Existence and Uniqueness of a Fractional Fokker-Planck Equation

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Abstract

Stochastic differential equations with Lévy motion arise the mathematical models for various phenomenon in geophysical and biochemical sciences. The Fokker-Planck equation for such a stochastic differential equations is a nonlocal partial differential equations. We prove the existence and uniqueness of the weak solution for this equation.

keywords: Fractional Laplacian operator, Fokker-Planck equation, Lax-Milgram theorem, Existence and uniqueness

1 introduction

For a system described by a stochastic differential equation with a $\alpha$-stable Lévy motion $L^\alpha_t$ for $\alpha \in (0, 2)$,

$$dX_t = b(X_t)dt + a dB_t + dL^\alpha_t,$$

the corresponding Fokker-Planck equation [1] contains a nonlocal Laplacian operator,

$$u_t = a \Delta u - \left(-\Delta\right)^{\alpha/2} u - div(b(x) \cdot u),$$

where $b$ is a two-dimensional vector function and $\left(-\Delta\right)^{\alpha/2}$ is a nonlocal Laplacian operator defined by

$$\left(-\Delta\right)^{\alpha/2} f(x) = \int_{\mathbb{R}^d \times [0]} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} dy.$$

Fractional partial differential equations arise the mathematical models for various phenomenon in physics and biology. Such as anomalous diffusion of particles [2] and the cell density evolution in certain biological processes [3].

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Volume constraints are natural extensions to the fractional case of boundary conditions for differential equations [4].

The Fokker-Planck equations have been used in the modeling of many physical phenomena, in particular, for the description of the evolution of plasmas [5]. Moreover, there are some settings in which particles may have long jumps [6]. Questions such as existence of solutions, hydrodynamic limits, and long-time behavior for the Fokker-Planck system has been extensively studied by many authors [7, 8, 9].

There are few works dealing with the existence and uniqueness of the Fokker-Planck equations. Wei and Tian [10] obtain the existence and uniqueness of weak $L^p$ solution on the whole space. AcevesSanchez and Cesbron [11] studied the nonlocal Fokker-Planck problem on $\mathbb{R}^d$ in fractional Sobolev space with the drift term is a direct proportion function. In [12], the weak solution is just considered in the Sobolev space $H^1$. Moreover, there is no proof of the well-posedness for equation with Laplace operator.

This paper is devoted to the study of the nonlocal Fokker-Planck equation on a bounded interval in $\mathbb{R}^2$. There are two main results: For $a \geq 0$, we obtain the existence and uniqueness in the Sobolev space $H^1$ with respect to spatial variables; for $a = 0$, we reach the conclusion in a fractional sobolev space $H^{\alpha/2}$ with respect to spatial variables. We can see that, This is more accurate.

2 Well-posedness of Fokker-Planck equation

In this section, we give some function spaces and obtain the existence and uniqueness of the weak solution of a fractional Fokker-Planck equation on a bounded domain in $\mathbb{R}^2$.

2.1 Function spaces

Denote $Q_T = D \times (0, T)$. Let $\partial_t Q_T$ and $\partial_p Q_T$ be the lateral boundary $D^c \times (0, T)$ and the parabolic boundary $\partial_t Q_T \cup \{(x, t); x \in D, t = 0\}$ of $Q_T$. Denote by $\dot{W}, \dot{W}$ the set of all functions in $W$ vanish on $\partial_p Q_T$ and $\partial_l Q_T$ respectively.

For the integer $\beta, r \in \{0, 1\}$, the set
\[
\{u; \nabla^\beta u, \partial^r_t u \in L^2(Q_T)\}
\]
edowed with the norm
\[
||u||_{W^{1,1}_2(Q_T)}^2 = \int \int_{Q_T} \sum_{\beta \in \{0, 1\}} |\nabla^\beta u|^2 + \sum_{r \in \{0, 1\}} |\partial^r_t u|^2 dxdt
\]
is denoted by $W^{1,1}_2(Q_T)$. Denote
\[
V(Q_T) := \{u \in \dot{W}^{1,1}_2(Q_T), \nabla u_t \in L^2(Q_T; \mathbb{R})\},
\]
and define the inner product as
\[
(u, v)_{V(Q_T)} = (u, v)_{W^{1,1}_2(Q_T)} + (\nabla u_t, \nabla v_t)_{L^2(Q_T)}.
\]
2.2 Main result

We consider the following nonlocal Fokker-Planck equation

\[
\begin{align*}
\frac{du}{dt} &= a\Delta u - (-\Delta)^{\alpha/2} u - \text{div}(b(x) \cdot u) \quad x \in D \subset \mathbb{R}^2, \ t \in (0, T), \\
\left. u \right|_{D^c} &= 0, \\
 u(x, 0) &= u_0(x),
\end{align*}
\]

(1)

where \(D = (0, 1)\) is an bounded domain in \(\mathbb{R}^2\), and \(D^c\) is the complement of \(D\). We will prove the existence and uniqueness of the solution to the equation (1).

**Remark 1.** We define nonlocal divergence operator \(\mathcal{D}\) on \(\beta\) as

\[
\mathcal{D}(\beta)(x) := \int_{\mathbb{R}^d} (\beta(x, z) + \beta(z, x)) \cdot \gamma(x, z) dz \quad \text{for } x \in D.
\]

For a function \(\phi(x)\), the adjoint operator \(\mathcal{D}^*\) corresponding to \(\mathcal{D}\) is the operator whose action on \(\phi\) is given by

\[
\mathcal{D}^*(\phi)(x, z) = -(\phi(z) - \phi(x)) \gamma(x, z) \quad \text{for } x, z \in D.
\]

Here we take \(\gamma(x, z) = \frac{1}{\sqrt{2}} \frac{1}{|z-x|^{\frac{d+2+\alpha}{2}}}\). we have

\[
\mathcal{D}\mathcal{D}^* = -(\Delta)^{\alpha/2}.
\]

**Definition 1.** Consider \(u_0\) in \(L^2(D)\). We say that \(u\) is a weak solution of equation (1), if for any \(\phi \in C^\infty_c([0, T] \times D),\)

\[
\int_{Q_T} u(\partial_t \phi + a\Delta \phi - (-\Delta)^{\alpha/2} \phi + b(x) \cdot \nabla \phi) dx dt + \int_{Q_T} u_0(x) \phi(0, x) dx = 0
\]

(2)

**Theorem 1.** Consider \(u_0 \in L^2(D),\)

1. If \(a = 0\) and \(\text{div} b(x) \geq 0\), there exists a unique weak solution and this solution satisfies

\[
f \in \mathcal{X} := \{ f : f \in L^2(Q_T), \frac{|f(t, x) - f(t, y)|}{|x - y|^{\frac{d+2}{2}}} \in L^2(Q_T \times \mathbb{R}^2) \}.
\]

2. For \(a \geq 0\), there exist a unique weak solution in \(\bar{W}^{1,1}_2(Q_T)\).

**Remark 2.** Note that this definition of \(\mathcal{X}\) is equivalent to saying that it is the set of functions which are in \(L^2([0, T])\) with respect to time and in \(H^{\alpha/2}(D)\) with respect to space.

**Proof.** 1. We consider the Hilbert space \(\mathcal{X}\) provided with the norm

\[
||u||_\mathcal{X}^2 = ||u||^2_{L^2(Q_T)} + ||D^*u||^2_{L^2(Q_T \times \mathbb{R}^2)}.
\]
Let us denote $T$ the operator, given by

$$T u = \partial_t u + \text{div}(b(x) \cdot u).$$

Moreover, we define the Hilbert space $Y$ as

$$Y = \{ f \in X : T u \in X' \}.$$

where $X'$ is the dual of $X$. From the fact that the space $C_c^\infty(Q_T)$ is a subspace of $X$ with a continuous injection, we define the pre-Hilbertian norm:

$$|\varphi|_{C_c^\infty(Q_T)} = ||\varphi||^2_X + \frac{1}{2}||\varphi(0, x)||^2_{L^2(D)}.$$

Now, we introduce the bilinear form $a : X \times C_c^\infty(Q_T) \rightarrow \mathbb{R}$ as

$$a(u, \varphi) = \int \int_{Q_T} -u \varphi_t + (D^* u, D^* \varphi)_{L^2(\mathbb{R})} - b(x) u \cdot \nabla \varphi dxdt.$$

(3)

and

$$L(\varphi) = -\int_D u_0(x) \varphi(0, x) dx.$$

Then find a solution $u$ in $X$ of (1) is equivalent to finding a solution $u$ in $X$ of $a(u, \varphi) = L(\varphi)$ for any $\varphi \in C_c^\infty(Q_T)$. First $a(u, \varphi)$ is continuous. Next we will obtain the coercivity of $a$.

$$a(\varphi, \varphi) = \int \int_{Q_T} -\varphi_t \varphi + (D^* \varphi, D^* \varphi)_{L^2(\mathbb{R})} - b(x) \varphi \cdot \nabla \varphi dxdt$$

$$= \frac{1}{2} \int_D \varphi^2(0, x) dx + \int \int_{Q_T} (D^* \varphi, D^* \varphi)_{L^2(\mathbb{R})} dxdt + \frac{1}{2} \int_D \text{div} b(x) \varphi^2 dxdt.$$

(4)

Then, there exists a positive constant $\delta$ such that $a(\varphi, \varphi) \geq \delta |\varphi|_{C_c^\infty(Q_T)}$. Thus the Lax-Milgram theorem implies the existence of $u$ in $X$ satisfying equation (1). That yields existence of a solution $u$ in $X$ of $a(u, \varphi) = L(\varphi)$ for any $\varphi \in C_c^\infty(Q_T)$. For $u \in X$, the linear bounded operator $T$ maps $u \in T$ to $-(\Delta)^{\alpha/2} u \in X'$, hence the weak solution $u$ is in $Y$.

Since the equation (1) is linear, to show the uniqueness, it is enough to show the unique solution with zero initial is the function $u \equiv 0$. Let $u$ be a solution of this problem on $Y$. Through integration by parts we have

$$2(\mathcal{T} u, u)_{X' X'} = (\partial_t u + \text{div}(b(x) \cdot u), u)_{X' X'}$$

$$= \int_D u^2(T, x) dx + \int_{Q_T} \text{div}(b(x) u)^2 dxdt \geq 0.$$

On the other hand, since $u$ satisfies equation (1), $\mathcal{T} u = -(\Delta)^{\alpha/2}$ in the weak sense, then

$$2(\mathcal{T} u, u)_{X' X'} = -(D^* u, D^* u) \leq 0.$$
That means \( u \equiv 0 \) a.e. on \( Q_T \). The solution is unique.

2. For the first part we set \( a > 0 \). To show the unique of the equation (1), it is enough to show the unique solution with zero initial is the function \( w \equiv 0 \).

\[
\begin{aligned}
    &\begin{cases}
    w_t = a\Delta w - (-\Delta)^{\alpha/2}w - \text{div}(b(x) \cdot w) \quad x \in D \subset \mathbb{R}^2, \\
    w|_{D^c} = 0, \\
    w(x,0) = 0.
    
    \end{cases}
\end{aligned}
\]

(5)

Set \( B[u,v,t] = \int_D a\nabla u \cdot \nabla v + (D^*u, D^*v)_{L^2(D \times \mathbb{R})} - b(x) \cdot u \cdot \nabla wdx \), then

\[
\frac{d}{dt} \left( \frac{1}{2}||w||^2_{L^2(D)} \right) + B[w,w,t] = (w,w') + B[w,w,t] = 0.
\]

This means

\[
a \int_D |\nabla w|^2 dx + (D^*w, D^*w)_{L^2(D \times \mathbb{R})}
= B[w,w,t] + \int_D b(x) \cdot w \cdot \nabla wdx 
\leq B[w,w,t] + \varepsilon \int_D |\nabla w|^2 dx + \frac{C}{4\varepsilon} \int_D |w|^2 dx.
\]

(6)

From Poincaré inequality

\[ ||w||_{L^2(D)} \leq C_1 ||D^*w||_{L^2(D \times \mathbb{R})}. \]

Then choose \( \varepsilon \) small enough, then

\[ \sigma||w||_{H^1} + \beta||w||_{H^{\alpha/2}} \leq B[w,w,t] + \gamma||w||^2_{L^2(D)}, \]

for positive constant \( \sigma, \beta, \) and \( \gamma \geq 0 \). We can see that

\[ B[w,w,t] \geq -\gamma||w||^2_{L^2(D)}. \]

By Gronwall inequality, we obtain

\[ w = 0. \]

The solution is unique.

Next, we will prove the existence of the problem. Without loss of generality, we just consider the case \( u_0 = 0 \).

\[
a(u,v) = \int_Q (u_t v_t + a\nabla u \nabla v_t + (D^*u, D^*v_t)_{L^2(D \times \mathbb{R})} + \text{div}(b(x) \cdot u)v_t) e^{-\theta t} dx dt, \theta > 0.
\]

Then

\[ |a(u,v)| \leq ||u||_{W^{1,1}(Q_T)} ||v||_{V(Q_T)}. \]

set

\[ a(v,v) := A + B + C + D \]
For \( v \in V(Q_T) \),

\[
B = a \int \int_{Q_T} \nabla v \cdot \nabla v e^{-\theta t} \, dx \, dt = a \int \int_{Q_T} \frac{1}{2} \frac{\partial}{\partial t} |\nabla v|^2 e^{-\theta t} \, dx \, dt
\]

\[
= a \int \int_{Q_T} \frac{1}{2} \frac{\partial}{\partial t}(|\nabla v|^2 e^{-\theta t}) \, dx \, dt + \frac{\theta}{2} \int \int_{Q_T} e^{-\theta t} |\nabla v|^2 \, dx \, dt
\]

\[
= \frac{a}{2} e^{-\theta t} \int_D |\nabla v|^2 |_{t=T} \, dx - \frac{a}{2} \int_D \gamma |\nabla v|^2 |_{t=0} \, dx + \frac{a\theta}{2} \int \int_{Q_T} e^{-\theta t} |\nabla v|^2 \, dx \, dt
\]

(7)

Since \( v \in V(Q_T) \), then \( \gamma |\nabla v|^2 |_{t=0} = 0 \), combine with Poincaré inequality

\[
B \geq \frac{\theta}{2} \int \int_{Q_T} e^{-\theta t} |\nabla v|^2 \, dx \, dt
\]

\[
\geq \frac{a\theta}{4} \int \int_{Q_T} e^{-\theta t} |\nabla v|^2 \, dx \, dt + \frac{a\theta}{4\mu} \int \int_{Q_T} e^{-\theta t} v^2 \, dx \, dt.
\]

(8)

Similarly,

\[
C \geq \frac{\theta}{2} \int \int_{Q_T} ||D^* v||_{L^2(R)}^2 \, dx \, dt.
\]

On the other hand, we have

\[
D = \int \int_{Q_T} \text{div}(b(x) \cdot v) v e^{-\theta t} \, dx \, dt
\]

\[
= \int \int_{Q_T} \text{div}b(x) \cdot v \cdot v e^{-\theta t} \, dx \, dt + \int \int_{Q_T} b(x) \cdot \nabla v \cdot v e^{-\theta t} \, dx \, dt
\]

\[
:= E + F,
\]

where

\[
|E| \leq \varepsilon \int \int_{Q_T} v_t^2 e^{-\theta t} \, dx \, dt + \frac{C_2}{\varepsilon} \int \int_{Q_T} v^2 e^{-\theta t} \, dx \, dt.
\]

\[
|F| \leq \varepsilon \int \int_{Q_T} v_t^2 e^{-\theta t} \, dx \, dt + \frac{C_3}{\varepsilon} \int \int_{Q_T} |\nabla v|^2 e^{-\theta t} \, dx \, dt.
\]

Then we have

\[
a(v, v) \geq (1 - 2\varepsilon) \int \int_{Q_T} v_t^2 e^{-\theta t} \, dx \, dt + \frac{\theta}{2} \int \int_{Q_T} ||D^* v||_{L^2(R)}^2 e^{-\theta t} \, dx \, dt
\]

\[
(\frac{a\theta}{4} - \frac{C_4}{\varepsilon}) \int \int_{Q_T} |\nabla v|^2 e^{-\theta t} \, dx \, dt + (\frac{a\theta}{4\mu} - \frac{C_5}{\varepsilon}) \int \int_{Q_T} v^2 e^{-\theta t} \, dx \, dt
\]

(10)

Choose \( \varepsilon \) small enough and \( \theta > 0 \) large enough, we have

\[
a(v, v) \geq \delta \|v\|_{W^{1,1}(Q_T)}^2
\]

where \( \delta \) is a positive constant. Then the Lax-Milgram theorem implies the existence of equation (1).

For \( a = 0 \), it is similar to the work that He and Duan did in [12].
3 Example

Langevin equation provide models of a diffusing particle. We consider the following system of stochastic differential equations

\[
\begin{aligned}
    dx_t &= v_t dt, \\
    m dv_t &= -\gamma v_t dt + \sigma dL^\alpha_t,
\end{aligned}
\]  

(11)

where \( m \) is the mass of the particle, \( \gamma \) and \( \sigma \) are the dissipation and diffusion coefficient, respectively, and \( L^\alpha_t \) is a Lévy process with generator \((-\Delta)^{\alpha/2})\).

Then the corresponding Lévy Fokker-Planck equation is

\[
\begin{aligned}
    u_t + v \cdot \nabla_x u &= -\frac{\sigma}{m}(-\Delta)^{\alpha/2}u + \frac{\gamma}{m} \text{div}_v(v \cdot u) \\
    u|_{D^c} &= 0, \\
    u(x, 0) &= u_0(x)
\end{aligned}
\]  

(12)

We can define the bilinear form as

\[
a_1(u, \varphi) = \int \int_{Q_T \times D} u t - vu \cdot \nabla_x \varphi + \frac{\sigma}{m} (D^\alpha_v u, D^\alpha_v \varphi)_{L^2(\mathbb{R})} + \frac{\gamma}{m} vu \cdot \nabla_v \varphi dxdtdv.
\]

and from the fact \( \text{div}_v(v) > 0 \), and

\[
\int \int_{Q_T \times D} v \varphi \cdot \nabla_x \varphi dxdtdv = 0,
\]

Then, there exists a positive constant \( \delta_1 \) such that \( a(\varphi, \varphi) \geq \delta_1 |\varphi|_{C^\infty_0(Q_T \times D)} \), we verified that the equation (12) has a unique weak solution in \( X_1 \) by Theorem 1, where,

\[
X_1 := \{ f : f \in L^2(Q_T \times D), \frac{|f(t, x, v) - f(t, x, w)|}{|v - w|^{2+\alpha}} \in L^2(Q_T \times D \times \mathbb{R}^2) \}.
\]

Clearly, the result of the example agree with the theoretical finding in this study.

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