INTERVAL-VALUED INTUITIONISTIC GRADATION OF OPENNESS

Chun-Kee Park

Abstract. In this paper, we introduce the concepts of interval-valued intuitionistic gradation of openness of fuzzy sets which is a generalization of intuitionistic gradation of openness of fuzzy sets and interval-valued intuitionistic gradation preserving mapping and then investigate their properties.

1. Introduction

After Zadeh [14] introduced the concept of fuzzy sets, there have been various generalizations of the concept of fuzzy sets. Chang [5] introduced the concept of fuzzy topology on a set $X$ by axiomatizing a collection $T$ of fuzzy subsets of $X$ and Coker [7] introduced the concept of intuitionistic fuzzy topology on a set $X$ by axiomatizing a collection $T$ of intuitionistic fuzzy subsets of $X$. In their definitions of fuzzy topology and intuitionistic fuzzy topology, fuzzyness in the concept of openness of fuzzy subsets and intuitionistic fuzzy subsets was absent. Chattopadhyay, Hazra and Samanta [6,8] introduced the concept of gradation of openness of fuzzy subsets. Zadeh [15] introduced the concept of interval-valued fuzzy sets and Atanassov [2] introduced the concept of intuitionistic fuzzy sets. Atanassov and Gargov [3] introduced the concept of interval-valued intuitionistic fuzzy sets which is a generalization of both interval-valued...
fuzzy sets and intuitionistic fuzzy sets. Mondal and Samanta [9,13] introduced the concept of intuitionistic gradation of openness and defined an intuitionistic fuzzy topological space and investigated their properties.

In this paper, we introduce the concepts of interval-valued intuitionistic gradation of openness of fuzzy sets which is a generalization of intuitionistic gradation of openness of fuzzy sets and interval-valued intuitionistic gradation preserving mapping and then investigate some properties of interval-valued intuitionistic gradation of openness of fuzzy sets and interval-valued intuitionistic gradation preserving mappings.

2. Preliminaries

Throughout this paper, let $X$ be a nonempty set, $I = [0, 1]$, $I_0 = (0, 1]$ and $I_1 = [0, 1)$. The family of all fuzzy sets of $X$ will be denoted by $I^X$. By $0_X$ and $1_X$ we denote the characteristic functions of $\phi$ and $X$, respectively. For any $A \in I^X$, $A^c$ denotes the complement of $A$, i.e., $A^c = 1_X - A$.

**Definition 2.1.** [4,6,12]. A gradation of openness (for short, GO) on $X$, which is also called a smooth topology on $X$, is a mapping $\tau : I^X \to I$ satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$,

(O2) $\tau(A \cap B) \geq \tau(A) \land \tau(B)$ for each $A, B \in I^X$,

(O3) $\tau(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau(A_i)$, for each subfamily $\{A_i : i \in \Gamma\} \subset I^X$.

The pair $(X, \tau)$ is called a smooth topological space (for short, STS).

**Definition 2.2.** [9]. An intuitionistic gradation of openness (for short, IGO) on $X$, which is also called an intuitionistic smooth topology on $X$, is an ordered pair $(\tau, \tau^*)$ of mappings from $I^X$ to $I$ satisfying the following conditions:

(IGO1) $\tau(A) + \tau^*(A) \leq 1$ for each $A \in I^X$,

(IGO2) $\tau(0_X) = \tau(1_X) = 1$ and $\tau^*(0_X) = \tau^*(1_X) = 0$,

(IGO3) $\tau(A \cap B) \geq \tau(A) \land \tau(B)$ and $\tau^*(A \cap B) \leq \tau^*(A) \lor \tau^*(B)$ for each $A, B \in I^X$,

(IGO4) $\tau(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau(A_i)$ and $\tau^*(\bigcup_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} \tau^*(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subset I^X$.

The triple $(X, \tau, \tau^*)$ is called an intuitionistic smooth topological space (for short, ISTS). $\tau$ and $\tau^*$ may be interpreted as gradation of openness and gradation of nonopenness, respectively.
Definition 2.3. [9]. Let \((X, \tau, \tau^*)\) and \((Y, \eta, \eta^*)\) be two ISTSs and \(f : X \to Y\) be a mapping. Then \(f\) is called a gradation preserving mapping (for short, a GP-mapping) if for each \(A \in \mathcal{I}^Y\), \(\eta(A) \leq \tau(f^{-1}(A))\) and \(\eta^*(A) \geq \tau^*(f^{-1}(A))\).

Let \(D(I)\) be the set of all closed subintervals of the unit interval \(I\). The elements of \(D(I)\) are generally denoted by capital letters \(M, N, \ldots\) and \(M = [M^L, M^U]\), where \(M^L\) and \(M^U\) are respectively the lower and the upper end points. Especially, we denote \(r = [r, r]\) for each \(r \in I\). The complement of \(M\), denoted by \(M^c\), is defined by \(M^c = 1 - M = [1 - M^U, 1 - M^L]\). Note that \(M = N\) iff \(M^L = N^L\) and \(M^U = N^U\) and that \(M \leq N\) iff \(M^L \leq N^L\) and \(M^U \leq N^U\).

Definition 2.4. [15]. A mapping \(A = [A^L, A^U] : X \to D(I)\) is called an interval-valued fuzzy set (for short, IVFS) on \(X\), where \(A(x) = [A^L(x), A^U(x)]\) for each \(x \in X\). \(A^L(x)\) and \(A^U(x)\) are called the lower and upper end points of \(A(x)\), respectively.

Definition 2.5. [10]. Let \(A\) and \(B\) be IVFSs on \(X\). Then
(a) \(A = B\) iff \(A^L(x) = B^L(x)\) and \(A^U(x) = B^U(x)\) for all \(x \in X\).
(b) \(A \subset B\) iff \(A^L(x) \leq B^L(x)\) and \(A^U(x) \leq B^U(x)\) for all \(x \in X\).
(c) The complement \(A^c\) of \(A\) is defined by \(A^c(x) = [1 - A^U(x), 1 - A^L(x)]\) for all \(x \in X\).
(d) For a family of IVFSs \(\{A_i : i \in \Gamma\}\), the union \(\bigcup_{i \in \Gamma} A_i\) and the intersection \(\bigcap_{i \in \Gamma} A_i\) are respectively defined by
\[
\bigcup_{i \in \Gamma} A_i(x) = [\vee_{i \in \Gamma} A_i^L(x), \vee_{i \in \Gamma} A_i^U(x)],
\]
\[
\bigcap_{i \in \Gamma} A_i(x) = [\wedge_{i \in \Gamma} A_i^L(x), \wedge_{i \in \Gamma} A_i^U(x)]
\]
for all \(x \in X\).

Definition 2.6. [3]. A mapping \(A = (\mu_A, \nu_A) : X \to D(I) \times D(I)\) is called an interval-valued intuitionistic fuzzy set (for short, IVIFS) on \(X\), where \(\mu_A : X \to D(I)\) and \(\nu_A : X \to D(I)\) are interval-valued fuzzy sets on \(X\) with the condition \(\sup_{x \in X} \mu_A^U(x) + \sup_{x \in X} \nu_A^U(x) \leq 1\). The intervals \(\mu_A(x) = [\mu_A^L(x), \mu_A^U(x)]\) and \(\nu_A(x) = [\nu_A^L(x), \nu_A^U(x)]\) denote the degree of belongingness and the degree of nonbelongingness of the element \(x\) to the set \(A\), respectively.

Definition 2.7. [11]. Let \(A = (\mu_A, \nu_A)\) and \(B = (\mu_B, \nu_B)\) be IVIFSs on \(X\). Then
(a) $A \subset B$ iff $\mu_A^U(x) \leq \mu_B^U(x)$, $\mu_A^L(x) \leq \mu_B^L(x)$ and $\nu_A^U(x) \geq \nu_B^U(x)$, $\nu_A^L(x) \geq \nu_B^L(x)$ for all $x \in X$.
(b) $A = B$ iff $A \subset B$ and $B \subset A$.
(c) The complement $A^c$ of $A$ is defined by $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$ for all $x \in X$.
(d) For a family of IVIFSs $\{A_i : i \in \Gamma\}$, the union $\bigcup_{i \in \Gamma} A_i$ and the intersection $\bigcap_{i \in \Gamma} A_i$ are respectively defined by
\[
\mu_{\bigcup_{i \in \Gamma} A_i}(x) = \bigcup_{i \in \Gamma} \mu_{A_i}(x), \quad \nu_{\bigcup_{i \in \Gamma} A_i}(x) = \bigcap_{i \in \Gamma} \nu_{A_i}(x),
\]
\[
\mu_{\bigcap_{i \in \Gamma} A_i}(x) = \bigcap_{i \in \Gamma} \mu_{A_i}(x), \quad \nu_{\bigcap_{i \in \Gamma} A_i}(x) = \bigcup_{i \in \Gamma} \nu_{A_i}(x)
\]
for all $x \in X$.

3. Interval-valued intuitionistic gradation of openness

**Definition 3.1.** An interval-valued intuitionistic gradation of openness (for short, IVIGO) on $X$, which is also called an interval-valued intuitionistic smooth topology on $X$, is an ordered pair $(\tau, \tau^*)$ of mappings $\tau = [\tau^L, \tau^U] : I^X \to D(I)$ and $\tau^* = [\tau^{*L}, \tau^{*U}] : I^X \to D(I)$ satisfying the following conditions:

- (IVIGO1) $\tau^L(A) \leq \tau^U(A)$, $\tau^{*L}(A) \leq \tau^{*U}(A)$ and $\tau^U(A) + \tau^{*U}(A) \leq 1$ for each $A \in I^X$,
- (IVIGO2) $\tau(0_X) = \tau(1_X) = 1$ and $\tau^*(0_X) = \tau^*(1_X) = 0$,
- (IVIGO3) $\tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B)$, $\tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B)$ and $\tau^{*L}(A \cap B) \leq \tau^{*L}(A) \vee \tau^{*L}(B)$, $\tau^{*U}(A \cap B) \leq \tau^{*U}(A) \vee \tau^{*U}(B)$ for each $A, B \in I^X$,
- (IVIGO4) $\tau^L(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau^L(A_i)$, $\tau^U(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau^U(A_i)$ and $\tau^{*L}(\bigcup_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} \tau^{*L}(A_i)$, $\tau^{*U}(\bigcup_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} \tau^{*U}(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subset I^X$.

The triple $(X, \tau, \tau^*)$ is called an interval-valued intuitionistic smooth topological space (for short, IVISTS). $\tau$ and $\tau^*$ may be interpreted as interval-valued gradation of openness and interval-valued gradation of nonopenness, respectively.

**Definition 3.2.** An interval-valued intuitionistic gradation of closedness (for short, IVIGC) on $X$, which is also called an interval-valued intuitionistic smooth cotopology on $X$, is an ordered pair $(\mathcal{F}, \mathcal{F}^*)$ of mappings $\mathcal{F} = [\mathcal{F}^L, \mathcal{F}^U] : I^X \to D(I)$ and $\mathcal{F}^* = [\mathcal{F}^{*L}, \mathcal{F}^{*U}] : I^X \to D(I)$ satisfying the following conditions:
(IVIGC1) $F^L(A) \leq F^U(A)$, $F^*(A) \leq F^*(U) (A)$ and $F^*(U) (A) + F^*(U) (A) \leq 1$ for each $A \in I^X$.

(IVIGC2) $F(0_X) = F(1_X) = 1$ and $F^*(0_X) = F^*(1_X) = 0$.

(IVIGC3) $F^L(A \cup B) \geq F^L(A) \land F^L(B)$, $F^U(A \cup B) \geq F^U(A) \land F^U(B)$ and $F^*(A \cup B) \leq F^*(A) \lor F^*(B)$, $F^U(A \cup B) \leq F^*(U) (A) \lor F^*(U) (B)$ for each $A, B \in I^X$.

(IVIGC4) $F^L(\bigcap_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} F^U(A_i)$, $F^U(\bigcap_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} F^U(A_i)$ and $F^*(\bigcap_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} F^*(A_i)$, $F^U(\bigcap_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} F^*(U) (A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subset I^X$.

**Theorem 3.3.** If $(\tau, \tau^*)$ is an IVIGO on $X$, then $(\tau^L, \tau^*)$ and $(\tau^U, \tau^*)$ are IGOs on $X$.

**Proof.** It follows immediately from Definition 2.2 and 3.1. \qed

For an IVIGO $(\tau, \tau^*)$ and an IVIGC $(F, F^*)$ on $X$, we define

$$
\tau_F(A) = F(A^c), \quad \tau^*_F(A) = F^*(A^c),
$$

$$
F_\tau(A) = \tau(A^c), \quad F^*_\tau(A) = \tau^*(A^c)
$$

for each $A \in I^X$.

**Theorem 3.4.** (a) $(\tau, \tau^*)$ is an IVIGO on $X$ if and only if $(F_\tau, F^*_\tau)$ is an IVIGC on $X$.

(b) $(F, F^*)$ is an IVIGC on $X$ if and only if $(\tau_F, \tau^*_F)$ is an IVIGO on $X$.

(c) $\tau_F = \tau$, $\tau^*_F = \tau^*$, $F_\tau = F$, $F^*_\tau = F^*$.

**Proof.** (a) Since $F^L(\tau) = F^L(\tau^c)$, $F^U(\tau) = F^U(\tau^c)$, $F^*(\tau) = F^*(\tau^c)$, we have

$$
F^L(\tau) \leq F^U(\tau), \forall A \in I^X \iff \tau^L(A^c) \leq \tau^U(A^c), \forall A \in I^X
$$

Similarly,

$$
F^*(\tau) \leq F^*(U) (A), \forall A \in I^X \iff \tau^*(L) \leq \tau^*(U) (A), \forall A \in I^X,
$$

$$
F^L(\tau) + F^*(\tau) \leq 1, \forall A \in I^X \iff F^L(\tau) + F^*(\tau) \leq 1, \forall A \in I^X.
$$

$F_\tau(0_X) = F_\tau(1_X) = 1, F^*_\tau(0_X) = F^*_\tau(1_X) = 0$

$$
\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0.
$$
$\mathcal{F}^L_r(A \cup B) \geq \mathcal{F}^L_r(A) \wedge \mathcal{F}^L_r(B), \forall A, B \in I^X$

$\Leftrightarrow \tau^L(A^c \cap B^c) \geq \tau^L(A^c) \wedge \tau^L(B^c), \forall A, B \in I^X$

$\Leftrightarrow \tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B), \forall A, B \in I^X$.

Similarly,

$\mathcal{F}^U_r(A \cup B) \geq \mathcal{F}^U_r(A) \wedge \mathcal{F}^U_r(B), \forall A, B \in I^X$

$\Leftrightarrow \tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B), \forall A, B \in I^X$,

$\mathcal{F}^*_r(A \cup B) \leq \mathcal{F}^*_r(A) \lor \mathcal{F}^*_r(B), \forall A, B \in I^X$

$\Leftrightarrow \tau^*(A \cap B) \leq \tau^*(A) \lor \tau^*(B), \forall A, B \in I^X$,

$\mathcal{F}^*_r(A \cup B) \leq \mathcal{F}^*_r(A) \lor \mathcal{F}^*_r(B), \forall A, B \in I^X$

$\Leftrightarrow \tau^*(A \cap B) \leq \tau^*(A) \lor \tau^*(B), \forall A, B \in I^X$.

Let $\{A_i : i \in \Gamma\} \subset I^X$. Then

$\mathcal{F}^L_r(\bigcap_{i \in \Gamma} A_i) = \tau^L((\bigcap_{i \in \Gamma} A_i)^c) = \tau^L(\bigcup_{i \in \Gamma} A_i^c)$,

$\mathcal{F}^U_r(\bigcap_{i \in \Gamma} A_i) = \tau^U((\bigcap_{i \in \Gamma} A_i)^c) = \tau^U(\bigcup_{i \in \Gamma} A_i^c)$,

$\mathcal{F}^*_r(A \cup B) \leq \mathcal{F}^*_r(A) \lor \mathcal{F}^*_r(B), \forall A, B \in I^X$

$\Leftrightarrow \tau^*(A \cap B) \leq \tau^*(A) \lor \tau^*(B), \forall A, B \in I^X$.

Hence we have

$\mathcal{F}^L_r(\bigcap_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \mathcal{F}^L_r(A_i), \forall \{A_i : i \in \Gamma\} \subset I^X$

$\Leftrightarrow \tau^L(\bigcup_{i \in \Gamma} A_i^c) \geq \bigwedge_{i \in \Gamma} \tau^L(A_i^c), \forall \{A_i : i \in \Gamma\} \subset I^X$

$\Leftrightarrow \tau^L(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau^L(A_i), \forall \{A_i : i \in \Gamma\} \subset I^X$.

Similarly,

$\mathcal{F}^U_r(\bigcap_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \mathcal{F}^U_r(A_i), \forall \{A_i : i \in \Gamma\} \subset I^X$

$\Leftrightarrow \tau^U(\bigcup_{i \in \Gamma} A_i^c) \geq \bigwedge_{i \in \Gamma} \tau^U(A_i^c), \forall \{A_i : i \in \Gamma\} \subset I^X$,

$\mathcal{F}^*_r(A \cup B) \leq \mathcal{F}^*_r(A) \lor \mathcal{F}^*_r(B), \forall A, B \in I^X$

$\Leftrightarrow \tau^*(A \cap B) \leq \tau^*(A) \lor \tau^*(B), \forall A, B \in I^X$.

$\mathcal{F}^*_r(A \cup B) \leq \mathcal{F}^*_r(A) \lor \mathcal{F}^*_r(B), \forall A, B \in I^X$

$\Leftrightarrow \tau^*(A \cap B) \leq \tau^*(A) \lor \tau^*(B), \forall A, B \in I^X$.
Thus \( (\tau, \tau^*) \) is an IVIGO on \( X \) if and only if \( (F_\tau, F^*_\tau) \) is an
IVIGC on \( X \).

(b) The proof is similar to (a).

(c) The proof is straightforward.

Let \( \{((\tau_i, \tau^*_i))_{i \in I} \} \) be a family of IVIGOs on \( X \). Then the intersection
of \( \{((\tau_i, \tau^*_i))_{i \in I} \} \) is defined by \( \cap_{i \in I} (\tau_i, \tau^*_i) = (\cap_{i \in I} \tau_i, \cup_{i \in I} \tau^*_i) \), where
\( (\cap_{i \in I} \tau_i)(A) = [\cap_{i \in I} \tau^L_i(A), \cap_{i \in I} \tau^U_i(A)] \) and \( (\cup_{i \in I} \tau^*_i)(A) = [\vee_{i \in I} \tau^L_i(A), \vee_{i \in I} \tau^U_i(A)] \) for each \( A \in I^X \).

**Theorem 3.5.** If \( \{((\tau_i, \tau^*_i))_{i \in I} \} \) is a family of IVIGOs on \( X \), then
\( \cap_{i \in I} (\tau_i, \tau^*_i) \) is an IVIGO on \( X \).

**Proof.** The proof is straightforward.

Let \( (\tau, \tau^*) \) be an IVIGO on \( X \). For \( [r, s] \in D(I) \), we define
\[
\tau_{[r,s]} = \{ A \in I^X : \tau(A) \geq [r, s] \},
\]
\[
\tau^*_{[r,s]} = \{ A \in I^X : \tau^*(A) \leq [1 - s, 1 - r] \},
\]
\[
(\tau, \tau^*)_{[r,s]} = \{ A \in I^X : \tau(A) \geq [r, s] \text{ and } \tau^*(A) \leq [1 - s, 1 - r] \}.
\]

**Theorem 3.6.** Let \( (\tau, \tau^*) \) be an IVIGO on \( X \) and \( [r, s] \in D(I) \). Then
\( \tau_{[r,s]}, \tau^*_{[r,s]} \) and \( (\tau, \tau^*)_{[r,s]} \) are Chang’s fuzzy topologies on \( X \).

**Proof.** Suppose that \( (\tau, \tau^*) \) is an IVIGO on \( X \) and \( [r, s] \in D(I) \).
We will prove that \( (\tau, \tau^*)_{[r,s]} \) is a Chang’s fuzzy topology on \( X \). Since
\( \tau(0_X) = r \) and \( \tau^*(0_X) = 1 \) and \( \tau^*(0_X) = 0 \), \( \tau^L(0_X) = 1 \geq r, \tau^U(0_X) = 1 \geq r \), \( \tau^L(1_X) \) \( = 1 \geq r, \tau^U(1_X) = 1 \geq r \), \( \tau^L(0_X) = 0 \), \( \tau^U(0_X) = 0 \), \( \tau^L(1_X) = 0 \), \( \tau^U(1_X) = 0 \). Thus \( \tau(0_X) \geq [r, s], \tau^L(1_X) \geq [r, s] \) and \( \tau^*(0_X) \leq [1 - s, 1 - r], \tau^*(1_X) \leq [1 - s, 1 - r] \). Hence \( 0_X, 1_X \in (\tau, \tau^*)_{[r,s]} \). Let \( A, B \in (\tau, \tau^*)_{[r,s]} \). Then
\( \tau^L(A) \geq r, \tau^U(A) \geq s, \tau^L(B) \geq r, \tau^U(B) \geq s \) and \( \tau^L(A) \leq 1 - s, \tau^U(A) \leq 1 - r, \tau^L(B) \leq 1 - s, \tau^U(B) \leq 1 - r \). So \( \tau^L(A \cap B) \geq \tau^L(A) \cap \tau^L(B) \geq r, \tau^U(A \cap B) \geq \tau^U(A) \land \tau^U(B) \geq s \) and \( \tau^L(A \cap B) \leq \tau^L(A) \lor \tau^L(B) \leq 1 - s, \tau^U(A \cap B) \leq \tau^U(A) \lor \tau^U(B) \leq 1 - r \). Thus \( \tau(A \cap B) \geq [r, s], \tau^L(A \cap B) \leq [1 - s, 1 - r] \). Hence \( A \cap B \in (\tau, \tau^*)_{[r,s]} \). Let \( \{ A_i : i \in I \} \subset (\tau, \tau^*)_{[r,s]} \). Then \( \tau^L(A_i) \geq r, \tau^U(A_i) \geq s \) and \( \tau^L(A_i) \leq 1 - s, \tau^U(A_i) \leq 1 - r \) for each \( i \in I \). So \( \tau^L(\bigcup_{i \in I} A_i) \geq \tau^L(A_i) \geq r, \tau^U(\bigcup_{i \in I} A_i) \geq \tau^U(A_i) \geq s \) and \( \tau^L(\bigcup_{i \in I} A_i) \leq \tau^L(A_i) \leq 1 - s, \tau^U(\bigcup_{i \in I} A_i) \leq \tau^U(A_i) \leq 1 - r \).
1 − r. Thus \( \tau(\cup_{i \in \Gamma} A_i) \geq [r, s] \) and \( \tau^*(\cup_{i \in \Gamma} A_i) \leq [1 - s, 1 - r] \). Hence \( \cup_{i \in \Gamma} A_i \in (\tau, \tau^*)_{[r, s]} \). Therefore \( (\tau, \tau^*)_{[r, s]} \) is a Chang’s fuzzy topology on \( X \).

Similarly, \( \tau_{[r, s]} \) and \( \tau^*_{[r, s]} \) are Chang’s fuzzy topologies on \( X \).

\[ \square \]

**Theorem 3.7.** Let \( (\tau, \tau^*) \) be an IVIGO on \( X \). Then \( \{\tau_{[r, s]}\}_{[r, s] \in D(I)} \) and \( \{\tau^*_{[r, s]}\}_{[r, s] \in D(I)} \) are two descending families of Chang’s fuzzy topologies on \( X \) such that \( \tau_{[r, s]} = \cap_{[p, q] < [r, s]} \tau_{[p, q]} \) and \( \tau^*_{[r, s]} = \cap_{[p, q] < [r, s]} \tau^*_{[p, q]} \) for each \( [r, s] \in D(I_0) \).

**Proof.** Let \([r, s], [t, u] \in D(I)\) with \([r, s] \leq [t, u]\). If \( A \in \tau_{[t, u]} \), then \( \tau^L(A) \geq t \) and \( \tau^U(A) \geq u \). So \( \tau^L(A) \geq r \) and \( \tau^U(A) \geq s \). Thus \( A \in \tau_{[r, s]} \). So \( \tau_{[t, u]} \subset \tau_{[r, s]} \). Similarly, \( \tau^*_{[t, u]} \subset \tau^*_{[r, s]} \). Therefore the families \( \{\tau_{[r, s]}\}_{[r, s] \in D(I)} \) and \( \{\tau^*_{[r, s]}\}_{[r, s] \in D(I)} \) are descending.

Let \([r, s] \in D(I_0)\). Since the family \( \{\tau_{[r, s]}\}_{[r, s] \in D(I)} \) is descending, \( \tau_{[r, s]} \subset \cap_{[p, q] < [r, s]} \tau_{[p, q]} \). If \( A \notin \tau_{[r, s]} \), then \( \tau^L(A) < r \) or \( \tau^U(A) < s \). Hence there exists \([p, q] \in D(I_0)\) with \([p, q] < [r, s] \) such that \( \tau^L(A) < p \) or \( \tau^U(A) < q \). Hence \( A \notin \cap_{[p, q] < [r, s]} \tau_{[p, q]} \). Thus \( \cap_{[p, q] < [r, s]} \tau_{[p, q]} \subset \tau_{[r, s]} \). Therefore \( \tau_{[r, s]} = \cap_{[p, q] < [r, s]} \tau_{[p, q]} \).

Similarly, \( \tau^*_{[r, s]} = \cap_{[p, q] < [r, s]} \tau^*_{[p, q]} \).

\[ \square \]

Let \( Y \subset X \). For each \( A \in I^X \), a fuzzy set \( A|_Y \), defined by \( A|_Y(x) = A(x), \ x \in Y \), is the restriction of \( A \) on \( Y \). For each \( B \in I^Y \), a fuzzy set \( B_X \), defined by \( B_X(x) = \begin{cases} B(x), & x \in Y \\ 0, & x \in X - Y \end{cases} \), is the extension of \( B \) on \( X \).

**Theorem 3.8.** Let \((X, \tau, \tau^*)\) be an IVISTS and \( Y \subset X \). Define two mappings \( \tau_Y, \tau^*_Y : I^Y \to D(I) \) by \( \tau_Y(A) = \cap \{\tau(B) : B \in I^X \text{ and } A \in I^Y \} \), \( \tau^*_Y(A) = \cap \{\tau^*(B) : B \in I^X \text{ and } B|_Y = A \} \) for each \( A \in I^Y \). Then \( (\tau_Y, \tau^*_Y) \) is an IVIGO on \( Y \) and \( \tau_Y(A) \geq \tau(A_X) \) and \( \tau^*_Y(A) \leq \tau^*(A_X) \) for each \( A \in I^Y \).

**Proof.** For each \( A \in I^Y \), let \( B \in I^X \) with \( B|_Y = A \). Since \( \tau^L(B) \leq \tau^L(B) \) and \( \tau^L(B) \leq \tau^L(B) \), \( \tau^L(B) = \cap \{\tau^L(B) : B \in I^X \text{ and } B|_Y = A \} \leq \cap \{\tau^L(B) : B \in I^X \text{ and } B|_Y = A \} = \tau_Y(A) \). Similarly, \( \tau^L(A) \leq \tau^L(A) \). Since \( 0 \leq \tau^U(B) + \tau^*U(B) \leq 1 \), \( \tau^U(B) \leq 1 - \tau^*U(B) \). Hence
we have
\[
\tau_Y^U(A) = \vee\{\tau^U(B) : B \in I^X \text{ and } B|_Y = A\}
\leq \vee\{1 - \tau^rU(B) : B \in I^X \text{ and } B|_Y = A\}
= 1 - \wedge\{\tau^U(B) : B \in I^X \text{ and } B|_Y = A\}
= 1 - \tau_Y^U(A).
\]
Therefore \(\tau_Y^U(A) + \tau_Y^rU(A) \leq 1\).

Clearly, \(\tau_Y(0_Y) = \tau_Y(1_Y) = 1\) and \(\tau_Y^r(0_Y) = \tau_Y^r(1_Y) = 0\).

Let \(A_1, A_2 \in I^Y\). Then \(\tau_Y^r(A_1 \cap A_2) = \wedge\{\tau^r(B) : B \in I^X \text{ and } B|_Y = A_1 \cap A_2\}\). If \(\tau_Y^r(A_1) \vee \tau_Y^r(A_2) = 1\), then \(\tau_Y^r(A_1 \cap A_2) \leq \tau_Y^r(A_1) \vee \tau_Y^r(A_2) = 1\). If \(\tau_Y^r(A_1) \vee \tau_Y^r(A_2) < 1\), take \([r, s]\) with \(\tau_Y^r(A_1) \vee \tau_Y^r(A_2) < [r, s] < 1\). Then there exists \(B_i \in I^X\) such that \(B_i|_Y = A_i\) and \(\tau^r(B_i) < [r, s]\) for \(i = 1, 2\). Since \((B_1 \cap B_2)|_Y = (B_1|_Y) \cap (B_2|_Y) = A_1 \cap A_2\) and \(\tau^r(B_1 \cap B_2) \leq \tau^r(B_1) \vee \tau^r(B_2) < [r, s]\), \(\tau_Y^r(A_1 \cap A_2) \leq \tau_Y^r(A_1) \vee \tau_Y^r(A_2)\). Thus \(\tau_Y^r(A_1) \vee \tau_Y^r(A_2) < [r, s]\) implies \(\tau_Y^r(A_1 \cap A_2) < [r, s]\). Hence \(\tau_Y^r(A_1 \cap A_2) \leq \tau_Y^r(A_1) \vee \tau_Y^r(A_2)\). Therefore \(\tau_Y^r(L(A_1 \cap A_2) \leq \tau_Y^r(L(A_1) \vee \tau_Y^r(L(A_2))\) and \(\tau_Y^U(A_1 \cap A_2) \leq \tau_Y^U(A_1) \vee \tau_Y^U(A_2)\).

Similarly, \(\tau_Y(L(A_1 \cap A_2) \geq \tau_Y^L(A_1) \wedge \tau_Y^L(A_2)\) and \(\tau_Y^U(A_1 \cap A_2) \geq \tau_Y^U(A_1) \wedge \tau_Y^U(A_2)\).

Let \(\{A_i : i \in \Gamma\} \subset I^Y\). Then \(\tau_Y^r(\bigcup_{i \in \Gamma} A_i) = \wedge\{\tau^r(B) : B \in I^X \text{ and } B|_Y = \bigcup_{i \in \Gamma} A_i\}\). If \(\bigwedge_{i \in \Gamma} \tau_Y^r(A_i) = 1\), then \(\tau_Y^r(\bigcup_{i \in \Gamma} A_i) \leq \bigwedge_{i \in \Gamma} \tau_Y^r(A_i) = 1\).

If \(\bigvee_{i \in \Gamma} \tau_Y^r(A_i) < 1\), take \([r, s]\) with \(\bigvee_{i \in \Gamma} \tau_Y^r(A_i) < [r, s] < 1\). Then \(\tau_Y^r(A_i) < [r, s]\) for each \(i \in \Gamma\). Hence there exist \(B_i \in I^X\) such that \(B_i|_Y = A_i\) and \(\tau^r(B_i) < [r, s]\) for each \(i \in \Gamma\). Since \((\bigcup_{i \in \Gamma} B_i)|_Y = \bigcup_{i \in \Gamma} (B_i|_Y) = \bigcup_{i \in \Gamma} A_i\) and \(\tau^r(\bigcup_{i \in \Gamma} B_i) \leq \bigvee_{i \in \Gamma} \tau^r(B_i) \leq [r, s]\), \(\tau_Y^r(\bigcup_{i \in \Gamma} B_i) \leq \tau^r(\bigcup_{i \in \Gamma} B_i) \leq [r, s]\). Thus \(\bigvee_{i \in \Gamma} \tau_Y^r(A_i) < [r, s]\) implies \(\tau_Y^r(\bigcup_{i \in \Gamma} A_i) \leq [r, s]\). Hence \(\tau_Y^r(\bigcup_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} \tau_Y^r(A_i)\). Therefore \(\tau_Y^L(\bigcup_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} \tau_Y^L(A_i)\) and \(\tau_Y^U(\bigcup_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} \tau_Y^U(A_i)\). Similarly, \(\tau_Y(L(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau_Y^L(A_i)\) and \(\tau_Y^U(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau_Y^U(A_i)\).

Therefore \((\tau_Y, \tau_Y^r)\) is an IVIGO on \(Y\).

Clearly, \(\tau_Y(A) \geq \tau(A_X)\) and \(\tau_Y^r(A) \leq \tau^r(A_X)\) for each \(A \in I^Y\).

\[\square\]

**Theorem 3.9.** Let \((\mathcal{F}, \mathcal{F}^*)\) be an IVIGO on \(X \subset Y \subset X\). Define two mappings \(\mathcal{F}_Y, \mathcal{F}_Y^* : I^Y \to D(1)\) by \(\mathcal{F}_Y(A) = \vee\{\mathcal{F}(B) : B \in I^X \text{ and } B|_Y = A\}\), \(\mathcal{F}_Y^*(A) = \wedge\{\mathcal{F}^*(B) : B \in I^X \text{ and } B|_Y = A\}\) for each \(A \in I^Y\). Then \((\mathcal{F}_Y, \mathcal{F}_Y^*)\) is an IVIGO on \(Y\) and \(\mathcal{F}_Y(A) \geq \mathcal{F}(A_X)\) and \(\mathcal{F}_Y^*(A) \leq \mathcal{F}^*(A_X)\) for each \(A \in I^Y\).
Proof. The proof is similar to Theorem 3.8.

When $\tau_Y$ and $\tau^*_Y$ are defined as in Theorem 3.8, $(Y, \tau_Y, \tau^*_Y)$ is called an interval-valued intuitionistic fuzzy subspace of the IVISTS $(X, \tau, \tau^*)$.

**Theorem 3.10.** Let $(Y, \tau_Y, \tau^*_Y)$ be an interval-valued intuitionistic fuzzy subspace of the IVISTS $(X, \tau, \tau^*)$. Then

(a) $F_{\tau_Y}(A) = \bigvee \{\tau(B) : B \in I^X \text{ and } B|_Y = A\}$ and

$b) F_{\tau^*_Y}(A) = \bigwedge \{\tau^*_Y(B) : B \in I^X \text{ and } B|_Y = A\}$

for each $A \in I^Y$.

(b) If $Z \subset Y \subset X$, then $\tau_Z = (\tau_Y)_Z$ and $\tau^*_Z = (\tau^*_Y)_Z$.

**Proof.** (a) For each $A \in I^Y$, we have

$F_{\tau_Y}(A) = \tau_Y(A^c)$

$= \bigvee \{\tau(B) : B \in I^X \text{ and } B|_Y = A^c\}$

$= \bigvee \{\tau(B) : B^c \in I^X \text{ and } B|_Y = A\}$

$= \bigvee \{\tau(B^c) : B^c \in I^X \text{ and } B|_Y = A\}$

$= \bigvee \{\tau(B) : B \in I^X \text{ and } B|_Y = A\}$.

Similarly, $F_{\tau^*_Y}(A) = \bigwedge \{\tau^*_Y(B) : B \in I^X \text{ and } B|_Y = A\}$

(b) For each $A \in I^Z$, we have

$(\tau_Y)_Z(A) = \bigvee \{\tau_Y(B) : B \in I^Y \text{ and } B|_Z = A\}$

$= \bigvee \{\bigvee \{\tau(C) : C \in I^X \text{ and } C|_Y = B\} : B \in I^Y \text{ and } B|_Z = A\}$

$= \bigvee \{\tau(C) : C \in I^X \text{ and } C|_Z = A\}$

$= \tau_Z(A)$.

Hence $\tau_Z = (\tau_Y)_Z$. Similarly, $\tau^*_Z = (\tau^*_Y)_Z$.

4. Interval-valued intuitionistic gradation preserving mappings

**Definition 4.1.** Let $(X, \tau, \tau^*)$ and $(Y, \eta, \eta^*)$ be two IVISTSs and $f : X \to Y$ be a mapping. Then $f$ is called an interval-valued intuitionistic gradation preserving mapping (for short, an IVIGP-mapping) if for each $A \in I^Y$, $\eta(A) \leq \tau(f^{-1}(A))$ and $\eta^*(A) \geq \tau^*(f^{-1}(A))$. 
Theorem 4.2. Let \((X, \tau, \tau^*)\) and \((Y, \eta, \eta^*)\) be two IVISTSs and \(f : X \to Y\) be a mapping. Then \(f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)\) is an IVIGP-mapping if and only if \(f : (X, \tau^L, \tau^{*L}) \to (Y, \eta^L, \eta^{*L})\) and \(f : (X, \tau^U, \tau^{*U}) \to (Y, \eta^U, \eta^{*U})\) are GP-mappings.

Proof. The proof is straightforward.

Definition 4.3. [1]. Let \((X, T, T^*)\) and \((Y, S, S^*)\) be two bitopological spaces of fuzzy subsets. Then a mapping \(f : (X, T, T^*) \to (Y, S, S^*)\) is said to be continuous if \(f : (X, T) \to (Y, S)\) and \(f : (X, T^*) \to (Y, S^*)\) are continuous.

Theorem 4.4. Let \((X, \tau, \tau^*)\) and \((Y, \eta, \eta^*)\) be two IVISTSs and \(f : X \to Y\) be a mapping. Then \(f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)\) is an IVIGP-mapping if and only if \(f : (X, \tau_{[r,s]}, \tau^*_{[r,s]}) \to (Y, \eta_{[r,s]}, \eta^*_{[r,s]})\) is continuous for each \([r, s] \in D(I_0)\).

Proof. Suppose that \(f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)\) is an IVIGP-mapping. Let \([r, s] \in D(I_0)\). If \(A \in \eta_{[r,s]}\), then \(\eta(A) \ge [r, s]\). By hypothesis, \(\eta(A) \le \tau(f^{-1}(A))\) and so \(\tau(f^{-1}(A)) \ge [r, s]\), i.e., \(f^{-1}(A) \in \tau_{[r,s]}\). Hence \(f : (X, \tau_{[r,s]}) \to (Y, \eta_{[r,s]})\) is continuous. If \(A \in \eta^*_{[r,s]}\), then \(\eta^*(A) \le [1-s, 1-r]\). By hypothesis, \(\eta^*(A) \ge \tau^*(f^{-1}(A))\) and so \(\tau^*(f^{-1}(A)) \le [1-s, 1-r]\), i.e., \(f^{-1}(A) \in \tau^*_{[r,s]}\). Hence \(f : (X, \tau^*_{[r,s]}) \to (Y, \eta^*_{[r,s]})\) is continuous. Therefore \(f : (X, \tau_{[r,s]}, \tau^*_{[r,s]}) \to (Y, \eta_{[r,s]}, \eta^*_{[r,s]})\) is continuous.

Conversely, suppose that \(f : (X, \tau_{[r,s]}, \tau^*_{[r,s]}) \to (Y, \eta_{[r,s]}, \eta^*_{[r,s]})\) is continuous for each \([r, s] \in D(I_0)\). Let \(A \in I^Y\). If \(\eta(A) = 0\), then \(\eta(A) \le \tau(f^{-1}(A))\). If \(\eta(A) = [r, s] \in D(I_0)\), then \(A \in \eta_{[r,s]}\). By hypothesis, \(f^{-1}(A) \in \tau_{[r,s]}\), i.e., \(\tau(f^{-1}(A)) \ge [r, s]\). Thus \(\eta(A) \le \tau(f^{-1}(A))\). If \(\eta^*(A) = 1\), then \(\eta^*(A) \ge \tau^*(f^{-1}(A))\). If \(\eta^*(A) = [r, s] < 1\), then \([1-s, 1-r] \in D(I_0)\) and \(\eta^*(A) = [r, s] = [1-(1-r), 1-(1-s)]\). Hence \(A \in \eta^*_{[1-s,1-r]}\). By hypothesis, \(f^{-1}(A) \in \tau^*_{[1-s,1-r]}\). Thus \(\tau^*(f^{-1}(A)) \le [1-(1-r), 1-(1-s)] = [r, s]\). Hence \(\eta^*(A) \ge \tau^*(f^{-1}(A))\). Therefore \(f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)\) is an IVIGP-mapping.

Definition 4.5. Let \((X, \tau, \tau^*)\) be an IVISTS and \(A \in I^X\). Then the \(([r, s], [t, u])\)-interval-valued intuitionistic fuzzy closure and \(([r, s], [t, u])\)-interval-valued intuitionistic fuzzy interior of \(A\) are defined by:

\[
closure_{[r,s],[t,u]}(A) = \cap \{K \in I^X : A \subseteq K, \ F_\tau(K) \ge [r, s], \ F_{\tau^*}(K) \le [t, u]\},
\]

\[
interior_{[r,s],[t,u]}(A) = \cap \{K \in I^X : A \subseteq K, \ F_\tau^*(K) \le [t, u], \ F_{\tau^*}(K) \ge [r, s]\}.
\]
int_{[r,s],[t,u]}(A) = \bigcup \{ G \in I^X : G \subset A, \ \tau(G) \geq [r,s], \ \tau^*(G) \leq [t,u] \},

where [r,s] \in D(I_0), [t,u] \in D(I_1) with s + u \leq 1.

Note that (cl_{[r,s],[t,u]}(A))^c = int_{[r,s],[t,u]}(A^c) and (int_{[r,s],[t,u]}(A))^c = cl_{[r,s],[t,u]}(A^c) for each A \in I^X.

**Theorem 4.6.** Let \((X, \tau, \tau^*)\) and \((Y, \eta, \eta^*)\) be two IVISTSs and \([r,s] \in D(I_0), [t,u] \in D(I_1) with s + u \leq 1\). If \(f : (X, \tau, \tau^*) \rightarrow (Y, \eta, \eta^*)\) is an IVIGP-mapping, then

(a) \(f(cl_{[r,s],[t,u]}(A)) \subset cl_{[r,s],[t,u]}(f(A))\) for each \(A \in I^X\).

(b) \(cl_{[r,s],[t,u]}(f^{-1}(A)) \subset f^{-1}(cl_{[r,s],[t,u]}(A))\) for each \(A \in I^Y\).

(c) \(f^{-1}(int_{[r,s],[t,u]}(A)) \subset int_{[r,s],[t,u]}(f^{-1}(A))\) for each \(A \in I^Y\).

**Proof.** (a) For each \(A \in I^X\), we have

\[
\begin{align*}
&f^{-1}(cl_{[r,s],[t,u]}(f(A))) \\
&= f^{-1}(\cap \{ K \in I^Y : f(A) \subset K, \ \mathcal{F}_\eta(K) \geq [r,s], \mathcal{F}_{\eta^*}(K) \leq [t,u] \}) \\
&= f^{-1}(\cap \{ K \in I^Y : f(A) \subset K, \ \eta(K^c) \geq [r,s], \eta^*(K^c) \leq [t,u] \}) \\
&\supset f^{-1}(\cap \{ K \in I^Y : f(A) \subset K, \ \tau(f^{-1}(K^c)) \geq [r,s], \tau^*(f^{-1}(K^c)) \leq [t,u] \}) \\
&= f^{-1}(\cap \{ K \in I^Y : f(A) \subset K, \ \tau((f^{-1}(K))^c) \geq [r,s], \tau^*((f^{-1}(K))^c) \leq [t,u] \}) \\
&\supset f^{-1}(\cap \{ K \in I^Y : f^{-1}(K) \subset A, \ \mathcal{F}_\tau(f^{-1}(K)) \geq [r,s], \mathcal{F}_{\tau^*}(f^{-1}(K)) \leq [t,u] \}) \\
&= \cap \{ f^{-1}(K) : K \in I^Y, \ A \subset f^{-1}(K), \ \mathcal{F}_\tau(f^{-1}(K)) \geq [r,s], \mathcal{F}_{\tau^*}(f^{-1}(K)) \leq [t,u] \} \\
&= cl_{[r,s],[t,u]}(A).
\end{align*}
\]

Hence \(f(cl_{[r,s],[t,u]}(A)) \subset f(f^{-1}(cl_{[r,s],[t,u]}(f(A)))) \subset cl_{[r,s],[t,u]}(f(A))\).

(b) Let \(A \in I^Y\). Then \(f^{-1}(A) \in I^X\). By (a), we have

\[
\begin{align*}
cl_{[r,s],[t,u]}(f^{-1}(A)) &\subset f^{-1}(f(cl_{[r,s],[t,u]}(f(A)))) \\
&\subset f^{-1}(cl_{[r,s],[t,u]}(f(f^{-1}(A)))) \\
&\subset f^{-1}(cl_{[r,s],[t,u]}(A)).
\end{align*}
\]
(c) Let $A \in I^Y$. By (b), $cl_{[r,s],[t,u]}(f^{-1}(A^c)) \subset f^{-1}(cl_{[r,s],[t,u]}(A^c))$ and so $(f^{-1}(cl_{[r,s],[t,u]}(A^c)))^c \subset (cl_{[r,s],[t,u]}(f^{-1}(A^c)))^c$. Hence

\[
\begin{align*}
    f^{-1}(int_{[r,s],[t,u]}(A)) &= (f^{-1}(cl_{[r,s],[t,u]}(A^c)))^c \\
    &\subset (cl_{[r,s],[t,u]}(f^{-1}(A^c)))^c \\
    &= int_{[r,s],[t,u]}(f^{-1}(A)).
\end{align*}
\]

\[\square\]

References

[1] A. S. Abu Safiya, A. A. Fora and M. W. Warner, Higher separation axioms in byfuzzy topological spaces, Fuzzy Sets and Systems 79 (1996), 367–372.
[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1) (1986), 87–96.
[3] K. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (3) (1989), 343–349.
[4] R. Badard, Smooth axiomatics, First IFSA Congress, Palma de Mallorca (July 1986).
[5] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 182–190.
[6] K. C. Chattopadhyay, R. N. Hazra and S. K. Samanta, Gradation of openness:fuzzy topology, Fuzzy Sets and Systems 49 (1992), 237-242.
[7] D. Coker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88 (1997), 81–89.
[8] R. N. Hazra, S. K. Samanta and K. C. Chattopadhyay, Fuzzy topology redefined, Fuzzy Sets and Systems 45 (1992), 79–82.
[9] T. K. Mondal and S. K. Samata, On intuitionistic gradation of openness, Fuzzy Sets and Systems 131 (2002), 323-336.
[10] T. K. Mondal and S. K. Samata, Topology of interval-valued fuzzy sets, Indian J. Pure Appl. Math. 30 (1) (1999), 23–38.
[11] T. K. Mondal and S. K. Samata, Topology of interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 119 (2001), 483–494.
[12] A. A. Ramadan, Smooth topological spaces, Fuzzy Sets and Systems 48 (1992), 371–375.
[13] S. K. Samata and T. K. Mondal, Intuitionistic gradation of openness: intuitionistic fuzzy topology, Busefal 73 (1997), 8–17.
[14] L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965), 338–353.
[15] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning I, Inform. Sci. 8 (1975), 199–249.
Chun-Kee Park
Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea
E-mail: ckpark@kangwon.ac.kr