SPECTRAL METHODS FOR TWO-DIMENSIONAL SPACE AND
TIME FRACTIONAL BLOCH-TORREY EQUATIONS

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Abstract. In this paper, we consider the numerical approximation of the space and time fractional Bloch-Torrey equations. A fully discrete spectral scheme based on a finite difference method in the time direction and a Galerkin-Legendre spectral method in the space direction is developed. In order to reduce the amount of computation, an alternating direction implicit (ADI) spectral scheme is proposed. Then the stability and convergence analysis are rigorously established. Finally, numerical results are presented to support our theoretical analysis.

1. Introduction. Although fractional calculus has a history almost as long as the integer calculus, only in the past decades does the fractional calculus undergo a rapid development. Recently, fractional differential equations can be used to describe some phenomena or process with hereditary and memory in physics and chemical processes, materials, control theory, biology, finance, and so on (see [7, 11, 13]). In physics, fractional derivatives are used to model anomalous diffusion, where diffusing particles are not locally homogeneous and the disorder is not well-approximated by assuming a unified change in diffusion constant (see [6, 10]).

In this paper, we consider the following two-dimensional space and time fractional Bloch-Torrey equations ([8, 9]):

\[ \frac{C}{0} D_t^\alpha u = K_x \frac{\partial^{2\beta} u}{\partial |x|^{2\beta}} + K_y \frac{\partial^{2\beta} u}{\partial |y|^{2\beta}} + f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \quad T > 0, \quad (1) \]

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with the initial and boundary conditions given by

\[
    u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega, \\
    u(x, y, t)|_{(x, y) \in \partial \Omega} = 0, \quad t \in (0, T],
\]

where \( 0 < \alpha < 1, \ 1/2 < \beta < 1, \ K_x, K_y > 0, \ \Omega = (a, b) \times (c, d). \) The functions \( f(x, y, t) \) and \( u_0(x, y) \) are known smooth functions. The Caputo fractional derivative

\[
    \frac{\partial^{\alpha} u}{\partial x^{\alpha}} u(x, y, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{d}{dt} u(x, y, t) dt,
\]

and \( \frac{\partial^{\beta}}{\partial |x|^{\beta}} \) and \( \frac{\partial^{\beta}}{\partial |y|^{\beta}} \) are the Riesz fractional derivatives are defined by

\[
    \frac{\partial^{\beta}}{\partial |x|^{\beta}} u = -c_1 \left( RL D_{a,x}^{\beta} u + RL D_{x,b}^{\beta} u \right) \quad \text{and} \quad \frac{\partial^{\beta}}{\partial |y|^{\beta}} u = -c_1 \left( RL D_{a,y}^{\beta} u + RL D_{y,c}^{\beta} u \right),
\]

where \( c_1 = \frac{1}{2 \cos(\beta \pi)} \). For \( n - 1 < \beta < n \), \( n \in \mathbb{N} \), the operators \( RL D_{a,x}^{\beta}, RL D_{x,b}^{\beta} \), \( RL D_{a,y}^{\beta} \) and \( RL D_{y,c}^{\beta} \) are defined as

\[
    RL D_{a,x}^{\beta} u = \frac{\partial^n}{\partial x^n} \left[ D_{a}^{-\beta} u \right] = \frac{1}{\Gamma(n - \beta)} \frac{\partial^n}{\partial x^n} \int_a^x (x-s)^{n-\beta-1} u(s, y, t) ds,
\]

\[
    RL D_{x,b}^{\beta} u = (-1)^n \frac{\partial^n}{\partial x^n} \left[ D_{b}^{-\beta} u \right] = \frac{(-1)^n}{\Gamma(n - \beta)} \frac{\partial^n}{\partial x^n} \int_x^b (s-x)^{n-\beta-1} u(s, y, t) ds,
\]

\[
    RL D_{a,y}^{\beta} u = \frac{\partial^n}{\partial y^n} \left[ D_{y}^{-\beta} u \right] = \frac{1}{\Gamma(n - \beta)} \frac{\partial^n}{\partial y^n} \int_c^y (y-s)^{n-\beta-1} u(x, s, t) ds,
\]

\[
    RL D_{y,c}^{\beta} u = (-1)^n \frac{\partial^n}{\partial y^n} \left[ D_{c}^{-\beta} u \right] = \frac{(-1)^n}{\Gamma(n - \beta)} \frac{\partial^n}{\partial y^n} \int_y^d (s-y)^{n-\beta-1} u(x, s, t) ds,
\]

where \( D_{a,x}^{-\nu} \) and \( D_{x,b}^{-\nu} \) are the left and right Riemann-Liouville integral operators defined by

\[
    D_{a,x}^{-\nu} u = RL D_{a,x}^{-\nu} u = \frac{1}{\Gamma(\nu)} \int_a^x (x-s)^{\nu-1} u(s, y, t) ds, \quad \nu > 0,
\]

\[
    D_{x,b}^{-\nu} u = RL D_{x,b}^{-\nu} u = \frac{1}{\Gamma(\nu)} \int_x^b (s-x)^{\nu-1} u(s, y, t) ds, \quad \nu > 0.
\]

Similarly, the left and right Riemann-Liouville integral operators \( D_{a,y}^{-\nu} \) and \( D_{y,c}^{-\nu} \) can be defined.

This space and time fractional Bloch-Torrey equations are a class of new fractional diffusion models. This model is more appropriate than the classical diffusion model to be applied to analysis the diffusion images of human brain tissues and provide new insights for further investigations of issue structures.

For the space and time fractional Bloch-Torrey equations, Yu et al. [4, 17] established the fundamental solutions and exact solutions. Yu et al. [18, 19] proposed a finite difference method for 3D and 2D space and time fractional Bloch-Torrey equations. Bu et al. [3, 20] proposed a finite element method for two-dimensional space and time fractional Bloch-Torrey equations and analysed its stability and convergence. In this paper, we aim to solve the problem (1)–(3) using a fully discrete numerical method. The method is based on a finite difference scheme in time and spectral approximation using Legendre functions in space. We give a detailed
analysis for the stability and convergence of the fully discrete scheme. The scheme is unconditionally stable and convergent with order \(O(\tau^2 + N^{3-r})\).

The remainder of the paper is organized as follows. In Section 2, some preliminaries and notations are shown. In Section 3, we propose a fully discrete scheme for the problem (1)-(3). In section 4, the stability and convergence of the fully discrete scheme are given. In section 5, we propose an ADI spectral method, and analyse its stability and convergence. We do some numerical experiments in Section 6. Finally, some conclusions are given in Section 7.

2. Preliminaries and notations. In this section, we show some notation and lemmas that are need in the following sections.

Let \(\Omega\) be a finite domain satisfying \(\Omega = I_x \times I_y = (a, b) \times (c, d)\), and denote by \((\cdot, \cdot)\) the inner product on the space \(L^2(\Omega)\) with the \(L^2\)-norm \(\| \cdot \|_{L^2(\Omega)}\) and the maximum norm \(\| \cdot \|_{L^\infty(\Omega)}\). If it does not cause confusion, we also define \((\cdot, \cdot)\) as the inner product on the interval \(I_x\) or \(I_y\). Let \(\mu\) be a nonnegative real number, and denote by \(H^\mu(\Omega)\) and \(H^\mu_0(\Omega)\) the usual Sobolev spaces with the norm \(\| \cdot \|_{H^\mu(\Omega)}\) and the seminorm \(\cdot \|_{H^\mu(\Omega)}\).

Firstly, we introduce some spaces that are used in the formulation of the numerical algorithms. We first introduce the space \(J^\mu_L(\Omega)\), \(J^\mu_R(\Omega)\) and \(J^\mu_S(\Omega)\) in \(\mathbb{R}^2\) (see [5]).

**Definition 2.1.** Let \(\mu > 0\). Define the seminorm
\[
|u|_{J^\mu_L(\Omega)} = \left( \| RL D^\mu_{a,x} u(x, y) \|^2_{L^2(\Omega)} + \| RL D^\mu_{c,y} u(x, y) \|^2_{L^2(\Omega)} \right)^{1/2}
\]
and the norm
\[
\| u \|_{J^\mu_L(\Omega)} = \left( \| u \|^2_{L^2(\Omega)} + |u|_{J^\mu_L(\Omega)} \right)^{1/2},
\]
and denote \(J^\mu_L(\Omega)\) (or \(J^\mu_L(\Omega)\)) as the closure of \(C^\infty(\Omega)\) (or \(C^\infty_0(\Omega)\)) with respect to \(\| \cdot \|_{J^\mu_L(\Omega)}\), where \(C^\infty_0(\Omega)\) is the space of smooth functions with compact support in \(\Omega\).

**Definition 2.2.** Let \(\mu > 0\). Define the seminorm
\[
|u|_{J^\mu_R(\Omega)} = \left( \| RL D^\mu_{a,b} u(x, y) \|^2_{L^2(\Omega)} + \| RL D^\mu_{b,y} u(x, y) \|^2_{L^2(\Omega)} \right)^{1/2}
\]
and the norm
\[
\| u \|_{J^\mu_R(\Omega)} = \left( \| u \|^2_{L^2(\Omega)} + |u|_{J^\mu_R(\Omega)} \right)^{1/2},
\]
and denote \(J^\mu_R(\Omega)\) (or \(J^\mu_R(\Omega)\)) as the closure of \(C^\infty(\Omega)\) (or \(C^\infty_0(\Omega)\)) with respect to \(\| \cdot \|_{J^\mu_R(\Omega)}\).

**Definition 2.3.** Let \(\mu \neq n - 1/2, n \in \mathbb{N}\). Define the seminorm
\[
|u|_{J^\mu_S(\Omega)} = \left( \| RL D^\mu_{a,x} u(x, y) \cdot RL D^\mu_{c,y} u(x, y) \| + |RL D^\mu_{a,y} u(x, y) \cdot RL D^\mu_{b,y} u(x, y) | \right)^{1/2}
\]
and the norm
\[
\| u \|_{J^\mu_S(\Omega)} = \left( \| u \|^2_{L^2(\Omega)} + |u|_{J^\mu_S(\Omega)} \right)^{1/2},
\]
and denote \(J^\mu_S(\Omega)\) (or \(J^\mu_S(\Omega)\)) as the closure of \(C^\infty(\Omega)\) (or \(C^\infty_0(\Omega)\)) with respect to \(\| \cdot \|_{J^\mu_S(\Omega)}\).

The fractional Sobolev space \(H^\mu(\Omega)\) can be defined by the Fourier transform approach. (see [12, 14]).
Lemma 2.8. If equivalent norms and seminorms. If and the norm 
and the constant $H$. In the following, $(\Omega, C_{\mu})$, $C_{\mu}(\mu, x)$ is the extension of $u(x,y)$, $\mathcal{u}$, $\mathcal{u}_{\mu}$, $\mathcal{u}_{\mu}$, $\mathcal{u}_{\mu}$, $\mathcal{u}_{\mu}$, $\mathcal{u}_{\mu}$. Furthermore, if $\mu \neq n - 1/2$, $n \in \mathbb{N}$, and $u \in J_{L,0}^\mu(\Omega)$, then there exists a positive constant $C$ independent of $u$ such that

$$|\hat{u}|_{F_\mu^\mu(\mathbb{R}^2)} \leq C|u|_{J_{L,0}^\mu(\Omega)}.$$ 

Similarly to Lemma 3.1.4 in [14], we can obtain the following lemma.

Lemma 2.6. Let $\mu_1, \mu_2 > 0$, $\Omega = (a,b) \times (c,d)$, $u \in J_{L,0}^{\max(\mu_1, \mu_2)}(\Omega) \cap J_{R,0}^{\max(\mu_1, \mu_2)}(\Omega)$. One has

\[
(RL\mathcal{D}_{a,x}^{\mu} u, RL\mathcal{D}_{x,b}^{\mu} u) = \cos(\mu \pi) \parallel RL\mathcal{D}_{a,x}^{\mu} u \parallel^2 + \parallel RL\mathcal{D}_{x,b}^{\mu} u \parallel^2,
\]

\[
(RL\mathcal{D}_{c,y}^{\mu} u, RL\mathcal{D}_{y,d}^{\mu} u) = \cos(\mu \pi) \parallel RL\mathcal{D}_{c,y}^{\mu} u \parallel^2 + \parallel RL\mathcal{D}_{y,d}^{\mu} u \parallel^2,
\]

where $\hat{u}$ is the extension of $u$ by zero outside $\Omega$. For simplicity, we denote $\parallel \cdot \parallel_0 = \parallel \cdot \parallel_{L^2(\Omega)}$ and $\parallel \cdot \parallel_\infty = \parallel \cdot \parallel_{L^\infty(\Omega)}$. We define a new seminorm $|\cdot|_\beta$ and norm $\parallel \cdot \parallel_\beta$ as

$$|u|_\beta = \left( |KL\mathcal{D}^{\beta}_{a,x} u|^2 + |KL\mathcal{D}^{\beta}_{x,b} u|^2 + |KL\mathcal{D}^{\beta}_{c,y} u|^2 + |KL\mathcal{D}^{\beta}_{y,d} u|^2 \right)^{\frac{1}{2}},$$

$$\parallel u \parallel_\beta = \left( |u|^2 + \parallel u \parallel^2 \right)^{\frac{1}{2}}.$$

The following lemmas from [5, 14] are needed in this paper.

Lemma 2.7. Let $\Omega = (a,b) \times (c,d)$, $\mu \neq n - 1/2$, $n \in \mathbb{N}$, and $u \in J_{L,0}^\mu(\Omega) \cap J_{R,0}^\mu(\Omega) \cap H_0^0(\Omega)$. Then there exist positive constants $C_1$ and $C_2$ independent of $u$ such that

$$C_1|u|_{H^\mu(\Omega)} \leq \max\{ |u|_{J_{L,0}^\mu(\Omega)}, |u|_{J_{R,0}^\mu(\Omega)} \} \leq C_2|u|_{H^\mu(\Omega)}.$$ 

Lemma 2.8. If $\mu > 0$, then $J_{L,0}^\mu(\mathbb{R}^2)$, $J_{R,0}^\mu(\mathbb{R}^2)$ and $H^\mu(\mathbb{R}^2)$ are equivalence with equivalent norms and seminorms. If $\mu > 0$, $\mu \neq n - 1/2$, $n \in \mathbb{N}$, then $J_{L,0}^\mu(\mathbb{R}^2)$, $J_{R,0}^\mu(\mathbb{R}^2)$ and $H^\mu(\mathbb{R}^2)$ are equivalence with equivalent norms and seminorms.

Lemma 2.9. If $u \in H_0^\mu(\Omega)$ and $0 < \gamma < \mu$, we have

$$\parallel u \parallel \leq C_1\parallel RL\mathcal{D}^{\gamma \mu}_{a,x} u \parallel \leq C_2\parallel RL\mathcal{D}^{\mu}_{a,x} u \parallel \text{ and } \parallel u \parallel \leq C_3\parallel RL\mathcal{D}^{\gamma \mu}_{c,y} u \parallel \leq C_4\parallel RL\mathcal{D}^{\mu}_{c,y} u \parallel,$$

where $C_1, C_2, C_3$ and $C_4$ are positive constants independent of $u$.

In the following, $C$ or $C_j$ denotes a generic positive constant independent $\tau, N$, and the constant $C$ will not be the same.
Lemma 2.10. Let $1 < \beta < 2$. Then for any $u \in H_0^\beta (\Omega)$ and $u \in H_0^{\beta/2} (\Omega)$, we have 
\[
 (RLD_{a,x}^\beta u, v) = (RLD_{a,x}^{\beta/2} u, RL D_{x,b}^{\beta/2} v) \quad \text{and} \quad (RLD_{x,b}^\beta u, v) = (RLD_{x,b}^{\beta/2} u, RL D_{a,x}^{\beta/2} v).
\]

Then, we introduce some function spaces. As in [15], the function spaces $V_N^{x,0}$ and $V_N^{y,0}$ can be expressed as
\[
 V_N^{x,0} = \text{span}\{ \phi_l(x) : l = 0, 1, \cdots, N-2 \}, \quad V_N^{y,0} = \text{span}\{ \varphi_l(y) : l = 0, 1, \cdots, N-2 \},
\]
in which $\phi_l(x)$ and $\varphi_l(y)$ are defined:
\[
\phi_l(x) = L_l(\hat{x}) - L_{l+2}(\hat{x}), \quad \hat{x} \in [-1, 1], \quad x = \frac{(b-a)\hat{x} + a + b}{2} \in [a, b],
\]
\[
\varphi_l(y) = L_l(\hat{y}) - L_{l+2}(\hat{y}), \quad \hat{y} \in [-1, 1], \quad y = \frac{(d-c)\hat{y} + c + d}{2} \in [c, d],
\]
where $L_l(\hat{z})(\hat{z} \in [-1, 1], l \in \mathbb{Z}^+)$ is the Legendre polynomial defined by the following recurrence relation in [16]:
\[
L_0(\hat{z}) = 1, \quad L_1(\hat{z}) = \hat{z}, \quad L_{l+1}(\hat{z}) = \frac{2l+1}{l+1} \hat{z} L_l(\hat{z}) - \frac{l}{l+1} \hat{z} L_{l-1}(\hat{z}), \quad l \geq 1.
\]

The Jacobi polynomials $P_l^{\mu,\nu}(\hat{z})(\mu, \nu > -1, \hat{z} \in [-1, 1], l \in \mathbb{Z}^+)$ are orthogonal with respect to the weight function $\omega^{\mu,\nu}(\hat{z}) = (1-\hat{z})^\mu (1+\hat{z})^\nu$. The explicit formula of the Jacobi polynomial is stated as follows (see [16]):
\[
P_l^{\mu,\nu}(\hat{z}) = 2^{-l} \sum_{j=1}^l \binom{l+\mu}{j} \binom{l+\nu}{l-j} (\hat{z} - 1)^{l-j} (\hat{z} + 1)^j.
\]
The Jacobi polynomials can also be generated by the three-term recurrence formula (see [16] for more details).

Therefore, the function space $V_N^0 = V_N^{x,0} \otimes V_N^{y,0}$ can be given by
\[
V_N^0 = \text{span}\{ \phi_k(x)\varphi_l(y) : k, l = 0, 1, \cdots, N-2 \}.
\]

At the end of this section, we introduce an interpolation operator and the projector $\Pi_N^{\beta,0}$. The Legendre-Gauss-Lobatto (LGL) interpolation operator $I_N : C(\bar{\Omega}) \to V_N^0$ is defined by
\[
I_N u(x_k, y_l) = u(x_k, y_l), \quad k, l = 0, 1, \cdots, N,
\]
where $x_k$ and $y_l$ are LGL points on the intervals $\bar{I}_x$ and $\bar{I}_y$, respectively. Denote
\[
A(u, v) = K_x c_I \left[ (RLD_{a,x}^\beta u, v) + (RLD_{x,b}^\beta u, v) \right] + K_y c_I \left[ (RLD_{c,y}^\beta u, v) + (RLD_{y,d}^\beta u, v) \right].
\]
Then the orthogonal projection operator $\Pi_N^{\beta,0} : H_0^\beta (\Omega) \to V_N^0$ is defined as
\[
A(u - \Pi_N^{\beta,0} u, v) = 0, \quad u \in H_0^\beta (\Omega), \quad \forall v \in V_N^0.
\]

From [2], we have the following properties of the projector $\Pi_N^{\beta,0}$.

Lemma 2.11. Let $s$ and $r$ be real numbers satisfying $0 < s < r$. Then there exist a projector $\Pi_N^{1,0}$ and a positive constant $C$ depending only on $r$ such that, for any function $u \in H_0^\beta (\Omega) \cap H_0^r (\Omega)$, the following estimate holds:
\[
\|u - \Pi_N^{1,0} u\|_{H^s(\Omega)} \leq C N^{s-r} \|u\|_{H^r(\Omega)}.
\]
Similarly, we obtain the following lemma about the projector $\Pi_N^{\beta,0}$.

**Lemma 2.12.** Let $\beta$ and $r$ be arbitrary real numbers satisfying $0 < \beta < 1$, $\beta < r$, $\beta \neq 1/2$. Then there exist a projector $\Pi_N^{\beta,0}$ and a positive constant $C$ independent of $N$ such that, for any function $u \in H_0^\beta(\Omega) \cap H_0^r(\Omega)$, the following estimate holds:

$$|u - \Pi_N^{\beta,0} u| \leq CN^{\beta - r}\|u\|_{H^r(\Omega)}.$$  

3. The fully discrete scheme. In this section, we present the fully discrete scheme for (1) applying the Legendre spectral method. Let $t_k = k\tau$, $k = 0, 1, \cdots, M$, where $\tau = T/M$ is the time step size. Denote $\sigma = 1 - \alpha/2$, $t_{k+\sigma} = (k + \sigma)\tau$ and $u^k = u(t_{k+\sigma}) = \sigma u^{k+1} + (1 - \sigma)u^k$. Let $L_x u = K_x \partial_x^2 u$, $L_y u = K_y \partial_y^2 u$. Then (1) can be rewritten as

$$C_0^D u(t_{k+\sigma}) = (L_x + L_y)u + f(x, y, t).$$

(6)

Suppose that $u(x, y, t)$ is sufficiently smooth with respect to time. At each time level $n$, the temporal derivative of (6) is discretized by the L2-1 formula (see [1]), i.e., for $k = 0, 1, \cdots, M - 1$,

$$C_0^D u(t_{k+\sigma}) = \frac{\tau^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{j=0}^{k} c_j^{(k,\alpha,\sigma)} (u(t_{k+1-j}) - u(t_{k-j})) + O(\tau^{3-\alpha}),$$

(7)

where $c_0^{(0,\alpha,\sigma)} = a_0^{(\alpha,\sigma)}$, for $k \geq 1$, the coefficient $c_j^{(k,\alpha,\sigma)}$ is expressed as

$$c_j^{(k,\alpha,\sigma)} = \begin{cases} a_0^{(\alpha,\sigma)} + b_1^{(\alpha,\sigma)}, & j = 0, \\ a_j^{(\alpha,\sigma)} + b_{j+1}^{(\alpha,\sigma)} - b_j^{(\alpha,\sigma)}, & 1 \leq j \leq k - 1, \\ a_k^{(\alpha,\sigma)} + b_{k}^{(\alpha,\sigma)}, & j = k, \end{cases}$$

(8)

where

$$a_0^{(\alpha,\sigma)} = \sigma^{1-\alpha}, a_l^{(\alpha,\sigma)} = (l + \sigma)^{-\alpha} - (l - 1 + \sigma)^{-\alpha}, \quad l \geq 1, \quad b_l^{(\alpha,\sigma)} = \frac{1}{2 - \alpha} [(l + \sigma)^{2-\alpha} - (l - 1 + \sigma)^{2-\alpha}]$$

$$- \frac{1}{2} [(l + \sigma)^{1-\alpha} - (l - 1 + \sigma)^{2-\alpha}], \quad l \geq 1.$$

For the the coefficients $\{c_j^{(k,\alpha,\sigma)}\}_{j=0}^k$, we have the following lemma (see [1]).

**Lemma 3.1.** For any $\alpha$ $(0 < \alpha < 1)$ and $0 \leq j \leq k$, $k \geq 1$, it holds

$$c_0^{(k,\alpha,\sigma)} > \frac{1 - \alpha}{2} (k + \sigma)^{-\alpha},$$

$$c_0^{(k,\alpha,\sigma)} > c_1^{(k,\alpha,\sigma)} > c_2^{(k,\alpha,\sigma)} > \cdots > c_{k-1}^{(k,\alpha,\sigma)} > c_k^{(k,\alpha,\sigma)},$$

(10)

$$2\sigma - 1 c_0^{(k,\alpha,\sigma)} - \sigma c_1^{(k,\alpha,\sigma)} > 0.$$  

(11)

Let $\delta_t^\sigma u^{k+1} = \frac{\tau^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{j=0}^{k} c_j^{(k,\alpha,\sigma)} (u(t_{k+1-j}) - u(t_{k-j}))$ and $q_j^{k+1} = \frac{\tau^{-\alpha}}{\Gamma(2 - \alpha)} c_j^{(k,\alpha,\sigma)}$, then we have $\delta_t^\sigma u^{k+1} = \sum_{j=0}^{k} q_j^{k+1} (u^{j+1} - u^j)$. Using the same arguments as in proof of Lemma 1 in [1], we obtain the following lemma.
Lemma 3.2. For $n = 0, 1, \cdots, M - 1$, one has
\[
(u^{k+1}, \delta_t^\alpha u^{k+\sigma}) \geq \frac{1}{2} \delta_t^\alpha \|u^{k+\sigma}\|^2 + \frac{1}{2g^1_k} \left( \delta_t^\alpha u^{k+\sigma}, \delta_t^\alpha u^{k+\sigma} \right),
\]
\[
(u^k, \delta_t^\alpha u^{k+\sigma}) \geq \frac{1}{2} \delta_t^\alpha \|u^{k+\sigma}\|^2 - \frac{1}{2g^1_k} \left( \delta_t^\alpha u^{k+\sigma}, \delta_t^\alpha u^{k+\sigma} \right),
\]
where $g^1_{-1} = 0$.

According to (11) and Lemma 3.2, we obtain the following corollary.

Corollary 3.3. For $n = 0, 1, \cdots, M - 1$, one has the following inequality
\[
(\sigma u^{k+1} + (1 - \sigma)u^k, \delta_t^\alpha u^{k+\sigma}) \geq \frac{1}{2} \delta_t^\alpha \|u^{k+\sigma}\|^2.
\]

We discretize the space using the basis function $\phi_k(x)\varphi_j(y)$, then the fully discrete scheme for problem (6) in weak formulation is as follows: find $u^k_{N+1} \in V^0_N$, such that
\[
(\delta_t^\alpha u^k_{N+1}, v_N) + A(u^k_{N+1}, v_N) = (I_N f^{k+\sigma}, v_N), \quad \forall v_N \in V^0_N,
\]
where $k = 0, 1, \cdots, M - 1$.

4. Stability and convergence of the fully discrete scheme. In this section, we study the stability and convergence of the scheme (12). For the stability, we have the following theorem.

Theorem 4.1. The scheme (12) is unconditionally stable in the sense that for all $\tau > 0$, it holds
\[
\|u^k_{N+1}\|^2 \leq \|u^k_N\|^2 + CT(1 - \alpha)T^\alpha \max_{0 \leq j \leq M - 1} \|f^{k+\sigma}\|^2, \quad k = 0, 1, \cdots, M - 1. \tag{13}
\]

Proof. Taking $v_N = u^k_N$ in (12) gives
\[
(\delta_t^\alpha u^k_{N+1}, u^k_{N+1}) + A(u^k_{N+1}, u^k_{N+1}) = (I_N f^{k+\sigma}, u^k_{N+1}). \tag{14}
\]

To estimate the second term on the left-hand side of (14), applying Lemma 2.10 and Lemma 2.5, we firstly deduce that
\[
A(u, u) = 2c_1 \left[ \left( K_x(\alpha L^2 D^\beta_{\alpha,x} u, \alpha L^2 D^\beta_{\alpha,x} u) \right) + \left( K_y(\alpha L^2 D^\beta_{\alpha,y} u, \alpha L^2 D^\beta_{\alpha,y} u) \right) \right] = K_x \|L^2 D^\beta_{\alpha,x} u\|^2_{L^2(\mathbb{R}^2)} + K_y \|L^2 D^\beta_{\alpha,y} u\|^2_{L^2(\mathbb{R}^2)} = |\hat{u}|^2_\beta. \tag{15}
\]

Using Young’s inequality and Lemma 2.9, we find
\[
(I_N f^{k+\sigma}, u^k_N) \leq \frac{1}{2} C \|I_N f^{k+\sigma}\|^2 + \frac{1}{2C} \|u^k_N\|^2 \leq \frac{1}{2} C \|f^{k+\sigma}\|^2 + \frac{1}{2} |u^k_N|^2. \tag{16}
\]

Applying Corollary 3.3, (15) and (16), from (14), we obtain
\[
\sum_{j=0}^k g^1_j (\|u^j_{N+1}\|^2 - \|u^j_N\|^2) \leq C \|f^{k+\sigma}\|^2. \tag{17}
\]

Let us rewrite the inequality (17) in the form
\[
g^1_k \|u^k_{N+1}\|^2 \leq \sum_{j=1}^k (g^1_j - g^1_{j-1}) \|u^j_N\|^2 + g^1_0 \|u^0_N\|^2 + C \|f^{k+\sigma}\|^2. \tag{18}
\]
Noticing that
\[ g^{k+1}_0 = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} c^{(k,\alpha,\sigma)}_k > \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \frac{1 - \alpha}{2} (k + \sigma)^{-\alpha} = \frac{1}{2\Gamma(1-\alpha)} (\tau(k + \sigma))^{-\alpha}, \]
then, we infer
\[ g^{k+1}_k \| u^{k+1}_N \| \leq \sum_{j=1}^{k} (g^{k+1}_j - g^{k+1}_{j-1}) \| u^{j}_N \|^2 + g^{k+1}_0 E. \]
(19)

Denote
\[ E = \| u^{0}_N \|^2 + CT(1-\alpha)T^\alpha \max_{0 \leq j \leq M-1} \| f^{k+\sigma} \|^2. \]
Then, (19) can be rewritten as
\[ g^{k+1}_k \| u^{k+1}_N \| \leq \sum_{j=1}^{k} (g^{k+1}_j - g^{k+1}_{j-1}) \| u^{j}_N \|^2 + g^{k+1}_0 E. \]
(20)

When \( k = 0 \), the estimate (13) holds immediately from the above inequality (20).
When \( k = 1, 2, \cdots, M - 1 \), we can obtain the inequality (13) applying the mathematical induction method. Suppose that we have proven that
\[ \| u^l_N \|^2 \leq E, \quad l = 1, 2, \cdots, k. \]
(21)

For \( l = k + 1 \), using the inequality (20) and the hypothesis (21), we deduce
\[ g^{k+1}_k \| u^{k+1}_N \| \leq \sum_{j=1}^{k} (g^{k+1}_j - g^{k+1}_{j-1}) \| u^{j}_N \|^2 + g^{k+1}_0 E \]
\[ \leq \sum_{j=1}^{k} (g^{k+1}_j - g^{k+1}_{j-1}) E + g^{k+1}_0 E = g^{k+1}_k E. \]
(22)
The proof is completed. \( \square \)

For the error analysis of the fully discrete scheme (12), we have

**Theorem 4.2.** Let \( u \) be the exact solution, \( \{u^k_N\}^M_{k=0} \) be the solution of the problem with the initial condition \( u^0_N = \Pi^0_N u_0 \). Suppose \( u \in \mathcal{C}^3([0,T];H^{23}_0(\Omega)) \), \( u \in H^r(\Omega) \), \( \mathcal{C}D^\alpha_t u \in H^r(\Omega) \), \( r \geq 1 \), then, one has
\[ \| u(t_k) - u^k_N \|^2 \leq C \alpha \left( CN^2(\beta-r) \| u \|^2_{W^{2r}(\Omega)} + CN^2(\beta-r) \| u \|^2_{H^r(\Omega)} + C \| u \|^2 \right). \]

**Proof.** (1) First we consider the case \( 0 < \alpha < 1 \). Let \( e^i_N = u(t_j) - u^i_N, \bar{e}^i_N = \Pi^{0,\tau} u(t_j) - u^i_N, \bar{\epsilon}^i_N = u(t_j) - \Pi^{0,\tau} u(t_j) \), thus we have \( e^i_N = \bar{e}^i_N + \bar{\epsilon}^i_N \), in particular \( e^0_N = u_0 - \Pi^{0,\tau} u_0 = \bar{e}^0_N, \bar{\epsilon}^0_N = 0 \). From the initial equation (1) and the fully discrete scheme (12), for all \( v_N \in V_0^N \), we have the following error equation,
\[ (\delta_{\bar{\epsilon}^k} e^{k+\sigma}_N, v_N) + A \left(e^{k}_N, v_N\right) = -(r^{k+\sigma}_{\bar{\delta}_{\bar{\epsilon}}}, v_N) + \sum_{i=1}^{3} R_i^{k+\sigma}, \]
(23)
where \( r^{k+\sigma}_{\bar{\delta}_{\bar{\epsilon}}} = \mathcal{C}D^\alpha_t u(t_{k+\sigma}) - \delta^\alpha u(t_{k+\sigma}) \), and
\[ R^{k+\sigma}_1 = A \left((u^k_N - u^{k+\sigma}), v_N\right), \quad R^{k+\sigma}_2 = -A \left(e^{k}_N, v_N\right), \quad R^{k+\sigma}_3 = -\left(\delta_{\bar{\epsilon}} e^{k+\sigma}_N, v_N\right). \]
Following the same lines as in the proof of Theorem 4.1, we obtain
\begin{equation}
\lVert(t_{u,\tau}^{k+\sigma}, v_N)\rVert \leq \lVert t_{u,\tau}^{k+\sigma}\rVert \cdot \lVert v_N\rVert \leq C_{1,u}\tau^{-2\alpha} + \frac{1}{8C}\lVert v_N\rVert^2
\end{equation}
(24)
where \(C_{1,u}\) is a positive constant depending on \(u\). For the term \(R_1^{k+\sigma}\), using Taylor’s expansion at \(t = t_{k+\sigma}\), \(u^k - u^{k+\sigma} = O(\tau^2)\), then we have
\begin{equation}
|R_1^{k+\sigma}| \leq C_{2,u}\tau^4 + \frac{1}{8C} \left(\lVert v_N\rVert^2_{\beta} + \lVert v_N\rVert^2_{\beta}\right) \leq C_{2,u}\tau^4 + \frac{1}{8}\lVert v_N\rVert^2_{\beta}.
\end{equation}
(25)
For the term \(R_2^{k+\sigma}\), applying Lemma 2.12, we deduce
\begin{equation}
|R_2^{k+\sigma}| \leq C\lVert e_{k+\sigma}^N\rVert^2_{\beta} + \frac{1}{8C} \left(\lVert v_N\rVert^2_{\beta} + \lVert v_N\rVert^2_{\beta}\right) \leq CN^{2(\beta-r)}\lVert u\rVert^2_{H^r(\Omega)} + \frac{1}{8}\lVert v_N\rVert^2_{\beta}.
\end{equation}
(26)
Using Lemma 2.9, Lemma 2.12 and (7), one has
\begin{equation}
\lVert \delta_t c_{k+\sigma}^N \rVert^2 = \lVert \delta_t \tilde{D}_t^\sigma (I - \Pi_N^\delta) u(t_{k+\sigma}) - \delta_t c_{k+\sigma}^N \rVert^2_{\beta} \leq 2\lVert \delta_t \tilde{D}_t^\sigma (I - \Pi_N^* \Pi_N^\delta) u(t_{k+\sigma})\rVert^2 + 2\lVert \delta_t c_{k+\sigma}^N \rVert^2_{\beta} \leq CN^{2(\beta-r)}\lVert \tilde{D}_t^\sigma u\rVert^2_{H^r(\Omega)} + C_3\tau^4.
\end{equation}
Thus, the term \(R_3^{k+\sigma}\) can be estimated as
\begin{equation}
|R_3^{k+\sigma}| \leq 2C\lVert \delta_t c_{k+\sigma}^N \rVert^2 + \frac{1}{8C}\lVert v_N\rVert^2 \leq CN^{2(\beta-r)}\lVert u\rVert^2_{H^r(\Omega)} + \frac{1}{8}\lVert v_N\rVert^2_{\beta}.
\end{equation}
(27)
Substituting (24)-(27) into (23), and taking \(v_N = e_{k}^N\), we obtain
\begin{equation}
\delta_t^\alpha \lVert e_{k+\sigma}^N \rVert^2 + \frac{1}{2}\lVert e_{k}^N \rVert^2_{\beta} \leq CN^{2(\beta-r)}\lVert u\rVert^2_{H^r(\Omega)} + CN^{2(\beta-r)}\lVert \tilde{D}_t^\sigma u\rVert^2_{H^r(\Omega)} + C_u\tau^4,
\end{equation}
(28)
where \(C_u\) is a positive constant dependent on \(u\). Let us rewrite (28) in the form
\begin{equation}
g_k^{k+1}\lVert e_{k+\sigma}^N \rVert^2 \leq \sum_{j=1}^{k} (g_j^{k+1} - g_{j-1}^{k+1}) \lVert e_{k}^N \rVert^2 + CN^{2(\beta-r)}\lVert u\rVert^2_{H^r(\Omega)} + CN^{2(\beta-r)}\lVert \tilde{D}_t^\sigma u\rVert^2_{H^r(\Omega)} + C_u\tau^4.
\end{equation}
(29)
Following the same lines as in the proof of Theorem 4.1, we obtain
\begin{equation}
\lVert e_{k+\sigma}^N \rVert^2 \leq 2\Gamma(1 - \alpha)T^\alpha \left(CN^{2(\beta-r)}\lVert u\rVert^2_{H^r(\Omega)} + CN^{2(\beta-r)}\lVert \tilde{D}_t^\sigma u\rVert^2_{H^r(\Omega)} + C_u\tau^4\right).
\end{equation}
Then applying the triangular inequality \(\lVert e_{k+1}^N \rVert \leq \lVert e_{k}^N \rVert + \lVert e_{k+\sigma}^N \rVert\), we deduce the desired result.

(2) Now we consider the case \(\alpha \to 1\). Denote
\[E_u = CN^{2(\beta-r)}\lVert u\rVert^2_{H^r(\Omega)} + CN^{2(\beta-r)}\lVert \tilde{D}_t^\sigma u\rVert^2_{H^r(\Omega)} + C_u\tau^4.\]
Taking into account the fact \(jT \leq T\) for all \(j = 1, 2, \ldots, M\), we are led to establish:
\[\lVert \tilde{e}_{j}^N \rVert^2 \leq 2\Gamma(2 - \alpha)j\tau^\alpha E_u, \quad j = 1, 2, \ldots, M.\]
(30)
From (29) and noticing \(g_j^1 = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \alpha_0(\alpha, \sigma) \geq \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\), inequality (30) is obvious for \(j = 1\). We will prove (30) by using the mathematical induction method. Suppose
Theorem 5.1. The scheme

\[ v \]

where spectral scheme inequality (31) by we get

This leads us to propose an ADI spectral scheme. method can significantly reduce the computation time and storage requirements.

If we add the term \( B(u_N^k, v_N) \) to the left-hand side of (12), we obtain the ADI spectral scheme

\[ (\delta_t u_N^{k+\sigma}, v_N) + A(u_N^k, v_N) + B(u_N^k, v_N) = (I_N f^{k+\sigma}, v_N), \]

(32)

where

\[ B(u, v) = \sigma^2 c_i^2 K_x K_y \left[ (RLD_{a,x}^\beta RLD_{c,y}^\beta u, RLD_{x,b}^\beta RLD_{y,d}^\beta v) + (RLD_{a,x}^\beta RLD_{a,x}^\beta RLD_{x,b}^\beta RLD_{c,y}^\beta v) + (RLD_{a,x}^\beta RLD_{a,x}^\beta RLD_{c,y}^\beta v) \right], \]

\[ v_N \in V_N^0 \text{ and } k = 0, 1, \cdots, M - 1. \]

For the stability of the scheme, we have the following theorem.

Theorem 5.1. The scheme (32) is unconditionally stable in the sense that for all \( \tau > 0, \) it holds

\[ \| u_N^{k+1} \| \leq \| u_N^k \| + C T \| f^{1+\sigma} \|, \quad k = 0, 1, \cdots, M - 1. \]

\[ (33) \]

Proof. The proof is almost the same as in the proof of Theorem 4.1, one just notices that

\[ B(u_N^k, v_N) = \frac{2 \sigma^2 c_i^2 K_x K_y}{g_{k+1}} (\cos \beta \pi)^2 \| RL \mathcal{D}_{-\infty,x}^\beta RLD_{-\infty,y}^\beta u_N^k \|_{L^2(\mathbb{R}^2)} \geq 0. \]
Theorem 5.2. Let $u$ be the exact solution, $\{u_N^k\}_{k=0}^M$ be the solution of the problem with the initial condition $u_N^0 = \Pi_N^{2,0} u_0$. Suppose $u \in C^3([0,T]; H_0^{2\beta}(\Omega))$, $u \in H^{3\beta+r}(\Omega)$, $r \geq 1$, then we have
\[
\|u(t_k) - u_N^k\|^2 \leq C_a \left( CN^{2(\beta-r)}\|u\|^2_{H^r(\Omega)} + CN^{2(\beta-r)}\|u\|^2_{H^{3\beta+r}(\Omega)} + C_u r^4 \right)
\]

Proof. In comparison with the proof of Theorem 4.2 for (12), one need only notice the additional term,
\[
B(u_N^k, \tilde{e}_N^k) = B(\Pi_N^{2,0} u^{k+\sigma}, \tilde{e}_N^k) - B(\Pi_N^{2,0} u^{k+\sigma}, \tilde{e}_N^k) \leq B(\Pi_N^{2,0} u^{k+\sigma}, \tilde{e}_N^k) + B(u^{k+\sigma}, \tilde{e}_N^k)
\]
\[
\leq C \left[ \|RLD_{a,x}^{2\beta} (RLD_{c,y}^{2\beta} (\Pi_N^{2,0} u^{k+\sigma} - u^{k+\sigma}))\| \|RLD_{a,y}^{2\beta} \tilde{e}_N^k\| + \|RLD_{a,x}^{2\beta} (RLD_{c,y}^{2\beta} (\Pi_N^{2,0} u^{k+\sigma} - u^{k+\sigma}))\| \|RLD_{a,y}^{2\beta} \tilde{e}_N^k\| + \|RLD_{a,b}^{2\beta} (RLD_{c,y}^{2\beta} (\Pi_N^{2,0} u^{k+\sigma} - u^{k+\sigma}))\| \|RLD_{a,y}^{2\beta} \tilde{e}_N^k\| \right]
\]
\[
\leq CN^{2(\beta-r)}\|u\|^2_{H^{3\beta+r}} + \frac{1}{8} \|\tilde{e}_N^k\|^2_{\beta}.
\]

Then following the same lines as in the proof of Theorem 4.2, we get the desired result. \qed

6. Numerical experiment. In this section, we give the detailed implementation of the proposed method and present two numerical examples to show the efficiency of our method. The algorithms are implemented by Matlab.

6.1. Implementation. One can rewrite (32) into the following equivalent form:
\[
\left( I + \frac{\sigma K_c c_1}{g_k^{k+1}} (RLD_{a,x}^{2\beta} + RLD_{x,b}^{2\beta}) \right) \left( I + \frac{\sigma K_c c_1}{g_k^{k+1}} (RLD_{c,y}^{2\beta} + RLD_{y,a}^{2\beta}) \right) u_N^{k+1}, v_N
\]
\[
= \left( I + \frac{\sigma K_c c_1}{g_k^{k+1}} (RLD_{a,x}^{2\beta} + RLD_{x,b}^{2\beta}) \right) \left( I + \frac{\sigma K_c c_1}{g_k^{k+1}} (RLD_{c,y}^{2\beta} + RLD_{y,a}^{2\beta}) \right) u_N^k, v_N
\]
\[
+ F^k, \quad \forall v_N \in V_N^0.
\]

(34)

where
\[
F^k = \frac{1}{g_k^{k+1}} (I_N f^{k+\sigma}, v_N) - \frac{1}{g_k^{k+1}} \left( \sum_{j=0}^{k-1} g_j^{k+1} (u_j^{k+1} - u_j^k), v_N \right) - \frac{1}{g_k^{k+1}} A(u_N^k, v_N)
\]
\[
- \frac{1}{g_k^{k+1}} B(u_N^k, v_N).
\]

Next, we give the matrix representation of the ADI method (34). The unknown function $u_N^{k+1} \in V_N^0$ has the following form
\[
u_N^{k+1} = \sum_{j=0}^{N-2} \sum_{l=0}^{N-1-j} d_{j,l}^{k+1} \phi_j(x) \varphi_l(y).
\]

(35)
Let the matrices $Φ_x, Φ_y, Ψ_x, Ψ_y ∈ \mathbb{R}^{(N−1)×(N−1)}$ that satisfy
\[
(Φ_x)_{j,l} = (φ_j, φ_l), \quad (Ψ_x)_{j,l} = (RL^β D^β_{x,δ} φ_j, RL^β D^β_{x,δ} φ_l),
\]
\[
(Φ_y)_{j,l} = (φ_j, φ_l), \quad (Ψ_y)_{j,l} = (RL^β D^β_{y,δ} φ_j, RL^β D^β_{y,δ} φ_l).
\]
Inserting $u^{k+1}_N$ into the ADI method (34) and letting $v_N = φ_j φ_l (j, l = 0, 1, \cdots, N−2)$, we obtain the matrix representation of the ADI method as follows:
\[
\left( Φ_x + \frac{σKx_c1}{g_k^{k+1}} (Ψ_x + Ψ^T_x) \right) C^{k+1} \left( Φ_y + \frac{σKy_c1}{g_k^{k+1}} (Ψ_y + Ψ^T_y) \right) = R^k + H^k + N^{k+1},
\]
where $C^{k+1}, R^k, H^k ∈ \mathbb{R}^{(N−1)×(N−1)}$, satisfying
\[
(C^{k+1})_{j,l} = d^{k+1}_j, \quad j, l = 0, 1, \cdots, N−2,
\]
\[
R^k = \left( Φ_x + \frac{σKx_c1}{g_k^k} (Ψ_x + Ψ^T_x) \right) C^k \left( Φ_y + \frac{σKy_c1}{g_k^k} (Ψ_y + Ψ^T_y) \right),
\]
\[
(H^k)_{j,l} = \frac{1}{g_k^{k-1}} (I_N k^{k+1,σ,φ_j φ_l} - \frac{1}{g_k^{k-1}} \left( \sum_{j=0}^{k-1} g_j^{k+1} (u_j + u_l), φ_j φ_l \right)
\]
\[
- \frac{1}{g_k^{k+1}} A(u_N, φ_j φ_l) - \frac{1}{g_k^{k+1}} B(u_N, φ_j φ_l), \quad j, l = 0, 1, \cdots, N−2.
\]

6.2. Numerical examples. In this subsection, we present numerical examples to verify our theoretical analysis.

**Example 6.1.** Consider the following two-dimensional fractional Bloch-Torrey equation
\[
\begin{align*}
&\frac{C_0}{\partial^α D^α_t} u = \frac{∂^{2β} u}{∂|x|^{2β}} + 2 \frac{∂^{2β} u}{∂|y|^{2β}} + f(x, y, t), \quad (x, y, t) ∈ Ω × (0, 0.5), \\
&u(x, y, 0) = 0, \quad (x, y) ∈ Ω, \\
&u(x, y, t)|(x, y) ∈ ∂Ω = 0, \quad t ∈ (0, 0.5),
\end{align*}
\]
with $α ∈ (0, 1), β ∈ (1/2, 1), Ω = (0, 1) × (0, 1)$ and
\[
f(x, y, t) = 10(x - x^2)(y - y^2) \frac{t^{1-α}}{Γ(2-α)} - \frac{5t(y^2 - y)}{cos(βπ)} \left[ \frac{x^{1-2β} + (1-x)^{1-2β}}{Γ(2-2β)} - \frac{2x^{2-2β} + 2(1-x)^{2-2β}}{Γ(3-2β)} \right] - \frac{10t(x^2 - x)}{cos(βπ)} \left[ \frac{y^{1-2β} + (1-y)^{1-2β}}{Γ(2-2β)} - \frac{2y^{2-2β} + 2(1-y)^{2-2β}}{Γ(3-2β)} \right].
\]
The exact solution of Example 6.1 is $u = 10t(x - x^2)(y - y^2)$.

**Example 6.2.** Consider the following two-dimensional fractional Bloch-Torrey equation
\[
\begin{align*}
&\frac{C_0}{\partial^α D^α_t} u = 4 \frac{∂^{2β} u}{∂|x|^{2β}} + 3 \frac{∂^{2β} u}{∂|y|^{2β}} + f(x, y, t), \quad (x, y, t) ∈ Ω × (0, 0.5), \\
&u(x, y, 0) = 0, \quad (x, y) ∈ Ω, \\
&u(x, y, t)|(x, y) ∈ ∂Ω = 0, \quad t ∈ (0, 0.5),
\end{align*}
\]
SPECTRAL METHODS FOR FRACTIONAL BLOCH-TORREY EQUATIONS

\[ \tau \alpha = 0 \quad \beta = 0 \]

\[ \tau \alpha = 0.3 \quad \beta = 0.6 \]

Con. rate \[ \alpha = 0.8 \quad \beta = 0.75 \]

Con. rate

| \( \tau \) | \( \alpha = 0.3 \) \( \beta = 0.6 \) | Con. rate | \( \alpha = 0.8 \) \( \beta = 0.75 \) | Con. rate |
|---|---|---|---|---|
| 1/10 | 1.4361e-004 | 2.1596 | 1.5512e-004 | 2.0231 |
| 1/20 | 3.2143e-005 | 2.0832 | 3.8163e-005 | 2.0046 |
| 1/40 | 7.5856e-006 | 2.0480 | 9.5104e-006 | 1.9942 |
| 1/80 | 1.8343e-006 | 1.9814 | 2.3872e-006 | 1.9743 |
| 1/160 | 4.6451e-007 | - | 6.0791e-007 | 1.9743 |

Table 1. \( L^2 \) errors and convergence rates for Example 6.1.

| \( \tau \) | \( \alpha = 0.3 \) \( \beta = 0.6 \) | Con. rate | \( \alpha = 0.8 \) \( \beta = 0.75 \) | Con. rate |
|---|---|---|---|---|
| 1/10 | 1.7946e-003 | 2.1361 | 1.6438e-003 | 2.0271 |
| 1/20 | 4.0827e-004 | 2.0638 | 4.0329e-004 | 2.0098 |
| 1/40 | 9.7653e-005 | 2.0361 | 1.0014e-005 | 1.9949 |
| 1/80 | 2.3810e-005 | 2.0142 | 2.5123e-005 | 1.9935 |
| 1/160 | 5.8943e-006 | - | 6.3092e-006 | 1.9763 |

Table 2. \( L^2 \) errors and convergence rates for Example 6.2.

The exact solution of Example 6.2 is

\[ u = 100(t^2 + 2t)x^2(1 - x)^2y^2(1 - y)^2. \]

To confirm the temporal accuracy, we choose \( N \) big enough to eliminate the error caused by spatial discretization. We take \( N = 125 \). Tables 1-2 show the errors and temporal convergence rate of \( \| u(T) - u_h(T) \| (T = 1) \) in \( L^2 \) discrete norm for different \( \alpha \) and \( \beta \). From which, we can see the temporal accuracy is second-order, independent of \( \alpha \), which is consistent with our theoretical analysis. Next, we check the spatial accuracy with respect to the polynomial degree \( N \). By fixing the time step small enough to avoid the contamination of the temporal error. We take \( \tau = 1.0 \times 10^{-3} \). Fig. 1 show the errors with respect to polynomial degree \( N \) in semi-log scale with different values of \( \alpha \) and \( \beta \) for Example 6.1. From which, we can see that the errors decay exponentially. Fig. 2 shows the errors with respect to polynomial degree \( N \) in log-log scale for Example 6.2.

7. Summary and discussion. In this work, we have developed a fully discrete scheme for two-dimensional space and time fractional Bloch-Torrey equations. The
scheme employs the spectral approximation using Legendre functions in space and the so-called $L2-1_\sigma$ formula for the discretization of the time Caputo fractional derivative. The proposed scheme has been proved to be unconditionally stable and convergent with order $O(\tau^2 + N^{\beta-r})$. We have presented some numerical experiments to confirm the theoretical analysis.

For the problems defined in unbounded spatial domain, one could use the Hermite functions or rational functions as basis functions to approximate the exact solutions. The fully discrete scheme proposed in this paper can be extended to solve the space and time fractional diffusion equations, but the computation work may be huge. In the future, we will try to solve this problem.

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