LOVÁSZ EXTENSION AND GRAPH CUT

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Abstract. A set-pair Lovász extension is established to construct equivalent continuous optimization problems for graph $k$-cut problems.

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1. Introduction

Motivated by the need of practical application in e.g., machine learning and big data, it is not only natural but also imperative for applied mathematicians to plug into valuable subjects emerged from well-established mathematics such as analytic techniques, topological views and algebraic structures. A firm bridge between discrete data world and continuous math field should be tremendously helpful. Along this direction, the Lovász extension [Lov83] provides a both explicit and equivalent continuous optimization problem for a discrete optimization problem. However, it deals with set-functions which admit only one input set and thus correspond to so-called 2-cut problems, for instance, the Cheeger cut problem [BRSH13]. Therefore it is not straightforward to apply the original Lovász extension into general $k$-cut problems, such as the dual Cheeger cut [Tre12, BJ13]. Accordingly, we ask

**Question 1.1.** How to write down a both explicit and equivalent continuous optimization problem for a graph $k$-cut problem?
Let us introduce some notations first. $G = (V, E)$ is an unweighted and undirected graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E$, and $w_{ij}$ the weight of the edge $i \sim j$. For two disjoint subsets $A$ and $B$ of $V$, let $E(A, B)$ denote the set of edges that cross $A$ and $B$. For $S \subset V$, let $S^c = V \setminus S$ be the complement of $S$. The edge boundary of $S$ is $\partial S = \partial S^c = E(S, S^c)$. The amount of edge set $E(A, B)$ is denoted by $|E(A, B)| = \sum_{i \in A} \sum_{j \in B} w_{ij}$, and the volume of $S$ is defined to be $\text{vol}(S) = \sum_{i \in S} d_i$, where $d_i = \sum_{j=1}^n w_{ij}$ is the degree of the vertex $i$.

**Definition 1.2** (dual Cheeger cut [Tre12, BJ13]). The dual Cheeger problem is devoted to solving

$$h^+(G) = \max_{S_1 \cap S_2 = \emptyset, S_1 \cup S_2 \neq \emptyset} \frac{2|E(S_1, S_2)|}{\text{vol}(S_1 \cup S_2)},$$

and we call $h^+(G)$ the dual Cheeger constant.

To say the least, before we discuss Question 1.1, the following specific question needs to be solved at the first place.

**Question 1.3.** Is there an explicit and equivalent continuous optimization for a graph 3-cut problem like the dual Cheeger cut (1.1)?

In this work, we propose a set-pair Lovász extension which not only provides a complete answer to Question 1.3 (even works for a series of graph 3-cut problems), but also enlarges the feasible region of resulting equivalent continuous optimization problems from the first quadrant $\mathbb{R}^+ \times \{0\}$ (see Theorem 2.2) to the entire space $\mathbb{R}^n \setminus \{0\}$ (see Theorem 1.5) for graph 2-cut problems like the Cheeger cut (see Theorem 1.17). This enlarged feasible region may have some advantages in designing solution algorithms.

**Definition 1.4** (set-pair Lovász extension). Let $V = \{1, \ldots, n\}$. For $x \in \mathbb{R}^n$, let $\sigma : V \cup \{0\} \to V \cup \{0\}$ be a bijection such that $|x_{\sigma(1)}| \leq |x_{\sigma(2)}| \leq \cdots \leq |x_{\sigma(n)}|$ and $\sigma(0) = 0$, where $x_0 := 0$. One defines the sets

$$V_{\sigma(i)}^\pm := \{j \in V : \pm x_j > |x_{\sigma(i)}|\}, \quad i = 0, 1, \ldots, n - 1.$$

Let

$$P_2(V) = \{(A, B) : A, B \subset V \text{ with } A \cap B = \emptyset\}.$$

Given $f : P_2(V) \to [0, +\infty)$, the set-pair Lovász extension of $f$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$f^L(x) = \sum_{i=0}^{n-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) f(V_{\sigma(i)}^+, V_{\sigma(i)}^-).$$

**Theorem 1.5.** Assume that $f, g : P_2(V) \to [0, +\infty)$ are two set-pair functions with $g(A, B) > 0$ whenever $A \cup B \neq \emptyset$. Then there hold both

$$\min_{(A, B) \in P_2(V) \setminus \{(\emptyset, \emptyset)\}} \frac{f(A, B)}{g(A, B)} = \min_{x \neq 0} \frac{f^L(x)}{g^f(x)}.$$
and

\[(1.6) \quad \max_{(A,B)\in \mathcal{P}_2(V)\setminus\{(\emptyset,\emptyset)\}} \frac{f(A,B)}{g(A,B)} = \max_{x\neq 0} \frac{f_L(x)}{g_L(x)}.\]

Theorem 1.5 and its applications listed below show a natural answer to Question 1.3.

**Theorem 1.6.**

\[(1.7) \quad 1 - h^+(G) = \min_{x\neq 0} \frac{I^+(x)}{\|x\|},\]

where

\[(1.8) \quad \|x\| = \sum_{i=1}^{n} d_i|x_i|,\]

\[(1.9) \quad I^+(x) = \sum_{i<j} w_{ij}|x_i + x_j|.\]

**Definition 1.7 (max 3-cut [FJ97]).** The max 3-cut problem is to determine a graph 3-cut by solving

\[(1.10) \quad h_{\max,3}(G) = \max_{A,B,C} \frac{2(|E(A,B)| + |E(B,C)| + |E(C,A)|)}{\text{vol}(V)},\]

and the associate \((A,B,C)\) is called a max 3-cut, where the subsets \(A, B, C\) satisfy \(A \cap B = B \cap C = C \cap A = \emptyset\) and \(A \cup B \cup C = V\).

**Theorem 1.8.**

\[(1.11) \quad h_{\max,3}(G) = \max_{x\neq 0} \frac{I(x) + \hat{I}(x)}{\text{vol}(V)\|x\|_\infty},\]

where

\[(1.12) \quad I(x) = \sum_{i<j} w_{ij}|x_i - x_j|,\]

\[(1.13) \quad \hat{I}(x) = \sum_{i<j} w_{ij}|x_i| - |x_j|,\]

\[(1.14) \quad \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.\]

**Definition 1.9 (ratio max 3-cut I).** The first ratio max 3-cut problem is to determine a graph 3-cut by solving

\[(1.15) \quad h_{\max,3,I}(G) = \max_{A,B,C} \frac{2(|E(A,B)| + |E(B,C)| + |E(C,A)|)}{\text{vol}(A) + \text{vol}(B)},\]

where \(A \cap B = B \cap C = C \cap A = \emptyset\) and \(A \cup B \cup C = V\).

**Theorem 1.10.**

\[(1.16) \quad 1 - h_{\max,3,I}(G) = \min_{x\neq 0} \frac{I^+(x) - 2\hat{I}(x)}{\|x\|}.\]
Definition 1.11 (ratio max 3-cut II). The second ratio max 3-cut problem is to determine a graph 3-cut by solving

\[
h_{\text{max},3,II}(G) = \max_{A,B,C} \frac{2(|E(A,B)| + |E(B,C)| + |E(C,A)|)}{\max\{\text{vol}(A \cup B), \text{vol}(C)\}},
\]

where \(A \cap B = B \cap C = C \cap A = \emptyset\) and \(A \cup B \cup C = V\).

Theorem 1.12.

\[
h_{\text{max},3,II}(G) = \max_{x \neq 0} \frac{2I(x) - \|x\| + I^+(x)}{\text{vol}(V)\|x\|_\infty - \min_{\alpha \in \mathbb{R}} \sum_{i=1}^n d_i |x_i - \alpha|}.
\]

In order to give a complete answer to Question 1.1, we propose an isomorphism to translate a \(k\)-cut (\(k > 3\)) problem to a 3-cut one on a graph of larger size, and then still utilize Theorem 1.5 to derive the corresponding continuous problem. On the other hand, Theorem 1.5 also works for graph 2-cut problems, although it produces a different form from that by the original Lovász extension, during which Lemma 3.6 plays a key role and translates a graph 2-cut with symmetric form into a graph 3-cut. The main difference lies in the feasible region.

Definition 1.13 (maxcut [Kar72]). The maxcut problem is to determine a graph cut by solving

\[
h_{\text{max}}(G) = \max_{S \subset \mathbb{V}} \frac{2|\partial S|}{\text{vol}(V)}.
\]

Definition 1.14 (Cheeger cut [BRSH13]). The Cheeger problem is to determine a graph cut by solving

\[
h(G) = \min_{S \subset \mathbb{V},S \neq \emptyset,\mathbb{V}} \frac{|\partial S|}{\min\{\text{vol}(S), \text{vol}(S^c)\}}.
\]

Definition 1.15 (anti-Cheeger cut [Xu16]). The anti-Cheeger constant \(h_{\text{anti}}(G)\) is defined as

\[
h_{\text{anti}}(G) = \max_{S \subset \mathbb{V}} \frac{|\partial S|}{\max\{\text{vol}(S), \text{vol}(S^c)\}}.
\]

The original Lovász extension (2.2) (vide post) yields the following equivalent continuous optimization problems:

\[
h_{\text{max}}(G) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{I(x)}{\text{vol}(V) \max x_i},
\]

\[
h(G) = \inf_{x \text{ nonconstant in } \mathbb{R}^n} \sup_{c \in \mathbb{R}} \frac{I(x)}{\sum_{i=1}^n d_i |x_i - c|},
\]

\[
h_{\text{anti}}(G) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{I(x)}{2\text{vol}(V) \max x_i - \min_{c \in \mathbb{R}} \sum_{i=1}^n d_i |x_i - c|}.
\]

In contrast, the proposed set-pair Lovász extension is capable of enlarging the feasible region from the first quadrant \(\mathbb{R}^n_+ \setminus \{0\}\) in Eqs. (1.20)-(1.22) to the entire space \(\mathbb{R}^n \setminus \{0\}\) in Eqs. (1.23)-(1.25).
Theorem 1.16.\[ h_{\text{max}}(G) = \max_{x \neq 0} \frac{I(x)}{\text{vol}(V)\|x\|_{\infty}}. \]

Theorem 1.17.\[ h(G) = \inf_{x \text{ nonconstant}} \sup_{c \in \mathbb{R}} \sum_{i=1}^{n} d_i |x_i - c|. \]

Theorem 1.18.\[ h_{\text{anti}}(G) = \max_{x \neq 0} \frac{I(x)}{2\text{vol}(V)\|x\|_{\infty} - \min_{\alpha \in \mathbb{R}} \|x - \alpha 1\|}. \]

Remark 1.19. Comparing (1.11) to (1.23), the continuous objective function for max 3-cut happens to only add a nonnegative term $\tilde{I}(x)$ to the numerator. Such slight formal discrepancy may imply some deep connections between maxcut and max 3-cut which deserves more efforts to explore.

In a word, we may summarize the research line for the graph cut problems into the picture:

![Diagram](combinatorial_optimization -> Lovász_extension -> continuous_optimization -> critical_point_theory -> spectral_theory)

The Lovász extension and its variants provide a systematic way to find explicit equivalent continuous optimization problems for discrete and combinatorial optimization ones. On the practical side, new possibilities immediately open up for designing continuous optimization algorithms for combinatorial problems. Several preliminary attempts have been tried [HB10, Ler12, CSZ15, CSZ16, CSZZ18] and more efforts are highly called for algorithm research. On the other hand, explicit objective functions of resulting continuous optimizations provide the possibility to develop and enrich the spectral graph theory [Chu97] as already did for the Cheeger cut [Cha16, CSZ15, CSZ17] and the dual Cheeger cut [CSZ16].

The rest of the paper is organized as follows. Section 2 collects basic properties of the Lovász extension including both continuity and convexity, and shows that the set-pair Lovász extension may be superior over the original one. Such superiority is further demonstrated in Section 3 by applying it into typical graph $k$-cut problems.

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2. Set-pair Lovász extension

Definition 2.1 (Lovász extension [Lov83]). Let \( V = \{1, \ldots, n\} \subset \mathbb{N} \). For \( x \in \mathbb{R}^n \), let \( \sigma : V \cup \{0\} \to V \cup \{0\} \) be a bijection such that \( x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)} \) and \( \sigma(0) = 0 \), where \( x_0 := 0 \). One defines the sets \( V_\sigma(i) := \{ j \in V : x_j > x_{\sigma(i)} \}, \quad i = 1, \ldots, n-1 \), \( V_0 = V \).

Let
\[
\mathcal{P}(V) = \{ A : A \subset V \}.
\]
and
\[
f^L_0(x) = \sum_{i=0}^{n-1} (x_{\sigma(i+1)} - x_{\sigma(i)}) f(V_\sigma(i)).
\]

Theorem 2.2 ([BRSH13], Theorem 1). Assume that \( f, g : \mathcal{P}(V) \to [0, +\infty) \) are two functions with \( g(A) > 0 \) whenever \( A \neq \emptyset \), then there hold both
\[
\min_{A \in \mathcal{P}(V) \setminus \{\emptyset\}} f(A) = \min_{x \in \mathbb{R}^n_+ \setminus \{0\}} \frac{f^L_0(x)}{g_0(x)},
\]
and
\[
\max_{A \in \mathcal{P}(V) \setminus \{\emptyset\}} f(A) = \max_{x \in \mathbb{R}^n_+ \setminus \{0\}} \frac{f^L_0(x)}{g_0(x)}.
\]

Remark 2.3. The proof of Theorem 2.2 (see [BRSH13]) heavily depends on the non-negativity of the terms \( x_{\sigma(1)} f(V_\sigma(1)) \) and \( (x_{\sigma(i+1)} - x_{\sigma(i)}) f(V_\sigma(i)) \) in the summation form (2.2), \( i = 1, \ldots, n-1 \), thereby indicating that one needs the constraint \( x_{\sigma(1)} \geq 0 \), i.e., \( x \in \mathbb{R}^n_+ \). The integral form (2.3) in Proposition 2.4 also manifests clearly such dependence through the last term. Indeed, the minor change of Eq. (1.20):
\[
-\infty = \inf_{x \neq 0} \frac{I(x)}{\text{vol}(V)} \max_i x_i \leq \min_{S \subset V} \frac{2|\partial S|}{\text{vol}(V)} \leq \max_{S \subset V} \frac{2|\partial S|}{\text{vol}(V)} < \sup_{x \neq 0} \frac{I(x)}{\text{vol}(V)} \max_i x_i = +\infty,
\]
shows an example in which Theorem 2.2 fails if we naively replace \( \mathbb{R}^n_+ \setminus \{0\} \) by \( \mathbb{R}^n \setminus \{0\} \) in Eqs. (2.3) and (2.4). Fortunately, as the fruitful results and discussions in this section, we can enlarge the feasible region \( \mathbb{R}^n_+ \setminus \{0\} \) to \( \mathbb{R}^n \setminus \{0\} \) using the proposed set-pair analog of Lovász extension.

Proposition 2.4 ([Bac13], Definition 3.1).
\[
f^L_0(x) = \int_{\min_{1 \leq i \leq n} x_i}^{\max_{1 \leq i \leq n} x_i} f(V_t) dt + f(V) \min_{1 \leq i \leq n} x_i,
\]
where \( V_t(x) = \{ i \in V : x_i > t \} \).
**Definition 2.5.** A set-function \( f : \mathcal{P}(V) \to \mathbb{R} \) is symmetric if \( f(A) = f(A^c) \) for any subset \( A \subset V \). A set-pair-function \( f : \mathcal{P}_2(V) \to \mathbb{R} \) is symmetric if \( f(A, B) = f(B, A) \) for any \((A, B) \in \mathcal{P}_2(V)\).

**Proposition 2.6** ([Bac13], Proposition 3.1). For \( f^L_0(x) \) by the Lovász extension, we have

(a) \( f^L_0(x + \alpha 1) = f^L_0(x) + \alpha f(V) \) for any \( \alpha \in \mathbb{R} \).

(b) \( f^L_0(x) \) is one-homogeneous.

(c) \( (f + g)^L_0 = f^L_0 + g^L_0 \), \( (\lambda f)^L_0 = \lambda f^L_0 \), \( \forall \lambda \geq 0 \).

(d) \( f^L_0(x) \) is even if and only if \( f \) is symmetric.

For \( A \subset V \), \( 1_A \) is the characteristic function of \( A \). For the set-pair case, we denote

\[
1_{A,B} = 1_A - 1_B, \quad \forall (A, B) \in \mathcal{P}_2(V).
\]

Accordingly, for any set-pair function \( f : \mathcal{P}_2(V) \to [0, +\infty) \), the following fact can be readily verified by Definition 1.4

\[
f^L(1_{A,B}) = f(A, B), \quad \forall (A, B) \in \mathcal{P}_2(V) \setminus \{ (\emptyset, \emptyset) \}.
\]

Particularly, the above equality is always true for any \((A, B) \in \mathcal{P}_2(V)\) if \( f(\emptyset, \emptyset) = 0 \).

Similarly, we can derive an integral form of the set-pair Lovász extension.

**Proposition 2.7.**

\[
f^L_0(x) = \int_0^{\|x\|_{\infty}} f(V^+_t(x), V^-_t(x)) dt,
\]

where \( V^+_t(x) = \{ i \in V : x_i > t \} \).

**Proof.** Let \( \sigma \) be a permutation defined in Definition 1.4. It is easy to check that if \( |x_{\sigma(i)}| \leq t < |x_{\sigma(i+1)}| \) then

\[
V^+_t(x) = \{ i \in V : x_i > t \} = V^+_t(x).
\]

Therefore,

\[
f^L_0(x) = \sum_{i=0}^{n-1} (|x_{\sigma(i)}| - |x_{\sigma(i+1)}|) f(V^+_{\sigma(i)}, V^-_{\sigma(i)})
\]

\[
= \sum_{i=0}^{n-1} \int_{|x_{\sigma(i)}|}^{x_{\sigma(i+1)}} f(V^+_t(x), V^-_t(x)) dt
\]

\[
= \int_0^{\|x\|_{\infty}} f(V^+_t(x), V^-_t(x)) dt.
\]

For simplicity, we denote by \((A, B) \subset (C, D)\) if \( A \subset C \) and \( B \subset D \) for \((A, B), (C, D) \in \mathcal{P}_2(V)\).
Proposition 2.8.

\[ f^L(x) = \sum_{i=1}^{p} \lambda_i f(V_i^+, V_i^-) \]

where \((V_p^+, V_p^-) \subset \cdots \subset (V_0^+, V_0^-) (p \in \mathbb{N}^+)\) is a chain satisfying \(V_0^+ \cap V_0^- = \emptyset\), \(\sum_{i=0}^{p} \lambda_i \mathbf{1}_{V_i^+, V_i^-} = x\) and \(\sum_{i=0}^{p} \lambda_i = \|x\|_\infty, \lambda_i \geq 0\).

Proof. Setting \(t_i = \sum_{j=0}^{i} \lambda_j, i = 0, \ldots, p, t_{p+1} = \|x\|_\infty\). Now we verify

\[ \lambda_i f(V_i^+, V_i^-) = \int_{t_{i-1}}^{t_i} f(V_t^+(x), V_t^-(x)) dt, \quad i = 0, \ldots, p, \]

where \(V_t^\pm(x) = \{t \in V | x_t > t\}\). It obviously holds for the case of \(\lambda_i = 0\), so we can assume that \(\lambda_i \neq 0\). Since

\[ \sum_{i=0}^{p} \lambda_i \mathbf{1}_{V_i^+, V_i^-} = x \]

we can assume that \(j \in V_i^\pm\) and thus \(j \notin V_i^\mp\) for any \(0 \leq i \leq p\). Consider the \(j\)-th component of (2.11) on both sides,

\[ \pm \sum_{V_i^\pm \ni j} \lambda_i = \pm \sum_{V_i^\pm \ni j} \lambda_i = \pm t_{p_j} = x_j, \]

where \(p_j\) is the largest integer such that \(j \in V_{p_j}^\pm\). Then

\[ V_t^\pm(x) = V_i^\pm, \quad \forall t \in [t_{i-1}, t_i), \]

and

\[ \int_{t_{i-1}}^{t_i} f(V_t^+(x), V_t^-(x)) dt = \int_{t_{i-1}}^{t_i} f(V_i^+, V_i^-) dt = \lambda_i f(V_i^+, V_i^-). \]

Therefore

\[ \sum_{i=0}^{p} \lambda_i f(V_i^+, V_i^-) = \sum_{i=0}^{p} \int_{t_{i-1}}^{t_i} f(V_t^+(x), V_t^-(x)) dt \]

\[ = \int_{0}^{t_p} f(V_t^+(x), V_t^-(x)) dt = f^L(x). \]

In particular, let \(p = n-1\), \(V_i^\pm = V_{\sigma(i)}^\pm\) and \(\lambda_i = |x_{\sigma(i+1)}| - |x_{\sigma(i)}|\), \(i = 0, 1, \ldots, n-1\), then (2.9) returns to (1.3).

Remark 2.9. A more detailed checking of the proof of Proposition 2.8 shows that, for given \(x \neq 0\), if we assume every \(\lambda_i > 0\), then the chain \((V_p^+, V_p^-) \subset \cdots \subset (V_0^+, V_0^-) (p \in \mathbb{N}^+)\) and \(\{\lambda_i\}_{i=0}^{p}\) in Proposition 2.8 are uniquely determined by \(x\) and thus independent of \(f\). That is, if there are two chains
(V_p^+ , V^-_p) \subset \cdots \subset (V_q^+ , V^-_q) \ (p \in \mathbb{N}^+) \text{ and } (\bar{V}_q^+ , \bar{V}_q^-) \subset \cdots \subset (\bar{V}_0^+ , \bar{V}_0^-) \ (q \in \mathbb{N}^+) \text{, as well as two sequences of positive numbers } \{ \lambda_i \}_{i=0}^p \text{ and } \{ \bar{\lambda}_i \}_{i=0}^q \text{, such that } V_0^+ \cap V_0^- = \bar{V}_0^+ \cap \bar{V}_0^- = \emptyset, \sum_{p=0}^q \lambda_i 1_{V_i^+} 1_{V_i^-} = \sum_{i=0}^q \bar{\lambda}_i 1_{\bar{V}_i^+} 1_{\bar{V}_i^-} = x \text{ and } \sum_{i=0}^p \lambda_i = \sum_{i=0}^q \bar{\lambda}_i = \|x\|_\infty, \text{ then } q = p, (V_i^+ , V_i^-) \ (V_i^+ , V_i^-) \text{ and } \bar{\lambda}_i = \lambda_i, \ i = 0, \ldots, p. \)

Propositions 2.7 and 2.8 provides respectively the integral (continuous) form and chain (combinatorial) form of the set-pair Lovász extension, both of which are very helpful.

The set-pair version of Proposition 2.6 reads as follows:

**Proposition 2.10.** For \( f^L(x) \) by the set-pair Lovász extension, we have

(a) \( f^L(x + \alpha \text{sign}(x)) = f^L(x) + \alpha f(V_0^+ , V_0^-) \) for any \( \alpha \geq 0 \).

(b) \( f^L(x) \) is one-homogeneous.

(c) \( (f + g)^L = f^L + g^L, (\lambda f)^L = \lambda f^L, \forall \lambda \geq 0 \).

(d) \( f^L(x) \) is even if and only if \( f \) is symmetric.

**Proof.** We will give the proof in turn.

(a) Let \( \tilde{x} = x + \alpha \text{sign}(x) \). Then \( \|\tilde{x}\|_\infty = \|x\|_\infty + \alpha \) and

\[
\begin{align*}
V_t^\pm(\tilde{x}) &= V_t^\pm(x), \quad \text{if } t \geq \alpha, \\
V_t^\pm(\tilde{x}) &= V_0^\pm(x), \quad \text{if } t \in [0, \alpha).
\end{align*}
\]

According to Proposition 2.7, we have

\[
f^L(\tilde{x}) = \int_0^{\|\tilde{x}\|_\infty} f(V_t^+(\tilde{x}), V_t^-(\tilde{x}))dt
= \int_0^\alpha f(V_0^+, V_0^-)dt + \int_{\alpha}^{\|\tilde{x}\|_\infty + \alpha} f(V_t^+(\tilde{x}), V_t^-(\tilde{x}))dt
\]

\[
= \alpha f(V_0^+, V_0^-) + \int_0^{\|\tilde{x}\|_\infty} f(V_t^+(x), V_t^-(x))dt
= \alpha f(V_0^+, V_0^-) + f^L(x).
\]

(b) For any \( \lambda > 0 \), we have

\[
f^L(\lambda x) = \int_0^{\lambda \|x\|_\infty} f(V_t^+(\lambda x), V_t^-(\lambda x))dt
= \int_0^{\|x\|_\infty} \lambda f(V_{\alpha t}^+(x), V_{\alpha t}^-(x))ds = \lambda f^L(x).
\]

(c) It can be obtained directly by the linearity of integral operators.

(d) On one hand, if \( f^L(x) \) is even, then for any \( A, B \in \mathcal{P}_2(V) \), there holds

\[
f(A, B) = f^L(1_{A,B}) = f^L(-1_{A,B}) = f^L(1_{B,A}) = f(B, A),
\]

due to (2.7). Hence \( f(A, B) \) is symmetric.

On the other hand, if \( f(A, B) \) is symmetric, then using Proposition 2.7
leads to
\[
f^L(x) = \int_0^{\|x\|_\infty} f(V_i^+(x), V_i^-(x))dt
\]
\[
= \int_0^{\|x\|_\infty} f(V_i^-(x), V_i^+(x))dt
\]
\[
= \int_0^{\|x\|_\infty} f(V_i^+(-x), V_i^-(x))dt = f^L(-x),
\]
i.e., \(f^L(x)\) is even.

Now we are in the position to give the proof of Theorem 1.5.

**Proof of Theorem 1.5.** On one hand, for any \((A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}\), we have \(f(A, B) = f^L(1_{A,B})\) and \(g(A, B) = g^L(1_{A,B})\) due to (2.7), and then (2.13)
\[
\min_{(A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}} \frac{f(A, B)}{g(A, B)} = \min_{(A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}} \frac{f^L(1_{A,B})}{g^L(1_{A,B})} \geq \inf_{x \neq 0} \frac{f^L(x)}{g^L(x)}.
\]
On the other hand, for any \(x \neq 0\), we have
\[
\frac{f^L(x)}{g^L(x)} = \frac{\sum_{i=0}^{n-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) f(V_{\sigma(i+1)}, V_{\sigma(i)})}{\sum_{i=0}^{n-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) g(V_{\sigma(i+1)}, V_{\sigma(i)})}.
\]
Let \((C, D) \in \{(V_{\sigma(i+1)}, V_{\sigma(i)}) | 0 \leq i \leq n - 1\}\) such that
\[
\frac{f(C, D)}{g(C, D)} = \min_{0 \leq i \leq n-1} \frac{f(V_{\sigma(i+1)}, V_{\sigma(i)})}{g(V_{\sigma(i+1)}, V_{\sigma(i)})},
\]
and thus
\[
\Pi_i := g(V_{\sigma(i+1)}, V_{\sigma(i)}) \left(\frac{f(V_{\sigma(i+1)}, V_{\sigma(i)})}{g(V_{\sigma(i+1)}, V_{\sigma(i)})} - \frac{f(C, D)}{g(C, D)}\right) \geq 0
\]
holds for any \(0 \leq i \leq n - 1\). Accordingly, we have
\[
\frac{f^L(x)}{g^L(x)} - \frac{f(C, D)}{g(C, D)} = \frac{\sum_{i=0}^{n-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) \Pi_i}{\sum_{i=0}^{n-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) g(V_{\sigma(i+1)}, V_{\sigma(i)})} \geq 0,
\]
which directly implies
\[
(2.14) \quad \min_{(A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}} \frac{f(A, B)}{g(A, B)} = \frac{f^L(1_{C,D})}{g^L(1_{C,D})} \leq \inf_{x \neq 0} \frac{f^L(x)}{g^L(x)}.
\]
Combining (2.13) and (2.14) finally yields
\[
\min_{(A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}} \frac{f(A, B)}{g(A, B)} = \min_{x \neq 0} \frac{f^L(x)}{g^L(x)}.
\]
The proof for the maximum problem (1.6) is similar and thus skipped. \(\square\)
A similar deduction to that for Theorem 1.5 leads to

**Proposition 2.11.** Assume that \( f, g : \mathcal{P}_2(V) \to [0, +\infty) \) are two set-pair functions satisfying \( f(\emptyset, V) = f(V, \emptyset) = 0 \) and \( g(A, B) > 0 \) whenever \((A, B) \notin \{(\emptyset, \emptyset), (\emptyset, V), (V, \emptyset)\}\), then there hold both

\[
\begin{equation}
(2.15) \quad \min_{(A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset), (\emptyset, V), (V, \emptyset)\}} \frac{f(A, B)}{g(A, B)} = \min_{x \text{ nonconstant}} \frac{f^L(x)}{g^L(x)},
\end{equation}
\]

and

\[
\begin{equation}
(2.16) \quad \max_{(A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset), (\emptyset, V), (V, \emptyset)\}} \frac{f(A, B)}{g(A, B)} = \max_{x \text{ nonconstant}} \frac{f^L(x)}{g^L(x)}.
\end{equation}
\]

Next, we study the continuity of \( f^L \).

**Theorem 2.12.** \( f^L \) is a Lipschitz continuous piecewise linear function.

**Proof.** For a mapping \( m : \{1, 2, \ldots, n\} \to \{-1, 1\} \) and a permutation \( \sigma \) of \( \{1, 2, \ldots, n\} \), one defines a closed convex cone as follows

\[
\Delta_{m, \sigma} := \{ x \in \mathbb{R}^n : |x_{\sigma(1)}| \leq \cdots \leq |x_{\sigma(n)}| \text{ with } x_{\sigma(i)}m(i) \geq 0 \},
\]

and it can be readily seen that \( \mathbb{R}^n = \bigcup_{m, \sigma} \Delta_{m, \sigma} \).

It suffices to prove that \( f^L \) is linear and Lipschitz continuous with a Lipschitz constant \( 2 \max_{(A, B) \in \mathcal{P}_2(V)} f(A, B) \) on each \( \Delta_{m, \sigma} \). In fact, for given \( m \) and \( \sigma \) and any \( x \in \Delta_{m, \sigma} \), we have

\[
f^L(x) = \sum_{i=0}^{n-1} (m(i + 1)x_{\sigma(i+1)} - m(i)x_{\sigma(i)}) f(V^+_{x_{\sigma(i)}}, V^-_{x_{\sigma(i)}})
\]

\[
= \sum_{i=1}^{n-1} x_{\sigma(i)}m(i) (f(V^+_{x_{\sigma(i-1)}}, V^-_{x_{\sigma(i-1)}}) - f(V^+_{x_{\sigma(i)}}, V^-_{x_{\sigma(i)}}))
\]

\[
+ x_{\sigma(n)}m(n) f(V^+_{x_{\sigma(n-1)}}, V^-_{x_{\sigma(n-1)}}).
\]

Since \( m(i) \) and \( f(V^+_{x_{\sigma(i)}}, V^-_{x_{\sigma(i)}}) \) are constants for given \( m \) and \( \sigma \), \( f^L \) is linear on \( \Delta_{m, \sigma} \). Moreover, for any \( x, y \in \Delta_{m, \sigma} \),

\[
|f^L(x) - f^L(y)| \leq \sum_{i=1}^{n-1} |x_{\sigma(i)} - y_{\sigma(i)}||f(V^+_{x_{\sigma(i-1)}}, V^-_{x_{\sigma(i-1)}}) - f(V^+_{x_{\sigma(i)}}, V^-_{x_{\sigma(i)}})|
\]

\[
+ |x_{\sigma(n)} - y_{\sigma(n)}||f(V^+_{x_{\sigma(n-1)}}, V^-_{x_{\sigma(n-1)}})|
\]

\[
\leq 2 \max_{(A, B) \in \mathcal{P}_2(V)} f(A, B) \|x - y\|_1.
\]

\[\square\]

The concept of submodular function was introduced by Lovász to characterize the convexity of its Lovász extension \cite{Lov83}. 
Definition 2.13 (submodular function \[Lov83\]). A set-function \( f : \mathcal{P}(V) \to \mathbb{R} \) is submodular if and only if, for all subsets \( A, B \subseteq V \),

\[
f(A) + f(B) \geq f(A \cup B) + f(A \cap B).
\]

Theorem 2.14 (\[Lov83\], Proposition 4.1). \( f^L \) is convex if and only if \( f \) is submodular.

Theorem 2.14 inspires us to consider the set-pair form of submodular function. A kind of set-pair submodular function was proposed \[BF08\].

Definition 2.15 (set-pair submodular function \[BF08\]). Let

\[
\mathcal{P}_2(V) = \{(X_I, X_O) : X_I \subset X_O \subset V\}.
\]

A function \( p : \mathcal{P}_2(V) \to \mathbb{R} \) is submodular if

\[
(2.17) \quad p(X_I, X_O) + p(Y_I, Y_O) \geq p(X_I \cap Y_I, X_O \cap Y_O) + p(X_I \cup Y_I, X_O \cup Y_O)
\]

for any \((X_I, X_O), (Y_I, Y_O) \in \mathcal{P}_2(V)\).

By taking \( p(X_I, X_O) = f(X_I, X_O \setminus X_I) \) and \( f(A, B) = p(A, A \cup B) \), we can transform from \( f : \mathcal{P}_2(V) \to \mathbb{R} \) to \( p : \mathcal{P}_2'(V) \to \mathbb{R} \) and vice versa. Such \( f \) and \( p \) are said to be equivalent.

Now we show necessary and sufficient conditions for the convexity of \( f^L \).

Theorem 2.16. Let \( f : \mathcal{P}_2(V) \to [0, +\infty) \) is a set-pair functions satisfying \( f(\emptyset, \emptyset) = 0 \). Then \( f^L \) is convex if and only if \( \forall (A, B), (C, D) \in \mathcal{P}_2(V) \)

\[
(2.18) \quad f(A, B) + f(C, D) \geq f((A \cup C) \setminus (B \cup D), (B \cup D) \setminus (A \cup C)) + f(A \cap C, B \cap D);
\]

if and only if \( \forall (X_I, X_O), (Y_I, Y_O) \in \mathcal{P}_2'(V) \) the equivalent function \( p \) satisfies

\[
(2.19) \quad p(X_I, X_O) + p(Y_I, Y_O) \geq p((X_I \cap Y_I, X_O \cap Y_O) \setminus Z) + p((X_I \cup Y_I) \setminus Z, (X_O \cup Y_O) \setminus Z),
\]

where \( Z = (X_O \cap Y_I \setminus X_I) \cup (Y_O \cap X_I \setminus Y_I) \).

Three lemmas below are needed in proving Theorem 2.16.

Lemma 2.17. For \( x \in \mathbb{R}^n \), \( N \in \mathbb{N}^+ \), and \( N > 2\|x\|_{\infty} \), let

\[
(2.20) \quad \hat{f}_N(x) = \min \left\{ \sum_{(A,B) \in \mathcal{P}_2(V)} \lambda_{A,B} f(A, B) \left| \begin{array}{c}
\sum_{(A,B) \in \mathcal{P}_2(V)} \lambda_{A,B} 1_{A,B} = x , \\
\sum_{(A,B) \in \mathcal{P}_2(V)} \lambda_{A,B} \leq N , \\
\lambda_{A,B} \geq 0 .
\end{array} \right. \right\}.
\]

Then \( \hat{f}_N(x) \) is convex.

Proof. The fact that \( \hat{f}_N(x) \) is well defined emerges from Proposition 2.8. Given \( x, y \in \mathbb{R}^n \), from (2.20), we deduce

\[
\hat{f}_N(x) = \sum_{(A,B) \in \mathcal{P}_2(V)} \alpha_{A,B} f(A, B)
\]

holds for some \( \alpha_{A,B} \geq 0 \) with \( \sum \alpha_{A,B} 1_{A,B} = x \). Similarly,

\[
\hat{f}_N(y) = \sum_{(A,B) \in \mathcal{P}_2(V)} \beta_{A,B} f(A, B)
\]
holds for some $\beta_{A,B} \geq 0$ with $\sum \beta_{A,B} 1_{A,B} = y$.

Let $\lambda_{A,B} = t\alpha_{A,B} + (1 - t)\beta_{A,B}$ with $t \in [0,1]$. Immediately, we have

$$z := t x + (1 - t) y = \sum_{(A,B) \in \mathcal{P}_2(V)} \lambda_{A,B} 1_{A,B}$$

with $\sum \lambda_{A,B} \leq N$ and $\lambda_{A,B} \geq 0$, and then

$$\hat{f}_N(z) \leq \sum_{(A,B) \in \mathcal{P}_2(V)} \lambda_{A,B} f(A, B) = t \hat{f}_N(x) + (1 - t) \hat{f}_N(y).$$

\[ \square \]

**Definition 2.18.** A set-pair function $f : \mathcal{P}_2(V) \to [0, +\infty)$ is said to be strictly submodular if, the inequality (2.18) holds. Moreover, the equality holds if and only if $(A, B) \subset (C, D)$ or $(A, B) \supset (C, D)$.

**Lemma 2.19.** If $f$ is strictly submodular, then $\hat{f}_N(x) = f^L(x)$ for $N > c\|x\|_\infty$ with $c > 1$.

**Proof.** Given $x \in \mathbb{R}^n$, according to Lemma 2.17 there exist $\lambda_{A,B} \geq 0$, $\forall (A, B) \in \mathcal{P}_2(V)$ with $\sum \lambda_{A,B} 1_{A,B} = x$ and $\sum \lambda_{A,B} \leq N$ such that

$$\hat{f}_N(x) = \sum_{(A,B) \in \mathcal{P}_2(V)} \lambda_{A,B} f(A, B).$$

No loss of generality, we can assume $\lambda_{\emptyset, \emptyset} = 0$.

We claim: if $\lambda_{A,B} \geq \lambda_{C,D} > 0$, then either $(A, B) \subset (C, D)$ or $(A, B) \supset (C, D)$. Suppose the contrary and let

$$\begin{align*}
\lambda'_{A,B} &= \lambda_{A,B} - \lambda_{C,D}, \\
\lambda'_{C,D} &= 0, \\
\lambda'_{A',B'} &= \lambda_{A',B'} + \lambda_{C,D}, \\
\lambda'_{C',D'} &= \lambda_{C',D'} + \lambda_{C,D}, \\
\lambda'_{E,F} &= \lambda_{E,F}, \forall (E, F) \in \mathcal{P}_2(V) \setminus \{(A, B), (C, D), (A', B'), (C', D')\},
\end{align*}$$

where $A' = (A \cup C) \setminus (B \cup D), B' = (B \cup D) \setminus (A \cup C), C' = A \cap C, D' = B \cap D$.

Then it can be easily verified that

$$\sum_{(P,Q) \in \mathcal{P}_2(V)} \lambda'_{P,Q} = \sum_{(P,Q) \in \mathcal{P}_2(V)} \lambda_{P,Q} \quad \text{and} \quad \sum_{(P,Q) \in \mathcal{P}_2(V)} \lambda'_{P,Q} 1_{P,Q} = x.$$

Direct calculation shows

$$\sum_{(P,Q) \in \mathcal{P}_2(V)} \lambda'_{P,Q} f(P, Q) - \sum_{(P,Q) \in \mathcal{P}_2(V)} \lambda_{P,Q} f(P, Q)$$

$$= \sum_{(P,Q) \in \mathcal{P}_2(V)} (\lambda'_{P,Q} - \lambda_{P,Q}) f(P, Q)$$

$$= \lambda_{C,D} (-f(A, B) - f(C, D) + f(A', B') + f(C', D')) < 0,$$
provided the strict submodularity of $f$. This contradicts the minimality of $\hat{f}(x)$. According to the mathematical induction we obtain $(\emptyset, \emptyset) \neq (V_p^+, V_p^-) \subset \cdots \subset (V_0^+, V_0^-)$ with

$$\hat{f}_N(x) = \sum_{i=0}^{p} \lambda_{V_i^+, V_i^-} f(V_i^+, V_i^-).$$

Moreover, we have $\sum_{i=0}^{p} \lambda_{V_i^+, V_i^-} = \|x\|_\infty$ via (2.12). After Proposition 2.8, $\hat{f}_N(x) = f^L(x)$. □

**Lemma 2.20.** The function

$$g(A, B) := \sqrt{|A| + |B|}$$

is strictly submodular.

**Proof.** Given $(A, B), (C, D) \in \mathcal{P}_2(V)$, let $A' = A \setminus D$, $B' = B \setminus C$, $C' = C \setminus B$ and $D' = D \setminus A$. Then

$$g((A \cup C) \setminus (B \cup D), (B \cup D) \setminus (A \cup C)) + g(A \cap C, B \cap D) = \sqrt{|A'| + |B'| + |C'| + |D'|} \leq \sqrt{|A| + |B| + |C| + |D|} = g(A, B) + g(C, D),$$

where the first inequality holds since the function $\sqrt{t}$ is strictly convex.

Meanwhile, we can easily see that the equality holds if and only if $(A, B) \subset (C, D)$ or $(C, D) \subset (A, B)$. The proof is thus completed. □

**Proof of Theorem 2.16** Suppose that $f$ satisfies (2.18). For any $\alpha > 0$, $f + \alpha g$ is strictly submodular according to Lemma 2.20. Thus, by Lemma 2.19, we have

$$f^L + \alpha g^L = (f + \alpha g)^L = \hat{f} + \alpha g \geq \hat{f}.$$  

(2.21)

Given $x \in \mathbb{R}^n$, set $\hat{f} = \hat{f}_N$ for fixed $N > 2\|x\|_\infty$. Hence, (2.20) leads to

$$\hat{f}(x) = \sum_{(A, B) \in \mathcal{P}_2(V)} \lambda_{A, B} f(A, B)$$

for some $\lambda_{A, B} \geq 0$ with $\sum \lambda_{A, B} 1_{A, B} = x$, $\sum \lambda_{A, B} < N$. Then

$$f^L(x) + \alpha g^L(x) = \hat{f}(x) + \alpha g(x) \leq \sum_{(A, B) \in \mathcal{P}_2(V)} \lambda_{A, B} (f(A, B) + \alpha g(A, B))$$

$$= \hat{f}(x) + \alpha \sum_{(A, B) \in \mathcal{P}_2(V)} \lambda_{A, B} \sqrt{n}$$

$$\leq \hat{f}(x) + \alpha N \sqrt{n}.$$  

(2.22)
Letting $\alpha \to 0$ in (2.21) and (2.22) yields

$$f^L(x) = \hat{f}(x),$$

and by Lemma 2.17, $\hat{f}(x)$ is convex, so is $f^L(x)$.

On the other hand, if $f^L(x)$ is convex, then

$$f((A \cup C) \setminus (B \cup D), (B \cup D) \setminus (A \cup C)) + f(A \cap C, B \cap D)$$

$$= f^L(1_{(A\cup C)\setminus(B\cup D), (B\cup D)\setminus(A\cup C)} + 1_{A\cap C, B\cap D})$$

$$= f^L(1_{A,B} + 1_{C,D})$$

$$= 2f^L((1_{A,B} + 1_{C,D})/2)$$

$$\leq f^L(1_{A,B}) + f^L(1_{C,D})$$

$$= f(A, B) + f(C, D),$$

where we have used $1_{A,B} + 1_{C,D} = 1_{(A\cup C)\setminus(B\cup D), (B\cup D)\setminus(A\cup C)} + 1_{A\cap C, B\cap D}$ in the third line. Thus, (2.18) is true.

Finally, let $X_I = A$, $X_O = A \cup B$, $Y_I = C$, $Y_O = C \cup D$, then

$$Z = (X_O \cap Y_I \setminus X_I) \cup (Y_O \cap X_I \setminus Y_I) = (B \cap C) \cup (D \cap A).$$

By taking $f(A, B) = p(A, A \cup B)$, we can translate inequality (2.19) into

$$f(A, B) + f(C, D) = p(X_I, X_O) + p(Y_I, Y_O)$$

$$\geq p(X_I \cap Y_I, X_O \cap Y_O \setminus Z) + p((X_I \cup Y_I) \setminus Z, (X_O \cup Y_O) \setminus Z)$$

$$= p(A \cap C, (A \cap C) \cup (B \cap D))$$

$$+ p((A \cup C) \setminus (B \cup D), ((A \cup C) \setminus (B \cup D)) \cup ((B \cup D) \setminus (A \cup C)))$$

$$= f(A \cap C, B \cap D) + f((A \cup C) \setminus (B \cup D), (B \cup D) \setminus (A \cup C)),$$

which means (2.18) and (2.19) are equivalent.

Hence $f^L$ is convex if and only if either (2.18) or (2.19) holds.

Comparing (2.19) of Theorem 2.16 to (2.17) of Definition 2.15, we are able to deduce that the submodularity introduced in Definition 2.15 for a set-pair function fails to ensure an extension of the equivalence stated in Theorem 2.14 between convexity and submodularity for $f^L$ into $f^L$ (i.e., Definition 2.15 is neither necessary nor sufficient for $f^L$ to be convex), whereas (2.18) succeeds. In such sense, we might call the set-pair function satisfying (2.18) or (2.19) to be submodular. However, both (2.18) and (2.19) are not so easy-looking that we give a concise necessary condition for $f^L$ to be convex.

**Definition 2.21.** A set-pair-function $f : \mathcal{P}_2(V) \to \mathbb{R}$ is partially submodular if and only if it is submodular for each component, i.e.,

$$f(A, B) + f(A, D) \geq f(A, B \cup D) + f(A, B \cap D),$$

$$f(A, B) + f(C, B) \geq f(A \cup C, B) + f(A \cap C, B),$$

for all subsets $A, B, C, D \subset V$ with $A \cap B = A \cap D = C \cap B = \emptyset$.

**Corollary 2.22.** If $f^L$ is convex, then $f$ must be partially submodular.
Table 1. Set-pair Lovász extension of five object functions.

| Object function | Set-pair Lovász extension |
|------------------|---------------------------|
| $F_1(A, B) = |\partial A| + |\partial B|$ | $F_1^L(x) = I(x)$ |
| $F_2(A, B) = |E(A, B)|$ | $F_2^L(x) = \frac{1}{2}\|x\| - \frac{1}{2}I^+(x)$ |
| $G_1(A, B) = \text{vol}(V)$ | $G_1^L(x) = \text{vol}(V)\|x\|_\infty$ |
| $G_2(A, B) = \text{vol}(A) + \text{vol}(B)$ | $G_2^L(x) = \|x\|$ |
| $G_3(A, B) = \sum_{X \in \{A, B\}} \min_{Y \subseteq \{X, X^c\}} \text{vol}(Y)$ | $G_3^L(x) = \min_{\alpha \in \mathbb{R}} \|x\| - \alpha 1$ |

**Proof.** If $f^L$ is convex, then $f$ must satisfy (2.18). Setting $C = A$ and $D = B$ respectively in (2.18), we can find that $f$ is partially submodular. \qed

Similar to Corollary 2.22 if $p$ is submodular, then its equivalent function $f$ must be partially submodular.

Finally, we compare the set-pair Lovász extension to the original one:

1. In contrast to the succinct integral form (2.8) of $f^L$, the integral form (2.5) of $f^L$ has an extra remainder term.
2. The original Lovász extension is unable directly to deal with graph 3-cut problems such as the dual Cheeger-cut problem, whereas the set-pair Lovász extension works.
3. The characterization of the convexity of $f^L$ is easier than $f^L$.

3. **Applications to graph cut**

A straightforward application of the original Lovász extension (2.2) into a graph 3-cut problem such as the dual Cheeger problem is not feasible. Instead, we will show in this section that the set-pair Lovász extension (1.4) can succeed to find an explicit and equivalent continuous optimization problem for graph 3-cut. To be more specific, the set-pair Lovász extension of the five set-pair functions is summarized in Table 1.

The first four functions in Table 1 can be calculated directly according to Proposition 2.7. In fact,

$$F_1^L(x) = \int_0^{\|x\|_\infty} |\partial V_t^+(x)| + |\partial V_t^-(x)|dt.$$  

Then substituting

$$|\partial V_t^+(x)| = \sum_{i,j} w_{ij}(\chi_{x_i \leq t < x_j} + \chi_{x_j \leq t < x_i}),$$

$$|\partial V_t^-(x)| = \sum_{i,j} w_{ij}(\chi_{x_i < -t \leq x_j} + \chi_{x_j < -t \leq x_i})$$
into Eq. (3.1) yields

\[
F_L^1(x) = \sum_{i<j} w_{ij} \int_0^{\|x\|_{\infty}} \chi_{x_i \leq t < x_j} + \chi_{x_j \leq t < x_i} + \chi_{x_i < -t \leq x_j} + \chi_{x_j < -t \leq x_i} dt
\]

\[
= \sum_{i<j} w_{ij} \int_{-\|x\|_{\infty}}^{\|x\|_{\infty}} \chi_{x_i \leq t < x_j} dt
\]

\[
= \sum_{i<j} w_{ij} |x_i - x_j| = I(x),
\]

where the integral equalities hold when '<' is replaced by '<'.

Hereafter the endpoints of intervals in the integral form (2.8) are dropped for convenience. Thus, it gives the form of set-pair Lovász extension of \(F_1\) in Table 1.

Applying Proposition 2.7 to \(F_2\), we get

\[
F_L^2(x) = \int_0^{\|x\|_{\infty}} F_2(V_t^+(x), V_t^-(x)) dt
\]

\[
= \int_0^{\|x\|_{\infty}} |E(V_t^+(x), V_t^-(x))| dt
\]

\[
= \int_0^{\|x\|_{\infty}} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \chi_{x_i > t \chi_{x_j < -t}} dt
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \int_0^{\|x\|_{\infty}} \chi_{x_i > t \chi_{x_j < -t}} dt
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \min\{x_i + |x_i|, |x_j| - x_j\}
\]

\[
= \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} (x_i + |x_i| + |x_j| - x_j - |x_i + x_j| + |x_i - |x_j||)
\]

where we used the fact that \(\min\{a, b\} = \frac{1}{2}(a+b-|a-b|)\) in the last equality. Further, we can obtain

\[
F_L^2(x) = \frac{1}{4} \sum_{i<j} w_{ij} (2(|x_i| + |x_j|) - |x_i + x_j + |x_i| - |x_j| - |x_i + x_j + |x_i| + |x_j||)
\]

\[
(3.2)
\]

\[
= \frac{1}{4} \sum_{i<j} w_{ij} (2|x_i| + 2|x_j| - 2 \max\{|x_i + x_j|, |x_i| - |x_j|\})
\]

\[
= \frac{1}{2} \|x\| - \frac{1}{2} \sum_{i<j} |x_i + x_j| = \frac{1}{2} \|x\| - \frac{1}{2} I^+(x),
\]

where Eq. (3.2) utilizes the fact that \(2 \max\{|a|, |b|\} = |a + b| + |a - b|\).
Correspondingly, the set-pair extensions of $G_1$ and $G_2$ in Table 1 can be verified in the following way:

$$G_1^L(x) = \int_0^{\|x\|_\infty} G_1(V_t^+(x), V_t^-(x))dt = \text{vol}(V)\|x\|_\infty,$$

$$G_2^L(x) = \int_0^{\|x\|_\infty} G_2(V_t^+(x), V_t^-(x))dt = \int_0^{\|x\|_\infty} \text{vol}(V_t^+(x)) + \text{vol}(V_t^-(x))dt = \int_0^{\|x\|_\infty} \sum_{i=1}^n d_i x_i |_t dt = \|x\|.$$

Now, let us focus on $G_3$. Direct calculation shows

$$G_3^L(x) = \int_0^{\|x\|_\infty} G_3(V_t^+(x), V_t^-(x))dt \leq \int_0^{\|x\|_\infty} \text{min}\{\text{vol}(V_t^+(x)), \text{vol}(V_t^+(x)^c)\} + \text{min}\{\text{vol}(V_t^-(x)), \text{vol}(V_t^-(x)^c)\}dt \leq \int_0^{\|x\|_\infty} \text{min}\{\text{vol}(V_t^+(x)), \text{vol}(V_t^+(x)^c)\}dt.$$

Let $\sigma$ be a permutation of $\{1, 2, \ldots, n\}$ such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}$. Then there exists $k_0 \in \{1, 2, \ldots, n\}$ satisfying

$$\sum_{i=1}^{k_0-1} d_{\sigma(i)} < \frac{1}{2}\text{vol}(V) \leq \sum_{i=1}^{k_0} d_{\sigma(i)}.$$  

Consequently, it reveals that

$$\text{min}\{\text{vol}(V_t^+(x)), \text{vol}(V_t^+(x)^c)\} = \begin{cases} \text{vol}(V_t^+(x)), & \text{if } t < x_{\sigma(k_0)}, \\ \text{vol}(V_t^+(x)^c), & \text{if } t \geq x_{\sigma(k_0)}, \end{cases}$$

and

$$G_3^L(x) = \int_{x_{\sigma(1)}}^{x_{\sigma(k_0)}} \text{vol}(V_t^+(x)^c)dt + \int_{x_{\sigma(k_0)}}^{x_{\sigma(n)}} \text{vol}(V_t^+(x))dt = \sum_{i=1}^{k_0-1} (x_{\sigma(i+1)} - x_{\sigma(i)}) \sum_{j=1}^{i} d_{\sigma(j)} + \sum_{i=k_0}^{n-1} (x_{\sigma(i+1)} - x_{\sigma(i)}) \sum_{j=i+1}^{n} d_{\sigma(j)} = \sum_{j=1}^{k_0-1} d_{\sigma(j)} \sum_{i=j}^{k_0-1} (x_{\sigma(i+1)} - x_{\sigma(i)}) + \sum_{j=k_0+1}^{n-1} d_{\sigma(j)} \sum_{i=k_0}^{j-1} (x_{\sigma(i+1)} - x_{\sigma(i)})$$
= \sum_{i=1}^{n} d_{\sigma(i)} |x_{\sigma(i)} - x_{\sigma(k_0)}| = \|x - x_{\sigma(k_0)}1\|.

On the other hand, \(\|x - \alpha 1\|\) is convex in \(\alpha\) and satisfies

\[
p_{\alpha} := -\sum_{i=1}^{n} d_{\sigma(i)} \text{sign}(x_{\sigma(i)} - \alpha) \in \partial_{\alpha} \|x - \alpha 1\|
\]

and then

\[
\begin{cases}
p_{\alpha} \leq \sum_{j=1}^{k_0} d_{\sigma(j)} - \sum_{j=k_0}^{n} d_{\sigma(j)} \leq 0, & \text{if } \alpha < x_{\sigma(k_0)}; \\
p_{\alpha} \geq \sum_{j=1}^{k_0} d_{\sigma(j)} - \sum_{j=k_0+1}^{n} d_{\sigma(j)} \geq 0, & \text{if } \alpha > x_{\sigma(k_0)}.\end{cases}
\]

This implies that \(\|x - \alpha 1\|\) is decreasing with respect to \(\alpha\) in \((-\infty, x_{\sigma(k_0)})\) and increasing in \((x_{\sigma(k_0)}, +\infty)\). Thus, we obtain that

\[
x_{\sigma(k_0)} \in \arg \min_{\alpha \in \mathbb{R}} \|x - \alpha 1\|.
\]

Therefore,

\[
(3.5) \quad G_{3}^{L}(x) = \|x - x_{\sigma(k_0)}1\| = \min_{\alpha \in \mathbb{R}} \|x - \alpha 1\|.
\]

3.1. **Graph 3-cut problems.** It is straightforward to utilize the proposed set-pair Lovász extension to deal with the combination optimizations in a set-pair form, for example, the dual Cheeger and max 3-cut problems. Now, we are in a position to answer Question 1.3.

**Proof of Theorem 1.6.** Applying Theorem 1.5 in the dual Cheeger cut problem (1.1) yields

\[
(3.6) \quad h^{+}(G) = \max_{x \neq 0} \frac{2F_{2}^{L}(x)}{G_{3}^{L}(x)} = 1 - \min_{x \neq 0} \frac{I^{+}(x)}{\|x\|},
\]

where we have used \(F_{2}^{L}\) and \(G_{3}^{L}\) in Table 1. □

**Proof of Theorem 1.8.** Since \(A \cup B = B \cup C = C \cup A = \emptyset\) and \(A \cup B \cup C = V\), we can suppose that \(A \cup B \neq \emptyset\). Then, by Theorem 1.5 and Proposition 2.10, we have

\[
\frac{1}{2} h_{\max,3}(G) = \max_{A,B,C} \frac{|E(A,B)| + |E(B,C)| + |E(C,A)|}{\text{vol}(V)} \\
= \max_{A,B,C} \frac{|E(A,B)| + |E(A \cup B, C)|}{\text{vol}(V)} \\
= \max_{(A,B) \in \mathcal{P}_{2}(V) \setminus \{(\emptyset,\emptyset)\}} \frac{|E(A,B)| + |\partial(A \cup B)|}{\text{vol}(V)} \\
= \max_{(A,B) \in \mathcal{P}_{2}(V) \setminus \{(\emptyset,\emptyset)\}} \frac{|\partial A| + |\partial B| - |E(A,B)|}{\text{vol}(V)} \\
= \max_{(A,B) \in \mathcal{P}_{2}(V) \setminus \{(\emptyset,\emptyset)\}} \frac{F_{1}(A,B) - F_{2}(A,B)}{G_{1}(A,B)}
\]
\[ h_{\text{max},3}(G) = \max_{x \neq 0} \frac{F_L(x) - F_L(x)}{G_L(x)}, \]

where \( F_L \) and \( G_L \) are given in Table I. Thus

\[(3.7)\]

\[ h_{\text{max},3}(G) = \max_{x \neq 0} \frac{2I(x) - ||x|| + I^+(x)}{\text{vol}(V)||x||_\infty}. \]

It follows from \(|a - b| + |a + b| = 2 \max\{ |a|, |b| \} = |a - b| + |a| + |b| \) for any \( a, b \in \mathbb{R} \) that

\[ I(x) = \sum_{i<j} w_{ij} |x_i - x_j| \]

\[ = \sum_{i<j} w_{ij} (|x_i| + |x_j| + ||x_i| - |x_j|| - |x_i + x_j|) \]

\[ = \sum_{i=1}^n d_i |x_i| + \sum_{i<j} w_{ij} ||x_i| - |x_j|| - \sum_{i<j} w_{ij} |x_i + x_j| \]

\[ = ||x|| + \hat{I}(x) - I^+(x), \]

and thus

\[ 2I(x) - ||x|| + I^+(x) = 2\hat{I}(x) + ||x|| - I^+(x) = I(x) + \hat{I}(x). \]

Finally, Eq. (3.7) turns out to be

\[ h_{\text{max},3}(G) = \max_{x \neq 0} \frac{I(x) + \hat{I}(x)}{\text{vol}(V)||x||_\infty}. \]

\[ \square \]

**Proof of Theorem 1.10.** According to Theorem 1.5, we have

\[ h_{\text{max},3,I}(G) = \max_{(A,B) \in \mathcal{P}_2(V)} \frac{F_1(A,B) - F_2(A,B)}{G_2(A,B)} \]

\[ = \max_{x \neq 0} \frac{||x|| - I^+(x) + 2\hat{I}(x)}{||x||}. \]

\[ \square \]

**Proof of Theorem 1.12.** Let \( (3.8) \)

\[ G^L(A, B) = \min\{ \text{vol}(A \cup B), \text{vol}((A \cup B)^c) \}, \]

and \( \sigma \) be a permutation of \( \{1, 2, \ldots, n\} \) such that \( |x_{\sigma(1)}| \leq |x_{\sigma(2)}| \leq \cdots \leq |x_{\sigma(n)}| \). By Definition 1.4, the set-pair Lovász extension of \( G(A, B) \) is

\[ G^L(x) = \sum_{i=0}^{n-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) G(V_{\sigma(i)}^+, V_{\sigma(i)}^-) \]

\[ = \sum_{i=0}^{n-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) \min\{ \text{vol}(V_{\sigma(i)}^+ \cup V_{\sigma(i)}^-), \text{vol}((V_{\sigma(i)}^+ \cup V_{\sigma(i)}^-)^c) \} \]
\[= \sum_{i=0}^{n-1} (|x_{\sigma(i)}| - |x_{\sigma(i+1)}|) G_3(V_{\sigma(i)}^+ \cup V_{\sigma(i)}^-, \emptyset)\]
\[= G_3^L(|x|) = \min_{\alpha \in \mathbb{R}} \| |x| - \alpha \|, \]

where \(|x| = (|x_1|, \ldots, |x_n|)\) and Eq. (3.5) is applied in the last line. Finally, applying Theorem 1.5 into Eq. (1.16) leads to

\[h_{\text{max}, 3, \mathcal{I}}(G) = \max_{x \neq 0} \frac{2F_1^L(x) - 2F_2^L(x)}{G_1^L(x) - G_2^L(x)} = \max_{x \neq 0} \frac{2I(x) - \|x\| + I^+(x)}{\text{vol}(V)||x||_\infty - \sum_{\alpha \in \mathbb{R}} \sum_{i=1}^{n} d_i |x_i| - \alpha |}. \]

\[\square\]

3.2. **Graph \(k\)-cut \((k > 3)\) problems.** In this section, we present a preliminary attempt to a graph \(k\)-cut problem. The main idea is to transfer a graph \(k\)-cut problem to a 3-cut one on a larger graph. To this end, let us start from

\[(3.9) \quad \mathcal{P}_k([n]) = \{(A_1, \ldots, A_k) | A_i \cap A_j = \emptyset, A_i \subset [n]\}, \]
\[(3.10) \quad \mathcal{H}_{k+1}([n]) = \{(A_1, \ldots, A_{k+1}) | A_i \cap A_j = \emptyset, \bigcup_{i=1}^{k+1} A_i = [n]\}, \]

where \([n] = \{1, 2, \ldots, n\}\). Obviously \(\mathcal{P}_k([n])\) is equivalent to \(\mathcal{H}_{k+1}([n])\), \(\mathcal{P}_k([n]) \simeq \mathcal{H}_{k+1}([n])\). Then a bijection between \(\mathcal{P}_2([ln])\) and \(\mathcal{H}_{3l}([n])\) can be obtained via

\[(3.11) \quad \mathcal{P}_2([ln]) \simeq \mathcal{H}_{3l}([ln]) \simeq \prod_{i=1}^{l} \mathcal{H}_3([n]) \simeq \mathcal{H}_{3l}([n]). \]

For a family \(\mathcal{P}\) consisting of set-tuples, we use \(C(\mathcal{P}) := \{f : \mathcal{P} \to \mathbb{R}\}\) to denote the collection of real valued functions on \(\mathcal{P}\), and then have the following commutative diagrams for any \(k < 3^l\):

**Figure 1.** The commutative diagram on the right is indeed the dual diagram of the left one in some sense.
In Fig. 1, \( h_1 \) is the natural injective mapping from \( H_3([n]) \) to \( \mathcal{P}_k([n]) \) by choosing only the last \( k \) parts from each element in \( H_3([n]) \). Therefore, given \( f \in C(\mathcal{P}_k([n])) \), there exists \( F \in C(\mathcal{P}_2([ln])) \) and an injective mapping \( h \) from \( \mathcal{P}_2([ln]) \) to \( \mathcal{P}_k([n]) \) such that \( F = f \circ h \). For convenience, the set-pair Lovász extension of \( F \) is again called the Lovász extension of \( f \). That is, there exist \( F_1 \in C(H_3([n])) \) and \( F_2 \in C(\prod_{i=1}^l H_3([n])) \) such that

\[
(3.12) \quad F_1 = f \circ h_1, \quad F_2(\prod_{i=1}^l (T_0^i, T_1^i, T_2^i)) = F_1((A_0, A_1, \ldots, A_{3^l-1})),
\]

where \((T_0^i, T_1^i, T_2^i) \in H_3([n]), i = 1, \ldots, l, \) and \( A_j = \bigcap_{i=1}^l T_{a_i}^i \) with \((a_1 \ldots a_l)_3\) being the ternary representation of \( j \) for \( j = 0, 1, \ldots, 3^l - 1 \). In other words, there exists an injection \( h_2 \) from \( \prod_{i=1}^l H_3([n]) \) to \( H_3([n]) \) such that

\[
(3.13) \quad F_2 = F_1 \circ h_2.
\]

Thus, the correspondence is well established between the function \( f \) defined on \( P_k([n]) \) and the function \( F \) on \( P_2([ln]) \) by letting \( h = h_1 \circ h_2 \circ h_3 \) and

\[
F = F_2 \circ h_3 = F_1 \circ h_2 \circ h_3 = f \circ h.
\]

Applying Theorem 1.5, we are able to give an equivalent continuous optimization for the max \( k \)-cut problem.

**Definition 3.1 (max \( k \)-cut [1397]).** Given a connected graph \( G = (V, E) \) with \( V = [n] \), the max \( k \)-cut problem is to determine a graph \( k \)-cut by solving

\[
(3.15) \quad h_k(G) = \min_{(A_1, A_2, \ldots, A_k) \in H_k([n])} \frac{\sum_{i=1}^k |\partial A_i|}{\sum_{i=1}^k \text{vol}(A_i)}.
\]

We can find \( F, G \in C(\mathcal{P}_2([ln])) \) such that

\[
(3.16) \quad F(T_1, T_2) = \sum_{j=3^l-k}^{3^l-1} |\partial A_j|, \quad \text{and} \quad G(T_1, T_2) = \sum_{j=3^l-k}^{3^l-1} \text{vol}(A_j),
\]

where \( A_j = \bigcap_{i=1}^l T_{a_i}^i, T_{a_i}^i \) is given in Eq. (3.14), and \((a_1 \ldots a_l)_3\) denotes the ternary representation of \( j \) for \( j \in \{0, 1, \ldots, 3^l - 1\} \).

Let us write down the functions \( F^L \) and \( G^L \) explicitly. In fact,

\[
(3.17) \quad F^L(\prod_{i=1}^l x^{(i)}) = \int_0^{\|x\|_\infty} F(V_i^+, V_i^-) dt,
\]
is a continuous function defined on $\mathbb{R}^{nl}$. Here $V^±_t = \{(i - 1)n + j | \pm x^{(i)}_j > t\}$ for any $x = \prod_{i=1}^l x^{(i)} \in \mathbb{R}^{nl}$, and $x^{(i)} = (x^{(i)}_1, \ldots, x^{(i)}_n)$.

Proposition 3.2.

$$F^L(\prod_{i=1}^l x^{(i)}) = \sum_{j=1}^n d_j z_j - 2 \sum_{i<j} w_{ij} \sum_{(a_l \ldots a_1)_3 = 3^l-k} z_{ij}^{(a_l \ldots a_1)_3},$$

where

$$z_j = \min \left\{ t \geq 0 | (a_l \ldots a_1)_3 < 3^l - k, \ a_i = 1_{x^{(i)}_j > t} + 21_{x^{(i)}_j > t} \right\},$$

$$z_{ij}^{(a_l \ldots a_1)_3} = \min \left\{ (-1)^{a_\alpha} x^{(a)}_{i'} - |x^{(b)}_{j'}| |a_\alpha > 0, a_\beta = 0, i', j' \in \{i, j\} \right\},$$

$$z_+ = \max \{z, 0\}.$$

Proof. A direct calculation leads to

$$F^L(\prod_{i=1}^l x^{(i)}) = \int_0^\|x\| F(V_t) dt = \int_0^\|x\| \sum_{(a_l \ldots a_1)_3 = 3^l-k} |\partial A_{a_l \ldots a_1}(t)| dt$$

$$= \int_0^\|x\| \sum_{(a_l \ldots a_1)_3 = 3^l-k} |\text{vol}(A_{a_l \ldots a_1}(t))| - 2 |E(A_{a_l \ldots a_1}(t))| dt$$

$$= I - II,$$

where $E(A)$ is the set of all edges with endpoints in $A$ and

$$I = \int_0^\|x\| \sum_{(a_l \ldots a_1)_3 = 3^l-k} |\text{vol}(A_{a_l \ldots a_1}(t))| dt,$$

$$II = 2 \int_0^\|x\| \sum_{(a_l \ldots a_1)_3 = 3^l-k} |E(A_{a_l \ldots a_1}(t))| dt.$$

It is easy to check the first part

$$I = \int_0^{\|x\|} \sum_{(a_l \ldots a_1)_3 = 3^l-k} \sum_{i<j} w_{ij} [1_{i \in A_{a_l \ldots a_1}(t)} + 1_{j \in A_{a_l \ldots a_1}(t)}] dt$$

$$= \sum_{i<j} w_{ij} \int_0^{\|x\|} \sum_{(a_l \ldots a_1)_3 = 3^l-k} 1_{i \in A_{a_l \ldots a_1}(t)} + 1_{j \in A_{a_l \ldots a_1}(t)} dt$$

$$= \sum_{i<j} w_{ij} \int_0^{\|x\|} 1_{i \in \cup_{(a_l \ldots a_1)_3 = 3^l-k} A_{a_l \ldots a_1}(t)} + 1_{j \in \cup_{(a_l \ldots a_1)_3 = 3^l-k} A_{a_l \ldots a_1}(t)} dt$$

$$= \sum_{i<j} w_{ij} (z_i + z_j) = \sum_{j=1}^n d_j z_j,$$
and the second part

\[ II = 2 \int_0 \| x \| \sum_{(a_l \cdots a_1) = 3^l - 1} \sum_{i < j} w_{ij} 1_{i,j \in A_{a_l \cdots a_1}(t)} dt \]

\[ = 2 \sum_{i < j} w_{ij} \int_0 \| x \| \sum_{(a_l \cdots a_1) = 3^l - 1} 1_{i,j \in A_{a_l \cdots a_1}(t)} dt \]

\[ = 2 \sum_{i < j} w_{ij} \int_0 \| x \| \prod_{\alpha : a_\alpha > 0, i' \in \{i,j\}} 1_{\prod_{\beta : a_\beta = 0, j' \in \{i,j\}} 1_{|x_{i'}| < t} dt} \]

\[ = 2 \sum_{i < j} w_{ij} \int_0 \| x \| \prod_{\alpha : a_\alpha > 0, i' \in \{i,j\}} z_{ij}^{(a_l \cdots a_1)}. \]

Thus, we complete the proof. \( \square \)

**Remark 3.3.** If \( k = 3^l - 1 \), then

\[ F_L(l \prod_{i=1}^l x^{(i)}) = \sum_{j=1}^n d_j \max_s |x_j^{(s)}| - 2 \sum_{i < j} w_{ij} \sum_{(a_l \cdots a_1) = 1} z_{ij}^{(a_l \cdots a_1)}. \]

It can be readily verified that:

**Proposition 3.4.**

\[ G_L(l \prod_{i=1}^l x^{(i)}) = \sum_{j=1}^n d_j z_j = I. \]

Accordingly, we get

**Proposition 3.5.**

\[ (3.18) \]

\[ h_k = \max_{x \in \mathbb{R}^n \setminus K} \frac{F_L(x)}{G_L(x)}, \]

where \( F_L \) and \( G_L \) are defined in Propositions 3.2 and 3.4, respectively, and \( K = \{ x | z_j = 0, \forall j = 1, 2, \ldots, n \} \).

### 3.3. Graph 2-cut problems

With the help of the following lemma, the proposed set-pair Lovász extension also works for some graph 2-cut problems.

**Lemma 3.6.** Suppose \( f, g : \mathcal{P}(V) \to [0, +\infty) \) are two symmetric functions with \( g(A) > 0 \) for any \( A \in \mathcal{P}(V) \). Let \( F(A, B) = f(A) + f(B) \) and \( G(A, B) = g(A) + g(B) \). Then

\[ \min_{A \in \mathcal{P}(V)} \frac{f(A)}{g(A)} = \min_{(A,B) \in \mathcal{P}_2(V)} \frac{F(A, B)}{G(A, B)}. \]
Accordingly, we have
\[
\max_{A \in \mathcal{P}(V)} \frac{f(A)}{g(A)} = \max_{(A,B) \in \mathcal{P}_2(V)} \frac{F(A,B)}{G(A,B)}.
\]

**Proof.** First we prove (3.19). On one hand, let \((A_0, B_0) \in \mathcal{P}_2(V)\) be the minimizer of \(\frac{f(A_0) + f(B_0)}{g(A_0) + g(B_0)}\). Without loss of generality, we may assume \(\frac{f(A_0)}{g(A_0)} \leq \frac{f(B_0)}{g(B_0)}\). Then
\[
\frac{f(A_0) + g(B_0)}{g(A_0) + g(B_0)} - \frac{f(A_0)}{g(A_0)} = \frac{f(B_0)g(A_0) - f(A_0)g(B_0)}{g(A_0)(g(A_0) + g(B_0))} \geq 0,
\]
which follows that
\[
\min_{(A,B) \in \mathcal{P}_2(V)} \frac{f(A) + f(B)}{g(A) + g(B)} = \frac{f(A_0) + g(B_0)}{g(A_0) + g(B_0)} \geq \frac{f(A_0)}{g(A_0)} \geq \min_{A \in \mathcal{P}(V)} \frac{f(A)}{g(A)}.
\]
On the other hand, let \(A_1 \subset V\) be the minimizer of \(\frac{f(A_1) + f(A_1^c)}{g(A_1) + g(A_1^c)}\). Then we have
\[
\min_{(A,B) \in \mathcal{P}_2(V)} \frac{f(A) + f(B)}{g(A) + g(B)} \leq \frac{f(A_1) + f(A_1^c)}{g(A_1) + g(A_1^c)} = \frac{f(A_1)}{g(A_1)} = \min_{A \in \mathcal{P}(V)} \frac{f(A)}{g(A)},
\]
and hence,
\[
\min_{A \in \mathcal{P}(V)} \frac{f(A)}{g(A)} = \min_{(A,B) \in \mathcal{P}_2(V)} \frac{f(A) + f(B)}{g(A) + g(B)}.
\]
It is also true if we replace ‘min’ by ‘max’, i.e., (3.20) holds. \(\square\)

**Proof of Theorem 1.16** The proof involves \(F_1\) and \(G_1\). Let
\[
f(A) = |\partial A| \quad \text{and} \quad g(A) = \frac{1}{2} \text{vol}(V).
\]
Since \(f\) and \(g\) are symmetric functions, by Lemma 3.6 and Theorem 1.5, we have
\[
h_{\max}(G) = \max_{S \in \mathcal{P}(V)} \frac{2|\partial S|}{\text{vol}(V)} = \max_{S \in \mathcal{P}(V)} \frac{f(S)}{g(S)} = \max_{(A,B) \in \mathcal{P}_2(V)} \frac{F_1(A,B)}{G_1(A,B)}
\]
\[
= \max_{(A,B) \in \mathcal{P}_2(V) \setminus \{(\emptyset,\emptyset)\}} \frac{F_1(A,B)}{G_1(A,B)} = \max_{x \neq 0} \frac{F_1^L(x)}{G_1^L(x)}.
\]
Accordingly, we have
\[
h_{\max}(G) = \max_{x \neq 0} \frac{F_1^L(x)}{G_1^L(x)} = \max_{x \neq 0} \frac{I(x)}{\text{vol}(V)\|x\|_\infty}.
\]
\(\square\)

**Proof of Theorem 1.17** Let \(f(A) = |\partial A|\) and \(g(A) = \min\{\text{vol}(A), \text{vol}(A^c)\}\). Since \(f\), \(g\) are symmetric functions, by Eq. (2.15), Lemma 3.6 and Proposition 2.11, we have
\[
h(G) = \min_{S \subset V} \frac{f(S)}{g(S)} = \min_{(A,B) \in \mathcal{P}_2(V) \setminus \{(\emptyset,\emptyset), (\emptyset,\emptyset), (V,\emptyset)\}} \frac{F_1(A,B)}{G_2(A,B)}
\]
\[
= \min_{x \text{ nonconstant}} \frac{F_1^L(x)}{G_2^L(x)} = \inf_{x \text{ nonconstant}} \sup_{c \in \mathbb{R}} \sum_{i=1}^n d_i |x_i - c|.
\]
where $F_1^L$ and $G_2^L$ have been presented in Table 1.

**Proof of Theorem 7.15**

Let

\[(3.21) \quad f(A) = |\partial A| \quad \text{and} \quad g(A) = \max\{\text{vol}(A), \text{vol}(A^c)\}.\]

Since $f$, $g$ are symmetric functions, by Lemma 3.6 and Theorem 1.5, we obtain

\[
h\text{anti}(G) = \max_{S \in \mathcal{P}(V)} \frac{f(S)}{g(S)} = \max_{(A,B) \in \mathcal{P}_2(V)} \frac{F_1(A,B)}{G(A,B)}
\]

\[
= \max_{(A,B) \in \mathcal{P}_2(V) \setminus \{(\emptyset,\emptyset)\}} \frac{F_1(A,B)}{G(A,B)} = \max_{x \neq 0} \frac{F_1^L(x)}{G^L(x)},
\]

where $G(A,B) = 2G_1(A,B) - G_3(A,B)$, and $G^L = 2G_1^L - G_3^L$. Thus,

\[
h\text{anti}(G) = \max_{x \neq 0} \frac{F_1^L(x)}{2G_1^L(x) - G_3^L(x)} = \max_{x \neq 0} \frac{I(x)}{2\text{vol}(V)\|x\|_\infty - \min_{\alpha \in \mathbb{R}} \|x - \alpha 1\|}.
\]

\[\square\]

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