Quiver gauge theories and integrable lattice models

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ABSTRACT: We discuss connections between certain classes of supersymmetric quiver gauge theories and integrable lattice models from the point of view of topological quantum field theories (TQFTs). The relevant classes include 4d $\mathcal{N} = 1$ theories known as brane box and brane tiling models, 3d $\mathcal{N} = 2$ and 2d $\mathcal{N} = (2,2)$ theories obtained from them by compactification, and 2d $\mathcal{N} = (0,2)$ theories closely related to these theories. We argue that their supersymmetric indices carry structures of TQFTs equipped with line operators, and as a consequence, are equal to the partition functions of lattice models. The integrability of these models follows from the existence of extra dimension in the TQFTs, which emerges after the theories are embedded in M-theory. The Yang–Baxter equation expresses the invariance of supersymmetric indices under Seiberg duality and its lower-dimensional analogs.
1 Introduction

The present work is motivated by an intriguing connection discovered in the past few years between supersymmetric quiver gauge theories and integrable lattice models in statistical mechanics.

In [1, 2], Bazhanov and Sergeev introduced an integrable spin model on a planar lattice, which generalizes many of previously known integrable lattice models such as the Kashiwara–Miwa [3] and chiral Potts [4–6] models. Soon after the first paper by Bazhanov and Sergeev appeared, Spiridonov [7] gave an interpretation of their model in terms of $\mathcal{N} = 1$ quiver gauge theories in four dimensions. The relation to gauge theory was further elucidated by Yamazaki [8, 9], who realized that the relevant quiver gauge theories arise naturally from a particular class of brane configurations in string theory, called brane tilings [10]. From the gauge theory viewpoint, the lattice on which the spin model is defined is the quiver diagram, the partition function is the supersymmetric index, and the
Yang–Baxter equation that guarantees the integrability of the model is a special instance of Seiberg duality.

This discovery, while quite remarkable, leaves us with a series of questions: Why are the supersymmetric indices of these theories captured by a lattice model? Why is this model integrable? Are there structures of integrable lattice models hidden in other theories?

In this paper we answer these questions, combining ideas from two equally stimulating connections uncovered in recent years. Of these connections, one lies between a certain class of quiver gauge theories and topological quantum field theories (TQFTs), whereas the other relates TQFTs and integrable lattice models. Our goal is to connect quiver gauge theories and integrable lattice models, and TQFTs provide a bridge between the two elements.

The first connection in question arises from the M5-brane construction of 4d $\mathcal{N} = 2$ theories [11–13]. Consider a stack of M5-branes wrapped on $S^1 \times S^3 \times \Sigma$, where $\Sigma$ is a compact Riemann surface. In addition, we introduce M5-branes that intersect with these branes along submanifolds of the form $S^1 \times S^3 \times \{p_i\}$, with $p_i$ being points on $\Sigma$. If $\Sigma$ is small compared to the $S^1$ and $S^3$, this brane system is described at low energies by an $\mathcal{N} = 2$ theory on $S^1 \times S^3$. Often this is a quiver gauge theory. The path integral on the geometry $S^1 \times S^3$ (after analytic continuation to Euclidean spacetime) computes the supersymmetric index of the theory. The most important property of this quantity is that it is invariant under continuous changes of the parameters of the theory. Since the geometry of $\Sigma$ encodes such parameters, it follows that the supersymmetric index is a topological invariant of $\Sigma$. In fact, it is equal to a correlation function $\langle \prod_i O_i(p_i) \rangle$ in a TQFT on $\Sigma$, where $O_i$ is a local operator representing the M5-brane inserted at $p_i$ [14].

If we consider a protected quantity different from the index on $S^3$ (such as the lens space index [15]), then we get another TQFT.

The second connection refers to a general construction of integrable lattice models from TQFTs of a special kind, due to Costello [16, 17]. Given a 2d TQFT equipped with line operators, we can place it on a torus $T^2$ and wrap line operators around various 1-cycles that form a lattice. The topological invariance of the theory implies that the correlation function for this lattice of line operators coincides with the partition function of a statistical mechanics model defined on the same lattice. Under this correspondence, the Yang–Baxter equation translates to the statement that the correlator remains the same when a line operator is moved past the intersection of two other line operators. The structure of a TQFT itself is not strong enough to ensure this property since the move is not topologically trivial: the system may undergo a phase transition as the moved line hits and crosses the intersection point. If, however, the theory has “extra dimensions” along which the line can be shifted, the collision can be avoided and hence the Yang–Baxter equation holds. What is more, the correlator then carries continuous parameters, namely the positions of the line operators in the extra dimensions. In the context of integrable models, continuous parameters on which the Boltzmann weight depends are called spectral parameters, and their presence is essential for integrability. Thus, a 2d TQFT with extra dimensions produces from line operators a solution of the Yang–Baxter equation with spectral parameter.
We wish to understand the connection between brane tilings and integrable lattice models in light of these independent, though apparently related, developments in the relevant areas. To this end, it proves helpful to first study the case of the brane box construction [18], which is a precursor of the brane tiling construction and conceptually simpler.

The brane box construction is similar to the M5-brane construction described above. In this construction, we start with a stack of D5-branes on $S^1 \times S^3 \times T^2$, and put NS5-branes on submanifolds of the form $S^1 \times S^3 \times C_\alpha$. Here $C_\alpha$ are 1-cycles of $T^2$, making up a lattice. (Actually, $T^2$ can be replaced by any Riemann surface without breaking the $\mathcal{N} = 1$ supersymmetry.) At low energies, this brane configuration realizes a 4d $\mathcal{N} = 1$ quiver gauge theory, which we refer to as a “brane box model,” placed on $S^1 \times S^3$. Just as we did for the M5-brane construction, we can relate this theory to a 2d TQFT by considering its supersymmetric index. Adapted to the present situation, the argument used there suggests that the supersymmetric index is given by a correlation function $\langle \prod_\alpha L_\alpha(C_\alpha) \rangle$ in a TQFT on $T^2$, where $L_\alpha(C_\alpha)$ is a line operator created by the NS5-brane wrapped around $C_\alpha$.

According to the construction of lattice models from TQFTs, this correlator coincides with the partition function of a lattice model. To establish the integrability of the model, we need extra dimensions along which line operators can move freely. An extra dimension indeed emerges as 11th dimension if we embed the brane system into M-theory by string dualities. Hence, we conclude that the supersymmetric index of a brane box model is equal to the partition function of an integrable lattice model. It turns out that the Yang–Baxter equation reduces to Seiberg duality for SU($N$) SQCD with $2N$ flavors.

Once the connection between the brane box construction and integrable lattice models is understood, the case of brane tilings is not so difficult. In this case we consider deformations of brane box configurations, in which we let the NS5-branes combine with the D5-branes over ribbon-shaped neighborhoods of the 1-cycles $C_\alpha$ in $T^2$. Such a deformed brane configuration still yields a 4d $\mathcal{N} = 1$ quiver gauge theory, provided that the deformation meets a certain criterion. By the same reasoning as above, we deduce that the supersymmetric index of this theory is given by the correlation function of “thickened” line operators representing the ribbon neighborhoods, and equal to the partition function of an integrable lattice model. In this way the Bazhanov–Sergeev model arises from brane tilings. Again, the Yang–Baxter equation boils down to Seiberg duality, though this time the equation involves a sequence of four basic duality transformations.

So we have answers to the first two of our questions. The supersymmetric indices of theories constructed from brane tilings are captured by a lattice model since they are given by correlation functions of line operators in a TQFT, which in turn is a consequence of the nature of the brane construction and the fact that the index is a protected quantity. Furthermore, the integrability of this lattice model is guaranteed by the hidden extra dimension which emerges after the brane system is embedded in M-theory.

In sections 2–4, we discuss in greater detail the connections summarized here among the brane box and brane tiling constructions, TQFTs with extra dimensions, and integrable lattice models. After reviewing in section 2 the construction of integrable lattice models from TQFTs with extra dimensions, we explain in section 3 how it can be applied to the supersymmetric indices of brane box models. We identify the associated integrable lattice
model, and show that the Yang–Baxter equation for this model takes the form of Seiberg duality. In section 4, we treat the case of brane tiling models.

Sections 5 and 6 are devoted to answering our third question, that is, finding more examples of quiver gauge theories whose supersymmetric indices are captured by integrable lattice models.

One way to produce more quiver gauge theories is to apply T-duality to the brane configurations discussed above. In this manner we get 3d $\mathcal{N} = 2$ and 2d $\mathcal{N} = (2, 2)$ theories. Being related to the 4d parents by T-duality, the supersymmetric indices of these theories are also given by the partition functions of some integrable lattice models. In section 5, we will see that for these theories, the Yang–Baxter equation follows from lower-dimensional analogs of Seiberg duality, namely a variant of Aharony duality \cite{19} in the 3d case and Hori–Tong duality \cite{20} in the 2d case.

Perhaps more unexpected is that there are 2d $\mathcal{N} = (0, 2)$ quiver gauge theories whose supersymmetric indices, or elliptic genera, exhibit integrability. They have half as many supercharges as the theories mentioned so far, and cannot be obtained by simple dimensional reduction from three or four dimensions.

In section 6, we discuss three classes of such $\mathcal{N} = (0, 2)$ theories. Two of them are $\mathcal{N} = (0, 2)$ counterparts of the classes of $\mathcal{N} = (2, 2)$ theories considered in section 5. The Yang–Baxter equation for these classes identifies two theories related by Seiberg-like triality, discovered by Gadde, Gukov and Putrov \cite{21}. The third class consists of $\mathcal{N} = (0, 2)$ theories constructed from brane cube configurations \cite{22}, and actually gives rise to a 3d lattice model. The integrability condition for this model is that its Boltzmann weight satisfies Zamolodchikov’s tetrahedron equation \cite{23, 24}, which is the 3d analog of the Yang–Baxter equation. Our analysis in this section will be somewhat incomplete, unfortunately. For the first two classes, we will demonstrate the integrability of the associated lattice models, but not identify the underlying brane constructions or TQFT structures whose existence is strongly suggested by the integrability. For the third class, on the other hand, we will describe the brane construction and associated lattice model, but not determine the corresponding solution of the tetrahedron equation.$^1$

Having established the structures of integrable lattice models in several classes of quiver gauge theories, we should now ask what we can do with these structures. It is likely that knowledge accumulated in the area of integrable models provides new insights into these theories or quiver gauge theories in general. Conversely, tools from the gauge theory side, such as localization and $1/N$ expansion, may help further elucidate the physics of integrable models.

Given the generality of the TQFT construction of integrable lattice models, we also expect that there are many more applications in addition to those discussed in this paper. Below we describe just a few possibilities.

The NS5-branes in a brane box configuration can be mapped by dualities to either M5- or M2-branes intersecting with a stack of M5-branes. These branes represent codimension-

\begin{footnote}{A different solution of the tetrahedron equation has been found recently by Gadde and Yamazaki \cite{25}. Their solution is based on $\mathcal{N} = (0, 2)$ SQCD \cite{21} with all flavor nodes having equal ranks. It remains to be seen whether this one may be understood from the perspective adopted in this paper.}

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2 and -4 defects in 6d $\mathcal{N} = (2, 0)$ superconformal field theory. From the 4d point of view, they are domain walls and line operators in an $\mathcal{N} = 2$ theory.\footnote{We can obtain any $\mathcal{N} = 2$ theory of class $\mathcal{S}$ \cite{fkm,OFK19} from the brane box configuration \eqref{eq:brane-box} by replacing $T^2$ with the relevant Riemann surface and introducing D5-branes in the 012357 directions.} This observation suggests that integrability plays a key role in the physics of defects in the 6d theory and 4d $\mathcal{N} = 2$ theories, and the ideas contained in this paper may be useful for understanding it. In this regard, we point out that the Yang–Baxter equation has made an unexpected appearance in studies of the moduli spaces of $\mathcal{N} = 2$ theories compactified on $S^1$ \cite{Bouwknegt:1993ja, Bouwknegt:1995vs}.

Another important appearance of integrability is found in the Bethe/gauge correspondence \cite{Beisert:2005tm, Beisert:2005di, Beisert:2006ez} between 2d $\mathcal{N} = (2, 2)$ theories and quantum integrable systems. It may be possible to connect that story to ours, by topologically twisting $\mathcal{N} = (2, 2)$ theories associated with brane box and brane tiling configurations.

Lastly, integrability features prominently in the AdS/CFT correspondence, and the constructions discussed in the present work may shed light on this aspect; after all, the developments of brane box and brane tiling techniques were motivated by the AdS/CFT correspondence. In fact, brane box models were studied at one-loop level in the planar limit in \cite{FGH}, and it was shown that at this level, their dilatation operator can be identified with the Hamiltonian of an integrable spin chain. This fact raises the hope that the integrability in the AdS/CFT correspondence may be understood in a framework that extends the one presented in this paper.

## 2 Integrable lattice models from TQFTs with extra dimensions

As explained in the introduction, the construction of integrable lattice models using line operators in TQFTs constitutes an essential ingredient for our argument. So let us begin by reviewing this construction. Our discussion mainly follows Costello’s paper \cite{Costello}, to which we refer the reader for more details.

### 2.1 Vertex models from TQFTs

Consider a 2d TQFT equipped with a family of line operators parametrized by some continuous set, say $\mathbb{R}$ or $\mathbb{C}$. We denote by $L_r(C)$ a line operator with parameter $r$, supported on an oriented closed curve $C$. Let us place this TQFT on a torus $T^2$, and choose 1-cycles $A_1, \ldots, A_m, B_1, \ldots, B_n$ that form an $m \times n$ lattice on $T^2$. The case with $(m, n) = (2, 3)$ is illustrated in figure 1. We are interested in the correlation function

$$Z(\{r_\alpha\}, \{s_\beta\}) = \left\langle \prod_{\alpha=1}^{m} L_{r_\alpha}(A_\alpha) \prod_{\beta=1}^{n} L_{s_\beta}(B_\beta) \right\rangle$$

of line operators wrapped around these cycles.

We compute this correlation function by breaking up the torus into smaller rectangular pieces, as shown with dotted lines in figure 1. The idea is that we first perform the path integral on each of these pieces, and then glue the results together to reconstruct the path integral on $T^2$. \footnote{We can obtain any $\mathcal{N} = 2$ theory of class $\mathcal{S}$ \cite{fkm,OFK19} from the brane box configuration \eqref{eq:brane-box} by replacing $T^2$ with the relevant Riemann surface and introducing D5-branes in the 012357 directions.}
Figure 1: A $2 \times 3$ lattice of line operators on a torus. The dotted lines divide the lattice into rectangular pieces, each of which contains two intersecting segments of line operators.

Each piece in this decomposition contains two segments of intersecting line operators. Every such piece is, topologically, a square with two line operators crossing in the middle:

\[
\begin{array}{c}
\text{r} \\
\text{s}
\end{array}
\quad .
\]

(2.2)

Using the topological invariance of the theory, we can deform it as

\[
\begin{array}{c}
\text{r} \\
\text{s}
\end{array}
\quad .
\]

(2.3)

Let $V_r$ be the space of states on a finite interval intersected by $L_r$. Intuitively, $V_r$ is the Hilbert space of an open string with a particle attached whose worldline is the line operator $L_r$. In this language, the above picture shows that given an initial state of two open strings in $V_r \otimes V_s$, the path integral outputs a final state in $V_s \otimes V_r$ resulted from interaction in the middle. Thus, the path integral on the square produces a linear map

\[
\hat{R}(r,s) : V_r \otimes V_s \longrightarrow V_s \otimes V_r.
\]

(2.4)

We call $\hat{R}$ the $R$-matrix of the TQFT with line operators.

Choosing a basis $\{e_{r,i}\}$ for $V_r$ for all $r$, we can represent $\hat{R}$ by its matrix elements. The matrix element $\hat{R}_{i_1 j_1}^{i_2 j_2}(r,s)$ is the scattering amplitude for the process in which the initial state $e_{r,i_1} \otimes e_{s,j_1}$ ends up in the final state $e_{s,j_2} \otimes e_{r,i_2}$. We represent it pictorially as

\[
\hat{R}_{i_1 j_1}^{i_2 j_2}(r,s) = \quad \begin{array}{c}
\text{r} \\
\text{s}
\end{array}
\quad .
\]

(2.5)
Then, the correlation function (2.1) is given by the formula

\[ Z(\{r_\alpha\}, \{s_\beta\}) = \sum_{\{i_\alpha,\beta\}} \sum_{\{j_\beta,\alpha\}} \prod_{\alpha=1}^{m} \prod_{\beta=1}^{n} r_\alpha \frac{j_{\beta,\alpha+1}}{i_{\alpha,\beta}} \frac{i_{\alpha,\beta+1}}{j_{\beta,\alpha}} , \]  

(2.6)

with the periodic boundary conditions \( i_{\alpha,n+1} = i_{\alpha,1} \) and \( j_{\beta,m+1} = j_{\beta,1} \).

By representing the R-matrix as above, we are treating the open strings as if their physical degrees of freedom are carried solely by the particles attached to them, or equivalently, by the edges of the lattice. In words, the formula (2.6) instructs us to do the following. First, choose a state on every edge of the lattice. Next, compute the probability amplitude for this configuration of states by multiplying the corresponding R-matrix elements. Finally, sum over all possible such configurations to find the answer.

This procedure is, in fact, precisely how the partition function of a vertex model in statistical mechanics is defined. In a vertex model, state variables are assigned to the edges of a lattice, and the interaction takes place at the vertices. The total energy \( E \) of the system is the sum of interaction energies at the vertices, so the Boltzmann weight \( e^{-\beta E} \) factorizes into weight factors associated to the vertices. These factors are matrix elements of \( \tilde{R} \). In this context, \( r \) is called the spectral parameter of the model.

In conclusion, we have found that the correlation function (2.1) for a lattice of line operators in a 2d TQFT coincides with the partition function of a vertex model defined on the same lattice, with the line operator parameter playing the role of a spectral parameter.

2.2 Integrability and extra dimensions

Let us fix the parameters \( \{s_\beta\} \) in the lattice of line operators under consideration. With the vertical direction of the torus viewed as the time direction, the Hilbert space of the theory is the tensor product \( \mathcal{H} = \bigotimes_{\beta=1}^{n} V_{s_\beta} \).\(^3\) We define the row-to-row transfer matrix \( T(r) \in \text{End}(\mathcal{H}) \) by

\[ T(r) = \text{Tr}_{V_r} \left( \tilde{R}(r,s_1) \circ V_r \cdots \circ V_r \tilde{R}(r,s_n) \right) . \]  

(2.7)

In components,

\[ T_{j_{1,1},\ldots,j_{n,1}}^{j_{1,2},\ldots,j_{n,2}}(r) = \sum_{\{i_\beta\}} \prod_{\beta=1}^{n} r_{i_\beta} \frac{j_{\beta,2}}{i_{\beta,1}} \frac{i_{\beta+1}}{j_{\beta,1}} . \]  

(2.8)

\(^3\)To be precise, the actual Hilbert space is smaller than \( \mathcal{H} \) since the degrees of freedom associated with the endpoints (“Chan–Paton factors”) must match between adjacent intervals. However, we can enlarge the Hilbert space to \( \mathcal{H} \) simply by assigning infinite energy to those states in \( \mathcal{H} \) that do not satisfy this condition. This is consistent with the path integral definition (2.3) of the R-matrix, since the scattering amplitude vanishes unless the Chan–Paton factors match.
In terms of $T(r)$, the partition function (2.6) is written as

$$Z(\{r_\alpha\}, \{s_\beta\}) = \text{Tr}_\mathcal{H} \prod_{\alpha=1}^m T(r_\alpha) = \sum_{\{j_\beta, \alpha\}} \prod_{\alpha=1}^m T^{j_1, \alpha+1, \ldots, j_n, \alpha+1}(r_\alpha).$$

(2.9)

In a TQFT, time evolution is trivial unless the state hits some operator at some point in time. We may think of $T(r)$ as the time evolution operator induced by $L_r$.

We say that the vertex model on the lattice of line operators is integrable if $T(r)$ is analytic in $r$ and

$$[T(r), T(r')] = 0$$

(2.10)

for all $r, r'$. The rationale for this terminology is that by Taylor expanding $T(r)$, we get an infinitely many operators that commute with the time evolution operator.

A sufficient condition for integrability is that the R-matrix has the following two properties. The first is that $\tilde{R}(r, s)$ is an isomorphism for $r \neq s$, with the inverse being $\tilde{R}(s, r)$:

$$\tilde{R}(r, s) \tilde{R}(s, r) = 1.$$ 

(2.11)

The second is that $\tilde{R}$ satisfies the Yang–Baxter equation:

$$\tilde{R}(s, t) \tilde{R}(r, t) \tilde{R}(r, s) = \tilde{R}(r, s) \tilde{R}(r, t) \tilde{R}(s, t).$$

(2.12)

Each factor on the two sides of the equation is regarded as an operator on a tensor product of $V_r$, $V_s$ and $V_t$. For example, $\tilde{R}(r, t)$ on the left-hand side is a map from $V_s \otimes V_r \otimes V_t$ to $V_s \otimes V_t \otimes V_r$.

In the TQFT language, the identity (2.11) means that two tangled line operators can be straightened out:

$$r = s.$$ 

(2.13)

The Yang–Baxter equation, on the other hand, expresses invariance under moving a line operator past the intersection of two other line operators:

$$r = s.$$ 

(2.14)
To see that these properties imply the commutativity of transfer matrices, consider the following move of line operators:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{move.png}
\end{array}
\]

Taking the trace over \(V_r \otimes V_{r'}\) of both sides, and using the identity (2.13) and the cyclic property of the trace, we find that the left- and right-hand sides become \(T(r)T(r')\) and \(T(r')T(r)\), respectively.

In general, a 2d TQFT does not have the above properties. Physics may change discontinuously under the relevant moves due to phase transitions, as topology of the line operators does not remain the same. Imagine, however, that there are “extra dimensions” hidden in the above pictures. For instance, the direction perpendicular to the page may provide one. The line operators can then sit at different points in the extra dimensions. If so, these moves are topologically trivial and hence the R-matrix satisfies the desired properties. An argument of this sort is familiar from studies of knot invariants, which may be approached either from the perspective of 3d TQFTs or lattice models [32, 33].

The above observation motivates us to introduce the notion of extra dimensions to TQFTs. Consider a \(D\)-dimensional quantum field theory formulated on \(\Sigma \times M\), where \(\Sigma\) is any \(d\)-manifold and \(M\) is a fixed \((D - d)\)-manifold. We say that the theory is a \textit{d-dimensional TQFT with extra dimensions} if it is topological on \(\Sigma\).\(^4\) We refer to \(M\) as the \textit{internal space} of the theory. This definition is very much in the same spirit as the notion of a 2d conformal field theory “valued in 4d quantum field theories” [49] developed in connection to the 6d construction of 4d \(\mathcal{N} = 2\) theories of class \(S\) [12, 13]. In that case, however, the choice of the 4d spacetime is not part of the definition. In contrast, here we demand the theory to be topologically invariant on \(\Sigma\) for a specific choice of \(M\).

Now we come to the main point of our discussion. Suppose that a 2d TQFT with extra dimensions has line operators. Place the theory on \(T^2\) and wrap line operators around 1-cycles \(A_\alpha\) and \(B_\beta\). In the internal space \(M\), the 1-cycles are located at some points \(x_\alpha\) and \(y_\beta\), respectively. The situation is similar to what we considered at the beginning of this section, but unlike that case, we do \textit{not} assume that the line operators carry a continuous parameter. To highlight the difference, let us consider the extreme case that the theory has only one type of line operator, \(L\). Repeating the same argument, we arrive at the

\[^4\text{Although such theories might sound exotic, actually many have already been studied before. Typically, they are constructed from supersymmetric theories by topological twisting along } \Sigma. \text{ 2d examples are Costello’s theory mentioned below, 4d } \mathcal{N} = 2 \text{ gauge theories with } M = S^2 \text{ [34], the } \Omega\text{-background } \mathbb{R}^2_\epsilon \text{ [35]} \text{ and (in the superconformal case) any Riemann surface [36], 5d } \mathcal{N} = 2 \text{ super Yang–Mills theory with } M = S^2 \text{ [37], and 6d } \mathcal{N} = (2, 0) \text{ superconformal field theory with } M = S^1 \times S^3 \text{ [14]. 3d examples include 5d } \mathcal{N} = 2 \text{ super Yang–Mills theory with } M = \mathbb{R}^2_\epsilon \text{ [38, 39] and } S^2 \text{ [40–42], 6d } \mathcal{N} = (2, 0) \text{ theory with } M = S^1 \times S^2 \text{ [43], } S^3 \text{ [44], and more general lens spaces } L(p, q) \text{ [45]. 4d examples are 6d } \mathcal{N} = (2, 0) \text{ theory with } M \text{ any Riemann surface [46–48].}\]
conclusion that the correlation function

\[ Z(\{x_\alpha\}, \{y_\beta\}) = \left\langle \prod_{\alpha=1}^{m} \mathcal{L}(A_\alpha \times \{x_\alpha\}) \prod_{\beta=1}^{n} \mathcal{L}(B_\beta \times \{y_\beta\}) \right\rangle \]  

(2.16)

is given by the partition function of a vertex model defined on the lattice of line operators. As we have just seen, the presence of extra dimensions guarantees that the R-matrix is invertible and satisfies the Yang–Baxter equation.

A beautiful insight of Costello [16, 17] is that even though the line operator lacks a continuous parameter, the R-matrix still depends on such parameters: the positions \(x_\alpha, y_\beta\) of line operators in \(M\). Thus, the vertex model is integrable if the transfer matrix varies analytically on \(M\). Based on this idea, it was shown in [17] that a special case of the 6-vertex model corresponding to the XXX spin chain, and its generalizations based on Lie algebras other than \(\mathfrak{sl}_2\), arise from a 2d TQFT with extra dimensions whose internal space \(M = \mathbb{CP}^1\). This TQFT is obtained from a deformed and topologically twisted version of 4d \(\mathcal{N} = 1\) super Yang–Mills theory, and the line operators used in the construction are Wilson lines. The extra dimensions therefore elegantly explain not only why the Yang–Baxter equation holds for these models, but also where the spectral parameter comes from.

Speaking of extra dimensions, it should be noted that although only line operators were considered in our discussion, we could as well use higher-dimensional operators that have codimension at least two in \(\Sigma \times M\). This is because after we wrap them on 1-cycles on \(\Sigma\), there is still room in \(M\) for them to avoid one another. In our main examples, we will make use of codimension-2 defects in a 6d theory.

2.3 IRF models

In the construction described above, we obtained a vertex model from line operators by letting the midpoints of open strings represent all physical degrees of freedom. In many cases, however, degrees of freedom really reside on (and only on) the endpoints. In such cases it is more efficient to reformulate the vertex model as an interaction-round-a-face (IRF) model.

In an IRF model, state variables live on the vertices of a lattice, and interaction takes place among vertices connected by edges that surround a face. We use the letters \(a, b,\) etc. to denote state variables. For a square lattice, the Boltzmann weight for the interaction is written as

\[ W\left( \begin{array}{cc} d & c \\ a & b \end{array} \bigg| r, s \right) = r^{a} \cdot s^{d} \]  

(2.17)

The dashed oriented lines are rapidity lines, and make up the dual lattice. The parameters \(r, s\) are spectral parameters, also called rapidities. An IRF model is a vertex model on the dual lattice, in which the state space assigned to an edge is the direct product of two spaces and many elements of the R-matrix vanish.
From the TQFT point of view, rapidity lines are line operators, and the rapidities are their parameters. The Yang–Baxter equation (2.14) for an IRF model is

\[ a \ b \ c \ d \ e \ f \ s \ t \ r = a \ b \ c \ d \ e \ f \ s \ t \ r. \]  

(2.18)

In terms of the Boltzmann weight, the equation reads

\[ \sum_g W(e \ d \ s, \ t) W(g \ c \ r, \ t) W(e \ g \ r, \ s) \]
\[ = \sum_g W(d \ c \ r, \ s) W(e \ d \ r, \ t) W(f \ g \ a \ b \ s, \ t). \]  

(2.19)

### 2.4 3d TQFTs and the tetrahedron equation

As a generalization, we can construct higher-dimensional lattice models using codimension-1 defects in a \(d\)-dimensional TQFT with \(d > 2\). Placing the theory on a \(d\)-torus \(T^d\) and putting these defects on \((d - 1)\)-cycles that form a \(d\)-dimensional lattice, we get a vertex model on this lattice. If the theory has extra dimensions, the R-matrix of the model satisfies a \(d\)-dimensional analog of the Yang–Baxter equation.

To be concrete, let us take \(d = 3\). In this case, codimension-1 defects are surface operators, and three of them intersect at a vertex of the lattice. A cubic neighborhood of a vertex looks like

\[ r \ s \ t \]

(2.20)

The orientations of the lattice edges specify those of the surface operators (drawn with dotted lines in the picture). Viewing this picture as the worldvolumes of three faces of the cube traveling in a diagonal direction, we see that the R-matrix \(R(r, s, t)\) produced by the path integral on this cube is an endomorphism of \(V_{r,s} \otimes V_{s,t} \otimes V_{t,s}\), where \(V_{r,s}\) is the Hilbert spaces on a square intersected by two surface operators with parameters \(r\) and \(s\).

\(^5\)In the present case there is no natural way to distinguish the different orderings of the tensor products since the positions of three squares can be permuted on a plane. This is to be contrasted with the \(d = 2\) case, where the positions of two intervals on a line cannot be interchanged without collision.
Pictorially, the R-matrix is represented as

\[ R(r, s, t) = (t, r) (s, t) (r, s) \]. \quad (2.21)

To avoid clutter, in what follows we will suppress spectral parameters in this kind of pictures.

The 3d analog of the Yang–Baxter equation is Zamolodchikov’s tetrahedron equation \([23, 24]\), obtained by moving one of four surface operators that form a tetrahedron:

\[
\begin{array}{c}
\text{Graphically, it takes the form of equivalence between two ways of dividing a dodecahedron.}
\end{array}
\]

The tetrahedron equation implies the commutativity of layer-to-layer transfer matrices at different values of the spectral parameter.

When the state variables may be considered as living on the cubes surrounded by surface operators, a vertex model can be reformulated as an interaction-round-a-cube (IRC) model \([50]\). The Boltzmann weight of an IRC model is represented by a cube:

\[
W(a|e, f, g|b, c, d|h) = \quad (2.23)
\]

The tetrahedron equation for an IRC model is

\[
\sum_d W(a_4|c_1, c_3, c_2|b_3, b_2, b_1|d)W(c_1|a_3, b_1, b_2|d, c_6, c_4|b_4)
\times W(b_1|c_4, c_3, d|b_3, b_4, b_1|c_2, c_6, c_5|a_1)
= \sum_d W(b_1|c_4, c_3, c_1|a_4, a_3, a_2|d)W(c_1|a_3, a_4, b_2|c_2, c_6, d|a_1)
\times W(a_4|d, c_3, c_2|b_3, a_1, a_2|c_5)W(d|a_3, a_2, a_1|c_5, c_6, c_4|b_4). \quad (2.24)
\]

Graphically, it takes the form of equivalence between two ways of dividing a dodecahedron.
into four hexahedra:

\[ \begin{array}{c}
\text{D5} & \times & \times & \times & \times & \times & \times \\
\text{NS5} & \times & \times & \times & \times & \times \\
\text{NS5} & \times & \times & \times & \times & \times \\
\end{array} \]  

(2.25)

3 4d $\mathcal{N} = 1$ quiver gauge theories

Now we wish to apply the above construction to 4d $\mathcal{N} = 1$ quiver gauge theories realized by brane box configurations. In this section, we explain how the elements that entered the construction – TQFT, lattice of line operators, and extra dimensions – arise nicely in the supersymmetric indices of brane box models. Furthermore, we determine the associated integrable lattice model, and demonstrate that the Yang–Baxter equation for this model is equivalent to the equality between the supersymmetric index of SU($N$) SQCD with $2N$ flavors and that of its Seiberg dual.

3.1 Brane box models

Consider Type IIB superstring theory, and suppose that we have $N$ coincident D5-branes extending in the directions 012346, and a number of NS5-branes in 012345 and 012367:

\[
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{D5} & \times & \times & \times & \times & \times & \times & \\
\text{NS5} & \times & \times & \times & \times & \times & \times & \\
\text{NS5} & \times & \times & \times & \times & \times & \times & \\
\end{array}
\]  

(3.1)

These branes are all located at the same point in the 89-plane. So each NS5-brane intersects with the D5-branes along a straight line parallel to the 4- or 6-axis.

Macroscopically, this brane system realizes a supersymmetric quiver gauge theory, much the same way as the brane constructions of 3d $\mathcal{N} = 4$ \[51\] and 4d $\mathcal{N} = 2$ \[11\] gauge theories. In those constructions, D3- or D4-branes are suspended between NS5-branes placed at finite intervals in the 6-direction. The difference here is that the D5-branes are divided by NS5-branes into finite segments in the 4-direction as well. The resulting theory therefore has less spacetime dimensions and supersymmetry, but a more complicated quiver: it is a 4d $\mathcal{N} = 1$ gauge theory described by a planar quiver diagram. Brane box models, first studied in \[18\], are theories constructed from brane configurations of this kind.

To avoid complications associated with semi-infinite branes, let us compactify the 45-plane to a torus $T^2$. Then, the intersections of the NS5-branes with the D5-branes form a lattice on the 46-torus. We draw the intersections by oriented dashed lines and call them rapidity lines, anticipating connection with integrable lattice models. We choose the
orientations for horizontal lines in such a way that they are parallel (and not antiparallel) to one another, and similarly for vertical lines. See figure 2a for an example of a brane box model, in which the rapidity lines make up a $2 \times 3$ lattice.

The rule for determining the quiver is as follows. We label the faces of the lattice by letters $a$, $b$, etc. To the $a$th face is associated a vector multiplet for gauge group $SU(N)_a$, which originates from open strings whose ends are attached on this part of the D5-branes. Since the rank of the gauge group is fixed throughout our discussion, in our quiver diagram we represent the vector multiplet simply by a circle node labeled $a$:

$$a$$  \hspace{2cm} (3.2)

From two faces $a$ and $b$ adjacent to each other or sharing a vertex, we get a chiral multiplet in a bifundamental representation of $SU(N)_a \times SU(N)_b$. This is represented by an arrow between the two nodes:

$$a \rightarrow b$$  \hspace{2cm} (3.3)

is a chiral multiplet in the representation $(\mathbf{N}, \mathbf{N})$ of $SU(N)_a \times SU(N)_b$, coupled to the vector multiplets. If four faces $a$, $b$, $c$, $d$ are placed around a vertex in the counterclockwise order, the corresponding part of the quiver is given by

$$ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} $$  \hspace{2cm} (3.4)

What makes brane box models physically interesting is that they are chiral: between nodes, arrows only point in one direction. Finally, to each triangle formed by three arrows, we get a superpotential term given by the trace of the product of the corresponding bifundamental multiplets, with sign determined by the orientation of the arrows. The quiver diagram for the $2 \times 3$ brane box model is shown in figure 2b.

The gauge group of a brane box model is the product of $SU(N)$s and not $U(N)$s. To understand why, let us suppose for simplicity that there are only NS5-branes spanning the 012345 directions. In this case we get a 5d $\mathcal{N} = 1$ theory. We are taking the string coupling
Figure 3: Sliding D5-brane segments along NS5-branes. Illustrated here is an ideal situation with \( g_s = 0 \).

Figure 4: For \( g_s > 0 \), taking apart two D5-brane segments creates either (a) a tilted \((N, 1)\) 5-brane or (b) a tilted \((N, -1)\) 5-brane.

\( g_s \) to be small, which means that the NS5-branes have tension much larger than that of the D5-branes and can be treated as rigid objects. These heavy NS5-branes “chop” the D5-branes into finite segments in the 6-direction. Imagine sliding these segments slightly in the 5-direction. If \( g_s \) were strictly zero, the NS5-branes would be infinitely heavier than the D5-branes and not affected by such a deformation at all. Then we would have a situation sketched in figure 3.

In reality, \( g_s \) is small but nonzero, and the deformation does affect the NS5-branes. What actually happens is that when a stack of \( N \) D5-branes hit on an NS5-brane, these branes combine to form a bound state – an \((N, 1)\) or \((N, -1)\) 5-brane, depending on the relative positions of the branes – which then goes off diagonally, as shown in figure 4. As a result, moving the D5-brane segments in the 5-direction entails shifts of semi-infinite NS5-branes in the 6-direction, and this costs an infinite amount of energy. The \( x^5 \)-coordinates of the D5-brane segments are therefore frozen. As the diagonal U(1) gauge field on each segment is in the same 5d multiplet as the scalar \( x^5 \), it is frozen as well. Thus, the gauge group of the 5d theory is the product of SU(\( N \))s, times a U(1) factor corresponding to the overall center of mass coordinate of the D5-branes. The U(1) factor is decoupled from the rest of the dynamics, so we ignore it.

3.2 R-symmetry and flavor symmetries

The brane system (3.1) is invariant under rotations in the 89-plane. The rotational symmetry appears in the effective 4d theory as an R-symmetry U(1)\(_R\). However, the R-charges of the bifundmanetal multiplets are not determined uniquely by the brane configuration,
because of the possibility of shifting them by U(1) flavor charges. This ambiguity plays a 
crucial role in the connection to integrable lattice models, so let us look at it closely.

We denote by $R_{ab}$ the R-charge of the bifundamental multiplet represented by an arrow 
going from node $a$ to $b$. There are two conditions for a set of R-charges $\{R_{ab}\}$ to be realized 
in a brane box model. The first is that the superpotential must have R-charge 2 so that it 
preserves $U(1)_R$. Thus, for every triangle

$$R_{ab} + R_{bc} + R_{ca} = 2.$$  \hspace{1cm} (3.6)

The second is that $U(1)_R$ must be nonanomalous. This requirement leads to the condition 
that at every node, the R-charges of the bifundamental multiplets starting from or ending 
on that node add up to the number of such multiplets minus two. In our case, for every

$$R_{ab} + R_{ca} + R_{ad} + R_{ea} + R_{af} + R_{ga} = 4.$$  \hspace{1cm} (3.8)

The second condition is equivalent to the vanishing of the one-loop $\beta$-functions for the 
gauge couplings, and one expects that the theory flows to a nontrivial IR fixed point when 
this condition is satisfied.

What is the dimension of the space of solutions to the constraint equations? The 
quiver of a brane box model provides a triangulation of $T^2$. The number of R-charges to 
be assigned is equal to the number of edges $E$, whereas the number of constraints is equal 
to the number of faces $F$ plus that of vertices $V$. Thus, naive counting would suggest that 
the dimension is given by minus the Euler characteristic $\chi = V - E + F$, which vanishes 
for $T^2$. This is not the case, though. For example, we can assign R-charge $R = 1 - r$ to 
the horizontal arrows, $R = 1 + r$ to the vertical ones and $R = 0$ to the diagonal ones, with 
r being a free parameter.

The reason for the mismatch is that the constraints are not all independent. Indeed, 
using the triangle condition (3.6), the anomaly cancellation condition (3.8) can be replaced 
by

$$R_{ad} - R_{bc} + R_{ga} - R_{fe} = 0.$$  \hspace{1cm} (3.9)

The sum of the left-hand side, with $a$ running over all nodes in the same row of the quiver, 
is zero due to the periodicity of $T^2$. So there is one relation for each row. Similarly, there is
one relation for each column. For an \( m \times n \) lattice, the dimension of the space of solutions is therefore at least \( m + n \), the total number of rapidity lines.

These \( m + n \) degrees of freedom in the R-charge assignment may be thought of as degrees of freedom associated to the rapidity lines. Concretely, we can parametrize the R-charges as follows. To each rapidity line we assign a real parameter, and call it the rapidity or spectral parameter of the rapidity line. Then, an arrow has \( R = 1 - r \) if it is horizontal and \( R = 1 + r \) if vertical, where \( r \) is the spectral parameter of the rapidity line intersecting with that arrow. The R-charges of the diagonal arrows are determined by the condition (3.6). The rule is summarized in figure 5. One readily sees that with this rule, the condition (3.8) is also satisfied.

In general, a U(1) R-symmetry can be shifted by U(1) flavor symmetries, that is, global U(1) symmetries that commute with the supersymmetry algebra. In view of this fact, the degrees of freedom in the R-charge assignment just found suggests that there are \( m + n \) U(1) flavor symmetries, each associated to a single rapidity line.

The origin of these global symmetries can be most clearly understood in the original brane picture. Let us pick an NS5-brane, say the one corresponding to the \( \alpha \)th rapidity line, and imagine, as before, sliding the D5-brane segments that end on it from left and right. This operation splits the NS5-brane into the upper and lower semi-infinite NS5-branes, joined by a finite segment of \((N, \pm 1)\) 5-brane; see figure 4. For definiteness, consider the case with an \((N, 1)\) 5-brane. Bifundamental multiplets arise from open strings whose ends are localized at the junctions where semi-infinite NS5-branes meet D5-branes and the \((N, 1)\) 5-brane. These strings may be viewed as fundamental strings \((N, 0)\)-strings), as in figure 6a, or D-strings \((0, 1)\)-strings) with the opposite orientation,\(^6\) as in figure 6b. As such, they are acted upon by global U(1) gauge transformations on the semi-infinite NS5-branes. In the effective 4d theory, gauge transformations on the upper and lower branes appear as elements in the same symmetry \(U(1)_\alpha\), just acting in opposite manners. Hence, a U(1) flavor symmetry is associated to each NS5-brane or rapidity line.

Equivalently, we may regard \(U(1)_\alpha\) as arising from the U(1) gauge symmetries on the

\(^6\)To determine the orientation, we draw an \((N, 0)\)-string attached on the D5-branes, and move the endpoints to the semi-infinite NS5-branes. When an endpoint passes a junction, an \((N, 1)\)-string is created due to charge conservation. When the other endpoint passes the other junction, the \((N, 1)\)-string is annihilated, leaving a \((0, 1)\)-string. The orientation of the \((0, 1)\)-string can be seen from the final picture.
D5-brane segments, which we have seen are nondynamical. From this point of view, \( U(1)_\alpha \) acts on all D5-brane segments on the “right” side of the \( \alpha \)th N5-brane, or on all of those on the “left” side in the opposite manner. Of course, the words “left” and “right” are not really meaningful when the 4- and 6-direction are periodic, but we can make sense of this interpretation by, roughly speaking, considering the D5-branes to be sections of a \( U(1)_\alpha \)-bundle on the 46-torus.

This point is probably easier to understand if we consider the situation where there are only NS5-branes in the 012345 directions, and look at the mass parameter \( m_\alpha \) associated to \( U(1)_\alpha \) in the corresponding 5d \( \mathcal{N} = 1 \) theory. \( m_\alpha \) is the value of the real scalar in the background vector multiplet for \( U(1)_\alpha \), and proportional to the difference of the \( x^5 \)-coordinates of the two D5-brane segments that end on the \( \alpha \)th NS5-brane. The periodicity in the 6-direction suggests that the sum of \( m_\alpha \) over the rapidity lines must vanish. However, these constraints can be modified if we compactify the 6-direction using a twisted boundary condition that involves a shift in the 5-direction. (The same twisted boundary condition is used in the brane construction of 4d \( \mathcal{N} = 2 \) elliptic models [11].) This modification introduces one more parameter, namely the amount of shift, leading to the same number of mass parameters as the rapidity lines.

We normalize the \( U(1)_\alpha \)-charge \( F_\alpha \) in such a way that horizontal and vertical bifundamental arrows intersecting with the \( \alpha \)th rapidity line have \( F_\alpha = -1 \) and \( +1 \), respectively, while \( F_\beta = 0 \) for \( \beta \neq \alpha \). The charges of the diagonal arrows are found by demanding that the superpotential have \( F_\alpha = 0 \) for all \( \alpha \). Using \( F_\alpha \), our R-charge parametrization can be concisely written as

\[
R = R_0 + \sum_\alpha r_\alpha F_\alpha,
\]

where \( R_0 \) is the R-charge defined by assigning \( R_0 = 1 \) to the horizontal and vertical arrows, and \( r_\alpha \) is the parameter of the \( \alpha \)th rapidity line. As expected, the degrees of freedom in the definition of \( R \) come from shifts by flavor symmetries.

We have seen above that there are at least \( m + n \) \( U(1) \) flavor symmetries, but have not determined the precise number. Actually, in some cases there are extra \( U(1) \) symmetries in addition to the ones discussed above. For example, for a \( 2 \times 2 \) lattice, the most general R-charge assignment has five parameters rather than four, as shown in figure 7. In fact, from the relation between the brane box and brane tiling constructions explained in section 4,
one can deduce that the number of U(1) flavor symmetries is equal to \( m + n + \gcd(m, n) - 1 \).

We will not consider these extra flavor symmetries. Alternatively, one may assume that \( m \) and \( n \) are coprime.

3.3 Structure of a TQFT with extra dimension

So far we have described the brane box configuration (3.1) from the point of view of the effective 4d \( \mathcal{N} = 1 \) theory. To make contact with the framework discussed in section 2, let us instead think of it as realizing 6d maximally supersymmetric Yang–Mills theory on the D5-branes, in the presence of codimension-1 defects created by the NS5-branes. We wish to turn this 6d theory into a theory that is topological on the 46-torus, and regard the defects as line operators with spectral parameter in the TQFT. This is achieved by applying an appropriate topological twist and restricting physical quantities to compute, as we now explain.

Type IIB superstring theory has \( \mathcal{N} = (2, 0) \) supersymmetry in 9 + 1 dimensions, generated by Majorana–Weyl spinors \( \epsilon_L, \epsilon_R \) such that

\[
\Gamma_{0123456789} \epsilon_L = \epsilon_L, \quad \Gamma_{0123456789} \epsilon_R = \epsilon_R.
\]

Here \( \Gamma_M, M = 1, \ldots, 9 \) are Gamma matrices, and \( \Gamma_{0123456789} = \Gamma_0 \cdots \Gamma_9 \) is the chirality operator. The D5-branes preserve supersymmetry generated by \( \epsilon_L, \epsilon_R \) satisfying

\[
\epsilon_L = \Gamma_{012346} \epsilon_R.
\]

Likewise, the NS5-branes impose conditions

\[
\epsilon_L = \Gamma_{012345} \epsilon_L, \quad \epsilon_R = -\Gamma_{012345} \epsilon_R
\]

and

\[
\epsilon_L = \Gamma_{012367} \epsilon_L, \quad \epsilon_R = -\Gamma_{012367} \epsilon_R
\]

on the preserved supersymmetry. The conditions on \( \epsilon_R \) coming from the NS5-branes actually follow from those on \( \epsilon_L \) and the D5-brane condition, hence are redundant. Since \( \Gamma_{012345} \) and \( \Gamma_{012367} \) square to the identity and are traceless, half of their eigenvalues are +1
and the other half are $-1$. Moreover, they commute with each other and are simultaneously diagonalizable. Each NS5-brane condition therefore halves the number of independent solutions for $\epsilon_L$. Given $\epsilon_L$, the D5-brane condition determines $\epsilon_R$. In total, the 5-branes preserve four supercharges, which generate the $\mathcal{N} = 1$ supersymmetry of the 4d theory we have been studying.

From the NS5-brane conditions, we see $\Gamma_{45} \epsilon_L = \Gamma_{67} \epsilon_L$, or

$$(\Gamma_{46} + \Gamma_{57}) \epsilon_L = 0.$$ (3.15)

This equation says that if we replace the rotation group SO(2)$_{46}$ of the 46-plane (which we decompactify for a moment) with the diagonal subgroup SO(2)$_{46}'$ of SO(2)$_{46} \times$ SO(2)$_{57}$, then the preserved supercharges become scalars under the new rotation group. From the same equation we also find that the NS5-brane conditions on $\epsilon_L$ can be generalized to

$$\epsilon_L = \Gamma_{0123} (\Gamma_4 \cos \theta + \Gamma_6 \sin \theta)(\Gamma_5 \cos \theta + \Gamma_7 \sin \theta) \epsilon_L,$$ (3.16)

where $\theta$ is any angle. This shows that the NS5-branes can be rotated in the 46- and 57-plane by the same angle $\theta$ without destroying the supercharges, or for that matter, they can intersect with the 46-plane along arbitrary closed curves as long as the slopes are correlated between the 46- and 57-plane.

From the point of view of the 6d theory on the D5-branes, the replacement of SO(2)$_{46}$ with SO(2)$_{46}'$ is a topological twist along the 46-plane. After the twisting, eight out of the sixteen supercharges are scalars on the 46-plane, and as such, can be preserved even when the 46-plane is replaced by a general 2-manifold $\Sigma$. Codimension-1 defects can be inserted along arbitrary closed curves on $\Sigma$ while preserving four of the eight supercharges. These defects carry a spectral parameter, as we have seen from the 4d point of view.

Now, to produce a desired TQFT from this twisted theory, we simply restrict ourselves to quantities that are independent of the metric on $\Sigma$, and depend on the codimension-1 defects inserted on $\Sigma$ only through their topology and spectral parameters. Then, the twisted theory becomes a TQFT on $\Sigma$, and the defects become line operators with spectral parameter in this TQFT.

A good example of such a topological quantity is the supersymmetric index of the corresponding 4d $\mathcal{N} = 1$ theory $[52-54]$. As the theory has the R-symmetry U(1)$_R$, it can be placed on the Euclidean spacetime $S^3 \times S^1$ without breaking supersymmetry $[52, 54, 55]$. Under the isometry group SU(2) $\times$ SU(2) of $S^3$, the four supercharges transform as a pair of doublets $(2, 1)$. Let $J_i, J'_i$ be the generators of the SU(2) factors. The supersymmetric index is defined by

$$\mathcal{I}(p, q, \{u_\alpha\}; \{r_\alpha\}) = \text{Tr}_{S^3} \left((-1)^F p^{J_3 + J'_3 - R/2} q^{J_3 - J'_3 - R/2} \prod_\alpha u_\alpha^{F_\alpha}\right),$$ (3.17)

where $p, q, u_\alpha$ are complex parameters, and the trace is taken over the Hilbert space on $S^3$. This quantity is topological for the following reason. Bosonic and fermionic states with energy $E \neq 2J_3 - R$ are paired by the action of a supercharge. (Here we are taking the radius of $S^3$ to be 1.) Due to the presence of the factor $(-1)^F$, pairwise cancellations
occur and the only contributions to the trace come from states with \( E = 2J_3 - R \). Under continuous changes of parameters of the theory (other than the rapidities \( r_\alpha \) on which the factors \( u^F_\alpha \) implicitly depend), the index remains invariant since states can be brought into or out of the right energy level only in boson-fermion pairs. The index is therefore protected against such changes.

There are other topological quantities. For example, we can replace the \( S^3 \) in the above definition of the supersymmetric index with \( S^2 \times S^1 \) or a lens space \( L(p, q) \) [56]. However, an explicit expression of the index is not known except for simple cases such as \( L(p, 1) \) [15]. We will focus on the \( S^3 \) index, for which the relevant mathematical results are available.

So we have obtained from brane box models a 2d TQFT that has a family of line operators with spectral parameter. We can now apply the construction discussed in section 2 and get a vertex model, by placing the TQFT on \( T^2 \) and making a lattice of line operators. However, we cannot tell whether the model is integrable just from the structure of a TQFT. For the integrability, it is necessary to show, in addition, that no phase transitions occur when two line operator are untangled or a line operator passes through the intersection of two other line operators. In Costello’s argument, these properties are guaranteed by the existence of extra dimensions in which the line operators can miss one another.

We can make a similar argument in the present case. Let us go back to the brane picture. To the configuration (3.1), we apply T-duality in the 3-direction (which we take to be the \( S_1 \)-direction of the spacetime \( S^3 \times S^1 \)) to convert the D5-branes to D4-branes, and then lift the resulting Type IIA brane system to M-theory:

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 11 \\
\hline
M5 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
M5 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
M5 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{array}
\] (3.18)

These dualities transform the 6d theory on the D5-branes into an \( \mathcal{N} = (2, 0) \) theory on a stack of M5-branes, and the codimension-1 defects into codimension-2 defects. There is now an extra dimension along which line operators can move, namely the M-theory circle. Its presence ensures the integrability of the vertex model.

The similarity to Costello’s argument suggests that the \( x^{11} \)-coordinates of line operators are related to the spectral parameters \( r_\alpha \). This is indeed true, as the following analysis shows.

If we trade off the flavor fugacities \( u_\alpha \) for new parameters \( s_\alpha \) given by \( u_\alpha = (pq)^{-s_\alpha/2} \), the supersymmetric index (3.17) depends on \( s_\alpha \) only through the combination \( R + \sum_\alpha s_\alpha F_\alpha \). Hence, the effect of turning on nonzero values for \( s_\alpha \) is equivalent to shifting \( r_\alpha \rightarrow r_\alpha + s_\alpha \), and \( s_\alpha \) represent the same degrees of freedom as \( r_\alpha \). We want to understand how \( s_\alpha \) are described in the language of branes, and keep track of them under the dualities.

Consider the situation depicted in figure 6. We saw that \( U(1)_\alpha \) can be interpreted as either the global gauge symmetry on the lower half of the \( \alpha \)th NS5-brane, or that on the upper half acting in the opposite manner. The conserved current \( J_\alpha \) for \( U(1)_\alpha \) therefore
couples to the 6d theory through a term of the form

$$i \int _{S^3 \times S^1 \times T^2} (A^-_\alpha - A^+_\alpha) \wedge \star J_\alpha. \quad (3.19)$$

Here $A^-_\alpha$ and $A^+_\alpha$ are the restrictions of the gauge field on the $\alpha$th NS5-brane to the lower and upper halves, respectively. Along the 46-torus, they are supported on the 1-cycle $C_\alpha$, and so is $J_\alpha$. We give a background value to the gauge field such that $A^-_\alpha - A^+_\alpha$ is constant along $S^3$ and $C_\alpha$. As $F_\alpha$ is the integral of $\star J_\alpha$ along $S^3$ and $T^2$, we find that the above term becomes

$$i \int _{S^1} (A^-_\alpha - A^+_\alpha) F_\alpha. \quad (3.20)$$

This term enters the path integral as an exponentiated factor, so $s_\alpha$ is proportional to the difference of the holonomies of $A^-_\alpha$ and $A^+_\alpha$ around the 3-circle.

These holonomies are mapped under the T-duality to the background values of a scalar field on the upper and lower halves of the corresponding Type IIA NS5-brane. Upon lifting to M-theory, the NS5-brane becomes an M5-brane, and the scalar is mapped to the $x^{11}$-coordinate. Therefore, the spectral parameter for a line operator is given by the difference in the $x^{11}$-coordinates of the upper and lower halves of the M5-brane.

We remark that if the holonomies of $A^+_\alpha$ and $A^-_\alpha$ are equal, then the whole $\alpha$th NS5-brane is mapped to a single point on the M-theory circle, and the supersymmetric index does not depend on its position at all. This observation is consistent with the fact that the $\mathcal{N} = (2,0)$ theory compactified on $S^3$ has a topological sector that is equivalent to Chern–Simons theory with complex gauge group at level $k = 1$ [42]. In the present setting, the NS5-brane creates a Wilson line in the Chern–Simons theory on $T^2 \times S^1$, and its precise position on the 3-manifold is irrelevant. When the holonomies are different, the Wilson line splits into two line operators, and apparently they no longer belong to the topological sector. It is clear from our viewpoint that the same consideration should apply to other indices obtained by replacing $S^3$ with appropriate 3-manifolds. Indeed, the $\mathcal{N} = (2,0)$ theory on $S^2 \times S^1$ has a topological sector equivalent to complex Chern–Simons theory at level $k = 0$ [40, 41], while the lens space $L(k, 1)$ gives complex Chern–Simons theory at level $k$ [45]. Similarly, a twisted product of $\mathbb{R}^2$ and $S^1$ leads to analytically continued Chern–Simons theory, which is the holomorphic sector of complex Chern–Simons theory [39, 57–59].

### 3.4 Integrable lattice model for brane box models

Finally, let us identify the integrable lattice model associated with brane box models. To do so, we look at the integral formula for the supersymmetric index.

Recall that the flavor fugacities $u_\alpha$ represent the same degrees of freedom as the spectral parameters $r_\alpha$. Since we are going to keep the dependence on $r_\alpha$, we set $u_\alpha = 1$. With this understood, the supersymmetric index of a brane box model is computed as follows [52, 60].

First, we assign the factor

$$I_V(z_\alpha) = ((p; p)_\infty (q; q)_\infty)^{-1} \prod_{i \neq j} \frac{1}{\Gamma(z_{\alpha,i}/z_{\alpha,j}; p, q)} \quad (3.21)$$

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$$I_V(z_\alpha) = ((p; p)_\infty (q; q)_\infty)^{-1} \prod_{i \neq j} \frac{1}{\Gamma(z_{\alpha,i}/z_{\alpha,j}; p, q)} \quad (3.21)$$
to the vector multiplet represented by node \( a \), and
\[
\mathcal{I}_B(z_a, z_b; R_{ab}) = \prod_{i,j} \Gamma((pq)^{R_{ab}/2}z_{b,j}/z_{a,i}; p, q)
\] (3.22)
to the bifundamental chiral multiplet with \( R = R_{ab} \) represented by an arrow from node \( a \) to \( b \). Here
\[
\Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1}{1 - z^{-1}p^{i+1}q^{j+1}}
\] (3.23)
is the elliptic Gamma function, \((z; q)_\infty = \prod_{i=0}^{\infty}(1 - zq^i)\), and \( z_a \) collectively denotes the fugacities \( z_{a,i} \), \( i = 1, \ldots, N \) for the gauge group \( SU(N)_a \), obeying \( \prod_{i=1}^{N} z_{a,i} = 1 \). Then, we multiply the factors from all nodes and arrows, and integrate over the fugacities:
\[
\mathcal{I} = \int \prod_{\oplus} \frac{1}{N!} \prod_{j=1}^{N-1} \frac{dz_{a,j}}{2\pi i z_{a,j}} \mathcal{I}_V(z_a) \prod_{\oplus \rightarrow \oplus} \mathcal{I}_B(z_a, z_b; R_{ab}).
\] (3.24)
The integration for each fugacity is performed over the unit circle \( \{|z_{a,i}| = 1\} \).

The above procedure clearly shows the statistical mechanical nature of the supersymmetric index. In the statistical mechanical interpretation, \( z_{a,i} \) are continuous state variables, and the factors \( \mathcal{I}_V \) and \( \mathcal{I}_B \) are the Boltzmann weights for self- and nearest-neighbor interactions, respectively. The partition function is defined by multiplying the weights and summing over all possible state configurations, and this is identified with the supersymmetric index.

Since the state variables \( z_{a,i} \) are assigned to the nodes of the quiver which correspond to the faces of the lattice on the 46-torus, it is natural to formulate this model as an IRF model. We can take its Boltzmann weight to be\(^7\)
\[
W\left( \begin{array}{c} z_d \\ z_c \\ z_a \\ z_b \end{array} \mid \begin{array}{c} r \\ s \end{array} \right) = \mathcal{I}_V(z_a)\mathcal{I}_B(z_a, z_b; 1 - s)\mathcal{I}_B(z_b, z_c; 1 + r)\mathcal{I}_B(z_c, z_a; s - r).
\] (3.26)
Graphically, it can be represented as
\[
\begin{array}{c}
A \\
B
\end{array}
=r

\begin{array}{c}
C \\
D
\end{array}
=s-r

\begin{array}{c}
E \\
F
\end{array}
=1+r
\]

\[
\begin{array}{c}
G \\
H
\end{array}
=1-s
\] (3.27)

(Notice a slight discrepancy in the notation compared to the original definition (2.17). Here \( a, b, \) etc. on the left-hand side really stands for \( z_a, z_b, \) etc.) A square node \( \square \) indicates

\[\text{Graphically, it can be represented as:}\]

\[\text{(Notice a slight discrepancy in the notation compared to the original definition (2.17). Here} \ a, \ b, \) etc. on the left-hand side really stands for \( z_a, z_b, \) etc.) A square node \( \square \) indicates}

\[\text{\(7\)This is not the only choice. For example, one may replace the right-hand side with the more symmetric expression}
\]
\[
\sqrt{\mathcal{I}_V(a)\mathcal{I}_B(a, b; 1 - s)\mathcal{I}_B(b, c; 1 + r)\mathcal{I}_B(c, a; s - r)\mathcal{I}_V(c)\mathcal{I}_B(c, a; s - r)\mathcal{I}_B(a, d; 1 + r)\mathcal{I}_B(d, c; 1 - s)}
\] (3.25)

\[\text{without spoiling the integrability.}\]
that the associated factor $I_V(z_a)$ is absent, and a node $\tilde{a}$ drawn with a dashed line means
that the factor is present but integration is not performed over $z_a$. The intersections of
NS5-branes with the D5-branes, which create line operators in the TQFT on the 46-torus,
play the roles of rapidity lines, as anticipated.

Let us show that this Boltzmann weight satisfies the Yang–Baxter equation (2.18). For
this purpose we will need the identity

$$I_B(z_a, z_b; R_{ab}) = 1$$

which holds for $R_{ab} + R_{ba} = 2$. In effect, it means that we can erase two arrows going in
opposite directions between the same pair of nodes:

$$a \rightarrow b = a \leftarrow b .$$

This property reflects the fact that if one has chiral multiplets $\Phi_1, \Phi_2$ in conjugate rep-
resentations whose R-charges add up to 2, one can give them masses by turning on the
superpotential term $m\Phi_1\Phi_2$. The supersymmetric index is independent of $m$, and in the
limit $m \rightarrow \infty$, these multiplets become infinitely massive and decouple, leaving no contri-
bution to the index.

Now, plugging the expression (3.27) into the Yang–Baxter equation, we find

$$I_B(z_a, z_b; R_{ab})I_B(z_b, z_c; R_{bc}) = 1$$

which holds for $R_{ab} + R_{bc} = 2$. In effect, it means that we can erase two arrows going in
opposite directions between the same pair of nodes:

$$a \rightarrow b = a \leftarrow b .$$

Some nodes and arrows cancel out between the two sides, while the arrows between nodes
$b$ and $g$ on the right-hand side cancel thank to the identity (3.29). Moving the arrows that
are not connected to node $g$ from the left- to right-hand side using the same identity, we
end up with the equality

$$I_B(z_a, z_b; R_{ab})I_B(z_b, z_c; R_{bc}) = 1$$

Interestingly, this equality expresses precisely the invariance of the supersymmetric index
under Seiberg duality.

Seiberg duality says that for $N_f > N_c + 1$, an $\mathcal{N} = 1$ SU($N_c$) gauge theory with $N_f$
flavors ($Q, \tilde{Q}$) and no superpotential has a dual description in the infrared as an $\mathcal{N} = 1$
SU\((N_f - N_c)\) gauge theory with \(N_f\) flavors \((q, \tilde{q})\) and \(N_f^2\) singlets \(M\), coupled through a superpotential \(W = qMq\). For both theories, the flavor symmetry is \(SU(N_f)_L \times SU(N_f)_R \times U(1)_B\) and the R-symmetry is \(U(1)_R\). Under these symmetries and the gauge group \(G = SU(N_c)\) or \(SU(N_f - N_c)\), the chiral multiplets transform as follows:

| \(G\) | \(SU(N_f)_L\) | \(SU(N_f)_R\) | \(U(1)_B\) | \(U(1)_R\) |
|------|---------------|---------------|------------|------------|
| \(Q\) | □             | □             | 1          | 1 - \(N_c/N_f\) |
| \(\tilde{Q}\) | □             | 1             | □          | -1         |
| \(q\) | □             | □             | 1          | \(N_c/(N_f - N_c)\) |
| \(\tilde{q}\) | □             | 1             | □          | \(N_c/N_f\) |
| \(M\) | 1             | □             | □          | 0          | 2(1 - \(N_c/N_f\)) |

\((3.32)\)

In the case at hand, \(N_f = 2N_c = 2N\) and the charged chiral multiplets all have \(R = 1/2\). This corresponds to the situation with \((r, s, t) = (-1/2, 0, 1/2)\). As we stressed, however, we can shift the R-charge by \(U(1)\) flavor charges. The quivers that appear in the equality \((3.31)\) split the \(2N\) flavors into a pair of \(N\) flavors. This breaks each of \(SU(2N)_L\) and \(SU(2N)_R\) down to an \(SU(N) \times SU(N) \times U(1)\) subgroup. So in total there are three \(U(1)\) flavor symmetries that can be used to shift the R-charge, and they give rise to the three parameters \(r, s\) and \(t\).

Since the supersymmetric index is protected from continuous changes of the parameters of the theory, it is invariant under renormalization group (RG) flow and therefore under Seiberg duality. Indeed, as observed in [60], the equality \((3.31)\) follows from an identity for the elliptic Gamma function proved in [61]. Thus, we have demonstrated the integrability of our model.\(^8\)

The appearance of Seiberg duality as an integrability condition might be unexpected from the point of view of lattice models. From our perspective, this is not entirely surprising. The integrability is simply a consequence of the topological invariance of the relevant TQFT with extra dimensions that has line operators, and these line operators are created by NS5-branes. It is well known that in the brane construction of 4d \(\mathcal{N} = 1\) gauge theories, Seiberg duality is realized by rearrangement of NS5-branes [63].

### 4 Brane tiling models

In the discussion of brane box models in the previous section, we found it helpful to consider a deformation of the brane configuration that splits the D5-branes across NS5-branes and creates \((N, \pm 1)\) 5-branes, as in figure 4. The bifundamental multiplets represented by the horizontal and vertical arrows in the quiver originate from open strings whose ends are

\(^{8}\) The invertibility of the R-matrix follows from the identity

\[
\frac{1}{N!} \oint \prod_{j=1}^{N-1} \frac{dz_{a,j}}{2\pi i z_{a,j}} \mathcal{I}_B(z_a, z_c; R)\mathcal{I}_V(z_c)\mathcal{I}_B(z_c, z_b; -R)\mathcal{I}_V(z_b) = K \sum_{\tau \in S_N} \delta(z_a, \tau(z_b)),
\]

where \(S_N\) is the symmetric group of degree \(N\) and the delta function \(\delta\) is defined with respect to the measure \(\oint dz_{a,j}/2\pi i z_{a,j}\). The overall factor \(K = \prod_{a = \pm 1} \Gamma((pq)^{N/2}; p, q)\) can be removed by rescaling the R-matrix. This identity is a special case of Theorem 11 of [62], as explained in Appendix A of [2].
Figure 8: The 46-space representations for the brane configurations in figure 4a and 4b. The number $q$ labeling a region indicates that there is an $(N, q)$ 5-brane on that region.

attached on the brane junctions. The original configuration is recovered in the limit where the D5-brane segments are joined together and the $(N, \pm 1)$ 5-branes disappear, and these multiplets become massless in this limit.

What happens if we split the D5-branes across every NS5-brane, and keep the distances between the segments finite? In such a situation, the horizontal and vertical multiplets are all massive and absent in the infrared theory. As far as RG invariant quantities such as the supersymmetric index are concerned, these multiplets can therefore be ignored. On the other hand, there are now far more ways in which diagonal multiplets arise. To see this clearly, let us represent brane junctions by oriented dash-dot lines, which we call zigzag paths. The orientation of a zigzag path is such that if it is going up, then there is an $(N, q)$ 5-brane on the left and $(N, q + 1)$ 5-brane on the right for some $q$. For example, the two configurations in figure 4 are represented on the 46-torus as in figure 8. Diagonal multiplets arise from open strings localized near intersections of rapidity lines or zigzag paths. When the D5-branes split, each rapidity line splits into two antiparallel zigzag paths. Correspondingly, each intersection point breaks apart into four intersection points. As the number of intersection points increases, we expect to get more diagonal multiplets.

The number of U(1) flavor symmetries also gets larger. Recall that these come from global gauge symmetries on the NS5-branes. In a brane box model, gauge transformations on the upper and lower halves of an NS5-brane give the same flavor symmetry in the 4d theory since they simply act in opposite manners on the horizontal and vertical multiplets (and the action on the diagonal multiplets is determined by the requirement that the superpotential be neutral). After the brane configuration is deformed, there are no horizontal or vertical multiplets anymore, so gauge transformations on the two halves can lead to different flavor symmetries.

We can make these statements more precise when the deformed system contains only $(N, 0)$ and $(N, \pm 1)$ 5-branes. In fact, in such a situation the system is still described by an $\mathcal{N} = 1$ quiver gauge theory [10]. The quiver is such that an SU($N$) node is assigned to
each \((N,0)\) region, and an arrow connects two nodes sharing a vertex:

\[
\begin{array}{c|c}
\text{1} & \text{0} \\
\hline
\text{0} & \text{-1}
\end{array}
\]

\(i\quad j\)

(An \((N,\pm 1)\) 5-brane is dual to a single D5-brane, so it supports a U(1) gauge field which is frozen at low energies.) To the \(i\)th zigzag path is associated a flavor symmetry U(1)\(_i\), and arrows intersecting with this path has nonzero U(1)\(_i\)-charge \(F_i\). We choose the normalization for the \(F_i\) such that the arrow in the above picture has \(F_i = +1\) and \(F_j = -1\).

\[
\sum_i F_i = 0.
\]

Hence, the number of U(1) flavor symmetries is equal to the number of the zigzag paths minus one.

Once we start thinking in terms of zigzag paths, we can construct many more \(\mathcal{N} = 1\) theories. We can consider various configurations of zigzag paths, not necessarily obtained by deforming brane box configurations. Such configurations of zigzag paths are called brane tilings. Provided that only \((N,0)\) and \((N,\pm 1)\) 5-branes appear on the graph of zigzag paths, brane tilings yield \(\mathcal{N} = 1\) quiver gauge theories, whose quivers are determined by the same rule as above. Actually, brane box models can also be described as a brane tiling model; see figure 2b for the brane tiling configuration for the 2 \(\times\) 3 brane box model, whose quiver is shown in figure 2b. In this sense, brane tilings generalize the brane box construction. The reader may consult the reviews \([65, 66]\) for more extensive discussions on brane tilings and their relations to other subjects such as the AdS/CFT correspondence.

Now, let us try to connect brane tilings to an integrable lattice model. Since the only difference compared to the brane box case is the type of defects inserted in the 6d theory on the D5-branes, much of the previous argument carries over. Using this 6d theory, we can define a 2d TQFT whose correlation function for a configuration of zigzag paths is given by the supersymmetric index of the corresponding 4d \(\mathcal{N} = 1\) theory. An extra
dimension emerges if we utilize dualities and embed the 6d theory in M-theory. Zigzag paths are mapped to semi-infinite M5-branes, and their $x^{11}$-coordinates provide spectral parameters. So we indeed have all ingredients required for the construction of an integrable lattice model. It is reassuring that we have found the relation (4.2), which implies that the supersymmetric index is invariant under an overall shift of the $x^{11}$-coordinates.

The next step would be to write down an integral expression of the supersymmetric index and read off the R-matrix. Clearly, we cannot do this for an arbitrary configuration of zigzag paths; in general, we find $(N, q)$ regions with $|q| > 1$, and do not know what the corresponding 4d theory is. So we need to restrict our attention to configurations without such regions. Doing this is sufficient if we just want to write down the partition function. However, if we want to actually verify that the lattice model is integrable, we get into trouble: the Yang–Baxter equation for three zigzag paths always involves regions with $|q| > 1$, as depicted in figure 10.

This difficulty stems from the fact that zigzag paths are not really line operators. They are more properly called domain walls since they separate regions with different physical properties. This is a problem we did not encounter for brane box models. In that case, rapidity lines were genuine line operators.

Our consideration at the beginning of this section suggests a resolution to this problem. We have seen that when a brane box model is deformed to a brane tiling model, a rapidity line splits into two antiparallel zigzag paths. Conversely, we can take two neighboring antiparallel zigzag paths, and regard them as one thick line (or “ribbon”) operator. In this way we can make two kinds of line operators, namely

$$ (r_1, r_2) = r_1 r_2 $$

and another one for which the orientations of the zigzag paths on the right-hand side are flipped. Below we will only use the first one.

If we use this line operator and give the “background charge” $q = -1$ to the TQFT,
then regions with $|q| > 1$ never arise:

$$
\begin{array}{cc}
-1 & -1 \\
-1 & -1 \\
\end{array}
= \begin{array}{ccc}
-1 & 0 & -1 \\
0 & 1 & 0 \\
-1 & 0 & -1 \\
\end{array}.
\tag{4.4}
$$

Thus, a brane tiling constructed with line operators of this type is always described by a quiver gauge theory, and we can readily write down an integral expression for its supersymmetric index. From this expression we can read off the R-matrix of the associated vertex model. Just as in the case for brane box models, we expect that the Yang–Baxter equation for this R-matrix can be understood as the equality between the supersymmetric indices of two dual gauge theories. Let us see if this is the case.

From the quiver rule (4.1), we find that the lattice model in question is a vertex model\footnote{The same model can also be described as an IRF model. However, the IRF description uses line operators of both types, and the Yang–Baxter equation involves twisting of one of the three ribbons. See \cite{9} for a discussion from the IRF viewpoint.} with the R-matrix given by

$$
\begin{array}{c}
(r_1, r_2) \\
(s_1, s_2) \\
\end{array}
\begin{array}{c}
\frac{1}{2} + r_1 - s_1 \\
\frac{1}{2} + s_1 - r_2 \\
\frac{1}{2} + s_2 - r_1 \\
\frac{1}{2} + r_2 - s_2 \\
\end{array}
= \begin{array}{c}
\frac{1}{2} + r_1 - s_1 \\
\frac{1}{2} + s_1 - r_2 \\
\frac{1}{2} + s_2 - r_1 \\
\frac{1}{2} + r_2 - s_2 \\
\end{array}.
\tag{4.5}
$$

Figure 11 shows the case when the model is placed on a $2 \times 3$ lattice. The R-charges of the bifundamental multiplets, specified by the numbers accompanying the arrows, are chosen to be consistent with the normalization of the flavor charges and add up to 2 around loops. With this choice, the R-charges satisfy the anomaly cancellation condition.

What we have just constructed is the integrable lattice model studied in \cite{2, 8}. (Its generalization to the lens space index was considered in \cite{9}.)}
Figure 12: A sequence of duality transformations that demonstrates the Yang–Baxter equation (4.6). The letter on an arrow between two graphs indicates the node at which the transformation is applied. Note that circle nodes can be relabeled freely since their fugacities are integrated over.

For the model is

\[
i_2 \rightarrow i_2, \quad j_2 \rightarrow j_2, \quad k_2 \rightarrow k_2, \quad i_1 \rightarrow i_1, \quad j_3 \rightarrow j_3, \quad k_3 \rightarrow k_3.
\]

(4.6)

For simplicity we have omitted the R-charges. As expected, this equation follows from Seiberg duality: applying the basic duality transformation (3.31) four times, we can turn the left-hand side into the right-hand side. This process is illustrated in figure 12.

Incidentally, a single Seiberg duality transformation is induced by the Yang–Baxter move involving two zigzag paths and one “twisted” ribbon operator:

\[
\begin{array}{c}
\text{zigzag paths} \\
\end{array}
\rightarrow
\begin{array}{c}
\text{twisted ribbon operator}
\end{array}
\]

(4.7)

This relation between the Yang–Baxter move and Seiberg duality was pointed out by Hanany and Vegh [67].
5 3d $\mathcal{N} = 2$ and 2d $\mathcal{N} = (2, 2)$ quiver gauge theories

The results from the previous sections imply, via T-duality, that there exist similar connections between lower-dimensional quiver gauge theories and integrable lattice models. In this section we discuss integrable lattice models associated with 3d $\mathcal{N} = 2$ and 2d $\mathcal{N} = (2, 2)$ theories. We will focus on those coming from brane box configurations, though our analysis can be adapted straightforwardly to the brane tiling case.

5.1 3d brane box models

Let us consider the following brane setup in Type IIA superstring theory, which is related to the 4d brane box model (3.1) by T-duality along the 2-direction:

\begin{align*}
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
D4 & \times & \times & \times & \times & \times & & & & \\
NS5 & \times & \times & \times & \times & \times & \times & & & \\
NS5 & \times & \times & \times & \times & \times & & \times & & \\
\end{array}
\end{align*}

This is a brane box configuration that realizes a 3d $\mathcal{N} = 2$ quiver gauge theory. Being related by T-duality, the theory is essentially the dimensional reduction of the corresponding 4d $\mathcal{N} = 1$ theory, and is described by the same quiver (with all Chern–Simons levels vanishing). We expect that the supersymmetric index of this theory is given by the partition function of an integrable lattice model.

As in the 4d case, the theory can be placed on $S^2 \times S^1$, with all four supercharges unbroken [68, 69]. We are interested in the supersymmetric index

\[ I(x; \{ r_\alpha \}) = \text{Tr}_{S^2} \left( (-1)^F x^{R + 2J_3} \right), \]

where $J_3$ is the Cartan generator of the isometry group $SU(2)$ of $S^2$. It receives contributions only from states with energy $E = R + J_3$. As usual, we have omitted the fugacities $u_\alpha$ for the flavor symmetries $U(1)_\alpha$, whose effects can be taken into account by keeping the dependence of the R-charge $R$ on the spectral parameters $r_\alpha$.

The 3d supersymmetric index is computed by an integral formula similar to the one in the 4d case. The novelty is that there can be nontrivial magnetic fluxes through $S^2$. For each gauge group $SU(N)_a$, the flux is specified by a set of integers $m_a = (m_{a,1}, \ldots, m_{a,N})$ subject to the condition $\sum_i m_{a,i} = 0$. Therefore, the formula involves a sum over fluxes as well as an integration over the fugacities for the gauge groups:

\[ I = \sum_{\{ m_a \}} \oint \prod_{\otimes} \frac{1}{|W_{m_a}|} \prod_{j=1}^{N-1} \frac{dz_{a,j}}{2\pi i z_{a,j}} I_V(z_{a,m_a}) \prod_{\otimes \rightarrow \otimes} I_B((z_a, m_a), (z_b, m_b); R_{ab}). \]

Here $|W_{m_a}|$ is the order of the Weyl group of the subgroup of $G$ that is left unbroken by
and the vector and bifundamental multiplet factors are given by \[ [68, 70, 71] \]

\[
\mathcal{I}_V(z_a, m_a) = \prod_{i \neq j} x^{-|m_{a,i} - m_{a,j}|/2} \left( 1 - \frac{z_{a,i}}{z_{a,j}} \right)^{|m_{a,i} - m_{a,j}|/2},
\]

\[ (5.4) \]

\[
\mathcal{I}_B((z_a, m_a), (z_b, m_b); R_{ab}) = \prod_{i,j} \left( x^{1-R_{ab} \frac{z_{a,i}}{z_{b,j}}} \right)^{|m_{a,i} - m_{b,j}|/2} \times \frac{(x^{m_{a,i} - m_{b,j}} + 2 - R_{ab} \frac{z_{a,i}}{z_{b,j}}; x^2)_{\infty}}{(x^{m_{a,i} - m_{b,j}} + R_{ab} \frac{z_{a,i}}{z_{b,j}}; x^2)_{\infty}}.
\]

\[ (5.5) \]

The bifundamental factor \(\mathcal{I}_B\) satisfies the identity

\[
\mathcal{I}_B((z_a, m_a), (z_b, m_b); R_{ab}) \mathcal{I}_B((z_b, m_b), (z_a, m_a); R_{ba}) = 1
\]

\[ (5.6) \]

for \(R_{ab} + R_{ba} = 2\).

We can connect the supersymmetric index to a lattice model by following the same procedure as before. We define an IRF model on the lattice drawn on the 46-torus by the NS5-branes. The state variables are \((z_a, m_a)\), and the Boltzmann weight is given by the same picture \((3.27)\). By construction, the supersymmetric index of a 3d brane box model coincides with the partition function of this IRF model. The Yang–Baxter equation again boils down to the equality \((3.31)\) between the supersymmetric indices of two gauge theories. This equality, if true, establishes the integrability of the IRF model.

The equality indeed follows from the SU(\(N\)) version \([72, 73]\) of the duality for 3d \(\mathcal{N} = 2\) gauge theories proposed by Aharony \([19]\). There is an interesting twist, though: the theories described by the quivers that appear in the equality are not dual to each other.

The correct duality takes the theory on the left-hand side to a theory whose matter content consists of the chiral multiplets described by the quiver on the right-hand side, plus two extra singlets \(V^+, V^-\). More generally, Aharony duality relates two 3d \(\mathcal{N} = 2\) gauge theories with the following matter contents:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
G & SU(N_f)_L & SU(N_f)_R & U(1)_B & U(1)_A & U(1)_R \\
\hline
Q & \boxed{1} & \boxed{1} & 1 & 1 & 1 - N_c/N_f \\
\tilde{Q} & \boxed{1} & \boxed{1} & -1 & 1 & 1 - N_c/N_f \\
q & \boxed{1} & \boxed{1} & N_c/(N_f - N_c) & -1 & N_c/N_f \\
\tilde{q} & \boxed{1} & \boxed{1} & N_c/(N_f - N_c) & -1 & N_c/N_f \\
M & 1 & \boxed{1} & 0 & 2 & 2(1 - N_c/N_f) \\
V_\pm & 1 & \boxed{1} & 0 & -N_f & 1 \\
\hline
\end{array}
\]

(5.7)

The gauge group \(G\) is SU(\(N_c\)) for the theory with \(Q, \tilde{Q}\), and SU(\(N_f - N_c\)) for its dual. Although a proof has not been given yet, there is strong evidence for the equality of the supersymmetric indices of these two theories in terms of series expansion in \(x\) \([73–75]\).

The above table contains one more global symmetry compared to its 4d counterpart \((3.32)\), namely the axial symmetry \(U(1)_A\) which is anomalous in four dimensions. In the definition of the index \((5.2)\), we did not include fugacities for axial symmetries. Thanks to this omission, \(V_\pm\) always have \(R = 1\) for whatever shift of \(R\) by the relevant
U(1) flavor symmetries, and their contributions to the index cancel since we can turn on a superpotential $W = mV_+V_-$. If we included these fugacities, the cancellation would no longer occur. In that case, the presence of $V_\pm$ would be crucial for the indices of the two theories to match.

Why does the logic fail if the fugacities for axial symmetries are turned on? The point is that when we apply T-duality to get a 3d brane box model from a 4d one, the 2-direction is compactified to a circle, however small its radius may be. In other words, we are really compactifying the 4d theory on $S^1$ with finite radius, not dimensional reducing it. As emphasized in [72], the effective 3d theory obtained by compactification has a superpotential that contains monopole operators. This superpotential breaks the axial symmetries, thereby reproducing the effect of anomalies in the 4d theory. So it is not that the logic fails. Rather, we have no fugacities to turn on to begin with. Upon inclusion of such superpotentials on both sides, the extra multiplets disappear from the duality.

We could as well decompactify the 2-direction so that the axial symmetries are recovered. However, the system is no longer related to a 4d brane box model then, and the analysis of the 3d theory becomes more difficult. In particular, additional chiral multiplets may arise from oscillations of D4-branes in the 2-direction. Presumably, with these additional multiplets taken into account, the Yang–Baxter equation for the associated lattice model reduces to Aharony duality or something similar.

### 5.2 2d brane box models

Applying further T-duality along the 1-direction, we obtain the following brane configuration realizing a 2d $\mathcal{N} = (2, 2)$ quiver gauge theory:

$$
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
\text{D3} & & & X & X & & & & & \\
\text{NS5} & X & X & X & X & & & & & \\
\text{NS5} & & & & & & X & X & & \\
\end{array}
$$

(5.8)

This theory is described by the same quiver diagram as in the 3d and 4d brane box cases. However, each node now represents a $U(N)$ gauge group, not $SU(N)$. The reason is that unlike the case with D4- or D5-branes, motion of D3-brane segments along NS5-branes create disturbances on the NS5-branes that fall off sufficiently rapidly, and costs only a finite amount of energy. (In the 3d brane box case, on the other hand, the required energy becomes arbitrarily large as we increase the radius of the 2-circle.)

The supersymmetric index of a 2d $\mathcal{N} = (2, 2)$ theory is known as the elliptic genus. It is given by

$$
I(q, y; \{r_\alpha\}) = \text{Tr}_{S^1} \left( (-1)^F q^{H_L} y^{R/2} \right),
$$

(5.9)

where $H_L$ is the left-moving Hamiltonian, and $R$ is defined as twice the left-moving R-charge so that a superpotential preserving the R-symmetry has $R = 2$. The trace receives contributions only from states with the right-moving Hamiltonian $H_R = 0$. The elliptic genus can be computed as the partition function with twisted boundary conditions on a torus whose complex parameter $\tau$ is given by $q = e^{2\pi \tau}$. As such, it enjoys a nice modular property.
An integral formula for the elliptic genus of an $N = (2, 2)$ gauge theory was derived in [76–78]. For the theory we are considering, the formula reads\footnote{For concreteness we will follow the treatment in [77, 78] where the elliptic genus is defined in the (R, R) sector, though the considerations given below equally apply to the other sectors. See [76] for the formula for the (NS, NS) sector.}

$$I = \int \prod \frac{1}{N!} \prod_{j=1}^{N} \frac{u_{a,j}}{2\pi i} I_V(u_a) \prod_{\underline{\theta} \to \underline{\theta}} I_B(u_a, u_b; R_{ab}),$$

with the vector and bifundamental multiplet factors given by

$$I_V(u_a) = \left( \frac{2\pi \eta(\tau)^3}{\theta_1(-z|\tau)} \right)^N \prod_{i \neq j} \frac{\theta_1(u_{a,i} - u_{a,j} + z|\tau)}{\theta_1(u_{a,i} - u_{a,j} - z|\tau)}.$$  

$$I_B(u_a, u_b; R_{ab}) = \prod_{i,j} \frac{\theta_1(u_{b,j} - u_{a,i} + (R_{ab}/2 - 1)z|\tau)}{\theta_1(u_{b,j} - u_{a,i} + R_{ab}z/2|\tau)}.$$  

Here $z$ is related to the fugacity $y$ by $y = e^{2\pi iz}$, and

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

$$\theta_1(z|\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n e^{2\pi iz(n+1/2)} e^{\pi i \tau (n+1/2)^2}$$

are the Dedekind eta function and a Jacobi theta function. Note that $\theta_1(-z|\tau) = -\theta_1(z|\tau)$. As a consequence, $I_B$ obeys the identity (3.28).

The above formula takes the same form as the 4d index formula (3.24) under the identification $z_{a,i} = e^{2\pi i u_{a,i}}$. The subtlety lies in the integration contour, as was elucidated in [77, 78] with a careful path integral analysis. Since $\theta_1(z|\tau)$ has a pole at $z = 0$, the integrand has poles at various places in the $u$-space, and the contour must pick up the residues at some but not all of these poles. There are many choices that lead to the correct answer. For us, a simple choice is such that for each node $a$, every $u_{a,i}$ encircles the pole located at $u_{a,i} = u_{b,j} - R_{ba}z/2$ for some $(b, j)$, which comes from the factor $I_B(u_b, u_a; R_{ba})$ associated to the incoming arrow

\begin{equation}
\begin{array}{c}
\circ\end{array}\hline\begin{array}{c}
\circ\end{array} \, (5.15)
\end{equation}

Moreover, $u_{a,i}$ and $u_{a,j}$ for $i \neq j$ must encircle different poles. We sum over the contributions from all such contours. Alternatively, we may choose the contours to encircle poles coming from outgoing arrows. The two choices give the same result, up to an overall sign. We refer the reader to [77, 78] for more details on the integration contour in general, and section 4.6 of [78] for details specific to theories closely related to ours.

As in the higher-dimensional cases, the supersymmetric index of a 2d brane box model coincides with the partition function of an IRF model. The Boltzmann weight is given by the formula (3.27), and the Yang–Baxter equation reduces to the equality (3.31). This time, let us demonstrate this equality explicitly.
The elliptic genus of the theory described by the quiver on the left-hand side is computed by the contour integral

\[
\mathcal{I}_{\text{LHS}} = \frac{1}{N!} \oint \prod_{j=1}^{N} \frac{du_{g,j}}{2\pi i} \mathcal{I}_V(u_g) \mathcal{I}_B(u_a, u_g; R_{ag}) \mathcal{I}_B(u_d, u_g; R_{dg}) \\
\times \mathcal{I}_B(u_g, u_c; R_{gc}) \mathcal{I}_B(u_g, u_f; R_{gf}).
\]  

(5.16)

We choose to evaluate it with contours that encircle \(N\) poles from the factors associated with the incoming arrows. For instance, suppose that we are evaluating the integral for contours such that the unordered set of these poles is \(\{u_{a,j} - R_{ag} z/2\}\), that is, each \(u_{g,i}\)-integral picks up the residue at \(u_{g,i} = u_{a,j} - R_{ag} z/2\) for some \(j\). The \(N!\) different ways to assign the poles to the \(u_{g,i}\) make equal contributions, so we only need to consider the case when \(u_{g,i} = u_{a,i} - R_{ag} z/2\) for all \(i\). Noting that the residues of \(N \theta_1\)s that give the poles cancel the factor of \((2\pi \eta(\tau)^3/\theta_1(-z))^N\) in \(\mathcal{I}_V\), we find that the sum of the contributions from these contours is

\[
\mathcal{I}_B(u_a, u_c; R_{ag} + R_{gc}) \mathcal{I}_B(u_a, u_f; R_{ag} + R_{gf}) \mathcal{I}_B(u_d, u_a; R_{dg} - R_{ag}).
\]  

(5.17)

On the other hand, the elliptic genus of the right-hand side is given by

\[
\mathcal{I}_{\text{RHS}} = \frac{1}{N!} \oint \prod_{j=1}^{N} \frac{du_{g,j}}{2\pi i} \mathcal{I}_V(u_g) \mathcal{I}_B(u_g, u_d; R_{gd}) \mathcal{I}_B(u_g, u_a; R_{ga}) \\
\times \mathcal{I}_B(u_c, u_g; R_{cg}) \mathcal{I}_B(u_f, u_g; R_{fg}) \\
\times \mathcal{I}_B(u_a, u_c; R_{ac}) \mathcal{I}_B(u_a, u_f; R_{af}) \mathcal{I}_B(u_d, u_c; R_{dc}) \mathcal{I}_B(u_d, u_f; R_{df}).
\]  

(5.18)

For the dual theory, we choose the poles from the outgoing arrows. The contours picking up the residues at the poles \(\{u_{d,i} + R_{gd}/2\}\) contribute

\[
\mathcal{I}_B(u_d, u_a; R_{ga} - R_{gd}) \mathcal{I}_B(u_c, u_d; R_{cg} + R_{gd}) \mathcal{I}_B(u_f, u_d; R_{fg} + R_{gd}) \\
\times \mathcal{I}_B(u_a, u_c; R_{ac}) \mathcal{I}_B(u_a, u_f; R_{af}) \mathcal{I}_B(u_d, u_c; R_{dc}) \mathcal{I}_B(u_d, u_f; R_{df}).
\]  

(5.19)

After canceling out some factors using the identity (3.28), we see that this is equal to the contribution (5.17) found above.

In a similar fashion, one can show that for each set of poles \(\{\hat{u}_{g,i}\} \subset \{u_{a,i} - R_{ag} z/2\}\) of \(\{\tilde{u}_{g,i}\} \subset \{u_{a,i} + R_{ga} z/2\}\) in the theory on the left-hand side, there is a set of poles \(\{\tilde{u}_{g,i}\} \subset \{u_{a,i} + R_{ga} z/2\}\) of \(\{u_{d,i} + R_{gd} z/2\}\) in the theory on the right-hand side such that the contours encircling these poles give the same contributions to the elliptic genera of the respective theories. (In the case that the R-charges vanish, \(\{\hat{u}_{g,i}\}\) is the complement of \(\{\tilde{u}_{g,i}\}\) in \(\{u_{a,i}\}\).) This is a one-to-one correspondence between the choices of poles in the two theories. Therefore, \(\mathcal{I}_{\text{LHS}} = \mathcal{I}_{\text{RHS}}\).

The equality just proved may be thought of as a consequence of a variant of Hori–Tong duality [20] proposed in [79]. There are important details to be noted, though. For the Yang–Baxter equation to hold, the fugacities \(u_a, u_c, u_d, u_f\) must be unconstrained since
they are associated to $U(N)$ gauge groups. However, when regarded as the elliptic genera of theories with gauge group $U(N)$, the two sides of the equality (3.31) must be evaluated with the constraint $\sum_i (u_{a,i} + u_{c,i} + u_{d,i} + u_{f,i}) = 0$ since the diagonal $U(1)$ subgroup of the flavor group $U(N)_a \times U(N)_c \times U(N)_d \times U(N)_f$ is gauged. So strictly speaking, the duality does not imply the Yang–Baxter equation. Also, the fugacities must be mapped trivially under the duality. This is true for the $u_a$, but not for the fugacity for the axial symmetry $U(1)_A$. Hence, we must set the latter to zero, as in the 3d case.

Finally, we remark that the duality does not work if we replace the gauge group with $SU(N)$. This is unlike the higher-dimensional cases, where one can gauge $U(1)_B$ to obtain the duality for $U(N)$ theories from that for $SU(N)$ theories. Had we started with the dimensional reduction of the 3d brane box model and simply discarded the $U(1)_A$ fugacity, we would have reached a wrong conclusion.

6 2d $\mathcal{N} = (0, 2)$ quiver gauge theories

Looking at how the duality relation (3.31) was verified for 2d brane box models, we notice that the numerator and denominator of the bifundamental factor $I_B$ played rather different roles. More specifically, while the denominator determines the pole structure and hence the possible choices for integration contours, the numerator only provides cancellations of some factors. The separation in their roles hints at the existence of integrability in less supersymmetric situations where an $\mathcal{N} = (2, 2)$ chiral multiplet decomposes into two multiplets that correspond to the numerator and denominator.

$\mathcal{N} = (0, 2)$ supersymmetry provides precisely the required decomposition: an $\mathcal{N} = (2, 2)$ chiral multiplet consists of an $\mathcal{N} = (0, 2)$ chiral multiplet and Fermi multiplet, with the former corresponding to the denominator and the latter to the numerator. In this section we introduce three classes of $\mathcal{N} = (0, 2)$ quiver gauge theories whose elliptic genera are captured by integrable lattice models. The first two classes lead to IRF models much like $\mathcal{N} = (2, 2)$ brane box and brane tiling models discussed in section 5.2. The third class gives rise to an IRC model on 3d lattices.

6.1 $\mathcal{N} = (0, 2)$ theories associated with brane box configurations

Let us take a quiver from some brane box model, and replace the vertical arrows with dotted arrows. (Of course, one may as well choose to replace the horizontal arrows instead.) For example, starting from the quiver for the $2 \times 3$ brane box model shown in figure 2b, we obtain the diagram in figure 13. We interpret the resulting diagram as a quiver of an $\mathcal{N} = (0, 2)$ gauge theory, letting dotted arrows represent bifundamental Fermi multiplets. As usual, circle nodes and solid arrows represent vector and bifundamental chiral multiplets. By this procedure we can associate an $\mathcal{N} = (0, 2)$ quiver gauge theory to every brane box model. The reader is referred to [21, 80] for details on $\mathcal{N} = (0, 2)$ gauge theories.

Although the left-moving R-symmetry is no longer present, the theory still has the flavor symmetries $U(1)_a$, at least classically. So we define the “R-charge” in the present case by

$$R = \sum_\alpha r_\alpha F_\alpha.$$ (6.1)
Our claim is that the elliptic genus (5.9) of this theory is equal to the partition function of an integrable IRF model.

Before we give a proof of this claim, cautionary remarks are in order. As it is, the theory defined above suffers from mixed anomalies for the diagonal U(1) factors of the U(N) gauge groups. Similarly, the flavor symmetry generated by $R$ is anomalous. These anomalies must be canceled by introduction of extra multiplets. For the moment let us ignore these issues; we will address them later.

The relevant formula for elliptic genera was derived in [77, 78]. The only difference compared to the $\mathcal{N} = (2,2)$ case is that it involves three kinds of factors, corresponding to the three kinds of multiplets:

$$I_{\mathcal{V}}(u_a) = \left( \frac{2\pi \eta(\tau)^2}{i} \right)^N \prod_{i \neq j} \frac{i \theta_1(u_{a,i} - u_{a,j} | \tau)}{\eta(\tau)},$$

$$I_{\mathcal{BC}}(u_a, u_b; R_{ab}) = \prod_{i,j} \frac{i \eta(\tau)}{\theta_1(2u_{b,j} - u_{a,i} + R_{ab}z/2 | \tau)},$$

$$I_{\mathcal{BF}}(u_a, u_b; R_{ab}) = \prod_{i,j} \frac{i \theta_1(u_{b,j} - u_{a,i} + R_{ab}z/2 | \tau)}{\eta(\tau)}.$$  

(6.3)

(6.4)

(6.5)

Up to an immaterial overall sign, $I_{\mathcal{BC}}(u_a, u_b; R_{ab})I_{\mathcal{BF}}(u_a, u_b; R_{ab} - 1)$ is equal to the $\mathcal{N} = (2,2)$ bifundamental chiral multiplet factor (5.12). Likewise, $I_{\mathcal{V}}(u_a)I_{\mathcal{BC}}(u_a, u_b; -2)$ is equal to the $\mathcal{N} = (2,2)$ vector multiplet factor (5.11), corresponding to the decomposition of an $\mathcal{N} = (2,2)$ vector multiplet into an $\mathcal{N} = (0,2)$ vector multiplet and adjoint chiral multiplet.

We have

$$I_{\mathcal{BC}}(u_a, u_b; R_{ab})I_{\mathcal{BF}}(u_b, u_a; R_{ba}) = 1$$

(6.6)

for $R_{ab} + R_{ba} = 0$, or more graphically,

$$\begin{array}{ccc}
\begin{array}{c}
\circ
\end{array} & \circ & \begin{array}{c}
\circ
\end{array} \end{array} = \begin{array}{c}
\circ
\end{array} \begin{array}{c}
\circ
\end{array} \begin{array}{c}
\circ
\end{array} \begin{array}{c}
\circ
\end{array}.$$  

(6.7)

---

12 See [21, 81] for the corresponding formula for the (NS,NS) sector.
This corresponds to giving masses to a pair of chiral and Fermi multiplets forming a loop. For the theory under consideration, the elliptic genus can be computed as the partition function of an IRF model with Boltzmann weight

\[ r_s a b c d = a b c d - s r s - r. \] (6.8)

Plugging this into the Yang–Baxter equation, we find that the integrability of the model is equivalent to the equality

\[ a c d f g r t s - r - s - r + r - s = a c d f g t s - s - r - t r t - r + r - s. \] (6.9)

Let us demonstrate it.

The left-hand side is given by the contour integral

\[ \mathcal{I}_{LHS} = \frac{1}{N!} \oint \prod_{j=1}^{N} \frac{du_{j,g}}{2\pi i} \mathcal{I}_V(u_g) \mathcal{I}_{BC}(u_d, u_g; R_{dg}) \mathcal{I}_B(u_a, u_g; R_{ag}) \times \mathcal{I}_{BC}(u_g, u_c; R_{gc}) \mathcal{I}_{BC}(u_d, u_f; R_{df}). \] (6.10)

If we choose the contours to pick up residues from the incoming arrow, then the set of relevant poles is \( \{u_{d,i} - R_{dg} z/2\} \). Thus we have

\[ \mathcal{I}_{LHS} = \mathcal{I}_B(u_a, u_d; R_{ag} - R_{dg}) \mathcal{I}_{BC}(u_d, u_c; R_{dg} + R_{gc}) \mathcal{I}_{BC}(u_d, u_f; R_{dg} + R_{gf}). \] (6.11)

The right-hand side of the equality (6.9) is

\[ \mathcal{I}_{RHS} = \frac{1}{N!} \oint \prod_{j=1}^{N} \frac{du_{j,g,i}}{2\pi i} \mathcal{I}_V(u_g) \mathcal{I}_{BC}(u_g, u_a; R_{ga}) \mathcal{I}_B(u_g, u_d; R_{gd}) \times \mathcal{I}_{BC}(u_c, u_g; R_{cg}) \mathcal{I}_{BC}(u_f, u_g; R_{fg}) \times \mathcal{I}_B(u_a, u_c; R_{ac}) \mathcal{I}_B(u_a, u_f; R_{af}) \mathcal{I}_{BC}(u_d, u_c; R_{dc}) \mathcal{I}_{BC}(u_d, u_f; R_{df}). \] (6.12)

The “complement” set of poles is \( \{u_{a,i} + R_{ga} z/2\} \), and they give

\[ \mathcal{I}_{RHS} = \mathcal{I}_B(u_a, u_d; R_{gd} - R_{ga}) \mathcal{I}_{BC}(u_c, u_a; R_{cg} + R_{ga}) \mathcal{I}_{BC}(u_f, u_a; R_{fg} + R_{ga}) \times \mathcal{I}_B(u_a, u_c; R_{ac}) \mathcal{I}_B(u_a, u_f; R_{af}) \mathcal{I}_{BC}(u_d, u_c; R_{dc}) \mathcal{I}_{BC}(u_d, u_f; R_{df}). \] (6.13)
Canceling some factors using the identity (6.6), we find $\mathcal{I}_{\text{LHS}} = \mathcal{I}_{\text{RHS}}$.

In fact, the quivers on the two sides of the equality (6.9) describe two theories related by the (0, 2) triality transformation [21]. The third theory in the triality is the free theory with quiver

This is manifest in the expression (6.11) of the elliptic genus.

Finally, we come back to the issues of anomalies. To cancel the mixed U(1) gauge anomalies and $R$-anomaly, we add to each solid arrow

a Fermi multiplet with $R = - NR_{ab} - \epsilon$ in the representation $\det_a \otimes \det_b^{-1}$ of $\text{SU}(N)_a \times \text{SU}(N)_b$, and to each dotted arrow

a chiral multiplet with $R = - NR_{ab} + \epsilon$ in $\det_a \otimes \det_b^{-1}$. Here $\epsilon$ is a nonzero parameter. Furthermore, to each node we introduce a singlet chiral multiplet with $R = \epsilon$, though this is not required for anomaly cancellation. We can visualize these extra multiplets as wavy arrows, as in figure 14.

The presence of the extra multiplets modifies the factors associated to vector and bifundamental multiplets to

Apart from this modification, the computations of the elliptic genera for the equality (6.9) remain unchanged. In particular, we can use the same integration contours, since the extra

Figure 14: The R-charges of the extra multiplets for anomaly cancellation.
Figure 15: The $\mathcal{N} = (0, 2)$ theory obtained from the $2 \times 3$ brane tiling model.

multiplets introduce neither solid incoming arrows to the left-hand side nor solid outgoing arrows to the right-hand side.

6.2 $\mathcal{N} = (0, 2)$ theories associated with brane tiling configurations

Quivers from brane tiling models can also be modified to produce $\mathcal{N} = (0, 2)$ theories that exhibit integrability. As in the brane box case discussed above, we obtain these theories by replacing one of the arrows of the R-matrix by a dotted arrow:

$$i_1 \quad i_2 \quad j_1 \quad j_2 \quad = \quad r_1 - s_1 \quad s_2 - r_1 \quad s_1 - r_2 \quad r_2 - s_2 \quad .$$

Extra multiplets for anomaly cancellations are implicit in this picture. Figure 15 shows the quiver for the $\mathcal{N} = (0, 2)$ theory obtained from the $2 \times 3$ brane tiling model, whose quiver is given in figure 11b.

The Yang–Baxter equation for this R-matrix reduces to the equality

$$i_1 \quad i_3 \quad j_1 \quad j_3 \quad = \quad r_1 - s_1 \quad s_2 - r_1 \quad s_1 - r_2 \quad r_2 - s_2 \quad .$$

Application of the $(0, 2)$ triality transformation (6.9) four times (more precisely, one triality transformation followed by three others in the opposite direction) turns the left-hand side into the right-hand side. This is illustrated in figure 16.

6.3 Brane cube models

Lastly, we briefly discuss the class of $\mathcal{N} = (0, 2)$ quiver gauge theories known as brane cube models [22]. These theories are constructed from brane configurations analogous to brane
Figure 16: A sequence of \((0, 2)\) triality transformations that relates the two sides of the Yang–Baxter equation (6.21). An arrow between two graphs represents the \((0, 2)\) triality transformation in the direction from the right- to left-hand side of the equality (6.9).

Note that brane cube models include 3d brane box models (5.1) as special cases. A brane box model is characterized by the 2d lattice drawn on the 46-torus by the NS5-branes. For a brane cube model, the NS5-branes make a 3d lattice in the 246-space which we take to be a 3-torus \(T^3\).

A brane cube configuration can be thought of as describing intersecting codimension-1 defects in the 5d theory on the D4-branes. We can topologically twist the theory along the 246-space so that some supercharges are preserved when this space is replaced by a general 3-manifold \(\Sigma\). (Concretely, we replace the holonomy group \(SU(2)_{246}\) of the 246-space with the diagonal subgroup of \(SU(2)_{246} \times SU(2)_{357}\).) The twisting preserves two supercharges which generate \(\mathcal{N} = (0, 2)\) supersymmetry in the 01-space. The codimension-1 defects preserve these supercharges. By taking the elliptic genus of the \(\mathcal{N} = (0, 2)\) theory, we get a 3d TQFT on \(\Sigma\). The defects represent surface operators in this TQFT. An extra dimension emerges upon lifting the system to M-theory. Therefore, the elliptic genus of a brane cube model is given by the partition function of a 3d integrable lattice model.

Like an IRF model is associated with brane box models, the lattice model associated with brane cube models is an IRC model. According to the quiver rule found in [22], one

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
D4 & \times & \times & \times & \times & \times & & & & \\
NS5 & \times & \times & \times & \times & \times & \times & & & \\
NS5 & \times & \times & \times & \times & \times & \times & \times & & \\
NS5 & \times & \times & \times & \times & \times & \times & \times & \times & \\
\end{array}
\]

(6.22)
way to describe this model is to set its Boltzmann weight

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\otimes
\end{array}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\otimes
\end{array}
\end{array}
\end{array}.
\end{array}
$$

(6.23)

This Boltzmann weight, or another one that leads to the same IRC model on cubic lattices, should solve the tetrahedron equation (2.24). It would be interesting to verify this statement.

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