Asymptotic Infrared Fractal Structure of the Propagator for a Charged Fermion

S. Gulzari, J. Swain and A. Widom

Physics Department
Northeastern University
110 Forsyth Street,
Boston, MA 02115, USA
E-mail: john.swain@cern.ch

I. INTRODUCTION

The propagator $\tilde{S}(x - y)$ for a free electron in a noninteracting theory describes the amplitude for an electron with a definite mass to propagate from $y$ to $x$. For a theory with electromagnetic interactions, the picture is not so simple. The electron moving along its world line will be constantly interacting with its surrounding cloud of virtual infrared photons. Truly free particles are labeled by irreducible representations of the Poincaré group in terms of mass and spin. However, charged particles cannot be consistently assigned precise masses since they continually interact with massless photons. The resulting electromagnetic fields are too long ranged to rigorously define “in” and “out” fixed mass states and one must introduce concepts such as infrared photons (For a review see [4]). In any case, additive perturbation theory is not sensible since it requires very little energy to create and/or destroy an infinite number of photons each with virtually zero energy. This invalidates computations which only include low orders in $\alpha = e^2/\hbar c$. It was shown long ago that the power series expansion in integer powers of $\alpha$ for quantum electrodyamics cannot be convergent.

There exist treatments of the photon cloud about a given electron in terms of coherent states including associated electromagnetic fields. The resulting wave functions lose the notion of a local single-particle electron concept. The wave functions describe the electromagnetic fields as superpositions of an infinite number of photon Fock states. The many particle (one electron plus infinitely many photons) wave functions are quite complicated. In this work we seek a modified single electron Dirac propagator which represents an electron with its associated infinite number virtual photons without the need to explicitly employ the many body wave functions in the final results.

There exist propagator calculations in the literature based on either infinite sums of logarithmic Feynman diagrams or non-perturbative Schwinger computations which argue that such an electron propagator should be of the form

$$S(k) = \left(\frac{\kappa}{i\Lambda}\right)^{\gamma} \Gamma(1 + \gamma) \left\{ \frac{\kappa - \kappa^2 + i\ell}{[k^2 + \kappa^2 - i\ell]^1 + \gamma}\right\},$$

wherein $\hbar\kappa = mc$, $\Lambda$ is a short distance length scale, the fractional exponent $\gamma$ is a function of the coupling strength $\alpha = (e^2/\hbar c)$ and the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds \quad \text{with} \quad \Re(z) > 0.$$  

However, there is not yet any universal agreement about what constitutes the correct function $\gamma(\alpha)$.

Appelquist and Carazzone take $\gamma = -(\alpha/\pi) + \ldots$ to leading order. This differs from older work based on summing logarithms. The fourth volume of the Landau and Lifschitz course of theoretical physics at first appears to be in agreement with $\gamma \neq 0$ but ultimately chooses $\gamma = 0$ by assigning a small mass to the photon. The photon mass implies a broken gauge symmetry, as well as the broken conformal invariance of the free Maxwell field. The null result $\gamma = 0$ is also in conflict with our physical understanding that no sharp mass can be assigned to a charged particle. The effect of assigning the photon a small mass is clear from the physical picture of photon processes. As one backs away from the world line of an electron, thereby going to larger and larger wavelengths, a photon mass vetoes the production of more and more photons without limit and the scaling behavior of the electromagnetic fields is thereby broken (see Supplement 4 of [11]). The broken gauge symmetry is physically unacceptable. It has been argued that the long-range nature of the Coulomb interaction makes the definition of asymptotic “in” and “out” states of charged particles problematic in quantum field theory. In particular, the notion of a simple particle pole in the vacuum charged particle propagator is untenable and should be replaced by a more complicated branch cut structure describing an electron interacting with a possibly infinite number of soft photons. Previous work suggests a Dirac propagator raised to a fractional power dependent upon the fine structure constant, however the exponent has not been calculated in a unique gauge invariant manner. It has even been suggested that the fractal “anomalous dimension” can be removed by a gauge transformation. Here, a gauge invariant non-perturbative calculation will be discussed yielding an unambiguous fractional exponent. The closely analogous case of soft graviton exponents is also briefly explored.

PACS numbers: 12.20.-m, 03.70.+k
that the change from a pole into a branch point has measurable physical implications in measurements of “1/ω” noise in the Schrödinger non-relativistic limit of the relativistic Dirac equation. A path integral approach using the Schwinger proper time representation of the propagator and some work by Bloch and Nordsieck on soft photon emission gives the same sort of result, but with the final answer given only for charged scalar fields, and generally considered to be gauge invariant in such a way that the singularity structure can be returned to a simple pole by a choice of gauge. Fried also discusses this problem as do Johnson and Zumino, and Stefanis and collaborators. Batalin, Fradkin and Schvartsman have made a similar gauge dependent calculation for scalar particles.

The major defects of the existing calculations are that they do not directly give a Dirac propagator raised to a fractional power. Existing calculations are ambiguous and not gauge invariant.

The fractional exponent result is rather dramatic. The fractional exponent implies some non-locality. The non-locality has been previously introduced ad-hoc as a regularization tool. Here it appears naturally and also locality has been previously introduced ad-hoc as a regularization tool. Here it appears naturally and also non-locality has been previously introduced ad-hoc as a regularization tool. Here it appears naturally and also

II. GAUGE INvariant CALCULATION

For an electron moving through an external electromagnetic field, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), the Dirac propagator

\[
(-i\partial + \kappa) G(x, y; A) = \delta(x - y),
\]

may be solved employing the function \( \Delta(x, y; A) \);

\[
\Delta(x, y; A) = \int \gamma_\mu G(x, z; A)\gamma_\nu G(z, y; A) d^3z,
\]

\[
G(x, y; A) = (i\partial + \kappa) \Delta(x, y; A),
\]

\[
\mathcal{L}(v, x; A) = \frac{1}{2} m (v^\mu v_\mu - c^2) + \frac{e}{c} v^\mu A_\mu(x),
\]

and employing the operator representation \( p_\mu = -i\hbar \partial_\mu \), one may define the amplitude for the electron to go from \( y \) to \( x \) in a proper time \( \tau \) as the matrix element

\[
\mathcal{G}(x, y, \tau; A) = \langle x | e^{-i\mathcal{H}_{tot} \tau/\hbar} | y \rangle.
\]  

From Eqs. 11, 16 and 17 follows the electron propagator expression

\[
\Delta(x, y; A) = \frac{i\hbar}{2m} \int_0^\infty \mathcal{G}(x, y, \tau; A) d\tau,
\]

\[
h \mathcal{G}(x, y; A) = \left( mc - \gamma^\mu A_\mu(x) \right) \Delta(x, y; A).
\]

The physical significance of \( \mathcal{H}(p, x) \) can be made manifest in the formal classical limit \( \hbar \to 0 \). Hamilton’s equations in proper time,

\[
v^\mu = \frac{dx^\mu}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_\mu} \quad \text{and} \quad f^\mu = \frac{dp^\mu}{d\tau} = -\frac{\partial \mathcal{H}}{\partial x^\mu},
\]

directly yield the the Lorentz force on a charge equation of motion

\[
m \frac{dv^\mu}{d\tau} = \frac{e}{c} F^{\mu\nu} v_\nu.
\]

Alternatively one may employ the Lagrangian formalism,

\[
\mathcal{L}(v, x; A) = \frac{1}{2} m (v^\mu v_\mu - c^2) + \frac{e}{c} v^\mu A_\mu(x),
\]

\[
\frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) = \left( \frac{\partial \mathcal{L}}{\partial x^\mu} \right);
\]

i.e. Eqs. 10 also imply Eq. 9.

For long wavelength modes of the electromagnetic field, one may with a sufficient degree of accuracy neglect the spin flip term in the Hamiltonian Eq. 9. In such a case we may approximate Eq. 11 by the Lagrangian path integral formulation

\[
\mathcal{G}(x, y, \tau; A) \approx \int_{X(0)=y}^{X(\tau)=x} e^{i\mathcal{S}[X;A]/\hbar} \prod_\sigma dX(\sigma),
\]

\[
\mathcal{S}[X; A] = \int_0^\tau \mathcal{L}(\dot{X}(\sigma), X(\sigma); A) d\sigma.
\]  

Now, let us consider an electron following a world line path \( P \) from \( x \) to \( y \) in a proper time \( \tau \). Since the general path \( P \) contributing to the functional integral Eq. 11 represents the virtual motion of an electron, one finds, in general, that \( c^2 \tau^2 \neq -(x - y)^2 \). It is only in the classical limit \( \hbar \to 0 \) that \( c^2 \tau^2 = -dX^\mu dX_\mu \). In quantum mechanics the amplitude for a process is the coherent sum of all amplitudes for all the different ways in which that process could happen. Thus, one integrates through all possible proper times the electron could accumulate while going from \( x \) to \( y \). Consider two different paths, \( P_1 \) and \( P_2 \), contributing to the path integral in Eqs. 11. Although the endpoints \( y \) and \( x \) are the same, the proper time of the two paths are different. The relativistic “toy model” analogy is to consider two twins starting at the same age at \( y \) taking two different paths, \( P_1 \) and \( P_2 \), and...
meeting again at \( x \) when their ages are in general different.

The interaction between the electron and the electromagnetic vector potential is described by the action

\[
S_{\text{int}}(P; A) = \int_0^\tau \mathcal{L}_{\text{int}}(\dot{X}(\sigma), X(\sigma); A)d\sigma, \\
S_{\text{int}}(P; A) = \frac{\hbar}{c} \int_0^\tau A_\mu(X(\sigma))\dot{X}^\mu(\sigma)d\sigma, \\
S_{\text{int}}(P; A) = \frac{\hbar}{e} \int_P A_\mu(X)dX^\mu, \tag{12}
\]

where the integral is along the world line \( P \). To describe an electron moving through a vacuum region with zero point electromagnetic fields, one averages over the vacuum field fluctuations according to the rule

\[
e^{iS_{\text{int}}(P;A)/\hbar} \rightarrow \langle 0| e^{iS_{\text{int}}(P;A)/\hbar} |0 \rangle_+, \\
e^{iS_{\text{int}}(P;A)/\hbar} \rightarrow e^{iS_{\text{self}}(P)/\hbar},
\]

\[
S_{\text{self}}(P) = \frac{\hbar\alpha}{2} \int_P \int_P D_{\mu\nu}(x_1 - x_2)d\sigma^\mu_1d\sigma^\nu_2. \tag{13}
\]

The action form in Eq. (13) is of a well known form \(^{24}\), and we have bypassed the usual Bloch-Nordsieck replacement of \( c\gamma^\mu \) by four velocity \( v^\mu \) by simply evaluating a phase in the soft photon infrared limit that we are considering. The propagator may be written

\[
D_{\mu\nu}(x_1 - x_2) = \left( \eta_{\mu\nu} - (1 - \xi) \frac{\partial_\mu \partial_\nu}{\partial^2} \right) D(x - y), \\
D(x - y) = \int \frac{4\pi}{k^2 - i0^+} e^{ik(x-y)} \frac{d^4k}{(2\pi)^4}, \\
D(x - y) = i \left\{ \frac{1}{(x-y)^2 + i0^+} \right\}. \tag{15}
\]

where the parameter \( \xi \) fixes a gauge. Because the world line of the electron never begins nor ends (charge conservation), the partial derivative terms in Eq. (15) do not contribute to the self action in Eq. (13); Independently of the gauge parameter \( \xi \) we have

\[
S_{\text{self}}(P) = \frac{\hbar\alpha}{2} \int_P \int_P D(x_1 - x_2)dx_1^\mu dx_2^\mu. \tag{16}
\]

In the absence of any external field (above and beyond the vacuum fluctuation operator \( \hat{A} \)) we have now derived expressions for the renormalized vacuum electron propagator

\[
\hat{G}(x - y) = \int S(k)e^{ik(x-y)} \frac{d^4k}{(2\pi)^4}, \\
\hat{G}(x - y) = (0 \langle G(x, y; \hat{A}) | 0 \rangle_+, \\
\hat{G}(x - y) = (i\theta + \kappa) \hat{\Delta}(x - y), \\
\hat{\Delta}(x - y) = \frac{i\hbar}{2m} \int_0^\infty \hat{G}(y - x, \tau)d\tau. \tag{17}
\]

The functional integral expression for \( \hat{G}(x - y, \tau) \) is given by

\[
\hat{G}(x - y, \tau) = \int_{X(0)=y}^{X(\tau)=x} e^{i\mathcal{S}[X; A]/\hbar} \prod_{\sigma} dX(\sigma), \\
\mathcal{S}[X] = \int_0^\tau L_0(\dot{X}(\sigma))d\sigma + S_{\text{self}}[X], \tag{18}
\]

wherein the free electron Lagrangian is

\[
L_0(\dot{X}(\sigma)) = \frac{1}{2} m_0(\dot{X}^\mu \dot{X}_\mu - c^2), \tag{19}
\]

and the self action is given by Eqs. (15) and (16) as

\[
S_{\text{self}}[X] = \frac{i\hbar\alpha}{2\pi} \int_0^\tau \int_0^\tau \int_0^\tau \int_0^\tau \frac{\dot{X}^\mu(\sigma_1)\dot{X}_\mu(\sigma_2)d\sigma_1d\sigma_2}{(X(\sigma_1) - X(\sigma_2))^2 + i0^+}. \tag{20}
\]

The divergent piece of the self action

\[
\Re S_{\text{self}}[X] = \frac{\Delta m}{2} \int_0^\tau (\dot{X}^\mu(\sigma)\dot{X}_\mu(\sigma) - c^2)d\sigma, \tag{21}
\]

The formally infinite self mass can be described by a finite physical mass \( 0 < m = (m_0 + \Delta m) < \infty \). Thus, Eq. (13) is renormalized to

\[
\tilde{G}(x - y, \tau) = \int_{X(0)=y}^{X(\tau)=x} e^{i\mathcal{S}[X; A]/\hbar} \prod_{\sigma} dX(\sigma), \\
\mathcal{S}[X] = \int_0^\tau L_m(\dot{X}(\sigma))d\sigma + iW[X] \\
L_m(\dot{X}(\sigma)) = \frac{1}{2} m(\dot{X}^\mu \dot{X}_\mu - c^2) \\
W[X; \tau] = 3m S_{\text{self}}[X], \\
W[X; \tau] = \frac{\hbar\alpha}{2\pi} \int_0^\tau \int_0^\tau \int_0^\tau \int_0^\tau \frac{\dot{X}^\mu(\sigma_1)\dot{X}_\mu(\sigma_2)d\sigma_1d\sigma_2}{(X(\sigma_1) - X(\sigma_2))^2}. \tag{22}
\]

For a straight line path \( X^\mu(\sigma) = V^\mu \sigma \) with \( V^\mu V_\mu = c^2 \), one finds

\[
W(\tau) = \frac{\hbar\alpha}{2\pi} \int_0^\tau \int_0^\tau d\sigma_1d\sigma_2 \frac{\dot{X}(\sigma_1)\dot{X}(\sigma_2)}{(X(\sigma_1) - X(\sigma_2))^2}, \tag{23}
\]

which can be made finite with the formal differential regularization\(^{27}\)

\[
\frac{d^2W(\tau)}{d\tau^2} = \frac{\hbar\alpha}{\pi\tau^2}. \tag{24}
\]

The solution to Eq. (24) with a logarithmic cut-off \( \Lambda \) is

\[
W(\tau) = -\left( \frac{\hbar\alpha}{\pi} \right) \ln \left( \frac{c\tau}{2\Lambda} \right). \tag{25}
\]
From Eqs. (22) and (25), one finds
\[ G(x - y, \tau) = e^{-iW(x-y)/\hbar} G_m(x - y, \tau) \] (26)
wherein \( G_m(x - y, \tau) \) is the proper time Green's function for a particle of mass \( m \) with the corresponding free Lagrangian \( L_m(X) \). In detail and to exponentially lowest order in \( \alpha \),
\[ \tilde{G}_m(x - y, \tau) = \int \left\{ e^{-\frac{i}{2} (k^2 + \kappa^2) \tau / 2m} e^{ik \cdot (x-y)} \right\} \frac{d^4 k}{(2\pi)^4}; \]
\[ \tilde{G}(x - y, \tau) = \left( \frac{c r}{2\Lambda} \right)^{\alpha/\pi} \tilde{G}_m(x - y, \tau). \] (27)

Eqs. (17) and (27) imply
\[ \Delta(x - y) = \int D(k) e^{ik \cdot (x-y)} \frac{d^4 k}{(2\pi)^4}; \]
\[ D(k) = \left( \frac{\kappa}{r\Lambda} \right)^{\alpha/\pi} \left\{ \frac{\Gamma(1 + (\alpha/\pi))}{[k^2 + \kappa^2 - i0^+]^{1+(\alpha/\pi)}} \right\} \],
\[ \tilde{G}(x - y) = \int S(k) e^{ik \cdot (x-y)} \frac{d^4 k}{(2\pi)^4}; \]
\[ S(k) = (\kappa - k^2) D(k). \] (28)

Eq. (28) is equivalent to the central Eq. 1 of this work with
\[ \gamma = \frac{\alpha}{\pi} + \ldots \] (29)
in agreement with the magnitude, but not the sign, of \( \gamma \) in Appelquist and Carazzone [13].

III. PHYSICAL INTERPRETATION

It is interesting to consider what the physical interpretation of the non-integer exponent in the radiatively corrected Dirac propagator means. First of all, the fact that the exponent is non-integer means that the renormalized Dirac operator is non-local [25]. This was, of course, to be expected since the electromagnetic field has infinite range.

One may also expect a degree of self-similarity at long wavelengths since the only scale-breaking term in QED is the mass of the electron. Intuitively, one might think of stepping back farther and farther from an electron world line and seeing contributions to its dressed structure from longer and longer wavelengths, i.e. softer and softer virtual photons. This would suggest a fractal structure, which is made precise by the above derivation. Such notions of scaling and fractality are not new in QED and in quantum field theory in general, but are often considered in the high energy, ultraviolet limit [24, 27]. In this case additional complications arise since more and more charged excitations must be included, but again, one sees fractional exponents in form of anomalous dimensions and the renormalization group – another reflection of a non-trivially realized scale invariance in the theory, but now at short distances.

If the photon is given a mass, however small, this structure will break down asymptotically, since now there is a minimum energy required to create a virtual photon, and at distances greater than the corresponding Compton wavelength one will get the non-interacting Dirac propagator. This was done in by Lifshitz et al. [14], and this argument makes clear how breaking gauge invariance, i.e. including a photon mass, removes the anomalous scaling behavior here derived for gauge invariant QED.

The fact that a particle is non-localizable, at least in part due to its electromagnetic field which extends over all space, is interesting. The feeling of this calculation is such that at least part of what one thinks of as quantum-mechanical about an otherwise point-particle (its lack of localizability) may arise from the non-perturbative quantum mechanics of its self-interaction [21].

Finally, the path integral expressions for the QED charged Dirac propagator employed here allow one to assign a fractal dimension to the paths taken by charged particles in the infrared limit. The fractal nature of particle paths has been discussed in the literature (see, for example [32, 33]). Abbott and Wise [34], working from nonrelativistic quantum mechanics, found 2 as the dimension of a quantum mechanical path, as opposed to 1 for a classical path. Cannata and Ferrari [35] extended this work for spin-1/2 particles and find different results not only in the classical and quantum mechanical limits, but also in the non-relativistic and relativistic limits. Intuitively one can understand the dimension 2 result for the nonrelativistic quantum mechanical case by thinking of the Schrödinger equation as a diffusion equation in imaginary time [32]. For diffusion one has the distance \( r \) a particle covers in time \( t \) satisfying a relationship of the form \( t \propto r^d \) wherein \( d \) is the fractal dimension of the “path”. For example, \( t \propto r^2 \) in the diffusion limit, and \( t \propto r \) in the ballistic (simple path) limit [37]. Here we have a closely analogous situation but with a 4-dimensional Hamiltonian \( H \) and with fractal diffusion in proper time. We find
\[ d = 2(1 + \gamma) \approx 2 + \frac{2\alpha}{\pi} + \ldots, \] (30)
i.e. the “path” through space-time as a function of internal proper time has a dimension slightly higher than that of a two-dimensional surface. This excess over dimension 2 can be thought of as due to an additional roughening of the path of a charged particle due to interactions with vacuum fluctuations. All previous discussions have ignored the effects of self-interaction via long-range fields.

IV. EXTENSIONS

It is interesting to ask to what extent one might expect that similar behaviour might occur with other inter-
actions. With the weak interactions, the finite mass of the gauge bosons will cause this analysis to break down at distances of the order of their Compton wavelengths. The weak interactions are not infinite range.

For QCD, there is an additional problem in that the gluons, unlike the photons, now couple to each other, and with increasing strength at larger and larger distances, so we do not expect the QED treatment to carry over very easily. That being said, one might well expect some sort of fractional exponent propagator with a to appear. In the QED case it appears for large values of \( \alpha \) radically different propagators would arise as the exponent hits zero and then changes sign. Some of this behaviour may be linked with confinement.

For quantum gravity one can do the analysis in much the same way as for quantum electrodynamics. The ultraviolet divergences to need not worry us since we are dealing with is a strictly infrared problem. There should be graviton-graviton interactions, but if we neglect these as small compared to graviton-electron interactions we can simply repeat what was done for electrostatics but now with the Newtonian limit of gravity. Note that this sort of approximation could not be done consistently in the QCD case for a quark propagator since gluon-gluon couplings are comparable to gluon-quark couplings, but there is evidence from other calculations of the appearance of anomalous dimensions for infrared propagators. (For a review see [38]. Very recent calculations can also be found in references [39].)

Recognizing that we need to replace the repulsive electrostatic self-interaction, say \( +(e^2/r) \), with the attractive gravitational self-interaction, say \( -Gm^2/r \), suggests an asymptotic form of the Dirac propagator exponent

\[
\gamma \approx \frac{1}{\pi \hbar c}(e^2 - Gm^2) + \ldots.
\] (31)

If \( m = |e/\sqrt{G}| \), which is the ADM [40, 41] mass of charged shell of charge \( e \), regularized by its own gravity, then one recovers an effectively free propagator. Given that one generally makes measurements using the electromagnetic interaction, and the suggestion that quantum mechanics might be linked to self-interaction [42], it is interesting to consider what this might imply for the role of gravity in the quantum measurement problem [42]. In particular, since one has a connection between mass and charge which is non-perturbative in Newton’s \( G \), and implies a mass near the Planck mass \( \sim 10^{-57} \) g, which may be thought to be in the neighborhood of a putative classical-quantum boundary.

V. CONCLUSIONS

We have provided a simple and intuitive path integral description of how the propagator for a charged Dirac particle is modified by soft self-energy radiative corrections. The result is a self-similar (fractal) object with the non-locality one would expect for a particle carrying an infinite range field. The extension of this treatment to other interactions was briefly explored, and the special case of soft graviton corrections quantitatively discussed.

VI. ACKNOWLEDGEMENTS

We would like to thank Yogi Srivastava for useful conversations. We would also like to thank Reinhard Alkofer for pointing out references [11] and [35] to us, and Christian Fischer for pointing out [39]. This work was supported in part by a grant from the National Science Foundation NSF-0457001.

[1] S. Weinberg, “The Quantum Theory of Fields I”, Cambridge University Press, Cambridge (1995).
[2] D. Buchholz, Phys. Lett. B174, 331 (1986).
[3] D. Buchholz, M. Porrmann, U. Stein, Phys. Lett. B267, 337 (1991).
[4] B. Schroer, Fortschr. d. Phys. 11, 1 (1963).
[5] M. Porrmann, Commun. Math. Phys. 248, 269 (2004); Commun. Math. Phys. 248, 305 (2004).
[6] F. J. Dyson, Phys. Rev. 85, 631 (1952).
[7] T.W.B. Kibble, J. Math. Phys. 9, 315 (1968); Phys. Rev. 173, 1527 (1968); Phys. Rev. 173, 1527 (1968); V. Chung, Phys. Rev. 140, 1110 (1965); D. Zwanziger, Phys. Rev. D7, 1082 (1973); Phys. Rev. Lett. 30, 934 (1973); Phys. Rev. D11 3481 and 3504, L. Fadeev and P. Kulish, Teor. Mat. Fiz. 4, 153 (1970); Theor. Math. Phys. 4, 745 (1970).
[8] A. A. Abrikosov, Sov. Phys. JETP 3, 71 (1956), A. V. Suidzuiski Sov. Phys. JETP 4, 179 (1957), L. D. Soloviev, Sov. Phys. JETP 17, 209 (1963), D. V. Shirkov, Il Nuovo Cim. 3, 845 (1956), L. D. Soloviev, Sov. Phys. Dokl. 1, 536 (1957).
[9] J. Schwinger, Phys. Rev. 82, 664 (1951).
[10] E.S. Fradkin, Nucl. Phys. 76, 588 (1966).
[11] J. M. Jauch and F. Rohrlich, “The Theory of Photons and Electrons, 2nd edition”, Springer Verlag, Berlin (1976).
[12] R.J. Rivers, “Path Integral Methods in Quantum Field Theory”, Cambridge University Press, Cambridge (1987).
[13] T. Appelquist and J. Carazzone, Phys. Rev. D11, 2856, (1975).
[14] E.M. Lifshitz, L.P. Pitaevskii and V.B. Berestetskii, “Quantum Electrodynamics”, Pergamon Press, Oxford (1980).
[15] P.H. Handel, Phys. Rev. Lett. 34, 1492 (1975); Phys. Rev. Lett. 34, 1495 (1975).
[16] F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).
[17] H. M. Fried, “Functional Methods and Models in Quantum Field Theory”, MIT Press, Cambridge (1972).
[18] K. Johnson and B. Zumino, Phys. Rev. Lett. 3, 351 (1959).
[19] A. Kernenmann and N. G. Stefanis, Phys. Rev. D40, 2103 (1989), A. I. Karanija, C. N. Ktorides and N. G. Stefanis, Phys. Rev. D52, 5898 (1995).
[20] I.A. Batalin, E.S. Fradkin, and SH.M. Shvartsman, Nucl. Phys. B258, 435 (1985).
[21] J. Zinn-Justin, Chinese Journal of Physics 38, 521 (2000).
[22] D. Evens, J.W. Moffat, G. Kleppe and R.P. Woodard, Phys. Rev. D43, 499 (1991), J.W. Moffat, Phys. Rev. D41, 1177 (1990).
[23] E.R. Speer, “Generalized Feynman Amplitudes”, Annals of Mathematics Studies 62, Princeton University Press, Princeton (1969).
[24] J.A. Wheeler and R.P. Feynman, Rev. Mod. Phys. 17, 157 (1945).
[25] D. Z. Freedman, K. Johnson, J. I. Latorre, Nucl. Phys. B371, 353 (1992).
[26] J. Schwinger, “Particles, Sources, and Fields”, Perseus Books, Reading (1970).
[27] I. Podlubny, “Fractional Differential Equations”, Academic Press, London (1999).
[28] B. B. Mandellrot, “The Fractal Geometry of Nature”, W.H. Freeman, New York (1982).
[29] J. Collins, “Renormalization : An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion”, Cambridge University Press, Cambridge (1984).
[30] M. Gell-Mann and F.E. Low, Phys. Rev. 95, 1300 (1954).
[31] D. Hestenes, Foundations of Physics, 15, 63 (1983).
[32] R.P. Feynman and A. R. Hibbs, “Quantum Mechanics and Path Integrals”, McGraw-Hill, New York (1965).
[33] S. Amir-Azizi, A.J.G. Hey, T.R. Morris, Complex Systems 1, 923 (1987).
[34] L.F. Abbott and M.B. Wise, Am. J. Phys. 49(1), 37 (1981).
[35] F. Cannata and L. Ferrari, Am. J. Phys. 56(8), 721 (1988).
[36] E. Nelson, Phys. Rev. 150, 1079 (1966).
[37] D. Sornette, Eur. J. Phys. 11, 334 (1990).
[38] R. Alkofer and L. von Smekal, Physics Reports 353, 281 (2001).
[39] C. S. Fischer, R. Alkofer, T. Dahm and P. Maris, Phys. Rev. D 70, 073007 (2004); C. S. Fischer, arXiv:hep-ph/0605173
[40] A. Ashtekar, “Lectures on Non-Perturbative Quantum Gravity”, Chapt. 1, World Scientific, Singapore (1991).
[41] R. Arnowitt, S. Deser and C.W. Misner, Phys. Rev. 120, 313 (1960); Ann. Phys. (N.Y.) 33, 88 (1965).
[42] R. Penrose, Gen. Rel. and Grav. 28, 581 (1996).