Yong Ren*, Wensheng Yin and Dongjin Zhu
Department of Mathematics, Anhui Normal University
Wuhu 241000, China
(Communicated by Björn Schmalfuß)

ABSTRACT. In this article, we discuss a class of impulsive stochastic function differential equations driven by G-Brownian motion with delayed impulsive effects (G-DISFDEs, in short). Some sufficient conditions for p-th moment exponential stability of G-DISFDEs are derived by means of G-Lyapunov function method, average impulsive interval approach and Razumikhin-type conditions. An example is provided to show the effectiveness of the theoretical results.

1. Introduction. Impulsive dynamical systems have been widely used in many branches of science and technology such as in the transmission of the impulse information, control systems with communication constraints etc. Recently, special interest has been focused on the issues of delayed impulsive. In this case, the impulsive states are related not only to the current states but also to the past states. Consequently, it is more realistic to impulsive systems with delayed impulsive effects, one can see Cheng et al. [1], Li et al. [9] and Yao et al. [22] for more details.

Recently, Peng [12] established the fundamental theory of time-consistent nonlinear G-expectation. Under the framework of the nonlinear G-expectation, Peng [12, 13] introduced the G-Gaussian distribution and the G-Brownian motion, which have very rich and interesting new structures which nontrivially generalize the classical ones. Since these notions were introduced, many investigators have studied the properties on G-Brownian motion (see Hu et al. [4, 5, 6]) and stochastic differential equation driven by G-Brownian motion (G-SDEs, in short). Gao [3] and Peng [15] have proved the existence and uniqueness of the solution for G-SDEs. Since then, this kind of G-SDEs have generated lots of developments. For more details, we refer the reader to Ren et al. [18, 19, 20, 21], Yin and Ren [23], Zhang and Chen [24, 25] and the references therein. For the updated developments on G-stochastic analysis and G-SDEs, one can see the survey paper by Peng [16].

Motivated by the aforementioned works, we aim to study the stability problem of stochastic function differential equations driven by G-Brownian motion with delayed...
impulsive effects (G-DISFDEs, in short). Some sufficient conditions for p-th moment exponential stability of G-DISFDEs are derived by means of G-Lyapunov function method, average impulsive interval approach and Razumikhin-type conditions. It should be mentioned that Ren et al. [19] derived some sufficient conditions for p-th moment exponential stability and quasi sure exponential stability of solutions to impulsive G-SDEs. However, their results are limited to the supremum or infimum of impulsive interval to some degree. Different from their works, delay and average impulsive interval method are considered in this paper, which are less conservative from the view of impulsive stabilization.

The paper is organized as follows. In Section 2, we introduce some preliminaries and notations of G-stochastic analysis for further discussion. In Section 3, we discuss the exponential stability of G-SDE with delayed impulsive effects by means of G-Lyapunov function, average impulsive interval approach and Razumikhin-type conditions. In the last Section, we apply the results to illustrate the stability of G-SDE with linear impulsive sequence and propose an example to verify the effectiveness of the obtained results.

2. Preliminaries. In this part, we recall some useful notions. For more details, one can refer to Denis et al. [2] and Peng [13, 15].

Let Ω denote the space of all \( \mathbb{R}^d \)-valued continuous functions \( \omega : t \in [0, +\infty) \to \omega_t \in \mathbb{R}^d \) with \( \omega_0 = 0 \), equipped with the distance

\[
\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i} \left[ \max_{t \in [0, i]} |\omega^1_t - \omega^2_t| \right] \wedge 1.
\]

It is easy to notice \((\Omega, \rho)\) is a metric space.

**Definition 2.1.** \( \mathbb{E} : Lip(\mathbb{R}^d) \to \mathbb{R} \) is called a sublinear expectation. If the following properties are satisfied.

1. (Monotonicity) \( \mathbb{E}(X) \geq \mathbb{E}(Y) \), if \( X \geq Y \)
2. (Cash translatability) \( \mathbb{E}(X + c) = \mathbb{E}(X) + c, \forall c \in \mathbb{R} \).
3. (Sub-additivity) \( \mathbb{E}(X + Y) \leq \mathbb{E}(X) + \mathbb{E}(Y) \).
4. (Positive homogeneity) \( \mathbb{E}(\lambda X) = \lambda \mathbb{E}(X), \forall \lambda \geq 0 \).

For each monotonic and sublinear function \( G : \mathbb{S}^d \to \mathbb{R} \) by

\[
G(A) = \frac{1}{2} \sup_{Q \in \Lambda} tr[AQ],
\]

where \( \Lambda \subset \mathbb{S}^d_+ \) is a bounded, convex and closed set. Peng [12] constructed a sublinear expectation space \((\Omega, L_{lip}(\Omega), \mathbb{E}, (\mathbb{E}_t)_{t \geq 0})\) called G-expectation space. To be exactly, for each \( \xi \in L_{lip}(\Omega) \) with the form of

\[
\xi = \varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_k}), \ 0 = t_0 < t_1 < \cdots < t_k < \infty,
\]

we define the conditional G-expectation by

\[
\mathbb{E}_t[\xi] := u_i(t, B_t : B_{t_1}, B_{t_2}, \cdots, B_{t_{i-1}})
\]

for each \( t \in [t_{i-1}, t_i) \), \( i = 1, \cdots, k \). Here, the function \( u_i(t, x; x_1, \cdots, x_{i-1}) \) parameterized by \( (x_1, \cdots, x_{i-1}) \in \mathbb{R}^{d \times (i-1)} \) is the viscosity solution of the following G-heat equation:

\[
\partial_t u_i(t, x : x_1, \cdots, x_{i-1}) + G(\partial_{xx}^2 u_i(t, x : x_1, \cdots, x_{i-1})) = 0, \ (t, x) \in [t_{i-1}, t_i) \times \mathbb{R}^d,
\]
with terminal conditions
\[ u_i(t, x : x_1, \ldots, x_{i-1}) = u_{i+1}(t, x : x_1, \ldots, x_{i-1}, x), \] for \( i < k \)
and \( u_k(t_k, x : x_1, \ldots, x_{k-1}) = \varphi(x_1, \ldots, x_{k-1}, x) \). The \( G \)-expectation of \( \xi \) is defined by \( E = E_0[\xi] \). In this space the corresponding canonical process \( B_t \) is called \( G \)-Brownian motion.

The Itô integral with respect to the \( G \)-Brownian motion is discussed as follows. We first consider the following set of step processes:
\[
M^0_b([0, T]) := \left\{ \eta(\omega) := \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})} ; \xi_j(\omega) \in B_b(\Omega_t), \right\}
\forall N \in \mathbb{N}, \ 0 = t_0 < t_1 < \cdots < t_N = T, j = 0, 1, \cdots, N - 1 \right\}.
\]
\( (1) \)

**Definition 2.2.** For \( p \geq 1 \), we denote by \( M^p_b([0, T]) \) the completion of \( M^0_b([0, T]) \) under the following norm
\[
||\eta||_{M^p_b([0, T])} = \left[ E \left( \frac{1}{T} \int_0^T |\eta|^p dt \right) \right]^{\frac{1}{p}}.
\]

Now, we propose the definition of the \( G \)-Itô integral.

**Definition 2.3.** (Itô Integral) Let \( B^a \) denote the inner product of \( a \in \mathbb{R}^d \) and \( B \) and Set \( \sigma_{aa} := E[(a, B_1)^2] \), we define the Itô integral
\[
I_{[0, T]}(\eta) = \int_0^T \eta_t dB^a_t := \sum_{j=0}^{N-1} \xi_j(B^a_{t_{j+1}} - B^a_{t_j}),
\]
for \( \eta(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})} \in M^0_b([0, T]). \)

**Remark 1.** It is easy to check that the linear mapping \( I : M^0_b([0, T]) \to L^2(\Omega_T) \) is continuous. Thus, it can be continuously extended to \( I : M^2_b([0, T]) \to L^2(\Omega_T). \)

One can learn the properties of the extended Itô integral from Li and Peng [10] and the references therein. And for any \( \eta \in M^2_b([0, T]), \) we also define \( \int_0^T \eta_t dB^a(t) = I_{[0, T]}(\eta). \)

**Definition 2.4.** Set \( \{\pi^N_t, N \geq 1\} \) be a sequence of partitions of \([0, t]\) with \( \mu(\pi^N_t) \to 0 \) as \( N \to \infty \). For any \( \eta \in M^{1,0}_b([0, T]), \) \( \langle B^a \rangle(t) \) is the quadratic variation process of \( G \)-Brownian motion \( B^a(t), \) define
\[
\langle B^a \rangle(t) := \lim_{N \to \infty} \sum_{j=0}^{N-1} (B^a(t^N_{j+1}) - B^a(t^N_j))^2 = (B^a(t))^2 - 2 \int_0^t B^a(s) dB^a(s).
\]

Furthermore, the mutual variation process of \( B^a \) and \( B^\alpha \) is defined by
\[
\langle B^a, B^\alpha \rangle_t := \frac{1}{4} (\langle B^{a+\alpha} \rangle_t - \langle B^{a-\alpha} \rangle_t),
\]
where \( a = (a_1, \ldots, a_d)^T, \alpha = (\bar{a}_1, \ldots, \bar{a}_d)^T. \)
Theorem 2.5. (Hu [8], Lemma 3.3). Let $B(\Omega)$ be the Borel $\sigma$-algebra of $\Omega$. Then, there exists a weakly compact set $\mathcal{P}$ of probability measures defined on $(\Omega, B(\Omega))$ such that

$$E[X] = \sup_{P \in \mathcal{P}} E_P[X],$$

where $E_P$ is the linear expectation with respect to $P$. For this $\mathcal{P}$, the associated capacity is defined by

$$C(A) := \sup_{P \in \mathcal{P}} P(A), A \in B(\Omega).$$

Remark 2. (Denis et al. [2], Definition 3). A set $A \in B(\Omega)$ is called polar if $C(A) = 0$. A property is said to hold quasi-surely (q.s., in short) if it holds outside a polar set.

Lemma 2.6. (Li and Peng [10], Theorem 5.4). Let $\phi \in C^{1,2}([0, T] \times \mathbb{R})$ and

$$X^n_t = X^n_0 + \int_t^T \alpha^n_s ds + \int_t^T \eta^{nij}_s d(B^i, B^j)(s) + \int_t^T \beta^n_s dB^j(s),$$

where $\alpha^n, \eta^{nij} \in M^1_\text{loc}(0, T), \beta^{nij} \in M^2_\text{loc}(0, T)$. Then for each $t \in [0, T]$, we have, quasi-surely,

$$\phi(t, X_t) - \phi(0, X_0) = \int_0^t \partial_x \phi(u, X_u) \beta^{nij}_u dB^j(u) + \int_0^t \partial_t \phi(u, X_u) + \partial_x \phi(u, X_u) \alpha^n_u du$$

$$+ \int_t^T \left\{ \partial_x \phi(u, X_u) \eta^{nij}_u + \frac{1}{2} \partial^2_{xx} \phi(u, X_u) \beta^{nij}_u \right\} d\langle B^i, B^j \rangle(u).$$

Lemma 2.7. (Li et al. [11], Lemma A.3.). Let $\eta^{ij} \in M^1_\text{loc}([0, T]; \mathbb{R}^d)$ and $M_t = \int_0^t \eta^{ij}(s)d\langle B^i, B^j \rangle(s) - \int_0^t 2G(\eta(s))ds$. Then, for each $t \in [0, T]$, $\mathbb{E}M_t \leq 0$.

Proposition 1. (Li and Peng [10], Lemma 3.4). Let $\eta \in M^2_\text{loc}[0, T]$. Then

$$\mathbb{E}\left( \int_0^T \eta^i dB^i(t) \right) = 0, \quad \mathbb{E}\left( \left\{ \int_0^T \eta^i dB^i(t) \right\}^2 \right) \leq \sigma^2_{aa} \mathbb{E}\left( \int_0^T \eta^2 dt \right).$$

3. System description. Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear space, $\mathbb{R}^n$ stands for the $n$-dimensional Euclidean space, $\mathbb{N}$ denotes the set of all positive real numbers, $\mathbb{N}$ denotes the set of positive integers. Letting $\tau > 0$, $C([-\tau, 0]; \mathbb{R}^n)$ denotes the family of bounded continuous $\mathbb{R}^n$-valued functions $\varphi$ with the norm $\|\varphi\| = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|$.

In this paper, we consider the following G-DISFDES

$$\left\{ \begin{array}{l}
dy(t) = f(t, x_k) dt + h_{ij}(t, x_k) dB^i(t) + \sigma_j(t, x_k) dB^j(t), t \neq k, t \geq 0, \\
x(t_k) - x(t_{k-}) = I_k(x(t_k), x(t_{k-})): \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}_+^{\mathbb{N}}, \end{array} \right. k \in \mathbb{N},$$

with $x_0 = \xi = \{\xi(\theta), -\tau \leq \theta \leq 0\} \in M^2_\text{loc}([-\tau, 0]; \mathbb{R}^n)$, where $x_t = \{x(t + \theta) : -\tau \leq \theta \leq 0\}, x_t, x_{t-} \in C([-\tau, 0]; \mathbb{R}^n)$, $f, h_{ij}, \sigma_j : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ and $f, h_{ij}, \sigma_j \in M^2_\text{loc}([0, T]; \mathbb{R}^n)$. The impulsive function $I_k(x(t_k), x(t_{k-})) : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}_+^{\mathbb{N}}$ and the impulsive moments $t_k(k = 1, 2, \cdots)$ satisfies $0 < t_1 < t_2 < \cdots, t_k \to \infty (as k \to \infty), x(t_k) = \lim_{t \to t_k} x(t), \langle B^i, B^j \rangle(\cdot)$ is the mutual variation process of the $(B^i)\cdot$.

As a standing hypothesis, we assume that $f, h_{ij}, \sigma_j, I_k, J_k$ satisfy locally Lipschitz and linear growth conditions. We can prove the existence and uniqueness of the
system (2) by applying the similar methods as in Hu and Ren [7]. In order to discuss the stability of system (2), we assume that \( f(t, 0) = 0, h_{ij}(t, 0) = 0, I_k(t_k, 0) = 0, J_k(t_k, 0) \equiv 0 \) for \( t \geq t_0 \), which implies that \( x(t) = 0 \) is a trivial solution when the initial value \( x_0 = 0 \).

**Remark 3.** In this paper, we use the Einstein convention, i.e. the above repeated indices of \( i \) and \( j \) imply the summation, i.e.,

\[
\int_0^t h_{ij}(s, x_s) d(B^i, B^j)(s) := \sum_{i,j=1}^d \int_0^t h_{ij}(s, x_s) d(B^i, B^j)(s),
\]

\[
\int_0^t \sigma_j(s, x_s) dB^j(s) := \sum_{j=1}^d \int_0^t \sigma_j(s, x_s) dB^j(s).
\]

**Definition 3.1.** The function \( V \) is a deterministic nonnegative Lyapunov function, if \( V(t, x) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+) \), i.e., \( V_t, V_x, V_{xx} \) are continuous on \( \mathbb{R}_+ \times \mathbb{R}^n \), where

\[
V_i(t, x) = \frac{\partial V(t, x)}{\partial t}, \quad V_x(t, x) = \left( \frac{\partial V(t, x)}{\partial x_1}, \frac{\partial V(t, x)}{\partial x_2}, \ldots, \frac{\partial V(t, x)}{\partial x_n} \right),
\]

\[
V_{xx}(t, x) = \left( \frac{\partial^2 V(t, x)}{\partial x_i \partial x_j} \right)_{n \times n}.
\]

**Definition 3.2.** Letting \( \varphi(\theta) = x(t + \theta) \) and \( \varphi = x_t \), it is obvious that \( \varphi(0) = x_t(0) = x(t) \). For \( V \), we define a G-Lyapunov function \( L \) from \( \mathbb{R}_+ \times \mathbb{R}^n \) to \( \mathbb{R} \) by

\[
L(t, \varphi) := V_i(t, x(t))) + \langle V_x(t, x(t)), f(t, \varphi) \rangle + G(\langle V_x(t, x(t)), h(t, \varphi) \rangle + \langle V_{xx}(t, x(t)) \sigma(t, \varphi), \sigma(t, \varphi) \rangle),
\]

where \( \langle V_x(t, x(t)), h(t, \varphi) \rangle + \langle V_{xx}(t, x(t)) \sigma(t, \varphi), \sigma(t, \varphi) \rangle \) is the symmetric in \( S^d(\mathbb{R}) \), with the form

\[
\langle V_x(t, x(t)), h(t, \varphi) \rangle + \langle V_{xx}(t, x(t)) \sigma(t, \varphi), \sigma(t, \varphi) \rangle := \left[ \langle V_x(t, x(t)), h_{ij}(t, \varphi) \rangle + \langle V_{xx}(t, x(t)) \sigma_i(t, \varphi), \sigma_i(t, \varphi) \rangle \right]_{i,j=1}^d.
\]

**Definition 3.3.** The system (2) is said to be \( p \)-th moment exponentially stable, if for any initial \( \xi \), there exist two positive constants \( \lambda \) and \( M \) such that

\[
E|\xi|^p \leq M|\xi|^p e^{-\lambda t}.
\]

**Definition 3.4.** The average impulsive interval of the impulsive sequence \( \{t_k\}_{k \in \mathbb{N}} \) is equal to a positive number \( \theta \) if there exists a positive integer \( N_0 \) such that

\[
\frac{t-s}{\theta} - N_0 \leq N(t, s) \leq \frac{t-s}{\theta} + N_0, \quad \forall t_0 \leq s < t,
\]

where \( N(t, s) \) denotes the number of impulsive times of the impulsive sequence \( \{t_k\}_{k \in \mathbb{N}} \) on the interval \( [s, t] \).

4. Main results.

**Theorem 4.1.** Assume that there exists a function \( V \), some constants \( c_1, c_2, p > 0, d_1 > 0, d_2 \geq 0, \eta_2 \geq 0, \theta > 0, N_0 \in \mathbb{N} \) and \( \eta_1 \in \mathbb{R} \) such that

1. for all \( (t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n \),

\[
c_1|\xi|^p \leq V(t, x) \leq c_2|\xi|^p,
\]

2. for all \( i, j \in \mathbb{N} \),

\[
\int_0^t \sigma_j(s, x_s) dB^j(s) := \sum_{j=1}^d \int_0^t \sigma_j(s, x_s) dB^j(s).
\]
(b) for all $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$, $t \geq t_0$, $t \neq t_k$, $k \in \mathbb{N}$,

$$V(t_k, \varphi(0) + I_k(t_k, \varphi(0)) + J_k(t_k, \varphi)) \leq d_1V(t^-_k) + d_2 \sup_{\theta \in [-\tau, 0]} V(t^-_k + \theta, \varphi(\theta)),$$  

(5)

(c) for all $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$, $k \in \mathbb{N}$,

$$LV(t, \varphi) \leq \eta_1 V(t, \varphi(0)) + \eta_2 \sup_{\theta \in [-\tau, 0]} V(t + \theta, \varphi(\theta)),$$  

(6)

(d) $\eta_1 + \frac{\ln(d_1 + \gamma_1 d_2)}{\theta} + \alpha \eta_2 < 0$, where $\gamma_1 = \max_{\theta \in [-\tau, 0]} e^{\eta_1 \theta}$ and

$$\alpha = \max \left\{ e^{-N_0 \ln(d_1 + \gamma_1 d_2)} \frac{1}{e^{N_0 \ln(d_1 + \gamma_1 d_2)}} \right\},$$

(e) the impulsive sequence satisfies for any $t_0 \leq s < t$, if $d_1 + \gamma_1 d_2 \geq 1$, then $N(t, s) \leq \frac{t \ln \lambda}{\theta} + N_0$, if $d_1 + \gamma_2 < 1$, then $\frac{t \ln \lambda}{\theta} + N_0 \leq N(t, s)$.

Then the solution of system (2) satisfies that

$$\mathbb{E}[x(t)]^p \leq \gamma_2 e^{-\eta_1(t-t_0)} e^{-\alpha \tau} \mathbb{E} \| \xi \|^p, t \geq t_0,$$

where $\gamma_2 = \sup_{\theta \in [-\tau, 0]} e^{-\eta_1 \theta}$ and $\lambda$ is the unique positive solution of the following equation

$$\lambda + \eta_1 + \frac{\ln(d_1 + \gamma_1 d_2)}{\theta} + \alpha \eta_2 e^\lambda = 0.$$

Proof. Let

$$W(t) = e^{-\eta_1(t-t_0)} V(t, x(t)), \quad t \in [t_0 - \tau, \infty].$$

For each $k \in \mathbb{N}$, it follows from condition (b), we get

$$W(t_k) = e^{-\eta_1(t_k-t_0)} V(t_k, x(t_k))$$

$$\leq e^{-\eta_1(t_k-t_0)} \left( d_1 V(t^-_k, x(t^-_k)) + d_2 \sup_{\theta \in [-\tau, 0]} V(t^-_k + \theta, x(t^-_k + \theta)) \right)$$

$$\leq d_1 e^{-\eta_1(t_k-t_0)} V(t^-_k, x(t^-_k)) + \gamma_1 d_2 \sup_{\theta \in [-\tau, 0]} \left( V(t^-_k + \theta, x(t^-_k + \theta)) e^{-\eta_1(t_k-t_0)} \right)$$

$$= d_1 W(t^-_k) + \gamma_1 d_2 \sup_{\theta \in [-\tau, 0]} W(t^-_k + \theta).$$  

(7)

Then, by using G-Itô’s formula to $e^{-\eta_1(t-t_0)} V(t, x(t))$, we obtain

$$\begin{aligned}
\int e^{-\eta_1(t-t_0)} V(t, x(t)) dt &= e^{-\eta_1(t-t_0)} \left( -\eta_1 V(t, x(t)) + V_\ell(t, x(t)) \right) dt + e^{-\eta_1(t-t_0)} \int \sigma_j(t, x(t)) dB^j(t) \\
&\quad + e^{-\eta_1(t-t_0)} \int \sigma(t, x(t)) dB^j(t) \\
&\quad + e^{-\eta_1(t-t_0)} \int \{V_{xx}(t, x(t)) \sigma(t, x(t)), \sigma_j(t, x(t)) \} dB^j(t) + \sigma(t, x(t)) dB^j(t).$$  

(8)
For $t \in [t_0, t_1)$, we have
\[
e^{-\eta_1(t-t_0)}V(t, x(t))
= V(t_0, x(t_0)) + \int_{t_0}^{t} e^{-\eta_1(s-t_0)} [LV(s, x(s)) - \eta_1 V(s, x(s))] ds
+ M^{t_0}_t + \int_{t_0}^{t} e^{-\eta_1(s-t_0)} \langle V_x(s, x(s)), \sigma_j(s, x(s)) \rangle dB^j(s),
\]
where
\[
M^{t_0}_t = \int_{t_0}^{t} e^{-\eta_1(s-t_0)} \langle V_x(s, x(s)), h_{ij}(s, x(s)) \rangle d(B^i, B^j)(s)
+ \frac{1}{2} \int_{t_0}^{t} e^{-\eta_1(s-t_0)} \langle V_{xx}(s, x(s)) \sigma_i(s, x(s)), \sigma_j(s, x(s)) \rangle d(B^i, B^j)(s)
- \int_{t_0}^{t} e^{-\eta_1(t-t_0)} G \left( \langle V_x(s, x(s)), h(s, x(s)) \rangle, \langle V_x(s, x(s)), \sigma(s, x(x(s)) \rangle \right) ds.
\]
From (6) and (9), we get
\[
e^{-\eta_1(t-t_0)}V(t, x(t))
\leq V(t_0, x(t_0)) + \int_{t_0}^{t} e^{-\eta_1(s-t_0)} \eta_2 \sup_{\theta \in [-\tau, 0]} V(s + \theta, x(s + \theta)) ds
+ M^{t_0}_t + \int_{t_0}^{t} e^{-\eta_1(s-t_0)} \langle V_x(s, x(s)), \sigma_j(s, x(s)) \rangle dB^j(s).
\]
Taking $G$-expectations on both sides of (11), we have
\[
\mathbb{E}W(t) \leq \mathbb{E}W(t_0) + \mathbb{E} \int_{t_0}^{t} e^{-\eta_1(s-t_0)} \eta_2 \sup_{\theta \in [-\tau, 0]} V(s + \theta, x(s + \theta)) ds,
\]
which implies that
\[
\mathbb{E}W(t_1^-) \leq \mathbb{E}W(t_0) + \mathbb{E} \int_{t_0}^{t_1} e^{-\eta_1(s-t_0)} \eta_2 \sup_{\theta \in [-\tau, 0]} V(s + \theta, x(s + \theta)) ds.
\]
Similarly, for $t \in [t_1, t_2)$, combining (7) with (13), we derived that
\[
\mathbb{E}W(t) \leq \mathbb{E}W(t_1) + \mathbb{E} \int_{t_1}^{t} \eta_2 e^{-\eta_1(s-t_0)} \sup_{\theta \in [-\tau, 0]} V(s + \theta, x(s + \theta)) ds
\leq d_1 \mathbb{E}W(t_1^-) + \gamma_1 d_2 \mathbb{E} \sup_{\theta \in [-\tau, 0]} W(t_1^- + \theta)
+ \mathbb{E} \int_{t_1}^{t} \eta_2 e^{-\eta_1(s-t_0)} \sup_{\theta \in [-\tau, 0]} V(s + \theta, x(s + \theta)) ds
\leq d_1 \left[ \mathbb{E}W(t_0) + \mathbb{E} \int_{t_0}^{t_1} \eta_2 e^{-\eta_1(s-t_0)} \sup_{\theta \in [-\tau, 0]} V(s + \theta, x(s + \theta)) ds \right]
If \( t > t \), by induction, for \( h \),

Thus, for \( t > t \), we have

Moreover, \( h_{E} \leq \gamma_{d} \) \( \int_{t}^{t_{1}} e^{-\eta_{2}(s-t_{0})} \sup_{\theta \in [-\tau,0]} V(s+\theta,x(s+\theta))ds \)

By induction, for \( t \in [t_{k-1}, t_{k}), k \in \mathbb{N} \), we have

Thus, for \( t > t_{0} \), we obtain

If \( d_{1} + \gamma_{1} d_{2} \geq 1 \), recalling the condition (e), we have

If \( d_{1} + \gamma_{1} d_{2} < 1 \), we get

Thus, for \( t > s \), we show that

Submitting this into (16), for \( t > t_{0} \), it follows that

Let \( h(\lambda) = \lambda + \eta_{1} + \frac{\ln(d_{1}+\gamma_{1} d_{2})}{\sup_{\theta \in [-\tau,0]} V(t_{0}+\theta,x(t_{0}+\theta))} \)

Moreover, \( h(\infty) = +\infty \) and \( h'(t) = 1 + \alpha \eta_{2} e^{\tau t} > 0 \). Hence, there exists a unique
positive constant $\lambda$ such that $h(\lambda) = 0$. Let $\epsilon \in (0, 1)$ be arbitrary. Next, we assume that

$$
EV(t, x(t)) \leq (\gamma_2 \alpha + \epsilon) e^{-\lambda(t-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, x(t_0 + \theta)), \; t \geq t_0 - \tau. \quad (21)
$$

Clearly,

$$
EV(t, x(t)) \leq \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, x(t_0 + \theta)) < (\gamma_2 \alpha + \epsilon) e^{-\lambda(t-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, x(t_0 + \theta)), \; t \in [t_0 - \tau, t_0].
$$

Hence, we only need to prove (21) for $t > t_0$. If (21) is not true. Set

$$
t^* = \inf_{t \in (t_0, \infty)} \{ t : EV(t, x(t)) \geq (\gamma_2 \alpha + \epsilon) e^{-\lambda(t-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, x(t_0 + \theta)) \}, \quad (23)
$$

then, we have

$$
EV(t^*, x(t^*)) = (\gamma_2 \alpha + \epsilon) e^{-\lambda(t^*-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, x(t_0 + \theta)). \quad (24)
$$

Thus, from (20), (24) and $h(\lambda) = 0$, it holds that

$$
EV(t^*, x(t^*)) \leq \gamma_2 \alpha e^{(n_1 + \frac{ln(d_1 + \gamma_2)}{\epsilon}) (t^*-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, x(t_0 + \theta)) + \int_{t_0}^{t^*} \alpha \eta_2 e^{(n_1 + \frac{ln(d_1 + \gamma_2)}{\epsilon}) (t-s)} \sup_{\theta \in [-\tau, 0]} EV(s + \theta, s + \theta) ds.
$$

$$
\leq \gamma_2 \alpha e^{(n_1 + \frac{ln(d_1 + \gamma_2)}{\epsilon}) (t^*-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, x(t_0 + \theta)) + \alpha \int_{t_0}^{t^*} (\gamma_2 \alpha + \epsilon) \eta_2 e^{\lambda (n_1 + \frac{ln(d_1 + \gamma_2)}{\epsilon}) (t-s) - \lambda (s-t_0)} e^{-\lambda(s-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, t_0 + \theta) ds.
$$

$$
= (\gamma_2 \alpha + \epsilon) e^{-\lambda(t^*-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, t_0 + \theta)) - \epsilon e^{(n_1 + \frac{ln(d_1 + \gamma_2)}{\epsilon}) (t^*-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, t_0 + \theta))
$$

$$
< (\gamma_2 \alpha + \epsilon) e^{-\lambda(t^*-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, t_0 + \theta)), \quad (25)
$$

which contradicts (24), therefore, (21) holds. Let $\epsilon \to 0$ in (21), we obtain

$$
EV(t, x(t)) \leq \gamma_2 \alpha e^{-\lambda(t-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, x(t_0 + \theta)), \; t \geq t_0 - \tau. \quad (26)
$$

Then it follows from (4) and (26) that

$$
\epsilon_1 E|x(t)|^p \leq EV(t, x(t)) \leq \gamma_2 \alpha e^{-\lambda(t-t_0)} \sup_{\theta \in [-\tau, 0]} EV(t_0 + \theta, t_0 + \theta)) \leq \gamma_2 \alpha \epsilon_2 e^{-\lambda(t-t_0)} E\|\xi\|^p, \; t \geq t_0, \quad (27)
$$
which implies that
\[ E|x(t)|^p \leq \frac{\gamma_2 \alpha c_2}{c_1} \|x\|^p e^{-\lambda(t-t_0)}, \quad t \geq t_0. \] (28)

This completes the proof. \hfill \Box

Letting \( J_k \equiv 0 \) in the system (2), we have
\[
\begin{aligned}
\left\{
\begin{array}{l}
dx(t) = f(t, x(t))dt + h_{ij}(t, x(t))dB^i(t) + \sigma_j(t, x(t))dB^j(t), \quad t \neq t_k, t \geq 0, \\
x(t_k) - x(t_k^-) = I_k(t_k^-, x(t_k^-)), \quad k \in \mathbb{N}, \\
x_{t_0} = \xi, \quad -\tau \leq \theta \leq 0,
\end{array}
\right.
\end{aligned}
\] (30)

For the system (29), we have the following result.

**Corollary 1.** Assume that there exists a function \( V \in \nu_0 \) and some constants \( c_1, c_2, p > 0, d > 0, \eta_1 \geq 0, \eta_2 > 0, N_0 \in \mathbb{N} \) and \( \eta_1 \in \mathbb{R} \) such that
\begin{itemize}
  \item[(a)] for all \( (t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n \),
  \[ c_1 |x|^p \leq V(t, x) \leq c_2 |x|^p, \]
  \item[(b)] for all \( \varphi \in C([-\tau, 0]; \mathbb{R}^n) \), \( t \geq t_0, t \neq t_k, k \in \mathbb{N}, \)
  \[ V(t, \varphi(0)) + I_k(t_k, \varphi(0)) \leq dV(t_k^-, \varphi(0)), \]
  \item[(c)] for all \( \varphi \in C([-\tau, 0]; \mathbb{R}^n), k \in \mathbb{N}, \)
  \[ \mathcal{L}V(t, \varphi) \leq \eta_1 V(t, \varphi(0)) + \eta_2 \sup_{\theta \in [-\tau, 0]} V(t + \theta, \varphi(\theta)), \]
  \item[(d)] \( \eta_1 + \frac{ln d}{\rho} + \alpha \eta_2 < 0 \), where \( \alpha = \max \left\{ \epsilon N_0^{\text{ind}}, \frac{1}{\epsilon N_0^{\text{ind}}} \right\}. \)
\end{itemize}

Then, when \( d < 1 \), the impulse times \( N(t, s) \geq \frac{t - s}{\rho} - N_0 \), when \( d > 1 \), the impulse times \( N(t, s) \leq \frac{t - s}{\rho} + N_0 \). Thus the trivial solution of system (29) is \( p \)-th moment exponentially stable.

**Proof.** We need to apply Theorem (4.1) with \( d_1 = d, d_2 = 0 \). \hfill \Box

5. **Applications and Example.** In this section, we consider the following system
\[
\begin{aligned}
\left\{
\begin{array}{l}
dx(t) = f(t, x(t), x(t - \tau(t)))dt + h_{ij}(t, x(t), x(t - \tau(t)))dB^i(t) + d^2(t, x(t), x(t - \tau(t)))dB^j(t), \quad t \neq t_k, t \geq 0, \\
x(t_k) - x(t_k^-) = I_k(t_k^-, x(t_k^-)) + J_k(t_k, x(t_k^-)), \quad k \in \mathbb{N}, \\
x_0 = \xi(\theta), \quad -\tau \leq \theta \leq 0,
\end{array}
\right.
\end{aligned}
\] (30)

where \( x(t) = (x_1(t), \cdots, x_n(t))^T \) and the mappings \( f, h_{ij}, \sigma : \mathbb{R}_+ \times \mathbb{R}^n \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n \) and \( f, h_{ij}, \sigma \in M^2([0, T]; \mathbb{R}^n) \). The impulsive function \( I_k(t_k^-, x(t_k^-)) \) and \( J_k(t_k, x(t_k^-)) : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n \) and the impulse moments \( t_k(k = 1, 2, \ldots) \) satisfies \( 0 < t_1 < t_2 < \cdots, t_k \to \infty \) (as \( k \to \infty \)), \( x(t_k^-) = \lim_{t \to t_k^-} x(t) \), \( (B^i, B^j)(\cdot) \) is the mutual variation process of the \( (B)^{(\cdot)} \). For simplicity, we just consider an appropriate linear impulsive sequence as follows
\[ x(t_k) - x(t_k^-) = Cx(t_k^-) + Dx(t_k^- - \tau(t_k)), k \in \mathbb{N}. \]

To establish the sufficient conditions ensuring the exponential stability of system (30), we need to the following hypothesis.
Assumption 5.1. There exist symmetric positive-define matrixes $Q$ and two constants $\beta_1, \beta_2$ such that

$$2x^T f(t, x, y) + 2G((x^T Q, h(t, x, y)) + (Q\sigma(t, x, y), \sigma(t, x, y))) \leq \beta_1 x^T Q x + \beta_2 y^T Q y,$$

for $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$.

Theorem 5.2. Under the above assumption, there exists a symmetrical positive matrixes $Q$ and some constants $d_1 > 0, d_2 \geq 0, \eta_2 \geq 0, N_0 \in \mathbb{N}$ and $\eta_1 \in \mathbb{R}$ such that

(a) the following matrix inequalities hold:

$$\begin{pmatrix} -d_1 Q & 0 & (I + C)^T Q \\ * & -d_2 Q & D^T Q \\ * & * & -Q \end{pmatrix} \leq 0,$$

$$\begin{pmatrix} \beta_1 Q - \eta_1 Q & 0 \\ * & \beta_2 Q - \eta_2 Q \end{pmatrix} \leq 0,$$

(b) $\eta_1 + \frac{\ln(d_1 + \gamma_1 d_2)}{\theta} + \alpha \eta_2 < 0$, where $\gamma_1 = \sup_{\theta \in [-\tau, 0]} e^{\eta_\theta}$ and

$$\alpha = \max \left\{ e^{N_0 \ln(d_1 + \gamma_1 d_2)}, \frac{1}{e^{N_0 \ln(d_1 + \gamma_1 d_2)}} \right\},$$

(c) the impulsive sequences satisfies for any $t_0 \leq s < t$, if $d_1 + \gamma_1 d_2 \geq 1$, then $N(t, s) \leq \frac{t-s}{\theta} + N_0$, if $d_1 + \gamma_1 d_2 < 1$, then $\frac{t-s}{\theta} + N_0 \leq N(t, s)$.

Then the solution of system (30) is exponentially stable in the mean square and $\lambda$ is the unique positive solution of the following equation

$$\lambda + \eta_1 + \frac{\ln(d_1 + \gamma_1 d_2)}{\theta} + \alpha \eta_2 e^{\lambda \theta} = 0.$$

Proof. Let $V(x) = x^T Q x$. According to matrix inequalities and Schur complement, it follows that

$$\begin{pmatrix} I + C & D \end{pmatrix}^T Q \begin{pmatrix} I + C & D \end{pmatrix} \leq \begin{pmatrix} d_1 Q & 0 \\ 0 & d_2 Q \end{pmatrix},$$

$$\begin{pmatrix} \beta_1 Q & 0 \\ * & \beta_2 Q \end{pmatrix} \leq \begin{pmatrix} \eta_1 Q & 0 \\ 0 & \eta_2 Q \end{pmatrix}.$$

On one hand, for $t = t_k, k \in \mathbb{N}$, we have

$$V(t_k, x(t)) + I_k(t_k, x(t) + J_k(t_k, \varphi)) = \begin{pmatrix} \varphi(0) \\ \varphi(-\tau(t_k)) \end{pmatrix}^T \begin{pmatrix} I + C & D \end{pmatrix}^T Q \begin{pmatrix} I + C & D \end{pmatrix} \begin{pmatrix} \varphi(0) \\ \varphi(-\tau(t_k)) \end{pmatrix}.$$

Thus, by using (34) and (35), we get

$$V(t_k, \varphi(0) + I_k(t_k, \varphi(0) + J_k(t_k, \varphi)) \leq d_1 V(t^-_k, x(t)) + d_2 \sup_{\theta \in [-\tau, 0]} V(t^-_k + \theta, \varphi(\theta)).$$

(37)
Example 5.3. We consider the following two-dimensional
Thus, the trivial solution of the system is

\[ C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]

Corollary 2. Assume that there exists a matrix \( Q > 0 \) and some constants \( d > 0, \eta_2 \geq 0, \varrho > 0, N_0 \in \mathbb{N} \) and \( \eta_1 \in \mathbb{R} \) such that

(a) the following matrix inequalities hold:

\[
(I + C)^T Q (I + C) \leq d Q,
\]

\[
\begin{pmatrix} \beta_1 Q - \eta_1 Q & 0 \\ 0 & \beta_2 Q - \eta_2 Q \end{pmatrix} \leq 0,
\]

(b) \( \eta_1 + \frac{h d}{\varrho} + \alpha \eta_2 < 0 \), where \( \gamma_1 = \max_{t \in [-\tau,0]} e^{\eta_1 t} \) and \( \alpha = \max \{ e^{N_0 \varrho d}, \frac{1}{e^{N_0 \varrho d}} \} \). Thus, the trivial solution of the system is \( p \)-th moment exponentially stable.

Example 5.3. We consider the following two-dimensional G-DISFDES

\[
\begin{cases}
    dx(t) = f(t, x(t), x(t - \tau(t)))dt + h_{11}(t, x(t), x(t - \tau(t)))dB^1(t) \\
    + \sigma(t, x(t), x(t - \tau(t)))dB^2(t), t \neq t_k, t \geq 0, \\
    x(t_k) - x(t_k^-) = C x(t_k^-) + D x(t_k^- - \tau(t)), t = t_k, k \in \mathbb{N},
\end{cases}
\]

where

\[
f(t, x(t)) = \begin{pmatrix} \sin x_1(t) \\ \sin x_2(t) \end{pmatrix}, h_{11}(t, x(t)) = \begin{pmatrix} \sin x_1(t) \\ \sin x_2(t) \end{pmatrix},
\]

\[
h_{12}(t, x(t)) = \begin{pmatrix} x_2(t) \sin x_1(t) \\ -0.5x_1(t) \sin x_1(t) \end{pmatrix}, h_{22}(t, x(t)) = \begin{pmatrix} \sin x_1(t) \\ 0.5 \sin x_2(t) \end{pmatrix},
\]

\[
\sigma_1(t, x(t - \tau)) = \begin{pmatrix} \sin x_1(t - 0.02) \\ \sin x_2(t - 0.02) \end{pmatrix}, \sigma_2(t, x(t)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

\[
\sum = \left\{ \lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{pmatrix} : \lambda_{11} \in \left[ \frac{4}{\mathbb{N}}, \frac{1}{2} \right], \lambda_{12} \in \left[ \frac{1}{2}, \frac{3}{\mathbb{N}} \right], \lambda_{22} \in \left[ \frac{1}{4}, \frac{1}{2} \right] \right\},
\]

\[
C = \begin{pmatrix} -0.7 & 0.2 \\ 0.2 & -0.8 \end{pmatrix}, D = \begin{pmatrix} -0.1 & 0.2 \\ 0.2 & 0.3 \end{pmatrix}.
\]

Letting \( V(t, x) = x^T Q x, x(t) = (x_1(t), x_2(t))^T \) and \( Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \), then, we have

\[
A = \{ V_x(t, x), 2h(t, x(t)) \}
\]

\[
= \begin{pmatrix} 4x_1 \sin x_1(t) + 8x_2 \sin x_2(t) & 0 \\ 0 & 4x_1 \sin x_1(t) + 4x_2 \sin x_2(t) \end{pmatrix},
\]

On the other hand, for \( t \neq t_k, k \in \mathbb{N} \), we obtain

\[
\mathcal{L} V(t, \varphi) = 2 \varphi(0)^T Q f(t, \varphi(-\tau(t))) + 2 G(\langle \varphi(0)^T Q h(t, \varphi(-\tau(t))) + Q \sigma(t, \varphi(-\tau(t))), \sigma(t, \varphi(-\tau(t))) \rangle)
\]

\[
\leq \beta_1 \varphi(0)^T Q \varphi(0) + \beta_2 \varphi(-\tau(t))^T Q \varphi(-\tau(t))
\]

\[
= \begin{pmatrix} \varphi(0) \\ \varphi(-\tau(t)) \end{pmatrix}^T \begin{pmatrix} \beta_1 Q & 0 \\ 0 & \beta_2 Q \end{pmatrix} \begin{pmatrix} \varphi(0) \\ \varphi(-\tau(t)) \end{pmatrix},
\]

Hence, by using (35) and (38), we derive that

\[
\mathcal{L} V(t, \varphi) \leq \eta_1 V(t, \varphi(0)) + \eta_2 \sup_{\theta \in [-\tau,0]} V(t + \theta, \varphi(\theta)).
\]

Consequently, the conclusion follows from Theorem 4.1, directly, which completes the proof. \( \square \)
\[ B = \langle V_{xx}(t, x)\sigma(t, x(t)), \sigma(t, x(t)) \rangle = \left( \begin{array}{ccc} 2\sin^2 x_1(t - 0.02) + 4\sin^2 x_2(t - 0.02) & 0 \\ 0 & 0 \end{array} \right), \]

\[ G(A) = \frac{1}{2}\sup_{t\in [0,2]}\text{tr}(A\Gamma) \leq \frac{3}{2}x(t)^TQ(t), \quad G(B) = \frac{1}{2}\sup_{t\in [0,2]}\text{tr}(B\Gamma) \leq \frac{1}{2}x(t - 0.02)^TQx(t - 0.02). \]

From Assumption 5.1, we have \( \beta_1 = 5 \) and \( \beta_2 = 1 \). From the LMIs (32) and (33), we get \( d_1 = 0.34, d_2 = 0.42, \eta_1 = 5 \) and \( \eta_2 = 1 \). Setting \( \gamma_1 = 1, N_0 = 2, \) and \( \alpha = 2.2277, \varrho < 0.0524, \) for all \( \epsilon > 0, \) setting \( \varrho = 0.0523, \epsilon = 0.001 \) and constructing the impulse sequence as

\[ \{\epsilon, N_0\varrho, N_0\varrho + \epsilon 2N_0\varrho, 2N_0\varrho + \epsilon, \cdots\}, \]

then the impulse sequence \( N(t, s) \geq \frac{\epsilon - s}{\varrho} - N_0. \) According to condition (b), (c) of Theorem 5.2 and the system (41), we know that system (41) is the exponential stability in mean square.

**Acknowledgments.** We would like to thank the anonymous referee and the editor for their careful reading and useful remarks.

**REFERENCES**

[1] P. Cheng, F. Deng and F. Yao, Exponential stability analysis of impulsive stochastic functional differential systems with delayed impulses, *Commun. Nonlinear Sci. Numer. Simul.*, 19 (2014), 2104–2114.

[2] L. Denis, M. Hu and S. Peng, Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths, *Potential Anal.*, 34 (2011), 139–161.

[3] F. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, *Stochastic Process. Appl.*, 119 (2009), 3356–3382.

[4] F. Hu, Z. Chen and P. Wu, A general strong law of large numbers for non-additive probabilities and its applications, *Statistics*, 50 (2016), 733–749.

[5] F. Hu, Z. Chen and D. Zhang, How big are the increments of G-Brownian motion?, *Sci. China Math.*, 57 (2014), 1687–1700.

[6] F. Hu and Z. Chen, General laws of large numbers under sublinear expectations, *Comm. Statist. Theory Methods*, 45 (2016), 4215–4229.

[7] L. Hu and Y. Ren, Impulsive stochastic differential equations driven by G-Brownian motion, *In Brownian Motion: Elements, Dynamics and Applications*, editors: Mark A. McKibben and Micah Webster, Nova Science Publishers, Inc, New York, 2015, Capter 13, 231–242.

[8] M. Hu and S. Peng, On the representation theorem of G-expectations and paths of G-Brownian motion, *Acta Math. Appl. Sin. Engl. Ser.*, 25 (2009), 539–546.

[9] D. Li, P. Cheng and S. Shu, Exponential stability of hybrid stochastic functional differential systems with delayed impulsive effects: average impulsive interval approach, *Math. Methods Appl. Sci.*, 40 (2017), 4197–4210.

[10] X. Li and S. Peng, Stopping times and related Itô’s calculus with G-Brownian motion, *Stochastic Process. Appl.*, 121 (2011), 1492–1508.

[11] X. Li, X. Lin and Y. Lin, Lyapunov-type conditions and stochastic differential equations driven by G-Brownian motion, *J. Math. Anal. Appl.*, 439 (2016), 235–255.

[12] S. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Itô type, *Stochastic Analysis and Applications*, in: *Abel Symp.*, Springer, Berlin, 2 (2007), 541–567.

[13] S. Peng, G-Brownian motion and dynamic risk measures under volatility uncertainty, preprint, *arXiv:0711.2834v1*.

[14] S. Peng, Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, *Stochastic Process. Appl.*, 118 (2008), 2223–2253.

[15] S. Peng, Nonlinear expectations and stochastic calculus under uncertainty-with robust central limit theorem and G-Brownian motion, preprint, *arXiv:1002.4546v1*.

[16] S. Peng, Theory, methods and meaning of nonlinear expectation theory (in Chinese), *Sci Sin Math.*, 47 (2017), 1223–1254.
[17] Y. Ren, Q. Bi and R. Sakthivel, Stochastic functional differential equations with infinite delay driven by G-Brownian motion, Math. Method. Appl. Sci., 36 (2013), 1746–1759.
[18] Y. Ren and L. Hu, A note on the stochastic differential equations driven by G-Brownian motion, Statist. Probab. Lett., 81 (2011), 580–585.
[19] Y. Ren, X. Jia and L. Hu, Exponential stability of solutions to impulsive stochastic differential equations driven by G-Brownian motion, discrete Conti. Dyn. Syst. Ser–B., 20 (2015), 2157–2169.
[20] Y. Ren, X. Jia and R. Sakthivel, The p-th moment stability of solution to impulsive stochastic differential equations driven by G-Brownian motion, Appl. Anal., 96 (2017), 988–1003.
[21] Y. Ren, J. Wang and L. Hu, Multi-valued stochastic differential equations driven by G-Brownian motion and related stochastic control problems, Internat. J. Control, 90 (2017), 1132–1154.
[22] F. Yao, J. Cao, L. Qiu and P. Cheng, Exponential stability analysis for stochastic delayed differential systems with impulsive effects: average impulsive interval approach, Asian J. Control, 19 (2017), 74–86.
[23] W. Yin and Y. Ren, Asymptotical boundedness and stability for stochastic differential equations with delay driven by G-Brownian motion, Appl. Math. Lett., 74 (2017), 121–126.
[24] D. Zhang and Z. Chen, Exponential stability for stochastic differential equations driven by G-Brownian motion, Appl. Math. Lett., 25 (2012), 1906–1910.
[25] D. Zhang and P. He, Functional solution about stochastic differential equations driven by G-Brownian motion, Discrete Conti. Dyn. Syst. Ser–B., 20 (2015), 281–293.

Received for publication January 2018.

E-mail address: brightry@hotmail.com, renyong@126.com
E-mail address: WenShengYin@126.com
E-mail address: zhudj@mail.ahnu.edu.cn