Cycle Connectivity and Automorphism Groups of Flag Domains

Alan Huckleberry

Dedicated to Arcady Onishchik on the occasion of his 80th birthday.

Abstract
A flag domain $D$ is an open orbit of a real form $G_0$ in a flag manifold $Z = G/P$ of its complexification. If $D$ is holomorphically convex, then, since it is a product of a Hermitian symmetric space of bounded type and a compact flag manifold, $\text{Aut}(D)$ is easily described. If $D$ is not holomorphically convex, then in previous work it was shown that $\text{Aut}(D)$ is a Lie group whose connected component at the identity agrees with $G_0$ except possibly in situations which arise in Onishchik’s list of flag manifolds where $\text{Aut}(Z)^0 = \hat{G}$ is larger than $G$. In the present work the group $\text{Aut}(D)^0 = \hat{G}_0$ is described as a real form of $\hat{G}$. Using an observation of Kollar, new and much simpler proofs of much of our previous work in the case where $D$ is not holomorphically convex are given.

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1 Introduction and statement of results

Recall that if $Z$ is a compact complex manifold, then its Lie algebra $\mathfrak{g} = \text{Vect}_0(Z)$ of holomorphic vector fields is finite-dimensional and that the fields in $\mathfrak{g}$ can be integrated to define a holomorphic action of the associated simply-connected complex Lie group $G$. If $Z$ is homogeneous in the sense that this group acts transitively, then we choose a base point $z_0 \in Z$, let $H = G_{z_0}$ denote the isotropy group at that point and identify $Z$ with the quotient $G/H$. If $G$ is projective algebraic with trivial Albanese, i.e., with $b_1(Z) = 0$, then $G$ is semisimple, the isotropy group $H$ is a so-called parabolic subgroup, which from now on we denote by $P$, and $Z = G/P$ is a $G$-orbit in the projective space $\mathbb{P}(V)$ of an appropriate $G$-representation space $V$. In this case we refer to $Z$ as a flag manifold.

A real form $G_0$ of $G$ is a real Lie subgroup of $G$ such that the complexification $g_0 + i g_0$ is the Lie algebra $g$. If $Z = G/P$ is a flag manifold, then any real form $G_0$ of $G$ has only finitely many orbits in $Z$ ([W], see also [FHW] for this as well as other background.). In particular, $G_0$ always has at least one open orbit $D$. We refer to such an open orbit as a flag domain. If $G_0$ is not simple, then, $D$ has product structure corresponding to the factors of $G_0$. Thus, for our considerations here there is no loss of generality in assuming that $G_0$ is simple which we do throughout. Note that if $G_0$ has the abstract structure of a complex Lie group, then its complexification $G$ is, however, not simple. Note also that $G_0$ could act transitively on $Z$, e.g., this is always the case for a compact real form. However, from the point of view of this article, in that case all phenomena are well understood and therefore we assume that $D$ is a proper subset of $Z$.

Since by assumption a flag domain $D$ is noncompact, there is no a priori reason to expect that Aut($D$) or Vect$_0(D)$ is finite-dimensional. In fact if $D$ possesses non-constant holomorphic functions, the latter is not the case and the former is often not the case as well. Let us begin here by reviewing this situation.

If $X$ is any complex manifold, then the equivalence relation,

$$x \sim y \Leftrightarrow f(x) = f(y) \text{ for all } f \in O(X),$$

is equivariant with respect to the full group Aut($X$) of holomorphic automorphisms. If $X = G/H$ is homogeneous with respect to a Lie group of holomorphic transformations, then the reduction $X \to X/\sim$ by this equivalence relation is a $G$-equivariant holomorphic homogeneous fibration $G/H \to G/I$. If $D = G_0/H_0$ is a flag domain, then this reduction has a particularly simple form ([W], [FHW],§4.4). For this let $D = G_0/z_0$ with $H_0$ (resp $P$) be the $G_0$-isotropy subgroup (resp. $G$-isotropy subgroup) at $z_0$. 
Theorem 1.1 If $D = G_0/z_0$ is a flag domain with $O(D) \neq \mathbb{C}$, then the holomorphic reduction $D = G_0/H_0 \to G_0/I_0 = \tilde{D}$ is the restriction of a fibration $Z = G/P \to G/P = \tilde{Z}$ of the ambient flag manifold with the properties

1. The fiber of $Z \to \tilde{Z}$, which is itself a flag manifold, agrees with the fiber of $D \to \tilde{D}$.
2. The base $\tilde{D}$ is a $G_0$-flag domain in $\tilde{Z}$. It is a Hermitian symmetric space of noncompact type embedded in a canonical way in its compact dual $\tilde{Z}$.

Recall that a symmetric space of noncompact type of a simple Lie group is a topological cell and that in the Hermitian case it is a Stein manifold. Thus Grauert’s Oka-principle implies that the fibration $D \to \tilde{D}$ is a (holomorphically) trivial bundle. As a consequence we have the following more refined version of the above result.

Corollary 1.2 A flag domain $D$ with $O(D) \neq \mathbb{C}$ is the product $\tilde{D} \times F$ of a Hermitian symmetric space $\tilde{D}$ of noncompact type and a compact flag manifold $F$. In particular, $D$ is holomorphically convex and $D \to \tilde{D}$ is its Remmert reduction.

As indicated above our goal here is to describe the connected component at the identity $\text{Aut}(D)^0$ of the group of holomorphic automorphisms of any given flag domain $D$. With certain exceptions which we cover in detail below, we carried out this project in [H1] by studying the associated action of $\text{Aut}(D)$ on a certain space (described below) $C_q(D)$ of holomorphic cycles. If $D = \tilde{D}$ is a Hermitian symmetric space of noncompact type, such cycles are just isolated points and $C_q(D) = D$. Thus the cycle space gives us no additional information. However, in this case $D$ possesses the invariant Bergman metric and as a result $\text{Aut}(D)$ is well-understood.

If $D = \tilde{D} \times F$ is a product with nontrivial base and fiber, then, although it is infinite-dimensional, $\text{Aut}(D)$ is in a certain sense easy to describe: The fibration $D \to \tilde{D}$ induces a surjective homomorphism $\text{Aut}(D) \to \text{Aut}(\tilde{D})$. The kernel is the space $\text{Hol}(\tilde{D}, \text{Aut}(F))$ of holomorphic maps from the base to the complex Lie group $\text{Aut}(F)$ and as a result $\text{Aut}(D) = \text{Hol}(\tilde{D}, \text{Aut}(F)) \rtimes \text{Aut}(D)$ has semidirect product structure.

Having settled the case where $O(D) \neq \mathbb{C}$, or equivalently where $D$ is holomorphically convex, we turn to the situation where $O(D) = \mathbb{C}$. In [H1] we showed that $\text{Aut}(D)$ is a (finite-dimensional) Lie group which, with certain exceptions which are handled below, $\text{Aut}(D)^0 = G_0$. Other than taking care of these exceptional cases, where in fact $\text{Aut}(D)^0$ contains $G_0$ as a proper Lie subgroup, here we also make use of an observation of Kollar ([K]) which leads to a simple proof of $\text{Aut}(D)^0 = G_0$ with the possible exceptions. This proof is given in §2.
Before going into the details of proofs, let us state the main result of the paper. For this the following classification theorem of A. Onishchik ([O1]) is the key first step for handling the exceptional cases mentioned above.

**Theorem 1.3** The following is a list of the flag manifolds $Z$ and (connected) complex simple Lie groups $G$ and $\hat{G}$ so that $Z = G/P$ and $\hat{G} := \text{Aut}(Z)^0$ properly contains $G$.

1. The manifold $Z$ is the odd-dimensional projective space $\mathbb{P}(\mathbb{C}^{2n})$ where, after lifting to simply-connected coverings, $G = \text{Sp}_{2n}(\mathbb{C})$ and $\hat{G} = \text{SL}_{2n}(\mathbb{C})$.
2. The 5-dimensional complex quadric $Z$ is equipped with the standard action of $\hat{G} = \text{SO}_7(\mathbb{C})$ and $G$ is the exceptional complex Lie group $G_2$ embedded in $\hat{G}$ as the automorphism group of the octonians.
3. Equipping $\mathbb{C}^{2n}$ with a non-degenerate complex bilinear form $b$, $Z$ is the space of $n$-dimensional $b$-isotropic subspaces, $\hat{G}$ is the $b$-orthogonal group $\text{SO}_{2n}(\mathbb{C})$ and $G$ is the complex orthogonal group $\text{SO}_{2n-1}(\mathbb{C})$ which is embedded in $\hat{G}$ as the connected component at the identity of the isotropy group of the $\hat{G}$-action at some nonzero point in $\mathbb{C}^{2n}$.

Referring to the above list of exceptions as Onishchik’s list, our main result can be stated as follows.

**Theorem 1.4** If $D$ is a $G_0$-flag domain in $Z = G/Q$, then $\text{Aut}(D)$ can be described as follows:

1. If $O(D) \neq \mathbb{C}$, or equivalently if it is holomorphically convex, $D$ is a product $D \times F$ of a Hermitian symmetric space $D$ of non-compact type and a compact flag manifold $F$, and $\text{Aut}(D)$ is correspondingly a semidirect product $\text{Aut}(D) = \text{Hol}(D, \text{Aut}(F)) \rtimes \text{Aut}(D)$.
2. If $O(D) = \mathbb{C}$, then $\text{Aut}(D)$ is a finite-dimensional Lie group of holomorphic transformations on $D$ and, if the complexification $G$ is the full group $\text{Aut}(Z)^0$, then $\text{Aut}(D)^0 = G_0$.
3. If $O(D) = \mathbb{C}$ and $G$ is a proper subgroup of $\hat{G} = \text{Aut}(Z)^0$, then in each case of Onishchik’s list $\text{Aut}(D)^0 = \hat{G}_0$ is a uniquely determined real form of $\hat{G}$ which contains $G_0$ as a proper subgroup.

It should be remarked that the simple proof given here of the fact that if $O(D) = \mathbb{C}$, then $\text{Aut}(D)^0$ is a Lie group acting on $Z$ does not yield a proof that in this case full group $\text{Aut}(D)$ is a Lie group. At the present time we have no other proof of this fact other than that in [H1].

### 2 Cycle connectivity

In [H2] we used chains of cycles to study the pseudoconvexity and pseudoconcavity of flag domains. We continued the use of these chains in our
study of Aut(D) in [H1]. Here, in particular compared to the chains in [K], it is
sufficient to consider chains of a very special type which we now introduce.

A basic fact, which is the tip of the iceberg of Matsuki duality, is that for a
flag domain D any given maximal compact subgroup $K_0$ of $G_0$ has exactly
one orbit $C_0 = K_0 z_0$ in D which is a complex submanifold. In fact it is
the (unique) orbit of minimal dimension. If $K$ is the complexification of $K_0$,
then since $C_0$ is complex, $K$ stabilizes it. Denoting $q := \dim_C C_0$, we usually
regard $C_0$ as a point in the Barlet cycle space $C_q (D)$, but for our purposes
here we may regard it as a point in the full Chow space $C_q (Z)$ where $G$
is acting algebraically. The group theoretical cycle space of $D$ is then defined
as connected open subset

$$M(D) = \{ g(C_0) : g \in G, \ g(C_0) \subset D \}$$

of the orbit of the base cycle $C_0$. One can show that $M(D)$ is a closed sub-
manifold of $C_q (D)$ (See [FHW] for background and a systematic study of
these cycle spaces.). For the purposes of this paper a chain of cycles is a finite
connected union of (supports of) cycles in $M(D)$. We often write such a chain
as $(C_1, \ldots, C_m)$ to indicate that $C_i \cap C_{i+1} \neq \emptyset$
Using such chains we have the cycle connection equivalence relation

$$x \sim y \iff x \text{ and } y \text{ are contained in a chain}.$$

Note that this relation is $G_0$-equivariant. In particular, if $D = G_0 / H_0$ then
there is a (possibly not closed) subgroup $I_0$ of $G_0$ which contains $H_0$ so that
the quotient of $D$ by this equivalence relation is given by $G_0 / H_0 \to G_0 / I_0$.
Now if $z_0 \in D$ is the base point where $H_0 : = G_{z_0}$ and $K_0 z_0 = K z_0 = C_0$
is the base cycle, then, since $C_0$ is by definition contained in the equivalence class
of $z_0$, it is immediate that $I_0 \supset K_0$. Since $K_0$ is a maximal subgroup of $G_0$, i.e.,
any (not necessarily closed) subgroup of $G_0$ which contains $K_0$ is either $K_0$ or
$G_0$, the following is immediate (see also [H1] and [H2] for the same proof).

**Proposition 2.1** The following are equivalent:

1. $O(D) = \emptyset$
2. $D$ is not holomorphically convex.
3. There is no nontrivial $G_0$-equivariant holomorphic map of $D$ to a Hermitian
   symmetric space $\tilde{D}$ of noncompact type.
4. $D$ is cycle connected.

**Proof.** The equivalence of the first three conditions follows from the discus-
sion in §1. If $D$ is cycle connected, then, since $\tilde{D}$ is Stein and therefore every
holomorphic map to $\tilde{D}$ is constant along every chain, 4.) $\Rightarrow$ 3.). Conversely,
if $D$ is not cycle connected, then the equivalence class containing the base
point $z_0$ is just the cycle $C_0$ which is therefore stabilized by the $G$-isotropy $P$
as well as $K$. Since the cycle connection reduction is given by $G_0 / H_0 \to G_0 / K_0$,
it follows that this fibration is the restriction of the fibration $G/P \to G/\tilde{P}$ of $Z$ where $\tilde{P} = KP$ and therefore the base $G_0/K_0$ is the Hermitian symmetric space $\tilde{D}$. In other words, the cycle connected reduction is just the holomorphic reduction and in particular $\mathcal{O}(D) \neq \mathbb{C}$.

**Remark.** For applications in another context, Griffiths, Robles and Toledo recently gave another proof a result which is essentially equivalent to Proposition 2.1. (see [GRT]).

Although it is well-known that $K_0$ is a maximal subgroup of $G_0$, for the convenience of the reader we would like to give the following nice proof of J. Brun which was pointed out to us by Keivan Mallahi Karai (see the Appendix of [B]).

**Theorem 2.2** If $G$ is a connected simple Lie group, $K$ is a maximal compact subgroup and $L$ is an abstract group which contains $K$, then $L$ is either $G$ or $K$.

**Proof.** Standard results in the theory of symmetric spaces show that $K$ is connected and the adjoint representation of $K$ on $\mathfrak{g}/\mathfrak{k}$ is irreducible. Thus if $\ell$ is a Lie subalgebra of $\mathfrak{g}$ which properly contains $\mathfrak{k}$, then $\ell = \mathfrak{g}$. Thus, if $L$ is closed, then the result is immediate. Furthermore, if $L$ is not closed and properly contains $K$, then its closure $cl(L)$ is the full group $G$. In that case we let $t'$ be the vector subspace of the Lie algebra $\mathfrak{g}$ of $G$ which is generated by $\text{Ad}(x)(\mathfrak{k})$ for all $x \in L$. Since $cl(L) = G$, it follows by continuity that $t'$ is $G$-invariant and since $G$ is simple, it is immediate that $t' = \mathfrak{g}$. Therefore there are finitely many elements $x_i \in L$ so that

$$\mathfrak{g} = \sum_{i=1}^{m} \text{Ad}(x_i)(\mathfrak{k})$$

and as a result the map

$$K^m \to G, \ (k_1, \ldots, k_m) \mapsto \prod (x_i k_i x_i^{-1})$$

has maximal rank at the origin. Thus $L$ contains a compact neighborhood of the origin and is therefore compact, i.e., contrary to assumption $L = K$.

**3 Finiteness Theorem**

Our original goal in this setting was to show that a flag domain $D$ is either pseudoconvex or pseudoconcave ([H2]). More precisely, we had hoped to show that if $D$ is not holomorphically convex, then $C_0$ has a pseudoconcave neighborhood which is filled out by cycles. If this would be possible, then using Andreotti’s finiteness theorem ([A]) we would be able to conclude that
the space of sections of any holomorphic vector bundle, in particular the space $\text{Vect}_0(D)$, is finite-dimensional. Although we have been successful in constructing such a neighborhood in a number of cases ([H2]), we have failed to do this in general. Recently, in a substantially more general setting, Kollar proved the desired finiteness theorem along with a number of equivalent properties which would follow from the pseudoconcavity of $D ([K])$. Here we make use of Kollar’s result, leaving the question of existence of the pseudoconcave neighborhood open.

Formulated in our setting, Kollar’s finiteness result can be stated as follows.

**Theorem 3.1** The space $\Gamma(D,E)$ of sections of any holomorphic vector bundle on a cycle connected flag domain is finite-dimensional.

This is an immediate consequence of the same result for line bundles which in turn is proved using the following Lemma (Lemma 15 in [K]), again formulated in our restricted context.

**Lemma 3.2** Let $L$ be a holomorphic line bundle on $D$. Then, given $d \in \mathbb{N}$ there exists $d_0 \in \mathbb{N}$ so that for every $C \in \mathcal{M}_0$ and any $z_0 \in C$ every section $s \in \Gamma(D,L)$ which vanishes of order $d_0$ at $z_0$ vanishes of order $d$ along $C$.

The proof is given by classical methods which are reminiscent of Siegel’s Schwarz Lemma. One key point is that $C$ can be filled out by rational curves which in our case are closures of orbits of 1-parameter groups.

Now, given a chain of cycles $(C_1, \ldots, C_m)$ with $z_i \in C_i \cap C_{i+1}$, and given $d_m \in \mathbb{N}$ we apply the Lemma to obtain $d_{m-1} \in \mathbb{N}$ so that if $s$ vanishes of order $d_{m-1}$ at $z_{m-1}$, then it vanishes of order $d_m$ along $C_m$. Working backwards to the first cycle in the chain, we see that the Lemma holds for chains.

**Corollary 3.3** Given $d \in \mathbb{N}$ there exists $d_1 \in \mathbb{N}$ so that for any chain $(C_1, \ldots, C_m)$ of length $m$ and any $z_1 \in C_1$ if $s$ vanishes of order $d_1$ at $z_1$, then it vanishes of order $d$ along $C_m$.

It should be emphasized that for a fixed $d$ the required vanishing order $d_1$ depends on $m$. Thus to apply this result we need some sort of uniform estimate for the length of a chain connecting two given points. This can be given as follows.

For example, let $C_1$ be a base cycle for a given maximal compact subgroup $K_0$. Recall that the complexification $K$ has only finitely many orbits in $Z$ and therefore has a (unique) open dense orbit $\Omega$. Take $z_1 \in C_1$ and any point $z \in \Omega$ and let $(C_1, \ldots, C_m)$ be a chain connecting $z_0$ to $z$. For $k \in K$ sufficiently close to the identity, the chain $(k(C_1), \ldots, k(C_m))$ is still contained in $D$. Thus, since $k(C_1) = C_1$ and $k(z)$ can be an arbitrary point in a sufficiently small neighborhood $U$ of $z$, we have the desired vanishing theorem.

**Corollary 3.4** If $s \in \Gamma(D,L)$ vanishes of sufficiently high order at a given point $z_1 \in C_1$, then it vanishes identically. In particular, $\Gamma(D,L)$ is finite-dimensional.
Proof. Since the required vanishing order \( d_1 \) only depends on the number \( d_m \) and the length \( m \), Corollary 3.3 implies that if \( s \) vanishes of order \( d_1 \) at \( z_1 \), then it vanishes at every point of the set \( U \) which was constructed above. The desired result then follows from the identity principle.

As we remarked above, the finiteness theorem for vector bundles is an immediate consequence of this Corollary (see [K], p. 8).

4 Integrability of vector fields

The following is the main result of this section.

**Theorem 4.1** Let \( Z = G/Q \) be a complex flag manifold and \( \hat{g} \) be a finite-dimensional complex Lie algebra which contains \( g := \text{Lie}(G) \). Let \( \hat{G} \) be a complex Lie group which contains \( G \) and is associated to \( \hat{g} \). If \( \hat{q} \) is a complex subalgebra of \( \hat{g} \) so that the quotient map \( \hat{g} / \hat{q} \to g/q \) induces an isomorphism

\[
\hat{g} / \hat{q} = g / q,
\]

then \( \hat{G} \) acts holomorphically on \( Z \) with

\[
Z = \hat{G} / \hat{Q} = G / Q.
\]

**Proof.** We apply a basic idea of Tits. For this regard \( x_0 : = \hat{q} \) as a point in the Grassmannian \( X := \text{Gr}_k(\hat{g}) \) of subspaces of dimension \( k = \dim \hat{q} \) in \( \hat{g} \). The isotropy group at \( x_0 \) of the \( \hat{G} \)-action on \( X \) is the normalizer

\[
\hat{N} = \{ \hat{g} \in \hat{G} : \text{Ad}(\hat{g})(\hat{q}) = \hat{q} \}.
\]

Denote by \( N = \hat{N} \cap G \) the \( G \)-isotropy at \( x_0 \) and note that if \( g \in N \) and \( \xi \in q \), it follows that \( \text{Ad}(\xi)g \in \hat{q} \cap g = q \). In other words \( N \) is contained in the normalizer of \( q \) in \( g \). Since the parabolic group \( Q \) is self-normalizing in \( G \), it follows that \( N \subset Q \). But \( \hat{n} \supset \hat{q} \) and \( \hat{q} \cap g = q \). Therefore \( n \supset q \). Consequently \( N = Q \) and the \( \hat{G} \)-orbit of \( \hat{q} \) is the compact manifold \( Z = G / Q \). Since \( \hat{g} / \hat{q} = g / q \), the \( \hat{G} \)-orbit of \( \hat{q} \) has dimension at most that of \( G / Q \). But on the other hand \( \hat{G} \supset G \) and therefore the \( \hat{G} \)-orbit has the same dimension as the \( G \)-orbit. Consequently \( G.x_0 \) is open in \( G x_0 \) and the compactness of \( G.x_0 \) implies that these orbits agree.

Applying the Finiteness Theorem, the following is now immediate.

**Corollary 4.2** Let \( G_0 \) be a simple real form of a complex semisimple Lie group \( G \) and \( D \) be a cycle connected \( G_0 \)-flag domain in a \( G \)-flag manifold \( Z = G / Q \). Let \( \overline{\mathfrak{g}} \) be the Lie algebra of holomorphic vector fields on \( D \). Then the restriction mapping
R : aut(Z) → G is an isomorphism and the action of G can be integrated to the action of a connected complex Lie group G which is thereby identified with Aut(Z)^0.

As a consequence we have the description of Aut(D) which was proved by other means in [H1].

Corollary 4.3 If D is a G_0-flag domain Z = G/Q, then one of the following holds:
1. If D is holomorphically convex, it is a product of a compact flag manifold and Hermitian symmetric space of noncompact type and Aut(D)^0 can be described as in §1.
2. If D is not holomorphically convex or equivalently it is cycle connected, then Aut(D)^0 is a finite-dimensional Lie group which is acting on Z and agrees with G_0 with the possible exceptions in the situations classified by Onishchik where G is a proper subgroup of G = Aut(Z)^0.

5 Exceptional cases

To complete our project of understanding the automorphism groups of flag domains, we must analyze the exceptional cases indicated in the above Corollary. We do this here, proving the following result.

Theorem 5.1 Suppose that D ⊂ Z = G/Q is a G_0-flag domain which is not holomorphically convex and that G is properly contained in complex Lie group G = Aut(Z)^0. Then there is a uniquely determined real form G_0 = Aut(D)^0 of G which properly contains G_0 and which stabilizes D.

Our proof of this fact amounts to a concrete discussion for each of the three classes of exceptions in Onishchik’s list which was given in §1. Below we show that these cases not only occur but also occur at the level of real forms. This is the content of (3) in Theorem 1.4 and Theorem 5.1 above.

5.1 Projective space

Here we consider the case where Z = P(V) is the projective space of an even-dimensional complex vector space V = C^{2n}. Define the complex bilinear form b by b(z, w) = z^t w. In the standard basis (e_1, ..., e_{2n}) define J : V → V by J(e_i) = e_{n+i}, i ≤ n, and J(e_i) = -e_{i-n}, i > n. Note J is b-orthogonal with J^2 = -Id and define a (complex, bilinear) symplectic form by ω(z, w) = z^t Jw. Define V_+ := Span{e_1, ..., e_n} and V_- = Span{e_{n+1}, ..., e_{2n}} and correspondingly E := +Id ⊕ -Id. If C : V → V denotes the standard complex conjugation given by z → z, define a non-degenerate (mixed-signature) Hermitian structure on V by h(z, w) = z^t EC(w). Finally, if the antilinear map ϕ : V → V is defined by
\[ z \mapsto -JEw, \text{ it follows that } h(z, w) = \alpha(z, \varphi(w)) \text{ and, since } \varphi^2 = -\text{Id}, \text{ that } \varphi \text{ is an } h\text{-isometry. Observe that if } P \text{ is a } \varphi\text{-invariant subspace of } V, \text{ then } P^{\perp_\varphi} = P^\perp. \]

\[ \text{In particular, } P \text{ is symplectic if and only if it is } h\text{-nondegenerate and in either of these cases } V = P \oplus P^\perp_h \text{ is a decomposition of } V \text{ into } h\text{-nondegenerate, symplectic subspaces.} \]

The complex symplectic group \( G = \text{Sp}_{2n}(\mathbb{C}) \) defined by \( \alpha \) has two types of real forms. The first case to be considered is where \( G_0 \) is the real form \( SU(n,n) \) of \( \tilde{G} = \text{SL}_{2n}(\mathbb{C}) \) which is defined as the group of \( h\)-isometries. In this case the real form \( G_0 = \text{Sp}_{2n}(\mathbb{R}) \) of \( G = \text{Sp}_{2n}(\mathbb{C}) \) is defined as the intersection \( G_0 \cap \text{Sp}_{2n}(\mathbb{C}) \). Considering the orbits of these groups on \( \mathbb{P}(V) \) we let \( D_+ \) (resp. \( D_- \)) be the open sets in \( \mathbb{P}(V) \) of \( h\)-positive (reps. \( h\)-negative) lines.

**Proposition 5.2** The open sets \( D_+ \) and \( D_- \) are both orbits of \( G_0 \) and \( \tilde{G}_0 \).

**Proof.** It is clear that \( D_+ \) and \( D_- \) are \( G_0 \)- and \( \tilde{G}_0 \)-invariant and that \( D_+ \cup D_- \) is dense in \( \mathbb{Z} \). Since \( G_0 \subset \tilde{G}_0 \), it is therefore enough to show that \( G_0 \) acts transitively on both sets. The proof for \( D_+ \) is exactly the same as for \( D_- \) and therefore we only give it for \( D_+ \). For this, given positive lines \( L = \mathbb{C}z \) and \( \tilde{L} = \mathbb{C}\tilde{z} \), we define \( P = \text{Span}[z, \varphi(z)] \) and \( \tilde{P} = \text{Span}[\tilde{z}, \varphi(\tilde{z})] \). These planes are \( h\)-nondegenerate and symplectic. We normalize \( z \) and \( \tilde{z} \) so that \( ||z||_{\tilde{L}}^2 = ||\tilde{z}||^2 = 1 \) and, since \( \varphi : E_+ \to E_- \), \( ||\varphi(z)||^2 = ||\varphi(\tilde{z})||^2 = -1 \). Applying this procedure to \( P^+ \) and \( \tilde{P}^+ \), we have \( h\)- and \( \alpha\)-orthogonal decompositions

\[ V = P_1 \oplus \ldots \oplus P_n = \tilde{P}_1 \oplus \ldots \oplus \tilde{P}_n \]

of \( V \). Furthermore, every \( P_i \) (resp. \( \tilde{P}_i \)) comes equipped with a basis \( (z_i, \varphi(z_i)) \) (resp. \( (\tilde{z}_i, \varphi(\tilde{z}_i)) \)) such that the mapping \( T_i : P_i \to \tilde{P}_i \) defined by \( z_i \mapsto \tilde{z}_i \) and \( \varphi(z_i) \mapsto \varphi(\tilde{z}_i) \) is both symplectic and an \( h\)-isometry. It follows that \( T = T_1 \oplus \ldots \oplus T_n \) is both a symplectic isomorphism and \( h\)-isometry of \( V \), i.e., \( T \in G_0 \). Since \( T(L) = \tilde{L} \), the proof is complete.

Now let us turn to the real form \( G_0 = \text{Sp}(2p,2q) \) of \( G = \text{Sp}_{2n}(\mathbb{C}) \). In this case we line up \( J \) and \( E \) in a different way. The decomposition \( V := V_+ \oplus V_- \) and \( J \) are the same, but now \( h \) has signature \( (p,q) \) on both spaces, being defined by the block diagonal matrix \( E_{p,q} = (\text{Id}_p, -\text{Id}_q) \). Then \( G_0 = SU(2p,2q) \) is defined as above by the Hermitian form \( h \) and \( G_0 = G \cap \tilde{G}_0 \). The proof of the following fact is exactly the same as that of Proposition 5.2 above.

**Zusatz.** Proposition 5.2 also holds for \( G_0 = \text{Sp}(2p,2q) \) and \( \tilde{G}_0 = SU(2p,2q) \). \( \square \)
5.2 5-dimensional quadric

Here we consider $V = \mathbb{C}^7$ equipped with the complex bilinear form $b$ defined by $||z||^2_b = (z_1^2 + z_2^2 + z_3^2) - (z_4^2 + \ldots + z_7^2)$ and Hermitian form $h$ defined by $||z||^2_h = (|z_1|^2 + |z_2|^2 + |z_3|^2) - (|z_4|^2 + \ldots + |z_7|^2)$. Denote by $\hat{G} = \text{SO}_7(\mathbb{C})$ the associated complex orthogonal group and by $\hat{G}_0 := \text{SO}(3,4)$ the associated group of Hermitian isometries.

We regard the exceptional complex Lie group $G = G_2$ as being embedded in $\hat{G}$ as the automorphism group $\text{Aut}(\mathbb{O})$ of the octonians. It has a unique noncompact real form $\hat{G}_0 = \text{Aut}(\mathbb{O})$, the automorphism group of the split octonians $\mathbb{O}$. In this way $\hat{G}_0$ is the intersection $G \cap \hat{G}_0$ of $G$ with the real from $\hat{G}_0 = \text{SO}(3,4)$ (see, e.g., [Ha] for details). Note that $\hat{G}_0$ is invariant by the standard complex conjugation $z \mapsto \bar{z}$.

The remainder of this paragraph is devoted to the proof of the following fact.

**Proposition 5.3** For every $z \in Z$ it follows that $G_0z = \hat{G}_0z$. In particular the open orbits of $G_0$ and $\hat{G}_0$ coincide.

We should note that, as indicated below, the open orbits of $\hat{G}_0$ are the spaces $D_+$ and $D_-$ of positive and negative lines, respectively.

For the proof of Proposition 5.3 we use Matsuki duality (see, e.g., Chapter 8 in Part II of [FHW]) which states that there is a 1–1 correspondence between the $G_0$-orbits and $K$-orbits in $Z$. This can be given as follows: For every $G_0$-orbit there is a unique $K$-orbit which intersects it in the unique $K_0$-orbit of minimal dimension and vice versa, i.e., given a $K$-orbit there is a unique $G_0$-orbit which intersects it in the unique $K_0$-orbit of minimal dimension. Due to our interest in the open $G_0$-orbits (resp. $\hat{G}_0$-orbits) in $Z$, we have stated the above result on that side of the duality. However, we have found it more convenient to prove the corresponding dual statement.

Let us fix the maximal compact subgroup $K_0 \cong (\text{SU}_2 \times \text{SU}_2)/(-\text{Id},-\text{Id})$ of $G_0$ being diagonally embedded in the maximal compact subgroup $\hat{K}_0 = \text{SO}(3) \times \text{O}(4))$ of $\hat{G}_0$. If $E_+ := \text{Span}(e_1,e_2,e_3)$ and $E_- := \text{Span}(e_4,\ldots,e_7)$, then we define $z_+ := e_1 + ie_2, z_- := e_4 + ie_5$ and observe that the base cycles $C_+$ and $C_-$ for the open orbits of the $\hat{G}_0$-action are the quadrics of $b$-isotropic lines in $E_+$ and $E_-$, respectively. The corresponding open orbits are the spaces $D_+ = \hat{G}_0z_+$ of positive lines in $Z$ and $D_- = \hat{G}_0z_-$ of negative lines, respectively. The complement of $D_+ \cup D_-$, which is the space of lines which are both $b$- and $h$-isotropic, consists of two $\hat{G}_0$-orbits, the real points $Z_\mathbb{R}$ and its complement.
The \( \hat{K} \)-orbits which correspond via Matsuki duality to the four \( \hat{G}_0 \)-orbits are the two base cycles \( C_+ \) and \( C_- \), the open \( \hat{K} \)-orbit of any point on \( Z_\mathbb{R} \) and a forth orbit \( O \) which has two ends, i.e., which has the two base cycles on its boundary. In fact this forth orbit is a \( \mathbb{C}^* \)-principal bundle over the 2-dimensional cycle \( C_- \). (See \[FHW\], §16.4 for a detailed discussion in the case of the K3-period domain which can be transferred verbatim to the case at hand.) To prove the above Proposition 5.3 we show that \( K \) acts transitively on each of these four \( \hat{K} \)-orbits.

Now the second factor of \( \hat{K} \) acts trivially on \( C_+ \) and the first factor acts trivially on \( C_- \) and vice versa. Since \( K \) is diagonally embedded in \( \hat{K} \) and projects onto both factors, it is immediate that it acts transitively on both \( C_+ \) and \( C_- \) as well. Since \( O \) is a \( \mathbb{C}^* \)-bundle over \( C_- \), \( K \) acts transitively on the base of this bundle and has an open orbit in the bundle space \( O \), it is immediate that it acts transitively on \( O \).

It remains to show that \( K \) acts transitively on the open \( \hat{K} \)-orbit. For this we first note that, since \( e_3 + e_4 \in Z_\mathbb{R} \) and the connected component at the identity of \( \hat{K}_0 \) is the product of the special orthogonal groups of \( E_+ \) and \( E_- \), it follows that up to finite group quotients \( Z_\mathbb{R} \) is the corresponding product \( S^2 \times S^3 \) of spheres. One immediately observes that \( G_0 \) acts transitively on \( Z_\mathbb{R} \), because every \( G_0 \)-orbit is at least half-dimensional over \( \mathbb{R} \). Thus \( K_0 \) acts transitively on \( Z_\mathbb{R} \) and if \( z_\mathbb{R} \) is an arbitrary point of \( Z_\mathbb{R} \), it follows that \( K.z_\mathbb{R} \) is open in \( \hat{K}.z_\mathbb{R} \).

To complete the proof of Proposition 5.3 we must show that \( K.z_\mathbb{R} = \hat{K}.z_\mathbb{R} \). For this we let \( \hat{K}_1 \) be the first factor of the product decomposition of the connected component of \( \hat{K} \) and consider the homogeneous fibration

\[
\hat{K}.z_\mathbb{R} = \hat{K}/L \to \hat{K}/\hat{K}_1L = \hat{K}_2/L_2 = B,
\]

Since \( Z_\mathbb{R} \) is essentially a product \( S^2 \times S^3 \) corresponding to the decomposition of the connected component \( \hat{K}_0 \), it follows that up to finite group quotients the base \( B \) is the complexification of \( S^3 \), i.e., the affine quadric \( Q^{(3)} = SO_4(\mathbb{C})/SO_3(\mathbb{C}) \). Since \( K \) projects surjectively onto both factors of \( \hat{K} \), it is immediate that \( K \) acts transitively on \( B = K/M \). Now the induced fibration of \( K.z_\mathbb{R} \) is a homogeneous bundle \( K/L \to K/M \) where the fiber \( M/L \) is an open \( M \)-orbit in the corresponding fiber \( \tilde{F} \) of the \( \hat{K} \)-bundle \( \hat{K}/L \to \hat{K}/M \). But \( M \) acts on this fiber as \( SO_3(\mathbb{C}) \) so that \( \tilde{F} \) is the affine quadric \( Q^{(3)} \). Since \( K/M \) is affine, \( M \) is reductive. But the only reductive subgroup of \( SO_3(\mathbb{C}) \) with an open orbit in \( Q^{(3)} \) is \( SO_3(\mathbb{C}) \) itself. Consequently \( K \) does indeed act transitively on the open \( \hat{K} \)-orbit and the proof of Proposition 5.3 is complete. \( \square \)
5.3 Space of isotropic $n$-planes in $\mathbb{C}^{2n}$

Now let $\mathring{V} = \mathbb{C}^{2n}$ be equipped with its standard basis $(e_1, \ldots, e_{2n})$ and complex bilinear form defined by $b(z, w) = z^t w$. The complex orthogonal group $SO_{2n}(\mathbb{C})$ of $b$-isometries is denoted by $\mathring{G}$. We let $G := \text{Fix}_{\mathring{G}}(e_{2n})$. In this way $G \cong SO_{2n-1}(\mathbb{C})$ is the orthogonal group of the the restriction of $b$ to $V := \text{Span}[e_1, \ldots, e_{2n-1}]$. We consider the action of these groups on the flag manifold $Z$ of $n$-dimensional $b$-isotropic subspaces of $\mathring{V}$.

**Proposition 5.4** The groups $G$ and $\mathring{G}$ act transitively on $Z$.

**Proof.** Note that the intersection $W := \mathring{W} \cap V$ of an isotropic $n$-plane in $\mathring{V}$ is an isotropic $(n-1)$-plane in $V$. It follows that $\mathring{W} = W \oplus \mathbb{C}(v + i e_{2n})$ for some $v \in V$. Applying an appropriate element of $G$, we may assume that $v = e_{2n-1}$ and it then follows that $W \subset \text{Span}[e_1, \ldots, e_{2n-2}]$. We then apply the induction assumption to obtain a transformation in the corresponding $SO_{2n-2}(\mathbb{C})$ to bring $W$ to the normal form with basis $(e_1 + i e_{n+1}, \ldots, e_{n-1} + i e_{2n-2})$ so that altogether we have found a transformation in $G$ which brings $W$ to the normal form with the basis $(e_1 + i e_{n+1}, \ldots, e_{2n-1} + i e_{2n})$.

Recall that up to conjugation the only real forms of $SO_{2n-1}(\mathbb{C})$ are the isometry groups $G_0 = SO(p, q)$ for the mixed signature Hermitian form defined by $h(z, w) = z^t E q(w)$ on $V$ where $E = E_{pq}$ is defined in the same way as in §5.1. Without loss of generality we may choose $h$ to be this form and note that an appropriately chosen arbitrarily small perturbation of an isotropic $n$-plane $\mathring{W}$ will result in the intersection $W = \mathring{W} \cap V$ being $h$-nondegenerate. Thus, if $G_0, z =: D$ is an open orbit in $Z$, the $(n-1)$-plane $W$ associated to $z$ is $h$-nondegenerate.

Note that if $p$ is even, then $q$ is odd and vice versa. To make the notation more explicit, we assume that $p$ is even. Now the space of $h$-positive $b$-isotropic lines in $V$ is an open $G_0$-orbit. Thus, given $W$ as above, we may apply an element $g \in G_0$ so that after replacing $\mathring{W}$ by $g(W)$ we have $L = \mathbb{C}(e_1 + i e_2) \subset W$. Notice that subspace of $V$ of vectors which are both $h$- and $b$-orthogonal to $L$ is simply $\text{Span}[e_3, \ldots, e_{2n-1}]$. Thus, after going to this smaller space, we have the same situation as before. Hence we may continue on by induction to obtain a maximal $h$-positive subspace $W_+$ of $W$ which is $\frac{n}{2}$-dimensional and which has a distinguished basis produced by our procedure. Applying the same argument as above to the $h$-complement $W_-^\perp$ in $W$, one obtains an element $g \in G_0$ so that $W_0 := g(W)$ has the distinguished basis

$$(e_1 + i e_2, e_3 + i e_4, \ldots, e_{p-1} + i e_p, e_{p+1} + i e_{p+2}, \ldots, e_{2n-3} + i e_{2n-2}).$$

**Proposition 5.5** If $\mathring{W}$ is a $b$-isotropic $n$-plane in $\mathring{V}$, then there exists an element $g \in G_0$ with $g(\mathring{W}) = W_0 \oplus \mathbb{C}(e_{2n-1} + i e_{2n}) =: \mathring{W}_0$. 


Proof. Let \( g \in G_0 \) be chosen as above with \( g(W) = W_0 \). It is then immediate that \( g(\hat{W}) = W_0 \oplus \mathbb{C}w \) where \( w = \pm e_{2n-1} + ie_{2n} \). We obtain the positive sign by, e.g., multiplying \( e_1 \) and \( e_2 \) by \( i \), \( e_{2n-1} \) by \( -1 \) and \( e_j \) by \( +1 \) otherwise. Since this transformation is also in \( G_0 \), the desired result follows.

**Theorem 5.6** The Hermitian form \( h \) can be naturally extended to a non-degenerate Hermitian form \( \hat{h} \) on \( \hat{V} \) with signature \((p, q+1)\) (resp. \((p+1, q)\)) if \( p \) is even (resp. odd) so that the unique open orbit \( D \) of the resulting real form \( \hat{G}_0 \) is the set of isotropic \( n \)-planes of signature \((\frac{p}{2}, \frac{q+1}{2})\) (resp. \((\frac{p+1}{2}, \frac{q}{2})\)). Furthermore, the \( h \)-isometry group \( G_0 \) in \( \text{SO}_{2n-1}(\mathbb{C}) \) also acts transitively on \( D \) which is also its unique open orbit in \( Z \).

**Proof.** It is enough to consider the case we where \( p \) is even and \( \|e_{2n-1}\|^2 = -1 \). Extending \( h \) to \( \hat{h} \) on \( \hat{V} \) with \( e_{2n} \) being orthogonal to \( V \) and \( \|e_{2n}\|^2 = -1 \), it follows that \( \hat{h} \) is of signature \((p, q+1)\). Let \( \hat{G}_0 = \text{SO}(p, q+1) \) be the real form of \( \hat{G} = \text{SO}_{2n}(\mathbb{C}) \) defined by \( \hat{h} \). Arguing as above we see that the unique open \( \hat{G}_0 \)-orbit \( D \) in \( Z \) is the set of isotropic \( n \)-planes \( \hat{W} \) with signature \((\frac{p}{2}, \frac{q+1}{2})\). A reformulation of Proposition 5.5 is that \( G_0 \) also acts transitively on \( D \).

The following is a less technical formulation of this fact.

**Corollary 5.7** If \( Z \) is the complex flag manifold of isotropic \( n \)-planes in \( \mathbb{C}^{2n} \) where both \( \hat{G} \) and \( G_0 \) act transitively, every real form \( G_0 \) of \( G \) has a unique open orbit \( D \) which is the unique open orbit of a canonically determined real form \( \hat{G}_0 = \text{Aut}(D)^0 \) of \( \hat{G} = \text{Aut}(Z)^0 \).

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**References**

[A] A. Andreotti: Théorèmes de dépendance algébrique sur les espaces complexes pseudo-concaves, Bull. Soc. Math. France 91 (1963) 1-38

[B] J. Brun: Sur la simplification par les variétés homogènes, Math. Ann. 230, (1977) 175-182

[FHW] G. Fels, A. Huckleberry, and J. A. Wolf: Cycles Spaces of Flag Domains: A Complex Geometric Viewpoint, Progress in Mathematics, Volume 245, Springer/Birkhäuser Boston, 2005

[GRT] P. Griffiths, C. Robles and D. Toledo: Quotients of non-classical flag domains are not algebraic (arXiv1303.0252)

[Ha] R. Harvey: Spinors and Calibrations, Academic Press (1990)
[H1] A. Huckleberry: Hyperbolicity of cycle spaces and automorphism groups of flag domains, American Journal of Mathematics, Vol. 136, Nr. 2 (2013) 291-310 (arXiv:1003:5974)

[H2] A. Huckleberry: Remarks on homogeneous manifolds satisfying Levi-conditions, Bollettino U.M.I. (9) III (2010) 1-23 (arXiv:1003:5971)

[K] J. Kollar: Neighborhoods of subvarieties in homogeneous spaces (arXiv1308.5603)

[O1] A. Onishchik: Transitive compact transformation groups, Math. Sb. (N.S.) 60 (1963) 447-485 English Trans: AMS Trans. (2) 55 (1966) 5-58

[O2] Arkadii L’vovich Onishchik (on his 70th birthday), Russian Math. Surveys 58.6 1245-1253

[W] J. A. Wolf: The action of a real semisimple Lie group on a complex manifold, I: Orbit structure and holomorphic arc components, Bull. Amer. Math. Soc. 75 (1969), 1121–1237.