Quantum strategies are better than classical in almost any XOR game
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Introduction
We consider a random instances of nonlocal games where two players plays against a referee. The rules of the game are specified by an \( n \times n \) matrix \( A \) whose entries are \(+1\) and \(-1\).

The game works as follows:
- The referee randomly chooses inputs \( i \in \{1, 2, \ldots, n\} \) and \( j \in \{1, 2, \ldots, n\} \) and sends them to the players;
- The players reply by sending bits \( x \) and \( y \);
- The players win if \( x = y \) and \( A_{ij} = +1 \) or \( x \neq y \) and \( A_{ij} = -1 \).

We consider the case when the matrix \( A \) that specifies the rules of the game is chosen randomly against all \( \pm 1 \)-valued \( n \times n \) matrices \( A \).

Results
- The maximum winning probability \( p_q \) that can be achieved by a quantum strategy is \( \frac{1}{2} + \frac{1}{\sqrt{n}} + o(1) \).
- The maximum winning probability \( p_{cl} \) that can be achieved by a classical strategy satisfies
  \[
  \frac{1}{2} - \frac{0.6394 \ldots - o(1)}{\sqrt{n}} \leq p_{cl} \leq \frac{1}{2} + \frac{0.8325 \ldots + o(1)}{\sqrt{n}},
  \]
where both winning probabilities can be achieved with probability \( 1 - o(1) \).

Let \( \Delta \) be the maximum of the winning probability minus the losing probability. We obtain that
\[
\Delta_q = 2p_q - 1 \quad \text{and} \quad \Delta_{cl} = 2p_{cl} - 1.
\]
Thus, our results imply that the advantage of quantum strategies is
\[
1.2011 \ldots < \frac{\Delta_q}{\Delta_{cl}} < 1.5638 \ldots
\]
for almost all games.

Methods
- The classical value of the game is equal to
  \[
  \Delta_{cl} = \frac{1}{n^2} \max_{u_1, \ldots, u_n \in \{+1, -1\}} \max_{v_1, \ldots, v_n \in \{+1, -1\}} \sum_{i,j=1}^{n} A_{ij} u_i v_j;
  \]
- The upper bound follows straightforwardly from Chernoff bounds;
- To prove the lower bound we give an algorithm for choosing \( u_i \) and \( v_j \). In the proof that the algorithm achieves the lower bound we analyze certain random walk;
- In the quantum case, Tsirelson's theorem implies that
  \[
  \Delta_q = \frac{1}{n^2} \max_{u_1, \ldots, u_n} \max_{v_1, \ldots, v_n} \sum_{i,j=1}^{n} A_{ij} (u_i v_j);
  \]
- The upper bound follows from the fact that \( \|A\| = (2 + o(1)) \sqrt{n} \) with a high probability;
- The lower bound can be obtained from the modified version of Marčenko-Pastur law.

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