The lattice world, quantum foam and the universe as a metamaterial

The use of black holes as hyper-resolving microscopes to probe the fine structure of space

David T. Crouse

1 Introduction

In this paper, the concepts of the fundamental length of space (i.e., the Planck length \( l_p = \sqrt{\hbar G/c^3} = 1.62 \times 10^{-35} \text{ m} \)), Werner Heisenberg’s “Lattice World” and John Wheeler’s quantum foam theory will be combined, resulting in a universe-wide gravity crystal (GC) that permeates all space (Fig. 1). The GC will manifest itself in the gravitational and inertial properties of point-like particles that have masses on the order of the Planck mass \( m_p = \sqrt{\hbar c/G} = 2.18 \times 10^{-8} \text{ kg} \). By using well-developed concepts and techniques from the fields of quantum electrodynamics and solid-state physics, it will be shown that the inertial masses \( m_i \) of black holes become different than their gravitational masses \( m_g \) when they have particular momenta. For particular ranges of momenta, \( m_i \) can have values either much larger or smaller than \( m_g \), be near zero, or even be negative. The possibility of non-equality of inertial and gravitational masses of particles—a violation of Einstein’s equivalence principle in appearance only—is not a surprising result given the fact that the particles are traveling through a material, namely the GC. Such a phenomenon occurs for electrons in regular crystals (e.g., silicon, germanium) and is a well understood and experimentally verified phenomenon. As will be discussed in this paper, the momentum-dependent inertial masses of black holes may have a connection to dark matter, dark energy and other anomalies observed in the motion of black holes and galaxies. Also discussed in this paper is how one can use black holes as hyper-resolving microscopes to study the properties of space at length scales comparable to \( l_p \).

This paper is organized as follows. In Sect. 2, a short summary is given of relevant episodes of the 2500-year-long history of the study of the fine structure of space. In Sect. 3, an introduction is given for the concept of
The crystal is cubic in structure with a lattice constant of $l_p$ (EPM) and a relativistic energy band diagrams in crystals will be used, including Democritus’s atomic hypothesis of materials [1–3]. Starting minimized) or continuous (i.e., infinitely divisible) is as old as 

The discussion of whether space itself is discrete (i.e., atomic) or continuous (i.e., infinitely divisible) is as old as Democritus’s atomic hypothesis of materials [1–3]. Starting in the fifth century BC, Parmenides and his student Zeno of Elea inquired about the nature of space, change, motion and plurality [4]. Their inquiries resulted in the famous Zeno’s paradoxes that address motion, time and change. Many later philosophers and physicists have tried to resolve these and other paradoxes concerning time and space; these philosophers include Rene Descartes, Leonhard Euler, Gottfried Leibniz, George Berkeley, David Hume, David Hilbert, and many others have contributed to this discussion, as described by Amit Hagar in [1]. However, despite over twenty-five centuries of philosophical musings on the matter by these and other philosophers, Hagar concludes that logic alone cannot disqualify either the discrete picture or the continuous picture of space [5].

If space–time is discretized, it is generally believed that the most probable atom of space (i.e., the smallest allowable length of space) is $l_p$, and the atom of time is $t_p = l_p/c = 5.39 \times 10^{-44}$ s [1]. Other Planck values include: the Planck mass $m_p$ being the most mass a point particle can have, Planck charge $q_p = \sqrt{\hbar \epsilon}$ (in Gaussian units), Planck temperature $T_p = m_p c^2/k_B$ (where $k_B$ is Boltzmann’s constant) and others. It is often believed that the question of the discretization of space is not testable, either now or in the future, because of the extraordinarily small size of $l_p$. However, as will be shown in this work, the effects of the discreteness of space will manifest itself in measurable phenomena, namely the motion of black holes.

In this work, the role played by Heisenberg’s lattice world is relatively minor but nonetheless important. In 1930, Heisenberg introduced the concept of space being discretized in unit cells of size of $l_o$ with $r_o = 10^{-15}$ m [6]. Along with this discretization, he proposed to replace the differential equations used in physics with difference equations. Niels Bohr and Wolfgang Pauli pointed out obvious problems with the concept, including the breaking of isotropy of space, and non-conservation of energy and momentum [7, 8]. Additionally, it appeared that the concept of a fundamental or smallest length breaks Lorentz invariance—this ostensibly smallest length would be Lorentz-contracted to a yet smaller value in a moving reference frame. In spite of these problems, the lattice world concept was further studied by Heisenberg and others, including Matvei Bronstein [9]. In this work, the concepts of a fundamental length and crystalline order will be used, but at a length scale twenty orders of magnitude smaller than what Heisenberg proposed.

The final and most important historical concept explored in this work is John A. Wheeler’s concept of quantum foam that he introduced in 1957 in an attempt to show that all of classical physics, particle physics included, is “purely geometrical and based throughout on the most firmly established principles of electromagnetism and general relativity” [10]. Wheeler made use of well-developed concepts in quantum electrodynamics, in which the probability amplitude of a transition of a system from an initial configuration to a final configuration is obtained by summing the Feynman–Huygens exponent over all possible histories:

$$
\langle C_2 \sigma_2 | C_1 \sigma_1 \rangle = \sum_H e^{iH_0/t} \tag{1}
$$

In order for a particular history to contribute to the transition probability, the phase of the exponent in Eq. (1), which is dependent on the metric $g$ and the electric vector

![A schematic of the Leopold crystal—the universe-wide GC. The basis (the blue spheres) is assumed to be particle of mass $m_p$ that produces a spherically symmetric 1/r potential energy profile. The crystal is cubic in structure with a lattice constant of $l_p$. Right The dual or reciprocal lattice, which is also cubic with a lattice constant of $2\pi/l_p$.](image-url)
potential $A$, should be small ($\sim 1$ radian or less) to avoid destructive interference. This limits fluctuations in $g$ (i.e., $\Delta g$) that can occur over a volume of space–time $L^d$ to be on the order of $\Delta g \sim \ell_p/L$, where $\ell_p$ is the Planck length, and fluctuations in $A$ to be on the order of $\sqrt{hc}/L$. The fluctuations in the metric $\Delta g$ remains small relative to $g$ until $L$ approaches $\ell_p$, at which point, $\Delta g \sim g$ and “the character of the space undergoes an essential change... and multiple connectedness develops” [10] that results in a disordered collection of “wormholes” in the topology of space with a spacing of approximately $\ell_p$; he calls this wormhole collection “quantum foam”.

Each wormhole has an associated pair of positive and negative electrical charges of charge $\pm q_{\text{Planck}} = \pm \sqrt{\hbar c}$ that produces an intense electromagnetic field energy $E$ that has associated with it a mass of $m = c^2E = m_p$. After further discussion, he summarizes his findings as follows [10]:

1. Quantization of the physics of Maxwell and Einstein forces upon space a foam-like structure... on the scale of $\ell_p$.
2. In the vacuum, virtual pairs of charges are being continually created and annihilated.
3. With these pairs are associated charges and especially masses “far larger than anything familiar from the elementary particle problem” [10].
4. The gravitational energy density is negative and approximately equal in magnitude to the positive electromagnetic energy density, and “circumstances are favorable for the local compensation of electromagnetic energy by gravitational energy,” such that “to the extent this compensation holds locally, nearby wormholes exert no gravitational attraction on remote concentrations of mass-energy.”

Item 4 states that the formation of the foam is an energy conserving process. This property makes these fluctuations highly probable to occur and contribute to the transition probability. However, in this work, it will not be assumed that “nearby wormholes exert no gravitational attraction on remote concentrations of mass-energy.” Also, in this work it will be assumed, contrary to Item 2, that these wormholes (i.e., Planck masses) are stable in time. And most importantly in this work, rather than a random distribution of these wormholes with an average spacing of $\ell_p$ as Wheeler assumed, a regular cubic lattice of lattice constant $\ell_p$ will be assumed (Fig. 1). Note that the GC does not need to have a cubic structure to produce the gravitational anomalies described later in this work—the same anomalies will occur for other lattice structures, such as dodecahedron [11].

Thus, we arrive at the GC described in this work—a lattice with a spatial period (i.e., lattice constant) that is the smallest that nature allows, namely the Planck length $\ell_p$, and a basis composed of an elementary point particle with a mass that is the most that nature allows, namely the Planck mass $m_p$. This crystal, which I call the Leopold crystal, is assumed to fill up all space with no defects of any kind, including edges. Any dislocations or interstitial sites would necessarily involve distances less than $\ell_p$—something thought not to be possible. It is well known that an elementary particle with mass $m_p$ has a reduced Compton wavelength $\lambda_c = \hbar/m_p c$ that is equal to $\ell_p$. With the $\lambda_c$ being the fundamental limit on measurements of the position of a particle, the question as to where the Planck particle is within unit cell of volume $\ell_p^3$ is unanswerable. Also, because the lattice constant $\ell_p$ is one-half of Schwarzschild radius ($R_c = 2Gm_p/c^2$), the unit cells (composed of a Planck particle) form an array of quantum black holes, each with the smallest allowable mass for a black hole.

To get a sense of what particles would experience the GC to any significant degree, one can compare the relative magnitudes of a particle’s kinetic energy $KE = \hbar^2|\mathbf{k}|^2/2m$ and gravitational potential energy $|V| = Gmmp/l_p$, with $m$ being the gravitational and inertial mass of the particle. In the physics of crystals, $|\mathbf{k}|$ can be assumed to be within the first Brillouin zone (BZ), i.e., $-\pi/l_p < |\mathbf{k}| < \pi/l_p$, and that the effects of the crystal most often manifest themselves at the BZ boundary of $|\mathbf{k}| = \pi/l_p$. Thus, when an elementary particle’s mass is large enough, namely when $m = (\pi/\sqrt{2})m_p = 2.22m_p$, $KE$ will equal $|V|$. Particles with less (more) mass experience the effects of the crystal to a lesser (greater) degree. Since we have already stated that masses greater than $m_p$ within a volume $\ell_p^3$ are not possible, let us consider particles within this crystal having a mass $m_p$. Such a mass, namely $m_p$, is very large—almost no elementary particle has a mass close to this value. The exceptions to this are black holes, and to a much lesser extent the Higgs Boson ($m = 2.25 \times 10^{-25}$ kg), top-quark ($m = 3.09 \times 10^{-25}$ kg), and exceptionally high-energy cosmic rays. All heavier entities (e.g., atoms, molecules, neutron stars) have their mass distributed over a volume such that the mass density is much less than $m_p/\ell_p^3$. Black holes are predicted to compress the mass of many stars to a single, infinitesimally small point. Even if this mass is not compressed to a singularity but instead limited to a density of $\rho_{\text{max}} = m_p/\ell_p^3$ (as assumed later in this work), the inertial properties of the black hole will be significantly influenced by the GC.
3 Gravitational flux and effective mass model of gravity

Non-relativistic gravitational fields produced by point masses have a spatial dependence the same as electric fields produced by point charges, and thus, the tools used in effective mass theory of crystals can be used for gravity. One can start with the equation:

$$\nabla \cdot \mathbf{D} = 4\pi G \rho_{\text{free}}$$

(2)

Because the right side of Eq. (2) only contains the free mass density $\rho_{\text{free}}$ produced by non-lattice entities (e.g., stars, planets, electrons) and not the masses comprising the crystal, $\mathbf{D}$ is not the gravitational field but is instead the gravitational displacement (or flux density). The total gravitational field $G_{\text{total}}$ is given by sum of the gravitational field $G_{\text{free}}$ produced by non-lattice entities (again stars, planets, electrons) and the gravitation field $G_{\text{lattice}}$ produced by the GC.

Similar to electromagnetics, we can relate the $\mathbf{D}$ and $G_{\text{total}}$ via a constitutive equation with a constant of proportionality we can call the gravitational permeability $\mu_g$. Also, if all the gravitational forces on a particle (that is traveling within the crystal) are taken into account and summed, then this sum (under non-relativistic conditions) is equal to the product of its gravitational mass $m_g$ and acceleration ($m_g \mathbf{a} = m_g G_{\text{total}}$) and is in agreement with Einstein’s equivalence principle. These two relations lead to:

$$\mathbf{D} = \mu_g G_{\text{total}} \quad \mathbf{a} = G_{\text{total}}$$

(3)

In effective mass theory, the forces due to external (non-lattice) sources, such as an electric field produced by an applied voltage, are equal to the product of the effective mass and acceleration. Similarly to gravitational forces, we can write:

$$\mathbf{F} = m_e \mathbf{a} = F_{\text{external}} = m_e \mathbf{D}$$

(4)

Substituting Eq. (3) in Eq. (4) leads to the following relation:

$$\mu_g = m_e / m_g$$

(5)

The value of $m_g$ is a constant, but the next section will describe methods to calculate $m_i$, and it will be shown that $m_i$ is dependent on the momentum of the particle and can take on values much different than unity (what $\mu_g$ would be absent the lattice), including positive values much greater than $m_g$, values near zero, and can also be negative.

4 The band diagram of the universal gravity crystal

One can draw upon the tools (long developed and experimentally verified) from the fields of solid-state physics and crystallography. For particles with $m < m_p$, the particle only weakly experiences the effects of the crystal and one can use the nearly free particle model or empirical pseudopotential method (EPM) to calculate the energy band diagram [12, 13]. For particles with $m > 2.22m_p$, one has to use the tight-binding (TB) model or another model that assumes that the particles are predominately localized around one lattice site and only weakly experiences the Planck particles at neighboring lattice sites. The TB model has been used to model particles with $m > 2.22m_p$, but due to space constraints, this work cannot be included in this paper but will be made public elsewhere. The same qualitative behavior of $m_i$ is seen using the TB model as compared to the EPM model. Namely, the TB model predicts non-equality of $m_i$ and $m_e$, and a momentum dependence for $m_i$ such that it can be much larger than $m_g$, be negative or near zero. With either the EPM or TB models, one starts with the time independent form of Schrodinger’s equation where $V(\mathbf{r})$ includes the gravitational potential energy profile produced by all the particles comprising the crystal:

$$\frac{\hbar^2}{2m_g} \nabla^2 \psi + V(\mathbf{r})\psi = E\psi$$

(6)

Identical to what is done with the periodic Coulombic potential energy profile created by the positively charged nuclei in crystal, $V(\mathbf{r})$ in Eq. (6) for an electrically neutral particle of mass $m_g$ is:

$$V(\mathbf{r}) = -G m_g m_p \sum_{\mathbf{R}} \frac{1}{|\mathbf{r} - \mathbf{R}|}$$

(7)

where $m_g$ is both the inertial mass and gravitational mass of the particle in all the terms in Eqs. (6) and (7), $m_p$ is the mass of each particle comprising the crystal, and $\mathbf{R}$ are the spatial translation vectors $\mathbf{R} = n_1 \mathbf{a}_x + n_2 \mathbf{a}_y + n_3 \mathbf{a}_z$ with $n_1, n_2$ and $n_3$ being integers, and $\mathbf{a}_x = l_p \hat{x}, \mathbf{a}_y = l_p \hat{y}, \mathbf{a}_z = l_p \hat{z}$ are the primitive spatial translation vectors for the cubic lattice.

The Fourier transform of $V(\mathbf{r})$ is:

$$V(|\mathbf{K}|) = \frac{1}{l_p^3} \int_{\text{unitecell}} e^{-i \mathbf{K} \cdot \mathbf{r}} V(\mathbf{r}) d\mathbf{r} = -\frac{2\pi m_g G}{l_p} \sin^2(|\mathbf{K}|)$$

(8)

with $\mathbf{K}$ being the reciprocal lattice vectors. Because the periodicity of the structure, the wave functions of the particles are Bloch functions:
\( \psi_k(\mathbf{r}) = e^{i \mathbf{k} \cdot \mathbf{r}} \sum_{\mathbf{K}'} U(\mathbf{K}') e^{i \mathbf{K}' \cdot \mathbf{r}} \)  \( \tag{9} \)

Inserting Eqs. (8)–(9) into Eq. (6) yields the matrix equation:

\[
\left[ \frac{\hbar^2 (\mathbf{k} + \mathbf{K})^2}{2m} \delta_{\mathbf{K}, \mathbf{K}'} + \sum_{\mathbf{K}''} V(|\mathbf{K} - \mathbf{K}'|) \right] U(\mathbf{K}') = EU(\mathbf{K})
\]

\( \tag{10} \)

One then simply solves Eq. (10) for the eigenvectors \( U(\mathbf{K}) \) and eigenvalues \( E \) along particular directions in \( \mathbf{k} \)-space (see Fig. 2) and plots the energy of the modes as a function of \(|\mathbf{k}|\). Once the dispersion curves \((\omega - \mathbf{k} \text{ curves})\) are calculated, the effective inertial mass can be calculated by the well-known formula \([\, 12, 13\, ]\) given below, and plotted in Fig. 3 as a function of \(|\mathbf{k}|\).

\[
\frac{1}{m_i} = \frac{1}{\hbar^2} \frac{\partial^2 E_n(\mathbf{k})}{\partial k_i \partial k_j}
\]

\( \tag{11} \)

It is seen from Fig. 2 that the bands are parabolic near \( k = 0 \) (i.e., the \( \Gamma \) point in momentum space), but near the BZ boundaries at \( X, M \) and \( R, \) \( \hbar^2 E/\partial k^2 \) changes sign, leading to sharp changes and negative values for \( m_i \). The salient point is that the mass given by Eq. (11) is the inertial mass \( m_i \) affecting properties such as acceleration, momentum and kinetic energy and not the gravitational mass \( m_g \) in Eqs. (7) and (8) involving the gravitational interaction between particles. Thus, for particles that experience the effects of the crystal, there will seemingly be a violation of Einstein’s equivalence principle, in that \( m_i \) can be different than \( m_g \). However, this violation is in appearance only and results from the fact that one typically calculates a particle’s acceleration in space using only non-lattice contributions to the gravitation field. Also, with the system being comprised of the particle and the crystal, overall conservation of energy and momentum is maintained; any energy or momentum given up by the particle is taken up by the crystal. Thus, the motion of a particle with negative effective inertial mass will be accompanied by gravity waves within the crystal that propagate at some speed away from the particle, presumably \( c \).

Once \( m_i \) has been calculated, all the effects of the crystal are contained within it—the potential energy profile created by the array of Planck masses can be ignored and only non-crystal sources of gravitational potential energy will contribute to \( V_{\text{external}}(\mathbf{r}) \) in Schrodinger’s equation, and \( m_i \) is used in the denominator of the kinetic energy term in Schrodinger’s equation:

\[
-\frac{\hbar^2}{2m_i} \nabla^2 \psi + V_{\text{external}}(\mathbf{r}) \psi = E \psi
\]

\( \tag{12} \)

Both the EPM model and the TB model presented above are non-relativistic models. Even though the goal of this work is not to provide precise values for \( m_i \) for particular particles but only to introduce the gravitational and inertial anomalies that the GC produces, doubt can be cast on validity of these non-relativistic models. This is because, for particles with momenta close to the BZ boundary of \( k = \pi/\ell_p \) and with a mass \( m_p \), their velocities approach \( c \). To demonstrate that the same phenomena are predicted using a relativistic method, a relativistic Kronig–Penney (KP) model described by Strange [14] will be used. The starting point is the one-dimensional Dirac equation:

\[
\left( \frac{\hbar}{i} \frac{\partial}{\partial x} - \beta m c^2 + V(x) \right) \psi(x) = W \psi(x)
\]

\( \tag{13} \)

where \( \alpha \) and \( \beta \) are given below,
\[ \hat{x}_k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \hat{\beta}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]  (14)

\( \psi \) is a Bloch wave similar in form to Eq. (9) with the additional complexity of being a 4-vector. \( \psi \) also has to obey the Bloch boundary conditions:

\[ e^{ikx} \psi(x) = \psi(x + a) \]  (15)

Defining a value \( E \) as \( E = W - mc^2 \) to eliminate the rest energy, and applying the all necessary boundary conditions throughout one period of the structure, leads to the following equation:

\[ \cos(ka) = \cos(k_1 b) \cos(k_2 (a - b)) - \frac{\Gamma^{-1} + \Gamma}{2} \sin(k_1 b) \sin(k_2 (a - b)) \]  (16)

where \( k_1, k_2 \) and \( \Gamma \) are given by:

\[ k_1^2 = \frac{(E + V_o + 2mc^2)(E + V_o)}{\epsilon^2 h^2} \quad k_2^2 = \frac{E(2mc^2)}{\epsilon^2 h^2} \]  (17)

\[ \Gamma = \frac{E + V_o}{k_1} \]  (18)

Shown in Fig. 4 is the energy–momentum band diagram for a one-dimensional period potential well structure with a period \( a \) equal to \( l_p \), a well width \( b \) of 0.952\( l_p \) and the well depth of \( V_o = Gm^2_p/l_p \). The same qualitative behavior observed with the non-relativistic EPM model is observed with the relativistic KP model, namely the occurrence of large positive, near zero and negative inertial mass as the momentum of the particle approaches the BZ boundary at \( k = \pi/l_p \), as shown in Fig. 5.

**5 Discussion**

For the purposes of presenting this work at the Meta’15 conference, similarities were described between the properties of GC and electromagnetic metamaterials. The accepted definition of metamaterials given by Rodger Walser is that they are “...macroscopic composites having a manmade, three-dimensional, periodic cellular architecture designed to produce an optimized combination, not available in nature, of two or more responses to specific excitation.” Additionally, the materials should: be periodic composite materials, have constituent elements that are substantially smaller than the wavelength and have properties that can be described by materials parameters (e.g., \( \epsilon, \mu \)). The GC is periodic and is a composite of Planck particles and the void, its constituent elements are smaller than the wavelengths of any excitation, and its properties can be described by the parameter \( \mu_g \), and \( \mu_g \) can take on values not commonly thought to occur in nature (i.e., negative and near zero values). Of course, the GC is not optimized by humankind for any application and not even “manmade.” In fact, this metamaterial is not “meta” at all—it is not beyond what nature provides but is the very fabric of nature, the very fabric of space–time. Nevertheless, the GC is an extraordinary material that is omnipresent. If it can be harnessed or exploited in some way by future technologies, it would allow extraordinary advancements in many scientific and technological fields, including propulsion systems.

At present, the GC may be affecting the motion of black holes and the evolution of the universe. It may be that regular black holes (i.e., not quantum black holes of \( m \sim m_p \)) do not concentrate their mass to an infinitely small singularity but instead to a mass density of \( m_p/l_p^3 \), such a thing would be consistent with the view that the maximum mass an elementary particle can have is \( m_p \) and any particle contained within a volume of \( l_p^3 \) is necessarily elementary.
In this case, the models described in this work for particles with $m_g = m_p$ traveling within the GC are applicable and black holes’ inertial properties will be significantly affected only when they have a very large velocity relative to the GC. Fast moving black holes may then have very large positive inertial masses and would therefore be relatively unresponsive to the gravitational field produce by all the stars and planets of the universe. At other momenta, a black hole’s $m_i$ may be negative. If, for example, a supermassive black hole at the center of a galaxy is moving fast away from the center of the observable universe, it may have a negative value for $m_i$ that is approximately equal in magnitude to its gravitational mass $m_g$. If this is the case, then the only observable effect may be that fast moving galaxies (perhaps at the edges of the observable universe) may be increasing their acceleration away from the center of the universe. Another example is for a supermassive black hole has a momentum such that its inertial mass is near zero; in this case the black hole may be ejected from the parent galaxy.

If black holes can compress significantly more mass than $m_p$ into a volume of $l_p^3$, then the models described in this work, and the accompanying work using the TB model, predict that the black hole will become trapped around one or several lattice points of the GC. If too massive, the black holes may generate defects in the GC or locally obliterate the GC. Either way, by observing the motion of fast moving black holes, one should be able to assess the structure and properties of any universe-wide GC, in effect using black holes as a probe to study the fine structure of space.

Additionally, one can look at large-scale structural features of the universe for experimental evidence of the discretization of space. Similar to regular crystals, the GC may possess crystal “facets” at its surface. The existence of facets of the universe will leave tell-tale patterns on the cosmic microwave background (CMB) from the Wilkinson Microwave Anisotropy Probe (WMAP) [11]. The orientations of these enormous facets can provide information about the minuscule lattice and unit cell as to whether the lattice type is cubic, dodecahedron or something else.

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