Weak-Field Spherically Symmetric Solutions in $f(T)$ gravity

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Abstract

We study weak-field solutions having spherical symmetry in $f(T)$ gravity; to this end, we solve the field equations for a non diagonal tetrad, starting from Lagrangian in the form $f(T) = T + \alpha T^n$, where $\alpha$ is a small constant, parameterizing the departure of the theory from GR. We show that the classical spherically symmetric solutions of GR, i.e. the Schwarzschild and Schwarzschild-deSitter solutions, are perturbed by terms in the form $\propto r^{2-2n}$ and discuss the impact of these perturbations in observational tests.
I. INTRODUCTION

Since the very discovery of the accelerated cosmic expansion \([1, 2]\), and its confirmation due to multiple observations \([3–5]\), it has been customary to investigate theories that extend general relativity (GR), in order to get an agreement with the observations, without requiring the existence of dark entities. Hence, motivated by this instance, which recognizes that GR fails in describing gravity at large scales (consider also the old issue of the rotation curves of spiral galaxies \([6]\)), several theories have been proposed to generalize Einstein’s theory. Some of these modified models of gravity are geometric extensions of GR, in other words they are based on a richer geometric structure, which is supposed to give the required ingredients to support the observations.

As a prototype of this approach, one can consider the \(f(R)\) theories, where the gravitational Lagrangian depends on a function \(f\) of the curvature scalar \(R\) (see \([7, 8]\) and references therein): when \(f(R) = R\) the action reduces to the usual Einstein-Hilbert action, and Einstein’s theory is obtained. Another example is given by the so-called \(f(T)\) theories, which have similarities and differences with respect to \(f(R)\). To begin with, they are based on teleparallel gravity (TEGR) \([9]\), where the gravitational interaction is determined by torsion, and the torsion scalar \(T\) appears in the Lagrangian instead of the curvature scalar. Furthermore, the underlying Riemann-Cartan space-time is endowed with the Weitzenböck connection (instead of the Levi-Civita connection), which is not commutative under the exchange of the lower indices, and has zero curvature while non-zero torsion. Actually, Einstein himself proposed such an alternative point of view on gravitation in terms of torsion and tetrads \([10]\). In fact, in the TEGR picture the tetrads field is promoted to be the dynamical field instead of the metric tensor. In spite of these differences, TEGR and GR have equivalent dynamics: in other words, every solution of GR is also solution of TEGR. However, when TEGR is generalized to \(f(T)\) by considering a gravitational Lagrangian that is a function of the torsion scalar, the equivalence breaks down \([11, 12]\). As a consequence \(f(T)\) theories can be considered potential candidates for explaining (on a purely geometric ground) the accelerated expansion of the universe, without requiring the existence of exotic cosmic fluids (see e.g. \([13]\)).

While \(f(R)\) theories gives fourth order equations (at least in the metric formalism, while they are still second order in the Palatini approach, see e.g. \([8]\)), the \(f(T)\) field equations are
second order in the field derivatives since the torsion scalar is a function of the square of the first derivatives of the tetrads field. Furthermore, as for \( f(R) \) theories, the generalized TEGR displays additional degrees of freedom (whose physical nature is still under investigation\[14\]) related to the fact that the equations of motion are not invariant under local Lorentz transformations\[15\]. In particular, this implies the existence of a preferential global reference frame defined by the autoparallel curves of the manifold that solve the equations of motion. Consequently, even though the symmetry can help in choosing suitable coordinates to write the metric in a simple way, this does not give any hint on the form of the tetrad. As discussed in\[16\], a diagonal tetrad -that could in principle be a good working-ansatz for dealing with diagonal metrics- is not a good choice to properly parallelize the spacetime both in the context of non-flat homogenous and isotropic cosmologies (Friedman-Lemaitre-Robertson-Walker universes) and in spherically symmetric space-times (Schwarzschild or Schwarzschild-de Sitter solutions).

The case of the Schwarzschild solution and, more in general, of the spherically symmetric solutions in \( f(T) \) gravity, is particularly important, because these solutions, which describe the gravitational field of point-like sources, allow to test \( f(T) \) theories at scales different from the cosmological ones, e.g. in the solar system. Indeed, \( f(T) \) theories can be used to explain the cosmic acceleration and observations on large scales (e.g. via galaxy clustering and cosmic shear measurements\[17\]), but we must remember that since GR is in excellent agreement with solar system and binary pulsar observations\[18\], every theory that aims at explaining the large scale dynamics of the Universe should reproduce GR in a suitable weak-field limit: the same holds true for \( f(T) \) theories. Recently, solar system data\[19, 20\] have been used to constrain \( f(T) \) theories; these results are based on the spherical symmetry solution found by Iorio&Saridakis\[19\], who used a diagonal tetrad. In this paper we follow the approach described in\[16\] to define a “good tetrad” in \( f(T) \) gravity - that is consistent with the equations of motion without constraining the functional form of the Lagrangian - and solve the field equations to obtain weak-field solutions with a power-law ansatz for an additive term to the TEGR Lagrangian, \( f(T) = T + \alpha T^n \).

This paper is organized as follows: in Section II we review the theoretical framework of \( f(T) \) gravity and write the field equations, whose solutions for spherically symmetric space-times, in weak-field approximation, are given in Section III. Eventually, discussion and conclusions are in Sections IV and V.
II. \textit{f(T) Gravity Field Equations}

We start by briefly discussing the \textit{f(T)} gravity framework that leads to the field equations. To begin with, we point out that, in this scenario, the metric tensor can be viewed as a subsidiary field, and the vierbein field is the dynamical object whose components in a given coordinate basis $e^a_\mu$ are related to the metric tensor by

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x),$$

(1)

where $\eta_{ab} = \text{diag}(1, -1, -1, -1)$. Notice that latin indexes refer to the tangent space while greek indexes label coordinates on the manifold. Hence, the dynamics is obtained by the action\footnote{We use units such as $c = 1$.}

$$S = \frac{1}{16\pi G} \int f(T) e \, d^4x + S_M,$$

(2)

where $e = \det e^a_\mu = \sqrt{-\det(g_{\mu\nu})}$ and $S_M$ is the action for the matter fields\footnote{Notice that many authors write the gravitational Lagrangian in the form $T + f(T)$, thus denoting the deviation from GR by means of the function $f(T)$; on the contrary, here $f(T)$ is the whole Lagrangian.}. Here $f(T)$ is a differentiable function of the torsion scalar $T$, which is defined as

$$T = S^\rho_{\mu\nu} T^\mu_\rho^\nu,$$

(3)

where the contorsion tensor $S^\rho_{\mu\nu}$ is defined by

$$S^\rho_{\mu\nu} = \frac{1}{4} (T^\rho_{\mu\nu} - T^\mu_{\rho\nu} + T^\nu_{\mu\rho}) + \frac{1}{2} \delta^\rho_\mu T^\sigma_{\sigma\nu} - \frac{1}{2} \delta^\rho_{\nu} T^\sigma_{\sigma\mu},$$

(4)

and the torsion tensor $T^\lambda_{\mu\nu}$ is

$$T^\lambda_{\mu\nu} = e^\lambda_a (\partial_\nu e^a_\mu - \partial_\mu e^a_\nu).$$

(5)

Varying the action with respect to the vierbein $e^a_\mu(x)$, one gets the field equations

$$e^{-1} \partial_\mu (e_a^\rho S_{\rho}^{\mu\nu}) f_T + e_a^\lambda S_{\rho}^{\nu\mu} T^\rho_{\mu\lambda} f_T + e_a^\rho S_{\rho}^{\mu\nu} \partial_\mu(T) f_T + e_a^\rho S_{\rho}^{\mu\nu} \partial_\mu(T) f_T + \frac{1}{4} e_a^\nu f = 4\pi G e_a^\mu T^\nu_{\mu},$$

(6)

where $T^\nu_{\mu}$ is the matter energy-momentum tensor and subscripts $T$ denote differentiation with respect to $T$.\footnote{Notice that many authors write the gravitational Lagrangian in the form $T + f(T)$, thus denoting the deviation from GR by means of the function $f(T)$; on the contrary, here $f(T)$ is the whole Lagrangian.}
We look for spherically symmetric solutions of the field equations, so we start from the metric
\[ ds^2 = e^{A(r)} dt^2 - e^{B(r)} dr^2 - r^2 d\Omega^2, \] (7)
where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). Due to the lack of Local Lorentz Invariance, tetrads connected by local Lorentz transformations lead to the same metric - i.e. the same causal structure - but different equations of motions, thus physically inequivalent solutions. This means that, even in the spherically symmetric case, for which symmetry helps us in choosing the coordinates and the metric tensor in a simple form, it is quite complicated to do an ansatz for the tetrad field. In particular, for the symmetry and coordinates we are dealing with, it turns out to be a mistake to choose a diagonal form for \( e^a_{\mu} \): it does not properly parallelize the static spherically symmetric geometry in the context of \( f(T) \) gravity.

Then, with this caveat in mind, it is possible to derive the field equations for the non-diagonal tetrad:

\[
e^a_{\mu} = \begin{pmatrix}
e^{A/2} & 0 & 0 & 0 \\
0 & e^{B/2} \sin \theta \cos \phi & e^{B/2} \sin \theta \sin \phi & e^{B/2} \cos \theta \\
0 & -r \cos \theta \cos \phi & -r \cos \theta \sin \phi & r \sin \theta \\
0 & r \sin \theta \sin \phi & -r \sin \theta \cos \phi & 0
\end{pmatrix}
\]
following the approach described in [16]: in doing so, the functional form of the Lagrangian and the specific form of the torsion scalar are not constrained \textit{a priori}. The field equations are

\[
\frac{f(T)}{4} - f_T \frac{e^{-B(r)}}{4r^2} \left(2 - 2e^{B(r)} + r^2 e^{B(r)} T - 2r B'(r)\right) + 
- f_T T'(r) \frac{e^{-B(r)}}{r} \left(1 + e^{B(r)/2}\right) = 4\pi \rho 
\] (8)

\[
\frac{f(T)}{4} + f_T \frac{e^{-B(r)}}{4r^2} \left(2 - 2e^{B(r)} + r^2 e^{B(r)} T - 2r A'(r)\right) = 4\pi p 
\] (9)

\[
f_T \left[-4 + 4e^{B(r)} - 2r A'(r) - 2rB'(r) + r^2 A'(r)^2 - r^2 A'(r)B'(r) + 2r^2 A''(r)\right] + 
+ 2r f_T T' \left(2 + 2e^{B(r)/2} + r A'(r)\right) = 0 
\] (10)

where \( \rho, \ p \) are the energy density and pressure of the matter energy-momentum tensor, and the prime denotes differentiation with respect to the radial coordinate \( r \). Moreover, the torsion scalar is

\[
T = \frac{2e^{-B(r)}(1 + e^{B(r)/2})}{r^2} \left[1 + e^{B(r)/2} + rA'(r)\right]. 
\] (11)
III. WEAK-FIELD SOLUTIONS

Exact solutions in vacuum ($\rho = p = 0$) and in presence of a cosmological constant ($\rho = -p$) of the above field equations are thoroughly discussed in [16]; here we are interested in weak-field solutions with non constant torsion scalar, i.e. $T' = dT/dr \neq 0$.

Indeed, for actual physical situations such as in the solar system, the gravitational field is expected to be just a small perturbation of a flat background Minkowski spacetime. As a consequence, we write

$$e^{A(r)} = 1 + A(r), \quad e^{B(r)} = 1 + B(r)$$

and confine ourselves to linear perturbations. Moreover, we consider Lagrangians of sufficient generality, that we write in the form $f(T) = T + \alpha T^n$, where $\alpha$ is a small constant, parameterizing the departure of these theories from GR, and $|n| \neq 1$.

To begin with, we consider the case $n = 2$, that has been already analyzed in [19]. From (8-10) we obtain the following solutions

$$A(r) = -32 \frac{\alpha}{r^2} - \frac{C_1}{r} \quad (13)$$

$$B(r) = 96 \frac{\alpha}{r^2} + \frac{C_1}{r} \quad (14)$$

where $C_1$ is an integration constant. Then, on setting $C_1 = 2M$, we get the weak-field limit of the Schwarzschild solution plus a correction due to $\alpha$:

$$ds^2 = \left(1 - \frac{2M}{r} - 32 \frac{\alpha}{r^2}\right) dt^2 - \left(1 + \frac{2M}{r} + 96 \frac{\alpha}{r^2}\right) dr^2 - r^2 d\Omega^2 \quad (15)$$

Eventually, the torsion scalar turns out to be

$$T(r) = 8 \frac{\alpha}{r^2} - 128 \frac{\alpha}{r^4}. \quad (16)$$

These results can be compared to those obtained in [19], where a Lagrangian in the form $f(T) = T + \alpha T^2$ was considered. While to lowest order approximation in both cases the perturbations are proportional to $1/r^2$, the numerical coefficients are different: this is not surprising, since the authors in [19] solve different field equations - in particular they use a diagonal tetrad; moreover, they argue to obtain solutions for the $T = \text{constant}$ case.

Likewise, if we look for solutions of the equations [8]-[10] with $\rho = k, p = -k$, which corresponds to a cosmological constant, we obtain

$$ds^2 = \left(1 - \frac{2M}{r} - 32 \frac{\alpha}{r^2} - \frac{1}{3} \Lambda r^2\right) dt^2 - \left(1 + \frac{2M}{r} + 96 \frac{\alpha}{r^2} + \frac{1}{3} \Lambda r^2\right) dr^2 - r^2 d\Omega^2 \quad (17)$$
where we set \( k = \frac{A}{8\pi} \) and \( \Lambda \) is the cosmological constant. The torsion scalar is the same as \([16]\). So, the weak-field limit of Schwarzschild - de Sitter solution is perturbed by terms that are proportional to \( \alpha \).

The previous results can be generalized to the case of a Lagrangian in the form \( f(T) = T + \alpha T^n \), and we get

\[
A(r) = -\frac{C_1}{r} - \alpha \frac{r^{2-2n}}{2n-3} 2^{3n-1} - \frac{1}{3} \Lambda r^2 \tag{18}
\]

\[
B(r) = \frac{C_1}{r} + \alpha \frac{r^{2-2n}}{2n-3} 2^{3n-1} (-3n + 1 + 2n^2) + \frac{1}{3} \Lambda r^2 \tag{19}
\]

In particular, if \( \Lambda = 0 \), we obtain vacuum solutions. Notice that, on setting \( C_1 = 2M \), we obtain a weak-field Schwarzschild - de Sitter solution perturbed by terms that are proportional to \( \alpha \) and decay with a power of the radial coordinate, the specific value depending on the power-law chosen in the Lagrangian. The torsion scalar is

\[
T(r) = \frac{8}{r^2} + 2\alpha r^{-2n} 2^{3n} (n + 1) \tag{20}
\]

while the perturbation terms due to the deviation from GR are in the form

\[
A_\alpha(r) = \alpha a_n r^{2-2n}, \quad B_\alpha(r) = \alpha b_n r^{2-2n} \tag{21}
\]

where \( a_n = \frac{2^{3n-1}}{2n-3} \), \( b_n = \frac{2^{3n-1}}{2n-3} (2n^2 - 3n + 1) \). A close inspection of the perturbation terms reveals that they go to zero both when \( r \to \infty \) with \( n > 1 \) and when \( r \to 0 \) with \( n < 1 \). In the latter case, in order to keep the perturbative approach self-consistent, a maximum value of \( r \) must be defined to consider these terms as perturbations of the flat space-time background.

We remark here that our linearized approach can be applied to arbitrary polynomial corrections to the torsion scalar: as a consequence, by writing an arbitrary function as a suitable power series, it is possible to evaluate its impact as a perturbation of the weak-field spherically symmetric solution in GR: the \( n \)-th term of the series gives a contribution proportional to \( r^{2-2n} \).

**IV. DISCUSSION**

It is useful to comment on the constraints one can infer for the parameters of our model from solar system data. But, before proceeding, it is important to emphasize a point about
the tests of $f(T)$ gravity. In theories with torsion, there is a sharp distinction between the test particles trajectories: autoparallels, or affine geodesics, are curves along which the velocity vector is transported parallel to itself, by the space-time connection; extremals, or metric geodesics, are curves of extremal space-time interval with respect to the space-time metric \[21\]. While in GR autoparallels and extremals curves do coincide and we can simply speak of geodesics, the same is not true when torsion is present. So, it is not trivial to define the actual trajectories of test particles. The results obtained by \[19\] and \[20\] strictly apply to the case of metric geodesics. According to us, this is a very important issue, that is often neglected in the literature pertaining to theories alternative to GR based to torsion: we will focus on this issue a forthcoming publication \[22\].

Bearing this in mind, it is possible to comment on our results and compare them to those already available in the literature. In particular, because of the different choice of the tetrad, our solution, even in the case of a quadratic deformation of the TEGR Lagrangian, differs from the one found by Iorio&Saridakis. Both corrections are proportional to $1/r^2$, but they have different numerical coefficients. In particular, since they differ by a factor 10, the same happens to the constraints that can be obtained from our solution, by applying the approach described in \[19\] and \[20\].

In particular, Iorio & Saridakis \[19\] derive constraints from the rate of change of perihelia of the first four inner planets, obtaining

$$|\Lambda| \leq 6.1 \times 10^{-42} \, m^{-2}$$
$$|\alpha| \leq 1.8 \times 10^4 \, m^2.$$  

Tighter results have been obtained in a subsequent paper \[20\], where the authors consider upper bounds deriving from different phenomena: perihelion advance, light bending, gravitational time delay. But the strongest constraints come from the perihelion advance, in particular from some supplementary advances constructed by considering that the effects due to the Sun’s quadrupole mass moment might represent possible unexplained parts of perihelion advance in GR \[23\]. This gives

$$|\Lambda| \leq 1.8 \times 10^{-43} \, m^{-2}$$
$$|\alpha| \leq 1.2 \times 10^2 \, m^2.$$  

The upper bound for the solution in Eqs. \[17\] would be of the order $\sim 10 \, m^2$ for our $\alpha$
parameter.

V. CONCLUSIONS

We studied spherically symmetric solutions in the weak-approximation of $f(T)$ gravity. In particular, we started from a Lagrangian in the form $f(T) = T + \alpha T^n$, with $|n| \neq 1$, where $\alpha$ is a small constant which parameterizes the departure of these theories from GR, and solved the field equations using a non diagonal tetrad, showing that, to lowest approximation order, the perturbations of the corresponding GR solutions (Schwarzschild or Schwarzchild-de Sitter) are in the form $\propto \alpha r^{2-2n}$. These results can be used to evaluate the impact of the non linearity of the Lagrangian, for instance in the solar system.

The case $n = 2$, corresponding to the Lagrangian $f(T) = T + \alpha T^2$ has been already analyzed by [19] and [20], who used solar system observations to set constraints on the parameter $\alpha$. It is important to point out that the latter results are based on the solution obtained by Iorio&Saridakis [19], where a diagonal tetrad was used. On the other hand, since because of the invariance properties of $f(T)$ the choice of the tetrad field is crucial, we performed our calculations by using a more general non-diagonal tetrad, according to the prescriptions given in [16], and obtained a new solution of the $f(T)$ quadratic model, for which the torsion scalar is not forced to be zero.

We used the results already available in the literature to set constraints on the $\alpha$ parameter for the Lagrangian $f(T) = T + \alpha T^2$, from solar system data, and we obtained a different value for the current upper bound, even if we pointed out that the distinction between autoparallels, or affine geodesics, and extramals, or metric geodesics, is crucial in $f(T)$ gravity, and deserves further investigation that we are going to carry out in forthcoming publications.
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