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Fractional stochastic modelling of COVID-19 under wide spread of vaccinations: Egyptian case study

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Abstract This work predicts the dynamics of the COVID-19 under widespread vaccination to anticipate the virus's current and future waves. We focused on establishing two population-based models for predictions: the fractional-order model and the fractional-order stochastic model. Based on dose efficacy, which is one of the main imposed assumptions in our study, some vaccinated people will probably be exposed to infection by the same viral wave. We validated the generated models by applying them to the current viral wave in Egypt. We assumed that the Egyptian current wave began on 10th September 2021. Using current actual data and varying our models’ fractional orders, we generate different predicted wave scenarios. The numerical solution of our models is obtained using the fractional Euler method and the fractional Euler Maruyama method. At the end, we compared the current predicted wave under a high vaccination rate with the previous viral wave. Through this comparison, the vaccination control effect is quantified.

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1. Introduction

All countries are moving as fast as possible to vaccinate their population to limit daily infections and deaths and make viral waves more smooth. Vaccination is assumed to be a worldwide mass process with a large number of units needed. It is estimated that 20 different manufacturers will produce around $10^{10}$ doses of COVID-19 vaccination by the end of 2021, which should be ready for global distribution [1,2]. About 68% of the population worldwide is interested in taking the COVID-19 vaccination. That means about $3.7 \times 10^9$ adult people need to be vaccinated [3]. The mathematical models suggest that doses will not be enough until 2023 due to limited sources and manufacturing capacity for population coverage [4]. In many recent studies, mathematical epidemiological modelling is used to predominantly develop mass-action models and suitable tools for analyzing COVID-19 dynamics [5–14]. Different mathematical models are used for this purpose as the transmission rates and virus behaviour depend on individuals’ precautionary measures, daily vaccination rates, and vaccination...
efficiencies [15–21]. In [5,6], a complete analysis of the actual number of deaths due to COVID-19 was studied by the authors using a new class of probability density functions and the exponentiated transformation of the Gumbel type-II distribution. The authors of [8] used numerical approaches and logistic modelling techniques to conduct a thorough analysis of COVID-19. In [9,10], COVID-19 daily confirmed cases in Egypt and Iraq are modelled using a Gaussian fitting model and a logistic model is reached. In [11], the authors model COVID-19 daily confirmed cases in different countries using Fourier and sine wave fitting models.

On the other hand, the classic susceptible exposed infectious recovered model (SEIR), using integer order derivatives, is the adopted model for characterizing the epidemic of the COVID-19 outbreak in different countries. The extension of the classical SEIR model with delays is another routine to simulate the incubation period and the period before recovery [12]. In [13], the authors created a multi-strain SEIR epidemic model for COVID-19 and wrote a complete analysis on how to control the value of the reproduction number for the next upcoming virus peaks. Fractional order deterministic and stochastic differential equations are also used for developing dynamical SEIR models to study the nature of the COVID-19 with nonuniform data much more clearly [22–30]. Such fractional order differential equations play a very important role in different fields, including engineering, physics, signal and image processing, mechanics and dynamical systems, biology, control theory, and environmental sciences [31–33]. The Caputo fractional derivative is one of the most commonly used operators in fractional differential equations’ applications because the Caputo derivative is suitable for initial value problems (IVPs) and has many characteristics similar to integer-order derivatives [34–36]. In [11], the authors established a fractional-order stochastic dynamical model for COVID-19 to describe the second virus wave’s behaviour in Egypt. The authors of [37,38] created stochastic epidemic modes with time delays for COVID-19 behaviour estimation. In [39], the authors constructed a stochastic mathematical model to investigate virus temporal dynamics in Oman. They divided the total population into four classes and considered vaccine effects in their model. In [40], the authors generated a fractional-order stochastic model to optimize daily vaccinations in Saudi Arabia.

The aim of this paper is to develop two dynamical epidemic models, the fractional-order model and the fractional-order stochastic model, to describe COVID-19 behaviour and wide vaccination spread using Caputo fractional derivative and additive white noise. The main contribution of this is to justify the complete effect of vaccines, through different vaccines types, number of doses, and efficiencies, on viral wave shape and peak through affecting the wave reproduction number and then applying them to the Egyptian case study to verify the vaccination effectiveness for the fourth viral wave in Egypt, which is assumed to start on 10th September 2021.

This paper is prepared as follows: In Section 2, we provide the used mathematical preliminaries in our work. In Section 3, we describe the COVID-19 dynamical models. In Section 4, we apply the dynamical models to the Egyptian case study and compare the predicted daily infections, deaths, and vaccinations with their actual values. In the end, conclusions and future work are written.

2. Preliminaries

Recently, fractional differential equations have gained much attention since the fractional-order system response ultimately converges to the integer-order system response. The advantages of fractional derivatives are that they have a greater degree of flexibility in the model and provide an excellent instrument for the description of the properties of various practical processes and dynamical systems. For high accuracy, fractional derivatives are then used to describe the dynamics of our generated epidemiological dynamic models, which will describe the COVID-19 virus behaviour.

Definition 2.1. Suppose that $\phi$, $t > 0$, $\phi$, $t \in R$. The fractional operator ($d^\phi_t$), defined as [41,42]:

$$
d^\phi_t f(t) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(n-\phi)} \int_0^t (t-\zeta)^{n-\phi-1} f^{(n)}(\zeta)d\zeta, & \text{where } t > 0, 0 \leq n - 1 < \phi < n \in N_R
\\
d^\phi_t f(t), & \phi = n \in N_R
\end{array} \right.
$$

(1)

is called the Caputo fractional derivative or Caputo fractional differential operator of order $\phi$ and $N_R$ is the set of positive integer numbers.

Lemma 2.1. Let us assume that both the function $f(t)$ and its Caputo fractional derivative $d^\phi_t f(t)$, $\phi \in (0, 1]$, be elements of the metric space $C[a, b]$. The function $f(t)$ is a monotone increasing if $d^\phi_t f(t) \geq 0$ and a monotone decreasing if $d^\phi_t f(t) \leq 0$ for all $t \in [a, b]$.

Lemma 2.2. Let us assume that the following fractional order differential equation driven by Caputo fractional derivative and order $\phi \in (0, 1]$:

$$
d^\phi_x(t) = f(t, x), t_0 > 0, x(t_0) = x_0
$$

(2)

where $f: [t_0, \infty) \times R \to R^n$, $f(x, \dot{x})$ be a function, if the Lipschitz condition is satisfied by $f(t, x)$ with respect to $x(t)$, then the system (2) has a unique solution on the interval $[t_0, \infty) \times R$ [43].

To find the solution of Equation (2) over the interval $[0, T]$, a set of points $\{(t_c, x(t_c))\}$ are produced which are approximated as the used numerical method. To achieve this approximation, the interval $[0, T]$ is partitioned into $n_i$ subintervals $[t_0, t_n]$ for $c = 0, 1, n_i$. The general formula for the fractional Euler method to solve Equation (2) numerically is defined as:

\[
x(t_{c+1}) = x(t_c) + \frac{h}{\Gamma(\phi + 1)} f(t_c, x(t_c)),
\]

\[
t_{c+1} = t_c + h, c = 1, 2, \cdots, n_i - 1
\]

(3)
Convergency of the fractional Euler method is detailed described in [27].

**Definition 2.2.** A standard one-dimensional Wiener process (Brownian motion) is a stochastic process \( \{ \mathbf{B}(t), t > t_0 \} \) indexed by nonnegative real numbers \( t \) with the following properties [26]:

1. \( \mathbf{B}(t_0) = 0 \) with probability 1.
2. The function \( t \to \mathbf{B}(t) \) is continuous on \( t \).
3. If \( t_1 \neq t_2 \) then \( \mathbf{B}(t_1) \) and \( \mathbf{B}(t_2) \) are independent.
4. For \( \forall t \geq t_0 \), all increments \( \Delta \mathbf{B}_t = \mathbf{B}_{t+h} - \mathbf{B}_t \) are normally distributed with mean 0 and variance \( \Delta t \).

\[ \mathbf{B}(t) = \mathbf{B}(t_0) + \sum_{t_0 < s \leq t} \Delta \mathbf{B}_s. \]

**Definition 2.3.** A fractional-order system of m-dimensional stochastic differential equations (SDEs), under using Caputo fractional derivative, indexed by nonnegative real numbers \( t \) can be solved step by step at each interval \( [t_i, t_{i+1}] \), i.e., \( \Delta \mathbf{B}_t \sim N(0, \Delta t) \).

\[ X(t) = \mathbf{X}(t_0) + \int_{t_0}^t d \mathbf{X}(s) \mathbf{Y}(s) \, ds, \]

where \( \mathbf{X}(t) \) is the state vector and the functions \( \ell(t, X) \) are the drift and diffusion terms respectively of the SDE and \( \mathbf{B} \) represents the stochastic Wiener process.

The fractional-order stochastic model driven by Caputo fractional derivative of order \( \phi \) is described in detail in [28]. Applying fractional-order Maruyama scheme to Equation (4) on the interval \( t \in [t_i, t_{i+1}] \), it yields:

\[ X_i^{(n)} = X_i + \frac{1}{\Gamma(\phi)} \int_{t_i}^{t_{i+1}} \frac{u_i(t, \mathbf{X}(s), \mathbf{X}^{(n)}(\tau_i(s)))}{(t-s)^{1-\phi}} \, ds + \frac{1}{\Gamma(\phi)} \int_{t_i}^{t_{i+1}} \frac{v_i(t, \mathbf{X}(s), \mathbf{X}^{(n)}(\tau_i(s)))}{(t-s)^{1-\phi}} \, dB(t, \tau_i(s)) \]

\[ = \frac{k}{n} \mathbf{T} + \sum_{s \in \frac{k}{n} \mathbf{T}} \mathbf{Y}(s) \quad \text{for } k = 0, 1, \ldots, n-1, i = 1, 2, \ldots, m. \]

where \( n \) is the total time discretization and belongs to the set of positive integer numbers. The previous iterative equation can be solved step by step at each interval \( [t_{i+1}, t_{i+2}] \). Convergency of the used fractional Euler Maruyama scheme is described in detail in [28].

### 3. Mathematical modelling

Based on several strategies enforced by governments to vaccinate as much as possible of their populations, to fight against COVID-19 successive waves, we generated two new dynamical models to describe such pandemic disease. In the new models, the population \( N \) have partitioned into eleven classes, denoted by \( \mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{V}_1, \mathcal{V}_2, \mathcal{E}_1, \mathcal{I}_1, \mathcal{E}_2, \mathcal{I}_2, R \) and \( D \).

\( S, E \) and \( I \) respectively symbolize the main daily susceptible, exposed and infected individuals. \( V_1 \) and \( V_2 \) are the daily first and second vaccinated individuals using vaccination type \( i \). \( E_N \) and \( I_N \) are the daily first-vaccinated exposed and first-vaccinated infected individuals while \( E_S \) and \( I_S \) are the daily second-vaccinated exposed and second-vaccinated infected individuals. Finally, \( R \) and \( D \) are the daily recovered and daily deaths. All proposed models satisfy the following assumptions:

1. All involved transmission rates are with positive values and vary with time.
2. Symptomatic patients are only those who transmit the virus.
3. Susceptible individuals can move into infected classes without passing by exposed classes.
4. Both first and second vaccinated classes pass by exposure and infectious track with lower rates.
5. For one dose vaccinations, first vaccinated individuals can move to recovered class.
6. Maximum number of vaccination doses per virus wave cycle equals two.
7. No second infections in the same wave cycle.

COVID-19 dynamics under using different vaccinations with different efficiencies are modelled with two dynamic models, which are the fractional-order deterministic model and the fractional-order stochastic model.

#### 3.1. Fractional-order deterministic model

The fractional-order model structure is completely indicated in Fig. 1. Through direct contact, the daily susceptible people \( \mathcal{S} \) transfer into the main exposed class \( \mathcal{E} \) with daily transmission rate \( \beta_f(t) \) and into the main infected class \( \mathcal{I} \) with daily transmission rate \( \beta_f(t) \). The main exposed \( \mathcal{E} \) moves to the main infected people \( \mathcal{I} \) at a daily transmission rate \( \gamma_E(t) \). A part of the infected class \( \mathcal{I} \) enters the recovered class \( \mathcal{R} \) with a daily cure rate \( \gamma_R(t) \). The daily first vaccinated class \( \mathcal{V}_1 \) is created by vaccinating the susceptible individuals with the first dose of vaccine type \( i \) at a daily rate \( \beta_f(t) \). A part of the first vaccinated people is assumed that they will pass through the same track of infection by passing by the exposure class of the first vaccinated \( \mathcal{E}_1 \) with a lower rate \( \beta_f(t) \) and the infected class of first vaccinated \( \mathcal{I}_1 \) with a lower rate \( \gamma_E(t) \). The first dose efficiency \( (\xi_1(t)) \) will affect directly increasing the cure rate to be \( (1 + \xi_1(t))\gamma_R(t) \) and decrease the death rate to be \( (1 - \xi_1(t))\mu(t) \). The daily susceptible people \( \mathcal{S} \) transfer to the exposed first vaccinated class \( \mathcal{E}_1 \) and the infected first vaccinated class \( \mathcal{I}_1 \) with daily transmission rates \( (1 - \xi_1(t)) \beta_f(t) \) and \( (1 - \xi_1(t)) \beta_f(t) \) respectively when nearly contact occurs between them.

The remaining part of the first vaccinated individuals moves to the second vaccinated class \( \mathcal{V}_2 \) with a constant daily rate \( \beta_s(t) \) in case of two-dose vaccines. In one-dose vaccines, the remaining part of the first vaccinated individuals will move to the recovered class at a daily rate \( (\xi_1(t)\mu(t) + a) \), where \( a \) is the number of days to make sure that fully vaccinated people are recovered. We assumed that a small part of the exposed class of first vaccinated will by mistake take the second dose
so moved to the second vaccinated class with a daily rate $\beta_2(t)$. The lower part of the second vaccinated people is assumed that will pass through the same track of infection with the second vaccinated exposed class ($E_2$) and second vaccinated infected class ($I_2$) with lower rates $\beta_1(t)$ and $\alpha_2(t)$ respectively. While the larger part of the second vaccinated class will move to the recovered class with the daily rate $(\zeta_{2i}/a)$. The second dose efficiency $(\zeta_{2i})$ will affect directly increasing the cure rate to be $(1 + \zeta_{2i}/\gamma(t)$ and decrease the death rate to be $(I - \zeta_{2i}) \mu(t)$.

Similarly, the daily susceptible people $(S)$ move to the exposed class of second vaccinated ($E_2$), and the infected class of second vaccinated ($I_2$) with lower daily transmission rates $(I - \zeta_{2i}) \beta_2(t)$ and $(I - \zeta_{2i}) \beta_2(t)$ respectively.

By adding the vaccination effect, the fractional-order dynamical model in [7,44,45] is described by the following equations, where $0 < \phi < 1$:

$$d^\phi S = - \left( \frac{\beta_1(t)c_1}{N} E(t) + \sum_{i=1}^n (E_i(t) + E_{2i}(t)) \right)$$

$$+ \frac{\beta_2(t)c_2}{N} (I + \sum_{i=1}^n (I_i(t) + I_{2i}(t))) + \sum_{i=1}^n \delta_1(t) \right) \right) S(t)$$

$$d^\phi E = \left( \frac{\beta_1(t)c_1}{N} E(t)S(t) + \frac{\beta_2(t)c_2}{N} I(t)S(t) - \alpha_E(t)E(t)$$

$$- \frac{\beta_3(t)c_3}{N} \sum_{i=1}^n V_i(t).E(t) \right)$$

$$d^\phi I = (\delta_{1i}(t)E(t) - (\gamma(t) + \mu(t))I(t))$$

$$d^\phi V_i = \left( \delta_{1i}(t)S(t) + \frac{\beta_1(t)c_3}{N} V_i(t).E(t)$$

$$- \frac{\beta_3(t)c_3}{N} (E_{ii}(t) + 0.5I_{ii}(t))V_i(t) - \delta_2(t)V_i(t) - \frac{\zeta_{1i-bad}}{a} V_i(t) \right)$$

$$d^\phi E_{ii} = \left( \frac{\beta_1(t)c_1}{N} \zeta_{1i}E_{ii}(t)S(t) + \frac{\beta_2(t)c_2}{N} \zeta_{1i}I_{ii}(t)S(t)$$

$$+ \frac{\beta_3(t)c_3}{N} (E_{ii}(t) + 0.5I_{ii}(t))V_{ii}(t) - \alpha_E(t)E_{ii}(t)$$

$$- \frac{\beta_4(t)c_4}{N} V_{ii}(t)E_{ii}(t) \right)$$

$$d^\phi I_{ii} = (\delta_{2i}(t)E_{ii}(t) - (\gamma(t) + \mu(t))I_{ii}(t)$$

$$d^\phi V_{ii} = \left( \delta_{2i}(t)S_{ii}(t) + \frac{\beta_2(t)c_3}{N} V_{ii}(t)S_{ii}(t)$$

$$- \frac{\beta_3(t)c_3}{N} (E_{ii}(t) + 0.5I_{ii}(t))V_{ii}(t) - \delta_2(t)V_{ii}(t) - \frac{\zeta_{1i-bad}}{a} V_{ii}(t) \right)$$

$$d^\phi S_{ii} = \left( \frac{\beta_1(t)c_1}{N} \zeta_{1i}S_{ii}(t)E_{ii}(t) + \frac{\beta_2(t)c_2}{N} \zeta_{1i}S_{ii}(t)I_{ii}(t)$$

$$+ \frac{\beta_4(t)c_4}{N} S_{ii}(t)R(t) \right)$$

$$d^\phi R = \frac{\gamma(t)I(t)}{a} \left( I(t) + \sum_{i=1}^n (\zeta_{1i} + 1)I_{ii}(t) + (\zeta_{2i} + 1)I_{2i}(t) \right)$$

$$+ \sum_{i=1}^n \frac{\zeta_{1i-bad}}{a} V_{ii}(t) + \frac{\zeta_{1i-bad}}{a} V_{ii}(t)$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are respectively the daily average numbers of closed contacts for susceptible per exposed, susceptible per infected, first vaccinated per exposed first vaccinated and second vaccinated per exposed first vaccinated. With different $\phi$ values, we can obtain different viral wave characteristics and peaks. The assumed model dynamic rates are defined as:
β₁(t) = \begin{cases} β_{10}, & t ≤ t₀, \\ β_{10}e^{-(t-t₀)}, & t > t₀, \end{cases}
β₂(t) = \begin{cases} β_{20}, & t ≤ t₀, \\ β_{20}e^{-(t-t₀)}, & t > t₀, \end{cases}
β₃(t) = \begin{cases} β_{30}, & t ≤ t₀, \\ β_{30}e^{-(t-t₀)}, & t > t₀, \end{cases}
β₄(t) = \begin{cases} β_{40}, & t ≤ t₀, \\ β_{40}e^{-(t-t₀)}, & t > t₀. \end{cases}
(17)

α_{E}(t) = \begin{cases} \mu_0 e^{-\gamma t}, & t ≤ t₁, \\ 0, & t > t₁. \end{cases}
α_{E2}(t) = \begin{cases} \mu_0 e^{-\gamma t}, & t ≤ t₂, \\ 0, & t > t₂. \end{cases}
(18)

where \(t₀\) is the virus incubation period, \(t₁\) is the time after which the first vaccinated case was recorded, \(t₂\) is the time after which the second vaccinated case was recorded. The controlling parameter \(j\) has a value changed from one country to another following the precautionary measures. The system initial rates \(\mu_0, \beta_{10}, \beta_{20}, \beta_{10}, \beta_{40}, \alpha_{E0}, \gamma₀, \) and \(\mu_0\).

3.1.1. Existence and uniqueness

We consider the region \([0, \infty) × \mathbb{R} \) to prove the existence and uniqueness criterion of the solution of our fractional-order deterministic model, where \(\mathbb{R} = \{x₁, x₂, \ldots, x_{11}\} \subset \mathbb{R}^n: \max\{|S|, |E₁|, |E₂|, |I₁|, |I₂|, |V₁|, |V₂|, |E₃|, |I₃|, |R|, |D|\} ≤ H\}, and \(U, H\) are two finite positive real numbers. Let \(X = (S₁, E₁, I₁, I₂, E₂, E₃, I₃, V₁, V₂, I₄, I₅, R₆, \) and \(Y = (S₂, E₂, I₂, E₂, I₂, E₃, I₃, V₂, V₃, E₄, I₄, E₅, R₅, \) be two points in \(\mathbb{R}\) and set the mapping \(B: \mathbb{R} \rightarrow \mathbb{R}^{11}\) by

\(B(X) = (B₁(X), B₂(X), \ldots, B₁₁(X))\)

\(B₁(X) = -\left(\frac{β₁(t)c₁}{N}(E(t) + \sum_{i=1}^{n}(E₁(t) + E₂(t)))\right) + \frac{β₄(t)c₄}{N}(I(t) + \sum_{i=1}^{n}(I₁(t) + I₂(t))) + \sum_{i=1}^{n}δ₁(t)\right)S(t)

\(B₂(X) = \left(\frac{β₁(t)c₁}{N}E(t) + \frac{β₄(t)c₄}{N}I(t) - α₁(t)E(t)\right)

\(B₃(X) = \left(δ₂(t)S(t) + \frac{β₂(t)c₂}{N}V₁(t) - \frac{β₃(t)c₃}{N}(E₁(t) + 0.5I₁(t)) + \frac{β₃(t)c₃}{N}(E₂(t) + 0.5I₂(t)) - \frac{β₃(t)c₃}{N}V₂(t)E(t)\right)

\(B₄(X) = \left(\frac{β₁(t)c₁}{N}S(t) + \frac{β₄(t)c₄}{N}S(t)\right)

\(B₅(X) = \left(\frac{β₁(t)c₁}{N}I₁(t) + \frac{β₄(t)c₄}{N}I₂(t)\right)\)

\(B₆(X) = \left(\frac{β₁(t)c₁}{N}Eᵢ(t) + \frac{β₄(t)c₄}{N}Eᵢ(t)\right)

Proof: Assume that all transmission rates and vaccination efficiencies are positive bounded coefficients such that \(β₁(t), β₂(t), β₃(t), β₄(t) \leq β_{max}\) and \(δ₁(t) \leq \delta_{max}\), \(\gamma₀ \leq \gamma_{max}\), \(\mu_0 \leq \mu_{max}\), \(\xi₁, \xi₂, \xi₃, \xi₄, \xi₅, \xi₆ \leq \xi_{max} \leq 1\) and hence:

\(||B(X) - B(Y)|| = ||B₁(X) - B₁(Y)|| + ||B₂(X) - B₂(Y)|| + \cdots + ||B₁₁(X) - B₁₁(Y)|| ≤ K||X - Y||\)

\(||B₁(X) - B₁(Y)|| = \left|\frac{β₁(t)c₁}{N}\left(E(t) + \sum_{i=1}^{n}(Eᵢ(t) + Eⱼ(t))\right)\right| ≤ \left|\frac{β₁(t)c₁}{N}\left(E(t) + \sum_{i=1}^{n}(Eᵢ(t) + Eⱼ(t))\right)\right|

+ \left|\frac{β₄(t)c₄}{N}(I(t) + \sum_{i=1}^{n}(Iᵢ(t) + Iⱼ(t))) + \sum_{i=1}^{n}δ₁(t)\right|S(t)

+ \left|\frac{β₂(t)c₂}{N}V₁(t) + \frac{β₃(t)c₃}{N}I(t) - α₁(t)E(t)\right| ≤ \left|\frac{β₁(t)c₁}{N}(E(t) + \sum_{i=1}^{n}(Eᵢ(t) + Eⱼ(t))\right|
\[ |B_2(X) - B_2(Y)| \leq \left( \frac{\beta_m c_1}{N} (E_S - E_L) + \frac{\beta_m c_2}{N} (I_S - I_L) \right) \]
\[ - x_{2m}(E - E) - \frac{\beta_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} E \right) \]
\[ \leq \frac{2H_m c_m}{N} |I - E| + \frac{2H_m c_m}{N} |S - S| + \frac{\beta_m c_m}{N} |I - I| + \frac{\beta_m c_m}{N} |E - E| \]
\[ + x_{2m}(E - E) + \frac{2H_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} \right) \]
\[ \leq \frac{2H_m c_m}{N} |S - S| + \frac{\beta_m c_m}{N} |I - I| + \frac{2H_m c_m}{N} |E - E| + \frac{2H_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} \right) |I - I| \]
\[ + \frac{2H_m c_m}{N} \max_{1 \leq s \leq n} |V_{s,2} - V_{s,1}| \]

\[ |B_3(X) - B_3(Y)| \leq \left( \frac{\beta_m c_1}{N} (E_S - E_L) + \frac{\beta_m c_2}{N} (I_S - I_L) \right) \]
\[ - x_{3m}(E - E) - \frac{\beta_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} E \right) \]
\[ = \frac{2H_m c_m}{N} |I - E| + \frac{2H_m c_m}{N} |S - S| + \frac{\beta_m c_m}{N} |I - I| + \frac{\beta_m c_m}{N} |E - E| \]
\[ + \frac{2H_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} \right) |I - I| \]
\[ + \frac{2H_m c_m}{N} \max_{1 \leq s \leq n} |V_{s,2} - V_{s,1}| \]

\[ |B_4(X) - B_4(Y)| \leq \left( \frac{\beta_m c_1}{N} (E_S - E_L) + \frac{\beta_m c_2}{N} (I_S - I_L) \right) \]
\[ - x_{4m}(E - E) - \frac{\beta_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} E \right) \]
\[ = \frac{2H_m c_m}{N} |I - E| + \frac{2H_m c_m}{N} |S - S| + \frac{\beta_m c_m}{N} |I - I| + \frac{\beta_m c_m}{N} |E - E| \]
\[ + \frac{2H_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} \right) |I - I| \]
\[ + \frac{2H_m c_m}{N} \max_{1 \leq s \leq n} |V_{s,2} - V_{s,1}| \]

\[ |B_5(X) - B_5(Y)| \leq \left( \frac{\beta_m c_1}{N} (E_S - E_L) + \frac{\beta_m c_2}{N} (I_S - I_L) \right) \]
\[ - x_{5m}(E - E) - \frac{\beta_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} E \right) \]
\[ = \frac{2H_m c_m}{N} |I - E| + \frac{2H_m c_m}{N} |S - S| + \frac{\beta_m c_m}{N} |I - I| + \frac{\beta_m c_m}{N} |E - E| \]
\[ + \frac{2H_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} \right) |I - I| \]
\[ + \frac{2H_m c_m}{N} \max_{1 \leq s \leq n} |V_{s,2} - V_{s,1}| \]

\[ |B_6(X) - B_6(Y)| \leq x_{6m} |E_{i,1} - E_{t,1}| \]
\[ + \left( \frac{\beta_m c_1}{N} (E_S - E_L) + \frac{\beta_m c_2}{N} (I_S - I_L) \right) |I - I| \]
\[ + \frac{2H_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} E \right) \]
\[ + \frac{2H_m c_m}{N} |S - S| + \frac{\beta_m c_m}{N} |I - I| + \frac{\beta_m c_m}{N} |E - E| \]
\[ + \frac{2H_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} \right) |I - I| \]
\[ + \frac{2H_m c_m}{N} \max_{1 \leq s \leq n} |V_{s,2} - V_{s,1}| \]

The Lipschitz condition is satisfied by the function \( B(X) \) with respect to \( X = (S, E, I, V_{i,1}, E_{i,1}, I_{i,1}, V_{i,2}, E_{i,2}, I_{i,2}, R_{i,1}, D_{i,1}) \) if

\[ x_{2m}(E - E) - \frac{\beta_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} E \right) \]
\[ \leq \frac{2H_m c_m}{N} |I - E| + \frac{2H_m c_m}{N} |S - S| + \frac{\beta_m c_m}{N} |I - I| + \frac{\beta_m c_m}{N} |E - E| \]
\[ + \frac{2H_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} \right) |I - I| \]
\[ + \frac{2H_m c_m}{N} \max_{1 \leq s \leq n} |V_{s,2} - V_{s,1}| \]

where \( k = \max(k_1, k_2, \ldots, k_9) \).

The Lipschitz condition is satisfied by the function \( B(X) \) with respect to \( X = (S, E, I, V_{i,1}, E_{i,1}, I_{i,1}, V_{i,2}, E_{i,2}, I_{i,2}, R_{i,1}, D_{i,1}) \) in \( \mathbb{R} \) if

\[ x_{6m} |E_{i,1} - E_{t,1}| \]
\[ + \left( \frac{\beta_m c_1}{N} (E_S - E_L) + \frac{\beta_m c_2}{N} (I_S - I_L) \right) |I - I| \]
\[ + \frac{2H_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} E \right) \]
\[ + \frac{2H_m c_m}{N} |S - S| + \frac{\beta_m c_m}{N} |I - I| + \frac{\beta_m c_m}{N} |E - E| \]
\[ + \frac{2H_m c_m}{N} \left( \sum_{i=1}^{n} V_{i,2} - \sum_{j=1}^{n} V_{i,j} \right) |I - I| \]
\[ + \frac{2H_m c_m}{N} \max_{1 \leq s \leq n} |V_{s,2} - V_{s,1}| \]

The Lipschitz condition is satisfied by the function \( B(X) \) with respect to \( X = (S, E, I, V_{i,1}, E_{i,1}, I_{i,1}, V_{i,2}, E_{i,2}, I_{i,2}, R_{i,1}, D_{i,1}) \) in \( \mathbb{R} \) if
3.1.2. Positiveness of solution

The positive solutions are the only useful solutions in mathematical epidemiology. Let us consider \( \mathscr{R}^+ = \{(S, Ey, I_1, I_2, V_1, \ldots, V_{26}, R, D) \in \mathbb{R} : S, Ey, I_1, I_2, V_1, \ldots, V_{26}, R, D \in \mathbb{R}^+ \} \).

**Theorem 3.1.** Each solution of the fractional-order system in Equations (6) to (16) starting in \( \mathbb{R}^+ \) is positive.

**Proof.** First consider that \( S_0 = (S_0, E_0, I_{10}, V_{10}, V_{15}, I_{15}, E_{26}, I_{26}, R_0, D_0) \in \mathscr{R}^+ \) be an initial solution of the fractional-order system. Then establish that any solution \( X(t) \in \mathbb{R}^+ \) is positive. Assume \( T \) be a real number such that \( t_0 \leq t < T \) and

\[
S(t) = \begin{cases} 
0, & t_0 \leq t < T \\
0, & t = T \\
< 0, & t = T^+ 
\end{cases}
\]

The first equation in the fractional-order system implies that \( d^\alpha_t S(t) |_{t=T} = 0 \). Then from Lemma 2.2, \( S(T^+) = 0 \), and this contradicts the assumption \( S(T^+) < 0 \). Hence for any \( t \in [t_0, \infty) \), we have \( S(t) \geq 0 \). Similarly, we can prove that \( E(t) \geq 0, I(t) \geq 0, V_i(t) \geq 0, I_2(t) \geq 0, V_2(t) \geq 0, I_{26}(t) \geq 0, R(t) \geq 0, \) and \( D(t) \geq 0. \)

3.1.3. Equilibrium points and stability analysis

To determine the equilibrium points of the proposed fractional-order model equate the right-hand side of the modelling equations to zero:

\[
d^\alpha_t S = -\left( \frac{\beta_1(t)c_1}{N} S(t) \right) E(t) + \sum_{i=1}^{n} \left( \frac{\beta_i(t)c_{2i}}{N} V_i(t) \right) \}
\]

\[
d^\alpha_t E = \left( \frac{\beta_1(t)c_1}{N} E(t) \right) S(t) + \frac{\beta_2(t)c_2}{N} S(t) I(t) - \left( \frac{\beta_1(t)c_1}{N} S(t) + \frac{\beta_2(t)c_2}{N} S(t) I(t) \right)
\]

\[
d^\alpha_t I = \left( \frac{\beta_1(t)c_1}{N} I(t) \right) E(t) - \left( \frac{\beta_2(t)c_2}{N} I(t) \right) V_i(t) - \left( \frac{\beta_1(t)c_1}{N} I(t) \right)
\]

\[
d^\alpha_t V_i = \left( \frac{\beta_1(t)c_1}{N} V_i(t) \right) I(t) + \left( \frac{\beta_2(t)c_2}{N} V_i(t) \right) I(t) - \left( \frac{\beta_1(t)c_1}{N} V_i(t) \right) - \left( \frac{\beta_2(t)c_2}{N} V_i(t) \right)
\]

\[
d^\alpha_t E_{10} = \left( \frac{\beta_1(t)c_1}{N} E_{10}(t) \right) S(t) + \left( \frac{\beta_2(t)c_2}{N} E_{10}(t) \right) V_i(t) - \left( \frac{\beta_1(t)c_1}{N} E_{10}(t) \right)
\]

\[
d^\alpha_t I_{10} = \left( \frac{\beta_1(t)c_1}{N} I(t) \right) E_{10}(t) - \left( \frac{\beta_2(t)c_2}{N} I(t) \right) V_i(t) - \left( \frac{\beta_1(t)c_1}{N} I(t) \right)
\]

3.2. Fractional-order stochastic model

Stochastic models are even used to discuss complex systems with a high degree of randomness due to applied external noise. As we talk about a new virus with a stochastic, nature, we developed the fractional-order stochastic model to produce realistic curves describing COVID-19 behaviour using different valid types of vaccination doses during the same viral wave. Through this modelling approach, an additional novelty is added to our article. The fractional-order stochastic model is generated by starting with the fractional-order deterministic model equations and then adding white noise diffusion terms, satisfying the Wiener process, to all these equations. The newly diffusion terms cover more probabilistic scenarios with a wide
range of probable viral wave dynamics. The developed model equations after adding the new stochastic terms are as explained in Eqs. (22)–(32).

$$d_i^2 S = -\left(\frac{\beta(t)_{ci}}{N} S(t)I(t) + \sum_{i=1}^{n} (E_i(t) + E_2(t)) \right)$$

$$+ \frac{\beta(t)_{c2}}{N} S(t)\left[I(t) + \sum_{i=1}^{n} (I_i(t) + I_2(t)) + \sum_{i=1}^{n} \delta_i(t) \right]$$

$$- \sigma_{EF} E(t)\frac{dB(t)}{dt}$$

$$d_i^2 E = \left(\beta(t)_{ci} E(t)I(t) + \frac{\beta(t)_{c2}^2}{N} I(t)S(t) - \chi E(t) E(t) \right)$$

$$- \beta_i(t)_{c2} \sum_{i=1}^{n} V_i(t)E(t)) + k_i \sigma_{EF} E(t)\frac{dB(t)}{dt}$$

$$d_i^2 I = (\chi E(t) E(t) - [\gamma(t) + \mu(t)] I(t)) - \sigma_{RI} I(t)R(t)\frac{dB(t)}{dt}$$

$$d_i^2 V_{li} = \left(\delta_i(t) S(t) + \frac{\beta_i(t)_{c1}}{N} E_i(t)I(t) \right)$$

$$- \beta_i(t)_{c2} \left(E_i(t) + 0.5I_2(t)\right) V_{li}(t) - \delta_2 V_{li}(t) + \frac{\xi_{li} - \mu_i}{a} V_{li}(t)$$

$$- \sigma_{EV_i} I_i(t)\frac{dB(t)}{dt}$$

$$d_i^2 E_{li} = \left(\beta_i(t)_{c1} E_i(t)I(t) + \frac{\beta_i(t)_{c2}^2}{N} I_i(t)S(t) \right)$$

$$+ \beta_i(t)_{c2} \left(E_i(t) + 0.5I_2(t)\right) V_{li}(t) - \frac{\xi_i - \mu_i}{a} V_{li}(t)$$

$$- \sigma_{EF_i} V_{li}(t)\frac{dB(t)}{dt}$$

$$d_i^2 I_i = (\chi E_i(t) E_i(t) - [\gamma_i(t) + \mu_i(t)] I_i(t))$$

$$- \sigma_{RI_i} I_i(t)R(t)\frac{dB(t)}{dt}$$

$$\xi_{li} V_{li} = \left(\delta_2 V_{li}(t) + \frac{\beta_i(t)_{c4}}{N} V_{li}(t) \right)$$

$$- \beta_i(t)_{c3} \left(E_i(t) + 0.5I_2(t)\right) V_{li}(t) - \frac{\xi_{li} - \mu_i}{a} V_{li}(t)$$

$$- \sigma_{EV_i} I_i(t)\frac{dB(t)}{dt}$$

$$d_i^2 E_{li} = \left(\beta_i(t)_{c1} E_i(t)I(t) + \frac{\beta_i(t)_{c2}^2}{N} I_i(t)S(t) \right)$$

$$+ \beta_i(t)_{c2} \left(E_i(t) + 0.5I_2(t)\right) V_{li}(t) - \frac{\xi_i - \mu_i}{a} V_{li}(t)$$

$$+ k_i \sigma_{EF_i} E_i(t)\frac{dB(t)}{dt}$$

$$d_i^2 I_i = (\chi E_i(t) E_i(t) - [\gamma_i(t) + \mu_i(t)] I_i(t))$$

$$- \sigma_{RI_i} I_i(t)R(t)\frac{dB(t)}{dt}$$

$$d_i^2 R = \left(\frac{\nu(t)}{a} \left[I(t) + \sum_{i=1}^{n} (\xi_i(t) + 1) I_i(t) + (\xi_2(t) + 1) I_2(t) \right] \right)$$

$$+ \sum_{i=1}^{n} \frac{\xi_{li} - \mu_i}{a} V_{li}(t) + \sum_{i=1}^{n} \frac{\xi_{li} - \mu_i}{a} V_{li}(t) + k_4 [\sigma_{RI} I(t)$$

$$+ \sigma_{RI_i} I_i(t) + \sigma_{RI_2} I_2(t)] R(t)\frac{dB(t)}{dt}$$

$$d_i^2 D = \left(\frac{\mu(t)}{a} \left[I(t) + \sum_{i=1}^{n} (\xi_i(t) + 1) I_i(t) + (\xi_2(t) + 1) I_2(t) \right] \right)$$

$$+ \sigma_{RI} I_i(t) + \sigma_{RI_2} I_2(t)\frac{dB(t)}{dt}$$

$$+ \sigma_{Ei} E_i(t) + \sigma_{EF_i} E_i(t)\frac{dB(t)}{dt}$$

$$= -\left(\frac{\beta_i(t)_{c1}}{N} E_i(t)I_i(t) + \frac{\beta_i(t)_{c2}^2}{N} I_i(t)S_i(t) \right)$$

$$- \beta_i(t)_{c2} \left(E_i(t) + 0.5I_2(t)\right) V_{li}(t) - \frac{\xi_i - \mu_i}{a} V_{li}(t)$$

$$- \sigma_{EV_i} I_i(t)\frac{dB(t)}{dt}$$
and then
\[ \begin{align*}
- \left( \frac{\beta_{c1}}{N} \left\{ [E + \sum_{i=1}^{n} (E_i + E_{i2})] \right\} \right) &> 0, \\
\left( \frac{\beta_{c3}}{N} \left\{ [S - \sum_{i=1}^{n} \left( I_{ii} + I_{i2} \right)] \right\} \right) &> 0, \\
\left( \frac{\beta_{c1}}{N} E - \frac{\beta_{c3}}{N} \left( E_{i2} + 0.5I_{i2} \right) \right) &> 0,
\end{align*}\]

Hence, the fractional stochastic model is stable to 0, when
\[ \begin{align*}
- \left( \frac{\beta_{c1}}{N} \left\{ [E + \sum_{i=1}^{n} (E_i + E_{i2})] \right\} \right) &> 0, \\
\left( \frac{\beta_{c3}}{N} \left\{ [S - \sum_{i=1}^{n} \left( I_{ii} + I_{i2} \right)] \right\} \right) &> 0, \\
\left( \frac{\beta_{c1}}{N} E - \frac{\beta_{c3}}{N} \left( E_{i2} + 0.5I_{i2} \right) \right) &> 0,
\end{align*}\]

Using both the fractional deterministic and stochastic models, we can study the vaccination effect on countries by comparing currently predicted waves using these two models and previous waves before vaccines. For real comparisons, the actual number of vaccination doses and types should be considered because different vaccinations have a different number of doses and efficiencies. In our models, we preferred that vaccination will not limit infections completely but it will reduce the infection rate according to vaccine efficiency and the number of shots.

Table 1: Main types of vaccines used in Egypt during the fourth viral wave.

| Vaccine Name | First dose efficiency | Second dose efficiency | Ref. |
|--------------|-----------------------|------------------------|------|
| AstraZeneca  | 48.7%                 | 74.5%                  | [17–19,48] |
| Sinopharm    | 50%                   | 79%                    | [17,20,48] |
| Johnson & Johnson | 67%     | –                     | [19,48] |

Table 2: Dynamic models’ estimated parameters for Egypt.

| Parameter | Estimated value | Parameter | Estimated value |
|-----------|----------------|-----------|----------------|
| \( \beta_{i0} \) | 0.0177 | \( \beta_{i0} \) | 1.24E-3 |
| \( \beta_{i1} \) | 0.07 | \( \beta_{i1} \) | 6.2E-4 |
| \( \epsilon_{i1} \) | 3.3 | \( \epsilon_{i1} \) | 9.57E-6 |
| \( \epsilon_{i2} \) | 2.9E-6 | \( \epsilon_{i2} \) | 9.57E-6 |
| \( J \) | 5.5E-3 | \( J \) | 14 |
| \( i_{11} \) | 28 | \( i_{11} \) | 56 |
| \( s_{0} \) | 0.048 | \( s_{0} \) | 0.014 |
| \( s_{1} \) | 1E-3 | \( s_{1} \) | 1E-3 |
| \( s_{2} \) | 0.011 | \( s_{2} \) | 0.088 |
| \( s_{3} \) | 0.011 | \( s_{3} \) | 3E-3 |
| \( \delta_{1} \) | 0.01 | \( \delta_{1} \) | 0.487 |
| \( \delta_{11} \) | 5E-5 | \( \delta_{11} \) | 0.67 |
| \( \delta_{12} \) | 1.8E-5 | \( \delta_{12} \) | 0.79 |
| \( \delta_{13} \) | 5E-5 | \( \delta_{13} \) | 0 |
| \( \delta_{14} \) | 0.03 | \( \delta_{14} \) | 0.035 |
| \( \delta_{15} \) | 2E-5 | \( \delta_{15} \) | 0.99 |
| \( \delta_{16} \) | 0.0035 | \( \delta_{16} \) | 0.97 |
| \( \delta_{17} \) | 7E-3 | \( \delta_{17} \) | 0.03 |
| \( \delta_{18} \) | 0.025 | \( \delta_{18} \) | 0.025 |

population on 1st February 2021 but at a low rate. While the total vaccination doses presented to the Egyptian citizens didn’t exceed 5 million doses till 10th July 2021 with an average daily consumed doses of 31,250 units [48,49]. Starting from 20th August 2021, the daily doses consumption rate exceeds 148,000 units. This means an effective vaccination rate with a strong effect on controlling the fourth virus wave assumed started on 14th September 2021. The total number of doses that have been given to the Egyptian citizen in the first 45 days of the fourth viral wave exceeds 14 million doses [48,49]. Assuming the same average daily vaccination till the fourth viral ends. This means that the number of consumed doses will exceed 50 million doses. The main types of vaccines used in Egypt and their efficiencies are as indicated in Table 1. There exists other used vaccines types but with very low daily rates because of not receiving large numbers of them. We applied our two generated models on the fourth viral wave of Egypt to predict the total number of exposed, infected, recovered and deaths under valid vaccine doses. All transmission rates are assumed that be all controlled under vaccines and don’t exceed their values in the Egyptian second viral wave [7].
We used the transmission rate values of the previous wave in Ref. [7] as maximum values to estimate the current wave transmission rates. To begin making trial and error estimations and reach estimated coefficients in Table 2, we used the interval defined with the maximum bound of previous wave rates and minimum bound with 50% of the maximum bound. The estimated parameters values are reached through trial-and-error random variations with specified initial intervals and with at least 0.7 determination coefficient value as a stopping criterion for both daily infections and daily deaths. In all dynamic models, the fourth viral wave is assumed to start in Egypt on 10th September 2021 (day 0). To solve the developed models numerically we used fractional Euler’s method [27] and Euler- Murayama method [28].

4. Results and discussion

The fractional-order deterministic model of orders 0.98, 0.96 and 0.94 is applied to the Egyptian case study to describe the fourth viral wave and time of the peak of each class. Through this study, the vaccination effect of controlling daily infections and deaths will be quantified. The valid vaccination doses for the Egyptian government during the fourth viral wave, to give to its population, is assumed to be 50 million doses. We assumed that 43% of the valid doses are of Sinopharm type and similarly for AstraZeneca type with 43% share. The valid vaccines for Johnson & Johnson type are assumed 13%. We assumed that the fourth wave began on 10th September 2021 (day 0). At day 0, the population (N) and initial values of daily infected (I), recovered (R) and deaths (D) are taken from references [48,49]. The proposed models’ parameters are assumed and estimated by trial and error based on the second viral wave transmission rates [7]. We used Euler’s method with step time equals 0.001 days to solve the different fractional-order deterministic models. After that, we applied stochastic fractional-order models of orders 0.98 and 0.96 to reach more realistic wave dynamics. The solution of the stochastic fractional-order models is justified using the Euler- Murayama method with a number of paths equals 20 and discretization of time equals 0.002 days, then the average path solution is taken. We used MATLAB program version 2018b to simulate our models’ dynamics.

Daily Susceptible class variation with time under using different dynamic models is as presented in Fig. 2a. The class is

![Fig. 2](https://example.com/f2.png)

a. Daily susceptible individuals

b. Daily main exposed individuals

c. Daily main infected individuals

Fig. 2 Daily susceptible, main exposed and main infected using different dynamical models.
decreasing with time and has minimum values when predicted using the fractional deterministic model of order 0.94. Fig. 2b shows the predicted values of the daily main exposed class. The high vaccination rate at the fourth wave beginning has a strong effect on this class. As it makes the class starts to decrease with time after the viral wave day no. 42. The stochastic fractional model of order 0.96 predicts the highest peak value of this class with a value equal to 2336. As the main exposed class is over-damped under vaccinations, this reflects on the predicted curve shapes of daily main infected individuals. As shown in Fig. 2c, the main infected class predicted viral wave peak equals 1112 in viral wave day no. 79 using the stochastic fractional model of order 0.96. The minimum class peak predicted values equals 934 in viral wave 46 using the deterministic model of order 1. The stochastic models of orders 0.98 and 0.96 give different curve peaks and with different times of actions than the deterministic models, so they justify more realistic curve shapes for this class.

The estimated curves of daily first vaccinated people using AstraZeneca type, using different deterministic and stochastic models, are as indicated in Fig. 3. The highest estimated curve values are reached using the stochastic fractional model of order 0.96 with a peak daily vaccination rate equals 126,000 doses in viral wave day no. 27. This is mainly based on our assumptions that the AstraZeneca type will represent 43% of valid doses. Considering AstraZeneca type first dose efficiency, a part of first vaccinated people will be exposed to the virus and then move to the infection area before taking the second dose. Reached results for the estimated first vaccinated exposed and infected due to using AstraZeneca type are presented in Fig. 3. The peak of the daily first vaccinated exposed will take values between 1077 and 1577 person. The deterministic fractional models give the same time of peak for the first vaccinated class at viral wave day no. 131 while the stochastic models of orders 0.98 and 0.96 give this class peak at viral wave day no 128 and 91 respectively. As indicated in Fig. 3, The estimated infected individual who has taken the first dose of AstraZeneca type will reach a maximum peak value equal to 168 infections in viral wave day no. 155 using the fractional model of order 0.94. In the case of the stochastic model of order 0.96, the estimated peak of daily infections will reach 110 people in viral wave day no. 129 while reaching a peak of 156 in viral wave day 163 using the same model of order 0.98. Fig. 4 indicates the estimated curves for daily first vaccinated people using Johnson and Johnson type using different deterministic and stochastic models. The maximum estimated peak for the daily vaccinated people using this type will equal 42,700 in day no. 45 using the deterministic model of order 0.94. Because this type is a one-shot type with high efficiency, then the numbers of daily exposure and infections taken this type of vaccine will be limited. As shown in Fig. 4, the estimated peak range of daily exposure using this type will be between 204 and 305 persons while the daily peak on infections will be between 7 and 12 people. Based on these curves, the one-shot vaccine type is more effective in controlling and reducing daily exposure and daily infections than the two-shot vaccine type. In the case of using the Sinopharm vaccine type, Fig. 5 shows the assumed daily first vaccinated curves using different dynamical models. The estimated maximum peak of daily first vaccinated will be 114,000 doses using the fractional model of order 0.94. Because this vaccination type has a first dose efficiency higher than the Aztrazeneca type
the peak of estimated daily exposure and infected individuals using this type will be lower. Using results in Fig. 5, the estimated peak range of daily exposure using this type will be between 821 and 1195 persons while the daily peak on infections will be between 72 and 101 people.

The estimated curves of second daily vaccinated individuals using AstraZeneca and Sinopharm are as indicated in Figs. 6 and 7. Using different dynamical models, the daily exposed class for persons who took the second dose of Astrazeneca type will have a peak range between 202 and 301 persons while taking a range between 129 and 161 persons in case of taking Sinopharm type. The highest estimated peak value of the exposed persons is reached by the stochastic fractional model of order 0.98 in AstraZeneca type and by the fractional model of order 0.94 in Sinopharm type. The estimated daily second vaccinated infections using both the stochastic and the deterministic models are presented in Figs. 6 and 7. The highest estimated peak value of the infected persons of Astrazeneca type will have a peak range between 5 and 11 persons while taking a range between 4 and 6 persons in case of taking Sinopharm type. The estimated daily second vaccinated infections in the case of Sinopharm type are lower than those produced by AstraZeneca type because Sinopharm type has a higher second dose efficiency.

Now to study the effect of using vaccination with high daily rates on the fourth viral wave, we compared the estimated stochastic and deterministic curves of the total daily infected, total daily deaths and total daily vaccinations using different models with both current and previously available data of Egypt [48,49]. Reached results are as indicated in Fig. 8. Firstly, through comparing actual current data of Egypt of actual daily deaths and actual daily confirmed cases, the
stochastic fractional-order dynamical models give more realistic scenarios than other remaining deterministic models. From obtained results, the stochastic fractional model of order 0.98 is the best model to predict the daily infections of the fourth viral wave and after it came the stochastic fractional model of order 0.96. From Fig. 8b, the best model to predict daily deaths is the stochastic model of order 0.96 and after it came the stochastic model of order 0.98. In Fig. 8c, the total daily vaccinations using the different dynamical models have very near curve shapes because the cumulative daily vaccinations are assumed to equal 50 million doses. Using current data of daily vaccinations of Egypt, all estimated curves using different models can be considered with the same accuracy of prediction. As the stochastic fractional model of order 0.98 is the best model to describe the current daily infections, we compared it with previous wave data to quantify the vaccination effect. High vaccination rate results in decreasing the peak of daily infections from 1418 in the previous wave to 1067 in the current wave. The cumulative daily infections will be decreased from 160,845 to 125,690 people. In the case of daily deaths and using the stochastic model of order 0.96, the current wave predicted peak will be decreased from 64 to 58 people. While the cumulative daily deaths decreased from 9163 to 8118 people.

5. Conclusions

In this paper, we developed a fractional dynamical model and stochastic fractional dynamical model to evaluate COVID-19 dynamics under wide vaccination spread. By using fractional derivative and stochastic white noise, the two proposed models studied some realistic scenarios for different expected classes’ dynamics. In the same viral wave, the proposed models considered daily infections of first and second vaccinated people. One-dose and two-dose vaccine combinations and each dose efficiency was considered in the suggested models to quantify the effect of each vaccine type. The numerical solution of the proposed models and simulation to study the Egyptian case study is valid through using fractional Euler’s method and
Euler-Maruyama method on MATLAB software program. From the results, a high vaccination rate will result in decreasing the peak of daily infections from 1418 in the previous wave to 1067 in the current wave. The cumulative daily infections will be decreased from 160,845 to 125,690 people. In the case of daily deaths and using the stochastic model of order 0.96, the current wave predicted peak will be decreased from 64 to 58 people. Based on the proposed models’ predicted scenarios of daily exposure, infections and deaths, the public health officials can move on to eradicate this contagious disease considering the negative aspects of vaccines with low efficiency in producing daily infections again. In our future work, we will expand our models to study the next waves after increasing the number of doses shots and considering reinfection of the recovered individuals.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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