Phase transition for percolation on randomly stretched lattice

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Abstract

Let $\xi_i \geq 1$ be a sequence of i.i.d. random variables. Starting from the usual square lattice replace each edge that links a site in $i$-th column to a site in the $(i+1)$-th column by a long edge having length $\xi_i$. Then declare independently each edge $e$ in the resulting lattice open with probability $p_e = p|e|$ where $p \in [0, 1]$ and $|e|$ is the length of $e$. We relate the occurrence of nontrivial phase transition for this model to moment properties of $\xi_1$. More precisely, we prove that the model undergoes a nontrivial phase transition when $E(\xi_1^\eta) < \infty$, for some $\eta > 1$ whereas, when $E(\xi_1^\eta) = \infty$ for some $\eta < 1$, no phase transition occurs.

1 Introduction

In this paper we discuss a model of percolation on a generalized version of $\mathbb{Z}^2_+$. Such generalization is described below. Fix $\Lambda = \{x_0, x_1, \cdots \} \subseteq \mathbb{R}$ an increasing sequence called environment and define the lattice $\mathcal{L}_\Lambda = (V(\mathcal{L}_\Lambda), E(\mathcal{L}_\Lambda))$ as

$$
V(\mathcal{L}_\Lambda) := \Lambda \times \mathbb{Z}_+ = \{(x, y) \in \mathbb{R}^2; x \in \Lambda, y \in \mathbb{Z}_+\} \quad \text{and} \quad
E(\mathcal{L}_\Lambda) := \left\{\{(x_i, n), (x_j, m)\} \subseteq V(\mathcal{L}_\Lambda); |i-j| + |n-m| = 1\right\}.
$$

Roughly speaking, $\mathcal{L}_\Lambda$ can be regarded as the lattice obtained from $\mathbb{Z}^2_+$ when the horizontal edges of $\mathbb{Z}^2_+$ are replaced by edges whose lengths are given by the difference of two consecutive elements in $\Lambda$ as illustrated in Figure 1.

For each $p \in [0, 1]$, denote $\mathbb{P}_p^\Lambda(\cdot)$ the probability measure on $\{0, 1\}^{E(\mathcal{L}_\Lambda)}$ under which the random variables $\{\omega(e)\}_{e \in E(\mathcal{L}_\Lambda)}$ are independent Bernoulli random variables with mean $p_e = p|e|$, where, for each edge $e := \{v_1, v_2\} \in E(\mathcal{L}_\Lambda)$, $|e| = \|v_1 - v_2\|$ denotes the Euclidean length of $e$. 

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We wish to allow for random environments $\Lambda$. For that we will assume that $\Lambda$ is distributed according to a renewal process as we describe next. Let $\xi$ be a positive random variable, and $\{\xi_i\}_{i \in \mathbb{Z}^+}$ be i.i.d. copies of $\xi$. Set

$$\Lambda = \left\{ \sum_{1 \leq i \leq k} \xi_i; \ k \in \mathbb{Z}^+ \right\} = \left\{ x_i \in \mathbb{R}; x_0 = 0 \text{ and } x_k = x_{k-1} + \xi_k \text{ for } k \in \mathbb{Z}^+ \right\},$$

which is called an renewal process with interarrival distribution $\xi$. We denote $\nu_\xi(\cdot)$ the law of this renewal process. An overview of renewal processes will be provided in Section 5.

We are now ready to state our main results which relate the occurrence of a phase transition with moment properties of $\xi$.

**Theorem 1.1.** Let $\xi$ be a positive random variable with $\mathbb{E}(\xi^\eta) < \infty$ for some $\eta > 1$. Then there is $p < 1$ such that

$$\mathbb{P}_p^\Lambda(o \leftrightarrow \infty) > 0, \text{ for } \nu_\xi\text{-almost every environment } \Lambda.$$

Also there is $p > 0$ such that

$$\mathbb{P}_p^\Lambda(o \leftrightarrow \infty) = 0, \text{ for } \nu_\xi\text{-almost every environment } \Lambda.$$

**Theorem 1.2.** Let $\xi$ be a positive random variable with $\mathbb{E}(\xi^\eta) = \infty$ for some $\eta < 1$. Then for any $p \in [0, \infty)$,

$$\mathbb{P}_p^\Lambda(o \leftrightarrow \infty) = 0, \text{ for } \nu_\xi\text{-almost every environment } \Lambda.$$

Theorem 1.2 rules out the occurrence of a nontrivial phase transition when the increments of the renewal process have sufficiently heavy tails. Not very surprisingly, this phenomenon stems from the fact that the consecutive columns are typically located very far apart. Its proof is presented in Section 4 and consists of an application of the Borel-Cantelli Lemma.

Theorem 1.1 states that the model undergoes a non-trivial phase transition when the tails are sufficiently light. Its proof is more intricate and relies on the control of the environment via a multiscale analysis. It is reminiscent of the main result in [4] where the authors study the survival of a contact process in random environments. Indeed the techniques developed therein, when translated to our context, seem only to apply to the case when the $\xi$’s have geometric distribution.

An interesting problem is to determine whether the phase transition occurs when the lattice is stretched both horizontally and vertically. This was done in [10] in the case where $\xi$ has geometric distribution. We are currently unable to tackle this problem when the stretching is made according to general renewal processes.
An alternative formulation of the percolation model defined above is the percolation on $\mathbb{Z}_+^2$ where, conditioned in $\xi_1, \xi_2, \ldots$, each edge $e \in E(\mathbb{Z}_+^2)$ is declared open independently with probability

$$
p_e = \begin{cases} 
p, & \text{if } e = \{(i, j), (i, j + 1)\} \\
p^{\xi_{i+1}}, & \text{if } e = \{(i, j), (i + 1, j)\} \end{cases}.
$$

(1)

The random variables $\xi_i$’s indicate how distant the columns of the stretched lattice lie from one another in the original formulation. In the alternate formulation, conditioned on $\xi_1, \xi_2, \ldots$, the resulting bond percolation process in $\mathbb{Z}_+^2$ is inhomogeneous, unless the distribution of $\xi$ is concentrated on 1.

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**Figure 1**: Illustration on the left of the lattice $\mathcal{L}_\Lambda$. On the right, the alternative formulation on $\mathbb{Z}_+^2$. Note that the environment $\Lambda$ can be seen as a sequence of $x_i$’s or as a sequence of $\xi_i$, since $\xi_i = x_i - x_{i-1}$.

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We finish this section presenting an overview of the paper. As already mentioned, Theorem 1.2 is proved in Section 4. Section 5 contains a brief review on renewal processes and some results that will be used for proving a decoupling inequality which is crucial in the proof of Theorem 1.1. In Section 6, we develop the multiscale scheme to be used to prove Theorem 1.1. First, in Section 6.1, we define a fast-growing sequence of numbers which correspond to the scales in which we analyze the model. Then we partition $\mathbb{Z}_+$ into the so-called blocks which are intervals whose length are related to the $k$-th scale. A block at scale $k$ will be labeled either bad or good hierarchically depending on whether or not the renewal process within it has arrivals that are close to each other. We will show that bad blocks are extremely rare. Section 6.2 is destined to the construction of crossing events in rectangles that extend very far vertically and whose basis project to blocks. Such crossings will occur with very high probability on good blocks. In Section 7, we finish the proof of Theorem 1.1 putting together the fact that the blocks are most likely good and that good blocks are easy to cross.
2 Notation

In this section we provide the notation that will be used in the following sections.

Consider the first quadrant of the square lattice $\mathbb{Z}_+^2 = (V(\mathbb{Z}_+^2), E(\mathbb{Z}_+^2))$ defined by
\[
V(\mathbb{Z}_+^2) = \{v = (v_1, v_2) \in \mathbb{R}^2; v_1, v_2 \in \mathbb{Z}_+\},
\]
\[
E(\mathbb{Z}_+^2) = \{\{v, w\} \subseteq V(\mathbb{Z}_+^2); |v_1 - w_1| + |v_2 - w_2| = 1\}.
\]

When there is no risk of ambiguity, we abuse notation and do not distinguish between $V(\mathbb{Z}_+^2)$ and $\mathbb{Z}_+^2$, and similarly for other graphs. Denote the origin of the lattice $\mathbb{Z}_+^2$ by $o = (0, 0)$. We write $v \sim w$ if $v$ is a neighbor of $w$, i.e. $\{v, w\} \in E(G)$.

A path in $A \subseteq \mathbb{Z}_+^2$ is a sequence of sites $v_0 \sim v_1 \sim \cdots \sim v_n$ such that $v_i \in A$ for all $i$.

Given a percolation configuration $\omega \in \{0, 1\}^{\mathbb{Z}_+^2}$, an edge $e \in E(\mathbb{Z}_+^2)$ is said to be open if $\omega(e) = 1$, otherwise it is said to be closed. For two sites $v, w \in \mathbb{Z}_+^2$, $v$ and $w$ are connected (denoted $v \leftrightarrow w$) if there exists a sequence $v = v_0 \sim v_1 \sim \cdots \sim v_n = w$ such that $\omega(\{v_i, v_{i+1}\}) = 1$ for every $0 \leq i < n$. The cluster of a site $v$ is the set of all sites $w$ such that $v \leftrightarrow w$, and we denote by $\{v \leftrightarrow \infty\}$ the event where the cluster of $v$ has infinite cardinality.

Let $a, b, c, d \in \mathbb{Z}_+$ with $a < b$ and $c < d$. Denote by
\[
R = R([a, b) \times [c, d))
\]
the subgraph of $\mathbb{Z}_+^2$ defined by
\[
V(R) = [a, b) \times [c, d] \quad \text{and} \quad E(R) = \{(x, y), (x + i, y + 1 - i); (x, y) \in [a, b - 1] \times [c, d - 1], i \in \{0, 1\}\},
\]
where $[a, b)$ denotes the set of all integers between $a$ and $b$, including them both. Note that $R$ is the rectangle $[a, b) \times [c, d]$ removed the right and top sides. See Figure 2. The horizontal and vertical crossing events in $R$ are defined respectively as
\[
C_h(R) = \{\{a\} \times [c, d] \leftrightarrow \{b\} \times [c, d] \text{ in } R\},
\]
\[
C_v(R) = \{[a, b) \times \{c\} \leftrightarrow [a, b) \times \{d\} \text{ in } R\},
\]
where $\{A \leftrightarrow B \text{ in } R\}$ indicates that there are sites $v \in A$ and $w \in B$ connected on the subgraph $R$.

In the Bernoulli bond percolation model on $\mathbb{Z}^2$, the $\omega(e)$’s are independent Bernoulli random variables with mean $p_e$. In the homogeneous case ($p_e = p \in [0, 1]$ for every $e \in E(\mathbb{Z}^2)$), let $\mathbb{P}_p$ be the probability measure corresponding. Denote by $p_c(\mathbb{Z}^2)$ the critical point of the homogeneous case, i.e.
\[
p_c(\mathbb{Z}^2) = \sup\{p \in [0, 1]; \mathbb{P}_p(o \leftrightarrow \infty) = 0\}.
\]
Kesten [12] proved that $p_c(\mathbb{Z}^2) = 1/2$. 

3 Related works

One of the main motivations for this work is the general question:

How does the introduction of inhomogeneities or dependencies along columns affect the nature of the phase transition or shift the critical point?

This type of question was posed in a number of different situations. Below we provide a brief overview of results obtained on this problem.

In several percolation models, inhomogeneities arise by introducing an environment which specifies how to assign weights $p_e$ for each edge $e$ of the graph. For instance, for the square lattice, $\mathbb{Z}^2$, one way to introduce inhomogeneities is by fixing columns (the environment) whose edges will have a probability $p$ of being open, while the other edges will be open with probability $q$. Formally let $\Lambda \subseteq \mathbb{Z}$, and set

$$E_{\text{vert}}(\Lambda) := \{((x, y), (x, y + 1)); \; x \in \Lambda, y \in \mathbb{Z}\},$$

the set of vertical edges in the columns that project to $\Lambda$. Let $\mathbb{P}^\Lambda_{p,q}(\cdot)$ be the distribution in $\{0, 1\}^{E(\mathbb{Z}^2)}$ of the percolation model whose edges are open independently with probability

$$p_e = \begin{cases} p, & \text{if } e \in E_{\text{vert}}(\Lambda), \\ q, & \text{if } e \notin E_{\text{vert}}(\Lambda). \end{cases}$$

In one extreme, one can consider $\Lambda = \{0\}$. It follows from the results on percolation in half-spaces of Barsky, Grimmett and Newman [3] that $\mathbb{P}^0_{p,q}(0 \leftrightarrow \infty) > 0$ whenever $q > p_c(\mathbb{Z}^2)$. In [20], Zhang nicely explores the ideas in [8] of constructing dual circuits around the origin, together with the RSW [17, 19] techniques, in order to prove that $\mathbb{P}^0_{p,q}(0 \leftrightarrow \infty) = 0$ for any $p \in [0, 1)$ and $q \leq p_c(\mathbb{Z}^2) = 1/2.$
In another extreme, for the case \( \Lambda = \mathbb{Z} \), it is known that \( P_{p,q}^\mathbb{Z}(o \leftrightarrow \infty) > 0 \) iff \( p + q > 1 \) (see page 54 in [13] or Section 11.9 in [7]).

Suppose now that \( \Lambda \) is such that, for some positive integer \( k \), every \( l \in \mathbb{Z} \), \( \Lambda \cap [l, l+k] \neq \emptyset \). A classical argument due to Aizenman and Grimmett [2] can be employed in order to show that for any \( \varepsilon > 0 \) there is \( \delta = \delta(k, \varepsilon) > 0 \) such that \( P_{p_c+\varepsilon, p_c-\delta}^\Lambda(o \leftrightarrow \infty) > 0 \), where \( p_c = p_c(\mathbb{Z}^2) \).

We now consider models for which the environment is random. Let \( \nu_\rho \) be the probability measure on \( \mathbb{Z} \) under which \( \{i \in \Lambda\} \) are independent events having probability \( \rho \). In [5] the authors have shown that for any \( \varepsilon > 0 \) and \( \rho > 0 \) there is \( \delta = \delta(\rho, \varepsilon) > 0 \) such that \( P_{p_c+\varepsilon, p_c-\delta}^\Lambda(o \leftrightarrow \infty) > 0 \) for \( \nu_\rho \)-almost every \( \Lambda \).

Using the arguments in Bramson, Durrett and Schonmann [4] one can prove that for any \( \rho \in [0, 1) \), there is \( p < 1 \) large enough such that \( P_{0,p}^\Lambda(o \leftrightarrow \infty) > 0 \) for \( \nu_\rho \)-almost every environment \( \Lambda \). This means that, even deleting edges along vertical columns of the square lattice according to Bernoulli trials, bond percolation on the remaining lattice still exhibits a phase transition. In [10], Hoffman studies the case where both rows and columns are deleted independently. He proves that a non-trivial phase transition still takes place.

In another variation, Kesten, Sidoravicius and Vares [14] considered a site percolation model on the square lattice with the edges oriented in the NE and SE directions. They open sites with a probability that is above or below the critical point depending on whether the site lies on \( \Lambda \) or not, respectively and prove that the model percolates if the density of \( \Lambda \) is sufficiently high.

Another related model is the Bernoulli line percolation on \( \mathbb{Z}^d \), \( d \geq 3 \), introduced in the physics literature by Kantor [11] and studied in [6, 9, 18]. It consists of studying the vacant set left by a set of lines parallel to the coordinate axes at random, selected according independent Bernoulli trials. In [9], the existence of a phase transition and the connectivity decay are studied. The presence of power-law decay within the subcritical phase and throughout all the supercritical phase contrasts sharply with the behavior of models with finite-range dependencies where the decay is exponential, see [16, 1].

4 Proof of Theorem 1.2

Proof of Theorem 1.2. Let \( \eta < 1 \) be such that \( \mathbb{E}(\xi^\eta) = \infty \) and set \( \varepsilon > 0 \) such that \( \eta^{-1} = 1 + 2\varepsilon \).

\[
\sum_{n=0}^\infty \mathbb{P}(\xi > n^{1+2\varepsilon}) = \sum_{n=0}^\infty \mathbb{P}(\xi^\eta > n) = \infty. \tag{5}
\]

Consider the events \( F_i = \{\xi_i \geq i^{1+2\varepsilon}\} \), with \( i \in \mathbb{Z}_+^* \) and recall that \( \{\xi_i\}_{i \in \mathbb{Z}_+^*} \) are independent copies of \( \xi \). Since the \( F_i \) are independent and \( \sum_i \mathbb{P}(F_i) = \infty \), we have
by Borel-Cantelli Lemma that $\nu_\xi(F_i \text{ i.o.}) = 1$.

Let us now fix $\Lambda \in \{F_i \text{ i.o.}\}$ and $p \in (0, 1)$. It is sufficient to show that $\mathbb{P}_p(o \leftrightarrow \infty) = 0$. Pick an increasing subsequence $i_k = i_k(\Lambda), k \in \mathbb{Z}_+$ such that $F_{i_k}$ occurs for every $k$. Roughly speaking, as we will see below, the vertical columns of $\mathcal{L}_\Lambda$ that project to $x_{i_k-1}$ and $x_{i_k}$ are too distant from each other to allow for paths to connect between them. We choose however, to use the equivalent formulation of the percolation model on $\mathbb{Z}_+^2$ whose parameters are given by $p_\nu$ in (1).

Recall the notation of the horizontal and vertical crossing events in rectangles introduced in (2), (3) and (4). For each $k \in \mathbb{Z}_+$ let

$$R_k = R\left(\left[0,i_k\right) \times \left[0, \lceil \exp(i_k^{1+\varepsilon}) \rceil\right)\right),$$

and note that

$$\mathbb{P}_p(o \leftrightarrow \infty) \leq \mathbb{P}_p^A(C_h(R_k)) + \mathbb{P}_p^A(C_v(R_k)). \quad (6)$$

The probability of $C_h(R_k)$ is bounded above by the probability that there is an open edge between the columns $\{i_k - 1\} \times \mathbb{Z}_+$ and $\{i_k\} \times \mathbb{Z}_+$. Since $\xi_{i_k}$ is very large, the height of $R_k$ is not large enough to ensure the existence of an open edge with good probability. In fact, let

$$J_k = \{0, 1, \cdots, \lceil \exp(i_k^{1+\varepsilon}) \rceil - 1\}$$

and note that

$$\mathbb{P}_p^A(C_h(R_k)) \leq \mathbb{P}_p^A\left(\bigcup_{j \in J_k} \{(i_k - 1, j), (i_k, j)\text{ is open}\}\right) \leq \lceil \exp(i_k^{1+\varepsilon}) \rceil p^{i_k} \leq \lceil \exp(i_k^{1+\varepsilon}) \rceil \exp(ln p \cdot i_k^{1+2\varepsilon}) \xrightarrow{k \to \infty} 0, \quad (7)$$

where we used the definition of $i_k(\Lambda)$ in the last inequality.

In order to bound the probability of $C_v(R_k)$, we note that, on this event there must be at least one vertical edge connecting the $j$-th and $(j+1)$-th row in $R_k$, for every $j \in J_k$. The height of $R_k$ is large enough to guarantee that this event has vanishing probability as $k$ grows. Note that

$$\mathbb{P}_p^A(C_v(R_k)) \leq \mathbb{P}_p^A\left(\bigcap_{j \in J_k} \{\{(l, j), (l, j + 1)\text{ is open}\}\right) = \left(1 - (1 - p)^{i_k}\right)^{|J_k|} \leq \exp\left(- \exp(ln(1 - p)i_k)\right) \cdot |J_k| \leq \exp\left(- \exp(ln(1 - p)i_k + i_k^{1+\varepsilon})\right) \xrightarrow{k \to \infty} 0. \quad (8)$$
In the second inequality sign above we used \(1 - x \leq \exp(-x)\).

Combining (6), (7) and (8), we get \(P^A_p(o \leftrightarrow \infty) = 0\), which concludes the proof. \(\square\)

## 5 Renewal processes

The purpose of this section is to prove a decoupling inequality (Lemma 5.1), which will be used as a fundamental tool in our multiscale analysis in Section 6.1. We start presenting a brief outline of some results on renewal processes.

### 5.1 Definition and notation

Let \(\xi\) and \(\chi\) be integer-valued random variables called *interarrival time* and *delay*, respectively. We assume that \(\xi \geq 1\) and \(\chi \geq 0\) a.s. Let \(\{\xi_i\}_{i \in \mathbb{Z}^*_+}\) be i.i.d. copies of \(\xi\), also independent of \(\chi\). We define the *renewal process*

\[
X = X(\xi, \chi) = \{X_i\}_{i \in \mathbb{Z}^*_+}
\]

recursively as:

\[
X_0 = \chi, \quad \text{and} \quad X_i = X_{i-1} + \xi_i \quad \text{for } i \in \mathbb{Z}^*_+.
\]

We say that the \(i\)-th renewal occurs at time \(t\) if \(X_{i-1} = t\). The law of \(X\) regarded as a random element on a probability space supporting \(\chi\) and the i.i.d. copies of \(\xi\) will be denoted by \(\nu^\chi_\xi\).

It is convenient to define two other processes

\[
Y = Y(\xi, \chi) = \{Y_n\}_{n \in \mathbb{Z}^*_+} \quad \text{and} \quad Z = Z(\xi, \chi) = \{Z_n\}_{n \in \mathbb{Z}^*_+},
\]

as

\[
Y_n = \begin{cases} 
1, & \text{if a renewal of } X \text{ occurs at time } n, \\
0, & \text{otherwise.}
\end{cases} \quad (9)
\]

and

\[
Z_n = \min\{X_i - n; \ i \in \mathbb{Z}^*_+ \text{ and } X_i - n \geq 0\}. \quad (10)
\]

Since each one of the processes \(X\), \(Y\) and \(Z\) fully determines the two others (see Figure 3), \(Y\) and \(Z\) will also be called renewal process with interarrival time \(\xi\) and delay \(\chi\). We abuse notation and write \(\nu^\chi_\xi\) for the law of \(Y\) and \(Z\). It is worth noting that \(Z\) is a Markov chain.
Figure 3: Illustration of the processes $X, Y, Z$. In this realization, $\chi = 1$, $\xi_1 = 2$, $\xi_2 = 5$, $\xi_3 = 3$ and $\xi_4 = 1$.

For $m \in \mathbb{Z}_+$ consider $\theta_m : \mathbb{Z}^\infty \mapsto \mathbb{Z}^\infty$, the shift operator given by

$$\theta_m(x_0, x_1, \cdots) = (x_m, x_{m+1}, \cdots).$$

It is desirable that $Z$ be invariant under shifts, i.e.

$$\theta_m Z \overset{d}{=} Z \quad \text{for any } m \in \mathbb{Z}_+. \quad (11)$$

If $\mathbb{E}(\xi) < \infty$, we can define a random variable $\rho = \rho(\xi)$ with distribution

$$\rho_k = \mathbb{P}(\rho = k) := \frac{1}{\mathbb{E}(\xi)} \sum_{i=k+1}^{\infty} \mathbb{P}(\xi = i), \text{ for any } k \in \mathbb{Z}_+. \quad (12)$$

Using $\rho$ as the delay, yields a Markov process $Z(\xi, \rho)$ satisfying $(11)$. For this reason, the random variable $\rho$ with distribution given by $(12)$ is called stationary delay. In particular,

$$Z_n \overset{d}{=} Z_0 \overset{d}{=} \rho. \quad (13)$$

Also note that

$$\mathbb{E}(\xi^{1+\varepsilon}) < \infty \Rightarrow \mathbb{E}(\rho^{\varepsilon}) < \infty. \quad (14)$$

We say that a random variable $\xi$ is aperiodic if

$$\gcd \{ k \in \mathbb{Z}_+^* ; \ P(\xi = k) > 0 \} = 1.$$ 

For the rest of this section we assume $\xi$ aperiodic.

Let $X = X(\xi, \chi)$ and $X' = X(\xi, \chi')$ be two independent renewal processes with interarrival time $\xi$ and delays $\chi$ and $\chi'$ respectively, and denote $v_{\xi}^{\chi,\chi'}(\cdot)$ the product measure $v_{\xi}^\chi \otimes v_{\xi}^{\chi'}$. Recall the definition of $Y, Y'$ in $(9)$ and define

$$T := \min\{ k \in \mathbb{Z}_+^* ; Y_k = Y'_k = 1 \},$$

the coupling time of $X$ and $X'$. 

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5.2 Decoupling

The goal of this section is to prove a decoupling inequality for stationary renewals. In order to do so, we will bound \( v_{\xi}^{x,x'}(T > n) \) above applying Markov’s inequality to \( T^\varepsilon \), where \( \varepsilon > 0 \). The next theorem whose proof was taken from [15] (Theorem 4.2, page 27), guarantees that \( T^\varepsilon \) has finite expectation. We write \( \mathbb{E}_{\xi}^{x,x'}(\cdot) \) for the expectation with respect to \( v_{\xi}^{x,x'} \).

**Theorem 5.1** (Theorem 4.2 in [15]). Let \( \xi \) be an aperiodic positive integer-valued random variable. Suppose that for some \( \varepsilon \in (0,1) \), \( \mathbb{E}(\xi^{1+\varepsilon}) < \infty \), and that \( \chi, \chi' \) are non-negative integer-valued random variables with \( \mathbb{E}(\chi^\varepsilon) \) and \( \mathbb{E}(\chi'^\varepsilon) \) finite. Then \( \mathbb{E}_{\xi}^{\chi,\chi'}(T^\varepsilon) < \infty \).

We can now prove the decoupling inequality for renewal processes.

**Lemma 5.1.** Let \( \xi \) be a positive integer-valued, aperiodic random variable with \( \mathbb{E}(\xi^{1+\varepsilon}) < \infty \), for some \( \varepsilon > 0 \), and consider the renewal process \( Y = Y(\xi, \rho(\xi)) \) defined in [19]. Then there exists \( c_1 = c_1(\xi, \varepsilon) \in (0, \infty) \) such that for all \( n, m \in \mathbb{Z}_+ \) and for every pair of events \( A \) and \( B \), with

\[
A \in \sigma(Y_i; 0 \leq i \leq m) \quad \text{and} \quad B \in \sigma(Y_i, i \geq m + n)
\]

we have

\[
v_{\xi}^\rho(A \cap B) \leq v_{\xi}^\rho(A)v_{\xi}^\rho(B) + c_1 n^{-\varepsilon}. \tag{15}
\]

**Proof.** For simplicity, \( \xi \) will be omitted in \( v_{\xi}^\rho \). If \( v_{\rho}(A) = 0 \) there is nothing to be proved. Suppose then that \( v_{\rho}(A) > 0 \). Recall the definition of \( Z \) in (10), and note that

\[
v_{\rho}(A \cap B) = v_{\rho}(A \cap B \cap \{Z_m > n/2\}) + v_{\rho}(A \cap B \cap \{Z_m \leq n/2\})
\]

\[
\leq v_{\rho}(Z_m > n/2) + v_{\rho}(A)v_{\rho}(B \cap \{Z_m \leq n/2\})|A|
\]

\[
\leq v_{\rho}(Z_m > n/2) + v_{\rho}(A) \sum_{0 \leq i \leq [n/2]} v_{\rho}(Z_m = i|A)\max_{0 \leq j \leq [n/2]} v_{\rho}^{\delta_{m+j}}(B)
\]

\[
\leq v_{\rho}(Z_m > n/2) + v_{\rho}(A) \max_{0 \leq j \leq [n/2]} v_{\rho}^{\delta_{m+j}}(B). \tag{16}
\]

Now we compare \( v_{\rho}^{\delta_{m+j}}(B) \) with \( v_{\rho}^{\rho}(B) \), when \( 0 \leq j \leq [n/2] \). Using that

\[
v_{\rho}^{\delta_{m+j}}(B) = v_{\rho}^{\delta_{0}}(\theta_{m+j}(B)) \quad \text{by the stationarity of } \rho
\]

and by the bound

\[
|v_{\rho}^{\delta_{m+j}}(B) - v_{\rho}^{\rho}(B)| = |v_{\rho}^{\delta_{0}}(\theta_{m+j}(B)) - v_{\rho}^{\rho}(\theta_{m+j}(B))|
\]

\[
\leq v_{\rho}^{\delta_{n}}(T > n - j)
\]

\[
\leq v_{\rho}^{\delta_{n}}(T > n/2). \tag{17}
\]

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By (16), (17) and by the fact $Z_m \overset{d}{=} Z_0 \overset{d}{=} \rho$

$$v^\rho(A \cap B) \leq v^\rho(A)v^\rho(B) + v^\rho(\rho > n/2) + v^\delta_{\rho}(T > n/2)$$

$$\leq v^\rho(A)v^\rho(B) + 2\varepsilon E(\rho^\varepsilon)n^{-\varepsilon} + 2\varepsilon E^\delta_{\rho}(T^\varepsilon)n^{-\varepsilon},$$

where the last inequality follows from the Markov inequality for $\rho^\varepsilon$ and $T^\varepsilon$. Finally take $c_1 = 2\varepsilon E(\rho^\varepsilon) + 2\varepsilon E^\delta_{\rho}(T^\varepsilon)$ which is finite by (14) and Theorem 5.1.

6 The multiscale scheme

Throughout this section we fix $\xi$ positive, integer-valued and aperiodic. We also assume $E(\xi^{1+\varepsilon}) < \infty$ for some $\varepsilon > 0$ and denote $\rho = \rho(\xi)$ the respective stationary delay given in (12). These conditions on $\xi$ allow us to redefine the percolation model on horizontally stretched square lattice to obtain an equivalent model on $\mathbb{Z}_2^+$ as follows:

Consider the environment $\Lambda \subseteq \mathbb{Z}_+$ distributed as $v_\xi$ and set

$$E_{\text{vert}}(\Lambda^c) := \{(x, y), (x, y + 1) \in E(\mathbb{Z}_2^+); x \not\in \Lambda, y \in \mathbb{Z}_+\}.$$

Let each edge $e \in E(\mathbb{Z}_2^+)$ be open independently with probability

$$p_e = \begin{cases} 0, & \text{if } e \in E_{\text{vert}}(\Lambda^c), \\ p, & \text{if } e \not\in E_{\text{vert}}(\Lambda^c). \end{cases}$$

Edges which are not open are called closed.

Geometrically, this formulation consists in preserving the columns of the $\mathbb{Z}_2^+$ lattice that project to $\Lambda$ while deleting the ones that project to $\Lambda^c$. The resulting graph is similar to the stretched lattice $L_\Lambda$ defined in Section 1, however, the edges are now split into unit length segments. Each one of these edges is open independently with probability $p$.

Note that, we can recover the original formulation on $L_\Lambda$ by declaring an edge open if all the corresponding unitary edges in $\mathbb{Z}_2^+$ are open in the new formulation. Therefore, these two formulations are equivalent and we slightly abusing notation, denoting $P^A_p(\cdot)$ the law of this new model.

Since the model is now defined on $\mathbb{Z}_2^+$, one can define the rectangle $R = R([a, b] \times [c, d])$ for any $a, b, c, d \in \mathbb{Z}_+$ and the corresponding (horizontal and vertical) crossing events as in (2), (3) and (4). In the remainder of this section only this new definition of the model will be adopted.
6.1 Environments

Let us fix constants
\[ \alpha \in \left(0, \frac{\varepsilon}{2}\right] \quad \text{and} \quad \gamma \in \left(1, 1 + \frac{\alpha}{\alpha + 2}\right) \]
which will appear as exponents in several expressions below. The exponent \( \gamma \) will
give the rate of growth for the scales in which we study the environment while \( \alpha \)
will give the rate of decay of probability that bad events occur in each scale (see
(19) and (24)).

Let us also fix \( L_0 = L_0(\xi, \varepsilon, \alpha, \gamma) \in \mathbb{Z}_+ \) sufficiently large so that

(i) \( L_0^{\gamma-1} \geq 3 \),
(ii) \( L_0^{\varepsilon-\alpha} \geq \mathbb{E}(\rho^s) \) and
(iii) \( L_0^2 \geq c_1 + 1 \), where \( c_1 \) is given by the Lemma 5.1 and
\[ c_2 = 2 + 2\alpha - \gamma\alpha - 2\gamma. \]

Note that \( c_2 > 0 \) by the choice of \( \gamma \) in (18).

Once \( L_0 \) is fixed, we can define recursively the sequence of scales \( (L_k)_{k \in \mathbb{Z}_+} \) by
\[ L_k = L_{k-1}[L_{k-1}^{-1}], \quad \text{for any } k \geq 1. \]

Item (i) in the definition of \( L_0 \) together with (18) and (19) implies that the scales
grow super-exponentially fast. In fact,
\[ \left(\frac{2}{3}\right)^k L_0^k \leq \cdots \leq \frac{2}{3} L_{k-1}^k \leq L_k \leq L_{k-1}^k \leq \cdots \leq L_0^k. \]

Items (ii) and (iii) are technical and will be used to prove the Lemma 6.1 below.

For \( k \in \mathbb{Z}_+ \), consider the partition of \( \mathbb{R}_+ \) into intervals of length \( L_k \):
\[ I_j^k := [jL_k, (j + 1)L_k), \quad \text{with } j \in \mathbb{Z}_+. \]

The interval \( I_j^k \) is called the \( j \)-th block at scale \( k \). For \( k \geq 1 \), each block at scale \( k \) can be split into \( [L_{k-1}^{-1}] \) disjoint blocks at scale \( k - 1 \):
\[ I_j^k = \bigcup_{i \in l_{k,j}} I_i^{k-1}, \]
where
\[ l_{k,j} := \{ i \in \mathbb{Z}_+; I_i^{k-1} \cap I_j^k \neq \emptyset \} = \{ j[L_{k-1}^{-1}], \cdots, (j + 1)[L_{k-1}^{-1}] - 1 \}. \]
Now fix an environment $\Lambda \subseteq \mathbb{Z}_+$. Blocks will be labeled good or bad depending on $\Lambda$ in a recursive way. For $k = 0$ declare the $j$-th block at scale 0, $I^0_j$, good if $\Lambda \cap I^0_j \neq \emptyset$, and bad otherwise. Once the blocks at scale $k - 1$ are all labeled good or bad we declare a block at scale $k$ bad if it contains at least two non-consecutive bad blocks at scale $k - 1$, and good otherwise. More precisely, for $j, k \in \mathbb{Z}_+$ consider the events $A^k_j$ defined recursively by

$$A^0_j = \{ \Lambda \subseteq \mathbb{Z}_+; \Lambda \cap I^0_j = \emptyset \} \text{ and } A^k_j = \bigcup_{i_1, i_2 \in l_{k,j}, |i_1 - i_2| \geq 2} (A^{k-1}_{i_1} \cap A^{k-1}_{i_2}) , \text{ for } k \geq 1. \quad (23)$$

Sometimes we will write $\{I^k_j \text{ is bad}\}$ instead of $A^k_j$ and $\{I^k_j \text{ is good}\}$ for the complementary set of environments. By (i), $l_{k,j}$ has at least three elements, so that the union in (23) always runs over a nonempty collection of indices.

We now define

$$p_k := \nu^\rho_\xi(A^0_k) = \nu^\rho_\xi(A^k_j).$$

where the equality follows from the stationarity of $\rho$.

The next lemma establishes an upper bound for the $p_k$’s, which is a power law in $L_k$ with exponent $\alpha$.

**Lemma 6.1.** For every $k \in \mathbb{Z}_+$ we have

$$p_k \leq L_k^{-\alpha}. \quad (24)$$

**Proof.** We proceed by induction on $k$. Using Markov’s inequality for $\rho^\epsilon$, we get

$$p_0 = \nu^\rho_\xi(A^0_0) \stackrel{(10)}{=} \nu^\rho_\xi(Z_0 > L_0) \stackrel{(13)}{=} \mathbb{P}(\rho > L_0) \leq \frac{\mathbb{E}(\rho^\epsilon)}{L_0},$$

which, together with (ii) implies $p_0 \leq L_0^{-\alpha}$.

Using (15)

$$p_{k+1} = \nu^\rho_\xi(A^{k+1}_0) \leq \sum_{i_1, i_2 \in l_{k+1,0}, |i_1 - i_2| \geq 2} \nu^\rho_\xi(A^k_{i_1} \cap A^k_{i_2}) \leq L_k^{2(\gamma - 1)} [p_k^2 + c_1 L_k^{-\epsilon}] \quad (25)$$

which is a recursive inequality relating $p_{k+1}$ to $p_k$.

Now assume that for some $k \in \mathbb{Z}_+$, $p_k \leq L_k^{-\alpha}$. Plugging this into (25), we get

$$p_{k+1} \leq L_k^{2\gamma - 2} (L_k^{-2\alpha} + c_1 L_k^{-\epsilon}) \stackrel{(13)}{=} (1 + c_1) L_k^{2\gamma - 2 - 2\alpha} \quad (26)$$
which implies
\[ \frac{p_{k+1}}{L_{k+1}^{-\alpha}} \leq (1 + c_1)L_k^{2\gamma - 2 - 2\alpha}L_{k+1}^{-\alpha} \leq (1 + c_1)L_k^{2\gamma - 2 - 2\alpha + \gamma}(iii) \leq 1. \]

That is to say that \( p_{k+1} \) is also bounded above by \( L_{k+1}^{-\alpha} \). This concludes the proof. \( \square \)

6.2 Crossings events

In this section we will define crossing events in certain rectangles of \( \mathbb{Z}_+^2 \). The base of such an rectangle will be a block at scale \( k \), for some \( k \). Also, the rectangles will be very elongated on the vertical direction, meaning that the height is a stretched exponential function of the length of the base. First fix

\[ \mu \in \left( \frac{1}{\gamma}, 1 \right) \]  

and define recursively the sequence of heights \((H_k)_{k \in \mathbb{Z}_+}\) by

\[ H_0 = 100 \] and \( H_k = 2\lceil \exp(L_k^\mu) \rceil H_{k-1} \), for \( k \geq 1 \).

The choice \( H_0 = 100 \) is arbitrary and we could have used any other positive integer.

Recall the crossing events defined in (3) and (4) and, for \( i, j, k \in \mathbb{Z}_+ \) denote

\[ C_{i,j}^k := C_h \left( (I_i^k \cup I_{i+1}^k) \times [jH_k, (j+1)H_k] \right) \]  

\[ D_{i,j}^k := C_v \left( I_i^k \times [jH_k, (j+2)H_k] \right). \]

See Figure 4.

The next lemma guarantees that the crossing events above occur with high probability, inside a rectangle that projects onto a good block. This fact will allow us to build an infinite cluster with positive probability. Before stating this result, let us define, for every \( i, j, k \in \mathbb{Z}_+ \) and \( p \in (0, 1) \),

\[ q_k(p; i, j) := \max \left\{ \max_{\Lambda; I_i^k \text{ and } I_{i+1}^k \text{ are good}} \mathbb{P}^\Lambda_p \left( \{C_{i,j}^k\}^c \right), \max_{\Lambda; I_i^k \text{ is good}} \mathbb{P}^\Lambda_p \left( \{D_{i,j}^k\}^c \right) \right\}. \]

Translation invariance, allows us to write, for every \( k \in \mathbb{Z}_+ \),

\[ q_k(p) := q_k(p; 0, 0) = q_k(p; i, j), \text{ for any } i, j \in \mathbb{Z}_+. \]  

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Figure 4: Illustration of the events $C_{i,j}^k$ and $D_{i,j}^k$.

We also fix

$$\beta \in (\gamma \mu - \gamma + 1, 1).$$  

(31)

(Note that this is possible because (27) yields $\gamma \mu - \gamma < 0$). The $\beta$ will give the rate of decay of $q_k$ (see Lemma 6.2).

**Lemma 6.2.** There exist $c_3 = c_3(\gamma, L_0, \mu, \beta) \in \mathbb{Z}_+$ and $p = p(\gamma, L_0, \mu, \beta, c_3)$ sufficiently close to 1 such that

$$q_k(p) \leq \exp(-L_k^\beta), \text{ for any } k \geq c_3.$$

This lemma is a straightforward consequence of the two following lemmas:

**Lemma 6.3.** Let $p > 1/2$. There exists $c_4 = c_4(\gamma, L_0, \mu, \beta) \in \mathbb{Z}_+$, such that for all $k \geq c_4$

if $q_k(p) \leq \exp(-L_k^\beta)$, then

$$\Pr^\Lambda_p(C_{0,0}^{k+1}) \leq \exp(-L_{k+1}^\beta)$$

for every environment $\Lambda \in \{I_{0}^{k+1} \text{ is good}\} \cap \{I_{1}^{k+1} \text{ is good}\}$.

**Lemma 6.4.** There is $c_5 = c_5(\gamma, L_0, \mu, \beta) \in \mathbb{Z}_+$, such that for all $k \geq c_5$, we have

if $q_k(p) \leq \exp(-L_k^\beta)$, then

$$\Pr^\Lambda_p(D_{0,0}^{k+1}) \leq \exp(-L_{k+1}^\beta),$$

for every environment $\Lambda \in \{I_{0}^{k+1} \text{ is good}\}$.
We now show how Lemma 6.2 follows from Lemmas 6.3 and 6.4.

**Proof of Lemma 6.2.** Let \( c_3 := \max\{c_4, c_5\} \) and choose \( p = p(\gamma, L_0, \mu, \beta, c_3) < 1 \) such that \( q_{c_3}(p) \leq \exp(-L_0^\beta) \). Lemmas 6.3 and 6.4 imply that \( q_k(p) \leq \exp(-L_0^\beta k) \), for any \( k \geq c_3 \).

Next we present the proofs of Lemmas 6.3 and 6.4.

**Proof of Lemma 6.3.** Fix an environment \( \Lambda \) for which \( I_{k+1}^{0,0} \) and \( I_{k+1}^{1,0} \) are good blocks. Both \( I_{k+1}^{0,0} \) and \( I_{k+1}^{1,0} \) may contain at most two (consecutive) bad blocks at scale \( k \). Even though it may seem hard to cross these bad blocks, there will be many attempts available to do so. Indeed, let us divide the rectangle \( R([0, 2L_{k+1}] \times [0, H_{k+1}]) \) into bands of height \( 2H_k \), and verify whether some crossings take place inside these bands (see Figure 5). For \( 0 \leq j \leq \lfloor \exp(L_0^\mu k) \rfloor - 1 \), define the events

\[
G_j := C_h \left( R([0, 2L_{k+1}] \times [j H_k, (j + 2) H_k]) \right)
\]

and note that if \( C_{k+1}^{0,0} \) does not occur, then none of the events \( G_j \) occur. Therefore,

\[
P^\Lambda_p(\{C_{k+1}^{0,0}\}^c) \leq (1 - P^\Lambda_p(G_0))^{\exp(L_0^\mu k_{k+1})}.
\]

where we have used independence and invariance of events \( G_j \)’s.

![Figure 5: The occurrence of \( G_2 \), implies the occurrence of \( \{C_{k+1}^{0,0}\} \).](image)

In order to bound below the probability of \( G_0 \) we will build horizontal crossings in \( R([0, 2L_{k+1}] \times [0, 2H_k]) \) using the events \( C_i^{k,0} \) and \( D_i^{k,0} \) when they are supported in rectangles that project to good blocks at scale \( k \). However, in case rectangles
that project to bad block of the type $I^k_i$ appear, we will try to cross them straight in their bottom. The strategy is illustrated in Figure 6.

In order to bound below the probability of crossing the possible bad blocks at scale $k$, we will introduce the events $B_0$ and $B_1$ as follows. If all the blocks at scale $k$ that form $I^k_0$ (resp. $I^k_1$) are good, define $B_0 = \emptyset$ (resp. $B_1 = \emptyset$). Otherwise, denote by $j_0$ (resp. $j_1$) the earliest index $i \in l_{k+1,0}$ (resp. $i \in l_{k+1,1}$) such that $I^k_{j_0}$ (resp. $I^k_{j_1}$) is a bad block at scale $k$. Now, for, $l = 0, 1$, define

$$I^*_l := (I^k_{j_l-1} \cup I^k_{j_l} \cup I^k_{j_l+1} \cup I^k_{j_l+2}) \cap (I^k_0 \cup I^k_1).$$

The intervals $I^*_0$ and $I^*_1$ are just enlarged versions of $I^k_{j_0}$ and $I^k_{j_1}$ that contain all the bad blocks at scale $k$ inside $I^k_0$ and $I^k_1$, plus the good blocks at scale $k$ that are adjacent to these bad blocks to the left and to the right (as long as the latter are still contained in $I^k_0 \cup I^k_1$). See Figure 6.

![Figure 6](image-url)

Figure 6: In this picture, $I^k_0$, $I^k_1$ and $I^k_{j_l+1}$ are the only bad blocks at scale $k$ inside $I^k_0 \cup I^k_1$. We use the crossings provided by the events $C^k_{t,0}$ and $D^k_{t,0}$ to traverse rectangles that project to good blocks at scale $k$. Rectangles that project to bad blocks at scale $k$ and their neighbors are traversed straight in their bottom, yielding the occurrence of $B_0$ and $B_1$. All together, the occurrence of the $C^k_{t,0}$, $D^k_{0,1}$, $B_0$ and $B_1$ implies the occurrence of $G_0$.

Now, for $l = 0, 1$, denote

$$B_l := \{\text{all edges of the form } \{(m,0), (m+1,0)\} \text{ with } m \in I^*_l \text{ are open}\}.$$ 

Since $p > 1/2$, and since $I^*_0$ and $I^*_1$ have length at most $4L_k$, we have

$$\mathbb{P}_p(A(B_0 \cap B_1) \geq p^{8L_k} \geq 2^{-8L_k}. \quad (33)$$

Note that

$$\left( \bigcap_{i: I^k_i \text{ are good}} C^k_{t,0} \right) \cap \left( \bigcap_{j: I^k_j \text{ is good}} D^k_{j,0} \right) \cap B_0 \cap B_1 \subseteq G_0 \quad (34)$$

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where \(0 \leq i, j \leq 2\lfloor L_k^{-1}\rfloor - 1\). Since the events \(B_t, C_{i,0}^k, D_{k,0}^k\) are increasing, it follows from the FKG inequality, (30), (33) and (34) that

\[
P^\Lambda_p(G_0) \geq (1 - q_k(p))^{4L_k^{-1}} \geq (1 - 4L_k^{-1} q_k(p)) 2^{-8L_k}.
\] (35)

Pick \(c_6 = c_6(L_0, \gamma, \beta)\) sufficiently large such that \(4L_k^{-1} e^{-L_k^\beta} \leq 1/2\) for any \(k \geq c_6\). Now if \(q_k(p) \leq \exp(-L_k^\beta)\), then (35) implies

\[
P^\Lambda_p(G_0) \geq (1 - 4L_k^{-1} e^{-L_k^\beta}) 2^{-8L_k} \geq 2^{-8L_k-1}.
\] (36)

Plugging (36) into (32) and dividing by \(\exp(-L_{k+1}^\beta)\) we get

\[
\frac{P^\Lambda_p(C_{0,0}^{k+1})}{\exp(-L_{k+1}^\beta)} \leq \exp(L_{k+1}^\beta) \left(1 - 2^{-8L_{k+1}}\right)^\exp(L_{k+1}^\mu)
\]

\[
\leq \exp(L_{k+1}^\beta - \exp(-8\ln 2 \cdot L_k - \ln 2 + L_{k+1}^\mu))
\]

\[
\leq \exp(L_{k+1}^\beta - \exp(-8\ln 2 \cdot L_k - \ln 2 + (\frac{\gamma}{3})^\mu L_{k+1}^\gamma)),
\] (37)

where in the second inequality we use that \(1 - x \leq \exp(-x)\).

By (27), \(\gamma \mu > 1\) therefore, we can take \(c_4 = c_4(\gamma, L_0, \mu, \beta, c_6) \geq c_6\) sufficiently large such that for any \(k \geq c_4\) the right-hand side in (37) is at most 1. This finishes the proof. \(\square\)

**Proof of Lemma 6.4** Fix an environment \(\Lambda\) such that \(I_0^{k+1}\) is good. We will estimate \(P^\Lambda_p(D_{0,0}^{k+1})\) using a Peierls-type argument in a renormalized lattice. Each rectangle \(I_k^i \times [jH_k, (j+1)H_k]\) will correspond to a vertex \((i, j)\) in this renormalized lattice. This renormalized lattice is then just the \(\mathbb{Z}_+^2\) lattice and the vertex \((i, j) \in \mathbb{Z}_+^2\) is declared open if the event \(C_{i,j}^k \cap D_{i,j}^k\) occurs, see Figure 7. This gives rise to a dependent percolation process in the the renormalized lattice.

Since \(I_0^{k+1}\) is good, either \(I_i^k\) is good for every \(i \in \{0, 1, \ldots, \lfloor 2L_k^{-1}\rfloor - 1\}\) or \(I_i^k\) is good for every \(i \in \{\lfloor \frac{1}{2}L_k^{-1}\rfloor + 1, \ldots, \lfloor L_k^{-1}\rfloor - 1\}\). Assume without loss of generality that the former holds and define

\[
L := \lfloor \frac{1}{3}L_k^{-1}\rfloor - 1.
\] (38)

Consider the rectangle

\[
R = R \left( \left[0, L\right] \times \left[0, 4\exp(L_{k+1}^\mu)\right] \right),
\]

and the event \(C_v(R)\) that this rectangle is crossed vertically. Note that

\[
P^\Lambda_p(D_{0,0}^{k+1}) \geq P(C_v(R)).
\]
Now we use the Peierls argument: suppose that the event $C_v(R)$ does not occur. Then there exists a sequence of distinct vertices $(i_0, j_0), (i_1, j_1), \ldots, (i_n, j_n)$ in $R$ such that

1. $\max \{|i_l - i_{l-1}|, |j_l - j_{l-1}|\} = 1$,
2. $(i_0, j_0) \in \{0\} \times [0, 4^\left[ \exp(L_{k+1}^\mu) \right]]$ and $(i_n, j_n) \in \{L\} \times [0, 4^\left[ \exp(L_{k+1}^\mu) \right]]$ and
3. $(i_k, j_k)$ is closed for every $k = 0, \ldots, n$.

Note that there are at most $4^\left[ \exp(L_{k+1}^\mu) \right] 8^n$ sequences with $n + 1$ vertices that satisfy 1 and 2. Also, the probability that a vertex of $R$ be open is at least $1 - 2q_k(p) \geq 1 - 2 \exp(-L_{k}^\beta)$.

By the geometry of the crossing events in the original lattice, for any $(i, j) \in \mathbb{Z}_+^2$, the event $\{(i, j)\text{ is open}\}$ in the renormalized lattice depends on $\{(i', j')\text{ is open}\}$ for, at most 7 distinct vertices $(i', j')$ (see Figure 7). Therefore, for every set containing $n + 1$ vertices, there are at least $\lfloor n/7 \rfloor$ vertices whose states are mutually independent.

Therefore,

$$
\mathbb{P}(C_v(R)^c) \leq \sum_n \mathbb{P}(\text{there is a sequence of } n + 1 \text{ vertices satisfying 1., 2. and 3.}) \\
\leq \sum_{n \geq L} 4^\left[ \exp(L_{k+1}^\mu) \right] 8^n \left(2 \exp(-L_{k}^\beta)\right)^\left[ n/7 \right] \\
\leq 4^\left[ \exp(L_{k+1}^\mu) \right] \sum_{n \geq L} \exp\left(n \ln 8 + \lfloor n/7 \rfloor \ln 2 - \lfloor n/7 \rfloor L_{k}^\beta\right) \\
\leq c_7 \exp\left(L_{k+1}^\mu - c_8 \cdot L_{k}^{\beta + \gamma - 1}\right),
$$

for some $c_7 = c_7(\gamma, L_0, \beta) > 0$ and $c_8 = c_8(\gamma, L_0, \beta) > 0$ sufficiently large.

Therefore,

$$
\frac{\mathbb{P}_p^\Lambda(\{D_{k+1}^{0,0}\} \cup \{\gamma\})}{\exp(-L_{k+1}^\beta)} \leq c_7 \exp\left(L_{k}^{\gamma\mu} + L_{k}^{\gamma\beta} - c_8 L_{k}^{\beta + \gamma - 1}\right)
$$

(39)

It follows from the choice of $\beta$ in (31), that

$$
\beta + \gamma - 1 > \max\{\gamma\beta, \gamma\mu\}.
$$

The proof now follows by choosing $c_5 = c_5(\gamma, L_0, \mu, \beta)$ sufficiently large so that the right-hand side of the (39) is less than 1 whenever $k \geq c_5$. \qed
Figure 7: On the left, we illustrate the occurrence of the event \( C_{i,j}^k \cap D_{i,j}^k \) on the original lattice. On the right, we depict the renormalized square lattice where the circles represent the sites. The occurrence of \( C_{i,j}^k \cap D_{i,j}^k \) in the original lattice implies that the site \((i, j)\) (represented as a black circle) is open in the renormalized lattice. The state of the site \((i, j)\) only depends on the state of the other six sites represented as circles with a dot inside.

7 Proof of Theorem 1.1

Proof of Theorem 1.1. In fact, let \( \xi \) be any positive random variable such that \( \mathbb{E}(\xi^\eta) < \infty \) for a given \( \eta > 1 \). Denote

\[
m := \gcd \{ k \in \mathbb{Z}_+^*; \mathbb{P}(\lceil \xi \rceil = k) \neq 0 \}.
\]

Now define the positive integer-valued, aperiodic random variable \( \xi' = \lceil \xi \rceil / m \) which also satisfies \( \mathbb{E}( (\xi')^\eta) < \infty \). If for some \( p < 1 \), we have \( \mathbb{P}^{\Lambda}_{p}(o \leftrightarrow \infty) > 0 \) for \( \nu_{\xi'}\text{-a.e.} \) environment \( \Lambda' \), then also \( \mathbb{P}^{\Lambda}_{p/m}(o \leftrightarrow \infty) \) for \( \nu_{\xi}\text{-a.e.} \) \( \Lambda \) as it can be seen by a simple coupling argument. In view of this, we will assume without loss of generality that \( \xi \) is a positive integer-valued, aperiodic random variable. In particular, we can apply Lemmas 6.1 and 6.2 from Section 6.

Lemma 6.1 implies

\[
u_{\xi}' \left( \bigcup_{0 \leq i \leq |L_k^{-1}|-1} \{ I_i^k \text{ is bad} \} \right) \leq L_k^{-1} L_k^{-\alpha} \leq L_k^{-\frac{\alpha}{2}}.\]

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By (20), $\sum_k L_k^{-\alpha/2} < \infty$, therefore, it follows from Borel-Cantelli’s Lemma, that for $\nu^\rho$-almost everywhere environment $\Lambda$, there exists $c_9 = c_9(\Lambda, c_3) > c_3$ such that for every $k \geq c_9$, all the blocks at scale $k$ inside $I_0^{k+1}$ are good. Fix such an environment $\Lambda$.

Let $c_{10} \geq c_9$ be any integer such that

$$
\sum_{k \geq c_{10}} 2L_k^{-1} \exp(-L_k^\beta) < 1/2.
$$

Recall the definition of $C^k_{i,0}$ and $D^k_{i,0}$ in (28) and (29) and note that

$$
\bigcap_{k \geq c_{10}} \left( \bigcap_{i=0}^{\lfloor L_k^{-1} \rfloor - 2} (C^k_{i,0} \cap D^k_{i,0}) \right) \subseteq \{ \text{there is an infinite cluster} \}.
$$
(see Figure 3). Using the FKG inequality and (30), (40) and Lemma 6.2, we get

\[ P_p^A \left( \text{there is an infinite cluster} \right) \geq \prod_{k \geq c_{10}} (1 - 2q_k(p))^{\left\lfloor L_k^{\gamma - 1} \right\rfloor - 1} \]

\[ \geq 1 - \sum_{k \geq c_{10}} 2L_k^{\gamma - 1} q_k(p) \]

\[ \geq 1 - \sum_{k \geq c_{10}} 2L_k^{\gamma - 1} \exp(-L_k^\beta) \geq \frac{1}{2}, \quad (41) \]

It follows that, for some vertex \( x \), the probability that \( x \) belong to an infinite connected component is positive. Also the probability that \( o \) is connected to this site \( x \) is positive. The FKG inequality guarantees that \( P_p^A(o \leftrightarrow \infty) > 0 \).

\[ \square \]

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