ASYMPTOTICALLY SECTIONAL-HYPERBOLIC ATTRACTORS

B. SAN MARTÍN AND KENDRY J. VIVAS

Departamento de Matemáticas, Universidad Católica del Norte
Antofagasta, Chile

(Communicated by Kuo-Chang Chen)

Abstract. The notion of asymptotically sectional-hyperbolic set was recently introduced. The main feature is that any point outside of the stable manifolds of its singularities has arbitrarily large hyperbolic times. In this paper we prove the existence, on any three-dimensional Riemannian manifold, of attractors with Rovella-like singularities satisfying this kind of hyperbolicity. Furthermore, we prove that asymptotically sectional-hyperbolic Lyapunov-stable sets, under certain conditions, have positive topological entropy.

1. Introduction. During the sixties, in an attempt to predict climate behavior, Lorenz [14] simplified Saltzman’s thermal convection equations obtains the following parameter-depending polynomial system:

\[
\begin{align*}
\dot{x} &= -\sigma x + \sigma y \\
\dot{y} &= \rho x - y - xz \\
\dot{z} &= -\beta z + xy.
\end{align*}
\] (1)

Lorenz’s observations on that systems show positive orbits moving erratically to a bounded domain for parameters close to \(\sigma = 10, \beta = 8/3\) and \(\rho = 28\), that is, the positive orbits behavior is chaotic and presents sensitive dependence to initial conditions. In other words, equations (1) appears to show an attractor with interesting dynamic properties which contains the equilibrium point \((0,0,0)\), making it a non-hyperbolic attractor.

In order to get a better understanding about Lorenz’s systems dynamic behavior, Guckenheimer in 1976 [10] introduced the Geometric Lorenz Attractor. Its structure was widely studied by Guckenheimer and Williams [11], Afraimovich, Bykov and Shilnikov [1] and Williams [28]. Although this attractor is non-hyperbolic, it presents hyperbolic features such as transitivity and dense periodic orbits in a robust way.

In 1986, Labarca and Pacífico in [13] introduced The Singular Horseshoe, the first variation ever considered of the geometric Lorenz attractor. It was conceived as a way to disprove Palis-Smale’s stability conjecture for flows on manifolds with boundary. Later, in 1993, Rovella in [26] introduced a second variation of geometric Lorenz attractor. This variation replaced the singularity by one with a central
contracting condition (three real eigenvalues $\lambda_1, \lambda_2$ and $\lambda_3$ which satisfy the relation $\lambda_2 < \lambda_3 < 0 < \lambda_1 < -\lambda_3$). This singularity is named Rovella-like for short. This model, called geometric Rovella attractor, although is non-robust and non-hyperbolic was proved to be persistent in a theoretical measure point of view, that is, for a two-parameter family of vector fields, the parameters set exhibiting Rovella attractor is a positive Lebesgue measure set. Several properties of these attractors were obtained [15], [16], [17], [24]. These include their appearance in the unfolding of certain homoclinic bifurcations [20], [23].

In 1997, Morales, Pacifico and Pujals, based on Shilnikov and Turaev’s construction [27], defined the notion of singular-hyperbolic system. This concept, in turn, generated the singular-hyperbolic systems which extended Smale’s hyperbolic theory, having the geometric Lorenz attractor as a typical example. The Rovella attractor does not belong to this class due to the contracting property of its singularity.

In 2004 Metzger and Morales introduced the notion of sectional-hyperbolic system [18], an attempt to explain the main features of certain higher dimensional systems like the multidimensional Lorenz attractor [5] and shows that every $C^r$-robust transitive and strongly homogeneous set satisfies this sort of hyperbolicity. It should be noted that, in a three-dimensional context, both concepts agree (sectional-hyperbolic and singular-hyperbolic).

A key feature of the sectional-hyperbolic sets is that they satisfy the Hyperbolic Lemma, that is, a compact invariant set of a given vector field which every compact invariant set without singularities contained in it is hyperbolic. Examples of systems satisfying this property are the star flows, i.e., flows which cannot be $C^1$ approximated by ones with non-hyperbolic periodic points or singularities [8] and the geometric Rovella attractor.

Another interesting property of sectional-hyperbolic systems is that every Lyapunov stable set with this kind of hyperbolicity has positive topological entropy. This was proved in 2015 by Arbieto, Barragán and Morales by using the Pesin entropy formula for $C^1$ diffeomorphisms (due to such sets support an SRB-like measure for the time-one map) and the volume expansion of its central subbundle [2].

In 2017, with the purpose to extend the sectional-hyperbolic theory, Morales and the first author in [21] introduced the notion of asymptotically sectional-hyperbolic set. This set is partially hyperbolic, with hyperbolic singularities and with an eventually asymptotically expanding central direction outside of stable manifolds of the singularities. This concept extends the sectional-hyperbolicity properly (i.e. a sectional-hyperbolic set is asymptotically sectional-hyperbolic but not conversely) and satisfies the Hyperbolic Lemma. In fact, they prove the existence of vector fields with compact and invariant sets satisfying this kind of hyperbolicity containing Rovella-like singularities (hence non sectional-hyperbolic). They exhibit the contracting singular horseshoe, which is a variant of Labarca and Pacifico’s singular horseshoe, replacing the existing singularity for a Rovella-like one. This set is transitive, but it is not an attractor. Therefore, it appears natural to ask: Are there asymptotically sectional-hyperbolic attractors which are not sectional-hyperbolic?

In this work, we will provide a positive answer for this question. Indeed, we construct a compact Riemannian manifold whose boundary is a 4-genus surface and a vector field that is inwardly transverse to the boundary, having an attractor with three singularities, one of them central dissipative. Furthermore, we will use
the technique given in [2] to prove that asymptotically sectional-hyperbolic attractors have positive topological entropy under certain hypothesis. More precisely, we assume the existence of a non-atomic SRB-like measure supported in \( \Lambda \) for the time-one map.

2. Statement of the results. From now on \( M \) will denote a differentiable manifold endowed with a Riemannian metric \( \| \cdot \| \). By a flow we mean the one-parameter family of maps \( X_t \) induced by a \( C^1 \) vector field \( X \) of \( M \). We will say that \( X \) is a three-dimensional vector field when \( \dim M = 3 \). We denote by \( \text{Sing}(X) \) the set of singularities (i.e. zeros of \( X \)). By a periodic point we mean a point \( x \in M \) for which there is a minimal \( T > 0 \) such that \( X_T(x) = x \). By an orbit we mean \( O(x) = \{X_t(x) : t \in \mathbb{R}\} \) and by a periodic orbit we mean to the orbit of a periodic point. We say that \( \Lambda \subset M \) is invariant if \( X_t(\Lambda) = \Lambda \) for all \( t \in \mathbb{R} \). We say that \( \Lambda \) is hyperbolic if there is a continuous invariant splitting \( T_\Lambda M = E^s \oplus E^u \) and numbers \( K, \lambda > 0 \) such that:

(a) \( E^s \) is contracting, i.e., \( \|DX_t(x)|_{E^s_x}\| \leq Ke^{-\lambda t} \) for every \( x \in \Lambda \) and \( t \geq 0 \).

(b) \( E^u_x \) is the subspace spanned by \( X(x) \) in \( T_x M \), for every \( x \in \Lambda \).

(c) \( E^u \) is expanding, i.e., \( m(DX_t(x)|_{E^u_x}) \geq Ke^{\lambda t} \) for every \( x \in \Lambda \) and \( t \geq 0 \),

where \( m(A) = \inf_{v \neq 0} \|Av\| / \|v\| \).

Invariance means here that \( DX_t(x)E^s_x = E^s_{X_t(x)} \) for every \( t \in \mathbb{R} \), \( x \in \Lambda \) and \( * = s, u \).

A singularity or periodic orbit is hyperbolic if it is hyperbolic as a compact invariant set of \( X \). The elements of a hyperbolic periodic orbit will be called hyperbolic periodic points.

We say that a compact invariant set \( \Lambda \) has a dominated splitting with respect to the tangent flow if there is a continuous invariant splitting \( T_\Lambda M = E \oplus F \) and numbers \( K, \lambda > 0 \) such that

\[
\frac{\|DX_t(x)|_{E_x}\|}{m(DX_t(x)|_{E_x})} \leq Ke^{-\lambda t},
\]

for every \( x \in \Lambda \) and every \( t \geq 0 \). In this case we say that \( F \) dominates \( E \).

A compact invariant set \( \Lambda \) is partially hyperbolic if it has a dominated splitting \( T_\Lambda M = E^s \oplus F \) with respect to the tangent flow whose dominated subbundle \( E^s \) is contracting (in the sense of (a) above).

According to [18] we say that a compact invariant partially hyperbolic set \( \Lambda \) is sectional-hyperbolic if its singularities are hyperbolic and if its central subbundle \( F \) is sectionally expanding, i.e., there are \( K, \lambda > 0 \) such that

\[
|\text{det} (DX_t(x)|_{L_x})| \geq Ke^{\lambda t},
\]

for every \( x \in \Lambda \), every \( t \geq 0 \) and every two-dimensional subspace \( L_x \) of \( F_x \).

It is well known that if \( \sigma \in M \) is a hyperbolic singularity for \( X \) then it has associated its stable and unstable manifolds \( W^s(\sigma) \), \( W^u(\sigma) \) which are tangent to \( \sigma \) to stable and unstable subspaces \( E^s_\sigma \) and \( E^u_\sigma \) of \( T_\sigma M \) respectively. Denote by \( W^s(\text{Sing}(X)) \) to the union of stable manifolds \( W^s(\sigma) \) of the hyperbolic singularities of \( X \).

With this notation we introduce the following definition:

**Definition 2.1.** Let \( \Lambda \) be a compact invariant partially hyperbolic set of a vector field \( X \) whose singularities are hyperbolic. We say that \( \Lambda \) is asymptotically sectional-hyperbolic if its central subbundle is eventually uniformly asymptotically expanding.
outside the stable manifolds of the singularities, i.e., there exists $C > 0$ such that

$$\limsup_{t \to \infty} \frac{\log \left| \det(DX_t(x)\mid_{L_x}) \right|}{t} \geq C,$$

(2)

for every $x \in \Lambda \setminus W^s(Sing(X))$ and every two-dimensional subspace $L_x$ of $F_x$.

**Remark 1.** In [4] the authors define the following 2-Riemannian metric [22]:

$$\langle u, v/w \rangle_x = \langle u, v \rangle_x \cdot \langle w, w \rangle_x - \langle u, w \rangle_x \cdot \langle v, w \rangle_x, \quad \forall x \in M, u, v, w \in T_x M,$$

which induces a 2-norm [9] (also called areal metric [12]) given by

$$\|u, v\| = \sqrt{\langle u, u/v \rangle_x}, \quad \forall x \in M, u, v \in T_x M.$$

(3)

By using the 2-norm (3) it is possible rewrite the Definition 2.1 as follows: A compact invariant partially hyperbolic set $\Lambda$ with hyperbolic singularities is asymptotically sectional-hyperbolic if there exists a positive constant $C$ such that

$$\limsup_{t \to \infty} \frac{\log \|DX_t(x)u, DX_t(x)v\|}{t} \geq C, \quad \forall x \in \Lambda \setminus W^s(Sing(X)), u, v \in F_x.$$

From [21] we can highlight the following interesting properties that satisfy the asymptotically sectional-hyperbolic sets:

**Remark 2.** Every asymptotically sectional-hyperbolic set satisfies the Hyperbolic Lemma.

**Remark 3.** Every sectional-hyperbolic set is asymptotically sectional-hyperbolic but not conversely: Take for instance the union of Rovella-like singularity $\sigma$, a hyperbolic saddle-type periodic orbit $O$ and a generic regular orbit in $W^s(\sigma) \cap W^u(O)$.

The example in the previous remark is far from being chaotic because it is not transitive. Therefore, it is natural to ask if there are chaotic regions which are asymptotically sectional-hyperbolic but not sectional-hyperbolic. A positive answer to that question is given by the first author and Morales in [21] exhibiting the Contracting Horseshoe, which is a slight modification of the Singular Horseshoe, replacing the Lorenz-like singularity by a Rovella-like singularity.

**Definition 2.2.** A compact invariant set $\Lambda \subset M$ for a $C^1$ vector field $X$ is:

- **transitive** if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the $\omega$–limit set defined by
  $$\omega(x) = \{y \in M : y = \lim_{n \to \infty} X_{t_n}(x) \text{ for some sequence } t_n \to \infty\}.$$

- **attracting** if there exists a neighborhood $U$ of $\Lambda$ such that
  $$\Lambda = \bigcap_{t \geq 0} X_t(U),$$

  (such a neighborhood $U$ is often called trapping region);

- **attractor** if $\Lambda$ is a transitive attracting set.

The attractors represent important objects in the dynamical literature. In fact, they play an important role in Smale’s Spectral Decomposition Theorem [25]. Examples of (nontrivial) hyperbolic attractors for flows can be obtained by suspending the classical Plykin attractor [25]. An example which is sectional-hyperbolic but not hyperbolic is precisely the geometric Lorenz attractor. Therefore, the following
question arises: Are there attractors which are asymptotically sectional-hyperbolic with Rovella-like singularities?

One of our main results gives a positive answer for the question above.

**Theorem 2.3.** On every three-dimensional manifold there exists a vector field $X$ exhibiting an asymptotically sectional-hyperbolic attractor having both dense periodic orbits and a Rovella-like singularity.

In order to state our second result we recall some facts about SRB-like measures. If $f : M \to M$ is a continuous map and $\mu$ is a Borel probability measure, we say that $\mu$ is an *invariant measure* if $\mu(f^{-1}(A)) = \mu(A)$ for any Borel set $A \subseteq M$. Denote by $\text{supp}(\mu)$ the support of the measure $\mu$. We say that $\mu$ is *non-atomic* if $\mu(\{x\}) = 0$ for all $x \in M$. For any point $x \in M$, the *empirical probabilities of orbit* of $x$ are defined by

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}, n \in \mathbb{N},$$

where $\delta_y$ is the Dirac measure supported at $y \in M$. We denote by $p_\omega(x)$ the set of measures which are accumulated for empirical probabilities associated to the orbit of $x$, i.e., is the set of weak$^*$-limits of subsequences of $(4)$. Given $\varepsilon > 0$ and a probability measure $\mu$, we define the *basin of $\varepsilon$-attraction* of $\mu$ as

$$B_\varepsilon(\mu) := \{x \in M : \text{there is } \nu \in p_\omega(x), d_*(\mu, \nu) < \varepsilon\},$$

where $d_*$ is the standard metric in the space of probability measures. An invariant measure $\mu$ is *SRB-like* if for all $\varepsilon > 0$, $B_\varepsilon(\mu)$ has positive Lebesgue measure. Finally, we say that an invariant set $\Lambda$ is *Lyapunov stable* if for every neighborhood $U$ of $\Lambda$ there is a neighborhood $V$ of $\Lambda$ such that $X_t(V) \subseteq U$ for all $t \geq 0$.

**Theorem 2.4.** Let $\Lambda$ be an asymptotically sectional-hyperbolic Lyapunov stable set of a vector field $X$ on a compact manifold $M$. If $\Lambda$ supports a non-atomic SRB-like measure for the time-one map $X_1$, then $\Lambda$ has positive topological entropy.

**Remark 4.** The hypothesis on $\mu$ in the statement of previous theorem is not superfluous. Indeed, we consider $M = S^2 \times [-1, 1]$ and the following three-dimensional vector field $X$ given in the Figure 1.

Denote $S^2_t = S^2 \times \{t\}$. Here, $P, S \in S^2_0$ are sinks, $O$ is a saddle periodic orbit and $\omega(x) = N$ for all $x$ in the north hemisphere (of $S^2_0$) and $\omega(x) = S$ for all $x$ in the south hemisphere (of $S^2_0$).

**Figure 1.** A vector field $X$ without non-atomic SRB-like measures
The construction above implies that \( \Lambda = S_0^2 = \bigcap_{t \geq 0} X_1(U) \), \( U = S^2 \times (-\varepsilon, \varepsilon) \), is an attracting set (so Lyapunov-stable) and asymptotically sectional-hyperbolic because \( \Lambda \setminus W^s(\text{Sing}(X)) = \mathcal{O} \) and \( E_p^u = E_p^u \oplus \langle X(p) \rangle \) for all \( p \in \mathcal{O} \), so that (2) is satisfied. Nevertheless, all SRB-like measures for \( X_1 \) supported in \( \Lambda \) are given by \( \mu = \lambda \delta_P + (1 - \lambda) \delta_S, \lambda \in [0,1] \). Moreover, we have \( h_{\text{top}}(X) = 0 \) by the construction of \( X \).

3. Proofs. This section will be divided in three subsections. In the section 3.1 we will construct the attractor \( \Lambda \) announced in Theorem 2.3. In Subsection 3.2 we prove that such attractor is asymptotically sectional-hyperbolic and in the last subsection we will give the proof of Theorem 2.4.

3.1. Construction of \( \Lambda \). We begin with three 3-cells: A cube \( Q \) and two copies of a cell \( T \) like in the Geometric Lorenz Attractor (as in [11]), say \( T_l \) and \( T_r \). All these sets are described in the Figure 2.

![Figure 2. The cube Q and the cells T_l and T_r](image)

As in [11] we describe the vector filed on \( T_l \) and \( T_r \) through the differential equation

\[
\begin{align*}
\dot{x} &= \lambda_l^u x \\
\dot{y} &= \lambda_l^{ss} y \\
\dot{z} &= \lambda_l^s z
\end{align*}
\]

and

\[
\begin{align*}
\dot{x} &= \lambda_r^u x \\
\dot{y} &= \lambda_r^{ss} y \\
\dot{z} &= \lambda_r^s z
\end{align*}
\]

In the same way, the vector field on \( Q \) is given by the differential equation

\[
\begin{align*}
\dot{x} &= \lambda_0^u x \\
\dot{y} &= \lambda_0^{ss} y \\
\dot{z} &= \lambda_0^s z
\end{align*}
\]

In that figure it is considered \( C_l \) and \( C_r \) as the up faces of \( T_l \) and \( T_r \) respectively, and \( B_l, B_r \) and \( S^\pm = \{ z = \pm 1 \} \) faces of \( Q \). In each cell, \((0,0,0)\) correspond to the singularities \( \sigma_l, \sigma_r \) and \( \sigma_0 \) respectively.

Denoting

\[
\alpha_i = -\frac{\lambda_i^s}{\lambda_i^u} \quad \text{and} \quad \beta_i = \frac{\lambda_i^{ss}}{\lambda_i^s} \quad \text{for} \quad i \in \{0,l,r\}
\]

we impose the following conditions:

\[
\alpha_0 > 1, \quad \alpha_i < 1 \quad \text{and} \quad \alpha_0 \alpha_i < 1 \quad \text{for} \quad i \in \{l,r\}.
\]
The first two inequalities in the above equation implies that $\sigma_0$ is a Rovella-like singularity while $\sigma_i$, $i \in \{l, r\}$, are Lorenz-like. The third inequality is imposed in order to the first return map has an expanding condition along the central subbundle.

Next, we glue the cells $T_l$ and $T_r$ to $Q$ by joining $A_l$ with $B_l$ and $A_r$ with $B_r$ by an appropriate transformation. This results in a 3-cell equiped with a flow as described in the Figure 3.

Figure 3. The flow in the resulting 3-cell

Now, we make flow in a smooth way so as to take the four “cusp triangles” in the 3-cells above go to $S = S^+ \cup S^- \subset Q$ as indicate in the Figure 4.

Figure 4. The flow from the cusp triangles to $S^+$ and $S^-$

Assuming that the vertical foliation given by the $y$–direction

$$\{ \ast \times [-\frac{1}{2}, \frac{1}{2}] : \ast \in [-\frac{1}{2}, \frac{1}{2}] \cup [-\frac{1}{2}, \frac{1}{2}] \}$$
is invariant, the resulting vector field denoted by $X$ has a Poincaré map $F : S \rightarrow S$ of the form

$$F(x, y) = (f(x), H(x, y)).$$

The map $H$ is assumed to satisfy the following properties:

(H1) $0 < |H(x, y)| < \frac{1}{2}$ for $x \neq 0$.

(H2) $|\frac{\partial H}{\partial y}(x, y)| < \frac{1}{2}$ for all $(x, y) \in S$.

To describe the map $f$ we write $S = S^+ \cup S^- = [-\frac{1}{2}, \frac{1}{2}]^2 \cup [-\frac{1}{2}, \frac{1}{2}]^2$, where $[-\frac{1}{2}, \frac{1}{2}]^2 = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ and $\cup$ denotes the disjoint union. Moreover, we set $(x^+, y^+)$ for points in $S^+$ and $(x^-, y^-)$ for points in $S^-$. With this notation, due to the third condition in (5), we can assume the following properties (where $f(\pm a) = \lim_{x \rightarrow a, x > a} f(x)$ and $f(-a) = \lim_{x \rightarrow a, x < a} f(x)$ are used for the lateral limits):

(f1) $f(-0^+) = \frac{1}{2}$, $f(0^+) = -\frac{1}{2}$, $f(-0^-) = \frac{1}{2}$ and $f(0^-) = -\frac{1}{2}$.

(f2) The third inequality in (5) implies that $f'(x) = \lambda > \sqrt{2}$ for every $x \in [-\frac{1}{2}, \frac{1}{2}] \cup [-\frac{1}{2}, \frac{1}{2}]$.

(f3) $-\frac{1}{2} < f(x^+) < \frac{1}{2}$ for every $x^+ \in [-\frac{1}{2}, \frac{1}{2}]$ and $-\frac{1}{2} < f(x^-) < \frac{1}{2}$ for every $x^- \in [-\frac{1}{2}, \frac{1}{2}]$.

The shape of the map $F$ is described below.

**Figure 5. The two-dimensional dynamics $F$**

Conditions (f1) to (f3) above imply that $f$ is weak locally eventually onto (weak-l.e.o). Indeed, for all subinterval $J$ there exists a natural number $n_0$ such that $f^{n_0}(J) \cup f^{n_0+1}(J) = [-\frac{1}{2}, \frac{1}{2}] \cup [-\frac{1}{2}, \frac{1}{2}]$. It follows that $f$ is transitive (i.e. it has a dense positive orbit). Let $\Lambda = \{z \in S : F^n(z) \text{ exists } \forall n \in \mathbb{Z}\}$. Using that the vertical direction is a contracting foliation (condition (H3)) and the transitivity of $f$ we have the transitivity of $F$ on $\Lambda$. Furthermore, the weak-l.e.o condition of $f$ implies the density of the periodic points of $F$. The proof is similar to geometric Lorenz attractor’s case [28]. We define $\Lambda = \bigcup_{t \in \mathbb{R}} X_t(\Lambda)$.

Now, as usual, by using the Tubular Flow Theorem, we can construct the vector field $X$ in Figure 4 in a such a way that it is inwardly transverse to the boundary.
It follows that $X$ has a trapping region $U$, i.e., $X_t(U) \subset U$ for $t \geq 0$. Topologically $U$ is a solid handlebody of genus $4$ as in the Figure 6.

**Figure 6.** The trapping region $U$

Therefore, $\Lambda$ is an attracting set (with trapping region $U$). By standard topological arguments we can embed the flow $X$ in $U$ on any three-dimensional manifold.

Finally, using that $F$ is transitive with dense periodic orbits we obtain that $\Lambda$ is an attractor for $X$. Actually $\Lambda$ is a homoclinic class of $X$ (by the arguments in [3]).

### 3.2. Asymptotically sectional-hyperbolicity of $\Lambda$.

We start with the discussion of the partially hyperbolicity of $\Lambda$. At first glance we note the existence of a strong stable direction $E^s$ in the trapping region $U$ which is parallel to the $y$ axis. To obtain the central direction $E^c$ on $\Lambda$ we use the canonical argument given by saturation of the central cone field defined in $\Lambda$, that is, we consider the complement of stable $\frac{\pi}{4}$-cone field at $\Lambda$ ($C_{\frac{\pi}{4}}(x, \{e_1, e_3\})$, where $x \in \Lambda$) and, afterwards, saturates it with the flow $X$. This defines the partially hyperbolic splitting $E^s \oplus E^c$ on $\Lambda$.

To prove that $\Lambda$ is asymptotically sectional-hyperbolic it remains to prove that $E^c$ is eventually asymptotically sectionally expanding outside the stable manifolds of the singularities, i.e., we need to find $C > 0$ such that

$$\limsup_{t \to \infty} \frac{\log |\det(DX_t(p)|_{E^c_p})|}{t} \geq C, \forall p \in \Lambda \setminus W^{s}(Sing(X)).$$

Let $p \in \Lambda$ such that $p$ does not belong to the stable manifold of any singularity. Then there is a positive finite time $t$ such that $X_t(p) \in S$. This time is irrelevant in the asymptotic behavior of $|\det(DX_t(p)|_{E^c_p})|$, so we can suppose that $p \in S$.

Denote $q = F(p) = X_t(p)$ for some positive time $t$. For every tangent vector $u = (A, B, 0) \in E^c_p \cap T_pS$, we denote $v = DF(p)u$ and $w = DX_t(p)u$. Since
On the one hand, if $p = (x, y, z_p)$, we observe that

$$\text{vol}< X(p); u > = \| X(p) \times u \|,$$

so

$$\text{vol} < X(p); u > = \sqrt{(B\lambda_0 z_p)^2 + (A\lambda_0 z_p)^2 + (B\lambda_0 x_p - A\lambda_0 y_p)^2},$$

where $p = (x_p, y_p, z_p)$. As $X(p) \times u \in C^* \left( p, \{ \epsilon_1, \epsilon_3 \} \right)$ (as in [6]) we have $(A\lambda_0 z_p)^2 \geq (B\lambda_0 z_p)^2 + (B\lambda_0 x_p - A\lambda_0 y_p)^2$, so $\text{vol} < X(p); u > \geq \sqrt{2|\lambda_0| |A|}$, because $z_p = \pm 1$.

On the other hand, if $w = (a, b, c)^t$ then

$$\text{vol} < X(q); w > = \sqrt{(c\lambda_s y_q - b\lambda_0)^2 + (c\lambda_s x_q - a\lambda_0)^2 + (b\lambda_0 x_q - a\lambda_0 y_q)^2},$$

where $q = (x_q, y_q, z_q)$ and $z_q = \pm 1$. Besides, by a straightforward computation we see that $v = DF(p)u = (a - \frac{\lambda_0}{\lambda_s} x_q c, b - \frac{\lambda_0}{\lambda_s} y_q c, 0)$, therefore $\text{vol} < X(q); w > \geq |\lambda_0| ||v||$. Now, as

$$DF(p)u = \begin{pmatrix} f'(p) & 0 \\ \partial_x H(p) & \partial_y H(p) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix},$$

then

$$DF(p)u = (f'(p)A, \partial_x H(p)A + \partial_y H(p)B).$$

So $||v|| = \sqrt{(f'(p)A)^2 + (\partial_x H(p)A + \partial_y H(p)B)^2} \geq \lambda |A|$. This implies that

$$|\det(DX_t(p)|_{E^p})| \geq \frac{\lambda}{\sqrt{2}} = \rho > 1.$$

Next, we will prove that successive return times give exponential growth for $|\det(DX_t(p)|_{E^p})|$. Let $V_{\delta_0}$ be a $\delta_0$ open neighborhood of $W^s(\sigma) \cap S$ and $p = p_0$. Let $t_i$ and $p_i$ the successive return times and return points for $p_0$ respectively. They satisfy $p_{i+1} = X_{t_{i+1}}(p_i)$ for all $i \geq 0$. Denote $A = \{ 0 \leq i \leq n - 1 : p_i \notin V_{\delta_0} \}$, $B = \{ 0 \leq i \leq n - 1 : p_i \in V_{\delta_0} \}$ and $T_n = \sum_{i=0}^{n-1} t_{i+1}$.

(a) Define $T_{\delta_0} = \max \{ t_{n+1} : p_i \in A \}$. Note that $T_{\delta_0}$ does not depend on $n$. Then

$$\sum_{i \in A} \log |\det(DX_t(p_i)|_{E^p})| \geq \frac{\log \rho}{T_{\delta_0}} \sum_{i \in A} t_{i+1}.$$

(b) On the other hand, for $p \in V_{\delta_0}$ the first return flight time $t$ can be decomposed as $t = \tau_1 + \tau_2 + \tau_3 + \tau_4$, where $\tau_1, \tau_2, \tau_3$ and $\tau_4$ are the times to go from one section to the next in the “flow order”, that is:

- $\tau_1$ is the flight time to go from $p$ to lateral side of $Q$.
- $\tau_2$ is the complement flight time to go from $p$ to the up face of $T_l$ or $T_r$ respectively.
- $\tau_3$ is the flight time to go from the up face of $T_l$ or $T_r$ to the lateral face of $T_l$ or $T_r$ respectively.
- $\tau_4$ is the remaining time necessary to return to $S$.

Now,

i) Let $p = (x, y, \pm 1)$, $|x| < \delta_0$. Then $\tau_1 = \frac{1}{\lambda_s} \log |x|$ and

$$|\det(DX_{\tau_1}(p)|_{E^p})| \geq \frac{e^{(\lambda_0 + \lambda_s)\tau_1}}{\sqrt{2}}.$$
We finish the proof by putting (a) and (b) together to obtain

\[ |\det(DX_{\tau_3}(\bar{p})|_{\mathcal{E}_p^c})| \geq \frac{e^{(\lambda^r_u + \lambda^l_u)\tau_3}}{\sqrt{2}}. \]

In the same way, if \( x < 0 \), then \( \bar{p} = X_{\tau_1+\tau_2}(p) = J^r \circ \Pi_{loc,0}(x, y, \pm 1) = (x^\alpha, yx^{\beta_0}, 1) \) for \( x > 0 \) then

\[ \tau_3 = -\frac{\alpha_0}{\lambda^l_u} \log x \quad \text{and} \quad \log |\det(DX_{\tau_3}(\bar{p})|_{\mathcal{E}_p^c})| \geq \frac{e^{(\lambda^r_u + \lambda^l_u)\tau_3}}{\sqrt{2}}. \]

From i) and ii) we have

\[ \log |\det(DX_{\tau_1}(p)|_{\mathcal{E}_p^c})| + \log |\det(DX_{\tau_3}(\bar{p})|_{\mathcal{E}_p^c})| \geq -\frac{\lambda^0_u \lambda^r_u - \lambda^0_u \lambda^l_u}{\lambda^0_u \lambda^r_u} \cdot \log |x| - \log 2 \]

or

\[ \log |\det(DX_{\tau_3}(\bar{p})|_{\mathcal{E}_p^c})| + \log |\det(DX_{\tau_3}(\bar{p})|_{\mathcal{E}_p^c})| \geq -\frac{\lambda^0_u \lambda^l_u - \lambda^0_u \lambda^l_u}{\lambda^0_u \lambda^r_u} \cdot \log |x| - \log 2. \]

Furthermore, for \( \delta_0 \) small enough we have

\[ \log |\det(DX_{\tau_1}(p)|_{\mathcal{E}_p^c})| + \log |\det(DX_{\tau_3}(\bar{p})|_{\mathcal{E}_p^c})| \geq K, \]

where \( K = \frac{1}{2} \min \left\{ -\frac{\lambda^0_u \lambda^r_u - \lambda^0_u \lambda^l_u}{\lambda^0_u \lambda^r_u}, -\frac{\lambda^0_u \lambda^l_u - \lambda^0_u \lambda^l_u}{\lambda^0_u \lambda^l_u} \right\} > 0. \)

iii) There exists a positive constant \( C' \) such that \( \max\{\tau_2, \tau_4\} \leq C' \). We denote \( q = X_{\tau_1}(p), \bar{q} = X_{\tau_3}(\bar{p}) \). For \( \delta_0 \) is small enough, depending on the case \( x > 0 \) or \( x < 0 \), we have that \( \log |\det(DX_{\tau_2}(p)|_{\mathcal{E}_p^c})| \approx \log \left( \frac{\lambda^r_u}{\lambda^l_u} \right) \) and \( \log |\det(DX_{\tau_3}(\bar{q})|_{\mathcal{E}_p^c})| \approx \log \left( \frac{\lambda^l_u}{\lambda^r_u} \right) \).

Joining all above, and shrinking \( \delta_0 \) if necessary we have that

\[ \sum_{i \in B} \log |\det(DX_{t_{i+1}}(p)|_{\mathcal{E}_p^c})| \geq \sum_{i \in B} \frac{K}{2} t_{i+1}. \]

We finish the proof by putting (a) and (b) together to obtain

\[ \frac{1}{T_n} \log |\det(DX_{\tau_n}(p)|_{\mathcal{E}_p^c})| \geq \frac{1}{T_n} \left( \frac{\log \rho}{T_{\delta_0}} \sum_{i \in A} t_{i+1} + \sum_{i \in B} \frac{K}{2} t_{i+1} \right) \]

\[ \geq \frac{1}{T_n} C \left( \sum_{i \in A} t_{i+1} + \sum_{i \in B} t_{i+1} \right) = C, \]

where \( C = \min \left\{ \frac{\log \rho}{T_{\delta_0}}, \frac{K}{2} \right\} > 0 \). Note that if \( \delta_0 \) is small enough then \( C = \frac{\log \rho}{T_{\delta_0}}. \)
3.3. **Proof of Theorem 2.4.** Before the proof of Theorem 2.3 it is necessary recall some previous facts. Let \( f : M \rightarrow M \) a diffeomorphism, \( \mu \) an invariant measure and \( \Lambda \) a compact invariant subset of \( M \) for \( f \). The Oseledets’s theorem guarantees, for every continuous invariant subbundle \( F \) of \( T\Lambda M \), the existence of a full measure set \( R \) (called regular points) with the following property: For every \( x \in R \) there exists a positive integer \( k(x) \), real numbers \( \chi_1(x) < \chi_2(x) < \cdots < \chi_{k(x)}(x) \) and a splitting \( F_x = F_x^1 \oplus F_x^2 \oplus \cdots \oplus F_x^{k(x)} \), depending measurably on \( x \in R \), such that

\[
\lim_{n \to \pm \infty} \frac{1}{n} \log \|Df^n(x)v^i\| = \chi_i(x), \forall v^i \in F_x^i \setminus \{0\}, 1 \leq i \leq k(x).
\]

The numbers \( \chi_i(x) \) are called Lyapunov exponents of \( \mu \) along \( F \). Catsigeras et al. in [7] give a \( C^1 \)-version of the Pesin Entropy Formula for diffeomorphisms:

**Lemma 3.1.** Let \( \Lambda \) be a Lyapunov stable set of a flow \( X \). If \( \Lambda \) has a dominated splitting \( T\Lambda M = E \oplus F \) with respect to the tangent flow and \( \mu \) is a SRB-like measure for the time-one map \( X_1 \), then

\[
h_{\mu}(X_1) \geq \int \sum_{i=1}^{\dim F_x} \chi_i(x) d\mu,
\]

where \( \chi_i(x) \), \( i = 1, \ldots, \dim F_x \), are the Lyapunov exponents along \( F_x \).

Finally, we have the following lemma:

**Lemma 3.2.** If \( \Lambda \) is a compact invariant set for a flow \( X \) with hyperbolic singularities and \( \mu \) is a probability invariant measure for \( X_1 \), then \( \mu(W^s(\sigma) \setminus \{\sigma\}) = 0 \).

**Proof.** First, notice that if \( \sigma \) is a sink, the sets \( W^s(\sigma) \setminus B(\sigma, \frac{1}{n}) \), \( n \geq 1 \), where \( B(\sigma, \frac{1}{n}) \) is the open ball of radius \( \frac{1}{n} \) around \( \sigma \) in \( M \), are measurable and

\[
W^s(\sigma) \setminus \{\sigma\} = \bigcup_{n \geq 1} \left( W^s(\sigma) \setminus B \left( \sigma, \frac{1}{n} \right) \right).
\]

Suppose that \( \mu \left( W^s(\sigma) \setminus B \left( \sigma, \frac{1}{n_0} \right) \right) > 0 \) for some natural number \( n_0 \). Then, by Poincaré’s recurrence theorem, \( \mu - a.e x \in W^s(\sigma) \setminus B \left( \sigma, \frac{1}{n_0} \right) \) satisfies \( X_k(x) \in W^s(\sigma) \setminus B \left( \sigma, \frac{1}{n} \right) \) for infinite values of \( k \), which is a contradiction. So, by monotony \( \mu(W^s(\sigma)) = 0 \). Now, if \( \sigma \) is a saddle, then the sets \( V_n = W^s(\sigma) \cap B \left( \sigma, \frac{1}{n} \right) \), \( n \geq 1 \), are measurable and \( V_n \subset V_{n+1} \) for all \( n \geq 1 \). So, proceeding as above, we have \( \mu(W^s(\sigma) \setminus \{\sigma\}) = 0 \) \( \square \)

**Proof of Theorem 2.4.** By Lemma 2.1 in [2], the Lyapunov measure supports an SRB-like measure. As \( \Lambda \) is partially hyperbolic, it follows of Lemma 3.1 that

\[
h_{\mu}(X_1) \geq \int \sum_{i=1}^{\dim E^c_p} \chi_i(p) d\mu.
\]

**Claim.** \( \sum_{i=1}^{\dim E^c_p} \chi_i(p) = \lim_{n \to \infty} \frac{1}{n} \log |\det(DX_n(p)|_{E^c_p})| > 0 \) in a \( \mu \)-positive measure set.
Since \( \Lambda \) is asymptotically sectional-hyperbolic there exists \( C > 0 \) such that for all \( p \in \Lambda \setminus W^s(Sing(X)) \)

\[
\limsup_{t \to \infty} \frac{1}{t} \log |\det(DX_t(p)|_{E_p^c})| \geq C,
\]
so there exists a subsequence \( \{ t_k \}_{k \geq 1} \subset \mathbb{R} \) such that

\[
\frac{1}{t_k} \log |\det(DX_{t_k}(p)|_{E_p^c})| \geq C.
\] (6)

Now, by a straightforward computation we have \( DX_{t_k}(p)|_{E_p^c} = DX_{t_k}(p_k)D\)

\( X_{-\tau_k}(x)|_{E_p^c} \), where \( p_k = X_{-\tau_k}(p) \in X_{[0,1]}(p) \subset \Lambda \setminus W^s(Sing(X)) \), \( \tau_k = t_k - [t_k] \in [0,1) \), where \([ \cdot ]\) denotes the integer part. So, from (6) we have

\[
\frac{1}{[t_k]} \log |\det(DX_{t_k}(p)|_{E_p^c})| \\
\geq \frac{1}{t_k} \log |\det(DX_{t_k}(p_k)|_{E_p^c})| + \frac{1}{t_k} \log |\det(DX_{-\tau_k}(p)|_{E_p^c})| \\
\geq C + \frac{1}{t_k} \log |\det(DX_{-\tau_k}(p)|_{E_p^c}).
\]

As \( f(t) = \log |\det(DX_{-\tau}(p)|_{E_p^c})|, t \in [0,1], \) is bounded we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log |\det(DX_n(p)|_{E_p^c})| \geq C.
\] (7)

Let \( A = \Lambda \setminus W^s(Sing(X)) \) and \( p \in R \cap A \). As \( \sum_{i=1}^{\dim(E_p^c)} \chi_i(p) = \lim_{n \to \infty} \frac{1}{n} \log |\det(DX_n(p)|_{E_p^c})| \) is valid for \( p \in R \), by (7) we have

\[
\sum_{i=1}^{\dim(E_p^c)} \chi_i(p) = \lim_{n \to \infty} \frac{1}{n} \log |\det(DX_n(p)|_{E_p^c})| \\
= \limsup_{n \to \infty} \frac{1}{n} \log |\det(DX_n(p)|_{E_p^c})| \geq C > 0.
\]

Since \( \mu \) is non-atomic it follows by Lemma 3.2 that \( \mu(A^c) = 0 \), which implies that \( \mu(A > 0) \) (because \( \mu \) is supported in \( \Lambda \)). This proves the claim. So, \( \mu(R \cap A) > 0 \) and

\[
h_{\mu}(X_1) \geq \int \sum_{i=1}^{\dim(E_p^c)} \chi_i(p) d\mu \\
= \int_{R \cap \Lambda} \sum_{i=1}^{\dim(E_p^c)} \chi_i(p) d\mu + \int_{R \cap A^c} \sum_{i=1}^{\dim(E_p^c)} \chi_i(p) d\mu \\
= \int_{R \cap \Lambda} \sum_{i=1}^{\dim(E_p^c)} \chi_i(p) d\mu \\
> 0.
\]

Therefore, by variational principle we have \( h_{top}(X_1) \geq h_{\mu}(X_1) > 0 \). Thus, as \( h_{top}(X) = h_{top}(X_1) \), we obtain the announced result. \( \Box \)
Acknowledgments. We would like to thank the referees very much for their valuable comments and suggestions.

REFERENCES

[1] V. S. Afraimovic, V. V. Bykov and L. P. Shilnikov, The origin and structure of the Lorenz attractor, Dokl. Akad. Nauk SSSR, 234 (1977), 336–339.
[2] A. Arbieto, A. M. Lopez and C. A. Morales, Homoclinic classes for sectional-hyperbolic sets, Kyoto J. Math, 56 (2016), 531–538.
[3] S. Bautista, The geometric Lorenz attractor is a homoclinic class, Bol. Mat., 11 (2004), 69–78.
[4] S. Bautista and C. A. Morales, On the intersection of sectional-hyperbolic sets, J. Mod. Dyn., 9 (2015), 203–218.
[5] C. Bonatti, A. Pumariño and M. Viana, Lorenz attractors with arbitrary expanding dimension, International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), 39–44, World Sci. Publ., River Edge, NJ, 2000.
[6] J. Carmona, D. Carrasco-Olivera and B. San Martín, On the $C^1$ robust transitivity of the geometric Lorenz attractor, J. Differential Equations, 262 (2017), 5928–5938.
[7] E. Catsigeras, M. Cerminara and H. Enrich, The Pesin entropy formula for $C^1$ diffeomorphisms with dominated splitting, Ergodic Theory Dynam. Systems, 35 (2015), 737–761.
[8] S. Gan and L. Wen, Nonsingular star flows satisfy Axiom A and the no-cycle condition, Invent. Math., 164 (2006), 279–315.
[9] S. Gähler, Lineare 2-normierte Räume, Math. Nachr., 28 (1964), 1–43.
[10] J. Guckenheimer, A strange, strange attractor, in The Hopf Bifurcation and Its Applications, Applied Mathematical Sciences, (1976), 368–381.
[11] J. Guckenheimer and R. F. Williams, Structural stability of Lorenz attractors, Inst. Hautes Études Sci. Publ. Math., 50 (1979), 59–72.
[12] A. Kawaguchi, On areal spaces. I. Metric tensors in $n$-dimensional spaces based on the notion of two-dimensional area, Tensor N.S., 1 (1950), 14–45.
[13] R. Labarca and M. J. Pacifico, Stability of singularity horseshoes, Topology, 25 (1986), 337–352.
[14] E. N. Lorenz, Deterministic nonperiodic flow, J. Atmospheric Sci., 20 (1963), 130–141.
[15] R. J. Metzger, Stochastic stability for contracting Lorenz maps and flows, Comm. Math. Phys., 212 (2000), 277–296.
[16] R. J. Metzger, Sinai-Ruelle-Bowen measures for contracting Lorenz maps and flows, Ann. Inst. H. Poincaré Anal. Non Linéaire, 17 (2000), 247–276.
[17] R. J. Metzger and C. A. Morales, The Rovella attractor is a homoclinic class, Bull. Braz. Math. Soc., 37 (2006), 89–101.
[18] R. J. Metzger and C. A. Morales, Sectional-hyperbolic systems, Ergodic Theory Dynam. Systems, 28 (2008), 1587–1597.
[19] C. A. Morales, M. J. Pacifico and E. R. Pujals, Singular hyperbolic systems, Proc. Amer. Math. Soc., 127 (1999), 3393–3401.
[20] C. A. Morales, M. J. Pacifico and B. San Martín, Contracting Lorenz attractors through resonant double homoclinic loops, SIAM J. Math. Anal., 38 (2006), 309–332.
[21] C. A. Morales and B. San Martín, Contracting singular horseshoe, Nonlinearity, 30 (2017), 4208–4219.
[22] C. A. Morales and M. Vilches, On 2-Riemannian manifolds, SUT J. Math., 46 (2010), 119–153.
[23] E. M. Muñoz, B. San Martín and J. A. Vera, Nonhyperbolic persistent attractors near the Morse-Smale boundary, Ann. Inst. H. Poincaré Anal. Non Linéaire, 20 (2003), 867–888.
[24] M. J. Pacífico and M. Todd, Thermodynamic formalism for contracting Lorenz flows, J. Stat. Phys., 139 (2010), 159–176.
[25] R. C. Robinson, An Introduction to Dynamical Systems: Continuous and Discrete, 2nd edition, American Mathematical Society, Providence, 2012.
[26] A. Rovella, The dynamics of perturbations of the contracting Lorenz attractor, Bol. Soc. Brasil. Mat., 24 (1993), 233–259.
[27] D. V. Turaev and L. P. Shilnikov, An example of a wild strange attractor, Sb. Math., 189 (1998), 291–314.
[28] R. F. Williams, The structure of Lorenz attractors, *Inst. Hautes Études Sci. Publ. Math.*, 50 (1979), 73–99.

Received August 2018; revised November 2018.

*E-mail address*: samart@ucn.cl

*E-mail address*: kendry.vivas01@ucn.cl