Enhanced the thermal Entanglement in Anisotropy Heisenberg $XY Z$ Chain

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The thermal entanglement in Heisenberg $XY Z$ chain is investigated in the presence of external magnetic field $B$. In the two-qubit system, the critical magnetic field $B_c$ is increased because of introducing the interaction of the $z$-component of two neighboring spins $J_z$. This interaction not only improves the critical temperature $T_c$, but also enhances the entanglement for particular fixed $B$. We also analyze the pairwise entanglement between nearest neighbors in three qubits. The pairwise entanglement, for a fixed $T$, can be strong by controlling $B$ and $J_z$.

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I. INTRODUCTION

Entanglement is an important resource in quantum information [1]. The ideal case which Quantum computing and quantum communication are put into use is to find entanglement resource in solid system at a finite temperature. The Heisenberg model is a simple but realistic and extensively studied solid state system [2] [3]. Recently, Heisenberg interaction is not localized in spin system. It can be realized in quantum dots [4], nuclear spins [5], cavity QED [6] [7]. This effect Hamiltonian can be used for quantum computation [8] and controlled NOT gate [7]. The thermal entanglement in isotropic Heisenberg spin chain has been studied in the absence [9,10,15] and in the presence of magnetic field [9,10,14]. The entanglement of two-qubit isotropic Heisenberg system decreases with the increasing $T$ and vanishes beyond a critical value $T_c$ [9,10], which is independent of $B$. Pairwise entanglement in $N$-qubit isotropic Heisenberg system in certain degree can be increased by increasing the temperature or the external field $B$ [9]. Anisotropic Heisenberg spin chain has been investigated in the case of $B = 0$ [10] and $B \neq 0$ [11]. For a two-qubit anisotropic Heisenberg $XY$ chain, one is able to produce entanglement for finite $T$ by adjusting the magnetic field strength [11]. However, the entanglement by increasing $T$ or $B$, in two-qubit anisotropic Heisenberg $XY$ chain [11] or in $N$-qubit isotropic Heisenberg chain [9], is very weak. How to produce strong entanglement is worthy to study. On the other hand, we have not find the work about two-qubit or the $N$-qubit anisotropic $XY Z$ Heisenberg chain in the presence of magnetic field. Although the $N$-qubit Heisenberg chain has been studied [12] [9], in Ref. [12] the authors studied the maximum possible nearest neighbor entanglement for ground state in a ring of $N$ qubits, and in [9] they just investigated the case of isotropic $N$ qubits Heisenberg chain. In this paper, we study the entanglement of two-qubit anisotropic Heisenberg $XY Z$ chain and the pairwise entanglement of three-qubit anisotropic Heisenberg $XYZ$ chain. Introducing the interaction of the $z$-component of two neighboring spins not only improve the critical temperature $T_c$ but also enhance the entanglement for fixed $B$ and $T$ in particular regions. In the case of anisotropic three-qubit Heisenberg $XYZ$ chain, the effect of partial anisotropy $\gamma$ make the revival phenomenon more apparent than in two-qubit chain; for a fixed $T$, one can obtain a robust entanglement by controlling $B$ and $J_z$.

The Hamiltonian of $N$-qubit anisotropic Heisenberg $XYZ$ model in an external magnetic field $B$ is [11]

$$H = \frac{1}{2} \sum_{i=1}^{N} [J_x^{i} \sigma_x^{i} \sigma_x^{i+1} + J_y^{i} \sigma_y^{i} \sigma_y^{i+1} + J_z^{i} \sigma_z^{i} \sigma_z^{i+1} + B(\sigma_z^{i} + \sigma_z^{i+1})],$$

(1)

where $\sigma_j = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices and $J_i (i = x, y, z)$ is real coupling coefficient. The coupling coefficient $J_i$ of arbitrary nearest neighbor two qubits is equal in value. For the spin interaction, the chain is said to be antiferromagnetic for $J_i > 0$ and ferromagnetic for $J_i < 0$.

For a system in equilibrium at temperature $T$, the density operator is $\rho = Z^{-1} \exp(-H/k_BT)$, where $Z = Tr[\exp(-H/k_BT)]$ is the partition function and $k_B$ is Boltzmann’s constant. For simplicity we write $k_B = 1$. Entanglement of two qubits can be measured by concurrence $C$ which is written as $C = \max\{0, 2 \max\{\lambda_i\} - \sum_{i=1}^{4} \lambda_i\}$ [16] [17] [13], where $\lambda_i$ is the square roots of the eigenvalues of the matrix $R = \rho S \rho^* S$, $\rho$ is the density matrix, $S = \sigma_y \otimes \sigma_y$ and $*$ stand for complex conjugate. The concurrence is available no matter what $\rho$ is pure or mixed.
II. TWO-QUBIT HEISENBERG $XY$ CHAIN

Now, we consider the Hamiltonian for anisotropic two-qubit Heisenberg $XYZ$ chain in an external magnetic field $B$. The Hamiltonian can be expressed as

$$H = J(\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) + J\gamma(\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) + \frac{J_x}{2} \sigma_1^z \sigma_2^z + \frac{B}{2}(\sigma_1^z + \sigma_2^z)$$

(2)

where $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ is raising and lowering operator respectively, and $J = J_x + J_y$, $\gamma = \frac{J_x - J_y}{J_x + J_y}$. The parameter $\gamma$ $(0 < \gamma < 1)$ measure the anisotropy (partial anisotropy) in $XY$ plane. When the Hamiltonian of the system has the form of Eq.(2), in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the density matrix of the system can be written as

$$\rho_{12} = \begin{pmatrix}
  u_1 & 0 & 0 & v
  \\
  0 & w & z & 0
  \\
  0 & z & w & 0
  \\
  v & 0 & 0 & u_2
\end{pmatrix}.$$  

(3)

These nonzero matrix element can be calculated through

$$u_1 = Tr(|00\rangle\langle00|\rho), u_2 = Tr(|11\rangle\langle11|\rho),$$

$$w = Tr(|01\rangle\langle01|\rho), v = Tr(|00\rangle\langle11|\rho), z = Tr(|01\rangle\langle10|\rho).$$

(4)

The square roots of the eigenvalues of the matrix $R$ are $\lambda_{1,2} = |w \pm z|$, $\lambda_{3,4} = |u_1u_2 \pm v|$. Therefore, we can calculate the concurrence.

The eigenvalues and eigenstates of $H$ are easily obtained as $H|\Psi^\pm\rangle = (-\frac{J}{2} \pm J)|\Psi^\pm\rangle$, $H|\Sigma^\pm\rangle = (\frac{J}{2} \pm \eta)|\Sigma^\pm\rangle$, with the eigenstates $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$, $|\Sigma^\pm\rangle = \frac{1}{\sqrt{2}\eta}[(\eta \mp B)|00\rangle \pm J_\gamma|11\rangle]$, where $\eta = \sqrt{B^2 + (J_\gamma)^2}$. One can notice that the eigenstates are the same as the case of $J_z = 0$ [11]. Because the basises $|01\rangle$ and $|10\rangle$ are the two degenerate eigenstates of $\sigma_1^z \sigma_2^z$ with eigenvalue $-1$, hence the superposition of the two degenerate states $|01\rangle$ and $|10\rangle$ still is the eigenstate of $\sigma_1^z \sigma_2^z$, that is, $|\Psi^\pm\rangle$ is the eigenstate of $J_z = 0$ as well as that of $J_z \neq 0$. The same reason account for $|\Sigma^\pm\rangle$ both as an eigenstate of Eq.(2) and as that of the case of $J_z = 0$. From Eq.(4), tracing on the eigenstates, we obtain the square roots of the eigenvalues of the matrix $R$

$$\lambda_{1,2} = Z^{-1} e^{\frac{\beta J_\gamma}{2}} e^{\pm \beta J},$$

$$\lambda_{3,4} = Z^{-1} e^{\frac{\beta J_\gamma}{2}} \sqrt{\frac{1}{\eta} + \frac{\lambda_{1,2}^2}{\eta} \sinh \beta \eta} = \frac{J_\gamma}{\eta} \sinh \beta \eta,$$

(5)

where the partition function $Z = 2(e^{-\beta J} \cosh \beta \eta} + e^{\beta J} \cosh \beta J)$. Because the concurrence is invariant under the substitutions $J \rightarrow -J$ and $\gamma \rightarrow -\gamma$ [11], we will consider the case $J > 0$ and $0 < \gamma < 1$. But with substitution $J_z \rightarrow -J_z$ the the concurrence is variant. We choose $J_z > 0$, and we will state the reason later.

We first review the circumstance of anisotropic Heisenberg $XY$ chain, which is analyzed in [11]. At $T = 0$, exist a critical magnetic field $B_c$. As $B$ cross $B_c$, the concurrence $C$ drops suddenly then undergoes a "revival" for sufficient large $\gamma$. However, we noticed that $B_c$ decrease with the increasing of the anisotropic parameter $\gamma$. Although with $\gamma$ increasing the critical temperature $T_c$ is improved, the entanglement, when temperature is in the revival region, is very weak.

With $\gamma = 0.3$, we show the concurrence as a function of $B$ and $T$ for two values of $J_z$ in Fig. 1. For $J_z = 0$ (Fig.1a) corresponding to the circumstance of anisotropic Heisenberg $XY$ chain [11], one can observe a revival phenomenon and the weak entanglement in revival region. For the convenience of representation, we define the main region in which concurrence $C$ keeping its constant and maximal value. Comparing Fig. 1 (a) with (b), we find that with the increasing of $J_z$, the main region is extended in terms of $B$ and $T$, i.e., the critical magnetic field $B_c$ is broadened and the critical temperature $T_c$ in main region is improved. That is to say, the range of concurrence $C$ keeping its constant and maximal is extended in terms of $B$ and $T$, so we can obtain strong entanglement in the extended range.

We can understand the effect of $J_z$ on $B_c$ from the case of $T = 0$. For $T = 0$ under the condition of $J_z \leq J$, $C$ can be written analytically as

$$C(T = 0) = \begin{cases}
  1 & \text{for } \eta < J + J_z \\
  (1 - J_\gamma/\eta)/2 & \text{for } \eta = J + J_z \\
  J_\gamma/\eta & \text{for } \eta > J + J_z
\end{cases}$$

(6)
The parameters $J$, $\eta$ and $\gamma$ are independent of $J_z$ in the case of two interacting qubits. Comparing Eq.(6) with Eq.(6) of Ref. [11], we can see clearly that if $J_z$ is positive, $J_z$ makes the intersection points of piecewise function shift. In this paper, we consider the case of $J_z > 0$. Fig. 2 shows the concurrence at $T = 0$ for three values of positive $J_z$. It show clearly that concurrence drops sharply at a finite value of magnetic field $B$, which is called critical magnetic field $B_c$, at which the quantum phase transition occurs [11]. But with the increasing of $J_z$, $B_c$ is increased. The interaction of the z-component of two neighboring spins $J_z$ causes a shift in the locations of the phase transitions. Namely, the presence of positive $J_z$ increases the region over which the concurrence $C$ attains its maximum value. This result means that in larger region of $B$ and $T$ we can obtain stronger entanglement. The effect of $J_z$ is different with that of $\gamma$ on changing $B_c$. In the case of $J_z = 0$ [11], although with the increasing of $\gamma$ the critical temperature $T_c$ is increased, the larger the values of $\gamma$, the smaller the critical magnetic field $B_c$. Here, introducing the z-component interaction of two neighboring spins not only extends critical magnetic field $B_c$ but also improves critical temperature $T_c$ and the entanglement (we will further show it in Fig. 3).

Let us consider concurrence changing with temperature for different values of $J_z$ in a fixed $B$ ($B = 1.1$). We plot it in Fig. 3 with $\gamma = 0.3$. We notice that existing a critical temperature $T_c$ at which the entanglement vanishes. Obviously, $T_c$ is improved monotonously with increasing of $J_z$. Under the condition $J_z = 0$ (corresponding to XY model [11]), the concurrence exhibit a revival phenomenon, but the maximal values of entanglement in both area are small. If introducing the $J_z$, the critical external magnetic field $B_c$ become larger so that $B = 1.1$ is less than $B_c$ (the critical magnetic when $J_z = 0.2, 0.5$ or $J_z = 0.9$), thus we observe the maximal value of entanglement $1$. In the temperature range $0 < T < 1.725$, the larger $J_z$ the stronger entanglement. Therefore, $J_z$ not only improve the critical temperature $T_c$, but also enhance the entanglement for particular fixed $B$ and $\gamma$.

### III. THE PAIRWISE ENTANGLEMENT IN THREE QUBITS

The calculation of pairwise entanglement in $N$ qubits is very complicated due to the anisotropy in Heisenberg $XYZ$ chain. Here we just calculate the pairwise entanglement in three qubits to show the effects of $J_z$. We now solve the eigenvalue problems of the three-qubit $XYZ$ Hamiltonian. We list the eigenvalues and the corresponding eigenvectors as follow

$$E_{1,2} = -J - \frac{J_z}{2} + B : |\Phi_{1,2}\rangle = \pm \frac{1}{2}(1 + \frac{1}{\sqrt{3}})|110\rangle + \frac{1}{\sqrt{3}}|101\rangle + \frac{1}{2}(1 \pm \frac{1}{\sqrt{3}})|011\rangle,$$

$$E_{4,4} = J + \frac{J_z}{2} - B \pm \eta_- : |\Phi_{3,4}\rangle = \frac{1}{\sqrt{2\eta_-[\eta_- + (J_z - 2B - J)]}}[(J_z - 2B - J \pm \eta_-)|000\rangle + J\gamma \sum_{n=0}^{2} \Upsilon^n|110\rangle];$$

$$E_{5,6} = -J - \frac{J_z}{2} - B : |\Phi_{5,6}\rangle = \pm \frac{1}{2}(1 + \frac{1}{\sqrt{3}})|010\rangle + \frac{1}{\sqrt{3}}|100\rangle + \frac{1}{2}(1 \pm \frac{1}{\sqrt{3}})|001\rangle;$$

$$E_{7,8} = J + \frac{J_z}{2} + B \pm \eta_+ : |\Phi_{7,8}\rangle = \frac{1}{\sqrt{2\eta_+[\eta_+ + (J_z + 2B - J)]}}[(J_z + 2B - J \pm \eta_+)|111\rangle + J\gamma \sum_{n=0}^{2} \Upsilon^n|010\rangle].$$

where $\eta_{\pm} = \sqrt{(J_z - J \pm 2B)^2 + 3(J\gamma)^2}$, $\Upsilon$ is the cyclic right shift operator [15]. The reduced density matrix of two nearest-neighbor qubits in $N$ qubits system also has the form of Eq.(3). Employing Eq.(4) and tracing on the basis of eigenstates shown in Eq. (7), one can get the density matrix $\mu_1, \mu_2, w, z, \nu$, then further obtain the concurrence. Here we do not write the expressions of $\lambda_i$ because it is very long. We will directly plot some curves to show the effect of $J_z$ on enhancing entanglement.

Fig.4 show concurrence as a function of $B$ and $T$ with $\gamma = 0.3, J_z = 0.9$ and $J = 1.0$ in three-qubit $XYZ$ Heisenberg chain. We see that with the same $\gamma = 0.3$, the effect of partial anisotropy $\gamma$ make the revival phenomenon more apparent than in two-qubit chain. When $B = 4$ in Fig. 1, the largest critical temperature $T_c$ produced by $\gamma$ is about 1.0 (Fig.1a); due to the restrain of $J_z$ the maximum temperature only caused by $\gamma$ is below 0.8(Fig.1b). However, in three-qubit system if $B = 4$ with the same set of parameters, comparing Fig.1b with Fig.4, the critical temperature $T_c$ in revival region almost equal to 1.8. The stronger effect of $\gamma$ implies that if we aim to obtain strong entanglement we can decrease $\gamma$ properly and increase $J_z$, otherwise increasing $\gamma$ can make the revival phenomenon more evident. Of course, the coupling constant $J_z$ also increase magnetic field $B_c$ and expend the region of concurrence keeping constant in terms of $B$ and $T$ as it do in two-qubit (for the limited of the page,we do not plotted here).

For $T = 0.6$, Fig.5 show concurrence as function of $B$ and $J_z$. There is no entanglement for $B = 0$, which corresponds with Fig.4. If $J_z$ is below a certain value, in case of Fig. 5 the value is about 0.2, the entanglement appears in one area corresponding to the "revival" [11] on condition that the magnetic field is larger than a certain value, and the
certain value of $B$ is increased with the enhanced of $J_z$. But, if $J_z$ is larger than 0.2, there are two areas appearing entanglement, and the entanglement appearing in the lower range of $B$ can be much stronger than that in higher magnetic field. In the lower range of $B$, for a certain $B$, the large $J_z$ the large concurrence. Thus, in the $N$-qubits $XYZ$ system, for a fixed $T$, one can obtain a robust entanglement by controlling $B$ and $J_z$.

IV. CONCLUSION

The thermal entanglement in anisotropic $XYZ$ Heisenberg chain is investigated. Through analyzing the two-qubit system, we find that with the increasing of $J_z$, the critical magnetic field $B_c$ is increased; the coupling along $Z$ not only improves the critical temperature $T_c$, but also enhances the entanglement for certain fixed $B$. We also analyze the entanglement between two nearest neighbors in three qubits and find that the effect of partial anisotropy is more evident than it do in two-qubit system. The pairwise entanglement exhibit a interesting phenomenon. For certain fixed $B$, if the coupling constant $J_z$ is small, the pairwise entanglement only exists in relative strong magnetic field $B$ and the entanglement is weak. By increasing $J_z$ in lower range of $B$, one can obtain a strong entanglement. Therefore, interaction constant of the z-component of two neighboring spins $J_z$ play important role in enhancing entanglement and in improving the critical temperature.

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The captions of the figure:

Fig. 1 Concurrence in two-qubit Heisenberg $XYZ$ chain is plotted vs $T$ and $B$, where (a): $J_z = 0$, (b): $J_z = 0.9$. For all plotted $J = 1.0$, $\gamma = 0.3$.

Fig. 2 Concurrence in two-qubit Heisenberg $XYZ$ chain vs $B$ at zero temperature for various values of $J_z$ with $\gamma = 0.3$ and $J = 1.0$. From left to right $J_z$ equal to 0, 0.5, 0.9, respectively.

Fig. 3 Concurrence in two qubits Heisenberg $XYZ$ chain is plotted vs $T$. For all plotted $J = 1.0$, $B = 1.1$, $\gamma = 0.3$. From top to bottom $J_z$ equal to 0.9, 0.5, 0.2, 0, respectively.

Fig. 4 Pairwise entanglement in three-qubit Heisenberg $XYZ$ chain is plotted as a function of $T$ and $B$, where $\gamma = 0.3$, $J = 1.0$, $J_z = 0.9$.

Fig. 5 Pairwise entanglement is plotted as a function of $B$ and $J_z$, where $T = 0.6$, $J = 1.0$, $\gamma = 0.3$.

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