Robust estimation of parameters in logistic regression via solving the Cramer-von Mises type $L_2$ optimization problem

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Abstract
This paper proposes a novel method to estimate parameters in a logistic regression model. After obtaining the estimators, their asymptotic properties are rigorously investigated.

Keywords: Contaminated distribution, Cramer-von Mises optimization, logistic function, maximum likelihood, robustness

1 Introduction
Since Cramér (1928) and von Mises (1928) proposed the Cramer-von Mises (CvM) criterion, it has been extensively used for the goodness-of-fit (GOF) test: see, e.g., Anderson and Darling (1952) and Anderson (1962). Since then, it has been called CvM test, instead of CvM criterion, which is circumstantial evidence of its popularity for the GOF test. The application of the CvM criterion was not limited to the GOF test: it was actively adopted to parametric and nonparametric estimation methods by statisticians. One of the well-known estimation methods in which the CvM criterion left its seminal mark in the statistical literature is the minimum distance (MD) estimation. In the literature of the MD estimation, the CvM type distance, ad litteram, means any measure of difference between two entities that resembles, and hence, is a variant of the original CvM criterion: for example, the original probability measure has been replaced by Lebesgue or degenerate integrating measure for the application purpose.

Wolfowitz (1953, 1957) published a series of research articles about the MD estimation, it attracted the attention of many researchers. Among them, Parr and Schucany (1980), Millar (1981, 1982, 1984), and Donoho and Liu (1988a, b) conducted exhaustive studies into the MD estimation wherein they riggled out the MD estimation with various distances including the CvM, Kolmogorov-Smirnov, Kuiper, Hellinger, Lévy, and Prohorov distances. After employing several robustness criteria proposed by Hampel (1968, 1974, 1986), they demonstrated the robustness of the MD estimation can be obtained from using the CvM type distance. But, all their work was applicable to the one sample model only. Extending the application of the MD estimation to the regression and autoregressive setups, Koul (1970, 1985a,b, 1986, 2002) rigorously investigated asymptotic properties of the MD estimators – obtained from the CvM type distance – of the parameters of regression and autoregressive models where the error and innovation in the models are independent. Further research on the extension of the MD estimation with the CvM type distance followed Koul (op.cit.) Kim (2020) studied the MD estimation of the parameters of the regression model with dependent errors. For the computational aspect, Dhar (1991, 1992) demonstrated the MD estimators under regression and autoregressive models exist under certain conditions. Kim (2018) proposed a novel algorithm through which a practitioner can obtain the MD estimator much faster than through other well-celebrated numerical methods.

To recapitulate the MD estimation briefly, it has been widely used, with its application starting from estimation of the location parameter of one sample model and culminating in estimation of the parameters of the regression models. However, all of the aforementioned applications are limited to continuous random variables; the MD estimation with the CvM type distance has been applied neither one sample of discrete random variables nor regression models of discrete response variables. Giving a second thought about why
it was not used at all for the discrete variable, one can easily find an answer. Consider, for example, a linear regression model

\[ Y_i = x_i' \beta_0 + \epsilon_i, \quad i = 1, 2, ..., n, \]  

(1.1)

where \( x_i, \beta \in \mathbb{R}^p \) and \( Y_i, \epsilon_i \in \mathbb{R} \). \( (x_i, Y_i) \) are observed while \( \epsilon_i \) are unobservable random variables. Assume that both \( x_i \) and \( Y_i \) are continuous variables. For the above model, the most popular and common CvM type \( L_2 \) distance is defined as

\[ T(\beta) := \int \left[ \sum_{i=1}^n c_i \left( I(Y_i - x_i' \beta_0 \leq s) - H(s) \right) \right]^2 dG(s), \]

where \( G \) is an integrating measure, \( H \) is a modeled distribution function of the error in (1.1), and \( c_i \in \mathbb{R}, 1 \leq i \leq n \) are real numbers. Subsequently, the MD estimator for \( \beta_0 \) in (1.1) is defined as the value that minimizes the above \( L_2 \) distance \( T \). The existence of the integrating measure \( G \) embedded in the distance function explains why the CvM type distance is intrinsically the least amenable of all distances to being combined with the MD estimation: it requires the distribution function \( H \) to be a continuous function that can be utterly impossible for discrete random variables.

Considering its all asymptotically desirable properties such as efficient and robustness, it is certainly regrettable that the MD estimation with the CvM type distance can not be applied to the parameter estimation of discrete distributions. Hence, its applicability to discrete distributions merits further investigation. Motivated by its desirable properties, Kim (2024) applied the MD methodology to the estimation of the success probability in the binomial distribution after replacing the continuous measure \( G \) with a discrete measure. He demonstrated the asymptotic convergence of the resulting MD estimator to a normal distribution, and, more importantly, that it is more robust than the maximum likelihood estimator in the presence of outliers. He also compared it with other well-celebrated robust estimators including E-estimator and divergence estimator based on the entropy: see, e.g., Ruckstuhl and Welsh (2001), Jones et al. (2001), and Kawashima and Fujisawa (2017) for more details. Kim (2024) demonstrated that the MD estimator compares favorably with other estimators. Then, as done in the literature of the MD estimation between 70s and 80s, we extend the domain of applicability of the MD estimation from the discrete one sample binomial distribution to its regression setup, viz. a logistic regression model.

The rest of this article is organized as follows. Section 2 introduces different version of the MD estimation method tailored to the current logistic regression setup, propose the MD estimator of the regression parameters, and present the asymptotic properties of the proposed estimators, which is the main result of this study. Then, Section 2 is followed by Section 3 where leaders can find empirical studies. The proposed estimator will be compared with the most famous estimators for logistic regression parameters: the generalized linear model (GLM) estimators. Then, Section 4 briefly recapitulates and conclude this article.

2 Minimum distance estimation

2.1 Logistic regression

This section provides a brief summary of logistic regression before we buckle down to the MD estimation method. Logistic regression is a statistical model that tries to explain a relationship between a categorical response variable \( Y_k \) and a (either categorical or continuous) predictor \( x_k \) where the probability that \( y_k \) is happening can be modelled as

\[ P(Y_k = y) = p(x_k; \beta) = \frac{1}{1 + e^{-x_k' \beta}}, \quad y = 0, 1. \]  

(2.1)

The name “logistic” comes from the fact the model employs the logistic function \( 1/(1+e^{-s}), \quad s \in \mathbb{R} \). Note that logistic regression builds a model the probability of the response variable, not itself, depends the predictor, which is the fundamental difference from the usual linear regression models. For \( n \) pairs of observations \( (y_1, x_1), ..., (y_n, x_k), \) the likelihood for a logistic regression is defined as

\[ L(\beta) = \prod_{k=1}^n p(x_k; \beta)^{y_k} (1 - p(x_k; \beta))^{1-y_k}, \]
or, equivalently, the log-likelihood can be defined as

\[ l(\beta) = \sum_{k=1}^{n} [y_k \log p(x_k; \beta) + (1 - y_k) \log(1 - p(x_k; \beta))]. \]

Consequently, the maximum likelihood (ML) estimator is defined as

\[ \hat{\beta}_{ML} = \arg \max_{\beta \in \mathbb{R}^{p}} l(\beta), \]

where the above optimization problem is solved through a numerical solution since the closed-form solution doesn’t exist.

### 2.2 Uniformly locally asymptotically quadratic

Now we consider the general case, that is, \( y \) is not binary but can take more integer values, that is, \( y = 0, 1, ..., m \). Let \( F_k(y; \beta) = P(Y_k \leq y) \) with \( P(Y_k = y) = \binom{m}{y} p^y_k (1 - p_k)^{m-y} \), \( y = 1, 2, ..., m \), where \( p_k \) is defined as in (2.1), that is, \( Y_k \) follows Binomial distribution with the success probability \( p_k \) and the number of trials \( m \). Note that \( F_k(y; \beta) \) can be rewritten as

\[ F_k(y; \beta) = \binom{m}{y} \int_0^{1-p_k(\beta)} s^{m-y+1} (1 - s)^y \, ds, \]

(2.2)

where \( m_y = (m - y) \binom{m}{y} \), \( g_y(s) = s^{m-y+1} (1 - s)^y \), and \( p_k(\beta) \) is as in (2.1). The following are plain beyond misapprehension:

(i) \( \sum_{y=0}^{m} m_y^r < \infty \) for \( r = 1, 2, \) (ii) \( \sup_{\beta \in \mathbb{R}^p} |p_k(\beta)| \leq 1 \), and (iii) \( \max_{0 \leq y \leq m, s \in [0,1]} |g_y(s)| \leq 1 \). (2.3)

Next, define a Cramer-von Mises type \( L_2 \) distance function

\[ \mathcal{L}(\beta) = \sum_{j=1}^{p} \sum_{y=1}^{m} \sum_{k=1}^{n} d_{kj} \left( I(Y_k \leq y) - F_k(y; \beta) \right)^2 \]

(2.4)

where \( I(\cdot) \) is an indicator function, and \( d_{kj} \in \mathbb{R} \) can be any real value. Koul (2002) demonstrated the proper choice of \( d_{kj} \) gives a rise to efficient estimators of parameters in the linear model for various error distributions including normal, logistics, and double exponential distributions: see Koul (op.cit, Chapter 5.6) for more details. Subsequently, define the estimator

\[ \mathcal{L}(\tilde{\beta}) = \inf_{\beta \in \mathbb{R}^{p}} \mathcal{L}(\beta), \]

(2.5)

where \( A \) is a \( p \times p \) nonsingular matrix that results in

\[ \| A^{-1}(\tilde{\beta} - \beta_0) \| = O_p(1). \]

(2.6)

Note that optimization problems with variants of the CvM type of distance usually don’t have a closed-form solution, and hence, finding an estimator purports to be somewhat daunting at first glance. After one contrives to procure the estimator, finding its asymptotic properties is even more challenging. Koul (2002) introduced some conditions that are referred to as uniformly locally asymptotic quadratic conditions.
under which the distance function can be closely approximated by another quadratic distance function \( Q \), and hence, the extent of difficulty to solve the complex optimization problem in (2.4) can be pared back to that to solve the optimization problem of a quadratic function. These ULAQ conditions that enabled the applicability of the MD estimation to other problems and, as a result, enlarged the domain of the MD estimation are stated below.

(U. 1) There exist a sequence of random vector \( S_n(\beta_0) \) and a sequence of real matrix \( W_n(\beta_0) \) such that for all \( 0 < b < \infty \)

\[
\sup \left| \mathcal{L}(\beta) - \mathcal{L}(\beta_0) - (\beta - \beta_0)'S_n(\beta_0) - \frac{1}{2}(\beta - \beta_0)'W_n(\beta_0)(\beta - \beta_0) \right| = o_p(1),
\]

where the supremum is taken over \( \{ \beta \in \mathbb{R}^p : \|A^{-1}(\beta - \beta_0)\| \leq b \} \).

(U. 2) For all \( \varepsilon > 0 \), there exists a \( 0 < c_\varepsilon < \infty \) such that

\[
\mathbb{P} \left( \|\mathcal{L}(\beta_0)\| \leq c_\varepsilon \right) \geq 1 - \varepsilon.
\]

(U. 3) For all \( \varepsilon > 0 \) and \( 0 < c < \infty \), there exists a \( 0 < b < \infty \) and \( N_{b, \varepsilon} \) (depending on \( b \) and \( \varepsilon \)) such that

\[
\mathbb{P} \left( \inf \{ \mathcal{L}(\beta) \} > c \right) \geq 1 - \varepsilon,
\]

where the infimum is taken over \( \{ \beta \in \mathbb{R}^p : \|A^{-1}(\beta - \beta_0)\| > b \} \).

After ascertaining the ULAQ conditions are all met, one can obtain the asymptotic distribution of the MD estimator by using the following lemma that is originally proposed by Koul (2002) but is reproduced here: see Theorem 5.4.1 from Koul (op.cit).

**Lemma 2.1.** Assume \( \mathcal{L} \) satisfies the ULAQ assumptions (U. 1)- (U. 3). Let \( B_n := AW_n(\beta_0)A \) where \( A \) is as in (2.2). Let \( \beta \) denotes the MD estimator that solves the optimization problem in (2.4). Then,

\[
B_n^{-1}A^{-1}(\hat{\beta} - \beta_0) = AS_n(\beta_0) + o_p(1).
\]

**Remark 2.1.** Lemma [2.1] says finding the asymptotic distribution of the MD estimator \( \hat{\beta} \) amounts to; (i) specifying \( S_n \) and \( W_n \); and (ii) finding the asymptotic distribution of \( S_n \).

### 2.3 Asymptotic properties of the MD estimator \( \hat{\beta} \)

This section keeps finding the asymptotic distribution of \( \beta \) at the top of the agenda. To that end, we introduce more assumptions to achieve the goal. It should be noted that the following assumptions have a root in Koul (2002, pp. 173-175). This study requires less assumptions than Koul (2002) since some assumptions in Koul (op.cit) are a priori satisfied: see, e.g., Remark 2.3. Let \( c_k := A x_k \) where \( x_k \in \mathbb{R}^p \) is the \( k \)th row vector of the design matrix \( X \). Write a \( n \times p \) matrix \( D := (\langle d_k \rangle) \), \( 1 \leq k \leq n \), \( 1 \leq j \leq p \). Let \( d_k \in \mathbb{R}^p \), \( 1 \leq k \leq n \) denote the \( k \)th row vector of \( D \).

(a.1) \( (X'X)^{-1} \) exists and \( A = (X'X)^{-1/2} \).

(a.2) For all \( 1 \leq j \leq p \), \( \sum_{k=1}^{n} d_{k,j}^2 = 1 \).

(a.3) \( \max_{1 \leq k \leq n} \|c_k\| = o(1) \).

(a.4) For \( 1 \leq j \leq p \), \( \sum_{k=1}^{n} \|d_k c_k\| = O(1) \).

(a.5) For all \( 1 \leq k \leq n \) and for all \( e \in \mathbb{R}^p \), either \( d_k c_k e' A e \geq 0 \) or \( d_k c_k e' A e \leq 0 \) holds true.

Under (a.1)-(a.5), we shall prove the \( \mathcal{L} \) satisfy the ULAQ conditions that are introduced in the previous section and derive the asymptotic distribution of the MD estimators. Let \( W := (W_1, ..., W_p)' \in \mathbb{R}^p \) where

\[
W_j(y, \beta) = \sum_{k=1}^{n} d_{k,j} \left\{ I(Y_k \leq y) - F_k(y; \beta) \right\}.
\]
Observe that the distance function can be rewritten as
\[ \mathcal{L}(\beta) = \sum_{j=1}^{p} \sum_{y=0}^{m} [W_j(y, \beta)]^2 = \sum_{y=0}^{m} \|W(y, \beta)\|^2. \]

Recall \( p_k(\beta) \) in (2.1) and let \( q_k(\beta) := \partial p_k(\beta)/\partial \beta \in \mathbb{R}^p \). Subsequently, let
\[ S_n(\beta) := -2 \sum_{j=1}^{p} \sum_{y=0}^{m} W_j(y, \beta) d_{k_j} m_y g_y(p_k(\beta)) q_k(\beta), \]
\[ W_n(\beta) = \sum_{j=1}^{p} \sum_{y=0}^{m} \left\| \sum_{k=1}^{n} d_{k_j} m_y g_y(p_k(\beta)) q_k(\beta) \right\|^2, \]
and
\[ Q(\beta) = \mathcal{L}(\beta_0) + (\beta - \beta_0)' S_n(\beta_0) + \frac{1}{2} (\beta - \beta_0)' W_n(\beta_0)(\beta - \beta_0). \] (2.7)

Note that showing (U.1) is met amounts to showing that
\[ \sup_{\|A^{-1}(\beta - \beta_0)\| \leq \delta} |\mathcal{L}(\beta) - Q(\beta)| = o_p(1). \]

Let \( \lambda_k(y, \beta) := m_y g_y(p_k(\beta))(1 - p_k(\beta)) \). To conserve the space, let \( p_k^0 \) and \( p_k \) denote \( p_k(\beta_0) \) and \( p_k(\beta) \), respectively. Accordingly, any term with superscript 0 implies it has \( \beta_0 \) as an argument while one without the superscript, instead, contains a general \( \beta \). For example,
\[ \lambda_k^0(y) := \lambda_k(y, \beta_0), \quad = m_y g_y(p_k(\beta_0))(1 - p_k(\beta_0)), \quad = m_y g_y(p_k^0(1 - p_k^0)), \]
while \( \lambda_k(y) := \lambda_k(y, \beta) = m_y g_y(p_k(\beta))(1 - p_k(\beta)) \). Next, for \( y \in \mathbb{R} \) define a \( n \times n \) diagonal matrix \( \Lambda(y, \beta) := \text{diag}[\lambda_1(y), ..., \lambda_n(y)] \) and define \( R_j(y, \beta) := \sum_{k=1}^{n} d_{k_j} \lambda_k(y) x_k \). Let \( \Gamma_n(y, \beta) := \Lambda X' \Lambda(y) \). Note that \( q_k(\beta) = (1 - p_k(\beta)) x_k \), and hence, one readily has
\[ W_n(\beta) = \sum_{j=1}^{p} \sum_{y=0}^{m} R_j(y) R_j(y)', \]
\[ = A^{-1} \Gamma_n(\beta) A^{-1}, \]
where \( \Gamma_n(\beta) := \sum_{y=0}^{m} \Gamma_n(y, \beta) \Gamma_n(y, \beta)' \) that is no other than \( B_n \) in Lemma (2.1) For \( 1 \leq i, j, k \leq n \), let
\[ \sigma_{ijk}(\beta) := \sum_{x=0}^{m} \sum_{y=0}^{m} \lambda_i(x, \beta) \lambda_j(y, \beta) \left[ F_k(x \wedge y; \beta) - F_k(x; \beta) F_k(y; \beta) \right]. \]

We are ready to state one of the main results in this study. Recall \( d'_i \) is the \( i \)th row vector of \( D \).

**Lemma 2.2.** Assume
\[ \max_{1 \leq k \leq n} \sum_{i \neq k} (d'_i d_k)^2 = o(1). \] (2.8)

Let \( \Sigma := ([\Sigma_{ij}]) \), \( 1 \leq i, j \leq n \) where
\[ \Sigma_{ij} = \sum_{k=1}^{n} d'_i (d_k d'_k) d_j \sigma_{ijk}. \]

Assume that for any \( u \in \mathbb{R}^p \),
\[ \liminf_{n \to \infty} u' \Sigma u > 0. \] (2.9)
Then,
\[ \mathbf{A}S_n(\beta_0) \Rightarrow_D N(0, \tilde{\Sigma}), \]
where
\[ \tilde{\Sigma} := 4\mathbf{A}\Sigma\mathbf{A}. \]

Remark 2.2. If \( \mathbf{D} := \mathbf{X}\mathbf{A}, \) \( \tilde{\Sigma} \) is a priori satisfied.

Proof. Let \( \mathcal{W}(y, \beta) := (\mathcal{W}_1, ..., \mathcal{W}_p)' \in \mathbb{R}^p \) where \( \mathcal{W}_j \) denote \( \mathcal{W}_j(y, \beta), \) \( 1 \leq j \leq p. \) Note that \( \mathbf{A}S_n \) can be rewritten as
\[
\mathbf{A}S_n(\beta) = -2\mathbf{A} \sum_{j=1}^{p} \sum_{y=0}^{m} \mathcal{W}_j(y) \mathcal{R}_j(y),
\]
\[
= -2 \sum_{y=0}^{m} \Gamma_n(y) \mathcal{W}(y),
\]
\[
= -2\mathbf{A}_T \sum_{y=0}^{m} \Lambda(y) \mathcal{W}(y),
\]
\[
= -2\mathbf{A}_T \mathcal{T}_n, \quad \text{say.}
\]
Thus, the proof of the lemma amounts to showing asymptotic convergence of \( \mathcal{T}_n \) to a normal distribution.

Let \( \eta_k(y, \beta) := I(Y_k \leq y) - F_k(y; \beta). \) Note that \( \mathbb{E}[\eta_k(y)] = 0 \) and \( \mathbb{E}[\eta_k(x)\eta_k(y)] = F_k(x \land y) - F_k(x)F_k(y) \) that is the third term of the summand of \( \sigma_{ijk}. \) Let \( \mathcal{T}_j \) denote the \( j \)th entry of \( \mathcal{T}_n. \) Recall \( \mathbf{d}_k \) the \( k \)th row vector of \( \mathbf{D} \) and observe that \( \mathcal{W}(y) \) can be expressed as
\[
\mathcal{W}(y) = \sum_{k=1}^{n} \mathbf{d}_k \eta_k(y),
\]
and hence,
\[
\mathcal{T}_j = \sum_{k=1}^{n} \sum_{y=0}^{m} \omega_{jk} \lambda_j(y) \eta_k(y),
\]
where \( \omega_{jk} := \mathbf{d}_k' \mathbf{d}_j. \) For \( \mathbf{a} := (a_1, ..., a_n) \in \mathbb{R}^n, \) one has \( \mathbf{a}' \mathcal{T}_n = \sum_{k=1}^{n} t_k \) where
\[
t_k := \sum_{j=1}^{n} a_j \omega_{jk} \sum_{y=0}^{m} \lambda_j(y) \eta_k(y).
\]
Note that
\[
t_k^2 \leq m \| \mathbf{a} \|^2 \max_{1 \leq k \leq n} \sum_{i=1}^{n} \omega_{ik}^2,
\]
where the inequality follows from \( |\lambda_j(y)| \leq 1 \) and \( |\eta_k(y)| \leq 1. \) Let \( \tau_n^2 := \sum_{k=1}^{n} \mathbb{E}(t_k^2). \) It is not difficult to show that
\[
\tau_n^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_j' \sum_{k=1}^{n} \omega_{ik} \omega_{jk} \sigma_{ijk}.
\]
We shall show that the Lindeberg-Feller (LF) condition for the convergence of \( \mathbf{a}' \mathcal{T}_n \) is met, that is, for all \( \varepsilon > 0 \)
\[
\tau_n^{-2} \sum_{k=1}^{n} \mathbb{E} \left[ t_k^2 : |t_k| > \varepsilon \tau_n \right] \leq \varepsilon^{-2} \sum_{k=1}^{n} \mathbb{E} \left[ |t_k|^2 \right] \mathbb{P}(|t_k| > \varepsilon \tau_n),
\]
\[
\leq \varepsilon^{-2} \tau_n^{-2} m \| \mathbf{a} \|^2 \max_{1 \leq k \leq n} \sum_{i=1}^{n} \omega_{ik}^2 \rightarrow 0.
\]
Lemma 2.3.
where the first and the second inequalities follow from (2.10) and the Chebyshev inequality, respectively, while the assumption (2.8) and (2.9) readily implies the convergence to 0. Thus, one has
\[ \tau_n^{-1} \sum_{k=1}^{n} t_k \Rightarrow N(0, 1). \]
Note that (2.11) implies
\[ \Phi = \sum_i \sum_j a_i \Sigma_{ij} a_j. \]
Consequently, the application of Cramer-Wold device yields
\[ \sum_1 \Phi \Rightarrow \mathcal{N}(0_{n \times 1}, I_{n \times n}), \]
thereby completing the proof of the lemma.
□

Lemma 2.3.
\[ \sup_{j=1}^{p} \sum_{y=0}^{m} \sum_{k=1}^{n} d_{kj} \left\{ F_k(y; \beta) - F_k(y; \beta_0) + m_y g_y(p_k^0)(\beta - \beta_0)^\prime q_k(\beta_0) \right\}^2 = o(1), \tag{2.12} \]
where the supremum is taken over \( \| A^{-1}(\beta - \beta_0) \| \leq b \).

Proof. Let \( u := A^{-1}(\beta - \beta_0) \in \mathbb{R}^p \). Recall \( c_k = A x_k \in \mathbb{R}^p \), 1 \( \leq k \leq n \), and \( q_k(\beta) = (1 - p_k(\beta)) x_k \). Then the summand of LHS of (2.12) can be rewritten as
\[ F_k(y; \beta) - F_k(y; \beta_0) + m_y g_y(p_k^0)(\beta - \beta_0)^\prime q_k(\beta_0) = -m_y u^\prime [g_y(\hat{s})(1 - p_k(\beta))] A x_k - g_y(p_k^0)(1 - p_k(\beta_0)) A x_k, \]
where \( \hat{s} = c_1 p_k^0 + (1 - c_1)p_k \) and \( \bar{\beta} = c_2 \beta_0 + (1 - c_2) \beta \) for some \( c_1, c_2 \in \mathbb{R} \). Therefore,
\[ \text{the supremum of LHS of (2.12)} \leq \sum_{j=1}^{p} \sum_{y=0}^{m} \sum_{k=1}^{n} d_{kj}^2 |u^\prime c_k|^2 [g_y(\hat{s})(1 - p_k(\bar{\beta})) - g_y(p_k^0)(1 - p_k(\beta_0))] \]
\[ \leq p b \left( \max_{1 \leq k \leq n} \| c_k \| \right) \left( \sum_{y=0}^{m} m_y^2 \right) \left( \sum_{k=1}^{n} d_{kj}^2 \right) \rightarrow 0, \]
where the first inequality follows from the Cauchy-Schwartz inequality; the second inequality follows from (ii) and (iii) of (2.3) and (a.2), (a.3) and (i) of (2.3) imply the convergence of zero, thereby completing the proof of the claim.

The next theorem demonstrates the first ULAQ condition is met. Recall \( Q \) from (2.7)

Theorem 2.1. Assume (a.1)-(a.5). Then, the distance function \( L \) in (2.7) satisfies (U.1), that is, for any \( 0 < c < \infty \),
\[ \mathbb{E} \left[ \sup_{\beta} |L(\beta) - Q(\beta)| \right] = o(1), \]
where the supremum is taken over \( \{ \beta \in \mathbb{R}^p : \| A^{-1}(\beta - \beta_0) \| \leq c \} \).

Proof. Recall \( R_j(y, \beta) \) and define a \( p \times p \) matrix \( R(y, \beta) := [R_1 \cdots R_p] \), that is, whose \( j \)th column is \( R_j(y, \beta) \). Note that \( L \) and \( Q \) can be rewritten in following quadratic forms
\[ L(\beta) = \sum_{j=1}^{p} \sum_{y=0}^{m} \left\{ W_j(y, \beta) - (\beta - \beta_0)^\prime R_j(y, \beta_0) \right\} \]
\[ - \sum_{k=1}^{n} d_{kj} \left\{ F_k(y; \beta) - F_k(y; \beta_0) - (\beta - \beta_0)^\prime R_j(y, \beta_0) \right\}^2, \]
and
\[
Q(\beta) = \sum_{j=1}^{p} \sum_{y=0}^{m} |W_j(y, \beta_0) - (\beta - \beta_0)' R_j(y, \beta_0)|^2,
\]
\[
= \sum_{y=0}^{m} \|W(y, \beta_0) - (\beta - \beta_0)' R\|^2.
\]

(a.2) readily implies \(E \sum_{y=0}^{m} \|W(y, \beta_0)\|^2 < \infty\), which, in turn, implies
\[
\sum_{y=0}^{m} \|W(y, \beta_0)\|^2 = O_p(1).
\]

Next, observe that
\[
u' A \left[ \sum_{y=0}^{m} R R' \right] A u = \sum_{j=1}^{p} \sum_{y=0}^{m} |u' A R_j(y, \beta_0)|^2 \leq b^2 \max_{0 \leq y \leq m} m_y^2 \sup_{\beta \in \mathbb{R}^p} |1 - p_k(\beta)|^2 \times \left\{ \max_{0 \leq y \leq m} \sup_{s \in (0,1)} |g_y(s)|^2 \right\} \sum_{j=1}^{p} \left( \sum_{k=1}^{n} \|d_{kj}c_k\|^2 \right)^2 < \infty,
\]
where the finiteness immediately follows from (2.3) and (a.4). Then, in view of Lemma 2.3, 2.4 and (2.14), expanding the quadratic of \(L\) and \(Q\) and applying the C-S inequality to the cross product terms will prove the claim, thereby completing the proof of the theorem.

**Remark 2.3.** In the case of a continuous \(F\), Koul (2002, pp. 173-175) assumed (2.13) and showed several pairs of integrating measure and distribution satisfy the assumption, which is a priori met in this study.

**Lemma 2.4.** Suppose the assumptions in Theorem 2.1 hold. Then, \(\mathcal{L}\) also satisfies (U.2) and (U.3).

**Proof.** Define \(V := (V_1, \ldots, V_p) \in \mathbb{R}^p\) and \(\hat{V} := (\hat{V}_1, \ldots, \hat{V}_p) \in \mathbb{R}^p\) where for \(1 \leq j \leq p\),
\[
V_j(u) := \sum_{y=0}^{m} W_j(y, \beta_0 + A u) \quad \text{and} \quad \hat{V}_j(u) := \sum_{y=0}^{m} \{W_j(y, \beta_0) + u' A R_j(y, \beta_0)\}.
\]

Also, let \(\hat{\Gamma}_n := \sum_{y=0}^{m} \Gamma(y, \beta_0)\) and \(k_n(e) := e' \hat{\Gamma}_n e\). Then, the proof of the lemma follows from the direct application of Lemma 5.5.4 in Koul (2002).

**Remark 2.4.** The proof of Lemma 5.5.4 in Koul (2002) used the fact that \(e' V(u)\) is monotone in \(\|u\|\) together with the assumption (a.5). Let \(u = re\) with \(\|e\| = 1\). Note that
\[
e' V(u) = \sum_{y=0}^{m} ^{n} \sum_{k=1}^{n} (e'd_k) \left[ I(Y_k \leq y) - m_y \int_0^{1} e^{-x_kh_y} e^{-r' A x_k} dx_k \right],
\]
where \(e' V(u)\) increases as \(r\) increases if \(d_k e x_k'A e \geq 0\) for all \(k\) while the opposite holds if the inequality changes.

We conclude this section by stating the main result of this article. Recall \(\Gamma_n(y, \beta) = A X' A(y) D\) and \(\hat{\Gamma}_n(\beta) = \sum_{y=0}^{m} \Gamma_n(y, \beta) \hat{\Gamma}_n(y, \beta)\).

**Theorem 2.2.** Suppose the assumptions of Theorem 2.1 hold. Then the MD estimator \(\hat{\beta}\) follows the asymptotic normal distribution
\[
A(\hat{\beta} - \beta) \Rightarrow D N(0, \hat{\Sigma}_n(\beta) \hat{\Sigma}_n(\beta)),
\]
where \(\hat{\Sigma}\) is as in Lemma 2.1.

**Proof.** Theorem 2.1 and Lemma 2.4 imply the distance function \(\mathcal{L}\) satisfies the ULAQ conditions. Consequently, Lemmas 2.1 and 2.2 yield the desired result, thereby completing the proof of the theorem.
3 Conclusion

In this study, we extended the application of the CvM type distance from a binomial distribution to logistic regression and proposed the MD estimator of the regression parameters. Current study can be a benchmark for the further application of the MD estimation to other regression analysis such as poisson regression that will form future research.

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