Size of quantum networks

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The metric structure of bosonic scale-free networks and fermionic Cayley-tree networks is analyzed focusing on the directed distance of nodes from the origin. The topology of the networks strongly depends on the dynamical parameter \( T \), called temperature. At \( T = \infty \) we show analytically that the two networks have a similar behavior: the distance of a generic node from the origin of the network scales as the logarithm of the number of nodes in the network. At \( T = 0 \) the two networks have an opposite behavior: the bosonic network remains very clustered (the distance from the origin remains constant as the network increases the number of nodes) while the fermionic network grows following a single branch of the tree and the distance from the origin goes as a power-law of the number of nodes in the network.

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Complex networks representing systems of interacting units have been studied and classified \([1,2]\) according to their different geometrical and topological properties. In between complex networks scale-free networks, with power-law connectivity distribution, have been found to describe different systems of nature and society \([3–5]\). Recently several models \([3–6]\) have been formulated that generate such structures, the prototype of them being the BA model \([7]\). Scale-free networks are particularly interesting because their highly inhomogeneous structure induces peculiar effects in the dynamical models that can be defined on them, as the absence of percolation \([8]\) and epidemic threshold \([9]\), the infinite Curie temperature for the paramagnetic to ferromagnetic transition in the frame of the Ising model \([10–13]\) and the good associative memory of the Hopfield model defined on network with large average connectivity \([14]\). Besides, the investigation of the metric structure \([15–19]\) of these networks is of great interest. It was first empirically found \([15]\) and then analytically derived \([16,17]\) that scale-free networks are characterized by having a mean distance \( < d > \) between nodes scaling like the logarithm of the system size \( N, < d > \sim \log(N) \).

The similarities between the structure of a BA network and a traditional Cayley tree have been recently studied. It has been found that they share high similarities. In fact they are both generated by the subsequent study of the Ising model \([10–13]\) and the good associative memory of the Hopfield model defined on network with large average connectivity \([14]\). Besides, the investigation of the metric structure \([15–19]\) of these networks is of great interest. It was first empirically found \([15]\) and then analytically derived \([16,17]\) that scale-free networks are characterized by having a mean distance \( < d > \) between nodes scaling like the logarithm of the system size \( N, < d > \sim \log(N) \).

The similarities between the structure of a BA network and a traditional Cayley tree have been recently studied. It has been found that they share high similarities. In fact they are both generated by the subsequent addition of a same elementary unit (a node connected to \( m \) links) attached to the rest of the network in the direct or in the reversed direction \([20]\). The symmetry between these two types of networks is evident if we assume that each node \( i \) has an innate quality or ‘energy’ \( \epsilon_i \) and that the dynamics is parametrized by a variable \( T \) called temperature that introduce a “thermal noise” as it has been done in self-organized models \([21–23]\). It is possible, then to observe that the BA model becomes a limiting case of a scale-free network described by a Bose distribution of the energies to which the incoming links point \([24]\), while the growing Cayley tree network is described by a Fermi distribution of the energies at the interface \([25]\). Quantum networks (the bosonic scale-free network and the fermionic Cayley tree network) evolve around a well defined core of initial nodes.

In this paper we focus our attention on the metric structure of quantum networks and in particular on their size, i.e. we estimated the distance measured over directed paths of a generic node \( i \) from the origin of the network and its dependence on the time \( t_i \) in which the node \( i \) has been added to the network. We derive the expression for the average value of the distance \( < \ell(t_i) > \) of node \( i \) introduced at time \( t_i \) from the origin, measured on directed paths, in quantum networks at \( T = \infty \). We find in agreement with \([15–17]\) that \( < \ell(t_i) > \sim \log(t_i) \) in the bosonic scale-free network and we show for \( T = \infty \) a similar behavior in fermionic networks.

At different values of \( T \) the topology of the network change drastically. For energy distribution functions \( p(\epsilon) \rightarrow 0 \) for \( \epsilon \rightarrow 0 \) the exponent \( \gamma \) of the power-law connectivity distribution \( P(k) \sim k^{-\gamma} \) goes from \( \gamma = 3 \) at \( T = \infty \) to \( \gamma = 2 \) for \( T = 0 \) and there is a phase transition at a critical temperature \( T_c \) below which a finite fraction of all the links is connected to a single node. Assuming that the bosonic network is a reasonable model for growing scale-free networks, we will have, for example, that the citation network \([26]\) with \( \gamma = 3 \) would correspond to a \( T = \infty \) dynamics while the incoming component of the World-Wide-Web with \( \gamma_m = 2.1 \) \([7,27]\) would correspond to a low temperature dynamics. As \( T \) decreases also the behavior of \( < \ell(t_i) > \) changes and the distance of node \( i \) from the origin depends less strongly on the time \( t_i \) of its arrival. At sufficiently low temperature \( \ell(t_i) \) remains constant as a function of \( t_i \). On the contrary in the fermionic network, at \( T = 0 \), when the dynamics becomes extremal, the network evolves far away from the origin and the distance of a node \( i \) from the origin of the network grows as a power-law of the time \( t_i \) of its arrival in the network.

A. Distances from the origin in the bosonic network

The bosonic network is a generalization of the well known BA network \([7]\). In this model, a new node with \( m \) links is added to the network at each time step. Each node \( i \) has an innate quality or ‘energy’ \( \epsilon_i \) extracted
from a probability distribution \( p(\epsilon) \). The way the new links are attached follows a generalized preferential attachment rule: the probability \( \Pi_j \) that an existing node \( j \) acquires a new link depends both on its connectivity \( k_j(t) \) and its energy \( \epsilon_j \), i.e.

\[
\Pi_j = \frac{e^{-\beta \epsilon_j}k_j(t)}{\sum_i e^{-\beta \epsilon_i}k_i(t)}
\]

(1)

with the parameter \( \beta = 1/T \) tuning the relevance of the energy \( \epsilon_j \) with respect to the connectivity \( k_j(t) \). This network displays a power law connectivity distribution \( P(k) \sim k^{-\gamma} \) with \( \gamma \in [2,3] \) depending on the \( p(\epsilon) \) distribution and the inverse temperature \( \beta \) [24]. In the \( T = \infty \) limit \( (\beta = 0) \) the network reduces to a scale-free BA network [7] with an average connectivity \( k_i \) of node \( i \) that grows in time as a power-law with exponent 1/2

\[
k_i(t) = m \sqrt{\frac{t}{t_i}}
\]

(2)

where \( t_i \) is the time in which node \( i \) was added to the network. The probability \( p_{i,j} \) that two nodes \( i \) and \( j \) are connected by a link can be calculated from Eq. (1). At \( T = \infty \) \( (\beta = 0) \), taking into account that at each time \( m \) new links are added to the network, the probability \( p_{i,j} \) is given by \( m \) times Eq. (1). After substituting (2) into Eq. (1) we obtain,

\[
p_{i,j} = \frac{m}{2} \frac{1}{\sqrt{t_i t_j}}
\]

(3)

The number of directed paths of length \( \ell \) connecting node \( i \), introduced in the network at time \( t_i \), to node \( i_0 \) belonging to the original core of the network \( (t_{i_0} = 1) \) is given by the mean value of the number of paths connecting node \( i \) to \( i_0 \) and passing through the points \( i_0, i_1, i_2, \ldots i_{\ell-1}, i_\ell = i \) with \( t_{n+1} > t_n \). Indicating each node with the time of its arrival in the network, the probability of any directed path is given by the product \( \Pi'_{n=1} \Pi_{\ell-1,n} \) with \( t_{n-1} < t_n \). In order to find \( n_\ell(t_i) \) we should sum over all possible paths. Replacing the sum over the nodes with the integrals over \( t_n \), we obtain for \( n_\ell(t_i) \),

\[
n_\ell(t_i) = \int_{t_1}^{t_i} dt_1 \int_{t_1}^{t_i} dt_2 \ldots \int_{t_{\ell-2}}^{t_i} dt_{\ell-1} p_{0,1} p_{1,2} \ldots p_{\ell-1,\ell}.
\]

(4)

Using \( t_0 = 1 \) and Eq. (3) valid in the \( T = \infty \) limit, we obtain

\[
n_\ell(t_i) = \left( \frac{m}{2} \right)^\ell \int_{1}^{t_i} dt_1 \int_{t_1}^{t_i} dt_2 \ldots \int_{t_{\ell-2}}^{t_i} dt_{\ell-1} \frac{1}{t_1 t_2 \ldots t_{\ell-1} \sqrt{t_i}}
\]

\[
= \frac{1}{(\ell - 1)!} \left( \frac{m}{2} \log(t_i) \right)^{\ell-1} \frac{m}{2 \sqrt{t_i}}
\]

(5)

This means that the mean distance between node \( i \) and a node \( i_0 \), such that \( t_{i_0} = 1 \), calculated only on directed paths, follows a Poisson distribution with average size

\[
< \ell(t_i) > = \frac{m}{2} \log(t_i).
\]

(6)

The distribution of the number of directed paths of length \( \ell \) starting from the origin of the network is proportional to the integral of \( n_\ell(t_i) \) over \( t_i \)

\[
P(\ell) \propto \int_{1}^{t} n_\ell(t') dt' = (-m)^\ell \left( 1 - \frac{\Gamma(\ell, \log(t)/2)}{\Gamma(\ell)} \right).
\]

(7)

In Fig. 2 we show the agreement between the numerical results and Eq. (7) for a bosonic network of \( 10^4 \) nodes at \( T = \infty \) and \( m = 1, 2 \).
The topology of the bosonic network changes as a function of the temperature $T = 1/\beta$. In particular for a distribution $p(\epsilon)$ such that $p(\epsilon) \to 0$ as $\epsilon \to 0$ we know [24] that there is a critical temperature $T_c$ below which the network has a topological transition and its structure is dominated by a single node that grabs a finite fraction of all the links. For $T > T_c$ ($\beta < \beta_c$) the network is in the so-called “fit-gets-rich” phase (FGR phase) while for $T < T_c$ ($\beta > \beta_c$) the network is in the so-called Bose-Einstein condensate phase (BE phase). To visualize this transition we have plotted the bosonic network with an energy distribution

$$p(\epsilon) = \frac{1}{\theta + 1} \epsilon^\theta, \ \epsilon \in (0, 1) \quad (8)$$

where $\theta = 0.5$, $m = 1$, above (Fig. 3a) and below (Fig. 3b) the phase transition.

\begin{figure}[h]
\centering
(a) \hspace{2cm} (b)
\caption{Graphic representation of the bosonic network. Graph (a) represents a network with $m = 1$, energy distribution given by Eq. (8) where $\theta = 0.5$ at $T = \infty$ ($\beta = 0$ -the BA network) and graph (b) represents a network with the same parameters $m$ and $\theta$ but with $T = 1/\beta = 0.33$ (network in the Bose-Einstein condensate phase). The number of nodes both networks is $N = 10^3$.}
\end{figure}

The network has been designed in order to underline the hierarchical structure of the network. Starting from the single node at the origin of the tree, we have placed all the nodes that are directly attached to it on a semicircle of unitary radius, each node $i$ separated from the next one by the angle $\Delta \alpha_i$ proportional to its connectivity, i.e.

$$\Delta \alpha_i = \frac{k_i}{\sum_{j \in N(i)} k_j} \quad (9)$$

where $N(i)$ are the nearest neighbors of node $i$ added at a time $t_j > t_i$. We have repeated the same construction for all the nodes of the network in such a way that all the nearest neighbors of node $i$ are on a semicircle of radius $r_i$ with

$$r_i = r_k \frac{k_i}{\sum_{j \in N(i)} k_j} \quad (10)$$

where $k$ is the node to which the node $i$ has been attached at time $t_i$. From Fig. 3 it is clear the change in the topology of the network at the critical temperature with the emergence of a single node that grabs a finite fraction of all the links in the Bose-Einstein condensate phase.

As the topology of the network changes, the behavior of $\ell(i)$ as a function of $t_i$ changes too. In fact we have

$$< \ell(i) > = a(\beta) \log(t_i) \quad (11)$$

where the coefficient $a(\beta)$ is a decreasing function of the inverse temperature $\beta = 1/T$.

\begin{figure}[h]
\centering
\caption{Distances from the origin in a bosonic network with $m = 2$ and $\theta = 0.5$ as a function of $\beta$.}
\end{figure}

In Fig. 4 we report $< \ell(i) >$ for a bosonic network with $p(\epsilon)$ of the type (8) with $\theta = 0.5$ and $m = 2$ at different values of the inverse temperature $\beta$, above and below the critical value $\beta_c = 1.7$ [24]. In order to illustrate the change in the topology of the network above and below the critical temperature in Fig. 5 we report the behavior of different relevant structural quantities for a network of size $N = 10^4$, $\theta = 0.5$ and with $m = 2$ around the critical inverse temperature $\beta_c = 1.7$. We report the fraction of links attached to the most connected node, the exponent of the power-law component of the connectivity distribution, the clustering coefficient and the coefficient $a(\beta)$. The fraction of links attached
to the most connected network $k_{\text{max}}(\beta)/N$ is the order parameter of the FGR-BE phase transition and increases as a function of $\beta$. The data reported in Fig. 5 are mediated over 100 runs. The connectivity distribution of the bosonic network contains a power-law component plus a peak indicating the condensation phenomena that appears for $\beta > \beta_c$. In Fig. 5 we report the exponent $\gamma(\beta)$ of the power-law component of the connectivity distribution that decreases as a function of $\beta$ with an asymptotic value of $\gamma = 2$. The data reported on Fig. 5 are mediated over 100 runs. The clustering coefficient $C(\beta)$ increases at the transition point while the coefficient $a(\beta)$ slowly decreases saturating toward a zero value for $\beta > 3.0$. The data reported in Fig. 5 for these last two quantities data are mediated over 10 run.

B. Distances from the origin in the fermionic network

The fermionic network [25] is a growing Cayley tree, where the innate qualities of the nodes (energies) define their different branching tendency. Starting at time $t = 1$ from a node $i_0$ at the origin of the network, the node $i_0$ at time $t = 2$ grows and $m$ new nodes are directly connected to it. Each node $i$ has an energy $\epsilon_i$ extracted from a given $p(\epsilon)$ distribution. At each timestep a new node with connectivity one (at the interface) is chosen to branch, giving rise to $m$ new nodes.

We assume that nodes with higher energy are more likely to grow than lower energy ones. In particular we take $\Pi_i$, the probability that a node $i$ of the interface (with energy $\epsilon_i$) grows at time $t$, to be

$$\Pi_i = \frac{e^{\beta \epsilon_i}}{\sum_{j \in \text{Int}(t)} e^{\beta \epsilon_j}}. \quad (12)$$

where the sum in the denominator is extended to all nodes $j$ at the interface $\text{Int}(t)$ at time $t$. The model depends on the inverse temperature $\beta = 1/T$. In the $\beta \to 0$ limit, high and low energy nodes are equally probable to grow and the model reduces to the Eden model. In the $\beta \to \infty$ limit the dynamics becomes extremal and only the nodes with the highest energy value are allowed to grow. In this case the model reduces to invasion percolation [28,29] on a Cayley tree.

Let us assume that $\beta = 0$ ($T = \infty$). The probability $\Pi_i$ that a node $i$ of the interface $\text{Int}(t)$ grows at time $t$ is given by

$$\Pi_i = \frac{1}{N_{\text{Int}}(t)}, \quad (13)$$

where $N_{\text{Int}}(t)$ is the total number of active nodes. Since at each timestep a node of the interface branches, and $m$ new nodes are generated, after $t$ timesteps the model generates an interface of $N_{\text{Int}}(t)$ nodes, with

$$N_{\text{Int}}(t) = (m - 1)t + 1. \quad (14)$$

Let us denote by $\rho(t, t_i)$ the probability that a node, born at time $t_i$, is still at the interface at time $t$. Since every node $i$ of the network grows with probability $\Pi_i$, Eq. (13) if it is at the interface, in mean field $\rho(t, t_i)$ follows

$$\frac{\partial \rho(t, t_i)}{\partial t} = - \frac{\rho(t, t_i)}{N_{\text{Int}}(t)}. \quad (15)$$

Replacing (14) in (15) in the limit $t \to \infty$ we get the solution

$$\rho(t, t_i) = \left(\frac{t_i}{t}\right)^{1/(m-1)}. \quad (16)$$

Consequently each node $i$ that arrives at the interface at time $t_i$, remains at the interface with a probability that decreases in time as a power-law. The probability $p_{i,j}$ that a node $i$ is attached to a node $j$ (arrived in the network at a later time $t_j > t_i$) is given by the r. h. s. of Eq. (13) calculated at time $t_j$. Taking into account Eq.(16) and the rate $m$ of the addition of new nodes, we obtain for $p_{i,j}$

$$p_{i,j} = \frac{m}{(m-1)t_j} \left(\frac{t_i}{t_j}\right)^{1/(m-1)}. \quad (17)$$

The number $n_{\ell}(t_i)$ of paths of length $\ell$ that connect a node $i$, introduced at time $t_i$, to the origin $i_0$ is given by the average number of paths connecting a node $i$ to a node $i_0$ and passing through the points $i_0, i_1, i_2, \ldots, i_{\ell-1}, i_{\ell} = i$ with $t_{\ell+1} > t_\ell$. Indicating each node with the time of its arrival in the network and the sum over the nodes with the integrals over $t_n$, we obtain for $n_{\ell}(t_i)$,

FIG. 5. Relevant structural quantities in a bosonic network of size $N = 10^4$ with $m = 2$ and $\theta = 0.5$ as a function of the inverse temperature $\beta$. 
\[ n_\ell(t_i) = \int_{t_i}^{t_{i+1}} dt_1 \int_{t_i}^{t_{i+1}} dt_2 \cdots \int_{t_{i-\ell+1}}^{t_{i-\ell+2}} dt_{\ell-1} p_{01} p_{12} \cdots p_{\ell-1,\ell} \]

and, using (17) we obtain, with a calculation analog to (5),

\[ n_\ell(t_i) = \frac{1}{(\ell - 1)!} \left( \frac{m}{m - 1} \log(t_i) \right)^{\ell - 1} \left( \frac{1}{t_i} \right)^{1/(m - 1)}. \]  

(19)

This means that the mean distance between a node \( t_i \) and the origin follows a Poisson distribution with average size

\[ \langle \ell(t_i) \rangle = \frac{m}{m - 1} \log(t_i). \]  

(20)

As in the bosonic network, in the fermionic network at infinite temperature \( (\beta = 0) \) the distance \( \langle \ell(t_i) \rangle \) of node \( i \) from the origin grows logarithmically with the time \( t_i \).

In Fig. 6 we report the analytical simulation of a fermionic network with \( p(\epsilon) \) uniform between zero and one, at \( T = \infty \) and \( m = 2, 3, 4 \).

As the temperature decreases, the topology of the network changes drastically, in Fig. 7 we show the Cayley tree with \( p(\epsilon) = 1, \epsilon \in (0, 1) \) and \( m = 2 \) at infinite temperature \( (\beta = 0.0) \) and at low temperature \( (\beta = 20.0) \).

At high temperature the network grows homogeneously in each direction while at low temperature it evolves following only a single branch of the tree.

The distance of a node \( i \) from the origin of the networks grows logarithmically with \( t_i \) at \( T = \infty \) \((\beta = 0)\). As the temperature decreases the behavior of \( \langle \ell(t_i) \rangle \) gets steeper. For \( p(\epsilon) \) uniformly distributed between zero and one, in the extremal case \( T = 0 \) \((\beta = \infty)\) when the node of highest energy grows deterministically at each time step, we have a dramatic change in the behavior and \( \langle \ell(i) \rangle \) grows as a power-law of \( t_i \).

\[ \langle \ell(i) \rangle \propto (t_i)^\zeta \]  

(21)

\( \zeta = 0.55 \pm 0.05 \) from the numerical results reported in Fig. 8.

In conclusion we have shown that bosonic and fermionic network are not only symmetrically built \([20]\) but also at \( T = \infty \) they are characterized by a distance \( \langle \ell(t_i) \rangle \) from the origin that grows like the logarithm of the time \( t_i \). On the contrary, in the limit \( T = 0 \) they behave in an opposite way: the bosonic network stays highly clustered with a distance from the origin that remains constant as the network evolves, in the fermionic network the distance \( \langle \ell(t_i) \rangle \) grows like a power-law of the time \( t_i \).
FIG. 8. Distance from the origin in a femionic network with $m = 2$ and $p(e)$ uniformly distributed between zero and one at different temperatures. At $\beta = 0$ ($T = \infty$) we have the predicted logarithmic behavior Eq. (20) while in the extremal case $\beta = \infty$ ($T = 0$) the network grows as a power-law of the network size. The solid line in the Inset is the power-law fit Eq. (21) with $\zeta = 0.55 \pm 0.05$.

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[1] L. A. N. Amaral, A. Scala, M. Barthélémy and H. E. Stanley PNAS 97, 11149 (2000).
[2] K.-I. Goh, E. OH, H. Jeong, B. Kahng and D. Kim, PNAS99,12583 (2002).
[3] R. Albert and A.-L. Barabási, Rev. Mod. Phys. 74, 47 (2002).
[4] S. N. Dorogovtsev and J. F. F. Mendes, Adv. in Phys. 51, 1079 (2002).
[5] S. Strogatz, Nature 410, 268 (2001).
[6] G. Caldarelli, A. Capocci, P. De Los Rios and M. A. Munoz, Phys. Rev. Lett. 89,258702 (2002).
[7] A.-L. Barabási and R. Albert, Science 286, 509 (1999).
[8] R. Cohen, K. Erez, D. ben-Avraham and S. Havlin, Phys. Rev. Lett. 85, 4626 (2000).
[9] R. Pastor-Satorras and A. Vespignani, Phys. Rev. Lett. 86, 3200 (2001).
[10] A. Aleksiejuk, J. A. Holyst and D. Stauffer, Physica A 310, 260 (2002).
[11] S. N. Dorogovtsev, A. V. Goltsev and J. F. F. Mendes, Phys. Rev. E 66, 016104 (2002).
[12] M. Leone, A. Vazquez, A. Vespignani and R. Zecchina, Eur. Phys. J. B 28, 191 (2002).
[13] G. Bianconi, Phys. Lett. A 303, 166 (2002).
[14] D. Stauffer, A. Aharony, L. da Fontoura and J. Adler, cond-mat/0212601 (2003).
[15] R. Albert, H. Jeong and A.-L. Barabási, Nature 401, 130 (1999).
[16] M. E. J. Newman, S. H. Strogatz and D. J. Watts, Phys. Rev. E 64, 026118 (2001).
[17] S. N. Dorogovtsev, J. F. F. Mendes and A. N. Samukhin, cond-mat/0210085 (2002).
[18] S. Wuchty and P. Stadler, http://www.santafe.edu/sfi/publications/Working-Papers/02-09-052.ps.gz (2002).
[19] R. Cohen and S. Havlin, Phys. Rev. Lett. 90, 058701 (2003).
[20] G. Bianconi, Phys. Rev. E 66,056123 (2002).
[21] M. Vergeles, Phys. Rev. Lett. 75, 1969 (1995).
[22] G. Caldarelli, A. Maritan and M. Vendr uscolo, Europhys. Lett. 35, 481 (1996).
[23] M. Vergeles, Phys. Rev. E 55, 6265 (1997).
[24] G. Bianconi and A.-L. Barabási, Phys. Rev. Lett. 86, 5632 (2001).
[25] G. Bianconi, Phys. Rev. E 66, 036116 (2002).
[26] S. Redner, Eur. Phys. J. B 4,131 (1998).
[27] A. Broder et al., Comput. Netw.33,309 (2000).
[28] D. Wilkinson and J. F. Wiesenfeld, J. Phys. A 16, 3365 (1983).
[29] N. Vanderwalle and M. Ausloos, Europhys. Lett. 37, 1 (1997).