Quasi-Monte Carlo integration with product weights for elliptic PDEs with log-normal coefficients

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Abstract

Quasi-Monte Carlo (QMC) integration of output functionals of solutions of the diffusion problem with a log-normal random coefficient is considered. The random coefficient is assumed to be given by an exponential of a Gaussian random field that is represented by a series expansion of some system of functions. Graham et al. [12] developed a lattice-based QMC theory for this problem and established a quadrature error decay rate $\approx 1$ with respect to the number of quadrature points. The key assumption there was a suitable summability condition on the aforementioned system of functions. As a consequence, product-order-dependent (POD) weights were used to construct the lattice rule. In this paper, a different assumption on the system is considered. This assumption, originally considered by Bachmayr et al. [3] to utilise the locality of support of basis functions in the context of polynomial approximations applied to the same type of the diffusion problem, is shown to work well in the same lattice-based QMC method considered by Graham et al.: the assumption leads us to product weights, which enables the construction of the QMC method with a smaller computational cost than Graham et al. A quadrature error decay rate $\approx 1$ is established, and the theory developed here is applied to a wavelet stochastic model. By a characterisation of the Besov smoothness, it is shown that a wide class of path smoothness can be treated with this framework.

Keywords: Quasi-Monte Carlo methods, Partial differential equations with random coefficients, Log-normal, Infinite dimensional integration

1 Introduction

This paper is concerned with quasi-Monte Carlo (QMC) integration of output functionals of solutions of the diffusion problem with a random coefficient of the form

$$-\nabla \cdot (a(x,\omega)\nabla u(x,\omega)) = f(x) \quad \text{in } D \subset \mathbb{R}^d, \quad u = 0 \quad \text{on } \partial D, \quad (1.1)$$

where $\omega \in \Omega$ is an element of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (clarified below), and $D \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. Our interest is in the log-normal case, that is, $a(\cdot,\cdot): D \times \Omega \to \mathbb{R}$ is assumed to have the form

$$a(x,\omega) = a_s(x) + a_0(x) \exp(T(x,\omega)) \quad (1.2)$$
with continuous functions $a_x \geq 0$, $a_0 > 0$, and Gaussian random field $T(\cdot, \cdot): D \times \Omega \to \mathbb{R}$ represented by a series expansion

$$T(x, \omega) = \sum_{j=1}^{\infty} Y_j(\omega) \psi_j(x), \quad \text{for all } x \in D,$$

with a suitable system of functions $(\psi_j)_{j \geq 1}$.

To handle a wide class of $a$ and $f$, we consider the weak formulation of the problem (1.1). By $V$ we denote the zero-trace Sobolev space $H^1_0(D)$ endowed with the norm

$$\|v\|_V := \left( \int_D |\nabla v(x)|^2 \, dx \right)^{\frac{1}{2}}, \quad (1.3)$$

and by $V' := H^{-1}(D)$ the topological dual space of $V$. For the given random coefficient $a(x, \omega)$, we define the bilinear form $\mathcal{A}(\omega; \cdot, \cdot): V \times V \to \mathbb{R}$ by

$$\Omega \ni \omega \mapsto \mathcal{A}(\omega; v, w) := \int_D a(x, \omega) \nabla v(x) \cdot \nabla w(x) \, dx \quad \text{for all } v, w \in V. \quad (1.4)$$

Then, for any $\omega \in \Omega$, the weak formulation of (1.1) reads: find $u(\cdot, \omega) \in V$ such that

$$\mathcal{A}(\omega; u(\cdot, \omega), v) = \langle f, v \rangle \quad \text{for all } v \in V, \quad (1.5)$$

where $f$ is assumed to be in $V'$, and $\langle \cdot, \cdot \rangle$ denotes the duality paring between $V'$ and $V$. We impose further conditions to ensure the well-posedness of the problem, which we will discuss later.

The ultimate goal is to compute $\mathbb{E}[G(u(\cdot))]$, the expected value of $G(u(\cdot, \omega))$, where $G$ is a linear bounded functional on $V$. The problem (1.1), and of computing $\mathbb{E}[G(u(\cdot))]$ often arises in many applications such as hydrology [9, 22, 23], and has attracted attention in computational uncertainty quantification (UQ). See, for example, [8, 25, 20] and references therein. Two major ways to tackle this problem are function approximation, and quadrature, in particular, quasi-Monte Carlo (QMC) methods.

Our interest is in QMC. It is now well known that the QMC methods beats the plain-vanilla Monte Carlo methods in various settings when applied to the problems of computing $\mathbb{E}[G(u(\cdot))]$ ([16, 20, 21]). Among the QMC methods, the algorithm we consider is randomly shifted lattice rules.

Graham et al. [16] showed that when the randomly shifted lattice rules are applied to the class of PDEs we consider, a QMC convergence rate, in terms of expected root square mean root, $\approx 1$ is achievable, which is known to be optimal for lattice rules in the function space they consider. More precisely, they showed that quadrature points for randomly shifted lattice rules that achieve such a rate can be constructed using an algorithm called component-by-component (CBC) construction. The algorithm uses weights, which represents the relative importance of subsets of the variables of the integrand, as an input, and the cost of it is dependent on the type of weights. The weights considered in [16] are so-called product-order-dependent (POD) weights, which were determined by minimising an error bound. For POD weights, the CBC construction takes $O(sn \log n + s^2 n)$ operations, where $n$ is the number of quadrature points and $s$ is the dimension of truncation $\sum_{j=1}^{s} Y_j(\omega) \psi_j(x)$. 

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The contributions of the current paper are twofold: proof of a convergence rate \( \approx 1 \) with product weights, and an application to a stochastic model with wavelets. In more detail, we show that for the currently considered problem, the CBC construction can be constructed with weights called product weights, and achieves the optimal rate \( \approx 1 \) in the function space we consider, and further, we show that the developed theory can be applied to a stochastic model which covers a wide class of wavelet bases.

Often in practice, we want to approximate the random coefficients well, and consequently \( s \) has to be taken to be large, in which case the second term of \( O(sn \log n + s^2n) \) becomes dominant. The use of the POD weights originates from the summability condition imposed on \( (\psi_j) \) by Graham et al. [16]. We consider a different condition, the one proposed by Bachmayr et al. [3] to utilise the locality of supports of \( (\psi_j) \) in the context of polynomial approximations applied to PDEs with random coefficients. We show that under this condition, the shifted lattice rule for the PDE problem can be constructed with a CBC algorithm with the computational cost \( O(sn \log n) \), the cost with the product weights as shown in [12]. Further, the stochastic model we consider broadens the range of applicability of the QMC methods to the PDEs with log-normal coefficients. One concern about the conditions, in particular the summability condition on \( (\psi_j) \), imposed in [16] is that it is so strong that only random coefficients with smooth realisations are in the scope of the theory. We show that at least for \( d = 1, 2 \), such random coefficients (e.g., realisations with just some Hölder smoothness) can be considered.

We note that the similar argument employed in the current paper is applicable to the randomly shifted lattice rules applied to PDEs with uniform random coefficients considered in [21]. One of the keys in the current paper is the estimate of the derivative given in Corollary 3.2. This result essentially follows from the results by Bachmayr et al. [3]. The paper [2], Part I of their work [3], considers the uniform case, and the similar argument as the one presented here turns out to work almost in parallel.

Upon finalising this paper, we learnt about the two papers, one by Gantner et al. [13], and the other by Herrmann and Schwab [17]. Our works share the same spirit in that we are all inspired by the work by Bachmayr et al. [2, 3]. The interest of Gantner et al. [13] is in the uniform case. They consider not only the randomly shifted lattice rules but also higher order QMCs. Since our interest was on the randomly shifted lattice rule in the uniform case, and our results are a proper subset of their work [13], we defer to [13] for the uniform case.

As for the log-normal case we provide a different, arguably simpler, proof for the same convergence rate with the exponential weight function, and we discuss the roughness of the realisations that can be considered.

Herrmann and Schwab [17] develops a theory under the setting essentially the same as ours. In contrast to our paper, they treat the truncation error in a general setting, and as for the QMC integration error, they consider both the exponential weight functions and the Gaussian weight function for the weighted Sobolev space. As for the exponential weight function, the current paper and [17] impose essentially the same assumptions (Assumption B below), and show the same convergence rate. However, our proof strategy is different, which turns out to result in different (product) weights, (and a different constant, although it does not seem to be easy to say which is bigger). Further, in contrast to [17], we provide a discussion of the roughness of the realisations of random coefficients as mentioned above. The log-normal case, in comparison to the uniform case where the “random parameters” can be uniformly bounded, is “intrinsically random” in the sense that the magnitude of each parameter can be arbitrarily large. As a consequence, the connection between the smoothness of the spatial basis and the
one of the smooth realisations are not immediately clear. In Section 5 we provide a discussion via the Besov characterisation of the realisations of the random coefficients and the embedding results.

The outline of the rest of the paper is as follows. In Section 2, we describe the problem we consider in detail. Then, in Section 4 we develop the QMC theory applied to the PDE problem with log-normal coefficients using the product weights. Section 5 provides an application of the theory: we consider a stochastic model represented by a wavelet Riesz basis. Then, we close this paper with concluding remarks in Section 6.

2 Setting

We assume that the Gaussian random field $T$ admits a series representation $T(x, \omega) = \sum_{j=1}^{\infty} Y_j(\omega) \psi_j(x)$, where $\{Y_j\}$ is a collection of independent standard normal random variables on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\psi_j)$ is a system of real-valued measurable functions on $\mathbb{R}^n$. For simplicity we fix $(\Omega, \mathcal{F}, \mathbb{P}) := (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), \mathbb{P}_Y)$, where $\mathbb{N} := \{1, 2, \ldots, \}$, $\mathcal{B}(\mathbb{R}^N)$ is the Borel $\sigma$-algebra generated by the product topology in $\mathbb{R}^N$, and $\mathbb{P}_Y := \prod_{j=1}^{\infty} \mathbb{P}_{Y_j}$ is the product measure on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ defined by the standard normal distributions $\{\mathbb{P}_{Y_j}\}_{j \in \mathbb{N}}$ on $\mathbb{R}$ (see, for example, [18, Chapter 2] for details). Then, for each $y \in \Omega$ we may see $Y_j(y)$ $(j \in \mathbb{N})$ as given by the projection (or the canonical coordinate function)

$$
\Omega = \mathbb{R}^N \ni y \mapsto Y_j(y) := y_j \in \mathbb{R}.
$$

Note in particular that from the continuity of the projection, the mapping $y \mapsto y_j$ is $\mathcal{B}(\mathbb{R}^N)/\mathcal{B}(\mathbb{R})$-measurable.

In the following, we write $T$ above as

$$
T(x, y) = \sum_{j=1}^{\infty} y_j \psi_j(x), \quad \text{for all } x \in D, \quad (2.1)
$$

and see it as a deterministically parametrised function on $D$. We will impose a condition considered by Bachmayr et al. [3] on $(\psi_j)$, see Assumption B below, that is particularly suitable for $\psi_j$ with local support.

To ensure the law on $\mathbb{R}^D$ is well defined, we suppose

$$
\sum_{j=1}^{\infty} \psi_j(x)^2 < \infty \text{ for all } x \in D, \quad (2.2)
$$

so that the covariance function $\mathbb{E}[T(x_1)T(x_2)] = \sum_{j \geq 1} \psi_j(x_1)\psi_j(x_2)$ $(x_1, x_2 \in D)$ is well-defined. We consider the parametrised elliptic partial differential equation

$$
- \nabla \cdot (a(x, y)\nabla u(x, y)) = f(x) \text{ in } D, \quad u = 0 \text{ on } \partial D, \quad (2.3)
$$

where

$$
a(x, y) = a_*(x) + a_0(x) \exp(T(x, y)), \quad (2.4)
$$

with continuous functions $a_*$, $a_0$ on $\bar{D}$. We assume $a_*$ is non-negative on $\bar{D}$, and $a_0$ is positive on $\bar{D}$.\n
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In accordance with the above formulation, we rewrite (1.5) as the parametrised variational problem: find \( u \in V \) such that
\[
\mathcal{A}(y; u(\cdot), y), v = \langle f, v \rangle \quad \text{for all } v \in V,
\] (2.5)
To prove well-posedness of the variational problem (2.5), we use the Lax–Milgram lemma. Conditions which ensure that the bilinear form \( \mathcal{A}(y; \cdot, \cdot) \) defined by the diffusion coefficient \( a \) is coercive and bounded are discussed later.

Motivated by UQ applications, we are interested in expected values of bounded linear functionals of the solution of the above PDEs. That is, given a continuous linear functional \( \mathcal{G} \in V' \) we wish to compute \( \mathbb{E}[\mathcal{G}(u(\cdot))] := \int_{\mathbb{R}^n} \mathcal{G}(u(\cdot), y)) \, d\mathcal{P}(y) \), where the measurability of the integrands will be discussed later. To compute \( \mathbb{E}[\mathcal{G}(u(\cdot))] \) we use a sampling method: generate realisations of \( a(x, y) \), which yields the solution \( u(x, y) \) via the PDE (2.3), and from these we compute \( \mathbb{E}[\mathcal{G}(u(\cdot))] \).

In practice, these operations cannot be performed exactly, and numerical methods need to be employed. This paper gives an analysis of the error incurred by the method outlined as follows. We compute the realisations by truncation, that is, for some integer \( s \geq 1 \) we generate \( a(x, (y_1, \ldots, y_s, 0, 0, 0, \ldots)) \). Further, the expectation is approximated by a QMC method.

Let \( u^s(x) = u^s(x, y) \) be the solution of (2.3) with \( y = (y_1, \ldots, y_s, 0, 0, 0, \ldots) \), that is, of the problem: find \( u^s \in V \) such that
\[
-\nabla \cdot (a(x, (y_1, \ldots, y_s, 0, 0, 0, \ldots)) \nabla u^s(x) = f(x) \quad \text{in } D, \quad u^s = 0 \text{ on } \partial D.
\] (2.6)
Here, even though the dependence of \( u^s \) on \( y \) is only on \( (y_1, \ldots, y_s) \), we abuse the notation slightly by writing \( u^s(x, y) := u^s(y_1, \ldots, y_s, 0, 0, 0, \ldots) \).

Let \( \Phi^{-1}_s : [0, 1]^s \ni v \mapsto \Phi^{-1}_s(v) \in \mathbb{R}^s \) be the inverse of the cumulative normal distribution function applied to each entry of \( v \). We write \( F(y) := F(y_1, \ldots, y_s) = \mathcal{G}(u^s(\cdot, y)) \) and
\[
I_s(F) := \int_{\mathbb{R}^s} F(\Phi^{-1}_s(v)) \, dv = \int_{\mathbb{R}^s} \mathcal{G}(u^s(\cdot, y)) \prod_{j=1}^s \phi(y_j) \, dy = \mathbb{E}[\mathcal{G}(u^s)],
\] (2.7)
where \( \phi \) is the probability density function of the standard normal random variable. The measurability of the mapping \( \mathbb{R}^s \ni y \mapsto \mathcal{G}(u^s(\cdot, y)) \in \mathbb{R} \) will be discussed later.

In order to approximate \( I_s(F) \), we employ a QMC method called a randomly shifted lattice rule. This is an equal-weight quadrature rule of the form
\[
Q_{s,n}(\Delta; F) := \frac{1}{n} \sum_{i=1}^n F\left( \phi^{-1}_s\left( \frac{iz}{n} + \Delta \right) \right),
\]
where the function \( \phi^{-1}_s : \mathbb{R}^s \ni y \mapsto \phi(y) \in [0, 1]^s \) takes the fractional part of each component in \( y \). Here, \( z \in \mathbb{N}^s \) is a carefully chosen point called the (deterministic) generating vector and \( \Delta \in [0, 1]^s \) is the random shift. We assume the random shift \( \Delta \) is a \( [0, 1]^s \)-valued uniform random variable defined on a suitable probability space different from \( (\Omega, \mathcal{F}, \mathbb{P}) \). For further details of the randomly shifted lattice rules, we refer to the surveys [11, 20] and references therein.

We want to evaluate the root-mean-square error
\[
\sqrt{\mathbb{E}[\mathcal{G}(u)] - Q_{s,n}(\Delta; F)}^2.
\] (2.8)
where $E^\Delta$ is the expectation with respect to the random shift. Note that in practice the solution $u^s$ needs to be approximated by some numerical scheme $\tilde{u}^s$, which results in computing $F(y) := G(\tilde{u}^s(y))$. Thus, the error $e_{s,n} := \sqrt{E^\Delta \left[ (E[G(u)] - Q_{s,n}(\Delta; \tilde{F}))^2 \right]}$ is what we need to evaluate in practice. Via the trivial decompositions we have, using $E^\Delta [Q_{s,n}(\Delta; \tilde{F})] = E[G(\tilde{u})]$ (see, for example, [11]),

$$e_{s,n}^2 = \left( E[G(u) - G(\tilde{u})]\right)^2 + E^\Delta \left[ (E[G(\tilde{u}^s)] - Q_{s,n}(\Delta; \tilde{F}))^2 \right]$$

$$\leq 2 \left( E[G(u) - G(\tilde{u})]\right)^2 + 2 \left( E[G(\tilde{u}) - G(\tilde{u}^s)] \right)^2 + E^\Delta \left[ (E[G(\tilde{u}^s)] - Q_{s,n}(\Delta; \tilde{F}))^2 \right],$$

(2.9)

(2.10)

where $\tilde{u}$ is an approximation of the solution $u$ of (2.3) with the same scheme as $\tilde{u}^s$.

For the sake of simplicity, we forgo the discussion on the numerical approximation of the solution of the PDE. Instead, we discuss the smoothness of the realisations of the random coefficient. Then, given a suitable smoothness of the boundary $\partial D$, the convergence rate of $E[G(u) - G(\tilde{u}^s)]$ is typically obtained from the smoothness of the realisations of the coefficients $a(\cdot, y)$, via the regularity of the solution $u$. See [16, 20, 21]. Therefore in the following, we concentrate on the truncation error and the quadrature error, the second and the third term of the above decomposition, and the realisations of $a$.

In the course of the error analyses, we assume $(\psi_j)$ satisfies the following assumption.

**Assumption B.** The system $(\psi_j)$ satisfies the following. There exists a positive sequence $(\rho_j)$ such that

$$\sup_{x \in D} \sum_{j \geq 1} \rho_j |\psi_j(x)| =: \kappa < \ln 2,$$

(\textbf{b1})

and further,

$$(1/\rho_j) \in \ell^q \quad \text{for some } q \in (0, 1].$$

(\textbf{b2})

We also use the following weaker assumption.

**Assumption B'.** The same as Assumption B, only with the condition (\textbf{b2}) being replaced with

$$(1/\rho_j) \in \ell^q \quad \text{for some } q \in (0, \infty).$$

(\textbf{b2}')

We note that (\textbf{b2'}), and thus also (\textbf{b2}), implies $\rho_j \to \infty$ as $j \to \infty$.

Some remarks on the assumptions are in order. First note that Assumption \textbf{H} implies $\sum_{j \geq 1} |\psi(x)| < \infty$ for any $x \in D$, and hence (2.2). Assumption \textbf{I'} is used to obtain an estimate on the mixed derivative with respect to the random parameter $y_j$, and further, ensures the almost surely well-posedness of the problem (2.5) — see Corollary 3.2 and Remark 1. Assumption \textbf{B} is used to obtain a dimension-independent QMC error estimate — see Theorems 4.4 and 5.1. The stronger the condition \textbf{B2} the system $(\psi_j)$ satisfies, that is, the smaller is $q$, the smoother the realisations of the random coefficient become. In Section 5.2 we discuss smoothness of realisations allowed by these conditions.
3 Bounds on mixed derivatives

In this section, we discuss bounds on mixed derivatives. In order to motivate the discussion in this section, first we explain how the derivative bounds come into play in the QMC analysis developed in the next section.

Application of QMC methods to elliptic PDEs with log-normal random coefficients was initiated with computational results by Graham et al. [15], and an analysis was followed by Graham et al. [16]. Following the discussion by [16], we assume the integrand \( F \) is in the space called the \emph{weighted unanchored Sobolev space} \( W_s \), consisting of measurable functions \( F: \mathbb{R}^s \to \mathbb{R} \) such that

\[
\| F \|_{W_s}^2 = \sum_{u \subseteq \{1:s\}} \frac{1}{\gamma_u} \int_{\mathbb{R}^{|u|}} \left( \int_{\mathbb{R}^{|u|-|u|}} \frac{\partial^{|u|} F}{\partial y_u} (y_u; y_{\{1:s\}\setminus u}) \prod_{j \in \{1:s\}\setminus u} \phi(y_j) \, dy_{\{1:s\}\setminus u} \right)^2 \prod_{j \in u} w_j^2(y_j) \, dy_u < \infty,
\]

where we assume, similarly to [16], that

\[
w_j^2(y_j) = \exp(-2\alpha_j |y|)
\]

for some \( \alpha_j > 0 \). Here, \( \{1:s\} \) is a shorthand notation for the set \( \{1,\ldots,s\} \), \( \frac{\partial^{|u|} F}{\partial y_u} \) denotes the mixed first derivative with respect to each of the “active” variables \( y_j \) with \( j \in u \subseteq \{1:s\} \), and \( y_{\{1:s\}\setminus u} \) denotes the “inactive” variables \( y_j \) with \( j \not\in u \). Further, weights \( (\gamma_u) \) describe the relative importance of the variables \( \{y_j\}_{j \in u} \). Note that the measure \( \int \, dy_u \) and \( \int \frac{1}{\gamma_u} \, dy_u \) differ by at most a constant factor depending on \( u \). Weights \( (\gamma_u) \) play an important role in deriving error estimates independently of the dimension \( s \), and further, in obtaining the generating vector \( z \) for the lattice rule via the component-by-component (CBC) algorithm.

Depending on the problem, different types of weights have been considered to derive error estimates. For the randomly shifted lattice rules, “POD weights” and “product-weights” have been considered ([11, 20]). When applied to the PDE parametrised with log-normal coefficients, the result in [16] suggests the use of POD weights for the problem.

We wish to develop a theory on the applicability of product weights, which has an advantage in terms of computational cost. The computational cost of the CBC construction is \( O(sn \log n + ns^2) \) in the case of POD weights, compared to \( O(sn \log n) \) for product weights [12]. Since we often want to approximate the random field well, and so necessarily we have large \( s \), the applicability of product weights is of clear interest.

Estimates of derivatives of the integrand \( F(y) \) with respect to the parameter \( y \), that is, the variable with which \( F(y) \) is integrated, are one of the keys in the error analysis of QMC. In [16], it was the estimates being of “POD-form” that led their theory to the POD weights. Under an assumption on the system \( (\psi_j) \), which is different from that in [16], we show that the derivative estimates turn out to be of “product-form”, and further that, under a suitable assumption, we achieve the same error convergence rate close to 1 with product weights.

Now, we derive an estimate of the product form. Let \( \mathcal{F} := \{\mu = (\mu_1, \mu_2, \ldots) \in \mathbb{N}_0^{\mathbb{N}} \mid \text{all but finite number of components of } \mu \text{ are zero}\} \). For \( \mu \in \mathcal{F} \) we use the notation \( |\mu| = \)
\[\sum_{j \geq 1} \mu_j \cdot \mu! = \prod_{j \geq 1} \mu_j!, \quad \rho^\mu = \prod_{j \geq 1} \rho_j^{\mu_j} \quad \text{for} \quad \rho = (\rho_j)_{j \geq 1} \in \mathbb{R}^N, \quad \text{and} \quad \partial^\mu u = \frac{\partial|\mu|}{y_j^{\mu_j(1)} \cdots y_j^{\mu_j(k)}} u, \tag{3.3}\]

where \( k = \# \{ j \mid \mu_j \neq 0 \} \).

We have the following bound on mixed derivatives of order \( r \geq 1 \) (although in our application we will need only \( r = 1 \)). The proof follows essentially the same argument as the proof by Bachmayr et al. [3, Theorem 4.1]. Here, we show a tighter bound by changing the condition from \( \frac{\ln 2}{\sqrt{r}} \) to \( \frac{\ln 2}{r} \) in [3, (91)], and we have \( \rho^2 \mu \) in (3.3) in place of \( \frac{\rho^2 \mu}{\mu^r} \) in the left hand side of [3, (92)].

**Proposition 3.1.** Let \( r \geq 1 \) be an integer. Suppose \( (\psi_j) \) satisfies the condition (1.1) with \( \ln 2 \) replaced by \( \frac{\ln 2}{r} \), with a positive sequence \( (\rho_j) \). Then, there exists a constant \( C_0 = C_0(r) \) that depends on \( k \) and \( r \), such that

\[\sum_{\mu \in \mathcal{F}} \rho^2 \mu \int_D a(y)|\nabla(\partial^\mu u(y))|^2 \, dx \leq C_0 \int_D a(y)|\nabla u(y)|^2 \, dx. \tag{3.4}\]

for all \( y \) that satisfies \( \| \sum_{j \geq 1} y_j \psi_j \|_{L^{\infty}(D)} < \infty \), where \( u(y) \) is the solution of (2.5) for such \( y \). The same bound holds also for \( u^r(y) \), the solution of (2.5) with \( y = (y_1, \ldots, y_k, 0, 0, \ldots) \).

**Proof.** Let

\[\Lambda_k := \{ \mu \in \mathcal{F} \mid |\mu| = k \quad \text{and} \quad \| \mu \|_{\mathcal{F}} \leq r \}, \quad \text{and} \quad \mathcal{S}_k := \{ \nu \in \mathcal{F} \mid \nu \leq \mu \quad \text{and} \quad \nu \neq \mu \} \]

for \( \mu \in \mathcal{F} \), with \( \leq \) denoting the component-wise partial order between multi-indices. Let us introduce the notation \( \| v \|_{a(y)}^2 := \int_D a(y)|\nabla v|^2 \, dx \) for all \( v \in V \), and let

\[\sigma_k := \sum_{\mu \in \Lambda_k} \rho^2 \mu \| \partial^\mu u(y) \|_{a(y)}^2. \]

We show below that we can choose \( \delta = \delta(r) < 1 \) such that

\[\sigma_k \leq \sigma_0 \delta^k \quad \text{for all} \quad k \geq 0. \tag{3.5}\]

Note that if this holds then we have

\[\sum_{|\mu|_{\mathcal{F}} \leq r} \rho^2 \mu \| \partial^\mu u(y) \|_{a(y)}^2 = \sum_{k=0}^{\infty} \sum_{\mu \in \Lambda_k} \rho^2 \mu \| \partial^\mu u(y) \|_{a(y)}^2 \leq \sum_{k=0}^{\infty} \sigma_k \leq \sigma_0 \sum_{k=0}^{\infty} \delta^k < \infty, \tag{3.6}\]

and the statement will follow with \( C_0 = C_0(r) = \sum_{k=0}^{\infty} \delta(r)^k \).

We now show \( \sigma_k \leq \sigma_0 \delta^k \). Note that from the assumption \( \| \sum_{j \geq 1} y_j \psi_j \|_{L^{\infty}(D)} < \infty \), in view of [3, Lemma 3.2] we have \( \partial^\mu u \in V \) for any \( \mu \in \mathcal{F} \). Thus, by taking \( v := \partial^\mu u \) (\( \mu \in \Lambda_k \)) in [3, (74)], we have

\[\sigma_k = \sum_{\mu \in \Lambda_k} \rho^2 \mu \int_D a(y)|\nabla \partial^\mu u(y)|^2 \, dx \leq \sum_{\mu \in \Lambda_k} \sum_{\nu \in \mathcal{S}_k} \left( \prod_{j \geq 1} \frac{\mu_j^{\nu_j - \nu_j}}{\nu_j!(\mu_j - \nu_j)!} \right) \int_D a(y) \left( \prod_{j \geq 1} |\psi_j|^{\mu_j - \nu_j} \right) |\nabla \partial^\nu u(y)||\nabla \partial^\nu u(y)| \, dx. \tag{3.7}\]
Using the notation
\[
\epsilon(\mu, \nu)(x) := \epsilon(\mu, \nu) := \frac{\mu! \rho^{\mu-\nu} |\psi|^{\mu-\nu}}{\nu! (\mu - \nu)!},
\] (3.8)
and the Cauchy–Schwarz inequality for the sum over \(S_\mu\), it follows that
\[
\sigma_k \leq \int_D \sum_{\mu \in \Lambda_k} \sum_{\nu \in S_\mu} \epsilon(\mu, \nu)a(y)|\rho^\nu \nabla \partial^\nu u(y)||\rho^\mu \nabla \partial^\mu u(y)| \, dx
\] (3.9)
\[
\leq \int_D \sum_{\mu \in \Lambda_k} \left( \sum_{\nu \in S_\mu} \epsilon(\mu, \nu)a(y)|\rho^\nu \nabla \partial^\nu u(y)| \right)^\frac{1}{2} \left( \sum_{\nu \in S_\mu} \epsilon(\mu, \nu)a(y)|\rho^\mu \nabla \partial^\mu u(y)|^2 \right)^\frac{1}{2} \, dx.
\] (3.10)

Let
\[
S_{\mu, \ell} := \{ \nu \in S_\mu \mid |\mu - \nu| = \ell \}.
\] Then, for \(\mu \in \Lambda_k\) we have
\[
S_\mu = \left\{ \nu \in \mathcal{F} \mid \nu \leq \mu, \nu \neq \mu \right\} = \bigcup_{\ell=1}^{|\mu|} \left\{ \nu \in \mathcal{F} \mid \nu \leq \mu, |\mu - \nu| = \ell \right\} = \bigcup_{\ell=1}^{|\mu|} S_{\mu, \ell},
\]
and further, from \(|\mu| = k\), we have
\[
\sum_{\nu \in S_\mu} \epsilon(\mu, \nu) = \sum_{\ell=1}^{k} \sum_{\nu \in S_{\mu, \ell}} \epsilon(\mu, \nu) = \sum_{\ell=1}^{k} \frac{\mu! \rho^{\mu-\nu} |\psi|^{\mu-\nu}}{\nu! (\mu - \nu)!}.
\] (3.11)

Since \(\nu \in S_{\mu, \ell}\) implies \(\sum_{j \in \text{supp } \mu} (\mu_j - \nu_j) = \ell\), there are \(\ell\) factors in \(\frac{\mu!}{\nu!} = \prod_{j \in \text{supp } \mu} (\mu_j - 1) \cdots (\nu_j + 1)\). From \(\mu_j \leq r\) \((j \in \text{supp } \mu)\), each of the factors is at most \(r\). Thus,
\[
\frac{\mu!}{\nu!} \leq r^\ell \quad \text{for } \mu \in \Lambda_k, \nu \in S_{\mu, \ell}.
\]
Therefore, from the multinomial theorem, for each \(x \in D\) it follows from (3.11) that
\[
\sum_{\nu \in S_\mu} \epsilon(\mu, \nu) \leq \sum_{\ell=1}^{k} r^\ell \sum_{\nu \in S_{\mu, \ell}} \frac{\rho^{\mu-\nu} |\psi|^{\mu-\nu}}{(\mu - \nu)!} \leq \sum_{\ell=1}^{k} r^\ell \sum_{|\tau|=\ell} \frac{\rho^\tau |\psi|^\tau}{\tau!} = \sum_{\ell=1}^{k} r^\ell \frac{1}{\ell!} \sum_{|\tau|=\ell} \frac{\ell!}{\tau!} \rho^\tau |\psi|^\tau
\] (3.12)
\[
= \sum_{\ell=1}^{k} r^\ell \frac{1}{\ell!} \left( \sum_{j=1}^{\infty} \rho_j |\psi_j|^\ell \right) \leq \sum_{\ell=1}^{k} r^\ell \frac{1}{\ell!} \kappa^\ell \leq e^{r\kappa} - 1 \leq e^{\ln 2} - 1 = 1.
\] (3.13)

Inserting into (3.10), we have
\[
\sum_{\mu \in \Lambda_k} \rho^{2\mu} \|\partial^\mu u(y)\|_{a(y)}^2 \leq \int_D \sum_{\mu \in \Lambda_k} \left( \sum_{\nu \in S_\mu} \epsilon(\mu, \nu)a(y)|\rho^\nu \nabla \partial^\nu u(y)| \right)\frac{1}{2} \left( \sum_{\nu \in S_\mu} \epsilon(\mu, \nu)a(y)|\rho^\nu \nabla \partial^\nu u(y)|^2 \right)^\frac{1}{2} \, dx.
\] (3.14)
Again applying the Cauchy–Schwarz inequality to the summation over $\Lambda_k$ and then to the integral, we have

$$\sigma_k \leq \int_D \left( \sum_{\mu \in \Lambda_k} \sum_{\nu \in S_\mu} \epsilon(\mu, \nu) a(y) |\rho^\nu \xi^\nu u(y)|^2 \right)^{\frac{1}{2}} \left( \sum_{\mu \in \Lambda_k} a(y) |\rho^\nu \xi^\nu u(y)|^2 \right)^{\frac{1}{2}} \, dx \leq \left( \sum_{\mu \in \Lambda_k} \sum_{\nu \in S_\mu} \epsilon(\mu, \nu) a(y) |\rho^\nu \xi^\nu u(y)|^2 \right)^{\frac{1}{2}} \sigma_k^\frac{1}{2},$$

and hence

$$\sigma_k \leq \int_D \sum_{\mu \in \Lambda_k} \sum_{\nu \in S_\mu} \epsilon(\mu, \nu) a(y) |\rho^\nu \xi^\nu u(y)|^2 \, dx. \quad (3.15)$$

Now, for any $k \geq 1$ and any $\nu \in \Lambda_k = \{ \nu \in \mathcal{F} \mid |\nu| = \ell, \| \nu \|_\infty \leq r \}$ with $\ell \leq k - 1$, let

$$R_{\nu, \ell, k} := \{ \mu \in \Lambda_k \mid \nu \in S_\mu \} = \{ \mu \in \mathcal{F} \mid |\mu| = k, \| \mu \|_\infty \leq r, \mu \geq \nu, \mu \neq \nu \}.$$

Then, for fixed $k \geq 1$ we can write

$$\bigcup_{\mu \in \Lambda_k} \bigcup_{\nu \in S_\mu} (\mu, \nu) = \bigcup_{\ell=0}^{k-1} \bigcup_{\nu \in \Lambda_\ell} \bigcup_{\mu \in R_{\nu, \ell, k}} (\mu, \nu). \quad (3.16)$$

Thus, we have

$$\sum_{\mu \in \Lambda_k} \sum_{\nu \in S_\mu} \epsilon(\mu, \nu) a(y) |\rho^\nu \xi^\nu u(y)|^2 = \sum_{\ell=0}^{k-1} \sum_{\nu \in \Lambda_\ell} a(y) |\rho^\nu \xi^\nu u(y)|^2 \sum_{\mu \in R_{\nu, \ell, k}} \epsilon(\mu, \nu). \quad (3.17)$$

Now, note that $k - \ell = \sum_{j \in \text{supp} \mu}^H - \sum_{j \in \text{supp} \mu}^\nu = |\mu - \nu|$. Thus, we have $\frac{\mu^\nu}{\nu^\nu} \leq r^{k-\ell}$. It follows that

$$\sum_{\mu \in R_{\nu, \ell, k}} \epsilon(\mu, \nu) = \sum_{\nu \in R_{\nu, \ell, k}} \frac{\mu^\nu}{\nu^\nu} \frac{\rho^\nu |\psi|^\mu |\psi|^\nu}{(\mu - \nu)!} \leq r^{k-\ell} \sum_{\nu \in R_{\nu, \ell, k}} \frac{\rho^\nu |\psi|^\mu |\psi|^\nu}{(\mu - \nu)!} \leq r^{k-\ell} \sum_{|\tau| = k-\ell} \frac{\rho^\tau |\psi|^\tau}{\tau!} \leq r^{k-\ell} \frac{1}{(k-\ell)!} k^{k-\ell}. \quad (3.18)$$

Then, substituting (3.14) into (3.17) we obtain from (3.15)

$$\sigma_k \leq \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell)!} (rK)^{k-\ell} \sigma_\ell. \quad (3.20)$$

From the assumption we have $\kappa < \frac{\ln 2}{r}$. Thus, we can take $\delta = \delta(r) < 1$ such that $\kappa < \delta \frac{\ln 2}{r}$. 

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We show \( \sigma_k \leq \sigma_0 \delta^k \) for all \( k \geq 0 \) by induction. This is clearly true for \( k = 0 \). Suppose \( \sigma_\ell \leq \sigma_0 \delta^\ell \) holds for \( \ell = 0, \ldots, k-1 \). Then, for \( \ell = k \) we have

\[
\sigma_k \leq \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell)!} (r\kappa)^{k-\ell} \sigma_\ell \leq \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell)!} (r\kappa)^{k-\ell} \sigma_0 \delta^\ell \leq \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell)!} (\ln 2)^{k-\ell} \sigma_0 \delta^\ell
\]

(3.21)
\[
= \sigma_0 \delta^k \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell)!} (\ln 2)^{k-\ell} \leq \sigma_0 \delta^k \left( e^{\ln 2} - 1 \right) = \sigma_0 \delta^k,
\]

(3.22)
which completes the proof. \( \square \)

With the notation

\[
\hat{a}(y) := \operatorname{ess inf}_{x \in D} a(x, y), \quad \text{and} \quad \underline{a}(y) := \operatorname{ess sup}_{x \in D} a(x, y),
\]

(3.23)
we have the following corollary, where here and from now on we set \( r = 1 \).

**Corollary 3.2.** Suppose \((\psi_j)\) satisfies Assumption \( B^r \) with a positive sequence \((\rho_j)\). Then, for \( C_0 = C_0(1) \) as in Proposition \( 3.1 \) for any \( u \subset N \) of finite cardinality we have

\[
\left\| \frac{\partial^{\mid u \mid} u}{\partial y_{u}} \right\|_{V'} \leq \sqrt{C_0} \frac{\| f \|_{V'}}{\underline{a}(y)} \prod_{j \in u} \frac{1}{\rho_j} < \infty, \quad \text{almost surely,}
\]

(3.24)
where \( \| \cdot \|_{V'} \) is the norm in the dual space \( V' \). The same bound holds also for \( \left\| \frac{\partial^{\mid u \mid} u}{\partial y_{u}} \right\|_{V'} \), with \( y = (y_1, \ldots, y_s, 0, 0, \ldots) \).

**Proof.** First, if \( y \in \mathbb{R}^N \) satisfies \( \| \sum_{j \geq 1} y_j \psi_j \|_{L^\infty(D)} < \infty \), then we have \( \frac{1}{\hat{a}(y)} < \infty \):

\[
\hat{a}(y) \geq \left( \inf_{x \in D} a_0(x) \right) \exp \left( - \operatorname{ess sup}_{x \in D} \sum_{j \geq 1} y_j \psi_j(x) \right),
\]

(3.25)
and thus

\[
\frac{1}{\hat{a}(y)} \leq \left( \inf_{x \in D} a_0(x) \right) \exp \left( \operatorname{ess sup}_{x \in D} \sum_{j \geq 1} y_j \psi_j(x) \right).
\]

(3.26)
Now, from \( (1/\rho_j) \in \ell^q \) for some \( q \in (0, \infty) \), in view of \ref{3}, Remark 2.2] we have

\[
\mathbb{E} \left[ \exp \left( k \sum_{j \geq 1} y_j \psi_j \right \|_{L^\infty(D)} \right) \] < \infty,
\]
for any \( 0 \leq k < \infty \). Thus, \( \| \sum_{j \geq 1} y_j \psi_j \|_{L^\infty(D)} < \infty \), and the right hand side of \( \ref{3.24} \) is bounded with full (Gaussian) measure. We remark that the \( B(\mathbb{R}^N)/B(\mathbb{R}) \)-measurability of the mapping \( y \mapsto \| \sum_{j \geq 1} y_j \psi_j \|_{L^\infty(D)} \) is not an issue. See \ref{3}, Remark 2.2] noting the continuity of norms, together with, for example, \ref{24}, Appendix to IV. 5].

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Now, noting that the standard argument regarding the continuous dependence of the solution of the variational problem (2.5) on \( f \), we have

\[
\int_D |\nabla (\frac{\partial |u|}{\partial y_a})|^2 \, dx \leq \sum_{\|\mu\|_\infty \leq 1} \rho^{2\mu} \int_D a(y)|\nabla (\partial^\mu u(y))|^2 \, dx.
\] (3.27)

**Remark 1.** We note that following a similar discussion to the above, \( \hat{a}(y) \) can be bounded almost surely. Thus, under the Assumption B\(^{\prime}\), the well-posedness of the problem (2.5) readily follows almost surely. Further, Assumption B\(^{\prime}\) implies the measurability of the mapping \( R_s \ni y \mapsto G(u_s(\cdot, y)) \in \mathbb{R} \). See [3, Corollary 2.1, Remark 2.2] noting \( G \in V^\prime \), together with the fact that a strongly \( F \)-measurable \( V \)-valued mapping is weakly \( F \)-measurable. For more details on the measurability of vector-valued functions, see for example, [24, 28].

4 QMC integration error with product weights

Based on the bound on mixed derivatives obtained in the previous section, now we derive a QMC convergence rate with product weights.

We first introduce some notations. Let

\[
\varsigma_j(\lambda) := 2 \left( \frac{\sqrt{2\pi} \exp(\alpha_j^2/\Lambda^*)}{\pi^{2-2\lambda^*}(1-\Lambda^*)^{\lambda^*}} \right)^\lambda \zeta(\lambda + \frac{1}{2}),
\] (4.1)

where \( \Lambda^* := \frac{2\lambda - 1}{4\lambda} \), and \( \zeta(x) := \sum_{k=1}^\infty k^{-x} \) denotes the Riemann zeta function.

**Theorem 4.1.** (\cite[Theorem 15]{16}) Let \( F \in W^s \). Given \( s, n \in \mathbb{N} \) with \( 2 \leq n \leq 10^3 \), weights \( \gamma = (\gamma_u)_{u \subseteq \mathbb{N}} \), and the standard normal density function \( \phi \), a randomly shifted lattice rule with \( n \) points in \( s \) dimensions can be constructed by a component-by-component algorithm such that,

\[
\sqrt{\mathbb{E} \| I_s(F) - Q_{s,n}(\Delta; F) \|^2} \leq 9 \left( \sum_{\emptyset \neq u \subseteq \{1:s\}} \gamma_u^\lambda \prod_{j \in u} \varsigma_j(\lambda) \right)^{\frac{1}{2}} n^{-\frac{1}{2}} \| F \|_{W^s}. \] (4.2)

For the weight function (3.2) we assume that the \( \alpha_j \) satisfy for some constants \( 0 < \alpha_{\min} < \alpha_{\max} < \infty \),

\[
\max \left\{ \frac{\ln 2}{\rho_j}, \alpha_{\min} \right\} < \alpha_j \leq \alpha_{\max}, \quad j \in \mathbb{N}.
\] (4.3)

For example, under Assumption B\(^{\prime}\) letting \( \alpha_j := 1 + \frac{\ln 2}{\rho_j} \) satisfies (4.3) with \( \alpha_{\min} := 1 \) and \( \alpha_{\max} := 1 + \sup_{j \geq 1} \frac{\ln 2}{\rho_j} \).

We have the following bound on \( \| F \|^2_{W^s} \). The argument is essentially by Graham et al. \cite[Theorem 16]{16}.
Proposition 4.2. Suppose Assumption [\(Z\)] is satisfied with a positive sequence \((\rho_j)\) such that
\[(1/\rho_j) \in \ell^1. \tag{4.4}\]
Then, we have
\[
\|F\|_{\mathcal{W}^s}^2 \leq (C^*)^2 \sum_{u \subseteq \{1:s\}} \frac{1}{\gamma_u} \left( \frac{1}{\prod_{j \in u} \rho_j} \right)^2 \prod_{j \in u} \frac{1}{\alpha_j} - (\ln 2)/\rho_j, \tag{4.5}\]
with a positive constant \(C^* := \frac{\|f\|_{\mathcal{W}^r} \|\mathcal{G}\|_{\mathcal{W}^r} \sqrt{C_0}}{\inf_{x \in D} a_0(x)} \left[ \exp \left( \frac{1}{7} \sum_{j \geq 1} \frac{(\ln 2)^2}{\rho_j} + \frac{2}{\sqrt{2\pi}} \sum_{j \geq 1} \frac{\ln 2}{\rho_j} \right) \right] < \infty. \]

Proof. In this proof we abuse the notation slightly and \(y\) always denotes \((y_1, \ldots, y_s, 0, 0, \ldots) \in \mathbb{R}^N.\) From (4.1) and (4.4), in view of Corollary 3.2 for \(\mathbb{P}_Y\)-almost every \(y\) we have
\[
\left| \frac{\partial^{[u]} F}{\partial y_u} \right| \leq \|\mathcal{G}\|_{\mathcal{W}^r} \left\| \frac{\partial^{[u]} \nu^s}{\partial y_u} \right\|_{\mathcal{V}} \leq \|\mathcal{G}\|_{\mathcal{W}^r} \sqrt{C_0} \prod_{j \in u} \frac{1}{\rho_j} \|f\|_{\mathcal{W}^r}. \tag{4.6}\]
Since
\[
\sup_{x \in D} \sum_{j \geq 1} |y_j| |\psi_j(x)| \leq \left( \sup_{j \geq 1} \frac{|y_j|}{\rho_j} \right) \sup_{x \in D} \sum_{j \geq 1} \rho_j |\psi_j(x)| \leq \sum_{j \geq 1} \frac{|y_j|}{\rho_j} \sum_{j \geq 1} \rho_j |\psi_j(x)|,
\]
the condition (b1) and equations (4.6) and (3.2) together with \(y_j = 0\) for \(j > s,\) imply
\[
\left| \frac{\partial^{[u]} F}{\partial y_u} \right| \leq \frac{K^*}{\prod_{j \in u} \rho_j} \prod_{j \in \{1:s\}} \exp \left( \frac{\ln 2}{\rho_j} |y_j| \right), \tag{4.7}\]
where \(K^* := \frac{\|f\|_{\mathcal{W}^r} \|\mathcal{G}\|_{\mathcal{W}^r} \sqrt{C_0}}{\inf_{x \in D} a_0(x)}.\) Then it follows from (3.1) that
\[
\|F\|_{\mathcal{W}^s}^2 = \sum_{u \subseteq \{1:s\}} \frac{1}{\gamma_u} \int_{\mathbb{R}^{|u|}} \left( \int_{\mathbb{R}^{-|u|}} \left| \frac{\partial^{[u]} F}{\partial y_u}(y_u; y_{\{1:s\}\setminus u}) \right| \prod_{j \in \{1:s\}\setminus u} \phi(y_j) \, dy_{\{1:s\}\setminus u} \right)^2 \prod_{j \in u} w_j^2(y_j) \, dy_u \tag{4.8}\]
\[
\leq \sum_{u \subseteq \{1:s\}} \frac{1}{\gamma_u} \int_{\mathbb{R}^{|u|}} \left( \prod_{j \in \{1:s\}\setminus u} \frac{K^*}{\rho_j} \prod_{j \in \{1:s\}\setminus u} \exp \left( \frac{\ln 2}{\rho_j} |y_j| \right) \prod_{j \in \{1:s\}\setminus u} \phi(y_j) \, dy_{\{1:s\}\setminus u} \right)^2 \prod_{j \in u} w_j^2(y_j) \, dy_u \tag{4.9}\]
\[
= (K^*)^2 \sum_{u \subseteq \{1:s\}} \frac{1}{\gamma_u} \left( \prod_{j \in u} \rho_j \right)^2 \times \left( \prod_{j \in \{1:s\}\setminus u} \exp \left( \frac{\ln 2}{\rho_j} |y_j| \right) \prod_{j \in \{1:s\}\setminus u} \phi(y_j) \, dy_{\{1:s\}\setminus u} \right)^2 \times \int_{\mathbb{R}^{|u|}} \prod_{j \in u} \exp \left( \frac{2 \ln 2}{\rho_j} |y_j| \right) \prod_{j \in u} w_j^2(y_j) \, dy_u. \tag{4.10}\]
Note that this takes essentially the same form as [16, (4.14)]. Thus, the rest of the proof is in parallel to that of [16, Theorem 16].

Noting that $2\alpha_j - \frac{4\ln^2 \rho_j}{\rho_j} < 0$, and following the same argument as in [16, (4.15)–(4.17)], we have

$$
\|F\|_{W^s}^2 \leq (K^*)^2 \sum_{u \subseteq \{1:s\}} \gamma_u \left( \prod_{j \in u} \rho_j \right)^2 \left( \prod_{j \in \{1:s\} \setminus u} 2 \exp \left( \frac{2 \ln 2}{2\rho_j^2} \right) \Phi \left( \frac{\ln 2}{\rho_j} \right) \right)^2 \prod_{j \in u} \frac{1}{\alpha_j - \ln 2/\rho_j},
$$

with $\Phi(\cdot)$ denoting the cumulative standard normal distribution function. Comparing this to [16, Equation (4.17)], the statement follows from the rest of the proof of [16, Theorem 16].

As in [16, Theorem 17], from Theorem 4.1 and Proposition 4.2 we have the following.

**Proposition 4.3.** For each $j \geq 1$, let $w_j(t) = \exp(-2\alpha_j|t|)$ ($t \in \mathbb{R}$) with $\alpha_j$ satisfying (4.3).

Given $s$, $n \in \mathbb{N}$ with $2 \leq n \leq 10^{30}$, weights $\gamma = (\gamma_u)_{u \subseteq \{1:s\}}$, and the standard normal density function $\phi$, a randomly shifted lattice rule with $n$ points in $s$ dimensions can be constructed by a component-by-component algorithm such that, for all $\lambda \in (1/2, 1)$,

$$
\sqrt{\mathbb{E}[\Delta | I_s(F) - Q_{s,n}(\Delta; F)]^2} \leq 9C^* C_{\gamma,s}(\lambda)n^{-\frac{1}{2}},
$$

with

$$
C_{\gamma,s}(\lambda) := \left( \sum_{u \subseteq \{1:s\}} \gamma_u^\lambda \prod_{j \in u} \varsigma_j(\lambda) \right)^{\frac{1}{\lambda}} \left( \sum_{u \subseteq \{1:s\}} \gamma_u \left( \prod_{j \in u} \rho_j \right)^2 \prod_{j \in u} \frac{1}{\alpha_j - \ln 2/\rho_j} \right)^{\frac{1}{2}},
$$

and $C^*$ defined as in Proposition 4.2.

We choose weights of the product form

$$
\gamma_u = \gamma_u^*(\lambda) := \left[ \left( \prod_{j \in u} \rho_j \right)^2 \prod_{j \in u} \varsigma_j(\lambda) [\alpha_j - \ln 2/\rho_j] \right]^{\frac{1}{1+\lambda}}.
$$

Then, it turns out that under a suitable value of $\lambda$ the constant (4.13) can be bounded independently of $s$, and we have the QMC error bound as follows.

**Theorem 4.4.** For each $j \geq 1$, let $w_j(t) = \exp(-2\alpha_j|t|)$ ($t \in \mathbb{R}$) with $\alpha_j$ satisfying (4.3). Let $\varsigma_{\max}(\lambda)$ be $\varsigma_j$ defined by (4.1) but $\alpha_j$ being replaced by $\alpha_{\max}$. Suppose $(\psi_j)$ satisfies Assumption B. Suppose further that, we choose $\lambda$ as

$$
\lambda = \begin{cases} 
\frac{1}{2} - \frac{\delta}{2} & \text{for arbitrary } \delta \in (0, 1/2] \\
\frac{2q}{2-q} & \text{when } q \in (0, \frac{2}{3}] \\
\frac{2}{3} & \text{when } q \in (\frac{2}{3}, 1],
\end{cases}
$$

and choose the weights $\gamma_u$ as in (4.14).
Then, given \( s, n \in \mathbb{N} \) with \( n \leq 10^{30} \), and the standard normal density function \( \phi \), a randomly shifted lattice rule with \( n \) points in \( s \) dimensions can be constructed by a component-by-component algorithm such that
\[
\sqrt{\mathbb{E}_{\Delta} |J_s(F)|} - Q_{s,n}(\Delta; F) \leq \begin{cases} 
9C_{\rho,q,\delta} C^* n^{-(1-\delta)} & \text{when } 0 < q \leq \frac{2}{3}, \\
9C_{\rho,q} C^* n^{-\frac{2}{3}} & \text{when } \frac{2}{3} < q \leq 1.
\end{cases}
\]
where the constant \( C_{\rho,q,\delta} \) (resp. \( C_{\rho,q} \)) is independent of \( s \) but depends on \( \rho := (\rho_j) \), \( q \) and \( \delta \) (resp. \( \rho \) and \( q \)), and \( C^* \) is defined as in Proposition 4.2.

In particular, with \( \alpha_j := 1 + \ln 2/\rho_j \) we have \( \gamma_u = \left[ \left( \frac{1}{\prod_{\alpha_j} \rho_j} \right)^2 \prod_{j \in u} \frac{1}{\sqrt{\lambda_j}} \right] \frac{1}{1+\lambda} \), and the same result as above holds with the finite constants \( C_{\rho,q,\delta} \), and \( C_{\rho,q} \) both given by
\[
C_{\rho,q,\delta} = C_{\rho,q} = \left( \prod_{j=1}^{\infty} \left( 1 + \left( \frac{\varsigma_j(\lambda)}{\rho_j^2} \right)^{\frac{1}{1+\lambda}} \right) \right) \left( \prod_{j=1}^{\infty} \left( 1 + \left( \frac{\varsigma_j(\lambda)}{\rho_j^{2\lambda}} \right)^{\frac{1}{1+\lambda}} \right) \right)^{\frac{1}{2}}.
\]
with \( \lambda \) given by \( 4.13 \).

Proof. Let \( \beta_j(\lambda) := \left( \frac{\varsigma_j(\lambda)}{\rho_j^2} \right)^{\frac{1}{1+\lambda}} \). Observe that with the choice of weights \( 4.14 \) we have
\[
C_{\gamma,s}(\lambda) = \left( \sum_{\emptyset \neq u \subseteq \{1:s\}} \prod_{j \in u} \beta_j(\lambda) \right)^{\frac{1}{1+\lambda}} \left( \sum_{u \subseteq \{1:s\}} \prod_{j \in u} \beta_j(\lambda) \right)^{\frac{1}{2}}
\]
\[
= \left( \prod_{j=1}^{s} (1 + \beta_j(\lambda)) \right)^{\frac{1}{1+\lambda}} \left( \prod_{j=1}^{s} (1 + \beta_j(\lambda)) \right)^{\frac{1}{2}}.
\]
Now let \( J := \inf_{j \geq 1} (\alpha_j - \ln 2/\rho_j) \), which is a positive value from \( 4.13 \). Further, note that \( \varsigma_j(\lambda) \leq \varsigma_{\max}(\lambda) \) for \( j \geq 1 \). Then, from \( \beta_j(\lambda) \geq 0 \) we have
\[
\prod_{j=1}^{s} (1 + \beta_j(\lambda)) \leq \prod_{j=1}^{s} \exp(\beta_j(\lambda)) \leq \exp \left( \sum_{j \geq 1} \beta_j(\lambda) \right) \leq \exp \left( \left[ \frac{\varsigma_{\max}(\lambda)}{J} \right] \sum_{j \geq 1} \left[ \frac{1}{\rho_j} \right]^{\frac{2\lambda}{1+\lambda}} \right).
\]
Thus, if \( \sum_{j \geq 1} \left[ \frac{1}{\rho_j} \right]^{\frac{2\lambda}{1+\lambda}} \) is finite we can conclude that \( C_{\gamma,s}(\lambda) \) is bounded independently of \( s \).

We discuss the relation between \( q \) and the exponent \( \frac{2\lambda}{1+\lambda} \). First note that from \( \lambda \in \left( \frac{1}{2}, 1 \right] \), we have \( \frac{2}{3} < \frac{2\lambda}{1+\lambda} \leq 1 \). Suppose \( 0 < q \leq \frac{2}{3} \). In this case, we always have \( q < \frac{2\lambda}{1+\lambda} \), and thus \( (1/\rho_j) \in \ell^{\frac{2\lambda}{1+\lambda}} \). Thus, \( \sum_{j \geq 1} \left[ \frac{1}{\rho_j} \right]^{\frac{2\lambda}{1+\lambda}} \) is finite follows. Letting \( \lambda := \frac{1}{2} - \frac{1}{2q} \) with an arbitrary \( \delta \in (0, \frac{1}{2}] \), we obtain the result for \( q \in (0, \frac{2}{3}] \). Next, consider the case \( \frac{2}{3} < q \leq 1 \). Then, letting \( \lambda := \lambda(q) = \frac{a}{2-q} \), we have \( \lambda \in (1/2, 1] \) and
\[
\frac{2\lambda}{1+\lambda} = \frac{2\frac{q}{2-q}}{1+\frac{2q}{2-q}} = \frac{2q}{2-q} = q,
\]
and thus \( \sum_{j \geq 1} \left[ \frac{1}{\rho_j} \right]^{\frac{2\lambda}{1+\lambda}} \) is finite.
5 Application to a wavelet stochastic model

Cioica et al. [6] considered a stochastic model in which users can choose the smoothness at will. In this section, we consider the Gaussian case, and show that the theory developed in Section 4 can be applicable for the model with a wide range of smoothness.

5.1 Stochastic model

For simplicity we assume \( D \) is a bounded convex polygonal domain. Consider a wavelet system \((\varphi_\xi)_{\xi \in \nabla}\) that is a Riesz basis for \( L^2(D) \)-space. We explain the notations and outline the standard properties we assume as follows. The indices \( \xi \in \nabla \) typically encode both the scale, often denoted by \(|\xi|\), and the spatial location, and also the type of the wavelet. Since our analysis does not rely on the choice of a type of wavelet, we often use the notation \( \xi = (\ell, k) \), \( \nabla = \{ (\ell, k) \mid \ell \geq \ell_0, k \in \nabla_\ell \} \) where \( \nabla_\ell \) is some countable index set. The scale level \( \ell \) of \( \varphi_\xi \) is denoted by \(|\xi| = |(\ell, k)| = \ell\). Furthermore, \((\tilde{\varphi}_\xi)_{\xi \in \nabla}\) denotes the dual wavelet basis, i.e.,

\[ \langle \varphi_\xi, \tilde{\varphi}_\xi' \rangle_{L^2(D)} = \delta_{\xi \xi'}, \xi, \xi' \in \nabla. \]

In the following, \( \alpha \lesssim \beta \) means that \( \alpha \) can be bounded by some constant times \( \beta \) uniformly with respect to any parameters on which \( \alpha \) and \( \beta \) may depend. Further, \( \alpha \sim \beta \) means that \( \alpha \lesssim \beta \) and \( \beta \lesssim \alpha \).

We list the assumption on wavelets:

(W1) the wavelets \((\varphi_\xi)_{\xi \in \nabla}\) form a Riesz basis for \( L^2(D) \);

(W2) the cardinality of the index set \( \nabla_\ell \) satisfies \( \#\nabla_\ell = C_{\nabla, 2^\ell} \) for some constant \( C_{\nabla} > 0 \);

(W3) the wavelets are local. That is, the supports of \( \varphi_{\ell, k} \) are contained in balls of diameter \( \sim 2^{-\ell} \), and do not overlap too much in the following sense: there exists a constant \( M > 0 \) independent of \( \ell \) such that for each given \( \ell \) for any \( x \in D \),

\[ \# \{ k \in \nabla_\ell \mid \varphi_{\ell, k}(x) \neq 0 \} \leq M; \quad (5.1) \]

(W4) the wavelets satisfy the cancellation property

\[ |\langle v, \varphi_\xi \rangle_{L^2(D)}| \lesssim 2^{-|\xi|} 2^{d(t+\tilde{m})} v|_{W^{\tilde{m}, \infty}(\text{supp}(\varphi_\xi))}, \]

for \( |\xi| \geq \ell_0 \) with some parameter \( \tilde{m} \in \mathbb{N} \), where \( |\cdot|_{W^{\tilde{m}, \infty}} \) denotes the usual Sobolev semi-norm. That is, the inner product is small when the function \( v \) is smooth on the support \( \text{supp}(\varphi_\xi) \);

(W5) the wavelet basis induces characterisations of Besov spaces \( B^t_q(L^p(D)) \) for \( 1 \leq p, q < \infty \) and all \( t \) with \( d \max\{1/p - 1, 0\} < t < t_* \) for some parameter \( t_* > 0 \). The upper bound \( t_* \) depends on the choice of wavelet basis. Since \( t \) we consider is typically small, here for simplicity we may define the Besov norm as

\[ \| v \|_{B^t_q(L^p(D))} := \left( \sum_{\ell = \ell_0}^{\infty} 2^{(\ell + d(\frac{1}{2} - \frac{1}{p})) q} \left( \sum_{k \in \nabla_\ell} |\langle v, \tilde{\varphi}_{\ell, k} \rangle_{L^2(D)}|^{q} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}, \quad (5.2) \]
(W6) the wavelets satisfy
\[ \sup_{x \in D} |\varphi_{\ell,k}(x)| = C_\varphi 2^{\beta_0 \ell^d} \] with some \( \beta_0 \in \mathbb{R}_+ \),

for some constant \( C_\varphi > 0 \). Typically we have \( \varphi_{\ell,k} \sim 2^{d\ell} \psi(2^\ell (x-x_{\ell,k})) \), for some bounded function \( \psi \). In this case we have \( \beta_0 = 1 \).

See [6, section 2.1] and references therein for further details. See also [7, 10, 27].

We now investigate a stochastic model expanded by the wavelet basis described above. Let \( \{Y_{\ell,k}\} \) be a collection of independent standard normal random variables on a suitable probability space \( (\Omega', \mathcal{F}', P') \). We assume the random field \((1.2)\) is given with \( T \) such that
\[ T(x,\omega') = \sum_{\ell=\ell_0}^{\infty} \sum_{k \in \nabla_{\ell}} Y_{\ell,k}(\omega') \sigma_{\ell} \varphi_{\ell,k}(x), \quad (5.4) \]
where
\[ \sigma_{\ell} := 2^{-\beta_1 \ell^d} \] with \( \beta_1 > 1 \).

From \( E_{P'} \left( \sum_{\ell=\ell_0}^{\infty} \sum_{k \in \nabla_{\ell}} Y_{\ell,k}(\omega')^2 \sigma_{\ell}^2 \right) = C_\nabla \sum_{\ell=\ell_0}^{\infty} 2^{-(\beta_1-1)\ell d} < \infty \), in view of (W1) the series \((5.4)\) converges \( P' \)-almost surely in \( L^2(D) \).

To replace \((1.2)\), we consider the following log-normal stochastic model:
\[ a(x,\omega') = a_*(x) + a_0(x) \exp \left( \sum_{\ell=\ell_0}^{\infty} \sum_{k \in \nabla_{\ell}} Y_{\ell,k}(\omega') \sigma_{\ell} \varphi_{\ell,k}(x) \right), \quad (5.6) \]

In the following, we argue that we can reorder \( \sigma_{\ell} \varphi_{\ell,k} \) lexicographically as \( \sigma_j \varphi_j \) and see it as \( \psi_j \), while keeping the law.

Throughout this section, we assume that the parameters \( \beta_0 \) and \( \beta_1 \) satisfy
\[ 0 < \beta_1 - \beta_0, \quad (5.7) \]
and that point evaluation \( \varphi_{\ell,k}(x) \) ((\( \ell, k \)) \( \in \nabla \)) is well-defined for any \( x \in D \). Under this assumption, reordering \( (Y_{\ell,k} \sigma_{\ell} \varphi_{\ell,k}) \) lexicographically does not change the law of \((5.4)\) on \( \mathbb{R}^D \). To see this, from the Gaussianity it suffices to show that the covariance function \( E_{P'}[T(\cdot)T(\cdot)] : D \times D \to \mathbb{R} \) is invariant under the reordering.

Fix \( x \in D \) arbitrarily. For any \( L, L' (L>L') \), from the independence of \( \{Y_{\ell,k}\} \) we have
\[ E_{P'} \left( \sum_{\ell=\ell_0}^{L} \sum_{k \in \nabla_{\ell}} Y_{\ell,k}(\omega') \sigma_{\ell} \varphi_{\ell,k}(x) - \sum_{\ell=\ell_0}^{L'} \sum_{k \in \nabla_{\ell}} Y_{\ell,k}(\omega') \sigma_{\ell} \varphi_{\ell,k}(x) \right)^2 = \sum_{\ell=L+1}^{L'} \sum_{k \in \nabla_{\ell}} \sigma_{\ell}^2 \varphi_{\ell,k}^2(x), \quad (5.8) \]
\[ \leq C_\varphi^2 M \sum_{\ell=L'+1}^{L} 2^{-(\beta_1-\beta_0)\ell d} < \infty. \quad (5.9) \]
Hence, the sequence \( \left\{ \sum_{\ell=0}^{L} \sum_{k \in \mathcal{V}_\ell} Y_{\ell,k}(\omega') \sigma_{\ell} \varphi_{\ell,k}(x) \right\}_L \) is convergent in \( L^2(\Omega', \mathbb{P}') \). The continuity of the inner product \( \mathbb{E}_{\nu}[\cdot, \cdot] \) on \( L^2(\Omega') \) in each variable yields

\[
\mathbb{E}_{\nu}[T(x_1)T(x_2)] = \sum_{\ell=0}^{\infty} \sum_{k \in \mathcal{V}_\ell} \sum_{k' \in \mathcal{V}_{\ell'}} \mathbb{E}_{\nu}[Y_{\ell,k}(\omega') \sigma_{\ell} \varphi_{\ell,k}(x_1) Y_{\ell',k'}(\omega') \sigma_{\ell'} \varphi_{\ell',k'}(x_2)] \quad (5.10)
\]

\[
= \sum_{\ell=0}^{\infty} \sum_{k \in \mathcal{V}_\ell} \sigma_{\ell}^2 \varphi_{\ell,k}(x_1) \varphi_{\ell,k}(x_2), \quad \text{for any } x_1, x_2 \in D. \quad (5.11)
\]

But we have \( \sum_{\ell=0}^{\infty} \sum_{k \in \mathcal{V}_\ell} \sigma_{\ell}^2 |\varphi_{\ell,k}(x_1)|^2 |\varphi_{\ell,k}(x_2)| \leq C_0^2 M \sum_{\ell=0}^{L} 2^{-(\beta_1 - \beta_0)d\ell}. \) Hence,

\[
\mathbb{E}_{\nu}[T(x_1)T(x_2)] = \sum_{j \geq 1} \sigma_j^2 \varphi_j(x_1) \varphi_j(x_2), \quad x_1, x_2 \in D.
\]

Following a similar discussion, we see that the series \( \sum_{j \geq 1} \sigma_j^2 y_j \varphi_j(x) \) converges in \( L^2(\Omega) \) for each \( x \in D \), and has the covariance function \( \sum_{j \geq 1} \sum_{k \in \mathcal{V}_j} \sigma_j^2 \varphi_{j,k}(x_1) \varphi_{j,k}(x_2) \). Hence the law on \( \mathbb{R}^D \) is the same. Thus, abusing the notation slightly we write \( T(\cdot, y) := T(\cdot, \omega'), y_{\ell,k} := Y_{\ell,k}(\omega'), \Omega = \mathbb{R}^N := \Omega', \mathcal{F} := \mathcal{F}', \mathbb{P}_y := \mathbb{P}', \) and \( \mathbb{E}_{\nu} := \mathbb{E}_{\nu}^y \).

Next, we discuss the applicability of the theory developed in Section 4 to the wavelet stochastic model above. We need to check Assumption B.

Take \( \theta \in (0, \frac{2}{d}(\beta_1 - \beta_0)) \), and for \( \xi = (\ell, k) \) let

\[
\rho_\xi := c 2^\theta |\ell| = c 2^\theta \ell,
\]

with some constant \( 0 < c < \ln 2 \left( MC_{\varphi} \sum_{\ell=0}^{\infty} 2^{\ell(\theta - \frac{2}{d}(\beta_1 - \beta_0))} \right)^{-1}. \)

Then, by virtue of the locality property (\ref{eq:locality}) we have \( (5.11) \) as follows:

\[
\sup_{x \in D} \sum_{\xi} \rho_\xi |\sigma_\xi \varphi_\xi(x)| \leq \sum_{\ell=0}^{\infty} \rho_\ell \sup_{x \in D} \sum_{k \in \mathcal{V}_j} |2^{-\frac{\beta_j \ell d}{2}} \varphi_{j,k}(x)| \leq c M C_{\varphi} \sum_{\ell=0}^{\infty} 2^{\ell 2 + \beta_1 - \beta_0} \ell^{d \ell} \ell^d < \ln 2. \quad (5.13)
\]

Further, note that by reordering for sufficiently large \( j \) we have \( (5.16) \)

\[
\sup_{x \in D} |\sigma_j \varphi_j(x)| \sim j^{-\frac{d}{2}}(\beta_1 - \beta_0). \quad (5.16)
\]

\[^1\] To see this, first recall that there are \( O(2^{d\ell}) \) wavelets at level \( \ell \). Thus, for an arbitrary but sufficiently large \( j \) we have

\[
2^{d \ell} \leq j \leq 2^{(\ell + 1)d}.
\]

for some \( \ell_j \).

Let \( \xi_j \in \nabla_{j} \) be the index corresponding to \( j \). Since \( |\xi_j| = \ell_j \), we have

\[
\sup_{x \in D} |\sigma_j \varphi_j(x)| = \sup_{x \in D} |\sigma_{j, \xi_j}(x)| \leq C_{\varphi} 2^{-\frac{\beta_j d}{2} \ell_j} 2^{\frac{\beta_j d}{2}} \leq C_{\varphi} 2^{d \ell_j} \beta_j - \frac{\beta_j d}{2}, \quad \text{for any } \beta^* > \beta_1 - \beta_0. \quad (5.14)
\]

The opposite direction can be derived as, from \( \beta_1 - \beta_0 > 0, \)

\[
2^{-\frac{\beta_j d}{2}} \leq 2^{-\ell_j \beta_j(\frac{d}{2} \beta_1 - \beta_0)} = \frac{1}{C_{\varphi}} \sup_{x \in D} |\sigma_j \varphi_j(x)|. \quad (5.15)
\]

The relation \( \rho_j \sim j^\frac{d}{2} \) can be checked similarly.
and

\[ \rho_j \sim j^{\theta}. \quad (5.17) \]

Thus, to have \( \sum_{j \geq 1} \frac{1}{\rho_j} < \infty \), the weakest condition on the summability on \( \frac{1}{\rho_j} \) for Assumption \( \mathcal{B} \) to be satisfied, it is necessary (and sufficient) to have \( \theta > d \).

The following proposition summarises the discussion above.

**Theorem 5.1.** Suppose the random coefficient (1.2) is given by (5.4) with \( (\varphi_{\ell,k}) \) that satisfies (5.3), and non-negative numbers \( (\sigma_{\ell}) \) that satisfy (5.5). Let \( (\rho_\xi) \) be defined by (5.12). Further, assume \( \beta_0 \) and \( \beta_1 \) satisfy

\[ \frac{2}{q} < \beta_1 - \beta_0, \quad (5.18) \]

for some \( q \in (0,1] \). Then, the reordered system \( (\sigma_j \varphi_j) \) with the reordered \( (\rho_j) \) satisfies Assumption \( \mathcal{B} \) and under the same conditions on \( w_j(t), \alpha_j, \) and \( \varsigma_j \) as in Theorem 4.4 we have the QMC error bound (4.16) with this \( q \).

**Proof.** Take \( \theta \in \left( \frac{d}{q}, \frac{d}{2} (\beta_1 - \beta_0) \right) \), and define \( (\rho_\xi) \) as in (5.12), reorder the components lexicographically, and denote the reordered \( (\rho_\xi) \) by \( (\rho_j) \). Then, we have

\[ \sum_{j \geq 1} \left( \frac{1}{\rho_j} \right)^q \lesssim \sum_{j \geq 1} \left( \frac{1}{j} \right)^{\frac{q \theta}{d}} < \infty. \quad (5.19) \]

Further, from \( \theta - \beta_1 d + \beta_0 d < 0 \) we have (5.13), and thus (5.1) holds. Hence, from the discussion in this section Assumption \( \mathcal{B} \) is satisfied, and thus in view of Theorem 4.4 we have (4.16). \( \square \)

### 5.2 Hölder smoothness of the realisations

Often, random fields \( T \) with realisations that are not smooth are regularly of interest. In this section, we see that the stochastic model we consider (5.6) allows reasonably rough random fields (Hölder smoothness) for \( d = 1, 2 \). The result is shown via Sobolev embedding results. We provide a necessary and sufficient condition to have specified Sobolev smoothness (Theorem 5.2). Recall that embedding results are in general optimal (see, for example, [1, 4.12, 4.40–4.44]), and in this sense, we have a sharp condition for our model to have Hölder smoothness. A building block is a Besov characterisation of the realisations which is essentially due to Cioica et al. [6, Theorem 6]. Here we define \( s := s(L) := \sum_{\ell=0}^{L} \mathbb{N}^{\ell}(\nabla_\ell) \), that is, the truncation is considered in terms of the level \( L \).

**Theorem 5.2.** ([6, Theorem 6]) Let \( p, q \in [1,\infty) \), and \( t \in \left( d \max\{1/p-1, 0\}, t_\ast \right) \), where \( t_\ast \) is the parameter in (W7). Then,

\[ t < d \left( \frac{\beta_1 - 1}{2} \right) \quad (5.20) \]

if and only if \( T \in B^{s(t)}_q(L_p(D)) \) a.s. Further, if (5.20) is satisfied, then the stochastic model (5.6) satisfies \( \mathbb{E}\left[ \|T^{s(t)}\|_{B^{s(t)}_q(L_p(D))}^q \right] \leq \mathbb{E}\left[ \|T\|_{B^{s(t)}_q(L_p(D))}^q \right] < \infty \) for all \( L \in \mathbb{N} \).
Proof. First, from the proof of [9, Theorem 6], we see that $T \in B_q^t(D_p(D))$ a.s., is equivalent to

$$\sum_{\ell=\ell_0}^{\infty} 2^{(t+d(1/2-1/p))q} \sigma_{\ell}^q(\# \nabla \ell)^{q/p} \sim \sum_{\ell=\ell_0}^{\infty} 2^{(t-\frac{d}{2}(\beta_1-1))} < \infty,$$

which holds from the assumption $t < d(\frac{\beta_1}{2})$. Similarly, from the proof of [9, Theorem 6] we have

$$\mathbb{E}[\|T\|_{B^t_q(D_p(D))}^q] \approx \sum_{\ell=\ell_0}^{\infty} 2^{(t+d(1/2-1/p))q} \sigma_{\ell}^q(\# \nabla \ell)^{q/p} < \infty.$$

Finally, from (W5) we have $\mathbb{E}[\|T^s\|_{B^t_q(D_p(D))}^q] = \sum_{\ell=\ell_0}^{\infty} 2^{(t+d(1/2-1/p))q} \mathbb{E}[(\sum_{k \in \nabla \ell} |Y_{\ell,k}|^p)^{q/p}] \leq \mathbb{E}[\|T\|_{B^t_q(D_p(D))}^q]$ completing the proof.

To establish the Hölder smoothness, we employ embedding results. To invoke them, we first establish that the realisations are continuous; we want the measurability, and want to keep the law of $T$ on $\mathbb{R}^D$.

The Hölder norm involves taking the supremum over the uncountable set $D$, and thus whether the resulting function $\Omega \ni y \rightarrow \|T(\cdot, y)\|_{c^t(D)} \in \mathbb{R}$, where $t_1 \in (0,1]$ is a Hölder exponent, is an $\mathbb{R}$-valued random variable is not immediately clear. We see that by the continuity the measurability is preserved.

Sobolev embeddings are achieved by finding a suitable representative by changing values of functions on measure zero sets of $D$. This change could affect the law on $\mathbb{R}^D$, since it is determined by the laws of arbitrary finitely many random variables $(T(x_1), \ldots, T(x_m))$ $(\{x_i\}_{i=1,\ldots,m} \subset D)$ on $\mathbb{R}^m$. To avoid this, we establish the existence of continuous modification, thereby taking the continuous element of a Besov function that respects the law of $T$ from the outset.

We make an assumption on the covariance function so that realisations of $T$ have continuous paths. We assume there exist positive constants $\iota_1$, $C_{KT}$, and $\iota_2(>d)$ satisfying

$$\mathbb{E}[\|T(x_1) - T(x_2)\|_1] \leq C_{KT} \|x_1 - x_2\|_2^\iota_1, \text{ for any } x_1, x_2 \in D. \quad (5.21)$$

Then, by virtue of Kolmogorov–Trotter’s theorem [19, Theorem 4.1] $T$ has a continuous modification. Further, the continuous modification is uniformly continuous on $D$ and it can be extended to the closure $\overline{D}$.

A Hölder smoothness of $(\varphi_{t,k})$ is sufficient for (5.21) to hold.

**Proposition 5.3.** Suppose that $(\sigma_{t})$ satisfies (5.5). Further, suppose that for each $(\ell,k) \in \nabla$, the function $\varphi_{t,k}$ is $t_0$-Hölder continuous on $D$ for some $t_0 \in (0,1]$. Then, $T$ has a modification that is uniformly continuous on $D$ and can be extended to the closure $\overline{D}$.

**Proof.** It suffices to show (5.21) holds. Fix $x_1, x_2 \in D$ arbitrarily. First note that

$$\sigma^2 := \mathbb{E}[\|T(x_1) - T(x_2)\|_2^2] = \sum_{\ell=\ell_0}^{\infty} \sum_{k \in \nabla \ell} \sigma^2_{t_0}((\varphi_{t,k}(x_1) - \varphi_{t,k}(x_2))^2 \leq C \|x_1 - x_2\|_2^{2t_0} \sum_{\ell=\ell_0}^{\infty} \sum_{k \in \nabla \ell} \sigma^2_t < \infty, \quad (5.23)$$
where \( C \) is the \( t_0 \)-Hölder constant. Then, since \( T(x_1) - T(x_2) \sim \mathcal{N}(0, \sigma^2_x) \) we observe that, with \( X_{\text{std}} \sim \mathcal{N}(0, 1) \) we have

\[
\mathbb{E}[|T(x_1) - T(x_2)|^{2m}] = \mathbb{E} \left[ |X_{\text{std}}\sigma_x|^{2m} \right] = \sigma_x^{2m} \mathbb{E}[|X_{\text{std}}|^{2m}] \tag{5.24}
\]

\[
\leq C^m \|x_1 - x_2\|_2^{2m\tau} \left( \sum_{\ell\in\mathbb{N}} \sum_{k\in\mathbb{N}} \sigma_k^2 \right)^m \mathbb{E}[|X_{\text{std}}|^{2m}], \text{ for any } m \in \mathbb{N}. \tag{5.25}
\]

Taking \( m > \frac{d}{2\tau_0} \), we have (5.21) with \( t_1 := 2m, C_{\text{KT}} := C^m \left( \sum_{\ell=0}^\infty \sum_{k\in\mathbb{N}} \sigma_k^2 \right)^m \mathbb{E}[|X_{\text{std}}|^{2m}] \), and \( t_2 := 2t_0m(\geq d) \), and thus the statement follows. \( \square \)

In the following, we assume \( \varphi_{\ell,k} \) is \( t_0 \)-Hölder continuous on \( D \) for some \( t_0 \in (0, 1] \). Note that under this assumption, we may assume \( \varphi_{\ell,k} \) is continuous on \( \overline{D} \).

Using the fact that \( T(. , y) \in B_{2}^r(\mathcal{L}(2(D))) = H^r(D) \) a.s., now we establish the Hölder smoothness of the random coefficients \( a \). From this result, for example, the convergence rate of the finite element method using the piecewise linear functions are readily obtained.

First, we argue that to analyse the Hölder smoothness of the realisations of \( a \), without loss of generality we may assume \( a_\ast \equiv 0 \) and \( a_0 \equiv 1 \). To see this, suppose \( a_\ast, a_0 \) in (5.6) satisfies \( a_\ast, a_0 \in C^{t_1}(\overline{D}) \) for some \( t_1 \in (0, 1] \). By virtue of

\[
|e^a - e^b| = \left| \int_a^b e^r dr \right| \leq \max\{e^a, e^b\}|b-a| \leq (e^a + e^b)|b-a| \text{ for all } a, b \in \mathbb{R}, \tag{5.26}
\]

for any \( x_0, x_1, x_2 \in \overline{D} \) \((x_1 \neq x_2) \) we have

\[
\left| e^{T(x_0)} \right| + \frac{|e^{T(x_1)} - e^{T(x_2)}|}{\|x_1 - x_2\|^t_1} \leq \left( \sup_{x \in \overline{D}} |e^{T(x)}| \right) \left( 1 + 2 \frac{|T(x_2) - T(x_3)|}{\|x_1 - x_2\|^t_1} \right). \tag{5.27}
\]

Noting that \( \|a_0 e^T\|_{C^{t_1}(\overline{D})} \leq C_{t_1} \|a_0\|_{C^{t_1}(\overline{D})} \|e^T\|_{C^{t_1}(\overline{D})} \) (see, for example [14, p. 53]) we have

\[
\|a\|_{C^{t_1}(\overline{D})} \leq \|a_\ast\|_{C^{t_1}(\overline{D})} + C_{t_1} \|a_0\|_{C^{t_1}(\overline{D})} \left( \sup_{x \in \overline{D}} |e^{T(x)}| \right) \left( 1 + 2 \|T\|_{C^{t_1}(\overline{D})} \right). \tag{5.28}
\]

Thus, given \( a_\ast, a_0 \in C^{t_1}(\overline{D}) \), it suffices to show \( \left( \sup_{x \in \overline{D}} |e^{T(x)}| \right) \left( 1 + 2 \|T\|_{C^{t_1}(\overline{D})} \right) < \infty \) for the Hölder smoothness of the realisations of \( a \). Therefore, in the rest of this subsection, for simplicity we assume \( a_\ast \equiv 0 \) and \( a_0 \equiv 1 \).

In order to invoke embedding results we assume \( t_\ast \) satisfies \( \frac{d}{2} < \lfloor t_\ast \rfloor \), and that we can take \( t \in (0, \frac{d}{2}(\beta_1 - 1)) \) such that \( \frac{d}{2} < |t| \). For the latter to hold, taking \( \beta_1 \geq 3 \), implying \( \frac{d}{2} < |\frac{d}{2}(\beta_1 - 1)| \), is sufficient, which is always satisfied for the presented QMC theory to be applicable. See Remark 2

Now, take \( t_1 \in (0, 1] \cap (0, |t| - \frac{d}{2}) \). Then, from \( B_{2}^r(\mathcal{L}(2(D))) = H^r(D) \) and the Sobolev embedding (for example, [1, Theorem 4.12]) we have

\[
\|a\|_{C^{t_1}(\overline{D})} \lesssim \left( \sup_{x \in \overline{D}} |a(x)| \right) \left( 1 + 2 \|T\|_{B_2^r(\mathcal{L}(2(D)))} \right). \tag{5.29}
\]

Similarly, we have \( \|a_\ast\|_{C^{t_1}(\overline{D})} \lesssim \left( \sup_{x \in \overline{D}} |a_\ast(x)| \right) \left( 1 + 2 \|T\|_{B_2^r(\mathcal{L}(2(D)))} \right) \).
We want to take the expectation of $\|a\|_{C^1(D)}$. To do this, we establish the $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurability of $y \mapsto \|a(\cdot, y)\|_{C^1(D)}$. Taking continuous modifications of $T$ if necessary, we may assume paths of $a$ are continuous on $D$. Then, from the continuity of the mapping
\[
\{(x_1, x_2) \in D \times D \mid x_1 \neq x_2\} \ni (x_1, x_2) \mapsto \frac{|a(x_1) - a(x_2)|}{\|x_1 - x_2\|_2^t} \in \mathbb{R},
\]
with a countable set $G$ that is dense in $\{(x_1, x_2) \in D \times D \mid x_1 \neq x_2\} \subset \mathbb{R}^d \times \mathbb{R}^d$ we have
\[
\sup_{x_1, x_2 \in D, x_1 \neq x_2} \frac{|a(x_1) - a(x_2)|}{\|x_1 - x_2\|_2^t} \leq \sup_{(x_1, x_2) \in G} \frac{|a(x_1) - a(x_2)|}{\|x_1 - x_2\|_2^t}.
\]

Thus, $y \mapsto \|a(\cdot, y)\|_{C^1(D)}$, and by the same argument, $y \mapsto \|a^*(\cdot, y)\|_{C^1(D)}$, are $\mathcal{B}(\mathbb{R}^N)/\mathcal{B}(\mathbb{R})$-measurable, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$.

From $E[|T^s|_{C(D)}] \leq E[|T^s|_{B^2(L_2(D))}] \leq E[|T|_{B^2(L_2(D))}] \leq (\sum_{t=0}^{\infty} 2^{(2t-d(\beta_1 - 1))})^{1/2} < \infty$ independently of $s$, and $E[|T|_{C(D)}] \leq (\sum_{t=0}^{\infty} 2^{(2t-d(\beta_1 - 1))})^{1/2} < \infty$, following the discussion by Charrier. Proof of Proposition 3.10 utilising the Fernique’s theorem there exists a constant $M_p > 0$ independent of $p$ such that
\[
E[\exp(p \|T^s(\cdot, y)\|_{C(D)})] \leq M_p,
\]
for any $p \in (0, \infty)$. Together with, $\sup_{x \in D}|a(x)| \leq \exp(\sup_{x \in D}|T(x)|)$, we have
\[
E[(\sup_{x \in D}|a^*(x)|)^{2p}], E[(\sup_{x \in D}|a(x)|)^{2p}] < M_{2p}, \quad \text{for any } p \in (0, \infty).
\]

Hence, from (5.29) we conclude that
\[
E[\|a\|_{C^1(D)}^p] \leq \max\{1, 2p^{-1/2}\} \sqrt{E\left[\left(\sup_{x \in D}|a(x)|\right)^{2p}\right]} \sqrt{1 + 4pE\left[\|T^s\|_{B^2(L_2(D))}^{2p}\right]} < \infty.
\]

Similarly, we have
\[
E[\|a^*\|_{C^1(D)}^p] \leq \max\{1, 2p^{-1/2}\} \sqrt{E\left[\left(\sup_{x \in D}|a^*(x)|\right)^{2p}\right]} \sqrt{1 + 4pE\left[\|T^s\|_{B^2(L_2(D))}^{2p}\right]} < \infty,
\]
where the right hand side can be bounded independently of $s$.

Remark 2. We provide a remark regarding the smoothness of the realisations that the currently developed theory permits. From the conditions imposed on the basis functions, e.g., the summability conditions, random fields with smooth realisations are easily in the scope of the QMC theory applied to PDEs. Here, the capability of taking reasonably rough random field into account is of interest. Typically, $L^2$ wavelet Riesz basis have growth rate $\beta_0 = 1$. Then, the condition $2 < \beta_1 - \beta_0$, the weakest condition on $\beta_1$ in Theorem 5.1 is equivalent to
\[
\beta_1 = 3 + \varepsilon, \quad \text{for any } \varepsilon > 0.
\]

(A1)
We discuss the smoothness of the realisations achieved by solving PDEs, the validity of point evaluations demands of absolutely continuous functions. Since in practice we employ a suitable numerical method which summarises the condition (5.20) with (A1).

For $B_t^2(L^2(D)) = H^t(D)$, in view of Theorem 5.2, $T(\cdot, y) \in H^t(D)$ a.s. if and only if the condition (5.20), holds. We recall the following embedding results. See, for example, [1, p. 85]. For $d = 1, 2, 3$ respectively, with $\beta_1 = 3 + \varepsilon$ the condition (5.20) reads $t < 1 + \varepsilon$, $t < 2 + \varepsilon$, and $t < 3 + \varepsilon$, where we rescaled $\varepsilon$ depending on $d$.

For $d = 2, 3$, this seems to be rough enough. For $d = 1$, $H^1(D)$ is characterised as a space of absolutely continuous functions. Since in practice we employ a suitable numerical method to solve PDEs, the validity of point evaluations demands $a(\cdot, y) \in C(D)$. For $d = 2$, we know $H^2(D)$ can be embedded to $C^{0, t}(\overline{D})$, $(t \in (0, 1))$. This is a standard assumption to have the convergence of FEM with the hat function elements on polygonal domains.

For $d = 3$, we know $H^3(D) = H^{1+2}(D)$ can be embedded to $C^{1, t}(\overline{D})$, $(t \in (0, 2 - \frac{2}{3}) = (0, \frac{4}{3}))$. In practice, we employ quadrature rules to compute the integrals in the bilinear form. That $a \in C^{1, t}(\overline{D})$ is a reasonable assumption to get the convergence rate for FEM with quadratures. As a matter of fact, we want $a(\cdot, y) \in C^{2r}(\overline{D})$ to have the $O(H^{2r})$ convergence of the expected $L^p(\Omega)$-moment of $L^2(D)$-error even for $C^{2, 2}$-bounded domains. See [4, Remark 3.14], and [20, Remark 3.2].

Finally, we note these embedding results are in general optimal (see, for example, [1], 4.12, 4.40–4.44), and in this sense, together with the characterisation (Theorem 5.2), the condition for our model to have Hölder smoothness is sharp.

### 5.3 Dimension truncation error

In this section we estimate the truncation error $E\|u - u^s\|_V$. Again, the truncation is considered in terms of the level $L$ and we let $s = s(L) = \sum_{i=\ell_0}^{L} \#(\nabla \ell_i)$. Let $a^s$ be $a(x, y)$ with $y_j = 0$ for $j > s$, and define $\tilde{a}^s(y), \hat{a}^s(y)$ accordingly. By a variant of Strang’s lemma, we have

$$\|u - u^s\|_V \leq \|a - a^s\|_{L^\infty(D)} \frac{\|f\|_V}{\tilde{a}(y)\hat{a}^s(y)}$$

(5.33)

for $y$ such that $\tilde{a}(y), \hat{a}^s(y) > 0$. This motivates us to derive an estimate on $\|a - a^s\|_{L^\infty(D)}$.

Assuming a differentiability and a further summability of $(\psi_j)$, Charrier [3] obtained estimates on the moments of $\|a - a^s\|_{C(\overline{D})}$ and thus $\|u - u^s\|_V$, by the inequality of the same form as (5.33). See [3, Proposition 3.4 and Theorem 4.2, together with Assumption 3.1]. A similar argument is employed in [16]. Bearing in mind the argument by Charrier uses the Fernique’s theorem for separable Banach spaces [3, Theorem 2.2, Proposition 2.3], the same argument is
applicable here by replacing $L^\infty(D)$ with $C(D)$, which can be done following the discussion in Section 5.2.

In the present paper, however, we impose no further smoothness condition of the wavelet basis functions. We note that from (5.10) and (5.18), we have $\sum_{j \geq 1} \sup_{x \in D} |\sigma_j \varphi_j|^p < \infty$ for some $p \in (0, 1]$. Thus, the theory developed by Graham et al. [16] can be applied to the scaled wavelet basis $\varphi_{\ell,k} \sigma_{\ell}^q$, which in turn, together with the truncation error estimate we obtain in the following, shows that in the theory developed in [16], the assumption A2 (b) that is used to obtain a truncation error estimate [16, Theorem 8] is in general, in particular, for a wide class of wavelets basis, is not necessary.

**Proposition 5.4.** Let $u$ be the solution of the variational problem (2.5) with the coefficient given by the stochastic model (5.6) defined with (5.4) and (5.5). Let $u^{s(L)}$ be the solution of the same problem but with $y_j := 0$ for $j > s(L)$. Suppose $t \in (0, t_s)$, where $t_s$ is the parameter in \( W2 \), satisfies $t < d \left( \frac{\beta_1 - 1}{2} \right)$. Then, we have

$$\mathbb{E} \left[ \|u - u^{s(L)}\|_V \right] \lesssim \left( \sum_{\ell=L+1}^{\infty} 2^{\ell(2\ell - d(\beta_1 - 1))} \right)^{1/2}.$$  \hspace{1cm} (5.34)

**Proof.** For $t \in (0, d \left( \frac{1}{2} \beta_1 - 1 \right))$, choose $p_0 \in [1, \infty)$ such that $\frac{d}{p_0} \leq t$ so that we can invoke the Besov embedding results. Since $\max \{ d(\frac{1}{p_0} - 1), 0 \} < t$, from Theorem 5.2 there exists a set $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and $T(\cdot, y) \in B_{d}^{q}(L^{p_0}(D))$ for all $y \in \Omega_0$ with any $q \in [1, \infty)$. Then, letting $T^L(x, y) := \sum_{\ell=0}^{L} \sum_{k \in \nabla_{\ell}} y_{\ell,k} \varphi_{\ell,k}(x)$, from the embedding result of Besov spaces [1, Chapter 7], and the characterisation by wavelets (W5) for any $L, L' \geq 1$ ($L \geq L'$) we have

$$\left\| T^L(\cdot, y) - T^L(\cdot, y') \right\|_{L^\infty(D)} \lesssim \left\| T^L(\cdot, y) - T^L(\cdot, y') \right\|_{B_{d}^{q}(L^{p_0}(D))}$$

$$\sim \left( \sum_{\ell=L+1}^{L} 2^{\ell(2\ell - d(1/2 - 1/p_0))} \left( \sum_{k \in \nabla_{\ell}} |\sigma_{\ell,y_{\ell,k}}|^{p_0} \right)^{q/p_0} \right)^{1/q} < \infty,$$  \hspace{1cm} (5.36)

for all $y \in \Omega_0$. Thus, the sequence $\{T^L(\cdot, y)\}_L$ ($y \in \Omega_0$) is Cauchy, and thus convergent in $L^\infty(D)$. Hence, we obtain

$$\left\| T(\cdot, y) - T^L(\cdot, y) \right\|_{L^\infty(D)}^{q} \lesssim \sum_{\ell=L+1}^{\infty} 2^{\ell(2\ell - d(1/2 - 1/p))} \left( \sum_{k \in \nabla_{\ell}} |\sigma_{\ell,y_{\ell,k}}|^{p} \right)^{q/p} \text{ a.s.,}$$  \hspace{1cm} (5.37)

for all $p \in [1, \infty)$ such that $\frac{d}{p} \leq t$. For such $p$ and any $q \in [1, \infty)$, from [3, Proof of Theorem 6], we have

$$\mathbb{E} \left[ \left\| T(\cdot, y) - T^L(\cdot, y) \right\|_{L^\infty(D)}^{q} \right] \lesssim \sum_{\ell=L+1}^{\infty} 2^{\ell(2\ell - d(1/2 - 1/p))} \left( \sum_{k \in \nabla_{\ell}} |\sigma_{\ell,y_{\ell,k}}|^{q} \right)^{q/p} \sim \sum_{\ell=L+1}^{\infty} 2^{\ell(q/2 - d(1/2 - 1)} < \infty.$$  \hspace{1cm} (5.38)
Further, from (5.20) we have
\[
\mathbb{E} \left[ \left\| a(x, y) - a^s(L)(x, y) \right\|_{L^\infty(D)}^2 \right]
\leq (\sup_{x \in D} |a_0(x)|^2) \mathbb{E} \left[ \exp(2 \|T(\cdot, y)\|_{L^\infty(D)}) + \exp(2\|T^L(\cdot, y)\|_{L^\infty(D)}) \right] \mathbb{E} \left[ \|T - T^L\|_{L^\infty(D)}^2 \right].
\]
(5.39)

The sequence \((\rho_j)\) defined by (5.12), when reordered, satisfies \((1/\rho_j) \in \ell^{d+\varepsilon}\) for any \(\varepsilon > 0\). Thus, from the proof of Corollary 4.2 as in [3, Remark 2.2], we have
\[
\max \left\{ \mathbb{E} [\exp(2 \|T(\cdot, y)\|_{L^\infty(D)})], \mathbb{E} [\exp(2\|T^L(\cdot, y)\|_{L^\infty(D)})] \right\} < M_2,
\]
(5.40)
where the constant \(M_2 > 0\) is independent of \(L\).

Together with (5.33), we have
\[
\mathbb{E}[\|u - u^s\|_V] \leq \|f\|_{\mathcal{V}'} \mathbb{E} \left[ \frac{1}{\bar{a}(y)} \right]^\frac{1}{4} \mathbb{E} \left[ \frac{1}{(\bar{a}^s(y))^4} \right] \frac{1}{4} \mathbb{E} [\|a - a^s\|_{L^\infty(D)}^2] < \infty,
\]
(5.41)
where Cauchy–Schwarz inequality is employed in the right hand side of (5.33). To see the finiteness of the right hand side of (5.41), note that
\[
\frac{1}{\bar{a}(y)} \leq \frac{1}{\inf_{x \in D} a_0(x)} \exp(\|T\|_{L^\infty(D)}), \quad \frac{1}{\bar{a}^s(y)} \leq \frac{1}{\inf_{x \in D} a_0(x)} \exp(\|T^L\|_{L^\infty(D)}),
\]
and further, from the same argument as above, we have
\[
\max \left\{ \mathbb{E} [\exp(4 \|T(\cdot, y)\|_{L^\infty(D)})], \mathbb{E} [\exp(4\|T^L(\cdot, y)\|_{L^\infty(D)})] \right\} < M_4,
\]
(5.42)
where the constant \(M_4 > 0\) is independent of \(L\).

Therefore, from (5.38), (5.39), and (5.41) we obtain
\[
\mathbb{E}[\|u - u^{s(L)}\|_V] \lesssim \mathbb{E} \left[ \|T - T^L\|_{L^\infty(D)}^2 \right] \lesssim \left( \sum_{\ell=L+1}^{\infty} 2^{\ell(2d-d(\beta_1-1))} \right)^{\frac{1}{2}}.
\]
(5.43)

We conclude this section with a remark on other examples to which the currently developed QMC theory is applicable. Bachmayr et al. [3] considered so-called functions \((\psi_j)\) with finitely overlapping supports, for example, indicator functions of a partition of the domain \(D\). It is easy to find a positive sequence \((\rho_j)\) such that Assumption [3] holds, and thus Theorem 4.4 readily follows. However, for these examples, due to the lack of smoothness it does not seem that it is easy to obtain a meaningful analysis as given above, and thus we forgo elaborating them.

6 Concluding remark

We considered a QMC theory for a class of elliptic partial differential equations with a log-normal random coefficient. Using an estimate on the partial derivative with respect to the parameter \(y_a\) that is of product form, we established a convergence rate \(\approx 1\) of randomly shifted lattice rules. Further, we considered a stochastic model with wavelets, and analysed the smoothness of the realisations, and truncation errors.
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References

[1] R. A. Adams and J. J. Fournier. Sobolev Spaces. Vol. 140. Academic Press, 2003 (pages 19, 21, 26, 24).
[2] M. Bachmayr, A. Cohen, and G. Migliorati. Sparse polynomial approximation of parametric elliptic PDEs. Part I: affine coefficients. ESAIM Math. Model. Numer. Anal. 51 (2017), pp. 321–339 (page 3).
[3] M. Bachmayr, A. Cohen, R. DeVore, and G. Migliorati. Sparse polynomial approximation of parametric elliptic PDEs. Part II: lognormal coefficients. ESAIM Math. Model. Numer. Anal. 51 (2017), pp. 341–363 (pages 13, 31, 8, 11, 12, 26).
[4] J. Charrier, R. Scheichl, and A. L. Teckentrup. Finite element error analysis of elliptic PDEs with random coefficients and its application to multilevel Monte Carlo methods. SIAM J. Numer. Anal. 51 (2013), pp. 322–352 (page 23).
[5] J. Charrier. Strong and weak error estimates for elliptic partial differential equations with random coefficients. SIAM J. Numer. Anal. 50 (2012), pp. 216–246 (pages 22, 23).
[6] P. A. Cioica, S. Dahlke, N. Döhring, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R. L. Schilling. Adaptive wavelet methods for the stochastic Poisson equation. BIT Numer. Math. 52 (2012), pp. 589–614 (pages 16, 17, 13, 20, 24).
[7] A. Cohen. Numerical Analysis of Wavelet Methods. Vol. 32. Elsevier, 2003 (page 17).
[8] A. Cohen and R. DeVore. Approximation of high-dimensional parametric PDEs. Acta Numer. 24 (2015), pp. 1–159 (page 2).
[9] G. Dagan. Solute transport in heterogeneous porous formations. J. Fluid Mech. 145 (1984), pp. 151–177 (page 2).
[10] R. A. DeVore. Nonlinear approximation. Acta Numer. 7 (1998), pp. 51–150 (page 17).
[11] J. Dick, F. Y. Kuo, and I. H. Sloan. High-dimensional integration: The quasi-Monte Carlo way. Acta Numer. 22 (2013), pp. 133–288 (pages 5, 7).
[12] J. Dick, F. Y. Kuo, Q. T. Le Gia, D. Nuyens, and C. Schwab. Higher order QMC Petrov–Galerkin discretization for affine parametric operator equations with random field inputs. SIAM J. Numer. Anal. 52 (2014), pp. 2676–2702 (pages 5, 7).
[13] R. N. Gantner, L. Herrmann, and C. Schwab. Quasi-Monte Carlo integration for affine-parametric, elliptic PDEs: local supports imply product weights. Seminar for Applied Mathematics, ETH Zürich Research Report No. 2016-32 (2016) (page 5).
[14] D. Gilbarg and N. S. Trudinger. Elliptic Partial Differential Equations of Second Order. Berlin Heidelberg: Springer, 1983 (page 21).
[15] I. G. Graham, F. Y. Kuo, D. Nuyens, R. Scheichl, and I. H. Sloan. Quasi-Monte Carlo methods for elliptic PDEs with random coefficients and applications. J. Comput. Phys. 230 (2011), pp. 3668–3694 (page 7).
[16] I. G. Graham, F. Y. Kuo, J. A. Nichols, R Scheichl, C. Schwab, and I. H. Sloan. Quasi-Monte Carlo finite element methods for elliptic PDEs with lognormal random coefficients. Numer. Math. 131 (2015), pp. 329–368 (pages [3] [5] [7] [12] [14] [23] [24]).

[17] L. Herrmann and C. Schwab. Quasi-Monte Carlo integration for lognormal-parametric, elliptic PDEs: local supports imply product weights. Seminar for Applied Mathematics, ETH Zürich Research Report No. 2016-39 (2016) (page 3).

[18] K. Itô. Introduction to Probability Theory. Cambridge: Cambridge University Press, 1984 (page 4).

[19] H. Kunita. “Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms”. Real and Stochastic Analysis. Ed. by M. M. Rao. Trends Math. Boston: Birkhäuser, 2004, pp. 305–373 (page 20).

[20] F. Y. Kuo and D. Nuyens. Application of quasi-Monte Carlo methods to elliptic PDEs with random diffusion coefficients: a survey of analysis and implementation. Foundations of Computational Mathematics (2016), (in press) (pages 24 30 47).

[21] F. Y. Kuo, C. Schwab, and I. H. Sloan. Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. SIAM J. Numer. Anal. 50 (2012), pp. 3351–3374 (pages 2 3 6).

[22] R. L. Naff, D. F. Haley, and E. A. Sudicky. High-resolution Monte Carlo simulation of flow and conservative transport in heterogeneous porous media: 1. Methodology and flow results. Water Resour. Res. 34 (1998), pp. 663–677 (page 2).

[23] R. L. Naff, D. F. Haley, and E. A. Sudicky. High-resolution Monte Carlo simulation of flow and conservative transport in heterogeneous porous media: 2. Transport results. Water Resour. Res. 34 (1998), pp. 679–697 (page 2).

[24] M. Reed and B. Simon. Methods of Modern Mathematical Physics. I. Second. Academic Press, 1980 (pages 11 12).

[25] C. Schwab and C. J. Gittelson. Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs. Acta Numer. 20 (2011), pp. 291–467 (page 2).

[26] A. Teckentrup, R Scheichl, M. Giles, and E. Ullmann. Further analysis of multilevel Monte Carlo methods for elliptic PDEs with random coefficients. Numer. Math. 125 (2013), pp. 569–600 (page 24).

[27] K. Urban. Wavelets in Numerical Simulation. Vol. 22. Lecture Notes in Computational Science and Engineering. Berlin Heidelberg: Springer, 2002 (page 17).

[28] K. Yosida. Functional Analysis. Classics in Mathematics. Berlin: Springer, 1995 (page 12).