I. INTRODUCTION

A dust grain suspended in a plasma collects ions and electrons (plasma particles), and gains a net electric charge that fluctuates in time. This type of fluctuations is intrinsic noise which occurs in systems consisting of discrete particles. The intrinsic noise is inherent in the actual mechanism that is responsible for the evolution of the system, and cannot be switched off. The discrete particles are ions and electrons in In the grain charging system, which are attached to the grain at random times, resulting in fluctuations of its charge.

Some phenomena observed in dusty plasmas are attributed to the intrinsic fluctuations of the grain charge. One reason to so-called dust heating, in which grains gain kinetic energy, is these fluctuations. The external electric and ion forces, and the Columb forces between the dust grains are dependent on the electric charges of the grains. As the grain charge fluctuate, so do these forces, leading to random oscillations of the grains. It is shown that if the intrinsic charge fluctuations are strong, the grain oscillations become unstable. It is also shown that the dust charge fluctuations are a source of dissipation and are responsible for the formation of the dust-ion-acoustic shock waves.

In the current study, the charge fluctuations are described for a grain which is suspended in a plasma containing electrons and different kinds of negative or positive ions that are singly or multiply charged. The assumed plasma is the most general case of a plasma in terms of the involved components. Some phenomena relevant to the problem considered here have been identified: In the solar system, interstellar dust gains are charged by electrons, protons, and alpha particles. In addition to these ions, other highly charged ions of carbon, oxygen, and nitrogen contribute to the charging of dust in the solar wind plasma. Dust in nuclear-induced plasmas is charged by alpha particles and multiply-charged fission fragments. Dust in tokamaks is charged as it collects electrons, deuterium ions, and multiply-charged impurity ions.

A number of approaches have been proposed to describe the grain charge intrinsic fluctuations. They are based on discrete stochastic modeling, master and Fokker-Planck equations, or Langevin equations. The approaches given by Matsoukas and Russell, , Matsoukas, Russell, and Smith, Matsoukas and Russell, , Shotorban, Matthews, Shotorban, and Hyde, , Khrapak et al. , Khrapak et al. , Khrapak et al. are valid only for cases in which all plasma particles are singly charged. The approach given by Cui and Goree is valid only for cases that all contained ions have an identical charge number. As will be seen in the following section, all these approaches are special cases of general approaches that are proposed in the present study for the description of grain charging in plasmas with various types of singly or multiply charged ions.

The outline of the paper is as follows: In sec. II A a master equation describing the grain charging in a general multi-component plasma through a Markovian approach is presented. In sec. II B the derivation of a Fokker-Planck equation from the master equation is detailed and a solution for the derived Fokker-Planck equation is obtained. Sec. II is continued with subsections on grain mean and variance equations, a Langevin equation and a discrete stochastic model that can be utilized to simulate the time advancement of the fluctuating grain charge. In sec. IIII results obtained by approaches given in sec. III to describe intrinsic charge fluctuations of grains suspended in a Maxwellian plasma containing electrons, protons, and alpha particles are presented. In sec. IV conclusions are drawn.

II. MATHEMATICAL FORMULATION

A. Master Equation

The differential form of the Chapman-Kolmogorov equation, known as the master equation, is used for the Markovian description of dust grain charging:

\[
\frac{dP(Z, t)}{dt} = \sum_{Z'} W(Z|Z')P(Z', t) - W(Z'|Z)P(Z, t),
\]

(1)
where $Z$ indicates the net number of elementary charges collected on the grain, $P(Z, t)$ is the probability density function (PDF) of the state being at $Z$ at time $t$, and $W(Z' | Z)$ is the transition probability per unit time from state $Z'$ to state $Z$. Eq. (11) can be regarded as a gain-loss equation for the probabilities of separate elementary charge states. Here, $W(Z' | Z)$ is expressed in terms of the rate of the attachment of plasma particles (electrons or ions) to the grain:

$$W(Z' | Z) = \sum_{j=1}^{N} W_j(Z) \delta_{\zeta_j, Z-Z'},$$  \hspace{1cm} (2)

where $\delta_{x,y}$ is the Kronecker delta, $\zeta_j$ indicates the charge number (valence) of the $j$-th plasma component particle, $W_j(Z')$ is the rate of the attachment of the $j$-th component plasma particles to the grain, and $N$ is the total number of the plasma components. The rates of attachment of plasma particles in the collisional charging of a grain are correlated with the their velocity distribution (see Appendix A). It is noted that the electric current of each component to the dust grain can be calculated by

$$I_j(Z) = \zeta_j W_j(Z).$$  \hspace{1cm} (3)

Sometimes in addition to the direct impingement of plasma particles onto the grain, there are other mechanisms such as the ultraviolet induced photoemission of electrons and the secondary emission of electrons that contribute to the grain charging. If these mechanisms are also present in the charging of the grain, their associated rates can be readily incorporated into eq. (2). Then, for instance, $W_{Z+1}(Z)$ and $W_{Z+2}(Z)$ can represent the rate of the detachment of electrons from the grain by photoemission and secondary emission, respectively.

Having substituted for $W(Z' | Z)$ in eq. (1) from eq. (2), one obtains:

$$\frac{dP(Z, t)}{dt} = \sum_{j=1}^{N} W_j(Z - \zeta_j) P(Z - \zeta_j, t) - W_j(Z) P(Z, t).$$  \hspace{1cm} (4)

Fig. 1 shows a diagram for the state of the grain charge according to this equation for a plasma with three components including electrons, singly charged positive ions, and doubly charged positive ions. The master equation given by Matsoukas and Russell, Matsoukas, Russell, and Smith for grain charging is a special case of eq. (4), applicable for plasmas containing only singly-charged components. From eq. (4), an equilibrium charge distribution is obtained at the stationary state when $t \to \infty$ and $dP/dt$ vanishes:

$$\sum_{j=1}^{N} W_j(Z - \zeta_j) P_s(Z - \zeta_j) - P_s(Z) \sum_{j=1}^{N} W_j(Z) = 0,$$  \hspace{1cm} (5)

where $P_s(Z)$ is the PDF at the stationary state.

The master equation (4) may be shown in a matrix form by

$$\dot{P}(t) = WP(t),$$  \hspace{1cm} (6)

where $P(t)$ is a vector with elements $P_m(t) \equiv P(m, t)$, and $W$ is the transition rate matrix with elements

$$W_{mn} = \sum_{j=1}^{N} W_j(n) \delta_{\zeta_j, m-n} - \delta_{m,n} \sum_{j=1}^{N} W_j(m).$$  \hspace{1cm} (7)

The off-diagonal elements of $W$ are the transition probability per unit time given in eq. (2), i.e., $W_{mn}$ = $W(m,n)$ if $n \neq m$ while its diagonal elements $W_{mm}$ = $-\sum_{j=1}^{N} W_j(m)$ is the net escape rate from state $m$. In the current work, eq. (6), which represents a system of ordinary differential equations, is solved numerically in order to find a solution for the master equation (4). An analytical solution to eq. (6) with $W$ given in eq. (7) is not known. It is noted that only in rare cases it is possible to solve the master equations explicitly.

### B. Fokker-Planck Equation

Because of the lack of an analytical solution to the master equation (4), a Fokker-Planck equation with an analytical solution is developed for $P(Z, t)$ from eq. (4). This development is achieved through the system-size expansion method, which is a systematic approximation approach for the expansion of master equations that describe systems with intrinsic fluctuations. This approach has been recently employed for the expansion of the master equation of the grain charging in plasmas containing only electrons and singly charged ions.
in the grain charging case as will be seen later. Following an ansatz proposed by Van Kampen, a change of variable is performed:

\[ Z = \Omega \phi(t) + \Omega^{1/2} \xi. \]  

(8)

According to this equation, \( Z \) is a combination of a deterministic part \( \phi(t) \) scaled by \( \Omega \), and a random part \( \xi \) scaled by \( \Omega^{1/2} \). The probability density function can be then expressed as

\[ P(Z, t) = P(\Omega \phi(t) + \Omega^{1/2} \xi, t) = \Pi(\xi, t). \]  

(9)

Expanding the terms in the master equation in powers of \( \Omega^{-1/2} \) and setting the coefficients of two highest powers of \( \Omega \) to zero, results in the following equations:

\[ \frac{d\phi(t)}{dt} = a_1[\phi(t)], \]

(10)

\[ \frac{\partial \Pi(\xi, t)}{\partial t} = -a_1'[\phi(t)] \frac{\partial \Pi(\xi, t)}{\partial \xi} + \frac{1}{2} \sum_{i=2}^{1} \frac{\partial^2 \Pi(\xi, t)}{\partial \xi^2}. \]

(11)

where \( a_k(x) = \Omega^{-1}a_k(x\Omega) \) and \( a_1'(x) = \partial a_1(x)/\partial x \).

Here, \( a_k(Z) \) is the \( k \)th jump moment calculated by \( a_k(Z) = \sum_{j} c_j^k W_j(Z) \), which by using eq. (12), is simplified to

\[ a_k(Z) = \sum_{j} c_j^k W_j(Z). \]

(12)

For \( k = 1 \), one obtains

\[ a_1(Z) = \sum_{j} I_j(Z) \]

(13)

where the right hand side is the net electric current from the plasma components to the grain according to eq. 3.

C. Mean and Variance Equations

Equation (10) is referred as the macroscopic equation by Van Kampen, and Eq. (11) is a Fokker-Planck equation with time dependent drift and diffusion coefficients. Since the objective is to solve for \( P(Z, t) \) with an initial condition of \( P(Z, 0) = \delta(Z - Z_0) \), where \( Z_0 \) is the initial charge on the grain, from eq. (5), it could be said that \( \phi(0) = Z_0/\Omega \), and for all times \( \langle \phi \rangle = \langle Z \rangle/\Omega \) and \( \langle \xi \rangle = 0 \). Thus, eq. (10), as the macroscopic equation, can be expressed by:

\[ \frac{d\langle Z \rangle}{dt} = a_1(\langle Z \rangle), \]

(14)

where \( \langle Z \rangle \) is the grain mean charge. According to eq. (13), the rate of change of the mean grain charge is equal to the net current evaluated at the mean grain charge. Eq. (14) describes the mean charge evolution with time and by itself is in closed form. According to this equation, the mean charge at the stationary state can be obtained by solving \( a_1(\langle Z \rangle) = 0 \). It can be shown that the stationary mean charge can be reached and will be stable if \( a_1'(\langle Z \rangle) < 0 \) (see fig. 2). The time evolution of the grain mean charge in a multicomponent plasma obtained by eq. (14) is displaced in Fig. 3.

The Fokker-Planck equation (11) has a Gaussian solution for \( \Pi(\xi, t) \) at all \( t \)'s, which means \( P(Z, t) \) is a Gaussian function, according to eq. (9), with a time-dependent mean and variance. The mean is governed by (14) while the variance is governed by the following equation:

\[ \frac{d\langle Z^2 \rangle}{dt} = 2a_1'(\langle Z \rangle) \langle Z^2 \rangle + a_2(\langle Z \rangle), \]

(15)

where \( \langle Z^2 \rangle \) is the variance of the grain charge, where \( Z = Z - \langle Z \rangle \), and

\[ a_1'(Z) = \frac{\partial a_1(Z)}{\partial Z} = \sum_{j} c_j^1 \frac{\partial W_j(Z)}{\partial Z}. \]

(16)

Eq. (15) models the time evolution of the variance, and is in closed form when coupled to Eq. (14). The time evolution of \( \langle Z \pm \langle Z^2 \rangle^{1/2} \rangle \), where \( \langle Z^2 \rangle \) is obtained from
eq. (15), is displayed in Fig. 3. Also, from Eq. (15), the variance of the grain charge at the stationary state can be obtained as

$$\langle Z^2 \rangle_s = -\alpha_2 \langle \langle Z \rangle_s \rangle / 2\alpha_1' \langle \langle Z \rangle_s \rangle. \quad (17)$$

It is noted that the Fokker-Planck equation derived by Matsoukas and Russell for the charging of a dust grain in plasmas containing only electrons and singly charged ions is a special case of eq. (11) where the drift and diffusion coefficients are given in terms of the grain charge stationary mean and variance. So their Fokker-Planck equation is valid only for the cases that the initial charge of the dust grain is in the vicinity of its mean value at the stationary state.

D. Langevin Equation

Equation (11) is statistically equivalent to the following Langevin type of stochastic differential equation:

$$d\tilde{Z}(t) = \alpha_1' \langle \langle Z \rangle \rangle \tilde{Z}(t) dt + \sqrt{\alpha_2 \langle \langle Z \rangle \rangle} dw(t), \quad (18)$$

where $w(t)$ is a Wiener process. This equation could have applications for the cases in which the dynamic behavior of the grain is of interest when charge fluctuations are significant. For the case of ions being only singly charged, eq. (18) is simplified to the Langevin equation developed by Matsoukas and Russell, Khrapak et al., for the stationary, and by Shotorban for the non-stationary charging of a grain. The time correlation of the grain charge fluctuations at the stationary state can be obtained as

$$\langle \tilde{Z}(t) \tilde{Z}(t + \tau) \rangle_s = -\frac{\alpha_2 \langle \langle Z \rangle_s \rangle}{2\alpha_1' \langle \langle Z \rangle_s \rangle} \exp \left[ -|\alpha_1'(\langle Z \rangle_s)| \tau \right], \quad (19)$$

based on which, the particle charging time scale can be defined as $\tau_{ch} = 1/|\alpha_1'(\langle Z \rangle_s)|$.

E. Discrete Stochastic Model

In the development of the Fokker-Planck equation, it is assumed that the net grain charge continuously changes over time. From eq. (11), a model can be developed for the discrete stochastic process of the grain charging (the model is referred by the discrete stochastic model in this work). That is to model the time evolution of $Z(t)$ which randomly changes over time while it only admits integer numbers. To do so, two issues must be dealt with: first, how to calculate the random time intervals at which the plasma particles are attached to the grain; and second, how to specify the type of the plasma particle that is attached at the time of the jump. That is the moment at which the charge of the grain changes as much as the charge of the attached plasma particle. According to the Markovian description, the jump event is based on that the time scale of the attachment process of plasma particles to the dust grain, i.e., the collision of plasma particles with the dust, is so small that the attachment is assumed to effectively instantaneously takes place. Here, the stochastic simulation algorithm given by Gillespie for simulation of chemical kinetics with master equations similar to eq. (11) is adapted for the discrete stochastic modeling of dust charging:

The probability, given at $Z(t) = Z$, that the attachment of a plasma particle to the grain will occur in a time between $t + \tau$ and $t + \tau + d\tau$, and that plasma particle belongs to the $j$th component of the plasma is $f(\tau, j|Z)d\tau$ where $f(\tau, j|Z)$ is a PDF obtained by

$$f(\tau, j|Z) = W_j(Z) \exp \left[ -\lambda(Z)\tau \right], \quad (20)$$

where $\lambda(Z) = \sum_{j=1}^{N} W_j(Z)$. Eq. (20) implies that the time interval $\tau$ is a random number with an exponential distribution with a mean $1/\lambda(Z)$ and variance $1/\lambda(Z)^2$, and $j$ is a random number with point probabilities $W_j(Z)/\lambda(Z)$. Thus, a single realization of $Z(t)$ can be constructed as follows: 1. At a given time with $Z = Z(t)$, evaluate $W_j(Z)$’s and their summation $\lambda(Z)$; 2. Generate the time interval according to $\tau = \ln(1/r_1)/\lambda(Z)$, where $r_1$ is a random number with a uniform distribution; 3. Generate $j$, which is the smallest integer satisfying $\sum_{k=1}^{j-1} W_k(Z) > r_2 \lambda(Z)$, where $r_2$ is a random number with a uniform distribution; 4. Repeat the procedure with changing $t$ to $t + \tau$ and $Z(t)$ to $Z(t + \tau) = Z(t) + \zeta_j$.

Fig. 3 shows a single realization of the grain charge obtained through this discrete stochastic model for a plasma containing electrons, protons, and alpha particles.

It is noted that the discrete stochastic model demonstrated here is simplified to the model proposed by Cui and Goree, which was developed for cases in which there are only singly charged negative plasma particles and multiply charged positive ions with an identical charge number.

III. EXAMPLE PROBLEM: GRAIN CHARGING IN A MAXWELLIAN MULTI-COMPONENT PLASMA

The proposed models and approaches in this work are general and not dependent on the form of the currents. Here, as an example, we consider a particular case in which plasma particles follow Maxwellian distributions with a velocity distribution given by:

$$f_j(v, \theta) = n_j \left( \frac{m_j}{2\pi k_B T_j} \right)^{\frac{3}{2}} \exp \left( -\frac{m_j v^2}{2k_B T_j} \right), \quad (21)$$

where $n_j$ and $T_j$ are the number density and temperature of the $j$th component particle, respectively. Substituting this equation in eq. (A1), one obtains
FIG. 3. Time evolution of the net number of elementary charges collected on a dust grain with a radius of (a) 3 nm; and (b) 30 nm, obtained by discrete stochastic model (step line) and the system-size-expansion method with a mean \( \langle Z \rangle \) (solid line) and \( \langle Z \rangle \pm \hat{Z}^{1/2} \) (dashed line). Here, \( \hat{t} = \Omega^{-1} \Gamma t \) is the dimensionless time where \( \Gamma \) is given by eq. (24). Refer to the caption of Fig. 2 for the grain and plasma properties.

\[
W_j(Z) = \Gamma \hat{T}_j \sqrt{\frac{T_j}{m_j}} \times \begin{cases} 
1 - \frac{\zeta_j Z}{\hat{T}_j \Omega} & \zeta_j Z \leq 0, \\
\exp \left( -\frac{\zeta_j Z}{\hat{T}_j \Omega} \right) & \zeta_j Z > 0,
\end{cases} \tag{22}
\]

where \( \hat{T}_j = T_j / T_e \), \( \hat{m}_j = m_j / m_e \), \( \hat{n}_j = n_j / n_e \),

\[
\Omega = \frac{4 \pi \epsilon_0 R k_B T_e}{e^2}, \tag{23}
\]

\[
\Gamma = \pi R^2 n_e \sqrt{\frac{8 k_B T_e}{\pi m_e}} = \frac{\Omega \omega_{pe} R}{\sqrt{2 \pi \lambda_{De}}}. \tag{24}
\]

where \( \lambda_{De} = \sqrt{\epsilon_0 k_B T_e / n_e e^2} \) is the electron Debye length and \( \omega_{pe} = \sqrt{n_e e^2 / \epsilon_0 m_e} \) is the electron plasma frequency.

In eq. (23), \( \Omega \) is the system size utilized in eq. (8), and it is a reference number of elementary charges. According to this definition, an \( R \)-radius conducting sphere charged with \( \Omega \) number of elementary charges will have an electric potential equal to two thirds of the mean kinetic energy of electrons of the surrounding plasma. It is noted that in an identical plasma environment, a larger dust grain gains more charge. However, the ratio of the root mean square (rms) of charge fluctuations (charge standard deviation) to the charge mean decreases as the size of the grain increases. In fact, the system size expansion of the master equation is based on that both mean and variance of the grain charge are scaled by \( \Omega \) according to

FIG. 4. Probability density function of grain charge at \( \hat{t} = 10 \) (circle symbols and dashed lines) and the stationary state (plus symbols and solid lines) for a grain radius of (a) 3 nm; and (b) 30 nm, obtained by solving the master equation (discrete points) and Gaussian solution (lines) through the system-size expansion method. Refer to the caption of Fig. 3 for the grain and plasma properties.
eq. \((8)\). It is seen in eq. \((23)\) that \(\Omega \propto R\) for Maxwellian distributions of plasma particles.

Figure 1 shows the probability density function of the grain charge in a plasma containing three components with Maxwellian distributions. Two grains with radius of 3 nm and 30 nm are considered and the PDFs are obtained by two models. In the first model, the master equation \((4)\) is numerically solved, using the form given in eq. \((6)\), and in the second model, the PDF is a Gaussian function whose mean and variance are determined by eqs. \((14)\) and \((15)\), respectively, via the system size expansion method. As can be seen, the discrepancy between the models for larger grain is negligible while it is more significant for smaller grain. The system size \(\Omega\) for 3 nm and 30 nm are calculated 3.59 and 35.9, respectively, so the system-size expansion approximation is more adequate for the larger grains.

### IV. CONCLUSIONS

Markovian description of charging of a dust grain suspended in a general multicomponent plasma containing electrons, negative and positive singly- or multiply-charged ions was done through the formulation of a master equation. An analytical solution is lacking for the master equation. A discrete stochastic model, based on the master equation, was proposed to simulate the time evolution of the dust grain charge. Moreover, a Fokker-Planck equation was derived from the master equation through the system-size expansion method of Van Kampen. The Fokker-Planck equation has an analytical Gaussian solution with a mean and variance governed by two differential equations valid at both stationary and non-stationary states. As a test problem, two grains with different sizes in a plasma containing electrons, protons and alpha particles with Maxwellian distributions were considered. The probability density functions of the grain charge were obtained by solving the master equation numerically and were compared against the Gaussian solutions obtained for the Fokker-Planck equation. There was very good agreement between two solutions. Results showed that the deviation from the Gaussian solution was more significant for the smaller grain. The system size, which is linearly correlated with the radius of the grain and used in the expansion of the master equation through the system-size expansion method, is smaller for the smaller grain. The approximation made in this method is less favorable for smaller system sizes.

### ACKNOWLEDGMENTS

The author acknowledges the support through the 2012 Junior Faculty Distinguished Research award by The University of Alabama in Huntsville.

---

**Appendix A: Rates of Attachment of Plasma Particles in Collisional Charging**

It is shown \((11, 30)\) that the rate of the attachment of the \(j\)-th component plasma particle to the grain in the collisional charging of the grain is correlated to the velocity distribution of the plasma particle \(f_j(v, \theta)\) by

\[
W_j(Z) = 2\pi \int_{v_0}^{\infty} dv \int_0^{\pi} d\theta \sigma_j(v, Z) f_j(v, \theta) v^3 \sin \theta, \quad (A1)
\]

where \(v\) is the thermal velocity of the impinging plasma particle, and \(v_0\) is the minimum velocity of the impinging plasma particle, which is \(v_0 = (\zeta_j Z^2/2\pi\epsilon_0 R m_j)^{1/2}\) if \(\zeta_j Z > 0\); otherwise, \(v_0 = 0\). Here, \(m_j\) is the mass of the \(j\)-th component particle and \(R\) is the radius of the grain. In equation \((A1)\), \(\sigma_j\) is the collisional cross section given by \(\sigma_j(v, Z) = \pi R^2 (1 - \zeta_j Z^2/2\pi\epsilon_0 R m_j v^2)\).

1. N. G. Van Kampen, Stochastic Processes in Physics and Chemistry (Elsevier Science Publishers, North Holland, Amsterdam, 2007).
2. V. E. Fortov, A. G. Khrapak, S. A. Khrapak, V. I. Molotkov, and O. F. Petrov, Physics-Uspekhi 47, 447 (2004).
3. O. S. Vaulina, S. A. Khrapak, A. P. Nefedov, and O. F. Petrov, Phys. Rev. E 60, 5959 (1999).
4. U. de Angelis, A. Ivlev, G. Morfill, and V. Tsytovich, Physics of Plasmas 12, 052301 (2005).
5. G. Norman, V. Stegailov, and A. Timofeev, Journal of Experimental and Theoretical Physics 113, 887 (2011).
6. G. Morfill, A. V. Ivlev, and J. R. Jokipii, Phys. Rev. Lett. 83, 971 (1999).
7. A. V. Ivlev, U. Konopka, and G. Morfill, Phys. Rev. E 62, 2739 (2000).
8. A. Mamun and P. Shukla, Plasma Science, IEEE Transactions on 30, 720 (2002).
9. Duha and A. Mamun, Physics Letters A 373, 1287 (2009).
10. H. Alinejad, Astrophysics and Space Science 327, 131 (2010).
11. H. Kimura and I. Mann, The Astrophysical Journal 499, 454 (1998).
12. H. Mann, Annual Review of Astronomy and Astrophysics 48, 173 (2010).
13. Mann and M. Hamrin, Ann. Geophys 31, 39 (2013).
14. V. Pines, M. Zlatkowski, and A. Chait, Advances in Space Research 45, 812 (2010).
15. V. Kharchenko and N. Lewkow, in Nanodust in the Solar System: Discoveries and Interpretations, Astrophysics and Space Science Library, Vol. 385, edited by I. Mann, N. Meyer-Vernet, and A. Czechowski (Springer Berlin Heidelberg, 2012) pp. 179–194.
16. V. Fortov, A. Nefedov, V. Vladimirov, L. Deputatova, A. Budnik, A. Khudyakov, and V. Rykov, Physics Letters A 284, 118 (2001).
17. R. Smirnov, A. Y. Pogarov, M. Rosenberg, S. Krasheninnikov, and D. Mendis, Plasma Physics and Controlled Fusion 49, 347 (2007).
18. S. Krasheninnikov, R. Smirnov, and D. Rudakov, Plasma Physics and Controlled Fusion 53, 083001 (2011).
19. C. Cui and J. Goree, IEEE Trans. Plasma Sci. 22, 151 (1994).
20. T. Matsoukas and M. Russell, J. Appl. Phys. 77, 4285 (1995).
21. T. Matsoukas, M. Russell, and M. Smith, J. Vac. Sci. Technol. A 14, 624 (1996).
22. T. Matsoukas and M. Russell, Phys. Rev. E 55, 991 (1997).
23. B. Shotorban, Phys. Rev. E 83, 066403 (2011).
24. S. Matthews, B. Shotorban, and T. W. Hyde, The Astrophysical Journal 776, 103 (2013).
25. A. Khrapak, A. P. Nefedov, O. F. Petrov, and O. S. Vaulina, Phys. Rev. E 59, 6011 (1999).
26. P. K. Shukla and B. Eliasson, Rev. Mod. Phys. 81, 25 (2009).
27. B. Shotorban, Physics of Plasmas 19, 053702 (2012).
28C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, New York, NY, 2004).

29D. Gillespie, Annu. Rev. Phys. Chem. 58, 35 (2007).

30E. Dwek and R. G. Arendt, Annual Review of Astronomy and Astrophysics 30, 11 (1992).