CLASSIFYING SPACES FOR FAMILIES OF SUBGROUPS OF 8-LOCATED GROUPS

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Abstract. We investigate the structure of the minimal displacement set in 8-located complexes with the SD'-property. We show that such set embeds isometrically into the complex. Since 8-location and simple connectivity imply Gromov hyperbolicity, the minimal displacement set in such complex is systolic. Using these results, we construct a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on an 8-located complex with the SD'-property.

1. Introduction

Curvature can be expressed both in metric and combinatorial terms. Metrically, one can refer to 'nonpositively curved' (respectively, 'negatively curved') metric spaces in the sense of Aleksandrov, i.e. by comparing small triangles in the space with triangles in the Euclidean plane (hyperbolic plane). These are the CAT(0) (respectively, CAT(-1)) spaces. Combinatorially, one looks for local combinatorial conditions implying some global features typical for nonpositively curved metric spaces.

A very important combinatorial condition of this type was formulated by Gromov [10] for cubical complexes, i.e. cellular complexes with cells being cubes. Namely, simply connected cubical complexes with links (that can be thought as small spheres around vertices) being flag (respectively, 5-large, i.e. flag-no-square) simplicial complexes carry a canonical CAT(0) (respectively, CAT(-1)) metric. Another important local combinatorial condition is local $k$-largeness, introduced by Januszkiewicz-Świątkowski [13] and Haglund [11]. A flag simplicial complex is locally $k$-large if its links do not contain 'essential' loops of length less than $k$. In particular, simply connected locally 7-large simplicial complexes, i.e. 7-systolic complexes, are Gromov hyperbolic [14]. The theory of 7-systolic groups, that is, groups acting geometrically on 7-systolic complexes, allowed to provide important examples of highly dimensional Gromov hyperbolic groups [12, 13, 22, 27, 29, 8].

However, for groups acting geometrically on CAT(-1) cubical complexes or on 7-systolic complexes, some very restrictive limitations are known. For example, 7-systolic groups are in a sense 'asymptotically hereditarily aspherical', i.e. asymptotically they can not contain essential spheres. This yields in particular that such groups are not fundamental groups of negatively curved manifolds of dimension...
above two; see e.g. [14, 20, 21, 27, 9, 25]. This rises need for other combinatorial conditions, not imposing restrictions as above. In [23, 5, 1, 4] some conditions of this type are studied – they form a way of unifying CAT(0) cubical and systolic theories.

On the other hand, Osajda [24] introduced a local combinatorial condition of 8-location, and used it to provide a new solution to Thurston’s problem about hyperbolicity of some 3-manifolds. In [16] a version of 8-location, suggested in [24, Subsection 5.1]. This 8-location says that homotopically trivial loops of length at most 8 admit filling diagrams with one internal vertex. However, in the new 8-location essential 4-loops are allowed. In [16] (Theorem 4.3) it is shown that simply connected, 8-located simplicial complexes are Gromov hyperbolic. In the current paper we give an application to this result.

We focus on the study of the minimal displacement set in an 8-located complex satisfying the $SD'$-property. One of the paper’s results states that such set is isometrically embedded into the complex. Moreover, we show that such set is Gromov hyperbolic. In particular, it is systolic. This follows as an application of the fact that 8-located complexes with the $SD'$-property are Gromov hyperbolic (see [16]).

For CAT(0) spaces and systolic complexes, however, studying the structure of the minimal displacement set is useful when constructing a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on a CAT(0) space, respectively on a systolic complex (see [2, 26]). We expect similar results in the 8-located case. Knowing that the minimal displacement set of an 8-located complex with the $SD'$-property embeds isometrically into the complex and it is systolic, we will be able to apply results proven in [7] and [26] on systolic complexes.

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2. Preliminaries

Let $X$ be a simplicial complex. We denote by $X^{(k)}$ the $k$-skeleton of $X$, $0 \leq k < \dim X$. A subcomplex $L$ in $X$ is called full as a subcomplex of $X$ if any simplex of $X$ spanned by a set of vertices in $L$, is a simplex of $L$. For a set $A = \{v_1, ..., v_k\}$ of vertices of $X$, by $\langle A \rangle$ or by $\langle v_1, ..., v_k \rangle$ we denote the span of $A$, i.e. the smallest full subcomplex of $X$ that contains $A$. We write $v \sim v'$ if $\langle v, v' \rangle \in X$ (it can happen that $v = v'$). We write $v \sim v'$ if $\langle v, v' \rangle \notin X$. We call $X$ flag if any finite set of vertices which are pairwise connected by edges of $X$, spans a simplex of $X$.

A cycle (loop) $\gamma$ in $X$ is a subcomplex of $X$ isomorphic to a triangulation of $S^1$. A full cycle in $X$ is a cycle that is full as a subcomplex of $X$. A $k$-wheel in $X$ $(v_0; v_1, ..., v_k)$ (where $v_i, i \in \{0, ..., k\}$ are vertices of $X$) is a subcomplex of $X$ such that $\gamma = (v_1, ..., v_k)$ is a full cycle and $v_0 \sim v_1, ..., v_k$. The length of $\gamma$ (denoted by $|\gamma|$) is the number of edges in $\gamma$. Given two cycles $\alpha, \beta$ of $X$, we denote by $\alpha \ast \beta$ their concatenation.

We define the combinatorial metric on the 0-skeleton of $X$ as the number of edges in the shortest 1-skeleton path joining two given vertices.
A ball (sphere) $B_i(v, X)$ ($S_i(v, X)$) of radius $i$ around some vertex $v$ is a full subcomplex of $X$ spanned by vertices at combinatorial distance at most $i$ (at combinatorial distance $i$) from $v$.

**Definition 2.1.** A simplicial complex is $m$-located, $m \geq 4$, if it is flag and every full homotopically trivial loop of length at most $m$ is contained in a 1-ball.

Let $\sigma$ be a simplex of $X$. The link of $X$ at $\sigma$, denoted by $X_\sigma$, is the subcomplex of $X$ consisting of all simplices of $X$ which are disjoint from $\sigma$ and which, together with $\sigma$, span a simplex of $X$. We call a flag simplicial complex $k$-large if there are no full $j$-cycles in $X$, for $j < k$. We say $X$ is locally $k$-large if all its links are $k$-large. We call $X$ $k$-systolic if it is connected, simply connected and locally $k$-large. For $k = 6$, we abbreviate $k$-systolic to systolic.

We introduce further a global combinatorial condition on a flag simplicial complex.

**Definition 2.2.** Let $X$ be a flag simplicial complex. For a vertex $O$ of $X$ and a natural number $n$, we say that $X$ satisfies the property $SD'_n(O)$ if for every $i \in \{1, \ldots, n\}$ we have:

1. (T) (triangle condition): for every edge $e \in S_{i+1}(O)$, the intersection $X_e \cap B_i(O)$ is non-empty;
2. (V) (vertex condition): for every vertex $v \in S_{i+1}(O)$, and for every two vertices $u, w \in X_v \cap B_i(O)$, there exists a vertex $t \in X_v \cap B_i(O)$ such that $t \sim u, w$.

We say $X$ satisfies the property $SD'(O)$ if $SD'_n(O)$ holds for each natural number $n$. We say $X$ satisfies the property $SD'$ if $SD'_n(O)$ holds for each natural number $n$ and for each vertex $O$ of $X$.

The following result is given in [24].

**Proposition 2.1.** A simplicial complex which satisfies the property $SD'(O)$ for some vertex $O$, is simply connected.

**Definition 2.3.** A group acting properly discontinuously and cocompactly, by automorphisms, on an $m$-located simplicial complex with the $SD'$ property, is called an $m$-located group, $m \geq 4$.

**Definition 2.4.** Given a path $\gamma = (v_0, v_1, \ldots, v_n)$ in a simplicial complex $X$, one can tighten it to a full path $\gamma'$ with the same endpoints by repeatedly applying the following operations:

- if $v_i$ and $v_j$ are adjacent in $X$ for some $j > i + 1$, then remove from the sequence all $v_k$ where $i < k < j$;
- if $v_i$ and $v_j$ coincide in $X$ for some $j > i$, then remove from the sequence all $v_k$ where $i < k \leq j$.

We call $\gamma'$ a tightening of $\gamma$. We allow the trivial case when $\gamma$ is already full. Then its tightening is the path itself.

2.1. Minimal displacement set for CAT(0) spaces. For CAT(0) spaces the minimal displacement set is studied in [2].

**Definition 2.5.** Let $X$ be a metric space and let $h$ be an isometry of $X$. The displacement function of $h$ is the function $d_h : X \rightarrow \mathbb{R}$ defined by $d_h(x) = d(h(x), x)$. The translation length of $h$ is the number $|h| = \inf\{d_h(x) | x \in X\}$. The set of points
where \(d_h\) attains this infimum is denoted by \(\text{Min}_X(h)\) and it is called the \textit{minimal displacement set}.

**Definition 2.6.** Let \(X\) be a metric space. An isometry \(h\) of \(X\) is called

1. \textit{elliptic} if \(h\) has a fixed point,
2. \textit{hyperbolic} if \(d_h\) attains a strictly positive minimum.

The following result concerns the structure of the minimal displacement set of a hyperbolic isometry \(h\) in a CAT(0) space.

**Theorem 2.2.** Let \(X\) be a CAT(0) space.

1. An isometry \(h\) of \(X\) is hyperbolic if and only if there exists a geodesic line \(c : \mathbb{R} \to X\) which is translated non-trivially by \(h\); namely \(h(c(t)) = c(t + a)\), for some \(a > 0\). The set \(c(\mathbb{R})\) is called an axis of \(h\). For any such axis, the number \(a\) is equal to \(|h|\).

   Let \(h\) be a hyperbolic isometry of \(X\).

2. The axes of \(h\) are parallel to each other and their union is \(\text{Min}_X(h)\).

3. \(\text{Min}_X(h)\) is isometric to a product \(Y \times \mathbb{R}\), and the restriction of \(h\) to \(\text{Min}_X(h)\) is of the form \((y, t) \to (y, t + |h|)\), where \(y \in Y, \ t \in \mathbb{R}\) (see [2], Theorem 6.8, page 231).

2.2. Minimal displacement set for systolic complexes. For systolic complexes the minimal displacement set is studied in [7].

Let \(h\) be an isometry of a simplicial complex \(X\). We define the \textit{displacement function} \(d_h : X^{(0)} \to \mathbb{N}\) by \(d_h(x) = d_X(h(x), x)\). The \textit{translation length} of \(h\) is defined as \(|h| = \min_{x \in X^{(0)}} d_h(x)\). If \(h\) does not fix any simplex of \(X\), then \(h\) is called \textit{hyperbolic}. In such case one has \(|h| > 0\). Otherwise we call the isometry \(h\) \textit{elliptic}. For a hyperbolic isometry \(h\), we define the minimal displacement set \(\text{Min}_X(h)\) as the subcomplex of \(X\) spanned by the set of vertices where \(d_h\) attains its minimum. Clearly \(\text{Min}_X(h)\) is invariant under the action of \(h\).

**Theorem 2.3.** Let \(h\) be a hyperbolic isometry of a systolic complex \(X\). Then the subcomplex \(\text{Min}_X(h)\) is a systolic subcomplex, isometrically embedded into \(X\) (see [7], Propositions 3.3 and 3.4).

Let \(h\) be an isometry of a simplicial complex \(X\). An \(h\)-invariant geodesic in \(X\) is called an \textit{axis} of \(h\). We say that \(\text{Min}_X(h)\) is the union of axes, if for every vertex \(x \in \text{Min}_X(h)\), there is an \(h\)-invariant geodesic passing through \(x\), i.e. \(\text{Min}_X(h)\) can be written as follows:

\[
\text{Min}_X(h) = \text{span}\{\bigcup \gamma | \gamma \text{ is an } h\text{-invariant geodesic} \} \quad (2.1)
\]

In this case, the isometry \(h\) acts on \(X\) as a translation along the axes by the number \(|h|\).

For two subcomplexes \(X_1, X_2 \subset X\), the distance \(d_{\text{min}}(X_1, X_2)\) is defined to be

\[
d_{\text{min}}(X_1, X_2) = \min\{d_X(x_1, x_2) | x_1 \in X_1, x_2 \in X_2\}.
\]

Next we define the \textit{graph of axes} denoted by \(Y_h\). For a hyperbolic isometry \(h\) satisfying (2.1), we define the simplicial graph \(Y_h\) as follows:

\[
Y_h^{(0)} = \{\gamma | \gamma \text{ is an } h\text{-invariant geodesic in } \text{Min}_X(h)\},
\]

\[
Y_h^{(1)} = \{\{\gamma_1, \gamma_2\} | d_{\text{min}}(\gamma_1, \gamma_2) \leq 1\}.
\]

Let \(d_{Y_h}\) denote the associated metric on \(Y_h^{(0)}\).
2.3. **Hyperbolicity.** One of the paper’s main results relies on the following theorem.

**Theorem 2.4.** Let \( X \) be an 8-located simplicial complex which satisfies the SD’ property. Then the 0-skeleton of \( X \) with a path metric induced from \( X^{(1)} \), is \( \delta \)-hyperbolic, for a universal constant \( \delta \) (see [16], Theorem 3.7).

We shall apply the following lemmas frequently.

**Lemma 2.5.** Let \( X \) be an 8-located simplicial complex which satisfies the SD’\( n \)\( (O) \) property for some vertex \( O \), \( n \geq 2 \). Let \( v \in S_{n+1}(O) \) and let \( y, z \in B_n(O) \) be such that \( v \sim y, z \) and \( d(y, z) = 2 \). Let \( w \in B_n(O) \) be a vertex such that \( w \sim y, v, z \), given by the vertex condition (V). We consider the vertices \( u_1, u_2 \in B_{n-1}(O) \) such that \( u_1 \sim y, w \) and \( u_2 \sim w, z \), given by the triangle condition (T). If \( u_1 \sim z \) and \( u_2 \sim y \), then \( u_1 \sim u_2 \) (see [15], Lemma 3.1).

**Lemma 2.6.** Let \( X \) be an 8-located simplicial complex which satisfies the SD’\( n \)\( (O) \) property for some vertex \( O \), \( n \geq 2 \). Let \( v_1, v_2, v_3 \in B_{n-1}(O) \) be such that \( v_1 \sim v_2 \sim v_3 \). Let \( w_1, w_2 \in B_{n-2}(O) \) be such that \( w_1 \sim v_1, v_2 \) and \( w_2 \sim v_2, v_3 \), given by the triangle condition (T). Let \( p_1, p_2 \in B_n(O) \) be such that \( p_1 \sim v_1, v_2 \) and \( p_2 \sim v_2, v_3 \), given by the triangle condition (T). Then \( w_1 \sim w_2 \) if and only if \( p_1 \sim p_2 \) (see [16], Lemma 3.2).

2.4. **Classifying spaces with finite or virtually cyclic stabilisers.** The main goal of this section is, given a group \( G \), to describe a method of constructing a model for a classifying space with virtually cyclic stabilisers out of a model for a classifying space with finite stabilisers. The presented method is due to W. Lück and M. Weiermann ([19]). First we give the necessary definitions.

A collection of subgroups \( \mathcal{F} \) of a group \( G \) is called a *family* if it is closed under taking subgroups and conjugation by elements of \( G \). Two examples which will be of interest to us are the family \( \mathcal{FLN} \) of all finite subgroups, and the family \( \mathcal{VCY} \) of all virtually cyclic subgroups.

**Definition 2.7.** Given a group \( G \) and a family of its subgroups \( \mathcal{F} \), a *model for the classifying space* \( E_{\mathcal{F}}G \) is a G-CW-complex \( X \) such that for any subgroup \( H \subset G \) the fixed point set \( X^H \) is contractible if \( H \in \mathcal{F} \), and empty otherwise.

Let \( E_G \) denote \( E_{\mathcal{FLN}}G \) and let \( E_G \) denote \( E_{\mathcal{VCY}}G \).

A model for \( E_{\mathcal{F}}G \) exists for any group and any family. Any two models for \( E_{\mathcal{F}}G \) are G-homotopy equivalent (see [18]). However, general constructions always produce infinite dimensional models.

We will describe a method of constructing a finite dimensional model for \( E_G \) out of a model for \( E_G \) and appropriate models associated to infinite virtually cyclic subgroups of \( G \). If \( H \subset G \) is a subgroup and \( \mathcal{F} \) is a family of subgroups of \( G \), let \( \mathcal{F} \cap H \) denote the family of all subgroups of \( H \) which belong to the family \( \mathcal{F} \). More generally, if \( \phi : H \to G \) is a homomorphism, let \( \phi^*\mathcal{F} \) denote the smallest family of subgroups of \( H \) that contains \( \phi^{-1}(F) \) for all \( F \in \mathcal{F} \).

Consider the collection \( \mathcal{VCY} \setminus \mathcal{FLN} \) of infinite virtually cyclic subgroups of \( G \). It is not a family since it does not contain the trivial subgroup. Define an equivalence relation on \( \mathcal{VCY} \setminus \mathcal{FLN} \) by

\[
H_1 \sim H_2 \iff |H_1 \cap H_2| = \infty
\]
Let \([H]\) denote the equivalence class of \(H\), and let \([\mathcal{VCY} \subset \mathcal{FIN}]\) denote the set of equivalence classes. The group \(G\) acts on \([\mathcal{VCY} \subset \mathcal{FIN}]\) by conjugation, and for a class \([H]\) \(\in [\mathcal{VCY} \subset \mathcal{FIN}]\) define the subgroup \(N_G(H) \subseteq G\) to be the stabiliser of \([H]\) under this action, i.e. 
\[
N_G(H) = \{ g \in G \mid |g^{-1}Hg \cap H| = \infty \}
\]

The subgroup \(N_G(H)\) is called the commensurator of \(H\), since its elements conjugate \(H\) to the subgroup commensurable with \(H\). For \([H]\) \(\in [\mathcal{VCY} \subset \mathcal{FIN}]\) define the family \(\mathcal{G}[H]\) of subgroups of \(N_G(H)\) as follows
\[
\mathcal{G}[H] = \{ K \subset G \mid K \in \mathcal{VCY} \subset \mathcal{FIN}, [K] = [H] \} \cup \{ K \in \mathcal{FIN} \cap N_G[H] \}.
\]

**Definition 2.8.** A group \(G\) satisfies condition (C) if for every \(g, h \in G\) with \(|h| = \infty\) (infinite order) and any \(k, l \in \mathbb{Z}\) we have
\[
g h^k g^{-1} = h^l \implies |k| = |l|
\]

**Lemma 2.7.** Let \(K \subset N_G[H]\) be a finitely generated subgroup that contains some representative of \([H]\) and assume that the group \(G\) satisfies condition (C). Choose an element \(h \in H\) such that \([h]\) ) = \([H]\) (any element of infinite order has this property). Then there exists \(k \geq 1\), such that \((h^k)\) is normal in \(K\).

For the proof see [26], Lemma 2.6, page 8.

3. **Minimal displacement set for 8-located complexes with the SD'-property**

We study the structure of the minimal displacement set in an 8-located complex with the SD'-property. The notations introduced in section 2 hold in this section as well.

**Lemma 3.1.** Let \(h\) be a simplicial isometry without fixed points of a simplicial complex \(X\). We choose a vertex \(v \in \text{Min}_X(h)\) and a geodesic \(\alpha \subset X^{(1)}\) joining \(v\) with \(h(v)\). Consider a simplicial path \(\gamma : \mathbb{R} \to X\) (where \(\mathbb{R}\) is given a simplicial structure with \(\mathbb{Z}\) as the set of vertices) being the concatenation of geodesics \(h^n(\alpha), n \in \mathbb{Z}\). Then \(\gamma\) is a \(|h|\)-geodesic i.e. \(d(\gamma(a), \gamma(b)) = |a - b|\) if \(a, b\) are such integers that \(|a - b| \leq |h|\). In particular, \(\text{Im}(\gamma) \subset \text{Min}_X(h)\).

**Proof.** The proof is similar to the one given in [7], Fact 3.2. We prove the statement for \(|a - b| = |h|\) (this implies the general case). Then, by the construction of \(\gamma\), either \(\gamma(b) = h(\gamma(a))\) or \(\gamma(a) = h(\gamma(b))\). Thus we have \(d(\gamma(a), \gamma(b)) \geq |h|\). The opposite inequality follows from the fact that \(\gamma\) is a simplicial map. \(\square\)

Next we prove one of the paper’s main results.

**Theorem 3.2.** Let \(h\) be a (simplicial) isometry with no fixed points of an 8-located complex \(X\) with the SD'-property. Assume \(|h| > 3\). Then the 1-skeleton of \(\text{Min}_X(h)\) is isometrically embedded into \(X\).

**Proof.** The construction is similar to the one given in [7], Proposition 3.3 for systolic complexes.

Suppose the 1-skeleton of \(\text{Min}_X(h)\) is not isometrically embedded. Then there exist vertices \(v, w \in \text{Min}_X(h)\) such that no geodesic in \(X\) with endpoints \(v\) and \(w\) is contained in \(\text{Min}_X(h)\). Choose \(v\) and \(w\) so that \(d(v, w)\) minimal (clearly \(d(v, w) > 1\)). Join \(v\) with \(h(v)\), \(w\) with \(h(w)\) and \(v\) with \(w\) by geodesics \(\alpha, \beta\) and \(\gamma\), respectively.
Then $h(v)$ is joined with $h(w)$ by $h(\gamma)$. Note that $l(\alpha) = l(\beta) = |h|$, $l(\gamma) = l(h(\gamma)) > 1$.

According to Lemma 3.1, we have $\alpha, \beta \subset \text{Min}_X(h)$. Then, by minimality of $d(v, w)$, geodesics $\alpha$ and $\gamma$ intersect only at the endpoints. The same holds for the geodesics $\alpha$ and $h(\gamma)$, $\beta$ and $\gamma$, $\beta$ and $h(\gamma)$, respectively. Suppose there is a vertex $x \in \gamma \cap h(\gamma)$. Then $h(x) \in h(\gamma)$ and $h(x) \neq x$, since $h$ has no fixed points.

We may assume, not losing generality, that $h(v), x, h(x)$ and $h(w)$ lie on $h(\gamma)$ in this order. Then $d(x, h(x)) = d(h(v), h(x)) - d(h(v), x) = d(v, x) - d(h(v), x) \leq d(v, h(v)) = |h|$. So $x \in \text{Min}_X(h)$, contradicting the minimality of $d(v, w)$. Thus the geodesics $\alpha, \beta, \gamma, h(\gamma)$ either are pairwise disjoint but the endpoints or $\alpha$ and $\beta$ have nonempty intersection. In both situations we proceed as follows.

Let $y, x$ be adjacent vertices on $\gamma$ such that $d(y, v) = d(x, v) - 1$. It may happen that $y = v$ or $x = w$ but not simultaneously due to the fact that $d(v, w) > 1$.

The vertex $y$ is the last vertex of $\gamma$ such that $d(y, h(y)) = d(y, v) + d(v, h(v)) + d(h(v), h(y))$ (i.e. $y$ is the last vertex of $\gamma$ to be joined with $h(y)$ by the left of the cycle $\gamma * \beta * h(\gamma) * \alpha$). The vertex $x$ is the first vertex of $\gamma$ such that $d(x, h(x)) = d(x, w) + d(w, h(w)) + d(h(w), h(x))$ (i.e. $x$ is the first vertex of $\gamma$ to be joined with $h(x)$ by the right of the cycle $\gamma * \beta * h(\gamma) * \alpha$). Let $y' \in \gamma, v', v \sim v$ (possibly with $v' = y$).

There are two cases: either $l(\gamma) = 2$ or $l(\gamma) \geq 3$.

Assume first $l(\gamma) = 2$. Then $y = x$. Note that $d(v, h(y)) = d(w, h(y)) = |h| + 1$.

Note that $y \in B_{-1} + |h| (h(y))$. Because $v, w \in X_y \cap B_{1 + |h|} (h(y))$, the (V) condition of the SD'(h(y))-property implies that there exists a vertex $t \in X_y \cap B_{1 + |h|} (h(y))$ such that $t \sim v, w$.

Because $v, t \in B_{1 + |h|} (h(y))$, $v \sim t$, the (E) condition of the SD'(h(y))-property implies that there exists a vertex $p \in B_{1 + |h|} (h(y))$ such that $p \sim v, t$.

Because $t, w \in B_{1 + |h|} (h(y))$, $t \sim w$, the (E) condition of the SD'(h(y))-property implies that there exists a vertex $q \in B_{1 + |h|} (h(y))$ such that $q \sim t, w$.

Let $y \in B_{2 + |h|} (h(y))$, $v, t, w \in X_y \cap B_{1 + |h|} (h(y))$, $p, q \in B_{1 + |h|} (h(y))$, $p \sim v, t; q \sim t, w$. Then Lemma 2.4 implies that $p \sim q$.

Let $l \in \beta$ such that $w \sim l$. Because $q, t \in X_w \cap B_{|h|} (h(y))$, the (V) condition of the SD'(h(y))-property implies that there exists a vertex $r \in X_w \cap B_{|h|} (h(y))$ such that $r \sim q, l$.

Because $p, q \in B_{|h|} (h(y))$, $p \sim q$, the (E) condition of the SD'(h(y))-property implies that there exists a vertex $m \in B_{|h| - 1} (h(y))$ such that $m \sim p, q$.

Because $q, r \in B_{|h|} (h(y))$, $q \sim r$, the (E) condition of the SD'(h(y))-property implies that there exists a vertex $n \in B_{|h| - 1} (h(y))$ such that $n \sim q, r$.

Note that $t, w \in B_{1 + |h|} (h(y))$, $p, q, r \in B_{|h|} (h(y))$, $m, n \in X_y \cap B_{|h| - 1} (h(y))$, $p, q \in X_t, q, r \in X_w$. Then, because $t \sim w$, Lemma 2.4 implies that $m \sim n$.

Let $\delta$ be the tightening of the cycle $(y, v, p, m, n, r, w)$. Note that $|\delta| \leq 7$ and the cycle $\delta$ is full. Then, by 8-location, there is a vertex $f$ such that $\delta \subset X_f$. Hence $d(y, m) = 2$. But $y \in B_{2 + |h|} (h(y))$ while $m \in B_{|h| - 1} (h(y))$. Therefore $d(y, m) = 3$. This yields a contradiction.

For the rest of the proof let $l(\gamma) \geq 3$.

Note that either $d(v', h(v')) = |h| + 2$ or $d(v', h(v')) = |h| + 1$ or $d(v', h(v')) = |h|$. We analyze these cases below.

Case A. Suppose $d(v', h(v')) = |h| + 2$. So there do not exist vertices $a, b \in \alpha$ such that $v' \sim a \sim v, h(v') \sim b \sim h(v)$.
Case A.1. Assume $|\gamma| = 2k, k \in \mathbb{N}^*$. 

Assume w.l.o.g. $d(y, v) = k$. Then, due to the choice of the vertices $x$ and $y$, we have $d(x, y) = k - 1$. Recall $y$ is the last vertex of $\gamma$ to be joined with $h(y)$ by the left of the cycle $\gamma \ast \beta \ast h(\gamma) \ast \alpha$; $x$ is the first vertex of $\gamma$ to be joined with $h(x)$ by the right of the cycle $\gamma \ast \beta \ast h(\gamma) \ast \alpha$.

Let $z \in \gamma$ such that $z \sim y$, $d(z, v) = d(y, v) - 1$. Note that $d(z, h(y)) = d(z, h(y)) = 2k - 1 + |h|$. Hence $z, x \in X_y \cap B_{2k-1+|h|}(h(y))$. Then the (V) condition of the $SD'(h(y))$-property implies that there exists a vertex $t \sim x, z$ such that $t \in X_y \cap B_{2k-1+|h|}(h(y))$.

Note that $z, t \in B_{2k-1+|h|}(h(y))$ and $z \sim t$. Then, by the (E) condition of the $SD'(h(y))$-property, there exists $p \in B_{2k-2+|h|}(h(y))$ such that $p \sim z, t$.

Note that $t, x \in B_{2k-2+|h|}(h(y))$ and $t \sim x$. Therefore, by the (E) condition of the $SD'(h(y))$-property, there exists $q \in B_{2k-1+|h|}(h(y))$ such that $q \sim t, x$.

Note that $y \in B_{2k+|h|}(h(y))$, $z, t, x \in X_y \cap B_{2k-1+|h|}(h(y))$, $p, q \in X_y \cap B_{2k-2+|h|}(h(y))$, $p \sim q, q \sim x$. Then Lemma 2.5 implies that $p \sim q$.

If $|\gamma| = 3$, let $u = u$. If $|\gamma| > 3$, let $u \in \gamma$ such that $x \sim u, d(u, w) = d(u, v) - 1$.

Note that $d(q, h(y)) = d(u, h(y)) = 2k - 2 + |h|$. Hence $q, u \in X_x \cap B_{2k-2+|h|}(h(y))$.

Then the (V) condition of the $SD'(h(y))$-property implies that there exists a vertex $r \sim q, u$ such that $r \in X_x \cap B_{2k-2+|h|}(h(y))$.

Note that $p, q \in B_{2k-2+|h|}(h(y))$ and $p \sim q$. Then by the (E) condition of the $SD'(h(y))$-property, there exists $m \in B_{2k-3+|h|}(h(y))$ such that $m \sim p, q$.

Note that $q, r \in B_{2k-3+|h|}(h(y))$ and $q \sim r$. Then by the (E) condition of the $SD'(h(y))$-property, there exists $n \in B_{2k-3+|h|}(h(y))$ such that $n \sim q, r$.

Note that $t, x \in B_{2k+|h|}(h(y))$, $p, q, r, \in B_{2k-2+|h|}(h(y))$, $m, n \in X_y \cap B_{2k-3+|h|}(h(y))$, $p, q \in X_x$, $q, r \in X_x$. Then, because $t \sim x$, Lemma 2.6 implies that $m \sim n$.

Let $\delta$ be the tightening of the cycle $(y, z, p, m, n, r, x)$. Note that $|\delta| \leq 7$ and the cycle $\delta$ is full. Then, by $8$-location, there is a vertex $f$ such that $\delta \subset X_f$. Hence $d(y, m) = 2$. But $y \in B_{2k+|h|}(h(y))$ while $y \in B_{2k-3+|h|}(h(y))$. Therefore $d(y, m) = 3$. This yields a contradiction.

Case A.2. Assume $|\gamma| = 2k + 1, k \in \mathbb{N}^*$. 

Assume first $d(v, y) = k + 1$. Then $d(y, v) = k$. Note that $d(y, h(y)) = 2k + 1 + |h|$ by the left of the cycle $\gamma \ast \beta \ast h(\gamma) \ast \alpha$ and $d(y, h(y)) = 2k + |h|$ by the right of the cycle $\gamma \ast \beta \ast h(\gamma) \ast \alpha$. So the geodesic from $y$ to $h(y)$ passes by the right of the cycle $\gamma \ast \beta \ast h(\gamma) \ast \alpha$. But the point $y$ is chosen such that the geodesic from $y$ to $h(y)$ passes by the left of the cycle $\gamma \ast \beta \ast h(\gamma) \ast \alpha$. The situation $d(v, y) = k + 1$ is therefore not possible. So the only possible case is when $d(v, y) = k$. Therefore $d(y, w) = k + 1, d(x, w) = k$.

Let $z \in \gamma$ such that $z \sim y$, $d(z, v) = d(y, v) - 1$. Note that $d(x, h(x)) = d(z, h(x)) = 2k + |h|$. Because $x, z \in X_y \cap B_{2k+|h|}(h(x))$, the (V) condition of the $SD'(h(x))$-property implies that there exists a vertex $t \in X_y \cap B_{2k+|h|}(h(x))$ such that $t \sim x, z$.

Note that $z, t \in B_{2k+|h|}(h(x))$ and $z \sim t$. Then, by the (E) condition of the $SD'(h(x))$-property, there exists $p \in B_{2k-1+|h|}(h(y))$ such that $p \sim z, t$.

Note that $t, x \in B_{2k+|h|}(h(x))$ and $t \sim x$. Then, by the (E) condition of the $SD'(h(x))$-property, there exists $q \in B_{2k-1+|h|}(h(x))$ such that $q \sim t, x$.

Note that $y \in B_{2k+1+|h|}(h(x))$, $z, t, x \in X_y \cap B_{2k+1+|h|}(h(x))$, $p, q \in X_x \cap B_{2k-1+|h|}(h(x))$, $p \sim q, q \sim x$. Then Lemma 2.5 implies that $p \sim q$.
If $|\gamma| > 5$, let $l \in \gamma$ such that $z \sim l$, $d(l, v) = d(z, v) - 1$. If $|\gamma| \in \{3, 5\}$, then $l \in \alpha$ such that $z \sim l$, $d(l, v) = d(z, v) - 1$. Note that $d(l, h(x)) = d(p, h(x)) = 2k - 1 + |h|$. Because $l, p \in X_2 \cap B_{2k-1+|h|}(h(x))$, the (V) condition of the SD$^\prime$(h(x))-property implies that there exists a vertex $s \in X_2 \cap B_{2k-1+|h|}(h(x))$ such that $s \sim l, p$.

Because $l, s \in B_{2k-1+|h|}(h(x))$, $l \sim s$, the (E) condition of the SD$^\prime$(h(x))-property implies that there exists a vertex $m \in B_{2k-2+|h|}(h(x))$ such that $m \sim l, s$.

Because $s, p \in B_{2k-1+|h|}(h(x))$, $s \sim p$, the (E) condition of the SD$^\prime$(h(x))-property implies that there exists a vertex $n \in B_{2k-2+|h|}(h(x))$ such that $n \sim s, p$.

Because $p, q \in B_{2k-1+|h|}(h(x))$, $p \sim q$, the (E) condition of the SD$^\prime$(h(x))-property implies that there exists a vertex $r \in B_{2k-2+|h|}(h(x))$ such that $r \sim p, q$.

Note that $z \in B_{2k+1+|h|}(h(x))$, $l, s, p \in X_2 \cap B_{2k-1+|h|}(h(x))$, $m, n \in X_2 \cap B_{2k-2+|h|}(h(x))$, $m \sim l, n \sim p$. Then Lemma 2.4 implies that $m \sim n$.

Note that $z, t \in B_{2k+1+|h|}(h(x))$, $s, p, q \in B_{2k-1+|h|}(h(x))$, $n, r \in X_3 \cap B_{2k-2+|h|}(h(x))$, $s, p \in X_3$, $p, q \in X_3$, $n \sim s, r \sim q$. Then, because $z \sim t$, Lemma 2.4 implies that $n \sim r$.

Let $\delta$ be the tightening of the cycle $(y, z, l, m, n, r, q, x)$. Note that $|\delta| \leq 8$ and the cycle $\delta$ is full. Then, by 8-location, there is a vertex $f$ such that $\delta \subset X_f$.

Hence $d(y, m) = 2$. But $y \in B_{2k+1+|h|}(h(y))$ while $m \in B_{2k-2+|h|}(h(y))$. Therefore $d(y, m) = 3$. This yields a contradiction.

In conclusion, we have $d(v', h(v')) \neq |h| + 2$. This completes case A.

Case B. There exists a vertex $x \in \alpha$, $v \sim a \sim v'$. Suppose $d(v', h(v')) = |h| + 1$.

Case B.1. Assume $|\gamma| = 2k, k \in \mathbb{N}^*$.

Assume w.l.o.g. $d(y, v) = k$. Then $d(y, h(y)) = 2k - 1 + |h|$. Due to the choice of the vertices $x, y \in \gamma$, we have $d(x, w) = k - 1$. Recall $y$ is the last vertex of $\gamma$ to be joined with $h(y)$ by the left of the cycle $\gamma*\beta*h(\gamma)*\alpha$: $x$ is the first vertex of $\gamma$ to be joined with $h(x)$ by the right of the cycle $\gamma*\beta*h(\gamma)*\alpha$.

Note that $d(y, h(y)) = d(x, h(y)) = 2k - 1 + |h|$. Then $y, x \in B_{2k-1+|h|}(h(y))$.

Because $y \sim x$, the (E) condition of the SD$^\prime$(h(y))-property implies that there exists a vertex $t \in B_{2k-2+|h|}(h(y))$ such that $t \sim y, x$.

Let $l \in \gamma$ such that $x \sim l$, $d(l, w) = d(x, w) - 1$. Note that $t, l \in X_2 \cap B_{2k-2+|h|}(h(y))$. Then the (V) condition of the SD$^\prime$(h(y))-property implies that there exists a vertex $m \in X_2 \cap B_{2k-2+|h|}(h(y))$ such that $m \sim t, l$.

Because $t, m \in B_{2k-2+|h|}(h(y))$, $t \sim m$, the (E) condition of the SD$^\prime$(h(y))-property implies that there exists a vertex $r \in B_{2k-3+|h|}(h(y))$ such that $r \sim t, m$.

Because $m, l \in B_{2k-2+|h|}(h(y))$, $m \sim l$, the (E) condition of the SD$^\prime$(h(y))-property implies that there exists a vertex $s \in B_{2k-3+|h|}(h(y))$ such that $s \sim m, l$.

Note that $x \in B_{2k-1+|h|}(h(y))$, $t, m, l \in X_2 \cap B_{2k-2+|h|}(h(y))$ and $r, s \in X_m \cap B_{2k-3+|h|}(h(y))$, $r \sim t, s \sim l$. Then Lemma 2.4 implies that $r \sim s$.

If $|\gamma| = 4$, then let $u = w$. If $|\gamma| > 4$, let $u \in \gamma$ such that $l \sim u$, $d(u, w) = d(l, w) - 1$. Note that $s, u \in X_3 \cap B_{2k-3+|h|}(h(y))$. Then the (V) condition of the SD$^\prime$(h(y))-property implies that there exists a vertex $p \in X_l \cap B_{2k-3+|h|}(h(y))$ such that $p \sim s, u$.

Because $r, s \in B_{2k-3+|h|}(h(y))$, $r \sim s$, the (E) condition of the SD$^\prime$(h(y))-property implies that there exists a vertex $c \in B_{2k-4+|h|}(h(y))$ such that $c \sim r, s$.

Because $s, p \in B_{2k-3+|h|}(h(y))$, $s \sim p$, the (E) condition of the SD$^\prime$(h(y))-property implies that there exists a vertex $d \in B_{2k-4+|h|}(h(y))$ such that $d \sim s, p$.

Note that $m, l \in B_{2k-2+|h|}(h(y))$, $r, s, p \in B_{2k-3+|h|}(h(y))$, $c, d \in X_3 \cap B_{2k-4+|h|}(h(y))$, $r, s \in X_m$, $s, p \in X_l$. Then, because $m \sim l$, Lemma 2.4 implies that $c \sim d$. 


Let $\delta$ be the tightening of the cycle $(x, t, r, c, d, p, l)$. Note that $|\delta| \leq 7$ and the cycle $\delta$ is full. Then, by 8-location, there is a vertex $f$ such that $\delta \subset X_f$. Hence $d(x, c) = 2$. But $x \in B_{2k-1+|h|}(h(y))$ while $c \in B_{2k-4+|h|}(h(y))$. Therefore $d(x, c) = 3$. This yields a contradiction.

Case B.2. Assume $|\gamma| = 2k + 1, k \in \mathbb{N}^+$. Assume first $d(v, y) = k + 1$. Then $d(y, w) = k$. Note that $d(y, h(y)) = 2k + 1 + |h|$ by the left of the cycle $\gamma * \beta * h(\gamma) * \alpha$ and $d(y, h(y)) = 2k + |h|$ by the right of the cycle $\gamma * \beta * h(\gamma) * \alpha$. But the point $y$ is chosen such that the geodesic from $y$ to $h(y)$ passes by the right of the cycle $\gamma * \beta * h(\gamma) * \alpha$. The situation $d(v, y) = k + 1$ is therefore not possible. So the only possible case is when $d(v, y) = k$. Therefore $d(y, w) = k + 1$, $d(x, w) = k$.

Note that $d(x, h(x)) = d(y, h(x)) = 2k + |h|$. Hence $y, x \in B_{2k+|h|}(h(x))$. Then, since $x \sim y$, the (E) condition of the SD$^\prime(h(x))$-property implies that there exists a vertex $t \in B_{2k-1+|h|}(h(x))$ such that $t \sim y, x$.

If $|\gamma| = 3$, let $z \in \alpha, z \sim y$. If $|\gamma| > 3$, let $z \in \gamma$ such that $z \sim y, d(z, v) = d(y, v) - 1$. Note that $z, t \in X_y \cap B_{2k-1+|h|}(h(x))$. Then the (V) condition of the SD$^\prime(h(x))$-property implies that there exists a vertex $u \in X_y \cap B_{2k-1+|h|}(h(x))$ such that $u \sim z, t$.

Because $z, u \in B_{2k-1+|h|}(h(x)), z \sim u$, the (E) condition of the SD$^\prime(h(x))$-property implies that there exists a vertex $p \in B_{2k-2+|h|}(h(x))$ such that $p \sim z, u$.

Because $u, t \in B_{2k-1+|h|}(h(x)), u \sim t$, the (E) condition of the SD$^\prime(h(x))$-property implies that there exists a vertex $q \in B_{2k-2+|h|}(h(x))$ such that $q \sim u, t$.

Note that $y \in B_{2k+|h|}(h(x)), z, u, t \in X_y \cap B_{2k-1+|h|}(h(x))$ and $p, q \in X_u \cap B_{2k-2+|h|}(h(x))$, $p \sim z, q \sim t$. Then Lemma 2.4 implies that $p \sim q$.

If $|\gamma| = 3$, let $l \in \alpha, l \sim z$. If $|\gamma| = 5$, let $l = v$. If $|\gamma| > 5$, let $l \in \gamma$ such that $l \sim z, d(l, v) = d(z, v) - 1$. Note that $l, p \in X_z \cap B_{2k-2+|h|}(h(x))$. Then, the (V) condition of the SD$^\prime(h(x))$-property implies that there exists a vertex $r \in X_z \cap B_{2k-2+|h|}(h(x))$ such that $r \sim l, p$.

Because $r, p \in B_{2k-2+|h|}(h(x)), r \sim p$, the (E) condition of the SD$^\prime(h(x))$-property implies that there exists a vertex $n \in B_{2k-3+|h|}(h(x))$ such that $n \sim r, p$.

Because $p, q \in B_{2k-2+|h|}(h(x)), p \sim q$, the (E) condition of the SD$^\prime(h(x))$-property implies that there exists a vertex $c \in B_{2k-3+|h|}(h(x))$ such that $c \sim p, q$.

Note that $z, n \in B_{2k-1+|h|}(h(x)), r, p, q \in B_{2k-2+|h|}(h(x)), r, p \in X_z, p, q \in X_u, n, c \in X_p \cap B_{2k-3+|h|}(h(x))$. Then, because $z \sim u$, Lemma 2.4 implies that $n \sim c$.

Let $\delta$ be the tightening of the cycle $(y, z, r, n, c, q, t)$. Note that $|\delta| \leq 7$ and the cycle $\delta$ is full. Then, by 8-location, there is a vertex $f$ such that $\delta \subset X_f$. Hence $d(y, n) = 2$. But $y \in B_{2k+|h|}(h(x))$ while $n \in B_{2k-3+|h|}(h(x))$. Therefore $d(y, n) = 3$. This yields a contradiction.

In conclusion we have $d(v', h(v')) \neq |h| + 1$. This completes case B.

Case C. There exists a vertex $b \in \alpha$ such that $h(v) \sim b \sim h(v')$. Suppose $d(v', h(v')) = |h| + 1$.

Case C.1. Assume $|\gamma| = 2k, k \in \mathbb{N}^+$. Assume w.l.o.g. $d(y, v) = k$. Then, due to the choice of the vertices $y, x \in \gamma$, we have $d(x, w) = k - 1$. Recall $y$ is the last vertex of $\gamma$ to be joined with $h(y)$ by the left of the cycle $\gamma * \beta * h(\gamma) * \alpha$; $x$ is the first vertex of $\gamma$ to be joined with $h(x)$ by the right of the cycle $\gamma * \beta * h(\gamma) * \alpha$. 


Note that $d(h(y), y) = d(h(x), y) = 2k - 1 + |h|$. Because $h(y), h(x) \in B_{2k-1+|h|}(y)$ and $h(y) \sim h(x)$, the (E) condition of the SD'(y)-property implies that there exists a vertex $t \in B_{2k-2+|h|}(y)$ such that $t \sim h(y), h(x)$.

If $|\gamma| = 3$, let $l = h(w)$. If $|\gamma| > 3$, let $l \in h(\gamma)$ such that $l \sim h(x), d(l, h(w)) = d(h(x), h(w)) - 1$. Note that $t, l \in X_{h(x)} \cap B_{2k-2+|h|}(y)$. Then the (V) condition of the SD'(y)-property implies that there exists a vertex $u \in X_{h(x)} \cap B_{2k-2+|h|}(y)$ such that $u \sim t, l$.

Note that $t, u \in B_{2k-2+|h|}(y), t \sim u$, the (E) condition of the SD'(y)-property implies that there exists a vertex $p \in B_{2k-3+|h|}(y)$ such that $p \sim t, u$.

Note that $u, l \in B_{2k-2+|h|}(y), u \sim l$, the (E) condition of the SD'(y)-property implies that there exists a vertex $q \in B_{2k-3+|h|}(y)$ such that $q \sim u, l$.

Note that $h(x) \in B_{2k-1+|h|}(y), t, u, l \in X_{h(x)} \cap B_{2k-2+|h|}(y)$ and $p, q \in X_u \cap B_{2k-3+|h|}(y), p \sim t, q \sim l$. Then Lemma 2.5 implies that $p \sim q$.

If $|\gamma| = 4$, then $z \in B \sim l$. If $|\gamma| = 6$, then $z = w$. If $|\gamma| > 6$, let $z \in h(\gamma)$ such that $z \sim l, d(z, h(w)) = d(l, h(w)) - 1$. Note that $q, z \in X_l \cap B_{2k-3+|h|}(y)$. Then the (V) condition of the SD'(y)-property implies that there exists a vertex $n \in X_l \cap B_{2k-3+|h|}(y)$ such that $n \sim q, z$.

Because $p, q \in B_{2k-3+|h|}(y), p \sim q$, the (E) condition of the SD'(y)-property implies that there exists a vertex $r \in B_{2k-4+|h|}(y)$ such that $r \sim p, q$.

Because $q, n \in B_{2k-3+|h|}(y), q \sim n$, the (E) condition of the SD'(y)-property implies that there exists a vertex $c \in B_{2k-4+|h|}(y)$ such that $c \sim q, n$.

Note that $u, l \in B_{2k-1+|h|}(y), p, q, n \in B_{2k-3+|h|}(y), p, q \in X_u, q, n \in X_l, r, c \in X_q \cap B_{2k-4+|h|}(y), p \sim r, n \sim c$. Then, because $u \sim l$, Lemma 2.4 implies that $r \sim c$.

Let $\delta$ be the tightening of the cycle $(h(x), t, p, r, c, n, l)$. Note that $|\delta| \leq 7$ and the cycle $\delta$ is full. Then, by $8$-location, there is a vertex $f$ such that $\delta \subset X_f$. Hence $d(h(x), r) = 2$. But $h(x) \in B_{2k-1+|h|}(y)$ while $r \in B_{2k-4+|h|}(y)$. Therefore $d(h(x), r) = 3$. This yields a contradiction.

Case C.2. Assume $|\gamma| = 2k + 1, k \in \mathbb{N}^*$. Assume first $d(v, y) = k + 1$. Then $d(y, w) = k$. Note that $d(y, h(y)) = 2k + 1 + |h|$ by the left of the cycle $\gamma \star \beta \star h(\gamma) \star \alpha$ and $d(y, h(y)) = 2k + |h|$ by the right of the cycle $\gamma \star \beta \star h(\gamma) \star \alpha$. So the geodesic from $y$ to $h(y)$ passes by the right of the cycle $\gamma \star \beta \star h(\gamma) \star \alpha$. But the point $y$ is chosen such that the geodesic from $y$ to $h(y)$ passes by the left of the cycle $\gamma \star \beta \star h(\gamma) \star \alpha$. The situation $d(v, y) = k + 1$ is therefore not possible. So the only possible case is when $d(v, y) = k$. Therefore $d(y, w) = k + 1, d(x, w) = k$.

Note that $d(h(x), y) = d(h(x), x) = 2k + |h|$. Because $h(y), h(x) \in B_{2k+|h|}(x)$ and $h(y) \sim h(x)$, the (E) condition of the SD'(x)-property implies that there exists a vertex $t \in B_{2k-1+|h|}(x)$ such that $t \sim h(y), h(x)$.

If $|\gamma| = 3$, let $l = h(w)$. If $|\gamma| > 3$, let $l \in h(\gamma)$ such that $l \sim h(x), d(l, h(w)) = d(h(x), h(w)) - 1$. Note that $t, l \in X_{h(x)} \cap B_{2k-1+|h|}(x)$. Then the (V) condition of the SD'(x)-property implies that there exists a vertex $u \in X_{h(x)} \cap B_{2k-1+|h|}(x)$ such that $u \sim t, l$.

Because $t, u \in B_{2k-1+|h|}(x), t \sim u$, the (E) condition of the SD'(x)-property implies that there exists a vertex $p \in B_{2k-2+|h|}(x)$ such that $p \sim t, u$.

Because $u, l \in B_{2k-1+|h|}(x), u \sim l$, the (E) condition of the SD'(x)-property implies that there exists a vertex $q \in B_{2k-2+|h|}(x)$ such that $q \sim u, l$. 
Note that \( h(x) \in B_{2k+|h|}(x) \), \( t, u, l \in X_{h(x)} \cap B_{2k+1+|h|}(x) \), \( p, q \in X_u \cap B_{2k−2+|h|}(x) \), \( p \sim t, q \sim l \). Then Lemma 2.3 implies that \( p \sim q \).

If \( |\gamma| = 3 \), let \( z \in \beta \). If \( |\gamma| = 5 \), let \( z = h(w) \). If \( |\gamma| > 5 \), let \( z \in h(\gamma) \) such that \( z \sim l, d(z, h(w)) = d(l, h(w)) − 1 \). Note that \( q, z \in X_l \cap B_{2k−2+|h|}(x) \).

Then the (V) condition of the SD'\( (x)\)-property implies that there exists a vertex \( s \in X_l \cap B_{2k−2+|h|}(x) \) such that \( s \sim q, z \).

Because \( p, q \in B_{2k−2+|h|}(x) \), \( p \sim q \), the (E) condition of the SD'\( (x)\)-property implies that there exists a vertex \( c \in B_{2k−3+|h|}(x) \) such that \( c \sim p, q \).

Because \( q, s \in B_{2k−2+|h|}(x) \), \( q \sim s \), the (E) condition of the SD'\( (x)\)-property implies that there exists a vertex \( d \in B_{2k−3+|h|}(x) \) such that \( d \sim q, s \).

Note that \( u, l \in B_{2k−1+|h|}(x) \), \( p, q, s \in B_{2k−2+|h|}(x) \), \( p, q \in X_u \), \( s \in X_l \), \( c, d \in X_q \cap B_{2k−3+|h|}(y) \). Then, because \( u \sim l \), Lemma 2.4 implies that \( c \sim d \).

Let \( \delta \) be the tightening of the cycle \( (h(x), t, p, c, d, s, l) \). Note that \( |\delta| \leq 7 \) and the cycle \( \delta \) is full. Then, by 8-location, there is a vertex \( f \) such that \( \delta \subset X_f \).

Hence \( d(h(x), e) = 2 \). But \( h(x) \in B_{2k+|h|}(x) \) while \( c \in B_{2k−3+|h|}(x) \). Therefore \( d(h(x), e) = 3 \). This yields a contradiction.

In conclusion we have \( d(v', h(v')) \neq |h| + 1 \). This completes case C.

Case D. There exist vertices \( a, b \in \alpha \) such that \( v' \sim a \sim v, h(v') \sim b \sim h(v) \).

Then \( d(v', h(v')) = |h| \) which yields a contradiction.

\[ \square \]

**Lemma 3.3.** Let \( h \) be a (simplicial) isometry with no fixed points of an 8-located complex \( X \) with the SD'\( (x)\)-property. Let \( Y = \text{Min}_X(h) \). Then \( Y = \text{Min}_Y(h) \).

**Proof.** Let \( x \in X \) such that \( d_X(x, h(x)) = |h| \). Then \( x \in Y \). Let \( y = h(x) \in Y \) such that \( d_Y(y, h(y)) = |h| \). So \( y \in \text{Min}_Y(h) \). Since \( d_X(x, h(x)) = d_Y(y, h(y)) \), we have \( Y = \text{Min}_Y(h) \).

\[ \square \]

The construction of a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on an 8-located complex with the SD'\( (x)\)-property relies on the following result.

**Theorem 3.4.** Let \( h \) be a (simplicial) isometry having no fixed points with \( |h| > 3 \), of an 8-located complex \( X \) with the SD'\( (x)\)-property. Then the set \( \text{Min}_X(h) \) is Gromov hyperbolic. In particular, \( \text{Min}_X(h) \) is systolic.

**Proof.** Theorem 2.3 implies that \( X \) is Gromov hyperbolic. Let \( Y = \text{Min}_X(h) \). Lemma 3.3 implies that \( Y = \text{Min}_Y(h) \). The proof is by contradiction. Suppose there exists a \( k \)-wheel \( \gamma = (z; x_1, \ldots, x_k) \subset Y \), \( 5 \leq k \leq 6 \). According to Lemma 3.2 the 1-skeleton of \( Y \) is isometrically embedded into \( X \). Then the \( k \)-wheel \( \gamma \) also belongs to \( X \). Due to the Gromov hyperbolicity of \( X \), this yields a contradiction. So there does not exist any \( k \)-wheel in \( Y \), \( 5 \leq k \leq 6 \). This implies that \( Y \) is Gromov hyperbolic. In particular, \( Y \) is systolic.

\[ \square \]

The following results on 8-located complexes with the SD'\( (x)\)-property are immediate consequences of the fact that the minimal displacement set of a nonelliptic isometry acting on such complex is a systolic subcomplex and it embeds isometrically into the complex. Their systolic analogues, also given below, imply these similarities. We shall refer to these results when constructing a low-dimensional
Theorem 3.5. Let $h$ be a nonelliptic simplicial isometry of a uniformly locally finite systolic complex $X$. Then there is an $h^n$-invariant geodesic for some $n \geq 1$.

For the proof see [7], Theorem 3.5, page 46.

Theorem 3.6. Let $h$ be a nonelliptic simplicial isometry of a uniformly locally finite 8-located complex $X$ with the SD'-property. Assume $|h| > 3$. Then in $X$ there is an $h^n$-invariant geodesic for some $n \geq 1$.

Proof. Let $Y = \text{Min}_X(h)$. Since $|h| > 3$, Theorem 3.4 implies that $Y$ is systolic. Then, by Theorem 3.5, there is in $Y$ an $h^n$-invariant geodesic $\gamma$ for some $n \geq 1$. Since, by Theorem 3.2, $Y^{(1)}$ is isometrically embedded into $X$, the $h^n$-invariant geodesic $\gamma$ also belongs to $X$. This completes the proof.

Theorem 3.7. Let $h$ be a simplicial isometry of a uniformly locally finite systolic complex $X$. Then either there is an $h$-invariant simplex (elliptic case) or there is an $h$-invariant thick geodesic (hyperbolic case).

For the proof see [7], Theorem 3.8, page 49.

Theorem 3.8. Let $h$ be a simplicial isometry of a uniformly locally finite 8-located complex $X$ with the SD'-property. Assume $|h| > 3$. Then either there is an $h$-invariant simplex (elliptic case) or there is an $h$-invariant thick geodesic (hyperbolic case).

Proof. Let $Y = \text{Min}_X(h)$. Since $|h| > 3$, Theorem 3.4 implies that $Y$ is systolic. Then, by Theorem 3.7, in $Y$ either there is an $h$-invariant simplex (elliptic case) or there is an $h$-invariant thick geodesic (hyperbolic case). Since, by Theorem 3.2, $Y^{(1)}$ is isometrically embedded into $X$, this $h$-invariant simplex (elliptic case), respectively this $h$-invariant thick geodesic (hyperbolic case) also belongs to $X$.

Theorem 3.9. Let $h$ be a nonelliptic simplicial isometry of a uniformly locally finite systolic complex $X$. If there exists an $h^n$-invariant geodesic in $X$, then for any vertex $x \in \text{Min}_X(h^n) \subset X$, there exists an $h^n$-invariant geodesic passing through $x$.

For the proof see [7], Remark page 48.

Theorem 3.10. Let $h$ be a nonelliptic simplicial isometry of a uniformly locally finite 8-located complex $X$ with the SD'-property. Assume $|h| > 3$. If there exists an $h^n$-invariant geodesic in $X$, then for any vertex $x \in \text{Min}_X(h^n) \subset X$, there exists an $h^n$-invariant geodesic passing through $x$.

Proof. Let $Y = \text{Min}_X(h)$. Theorem 3.4 implies that $Y$ is systolic. According to Theorem 3.5 in $Y$ (and then, by Theorem 3.6, also in $X$) there exists an $h^n$-invariant geodesic for some $n \geq 1$. Hence, by Theorem 3.9, for any vertex $x \in \text{Min}_X(h^n) \subset Y$, there exists an $h^n$-invariant geodesic passing through $x$. Since, by Theorem 3.2, $Y^{(1)}$ is isometrically embedded into $X$, this implies that for any vertex $x \in \text{Min}_X(h^n) \subset Y \subset X$, there exists an $h^n$-invariant geodesic passing through $x$. 

□
4. Classifying spaces with virtually cyclic stabilisers for 8-located groups

In this section we construct a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on an 8-located complex with the $SD’$-property. The proof relies on the fact that the minimal displacement set of such complex is a systolic subcomplex that embeds isometrically into the complex. We start by giving the systolic analogue of one of the main results the construction will be based on.

**Theorem 4.1.** Let $X$ be a systolic locally finite simplicial complex. For a hyperbolic isometry $h$ whose minimal displacement set is a union of axes (that is, for $h$ satisfying (2.1)) and $|h| > 3$, the graph of axes $(Y(h), d_{Y(h)})$ is quasi-isometric to a simplicial tree.

For the proof see [26], Corollary 4.6, page 21.

**Theorem 4.2.** Let $X$ be a locally finite 8-located complexes with the $SD’$-property. For a hyperbolic isometry $h$ whose minimal displacement set is a union of axes (that is, for $h$ satisfying (2.1)) and $|h| > 3$, the graph of axes $(Y(h), d_{Y(h)})$ is quasi-isometric to a simplicial tree.

**Proof.** Let $Y = \text{Min}_X(h)$. Lemma 3.3 implies that $Y = \text{Min}_Y(h)$. If there do not exist $h$-invariant geodesics in $X$, take an $h^n$-invariant geodesic in $X$, $n > 1$ (see Lemma 4.9). Assume there exist $h$-invariant geodesics in $X$. These geodesics are also in $Y$ because, according to (2.1),

$$Y = \text{span}(\bigcup |\gamma| \text{ is an } h\text{-invariant geodesic }).$$

Theorems 3.2 and 3.3 imply that $Y$ is systolic and $Y^{(1)}$ embeds isometrically into $X$. Then, by Theorem 4.1, the result follows.

For the rest of the section, let $G$ be a group acting properly discontinuously on a uniformly locally finite 8-located complex $X$ with the $SD’$-property of dimension $d$.

**Theorem 4.3.** The systolic complex $X$ is a model for $\mathbb{E}G$.

For the proof see [5], Theorem E.

In order to construct models for the commensurators $N_G[H]$, first we show that the group $G$ satisfies condition (C). Using this, in every finitely generated subgroup $K \subseteq N_G[H]$ that contains $H$ we find a suitable normal cyclic subgroup. As shown in [26] for systolic complexes, the quotient group acts properly on a quasi-tree.

**Lemma 4.4.** The group $G$ satisfies condition (C).

**Proof.** The proof is similar to the one given in [26] (Lemma 5.2). Take arbitrary $g, h \in G$ such that $|h| = \infty$, and assume there are $k, l \in \mathbb{Z}$ such that $g^{-1} h^k g = h^l$.

We have to show that $|k| = |l|$. Since the action of $G$ on $X$ is proper, the element $h$ acts as a hyperbolic isometry. By Theorem 4.2, there is in $X$ an $h^n$-invariant geodesic for some $n \geq 1$. We get the claim by considering the following sequence of equalities for the translation length:

$$|k| \cdot |h^n| = |h^{k,n}| = |g^{-1} \cdot h^{n,k} \cdot g| = |h^{k \pm n}| = |l| \cdot |h^n|.$$
The first and the last of the equalities follow from the fact that the translation length of an element can be measured on an invariant geodesic, the second one is an easy calculation and the third one is straightforward. □

**Lemma 4.5.** Let $K$ be a finitely generated subgroup of $G$, and $h \in K$ a hyperbolic isometry satisfying (2.1), such that $\langle h \rangle$ is normal in $K$. Then the proper action of $G$ on $X$ induces a proper action of $K/\langle h \rangle$ on the graph of axes $Y(h)$.

**Proof.** The proof is similar to the one given in [26], Lemma 5.3 in the systolic case. □

**Lemma 4.6.** Let $h$ be a hyperbolic isometry of an 8-located complex with the $SD^\prime$-property $X$. Assume that $|h| > 3$. Then if $h$ satisfies (2.1) then so does $h^n$ for any $n \in \mathbb{Z} \setminus \{0\}$.

**Proof.** The result follows by Lemma 3.10 and the fact that an $h$-invariant geodesic is $h^n$-invariant. □

**Lemma 4.7.** Let $K$ be a finitely generated subgroup of $NG[H]$ that contains $H$. Then there is a short exact sequence

\[ 0 \rightarrow \langle h \rangle \rightarrow K \rightarrow K/\langle h \rangle \rightarrow 0, \]

such that $h \in H$ is of infinite order and the group $K/\langle h \rangle$ is virtually free.

**Proof.** The proof is similar to the one given in [26], Lemma 5.4. Choose an element of infinite order $\tilde{h} \in H$ satisfying the following two conditions:

1. the set $\text{Min}_X(\tilde{h})$ is the union of axes (see (2.1));
2. the translation length $|\tilde{h}| > 3$.

Both conditions above can be ensured by rising $\tilde{h}$ to a sufficiently large power. Indeed, by Lemma 3.9 there exists $n \geq 1$ such that $\tilde{h}^n$ satisfies the first condition above. If $|\tilde{h}^n| \leq 3$ then replace it with $\tilde{h}^{4n}$. The element $\tilde{h}^{4n}$ satisfies both conditions. If an element satisfies the conditions above then, by Lemma 4.6, so does any of its powers. Since $G$ satisfies condition (C), by Lemma 2.7 there exists an integer $k \geq 1$ such that $\langle \tilde{h}^k \rangle$ is normal in $K$.

Put $h = \tilde{h}^k$. By Lemma 4.6, the group $K/\langle h \rangle$ acts properly by isometries on the graph of axes $(Y(h), d_Y(h))$, which, by Theorem 4.2 is a quasi-tree. In conclusion the group $K/\langle h \rangle$ is virtually free. □

The proofs of the next results are similar to the one given for systolic complexes in [26] (see Lemma 5.5, Theorem C).

**Lemma 4.8.** For every $[H] \in \mathcal{VCY} \setminus \mathcal{FIN}$ there exist:

1. a 2-dimensional model for $EG[H]NG[H]$;
2. a 3-dimensional model for $EN_G[H]$.

**Theorem 4.9.** There exists a model for $EG$ of dimension

\[ \dim EG = \begin{cases} 
  d+1, & \text{if } d \leq 3, \\
  d, & \text{if } d \geq 4.
\]
References

[1] B. Brešar, J. Chalopin, V. Chepoi, T. Gologranc and D. Osajda, Bucolic complexes Adv. Math., 243, 2013, pp. 127-167.
[2] M. Bridson, A. Haefliger, Metric spaces of non-positive curvature Springer, New York, 1999.
[3] V. Chepoi, Graphs of some CAT(0) complexes, Adv. in Appl. Math., 24, 2000, no. 2, pp. 125-179.
[4] J. Chalopin, V. Chepoi, H. Hirai and D. Osajda, Weakly modular graphs and nonpositive curvature, preprint, arXiv:1409.5892, 2014.
[5] V. Chepoi and D. Osajda, Dismantlability of weakly systolic complexes and applications, Trans. Amer. Math. Soc., 367, 2015, no. 2, pp. 1247-1272.
[6] R. Diestel, Graph theory, Graduate Texts in Mathematics, 173, Springer, Heidelberg, 2010.
[7] T. Elsener, Isometries of systolic spaces, Fundamenta Mathematicae, 204, 2009, 39-55.
[8] T. Elsener, Flats and the flat torus theorem for systolic spaces, Geometry and Topology, 13, 2009, 661-698.
[9] R. Gómez-Ortells, Compactly supported cohomology of systolic 3-pseudomanifolds, Colloq. Math., 135, 2014, no. 1, pp. 103-112.
[10] M. Gromov, Hyperbolic groups, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987, pp. 75-263.
[11] I.-C. Lazăr, Systolic simplicial complexes are collapsible, Bull. Math. Soc. Sci. Math. Roumanie,, 56(104), 2, 2013, pp. 229-236.
[12] I.-C. Lazăr, A combinatorial negative curvature condition implying Gromov hyperbolicity, preprint, arXiv:1501.05487v1, 2015.
[13] I.-C. Lazăr, Minimal disc diagrams of 5/9-simplicial complexes, Michigan Math. J., 69, 2020, pp. 793-829.
[14] W. Lück, Survey on classifying spaces for families of subgroups, L. Bartholdi, T. Ceccherini-Silberstein, T. Smirnova-Nagnibeda and A. Zuk (eds.), Infinite groups: geometric, combinatorial and dynamical aspects. (Gaeta, 2003.) Progress in Mathematics, 248, Birkhäuser, Basel, 2005, 269-322.
[15] T. Prytuła, Inﬁnite systolic groups are not torsion, Colloquium Mathematicum 153, 2018, no. 2, pp. 169-194.
