Phantom Accretion onto the Schwarzschild de-Sitter Black Hole *

M Sharif**, G Abbas

Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan

(Received 20 April 2011)

We deal with phantom energy accretion onto the Schwarzschild de-Sitter black hole. The energy flux conservation, relativistic Bernoulli equation and mass flux conservation equation are formulated to discuss the phantom accretion. We discuss the conditions for critical accretion. It is found that the mass of the black hole decreases due to phantom accretion. There exist two critical points which lie in the exterior of horizons (black hole and cosmological horizons). The results for the phantom energy accretion onto the Schwarzschild black hole can be recovered by taking $\Lambda \to 0$.

PACS: 04.70.Bw, 04.70.Dy, 95.35.+d DOI:10.1088/0256-307X/28/9/090402

Recent developments in observational cosmology reveal that our universe is in an accelerating phase. This was first confirmed by the data of type-Ia supernova and a large-scale structure, [1–4] Also, the anisotropies in cosmic microwave background (CMB) radiations as observed by WMAP,[5–7] favor the accelerating behavior of the universe. The exotic energy with negative pressure, known as dark energy (DE), is thought to be responsible for this behavior of the universe. Despite the observed facts, the nature of DE is still a challenging problem in theoretical physics.

Different models such as quintessence,[8] phantom,[9] tachyon field,[10] holographic[11] and braneworld[12] models were proposed to understand the nature of DE. The simplest form of DE is vacuum energy (the cosmological constant) for which the equation-of-state (EoS) parameter is $\omega = -1$. The quintessence (dynamical evolving scalar field with negative pressure) and phantom models are a hypothetical form of DE for which $\omega > -1$ and $\omega < -1$, respectively.[13–15] Phantom energy violates the dominant energy condition (i.e., $\rho + p < 0$, where $\rho$ is energy density and $p$ is pressure) which results in the existence of wormholes. The expansion of the universe is dominated by phantom energy, which diverges to approach the future singularity (the big rip). In this case, the phantom energy density $\rho \to \infty$ for $t < \infty$.

Bondi[16] investigated the problem of matter accretion onto compact objects in Newtonian gravity. Michel[17] studied the steady-state accretion of gas onto the Schwarzschild black hole (BH) in relativistic physics. Many researchers studied the accretion of different forms of fluid onto the BH. Babichev et al.[18] have shown that accretion of phantom energy onto the Schwarzschild BH diminishes the BH mass. Jamil et al.[19] have explored the effects of phantom accretion onto a charged BH. They pointed out that if the mass of the BH becomes smaller (due to accretion of phantom energy) than its charge, then the BH is converted to a naked singularity. This is a violation of the cosmic censorship hypothesis. The same conclusion was deduced by Babichev et al.[20] by studying phantom accretion onto a charged BH with the generalized linear EoS and a Chaplygin gas. Madrid et al.[21] explored that a Kerr-Newmann BH could be transformed to a naked singularity by the accretion of phantom energy.

In a recent study,[22] we have examined the phantom accretion by a 5D charged BH. The Schwarzschild de-Sitter (SdS) BH is a solution with vacuum energy (the cosmological constant) which helps to describe BH formation in the process of the birth of the universe.[23] This is important for the consideration of quantum effects near the BH in the universe models. Many authors,[24–26] have considered various cosmological phenomena in SdS spacetime. Also, our universe is in a phase of accelerated expansion due to a positive cosmological constant (DE) and might approach a de-Sitter phase for $t < \infty$.[4] Martian-Moruno et al.[27] studied DE accretion on the SdS BH with FRW background. They have found that the BH mass vanishes at big rip time. However, they do not explore the location of critical points of accretion.

In this Letter, we investigate the phantom accretion onto a static SdS BH by using the procedure of Jamil et al.[19] and discuss the locations of the critical points of accretion. Further, the relationships between critical points and horizons are found, which were not given by Martian-Moruno et al.[27] The gravitational units (i.e., the gravitational constant $G = 1$ and speed of light in vacuum $c = 1$) are used. All the Latin and Greek indices vary from 0 to 3, unless otherwise.

*Supported by the Higher Education Commission, Islamabad, Pakistan through the Indigenous Ph.D. 5000 Fellowship Program Batch-IV.

**Email: msharif.math@pu.edu.pk
© 2011 Chinese Physical Society and IOP Publishing Ltd

090402-1
We consider a static spherically symmetric SdS BH given by
\[ ds^2 = (1 - \frac{2m}{r} - \frac{r^2}{a^2})dt^2 - \frac{1}{(1 - \frac{2m}{r} - \frac{r^2}{a^2})}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]
(1)
where \( a = \sqrt{\frac{\Lambda}{3}} \), \( m \) and \( \Lambda \) are constants. This metric has essential singularity at \( r = 0 \), which is covered by the BH horizons. Such horizons can be found by solving \( g_{tt} = 1 - \frac{2m}{r} - \frac{r^2}{a^2} = 0 \) for \( r \) whose positive real roots will give horizons. Using the approach discussed in Ref. [23] for solving the cubic polynomial, we explore the solution in the following three cases.

\textbf{Case i:} For \( \frac{m}{a} < \frac{1}{\sqrt{27}} \), there are three real roots of which two are positive and one is negative (neglected). The positive and negative roots are given by
\[ r_{bh} = \frac{2a}{\sqrt{3}} \sin \varphi, \quad r_{ch} = a(\cos \varphi - \frac{1}{\sqrt{3}} \sin \varphi), \quad r_o = -(r_{bh} + r_{ch}), \]
(2)
where \( \sin 3\varphi = \sqrt{27} \frac{m}{a} \). The subscripts bh and ch in the above equation stand for the BH and cosmological horizons. When \( \Lambda \to 0 \) (\( a \to \infty \)), we obtain \( r_{bh} \to 2m \) (Schwarzschild horizon) and \( r_{ch} \to \infty \) (cosmological horizon does not exist). Also, for \( m \to 0 \), \( r_{bh} \to 0 \) and \( r_{ch} \to a \equiv \sqrt{\frac{3}{2}} \) (de-Sitter horizon). We would like to mention here that \( r_{ch} > r_{bh} \) for \( 0 \leq \varphi < \frac{\pi}{3} \), which implies that the exterior of the Schwarzschild BH is covered by the de-Sitter universe and \( \varphi = \frac{\pi}{6} \), \( r_{ch} = r_{bh} = \frac{a}{\sqrt{3}} \). For \( \frac{\pi}{3} < \varphi < \frac{2\pi}{3} \), we have \( r_{ch} < r_{bh} \), which implies that the de-Sitter spacetime is the interior structure of the Schwarzschild BH.

\textbf{Case ii:} When \( \frac{m}{a} > \frac{1}{\sqrt{27}} \), there are three real roots of which two are positive (repeated) and one is negative. Since the negative root is neglected, so the positive roots give unique horizons, i.e.,
\[ r = r_{bh} = r_{ch} = \frac{a}{\sqrt{3}} = \sqrt{\frac{1}{\Lambda}}. \]
(3)

\textbf{Case iii:} For \( \frac{m}{a} > \frac{1}{\sqrt{27}} \), there are two imaginary roots and one is a negative real root, hence no horizon exists in this case.

Now we consider the phantom energy in the form of a perfect fluid whose energy-momentum tensor is
\[ T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} - pg_{\mu\nu}, \]
(4)
where \( \rho \) is the energy density, \( p \) is the pressure and \( u^\mu = (u^t, u^r, 0, 0) \) is the four-vector velocity. It is mentioned here that \( u^\mu \) satisfies the normalization condition, i.e., \( u^\mu u_\mu = 1 \).

The relativistic Bernoulli energy conservation equation for accretion onto the SdS BH (using energy-momentum-tensor conservation) is given by
\[ r^2 u(p + p)(1 - \frac{2m}{r} - \frac{r^2}{a^2})\frac{1}{2} = C_0, \]
(5)
where \( C_0 \) is an integration constant and \( u^r = u < 0 \) for inward flow. Further, the energy flux equation can be derived by projecting the energy-momentum conservation law on the four-velocity, i.e., \( u_{\mu}T^\nu{}_{\mu\nu} = 0 \) for which Eq. (4) leads to
\[ r^2 u \exp \left[ \int_{\rho_{\infty}}^\rho \frac{dp'}{\rho' + p(p')} \right] = -C_1, \]
(6)
where \( C_1 > 0 \) is another integration constant which is related to the energy flux. Also, \( \rho \) and \( \rho_{\infty} \) are densities of the phantom energy at finite and infinite \( r \).

From Eqs. (5) and (6), we obtain
\[ (\rho + p)(1 - \frac{2m}{r} - \frac{r^2}{a^2} + u^2)\frac{1}{2} \exp \left[ -\int_{\rho_{\infty}}^\rho \frac{dp'}{\rho' + p(p')} \right] = C_2, \]
(7)
where \( C_2 = \frac{C_0 r^2}{C_1} = \rho_{\infty} + p(\rho_{\infty}). \)

The rate of change of BH mass due to fluid accretion onto it is given by \( \frac{\dot{m}}{C_1^2} \)
\[ \frac{\dot{m}}{C_1} = 4\pi r^2 T r_0. \]
(8)
Using Eqs. (6) and (7) in the above equation yields
\[ \frac{\dot{m}}{C_1} = 4\pi C_1[\rho_{\infty} + p(\rho_{\infty})]. \]
(9)
It is clear that \( \dot{m} < 0 \) if \( (\rho_{\infty} + p_{\infty}) < 0 \). Thus the accretion of phantom energy onto a BH causes the mass of BH to decrease. Moreover, one can solve Eq. (8) for \( m \) using the EoS \( \rho = k p \). Since all \( p \) and \( \rho \) violating dominant energy conditions must satisfy Eq. (9), hence it holds in general. Note that if in Eq. (9) the matter contributes to the sum \( (\rho_{\infty} + p_{\infty}) \) instead of phantom energy, then the accretion of matter would increase the mass of the BH. Since this is not the case for matter, there is a decrease of mass.

Now we analyze the critical points (points at which flow speed is equal to the speed of sound) during the accretion of fluid on the BH. The fluid falls onto the BH with increasing velocity along the particle trajectories. For any critical point \( r = r_c \), we have the following possibilities:\[ \frac{\dot{m}}{C_1} \]
1. \( u^2 = V_s^2 \) at \( r = r_c \), \( u^2 < V_s^2 \) for \( r > r_c \) and \( u^2 > V_s^2 \) for \( r < r_c \). When \( r \to \infty \), the flow speed is negligible, it is equal to the speed of sound at a critical value of \( r \) while it is supersonic inside a region interior to \( r_c \).
2. \( u^2 \geq V_s^2 \) for all \( r \) which are the non-realistic cases as they describe the supersonic and subsonic solution for all values of \( r \).
3. \( u^2 = V_s^2 \) for all the values of \( r < r_c \) or \( r > r_c \). It is also a non-physical case because it is impossible to have the same flow speed inside and outside the critical points.

Thus the solution is the only physical solution.
For the discussion of critical points of accretion onto a BH, we follow the procedure introduced by Michel.\cite{14} The conservation of mass flux, $J^\mu, \mu = 0$, yields

$$\rho u r^2 = k,$$

(10)

where $k$ is the constant of integration. Dividing and squaring Eqs. (5) and (10), we get

$$\left(\frac{\rho + p}{\rho}\right)^2 \left(1 - \frac{2m}{r} - \frac{r^2}{a^2} + u^2\right) = k_1,$$

(11)

where $k_1 = (\frac{C_0}{k})^2$ is a positive constant. Differentiating Eqs. (10) and (11) and eliminating $dp$, we get

$$\frac{dr}{r} \left(2V^2 - \frac{\frac{ma^2}{r} + \frac{r^2}{a^2}}{1 - \frac{2m}{r} - \frac{r^2}{a^2} + u^2}\right) + \frac{du}{u} \left(V^2 - \frac{u^2}{1 - \frac{2m}{r} - \frac{r^2}{a^2} + u^2}\right) = 0,$$

(12)

where $V^2 = \frac{4n(r+p)}{dln\rho} - 1$.

This equation shows that turn-around points (critical points) are located where both the square brackets vanish. Thus

$$u_e^2 = \frac{ma^2 + r_e^3}{2a^2 r_e}, \quad V_e^2 = \frac{ma^2 + r_e^3}{2a^2 r_e - 3ma^2 - r_e^3}.$$ 

(13)

We see that the physically acceptable solutions of the above equation are obtained if $u_e^2 > 0$ and $V_e^2 > 0$ implying that

$$2a^2 r_e - 3ma^2 - r_e^3 > 0,$$

(14)

$$ma^2 + r_e^3 > 0.$$ 

(15)

It is worthwhile to mention here that when $a \to \infty$, i.e., $A \to 0$, the above equations reduce to the results of accretion onto the Schwarzschild BH. The subscript $c$ is used to represent a quantity at a point where the speed of flow is equal to the speed of sound, such a point is called a critical point. The fluid that moves towards a BH hole initially has a speed less than the speed of sound but as it comes closer to BH horizons, its speed may transit to a supersonic level. The circular region around the BH where flow speed is equal to the speed of sound is called a sound horizon. The flow speed is supersonic (subsonic) inside (outside) the sound horizons. Inside the sound horizon $r < r_s$, the flow speed is supersonic but less than the speed of light, as fluid reaches the BH horizon, the flow speed approaches to the speed of light. After crossing the BH horizon, it becomes greater than the speed of light.

Now we discuss the EoS and its parameter for understanding the accretion of phantom energy onto the SdS BH. It is obvious that for polytropic EoS, $p = k \rho$ and $k < 0$, the speed of sound (i.e., $V_s^2 = \frac{dp}{d\rho}$) becomes meaningless and fluid acts as an exotic cosmic fluid that cannot be accreted onto a BH. This problem was solved by Babichev et al.\cite{15} They introduced a generalized linear EoS, i.e., $p = \alpha (r - \rho_0)$, where $\alpha$ and $\rho_0$ are constants and $\alpha > (<) 0$ corresponds to a stable (unstable) fluid. This EoS leads to $V_s^2 = \alpha$. The constant $\alpha$ is related to the EoS parameter $k$ by $k = \alpha (\frac{\rho}{r} - \rho_0)$. This implies that $k < 0$ corresponds to $\alpha > 0$ and $\rho < \rho_0$ and phantom energy acts as a hydrodynamically stable fluid. Thus it accretes onto the BH and diminishes its mass.

Using the procedure for solving the cubic equation mentioned in Ref.\cite{23}, Eq. (14) can be solved in the following cases.

Case 1: For $\frac{m}{a} < \frac{4\sqrt{2}}{9\sqrt{3}}$, there are three real roots of which two are positive and one is negative (neglected). The positive roots are given by

$$r_{c_1} = \frac{2a\sqrt{\frac{\pi}{3}} \sin \chi}{\sqrt{3}}, \quad r_{c_2} = \sqrt{2a} (\cos \chi - \frac{1}{\sqrt{3}} \sin \chi),$$

(16)

where $\sin 3\chi = \frac{m}{a} \left(\frac{1}{2}\right)^\frac{1}{3}$. For $0 < \chi < \frac{\pi}{6}$, we have $0 < r_{c_1} < r_{c_2}$, thus $r_{c_1}$ and $r_{c_2}$ are inner and outer critical points of flow. For $\chi = \frac{\pi}{6}$, both the critical points coincide, i.e., $r_{c_1} = r_{c_2} = r = \frac{2a}{\sqrt{3}}$, hence $r_c > r$. This is not physical and will be discussed in the following case 3. Further, taking $0 < \chi = \varphi < \frac{\pi}{6}$ and comparing Eqs. (2) and (16), we obtain a relationship between horizons and critical values of $r$, i.e., $r_{c_2} > r_{c_1} > r_{c_{ch}} > r_{bh} > 0$. This implies that curvature singularity at $r = 0$ is covered by different circular boundaries of radii $\tilde{r}(>0) = r_{bh} < r_{c_{ch}} < r_{c_1} < r_{c_2}$. We conclude from here that both the critical points lie outside the horizons. In order to obtain the critical points, we use the solution of Eq. (14), i.e., Eq. (16). Substituting Eq. (16) into Eq. (15) yields

$$\frac{m}{a} = \frac{16}{3} \frac{1}{\sqrt{3}} \sin \chi (\cos^2 \chi - 1),$$

(17)

$$\frac{m}{a} = \frac{2}{27} \sqrt{2} (\sqrt{3} \sin \chi - 3 \cos \chi).$$

(18)

Since $-1 \leq |\cos \chi| \leq 1$, Eq. (17) implies that $\frac{m}{a} \leq 0$, which is in contradiction, hence $r_{c_1}$ is not a physical accretion solution. Further, Eq. (18) gives $\frac{m}{a} > 0$ for $\chi > \frac{\pi}{6}$, thus $r_{c_2}$ is a possible critical accretion solution.

The above equations are valid, i.e., $\frac{m}{a} > 0$, so $r_{c_1}$ and $r_{c_2}$ are inner and outer critical points of flow.

Case 2: When $\frac{m}{a} > \frac{4\sqrt{2}}{9\sqrt{3}}$, there is only one negative real root and the other two are complex. Consequently, this case has no physical solution and hence no critical points.

Case 3: For $\frac{m}{a} = \frac{4\sqrt{2}}{9\sqrt{3}}$, there are two positive repeated roots and one is a negative root. The positive root is

$$r_{c_1} = r_{c_2} = \sqrt{\frac{2}{3}} a = r_c.$$ 

(19)
These roots are not physically critical points because applications of these roots in Eq. (15) lead to \( m_a < 0 \), which is in contradiction, hence this case is also discarded.

We have been devoted to studying phantom accretion onto the SdS BH. Using the energy flux conservation, the relativistic Bernoulli equation and the mass flux conservation equation, we formulate the equations of motions for a steady-state spherically symmetric phantom flow onto the SdS BH. The results reduce to the Schwarzschild BH case\(^{[17]}\) when \( a \to \infty \) (i.e., \( A \to 0 \)). The summary of the results is given as follows: (1) There are two horizons (BH and cosmological) and two critical points, such that \( r_{bh} < r_{ch} < r_c \). (2) The particular case \( r_{c1} = r_{c2} = r_c \) is not a physical critical point because it gives \( m_a < 0 \), which is in contradiction. (3) The solution \( r_{c2} \) provides a physical critical point because it yields \( m_a > 0 \) for \( \chi > \frac{\pi}{6} \). (4) Analytically, we can also determine (by using \( \chi > \frac{\pi}{6} \) in Eq. (16)) that \( r_{c2} > r_{c1} \), but physically it is impossible to reverse the solution, so this case is discarded. (5) A physically possible critical point always lies outside the horizons. This is according to the cases of accretion onto the Schwarzschild BH\(^{[17]}\) and charged BH\(^{[19]}\).

References

[1] Perlmutter S et al 1997 Astrophys. J. 483 565
[2] Perlmutter S et al 1998 Nature 391 51
[3] Perlmutter S et al 1999 Astrophys. J. 517 565
[4] Riess A G et al 1998 Astron. J. 116 1009
[5] Bennett C L et al 2003 Astrophys. J. Suppl. 148 1
[6] Spergel D N et al 2003 Astrophys. J. Suppl. 148 175
[7] Verde L et al 2002 Mon. Not. R. Astron. Soc. 335 432
[8] Sahni V and Starobinsky A A 2000 Int. J. Mod. Phys. D 9 373
[9] Caldwell R R 2002 Phys. Lett. B 23 545
[10] Sen A J 2002 High Energy Phys. 48 204
[11] Wang B, Gong Y G and Abdalla E 2005 Phys. Lett. B 624 141
[12] Li M 2004 Phys. Lett. B 603 1
[13] Frieman J A et al 1995 Phys. Rev. Lett. 75 2077
[14] Coble K, Dodelson S and Frieman J A 1997 Phys. Rev. D 55 1851
[15] Haines H et al 2009 Astrophys. J. Suppl. 180 225
[16] Bondi H 1952 Mon. Not. Roy. Astron. Soc. 112 195
[17] Michel F C 1972 Astrophys. Space Sci. 15 153
[18] Babichev E, Dokuchaev V and Eroshenko Y 2004 Phys. Rev. Lett. 93 021102
[19] Jamil M, Rashid M and Qadir A 2008 Eur. Phys. J. C 58 325
[20] Babichev E, Dokuchaev V and Eroshenko Y J 2011 Exp. Theor. Phys. 112 793
[21] Madrid J A Jimenez and Gonzalez-Dias P F 2008 Gravit. Cosmol. 14 213
[22] Sharif M and Abbas G 2011 Mod. Phys. Lett. A 26 1731
[23] Sung-Won K 1991 J. Korean Phys. Soc. 24 118
[24] Gibbons G W and Hawking S W 1997 Phys. Rev. D 55 1851
[25] Khan U and Panchapakesan N 1981 Phys. Rev. D 24 829
[26] Nagai H 1972 Prog. Theor. Phys. 82 322
[27] Martin-Moruno P et al 2009 Gen. Relativ. Gravit. 41 2797
[28] Padmanabhan T 2000 Theoretical Astrophysics: Astrophysical Processes (Cambridge: Cambridge University) vol 1