We prove two relations for the antisymmetrizer in the Hecke algebra and derive certain restrictions imposed by these relations on unitary representations of the Hecke algebra on tensor powers of the space \( C^n \). Bibliography: 5 titles.

1. Introduction

The Hecke algebra \( H_N(q) \) is the associative algebra over \( \mathbb{C}(q) \) with generators \( R_1, \ldots, R_{N-1} \) satisfying the relations

\[
R_k^2 = 1 + (q - q^{-1}) R_k, \quad k = 1, \ldots, N - 1, \tag{1}
\]
\[
R_k R_m R_k = R_m R_k R_m, \quad |k - m| = 1, \tag{2}
\]
\[
R_k R_m = R_m R_k, \quad |k - m| \geq 2. \tag{3}
\]

The Hecke algebra can be regarded as a \( q \)-deformation of the group algebra of the symmetric group \( S_N \). Relation (2) coincides with the Yang–Baxter equation in the braid group form. For this reason, representations of the Hecke algebra play an important role in the theory of quantum groups \([2, 3]\) and in the theory of quantum integrable systems.

Recall that a \( q \)-deformation of a number \( k \in \mathbb{Z} \) is defined by

\[
[k] = \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{4}
\]

We will need the following easily verifiable identities

\[
[k - 1] + [k + 1] = [2][k], \quad [k - 1][k + 1] + 1 = [k]^2. \tag{5}
\]

The \( q \)-antisymmetrizer \( P_N \in H_N(q) \) is defined recursively by the relations (where by \( P_N \in H_{N+1}(q) \) is meant the element \( P_N \otimes \text{id} \))

\[
P_1 = 1, \quad P_{N+1} = \frac{1}{[N+1]} P_N (q^N - [N] R_N) P_N. \tag{6}
\]

\( P_N \) is an idempotent

\[
P_N^2 = P_N, \tag{7}
\]

and it satisfies the following relations

\[
(R_k + q^{-1} \cdot 1) P_N = P_N (R_k + q^{-1} \cdot 1) = 0, \quad k = 1, \ldots, N - 1. \tag{8}
\]

Let \( P'_N \) denote the image of the element \( P_N \in H_{N+1}(q) \) under the shift of generators

\[
R'_k = R_{k+1}, \quad k = 1, \ldots, N - 1. \tag{9}
\]

In the present work, we will establish the relation

\[
(P_N - P'_N)^3 = \frac{[N - 1][N + 1]}{[N]^2} (P_N - P'_N). \tag{10}
\]
Besides, we will show that the generators $T_k = q^{-1} \cdot 1 + R_k$ satisfy the relation
\[
([2]^2 + 1) T_N P_N T_N = \frac{[2][N + 2][2N]}{[N]} T_N P_{N-1} - \frac{[N-1]}{[N]} P_{N-1} T_N T_{N-1} T_N T_{N-1} T_N P_{N-1}
\] (11)
and we will consider some restrictions that relations (10) and (11) impose on unitary representations of the Hecke algebra on spaces that are tensor powers of $\mathbb{C}^n$. In [1], relation (10) was obtained for the Jones–Wenzl projector of the Temperley–Lieb algebra $T L_N(Q)$. We remark that, in the Temperley–Lieb algebra case, relation (11) takes the following simpler form
\[
T_N P_N T_N = ([N] + 2) T_N P_{N-1}.
\] (12)

2. Derivation of the relations for $P_N$

Introduce new generators
\[
T_k = q^{-1} \cdot 1 + R_k.
\] (13)

For these generators, relations (1)–(3) are equivalent to the following ones
\[
T_k^2 = (q + q^{-1}) T_k, \quad k = 1, \ldots, N - 1, \tag{14}
\]
\[
T_k T_m T_k - T_m T_k T_m = T_k - T_m, \quad |k - m| = 1, \tag{15}
\]
\[
T_k T_m = T_m T_k, \quad |k - m| \geq 2, \tag{16}
\]
and formulae (6), (8), and (9) acquire the form
\[
P_1 = 1, \quad P_{N+1} = P_N - \rho_N P_N T_N P_{N}, \quad \rho_N = \frac{[N]}{[N+1]}, \tag{17}
\]
\[
T_k P_N = P_N T_k = 0, \quad k = 1, \ldots, N - 1, \tag{18}
\]
\[
T'_k = T_{k+1}, \quad k = 1, \ldots, N - 1. \tag{19}
\]

Let $\phi_N$ be an automorphism of the algebra $H_N(q)$ defined on the generators as follows
\[
\phi_N(T_k) = T_{N-k}, \quad k = 1, \ldots, N - 1. \tag{20}
\]

From relations (18) we infer that $T_k \phi_N(P_N) = \phi_N(P_N) T_k = 0$ for all $k = 1, \ldots, N - 1$. Relations (17) imply that $(P_N - 1)$ is a sum of monomials in $T_1, \ldots, T_{N-1}$. These properties allow us to check that $P_N$ is invariant under the action of the automorphism $\phi_N$,
\[
\phi_N(P_N) = \phi_N(P_N)(1 - P_N + P_N) = \phi_N(P_N) P_N = (\phi_N(P_N - 1) + 1) P_N = P_N. \tag{21}
\]

Note that $\phi_{N+1}(X \otimes id) = X'$ for any $X \in H_N(q)$. In particular, taking into account the property (21), we see that $\phi_{N+1}(P_N \otimes id) = P'_N$. Therefore, applying the automorphism $\phi_{N+1}$ to the relation (17), we obtain
\[
P_{N+1} = P'_N - \rho_N P'_N T_1 P'_N. \tag{22}
\]
Lemma 1. In the algebra $H_{N+2}(q)$, the following relations hold
\begin{equation}
P_{N+1} - P'_{N+1} = \rho_N P_N (T_{N+1} - T_1) P'_N,
\end{equation}
\begin{equation}
\rho_N P'_N T_1 P'_N T_1 P'_N = P'_N T_1 P'_N,
\end{equation}
\begin{equation}
\rho_N P'_N T_{N+1} P'_N T_{N+1} P'_N = P'_N T_{N+1} P'_N,
\end{equation}
\begin{equation}
P'_N (T_1 P'_N T_{N+1} + P'_N T_{N+1} P'_N T_{N+1}) P'_N = -\frac{[N+1]}{[N]^3} (P_{N+1} - P'_{N+1}).
\end{equation}

Proof. Applying the shift (19) to the relation (17), we obtain
\begin{equation}
P'_{N+1} = P'_N - \rho_N P'_N T_{N+1} P'_N.
\end{equation}
Relation (23) is the difference of relations (22) and (27).

Note that, as a consequence of relations (18), we have for $N \geq 2$ the equalities
\begin{equation}
P_N P'_{N-1} = P'_N P_N = P'_N, \quad T_{N+1} P'_N P_{N-1} = P'_{N-1} T_{N+1}, \quad T_N P'_N = P'_N T_N = 0.
\end{equation}
Let us prove relation (25). For $N = 1$, it coincides with relation (14). For $N \geq 2$, we have
\begin{equation}
P'_N T_{N+1} P'_N T_{N+1} P'_N \overset{(27)}{=} P'_N (T_{N+1} + P'_{N-1} T_{N+1} - \rho_{N-1} T_{N+1} P'_{N-1} T_{N+1}) P'_N
\end{equation}
\begin{equation}
\overset{(28)}{=} P'_N (T'_{N+1} - \rho_{N-1} T_{N+1} T_N T_{N+1}) P'_N
\end{equation}
\begin{equation}
\overset{(14),(15)}{=} P'_N ([2] T_{N+1} - \rho_{N-1} (T_N T_{N+1} T_N + T_{N+1} - T_N)) P'_N
\end{equation}
\begin{equation}
\overset{(28)}{=} ([2] - \rho_{N-1}) P'_N T_{N+1} P'_N = \frac{1}{\rho_N} P'_N T_{N+1} P'_N.
\end{equation}
The last equality uses the identity
\begin{equation}
[2] - \rho_{N-1} = \frac{[2][N] - [N - 1]}{[N]} \overset{(5)}{=} \frac{[N+1]}{[N]} = \frac{1}{\rho_N}.
\end{equation}
Relation (24) can be obtained from relation (25) by applying the automorphism $\phi_{N+2}$ and taking into account that $\phi_{N+2}(P'_N) = P'_N$.

Let us prove relation (26). For $N = 1$, it coincides with relation (15) for $k = 1$ and $m = 2$. For $N \geq 2$, we consider the difference of relations (17) and (22) (where $N$ is replaced with $(N + 1)$):
\begin{equation}
\frac{1}{\rho_{N+1}} (P_{N+1} - P'_{N+1}) = P_{N+1} T_{N+1} P_{N+1} - P'_{N+1} T_1 P'_{N+1}
\end{equation}
\begin{equation}
\overset{(22),(27)}{=} (P'_N - \rho_N P'_N T_1 P'_N) T_{N+1} (P'_N - \rho_N P'_N T_1 P'_N)
\end{equation}
\begin{equation}
- (P'_N - \rho_N P'_N T_{N+1} P'_N) T_1 (P'_N - \rho_N P'_N T_{N+1} P'_N)
\end{equation}
\begin{equation}
= P'_N (T_{N+1} - T_1) P'_N + \rho_N^2 P'_N (T_1 P'_N T_{N+1} P'_N T_1 - T_{N+1} P'_N T_1 P'_N T_{N+1} P'_N)
\end{equation}
\begin{equation}
\overset{(23)}{=} \frac{1}{\rho_N} (P_{N+1} - P'_{N+1}) + \rho_N^2 P'_N (T_1 P'_N T_{N+1} P'_N T_1 - T_{N+1} P'_N T_1 P'_N T_{N+1} P'_N).
\end{equation}
The equality of the first and the last expressions is equivalent to relation (26) because we have the following identity
\begin{equation}
\frac{1}{\rho_N} - \frac{1}{\rho_{N+1}} = \frac{[N+1]}{[N]} - \frac{[N+2]}{[N+1]} = \frac{[N+1]^2 - [N][N+2]}{[N][N+1]} \overset{(5)}{=} \frac{1}{[N][N+1]}.
\end{equation}
Proposition 1. In the algebra $H_{N+2}(q)$, relation (10) holds.

Proof. 

\[(P_{N} - P'_{N+1})^3 = \rho_N P_N (P_{N+1} - P'_{N+1})^2 P'_N (T_{N+1} - T_1) P'_N\]

\[= \rho_N^2 P'_N (T_{N+1} P'_N T_{N+1} + T_1 P'_N T_{N+1} - T_{N+1} P'_N T_{N+1} + T_1 P'_N T_{N+1}) P'_N\]

\[= \rho_N P'_N (T_{N+1} - T_1 - \rho_N T_{N+1} P'_N T_{N+1} - \rho_N T_1 P'_N T_{N+1}) P'_N\]

With the help of this relation, we obtain

\[(P_{N+1} - P'_{N+1})^3 = \rho_N (P_{N+1} - P'_{N+1})^2 P'_N (T_{N+1} - T_1) P'_N\]

\[= \rho_N^2 P'_N (T_{N+1} P'_N T_{N+1} - T_1 P'_N T_{N+1} - T_{N+1} P'_N T_{N+1} + T_1 P'_N T_{N+1}) P'_N\]

\[= \rho_N P'_N (T_{N+1} - T_1 + \rho_N T_1 P'_N T_{N+1} P'_N T_{N+1} - \rho_N T_{N+1} P'_N T_{N+1} P'_N T_{N+1}) P'_N\]

\[= \rho_N P'_N (T_{N+1} - T_1 + \frac{1}{[N+1]^2} (P_{N+1} - P'_{N+1})^3 = \frac{[N][N+2]}{[N+1]^2}) (P_{N+1} - P'_{N+1})\]. □

Let us remark that all the relations of Lemma 1 hold as well for the Temperley–Lieb algebra $TL_N(Q)$ with the parameter $Q = [2]$, i.e., in the case when the generators $\mathbf{T}_k$ satisfy instead of relation (15) the relation $\mathbf{T}_k \mathbf{T}_m \mathbf{T}_k = \mathbf{T}_k$ for $|k-m| = 1$. Therefore, Proposition 1 holds as well for the algebra $TL_N(Q)$. However, for the generators of the algebra $TL_N(Q)$ there are simpler relations (cf. equations (28)-(31) in [1]), which allow to give a simpler derivation of relation (10) for the Jones-Wenzl projector.

Proposition 2. In the algebra $H_{N+1}(q)$, $N \geq 2$, relation (11) holds.

Proof. 

\[T_N P_N T_N + P_N (17) = T_N (P_{N-1} - \rho_{N-1} P_{N-1} T_{N-1} P_{N-1}) T_N + P_N\]

\[= [2] T_N P_{N-1} - \rho_{N-1} P_{N-1} T_N T_{N-1} T_{N} P_{N-1} + P_N\]

\[= [2] T_N P_{N-1} - \rho_{N-1} P_{N-1} T_N T_{N-1} T_{N} P_{N-1} - \rho_{N-1} P_{N-1} T_N P_{N-1}\]

\[+ \rho_{N-1} P_{N-1} T_N T_{N-1} P_{N-1} + P_N\]

\[= [2] - \rho_{N-1} T_N P_{N-1} + P_{N-1} - \rho_{N-1} P_{N-1} T_N T_{N-1} T_{N} T_{N-1} P_{N-1}.\] (32)

Multiplying the derived equality by $T_N$ from both sides and using relations (14) and (28), we obtain

\[([2]^2 + 1) T_N P_N T_N = ([2]^2 (2) - \rho_{N-1}) T_N P_{N-1}\]

\[= [2] - \rho_{N-1} T_N T_{N-1} T_N P_{N-1}.\] (33)

Relations (11) and (33) are equivalent because the following identity holds

\[2 (2) - \rho_{N-1} + 1 = \frac{[N]}{[N]} + \frac{[N]}{[N]} + \frac{[N]}{[N]} + \frac{[N]}{[N]} + \frac{[N]}{[N]}.\] (34) □
3. On unitary representations on \((\mathbb{C}^n)^{\otimes N}\)

Let us regard the algebra \(H_N(q)\) as a complex algebra with a parameter \(q \in \mathbb{C}\), \(|q| = 1\) and an involution \(*\) such that
\[
q^* = q^{-1}, \quad R_k^* = R_k^{-1}.
\]
(35)
The unitarity of the generators \(R_k\) is equivalent to the hermiticity of the generators \(T_k\),
\[
T_k^* = (q^{-1} \cdot 1 + R_k)^* = q \cdot 1 + R_k^{-1} \equiv (1) q \cdot 1 + R_k - (q - q^{-1}) \cdot 1 = T_k.
\]
(36)

Let \(I \in \text{Mat}(n, \mathbb{C})\) denote the identity matrix and \(T \in \text{Mat}(n^2, \mathbb{C})\) be a Hermitian matrix satisfying the relations
\[
T^2 = (q + q^{-1}) T, \quad T_1 T_2 T_1 - T_2 T_1 T_2 = T_1 - T_2,
\]
(37)
where \(T_1 \equiv T \otimes I\), \(T_2 \equiv I \otimes T\), and \(\otimes\) stands for the Kronecker product. Then a homomorphism \(\tau : H_N(q) \rightarrow \text{Mat}(n^N, \mathbb{C})\) such that
\[
\tau(T_k) = T_k \equiv I^\otimes(k-1) \otimes T \otimes I^\otimes(N-k-1)
\]
(38)
is a \(*\)-representation of the algebra \(H_N(q)\) on the tensor product space \((\mathbb{C}^n)^{\otimes N}\). In this representation, \(\tau(R_k) = R_k\) are unitary matrices and they provide solutions to the Yang–Baxter equation
\[
R_k R_{k+1} R_k = R_{k+1} R_k R_{k+1}.
\]
(39)
Note that the matrices \(R_k\) are unitary and involutory if \(q = 1\). A classification of solutions of the Yang–Baxter equation (39) for this case was obtained in [4].

Since we have \((q + q^{-1}) \in \mathbb{R}\) for \(|q| = 1\) and relations (37) are invariant under the substitution \(T_k \rightarrow -T_k\), \(q \rightarrow -q\), it suffices to consider only the case \((q + q^{-1}) > 0\). (If \((q + q^{-1}) = 0\), then the only Hermitian matrix such that \(T^2 = 0\) is \(T = 0\).)

**Proposition 3.** Let \(N \in \mathbb{N}\), \(N \geq 2\) and \(q = e^{i\gamma}\), where \(\frac{\pi}{N+1} \leq \gamma < \frac{\pi}{N}\). In this case, if a Hermitian matrix \(T \neq 0\) satisfies relations (37) and \(\tau\) is the corresponding representation (38), then \(\tau(P_N) = 0\).

**Proof.** Set \(P_N = \tau(P_N)\). Since \(0 < \gamma < \frac{\pi}{N}\), then \(p_n = \frac{|n|}{|n+1|} = \frac{\sin(n\gamma)}{\sin((n+1)\gamma)}\) are well defined (and positive) for all \(n = 1, \ldots, N - 1\). Therefore, the projectors \(P_k\) defined recursively by the formula (17) exist for all \(k = 1, \ldots, N\). The same formula (17) implies also that all the projectors \(P_k\) are Hermitian and hence that \((P_N - P_N')^4\) and \((P_N - P_N')^2\) are positive semi-definite matrices. But relation (10) implies the equality
\[
(P_N - P_N')^4 = \frac{[N-1][N+1]}{[N]^2} (P_N - P_N')^2,
\]
(40)
where \(\frac{[N-1][N+1]}{[N]^2} \leq 0\) because \((N-1)\gamma < \pi\) and \(\pi \leq (N+1)\gamma < 2\pi\). That is, the right hand side of (40) is a negative semi-definite matrix. Thus, equality (40) can hold only if \(P_N = P_N'\). But then we have \(0 = T_1 P_N = T_1 P_N' = T \otimes P_N\). Whence \(P_N = 0\) because \(T \neq 0\). \(\square\)

Let us remark that the statement proved above is similar to the statement about \(C^*\)-representations of the Hecke algebra on a Hilbert space obtained in [5]. In the case that we consider here, the translational invariance of representation (38) leads to additional restrictions. In particular, it was shown in [1] that unitary representations of the Temperley–Lieb algebra on the tensor product space \((\mathbb{C}^n)^{\otimes N}\) exist only if \(|q + q^{-1}| = 1, \sqrt{2}, \sqrt{3}, 2\).
**Proposition 4.** Let \( N \in \mathbb{N}, \ N \geq 3 \) and \( q = e^{i\gamma} \), where \( \frac{\pi}{N+1} < \gamma < \frac{\pi}{N} \). Suppose, in addition, that
\[
[N + 2] + 2[N] < 0. \tag{41}
\]
In this case, if a Hermitian matrix \( T \neq 0 \) satisfies relations (37) and \( \tau \) is the corresponding representation (38), then \( \tau(P_{N-1}) = 0 \).

**Proof.** By Proposition 3, we have \( P_N = 0 \). Therefore, taking into account that \( [N] > 0 \), relation (11) acquires the form
\[
(2)[(N + 2] + 2[N]) T_N P_{N-1} = [N - 1] P_{N-1} T_N T_{N-1} T_{N-1} T_N P_{N-1}. \tag{42}
\]
The right hand side of (42) is a positive semi-definite matrix since \([N - 1] > 0 \). The matrix \([2]T_N P_{N-1} = T_N P_{N-1}(T_N P_{N-1})^* \) is also positive semi-definite because \( T_N \) commutes with \( P_{N-1} \). Therefore, if the condition (41) is satisfied, then the left hand side of (42) is a negative semi-definite matrix. Thus, relation (42) can hold only if \( T_N P_{N-1} = 0 \). But \( T_N P_{N-1} = P_{N-1} \otimes T, \) where \( T \neq 0 \). Whence \( P_{N-1} = 0 \). \( \square \)

Let us remark that we have \([N + 2] + 2[N] = 1 \) for \( \gamma = \frac{\pi}{N+1} \) and \([N + 2] + 2[N] = -[2] < 0 \) for \( \gamma = \frac{\pi}{N} \). Therefore, on every interval \( \left( \frac{\pi}{N+1}, \frac{\pi}{N} \right) \), \( N \geq 3 \), there exist values of \( \gamma \) for which the condition (41) is satisfied.

**Proposition 5.** Let \( q = e^{i\gamma} \), where \( \gamma \in (\arccos(\sqrt{\frac{1}{8}(1 + \sqrt{5})}, \frac{1}{2}\pi)) \). If a Hermitian matrix \( T \neq 0 \) satisfies relations (37), then \( T = [2](I \otimes I) \).

**Proof.** Consider the corresponding representation (38). For \( \gamma \in [\frac{1}{3}\pi, \frac{1}{2}\pi) \), we have \( T = [2](I \otimes I) \), because \( P_2 = 0 \) by Proposition 3.

For \( N = 3 \), the condition (41) is satisfied if \( f(\gamma) \equiv \sin 5\gamma + 2 \sin 3\gamma < 0 \). The equation \( f(\gamma) = (16 \cos^4 \gamma - 4 \cos^2 \gamma - 1) \sin \gamma = 0 \) has on the interval \( (\frac{1}{3}\pi, \frac{1}{2}\pi) \) a single root which corresponds to the value \( \cos^2 \gamma_0 = \frac{1}{8}(1 + \sqrt{5}) \). Taking into account that \( f(\frac{5}{4}\pi) < 0 \), we conclude that, for \( \gamma \in (\gamma_0, \frac{1}{3}\pi) \), the condition (41) is satisfied. But then we have \( P_2 = 0 \) by Proposition 4 and, hence, \( T = [2](I \otimes I) \). \( \square \)

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