Subordination scenario of anomalous relaxation

Aleksander Stanislavsky\textsuperscript{a,*}, Karina Weron\textsuperscript{b,**}

\textsuperscript{a}Institute of Radio Astronomy, Ukrainian National Academy of Sciences, 4 Chervonopraporna St., 61002 Kharkov, Ukraine

\textsuperscript{b}Institute of Physics, Wroclaw University of Technology, Wyb. Wyspianskiego 27, 50-370 Wroclaw, Poland

Abstract

The non-exponential relaxation is shown to result from subordination by inverse tempered $\alpha$-stable processes. The main feature of tempered $\alpha$-stable processes is a finiteness of their moments, and the class of random processes includes ordinary $\alpha$-stable processes as a particular case. Starting with the Cole-Cole response this subordination approach establishes its direct link with the Cole-Davidson law. We derive the relaxation function describing the tempered relaxation. The meaning of the empirical response function is clarified.

Key words:
Lévy-stable process, Subordination, Non-exponential relaxation

1 Introduction

The major feature of dynamical processes in many complex relaxing systems is their stochastic background \cite{1,2}. Particularly, in any dielectric (complex) system under an week external electric field (external action) only a part (active dipoles or objects) of the total number of dipoles is directly governed by changes of the field. But even those dipoles, not contributing to the relaxation dynamics, can have an effect on the behavior of active dipoles \cite{3}. If the dipoles interact with each other, then their evolution has a random character. Consequently, the behavior of such a relaxing system as a whole will not be exponential in nature. In this case the macroscopic behavior of the complex

\textsuperscript{*}E-mail address: alexstan@ira.kharkov.ua

\textsuperscript{**}E-mail address: Karina.Weron@pwr.wroc.pl
systems is governed by “averaging principles” like the law of large numbers following from the theory of probability [4]. The macroscopic dynamics of complex systems is not attributed to any particular object taken from those forming the systems. The finding out an “averaged” object representity for the entire relaxing system is not simple. The relation between the local random characteristics of complex systems and the universal deterministic empirical laws requires a probabilistic justification. There are some points of view on this problem. One of well-known them is based on randomizing the parameters of distributions that describes the relaxation rates in disordered systems. With regard to the dielectric relaxation, each individual dipole in a complex system relaxes exponentially, but their relaxation rates are different and obey a probability distribution (continuous function) [3,5]. This approach is successive for getting many empirical response laws and their classification, but it sometime becomes enough complicated to interpret their interrelations and to derive macroscopic response equations.

In this paper we suggest an alternative approach to the analysis of non-exponential relaxation. It is based on subordination of random processes. Recall that in the theory of anomalous diffusion the notation of subordination occupies one of the most important places (see, for example, [6] and references therein). So, a subordinated process \( Y(U(t)) \) is obtained by randomizing the time clock of a random (parent) process \( Y(t) \) by means of a random process \( U(t) \) called the directing process. The latter process is also often referred to as the randomized time or operational time [7]. In the common case the process \( Y \) may be both random and deterministic in nature. The subordination of random processes is a starting point for the anomalous diffusion theory.

We develop this approach to relaxation processes. It gives an efficient method for calculating the dynamical evaluating averages of the relaxation processes. In this connection Section 2 is devoted a presentation of recent achievements of this method. Starting with the description of the two-state system evolution as a Markovian process, we develop the analysis on subordinated random processes. The processes differ from the Markovian ones by the temporal variable becoming random. In this context the Cole-Cole relaxation is an evident example. In Section 3 we consider the tempered \( \alpha \)-stable processes. They overcome the infinite-moment difficulty of the usual (not tempered) \( \alpha \)-stable processes. As applied to the anomalous diffusion, the tempering gives a preserving the subdiffusive behavior for short times whereas for long times the diffusion is something like normal. Using the processes in Section 4 we develop a subordination scheme for the description of the tempered relaxation. Section 5 formulates major properties of such relaxation. We show that it has a direct relation to the well-known experimental laws of relaxation, in particular, to the Cole-Davidson law. Finally, the conclusions are drawn in Section 6.
2 Relaxation in Two-State Systems

The simplest ordinary interpretation of relaxation phenomena is based on the concept of a system of independent exponentially relaxing objects (for example, dipoles) with different (independent) relaxation rates. The relaxation process, following this law (called Debye’s), may be represented by behavior of a two-state system. Let \( N \) be the common number of dipoles in a dielectric system. If \( N_\uparrow \) is the number of dipoles in the state \( \uparrow \), \( N_\downarrow \) is the number of dipoles in the state \( \downarrow \) so that \( N = N_\uparrow + N_\downarrow \). Assume that for \( t = 0 \) the system is stated in order so that the states \( \uparrow \) dominate, namely

\[
\frac{N_\uparrow(t = 0)}{N} = n_\uparrow(0) = 1, \quad \frac{N_\downarrow(t = 0)}{N} = n_\downarrow(0) = 0,
\]

where \( n_\uparrow \) is the part of dipoles in the state \( \uparrow \), \( n_\downarrow \) the part in the state \( \downarrow \). Denote the transition rate by \( w \) defined from microscopic properties of the system (for instance, according to the given Hamiltonian of interaction and the Fermi’s golden rule). In the simplest case (\( D \) relaxation) the kinetic equation takes the form

\[
\begin{align*}
\dot{n}_\uparrow(t) - w \{n_\downarrow(t) - n_\uparrow(t)\} &= 0, \\
\dot{n}_\downarrow(t) - w \{n_\uparrow(t) - n_\downarrow(t)\} &= 0,
\end{align*}
\]

(1)

where, as usual, the dotted symbol means the first-order derivative. The relaxation function for the two-state system is

\[
\phi_D(t) = 1 - 2n_\downarrow(t) = 2n_\uparrow(t) - 1 = \exp(-2wt).
\]

It is easy see that the steady state of the system corresponds to equilibrium with \( n_\uparrow(\infty) = n_\downarrow(\infty) = 1/2 \). Clearly, its response has also an exponential character. However, this happens to be the case of dipoles relaxing irrespective of each other and of their environment. If the dipoles interact with their environment, and the interaction is complex (or random), their contribution in relaxation already will not result in any exponential delay.

Assume that the interaction of dipoles with environment is taken into account with the aid of the temporal subordination. We will consider the evolution of the number of dipoles in the states \( \downarrow \) and \( \uparrow \). This are parent random processes in the sense of subordination. They may be subordinated by another random process with a probability density, say \( p(\tau, t) \). If \( n_\uparrow(\tau) \) and \( n_\downarrow(\tau) \), taking from Eq. (1) as probability laws of the parent processes, depend now on a local time \( \tau \), then the resulting \( n_\uparrow(t) \) and \( n_\downarrow(t) \) after the subordination is determined by
the integral relation
\[ n_\uparrow(t) = \int_0^\infty n_\uparrow(\tau) p(\tau, t) \, d\tau, \quad n_\downarrow(t) = \int_0^\infty n_\downarrow(\tau) p(\tau, t) \, d\tau. \]

If the directing process (a new time clock or stochastic time arrow [9]) is an inverse \( \alpha \)-stable process, its probability density has the following Laplace image
\[ p^S(\tau, t) = \frac{1}{2\pi j} \int_{Br} e^{ut-\tau u^\alpha} u^{\alpha-1} \, du = t^{-\alpha} F_\alpha(\tau/t^{\alpha}), \quad (2) \]

where \( Br \) denotes the Bromwich path. This probability density has a simple physical interpretation. It determines the probability to be at the internal time (or so-called operational time) \( \tau \) on the real (physical) time \( t \). The function \( F_\alpha(z) \) can be expanded as a Taylor series. Besides, it has the Fox’ H-function representation
\[ F_\alpha(z) = H_{10}^{11}\left(z \left| \begin{array}{c} (1 - \alpha, \alpha) \\ (0, 1) \end{array} \right. \right) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k! \Gamma(1 - \alpha(1 + k))}, \]

where \( \Gamma(x) \) is the ordinary gamma function. In the theory of anomalous diffusion the random process \( S(t) \) is applied for the subordination of Lévy (or Gaussian) random processes [10]. The inverse \( \alpha \)-stable process accounts for the amount of time that a walker does not participate in the motion process [11]. If the walker participated all time in the motion process, the internal time and the physical (external) time would coincide.

As was shown in [9,12,13], the stochastic time arrow can be applied to the general kinetic equation. Then the equation describing a two-state system takes the following form
\[ \begin{cases} D^\alpha n_\uparrow(t) - w \{ n_\downarrow(t) - n_\uparrow(t) \} = 0, \\ D^\alpha n_\downarrow(t) - w \{ n_\uparrow(t) - n_\downarrow(t) \} = 0, \end{cases} \quad 0 < \alpha \leq 1, \quad (3) \]

where \( D^\alpha \) is the \( \alpha \)-order fractional derivative with respect to time. Here we use the Caputo derivative [14], namely
\[ D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \, d\tau, \quad n-1 < \alpha < n, \]
where \( x^{(n)}(t) \) means the \( n \)-derivative of \( x(t) \). The relaxation function for the two-state system is written as

\[
\phi_{CC}(t) = 1 - 2n_\uparrow(t) = 2n_\uparrow(t) - 1 = E_\alpha(-2wt^{\alpha}),
\]

where \( E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\alpha)} \) is the one-parameter Mittag-Leffler function \[15\]. It is important to notice that the relaxation function corresponds to the Cole-Cole (CC) law. With reference to the theory of subordination the CC law shows that the dipoles tend to equilibrium via motion alternating with stops so that the temporal intervals between them is random. The random values are governed by an inverse \( \alpha \)-stable subordinator.

The evolution of \( n_\uparrow(t) \) and \( n_\downarrow \) in Eq. (2) can be connected with the Mittag-Leffler distribution. If \( Z_n \) denotes the sum of \( n \) independent random values with the Mittag-Leffler distribution, then the Laplace transform of \( n^{-1/\alpha}Z_n \) is \((1 + s^{\alpha}/n)^{-n} \), which tends to \( e^{-s^{\alpha}} \) as \( n \) tends to infinity. Following the arguments of Pillai \[16\], this supports an infinity divisibility of the Mittag-Leffler distribution. Due to the power asymptotic form (long tail) the distribution with parameter \( \alpha \) is attracted to the stable distribution with exponent \( \alpha \), \( 0 < \alpha < 1 \). The property of the Mittag-Leffler distribution permits one to determine a stochastic process. The process (called Mittag-Leffler’s) arises of subordinating a stable process by a directing (generalized) gamma process \[16\]. In this case the relaxation function has the form

\[
\phi_{HN}(t) = 1 - \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(b + k)}{k! \Gamma(b) \Gamma(1 + ab + ak)} \left( \frac{t}{\tau_{HN}} \right)^{ab+ak}, \tag{4}
\]

where \( a, b, \tau_{HN} \) are constant. The one-side Fourier transformation of the relaxation function gives the Havriliak-Negami (HN) law

\[
\chi_{HN}(\omega) = \int_0^{\infty} e^{-i\omega t} \left( -\frac{d\phi_{HN}(t)}{dt} \right) dt = \frac{1}{(1 + (i\omega\tau_{HN})^{a})^{b}}. \tag{5}
\]

This result also corresponds to the well-know HN empirical law. Thus, the HN relaxation can be explained from the subordination approach, if the hitting time process of dipole orientations transforms into the Mittag-Leffler process \[17\]. For that the hitting time process has an appropriate distribution attracted to the stable distribution. The subordination of the latter results just in the Mittag-Leffler process. It is interesting to observe that the Lévy process subordinated by another Lévy one leads again to the Lévy process, but with other index \[18\]. Unfortunately, the description from the Mittag-Leffler process gives nothing for the derivation of any macroscopic response equation like Eq. (3).
3 Tempered $\alpha$-stable Process and Its Inverse

The relaxation model based on the inverse $\alpha$-stable process starts with the consideration of $\alpha$-stable processes having the infinite-moment difficult. To overcome it, one can develop an approach stated on tempered $\alpha$-stable processes. The tempered $\alpha$-stable process \cite{19, 20} has the Laplace image of its distribution in the form

$$\tilde{f}(u) = \exp(\delta^\alpha - (u + \delta)^\alpha).$$

(6)

When $\delta$ equals to zero, the tempered $\alpha$-stable process becomes simply $\alpha$-stable.

However, the distribution \cite{3} describes only probabilistic properties in terms of internal time. For subordination we need the probability distribution of the inverse tempered $\alpha$-stable process. If $f(\tau, t)$ is the p.d.f. of internal time, then the p.d.f. of its inverse $g(\tau, t)$ can be represented as

$$g(\tau, t) = -\frac{\partial}{\partial \tau} \int_{-\infty}^{t} f(t', \tau) \, dt'.$$

Taking the Laplace transform of $g(\tau, t)$ with respect to $t$, we get

$$\tilde{g}(\tau, u) = -\frac{1}{u} \frac{\partial}{\partial \tau} \tilde{f}(u, \tau) = \frac{(u + \delta)^{\alpha} - \delta^{\alpha}}{u} e^{-\tau[(u+\delta)^\alpha - \delta^\alpha]}.$$

(7)

When $\delta \to 0$, Eq. \cite{7} tends to

$$\tilde{g}(\tau, u) = u^{\alpha-1} e^{-\tau u^\alpha}.$$

This expression corresponds to the Laplace image of an inverse $\alpha$-stable p.d.f. describing a directing process in the theory of Cole-Cole relaxation. After the inverse Laplace transform we have Eq. \cite{2}.

In the Laplace space the function $g(\tau, u)$ has a simple form. Because of general properties of the Laplace transform we can find $g(\tau, t)$ explicitly, but its representation is enough complicated and expressed through a integral of the Wright functions \cite{21}. For our calculation it will be sufficient to know only the function $g(\tau, u)$. Therefore, we will not present any explicit form $g(\tau, t)$ here.
4 Macroscopic Response Equation of Tempered Relaxation

If in Eq. (1) the value $w$ will depend on time as $a A^t t^{\alpha-1}$, we come to the description of the Kohlrausch-Williams-Watts (KWW) relaxation. Although such a equation does not contain any (for example, micro/mesoscopic and so on) details about relaxation processes, it is convenient for practical purpose because of its simplicity. When the relaxation follows the CC, CD (Cole-Davidson), HN laws, the equation (1) is not so simple as in the case of D and KWW relaxation. Recall that the CC relaxation and response functions can be expressed in terms of a solution of the fractional differential equation [18]. With macroscopic equations for the CD and HN responses the situation becomes more else complicated. Consider the derivation of the macroscopic response equation of tempered relaxation in more details.

For a general type of a Markovian process the general kinetic equation is

$$\frac{dp(t)}{dt} = \hat{W} p(t),$$

(8)

where $\hat{W}$ denotes the transition rate operator (see details, in [22]). This equation defines the probability $p$ for the system transition from one state into others. Next, we determine a new process governed by an inverse tempered $\alpha$-stable process with the Laplace image (7), namely

$$p_\alpha(t) = \int_0^\infty g(t, \tau) p(\tau) \, d\tau.$$

The Laplace transform $\tilde{p}_\alpha(s)$ is given by

$$\tilde{p}_\alpha(s) = \int_0^\infty e^{-st} p_\alpha(t) \, dt$$

and leads to

$$\hat{W} \tilde{p}_\alpha(s) = \frac{(s + \delta)^\alpha - \delta^\alpha}{s} \hat{W} \tilde{p}((s + \delta)^\alpha - \delta^\alpha)$$

$$= \frac{(s + \delta)^\alpha - \delta^\alpha}{s} \left\{ ((s + \delta)^\alpha - \delta^\alpha) \tilde{p}((s + \delta)^\alpha - \delta^\alpha) - p(0) \right\}$$

$$= ((s + \delta)^\alpha - \delta^\alpha) \tilde{p}_\alpha(s) - \frac{(s + \delta)^\alpha - \delta^\alpha}{s} p(0).$$

(9)
From this it follows
\[ p_\alpha(t) = p(0) + \int_0^t d\tau M(t-\tau) \dot{W} p_\alpha(t), \] (10)

where the kernel \( M(t) \) is written as
\[ M(t) = e^{-\delta t} t^{\alpha-1} E_{\alpha, \alpha}(\delta^\alpha t^\alpha), \]

where \( E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} z^n / \Gamma(\beta + n\alpha) \) is the two-parameter Mittag-Leffler function \(^1\). This equation covers a number of particular cases known earlier. For \( \alpha = 1 \) we obtain Eq. (8), and for \( \delta = 0 \) it becomes fractional. If a system has discrete states, then the generating function is of the form
\[ G(\zeta, t) = \sum_{k=0}^{\infty} \zeta^k p_k(t), \]

where \( \zeta \) takes values \( |\zeta| \leq 1 \) for a series to converge. With the help of the generating function, one can find the moments by taking the derivative with respect to \( \zeta \) and then setting \( \zeta = 1 \). The generating function of the process governed by the stochastic time clock is given by the relation
\[ G_\alpha(\zeta, t) = \int_0^\infty g(\tau, t) G(\zeta, \tau) d\tau. \] (11)

Thus, the generating function for a discrete Markovian process directed by a subordinated process can be obtained from the appropriate generating function of the parent process by immediate integration.

The relaxation in a two-state system under the inverse tempered \( \alpha \)-stable subordinator gives
\[ n_\uparrow(t) = n_\uparrow(0) + w \int_0^t M(t-\tau) \{ n_\downarrow(t) - n_\uparrow(t) \} d\tau, \]
\[ n_\downarrow(t) = n_\downarrow(0) + w \int_0^t M(t-\tau) \{ n_\uparrow(t) - n_\downarrow(t) \} d\tau. \] (12)

For \( t \ll 1 \) (or \( \delta \to 0 \)) the kernel \( M(t) \) takes the power form \( t^\alpha / \Gamma(\alpha) \) as a fractional kernel in the integral representation of Eq. (3). However, for \( t \gg 1 \) (or \( \alpha \to 1 \)) \( M(t) \) becomes constant and, as a result, Eq. (12) transforms into
the integral form of the ordinary equations (1). From the linearity of these equations it just follows

\[ n_\uparrow(t) + n_\downarrow(t) = n_\uparrow(0) + n_\downarrow(0) , \quad n_\uparrow(t) - n_\downarrow(t) = n_\uparrow(0) - n_\downarrow(0) - 2w \int_0^t M(t-\tau) \{n_\uparrow(\tau) - n_\downarrow(\tau)\} d\tau. \]

Consequently, we obtain

\[ n_\uparrow(t) = 1 - w \int_0^t e^{-\delta \tau} \tau^{\alpha-1} E_{\alpha,\alpha}(\pm \delta^{\alpha} - 2w \tau^{\alpha}) d\tau, \]

\[ n_\downarrow(t) = w \int_0^t e^{-\delta \tau} \tau^{\alpha-1} E_{\alpha,\alpha}(\pm \delta^{\alpha} - 2w \tau^{\alpha}) d\tau. \]

The equations have steady states (\( t \to \infty \)) corresponding to equilibrium in this system. According to [23], we know

\[ \int_0^\infty e^{-ax} x^{\alpha-1} E_{\alpha,\alpha}(\pm bx) dx = \frac{1}{a^{\alpha} \mp b} , \quad (\text{Re}(a) > |b|^{1/\alpha}), \]
then \( n_i(\infty) = n_j(\infty) = 1/2 \) for any \( \delta \geq 0 \) and \( w > 0 \) (see, for example, Fig. 1). The transition rate \( w \) is again defined by microscopic properties of the system. Thus the relaxation function takes the form

\[
\phi_{\text{temp}}(t) = 1 - 2n_i(t) = 1 - 2w \int_0^t e^{-\delta \tau} \tau^{\alpha-1} E_{\alpha,\alpha}([\delta^\alpha - 2w] \tau^\alpha) \, d\tau.
\]

The response function \( f_{\text{temp}}(t) = -d\phi_{\text{temp}}(t)/dt \) is written as

\[
f_{\text{temp}}(t) = 2w e^{-\delta t} \tau^{\alpha-1} E_{\alpha,\alpha}([\delta^\alpha - 2w] \tau^\alpha).
\]

Fig. 2 demonstrates how to change the response function with the increase of \( \delta \).

For the experimental study it is interesting to get the frequency-domain representation of the latter function. Its real part describes a dispersion of the relaxing medium, and its imaginary part is an absorption. The values explicitly can be measured in experiments. The one-side Fourier transformation of the response function gives

\[
\chi_{\text{temp}}(\omega) = \int_0^\infty e^{-i\omega t} \left( -\frac{d\phi_{\text{temp}}(t)}{dt} \right) \, dt = \frac{1}{1 - \sigma^\alpha + (i\omega/\omega_p + \sigma)^\alpha}, \tag{13}
\]

where \( \omega_p = (2w)^{1/\alpha} \) and \( 0 \leq \sigma = \delta/(2w)^{1/\alpha} < \infty \) are constant. The parameter \( \omega_p \) is the characteristic frequency of the relaxing system. It is easy to notice that the expression (13) for \( \alpha = 1 \) is reduced to the D law, for \( \delta = 0 \) (or \( \sigma = 0 \)) it describes the CC relaxation, and for \( \sigma = 1 \) it does the CD law (see Fig. 3).

### 5 Major Properties of Tempered Relaxation

The frequency dependence of the dielectric susceptibility for orientational polarization of dipoles has been the subject of experimental and theoretical studies for many years, but still there is no any generally accepted theory capable of explaining the observed phenomena. Experimentally it is well known that the complex dielectric susceptibility \( \chi(\omega) = \chi'(\omega) - i\chi''(\omega) \) of most dipolar substances demonstrates a peak in the loss component \( \chi''(\omega) \) at a characteristic frequency \( \omega_p \), and it is characterized by high- \((\omega \gg \omega_p)\) and low-frequency \((\omega \ll \omega_p)\) dependencies. The tempered relaxation shows
Fig. 2. Response function with different $\delta$ ($\omega = 0.5$, $\alpha = 0.5$).

Fig. 3. Imaginary term of the frequency-domain relaxation function $\chi(\omega) = \chi'(\omega) - i\chi''(\omega)$. 
\[ \chi'(\omega) = \frac{A + B \cos(C)}{A^2 + 2AB \cos(C) + B^2} \]
\[ \chi''(\omega) = \frac{B \sin(C)}{A^2 + 2AB \cos(C) + B^2} \]

where \( A = 1 - \sigma^\alpha \), \( B = (\sigma^2 + \omega^2/\omega_p^2)^{\alpha/2} \) and \( C = \alpha \arctan(\omega/\delta) \). For small \( \omega \) and any positive \( \sigma \neq 0 \) it is easy to see that \( \lim_{\omega \to 0} \chi'(\omega) \sim \omega \) and \( \lim_{\omega \to 0} \chi''(\omega) \sim 1 \) whereas for large \( \omega \) we get \( \lim_{\omega \to \infty} \chi'(\omega) \sim \omega^{-\alpha} \) and \( \lim_{\omega \to \infty} \chi''(\omega) \sim \omega^{-\alpha} \). This implies that

\[ \lim_{\omega \to \infty} \frac{\chi''(\omega)}{\chi'(\omega)} = \tan \left( \frac{\alpha \pi}{2} \right) = \cot \left( \frac{n \pi}{2} \right), \]

where \( n = 1 - \alpha \), that is in agreement with the experimental results [12]. However, for small \( \omega \) we come to

\[ \lim_{\omega \to 0} \frac{\chi''(\omega)}{\chi'(\omega) - \chi'_0} = \infty. \]

This means that the energy lost per cycle does not have a constant relationship to the extra energy that can be stored by a static field. According to such an asymptotic behavior and in Fig. 3 it is seen that the tempered relaxation takes an intermediate place between the D, CC and CD types of relaxation.

6 Conclusions

In this paper we have represented our progress in the subordination analysis of relaxation phenomena in the complex systems. The approach permits ones to consider many relaxation laws on the unique theoretical base originating from the stochastic nature of relaxation. The general probabilistic formalism treats the relaxation of the complex systems regardless of the precise nature of local interactions. Following this approach, we have derived the empirical relaxation laws and their macroscopic equations, have characterized their parameters, have connected the parameters with local random characteristics of the relaxation processes, have demonstrated how to make the transition from the microscopic random dynamics in the complex stochastic systems to the macroscopic deterministic description by integro-differential equations. It should be pointed out that the form of these equations is a direct sign of complexity evolution in such systems. Although we restricted only by the detailed analysis of two-state systems, due to Eq.(10), this consideration can be developed to the study of many-state systems (as an example, see the analysis of the three-state fractional system in [13])). The tempered relaxation establishes
a connection between several types of relaxation (D, CC and CD). Its asymptotic behavior earnestly shows that starting as a non-exponential relaxation, latter it tends to the D law. Moreover, the theory of subordination suggests a clear interpretation of the tempered relaxation. As applied to the dielectric relaxation, the interaction of dipoles with each other and their environment has a confined time of action on the relaxation process near a starting point of relaxation. Latter they behave independently just as this is the case of exponential (D) relaxation. To put it in another way, the dipoles are strongly connected with each other in the initial stage of relaxation (and, therefore, their response function obeys a non-exponential decay), but the connection is not long-lived, and in an interval of time each dipole evaluates on its own.

Acknowledgments

AS is grateful to the Institute of Physics and the Hugo Steinhaus Center for pleasant hospitality during his visit in Wroclaw University of Technology.

References

[1] A.K. Jonscher, *Dielectric Relaxation in Solids* (Chelsea Dielectric Press, London, 1983).

[2] A.K. Jonscher, *Universal Relaxation Law* (Chelsea Dielectric Press, London, 1996).

[3] A. Jurlewicz, K. Weron, J. Stat. Phys. **73**, 69(1993); K. Weron, M. Kotulski, Ibid. **88**, 69(1997); A. Jurlewicz, K. Weron, A. K. Jonscher, IEEE Trans. Dielectrics **E18**, 352(2001); A. Jurlewicz, K. Weron J. of Non-Cryst. Sol. **305**, 112(2002); A. K. Jonscher, A. Jurlewicz, K. Weron, Contemp. Phys. **44**, 329(2003).

[4] A. Jurlewicz, Dissertationes Math. **431**, 1(2005).

[5] A. Jurlewicz, K. Weron, Cell. Molec. Biol. Lett. **4**, 55(1999); A. Jurlewicz, K. Weron, Acta Phys. Polon. B **31**, 1077(2000).

[6] R. Metzler, J. Klafter, J. Phys. A **37**, R161(2004).

[7] W. Feller, *An Introduction to Probability and Its Applications* (Wiley, New York, 1996).

[8] C.J.F. Böttcher, P. Bordewijk, *Theory of Electronic Polarization* (Elsevier, Amsterdam, 1978).

[9] A.A. Stanislavsky, Phys. Rev. E **67**, 021111(2003).
[10] M.M. Meerschaert, H.-P. Scheffler, J. Appl. Probab. 41, 623 (2004); P. Becker-Kern, M.M. Meerschaert, H.-P. Scheffler, Ann. Probab. 32(1B), 730(2004).

[11] B. Baeumer, D. A. Benson, M. M. Meerschaert, Physica A 350, 245(2005).

[12] A.A. Stanislavsky, Acta Phys. Polon. B 34(7), 3649(2003).

[13] A.A. Stanislavsky, Theor. and Math. Phys. 138, 418(2004).

[14] M. Caputo, J. Acousti. Soc. Am. 66(1), 176(1979); R. Gorenflo, F. Mainardi, “Fractional calculus: integral and differential equations of fractional order”, In: A. Carpinteri and F. Mainardi (eds.) Fractals and Fractional Calculus in Continuum Mechanics (Springer-Verlag, New York, 1997), pp. 223-276.

[15] Higher Transcendental Functions, edited by A. Erdélyi (McGraw-Hill, New York, 1955), Vol. 3, Chap. 18.

[16] R.N. Pillai, Ann. Inst. Statist. Math. 42(1), 157(1990).

[17] I.M. Sokolov, Phys. Rev. E 63, 056111(2001).

[18] A.A. Stanislavsky, Chaos, Solitons and Fractals 34(1), 51(2007).

[19] A.I. Saichev, S.G. Utkin, Modern Problems of Statistical Physics bf 1, 1(2002) (in Russian); A. Piryatinska, A.I. Saichev, and W.A. Woyczynski, Physica A 349, 375(2005).

[20] J. Rosiński, Stoch. Proc. Appl. 117, 677 (2007).

[21] R. Gorenflo, Yu. Luchko, and F. Mainardi, Fractional Calc. Appl. Anal. 2, 383(1999).

[22] N. G. van Kampen, Stochastic Processes in Physics and Chemistry (North-Holland Physics Publishing, Amsterdam, 1984).

[23] I. Podlubny, Fractional Differential Equations (Academic Press, San Diego, 1999).