FANO VERSUS CALABI - YAU

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Abstract. In this article we discuss some numerical parts of the mirror conjecture. For any 3-dimensional Calabi-Yau manifold author introduces a generalization of the Casson invariant known in 3-dimensional geometry, which is called Casson-Donaldson invariant. In the framework of the mirror relationship it corresponds to the number of SpLag cycles which are Bohr-Sommerfeld with respect to the given polarization. To compute the Casson-Donaldson invariant the author uses well known in classical algebraic geometry degeneration principle. By it, when the given Calabi-Yau manifold is deformed to a pair of quasi Fano manifolds glued upon some K3-surface, one can compute the invariant in terms of "flag geometry" of the pairs (quasi Fano, K3-surface).

Introduction

According to the "Oxford program" proposed by S. Donaldson and R. Thomas ([4]), some constructions of real gauge theories can be repeated after generalization to the complex case. In a sense, it is a variant of Arnold sentence which claims that every real notion has (or should have) an analogy in the complex case ([1]). So it is a kind of complexification. The authors of [4] propose two possible ways allowing to translate the Chern-Simons theory and the Yang-Mills theory originally defined over 3-dimensional and 4-dimensional real manifolds respectively to the cases of 3-dimensional and 4-dimensional complex manifolds. The first move comes with the question what should substitute the notion of orientation in the complex case. Reasoning in the way parallel to the original real case, authors of [4] show that compact complex manifold is c-orientable iff its canonical class is trivial. Thus they restrict the investigation by the condition which imposes that we deal with the case of Calabi-Yau manifolds. Thus a c-orientation is defined by an appropriate trivialization of the canonical class. Here we study the 3-dimensional complex case which contains 3-dimensional Calabi-Yau manifolds.

This case is of an enormous interest because of some physical applications: the Calabi-Yau realm produces collections of non negative numbers (so called Casson-Donaldson invariant) which enter in the computations of physical theories. The combinatorical structure of these collections is the following: the set $\mathcal{CY}_3$ of all possible topological types of compact non singular Calabi-Yau threefolds can be coded by the set of lattices (so called Mukai lattices) equipped with bilinear forms

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1The last lecture of Professor A.N. Tyurin (24.02.1940 - 27.10.2002) given at the Fano conference (Turin, October 2002). The text was prepared and edited by Nik. Tyurin and Yu. Tiourina. We'd like to thank Max-Planck Institute for Mathematics (Bonn, Germany).
(so called Mukai forms); any pair \((M, \chi)\) of the objects represents the corresponding topological \(K\)-algebra of the vector bundles (in the broad sense of the reflexive sheaves) over a given Calabi-Yau threefold; and the Casson-Donaldson invariant is a map

\[
CD : \{(M, \chi)\} \rightarrow \mathbb{Z},
\]

which gives us a collection of numbers derived from the given topological type. Thus one of the problems posed by physics is to compute that numbers which is a very hard task in general.

The original real construction gives a hint about what can be exploited to perform the computations in some special cases when the given Calabi-Yau manifolds are constructive. It means that for a given \(M\) there exists its birational model such that a deformation to a reducible threefold is possible:

\[
M_0 = F_+ \cup S F_- , \tag{0.1}
\]

where \(F_\pm\) are two quasi Fano components, glued together transversally along a K3 - surface \(S\) which belongs simultaneously to both the anticanonical systems of the components. In this setup:

1) the Mukai lattice of \(M\) can be reproduced from the Mukai lattices of the components \(F_\pm\)

and

2) CD - invariant can be derived from the geometry of the Fano varieties.

Thus in the constructive case the knowledge of the Fano geometry is sufficient to perform the desired computations.

Here we have to note that as it was emphasized by the authors of [4] the collection of CD - invariants is an analogy of the classical Casson invariant of compact oriented real 3 - manifolds. Since the classical one is computed using the Heegard decomposition, in the complex case one could exploit the same idea. As an intermediate step in the decomposition for the real manifolds one considers the following picture: a Riemannian surface \(\Sigma\) is included into a given 3 - dimensional manifold \(X\), cutting it into two pieces glued over the surface. Thus the picture is

\[
X = H_+ \cup_\Sigma H_- ,
\]

where \(H_\pm\) is an oriented 3 - manifold with boundary such that

\[
\partial H_\pm = \pm \Sigma.
\]

This is exactly the same as in (0.1) since the complex analogy of real 3 - dimensional oriented manifold with boundary is quasi Fano variety with a fixed K3 - surface from the anticanonical system. This K3 - surface plays the role of the complex boundary: since it should be \(c\) - oriented it is a K3 - surface.

This parallel imposes the following sentence: since every real 3 - dimensional oriented compact manifold can be represented by an appropriate Heegard diagram, so we can expect (basing on our practical experience) that every Calabi-Yau manifold in dimension 3 is constructive and thus we can restrict the investigations of non linear super \(\sigma\) - models with CY - target spaces to the Fano realm. Also, if it is true it should be just a finite set of different topological types for 3 - dimensional Calabi-Yau manifolds.
1. Mirror picture: LHS and RHS

As it was observed first by theoretical physicists, for some Calabi-Yau 3-manifolds one has so-called mirror partners such that if \( X \) and \( X' \) are in the "mirror" correspondence then the deformation properties of the first one correspond to some properties of the cubic intersection form restricted to the Picard lattice over the second one ([14]). It implies in particular that the Hodge diamonds of \( X \) and \( X' \) are related by some reflection. Even more rough description recognizes \( X \) and \( X' \) as mirror partners if the equality holds only for two entries of the Hodge diamonds namely

\[
h^{1,1}(X) = h^{1,2}(X')
\]

and vice versa. While this weakening of the mirror conditions leads to pure numerical coincidences only, we shall understand the mirror correspondence in the following way: two 3-dimensional Calabi-Yau manifolds are related by the mirror duality if the algebraic geometry over the first one is isomorphic to the symplectic geometry over the second one and vice versa. Of course it isn’t quite clear how to treat both the geometries so one should explain first what do they mean as well as define what kind of equivalence is desired to claim that these geometries are isomorphic. A slightly abstract approach to this problem is given by so-called homological mirror symmetry proposed by M. Kontsevich which suggests some equivalence between two derivations from the geometries: on the LHS (algebraic geometry) one takes the derived category of coherent sheaves while on the RHS (symplectic geometry) it should be the Fukaya category — and these two are isomorphic as categories. On the other hand, the algebraic geometry contains some "real" objects, which have been constructed in boundaries of its framework and which are well understood and more than habitual to a number of the world experts who gathered at our conference. We mean the moduli spaces of vector bundles and sheaves and the systems of submanifolds (f.e. complete linear systems or related objects such as the Chow groups etc.). At the same time one could expect that the RHS of the mirror picture admits some objects of the same reality — a moduli space and systems of some special submanifolds. From this point of view the RHS used to be "terra incognita" up to the middle of nineties. At that time, different ways of developing the "classical" symplectic geometry were consolidated and united such that the Floer homology of Lagrangian submanifolds, pseudoholomorphic curves (in dimension 4), special Lagrangian cycles (for the Calabi-Yau case) etc. turned to be the parts of some unified consistent theory. Thus now one has the ingredients at the RHS which can be compared to the known objects existing on the LHS. In this section we briefly recall the generating objects for some numerical derivations on both the sides of the mirror picture.

2.1. LHS.

Let \( M \) be a smooth complete algebraic Calabi-Yau threefold which is equipped with a Ricci flat Kahler metric \( g \), giving the corresponding Kahler form \( \omega \). At the beginning point, one fixes an additional data: a complex orientation — a choice of a holomorphic \((3,0)\)-form \( \Omega \) which is defined up to the phase scaling \( e^{i\phi} \) since the norm is fixed by the metric. We collect these data in the quadruple

\[(M, g, \omega, \Omega).\]
The ring of even dimensional cohomology of $M$

$$H^{2i}(M, \mathbb{Z}) = \bigoplus_{i=0}^{3} H^{2i}(M, \mathbb{Z})$$

can be equipped with an involution $*$ acting componentwise by the following formulas:

$*$|_{H^0(M, \mathbb{Z}) \oplus H^4(M, \mathbb{Z})} = id,$

$*$|_{H^2(M, \mathbb{Z}) \oplus H^6(M, \mathbb{Z})} = -id.$

The definition implies that the bilinear form

$$(u, v) = -[v^* \cdot u]_6$$

is skew-symmetric (one could recognize the reason to introduce $*$ exactly in the derivation of this skew-symmetric property). At the same time one has a similar natural involution on the algebraic $K$-functor $K^0_{alg}$ on $M$ which sends each vector bundle $E$ to the dual bundle $E^*$. It’s easy to see that these two involutions are related by the homomorphism

$$ch : K^0_{alg} \rightarrow H^{2*}(M, \mathbb{Q})$$

such that the diagram

$$
\begin{array}{ccc}
K^0_{alg} & \xrightarrow{ch} & H^{2*}(M, \mathbb{Q}) \\
\downarrow^* & & \downarrow^* \\
K^0_{alg} & \xrightarrow{ch} & H^{2*}(M, \mathbb{Q})
\end{array}
$$

commutes (here $ch$ is the Chern character map).

On the other hand one has on $K^0_{alg}$ the following bilinear form

$$-\chi(E_1, E_2) = \sum_{i=0}^{3} (-1)^{i+1} \text{rk} \text{Ext}^i(M, E_1^* \otimes E_2),$$

where the homological spaces are the coherent cohomology of sheaves. This form can be represented as the image of a bilinear form over $H^{2*}(M, \mathbb{Z})$; recall that by the Riemann - Roch - Hirzebruch theorem we have

$$\chi(E_1, E_2) = [chE^2 \cdot chE^*_1 \cdot td_M]_6,$$

where $td_M$ is the Todd class of $M$. In the special situation of the Calabi - Yau manifolds bilinear form $\chi$ is skew symmetric:

$$H^0(M, E_1^* \otimes E_2)^* = H^3(E_2^* \otimes E_1)$$

$$H^1(E_1^* \otimes E_2)^* = H^2(E_2^* \otimes E_1)$$

$$H^2(E_1^* \otimes E_2)^* = H^1(E_2^* \otimes E_1)$$

$$H^3(E_1^* \otimes E_2)^* = H^0(E_2^* \otimes E_1)$$
by the Serre duality which in this case reads as
\[ H^i(E^* \otimes F) = H^{3-i}(F^* \otimes E) \]
since the canonical class is trivial. Moreover, the Todd class of \( M \) equals
\[ td_M = 1 + \frac{1}{12}c_2(M) \]
and consequently
\[ td^*_M = td_M \]
since \( c_2(M) = c_2^*(M) \) by the definition. Thus we have a special class in \( H^{2*}(M, \mathbb{Q}) \) namely
\[ \sqrt{td_M} = 1 + \frac{1}{24}c_2(M) \]
defined uniquely by the condition that the leading term is 1.

Following S. Mukai we slightly correct the Chern character map twisting it by the element
\[ m(E) = chE \cdot \sqrt{td_M} \in H^{2*}(M, \mathbb{Z}) \]
for a vector bundle \( E \); we call it the Mukai vector of \( E \). The Mukai vectors form a lattice \( L_M \) which is called the Mukai lattice, equipped with the bilinear skew symmetric form \( \chi \); thus the topological type of the underlayng Calabi - Yau 3 - manifold can be encoded by the pair \( (L_M, \chi) \).

On the other hand, if \( E \) is a holomorphic bundle over \( M \), then one has the following derivations from the deformation theory:
1) the space \( Ext^1(E, E) \) is the space of infinitesimal deformations of \( E \);
2) the space \( Ext^2(E, E) \) is the space of obstructions;
and
3) locally the moduli space of holomorphic vector bundles of type \( m(E) \) is presented by the Kuranishi map
\[ k : Ext^1(E, E) \rightarrow Ext^2(E, E) \]
such that
\[ \mathcal{M}_E|_{O(E)} = k^{-1}(0) \]
in a small neighborhood \( O(E) \) of the given point \( E \in \mathcal{M}_E \).

But in the Calabi - Yau realm
\[ Ext^1(E, E) = Ext^2(E, E)^* \]
by the Serre duality. Thus one could expect in general that the moduli spaces are zero dimensional. We claim here that at least the virtual (expected) dimension of the moduli space of simple vector bundles of a topological type \( m(E) \in L_M \) is zero.

Further, the Kahler form \( \omega \) fixed on \( M \) determines a polarization \( H \) on \( M \) which separates special class of holomorphic vector bundles — the bundles which are stable
with respect to the given polarization. The stability condition imposes one additional requirement on the topological types: for a vector bundle (or reflexive sheaf) \( E \) the expression
\[
c_2(E) - \frac{rk(E) - 1}{2 \cdot rk(E)} \cdot c_1^2(E) = \Delta(E)
\]
is called the discriminant of the bundle. Then the Bogomolov inequality necessary for any stable bundle is given by the constraint
\[
\Delta(E) \cdot H > 0
\]
for the polarization \( H \) over \( M \).

Roughly the idea of the Casson - Donaldson invariant is as follows: the moduli space of \( H \)-stable holomorphic vector bundles of a given type \( m \in H^2(M, \mathbb{Z}) \) is expected to be a finite set
\[
\mathcal{M}_H(m) = \{p_1, \ldots, p_k\},
\]
Thus, counting the number, one can define an integer function on the Mukai lattice
\[
CD_H(m) = \deg \mathcal{M}_H(m) \in \mathbb{Z}
\]
and call it the Casson - Donaldson invariant. Of course it is an ideal picture: for example, we added the stability condition since otherwise the set of holomorphic vector bundles of a fixed type doesn’t in general possess any good structure. For stable bundles of a fixed topological type \( m \in \mathcal{L}_M \) the moduli space admits a natural compactification \( \overline{\mathcal{M}}_H^*(m) \) which in the transversal case carries the structure of zero dimensional scheme. As the length of the scheme the number
\[
CD_H(m) = \deg \overline{\mathcal{M}}_H^*(m)
\]
is well defined. It is quite reasonable to call this number the Casson - Donaldson invariant: on the one hand, it is obviously analogous to the Casson invariant as it was proposed in [4]; on the other hand, it is an obvious analogy of the Donaldson polynomial of degree zero for real 4-manifolds ([3]).

The problem is that in the real life the transversal situation can not occur for some bundles: deformations can be unobstructed what happens, for example, in the case of the tangent bundle of \( M \). Then the geometrical dimension of the moduli space of the topological type of \( TM \) is equal to the dimension of the component of \( CY_3 \)-moduli space. To get the number in this case we have to apply the usual “deformation to the normal cone” trick.

The difference between the virtual and the infinitesimal dimensions can be measured as follows. Let us consider on the set of all vector bundles some other bilinear form \( h \) defined as
\[
h(E_1, E_2) = rkH^1(E_1^* \otimes E_2) - rkH^0(E_1^* \otimes E_2).
\]
Then it is easy to see that \( \chi \) is exactly the skew symmetric part of \( h \). At the same time the symmetrical part of \( h \) could be viewed as an analogy of the sum of the Betti
numbers. We use it instead of the usual Euler characteristics of coherent sheaves taking in mind the real case where the sum of the Betti numbers gives a numerical estimation for the statement of the Arnold’s conjecture, instead of pure topological Euler characteristics of the based real manifold. As in the symplectic geometry, in our case the number given by \( h \) is not a topological invariant. Now we see that the difference between the virtual and the infinitesimal dimensions is measured by the bilinear form \( h \). It is fruitful to consider the vector bundles for which these dimensions coincide. Such simple bundles are called expectional, aspheric or spheric depending on the context (we will see at the next subsection that they are analogous to some spheres on the RHS). We will call such bundles expectional because they are simple and infinitesimally rigid. S. Mukai noted that the bundles play the role of roots in the Mukai lattice. In particular, for every such a bundle one has

\[ h(E, E) = -1 \]

which is the minimal number for simple bundles. At the same time the Casson-Donaldson invariant is well defined so these objects are “right” from the point of view of theoretical physics. One can really derive a collection of numbers for Mukai vectors corresponding to these expectional bundles such that \( CD_H(m) \) is the degree \((1.1)\) above.

As we’ve mentioned, the bilinear form \((1.2)\) is not purely topological. One could answer the following natural question about it: is there an equivalence relation \( \sim \) on \( K_{alg}^0 \) such that the form \( h \) is equal to the lifting of a bilinear form on the lattice \( K_{alg}^0 / \sim = F \). Obviously, \( F \) can not be equal to \( H^{2\ast}(M, \mathbb{Z}) \) and if the answer to the question posed here is “Yes” then it should be a lattice \( F \) between the Chow ring \( CH^\ast(M) = A^\ast(M)/(\text{rational equivalence}) \) and \( H^{2\ast}(M, \mathbb{Z}) \) (for example, \( A(M) = A^\ast(M)/(\text{algebraic equivalence}) \)). At the same time by the Griffits theorem for 3-dimensional quintic \( M \subset \mathbb{CP}^4 \) we have \( A(M) \neq H^{2\ast}(M, \mathbb{Z}) \), thus our question is of the ”Lefshetz problem” type.

Further, recall that coherent sheaves and bundles over any variety can be transformed by so called modular operations: universal extensions, universal divisions and some of their combinations ([11], Ch. 2). These operations preserve the properties of the moduli spaces. The regular application of these transformations have begun with S. Mukai paper [8] where the author described the ”reflection” operation which can be decomposed into a combination of the divisions and the extensions (or ”returns” in the special terminology used by the experts). The spectrum of the possible applications of these operations is quite wide: from the derived categories of coherent sheaves to the categories of the representations of algebras and quivers. We can extract some combinatorical part from the purely geometric investigations using such operation (and almost always we obtain some representation of a braid group). And this usually gives some mutual underlayng rules for a priori different theories.

Every Mukai vector \( m \in L_M \) defines a transformation of the full lattice given by the natural formula

\[ \alpha_m(m') = -m' - \chi(m, m') \cdot m, \]

but we prefer to define the following corrected version

\[ \alpha_m(m') = -m' - h(m, m') \cdot m. \]
From the first viewpoint it seems to be ill defined, because the form $h(1.2)$ doesn’t descend to the even cohomology ring. However, extending the discussion to the derived category $D^b(M)$ of coherent sheaves on $M$ one could go this way: let us consider the map to the Atiyah ring

$$r : D^b(M) \to K^0_{alg}.$$ 

then if $m$ can be realised by an exeptional bundle $E$ it can be shown that the desired transformation can be defined as a functor

$$\alpha_E : D^b(M) \to D^b(M)$$

on the derived category. Of course, such lifting heavily depends on the choice of this exceptional bundle $E$ in the fixed topological class $m(E)$. Let us choose and fix it. Then for any bundle $E'$ we have the following sheaf

$$\alpha_E(E') = \ker(\text{can} : H^0(E^* \otimes E') \otimes E \to E').$$

If this one is a stable bundle:

$$[\alpha_E(E')] = \alpha_{[E]}([E']),$$

then this bundle is the result of a modular operation, namely of the universal division.

Let $H \in \text{Pic}M$ be a polarization of $M$. This one as a bundle of rank 1 defines an automorphism of the Mukai lattice:

$$[T_{H^k}(E)] = T^k_H = [E \otimes H^k],$$

which can be lifted to the derived category or to any other "category" which could be defined by vector bundles. Using the standard homological technique and wordwise the same arguments as in Ch. 2 of [11], one can easily prove the following

**Proposition (1.1).** For any two topological types $m$ and $m'$ with holomorphic realizations $\{E_1, \ldots, E_{CD_H(m)}\}$ and $\{E'_1, \ldots, E'_{CD_H(m')}\}$ by exceptional stable bundles there exists a level $k_0(m,m') \in \mathbb{N}$ such that

1) for any $k > k_0(m,m')$ all modular operations $\alpha_{E_i}(E'_j \otimes H^k)$ are correctly defined;

2) all the bundles from the collection $\{\alpha_{E_i}(E'_j \otimes H^k)\}$ are exeptional and stable and form the complete collection of the holomorphic realizations of a topological class $\alpha_m(T^k_H(m'))$;

3) consequently, the Casson - Donaldson number for the last topological type is given by the formula

$$CD_H(\alpha_m(T^k_H(m'))) = CD_H(m) \cdot CD_H(m').$$

From this observation one immediately gets
Corollary 1.1. The Casson - Donaldson invariant is unbounded as a function.

Of course, there are special vector bundles for which the number $CD_H$ is equal to 1. For example,

$$CD_H(m(L)) = 1$$

for any line bundle $L$ over $M$ (recall that by the definition $h^{1,0} = 0$). On the other hand, there exist many types of sheaves with special Mukai vectors whose moduli spaces are compact and smooth (but of positive dimensions). For example, take $F = \mathcal{O}_p$ where $p \in M$ is a point. In this case the deformation to the normal cone gives us

$$CD_H(m(\mathcal{O}_p)) = \chi_{\text{top}}(M) = \sum (-1)^i rk H^i(M, \mathbb{R})$$

where the right hand side is just the topological Euler characteristics. As well the interpretation of the moduli space $\mathcal{M}(m)$ as the zero set of a holomorphic differential on the space $\mathcal{D}_M''(E)$ of $\bar{\partial}$ - connections (see [13]) shows that

$$CD(m(E)) = \chi_{\text{top}}(\mathcal{D}_M''(E))$$

(in spite of the fact that the space $\mathcal{D}_M''(E)$ is infinite dimensional itself). This space depends on the topological type of $E$ only, that is, on the Chern character $ch(E)$ or on the Mukai vector $m(E)$.

At the end of the LHS short description we have to mention that obviously the Casson - Donaldson invariant is the imitation of a well known one: let $m \in H^2(M, \mathbb{Z})$ and $R(m)$ be the family of the rational curves which represent this cohomology class. Then by the same reason from the deformation theory we can expect that scheme $R(m)$ is zero dimensional of length

$$R_M(m) = \deg R(m).$$

The function

$$R_M : H^2(M, \mathbb{Z}) \rightarrow \mathbb{Z}$$

underlies the physical parameters in the same way as $CD_H$ from (1.1). For example, if $M \subset \mathbb{C}P^4$ is a quintic and $m \in H^2(M, \mathbb{Z})$ is the class of the projective line in $\mathbb{C}P^4$, then $R_M(m) = 2875$. Moreover, the Clemens conjecture predicts that our expectations of "finitness" are true for a generic quintic.

We propose here the following conjecture closely analogous to the Clemens’ one, namely:

Conjecture. Over generic quintic in $\mathbb{C}P^4$ every stable rank 2 vector bundle $E$ is infinitesimally rigid, that is, $H^1(\text{ad}E) = 0$.

1.2. RHS.

On the symplectic side of the mirror picture one deals with the same type quadruple $(M, g, \omega, \Omega)$ understanding it first as a symplectic manifold with integer symplectic structure $\omega$. The integrability condition means that there is a line bundle $H$ with the first Chern class

$$c_1(H) = [\omega]$$

(1.3)
equipped with a hermitian connection \( a \in A_h(H) \) such that

\[
F_a = 2\pi i \omega
\]  

(1.4)

and the last condition defines this connection almost uniquely up to the gauge transformations.

In the symplectic setup we consider Lagrangian cycles as the geometric objects instead of vector bundles (the term "cycle" used instead of "submanifold" is discussed in [6]). Like for the vector bundles, one can impose some conditions on the Lagrangian cycles to get some appropriate objects (moduli spaces). In our setup (the case of so-called fixed complex polarization) there are two basic definitions which distinguish from the space \( \mathcal{L} \) of all Lagrangian cycles two classes. We recall briefly what they are. The first condition is more universal: it can be imposed in general situation when the symplectic form is integer. Namely one calls a Lagrangian submanifold \( S \) Bohr - Sommerfeld if the restriction to it of the prequantization pair \( (H, a) \) defined by (1.3) and (1.4) admits a covariantly constant section ([6]). Indeed, since \( S \) is Lagrangian and the curvature form of \( a \) is proportional to \( \omega \), the restriction to any Lagrangian submanifold should be isomorphic to a pair (trivial bundle, flat connection). But if this flat connection is gauge equivalent to the trivial product connection then one says that \( S \) is Bohr - Sommerfeld. In [6] one constructs some moduli spaces of such Bohr - Sommerfeld Lagrangian cycles.

Besides fixing some hermitian structure on \( H \) to define the Bohr - Sommerfeld condition on the Lagrangian cycles, the definition of SpLag cycles needs some complex polarization as the ruling ingredient. Generally complex polarization means that some appropriate integrable complex structure compatible with the given symplectic structure is fixed over the base manifold. At the same time over our Calabi - Yau threefold the complex structure is defined by the choice of \( \Omega \) which was made at the beginning of our discussion. Thus for the oriented Calabi - Yau threefold \((M, \Omega)\) special Lagrangian cycle (spLag cycle for short) is a 3-dimensional Lagrangian submanifold \( S \) such that the restriction \( \Omega|_S \) satisfies

\[
\text{Im} \Omega|_S \cong 0 \quad \text{or} \quad \text{Re} \Omega|_S \cong \text{Vol}_g(S),
\]

and these two conditions are equivalent ([12]). On the other hand, there is another way to define the cycles based on considerations of Gauss map and Gauss vector field over the space of all Lagrangian cycles ([12]); in the same paper one finds some intermediate condition on Lagrangian cycles which can be exploited in geometric quantization.

The local deformation theory for both types of Lagrangian cycles is well understood: its basic fact is that every symplectic manifold considered near some Lagrangian submanifold looks like a small neighborhood of the zero section of the cotangent bundle of the submanifold (the Darboux - Weinstein theorem). Thus in this description the deformations of Bohr - Sommerfeld cycles are presented by the graphs of exact forms while in the second case the deformations are given by harmonic forms. This means that at least dimensionally these two families of deformations are complement.
On the space $\mathcal{L}$ of all Lagrangian cycles one has the following bilinear forms. First of all there is pure topological intersection form

$$< S_1, S_2 >= [S_1], [S_2] \in \mathbb{Z}$$

where $[S_i] \in H_3(M, \mathbb{Z})$. It’s clear that it is skew symmetric. At the same time for every two Lagrangian cycles intersecting transversally, one can define the Floer homologies

$$FH^*(S_1, S_2),$$

([5]). For the pair $S_1, S_2$ we have a complex

$$\delta : C^{S_1 \cap S_2} \rightarrow C^{S_1 \cap S_2},$$

defined by the set of the intersection points (we require transversality of the cycles). This construction results in a collection of finite abelian groups labelled by integers and this collection of the Floer homology groups gives us the following ”bilinear” form

$$\theta(S_1, S_2) = \sum_{i=0}^{3} (-1)^{i+1} \text{rk} FH^i(S_1, S_2),$$

which is a symplectic analogue of $\chi$. If $S_1$ and $S_2$ are not transversal in $M$, then one can deform one of them using some appropriate Hamiltonian deformation to establish the transversal picture and then apply the same arguments. For example, it’s possible to generalize the definition to the case

$$S_1 = S_2 = S$$

for a single cycle. Then in general the group concides with the usual de Rahm cohomology of $S$ (at least dimensionally). All the details of the long discussion can be found in [9] and other papers mentioned there. For our story it’s important that one can describe the deformation theory of the special Lagrangian cycles in a way similar to the one described in LHS when we discussed the case of coherent sheaves:

1) the space $FH^1(S, S)$ is the space of infinitesimal deformations of spLag cycle $S$;

2) the space $FH^2(S, S)$ is the space of obstructions;

3) the local model of the moduli space is given by some version of the Kuranishi map

and

4) the spaces of the deformations and the obstructions are dual and hence equidimensional.

As we’ve mentioned the deformation theory hardly depends on the topology of the cycle itself and the topology of the embedding $S \rightarrow M$. But the real situation is much more closer to the LHS: it was proved by McLean that all spLag deformations are unobstructed. Together with the identification

$$FH^1(S, S) = H^1(S, \mathbb{R})$$
(which takes place if the embedding $S \to M$ possesses some natural property which is generic for the setup) it gives us that f.e. if $S$ is a homological sphere then the moduli space of spLag cycles is presented by points.

Now we are in position to formulate the Vafa conjecture which predicts more precise variant of the mirror symmetry. For mirror partners $M, M'$ it should be a map

$$\text{mir} : H^{2*}(M, \mathbb{Z}) \to H^{3}(M', \mathbb{Z})$$

induced by some one - to - one correspondence between $E$’s and $S$’s i.e. between stable vector bundles over $M$ and spLag cycles inside the mirror partner such that

$$\text{Ext}^i(E_1, E_2) = F H^i(\text{mir}(E_1), \text{mir}(E_2)).$$

One can draw this parallel further, considering the natural operations which can be applied for both objects: under the correspondence between $E$ and $S$ the operation of the Lagrangian connected sum

$$S_1 \# S_2$$

is dual to the extension operation:

$$0 \to E_1 \to E \to E_2 \to 0$$

for stable bundles $E_1$ and $E_2$. At the same time, a notion mentioned in LHS, is clarified: a stable vector bundle $E$ over $M$ is called spherical (or by often used term "exceetional") if

$$\text{Ext}^1(E, E) = 0.$$

By this strong version of the mirror conjecture, $S = \text{mir}(E)$ has to be a homological sphere

$$H^1(\text{mir}(E)) = H^2(\text{mir}(E)) = 0.$$

Again it is reasonable to expect that the moduli space $\mathcal{M}_S$ of spLag realizations of this sphere is just a finite set of points:

$$\mathcal{M}_S = \{p_1, \ldots, p_d\}$$

and the number of these points is a symplectic invariant

$$SC_M(S) = \#(\mathcal{M}_S).$$

Then it would be natural to expect this number to be equal to the Casson - Donaldson invariant of the stable bundle.

The symplectic part (or RHS) is not quite well understood yet. At the same time, the meaning of LHS is much more clear and even suited for computations. Here we will present some method to deal with the LHS hoping that in the nearest future matching investigations of RHS will appear.
2. Degeneration method for the computation of the Casson-Donaldson invariant

2.1. Degeneration principle.

One of the useful methods in algebraic geometry is based on "degeneration principle": if one can reduce the situation to some appropriate degenerated case, compute what is desired and then prove that the number is invariant under the deformation, then the problem is solved. As an example, we take an old standard problem which was known 200 years ago: the question is how many lines intersect 4 given skew ones in $\mathbb{CP}^3$. The regular construction gives that there are exactly 2 such projective lines. But one can compute this number using the degeneration principle as follows.

Let us move this couple of skew lines to the case when they are divided into two pairs $(l_1, l_2), (m_1, m_2)$ such that

\[ < l_1, l_2 > = \pi_1 = \mathbb{CP}^2 \subset \mathbb{CP}^3 \]
\[ < m_1, m_2 > = \pi_m = \mathbb{CP}^2 \subset \mathbb{CP}^3 \]
\[ l_i \cap m_j = \emptyset. \]

Then we see that the answer is 2: the first desired line is given by the intersection points $l_1 \cap l_2$ and $m_1 \cap m_2$ while the second one is given by the intersection $\pi_1 \cap \pi_m$.

Now we would like to apply the degeneration principle to the computation of the Casson-Donaldson invariant over Calabi-Yau 3-manifolds. Namely, let us solve the problem for some special type of Calabi-Yau degeneration when a given 3-manifold $M$ can be deformed to a manifold $M_0 = Y_+ \cup_S Y_-$ (2.1) where $Y_\pm$ are smooth quasi Fano 3-manifolds glued transversally along a non-singular surface $S$ of K3-type which belongs simultaneously to both the anticanonical systems:

\[ S \in | - K_{Y_\pm} |. \]

Then if we compute the numbers in the framework of this reduced picture and then prove that these numbers do not change under the deformation, it will give us the answer in more general situation.

Let us set up the framework for the application of the degeneration principle giving the following

**Definition 2.1.** Calabi-Yau 3-manifold is called constructive iff there exists an appropriate deformation of $M$ to $M_0$ which is decomposed as in (2.1).

The class of the constructive Calabi-Yau threefolds is wide enough: of course, all complete intersections in every weighted projective space are constructive as well as any elliptic net, etc. Moreover, one can observe that many rigid Calabi-Yau threefolds are constructive (and it is a really amazing fact!). Here we place the following
Example. The rigid Barth - Nieto - van Straten quintic ([10]) is the moduli space of abelian surfaces with the polarization of type (2,6) and fixed theta - structure. But it could be realised using the following model: consider the projective space $\mathbb{CP}^5$ equipped with homogeneous coordinates $(z_0, ..., z_5)$ and the corresponding system of the Newton hypersurfaces

$$S_k = \sum_{i=0}^{5} z_i^k.$$  

Then the pencil of quintics

$$< S_5, S_2 \cdot S_3 >$$

in $\mathbb{CP}^4$ given by the linear equation $S_1 = 0$ contains unique quintic with 130 nodes. It is the Barth - Nieto - van Straten quintic.  

To test the main idea of this article we will go in two different directions: at the rest of Section 2 we will discuss how one can compute the invariant, passing in the constructive case to some holomorphic symplectic setup and reducing the computation to a usual computation in this setup. In Section 3 we will show how one can construct ”real” examples of the computation when we will develop the corresponding gluing technique getting constructive Calabi - Yau manifolds together with the results of the computations.  

So the next natural question arises after the definition of the constructive Calabi - Yau manifolds is understood: what is the vector bundles over such reduced Calabi - Yau 3 - manifold $M_0$? A vector bundle $E$ on $M_0$ is a pair of vector bundles $E_{\pm}$ over $Y_{\pm}$ such that their restrictions coincide:

$$E_{+}|_S = E_{-}|_S.$$  

So the next step for us is to describe the geometry of vector bundles over the flags of type $(S \subset Y)$ where $Y$ is a quasi Fano variety (its definition see below) and $S$ is a K3 - surface from the anticanonical system.  

2.2. The geometry of the vector bundles on the flags.

We start with the following natural

**Definition 2.2.** A variety $Y$ is called a quasi Fano variety if the anticanonical linear system contains a smooth K3 - surface and $\chi(O_Y) = 1$.  

Well known examples of quasi Fano varieties are given by the blowing up of the classical Fano varieties with centers on fixed surfaces of the anticanonical systems.  

Let $Y$ be a quasi Fano variety and $S$ be a fixed K3 - surface from the anticanonical system. We will call such a pair a flag. This is a complex analogue of a real 3 - manifold with boundary.  

For the quasi Fano varieties the arithmetical properties of the Mukai lattice are slightly different from the CY - case, so f.e. the bilinear form $\chi$ is not skew symmetric and one has to decompose it into the symmetric part and the skew - symmetric part denoting these as $\chi_{\pm}$ respectively. Returning to the vector bundles on $Y$ we see that the symmetric form $\chi_+(E_1, E_2)$ depends only on the first three components of the ring $H^{2*}(Y, Z)$. Further, by the definition

$$O_Y(K_Y) = O_Y(-S).$$
The canonical class (like any invertible sheaf) defines an automorphism $T_{K_Y}$ of the Mukai lattice $L_Y$ equipped with the Mukai form by the formula

$$T_{K_Y}(m) = m \cdot e^{K_Y}.$$ 

Restricting each vector bundle to the surface $S$ we obtain the following map of the Mukai lattices:

$$res : L_Y \to L_S = H^{2*}(S, \mathbb{Z}).$$

The image of this map in the Mukai lattice of $S$ coincides with the image of the following operator

$$im(id - T_S) \subset L_M,$$

where the last operator can be defined as follows

$$(id - T_S)(u_0, u_2, u_4, u_6) = (0, -K_Y \cdot u_0, u_2 \cdot S, u_4 \cdot S) = (u_0, u_2 \cdot S, u_4 \cdot S) \in H^*(S).$$

Then the bilinear form

$$<,> = \frac{1}{2} res^* (,)$$

is the preimage of the standard symmetric bilinear form on $H^{2*}(S, \mathbb{Z})$ given by

$$(u, v) = - [v^* \cdot u]_4.$$ 

Thus the restriction, considered as a transformation, maps the root

$$\sqrt{td_Y^+}$$

to the root of the Todd class of the K3 - surface

$$(1, 0, 1) = \sqrt{td_S}$$

since by the definition of quasi Fano variety

$$\frac{c_2(TY) \cdot K_Y}{24} = \chi(\mathcal{O}_Y) = 1.$$ 

The point is that for every Mukai vector $m$ corresponding to a vector bundle $E$ on $Y$ the Mukai vector of the restricted bundle is given by

$$m(E|_S) = ch(E|_S) \cdot \sqrt{td_S} = (id - T_S)(m(E))$$

over this K3 - surface. Moreover, the symmetric bilinear form $\chi_+(E_1, E_2)$ on $Y$ is the lifting of the classical symmetric form

$$\chi(E_1|_S, E_2|_S)$$
over $S$. Consider now "a geometric realization" of $res$. Namely, since every vector bundle $E$ on $Y$ can be restricted to $S$, we have a map

$$r : \mathcal{M}_E \to \mathcal{M}_{E|S}$$

of the moduli spaces. The first result about these moduli spaces is purely arithmetical: Proposition 11.2 of [13] ensures us that for any simple vector bundle on $Y$ one has

$$v.dim \mathcal{M}_E = \frac{1}{2} v.dim \mathcal{M}_{E|S}.$$ 

We will see in a moment that this fact is much deeper than just a purely numerical coincidence.

The Bertini theorem gives us

$$H^2(Y, \mathbb{Z}) = H^{1,1}(S) \cap H^2(S, \mathbb{Z}).$$

A Mukai theorem says that for any primitive vector

$$m = (u_0, u_2, u_4) \in L_S$$

we have

$$\{u_0 > 0; \quad u_2 \in H^{1,1}(S); \quad m^2 \geq -2\} \implies \mathcal{M}_S(m) \neq \emptyset$$

that is for this vector there exists a stable bundle $E_0$ over $S$ such that for $m = m(E_0)$ the moduli space exists and

$$\dim \mathcal{M}_S(m) \geq m^2 + 2.$$ 

Suppose that there exists a $-K_Y$-stable bundle $E$ over $Y$ such that

$$E_0 = E|_S.$$ 

Then

$$\mathcal{M}_Y(m) \neq \emptyset \; \text{and} \; \dim \mathcal{M}_Y(m) \geq \frac{1}{2} m^2 + 1$$

([8]). To go further we need the following

**Definition 2.3.**

1) A vector bundle $E$ on $Y$ is called regular if $H^2(adE) = 0$.

2) An irreducible component $\mathcal{M}_Y(m)_0$ is called regular if a generic bundle $E$ which belongs to this component is regular.

These bundles are very important to us: for them we have

$$\dim \mathcal{M}_Y(m)_0 = v.dim \mathcal{M}_Y(m)_0 = \frac{1}{2} m^2 + 1.$$ 

Moreover, the restriction map

$$r : \mathcal{M}_Y(m)_0 \to \mathcal{M}_S(m) \quad (2.2)$$
is an immersion at any generic point. Both statements follow from the short exact cohomology sequence

\[ 0 \to H^1(\text{ad}E) \to H^1(\text{ad}|_S) \to H^2(\text{ad}E(K_Y)) \to 0 \]  

(2.3)

defined for the restriction sequence of \( \text{ad}E \):

\[ 0 \to \text{ad}E(K_Y) \to \text{ad}E \to \text{ad}|_S \to 0 \]

(because \( H^1(\text{ad}E(K_Y)) \) is zero by the Serre duality). Moreover, the continuation of the cohomological sequence gives

\[ 0 \to H^2(\text{ad}|S) \to H^3(\text{ad}E(K_Y)) \]

which ensures us that \( H^2(\text{ad}|S) \) is trivial if \( E \) is simple:

\[ H^0(\text{ad}E) = 0 \implies H^3(\text{ad}E(K_Y)) = H^0(\text{ad}E)^* = 0. \]

Therefore for simple bundles over \( Y \) we have the following implication: if \( E \) is regular then \( E|_S \) is regular. This gives us

**Proposition 2.1.** The restriction map \( r \) (2.2) is an embedding into the regular component \( \mathcal{M}_Y(m)_0 \) such that

\[ \dim \mathcal{M}_S(m) = 2\dim \mathcal{M}_Y(m)_0. \]

Now we invoke the holomorphic symplectic geometry: every regular component \( \mathcal{M}_S(m)_0 \) of vector bundles over any K3 - surface admits the Mukai holomorphic symplectic structure

\[ \omega_S : T\mathcal{M}_S(m)_0 \to T^*\mathcal{M}_S(m)_0. \]  

(2.4)

The crucial fact which underlies our further investigations is that

**Proposition 2.2.** The image of the restriction map

\[ r(\mathcal{M}_Y(m)_0) \subset \mathcal{M}_S(m)_0 \]

is a Lagrangian subvariety of \( \mathcal{M}_S(m)_0 \) with respect to \( \omega_S \).

The proof is very simple but quite illustrative. The tangent space of \( \mathcal{M}_S(m)_0 \) at any regular point \( r(E) \) is isomorphic to \( H^1(\text{ad}|_S) \). Thus we can understand the second arrow in the exact sequence (2.3) as the differential of the restriction map

\[ 0 \to T_E\mathcal{M}_Y(m)_0 \xrightarrow{dr} T_{E|_S}\mathcal{M}_S(m)_0 \]  

(2.5)

and it is a monomorphism. On the other hand according to [8] the Serre duality over \( S \) is nothing but the restriction of the symplectic structure \( \omega_S \) to the fiber of the tangent bundle that is

\[ T_{E|_S}\mathcal{M}_S(m)_0 = H^1(\text{ad}|_S) = H^1(\text{ad}|_S)^* = T^*_{E|_S}\mathcal{M}_S(m)_0 \]
and we can continue the sequence (2.5) dualizing it and using the identification (2.4) which gives

$$0 \to T_{E\mathcal{M}_Y(m)_0} \xrightarrow{dr} T_{E|_S\mathcal{M}_S(m)_0} = T^*_{E|_S\mathcal{M}_S(m)_0} \xrightarrow{(dr)^*} T^*_{E\mathcal{M}_Y(m)_0} \to 0. \quad (2.6)$$

The sequence (2.6) is exact, as it is equivalent to (2.3), so

$$\omega_S|_{T_{E\mathcal{M}_Y(m)_0}} \cong 0$$

and we are done. Moreover, the same sequence (2.6) shows that in the regular case the normal bundle should be isomorphic to the cotangent bundle, so

$$N_r(\mathcal{M}_Y(m)_0) \subset \mathcal{M}_S(m)_0 = T^*\mathcal{M}_Y(m)_0. \quad (2.7)$$

Proposition 2.2 suggests the introduction of a new integer invariant for Mukai vector $m$ over pair $(S, Y)$. Since $r(\mathcal{M}_Y(m)_0)$ is a cycle of middle dimension in $\mathcal{M}_S(m)_0$, one can define its self intersection index

$$CD_{(S,Y)}(m) = [r(\mathcal{M}_Y(m)_0)]^2$$

which we call the relative Casson-Donaldson invariant of the pair $(S, Y)$. In the compact and non-singular cases this number can be computed as the top Chern class of the normal bundle:

$$CD_{(S,Y)}(m) = c_{\text{top}}(N_r(\mathcal{M}_Y(m)_0), \mathcal{M}_S(m)_0),$$

and (2.7) shows that in this case

$$CD_{S,Y}(m) = \pm \chi(\mathcal{M}_Y(m)_0). \quad (2.8)$$

It is natural to exploit this relative version for the computations of the absolute Casson-Donaldson invariant.

Thus the general strategy should be as follows: for some Calabi-Yau threefold we are looking for the deformation to a reducible ”double” which is glued from two quasi Fano varieties along the K3 - surface. Then we reduce the question to the holomorphic symplectic geometry over the K3 - surface $S$ where the moduli spaces of vector bundles over the quasi Fanos live as holomorphic Lagrangian submanifolds.

At the same time this recipe can be adapted for computations of some other type numbers: f.e. by the same procedure we can compute the number of lines on generic $M_8$ which is a double cover of $\mathbb{CP}^3$ ramified at a generic surface of degree 8 (we will study this case in details in the next section). Then deforming this $M_8$ to a pair of $\mathbb{CP}^3_{\pm}$ glued along a quartic $S$ we have two families of lines on each $\mathbb{CP}^3_{\pm}$ which are the Grassmannian $G(2, 4)_{\pm}$. The intersection with $S$ defines maps

$$r_{\pm} : G(2, 4)_{\pm} \to \text{Hilb}^4(S).$$
These maps obviously are embeddings. But the smooth variety \( Hilb^4(S) \) has the Mukai holomorphic symplectic form \( \omega_S \). The images \( r_\pm(G(2, 4)_\pm) \) are homotopy equivalent smooth Lagrangian subvarieties. Hence the desired number is

\[
R_{M_8}(1) = 2r(G(2, 4))^2 = 2c_{top}T^*G(2, 4) = 12.
\]

Now we can ask: is it possible to compute some other "classical" numbers using this degeneration method? For example, let us compute the number \( R_{Q_5}(k) \) of rational curves of degree \( k \) on a generic quintic \( Q_5 \subset \mathbb{CP}^4 \). One can deform \( Q_5 \) to

\[
M_0 = Q_2 \cup_S Q_3
\]

Then a rational curve \( \gamma \) of a degree, say, 10 degenerates to a reducible curve \( \gamma_3 \cup \gamma_2 \) where \( \gamma_3 \) is a cubic on \( Q_3 \) and \( \gamma_2 \) is a conic on \( Q_2 \). The intersection maps

\[
r_+: \{\gamma_2\} \to Hilb^6(S) \\
r_-: \{\gamma_3\} \to Hilb^6(S)
\]

are embeddings. So what is the number

\[
\#[r_+(\{\gamma_2\}) \cap r_-(\{\gamma_3\})] = ?
\]

Note that the answer is unknown even for smaller \( k \)s: the record is

\[
R_{Q_5}(5) = 229305888887625.
\]

However, the degeneration method gives some reason why these numbers as coefficients of a generating function are wrong (R. Pandharipande observed recently that the coefficient \( n_{10} \) of the generating function doesn’t give the number \( R_{Q_5}(10) \)).

In the next section we discuss the construction starting from the end: this way we will find some particular examples of constructive Calabi - Yau threefolds and study the vector bundles over these ones.

3. Constructive Calabi-Yau threefolds

3.1. Deformations of flags and vector bundles.

The deformation theory of the pairs \( (K3 \subset Fano) \) is quite similar to the deformation theory for the complex manifold: one can construct over any flag a bundle \( T(S, Y) \) (or a coherent sheaf) such that

1) the space \( H^1(T(S, Y)) \) is the space of formal deformations;
2) the space \( H^2(T(S, Y)) \) is the space of obstructions;
3) and the corresponding Kuranishi map

\[
\Phi : H^1(T(S, Y)) \to H^2(T(S, Y))
\]

gives us a local model of the moduli space of deformations which is \( \Phi^{-1}(0) \). To construct the sheaf, consider the restriction sequence for the tangent bundle

\[
0 \to TY(-S) \to TY \to TY|_S
\] (3.1)
together with the standard exact sequence on $S$

$$0 \rightarrow TS \rightarrow TY|_S \rightarrow N_{(S,Y)} \rightarrow 0,$$

(3.2)

where the last line bundle is the normal bundle to the surface in the threefold. Let us combine two last epimorphisms from (3.1) and (3.2) getting

$$TY \rightarrow N_{(S,Y)} \rightarrow 0$$

(3.3)

and complete (3.3) to an exact sequence

$$0 \rightarrow T(S,Y) \rightarrow TY \rightarrow N_{(S,Y)} \rightarrow 0.$$

$T(S,Y)$ is the bundle (or the sheaf) which describes the local deformation theory.

Suppose that

$$H^1(S, N_{(S,Y)}) = 0.$$

Then we can compare two Kuranishi maps, combining them in one diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & H^0(N_{S,Y}) & \rightarrow & H^1(T(S,Y)) & \rightarrow & H^1(TY) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^2(T(S,Y)) & \rightarrow & H^2(TY) & \rightarrow & 0
\end{array}
$$

This gives us the following

**Proposition 3.1.** If the deformation of $Y$ is unobstructed then the same is true for the deformation of any pair $(S,Y)$.

The definition of $T(S,Y)$ implies

$$0 \rightarrow TY(-S) \rightarrow T(S,Y) \rightarrow TS \rightarrow 0.$$  

(3.4)

Since our pair is of the type (K3 in Fano), the long cohomological sequence gives for (3.4)

$$0 \rightarrow H^1(\Lambda^2\Omega Y) \rightarrow H^1(T(S,Y)) \rightarrow H^1(\Omega S)^*$$

$$\rightarrow H^1(\Omega Y)^* \rightarrow H^2(T(S,Y)) \rightarrow 0$$

(here we use the equality

$$E \otimes \Lambda^3 E^* = \Lambda^2 E^*$$

for $rkE = 3$ - case together with the Serre duality). It’s easy to see that the homomorphism

$$H^1(\Omega S)^* \rightarrow H^1(\Omega Y)^*$$

from the last sequence is dual to the restriction map

$$r : H^{1,1}(Y) \rightarrow H^{1,1}(S)$$

(via the Dolbeaut isomorphism). This gives us
Proposition 3.2. The obstruction space is given by $(\ker r)^* \subset H^{2,2}(Y)$. In particular, if Pic $Y = \mathbb{Z}$, then the deformation of $(S,Y)$ is unobstructed and according to Proposition 3.1 this implies that the deformation of $Y$ is unobstructed to.

Indeed, the homomorphism

$$H^1(\Omega S)^* \to H^1(\Omega Y)^*$$

must be nontrivial. Thus it has to be an epimorphism. Moreover, the space

$$H^1(TY(K_Y)) = H^1(\Lambda^2 \Omega Y)$$

is the space of the deformations of the pair which preserves the complex structure on $S$.

Recall that there exists a collection of obstructions for the equivalence of n-th order thickening of our given K3 - surface $S$ in the quasi Fano variety $Y$ and its flat model. The first obstruction is given by the class

$$\omega_1 \in H^1(TS \otimes N_{(S,Y)}^*).$$

From the standard exact sequence we have

$$H^1(TS \otimes N_{(S,Y)}^*) = H^1(\Lambda^2 \Omega Y|_S).$$

On the other hand, in our special case we have

$$H^1(TS \otimes N_{(S,Y)}^*) = H^1(TS \otimes N_{(S,Y)})^*$$

by the Serre duality. We will use these identifications for the ”gluing procedure”.

3.2. Gluing procedure.

Starting with the configuration (2.1) we get over the fixed K3 - surface $S$ two normal bundles

$$N_{S,Y\pm}$$

which are completely different. The topological smoothing procedure is very similar to the topological surgery in the real case: first of all, we cut a small neighborhood of the singular locus $S$ in $X_0$ considering small tubes

$$S^1(N_{S,Y\pm}) \subset N_{S,Y\pm}$$

in the normal bundles which are

$$N_{S,Y\pm} = L_{-K_{Y\pm}}.$$

Removing small disc - bundles

$$D^2(N_{S,Y\pm})$$

with the boundaries

$$\partial D^2(N_{S,Y\pm}) = -S^1(N_{S,Y\pm})$$
from $Y_\pm$ we get some open singular threefold

$$V_0 = D^2(N_{S,Y_+}) \cup_S D^2(N_{S,Y_-}).$$

Thus one gets three real 6 - manifolds with boundaries:

$$V_0 \quad | \quad \partial V_0 = S^1(N_{S,Y_+}) \cup S^1(N_{S,Y_-});$$

$$Y_\pm^0 = Y_\pm - D^2(N_{S,Y_\pm}) \quad | \quad \partial Y_\pm^0 = S^1(N_{S,Y_\pm});$$

and the resulting $X_0$ is glued from these three pieces.

Now we can deform slightly the singular real 6 - manifold $V_0$ preserving the boundary $\partial V_0$ by the following construction ([2]). Consider the following quadratic map of the bundles over $S$:

$$q : N_{S,Y_+} \oplus N_{S,Y_-} \to N_{S,Y_+} \otimes N_{S,Y_-} = L_{-K_{Y_+} - K_{Y_-}}.$$

Any section $s \in H^0(L_{-K_{Y_+} - K_{Y_-}})$ can be regarded as an embedding

$$i_s : S \to L_{-K_{Y_+} - K_{Y_-}}$$

thus for any section we get a manifold

$$V_s = q^{-1}(i_s(S)) \cap D^2(N_{S,Y_+}) \times_S D^2(N_{S,Y_-}).$$

Choosing the neighborhoods small enough one can make the picture such that the boundary becomes diffeomorphic to

$$\partial V_s = S^1(N_{S,Y_+}) \cup S^1(N_{S,Y_-}).$$

Then we can glue this $V_s$ with $Y_\pm^0$ along the components of the boundaries and get a new compact real 6 - manifold $X_s$. Moreover, if the zero set

$$(s)_0 = C \in | - K_{Y_+} - K_{Y_-}|$$

of the section $s$ is a smooth curve in $S$ then $V_s$ is non - singular and the construction gives us a topomodel of Calabi - Yau threefold. If the curve $C$ admits some simple singularities then $V_s$ would be singular in these points but applying the small resolution of these singular points one can get another topomodel of Calabi - Yau manifold of different topological type. It will be very usefull to get the complete list of the topomodels which can be reached by this procedure.

Until now we discussed the gluing procedure from the point of view of smooth real 6 - manifolds. But it is important that we can do this smoothing surgery preserving almost complex structures.

Now we go further describing the deformations of the complex structures over the constructed topomodels. Recall ([2], [7]) that in our situation there exists a sheaf $T(Y_+, S, Y_-)$ which can be constructed in terms of $T(S, Y_\pm)$ such that its first cohomology space presents the infinitesimal deformations of the reducible threefold to
reducible threefolds of the same topological type but the space \( \mathcal{H}^1 \) of all infinitesimal deformations is more complicated: it is included in the following exact sequence

\[
0 \to H^1(T(Y_+, S, Y_-)) \to \mathcal{H}^1 \to H^0(N_{S,Y_+} \otimes N_{S,Y_-}) \to 0. \tag{3.5}
\]

However the obstruction space has precisely the same type

\[
\mathcal{H}^2 = H^2(T(Y_+, S, Y_-))
\]

(5.1 of [2]). We can describe the sheaf \( T(Y_+, S, Y_-) \) considering the entires as disjoint flags \( (S_\pm \subset Y_\pm) \) together with the maps

\[
n_\pm : S_\pm \to S = SingX_0.
\]

Constructing exact sequence (3.4) for every flag component one gets the desired sheaf from the following exact sequence

\[
0 \to T(Y_+, S, Y_-) \to T(S_+, Y_+) \oplus T(S_-, Y_-) \to TS \to 0,
\]

where at the prefinal step we use \( \frac{1}{2}(n_+ + n_-)_* \). Thus one has for \( T(Y_+, S, Y_-) \) the exact sequence

\[
0 \to \oplus_\pm TY_\pm(-S) \to T(Y_+, S, Y_-) \to TS = ker\frac{1}{2}(n_+ + n_-)_* \to 0
\]

which induces the long cohomology sequence

\[
0 \to \oplus_\pm H^1(\Lambda^2\Omega Y_\pm) \to H^1(T(Y_+, S, Y_-)) \to H^1(\Omega S)^* \\
\to \oplus_\pm H^1(\Omega Y_\pm)^* \to H^2(T(Y_+, S, Y_-)) \to 0.
\]

The direct sum of the compositions \( (r_\pm \cdot (n_\pm)_*)_* \) gives the map

\[
R^\pm_\pm = (r_+ \cdot (n_+)_*) \oplus (r_- \cdot (n_-)_*) : H^{1,1}(Y_+) \oplus H^{1,1}(Y_-) \to H^{1,1}(S).
\]

In terms of this map we can formulate the following

**Proposition 3.3.** The obstruction space is given by

\[
H^2(T(Y_+, S, Y_-)) = (kerR^\pm_\pm)^* \subset H^{1,1}(Y_+)^* \oplus H^{1,1}(Y_-)^*.
\]

Turning back to the definition of \( \mathcal{H}^1 \) given by (3.5) one can see that the deformation complex inducing the Kuranishi map is

\[
0 \to H^1(T(Y_+, S, Y_-)) \to \mathcal{H}^1 \to H^0(N_{S,Y_+} \otimes N_{S,Y_-}) \xrightarrow{\Psi} H^2(T(Y_+, S, Y_-)) = (kerR^\pm_\pm)^* \subset H^{2,2}(Y_+) \oplus H^{2,2}(Y_-).
\]

The precise description of \( \Psi \) is contained in [2].
Corollary 3.1.

1) Let $\text{Pic} Y_\pm = \mathbb{Z}$ and let $N_{S,Y_+} \otimes N_{S,Y_-}$ be generated by sections and nontrivial, then the dimension of the space $\mathcal{M}_{X_0}$ of non singular deformations is

$$v \dim \mathcal{M}_{X_0} = h^{1,2}(Y_+) + h^{1,2}(Y_-) + h^0(N_{S,Y_+} \otimes N_{S,Y_-}) - 1;$$

2) if $N_{S,Y_+} \otimes N_{S,Y_-} = \mathcal{O}_S$ then the deformation is unobstructed,

$$v \dim \mathcal{M}_{X_0} = h^{1,2}(Y_+) + h^{1,2}(Y_-) + 1$$

and the body of the deformation family is smooth.

The second statement is known from [2], [7]. The first statement can be proven as follows: consider the 1st order jet with the first obstruction classes of the pairs $(S_\pm, Y_\pm)$

$$\omega_1^\pm \in H^1(TS \otimes N_{S_\pm,Y_\pm}^*),$$

described at the end of the previous subsection, and consider the natural homomorphism

$$H^0(N_{S_+,Y_+} \otimes N_{S_-,Y_-}) \otimes H^1(TS \otimes N_{S_\pm,Y_\pm}^*) \xrightarrow{\cdot \omega} H^1(TS \otimes N_{S_\pm,Y_\pm}^*)^*. $$

It’s easy to see that 1 - extension of the deformation given by a section $s \in H^0(N_{S_+,Y_+} \otimes N_{S_-,Y_-})$ is constrained by only one condition

$$\omega_1^-(c(s \otimes \omega)^+)) = 0.$$

This implies the first statement.

Now we can illustrate the results as follows: suppose that some K3 - surface $S$ contained in a quasi Fano flag $S, Y$ admits an involution

$$i : S \rightarrow S$$

such that

$$i^*(K_Y|_S) = K_Y|_S.$$ 

Then take the reducible CY - threefold given by the gluing map with

$$n_+ = id, \quad n_- = i.$$

and obtain

Corollary 3.2. Under the present conditions this double manifold $X_0$ can be deformed to a smooth Calabi - Yau threefold.

At the next subsection we add the vector bundle ingredient to our constructions and (since it’s not too hard) consider some particular examples.
3.3. Vector bundles over constructive manifolds.

The description of the vector bundles over the glued and deformed threefold $X$ is very simple. If we denote the composition
\[ n_+ \cdot (n_-)^{-1} \]
as $g$ (thus $g$ is an automorphism of K3 - surface $S$) then

**Proposition 3.4.** A pair of stable vector bundles $E_{\pm}$ over the pair of quasi Fano threefolds $Y_\pm$ can be glued and deformed to a vector bundle over $X$ if and only if
\[ E_+|_S = g^*(E_-|_S). \]

We may expect the infinitesimal rigidity for the vector bundles on $X$. Our task is to describe the deformations of pairs $(X + a$ vector bundle). According to [2] this problem can be reduced to the problem of the deformations for the projectivizations of the vector bundle components. But for this it’s easy to consider only the deformations with smooth total spaces. The criterium for this one is the following

**Proposition 3.5.** The total space of the deformations of a reducible CY - threefold is smooth iff the tensor product of the normal bundles $N_{S,Y_\pm}$ is trivial.

Now we can reduce the general situation to the trivial tensor product case just blowing up, say, the Fano variety $Y_+$ along the curve $C$. Thus we can expect that the construction can be performed in a sufficiently general case.

The simplest way to obtain a constructive threefold is to take double of a flag. Let us add to a flag $S \subset Y$ its copy and glue them along the fixed K3 - surface:
\[ 2SY = (Y, S, Y) \]
so
\[ n^\pm = id, \quad g = id. \]
Certainly this double can be deformed to a smooth Calabi - Yau threefold $X$ which type we will denote by the same symbol $2SY$. Any vector $m$ with non negative square from the Mukai lattice $M_Y$ defines the corresponding vector $m \in M_{2SY}$ satisfies $m^2 = 0$ of the induced Mukai lattice of the double. For every regular component $\mathcal{M}_Y(m)_0$ of the stable vector bundle moduli space over $Y$ and for every vector bundle $E \in \mathcal{M}_Y(m)_0$ we can construct its double $2SE$ gluing two copies of $E$ along the restriction $E|_S$. As a result, instead of the expected finite set of vector bundles we obtain some non transversal component of the moduli space
\[ \mathcal{M}_{2SY}(m) = \{2SE \mid E \in \mathcal{M}_Y(m)_0\} \]
which obviously has positive dimension being parametrized by $\mathcal{M}_Y(m)_0$. Thus from the first viewpoint the situation is not quite good. But this large component decays to a finite set of vector bundles when we deform the reducible double $2SY$ to a smooth $X$. The degree of this finite set coincides with the number $CD_{S,Y}(m)$, defined above in (2.8). The source of the coincidence is hidden in the following equality of two topological Euler characteristics
\[ \chi(D^*(2SE)) = \chi(\mathcal{M}_Y(m)_0). \]

At the rest of this section we place two examples of the construction which are concentrated on objects well known in algebraic geometry.
**Example 1.** We consider for a smooth quartic surface $S$ in $\mathbb{CP}^3$ the corresponding double of the flag. The resulting double $2S\mathbb{CP}^3$ is the degeneration of a smooth cover of $\mathbb{CP}^3$ ramified along a smooth surface of degree 8. For a vector bundle $E$ over $\mathbb{CP}^3$ of rank 2 with the Chern classes $c_1 = 0$, $c_2 = 1$ its Mukai vector is presented by

$$m(E) = 2 - \frac{1}{2} P.D.(H);$$

then the virtual dimension of the moduli space is given by

$$v.dim\mathcal{M}_{\mathbb{CP}^3}(2, 0, \frac{1}{2} P.D.(H), 0) = m^2 + 1 = 5.$$  

It’s well known that every stable bundle of the type is a mathematical instanton given by a section of $\Omega\mathbb{CP}^3(2)$ that is by a monad

$$0 \rightarrow \mathcal{O}_{\mathbb{CP}^3}(-1) \xrightarrow{s(-1)} \Omega\mathbb{CP}^3(-1) \rightarrow E \rightarrow 0.$$  

Therefore the compactification of the moduli space is constructed as

$$\mathcal{M}_{\mathbb{CP}^3}(2, 0, -\frac{1}{2} P.D.(H), 0) = \mathbb{CP}^5 = \Lambda^2\mathbb{CP}^3.$$  

Now restricting every such vector bundle on our quartic $S$ one gets another monad with the following display

$$0 \rightarrow \mathcal{O}_S(-1) \xrightarrow{s(-1)} \Omega\mathbb{CP}^3(-1)|S \rightarrow E|_S \rightarrow 0.$$  

¿From this it is easy to see that the restriction map

$$res : \mathcal{M}_{\mathbb{CP}^3}(2, 0, -\frac{1}{2} P.D.(H), 0) \rightarrow \mathcal{M}_S(2, 0, -2)$$

is an embedding. Thus we get

$$CD_{(S, \mathbb{CP}^3)}(2, 0, -\frac{1}{2} P.D.(H), 0) = 6$$

since it is the Euler characteristic of the projective space. More geometrically, consider some general linear transformation

$$g : \mathbb{CP}^3 \rightarrow \mathbb{CP}^3$$

and the induced skew double $\mathbb{CP}^3 \cup_S g(\mathbb{CP}^3)$. Then a vector bundle $E$ from the moduli space

$$\mathcal{M}_{\mathbb{CP}^3}(2, 0, -\frac{1}{2} P.D.(H), 0)$$

defines the corresponding vector bundle on the skew double iff

$$g^*(E) = E.$$  

Taking $\Lambda^2\mathbb{CP}^3$ as a model of the compactification consider the transformation of the wedge square induced by $g$ and denote it as $\Lambda^2g$. Then it’s clear that the condition
$g^*(E) = E$ implies that $E$ corresponds to a fixed point of $\Lambda^2 g$. Thus again we get that there are 6 vector bundles of this sort: the set $E_1, ..., E_6$ is just the six edges of the simplex with the vertices in the fixed point of $g$ in $\mathbb{CP}^3$.

Now consider any smooth curve

$$C = (s)_0, \quad s \in H^0(S, \mathcal{O}_S(8))$$

and blow it up as a curve in $\mathbb{CP}^3$, getting

$$\sigma : \tilde{\mathbb{CP}}^3 \to \mathbb{CP}^3.$$

The gluing procedure then gives us an irreducible constructive Calabi-Yau threefold

$$\tilde{\mathbb{CP}}^3 \cup_S g(\mathbb{CP}^3)$$

which can be deformed to a smooth threefold $X$ with smooth total deformation space. Then our six vector bundles give six doubles

$$\sigma^*(E_i) \cup_S E_i.$$

One can prove that the resulting vector bundles are infinitesimally rigid. It implies the fact that for any smooth double $X$ from the smooth deformation family

$$CD_X((2, 0, \sigma^*(\frac{1}{2} P.D.(H))), 0), (2, 0, -\frac{1}{2} P.D.(H), 0)) = 6.$$

**Example 2.** We consider now the moduli space $MI_k$ of the mathematical instantons, that is, the moduli space of the stable vector bundles over $\mathbb{CP}^3$ of rank 2 with the Chern classes $c_1 = 0, c_2 = k$ satisfying the instanton equation

$$h^1(E(-2)) = 0.$$

Then, just like in the previous example, the restriction to $S$ must be an embedding by the same reason. The monad description shows that for general linear transformation $g$ the induced action

$$g^*: MI_k \to MI_k$$

admits a finite set of fixed points

$$E_1, ..., E_N$$

which give the corresponding finite set of vector bundles over the double. Note that any generic linear transformation defines a $\mathbb{C}^*$-action on $\mathbb{CP}^3$ and on the moduli space $MI_k$ as well. The computation of the number $N$ (equal to the number of fixed points on $MI_k$) and of the ”instanton” component of the vector bundles over the double is parallel to the computation of the Euler characteristic of $MI_k$ using the Bott formula. Of course, this number $N$ is a term of CD-invariant of the double. However there exist some other components of the moduli space on $\mathbb{CP}^3$. Recall
that any holomorphic vector bundle with topologically trivial determinant over any Fano variety of even index has the Atiyah - Rees invariant

\[ AR(E) = h^1(E(\frac{1}{2}K_Y)) \mod 2 \]

which distinguishes components of the moduli spaces. On the other hand, if

\[ c_1(E) = 0 \]

then \( E \) is anti self dual and the Serre duality induces the natural non degenerated skew symmetrical form on \( H^1(E) \). Thus the equality

\[ h^1(E|_S) = 0 \mod 2 \]

always holds. Therefore for every mathematical instanton \( E \)

\[ AR(E) = 0 \]

and besides of \( MI_k \) there exists another component \( M_k \) (as a good example we take \( k = 3 \)). For a small \( k \) it can be checked that the restriction map embeds this component into the moduli space of vector bundles over \( S \). Thus

\[ res(MI_k) \cap res(M_k) = \emptyset. \]

Hence the Euler characteristics of the moduli spaces computed by the Bott formula give the answer

\[ CD_X(\langle 2, 0, -\sigma^*(\frac{5}{2}P.D.(H)), 0 \rangle, \langle 2, 0, -\frac{5}{2}P.D.(H), 0 \rangle) = \chi(MI_3) + \chi(M_3). \]

Some other examples are considered in [13].

**Conclusion**

The author would propose a number of examples illustrating the algebraic geometry of the constructions presented here. But let us emphasize just the questions which are deeply important for the modern theoretical physics.

A standard question which is asked by physicists last time is about the possible topological types of 3 - dimensional Calabi - Yau manifolds. More precisely the question is just on the number of these types. And even more concretely: is this number bounded or not? The way proposed here could give an answer. Namely as it was already mentioned above if any 3 - dimensional Calabi - Yau manifold is constructive then it were only finite number of different topological types. Emphasize again that yet nobody knows is this number finite or not.

On the other hand, one could ask an "adjoint" question: *how many topological types can be derived by this construction of smooth deformations of threefolds?* This question is quite natural in the framework of algebraic geometry (and much more simpler than the previos one). But turning back one can ask a question of even
more higher level: how many topological types of stable vector bundles can one get by this construction?

At the same time the mirror conjecture dictates that our constructions presented here should be compatible with some other aspects of mirror. We mean that in the setup of Landau-Ginzburg models any Fano variety admits a mirror partner. Very briefly, it is a pair

$$((\mathbb{C}^*)^n, W_n),$$

where $W_n$ is a function which is called potential. If any 3-dimensional Calabi-Yau manifold could be deformed to a pair of flags then its mirror partner should be expressible in terms of the Landau-Ginzburg models. Thus one can exploite the duality to check the problem.

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