EQUIVARIANT VECTOR BUNDLES ON COMPLETE SYMMETRIC VARIETIES OF MINIMAL RANK

INDRANIL BISWAS, S. SENTHAMARAI KANNAN, AND D. S. NAGARAJ

ABSTRACT. Let $X$ be the wonderful compactification of a complex symmetric space $G/H$ of minimal rank. For a point $x \in G$, denote by $Z$ the closure of $BxH/H$ in $X$, where $B$ is a Borel subgroup of $G$. The universal cover of $G$ is denoted by $\tilde{G}$. Given a $\tilde{G}$ equivariant vector bundle $E$ on $X$, we prove that $E$ is nef (respectively, ample) if and only if its restriction to $Z$ is nef (respectively, ample). Similarly, $E$ is trivial if and only if its restriction to $Z$ is so.

1. Introduction

Let $\sigma$ be an involution of a semisimple adjoint type algebraic group $G$ over $\mathbb{C}$, and let $H = G^\sigma$ be the corresponding fixed point locus. De Concini and Procesi constructed a smooth projective variety $X = \overline{G/H}$ equipped with an action of $G$, that contains an open dense $G$–orbit $G/H$ [DP]. This $X$ is known as the wonderful compactification of the symmetric space $G/H$.

Richardson and Springer described the $B$–orbits in $G/H$ in terms of the combinatorics of the Weyl group $W$, where $B$ is a Borel subgroup of $G$ (see [RS]). The rank of $G/H$ is defined by Panyushev [Pa] and Knop [Kn1]. The minimal rank symmetric spaces were introduced by Brion [Br]. Brion and Joshua have studied the geometry of the closures in $X$ of the $B$–orbits in $G/H$, whenever $G/H$ is of minimal rank [BJ]. Tchoudjem has also studied the closures in $X$ of the $B$–orbits in $G/H$, whenever $G/H$ is of minimal rank [TC].

This paper deals with the restriction of equivariant vector bundles on $X$ to some natural class of subvarieties of $X$, like $B$–orbit closures.

Let $\tilde{G}$ be the simply connected covering of $G$. The action of $G$ on $X$ produces an action of $\tilde{G}$ on $X$ using the natural projection $\tilde{G} \to G$. Given an algebraic vector bundle $E$ on $X$, we can get a class of vector bundles on $X$ by pulling back $E$ using the automorphisms of $X$ given by the action of $G$. It can be shown that the isomorphism classes of these pullbacks remain constant if and only if $E$ admits a $\tilde{G}$–equivariant structure (meaning the action of $\tilde{G}$ on $X$ admits a lift to an action on $E$).

We prove the following (see Theorem 3.5):

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Theorem 1.1. Assume that $G/H$ is of minimal rank. Fix a point $x \in G$. Let $Z$ be the closure of $BxH/H$ in $X$. Let $E$ be a $\tilde{G}$ equivariant vector bundle on $X$. Then, $E$ is nef (respectively, ample) if and only if the restriction of $E$ to $Z$ is nef (respectively, ample). Similarly, $E$ is trivial if and only if its restriction to $Z$ is trivial.

In [HMP], a similar result is proved for vector bundles on toric varieties.

Before stating the next result, we recall that for the conjugation action of $\tilde{G}$ on itself, Steinberg proved that for a maximal torus $T$ of $G$, the restriction homomorphism

$$\mathbb{C}[\tilde{G}] \longrightarrow \mathbb{C}[\tilde{T}]^{W(G,T)}$$

is an isomorphism, where $\tilde{T}$ is the inverse image of $T$ in $\tilde{G}$ and $W(G,T)$ is the Weyl group of $G$ with respect to $T$ [St1]. Hence, we have the Steinberg map

$$\tau : \tilde{G} \longrightarrow \tilde{T}/W(G,T) = \mathbb{A}^n.$$ 

Let $c$ be a Coxeter element in the Weyl group $W(G,T)$, and let $F$ be the fiber of the Steinberg map $\tau$ containing a representative $n_c$ of $c$ in $N_G(\tilde{T})$. Let $F'$ be the image of $F$ in $G$. Set $Z = Z_1 \cup Z_2$, where $Z_1$ is the closure of $F'$ in the wonderful compactification $\tilde{G}$ of $G$, and $Z_2$ is the unique closed $G \times G$ orbit in $\tilde{G}$ [DP].

The group $\tilde{G} \times \tilde{G}$ acts on $\tilde{G}$ which factors through the action of $G \times G$ on $\tilde{G}$. Given an algebraic vector bundle $E$ on $\tilde{G}$, the isomorphism classes of its translates by the elements of $G \times G$ remain constant if and only if $E$ admits a $\tilde{G} \times \tilde{G}$-equivariant structure.

We also prove the following (see Theorem 1.2):

Theorem 1.2. Let $E$ be a $\tilde{G} \times \tilde{G}$ equivariant vector bundle on $\tilde{G}$. Then, $E$ is nef (respectively, ample) if and only if the restriction of $E$ to $Z$ is nef (respectively, ample). Similarly, $E$ is trivial if and only if $E|_Z$ is trivial.

2. Preliminaries

2.1. Lie algebras and Algebraic groups. In this subsection we recall some basic facts and notation on Lie algebras and algebraic groups (see [Hu], [Hu1] for details). Throughout $G$ denotes a semisimple adjoint-type algebraic group over the field $\mathbb{C}$ of complex numbers. In particular, the center of $G$ is trivial. For a maximal torus $T$ of $G$, the group of all characters of $T$ will be denoted by $X(T)$. The normalizer of $T$ in $G$ will be denoted by $N_G(T)$, while

$$W(G,T) := N_G(T)/T$$

is the Weyl group of $G$ with respect to $T$. Let $R \subset X(T)$ be the root system of $G$ with respect to $T$. For a Borel subgroup $B$ of $G$ containing $T$, let $R^+(B)$ denote the set of positive roots determined by $T$ and $B$. Further,

$$S = \{\alpha_1, \ldots, \alpha_n\}$$
denotes the set of simple roots in \( R^+(B) \). For \( \alpha \in R^+(B) \), let \( s_\alpha \in W(G, T) \) be the reflection corresponding to \( \alpha \). The Lie algebras of \( G, T \) and \( B \) will be denoted by \( \mathfrak{g}, \mathfrak{t} \) and \( \mathfrak{b} \) respectively. The dual of the real form \( \mathfrak{t}_R \) of \( \mathfrak{t} \) is \( X(T) \otimes \mathbb{R} = \text{Hom}_\mathbb{R}(\mathfrak{t}_R, \mathbb{R}) \).

The positive definite \( W(G, T) \)-invariant form on \( \text{Hom}_\mathbb{R}(\mathfrak{t}_R, \mathbb{R}) \) induced by the Killing form on \( \mathfrak{g} \) is denoted by \( (\cdot, \cdot) \). We use the notation

\[
\langle \nu, \alpha \rangle := \frac{2(\nu, \alpha)}{\alpha, \alpha}.
\]

In this setting one has the Chevalley basis

\[
\{x_\alpha, h_\beta \mid \alpha \in R, \beta \in S\}
\]

of \( \mathfrak{g} \) determined by \( T \). For a root \( \alpha \), we denote by \( U_\alpha \) (respectively, \( \mathfrak{g}_\alpha \)) the one–dimensional \( T \)-stable root subgroup of \( G \) (respectively, the subspace of \( \mathfrak{g} \)) on which \( T \) acts through the character \( \alpha \).

Let \( \sigma \) be an algebraic automorphism of \( G \) of order two. Let \( H = G^\sigma \) be the subgroup consisting of all fixed points of \( \sigma \) in \( G \). The connected component of \( H \) containing the identity element will be denoted by \( H^0 \). We refer to \([R]\) and \([RS]\) for following facts.

A torus \( T' \) of \( G \) is said to be \( \sigma \)-anisotropic if \( \sigma(t) = t^{-1} \) for every \( t \in T' \). Recall that the rank of \( G/H \) is defined to be the dimension of a maximal dimensional anisotropic torus.

If \( T \) is a \( \sigma \)-stable maximal torus of \( G \), then \( \sigma \) induces an automorphism of \( X(T) \) of order two. Note that we have \( \sigma(R) = R \). Further, one has \( T = T_1T_2 \), where \( T_1 \) is a torus such that \( \sigma(t) = t \) for every \( t \in T_1 \), and \( T_2 \) is a \( \sigma \)-anisotropic torus. Clearly \( T_1 \cap T_2 \) is finite. Hence, we have \( \text{rank}(G/H) \geq \text{rank}(G) - \text{rank}(H) \).

Throughout, we assume that \( G/H \) is of minimal rank, or in other words

\[
\text{rank}(G/H) = \text{rank}(G) - \text{rank}(H) .
\]

We refer to \([Br]\) and \([Kn2]\) for facts about minimal rank.

The following lemma may be known, but for the sake of completeness we provide a proof here.

**Lemma 2.2.**

1. Any two \( \sigma \)-stable maximal tori of \( G \) are conjugate by an element of the connected component \( H^0 \) of \( H \) containing the identity element.

2. Any maximal torus \( S \) of \( H^0 \) is contained in a unique maximal torus \( T \) of \( G \). Further, \( T \) is \( \sigma \)-stable.

3. Any Borel subgroup \( Q \) of \( H^0 \) is contained in a \( \sigma \)-stable Borel subgroup \( B \) of \( G \). Further, the Borel subgroup of \( G \) containing \( Q \) is unique.

**Proof.** Proof of (1). Let \( T_1 \) and \( T_2 \) be two \( \sigma \)-stable maximal tori in \( G \). Define

\[
S_i := (T_i \cap H)^0,
\]
$i = 1, 2$. Since $G/H$ is of minimal rank, $S_1$ and $S_2$ are maximal tori in $H^0$. Hence, there is an element $h \in H^0$ such that $hS_1h^{-1} = S_2$. Also, $T_i = C_G(S_i)$ (see [Ri p. 295, Lemma 5.3 and Lemma 5.4]).

Proof of (2). Take $T = C_G(S)$. Proof of (3). We will first prove the existence of a stable Borel subgroup containing $Q$.

By [St2 p. 51, Lemma 7.5], there is a $\sigma$ stable Borel subgroup $B'$ of $G$. By [Ri p. 295, Lemma 5.1], the intersection $(B' \cap H)^0$ is a Borel subgroup of $H^0$. Hence, there is a $h \in H^0$ such that $Q = h(B' \cap H)^0h^{-1}$. Now take $B = hB'h^{-1}$.

To prove the uniqueness of $B$, let $B_1$ be a Borel subgroup of $G$ containing $Q$. As shown above, there is a $\sigma$ stable Borel subgroup $B$ of $G$ containing $Q$. Choose a maximal torus $S$ of $H^0$ lying in $Q$. From part (2) of the lemma we know that $T = C_G(S)$ is the unique maximal torus of $G$ containing $S$. Hence, $T$ is contained in both $B_1$ and $B$. Thus, there is a $w \in W(G, T)$ such that $wBw^{-1} = B_1$.

We now prove that $R^+(B_1) = R^+(B)$. Let $\alpha \in R^+(B) \setminus R^+(B)^\sigma$. Then the $\sigma$ invariant vector $x_\alpha + \sigma(x_\alpha)$ is in the Lie algebra of $Q$. Hence, $x_\alpha + \sigma(x_\alpha)$ is in the Lie algebra of $B_1$. Thus, both $\alpha$ and $\sigma(\alpha)$ are in $R^+(B_1) \setminus R^+(B_1)^\sigma$. Hence, we have

$$R^+(B) \setminus R^+(B)^\sigma = R^+(B_1) \setminus R^+(B_1)^\sigma.$$ 

Now, let $\alpha \in R^+(B)^\sigma$. We will show that $\sigma$ acts trivially on $U_\alpha$. Let $T_\alpha \subset T$ be the connected component, containing the identity element, of the kernel of $\alpha$. Consider the restriction of $\sigma$ to $C_G(T_\alpha)$. Let $C'$ be the commutator subgroup of $C_G(T_\alpha)$. If the action of $\sigma$ on $U_\alpha$ is not trivial, then there is a one-dimensional $\sigma$ stable anisotropic torus $S'$ in $C'$. Let $T_1 = S'T_\alpha$. Then we have $T_1^\sigma = T_\alpha^\sigma$. Hence by [Ri, Lemma 5.4] we have $T_1 = C_G(T_\alpha^\sigma)$. But this contradicts the fact that $T_\alpha^\sigma$ is a singular torus. Hence we have $U_\alpha \subset (B)^\sigma = Q \subset B_1$.

Thus, we have shown that $R^+(B) = R^+(B_1)$. Hence, we have $B = B_1$. This completes proof. \qed

### 2.3. Nef vector bundle.

Let $E$ be an algebraic vector bundle over a complex projective variety $Y$. Let $\mathbb{P}(E)$ denote the associated projective bundle over $Y$ whose fiber over any point $y \in Y$ is the space of all one-dimensional quotients of the fiber $E_y$ of $E$ over $y$. The line bundle over $\mathbb{P}(E)$ whose fiber over any one-dimensional quotient is the one-dimensional quotient itself, will be denoted by $\mathcal{O}_{\mathbb{P}(E)}(1)$.

A line bundle $L$ over $Y$ is called nef if for every pair $(C, \varphi)$, where $C$ is an irreducible smooth complex projective curve and $\varphi : C \to Y$ is a morphism, the degree of the pullback $\varphi^*L$ is nonnegative. A vector bundle $E \to Y$ is called nef if the above line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$ is nef.
3. Restriction of equivariant vector bundles to $B$-orbit closure

Let $T$ be a $\sigma$ stable maximal torus of $G$. Let $B$ be a Borel subgroup of $G$ containing $T$ such that for any root $\alpha \in R^+(B)$, either $\sigma(\alpha) = \alpha$ or $\sigma(\alpha) \in -R^+(B)$.

Let

$$X := \frac{G}{H}$$

be the wonderful compactification of the symmetric space $G/H$ constructed in [DP]. Let $Z$ be the closure in $X$ of the $B$–orbit of a point in $G/H$.

Let $P$ be the parabolic subgroup of $G$ containing $B$ such that the $G/P$ is the unique closed $G$ orbit in $X$ (see [DP]). In this case, $\sigma(P)$ is opposite to $P$ and $P \cap \sigma(P)$ is the Levi subgroup $L$ of $P$. Let $R(L)$ denote the roots of $L$ with respect to $T$.

The following lemma is about a $B$–orbit in $G/H$. We refer to [RS] for information on $B$–orbit closures in $G/H$. For any algebraic group acting on variety, it is well known that there is always a closed orbit. For instance, any orbit of minimal dimension is closed (see, [Hu] p. 60. Proposition)).

**Lemma 3.1.** Let $x \in G$ be such that $B \cdot xH/H$ is closed in $G/H$. Then, $x^{-1}Bx$ is $\sigma$ stable and there is a $w \in W(G,T)$ such that $B \cdot xH/H = B \cdot wH/H$.

**Proof.** Let $Q := (x^{-1}Bx \cap H)^0$. Since $B \cdot xH/H$ is closed in $G/H$, this $Q$ is a Borel subgroup of $H^0$. Further, we have $Q \subset x^{-1}Bx$. Hence, by Lemma 2.2, $x^{-1}Bx$ is $\sigma$ stable.

Now, let $S = (T \cap H)^0$. Since $G/H$ is of minimal rank, this $S$ is a maximal torus in $H^0$, and hence we choose a Borel subgroup $Q'$ of $H^0$ containing $S$. Thus, there is a $h \in H^0$ such that $hQh^{-1} = Q'$. Consequently, $hx^{-1}Bxh^{-1}$ is a Borel subgroup of $G$ containing $S$. Thus, there is a $w \in W(G,T)$ and a $b \in B$ such that $xh^{-1} = bw$, and we have $B \cdot xH/H = B \cdot wH/H$. \hfill \square

An interesting fact in case of minimal rank is the following uniqueness of the closed $B$–orbit (see, [Re] p. 1788, Proposition 2.2)).

**Lemma 3.2.** There is a unique closed $B$–orbit in $G/H$.

**Proof.** Clearly, there is a minimal dimensional $B$–orbit in $G/H$ and it is closed. For its uniqueness, let $Bx_1H/H$ and let $Bx_2H/H$ be two closed $B$–orbits in $G/H$. Then, by Lemma 3.1 there are $w_1$ and $w_2$ in $W$ such that $B \cdot x_iH/H = B \cdot w_iH/H$ for $i = 1, 2$.

Let $S = (T \cap H)^0$. Set $B_i := w_i^{-1}Bw_i$, and $Q_i = (B_i \cap H)^0$ for $i = 1, 2$. Both $Q_1$ and $Q_2$ are Borel subgroups of $H^0$ containing $S$. Therefore, there is a $\phi \in W(H^0, S)$ such that $\phi Q_1 \phi^{-1} = Q_2$. Hence both $\phi B_1 \phi^{-1}$ and $B_2$ are Borel subgroups of $G$ containing $Q_2$. By Lemma 2.2 we have $\phi B_1 \phi^{-1} = B_2$, and hence $w_1 = w_2 \phi$. Thus $Bx_1H/H = Bx_2H/H$. \hfill \square

We now recall from [BJ] a result of Brion and Joshua.
Lemma 3.3 ([BJ] p. 482, Lemma 2.1.1). Let $Y$ be the closure of $T H/H$ in $X$, and let $z$ denote the unique $B$-fixed point in $X$. Then, every $T$ stable curve in $X$ is one of the following:

1. There is a positive root $\alpha \in R^+(B) \setminus \overline{R(L)}$ and an element $\phi \in W(G,T)$ such that $\phi(C) = C_\alpha = U_\alpha s_\alpha z$. In this case $\alpha$ and $\sigma(\alpha)$ are orthogonal, and $s_\alpha s_{\sigma(\alpha)}$ is in $W(H^0, (T \cap H)^0)$.

(2) There is a restricted root $\gamma = \alpha - \sigma(\alpha)$ and an element $\phi \in W(G,T)$ such that $\phi(C) = C_{z,\gamma}$, where $C_{z,\gamma}$ is the unique $T$-stable curve containing $z$ and on which $T$ acts through the character $\gamma$. Moreover, the curve $C_{z,\gamma}$ lies in $Y$.

Lemma 3.4. Take $x \in G$, and let $Z$ be the closure of $B x H/H$ in $X$. Then every irreducible $T$ stable curve in $X$ lies in $W(G,T) \cdot Z$.

Proof. Note that the closure of $B \cdot x H/H$ in $G/H$ contains a closed $B$ orbit. Therefore we assume that $B \cdot x H/H$ is the unique closed $B$ orbit in $G/H$.

By Lemma 3.3, there is an element $w \in W(G,T)$ such that $B \cdot x H/H = B \cdot w H/H$. Let $C$ be an irreducible $T$ stable curve in $X$. By Lemma 3.3,

- either there is a positive root $\alpha \in R^+(B) \setminus \overline{R(L)}$ and a $\phi \in W(G,T)$ such that $\phi(C) = C_\alpha = U_\alpha s_\alpha z$,
- or there is a restricted root $\gamma$ and a $\phi \in W(G,T)$ such that $\phi(C) = C_{z,\gamma}$.

Recall that $Y = \overline{TH/H}$ and $S = (T \cap H)^0$. Now, since $s_\alpha s_{\sigma(\alpha)} \in W(H^0, S)$, and $z \in Y$ (see, Lemma 3.3 (2)), we have

$$s_\alpha s_{\sigma(\alpha)} \cdot z \in Y.$$ 

Hence, $w s_\alpha s_{\sigma(\alpha)} \cdot z \in w \cdot Y = \overline{TwH/H}$. Since $\alpha$ and $\sigma(\alpha)$ are orthogonal, $s_\alpha s_{\sigma(\alpha)}(\alpha) = -\alpha$. Hence, either $w(\alpha)$ is positive or $w s_\alpha s_{\sigma(\alpha)}(\alpha) = w(-\alpha)$ is positive. Further, $s_\alpha s_{\sigma(\alpha)} \in W(H^0, S)$. Hence $B w H/H = B w s_\alpha s_{\sigma(\alpha)} H/H$.

Now, if $w(\alpha)$ is positive, then $U_{w(\alpha)} w s_\alpha s_{\sigma(\alpha)} \cdot z$ is contained in $\overline{BwH/H}$. Hence,

$$w s_{\sigma(\alpha)} (C_\alpha) = w s_{\sigma(\alpha)} U_\alpha s_\alpha \cdot z = U_{w(\alpha)} w s_\alpha s_{\sigma(\alpha)} \cdot z$$

is contained in $\overline{BwH/H}$.

If $w s_{\sigma(\alpha)}(\alpha) = w(-\alpha)$ is positive, then $w s_\alpha (C_\alpha) = U_{w(-\alpha)} w \cdot z$ is contained in $\overline{BwH/H}$.

Thus, in either case, the curve $C_\alpha$ lies in $W(G,T) \cdot Z$.

Since $C_{z,\gamma} \subset Y$, we have $w(C_{z,\gamma}) \subset \overline{TwH/H}$. Hence, both type of curves in Lemma 3.3 lie in the union of the $W(G,T)$ translates of $\overline{BwH/H} = B x H/H$. This completes the proof.  \(\square\)
Notation: Let $G$ be a semi-simple adjoint group over the field $\mathbb{C}$ of complex numbers as above, and let $\tilde{G}$ be its universal cover. For a maximal torus $T$ in $G$, we denote its inverse image in $\tilde{G}$ by $\tilde{T}$.

Note that $\tilde{G}$ acts on $X$ and hence we can consider $\tilde{G}$ equivariant vector bundles on $X$.

Theorem 3.5. Fix a point $x \in G$. Let $Z$ be the closure of $BxH/H$ in $X$, where $B$ is a $\sigma$ stable Borel subgroup of $G$. Let $E$ be a $\tilde{G}$ equivariant vector bundle on $X$. Then, $E$ is nef (respectively, ample) if and only if the restriction of $E$ to $Z$ is nef (respectively, ample). Similarly, $E$ is trivial if and only if its restriction to $Z$ is trivial.

Proof. Since the restriction of a nef or ample or trivial vector bundle to a subvariety is nef or ample or trivial respectively, we have only to prove the “if” part of the theorem.

First assume that the restriction $E|_Z$ is nef. We need to show that for any irreducible closed curve $C$ in $\mathbb{P}(E)$, the degree of the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)|_C$ is nonnegative, where $\mathcal{O}_{\mathbb{P}(E)}(1) \rightarrow \mathbb{P}(E)$ is the line bundle defined in Section 2.3.

Let $Y(\tilde{T})$ denote the group of all one-parameter subgroups of $\tilde{T}$, where $\tilde{T}$, as before, is the inverse image in $\tilde{G}$ of a $\sigma$ stable maximal torus $T$ of $G$ lying in $B$. Choose a $\mathbb{Z}$-basis $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $Y(\tilde{T})$.

Let $\tilde{C}$ be an irreducible closed curve in the projective bundle $\mathbb{P}(E)$ over $X$. If the image of $C$ in $X$ is a point, then the degree of $\mathcal{O}_{\mathbb{P}(E)}(1)$ restricted to $\tilde{C}$ is positive, because $\mathcal{O}_{\mathbb{P}(E)}(1)$ is relatively ample. Hence we can assume that image of $\tilde{C}$ in $X$ is a curve $C$. Let $\tilde{C}_1$ be the flat limit of $\lambda_1(t)\tilde{C}$ as $t$ goes to zero (i.e., the one dimensional cycle corresponding to the limit point in the Hilbert Scheme of $\mathbb{P}(E)$). Then $\tilde{C}_1$ is a 1-dimensional cycle in $\mathbb{P}(E)$ linearly equivalent to $\tilde{C}$, and the image $C_1$ of $\tilde{C}_1$ in $X$ is invariant under $\lambda_1$. Inductively, define $\tilde{C}_i$ to be the flat limit of $\lambda_i(t)\tilde{C}_{i-1}$ as $t$ tends to zero, where $2 \leq i \leq n$. Then $\tilde{C}_i$ is linearly equivalent to $\tilde{C}$, and the image $C_i$ of $\tilde{C}_i$ in $X$ is invariant under the action on $X$ of the sub-torus of $T$ generated by the images of $\{\lambda_1, \lambda_2, \ldots, \lambda_i\}$.

In particular, $\tilde{C}_n$ is linearly equivalent to $\tilde{C}$, and every irreducible component of $\tilde{C}_n$ lies in the preimage of the $T$ invariant curve $C_n \subset X$. But $C_n$ can be conjugated to a curve in $Z$ (see Lemma 3.4), hence, by our assumption, $E|_{C_n}$ is nef. Therefore, the degree of the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)|_{\tilde{C}_n}$ is nonnegative (recall that degree$(\mathcal{O}_{\mathbb{P}(E)}(1)|_{\tilde{C}}) = \text{degree}(\mathcal{O}_{\mathbb{P}(E)}(1)|_{\tilde{C}_n})$). This proves that $E$ is nef.

Next assume that $E|_Z$ is ample.

For any positive integer $n$, let Sym$^n(E)$ denote the $n$-th symmetric power of the equivariant vector bundle $E$. To prove that $E$ is ample, we first note that there are only finitely many $T$ stable curves in $X$, and all of them lie in $W(G,T) \cdot Z$ (see Lemma 3.4). Thus the assumption implies that Sym$^n(E)|_C$ is ample for any $T$ stable curve $C$ in $X$ and for any $n \geq 1$.

Since line bundles on $X$ are equivariant for the $\tilde{G}$ action on $X$, the vector bundles Sym$^n(E) \otimes L$ are all $\tilde{G}$ equivariant vector bundles on $X$, where $L$ is any line bundle on
X. Fix an ample line bundle $L$ on $X$, and let $n$ be an integer such that $n > \text{degree}(L|_C)$ for every $T$ invariant curve $C$ in $X$. Then it follows from the argument in the first part of the proof of the theorem that $\text{Sym}^n(E) \otimes L^{-1}|_Z$ is nef, and hence $\text{Sym}^n(E) \otimes L^{-1}$ is nef. This implies $\text{Sym}^n(E)$ is ample and hence $E$ is ample (see, [Ha, p. 67, Proposition 2.4]).

Finally assume that $E|_Z$ is trivial.

Since $E|_Z$ is trivial, the dual $(E|_Z)^* = E^*|_Z$ is also trivial. Note that a trivial vector bundle is nef. Therefore, from the first part of the theorem we conclude that both $E$ and its dual $E^*$ are nef. Therefore, by [DPS, p. 311, Theorem 1.18] the vector bundle $E$ admits a filtration of holomorphic subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E$$

such that each successive quotient $E_i/E_{i-1}$, $1 \leq i \leq \ell$, admits a unitary flat connection. This implies that $E$ is semistable and $c_j(E) = 0$ for all $j \geq 1$, where $c_j$ is the rational Chern class. Now, by [Si, p. 40, Corollary 3.10] the vector bundle $E$ admits a flat holomorphic connection.

The variety $X$ is simply connected, because it is unirational (see, [Se, p. 483, Proposition 1]). Therefore, any holomorphic vector bundle on $X$ admitting a flat holomorphic connection is a holomorphically trivial vector bundle. In particular, the vector bundle $E$ is trivial. \hfill \Box

The proof of first two parts of the above theorem closely follows that of [HMP, p.610, Theorem 2.1].

4. A special Steinberg fiber

As before, $G$ be a semisimple adjoint group. Let $T$ be a maximal torus of $G$, $W(G, T)$ the Weyl group of $G$ with respect to $T$ and $B$ a Borel subgroup of $G$ containing $T$. Let $\tilde{G}$ be the simply connected covering of $G$, and let $\tilde{T}$ (respectively, $\tilde{B}$) be the inverse image of $T$ (respectively, $B$) in $\tilde{G}$. Let $c$ be a Coxeter element in $W$. We fix a representative $n_c$ of $c$ in $N_{\tilde{G}}(\tilde{T})$.

Lemma 4.1. The homomorphism $\phi_c : \tilde{T} \longrightarrow \tilde{T}$ given by $\phi_c(t) = t n_c t^{-1} n_c^{-1}$ is surjective.

Proof. It is enough to prove that the kernel of $\phi_c$ is finite. We can choose a reduced expression $c = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n}$ for $c$ such that $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is the set of simple roots labeled in some ordering. Let $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$. Then, the set $\{\beta_1, \beta_2, \cdots, \beta_n\}$ is the set of positive roots which are made negative by $c^{-1}$.

By [YZ, p. 862, Lemma 2.1], we have $\omega_i - c(\omega_i) = \beta_i$. Now, let $t$ be an element of the kernel of $\phi_c$. Then, $\beta_i(t) = 1$ for every $i = 1, 2, \cdots, n$. Hence,

$$\ker(\phi_c) \subset \bigcap_{i=1}^{n} \ker(\beta_i).$$
Since \( \{ \beta_1, \beta_2, \cdots, \beta_n \} \) is a basis of the root lattice of \( \tilde{G} \) with respect to \( \tilde{T} \), the kernel of \( \phi_c \) lies in the center of \( \tilde{G} \). Thus, it is finite.

Now, let \( \sigma \) be the involution of \( G \times G \) defined by \( \sigma(x, y) = (y, x) \). Note that the diagonal subgroup \( \Delta(G) \) of \( G \times G \) is the subgroup of fixed points, \( T \times T \) is a \( \sigma \)-stable maximal torus of \( G \times G \) and \( B \times B^- \) is a Borel subgroup having the property that \( \sigma(\alpha) \in -R^+(B \times B^-) \) for every \( \alpha \in R^+(B \times B^-) \).

Let \( \overline{G} \) denote the wonderful compactification of the group \( G \), where \( G \) is identified with the symmetric space \( (G \times G)/\Delta(G) \).

Now, consider the action of \( \tilde{G} \) on \( \tilde{G} \) by conjugation. We note that \( \tilde{T} \) is stable under the action of \( N_{\tilde{G}}(\tilde{T}) \).

It is proved in [SL[1] that the restriction
\[
\mathbb{C}[\tilde{G}] \to \mathbb{C}[\tilde{T}]^{W(G, T)}
\]
is an isomorphism, and the latter is a polynomial ring. Hence we have the Steinberg map
\[
\tau : \tilde{G} \to \tilde{T}/W(G, T) = \mathbb{A}^n.
\]

Let \( F \) be the fiber of the Steinberg map \( \tau \) containing a representative \( n_c \) of \( c \) in \( N_{\tilde{G}}(\tilde{T}) \). By an abuse of notation, we denote by \( n_c \) the image of \( n_c \) in \( N_G(T) \). Let \( F' \) be the image of \( F \) in \( G \), and let \( Z = Z_1 \cup Z_2 \), where \( Z_1 \) is the closure of \( F' \) in \( \overline{G} \) and \( Z_2 \) is the unique closed \( G \times G \) orbit in \( \overline{G} \).

**Theorem 4.2.** Let \( E \) be a \( \tilde{G} \times \tilde{G} \)-equivariant vector bundle on \( \overline{G} \). Then, \( E \) is nef (respectively, ample) if and only if the restriction of \( E \) to \( Z \) is nef (respectively, ample).

Similarly, \( E \) is trivial if and only if its restriction to \( Z \) is so.

**Proof.** Set \( W = W(G, T) \). By the proof of Theorem 3.5, it is sufficient to prove that every \( T \times T \) stable curve in \( \overline{G} \) lies in \( (W \times W) \cdot Z \). It is easy to see that, for every root \( \alpha \in R^+(B) \), the \( T \times T \) stable curve \( (\{1\} \times U_{-\alpha}) \cdot (1, s_\alpha) \cdot z \) lies in \( Z_2 \). Similarly, \( (U_\alpha \times \{1\}) \cdot (s_\alpha, 1) \cdot z \) lies in \( Z_2 \) for every \( \alpha \in R^+(B) \). Thus, every \( T \times T \) stable curve of type 1 in Lemma 3.3 lies in \( (W \times W) \cdot Z \).

On the other hand, by Lemma 4.1, the homomorphism \( \phi_c \) is onto and hence, the closure of \( Tn_c = \{tn_c t^{-1} | t \in T \} \) in \( \overline{G} \) is contained in \( Z_1 \). Therefore every \( T \times T \) stable curve of type 2 in Lemma 3.3 as well lies in \( (W \times W) \cdot Z \). This completes the proof of the theorem. \( \square \)

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Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 500004, India

E-mail address: indranil@math.tifr.res.in

Chennai Mathematical Institute, Plot H1, Sipcot IT Park, Siruseri, Kelambakam, Chennai 603103, India

E-mail address: kannan@cmi.ac.in

Institute of Mathematical Sciences, C.I.T. Campus, Taramani, Chennai 600113, India

E-mail address: dsn@imsc.res.in