GROUP SCHEMES WITH $\mathbb{F}_q$-ACTION

THOMAS POGUNTKE

Abstract. We prove an equivalence of categories, generalizing the equivalence between commutative flat group schemes in characteristic $p$ with trivial Verschiebung and Dieudonné $\mathbb{F}_p$-modules to group schemes with an action of $\mathbb{F}_q$.

Contents

1. Introduction 1
2. Preliminaries on Group Schemes 4
3. Group Schemes of Additive Type 7
4. The Category of $\mathbb{F}_q$-Shtukas 10
5. $\mathbb{F}_q$-Actions on Group Schemes 11
6. The Functors $G$ and $M$ 14
7. Quasi-Balanced Group Schemes 17
8. The Infinite Case 20
References 23

1. introduction

Let $p$ be a prime and let $k$ be a field of characteristic $p$. Denote by $\text{Gr}_k^+$ the category of affine commutative group schemes over $k$ which can be embedded into $G_a^N$ for some set $N$. We assign to $G \in \text{Gr}_k^+$ its Dieudonné $\mathbb{F}_p$-module $M(G) = \text{Hom}_{\text{Gr}_k^+}(G, G_a)$, with the obvious left module structure over $\text{End}_{\text{Gr}_k^+}(G_a) \cong k[F]$, the non-commutative polynomial ring with $F \lambda = \lambda^p F$ for $\lambda \in k$.

These Frobenius modules completely classify group schemes of the above type, as shown by the following theorem ([2], IV, §3, 6.7).

Theorem 1.1. The contravariant functor $M$ defines an anti-equivalence of categories

$$M: \text{Gr}_k^+ \to k[F] \text{-Mod}. \quad (1.1)$$

Under the above duality, group schemes of finite presentation correspond to finitely generated $k[F]$-modules, and finite group schemes to finite-dimensional $k$-vector spaces.

The above result allows us to describe the structure of our category over a perfect field, and its simple objects if the base is algebraically closed ([2], IV, §3, 6.9).

Theorem 1.2. Let $k$ be a perfect field. Every algebraic $G \in \text{Gr}_k^+$ can be written as a product

$$G \cong G_a^n \times \pi_0(G) \times H,$$

where $n \in \mathbb{N}$, $H$ is a product of group schemes of the form $\alpha_{p^r}$, and $\pi_0(G)$ is an étale sheaf of $\mathbb{F}_p$-vector spaces. If $k$ is algebraically closed, we have

$$\pi_0(G) \cong (\mathbb{F}_p)^m, \ m \in \mathbb{N}.$$
On the other hand, let $S$ be a scheme of characteristic $p$. Consider the category $\text{gr}_S^{+\vee}$ of finite flat group schemes over $S$ of height $\leq 1$ (i.e., killed by the Frobenius). Let $p$-Lie$_S$ denote the category of finite flat $\mathcal{O}_S$-Lie algebras. Then we have the following classification theorem ([7], 7.5).

**Theorem 1.3.** The functor

$$\mathcal{L}: \text{gr}_S^{+\vee} \to p$$.Lie$_S$, $G \mapsto \text{Lie}(G),$$

defines an equivalence of categories.

Our main results generalize Theorem 1.1, and reduce (for “$q = p$”) to Theorem 1.3 via Cartier duality, respectively. Moreover, we formulate two conjectures, under which they define an equivalence of categories.

**Theorem 1.4.** The category of finite flat group schemes over $S$ of height $\leq 1$ decomposes into eigenspaces for the $F$-action. We construct an analogue of the contravariant functor (1.1),

$$\mathcal{M}_q = \mathcal{M}: F_q\text{-gr}_A^+ \to F_q\text{-sht}_A, \quad G \mapsto \text{Hom}_{F_q\text{-gr}_A^+}(G, G_a).$$

We also give the construction of a functor in the other direction,

$$G_q = \mathcal{G}: F_q\text{-sht}_A \to F_q\text{-gr}_A^+,$$

which is fully faithful and left-adjoint to $\mathcal{M}$. However, $G_q$ does not define an equivalence of categories for $q \neq p$. We describe a full subcategory $F_q\text{-gr}_A^{+,b}$ of balanced group schemes in $F_q\text{-gr}_A^+$, and prove that it is the essential image of $\mathcal{G}$. Let $G = \text{Spec}(B_G) \in F_q\text{-gr}_A^+$. The space of primitive elements in the affine algebra of $G$ decomposes into eigenspaces for the $F_q^+$-action as

$$\text{Prim}(B_G) = \bigoplus_{n=0}^{r-1} \text{Prim}_{F_q^n}(B_G).$$ (1.2)

Now $G$ is called balanced, if the $p$-power maps $f_t : \text{Prim}_{F_q^n}(B_G) \to \text{Prim}_{F_q^{n+1}}(B_G), \quad x \mapsto x^p$, are isomorphisms for all $0 \leq t < r - 1$. Note that when $q = p$, we have $F_p\text{-gr}_A^{+,b} = \text{gr}_A^+$. Our first main result is:

**Theorem 1.5.** The functor $\mathcal{G}: F_q\text{-sht}_A \to F_q\text{-gr}_A^{+,b}$ defines an anti-equivalence of categories with quasi-inverse $\mathcal{M}$.

Our definition of the balanced subcategory of $F_q\text{-gr}_A^+$ is inspired by Raynaud’s paper [12]. He considers finite commutative group schemes $G$ with an action of $F_q$, and the decomposition of the augmentation ideal into eigenspaces for the $F_q^+$-action,

$$I_G = \bigoplus_{j=1}^{g-1} I_j,$$

similarly to (1.2). Note that all summands $I_j$ are finite flat $A$-modules. Raynaud imposes the condition that $\text{rk}(I_j) = 1$, for all $j$. We define a group scheme $G \in F_q\text{-gr}_A^+$ to be quasi-balanced if $\text{rk}(I_j)$ is the same for all $j$. This turns out to be almost the same as being balanced. The following theorem is our second main result.

**Theorem 1.6.** Let $G \in F_q\text{-gr}_A^+$. If $G$ is balanced, then it is quasi-balanced. For $q \neq 4$, the converse holds.
Finally, we consider the question whether \( \mathcal{M}: \mathbb{F}_q\text{-Gr}_A^{+,b} \to \mathbb{F}_q\text{-Sht}_A \) defines an equivalence of categories in general. In order to make sense of this, we have to assume the following.

**Conjecture 1.6.** For any \( G \in \mathbb{F}_q\text{-Gr}_A^+ \), the \( A \)-module \( \mathcal{M}(G) \) is flat.

In fact, we only need a somewhat weaker statement. Also, recall from above that it holds for finite \( G \). Let us moreover assume the following key statement.

**Conjecture 1.7.** For \( G \in \mathbb{F}_q\text{-Gr}_A^+ \) such that there is an embedding \( G \hookrightarrow \mathbb{G}_a^N \) with \( N \in \mathbb{N} \), the induced morphism \( \mathcal{M}_p(\mathbb{G}_a^N) = A[F]_p^N \to \mathcal{M}_p(G) \) is surjective.

By Theorem 1.1 this holds at least if \( A = k \) is a field. We obtain the conjectural main result that \( \mathcal{M} \) defines an equivalence of categories if we restrict ourselves to finitely presented group schemes and finitely generated \( A[F^r] \)-modules, respectively. In particular, we obtain the following analogue of Theorem 1.2.

**Theorem 1.8.** Let \( k \) be a perfect field. Every finitely presented \( G \in \mathbb{F}_q\text{-Gr}_k^{+,b} \) is a product
\[
G \cong \mathbb{G}_a^m \times \pi_0(G) \times H,
\]
with \( n \in \mathbb{N} \) and \( H \) a product of group schemes of the form \( \alpha_q \cdot r \), and where \( \pi_0(G) \) is an étale sheaf of \( \mathbb{F}_q \)-vector spaces. If \( k \) is algebraically closed, \( \pi_0(G) \cong (\mathbb{F}_q)^m \), for some \( m \in \mathbb{N} \).

Theorem 1.3 has an interesting history. In his article [3], §2, Drinfel’d defines the functor \( \mathcal{G}: \mathbb{F}_q\text{-sht}_A \to \mathbb{F}_q\text{-gr}_A^+ \) and shows that it is fully faithful and exact. Furthermore, he proves that the étale group schemes in \( \mathbb{F}_q\text{-gr}_A^+ \) lie in the essential image of \( \mathcal{G} \).

In Laumon’s book [9], App. B, he claims that \( \mathbb{F}_q\text{-sht}_k \) is anti-equivalent to \( \mathbb{F}_q\text{-gr}_k^+ \), where \( k \) is a perfect field of characteristic \( p \). However, \( \mathbb{F}_q\text{-gr}_k^+ \) is not an abelian category for \( q \neq p \), and \( \alpha_p \) is of \( \mathbb{F}_q \)-additive type but not balanced. This error was pointed out to us by Hartl.

Laumon’s argument is sufficiently detailed to locate the mistake in his reasoning.

In [14], 1.7, Taguchi gives a (rather brief) proof of Theorem 1.3. He describes \( \mathbb{F}_q\text{-gr}_k^{+,b} \) by a condition on the order of the group schemes however, which precludes a generalization to the category \( \mathbb{F}_q\text{-Gr}_A^+ \) as above.

Abrashkin [1] considers a category \( \text{DGr}^*(\mathbb{F}_q)_A \), based on a definition of Faltings [4]. Roughly, the \( \mathbb{F}_q \)-action on \( G \in \mathbb{F}_q\text{-gr}_A^+ \) is strict, if \( G \) has a deformation \( G' \) (which is then universal with respect to its \( \mathbb{F}_q \)-action) such that \( \mathbb{F}_q \) acts via scalar multiplication on the associated representative of the cotangent complex.

In [4], 2.3, Abrashkin constructs an equivalence of categories \( D_q: \mathbb{F}_q\text{-sht}_A \xrightarrow{\sim} \text{DGr}^*(\mathbb{F}_q)_A \).

Moreover, he shows in [4], 2.3.2., that a group scheme with strict \( \mathbb{F}_q \)-action is balanced. Hence, the obvious functor \( \text{DGr}^*(\mathbb{F}_q)_A \to \mathbb{F}_q\text{-gr}_A^{+,b} \) is well-defined, and it is clear from the constructions that the following diagram commutes.

\[
\begin{array}{ccc}
\mathbb{F}_q\text{-sht}_A & \xrightarrow{D_q} & \text{DGr}^*(\mathbb{F}_q)_A \\
\downarrow{\mathcal{G}_q} & & \\
\mathbb{F}_q\text{-gr}_A^{+,b}
\end{array}
\]

Let us summarize the structure of the paper. In [2] we provide some basic definitions and results on group schemes. Section [3] specializes to group schemes of additive type, and culminates in the proof of Theorem 1.3 in the crucial case \( q = p \). Some details are postponed to avoid repetition and streamline the argument.

In [4] and [5] we define the categories \( \mathbb{F}_q\text{-sht}_A \) and \( \mathbb{F}_q\text{-gr}_A^+ \), respectively, and study their internal structure. Section [6] is concerned with the construction of the functors \( \mathcal{M} \) and \( \mathcal{G} \), a more detailed analysis of their properties, and the proof of Theorem 1.3.

In [7] we introduce quasi-balanced group schemes, and compare the two balance conditions. Finally, Section [8] concerns the question what we can still say in the case of infinite group schemes.
2. Preliminaries on Group Schemes

Let $p$ be a prime number, and $S$ a scheme of characteristic $p$.

**Definition 2.1.** For an $S$-scheme $X$, denote by $\text{Frob}_X : X \to X$ its Frobenius endomorphism. Let $X^{(p)} = X \times_{S, \text{Frob}_S} S$. The relative Frobenius $F_X : X \to X^{(p)}$ of $X$ is defined by the following diagram with cartesian square.

\[
\begin{array}{ccc}
X & \xrightarrow{F_X} & X^{(p)} \\
\downarrow \text{Frob}_X & & \downarrow \text{Frob}_S \\
X & \xrightarrow{\text{Frob}_S} & S,
\end{array}
\]

In particular, $F_X$ is a morphism of $S$-schemes.

**Definition 2.2.** We denote by $\text{Gr}_S$ the category of affine commutative flat group schemes over $S$, and its full subcategory of finite group schemes by $\text{gr}_S$.

**Convention:** All of our considerations take place locally on $S$. To emphasize when we assume $S = \text{Spec } A$, we will write $\text{Gr}_A = \text{Gr}_S$. We also fix $G = \text{Spec } B_G$ as a notation.

**Definition 2.3.** We write $\text{Hopf}_A$, resp. $\text{hopf}_A$, for the opposite category of $\text{Gr}_A$, resp. $\text{gr}_A$.

**Definition 2.4.** Let $G = \text{Spec } B_G \in \text{Gr}_A$. Consider the symmetrization morphism

\[ s : B_G^{\otimes p} \to \text{TS}^p(B_G), \quad x_1 \otimes \ldots \otimes x_p \mapsto \sum_{\pi \in S_p} x_{\pi(1)} \otimes \ldots \otimes x_{\pi(p)}, \quad (2.1) \]

where $\text{TS}^p(B_G) := (B_G^{\otimes p})^{S_p}$. Since $G$ is flat, $x \mapsto x^{\otimes p}$ induces an isomorphism

\[ \sigma^*_p B_G := B_G \otimes_{A, \sigma_p} A \xrightarrow{s} \text{TS}^p(B_G)/s(B_G^{\otimes p}), \]

see [2], IV, §3, 4.1. Here, $\sigma_p$ denotes the Frobenius of $A$, and so we have $G^{(p)} = \text{Spec}(\sigma^*_p B_G)$, which is by the above a closed subscheme of $S^p G := \text{Spec}(\text{TS}^p(B_G))$. The Verschiebung of $G$ is then defined as the composition

\[ V_G : G^{(p)} \longrightarrow S^p G \xrightarrow{\text{mult}} G, \]

where $\text{mult}$ is the $p$-fold multiplication on $G$, which factors over $S^p G$, since $G$ is commutative.

**Remark 2.5.** We have $F_G \circ V_G = p \cdot \text{id}_{G^{(p)}}$, and $V_G \circ F_G = p \cdot \text{id}_{G^{(p)}}$, by [2], IV, §3, 4.6. On affine algebras, $V_G$ acts by taking $p$-th “copowers”. In this sense, it is dual to the (relative) Frobenius, which we make precise below. The name (German for “shift”) comes from the Verschiebung on Witt (co-)vectors, where it acts as an index shift (cf. [6], III.3.1).

**Definition 2.6.** For $G \in \text{Gr}_A$, let $\eta : B_G \to A$ be the augmentation – or counit – of $B_G$, given by the unit section of $G$. The augmentation ideal of $G$ is defined by $I_G = \ker(\eta)$.

**Remark 2.7.** The short exact sequence

\[ 0 \to I_G \to B_G \xrightarrow{\eta} A \to 0 \]

is split on the right by the unit $\varepsilon : A \to B_G$ of $B_G$, so that in fact $B_G = A \oplus I_G$. In particular, the $A$-module $I_G$ is flat.
Definition 2.8. Let $G \in \text{Gr}_A$. The space of primitive elements in $B_G$ is defined by

$$\text{Prim}(B_G) := \{ x \in I_G \mid \Delta(x) = x \otimes 1 + 1 \otimes x \},$$

where $\Delta$ is the comultiplication on $B_G$, i.e. the map induced by the multiplication of $G$. The subgroup of group-like elements of $B_G$ is defined by

$$\text{Grp}(B_G) := \{ x \in B_G \mid \Delta(x) = x \otimes x, \; \eta(x) = 1 \},$$

where $\eta: B_G \to A$ is the counit of $B_G$.

Example 2.9. The group structure on $G_a = \text{Spec} A[x]$ is defined by $x \in \text{Prim}(B_{G_a})$. This also yields for $G \in \text{Gr}_A$ that

$$\text{Prim}(B_G) \cong \text{Hom}(G, G_a)$$

by the universal property of the polynomial algebra.

Remark 2.10. Let $G \in \text{Gr}_A$ and $x \in I_G$. Then

$$\Delta(x) \equiv x \otimes 1 + 1 \otimes x \mod I_G \otimes I_G,$$

cf. [15], 2.3. This explains the name “primitive element”.

Example 2.11. Let $x \in B_G$. If $x \in \text{Prim}(B_G)$, we have

$$V_G^a(x) = x \otimes 1 \otimes \ldots 1 + \ldots 1 \otimes x = \frac{1}{(p - 1)!} s(x \otimes 1 \otimes \ldots 1) \equiv 0,$$

where $s$ is the morphism [2.1]. On the other hand, if $x \in \text{Grp}(B_G)$, then

$$V_G^a(x) = x^{\otimes p} \equiv x \otimes x_{p1}.$$

Proposition 2.12. Let $B, C$ be Hopf algebras over $A$. Then we have

$$\text{Prim}(B \otimes C) = \text{Prim}(B) \otimes 1 + 1 \otimes \text{Prim}(C).$$

Proof. Let $G = \text{Spec} B$ and $H = \text{Spec} C$ be the corresponding group schemes. We have to show that the isomorphism

$$\text{Prim}(B) \times \text{Prim}(C) \sim \to \text{Hom}(G, G_a) \times \text{Hom}(H, G_a) \sim \to \text{Hom}(G \times H, G_a) \sim \to \text{Prim}(B \otimes C)$$

is given by $(y, z) \mapsto y \otimes 1 + 1 \otimes z \in \text{Prim}(B \otimes C)$. By definition, the image of $(y, z)$ in $\text{Hom}(G \times H, G_a)$ is induced by the unique Hopf algebra morphism $f: A[x] \to B \otimes C$ so that

$$x \in A[x] \ni x$$

$$y \in B \ni y \otimes 1 + 1 \otimes z \ni C \ni z$$

commutes. Here, the projection $\pi_B: B \otimes C \to B$ (and similarly $\pi_C$) is given by the injection $G \to G \times H, \ g \mapsto (g, 0)$. Explicitly, $\pi_B(b \otimes c) = b \epsilon(\eta(c))$, where $\eta: C \to A$ is the counit and $\epsilon: A \to C$ the unit of $C$. But now indeed,

$$\pi_B(f(x)) = \pi_B(y \otimes 1 + 1 \otimes z) = y \epsilon(\eta(x)) + \epsilon(\eta(z)) = y,$$

because $z \in I_H = \ker(\eta)$. We have $\pi_C(f(x)) = z$ by the same argument.

Definition 2.13. Let $G \in \text{Gr}_S$. The Cartier dual $G^\vee = \overline{\text{Hom}}(G, G_m)$ of $G$ is defined by

$$G^\vee: R \mapsto \text{Hom}_{\text{Gr}_A}(G \otimes R, G_m \otimes R)$$

as a functor of points.
Example 2.14. The group scheme $\alpha_p = \text{Spec } A[x]/(x^p)$, with $x \in \text{Prim}(B_{\alpha_p})$, is self-dual,

$$\alpha_p^\vee = \alpha_p.$$ 

On the other hand, consider the constant group scheme

$$\mathbb{F}_p = \text{Spec } A[x]/(x^p - x),$$

with $x$ primitive. Its dual is given by the roots of unity

$$\mu_p = \text{Spec } A[T]/(T^p - 1),$$

where $T$ is group-like, cf. e.g. [11], §5, p.11.

Lemma 2.15. For $G = \text{Spec } B_G \in \text{gr}_A$, we have $G^\vee = \text{Spec } B_G^\vee$, where $B_G^\vee = \text{Hom}_A(B_G, A)$. The (co-)multiplication is given by the transpose of the (co-)multiplication on $B_G$, i.e.

$$\nabla_{G^\vee} : B_G^\vee \otimes B_G^\vee \cong \text{Hom}(B_G \otimes B_G, A) \xrightarrow{\Delta_G^\vee} \text{Hom}(B_G, A),$$

$$\Delta_{G^\vee} : B_G^\vee \xrightarrow{\nabla_G} \text{Hom}(B_G \otimes B_G, A) \cong B_G \otimes B_G^\vee,$$

and similarly for the unit and counit,

$$\varepsilon_{G^\vee} : A \cong \text{Hom}_A(A, A) \xrightarrow{\eta_G^\vee} B_G^\vee,$$

$$\eta_{G^\vee} : B_G^\vee \xrightarrow{\varepsilon_G} \text{Hom}_A(A, A) \cong A.$$

Proof. (cf. [13], 3.2.2) By Yoneda, we have to show that there is a natural group isomorphism

$$\text{Hom}_{A-\text{Alg}}(B_G^\vee, R) \xrightarrow{3.2.2} \text{Hom}_{\text{Hopf}_A}(R[T^{\pm 1}], B_G \otimes R) \cong \text{Hom}_{G_{A-\text{Gr}}}(G \otimes R, \mathbb{G}_m \otimes R) = G^\vee(R).$$

First, consider $R = A$. By definition, $T$ is group-like, therefore we have an isomorphism

$$\text{Hom}_{\text{Hopf}_A}(A[T^{\pm 1}], B_G) \cong \text{Grp}(B_G).$$

On the other hand, we can consider

$$\text{Hom}_{A-\text{Alg}}(B_G^\vee, A) \subseteq \text{Hom}_A(B_G^\vee, A) \xrightarrow{\text{ev}_x} B_G$$

But $x \in B_G$ is group-like if and only if

$$\text{ev}_x(\alpha \beta) = (\nabla_A \circ (\alpha \otimes \beta) \circ \Delta_G)(x) = \alpha(x) \beta(x) = \text{ev}_x(\alpha) \text{ev}_x(\beta)$$

for all $\alpha, \beta \in B_G^\vee$. Thus the group-like elements in $B_G$ correspond precisely to the algebra morphisms in $\text{Hom}_A(B_G^\vee, A)$, as desired. The argument is stable under base change to $R$. □

Remark 2.16. In the case of Lemma 2.15, it is easy to see that Frobenius and Verschiebung are dual to one another, as Cartier duality exchanges multiplication and comultiplication. The same is indeed true for any $G \in \text{Gr}_A$ by [2], IV, §3, 4.9,

$$F_G^\vee = (V_G)^\vee, \text{ and } V_G^\vee = (F_G)^\vee. \quad (2.2)$$

The following result is crucial for our main theorem in the finite case.

Lemma 2.17. For $G = \text{Spec } B_G \in \text{gr}_A$, we have isomorphisms of $A$-modules

$$\text{Lie } G^\vee \cong \text{Hom}_A(I/I^2, A) \cong \text{Der}_A(B^\vee, A) \cong \text{Prim } B_G,$$

where $B^\vee = B_G^\vee$ and $I = \ker(\eta^\vee : B^\vee \to A)$ is the augmentation ideal of $G^\vee$.  

Proof. (cf. [6], I.8.3 ff.) Let $A(\varepsilon) = A[t]/(t^2)$ be the algebra of dual numbers, and $\pi : A(\varepsilon) \twoheadrightarrow A$ the projection. For $u \in G^\vee(A(\varepsilon))$, we have by definition

$$u \in \text{Lie}G^\vee = \ker G^\vee(\pi) \iff (B^\vee \twoheadrightarrow A(\varepsilon) \xrightarrow{\varepsilon} A) = \eta^\vee \iff u(I) \subseteq \varepsilon A(\varepsilon).$$

In that case, we get $u(I^2) \subseteq \varepsilon^2 A(\varepsilon) = 0$, hence an element in the tangent space of $G^\vee$,

$$\tau : I/I^2 \to A, \quad \alpha \mapsto \frac{u(\alpha)}{\varepsilon}.$$

The second isomorphism is just the universal property (cf. [15], 2.11)

$$\text{Der}_A(B^\vee, A) \cong \text{Hom}_{B^\vee}(\Omega^1_{B^\vee|A}, A) \cong \text{Hom}_{B^\vee}(I/I^2 \otimes_A B^\vee, A) \cong \text{Hom}_A(I/I^2, A),$$

where the $B^\vee$-module structure on $A$ is given by $\eta^\vee$. Finally, consider the natural pairing

$$\langle \cdot, \cdot \rangle : B^\vee \times B_G \to A, \quad (\alpha, x) \mapsto \alpha(x).$$

For $x \in B_G$, recalling Lemma 2.15 we have $x \in \text{Prim}(B_G)$ if and only if

$$\langle \alpha \beta, x \rangle = (\alpha \otimes \beta)((\Delta(x)) = (\alpha \otimes \beta)(x \otimes 1 + 1 \otimes x) = \langle \alpha, x \rangle \eta^\vee(\beta) + \eta^\vee(\alpha)\langle \beta, x \rangle,$$

that is to say $\langle \cdot, x \rangle \in \text{Der}_A(B^\vee, A)$.

\[\square\]

Remark 2.18. Lemma 2.15 will also allow us to dualize our theory, in the sense that

$$\text{Lie} \text{Hom}(G, G_m) = \text{Hom}(G, G_a)$$

for $G \in \text{gr}_A$. Therefore, Cartier duality reduces Theorem 3.12 to Theorem 1.3, i.e. [7], 7.5.

3. Group Schemes of Additive Type

Definition 3.1. A group scheme $G \in \text{Gr}_{S}$ is of additive type if there exists a closed embedding of $G$ into $\mathbb{G}_a^N$ for some set $N$, locally on $S$. We define $\text{Gr}_A^+$, resp. $\text{gr}_A^+$, to be the full subcategory of $\text{Gr}_A$, resp. $\text{gr}_A$, of group schemes of additive type.

Theorem 3.2. Let $G \in \text{Gr}_S$. The following conditions are equivalent.

\begin{enumerate}[(i)]
  \item $G \in \text{Gr}_S^+.$
  \item $I_G = (\text{Prim} B_G)$, i.e. $\text{Prim} B_G$ generates $I_G$ as an ideal, locally on $S$.
\end{enumerate}

Moreover, the above conditions imply the following.

\begin{enumerate}[(i)]
  \item $V_G = 0$.
  \item For finite $G \in \text{gr}_S^+$, all three conditions are equivalent.
\end{enumerate}

Proof. (Raynaud) The equivalence of (i) and (ii) is clear. Indeed, $G \hookrightarrow \mathbb{G}_a^N$ is the same as a Hopf algebra epimorphism $A[x_n \mid n \in N] \twoheadrightarrow B_G$, and the $x_n$ are primitive by definition. The implication “(ii) $\Rightarrow$ (iii)” is settled by Example 2.11.

Now it remains to show that if $G \in \text{gr}_A$ and $A$ is a local ring, then

$$V_G = 0 \Rightarrow \exists G \hookrightarrow \mathbb{G}_a^N \text{ for some } N \in \mathbb{N}.$$

Consider the Cartier dual $G^\vee$ of $G$, with affine algebra $B^\vee = B_{G^\vee}$ (cf. Lemma 2.15) and augmentation ideal $I = I_{G^\vee}$. Since $G$ is finite and flat, $B_G$ is reflexive. Thus on $R$-valued points, we have

$$G(R) \cong \text{Hom}_{Gr_m}(G^\vee \otimes R, G_m \otimes R) \cong \text{Grp}(B^\vee \otimes R) \hookrightarrow (B^\vee \otimes R)^{\times} =: B^{\vee \times}(R),$$

hence an embedding $G \hookrightarrow B^{\vee \times} \cong G_m \times (1 + I)$. Here, $(1 + I)(R) := 1 + \ker(\eta_{G^\vee} \otimes R)$. Moreover, consider the kernel $H$ of the composition

$$G \hookrightarrow G_m \times (1 + I) \twoheadrightarrow (1 + I).$$

Then $H \hookrightarrow G_m$, so the fibres of $H$ are of multiplicative type. But they are also killed by the Verschiebung, hence vanish ([2], IV, §3, 4.11). By Nakayama, $H = 0$.

Finally, since $F_{G^\vee} = (V_G)^{\vee} = 0$ by (2.2), we have $I^p = 0$. Thus $(1 + I)$ is isomorphic via truncated log and exp to the finite free additive group $I$, a power of $\mathbb{G}_a$. \[\square\]
Remark 3.3. Over a field $A = k$, all three conditions in Theorem 3.2 are equivalent by [2], IV, §3, 6.6. We conjecture that this in fact holds over an arbitrary base scheme $S$, at least assuming that $G$ is locally of finite presentation over $S$. 

Remark 3.4. If condition (ii) in Theorem 3.2 holds, the Hopf algebra $B_G$ is also called primitively generated, i.e. it is generated as an algebra by its primitive elements.

Example 3.5. The constant group scheme $\mathbb{F}_p$ embeds into $G_a$ via the projection

$$A[x] \rightarrow A[x]/(x^p - x).$$

The same holds for the group schemes $\alpha_{p^r} = \text{Spec } A[x]/(x^p)$, for $s \in \mathbb{N}$, since $x$ is primitive by definition. Hence they are all of additive type.

We now compute the order of a finite group scheme $G$ of additive type. This is the essential step towards our main theorem.

Proposition 3.6. Let $G \in \text{gr}_A^+$, and consider its dual $B^\vee = B^\vee_G$. There exists an algebra isomorphism

$$B^\vee \cong A[t_1, \ldots, t_n]/(t_1^p, \ldots, t_n^p),$$

locally on $\text{Spec } A$. Moreover, the $A$-module $\text{Prim}(B_G)$ is flat, and

$$\text{ord}(G) = p^{\text{rk}(\text{Prim } B_G)}.$$

Proof. (cf. [7], 7.4.3 and [9], B.3.14) Let $A$ be a local ring with residue field $k$. Let $I$ be the augmentation ideal of $B^\vee$, which is then free of finite rank $d$. Choose a basis $e_1, \ldots, e_d$ of $I_k := I \otimes k$, such that $e_{n+1}, \ldots, e_d$ is a basis of $I_k^2$. Let $t_i \in I$ be a lift of $e_i$ for $1 \leq i \leq d$, so that $t_1, \ldots, t_d$ is an $A$-basis of $I$ by Nakayama.

Now consider the free $A$-submodule $M := \text{span}_A(t_1, \ldots, t_n) \subseteq I$, and define

$$B' := \text{Sym}(M)/(t \otimes t | t \in M) \cong A[t_1, \ldots, t_n]/(t_1^p, \ldots, t_n^p).$$

Since $F_{G^\vee} = (V_G)^\vee = 0$, we have $I^p = 0$, and the canonical morphism

$$\psi: B' \rightarrow B^\vee, t_i \mapsto t_i,$$

is well-defined. Surjectivity of $\psi$ is easy to check along the filtration

$$0 = I^p \subseteq I^{p-1} \subseteq \ldots \subseteq I \subseteq B^\vee.$$

We claim that in fact $\dim_k(B' \otimes k) = \dim_k(B^\vee \otimes k)$, so that $\psi \otimes k$ is an isomorphism. To show this, we may assume $k$ to be perfect. Using $I^p_k = 0$, we then know by [2], III, §3, 6.3, that there is an algebra isomorphism

$$B^\vee \otimes k \cong k[T_1, \ldots, T_n]/(T_1^p, \ldots, T_n^p).$$

But then in particular $N = \dim_k(I_k/I_k^2) = n$. Since both $B^\vee$ and $B'$ are finite flat $A$-modules, $\psi$ is an isomorphism.

For the second part, it suffices by Lemma 2.17 to show that $I/I^2$ is free. But $\psi^{-1}$ induces an isomorphism of $A$-modules

$$I/I^2 \overset{\sim}{\rightarrow} J/J^2 \cong M,$$

where $J$ denotes the augmentation ideal of $B'$. Finally, Lemma 2.17 then tells us that $\text{rk}(\text{Prim } B_G) = \text{rk}(I/I^2) = n$, and therefore indeed

$$\text{ord } G = \text{rk}(B^\vee) = p^n = p^{\text{rk}(\text{Prim } B_G)},$$

as desired. \hfill \Box

Remark 3.7. It is easy to see that for any $A$-algebra $R$, and any $G \in \text{Gr}_A$, we have the canonical map

$$\text{Prim}(B_G) \otimes_A R \rightarrow \text{Prim}(B \otimes_A R). \quad (3.1)$$

Now let $G \in \text{gr}_A^+$, and assume that $R$ is the residue field at some point of $\text{Spec } A$. To show that then (3.1) is an isomorphism, we may assume $A$ to be local. But (3.1) is clearly injective, and both sides of (3.1) are finite $R$-modules of the same rank by Proposition 3.6.
Definition 3.8. Let $A[F]$ be the non-commutative polynomial ring over $A$ with $F\lambda = \lambda^p F$ for any $\lambda \in A$. The category of $A[F]$-modules, which are finite and flat as $A$-modules, will be denoted by $\text{sht}_A$.

Proposition 3.9. The primitive elements in the affine algebra of $\mathbb{G}_a^N$ are given by

$$
\text{Prim}(A[x_n \mid n \in N]) = \text{span}_A(x_n^p - x_n^e \mid n \in N, e \in \mathbb{N}) =: p^{(N)},
$$

the space of additive polynomials. In other words, for any set $N$,

$$\text{Hom}(\mathbb{G}_a^N, \mathbb{G}_a) \cong A[F]^\oplus N$$

as $A[F]$-modules. In particular, $\text{End}(\mathbb{G}_a) = A[F]$.

Proof. Of course, “$\supseteq$” is obvious. Now it suffices to see that $\text{Prim } A[x] \subseteq \text{span}_A(x^p - x^e \mid e \in \mathbb{N})$. Let thus $z = \sum_{n \in \mathbb{N}} \lambda_n x^n \in \text{Prim } A[x]$. Then

$$
\Delta(z) = \sum_{n \in \mathbb{N}} \lambda_n (x^n \otimes 1) + \sum_{n \in \mathbb{N}} \lambda_n (1 \otimes x^n).
$$

On the other hand, since $\Delta$ is an algebra morphism and $x$ is primitive,

$$
\Delta(z) = \sum_{n \in \mathbb{N}} \lambda_n (x \otimes 1 + 1 \otimes x)^n = \sum_{n \in \mathbb{N}} \lambda_n \sum_{k \leq n} \binom{n}{k} (x^k \otimes x^{n-k}).
$$

Comparing coefficients, we see that if $\lambda_n \neq 0$, then $\binom{n}{k} \equiv 0 \mod p$ for all $0 < k < n$. But this implies that $n = p^e$ for some $e \in \mathbb{N}$, cf. [5], Theorem 3. \qed

Definition 3.10. We denote the functor associating to a finite group scheme of additive type its Dieudonné $\mathbb{F}_p$-module by

$$
\mathcal{M}: \text{gr}_A^+ \to \text{sht}_A, \ G \mapsto \text{Hom}(G, \mathbb{G}_a).
$$

Here, the $A[F]$-module structure on $\mathcal{M}(G)$ is given by

$$
Fx = x^p \text{ for } x \in \text{Prim}(B_G).
$$

Equivalently, this is the obvious left module structure on $\text{Hom}(G, \mathbb{G}_a)$ over $\text{End}(\mathbb{G}_a) \cong A[F]$, cf. Proposition 3.9. Conversely, let us define for $M \in \text{sht}_A$ the corresponding group scheme

$$
\mathcal{G}(M) = \text{Spec}(\text{Sym}(M)/\mathfrak{f}),
$$

where $\mathfrak{f}$ is the ideal $\mathfrak{f} = (x^{\otimes p} - Fx \mid x \in M)$ in the symmetric algebra over $A$. The group structure on $\mathcal{G}(M)$ is defined by $M \subseteq \text{Prim}(B_{\mathcal{G}(M)})$.

Remark 3.11. The easy calculations verifying that the functors $\mathcal{G}$ and $\mathcal{M}$ are well-defined, are performed at the beginning of [48].

Theorem 3.12. The functor $\mathcal{M}$ defines an anti-equivalence of categories.

Proof. Choose an $A$-basis $x_1, \ldots, x_N$ of $M \in \text{sht}_A$. This yields the basis

$$
\prod_{i=1}^N x_i^{e_i} \mid 0 \leq e_i < p
$$

of $B_{\mathcal{G}(M)} = \text{Sym}(M)/\mathfrak{f}$. Therefore, we obtain

$$
\text{ord } \mathcal{G}(M) = p^{\text{rk } M}. \quad (3.2)
$$

Now, it is not hard to see that the functor $\mathcal{G}: \text{sht}_A \to \text{gr}_A^+$ is left-adjoint to $\mathcal{M}$. We will give the details in Lemma 8.4. For $G \in \text{gr}^+_A$ and $M \in \text{sht}_A$, consider the adjunction morphisms

$$
u_G: G \to \mathcal{G}(\mathcal{M}(G)) \text{ and } \upsilon_M: M \to \mathcal{M}(\mathcal{G}(M)).$$
By construction, \( v_M \) is the inclusion \( v_M : M \hookrightarrow \text{Prim}(B_G(M)) \). By Proposition [3.6 and 3.2], as well as base change to the fibre (Remark 4.1), we see that \( v_M \) is an isomorphism.

Now consider the map
\[
u_G^*: \text{Sym}(\text{Prim}(B_G))/\langle x^p - x^p \mid x \in \text{Prim}(B_G) \rangle \twoheadrightarrow B_G,
\]
which is the identity on \( \text{Prim}(B_G) \). Since \( B_G \) is primitively generated, we see that \( \nu_G^* \) is surjective. By Proposition [3.6 and 3.2], we have
\[
\text{ord} \, \mathcal{G}(\mathcal{M}(G)) = p^{\text{rk} \, \text{Prim} \, B_G} = \text{ord} \, G.
\]
Thus \( \nu_G^* \) is an epimorphism between finite flat modules of the same rank, hence bijective. □

4. THE CATEGORY OF \( \mathbb{F}_q \)-SHTUKAS

Let \( p \) be a prime, \( q = p^r \), and \( S \) an \( \mathbb{F}_q \)-scheme.

**Definition 4.1.** A finite \( \mathbb{F}_q \)-shtuka over \( S \) is a pair \((M, f)\), where \( M \) is a flat \( \mathcal{O}_S \)-module of finite rank and \( f \) is a \( q \)-linear endomorphism of \( M \). That is, \( f \) is a (linearized) map
\[
f : \sigma_q^* M = M \otimes_{\mathcal{O}_S, \sigma_q} \mathcal{O}_S \to M,
\]
where \( \sigma_q = \sigma_q^r \) is the Frobenius of \( \mathcal{O}_S \). A morphism of finite \( \mathbb{F}_q \)-shtukas \( \Phi : (M, f) \to (M', f') \) is an \( \mathcal{O}_S \)-module morphism such that the diagram
\[
\begin{array}{ccc}
M & \xrightarrow{\Phi} & M' \\
\downarrow f & & \downarrow f' \\
M & \xrightarrow{\Phi} & M'
\end{array}
\]
commutes. The category of finite \( \mathbb{F}_q \)-shtukas over \( S \) will be denoted by \( \text{F}_q \text{-sht}_S \). We shall write \( \text{F}_q \text{-sht}_A = \text{F}_q \text{-sht}_S \), when \( S = \text{Spec} \, A \).

**Remark 4.2.** Note that \((M, f) \in \text{F}_q \text{-sht}_A \) is the same as the left \( A[F] \)-module \( M \), which is flat over \( A \), defined by \( Fx = f(x) \) for \( x \in M \). Thus \( \text{F}_q \text{-sht}_A = \text{sht}_A \).

The definition comes from the following geometric example.

**Example 4.3.** Let \( X \) be a smooth projective geometrically irreducible curve over \( \mathbb{F}_q \). A (right) shtuka (or \( \mathcal{F} \)-sheaf) of rank \( d \in \mathbb{N} \) over \( S \) is a diagram
\[
\begin{array}{c}
\mathcal{L} \\
\downarrow i \\
\mathcal{E}
\end{array}
\]
with \( \mathcal{L} \) and \( \mathcal{E} \) flat sheaves of \( \mathcal{O}_{X \times S} \)-modules of rank \( d \), injective homomorphisms \( \tau \) and \( i \), and such that \( \text{coker}(\tau) \), resp. \( \text{coker}(i) \), is supported on the graph \( \Gamma_{\alpha} \), resp. \( \Gamma_{\beta} \), of some sections \( \alpha, \beta : S \to X \) (called the zero, resp. the pole, of the shtuka).

Let \( D \subseteq X \) be a finite subscheme away from the pole, i.e. \( \beta : S \to X \setminus D \), then \( i|_{D \times S} \) is an isomorphism. Setting \( \mathcal{L}_D := \mathcal{L}|_{D \times S} \), we get a morphism completing diagram (4.1),
\[
f' : (\text{id}_D \times F_S)^* \mathcal{L}_D \longrightarrow \mathcal{L}_D.
\]

Denote by \( \pi : D \times S \to S \) the projection, then \((\pi_* \mathcal{L}_D, f)\) is a finite \( \mathbb{F}_q \)-shtuka over \( \mathcal{O}_S \), where
\[
f : F_S^* \pi_* \mathcal{L}_D \cong \pi_* ((\text{id}_D \times F_S)^* \mathcal{L}_D - \pi_* f' \mathcal{L}_D).
\]

Drinfel’d [3] introduced \( \mathcal{F} \)-sheaves in the proof of the Langlands conjecture for \( GL_2 \) over a global field of characteristic \( p \).

Let us remark here a simple dichotomy in the category \( \text{F}_q \text{-sht}_k \), where \( k \) is a perfect field (see [9], B.3.10). We will later use it to generalize Theorem 1.2.
**Lemma 4.4.** Let $k$ be a perfect field. For $(M,f) \in {\mathbb F}_q$-sht$_k$, we have a unique decomposition

$$(M,f) = (M_{ss}, f_{ss}) \oplus (M_{nil}, f_{nil})$$

such that $f_{ss} = f|_{M_{ss}}$ is an isomorphism and $f_{nil} = f|_{M_{nil}}$ is nilpotent.

**Proof.** We define

$$M_{ss} = \bigcap_{n \in \mathbb{N}} \text{im}(f^n), \quad \text{and} \quad M_{nil} = \bigcup_{n \in \mathbb{N}} \ker(f^n).$$

Then there is some $N \in \mathbb{N}$ with $M_{ss} = \text{im}(f^N)$ and $M_{nil} = \ker(f^N)$, so that in particular

$$\dim(M) = \dim(M_{ss}) + \dim(M_{nil}).$$

Now suppose that $m \in M_{ss} \cap M_{nil}$. Then we have $m = f^N(m')$ for some $m' \in M$, and we obtain $f^{2N}(m') = f^N(m) = 0$. But since $\ker(f^{2N}) = \ker(f^N)$, in fact $m = f^N(m') = 0$. \hfill $\Box$

## 5. \(\mathbb{F}_q\)-Actions on Group Schemes

**Definition 5.1.** Let $S$ be an $\mathbb{F}_q$-scheme, and $G \in \text{Gr}_S$. An $\mathbb{F}_q$-action on $G$ is a ring morphism

$$[\_ | \_] : \mathbb{F}_q \to \text{End}_{\text{Gr}_S}(G), \quad \alpha \mapsto [\alpha]_G.$$

A morphism of group schemes with $\mathbb{F}_q$-action $\varphi : (G, [\_ | \_]) \to (H, [\_ | \_])$ is a morphism of group schemes over $S$ such that the diagram

$$\begin{array}{ccc}
G & \xrightarrow{\varphi} & H \\
\downarrow{[\alpha]_G} & & \downarrow{[\alpha]_H} \\
G & \xrightarrow{\varphi} & H
\end{array}$$

commutes for all $\alpha \in \mathbb{F}_q$. When there is no ambiguity, we will just write $[\alpha]$ for the action.

We denote by $\mathbb{F}_q$-Gr$_S$ the category of group schemes in Gr$_S$, together with an $\mathbb{F}_q$-action. For its objects, we will write $G$ instead of $(G, [\_ | \_])$. The full subcategory of $\mathbb{F}_q$-Gr$_S$ of finite group schemes is $\mathbb{F}_q$-Gr$_S$. We will replace $S$ by $\mathbb{A}$ in the notation, when $S = \text{Spec} \mathbb{A}$. Similarly, we consider the dual categories $\mathbb{F}_q$-Hopf$_\mathbb{A}$, resp. $\mathbb{F}_q$-hopf$_\mathbb{A}$.

**Example 5.2.** When we consider $G_\alpha = \text{Spec} \mathbb{A}[x]$ as an object of $\mathbb{F}_q$-Gr$_\mathbb{A}$, we mean that

$$[\alpha]^*x = \alpha x$$

for all $\alpha \in \mathbb{F}_q$, unless explicitly stated otherwise. The same extends to the group schemes $\alpha_{\mathbb{F}_q} \subseteq G_\alpha$, and the constant group $\mathbb{F}_q \subseteq G_\alpha$, as well as any product of these groups.

**Remark 5.3.** Let $G, H \in \mathbb{F}_q$-Gr$_\mathbb{A}$. Then $G \times H$ is endowed with the product $\mathbb{F}_q$-action

$$[\alpha]_{G \times H} = [\alpha]_G \times [\alpha]_H.$$

Let $\varphi, \psi : G \to H$ be morphisms in $\mathbb{F}_q$-Gr$_\mathbb{A}$. Then the diagram

$$\begin{array}{ccc}
G & \xrightarrow{\text{diag}} & G \times G \\
\downarrow{[\alpha]_G} & & \downarrow{[\alpha]_{G \times G}} \\
G & \xrightarrow{\varphi \times \psi} & H \times H \\
\downarrow{[\alpha]_H} & & \downarrow{[\alpha]_{H \times H}} \\
G & \xrightarrow{\text{diag}} & G \times G \\
\downarrow{[\alpha]_G} & & \downarrow{[\alpha]_{G \times G}} \\
G & \xrightarrow{\varphi \times \psi} & H \times H \\
\downarrow{[\alpha]_H} & & \downarrow{[\alpha]_{H \times H}}
\end{array}$$

commutes for all $\alpha \in \mathbb{F}_q$, so that $\varphi + \psi \in \text{Hom}_{\mathbb{F}_q\text{-Gr}_\mathbb{A}}(G, H)$ again.

Moreover, $\mathbb{F}_q$-Gr$_\mathbb{A}$ is an $\mathbb{F}_q$-linear category. The vector space structure on $\text{Hom}_{\mathbb{F}_q\text{-Gr}_\mathbb{A}}(G, H)$ is given by the obvious actions of

$$[\alpha]_G \in \text{End}_{\mathbb{F}_q\text{-Gr}_\mathbb{A}}(G), \quad \text{resp.} \quad [\alpha]_H \in \text{End}_{\mathbb{F}_q\text{-Gr}_\mathbb{A}}(H), \quad \text{for} \ \alpha \in \mathbb{F}_q,$$

which agree by definition.
The kernel of $\varphi: G \to H$ in $\mathbb{F}_q\text{-Gr}_A$ is given by $\ker(\varphi) = \text{Spec } B$, where $B = B_G/\varphi^*(I_H)B_G$ (see [10], VII, 4.1). Indeed, the $\mathbb{F}_q$-action of $B_G$ descends to $B$, simply because the diagram

\[
\begin{array}{c}
B_H \xrightarrow{\varphi^*} B_G \\
\downarrow \quad \quad \quad \downarrow \\
B_H \xrightarrow{\alpha} B_G
\end{array}
\]

commutes. Similarly, the $\mathbb{F}_q$-action on $H$ restricts to $\ker(\varphi) = \text{Spec } C$, where
\[
C = \{ x \in B_H \mid \Delta(x) - 1 \otimes x \in \ker(\varphi^*) \otimes B_H \},
\]

cf. [9], I.6.3. Indeed, by (5.1), we have
\[
\Delta([\alpha]^*x) - 1 \otimes [\alpha]^*x = ([\alpha]^* \otimes [\alpha]^*)(\Delta(x) - 1 \otimes x) \in \ker(\varphi^*) \otimes B_H
\]
for all $x \in C$.

**Definition 5.4.** Let $G = \text{Spec } B_G \in \mathbb{F}_q\text{-Gr}_A$. The eigenspaces of the $\mathbb{F}_q^\times$-action on $I = I_G$ are given by
\[
I_j := I_j(G) := \{ x \in I_G \mid [\alpha]^*x = \alpha^jx \ \forall \alpha \in \mathbb{F}_q \},
\]
for $0 < j < q$, noting that $\mathbb{F}_q^\times \cong \mathbb{Z}/(q - 1)$. We also set $\text{Prim}_j(B_G) := \text{Prim}(B_G) \cap I_j$.

**Remark 5.5.** Since $\text{ord}(\mathbb{F}_q^\times)$ is prime to $p$, the ideal $I_G$ decomposes into its eigenspaces as
\[
I_G = \bigoplus_{j=1}^{q-1} I_j.
\]

Indeed, we can write $\mathbb{F}_q[\mathbb{F}_q^\times] = \mathbb{F}_q[X]/(X^{q-1} - 1) = \bigoplus_{j=1}^{q-1} \mathbb{F}_q \chi_j$, with $\chi_j(\alpha) = \alpha^j$. This yields a system of orthogonal idempotents of $\text{End}_A(I_G)$,
\[
e_j = \frac{1}{q - 1} \sum_{\alpha \in \mathbb{F}_q^\times} \chi_j^{-1}(\alpha)[\alpha]^*, \quad 0 < j < q.
\]

Hence we obtain (5.2), since $I_j = e_j(I_G)$. In particular, it follows that the $I_j$ are flat over $A$ as direct summands of the flat $A$-module $I$. The analogous statements hold for the $\text{Prim}_j(B_G)$, if $\text{Prim}(B_G)$ is flat, noting that $\text{Prim}(B_G)$ is stable under $[\alpha]^* \in \text{End}_{\text{Hopf}_A}(B_G)$, $\alpha \in \mathbb{F}_q^\times$.

**Definition 5.6.** Let $G \in \mathbb{F}_q\text{-Gr}_S$. We say that $G$ is of $\mathbb{F}_q$-additive type, if there exists an $\mathbb{F}_q$-equivariant closed embedding
\[
G \hookrightarrow G_a^N \text{ for some set } N, \text{ locally on } S.
\]

The full subcategory of $\mathbb{F}_q\text{-Gr}_A$, resp. $\mathbb{F}_q\text{-gr}_A$, of group schemes $G$ of $\mathbb{F}_q$-additive type is denoted by $\mathbb{F}_q\text{-Gr}_A^+$, resp. $\mathbb{F}_q\text{-gr}_A^+$. For $q = p$, we drop $\mathbb{F}_q$ from the notation, as before.

**Remark 5.7.** Let $G, H \in \mathbb{F}_q\text{-Gr}_A^+$. If we have $G \hookrightarrow G_a^N$ and $H \hookrightarrow G_a^L$ in $\mathbb{F}_q\text{-Gr}_A$, then
\[
G \times H \hookrightarrow G_a^{N \cup L},
\]

cf. Remark [5.3]. Therefore, $G \times H \in \mathbb{F}_q\text{-Gr}_A^+$. Conversely, the embeddings $G, H \hookrightarrow G \times H$ respect the $\mathbb{F}_q$-actions. Hence if $G \times H$ is of $\mathbb{F}_q$-additive type, then so are $G$ and $H$.

**Remark 5.8.** Let $G \in \mathbb{F}_q\text{-Gr}_S$. Then
\[
G \text{ is of } \mathbb{F}_q\text{-additive type } \iff I_G = (\text{Prim}_1(B_G)), \text{ locally on } S,
\]
in analogy to Theorem [3.2].
Theorem 5.9. Let $G \in \mathbb{F}_q \mathrm{-gr}_A$. Then $G$ is of $\mathbb{F}_q$-additive type if and only if it is of additive type and the $p$-power maps $f_t : \mathrm{Prim}_{p^t}(B_G) \to \mathrm{Prim}_{p^{t+1}}(B_G)$, $x \mapsto x^{p^t}$, are surjective for $0 \leq t < r - 1$. Moreover, if $G \in \mathbb{F}_q \mathrm{-gr}^+_A$, we have the decomposition

$$
\mathrm{Prim}(B_G) = \bigoplus_{s=0}^{r-1} \mathrm{Prim}_{p^s}(B_G)
$$

into eigenspaces. Equivalently, $\mathrm{Prim}_{p^j}(B_G) = 0$ for all $j \neq p^s$, $s \in \mathbb{N}$.

Proof. Assume $A$ to be local, and let $\iota : G \hookrightarrow \mathbb{G}_a^N$ in $\mathbb{F}_q \mathrm{-Gr}_A$. Proposition 3.9 implies that the additive polynomials decompose as desired,

$$
\mathcal{P} = \mathrm{Prim}(A[x_1, \ldots, x_N]) = \bigoplus_{s=0}^{r-1} \mathcal{P}_s,
$$

with $\mathcal{P}_s = \mathrm{Prim}_{p^s}(A[x_1, \ldots, x_N])$. Let $k$ be the residue field of $A$. We have the epimorphism

$$
\iota^*|_{\mathcal{P}} \otimes_A k : \mathcal{P} \otimes_A k = \mathrm{Prim}(k[x_1, \ldots, x_N]) \twoheadrightarrow \mathrm{Prim}(B_G \otimes_A k)
$$

by Theorem 1.1. Thus $\mathcal{P} \otimes_A k \twoheadrightarrow \mathrm{Prim}(B_G) \otimes_A k$, by Remark 5.7. Hence by Nakayama

$$
\iota^*|_{\mathcal{P}} : \mathrm{Prim}(A[x_1, \ldots, x_N]) = \mathrm{Prim}(B_G).
$$

But $\iota^*$ respects the $\mathbb{F}_q$-action, so in fact $\iota^*(\mathcal{P}_s) = \mathrm{Prim}_{p^s}(B_G)$ for all $0 \leq s < r$. This settles the second part, and moreover implies that under the epimorphism $\iota^*$, we have

$$
0 = \mathcal{P}_{t+1}/(\mathcal{P}_t)^p \twoheadrightarrow \mathrm{Prim}_{p^{t+1}}(B_G)/f_t(\mathrm{Prim}_{p^t}(B_G)).
$$

Conversely, if all the $f_t$ are surjective, we have

$$
I_G = (\mathrm{Prim}(B_G)) = (\mathrm{Prim}_1(B_G)),
$$

and $G$ is of $\mathbb{F}_q$-additive type by Remark 5.8. \hfill \Box

Remark 5.10. In the previous proof, we do not need to invoke Theorem 1.1. With the above notation, we may assume $k$ algebraically closed. Then $\mathrm{Ext}^1_{\mathbb{F}_q}(\mathrm{coker} \ \iota \otimes k, \mathbb{G}_a,k) = 0$, cf. [9], B.3.15. This implies as before the surjectivity of

$$
\iota^*|_{\mathcal{P}} : \mathrm{Hom}(\mathbb{G}_a^N, \mathbb{G}_a) \twoheadrightarrow \mathrm{Hom}(G, \mathbb{G}_a).
$$

Lemma 5.11. Let $G \in \mathbb{F}_q \mathrm{-gr}^+_A$, and let $f_t$ be the $p$-power maps from Theorem 5.9. The following conditions are equivalent.

(i) All the maps $f_t$ are isomorphisms, for $0 \leq t < r - 1$.

(ii) The map $f' : \mathrm{Prim}_1(B_G) \to \mathrm{Prim}_{p^{r-1}}(B_G)$, $x \mapsto x^{p^{r-1}}$, is injective.

(iii) The rank of $\mathrm{Prim}_{p^s}(B_G)$ is the same for all $0 \leq s < r - 1$.

(iv) $\mathrm{ord}(G) = q^{\mathrm{rk Prim}_1(B_G)}$.

Proof. We have (i) $\Leftrightarrow$ (ii) by Theorem 5.9. Furthermore, (iii) is equivalent to the first two conditions by Nakayama. Now, Proposition 3.9 tells us that $\mathrm{rk Prim}_{p^s}(B_G) = \mathrm{rk Prim}_1(B_G)$ for all $0 \leq s \leq r - 1$ if and only if

$$
\mathrm{ord}(G) = p^{\mathrm{rk Prim}_1(B_G)} = q^{\mathrm{rk Prim}_1(B_G)}.
$$

Hence (iii) $\Leftrightarrow$ (iv). \hfill \Box

Definition 5.12. If the conditions in Lemma 5.11 hold for $G \in \mathbb{F}_q \mathrm{-gr}^+_A$, we say that $G$ is balanced. The full subcategory of $\mathbb{F}_q \mathrm{-gr}^+_A$ of balanced group schemes will be called $\mathbb{F}_q \mathrm{-gr}^+_A$.\b

Remark 5.13. In [13], Taguchi defines $\mathbb{F}_q \mathrm{-gr}^+_A$ using condition (iv) from Lemma 5.11. The first two conditions will be useful to generalize the definition to infinite group schemes.
Remark 5.14. We have \( F_{p^r} \text{gr}^+_A = \text{gr}^+_A \), since the condition on the \( p \)-power maps is empty.

Example 5.15. Consider \( G = \alpha_{p^r} = \text{Spec} A[x]/(x^{p^r}) \) with the usual \( F_q \)-action \( [\alpha] x = ax \) for \( \alpha \in F_q \). Then
\[
\alpha_{p^r} \text{ is balanced } \iff r|s. 
\]
Indeed, \( \text{Prim}(B_{\alpha_{p^r}}) = \text{span}_A(x, x^{p}, \ldots, x^{p^{r-1}}) \) by Proposition 5.10, and therefore
\[
\text{Prim}_1(B_{\alpha_{p^r}}) = \text{span}_A(x^a | 0 \leq a < s/r), 
\]
since \( p^s \equiv 1 \mod q - 1 \iff \frac{p^{s-1}}{q-1} \in \mathbb{Z} \iff r|s. \)

Remark 5.16. Let \( G \in \mathbb{F}_q \text{gr}^+_S \). If \( S \) is connected, then
\( G \) is balanced \( \iff G_s \) is balanced for some point \( s \in S \), because \( \text{Prim}(B_G) \) is flat and stable under base change (Remark 5.7). Hence the balanced locus of \( G \in \mathbb{F}_q \text{gr}^+_S \) in \( S \) is a union of connected components. If \( S \) is noetherian, it is thus closed and open.

Remark 5.17. If \( G \in \mathbb{F}_q \text{gr}^+_S \) is étale, its Frobenius is an isomorphism. Therefore, \( G \) is balanced by Lemma 5.11 (i).

Lemma 5.18. Let \( G, H \in \mathbb{F}_q \text{gr}^+_A \). If two of \( G, H \) and \( G \times H \) are balanced, then so is the third.

Proof. Proposition 2.12 implies that
\[
\text{Prim}_{s} (B_G \otimes_B H) = \text{Prim}_{s} (B_G) \otimes 1 + 1 \otimes \text{Prim}_{s} (B_H) 
\]
for all \( 0 \leq s \leq r - 1 \). Then it is clear that if two of \( G, H \) and \( G \times H \) satisfy condition (i) from Lemma 5.11 the third does as well. \( \square \)

6. THE FUNCTORS \( \mathcal{G} \) AND \( \mathcal{M} \)

We keep the notation of \( A \) denoting an \( \mathbb{F}_q \)-algebra, for \( q = p^r \).

Definition 6.1. The Dieudonné \( \mathbb{F}_q \)-functor is the contravariant functor
\[
\mathcal{M} : \mathbb{F}_q \text{gr}^+_A \to \mathbb{F}_q \text{sh}t_A, \quad G \mapsto (\text{Prim}_1(B_G), x \mapsto x^q). 
\]
Recall that \( \text{Prim}_1(B_G) \cong \text{Hom}_{\mathbb{F}_q \text{gr}_A}(G, \mathbb{G}_a) \), and that it is a flat \( A \)-module, cf. Remark 5.5.

Remark 6.2. \( \mathcal{M} \) is well-defined, since \( \Delta(x^q) = (x \otimes 1 + 1 \otimes x)^q = x^q \otimes 1 + 1 \otimes x^q \), and
\[
[\alpha]^* x^q = \alpha^q x^q = \alpha x^q \text{ for } x \in M, \, \alpha \in \mathbb{F}_q. 
\]

Definition 6.3. The Drinfeld’s \( \mathbb{F}_q \)-functor is defined to be the contravariant functor
\[
\mathcal{G} : \mathbb{F}_q \text{sh}t_A \to \mathbb{F}_q \text{gr}^+_A, \quad (M, f) \mapsto \text{Spec}(\text{Sym}(M)/\mathfrak{f}), 
\]
where \( \mathfrak{f} \) is the ideal \( \mathfrak{f} = (x^\mathfrak{q} - f(x) | x \in M) \). Comultiplication and \( F_q \)-action are given by
\[
\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \text{and } [\alpha]^* x = \alpha x \text{ for } x \in M, \, \alpha \in \mathbb{F}_q, 
\]
extended to the whole algebra.

Remark 6.4. The \( \mathbb{F}_q \)-action on \( \mathcal{G}(M, f) \) is well-defined, since we have
\[
[\alpha]^* x^\mathfrak{q} = \alpha^q x^\mathfrak{q} = \alpha f(x) = [\alpha]^* f(x) \text{ for } \alpha \in \mathbb{F}_q, \, x \in M. 
\]
If \((M, f) \in \mathbb{F}_q \text{sh}t_A \), we can take an \( A \)-basis \( x_1, \ldots, x_N \) of \( M \) and the projection
\[
\text{Sym}(M) \twoheadrightarrow \text{Sym}(M)/\mathfrak{f} 
\]
will define an embedding \( \mathcal{G}(M, f) \hookrightarrow \mathcal{G}^N_a \), which respects the \( \mathbb{F}_q \)-actions. Moreover, note that the products \( \{ x_i^{e_i} | 0 \leq e_i < \mathfrak{q} \} \) form a basis of \( B^\mathfrak{q}_{\mathcal{G}(M, f)} = \text{Sym}(M)/\mathfrak{f} \). Therefore
\[
\text{ord}(\mathcal{G}(M, f)) = q^{k_M}, \tag{6.1} 
\]
and indeed \( \mathcal{G}(M, f) \in \mathbb{F}_q \text{gr}^+_A \) by Lemma 5.11 (iv).
We are now prepared to state our main result.

**Theorem 6.5.** The Drinfel’d $\mathbb{F}_q$-functor $\mathcal{G}: \mathbb{F}_q$-$\text{sht}_A \to \mathbb{F}_q$-$\text{gr}_A^+$ defines an anti-equivalence of categories with quasi-inverse $\mathcal{M}$.

**Proof.** The proof is the same as for Theorem 6.4, where we use (6.1) in place of (3.2). This yields that the adjunction morphisms $v_{(M,f)}$ are bijective. On the other hand, $u_{G}: \text{Sym}(\text{Prim}_1(B_G))/(x^{qg} - x^q | x \in \text{Prim}_1(B_G)) \to B_G$ is surjective, since $B_G$ is generated as an algebra by $\text{Prim}_1(B_G)$, cf. Remark 5.8. Finally, $\text{ord} \mathcal{G}(\mathcal{M}(G)) = q^{\text{rk}(\text{Prim}_1(B_G))} = p^{\text{rk}(\text{Prim}_1(B_G))} = \text{ord} G$, by Proposition 5.6 and because $G$ is balanced. 

Let us note some properties of the functor $\mathcal{G}$ (cf. [3], 2.1).

**Proposition 6.6.** Let $(M, f: \sigma_q^* M \to M) \in \mathbb{F}_q$-$\text{sht}_A$, and $I = I_{\mathcal{G}(M,f)}$ the augmentation ideal of $\mathcal{G}(M,f)$. The cotangent space of $\mathcal{G}(M,f)$ is described by $I/I^2 \cong \text{coker}(f)$ as an $A$-module. Moreover,

(a) The group scheme $\mathcal{G}(M,f)$ is étale $\iff f$ is a nilpotent, locally on $\text{Spec } A$.

(b) The fibres of $\mathcal{G}(M,f)$ are connected $\iff f$ is nilpotent, locally on $\text{Spec } A$.

**Proof.** For the first part, we see that the composition $\tau: M \hookrightarrow I \twoheadrightarrow I/I^2$ is surjective, since every element of $I/I^2$ is represented by a linear polynomial in $M$. But $\ker(\tau) = M \cap I^2 = f(M)$.

Indeed, an element of $M$ lies in $I^2$ if and only if it is of the form $f(x) \equiv x^{qg} \in I^2$ for some $x \in M$.

The statements (a), (b) follow from the fact that $f$ is a power of the Frobenius on $\text{Sym}(M)/f$. Hence $f$ is bijective $\iff \text{Frob}_{\mathcal{G}(M,f)}$ is an isomorphism $\iff \mathcal{G}(M,f)$ is étale, because $\mathcal{G}(M,f)$ is finite and flat (cf. [2], IV, §3, 5.3).

Analogously, $f$ is locally nilpotent if and only if $\text{Frob}_{\mathcal{G}(M,f)}$ is. But each fibre of $\mathcal{G}(M,f)$ is connected if and only if its Frobenius is nilpotent (loc.cit.).

**Example 6.7.** For $q \neq p$, the category $\mathbb{F}_q$-$\text{gr}_A^+$ is not abelian (even if $A$ is a field). Indeed, consider the short exact sequence $0 \to \alpha_p \to \alpha_q \to H \to 0$ in $\mathbb{F}_q$-$\text{gr}_A$. Applying $\text{Hom}_{\mathbb{F}_q}$-$\text{Gr}_A(-, G_0)$, we get $0 \to \text{Prim}_1(B_H) \to \text{Prim}_1(B_{\alpha_q}) \twoheadrightarrow \text{Prim}_1(B_{\alpha_p})$. Hence $I_H \neq (\text{Prim}_1(B_H)) = 0$, and $H$ is not of $\mathbb{F}_q$-additive type by Remark 5.8. Of course, $\alpha_p$ is not balanced (cf. Example 5.15).

**Example 6.8.** Let $q \neq p$. Take $B_G = A[x_1, \ldots, x_r]/(x^p)$ with $x_i \in \text{Prim}_{p-1}(B_G)$. Clearly, $\text{rk Prim}_{p}(B_G)$ is the same for all $0 \leq s < r$ and $\text{Prim}_1 B_G = 0 \forall j \neq p^s$. The Hopf algebra morphism $u^*: A[x]/{x^q} \to B_G$, $x \mapsto x_1$ is compatible with the $\mathbb{F}_q$-actions, i.e. it induces $u: G \to \alpha_q$ in $\mathbb{F}_q$-$\text{gr}_A$. Setting $I = (x)$ to be the augmentation ideal of $\alpha_q$, we see that $\ker(u) = \text{Spec}(B_G/u^*(I)B_G) = \text{Spec } A[x_2, \ldots, x_r]/(x_2^p, \ldots, x_r^p)$ is not balanced. Of course, $G$ is not of $\mathbb{F}_q$-additive type.
In Theorem 8.13, we will be able to describe the structure of our category over a perfect field \( k \). This allows the following fibrewise characterization.

**Corollary 6.9.** Let \( G \in \mathbb{F}_q \text{-} \text{gr}^+_A \) be connected. Then \( G \) is balanced if and only if it is of the form \( \prod \alpha_{q^i} \) on all geometric fibres over \( S \).

**Proof.** We can check \( G \) to be balanced on the fibres, by Remark 5.16 and since the condition is stable under base change (cf. Remark 5.7). By Theorem 8.14 a geometric fibre of \( G \) is balanced if and only if it is of the form \( \prod \alpha_{q^i} \).

The following result gives another perspective on the balance property (Remark 6.11).

**Proposition 6.10.** Assume that \( A \) is an \( \mathbb{F}_q \)-algebra, \( n \geq 1 \). Let \( F: \mathbb{F}_q \text{-} \text{gr}^+_A \to \mathbb{F}_q \text{-} \text{gr}^+_A \) be the forgetful functor. Then the following diagram commutes:

\[
(M, f) \in \mathbb{F}_q^n \text{-} \text{sht}_A \xrightarrow{\varrho^n} \mathbb{F}_q^n \text{-} \text{gr}^+_A
\]

(6.2)

\[
\left( \bigoplus_{i=0}^{n-1} (\sigma_q^i)^* M, F \right) \in \mathbb{F}_q^n \text{-} \text{sht}_A \xrightarrow{\varrho^n} \mathbb{F}_q^n \text{-} \text{gr}^+_A,
\]

where \( F: (x_0, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, f(x_0)) \) is the matrix

\[
\begin{pmatrix}
0 & 1 & 0 \\
\vdots & \ddots & \vdots \\
0 & & 1 \\
\end{pmatrix}.
\]

**Proof.** We denote the scalar multiplication on \((\sigma_q^i)^* M\) by \( \lambda \cdot x = \lambda^q \cdot x \) for \( \lambda \in A \), \( x \in M \) (the usual action without the dot). Since \( f \) is \( q^n \)-linear, we get

\[
F(\lambda (x_0, \ldots, x_{n-1})) = (\lambda^q x_1, \ldots, \lambda^{q^{n-1}} x_{n-1}, \lambda^q f(x_0)) = \lambda^q F(x_0, \ldots, x_{n-1}),
\]

for all \((x_0, \ldots, x_{n-1}) \in M' := \bigoplus_{i=0}^{n-1} (\sigma_q^i)^* M\). We have to show that

\[
\text{Sym}(M)/\bar{f}_M \cong \text{Sym}(M')/\bar{f}'_M \text{ in } \mathbb{F}_q \text{-} \text{hopf}_A,
\]

where \( \bar{f}_M = (x \cdot q^n - f(x) \mid x \in M) \) and \( \bar{f}'_M = (x \cdot q^n - F(x) \mid x \in M') \). We define \( \varphi: \text{Sym}(M) \to \text{Sym}(M')/\bar{f}_M \) via \( M \ni x \mapsto (x, 0, \ldots, 0) \in M' \), extended to an algebra morphism. Note that for \( x \in M \), we have

\[
\varphi(x^q) = (x, 0, \ldots, 0)^q = F(x, 0, \ldots, 0) = (\ldots, 0, x, 0, \ldots) \in (\sigma_q^i)^* M \subseteq M'.
\]

Now, \( \varphi \) factors through the quotient, because

\[
\varphi(x^q - f(x)) = F^n(x, 0, \ldots, 0) - (f(x), 0, \ldots, 0) = 0.
\]

Finally, \( \overline{\varphi}: \text{Sym}(M)/\bar{f}_M \to \text{Sym}(M')/\bar{f}'_M \) is an isomorphism in \( \mathbb{F}_q \text{-} \text{Hopf}_A \), since by definition,

\[
\overline{\varphi}(\text{Prim}_1(\text{Sym}(M)/\bar{f}_M)) = \text{Prim}_1(\text{Sym}(M')/\bar{f}'_M),
\]

and it maps bases to bases.

**Remark 6.11.** Consider the following diagram,

\[
\mathbb{F}_q \text{-} \text{gr}^+_A \xrightarrow{\mathcal{M}_q} \mathbb{F}_q \text{-} \text{sht}_A \ni (M, f)
\]

(6.3)

\[
\mathbb{F}_p \text{-} \text{gr}^+_A \xrightarrow{\mathcal{M}_p} \mathbb{F}_p \text{-} \text{sht}_A \ni \left( \bigoplus_{i=0}^{r-1} (\sigma_p^i)^* M, F \right)
\]

corresponding to the commutative diagram (6.2).
where \( n \) are isomorphisms. This is equivalent to condition (ii) in Lemma 5.11.

\[
\text{Prim}(B_G) \cong \bigoplus_{i=0}^{r-1} (\sigma_p^i)^* \text{Prim}(B_G)
\]

if and only if all the (linearized) maps

\[
(\sigma_p^i)^* \text{Prim}(B_G) \to \text{Prim}(B_G), \ x \mapsto x^{p^i}, \ (0 \leq t < r)
\]

are isomorphisms. This is equivalent to condition (ii) in Lemma 5.11.

7. Quasi-Balanced Group Schemes

Let \( A \) be an \( \mathbb{F}_q \)-algebra, \( q = p^r \). For \( G \in \mathbb{F}_q \text{-gr}_A^+ \), consider the eigenspace decomposition

\[
I_G = \bigoplus_{j=1}^{q-1} I_j
\]

for the \( \mathbb{F}_q^* \)-action on the augmentation ideal of \( G \) (cf. Lemma 5.2).

**Definition 7.1.** A group scheme \( G \in \mathbb{F}_q \text{-gr}_A^+ \) is called quasi-balanced if \( \text{rk}(I_j) \) is the same for all \( 1 \leq j \leq q-1 \).

**Remark 7.2.** Let \( G \in \mathbb{F}_q \text{-gr}_A^+ \) be quasi-balanced. By Proposition 3.6, \( \text{rk}(I_j) = p^N - 1 \), where \( N = \text{rk}(\text{Prim}(B_G)) \). Then we have \( \frac{p^N - 1}{q-1} \in \mathbb{Z} \), and thus \( r|N \), say \( rn = N \). This yields

\[
\text{rk}(I_j) = q^{n-1} + \ldots + q + 1 \text{ for all } 1 \leq j \leq q-1,
\]

and of course \( \text{ord} G = q^N \). Note that the analogue of Remark 5.18 holds, i.e. for \( G \in \mathbb{F}_q \text{-gr}_S^+ \), the quasi-balanced locus of \( G \) in \( S \) is closed and open (if the base is noetherian).

**Lemma 7.3.** Every \( G \in \mathbb{F}_q \text{-gr}_A^{+b} \) is quasi-balanced.

**Proof.** If \( x_1, \ldots, x_n \) is a basis of \( \text{Prim}(B_G) \), then, since \( G \) is balanced,

\[
\{ \prod_{i=1}^n x_i^{e_i} \mid 0 \leq e_i < q, \ (e_1, \ldots, e_n) \neq 0 \}
\]

is a basis of \( I_G \) (cf. Remark 6.4). On this basis, \( \alpha \in \mathbb{F}_q \) acts via

\[
[\alpha]^* \prod_{i=1}^n x_i^{e_i} = \alpha^{\sum_{i=1}^n e_i} \prod_{i=1}^n x_i^{e_i}.
\]

Therefore, it decomposes into eigenbases for \( I_j \),

\[
\{ \prod_{i=1}^n x_i^{e_i} \mid 0 \leq e_i < q, \ (e_1, \ldots, e_n) \neq 0, \ \sum_{i=1}^n e_i \equiv j \mod q-1 \}.
\]

In order to count the ranks, we identify the bases with

\[
E_j^{(n)} := \{ 0 \neq e = (e_1, \ldots, e_n) \mid 0 \leq e_i < q, \ \sum_{i=1}^n e_i \equiv j \mod q-1 \}.
\]

We claim that

\[
\#E_j^{(n)} = q^{n-1} + \ldots + q + 1 \text{ for all } 1 \leq j \leq q-1.
\]

Let us prove this by induction on \( n \). For \( n = 1 \), there is nothing to show. For \( n > 1 \), we have

\[
E_j^{(n)} = \prod_{e=0}^{q-1} \{ e \in E_j^{(n)} \mid e_n = e \} \cong \prod_{e=0}^{q-1} \{ (e, e) \mid e \in E_j^{(n-1)} - \{(0, \ldots, 0)\} \} \cong \prod \{ (0, \ldots, 0, j) \}.
\]

Therefore indeed: \( \#E_j^{(n)} = q(q^{n-2} + \ldots + q + 1) + 1 = q^{n-1} + \ldots + q + 1. \)
Remark 7.4. For $G \in \text{gr}_+^\circ A$, i.e. $q = p$, the condition of being quasi-balanced is therefore automatic. We can also see this concretely via $p$-adic expansion. Namely, let

$$\rho: \mathbb{Z}/(p^n - 1) \longrightarrow \mathbb{Z}/(p - 1)$$

be the projection, where $\text{ord} G = p^n$, as above. Then we have the bijection

$$E_j = E_j^{(n)} \xrightarrow{\sim} \rho^{-1}(j), \ (e_1, \ldots, e_n) \mapsto \sum_{i=1}^n e_ip^i,$$

and thus again $\#E_j = \frac{p^n-1}{p-1} = p^{n-1} + \ldots + p + 1$.

Lemma 7.5. Let $G, H \in \mathbb{F}_q \cdot \text{gr}_A^+$. Then for $1 \leq j \leq q - 1$, we have

$$\text{rk} I_j(G \times H) = \sum_{k+l \equiv j(q-1)} \text{rk} I_k(G) \cdot \text{rk} I_l(H),$$

(7.1)

where we have set $I_0(-) := A$ for simplicity.

Proof. The product decomposes as follows:

$$B_G \otimes_A B_H = \left( \bigoplus_{k=0}^{q-1} I_k(G) \right) \otimes_A \left( \bigoplus_{l=0}^{q-1} I_l(H) \right) = \bigoplus_{k,l} \left( I_k(G) \otimes_A I_l(H) \right),$$

and, of course, whenever $k + l \equiv j(q-1)$, we have

$$I_k(G) \otimes_A I_l(H) \subseteq I_j(G \times H),$$

by definition of the product $\mathbb{F}_q$-action. \hfill \qed

Corollary 7.6. Let $G, H \in \mathbb{F}_q \cdot \text{gr}_A$. If two of $G$, $H$ and $G \times H$ are quasi-balanced, then so is the third.

Proof. Let $G, H$ be quasi-balanced. By Remark 7.2, we have $\text{ord} G = q^n$ and $\text{ord} H = q^m$ for some $n, m \in \mathbb{N}$. Then for any $1 \leq j \leq q - 1$, the product formula (7.1) becomes

$$\text{rk} I_j(G \times H) = \sum_{k+l \equiv j(q-1)} \text{rk} I_k(G) \cdot \text{rk} I_l(H) + \text{rk} I_j(G) + \text{rk} I_j(H)$$

$$= (q - 1)(q^{m-1} + \ldots + q + 1)(q^{n-1} + \ldots + q + 1) + \text{rk} I_j(G) + \text{rk} I_j(H)$$

$$= (q^n - 1)(q^{m-1} + \ldots + q + 1) + (q^{n-1} + \ldots + q + 1) + (q^{m-1} + \ldots + q + 1)$$

$$= q^n(q^{m-1} + \ldots + q + 1) + q^{n-1} \ldots + q + 1$$

$$= q^{n+m-1} + \ldots + q^n + q^{n-1} \ldots + q + 1.$$

We conclude that $G \times H$ is quasi-balanced.

Conversely, assume that $H$ and $G \times H$ are quasi-balanced. Since by Remark 7.2

$$\text{ord} H = q^m, \text{ and } \text{ord}(G \times H) = q^{n+m},$$

for some $m, n \in \mathbb{N}$, we have $\text{ord} G = q^n$. Applying (7.1) again, we get for $1 \leq j \leq q - 1$ that

$$q^{n+m-1} + \ldots + q + 1 = \sum_{1 \leq l \leq q-1} \text{rk} I_k(G)(q^{m-1} + \ldots + q + 1) + \text{rk} I_j(G) + \text{rk} I_j(H)$$

$$= (q^n - 1)(q^{m-1} + \ldots + q + 1) + \text{rk} I_j(G) + (q^{m-1} + \ldots + q + 1)$$

$$= q^{n+m-1} + \ldots + q^n + \text{rk} I_j(G),$$

hence the claim. \hfill \qed
Remark 7.7. Let us consider group schemes of the form

\[ G = \text{Spec}(A[x_1, \ldots, x_h]/(x_1^{p_1}, \ldots, x_h^{p_y})) = \prod_{i=1}^{h} \alpha_{p_i}, \]

so that \( x_i \in \text{Prim}_1(B_G) \). Consider the standard basis of \( I_G \),

\[ \{ \prod_{i=1}^{h} x_i^{| \leq e_i < p_i}, (e_1, \ldots, e_h) \neq 0 \}. \]

As always, it decomposes into eigenbases for the \( I_j \). This yields

\[ \text{rk} I_j = \sum_{a \equiv j (q-1)} n_a, \]

where

\[ n_a := \# \{ 0 \neq (e_1, \ldots, e_h) | 0 \leq e_i < p_i, \sum_{i=1}^{h} e_i = a \}. \]

Note that \( n_a \) is precisely given by the coefficient of \( X^a \) in the polynomial

\[ S(X) = (X^{p_1-1} + \ldots + X + 1) \cdot \ldots \cdot (X^{p_y-1} + \ldots + X + 1) \in \mathbb{Z}[X]. \] (7.2)

Example 7.8. Let \( q = 4 \), and consider the special case

\[ G = \text{Spec}(A[x_1, \ldots, x_6]/(x_1^p, \ldots, x_6^p)). \]

Then \( G \in \mathbb{F}_q \text{-gr}_A \) is quasi-balanced, since

\[ S(X) = (X + 1)^6 = X^6 + 6X^5 + 15X^4 + 20X^3 + 15X^2 + 6X + 1. \]

Hence \( \text{rk} I_j = 21 \) for all \( 1 \leq j \leq 3 \). But obviously \( G \notin \mathbb{F}_q \text{-gr}_{A,b} \), because \( \text{Prim}_p(B_G) = 0 \).

Thus the converse to Lemma 7.3 is false in general.

However, this is essentially the “only” counter-example, as the following results show.

Lemma 7.9. Let \( G = \alpha_{p_1} \times \ldots \times \alpha_{p_y} \) as in Remark 7.7.

1. If \( q \neq 4 \), then \( G \) is quasi-balanced if and only if \( r|s_i \) for all \( 1 \leq i \leq h \).

2. If \( q = 4 \), then \( G \) is quasi-balanced if and only if \( 6 | \# \{ i | s_i \neq 0 \text{ mod } r \} \).

Proof. (with Sauermann) By Corollary 7.8 \( G \) is quasi-balanced if and only if \( G \times \alpha_{q^m}, m \in \mathbb{N} \), is quasi-balanced. Eliminating the factors of the form \( \alpha_{q^m} \) from \( G \), we can therefore assume that we have \( s_i \neq 0 \text{ mod } r \) for all \( i \). Then we have to show that

\[ G \text{ is quasi-balanced if and only if } h = 0 \text{ in case (1), and } 6|h \text{ in case (2).} \]

Note that for \( r = 1 \), the claims are vacuous.

First, let us assume that \( G \) is quasi-balanced. We keep the notation from Remark 7.7. Then evaluating (7.2) at the primitive root of unity \( \zeta = e^{\frac{2\pi i}{q-1}} \) yields

\[ S(\zeta) = \sum_{a \geq 0} n_a \zeta^a = \sum_{j=0}^{q-1} \text{rk}(I_j) \zeta^j = 1 + \text{rk}(I_1) \sum_{j=1}^{q-1} \zeta^j = 1 + \text{rk}(I_1) \frac{\zeta^{q-1} - 1}{\zeta - 1} = 1. \]

Let \( 0 < t_i < r \) with \( s_i \equiv t_i \text{ mod } r \). Then

\[ 1 = S(\zeta) = \prod_{i=1}^{h} (\zeta^{p_i-1} + \ldots + \zeta + 1) = \prod_{i=1}^{h} \frac{\zeta^{p_i-1} - 1}{\zeta - 1} = \prod_{i=1}^{h} \frac{\zeta^{p_i-1} - 1}{\zeta - 1}. \] (7.3)

On the other hand, \( |\zeta^j - 1| \geq |\zeta - 1| \) for all \( 1 \leq j < q - 1 \). Hence by (7.3) we must have

\[ |\zeta^{p_i-1} - 1| = |\zeta - 1| \text{ for all } 1 \leq i \leq h. \]

Since \( 0 < t_i < r \), this implies that \( p_i \equiv -1 \text{ mod } (q - 1) \), and therefore

\[ q = p_i + 2, \text{ for } 1 \leq i \leq h. \]
Thus \( q = 4 \) (unless \( h = 0 \)), and \( t_i = 1 \) for all \( i \). In that case then, equation (7.3) reads
\[
(\zeta + 1)^h = 1.
\]

But \( \zeta + 1 = e^{\frac{2\pi i}{q}} + 1 = e^{\frac{2\pi i}{q}} \), and therefore \( 6|h \).
Conversely, the “if”-direction in (1) is trivial (\( h = 0 \)). In case (2), keeping in mind (7.3),
\[
\text{rk}(I_1) + \text{rk}(I_2) + \text{rk}(I_3) = S(\zeta - 1) - (\zeta + 1)^h - 1 = 0.
\]
Conjugating this equation, we see that indeed \( \text{rk}(I_1) = \text{rk}(I_2) = \text{rk}(I_3) \).
\( \square \)

**Theorem 7.10.** Let \( q \neq 4 \). Then \( G \in \mathbb{F}_q\cdot\text{gr}_{A} \) is quasi-balanced if and only if it is balanced.

**Proof.** We have already seen “\( \Leftarrow \)” in Lemma (7.3). Now let \( G \) be quasi-balanced. Since the condition is stable under base change, we can assume \( A = k \) to be a perfect field. As in (the proof of) Theorem 8.14, we have an \( \mathbb{F}_q \)-equivariant decomposition
\[
G = \pi_0(G) \times H.
\]
By Remark 5.17, \( \pi_0(G) \) is balanced, hence quasi-balanced, and so \( H \) is quasi-balanced as well by Corollary 7.6. Again as in Theorem 8.14,
\[
B_H \cong k[x_1, \ldots, x_h]/(x_1^{p^h}, \ldots, x_h^{p^h}) \in \mathbb{F}_q\cdot\text{hopf}_{A}.
\]
Therefore, by Lemma 7.9 since \( H \) is quasi-balanced, \( r|s_i \) for all \( 1 \leq i \leq h \), and thus \( H \) is balanced, which means so is \( G \).
\( \square \)

8. The Infinite Case

In this section, we detail some of the difficulties one encounters when trying to transfer the theory from the finite to the infinite case.

**Definition 8.1.** The category \( \mathbb{F}_q\cdot\text{Sht}_{A} \) consists of pairs \((M, f)\) of a flat \( A \)-module \( M \) and a \( q \)-linear endomorphism \( f \) of \( M \). Morphisms in \( \mathbb{F}_q\cdot\text{Sht}_{A} \) are defined as in \( \mathbb{F}_q\cdot\text{sht}_{A} \). Note that again \( \mathbb{F}_q\cdot\text{Sht}_{A} = \text{Sht}_{A} \).

The shtuka \((M, f)\) is called finitely generated if there exist some \( x_1, \ldots, x_N \in M \) such that
\[
M = \{ \sum_{i=1}^{N} \sum_{a=0}^{d} \lambda_a f^a(x_i) \mid \lambda_a \in A, \ d \in \mathbb{N} \}.
\]

We denote by \( \mathbb{F}_q\cdot\text{Sht}_{A, f.g.} \) the full subcategory of \( \mathbb{F}_q\cdot\text{Sht}_{A} \) of finitely generated shtukas.

Trying to define the functor \( \mathcal{M} \) in general, the first problem to arise is the following.

**Conjecture 8.2.** Let \( G \in \text{Gr}_{A}^{\mathbb{F}} \). Then the \( A \)-module \( \text{Prim}(B_G) \) is flat.

Let us provisionally include the condition in the definition, and denote the corresponding full subcategories by \( \text{Gr}_{A}^{\mathbb{F}} \), \( \mathbb{F}_q\cdot\text{Gr}_{A}^{\mathbb{F}} \), and so forth. Then we may define
\[
\mathcal{M}_q = \mathcal{M} : \mathbb{F}_q\cdot\text{Gr}_{A}^{\mathbb{F}} \to \mathbb{F}_q\cdot\text{Sht}_{A}, \quad \mathcal{G}_q = \mathcal{G} : \mathbb{F}_q\cdot\text{Sht}_{A} \to \mathbb{F}_q\cdot\text{Gr}_{A}^{\mathbb{F}},
\]
as before. Recall that \( \mathcal{M}_q(G) \) is flat if \( \text{Prim}(B_G) \) is, cf. Remark 5.9.

**Remark 8.3.** The routine verifications in [90] together with the following, show that the two functors are well-defined. For \((M, f) \in \mathbb{F}_q\cdot\text{Sht}_{A} \), choose a basis \( \{x_n \mid n \in N \} \) of \( M \). Then
\[
A[x_n \mid n \in N] \longrightarrow \text{Sym}(M)/f; \quad \text{hence } G \hookrightarrow \mathbb{G}_a^N.
\]
This is of course an \( \mathbb{F}_q \)-equivariant embedding, so \( G(M, f) \) is of \( \mathbb{F}_q \)-additive type.

The adjunction (cf. [90], B.3.9) holds in this generality.
Lemma 8.4. The functors $G$ and $M$ form an adjoint pair $(G, M)$ of $\mathbb{F}_q$-linear functors, that is, we have bifunctorial isomorphisms of $\mathbb{F}_q$-vector spaces

$$\text{Hom}_{\mathbb{F}_q\text{-Gr}_A}(G, G(M, f)) \cong \text{Hom}_{\mathbb{F}_q\text{-Sht}_A}((M, f), M(G)).$$

In particular, $G$ is right-exact and $M$ is left-exact.

Proof. We use $\text{Hom}_{\mathbb{F}_q\text{-Gr}_A}(G, G(M, f)) \cong \text{Hom}_{\mathbb{F}_q\text{-Hopt}_A}(\text{Sym}(M)/f, B_G)$. Now the map

$$\text{Hom}_{\mathbb{F}_q\text{-Hopt}_A}(\text{Sym}(M)/f, B_G) \rightarrow \text{Hom}_{\mathbb{F}_q\text{-Sht}_A}((M, f), M(G)), \varphi \mapsto \varphi|_M, \quad (8.1)$$

is well-defined and bijective. Indeed, by definition, we have $M \subseteq \text{Prim}_1(\text{Sym}(M)/f)$, and hence $\varphi(M) \subseteq \text{Prim}_1(B_G)$. Furthermore, $\varphi|_M$ is a morphism of shtukas. Namely, the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\varphi|_M} & \text{Prim}_1(B_G) \\
\downarrow f & & \downarrow z \rightarrow z^a \\
M & \xrightarrow{\varphi|_M} & \text{Prim}_1(B_G)
\end{array}$$

commutes, because $\varphi(x^{\otimes q} - f(x)) = 0$ for all $x \in M$, by definition.

The inverse map of (8.1) is given by

$$\text{Hom}_{\mathbb{F}_q\text{-Sht}_A}((M, f), M(G)) \rightarrow \text{Hom}_{\mathbb{F}_q\text{-Hopt}_A}(\text{Sym}(M)/f, B_G), \Phi \mapsto \hat{\Phi},$$

where $\hat{\Phi}$ is the extension of $\Phi$ to $\text{Sym}(M)$, which descends to $\text{Sym}(M)/f$. Indeed, the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\Phi} & \text{Prim}_1(B_G) \\
\downarrow f & & \downarrow z \rightarrow z^a \\
M & \xrightarrow{\Phi} & \text{Prim}_1(B_G)
\end{array}$$

commutes, and thus $\hat{\Phi}(x^{\otimes q} - f(x)) = \Phi(x)^q - \Phi(f(x)) = 0$ for $x \in M$. Finally, $\hat{\Phi}$ clearly respects the coalgebra structure and $\mathbb{F}_q$-action.

Now, $\text{Hom}_{\mathbb{F}_q\text{-Gr}_A}(-, G_a)$ is additive, and it follows from Proposition 2.12 that

$$\text{Prim}_1(B_G \otimes B_H) = \text{Prim}_1(B_G) \otimes 1 + 1 \otimes \text{Prim}_1(B_H).$$

Hence $z \mapsto z^q$ in $\mathcal{M}(G \times H)$ is given by $(x, y) \mapsto (x^q, y^q)$ in $\mathcal{M}(G) \oplus \mathcal{M}(H)$, and so $\mathcal{M}$ is additive. By adjunction then so is $G$. The $\mathbb{F}_q$-linearity is trivial, as it reduces to the statement that $[\alpha]^*$, $\alpha \in \mathbb{F}_q$, acts on $\text{Prim}_1$ by scalar multiplication.

In order to show the analogue of Theorem 6.5 we would like to restrict ourselves to group schemes which are locally of finite presentation over the base. However, the following question remains open.

Conjecture 8.5. Let $G \in \text{Gr}_A$, and assume that there exists a closed embedding $\iota: G \hookrightarrow G_a^N$, for some $N \in \mathbb{N}$. Then the morphism

$$\iota^* : \mathcal{M}_p(G_a^N) = \mathcal{A}[F]^N \rightarrow \mathcal{M}_p(G)$$

is surjective.

Note that if $A = k$ is a field, then Conjecture 8.5 holds by Theorem 1.1.

Remark 8.6. The restricted functor $\mathcal{M} : \mathbb{F}_q\text{-Gr}_A^{f,p} \rightarrow \mathbb{F}_q\text{-Sht}_A$ is well-defined, assuming Conjecture 8.5 holds. For this, we have to see that if

$$\iota^* : \mathcal{A}[x_1, \ldots, x_N] \twoheadrightarrow B_G$$

in $\mathbb{F}_q\text{-Hopt}_A$, then $\iota^*(x_1), \ldots, \iota^*(x_N) \in \text{Prim}_1(B_G)$ is a system of generators. Indeed, recall that

$$\text{Prim}_1(A[x_1, \ldots, x_N]) = \text{span}_A(x_i^{q^a} \mid 1 \leq i \leq N, a \in \mathbb{N}),$$

by Proposition 3.9. On the other hand, Conjecture 8.5 implies that $\mathcal{M}(\iota) = \iota^*|_{\mathcal{M}}$ surjects onto $\text{Prim}_1(B_G)$, as in the proof of Theorem 5.9.
Lemma 8.7. Assume Conjecture \[8.3\]. For \( G \in \mathbb{F}_q^*\text{-Gr}_{A,f.p.} \), the following are equivalent.

(i) \( f_t: \text{Prim}_t(B_G) \to \text{Prim}_{t+1}(B_G), \quad x \mapsto x^p, \) is an isomorphism, for \( 0 \leq t < r - 1 \).

(ii) The map \( f^t: \text{Prim}_t(B_G) \to \text{Prim}_{r-1}(B_G), \quad x \mapsto x^{p^{r-1}}, \) is injective.

Proof. The claim tautologically follows from the analogue of Theorem \[5.9\] the key ingredient in the proof of which is precisely Conjecture \[8.5\]. \( \square \)

Definition 8.8. A group scheme \( G \in \mathbb{F}_q^*\text{-Gr}_{A,f.p.} \) is called balanced if the conditions in Lemma \[8.7\] hold. We denote by \( \mathbb{F}_q^*\text{-Gr}_{A,f.p.} \) the full subcategory of \( \mathbb{F}_q^*\text{-Gr}_{A,f.p.} \) of balanced group schemes.

Remark 8.9. The functor \( G: \mathbb{F}_q^*\text{-Sht}_{A,f.g.} \to \mathbb{F}_q^*\text{-Gr}_{A,f.p.} \) is well-defined. To see this, choose a system of generators \( x_1, \ldots, x_N \) of \( (M, f) \in \mathbb{F}_q^*\text{-Sht}_{A,f.g.} \), so that

\[
A[x_1, \ldots, x_N] \to \text{Sym}(M)/f.
\]

Indeed, all elements of \( \text{Sym}(M)/f \) are polynomials in the \( x_i \), as we can write any \( x \in M \) as

\[
x = \sum_{i=1}^N \sum_{a=0}^d \lambda_a f^a(x_i) = \sum_{i=1}^N \sum_{a=0}^d \lambda_a x_i^{a+q^a} \quad \text{in} \quad \text{Sym}(M)/f.
\]

Keeping in mind Theorem \[5.9\] it then moreover follows that for all \( 0 \leq s < r \), the maps

\[
M = \text{Prim}(\text{Sym}(M)/f) \to \text{Prim}(\text{Sym}(M)/f), \quad x \mapsto x^{p^s},
\]

are isomorphisms. We conclude that \( G(M, f) \) is balanced, by Lemma \[8.7\] (ii).

Remark 8.10. Let \( G \in \mathbb{F}_q^*\text{-Gr}_{A,f.p.} \), and \( i: G \hookrightarrow \mathbb{G}_a^N \). Let further \( R \) be the residue field at a point \( s \in \text{Spec } A \). Then \( t_s: G_s \hookrightarrow \mathbb{G}_a^N \) as well. Therefore,

\[
\begin{array}{ccc}
\text{Prim}(A[x_1, \ldots, x_N]) \otimes_A R & \longrightarrow & \text{Prim}(B_G) \otimes_A R \\
\downarrow \text{(\*)} & & \downarrow \\
\text{Prim}(R[x_1, \ldots, x_N]) & \longrightarrow & \text{Prim}(B_G) \otimes_A R
\end{array}
\]

by Conjecture \[8.3\] (and recalling Remark \[5.7\]). We can see for example from Proposition \[8.9\] that (\*\*) is an isomorphism. We conclude that the primitive elements are stable under base change to the fibre.

Note that if we assume Conjecture \[8.3\] then we only need Conjecture \[8.2\] to hold for locally finitely presented \( G \). Indeed, we can then prove the adjunction of \( G \) and \( M \) as functors between \( \mathbb{F}_q^*\text{-Gr}_{A,f.p.} \) and \( \mathbb{F}_q^*\text{-Sht}_{A,f.g.} \) by the identical argument.

Theorem 8.11. Assume that Conjectures \[8.2\] and \[8.3\] are true. The functor

\[
G: \mathbb{F}_q^*\text{-Sht}_{A,f.g.} \to \mathbb{F}_q^*\text{-Gr}_{A,f.p.}
\]

defines an anti-equivalence of categories with quasi-inverse \( M \).

Proof. We follow [2], IV, §3, 6.5. First of all, we may assume that \( A \) is a local ring, and in fact even an Artin local ring, by Remark \[8.10\]. We consider the short exact sequence

\[
0 \to G \overset{u_G}{\longrightarrow} G(M(G)) \to Q \to 0,
\]

(8.2)

defined by the adjunction morphism. On the other hand, we once again have the isomorphism

\[
v_{M(G)}^* = \text{Hom}_{Gr_A}(u_G): \text{Hom}_{Gr_A}(G(M(G)), \mathbb{G}_a) \overset{\sim}{\longrightarrow} \text{Hom}_{Gr_A}(G, \mathbb{G}_a).
\]

Since \( G \) is balanced, \( M(u_G) \) extends to all primitive elements, i.e.

\[
\text{Hom}_{Gr_A}(u_G, \mathbb{G}_a): \text{Hom}_{Gr_A}(G(M(G)), \mathbb{G}_a) \overset{\sim}{\longrightarrow} \text{Hom}_{Gr_A}(G, \mathbb{G}_a).
\]

Then applying \( M_p = \text{Hom}_{Gr_A}(\text{ }, \mathbb{G}_a) \) to \[8.2\] yields the exact sequence

\[
0 \to \text{Hom}_{Gr_A}(Q, \mathbb{G}_a) \to \text{Hom}_{Gr_A}(G(M(G)), \mathbb{G}_a) \overset{\sim}{\longrightarrow} \text{Hom}_{Gr_A}(G, \mathbb{G}_a).
\]

(8.3)
The fibre of $Q$ over the closed point of $\text{Spec } A$ is unipotent by [2], IV, §2, 2.3, i.e. there is a non-trivial morphism to the additive group. Since $A$ is an Artin ring, this lifts to $Q \to \mathbb{G}_a$, up to a Frobenius twist on $\mathbb{G}_a$.

But we have $\text{Hom}_{\text{Gr}_A}(Q, \mathbb{G}_a) = 0$ by (8.3), hence $Q$ must be trivial. □

Remark 8.12. In particular, $\mathbb{G}_a$ is an injective object of $\mathbb{F}_q \text{-Gr}_{\mathbb{A}^1, \text{f.p.}}$, that is, $M$ maps short exact sequences to short exact sequences. Moreover, if $A = k$ is a field, then $\mathbb{F}_q \text{-Gr}_{\mathbb{A}^1, \text{f.p.}}$ is an abelian category.

Remark 8.13. Over a field $A = k$, we can drop the condition that our group schemes $G$ are finitely presented and that the $k[F^r]$-modules are finitely generated, respectively. Indeed, we may argue as above, applying Theorem 1.1 on the way. Alternatively, by [2], III, §3, 7.5, we can write $G = \varprojlim_i G_i$ with $G_i \in \text{Gr}_{k}$ finitely presented.

Finally, let us prove the following structure theorem, generalizing Theorem 1.2.

Theorem 8.14. If $k$ is a perfect field, every $G \in \mathbb{F}_q \text{-Gr}_{\mathbb{A}^1, \text{f.p.}}$ can be written as a product

$$G \cong G^\alpha_0 \times \pi_0(G) \times H,$$

with $H$ a product of group schemes of the form $\alpha_q^r$, and where $\pi_0(G)$ is an étale sheaf of $\mathbb{F}_q$-vector spaces. If $k$ is algebraically closed, we have

$$\pi_0(G) \cong (\mathbb{F}_q)^m$$

for some $m \in \mathbb{N}$.

Proof. As in [2], IV, §3, 6.9, we use the fact that $k[F^r]$ is left-Euclidean to decompose $M(G)$ into its $k[F^r]$-torsion submodule $M = T(M(G))$ and its torsionfree part. Furthermore, applying Lemma 4.4 to $M$ altogether yields the decomposition

$$G = G(M(G)) = G(M(G)/(M, f)) \times G(M_{ss}, f_{ss}) \times G(M_{nil}, f_{nil}).$$

Then $M(G)/(M, f)$ is free of finite rank $n \in \mathbb{N}$, and $G(M(G)/(M, f)) \cong G^\alpha_0$. The finite part of $G$ consists of the (by Lemma 4.4 maximal) étale part $\pi_0(G) = G(M_{ss}, f_{ss})$ and the connected part $H = G(M_{nil}, f_{nil})$, cf. Proposition 8.10. By Theorem 1.2 as a group scheme,

$$H \cong \alpha_{p^n_1} \times \ldots \times \alpha_{p^n_m} = \text{Spec}(k[x_1, \ldots, x_h]/(x_1^{n_1}, \ldots, x_h^{n_h})).$$

Since $H \in \mathbb{F}_q \text{-Gr}_{\mathbb{k}^+}$, the maps $\text{Prim}_1(B_H) \to \text{Prim}_{p^n}(B_H)$, $x \mapsto x^{p^n}$, are surjective (Theorem 5.9). Thus $H$ can be written as a product as above even in $\mathbb{F}_q \text{-Gr}_{\mathbb{k}^+}$. But then $r|n_i$ for all $i$, cf. Example 5.15.

If $k$ is algebraically closed, then $\pi_0(G)$ is constant. Furthermore, it is killed by $p = F \circ V$ (cf. Remark 2.5) and of $q$-power order. Hence it is indeed a power of $\mathbb{F}_q = \text{Spec}(k[x]/(x^q - x))$. Again as above, the $\mathbb{F}_q$-action must be the canonical one.

(Note that there is no well-defined $\mathbb{F}_q$-action on $\mathbb{F}_q$ unless $r|t$).

References

[1] V. Abrashkin, Galois modules arising from Faltings’s strict modules. Compositio Mathematica 142, pp. 867-888, 2006.
[2] M. Demazure, P. Gabriel, Groupes Algébriques. North-Holland Publishing Company, 1970.
[3] V. G. Drinfel’d, Varieties of modules of $F$-sheaves. Functional Analysis and Its Applications 21, no. 2, pp. 107-122, 1987.
[4] G. Faltings, Group Schemes with Strict $O$-Action. Moscow Mathematical Journal 2, no. 2, pp. 249-279, 2002.
[5] N. J. Fine, Binomial Coefficients Modulo a Prime. The American Mathematical Monthly 54, no. 10, pp. 589-592, 1947.
[6] J.-M. Fontaine, Groupes $p$-divisibles sur les corps locaux. Astérisque 47-48, 1977.
[7] P. Gabriel, Étude infinitésimale des schémas en groupes. SGA 3, Schémas en Groupes I, Exposé VII_A (new edition). Available online at https://www.imj-prg.fr/~patrick.polo/SGA3/ (May 23, 2011).

[8] U. Hartl, R. Kumar Singh, Local Shtukas and Divisible Local Anderson-Modules. Unpublished Manuscript (2014).

[9] G. Laumon, Cohomology of Drinfeld Modular Varieties, Part I. Cambridge Studies in Advanced Mathematics 41, Cambridge University Press, 1996.

[10] J. S. Milne, Basic Theory of Affine Group Schemes. Available online at http://www.jmilne.org/math/ (March 11, 2012).

[11] R. Pink, Finite group schemes. Notes, ETH Zürich 2004/05.

[12] M. Raynaud, Schémas en groupes de type $(p,\ldots,p)$. Bulletin de la Société Mathématique de France 102, pp. 241-280, 1974.

[13] J. Stix, A course on finite flat group schemes and $p$-divisible groups. Notes, Heidelberg 2009.

[14] Y. Taguchi, A duality for finite $t$-modules. J. Math. Sci. Univ. Tokyo 2, pp. 563-588, 1995.

[15] J. Tate, Finite Flat Group Schemes. In: G. Cornell, J. H. Silverman, G. Stevens, Modular Forms and Fermat’s Last Theorem, pp. 121-154, Springer Verlag 1997.

Hausdorff Center for Mathematics
Villa Maria
Endenicher Allee 62
53115 Bonn, Germany
email: thomas.poguntke@hcm.uni-bonn.de