Decoupling of Field Equations in Einstein and Modified Gravity

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Abstract. This paper is concerned with giving the proof that there is a general decoupling property of vacuum and nonvacuum gravitational field equations in Einstein gravity and $f(R, T)$-modifications. The constructions are possible in terms of geometric and physical objects adapted to certain nonholonomic 2+2 splitting with local fibred structure. This allows us to generate exact and/or parametric depending solutions with generic off-diagonal metrics and generalized, or Levi-Civita, connections. Different classes of modified spacetimes are determined by corresponding generating and integration functions depending, in general, on all space and time coordinates and may possess, or not, Killing symmetries. The initial data sets for the Cauchy problem and their global properties are analyzed. There are formulated the criteria of evolution with spacetime splitting and decoupling of fundamental field equations. Examples of exact solutions defining ellipsoid deformations of black hole metrics and solitonic configurations are provided.

1. Introduction

The issue of constructing exact and approximate solutions of (modified) gravitational and matter field equations is of interest in mathematical physics and for various applications in modern cosmology and astrophysics. Mathematically, the fundamental field equations in gravitational physics are defined by sophisticate systems of nonlinear partial differential equations (PDE) which are very difficult to be integrated and studied in general forms.

In this work, we address again and develop a geometric approach (the so-called anholonomic frame deformation method, AFDM, see [20, 21, 22]) to constructing exact solutions in gravity described by generic off-diagonal and nonlinear gravitational, gauge and scalar field interactions. Such a geometric method allows us to generate exact solutions when the coefficients of generic off-diagonal metrics and various classes of connections depend on all spacetime coordinates. Our goal is to provide new results on the decoupling property\(^1\) of modified gravity equations and formal integration of such nonlinear systems of PDEs. Certain issues on the Cauchy problem and decoupling of gravitational equations and the initial data sets and nonholonomic evolution (with non-integrable constraints and/or with respect to anholonomic frames) will be analyzed. Examples of off-diagonal solutions for modified black holes/ellipsoids and solitonic configurations will be given.

\(^{1}\) It is used also the term “separation” of equations, which should not be confused with separation of variables.
Section 2 contains geometric preliminaries on modified gravity theories written in nonholonomic variables. In Section 3 we provide the main Theorems on decoupling the gravitational field equations for generic off–diagonal metrics with one Killing symmetry. We consider extensions to “non–Killing” configurations with coefficients depending on all set of four coordinates in section 4. Section 5 contains a study of the Cauchy problem in connection to the decoupling property of gravitational field equations. Explicit examples of generic off–diagonal exact solutions are studied in the next two sections. In Section 6, there are constructed nonholonomic vacuum deformations of black holes determined by off–diagonal gravitational interactions. Ellipsoid–solitonic configurations are analyzed in Section 7. The most important formulas and computations which are necessary to prove the decoupling property are given in Appendix A.

2. Einstein and Modified Gravity in Nonholonomic Variables

We provide geometric preliminaries on $f(\ast R, T)$–gravity models written in nonholonomic variables.\(^2\) In this work, the scalar curvature $\ast R$ is constructed for an auxiliary connection $D$ completely defined by the structure $g$ and $T$ is the trace of the energy–momentum tensor for matter fields. If $D = \nabla$, where $\nabla$ is the Levi–Civita connection, we get as particular cases the theories studied, for instance, in Ref. [11], when $\ast R = R$ is just the scalar curvature of a pseudo–Riemannian space. For $f(\ast R, T) = R$, such a modified gravity model transforms into the “standard” Einstein gravity theory.

2.1. Nonholonomic $2+2$ splitting

In general relativity (GR), the curved spacetime $(V, g)$ is defined by a pseudo–Riemannian manifold $V$ endowed with a Lorentzian metric $g$ as a solution of the Einstein equations\(^3\). The space of classical physical events is modelled as a Lorentzian four dimensional, 4-d, manifold $\mathcal{V}$ (of necessary smooth class, Housdorff and paracompact one) when the symmetric 2–covariant tensor $g = \{g_{\alpha\beta}\}$ defines in each point $u \in \mathcal{V}$ and nondegenerate bilinear form on the tangent space $T_u \mathcal{V}$, for instance, of signature $(+++–)$. The assumption that $T_u \mathcal{V}$ has its prototype (local fiber) the Minkowski space $\mathbb{R}^{3,1}$ leads at a causal character of positive/negative/null vectors on $\mathcal{V}$, i.e. for the module $\mathcal{X}$ of vectors fields $X, Y, ... \in \mathcal{X}(\mathcal{V})$, which is similar to that in special relativity.

Let us denote by $e_{\alpha}$ and $e^\beta$ a local frame and, respectively, its dual frame [we can consider orthonormal (co) bases], where Greek indices $\alpha, \beta, ...$ may be abstract ones, or running values 1, 2, 3, 4. For a coordinate base $u = \{u^\alpha\}$ on a chart $U \subset \mathcal{V}$, we can write $e_{\alpha} = \partial_u = \partial/\partial u^\alpha$ and $e^\beta = du^\beta$ and, for instance, define the coefficients of a vector $X$ and a metric $g$, respectively, in the forms $X = X^\alpha e_{\alpha}$ and

$$g = g_{\alpha\beta}(u)e^\alpha \otimes e^\beta, \quad (1)$$

where $g_{\alpha\beta} := g(e_{\alpha}, e_{\beta})$.\(^4\) We consider bases with non–integrable (equivalently, nonholonomic/anholonomic) $2 + 2$ splitting for conventional, horizontal, $h$, and vertical, $v$, decomposition, when for the tangent bundle $TV := \bigcup_u T_u \mathcal{V}$ a Whitney sum

$$N : TV = h\mathcal{V} \oplus v\mathcal{V} \quad (2)$$

is globally defined. Such a nonholonomic distribution is determined locally by its coefficients $N^h_\beta(u)$, when $N = N^h_\beta(x, y)dx^\beta \otimes \partial/\partial y^\beta$, where $u^\alpha = (x^1, y^\alpha)$ splits into $h$–coordinates, $x = (x^i)$.

\(^2\) Such a formulation is necessary for proving the main results on decoupling and integrating in certain general forms the gravitational field equations, see sections 3 and 4.

\(^3\) We assume that readers are familiar with basic concepts and results on mathematical relativity and methods of constructing exact solutions outlined, for instance, in above mentioned monographs and reviews.

\(^4\) The summation rule on repeating low–up indices will be applied if the contrary is not stated.
and $v$–coordinates, $y = (y^a)$, with indices running, respectively, values $i, j, k, \ldots = 1, 2$ and $a, b, c, \ldots = 3, 4, 5$.

A spacetime $(V, g)$ can be equipped with a non–integrable fibred structure (2) and such a manifold is called nonholonomic (equivalently, N–anholonomic). We use “boldface” letters in order to emphasize that certain spaces and geometric objects/constructions are “N–adapted”, i.e., adapted to a $h$–$v$–splitting. The geometric objects are called distinguished (in brief, d–objects, d–vectors, d–tensors etc). For instance, we write a d–vector as $X = (hX, vX)$ for a nonholonomic Lorentz manifold/spacetime $(V, g)$.

On a spacetime $(V, g)$, we can perform/adapt the geometric constructions using “N–elongated” local bases (partial derivatives), $e_\nu = (e_i, e_a)$, and cobases (differentials), $e^\mu = (e^i, e^a)$, when
\[ e_i = \partial / \partial x^i - N^a_i (u) \partial / \partial y^a, \quad e_a = \partial / \partial y^a, \quad e^i = dx^i, \quad e^a = dy^a + N^a_i (u) dx^i. \]

Such (co) frame structures depend linearly on N–connection coefficients being, in general, nonholonomic. For instance, the basic vectors (3) satisfy certain nontrivial nonholonomy relations
\[ [e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha \beta} e_\gamma, \]
with nontrivial anholonomy coefficients
\[ W^h_{3a} = \partial_a N^h_j, W^a_{ij} = \Omega^a_{ij} = e_j (N^a_i) - e_i (N^a_j). \]

Any spacetime metric $g = \{ g_{\alpha \beta} \} (1)$, via frame/coordinate transforms can be represented equivalently in N–adapted form as a d–metric
\[ g = g_{ij} (x, y) e^i \otimes e^j + g_{ab} (x, y) e^a \otimes e^b, \]

or, with respect to a coordinate local cobasis $du^\alpha = (dx^i, dy^a)$, as an off–diagonal metric
\[ g = g^\alpha_\beta du^\alpha \otimes du^\beta, \]
where
\[ g^\alpha_\beta = \begin{bmatrix} g_{ij} + N^a_i N^b_j g_{ab} & N^c_i g_{ae} \\ N^d_j g_{be} & g_{ab} \end{bmatrix}. \]

A metric $g$ is generically off–diagonal, if (7) can not be diagonalized via coordinate transforms. Ansatzes of this type are used in Kaluza–Klein gravity when $N^a_i (x, y) = \Gamma^a_{ij} (x) y^j$ and $y^a$ are “compactified” extra–dimensions coordinates, or in Finsler gravity theories, see details in [17, 22]. In this work, we restrict our considerations only to the 4–d gravity theories. The principle of general covariance in GR, allows us to consider any frame/coordinate transforms and write a spacetime metric $g$, equivalently, in any of the above form (1), (7) and/or (6). The last mentioned parametrization will allow us to prove a very important property of decoupling of gravitational field equations with respect to N–adapted bases (3) and (4).

Via frame/coordinate transforms $e_\alpha = e^\alpha_\alpha (x, y) e_\alpha, \quad g^\alpha_\beta = e^\alpha_\alpha e^\beta_\beta g^\alpha_\beta$, a metric $g$ (6) can be written in a form with separation of $v$–coordinates and nontrivial vertical conformal transforms,
\[ g = g_i dx^i \otimes dx^i + \omega^2 h_i h_a e^a \otimes e^a, \]
\[ e^i = dy^3 + (w_i + w_a^i) \, dx^i, \quad e^4 = dy^4 + (n_i + n_a^i) \, dx^i, \]

were
\[ g_i = g_i (x^k), \quad g_a = \omega^2 (x^i, y^c) h_a(x^k, y^3) h_b(x^k, y^4), \]
\[ N^3_i = w_i (x^k, y^3) + w_a^i (x^k, y^4), \quad N^4_i = n_i (x^k, y^3) + n_a^i (x^k, y^4). \]

5 We note that the 2+2 splitting can be considered as an alternative to the well known 3+1 splitting. The first one is convenient, for instance, for constructing generic off–diagonal solutions and elaborating models of deformation and/or A–brane quantization of gravity, but the second one is more important for canonical/loop quantization etc.
are functions of necessary smooth class which will be defined in a form to generate solutions of gravitational field equation\textsuperscript{6}.

The aim of this section is to prove that the gravitational field equations in the vacuum cases and certain very general classes of matter field sources decouple for parameterizations of metrics in the form (9).

2.2. Torsions and curvatures

In a general case, a metric–affine manifold \( V \) is endowed with a metric structure \( g \) and an affine (linear) connection structure \( D \) (as a covariant differentiation operator). A linear connection gives us the possibility to compute the directional derivative \( D_X Y \) of a vector field \( Y \) in the direction of \( X \). It is characterized by three fundamental geometric objects

\( (i) \) the torsion field is (by definition) \( \mathcal{T}(X, Y) := D_X Y - D_Y X - [X, Y]; \)

\( (ii) \) the curvature field is \( \mathcal{R}(X, Y) := D_X D_Y X - D_Y D_X X - D[D_X Y]; \)

\( (iii) \) the nonmetricity field is \( \mathcal{Q}(X) := D_X g. \)

Introducing \( X = e_a \) and \( Y = e_b \), defined by (3), into above formulas, we compute the \( N \)-adapted coefficients \( D = \{ \Gamma^\gamma_{\alpha\beta} \} \) and corresponding fundamental geometric objects,

\[ \mathbf{T} = \{ T^\gamma_{\alpha\beta} = (T^\gamma_{j,k}, T^\gamma_{j,a}, T^\gamma_{j,i}, T^\gamma_{b,i}, T^\gamma_{b,c}) \}; \]

\[ \mathcal{R} = \{ R^\alpha_{\beta\gamma\delta} = (R^a_{hjk}, R^a_{bjk}, R^a_{hja}, R^a_{hba}, R^a_{bea}) \}; \]

\[ \mathcal{Q} = \{ Q^\gamma_{\alpha\beta} \}. \]

Every (pseudo) Riemannian manifold \( (V, g) \) is naturally equipped with a Levi–Civita connection \( D = \nabla = \{ \nabla \Gamma^\gamma_{\alpha\beta} \} \) completely defined by \( g = \{ g_{\alpha\beta} \} \) if and only if the metric compatibility, i.e., \( \nabla Q(X) = \nabla_X g = 0 \), and zero torsion, i.e., \( \nabla \mathbf{T} = 0 \), conditions are satisfied. Hereafter, we shall write, for simplicity, \( \nabla \Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} \). It should be emphasized that \( \nabla \) does not preserve under parallelism and general frame/coordinate transforms a \( N \)-splitting (2). Nevertheless, it is possible to construct a unique distortion relation

\[ \nabla = \tilde{D} + \tilde{Z}, \tag{10} \]

where both linear connections \( \nabla \) and \( \tilde{D} \) (the second one can be considered as an auxiliary linear connection, which in literature is called the canonical distinguished connection; in brief, \( d \)-connection) and the distortion tensor \( \tilde{Z} \), i.e. all values in the above formula, are completely defined by \( g = \{ g_{\alpha\beta} \} \) for a prescribed \( N = \{ N^a_i \} \), see details in [20, 21, 22].

**Theorem 2.1** With respect to \( N \)-adapted frames (3) and (4), the coefficient of distortion relation (10) are computed

\[ \Gamma^\gamma_{\alpha\beta} = \tilde{\Gamma}^\gamma_{\alpha\beta} + \tilde{Z}^\gamma_{\alpha\beta}, \tag{11} \]

where the canonical \( d \)-connection \( \tilde{D} = \{ \tilde{L}^i_{jk}, \tilde{L}^a_{bk}, \tilde{C}^i_{jc}, \tilde{C}^a_{be} \} \) is defined by coefficients

\[ \tilde{L}^i_{jk} = \frac{1}{2} g^{ir} \left( e_k g_{jr} + e_j g_{kr} - e_r g_{jk} \right), \]

\[ \tilde{L}^a_{bk} = e_b (N^a_k) + \frac{1}{2} g^{ac} \left( e_k g_{bc} - g_{bc} e_b N^a_k - g_{db} e_c N^d_k \right), \]

\[ \tilde{C}^i_{jc} = \frac{1}{2} g^{ik} e_c g_{jk}, \]

\[ \tilde{C}^a_{be} = \frac{1}{2} g^{ad} (e_c g_{bd} + e_b g_{cd} - e_d g_{bc}). \tag{12} \]

\textsuperscript{6} There is not summation on repeating “low” indices \( a \) in formulas (9) but such a summation is considered for crossing “up–low” indices \( i \) and \( a \) in (8). We shall underline a function if it positively depends on \( y^i \) but not on \( y^a \).
and the distortion tensor \( \hat{Z}^{\gamma}_{\alpha\beta} \) is

\[
Z^{\gamma}_{jk} = -\tilde{\zeta}^{\gamma}_{jk}g^{ab} - \frac{1}{2} \Omega^{\gamma}_{jk}, \quad Z^{\gamma}_{bk} = \frac{1}{2} \Omega^{\gamma}_{jk}g^{ab} - \zeta^{\gamma}_{bk}C_{bh}, \quad Z^{\gamma}_{ab} = \pm \zeta^{\gamma}_{ab}D_{ab}, \quad Z^{\gamma}_{jk} = 0, \quad (13)
\]

\[
Z^{\gamma}_{kb} = \frac{1}{2} \Omega^{\gamma}_{jk}g^{ab} - \zeta^{\gamma}_{bk}C_{bh}, \quad Z^{\gamma}_{ab} = -\zeta^{\gamma}_{ab}D_{ab}, \quad Z^{\gamma}_{ab} = -\frac{g^{ij}}{2}[\hat{\nabla}_{ja}g_{kb} + \hat{\nabla}_{jb}g_{ka}],
\]

for \( \zeta^{\gamma}_{jk} = \frac{1}{2}(\partial_{j}^{\gamma}_{k} - \partial_{k}^{\gamma}_{j} - g_{jk}g^{dh}) \) and \( \pm \zeta^{\gamma}_{ab} = \frac{1}{2}(\partial_{a}^{\gamma}_{b} + g_{ab}g^{dh}) \). The nontrivial coefficients \( \Omega^{\gamma}_{jk} \) and \( \hat{T}^{\gamma}_{\alpha\beta} \) are given, respectively, by \( W^{\gamma}_{\beta\alpha} \) as formulas (5) and, see below, (14).

**Proof.** It follows from a straightforward verification in \( N \)-adapted frames that the sums \( \alpha \), \( \beta \), \( \gamma \) result in the coefficients of the Levi–Civita connection \( \Gamma^{\gamma}_{\alpha\beta} \) for a general metric parametrized as a \( \text{d} \)-metric \( g = [g_{ij}] \) (6). \( \square \)

All geometric constructions and physical theories derived for geometric data \( (g, \nabla) \) can be equivalently modeled by geometric data \( (g, N, D) \) because of unique distortion relation (10).

**Theorem 2.2** The nonholonomically induced torsion \( \hat{T} = \{\hat{T}^{\gamma}_{\alpha\beta}\} \) of \( \hat{D} \) is determined in a unique form by the metric compatibility condition, \( \hat{D}g = 0 \), and zero horizontal and vertical torsion coefficients, \( \hat{T}^{\gamma}_{ij} = 0 \) and \( \hat{T}^{\gamma}_{ab} = 0 \), but with nontrivial \( h-v \)–coefficients

\[
\hat{T}^{\gamma}_{ij} = \hat{L}^{\gamma}_{ij} - \hat{L}^{\gamma}_{kj} + \hat{L}^{\gamma}_{ja} = \hat{C}^{\gamma}_{ija}T^{\gamma}_{ij} = -\Omega^{\gamma}_{ij}, \quad \hat{T}^{\gamma}_{ab} = \hat{L}^{\gamma}_{ab} - \epsilon a(N^{\gamma}_{a}), \quad \hat{T}^{\gamma}_{bc} = \hat{C}^{\gamma}_{bca} - \hat{C}^{\gamma}_{cab}.
\]

**Proof.** The coefficients (14) are computed by introducing \( D = \hat{D} \), with coefficients (12), and \( X = e_{a}, Y = e_{b} \) (for \( N \)-adapted frames (3)) into standard formula for torsion, \( T(X, Y) := D_{X}Y - D_{Y}X - [X, Y] \). \( \square \)

In a similar form, introducing \( \hat{D} \) and \( X = e_{a}, Y = e_{b}, Z = e_{c} \), into \( \hat{R}(X, Y) := D_{X}D_{Y} - D_{Y}D_{X} - D_{[X, Y]} \), we prove

**Theorem 2.3** The curvature \( \hat{R} = \{\hat{R}^{\gamma}_{\alpha\beta\gamma}\} \) of \( \hat{D} \) is characterized by \( N \)-adapted coefficients

\[
\hat{R}^{\gamma}_{hjk} = e_{h}\hat{L}^{\gamma}_{ijk} - e_{j}\hat{L}^{\gamma}_{ikh} + \hat{L}^{\gamma}_{mk} - \hat{L}^{\gamma}_{hk}\hat{L}^{\gamma}_{mj} - \hat{C}^{\gamma}_{mjh} - \hat{C}^{\gamma}_{mj} - \hat{C}^{\gamma}_{hjk},
\]

\[
\hat{R}^{\gamma}_{bkj} = e_{k}\hat{L}^{\gamma}_{jbk} - e_{j}\hat{L}^{\gamma}_{bkl} + \hat{C}^{\gamma}_{bkl} - \hat{C}^{\gamma}_{hjk} - \hat{C}^{\gamma}_{hk} - \hat{C}^{\gamma}_{hjk},
\]

\[
\hat{R}^{\gamma}_{jka} = e_{k}\hat{L}^{\gamma}_{jka} - \hat{D}_{k}\hat{C}^{\gamma}_{ia} + \hat{C}^{\gamma}_{jki}, \quad \hat{R}^{\gamma}_{kab} = e_{a}\hat{L}^{\gamma}_{kh} - D_{k}\hat{C}^{\gamma}_{ba} + \hat{C}^{\gamma}_{bda} - \hat{C}^{\gamma}_{bdc} + \hat{C}^{\gamma}_{bd} - \hat{C}^{\gamma}_{bdc}.
\]

We can re–define the differential geometry of a (pseudo) Riemannian space \( V \) in nonholonomic form in terms of geometric data \( (\hat{g}, \hat{D}) \), which is equivalent to the formulation with \( (g, \nabla) \).

**Corollary 2.1** The Ricci tensor \( \hat{R}^{\gamma}_{\alpha\beta} := \{\hat{R}^{\gamma}_{\alpha\beta}\} \) of \( \hat{D} \) is characterized by coefficients

\[
\hat{R}^{\gamma}_{\alpha\beta} = \{\hat{R}^{\gamma}_{ij} := \hat{R}^{\gamma}_{ijk}, \hat{R}^{\gamma}_{ia} := -\hat{R}^{\gamma}_{ika}, \hat{R}^{\gamma}_{ai} := \hat{R}^{\gamma}_{aib}, \hat{R}^{\gamma}_{ab} := \hat{R}^{\gamma}_{abc}\}.
\]

**Proof.** The formulas for \( h-v \)-components (16) are obtained by contracting, respectively, the coefficients (15). Using \( \hat{D} \) (12), we express such formulas in terms of partial derivatives of coefficients of metric \( g \) (1) and any equivalent parametrization in the form (6), or (7). \( \square \)

The scalar curvature \( \hat{s}\hat{R} \) of \( \hat{D} \) is by definition

\[
\hat{s}\hat{R} = g^{\alpha\beta}\hat{R}^{\gamma}_{\alpha\beta} = g^{ij}\hat{R}^{\gamma}_{ij} + g^{ab}\hat{R}^{\gamma}_{ab}.
\]
In order to elaborate models of gravity theories for \( \nabla \) and/or \( \hat{D} \), we have to consider the corresponding Ricci tensors,

\[
\hat{R}ic = \{ R_{\beta\gamma} := R^\alpha_{\beta\gamma\alpha} \}, \quad \text{for } \nabla = \{ \Gamma^\gamma_{\alpha\beta} \},
\]

(18)

and \( \hat{R}ic = \{ \hat{R}_{\beta\gamma} := \hat{R}^\alpha_{\beta\gamma\alpha} \}, \quad \text{for } \hat{D} = \{ \hat{\Gamma}^\gamma_{\alpha\beta} \}. \]

(19)

For instance, using (16) and (17), we can compute the Einstein tensor \( \hat{E}_{\alpha\beta} \) of \( \hat{D} \),

\[
\hat{E}_{\alpha\beta} = \hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{s} \hat{R}.
\]

(20)

In general, this tensor is different from a similar one constructed with (18) for the Levi–Civita connection \( \nabla \).

**Proposition 2.1** The \( N \)-adapted coefficients \( \hat{\Gamma}^\gamma_{\alpha\beta} \) of \( \hat{D} \) are identic to the coefficients \( \Gamma^\gamma_{\alpha\beta} \) of \( \nabla \), both sets being computed with respect to \( N \)-adapted frames (3) and (4), if and only if the conditions \( \hat{\Sigma}^c_{aj} = e_a(N^c_j), \hat{C}^i_{jb} = 0 \) and \( \Omega^a_{ji} = 0 \) are satisfied.

**Proof.** If the conditions of the Proposition, i.e., constraints (27), are satisfied, all \( N \)-adapted coefficients of the torsion \( \hat{T}^\gamma_{\alpha\beta} \) (14) are zero. In such a case, the distortion tensor \( \hat{Z}^\gamma_{\alpha\beta} \) (13) is also zero. Following formula (11), we get \( \Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta} \). Inversely, if the last equalities of coefficients are satisfied for a chosen splitting (2), we get trivial torsions and distortions of \( \nabla \). We emphasize that \( \hat{D} \neq \nabla \) because such connections have different transformation rules under frame/coordinate transforms. Nevertheless, if \( \Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta} \) in a \( N \)-adapted frame of reference, we get corresponding equalities for the Riemann and Ricci tensors. This means that the \( N \)-coefficients are such way fixed via frame transforms that the nonholonomic distribution became integrable even, in general, the frames (3) and (4) are nonholonomic (because not all anholonomy coefficients are not obligatory zero, for instance, \( w^b_{ia} = \partial_a N^b_i \) may be nontrivial. \( \square \)

2.3. Field equations in nonholonomically modified gravity

We study modified gravity theories derived for the action

\[
S = \frac{1}{16\pi} \int \delta u^4 \sqrt{|g_{\alpha\beta}|} \left( f(\hat{R}, T) + m L \right)
\]

(21)

where \( m L \) is the matter Lagrangian density for which the stress–energy tensor of matter is defined via variation on inverse metric tensor as \( T_{\alpha\beta} = -\frac{2}{\sqrt{|g_{\mu\nu}|}} \frac{\delta (\sqrt{|g_{\mu\nu}|} m L)}{\delta g^{\mu\nu}} \), trace \( T := g^{\alpha\beta} T_{\alpha\beta} \), and \( f(\hat{R}, T) \) is an arbitrary functional on \( \hat{R} \) (17) and \( T \). For simplicity, we assume that the stress–energy tensor of the matter is given by

\[
T_{\alpha\beta} = (\rho + p)v_\alpha v_\beta - pg_{\alpha\beta},
\]

(22)

where in the approximation of perfect fluid matter \( \rho \) is the energy density, \( p \) is the pressure and the four–velocity \( v_\alpha \) is subjected to the conditions \( v_\alpha v^\alpha = 1 \) and \( v^\alpha D_\beta v_\alpha = 0 \), for \( m L = -p \) in a corresponding local frame. We also consider approximations for type

\[
f(\hat{R}, T) = f(\hat{R}) + 2f(T)
\]

(23)

and denote by \( 1F(\hat{R}) := \partial f(\hat{R})/\partial \hat{R} \) and \( 2F(T) := \partial^2 f(T)/\partial T \).
Theorem 2.4 The gravitational field equations for a modified gravity model (21) with \( f \)-functional (23) and perfect fluid stress–energy tensor (22) can be re–written equivalently using the canonical d–connection \( \hat{\nabla} \),

\[
\hat{\nabla}_{\beta\delta} - \frac{1}{2} g_{\beta\delta} \hat{\nabla} R = \Upsilon_{\beta\delta},
\]

where the source d–tensor \( \Upsilon_{\beta\delta} \) is such way constructed that \( \Upsilon_{\beta\delta} \rightarrow 8\pi G T_{\beta\delta} \) for \( \hat{\nabla} \rightarrow \nabla \), where \( T_{\beta\delta} \) is the energy–momentum tensor in GR with coupling gravitational constant \( G \).

Proof. By varying the action \( S \) (21) with respect to \( g^{\alpha\beta} \) and following a N–adapted covariant differential calculus with respect bases (3) and (4) (see similar results for the Levi–Civita connection in [11]) we obtain the gravitational field equations (24) and respective effective sources. \[ \Box \]

We consider matter field sources in (24) which can be diagonalized with respect to N–adapted frames,

\[
\Upsilon^\beta_\delta = \text{diag}[\Upsilon_\alpha : \Upsilon^1_1 = \Upsilon^2_2 = \Upsilon(x^k, y^\beta) + \Upsilon(x^k, y^4); \Upsilon^3_3 = \Upsilon^4_4 = \nu\Upsilon(x^k)].
\]

Such a formal diagonalization can be performed via corresponding frame/ coordinate transforms for very general distributions of matter fields. Such effective sources can be considered as nonholonomic constraints on the Ricci tensor (see Theorem 3.1) computed for certain general assumptions on modified off–diagonal gravitational interactions.

Corollary 2.2 The gravitational field equations (24) transform into the Einstein equations in GR, in “standard” form for \( \nabla \),

\[
E_{\beta\delta} = R_{\beta\delta} - \frac{1}{2} g_{\beta\delta} R = \kappa T_{\beta\delta},
\]

where \( R := g^{\beta\delta} R_{\beta\delta} \), if \( 2f = 0 \), \( 1f (\hat{\nabla} R) = R \), for the same N–adapted coefficients for \( \hat{\nabla} \) and \( \nabla \) if

\[
\hat{\Gamma}_{\alpha\beta} = e_\alpha (N^\gamma_j), \quad \hat{\nabla}^i_{\alpha\beta} = 0, \quad \Omega^\alpha_{ji} = 0.
\]

Proof. In general, the systems of PDEs (24) and (26) are very different. But if the constraints (27) are imposed additionally on \( \hat{\nabla} \), we satisfy the conditions of Proposition 2.1, when \( \Gamma^{\gamma}_{\alpha\beta} = \hat{\Gamma}^{\gamma}_{\alpha\beta} \) results in \( R_{\beta\delta} = \hat{\nabla} R_{\beta\delta} \) and \( E_{\alpha\beta} = \hat{\nabla} E_{\alpha\beta} \).

3. Decoupling of Gravitational Field Equations

The Einstein equations and their generalizations to various models of commutative and/or noncommutative gravity (for instance, in Finsler variables) possess a very important property that they decouple with respect to certain N–adapted frames of reference (3) and (4) and for the canonical d–connection \( \hat{\nabla} \). The aim of this section is to show how the anholonomic frame deformation method [20, 21, 22], AFDM, can be applied to decouple the fundamental field equations (24) for modified gravity.
3.1. Off–diagonal spacetimes with Killing symmetry

We use the ansatz (8) when \( \omega = 1, h_3 = 1, w_i = 0 \) and \( n_i = 0 \) in data (9) and \( \Upsilon = 0 \) for (25). Such a generic off–diagonal metric does not depend on variable \( y^4 \), i.e., \( \partial / \partial y^4 \) is a Killing vector, if \( h_4 = 1 \). Nevertheless, the decoupling property can proven for the same assumptions but arbitrary \( h_4(x^k, y^4) \) with nontrivial dependence on \( y^4 \). We call this class of metrics to be with weak Killing symmetry because they result in systems of PDEs (24) as for the Killing case but there are differences in (27) if \( h_4 \neq 1 \). It will be convenient to use brief denotations for partial derivatives, \( a^* = \partial a / \partial x^i \), \( a^* = \partial a / \partial x^2 \), \( a^* = \partial a / \partial y^3 \), \( a^* = \partial a / \partial y^4 \). The equations will be written with respect to \( N \)–adapted frames of type (3) and (4).

**Theorem 3.1** The gravitational field equations (24) with possible constraints (27) for a metric \( g \) (9) with \( \omega = h_3 = 1 \) and \( w_i = n_i = 0 \) and \( \Upsilon = 0 \) in matter source \( \Upsilon^\beta_\gamma \) (25) are equivalent, respectively, to

\[
\hat{R}_i^1 = \hat{R}_2^2 = \frac{-1}{2g_1 g_2} [g_1 g_2^* - \frac{g_1 g_2}{2g_1} - \frac{(g_2)^2}{2g_2} + g_1' - \frac{g_1 g_2'}{2g_2}] = - v \Upsilon, \tag{28}
\]

\[
\hat{R}_3^3 = \hat{R}_4^4 = - \frac{1}{2h_3 h_4} [h_3^* - \frac{(h_3)^2}{2h_4} - \frac{h_3' h_3^*}{2h_3}] = - \Upsilon, \tag{29}
\]

\[
\hat{R}_{3k} = \frac{w_k}{2} [h_4^* - \frac{(h_4)^2}{2h_4} - \frac{h_4' h_4^*}{2h_3}] + \frac{h_4^*}{2h_4} \left( \frac{\partial h_3}{h_3} + \frac{\partial h_4}{h_4} \right) - \frac{\partial k h_4^*}{2h_4} = 0, \tag{30}
\]

\[
\hat{R}_{4k} = \frac{h_4}{2h_3} n_k^* + \left( \frac{h_4^*}{h_3} n_k^* - \frac{3}{2} h_4^* \right) n_k^* = 0, \tag{31}
\]

and \( w_i^* = (\partial_i - w_i \partial_3) \ln |h_4|, \partial_3 w_i = \partial_i w_k, \tag{32} \]

\( n_k h_4^* = \partial_k h_4, n_i^* = 0, \partial_i n_k = \partial_k n_i. \)

**Proof.** See Appendix A. □

Let us discuss the decoupling (splitting) property of the modified and Einstein equations with respect to certain classes of \( N \)–adapted frames which is contained in the system of PDEs (28)–(32). For instance, the first equation is for a 2–d metric which always can be diagonalized, \( [g_1, g_2] \), and/or made to be conformally flat. Prescribing a function \( g_1 \) and source \( v \Upsilon \), we can find \( g_2 \), or inversely. The equation (29) contains only the first and second derivatives on \( \partial / \partial y^3 \) and relates two functions \( h_3 \) and \( h_4 \). Prescribing one of such functions and source \( \Upsilon \), we can define the second one taking, respectively, one or two derivations on \( y^3 \). The equation (30) is a linear algebraic system for \( w_k \) if the coefficients \( h_a \) have been already defined as a solution of (29). Nevertheless, we have to solve a system of first order PDEs on \( x^k \) and \( y^3 \) in order to find \( w_k \) resulting in zero torsion conditions (32). Such conditions do not allow a complete decoupling because the first equations relate \( w_i \) to \( H = \ln |h_4| \) via corresponding first order PDEs. Nevertheless, it is possible, for instance, to integrate such solutions for any prescribed \( H \) (see more details below, in Remark 4.1). The fourth equation (31) becomes trivial for any \( n_i^* = 0 \) if we want to satisfy completely such zero torsion conditions\(^7\). A nontrivial function \( \odot_1 \) is explicitly present in the conditions (32). If such restrictions are satisfied, this allows us to eliminate \( h_4 \) from (30).

We conclude that the modified equations for metrics with one Killing symmetry can be in such way parametrized with respect to \( N \)–adapted frames that they decouple and separate into “quite simple” PDEs for \( h \)–components, \( g_i \), and then for \( v \)–components, \( h_a \). The \( N \)–connection

\(^7\) Nontrivial solutions and nonzero torsion configurations present interest in modified theories of gravity and brane physics, see such examples in Ref. [24].
coefficients also separate and can be defined from corresponding algebraic and/or first order PDE. The “zero torsion” conditions impose certain additional constraints (as some simple first order PDE with possible separation of variables) on N–coefficients and coefficients of v–metric.

In a similar form, we can decouple the modified gravitational field equations for spacetimes with one Killing symmetry on \( \partial / \partial y_3 \).

**Corollary 3.1** The equations (24) and (27) for a metric \( g (9) \) with \( \omega = h_4 = 1 \) and \( w_i = n_i = 0 \) and \( v = 0 \) in matter source \( \Upsilon^3_3 \) (25) are equivalent, respectively, to

\[
\begin{align*}
\hat{R}_1^1 &= \hat{R}_2^2 = -\frac{1}{2g_1g_2} \left[ g_1^* g_2^* + \frac{g_1^*}{2g_1} \left( \frac{g_2^*}{2g_2} \right)^2 + g_1^* - \frac{g_1^* g_2^*}{2g_2} \right] = \nabla \Upsilon, \\
\hat{R}_3^3 &= \hat{R}_4^4 = -\frac{1}{2h_3 h_4} \left[ h_3^* \left( \frac{h_4^*}{2h_4} \right)^2 - \frac{h_3^* h_4^*}{2h_4} \right] = -\Upsilon, \\
\hat{R}_{3k} &= +\frac{h_k}{2h_4} w_{k^*} + \left( 2h_3 - \frac{3}{2} h_3^* \right) \frac{h_{k^*}}{2h_4} = 0, \\
\hat{R}_{4k} &= \frac{n_k}{2h_3} \left[ h_{k^*} - \frac{(h_3)^2}{2h_3} - \frac{h_{k^*} h_3^*}{2h_4} \right] + \frac{h_{k^*}}{4h_3} \left( \partial_k h_4 + \partial_k h_4 \frac{h_3^*}{h_4} \right) - \frac{\partial_k h_3^*}{2h_3} = 0, 
\end{align*}
\]

and \( w_k h_{k^*} = (\partial_k - n_k \partial_i) \ln h_3, (\partial_k - n_k \partial_i) n_k = (\partial_i - n_i \partial_k) n_k, \)

\[
\begin{align*}
\partial_k w_{k^*} &= \partial_k w_{k^*} = 0, \partial_1^a w_{k^*} = \delta_{k} w_{k^*}.
\end{align*}
\]

**Proof.** It is similar to that for Theorem 3.1 provided in Appendix Appendix A. We do not repeat such computations. \( \square \)

Using the above Theorem and Corollary, we can state:

**Conclusion 3.1** The nonlinear systems of PDEs corresponding to (modified) gravitational equations (24) and (27) for metrics \( g (9) \) with Killing symmetry on \( \partial / \partial y_4 \), when \( \omega = h_3 = 1 \) and \( w_i = n_i = 0 \) and \( v = 0 \) in matter source \( \Upsilon^3_3 \) (25), can be transformed into respective systems of PDEs for data with Killing symmetry on \( \partial / \partial y_3 \), when \( \omega = h_4 = 1 \) and \( w_i = n_i = 0 \) and \( v = 0 \), if \( h_3 (x^1, y^3) \rightarrow h_4 (x^1, y^3), h_4 (x^1, y^3) \rightarrow h_3 (x^1, y^3), w_k (x^1, y^3) \rightarrow n_k (x^1, y^3) \) and \( n_k (x^1, y^3) \rightarrow w_k (x^1, y^3) \).

The above presented method of nonholonomic deformations can be used for decoupling gravitational field equations if some metrics do not possess, in general, any Killing symmetries. The generic nonlinear character of such systems of PDEs does not allow us to use a principle of superposition of solutions. Nevertheless, certain classes of conformal transforms for the v–components of d–metrics and nonholonomic constraints of integral varieties give us the possibility to extend the anholonomic deformation method to “non–Killing” vacuum and nonvacuum gravitational interactions. In the next two subsections, we analyze two possibilities to decouple the effective Einstein equations for metrics with coefficients depending on all spacetime coordinates.

### 3.2. Preserving decoupling under \( v \)-conformal transforms

This property is stated by

**Lemma 3.1** The gravitational field equations (24) for geometric data (A.1), i.e., the system of PDEs (29)–(31), do not change under a “vertical” conformal transform with nontrivial \( \omega (x^k, y^a) \) to a d–metric (9) if the following conditions are satisfied,

\[
\begin{align*}
\partial_k \omega - w_i \omega^* - n_i \omega^c &= 0 \quad \text{and} \quad \hat{T}_{kk}^a = 0.
\end{align*}
\]
Proof. It follows from straightforward computations when coefficients
\[ g_i(x^k), g_3 = h_3(x^k, y^3), \quad g_4 = h_4(x^k, y^3) \text{,} \]
are generalized to a nontrivial \( \omega(x^k, y^3) \) with \( \omega_3 = \omega^3 h_3 \) and \( \omega_4 = \omega^4 h_4 \). Using, respectively, formulas (11), (12), (15) and (16), we get distortion relations for the Ricci d–tensors (19), \[ \hat{R}^a_b = R^a_b + \hat{Z}^a_b \text{ and } \hat{R}^a_b = \hat{R}^a_b = 0, \]
where \( \hat{R}^a_b \) and \( \hat{Z}^a_b \) are defined by a nontrivial \( \omega \) and computed using the same formulas. We do not provide certain similar details from Refs. [21, 22] for a nontrivial \( h_4 \) does not modify substantially the proof that \( \hat{Z}^a_b = 0 \) if the conditions (38) are satisfied. □

Using the Theorem 3.1, Corollary 3.1, Conclusion 3.1, Lemma 3.1, we prove

**Theorem 3.2** A d–metric

\[
\begin{align*}
g &= g_i(x^k)dx^i \otimes dx^i + \omega^2(x^k, y^3) \left( h_3 e^3 \otimes e^3 + h_4 \frac{\partial}{\partial e^4} \otimes e^4 \right), \\
e^3 &= dy^3 + \omega_i(x^k, y^3)dx^i, \quad e^4 = dy^4 + n_i(x^k)dx^i, \quad (39)
\end{align*}
\]

satisfying the PDEs (28)– (32) and \( \partial_k \omega - \omega_i \omega^*, - n_i \omega^* = 0 \), or a d–metric

\[
\begin{align*}
g &= g_i(x^k)dx^i \otimes dx^i + \omega^2(x^k, y^3) \left( h_3 e^3 \otimes e^3 + h_4 e^4 \otimes e^4 \right), \\
e^3 &= dy^3 + \omega_i(x^k)dx^i, \quad e^4 = dy^4 + n_i(x^k, y^3)dx^i, \quad (40)
\end{align*}
\]

satisfying the PDEs (33)– (37) and \( \partial_k \omega - \omega_i \omega^* - n_i \omega^* = 0 \), define, in general, two different classes of generic off–diagonal solutions of Einstein equations (24) and (27) with respective sources of type (25).

Both ansatzes of type (39) and (40) consist particular cases of parametrizations of metrics in the form (8). Via frame/coordinate transform into a finite region of a point \( 0^u \in V \) any spacetime metric in GR and f–modifications can be represented in an above mentioned d–metric form. If only one of coordinates \( y^k \) is timelike, the solutions of type (39) and (40) can not be transformed mutually via nonholonomic frame deformations preserving causality.

### 3.3. Decoupling with effective linearization of the Ricci tensors

The explicit form of field equations for vacuum and nonvacuum gravitational interactions depends on the type of frames and coordinate systems we consider for decoupling such PDEs. We can split such systems for more general parameterizations (than ansatzes (39) and (40)) in a form (8) with nontrivial \( \omega \) and N–coefficients in (9). This is possible in any open region \( U \subset V \) where for computing the N–adapted coefficients of the Riemann and Ricci d–tensors, see formulas (15) and (16), we can neglect contributions from quadratic terms of type \( \hat{\Gamma} \cdot \hat{\Gamma} \) but preserve values of type \( \partial_k \hat{\Gamma} \). For such constructions, we have to introduce a class of N–adapted normal coordinates when \( \hat{\Gamma}(u_0) = 0 \) for points \( u_0 \), for instance, belonging to a line on \( U \). Such conditions can be satisfied for decompositions of metrics and connections on a small parameter like it is explained in details in Ref. [20] (see decompositions on a small eccentricity parameter \( \varepsilon \) in Section 5). Other possibilities can be found if we impose nonholonomic constraints, for instance, of type \( h_4^* = 0 \) but for nonzero \( h_4(x^k, y^3) \) and/or \( h_4^*(x^k, y^3) \); such constraints can be solved in non–explicit form and define a corresponding subclass of N–adapted frames. Considering further nonholonomic deformations with a general decoupling with respect to a “convenient” system of reference/coordinates, we can deform the equations and solutions to configurations when terms of type \( \hat{\Gamma} \cdot \hat{\Gamma} \) become important.
Theorem 3.3 (“Non–quadratic” decoupling) The effective Einstein equations in modified gravity (for instance, in the form (24) and (27)), via nonholonomic frame deformations to a metric \( g \) (9) and matter source \( \Gamma^\beta_\delta \) (25), when contributions from terms of type \( \hat{r} \cdot \hat{r} \) are considered small for an open region \( U \subset \mathbb{V} \), can be transformed equivalently into a system of PDEs with \( h–v \)–decoupling

\[
\begin{align*}
\hat{R}_1^1 &= \frac{-1}{g_1 g_2} \left[ g_2^2 - \frac{g_1^2 g_2^2}{2 g_1} - \frac{(g_1^2)^2}{2 g_2^2} + g_1 - \frac{g_1 g_2}{2 g_2} - \frac{(g_1^2)^2}{2 g_1} \right ] = -v^\gamma, \\
\hat{R}_3^1 &= -\frac{1}{2 h_3} \left[ h_4^{**} - \frac{(h_3)^2}{2 h_4} - \frac{h_3^2 h_4}{2 h_3} \right ] - \frac{1}{2 h_3 h_4} \left[ h_3^{**} - \frac{(h_3)^2}{2 h_3} \right ] = -\gamma - \hat{\gamma}, \\
\hat{R}_{3k} &= \frac{h_3}{2 h_4} \left[ h_4^{**} - \frac{(h_3)^2}{2 h_4} - \frac{h_3^2 h_4}{2 h_3} \right ] + \frac{h_3}{4 h_4} \left[ \frac{\partial h_3}{h_3} + \frac{\partial h_4}{h_4} \right ] - \frac{\partial h_4}{h_4} = 0, \\
\hat{R}_{4k} &= \frac{h_3}{2 h_4} \left[ h_4^{**} - \frac{(h_3)^2}{2 h_4} - \frac{h_3^2 h_4}{2 h_3} \right ] + \frac{h_3}{4 h_4} \left[ \frac{\partial h_3}{h_3} + \frac{\partial h_4}{h_4} \right ] - \frac{\partial h_4}{h_4} = 0, \\
w^* &= (\partial_\lambda - w_1 \partial_1) \ln [h_4], (\partial_\lambda - w_k \partial_k) w_i = (\partial_\lambda - w_1 \partial_1) w_k, n_i^* = 0, \partial_\lambda n_k = \partial_\lambda n_i, \\
w^0 &= 0, \partial_\lambda w_k = \partial_\lambda w_i, n_i^0 = (\partial_\lambda - n_k \partial_k) n_i = (\partial_\lambda - n_k \partial_k) n_k, \\
\hat{e}_k = \partial_\lambda &- (w_i + w_\lambda) \omega^* - (n_i + n_\lambda) \omega^0 = 0.
\end{align*}
\]

\textbf{Proof.} It is a consequence of Conclusion 3.1 and Theorems 3.1 and 3.2 for superpositions of ansatzes (39) and (40) resulting into (8). If we repeat the computations from Appendix A for geometric data (9) considering that contributions of type \( \Gamma \cdot \hat{r} \) are small, we can see that the equations (42)–(44) are derived to be, respectively, equivalent to sums of (29)–(31) and (34)–(36). The torsionless conditions (45) consist a sum of similar conditions (32) and (37). □

In general, the solutions defined by a system (41)–(45) can not be transformed into solutions parameterized by an ansatz (39) and/or (40). As we shall prove in Section 4, the general solutions of the such systems of PDEs are determined by corresponding sets of generating and integration functions. A solution for (41)–(45) contains a larger set of \( h–v \)–generating functions than those with some \( N \)–coefficients stated to be zero.

4. Generic Off–Diagonal Solutions

The goal of this section is to show how decoupling effective Einstein equations we can integrate such PDEs in very general forms depending on the properties of the coefficients of an ansatz for metrics. Some similar theorems for 4–d, 5–d and higher dimension modifications of GR were proven in Refs [21, 22, 20]. In this work, we generalize those results for \( f \)–modifications.

4.1. Generating solutions with weak one Killing symmetry

We prove that the modified gravitational equations encoding gravitational and generic off–diagonal nonlinear interactions and satisfying the conditions of Theorem 3.1 can be integrated in general forms for \( h^*_a \neq 0 \) and certain special cases with zero and non–zero sources (25). In general, such generic off–diagonal metrics are determined by generating functions depending on three/four coordinates. The bulk of known exact solutions with diagonalizable metrics and coefficients depending on two coordinates (in certain special frames of references) can be included as special cases for more general nonholonomic configurations.
4.1.1. (Non) vacuum metrics with $h^*_a \neq 0$. For ansatz (8) with data $\omega = 1$, $h_3 = 1$, $w_i = 0$ and $n_s = 0$ for (9), when $h^*_a \neq 0$, and the condition that the source

$$\mathbf{Y}^i_3 = \text{diag}[\mathbf{Y}_a] : \mathbf{Y}^1_1 = \mathbf{Y}^2_2 = \mathbf{Y}(x^k, y^3) ; \mathbf{Y}^3_3 = \mathbf{Y}^4_4 = \nu \mathbf{Y}(x^k)],$$

is not zero, the solutions of gravitational field equations can be constructed as follows

**Theorem 4.1** The system (28)–(31) with source (46) can be integrated in general forms by

$$g = \epsilon_1 e^{\psi(x^k)} dx^i \otimes dx^i + h_3(x^k, y^3) e^3 \otimes e^3 + h_4(x^k, y^3) h^0_4(x^k, y^4) e^4 \otimes e^4,$$

$$e^3 = dy^3 + w_i(x^k, y^3) dx^i, \quad e^4 = dy^4 + n_i(x^k) dx^i,$$

with coefficients determined by generating functions $\psi(x^k), \phi(x^k, y^3), \phi^* \neq 0, n_i(x^k)$ and $h^0_4(x^k, y^4)$, and integration functions $\nu \phi(x^k)$ following recurrent formulas and conditions

$$\epsilon_1 \psi'' + \epsilon_2 \psi'' = 2 \nu \mathbf{Y},$$

$$h_4 = \pm \frac{1}{4} \int |\mathbf{Y}|^{-1} (e^{2\phi})^* dy^3, \text{ or } = \pm \frac{1}{4\Lambda} e^{2[\phi - \nu \phi]}, \text{ if } \mathbf{Y} = \Lambda = \text{const};$$

$$h_3 = \pm \left(\sqrt{|h_4|}\right)^2 e^{-2\phi} = \frac{\phi^*}{2 |\mathbf{Y}|} (\ln |h_4|)^* \text{ or } = \pm (\frac{\phi^*}{4\Lambda})^2, \text{ if } \mathbf{Y} = \Lambda = \text{const};$$

$$w_i = -\partial_i \phi / \phi^*,$$

(50)

where constraints

$$w_i^* = (\partial_i - w_i \partial_3) \ln |h_4|, \partial_k w_i = \partial_i w_k, \quad n_k h^0_4 = \partial_k h^0_4, \partial_i n_k = \partial_k n_i.$$ (51)

must be imposed in order to satisfy the zero torsion conditions (32); we should take respective values $\epsilon_1 = \pm 1$ and $\pm$ in (49) and/or (50) if we want to fix a necessary spacetime signature.

**Proof.** We sketch a proof which transforms into similar ones in [21, 22, 20] if $h_4 = 1$.

- A horizontal metric $g_i(x^2)$ is for 2-d and can be always represented in a conformally flat form $e^\epsilon e^{\psi(x^k)} dx^i \otimes dx^i$. For such an h-metric, the equation (28) is a 2-d Laplace/wave equation (48) which can be solved exactly if a source $\nu \mathbf{Y}(x^k) - 4s^2$ is prescribed from certain physical conditions.

- If $h^*_a \neq 0$, we can define nontrivial functions

$$\phi = \ln \left| \frac{h^*_a}{\sqrt{|h_3 h_4|}} \right|, \quad \gamma := \left( \ln \left| \frac{|h_4|^{3/2}}{|h_3|} \right| \right)^*,$$

$$\alpha_i = h^*_a \partial_i \phi, \quad \beta = h^*_a \phi^*$$ (52)

for a function $\phi(x^k, y^3)$. If $\phi^* \neq 0$, we can write, respectively, the equations (29)–(31) in the forms

$$\phi^* h^*_4 = 2 h_3 h_4 \mathbf{Y},$$

$$\beta w_i + \alpha_i = 0,$$

$$n_i^* + \gamma n_i = 0.$$ (54)

For the last equation, we must take any trivial solution given by functions $n_i(x^k)$ satisfying the conditions $\partial_j n_j = \partial_j n_i$ in order to solve the constraints (32). Using coefficients (52) with $\alpha_i \neq 0$ and $\beta \neq 0$, we can always express $w_i$ via derivatives of $\phi$, i.e., in the form (50).
We can choose any $\phi(x^k, y^3)$ with $\phi^* \neq 0$ as a generating function and express $h_4$ and, after that, $h_3$ as some integrals/derivatives of functions depending on $\phi$ and source $\Upsilon(x^k, y^3)$, see corresponding formulas (49) and (50). The integrals can be computed in a general explicit form if $\Upsilon(x^k, y^3) = \Lambda = \text{const}$, when possible matter field interactions and $f$-modifications are approximated by an energy–momentum tensor as a cosmological constant,

$$\Upsilon^\beta_\delta = \delta^\beta_\delta \Upsilon = \delta^\beta_\delta \Lambda. \quad (55)$$

- A possible dependence on $y^4$ is present in function $h_4$, which must satisfy conditions of type (A.6) in order to be compatible with (32). It is not possible to write, in explicit form, the solutions for the zero torsion condition if the source $\Upsilon^\beta_\delta$ is parameterized by arbitrary functions. Nevertheless, if $\Upsilon^\beta_\delta$ is of type (55), we get $h_4 \sim e^{2\phi}$ and $w_i \sim \partial_i \phi / \phi^*$ positively solve the constraints

$$w_i^* = (\partial_i - w_i \partial_3) \ln |h_4| \quad \text{and} \quad \partial_i w_j = \partial_j w_i, \quad (56)$$

transformed into $\phi^* \partial_i \phi - \phi^* (\partial_i \phi)^* = 0$. By straightforward computation, we can check that (32) are satisfied by (51) when $n_i^* = 0$ and $w_i$ is determined by (50). \[\square\]

The solutions constructed in Theorem 4.1, and those which can be derived following Corollary 3.1 are very general ones and contain as particular cases (perhaps) all known exact solutions for (non) holonomic effective Einstein spaces with Killing symmetries. They also can be generalized to include arbitrary finite sets of parameters as it is proven in Ref. [20].

**Remark 4.1 (−Example)** Introducing $H = -\ln |h_4|$, the system of equations (53), (54) and (56) can be written in the form

$$h_3 = -\phi^* H^* / 2\Upsilon_2, \quad (57)$$

$$\phi^* w_i + \partial_i \phi = 0, \quad (58)$$

$$w_i^* + \partial_i H - w_i H^* = 0. \quad (59)$$

Considering $\Upsilon(x^k, y^3)$ as a generating function, we can integrate (59) in explicit form using parameterizations $w_i = w_i(x^k, y^3) \times 2w_i(x^k, y^3)$ (in this formula, we do not consider summation on $i$). We choose $\phi(x^k, y^3)$ as a nontrivial solution of the system of first order PDEs (58) for any found $w_i$. So, we can define $h_3$ for any prescribed source $\Upsilon_2$ in (57). We conclude that the LC-conditions (32) (relating $w_i$ to $h_4$, which does not allow a “complete” decoupling) can be solved also in very general forms by further fixing of parameterizations, classes of functions and boundary conditions for $H, \phi$ and $\Upsilon_2$.

Nevertheless, we note that an explicit example of exact solutions can be derived by taking derivative on $*$ for (58) and using a subclass of functions $\phi$ when $(\partial_i \phi)^* = \partial_i (\phi^*)$. We obtain

$$w_i \phi^* + w_i^* + \partial_i \phi^* = 0. \quad \text{This system is equivalent to} \quad (59) \text{if, for instance,} \quad \Phi := \ln |\phi^*|, \quad H^* = -\Phi^*, \quad \partial_i H = \partial_i \Phi. \quad \text{Such equations are satisfied by any functions} \quad H(-\varsigma) \quad \text{and} \quad \Phi(+\varsigma), \quad \text{where} \quad \pm \varsigma := \pm v - x^1 - x^2.$$

4.1.2. Effective modified vacuum gravitational configurations We can consider a subclass of generic off–diagonal modified gravitation interactions which can be encoded as effective Einstein manifolds. In general, such classes of solutions do not have a smooth limit from non-vacuum to vacuum models.
Corollary 4.1 The effective vacuum solutions for the modified gravitational equations with ansatz for metrics of type (47) with vanishing source (46) are parametrized in the form

\[
\begin{align*}
    g &= \epsilon_i e^{\psi(x^i)} dx^i \otimes dx^i + h_3(x^k, y^3) e^3 \otimes e^3 + h_4(x^k, y^3) h_4(x^k, y^4) e^4 \otimes e^4, \\
    e^3 &= dy^3 + w_i(x^k, y^3) dx^i, \\
    e^4 &= dy^4 + n_i(x^k) dx^i,
\end{align*}
\]  

(60)

where coefficients are defined by solutions of the system

\[
\begin{align*}
    \ddot{\psi} + \psi'' &= 0, \\
    \phi^* h_4^* &= 0, \\
    \beta w_i + \alpha_i &= 0,
\end{align*}
\]  

(61)–(63)

and with coefficients computed following formulas (52) for nonzero \( \phi^* \) and \( h_4^* \) with possible further zero limit; such coefficients and \( h_3 \) and \( h_4 \) are subjected, additionally, to the zero–torsion conditions (51).

Proof. Considering the system of equations (28)–(31) with zero right sides, we obtain, respectively, the equations (61)–(63). For positive signatures on h–subspace and equation (61), we can take \( \psi = 0 \), or consider a trivial 2-d wave equation if one of coordinates \( x^k \) is timelike. There are two possibilities to satisfy the condition (62). The first one is to consider that \( h_4 = h_4(x^k) \), i.e., \( h_4^* \) which states that the equation (62) has solutions with zero source for arbitrary function \( h_3(x^k, y^3) \) and arbitrary N–coefficients \( w_i(x^k, y^3) \) as follows from (52). For such vacuum configurations, the functions \( h_3 \) and \( w_i \) can be taken as generation ones which should be constrained only by the conditions (51). Equations of type (56) constrain substantially the class of admissible \( w_i \) if \( h_4 \) depends only on \( x^k \). Nevertheless, \( h_3 \) can be an arbitrary one generating solutions which can be extended for nontrivial sources \( \Upsilon \) and systems (33)–(37) and/or (41)–(45).

A different class of solutions can be generated if we state, after corresponding coordinate transforms, \( \phi = \ln |h_4^*/\sqrt{|h_3 h_4|}| = 0 \phi = const, \phi^* = 0 \). For such configurations, we can consider \( h_4^* \neq 0 \), and solve (62) as

\[
\sqrt{|h_3|} = 0 h(\sqrt{|h_4|})^*,
\]

(64)

for \( 0 h = const \neq 0 \). Such \( v \)–metrics are generated by any \( f(x^i, y^3) \), \( f^* \neq 0 \), when

\[
h_4 = -f^2 (x^i, y^3) \quad \text{and} \quad h_3 = (0 h)^2 \left[ f^* (x^i, y^3) \right]^2,
\]

(65)

where the signs are such way fixed that for \( N_i^a \to 0 \) we obtain diagonal metrics with signature \((+, +, +, -)\). The coefficients \( \alpha_i = \beta = 0 \) in (63) and \( w_i(x^k, y^3) \) can be any functions solving (51). This is equivalent to

\[
\begin{align*}
    w_i^* &= 2 \partial_i \ln |f| - 2 w_i(\ln |f|)^*, \\
    \partial_k w_i - \partial_i w_k &= 2(w_k \partial_i - w_i \partial_k) \ln |f|,
\end{align*}
\]  

(66)

for any \( n_i(x^k) \) when \( \partial_i n_k = \partial_k n_i \). Constraints of type \( n_k h_4^* = \partial_k h_4 \) (A.6) have to be imposed for a nontrivial multiple \( h_4^* \). □

Using Corollary 3.1, the ansatz (60) can be “dualized” to generate effective vacuum solutions with weak Killing symmetry on \( \partial/\partial y^3 \). Finally, we note that the signature of the generic off–diagonal metrics generated in this subsection depend on the fact which coordinate \( x^1, x^2, y^3 \) or \( y^4 \) is chosen to be a timelike one.
4.2. Non–Killing effective Einstein configurations

The Theorem 3.2 can be applied for constructing non–vacuum and effective vacuum solutions of the modified gravitational equations depending on all coordinates without explicit Killing symmetries.

4.2.1. Non–vacuum off–diagonal solutions  We can generate such effective Einstein manifolds as follows

**Corollary 4.2** An ansatz of type (39) with \(d\)-metric

\[
\begin{align*}
g &= \epsilon e^\phi dx^i \otimes dx^i + \omega^2 [e^\phi + \frac{1}{4\Lambda} - \frac{e^{2\phi - \alpha\phi}}{4\Lambda}] e^3 \otimes e^3 + \frac{1}{4\Lambda} e^{2\phi - \alpha\phi} [\mathbb{L}_4 e^4 \otimes e^4], \\
e^3 &= dy^3 - (\partial \phi/\phi^*)_dx^i, \\
e^4 &= dy^4 + n_4(x^k) dx^i,
\end{align*}
\]

where the coefficients are subjected to conditions (48)–(51) and \(\partial_{x} \omega + (\partial \phi/\phi^*) \omega^* - n_4 \omega^o = 0\)

defines solutions of the Einstein equations \(R_{\alpha\beta} = \Lambda g_{\alpha\beta}\) with nonholonomic interactions and modifications encoded effectively into the vacuum structure of GR with nontrivial cosmological constant, \(\Lambda \neq 0\).

**Proof.** It is a consequence of Theorem 3.2 and Corollary 4.1. \(\Box\)

In a similar form, we can generate solutions of type (40) when the conformal factor is a solution of \(\partial_{x} \omega - w_i(x^k) \omega^* + (\partial \phi/\phi^*) \omega^o = 0\) with respective “dual” generating functions \(\omega\) and \(\phi\) when the data (48)–(51) are re–defined for solutions with weak Killing symmetry on \(\partial/\partial y^3\).

4.2.2. Effective vacuum off–diagonal solutions Vacuum Einstein spaces encoding nonholonomic interactions and \(f\)--modifications can be constructed using

**Corollary 4.3** An ansatz of type (39) with \(d\)-metric

\[
\begin{align*}
g &= \epsilon e^{\psi(x^i)} dx^i \otimes dx^i + \omega^2 (x^k, y^a) [e^\psi] [f^* (x^i, y^3)]^2 e^3 \otimes e^3 - f^2 (x^i, y^3) [\mathbb{L}_4 (x^k, y^4) e^4 \otimes e^4], \\
e^3 &= dy^3 + w_4(x^k, y^3) dx^i, \\
e^4 &= dy^4 + n_4(x^k) dx^i,
\end{align*}
\]

where the coefficients are subjected to conditions (64)–(66), (51) and \(\partial_{x} \omega - w_i \omega^* - n_4 \omega^o = 0\)

defines generic off–diagonal solutions of \(R_{\alpha\beta} = 0\).

**Proof.** It is a consequence of Theorem 3.2 and Corollary 4.1. \(\Box\)

Solutions of type (40) can be defined if the conformal factor is a solution of \(\partial_{x} \omega - w_i(x^k) \omega^* - n_4(x^k, y^4) \omega^o = 0\) with respective “dual” generating functions \(\omega(x^k, y^a)\) and \(\phi(x^k, y^4)\) when the data (64)–(66) and (51) are re–defined for ansatz with weak Killing symmetry on \(\partial/\partial y^3\).

Summarizing the results of this section, we state

**Claim 4.1** All generic off–diagonal non–vacuum and effective vacuum solutions of the modified gravitational equations determined, respectively, by Theorem 4.1 and Corollaries 4.1, 4.2 and 4.3 can be generalized to metrics of type (8). For such constructions, there are used nonlinear superposition of metrics and their “duals” on \(v\)-coordinates in order to define non–Killing solutions of respective systems of PDEs from Theorems 3.2 and/or 3.3.

The statements of this Claim are formulated following our experience on constructing generic off–diagonal vacuum and non–vacuum solutions. For such nonlinear systems, it is not possible to formulate certain general uniqueness and exhaustive criteria. Sure, not all solutions of modified gravitational equations can be constructed in such forms, or related to any sets of prescribed solutions via “nondegenerate” nonholonomic deformations. The length of this paper does not
allow us to present all technical details and general formulas for coefficients for ansatzes for d–metrics\(^8\) which can be constructed following this Claim. Explicit examples supporting our approach are given in sections 5 and 6.

5. The Cauchy Problem and Decoupling

The gravitational interactions in modified gravity studied in this work are described by off–diagonal solutions of

\[
R_{\alpha\beta} = \Lambda g_{\alpha\beta},
\]

which can be found in very general forms with respect to N–adapted frames for certain nonintegrable spacetime \(2 + 2\) splitting of type \(N(2)\). An effective cosmological constant \(\Lambda\) encodes a gravitational “vacuum” cosmological constant \(\Lambda\) in GR and \(f\)–modifications. Pseudo–Riemannian manifolds with metrics \(g_{\alpha\beta}\) adapted to chosen non–integrable distribution with \(2 + 2\) splitting and satisfying \(67\) are called \textit{nonholonomic effective Einstein manifolds}. Hereafter, we shall refer to such systems of PDEs as \textit{nonholonomic effective vacuum spacetimes}, regardless of whether, or not, an (effective) cosmological constant vanishes or can be “polarized” by gravitational and/or matter field interactions into some \(N\)–adapted diagonal sources admitting formal integration of gravitational field equations.

The equations \(67\), and their \(N\)–adapted equivalents \(24\)–\(27\), constitute a second–order system quasi–linear PDEs for the coefficients of spacetime metric \(g = \{g_{\alpha\beta}\}\). This means that given a manifold \(V\) of necessary smooth class such a quasi–linear system is linear in the second derivatives of the metric and quadratic in the first derivatives \(\partial_{\alpha}g_{\beta\gamma}\) (the coefficients of such PDEs are rational functions of \(g_{\alpha\beta}\)). To be able to decouple and formally integrate such systems is necessary to consider special classes of nonholonomic frames and constraints. For GR, this type of equations do not fall in any of the standard cases of hyperbolic, parabolic, or elliptic systems which typically lead to unique solutions. It is important to formulate criteria when such general solutions would be unique ones with a topology and differential structure determined by some initial data. How the diffeomorphism, or coordinate, invariance and arbitrary frame transforms (principle of relativity) would be taken into account for \(N\)–splitting?

In mathematical relativity \([5, 14, 8, 7, 9]\), it was proven a fundamental result (due to Choquet–Bruhat, 1952) that there exists a set of hyperbolic equations underlying \(67\). The goal of this section is to study the evolution (Cauchy) problem for the system \(24\)–\(27\) in \(N\)–adapted form and preserving the decoupling property.

5.1. The local \(N\)–adapted evolution problem

In the evolutionary approach, the topology of spacetime manifold is chosen in the form \(V = \mathbb{R} \times ^3V\), where \(^3V\) is a 3–d manifold carrying initial data. It should be noted here that there is no a priori known natural time–coordinate even we may fix a signature for metric and chose certain coordinates to be time or space like ones.

\textbf{Definition 5.1} A set of coordinates \(\{\hat{u}^\mu = (\hat{x}^i, \hat{y}^a)\}\) is canonically \(N\)–harmonic, i.e., it is both harmonic and adapted to a splitting \(N\) (2), if each of the functions \(\hat{u}^\mu\) satisfies the wave equation

\[
\Box \hat{u}^\mu = 0,
\]

where the canonical d’Alembert operator \(\Box := \hat{D}_\alpha \hat{D}^\alpha\) acts on a scalar \(f(x, y)\) in the form

\[
\Box f := (\sqrt{|g_{\alpha\beta}|})^{-1}e_\nu \left(\sqrt{|g_{\alpha\beta}|}g^{\mu\nu}e_\nu f\right) = (\sqrt{|g_{kl}|})^{-1}e_i \left(\sqrt{|g_{kl}|}g^{ij}e_j f\right) + (\sqrt{|g_{cd}|})^{-1}\partial_a \left(\sqrt{|g_{kl}|}g^{ab}e_b f\right),
\]

\(^8\) Such formulas are, for instance, of type \((48)\)–\((51)\), with functions \(h_3(., y^3)\) and \(h_4(., y^4)\) and further “dualization” \(h_3 \rightarrow \hat{h}_4(., \hat{y}^4)\) and \(h_4 \rightarrow \hat{h}_3(., \hat{y}^3)\) with corresponding re–definition of \(N\)–connection coefficients.
for a d–metric $g_{\alpha\beta} = (g_{ij}, g_{ab})$ (6) defined with respect to N–adapted (co) frames (3)–(4) and canonical d–connection $\mathbf{D}_\alpha$ (12).

We can say that such coordinates $\tilde{u}^\mu = (\tilde{x}^i, \tilde{y}^a)$ are N–adapted wave–coordinates.

**Lemma 5.1** In canonical N–harmonic coordinates, the effective Einstein equations (67) re–defined in canonical d–connection variables (24) can be written, equivalently,

$$\hat{E}^\alpha{}_{\beta} = \hat{g}^{\alpha\beta} - g^{\gamma\theta} \left[ (\hat{g}^{\alpha\mu} \hat{T}^\beta_{\mu\nu} + g^{\alpha\mu} \hat{T}^\beta_{\mu\nu}) \hat{\Gamma}^\nu_{\tau\theta} + 2g^{\gamma\mu} \hat{T}^\alpha_{\mu\theta} \hat{T}^\beta_{\gamma\tau} \right] - 2\Lambda g^{\alpha\beta} = 0, \quad (69)$$

i.e., such PDEs, for $g^{\alpha\beta}$ (using algebraic transforms, for $g_{\alpha\beta}$) form a system of second–order quasi–linear N–adapted wave–type equations.

**Proof.** It is a standard computation with respect to N–adapted frames by using formulas (12), (20) and (16). If the zero torsion conditions (27) are imposed, we get well known results from GR but (in our case) adapted to $2 + 2$ non–integrable splitting.

This Lemma allows us to apply the standard theory of hyperbolic PDEs (see, for instance, [10]). Let us denote by $H^{k}_{loc}$ the Sobolev spaces of functions which are in $L^2(K)$ for any compact set $K$ when their distributional derivatives are considered up to an integer order $k$ also in $L^2(K)$. We shall also use N–adapted wave coordinates with additional formal $3 + 1$ splitting, for instance, in a form $\tilde{u}^\mu = (\tilde{t}\tilde{u}, \tilde{u}^i)$, where $\tilde{t}\tilde{u}$ is used for the timelike coordinate and $\tilde{u}^i$ are for 3 spacelike coordinates. “Hats” can be eliminated if such a splitting is considered for arbitrary local coordinates. Standard results from the theory of PDEs give rise to this

**Theorem 5.1** The field equations (24) for nonholonomic effective Einstein manifolds have a unique solution $g^{\alpha\beta}$ defined by PDEs (69) stated on an open neighborhood $U \subset \mathbb{R} \times \mathbb{R}^3$ of $\mathcal{O} \subset \{0\} \times \mathbb{R}^3$ with any initial data

$$g^{\alpha\beta}(0, \tilde{u}^i) \in H^{k+1}_{loc} \quad \text{and} \quad \frac{\partial g^{\alpha\beta}(0, \tilde{u}^i)}{\partial (\tilde{t}\tilde{u})} \in H^{k+1}_{loc}, \quad k > 3/2. \quad \quad (70)$$

The set $U$ can be chosen in a form that $(U, g^{\alpha\beta})$ is globally hyperbolic with Cauchy surface $\mathcal{O}^0$.

There is no reason that a solution constructed using the anholonomic deformation method as we considered in section 3 will satisfy the wave conditions (68), even if we give certain initial data for equation (69). In order to establish an hyperbolic (and evolutionary) form of such nonholonomic vacuum gravitational equations we should re–define the Choquet–Bruhat problem (see details and references in [5, 14, 8, 18, 9]) with respect to N–adapted frames. Using a $3+1$ splitting of N–adapted coordinates, we write $g^{\alpha\beta} = (g^{t\beta}, g^{ij})$ and $e_\alpha = (e_t, e_i)$ and consider the d–vector field $n^\mu(x, y)$ of unit timelike normals to the hypersurface $\{\tilde{t}\tilde{u} = 0\}$\footnote{Choosing corresponding classes of nonholonomic distributions $\mathbb{N}$ (2), we can relax the conditions of differentiability as in Refs. [15, 19] (we omit such constructions in this work).}. There is no loss of generality if we assume that on such a hypersurface we have

$$g^{tt} = -1 \quad \text{and} \quad g^{ti} = 0. \quad \quad (71)$$

We can state such conditions via additional N–adapted frame/coordinate transform for any d–metric (6) with prescribed signature. It is also possible to re–define the generating functions for

\footnote{We do not use labels for coordinates like 0, 1, 2, 3 because the decoupling property of the Einstein equations and general solutions can be proven for arbitrary signature, for instance $(- + + +), (+ + - +)$, etc. and for any set of local coordinates $x^\alpha$ with $\alpha = 1, 2, 3, 4$.}
any class of off–diagonal solutions with decoupling of (modified) Einstein equations in order to satisfy (71).

Another necessary condition for the vanishing of a $\hat{\square} u^\mu$ is that this value is stated zero at $t^\alpha u = 0$. This allows us to express in N–adapted form the timelike derivatives through N–elongated space type ones,

$$e_i \left( \sqrt{|g_{\alpha\beta}|} g^\alpha_{\beta} \right) = -e_i \left( \sqrt{|g_{\alpha\beta}|} g^\beta_{\alpha} \right)$$

(72)

with N–elongated operators. So, the initial data from Theorem 5.1 can not be fixed in arbitrary form if we want to establish a hyperbolic (evolutionary) character for nonholonomic vacuum Einstein equations, i.e., to satisfy both systems (69) and (68). Really, the last system of first order PDEs allows us to compute the time–like derivatives $\partial g^\alpha_{\beta}(0, \hat{u}^\alpha) / \partial t^\alpha \big|_{\{ t^\alpha = 0 \}}$ if $g_{\alpha\beta} \big|_{\{ t^\alpha = 0 \}}$ and $\partial g_{\alpha\beta} / \partial t^\alpha \big|_{\{ t^\alpha = 0 \}}$ have been defined. We conclude that the essential data for formulating a N–adapted evolution problem should be formulated for a 3–d space d–metric

$$[3] g := g_{\alpha\beta}(x, y) e^\alpha \otimes e^\beta,$$

(73)

where $e^\alpha$ are N–elongated differentials of type (4), and its N–elongated (3) time–like derivatives.

Using the Theorem 5.1 and above presented considerations, we get the proof of the following theorem

**Theorem 5.2** If the initial data (70) satisfy the conditions (71) and (72), and the so–called effective Einstein N–adapted constraint equations,

$$\left( \bar{E}_{\alpha\beta} + \Lambda g_{\alpha\beta} \right) n^\alpha = 0,$$

(74)

are computed for zero distortion in (10), then the d–metric stated by Theorem 5.1 defines solutions of the nonholonomic vacuum equations (67) and/or (24)–(27).

This theorem gives us the possibility to state the Cauchy data for decoupled modified gravity systems and their generic off–diagonal solutions in order to generate N–adapted evolutions.

### 5.2. On initial data sets and global nonholonomic evolution

We adopt this convention for spacetime nonholonomic manifolds $V = \mathbb{R} \times 3V$, were $3V$ is a N–adapted 3-d manifold, when a Whitney sum $T 3V = h 3V \oplus v 3V$ is stated by a spacelike nonholonomic distribution $\mathcal{N}$ of type (2) and there is an embedding $e : 3V \rightarrow V$.

Using the d–metric $[3] g$ (73), we can define the second fundamental form $K$ of a spacelike hypersurface $3V$ in $V$, $\hat{K}(X, Y) := g(\hat{D}_X n, Y), \forall X \in T 3V$. The unity d–vector $n = n^\alpha e_\alpha = n_\alpha e^\alpha = n_\alpha e^\alpha = \left( |g^{t\alpha}| \right)^{-1} e^\alpha$, with normalization $g(n, n) = g^{\alpha\beta} n_\alpha n_\beta = g^{tt}(n_t)^2 = -1$, is time–future and normal to $3V$. The value $\hat{K} = \{ \hat{K}_{\alpha\beta} \}$ is the extrinsic canonical curvature d–tensor of $3V$. Imposing the zero torsion conditions (27), $\hat{K} \rightarrow \{ K_{\alpha\beta} \}$, where $K_{\alpha\beta} = -\frac{1}{2} g^{ta} \left( e_\alpha g_{ta} + e_\alpha g_{ta} - e_a g_{tt} \right) n_t$ are components computed in standard form using the Levi–Civita connection, but with respect to N–adapted frames. We can invert the last formula and write $\partial g_{\alpha\beta} = 2(g^{tt} n_t)^{-1} K_{\alpha\beta} + \{ \text{terms determined by } g_{\alpha\beta} \text{ and their space–derivatives} \}$. Such formulas show that $K_{\alpha\beta}$ and $\partial g_{\alpha\beta}$ are geometric counterparts on hypersurfaces $\{ t^\alpha = 0 \}$.

For the canonical d–connection $\hat{\nabla}_\alpha = \left( \hat{D}_t, \hat{D}_Y \right)$ and d–metric $[3] g$ induced on a spacelike hypersurface in a Lorentzian nonholonomic manifold $V$, we can derive a N–adapted variant of Gauss–Codazzi equations

$$[3] \hat{R}_{jk\ell}^\alpha = \hat{R}_{jk\ell}^\alpha + \hat{K}_{j\ell}^\alpha \hat{K}_{jk} - \hat{K}_{j\ell}^\alpha \hat{K}_{jk} \, \hat{D}_t \hat{K}_{jk} - \hat{D}_j \hat{K}_{jk} = \hat{R}_{jk\ell}^\alpha n^\alpha.$$
In these formulas, \[^{[3]}R\] is the canonical curvature d–tensor of \(^{[3]}g\). \(^{[3]}R\) is computed following formulas (15) as the spacetime canonical d–curvature tensor, \(n\) is the timelike normal to hypersurface \(V\) when the local \(N\)–adapted coordinate system is such way chosen that d–vectors \(e_7\) are tangent to \(V\). Contracting indices, introducing divergence operator \(\text{div}\) determined by \(N\)–elongated partial derivatives (3), trace operator \(tr\) and absolute differential \(d\), we derive from above equations and (24) the following system of equations

\[
\text{div} \hat{K} - d(tr \hat{K}) = 8\pi \hat{J}, \quad \text{momentum constraint; (75)}
\]

\[
\hat{R} - 2\lambda - \hat{g}(\hat{K}^2 + (tr \hat{K})^2 = 16\pi \hat{\rho}, \quad \text{Hamiltonian constraint; (75)}
\]

\[C(\hat{F}, \hat{g}) = 0, \quad \text{Einstein constraint eqs; (75)}\]

where \(\hat{R}\) is computed as the scalar (17) but for \(g\). In a general context, we consider that \(V\) is embedded in a nonholonomic spacetime with induced data \(\left( \begin{array}{c} V \\ \hat{g}, \hat{K}, \hat{F} \end{array} \right)\), we have \(\hat{J} := -(\hat{n}, \cdot)\) and \(\hat{\rho} := (\hat{n}, n)\). The term \(C(\hat{F}, \hat{g})\) denotes the set of additional constraints resulting from the non–gravitational part, including nonholonomic distributions \(N\) (2). If, in such a set, there are included the zero torsion conditions (27), we can omit “hats” on geometric/physical objects if they are written in “not” \(N\)–adapted frames of reference. The equations (75) form an undetermined system of PDEs. For 3-d, there are locally four equations for twelve unknown values given by the components of d–tensors \(g\) and \(K\). Using the conformal method with the Levi–Civita connection, see details and references in Ref. [6], we can study the existence and uniqueness of solutions to such systems.

The above considerations motivate the following definition

**Definition 5.2** A canonical vacuum initial \(N\)–adapted data set is defined by a triple \(\left( \begin{array}{c} V \\ \hat{g}, \hat{K} \end{array} \right)\), where \((\hat{g}, \hat{K})\) are defined as a solution of (75).

If the conditions (27) are imposed, the data \((\hat{g}, \hat{K})\) are equivalent to similar ones \((g, K)\) with \(K\) computed for the Levi–Civita connection. Covering \(V\) by coordinate neighborhoods \(U_u\) of points \(u \in V\), we can use Theorem 5.1 to construct globally hyperbolic \(N\)–adapted developments \((U_u, g_u)\) of an initial data set \(\left( \begin{array}{c} U_u \\ g_u \end{array} \right)\) as in the above definition. The d–metrics generated in such forms will coincide after performing suitable \(N\)–adapted frame/coordinated transforms on charts covering such a spacetime wherever such solutions are defined. We can patch all data \((U_u, g_u)\) together to a globally hyperbolic Lorentzian nonholonomic spacetime containing a Cauchy surface \(V\). We prove that

**Theorem 5.3** Any \(N\)–adapted initial data \(\left( \begin{array}{c} V \\ \hat{g}, \hat{K} \end{array} \right)\) of differentiability class \(H^{s+1} \times H^s, s > 3/2\), admits a globally hyperbolic, \(N\)–adapted and unique development (in the sense of Theorem 5.1 and of the above considered assumptions and proofs).

### 6. Anholonomic Modifications of Black Holes

6.1. (Non) holonomic non–Abelian effective vacuum spaces

This class of effective vacuum solutions are generated not just as a simple limit \(\Lambda \to 0\), for instance, for a class of solutions (47) with coefficients (48)–(50). We have to construct off–diagonal solutions of the effective Einstein equations for the canonical d–connection taking
the vacuum equations $\hat{R}_{\alpha\beta} = 0$ and an ansatz $g$ with coefficients satisfying the conditions

$$\epsilon_1 \psi''(r, \theta) + \epsilon_2 \psi''(r, \theta) = 0;$$

$$h_3 = \pm e^{-2 \phi^2} \left( \frac{h_4}{h_4} \right)^2 \text{ for a given } h_4(r, \theta, \varphi), \phi(r, \theta, \varphi) = 0;$$

$$w_i = w_i(r, \theta, \varphi), \text{ for any such functions if } \lambda = 0;$$

$$n_i = \begin{cases} 1n_i(r, \theta) + 2n_i(r, \theta) \int (h_4^2)^{5/2} dv, & \text{if } n_i^* \neq 0; \\ 1n_i(r, \theta), & \text{if } n_i^* = 0. \end{cases}$$

Vacuum solutions of the effective Einstein equations for the Levi–Civita connection, i.e., of $R_{\alpha\beta} = 0$, are generated if we impose additional constraints on coefficients of d–metric, for $e^{-2 \phi^2} = 1$, as solutions of (51),

$$h_3 = \pm 4 \left( \frac{h_4}{h_4} \right)^2, \quad h_4^* \neq 0;$$

$$w_1w_2 \left( \ln \left| \frac{w_1}{w_2} \right| \right)^* = w_2^* - w_1^*; \quad w_2^* - w_1^* = 0; \quad w_2^* = 0; \quad w_1^* = 0;$$

$$1n_1(r, \theta)^* - 1n_2^*(r, \theta) = 0, \quad n_i^* = 0.$$ (78)

The constructed class of vacuum solutions with coefficients subjected to conditions (76)–(78) is of type (60) for (61)–(63). Such metrics consist a particular case of vacuum ansatz defined by Corollary 4.3 with $h_4 = 1$ and $\omega = 1$.

Here we note that former analytic and numeric programs (for instance, standard ones with Maple/ Mathematica) for constructing solutions in gravity theories can not be directly applied for alternative verifications of our solutions. Those approaches do not encode the nonholonomic constraints which we use for constructing integral varieties. Nevertheless, it is possible to check, in general, the analytic form, see all details summarized in Refs. [20, 21, 22] (and formulas from Appendix Appendix A), that the vacuum effective Einstein equations (41)–(32) with zero effective sources, $\nabla \Upsilon = 0$ and $\Upsilon = 0$, can be solved by using the above presented off–diagonal ansatz for metrics.

6.2. $f$–deformations of the Schwarzschild metric

We can consider a “prime” metric which, in general, is not a solution of (modified) Einstein equations,

$$\check{\epsilon} g = -d\xi \otimes d\xi - r^2(\xi) \; d\theta \otimes d\theta - r^4(\xi) \sin^2 \theta \; d\varphi \otimes d\varphi + \omega^2(\xi) \; dt \otimes dt.$$ (79)

We shall deform nonholonomically this metric into a “target” off–diagonal one which will be a solution of the vacuum Einstein equations. The nontrivial metric coefficients in (79) are stated in the form

$$\check{g}_1 = -1,$$

$$\check{g}_2 = -r^2(\xi), \quad \check{h}_3 = -r^2(\xi) \sin^2 \theta, \quad \check{h}_4 = \omega^2(\xi),$$

$$\check{g}_1 = -1, \quad \check{g}_2 = -r^2(\xi), \quad \check{h}_3 = -r^2(\xi) \sin^2 \theta, \quad \check{h}_4 = \omega^2(\xi),$$ (80)

for local coordinates $x^1 = \xi, \; x^2 = \theta, \; x^3 = \varphi, \; x^4 = t$, where $\xi = \int dr \left| 1 - \frac{2m}{r} - \frac{\varphi^2}{r^2} \right|^{1/2}$ and $\omega^2(r) = 1 - \frac{2m}{r} + \frac{\varphi^2}{r^2}$. If we put $\epsilon = 0$ with $\mu_0$ considered as a point mass, the metric $\check{g}$ (79) determines the Schwarzschild solution. For simplicity, we analyze only the case of “pure” gravitational vacuum solutions, not considering a more general construction when $\epsilon = \epsilon^2$ can be related to the electric charge for the Reissner–Nordström metric. In our approach, $\epsilon$ is a small parameter (eccentricity) defining a small deformation of a circle into an ellipse.
We generate exact solutions of the system (61)–(63) with effective $\Lambda = 0$ via nonholonomic deformations $\varepsilon g - \eta g$, when $g_i = \eta_i \tilde{g}_i$ and $h_a = \eta_a \tilde{h}_a$ and $w_i, n_i$ define a target metric

$$\varepsilon g = \eta_1(\xi) d\xi \otimes d\xi + \eta_2(\xi) r^2(\xi) d\vartheta \otimes d\varphi + \eta_3(\xi, \vartheta, \varphi) r^2(\xi) \sin^2 \vartheta \delta \varphi \otimes \delta \varphi - \eta_4(\xi, \vartheta, \varphi) \omega^2(\xi) \delta t \otimes \delta t, \quad \delta \varphi = d\varphi + w_1(\xi, \vartheta, \varphi) d\xi + \omega_2(\xi, \vartheta, \varphi) d\vartheta, \quad \delta t = dt + n_1(\xi, \vartheta) d\xi + n_2(\xi, \varphi) d\varphi. \quad (81)$$

The gravitational field equations for zero source relate the coefficients of the vertical metric and polarization functions,

$$h_3 = h_0^2(b^*)^2 = \eta_3(\xi, \vartheta, \varphi) r^2(\xi) \sin^2 \vartheta, \quad h_4 = -b^2 = -\eta_4(\xi, \vartheta, \varphi) \omega^2(\xi), \quad (82)$$

for $|\eta_3| = (h_0)^2 \tilde{h}_4/\tilde{h}_3 [(\sqrt{|\eta_4|})^*]^2$. In these formulas, we have to choose $h_0 = \text{const}$ ($h_0 = 2$ in order to satisfy the first condition (78)), where $\tilde{h}_a$ are stated by the Schwarzschild solution for the chosen system of coordinates and $\eta_4$ can be any function satisfying the condition $\eta_4^* \neq 0$. We generate a class of solutions for any function $b(\xi, \vartheta, \varphi)$ with $b^* \neq 0$. For different purposes, it is more convenient to work directly with $\eta_4$, for $\eta_4^* \neq 0$, and/or $h_4$, for $h_4^* \neq 0$. The gravitational polarizations $\eta_1$ and $\eta_2$, when $\eta_1 = \eta_2 r^2 = e^{\psi(\xi, \vartheta)}$, are found from (41) with zero source, written in the form $\psi^{**} + \psi'' = 0$.

Introducing the defined values of the coefficients in the ansatz (81), we find a class of exact off–diagonal vacuum solutions of the Einstein equations defining stationary nonholonomic deformations of the Schwarzschild metric,

$$\varepsilon g = -\psi^*(d\xi \otimes d\xi + d\vartheta \otimes d\varphi) - 4 \left(\sqrt{|\eta_4|}\right)^* \omega^2 \delta \varphi \otimes \delta \varphi + \eta_4 \omega^2 \delta t \otimes \delta t, \quad \delta \varphi = d\varphi + w_1(\xi, \vartheta, \varphi) d\xi + \omega_2(\xi, \vartheta, \varphi) d\vartheta, \quad \delta t = dt + n_1(\xi, \vartheta) d\xi + n_2(\xi, \varphi) d\varphi. \quad (83)$$

The N–connection coefficients $w_1(\xi, \vartheta, \varphi)$ and $n_1(\xi, \vartheta)$ must satisfy the conditions (78) in order to get vacuum metrics in GR.

It should be emphasized here that, in general, the bulk of solutions from the set of target metrics do not define black holes and do not describe obvious physical situations. They preserve the singular character of the coefficient $\omega^2$ vanishing on the horizon of a Schwarzschild black hole if we take only smooth integration functions for some small deformation parameters $\varepsilon$.

### 6.3. Linear parametric polarizations induced by f–modifications

We may select some locally anisotropic configurations with possible physical interpretation of gravitational vacuum configurations with spherical and/or rotoid (ellipsoid) symmetry if it is considered a generating function

$$b^2 = q(\xi, \vartheta, \varphi) + \varepsilon \varphi(\xi, \vartheta, \varphi). \quad (84)$$

For simplicity, we restrict our analysis only to linear decompositions on a small parameter $\varepsilon$, with $0 < \varepsilon << 1$.

Using (84), we compute $(b^*)^2 = [(\sqrt{|q|})^*]^2 \left[1 + \varepsilon \frac{1}{(\sqrt{|q|})^*} \frac{q}{\sqrt{|q|}} \right]$ and the vertical coefficients of d–metric (83), i.e., $h_3$ and $h_4$ (and corresponding polarizations $\eta_3$ and $\eta_4$), see formulas (82)\textsuperscript{11}. We model rotoid configurations if we choose

$$q = 1 - \frac{2\mu(\xi, \vartheta, \varphi)}{r} \quad \text{and} \quad q = \frac{q_0(r)}{4\mu^2} \sin(\omega_0 \varphi + \varphi_0). \quad (85)$$

\textsuperscript{11} Nonholonomic deformations of the Schwarzschild solution (not depending on $\varepsilon$) can be generated if we consider $\varepsilon = 0$ and $b^2 = q$ and $(b^*)^2 = [(\sqrt{|q|})^*]^2$. 
for $\mu(\xi, \vartheta, \varphi) = \mu_0 + \varepsilon \mu_1(\xi, \vartheta, \varphi)$ (supposing that the mass is locally anisotropically polarized) with certain constants $\mu, \omega_0$ and $\varphi_0$ and arbitrary functions/ polarizations $\mu_1(\xi, \vartheta, \varphi)$ and $q_0(r)$ to be determined from some boundary conditions, with $\varepsilon$ being the eccentricity. This condition defines a small deformation of the Schwarzschild spherical horizon into an ellipsoidal one (rotoid configuration with eccentricity $\varepsilon$).

The resulting off–diagonal solution with rotoid type symmetry is

$$r^{\text{rot}} g = -e^{\nu} (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta) + (q + \varepsilon q) \delta t \otimes \delta t - 4[\sqrt{|q|}]^2 [1 + \varepsilon \frac{1}{(\sqrt{|q|})^2} (q/\sqrt{|q|})] \delta \varphi \otimes \delta \varphi, \quad \delta \varphi = d\varphi + w_1 d\xi + w_2 d\vartheta, \quad \delta t = dt + 1n_1 d\xi + 1n_2 d\vartheta. \quad (86)$$

The functions $q(\xi, \vartheta, \varphi)$ and $\varrho(\xi, \vartheta, \varphi)$ are given by formulas (85) and the N–connection coefficients $w(\xi, \vartheta, \varphi)$ and $n_1 = 1n_1(\xi, \vartheta)$ are subjected to conditions of type (78),

$$w_1 w_2 \left( \ln \frac{w_1}{w_2} \right)^* = w_2^* - w_1^*, \quad w_i^* \neq 0; \quad (87)$$

or $w_2 - w_1 = 0$, $w_1^* = 0$, $1n_1(\xi, \vartheta) - 1n_2(\xi, \vartheta) = 0$

and $\psi(\xi, \vartheta)$ being any function for which $\psi^{**} + \psi''' = 0$.

7. Ellipsoidal $f$–Configurations and Solitons

We can consider nonholonomic deformations in modified gravity for arbitrary signs of the cosmological constant $\Lambda \neq 0$ containing contributions from nonholonomic $f$–modifications. Such classes of solutions can be constructed in general form for a system (41)–(44) and (51) with coefficients of metric of type (47). Such metrics consist a particular case of non–vacuum ansatz defined by Corollary 4.2 with $\lambda_4 = 1$ and $\omega = 1$.

7.1. Nonholonomic rotoid deformations

Let us consider a diagonal metric of type

$$\gamma g = d\xi \otimes d\xi + r^2(\xi) d\vartheta \otimes d\vartheta + r^2(\xi) \sin^2 \vartheta d\varphi \otimes d\varphi + \lambda \varrho^2(\xi) dt \otimes dt, \quad (88)$$

where nontrivial metric coefficients are parametrized in the form $\gamma_1 = 1$, $\gamma_2 = r^2(\xi)$, $\gamma_3 = r^2(\xi) \sin^2 \vartheta$, $\gamma_4 = \lambda \varrho^2(\xi)$, for local coordinates $x^2 = \xi, x^2 = \vartheta, y^2 = \varphi, y^2 = t$, with $\xi = \int dr/|q(r)|^2$, and $\lambda \varrho^2(r) = -\sigma^2(t) q(r), \quad q(r) = 1 - 2m(r)/r - \Lambda r^2/3$.

The ansatz for such classes of solutions is chosen to be of the form

$$\gamma \tilde{g} = e^{\frac{\varepsilon}{2}(\xi, \theta)} \left( d\xi \otimes d\xi + d\vartheta \otimes d\vartheta + h_3(\xi, \theta, \varphi) \delta \varphi \otimes \delta \varphi + h_4(\xi, \theta, \varphi) \delta t \otimes \delta t, \quad \delta \varphi = d\varphi + w_1(\xi, \theta, \varphi) d\xi + w_2(\xi, \theta, \varphi) d\vartheta, \quad \delta t = dt + n_1(\xi, \theta, \varphi) d\xi + n_2(\xi, \theta, \varphi) d\vartheta, \quad \delta \varphi = \delta \varphi \right),$$

for $h_3 = -h_3(\xi, \theta, \varphi) r^2(\xi) \sin^2 \vartheta$, $h_4 = b^2 = \eta_4(\xi, \theta, \varphi) \lambda \varrho^2(\xi)$. The coefficients of this metric determine exact solutions if

$$\frac{\phi^{**}(\xi, \theta)}{\phi''(\xi, \theta)} = 2\Lambda; \quad h_3 = \pm \frac{(\phi^*)^2}{4\lambda} e^{2\varphi(\xi, \theta)}; \quad h_4 = \mp \frac{1}{4\lambda e^{2\varphi(\xi, \theta)}}; \quad w_i = -\partial_i \phi / \phi^*; \quad (89)$$

$$n_1 = 1n_1(\xi, \theta) + 2n_1(\xi, \theta) \int (\phi^*)^2 \lambda \varepsilon^{2\varphi(\xi, \theta)} d\varphi,$$

$$= \begin{cases} 
1n_1(\xi, \theta) + 2n_1(\xi, \theta) \int e^{-4\varphi(\xi, \theta)} d\varphi, & \text{if } n_1^* \neq 0; \\
1n_1(\xi, \theta), & \text{if } n_1^* = 0; 
\end{cases}$$

We may treat $\varepsilon$ as an eccentricity imposing the condition that the coefficient $h_4 = b^2 = \eta_4(\xi, \theta, \varphi) \varrho^2(\xi)$ becomes zero for data (85), if $r_+ \approx 2\mu_0 / \left(1 + \varepsilon \frac{\varrho_0(\xi)}{\mu_0} \sin(\omega_0 \varphi + \varphi_0)\right).$
for any nonzero \( h_4 \) and \( h_4^* \) and (integrating) functions \( 1n_i(\xi, \theta), \) \( 2n_i(\xi, \theta) \), generating function \( \phi(\xi, \theta, \varphi) \), and \( 0\phi(\xi, \theta) \) to be determined from certain boundary conditions for a fixed system of coordinates.

For nonholonomic effective ellipsoid de Sitter configurations, we parameterize

\[
\begin{align*}
\chi_{\text{rot}}^\text{rot} & = -e^{\phi(\xi, \theta)} (d\xi \otimes d\xi + d\theta \otimes d\theta) + (q + \varepsilon \rho) \delta t \otimes \delta t \\
& \quad - h_0^2 \left[ (\sqrt{|q|})^2 \left[ 1 + \varepsilon \frac{1}{(\sqrt{|q|})^2} \right] \right] \delta \varphi \otimes \delta \varphi,
\end{align*}
\]

\[
\delta \varphi = d\varphi + w_1 d\xi + w_2 d\theta, \quad \delta t = dt + n_4 d\xi + n_2 d\theta,
\]

(90)

where \( q = 1 - 2 \frac{\mu(r, \theta, \varphi)}{r} \), \( q = \frac{q(r)}{q_0(r)} \) \( \sin(\omega_0 \varphi + \varphi_0) \), are chosen to generate an anisotropic rotoid configuration for the smaller “horizon” (when \( h_4 = 0 \)), \( r_+ \approx 2 \frac{1}{\mu} \left( 1 + \varepsilon \frac{q(r)}{q_0(r)} \sin(\omega_0 \varphi + \varphi_0) \right) \), for a corresponding \( q_0(r) \).

We have to impose the condition that the coefficients of the above \( d \)-metric induce a zero torsion in order to generate solutions of the Einstein equations for the Levi–Civita connection. Using formula (87), for \( \phi^* \neq 0 \), we obtain that \( \phi(r, \varphi, \theta) = \ln[h_2^2/\sqrt{|h_3 h_4|}] \) must be any function defined in non–explicit form from equation \( 2e^{2\phi} \phi = \Lambda \). The set of constraints for the \( N \)-connection coefficients is solved, if the integration functions in (89) are chosen in a form

when \( w_1 w_2 \left( \ln \frac{w_1}{w_2} \right)^* = w^*_2 - w^*_1 \) for \( w^*_1 \neq 0; w^*_2 - w^*_1 = 0 \) for \( w^*_1 = 0; \) and take \( n_i = 1n_i(x^k) \) for \( 1n_i(x^k) - 1n_2(x^k) = 0 \).

In a particular case, in the limit \( \varepsilon \rightarrow 0 \), we get a subclass of solutions of type (90) with spherical symmetry but with generic off–diagonal coefficients induced by the \( N \)-connection coefficients. This class of spacetimes depend on cosmological constants polarized nonholonomically by \( f \)-modifications. We can extract from such configurations the Schwarzchild solution, if we select a set of functions with the properties \( \phi \rightarrow \text{const}, w_i \rightarrow 0, n_i \rightarrow 0 \) and \( h_4 \rightarrow \infty \).

### 7.2. Effective vacuum solitonic configurations

It is possible to construct off–diagonal vacuum spacetimes generated by 3–d solitons as examples of generic off–diagonal solutions with nontrivial vertical conformal factor \( \omega \). We consider that there are satisfied the conditions of Corollary 4.3 with \( h_4 = 1 \) for effective vacuum solutions (such configurations may encode \( f \)-modifications) and the Cauchy problem is stated as in Section 3.

#### 7.2.1. Solutions with solitonic factor \( \omega(x^1, y^3, t) \)

We take \( \omega = \eta(x^1, y^3, t) \), when \( y^4 = t \) is a timelike coordinate, as a solution of KdP equation [13]

\[
\pm \eta^{**} + (\partial_t \eta + \eta \eta^* + e \eta^{****})^* = 0,
\]

(91)

with dispersion \( \varepsilon \) and possible dependencies on a set of parameters \( \theta \). It is supposed that, in the dispersionless limit \( \varepsilon \rightarrow 0 \), the solutions are independent on \( y^3 \) and determined by Burgers’ equation \( \partial_t \eta + \eta \eta^* = 0 \). For such 3–d solitonic configurations, the conditions (51) are written in the form

\[
\mathbf{e}_1 \eta = \eta^* + w_1(x^1, y^3) \eta^* + n_1(x^1) \partial_t \eta = 0.
\]

For \( \eta' = 0 \), we can impose the condition \( w_2 = 0 \) and \( n_2 = 0 \).
Such vacuum solitonic metrics can be parametrized in the form
\[
\begin{align*}
g &= e^{\psi(x^k)}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) + \left[ \eta(x^1, y^3, t) \right]^2 h_a(x^1, y^3) \ e^a \otimes e^a, \\
e^3 &= dy^3 + w_1(x^k, y^3)dx^1, \quad e^4 = dy^4 + n_1(x^k)dx^1.
\end{align*}
\]

This class of metrics does not have (in general) Killing symmetries but may possess symmetries determined by solitonic solutions of (91). Alternatively, we can consider that \( \eta \) is a solution of any three dimensional solitonic and/or other nonlinear wave equations; in a similar manner, we can generate solutions for \( \omega = \eta(x^2, y^3, t) \).

7.2.2. Solitonic metrics with factor \( \omega(x^i, t) \) There are effective vacuum metrics when the solitonic dynamics does not depend on anisotropic coordinate \( y^3 \). In this case, \( \omega = \hat{\eta}(x^k, t) \) is a solution of KdP equation
\[
\pm \hat{\eta}^{(**)} + (\partial_t \hat{\eta} + \hat{\eta} \ \hat{\eta}' + e\hat{\eta}'''')' = 0. \quad (92)
\]

In the dispersionless limit \( \epsilon \to 0 \), the solutions are independent on \( x^1 \) and determined by Burgers’ equation \( \partial_t \hat{\eta} + \hat{\eta} \hat{\eta}' = 0 \). Alternatively, we can consider that \( \omega \) is determined by solitonic solutions of (91). This class of vacuum solitonic modified gravity configurations is given by
\[
\begin{align*}
2g &= e^{\psi(x^k)}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) + \left[ \hat{\eta}(x^k, t) \right]^2 h_a(x^k, y^3) \ e^a \otimes e^a, \\
e^3 &= dy^3 + w_1(x^k, y^3)dx^1, \quad e^4 = dy^4 + n_1(x^k)dx^1; \quad \text{the conditions (51) are} \\
& e_1 = n_1(x^1)\partial_1 \hat{\eta} = 0, \quad e_2 = \hat{\eta} + n_2(x^1)\partial_1 \hat{\eta} = 0.
\end{align*}
\]

It is possible to derive an infinite number of vacuum gravitational 2-d and 3-d configurations characterized by corresponding solitonic hierarchies and bi–Hamilton structures, for instance, related to different KdP equations (92) with possible mixtures with solutions for 2-d and 3-d sine–Gordon equations etc; see details in Ref. [23].

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Appendix A. Proof of Theorem 3.1

For \( \omega = 1 \) and \( \hat{b}_a = \text{const} \), such proofs can be obtained by straightforward computations [20]. The approach was extended for \( \omega \neq 1 \) and higher dimensions in [21, 22]. In this section, we sketch a proof for ansatz (8) with nontrivial \( \hat{b}_a \) depending on variable \( y^4 \) when \( \omega = 1 \) in data (9). At the next step, the formulas will be completed for nontrivial values \( \omega \neq 1 \).

If \( \hat{R}_1 = \hat{R}_2 = \hat{R}_3 = \hat{R}_4 \), the effective Einstein equations (24) for \( \hat{D} \) and data (A.1) (see below) can be written for any source (25) in the form
\[
\hat{E}^1 = \hat{E}^2 = -\hat{R}^3 = \hat{\Upsilon}(x^k, y^3) + \hat{\Upsilon}(x^k, y^3, y^4), \quad \hat{E}^3 = \hat{E}^4 = -\hat{R}^1 - \hat{\Upsilon}(x^k).
\]

The geometric data for the conditions of Theorem 3.1 are \( g_i = g_i(x^k) \) and
\[
g_3 = h_3(x^k, y^3), g_4 = h_4(x^k, y^3)\hat{b}_4(x^k, y^4), N^3 = w_1(x^k, y^3), N^4 = n_1(x^k, y^3), \quad \text{(A.1)}
\]
for \( \hat{b}_4 = 1 \) and local coordinates \( \omega^4 = (x^1, y^3) = (x^1, x^2, y^3, y^4) \). For such values, we shall compute respectively the coefficients of \( \Omega^a_{\alpha \beta} \) canonical d–connection \( \hat{\Gamma}^a_{\alpha \beta} \) (12), d–torsion \( \hat{T}^\gamma_{\alpha \beta} \) (14), necessary coefficients of d–curvature \( \hat{R}^r_{\alpha \beta \gamma} \) (15) with respective contractions for \( \hat{R}^r_{\alpha \beta} \) and resulting \( \hat{R}(17) \) and \( \hat{E}_{\alpha \beta}(20) \). Finally, we shall state the conditions (27) when general coefficients (A.1) are considered for d–metrics.
A.1. Coefficients of the canonical d–connection

There are horizontal nontrivial coefficients of $\tilde{\Gamma}_{ij}^k (12)$,

$$\tilde{L}_{jk} = \frac{1}{2} g^{ij} (e_k g_{j1} + e_j g_{k1} - e_1 g_{jk}) + \frac{1}{2} g^{ij} (e_k g_{j2} + e_j g_{k2} - e_2 g_{jk}),$$

i.e., $\tilde{L}_{jk}^1 = \frac{1}{2g_{j1}} (\partial_k g_{j1} + \partial_j g_{k1} - g_{jk}^*)$, $\tilde{L}_{jk}^2 = \frac{1}{2g_{j2}} (\partial_k g_{j2} + \partial_j g_{k2} - g_{jk}^*).$

The h–v–components $\tilde{L}_{bk}^a$ are computed following formulas

$$\tilde{L}_{a0}^3 = e_b (N_3^a) + \frac{1}{2g_{a}^3} [e_k g_{b3} - e_b N_3^k - g_{ab}(N_3^k)^\ast - g_{ab}(N_4^k)^\circ] = e_b (N_3^a)$$

$$+ \frac{1}{2g_{a}^3} [\partial_k g_{b3} - N_3^k g_{b3}^\ast - N_4^k g_{b4}^\ast - g_{ab}(N_3^k)^\ast - g_{ab}(N_4^k)^\circ]$$

$$= e_b (w_k + w_a) + \frac{1}{2g_{a}^3} [\partial_k g_{b3} - (w_k + w_a) g_{b3}^\ast - (n_k + n_a) g_{b3}^\ast - g_{ab}(w_k + w_a)^* - g_{ab}(N_3^k)^\circ].$$

$$\tilde{L}_{a4}^2 = e_b (N_4^a) + \frac{1}{2g_{a}^4} [e_k g_{b4} - g_{ab}(N_3^k)^\ast - g_{ab}(N_4^k)^\circ] = e_b (N_4^a)$$

$$+ \frac{1}{2g_{a}^4} [\partial_k g_{b4} - N_4^k g_{b4}^\ast - N_4^k g_{b4}^\ast - g_{ab}(N_3^k)^\ast - g_{ab}(N_4^k)^\circ]$$

$$= e_b (n_k + n_a) + \frac{1}{2g_{a}^4} [\partial_k g_{b4} - (w_k + w_a) g_{b4}^\ast - (n_k + n_a) g_{b4}^\ast - g_{ab}(n_k + n_a)^* - g_{ab}(N_3^k)^\circ].$$

In explicit form, we obtain the nontrivial values

$$\tilde{L}_{3k}^3 = w_k^* + \frac{1}{2g_{a}^3} [\partial_k g_{3a} - w_k g_{3}^* - n_k g_{3}^* - g_{a}^3 w_k^*] = \frac{1}{2g_{a}^3} [\partial_k g_{3a} - w_k g_{3}^*] = \partial_k h_{3}^a - \tilde{h}_{3}^a,$$

$$\tilde{L}_{4k}^3 = \frac{1}{2g_{a}^4} [-g_{a}^4 n_k^*] = - \frac{h_{4}^k}{2h_{3}^4} n_k^*, \tilde{L}_{4k}^4 = \tilde{h}_{4}^k = \frac{1}{2g_{a}^4} [-g_{a}^4 n_k^*] = \frac{1}{2} h_{4}^k,$$

$$\tilde{L}_{4k}^4 = \frac{1}{2g_{a}^4} [\partial_k g_{4a} - w_k g_{4}^* - n_k g_{4}^*] = \partial_k (h_{4}^k h_{4}^a) - \tilde{h}_{4}^k - \frac{h_{4}^k}{2h_{4}^4} - n_k h_{4}^k$$

For the set of h–v C–coefficients, we get $\tilde{C}_{hc}^k = \frac{1}{2} g^{ik} \partial_y h_{yk} = 0$. The v–components of C–coefficients are computed using the following formulas

$$\tilde{C}_{bc}^3 = \frac{1}{2g_{a}^3} (e_c g_{b3} + e_c g_{c3} - g_{bc}^3), \tilde{C}_{bc}^4 = \frac{1}{2g_{a}^4} (e_c g_{b4} + e_b g_{c4} - g_{bc}^4),$$

i.e., $\tilde{C}_{33}^3 = \frac{g_{3}}{2g_{3}^3}$, $\tilde{C}_{34}^3 = \frac{g_{4}}{2g_{3}^4}$, $\tilde{C}_{44}^3 = \frac{g_{4}}{2g_{3}^4} = 0$, $\tilde{C}_{34}^3 = - \frac{g_{4}}{2g_{3}^4} = - \frac{h_{3}^a}{2h_{3}^4}$, $\tilde{C}_{33}^4 = - \frac{g_{4}}{2g_{3}^4} = 0$, $\tilde{C}_{44}^4 = \frac{g_{4}}{2g_{3}^4} = \frac{h_{3}^a}{2h_{3}^4}$.

Putting together the above formulas, we find all nontrivial coefficients,

$$\tilde{L}_{11}^1 = \frac{g_{1}}{2g_{1}^1}, \tilde{L}_{12}^1 = \frac{g_{1}}{2g_{2}^1}, \tilde{L}_{12}^2 = - \frac{g_{2}}{2g_{2}^1}, \tilde{L}_{12}^2 = - \frac{g_{2}}{2g_{2}^2}, \tilde{L}_{12}^2 = - \frac{g_{2}}{2g_{2}^2}, \tilde{L}_{12}^2 = - \frac{g_{2}}{2g_{2}^2},$$

$$\tilde{L}_{4k}^3 = \frac{1}{2h_{3}^4} - \frac{h_{4}^k}{2h_{4}^4} - n_k h_{4}^k, \tilde{L}_{4k}^4 = \frac{1}{2h_{3}^4} - \frac{h_{4}^k}{2h_{4}^4} - n_k h_{4}^k,$$

We shall need also the values

$$\tilde{C}_{3} = \tilde{C}_{33}^3 + \tilde{C}_{34}^3 = \frac{h_{3}^3}{2h_{3}^4} + \frac{h_{3}^4}{2h_{4}^4}, \tilde{C}_{4} = \tilde{C}_{44}^4 = \frac{h_{4}^3}{2h_{4}^4}.$$
A.2. Coefficients for torsion of $\hat{D}$

Using data (A.1) for $w_i = n_i = 0$, the coefficients $\Omega_{ij}^a = e_j(N_i^a) - e_i(N_j^a)$, are computed

$$\Omega_{ij}^a = \partial_j(N_i^a) - \partial_i(N_j^a) - N_i^b \partial_b N_j^a + N_j^b \partial_b N_i^a = \partial_j(N_i^a) - \partial_i(N_j^a) - N_i^b (N_j^a)^* - N_j^b (N_i^a)^* + N_j^b (N_i^a)^o + N_j^b (N_i^a)^o$$

and the nontrivial values for coefficients are

$$\begin{align*}
\Omega_{12}^3 &= -\Omega_{12}^4 = \partial_2 w_1 - \partial_1 w_2 - w_1 w_2 + w_2 w_1 = w'_1 - w'_2 - w_1 w_2^* + w_2 w_1^*; \\
\Omega_{12}^4 &= -\Omega_{12}^3 = \partial_2 n_1 - \partial_1 n_2 - w_1 n_2^* + w_2 n_1^*.
\end{align*}$$

(A.4)

The nontrivial coefficients of $d$–torsion (14) are $\hat{T}_{ij}^a = -\Omega_{ji}^a$ (A.4) and $\hat{T}_{aj}^c = \hat{L}_a^c - e_a(N_j^c)$. We find for other types of coefficients that

$$\hat{T}_{jk} = \hat{L}_{jk} - \hat{L}_{kj} = 0, \quad \hat{T}_{ju} = \hat{C}_{ju} = 0, \quad \hat{T}_{bc} = \hat{C}_{bc} = \hat{C}_{cb} = 0.$$

We have such nontrivial $N$–adapted coefficients of $d$–torsion

$$\begin{align*}
\hat{T}_{3k}^i &= \hat{L}_{3k} - e_3(N_k^3) = \frac{\partial_k h_3}{2h_3} w_k - \frac{h_3}{2h_3} - w_k^*, \\
\hat{T}_{4k}^i &= \hat{L}_{4k} - e_4(N_k^4) = \frac{h_4 k}{2h_4} n_k^*, \\
\hat{T}_{4k}^i &= \hat{L}_{4k} - e_4(N_k^4) = \frac{\partial_k (h_4 k)}{2h_4} - \frac{h_4}{2h_4} - n_k^* - \frac{h_4 k}{2h_4}, \\
-\hat{T}_{12}^i &= w_1 - w_2^* - w_1 w_2^* + w_2 w_1^*, \\
-\hat{T}_{12}^i &= n_1 - n_2^* - n_1 n_2^* + n_2 n_1^*.
\end{align*}$$

(A.5)

If all coefficients (A.5) are zero, then $\Gamma_{\alpha\beta}^\gamma = \hat{\Gamma}_{\alpha\beta}^\gamma$.

A.3. Calculation of the Ricci tensor

Let us compute the values $\hat{R}_{ij} = \hat{R}_{ij}^k$ from (19) using (15),

$$\hat{R}_{hjk} = e_h \hat{L}_{hk} - e_j \hat{L}_{hk} + \hat{L}_{hjk}^m \hat{L}_{mk} - \hat{L}_{hjk}^m \hat{L}_{mk} - \hat{C}_{hk} \Omega_{jk}^a = \partial_h \hat{L}_{hk} - \partial_j \hat{L}_{hk} + \hat{L}_{hjk}^m \hat{L}_{mk} - \hat{C}_{hk} \Omega_{jk}^a,$$

where $\hat{C}_{hk} = 0$ and $e_h \hat{L}_{hk} = \partial_h \hat{L}_{hk} + N_k \partial_a \hat{L}_{hk} = \partial_h \hat{L}_{hk} + w_k \left(\hat{L}_{hk}^m\right)^* + n_k \left(\hat{L}_{hk}^m\right)^o = \partial_h \hat{L}_{hk}^m$. Taking derivatives of (A.2), we obtain

$$\begin{align*}
\partial_1 \hat{L}_{11} &= g_1^* \frac{g_1^*}{2g_1} - \frac{\left(g_1^*\right)^2}{2(g_1)^2}, \\
\partial_1 \hat{L}_{12} &= g_1^* \frac{g_1^*}{2g_1} - \frac{\left(g_1^*\right)^2}{2(g_1)^2}, \\
\partial_1 \hat{L}_{22} &= -g_2^* \frac{g_2^*}{2g_2} + \frac{\left(g_2^*\right)^2}{2(g_2)^2}, \\
\partial_1 \hat{L}_{12} &= -g_2^* \frac{g_2^*}{2g_2} + \frac{\left(g_2^*\right)^2}{2(g_2)^2}, \\
\partial_2 \hat{L}_{11} &= g_1^* \frac{g_1^*}{2g_1} - \frac{\left(g_1^*\right)^2}{2(g_1)^2}, \\
\partial_2 \hat{L}_{12} &= g_1^* \frac{g_1^*}{2g_1} - \frac{\left(g_1^*\right)^2}{2(g_1)^2}, \\
\partial_2 \hat{L}_{22} &= -g_2^* \frac{g_2^*}{2g_2} + \frac{\left(g_2^*\right)^2}{2(g_2)^2}, \\
\partial_2 \hat{L}_{12} &= -g_2^* \frac{g_2^*}{2g_2} + \frac{\left(g_2^*\right)^2}{2(g_2)^2}.
\end{align*}$$
For these values, there are only two nontrivial components,
\[
\hat{R}^1_{212} = \frac{g^*}{2g_1} - \frac{g^* g^*}{4 (g_1)^2} - \frac{(g^*)^2}{4 g_1 g_2} + \frac{g''}{2g_1} - \frac{g'' g'}{2 g_1 g_2} - \frac{(g')^2}{4 (g_1)^2},
\]
\[
\hat{R}^2_{112} = -\frac{g^*}{2g_2} + \frac{g^* g^*}{4 g_1 g_2} + \frac{(g^*)^2}{4 (g_2)^2} - \frac{g''}{2g_2} + \frac{g'' g'}{2 g_2 g_1} + \frac{(g')^2}{4 (g_2)^2}.
\]
Considering \(\hat{R}_{11} = -\hat{R}^2_{112}\) and \(\hat{R}_{22} = \hat{R}^1_{212}\), when \(g^i = 1/g_i\), we find
\[
\hat{R}^1 = \hat{R}^2 = -\frac{1}{2g_1 g_2} [g^*]^2 - \frac{(g^*)^2}{2g_1} + \frac{g''}{2g_2} - \frac{(g')^2}{2g_1},
\]
which can be found in equations (42).

The next step is to derive the equations (43). We consider the third formula in (15),
\[
\hat{R}^c_{bbk} = \frac{\partial \hat{L}_c}{\partial g^b} - \left( \frac{\partial \hat{C}_{c}^a}{\partial g^b} + \hat{L}_{d}^c \hat{C}_{da} - \hat{L}_{da}^c \hat{C}_{bd} + \hat{C}_{bd}^d \hat{T}_{ba} \right) + \frac{\partial \hat{L}_c}{\partial g^a} - \hat{C}_{ba}^a \hat{T}_{dk}.
\]
Contracting indices, we get \(\hat{R}_{bk} = \hat{R}^c_{bbk} = \frac{\partial \hat{L}_c}{\partial g^b} - \hat{C}_{ba}^a \hat{T}_{dk} + \hat{C}_{bd}^d \hat{T}_{ka}\). For \(\hat{C}_b = \hat{C}^c_b\), we write
\[
\hat{C}_{bk} = \hat{C}_b \hat{C}_d = \hat{L}_{d}^b \hat{C}_d = \partial_k \hat{C}_b + \hat{k} \hat{C}_b = \hat{T}_{bk} \hat{C}_d = \partial_k \hat{C}_b + \hat{k} \hat{C}_b = \hat{T}_{bb} \hat{C}_d + \hat{k} \hat{C}_b.
\]
We split conventionally \(\hat{R}_{bk} = [1] R_{bk} + [2] R_{bk} + [3] R_{bk}\), where
\[
[1] R_{bk} = \left( \hat{L}_{bk}^3 \right)^* + \left( \hat{L}_{bk}^4 \right)^*, \quad [2] R_{bk} = -\partial_k \hat{C}_b + \hat{k} \hat{C}_b + \hat{T}_{bb} \hat{C}_d,
\]
\[
[3] R_{bk} = \hat{C}_{ba}^a \hat{T}_{dk} = \hat{C}_{ba}^a \hat{T}_{dk} + \hat{C}_{ba}^a \hat{T}_{dk} = \hat{C}_{ba}^a \hat{T}_{dk} + \hat{C}_{ba}^a \hat{T}_{dk}.
\]
Using formulas (A.2), (A.5) and (A.3), we compute
\[
[1] R_{3k} = \left( \hat{L}_{3k}^3 \right)^* + \left( \hat{L}_{3k}^4 \right)^* = \left( \frac{\partial_k h_3}{2 h_3} - w_k \frac{h_3}{2 h_3} \right)^* = -w_k \frac{h_3}{2 h_3} - w_k \left( \frac{h_3}{2 h_3} \right)^* + \frac{1}{2} \left( \frac{\partial_k h_3}{h_3} \right)^*,
\]
\[
[2] R_{3k} = -\partial_k \hat{C}_3 + \hat{k} \hat{C}_3 + \hat{T}_{3k} \hat{C}_3 + \hat{T}_{3k} \hat{C}_4 = -\partial_k \left( \frac{h_3}{2 h_3} + \frac{h_3}{2 h_3} \right) + \frac{w_k}{2 h_3} \left( \frac{h_3}{2 h_3} + \frac{h_3}{2 h_3} \right)^* + \frac{1}{2} \frac{\partial_k h_3}{h_3} - \frac{1}{2} \frac{\partial_k h_3}{h_3},
\]
\[
[3] R_{3k} = \hat{T}_{3k} \hat{T}_{kk} = \frac{\partial_k h_3}{2 h_3} - w_k \frac{h_3}{2 h_3} - w_k \left( \frac{h_3}{2 h_3} \right)^*,
\]
Summarizing, we get
\[
\hat{R}_{3k} = w_k \left[ \frac{h_3}{2 h_3} - \frac{1}{2} \left( \frac{h_3}{2 h_3} \right)^* - \frac{1}{2} \frac{\partial_k h_3}{h_3} + \frac{1}{2} \frac{\partial_k h_3}{h_3} \right] + \frac{1}{2} \frac{\partial_k h_3}{h_3} + \frac{1}{2} \frac{\partial_k h_3}{h_3}.
\]
which is equivalent to (43) if the conditions \( n_k h_k^o = \partial_k h_4 \), see below formula (A.6), are satisfied.

In a similar way, we compute \( \hat{R}_{4k} = [1] R_{4k} + [2] R_{4k} + [3] R_{4k} \), where

\[
\begin{align*}
[1] R_{4k} &= (\hat{L}^3_{4k})^* + (\hat{F}^4_{4k})^o, \\
[2] R_{4k} &= -\partial_k \hat{C}_4 + w_k \hat{C}_4^o + n_k \hat{C}_4^o + \hat{L}^4_{4k} \hat{C}_3 \\
+ \hat{L}^4_{4k} \hat{C}_4, \\
[3] R_{4k} &= \hat{C}_a_{4d} \hat{L}_{kd} = \hat{C}_{a_{43}} \hat{L}_{kd} + \hat{C}_{a_{44}} \hat{L}_{kd} + \hat{C}_{a_{43}} \hat{L}_{kd} + \hat{C}_{a_{44}} \hat{L}_{kd}.
\end{align*}
\]

We get

\[
\begin{align*}
[1] R_{4k} &= -n_k h_k^o h_4^{\frac{h}{2}} + n_k^o \left( -h_k^{(4)} \frac{h_k}{2h_3} + \frac{h_k^2 h_4^o}{2(3h_3)^2} \right) h_4 + \partial_k h_k^o - h_k^o \frac{\partial_k h_4}{2h_4}, \\
[2] R_{4k} &= w_k \left( \frac{h_k^o}{2h_4^2} + \frac{(h_k^o)^2}{2h_4} \right) - n_k^o \frac{h_k h_4}{2h_3} \left( -h_k \frac{h_4^o}{2h_4} \right) + n_k \left( \frac{h_k^o}{2h_4} \right)^o - \frac{\partial_k h_4}{2h_4}, \\
[3] R_{4k} &= w_k \left( \frac{h_k^o}{2h_4^2} + h_k^o \right) + n_k \left( \frac{h_4^o}{2h_4} \right)^2 - \partial_k h_4 h_k^o - \frac{\partial_k h_4}{2h_4} - \frac{\partial_k h_4^o}{2h_4}.
\end{align*}
\]

We summarize the above three terms as follows

\[
\hat{R}_{4k} = -n_k h_k^o h_4^{\frac{h}{2}} + n_k^o \left( -h_k^{(4)} \frac{h_k}{2h_3} + \frac{h_k^2 h_4^o}{2(3h_3)^2} \right) h_4 + \partial_k h_k^o - h_k^o \frac{\partial_k h_4}{2h_4},
\]

and thus we have proved equations (44).

For \( \hat{R}^{jka}_{jk} = \frac{\partial L^j_{ka}}{\partial y^k} - \left( \frac{\partial C^a}{\partial y^k} + \hat{L}^a_{jk} \hat{C}_{ja} - \hat{L}^a_{jk} \hat{C}_{ja} - \hat{L}^a_{jk} \hat{C}_{ja} \right) + \hat{C}^a_{jka} \) from (15), we obtain zero values because \( \hat{C}^a_{jka} = 0 \) and \( \hat{L}^a_{jk} \) do not depend on \( y^k \). So, \( \hat{R}^{jka}_{jk} = 0 \).

Taking \( \hat{R}^{abcd}_{o} = \frac{\partial C^a_{o}}{\partial y^b} - \frac{\partial C^b_{o}}{\partial y^a} = \hat{C}^a_{o} - \hat{C}^b_{o} \) from (15) and contracting the indices in order to obtain the Ricci coefficients, \( \hat{R}_{bc} = \frac{\partial C_{o}}{\partial y^b} - \frac{\partial C_{o}}{\partial y^a} = \hat{C}^c_{o} - \hat{C}^d_{o} \), we compute \( \hat{R}_{bc} = (\hat{C}^a_{o})^* + (\hat{C}^b_{o})^o - \partial_k \hat{C}_b + \hat{C}^b_{c} \hat{C}_3 + \hat{C}^b_{c} \hat{C}^c_4 - \hat{C}^b_{3c} \hat{C}_{4c} - \hat{C}^b_{3c} \hat{C}_{4c} - \hat{C}^b_{3c} \hat{C}_{4c} - \hat{C}^b_{3c} \hat{C}_{4c} \). There are nontrivial values,

\[
\begin{align*}
\hat{R}_{33} &= \left( \hat{C}^a_{33} \right)^* + \left( \hat{C}^a_{33} \right)^o - \hat{C}^c_{3c} \hat{C}^c_3 + \hat{C}^c_{3c} \hat{C}^3_4 - \hat{C}^3_{3c} \hat{C}^3_4 - 2\hat{C}^a_{34} \hat{C}^a_{34} - \hat{C}^a_{34} \hat{C}^a_{34} \left( \frac{h_4}{2h_3} \right)^* \\
- \left( \frac{h_4}{2h_3} \right)^{\frac{h}{2}} \frac{h_4^o}{2h_3} \left( \frac{h_3}{2h_4} \right)^2 - \left( \frac{h_3}{2h_4} \right)^2 - \frac{1}{2} \frac{h_3^o}{h_4} + \frac{1}{4} \frac{h_4^o}{h_3} + \frac{1}{4} \frac{h_3^o}{h_4} + \frac{1}{4} \frac{h_3^o}{h_4} \frac{h_4^o}{h_3}.
\end{align*}
\]

\[
\begin{align*}
\hat{R}_{44} &= \left( \hat{C}^a_{44} \right)^* + \left( \hat{C}^a_{44} \right)^o - \partial_k \hat{C}_4 + \hat{C}^a_{4} \hat{C}^c_3 + \hat{C}^a_{4} \hat{C}^c_4 - \hat{C}^a_{43} \hat{C}^3_4 - \hat{C}^a_{44} \hat{C}^4_4 - \hat{C}^a_{44} \hat{C}^4_4 \left( \frac{h_4}{2h_3} \right)^* \\
- \left( \frac{h_4}{2h_3} \right)^{\frac{h}{2}} \frac{h_4^o}{2h_3} \left( \frac{h_3}{2h_4} \right)^2 - \left( \frac{h_3}{2h_4} \right)^2 = -\frac{1}{2} \frac{h_3^o}{h_4} \frac{h_4^o}{h_3} + \frac{1}{2} \frac{h_3^o}{h_4} \frac{h_4^o}{h_3} + \frac{1}{2} \frac{h_3^o}{h_4} \frac{h_4^o}{h_3} \frac{h_4^o}{h_3}.
\end{align*}
\]

We get the nontrivial v–coefficients of the Ricci d–tensor,

\[
\begin{align*}
\hat{\mathbf{R}}_3 &= \frac{1}{2} \frac{h_3^o}{h_4} \frac{h_4^o}{h_3} \frac{h_4^o}{h_3}, \\
\hat{\mathbf{R}}_4 &= \frac{1}{2} \frac{h_3^o}{h_4} \frac{h_4^o}{h_3} \frac{h_4^o}{h_3}.
\end{align*}
\]

i.e., equation (42).
A.4. Zer torsion conditions

Let us analyze how to solve the equation

$$\hat{T}_{4k}^4 = \hat{L}_{4k}^4 - e_4(N_k^4) = \frac{\partial_k (h_4 h_4)}{2 h_4} - w_k \frac{h_4^3}{2 h_4} - n_k \frac{h_4^2}{2 h_4} = 0,$$

which follows from formulas (A.5) for a vanishing torsion for $\hat{D}$. Taking any $h_4$ for which

$$n_k h_4^2 = \partial_k h_4,$$  \hspace{1cm} (A.6)

the condition $n_k h_4^2 = \frac{h_4^3}{2 h_4} - \frac{h_4^3}{2 h_4} = 0$ is satisfied. For instance, parameterizing $h_4 = h_4(x^k) h_4(y^4)$, the equation (A.6) is solved by any $h(y^4) = e^{\kappa y^4}$ and $n_k = \kappa \partial_k [h_4(x^k)]$, for $x = \text{cont}$.

We conclude that for any $n_k$ and $h_4$ related by conditions (A.6) the zero torsion conditions (A.5) are the same as for $h_4 = \text{const}$. Using a similar proof from [21, 22], it is possible to verify by straightforward computations that $\hat{T}_{\beta\gamma}^\alpha = 0$ if the equations (32) are solved.

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