Random zero sets for Fock type spaces

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Abstract
Given a nondecreasing sequence \( \Lambda = \{\lambda_n > 0\} \) such that \( \lim_{n \to \infty} \lambda_n = \infty \), we consider the sequence \( N_{\Lambda} := \{\lambda_n e^{i\theta_n}, n \in \mathbb{N}\} \), where \( \theta_n \) are independent random variables uniformly distributed on \([0, 2\pi]\). We discuss the conditions on the sequence \( \Lambda \) under which \( N_{\Lambda} \) is a zero set (a uniqueness set) of a given weighted Fock space almost surely. The critical density of the sequence \( \Lambda \) with respect to the weight is found.

Keywords
Entire function · Fock space · Zero set · Random rotations

1 Introduction

Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a positive increasing function such that
\[
\lim_{t \to \infty} \varphi(t) = \infty.
\]
An entire function \( f \) is said to belong to the Fock type space \( \mathcal{F}_p^{\varphi} \), where \( 0 < p < \infty \), if
\[
\|f\|_{p, \varphi}^p := \int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(|z|)} \, dm(z) < \infty,
\]
where \( m \) is Lebesgue measure on \( \mathbb{C} \). The Fock spaces corresponding to the function \( \varphi(r) = r^2/2 \) will be called the classical Fock spaces and denoted by \( \mathcal{F}^p \).

A sequence \( \mathcal{N} = \{z_k \in \mathbb{C}\} \) is called a zero set of the space \( \mathcal{F}_p^{\varphi} \) if there exists a nonzero function \( f \in \mathcal{F}_p^{\varphi} \) whose zeroes are exactly the points from the sequence \( \mathcal{N} \).

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A sequence $\mathcal{N}$ is a uniqueness set of $\mathcal{F}_p^\varphi$, if the only function that vanishes at $\mathcal{N}$ is the zero function.

Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ be a nondecreasing sequence of positive numbers such that $\lim \lambda_n = \infty$.

**Definition 1.1** The random sequence $\mathcal{N}_\Lambda$, obtained by rotating each element $\lambda_n$ of the sequence $\Lambda$ by a random angle:

$$
\mathcal{N}_\Lambda := \left\{ z_n = \lambda_n e^{i\theta_n} : n \in \mathbb{N} = \{1, 2, 3, \ldots\} \right\},
$$

where $\theta_n$ are independent random variables uniformly distributed on $[0, 2\pi]$, will be called the randomization of $\Lambda$.

In this note, we will be concerned with the following question: under which conditions on $\Lambda$ the random sequence $\mathcal{N}_\Lambda$ is a zero set of the Fock space $\mathcal{F}_p^\varphi$ almost surely?

Speaking of a similar question for other spaces of analytic functions, let us start with the Hardy spaces $H^p(\mathbb{D})$. Here the situation is fully described by the Blaschke condition: the sequence $\mathcal{N} = \{z_n : |z_n| < 1\}$ is a zero set of the space $H^p(\mathbb{D})$ if and only if $\sum (1 - |z_n|) < \infty$, for any $p \in (0, \infty)$. The situation with the Bergman spaces $A^p(\mathbb{D})$ occurs to be quite different, as was originally shown in 1974 by Horowitz [7]. The random approach to this question was firstly considered in 1990 by LeBlanc [10], who obtain a sufficient condition for the set $\mathcal{N}_\Lambda$ with random angles to be almost surely a zero set of $A^2(\mathbb{D})$, using the Blaschke-type product introduced earlier by Horowitz. Later this result was improved and extended by Bomash [2] and Horowitz [8].

Turning to the Fock spaces, we refer to the book [15] by Zhu for some general results and to the paper by Lyons and Zhai [12], where a discussion of the zero sets for Bergman and Fock spaces can be found.

While it is a long standing problem to characterize deterministic zero sets of the Fock space, the probabilistic approach has some advantages.

Recently X.Fang and P.T.Tien in [5] found a sufficient condition on the sequence $\Lambda$ under which the sequence $\mathcal{N}_\Lambda$ is almost surely a zero set of the classical Fock space $\mathcal{F}_{\alpha^2/2}$. In particular, they showed that if $\Lambda = \{\lambda_n\}$, where $\lambda_n \sim c\sqrt{n}$ as $n \to \infty$, then for all $\alpha > \frac{16}{(\sqrt{15}-3)c^2}$ and $p > 0$ the randomized sequence $\mathcal{N}_\Lambda$ is a zero set of $\mathcal{F}_{\alpha^2/2}$ almost surely, while for $\alpha < 1/c^2$ and $p > 0$ the randomized sequence is a uniqueness set for $\mathcal{F}_{\alpha^2/2}$. Since the bounds in these conditions do not match, the question of the critical value has been raised in the paper.

It is also worth mentioning the paper [3] by Chistyakov, Lyubarskii and Pastur, who dealt with another kind of randomization and showed that a random perturbation of the lattice $a(\mathbb{Z} + i\mathbb{Z}) = \{am + ian, m, n \in \mathbb{Z}\}$ is a.s. a zero set of the classical Fock space if $a > \frac{1}{\sqrt{\pi}}$, and not a zero set if $a < \frac{1}{\sqrt{\pi}}$, under some natural conditions on the perturbation.

We will use the following notation and terminology.
**Definition 1.2** By \( n(t) \) we denote the number of the elements of the sequence \( N \) in the open disk of radius \( t \):

\[
n(t) = \# \{ z_n \in N : |z_n| < t \}.
\]

**Definition 1.3** Given an increasing differentiable function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) and a sequence \( N = \{ z_n \in \mathbb{C} \} \) such that

\[
\lim_{t \to \infty} \frac{n(t)}{t \varphi'(t)} = A,
\]

we say that \( N \) is

- **of critical density** with respect to the Fock space \( F^p_\varphi \), if \( A = 1 \);
- **of subcritical density** with respect to the Fock space \( F^p_\varphi \), if \( A < 1 \);
- **of supercritical density** with respect to the Fock space \( F^p_\varphi \), if \( A > 1 \).

The results of the present paper are divided into three parts. In Theorem 2.1 we propose a (nonrandom, Jensen type) sufficient condition for the sequence \( N \) to be a uniqueness set of the Fock space \( F^p_\varphi \). In particular, it yields that any sequence \( N \) of supercritical density with respect to \( F^p_\varphi \) is not a zero set of this space. Some examples of the uniqueness sets with critical density are also given.

In Theorem 3.3 we show that, provided some regularity of \( \varphi \), for subcritical sequences \( \Lambda \), the randomization \( N_{\Lambda} \) is a zero set of the Fock space \( F^p_\varphi \) almost surely. In particular, it follows that for the sequence \( \Lambda = \{ \lambda_n \} \) such that \( \lim_{n \to \infty} \frac{\lambda_n}{\sqrt{n}} > 1 \), the random sequence \( N_{\Lambda} \) is almost surely a zero set for the classical Fock space \( F^p \), which settles the question on the mismatching bounds in [5].

The critical case is more subtle, and we cannot determine whether the random sequence \( N_{\Lambda} \) is a zero set almost surely in terms of the asymptotic behavior of \( n(t) \). In Theorem 4.2 we consider only the classical Fock space \( F^p \), and give examples of random a.s. zero sets with critical density with respect to the space \( F^p \).

In particular, it follows from our results that

1) if \( \Lambda \sim a \sqrt{n} \), then the sequence \( N_{\Lambda} \) is

(a) a zero set of \( F^2 \) for \( a > 1 \) almost surely;
(b) a uniqueness set of \( F^2 \) for \( a < 1 \);
(c) a uniqueness set for \( \lambda_n = \sqrt{n} + \alpha, \ \alpha \leq 1/2 \).

2) if \( \Lambda \) is a nondecreasing sequence consisting of the moduli of all elements of the lattice \( \mathbb{Z} + i\mathbb{Z} \):

\[
\Lambda = \sqrt{a\pi} |\mathbb{Z} + i\mathbb{Z}| = \sqrt{a\pi} \{ 0, 1, 1, 1, \sqrt{2}, \sqrt{2}, \ldots \},
\]

then the sequence \( N_{\Lambda} \) is

(a) a zero set of \( F^2 \) for \( a > 1 \) almost surely;
(b) a uniqueness set of \( F^2 \) for \( a < 1 \);
(c) **Open question:** is $\mathcal{N}_\Lambda$ a uniqueness set in the critical case $a = 1$?

For the convenience of the reader, we add an Appendix at the end of the paper, which collects the results used in the proofs.

## 2 Sufficient condition for the uniqueness set of the Fock space

In this section we give a (nonrandom) condition on any sequence $\Lambda$ sufficient for the sequence $\mathcal{N}$ such that $|z_n| = \lambda_n, n \in \mathbb{N}$, to be a uniqueness set of $\mathcal{F}_\varphi^p$. The following theorem is an immediate consequence of Jensen formula and Jensen inequality.

**Theorem 2.1** Let $g : \mathbb{R}_+ \to (0, 1]$ be a function such that $(\log g + \varphi)$ is absolutely continuous, and let

$$
\int_1^\infty g^p(t)dt = \infty.
$$

A sequence $\mathcal{N} = \{z_n : |z_n| = \lambda_n, \forall n \in \mathbb{N}\}$ is a uniqueness set for the Fock space $\mathcal{F}_\varphi^p$, provided that there exists $M > 0$ such that

$$
n(t) \geq t \cdot (\log g(t) + \varphi(t))' - \frac{1}{p}, \forall t \geq M. \quad (1)
$$

If $\varphi$ is a differentiable function, taking $g(t) = t^{-1/p}$ we get

**Corollary 2.2** If there exists $M > 0$ such that

$$n(t) + 2/p \geq t\varphi'(t), \forall t \geq M,$

then every sequence $\mathcal{N} = \{z_n : |z_n| = \lambda_n, \forall n \in \mathbb{N}\}$ is a uniqueness set for the Fock space $\mathcal{F}_\varphi^p$.

**Corollary 2.3** A sequence $\mathcal{N}$ of the supercritical density with respect to $\mathcal{F}_\varphi^p$ is a uniqueness set for the Fock space $\mathcal{F}_\varphi^p$.

**Proof of Theorem 2.1** We will prove it by contradiction. Suppose that there is a function $f \in \mathcal{F}_\varphi^p$ such that the number of zeros $n(t)$ of this function satisfies the conditions of the theorem. Without loss of generality we can assume that $|f(0)| = 1$. Indeed, if $f(0) = 0$ and $z = 0$ is a zero of multiplicity $k$, we can consider a function $\tilde{f}(z) := C \cdot f(z) \frac{(1 - z)^k}{z^k}$ instead, where $C = \frac{k!}{f^{(k)}(0)}$ is the normalizing constant.

The numbers of zeroes of both functions in the disk $\{z : |z| < t\}$ coincide for $t > 1$, and the conditions $f \in \mathcal{F}_\varphi^p$ and $\tilde{f} \in \mathcal{F}_\varphi^p$ are equivalent.

Furthermore, by Jensen inequality for $R > M$ we have
\[
\log \int_{[0,2\pi]} |f(Re^{i\theta})|^p \frac{d\theta}{2\pi} \geq p \int_{[0,2\pi]} \log |f(Re^{i\theta})| \frac{d\theta}{2\pi} \\
= p \int_0^R \frac{n(t)}{t} \, dt \geq \int_M \left( p \log g(t) + \varphi(t)' - \frac{1}{t} \right) \, dt \\
= p \log g(R) + p\varphi(R) - \log R - p(\log g(M) + \varphi(M)) + \log M.
\]

Hence for some \(C > 0\) and for all \(R > M\)

\[
\int_{[0,2\pi]} |f(Re^{i\theta})|^p e^{-p\varphi(R)} \frac{d\theta}{2\pi} \geq C \frac{g^p(R)}{R}.
\]

Consequently, for \(f \in \mathcal{F}_\varphi^p\) we get

\[
\| f \|^p_{p,\varphi} = \int_C |f(z)|^p e^{-p\varphi(|z|)} \, dm(z) \\
\geq \int_M \int_{[0,2\pi]} |f(Re^{i\theta})|^p e^{-p\varphi(R)} \, Rd\theta \, dR \geq C \int_M g^p(R) \, dR = \infty.
\]

The contradiction proves the theorem. \(\square\)

Although the following corollary can be derived from the Theorem 2.1, it is simpler to give a direct argument. We use the standard notation \([x]\) to denote the integer part of \(x\).

**Corollary 2.4** Every sequence \(\mathcal{N} = \{z_n : |z_n| = \sqrt{n+\alpha}, \ n \in \mathbb{N}\}, \ \alpha \leq 1/2,\) is a uniqueness set for the classical Fock space \(\mathcal{F}^2\).

**Proof of Corollary 2.4** Let \(f \in \mathcal{F}^2\) and \(\lambda_n = |z_n| = \sqrt{n+\alpha}\). Assume, without loss of generality, that \(|f(0)| = 1\). Then

\[
\log \int_{[0,2\pi]} |f(Re^{it})|^2 \frac{dt}{2\pi} \geq 2 \int_{[0,2\pi]} \log |f(Re^{it})| \frac{dt}{2\pi} \\
= 2 \sum_{|z_n| \leq R} \log \frac{R}{|z_n|} \geq 2[R^2 - \alpha] \log R - \log \Gamma\left([R^2 - \alpha] + \alpha + 1\right) \\
= 2[R^2 - \alpha] \log R - \left([R^2 - \alpha] + \alpha + 1/2\right) \log([R^2 - \alpha] + \alpha + 1) \\
+ [R^2 - \alpha] + O(1) \\
= - (2\alpha + 1) \log R + R^2 + O(1).
\]

Hence,

\[
\int_{[0,2\pi]} |f(Re^{it})|^2 e^{-R^2} \frac{dt}{2\pi} \geq C / R^{1+2\alpha},
\]
and
\[
\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dm(z) \geq \int_{1}^{\infty} \int_{[0,2\pi]} |f(Re^{it})|^2 e^{-R^2} R dt \ dR \\
\geq \int_{1}^{\infty} \frac{C}{R^{2\alpha}} dR = +\infty.
\]

\[\square\]

Open question: How to describe the set of all \( \alpha > 0 \) such that every sequence \( \mathcal{N} = \{z_n : |z_n| = \sqrt{n + \alpha}, \ n \in \mathbb{N} \} \) is a uniqueness set for the classical Fock space \( \mathcal{F}_2 \)?

3 Sufficient condition for random zero set: subcritical density

The results of this section are based on the Levin-Pfluger theory of entire functions of so called completely regular growth (Appendix 5.6, 5.7), a detailed presentation of these results can be found in Chapter II of Levin’s book [11]. We recall some definitions that will be used.

Definition 3.1 A function \( \rho(t) \) is called a \textbf{proximate order}, if it satisfies the following conditions

\[
0 < \lim_{t \to \infty} \rho(t) = \rho < \infty; \\
\lim_{t \to \infty} t \rho'(t) \log t = 0.
\]

Given a sequence \( \mathcal{N} \subset \mathbb{C} \) denote by \( n(t, \alpha, \beta) \) the number of the elements of \( \mathcal{N} \) in the sector \( S(t, \alpha, \beta) := \{z \in \mathbb{C} : |z| < t, \ \arg z \in (\alpha, \beta)\} \).

Definition 3.2 The sequence \( \mathcal{N} \) is said to have an \textbf{angular density} with respect to the proximate order \( \rho(r) \) if for any \( \alpha \) and \( \beta \) except for some countable set there exists a finite limit

\[
\lim_{r \to \infty} \frac{n(r, \alpha, \beta)}{r^{\rho(r)}}.
\]

The main result of this section is the following theorem.

Theorem 3.3 Let \( \varphi(t) = t^{\rho(t)} \) where \( \rho(t) \) is a proximate order. Suppose that a sequence \( \Lambda = \{\lambda_k : \lambda_k \geq 1\} \) is nondecreasing and of subcritical density with respect to the Fock space \( \mathcal{F}_p^\varphi \). Then the random sequence \( \mathcal{N}_\Lambda \) is almost surely a zero set of the Fock space \( \mathcal{F}_p^\varphi \).

Corollary 3.4 Let \( \Lambda := \{\lambda_n\}, \) where \( \lambda_n \sim a\sqrt{n}, \ n \to \infty, \) with \( a > 1 \). Then the sequence \( \mathcal{N}_\Lambda \) is almost surely a zero set of the classical Fock space \( \mathcal{F}_p \).
Corollary 3.5 The sequence $N_\Lambda$, where $\Lambda = \sqrt{a\pi} |\mathbb{Z} + i\mathbb{Z}|$ with $a > 1$, is almost surely a zero set of the classical Fock space $F_p$.

To show that $\Lambda$ from the Corollary 3.5 satisfies the conditions of Theorem 3.3, note that

$$n(t) = \# \{(b, c) \in \mathbb{Z}^2 : \sqrt{a\pi} |b + ic| < t\} = \# \{(b, c) \in \mathbb{Z}^2 : |b + ic| < \frac{t}{\sqrt{a\pi}}\} = N\left(\frac{t}{\sqrt{a\pi}}\right),$$

here we use the standard notation from the Gauss circle problem and denote by $N(t)$ the number of integer lattice points inside a circle of radius $t$. It is well known that

$$\lim_{t \to \infty} \frac{N(t)}{t^2} = \pi,$$

and so we have

$$\lim_{t \to \infty} \frac{n(t)}{t^2} = \frac{1}{a} < 1.$$

Remark 3.6 Since $\phi(t) = t^{\rho(t)}$, where $\rho(t)$ is a proximate order, we have

$$\lim_{r \to \infty} \frac{r \phi'(r)}{\phi(r)} = \lim_{r \to \infty} \left(\rho(r) + r \rho'(r) \log r\right) = \rho,$$

and so the subcriticality of the sequence $\Lambda$ is equivalent to the condition

$$\lim_{r \to \infty} \frac{n(r)}{\phi(r)} = a < \rho. \quad (2)$$

Let us first prove some auxiliary results.

3.1
In this section we show that a random sequence $N_\Lambda$ in Theorem 3.3 almost surely obeys some regularity properties.

The following lemma is to show that $N_\Lambda$ with probability one has an angular density of index $\rho(r)$, which is equivalent, by Remark 3.6, to the statement that with probability one for any $\alpha$ and $\beta$ the limit $\lim_{r \to \infty} \frac{n(r, \alpha, \beta)}{n(r)}$ exists and is finite.

Lemma 3.7 With probability one the random sequence $N_\Lambda$ obeys the following property:

$$\lim_{r \to \infty} \frac{n(r, 2\pi x, 2\pi y)}{n(r)} = y - x, \quad 0 \leq x < y \leq 1.$$
Proof This lemma follows immediately from the Weyl-type criterion (Appendix, 5.4). Indeed, applying this criterion to the sequence \( X_n = \frac{\theta_n}{2\pi} \) with \( \phi_n(x) = \frac{e^{i\pi x} - 1}{ix} \), \( n \in \mathbb{N} \), we get for \( x \in [0, 1] \)

\[
\lim_{n \to \infty} \frac{n(r, 0, 2\pi x)}{n(r)} = \lim_{N \to \infty} \frac{\#\{j : X_j < x\}}{N} = x \quad \text{a.s.}
\]

\( \square \)

Lemma 3.8 Under the conditions of Theorem 3.3, if \( \rho \in \mathbb{N} \), then the series \( \sum_k \frac{1}{z_k^\rho} \) converges almost surely. Furthermore, almost surely

\[
\lim_{r \to \infty} r^{\rho - \rho(r)} \cdot \sum_{|z_k| > r} \frac{1}{z_k^\rho} = 0. \tag{3}
\]

Proof To justify the convergence of the series of variances \( \text{Var} \frac{1}{z_k^\rho} \), recall that by subcriticality of \( \Lambda \), or, equivalently, by (2) we have for some \( C > 0 \)

\[
n(r) \leq Cr^{3\rho/2}, \quad r \geq 1.
\]

Since the sequence \( \Lambda \) is increasing, it follows that

\[
k \leq n(\lambda_k + 1) \leq C \cdot (\lambda_k + 1)^{3\rho/2}, \quad k \geq 1. \tag{4}
\]

Finally,

\[
\sum \text{Var} \frac{1}{z_k^\rho} = \sum \lambda_k^{-2\rho} \leq C_1 \sum k^{-4/3} < \infty.
\]

Now we can use the Khinchine-Kolmogorov theorem (Appendix, 5.2) for the random variables \( Y_k = \text{Re} \frac{1}{z_k^\rho} \) and \( Y_k = \text{Im} \frac{1}{z_k^\rho} \). It follows that the series \( \sum_k \frac{1}{z_k^\rho} \) converges almost surely.

Now we estimate the rate of convergence of this series. Since \( \mathbb{E} \left( \sum_{|z_k| > r} \frac{1}{z_k^\rho} \right) = 0 \) and

\[
\text{Var} \left( \sum_{|z_k| > r} \frac{1}{z_k^\rho} \right) = \sum \lambda_k^{-2\rho} \leq C_2 \int_r^\infty \frac{dn(t)}{t^{2\rho}} = C_3 \left( \frac{n(r)}{r^{2\rho}} + \int_r^\infty \frac{t^{\rho(t)}}{t^{2\rho+1}} dt \right) \leq \frac{C_4}{\sqrt{r^\rho}},
\]
by Chebyshev’s inequality we have
\[ \mathbb{P}\left( \left| \sum_{|z_k| > r} \frac{1}{z_k^\rho} \right| \geq r^{-\rho/8} \right) \leq C_5 r^{-\rho/4}. \]

Put \( R_n = n^{5/\rho} \). Then
\[ \mathbb{P}\left( \left| \sum_{|z_k| > R_n} \frac{1}{z_k^\rho} \right| \geq n^{-5/8} \right) \leq C_5 n^{-5/4}. \]

Hence, by the Borel-Cantelli lemma, almost surely there exists (a random) \( N \in \mathbb{N} \) such that for \( n > N \)
\[ \left| \sum_{|z_k| > R_n} \frac{1}{z_k^\rho} \right| < R_n^{-\rho/8}. \quad (5) \]

Let now \( r \in (R_n, R_{n+1}), \ n > N \). Then
\[ \left| \sum_{R_n < |z_k| \leq r} \frac{1}{z_k^\rho} \right| \leq \sum_{R_n < |z_k| \leq r} \frac{1}{|z_k|^\rho} \leq \frac{n(R_{n+1}) - n(R_n)}{R_n^\rho}. \]

Recall that
\[ \lim_{r \to \infty} \frac{n(r)}{\phi(r)} = a < \rho. \]

Hence,
\[ r^{\rho - \rho(r)} \left| \sum_{R_n < |z_k| \leq r} \frac{1}{z_k^\rho} \right| \leq \frac{(a + o(1))\phi(R_{n+1}) - (a + o(1))\phi(R_n)}{(1 + o(1)) \phi(R_n)}. \]

By Remark 3.6
\[ \log \frac{\phi(R_{n+1})}{\phi(R_n)} = \int_{R_n}^{R_{n+1}} \frac{\phi'(t)dt}{\phi(t)} \leq C_6 \int_{R_n}^{R_{n+1}} \frac{dt}{t} = C_6 \log \frac{R_{n+1}}{R_n} = o(1), \]

and hence
\[ \lim_{n \to \infty} \frac{\phi(R_{n+1})}{\phi(R_n)} = 1. \]
It follows that

\[ r^{\rho - \rho(r)} \left| \sum_{R_n < |z_k| \leq r} \frac{1}{z_k^\rho} \right| = o(1), \quad r \to \infty. \quad (6) \]

Finally, summing up (5) and (6), we have almost surely

\[
\lim_{r \to \infty} r^{\rho - \rho(r)} \left| \sum_{r < |z_k|} \frac{1}{z_k^\rho} \right| \leq \lim_{r \to \infty} r^{\rho - \rho(r)} \left( \left| \sum_{R_n < |z_k|} \frac{1}{z_k^\rho} \right| + \left| \sum_{R_n < |z_k| \leq r} \frac{1}{z_k^\rho} \right| \right) = 0.
\]

The lemma is proved. \(\square\)

3.2

**Proof of Theorem 3.3** Given a random sequence \( N_\Lambda \) consider the corresponding Weierstrass canonical product

\[ W(z) := \prod_{z_k \in N_\Lambda} G \left( \frac{z}{z_k}; [\rho] \right), \]

where \( G(w; d) \) are the elementary factors:

\[ G(w; d) = (1 - w)e^{w + \frac{w^2}{2} + \cdots + \frac{w^d}{d}}. \]

It is well known (see [11], Chapter I), that the zero set of \( W \) coincides with the set \( \Lambda \).

By Lemma 3.7 the sequence \( N_\Lambda \) almost surely has angular density with index \( \rho(r) \), and in case \( \rho \notin \mathbb{N} \) we can apply the Levin-Pfluger theorem I (Appendix, 5.6). By this theorem, there exists a subset of a complex plane \( E = \bigcup_j D_j \), where \( \{D_j\}_j \) is a set of disks with zero linear density (for the definition of this notion see (Appendix, 5.5)), such that the following relation holds:

\[
\lim_{r \to \infty, \ r e^{i t} \notin E} \frac{\log |W(re^{i t})|}{r^{\rho(r)}} = \frac{a}{\rho} < 1.
\]

(7)

In case \( \rho \in \mathbb{N} \) we should take extra care about some kind of symmetry in the distribution of zeros of \( W \), which is provided by Lemma 3.8. By this lemma we know that there exists a (random, finite) number \( S := \sum \varepsilon_k^{-\rho} \) and the series converges fast enough so that

\[ \delta := \lim_{r \to \infty} r^{\rho - \rho(r)} \sum_{|z_k| > r} \frac{1}{z_k^\rho} = 0 \]
almost surely. Now, consider the entire function

\[ \tilde{W}(z) = e^{S \cdot z^p} \prod G \left( \frac{z}{z_k}; \rho \right). \]

By the Levin-Pfluger theorem II (Appendix, 5.7) there exists a subset of a complex plane \( E = \bigcup_j D_j \), where \( \{D_j\}_j \) is a set of disks with zero linear density (Appendix, 5.5), such that the following relation holds:

\[ \lim_{r \to \infty, \ re^{it} \notin E} \frac{\log |\tilde{W}(re^{it})|}{r^{\rho(r)}} = \frac{a}{\rho} < 1. \]

Thus, by (7) and (8), by the maximum principle, with probability one there exists a nonzero function, \( W \in F_p^{\phi} \) for \( \rho \notin \mathbb{N} \), and \( \tilde{W} \in F_p^{\phi} \) for \( \rho \in \mathbb{N} \), with zeros exactly at the points of the random sequence \( \mathcal{N}_\Lambda \) (by the construction). That is, \( \mathcal{N}_\Lambda \) is almost surely a zero set of \( F_p^{\phi} \).

\[ \Box \]

\section{4 Critical density}

In this section we produce sequences \( \Lambda \) of critical density with respect to the classical Fock space \( F_p \) such that the random sequence \( \mathcal{N}_\Lambda \) with independent uniformly distributed arguments is a zero set of \( F_p \).

\textbf{Remark 4.1} Note, that Corollary 2.4 provides us with an example of another nature — a sequence of critical density that is not a zero set under any rotation of the arguments.

\textbf{Theorem 4.2} Let \( \mathcal{N} = \{z_n = \lambda_n e^{i\theta_n}\} \), where the sequence \( (\lambda_n) \) is strictly monotone, \( \lambda_n \geq 1 \) and

\[ \lambda_n^2 = n + a \sqrt{n} \cdot \log^b n + o(1), \quad n \to \infty, \]

for some \( a > 0 \), \( b > 3/2 \), and \( \theta_n \) are independent random variables uniformly distributed on \([0, 2\pi]\). Then almost surely \( \mathcal{N} \) is a zero set for the classical Fock space \( F_p \) with any \( p > 0 \).

We will need the following technical lemma.

\textbf{Lemma 4.3} Under the conditions of the theorem

1. \( \lambda_{n+1}^2 - \lambda_n^2 \to 1, \quad \text{as} \quad n \to \infty, \)
2. \( \lambda_{n+1} - \lambda_n \sim \frac{1}{2\sqrt{n}}, \quad \text{as} \quad n \to \infty, \)
3. \( n(t) = t^2 - (a + o(1)) \cdot t \cdot \log^b t^2, \quad \text{as} \quad t \to \infty; \)

\textbf{Proof of Lemma 4.3} The first two claims follow immediately from (9).
To verify the third claim, note that if $n = n(t)$, then $\lambda_n < t \leq \lambda_{n+1}$, and
\[
t = \lambda_n + o(1) = \sqrt{n} + o(\sqrt{n}),
\]
therefore
\[
\lim_{t \to \infty} \frac{t^2 - n(t)}{t \cdot \log^b t^2} = \lim_{n \to \infty} \frac{(\lambda_n + o(1))^2 - n}{(\sqrt{n} + o(\sqrt{n})) \cdot \log^b(\sqrt{n} + o(\sqrt{n}))^2} = \lim_{n \to \infty} \frac{a\sqrt{n} \cdot \log^b n + o(\sqrt{n})}{(\sqrt{n} + o(\sqrt{n})) \cdot \log^b(\sqrt{n} + o(\sqrt{n}))^2} = a.
\]
The lemma is proved. \hfill \Box

**Proof of Theorem 4.2** For $k \in \mathbb{N}$ denote
\[
H_k(z) := G\left(\frac{z}{z_k}; 1\right) = \left(1 - \frac{z}{z_k}\right) e^{z/z_k};
\]
\[
h_k(z) := \log |H_k(z)| = \frac{1}{2} \log \left(1 - 2 \frac{|z|}{\lambda_k} \cos(\beta - \theta_k) + \frac{|z|^2}{\lambda_k^2}\right) + \frac{|z|}{\lambda_k} \cos(\beta - \theta_k),
\]
where $\beta = \arg z$.

Furthermore, for $|z| < |z_k|$ we have
\[
h_k(z) = \text{Re} \left(\log \left(1 - \frac{z}{z_k}\right) + \frac{z}{z_k}\right) = -\text{Re} \sum_{j \geq 2} \frac{z^j}{j z_k^j}.
\]

We start with a calculation of the expectation of $h_k(z)$, with $|z| = R$ and $\arg z = \beta$:
\[
\mathbb{E}(h_k(z)) = \mathbb{E}\left(\log \left|1 - \frac{R}{\lambda_k} e^{i(\beta - \theta_k)}\right| + \mathbb{E}\left(\frac{R}{\lambda_k} \cos(\beta - \theta_k)\right)\right)
\]
\[
= \frac{1}{2\pi} \int_{[-\pi,\pi]} \log \left|1 - \frac{R}{\lambda_k} e^{i(\beta - \theta_k)}\right| d\theta_k + \frac{1}{2\pi} \int_{[-\pi,\pi]} \frac{R}{\lambda_k} \cos(\beta - \theta_k) d\theta_k
\]
\[
= \begin{cases} 
0, & R \leq \lambda_k \\
\log \frac{R}{\lambda_k}, & R > \lambda_k.
\end{cases} \quad (10)
\]
\hfill \Box

**Lemma 4.4** Under the conditions of Theorem 4.2 the function $\prod_{k=1}^{\infty} H_k(z)$ is an entire function almost surely.

**Proof** Consider the random variables $\overline{z_k}^2$. Since $\mathbb{E}(\overline{z_k}^2) = 0$ and $\text{Var}(\overline{z_k}^2) = \lambda_k^{-4} \sim k^{-2}$, by the Khinchin-Kolmogorov theorem (Appendix, 5.2) it follows that almost surely the series $\sum \overline{z_k}^2$ converges, so that $\sigma := \sum \overline{z_k}^2$ is a random number which is almost surely finite.
Consider an entire function determined by the Weierstrass canonical product

\[ K(z) := \prod_{z_k \in \Lambda_1} \prod_{z_k} G \left( \frac{z}{z_k}; 2 \right). \]

Note, that for all \( k \) such that \( |z_k| \geq 2|z| \) we have

\[ \log \left| G \left( \frac{z}{z_k}; 2 \right) \right| = \Re \left( \log G \left( \frac{z}{z_k}; 2 \right) \right) \leq \sum_{j=3}^{\infty} \left| \frac{z}{z_k} \right|^j \leq 2 \left| \frac{z}{z_k} \right|^3. \]  

(11)

Now,

\[ \prod_{k=1}^{\infty} H_k(z) = \prod_{z_k \in \Lambda_1} \prod_{z_k} G \left( \frac{z}{z_k}; 1 \right) = K(z) \exp \left( -z^2 \sum_k z_k^{-2} \right) = K(z) \exp \left( -\sigma z^2 \right), \]

hence it is almost surely an entire function.

The lemma is proved. \( \square \)

Put

\[ W(z) := \prod_{k=1}^{\infty} H_k(z). \]

By Lemma 4.4 it is an entire function with probability one. We claim that almost surely

\[ \limsup_{R \to \infty} \left( \max_{|z|=R} |W(z)| \right) \exp \left( -\frac{R^2}{2} \right) (R + 1)^{3/p} \leq 1, \]

(12)

and, hence, \( W \in \mathcal{F}^p \) for \( 0 < p < \infty \) with probability one.

To prove relation (12) we will estimate the probability of the large deviation of \( W(z) \) from its expected value. First, by (10), the expected value of the random function \( \log |W(z)| \) at the point \( z \), \( |z| = R \), is

\[ \mathbb{E} (\log |W(z)|) = \mathbb{E} \left( \sum_{k=1}^{\infty} h_k(z) \right) = \sum_{\lambda_k < R} \log \frac{R}{\lambda_k} = \int_1^R \frac{n(t)dt}{t} \]

\[ = \int_1^R \frac{(t^2 - (a + o(1)) t \log t^2) dt}{t} = \frac{R^2}{2} - \int_1^R (a + o(1)) \log t^2 dt \]

\[ = \frac{R^2}{2} - (a + o(1)) R \log R^2. \]
So, for $R$ large enough

$$
\mathbb{E} (\log |W(z)|) \leq \frac{R^2}{2} - \frac{a}{2} R \log^b R \leq \frac{R^2}{2} - \frac{a}{2} R \log^b R.
$$

Set $R_k = \frac{\lambda_k + \lambda_{k+1}}{2}, \varepsilon_k = \frac{\lambda_{k+1} - \lambda_k}{4(\lambda_k + \lambda_{k+1})}, k \geq 1$.

By Lemma 4.3

$$
\lim_{k \to \infty} \frac{k}{R_k^2} = \lim_{k \to \infty} \frac{k}{\lambda_k^2} = 16 \lim_{k \to \infty} k \varepsilon_k = 1. \tag{13}
$$

Fix $k \geq 1$ and $z$ such that $|z| = R_k$. The function $W$ is zero-free in the annulus $(1 - \varepsilon_k)R_k \leq |z| \leq (1 + \varepsilon_k)R_k$, and we may represent $W(z)$ as the product of two factors:

$$
W_0(z) := \prod_{1 \leq s \leq k} H_s(z); \quad W_\infty(z) := \prod_{s > k} H_s(z).
$$

As above,

$$
\mathbb{E} (\log |W(z)|) = \mathbb{E} (\log |W_0(z)|) \leq \frac{R^2}{2} - \frac{a}{2} R \log^b R.
$$

We will estimate the probability of the events

$$
\sup_{|z|=R_k} \log |W(z)| \geq \frac{R_k^2}{2} - \frac{3}{p} \log R_k.
$$

As the main tool we will use the Bernstein-Hoeffding concentration inequality (Appendix, 5.1), applying it to $W_0$ and $W_\infty$ separately. As the first step we use the concentration inequality for the value of $\log |W_0(z)|$ (log $|W_\infty(z)|$) at an individual point $z$ on the circle $|z| = R_k$, then we turn to the estimate of the maximum of $\log |W_0(z)|$ (log $|W_\infty(z)|$) over a finite number of points on this circle, and finally we prove that the deviation of $\log |W_0(z)|$ (log $|W_\infty(z)|$) from its expectation is small enough with probability close to one on the whole circle $|z| = R_k$.

### 4.1 Concentration inequality for $\log |W_0(z)|$.

In this section, given $k \geq 1$, we estimate the probability of the large upward deviation of the quantity $\sup_{|z|=R_k} \log |W_0(z)|$ from the expectation of $\log |W_0(z)|$, $|z| = R_k$. 

4.1.1 One-point inequalities

Put

\[ X_s := h_s(z) - \mathbb{E}(h_s(z)) = \begin{cases} h_s(z) - \log \frac{|z|}{\lambda_s}, & 1 \leq s \leq k, \\ h_s(z), & s > k, \end{cases} \]

where \(|z| = R_k\). First we estimate the size of \(X_s\) for \(s \leq k\):

\[ |X_s| = \left| h_s(z) - \log \frac{R_k}{\lambda_s} \right| = \left| \log \frac{\lambda_s}{R_k} - e^{i(\beta - \theta_s)} \right| + \left| \frac{R_k}{\lambda_s} \cos(\beta - \theta_s) \right| \]

\[ \leq \left| \log \left( 1 - \frac{\lambda_s}{R_k} \right) \right| + \frac{R_k}{\lambda_s} \leq \log \frac{1}{\varepsilon_k} + \frac{R_k}{\lambda_s} := c_s. \]

Next,

\[ \sum_{s=1}^{k} c_s^2 \leq \sum_{s=1}^{k} 2(\log^2 \varepsilon_k + R_k^2/\lambda_s^2) = 2k \log^2 \varepsilon_k + 2R_k^2 \int_1^{R_k} \frac{dn(t)}{t^2} \leq CR_k^2(\log^2 \varepsilon_k + \log R_k). \]

By the Bernstein-Hoeffding inequality (Appendix, Sect. 5.1) we have

\[ \mathbb{P} \left( \left| \log |W_0(z)| - \mathbb{E}(\log |W_0(z)|) \right| \geq \frac{a}{16} R_k \log^b R_k \right) \leq \exp \left( -\frac{a^2 R_k^2 \log^2 R_k}{C_1 R_k^2(\log^2 \varepsilon_k + \log R_k)} \right) \leq \exp \left( -\frac{a^2 \log^2 R_k}{C_1(\log^2 \varepsilon_k + \log R_k)} \right). \]

Therefore, using relation (13), we conclude that

\[ \log \mathbb{P} \left( \left| \log |W_0(z)| - \mathbb{E}(\log |W_0(z)|) \right| \geq \frac{a}{16} R_k \log^b R_k \right) \lesssim -\frac{\log^2 R_k}{(\log^2 \varepsilon_k + \log R_k)} \lesssim -\log^{2(b-1)} R_k. \]

4.1.2 Multi-point inequalities

Consider now \(N_k\) equidistant points on the circle \(|z| = R_k\) (the number \(N_k\) is to be defined later):

\[ \zeta_s := R_k \exp \left( 2\pi i \frac{s}{N_k} \right), \quad s = 0, \ldots, N_k - 1. \]
For each of the points \( \zeta_s \) we have
\[
\log \mathbb{P} \left\{ \log |W_0(\zeta_k)| \geq \frac{R_k^2}{2} - \frac{7a}{16} R_k \log^b R_k \right\} \lesssim -\log^{(b-1)} R_k.
\]

Since
\[
\mathbb{P} \left\{ \sup_{0 \leq s < N_k} \log |W_0(\zeta_s)| \geq \frac{R_k^2}{2} - \frac{7a}{16} R_k \log^b R_k \right\} \leq \sum_{0 \leq s < N_k} \mathbb{P} \left\{ \log |W_0(\zeta_s)| \geq \frac{R_k^2}{2} - \frac{7a}{16} R_k \log^b R_k \right\},
\]
we obtain
\[
\log \mathbb{P} \left\{ \sup_{0 \leq s < N_k} \log |W_0(\zeta_s)| \geq \frac{R_k^2}{2} - \frac{7a}{16} R_k \log^b R_k \right\} \lesssim -\log^{(b-1)} R_k,
\]
if \( \log N_k = o(\log^{(b-1)} R_k) \).

### 4.1.3 Sup–inequalities

Now we are ready to estimate the supremum of \( \log |W_0(z)| \) on the whole circle \( |z| = R_k \). For \( 1 \leq s \leq k \) we have
\[
1 < \lambda_s \leq \lambda_k < R_k (1 - \epsilon_k),
\]
and, hence,
\[
\left| \frac{\partial}{\partial \beta} \log |W_0(R_k e^{i\beta})| \right| = \sum_{s=1}^{k} \left( \frac{R_k \sin(\beta - \theta_s)}{1 + \frac{R_k^2}{\lambda_s^2} - 2 \frac{R_k}{\lambda_s} \cos(\beta - \theta_s)} - \frac{R_k \sin(\beta - \theta_s)}{\lambda_s} \right)
\]
\[
\leq \sum_{s=1}^{k} \frac{R_k}{\lambda_s} \left( \frac{1}{\left( \frac{R_k}{\lambda_s} - 1 \right)^2 + 1} \right) \leq k R_k \left( \frac{(1 - \epsilon_k)^2}{\epsilon_k^2} + 1 \right) \leq k R_k \frac{2}{\epsilon_k^2}.
\]

Taking \( N_k = k^3 \) and using (13) we obtain for \( k > K \), where \( K \) is some deterministic constant,
\[
\frac{2\pi}{N_k} \cdot \frac{2k R_k}{\epsilon_k^2} = 4\pi \cdot \frac{R_k}{(k\epsilon_k)^2} \leq 15 R_k \leq \frac{a}{16} R_k \log^b R_k.
\]

Since
\[
\mathbb{P} \left\{ \sup_{|z|=R_k} \log |W_0(z)| \geq \frac{R_k^2}{2} - \frac{6\alpha}{16} R_k \log^b R_k \right\} \\
\leq \mathbb{P} \left\{ \sup_{0 \leq s < N_k} \log |W_0(\zeta_s)| \geq \frac{R_k^2}{2} - \frac{6\alpha}{16} R_k \log^b R_k - \frac{2\pi}{N_k} \cdot \frac{2k R_k}{\epsilon_k^2} \right\} \\
\leq \mathbb{P} \left\{ \sup_{0 \leq s < N_k} \log |W_0(\zeta_s)| \geq \frac{R_k^2}{2} - \frac{7\alpha}{16} R_k \log^b R_k \right\},
\]

we get

\[
\log \mathbb{P} \left\{ \sup_{|z|=R_k} \log |W_0(z)| \geq \frac{R_k^2}{2} - \frac{6\alpha}{16} R_k \log^b R_k \right\} \lesssim -\log^{2(b-1)} R_k. \quad (14)
\]

4.2 Estimating \( \log |W_\infty(z)| \).

We still keep the number \( k \geq 1 \) fixed. Our next step is to estimate the probability that \( \sup_{|z|=R_k} \log |W_\infty(z)| \) is “too large”.

4.2.1 One-point inequalities

Consider a point \( z \) on the circle \( |z| = R_k \). By Lemma 4.4 the series \( \sum_{s=k}^{\infty} h_s(z) \) converges almost surely, let regroup it in the following way

\[
\sum_{s>k} h_s(z) = \sum_{m=0}^{\infty} S_{m,k}, \quad \text{where} \quad S_{m,k} = \sum_{s=k^{2m+1}}^{k^{2m+1}} h_s(z),
\]

and estimate the probability of large deviation from zero of each \( S_{m,k}, m = 0, 1, \ldots \).

Since

\[
|h_s(z)| = \left| -\text{Re} \sum_{m \geq 2} \frac{z^m}{m! z_s^m} \right| \leq \frac{R_k^2}{2 \lambda_s^2} \left( 1 + \sum_{m \geq 1} \frac{|z|^m}{m! z_s^m} \right) \\
\leq C \frac{R_k^2}{s} \log \frac{1}{\epsilon_k} \leq C \frac{R_k^2 \log R_k}{s} =: c_{k,s},
\]

where \( C \) is a deterministic constant, and

\[
\sum_{s=k^{2m+1}}^{k^{2m+1}} c_{k,s}^2 \leq \frac{C_1 R_k^4 \log^2 R_k}{k^{2m+1}},
\]
using the Bernstein-Hoeffding inequality (Appendix, 5.1) we get

\[
P_{m,k} := \mathbb{P} \left( |S_{m,k}| \geq \frac{6}{\pi^2} \cdot \frac{a R_k \log^b R_k}{8 \cdot (m + 1)^2} \right) \leq \exp \left( -C_2 \frac{2^{m+1}}{(m + 1)^3} \log^{2(b-1)} R_k \right),
\]

where \( C_1, C_2 \) are some deterministic constants. Hence,

\[
\mathbb{P} \left( \left| \sum_{s > k} h_s(z) \right| \geq \frac{a}{8} R_k \log^b R_k \right) \leq \sum_{m=0}^{\infty} P_{k,m} \lesssim \exp \left( -C_3 \log^{2(b-1)} R_k \right),
\]

and finally

\[
\log \mathbb{P} \left( \log |W_\infty(z)| \geq \frac{a}{8} R_k \log^b R_k \right) \lesssim - \log^{2(b-1)} R_k.
\]

### 4.2.2 Multi-point inequalities

Consider \( N_k \) equidistant points on the circle \(|z| = R_k|:

\[
\zeta_s := R_k \exp \left( 2\pi i \frac{s}{N_k} \right), \quad 0 \leq s < N_k.
\]

For each of these points we have

\[
\log \mathbb{P} \left( \log |W_\infty(\zeta_s)| \geq \frac{a}{8} R_k \log^b R_k \right) \lesssim - \log^{2(b-1)} R_k,
\]

hence

\[
\log \mathbb{P} \left( \sup_{0 < s \leq N_k} |W_\infty(\zeta_s)| \geq \frac{a}{8} R_k \log^b R_k \right) \lesssim - \log^{2(b-1)} R_k
\]

if \( \log N_k = o(\log^{2(b-1)} R_k) \).

### 4.2.3 Sup–inequalities

Here we estimate the supremum of \( \log |W_\infty(z)| \) on the whole circle \(|z| = R_k|.

We represent the angular derivative of \( h_s, s > k \), in the form

\[
\frac{\partial}{\partial \varphi} h_s(z) = -\text{Re} \sum_{l=2}^{\infty} \frac{i z^l}{z_s^l} = -\text{Re} \left( \frac{i z^2}{z_s^2} + \frac{i z^3}{z_s^3} \cdot \frac{z_s}{z - z_s} \right).
\]

Therefore, (here \( z = R_k e^{i \beta} \))
\[
\left| \frac{\partial}{\partial \beta} \log |W_\infty(R_k e^{i\beta})| \right| \\
\leq \left| \sum_{s > k} \frac{z^2}{z_s^2} \right| + \left| \sum_{s > k} \frac{z^3}{z_s^2} \cdot \frac{z_s}{z_s - z} \right| \\
\leq R_k^2 \left| \sum_{s > k} \frac{1}{z_s^2} \right| + R_k^3 \left| \sum_{s > k} \frac{1}{z_s^3} \right| + \frac{1}{\lambda_s^2(\lambda_s - R_k)} \\
= R_k^2 \left| \sum_{s > k} \frac{1}{z_s^2} \right| + R_k^3 \int_0^{\infty} \frac{dn(t)}{t^2(t - R_k)} \\
\leq R_k^2 \left| \sum_{s > k} \frac{1}{z_s^2} \right| + CR_k^2 \log \left(1 + \frac{1}{\varepsilon_k}\right).
\]

Now we estimate the probability of the event \( \left| \sum_{s > k} \frac{1}{z_s^2} \right| > 1 \). Using the Bernstein-Hoeffding inequality (Appendix, 5.1), we have, by the same way as in the Sect. 4.2.1,

\[
\mathbb{P} \left( \sum_{s > k} \frac{z_s^{-2}}{z_s} \geq \frac{1}{z_s^2} \geq 1 \right) \leq \frac{1}{2} \mathbb{P} \left( \sum_{s = k2^{m+1}}^{2^{m+1}} \sum_{s = k2^{m+1}} \frac{z_s^{-2}}{z_s} \geq \frac{6}{\pi^2(m + 1)^2} \right) \\
\leq \sum_{m=0}^{\infty} \exp \left( -\frac{C_1k2^{m+1}}{(m + 1)^4} \right) \leq \exp(-C_2k),
\]

where \( C_1, C_2 \) are some deterministic constants. Taking \( N_k = k \) we obtain for \( k > K \), where \( K \) is some deterministic constant, that

\[
\log \mathbb{P} \left( \frac{2\pi}{N_k} \sup_{|z| = R_k} \left| \frac{\partial}{\partial \beta} \log |W_\infty(z)| \right| \geq \frac{a}{8} R_k \log b R_k \right) \lesssim -k.
\]

Now, since

\[
\mathbb{P} \left\{ \sup_{|z| = R_k} \log |W_\infty(z)| \geq \frac{a}{4} R_k \log b R_k \right\} \\
\leq \mathbb{P} \left\{ \sup_{0 \leq s < N_k} \log |W_\infty(\zeta_s)| + 2\pi N_k \sup_{|z| = R_k} \left| \frac{\partial}{\partial \beta} \log |W_\infty(z)| \right| \geq \frac{a}{4} R_k \log b R_k \right\} \\
\leq \mathbb{P} \left\{ \sup_{0 \leq s < N_k} \log |W_\infty(\zeta_s)| \geq \frac{a}{8} R_k \log b R_k \right\} \\
+ \mathbb{P} \left\{ \frac{2\pi}{N_k} \sup_{|z| = R_k} \left| \frac{\partial}{\partial \beta} \log |W_\infty(z)| \right| \geq \frac{a}{8} R_k \log b R_k \right\},
\]

for \( k > K \) we get

\[
\log \mathbb{P} \left\{ \sup_{|z| = R_k} \log |W_\infty(z)| \geq \frac{a}{4} R_k \log b R_k \right\} \lesssim -\log^{2(b-1)} R_k. \tag{15}
\]
Finally, by inequalities (14) and (15), there exists some deterministic constant $K_1$ such that for $k > K_1$ we get

$$\log \mathbb{P} \left\{ \sup_{|z|=R_k} \log |W(z)| \geq \frac{R_k^2}{2} - \frac{a}{8} R_k \log^b R_k \right\} \leq -\log^{2(b-1)} R_k.$$  (16)

### 4.3 The end of the proof

Given $C > 0$, there exists $M \in \mathbb{N}$ such that for $k \geq M$ we have

$$\exp \left( -C \log^2 R_k \right) \lesssim R_k^{-4} \lesssim k^{-2}.$$  

Hence, by the Borel-Cantelli lemma it follows from (16) that with probability one for all but a finite number of $k$ we have

$$\sup_{|z|=R_k} \log |W(z)| = \frac{R_k^2}{2} - \frac{a}{8} R_k \log^b R_k.$$  (17)

From now on we assume that (17) holds for $k \geq k_0$.

Now let $R_{k-1} \leq R \leq R_k$, $k \geq k_0$. Then for every $z$ on the circle $|z| = R$ almost surely

$$\sup_{|z|=R} \log |W(z)| \leq \sup_{|z|=R_k} \log |W(z)| \leq \frac{R_k^2}{2} - \frac{a}{8} R_k \log^b R_k.$$  

Recall that $R_k \sim \sqrt{k}$ and, by Lemma 4.3, $R_k - R_{k-1} \sim \frac{C}{\sqrt{k}}$ as $k \to \infty$. Hence, almost surely, for large $R$ we have

$$\sup_{|z|=R} \log |W(z)| < \frac{1}{2} \left( R_{k-1} + \frac{C_1}{\sqrt{k}} \right)^2 - \frac{a}{8} R_k \log^b R_k$$

$$= \frac{R_{k-1}^2}{2} + \frac{R_{k-1} C_1}{\sqrt{k}} + \frac{C_1^2}{2k} - \frac{a}{8} R_k \log^b R_k$$

$$\leq \frac{R^2}{2} - \frac{3}{p} \log(R + 1).$$

Thus, with probability one we have

$$\|W\|_{F^p} = \int_{\mathbb{C}} |W(z)|^p \exp \left( -p \frac{|z|^2}{2} \right) dm(z) \leq C \int_{\mathbb{C}} (|z| + 1)^{-3} dm(z) < \infty.$$  \qed
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5 Appendix

Here we recall some notions and results which we have used in this note.

5.1 Bernstein-Hoeffding concentration inequality [14, Theorem 2.2.6]

Let $X_1, X_2, \ldots, X_N$ be independent random variables such that $a_i \leq X_i \leq b_i$ for every $i$. Then, for any $t > 0$ we have

$$P \left( \sum_{n=1}^{N} (X_n - E(X_n)) \geq t \right) \leq \exp \left( -\frac{2t^2}{\sum_{i=1}^{N} (b_i - a_i)^2} \right).$$

5.2 Khinchine-Kolmogorov theorem on convergence of random series [1, Theorem 22.6]

Let $(Y_k)_{k \in \mathbb{N}}$ be a sequence of independent real-valued random variables with zero expectation. If

$$\sum \text{Var} (Y_k) < \infty,$$

then the series $\sum Y_k$ converges a.s.
5.3 Kolmogorov maximal inequality [1, Theorem 22.4]

Let $Y_1, Y_2, \ldots, Y_N$ be a sequence of independent real-valued random variables with zero expectation and finite variances. For $\gamma > 0$

$$P \left( \max_{1 \leq k \leq N} \left| \sum_{m=1}^{k} Y_m \right| \geq \gamma \right) \leq \frac{1}{\gamma^2} \sum_{m=1}^{N} \text{Var}(Y_m).$$

5.4 Weyl-type criterion of uniform distribution of random sequence [6, Theorem 2]

Let $X_1, X_2, \ldots$ be a sequence of independent real-valued random variables with characteristic functions $\phi_1, \phi_2, \ldots$. Then this sequence is uniformly distributed modulo 1 almost surely, that is

$$\lim_{N \to \infty} \frac{\# \{k : X_k - \lfloor X_k \rfloor < x \}}{N} = x, \quad \forall x \in [0, 1)$$

with probability one, if and only if for every $k \in \mathbb{N}$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \phi_n(2\pi k) = 0.$$

5.5 Definition of a set of disks with zero linear density [11, Chapter II, §1]

A set $D$ of disks $D_j$ in the complex plane is said to have zero linear density, if

$$\lim_{r \to \infty} \frac{1}{r} \sum_{|c_j| < r} \lambda_j = 0,$$

where $\lambda_j$ is the radius and $c_j$ is the center of $D_j$.

5.6 Levin-Pfluger theorem I [11, Chapter II, §1, Theorem 1]

Let a discrete set $\mathcal{N} \subset \mathbb{C}$ have an angular density of index $\rho(r)$, where $\rho(r)$ is a proximate order with $\rho = \lim_{r \to \infty} \rho(r) \notin \mathbb{N}$. Let $\Delta$ be a nondecreasing function such that for all but a countable set of angles

$$\Delta(\beta) - \Delta(\alpha) = \lim_{r \to \infty} \frac{n(t, \alpha, \beta)}{r^\rho(t)}.$$
Then for \( z \in \mathbb{C} \setminus E \), where \( E \) is a set of disks with zero linear density, the canonical product

\[
W(z) = \prod_k G \left( \frac{z}{z_k}; \lfloor \rho \rfloor \right)
\]

satisfies the asymptotic relation

\[
\lim_{r \to \infty} \frac{\log |W(re^{i\theta})|}{r^{\rho(r)}} = \frac{\pi}{\sin(\pi \rho)} \int_{\theta}^{\theta + 2\pi} \cos (\rho(\psi - \theta - \pi)) \, d\Delta(\psi).
\]

**5.7 Levin-Pfluger theorem II** [11, Chapter II, §1, Theorem 2]

Let a set \( N \subset \mathbb{C} \) have an angular density of index \( \subset (r) \), where \( \rho(r) \) is a proximate order with \( \rho = \lim_{r \to \infty} \rho(r) \in \mathbb{N} \). Let \( \Delta \) be a nondecreasing function such that for all but a countable set of angles

\[
\Delta(\beta) - \Delta(\alpha) = \lim_{t \to \infty} \frac{n(t, \alpha, \beta)}{t^{\rho(t)}}.
\]

Let the following limits exist and be finite

\[
S := \sum z_n^{-\rho}, \\
\delta := \lim_{r \to \infty} r^{\rho - \rho(r)} \sum_{|z_n| > r} \frac{1}{z_n^{\rho}}.
\]

Then for \( z \in \mathbb{C} \setminus E \), where \( E \) is a set of disks with zero linear density, the entire function

\[
W(z) = e^{S \cdot z} \prod_k G \left( \frac{z}{z_k}; \rho \right)
\]

satisfies the following asymptotic relation

\[
\lim_{r \to \infty} \frac{\log |W(re^{i\theta})|}{r^{\rho(r)}} = - \int_{\theta}^{\theta + 2\pi} (\psi - \theta) \sin (\rho(\psi - \theta)) \, d\Delta(\psi) \\
+ \frac{\delta}{\rho} \cos (\rho(\theta - \arg \delta)).
\]

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