Abstract. A contact distribution on $\mathbb{P}^3$ is defined by the 1-form $
abla := x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4$, up to a change of projective coordinates. The family of contact distributions is parameterized by the complement of the Pfaff-Plücker quadric in the projective 5-space of antisymmetric $4 \times 4$ matrices. A foliation of dimension 1 and degree $d$ is specified by a polynomial vector field $\varphi := \sum p_i \partial_{x_i}$, $p_i$ homogeneous of degree $d$. The foliation is called Legendrian if tangent to some distribution of contact. Our goal is to give formulas for the dimensions and degrees of the varieties of Legendrian foliations, and of the varieties of foliations tangent to a pencil of planes.

To Alberto Collino, in memoriam.

1. Introduction

A holomorphic contact structure on a complex manifold $X$ is a codimension one maximally non-integrable holomorphic sub-bundle $\mathcal{F} \subset TX$. The pair $(X, \mathcal{F})$ is called a contact complex manifold. Such structures have appeared in a link between Riemannian and Algebraic Geometries. For instance, Salamon [29] and LeBrun [21] proved that Kähler-Einstein Fano contact manifolds are twistor spaces of positive quaternion-Kähler manifolds. See also Beauville [2]. Demailly [10] and Kebekus et al. [19] have shown that projective contact manifolds are either Fano with $b_2 = 1$ or a projectivized tangent bundle of a projective manifold. We refer the reader to Druel [11, 12] and Ye [35] for further classification results in low dimensions and toric manifolds. In [18] Kebekus shows that if $(X, \mathcal{F})$ is a Fano contact manifold with $b_2 = 1$ which is not a projective space, then $X$ is covered by lines tangent to the contact distribution $\mathcal{F}$. Such lines are examples of Legendrian
curves, \( f : C \to (X, \mathcal{F}) \) with \( f_*T\mathcal{C} \subset \mathcal{F}|_C \). Enumerative aspects of Legendrian curves of contact in \( \mathbb{P}^{2n+1} \) and their moduli spaces have been studied in [1, 3, 16, 20, 23, 25].

We investigate Legendrian foliations on \( \mathbb{P}^3 \). These are foliations of dimension one (cf. 4.5.1) whose leaves are tangent to a contact structure (4.6). They appear in the classification of foliations of low degree for which the singular schemes are of pure dimension one, cf. [8].

We present a description of the spaces of Legendrian foliations enabling us to determine a formula for their degrees, in the spirit of [13, 22, 27, 33].

A distribution of degree \( m \) and codimension one on \( \mathbb{P}^3 \) is defined by a 1-form \( \sum a_i dx_i \) where the \( a_i \) are homogeneous polynomials of degree \( m + 1 \) such that \( \sum a_i x_i = 0 \). We assume henceforth \( m = 0 \). As syzygies-trained minds will recognize, this entails the vector of coefficients \( (a_1, \ldots, a_4) \) is a linear combination \( \sum a_{ij} \kappa_{ij} \) of the six basic Koszul relations \( \kappa_{12} := (x_2, -x_1, 0, 0), \ldots, \kappa_{34} := (0, 0, x_4, -x_3) \). Each distribution of degree 0 is specified by a point in the projectivization, \( \mathbb{P}^5 \), of the space of anti-symmetric \( 4 \times 4 \) matrices, cf. §4.2. Matrices of maximal rank correspond to the distributions of contact. These distributions form the Zariski open subset of indecomposable 1-forms, complement of the Pfaff-Plücker quadric \( \mathbb{G} \subset \mathbb{P}^5 \). The anti-symmetric matrices of rank 2 i.e., elements in \( \mathbb{G} \), correspond to pencils of planes; this has lead us to study also foliations tangent to a pencil of planes, cf. 5.0.1.

Grosso modo, a foliation of dimension one is a (polynomial) recipe to draw a direction at each point. Precisely, for each integer \( d \geq 0 \) we call

\[
\text{Fol}_d := \mathbb{P}(H^0(\mathbb{P}^3, T\mathbb{P}^3(d - 1))),
\]

the space of foliations of dimension 1 and degree \( d \). Each element \( \varphi \in \text{Fol}_d \) is defined by a polynomial vector field \( \varphi := \sum p_i \partial x_i \), with \( p_i \) homogeneous of degree \( d \): think of the line joining \( P, \varphi(P) \) cf. 4.5.1. It corresponds to a map of vector bundles, \( \mathcal{O}_{\mathbb{P}^3}(1 - d) \to T\mathbb{P}^3 \). This defines a subspace of dimension one of the tangent space \( T_P\mathbb{P}^3 \) whenever \( \varphi(P) \neq 0 \), i.e., \( P \) is not a singular point of the foliation \( \varphi \).

A foliation is called Legendrian whenever tangent to some (variable) distribution of contact, say \( \omega \). This means the line specified by the foliation \( \varphi \) at a general point is contained in the plane assigned by \( \omega \), that is, \( \omega \cdot \varphi = 0 \). For fixed \( \omega \), the condition is linear on \( \varphi \), thus defining a subspace \( \omega^\perp_d := \{ \varphi \in \text{Fol}_d | \omega \cdot \varphi = 0 \} \). The main technical difficulty stems from the fact that the dimension of \( \omega^\perp_d \) jumps as the 1-form of contact \( \omega \in \mathbb{P}^5 \setminus \mathbb{G} \) specializes to a decomposable 1-form \( \omega_0 \in \mathbb{G} \) (i.e.,
a pencil of planes, see §4.4). The analogous question for tangency to other families of distributions seems out of reach for our toolbox.

Our main results are formulas for the degrees of the subvarieties of Legendrian foliations (and friends: the subvarieties of foliations tangent to a pencil of planes), see (12), (21). It turns out that the answers are given by polynomial functions on the degree of the foliation. This fits a trend of the last few decades in Enumerative Geometry. We think of it as a sort of Schubert Calculus programme for exploring the geometry of parameter spaces of foliations by curves.

It is reminiscent of Alberto Collino’s interest on Enumerative Geometry of the Hilbert schemes of lines (resp. conics) contained in certain hypersurfaces, cf. [6, 7]. The 2nd author misses the enlightening company of Alberto pacing the corridors at MIT back in the early 70’s.

2. A couple of pictures for starters

Each line in \( \mathbb{P}^3 \) gives a recipe to draw a plane through a general point: just take their linear span.

A distribution of planes

Another distribution of planes

Their intersections yield a foliation of dimension 1. The distributions depicted above are pencils of planes; they are defined by integrable 1-forms such as \( \omega := udv - vdu \), with \( u, v \) homogeneous polynomials of degree 1. The axis \( u = v = 0 \) is the singular locus of this distribution.

2.1. Another recipe for choosing a plane through each point in 3-space. Given \((x_0, y_0, z_0) \in \mathbb{A}^3\), le-voilà:

\[
y - y_0 = z_0x - x_0z.
\]

We have a bijection,

point \((x_0, y_0, z_0) \leftrightarrow \text{plane } z_0x - y - x_0z + y_0\).
This bijection extends to the so-called null correlation on $\mathbb{P}^3$,
$$\mathbb{P}^3 \ni [x_0, y_0, z_0, w_0] \leftrightarrow z_0x - w_0y - x_0z + y_0w \in \mathbb{P}^3,$$
which is related to a nice, classical construction sketched below for the reader’s benefit.

### 2.2. Revisit twisted cubics.

Eleven out of ten geometers cherish the twisted cubic as the most beloved rational curve in space. It is at center stage in the above null correlation and contact distribution.

For each point $P$ there are three osculating planes drawn from $P$ to the cubic curve. Take the connecting plane $p$ of the osculating points: $P \leftrightarrow p = (O_1, O_2, O_3)$.

![Diagram of twisted cubic with osculating planes](image)

Conversely, to every plane corresponds the point $P$ of intersection of the osculating planes of the twisted cubic curve, drawn in the 3 points of intersection of the plane with the curve.

### 3. Distributions, 1-forms

Let $S_k$ denote the space of homogeneous polynomials of degree $k$. A holomorphic distribution of degree $m$ of planes in $\mathbb{P}^3$ is defined by a projective 1-form

\[
\omega := a_1 dx_1 + \cdots + a_4 dx_4, \quad a_i \in S_{m+1},
\]

such that $\sum a_i x_i = 0$. For $P \in \mathbb{P}^3$, the plane selected by $\omega$ is $a_1(P)x_1 + \cdots + a_4(P)x_4 = 0$, (assuming $P$ is not a singularity of $\omega$, i.e., not all $a_i(P) = 0$). The reader is kindly referred to [4, 5, 15] for generalities on the subject.
4. SYZYGIES

The condition $\sum a_i x_i = 0$ in (1) ensures the 1-form descends from $\mathbb{C}^4$ to a section of $H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(m + 2))$. It also tells us that the vector of coefficients, $a := (a_1, \ldots, a_4)$, belongs to the module of syzygies of $x_1, x_2, x_3, x_4$. We register for later use the following elementary

4.1. **Lemma.** The syzygies of the regular sequence $x_1, \ldots, x_4$ are the linear combinations $a := (a_1, \ldots, a_4) = \alpha_{12} \kappa_{12} + \cdots + \alpha_{34} \kappa_{34}$ of the basic six Koszul relations $\kappa_{12} := (x_2, -x_1, 0, 0)$, $\ldots$, $\kappa_{34} := (0, 0, x_4, -x_3)$.

4.2. For the case of 1-forms of degree $m = 0$, assumed henceforth, we have

$$\omega := \sum a_i dx_i, \, \deg a_i = 1, \, \sum a_i x_i = 0,$$

$$a := (a_1, \ldots, a_4) = \sum_{i<j} \alpha_{ij} \kappa_{ij}, \, \alpha_{ij} \in \mathbb{C}.$$ 

Hence $\omega$ can be rewritten as

$$\omega = \alpha_{12} (x_2 dx_1 - x_1 dx_2) + \cdots + \alpha_{34} (x_4 dx_3 - x_3 dx_4), \, \alpha_{ij} \in \mathbb{C}.$$ 

It corresponds to an anti-symmetric $4 \times 4$ matrix,

$$\omega \leftrightarrow \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ -\alpha_{12} & 0 & \alpha_{23} & \alpha_{24} \\ -\alpha_{13} & -\alpha_{23} & 0 & \alpha_{34} \\ -\alpha_{14} & -\alpha_{24} & -\alpha_{34} & 0 \end{pmatrix} \leftrightarrow \text{bivector in } \wedge \mathbb{S}_1, \, \text{e.g.,} \,$$

$$\kappa_{ij} \leftrightarrow x_j dx_i - x_i dx_j \leftrightarrow x_i \wedge x_j.$$ 

4.3. **Definition.** We say $\omega$ is a 1-form of contact if $\det(\alpha_{ij}) \neq 0$.

In the sequel we identify the 1-form (2) with either the $4 \times 4$ anti-symmetric matrix $(\alpha_{ij})$ or the bivector $\sum \alpha_{ij} x_i \wedge x_j$, as elements of the projective space $\mathbb{P}^5 = \mathbb{P}^2(\wedge \mathbb{S}_1)$.

4.4. **Normal forms.** Linear algebra tells us that the rank of a $4 \times 4$ nonzero anti-symmetric matrix is either 4 or 2. Accordingly there are 2 orbits in $\mathbb{P}^2(\wedge \mathbb{S}_1)$ under the induced action of $\text{GL}(\mathbb{S}_1)$:

- the open orbit, formed by the anti-symmetric $4 \times 4$ matrices of rank 4, corresponding to contact forms = indecomposable bivectors, and
- the closed orbit, $\mathcal{G} := \text{Pfaff-Pl"ucker quadric}$, corresponding to integrable 1-forms $\leftrightarrow$ decomposable bivectors.

\begin{align*}
\text{rank 2:} & \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
& \downarrow \quad x_2 dx_1 - x_1 dx_2 \quad \text{closed orbit} \\
\text{rank 4:} & \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\
& \downarrow \quad x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 \quad \text{open orbit}
\end{align*}
We recall these 1-forms define distributions of planes with nice geometric interpretations, cf. 2, 2.2.

We’ll focus on nested pairs \((\mathcal{D}, \phi)\) consisting of a foliation \(\phi\) of dimension one and a distribution \(\mathcal{D}\) of codimension one of fixed degrees such that \(\phi(P) \subset \mathcal{D}(P)\) \(\forall P \in \mathbb{P}^3\) off singularities.

4.5. **The space of foliations.** A foliation of dimension one and degree \(d\) on \(\mathbb{P}^3\) is defined by a homogeneous polynomial vector field

\[
\phi := p_1 \partial x_1 + \cdots + p_4 \partial x_4, \quad p_i \in S_d
\]

up to scalar multiple: For each point \(P \in \mathbb{P}^3\), draw the line joining \(P, [p_1(P), \ldots, p_4(P)]\). The line is undefined at the points where the \(2 \times 4\) matrix

\[
\begin{pmatrix}
  x_1 & \cdots & x_4 \\
p_1(x) & \cdots & p_4(x)
\end{pmatrix}
\]

has rank < 2; these are the *singular points* of the foliation. The foliation remains the same if you add to the vector field \(\phi\) a multiple of the radial vector field, \(\partial_R := x_1 \partial x_1 + \cdots + x_4 \partial x_4\). The above matrix changes to

\[
\begin{pmatrix}
  x_1 & \cdots & x_4 \\
p_1(x)+x_1q & \cdots & p_4(x)+x_4q
\end{pmatrix};
\]

getsame Plücker coordinates (\(2 \times 2\) minors).

We often abuse notation and use the same symbol \(\phi\) for the vector field (assumed \(\neq 0\)) and the foliation of dimension one it defines.
4.5.1. **Definition.** The space of foliations of dimension one and degree $d$ on $\mathbb{P}^3$ is the projectivization

$$
\mathbb{P}^N = \mathbb{P} \left( \frac{S_d \otimes S^*_1}{S_{d-1} \cdot \partial_R} \right) =: \text{Fol}_d,
$$

where

$$
S_d = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)), \\
S^*_1 = \langle \partial_{x_1}, \ldots, \partial_{x_4} \rangle = \mathbb{C}^4,
$$

$$
N + 1 = 4 \left( \frac{d+3}{3} \right) - \left( \frac{d+2}{3} \right) = \frac{(d+4)(d+2)(d+1)}{2},
$$

$$(S_d \otimes S^*_1 \ni \varphi = p_1 \partial_{x_1} + \cdots + p_4 \partial_{x_4}, p_i \in S_d, \text{mod } \partial_R).$$

Taking cohomology in the twisted Euler’s sequence

$$
\mathcal{O}_{\mathbb{P}^3}(d-1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(d) \otimes S^*_1 \rightarrow T\mathbb{P}^3(d-1)
$$

we have the natural identification

$$
\tilde{\Phi}_d := H^0(\mathbb{P}^3, T\mathbb{P}^3(d-1)) = \frac{S_d \otimes S^*_1}{S_{d-1} \cdot \partial_R}.
$$

Our goal is to find the dimensions and degrees of certain subvarieties of the space of foliations $\text{Fol}_d = \mathbb{P}(\tilde{\Phi}_d)$ defined by imposing the condition of tangency to a varying distribution of planes in the following sense.

4.6. **Definition.** A foliation defined by a vector field $\varphi = \sum p_i \partial_{x_i}$ is *tangent* to a distribution $\omega = \sum a_i dx_i$ whenever

$$
\omega \cdot \varphi := \sum a_i p_i = 0.
$$

For fixed $\omega$, this defines a vector (resp. projective) subspace

$$
\tilde{\omega}_d^\perp := \{ \varphi \in \tilde{\Phi}_d | \omega \cdot \varphi = 0 \} \quad \text{ (resp. } \omega_d^\perp := \mathbb{P}(\tilde{\omega}_d^\perp) \subset \text{Fol}_d).
$$

The game now is to move the 1-form $\omega$ in a suitable family $\mathbb{D}$ of distributions, i.e., pick some closed, irreducible subvariety

$$
\mathbb{D} \subseteq \mathbb{P}(H^0(\mathbb{P}^3, \Omega^1(m+2)))
$$

As customary, we look at the correspondence

$$
\omega_d^\perp \subset \tilde{\mathbb{D}}_d := \{ (\omega, \varphi) \in \mathbb{D} \times \text{Fol}_d | \omega \cdot \varphi = 0 \}
$$

$$
\omega \in \mathbb{D} \quad \xrightarrow{pr_1} \tilde{\mathbb{D}}_d \quad \xrightarrow{pr_2} \mathbb{D}_d \subset \text{Fol}_d.
$$

The union

$$
\mathbb{D}_d := pr_2(\tilde{\mathbb{D}}_d) = \bigcup_{\omega \in \mathbb{D}} \omega_d^\perp \subset \text{Fol}_d.
is a first approximation to our object of desire. Attention must be payed to the variation of \( \dim \omega^\perp_d \) as \( \omega \) runs in \( D \). It may prevent irreducibility of \( \hat{D}_d \). This is the situation envisaged below, setting \( m = 0 \) and taking \( D = \mathbb{P}^5 \). Now the fiber dimensions of \( \text{pr}_1 : \hat{D}_d \longrightarrow D \) are given by

\[
\dim \omega^\perp_d = \begin{cases} 
2\left( \frac{d+3}{3} \right) - 1 & \text{for } \omega \in \mathbb{G} \ (\text{cf. (13)}), \\
(d + 4)(d + 2)d/3 - 1 & \text{for } \omega \in \mathbb{P}^5 \setminus \mathbb{G} \ (\text{cf. 7.3}).
\end{cases}
\]

Clearly \( \hat{D}_d \) is closed in \( D \times \text{Fol}_d \), defined by a bunch of bilinear equations to be shown in a moment. The fiber of \( \text{pr}_1 \) over \( \omega \in D \) is \( \omega^\perp_d \), cf. (4).

\section{4.6.1. Equations for \( \hat{D}_d \).}

We have a natural diagram of maps of vector bundles over \( D \times \mathbb{P}^3 \), (abusing notation suppressing pullbacks)

\[
\begin{array}{ccc}
\mathcal{O}_D(-1) & \longrightarrow & H^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}^1(m + 2)) \times \mathbb{P}^3 \\
\downarrow & & \downarrow \\
\Omega_{\mathbb{P}^3}^1(m + 2) & \longrightarrow & \text{Hom}(T\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m + 2)).
\end{array}
\]

The slant arrow \( \mathcal{O}_D(-1) \rightarrow \text{Hom}(T\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m + 2)) \) yields the universal (twisted) differential form

\[
(5) \quad T\mathbb{P}^3 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(m + 2) \otimes \mathcal{O}_D(1)
\]

over \( D \times \mathbb{P}^3 \). On the other hand, pulling back via \( \mathbb{P}^3 \times \text{Fol}_d \rightarrow \text{Fol}_d \), we have

\[
\begin{array}{ccc}
\mathcal{O}_{\text{Fol}_d}(-1) & \longrightarrow & H^0(\mathbb{P}^3, T\mathbb{P}^3(d - 1)) \times \mathbb{P}^3 \\
\downarrow & & \downarrow \\
T\mathbb{P}^3(d - 1) & = & \text{Hom}(\mathcal{O}_{\mathbb{P}^3}(1 - d), T\mathbb{P}^3)
\end{array}
\]

whence the universal vector field of degree \( d \),

\[
\mathcal{O}_{\text{Fol}_d}(-1) \otimes \mathcal{O}_{\mathbb{P}^3}(1 - d) \longrightarrow T\mathbb{P}^3 \times \text{Fol}_d
\]

over \( \mathbb{P}^3 \times \text{Fol}_d \). Composing with the universal differential form (5) we get a section

\[
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(1 - d) \otimes \mathcal{O}_{\text{Fol}_d}(-1) & \xrightarrow{\sigma} & \mathcal{O}_D(1) \otimes \mathcal{O}_{\mathbb{P}^3}(m + 2) \\
\downarrow & & \downarrow \\
T\mathbb{P}^3
\end{array}
\]
over $\mathbb{D} \times \mathbb{P}^3 \times \text{Fol}_d$. Twisting by $\mathcal{O}_\mathbb{D}(-1) \otimes \mathcal{O}_{\mathbb{P}^3}(d-1)$ we find a diagram

\begin{equation}
(\sum a_jdx_j) \otimes (\sum p_i\partial x_i) \xrightarrow{\sigma'} \sum a_ip_i
\end{equation}

\begin{equation}
(\sigma')^* \mathcal{O}_{\mathbb{P}^3}(m + d + 1)
\end{equation}

\begin{equation}
\mathcal{O}_\mathbb{D}(-1) \otimes \mathcal{O}_{\text{Fol}_d}(1) \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^3}(m + d + 1)
\end{equation}

\begin{equation}
\mathcal{O}_\mathbb{D}(-1) \otimes T\mathbb{P}^3(d-1)
\end{equation}

Taking direct image under $pr_3 : \mathbb{D} \times \text{Fol}_d \times \mathbb{P}^3 \rightarrow \mathbb{D} \times \text{Fol}_d$, we get

\begin{equation}
\mathcal{O}_\mathbb{D}(-1) \otimes \mathcal{O}_{\text{Fol}_d}(1) \xrightarrow{\sigma} (pr_3)^* (\mathcal{O}_{\mathbb{P}^3}(m + d + 1))
\end{equation}

\begin{equation}
\mathcal{O}_\mathbb{D}(-1) \otimes H^0(\mathbb{P}^3, T\mathbb{P}^3(d-1)) \ni \omega \otimes \varphi \quad (\omega = \sum a_idx_i, \varphi = \sum p_i\partial x_i).
\end{equation}

On the fiber over $(\omega = \sum a_idx_i, \varphi = \sum p_i\partial x_i) \in \mathbb{D} \times \text{Fol}_d$ we have $\sigma(\omega \otimes \varphi) = \sum a_ip_i \in S_{d+m+1}$. Set

\begin{equation}
\begin{cases}
J^\omega = \langle a_1, \ldots, a_4 \rangle, \text{ homogeneous ideal of singularities of } \omega,
\end{cases}
\end{equation}

\begin{equation}
(J^\omega)_k = S_k \cap J^\omega, \text{ subspace of forms of degree } k \text{ in } J^\omega.
\end{equation}

For fixed $\omega = \sum a_idx_i \in \mathbb{D}$, the vector fields tangent to $\omega$ form the vector subspace as in (4),

\begin{equation}
\tilde{\omega}^d = \ker \begin{array}{c}
H^0(\mathbb{P}^3, T\mathbb{P}^3(d-1)) \rightarrow (J^\omega)_{d+m+1}
\end{array} \xrightarrow{\omega} \sum p_i\partial x_i \rightarrow \sum a_ip_i
\end{equation}

As said before, we specialize to distributions of degree $m = 0$. That’s for now the sole situation we know how to control the ranks of $(J^\omega)_d$. Presently,

$H^0(\mathbb{P}^3, \Omega^1(2)) = \mathbb{P}(\wedge^2 S_1)$, $\mathbb{D} \subseteq \mathbb{P}(\wedge^2 S_1) = \mathbb{P}^5 = \{\text{distributions of degree } 0\}$.

Recall there are just two orbits (cf. 4.4):

4.6.2. **closed orbit.** $^c\omega := x_2dx_1 - x_1dx_2 \in \mathbb{G}$, pencil of planes. We have $^c\omega \cdot (H^0(\mathbb{P}^3, T\mathbb{P}^3(d-1))) = (J^{^c\omega})_{d+1} = \langle x_1, x_2 \rangle_{d+1}$.

4.6.3. **open orbit.** $^o\omega := x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4$, 1-form of contact (4.3). Now $^o\omega \cdot (H^0(\mathbb{P}^3, T\mathbb{P}^3(d-1))) = S_{d+1}$. We look at each at a time, starting with $\mathbb{D} = \mathbb{G}$. 

5. Foliations tangent to a pencil of planes

The distributions pictured in §2 are given by a 1-form like \( \omega := udv - vdu, u, v \in S_1 \). Now \( \omega \) lies in the grassmannian \( G \)-closed orbit in \( \mathbb{P}^5 \).

The rank of the evaluation map \( \omega \cdot \) defined in (7) is \( \left( \frac{d+4}{3} \right) - (d + 2) \), independent of \( \omega \in G \):

\[
(J^\omega)_{d+1} = \langle u, v \rangle S_d = \ker \left( S_{d+1} \longrightarrow \text{Sym}_{d+1}(S_1/\langle u, v \rangle) \right).
\]

Recall the tautological sequence of vector bundles of rank 2 over \( G \),

\[
\begin{array}{cccc}
R & \longrightarrow & S_1 \times G & \longrightarrow \mathcal{Q} \\
\end{array}
\]

where the fibers \( R_{(u,v)} = \langle u, v \rangle \), \( Q_{(u,v)} = S_1/\langle u, v \rangle \). It induces the exact sequence

\[
\begin{array}{cccc}
\Pi_d & \longrightarrow & \wedge^2 R \otimes H^0(\mathbb{P}^3, T\mathbb{P}^3(d - 1)) & \longrightarrow \mathcal{P}_d \\
\end{array}
\]

where

\[
\mathcal{P}_d := \ker \left( G \times S_{d+1} \longrightarrow \text{Sym}_{d+1} \mathcal{Q} \right).
\]

Tensoring the top row of (9) by \( \wedge^2 R^* = \wedge^2 Q \) we get the exact sequence

\[
\begin{array}{cccc}
\Pi'_d := \Pi_d \otimes \wedge^2 \mathcal{Q} & \longrightarrow & H^0(\mathbb{P}^3, T\mathbb{P}^3(d - 1)) & \longrightarrow \mathcal{P}_d \otimes \wedge^2 \mathcal{Q}.
\end{array}
\]

5.0.1. Definition. The image \( \Pi_d \subset \text{Fol}_d \) of the projective subbundle \( \mathbb{P}(\Pi'_d) \subset G \times \text{Fol}_d \) is the variety of foliations of degree \( d \) tangent to some pencil of planes.

\[
\begin{array}{ccc}
\mathbb{P}(\Pi'_d) & \subset & G \times \text{Fol}_d \\
\Pi'_d & \subset & \text{Fol}_d.
\end{array}
\]

5.1. Proposition. The map \( p: \mathbb{P}(\Pi'_d) \longrightarrow \Pi_d \subset \text{Fol}_d \) in the diagram (11) is generically bijective for \( d > 1 \).

Proof. Let \( \varphi \) be a general point in \( \Pi_d \). We must show \( \varphi \) is tangent to one and only one distribution \( \omega \in G \). Say \( \varphi = \sum p_i \partial_{x_i}, \omega = x_2 dx_1 - x_1 dx_2 \). Now \( 0 = \omega \cdot \varphi = x_2 p_1 - x_1 p_2 \) implies \( p_1 = x_1 q, p_2 = x_2 q \) for some \( q \in S_{d-1} \), and no condition is imposed on \( p_3, p_4 \). If \( \varphi \) were tangent to another \( \eta \in G \), we may assume \( \eta = x_4 dx_3 - x_3 dx_4 \) (resp.
\( \eta' = x_4dx_2 - x_2dx_4 \). This forces \( p_3, p_4 \) (resp. \( p_2, p_4 \)) share a common nonconstant factor (as \( d > 1 \)). We have used the fact that the natural action of \( \text{GL}(S_1) \) on \( \mathbb{G} \times \mathbb{G} \) has only the orbits of \( (\omega, \omega), (\omega, \eta), (\omega, \eta') \) corresponding to the diagonal, the open orbit and the “incidence”. \( \square \)

6. A polynomial formula

6.1. **Corollary.** For all \( d > 1 \), the degree of the variety \( \Pi_d \) of foliations tangent to a pencil of planes is given by the top dimensional Segre class \( s_4\Pi'_d \).

**Proof.** The assertion comes pretty much from Fulton’s construction of Segre & Chern classes, [14, p. 47] taking (5.1) into account. Setting \( h := \) hyperplane class in \( \text{Fol}_d \), we have from (11)

\[
\deg \Pi_d = \deg (\Pi_d \cap h^{\dim \Pi_d}) = \deg p^*h^{\dim \Pi_d} = \deg q_*h^{\dim \text{F}(\Pi_d)} = s_4\Pi'_d. 
\]

\( \square \)

6.2. **Explicit calculation.** The exact sequences (10), (9) yield

\[
s_4\Pi'_d = c_4(\mathcal{P}_d \otimes \wedge^2 \mathcal{Q}).
\]

Using maple, we may invoke [17] Katz & Strømme’s Schubert:

```maple
with(schubert): grass(2,4,q,all);
w2q:=wedge(2,Qq); # Plücker line bdle \( O_G(1) \)
Pd:=Symm(d+1,4)-Symm(d+1,Qq);
Pd:=Pd*w2q;
chern(4,Pd);
integral(%);
factor(%);
athus:=unapply(%,d);
```

We find the formula for the degree of the variety of foliations of dimension 1 and degree \( d \), tangent to a (varying) pencil of planes in \( \mathbb{P}^3 \):

\[
(12) \quad \deg \Pi_d = 5 \binom{d+4}{5} \binom{d+3}{3} (d^2 + 2d + 3)(d^2 + 6d + 11)/108.
\]

Note the degree (=12) in \( d \) is thrice the dimension of the family of distributions at hand. This is expected by the argument in § 8 below.

We also register the calculation of the dimension,

\[
(13) \quad \left[ \begin{array}{l}
\dim \Pi_d = \dim \mathbb{G} + \text{rank}\Pi_d - 1 = \\
(\text{known from (9)}) \quad 3 + h^0(T\mathbb{P}^3(d - 1)) - \text{rank}\mathcal{P}_d = \\
3 + (d + 3)(d + 2)(d + 1)/3 = 3 + 2\binom{d+3}{3}.
\end{array} \right.
\]
7. Legendrian vector fields

These are polynomial vector fields in $\mathbb{P}^3$ (more generally, $\mathbb{P}^{2n+1}$) which are tangent to a distribution of contact (4.6.3).

7.1. Example. The vector field $x_1^2 \partial_{x_1} + x_2x_1 \partial_{x_2} + x_3x_4 \partial_{x_3} + x_4^2 \partial_{x_4}$ is tangent to $\omega := x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4$.

7.2. Universal anti-symmetric map. We’ll need the universal anti-symmetric map $\alpha$ as defined at the bottom of the diagram below:

$$
\begin{array}{ccc}
\left( S^*_1 \wedge S_1 \right)_{\mathbb{P}^5} & \longrightarrow & S_1_{\mathbb{P}^5} \\
\bigg\uparrow & & \bigg\uparrow \\
S^*_1 \otimes \mathcal{O}_{\mathbb{P}^5}(-1) & \longrightarrow & S_1 \otimes \mathcal{O}_{\mathbb{P}^5}
\end{array}
$$

(14)

$$
\partial_{x_i} \otimes \sum u_j \wedge v_j \longrightarrow \sum (\partial_{x_i} u_j) \cdot v_j - (\partial_{x_i} v_j) \cdot u_j
$$

The rank of $\alpha$ drops from 4 to 2 over $G = Pfaff–Plücker quadric in \mathbb{P}^5 = \mathbb{P}^{\wedge 2} \mathbb{P}^1).$ Recall $G$ is the locus of decomposable 1-forms (bivectors) $\omega = u dv - v du (\leftrightarrow u \wedge v)$.

Tensoring (14) with $S_d = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))$, we get the diagram

$$
\begin{array}{ccc}
S_d \otimes S^*_1 \otimes \mathcal{O}_{\mathbb{P}^5}(-1) & \longrightarrow & S_d \otimes S_1_{\mathbb{P}^5} \\
\bigg\uparrow & & \bigg\uparrow \\
S_{d-1} \cdot \partial_R \otimes \mathcal{O}_{\mathbb{P}^5}(-1) & \longrightarrow & S_{d+1}_{\mathbb{P}^5}
\end{array}
$$

(15)

The map $\tau_d$ passes to the quotient and induces

$$
\begin{array}{ccc}
\left( \frac{S_d \otimes S^*_1}{S_{d-1} \cdot \partial_R} \right) \otimes \mathcal{O}_{\mathbb{P}^5}(-1) & \longrightarrow & \frac{S_{d+1}_{\mathbb{P}^5}}{}
\end{array}
$$

(16)

$$
\varphi \text{ mod } \partial_R \otimes \omega \longmapsto a_1 p_1 + \cdots + a_4 p_4
$$

with notation as in display (3).

7.2.1. Remarks. We register the map $\overline{\tau}_d$ in display (16) arises via direct image of

$$
T \mathbb{P}^3(d - 1) \otimes \mathcal{O}_{\mathbb{P}^5}(-1) \xrightarrow{\overline{\tau}_d} \mathcal{O}_{\mathbb{P}^5}(d + 1) \text{ by } pr_1 : \mathbb{P}^5 \times \mathbb{P}^3 \rightarrow \mathbb{P}^5
$$
as in (6). A crucial observation is the fact that the image sheaf of $\overline{\tau}_d$ is the direct image of the image sheaf of $\overline{\tau}_d$, $d >> 0$: kill the appropriate
The map $\tau_d$ is surjective off the Pfaffian $\subseteq \mathbb{P}(\wedge S_1)$ because so is $\tau_d$.

Let

$$\begin{align*}
\mathcal{L}_d := \ker (\tau_d \otimes \mathcal{O}_{\mathbb{P}^5}(1)) & \subset H^0(\mathbb{P}^3, T\mathbb{P}^3(d-1)); \\
\mathcal{M}_d := \text{image} (\tau_d \otimes \mathcal{O}_{\mathbb{P}^5}(1)) & \subset S_{d+1} \otimes \mathcal{O}_{\mathbb{P}^5}(1).
\end{align*}$$

(17)

$\mathcal{L}_d$ and $\mathcal{M}_d$ are vector bundles over the open subset $\mathbb{P}^5 \setminus G$.

### 7.3. Lemma

$L_d, M_d$ both extend as vector bundles over $\mathbb{P}^5$ and fit into the exact sequence

$$\xymatrix{ \mathcal{L}_d \ar[r] & H^0(\mathbb{P}^3, T\mathbb{P}^3(d-1)) \ar[r] & \mathcal{M}_d \ar[r] & S_{d+1} \otimes \mathcal{O}_{\mathbb{P}^5}(1). }$$

Moreover,

(i) the inclusion $\mathcal{M}_d \subset S_{d+1} \otimes \mathcal{O}_{\mathbb{P}^5}(1)$ is an equality over $\mathbb{P}^5 \setminus G$

(ii) we have $\text{rank } \mathcal{L}_d = (d+4)(d+2)d/3$.

**Proof.** This is because the rank of $\tau_d$ drops precisely along a hypersurface. Indeed, looking at the diagram (15), notice the rightmost vertical map is surjective, whereas the top horizontal one drops rank precisely along the Pfaffian. Now use the following (inspired by Raynaud-Gruson’s [26, (5.4.3)])

### 7.3.1. Claim

Let $\theta : \mathcal{E}_1 \to \mathcal{E}_2$ be a map of locally free sheaves over a variety $X$. Let $r$ denote the generic rank of $\theta$. Let $\mathcal{J}$ be the image sheaf of $\wedge \mathcal{E}_1 \otimes \wedge \mathcal{E}_2^\vee \to \mathcal{O}_X$ induced by $\theta$. Let $\mathcal{G}_{\mathcal{E}_1}^r$ be the Grassmann bundle of rank $r$ quotients of $\mathcal{E}_1$. There is a rational section $X \dashrightarrow \mathcal{G}_{\mathcal{E}_1}^r$ induced by $\theta$. It extends to a morphism if $\mathcal{J}$ is invertible. In this case, the image sheaf of $\theta$ is locally free.

### 7.3.2. Proof of Claim

Let $X' \to X$ be the blowup of $\mathcal{J}$. Arguing with the Plücker embedding $\mathcal{G}_{\mathcal{E}_1}^r \subset \mathbb{P}(\wedge \mathcal{E}_1)$, we see that the composition $X' \to X \dashrightarrow \mathcal{G}_{\mathcal{E}_1}^r$ is a morphism. Since $X' \to X$ is an isomorphism for $\mathcal{J}$ invertible, we are done.

Over the open orbit, $\mathcal{L}_d$ is the set of pairs $(\omega, \varphi)$ such that the vector field $\varphi$ is Legendrian w.r.t. the form of contact $\omega$.

The image $\mathcal{M}_d$ is a locally free subsheaf of $S_{d+1} \otimes \mathcal{O}_{\mathbb{P}^5}(1)$ (of same rank), though not locally split.
7.3.3. **Definition.** Notation as in (3), the image $L_d \subset \text{Fol}_d$ of the projective subbundle $P(L_d) \hookrightarrow \mathbb{P}^5 \times \text{Fol}_d$ is the variety of Legendrian foliations.

7.4. **Proposition.** The map $p : \mathbb{P}(L_d) \rightarrow L_d \subset \text{Fol}_d$ is generically bijective for $d > 1$.

*Proof.* Let $\varphi$ be a general point in $L_d$. We must show $\varphi$ is tangent to one and only one distribution $\omega \in \mathbb{P}^5 \setminus G$. We have from [8, Theorem 7.2] that $\varphi$ corresponds to a decomposable 2-form $\alpha \wedge \omega$, where $\alpha$ is a polynomial 1-form of degree $d$. If $\varphi$ were tangent to another $\eta \in \mathbb{P}^5 \setminus G$, then $\alpha \wedge \omega \wedge \eta = 0$. By [8, Theorem 5.1] the singular points of $\omega \wedge \eta$ consist of two skew lines, in particular has codimension 2. So Saito’s division Lemma [28] applies and we get that $\alpha = f \omega + g \eta$, where $f, g \in \mathcal{S}_{d-1}$. Therefore, $\alpha \wedge \omega = g \cdot (\eta \wedge \omega)$. We arrive at a contradiction, since $\deg(g) = d - 1 > 0$ and $\alpha \wedge \omega$ has no zeros of codimension one. \qed

7.4.1. **Remark.** Alan Muniz has communicated to the authors that the decomposability result [8, Theorem 7.2] also follows from Saito’s division Lemma [28]. In fact, let $\theta$ be a homogeneous polynomial 2-form on $\mathbb{C}^4$ which induces a Legendrian foliation on $\mathbb{P}^3$. Consider the linear contact form $\omega$ such that $\theta \wedge \omega = 0$. Since $\{\omega = 0\} = \{0\} \subset \mathbb{C}^4$, then by Saito’s division Lemma there is a polynomial 1-form $\alpha$ such that $\theta = \alpha \wedge \omega$.

7.4.2. **Corollary.** The degree of $L_d$ ($d > 1$) is given by the Segre class $s_5 L_d = c_5 M_d$.

*Proof.* Same as for tangency to pencils of planes, cf. 6.1. \qed

7.5. **Calculation.** We proceed to the actual calculation. Recall the exact sequence (7.3). Since the middle term is trivial, it implies

$$\text{Segre} L_d = (\text{Chern} L_d)^{-1} = \text{Chern} M_d.$$

The degree of the latter class can be found using Bott’s formula in the equivariant context as we learn from [24].

As kindly enticed by the referee, a practical summary of the main ingredients is included in the sequel for the reader’s convenience. We are given a smooth, projective variety $X$ endowed with an action of the
torus $T = \mathbb{G}_m$. Let $E \xrightarrow{\pi} X$ be an equivariant vector bundle. So there is an action $\mathbb{G}_m \times E \xrightarrow{\psi} E$ yielding a natural commutative diagram

$$
\begin{array}{c}
\mathbb{G}_m \times E \\
\downarrow
\end{array}
\begin{array}{c}
E
\end{array}
\xrightarrow{\psi}
\begin{array}{c}
\mathbb{G}_m \times X
\end{array}
\xrightarrow{\pi}
\begin{array}{c}
X.
\end{array}
$$

If $P \in X$ is a point fixed by the action, it follows that $\mathbb{G}_m$ acts on the fiber $E_P$. Any such action splits $E_P = \bigoplus E^\chi_P$, a direct sum of eigenspaces with character $\chi$. This means $\mathbb{G}_m \times E_P \ni (t,v) \mapsto t \cdot v = \sum \chi(t)v_\chi$.

Each $\chi(t) = t^w_\chi$ for some integers $w_\chi$, called weights. Bott’s formula expresses integrals of polynomials in the Chern classes of equivariant vector bundles in terms of the corresponding classes in equivariant cohomology. With the simplifying assumption that the set of points in $X$ left fixed by the action is finite, the equivariant cohomology classes are just symmetric functions on the weights. Next we make this explicit for $c_5\mathcal{M}_2$.

This requires the description of the fibers of $\mathcal{M}_d$ over the 6 fixed points of $\mathbb{P}(\wedge^2 S^1)$ under an action of $\mathbb{C}^*$ induced from a natural action on $S^1$: $x_i \mapsto t^{w_i}x_i$, where the weights $w_i \in \mathbb{N}$ will be chosen appropriately.

The fixed points are the “canonical” anti-symmetric matrices of rank 2, namely

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Let us work out the fiber of $\mathcal{M}_d$ say over the last matrix in the above list. That matrix corresponds to the 1-form

$$
(18) \quad \omega_0 := x_4dx_3 - x_3dx_4.
$$

We perturb $\omega_0$ to full rank, moving away from the Pfaffian. Define

$$
(19) \quad \omega_t := x_4dx_3 - x_3dx_4 + t(x_1dx_2 - x_2dx_1).
$$

The fiber of $\mathcal{M}_d$ over $\omega_0$ is found by evaluating $\omega_t$ on vector fields of degree $d$ and taking $\lim_{t \to 0}$. In the sequel, set for simplicity, $d = 2$. The calculations reported below were performed using SINGULAR [31]; script available in [34].

We construct a basis for $H^0(\mathbb{P}^3, T\mathbb{P}^3(2-1))$ consisting of eigenvectors for the given $\mathbb{C}^*$-action. Consider the subspace $S^\circ_2 := \{ \varphi := \Sigma p_i \partial_{x_i} \in S_2 \otimes S_1^* \mid p_i \in S_2, \Sigma \partial_{x_i} p_i = 0 \}$ consisting of vector fields with zero divergence.

We have $(S_1 \cdot \partial_R) \oplus S^\circ_2 = S_2 \otimes S_1^*$, whence $H^0(\mathbb{P}^3, T\mathbb{P}^3(1)) \simeq S^\circ_2$. 


We want a basis for the space

\[ S_2^* = \ker \left( \begin{array}{cc}
S_2 \otimes S_1^* & \text{tr(jac)} \\
\Sigma \gamma_i \otimes \partial x_i & \rightarrow \Sigma \partial x_i \gamma_i
\end{array} \right) . \]

It’s formed, when \( d = 2 \), by 40 - 4 vectors like (cf. [34, L245])

\[
\begin{align*}
[x_2^2, 0, 0, 0], & \ldots \quad [-x_1 x_3, 0, 0, x_3 x_4], & \ldots \\
\| & \ldots \quad \| & \ldots \\
x_2^2 \partial x_1, & \ldots \quad -x_1 x_3 \partial x_1 + x_3 x_4 \partial x_4, & \ldots \\
2w_2 - w_1, & \ldots \quad w_1 + w_3 - w_1 = w_3 + w_4 - w_4, & \ldots \\
\end{align*}
\]

(20)

The bottom row in the above display lists the weights of the elements in the basis. Compute next the matrix of the map \( S \circ \) the fiber of \( M \) a column with a pivot. Then the fiber of \( \omega \) is a direct summand with weight \( w \)

\[
\begin{align*}
\omega_t[x_2^2, 0, 0, 0] &= -t x_2^3, \ldots , \\
\ldots \\
\omega_t[-x_1 x_3, 0, 0, x_3 x_4] &= -x_3^2 x_4 - t x_2 x_1 x_3, \ldots .
\end{align*}
\]

The sought for fiber of the image \( \mathcal{M} := \mathcal{M}_2 \) is computed via saturation:

- perform elementary row operations on the matrix of \( \omega_t \);
- divide each row by its gcd;
- last but not least, set \( t = 0 \).

We get a matrix with the correct rank 20. The columns of pivots form a basis for the fiber of \( \mathcal{M}_2 \) at the fixed point \( \omega_0 \) cf.(18).

Say for instance the basic vector \([x_2^2, 0, 0, 0] = x_2^2 \partial x_1\) corresponds to a column with a pivot. Then the fiber of \( \mathcal{M}_2 \) acquires an eigenspace which is a direct summand with weight \( 2w_2 - w_1 \). That’s all we care to collect and eventually obtain numbers feeding into Bott’s formula.

For \( \omega_t \) as in (19), letting \( t \to 0 \), we find the fiber of \( \mathcal{M}_2 \) is the direct sum of 20 (=rank) eigenspaces with corresponding weights

\[
2w_1 - w_2, w_1, w_2, -w_1 + 2w_2, -w_3 + 2w_4, w_3, 2w_3 - w_4, \\
w_2 - w_3 + w_4, w_2, w_2 + w_3 - w_4, 2w_2 - w_3, 2w_2 - w_4, w_1 - w_3 + w_4, \\
w_1, w_1 + w_3 - w_4, w_1 + w_2 - w_3, w_1 + w_2 - w_4, 2w_1 - w_3, 2w_1 - w_4
\]

Plugging in numerical values for the \( w_i \), e.g., 0,2,7,10, we find the list of 20 weights

\[-2, 0, 2, 4, 13, 10, 7, 4, 5, 2, -1, -3, -6, 3, 0, -3, -5, -8, -7, -10.\]

The equivariant (top=5=DIM) Chern class at the chosen fixed point \( \omega_0 \) is the value of the 5th elementary symmetric function of those weights, i.e., the coefficient of \( t^5 \) in the product \((1 - 2t) \cdot (1 + 0t) \cdot (1 + 2t) \cdots (1 - 7t)(1 - 10t)\), to wit,

\[105534.\]
Repeating the calculation for each of the 6 fixed points, Bott’s formula reads

\[
\deg \text{Chern} M_2 = \sum_P c_5^{\text{equiv}} M_P / c_5^{\text{equiv}} T_P \mathbb{P}^5 = \frac{833800359}{42000} - \frac{38740434}{1500} + \frac{4199874}{336} + \frac{7716777}{336} - \frac{3398841}{1500} - \frac{105534}{42000} = 2224.
\]

This is the value of the new **Athus polynomial** \((21)\) for \(d = 2\).

8. Another polynomial formula

The calculation for higher degrees are better left for a script, cf. appendix of arXiv version \([34]\) of this article. We find by interpolation,

\[
\begin{aligned}
(d^4 + 2d^3 + 9d^2 + 14d + 24) \\
\left( d^8 + 34d^7 + 475d^6 + 3430d^5 + 13480d^4 + \\
29872d^3 + 45444d^2 + 44856d + 29808 \right) / (2^5 3^5 5^3).
\end{aligned}
\]

As we shall argue, the formula for the degree is a polynomial in \(d\) of degree \(\leq 15\), so it suffices to compute for the 16 values \(d \in \{2, 3, \ldots, 17\}\). Polynomials are deduced from the fact that \(M_d\) is a direct image in \(\mathbb{P}^5\) of a sheaf on \(\mathbb{P}^5 \times \mathbb{P}^3\) (cf. 7.2.1) whose Chern character is a polynomial in \(d\) of degree \(\leq 3\). Using Grothendieck-Riemann-Roch \([14, (5)\ p. 283]\) it follows that the Chern character of \(M_d\) is a polynomial in \(d\) of degree \(\leq 3\). Hence the top Chern class \(c_5 M_d\) is a polynomial with degree \(\leq 15\) as asserted.

9. Final remarks

The case of Legendrian vector fields on \(\mathbb{P}^5\) can be treated similarly, except for the need to perform an initial blowup of \(\mathbb{P}^{14}\), the space of anti-symmetric \(6 \times 6\) matrices, along the grassmannian of lines in \(\mathbb{P}^5\). Ditto for higher odd dimensional. However, the computational load as of now seems unfeasible: It took several many days to work out the very first few values.

\[
\begin{align*}
2 & : 310143368560 \\
3 & : 241876654493880936 \\
4 & : 11354802732747615971781 \\
5 & : 96189569307604075178197866 \\
6 & : 249521135387730096977922116592 \\
7 & : 269205473509858802653259153925591 \\
8 & : 146758500496340866823126747040755640
\end{align*}
\]

Code available as an appendix to the arXiv version of this note \([34]\).
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