Abstract. Characterising the correlations that arise when locally measuring a single joint quantum system is one of the main problems of quantum information theory. The seminal work [M. Navascués et al, NJP 10,7,073013 (2008)], known as the NPA hierarchy, reformulated this question as a polynomial optimisation problem over noncommutative variables and proposed a convergent hierarchy of necessary conditions, each testable using semidefinite programming. More recently, the problem of characterising the quantum network correlations, which arise when locally measuring several independent quantum systems distributed in a network, has received considerable interest. Several generalisations of the NPA hierarchy, such as the Scalar Extension [Pozas-Kerstjens et al, Phys. Rev. Lett. 123, 140503 (2019)], were introduced while their converging sets remain unknown. In this work, we introduce a new hierarchy and characterise its convergence in the case of the simplest network, the bilocal scenario, and explore its relations with the known generalisations. We also provide arguments against a commutator-based definition of the set of quantum correlations in networks.

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1. Introduction

Quantum correlations (the joint probability distribution established between independent parties measuring a same shared quantum state) are important both for foundational and applications of quantum theory. They underlie the Bell theorem [5, 11], a theoretical physics milestone [9]. They are used in the device independent framework to obtain practical application in terms of certification [41], e.g. of quantum devices [29, 40, 44], randomness [14, 34] or cryptographic protocols [1, 47]. In all these approaches, no assumption is made on the state and measurements. Hence, being able to characterise the set of probability distributions allowed by quantum theory is
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crucial. This can be done with the NPA hierarchy [25,31,33], which provides a converging hierarchy of Semi-Definite Programs (SDPs) to this set, using non-commutative polynomial optimisation theory.

More recently, a generalised framework in which several independent quantum sources are distributed and measured by several party in a network scenario received considerable interest [7, 16, 36, 37, 45]. It offered new insights on foundations and applications of quantum theory [12, 13, 21, 43, 51], in particular the simple bilocal scenario in which two independent sources are distributed to three parties in a line [3, 26, 38, 39, 48]. This calls for a generalisation of the NPA hierarchy. Several methods where proposed, but to our knowledge no proof of convergence exists.

In this paper, we introduce a new factorisation NPA hierarchy and prove convergence to the set of Bilocal Commutator Quantum Distributions, a relaxation of the quantum correlations obtainable in the bilocal scenario. We discuss the relation of the hierarchy with the inflation-NPA hierarchy of [49] and generalisation of our result to generic networks. Also, we discuss the scalar extension NPA hierarchy of [35] in our framework and its relation to Bilocal Commutator Quantum Distributions. We conclude with arguments in favor of a tensor (and not commutator) approach for the postulates of quantum information theory [46].

Besides the natural implications of our results to the study of quantum correlations in networks, we believe that our work can be of broader interest. We concentrate over the physically motivated problem of the characterisation of the correlations that can be obtained in the bilocal scenario given specific polynomial constraints, but our method can be extended to more general polynomial optimisation problems. Our method is adaptable to any (commutative or noncomutative) polynomial optimisation problem in which the optimisation is performed on a linear functional that is constrained to factorise as a product of two independent linear functionals.

Noncommutative polynomial optimisation with traces was already tackled in [24], but in general is not applicable to the problem of quantum correlations in networks. In particular, they provided an optimisation scheme for pure trace polynomial where cyclic equivalence assumed, while our framework should be thought as eigenvalue optimisation without cyclic equivalence with new constraints (see Appendix D).

1.1. Motivation

Bilocal scenario quantum correlations
In this article, we focus on the quantum bilocal scenario involving three parties,

\footnote{We also refer to the errata regarding scalar extension hierarchy in the previous version of the manuscript at the end of the main text.}
$A, B, C$ measuring two independent sources $\rho, \sigma$ distributed through the bilocal network (see Figure 1b). We focus on the set of quantum correlations $\hat{P} = \{p(abc|xyz)\}$ which can be obtained in this scenario, where $p(abc|xyz)$ is the probability of measurement results $a, b, c$ given that $A, B, C$ respectively performed measure $x, y, z$. According to the Born rule, they write

$$p(abc|xyz) = \text{Tr}_{\rho \otimes \sigma} (A_{a|x} \otimes B_{b|y} \otimes C_{c|z}),$$

where $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_{B_L}), \sigma \in \mathcal{B}(\mathcal{H}_{B_R} \otimes \mathcal{H}_C)$ are projectors over pure states, $A_{a|x} \in \mathcal{B}(\mathcal{H}_A), B_{b|y} \in \mathcal{B}(\mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}), C_{c|z} \in \mathcal{B}(\mathcal{H}_C)$ are PVMs, and $\mathcal{H}_A, \mathcal{H}_{B_L}, \mathcal{H}_{B_R}, \mathcal{H}_C$ are arbitrary Hilbert spaces.

Such correlations, which we call Tensor Bilocal Quantum Distributions, are the most basic example of quantum network correlations, the correlations which arise when several independent quantum states are distributed to several parties in a network [45].

**Standard Bell scenario quantum correlations and NPA hierarchy**

We first quickly recall an existing result already known for the more standard Bell scenario involving three parties, $A, B, C$ measuring a unique source $\tau$ (see Figure 1a). In this scenario, a quantum correlation $\hat{P} = \{p(abc|xyz)\}$ (a tensor quantum correlation) is given through the Born rule

$$p(abc|xyz) = \text{Tr}_\tau (A_{a|x} \otimes B_{b|y} \otimes C_{c|z}),$$

where the projector over a pure state $\tau \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ represents the state, the PVMs $A_{a|x} \in \mathcal{B}(\mathcal{H}_A), B_{b|y} \in \mathcal{B}(\mathcal{H}_B), C_{c|z} \in \mathcal{B}(\mathcal{H}_C)$ represent the performed measurement, and $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ are arbitrary Hilbert spaces.

Inner approximations of this set of correlations can easily be obtained, e.g. by sampling Hilbert spaces, states and measurement operators. An outer approximation was found by Navascués, Pironio, and Acín in [31]. Their method, called the NPA hierarchy, is the noncommutative counterpart of the Parrilo-Lasserre hierarchy [25, 33]. It provides a converging hierarchy of Semi-Definite Programs (SDPs) to any noncommutative polynomial optimisation problem over an archimedean semialgebraic set (a condition always satisfied in our context).

For the set of quantum correlation $\hat{P}$ obtainable in the scenario of Figure 1a, for a hierarchy level $n$, the NPA method asks for the existence of a moment matrix (or Hankel matrix) $\Gamma_n$ compatible with $\hat{P}$ (see Equation 2.1). The non-existence of such

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2When $\tau$ is a state, we use notation $\text{Tr}_\tau(M) := \text{Tr}(\tau \cdot M)$. In polynomial optimisation, the function $\text{Tr}_\tau(\cdot)$ is called a unital symmetric positive linear functional, often written $L(\cdot)$. 
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Figure 1. (a) Standard three-party Bell scenario. A three-particles quantum state, mathematically represented by a projector over a pure state \( \tau \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C) \) (a positive operator such that \( \text{Tr}(\tau) = 1 \) and \( \tau^2 = \tau \)), is created, each particle is sent to one of three separated parties \( A, B, C \). \( A \) measures the received particle according to some input \( x \), obtaining an output \( a \), mathematically represented by a PVMs \( A_a^x \in \mathcal{B}(\mathcal{H}_A) \) (a set of positive operators such that \( \sum_a A_a^x = 1 \)), and \( B, C \) do the same. The behavior of the experiment is described by a probability distribution \( \tilde{P} = \{ p(abc|xyz) \} \) with \( p(abc|xyz) = \text{Tr}_x (A_a^x \otimes B_b^y \otimes C_c^z) \).

(b) Bilocal scenario. Two two-particles quantum state (mathematically represented by two projectors over pure states \( \rho, \sigma \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \) such that \( \text{Tr}(\rho) = \text{Tr}(\sigma) = 1 \) and \( \rho^2 = \rho, \sigma^2 = \sigma \)) are created, \( A \) (resp. \( C \)) receiving one particle from \( \rho \) (resp. \( \sigma \)) and \( B \) one of each state, as depicted. \( A, B, C \) measurement operators are mathematically represented by positive operators \( A_{a|x} \in \mathcal{B}(\mathcal{H}_A), B_{b|y} \in \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_B), C_{c|z} \in \mathcal{B}(\mathcal{H}_C) \) such that \( \sum_a A_{a|x} = \sum_b B_{b|y} = \sum_c C_{c|z} = 1 \). The behavior of the experiment is described by a probability distribution \( \tilde{P} = \{ p(abc|xyz) \} \) with \( p(abc|xyz) = \text{Tr}_{\rho \otimes \sigma} (A_{a|x} \otimes B_{b|y} \otimes C_{c|z}) \).

\( \Gamma_n \) proves that \( \tilde{P} \) cannot be a tensor quantum correlation. Importantly, this existence problem is an SDP, which is easily solvable on a computer, with certificates of non-existence. Increasing the hierarchy level \( n \) (which indexes the size of \( \Gamma_n \)) gives stronger tests, and as \( n \) increases, this sequence of tests singles out the larger set of commutator quantum correlations \( \tilde{Q} = \{ q(abc|xyz) \} \) writing

\[
q(abc|xyz) = \text{Tr}_x (A_{a|x} B_{b|y} C_{c|z}),
\]

where \( \tau \) is a projector over a pure state, \( A_{a|x}, B_{b|y}, C_{c|z} \in \mathcal{B}(\mathcal{H}) \) are PVMs commuting together and all these operators are now part of the same global Hilbert spaces \( \mathcal{H} \). It was recently proven that for arbitrary infinite dimensional Hilbert spaces, the set of tensor quantum correlations is strictly included in the set of commutator quantum correlations [22]: informally, \( \{ \tilde{P} \} \subsetneq \{ \tilde{Q} \} \). However, in the case where the Hilbert spaces are restricted to be finite dimensional, the two sets coincide [42].
1.2. Content

After a more formal introduction of the standard NPA hierarchy to fix notations (see Section 2), we provide the first proof of convergence for the hierarchy of outer approximation of the set of bilocal quantum correlations (see Section 3). We discuss this result in light of the inflation-NPA hierarchy and its generalisation to larger networks (see Section 4). We conclude by discussing the tensor and commutator approach for the postulates of quantum information theory (see Section 5).

1.3. Main Contribution

Our hierarchy, the factorisation bilocal NPA Hierarchy, is introduced in Section 3.2. It is based on the same moment matrix $\Gamma$ as in the standard NPA hierarchy, to which we impose an additional nonlinear factorisation constraint corresponding to the relation

$$\text{Tr}_{\tau}(\hat{a}\hat{\gamma}) = \text{Tr}_{\tau}(\hat{a}) \cdot \text{Tr}_{\tau}(\hat{\gamma}),$$

where $\hat{a}$ (resp. $\hat{\gamma}$) is a monomial in Alice’s PVMs (resp. Charlie’s PVMs), and $\tau$ should be thought of as $\tau = \rho \otimes \sigma$: see Definition 3.3. This results in a hierarchy of (unpractical) non-SDP problems.

Our main result is the following theorem (see Theorems 3.2 and C.1) of convergence of this hierarchy to the set of commutator bilocal quantum correlations, introduced below after the theorem.

**Theorem** (Convergence of the factorisation bilocal NAP hierarchy). Let $\tilde{Q} = \{q(abc|xyz)\}$ be a probability distribution. The following are equivalent:

(i) $\tilde{Q}$ is a commutator bilocal quantum correlation,

(ii) $\tilde{Q}$ passes all factorisation bilocal NPA hierarchy tests,

where the set of commutator bilocal quantum correlations (see Definition 3.2) is a strict relaxation of the set of tensor bilocal quantum, defined as

**Definition** (Commutator Bilocal Quantum Distributions). $\tilde{Q}$ is a Commutator Bilocal Quantum Distribution iff there exist a Hilbert space $\mathcal{H}$, projectors (possibly infinite trace) $\rho, \sigma$, and PVMs $\hat{A}_{a|x}, \hat{B}_{b|y}, \hat{C}_{c|z} \in \mathcal{B}(\mathcal{H})$ such that

(i) $q(abc|xyz) = \text{Tr} \left( \rho \sigma \cdot \hat{A}_{a|x} \hat{B}_{b|y} \hat{C}_{c|z} \right)$

(ii) $\forall \hat{A}_{a|x}, \hat{B}_{b|y}, \hat{C}_{c|z}, [\hat{A}_{a|x}, \hat{B}_{b|y}] = [\hat{B}_{b|y}, \hat{C}_{c|z}] = [\hat{C}_{c|z}, \hat{A}_{a|x}] = 0$

(iii) $\tau := \rho \sigma = \sigma \rho$ is a projector over a pure state (i.e. $\text{Tr}(\tau) = 1$ and $\tau^2 = \tau$)

(iv) $\forall \hat{A}_{a|x}, \hat{C}_{c|z}, [\hat{A}_{a|x}, \sigma] = [\rho, \hat{C}_{c|z}] = 0.$
In this definition, the projective property of $\tau$ is necessary: as we discuss in Appendix F.2, it cannot be obtained by purification, contrary to tensor bilocal quantum distributions. We also provide a stopping criterion for our hierarchy corresponding to the case where the reconstructed Hilbert space $\mathcal{H}$ is finite-dimensional (see Theorem 3.3 of Section 3.3).

1.4. Discussion and open problems

The set of Commutator Bilocal Quantum Distribution is a strict relaxation of the physical set of Tensor Bilocal Quantum Distribution. Is it problematic?

Remark first that the standard NPA hierarchy also does not converge to the physical set of Tensor Quantum Distributions predicted by quantum theory for the Bell scenario, but to its natural approximation when only considering the commutation algebraic relations of the PVMs (see Definition 2.2 and Theorem 2.1), this relaxation is also strict [22]. As the set of Commutator Bilocal Quantum Distribution is the natural approximation of the set of Tensor Bilocal Quantum Distribution when only considering commutation algebraic relations of the PVMs and states, it is tempting to interpret our result as generalising the proof of convergence of the standard NPA hierarchy. However, we believe that this affirmation lacks justifications for the following three reasons, all associated with a problem we leave open:

- Restricting to finite-dimension Hilbert spaces, the standard NPA hierarchy recovers the physical set of distributions (i.e. the tensor and commutator definitions become equivalent). We were unable to prove a similar result for our hierarchy (see Appendix H).

- We wish to find a stronger definition for the scalar extension hierarchy (see Appendix C) such that a proof of convergence to the set of Commutator Bilocal Quantum Distributions can be achieved. As a related problem and an alternative solution, we seek a hierarchy that can be solved by SDP which is equivalent to our bilocal factorisation hierarchy.

- A third inflation-NPA hierarchy introduced in [49] can also be used to approximate the set of physical correlations in the bilocal scenario (see Section 4.1). The relation of it with our factorisation bilocal hierarchy remains to be clarified, and we conjecture that inflation-NPA hierarchy provides a stronger (possibly equivalent) test than our factorisation bilocal test.

- These three hierarchies can be generalised to the case of larger network scenarios (see Figure 3). In the triangle scenario, we show that the generalised inflation-NPA hierarchy is strictly more accurate than the generalisation of our two hierarchies (which are trivial as no factorisation constraint exists anymore, see Section 4.2).
The characterisation of the convergence of the generalised inflation-NPA hierarchy is a significant open problem in the theory of quantum correlations.

We conclude our paper in Section 5 by discussing the formulation of quantum information theory in terms of a commutator postulate. We first remark that if the inflation-NPA hierarchy is stricter than our two hierarchies for the bilocal scenario (problem we left open), then redefining quantum theory by postulating that Commutator Bilocal Quantum Distributions are the correlations feasible in the bilocal scenario would imply that inflation-NPA hierarchy is not an allowed method to characterise quantum correlations in the bilocal scenario.

At last, we explain in Appendix F why the consideration of other networks might give even stronger arguments against this commutator-based postulate, which could result in a quantum theory that is either a noncausal theory, or does not allow to consider independent copies of systems.

2. Tripartite quantum distributions and standard NPA hierarchy

2.1. Overview

As a preliminary background and to fix the notation, we briefly discuss the standard NPA hierarchy introduced by Navascués, Pironio, and Acín [31] for outer approximation of the quantum distribution obtainable in the standard tripartite Bell scenario in Figure 1a.

We first introduce the set of Tensor Tripartite Quantum Distributions, which are all the behaviours feasible in this scenario according to quantum theory:

**Definition 2.1** (Tensor Tripartite Quantum Distributions). Let $\tilde{P} = \{p(abc|xyz)\}$ be a three-party probability distribution. We say that $\tilde{P}$ is a Tensor Tripartite Quantum Distribution iff there exist a composite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, a projector over a pure state $\tau$ acting over $\mathcal{H}$, and PVMs $\{\hat{A}_{a|x}\}$ over $\mathcal{H}_A$, $\{\hat{B}_{b|y}\}$ over $\mathcal{H}_B$ and $\{\hat{C}_{c|z}\}$ over $\mathcal{H}_C$ such that

$$p(abc|xyz) = \text{Tr}_\tau\left(\hat{A}_{a|x} \otimes \hat{B}_{b|y} \otimes \hat{C}_{c|z}\right).$$

The outer approximations of this set are given by the NPA hierarchy method, which we now sketch. First, given that $\tilde{P}$ is a Tensor Tripartite Quantum Distribution with $p(abc|xyz) = \text{Tr}_\tau\left(\hat{A}_{a|x} \otimes \hat{B}_{b|y} \otimes \hat{C}_{c|z}\right)$, one can construct the $n \geq 3$ order
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moment matrix $\Gamma^n$ as:

$$
\begin{bmatrix}
\begin{array}{cccc}
A_{a|x} & B_{b|y} & C_{c|z} & A_{a|x}A_{a'|x'} & A_{a|x}B_{b|y} & \cdots \\
1 & & & \text{Tr}_\tau(\hat{A}_{a|x}\hat{B}_{b|y}) & & \\
(A_{a|x})^\dagger & \text{Tr}_\tau(\hat{A}_{a|x}\hat{B}_{b|y}) & & & \text{Tr}_\tau(\hat{B}_{b|y}) & \\
(B_{b|y})^\dagger & & \text{Tr}_\tau(\hat{B}_{b|y}) & & & \\
(C_{c|z})^\dagger & & & \text{Tr}_\tau(\hat{A}_{a'|x'}\hat{A}_{a|x}\hat{B}_{b|y}) & & \\
(A_{a|x}A_{a'|x'})^\dagger & & & & & \\
(A_{a|x}B_{b|y})^\dagger & & & & & \\
\cdots & & & & & \\
\end{array}
\end{bmatrix}
$$

(2.1)

where the lines and columns are labelled by all formal monomials in $\{A_{a|x}, B_{b|y}, C_{c|z}\}$ of length at most $n$, and the entries are the trace over $\tau$ of the product of the monomials labelling the entries (we only indicated some of the coefficients and omitted tensor products for readability, and used the relation $\hat{B}_{b|y}^2 = \hat{B}_{b|y}$).

$\Gamma^n$ satisfies several properties:

(i) $\Gamma^n_{1,1} = 1$.

(ii) $\Gamma^n$ is symmetric positive.

(iii) $\Gamma^n$ satisfies several linear constraints, such as $\Gamma^n_{(A_{a|x})^\dagger, B_{b|y}} = \Gamma^n_{\|, A_{a|x}B_{b|y}}$. Moreover, $\Gamma^n_{\|, A_{a|x}B_{b|y}C_{c|z}} = p(abc|xyz)$: we say that $\Gamma^n$ is compatible with $\tilde{P}$.

Hence, if $\tilde{P}$ is a Tensor Tripartite Quantum Distribution, then for all $n \geq 3$ there exists a moment matrix $\Gamma^n$ compatible with $\tilde{P}$. This provides a hierarchy of outer approximations of the set of Tensor Tripartite Quantum Distributions. Given $\tilde{P}$ and some $n \geq 3$, if there is no moment matrix $\Gamma^n$ compatible with $\tilde{P}$, then $\tilde{P}$ is not a Tensor Tripartite Quantum Distributions. Importantly, the problem of finding such $\Gamma_n$ is an SDP, which can be solved on a computer, with certificates of nonexistence.

A higher hierarchy level $n$ provides more accurate tests (as a compatible matrix $\Gamma^{n-1}$ can always be extracted from $\Gamma^n$). However, as this hierarchy is only sensitive to the algebraic relations between the operators (such as $\hat{B}_{b|y}^2 = \hat{B}_{b|y}$ or $[\hat{A}_{a|x}, \hat{C}_{c|z}] = 0$), this hierarchy of outer approximation does not exactly characterise the set of Tensor Tripartite Quantum Distributions. It was shown to converge to the larger set of Commutator Tripartite Quantum Distributions ([31]):

Theorem 2.1 (Convergence of the standard NPA hierarchy). Suppose that $\tilde{Q} = \{q(abc|xyz)\}$ is a probability distribution for Alice, Bob, and Charlie. Then $\tilde{Q}$ passes all the standard NPA hierarchy tests if $\tilde{Q}$ is a Commutator Tripartite Quantum Distribution,

where the Commutator Tripartite Quantum Distributions are defined as:
**Definition 2.2** (Commutator Tripartite Quantum Distributions). \( \mathcal{Q} \) is a **Commutator Tripartite Quantum Distribution** iff there exist a Hilbert space \( \mathcal{H} \), a projector over a pure state \( \tau \), and PVMs \( \{ \hat{A}_{a|x} \} \), \( \{ \hat{B}_{b|y} \} \) and \( \{ \hat{C}_{c|z} \} \) over \( \mathcal{H} \) such that

(i) \[ q(abc|xyz) = \text{Tr} \tau \left( \hat{A}_{a|x} \hat{B}_{b|y} \hat{C}_{c|z} \right) \]

(ii) \[ \forall \hat{A}_{a|x}, \hat{B}_{b|y}, \hat{C}_{c|z} \text{, we have } [\hat{A}_{a|x}, \hat{B}_{b|y}, \hat{C}_{c|z}] = [\hat{C}_{c|z}, \hat{A}_{a|x}] = 0. \]

Any Tensor Tripartite Quantum Distribution is trivially a Commutator Tripartite Quantum Distribution\(^3\). The converse, known as Tsirelson’s problem, was first conjectured but turned out to be false (see [22], where it was also proven that the set of physical Tensor Tripartite Quantum Distributions is not computable). Nevertheless, for finite-dimensional quantum systems, these two definitions are equivalent, thanks to an inductive result of Tsirelson’s Theorem (see Theorem H.1 and Remark H.2 in Appendix H).

### 2.2. Noncommutative polynomial formulation

Given a probability distribution \( \mathcal{Q} \), the NPA hierarchy tests the existence of operators in a Hilbert space, allowing us to express \( \mathcal{Q} \) as a Commutator Tripartite Quantum Distribution. Hence, this hierarchy cannot be formalised using operators or Hilbert spaces, as such objects may not exist. We now present the standard **canonical abstraction** procedure capable of performing this formalisation of the NPA hierarchy (based on noncommutative polynomials, see [8]), introducing abstract **letters** for each potential PVMs of the parties.

We introduce the alphabets \( a = (a_1, \ldots, a_I) \), \( b = (b_1, \ldots, b_J) \), and \( c = (c_1, \ldots, c_K) \) respectively composed of the letters \( \{ A_{a|x} \} \), \( \{ B_{b|y} \} \), and \( \{ C_{c|z} \} \), one letter for each PVMs \( \{ \hat{A}_{a|x} \} \), \( \{ \hat{B}_{b|y} \} \), and \( \{ \hat{C}_{c|z} \} \) (\( I, J, K \) are the respective numbers of Alice, Bob, and Charlie PVMs operators).

The products of multiple measurement operators then correspond to combinations of letters \( l_i \in a \cup b \cup c \), referred to as words denoted by bold Greek letters \( \alpha \), forming the set \( \langle a, b, c \rangle \). A word consisting of the letters in \( a \) is written as \( \alpha \), forming the set \( \langle a \rangle \). Similarly, we write \( \beta \in \langle b \rangle \) associated with the alphabet \( b \) and \( \gamma \in \langle c \rangle \) associated with \( c \). \( 1 \) is the empty word with no letters.

We further impose the algebraic relations satisfied by the measurement operators on the letters. More precisely, as the measurement operators are projectors, measurement operators of distinct parties commute together and are self-adjoint, and hence

\(^3\)The new operators \( \hat{A}_{a|x}, \hat{B}_{b|y}, \hat{C}_{c|z} \) are constructed as the original ones, padding with identities over the other local Hilbert spaces
we impose
\[ l_i^2 = l_i, \]
\[ [a_i, b_j] = [a_i, c_k] = [b_j, c_k] = 0, \]
\[ l_i^\dagger = l_i, (l_i l_j)^\dagger = l_j^\dagger l_i^\dagger = l_j l_i, \]

where the operation \( \dagger \) is an involution on the letters of a word, which stands for the conjugate transpose operation on operators. These rules imply that the words \( \alpha, \beta, \gamma \) commute with each other. Therefore, any word \( \omega \) admits a unique minimal form \( \omega = \alpha \beta \gamma \) where no letter is squared. Equality over words is checked in this minimal form. The length of \( \omega \), denoted by \( |\omega| \), is the number of letters in this minimal form.

The entries in the moment matrix \( \Gamma^n \) are indexed with the words \( \omega \) such that \( |\omega| \leq n \). Additionally, the ring of noncommutative polynomials \( \mathbb{T}^{abc} = \mathbb{R}(a, b, c) \) arises; see Appendix A.1.

3. Bilocal quantum distribution and two converging hierarchies

This section contains the main results of our paper. We first introduce the two sets of Bilocal Quantum Tensor Distributions and Bilocal Quantum Commutator Distributions (see Figure 1b and Section 3.1). Then, we introduce the factorisation bilocal NPA hierarchies and show its convergence to the set of Bilocal Quantum Commutator Distributions (Section 3.2). Afterwards, we provide a finite-level stopping criterion for the two hierarchies, corresponding to finite-dimensional Hilbert spaces (Section 3.3). At last, we introduce the much weaker scalar extension bilocal NPA hierarchy and show its partial convergence to Bilocal Quantum Commutator Distributions (Appendix C), under a very impractical scalar property assumption (Remark C.2).

3.1. Bilocal quantum distribution

Here, we first introduce the set of Bilocal Quantum Tensor Distributions (Definition 3.1), which according to quantum theory is the set of correlations that can be obtained in the bilocal scenario, in which two independent quantum sources are distributed to three parties (see Figure 1b). Then, we introduce the set of Bilocal Quantum Commutator Distributions (Definition 3.2), which is an outer approximation of this set, which will be characterised by our two hierarchies introduced in the next sections.

According to quantum theory, the set of correlations that can be obtained in the bilocal scenario is
Definition 3.1 (Tensor Bilocal Quantum Distributions). Let $\tilde{P} = \{p(abc|xyz)\}$ be a three-party probability distribution. We say that $\tilde{P}$ is a Tensor Bilocal Quantum Distribution iff there exist a composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_{BL} \otimes \mathcal{H}_{BR} \otimes \mathcal{H}_C$, two projectors over a pure state $\rho_{ABL}$ acting over $\mathcal{H}_A \otimes \mathcal{H}_{BL}$ and $\sigma_{BRC}$ acting over $\mathcal{H}_{BR} \otimes \mathcal{H}_C$, and some PVMs $\{\hat{A}_{a|x}\}$ over $\mathcal{H}_A$, $\{\hat{B}_{b|y}\}$ over $\mathcal{H}_{BL} \otimes \mathcal{H}_{BR}$, and $\{\hat{C}_{c|z}\}$ over $\mathcal{H}_C$ such that

$$p(abc|xyz) = \text{Tr}_{\rho_{ABL} \otimes \sigma_{BRC}}(\hat{A}_{a|x} \otimes \hat{B}_{b|y} \otimes \hat{C}_{c|z}).$$

As for the standard NPA hierarchy, we will only consider the abstract algebraic commutativity relations among the states and measurement operators. This leads to the following definition of Commutator Bilocal Quantum Distribution.

Definition 3.2 (Commutator Bilocal Quantum Distributions). Let $\tilde{Q} = \{q(abc|xyz)\}$ be a three-party probability distribution. We say that $\tilde{Q}$ is a Commutator Bilocal Quantum Distribution iff there exist a Hilbert space $\mathcal{H}$, a projector over a pure state $\tau$, two projectors (possibly infinite trace) $\rho$ and $\sigma$, and some PVMs $\{\hat{A}_{a|x}\}$, $\{\hat{B}_{b|y}\}$, and $\{\hat{C}_{c|z}\}$ over $\mathcal{H}$ such that

(i) $\tau = \rho \cdot \sigma = \sigma \cdot \rho$,
(ii) $q(abc|xyz) = \text{Tr}_\tau(\hat{A}_{a|x} \hat{B}_{b|y} \hat{C}_{c|z})$,
(iii) $[\hat{A}_{a|x}, \hat{B}_{b|y}] = [\hat{B}_{b|y}, \hat{C}_{c|z}] = [\hat{C}_{c|z}, \hat{A}_{a|x}] = 0$,
(iv) $[\hat{A}_{a|x}, \sigma] = [\rho, \hat{C}_{c|z}] = 0$,

for all $\hat{A}_{a|x}, \hat{B}_{b|y}, \hat{C}_{c|z}$.

The set of Tensor Bilocal Quantum Distributions of Definition 3.1 is trivially contained in the set of Commutator Bilocal Quantum Distributions of Definition 3.2. Due to [22], the converse is wrong, and the physical set of Tensor Bilocal Quantum Distributions is not computable. We were unable to prove the equality or not between these two sets for finite Hilbert spaces and leave this question open (see Appendix H).

Remark 3.1. In Definition 3.2, we ask that $\tau$ is a projector over a pure state, which can be deduced from the fact that $\rho, \sigma$ are projectors and $\text{Tr}(\tau) = 1$. In Definition 2.1, we already asked that $\rho, \sigma$, hence $\tau$, should be projectors over pure states. Note that with tensor products this condition is not necessary: due to purification, it is sufficient to ask that these states are density operators. Indeed, if $\tilde{P}$ satisfies the conditions of Definition 2.1 but with $\rho, \sigma$ only positive and trace 1, one can always enlarge the Hilbert spaces to obtain that $\tilde{P}$ satisfies the conditions of Definition 2.1 with purifications $\rho', \sigma'$ of $\rho, \sigma$ which are now projectors over pure states. In Appendix F.2, we show that such a purification procedure is no longer possible in the commutator context. In
other words, the density operator (mixed state) formulation is *not* equivalent to the pure state formulation in the commutator-based quantum model.

### 3.2. Factorisation bilocal NPA hierarchy

We now introduce the factorisation bilocal NPA hierarchy (Definitions 3.3 and 3.4) and show its convergence to the set of Commutator Bilocal Quantum Distributions (Theorem 3.2).

We use the canonical abstraction formalism of Section 2.2, with alphabets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in the letters $\{A_a|x\}$, $\{B_b|y\}$, and $\{C_c|z\}$, which are in one-to-one correspondence with the sets of PVMs $\{\hat{A}_{a|x}\}$, $\{\hat{B}_{b|y}\}$, and $\{\hat{C}_{c|z}\}$, respectively. Note that in the bilocal scenario, for $\hat{\alpha}$–$\hat{\gamma}$ respectively monomials in the PVMs of Alice and Charlie, with $\tau = \rho_{ABL} \otimes \sigma_{BRC}$, we have nonlinear constraints of the form

$$\text{Tr}_\tau(\hat{\alpha}\hat{\gamma}) = \text{Tr}_\tau(\hat{\alpha}) \cdot \text{Tr}_\tau(\hat{\gamma}).$$

This motivates the following definition:

**Definition 3.3** (Factorisation Bilocal Moment Matrix). Fix $n \in \mathbb{N}$, let $\Gamma^n$ be a square matrix indexed by all words $\omega \in \langle a, b, c \rangle$ of length $|\omega| \leq n$. We say that $\Gamma^n$ is a *Factorisation Bilocal Moment Matrix of order* $n$ if

1. $\Gamma^n_{1,1} = 1$.
2. $\Gamma^n$ is positive.
3. it satisfies all the linear constraints
   $$\Gamma^n_{\omega,\nu} = \Gamma^n_{\omega',\nu'}$$
   whenever $\omega^\dagger \nu = \omega'^\dagger \nu'$.
4. it satisfies the additional nonlinear factorisation constraints
   $$\Gamma^n_{\alpha,\gamma} = \Gamma^n_{\alpha,1} \cdot \Gamma^n_{1,\gamma},$$
   where $\alpha \in \langle a \rangle$ and $\gamma \in \langle c \rangle$.

We further say that $\Gamma^n$ is compatible with the tripartite distribution $\tilde{Q}$ iff $\Gamma^n_{1,Aa|x} \rho_{Bb|y} \sigma_{Cc|z} = q(abc|xyz)$. An infinite matrix $\Gamma^\infty$ is said to be a *Factorisation Bilocal Moment Matrix* iff all of its principal extracted matrices are Factorisation Bilocal Moment Matrices of some finite order.

Note that if one drops condition (iv), this definition becomes the standard NPA Moment Matrix of Section 2.1. It is straightforward to show that any Bilocal Tensor Quantum Distribution admits a Factorisation Bilocal Moment Matrix $\Gamma^n$ for all $n$. This yields the following hierarchy:
**Definition 3.4** (Factorisation bilocal NPA hierarchy). Let $\mathcal{Q} = \{q(abc|xyz)\}$ be a tripartite probability distribution. We say that $\mathcal{Q}$ passes factorisation bilocal NPA hierarchy if for all integers $n \geq 3$, there exists a Factorisation Bilocal Moment Matrix $\Gamma^n$ of order $n$ that is compatible with $\mathcal{Q}$.

We are ready for our first main result.

**Theorem 3.2** (Convergence of the factorisation bilocal NPA hierarchy). Let $\mathcal{Q} = \{q(abc|xyz)\}$ be a tripartite probability distribution. Then $\mathcal{Q}$ passes all the factorisation bilocal NPA hierarchy tests iff $\mathcal{Q}$ is a Commutator Bilocal Quantum Distribution.

**Proof.** We present here a sketch of the proof that is detailed in the Appendix B.1. It is straightforward to show that if $\mathcal{Q}$ is a Commutator Bilocal Quantum Distribution, then it admits a compatible Factorisation Bilocal Moment Matrix of any order (the nonlinear constraint (iv) is derived from the projectivity and commutativity of $\rho$ and $\sigma$ and the properties of the operator trace).

For the converse direction, suppose there exists a compatible Factorisation Bilocal Moment Matrix $\Gamma^n$ for each $n \geq 3$. The first part of our proof is similar to the proof of convergence of the standard NPA hierarchy (see [31]). We first construct a compatible infinite-size factorisation bilocal moment matrix $\Gamma^\infty$ by extracting a convergent sequence from the set of compatible $\Gamma^n$. Then we construct the Hilbert space $\mathcal{H}$ and the operators $\tau = |\phi_1\rangle\langle\phi_1|$, $\hat{A}_{a|x}$, $\hat{B}_{b|y}$, $\hat{C}_{c|z}$ through a Gelfand–Naimark–Segal (GNS) representation from $\Gamma^\infty$. The constructed operators are, respectively, a state and PVMs that automatically satisfy constraints (ii) and (iii) in Definition 3.2.

In the last part of our proof, we construct the operators $\rho$, $\sigma$. Remarking that $\rho$ should commute with all polynomials in all operators $\hat{y}$ associated with a word $\gamma \in \langle\mathcal{C}\rangle$, we construct $\rho$ as the orthogonal projector on the subspace $V_{B_RC}$ generated by all these polynomials in $\hat{y}$. We similarly define the operator $\sigma$ as the orthogonal projector on the subspace $V_{A_BL}$ generated by all polynomials in the operators $\hat{a}$. Note that both $V_{B_RC}$ and $V_{A_BL}$ contain $|\phi_1\rangle$. We orthogonally decompose them as $V_{B_RC} = \text{Span} \{|\phi_1\rangle\} \bigoplus W_{B_RC}$ and $V_{A_BL} = \text{Span} \{|\phi_1\rangle\} \bigoplus W_{A_BL}$. Then, we show that the factorisation condition (iv) in the moment matrices imposes that the two complement subspaces are orthogonal to each other, that is, $W_{A_BL} \perp W_{B_RC}$.

As $\rho$ and $\sigma$ are, respectively, orthogonal projectors over $V_{B_RC}$ and $V_{A_BL}$, this directly implies that they commute and their product is the projector over $\text{Span} \{|\phi_1\rangle\}$, that is, $\tau$, which proves constraints (i) of Definition 3.2. The last constraint (iv) is proven similarly using this orthogonality relation, which concludes the proof.

Although Theorem 3.2 gives a convergence to Commutator Bilocal Quantum Distributions, as the name suggests, it does not support the application of semi-definite programming (SDP) due to the nonlinearity in the constraints (iv) of Definition 3.3.
In other words, while Theorem 3.2 provides a theoretical characterisation, its practical usefulness to characterise bilocal quantum distributions is unclear. We solve this problem in the next section, considering a new hierarchy in which new scalar variables linearising these constraints are added.

3.3. Stopping criteria and finite-dimensionality

So far, we have shown that the factorisation bilocal NPA hierarchy characterises the set of Commutator Bilocal Quantum correlations (Definition 3.2) in the asymptotic limit. The reconstructed Hilbert space $\mathcal{H}$ is a priori infinite-dimensional, and there is no promise in the speed of convergence of these two hierarchies. In this section, we give a stopping criterion for our hierarchy, and show that it is associated with a finite-dimensional reconstructed Hilbert space $\mathcal{H}$. Our stopping criterion is similar to the one for the NPA hierarchy given in [31]. However, as we will discuss later in Appendix H, we do not prove that the set of finite-dimensional Hilbert space Tensor Bilocal Quantum correlations is recovered: we will leave this question open.

**Definition 3.5** (Rank loop). For any $N \geq 3$, the Factorisation Moment Matrix $\Gamma^N$ is said to have a rank loop if

$$\text{rank}(\Gamma^{N-1}) = \text{rank}(\Gamma^N).$$

The rank loop condition is defined similarly for Scalar Extension Bilocal Matrix $\Omega^n$.

**Theorem 3.3** (Stopping criteria). Let $\tilde{Q} = \{q(abc|xyz)\}$ be a tripartite probability distribution. The following are equivalent:

(i) $\tilde{Q}$ admits a finite-dimensional Commutator Bilocal Quantum representation.

(ii) $\tilde{Q}$ admits a Factorisation Bilocal Moment Matrix $\Gamma^N$ which has a rank loop for some $N \in \mathbb{N}$.

**Proof.** We give a sketch proof with the detailed proof in Appendix B.2.

It is straightforward to prove that (i) $\implies$ (ii). From a Commutator Bilocal quantum representation in a Hilbert space of finite dimension $d$, one can construct a sequence of Factorisation Bilocal Moment Matrices $\Gamma^n$. Then, the sequence of $\text{rank}(\Gamma^n)$ is increasing (as $\Gamma^{n-1}$ is a submatrix of $\Gamma^n$) and is upper bounded by $d$, which implies the rank loop.

Next, let us give a sketch proof of (ii) $\implies$ (i). Assume that $\Gamma^N$ admits a rank loop. Following the proof of the stopping criteria for the standard NPA case, from $\Gamma^N$ we construct a finite-dimensional Hilbert space $\mathcal{H}$ with operators $\tau, \hat{A}_{a|x}, \hat{B}_{b|y}, \hat{C}_{c|z}$, which satisfy the desired properties thanks to the rank loop condition. Next, we
introduce the operator \( \rho \) as the projector on the subspace generated by all polynomials taking the variables \( \hat{\gamma} \) associated with all words in \( \gamma \in \langle \mathcal{E} \rangle \) such that \( |\gamma| \leq N \). We similarly define \( \sigma \) as the projector on the subspace generated by polynomials evaluating in \( \hat{\alpha} \) such that \( |\alpha| \leq N \). Applying the factorisation constraints, we show that \( \rho \) and \( \sigma \) satisfy the conditions in Definition 3.2.



4. Inflation-NPA hierarchy and generalisation to other Networks

In this section, we first introduce a third hierarchy, which can be used to approximate quantum distributions in the bilocal scenario: the inflation-NPA hierarchy of [49]. This hierarchy was obtained by combining the standard inflation technique [19, 30, 50] to the NPA method [31], see [27] for an attempt to characterise its convergence. We prove that any distribution compatible with this inflation-NPA method is a Commutator Bilocal Quantum Distribution (i.e. it is compatible with the two hierarchies introduced in Section 3), and leave the question of equality or strict inclusion (in full generality or in the finite-dimensional case) between these two sets open.

Then, we introduce more general network scenarios (see Figure 3) and discuss the generalisations of factorisation, scalar extension, and inflation-NPA hierarchies to these scenarios. We remark that for the triangle scenario, the factorisation and scalar extension hierarchies are trivial (they correspond to the standard NPA hierarchy), while the NPA inflation hierarchy is nontrivial, showing a clear gap in the two sets which can be identified by these hierarchies in the context of generic network scenarios.

4.1. Inflation-NPA hierarchy

Consider a Tensor Bilocal Quantum Distribution \( \tilde{P} \) (Definition 3.1). Following [49], we now show that \( \tilde{P} \) can be associated with a new two-parameter sequence of increasing size moment matrices \( \Xi^{n,m} \) satisfying some additional symmetry constraints. This can be used to construct a new hierarchy capable of showing that some given distribution cannot be a Tensor Bilocal Quantum Distribution, called the inflation-NPA hierarchy (also called quantum inflation hierarchy). We then prove that this method is more (or equally) precise than the two factorisation and scalar extension hierarchies.

4.1.1. Inflation-NPA Hierarchy for the Bilocal scenario. Let \( \tilde{P} \) be a Tensor Bilocal Quantum Distribution: there exist a composite Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_{BL} \otimes \mathcal{H}_{BR} \otimes \mathcal{H}_C \) and operators such that

\[
p(abc|xyz) = \text{Tr}_{\rho_{ABL} \otimes \sigma_{BR} \mathcal{C}} \left( \hat{A}_{a|x} \otimes \hat{B}_{b|y} \otimes \hat{C}_{c|z} \right).
\]
Consider two integers \( n \geq 3, m \geq 1 \). Introducing \( m \) copies of the Hilbert spaces and copies of the operators in these Hilbert spaces, we now construct a new inflation-NPA moment matrix \( \Xi^{n,m} \) that satisfies some SDP constraints, notably the linear symmetry conditions corresponding to the fact that the copies should be interchangeable.

We introduce \( m \) independent copies of each Hilbert spaces, called \( \mathcal{H}^i_A, \mathcal{H}^j_B, \mathcal{H}^k_C \). We consider copies of all PVMs measurement operators in every possible (combination of) Hilbert space. More precisely, we consider \( m \) operators \( \{ \hat{A}^i_{a^i|x^i} \} \) (resp. \( \{ \hat{C}^l_{c^l|z^l} \} \), \( m^2 \) operators \( \{ \hat{B}^{i,j}_{b^i,j|y^i,j} \} \) acting on Hilbert spaces \( \mathcal{H}^i_B \otimes \mathcal{H}^j_C \). We also consider \( m \) independent copies of \( \rho_{AB} \) and \( \sigma_{BC} \), denoted by \( \rho^i_{AB} \) acting on \( \mathcal{H}^i_A \otimes \mathcal{H}^i_B \) and \( \sigma^i_{BC} \) acting on \( \mathcal{H}^i_B \otimes \mathcal{H}^i_C \). We introduce \( \tau_m \) the global state tensor product of all states, \( \mathcal{H}^i = \mathcal{H}^i_A \otimes \mathcal{H}^i_B \otimes \mathcal{H}^i_C \) the \( i \)th diagonal spaces, \( \hat{B}^{i,j}_{b^i,j|y^i,j} \) the \( i \)th diagonal PVM and \( \mathcal{H} \) the global Hilbert space tensor product of all Hilbert spaces.

In the following, to simplify the notation, we leave the inputs and the outputs of the operators implicit (e.g. \( A^i \) should be thought of as \( A^i_{a^i|x^i} \)). As in the standard NPA case, we also identify an operator over a local Hilbert space with the operator acting over the full Hilbert space padding with identity operators, which allows to replace all tensor products with simple products\(^4\).

Physically, \( \Xi^{n,m} \) corresponds to the “standard” moment matrix of a new hypothetical scenario in which the parties now have access to several copies of the sources and decide which one to measure according to the additional input \( i \) for \( A, (j, k) \) for \( B \) and \( l \) for \( C \) (see Figure 2), to which we additionally impose symmetry conditions deduced from the interchangeability of the copies of the states. More precisely, \( \Xi^{n,m} \) is indexed by words \( \omega, \nu \) (of size \( \leq n \)) composed of letters \( \{ A^i \}, \{ B^{j,k} \}, \{ C^l \} \), and its coefficients are \( \Xi^{n,m}_{\omega,\nu} = \text{Tr}_{\tau_m} (\hat{\omega}^{\dagger} \hat{\nu}) \). It satisfies four types of conditions:

1. \( \Xi_{1,1}^{n,m} = 1 \) and \( \Xi^{n,m} \) is symmetric positive.
2. Commutation of operators in different Hilbert spaces implies that \( \Xi^{n,m} \) satisfies several linear constraints (e.g. \( A^1 \) commutes with \( A^2 \) and \( B^{1,1} \) commutes with \( B^{2,3} \) but not with \( B^{1,2} \)).
3. The invariance of \( \tau_m \) under permutations \( \theta, \theta' \in S_m \times S_m \) of the copies of \( \rho_{AB} \) and \( \sigma_{BC} \) implies new linear constraints on \( \Xi^{n,m} \). Acting with \( \theta \) (resp. \( \theta' \)) over \( \tau_m \) is equivalent to permuting the index \( i, j \) (resp. \( k, l \)) in

\(^4\text{e.g., } A^i \text{ acting over } \mathcal{H}^i_A \text{ is identified with } A^i \otimes 1 \text{ acting over } \mathcal{H} \text{ where the identity acts over the tensor product of all Hilbert spaces except } \mathcal{H}^i_A \text{. Then, for instance, the operator } A^i \otimes A'^i \otimes C^l \text{ is identified with } A^i \cdot A'^i \cdot C^l \).
Figure 2. Physical scenario corresponding to the inflation technique (to simplify, inputs $x, y, z$ and outputs $a, b, c$ are omitted). In case a distribution $\bar{\mathcal{P}}$ is obtained in the bilocal scenario, one can consider the new (thought) scenario in which the sources are duplicated, and each party is given additional inputs $i, j, k,$ and $l$. The parties apply the measurement specified in the original scenario to the copies of the states given by these extra inputs. Here, we represented the case where $A$ performs here measurement $A_{a|x}$ over her share of state $\rho^1$, corresponding to the new operator $A_{a|x}^1$. Similarly, here Bob measures operator $B_{b|y}^{12}$ over his shares of states $\rho^1, \sigma^2$ and Charlie measures $C_{c|z}^1$ over his share of state $\sigma^2$. Note that in that case Charlie measures a state not considered by the other parties: his behavior factorises from the rest, which is imposed by the condition (iv) of the main text. Moreover, as the states and PVMs are copies of each other, the behavior should be invariant under $S_2 \times S_2$ permutation group, as specified by condition (iii) of the main text. Condition (iv) corresponds to the situation in which all parties measure all diagonal operators, that is Alice measures $A_{a|x}^1$ and $A_{a|x}^2$, Bob measures $B_{b|y}^{11}$ and $B_{b|y}^{22}$, Charlie measures $C_{c|z}^1$ and $C_{c|z}^2$. $\Xi_{n,m}^{n,m}$ is the moment matrix associated to this scenario, satisfying conditions (iii) and (iv) due to the specificities of this scenario where states and operators are duplicated.

Moreover, note that when only diagonal operators $A^i, B^{i,j}, C^l$ are involved, the trace over $\tau_m$ factorises as a product of $m$ traces over each diagonal space $\mathcal{H}^i$. Hence $\Xi_{n,m}^{n,m}$ satisfies $\Xi_{n,m}^{n,m}_{1,\prod_{i=1}^{m} A_iB^{i,j}C^l} = \prod_{i=1}^{m} \sum_{a^i b^i j^l c^l} p(a^i b^i j^l c^l) | x^i y^l j^l c^l \rangle \langle x^i y^l j^l c^l |$ with $\bar{\mathcal{P}}^5$. We say that $\Xi_{n,m}^{n,m}$ is compatible with $\bar{\mathcal{P}}$.

As for the NPA and our previous hierarchies, the existence of a hierarchy of inflation-NPA moment matrices $\Xi_{n,m}^{n,m}$ compatible with any tensor bilocal distribution $\bar{\mathcal{P}}$ can be used as a way to test whether some given distribution $\bar{\mathcal{Q}}$ has such a model. If for some $n, m$, there does not exist any $\Xi_{n,m}^{n,m}$ compatible with $\bar{\mathcal{Q}}$ (this is an SDP problem), then $\bar{\mathcal{Q}}$ is not a Tensor Bilocal Quantum Distribution.

\footnote{Physically, this terms corresponds to the parties $m$ parallel implementations of the initial experiment, one for each $(\rho^i, \sigma^i)$.}
Remark 4.1. Note that a moment matrix constructed from a Tensor Bilocal Quantum Distribution $\Xi_{n-m}^{n,m}$ satisfies additional nonlinear factorisation constraints. More precisely, for some words $\alpha^i$, $\beta^i$, $\gamma^i$ in the letters $\{A^i\}$, $\{B^i\}$ and $\{C^i\}$, we have the factorisation constraint $\Xi_{1,1}^{n,m} \alpha^i \beta^i \gamma^i = \prod_i \Xi_{1,1}^{n,m} \alpha^i \beta^i \gamma^i$. This condition can be imposed in the tests, but it is not practical, as it results in non-SDP problems. It is likely that these conditions are imposed anyway when $m, n$ go to infinity (the proof would be related to the de Finetti-like theorems). A formal proof of convergence of the inflation-NPA hierarchy could first exploit these additional nonlinear conditions.

4.1.2. Commutator quantum distribution and inflation-NPA hierarchy. We now wish to understand the relation between the inflation-NPA hierarchy and the factorisation bilocal hierarchy. As in Remark 4.1, there might exist a de Finetti-like argument implying that $\Xi_{n-m}^{n,m} \alpha^i \beta^i \gamma^i = \prod_i \Xi_{1,1}^{n,m} \alpha^i \beta^i \gamma^i$, in as $m, n$ go to infinity. Thus we conjecture such an asymptotic factorisation property of inflation moment matrices should be true, and hence below relation of inflation-NPA hierarchy with factorisation bilocal hierarchy would also follow.

Conjecture 4.2. Consider a distribution $\tilde{Q} = \{q(abc|xyz)\}$. If $\tilde{Q}$ passes the bilocal inflation-NPA hierarchy, then it also passes the factorisation, that is, it is a commutator bilocal distribution.

Showing the converse of Conjecture 4.2, or finding a counterexample, is also open (also restricting to finite-dimensional Hilbert spaces). The difficulty lies in the off-diagonal measurements of Bob. That is, given a Factorisation Moment Matrix $\Gamma^m$, it is natural to define $\Gamma_{1,1}^m \alpha \beta \gamma = \Xi_{1,1}^{n,m} \alpha^i \beta^i \gamma^i$. With the $S_m \times S_m$ symmetry and diagonalisation constraint, we are able to define many entries of $\Xi_{n,m}^{n,m}$. However, entries such as $\Xi_{1,1}^{n,m} \beta^i \gamma^i$ cannot be obtained in this way. With a tensor product structure, the structure $\mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}$ can be used; but this is not possible with only the commutator relations.

We now prove a much weaker result, that is, if $\tilde{Q}$ is detected as infeasible by the scalar extension hierarchy (see Appendix C), then the inflation-NPA hierarchy will also detect it.

Theorem 4.3. Consider a distribution $\tilde{Q} = \{q(abc|xyz)\}$. The existence of $\Xi_{n,m}^{n,m}$ compatible with $\tilde{Q}$, where $m$ is sufficiently large, implies the existence of a scalar extension moment matrix $\Omega^n$. That is, if $\tilde{Q}$ passes the bilocal inflation-NPA hierarchy, then it also passes the scalar extension bilocal NPA hierarchy.

Proof. We first identify the PVMs of the first diagonal space with $\{\hat{A}_{a|x}\}$, $\{\hat{B}_{b|y}\}$, $\{\hat{C}_{c|z}\}$ and suppose $m$ to be of order $d^n$, where $d$ is the number of operators in $\{\hat{A}_{a|x}\}$. For each distinct word $A^i$, we assign a distinct $i \geq 2$ and identify the associated scalar extension with $A^i$ from the $i$-th diagonal space. Then, it is straightforward to construct a Scalar Extension Bilocal Moment Matrix. See Appendix E.1 for more details.
Figure 3. Some generalised network scenario: the triangle scenario (a) and the four party star network (b).

However, as discussed in Appendix C, the scalar extension bilocal hierarchy does not converge to the set of commutator bilocal quantum correlations, hence the result above does not clarify the relation between inflation-NPA and factorisation hierarchy.

4.2. Generalisation to other network scenarios

Up to now, we have considered quantum correlations which can be obtained in the three-parties Bell and Bilocal scenarios as in Figure 1. Quantum theory can also be used to define what correlations can be obtained in more complex scenarios, such as the four party star network or the triangle scenario of Figure 3, and all the three hierarchies we introduced can be generalised to these scenarios.

Let us now discuss the generalisation of our results to these generalised scenarios. We will first remark that the generalisation of our factorisation and scalar extension hierarchies is sometimes strictly less efficient to the inflation-NPA hierarchy, providing a clear gap in the case of the triangle scenario. Then, we evoke the potential generalisation of our main result to the star network scenario.

Let us first concentrate on the triangle scenario of Figure 3c, where the correlations predicted by quantum theory are:

**Definition 4.1** (Tensor Triangle Quantum Distributions). Let $\tilde{P} = \{p(abc|xyz)\}$ be a three-party probability distribution. We say that $\tilde{P}$ is a **Tensor Triangle Quantum Distribution** iff there exist a composite Hilbert space $\mathcal{H}_{AL} \otimes \mathcal{H}_{AR} \otimes \mathcal{H}_{BL} \otimes \mathcal{H}_{BR} \otimes \mathcal{H}_{CL} \otimes \mathcal{H}_{CR}$, projector over pure states $\rho_{ARBL}$ acting over $\mathcal{H}_{AR} \otimes \mathcal{H}_{BL}$, $\sigma_{BRCL}$ acting...
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over $\mathcal{H}_{BR} \otimes \mathcal{H}_{CL}$, $\pi_{CRAL}$ acting over $\mathcal{H}_{CR} \otimes \mathcal{H}_{AL}$, and some PVMs $\{\hat{A}_{a|x}\}$ over $\mathcal{H}_{AL} \otimes \mathcal{H}_{AR}$, $\{\hat{B}_{b|y}\}$ over $\mathcal{H}_{BL} \otimes \mathcal{H}_{BR}$, and $\{\hat{C}_{c|z}\}$ over $\mathcal{H}_{CL} \otimes \mathcal{H}_{CR}$ such that

$$p(abc|xyz) = \text{Tr}_{\rho_{ARBL} \otimes \sigma_{BRCL} \otimes \pi_{CRAL}} \left( \hat{A}_{a|x} \otimes \hat{B}_{b|y} \otimes \hat{C}_{c|z} \right).$$

The generalisation of our factorisation or scalar extension hierarchies to this new set of correlation is trivial. Indeed, no factorisation condition can be imposed anymore, as any pair of parties is sharing a quantum source. Hence, they consist of testing the existence of a hierarchy of moment matrices subject to the same conditions as in the standard NPA hierarchy. Then, they are unable to distinguish triangle quantum correlations from the correlation that can be obtained in the standard tripartite Bell scenario of Figure 1a.

In particular, the shared random bit distribution (in which the parties coordinately output 0 or 1 with probability 1/2, there are no inputs) is clearly feasible in the standard tripartite Bell scenario and hence cannot be detected as infeasible by the generalised factorisation or scalar extension hierarchies. However, we prove in Appendix E.2 that the inflation-NPA shows that it is infeasible in the triangle scenario. This proves that in the context of generic networks, the two generalised factorisation and scalar extension hierarchies can detect strictly fewer distributions as infeasible compared to the inflation-NPA hierarchy.

We also discuss in Appendix G a potential generalisation of our main results to the case of the star network scenario of Figure 3a.

5. Conclusion

Let us conclude our paper with a discussion on the tensor and commutator alternatives for defining quantum correlations. Throughout our paper, we introduced the correlations predicted by quantum theory in various networks using tensor-based definitions (Definition 2.1, 3.1 and 4.1). Then, we introduced several commutator-based relaxations of these definitions (Definition 2.2, 3.2, see also Definition F.1 of Appendix F). We presented the tensor-based definitions as the one postulated by quantum theory and the commutator-based definitions as nonphysical (but practical) relaxations.

However, this categorisation is still debated [46]. In the case of standard Bell scenarios, the commutator-based distributions of Definition 2.2 is sometimes taken as the set of correlations postulated by quantum theory, tensor-based distributions becoming an inner relaxation of this now physical set. Note that, as the two are known to

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For instance, [31] introduces the two definitions on equal footage, and then adopt the commutator one as the one postulated by quantum theory.
be equivalent for finite-dimensional Hilbert spaces (even though different for general ones [22]), the choice of a tensor-based or a commutator-based postulate may not result in fundamentally different predictions for realistic experiments.

This choice becomes more problematic once the bilocal scenario in which two independent sources are prepared is considered, assuming that the postulates should include a rule to encode this independent preparation\(^7\). Our tensor-based Definition 2.1 is the standard way to introduce quantum bilocal correlations. We are not aware of any concrete proposal with a commutator-based postulate, but see our commutator-based Definition 2.2 as a natural candidate. Adopting it may have significant consequences, even in the finite-dimensional case. Indeed, assume now that there exists a (maybe finite-dimensional Hilbert space) Commutator Bilocal Quantum Distribution \(\tilde{Q}_0\) rejected by the inflation-NPA hierarchy\(^8\). As \(\tilde{Q}_0\) is feasible in the bilocal scenario according to our new commutator-based postulate, the existence of such \(\tilde{Q}_0\) would imply that the inflation-NPA hierarchy is not an allowed method to characterise quantum correlations in networks. We explain in Appendix F why the consideration of other networks might give even stronger arguments against this commutator-based postulate, which could imply that quantum theory is either a noncausal theory, or does not allow one to consider independent copies of systems.

**Errata in the earlier version on the scalar extension hierarchy**

A few months after making our first manuscript available online, we were contacted by Laurens T. Ligthart and David Gross who recently demonstrated the equivalence among several bilocal quantum models and the completeness of inflation-NPA hierarchy in bilocal scenario [28]. They also answered some of the questions we proposed in this manuscript.

Thanks to their observation, we realised a mistake in our proof that scalar extension bilocal hierarchy converges to the set of all commutator bilocal quantum correlations (see Appendix C in the previous arXiv version), which we have removed. The current discussion of the scalar extension hierarchy (see Appendix C) contains much weaker results. The discussion of the relations between inflation-NPA hierarchy and our factorisation hierarchy is edited accordingly (see Section 4.1.2).

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\(^7\)Note that this is not necessary: in practise the tripartite Bell scenario formalism captures the bilocal scenario, as the two sources can be modelled by a unique global source, hence a formalism with no rule for independent preparation is possible.

\(^8\)We left that question open, but we conjecture it exists. Note that if not, our two hierarchies are equivalent (maybe only in finite-dimensional Hilbert spaces) to the inflation-NPA hierarchy for the bilocal scenario, but not for the triangle scenario.
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A. A formal construction of noncommutative polynomials

In this section, we provide a more mathematically rigorous construction of noncommutative polynomials such as $T_{abc}$ and $T_{a}^{abc}$. We also discuss the relation between the moment matrix and the linear functional. Lastly, we discuss the difference between our work and the existing results.

A.1. Noncommutative polynomials generated by finite alphabets

Fixed some $I \in \mathbb{N}$, we consider an alphabet $a = (a_1, \cdots, a_I)$ consisting of letters $a_i$ for $i = 1, \cdots, I$. The reduced monoid generated by the alphabet $a$ is denoted by $\langle a \rangle$ with reducibility condition

$$a_i^2 = a_i$$

for each letter $a_i$. All elements of $\langle a \rangle$ are called words and are assumed to be in minimal forms; that is, there is no word with two equal consecutive letters. The empty word is denoted by $1$. The length of the word $\omega$, denoted by $|\omega|$, is the number of letters in the word. The ring of noncommutative polynomials associated with $a$ is then written as $\mathbb{T}^a = \mathbb{R}\langle a \rangle$.

We now endow a $\star$-structure on $\mathbb{T}^a$ by introducing an involution\footnote{In mathematical literature, this involution is usually denoted by $\dagger$. Instead, we use $\dagger$ to simplify the notation and note the intuition of the conjugate transpose.} such that $\forall i, j = 1, \cdots, I$,

$$\mathbb{R}^\dagger = \mathbb{R}, a_i^\dagger = a_i,$$

$$(a_ia_j)^\dagger = a_j^\dagger a_i^\dagger = a_ja_i.$$ 

The subset of polynomials that are invariant under involution is called symmetric and is denoted by $\text{Sym}\mathbb{T}^a = \{ p \in \mathbb{T}^a \mid p^\dagger = p \}$. It follows that $\mathbb{T}^a = \mathbb{R}\langle a \rangle$ is a $\star$-algebra.

To make sense of the terminology “polynomial”, we need a notion of evaluation. Fix a finite-dimensional Hilbert space $\mathcal{H}$ with a set of variables $\{A = (A_1, \ldots, A_I) \mid A_i \in \mathcal{B}(\mathcal{H})\}$. Define evaluation for any $p \in \mathbb{T}^a_a$ to be the matrix polynomial $p(A)$, by replacing each letter $a_i$ with the operator $A_i$, where $\hat{a}$ is the operator corresponding to the word $\alpha$.

Example A.1. Consider $a = (a_1, a_2, a_3)$, and $p = a_1a_2a_3 + a_2^2 \in \mathbb{T}^a$. For evaluation, we may choose the Hilbert space of a qubit with the variables being Pauli matrices.
\[ \tilde{\sigma} = (\sigma_x, \sigma_y, \sigma_z). \] Whence,

\[ p_+ (\tilde{\sigma}) = \sigma_x \sigma_y \sigma_z + \sigma_y^2 \]

We can easily generalise the above definition to the case of having multiple alphabets. For our purposes, it suffices to consider only two more alphabets \( b = (b_1, \cdots, b_J) \) and \( c = (c_1, \cdots, c_K) \) for some \( J, K \in \mathbb{N} \). In this case, the elements of the reduced monoid \( \langle a, b, c \rangle \) are again called words, assuming the minimal forms due to the additional reducibility conditions

\[ b_j^2 = b_j, \ c_k^2 = c_k \]

for each letter \( b_j \) and \( c_i \). Moreover, we require all letters from different alphabets to commute with each other, that is,

\[ [a_i, b_j] = [a_i, c_k] = [b_j, c_k] = 0, \]

for any \( i = 1, \cdots, I, j = 1, \cdots, J, \) and \( k = 1, \cdots, K \). The ring of bilocal noncommutative polynomials is defined as \( \mathbb{T}^{abc} = \mathbb{R}\langle a, b, c \rangle \) with the same \( \star \)-structure, where the superscripts of \( \mathbb{T} \) are used to refer to the alphabets used. Given any Hilbert space \( \mathcal{H} \), the evaluation can be generalised by introducing more variables \( \{ B = (B_1, \ldots, B_J) \mid B_j \in \mathcal{B}(\mathcal{H}) \} \) and \( \{ C = (C_1, \ldots, C_L) \mid C_k \in \mathcal{B}(\mathcal{H}) \} \).

We remark that both \( \mathbb{T}^a \) and \( \mathbb{T}^{abc} \) are topological vector spaces; see Appendix A.4. The significance of this observation is subtle yet important, which allows for the application of the Banach–Alaoglu Theorem that shall be needed.

As a convention, we use bold Greek letters to refer to words, while bold letters such as \( p \) to refer to noncommutative polynomials. In particular, \( \alpha \) refers to the words in \( \langle a \rangle \), \( \beta \) for \( \langle b \rangle \), and \( \gamma \) for \( \langle c \rangle \). Incidentally, any word \( \omega \in \langle a, b, c \rangle \) can be uniquely decomposed \( \omega = \alpha \beta \gamma \). This coincides with our discussion in Section 2.2, and, by construction, the canonical abstraction of any Commutator Tripartite Quantum Distribution gives rise to the ring \( \mathbb{T}^{abc} \).

A.2. Scalar extension polynomials

Consider the alphabet \( a \). For any word \( \alpha \in \langle a \rangle \), we define scalar extension of \( \alpha \) as \( \kappa_{\alpha} \), imposing the commutation condition:

\[ [\kappa_{\alpha}, \alpha'] = [\kappa_{\alpha}, \kappa_{\alpha'}] = 0 \quad \forall \alpha, \alpha' \in \langle a \rangle. \]

Despite being fully commuting, note that \( \kappa_{\alpha \alpha'} \neq \kappa_{\alpha' \alpha} \) in general. Denote the infinite alphabet generated by the scalar extension of \( a \) by \( \overline{A} \), and let \( \overline{A}^n \) be the finite alphabet
consisting of the letters $\kappa_\alpha$ with $|\alpha| \leq n$. The elements of the free monoid $\langle \kappa_A \rangle$ are said to be pure scalar words and are generally represented by $\kappa$. The length of the word $\kappa$ is defined as the number of letters in the word, denoted by $|\kappa|$.

Observe also that, different from the reduced monoid $\langle a \rangle$, in the free monoid $\langle \kappa_A \rangle$ we have $\kappa_2^2 \neq \kappa_\alpha$ in general.

Following the commutation relations, define the commutative ring of pure scalar extension polynomials associated with $a$ as $T_a = \mathbb{R}[\kappa_\alpha, \alpha \in \langle a \rangle]$. We define the ring of scalar extension polynomials as $T_a = T_a \langle \alpha \rangle$.

Similarly to the previous section, we can generalise $T_a$ to the case of having multiple alphabets. In particular, given the alphabets $b$ and $c$, define the ring of bilocal scalar extension polynomials to be $T_{ab}^c = T_a \langle a, b, c \rangle$, while imposing commutativity

$$[\kappa_\alpha, \beta] = [\kappa_\gamma, \gamma] = 0.$$ It is clear that $T_a^a$ contains $T_a$ and $T_{ab}^c$ contains $T_{ab}^c$. The $*$-structure can then be extended to $T_a^a$ and $T_{ab}^c$ by requiring $T_a^1 = T_a$.

Note that any word in $\langle a, \kappa_A \rangle$ is of the form $\kappa_\alpha$. Analogously, any word in $\langle a, b, c, \kappa_A \rangle$ is of the form $\kappa_\alpha \beta \gamma$.

We can evaluate the polynomials in $T_a^a$. Fix a finite-dimensional Hilbert space $\mathcal{H}$ with a set of variables $\{A = (A_1, \ldots, A_I) \mid A_i \in \mathcal{B}(\mathcal{H})\}$, and choose a normalised positive operator $\tau \in \mathcal{B}(\mathcal{H})$. Define $\tau$-evaluation for any $p \in T^a_a$ to be the matrix trace polynomial $p_\tau(A)$, by replacing each letter $a_i$ with operator $A_i$ and each $\kappa_\alpha$ with the number $\text{Tr}(\tau a_i) = \text{Tr}(\tau (a_\alpha))$, where $a_\alpha$ is the operator corresponding to the word $\alpha$. The generalisation to $T_{ab}^c$ is straightforward.

Here is an example of a $\tau$-evaluation.

**Example A.2.** Consider $a = (a_1, a_2, a_3)$. We have $p = \kappa_a^2 + \kappa_2^2 a_2 \in T_a$ and $q = a_1^2 \kappa_a^2 a_2^2 a_3 \in T_a^a$. For evaluation, we may choose the Hilbert space of a qubit with the variables being Pauli matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Take $\tau$ as an arbitrary rank-1 projector. Whence,

$$p_\tau(\vec{\sigma}) = \text{Tr}(\tau \sigma_x) \text{Tr}(\tau \sigma_x \sigma_z) + \text{Tr}(\tau \sigma_y)^2$$

$$q_\tau(\vec{\sigma}) = \sigma_x \text{Tr}(\tau \sigma_x) \text{Tr}(\tau \sigma_y) \sigma_z.$$}

Finally, due to Appendix A.4, both $T^a_a$ and $T_{ab}^c$ are topological vector spaces. As mentioned before, this observation is important because it allows us to apply the Banach–Alaoglu Theorem.

**Remark A.3.** We can generalise the notion of scalar extension to more alphabets. For example, we can enlarge $T_{ab}^c$ via the scalar extension of $c$, which is identified by $\nu$. Using subscripts to refer to alphabets that admit scalar extensions, define
\[ T_{ac} = \mathbb{R}[\kappa_\alpha, \nu_\gamma], \alpha \in \langle a \rangle, \gamma \in \langle c \rangle \) and \( \mathbb{T}_{a}^{abc} = T_{ac} \langle a, b, c \rangle \). In addition, we need the commutation relations \( [\kappa_\alpha, \nu_\gamma] = 0 \).

Also, note that it is possible to generalise \( \tau \)-evaluation to noncommutative polynomials with multiple scalar extensions by considering a tensor product of Hilbert spaces and associating scalar extension operators to suitable partial traces. In the context of bilocal scenarios, this extra scalar extension does not provide any new information.

### A.3. Moment matrix and unital symmetric linear functional

We discuss the equivalence between moment matrices, such as \( \Gamma^n \) in Section 2.1, and linear functionals of \( \mathbb{T}^{abc} \). The discussion here has an obvious generalisation to \( \mathbb{T}_a^{abc} \).

Without loss of generality, we work with \( \mathbb{T}_a^{2d} = \mathbb{R}\langle a \rangle \) for convenience. Fix \( d \in \mathbb{N} \), denote \( \mathbb{T}_a^{2d} \) as all polynomials with words of length \( \leq 2d \).

A matrix \( H \) is indexed by words of length \( \leq d \) is said to satisfy noncommutative Hankel condition if

\[ H_{\omega, \nu} = H_{\omega', \nu'}, \text{ whenever } \omega^\dagger \nu = \omega'^\dagger \nu'. \]

The matrix \( H \) indexed by words of length \( \leq d \) is a moment matrix of order \( d \), also known as Hankel matrix, if it satisfies the noncommutative Hankel condition, is positive semi-definite, and \( H_{1,1} = 1 \). We note that this coincides with the properties of the moment matrix \( \Gamma^n \) in the standard NPA hierarchy, which justifies the terminology “moment matrix”.

To study the properties of \( \mathbb{T}^{abc} \), a natural object to consider is its dual, which is the space of its linear functionals. In particular, we focus on unital symmetric positive linear functional \( L : \mathbb{T}_{\leq 2d}^d \rightarrow \mathbb{R} \). A linear functional \( L \) is said to be unital if \( L(1) = 1 \) and symmetric if \( L(p) = L(p^\dagger) \) for all \( p \) in the domain of \( L \). If \( L(p^\dagger p) \geq 0 \) for all \( p \in \mathbb{T}_{\leq 2d}^d \), we say that \( L \) is positive.

These two objects are, in fact, equivalent. Given a unital symmetric positive linear functional \( L : \mathbb{T}_{\leq 2d}^d \rightarrow \mathbb{R} \), we can define a moment matrix \( H \) of order \( d \) via formula \( H_{\omega, \nu} = L(\omega^\dagger \nu) \), and vice versa. To conclude the equivalence, the only nontrivial thing that needs to be checked is the positiveness.

Indeed, for any two polynomials \( p = \sum_{\omega} p_\omega \omega, q = \sum_{\nu} q_\nu \nu \in \mathbb{T}_{\leq 2d}^d \), we obtain two vectors \( \vec{p} \) and \( \vec{q} \) given by the real coefficients \( p_\omega \) and \( q_\nu \). We then compute

\[ L(p^\dagger q) = \sum_{\omega, \nu} p_\omega^\dagger q_\nu L(\omega^\dagger \nu) = \sum_{\omega, \nu} p_\omega q_\nu H_{\omega, \nu} = \vec{p}^T H \vec{q}, \]

and the equivalence follows.

We note that a general unital symmetric positive linear functional \( L \) corresponds only to a moment matrix without extra constraints (other than the Hankel conditions).
To recover constraints such as in Definition 3.3, we require the positivity of $L$ on some suitable subsets of the symmetric polynomials.

### A.4. Noncommutative polynomials form topological vector spaces

We show that both $T_a^a$ and $T_a^{abc}$ are topological vector spaces, which implies that $T_a^a$ and $T_a^{abc}$ are also topological vector spaces. Let us focus on the case of $T_a^a$, as the subsequent generalisation is rather straightforward. While the vector space structure is clear, we need to endow a reasonable topology. To this end, we seek a suitable norm for $T_a^a$.

Recalling the notion of $\tau$-evaluation in Section A.2, we endow the topology with the following norm: for any $p \in T_a^a$, let

$$k_p = \max \{ k : \langle A \rangle_{\tau}^{\tau} A_k \leq 1, \forall i = 1, \cdots, n \}$$

where $\text{Sym}_k^n(\mathbb{R})$ is the set of $n$-tuples of $k \times k$ symmetric real matrices (the variables) and $\|A\|_{\text{op}}$ is the usual matrix norm.

This is indeed a norm: Both the triangular inequality and the absolute homogeneity are clear from the definition. Positive-definiteness is due to the fact that no non-zero trace polynomial vanishes on matrices of all finite sizes. The topological structure follows.

### B. Detailed proofs for convergence of bilocal NPA hierarchy

In this section, we prove in detail that both the NonSDP bilocal NPA hierarchy and the scalar extension bilocal NPA hierarchy converge to the Commutator Bilocal Quantum Distribution. In addition, we show that the rank loop condition indicates finite stopping criteria.

#### B.1. Proof for factorisation bilocal NPA hierarchy

Let us first prove Theorem 3.2.

**Proof.** We start with the easier direction, given any $n \geq 3$ and suppose that we have a Commutative Bilocal Quantum Distribution $\bar{Q}$ as in Definition 3.2. By purity we assume $\tau = |\phi_1\rangle\langle\phi_1|$ for some normalised vector $|\phi_1\rangle \in \mathcal{H}$. We now wish to show the factorisation bilocal NPA hierarchy of order $n$ is satisfied for all $n \geq 3$. 

To this end, we use the canonical abstraction and define the $n$th order moment matrix $\Gamma^n$ through $\Gamma^n_{\omega, \nu} = \text{Tr}_\tau(\hat{\omega} \hat{\nu})$, where $\hat{\omega}$ and $\hat{\nu}$ are the operators corresponding to the words $\omega, \nu \in \langle a, b, c \rangle$.

By construction, $\Gamma^n$ clearly satisfies the linear constraints and recovers $\tilde{Q}$ with the appropriate choice of words. Positivity is ensured since $\Gamma^n$ is a Gram matrix. We only need to check the nonlinear factorisation constraint. Let $\alpha$ and $\gamma$ be words in $h_a i$ and $h_c i$, respectively. We compute

$$\text{Tr}_\tau(\hat{\alpha} \tau \hat{\gamma}) = \text{Tr}\left(\tau \hat{\alpha} \tau \hat{\gamma}\right)$$

where the commutativity and cyclicity of the trace are applied several times. Then

$$\Gamma^n_{\alpha, \gamma} = \text{Tr}_\tau(\hat{\alpha} \tau \hat{\gamma})$$

as desired.

Conversely, we construct a GNS representation of $\mathcal{T}^{abc}$ given the existence of Factorisation Bilocal Moment Matrix $\Gamma^n$ of order $n$ for any natural number $n \geq 3$, exactly as in [31]. We then construct the operators $\rho$ and $\sigma$ with the help of nonlinear factorisation constraints.

We only provide a quick sketch for the GNS representation here since it was explicitly done in [31].

According to the Banach–Anaoglu Theorem, there exists a subsequence $\{\Gamma_n^m\}$ of $\Gamma^n$ that converges to a limit $\Gamma^\infty$ in the weak-$\star$ topology. In particular, the convergence is pointwise, i.e. $\lim_{i \to \infty} \Gamma_{\omega, \nu}^n = \Gamma^\infty_{\omega, \nu}$ for all $\omega, \nu$, and hence the limit $\Gamma^\infty$ satisfies all linear constraints and nonlinear factorisation constraints. Let $\Gamma_N \in \mathbb{R}^{N \times N}$ denote the extracted submatrix of $\Gamma^\infty$ indexed by words of length $\leq N$, it follows that $\Gamma_N^\infty$ is positive for all $N$. 
Now we apply a sequential Cholesky decomposition of the matrices $\Gamma_1^N$, which realises $\Gamma^\infty$ as a Gram matrix of an infinite family of vectors $\{|\phi_\omega\rangle\}$ indexed by all words $\omega$ and

$$\Gamma^\infty_{\omega,\nu} = \langle \phi_\omega | \phi_\nu \rangle.$$ 

The desired Hilbert space $\mathcal{H}$ is then defined as the closure of the linear span of $\{|\phi_\omega\rangle\}$, with respect to the inner product induced by the above formula. Let $|\phi_1\rangle = |v^{00}\rangle$ denote the vector corresponding to the null word 1, we define the global state $\tau = |\phi_1\rangle \langle \phi_1| = |v^{00}\rangle \langle v^{00}|$ which is pure.

Now, for each word $\omega = \alpha \beta \gamma \in \langle a, b, c \rangle$, we can construct the corresponding operators with the same decomposition $\hat{\omega} = \hat{\alpha} \hat{\beta} \hat{\gamma}$, where the commutation relation

$$[\hat{\alpha}, \hat{\beta}] = [\hat{\alpha}, \hat{\gamma}] = [\hat{\beta}, \hat{\gamma}]$$

is satisfied. For each letter $a_i \in a$, the operator $\hat{a}_i$ is a Hermitian projector acting on $\mathcal{H}$ via the formula

$$\hat{a}_i |\phi_\nu\rangle = |\phi_{\hat{a}_i \nu}\rangle,$$

and meeting all standard NPA hierarchy constraints. Similarly to the letters $b_j \in b$ and $c_k \in c$.

To construct operators $\rho$ and $\sigma$, define $V_{ABL} = \text{Span} \{ P(|\hat{\alpha}\rangle) \}_\rho$ as the subspace generated by all polynomials that only evaluate operators $\hat{\alpha}$. Clearly, the space $V_{ABL}$ is not empty since it contains $|v^{00}\rangle$. Starting from $|v^{00}\rangle$, we use the Gram–Schmidt process to construct an orthogonal basis $\{ P_i(|\hat{\alpha}\rangle) |v^{00}\rangle = |v^{i0}\rangle, i \in I \}$ for some polynomials $P_i$ and some countable index set $I$. We similarly introduce $V_{BCR} = \text{Span} \{ Q(|\hat{\gamma}\rangle) \}_Q$ with the orthogonal basis $\{ Q_j(|\hat{\gamma}\rangle) |v^{00}\rangle = |v^{0j}\rangle, j \in J \}$ for some countable index set $J$. We may assume $0 \in I \cap J$. Now, we define

$$\rho = \sum_j |v^{0j}\rangle \langle v^{0j}|,$$

$$\sigma = \sum_i |v^{i0}\rangle \langle v^{i0}|. \quad (B.1) \quad (B.2)$$

It remains to check that both $\rho$ and $\sigma$ satisfy the desired properties in Definition 3.2. Clearly we have $\rho^2 = \rho$, $\sigma^2 = \sigma$.

We claim that $\langle v^{i0}|v^{0j}\rangle = 0$ for all $i \in I \setminus \{0\}$ and $j \in J \setminus \{0\}$. Indeed, if $i, j \neq 0$, we may assume $P_i(|\hat{\alpha}\rangle) = \sum_{\hat{\alpha}} p_i^{\hat{\alpha}} \hat{\alpha}$ and $Q_j(|\hat{\gamma}\rangle) = \sum_{\hat{\gamma}} q_j^{\hat{\gamma}} \hat{\gamma}$, for some real coefficients $p_i^{\hat{\alpha}}$ and $q_j^{\hat{\gamma}}$. Then

$$\langle v^{i0}|v^{0j}\rangle = \langle v^{00}|P_i(|\hat{\alpha}\rangle)^\dagger \cdot Q_j(|\hat{\gamma}\rangle)|v^{00}\rangle$$
where both nonlinear factorisation constraint and orthogonality are applied. It follows that

$$\rho \cdot \sigma = \sum_{ij} |v^{0j}| \langle v^{0j} | v^{0i} \rangle \langle v^{0i} | v^{00} \rangle = |v^{00}\rangle \langle v^{00} | v^{00} \rangle = \tau = \sigma \cdot \rho.$$ 

Finally, we check $[\rho, \mathcal{H}] = [\mathcal{H}, \sigma] = 0$. Observe that the support of $[\rho, \mathcal{H}]$ is within $V_{BC}$ and

$$\langle v^{0j}|[\mathcal{H}, \rho]|v^{0j'} \rangle = \sum_{k} \langle v^{0j}|\mathcal{H}|v^{0k}\rangle \langle v^{0k}|v^{0j'} \rangle - \sum_{k} \langle v^{0j}|v^{0k}\rangle \langle v^{0k}|v^{0j'} \rangle$$

for all $j, j' \in J$. It follows that $[\rho, \mathcal{H}] = 0$. The analogous argument with the basis on $V_{AB}$ shows that $[\mathcal{H}, \sigma] = 0$. It is straightforward that Definition 3.2.(iv) is true, which finishes the proof.

**Remark B.1.** In mathematical notation, the GNS representation we constructed in the above proof is the triple $(\mathcal{H}, \pi, \tau)$, where the $\star$-representation

$$\pi: T^{abc} \rightarrow \mathcal{H}$$

$$\omega \mapsto \hat{\omega}$$

satisfies all the desired NPA constraints. The state (in the functional analysis sense) $\tau$ is defined by the linear completion of the formula

$$\tau(\hat{\omega}) = \tau(\pi(\omega)) = \Gamma_{1,\omega}^\infty = \langle \phi_1|\phi_\omega \rangle,$$

which is in bijective correspondence to $|\phi_1\rangle \langle \phi_1|$, the projector over the pure state $|\phi_1\rangle$. 
B.2. Proof for hierarchies with rank loop

In this section, we prove Theorem 3.3, which is restated for readability.

**Theorem B.2 (Stopping criteria).** Let $\mathcal{Q} = \{q(abc|xyz)\}$ be a tripartite probability distribution. The following are equivalent:

(i) $\mathcal{Q}$ admits a finite-dimensional Commutator Bilocal Quantum representation.

(ii) $\mathcal{Q}$ admits a Factorisation Bilocal Moment Matrix $\Gamma^N$ which has a rank loop for some $N \in \mathbb{N}$.

**Proof.** Assume (i), that is, we consider $\mathcal{P}$ having a finite-dimensional commutator representation: there exists a Hilbert space $\mathcal{H}$ of dimension $d \leq 1$, a pure state with decomposition $\tau = |\phi_1\rangle\langle\phi_1| = \rho \cdot \sigma = \sigma \cdot \rho$, and PVMs $\{A_{a|x}\}, \{B_{b|y}\}$, and $\{C_{c|z}\}$ such that the conditions of Definition 3.2 are satisfied. We need to prove (i) $\implies$ (ii), that is, $\mathcal{P}$ admits a Factorisation Bilocal Moment Matrix $\Gamma^N$ which has a rank loop for some $N \in \mathbb{N}$.

For any $n \geq 3$, we construct the alphabets as in Definition 3.4 and define the moment matrix of order $n$ through

$$
\Gamma^n_{\omega,\nu} = \text{Tr}_\tau(\hat{\omega}\hat{v}) = \langle \phi_1 | \hat{\omega}^\dagger \hat{v} | \phi_1 \rangle,
$$

for length $|\omega|, |\nu| \leq n$.

Note that $\text{rank}(\Gamma^n) \leq d$ since for any $\omega$ with length $\leq n$, $\text{Span} \{\hat{\omega} | \phi_1 \} |_{|\omega| \leq n}$ has dimension $\leq d$. Additionally, by construction $\Gamma^n$ is compatible with $\mathcal{P}$. It is also a submatrix of $\Gamma^{n+1}$. Hence

$$
\text{rank}(\Gamma^n) \leq \text{rank}(\Gamma^{n+1}) \leq d < \infty.
$$

It follows that for some sufficiently large $N$ there exists a Factorisation Moment Matrix $\Gamma^N$ such that

$$
\text{rank}(\Gamma^{N-1}) = \text{rank}(\Gamma^N).
$$

Let us now prove that (ii) $\implies$ (i). Suppose that we have $\Gamma^N$ that satisfies the rank loop condition and $\text{rank}(\Gamma^N) = d$. Similarly to Section B.1, we perform a Cholesky decomposition to realise that $\Gamma^N$ is a Gram matrix given by

$$
\Gamma^N_{\omega,\nu} = \langle \phi_\omega | \phi_\nu \rangle
$$

for a finite family of vectors $\{|\phi_\omega\rangle\} |_{|\omega| \leq N}$. We define the Hilbert space

$$
\mathcal{H} = \text{Span} \{|\phi_\omega\rangle\} |_{|\omega| \leq N}
$$
with the inner product induced by the above equation. It follows that \( \dim(\mathcal{H}) = d \).

Let \( |\phi_1\rangle \) be the vector associated with the empty word 1 and let \( \tau = |\phi_1\rangle\langle \phi_1| \). We construct the operator \( \hat{\omega} = \hat{\alpha} \hat{\beta} \hat{\gamma} \) corresponding to \( \omega = \alpha \beta \gamma \) in the same way as in Section B.1. For each letter \( a_i \in A, b_j \in B, \) and \( c_k \in C \), the associated operators are self-adjoint.

Now we check the commutativity of \( \hat{a}_i, \hat{b}_j, \) and \( \hat{c}_k \), and subsequently \( \hat{\alpha}, \hat{\beta}, \) and \( \hat{\gamma} \). Since we only have words defined up to length \( N \), we let \( \ldots \chi \ldots \) be arbitrary words of length \( N \) and calculate

\[
\langle \phi_\omega | [\hat{a}_i, \hat{b}_j] | \phi_\omega \rangle = \Gamma^N_{a_i \omega, b_j \nu} - \Gamma^N_{b_j \omega, a_i \nu} = 0,
\]
due to linear constraints and self-adjointness. Similarly, we have

\[
\langle \phi_\omega | [\hat{a}_i, \hat{c}_k] | \phi_\omega \rangle = \langle \phi_\omega | [\hat{b}_j, \hat{c}_k] | \phi_\omega \rangle = 0.
\]

Note that, in general, these equations are not enough to show the commutativity, as we also need the information from \( f_j \phi_\omega | \chi \rangle = \Gamma^N \). But with the rank loop condition \( \text{rank} \Gamma^{N-1} = \text{rank} \Gamma^N \) we may write

\[
\mathcal{H} = \text{Span} \{|\phi_\omega\rangle\}_{|\omega| \leq N} = \text{Span} \{|\phi_\omega\rangle\}_{|\omega| \leq N-1}.
\]

That is, the vectors in RHS also span \( \mathcal{H} \) and thus the support of \( \hat{a}_i, \hat{b}_j, \) and \( \hat{c}_k \). The commutation relation follows.

Let us now construct the operators \( \rho \) and \( \sigma \). As before, define

\[
V_{ABL} = \text{Span} \{P(\{\hat{a}\})\}_P = \text{Span} \{P_i(\{\hat{a}\}) | v^{0i}_0 \rangle = | v^{i0}_0 \rangle, i \in I\}
\]

for some index set \( I \) and

\[
V_{BRC} = \text{Span} \{Q(\{\hat{\gamma}\})\}_Q = \text{Span} \{Q_j(\{\hat{\gamma}\}) | v^{0j}_0 \rangle = | v^{0j}_0 \rangle, j \in J\}
\]

for some index set \( J \). Then we define

\[
\rho = \sum_j | v^{0j}_0 \rangle \langle v^{0j}_0 | \quad (B.5)
\]
\[
\sigma = \sum_i | v^{i0}_0 \rangle \langle v^{i0}_0 | \quad (B.6)
\]

As subspaces of \( \mathcal{H} \), both \( V_{ABL} \) and \( V_{BRC} \) are of dimension \( \leq N \), and thus \( \rho, \sigma \) are finite-dimensional operators.

Again, \( | v^{i0}_0 \rangle \) and \( | v^{0j}_0 \rangle \) are orthogonal if \( i, j \neq 0 \). Suppose that \( P_i(\{\hat{a}\}) = \sum_\alpha p^{i}_\alpha \hat{a} \) and \( Q_j(\{\hat{\gamma}\}) = \sum_\gamma q^{j}_\gamma \hat{\gamma} \) for some real coefficients \( p^{i}_\alpha \) and \( q^{j}_\gamma \). We compute

\[
\langle v^{i0}_0 | v^{0j}_0 \rangle = \langle v^{i0}_0 | P_i(\{\hat{a}\})^\dagger \cdot Q_j(\{\hat{\gamma}\}) | v^{0j}_0 \rangle
\]
\[
= \sum_{\hat{\alpha}, \hat{\gamma}} (p^{i}_{\hat{\alpha}})^{\dagger} q^{j}_{\hat{\gamma}} \cdot \langle v^{00} | \hat{\alpha}^{\dagger} \hat{\gamma} | v^{00} \rangle \\
= \sum_{\hat{\alpha}, \hat{\gamma}} (p^{i}_{\hat{\alpha}})^{\dagger} q^{j}_{\hat{\gamma}} \cdot \Gamma_{\alpha, \gamma}^{N} \\
= \sum_{\hat{\alpha}, \hat{\gamma}} (p^{i}_{\hat{\alpha}})^{\dagger} q^{j}_{\hat{\gamma}} \cdot \Gamma_{\alpha, 1}^{N} \cdot \Gamma_{1, \gamma}^{N} \\
= \sum_{\hat{\alpha}, \hat{\gamma}} (p^{i}_{\hat{\alpha}})^{\dagger} q^{j}_{\hat{\gamma}} \cdot \langle v^{00} | \hat{\alpha}^{\dagger} | v^{00} \rangle \langle v^{00} | \hat{\gamma} | v^{00} \rangle \\
= \langle v^{00} | P_{i}(\{\hat{\alpha}\})^{\dagger} | v^{00} \rangle \langle v^{00} | Q_{j}(\{\hat{\gamma}\}) | v^{00} \rangle \\
= \langle v^{00} | v^{00} \rangle \langle v^{00} | v^{00} \rangle = 0.
\]

Then the same argument as in Section B.1 shows that \( \rho, \sigma \) satisfy Definition 3.2.

C. Scalar extension technique and linear bilocal NPA hierarchy

We now show that in some (not yet fully characterised) cases, the nonlinearity issue of the factorisation bilocal NPA hierarchy can be circumvented thanks to the scalar extension method proposed in [35]. We introduce the scalar extension bilocal NPA hierarchy (Definition C.1 and C.2), and discuss its connection to the set of Commutator Bilocal Quantum Distributions (Theorem C.1).

As we will see in this section, the main idea of the scalar extension method is to introduce, for any word \( \alpha \) corresponding to the operator \( \hat{\alpha} \), its scalar extension letter \( \kappa_{\alpha} \) that ideally corresponds to the scalar operator \( \text{Tr}_{\tau}(\hat{\alpha}) \mathbb{1}_{H} \). We then construct the Scalar Extension Moment Matrix \( \Omega^{n} \) with all words of length \( \leq n \) written from the alphabets \( a, b, c \) and \( \kappa_{A} \), the set of all scalar extension letters (truncating \( \kappa_{A} \) to keep it finite). We will require \( \kappa_{\alpha} \) to commute with all other letters and with the condition that \( \Omega^{n}_{1, \alpha} = \Omega^{n}_{1, \kappa_{\alpha}} \).

Now, suppose that the reconstructed operators \( \hat{k}_{\alpha} \) from the GNS-representation (introduced in the proof of Theorem 3.2) are indeed proportional to the identity. Then \( \Omega^{n}_{1, \alpha} = \Omega^{n}_{1, \kappa_{\alpha}} \) imposes that \( \hat{k}_{\alpha} = \text{Tr}_{\tau}(\hat{\alpha}) \mathbb{1}_{H} \), and condition \( \Omega^{n}_{1, \alpha, \gamma} = \Omega^{n}_{1, \kappa_{\alpha} \gamma} \) corresponds to a linearised version of Equation 3.1 rewritten as \( \text{Tr}_{\tau}(\hat{\alpha} \hat{\gamma}) = \text{Tr}_{\tau}(\text{Tr}_{\tau}(\hat{\alpha}) \mathbb{1}_{H} \hat{\gamma}) = \text{Tr}_{\tau}(\hat{k}_{\alpha} \hat{\gamma}). \)

However, in general, it is not true that \( \hat{k}_{\alpha} \) are scalar multiples of the identity and this extra scalar assumption makes the scalar extension hierarchy a much weaker (and impractical) result.
C.1. Scalar extension with canonical abstraction

In this section, we extend the noncommutative polynomial formalism of Section 2.2 to include the scalars. In addition to the alphabets and rules introduced in Section 2.2, we introduce the new (infinite size) alphabet $\kappa_A$ composed of the letters $\kappa_\alpha$ for all $\alpha \in \langle a \rangle$.

We impose that $\kappa_\alpha$ commutes with all letters and is self-adjoint, that is, for all $\alpha \in \langle a \rangle$ and letter $l_i \in a \cup b \cup c \cup \kappa_A$:

$$[\kappa_\alpha, l_i] = 0.$$  

Note that $\kappa_\alpha \neq \kappa_\alpha'$ and $\kappa_\alpha^2 \neq \kappa_\alpha$ in general. These rules imply that any word $\omega$ in $\langle a, b, c, \kappa_A \rangle$ admits a unique minimal form $\kappa \beta \gamma$ where $\kappa \in \langle \kappa_A \rangle$ (and $\alpha \in \langle a \rangle$, $\beta \in \langle b \rangle$, $\gamma \in \langle c \rangle$ have no squared letters), used to define its length and the equality of words. Note that the alphabet $\kappa_A$ is infinite in size. Therefore, to keep the size of the moment matrix finite, we introduce $\kappa_A^n$ as the alphabet (of finite size) consisting only of letters $\kappa_\alpha$ with $|\alpha| \leq n$.

With scalar extension, the ring of bilocal scalar extension polynomials $T_{\alpha abc}^{\kappa_A}$ naturally arises; see Appendix A.2 for a formal discussion.

C.2. Scalar extension bilocal NPA hierarchy

We now construct the Scalar Extension Bilocal Moment Matrix $\Omega^n$ whose entries are indexed by words in $\langle a, b, c, \kappa_A \rangle$.

**Definition C.1** (Scalar Extension Bilocal Moment Matrix). Fix $n \in \mathbb{N}$, let $\Omega^n$ be a square matrix indexed by all words $\omega \in \langle a, b, c, \kappa_A^n \rangle$ of length $|\omega| \leq n$. We say that $\Omega^n$ is a Scalar Extension Moment Matrix of order $n$ if

(i) $\Omega^n_{1,1} = 1$.

(ii) $\Omega^n$ is positive.

(iii) it satisfies all the linear constraints

$$\Omega^n_{\omega, \nu} = \Omega^n_{\omega', \nu'}$$  \hspace{1cm} (C.1)

whenever $\omega^+ \nu = \omega'^+ \nu'$.

(iv) for all $\alpha \in \langle a \rangle$ and $\gamma \in \langle c \rangle$,

$$\Omega^n_{1, \alpha \gamma} = \Omega^n_{1, \kappa_\alpha \gamma}.$$  \hspace{1cm} (C.2)
We further say that $\Omega^n$ is compatible with the tripartite distribution $\tilde{Q}$ iff $\Omega^n_{1,A_{ai}B_{bij}C_{ciz}} = q(abc|xyz)$. An infinite matrix $\Omega^\infty$ is said to be a Scalar Extension Bilocal Moment Matrix iff all of its principal extracted matrices are Scalar Extension Bilocal Moment Matrix of some finite order.

**Definition C.2** (Scalar extension bilocal NPA hierarchy). Let $\tilde{Q} = \{q(abc|xyz)\}$ be a tripartite probability distribution. We say that $\tilde{Q}$ passes the scalar extension bilocal NPA hierarchy if for any integer $n \geq 3$, there exists a Scalar Extension Bilocal Moment Matrix $\Omega^n$ of size $n$ which is compatible with $\tilde{Q}$.

**Theorem C.1** (Scalar extension bilocal NPA hierarchy and commutator bilocal quantum correlations). Suppose that $\tilde{Q} = \{q(abc|xyz)\}$ is a tripartite probability distribution. If $\tilde{Q}$ is a Commutator Bilocal Quantum Distribution then $\tilde{Q}$ passes all the scalar extension bilocal NPA hierarchy tests.

Conversely, if $\tilde{Q}$ passes all the scalar extension bilocal NPA hierarchy tests, then there exists an associated GNS-representation $(\mathcal{H}, \pi, \tau)$, where $\mathcal{H}$ is a Hilbert space, $\pi : T^{abc} \rightarrow \mathcal{H}$ is a $*$-representation, and $\tau$ is a state. Moreover, if $\pi(T_a) = \mathbb{R}1_{\mathcal{H}}$, then $\tilde{Q}$ is a Commutator Bilocal Quantum Distribution.

**Proof.** Proving the necessity of the hierarchy is again straightforward. For the converse, we assume the existence of a hierarchy of compatible $\Omega^n$. We repeat the GNS construction as in the standard NPA hierarchy proof of convergence to construct the Hilbert space $\mathcal{H}$, the operators $\tau, \hat{A}_{a|x}, \hat{B}_{b|y}, \hat{C}_{c|z}$, and the operators $\hat{k}_a$ from a compatible infinite size Factorisation Bilocal Moment matrix $\Omega^\infty$ (obtained by extracting a convergent sequence from the set of compatible $\Omega^n$), obtaining an associated GNS-triple $(\mathcal{H}, \pi, \tau)$. Consequently, we obtain operators that meet constraints (ii) and (iii) in Definition 3.2.

Moreover, the GNS construction obtains operators $\hat{k}_a$ that commute with all operators $\hat{\omega}$ and the assumption that $\pi(T_a) = \mathbb{R}1_{\mathcal{H}}$ imposes all $\hat{k}_a$ must be scalar multiples of the identity. It follows from Equation (C.2), taking $\gamma = 1$, that $\hat{k}_a = \text{Tr}_\tau(\hat{\omega})1_{\mathcal{H}}$. Then, the condition Equation (C.2) imposed on the moment matrix gives: $\text{Tr}_\tau(\hat{\omega}\hat{\gamma}) = \Omega^n_{1,ay_1} = \Omega^n_{1,\alpha_1\gamma} = \text{Tr}_\tau(1_{\mathcal{H}})\hat{\gamma}) = \text{Tr}_\tau(\hat{\omega})\text{Tr}_\tau(\hat{\gamma})$, which imposes the nonlinear factorisation condition of Equation (3.1).

Therefore, we can introduce $\rho$ and $\sigma$ in the same way as in the proof of Theorem 3.2, which satisfy conditions (i) and (iv) of Definition 3.2 by the same calculation.

**Remark C.2.** We remark that there is no reason to believe that condition $\pi(T_a) = \mathbb{R}1_{\mathcal{H}}$ holds in general, and that the scalar assumption is essential to the bilocal feasibility problem.
Consider the example of shared random bit distribution provided in Appendix F.2. Since all measurements $A_{a|x}, B_{b|y}, C_{c|z}$ commute, we may simply define $\hat{k}_a = \hat{\alpha}$ and construct the associated $\Omega^n$ for all $n \geq 3$. However, the shared random bit distribution is not bilocal, which is a counterexample to the claim that the scalar extension bilocal hierarchy converges to the set of all Commutator Bilocal Quantum Distributions. In fact, without the scalar assumption, the hierarchy cannot exclude any classical distribution by the same line of reasoning.

One may wish to characterise the condition in which the scalar assumption can be satisfied. First, we may take advantage of the convexity and use the extremal condition. Indeed if the weak-$\star$ limit of the hierarchy $\Omega^n$ is extremal, then $\pi(T_a)$ must be a set of scalar multiples of the identity, being a subset of the centre of $\pi(T_{a|b|c})$. Alternatively, one can demand the weak-$\star$ limit of any $\kappa_\alpha$ under $\pi$ that squares to itself must be the identity. Then one can show that any projection in the weak-$\star$ closure of $\pi(T_a)$ must be trivial, and consequently $\pi(T_a) = \mathbb{R}_{+1}$.\mathcal{H}.

Note that these characterisations are very impractical in application. Finding an equivalent representation-independent condition that can strengthen Definition C.1 is still an open question.

D. Comparison of our framework with [24]

In this section, we discuss our results within the framework of the theory of noncommutative polynomial optimisation [10], and compare them with [24].

The standard NPA hierarchy [31] can be understood as an eigenvalue optimisation problem, where the aim is to minimise the lowest eigenvalue of a noncommutative polynomial under noncommutative polynomial constraints. For instance,

$$\min_{A, B, C} \text{Eig} \left( ABC^2 - AB \right)$$

subject to $A^2 + B^2 + C^2 \leq 1$,

where $A, B, C$ are self-adjoint operators acting on some Hilbert space $\mathcal{H}$. Equivalently, one can minimise the trace of the polynomial over any normalised state $\tau$. Hence the optimisation problem above is equivalent to

$$\min_{A, B, C, \tau} \text{Tr}_{\tau} \left( ABC^2 - AB \right)$$

subject to $A^2 + B^2 + C^2 \leq 1$

and $\tau \geq 0, \text{Tr}(\tau) = 1$. 

In the standard eigenvalue optimisation problem, one performs the optimisation on the operator semialgebraic set associated with the constraints, which consists of all bounded self-adjoint operators satisfying the constraints.

In our case, we deal with a “pseudo-eigenvalue optimisation problem” with the constraints given by scalar extension polynomials as defined in Appendix A.2. Now, \( \tau \) is constrained to explore a restricted space, and hence the minimum is a priori not obtained for \( \tau \) being an eigenvector of the polynomial. Note also that, by construction, scalar extension polynomials are different from trace polynomials, as they do not satisfy cyclic equivalence; for example, \( \text{Tr}_\tau(ABC^2) \neq \text{Tr}_\tau(C^2AB) \) in general. The goal of our paper is to solve a feasibility problem: given some correlations \( \vec{P} \), we ask if it is compatible with a quantum model in some given network, which corresponds to optimising over a trivial polynomial. An example of a feasibility optimisation problem is:

\[
\min_{A,B,C,\tau} \text{Tr}_\tau(\mathbb{I}) \\
\text{s.t. } A^2 + B^2 + C^2 \leq 1 \\
\text{and } \tau \geq 0, \text{Tr}(\tau) = 1, \\
\text{and } \forall k, l, \text{Tr}_\tau(A^k C^l) = \text{Tr}_\tau(A^k) \text{Tr}_\tau(C^l),
\]

Note that one could also look for the minimisation of some linear Bell-like expression, in which case the optimisation can be seen as a pseudo-eigenvalue optimisation problem of some polynomial with the pseudo-eigenvector \( \tau \) constrained to explore a restricted space. An example of such a pseudo-eigenvalue optimisation problem is:

\[
\min_{A,B,C,\tau} \text{Tr}_\tau(ABC^2 - AB) \\
\text{s.t. } A^2 + B^2 + C^2 \leq 1 \\
\text{and } \tau \geq 0, \text{Tr}(\tau) = 1, \\
\text{and } \forall k, l, \text{Tr}_\tau(A^k C^l) = \text{Tr}_\tau(A^k) \text{Tr}_\tau(C^l),
\]

Remark finally that we could also perform optimisation over more general polynomial Bell-like expression, in which case the optimisation cannot be seen anymore as a pseudo minimal eigenvalue problem. For instance:

\[
\min_{A,B,C,\tau} \text{Tr}_\tau(ABC^2) - \text{Tr}_\tau(A) \text{Tr}_\tau(AB) + \text{Tr}_\tau(B) \\
\text{s.t. } \text{Tr}_\tau(A^2) + A^2 + B^2 + C^2 \leq 1 \\
\text{and } \tau \geq 0, \text{Tr}(\tau) = 1, \\
\text{and } \forall k, l, \text{Tr}_\tau(A^k C^l) = \text{Tr}_\tau(A^k) \text{Tr}_\tau(C^l).
\]
Importantly, all these optimisation tasks are performed on the operator semialgebraic set associated with scalar extension polynomial constraints, which consists of all bounded self-adjoint operators satisfying the constraints, similarly to the standard NPA hierarchy.

Contrary to our work, [24] focuses on the trace optimisation problem, which is very different from our framework. Instead of minimising the eigenvalue, their aim is to minimise the normalised trace of a trace polynomial under constraints given by trace polynomials. For example,

$$\min_{A,B,C} \text{Tr}(ABC^2 - \text{Tr}(AB)AB + \text{Tr}(B))$$

s.t. $\text{Tr}(A^2) + A^2 + B^2 + C^2 \leq 1$,

where the trace should be understood as the normalised trace (normalised by the “dimension of the space” so that $\text{Tr}(\mathbb{1}) = 1$). Observe that, due to the property of trace operators, the cyclic equivalence is assumed for trace polynomials; e.g. $\text{Tr}(AB) = \text{Tr}(BA)$ and $\text{Tr}(ABC^2) = \text{Tr}(C^2AB)$. This differs from our framework for the scalar extension polynomials.

Moreover, note that since the positivity of trace is a stronger notion than positive semi-definiteness, trace optimisation problem is harder than the eigenvalue optimisation. Indeed, to ensure that the notion of trace is well-defined for the self-adjoint bounded operators (in possibly infinite dimension), the optimisation must be taken on the semialgebraic set composed of Type-II$_1$ von Neumann factors$^{10}$. Note that the Type-II$_1$ semialgebraic set is a subset of the operator semialgebraic set of all bounded self-adjoint operators satisfying the constraints; namely, our framework has less restriction on the set of operators where optimisation is performed.

Finally, contrary to our case, the approach of [24] is only valid for pure trace polynomials such as $\text{Tr}(ABC^2 - \text{Tr}(AB)AB + \text{Tr}(B))$, but not for general trace polynomials such as $ABC^2 - \text{Tr}(AB)AB + \text{Tr}(B)^{11}$. The difficulty is indicated by [24, Example 4.6], that is, unlike pure trace polynomials, nonpure trace polynomials lack an analogous version of Helton–McCullough Positivstellensatz.

$^{10}$A Type-II$_1$ factor is an infinite-dimensional finite von Neumann algebra, which admits a unique faithful tracial state. Thus, the trace operator is well-defined.

$^{11}$Note that optimisation over such non-pure trace polynomials corresponds to some eigenvalue optimisation problem. It could be reformulated in terms of a state $\tau$, which would involve two traces: normalised $\text{Tr}(\cdot)$ and trace over state $\text{Tr}_\tau(\cdot)$. 
E. The inflation-NPA hierarchy

E.1. Formalism of inflation-NPA hierarchy in bilocal scenario

We use noncommutative polynomial formalism to encode the inflational data of inflation order $m$. That is, we associate POVMs $\{\hat{A}_i^i | x^i \}$, $\{\hat{B}_{b_{j,k}^i} | y_{j,k} \}$, and $\{\hat{C}_{c_{l}^l} | z^l \}$ with alphabets $a_i$, $b_{j,k}^i$, and $c_{l}^l$, for $i, j, k, l = 1, \ldots, m$. We impose the same algebraic rules on all the letters inherited from the underlying operators and inflational structure.

Therefore, any word $\omega \in \langle a_i, b_{j,k}^i, c_{l}^l | i, j, k, l = 1, \ldots, m \rangle$ admits a decomposition

$$\omega = \alpha^1 \cdots \alpha^m \beta^1 \cdots \gamma^m,$$

where $\beta \in \langle b_{j,k}^i | j, k = 1, \ldots, m \rangle$. Note that we cannot further write $\beta = \beta^{11} \cdots \beta^{jk} \cdots \beta^{mm}$ since $\beta^{jk}$ does not commute with $\beta^{j'k'}$ in general.

The resulting ring of noncommutative polynomials is

$$\mathbb{R}a^i \cdots a^m b_{j,k}^{i1} \cdots b_{j,k}^{im} c_{l}^{1} \cdots c_{l}^{m} = \mathbb{R}\langle a_i, b_{j,k}^i, c_{l}^l | i, j, k, l = 1, \ldots, m \rangle.$$

The product symmetric group $S_m \times S_m$ naturally acts on the inflation words by permuting the superscripts. For example, let $\theta, \theta' \in S_m$ and consider the word $\omega = \alpha^1 \cdots \alpha^m \beta^{11} \cdots \beta^{jk} \cdots \beta^{mm} \gamma^{1} \cdots \gamma^{m}$. Then

$$(\theta \times \theta')(\omega) = \alpha^{\theta(1)} \cdots \alpha^{\theta(m)} \beta^{\theta(1)\theta'(1)} \cdots \beta^{\theta(j)\theta'(k)} \cdots \beta^{\theta(m)\theta'(m)} \gamma^{\theta'(1)} \cdots \gamma^{\theta'(m)}.$$

Now we can define Inflation Bilocal Moment Matrix and inflation-NPA bilocal hierarchy.

Definition E.1 (Inflation Bilocal Moment Matrix). Fix $n, m \in \mathbb{N}$, let $\Xi^{n,m}$ be a square matrix indexed by all words $\omega \in \langle a_i, b_{j,k}^i, c_{l}^l | i, j, k, l = 1, \ldots, m \rangle$ of length $|\omega| \leq n$. We say that $\Xi^{n,m}$ is a $m$th-Inflation Moment Matrix of order $n$ if

(i) $\Xi^{n,m}_{1,1} = 1$.

(ii) $\Xi^{n,m}$ is positive.

(iii) it satisfies all the linear constraints

$$\Xi^{n,m}_{\omega,\gamma} = \Xi^{n,m}_{\omega',\gamma}$$

whenever $\omega^\dagger \gamma = \omega'^\dagger \gamma'$.

\[12\]Unlike the decomposition of words in non-inflated scenario, the expression here is not unique. Nonetheless, it is always possible to choose a canonical representation by fixing a rule on the ordering of all $\beta^{jk}$ that respect the partial commutativity. We omit proposing such a rule since it is cumbersome and irrelevant to the following discussion.
(iv) it satisfies the $S_m \times S_m$ symmetry

$$\Xi^{n,m}_{1,\omega} = \Xi^{n,m}_{1,(\theta \times \theta')(\omega)},$$

(E.1)

for all permutations $\theta, \theta' \in S_m$.

We further say that $\Xi^{n,m}$ is compatible with the tripartite distribution $\tilde{Q}$ iff

$$\Xi^{n,m}_{1,\prod_{i=1}^m A^i_{\alpha^i|x^i} B^i_{\beta^{ii}|y^i} C^{i}_{\gamma^i}} = \prod_{i=1}^m q(a^i b^{ii} c^i | x^i y^i z^i).$$

An infinite matrix $\Xi^{\infty,m}$ is said to be a $m$th-Inflation Moment Matrix iff all of its principal extracted matrices are $m$th-Inflation Moment Matrix of some finite order.

**Remark E.1.** As noted in the main text, there are extended nonlinear constraints that can also be imposed. That is,

$$\Xi^{n,m}_{1,\prod_{i=1}^m A^i_{\alpha^i|x^i} B^i_{\beta^{ii}|y^i} C^{i}_{\gamma^i}} = \prod_{i=1}^m q(a^i b^{ii} c^i | x^i y^i z^i).$$

(E.2)

for $1 \leq t \leq s \leq m$ and arbitrary diagonal words $\omega^i = \alpha^i \beta^{ii} \gamma^i$ for all $i = 1, \cdots, n$ with length $\leq n$. We do not include this extended version since it is nonSDP and hence impractical.

It is likely that with a de Finetti-type argument, the extended nonlinear constraints follow from Definition E.1 in the asymptotic limit of $n, m$.

**Definition E.2** (Bilocal inflation-NPA hierarchy). Let $\tilde{Q} = \{q(abc|xyz)\}$ be a tripartite probability distribution. We say that $\tilde{Q}$ passes bilocal inflation-NPA hierarchy if for all integers $m \geq 1$ and $n \geq 3$, there exists a $m$th-Inflation Bilocal Moment Matrix $\Xi^{n,m}$ of order $n$ that is compatible with $\tilde{Q}$.

As the archetype for quantum inflation for bilocal scenario, the Tensor Bilocal Quantum Distribution passes the bilocal inflation-NPA hierarchy, as expected.

**Lemma E.2.** All Tensor Bilocal Quantum Distributions $\tilde{P}$ passes the bilocal inflation-NPA hierarchy.

**Proof.** With inflation canonical abstraction we may define

$$\Xi^{n,m}_{\omega,\nu} = \text{Tr}_{\tau_n}(\hat{\omega}^\dagger \hat{\nu})$$

for suitable words $\omega, \nu$. It is clear that $\Xi^{n,m}$ satisfies Definition E.1.

Next, we restate and prove Theorem 4.3.
Theorem E.3. If the probability distribution \( \{\tilde{P} = \{p(abc|xyz)\}\} \) passes the bilocal inflation-NPA hierarchy, then it also passes the factorisation and scalar extension bilocal NPA hierarchies.

Proof. By Theorem 3.2 and Theorem C.1 it suffices to consider the scalar extension bilocal NPA hierarchy. For any \( n \geq 3 \) and suppose that the size of alphabet \( a \) is \( d \). We show that, for \( m \geq \sum_{i=0}^{n} d^i \), we can obtain the Scalar Extension Moment Matrix \( \Omega^n \) from \( m \)th-Inflation Moment Matrix \( \Xi^{n,m} \).

First, we identify the alphabets \( a^1, b^{11}, \) and \( c^{12} \) of \( \Xi^{n,m} \) with \( a, b, \) and \( c \) for \( \Omega^n \). Furthermore, for each distinct word \( \alpha \) with \( |\alpha| \leq n \) in \( a \), we assign a distinct index \( i \geq 2 \) and then identify the scalar extension \( \kappa_\alpha \) with \( \alpha^i \). By construction, \( \alpha^i \) with \( i \geq 2 \) commutes with \( a^1, b^{11}, \) and \( c^{12} \), and since \( i \) are distinct for distinct words, all scalar extensions commute with each other. Additionally, the choice of \( m \) is large enough for index assignment.

Now, define \( \Omega^n \) by
\[
\Omega^n_{1,\alpha \beta \gamma \kappa_\alpha} = \Xi^{n,m}_{1,\alpha^i \beta^{11} \gamma^{12} \alpha^k},
\]
where \( k \) is the unique index assigned to \( \alpha \). By construction, condition (i), (ii), and (iii) of Definition C.1 are evident. Condition (iv) follows easily due to the \( S_m \times S_m \) symmetry
\[
\Omega^n_{1,\alpha \gamma} = \Xi^{n,m}_{1,\alpha^1 \gamma^{1}} = \Xi^{n,m}_{1,\alpha^2 \gamma^{1}} = \Omega^n_{1,\kappa_\alpha \gamma}.
\]

However, the converse remains an open question.

E.2. Inflation-NPA hierarchy excludes the shared random bit distribution in the triangle scenario

In this section, we briefly explain why the inflation-NPA hierarchy can be used to prove the impossibility of obtaining a shared random bit distribution in the triangle scenario. This proof is directly inspired by the standard proof of [49].

Consider the uniform shared random bit distribution \( \tilde{P}_{ABC} = 1/2([000] + [111]) \), where [000] (resp. [111]) is the deterministic distribution of all parties outputting 0 (resp. 1). Assume, for contradiction, that it admits an inflation-NPA hierarchy of compatible moment matrices \( \Xi^{n,m} \). We now show that the existence of \( \Xi^{2,2} \) already provides a contradiction.

First, note that the distribution \( \tilde{Q}_{A^{11}B^{11}C^{12}} \) with the coefficients
\[
q(a_{11}b_{11}c_{12}) = \Xi^{2,2}_{A^{11}B^{11}C^{12}}
\]
is a probability distribution (for simplicity in the notation, we omitted the subscripts; e.g., $A_{11}$ should be interpreted as $A_{a_11}$), as these numbers are positive (they can be seen as concrete measurement probabilities in a quantum experiment) and sum to one according to the constraints satisfied by $\Sigma^{2,2}$.

According to the diagonalisation constraint, we have the marginal $\tilde{Q}_{A_{11}B_{11}}$ over the first two outputs satisfying $\tilde{Q}_{A_{11}B_{11}} = P_{AB} = 1/2([00] + [11])$, which means that we have $a_{11} = b_{11}$ is a shared uniform random bit. Moreover, we have $\tilde{Q}_{B_{11}C_{12}} = \tilde{Q}_{B_{11}C_{11}} = 1/2([00] + [11])$, where we first used the symmetry condition and then the diagonalisation constraint, meaning that we have $b_{11} = c_{12}$ is a shared uniform random bit. These two conditions imply that we must have $a_{11} = c_{12}$ is a shared random bit, that is, $\tilde{Q}_{A_{11}C_{12}} = 1/2([00] + [11])$.

However, the symmetry condition also imposes $\tilde{Q}_{A_{11}C_{12}} = \Sigma^{2,2}_{A_{11},C_{12}} = \Sigma^{2,2}_{A_{11},C_{22}}$, and the diagonalisation constraint implies $\Sigma^{2,2}_{A_{11},C_{22}} = P_A \cdot P_C = 1/4([00] + [01] + [10] + [11])$, which is contradictory.

F. Commutator-based definition of quantum correlations in networks

In this section, we propose a general commutator-based definition of accessible quantum correlations $Q$ feasible in an arbitrary network, generalising the Definition 3.2 of Commutator Bilocal Quantum Distributions $\tilde{Q}$. Then, we discuss potential issues with this definition. At last, we show that purification of mixed states cannot be done with this definition.

F.1. Definition of Commutator Network Quantum Distributions

We propose the following natural general commutator-based definition of the accessible quantum correlations $Q$ feasible in an arbitrary network:

(i) The global state, represented by a projector over a pure state $\tau$, acts over a global Hilbert space $\mathcal{H}$.

(ii) Each party $A, B, C, \ldots$ PVMs operators $\hat{A}_{a|x}, \hat{B}_{b|y}, \hat{C}_{c|z}, \ldots$ acts over $\mathcal{H}$, two operators associated with different parties commute.

(iii) $q(abc \cdots |xyz \cdots) = \text{Tr}_\tau(\hat{A}_{a|x}\hat{B}_{b|y}\hat{C}_{c|z} \cdots)$.

(iv) Each source is associated with a projector $\rho, \sigma, \pi, \ldots$ acting over $\mathcal{H}$, and two projectors associated with different sources commute. Moreover, the product of these operators is $\tau = \rho \cdot \sigma \cdot \pi \cdots$.

(v) For any party (say, $A$) not connected to a source (say, corresponding to source $\rho$), the associated operators commute with the source projector (here, $[\rho, \hat{A}_{a|x}] = 0$).
Note that if we restrict to only (i), (ii), and (iii), we recover commutator multipartite quantum correlations as in the standard Bell scenarios.

As implied by Conjecture 4.2, inflation-NPA method may converge to a subset of the set of Commutator Bilocal Quantum Correlations. We leave the question of equality, or strict inclusion, open. Strict inclusion would imply that either the definition of Commutator Bilocal Quantum Correlations cannot be taken as the quantum theory postulated correlations in the bilocal scenario, or that the inflation-NPA method is not an allowed method to characterise quantum correlations in the bilocal scenario.

In networks beyond the bilocal scenario, this commutator-based postulate becomes even more suspicious. For example, in the triangle scenario in Figure 3a, it reads:

**Definition F.1 (Commutator Triangle Quantum Distributions).** Let \( \mathcal{Q} = \{q(abc|xyz)\} \) be a three-party probability distribution. We say that \( \mathcal{Q} \) is a Commutator Triangle Quantum Distribution iff there exist a Hilbert space \( \mathcal{H} \), a projector over a pure state \( \tau \), three mutually commuting projectors (of possibly infinite trace) \( \rho, \sigma, \) and \( \pi \), and some PVMs \( \hat{A}_{a|x}, \hat{B}_{b|y}, \) and \( \hat{C}_{c|z} \) over \( \mathcal{H} \) such that

(i) \( \tau \) acts on \( \mathcal{H} \).

(ii) \( [\hat{A}_{a|x}, \hat{B}_{b|y}] = [\hat{B}_{b|y}, \hat{C}_{c|z}] = [\hat{C}_{c|z}, \hat{A}_{a|x}] = 0 \).

(iii) \( q(abc|xyz) = \text{Tr}_{\tau}(\hat{A}_{a|x} \hat{B}_{b|y} \hat{C}_{c|z}) \).

(iv) \( [\rho, \sigma] = [\sigma, \pi] = [\pi, \rho] = 0 \) and \( \tau = \rho \cdot \sigma \cdot \pi \).

(v) \( [\hat{A}_{a|x}, \rho] = [\hat{B}_{b|y}, \pi] = [\hat{C}_{c|z}, \sigma] = 0 \).

In that case, there exists a weaker method than inflation-NPA to find constraints on the obtainable correlations in the triangle network, called the non-fanout inflation method \([21, 50]\). Contrary to the factorisation bilocal NPA, scalar extension, and inflation-NPA hierarchies, it is not based on the mathematical Hilbert space formulation of quantum theory. It is solely based on the assumptions of causality and device replication (any device should be replicable in independent copies) \([2,4,12,13,18]\), two very weak assumptions. For network scenarios without loops, the non-fanout inflation does not provide nontrivial constraints beyond the fact that marginal correlations between groups of parties not sharing a source should factorise (as there does not exist any nontrivial non-fanout inflation of no loop networks): this method cannot be used in the bilocal scenario to restrict correlations.

However, in other networks, it implies that a feasible \( \mathcal{Q} \) should be associated with compatible distributions in all the non-fanout inflation of this network. For instance, if \( \mathcal{Q} \) is feasible in the triangle network, it should be associated with an other distribution \( \mathcal{R} \) defined in the hexagon inflation of the triangle, compatible with \( \mathcal{Q} \). As shown in \([21, 50]\), this allows one to reject the feasibility of some distributions such as the shared random bit in the triangle scenario. It is uncertain whether any \( \mathcal{Q} \) satisfying
Definition F.1 above is always associated with such distributions $\tilde{R}$ in the hexagon (or larger) inflation of the triangle. In case it is not (which would be the case if this definition allows for the shared random bit distribution), this would imply that such a version of quantum theory based on a commutator-based postulate does not allow one to consider independent copies of systems, which we would see as a strong argument against this version of quantum theory.

F.2. Pure-mixed state inequivalence in commutator model

In this section, we show that the condition of $\tau$ being pure in Definition 3.2 is non-trivial, as stated in Remark 3.1. That is, we cannot reformulate Definition 3.2 in terms of demanding $\tau$ to be only a mixed state.

Indeed, consider six qubit spaces $H_{A_i}$, $H_{B_i}$, and $H_{C_i}$, for $i = 0, 1$. Let $H_i = H_{A_i} \otimes H_{B_i} \otimes H_{C_i}$, and define the global Hilbert space by direct sum decomposition $H = H_0 \oplus H_1$. Let $H_i = H_{A_i} \otimes H_{B_i} \otimes H_{C_i}$, and define the global Hilbert space by direct sum decomposition $H = H_0 \oplus H_1$. The global state $\tau$ is defined by

$$\tau = \frac{1}{2} (|0\rangle \langle 0|_{A_0} \otimes |0\rangle \langle 0|_{B_0} \otimes |0\rangle \langle 0|_{C_0} \oplus |1\rangle \langle 1|_{A_1} \otimes |1\rangle \langle 1|_{B_1} \otimes |1\rangle \langle 1|_{C_1}).$$  (F.1)

In particular, we have a product decomposition $\tau = \rho \sigma$, where

$$\sqrt{2} \rho = |0\rangle \langle 0|_{A_0} \otimes 1_{B_0 C_0} \oplus |1\rangle \langle 1|_{A_1} \otimes 1_{B_1 C_1},$$

$$\sqrt{2} \sigma = |0\rangle \langle 0|_{B_0} \otimes |0\rangle \langle 0|_{C_0} \otimes 1_{A_0} \oplus |1\rangle \langle 1|_{B_1} \otimes |1\rangle \langle 1|_{C_1} \otimes 1_{A_1}.$$  (F.2)

For measurements, we take the PVMs

$$A_{a=0} = |0\rangle \langle 0|_{A_0} \otimes 1_{B_0 C_0} \oplus |0\rangle \langle 0|_{A_1} \otimes 1_{B_1 C_1},$$

$$A_{a=1} = |1\rangle \langle 1|_{A_0} \otimes 1_{B_0 C_0} \oplus |1\rangle \langle 1|_{A_1} \otimes 1_{B_1 C_1},$$

$$B_{b=0} = |0\rangle \langle 0|_{B_0} \otimes 1_{A_0 C_0} \oplus |0\rangle \langle 0|_{B_1} \otimes 1_{A_1 C_1},$$

$$B_{b=1} = |1\rangle \langle 1|_{B_0} \otimes 1_{A_0 C_0} \oplus |1\rangle \langle 1|_{B_1} \otimes 1_{A_1 C_1},$$

$$C_{c=0} = |0\rangle \langle 0|_{C_0} \otimes 1_{A_0 B_0} \oplus |0\rangle \langle 0|_{C_1} \otimes 1_{A_1 B_1},$$

$$C_{c=1} = |1\rangle \langle 1|_{C_0} \otimes 1_{A_0 B_0} \oplus |1\rangle \langle 1|_{C_1} \otimes 1_{A_1 B_1}.$$  (F.3)

It can be easily checked that Definition F.1 is satisfied and that

$$\text{Tr}_\tau (A_a B_b C_c) = \begin{cases} \frac{1}{2} & a = b = c \\ 0 & \text{otherwise} \end{cases}$$

is precisely the shared random bit distribution $\tilde{P}_{ABC} = 1/2([000] + [111]).$
Note that introducing an extra qubit space $\mathcal{H}_S$ of basis $|0\rangle_S, |1\rangle_S$ corresponding to cases $i = 0, 1$, respectively, one can state this example differently. In the global Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_S$:

$$\tau = \frac{1}{2} \sum_{i=0,1} |i\rangle \langle i|_A \otimes |i\rangle \langle i|_B \otimes |i\rangle \langle i|_C \otimes |i\rangle \langle i|_S,$$

$$\sqrt{2}\rho = \sum_{i=0,1} |i\rangle \langle i|_A \otimes |i\rangle \langle i|_S \otimes 1_{BC},$$

$$\sqrt{2}\sigma = \sum_{i=0,1} |i\rangle \langle i|_B \otimes |i\rangle \langle i|_C \otimes |i\rangle \langle i|_S \otimes 1_{A},$$

and symmetrically for $B_0, B_1, C_0, C_1$.

The quantum strategies provided above obviously satisfy Definition 3.2 except for the purity condition. However, it is also clear that $\tilde{P}_{ABC}$ cannot be a bilocal quantum distribution because it cannot factorise between Alice and Charlie.

The deep reason of pure-mixed state inequivalence is that the usual purification does not preserve commutator structure\(^\text{13}\), we provide an alternative definition in commutator model. To this end, consider Definition 3.2 with distribution $\tilde{Q}$, but we instead require that $\rho, \sigma$ are only (possibly infinite trace) positive operators for $\tau = \rho \sigma$. We say that $\tau$ is $c$-purifiable if there exists some (pure) density operator $\tau' = \rho' \sigma'$ satisfying all the constraints in Definition 3.2 with $\tilde{Q}$, and $\tau'$ is said to be the $c$-purification of $\tau$. The generalisation of $c$-purification to an arbitrary network in Appendix F is straightforward. We have the following criterion for $c$-purifiability in bilocal scenario.

**Corollary F.1.** Consider the notation above. Then $\tau$ is $c$-purifiable iff all the factorisation constraints hold, that is, $\text{Tr}_\tau(\hat{\alpha} \hat{\gamma}) = \text{Tr}_\tau(\hat{\alpha}) \text{Tr}_\tau(\hat{\gamma})$, for any $\alpha \in \langle a \rangle$ and $\gamma \in \langle c \rangle$.

**Proof.** This is a direct consequence of the proof for Theorem 3.2. \hfill \blacksquare

Note that the quantum strategies provided above are examples of Corollary F.1. Also note that, with the notion of $c$-purifiability, it is possible to formulate commutator-based quantum correlations in general networks in terms of mixed-state. Nevertheless, such a reformulation is cumbersome and redundant; hence we omitted it. Finally, we observe that the inequivalence between the pure state formulation and the mixed state formulation, and the need for an alternative notion of purification, suggest extra subtlety in the commutator-based model compared to the tensor-based model.

\(^{\text{13}}\text{Note that in tensor-based model, the purification of a mixed state } \tau = \rho \otimes \sigma \text{ always assumes the form } |\phi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle, \text{ where } |\psi_1\rangle \text{ and } |\psi_2\rangle \text{ are the purification of } \rho \text{ and } \sigma, \text{ respectively.} \)
G. The four party star network

The set of distributions that can be obtained in the four-party star network of Figure 3a (according to the tensor postulate) is a direct generalisation of Definition 3.1, with an extra state $\pi_{DBD}$ acting over $\mathcal{H}_B \otimes \mathcal{H}_D$ and additional PVMs for $D$. We now discuss the potential generalisation of our proof in Section 3 to this new scenario. We show that when a distribution $\tilde{Q}$ admits a Factorisation Star-shaped Trilocal Moment Matrix (see Definition G.1), then it is an Anti-state Commutator Star-shaped Quantum Distribution (which is weaker than what we could expect following the definition of Appendix F). As we do not use all the assumptions at our disposition, we conjecture that a more restricted model, maybe the one of Section F, can be proven from the existence of such a hierarchy of Factorisation Star-shaped Trilocal Moment Matrix.

Let us first generalise the factorisation hierarchy by introducing pairwise factorisation constraints:

**Definition G.1** (Factorisation Star-shaped Trilocal Moment Matrix). Fix $n \in \mathbb{N}$, let $\Gamma^n$ be a square matrix indexed by all words $\omega \in \langle a, b, c, d \rangle$ of length $|\omega| \leq n$. We say that $\Gamma^n$ is a Factorisation Star-shaped Trilocal Moment Matrix of order $n$ if

(i) $\Gamma^n_{1,1} = 1$.

(ii) $\Gamma^n$ is positive.

(iii) it satisfies all the linear constraints

$$\Gamma^n_{\omega,\nu} = \Gamma^n_{\omega',\nu'}$$

whenever $\omega^\dagger \nu = \omega'^\dagger \nu'$.

(iv) it satisfies the pairwise nonlinear factorisation constraints

$$\Gamma^n_{\alpha,\gamma} = \Gamma^n_{\alpha,1} \cdot \Gamma^n_{1,\gamma},$$
$$\Gamma^n_{\alpha,\delta} = \Gamma^n_{\alpha,1} \cdot \Gamma^n_{1,\delta},$$
$$\Gamma^n_{\gamma,\delta} = \Gamma^n_{\gamma,1} \cdot \Gamma^n_{1,\delta},$$

where $\alpha \in \langle a \rangle$, $\gamma \in \langle c \rangle$, and $\delta \in \langle d \rangle$.

(v) it satisfies the nonlinear constraints

$$\Gamma^n_{\alpha\gamma,\delta} = \Gamma^n_{\alpha\gamma,1} \cdot \Gamma^n_{1,\delta},$$

equivalent to the three-factorisation $\Gamma^n_{\alpha\gamma,\delta} = \Gamma^n_{\alpha,1} \cdot \Gamma^n_{\gamma,1} \cdot \Gamma^n_{1,\delta}$ due to (iv).

An infinite matrix $\Gamma^\infty$ is said to be a Factorisation Star-shaped Trilocal Moment Matrix if all of its extracted matrices are Factorisation Trilocal Moment Matrices of some
finite order. We further say that \( \Gamma^n \) is compatible with the four-party distribution \( \tilde{Q} \) iff

\[
\Gamma^n_{A|x, B|y, C|z, D|w} = q(abc|xyz) = q(abcd|xyzw). 
\]

Observe that condition (iv) in Definition G.1 does not imply condition (v). In other words, condition (iv) alone does not fully contain the consequences in terms of factorisation of the topology of the star-shaped network. This definition is associated with the following hierarchy of tests that some distribution \( \tilde{Q} \) is feasible in the four-party star network:

**Definition G.2** (Factorisation star-shaped trilocal NPA hierarchy). Let \( \tilde{Q} = \{ q(abc|xyzw) \} \) be a probability distribution for Alice, Bob, Charlie, and Dave. We say that \( \tilde{Q} \) passes the factorisation star-shaped trilocal NPA hierarchy if for all integers \( n \geq 4 \), there exists a Factorisation Star-shaped Trilocal Moment Matrix \( \Gamma^n \) of order \( n \) that is compatible with \( \tilde{Q} \).

We now show that if \( \tilde{Q} \) admits such a hierarchy, then it is an Anti-state Commutator Star-shaped Quantum Distribution (defined below). However, contrary to the bilocal scenario, we do not prove the converse and conjecture that it does not hold, as we suspect that our proof does not use condition (v) in Definition G.1.

**Definition G.3** (Anti-state Commutator Star-shaped Trilocal Quantum Distributions). Let \( \tilde{Q} = \{ q(abc|xyzw) \} \) be a probability distribution for Alice, Bob, Charlie, and Dave. We say that \( \tilde{Q} \) is a Anti-state Commutator Star-shaped Trilocal Quantum Distribution iff there exist a Hilbert space \( \mathcal{H} \), a projector over a pure state \( \tau \), three positive operators (possibly infinite trace) \( \tilde{\rho}, \tilde{\sigma}, \tilde{\pi} \) commuting with each other, and PVMs \( \{ \hat{A}_{a|x} \}, \{ \hat{B}_{b|y} \}, \{ \hat{C}_{c|z} \}, \{ \hat{D}_{d|w} \} \) over \( \mathcal{H} \) such that

(i) \( q(abc|xyz) = \text{Tr} \left( \tau \cdot \hat{A}_{a|x} \cdot \hat{B}_{b|y} \cdot \hat{C}_{c|z} \cdot \hat{D}_{d|w} \right) \)

(ii) \( [\hat{A}_{a|x}, \hat{B}_{b|y}] = [\hat{A}_{a|x}, \hat{C}_{c|z}] = [\hat{A}_{a|x}, \hat{D}_{d|w}] = [\hat{B}_{b|y}, \hat{C}_{c|z}] = [\hat{B}_{b|y}, \hat{D}_{d|w}] = [\hat{C}_{c|z}, \hat{D}_{d|w}] = 0 \)

(iii) \( \tau = \tilde{\rho} \cdot \tilde{\sigma} = \tilde{\sigma} \cdot \tilde{\pi} = \tilde{\pi} \cdot \tilde{\rho} \)

(iv) \( [\tilde{\rho}, \tilde{\sigma}] = [\tilde{\rho}, \tilde{\pi}] = [\tilde{\sigma}, \tilde{\pi}] = 0 \)

(v) \( [\hat{A}_{a|x}, \tilde{\rho}] = [\hat{C}_{c|z}, \tilde{\sigma}] = [\hat{D}_{d|w}, \tilde{\pi}] = 0 \)

for all \( \hat{A}_{a|x}, \hat{B}_{b|y}, \hat{C}_{c|z}, \) and \( \hat{D}_{d|w} \).

Note that this definition is not the natural extension of Definition 3.2. Indeed, here the operators \( \tilde{\rho}, \tilde{\sigma}, \tilde{\pi} \) should not be understood as the original states \( \rho, \sigma, \pi \), but as the corresponding “anti-states” \( \tilde{\rho} = \sigma \cdot \pi, \tilde{\sigma} = \pi \cdot \rho \) and \( \tilde{\pi} = \rho \cdot \sigma \). The bilocal scenario can be seen as a degenerate case: As there are only two states, we could interpret in the proof of Theorem 3.2 our constructed \( \sigma \) as \( \tilde{\rho} \) and \( \rho \) as \( \tilde{\sigma} \).
Theorem G.1 (Convergence of the factorisation star-shaped trilocal NPA hierarchy). Suppose that $\overline{Q} = \{ q(abcd|x\ yzw) \}$ is a probability distribution for Alice, Bob, Charlie, and Dave. If $\overline{Q}$ passes all the factorisation star-shaped trilocal NPA hierarchy tests, then $\overline{Q}$ is a Anti-state Commutator Star-shaped Trilocal Quantum Distribution.

Moreover, if we forget the condition (v) of Definition G.3 (three-factorisation), then the above two statements are equivalent.

Proof. The proof is completely analogous to the proof of Theorem 3.2 in Appendix B.1. Suppose that we have an Anti-state Commutator Star-shaped Trilocal Quantum Distribution $\overline{Q}$ and we wish to show that Definition G.3 can be satisfied modulo (v). The only nontrivial conditions that need to be shown are (iv).

Condition (iv) is done similarly to Appendix B.1. Indeed,

$$\text{Tr}_\tau(\alpha^\dagger \tau \gamma) = \text{Tr}(\tau \alpha^\dagger \gamma) = \text{Tr}(\alpha^\dagger \rho \gamma) = \text{Tr}(\alpha^\dagger \rho \gamma) = \text{Tr}(\alpha^\dagger \rho \gamma) = \text{Tr}(\alpha^\dagger \rho \gamma),$$

where we used the commutativity and cyclicity of the trace. This implies $\Gamma_{\alpha,\gamma}^n = \Gamma_{\alpha,1,\gamma}^n$ with the same argument as in Appendix B.1. The other two constraints in (iv) follow analogously.

For the converse, we only comment on the construction of $\tilde{\rho}, \tilde{\sigma}, \tilde{\pi}$ since the rest are the same. Let the vector corresponding to the null word 1 be $|\phi_1\rangle = |\psi^{000}\rangle$. Define the subspaces $V_{AB_1} = \text{Span} \{ P_1(\{\hat{\alpha}\}) \}_P$, $V_{B_2C} = \text{Span} \{ Q(\{\hat{\gamma}\}) \}_Q$, and $V_{B_3D} = \text{Span} \{ R(\{\hat{\delta}\}) \}_R$ each with orthogonal basis $\{ P_i(\{\hat{\alpha}\}) |\psi^{000}\rangle = |\psi^{000}\rangle, i \in I \}$, $\{ Q_j(\{\hat{\gamma}\}) |\psi^{000}\rangle = |\psi^{000}\rangle \}$, and $\{ R_k(\{\hat{\delta}\}) |\psi^{000}\rangle = |\psi^{000}\rangle \}$ for some countable index set $I, J, K$, respectively. Using nonlinear factorisation constraints, the analogous calculation as in Appendix B.1 shows

$$\tilde{\rho} = \sum_i |\psi^{000}\rangle \langle \psi^{000}|$$  \hspace{1cm} (G.1)

$$\tilde{\sigma} = \sum_j |\psi^{0j0}\rangle \langle \psi^{0j0}|$$  \hspace{1cm} (G.2)

$$\tilde{\pi} = \sum_k |\psi^{00k}\rangle \langle \psi^{00k}|$$  \hspace{1cm} (G.3)

satisfying all the desired properties.

Remark G.2. We remark again that the converse of Theorem G.1 is not true in general, i.e. an Anti-state Commutator Star-shaped Trilocal Quantum Distribution may
not admit a hierarchy of Factorisation Star-shaped Trilocal Moment Matrices satisfying the three-factorisation constraints. Indeed, to this end, we would need to show that the equation

$$\text{Tr}_\tau(\hat{\alpha}^\dagger\hat{\gamma}^\dagger\hat{\delta}) \neq \text{Tr}(\tau\hat{\alpha}^\dagger\hat{\gamma}^\dagger\tau\hat{\delta})$$

holds. But this is not true since we cannot “merge” two $\tau$’s on the right-hand side of the equation with the commutation relations and cyclicity of trace given by the definition of Anti-state Commutator Star-shaped Trilocal Quantum Distributions.

This observation suggests that the factorisation star-shaped trilocal NPA hierarchy converges to a proper subset of all Anti-state Commutator Star-shaped Trilocal Quantum Distributions. It may converge to the model described in Appendix F.

### H. Bilocal Tsirelson’s problem

Two alternatives postulates exists to introduce the distributions predicted by quantum theory: the tensor-based postulate of Definition 2.1 and the commutator-based postulate of Definition F.1. While it is trivial that a tensor-based model implies the existence of a commutator-based model, the question of the converse implication was a long-standing problem (equivalent to Kirchberg’s Conjecture and Connes’ Embedding Conjecture [15, 17, 20, 32]) that was only recently disproven by [22]. However, Tsirelson proved that, restricting to finite-dimensional Hilbert spaces, the two definitions are equivalent [42, 46]. He proved the following theorem:

**Theorem H.1** (Tsirelson). *Given a Hilbert space $\mathcal{H}$, let $\{\hat{A}_{a|x}\}, \{\hat{B}_{b|y}\}$ be two finite commuting sets of positive operators in $\mathcal{B}(\mathcal{H})$, each generating a finite-dimensional von Neumann algebra $\mathcal{A}$ and $\mathcal{B}$. Let $\bar{Q} = \{q(ab|xy)\}$ be a probability distribution associated with $\{\hat{A}_{a|x}\}, \{\hat{B}_{b|y}\}$ and a global state $\tau$.

Then there exists a Hilbert space $\mathcal{H}$ with a decomposition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, such that $\{\hat{A}_{a|x}\}$ can be mapped into $\mathcal{B}(\mathcal{H}_A)$ and $\{\hat{B}_{b|y}\}$ can be mapped into $\mathcal{B}(\mathcal{H}_B)$ via injective $*$-homomorphisms. That is,

$$\{\hat{A}_{a|x}\} \mapsto \{\hat{A}_{a|x} \otimes 1_B\}$$

$$\{\hat{B}_{b|y}\} \mapsto \{1_A \otimes \hat{B}_{b|y}\}.$$ 

Moreover, there exists a state $\bar{\tau}$ on $\mathcal{H}$ recovering the probability distribution $\bar{Q}$ with the operators $\{\hat{A}_{a|x} \otimes 1_B\}$ and $\{1_A \otimes \hat{B}_{b|y}\}$, namely,

$$q(ab|xy) = \text{Tr}_\tau(\hat{A}_{a|x}\hat{B}_{b|y}) = \text{Tr}_{\bar{\tau}}(\hat{A}_{a|x} \otimes \hat{B}_{b|y})$$
Remark H.2. Observe that for some given Commutator Bilocal Quantum Distribution in a finite-dimensional Hilbert space, an inductive application of Theorem H.1 can easily yield a Tensor Tripartite Quantum model for that distribution. Thus, Definition 2.1 is equivalent to Definition 2.2 under the assumption of finite-dimensionality. By the same line of reasoning, we can argue the finite-dimensional equivalence of any multipartite quantum correlations.

Does the bilocal version of Remark H.2 hold? That is, if restricting to finite-dimensional Hilbert spaces, are Definition 3.1 and 3.2 equivalent? We leave this question open and now discuss why the proof is not a straightforward application of Tsirelson’s Theorem H.1. One of the impasses is due to the fact that the new global state \( \tilde{\tau} \) constructed does not necessarily admit a bilocal decomposition. That is, the equality \( \tilde{\tau} = \rho'_{ABL} \otimes \sigma'_{BRC} \), where \( \rho'_{ABL} \) and \( \sigma'_{BRC} \) are some operators, does not always hold. In addition, there is a more subtle problem when embedding the PVMs into the split Hilbert space in bilocal scenarios.

H.1. A sketch proof of Tsirelson’s theorem

Let us elaborate by discussing the proof of Theorem H.1. To this end, we introduce the following known facts in the theory of operator algebra; for more details see [6] and [23].

Definition H.1. Given a Hilbert space \( \mathcal{H} \), a von Neumann algebra \( \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \) is called a factor if it has a trivial centre, that is

\[
Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}' = \mathbb{C}1,
\]

where \( \mathcal{A}' \) denotes the commutant of \( \mathcal{A} \).

The significance of factors is that any von Neumann algebra admits a direct integral decomposition. By assuming finite-dimensionality, we have the following structural result.

Theorem H.3. Every finite-dimensional von Neumann algebra \( \mathcal{A} \) is a direct sum of \( m \) factors, where \( m = \dim Z(\mathcal{A}) \).

Let \( p \) be a projection on some subspace \( \mathcal{K} \) of \( \mathcal{H} \) and \( u \in \mathcal{B}(\mathcal{H}) \), the compression of \( u \) to \( \mathcal{K} \), denoted by \( u_p \), is an element of \( \mathcal{B}(\mathcal{K}) \) such that \( u_p : v \mapsto pwv \). Then we have the following lemma.

\footnote{Note that a positive answer would then prove the converse of Theorem 4.3 and Conjecture 4.2 in finite-dimensional case.}

\footnote{For our purpose, all factors are automatically of Type-I. We shall omit the terminology for simplicity.}
Lemma H.4. Suppose that $\mathcal{A} \subset \mathcal{B}(H)$ is a $\star$-algebra and $p \in \mathcal{A}'$ is a projection. Then $p\mathcal{A}p$ and $\mathcal{A}_p = \{u_p \mid u \in \mathcal{A}\}$ are both $\star$-algebra and the map

$$p\mathcal{A}p \to \mathcal{A}_p, \ p u p \leftrightarrow u_p$$

is a $\star$-isomorphism. Moreover, if we also have $p \in \mathcal{A}''$, then

$$(\mathcal{A}')_p = (\mathcal{A}_p)'$$

Theorem H.5. Let $\mathcal{A}$ be a factor in $\mathcal{B}(H)$, where $H$ is finite-dimensional. Then there exists a spatial isomorphism $^{16} H \simeq H_1 \otimes H_2$, represented by a unitary map $u \in \mathcal{B}(H)$ such that $u\mathcal{A}u^\dagger = \mathcal{B}(H_1) \otimes 1_{H_2}$.

Furthermore, for the commutator $\mathcal{A}'$ we have $u\mathcal{A}'u^\dagger = 1_{H_1} \otimes \mathcal{B}(H_2)$ $^{17}$.

We are ready to give a sketch proof of Theorem H.1 referencing [42] and [46].

Proof. The technique is known as “doubling the centre”. Note that $\mathcal{B} \subset \mathcal{A}$. If $\mathcal{A}$ is a factor, then we are done by Theorem H.5. Otherwise, since the centre $Z(\mathcal{A})$ is a finite-dimensional unital commutative $C^*$-algebra, the Gelfand–Naimark Theorem gives rise to a finite set of minimal orthogonal central projections $\{p_i\} \subset Z(\mathcal{A})$ that sum up to the identity $1_H$.

Therefore, we may identify $\mathcal{A}$ and $\mathcal{A}'$ with direct sum representations $\bigoplus_i \mathcal{A}_i$ and $\bigoplus_i \mathcal{A}'_i$, respectively, where $\mathcal{A}_i = \mathcal{A}p_i$. It can be checked that $\mathcal{A}_i$ is a factor for all $i$. Consequently, we have a direct sum decomposition $H \cong \bigoplus_i p_i(H) = \bigoplus_i H_i$.

Then, applying Theorem H.5 we split $\mathcal{A}_i$ and $\mathcal{A}'_i$ with tensor products and, subsequently, we have $H_i = H_{1i} \otimes H_{2i}$. It can be checked that the natural embedding of $H$ in $(\bigoplus_i H_{1i}) \otimes (\bigoplus_i H_{2i})$ induces the desired injective $\star$-homomorphism for $\mathcal{A}$ and $\mathcal{A}'$. Moreover, the ($C^*$-algebraic) functional on $(\bigoplus_i H_{1i}) \otimes (\bigoplus_i H_{2i})$ induced from $\text{Tr}_{\tau}(\cdot)$ can be checked as a state. Hence, finite-dimensionality implies the existence of an associated density operator $\tilde{\tau}$, which we may purify.

Remark H.6. Now we are able to describe the difficulty of showing a bilocal version of Theorem H.1, which is due to the existence of operators $\rho$ and $\sigma$. Indeed, for example, we may consider the fact that $\mathcal{B}, \mathcal{C}, \{\sigma\} \subset \mathcal{A}'$ and perform a minimal central decomposition of $\mathcal{A}$ and $\mathcal{A}'$. But then we cannot say the same about $\rho$ since $\rho \notin \mathcal{A}'$ in general, and hence it is not clear how to embed $\rho$ isomorphically in the new Hilbert space. A similar problem arises when we attempt to start with the algebra $\mathcal{C}$.

$^{16}$In the theory of operator algebra, it refers to the isomorphism induced by the adjoint action of some unitary.

$^{17}$This is a direct consequence of Lemma H.4

$^{18}$Note there is no ambiguity in the notation $\mathcal{A}'_i$ due to Lemma H.4.
One may also consider the algebra generated by $\mathcal{A}$ and $\rho$, with $C$ and $\sigma$ lying in its commutant, to obtain a splitting $\mathcal{H}_{ABL} \otimes \mathcal{H}_{BR\bar{C}}$. Although seemingly recovering Tensor Bilocal Quantum correlations in Definition 3.1, the algebra $\mathcal{B}$ may not be embedded into the splitting Hilbert space isomorphically. The failure of this example is also related to the difficulty of proving the converse of Theorem 4.3 in finite dimension, as it is unclear how to find a sensible splitting of Bob’s measurement $\mathcal{B}$.

Furthermore, evident from the proof above, we are not guaranteed that the new global state $\bar{\tau}$ admits the desired bilocal decomposition.

References

[1] A. Acín, N. Gisin, and L. Masanes, From Bell’s Theorem to Secure Quantum Key Distribution. Phys. Rev. Lett. 97 (2006), no. 12, 120405

[2] J.-D. Bancal and N. Gisin, Nonlocal boxes for networks. Physical Review A 104 (2021), no. 5

[3] J.-D. Bancal, N. Sangouard, and P. Sekatski, Noise-resistant device-independent certification of bell state measurements. Physical Review Letters 121 (2018), no. 25

[4] S. Beigi and M.-O. Renou, Covariance decomposition as a universal limit on correlations in networks. IEEE Transactions on Information Theory 68 (2022), no. 1, 384–394

[5] J. S. Bell, On the einstein podolsky rosen paradox. Physics Physique Fizika 1 (1964), 195–200

[6] B. Blackadar, Operator algebras: Theory of c*-algebras and von neumann algebras. Encyclopaedia of Mathematical Sciences, Springer Berlin Heidelberg, 2006

[7] C. Branciard, N. Gisin, and S. Pironio, Characterizing the nonlocal correlations created via entanglement swapping. Phys. Rev. Lett. 104 (2010), 170401

[8] O. Bratteli and D. W. Robinson, Operator algebras and quantum statistical mechanics. Theoretical and Mathematical Physics, Springer Berlin, Heidelberg, Berlin, 1997

[9] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality. Rev. Mod. Phys. 86 (2014), 419–478

[10] S. Burgdorf, I. Klep, and J. Povh, Optimization of polynomials in non-commuting variables. SpringerBriefs in Mathematics, Springer International Publishing, 2016

[11] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. 23 (1969), 880–884

[12] X. Coiteux-Roy, E. Wolfe, and M.-O. Renou, Any physical theory of nature must be boundlessly multipartite nonlocal. Physical Review A 104 (2021), no. 5

[13] X. Coiteux-Roy, E. Wolfe, and M.-O. Renou, No bipartite-nonlocal causal theory can explain nature’s correlations. Physical Review Letters 127 (2021), no. 20

[14] R. Colbeck and R. Renner, Free randomness can be amplified. Nat. Phys. 8 (2012), no. 6, 450–453

[15] A. Connes, Classification of injective factors cases II 1 , II $\infty$ , III $\lambda$ , $\lambda$ 1. The Annals of Mathematics 104 (1976), no. 1, 73
[16] T. Fritz, Beyond bell’s theorem: correlation scenarios. *New Journal of Physics* 14 (2012), no. 10, 103001

[17] T. Fritz, Tsirelson’s problem and kirchberg’s conjecture. *Reviews in Mathematical Physics* 24 (2012), no. 05, 1250012

[18] N. Gisin, J.-D. Bancal, Y. Cai, P. Remy, A. Tavakoli, E. Z. Cruzeiro, S. Popescu, and N. Brunner, Constraints on nonlocality in networks from no-signaling and independence. *Nature Communications* 11 (2020), no. 1

[19] V. Gittion, Outer approximations of classical multi-network correlations. 2022, URL https://arxiv.org/abs/2202.04103

[20] I. Goldbring, The connes embedding problem: A guided tour. *Bulletin of the American Mathematical Society* 59 (2022), no. 4, 503–560

[21] J. Henson, R. Lal, and M. F. Pusey, Theory-independent limits on correlations from generalized Bayesian networks. *New J. Phys.* 16 (2014), no. 11, 113043

[22] Z. Ji, A. Natarajan, T. Vidick, J. Wright, and H. Yuen, Mip*=re. *Communications of the ACM* 64 (2021), no. 11, 131–138

[23] R. Kadison and J. Ringrose, *Fundamentals of the theory of operator algebras. volume ii*. Fundamentals of the Theory of Operator Algebras, American Mathematical Society, 1997

[24] I. Klep, V. Magron, and J. Volčič, Optimization over trace polynomials. *Annales Henri Poincaré* 23 (2022), no. 1, 67–100

[25] J. B. Lasserre, A sum of squares approximation of nonnegative polynomials. *SIAM Review* 49 (2007), no. 4, 651–669

[26] C. M. Lee and M. J. Hoban, Towards device-independent information processing on general quantum networks. *Phys. Rev. Lett.* 120 (2018), 020504

[27] L. T. Ligthart, M. Gachechiladze, and D. Gross, A convergent inflation hierarchy for quantum causal structures. 2021, URL https://arxiv.org/abs/2110.14659

[28] L. T. Ligthart and D. Gross, The inflation hierarchy and the polarization hierarchy are complete for the quantum bilocal scenario. *arXiv preprint arXiv:2212.11299* (2022)

[29] D. Mayers and A. Yao, Quantum cryptography with imperfect apparatus. In *Proc. 39th symposium on foundations of computer science*, pp. 503–509, 1998

[30] M. Navascués and E. Wolfe, The inflation technique completely solves the causal compatibility problem. *Journal of Causal Inference* 8 (2020), no. 1, 70–91

[31] M. Navascués, S. Pironio, and A. Acín, A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New Journal of Physics* 10 (2008), no. 7, 073013

[32] N. Ozawa, About the connes embedding conjecture. *Japanese Journal of Mathematics* 8 (2013), no. 1, 147–183

[33] P. A. Parrilo, Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization. *PhD thesis* (2000)

[34] S. Pironio et al., Random numbers certified by Bell’s theorem. *Nature* 464 (2010), no. 7291, 1021–1024

[35] A. Pozas-Kerstjens, R. Rabelo, L. Rudnicki, R. Chaves, D. Cavalcanti, M. Navascués, and A. Acín, Bounding the sets of classical and quantum correlations in networks. *Phys. Rev. Lett.* 123 (2019), 140503
[36] M.-O. Renou and S. Beigi, Nonlocality for generic networks. *Physical Review Letters* **128** (2022), no. 6

[37] M.-O. Renou, E. Bäumer, S. Boreiri, N. Brunner, N. Gisin, and S. Beigi, Genuine quantum nonlocality in the triangle network. *Physical Review Letters* **123** (2019), no. 14

[38] M. O. Renou, J. Kaniewski, and N. Brunner, Self-testing entangled measurements in quantum networks. *Physical Review Letters* **121** (2018), no. 25

[39] M.-O. Renou, D. Trillo, M. Weilenmann, T. P. Le, A. Tavakoli, N. Gisin, A. Acín, and M. Navascués, Quantum theory based on real numbers can be experimentally falsified. *Nature* **600** (2021), no. 7890, 625–629

[40] T. V. Rotem Arnon-Friedman, Renato Renner, Simple and tight device-independent security proofs. *arXiv:1607.01797* (2016)

[41] V. Scarani, The Device-Independent Outlook on Quantum Physics. *Acta Physica Slovaca* **62** (2012), no. 4, 347

[42] V. B. Scholz and R. F. Werner, Tsirelson’s Problem. *arXiv e-prints* (2008), arXiv:0812.4305

[43] P. Sekatski, S. Boreiri, and N. Brunner, Partial self-testing and randomness certification in the triangle network. 2022, URL https://arxiv.org/abs/2209.09921

[44] I. Šupić and J. Bowles, Self-testing of quantum systems: a review. *Quantum* **4** (2020), 337

[45] A. Tavakoli, A. Pozas-Kerstjens, M.-X. Luo, and M.-O. Renou, Bell nonlocality in networks. *Reports on Progress in Physics* **85** (2022), no. 5, 056001

[46] B. Tsirelson, Bell inequalities and operator algebras. 2006, URL https://www.tau.ac.il/~tsirel/Research/bellopalg/main.html

[47] U. Vazirani and T. Vidick, Fully device-independent quantum key distribution. *Physical Review Letters* **113** (2014), no. 14

[48] M. Weilenmann and R. Colbeck, Self-testing of physical theories, or, is quantum theory optimal with respect to some information-processing task? *Physical Review Letters* **125** (2020), no. 6

[49] E. Wolfe, A. Pozas-Kerstjens, M. Grinberg, D. Rosset, A. Acín, and M. Navascués, Quantum inflation: A general approach to quantum causal compatibility. *Phys. Rev. X* **11** (2021), 021043

[50] E. Wolfe, R. W. Spekkens, and T. Fritz, The inflation technique for causal inference with latent variables. *Journal of Causal Inference* **7** (2019), no. 2

[51] I. Šupić, J. Bowles, M.-O. Renou, A. Acín, and M. J. Hoban, Quantum networks self-test all entangled states. 2022, URL https://arxiv.org/abs/2201.05032