The BRST Double Complex for the Coupling of Gravity to Gauge Theories

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Abstract

We consider (effective) Quantum General Relativity coupled to the Standard Model (QGR-SM) and study the corresponding BRST double complex. This double complex is generated by infinitesimal diffeomorphisms and infinitesimal gauge transformations. To this end, we define the respective BRST differentials $P$ and $Q$ and their corresponding anti-BRST differentials $\overline{P}$ and $\overline{Q}$. Then we show that all differentials mutually anticommute, which allows us to introduce the total BRST differential $D := P + Q$ and the total anti-BRST differential $\overline{D} := \overline{P} + \overline{Q}$. Furthermore, we characterize all non-constant Lagrange densities that are essentially closed with respect to the diffeomorphism differentials $P$ and $\overline{P}$ as scalar tensor densities of weight $w = 1$. As a consequence thereof, we obtain that graviton-ghosts decouple from matter if the Yang–Mills theory gauge fixing and ghost Lagrange densities are tensor densities of weight $w = 1$. In particular, we show that every such Lagrange density can be modified uniquely to satisfy said condition. Finally, we introduce a total gauge fixing fermion $\Upsilon$ and show that we can generate the complete gauge fixing and ghost Lagrange densities of QGR-SM via $D \Upsilon$.

1 Introduction

BRST cohomology is a powerful tool to study quantum gauge theories together with their gauge fixings and corresponding ghosts via homological algebra [1, 2, 3, 4]. More precisely, a nilpotent operator $D$ is introduced that performs an infinitesimal gauge transformation in direction of the ghost field. This so-called BRST operator $D$ can be seen either as an odd vector field on the super vector bundle for particle fields or as an odd derivation on the superalgebra of particle fields. The nilpotency of $D$ can then be used to compute its cohomology. This is useful, as physical states of the system can be identified with elements in its 0-th cohomology class. Furthermore, this formalism can be used to unify the gauge fixing and ghost Lagrange densities as follows: First of all, we understand a quantum gauge theory Lagrange density $L_{QGT}$ as the sum of the classical gauge theory Lagrange density $L_{GT}$ together with a gauge fixing Lagrange density $L_{GF}$ and its corresponding ghost Lagrange density $L_{\text{Ghost}}$, i.e.,

$$L_{QGT} := L_{GT} + L_{GF} + L_{\text{Ghost}}.$$  

(1)

By construction, the gauge fixing and ghost Lagrange densities are not independent: In the Faddeev–Popov setup the ghost Lagrange density is designed such that the ghost field satisfies residual gauge transformations of the chosen gauge fixing as equations of motion, with the antighost as Lagrange multiplier. In the BRST framework, both the gauge fixing and ghost

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\[1\text{We remark that it is possible to extend this setup with anti-BRST operators, so that ghosts and antighosts can be treated on an equal footing, cf. [5] and the analysis in this article.}\]
Lagrange densities can be generated from a so-called gauge fixing fermion \( \Upsilon \) via the action of \( D \), i.e.
\[
\mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}} \equiv D \Upsilon.
\] (2)

Since the term \( D \Upsilon \) is \( D \)-exact, it is also \( D \)-closed and thus does not contribute to the 0th cohomology class. Thus, in particular, it does not affect physical observables. To incorporate the gauge fixing, we additionally add the corresponding Lautrup–Nakanishi auxiliary fields \([6,7]\). These are Lie algebra valued fields that act as Lagrange multipliers and whose equations of motion are given by the gauge fixing conditions. We remark that it is also possible to define anti-BRST operators, which are homological differentials, by essentially replacing ghosts with antighosts in addition to a slightly modified action on the corresponding ghost, antighost and Lautrup–Nakanishi auxiliary fields \([5,8,9]\). This construction then allows to generate a large class of gauge fixing fermions via the action of \( D \) on an even functional \( F \) in ghost-degree 0, i.e.
\[
\Upsilon \equiv DF.
\] (3)

We refer the interested reader to the introductory texts \([10,11,12]\), the historical overview \([13]\) and the generalizations of this framework to anti-BRST operators \([5,8,9]\) and perturbative quantum gravity \([8,9,14]\).

In recent articles, we have studied several aspects of (effective) Quantum General Relativity coupled to the Standard Model: This includes a proper treatment of its geometric foundation \([15]\), a generalization of Connes–Kreimer renormalization theory to gauge theories and gravity \([16]\) and the complete gravity-matter Feynman rules \([17]\). In this article, we introduce and study the corresponding BRST double complex. The invariance of the theory under diffeomorphisms and gauge transformations implies first of all the existence of two such operators, \( P \) and \( Q \): The first performs infinitesimal diffeomorphisms in direction of the graviton-ghost and the second performs infinitesimal gauge transformations in direction of the gauge ghost, cf. Definitions 3.1 and 4.1. Then we provide the two gauge fixing fermions \( \zeta \) and \( \zeta_{\L} \): The first implements the de Donder gauge fixing together with the corresponding graviton-ghosts and the second implements the Lorenz gauge fixing together with the corresponding gauge ghosts, cf. Definitions 3.4 and 4.3. In particular, we have reworked the conventions such that the quadratic gauge fixing and ghost Lagrange densities are rescaled by the inverses of the respective gauge fixing parameters, \( \zeta \) and \( \xi \): This induces that unphysical propagators are rescaled by these parameters directly, which introduces an additional grading on the algebra of Feynman diagrams, cf. \([16,18,19]\).

Furthermore, we show that all non-constant functionals on the superalgebra of particle fields that are essentially closed with respect to the diffeomorphism BRST operator \( P \) are scalar tensor densities of weight \( w = 1 \), cf. Lemma 3.3\(^2\). This allows us to show that graviton-ghosts decouple from matter of the Standard Model if the gauge fixing fermion of Yang–Mills theory is a tensor density of weight \( w = 1 \), cf. Theorem 5.3. In particular, we show that every such gauge fixing fermion can be modified uniquely to satisfy said condition. Moreover, we introduce the corresponding anti-BRST operators \( \overline{P} \) and \( \overline{Q} \) in Definitions 3.7 and 4.5 and show that all BRST operators mutually anticommute, i.e.
\[
[ P, Q ] = [ P, \overline{Q} ] = [ \overline{P}, Q ] = [ \overline{P}, \overline{Q} ] = 0
\] (4)

cf. Theorem 5.1 and Corollary 5.2\(^3\). This is a non-trivial observation, as infinitesimal diffeomorphisms concern all particle fields and thus in particular the operators \( Q \) and \( \overline{Q} \). As a result, their sums
\[
D := P + Q
\] (5a)

\(^2\)We remark that this statement applies also to the diffeomorphism anti-BRST operator \( \overline{P} \).

\(^3\)We emphasize that we use the symbol \([\cdot,\cdot]\) for the supercommutator: In particular, it denotes the anticommutator if both arguments are odd, cf. Definition 2.2.
are also differentials, which we call total BRST operator and total anti-BRST operator. This allows us again to identify the physical states of the theory as elements in the respective 0-th (co)homology classes. In addition, we show that the sum over the gauge fixing fermion \( \zeta \) for the linearized de Donder gauge fixing and the gauge fixing fermion \( I_{\{1\}} \) for the covariant Lorenz gauge fixing

\[ \Upsilon := \zeta^{(1)} + I_{\{1\}} \]  

is again a gauge fixing fermion, which we call total gauge fixing fermion, cf. Theorem 5.4.

In particular, we obtain the complete gauge fixing and ghost Lagrange densities of (effective) Quantum General Relativity coupled to the Standard Model via \( DT \). Finally, we also introduce the corresponding anti-BRST operator in Definitions 3.7 and 4.5 and show that all BRST and anti-BRST operators mutually anticommute in Corollaries 3.8 and 4.6, together with the already mentioned results Theorem 5.1 and Corollary 5.2. We believe that this analysis provides an important contribution to the quantization of gravity coupled to gauge theories.

In this article, we consider (effective) Quantum General Relativity coupled to the Standard Model. Specifically, we consider the following Lagrange density for Quantum Yang–Mills theory:

\[ \mathcal{L}_{QGR} := \mathcal{L}_{GR} + \mathcal{L}_{GF} + \mathcal{L}_{\text{Ghost}} \]

\[ = -\frac{1}{2\pi^2} \left( \sqrt{-\text{Det}(g)} R + \frac{1}{2\zeta} \eta^{\mu\nu} \eta^{\rho\sigma} \frac{1}{\zeta} \left( \frac{1}{\zeta} \delta_{\mu\nu} \eta^{\rho\sigma} \right) \frac{1}{\zeta} \left( \frac{1}{\zeta} \delta_{\rho\sigma} \eta^{\mu\nu} \right) \right) dV_\eta \]

In particular, we consider Linearized General Relativity with respect to the metric decomposition \( g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu} \), where \( h_{\mu\nu} \) is the graviton field and \( \kappa := \sqrt{\kappa} \) the graviton coupling constant (with \( \kappa := 8\pi G \) the Einstein gravitational constant). In addition, \( R := g^{\alpha\sigma} R^\mu_\nu_{\alpha\sigma} \) is the Ricci scalar (with \( R^\mu_\nu_{\alpha\sigma} := \partial_\sigma \Gamma^\mu_\nu_{\alpha\rho} - \partial_\rho \Gamma^\mu_\nu_{\alpha\sigma} + \Gamma^\mu_\rho_{\alpha\sigma} \Gamma^\rho_\nu_{\mu\delta} - \Gamma^\rho_\nu_{\mu\delta} \Gamma^\delta_\rho_{\alpha\sigma} \) the Riemann tensor). Furthermore, \( dV_\eta := dt\wedge dx\wedge dy\wedge dz \) denotes the Minkowskian volume form, which is related to the Riemannian volume form \( dV_g \) via \( dV_g \equiv -\sqrt{-\text{Det}(g)} dV_\eta \). Moreover, \( \eta^{\mu\nu} \Gamma_{\rho\sigma} \equiv 0 \) is the linearized de Donder gauge fixing functional and \( \zeta \) denotes the gauge fixing parameter. Finally, \( C_\mu \) and \( \overline{C}_\mu \) are the graviton-ghost and graviton-antighost, respectively. We refer to [15, 17] for more detailed introductions and further comments on the chosen conventions. Then, additionally, we consider the following Lagrange density for Quantum Yang–Mills theory:

\[ \mathcal{L}_{QYM} := \mathcal{L}_{YM} + \mathcal{L}_{GF} + \mathcal{L}_{\text{Ghost}} \]

\[ = -\frac{1}{2g} \sqrt{\delta_{ab} \left( g^{\mu\nu} \Gamma^\rho_\nu_{\mu\beta} F^\beta_\rho + \frac{1}{\xi} L^a L^b \right) dV_g } \]

\[ + g^{\mu\nu} \left( \frac{1}{\xi} e^a \left( \nabla^T_{\mu} \partial_\nu e^a \right) + g f^a_{bc} \tilde{\varepsilon}_a \left( \nabla^T_{\mu} (c^b A^c_\nu) \right) \right) dV_g \]

We remark that \( F^a_{\mu\nu} := g \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right) - \frac{1}{2} f^a_{bc} A^b_\mu A^c_\nu \) is the local curvature form of the gauge boson \( A^a_\mu \). Additionally, \( L^a := g g^{\mu\nu} \left( \nabla^T_{\mu} A^a_\nu \right) \equiv 0 \) is the covariant Lorenz gauge fixing functional and \( \xi \) denotes the gauge fixing parameter. Finally, \( e^a \) and \( \tilde{\varepsilon}_a \) are the gauge ghost and gauge antighost, respectively. These two Lagrange densities are then completed with the Lagrange
densities for a vector of complex scalar fields and a vector of spinor fields, both subjected to the action of the gauge group:

$$\mathcal{L}_{\text{Matter}} := \left( g^{\mu \nu} \left( \nabla_\mu H \right)^A \left( \nabla_\nu H \right)^\dagger_A + \sum_{i \in I} \frac{\alpha_i}{2} \left( \Phi^i \right)^* \left( \Phi^i \right) + \overline{\Psi} \left( \gamma^\Sigma M - m_\Psi \right) \Psi \right) dV_g $$

(9)

Here, $\Phi$ and $\Psi$ denote the respective vectors of complex scalar fields and spinor fields, with corresponding dual vectors $\Phi^i$ and $\overline{\Psi}$. Furthermore, $\nabla_\mu H := \partial_\mu + ig A_\mu^a H_a$ and $\nabla_\mu \Sigma M := \partial_\mu + \varpi_\mu + ig A_\mu S_a$ denote the respective covariant derivatives, where $H_a$ and $S_a$ denote the infinitesimal actions of the gauge group $G$ on the Higgs bundle $H$ and the twisted spinor bundle $\Sigma M$, respectively, and $\varpi_\mu$ is the spin connection on the twisted spinor bundle. In addition, $\nabla^m \Sigma M := e^{\mu m} \gamma_m \left( \partial_\mu + \varpi_\mu + ig A_\mu S_a \right)$ denotes the corresponding twisted Dirac operator, where $e^{\mu m}$ is the inverse vielbein and $\gamma_m$ the Minkowski space Dirac matrix. Moreover, $I_\Phi$ denotes the set of scalar field interactions, with respective coupling constants $\alpha_i$ (and possible mass $\alpha_2 := -m_\Phi$).

Finally, $m_\Psi$ denotes the diagonal matrix with all fermion masses as entries. We refer to [17, Subsection 4.2] for a detailed discussion thereon.

We will continue this topic in future work as follows: We start with an analysis on the transversality of gravity coupled to gauge theories in [18]. In addition, we will also study the corresponding cancellation identities via the introduction of perturbative BRST cohomology in [19]. This will be a modified version of the Feynman graph cohomology introduced by Kreimer et al. in the realm of the Corolla polynomial [20, 21, 22, 23, 24, 25].

## 2 Geometric setup

We start this article with a section on the geometric underpinnings of BRST cohomology. This includes in particular graded supergeometry, as the BRST operator can be seen as a cohomological super vector field on the spacetime-matter bundle. Equivalently, the BRST operator can also be seen as a cohomological superderivation on the algebra of particle fields. After providing the relevant definitions for graded supergeometry, we then discuss spacetimes and the spacetime-matter bundle as a vector bundle whose sections describe the particle fields of (effective) Quantum General Relativity coupled to the Standard Model. Next we define metric decompositions and the graviton field. Finally, we close this section with a discussion on the diffeomorphism and gauge groups together with their infinitesimal actions.

**Definition 2.1** ($\mathbb{Z}^2$-graded supermanifold). Let $\mathcal{M}$ be a topological manifold. We call $\mathcal{M}$ a $\mathbb{Z}^2$-graded supermanifold, if it is isomorphic to a vector bundle $\pi: \mathcal{M} \to M$ that splits into a direct sum bundle such that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\sim} & \bigoplus_{(i,j) \in \mathbb{Z}^2} \mathcal{M}_{(i,j)} \\
\downarrow \pi & & \uparrow \hat{\pi} \\
M & & 
\end{array}
$$

(10)

where $(i, j) \in \mathbb{Z}^2$ denotes the degree of the subbundles and $\mathcal{M}_{(0,0)} \cong \{0\}$, i.e. the degree $(0,0)$ is concentrated in the so-called body $M$. We call the first integer $i$ the graviton-ghost degree, the second integer $j$ the gauge ghost degree and their sum $k := i + j$ the total ghost degree. Additionally, we assume that the grading is compatible with the super structure of $\mathcal{M}$, which
means that the parity of a subbundle is given via
\[ p \equiv i + j \mod 2, \]  
(11)
where \( 0 \in \mathbb{Z}_2 \) denotes even coordinates and \( 1 \in \mathbb{Z}_2 \) denotes odd coordinates. Concretely, on the level of graded super functions \( C(\mathcal{U}) \) for \( \mathcal{U} \subseteq \mathcal{M} \) this means that
\[
C(\mathcal{U}_{(i,j)}) \cong \begin{cases} 
C^\infty(\mathcal{U}_{(0,0)}) & \text{if } (i,j) = (0,0) \\
S(\mathcal{U}_{(i,j)}) & \text{if } p = 0 \\
A(\mathcal{U}_{(i,j)}) & \text{if } p = 1
\end{cases},
\]
(12)
where \( \mathcal{U}_{(i,j)} \subseteq \mathcal{M}_{(i,j)} \) is an open subset, \( C^\infty(\mathcal{U}_{(0,0)}) \) denotes real smooth functions on \( U \subseteq M \), \( S(\mathcal{U}_{(i,j)}) \) denotes symmetric formal power series and \( A(\mathcal{U}_{(i,j)}) \) denotes antisymmetric formal power series. Finally, we define the grade shift via
\[
\mathcal{M}_{(i,j)}[m,n] := \mathcal{M}_{(i+m,j+n)},
\]
(13)
which additionally implies a potential shift in parity according to Equation (11). We refer to [26] for more details in this direction.

**Definition 2.2 (Supercommutator).** Let \( \mathcal{M} \) be a supermanifold and \( X_1, X_2 \in \mathfrak{x}(\mathcal{M}) \) be two super vector fields of distinct parity \( p_1, p_2 \in \mathbb{Z}_2 \). Then we introduce the supercommutator as follows:
\[
[X_1, X_2] := X_1(X_2) - (-1)^{p_1p_2} X_2(X_1)
\]
(14)
This turns the module \( (\mathfrak{x}(\mathcal{M}), [\cdot, \cdot]) \) into a Lie superalgebra.

**Assumption 2.3.** In the following, we assume that the grading is compatible with the super structure in the sense of Equation (11).

**Definition 2.4 (Homological and cohomological vector fields).** Let \( \mathcal{M} \) be a \( \mathbb{Z} \)-graded supermanifold with compatible super structure. Then we denote the subspace of pure super vector fields \( X^\mu \) with degree \( z \in \mathbb{Z} \) by \( \mathfrak{x}_z(\mathcal{M}) \). Then an odd vector field \( \Xi \in \mathfrak{x}(\mathcal{M}) \) with the property
\[
[X, \Xi] \equiv 2 \Xi^2 \equiv 0
\]
is called homological if \( \Xi \in \mathfrak{x}_{(-1)}(\mathcal{M}) \) and cohomological if \( \Xi \in \mathfrak{x}_{(1)}(\mathcal{M}) \). This turns \( (\mathcal{C}_*(\mathcal{M}), \Xi) \) into a chain complex and the pair \( (\mathcal{M}, \Xi) \) is called a differential-graded manifold.

**Example 2.5.** Let \( M \) be a manifold with \( \Omega^\bullet(M) \) its sheaf of differential forms. Let furthermore \( M := T[1]M \) denote its degree shifted tangent bundle. Then we can identify \( \mathcal{M} \cong \Omega^\bullet(M) \) and obtain a cohomological vector field via the de Rham differential \( d \in \mathfrak{x}_{(1)}(\mathcal{M}) \) and a homological vector field via the de Rham codifferential \( \delta \in \mathfrak{x}_{(-1)}(\mathcal{M}) \).

**Definition 2.6 (Spacetime).** Let \( (M, g) \) be a \( d \)-dimensional Lorentzian manifold. We call \( (M, g) \) a spacetime, if it is smooth, connected and time-orientable.

**Definition 2.7 (Spacetime-matter bundle).** Let \( (M, g) \) be a spacetime. Then we define the spacetime-matter bundle of (effective) Quantum General Relativity coupled to the Standard
Model as the $\mathbb{Z}^2$-graded super bundle $\beta_M : \mathcal{B}_Q \to M$, where $\mathcal{B}_Q := M \times_M \mathcal{V}_Q$ is the fiber product over $M$ with

$$
\mathcal{V}_Q := \text{LorMet} (M) \times (T^* M \otimes_{\mathbb{R}} E) \times \left( G \times_{\rho} \left( H^{i} \oplus \Sigma M^{\mathbb{R}^j} \right) \right) \\
\times \text{Conn} (M, g) \times \left( T^*[1, 0] M \oplus T[-1, 0] M \oplus TM \right) \times \left( g[0, 1] \oplus g^*[0, -1] \oplus g^* \right),
$$

where $\text{LorMet} (M) \subset \text{Sym}^2_{\mathbb{R}} (T^* M)$ is the vector bundle of Lorentzian metrics with signature $(1, d - 1)$ and $T^* M \otimes_{\mathbb{R}} E$ is the vector bundle of vielbein fields with $E$ a real $d$-dimensional vector bundle. Furthermore, $G \times_{\rho} \left( H^{i} \oplus \Sigma M^{\mathbb{R}^j} \right)$ is the fiber product with respect to the action $\rho$ of the gauge group $G$ on the Higgs bundle $H^{i} := \mathbb{C}^i$ and the vector of spinor bundles $\Sigma M^{\mathbb{R}^j} \cong \mathbb{C}^j$. Moreover, $\text{Conn} (M, g) \subset \Omega^1 (M, g)$ denotes the vector bundle of local connection forms. Finally, $T^*[1, 0] M \oplus T[-1, 0] M$ and $g[0, 1] \oplus g^*[0, -1]$ denote the degree-shifted vector bundles for graviton-ghosts and gauge ghosts, respectively, and the additional bundles $TM$ and $g^*$ are for the Lautrup–Nakanishi auxiliary fields.

**Assumption 2.8.** We assume from now on that diffeomorphisms are homotopic to the identity, i.e. $\varphi \in \text{Diff}_0 (M)$.

**Remark 2.9.** Assumption [2.8] is motivated by the fact that diffeomorphisms homotopic to the identity are generated via the flows of compactly supported vector fields, $X \in \mathcal{X}_c (M)$, and differ from the identity only on compactly supported domains. Thus, diffeomorphisms homotopic to the identity preserve the asymptotic structure of spacetimes. We remark that, different from finite dimensional Lie groups, the Lie exponential map

$$
\exp : \mathcal{X}_c (M) \to \text{Diff}_0 (M)
$$

is no longer locally surjective, which leads to the notion of an evolution map

$$
\text{Evol} : C^\infty ([0, 1], \mathcal{X}_c (M)) \to C^\infty ([0, 1], \text{Diff}_0 (M))
$$

that maps smooth curves in the Lie algebra to smooth curves in the corresponding Lie group. We refer to [27] for further details.

**Definition 2.10** (Metric decomposition and graviton field). Let $(M, g)$ be a spacetime with background metric $b$. Then we define the graviton field as follows

$$
h_{\mu \nu} := \frac{1}{\kappa} (g_{\mu \nu} - b_{\mu \nu}) \iff g_{\mu \nu} \equiv b_{\mu \nu} + \kappa h_{\mu \nu},
$$

where $\kappa := \sqrt{\kappa}$ is the graviton coupling constant (with $\kappa := 8\pi G$ the Einstein gravitational constant). Thus, the graviton field $h_{\mu \nu}$ is given as a rescaled, symmetric $(0, 2)$-tensor field, i.e. as the section $\kappa h \in \Gamma (M, \text{Sym}^2 T^* M)$, provided that the background metric $b$ is also a symmetric $(0, 2)$-tensor field.

**Definition 2.11** (Sheaf of particle fields). Let $(M, \gamma)$ be a simple spacetime with topology $T_M$ and $\beta_M : \mathcal{B}_Q \to M$ the spacetime-matter bundle from Definition [2.7]. Then we define the sheaf of particle fields via

$$
\mathcal{F}_Q : T_M \to \Gamma (M, \mathcal{B}_Q), \quad U \mapsto \Gamma (U, B),
$$

where $B \subset \mathcal{B}_Q$ is one of the subbundles from Equation [16]. More precisely, we consider the following particle fields:
• Lorentzian metrics $\gamma \in \text{LorMet}(M)$
• Vielbein fields $e \in \Gamma(M,T^*M \otimes \mathbb{R} E)$
• Vector of $2i$ Higgs and Goldstone fields $\Phi \in \Gamma(M,H^{(i)})$
• Vector of $j$ fermion fields $\Psi \in \Gamma(M,\Sigma M^{\otimes j})$
• Gauge fields $i_g \in \text{Conn}(M,g)$
• Graviton-ghost fields $C \in \Gamma(M,T^*[1,0]M)$
• Graviton-antighost fields $\overline{C} \in \Gamma(M,T[-1,0]M)$
• Gauge ghost fields $c \in \Gamma(M,g[0,1])$
• Gauge antighost fields $c^\ast \in \Gamma(M,g^*[0,-1])$
• Graviton-Lautrup–Nakanishi auxiliary fields $B \in \Gamma(M,TM)$
• Gauge Lautrup–Nakanishi auxiliary fields $b \in \Gamma(M,g^*)$

**Definition 2.12** (Diffeomorphism group and group of gauge transformations). Given the situation of Definition 2.13, the physical theories that we are studying are invariant under the action of two groups: The diffeomorphism group homotopic to the identity $\mathcal{D} := \text{Diff}_0(M)$ and under the group of gauge transformations $\mathcal{G} := \Gamma(M,M \times G)$, where $G \cong U(1) \times \tilde{G}$ is the gauge group with $\tilde{G}$ compact and semisimple, cf. [28, 29, 30, 31] for a discussion on the Standard Model gauge group. The diffeomorphism group homotopic to the identity acts via

$$\varrho : \mathcal{D} \times \mathcal{B}_Q \rightarrow \mathcal{B}_Q \quad (\phi, \varphi) \mapsto \phi \ast \varphi,$$

where $\varrho$ acts naturally on $M$ and via push-forward on the corresponding particle bundles.\(^5\) Furthermore, the group of gauge transformations acts via

$$\rho : \mathcal{G} \times \mathcal{B}_Q \rightarrow \mathcal{B}_Q \quad (\gamma, \varphi) \mapsto \gamma \cdot \varphi,$$

where $\rho$ acts via the matrix representation on the vectors of Higgs and spinor fields. Additionally, we also consider the action of infinitesimal diffeomorphisms via

$$\rho : \mathfrak{D} \times \mathcal{B}_Q \rightarrow \mathcal{B}_Q \quad (X, \varphi) \mapsto \mathcal{L}_X \varphi,$$

where $\mathfrak{D} := \text{diff}(M) \cong \mathfrak{X}_c(M)$ is the Lie algebra of compactly supported vector fields and $\mathcal{L}_X$ denotes the Lie derivative of the geodesic exponential map. Moreover, we also consider the action of infinitesimal gauge transformations via

$$\rho : \mathfrak{G} \times \mathcal{B}_Q \rightarrow \mathcal{B}_Q \quad (Z, \varphi) \mapsto \ell_Z \varphi,$$

where $\mathfrak{G} := \Gamma(M,M \times g)$ is the Lie algebra of $g$-valued vector fields, with $g$ the Lie algebra of the gauge group $G$, and $\ell_Z$ denotes the Lie derivative of the Lie exponential map.

**Definition 2.13** (Transformation under (infinitesimal) diffeomorphisms). Given the situation of Definition 2.10 and Assumption 2.8, we define the action of diffeomorphisms $\phi \in \text{Diff}_0(M)$ on the graviton field via

$$(\tau \circ \phi)_\ast (\kappa h) := (\tau \circ \phi)_\ast g,$$

\(^5\)The action on the spinor bundle is more involved, as its construction depends crucially on the metric $g$. We refer to [32] for an explicit construction.
such that the background Minkowski metric can be conveniently defined to be invariant, i.e.
\[(\tau \circ \phi)_\ast \eta := 0,\] (24)
and on the other particle fields \(\varphi \in \Gamma (\mathcal{M}, E)\) as usual, i.e. via
\[(\tau \circ \phi)_\ast \varphi.\] (25)
In particular, the action of infinitesimal diffeomorphisms is given via the Lie derivative with respect the generating vector field \(X \in \mathfrak{X}_c (M)\), i.e.
\[\delta_X h_{\mu \nu} \equiv 1 \zeta \left( \nabla^TM X_\nu + \nabla^TM X_\mu \right),\] (26)
\[\delta_X \eta_{\mu \nu} \equiv 0,\] (27)
and
\[\delta_X \varphi \equiv \mathcal{L}_X \varphi,\] (28)
where \(\nabla^TM\) denotes the covariant derivative with respect to the Levi-Civita connection \(\Gamma\), i.e.
\[\Gamma^\rho_{\; \mu \nu} := \frac{1}{2} g^{\rho \sigma} \left( \partial_\mu g_{\nu \sigma} + \partial_\nu g_{\mu \sigma} - \partial_\sigma g_{\mu \nu} \right).\] (29)

3 The diffeomorphism BRST complex

In this section we study the diffeomorphism BRST operator \(P\) together with the de Donder gauge fixing fermion \(\zeta\) and its linearized variant \(\zeta^{(1)}\). To this end we start with the graviton BRST operator in Definition 3.1 as an odd super vector field on the spacetime-matter bundle and equivalently as an odd superderivation on the algebra of particle fields. Notably, we have reworked the conventions such that the unphysical quadratic Lagrange densities are rescaled by the inverse of the gauge fixing parameter \(\zeta\). In particular, we state that it is cohomological with respect to the graviton-ghost degree in Proposition 3.2. Then we characterize all non-constant functionals on the spacetime-matter bundle that are essentially closed with respect to \(P\) as tensor densities of weight \(w = 1\). This applies in particular to Lagrange densities, which we will use in Theorem 5.3. Then we show that the Lagrange densities for the de Donder gauge fixing and linearized de Donder gauge fixing together with their respective graviton-ghosts can be obtained via the gauge fixing fermions \(\zeta\) and \(\zeta^{(1)}\) in Proposition 3.4 and Corollary 3.5. Finally, we also set the diffeomorphism anti-BRST operator \(\bar{P}\) in Definition 3.7 and state that it is homological and anticommutes with \(P\) in Corollary 3.8.

Definition 3.1. We define the diffeomorphism BRST operator \(P \in \mathfrak{X}_{(1,0)} (\mathcal{B}_Q)\) as the following odd vector field on the spacetime-matter bundle with graviton-ghost degree 1:
\[P := \left( \frac{1}{\zeta} \partial_\mu C_\nu + \frac{1}{\zeta} \partial_\nu C_\mu - 2 \Gamma^\rho_{\mu \nu} C_\rho \right) \frac{\partial}{\partial h_{\mu \nu}} - \kappa C^\rho \left( \partial_\rho C_\sigma \right) \frac{\partial}{\partial C_\sigma} + \left( C^\rho \left( \partial_\rho A^a_\mu \right) + \left( \partial_\mu C^\rho \right) A^a_\rho \right) \frac{\partial}{\partial A^a_\mu}
+ \kappa \left( C^\rho \left( \partial_\mu e^a \right) + \left( \partial_\mu C^\rho \right) e^a \right) \frac{\partial}{\partial e^a} + \kappa C^\rho \left( \partial_\mu b^a \right) \frac{\partial}{\partial b^a}
+ \kappa C^\rho \left( \partial_\mu \Phi \right) \frac{\partial}{\partial \Phi} + \kappa C^\rho \left( \nabla^S_M \Psi + \frac{i}{4} \left( \partial_\mu X_\nu - \partial_\nu X_\mu \right) e^{mn} e^{\nu m} \sigma_{mn} \Psi \right) \frac{\partial}{\partial \Psi}\] (30)
Equivalently, its action on fundamental particle fields is given as follows:
\[P h_{\mu \nu} := \frac{1}{\zeta} \partial_\mu C_\nu + \frac{1}{\zeta} \partial_\nu C_\mu - 2 \Gamma^\rho_{\mu \nu} C_\rho\] (31a)
\[ PC_\rho := - \chi C^\sigma (\partial_\sigma C_\rho) \]  
\[ PC^\rho := \frac{1}{\zeta} B^\rho \]  
\[ PB^\rho := 0 \]
\[ P \eta_{\mu\nu} := 0 \]  
\[ P \partial_\mu := 0 \]  
\[ P \Gamma^\rho_{\mu\nu} := \left( C^\sigma (\partial_\sigma \Gamma^\rho_{\mu\nu}) + (\partial_\sigma C^\rho) \Gamma^\sigma_{\mu\sigma} - (\partial_\sigma C^\rho) \Gamma^\sigma_{\mu\nu} + \partial_\mu \partial_\nu C^\rho \right) \]  
\[ PA^a_\mu := \chi \left( C^\rho (\partial_\rho A^a_\mu) + (\partial_\rho C^\rho) A^a_\rho \right) \]  
\[ P e^a := \chi C^\rho (\partial_\rho e^a) \]  
\[ P \sigma^a := \chi C^\rho (\partial_\rho \sigma^a) \]  
\[ P b^a := \chi C^\rho (\partial_\rho b^a) \]  
\[ P \delta_{ab} := 0 \]  
\[ P \Phi := \chi C^\rho \left( \nabla^\Sigma M \Phi + \frac{1}{4} (\partial_\mu X_\nu - \partial_\nu X_\mu) e^{\mu m} e^{\nu n} (\sigma_{mn} \cdot \Psi) \right) \]

We remark that the action of \( P \) on all fields \( \varphi \notin \{ h, C, C, B, \eta \} \) is given via the Lie derivative with respect to \( C \), i.e. \( P \varphi \equiv \mathcal{L}_C \varphi \), rescaled by the gauge parameter \( \zeta \).  

**Proposition 3.2.** Given the situation of Definition 3.1, we have

\[ [P, P] \equiv 2P^2 \equiv 0 \]  

i.e. \( P \) is a cohomological vector field with respect to the graviton-ghost degree.

**Proof.** This follows immediately after a short calculation using the Jacobi identity. ■

**Lemma 3.3.** Let \( F \in C^\infty_{(w),(0,0)} (\mathcal{B}_Q, \mathbb{R}) \) be a non-constant local functional of tensor density weight \( w \in \mathbb{R} \). Then we have

\[ PF \simeq_{TD} 0 \]

if and only if \( w = 1 \), where \( \simeq_{TD} \) means equality modulo total derivatives.

**Proof.** We calculate

\[ PF = \mathcal{L}_C F = C^\rho (\partial_\rho F) + w (\partial_\rho C^\rho) F \]

\[ = \partial_\rho (C^\rho F) + (w - 1) (\partial_\rho C^\rho) F , \]

which is a total derivative if and only if \( w = 1 \), and thus proves the claimed statement. ■

---

6We remark that the Lie derivative of spinor fields is a non-trivial notion: This is due to the fact that the construction of the spinor bundle depends directly on the metric, which is itself affected by the Lie derivative. We use the formula of Kosmann [33], which uses the connection on the spinor bundle. It can be shown, however, that the result is indeed independent of the chosen connection. We remark that this formula can be embedded into the construction of a universal spinor bundle cf. [32].

7I.e. \( F \equiv (\text{Det} (g))^{w/2} f \) for an ordinary functional \( f \in C^\infty_{(0),(0,0)} (\mathcal{B}_Q, \mathbb{R}) \).

8The same statement also holds for the diffeomorphism anti-BRST operator \( \mathcal{T} \), cf. Definition 3.7.
**Proposition 3.4.** The Quantum General Relativity gauge fixing Lagrange density and its accompanying ghost Lagrange density

\[
L_{\text{QGR-GF}} + L_{\text{QGR-Ghost}} = -\frac{1}{4\pi^2\zeta} g^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu V_g \\
- \frac{1}{2\zeta} g_{\mu\nu} g^{\rho\sigma} \bar{C}^\mu \left( \partial_\mu \partial_\sigma C^\nu \right) V_g \\
- \frac{1}{2} g^{\mu\nu} \left( \left( \partial_\mu \mathcal{D}_\nu \right) C^\nu + d\mathcal{D}_\nu \left( \partial_\mu C^\nu \right) \right) V_g
\]  

(35)

for the de Donder gauge fixing functional \( \mathcal{D}_\mu := \eta^{\rho\sigma} \bar{\Gamma}_{\rho\sigma\mu} \) can be obtained from the following gauge fixing fermion \( \zeta \in \mathcal{C}_{(-1,0)} (B_Q) \)

\[
\zeta := \frac{1}{2} \bar{C}^{\rho} \left( \frac{1}{2} \mathcal{D}_\rho - \frac{1}{2} B_\rho \right) V_g
\]  

(36)

via \( P\zeta \).

**Proof.** The claimed statement follows directly from the calculations:

\[
P\zeta = \frac{1}{2\zeta} B^{\rho} \left( \frac{1}{2} \mathcal{D}_\rho - \frac{1}{2} B_\rho \right) V_g + \frac{1}{2} \bar{C}^{\rho} \left( P\mathcal{D}_\rho \right) V_g
\]  

(37a)

with

\[
P\mathcal{D}_\rho = P \left( g^{\mu\nu} \bar{\Gamma}_{\rho\mu\nu} \right) = C^{\sigma} \left( \partial_\sigma \mathcal{D}_\rho \right) + \left( \partial_\rho C^{\sigma} \right) \mathcal{D}_\sigma + g^{\mu\nu} g_{\rho\sigma} \left( \partial_\mu C^{\sigma} \right)
\]  

(37b)

along with the total derivative

\[
P \, dV_g = \partial_\rho \left( C^{\rho} \, dV_g \right)
\]  

(37c)

and then finally eliminating the Lautrup–Nakanishi auxiliary field \( B^\rho \) by inserting its equation of motion

\[
\text{EoM} \left( B_\rho \right) = \frac{1}{2} \mathcal{D}_\rho ,
\]  

(37d)

which are obtained as usual via an Euler–Lagrange variation of Equation (37a). □

**Corollary 3.5.** Given the situation of Proposition 3.4. Then the linearized de Donder gauge fixing and ghost Lagrange densities read

\[
L_{\text{QGR-GF}} + L_{\text{QGR-Ghost}} = -\frac{1}{4\pi^2\zeta} \eta^{\mu\nu} \mathcal{D}^{(1)}_{\mu} \mathcal{D}^{(1)}_{\nu} V_\eta \\
- \frac{1}{2\zeta} \eta^{\rho\sigma} \bar{C}^{\mu} \left( \partial_\mu \partial_\sigma C_\nu \right) V_\eta \\
- \frac{1}{2} \eta^{\rho\sigma} \bar{C}^{\mu} \left( \partial_\mu \left( \Gamma^{\nu}_{\rho\sigma} C_\nu \right) - 2 \partial_\mu \left( \Gamma^{\nu}_{\mu\sigma} C_\nu \right) \right) V_\eta
\]  

(38)

with the linearized de Donder gauge fixing functional \( \mathcal{D}^{(1)}_\mu := \eta^{\rho\sigma} \bar{\Gamma}_{\rho\sigma\mu}^{(1)} \). They can be obtained from the following gauge fixing fermion \( \zeta^{(1)} \in \mathcal{C}_{(-1,0)} (B_Q) \)

\[
\zeta^{(1)} := \frac{1}{2} \bar{C}^{\rho} \left( \frac{1}{2} \mathcal{D}^{(1)}_{\rho} - \frac{1}{2} B_\rho \right) V_\eta
\]  

(39)

via \( P\zeta^{(1)} \).
Proof. This can be shown analogously to the proof of Proposition 3.4.

Remark 3.6. In perturbative quantum gravity, it is sensible to use the linearized de Donder gauge fixing and ghost Lagrange densities from Corollary 3.5. The reason is that the perturbative expansion becomes simpler if the gauge fixing functional does only contribute to the propagator. Nevertheless, this also requires the choice of a background metric, cf. Definition 2.10. Thus, the complete de Donder gauge fixing can also be useful, as it does not depend on this choice.

Definition 3.7. Given the situation of Definition 3.1 we additionally define the diffeomorphism anti-BRST operator $\overline{P} \in \mathfrak{x}_{(-1,0)}(BQ)$ as the following odd vector field on the spacetime-matter bundle with graviton-ghost degree -1:

$$\overline{P} := \left. P \right|_{C \to \overline{C}}$$

(40a)

Together with the following additional changes

$$\overline{P}C_\rho := -\frac{1}{\xi}B^\rho + \kappa \overline{C}^{\rho} (\partial_\sigma C_\rho) + \kappa (\partial_\rho \overline{C}^\sigma) C_\sigma$$

(40b)

$$\overline{PC}^\rho := -\kappa C^\sigma (\partial_\sigma \overline{C}_\rho)$$

(40c)

$$\overline{PB}^\rho := -\kappa B^\sigma (\partial_\sigma B_\rho) + \kappa (\partial_\rho \overline{C}^\sigma) B^\sigma$$

(40d)

Corollary 3.8. Given the situation of Definition 3.7 we have

$$[\overline{P}, \overline{P}] \equiv 2\overline{P}^2 \equiv 0$$

and

$$[P, \overline{P}] = P \circ \overline{P} + \overline{P} \circ P \equiv 0,$$

i.e. $\overline{P}$ is a homological vector field with respect to the graviton-ghost degree that anticommutes with $P$.

Proof. This statement can be shown analogously to Proposition 3.2.

4 The gauge BRST complex

In this section we study the gauge BRST operator $Q$ together with the Lorenz gauge fixing fermion $F$ and its density variant $F_{\{1\}}$. To this end we start with the gauge BRST operator in Definition 4.1 as an odd super vector field on the spacetime-matter bundle and equivalently as an odd superderivation on the algebra of particle fields. Notably, we have reworked the conventions such that the unphysical quadratic Lagrange densities are rescaled by the inverse of the gauge fixing parameter $\xi$. In particular, we state that it is cohomological with respect to the gauge ghost degree in Proposition 4.2. Then we show that the Lagrange densities for the Lorenz gauge fixing and covariant Lorenz gauge fixing together with their respective gauge ghosts can be obtained via the gauge fixing fermions $F$ and $F_{\{1\}}$ in Proposition 4.3 and Corollary 4.4. Finally, we also set the gauge anti-BRST operator $\overline{Q}$ in Definition 4.5 and state that it is homological and anticommutes with $Q$ in Corollary 4.6.
Definition 4.1. We define the gauge BRST operator $Q \in \mathfrak{x}_{(0,1)}(\mathcal{B}_Q)$ as the following odd vector field on the spacetime-matter bundle with gauge ghost degree 1:

$$Q := \left( \frac{1}{\xi} \partial_{\mu} c^a + g f^a_{\ bc} c^b A^c_{\mu} \right) \frac{\partial}{\partial A^a_{\mu}} - \frac{g}{2} f^a_{\ bc} c^b c^c \frac{\partial}{\partial c^a} + \frac{1}{\xi} b_a \frac{\partial}{\partial b_a}$$

(43)

Equivalently, its action on fundamental particle fields is given as follows:

$$QA^a_{\mu} := \frac{1}{\xi} \partial_{\mu} c^a + g f^a_{\ bc} c^b A^c_{\mu}$$

(44a)

$$Qc^a := -\frac{g}{2} f^a_{\ bc} c^b c^c$$

(44b)

$$Q\bar{c}_a := \frac{1}{\xi} b_a$$

(44c)

$$Qb_a := 0$$

(44d)

$$Q\delta_{ab} := 0$$

(44e)

$$Q\delta_{\mu\nu} := 0$$

(44f)

$$QC_{\rho} := 0$$

(44g)

$$Q\bar{C}_{\rho} := 0$$

(44h)

$$QB_{\rho} := 0$$

(44i)

$$Q\eta_{\mu\nu} := 0$$

(44j)

$$Q\partial_{\mu} := 0$$

(44k)

$$Q\Gamma_{\rho\mu\nu} := 0$$

(44l)

$$Q\Phi := igc^a (\Phi_a \cdot \Phi)$$

(44m)

$$Q\Psi := igc^a (\bar{\Psi}_a \cdot \Psi)$$

(44n)

We remark that the action of $Q$ on all fields $\varphi \notin \{A, c, \bar{c}, b, \delta\}$ is given via the Lie derivative with respect to $c$, i.e. $Q\varphi \equiv \ell_c \varphi$, rescaled by the gauge parameter $\xi$.

Proposition 4.2. Given the situation of Definition 4.1, we have

$$[Q, Q] \equiv 2Q^2 \equiv 0,$$

(45)

i.e. $Q$ is a cohomological vector field with respect to the gauge ghost degree.

Proof. This follows immediately after a short calculation using the Jacobi identity. ■

Proposition 4.3. The Quantum Yang–Mills theory gauge fixing Lagrange density and its accompanying ghost Lagrange density

$$\mathcal{L}_{QYM-GF} + \mathcal{L}_{QYM-Ghost} = \frac{1}{2g^2\xi} \delta_{ab} W^a W^b dV_\eta$$

$$+ \frac{1}{\xi} \eta^{\mu\nu} \tau_a \left( \partial_{\mu} c^a \right) dV_\eta$$

$$+ g \eta^{\mu\nu} f^a_{\ bc} \bar{c}_a \left( \partial_{\mu} (c^b A^c_{\nu}) \right) dV_\eta$$

(46)

for the Minkowski metric Lorenz gauge fixing functional $W^a := g\eta^{\mu\nu} \left( \partial_\mu A^a_{\nu} \right)$ can be obtained from the following gauge fixing fermion $F \in \mathcal{C}_{(0),(0,-1)}(\mathcal{B}_Q)$

$$F := \bar{c}_a \left( \frac{1}{8} W^a - \frac{1}{2} b^a \right) dV_\eta$$

(47)
Proof. The claimed statement follows directly from the calculations:

\[
QF = \frac{1}{\xi} b_a \left( \frac{1}{g} L^a - \frac{1}{2} b^a \right) dV_\eta + \frac{1}{g} \tau_a (QL^a) dV_\eta \tag{48a}
\]

with

\[
QL^a = g\eta^{\mu\nu} \partial_\mu (QA^a\nu) = g\eta^{\mu\nu} \partial_\mu \left( \frac{1}{\xi} \partial_\nu c^a + g f_{bc} c^b A^c\nu \right) \tag{48b}
\]

and then finally eliminating the Lautrup–Nakanishi auxiliary field \( b_a \) by inserting its equation of motion

\[
\text{EoM} (b_a) = \frac{1}{g} L^a, \tag{48c}
\]

which are obtained as usual via an Euler–Lagrange variation of Equation (48a).

Corollary 4.4. Given the situation of Proposition 4.3. Then the covariant Lorenz gauge fixing and ghost Lagrange densities read

\[
\mathcal{L}_{QYM-GF} + \mathcal{L}_{QYM-Ghost} = \frac{1}{2g^2\xi} \delta_{ab} L^a L^b dV_g + \frac{1}{\xi} g^{\mu\nu} \tau_a \left( \nabla^M \partial_\mu (c^a) \right) dV_g + gg^{\mu\nu} f_{bc} c^b \left( \nabla^M (c^b A^c\nu) \right) dV_g, \tag{49}
\]

with the covariant Lorenz gauge fixing functional \( L^a := g g^{\mu\nu} \left( \nabla^M A^a\mu \right) \). They can be obtained from the following gauge fixing fermion \( I_{(1)} \in \mathcal{C}_{(1),(0,-1)} (\mathcal{E}Q) \)

\[
I_{(1)} := \tau_a \left( \frac{1}{g} L^a - \frac{1}{2} b^a \right) \sqrt{-\text{Det} (g)} dV_\eta \tag{50}
\]

via \( QI_{(1)} \).

Proof. This can be shown analogously to the proof of Proposition 4.3.

Definition 4.5. Given the situation of Definition 4.1, we additionally define the gauge anti-BRST operator \( \overline{Q} \in \mathcal{X}_{(0,-1)} (\mathcal{E}Q) \) as the following odd vector field on the spacetime-matter bundle with gauge ghost degree -1:

\[
\overline{Q} := Q \bigg|_{c \mapsto \overline{c}} \tag{51a}
\]

together with the following additional changes

\[
\overline{Q} c^a := -\frac{1}{\xi} b_a + g f_{bc} \overline{c}^b c^c \tag{51b}
\]
\[
\overline{Q} \tau_a := -\frac{g}{2} f_{bc} \overline{c}^b \overline{c}^c \tag{51c}
\]
\[
\overline{Q} b_a := -\frac{g}{2} f_{bc} \overline{c}^b b^c \tag{51d}
\]
Corollary 4.6. Given the situation of Definition 4.5, we have
\[ [\mathcal{Q}, \mathcal{Q}] \equiv 2\mathcal{Q}^2 \equiv 0 \] (52)
and
\[ [Q, \mathcal{Q}] \equiv Q \circ \mathcal{Q} + \mathcal{Q} \circ Q \equiv 0, \] (53)
i.e. \( \mathcal{Q} \) is a homological vector field with respect to the gauge ghost degree that anticommutes with \( Q \).

Proof. This statement can be shown analogously to Proposition 4.2. ■

5 The diffeomorphism-gauge BRST double complex

In this section we show that the BRST operators \( P \) and \( Q \) as well as the anti-BRST operators \( \mathcal{P} \) and \( \mathcal{Q} \) mutually anticommute, cf. Theorem 5.1 and Corollary 5.2. Thus, we can introduce the total BRST operator \( D := P + Q \) and total anti-BRST operator \( \mathcal{D} := \mathcal{P} + \mathcal{Q} \). Additionally, we show that each gauge theory gauge fixing fermion \( \chi \) can be modified uniquely to a tensor density of weight \( w = 1 \). This is a useful choice, as we also show that in this case the graviton-ghosts decouple from matter of the Standard Model, cf. Theorem 5.3. Finally, we introduce the total gauge fixing fermion as the sum \( \Upsilon := \chi^{(1)} + F_{(1)} \), where \( \chi^{(1)} \) is the gauge fixing fermion corresponding to the linearized de Donder gauge fixing and \( F_{(1)} \) is the gauge fixing fermion corresponding to the covariant Lorenz gauge fixing. Finally, we show that this allows us to create the complete gauge fixing and ghost Lagrange densities of (effective) Quantum General Relativity coupled to the Standard Model via \( D \Y \) in Theorem 5.4.

Theorem 5.1. Given the two BRST operators \( P \in \mathfrak{k}_{(1,0)}(\mathcal{B}Q) \) and \( Q \in \mathfrak{k}_{(0,1)}(\mathcal{B}Q) \) from Definition 3.1 and Definition 4.1, respectively. Then we have
\[ [P, Q] \equiv P \circ Q + Q \circ P \equiv 0, \] (54)
i.e. their sum
\[ D := P + Q \] (55)
is also a cohomological vector field with respect to the total ghost degree. We call \( D \) the total BRST operator.

Proof. We show this statement by an explicit calculation:
\[
P \circ Q = \chi \left( \frac{1}{\xi} \left( \partial_\mu C^\rho \right) \left( \partial_\nu c^a \right) + \frac{1}{\xi} C^\rho \left( \partial_\mu \partial_\nu c^a \right) + g f^c_{bc} C^\rho \left( \partial_\mu c^b \right) A^c_\mu \right.
\]
\[
+ g f^c_{bc} C^\rho c^b \left( \partial_\mu A^c_\mu \right) + g f^c_{bc} \left( \partial_\nu C^\rho \right) c^b A^c_\mu \frac{\partial}{\partial A^a_\mu} \left( \partial_\mu \partial_\nu c^a \right)
\]
\[
- \frac{2g}{\xi} f^c_{bc} C^\rho \left( \partial_\mu c^b \right) c^c \left( \partial_\nu c^c \right) \frac{\partial}{\partial c^a} + \frac{\chi}{\xi} C^\rho \left( \partial_\mu b^a \right) \frac{\partial}{\partial c^a}
\]
\[
+ i \xi C^\rho c^a \mathcal{S}_a \cdot \left( \partial_\mu \Phi \right) \frac{\partial}{\partial \Phi}
\]
\[
+ i \xi C^\rho c^a \mathcal{S}_a \cdot \left( \nabla^M \Psi + \frac{i}{4} \left( \partial_\mu X_\nu - \partial_\nu X_\mu \right) e^{\nu m} e^{\nu n} \left( \sigma_{mn} \cdot \Psi \right) \right) \frac{\partial}{\partial \Psi}
\]
\[
= -Q \circ P,
\]
where we have used \( C^\rho c^a \equiv -c^\rho C^a \) and \([\mathcal{S}_a, \sigma_{mn}] \equiv 0\). ■
Corollary 5.2. Given the two anti-BRST operators $\overline{P} \in \mathfrak{X}_{(−1,0)}(\mathcal{B}Q)$ and $\overline{Q} \in \mathfrak{X}_{(0,−1)}(\mathcal{B}Q)$ from Definition 3.7 and Definition 4.5, respectively. Then we have

$$[\overline{P}, \overline{Q}] \equiv \overline{P} \circ \overline{Q} + \overline{Q} \circ \overline{P} \equiv 0,$$

i.e. their sum

$$\overline{D} := \overline{P} + \overline{Q}$$

is also a homological vector field with respect to the total ghost degree. We call $D \in \mathfrak{X}_{(−1)}(\mathcal{B}Q)$ the total anti-BRST operator. Furthermore, given the two BRST operators $P \in \mathfrak{X}_{(1,0)}(\mathcal{B}Q)$ and $Q \in \mathfrak{X}_{(0,1)}(\mathcal{B}Q)$ from Definition 3.1 and Definition 4.1, respectively. Then we have additionally

$$[P, \overline{Q}] \equiv P \circ \overline{Q} + \overline{Q} \circ P \equiv 0, \tag{59}$$

and

$$[\overline{P}, Q] \equiv \overline{P} \circ Q + Q \circ \overline{P} \equiv 0, \tag{60}$$

i.e. all BRST and anti-BRST operators mutually anticommute.

Proof. This statement can be shown analogously to Theorem 5.1. ■

Theorem 5.3. The graviton-ghosts decouple from matter of the Standard Model if and only if the gauge fixing fermion of Yang–Mills theory is a tensor density of weight $w = 1$. In particular, every such gauge fixing fermion can be modified uniquely to satisfy said condition; the case of the Lorenz gauge fixing is given via $\mathcal{F}\{1\}$ in Equation (50).

Proof. We start with the first assertion: Let $\chi$ be a tensor density of weight $w = 1$. Then, due to Lemma 3.3 we have

$$P\chi \simeq_{TD} 0, \tag{61}$$

where $\simeq_{TD}$ means equality modulo total derivatives. Furthermore, due to Theorem 5.1 we have

$$(P \circ Q) = - (Q \circ P), \tag{62}$$

which thus implies

$$(P \circ Q) \chi \simeq_{TD} 0. \tag{63}$$

Thus, on the level of Lagrange densities we obtain

$$P\mathcal{L}_{YM} \simeq_{TD} 0, \tag{64}$$

where

$$\mathcal{L}_{YM} := \mathcal{L}_{YM} + Q\chi. \tag{65}$$

It directly follows that there are no interactions between graviton-ghosts and the fields $\varphi \in \{A, c, \bar{c}, b, \Phi, \Psi\}$, given that $\mathcal{L}_{YM}$ and the matter Lagrange densities are covariant, and thus tensor densities of weight $w = 1$. The second assertion follows directly from the fact that any functional on the spacetime-matter bundle $f \in C^{\infty}_{(0,0)}(\mathcal{B}Q, \mathbb{R})$ can be modified uniquely to obtain tensor density weight $w = 1$ by the following replacements

$$f \mapsto \sqrt{-\det (g)} f \left. \frac{\eta_{\mu\nu} \sim g_{\mu\nu}}{\partial_{\mu} \sim \nabla_{\mu}^{TM}} \right|,$$

which concludes the proof. ■

Equivalently: If the gauge fixing and gauge ghost Lagrange densities of Yang–Mills theory are tensor densities of weight $w = 1$.  

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15
Theorem 5.4. Given the total BRST operator $D$ from Theorem 5.1 and let $\Upsilon \in C_{(-1)}(\mathcal{B}_Q)$

$$\Upsilon := \zeta^{(1)} + I_{\{1\}}$$

be the sum of the gauge fixing fermions from Corollaries 3.5 and 4.4, respectively. Then the complete gauge fixing and ghost Lagrange densities can be generated via $D\Upsilon$.

Proof. This follows directly from the calculation

$$D\Upsilon = (P + Q)(\zeta^{(1)} + I_{\{1\}})$$
$$= P\zeta^{(1)} + PI_{\{1\}} + Q\zeta^{(1)} + QI_{\{1\}}$$
$$\simeq_{TD} P\zeta^{(1)} + QI_{\{1\}}$$

where we have used Lemma 3.3 and $\simeq_{TD}$ means equality modulo total derivatives.

6 Conclusion

We have studied the BRST double complex of (effective) Quantum General Relativity coupled to the Standard Model. To this end, we started with a review of the geometric underpinnings, notably graded supergeometry, in Section 2. Then, we have studied the diffeomorphism and gauge complexes separately in Sections 3 and 4. In particular, we have recalled that the BRST and anti-BRST operators are nilpotent and thus cohomological or homological, respectively. In addition, we have discussed the gauge fixing fermions for the de Donder, linearized de Donder, Lorenz and covariant Lorenz gauge fixing conditions. A particularly important result is Lemma 3.3, which characterizes all Lagrange densities that are essentially closed with respect to the diffeomorphism BRST operator and diffeomorphism anti-BRST operator as scalar tensor densities of weight $w = 1$. Finally, we study the corresponding double complex in Section 5. Our main results are that all BRST and anti-BRST operators anticommute and thus give rise to the corresponding total BRST operator and total anti-BRST operator, cf. Theorem 5.1 and Corollary 5.2. Furthermore, we have shown that graviton-ghosts decouple from matter of the Standard Model if the gauge fixing fermion of Yang–Mills theory is a tensor density of weight $w = 1$, cf. Theorem 5.3. Finally, we have shown that all gauge fixing and ghost Lagrange densities of (effective) Quantum General Relativity coupled to the Standard Model can be derived from a total gauge fixing fermion, via the action of the total BRST operator, cf. Theorem 5.4. In future work, we want to apply these findings to study the transversal structure of (effective) Quantum General Relativity coupled to the Standard Model in [18]. In addition, we aim to derive the symmetric (hermitian) ghost Lagrange densities for pure (effective) Quantum General Relativity and its coupling to Quantum Yang–Mills theory, as was first introduced in [5] for the case of pure Quantum Yang–Mills theory.

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10 We remark that this works in fact for any gauge fixing condition, as long as the gauge fixing fermion of the gauge theory is a tensor density of weight $w = 1$, cf. Lemma 3.3 and the proof of this theorem.

11 We remark that the analysis of our aimed generalizations is much more involved than the case of pure Quantum Yang–Mills theory in [5]: In the case of pure (effective) Quantum General Relativity we need to analyze a priori infinitely many possible monomials. Further, in its coupling to Quantum Yang–Mills theory, we need to find the appropriate covariant generalization.
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