SU(2) Charges as Angular-momentum in $N = 1$ Self-dual Supergravity

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Abstract

The $N = 1$ self-dual supergravity has $SL(2, \mathbb{C})$ symmetry. This symmetry results in $SU(2)$ charges as the angular-momentum. As in the non-supersymmetric self-dual gravity, the currents are also of their potentials and are therefore identically conserved. The charges are generally invariant and gauge covariant under local $SU(2)$ transforms approaching to be rigid at spatial infinity. The Poisson brackets constitute $su(2)$ algebra and hence can be interpreted as the generally covariant conservative angular-momentum.

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The study of self-dual gravities has drawn much attention in the past decade since the discovery of Ashtekar’s new variables, in terms of which the constraints can be greatly

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simplified\[1\]-\[2\]. The new phase variables consist of densitized $SU(2)$ soldering forms $\tilde{e}^{iA}_{\;B}$ from which a metric density is obtained according to the definition $q_{ij} = -\text{Tr} \tilde{e}_i \tilde{e}_j$, and a complexified connection $A_{iA}^{\;B}$ which carries the momentum dependence in its imaginary part. The original Ashtekar’s self-dual canonical gravity permits also a Lagrangian formulation\[3\] \[4\]. The supersymmetric extension of this Lagrangian formulation, which is equivalent to the simple real supergravity, was proposed by Jacobson\[5\], and the corresponding Ashtekar complex canonical transform was given by Gorobey et al\[6\]. The Lagrangian density is\[5\]

$$L_J = \frac{1}{\sqrt{2}}(e^{AA'} \wedge e_{BA'} \wedge F^B_A + i e^{AA'} \wedge \bar{\psi}_{A'} \wedge D\psi_A)$$

(1)

The dynamical variables are the real tetrad $e^{AA'}$ (the ”real” means $\bar{e}^{A'A} = e^{AA'}$), the traceless left-handed $SL(2,\mathbb{C})$ connection $A_{\mu MN}$ and the complex anticommuting spin-$\frac{3}{2}$ gravitino field $\psi_{\mu A}$. The $SL(2,\mathbb{C})$ covariant exterior derivative is defined by

$$D\psi_M := d\psi_M + A_M^\;N \wedge \psi_N$$

(2)

and the curvature 2-form is

$$F^M_N := dA_M^\;N + A_M^\;P \wedge A_P^\;N$$

(3)

The indices are lowered and raised with the antisymmetric $SL(2,\mathbb{C})$ spinor $\epsilon^{AB}$ and its inverse $\epsilon_{AB}$ according to the convention $\lambda_B = \lambda^A \epsilon_{AB}$, $\lambda^A = \epsilon^{AB} \lambda_B$, and the implied summations are always in north-westerly fashion: from the left-upper to the right-lower. The Lagrangian eq.(1) is a holomorphic functions of the connection and the equation for $A_{\mu A}^{\;B}$ is equivalent to

$$D e^{AA'} = \frac{i}{2} \psi^A \wedge \bar{\psi}^{A'}$$

(4)

provided $e^{AA'}$ is real. The Lagrangian $\frac{1}{2}(L_J + \bar{L}_J)$ for real supergravity is a non-holomorphic function but leads to no surfeit of field equations. Under the left-handed local supersymmetric transform generated by anticommuting parametres $\epsilon_A$

$$\delta \psi_A = 2D\epsilon_A, \quad \delta \bar{\psi}_{A'} = 0, \quad \delta e_{AA'} = -i \bar{\psi}_{A'} \epsilon_A$$

(5)

the Lagrangian $L_J$ is invariant without using any one of the Euler-Lagrangian equations while under the right-handed transform

$$\delta \psi_A = 0, \quad \delta \bar{\psi}_{A'} = 2D\bar{\epsilon}_{A'}, \quad \delta e_{AA'} = -i \psi_A \bar{\epsilon}_{A'}$$

(6)
\( \mathcal{L}_J \) is invariant *modulo* the field equations.

The \((3+1)\) decomposition is effected as

\[
\mathcal{L}_J = \tilde{e}^{kAB} \dot{A}_{kAB} + \tilde{\pi}^{kA} \dot{\psi}_{kA} - \mathcal{H}
\]

\( \mathcal{H} := \epsilon_{0A'J} \mathcal{H}^{JAA'} + \psi_{0A} \mathcal{S}^A + \dot{\mathcal{S}}^{A'} \psi_{0A'} + A_{0AB} \mathcal{J}^{AB} + \) (total divergence)  

The canonical momenta are

\[
\tilde{e}^{kAB} := -\frac{1}{\sqrt{2}} \epsilon^{ijk} e_i^{AA'} e_j^{A'}
\]

\[
\tilde{\pi}^{kA} := \frac{i}{\sqrt{2}} \epsilon^{ijk} e_i^{AA'} \bar{\psi}_{jA'}
\]

and the constraints are

\[
\mathcal{H}^{JAA'} := \frac{1}{\sqrt{2}} \epsilon^{ijk} (e_i^{BA'} F_{jKB} A - i \bar{\psi}_i^{A'} D_j \psi_k^A)
\]

\[
\mathcal{S}^A := D_k \tilde{\pi}^{kA}
\]

\[
\dot{\mathcal{S}}^{A'} := \frac{i}{\sqrt{2}} \epsilon^{ijk} e_i^{AA'} D_j \psi_k A
\]

\[
\mathcal{J}^{AB} := D_k \tilde{e}^{kAB} - \bar{\pi}^k (A \psi_k^B)
\]

The 0-components \( \epsilon_{0A'J}, \psi_{0A}, \bar{\psi}_{0A'} \) and \( A_{0AB} \) are just the Lagrange multipliers and the dynamical conjugate pairs are \((\tilde{e}^{kAB}, A_{jAB}), (\tilde{\pi}^{kA}, \psi_{kA})\). The constraints \( \mathcal{H}^{JAA'} = 0 \) and \( \dot{\mathcal{S}}^{A'} = 0 \) generate the following two

\[
\dot{\mathcal{H}}^{AB} := (\tilde{e}^j \tilde{e}^k F_{jk})^{AB} + 2 \bar{\pi}^j \tilde{e}^k D_{[j} \psi_{k]} \epsilon^{AB} + 2 (\bar{\pi}^j D_{[j} \psi_{k]}) \tilde{e}^{kAB} = 0
\]

\[
\mathcal{S}^{jA} := \frac{1}{\sqrt{2}} \epsilon^{ijk} \tilde{e}_i^{AB} D_j \psi_{kB} = 0
\]

The equations of motion will be properly expressed in Hamiltonian form \( \hat{f} = \{H, f\} \) if we assign the Poisson brackets

\[
\{\tilde{e}^{kAB}(x), A_{jAB}(y)\} = \delta_j^k \delta_{(M}^A \delta_{N)}^B \delta^3(x, y)
\]

\[
\{\tilde{\pi}^{kA}(x), \psi_{jA}(y)\} = -\delta_j^k \delta_M^A \delta^3(x, y)
\]

all other brackets among these quantities being zero. This is the outline of the theory.

In our previous works, we have obtained the \( SU(2) \) charges and the energy-momentum
SU(2) charge, self-dual supergravity

in the Ashtekar’s formulation of Einstein gravity[7]-[8] and they are closely related to the angular-momentum[9]-[11] and the energy-momentum [12] in the vierbein formalism of Einstein gravity. The fact that the algebra formed by their Poisson brackets do constitute the 3-Poincare algebra on the Cauchy surface supports from another aspect that their definitions are reasonable. Similarly, the study of SU(2) charges in the self-dual supergravity considered is also an interesting subject. In the following, we will employ the SL(2, C) invariance to obtain the conservative charges as we did previously[8] Under any SL(2, C) transform

\[ e_{\mu A A'} \rightarrow L_A^B \bar{R}_{A'}^B e_{\mu B B'}, \quad \psi_A \rightarrow L_A^B \psi_B, \quad \bar{\psi}_{A'} \rightarrow \bar{R}_{A'}^B \bar{\psi}_{B'} \]

\[ A_{\mu MN} \rightarrow L_M^A A_{\mu A} B (L^{-1})_{BN} + L_M^A \partial_\mu (L^{-1})_{AN} \tag{19} \]

\( \mathcal{L}_J \) is invariant. \( L \) and \( \bar{R} \) may not necessarily related by complex conjugation. Note that \( L_{AB} = -(L^{-1})_{BA} \), the transform of \( A \) may also be written as

\[ A_{\mu MN} \rightarrow L_M^A L_N^B A_{\mu AB} - L_M^A \partial_\mu L_{NA} \tag{20} \]

For infinitesimal transform, \( L_A^B = \delta A^B + \xi_A^B \) where \( \xi_{AB} = -\xi_{BA} \) are infinitesimal parametres. Thus we have

\[ \delta \xi A = [\xi, A] - d\xi, \quad \delta \psi = \xi \psi \tag{21} \]

When calculating the variation of the Lagrangian, one must take into consideration of the anticommuting feature of the gravitino field. We write the variation in the way that

\[ \delta \mathcal{L}_J = \delta \phi^A (\frac{\partial}{\partial \phi^A} - \partial_\mu \frac{\partial}{\partial \partial_\mu \phi^A}) \mathcal{L}_J + \partial_\mu (\delta \phi^A \frac{\partial}{\partial \partial_\mu \phi^A} \mathcal{L}_J) \tag{22} \]

where \( \phi^A \) denotes any field involved in the first order Lagrangian. Now both \( \frac{\partial}{\partial \phi^A} \) and \( \frac{\partial}{\partial \partial_\mu \phi^A} \) are (anti-)commuting if \( \phi^A \) is (anti-)commuting, and so there is no ordering problem.

The invariance of \( \mathcal{L}_J \) under the infinitesimal SL(2, C) transform is equivalent to the following modulo the field equations

\[ \partial_\rho (\delta A_{\sigma A}^B \frac{\partial \mathcal{L}_J}{\partial \partial_\rho A_{\sigma A}^B} + \delta \psi_{\sigma A} \frac{\partial \mathcal{L}_J}{\partial \partial_\rho \psi_{\sigma A}}) = 0 \tag{23} \]

For constant \( \xi \), we have

\[ \partial_\rho (\frac{1}{\sqrt{2}} \epsilon^{\mu \rho \sigma} e_{\mu}^{AA'} e_{\nu BA'} [\xi, A_{\sigma}]_A^B + \frac{i}{\sqrt{2}} \epsilon^{\mu \rho \sigma} e_{\mu}^{AA'} \psi_{BA'} (\xi \psi_{\sigma})_A = 0 \tag{24} \]
we have therefore the conservation of $SU(2)$ charges

$$
\partial_\mu \tilde{J}^\mu_{AB} = 0
$$

where

$$
\tilde{J}^\rho_{AB} = \frac{1}{\sqrt{2}} \epsilon^{\mu\nu\rho\sigma} (e_{\mu A} A^{\nu M} A_{\sigma B} - e_{\mu M} A^{\nu A} A_{\sigma B} + e_{\mu A} A^{\nu M} e_{\nu A} A_{\sigma B})
$$

Thus

$$
\tilde{J}^0_{AB} = \int_{\Sigma} \tilde{J}^0_{AB} d^3 x
$$

where

$$
\tilde{J}^0_{AB} = \frac{1}{\sqrt{2}} \epsilon^{i j k} (e_{i A} A^{j M} A_{k B} - e_{i M} A^{j A} A_{k B} + e_{i A} A^{j M} e_{j A} A_{k B})
$$

Using eq(9) and eq(10), $\tilde{J}^0_{AB}$ can be written as

$$
\tilde{J}^0_{AB} = [\tilde{e}^k, A_k]_{AB} + \pi_k (A \psi^k)
$$

The constraint $J_{AB} = 0$ guarantees that

$$
J_{AB} \approx \int_{\Sigma} \partial_k \tilde{e}^k_{AB} = \int_{\partial \Sigma} \tilde{e}^k_{AB} ds_i
$$

where $ds_i = \frac{1}{2} \epsilon_{i j k} dx_j \wedge dx_k$. It can also be obtained in the following way. Using the field equation $e^{A}(A \wedge (D e^{B} A - \frac{i}{2} \psi^{B}) \wedge \bar{\psi}^{A}) = 0$, we have

$$
e^{\mu \nu \rho \sigma} (e_{\mu A} A^{\nu M} A_{\sigma B} + \frac{i}{2} \bar{\psi}_{\nu A} \psi_{\sigma B})
$$

so

$$
\tilde{J}^\rho_{AB} = -\frac{1}{\sqrt{2}} e^{\mu \nu \rho \sigma} \partial_\sigma (e_{\mu A} A^{\nu B})
$$

Using

$$
e_{[\mu A} A^{\nu B]} = e_{[\mu AC} e_{\nu] B} C - i \sqrt{2} n_{[\mu e_{\nu]} AB}
$$

we have

$$
\tilde{J}^0_{AB} = -\frac{1}{\sqrt{2}} \epsilon^{i j k} \partial_k (e_{[i A} A^{j B]}) = -\frac{1}{\sqrt{2}} \epsilon^{i j k} \partial_k (e_{[i AC} e_{j] B} C - i \sqrt{2} n_{[i e_{j]} AB})
$$

$$
= \frac{1}{\sqrt{2}} \epsilon^{i j k} \partial_k (e_i e_j)_{AB} = \partial_k \tilde{e}^k_{AB}
$$
which is exactly the same as eq.(30) We can thus have the Poisson brackets
\[
\{J_{AB}, J_{MN}\} = \left\{ \int_{\Sigma} \tilde{e}^A dB, \int_{\Sigma} (\tilde{e}_M^P A_{iPN} + \tilde{e}_N^P A_{iPN}) d^3x \right\} = \frac{1}{2} (J_{MA\epsilon NB} + J_{MB\epsilon NA} + J_{NA\epsilon MB} + J_{MA\epsilon NB})
\]
(35)
Now the flat dreibein on Σ is needed in order to find the angular- momentum \(J_i\). To clarify the notions, we use the following conventions: \(\mu, \nu, \ldots\) denote the 4-dim curved indices and \(i, j, k, \ldots\) denote the 3-dim curved indices on \(\Sigma\); \(a, b, c, \ldots\) denote the flat 4-dim indices and \(l, m, n, \ldots\) denote the flat 3-dim indices on \(\Sigma\). The rigid flat vierbein is denoted as \(E^a_{AA'}\) and the rigid flat dreibein is denoted by \(E^m_{AB}\). Then define
\[
J_m := \frac{1}{\sqrt{2}} E^m_{AB} J_{AB}
\]
(36)
and using the relation \(\epsilon^{mnl} E_m E_n = \sqrt{2} E_l\) we have
\[
\{J_m, J_n\} = \epsilon^{mnl} J_l
\]
(37)
Therefore the su(2) algebra is restored. As in the non-supersymmetric case[8], we can also obtain only the SU(2) charges instead of the whole SL(2, C) charges. Yet, the angular-momentum \(J_{ab}\) obtained in [9]-[10] is completely contained in \(J_{MN}\) since we have from eq(32) that
\[
\tilde{j}_\rho^{\alpha} = -\frac{1}{2} j^{\rho}_{ab} E^a_{\alpha'} E^b_{\beta} A' \quad \text{where } \tilde{j}_\rho^{\alpha}_{ab}
\]
is the angular -momentum current obtained in [9]-[10].
\[
\tilde{j}_\rho^{\alpha}_{ab} = \sqrt{2} \epsilon^{\rho\sigma\mu\nu} \partial_\sigma (\epsilon_{\mu a} \epsilon_{\nu b})
\]
(39)
and the angular-momentum is
\[
J_{ab} = \int_{\Sigma} \tilde{j}_0^{\rho}_{ab} d^3x
\]
(40)
Hence
\[
J_{MN} = -\frac{1}{2} J_{ab} E_{[aM} A' \epsilon_{b]NA'} = \frac{1}{2} (J_{ij} E_{i[AC} E_{j]BC} - i \sqrt{2} J^{0i} r_0 E_{iA B})
\]
(41)
where \(L_i = \frac{1}{2} \epsilon_{ijk} J^{jk}\) are the spatial rotations and \(K_i = J_{0i} = -J^{0i}\) are the Lorentz boosts. Therefore
\[
J_i = \frac{1}{2} (L_i - iK_i)
\]
(42)
Bear in mind that both $\frac{1}{2}(L_i - iK_i)$ and $\frac{1}{2}(L_i + iK_i)$ obey the $su(2)$ algebra\[13\]. Actually, the boost charges are vanishing as can be seen from eq(30). Thus we can obtain the angular-momentum, in the self-dual simple supergravity once $J_{MN}$ is known.

We make a few remarks finally. The total charges take the same integral form as those in the non-supersymmetric case. Though we can obtain the $SU(2)$ sector of the $SL(2, \mathbb{C})$ charges, the information of the angular-momentum is completely contained in the $SU(2)$ charges. It can be seen from the surface integrals that the angular-momentum is governed by the $r^{-2}$ part of $\tilde{e}^i$. As in \[1\]-\[2\], we always assume that the phase space variables are subject to the boundary conditions.

$$e^\mu_{AB\mid\partial\Sigma} = (1 + \frac{M(\theta, \phi)}{r})^2 e^0_{AB} + O(1/r^2), \quad A_{\mu MN \mid\partial\Sigma} = O(1/r^2)$$

(43)

$$\tilde{\pi}_A^i = O(1/r), \quad \psi_{\mu A} = O(1/r)$$

(44)

where $e^\mu_{AB}$ denote the flat $SU(2)$ soldering forms. As a consequence, under the $SL(2, \mathbb{C})$ transforms behaving as

$$L_A^B = \Lambda_A^B + O(1/r^{1+\epsilon}), \quad (\epsilon > 0)$$

(45)

where $\Lambda$ are rigid transforms. The charges transform as

$$J_{MN} \to \Lambda_M^A \Lambda_N^B J_{AB}.$$  

(46)

i.e., they gauge covariant. Their conservation is generally covariant. As in the non-supersymmetric case\[7\]-\[8\], the currents have also potentials, i.e., can be expressed as a divergence of an antisymmetric tensor density. So they are identically conserved. Upon quantization, the Poisson brackets correspond to the quantal commutators and their algebra realizes indeed the $su(2)$ algebra. This shows that their interpretations reasonable.

It is novel that the relation between $J_{MN}$ and the constraint $\mathcal{J}^{AB}$ is the same as that between the electric charge and the Gauss law constraint in QED\[14\]

$$\nabla \cdot E - e \bar{\psi} \gamma_0 \gamma \psi = 0$$

(47)

$$q = \int_{\partial\Sigma} E \cdot dS$$

(48)

So the $J_{MN}$ is a kind of gauge charge.
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