A Model for Randomized Resource Allocation in Decentralized Wireless Networks

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Abstract

In this paper, we consider a decentralized wireless communication network with a fixed number $u$ of frequency sub-bands to be shared among $N$ transmitter-receiver pairs. It is assumed that the number of active users is a random variable with a given probability mass function. Moreover, users are unaware of each other’s codebooks and hence, no multiuser detection is possible. We propose a randomized Frequency Hopping (FH) scheme in which each transmitter randomly hops over a subset of $u$ sub-bands from transmission to transmission. Assuming all users transmit Gaussian signals, the distribution of the noise plus interference is mixed Gaussian, which makes calculation of the mutual information between the transmitted and received signals of each user intractable. We derive lower and upper bounds on the mutual information of each user and demonstrate that, for large Signal-to-Noise Ratio (SNR) values, the two bounds coincide. This observation enables us to compute the sum multiplexing gain of the system and obtain the optimum hopping strategy for maximizing this quantity. We compare the performance of the FH system with that of the Frequency Division (FD) system in terms of the following performance measures: average sum multiplexing gain ($\eta^{(1)}$), average minimum multiplexing gain per user ($\eta^{(2)}$), minimum nonzero multiplexing gain per user ($\eta^{(3)}$) and service capability ($\eta^{(4)}$). We show that (depending on the probability mass function of the number of active users) the FH system can offer a significant improvement in terms of $\eta^{(1)}$, $\eta^{(2)}$ and $\eta^{(4)}$ (implying a more efficient usage of the spectrum). It is also shown that $\frac{1}{e} \leq \frac{\eta^{(3)}_{FH}}{\eta^{(3)}_{FD}} \leq 1$, i.e., the loss incurred in terms of $\eta^{(3)}$ is not more than $\frac{1}{e}$. 
Index Terms

Randomized Signaling, Frequency Hopping, Spectrum Sharing, Decentralized Networks, Mixed Gaussian
Interference, Multiplexing Gain.

I. INTRODUCTION

A. Motivation

Increasing demand for wireless applications on one hand, and the limited available resources on the
other hand, provoke more efficient usage of such resources. Due to its significance, many researchers
have addressed the problem of resource allocation in wireless networks. One major challenge in wireless
networks is the destructive effect of multi-user interference, which degrades the performance when multiple
users share the spectrum. As such, an efficient and low complexity resource allocation scheme that
maximizes the quality of service while mitigating the impact of the multi-user interference is desirable. The
existing resource allocation schemes are either centralized, i.e., a central controller manages the resources,
or decentralized, where resource allocation is performed locally at each node. Due to the complexity of
adapting the centralized schemes to the network structure (e.g. number of active users), these schemes are
usually designed for a fixed network structure. This makes inefficient usage of resources because, in most
cases, the number of active users may be considerably less than the value assumed in the design process.
On the other hand, most of the decentralized resource allocation schemes suffer from the complexity,
either in the algorithm (e.g. game-theoretic approaches involving iterative methods) or in the hardware
(e.g. cognitive radio). Therefore, it is of interest to devise an efficient and low-complexity decentralized
resource allocation scheme, which is the main goal of this paper.

B. Related Works

1) Centralized Schemes: In recent years, many centralized power and spectrum allocation schemes
have been studied in cellular and multihop wireless networks [1]–[8]. Clearly, centralized schemes perform
better than the decentralized (distributed) approaches, while requiring extensive knowledge of the network
configuration. In particular, when the number of nodes is large, deploying such centralized schemes may
not be practically feasible.

Traditional wireless systems aimed to avoid the interference among users by using orthogonal transmis-
sion schemes. The most common example is the Frequency Division (FD) system, in which different users
transmit over disjoint frequency sub-bands. The assignment of frequency sub-bands is usually performed by a central controller. Despite its simplicity, FD is shown to achieve the highest throughput in certain scenarios. In particular, [9] proves that in a wireless network where interference is treated as noise (no multi-user detection is performed), if the crossover gains are sufficiently larger than the forward gains, FD is Pareto-rate-optimal. Due to practical considerations, such FD systems usually rely on a fixed number of frequency sub-bands. Hence, if the number of users changes, the system is not guaranteed to offer the best possible spectral efficiency because, most of the time, the majority of the potential users may be inactive.

2) Decentralized Schemes: In decentralized schemes, decisions concerning network resources are made by individual nodes based on their local information. Most of decentralized schemes reported in the literature rely on either game-theoretic approaches or cognitive radios. Cognitive radios [10] have the ability to sense the unoccupied portion of the available spectrum and use this information in resource allocation. Fundamental limits of wireless networks with cognitive radios are studied in [11]–[14]. Although cognitive radios avoid the use of a central controller, they require sophisticated detection techniques for sensing the spectrum holes and dynamic frequency assignment, which add to the overall system complexity [15]. Noting the above points, it is desirable to have a decentralized frequency sharing strategy without the need for cognitive radios, which allows the users to coexist while utilizing the spectrum efficiently and fairly.

Being a standard technique in spread spectrum communications and due to its interference avoidance nature, hopping is the simplest spectrum sharing method to use in decentralized networks. As different users typically have no prior information about the codebooks of the other users, the most efficient method is avoiding interference by choosing unused channels. As mentioned earlier, searching the spectrum to find spectrum holes is not an easy task due to the dynamic spectrum usage. As such, Frequency Hopping (FH) is a realization of a transmission scheme without sensing, while avoiding the collisions as much as possible. Frequency Hopping is one of the standard signaling schemes [16] adopted in ad-hoc networks. In short range scenarios, bluetooth systems [17]–[19] are the most popular examples of a Wireless Personal Area Network (WPAN). Using FH over the unlicensed ISM band, a bluetooth system provides robust communication to unpredictable sources of interference. A modification of Frequency Hopping, called Dynamic Frequency Hopping (DFH), selects the hopping pattern based on interference measurements in order to avoid dominant interferers. The performance of a DFH scheme when applied to a cellular system
is assessed in [20]–[22].

In [23], the authors consider the problem of bandwidth partitioning in a decentralized wireless network where different transmitters are connected to different receivers through channels with similar path loss exponent. Assuming the transmitters are scattered over the two dimensional plane according to a Poisson point process, a fixed bandwidth is partitioned into a certain number of sub-bands such that the so-called transmission intensity in the network is maximized while the probability of outage per user is below a certain threshold. The transmission strategy is based on choosing one sub-band randomly per transmission, which is a special case of FH.

Frequency hopping is also proposed in [14] in the context of cognitive radios, where each cognitive transmitter selects a frequency sub-band but quits transmitting if the sub-band is already occupied by a primary user.

Recently, Orthogonal Frequency Division Multiplexing (OFDM) has been considered as a promising technique in many wireless technologies. OFDM partitions a wide-band channel to a group of narrow-band orthogonal sub-channels. The popularity of OFDM motivates us to consider a Frequency Hopping scheme operating over $u$ narrow-band orthogonal frequency sub-bands. We note that the results of the paper are valid in a general setup where hopping is performed over an arbitrary orthogonal basis. To make the presentation as simple as possible, we take the orthogonal basis in frequency, which can be realized in practice using OFDM systems.

C. Contribution

In this paper, we consider a decentralized wireless communication network with a fixed number $u$ of frequency sub-bands to be shared among $N$ transmitter-receiver pairs. It is assumed that the number of active users is a random variable with a given probability mass function. Moreover, users are unaware of each other’s codebooks, and hence, no multiuser detection is possible. We propose a randomized Frequency Hopping scheme in which the $i^{th}$ transmitter randomly hops over $v_i$ out of $u$ sub-bands from transmission to transmission. Assuming i.i.d. Gaussian signals are transmitted over the chosen sub-bands, the distribution of the noise plus interference becomes mixed Gaussian, which makes the calculation of the achievable rate complicated. The main contributions of the paper are:

- We derive lower and upper bounds on the mutual information between the transmitted and received signals of each user and demonstrate that, for large SNR values, the two bounds coincide. Thereafter,
we are able to show that the achievable rate of the $i^{th}$ user scales like $\frac{v_i}{2} \prod_{j=1}^{N} (1 - \frac{v_j}{u}) \log \text{SNR}$.

- We show that each transmitter only needs the knowledge of the number of active users in the network, the forward channel gain and the maximum interference at its desired receiver to regulate its transmission rate. Knowing these quantities, we demonstrate how the $i^{th}$ user can achieve a multiplexing gain of $\frac{v_i}{2} \prod_{j \neq i}^{N} (1 - \frac{v_j}{u})$.

- We obtain the optimum design parameters $\{v_i\}_{i=1}^{N}$ in order to maximize various performance measures.

- We compare the performance of the FH with that of the Frequency Division in terms of the following performance measures: average sum multiplexing gain ($\eta^{(1)}$), average minimum multiplexing gain per user ($\eta^{(2)}$), minimum nonzero multiplexing gain per user ($\eta^{(3)}$) and service capability ($\eta^{(4)}$). We show that (depending on the probability mass function of the number of active users) the FH system can offer a significant improvement in terms of $\eta^{(1)}$, $\eta^{(2)}$, and $\eta^{(4)}$ (implying a more efficient usage of the spectrum). It is also shown that $\frac{1}{e} \leq \frac{\eta^{(3)}_{\text{FH}}}{\eta^{(3)}_{\text{FD}}} \leq 1$, i.e., the loss incurred in terms of $\eta^{(3)}$ is not more than $\frac{1}{e}$.

The paper outline is as follows. The system model is given in section II. Section III offers an analysis of the achievable rates. Upper bounds and lower bounds on the achievable rates of users are presented in this section. In section IV, based on the results in section III, we discuss how users in the FH system can fairly share the spectrum. Comparison with the FD scheme in terms of different performance measures is discussed in section V. Section VI offers a comparison between two versions of the proposed FH, i.e., the robust frequency hopping and adaptive frequency hopping. Finally, section VII states the concluding remarks.

D. Notation

Throughout the paper, we use the notation $E\{.\}$ for the expectation operator. $\Pr\{E\}$ denotes the probability of an event $E$, $\mathbb{1}(E)$ the indicator function of an event $E$ and $p_X(.)$ the probability density function (PDF) of a random variable $X$. Also, $I(X;Y)$ denotes the mutual information between random variables $X$ and $Y$ and $h(X)$ denotes the differential entropy of a continuous random variable $X$. Finally, the notation $f(\gamma) \sim g(\gamma)$ implies $\lim_{\gamma \to \infty} \frac{f(\gamma)}{g(\gamma)} = 1$. 
II. SYSTEM MODEL AND ASSUMPTIONS

We consider a wireless network with $N$ users operating over a spectrum consisting of $u$ orthogonal sub-bands. The number of active users is assumed to be a random variable with a given distribution, however, it is fixed during the whole transmission once it is set first. The transmission blocks of each user comprise of an arbitrarily large number of transmission slots. We remark that the results of this paper are valid regardless of having block synchronization among the users, however, we assume synchronization at the symbol level. It is assumed that the $i^{th}$ user exploits $v_i (\leq u)$ out of the $u$ sub-bands in each transmission slot and hops randomly to another set of $v_i$ frequency sub-bands in the next transmission slot. This user transmits independent real Gaussian signals of variance $\frac{P}{v_i}$ over the chosen sub-bands, in which $P$ denotes the total average power for each transmitter. Each receiver is assumed to know the hopping pattern of its affiliated transmitter. It is assumed that the users are not aware of each other’s codebooks and hence, no multiuser detection or interference cancelation is possible at the receiver sides.

The static and non-frequency selective channel gain of the link connecting the $i^{th}$ transmitter to the $j^{th}$ receiver is shown by $h_{i,j}$. As it will be shown in (45), the only information each transmitter needs in order to regulate its transmission rate (focusing on the achieved multiplexing gain) is its forward channel gain, the maximum interference level at its associated receiver and the number of active users in the network. This information can be obtained at the receiver side by investigating the interference PDF and provided to the corresponding receiver via a feedback link.

As all users hop over different portions of the spectrum from transmission slot to transmission slot, no user is assumed to be capable of tracking the instantaneous interference level. This assumption makes the interference plus noise PDF at the receiver side of each user be mixed Gaussian. In fact, depending on different choices the other users make to select the frequency sub-bands and values of the crossover gains, the interference on each frequency sub-band at any given receiver can have up to $2^{N-1}$ power levels. The vector consisting of the received signals on the frequency sub-bands at the $i^{th}$ receiver in a typical transmission slot is

$$\vec{Y}_i = h_{i,i}\vec{X}_i + \vec{Z}_i,$$

where $\vec{X}_i$ is the $u \times 1$ transmitted vector and $\vec{Z}_i$ is the noise plus interference vector at the receiver side of the $i^{th}$ user. Due to the fact that each transmitter hops randomly from slot to slot, one may write $p_{\vec{X}_i}(\cdot)$

$^1$Each user consists of a transmitter-receiver pair.
as

\[ p_{\tilde{X}_i}(\tilde{x}) = \sum_{C \in \mathcal{C}} \frac{1}{(u!)^u} g_u(\tilde{x}, C), \]  

which corresponds to the mixed Gaussian distribution. In the above equation, \( g_u(\tilde{x}, C) \) denotes the PDF of a zero-mean \( u \times 1 \) jointly Gaussian vector of covariance matrix \( C \) and the set \( \mathcal{C} \) includes all \( u \times u \) diagonal matrices in which \( v_i \) out of the \( u \) diagonal elements are \( \frac{P}{v_i} \) and the rest are zero. Denoting the noise plus interference on the \( j^{th} \) sub-band at the receiver side of the \( i^{th} \) user by \( Z_{i,j} \) (the \( j^{th} \) component of \( \tilde{Z}_i \)), it is clear that \( p_{Z_{i,j}}(.) \) is not dependent on \( j \). This is due to the fact that the crossover gains are not frequency selective and there is no particular interest to a specific frequency sub-band by any user. We assume there are \( L_i + 1 \) (\( L_i \leq 2^{N-1} - 1 \)) possible non-zero power levels for \( Z_{i,j} \), say \( \{\sigma^2_{i,l}\}_{l=0}^{L_i} \). Denoting the occurrence probability of \( \sigma^2_{i,l} \) by \( a_{i,l} \), \( p_{Z_{i,j}}(.) \) identifies a mixed Gaussian PDF as

\[ p_{Z_{i,j}}(z) = \sum_{l=0}^{L_i} \frac{a_{i,l}}{\sqrt{2\pi\sigma^2_{i,l}}} \exp\left(-\frac{z^2}{2\sigma^2_{i,l}}\right), \]  

where \( \sigma^2 = \sigma^2_{i,0} < \sigma^2_{i,1} < \sigma^2_{i,2} < \ldots < \sigma^2_{i,L_i} \) (\( \sigma^2 \) is the ambient noise power). We notice that for each \( l \geq 0 \), there exists a \( c_{i,l} \geq 0 \) such that \( \sigma^2_{i,l} = \sigma^2 + c_{i,l}P \) where \( 0 = c_{i,0} < c_{i,1} < c_{i,2} < \ldots < c_{i,L_i} \). One may write \( Z_{i,j} = \sum_{k=1}^{N} \epsilon_{k,j} h_{k,j} X_{k,j} + \nu_{i,j} \) where \( \epsilon_{k,j} \) is a Bernoulli random variable showing if the \( k^{th} \) user has utilized the \( j^{th} \) sub-band, \( X_{k,j} \) is the signal of the \( k^{th} \) user sent on the \( j^{th} \) sub-band (assuming it has utilized the \( j^{th} \) sub-band), and \( \nu_{i,j} \) is the ambient noise which is a zero-mean Gaussian random variable with variance \( \sigma^2 \). The ratio \( \frac{P}{\sigma^2} \) is taken as a measure of SNR and is denoted by \( \gamma \) throughout the paper.

### III. Analysis of the Achievable Rate

Let us denote the achievable rate of the \( i^{th} \) user by \( R_i \). It can be observed that the communication channel of this user is a channel with state \( S_i \), the hopping pattern of the \( i^{th} \) user, which is independently changing over different transmission slots, and is known to both the transmitter and the receiver. The achievable rate of such a channel is given by

\[ R_i = I(\tilde{X}_i; \tilde{Y}_i|S_i) = \sum_{s_i \in \mathcal{S}_i} \Pr(S_i = s_i)I(\tilde{X}_i; \tilde{Y}_i|S_i = s_i), \]  

where \( I(\tilde{X}_i; \tilde{Y}_i|S_i = s_i) \) is the mutual information between \( \tilde{X}_i \) and \( \tilde{Y}_i \) for the specific sub-band selection corresponding to \( S_i = s_i \). The set \( \mathcal{S}_i \) denotes all possible selections of \( v_i \) out of the \( u \) sub-bands. As \( p_{Z_i}(.) \) is a symmetric density function, meaning all its components have the same PDF given in (3), we
deduce that $I(\vec{X}_i; \vec{Y}_i|S_i = s_i)$ is independent of $s_i$. Therefore, to calculate $R_i$, we may assume any specific sub-band selection for the $i^{th}$ user in $S_i$, say the first $v_i$ sub-bands. Denoting this specific state by $s_i^*$, we get

$$R_i = I(\vec{X}_i; \vec{Y}_i|S_i = s_i^*). \tag{5}$$

In this case, we denote $\vec{Y}_i$ and $\vec{X}_i$ by $\vec{Y}_i(s_i^*)$ and $\vec{X}_i(s_i^*)$, respectively. Obviously, we have

$$R_i = I(\vec{X}_i(s_i^*); \vec{Y}_i(s_i^*)) = h(\vec{Y}_i(s_i^*)) - h(\vec{Z}_i). \tag{6}$$

Because $\vec{Y}_i(s_i^*)$ and $\vec{Z}_i$ have mixed Gaussian distribution, there is no closed-form expression for the differential entropy of these vectors. As such, we provide an upper bound and a lower bound on the achievable rate of each user in the following subsections and show that these bounds converge in the asymptotic high SNR regime.

A. Upper Bound on The Achievable Rates

In this section, we develop an upper bound $R_{i,\text{ub}}$ on the achievable rate of the $i^{th}$ user that is tight enough to ensure that $R_{i,\text{ub}} - R_i$ does not increase unboundedly as SNR increases. The idea behind this upper bound is the convexity of $R_i$ in terms of $p(\vec{Y}_i(s_i^*)|\vec{X}_i(s_i^*)$.

**Theorem 1** There exists an upper bound on the achievable rate of the $i^{th}$ user given by

$$R_{i,\text{ub}} = \frac{1}{2} v_i \prod_{k=1}^{N} \left(1 - \frac{v_k}{u}\right) \log \left(1 + \frac{|h_{i,i}|^2 \gamma}{v_i}\right) + \tilde{R}_{i,\text{ub}} \tag{7}$$

where $\lim_{\gamma \to \infty} \tilde{R}_{i,\text{ub}} < \infty$. In particular, $R_{i,\text{ub}} \sim \frac{1}{2} v_i \prod_{k=1}^{N} \left(1 - \frac{v_k}{u}\right) \log \gamma$.

**Proof:** Let $\vec{W}_i$ be the $u \times 1$ interference vector where its $j^{th}$ component $W_{i,j}$ is a random variable showing the interference term on the $j^{th}$ frequency sub-band at the receiver of the $i^{th}$ user. We have $W_{i,j} = \sum_{k=1, k \neq i}^{N} \epsilon_{k,j} h_{k,i} X_{k,j}$. Clearly, $\vec{W}_i$ is a mixed Gaussian random vector where the Gaussian components in its PDF represent different choices the other users make in selecting their sub-bands. In fact, we have $p_{\vec{W}_i}(\vec{w}) = \frac{1}{M_i} \prod_{m=1}^{M_i} g_u(\vec{w}, D_{i,m})$, where $M_i = \prod_{k \neq i}^{u}$ and $D_{i,m} = \text{diag}(d_{i,m}^{(1)}, \ldots, d_{i,m}^{(u)})$, in which $d_{i,m}^{(j)} = \sum_{k=1, k \neq i}^{N} \epsilon_{k,j}^2 |h_{k,i}|^2 \frac{P}{v_k}$ denotes the variance of $W_{i,j}$ for the $m^{th}$ realization of $\{\epsilon_{k,j}\}_{k \neq i}$ out of $M_i$ possible realizations.\(^2\) If the probability density function of the interference vector consisted only of

\(^2\)Note that as each user transmits independent Gaussian signals over its chosen sub-bands, the matrices $(D_{i,m})_{m=1}^{M_i}$ are diagonal.
The following Lemma offers an explicit expression for the achievable rate of such a virtual channel is given by

\[
\mathcal{R}_{i,m} = \frac{1}{2} \log \left( \frac{\det \left( \text{Cov}(\vec{X}_i(s^*_i)) + D_{i,m} + \sigma^2 I_u \right)}{\det (D_{i,m} + \sigma^2 I_u)} \right)
\]

\[
= \frac{1}{2} \log \left( \prod_{j=1}^{v_i} \left( \frac{|h_{i,j}|^2 P + d_{i,m}^{(j)} + \sigma^2}{v_i} \right) \right)
\]

\[
= \frac{1}{2} \sum_{j=1}^{v_i} \log \left( 1 + \frac{|h_{i,j}|^2 P}{v_i(d_{i,m}^{(j)} + \sigma^2)} \right).
\]

One may also state this as follows. Let \( T_{i,m} \triangleq \{ j : 1 \leq j \leq v_i, d_{i,m}^{(j)} = 0 \} \). Then,

\[
\mathcal{R}_{i,m} = \frac{|T_{i,m}|}{2} \log \left( 1 + \frac{|h_{i,j}|^2 \gamma}{v_i} \right) + \tilde{\mathcal{R}}_{i,m},
\]

where

\[
\tilde{\mathcal{R}}_{i,m} = \frac{1}{2} \sum_{1 \leq j \leq v_i, d_{i,m}^{(j)} \neq 0} \log \left( 1 + \frac{|h_{i,j}|^2 P}{v_i(d_{i,m}^{(j)} + \sigma^2)} \right)
\]

and \(|T_{i,m}|\) denotes the cardinality of the set \( T_{i,m} \). As each nonzero \( d_{i,m}^{(j)} \) is proportional to \( P \), it is clear that \( \lim_{\gamma \to \infty} \mathcal{R}_{i,m} < \infty \). We know that \( \mathcal{R}_i \) is convex in terms of \( p_{\gamma_i(s^*_i)}(\vec{X}_i(s^*_i)) \) [25].

Noting this and the fact that \( p_{\vec{Z}_i}(\vec{z}) = \frac{1}{M_i} \sum_{m=1}^{M_i} g_u(\vec{z}, D_{i,m} + \sigma^2 I_u) \), we have

\[
\mathcal{R}_i \leq \frac{1}{M_i} \sum_{m=1}^{M_i} \mathcal{R}_{i,m} = \left( \frac{1}{M_i} \sum_{m=1}^{M_i} |T_{i,m}| \right) \frac{1}{2} \log \left( 1 + \frac{|h_{i,j}|^2 \gamma}{v_i} \right) + \tilde{\mathcal{R}}_{i,ub},
\]

where \( \tilde{\mathcal{R}}_{i,ub} = \frac{1}{M_i} \sum_{m=1}^{M_i} \tilde{\mathcal{R}}_{i,m} \). Clearly, as each \( \tilde{\mathcal{R}}_{i,m} \) saturates by increasing \( \gamma \), one has \( \lim_{\gamma \to \infty} \tilde{\mathcal{R}}_{i,ub} < \infty \).

The following Lemma offers an explicit expression for \( \frac{1}{M_i} \sum_{m=1}^{M_i} |T_{i,m}| \).

**Lemma 1**

\[
\frac{1}{M_i} \sum_{m=1}^{M_i} |T_{i,m}| = v_i \prod_{k=1}^{N} \left( 1 - \frac{v_k}{u} \right).
\]

**Proof:** Defining \( A_{i,j} \triangleq \{ m : 1 \leq m \leq M_i, |T_{i,m}| = j \} \) for each \( 1 \leq i \leq N \) and \( 1 \leq j \leq v_i \), one may express the left-hand side of (12) as

\[
\frac{1}{M_i} \sum_{m=1}^{M_i} |T_{i,m}| = \frac{1}{M_i} \sum_{j=1}^{v_i} j |A_{i,j}|.
\]
Let $F_i$ be a random variable showing the number of interference-free sub-bands among the $v_i$ sub-bands selected by the $i^{th}$ user. Using (48) and noting that $\Pr\{F_i = j\} = \frac{|A_{i,j}|}{M_i}$,

$$\frac{1}{M_i} \sum_{m=1}^{M_i} |T_{i,m}| = \sum_{j=1}^{v_i} j \Pr\{F_i = j\} = \mathbb{E}\{F_i\}. \quad (14)$$

Let us define

$$F_{i,j} \triangleq \begin{cases} 1 & W_{i,j} = 0 \\ 0 & W_{i,j} \neq 0 \end{cases} \quad (15)$$

for any $1 \leq i \leq N$ and $1 \leq j \leq v_i$. Obviously, $F_i = \sum_{j=1}^{v_i} F_{i,j}$. As such,

$$\mathbb{E}\{F_i\} = \sum_{j=1}^{v_i} \mathbb{E}\{F_{i,j}\} = \sum_{j=1}^{v_i} \Pr\{W_{i,j} = 0\}. \quad (16)$$

Since $\Pr\{\epsilon_{k,j} = 1\} = \frac{u_k}{u}$,

$$\Pr\{W_{i,j} = 0\} = \Pr\{Z_{i,j} \text{ contains no interference}\} = \prod_{k=1}^{N} \Pr\{\epsilon_{k,j} = 0\} = \prod_{k=1}^{N} \left( 1 - \frac{v_k}{u} \right). \quad (17)$$

This yields

$$\mathbb{E}\{F_i\} = v_i \prod_{k=1, k \neq i}^{N} \left( 1 - \frac{v_k}{u} \right), \quad (18)$$

which completes the proof of Lemma 1.

Based on (11) and Lemma 1, the proof of Theorem 1 is complete.

**B. Lower Bound on the Achievable Rates**

In this section, we derive a lower bound on the achievable rates of users. The idea behind deriving this lower bound is to invoke the classical entropy power inequality (EPI). As we will see, this initial lower bound is not in a closed form as it depends on the differential entropy of a mixed Gaussian random variable. In appendix A, we obtain an appropriate upper bound on such an entropy which leads to the final lower bound on $R_i$.

**Theorem 2** There exists a lower bound $R_{i,lb}$ on the achievable rate of the $i^{th}$ user which can be written as

$$R_{i,lb} = \frac{1}{2} v_i \prod_{k=1}^{N} \left( 1 - \frac{v_k}{u} \right) \log \gamma + \tilde{R}_{i,lb}, \quad (19)$$
such that $\lim_{\gamma \to \infty} \tilde{R}_{i,lb} < \infty$. In particular, $\tilde{R}_{i,lb} \sim \frac{1}{2} v_i \prod_{k=1 \atop k \neq i}^N \left(1 - \frac{v_k}{u_k}\right) \log \gamma$.

**Proof:** We define $\vec{X}_i'$ to be the $v_i \times 1$ signal vector corresponding to the first $v_i$ elements of $\vec{X}_i(s^*_i)$. Clearly, $\vec{X}_i'$ is a Gaussian vector with covariance matrix $P \frac{v_i}{v_i}$. Let $\vec{Y}_i' = h_{i,i} \vec{X}_i' + \vec{Z}_i'$ where $\vec{Z}_i'$ is the noise plus interference vector at the receiver side of the $i^{th}$ user on the first $v_i$ sub-bands. Using entropy power inequality, we have

$$2^{\frac{2}{v_i} h(\vec{Y}_i')} \geq 2^{\frac{2}{v_i} h(h_{i,i} \vec{X}_i') + 2^{\frac{2}{v_i} h(\vec{Z}_i')}}.$$  

(20)

Dividing both sides by $2^{\frac{2}{v_i} h(\vec{Z}_i')}$, we get

$$h(\vec{Y}_i') - h(\vec{Z}_i') \geq \frac{v_i}{2} \log \left(2^{\frac{2}{v_i} (h(h_{i,i} \vec{X}_i') - h(\vec{Z}_i')) + 1}\right).$$  

(21)

On the other hand, since $\vec{Y}_i'$ is a subvector of $\vec{Y}_i(s^*_i)$, we have

$$R_i = I(\vec{X}_i(s^*_i); \vec{Y}_i(s^*_i)) \geq I(\vec{X}_i'; \vec{Y}_i') = h(\vec{Y}_i') - h(\vec{Z}_i').$$  

(22)

Comparing (21) and (22) yields

$$R_i \geq \frac{v_i}{2} \log \left(2^{\frac{2}{v_i} (h(h_{i,i} \vec{X}_i') - h(\vec{Z}_i')) + 1}\right).$$  

(23)

Clearly, $h(h_{i,i} \vec{X}_i') = \frac{v_i}{2} \log \left(2\pi e \frac{|h_{i,i}|^2 P}{v_i}\right)$. As $\vec{Z}_i'$ is a mixed Gaussian random vector, there is no closed-form formula for $h(\vec{Z}_i')$. Hence, we have to find an appropriate upper bound on $h(\vec{Z}_i')$ to further simplify (23). Using the chain rule for the differential entropy, we obtain

$$h(\vec{Z}_i') \leq \sum_{j=1}^{v_i} h(Z_{i,j}).$$  

(24)

Recalling the definitions of $\{a_{i,t}\}_{t=0}^{L_i}$ and $\{c_{i,t}\}_{t=0}^{L_i}$ in the system model, the following Lemma yields an upper bound on $h(Z_{i,j})$ for each $1 \leq j \leq v_i$.

**Lemma 2** For every $1 \leq j \leq v_i$ and for all values of $\gamma$, there exists an upper bound on $h(Z_{i,j})$ given by

$$h(Z_{i,j}) \leq \frac{1 - a_{i,0}}{2} \log(c_{i,L_i} \gamma + 1) + \log(\sqrt{2\pi e \sigma}) + \mathcal{H}_i$$  

(25)

where $\mathcal{H}_i \triangleq - \sum_{t=0}^{L_i} a_{i,t} \log a_{i,t}$ is the discrete entropy of $\{a_{i,t}\}_{t=0}^{L_i}$.

**Proof:** See Appendix A.
By (25), (24) and (23),

\[ R_i \geq R_{i,lb} \triangleq \frac{v_i}{2} \log \left( \frac{2^{-2\gamma_i^2} |h_{i,i}|^2 \gamma_i}{v_i (c_i L_i \gamma_i + 1)^{1-a_{i,0}} + 1} \right) \]

\[ = \frac{1}{2} v_i a_{i,0} \log \gamma + \frac{v_i}{2} \log \left( \frac{2^{-2\gamma_i^2} |h_{i,i}|^2}{v_i (c_i L_i \gamma_i + 1)^{1-a_{i,0}} + \gamma^{-a_{i,0}}} \right). \]  

(26)

Defining \( \tilde{R}_{i,lb} \triangleq \frac{v_i}{2} \log \left( \frac{2^{-2\gamma_i^2} |h_{i,i}|^2}{v_i (c_i L_i \gamma_i + 1)^{1-a_{i,0}} + \gamma^{-a_{i,0}}} \right) \), we note that \( \lim_{\gamma \to \infty} \tilde{R}_{i,lb} < \infty \). Combining this with the fact that \( a_{i,0} = \prod_{k=1}^{N} (1 - \frac{v_k}{u}) \) completes the proof of Theorem 2.

In [24], we address another approach to propose a lower bound on the achievable rate of the \( i^{th} \) user with the same SNR scaling as \( R_{i,lb} \).

One may consider the following generalization of the FH scheme. Let us assume that the users are not restricted to choose a fixed number of frequency sub-bands in each transmission slot. In fact, in each transmission slot the number of selected sub-bands can be any integer between 0 and \( u \), and the probability of choosing \( v \in [0, u] \cap \mathbb{Z} \) sub-bands by the \( i^{th} \) user is denoted by \( \mu_{i,v} \). Therefore, the \( i^{th} \) user has two random generators. The first random generator selects a number \( 0 \leq v \leq u \) according to the probability mass function \( \{\mu_{i,v}\}_{v=0}^{u} \), while the other generator selects \( v \) sub-bands among the whole available \( u \) sub-bands. This repeats independently from transmission slot to transmission slot. Based on the arguments made in section II, the achievable rate of the \( i^{th} \) user can be written as

\[ R_i = \sum_{v=0}^{u} \mu_{i,v} I(\bar{X}_i(s_{i,v}^*); \bar{Y}_i(s_{i,v}^*)), \]  

(27)
where $s_{i,v}^*$ denotes the state where the $i^{th}$ user selects the first $v$ sub-bands. Clearly, $I(\vec{X}_i(s_{i,0}^*); \vec{Y}_i(s_{i,0}^*)) = 0$ for any $1 \leq i \leq N$. Furthermore,

$$a_{i,0} = \Pr \left\{ \text{A given component of } \vec{Z}_i \text{ contains no interference} \right\}$$

$$= \sum_{v_1=0}^{u} \cdots \sum_{v_{i-1}=0}^{u} \sum_{v_{i+1}=0}^{u} \cdots \sum_{v_N=0}^{u} \prod_{k=1}^{N} \mu_{k,v_k} \left( 1 - \frac{v_k}{u} \right)$$

$$= \prod_{k=1}^{N} \sum_{v_k=0}^{u} \mu_{k,v_k} \left( 1 - \frac{v_k}{u} \right)$$

$$= \prod_{k=1}^{N} \sum_{v=0}^{u} \mu_{k,v} \left( 1 - \frac{v}{u} \right)$$

$$= \prod_{k=1}^{N} \left( 1 - \frac{\bar{v}_k}{u} \right)$$

(28)

where $\bar{v}_k \triangleq \sum_{v=0}^{u} v \mu_{k,v}$. Based on the results of this section, we get

$$I(\vec{X}_i(s_{i,v}^*); \vec{Y}_i(s_{i,v}^*)) \sim \frac{1}{2} v a_{i,0} \log \gamma.$$  \hspace{1cm} (29)

Using (27), (28) and (29) yields

$$R_i \sim \sum_{v=0}^{u} \frac{1}{2} \mu_{i,v} \prod_{k=1}^{N} \left( 1 - \frac{\bar{v}_k}{u} \right) \log \gamma$$

$$= \frac{1}{2} \bar{v}_i \prod_{k=1}^{N} \left( 1 - \frac{\bar{v}_k}{u} \right) \log \gamma.$$  \hspace{1cm} (30)

In fact, (30) demonstrates that the generalized FH scheme is equivalent to the FH scheme through substituting $\{v_i\}_{i=1}^{N}$ by $\{\bar{v}_i\}_{i=1}^{N}$. However, it is remarkable that in contrast to the FH scheme in which $\{v_i\}_{i=1}^{N}$ are integer values, in the generalized FH scheme $\{\bar{v}_i\}_{i=1}^{N}$ are real values. This provides more flexibility in system design. The above observation motivates us to use the generalized scenario in the sequel and we simply refer to it as the FH scheme. In this scheme, the $i^{th}$ user has a parameter $\bar{v}_i$, which can be chosen to be any real number in the interval $[0, u]$. 
IV. System Design

In this section, we find the optimum operation point for the FH scheme. This requires finding the optimum values of \( \bar{v}_i \) for \( i = 1, \ldots, N \). Based on the results established in the previous section, there exist upper and lower bounds on the achievable rate of each user that coincide in the high SNR regime. As such, the achievable rate itself must be asymptotically equivalent to each of these bounds, i.e.,

\[
R_i \sim \frac{1}{2} \bar{v}_i \prod_{k=1 \atop k \neq i}^{N} \left( 1 - \frac{\bar{v}_k}{u} \right) \log \gamma,
\]

(31)

where, based on the conclusion made at the end of section III, the parameters \( \{\bar{v}_i\}_{i=1}^{N} \) can be adjusted to be any real number in the range \([0, u]\). By (31), the network sum-rate can be asymptotically written as

\[
\sum_{i=1}^{N} R_i \sim SMG(\bar{v}_1, \ldots, \bar{v}_N) \log \gamma,
\]

(32)

where

\[
SMG(\bar{v}_1, \ldots, \bar{v}_N) \triangleq \sum_{i=1}^{N} \frac{1}{2} \bar{v}_i \prod_{k=1 \atop k \neq i}^{N} \left( 1 - \frac{\bar{v}_k}{u} \right).
\]

(33)

We call \( SMG(\bar{v}_1, \ldots, \bar{v}_N) \) the sum multiplexing gain of the system. \( SMG(\bar{v}_1, \ldots, \bar{v}_N) \) is a symmetric function of \( (\bar{v}_1, \ldots, \bar{v}_N) \) and has a saddle point at \( \bar{v}_i = \frac{u}{N} \) for \( 1 \leq i \leq N \). In a fair FH system, it is required that \( \bar{v}_i = v \) for all \( 1 \leq i \leq N \) where \( v \) is any real number in the interval \([0, u]\). Hence, we define

\[
SMGV(u, N) \triangleq SMG(\bar{v}_1, \ldots, \bar{v}_N) \bigg|_{\forall i \colon \bar{v}_i = v} = \frac{N}{2} v \left( 1 - \frac{v}{u} \right)^{N-1}.
\]

(34)

Maximizing this in terms of \( v \) yields

\[
v_{\text{opt}} = \frac{u}{N}.
\]

(35)

Computation of \( v_{\text{opt}} \) requires that all transmitters know the number of active users \( N \) in the network. As far as all channel gains are realizations of independent and continuous random variables, the number of power levels in the PDF of noise plus interference on any frequency sub-band at the receiver side of any user is almost surely equal to \( 2^{N-1} \). Therefore, any receiver can identify \( N \) and send it to its corresponding transmitter through a feedback link.
Setting $v = v_{\text{opt}}$, the highest sum multiplexing gain of the fair FH scheme is given by

$$
\sup_v \text{SMG}(v, N) = \frac{1}{2} u \left( 1 - \frac{1}{N} \right)^{N-1}.
$$

It is remarkable that $\frac{u}{N}$ may not be a positive integer. If we do not adopt the generalized FH scheme, then all users must hop randomly over sets of $\tilde{v} = \max \left\{ \lfloor \frac{w}{N} \rfloor, 1 \right\}$ frequency sub-bands. This results in a sum multiplexing gain of $\frac{N}{2} \tilde{v} \left( 1 - \frac{\tilde{v}}{u} \right)^{N-1}$. This is, in general, less than $\frac{1}{2} u \left( 1 - \frac{1}{N} \right)^{N-1}$. By adopting the generalized FH scheme in case $\frac{u}{N} \notin \mathbb{Z}$, each user only needs to hop randomly over different sets of frequency sub-bands of cardinality $\lfloor \frac{w}{N} \rfloor$ or $\lceil \frac{u}{N} \rceil$. In fact, each user has two random generators. The first random generator selects one of the numbers $\lfloor \frac{w}{N} \rfloor$ and $\lceil \frac{u}{N} \rceil$ with probabilities $\mu$ and $1 - \mu$, respectively, such that $\mu \lfloor \frac{w}{N} \rfloor + (1 - \mu) \lceil \frac{u}{N} \rceil = \frac{w}{N}$ or equivalently $\mu = \lfloor \frac{w}{N} \rfloor - \frac{w}{N}$. Let us assume the first random generator has selected a number $a \in \{ \lfloor \frac{w}{N} \rfloor, \lceil \frac{u}{N} \rceil \}$. Then, the second random generator selects a subset of cardinality $a$ among the $u$ frequency sub-bands. Doing this independently from transmission slot to transmission slot, the sum multiplexing gain given in (36) is achieved.

**Observation 1**- One might suggest another well-known utility function that is popular in the game theory context, namely the proportional fair function, which is defined as $\sum_{i=1}^{N} \log R_i$. We have

$$
\sum_{i=1}^{n} \log R_i \sim \sum_{i=1}^{N} \log \left( \frac{1}{2} \tilde{v}_i \prod_{k=1, k \neq i}^{N} \left( 1 - \frac{\tilde{v}_k}{u} \right) \log \gamma \right)
$$

$$
= \sum_{i=1}^{N} \log \left( \frac{1}{2} \tilde{v}_i \prod_{k=1, k \neq i}^{N} \left( 1 - \frac{\tilde{v}_k}{u} \right) \right) + N \log \log \gamma.
$$

(37)

It can be easily verified that $\sum_{i=1}^{N} \log \left( \frac{1}{2} \tilde{v}_i \prod_{k=1, k \neq i}^{N} \left( 1 - \frac{\tilde{v}_k}{u} \right) \right)$ has an absolute maximum at $\tilde{v}_i = \frac{w}{N}$ for $1 \leq i \leq N$.

**Observation 2**- As we will discuss in more detail in the next section, the number of active users in the system is in general a random variable. Although users in the FH system can use their knowledge about the number of active users to adjust the hopping parameter (as explained earlier in this section), one may devise a sub-optimal rule to fix $v = v^*$ given by

$$
v^* = \arg \max_{v \in [0, u]} \mathbb{E} \{ \text{SMG}(v, N) \},
$$

(38)

where the expectation is with respect to the number of active users in the network. This selection of $v$
by all users makes the system robust against changes in the number of active users in the network. We call this version of the FH system the robust Frequency Hopping. In fact, in the robust FH scenario, there exists a global hopping parameter \( v \) where all users hop over a number \( v \) of the \( u \) frequency sub-bands. We remark that the rule in (38) is a particular design approach for the robust FH system. In the next section, we consider another design rule based on maximizing the average of the minimum multiplexing gain per user in term of the number of active users in the network.

V. COMPARISON OF THE ROBUST FH SCENARIO WITH THE FD SCHEME

In a centralized setup, under the condition that no user is aware of the other users’ codebooks and the number of users is fixed and known to the central controller, it is shown in [9] that if the crossover channel gains are sufficiently larger than the forward channel gains, then every Pareto optimal rate vector is realized by Frequency Division for all ranges of SNR. However, in realistic scenarios, the number of active users is not fixed. This degrades the performance of the FD scheme as it is designed for a specified number of users. In particular, if the number of active users is less than the designed target of the FD scheme, a considerable portion of the spectrum may remain unused. This encourages us to compare the performance of the proposed robust FH scheme with that of the FD scheme in a setup where the number of active users is a random variable with a given distribution.

To perform the comparison, we introduce four different performance measures. In the following definitions, the \( \sup \) operation is over possible adjustable parameters in the system, e.g., the hopping parameter in the FH scenario. All expectations are taken with respect to \( N \). We define \( q_n \triangleq \Pr\{N = n\} \) for all \( n \geq 0 \). It is assumed that the maximum number of active users in the network is \( n_{\text{max}} \), i.e., \( \Pr\{N > n_{\text{max}}\} = 0 \). We usually take \( q_0 = 0 \) unless otherwise stated.

- **Average sum multiplexing gain**, which is defined as

\[
\eta^{(1)} \triangleq \sup \lim_{\gamma \to \infty} \frac{E\left\{ \sum_{i=1}^{N} \mathcal{R}_i \right\}}{\log \gamma} = \sup E\{\text{SMG}\},
\]

where \( \text{SMG} = \lim_{\gamma \to \infty} \frac{\sum_{i=1}^{N} \mathcal{R}_i}{\log \gamma} \) is the sum multiplexing gain.

---

3 In fact, the transmitters use their knowledge about the instantaneous number of active users only to regulate their transmission rate. This is explained in more details in (45).

4 Note that, as explicitly mentioned in the system model, the number of users is assumed to be fixed for the whole transmission period of interest.
• **Average minimum multiplexing gain per user**, which is defined as

\[
\eta^{(2)} \triangleq \sup \lim_{\gamma \to \infty} \frac{E\{\min_{1 \leq i \leq N} \mathcal{R}_i\}}{\log \gamma}.
\]  

(40)

• **Minimum nonzero multiplexing gain per user**, which is defined as

\[
\eta^{(3)} \triangleq \min_{n: q_n \neq 0} \min_{1 \leq i \leq N_{\text{serv}}} \lim_{\gamma \to \infty} \frac{\mathcal{R}_i}{\log \gamma}
\]  

(41)

where \(N_{\text{serv}}\) denotes the number of active users receiving service (i.e., their multiplexing gain is strictly positive).

• **Service capability**, which is defined as

\[
\eta^{(4)} \triangleq \sup E \left\{ \frac{N_{\text{serv}}}{N} \right\}
\]  

(42)

The FD system is designed to service, at most, a certain number of active users. We denote this design target in the FD scheme by \(n_{\text{des}}\). Therefore, the spectrum is divided to \(n_{\text{des}}\) bands where each band contains \(\frac{u}{n_{\text{des}}}\) frequency sub-bands. This requires that \(u\) is divisible by \(n_{\text{des}}\), which is assumed to be the case to guarantee fairness. Each user that becomes active occupies an empty band. If there is no empty band, no service is available. In case \(n_{\text{max}}\) is finite, the central controller in the FD system sets \(n_{\text{des}} = n_{\text{max}}\) to ensure that all users can receive service upon activation. In case \(n_{\text{max}}\) is not a finite number, the central controller sets \(n_{\text{des}} = u\) to guarantee that as many users receive service as possible. Therefore, \(n_{\text{des}} = \min\{n_{\text{max}}, u\}\). In fact, we will show that selecting \(n_{\text{des}} = \min\{n_{\text{max}}, u\}\) maximizes the service capability in the FD system.

We remark that due to the nature of the robust FH scheme, as far as users hop over a proper subset of size \(v\) of the \(u\) sub-bands, all users receive service, while if \(v = u\) and \(N > 1\), no user receives service, i.e., the multiplexing gain achieved by any active user is zero. As such, to get the largest service capability in the FH scenario, we require \(v \in (0, u)\). As an example, if \(v^*\) in (38) is equal to \(u\), the service capability will be less than 1. To avoid this, we set the global hopping parameter \(v = v^* - \varepsilon = u - \varepsilon\) for sufficiently small \(\varepsilon\) such that the performance of the robust FH is still above the performance of the FD scenario.

• **Average sum multiplexing gain**

This measure is a meaningful tool of comparison if \(n_{\text{max}} < \infty\). Hence, we assume \(n_{\text{max}}\) is a finite number and \(u\) is a multiple of \(n_{\text{max}}\) in this subsection. It is easily seen that the sum multiplexing gain in
the FD scenario is

$$\text{SMG}_{\text{FD}}(n_{\text{des}}, N) = \begin{cases} \frac{N - u}{2} n_{\text{des}} & N \leq n_{\text{des}} \\ \frac{u}{2} n_{\text{des}} & N > n_{\text{des}} \end{cases}.$$  \tag{43}$$

Noting (34), \(\text{SMG}_{\text{FH}}(v, N)\) is given by

$$\text{SMG}_{\text{FH}}(v, N) = \frac{1}{2} N v \left( 1 - \frac{v}{u} \right)^{N-1}. \tag{44}$$

Since the number of active users \(N\) is a global knowledge, all users can choose \(v = v_{\text{opt}} = \frac{u}{N}\) to achieve the maximum sum multiplexing gain. However, as mentioned earlier, a robust hopping strategy against changes in the number of active users is the one given in (38). It is notable that although the value of \(v\) is fixed at \(v^*\), all users regulate their rates based on the instantaneous number of active users to avoid transmission failure. Using the lower bound on the achievable rate of the \(i^{th}\) user given in (26), the \(i^{th}\) user selects its actual rate \(R_i\) as

$$R_i = \frac{u^*}{2} \log \left( \frac{\frac{u^*}{w^*}}{1 - \frac{u^*}{u}} \left( 1 - \frac{u^*}{w^*} \right)^{2(N-1)(1 - \frac{u^*}{w^*}) |h_{i,i}|^2 \gamma} + 1 \right). \tag{45}$$

It is seen that the quantities the \(i^{th}\) transmitter needs to evaluate \(R_i\) are \(|h_{i,i}|, \sum_{j \neq i} |h_{j,i}|^2\) and \(N\). The \(i^{th}\) receiver sends these required data to the transmitter via a feedback link.

We present an example to compare the performance of FH with that of FD in terms of \(\eta^{(1)}\).

**Example 1** - Let us consider a network where \(n_{\text{max}} = 2\). The central controller in the FD system sets \(n_{\text{des}} = 2\), and according to (43),

$$\eta^{(1)}_{\text{FD}} = \mathbb{E}\{\text{SMG}_{\text{FD}}(2, N)\} = q_1 \frac{u}{4} + q_2 \frac{u}{2} = \frac{q_1 + 2q_2}{4} u. \tag{46}$$

Based on (44),

$$\mathbb{E}\{\text{SMG}_{\text{FH}}(v, N)\} = \frac{1}{2} q_1 v + q_2 v \left( 1 - \frac{v}{u} \right). \tag{47}$$

Using this in (38),

$$v^* = \arg \max_{v \in [0, u]} \mathbb{E}\{\text{SMG}_{\text{FH}}(v, N)\} = \begin{cases} \frac{q_1 + 2q_2}{4q_2} u & q_1 \leq 2q_2 \\ u & q_1 > 2q_2 \end{cases}.$$  \tag{48}$$

Therefore,

$$\eta^{(1)}_{\text{FH}} = \sup_{v \in [0, u]} \mathbb{E}\{\text{SMG}_{\text{FH}}(v, N)\} = \mathbb{E}\{\text{SMG}_{\text{FH}}(v^*, N)\} = \begin{cases} \frac{(q_1 + 2q_2)^2}{16q_2} u & q_1 \leq 2q_2 \\ \frac{q_1}{2} u & q_1 > 2q_2 \end{cases}.$$  \tag{49}$$

It is easy to see that \(\eta^{(1)}_{\text{FH}} > \eta^{(1)}_{\text{FD}}\) if and only if \(q_1 > 2q_2\), or equivalently, \(q_1 > 2q_2\). We note that in this case \(v^* = u\), i.e., all users spread their power on the whole spectrum and no hopping is performed. This makes
service capability be strictly less than 1 because, if both users are active, none of them receive service. As such, we take \( v = u - \varepsilon \). To ensure that the performance of the robust FH scenario is above that of the FD system, we require

\[
\frac{1}{2} q_1 (u - \varepsilon) + q_2 (u - \varepsilon) \left( 1 - \frac{u - \varepsilon}{u} \right) > \frac{q_1 + 2q_2}{4} u.
\]

(48)

As far as \( \varepsilon < \frac{u}{2} \), (48) is equivalent to \( q_1 > 2q_2 \frac{1 - \frac{u - \varepsilon}{u}}{1 - \frac{u - \varepsilon}{u}} \). This is a more restrictive condition than \( q_1 > 2q_2 \) which is the cost paid for having full service capability. However, for \( \varepsilon \ll u \) the two regions of \((q_1, q_2)\) are almost the same. □

In [24], it is shown that in case \( n_{\text{max}} = 3 \), there exists a probability set of \((q_1, q_2, q_3)\) on the number of active users that makes FH achieve a higher performance compared to FD in terms of \( \eta^{(1)} \) while \( v^* \) is strictly less than \( u \).

- **Average minimum multiplexing gain per user**

This measure can also be written as

\[
\eta^{(2)} = \sup \mathbb{E} \left\{ \frac{\text{SMG}}{N} \mathbb{I}(N_{\text{serv}} = N) \right\}.
\]

(49)

In fact, if \( N_{\text{serv}} \neq N \), there exists at least one user that achieves no multiplexing gain. Therefore, the minimum multiplexing gain per user is zero in this case. However, if \( N_{\text{serv}} = N \), all users achieve a nonzero multiplexing gain. This measure can be used whether \( n_{\text{max}} \) is finite or infinite.

In case of the FH scenario, the rule to choose the optimum value of the global hopping parameter \( v \), denoted by \( v^\dagger \), is given by

\[
v^\dagger = \arg \max_{v \in [0, u]} \mathbb{E} \left\{ \frac{\text{SMG}_{FH}(v, N)}{N} \mathbb{I}(N = N_{\text{serv}}) \right\}.
\]

(50)

In this case, the actual transmission rate of the \( i^{th} \) user is given by (45) where \( v^* \) is replaced by \( v^\dagger \).

**Example 2** - Considering the same setup in example 1, as \( n_{\text{max}} < \infty \), we have \( N_{\text{serv,FD}} = N \). Hence, we have \( \eta^{(2)}_{\text{FD}} = \frac{1}{2} q_1 + \frac{1}{2} q_2 = \frac{u}{4} \). In case of the FH scheme,

\[
\mathbb{I}(N_{\text{serv,FH}} = N) = \begin{cases} 1 & N = 1 \text{ or } (N > 1 \text{ and } v \neq u) \\ 0 & \text{oth.} \end{cases}
\]

(51)
Hence,

\[
E \left\{ \frac{\text{SMG}_{\text{FH}}(v, N)}{N} \mathbb{1}(N_{\text{serv}} = N) \right\} = E \left\{ \frac{\text{SMG}_{\text{FH}}(v, N)}{N} \mathbb{1}(N_{\text{serv}} = N) \middle| N = 1 \right\} \Pr \{N = 1\} \\
+ E \left\{ \frac{\text{SMG}_{\text{FH}}(v, N)}{N} \mathbb{1}(N_{\text{serv}} = N) \middle| N = 2 \right\} \Pr \{N = 2\} \\
= \frac{1}{2} q_1 v + \frac{1}{2} q_2 v \left( 1 - \frac{v}{u} \right) \mathbb{1}(v \neq u) \\
= \frac{1}{2} q_1 v + \frac{1}{2} q_2 v \left( 1 - \frac{v}{u} \right). 
\]  

Hence,

\[
v^\dagger = \arg \max_{v \in (0, u]} \left\{ q_1 v + q_2 v \left( 1 - \frac{v}{u} \right) \right\}, 
\]

which yields

\[
v^\dagger = \begin{cases} 
\frac{u}{2q_2} & 2q_2 > 1 \\
u & 2q_2 \leq 1
\end{cases}.
\]

As such,

\[
\eta^{(2)}_{\text{FH}} = \begin{cases} 
\frac{1}{8q_2} u & 2q_2 > 1 \\
\frac{1}{2} q_1 u & 2q_2 \leq 1
\end{cases}.
\]

It is easy to see that \( \eta^{(2)}_{\text{FH}} > \eta^{(2)}_{\text{FD}} \) if and only if \( 2q_2 < 1 \), or equivalently \( q_1 > \frac{1}{2} \). However, in this case \( v^\dagger = u \). Hence, to make the service capability be 1, we choose the global hopping parameter \( v = u - \varepsilon \).

To ensure that FH still outperforms FD in terms of the average minimum multiplexing gain per user, we require,

\[
\frac{1}{2} q_1 (u - \varepsilon) + \frac{1}{2} q_2 (u - \varepsilon) \left( 1 - \frac{u - \varepsilon}{u} \right) > \frac{u}{4}.
\]

This is equivalent to \( 2q_2 < \frac{2}{1 - \varepsilon} \left( 1 - \frac{1}{2(1 - \varepsilon)} \right) \). □

In the next example, we provide a case where \( \eta^{(2)}_{\text{FH}} > \eta^{(2)}_{\text{FD}} \) while \( v^\dagger \) is strictly less than \( u \).

**Example 3** - Let \( n_{\text{max}} < \infty \). In this example, we aim to derive a sufficient condition on \( \{q_n\}^{n_{\text{max}}} \) such that \( \eta^{(1)}_{\text{FH}} > \eta^{(1)}_{\text{FD}} \) or \( \eta^{(2)}_{\text{FH}} > \eta^{(2)}_{\text{FD}} \).

**Case 1** - Let us consider the measure \( \eta^{(1)} \). We have the following result.

**Proposition 1** As far as

\[
E \{N\} < \frac{1}{2} \ln \left( (e^2 - 1)n_{\text{max}} \right),
\]

where \( n_{\text{max}} \) is the maximum number of users.
we have $\eta_{FD}^{(1)} < \eta_{FH}^{(1)}$.

**Proof:** See Appendix B.

For example, if $n_{\text{max}} = 2$, (57) gives $E\{N\} \leq 1.274$, or equivalently $q_1 \geq 0.726$. By example 1, we notice that $\eta_{FH}^{(1)} \geq \eta_{FD}^{(1)}$ if and only if $q_1 \geq 0.667$.

**Case 2-** As for $\eta^{(2)}$, along the same lines leading to (57), a sufficient condition for $\eta_{FH}^{(2)} > \eta_{FD}^{(2)}$ is given in the following Proposition.

**Proposition 2** As far as

$$\frac{1}{E\{N\}} \left( 1 - \frac{1}{E\{N\}} \right)^{E\{N\} - 1} > \frac{1}{n_{\text{max}}},$$

we have $\eta_{FD}^{(2)} < \eta_{FH}^{(2)}$.

**Proof:** See Appendix C.

For example, if $n_{\text{max}} = 10$, $q_1 = 0.22$, $q_2 = q_3 = q_4 = 0.24$ and $q_5 = q_6 = \cdots = q_{10} = 0.01$, one has $E\{N\} = 2.78$, which satisfies (58). Therefore, we conclude $\eta_{FH}^{(2)} > \eta_{FD}^{(2)}$. Computing these quantities directly, we get $\eta_{FD}^{(2)} = \frac{u}{16}$ and

$$\eta_{FH}^{(2)} = \frac{1}{2} \max_{v \in [0, u]} \left\{ v \sum_{n=1}^{10} q_n \left( 1 - \frac{v}{u} \right)^{n-1} \right\}$$

$$\overset{(a)}{=} \frac{1}{2} u \max_{\omega_v \in [0,1]} (1 - \omega_v) \left( 0.22 + 0.24(\omega_v + \omega_v^2 + \omega_v^3) + 0.01(\omega_v^4 + \omega_v^5 + \omega_v^6 + \omega_v^7 + \omega_v^8 + \omega_v^9) \right)$$

$$\overset{(b)}{=} 0.1121u$$

(59)

where in (a), we define $\omega_v \overset{\Delta}{=} 1 - \frac{v}{u}$ and (b) is obtained by setting $\omega_v = 0.28$, or equivalently $v = v^t = 0.72u$. This yields $\frac{\eta_{FH}^{(2)}}{\eta_{FD}^{(2)}} = 1.7936$.

**Example 4-** In this example, we assume a Poisson distribution on the number of active users, i.e.,

$q_n = e^{-\lambda} \frac{\lambda^n}{n!}, \ n \geq 0$. This assumption corresponds to the scenario where potentially a large number $n_{\text{max}}$ of users may share the spectrum. However, the activation probability $p$ of each user is very small. One can well approximate the number of active users in the network by a Poisson random variable with parameter
\( \lambda = pn_{\text{max}} \). We have

\[
E \left\{ \frac{\text{SMG}_{\text{FH}}(v,N)}{N} \mathbb{1}(N_{\text{serv,FH}} = N) \right\} \overset{(a)}{=} E \left\{ \frac{\text{SMG}_{\text{FH}}(v,N)}{N} \right\} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-\lambda n}}{n!} \left( v \left( 1 - \frac{v}{u} \right)^{n-1} \right) = \frac{1}{2} \frac{v}{1 - \frac{v}{u}} \sum_{n=1}^{\infty} \frac{e^{-\lambda n}}{n!} \left( 1 - \frac{v}{u} \right)^n
\]

\[
\overset{(b)}{=} \frac{1}{2} \frac{v}{1 - \frac{v}{u}} \left( e^\lambda (e^{\ln(1 - \frac{u}{v})} - 1) - e^{-\lambda} \right) = \frac{e^{-\lambda}(1 - \omega_v)}{2\omega_v} (e^{\lambda \omega_v} - 1) u. \tag{60}
\]

In the above equation, \( (a) \) results from the fact that \( \mathbb{1}(N_{\text{serv,FH}} = N) = 0 \) whenever \( v = u \) and \( N > 1 \), however, \( \text{SMG}_{\text{FH}}(v,N) = 0 \) in this case. \( (b) \) follows by the fact that \( E\{e^{tN}\} = e^{\lambda(e^t - 1)} \) for any \( t \) and \( \omega_v = 1 - \frac{v}{u} \) as defined in example 3. It can be easily seen that the optimal \( \omega_v \) satisfies the nonlinear equation

\[
e^{-\lambda \omega_v} = 1 - \lambda \omega_v + \lambda \omega_v^2.
\]

Solving this for \( \omega_v \), we find out that \( v^\dagger \) is not equal to \( u \) for all \( \lambda > 2 \). The following table lists the optimum values of \( \omega_v \), i.e., \( \omega_v^\dagger \), the values of \( v^\dagger \) and also the corresponding average minimum multiplexing gain per user \( \eta^{(2)}_{\text{FH}} \) for \( \lambda \in \{3, \cdots, 10\} \).

| \( \lambda \) | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|---------------|------|------|------|------|------|------|------|------|
| \( \omega_v^\dagger \) | 0.4536 | 0.6392 | 0.7347 | 0.7912 | 0.828 | 0.8537 | 0.8727 | 0.8873 |
| \( v^\dagger \) | 0.5464u | 0.3608u | 0.2653u | 0.2088u | 0.1720u | 0.1463u | 0.1273u | 0.1127u |
| \( \eta^{(2)}_{\text{FH}} \) | 0.0869u | 0.0615u | 0.0467u | 0.0374u | 0.0311u | 0.0266u | 0.0232u | 0.0206u |

In order to provide fairness among the users, the FD system tries to serve as many users as it can. Since it is not possible to serve more than \( u \) users and \( n_{\text{max}} \gg u \), the central controller sets \( n_{\text{des}} = u \). Therefore, \( N_{\text{serv,FD}} < N \) if and only if \( N > u \). Using this and by \( (43) \),

\[
\eta^{(2)}_{\text{FD}} = E \left\{ \frac{\text{SMG}_{\text{FD}}(n_{\text{des}},N)}{N} \mathbb{1}(N_{\text{serv,FD}} = N) \right\} = E \left\{ \frac{\text{SMG}_{\text{FD}}(u,N)}{N} \mathbb{1}(N \leq u) \right\} \Pr\{N \leq u\}
\]

\[
= \frac{1}{2} \sum_{n=1}^{u} \frac{e^{-\lambda n}}{n!}.
\tag{62}
\]

We have sketched \( \eta^{(2)}_{\text{FH}} \) and \( \eta^{(2)}_{\text{FD}} \) in terms of \( \lambda \) in fig. 1 and fig. 2 for the cases \( u = 7 \) and \( u = 20 \).
respectively. It is noticeable that $\eta_{FH}^{(2)}$ scales linearly with $u$. However, $\eta_{FD}^{(2)}$ is always less than $\frac{1}{2}$ no matter how large $u$ is. Thus, as $u$ increases, the advantage of FH over FD becomes more apparent. □

Fig. 1. Curves of $\eta_{FH}^{(2)}$ and $\eta_{FD}^{(2)}$ in terms of $\lambda$ in a network with $u = 7$ sub-bands.

Fig. 2. Curves of $\eta_{FH}^{(2)}$ and $\eta_{FD}^{(2)}$ in terms of $\lambda$ in a network with $u = 20$ sub-bands.
• Minimum nonzero multiplexing gain per user

The minimum nonzero multiplexing gain per user is the smallest nonzero multiplexing gain that a user in the network attains for different realizations in terms of the number of active users. Assuming \( n_{\text{max}} < \infty \), this happens when there are exactly \( n_{\text{max}} \) active users in the system. As the FD system is already designed to handle the case where \( n_{\text{max}} \) users are present in the network, the minimum multiplexing gain per user is automatically higher in FD as compared to FH. Setting \( n_{\text{des}} = n_{\text{max}} \), we have

\[
\eta_{\text{FD}}^{(3)} = \frac{\text{SMG}_{\text{FD}}(u,n_{\text{max}})}{n_{\text{max}}} = \frac{u}{2n_{\text{max}}}. \tag{56}
\]

In the case of FH, we assume that all users select \( v = \frac{u}{n_{\text{max}}} \). Hence,

\[
\eta_{\text{FH}}^{(3)} = \frac{\text{SMG}_{\text{FH}}(\frac{u}{n_{\text{max}}},n_{\text{max}})}{n_{\text{max}}} = \frac{u}{2n_{\text{max}}} \left( 1 - \frac{1}{n_{\text{max}}} \right)^{n_{\text{max}}-1}. \tag{57}
\]

Clearly, \( \frac{1}{e} \leq \frac{\eta_{\text{FH}}^{(3)}}{\eta_{\text{FD}}^{(3)}} \leq 1 \) as \( \left( 1 - \frac{1}{n_{\text{max}}} \right)^{n_{\text{max}}-1} \) approaches \( \frac{1}{e} \) from above by increasing \( n_{\text{max}} \). Therefore, the loss incurred in terms of \( \eta^{(3)} \) for the FH system is always less than \( \frac{1}{e} \).

• Service capability

Service capability demonstrates the fraction of users receiving service among the whole active users in the network. As mentioned earlier, a user is said to receive service whenever the achieved multiplexing gain of the user is nonzero. In the FD scenario, if \( N > u \), then a fraction of users cannot share the spectrum. However, in case \( \Pr\{N \leq u\} = 1 \), the FD scheme achieves the full service capability. As for the FH scheme, we already know that as far as all users hop over proper subsets of the sub-bands, every user achieves a nonzero multiplexing gain. The following examples offer comparisons between FD and FH in terms of the service capability.

Example 5- In this example, we consider a setup where \( n_{\text{max}} < \infty \). The central controller in FD simply sets \( n_{\text{des}} = n_{\text{max}} \) and the service capability is always equal to 1. The number of served users \( N_{\text{serv,FH}} \) in the FH scenario can be written as

\[
N_{\text{serv,FH}} = \begin{cases} 
N & N = 1 \text{ or } (N > 1 \text{ and } v \neq u) \\
0 & \text{oth.}
\end{cases} \tag{63}
\]

Therefore, as far as \( v \neq u \), we have \( N_{\text{serv,FH}} = N \) and the service capability is one. This shows that to achieve the maximum service capability in a system where \( n_{\text{max}} > 1 \), the hopping parameter \( v \) must be strictly less than \( u \). □

Example 6- In this example, we provide a case where \( n_{\text{max}} \) is not finite. Let us assume the distribution of the number of active users in the network is a Poisson distribution with parameter \( \lambda \), i.e.,

\[
q_n = \frac{\lambda^n e^{-\lambda}}{n!}
\]
for \( n \geq 0 \) where \( \lambda > 1 \). Let us compute \( v^* \) for the FH scenario. We have,

\[
E\{\text{SMG}_{\text{FH}}(v, N)\} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \left( n v \left( 1 - \frac{v}{u} \right)^{n-1} \right)
\]

\[
= \frac{1}{2} v \sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{(n-1)!} \left( 1 - \frac{v}{u} \right)^{n-1}
\]

\[
= \frac{1}{2} \lambda v \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \left( 1 - \frac{v}{u} \right)^n
\]

\[
= \frac{1}{2} \lambda v e^{-\frac{\lambda v}{u}}. \quad (64)
\]

Thus,

\[
v^* = \arg \max_v E\{\text{SMG}_{\text{FH}}(v, N)\} = \frac{u}{\lambda}. \quad (66)
\]

Since \( \lambda \neq 1 \), we get \( v^* \neq u \). Thus, choosing \( v = v^* \) maximizes \( E\left\{ \frac{N_{\text{serv}}}{N} \right\} \) and \( E\{\text{SMG}_{\text{FH}}(v, N)\} \) simultaneously, i.e., \( \eta_{\text{FH}}^{(4)} = 1 \).

In the FD system, \( N_{\text{serv,FD}} \) is given by

\[
N_{\text{serv,FD}} = \begin{cases} 
N & N \leq n_{\text{des}} \\
n_{\text{des}} & N > n_{\text{des}}
\end{cases} \quad (67)
\]

Thus,

\[
E\left\{ \frac{N_{\text{serv,FD}}}{N} \right\} = \Pr\{N \leq n_{\text{des}}\} + n_{\text{des}} \sum_{n=n_{\text{des}}+1}^{\infty} \frac{q_n}{n} 
\]

\[
= 1 - \Pr\{N \geq n_{\text{des}} + 1\} + n_{\text{des}} \sum_{n=n_{\text{des}}+1}^{\infty} \frac{q_n}{n}
\]

\[
= 1 - \sum_{n=n_{\text{des}}+1}^{\infty} q_n \left( 1 - \frac{n_{\text{des}}}{n} \right). \quad (68)
\]

By this expression, it is clear that to maximize \( E\left\{ \frac{N_{\text{serv,FD}}}{N} \right\} \), one must select \( n_{\text{des}} \) as large as possible. This basically justifies the assumption we made about selecting \( n_{\text{des}} = u \) in the FD scheme in the case where \( n_{\text{max}} \) is not finite. Thus,

\[
\eta_{\text{FD}}^{(4)} = 1 - \sum_{n=u+1}^{\infty} q_n \left( 1 - \frac{u}{n} \right) \quad (69)
\]

For instance, in the case of \( u = 5 \) and \( \lambda = 3 \), we have \( \eta_{\text{FD}}^{(4)} = 0.9806 \). \( \square \)
VI. ADAPTIVE FREQUENCY HOPPING

The results of the previous section are obtained based on the assumption that the hopping parameter \( v \) is fixed and is not adaptively changed based on the number of active users. The performance of the FH system can be improved by letting the transmitters adapt their hopping parameter based on the number of active users using (35). We refer to this scenario as Adaptive Frequency Hopping (AFH). In the following example, we study the performance improvement offered by AFH over FH in terms of \( \eta^{(1)} \) and \( \eta^{(2)} \).

**Example 7** - Let us assume that the number of active users is a Poisson random variable with parameter \( \lambda > 1 \). We already have

\[
\eta^{(1)}_{FH} = \frac{\sqrt{u}}{2e},
\]

while by (36),

\[
\eta^{(1)}_{AFH} = \frac{u}{2} \sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \left( 1 - \frac{1}{n} \right)^{n-1}.
\]

Figure 3 shows the plots of \( \eta^{(1)}_{FH} \) and \( \eta^{(1)}_{AFH} \) versus \( \lambda \) for \( u = 10 \). It is observed that \( \eta^{(1)}_{FH} \) does not change with \( \lambda \), while \( \eta^{(1)}_{AFH} \) decreases by increasing \( \lambda \). This indicates that in a crowded network (large \( \lambda \)), AFH does not provide any significant advantage over FH in terms of \( \eta^{(1)} \).

![Graph showing \( \eta_{AFH}^{(1)} \) versus \( \eta_{FH}^{(1)} \) for \( u = 10 \).](image)
We have already calculated $\eta_{FH}^{(2)}$ in example 4 in a system where $3 \leq \lambda \leq 10$. However, in case of AFH,

$$\eta_{AFH}^{(2)} = \frac{u}{2} \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \frac{1}{n} \left(1 - \frac{1}{n}\right)^{n-1}. \quad (72)$$

Figure 4 presents the plots of $\eta_{FH}^{(2)}$ and $\eta_{AFH}^{(2)}$ versus $\lambda$ for $u = 10$. Both $\eta_{FH}^{(2)}$ and $\eta_{AFH}^{(2)}$ decrease by increasing $\lambda$. However, the ratio $\frac{\eta_{AFH}^{(2)}}{\eta_{FH}^{(2)}}$ decreases as $\lambda$ increases. This indicates that for large values of $\lambda$, AFH does also not provide any significant advantage over FH in terms of $\eta^{(2)}$. □

**Fig. 4.** $\eta_{AFH}^{(2)}$ versus $\eta_{FH}^{(2)}$ for $u = 10$.

**VII. CONCLUSION**

We have addressed a decentralized wireless communication network with a fixed number $u$ of frequency sub-bands to be shared among $N$ transmitter-receiver pairs. It is assumed that the number of active users is a random variable with a given distribution. Moreover, users are assumed to be unaware of each other’s codebooks and hence, no multiuser detection is possible. We proposed a randomized Frequency Hopping (FH) scheme in which each transmitter randomly hops over subsets of the $u$ sub-bands from transmission to transmission. Assuming all users transmit Gaussian signals, the distribution of noise plus interference is mixed Gaussian, which makes the calculation of the mutual information between the input and output of each user intractable. We derived lower and upper bounds on this mutual information and demonstrated
that for large SNR values, the two bounds coincide. This observation enabled us to compute the sum multiplexing gain of the system and obtain the optimum hopping strategy for maximizing this value. We compared the performance of the FH with that of the FD in terms of the following performance measures: average sum multiplexing gain \( \eta(1) \), average minimum multiplexing gain per user \( \eta(2) \), minimum nonzero multiplexing gain per user \( \eta(3) \) and service capability \( \eta(4) \). We showed that (depending on the probability mass function of the number of active users) the FH system can offer a significant improvement in terms of \( \eta(1) \) and \( \eta(2) \) (implying a more efficient usage of the spectrum). It was also shown that 
\[
\frac{1}{e} \leq \frac{\eta(3)}{\eta(3)_F} \leq 1,
\]
i.e., the loss incurred in terms of \( \eta(3) \) is not more than \( \frac{1}{e} \). Moreover, computation of the so-called service capability showed that in the FH system any number of users can coexist fairly, while the maximum number of users in the FD system is limited by the number of sub-bands.

**APPENDIX A: PROOF OF LEMMAS 2**

Let us consider a general \( t \times 1 \) vector mixed Gaussian distribution \( p_{\vec{\Theta}}(\vec{\theta}) \) with different covariance matrices \( \{C_l\}_{l=1}^L \) and associated probabilities \( \{p_l\}_{l=1}^L \) given by
\[
p_{\vec{\Theta}}(\vec{\theta}) = \sum_{l=1}^L p_l g_l(\vec{\theta}, C_l),
\]
where 
\[
g_l(\vec{\theta}, C_l) = \frac{1}{(2\pi)^{\frac{t}{2}}(\det C_l)^{\frac{t}{2}}} \exp \left\{ -\frac{1}{2} \vec{\theta}^T C_l^{-1} \vec{\theta} \right\}.
\]
Hence,
\[
\int p_{\vec{\Theta}}(\vec{\theta}) \log p_{\vec{\Theta}}(\vec{\theta}) d\vec{\theta} = \sum_{l=1}^L J_l
\]
where 
\[
J_l \triangleq p_l \int g_l(\vec{\theta}, C_l) \log p_{\vec{\Theta}}(\vec{\theta}) d\vec{\theta}
\]
for \( 1 \leq l \leq L \). To find a lower bound on \( J_l \), we observe that
\[
J_l = p_l \int g_l(\vec{\theta}, C_l) \log \left( \sum_{m=1}^L p_m g_m(\vec{\theta}, C_m) \right) d\vec{\theta}
\]
\[
\geq p_l \int g_l(\vec{\theta}, C_l) \log (p_l g_l(\vec{\theta}, C_l)) d\vec{\theta}
\]
\[
= (p_l \log p_l) \int g_l(\vec{\theta}, C_l) d\vec{\theta} + p_l \int g_l(\vec{\theta}, C_l) \log g_l(\vec{\theta}, C_l) d\vec{\theta}
\]
\[
= p_l \log p_l + p_l \int g_l(\vec{\theta}, C_l) \log g_l(\vec{\theta}, C_l) d\vec{\theta}
\]
\[
(75)
\]
Using this together with (74) yields

\[
\begin{align*}
    h(\vec{\Theta}) &= -\int p_{\vec{\Theta}}(\vec{\theta}) \log p_{\vec{\Theta}}(\vec{\theta}) d\vec{\theta} \\
    &= - \sum_{l=1}^{L} J_l \\
    &\leq -p_l \log p_l - p_l \int g_l(\vec{\theta}, C_l) \log g_l(\vec{\theta}, C_l) d\vec{\theta} \\
    \quad \overset{(a)}{=} - \sum_{l=1}^{L} p_l \log p_l + \frac{1}{2} \sum_{l=1}^{L} p_l \log ((2\pi e)^{t} \det C_l)
\end{align*}
\]

where in (a), we have used the fact that the differential entropy of a \(t \times 1\) Gaussian vector with covariance matrix \(C_l\) is \(\frac{1}{2} \log ((2\pi e)^{t} \det C_l)\).

Let \(t = 1\) and \(\vec{\Theta} = Z_{i,j}\). Therefore,

\[
\begin{align*}
    h(Z_{i,j}) &\leq \frac{1}{2} \sum_{l=0}^{L_i} a_{i,l} \log(2\pi e \sigma_{i,l}^2) - \sum_{l=0}^{L_i} a_{i,l} \log a_{i,l} \\
               &\quad = \frac{1}{2} \sum_{l=0}^{L_i} a_{i,l} \log(2\pi e \sigma_{i,0}^2) + \frac{1}{2} \sum_{l=0}^{L_i} a_{i,l} \log \frac{\sigma_{i,l}^2}{\sigma_{i,0}^2} - \sum_{l=0}^{L_i} a_{i,l} \log a_{i,l} \\
               &\quad = \log(\sqrt{2\pi e} \sigma_{i,0}) + \frac{1}{2} \sum_{l=0}^{L_i} a_{i,l} \log \frac{\sigma_{i,l}^2}{\sigma_{i,0}^2} - \sum_{l=0}^{L_i} a_{i,l} \log a_{i,l}
\end{align*}
\]

However, for all \(l \geq 1\), we have \(\sigma_{i,l}^2 \leq \frac{\sigma_{i,L_i}^2}{\sigma_{i,0}^2} = 1 + c_{i,L_i} \gamma\). Thus,

\[
\begin{align*}
    h(Z_{i,j}) &\leq \frac{1}{2} \sum_{l=1}^{L_i} a_{i,l} \log(1 + c_{i,L_i} \gamma) + \log(\sqrt{2\pi e} \sigma_{i,0}) - \sum_{l=0}^{L_i} a_{i,l} \log a_{i,l} \\
               &\quad = \frac{1}{2} (1 - a_{i,0}) \log(1 + c_{i,L_i} \gamma) + \log(\sqrt{2\pi e} \sigma_{i,0}) - \sum_{l=0}^{L_i} a_{i,l} \log a_{i,l}
\end{align*}
\]

This concludes the proof of Lemma 2.

**APPENDIX B; PROOF OF PROPOSITION 1**

We have \(\eta_{FD}(1) = \frac{E(N)_{u}}{2n_{max}}\) and \(\eta_{FH}(1) = \frac{1}{2} \max_{v} \left\{ v E \left\{ N \left(1 - \frac{v}{u}\right)^{N-1} \right\} \right\} \). Let us define \(\Omega(v, N) = N \omega_v^{N-1}\) where \(\omega_v = 1 - \frac{v}{u}\). Thinking of \(N\) as a real parameter for the moment, we have \(\frac{\partial^2}{\partial N^2} \Omega(v, N) = \omega_v^{N-1} \left(N (\ln \omega_v)^2 + 2 \ln \omega_v\right)\). As \(N \geq 1\), we have \(\frac{\partial^2}{\partial N^2} \Omega(v, N) \geq \omega_v^{N-1} \left((\ln \omega_v)^2 + 2 \ln \omega_v\right)\). But, \((\ln \omega_v)^2 + 2 \ln \omega_v \geq 0\) if and only if \(\omega_v \leq \frac{1}{e^2}\) or \(\omega_v \geq 1\). Since \(\omega_v \leq 1\), we get \(\omega_v \leq \frac{1}{e^2}\). This implies that the function
\( \Omega(v, N) \) is a convex function of \( N \) as far as \( \omega_v \leq \frac{1}{e^2} \). Therefore, by Jensen’s inequality,

\[
E \left\{ N \left( 1 - \frac{v}{u} \right)^{N-1} \right\} = E\{\Omega(v, N)\} \geq \Omega(v, E\{N\}) = E\{N\} \left( 1 - \frac{v}{u} \right)^{E\{N\}-1}
\]  \hspace{1cm} (79)

which is valid as far as \( v \geq (1 - \frac{1}{e^2}) u \). Hence,

\[
\eta_{FH}^{(1)} = \frac{1}{2} \max_v \left\{ v E \left\{ N \left( 1 - \frac{v}{u} \right)^{N-1} \right\} \right\} \\
\geq \frac{1}{2} \max_{v \in \left( (1 - \frac{1}{2}) u, u \right]} \left\{ v E \left\{ N \left( 1 - \frac{v}{u} \right)^{N-1} \right\} \right\} \\
\geq \frac{1}{2} E\{N\} \max_{v \in \left( (1 - \frac{1}{2}) u, u \right]} \left\{ v \left( 1 - \frac{v}{u} \right)^{E\{N\}-1} \right\}.
\]  \hspace{1cm} (80)

The function \( v \left( 1 - \frac{v}{u} \right)^{E\{N\}-1} \) is a concave function in terms of \( v \) that achieves its absolute maximum at \( \frac{u}{E\{N\}} \). Therefore,

\[
\max_{v \in \left( (1 - \frac{1}{2}) u, u \right]} \left\{ v \left( 1 - \frac{v}{u} \right)^{E\{N\}-1} \right\} = \max \left\{ 1 - \frac{1}{e^2}, \frac{1}{E\{N\}} \right\} \left( 1 - \max \left\{ 1 - \frac{1}{e^2}, \frac{1}{E\{N\}} \right\} \right)^{E\{N\}-1} u.
\]  \hspace{1cm} (81)

Using (80) and (81),

\[
\eta_{FH}^{(1)} \geq \frac{1}{2} \max \left\{ 1 - \frac{1}{e^2}, \frac{1}{E\{N\}} \right\} \left( 1 - \max \left\{ 1 - \frac{1}{e^2}, \frac{1}{E\{N\}} \right\} \right)^{E\{N\}-1} E\{N\} u.
\]  \hspace{1cm} (82)

Hence, a sufficient condition for \( \eta_{FH}^{(1)} > \eta_{FD}^{(1)} \) to hold is that

\[
\max \left\{ 1 - \frac{1}{e^2}, \frac{1}{E\{N\}} \right\} \left( 1 - \max \left\{ 1 - \frac{1}{e^2}, \frac{1}{E\{N\}} \right\} \right)^{E\{N\}-1} > \frac{1}{n_{\max}}.
\]  \hspace{1cm} (83)

If \( E\{N\} \geq \frac{e^2}{e^2 - 1} \), we have \( \max \left\{ 1 - \frac{1}{e^2}, \frac{1}{E\{N\}} \right\} = 1 - \frac{1}{e^2} \). Hence, (83) reduces to the inequality \( E\{N\} < \frac{1}{2} \ln (\left( e^2 - 1 \right) n_{\max}) \). Therefore, if \( \frac{e^2}{e^2 - 1} \leq E\{N\} < \frac{1}{2} \ln (\left( e^2 - 1 \right) n_{\max}) \), then (83) is satisfied. On the other hand, if \( E\{N\} \leq \frac{e^2}{e^2 - 1} = 1.1565 \), we get \( \max \left\{ 1 - \frac{1}{e^2}, \frac{1}{E\{N\}} \right\} = \frac{1}{E\{N\}} \). Thus, (83) reduces to the inequality \( \frac{1}{E\{N\}} \left( 1 - \frac{1}{E\{N\}} \right)^{E\{N\}-1} > \frac{1}{n_{\max}} \). For each \( n_{\max} \geq 2 \), this yields an upper bound on \( E\{N\} \).

Since \( \frac{1}{E\{N\}} \left( 1 - \frac{1}{E\{N\}} \right)^{E\{N\}-1} \) is a decreasing function of \( E\{N\} \), the smallest of these upper bounds is obtained for \( n_{\max} = 2 \) and is equal to 1.2938. This means that for \( E\{N\} \leq 1.1565 \), (83) is automatically satisfied. Thus, (83) is equivalent to

\[
E\{N\} < \frac{1}{2} \ln \left( (e^2 - 1) n_{\max} \right).
\]  \hspace{1cm} (84)
APPENDIX C: PROOF OF PROPOSITION 2

We have $\eta_{FD}^{(2)} = \frac{u}{2n_{\text{max}}}$ and $\eta_{FH}^{(2)} = \frac{1}{2} \max_v \left\{ v \mathbb{E} \left\{ (1 - \frac{v}{u})^{N-1} \right\} \right\}$. The function $(1 - \frac{v}{u})^{N-1}$ is convex in terms of $N$. Using Jenson’s inequality,

$$\eta_{FH}^{(2)} \geq \frac{1}{2} \max_v \left\{ v \left( 1 - \frac{v}{u} \right)^{E\{N\}-1} \right\} = \frac{u}{2E\{N\}} \left( 1 - \frac{1}{E\{N\}} \right)^{E\{N\}-1}.$$  \hspace{1cm} (85)

Hence, a sufficient condition for $\eta_{FH}^{(2)} > \eta_{FD}^{(2)}$ to hold is

$$\frac{1}{E\{N\}} \left( 1 - \frac{1}{E\{N\}} \right)^{E\{N\}-1} > \frac{1}{n_{\text{max}}}.$$  \hspace{1cm} (86)

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