EXTENSIONS AND WELL’S TYPE EXACT SEQUENCE OF SKEW BRACES

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Abstract. In this article, we give a description of the split exact sequences of left skew braces. We define a free action of the second cohomology group of a left skew brace $H$ by $\text{Ann}(I)$ on $\text{Ext}_\alpha(H, I)$ and show that this action becomes transitive if $I$ is a trivial skew brace. We also generalize the Well’s type exact sequence for extensions by the trivial skew brace.

1. Introduction

A triple $(E, +, \circ)$, where $(E, +)$ and $(E, \circ)$ are groups is said to be a left skew brace if

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds for all $a, b, c \in E$, where $-a$ denotes the inverse of $a$ in $(E, +)$. In 2007, Rump [20] introduced classical braces to study involutive and non-degenerate solutions of the Yang-Baxter equation. Later, Guarnieri and Vendramin [14] generalized this concept to skew brace to study the non-degenerate solution of the Yang-Baxter equation, which is further generalized to semi-braces by Catino, Colazzo, and Stefanelli in [11] to study non-bijectional solutions of the Yang-Baxter equation. In [7], Ben David and Ginosar investigated extensions of bijective 1-cocycles. Carter, Elhambadi and Satio in [8] developed homology and cohomology theories for solution sets of the Yang-Baxter equations. Different homology theories for various structures related to solutions of the Yang-Baxter equations were investigated extensively by Lebed and Vendramin [17]. Cohomology and extensions of linear cycle sets with trivial actions is studied by Lebed and Vendramin [18]. Recently generalized by Jorge A. Guccione, Juan J. Guccione and Christian Valqui [13] to non trivial actions. Various type of products like matched product, semi-direct product, asymmetric product has been defined for the solutions of Yang-Baxter equation [see [3], [5], [9], [10], [12], [21]]. In [23], M. K. Yadav and author developed the theory of skew brace extensions for skew brace extensions by an abelian group and developed the Well’s type exact sequence for skew braces. This work can be thought as a generalization of [9], [15] at the level of extensions. The fundamental exact sequence of Wells for groups was introduced by C Wells in [22]. The fundamental exact sequence of Wells with various applications is carried out in all fine details in [19, Chapter 2]. A similar exact sequence for cohomology, extensions and automorphisms of quandles was constructed in [6]. In this paper, we define a new product for the skew braces and construct few examples. We give constructions for skew braces similar to that of group theory and generalize the Well’s type exact sequence for the trivial skew brace.

2. Preliminaries

An algebraic structure $(E, +, \circ)$ is said to be a left skew brace if $(E, +)$ and $(E, \circ)$ are a group and the following compatibility condition holds:

$$a \circ (b + c) = a \circ b - a + a \circ c$$

(1)

for all $a, b, c \in E$, where $-a$ denotes the inverse of $a$ with respect to ‘+’. Notice that the identity element $\theta$ of $(E, +)$ coincides with the identity element of $(E, \circ)$.

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For a left skew brace $E$ and $a \in E$, define a map $\lambda_a : E \to E$ by
$$\lambda_a(b) = -a + (a \circ b)$$
for all $b \in E$. The automorphism group of a group $G$ is denoted by $\text{Aut}(G)$. We have the following result for skew braces.

**Lemma 2.1.** Let $(E, +, \circ)$ be a left skew brace, then for each $a \in E$, the map $\lambda_a$ is an automorphism of $(E, +)$ and the map $\lambda : (E, \circ) \to \text{Aut}(E, +)$ given by $\lambda(a) = \lambda_a$ is a group homomorphism.

A sub skew brace $I$ of a left skew brace $E$ is said to be a left ideal of $E$ if $\lambda_a(y) \in I$ for all $a \in E$ and $y \in I$. A left ideal of $E$ is said to be an ideal if $(I, \circ)$ is a normal subgroup of $(E, \circ)$. The Socle of a skew brace $E$ is defined as $\text{Soc}(E) = \text{Ker} \lambda \cap Z(E, +)$, where $Z(E, +)$ represents the centre of the group $(E, +)$ and the annihilator of $E$ is defined as $\text{Ann}(E) = \text{Soc}(E) \cap Z(E, \circ)$.

The following is an easy but important observation, which will be used several times in what follows.

**Lemma 2.2.** Let $E$ be a left skew brace. Then for all $a, b \in E$, the following hold:

(i) $a + b = a \circ \lambda_a^{-1}(b)$.
(ii) $a \circ b = a + \lambda_a(b)$.

Let $E_1$ and $E_2$ be two left skew braces. A map $f : E_1 \to E_2$ is said to be a skew brace homomorphism if $f(a + b) = f(a) + f(b)$ and $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in E_1$. A one-to-one and onto skew brace homomorphism from $E_1$ to itself is called an automorphism of $E_1$. The kernel of a homomorphism $f : E_1 \to E_2$ is defined to be the subset $\{a \in E_1 \mid f(a) = 0\}$ of $E_1$. It turns out that $\text{Ker}(f)$, the kernel of $f$, is an ideal of $E_1$. The set of all skew brace automorphisms of a left skew brace $E$, denoted by $\text{Aut}(E)$, is a group.

Let $H$ and $I$ be two left skew braces. By an extension of $H$ by $I$, we mean a left skew brace $E$ with an exact sequence
$$0 \to I \to E \xrightarrow{i} H \to 0,$$
where $i$ and $\pi$ are injective and surjective brace homomorphisms, respectively. Thereafter, we denote the image of $y$ under $i$ by $y$ itself for all $y \in I$. A set map $s : H \to E$ is called a set-theoretic section of $E$ if $\pi(s(h)) = h$ for all $h \in H$ and $s(0) = 0$. The abbreviation ‘st-section’ will be used for ‘set-theoretic section’ throughout. We call $E$ to be split exact sequence of skew braces if there exist a st-section of $E$ which is a skew brace homomorphism.

### 3. Split Extensions of Skew Brace

Let $H$ and $I$ be two left skew braces. Let $\mu : (H, +) \to \text{Aut}(I, +)$, $\sigma : (H, \circ) \to \text{Aut}(I, \circ)$ be anti-homomorphisms, and $\nu : (H, \circ) \to \text{Aut}(I, \circ)$ be a homomorphism. Let $\mu, \sigma$ and $\nu$ satisfy the following compatibility condition

$$\nu_{h_1 \circ(h_2 + h_3)}(\sigma_{h_2 + h_3}(\nu_{h_1}^{-1}(y_1)) \circ \nu_{h_2}^{-1}(\mu_{h_3}(y_2) + y_3)) = \mu_{-h_1 + (h_2 \circ h_3)}(\nu_{h_1 \circ h_2}(\sigma_{h_2}(\nu_{h_1}^{-1}(y_1)) \circ \nu_{h_2}^{-1}(y_2)) - y_1)$$
$$+ \nu_{h_1 \circ h_3}(\sigma_{h_3}(\nu_{h_1}^{-1}(y_1)) \circ \nu_{h_3}^{-1}(y_3))$$

for all $y_1, y_2, y_3 \in I$ and $h_1, h_2, h_3 \in H$.

**Theorem 3.1.** Let $H$ and $I$ be two skew braces with $(\nu, \mu, \sigma)$, as defined above and satisfying (2), then the operations

$$(h_1, y_1) + (h_2, y_2) = (h_1 + h_2, \mu_{h_1}(y_1) + y_2),$$

$$(h_1, y_1) \circ (h_2, y_2) = (h_1 \circ h_2, \nu_{h_1 \circ h_2}(\sigma_{h_2}(\nu_{h_1}^{-1}(y_1)) \circ \nu_{h_2}^{-1}(y_2))$$

define a left skew brace structure on $H \times I$. 

Proof. It is easy to check that the given operations define group structure on $H \times I$ and the condition (1) follows from the compatibility condition of $(\nu, \mu, \sigma)$.

We call this structure a split semi-direct product of $H$ by $I$ with respect to the triplet $(\nu, \mu, \sigma)$ and denote it by $(H, I, \nu, \mu, \sigma)$.

Lemma 3.2. Let $(H, I, \nu, \mu, \sigma)$ be split semi-direct product of $H$ by $I$ with respect to some triplet $(\nu, \mu, \sigma)$. Then the following short exact sequence of skew braces

$$E := 0 \to I \xrightarrow{i} (H, I, \nu, \mu, \sigma) \xrightarrow{\pi} H \to 0$$

splits, where $i$ and $\pi$ are natural injection and projection respectively.

Proof. It is easy to check that the map $s : H \to (H, I, \mu, \sigma, \nu)$ given by $s(h) = (h, 0)$ is both a homomorphism of skew braces and a st-section of $E$ simultaneously.

Theorem 3.3. Let $E := 0 \to I \xrightarrow{i} E \xrightarrow{\pi} H \to 0$ be a split short exact sequence of skew braces. Then $E$ is a split semi-direct product of $H$ by $I$.

Proof. Let the short exact sequence $E := 0 \to I \xrightarrow{i} E \xrightarrow{\pi} H \to 0$ split. Then there exists a st-section $s : H \to (H, I, \mu, \sigma, \nu)$, which is also a skew brace homomorphism. Define $\mu : H \to Aut(I, +)$, $\sigma : H \to Aut(I, \circ)$, and $\nu : H \to Aut(I, +)$ by

$$\begin{align*}
\nu_h(y) &= -s(h) + (s(h) \circ y), \\
\mu_h(y) &= -s(h) + y + s(h), \\
\sigma_h(y) &= s(h)^{-1} \circ y \circ s(h).
\end{align*}$$

(5)

Since $E$ is a skew brace, we have

$$(s(h_1)+y_1)\circ(s(h_2)+y_2+s(h_3)+y_3) = (s(h_1)+y_1)\circ(s(h_2)+y_2)-(s(h_1)+y_1)+(s(h_1)+y_1)\circ(s(h_3)+y_3).$$

(6)

Using (5) and linearity of $s$ in ‘+’ and ‘\circ’, we can easily establish that $(\nu, \mu, \sigma)$ satisfies (2). Hence we have semi-direct product $(H, I, \mu, \sigma, \nu)$. We know that every element $x \in E$ can be uniquely written as $x = s(h) + y$. Define $\phi : E \to (H, I, \mu, \sigma, \nu)$ by $\phi(s(h) + y) = (h, y)$. Then $\phi$ is an isomorphism of skew braces and the diagram

$$\begin{array}{cccccc}
0 & \xrightarrow{i} & I & \xrightarrow{i'} & (H, I, \nu, \mu, \sigma) & \xrightarrow{\pi'} & H & \xrightarrow{\pi} & 0 \\
\downarrow 1d & & \downarrow \phi & & \downarrow 1d & & \downarrow \phi & & \downarrow 1d \\
0 & \xrightarrow{i} & I & \xrightarrow{i'} & (H, I, \nu, \mu, \sigma) & \xrightarrow{\pi'} & H & \xrightarrow{\pi} & 0
\end{array}$$

commutes, where $i'$ and $\pi'$ are natural injection and projection, respectively. This completes the proof.

4. Examples

In this section we provide some examples of split semi direct product of skew braces. We have used GAP to compute $\nu, \mu$ and $\sigma$. 
Example 1 Let \( Z \) and \( \mathbb{C} \) be trivial skew braces. Define \( \nu, \mu, \sigma : Z \to Aut(\mathbb{C}) \) by \( \nu_1(x) = \mu_1(x) = \sigma_1(x) = -x \). Using (3) and (4), we can define a skew brace structure on \( Z \times C \) by
\[
(l, y_1) + (m, y_2) = (l + m, (-1)^{m+n}y_1 + y_2),
\]
\[
(l, y_1) \circ (m, y_2) = (l + m, y_1 + (-1)^{m+n}y_2).
\]

Example 2 Let \( H = D_{2n} = \langle a, b \mid a^{2n} = b^2 = e, bab = a^{-1} \rangle \) and \( I = \mathbb{Z}_p \) be trivial skew braces, where \( D_{2n} \) and \( \mathbb{Z}_p \) denote dihedral group of order \( 4n \) and cyclic group of order \( p \) respectively. Define \( \nu, \mu, \sigma : D_{2n} \to Aut(\mathbb{Z}_p) \) by \( \nu_a(x) = \mu_a(x) = \sigma_a(x) = -x \) and \( \nu_b(x) = \mu_b(x) = \sigma_b(x) = -x \). Using (3) and (4), we can define a skew brace structure on \( D_{2n} \times \mathbb{Z}_p \) by
\[
(a^i b^j, y_1) + (a^m b^n, y_2) = (a^{i+m} b^{j+n}, y_2 + (-1)^{m+n} y_1),
\]
\[
(a^i b^j, y_1) \circ (a^m b^n, y_2) = (a^{i+m} b^{j+n}, y_1 + (-1)^{i+j} y_2).
\]

If we have trivial skew brace \( H = D_n \), where \( n \) is odd and \( I \) be the same as above, then we can define \( \nu, \mu, \sigma : D_n \to Aut(\mathbb{Z}_p) \) by \( \nu_a(x) = \mu_a(x) = \sigma_a(x) = x \) and \( \nu_b(x) = \mu_b(x) = \sigma_b(x) = -x \).

Example 3 Let \( H = \mathbb{Z}_8 \) be trivial skew brace and \( I = \mathbb{Z}_3 \times \mathbb{Z}_2 \) be skew brace of order 6 defined in [1] by the following operations
\[
(n, m) + (s, t) = (n + 2^m s, m + t),
\]
\[
(n, m) \circ (s, t) = (2^m n + 2^m s, m + t).
\]

We have \( (I, +) = ((1, 0), (0, 1)) \cong S_3 \) and \( (I, \circ) = ((1, 1), (0, 1)) \cong \mathbb{Z}_6 \). We take \( \mu_a(n, m) = (n, m) \), \( \sigma_a(n, m) = (n, m)^{-1} = (2n, m) \), \( \nu_a(n, m) = (2n, m) \) for all \( (n, m) \in I \), where \( a \) is a generator of \( H \). Hence the additive group of skew brace structure on \( (H, I, \nu, \mu, \sigma) \) is just direct product of their respective additive groups and multiplicative group is given as follows
\[
(a^k, (n, m)) \circ (a^i, (s, t)) = (a^{k+i}, ((2n, m) \circ (2s, t))^k).
\]

Example 4 Let \( H \) be brace of order 4 defined in [2] by \( (H, +) = \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( (H, \circ) = ((0, 1), (0, 0)) \cong \mathbb{Z}_4 \) and \( I \) be a brace such that \( (I, +) = \mathbb{Z}_4 \), \( (I, \circ) = (1, 2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Consider \( \mu_a = Id \) for all \( x \in H \), where \( Id \) denotes the identity mapping on \( I \) and \( \nu_{(0, 1)}(x) = \sigma_{(0, 1)}(x) = -x \). Then split semi-direct product of \( I \) by \( H \) is given by the skew brace with additive group as direct product of \( H \) and \( I \) and multiplicative group structure is given as follows
\[
((0, 1)^k, l) \circ ((0, 1)^n, m) = ((0, 1)^{k+n}, l + (-1)^k m + (-1)^n l m).
\]

Example 5 Let \( H \) be the brace of order 8 with additive group \( \mathbb{Z}_8 \) having Socle of order 2 and \( (H, \circ) = (1, 2) \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \) defined in [2] and \( I \) be brace of order 4 as defined in Example 3. Then we have total 8 different split semi-direct products of \( I \) by \( H \), interestingly with \( \mu_x = Id \) for all \( x \in H \) in all cases. We list few cases

(i) \( \nu_1(x) = x^{-1}, \nu_2(x) = x \) and \( \sigma_h(x) = x \) for all \( x \in I, \ h \in H \).

(ii) \( \nu_1(x) = x^{-1}, \nu_2(x) = x^{-1} \) and \( \sigma_2(2) = 3, \ \sigma_1(x) = x \) for all \( x \in I \).

5. General Extensions of Skew Braces

Let \((H, +, \circ)\) and \((I, +, \circ)\) be two skew braces, \( E : 0 \to I \overset{i}{\to} E \overset{p}{\to} H \to 0 \) be an extension of \( H \) by \( I \) and let \( s : H \to E \) be an st-section of \( E \). Corresponding to \( s \), consider the pair \((\beta, \tau)\), where \( \beta \) and \( \tau \) are
defined as
\[ \beta(h_1, h_2) := -s(h_1 + h_2) + s(h_1) + s(h_2), \]
\[ \tau(h_1, h_2) := s(h_1 \circ h_2)^{-1} \circ s(h_1) \circ s(h_2). \]

It is easy to see that \( \nu, \mu, \) and \( \sigma, \) defined in (5), need not be homomorphisms in general, but they satisfy

the following identities

\[ \nu_{h_1 \circ h_2} = \nu_{h_1} \nu_{h_2} \lambda_{\tau(h_1, h_2)}, \]
\[ \mu_{h_1 + h_2} = \mu_0 \beta(h_1, h_2) \mu_{h_1}, \]
\[ \sigma_{h_1 \circ h_2} = \sigma_{\tau(h_1, h_2)} \sigma_{h_1}, \]

where

\[ i^+_y(z) := y + z - y, \]
\[ i^+_y(z) := y \circ z \circ y^{-1}, \]

are inner automorphisms of \((H, +)\) and \((H, \circ)\), respectively, and \( \beta \) and \( \tau \) are as defined above in (7) and (8).

Let \( N \) be the smallest normal subgroup of \( \text{Aut}(I, +) \) generated by the set \( \{ \lambda_y \mid y \in I \} \). Let \( \text{Inn}(I, +) \) and \( \text{Inn}(I, \circ) \) be the inner automorphism subgroups of \( \text{Aut}(I, +) \) and \( \text{Aut}(I, \circ) \) respectively. Then we have the maps \( \bar{\nu} : (H, \circ) \to \text{Aut}(I, +)/N, \bar{\mu} : (H, +) \to \text{Aut}(I, \circ)/\text{Inn}(I, +) \) and \( \bar{\sigma} : (H, \circ) \to \text{Aut}(I, \circ)/\text{Inn}(I, \circ) \) defined by \( \nu, \mu, \) and \( \sigma \) composing with natural projections respectively. We call the triplet \( \chi := (\nu, \mu, \sigma) \) satisfying (9), (10), and (11) an action of \( H \) on \( I \) and corresponding triplet \( \bar{\chi} := (\bar{\nu}, \bar{\mu}, \bar{\sigma}) \) will be called a coupling from \( H \) to \( I \) corresponding to \( (\nu, \mu, \sigma) \). Let \( \chi = (\nu, \mu, \sigma) \) and \( \chi' = (\nu', \mu', \sigma') \) be two actions. Then we say that \( \chi' \approx \bar{\chi} \) if there exists a map \( \theta : H \to I \) such that \( \theta(0) = 0 \) and \( \nu'_h = \nu_h \lambda_{\theta(h)}, \mu'_h = \mu_0 \beta(h) \mu_h \) and \( \sigma' = i^+_{\nu_h} \sigma_h \).

Remark: If \( \chi' \approx \bar{\chi} \) then \( \chi' \approx \bar{\chi} \) but converse need not be true. Note that the map \( \theta : H \to I \) mentioned above need not be unique.

With this setting, we have

Proposition 5.1. Let \( 0 \to I \to E \xrightarrow{\pi} H \to 1 \) be an extension of a left skew brace \( I \) by \( H \). Then the following hold:

(1) The coupling \( \bar{\chi} \) is independent of the choice of an st-section.

(2) Equivalent extensions have the same coupling.

Proof. (1) Let \( s_1 \) and \( s_2 \) be two st-sections of \( \pi \). We know that two sections differ by an element of \( I \), hence for an element \( h \in H \), there exist \( y_h \in I \) such that \( s_2(h) = s_1(h) \circ y_h \). Let \( \chi = (\nu, \mu, \sigma) \) and \( \chi' = (\nu', \mu', \sigma') \) be actions corresponding to \( s_1 \) and \( s_2 \) respectively. Define \( \theta : H \to I \) be \( \theta(h) = y_h \). It can be easily seen that \( \chi' \approx \bar{\chi} \) using \( \theta \) as a required map.

(2) Let \( E \) and \( E' \) be two equivalent extensions. Then there exist a skew brace homomorphism \( \phi : E' \to E \) such that the following diagram commutes

\[
\begin{array}{cccccc}
0 & \longrightarrow & I & \longrightarrow & E' & \longrightarrow & H \quad \longrightarrow & 0 \\
& & \downarrow{\text{Id}} & & \downarrow{\phi} & \downarrow{\text{Id}} & \downarrow{\text{Id}} & \\
0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & H \quad \longrightarrow & 0.
\end{array}
\]

Let \( s : H \to E' \) be any st-section of the extension \( E' \). Then \( \phi s : H \to E \) is a st-section of extension the \( E \). Let \( \chi = (\nu, \mu, \sigma) \) be actions of \( E \) corresponding to \( \phi s \) and \( \chi' = (\nu', \mu', \sigma') \) be actions of \( E' \) corresponding
to \( s \), respectively. Then we have \( \nu = \nu', \mu = \mu' \) and \( \sigma = \sigma' \). Hence \( \tilde{\chi} \approx \tilde{\chi}' \) by taking \( \theta : H \to I \) to be \( \theta(h) = 0 \) for all \( h \in H \). As we have already proved that coupling is independent of an st-section, so this holds for all st-sections of \( E \) and \( E' \).

\[ \square \]

Let \( \text{Ext}(H, I) \) denote the set of equivalence classes of all extensions of \( H \) by \( I \). Equivalence class of an extension \( \mathcal{E} : 0 \to I \to E \to H \to 0 \) is denoted by \([\mathcal{E}]\). As a consequence of the preceding proposition, it follows that each equivalence class of extension of \( H \) by \( I \) admits a unique coupling \( \tilde{\chi} = (\tilde{\nu}, \tilde{\mu}, \tilde{\sigma}) \) corresponding to actions \( \chi = (\nu, \mu, \sigma) \) of \( H \) on \( I \). Let \( \text{Ext}((\tilde{\nu}, \tilde{\mu}, \tilde{\sigma})(H, I) \) denote the equivalence class of those extensions of \( H \) by \( I \) whose corresponding coupling is \((\tilde{\nu}, \tilde{\mu}, \tilde{\sigma})\). We can easily establish

**Corollary 5.2.** \( \text{Ext}(H, I) = \bigsqcup_{(\tilde{\nu}, \tilde{\mu}, \tilde{\sigma})} \text{Ext}((\tilde{\nu}, \tilde{\mu}, \tilde{\sigma})(H, I). \)

**Proposition 5.3.** Let \( \mathcal{E} \) be an extension of \( H \) by \( I \). Then the following hold

1) Let \( s \) be an st-section of \( \mathcal{E} \). Then the pair \((\beta, \tau)\) corresponding to \( s \) together with action defined in \((\mathcal{E})\) satisfies

\[
\beta(h_1, h_2 + h_3) + \beta(h_2, h_3) - \beta(h_1 + h_2, h_3) - \mu h_3(\beta(h_1, h_2)) = 0, \tag{14}
\]

and

\[
\tau(h_1, h_2 \circ h_3) \circ \tau(h_2, h_3) \circ \tau(h_1 \circ h_2, h_3)^{-1} \circ (\sigma h_3(\tau(h_1, h_2)))^{-1} = 0. \tag{15}
\]

2) Let \( s_1 \) and \( s_2 \) be two an st-sections of \( \mathcal{E} \), and let \((\beta_1, \tau_1)\) and \((\beta_2, \tau_2)\) be the pairs corresponding to \( s_1 \) and \( s_2 \), respectively. Then there exists a map \( \theta : H \to I \) such that

\[
s_2(h) = s_1(h) \circ \theta(h) = s_1(h) + \nu h(\theta(h)), \tag{16}
\]

and

\[
\nu h_1(h_2(-\theta(h_1 + h_2)) + \beta_1(h_1, h_2) + \mu h_2(\nu h_1(\theta(h_1))) + \nu h_2(\theta(h_2)) = \beta_2(h_1, h_2), \tag{17}
\]

for all \( h, h_1, h_2, h_3 \in H \).

3) Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be two equivalent extensions of \( H \) by \( I \), and let \( s_1 \) and \( s_2 \) be st-sections of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), respectively. Let \((\beta_1, \tau_1)\) and \((\beta_2, \tau_2)\) be the pairs corresponding to \( s_1 \) and \( s_2 \), respectively. Then there exists a map \( \theta : H \to I \) satisfying \((16) \) and \((17) \).

**Proof.** It is easy to see that \((1) \) and \((2) \) follows directly from definitions. Now we will prove \((3) \). Since \( \mathcal{E}_1 := 0 \to I \to E_1 \xrightarrow{\pi_1} H \to 0 \) and \( \mathcal{E}_2 := 0 \to I \xrightarrow{\phi} E_2 \xrightarrow{\pi_2} H \to 0 \) are two equivalent extensions, there exists an isomorphism \( \phi : E_1 \to E_2 \) such that the following diagram commutes

\[
\begin{array}{c}
0 \xrightarrow{} I \xrightarrow{} E_1 \xrightarrow{\pi_1} H \xrightarrow{} 0 \\
\downarrow \text{Id} \quad \downarrow \phi \quad \downarrow \text{Id} \\
0 \xrightarrow{} I \xrightarrow{} E_2 \xrightarrow{\pi_2} H \xrightarrow{} 0.
\end{array}
\]

Let \( s_1 \) be an st-section of \( \mathcal{E}_1 \). Then \( \phi s_1 \) is an st-section of \( \mathcal{E}_2 \). Let \((\beta_1, \tau_1)\) and \((\beta', \tau')\) be the pairs corresponding to \( s_1 \) and \( \phi s_1 \), respectively. By the commutativity of the above diagram, we have \( \phi(y) = y \), for all \( y \in I \), hence \( \beta_1 = \beta' \) and \( \tau_1 = \tau' \). Let \( s_2 \) be an st-section of \( \mathcal{E}_2 \) and \((\beta_2, \tau_2)\) be the pair corresponding to \( s_2 \). Using \((16), (17)\) for \( s_2 \) and \( \phi s_1 \), we get the desired result. \( \square \)
Definition 5.4. Let $\chi = (\nu, \mu, \sigma)$ be an action of $H$ on $I$, and $\beta, \tau : H \times H \to I$ such that $\beta$ and $\tau$ together with $\chi$ satisfies (14) and (15), respectively. Then the ordered pair $(\beta, \tau)$ is a 2-cocycle with action $\chi$.

Remark: Note that the pair $(\beta, \tau)$ defined by the (7) and (8) is a 2-cocycle corresponding to st-section $s$ with action defined by (5).

Let $E := 0 \to I \to E \xrightarrow{\bar{\nu}} H \to 0$ be an skew brace extension of $H$ by $I$. Let $s : H \to E$ be an st-section of $E$. Due to the compatibility condition of a left skew brace $E$, we have

\[
(s(h_1) \circ y_1) \circ (s(h_2) \circ y_2) = s(h_1) \circ (s(h_2) \circ y_2) - (s(h_1) \circ y_1) \circ (s(h_2) \circ y_3).
\]

From the above equality we see that the triple $(\chi, h, \beta)$ satisfy

\[
\nu_{h_1 \circ (h_2 + h_3)}(\tau(h_1, h_2 + h_3) \circ \sigma_{h_2 + h_3}(y_1) \circ \nu_{h_2 + h_3}^{-1}(\beta(h_2, h_3) + \mu_{h_3}(\nu_{h_2}(y_2)) + \nu_h(y_3))) = A,
\]

(18)

where

\[
A = \beta(h_1 - h_1, h_1 \circ h_3) + \mu_{h_3 + h_3}(\beta(h_1 \circ h_2, h_1) + \mu_{h_1 \circ h_2}(\tau(h_1, h_2) \circ \sigma_{h_2}(y_1) \circ y_2))
\]

\[- \nu_{h_1}(y_1) - \beta(h_1, h_1) + \nu_{h_1 \circ h_3}(\tau(h_1, h_3) \circ \sigma_{h_3}(y_1) \circ y_3),
\]

for all $h_1, h_2, h_3 \in H$ and $y_1, y_2, y_3 \in I$.

Remark: Note that if $I$ is an abelian group equipped with trivial skew brace structure, then the above condition will simplify to the condition defined for good triplet of actions in [24 Pg.5].

For $\alpha = (\bar{\nu}, \bar{\mu}, \bar{\sigma})$ a coupling from $H$ to $I$. Define

\[
Z^2_{\alpha}(H, I) := \left\{ (\chi, \beta, \tau) \mid \chi \text{ is an action of } H \text{ on } I, \chi \approx \alpha, \text{ and } (\beta, \tau) \text{ is a 2-cocycle with action } \chi \text{ and satisfy (12)} \right\}.
\]

Let $(\chi_1, \beta_1, \tau_1)$ and $(\chi_2, \beta_2, \tau_2)$ be two elements of $Z^2_{\alpha}(H, I)$, where $\chi_1 = (\nu_1, \mu_1, \sigma)$ and $\chi_2 = (\nu_2, \mu_2, \sigma)$. We say that $(\chi_1, \beta_1, \tau_1) \sim (\chi_2, \beta_2, \tau_2)$ if there exits a map $\theta : H \to I$ such that $\chi_2 \approx \chi_1$ by $\theta$ and $\beta_1, \beta_2$ satisfy (16), $\tau_1, \tau_2$ satisfy (17) with respect to $\theta$.

Proposition 5.5. The relation `$\sim$' defined in above para is an equivalence relation.

Proof. Reflexivity is easy to see by taking $\theta : H \to I$ given by $\theta(h) = 0$ for all $h \in H$. Now we will show that the above relation is symmetric. Let $(\chi_1, \beta_1, \tau_1) \sim (\chi_2, \beta_2, \tau_2)$, where $\chi_i = (\nu_i, \mu_i, \sigma)$ for $i = 1, 2$. We know that there exist a map $\theta : H \to I$ such that $\chi_1 \approx \chi_2$ by $\theta$ and (16), (17). Define $\psi : H \to I$ by $\psi(h) = \theta(h)^{-1}$. Then we have $\nu_{h_1}(h_2) = \nu_{h_2}(h_1)$, and hence $\nu_{h_1}(\psi(h)) = \nu_{h_2}(\theta(h))$. This proves that $\chi_1 \approx \chi_2$ by $\psi$. Similarly we can prove that $\beta_2, \beta_1$ satisfy (16) and $\tau_1, \tau_2$ satisfy (17) with respect to $\psi$. Next we prove transitivity. Let $(\chi_1, \beta_1, \tau_1) \sim (\chi_2, \beta_2, \tau_2)$ (by $\theta_1$) and $(\chi_2, \beta_2, \tau_2) \sim (\chi_3, \beta_3, \tau_3)$ (by $\theta_2$), where $\chi_i = (\nu_i, \mu_i, \sigma)$ for $i = 1, 2, 3$. We claim that $\chi_3 \approx \chi_1$ by $\phi : H \to I$ defined by $\psi(h) = \theta_1(h) \circ \theta_2(h)$. We have

\[
2\nu_{h_1} = 1_{\nu_{h_1}} \lambda_{\theta_1(h)}(h),
\]

(19)

and

\[
3\nu_{h} = 2\nu_{h} \lambda_{\theta_2(h)}.
\]

Combining these two equations we have

\[
3\nu_{h} = 1_{\nu_{h}} \lambda_{\theta_1(h)} \lambda_{\theta_2(h)} = 1_{\nu_{h}} \lambda_{\phi(h)}.
\]

For additive action we have

\[
2\mu_{h} = 1_{\nu_{h}} \lambda_{\theta_1(h)}(h).
\]
we have

\[ 3 \mu_h = i^+_{\nu_h(-\theta_2(h))} 2 \mu_h. \]

Combining the above two equations we have

\[ 3 \mu_h = i^+_{\nu_h(-\theta_2(h))} i^+_{\nu_h(-\theta_1(h))} 1 \mu_h. \]

Using (19) we have

\[ 3 \mu_h = i^+_{\nu_h(-\theta_2(h))} i^+_{\nu_h(-\theta_1(h))} 1 \mu_h. \]

Finally, using the relation \( a + \lambda_n(b) = a \circ b \), we get

\[ 3 \mu_h = i^+_{\nu_h(-\phi(h))} 1 \mu_h. \]

Similarly we can prove that \( 3 \sigma_h = i^\circ_{\phi(h)} -1 \sigma_h \), which shows that \( \chi_3 \approx \chi \) by \( \phi \). Next we prove that \( \beta \) and \( \beta_3 \) also satisfy \( \sigma \) with respect to \( \phi \). We have

\[ 1 \nu_{h_1 + h_2}(-\theta_1(h_1 + h_2)) + \beta_1(h_1, h_2) + 1 \mu_{h_2} (1 \nu_{h_1} (\theta_1(h_1))) + 1 \nu_{h_2} (\theta_1(h_2)) = \beta_2(h_1, h_2), \]

\[ 2 \nu_{h_1 + h_2}(-\theta_2(h_1 + h_2)) + \beta_2(h_1, h_2) + 1 \mu_{h_2} (2 \nu_{h_1} (\theta_2(h_1))) + 2 \nu_{h_2} (\theta_2(h_2)) = \beta_3(h_1, h_2). \]

Combining these two equations and using the fact that \( 1 \nu_{h_1} (\theta_1(h_1)) + 2 \nu_{h_2} (\theta_2(h_2)) = 1 \nu_{h} (\theta_1(h) \circ \theta_2(h)) \), we have

\[ \beta_3(h_1, h_2) = 2 \nu_{h_1 + h_2}(-\theta_2(h_1 + h_2)) + \nu_{h_1 + h_2}( -\theta_1(h_1 + h_2)) + \beta_1(h_1, h_2) + 1 \mu_{h_2} (1 \nu_{h_1} (\theta_1(h_1))) + 1 \nu_{h_2} (\theta_1(h_2)) + 2 \mu_{h_2} (2 \nu_{h_1} (\theta_2(h_1))) + 2 \nu_{h_2} (\theta_2(h_2)) \]

\[ = 1 \nu_{h_1 + h_2}(-\theta_2(h_1 + h_2) \circ \theta_1(h_1 + h_2) ) + \beta_1(h_1, h_2) + 1 \mu_{h_2} (1 \nu_{h_1} (\theta_1(h_1))) + 1 \nu_{h_2} (\theta_1(h_2)) + 2 \mu_{h_2} (2 \nu_{h_1} (\theta_2(h_1))) + 2 \nu_{h_2} (\theta_2(h_2)) \]

\[ = 1 \nu_{h_1 + h_2}(-\theta_2(h_1 + h_2) \circ \theta_1(h_1 + h_2) ) + \beta_1(h_1, h_2) + 1 \mu_{h_2} (1 \nu_{h_1} (\theta_1(h_1) \circ \theta_2(h_1))) + 1 \nu_{h_2} (\theta_1(h_2)) + 2 \mu_{h_2} (2 \nu_{h_1} (\theta_2(h_1))) + 2 \nu_{h_2} (\theta_2(h_2)) \]

\[ = 1 \nu_{h_1 + h_2}(-\phi(h_1 + h_2)) + \beta_1(h_1, h_2) + 1 \mu_{h_2} (1 \nu_{h_1} (\phi(h_1))) + 1 \nu_{h_2} (\phi(h_2)). \]

Similar calculation shows that \( \tau_1, \tau_3 \) satisfy \( \sigma \) with respect to \( \phi \). Hence the relation ‘\( \sim \)’ is an equivalence relation.

Define

\[ \mathcal{H}_\alpha^2(H, I) := Z_\alpha^2(H, I)/\sim \]

and denote \( [(\chi, \beta, \tau)] \in \mathcal{H}_\alpha^2(H, I) \), the equivalence class of \( (\chi, \beta, \tau) \). This concept will be used in next section.

6. ACTION OF COHOMOLOGY GROUP ON EXTENSIONS

In this section, we define a faithful group action of \( H_2^\alpha(H, Z(I)) \) \([23] \text{Pg.6}\) on \( \text{Ext}_\alpha(H, I) \) and we will show that this action is transitive whenever \( I \) is trivial skew brace.

**Theorem 6.1.** Let \( \alpha \) be a coupling from \( H \) to \( I \). Then there exists a bijection between \( \text{Ext}_\alpha(H, I) \) and \( \mathcal{H}_\alpha^2(H, I) \).
Proof. Define $\phi : \text{Ext}_\alpha(H, I) \to \mathcal{H}_\alpha^2(H, I)$ as follows. Let $\mathcal{E} := 0 \to I \xrightarrow{i} E \xrightarrow{\pi} H \to 0$ be an extension with coupling $\alpha$. Fix an $st$-section $s$, then there exists an action $\chi = (\nu, \mu, \sigma)$ as we defined in (5) such that $\bar{\chi} = \alpha$; also we have $(\beta, \tau)$ as we defined in (7) and (8), together they satisfy equation (18). Set $\phi([\mathcal{E}]) = [(\chi, \beta, \tau)]$.

Then by Proposition 5.1 and Proposition 5.3 the map $\phi$ is well defined. Next we define a map $\psi : \mathcal{H}_\alpha^2(H, I) \to \text{Ext}_\alpha(H, I)$ as follows. Given an element $(\chi, \beta, \tau)$ of $\mathcal{H}_\alpha^2(H, I)$, we define binary operations on the set $H \times I$ by setting

\[
\begin{align*}
(1) \ (h_1, y_1) \ + \ (h_2, y_2) &= (h_1 + h_2, \nu_{h_1 + h_2}^{-1} \beta(h_1, h_2) + \mu_{h_2}(\nu_{h_1}(y_1)) + \nu_{h_2}(y_2)), \\
(2) \ (h_1, y_1) \ \circ \ (h_2, y_2) &= (h_1 \circ h_2, \tau(h_1, h_2) \circ \sigma_{h_2}(y_1) \circ y_2).
\end{align*}
\]

It is easy to check that (14) and (15) gives the associativity of `+' and `·', respectively, which is enough to see that $(H \times I, +)$ and $(H \times I, \circ)$ are groups and (18) proves that `+' and `·' defined here satisfy (1). We denote this left skew brace structure by $(H, \circ)$. Now Consider the extension

$\mathcal{E}(\chi, \beta, \tau) := 0 \to I \xrightarrow{i} (H, I, \chi, \beta, \tau) \xrightarrow{\pi} H \to 0$,

where $i(y) = (0, y)$ and $\pi(h, y) = h$ for all $h \in H$ and $y \in I$. Define $\psi$ by setting

$$\psi(\langle \chi, \beta, \tau \rangle) = \mathcal{E}(\chi, \beta, \tau).$$

We show that the map $\psi$ is well defined. Let $(\chi_1, \beta_1, \tau_1) \sim (\chi_2, \beta_2, \tau_2)$, then there exist a map $\theta : H \to I$ such that $\chi_1 \approx \chi_2$ by $\theta$ and $\beta_1, \beta_2$ satisfy (16) and $\tau_1, \tau_2$ satisfy (17), respectively. Define $\zeta : \mathcal{E}(\chi_2, \beta_2, \tau_2) \to \mathcal{E}(\chi_1, \beta_1, \tau_1)$ given by

$$\zeta(h, y) = (h, \theta(h) \circ y).$$

We have

$$\zeta((h_1, y_1) + (h_2, y_2)) = (h_1 + h_2, \theta(h_1 + h_2) \circ \nu_{h_1 + h_2}^{-1} \beta(h_1, h_2) + 2\mu_{h_2}(\nu_{h_1}^{-1}(y_1)) + 2\nu_{h_2}(y_2)))$$

$$= (h_1 + h_2, \theta(h_1 + h_2) + \nu_{h_1 + h_2}^{-1} \beta(h_1, h_2) + 2\mu_{h_2}(\nu_{h_1}^{-1}(y_1)) + 2\nu_{h_2}(y_2)))$$

$$= (h_1 + h_2, \nu_{h_1 + h_2}^{-1} \beta(h_1, h_2) + \mu_{h_2}(\nu_{h_1}^{-1}(\theta(h_1))) + \nu_{h_2}(\theta(h_2))$$

$$+ 2\mu_{h_2}(\nu_{h_1}^{-1}(y_1)) + 2\nu_{h_2}(y_2)))$$

$$= (h_1 + h_2, \nu_{h_1 + h_2}^{-1} \beta(h_1, h_2) + \mu_{h_2}(\nu_{h_1}(\theta(h_1))) + \mu_{h_2}(\nu_{h_2}(y_1))$$

$$+ \nu_{h_2}(\theta(h_2) \circ y_2)))$$

$$= (h_1 + h_2, \nu_{h_1 + h_2}^{-1} \beta(h_1, h_2) + \mu_{h_2}(\nu_{h_1}(\theta(h_1) \circ y_1))$$

$$+ \mu_{h_2}(\nu_{h_2}(h_2) \circ y_2)))$$

$$= (h_1, \theta(h_1) \circ y_1) + (h_2, \theta(h_2) \circ y_2).$$

Which shows that $\zeta$ is linear in `+' . Similarly we can show that $\zeta$ is linear in `·' as well. It is easy to see that $\zeta$ is an isomorphism and the following diagram commutes

\[
\begin{array}{ccc}
0 & \longrightarrow & I \\
\downarrow \text{Id} & & \downarrow \zeta \\
0 & \longrightarrow & \mathcal{E}(\chi_2, \beta_2, \tau_2) \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{E}(\chi_2, \beta_2, \tau_2) & \longrightarrow & H \\
\pi_1 & & \downarrow \text{Id} \\
0 & \longrightarrow & H \\
\end{array}
\]

where $\pi_1$ and $\pi_2$ are natural projections. Hence, $\mathcal{E}(\chi_1, \beta_1, \tau_1)$ and $\mathcal{E}(\chi_2, \beta_2, \tau_2)$ are equivalent extensions. That shows that the map $\psi$ is well defined. It is easy to check that $\psi$ is well-defined and $\phi$ and $\psi$ are inverses of each other. The proof is now complete. \(\square\)

The elements of $\mathcal{Z}_\alpha^2(H, I)$ are called associated triplets as every element of $\mathcal{Z}_\alpha^2(H, I)$ is associated to some extension in view of Theorem 6.3.
Theorem 6.2. Let $H$ and $I$ be two skew braces and let $(\chi, \beta, \tau) \in Z^2_\alpha(H, I)$ be an associated triplet. If $\chi'$ is an action of $H$ on $I$ for which $\bar{\chi} \approx \chi'$, then there exist maps $\beta', \tau': H \to I$ such that $(\chi', \beta', \tau')$ is an associated triplet and $[(\chi, \beta, \tau)] = [(\chi', \beta', \tau')]$.

Proof. In the view of Theorem 6.1 there exists an extension
\[ \mathcal{E}(\chi, \beta, \tau) := 0 \to I \xrightarrow{i} E \xrightarrow{\pi} H \to 0, \]
corresponding to the associated triplet $(\chi, \beta, \tau)$. Let $s : H \to E$ be a st-section inducing $(\chi, \beta, \tau)$. Since $\bar{\chi} \approx \chi'$, there exist a map $\theta : H \to I$ such that $\theta(0) = 0$, $\nu_h = \nu_{h\theta(h)}$, $\mu_h = i^\theta_{\nu_{h\theta(h)}}^{-1}$ and $\sigma' = i^\theta_{\mu_{h\theta(h)}}^{-1}$. Define an st-section $s'(h) = s(h) \circ \theta(h)$ for all $x \in H$. Consequently we have an associated triplet $(\chi_1, \beta', \tau')$ corresponding to the st-section $s'$ and it is easy to see that $\chi' = \chi_1$. Hence $[(\chi, \beta, \tau)] = [(\chi_1, \beta', \tau')]$ as $(\chi, \beta', \tau')$ and $(\chi_1, \beta', \tau')$ are associated triplet of the same extension by different st-sections. This completes the proof. \qed

We now state a theorem analogous to [23, Theorem 3.6]. Let $H$ be a skew brace and $I$ be an abelian group.

Define
\[ Z^3_N(H, I) = \left\{ (g, f) \mid g, f : H \times H \to I, \text{ satisy properties \[14\] and \[15\], respectively, and vanish on degenerate tuples} \right\}, \]
and $B^3_N(H, I)$ is the collection of the pairs $(g, f) \in Z^3_N(H, I)$ such that there exists a map $\theta$ from $H$ to $I$ with $g = \nu_{h_1+h_2}((-\theta(h_1+h_2))+\mu_{h_2}(\nu_{h_1}(\theta(h_1)))+\nu_{h_2}(\theta(h_2)))$ and $f = -\theta(h_1 \circ h_2) + \sigma_h \theta(h_1) + \theta(h_2)$.

Put
\[ Z^3_N(H, I) = \left\{ \theta \text{ is a map from } H \to I \text{ such that } \begin{array}{c} \theta(h_1\circ h_2) = \nu_{h_2}(\theta(h_1)) + \nu_{h_2}(\theta(h_2)) \\ \text{and } \nu_{h_1+h_2}(\theta(h_1+h_2)) = \mu_{h_2}(\nu_{h_1}(\theta(h_1)) + \nu_{h_2}(\theta(h_2))) \end{array} \right\}, \]
the set $Z^3_N(H, I)$ is called as the set of derivations, and
\[ H^3_N(H, I) := Z^3_N(H, I)/B^3_N(H, I) \]
is the second cohomology group of $H$ by $I$.

Theorem 6.3. Let $H$ be a skew brace and let $I$ be an abelian group equipped with trivial brace structure. Let $\mathcal{E} := 0 \to I \xrightarrow{i} E \xrightarrow{\pi} H \to 0$ be an extension. Then the coupling and action are same, and there is a bijection between $\text{Ext}_N(\nu_{\mu}, \sigma)(H, I)$ and $H^3_N(H, I)$.

Theorem 6.4. Let $[\mathcal{E}] \in \text{Ext}_N(H, I)$ and $(\chi, \beta, \tau)$ be an associated triplet of $\mathcal{E}$. Then for $[(\beta_1, \tau_1)] \in H^1_N(H, \text{Ann}(I))$, the operation
\[ [(\beta_1, \tau_1)] [\mathcal{E}(\chi, \beta, \tau)] = [\mathcal{E}(\chi, \beta_1 + \beta, \tau_1 + \tau)] \]
defines a free action of the group $H^1_N(H, \text{Ann}(I))$ on the set $\text{Ext}_N(H, I)$. If $I$ is trivial skew brace, then this action becomes transitive.

Proof. It is easy to check that the action under consideration is well defined. Let $[\mathcal{E}(\chi, \beta, \tau)] \in \text{Ext}_N(H, I)$ and $[(\beta_1, \tau_1)] \in H^1_N(H, \text{Ann}(I))$ be such that $[(\beta_1, \tau_1)][\mathcal{E}(\chi, \beta, \tau)] = [\mathcal{E}(\chi, \beta_1 + \beta, \tau_1 + \tau)]$. Then
\[ [(\chi, \beta_1 + \beta, \tau_1 + \tau)] = [(\chi, \beta, \tau)], \]
and therefore there exist a map $\theta : H \to I$ such that $\theta(0) = 0$ and $\tilde{\chi} \approx \chi$ by $\theta$, which implies that $\theta(h) \in \text{Ann}(I)$ for all $h \in H$, and
\[ -\nu_{h_1+h_2}(\theta(h_1+h_2)) + \beta(h_1, h_2) + \mu_{h_2}(\nu_{h_1}(\theta(h_1))) + \nu_{h_2}(\theta(h_2)) = \beta(h_1, h_2) + \beta_1(h_1, h_2) \]
and
\[ \theta(h_1 \circ h_2)^{-1} \circ \tau(h_1, h_2) \circ \sigma_{h_2} \theta(h_1) \circ \theta(h_2) = \tau(h_1, h_2) + \tau_1(h_1, h_2) \]
for all \( h_1, h_2, h_3 \in H \) (using the fact that \( \theta(h) \in \text{Ann}(I) \) for all \( h \in H \)).

We have
\[ \beta_1(h_1, h_2) = \nu_{h_1} + \mu_{h_2}(-\theta(h_1 + h_2)) + \mu_{h_2}((\nu_{h_1}(\theta(h_1))) + \nu_{h_2}(\theta(h_2))), \]
\[ \tau_1(h_1, h_2) = -\theta(h_1 \circ h_2) + \sigma_{h_2} \theta(h_1) + \theta(h_2). \]
Thus \( [(\tau_1, \beta_1)] = 1 \), and hence the action is free.

Let \( E_1 \) and \( E_2 \) be two elements in \( \text{Ext}_\alpha(H, I) \). Then for \( i = 1, 2 \), \( |E_i| = |E_i(\chi_1, \beta_1, \tau_1)| \) for some associated triplet \( (\chi_i, \beta_i, \tau_i) \), where \( \chi_i = (\nu_i, \mu_i, \sigma_i) \). By Theorem 6.5.2, we can construct \( (\beta', \tau') \) such that \( (\chi_2, \beta', \tau') \) is an associated triplet with
\[ |E_1(\chi_1, \beta_1, \tau_1)| = |E_1(\chi_2, \beta', \tau')|. \]
We set
\[ \beta_3(h_1, h_2) = \beta'(h_1, h_2) - \beta_2(h_1, h_2) \]
and
\[ \tau_3(h_1, h_2) = \tau'(h_1, h_2) \circ \tau_2(h_1, h_2)^{-1}. \]
Now \( (\beta', \tau') \) and \( (\beta_2, \tau_2) \) are 2-cocycles with the same action \( \chi_2 \). Then from [6], [10] and [11] it follows that \( \beta_3(h_1, h_2) \in Z(I, +) \) and \( \tau_3(h_1, h_2) \in \text{Soc}(I) \) for all \( h_1, h_2 \in H \). If we take \( I \) to be trivial skew brace, then \( Z(I, +) = \text{Soc}(I) = \text{Ann}(I) \). Finally we get \( \beta_3, \tau_3 : H \rightarrow \text{Ann}(I) \). It is easy to see that \( (\beta_3, \tau_3) \) is 2-cocycle with respect to the action \( \chi_2 \) and \( (\beta_3, \tau_3)[E(\chi_2, \beta_2, \tau_2)] = [E(\chi_2, \beta', \tau')] = [E(\chi_1, \beta_1, \tau_1)]. \)
Hence the action is transitive, which completes the proof. \( \square \)

As a consequence, we get

**Theorem 6.5.** Let \( H \) be a skew brace and let \( I \) be a trivial skew brace with a fixed coupling \( \alpha \). Then there exists a bijection between \( \text{Ext}_\alpha(H, I) \) and \( \text{Ext}_\alpha(H, \text{Z}(I)) \).

### 7. Action of Automorphism Group on Extensions

Throughout this section we consider \( I \) to be a trivial skew brace and \( \mathcal{E} := 0 \rightarrow I \xrightarrow{\iota} E \xrightarrow{\pi} H \rightarrow 0 \) be an extension of skew braces. Then \( \nu : H \rightarrow \text{Aut}(I, +) \) as defined in [6], is independent of the choice of an s-section. Also \( \nu = \nu \) and \( \mu : (H, +) \rightarrow \text{Out}(I) \), \( \sigma : (H, \circ) \rightarrow \text{Out}(I) \), where \( \text{Out}(I) \) represents the group of outer-automorphisms of \( I \). For a pair \((\phi, \theta) \in \text{Aut}(H) \times \text{Aut}(I)\) of skew brace automorphisms and an extension
\[ \mathcal{E} : 0 \rightarrow I \xrightarrow{\iota} E \xrightarrow{\pi} H \rightarrow 0 \]
of \( H \) by \( I \), we can define a new extension
\[ \mathcal{E}^{(\phi, \theta)} : 0 \rightarrow I \xrightarrow{\iota^{\phi, \theta}} E \xrightarrow{\pi^{\phi, \theta}} H \rightarrow 0 \]
of \( H \) by \( I \). Thus, for a given \((\phi, \theta) \in \text{Aut}(H) \times \text{Aut}(I)\), we can define a map from \( \text{Ext}(H, I) \) to itself given by
\[ [\mathcal{E}] \mapsto [\mathcal{E}^{(\phi, \theta)}]. \]
If \( \phi \) and \( \theta \) are identity automorphisms, then obviously \( \mathcal{E}^{(\phi, \theta)} = \mathcal{E} \). It is also easy to see that
\[ [\mathcal{E}]^{(\phi_1, \theta_1)}(\phi_2, \theta_2) = ([\mathcal{E}]^{(\phi_1, \theta_1)}(\phi_2, \theta_2)). \]
We conclude that the association \( (20) \) gives an action of the group \( \text{Aut}(H) \times \text{Aut}(I) \) on the set \( \text{Ext}(H, I) \). From Corollary 6.2, we know that \( \text{Ext}(H, I) = \bigcup_{(\nu, \mu, \sigma)} \text{Ext}_{(\nu, \mu, \sigma)}(H, I) \). Let \( (\nu, \mu, \sigma) \) be an arbitrary but fixed coupling from \( H \) to \( I \). Let \( C_{(\nu, \mu, \sigma)} \) denote the stabiliser of \( \text{Ext}_{(\nu, \mu, \sigma)}(H, I) \) in \( \text{Aut}(H) \times \text{Aut}(I) \); more explicitly
\[ C_{(\nu, \mu, \sigma)} = \{(\phi, \theta) \in \text{Aut}(H) \times \text{Aut}(I) | \nu_h = \theta^{-1} \nu_{\phi(h)} \theta, \mu_h = \theta^{-1} \mu_{\phi(h)} \theta, \text{and } \sigma_h = \theta^{-1} \sigma_{\phi(h)} \theta \}. \]
It is easy to see that $C_{(\nu,\mu,\sigma)}$ is a subgroup of $\text{Autb}(H) \times \text{Autb}(I)$. For details see [23] Pg.15.

Next we consider an action of $C_{(\nu,\mu,\sigma)}$ on $H^2(N(H, Z(I)))$ (same as in [23]) by

\begin{equation}
[g,f] \mapsto [(g^{(\phi,\theta)}, f^{(\phi,\theta)})],
\end{equation}

where $g^{(\phi,\theta)}(h_1, h_2) = \theta^{-1}(g(\phi(h_1), \phi(h_2)))$. This action of $C_{(\nu,\mu,\sigma)}$ on $H^2(N(H, Z(I)))$ is same as the action of $C_{(\nu,\mu,\sigma)}$ on $\text{Ext}_{(\nu,\mu,\sigma)}(H, I)$ transferred on $H^2(N(H, Z(I)))$ through bijection of Theorem 6.5. Using this action we can define the semi-direct product $\Gamma = C_{(\nu,\mu,\sigma)} \times H^2(N(H, Z(I)))$. We wish to define an action of $\Gamma$ on $\text{Ext}_{(\nu,\mu,\sigma)}(H, I)$. For $(c, h) \in \Gamma$ and $[E] \in \text{Ext}_{(\nu,\mu,\sigma)}(H, I)$, define

\begin{equation}
[E]^c = [E]^h.
\end{equation}

**Lemma 7.1.** The rule in (22) gives an action of $\Gamma$ on $\text{Ext}_{(\nu,\mu,\sigma)}(H, I)$.

**Proof.** The proof follows on the lines of [23] Lemma 5.2.

Let $[E] \in \text{Ext}_{(\nu,\mu,\sigma)}(H, I)$ be a fixed extension. Since the action of $H^2(N(H, Z(I)))$ on $\text{Ext}_{(\nu,\mu,\sigma)}(H, I)$ is transitive and faithful, for each $c \in C_{(\nu,\mu,\sigma)}$, there exists a unique element (say) $h_c$ in $H^2(N(H, Z(I)))$ such that

\begin{equation}
[E]^c = [E]^{h_c}.
\end{equation}

We thus have a well defined map $\omega(E) : C_{(\nu,\mu,\sigma)} \to H^2(N(H, Z(I)))$ given by

\begin{equation}
\omega(E)(c) = h_c
\end{equation}

for $c \in C_{(\nu,\mu,\sigma)}$.

**Lemma 7.2.** The map $\omega(E) : C_{(\nu,\mu,\sigma)} \to H^2(N(H, Z(I)))$ defined in (23) is a derivation with respect to the action of $C_{(\nu,\mu,\sigma)}$ on $H^2(N(H, Z(I)))$ given in (21).

**Proof.** The proof follows on the lines of [23] Lemma 5.3.

Let $\mathcal{E} : 0 \to I \to E \xrightarrow{\gamma} H$ be an extension of a left skew brace $H$ by a trivial skew brace $I$ such that $[E] \in \text{Ext}_{(\nu,\mu,\sigma)}(H, I)$. Let $\text{Autb}_I(E)$ denote the subgroup of $\text{Autb}(E)$ consisting of all automorphisms of $E$ which normalize $I$, that is,

\begin{equation}
\text{Autb}_I(E) := \{\gamma \in \text{Autb}(E) \mid \gamma(y) \in I \text{ for all } y \in I\}.
\end{equation}

For $\gamma \in \text{Autb}_I(E)$, set $\gamma_I := |\gamma|_I$, the restriction of $\gamma$ to $I$, and $\gamma_H$ to be the automorphism of $H$ induced by $\gamma$. More precisely, $\gamma_H(h) = \pi(\gamma(s(h)))$ for all $h \in H$, where $s$ is an st-section of $\pi$. Notice that the definition of $\gamma_H$ is independent of the choice of an st-section. Define a map $\rho(\mathcal{E}) : \text{Autb}_I(E) \to \text{Autb}(H) \times \text{Autb}(I)$ by

\begin{equation}
\rho(\mathcal{E})(\gamma) = (\gamma_H, \gamma_I).
\end{equation}

Although $\omega(\mathcal{E})$ is not a homomorphism, but we can still talk about its set theoretic kernel, that is,

\begin{equation}
\text{Ker}(\omega(\mathcal{E})) = \{c \in C_{(\nu,\mu,\sigma)} \mid [E]^c = [E]\}.
\end{equation}

**Proposition 7.3.** For the extension $\mathcal{E}$, $\text{Im}(\rho(\mathcal{E})) \subseteq C_{(\nu,\mu,\sigma)}$ and $\text{Im}(\rho(\mathcal{E})) = \text{Ker}(\omega(\mathcal{E}))$.

**Proof.** The proof follows on the lines of [23] Proposition 5.4.

Continuing with the above setting, set $\text{Autb}^{H,I}(E) := \{\gamma \in \text{Autb}_I(E) \mid \gamma_I = \text{Id}, \gamma_H = \text{Id}\}$. Notice that $\text{Autb}^{H,I}(E)$ is precisely the kernel of $\rho(\mathcal{E})$. Hence, using Proposition 7.3 we get
Theorem 7.4. Let $\mathcal{E} : 0 \to I \to E \xrightarrow{\gamma} H$ be an extension of a left skew brace $H$ by a trivial skew brace $I$ such that $[\mathcal{E}] \in \text{Ext}_{(\nu, \bar{\mu}, \sigma)}(H, I)$. Then we have the following exact sequence of groups

$$0 \to \text{Aut}^H I(E) \to \text{Aut}_I(E) \xrightarrow{\mu(E)} C_{(\nu, \bar{\mu}, \sigma)} \xrightarrow{\omega(E)} H^2_N(H, Z(I)),$$

where $\omega(E)$ is, in general, only a derivation.

Further we have

Proposition 7.5. Let $\mathcal{E} : 0 \to I \to E \xrightarrow{\gamma} H$ be an extension of $H$ by $I$ such that $[\mathcal{E}] \in \text{Ext}_{(\nu, \bar{\mu}, \sigma)}(H, I)$. Then $\text{Aut}^H I(E) \cong Z_N^I(H, Z(I))$.

Proof. The map $\psi : Z_N^I(H, Z(I)) \to \text{Aut}^H I(E)$ defined by $\psi(\lambda)(s(h) \circ y) = s(h) \circ \lambda(h) \circ y$ is the required isomorphism. Rest proof follows on the lines of [23 Proposition 5.6]

We finally get the following Wells’ like exact sequence for skew braces.

Theorem 7.6. Let $\mathcal{E} : 0 \to I \to E \xrightarrow{\gamma} H$ be an extension of a left skew brace $H$ by a trivial skew brace $I$ such that $[\mathcal{E}] \in \text{Ext}_{(\nu, \bar{\mu}, \sigma)}(H, I)$. Then we have the following exact sequence of groups

$$0 \to Z^I_N(H, Z(I)) \to \text{Aut}_I(E) \xrightarrow{\mu(E)} C_{(\nu, \bar{\mu}, \sigma)} \xrightarrow{\omega(E)} H^2_N(H, Z(I)),$$

where $\omega(E)$ is, in general, only a derivation.

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