Veech groups are discrete subgroups of $\text{SL}(2, \mathbb{R})$ which play an important role in the theory of translation surfaces. For a special class of translation surfaces called origamis or square-tiled surfaces their Veech groups are subgroups of finite index of $\text{SL}(2, \mathbb{Z})$. We show that each stratum of the space of translation surfaces contains infinitely many origamis whose Veech group is a totally non congruence group, i.e. it surjects to $\text{SL}(2, \mathbb{Z}/n\mathbb{Z})$ for any $n$.

1 Introduction

Within the last thirty years the study of translation surfaces has become an active field in mathematics. Their moduli spaces come equipped with a natural action of $\text{SL}(2, \mathbb{R})$. It is one of the principal goals in this domain to understand the orbits of this action. This study culminated in the famous breakthrough result of Eskin, Mirzakhani and Mohammadi, namely the so-called magic wand theorem (cf. [EMM15, EM13]). The Veech group $\Gamma(X, \mu)$ associated to a translation surface $(X, \mu)$ plays a crucial role in this topic. $\Gamma(X, \mu)$ is the stabiliser of $(X, \mu)$ under the action of $\text{SL}(2, \mathbb{R})$. It turns out to be a discrete subgroup of $\text{SL}(2, \mathbb{R})$ and it carries a lot of information about the dynamical flow on the translation surface and about the Teichmüller flow defined by $(X, \mu)$. Origamis or square-tiled surfaces are a particularly important class of translation surfaces. These surfaces are tessellated by finitely many Euclidean unit squares. Their Veech groups are especially easy to handle. They are subgroups of finite index of $\text{SL}(2, \mathbb{Z})$ and can be calculated explicitly from the combinatorial data which define the origami. Furthermore, the set of origamis is dense in the moduli space of translation surfaces. The action of $\text{SL}(2, \mathbb{R})$ on the set of translation surfaces restricts to an action of $\text{SL}(2, \mathbb{Z})$ on origamis. It is still open whether all subgroups of $\text{SL}(2, \mathbb{Z})$ of finite index occur as Veech groups of origamis. A major result in this direction was achieved in [EM12] where it is proved that all subgroups of finite index (satisfying a slight condition) of the principal congruence group $\Gamma(2)$ occur as Veech groups, where $\Gamma(2)$ is the group of matrices which are congruent to the identity matrix modulo 2. As a result in some sense in the opposite direction it is shown in [Sch05] that all congruence groups (cf. below) of prime level except five exceptions occur as Veech groups.

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It is particularly interesting to study Veech groups of origamis that lie in the same fixed stratum, i.e. we fix the genus and the cone angles of the singularities (see below). [HL06, McM05] succeeded to give a complete classification of the \( SL(2, \mathbb{Z}) \)-orbits of origamis in the stratum \( \mathcal{H}_2(2) \) of translation surfaces of genus 2 with one singularity of angle \( 6\pi \). In this case, the set of origamis with \( d \) squares decomposes depending on \( d \) in one or two orbits. There are only a few further classification results for certain subloci of strata (cf. [LN14b, LN14a, LN18]). For general strata the classification problem is open. However, there exists a conjecture for a precise description of the orbits in each stratum by Delecroix and Lelièvre based on computer experiments.

A congruence subgroup \( \Gamma \) of \( SL(2, \mathbb{Z}) \) is a subgroup which is fully determined by its image in \( SL(2, \mathbb{Z}/n\mathbb{Z}) \) for some \( n \in \mathbb{N} \), i.e. it is the preimage of its image in \( SL(2, \mathbb{Z}/n\mathbb{Z}) \) under the canonical projection \( SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/n\mathbb{Z}) \). It turns out that such groups are rare among all finite index subgroups of \( SL(2, \mathbb{Z}) \). Turning to Veech groups of origamis: there are several families of origamis whose Veech groups could be explicitly determined as congruence groups in [Sch06, HS07, Her06]. In [Sch07] first examples of Veech groups that are non congruence groups were detected. Hubert and Lelièvre proved in [HL05] that for all but one origami of genus 2 with one singularity their Veech group is a non congruence group.

For an arbitrary subgroup \( \Gamma \) of \( SL(2, \mathbb{Z}) \) of finite index we may measure how much information we lose, if we consider all its images in the finite quotient groups \( SL(2, \mathbb{Z}/n\mathbb{Z}) \). In particular, all information is lost if for all \( n \) the image is the full group \( SL(2, \mathbb{Z}/n\mathbb{Z}) \). In this case, we call \( \Gamma \) a totally non congruence group. In [WS15] a criterion is given which detects totally noncongruence groups (cf. [WS15, Theorem 2]). It was further shown that for each stratum \( \mathcal{H}_2(2) \) all Veech groups of origamis are totally non congruence groups or almost totally non congruence groups (cf. [WS15, Theorem 3]). Finally, it was shown that for each stratum \( \mathcal{H}_{k+1}(2k) \) of translation surfaces with only one singularity of cone angle \( (k + 1)2\pi \) there are infinitely many origamis whose Veech group is a totally non congruence group (cf. [WS15, Theorem 4]). In this article we generalise this statement to all strata. For this we first improve the criterion for totally non congruence groups from [WS15, Theorem 2] and get the following very handy conditions which assure that a group \( \Gamma \) is a totally non congruence group.

**Theorem 1.** Let \( \Gamma \) be a finite index subgroup of \( SL(2, \mathbb{Z}) \). Denote \( e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Suppose that for each prime \( p \) there exist matrices \( A_1, A_2 \in SL(2, \mathbb{Z}) \) with the following properties:

A) \( \forall j \in \mathbb{N} : A_1 e_1 \neq j \cdot A_2 e_1 \text{ modulo } p \).

B) There exist \( m_1, m_2 \in \mathbb{N} \) with

\[ A_1 T^{m_1} A_1^{-1} \text{ and } A_2 T^{m_2} A_2^{-1} \text{ are contained in } \Gamma, \]

such that \( p \) divides neither \( m_1 \) nor \( m_2 \).

Then \( \Gamma \) is a totally non congruence group.

We then describe a method to construct one-cylinder origamis in each stratum for which we have a good control over the cylinder decompositions in horizontal and vertical direction and in the diagonal direction given by the vector \( v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Choosing special elements of this family we finally prove the following theorem.
Theorem 2. Every stratum contains an infinite family of origamis whose Veech groups are totally non-congruence groups.

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2 Preliminaries

In this section we give a concise introduction to translation surfaces, origamis and Veech groups suited to the purpose of this article. You can find more elaborate introductions to this topic for example in [HS06, EG97, MT02, Sch05, Vor97]. For the proofs of the facts that we state here we refer to these references.

Translation surfaces, origamis and strata A (finite) translation surface is a surfaces $X$ with an atlas $\mu$ to $\mathbb{R}^2$ such that all transition maps of the atlas $\mu$ are translations. The translation surface inherits a natural metric from the Euclidean metric in $\mathbb{R}^2$. Furthermore we have a well-defined notion of directions, since they are invariant under translations. Thus we may speak for example of horizontal and vertical geodesics, or more general of geodesics in direction $v \in \mathbb{R}^2$. Moreover, using local charts we can assign to each geodesic segment a vector in the plane $\mathbb{R}^2$ which is its development vector. Let $\overline{X}$ be the metric completion of $X$. The points in $\overline{X} \setminus X$ are called the singularities of $X$. In this article, we consider the classical situation of finite translation surfaces, i.e. translation surfaces $(X, \mu)$ such that the metric completion is compact, the set of singularities is discrete and all singularities are cone points of finite cone angle $k \cdot 2\pi$ $(k \in \mathbb{N})$. A geodesic segment between two (possible equal) singularities which does not contain any further singularity is called a saddle connection. Further important geometric data of the translation surface $(X, \mu)$ are its set of closed geodesics and its set of maximal cylinders in a given direction $v \in \mathbb{R}^2$. For genus $g \geq 2$, every closed geodesic lies in a unique maximal cylinder in the direction $v$ of the geodesic which is bounded by saddle connections, since we may move the geodesic transversely to $v$ until we hit singularities.

Finite translation surfaces are naturally distinguished into strata by their type of singularities. More precisely, a finite translation surface $(X, \mu)$ is said to be of type $(\alpha_1, \ldots, \alpha_n)$, if $\overline{X}$ has $n$ singularities of cone angle $(\alpha_1 + 1) \cdot 2\pi, \ldots, (\alpha_n + 1) \cdot 2\pi$. The usage of $\alpha_i$ instead of $\alpha_i + 1$ relates to the fact that a finite translation surface can equivalently be defined as a closed Riemann surface $X$ together with a holomorphic differential $\omega$. The charts of the atlas are then obtained by integrating with respect to $\omega$, the singularities are the zeroes of $\omega$ and $\alpha_i$ is the order of the zero. We then define the stratum $\mathcal{H}_g(\alpha_1, \ldots, \alpha_n)$ as the set of all equivalence classes of translation surfaces of type $(\alpha_1, \ldots, \alpha_n)$ of genus $g$. Two translation surfaces $(X_1, \mu_1)$ and $(X_2, \mu_2)$ are equivalent, if there exists a translation $f : X_1 \to X_2$, i.e. a homeomorphism which is a translation on each chart. We will usually write $(X, \mu) \in \mathcal{H}_g(\alpha_1, \ldots, \alpha_r)$ for the equivalence class defined by $(X, \mu)$. The set $\mathcal{H}_g(\alpha_1, \ldots, \alpha_n)$ is endowed with a topology itself, more precisely there is a natural way to define local coordinates as a manifold on a covering of it (cf. [Yoc10, Section 6.3]). Furthermore, $\mathcal{H}_g(\alpha_1, \ldots, \alpha_n)$ is endowed with a natural action of $\text{SL}(2, \mathbb{R})$ as follows. For a translation surface $(X, \mu)$ and a matrix $A \in \text{SL}(2, \mathbb{R})$, we define $A \cdot (X, \mu) = (X, \mu_A)$ to be the translation surface obtained from $(X, \mu)$ by composing each
chart of $\mu$ with the linear map $z \mapsto A \cdot z$. It is one of the main objectives in the field to understand the orbits of this action. There is yet another way how to define finite translation surfaces: Take finitely many polygons in the plane such that their edges come in pairs of edges of same length and same direction. Glue for each pair its two edges by a translation. In this way we obtain a closed surface $X$. The points which come from the vertices of the polygons may be cone points. Removing them defines a translation surface $X$. If all the polygons which form the translation surface are copies of the Euclidean unit square, the translation surface is called an origami or a square-tiled surface (cf. Figure 1).

![Figure 1: Gluing edges with same labels defines an origami of genus 3. This origami stems from [HS08].](image)

**Veech groups and the action of $\text{SL}(2, \mathbb{R})$** Let $(X, \mu)$ be a finite translation surface of genus $g$ in some stratum $\mathcal{H}_g(\alpha_1, \ldots, \alpha_n)$. The Veech group $\Gamma(X, \mu)$ is the stabiliser of $(X, \mu)$ for the action of $\text{SL}(2, \mathbb{R})$ on $\mathcal{H}_g(\alpha_1, \ldots, \alpha_n)$. It can equivalently be defined in the following way. Consider the group Aff$(X, \mu)$ of all affine homeomorphisms of $X$, i.e. homeomorphisms which are with respect to charts of the form $z \mapsto A \cdot z + b$ with $A \in \text{SL}(2, \mathbb{R})$ and $b \in \mathbb{R}^2$. It turns out that since all transition maps are translations the matrix $A$ is independent of the chosen charts. We obtain a group homomorphism $D : \text{Aff}(X, \mu) \to \text{SL}(2, \mathbb{R})$ which maps the affine homeomorphism $f$ to the matrix $A$, i.e. to its derivative. The Veech group is the image of $D$, hence it consists of all matrices $A$ which occur as derivative of some affine homeomorphism of the surface. It was already shown by Veech himself that $\Gamma(X, \mu)$ is a discrete subgroup of $\text{SL}(2, \mathbb{R})$ (cf. [Vee89, Proposition 2.7] or [Vor97, Proposition 3.3] for a very nice presentation). Furthermore, two translation surfaces in the same $\text{SL}(2, \mathbb{R})$-orbit have conjugated Veech groups.

Let us consider the example of the torus $\mathbb{R}^2/\mathbb{Z}^2$ endowed with the translation structure of its universal covering $\mathbb{R}^2$. Observe that the affine homeomorphisms lift to affine homeomorphisms of $\mathbb{R}^2$ which preserve the lattice $\mathbb{Z}^2$ up to a translation. And all such maps descend to the torus. Therefore the Veech group is in this case $\text{SL}(2, \mathbb{Z})$. 

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Special properties of origamis. We will use three equivalent ways to describe origamis, as explained in the following. The equivalences are described in more details in [Sch07, Section 1]. Recall that we obtain an origami by gluing copies of the Euclidean unit square along their edges which leads to a closed surface $\overline{X}$ tiled by squares. Hence, an origami made from $d$ unit squares is fully determined by a pair of permutations $(\sigma_a, \sigma_b)$ as follows. We label the squares with \{1, \ldots, d\}, then $\sigma_a(i)$ and $\sigma_b(i)$ denote the right and the upper neighbour of the square labelled by $i \in \{1, \ldots, d\}$. The fact that the surface is connected is equivalent to the fact that the subgroup of $S_d$ generated by the two permutations $\sigma_a$ and $\sigma_b$ acts transitively on the set \{1, \ldots, d\}. If we choose another labelling of the squares this leads to a simultaneous conjugation of the pair of permutations $(\sigma_a, \sigma_b)$. Altogether, we obtain an equivalence between the set of origamis up to translation equivalence and the set of pairs $(\sigma_a, \sigma_b)$ in $S_d^2$ up to simultaneous conjugation. There is yet another equivalent description of origamis which we will use. Observe that the surface $\overline{X}$ comes with a covering $p$ to the square-torus $T$ obtained by gluing parallel edges of the unit square. Namely, we map each square on $\overline{X}$ to the one square forming $T$ and this map is well-defined with respect to the gluings. The map $p$ is an unramified covering for all points which are not vertices. Hence if $\infty \in T$ is the point obtained from the vertex of the unit square, then $p : \overline{X} \rightarrow T$ is ramified at most over $\infty$.

For an origami $(X, \mu)$ the Veech group is always a finite index subgroup of $\text{SL}(2, \mathbb{Z})$. Here we should point to a subtlety in the definition of origami. Recall that we obtain the origami by gluing copies of the Euclidean unit square along their edges. More precisely, this gives us the metric completion $\overline{X}$ of the translation surface. The singularities of the translation surface stem from the vertices of the squares. However not every vertex has to be a singularity. Now there are two different natural ways how two define the translation surfaces $X$. We might either remove only the singularities of $\overline{X}$, or we might remove all points which come from a vertex. In the second case the Veech group is indeed a subgroup of $\text{SL}(2, \mathbb{Z})$ of finite index, in the first case it is commensurable to $\text{SL}(2, \mathbb{Z})$. However it turns out that for reduced origamis one obtains equal Veech groups for the translation surface with only singularities removed and for the surface with all vertex points removed (cf. [Kap11, Remark 2.9]). Following [MMY15, Section 1.2], we call an origami reduced, if the set of development vectors of all saddle connections generate $\mathbb{Z}^2$. This is a very mild restriction, since any origami $O$ is affine equivalent to a reduced origami $O'$, i.e. there is some matrix $A \in \text{GL}(2, \mathbb{R})$ such that $O' \sim A \cdot O$ and thus their Veech groups are conjugated in $\text{GL}(2, \mathbb{R})$. Here the action of $\text{GL}(2, \mathbb{R})$ on translation surfaces is defined just in the same way as the action of $\text{SL}(2, \mathbb{R})$. In this article we will restrict to reduced origamis and thus all Veech groups are subgroups of $\text{SL}(2, \mathbb{Z})$ of finite index.

Suppose that an origami is now given by the pair of permutations $(\sigma_a, \sigma_b)$. We obtain the stratum in which the associated translation surface lives in the following way: Let $p : \overline{X} \rightarrow T$ be the corresponding ramified cover of the torus. Let us choose a loop around the vertex of $T$, namely $x y x^{-1} y^{-1}$, where $x$ and $y$ are the closed curves on $T$ shown in Figure 2. The connected components of the preimage of this curve are loops around the singularities. Hence the number of the connected components are the number of singularities. Furthermore if the multiplicity of a component is $k$, then the corresponding singularity is of angle $2k\pi$. Hence the commutator $[\sigma_b^{-1}, \sigma_a^{-1}]$ determines the type of the singularities that we obtain. More precisely, each cycle of
length $k$ in the commutator corresponds to a singularity of cone angle $k \cdot 2\pi$.

In the proof our result the following two facts are crucial which are described in more detail e.g. in [WS15, 2.2,2.3]:

1. The action of $\text{SL}(2, \mathbb{R})$ on translation surfaces restricts to an action of $\text{SL}(2, \mathbb{Z})$ on origamis.

   The action can be explicitly given as described in the following. The two generators
   
   $$ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} $$

   act on an origami given as pair of permutations $(\sigma_a, \sigma_b)$ in the following way:
   
   $$ S : (\sigma_a, \sigma_b) \mapsto (\sigma_b^{-1}, \sigma_a^{-1}) \quad \text{and} \quad T : (\sigma_a, \sigma_b) \mapsto (\sigma_a, \sigma_b \sigma_a^{-1}) $$

   

2. Suppose that the translation surface $(X, \mu)$ defined by a primitive origami $O$ decomposes in the horizontal direction into $k$ cylinders of height 1 and length $m_1, \ldots, m_k$ and let $m$ be a multiple of $m_1, \ldots, m_k$. Then $T^m$ is in the Veech group $\Gamma(O)$. Similarly, if $(X, \mu)$ decomposes in the vertical direction into $l$ cylinders of length $m'_1, \ldots, m'_l$ and $m'$ is a multiple of $m'_1, \ldots, m'_l$, then $\Gamma(X, \mu)$ contains $T'^{m'}$. Here

   $$ T' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. $$

   If $O$ is given by the pair of permutations $(\sigma_a, \sigma_b)$, then the numbers $m_1, \ldots, m_k$ are precisely the cycle lengths of $\sigma_a$ and $m'_1, \ldots, m'_l$ are the cycle lengths of $\sigma_b$.

3 A criterion for being a totally non congruence group

   We denote

   $$ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1} $$

   Furthermore $p_n : \text{SL}(2, \mathbb{Z}) \rightarrow \text{SL}(2, \mathbb{Z}/n\mathbb{Z})$ is the canonical projection. We denote the images of the matrices $T$ and $T'$ in $\text{SL}(2, \mathbb{Z}/n\mathbb{Z})$ also by $T$ and $T'$. Finally, we denote by $I$ the $2 \times 2$-identity matrix over the respective ring.

   We start with a small but very useful calculation.
Lemma 1. Let \( A, B \in \text{GL}(2, \mathbb{Z}/n\mathbb{Z}) \) with \( A \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Then we have that
\[
ATA^{-1} = BT^{\det(B)/\det(A)}B^{-1}.
\]
Observe for the statement in Lemma 1 that \( T^n \) with \( n \in \mathbb{Z}/n\mathbb{Z} \) gives a well-defined matrix in \( \text{GL}(2, \mathbb{Z}/n\mathbb{Z}) \) and we have for any \( A \in \text{GL}(2, \mathbb{Z}/n\mathbb{Z}) \) that \( AT^nA^{-1} = (ATA^{-1})^n \).

Proof. Suppose first that \( A \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Hence we can write
\[
A = \begin{pmatrix} 1 & x \\ 0 & \det(A) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & y \\ 0 & \det(B) \end{pmatrix}
\]
with \( x, y \in \mathbb{Z}/n\mathbb{Z} \). A short calculation gives:
\[
ATA^{-1} = \begin{pmatrix} 1 & \det(A)^{-1} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad BTB^{-1} = \begin{pmatrix} 1 & \det(B)^{-1} \\ 0 & 1 \end{pmatrix}
\]
Thus the claim holds in this case. In the general situation we consider the two matrices \( A^{-1}B \) and \( I \) satisfying \( A^{-1}B \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and obtain from the preceding consideration
\[
T = (A^{-1}B)T^{\det(A^{-1}B)}(B^{-1}A) = A^{-1}BT^{\det(B)/\det(A)}B^{-1}A,
\]
which implies the claim. \( \square \)

We now deduct from Lemma 1 a criterion whether two conjugates of \( T \) generate the full group \( \text{SL}(2, \mathbb{Z}/p^r\mathbb{Z}) \).

Lemma 2. Let \( p \) be prime and \( r \in \mathbb{N} \). Let \( \Gamma \) be a subgroup of \( \text{SL}(2, \mathbb{Z}/p^r\mathbb{Z}) \). Suppose that \( \Gamma \) contains \( A_1TA_1^{-1} \) and \( A_2TA_2^{-1} \) with \( A_1, A_2 \in \text{SL}(2, \mathbb{Z}/p^r\mathbb{Z}) \) such that
\[
\forall m \in \mathbb{N} : \quad mA_1e_1 \neq A_2e_1 \mod p, \quad \text{where} \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in (\mathbb{Z}/p^r\mathbb{Z})^2
\]
Then \( \Gamma = \text{SL}(2, \mathbb{Z}/p^r\mathbb{Z}) \).

Proof. By conjugation we may assume that \( A_1 = I \) is the identity matrix. Consider the vector \( \begin{pmatrix} a \\ c \end{pmatrix} = A_2 \cdot e_1 \). By assumption \( c \) is not divisible by \( p \), hence \( c \) is in the multiplicative group \( (\mathbb{Z}/p^r\mathbb{Z})^\times \). Consider the following matrix \( B \in \text{GL}(2, \mathbb{Z}/p^r\mathbb{Z}) \) and its inverse \( B^{-1} \):
\[
B = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \quad \text{and} \quad B^{-1} = c^{-1} \cdot \begin{pmatrix} c & -a \\ 0 & 1 \end{pmatrix}.
\]
It follows directly from the definition of \( B \) that
\[
B^{-1}e_1 = e_1 \quad \text{and} \quad B^{-1}A_2e_1 = e_2 = Se_1, \quad \text{where} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
Hence we obtain from Lemma 1:
\[
(B^{-1}TB)^{\det(B^{-1})} = T \quad \text{and} \quad (B^{-1}A_2TA_2^{-1}B)^{\det(B^{-1})} = STS^{-1} = T^{r-1}
\]  (2)
It follows that
\[ \text{SL}(2, \mathbb{Z}/p^r \mathbb{Z}) = \langle T, T' \rangle \subseteq B^{-1} \Gamma B \subseteq \text{SL}(2, \mathbb{Z}/p^r \mathbb{Z}). \]

Hence we have \( B^{-1} \Gamma B = \text{SL}(2, \mathbb{Z}/p^r \mathbb{Z}) \) and thus \( \Gamma = \text{SL}(2, \mathbb{Z}/p^r \mathbb{Z}) \). Here it is crucial that \( \text{SL}(2, \mathbb{Z}/p^r \mathbb{Z}) \) is a normal subgroup in \( \text{GL}(2, \mathbb{Z}/p^r \mathbb{Z}) \).

Lemma 2 is the main ingredient that we need to prove Theorem 1, which provides us with a criterion whether a group is a totally non congruence group.

**Theorem 1.** Let \( \Gamma \) be a finite index subgroup of \( \text{SL}(2, \mathbb{Z}) \). Denote \( e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Suppose that for each prime \( p \) there exist matrices \( A_1, A_2 \in \text{SL}(2, \mathbb{Z}) \) with the following properties:

A) \( \forall j \in \mathbb{N} : A_1 e_1 \not\equiv j \cdot A_2 e_1 \) modulo \( p \).

B) There exist \( m_1, m_2 \in \mathbb{N} \) with

\[ A_1 T^{m_1} A_1^{-1} \text{ and } A_2 T^{m_2} A_2^{-1} \]

are contained in \( \Gamma \), such that \( p \) divides neither \( m_1 \) nor \( m_2 \).

Then \( \Gamma \) is a totally non congruence group.

**Proof.** We have to show that \( \text{pr}_n(\Gamma) = \text{SL}(2, \mathbb{Z}/n \mathbb{Z}) \) for all \( n \in \mathbb{N} \).

Let \( n = p_1^{r_1} \cdots p_k^{r_k} \) be the prime factorisation of \( n \). We thus have by the Chinese remainder theorem:

\[ \text{SL}(2, \mathbb{Z}/n \mathbb{Z}) = \text{SL}(2, \mathbb{Z}/p_1^{r_1} \mathbb{Z}) \times \cdots \times \text{SL}(2, \mathbb{Z}/p_k^{r_k} \mathbb{Z}) \]

We show:

\[ \forall i \in \{1, \ldots, k\} : \text{pr}_n(\Gamma) \supseteq \{I\} \times \cdots \times \{I\} \times \text{SL}(2, \mathbb{Z}/p_i^{r_i} \mathbb{Z}) \times \{I\} \times \cdots \times \{I\}. \quad (3) \]

For \( p = p_i \) we decompose \( n = p^r \cdot n_2 \) with \( \gcd(p, n_2) = 1 \). Choose \( m_1, m_2 \) such that they satisfy assumptions A) and B) with respect to \( p \). In particular, \( m_1 \) and \( m_2 \) are coprime to \( p \). By Bézout’s identity we find \( a, b \in \mathbb{Z} \) with \( 1 = a \cdot p^r + b \cdot m_1 m_2 n_2 \).

We then have for \( K = bm_1 m_2 n_2 \) that

\[ \Gamma \ni A_1 T^K A_1^{-1} = A_1 \begin{pmatrix} 1 \\ 0 \\ bm_1 m_2 n_2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} A_1^{-1}. \]

Furthermore, we have:

\[ A_1 T^K A_1^{-1} = A_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A_1^{-1} = A_1 T A_1^{-1} \mod p^r \] and

\[ A_1 T^K A_1^{-1} = I \mod n_2. \]

Hence the group \( \text{pr}_n(\Gamma) \) contains

\[ \text{pr}_n(A_1 T^K A_1^{-1}) = (A_1 T A_1^{-1}, I) \in \text{SL}(2, \mathbb{Z}/p^r \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}/n_2 \mathbb{Z}) = \text{SL}(2, \mathbb{Z}/n \mathbb{Z}). \]
Similarly, we obtain that
\[ \text{pr}_n(\Gamma) \ni \text{pr}_n(A_2 T A_2^{-1}) = (A_2 T A_2^{-1}, I) \in \SL(2, \Z/p^r \Z) \times \SL(2, \Z/n_2 \Z) = \SL(2, \Z/n \Z). \]

It follows from Lemma 2 that
\[ \text{pr}_n(\Gamma) \supseteq \SL(2, \Z/p^r \Z) \times \{I\}. \]

This implies the claim. \( \square \)

Theorem 1 is a generalisation of [WS15, Theorem 2] which we restate adapted to our context in Corollary 3.

**Corollary 3.** [WS15, Theorem 2]
*Let \( \Gamma \) be a finite index subgroup of \( \SL(2, \Z) \). Suppose there exist matrices \( C_1, C_2 \in \SL(2, \Z) \) and \( m_1, m_1', m_2, m_2' \in \N \) with
\[ \Gamma \ni C_1 T^{m_1} C_1^{-1}, C_1 T^{m_1'} C_1^{-1} \text{ and } \Gamma \ni C_2 T^{m_2} C_2^{-1}, C_2 T^{m_2'} C_2^{-1}, \]
such that \( \gcd(m_1 m_1', m_2 m_2') = 1 \). Then \( \Gamma \) is a totally non-congruence group.*

**Proof.** We show that the assumptions of Theorem 1 are fulfilled. Let \( p \) be prime. If \( p \) does not divide \( m_1 m_1' \), then we may choose \( A_1 = C_1, A_2 = C_1 S \). Denote \( e_1 = (0, 1) \) and \( e_2 = (1, 0) \). Since \( e_1 \neq j \cdot e_2 \mod p \) for all \( j \in \N \), we have that \( A_1 e_1 \neq j \cdot A_2 e_1 = j \cdot A_1 e_2 \mod p \). Thus in this case the assumptions are satisfied. If \( p \) divides \( m_1 m_1' \), then it does not divide \( m_2 m_2' \) and we can use the same arguments with \( C_2 \) instead of \( C_1 \). \( \square \)

**4 Nice one-cylinder origamis**

In this section we give explicit examples for one-cylinder origamis in each stratum. The following examples will provide building blocks for them.

**Example 4.** In the following we construct special one-cylinder origamis in \( \mathcal{H}(\alpha) \) with \( \alpha \) even and in \( \mathcal{H}(\alpha_1, \alpha_2) \) with \( \alpha_1, \alpha_2 \) odd.

i) *A family of origamis in \( \mathcal{H}(\alpha) \):*

Let \( \alpha = 2k \) be an even number. Define the origami \( O(\alpha) \) with \( N = 3k + 1 = \frac{3}{2} \alpha + 1 \) squares by the following permutations (cf. Figure 3):
\[
\sigma_a(\alpha) = (1, \ldots, N), \quad \sigma_b(\alpha) = (1, 2, 3) \circ (4, 5, 6) \circ \ldots \circ (3(k - 1) + 1, 3(k - 1) + 2, 3(k - 1) + 3)
\]

Observe that we obtain the commutator
\[
[\sigma_b^{-1}, \sigma_a^{-1}] = (3, 6, 9, \ldots, 3k - 1, 3k, 3k - 1, 3k - 4, 3k - 7, \ldots, 8, 5, 2, N)
\]

In particular the commutator consists of one cycle of length \( 2k + 1 \). Hence the origami has one singularity with cone angle \((2k + 1) \cdot 2\pi = (\alpha + 1) \cdot 2\pi \) and thus lies in \( \mathcal{H}(\alpha) \).
A family of origamis in $\mathcal{H}(\alpha)$

Hence $O_s$ squares and is defined by the permutations $O_{\alpha}$

Observe that $O_{\alpha}$ lies in $\mathcal{H}(\alpha)$ in the following way (cf. Figure 4).

We now define for arbitrary $l \geq 1$ the one-cylinder origami $O(\alpha;l)$ in $\mathcal{H}(\alpha)$ as a deformation of $O(\alpha)$ in the following way (cf. Figure 4). $O(\alpha;l)$ has $N' = N + l - 1 = \frac{3}{2} \alpha + l$

squares and is defined by the permutations $O_{\alpha;l}$

\[
\sigma_a(\alpha;l) = (1, \ldots, N'), \\
\sigma_b(\alpha;l) = \sigma_b(\alpha) = (1,2,3) \circ (4,5,6) \circ \ldots (3(k-1)+1,3(k-1)+2,3(k-1)+3)
\]

Observe that $O(\alpha;l)$ has again one singularity and lies in $\mathcal{H}(\alpha)$.

ii) A family of origamis in $\mathcal{H}(\alpha_1, \alpha_2)$ (cf. Figure 5):

Let $\alpha_1 = 2k_1 + 1$, $\alpha_2 = 2k_2 + 1$ be odd numbers. Define the origami $O(\alpha_1, \alpha_2)$ with $N = 3(k_1 + k_2) + 6 = \frac{3}{2}(\alpha_1 + \alpha_2) + 3$

squares by the following permutations $O_{\alpha_1, \alpha_2}$

\[
\sigma_a(\alpha_1, \alpha_2) = (1, \ldots, N), \\
\sigma_b(\alpha_1, \alpha_2) = \sigma_1 \circ \sigma_2 \circ \sigma_3
\]

where $\sigma_1 = (1,2,3) \circ (4,5,6) \circ \ldots (3(k_1 - 2, 3k_1 - 1,3k_1),$

$\sigma_2 = (3k_1 + 1,3k_1 + 5,3k_1 + 2,3k_1 + 3,3k_1 + 4),$

$\sigma_3 = (3k_1 + 6,3k_1 + 7,3k_1 + 8) \circ (3k_1 + 9,3k_1 + 10,3k_1 + 11) \circ \ldots$

$\ldots \circ (3(k_1 + k_2) + 3,3(k_1 + k_2) + 4,3(k_1 + k_2) + 5)$

In this case we obtain for the commutator:

\[
[\sigma_b^{-1}, \sigma_a^{-1}] = (3,6,9, \ldots, 3k_1,3k_1 + 3,3k_1 + 1,3k_1 - 1,3k_1 - 4,3k_1 - 7, \ldots, 5,2,N) \circ \\
(3k_1 + 1,3k_1 + 5,3k_1 + 8,3k_1 + 11, \ldots, N - 1, \\
N - 2, N - 5, N - 8, \ldots, 3k_1 + 5 + 2)
\]

In particular it consists of two cycles of length $2k_1 + 2 = \alpha_1 + 1$ and $2k_2 + 2 = \alpha_2 + 1$. Hence $O(\alpha_1, \alpha_2)$ lies in $\mathcal{H}(\alpha_1, \alpha_2)$. Similarly as in i), we define for $l \geq 1$ the origami $O(\alpha_1, \alpha_2;l)$ in $\mathcal{H}(\alpha_1, \alpha_2)$ with $N' = 3(k_1 + k_2) + 5 + l = \frac{3}{2}(\alpha_1 + \alpha_2) + 2 + l$

squares by the two permutations (cf. Figure 5)

\[
\sigma_a(\alpha_1, \alpha_2;l) = (1, \ldots, N'), \\
\sigma_b(\alpha_1, \alpha_2;l) = \sigma_b(\alpha_1, \alpha_2)
\]
We may now construct one-cylinder origamis in a general stratum $\mathcal{H}(\alpha_1, \ldots, \alpha_k)$ by cutting and pasting the origamis from Example 4 as described in the following. We assume that the numbers $\alpha_1, \ldots, \alpha_k$ are ordered such that the first part consists of even numbers and the second part of odd numbers. Recall that $\alpha_1 + 1, \ldots, \alpha_k + 1$ are the cycle lengths of the commutator $[\sigma_b^{-1}, \sigma_a^{-1}]$. Since the commutator is an even permutation, the number of odd $\alpha_i$ is even.

**Lemma 5.** Let $\alpha_1, \ldots, \alpha_p$ be even, $\alpha_{p+1}, \ldots, \alpha_{p+2q}$ be odd numbers. Let further $l$ be a natural number. We obtain a one-cylinder origami $O$ in $\mathcal{H}(\alpha_1, \ldots, \alpha_{p+2q})$ with

$$L = \frac{3}{2}(\alpha_1 + \ldots + \alpha_{p+2q}) + p + 3q + l - 1$$

squares as follows (cf. Figure 6). If $q \neq 0$, we take the origamis

$$O(\alpha_1), \ldots, O(\alpha_p), O(\alpha_{p+1}, \alpha_{p+2}), \ldots, O(\alpha_{p+2q-3}, \alpha_{p+2q-2}) \text{ and } O(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$$

defined in Example 4. We cut them along the left vertical edge of their first square which is equal to the right vertical edge of their last square. We then glue them in the stated order along these slits. If $q = 0$, we take the origamis $O(\alpha_1), \ldots, O(\alpha_{p-1}), O(\alpha_{p}; l)$ and do the same procedure. This means the origami $O$ is defined by the two permutations $(\sigma_a, \sigma_b)$ given as follows: If $q \neq 0$, we have

$$\begin{align*}
\sigma_a &= (1, \ldots, L), \\
\sigma_b &= \hat{\sigma}_b(\alpha_1) \circ \ldots \circ \hat{\sigma}_b(\alpha_p) \circ \hat{\sigma}(\alpha_{p+1}, \alpha_{p+2}) \circ \ldots \circ \hat{\sigma}(\alpha_{p+2q-3}, \alpha_{p+2q-2}) \quad (4)
\end{align*}$$

Here $\hat{\sigma}_b(\alpha_i)$, $\hat{\sigma}_b(\alpha_i, \alpha_{i+1})$, and $\hat{\sigma}(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ are conjugates of $\sigma_b(\alpha_i)$, $\sigma_b(\alpha_i, \alpha_{i+1})$, and $\sigma(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ which shift the labels of $O(\alpha_i)$, $O(\alpha_i, \alpha_{i+1})$ and $O(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ by the sum of the lengths of the origamis before them. More precisely, we define these permutations in the following way. Let $s_i = \frac{3}{2}\alpha_i + 1$ if $i \leq p$ and $s_i = \frac{3}{2}\alpha_i + \frac{3}{2}$ if $p + 1 \leq i \leq p + 2q - 1$. Then $O(\alpha_i)$ is of length $s_i$ for $i \leq p$ and $O(\alpha_i, \alpha_{i+1})$ is of length $s_i + s_{i+1}$ for $p+1 \leq i \leq p+2q-3$. Define $S_i = \sum_{j=1}^{i-1} s_j$. Let furthermore $sh(n) : \mathbb{N} \to \mathbb{N}$ be the map $n \mapsto n + a$. Then

$$\begin{align*}
\hat{\sigma}_b(\alpha_i) &= sh(S_i) \circ \sigma_b(\alpha_i) \circ sh(S_i)^{-1}, \\
\hat{\sigma}_b(\alpha_i, \alpha_{i+1}) &= sh(S_i) \circ \sigma_b(\alpha_i, \alpha_{i+1}) \circ sh(S_i)^{-1}, \text{ and} \\
\hat{\sigma}(\alpha_{p+2q-1}, \alpha_{p+2q}; l) &= sh(S_{p+2q-1}) \circ \sigma_b(\alpha_{p+2q-1}, \alpha_{p+2q}; l) \circ sh(S_{p+2q-1})^{-1}
\end{align*}$$

If $q = 0$, we similarly have

$$\sigma_a = (1, \ldots, L) \text{ and } \sigma_b = \hat{\sigma}_b(\alpha_1) \circ \ldots \circ \hat{\sigma}_b(\alpha_{p-1}) \circ \hat{\sigma}_b(\alpha_p; l),$$

Figure 5: The origami $O(\alpha_1, \alpha_2; l)$ from Example 4.
Lemma 6. Let $\Gamma$ be the Veech group of the origami $O = O(l)$ with $L = \frac{3}{2}(\alpha_1 + \ldots + \alpha_{p+2q}) + p+3q+l-1$ squares constructed in Lemma 5. Then $\Gamma$ contains the following parabolic matrices:

$$T^L, T'^{15}, \text{ and } T''^{(L-4q)},$$

with $T$ and $T'$ defined in (1) and $T'' = TT'T^{-1}$.

Proof. It follows from its definition that $O$ consists of one horizontal cylinder which has length $L$ and height 1. Thus the Veech group contains the matrix $T^L$. Furthermore, since all cycles of $\sigma_b$ are of length 1, 3 or 5, we have that $O$ decomposes into vertical cylinders of height 1 and length 1, 3 or 5. Hence $T'^{15}$ is contained in $\Gamma$. Finally, the origami $T''^{-1} \cdot O$ is given by the two permutations $(\sigma_b\sigma_a, \sigma_b)$. We will show below that $\sigma_b\sigma_a$ consists of one cycle of length $L-4q$ and further cycles of length 2. Hence $T''^{-1} \cdot O$ composes into horizontal cylinders of length $L-4q$ and of length 2. Therefore $T''^{(L-4q)} \in \Gamma(T''^{-1}O) = T''^{-1}T'T''$ and thus $T''T''^{(L-4q)}T''^{-1} = T''^{(L-4q)} \in \Gamma$. This finishes the claim.

Let us now show that the permutation $\sigma_b\sigma_a$ is of the desired form. We assume that $q \neq 0$. The case $q = 0$ works in the same way. Recall that $O$ consists of the origamis $O(\alpha_1), \ldots,
$O(\alpha_p), O(\alpha_{p+1}, \alpha_{p+2}), \ldots, O(\alpha_{p+2q-3}, \alpha_{p+2q-2}), O(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ which are glued in a row along slits. We label the squares of $O$ from left to right by $1, \ldots, L \in \mathbb{Z}/L\mathbb{Z}$. Let us consider how the permutation $\sigma_b \sigma_a$ acts on the labels of the squares.

Recall the definition of $S_i$ and $s_i$ in Lemma 5. The origamis $O(\alpha_i)$ are then of length $s_i$ and the origamis $O(\alpha_i, \alpha_{i+1})$ are of length $s_i + s_{i+1}$. Let us consider the squares belonging to the origami $O(\alpha_i)$ $(i \in \{1, \ldots, p\})$. The first square of the origami $O(\alpha_i)$ is labelled by $S_i + 1$ and the last one is labelled by $S_i + s_i$. Observe (cf. Figure 3) that the permutation $\sigma_b \sigma_a$ acts in the following way:

$$S_i \mapsto S_i + 2 \mapsto S_i + 3 \mapsto S_i + 5 \mapsto S_i + 6 \mapsto \ldots$$

In particular all squares of the origamis $O(\alpha_1), \ldots, O(\alpha_p)$, i.e. all squares labelled by $1, 2, \ldots, S_{p+1}$, lie in the same orbit.

Let us now consider the origamis $O(\alpha_i, \alpha_{i+1})$ $(i - p$ odd, $1 \leq i \leq 2q - 3)$. The first square of $O(\alpha_i, \alpha_{i+1})$ is labelled by $S_i + 1$, the last one by $S_i + s_i$. Observe that $\sigma_b \sigma_a$ acts in the following way (cf. Figure 5):

Denote $k_i = \frac{\alpha_i-1}{2}$ and $k_{i+1} = \frac{\alpha_{i+1}-1}{2}$.

$$S_i \mapsto S_i + 2 \mapsto S_i + 1 \mapsto S_i + 3 \mapsto S_i + 5 \mapsto S_i + 6 \mapsto \ldots$$

The remaining squares of $O(\alpha_i, \alpha_{i+1})$ which do not belong to this orbit are $S_i + 3k_i + 1, S_i + 3k_i + 2, S_i + 3k_i + 3$ and $S_i + 3k_i + 4$. They form two cycles $(S_i + 3k_i + 1, S_i + 3k_i + 3)$ and $(S_i + 3k_i + 2, S_i + 3k_i + 4)$ of length two.

Similarly, the permutation $\sigma_b \sigma_a$ acts on the squares of the origami $O(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ by:

Denote $i = p + 2q - 1$.

$$S_i \mapsto S_i + 2 \mapsto S_i + 1 \mapsto S_i + 3 \mapsto S_i + 5 \mapsto S_i + 6 \mapsto \ldots$$

 Altogether, we obtain for the permutation $\sigma_b \sigma_a$ one long cycle containing all squares except the squares $S_i + 3k_i + 1, S_i + 3k_i + 2, S_i + 3k_i + 3$ and $S_i + 3k_i + 4$ with $i - p$ odd and $p + 1 \leq i \leq p + 2q$. This circle has length $L - 4q$. Furthermore, we obtain $2q$ cycles of length 2. Hence $\sigma_b \sigma_a$ has the form which we claimed.

We are now able to obtain explicit origamis in each stratum whose Veech groups are totally non congruence groups.
Proposition 7. Let \( \alpha_1, \ldots, \alpha_p \) be even, \( \alpha_{p+1}, \ldots, \alpha_{p+2q} \) be odd numbers. Recall that in Lemma 5 we constructed an origami \( O \) in \( H(\alpha_1, \ldots, \alpha_{p+2q}) \) with \( L \) squares, where
\[
L = \frac{3}{2}(\alpha_1 + \ldots + \alpha_{p+2q}) + p + 3q + l - 1.
\]
Choose \( l \in \mathbb{N} \) such that:

i) \( \gcd(L, 30q) = 1 \).

ii) 3 and 5 do not divide \( L - 4q \).

Then the Veech group \( \Gamma = \Gamma(O) \) of \( O \) is a totally non congruence group.

Proof. We know from Lemma 6 that the matrices
\[
T^L, T^{15} \text{ and } T^{2(L-4q)} \text{ with } T'' = T'TT'^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}
\]
are contained in \( \Gamma \). We apply Theorem 1. Observe firstly that each pair \((A_1, A_2)\) of two matrices in \( \{T, T', T''\} \) satisfies property A) in Theorem 1 for any prime \( p \). We distinguish now three cases. Suppose as first case that \( p \) is neither a divisor of \( L \) nor of 15. Then we choose \( A_1 = T, A_2 = T', m_1 = L \) and \( m_2 = 15 \). By the assumption on \( p \) we have that \( p \) does neither divide \( m_1 \) nor \( m_2 \). As second case we consider that \( p \) divides \( L \). Then we choose \( A_1 = T', A_2 = T'', m_1 = 15 \) and \( m_2 = 2(L - 4q) \). Now, \( p \) does not divide \( m_1 \) by i). Furthermore, it follows from i) that \( p \) does not divide \( 4q \). Thus since it is a divisor of \( L \), it does not divide \( m_2 = L - 4q \).

In the remaining case, namely \( p = 3 \) or \( p = 5 \), we choose \( A_1 = T, A_2 = T'', m_1 = L \) and \( m_2 = 2(L - 4q) \). In this case \( p \) does neither divide \( m_1 \) (by i)) nor \( m_2 \) (by ii)). Hence, in all three cases we obtain that also property B) in Theorem 1 holds. This finishes the proof.

In particular, Proposition 7 defines in each stratum an infinite family of origamis.

Theorem 2. Every stratum contains an infinite family of origamis whose Veech groups are totally non-congruence groups.

Proof. The theorem directly follows from Proposition 7. Namely, we can choose \( l \) for example such that \( L \) is a prime with \( L > 4q \) which satisfies the following conditions:

\[
L \equiv \begin{cases} 
4q + 1 & \text{mod } 3, \text{ if } 3 \text{ does not divide } 4q + 1 \\
4q + 2 & \text{mod } 3, \text{ elsewise}
\end{cases}
\]

\[
L \equiv \begin{cases} 
4q + 1 & \text{mod } 5, \text{ if } 5 \text{ does not divide } 4q + 1 \\
4q + 2 & \text{mod } 5, \text{ elsewise}
\end{cases}
\]

By Dirichlet’s theorem on arithmetic progressions there are infinitely many primes which satisfy these conditions.
References

[EG97] Clifford J. Earle and Frederick P. Gardiner, *Teichmüller disks and Veech’s \( F \)-structures*, Extremal Riemann surfaces (San Francisco, CA, 1995), Contemp. Math., vol. 201, Amer. Math. Soc., Providence, RI, 1997, pp. 165–189. MR 1429199

[EM12] Jordan S. Ellenberg and D. B. McReynolds, *Arithmetic Veech sublattices of \( SL(2, \mathbb{Z}) \)*, Duke Math. J. 161 (2012), no. 3, 415–429. MR 2881227

[EM13] A. Eskin and M. Mirzakhani, *Invariant and stationary measures for the \( SL(2,\mathbb{R}) \) action on Moduli space*, ArXiv e-prints (2013).

[EMM15] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi, *Isolation, equidistribution, and orbit closures for the \( SL(2,\mathbb{R}) \) action on moduli space*, Ann. of Math. (2) 182 (2015), no. 2, 673–721. MR 3418528

[Her06] Frank Herrlich, *Teichmüller curves defined by characteristic origamis*, The geometry of Riemann surfaces and abelian varieties, Contemp. Math., vol. 397, Amer. Math. Soc., Providence, RI, 2006, pp. 133–144. MR 2218004

[HL05] Pascal Hubert and Samuel Lelièvre, *Noncongruence subgroups in \( H(2) \)*, International Math. Research Notices 2005:1 (2005), 47–64.

[HL06] ____, *Prime arithmetic Teichmüller discs in \( H(2) \)*, Israel J. Math. 151 (2006), 281–321. MR 2214127

[HS06] Pascal Hubert and Thomas Schmidt, *An introduction to Veech surfaces*, Handbook of dynamical systems 1B (2006), 501–526.

[HS07] Frank Herrlich and Gabriela Schmithüsen, *A comb of origami curves in the moduli space \( M_3 \) with three dimensional closure*, Geom. Dedicata 124 (2007), 69–94. MR 2318538

[HS08] ____, *An extraordinary origami curve*, Math. Nachr. 281 (2008), no. 2, 219–237. MR 2387362

[Kap11] André Kappes, *Monodromy representations and lyapunov exponents of origamis*, Ph.D. thesis, Karlsruhe Institute of Technology, EVA - Volltextarchiv der Universitätbibliothek Karlsruhe, 2011.

[LN14a] E. Lanneau and D.-M. Nguyen, *Connected components of Prym eigenform loci in genus three*, ArXiv e-prints (2014).

[LN14b] Erwan Lanneau and Duc-Manh Nguyen, *Teichmüller curves generated by Weierstrass Prym eigenforms in genus 3 and genus 4*, J. Topol. 7 (2014), no. 2, 475–522. MR 3217628
[LN18] ______, Weierstrass Prym eigenforms in genus four, preprint, see https://www-fourier.ujf-grenoble.fr/~lanneau/publications.html, 2018.

[McM05] Curtis T. McMullen, Teichmüller curves in genus two: Discriminant and spin, Math. Ann. 333 (2005), no. 1, 87–130.

[MMY15] Carlos Matheus, Martin Möller, and Jean-Christophe Yoccoz, A criterion for the simplicity of the Lyapunov spectrum of square-tiled surfaces, Invent. Math. 202 (2015), no. 1, 333–425. MR 3402801

[MT02] Howard Masur and Serge Tabachnikov, Rational billiards and flat structures, Handbook of dynamical systems, Vol. 1A, North-Holland, Amsterdam, 2002, pp. 1015–1089. MR 1928530

[Sch05] Gabriela Schmithüsen, Veech groups of origamis, Ph.D. thesis, Dissertation Universität Karlsruhe, 2005.

[Sch06] Gabriela Schmithüsen, Examples for Veech groups of origamis, The geometry of Riemann surfaces and abelian varieties, Contemp. Math., vol. 397, Amer. Math. Soc., Providence, RI, 2006, pp. 193–206. MR 2218009

[Sch07] ______, Origamis with non congruence Veech groups, Proceedings of 34th Symposium on Transformation Groups, Wing Co., Wakayama, 2007, pp. 31–55. MR 2313384

[Vee89] W. A. Veech, Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards, Invent. Math. 97 (1989), no. 3, 553–583.

[Vor97] Ya. B. Vorobets, Billiards in rational polygons: periodic trajectories symmetries and $d$-stability, Mat. Zametki 62 (1997), no. 1, 66–75. MR 1619976

[WS15] Gabriela Weitze-Schmithüsen, The deficiency of being a congruence group for Veech groups of origamis, Int. Math. Res. Not. IMRN (2015), no. 6, 1613–1637. MR 3340368

[Yoc10] Jean-Christophe Yoccoz, Interval exchange maps and translation surfaces, Homogeneous flows, moduli spaces and arithmetic, Clay Math. Proc., vol. 10, Amer. Math. Soc., Providence, RI, 2010, pp. 1–69. MR 2648692