}_1$. Such a radiation is practically defined by a couple $(g, n)$ with the black hole metric $g$ of (1.1) and the Chern number $n$ in the sense that these data are sufficient to describe the physical quantities (for instance, luminosity $L(n)$) characterizing the radiation process for TICs with the Chern number $n$ \cite{11, 12}. On the other hand, as is known (see, e. g., Ref.\cite{14}), the Hawking effect is being obtained when considering the system (black hole + matter field near it) semiclassically: the black hole is being described classically while the matter field is being quantized. All mentioned above suggests that the Hawking process occurs for the given pair $(g, n)$ when the black hole is in a quantum state which can be
characterized by the *semiclassical* energy

\[ E_n = \frac{M}{2} + \mathcal{E}(n) = \frac{M}{2} + \frac{1}{12M} \left( \frac{n}{e} \right)^2 \]  

(6.1)

with \( \mathcal{E}(n) = m_{\text{mon}}(n) \) of (5.1), so long as \( \mathcal{E}(n) \) is a natural energy of the monopole with the Chern number \( n \) residing in black hole, since the additional contribution to the Hawking radiation is conditioned actually by the same monopole \([11, 12]\). We call \( E_n \) semiclassical because the first term of (6.1) in usual units does not depend on \( \hbar \) while the second one does (see Sec. 5).

Under the circumstances there arises an infinite set of quantum states \((g, n)\) with the energy spectrum (6.1) for black hole. After this, the Gibbs average takes the form

\[ Z = \sum_{n \in \mathbb{Z}} e^{-\frac{E_n}{M}} = e^{-\frac{M}{2}} \sum_{n \in \mathbb{Z}} e^{-\mathcal{E}(n)/M} = e^{-4\pi M^2} \sum_{n \in \mathbb{Z}} q^{n^2} = e^{-4\pi M^2} \vartheta_3(0, q) \]  

(6.2)

with the Jacobi theta function \( \vartheta_3(v, q) \) and \( q = \exp \left( -\frac{2\pi}{3\alpha^2} \right) \). As a result, we obtain an inessential constant additive correction \( S_1 = \ln \vartheta_3(0, q) \) independent of \( M \) to the black hole entropy \( S = 4\pi M^2 \) but now the latter is the result of averaging over an infinite ensemble which should be considered as inherent to black hole due to its nontrivial topological properties.

It is clear that one can also consider all the triplets \((g, r_1, r_2)\), where the pair \((r_1, r_2)\) parametrizes the moduli space of \( U(2) \)-monopoles \( \mathfrak{M}_2 \) (see Sec. 2), so that the Gibbs average should be accomplished over \( \mathfrak{M}_2 \) which will again lead to some inessential additional correction to the entropy \( S \) due to dependence (5.2). Moreover, this scheme will obviously hold true for \( U(N) \)-monopoles at any \( N > 1 \) if the Gibbs average is accomplished over the moduli space of \( U(N) \)-monopoles \( \mathfrak{M}_N \) (see Sec. 2).

### 6.2 Reissner-Nordström black hole with \( Q \neq M \)

The formal partition function for the given case is

\[ Z = \exp \left( -\frac{M}{T} \left( 1 - \frac{\sqrt{1 - \alpha^2}}{2} \right) \right) \]  

(6.3)

with \( T = \sqrt{1 - \alpha^2}/[2\pi M(1 + \sqrt{1 - \alpha^2})] \). Under the circumstances the energy spectrum for black hole which is tied, for example, with \( U(1) \)-monopoles can be chosen according to (5.4) in the form

\[ E_n \approx M \left( 1 - \frac{\sqrt{1 - \alpha^2}}{2} \right) + m_{\text{mon}}(n) \frac{\sqrt{1 - \alpha^2}}{1 + \sqrt{1 - \alpha^2}} \]  

(6.4)
with $m_{\text{mon}}(n)$ of (5.4), so again we shall get an inessential constant additive correction independent of $\alpha$ to the entropy $S = \pi r_+^2$ after accomplishing the Gibbs average over the moduli space $\mathfrak{m}_1$. Obviously, the same holds true for any $U(N)$-monopoles provided that we shall accomplish the Gibbs average over the moduli space $\mathfrak{m}_N$.

### 6.3 Extremal Reissner-Nordström black hole

In this case the monopole massess are well defined and exist (see (5.8)–(5.10)), but we face the general difficulty of defining the entropy $S$ and temperature $T$ for this extremal case. At present there is no generally accepted consistent definition for the given quantities though in literature there have been done many attempts of analysing this situation (see, e. g., Ref.[16]). We shall not, therefore, dwell upon the given case.

Finally, as for the general Kerr-Newman case, at our disposal there are not yet any expressions of monopole masses in dependence of black hole parameters $M$, $Q$, $J$ for this case, so that the reasonings of the given section should be specified for the latter case after evaluating the necessary quantities that without doubt exist (see Ref.[1]).

As is clear from all the above, the black hole topology can bring many possibilities for producing a huge amount of new quantum numbers. So far we have, however, mainly spoken about bosons. As to the TICs of fermions, it is known [7, 16] that the given topology admits a countable number of the so-called Spin$^c$-structures and this might generate a range of new quantum charges for fermions on black holes. It should be noted that there exists topological duality between TICs of real scalar fields and usual spinorial structures (and the corresponding spinor fields) in the sense that both classes are classified by the same cohomology group $H^1(B, \mathbb{Z})$, the first cohomology group with coefficients in $\mathbb{Z}$ for the given manifold $B$, and this duality has nontrivial applications in quantum field theory, cosmology and $p$-branes (see our review of Ref.[17]). On the other hand, the classifying group for Spin$^c$-structures is $H^2(B, \mathbb{Z})$, the second cohomology group with coefficients in $\mathbb{Z}$ for manifold $B$, that is, the same as for TICs of complex scalar fields on the manifold $B$ (see, e. g., Refs.[5, 7]). For manifolds underlying the 4D black hole physics this group is equal to $\mathbb{Z}$[4, 5], so we obviously in the 4D black hole physics deal with complex analog of the above duality.

It seems to us, all the mentioned possibilities should be investigated from physical point of view, in particular, the influence of topological quantum numbers on quantum effects for fields near black holes. It is quite plausible that such a study will allow us to come to the conclusion that in black hole physics we deal with some (quantum) analog of the famous $K$-theory in algebraic topology. One can recall that $K$-theory takes into account all vector bundles over one or another manifold to build an appropriate topological invariant ($K$-ring) for manifolds (see, e. g., Refs.[4, 18]). At this, however, there exist a number of bundles which are more important for constructing $K$-ring (and
they, in essence, define it) than other ones over the given manifold. Perhaps, in black hole physics also topological quantum numbers which are tied with nontrivial vector bundles over the black hole topology can be split into more important and less important ones. The former could, for example, correspond to the observed physical fields, the latter could be directly unobserved but might help to build a statistical ensemble necessary to generate black hole entropy. But both classes, at any rate, should probably be certain relics from quantum gravity processes within black holes.

7 Concluding Remarks

The results of both the present paper and Refs.\[\text{1, 5, 11, 12}\] show that the 4D black hole physics can have a rich fine structure connected with the topology $\mathbb{R}^2 \times S^2$ underlying the 4D black hole spacetime manifolds. It seems to be quite probable that this fine structure could manifest itself in solving the whole number of problems within black hole physics so that one should seemingly thoroughly study the arising possibilities.

In view of all the above, one can a little touch upon the miscellaneous attempts of modelling the 4D black hole physics by considering the $D = 2$ and $D > 4$-cases that are at present popular enough in literature.

Generally speaking, from our point of view, the 2D black hole physics is plausible not very good laboratory for modelling the 4D case. Indeed, as a rule, the solutions describing a black hole within various 2D theories (see, e. g., Refs.\[\text{19}\]) are defined on trivial $\mathbb{R}^2$ or semitrivial $\mathbb{R} \times S^1$ background and, accordingly, lose essential topological features of the 4D case.

In contrast to $2D$ case, the one of higher dimensions is seemingly far more interesting. Really, in a number of the gravitation theories one has been demonstrated (see, e. g., Refs.\[\text{20}\]) that there exist the solutions describing black holes in any dimension $D > 2$. These black hole solutions are naturally defined on the manifolds with topologies $\mathbb{R}^2 \times S^{D-2}$, i. e., they are some reasonable extensions of the 4D black hole solutions, possess thermodynamic properties and, as a result, an entropy. The black hole information problem can, therefore be posed for these solutions as well. But it is clear that the given solutions can also carry nonclassical hair in the sense described in the present paper. Indeed, it is obvious that the $\mathbb{R}^2 \times S^{D-2}$-topologies admit a huge number of nontrivial complex and real vector bundles whose classification is evidently reduced to that for the $n$-sphere $S^n$, $n = D - 2$. As to the latter, standard results of algebraic topology (see, e. g., Ref.\[\text{4}\]) say that the $G$-vector bundles over $S^n$ for any Lie group $G$ are classified by $\pi_{n-1}(G)$, the $(n-1)$-th homotopic group of $G$. On the other hand, by virtue of the famous Bott periodicity \[\text{[21]}\], we have $\pi_{2k-1}(U(m)) = \mathbb{Z}$, $m \geq 1$, $k = 1, 2, \ldots$ and, as a consequence, over each even-dimensional sphere there exist a huge number of nontrivial complex vector bundles of any rank $m > 1$. As for the odd-dimensional spheres then, according to the Bott periodicity, $\pi_{2k}(U(m)) = 0$, i. e., no nontrivial complex vector bundles exist and, besides, in accordance
with the Bott periodicity for the orthogonal group $O(m)$, $\pi_{2k}(O(m)) = 0$ as well, i.e., there are not nontrivial real vector bundles either. As a result, we have the hair of type described in the present paper for black holes of Refs. [20] with topologies $\mathbb{R}^2 \times S^{2k}$, $k = 1, 2, \ldots$

In conclusion, it should be noted that some part of the above has been told by the author within the framework of scientific meeting [23].

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Appendix

We give here for inquiry some information about functions $P_{mn}^l$ mentioned in Sec. 2. The explicit form of the functions $P_{mn}^l$ can be chosen by miscellaneous ways, for instance, as follows (see, e.g., Ref. [22])

$$P_{mn}^l = (-1)^{-m-l} \sqrt{(l-m)!(l-n)!/(l+m)!(l+n)!} \cot^{m+n} \frac{\vartheta}{2} \times$$

$$\times \sum_{k=\max(m,n)}^l \frac{(l+k)l!2^k}{(l-k)!(k-m)!(k-n)!} \sin^{2k} \frac{\vartheta}{2}, \quad (A.1)$$

which holds true for integer and half-integer $l, n$, $|n| \leq l$. There is an orthogonality relation at $n$ fixed

$$\int_0^\pi P_{mn}^l(\cos \vartheta) P_{mn'}^{l'}(\cos \vartheta) \sin \vartheta d\vartheta = \frac{2}{2l+1} \delta_{ll'} \delta_{nn'} \quad (A.2)$$

$P_{mn}^l$ obeys the differential equation

$$\frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} \right] P_{mn}^l(x) = -\frac{m^2 + n^2 - 2mnx}{1-x^2} P_{mn}^l(x) = -l(l+1) P_{mn}^l(x) \quad (A.3)$$

with $x = \cos \vartheta$.

As was mentioned in Sec. 2, in physical literature devoted to the Dirac monopoles (see, e.g., Refs. [3]), the combinations $e^{im\varphi} P_{mn}^l(\cos \vartheta) = Y_{nm}(\vartheta, \varphi)$ are called the monopole (spherical) harmonics that coincide with the ordinary
ones at $n = 0$, i.e., $Y_{0lm}(\vartheta, \varphi) = Y_{lm}(\vartheta, \varphi)$, that is, the relations (A.1)–(A.3) pass on to the standard relations for ordinary spherical harmonics [22]. It should be noted, however, that in mathematical literature the monopole harmonics have been investigated in more depth and independently of physicists (see Ref. [22] and references quoted therein). From point of view of the global differential geometry, at $n \in \mathbb{Z}$ the monopole harmonics are the sections (written in local form on the bundle chart of coordinates $\vartheta, \varphi$ covering almost the whole $S^2$) of the complex line bundle with the Chern number $n$ over $S^2$, i.e., the ordinary ones are the sections of trivial line bundle. If $n$ is not integer then, as far as is known to the author, some geometric interpretation for monopole harmonics is absent.

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