ABSTRACT. We study the convergence to stationarity for random walks on dynamic random digraphs with given degree sequences. The digraphs undergo full regeneration at independent geometrically distributed random time intervals with parameter $\alpha$. Relaxation to stationarity is the result of a competition between regeneration and mixing on the static digraph. When the number of vertices $n$ tends to infinity and the parameter $\alpha$ tends to zero, we find three scenarios according to whether $\alpha \log n$ converges to zero, infinity or to some finite positive value: when the limit is zero, relaxation to stationarity occurs in two separate stages, the first due to mixing on the static digraph, and the second due to regeneration; when the limit is infinite, there is not enough time for the static digraph to mix and the relaxation to stationarity is dictated by the regeneration only; finally, when the limit is a finite positive value we find a mixed behaviour interpolating between the two extremes. A crucial ingredient of our analysis is the control of suitable approximations for the unknown stationary distribution.

1. INTRODUCTION

The analysis of stochastic processes on dynamic networks constitutes a fundamental theme of current and future research [24]. In this paper we are interested in the mixing time of random walks on dynamic graphs. In contrast with the case of static graphs, the theory for graphs evolving with time is far from being fully developed. Related problems have been considered in the literature on the so-called random walks in dynamic random environments; see e.g. [8, 18, 5, 2, 19] and references therein. More closely linked to our questions here is the analysis of mixing times of random walks on dynamical percolation [25, 27], and of random walks on evolving configuration models [3, 4]; see also [26] for recent results on some related general problems. These works are all concerned with the case of undirected graphs, and as far as we know there has been essentially no analysis of mixing times for dynamically evolving directed graphs up to now. Even before discussing the mixing properties, a key problem in the directed setting is the identification of the stationary distribution. In this paper we resolve these difficulties and obtain a precise description of the mixing times for a class of digraphs undergoing a particularly simple evolution, namely for digraphs with given degree sequences that are fully regenerated at independent geometrically distributed random time intervals.

We shall consider two families of directed graphs. Both are obtained via the so-called configuration model, with the difference that in the first case we fix both in and out degrees, while in the second case we only fix the out degrees. The models are sparse in that the degrees are bounded.

1.1. Directed configuration model. Let $V$ be a set of $n$ vertices. For simplicity we often write $V = [n]$, with $[n] = \{1, \ldots, n\}$. For each $n$, we have two finite sequences $d^+ = ...
(d^+_x)_{x \in [n]}$ and $d^- = (d^-_x)_{x \in [n]}$ of non negative integers such that

$$m = \sum_{x \in V} d^+_x = \sum_{x \in V} d^-_x. \quad (1.1)$$

In the directed configuration model $DCM(d^{\pm})$, a random graph $G$ is obtained as follows: 1) equip each node $x$ with a set $E^+_x$ of $d^+_x$ tails and a set $E^-_x$ of $d^-_x$ heads; 2) pick uniformly at random one of the $m!$ bijections from the set of all tails $\cup_x E^+_x$ into the set of all heads $\cup_x E^-_x$, call it $\sigma$; 3) for all $x, y \in V$, add a directed edge $(x, y)$ every time a tail from $x$ is mapped into a head from $y$ through $\sigma$. We call $C$ the set of all bijections $\sigma$, so that $|C| = m!$. The resulting graph $G = G(\sigma)$ may have self-loops and multiple edges, however it is classical that by conditioning on the event that there are no multiple edges and no self-loops one obtains a uniformly random simple digraph with in degree sequence $d^-$ and out degree sequence $d^+$.

Structural properties of random graphs obtained in this way have been studied in [14]. Here we shall consider the sparse case corresponding to bounded degree sequences, and in order to avoid non irreducibility issues, we shall assume that all degrees are at least 2. Thus, throughout this work it will always be assumed that

$$\min_{x \in [n]} d^+_x \geq 2, \quad \max_{x \in [n]} d^+_x = O(1). \quad (1.2)$$

We often use the notation $\Delta = \max_{x \in [n]} \max\{d^-_x, d^+_x\}$. Under the first assumption it is known that $DCM(d^{\pm})$ is strongly connected with high probability, while under the second assumption, it is known that $DCM(d^{\pm})$ has a uniformly (in $n$) positive probability of having no self-loops nor multiple edges. In particular, any property that holds with high probability for $DCM(d^{\pm})$ will also hold with high probability for a uniformly chosen simple graph subject to the constraint that in and out degrees be given by $d^-$ and $d^+$ respectively. Here and throughout the rest of the paper we say that a property holds with high probability (w.h.p. for short) if the probability of the corresponding event converges to 1 as $n \to \infty$.

1.2. Out configuration model. To define the second model, for each $n$ let $d^+ = (d^+_x)_{x \in [n]}$ be a finite sequence of non negative integers. In the out-configuration model $OCM(d^+)$ a random graph $G$ is obtained as follows: 1) equip each node $x$ with $d^+_x$ tails; 2) pick, for every $x$ independently, a uniformly random injective map $\sigma_x$ from the set of tails at $x$ to the set of all vertices $V$; 3) for all $x, y \in V$, add a directed edge $(x, y)$ if a tail from $x$ is mapped into $y$ through $\sigma_x$. Equivalently, $G$ is the graph whose adjacency matrix is uniformly random in the set of all $n \times n$ matrices with entries 0 or 1 such that every row $x$ sums to $d^+_x$. Notice that $G$ may have self-loops. We write $\sigma = (\sigma_x)_{x \in [n]}$, and let $C$ denote the set of all distinct such maps, so that $|C| = \prod_{x=1}^n \frac{n!}{(n-d^+_x)!}$. As above we make the assumptions

$$\min_{x \in [n]} d^+_x \geq 2, \quad \max_{x \in [n]} d^+_x = O(1), \quad (1.3)$$

and use the notation $\Delta = \max_{x \in [n]} d^+_x$.

1.3. Mixing of static digraphs. In what follows $\sigma \in C$ denotes a given realisation of either the directed configuration model $DCM(d^\pm)$ or the out-configuration model $OCM(d^+)$, and we write $G(\sigma)$ for the corresponding realisation of the digraph. We will treat both models on an equal footing as much as possible, and when we need to distinguish between them we often refer to these as model 1 and model 2 respectively. When the underlying graph is static and is given by one of these two models, the mixing time of the
random walk has been studied in [9, 10]. For a fixed configuration \( \sigma \in \mathcal{C} \), we consider the transition matrix

\[
P_\sigma(x, y) = \frac{\#(\sigma; x \rightarrow y)}{d_x^+}, \quad x, y \in [n],
\]

where \( \#(\sigma; x \rightarrow y) \) denotes the number of directed edges from \( x \) to \( y \) in \( G(\sigma) \). The random walk on \( G(\sigma) \) is thus the Markov chain \((X_0, X_1, \ldots)\) with state space \([n]\) and with transition probabilities \( P_\sigma(x, y) \). We use the notation \( P_\sigma^t(\cdot) \) for the law of the trajectory \((X_0, X_1, \ldots)\) when \( X_0 = x \), so that in particular, for any \( x, y \in [n] \), and \( t \in \mathbb{N} \):

\[
P_\sigma^t(X_t = y) = P_\sigma^t(x, y).
\]

We remark that in each of the two models above, the random walk on the digraph \( G = G(\sigma) \) has with high probability a unique stationary distribution \( \pi_\sigma \). In model 1 this follows from the fact that \( G(\sigma) \) is w.h.p strongly connected [14]. In model 2 on the other hand \( G(\sigma) \) may have vertices with in-degree zero (or, more generally, with a bounded in-neighborhood) and one cannot conclude that \( G(\sigma) \) is strongly connected. However, it is still the case that w.h.p. there exists a unique stationary distribution; see e.g. [1, 10] for more details. Let us now recall the main results of [9, 10]. Convergence to equilibrium will be quantified using the total variation distance. Given two probability measures \( \mu, \nu \), the latter is defined by

\[
\|\mu - \nu\|_{TV} = \max_E |\mu(E) - \nu(E)|,
\]

where the maximum ranges over all possible events in the underlying probability space. Let the entropy \( H \) and the associated entropic time \( T_{\text{ENT}} \) be defined by

\[
H = \sum_{x \in V} \mu_{\text{in}}(x) \log d_x^+, \quad T_{\text{ENT}} = \frac{\log n}{H},
\]

where the probability \( \mu_{\text{in}} \) is defined as \( \mu_{\text{in}}(x) = d_x^- / m \) for model 1 and as \( \mu_{\text{in}}(x) = 1/n \) for model 2. Note that our assumptions on \( d^\pm \) imply that the deterministic quantities \( H, T_{\text{ENT}} \) satisfy \( H = \Theta(1) \) and \( T_{\text{ENT}} = \Theta(\log n) \).

**Theorem 1** (Uniform cutoff at the entropic time [9, 10]). Let \( G(\sigma) \) be a random graph from either the directed configuration model \( \text{DCM}(d^\pm) \) or the out-configuration model \( \text{OCM}(d^+) \). For each \( \beta > 0, \beta \neq 1 \) one has:

\[
\max_{x \in [n]} \left| \|P_\sigma^t(x, \cdot) - \pi_\sigma\|_{TV} - \vartheta(\beta) \right| \overset{\mathbb{P}}{\rightarrow} 0, \quad t = \lfloor \beta T_{\text{ENT}} \rfloor,
\]

where \( \vartheta(\beta) \) is the step function \( \vartheta(\beta) = 1(\beta < 1) \).

In (1.8) we use the notation \( \overset{\mathbb{P}}{\rightarrow} \) for convergence in probability as \( n \to \infty \) with respect to the random choice of the configuration \( \sigma \in \mathcal{C} \). If we define the mixing time \( T_\sigma(\varepsilon) \) as the first \( t \in \mathbb{N} \) such that \( \max_{x \in [n]} \|P_\sigma^t(x, \cdot) - \pi_\sigma\|_{TV} \leq \varepsilon \) then Theorem 1 establishes that the mixing time of the random walk on the static digraph satisfies with high probability \( T_\sigma(\varepsilon) = (1 + o(1))T_{\text{ENT}} \), for any fixed \( \varepsilon \in (0, 1) \). This is an instance of the so-called cutoff phenomenon [16]. We refer to [23, 6, 7] for related results in the context of undirected graphs. Another important fact established in [9, 10] is that relaxation to equilibrium occurs much earlier than the mixing time if one starts from a delocalized initial state. See also [15] for a related result in terms of the spectrum of the matrix \( P_\sigma \). The precise version of this fact that we shall need reads as follows. Call widespread a sequence of
probability measures $\lambda = \lambda_n$ on $[n]$ such that for some $\varepsilon > 0$:
\[
\max_{x \in [n]} \lambda(x) \leq n^{-\frac{1}{2} - \varepsilon}, \quad \text{and} \quad \limsup_{n \to \infty} \sum_{x \in [n]} n\lambda_n(x)^2 < \infty. \tag{1.9}
\]

For any probability measure $\lambda$ on $[n]$ we write $\lambda P^t_\sigma$ for the probability $\sum_x \lambda(x)P^t_\sigma(x, y)$. The following result is proven in [11, Lemma 1].

**Lemma 2.** Let $G(\sigma)$ be a random graph from either the directed configuration model $DCM(d^\pm)$ or the out-configuration model $OCM(d^+)$. If $\lambda$ is widespread, then for any sequence $t = t(n) \to \infty$,
\[
\|\lambda P^t_\sigma - \pi_\sigma\|_{TV} \xrightarrow{P} 0. \tag{1.10}
\]

As we shall see in Corollary 13 below, the probability $\pi_\sigma$ is itself widespread with high probability.

1.4. **Mixing of dynamic digraphs.** We now introduce the joint evolution of the digraph and the random walk. Given $\alpha \in (0, 1)$, we consider the Markov chain with state space $\mathcal{C} \times [n]$ and with transition matrix
\[
P_\alpha((\sigma, x), (\eta, y)) = (1 - \alpha)P_\sigma(x, y)1_\sigma(\eta) + \alpha u(\eta)1_x(y), \tag{1.11}
\]
where $1_a(b)$ stands for 1 if $a = b$ and 0 otherwise and $u(\eta) = |\mathcal{C}|^{-1}$ denotes the uniform distribution over the set $\mathcal{C}$. In words, at each time $t \in \mathbb{N}$ independently, we sample a Bernoulli($\alpha$) random variable $J_t$; if $J_t = 1$ we pick a uniformly random $\eta \in \mathcal{C}$ and move from the current state $(\sigma, x)$ to the new state $(\eta, x)$, while if $J_t = 0$ we move to the new state $(\sigma, y)$ where $y$ is chosen uniformly at random among the out-neighbours of $x$ in the digraph $G(\sigma)$. We write $\{(\xi_t, X_t), t \geq 0\}$ for the trajectory of the Markov chain and write $P^t_{\sigma,x}(\cdot)$ for its law when started at $\xi_0 = \sigma$ and $X_0 = x$. It is not hard to check that this is an irreducible and aperiodic Markov chain and therefore it admits a unique stationary distribution $\pi^t$. A consequence of our results, see Remark 4 below, is that $\pi^t$ is well approximated in total variation distance by the probability measure $\nu$ on $\mathcal{C} \times [n]$ defined by
\[
\nu(\sigma, x) = u(\sigma)\pi_\sigma(x). \tag{1.12}
\]

We know that $\pi_\sigma$ is uniquely defined for all $\sigma$ in a set $\Omega_n \subset \mathcal{C}$ with $u(\Omega_n) \to 1$ as $n \to \infty$. To extend $\nu$ to all $\mathcal{C} \times [n]$ we may define e.g. $\pi_\sigma = \mu_{\mathcal{C}_n}$ for $\sigma \in \mathcal{C} \setminus \Omega_n$. We define
\[
D_{\sigma,x}^{1,\alpha}(t) = \|P_{\sigma,x}^{1,\alpha}(\xi_t = \cdot, X_t = \cdot) - \nu\|_{TV}. \tag{1.13}
\]

For each $t \in \mathbb{N}$, the quantity $D_{\sigma,x}^{1,\alpha}(t)$ is regarded as a random variable with respect to the uniform choice of the configuration $\sigma \in \mathcal{C}$. Moreover, we extend $D_{\sigma,x}^{1,\alpha}(\cdot)$ to all positive reals by taking the integer part of the argument.

**Theorem 3.** Fix a sequence $\alpha = \alpha_n$ such that $\alpha_n \to 0$ as $n \to \infty$. Then, for all $\beta > 0$
\[
\limsup_{n \to \infty} \max_{\sigma \in \mathcal{C}, x \in [n]} D_{\sigma,x}^{1,\alpha}(\beta \alpha^{-1}) \leq (1 + \beta)e^{-\beta}. \tag{1.14}
\]

Next, assume that
\[
\gamma = \lim_{n \to \infty} \alpha T_{\text{ENT}} \in [0, \infty]. \tag{1.15}
\]

Then, according to the value of $\gamma$ there are three scenarios:

1. If $\gamma = 0$ then for all $\beta > 0$:
\[
\max_{x \in [n]} \left| D_{\sigma,x}^{1,\alpha}(\beta \alpha^{-1}) - e^{-\beta} \right| \xrightarrow{P} 0. \tag{1.16}
\]
Figure 1. The asymptotic behavior on the scale $\alpha^{-1}$ of the quantity $D_{\alpha,\sigma,x}^J(t)$ for a typical starting environment $\sigma$ and arbitrary $x \in [n]$ in the case $\gamma = 0$ (left), $\gamma = \infty$ (center) and $\gamma \in (0, \infty)$ (right). The transition point in this last scenario is $t = \gamma \alpha^{-1} \sim T_{\text{ENT}}$, and we set $\gamma = 1$.

(2) If $\gamma = \infty$ then for all $\beta > 0$:

$$\max_{x \in [n]} \left| D_{\alpha,\sigma,x}^J (\beta \alpha^{-1}) - (1 + \beta) e^{-\beta} \right| \xrightarrow{P} 0. \quad (1.17)$$

(3) If $\gamma \in (0, \infty)$ then for all $\beta > 0$, $\beta \neq \gamma$:

$$\max_{x \in [n]} \left| D_{\alpha,\sigma,x}^J (\beta \alpha^{-1}) - \psi(\beta) \right| \xrightarrow{P} 0. \quad (1.18)$$

where

$$\psi(\beta) = \begin{cases} 
(1 + \beta) e^{-\beta} & \text{if } \beta < \gamma \\
 e^{-\beta} & \text{if } \beta > \gamma.
\end{cases}$$

The trichotomy displayed in Theorem 3 can be interpreted as follows; see also Figure 1. On the time scale $\alpha^{-1}$ the regeneration times, that is the $t \in \mathbb{N}$ such that $J_t = 1$, converge to a Poisson process of intensity 1. Then $e^{-\beta}$ and $(1 + \beta) e^{-\beta}$ represent the probability of having no regeneration and at most one regeneration up to time $\beta \alpha^{-1}$ respectively. Thus Theorem 3 essentially says that when the walk is far from being mixed within the current digraph then two regenerations are necessary and sufficient for a complete loss of memory of the initial state, whereas if the walk has already mixed within the current digraph then all it is required to reach stationarity is one regeneration.

Remark 4. From Theorem 3 it follows that

$$\lim_{n \to \infty} \| \nu - \pi^J \|_{TV} = 0, \quad (1.19)$$

which in turn implies that all statements in Theorem 3 hold with $\nu$ replaced by $\pi^J$. Indeed, to prove (1.19) observe that the invariance $\pi^J P_\alpha = \pi^J$ implies that for any $t \in \mathbb{N}$

$$\| \nu - \pi^J \|_{TV} \leq \max_{\sigma \in C, x \in [n]} D_{\alpha,\sigma,x}^J (t).$$

Taking $t = \beta \alpha^{-1}$, (1.14) implies that $\limsup_{n \to \infty} \| \nu - \pi^J \|_{TV} \leq (1 + \beta) e^{-\beta}$, and letting $\beta \to \infty$ we obtain (1.19).

The proof of Theorem 3 will be crucially based on Theorem 1 and Lemma 2. The dynamic setting however requires an important extension of these results that can be formulated as follows. For any $(\sigma, \eta) \in C \times C$ and integers $0 \leq s \leq t$, define

$$Q_{\sigma,\eta}^s(x,y) = \sum_{z \in [n]} P_{\sigma}^s(x,z) P_{\eta}^{t-s}(z,y). \quad (1.20)$$

Notice that the following theorem reduces to Theorem 1 if $s = 0$. 
Theorem 5 (Cutoff on double digraphs). Fix $\beta > 0$, take $t = \beta T_{\text{ENT}}$, and let $s = s(n)$ be any sequence such that $0 \leq s \leq t$. Let $\sigma$ and $\eta$ be two independent uniformly random configurations in $C$, and let $\mathbb{P}$ denote the associated probability. Then for fixed $\beta > 0$:

1. If $\beta < 1$:
   \[
   \min_{x \in [n]} \| Q_{\sigma,\eta}^{s,t}(x, \cdot) - \pi_{\eta} \|_{TV} \xrightarrow{\mathbb{P}} 1.
   \]

2. If $\beta > 1$ and $t - s \to \infty$ as $n \to \infty$:
   \[
   \max_{x \in [n]} \| Q_{\sigma,\eta}^{s,t}(x, \cdot) - \pi_{\eta} \|_{TV} \xrightarrow{\mathbb{P}} 0.
   \]

Theorem 5 will be proved in Section 4. Another key ingredient for the proof of Theorem 3 is the control of the annealed walk. By this we mean the law
\[
\mathbb{P}_{\text{an}}^x(\cdot) = \sum_{\eta \in C} u(\eta) \mathbb{P}_\eta^x(\cdot),
\]
where $\mathbb{P}_\eta^x$ is defined before (1.5).

Lemma 6. The annealed law satisfies
\[
\lim_{n \to \infty} \sup_{x \in [n], t \geq 1} \| \mathbb{P}_{\text{an}}^x(X_t = \cdot) - \mu_{\text{in}} \|_{TV} = 0.
\]

Lemma 6 will be proved in Section 2. Once Lemma 6 and Theorem 5 are available, we shall obtain Theorem 3 by a decomposition of the law at time $t$ according to the location of the regeneration times; see Section 2.2.

Finally, our last main result concerns the marginal distribution of the position of the walk, namely the non-Markovian process obtained by projecting the chain $(\xi_t, X_t)_{t \geq 0}$ on the second coordinate. According to Theorem 3 and Lemma 6 the law of $X_t$, for $t$ and $n$ suitably large, should be well approximated by $\mu_{\text{in}}$. The next result quantifies this statement by exhibiting once again a trichotomy. Define
\[
D_\alpha^{\sigma,x}(t) := \| \mathbb{P}^\sigma_{\sigma,x}(X_t = \cdot) - \mu_{\text{in}} \|_{TV}, \quad q := \mathbb{E} \| \pi_{\sigma} - \mu_{\text{in}} \|_{TV}.
\]

We remark that if the sequences $d^\pm$ are eulerian, that is $d^+_x = d^-_x$ for all $x \in [n]$, then $\pi_{\sigma} = \mu_{\text{in}}$ is stationary for all $\sigma \in C$. Thus in this case $q = 0$. On the other hand, results from [21, 22] imply that if the sequence is not eulerian then $q$ is bounded away from 0 and 1; see [9, Theorem 4] and [11, Remark 1] for more details.

Theorem 7. Fix a sequence $\alpha = \alpha_n$ such that $\alpha_n \to 0$ as $n \to \infty$. Then, for all $\beta > 0$
\[
\limsup_{n \to \infty} \max_{\sigma \in C, x \in [n]} D_\alpha^{\sigma,x}(\beta \alpha^{-1}) \leq e^{-\beta}.
\]

Next, assume that
\[
\gamma = \lim_{n \to \infty} \alpha T_{\text{ENT}} \in [0, \infty].
\]

Then, according to the value of $\gamma$ there are three scenarios:

1. If $\gamma = 0$ then for all $\beta > 0$:
   \[
   \max_{x \in [n]} | D_\alpha^{\sigma,x}(\beta \alpha^{-1}) - q e^{-\beta} | \xrightarrow{\mathbb{P}} 0,
   \]

   and, if $\beta \neq 1$ then
   \[
   \max_{x \in [n]} | D_\alpha^{\sigma,x}(\beta T_{\text{ENT}}) - \varphi(\beta) | \xrightarrow{\mathbb{P}} 0,
   \]
where

$$\varphi(\beta) := \begin{cases} 
1 & \text{if } \beta < 1 \\
q & \text{if } \beta > 1.
\end{cases} \quad (1.28)$$

(2) If $\gamma = \infty$, then for all $\beta > 0$:

$$\max_{x \in [n]} \left| D_{\sigma,x}^{\alpha}(\beta \alpha^{-1}) - e^{-\beta} \right| \xrightarrow{P} 0. \quad (1.29)$$

(3) If $\gamma \in (0, \infty)$ then for all $\beta > 0$, $\beta \neq \gamma$:

$$\max_{x \in [n]} \left| D_{\sigma,x}^{\alpha}(\beta \alpha^{-1}) - \varphi(\beta/\gamma)e^{-\beta} \right| \xrightarrow{P} 0. \quad (1.30)$$

**Figure 2.** The asymptotic behavior on the scale $\alpha^{-1}$ of the quantity $D_{\sigma,x}^{\alpha}(t)$ for a typical starting environment $\sigma$ and arbitrary $x \in [n]$ in the case $\gamma = 0$ (left), $\gamma = \infty$ (center) and $\gamma \in (0, \infty)$ (right). The transition point in the latter case is $t = \gamma \alpha^{-1} \sim T_{\text{ENT}}$. In this picture we take $\gamma = 1$ and $q = 1/2$.

The above results can be roughly interpreted as follows. If we follow only the position of the particle then after the first regeneration time the walk has the annealed law, and by Lemma 6 this is approximately $\mu_{\text{in}}$. Thus, a complete loss of memory of the initial state with relaxation to the limiting state $\mu_{\text{in}}$ occurs essentially at the time of the first regeneration of the digraph. On the other hand, if no regeneration occurs, then a partial loss of memory occurs at time $T_{\text{ENT}}$ because of the static mixing cutoff phenomenon, and this is quantified by the drop by a factor $q$ in total variation. The competition between these two effects explains the above triad; see Figure 2.

We conclude this introduction with some remarks on related work and a comment on possible extensions. The first instance of a trichotomy in the relaxation to equilibrium for random walk on dynamic graphs was revealed in [3, 4]. The authors studied non-backtracking walks on undirected graphs undergoing partial regenerations and obtained results that are qualitatively similar to our Theorem 7, with the difference that the quantity analogous to $q$ is zero in their case. While their model allows for more general regeneration mechanisms than the one considered here, a simplifying feature of their setting with respect to ours is that the stationary distribution is not altered when the underlying graph is updated. Inspired by these works, analogous trichotomy results were obtained for the PageRank surfer on static digraphs [11]. As we observed in [11], teleportation in the PageRank process plays a role similar to the dynamic regeneration; see also [28, 29] for related developments. An interesting extension of the results presented here would be to consider partial regenerations of the underlying digraph instead of full regenerations. For instance, a natural dynamic model for the DCM($d^{\pm}$) can be obtained by updating the permutation $\sigma$ using random transpositions only. Mixing time and the cutoff phenomenon are well understood for random transpositions [17], and it is tempting to conjecture that results of the same kind of those obtained here would hold in that finer setting.
2. TRICHOTOMY FOR THE JOINT PROCESS

We start with the proof of Lemma 6, and then prove Theorem 3 assuming the validity of Theorem 5. The proof of the latter is given in Section 4 below.

2.1. Proof of Lemma 6. We divide the proof in two cases: \( t \leq 2T_{\text{ENT}} \), and \( t > 2T_{\text{ENT}} \). If \( t \leq 2T_{\text{ENT}} \), in particular one has \( t = O(\log n) \), and we know from [12, Lemma 3.10] that

\[
P^\infty_t(X_t = y) = \mu_{\infty}(y)(1 + o(1)), \quad P^\infty_t(X_t = x) = O\left( n^{-1} \log n \right),
\]

(2.1)

for \( t = O(\log n) \), uniformly in \( x, y \in [n] \). The proof of (2.1) is carried out in detail in [12, Lemma 3.10] for model 1 only, but the very same arguments imply the validity of the statements for model 2 as well. The estimates in (2.1) are enough to conclude that uniformly in \( x \in [n] \):

\[
\|P^\infty_t(X_t = \cdot) - \mu_{\infty}\|_{TV} = \frac{1}{2} \sum_{y \in [n]} |P^\infty_t(X_t = y) - \mu_{\infty}(y)| = o(1), \quad t = O(\log n). \quad (2.2)
\]

We now turn to the case \( t > 2T_{\text{ENT}} \). By the triangle inequality we have

\[
\|P^\infty_t(X_t = \cdot) - \mu_{\infty}\|_{TV} \leq \mathbb{E}\|P^t_\sigma(x, \cdot) - \pi_{\sigma}\|_{TV} + \mathbb{E}\|\pi_{\sigma} - \mu_{\infty}\|_{TV}. \quad (2.3)
\]

Concerning the first term on the right hand side, we use Theorem 1 to obtain

\[
\mathbb{E}\|P^t_\sigma(x, \cdot) - \pi_{\sigma}\|_{TV} = o(1), \quad (2.4)
\]

uniformly in \( x \in [n] \), and \( t > 2T_{\text{ENT}} \). The second term on the right hand side of (2.3) can be bounded by a combination of the arguments in (2.2) and (2.4). Indeed, using again the triangle inequality and setting \( s = [2T_{\text{ENT}}] \):

\[
\|\pi_{\sigma} - \mu_{\infty}\|_{TV} \leq \mathbb{E}\|P^t_\sigma(x, \cdot) - \pi_{\sigma}\|_{TV} + \|P^\infty_t(X_t = \cdot) - \mu_{\infty}\|_{TV} = o(1). \quad (2.5)
\]

This ends the proof of Lemma 6.

2.2. Proof of Theorem 3. For every \((\eta, y) \in C \times [n]\) define

\[
\mu^\sigma_t(\eta, y) = \mathbf{P}^t_{\sigma,x}(\xi_t = \eta, X_t = y).
\]

Recall that \( J_s, s \in \mathbb{N} \) are i.i.d. Bernoulli(\(\alpha\)) random variables indicating the occurrence of the regeneration event. For each \( t \geq 1 \), consider the random variable \( \tau = \tau(t) \) defined by

\[
\tau = 1 \left( \exists s \in \{1, \ldots, t\} : J_s = 1 \right) \sup\{s \leq t \mid J_s = 1\}. \quad (2.6)
\]

We may write

\[
\mu^\sigma_t(\eta, y) = \sum_{s=0}^{t} \mathbf{P}^t(\tau = s) \mathbf{P}^t_{\sigma,x}((\xi_t, X_t) = (\eta, y) \mid \tau = s) = (1 - \alpha)^t \mathbf{1}_\sigma(\eta) P^t_\sigma(x, y) + \sum_{s=1}^{t} \alpha(1 - \alpha)^{t-s} \sum_{z \in [n]} \sum_{\xi \in C} \mathbf{u}(\eta) \mu^\sigma_{s-1}(\xi, z) P^t(\tau = s\xi, z, y).
\]

Since \( \mu^\sigma_{s-1}(\xi, z) \) admits the same decomposition we obtain the expansion:

\[
\mu^\sigma_t(\eta, y) = A^\sigma_t(\eta, y) + B^\sigma_t(\eta, y) + C^\sigma_t(\eta, y),
\]
where

\[ A_t^{\sigma,x}(\eta, y) = (1 - \alpha)^t \mathbf{1}_\sigma(\eta) P^t_\sigma(x, y), \]

\[ B_t^{\sigma,x}(\eta, y) = \alpha(1 - \alpha)^{t-1} \sum_{s=1}^t \sum_{z \in [n]} u(\eta) P^{s-1}_\sigma(x, z) P^{t-s}_\eta(z, y), \]

\[ C_t^{\sigma,x}(\eta, y) = \sum_{s=1}^t \sum_{r=1}^{s-1} \alpha^2 (1 - \alpha)^{t-1-r} \sum_{v, z \in [n]} u(\eta) u(\xi) \mu_r^{\sigma,x}(\omega, v) P^{s-1-r}_\xi(v, z) P^{t-s}_\eta(z, y). \]

Notice that for every choice of \( W = \nu, B_t^{\sigma,x}, C_t^{\sigma,x} \), for any fixed choice of \( \sigma \in \mathcal{C} \) one has

\[ \sum_{\eta \in \mathcal{C}} \sum_{\eta \neq \sigma \in [n]} W(\eta, y) = \sum_{\eta \in \mathcal{C}} \sum_{y \in [n]} W(\eta, y) + O(|\mathcal{C}|^{-1}). \]

Therefore,

\[ 2\|\mu_t^{\sigma,x} - \nu\|_{TV} = \sum_{\eta \in \mathcal{C}} \sum_{y \in [n]} |\mu_t^{\sigma,x}(\eta, y) - \nu(\eta, y)| \]

\[ = \sum_{y \in [n]} |\mu_t^{\sigma,x}(\sigma, y) - \nu(\sigma, y)| + \sum_{\eta \neq \sigma} \sum_{y \in [n]} |\mu_t^{\sigma,x}(\eta, y) - \nu(\eta, y)| \]

\[ = (1 - \alpha)^t + \sum_{\eta \in \mathcal{C}} \sum_{y \in [n]} |B_t^{\sigma,x}(\eta, y) + C_t^{\sigma,x}(\eta, y) - u(\eta) \pi(\eta)| + o(1). \] (2.7)

We may rewrite \( C_t^{\sigma,x}(\eta, y) = \chi u(\eta) \hat{C}_t^{\sigma,x}(\eta, y) \), where

\[ \chi = 1 - (1 - \alpha)^t - \alpha t(1 - \alpha)^{t-1} = \alpha^2 \sum_{s=1}^t \sum_{r=1}^{s-1} (1 - \alpha)^{t-r-1}, \]

\[ \hat{C}_t^{\sigma,x}(\eta, y) = \frac{1}{\chi} \alpha^2 \sum_{s=1}^t \sum_{r=1}^{s-1} (1 - \alpha)^{t-r-1} \sum_{z \in [n]} \sum_{v \in [n]} \mu_r^{\sigma,x}(v) P^s_\xi(X_{s-1-r} = z) P^{t-s}_\eta(z, y), \]

and we use the notation \( \mu_r^{\sigma,x}(v) := \sum_{\omega \in \mathcal{C}} \mu_r^{\sigma,x}(\omega, v) \). Notice that \( \hat{C}_t^{\sigma,x}(\eta, \cdot) \) is a probability on \([n]\). Define also the probability \( \lambda(\eta) \) by

\[ \lambda(\eta)(y) = \frac{1}{\chi} \alpha^2 \sum_{s=1}^t \sum_{r=1}^{s-1} (1 - \alpha)^{t-r-1} \mu_r^{\sigma,x}(y). \]

Lemma 6 implies that uniformly in \( \eta \in \mathcal{C} \):

\[ \|\hat{C}_t^{\sigma,x}(\eta, \cdot) - \pi(\eta)\|_{TV} = \|\lambda(\eta) - \pi(\eta)\|_{TV} + o(1). \] (2.8)

Moreover, Lemma 2 implies that whenever \( t - s \to \infty \):

\[ \sum_{\eta} u(\eta) \|\mu(\eta) P^{t-s}_\eta - \pi(\eta)\|_{TV} = o(1). \] (2.9)

Since \( \alpha \to 0 \), (2.9) implies

\[ \sum_{\eta} u(\eta) \|\lambda(\eta) - \pi(\eta)\|_{TV} = o(1). \] (2.10)

Inserting (2.8), (2.10) in (2.7) we obtain

\[ 2\|\mu_t^{\sigma,x} - \nu\|_{TV} = (1 - \alpha)^t + \sum_{\eta \in \mathcal{C}} \sum_{y \in [n]} |B_t^{\sigma,x}(\eta, y) + (1 - \chi) u(\eta) \pi(\eta)| + o(1). \] (2.11)
Let us now take $t = \beta \alpha^{-1}$, for some fixed constant $\beta > 0$. Since $\alpha \to 0$ we have $1 - \chi \to e^{-\beta}(1 + \beta)$ and

$$2\|\mu_t^{\sigma,x} - \nu\|_{TV} = e^{-\beta} + \sum_{\eta \in \mathcal{C}} u(\eta) \psi_t(\eta) + o(1),$$

(2.12)

where we define

$$\psi_t(\eta) = \sum_{y \in [n]} \left| \beta e^{-\beta} \widehat{B}_{t}^{\sigma,x}(\eta, y) - e^{-\beta}(1 + \beta) \pi_\eta(y) \right|,$$

(2.13)

with $\widehat{B}_{t}^{\sigma,x}(\eta, \cdot)$ the probability on $[n]$ defined by

$$\widehat{B}_{t}^{\sigma,x}(\eta, y) = \frac{1}{t} \sum_{s=1}^{t} \sum_{z \in [n]} P_{\sigma}^{s-1}(x, z) P_{\eta}^{t-s}(z, y).$$

We start by noting that

$$\psi_t(\eta) \leq 2\beta e^{-\beta} \|\widehat{B}_{t}^{\sigma,x}(\eta, \cdot) - \pi_\eta\|_{TV} + e^{-\beta}.$$  

In particular, (2.12) shows that uniformly in $(\sigma, x) \in \mathcal{C} \times [n]$

$$\|\mu_t^{\sigma,x} - \nu\|_{TV} \leq (1 + \beta)e^{-\beta} + o(1),$$

which proves (1.14). At this point we split the analysis in four cases.

2.2.1. $\alpha_{\text{ENT}} \to \gamma = \infty$. In this case we notice that $t = o(\log n)$. Therefore, for every $\sigma \in \mathcal{C}$, $x \in [n]$ there must exist a set $\mathcal{I} \subset [n]$ such that for all $s \leq t$:

$$P_{\sigma}^{s-1}(x, \mathcal{I}) = 1, \quad |\mathcal{I}| \leq \Delta t = n^{o(1)}.$$

Moreover, for every $\eta \in \mathcal{C}$ and for every $z \in [n]$ there exists a set $\mathcal{J}_z \subset [n]$ such that for every $s \leq t$

$$P_{\eta}^{t-s}(z, \mathcal{J}_z) = 1, \quad |\mathcal{J}_z| = n^{o(1)}.$$

Therefore, setting $\mathcal{J} = \cup_{z \in \mathcal{I}} \mathcal{J}_z$,

$$|\mathcal{J}| = n^{o(1)}, \quad \widehat{B}_{t}^{\sigma,x}(\eta, \mathcal{J}) = 1 - o(1).$$

Moreover, w.h.p. with respect to $\eta$ one has $\pi_\eta(\mathcal{J}) = o(1)$. Indeed, we know that for some constant $C > 0$, $\sum_{x \in [n]} \pi_\eta(x)^2 \leq C n^{-1}$ by Lemma 11 below, and for any $U \subset [n]$, Cauchy-Schwarz implies

$$\pi_\eta(U)^2 \leq |U| \sum_{x \in [n]} \pi_\eta(x)^2 \leq C |U| n^{-1}.$$  

(2.14)

It follows that w.h.p.

$$\psi_t(\eta) = 2\beta e^{-\beta} + e^{-\beta} + o(1).$$

In conclusion, (2.12) implies

$$\|\mu_t^{\sigma,x} - \nu\|_{TV} = (1 + \beta)e^{-\beta} + o(1),$$

which proves (1.17). Note that because of the uniform average over $\eta \in \mathcal{C}$ the convergence in (1.17) actually holds uniformly in $\sigma \in \mathcal{C}$ rather than in $\mathbb{P}$-probability as stated.
2.2.2. $\alpha T_{\text{ENT}} \rightarrow \gamma = 0$. In this case it possible to find a sequence $\nu = \nu(n) = o(1)$ that vanishes sufficiently slowly that

$$
\nu t = \nu \beta \alpha^{-1} = \omega(T_{\text{ENT}}).
$$

If $\hat{E}_{t}^{\sigma,x}(\eta, \cdot)$ denotes the probability on $[n]$

$$
\hat{E}_{t}^{\sigma,x}(\eta, y) = \frac{1}{(1 - 2\nu)t} \sum_{s = t}^{(1 - \nu)t} \sum_{z \in [n]} P_{\sigma}^{s - 1}(x, z) P_{\eta}^{t - s}(z, y),
$$

then

$$
\|\hat{E}_{t}^{\sigma,x}(\eta, \cdot) - \hat{E}_{t}^{\sigma,x}(\eta, \cdot)\|_{TV} = O(\nu).
$$

Let us write

$$
\sum_{z \in [n]} P_{\sigma}^{s - 1}(x, z) P_{\eta}^{t - s}(z, \cdot) =: \lambda P_{\eta}^{t - s}(\cdot),
$$

and notice that

$$
\|\lambda P_{\eta}^{t - s} - \pi_{\eta}\|_{TV} \leq \max_{x \in [n]} \|P_{\eta}^{t - s}(x, \cdot) - \pi_{\eta}\|_{TV}.
$$

Since $t - s = \omega(T_{\text{ENT}})$, from Theorem 1 we conclude that w.h.p. with respect to $\eta$:

$$
\|\hat{E}_{t}^{\sigma,x}(\eta, \cdot) - \pi_{\eta}\|_{TV} = o(1).
$$

Therefore, w.h.p.

$$
\|\hat{E}_{t}^{\sigma,x}(\eta, \cdot) - \pi_{\eta}\|_{TV} = o(1).
$$

By the triangular inequality and (2.12),

$$
\|\mu_{t}^{\sigma,x} - \nu\|_{TV} = e^{-\beta} + o(1).
$$

This proves (1.16). As in the previous case, it is worth noting that the convergence in (1.16) actually holds uniformly in $\sigma \in \mathcal{C}$ rather than in $\mathbb{P}$-probability.

2.2.3. $\alpha T_{\text{ENT}} \rightarrow \gamma \in (0, \infty) \text{ and } \beta < \gamma$. We want to control $\psi_{t}(\eta)$ as defined in (2.13). If $\beta < \gamma$ then $t = (1 - \epsilon)T_{\text{ENT}}$ for some $\epsilon \in (0, 1)$. We argue that w.h.p. with respect to the independent pair $(\sigma, \eta)$,

$$
\|\hat{E}_{t}^{\sigma,x}(\eta, \cdot) - \pi_{\eta}\|_{TV} = 1 - o(1). \tag{2.15}
$$

Call $Y_{s,t}$ the set of $y \in [n]$ such that

$$
\sum_{z \in [n]} P_{\sigma}^{s - 1}(x, z) P_{\eta}^{t - s}(z, y) \geq e^{-(1 + \epsilon)Ht}
$$

Summing over $y \in Y_{s,t}$, we must have

$$
|Y_{s,t}| \leq e^{(1 + \epsilon)Ht} = n^{1 - \epsilon^2}
$$

Lemma 15 below implies in particular that

$$
\sum_{y \in Y_{s,t}} \sum_{z \in [n]} P_{\sigma}^{s - 1}(x, z) P_{\eta}^{t - s}(z, y) = 1 - o(1).
$$

Setting $Y = \cup_{s=t}^{t} Y_{s,t}$ and noticing that $|Y| \leq T_{\text{ENT}} n^{1 - \epsilon^2} = o(n)$, we have

$$
\sum_{y \in Y} \hat{E}_{t}^{\sigma,x}(\eta, y) \geq \frac{1}{t} \sum_{s=1}^{t} \sum_{y \in Y_{s,t}} \sum_{z \in [n]} P_{\sigma}^{s - 1}(x, z) P_{\eta}^{t - s}(z, y) = 1 - o(1).
$$
Since $|Y| = o(n)$, $\hat{B}_t^{\sigma,x}(\eta, \cdot)$ is w.h.p. asymptotically singular with respect to $\pi_\eta$; see the argument in (2.14). This proves (2.15). Inserting this in (2.12)-(2.13), it follows that w.h.p. with respect to $\sigma \in C$:
$$\|\mu_t^{\sigma,x} - \nu\|_{TV} = (1 + \beta)e^{-\beta} + o(1).$$

2.2.4. $\alpha T_{\text{ENT}} \to \gamma \in (0, \infty)$ and $\beta > \gamma$. By definition there must exist some $\epsilon > 0$ such that
$$t = \beta \alpha^{-1} = \frac{\beta}{H_\gamma} \log n > (1 + \epsilon)T_{\text{ENT}}.$$ 

For every $v \in (0, \epsilon/2)$, at the price of an additive error $O(v)$ in total variation, we can replace $\hat{B}_t^{\sigma,x}(\eta, \cdot)$ by the probability $\hat{B}_1(\cdot)$ defined as
$$\hat{B}_1(y) = \frac{1}{(1 - 2v)\epsilon} \sum_{s=v}^{1-v} \sum_{z \in [n]} P^{s-1}(x, z) P^{t-s}_\eta(z, y).$$

Since $t > (1 + \epsilon)T_{\text{ENT}}$ and $t - s \to \infty$, we can use Theorem 5 to obtain that w.h.p. with respect to the independent pair $(\sigma, \eta)$,
$$\|\hat{B}_1 - \pi_\eta\|_{TV} = o(1) \quad (2.16)$$

From (2.13),
$$\psi_t(\eta) = e^{-\beta} \sum_{y \in [n]} |\beta \hat{B}_1(y) - (1 + \beta)\pi_\eta(y)| + O(v) = e^{-\beta} + O(v) + o(1).$$

Since $v$ is arbitrarily small, from (2.12) we obtain that w.h.p. with respect to $\sigma \in C$:
$$\|\mu_t^{\sigma,x} - \nu\|_{TV} = e^{-\beta} + o(1).$$

3. TRICHOTOMY FOR THE RANDOM WALK

Here we prove Theorem 7. The main observation can be stated as follows.

**Proposition 8.** Let $\tau = \tau(t)$ denote the random variable in (2.6). Then, uniformly in $t \geq 2$:
$$\lim_{n \to \infty} \max_{\sigma, x} \|P^{\sigma,x}_{\tau}(X_t = \cdot | 1 \leq \tau < t) - \mu_{\text{in}}\|_{TV} = 0$$

**Proof.** Observe that
$$P^{\sigma,x}_{\tau}(\tau \in \{0, t\}) = 1 - P^{\sigma,x}_{\tau}(1 \leq \tau < t) = (1 - \alpha)^t + \alpha(1 - \alpha)^{t-1} = (1 - \alpha)^{t-1}. \quad (3.1)$$

Moreover, if $1 \leq s < t$,
$$P^{\sigma,x}_{\tau}(X_t = y; \tau = s) = \alpha(1 - \alpha)^{t-s} \sum_{\eta \in C} u(\eta) \lambda P^{t-s}_\eta(y)$$

where $\lambda$ is the probability measure
$$\lambda(z) = P^{\sigma,x}_{\tau}(X_z = z | \tau = s).$$

We then compute the conditional probability
$$P^{\sigma,x}_{\tau}(X_t = y | 1 \leq \tau < t) = \frac{1}{1 - (1 - \alpha)^{t-1}} \sum_{s=1}^{t-1} P^{\sigma,x}_{\tau}(X_t = y; \tau = s) \quad (3.2)$$
$$= \frac{1}{1 - (1 - \alpha)^{t-1}} \sum_{s=1}^{t-1} \alpha(1 - \alpha)^{t-s} \sum_{\eta \in C} u(\eta) \lambda P^{t-s}_\eta(y). \quad (3.3)$$
Now we can rely on the uniform bound of Lemma 6 to conclude
\[
\|P^t_{\sigma,x}(X_t = \cdot | 1 \leq \tau < t) - \mu_{\text{in}}\|_{TV}
\leq \sum_{s=1}^{t-1} \frac{\alpha(1-\alpha)^{t-s}}{1-(1-\alpha)^{t-1}} \max_{x \in [n]} \|P^s_x(X_{t-s} = \cdot) - \mu_{\text{in}}\|_{TV} = o(1).
\]

□

**Corollary 9.** Uniformly in \(t \geq 1\):
\[
\|P^t_{\sigma,x}(X_t = \cdot) - \mu_{\text{in}}\|_{TV} = (1-\alpha)^t \|P^t_{\sigma}(x,\cdot) - \mu_{\text{in}}\|_{TV} + o(1).
\]

**Proof.** Note that
\[
P^t_{\sigma,x}(X_t = \cdot | \tau = 0) = P^t_{\sigma}(x,\cdot), \quad P^t_{\sigma,x}(X_t = \cdot | \tau = t) = P^{t-1}_{\sigma}(x,\cdot).
\]
Using Proposition 8, and \(\alpha \to 0\), the triangle inequality shows that
\[
\|P^t_{\sigma,x}(X_t = \cdot) - \mu_{\text{in}}\|_{TV}
= \|(1-\alpha)^t - \mu_{\text{in}}\|_{TV} + o(1)
= (1-\alpha)^t \|P^t_{\sigma}(x,\cdot) - \mu_{\text{in}}\|_{TV} + o(1).
\]

□

All statements in Theorem 7 follow from Corollary 9 provided we establish the next lemma.

**Lemma 10.** If \(t = \beta T_{\text{ENT}}\) then for any fixed \(\beta > 0, \beta \neq 1\):
\[
\max_{x \in [n]} \|P^t_{\sigma}(x,\cdot) - \mu_{\text{in}}\|_{TV} - \varphi(\beta) \xrightarrow{p} 0,
\]
where \(\varphi(\beta) = 1\) if \(\beta < 1\) and \(\varphi(\beta) = q\) for \(\beta > 1\), and \(q\) is defined in (1.23).

**Proof.** From Theorem 1 it is sufficient to show that if \(t = \beta T_{\text{ENT}}\) with \(\beta < 1\), then for any \(\varepsilon > 0\) w.h.p.
\[
\min_{x \in [n]} \|P^t_{\eta}(x,\cdot) - \mu_{\text{in}}\|_{TV} \geq 1 - \varepsilon,
\]
and that
\[
q - \|\pi_{\eta} - \mu_{\text{in}}\|_{TV} \xrightarrow{p} 0.
\]

The concentration (3.6) has been already proved in [9, Lemma 17] (see also [11, Proposition 6]). Concerning the estimate (3.5), we can use Lemma 15 below to show that if \(t = \beta T_{\text{ENT}}\) with \(\beta < 1\) then there exists a set \(U_x \subset [n]\) with \(|U_x| = o(n)\) such that w.h.p.
\[
P^t_{\eta}(x,U_x) \geq 1 - o(1).
\]
Since \(\mu_{\text{in}}(U_x) = o(1)\), this ends the proof. □
4. CUTOFF IN DOUBLE DIGRAPHS

We start by showing that w.h.p. the stationary distribution of a random digraph in any of the two models is a widespread measure.

**Lemma 11.** There exists a constant \( C \equiv C(\Delta) > 0 \) such that

\[
\lim_{n \to \infty} \mathbb{P} \left( n \sum_{z \in [n]} \pi_\sigma(z)^2 \leq C \right) = 1.
\]

**Proof.** Call \( Z = n \sum_{z \in [n]} \pi_\sigma(z)^2 \). Let \( t = \log^3(n) \) and consider the event

\[
D = \left\{ \sigma \in C : \max_{x,z \in [n]} |\pi_\sigma(z) - P_\sigma^t(x,z)| = o(n^{-3}) \right\}. \tag{4.1}
\]

A simple consequence of Theorem 1 (see [12, Lemma 3.11]) is that \( \mathbb{P}(D) = 1 - o(1) \). Therefore,

\[
\mathbb{P}(Z > C) \leq \mathbb{P}(Z > C; D) + o(1).
\]

By Markov’s inequality

\[
\mathbb{P}(Z > C; D) \leq \frac{\mathbb{E}[Z^K 1_D]}{C^K}, \quad \forall K \geq 1.
\]

Therefore, it is sufficient to show that \( \mathbb{E}[Z^K 1_D] \leq (C/2)^K \) for some \( K = \omega(1) \). Choose for example \( K = \log n \). Then,

\[
\mathbb{E}[Z^K 1_D] \leq n^K \mathbb{E} \left[ \left( \sum_{z \in [n]} \sum_{x \in [n]} \sum_{y \in [n]} \frac{1}{n^2} \left( P_\sigma^t(x,z) + o(n^{-3}) \right) \left( P_\sigma^t(y,z) + o(n^{-3}) \right) \right)^K \right]
\]

\[
\leq n^K \mathbb{E} \left[ \left( o(n^{-1}) + \sum_{z \in [n]} \sum_{x \in [n]} \sum_{y \in [n]} \frac{1}{n^2} P_\sigma^t(x,z) P_\sigma^t(y,z) \right)^K \right] + o(1)
\]

\[
= (2n)^K \mathbb{P}_{\text{unif}} \left( X_{t}^{(\ell)} = Y_{t}^{(\ell)}, \forall \ell \leq K \right) + o(1),
\]

where \( \mathbb{P}_{\text{unif}} \) denotes the annealed law of the \( 2K \) independent walks \((X_{s}^{(k)}, Y_{s}^{(k)})_{s \leq t}\) for \( k \leq K \), each starting at a uniformly random vertex:

\[
\mathbb{P}_{\text{unif}} = \frac{1}{n^{2K}} \sum_{x_1, \ldots, x_K \in [n]} \sum_{y_1, \ldots, y_K \in [n]} \sum_{\eta \in C} u(\eta) P_{x_1}^{\eta} \cdots P_{x_K}^{\eta} P_{y_1}^{\eta} \cdots P_{y_K}^{\eta}. \tag{4.2}
\]

For an explicit construction, we can generate recursively the walks and the environment, letting the trajectories reveal the configuration \( \eta \), the \( \ell \)-th trajectory living in the environment discovered by the previous \( \ell - 1 \) trajectories; see [12, Lemma 3.11] for more details. Therefore, it is sufficient to show that it is possible to find a constant \( C > 0 \) such that for every sufficiently large \( n \)

\[
\mathbb{P}_{\text{unif}} \left( X_{t}^{(\ell)} = Y_{t}^{(\ell)}, \forall \ell \leq K \right) \leq \left( \frac{C}{4n} \right)^K. \tag{4.3}
\]
For $k = 1, \ldots, K$, define the events
\begin{equation}
B_k = \bigcap_{t \leq k} \{X_t^{(k)} = Y_t^{(k)}\},
\end{equation}
and call $A_k$ the set of vertices which have at least one tail/head revealed by the trajectories $(X_t^{(k)}, Y_t^{(k)})_{t \leq k}$. Call $\Xi_k$ a realization of the trajectories $(X_s^{(k)}, Y_s^{(k)})_{s \leq t, t \leq k}$ satisfying $B_k$. We are going to prove that
\begin{equation}
P_{\text{unif}}(B_{k+1} \mid \Xi_k) = \sum_{z \in [n]} \sum_{x \in [n]} \sum_{y \in [n]} \frac{1}{n^2} P_{x,y}^{\text{unif}}(X_t^{(k+1)} = Y_t^{(k+1)} = z \mid \Xi_k) = O(1/n).
\end{equation}

We first show that if $x, y, z$ are three distinct vertices all in $A_k^c = [n] \setminus A_k$ then, uniformly in $\Xi_k$,
\begin{equation}
P_{x,y}^{\text{unif}}(X_t^{(k+1)} = Y_t^{(k+1)} = z \mid \Xi_k) = O\left(\frac{1}{n^2}\right).
\end{equation}

Consider the event $E_k$ that the trajectory $X^{(k)} = \{X_s^{(k)}\}$, $0 \leq s \leq t$ has no collision with itself nor with the environment previously discovered by $X^{(1)}, Y^{(1)}, \ldots, X^{(k-1)}, Y^{(k-1)}$, and let $Y_k$ denote the event that the walk $X^{(k)}$ does not visit $y$. At any given time, any given walk has probability at most $\Delta/(m - Kt) = O(1/n)$ of hitting a given vertex by generating a fresh new edge. Thus, by a union bound, the event $E_k^c$ has an intersection $\Xi_k$ has probability $O(Kt^2/n)$ uniformly in $k \leq K$.

We prove (4.6) by decomposing the event $X_t^{(k+1)} = Y_t^{(k+1)} = z$ along the four cases: $E_k \cap Y_k, E_k^c \cap Y_k, E_k \cap Y_k^c, E_k^c \cap Y_k^c$. Consider first the case $E_k \cap Y_k^c$. The probability of $E_k^c$ cannot exceed $O(Kt^2/n)$. Moreover, the probability of visiting $y \in A_k^c$ and $z \in A_k^c$ can each be bounded by $O(t/n)$. Thus,
\begin{equation}
P_{x,y}^{\text{unif}}(X_t^{(k+1)} = z; E_{k+1}^c; Y_{k+1} \mid \Xi_k) = O(t/n)O(t/n)O(Kt^2/n) = o(n^{-2}).
\end{equation}

Similarly,
\begin{equation}
P_{x,y}^{\text{unif}}(X_t^{(k+1)} = Y_t^{(k+1)} = z; E_{k+1}^c; Y_{k+1} \mid \Xi_k) = O(Kt^2/n)O(t/n)O(Kt^2/n) = o(n^{-2}).
\end{equation}

Indeed, the walk $X^{(k+1)}$ must visit $z \in A_k^c$ and also one of the previously discovered vertices, which has probability $O(Kt^2/n) \times O(t/n)$, and, if $Y_{k+1}$ holds, in order for the walk $Y^{(k+1)}$ to arrive in $z$ at time $t$ it is necessary to visit a vertex that was already discovered (e.g., $z$ itself). The latter event has probability $O(Kt^2/n)$.

If $E_{k+1} \cap Y_{k+1}$ holds, the for $X_t^{(k+1)} = z$, $Y_t^{(k+1)} = z$ to occur there must be a time $s \leq t$ such that $Y^{(k+1)}$ collides at time $s$ with the trajectory of $X^{(k+1)}$, then $Y^{(k+1)}$ stays on this trajectory for $t - s$ units of time, and then finally hits $z$ at time $t$. On the event $E_{k+1}$ the probability of $X_t^{(k+1)} = z$ is bounded by $d_z/m(1 + o(1))$, and the event that $Y^{(k+1)}$ spends $h$ units of time in the path $X^{(k+1)}$ is at most $2^{-h}$. Therefore,
\begin{equation}
P_{x,y}^{\text{unif}}(X_t^{(k+1)} = Y_t^{(k+1)} = z; E_{k+1}; Y_{k+1} \mid \Xi_k) \leq \frac{d_z}{m} (1 + o(1)) \cdot \frac{2\Delta}{m} \sum_{h=1}^{t} 2^{-h} \leq \frac{\Delta^2}{n^2}.
\end{equation}

Finally, if $x \neq y$, then the event $E_{k+1} \cap \{X_t^{(k+1)} = z\} \cap Y_{k+1}$ has probability $O(t/n) \times O(1/n)$. Under this event, when the walk $Y^{(k+1)}$ starts at $y$ the revealed in-neighborhood of $z$ consist of a unique path of length $t$ from $x$ to $z$ and $y$ is a vertex in this path. Since $y \neq x$, to achieve $Y^{(k+1)} = z$ it is necessary that $Y^{(k+1)}$ exits and re-enters the path.
This requires hitting the path by creating a fresh edge, which has probability $O(Kt^2/n)$. Hence,

$$P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)} = z; \mathcal{E}_{k+1}; \mathcal{Y}_{k+1}^e | \Xi_k \right) = O(t/n) O(1/n) O(Kt^2/n) = o(n^{-2}).$$

In conclusion, we have proved (4.6). In particular, we have obtained

$$\sum \sum \sum \frac{1}{n^2} P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)} = z | \Xi_k \right) \leq n^3 \frac{1}{n^2} \frac{\Delta^2}{n} = \frac{\Delta^2}{n}. \tag{4.7}$$

We now deal with the probability

$$P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)} | \Xi_k \right),$$

when $x \in A_k$ and $y \in A_k^c$ or viceversa. By symmetry we can restrict to the former case. We observe that

$$P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)} | \Xi_k \right) = O \left( \frac{Kt^2}{n} \right). \tag{4.8}$$

Indeed,

$$P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)}; \mathcal{Y}_{k+1} \mid \Xi_k \right) = O \left( \frac{Kt^2}{n} \right), \tag{4.9}$$

since the latter event requires that the walk $Y_t^{(k+1)}$ visits a vertex that has been already discovered by $X^{(1)}, Y^{(1)}, \ldots, X^{(k)}, Y^{(k)}, X^{(k+1)}$, while

$$P_{x,y}^{an} ( Y_{k+1}^c | \Xi_k ) = O \left( \frac{t}{n} \right). \tag{4.10}$$

Hence, using $|A_k| \leq Kt$,

$$\sum \sum \sum \frac{1}{n^2} P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)} | \Xi_k \right) \leq \frac{Kt \cdot n}{n^2} \times O \left( \frac{Kt^2}{n} \right) = o(n^{-1}). \tag{4.11}$$

The case $x = y$ in (4.5) is handled by the obvious bound

$$P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)} | \Xi_k \right) \leq 1.$$

The same can be done for the case $x \in A_k$ and $y \in A_k$. Indeed, $|A_k| \leq Kt$ implies

$$\sum \sum \sum \frac{1}{n^2} P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)} | \Xi_k \right) \leq \frac{Kt^2}{n^2} = o(n^{-1}). \tag{4.12}$$

Finally, if $z \in A_k$ and $x, y \in A_k^c$, $x \neq y$ then

$$P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)} \in A_k; \mathcal{Y}_{k+1} | \Xi_k \right) = O \left( \frac{K^2 t^4}{n^2} \right),$$

since both walks have to hit the cluster $A_k$ in order to visit $z$. On the other hand

$$P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)} \in A_k; \mathcal{Y}_{k+1}^c | \Xi_k \right) = O \left( \frac{Kt^2}{n} \right) O \left( \frac{t}{n} \right),$$

since $X^{(k+1)}$ needs to visit both $y$ and the cluster $A_k$. Hence

$$\sum \sum \frac{1}{n^2} P_{x,y}^{an} \left( X_t^{(k+1)} = Y_t^{(k+1)} \in A_k | \Xi_k \right) = O \left( \frac{K^2 t^4}{n^2} \right) = o(n^{-1}). \tag{4.13}$$
Therefore, putting together the bounds (4.7), (4.11), (4.12) and (4.13), and recalling (4.5), we showed that
\[
P_{\text{an}}(B_{k+1} \mid \Xi_k) \leq \frac{\Delta^2}{n} + o\left(\frac{1}{n}\right) \leq \frac{2\Delta^2}{n}.
\]
The same proof shows that
\[
P_{\text{an}}(B_1) \leq \frac{2\Delta^2}{n}.
\]
By the uniformity in \( k \leq K \) and in \( \Xi_k \) of the previous argument, we conclude that
\[
P_{\text{an}}(B_K) = P_{\text{an}}(B_1) \prod_{k=1}^{K-1} P_{\text{an}}(B_{k+1} \mid B_k) \leq \left(\frac{2\Delta^2}{n}\right)^K.
\]
Therefore it is sufficient to choose e.g. \( C = 8\Delta^2 \) to conclude that (4.3) holds.

\[\square\]

Lemma 12. We have
\[
\lim_{n \to \infty} \mathbb{P} \left( \max_{z \in [n]} \pi_{\sigma}(z) \leq \frac{\log^8(n)}{n} \right) = 1. \tag{4.14}
\]

Proof. For the DCM ensemble we may refer to [12, Theorem 1.5] for a much more precise result, where \( 8 \) is replaced by a constant \( a \in [0, 1] \). We give here an alternative proof of the weaker bound (4.14) that holds for the OCM as well. We show that if \( t = \log^3(n) \), then uniformly in \( z \in [n] \)
\[
\mathbb{P} \left( \sum_{x \in [n]} \frac{1}{n} P_{\sigma}^t(x, z) \geq \frac{\log^8(n)}{2n} \right) = o(n^{-1}). \tag{4.15}
\]

By the union bound, and the fact that the event \( D \) in (4.1) occurs w.h.p., (4.15) is sufficient to prove (4.14). Define
\[
W := \sum_{x \in [n]} \frac{1}{n} P_{\sigma}^t(x, z).
\]
By Markov inequality, for every \( K \geq 1 \)
\[
\mathbb{P} \left( W \geq \frac{\log^8(n)}{2n} \right) \leq \frac{2^K n^K}{\log^{8K}(n)} \mathbb{E} [W^K].
\]

As in the proof of Lemma 11, the term in the right hand side of the latter display can be read in terms of the annealed walks. In conclusion, to prove (4.15) it is sufficient to show that for \( K = \log n \)
\[
\mathbb{E} [W^K] = P^\text{an}_{\text{unif}} \left( X^{(k)}_t = z, \forall k \leq K \right) \leq \left( \frac{C \log^7(n)}{n} \right)^K, \tag{4.16}
\]
for some constant \( C > 0 \), where \( P^\text{an}_{\text{unif}} \) denotes the annealed law of \( K \) independent walks
\[
P_{\text{an}}^\text{unif} = \frac{1}{n^K} \sum_{x_1, \ldots, x_K} \sum_{\eta \in C} u(\eta) P_{\eta}^n_1 \cdots P_{\eta}^n_K. \tag{4.17}
\]
Reasoning as in the proof of Lemma 11, similarly to (4.4) we call
\[
B_k = \bigcap_{\ell \leq k} \{ X^{(\ell)}_t = z \}.
\]
The proof is completed by observing that uniformly in \( k \leq K \),
\[
P^{unif}_{\text{an}}(B_{k+1} \mid B_k) = O \left( \frac{Kt^2}{n} \right) = O \left( \frac{\log^7 n(n)}{n} \right),
\]
which is sufficient to prove (4.16). The above estimate simply follows by observing that
\( X_t^{(k+1)} = z \) implies that \( X^{(k+1)} \) hits at some time \( s \in [0,t] \) for the first time a vertex already discovered by the walks \( X^{(\ell)} \), \( \ell \leq k \).

Lemma 11 and Lemma 12 provide the result mentioned at the beginning of the section.

**Corollary 13.** W.h.p. \( \pi_\eta \) is a widespread measure.

### 4.1. Proof of Theorem 5

We now turn to the proof of Theorem 5. Let \( \sigma, \eta \) be two independent uniformly random configurations in \( C \). In this section we will assume that \( t = \Theta(\log n) \) and \( s \leq t \). Let \( Q^s_{x,\sigma,\eta} \equiv Q^s_{x,s,t} \) denote the quenched law of the walker that starts at \( X_0 = x \), goes for \( s \) steps trough \( \sigma \) and then, starting at \( X_s \), goes for \( t-s \) steps trough \( \eta \). We use the notation

\[
Q^s_{x}(x,y) = Q^s_{x,\sigma,\eta}(X_t = y).
\]

**Definition 14.** We define path of length \( t \) an arbitrary sequence of vertices \( p = (v_0, \ldots, v_t) \). We call weight of the path \( w(p) \) the product

\[
w(p) = \prod_{j=0}^{s-1} P_\sigma(v_j, v_{j+1}) \prod_{i=s}^{t-1} P_\eta(v_i, v_{i+1}).
\]

**Lemma 15.** If \( s \in [0,t] \) and \( t = \Theta(\log n) \), for every \( \varepsilon \in (0,1) \)

\[
\min_{x \in [n]} Q^s_{x,\sigma,\eta}(w(X_0, X_1, \ldots, X_t) \in \left[ e^{-(1+\varepsilon)Ht}, e^{-(1-\varepsilon)Ht} \right]) \xrightarrow{\text{P}} 1.
\]

**Proof.** The case \( s = 0 \) is exactly [9, Proposition 8] for the DCM and [10, Theorem 4] for the OCM. Since \( w \) is a product, if \( s, t-s = \Theta(\log n) \) the claim follows by these results in [9, 10] and the independence of \( \sigma \) and \( \eta \). It remains to consider the case \( s = o(\log n) \) and the case \( t-s = o(\log n) \). If \( s = o(\log n) \) then w.h.p. any path of length \( s \) has weight between \( \Delta^{-s} \) and \( 2\delta^{-s} \). We refer to [12, Lemma 3.1] for a proof of this fact. Since \( \Delta^{-s} \) and \( 2\delta^{-s} \) are \( e^{-o(t)} \) in this case, the result follows again by the case \( s = 0 \). The same argument works also in the case \( t-s = o(\log n) \).

**Proof of the Lower Bound of Theorem 5.** Let \( t = (1-\varepsilon)T_{\text{ENT}} \) for some \( \varepsilon > 0 \). Fix any \( x \in [n] \) and call \( U_x \) the set of vertices \( y \) such that \( Q^t_{x}(x,y) > e^{-(1+\varepsilon)Ht} = n^{-1+\varepsilon^2} \). It follows from Lemma 15, that \( Q^t_{x}(x,U_x) = 1-o(1) \) uniformly in \( x \). Moreover,

\[
n^{-1+\varepsilon^2}|U_x| \leq \sum_{y \in U_x} Q^t_{x}(x,y) \leq 1
\]

shows that \( |U_x| = o(n) \). Since \( \pi_\eta \) is widespread by Corollary 13, from the argument in (2.14) we have \( \pi_\eta(U_x) = o(1) \). Hence, w.h.p. uniformly in \( x \in [n] \) the measure \( Q^t_{x}(x,\cdot) \) is asymptotically singular with respect to \( \pi_\eta \). This concludes the proof of part (1) of Theorem 5.

We now turn to proving the upper bound in Theorem 5, which is more involved. In fact, an adaptation of the arguments of [9, 10] is not straightforward in this case. Below we present the details in the case of the DCM only. The proof for the OCM is very similar.
**Remark 16.** For what concerns the upper bound, we can restrict to the case \( s, t - s = \Theta(\log n) \) because of the following argument. If \( s = o(\log n) \) or \( t - s = o(\log n) \) then the upper bound in Theorem 5 holds as a consequence of Theorem 1. Indeed, if \( s = o(\log n) \) and \( t = (1 + \varepsilon) T_{\text{ENT}} \) for some \( \varepsilon > 0 \), we have \( t - s \geq (1 + \varepsilon/2) T_{\text{ENT}} \). Hence,

\[
\|Q_s^t(x, \cdot) - \pi_\eta\|_{TV} \leq \max_{x \in [n]} \|P^{t-s}_\eta(x, \cdot) - \pi_\eta\|_{TV} \xrightarrow{P} 0.
\]

On the other hand, if \( t - s = o(\log n) \) then \( s \geq (1 + \varepsilon/2) T_{\text{ENT}} \), and therefore

\[
\|Q_s^t(x, \cdot) - \pi_\eta\|_{TV} \leq \|Q_s^t(x, \cdot) - \pi_\sigma P^{t-s}_\eta\|_{TV} + \|\pi_\sigma P^{t-s}_\eta - \pi_\eta\|_{TV} \\
\leq \|P^{t-s}_\sigma(x, \cdot) - \pi_\sigma\|_{TV} + \|\pi_\sigma P^{t-s}_\eta - \pi_\eta\|_{TV} \xrightarrow{P} 0.
\]

where we used that \( \|P^{t-s}_\sigma(x, \cdot) - \pi_\sigma\|_{TV} \xrightarrow{P} 0 \) by Theorem 1 and that

\[
\|\pi_\sigma P^{t-s}_\eta - \pi_\eta\|_{TV} \xrightarrow{P} 0
\]

by Corollary 13 and Lemma 2, since we assume \( t - s \to \infty \).

In what follows we will assume \( t, s, t - s = \Theta(\log n) \) and \( t = (1 + \nu) T_{\text{ENT}} \) for some sufficiently small \( \nu > 0 \). The general case \( t = \beta T_{\text{ENT}} \) for any \( \beta > 1 \) follows by monotonicity of the \( TV \)-distance with respect to \( t \). Call

\[
h := \frac{1}{\nu} \log \Delta(n), \quad h := h \wedge \frac{t - s}{2} = \Theta(T_{\text{ENT}}), \quad r := t - s - h = \Theta(T_{\text{ENT}}),
\]

and notice that there exists some \( \epsilon \in (0, 1) \) such that

\[
r + s \leq (1 - \epsilon) T_{\text{ENT}}.
\]

**Strategy of proof.** The overall strategy of proof is the same as in [9, 10]. We recall here the main steps and then give the details of its implementation in our general setting. We can replace \( \pi_\eta \) by \( \mu_{\text{in}} P^h_\eta \) since we know by Lemma 2 that w.h.p.

\[
\|\mu_{\text{in}} P^h_\eta - \pi_\eta\|_{TV} = o(1).
\]

We will focus on a particular set of starting states. We call \( S^\sigma_\nu \) the set of vertices for which the out-neighborhood in \( G(\sigma) \) is a tree up to height \( h \). By [9, Proposition 6] (or [10, Lemma 9]) w.h.p. with respect to the configuration \( \sigma \), most of the vertices are in \( S^\sigma_\nu \), and the quenched probability that the walk is out of the set \( S^\sigma_\nu \) vanishes exponentially fast in time. More precisely, if \( \ell = \log \log n < s \), then w.h.p.

\[
\max_{x \in [n]} Q^\sigma_{\nu, \eta}(X_\ell \not\in S^\sigma_\nu) \leq 2^{-\ell}.
\]

Therefore, by the triangle inequality

\[
\max_{x \in [n]} \|Q^t_s(x, \cdot) - \mu_{\text{in}} P^h_\eta\|_{TV} \leq \max_{x \in [n]} Q^\sigma_{\nu, \eta}(X_\ell \not\in S^\sigma_\nu) + \max_{x \in [S^\sigma_\nu]} \|Q^{t-\ell}_s(x, \cdot) - \mu_{\text{in}} P^h_\eta\|_{TV}. \quad (4.19)
\]

Thus, in order to show the uniform upper bound in Theorem 5 it is sufficient to show an upper bound that holds uniformly in the random set \( S^\sigma_\nu \).

Below we will define a set of *nice paths* for the trajectory of the walk. For every couple of vertices \( x, y \) we let \( N_{x,y} \) denote the set of *nice paths* from \( x \) to \( y \) of length \( t \). Consequently, we define

\[
\tilde{Q}^t_s(x, y) := \sum_{p \in N_{x,y}} w(p)
\]
the probability to go from \( x \) to \( y \) along a nice path. Notice that for any \( \varepsilon > 0 \),
\[
\|\mu_{\eta}P_{\eta}^{h} - Q_{s}^{\ell}(x, \cdot)\|_{TV} = \sum_{y \in [n]} \left[ \mu_{\eta}P_{\eta}^{h}(y) - Q_{s}^{\ell}(x, y) \right]_{+} \\
\leq \sum_{y \in [n]} \left[ \mu_{\eta}P_{\eta}^{h}(y)(1 + \varepsilon) + \frac{\varepsilon}{n} - \bar{Q}_{s}^{\ell}(x, y) \right]_{+}.
\]

(4.20)

Therefore, if we can show that
\[
\mu_{\eta}P_{\eta}^{h}(y)(1 + \varepsilon) + \frac{\varepsilon}{n} \geq \bar{Q}_{s}^{\ell}(x, y),
\]
then the positive part in (4.20) can be neglected, and summing over \( y \in [n] \) one obtains
\[
\|\mu_{\eta}P_{\eta}^{h} - Q_{s}^{\ell}(x, \cdot)\| \leq \sum_{y \in [n]} \left( (1 + \varepsilon)\mu_{\eta}P_{\eta}^{h}(y) + \frac{\varepsilon}{n} - \bar{Q}_{s}^{\ell}(x, y) \right) \\
= 2\varepsilon + Q_{s}^{\eta}\left( (X_{0}, \ldots, X_{\ell}) \notin \cup_{y \in [n]} N_{x,y} \right).
\]

(4.22)

At this point we are left with showing that the probability of following a path that is not nice is arbitrarily small uniformly in the starting point \( x \in S_{s}^{\sigma} \), namely
\[
\max_{x \in S_{s}^{\sigma}} Q_{s}^{\eta}\left( (X_{0}, \ldots, X_{\ell}) \notin \cup_{y \in [n]} N_{x,y} \right) < \varepsilon \quad \text{w.h.p.}
\]

(4.23)

We first introduce the notation required to define the set of nice paths. Then we will present a proof of the validity of (4.21) and (4.23).

We start by constructing the subgraph \( G_{x}^{\sigma}(s) \) of \( G(\sigma) \) spanned by the paths of length at most \( s \), starting at \( x \), and with weight at least \( e^{-(1+\varepsilon)H_{s}} \). We construct \( G_{x}^{\sigma}(s) \) together with a spanning tree \( T_{x}^{\sigma}(s) \) of \( G_{x}^{\sigma}(s) \) in the following way.

**Definition 17. Construction of \( G_{x}^{\sigma}(s) \) and \( T_{x}^{\sigma}(s) \).**
- Call \( G^{\sigma}[0] \) the empty graph on \( \{ x \} \) and \( E_{1}^{x} = E_{x}^{s} \).
- To every \( e_{1} \in E_{1}^{x} \) associate the weight \( \hat{w}_{\sigma}(e_{1}) := (d_{2}^{+})^{-1} \).
- Recursively, for every \( \ell \geq 1 \):
  1. Choose a tail \( e_{\ell} \in E_{\ell}^{x} \) with maximal weight and reveal \( \sigma(e_{\ell}) = f_{\ell} \).
  2. Add the edge \( (e_{\ell}, f_{\ell}) \) to \( G^{\sigma}[\ell - 1] \) and call the resulting graph \( G^{\sigma}[\ell] \).
  3. Call the edge \( (e_{\ell}, f_{\ell}) \) open if \( \nu(f_{\ell}) \notin G^{\sigma}[\ell - 1] \).
  4. Call \( T^{\sigma}[\ell] \) the open subgraph of \( G^{\sigma}[\ell] \).
  5. If \( \nu(f_{\ell}) \notin G^{\sigma}[\ell - 1] \), then associate to any \( e' \in E_{\nu(f_{\ell})}^{+} \) the weight \( \hat{w}_{\sigma}(e') := \hat{w}_{\sigma}(e_{\ell})(d_{\nu(f_{\ell})}^{+})^{-1} \), and if
    \[
    \hat{w}_{\sigma}(e') \geq e^{-(1+\nu)H_{s}},
    \]
    then let \( E_{\ell+1}^{x} = E_{\ell}^{x} \setminus \{ e_{\ell} \} \cup E_{\nu(f_{\ell})}^{+} \). Otherwise, set \( E_{\ell+1}^{x} = E_{\ell}^{x} \setminus \{ e_{\ell} \} \).
  6. Remove from \( E_{\ell+1}^{x} \) the tails \( e' \) such that the vertex \( \nu(e') \) is at distance greater than \( s \) from \( x \) in \( T^{\sigma}[\ell] \).
- Iterate the instructions above up to the random time \( \kappa_{\sigma} \) at which \( E_{\kappa_{\sigma}}^{x} = \emptyset \), and call
  \[
  T_{x}^{\sigma}(s) := T^{\sigma}[\kappa_{\sigma}], \quad G_{x}^{\sigma}(s) := G^{\sigma}[\kappa_{\sigma}].
  \]

The definition given above of the subgraphs \( T_{x}^{\sigma}(s) \) and \( G_{x}^{\sigma}(s) \) coincides with that given in [9, 10]. It was shown in [9, 10] that the random walk on the static environment \( \sigma \), starting at \( x \in S_{s}^{\sigma} \) and of length \( s \) will stay on the tree \( T_{x}^{\sigma}(s) \) w.h.p.. Hence, in the double environment case, the walk will be w.h.p. in one of the leaves of \( T_{x}^{\sigma}(s) \) at
time \( s \). Call \( \mathcal{L}_s^{x,\sigma} \) the set of leaves of \( T_x(\sigma) \) at distance \( s \) from \( x \). We now construct the subgraph \( G_{x,\sigma}^{\mathcal{L}}(r) \) of \( G(\eta) \) consisting of all the paths in \( G(\eta) \) which start at some \( z \in \mathcal{L}_s^{x,\sigma} \), have length \( r \), and cumulative weight larger than \( e^{-(1+\nu/2)H(t-h)} \). Similarly to the construction in Definition 17, together with \( G_{x,\sigma}^{\mathcal{L}}(r) \) we are going to construct a collection \( W_{x,\sigma}^{\mathcal{L}}(r) \) of disjoint rooted directed trees, each rooted at some \( z \in \mathcal{L}_s^{x,\sigma} \) and with depth \( r \). The forest \( W_{x,\sigma}^{\mathcal{L}}(r) \), seen as a collection of edges, will be our candidate for the support of the walk from time \( s \) to time \( t-h \).

**Definition 18. Construction of \( G_{x,\sigma}^{\mathcal{L}}(r) \) and \( W_{x,\sigma}^{\mathcal{L}}(r) \).**

- Call \( G_{x,\sigma}^{\mathcal{L}}[0] \) the empty graph on \( \mathcal{L}_s^{x,\sigma} \) and call \( E_{1}^{\mathcal{L}} = \bigcup_{z \in \mathcal{L}_s^{x,\sigma}} E_{z}^{+} \).
- To every \( e_1 \in E_{1}^{\mathcal{L}} \) associate the weight
  \[
  \tilde{w}_{\sigma,\eta}(e_1) := \tilde{w}_{\sigma}(e_1),
  \]
  of the unique path in \( T_x(\sigma) \) joining \( x \) to \( v(e_1) \) times the inverse of the out degree of \( v(e_1) \); see Definition 17.
- Recursively, for every \( \ell \geq 1 \)
  1. Choose a tail in \( e_\ell \in E_{\ell}^{\mathcal{L}} \) with maximal weight and reveal \( \eta(e_\ell) = f_\ell \).
  2. Add the edge \((e_\ell, f_\ell)\) to \( G_{x,\sigma}^{\mathcal{L}}[\ell - 1] \) and call the resulting graph \( G_{x,\sigma}^{\mathcal{L}}[\ell] \).
  3. Call the edge \((e_\ell, f_\ell)\) open if \( v(f_\ell) \not\in G_{x,\sigma}^{\mathcal{L}}[\ell - 1] \).
  4. Call \( W_{x,\sigma}^{\mathcal{L}}[\ell] \) the open subgraph of \( G_{x,\sigma}^{\mathcal{L}}[\ell] \).
  5. If \( v(f_\ell) \not\in G_{x,\sigma}^{\mathcal{L}}[\ell - 1] \), then associate to any \( e' \in E_{v(f_\ell)}^{+} \) the weight \( \tilde{w}_{\sigma,\eta}(e') := \tilde{w}_{\sigma,\eta}(e_\ell)(d_{v(f_\ell)}^{+})^{-1} \), and if
  \[
  \tilde{w}_{\sigma,\eta}(e') \geq e^{-(1+\nu/2)H(t-h)} := \tilde{w}_{\text{min}},
  \]
  then set \( E_{\ell+1}^{\mathcal{L}} = E_{\ell}^{\mathcal{L}} \setminus \{e_\ell\} \cup E_{v(f_\ell)}^{+} \). Otherwise, set \( E_{\ell+1}^{\mathcal{L}} = E_{\ell}^{\mathcal{L}} \setminus \{e_\ell\} \).
  6. Remove from \( E_{\ell+1}^{\mathcal{L}} \) the tails \( e' \) such that the vertex \( v(e') \) is at distance greater than \( r \) from the corresponding root in \( W_{x,\sigma}^{\mathcal{L}}[\ell] \).
- Iterate the instructions above up to the random time \( \kappa_{\sigma,\eta} \) at which \( E_{\kappa_{\sigma,\eta}}^{\mathcal{L}} = \emptyset \), and call
  \[
  W_{x,\sigma}^{\mathcal{L}}(r) := W_{x,\sigma}^{\mathcal{L}}[\kappa_{\sigma,\eta}], \quad G_{x,\sigma}^{\mathcal{L}}(r) := G_{x,\sigma}^{\mathcal{L}}[\kappa_{\sigma,\eta}].
  \]

We know by [9, 10] that the random number of edges revealed by the construction in Definition 17, \( \kappa_{\sigma,\eta} \), is a.s. \( o(n) \). We need an analogous result for the quantity \( \kappa_{\sigma,\eta} \) in Definition 18.

**Lemma 19.** For any \( \sigma, \eta \in C \) and \( x \in [n] \),

\[
\tilde{w}_{\sigma,\eta}(e_\ell) \leq \frac{r}{\ell}, \quad \forall \ell < \kappa_{\sigma,\eta}.
\]

In particular, recalling that \( t - h = r + s \leq (1 - \epsilon)T_{\text{ENT}} \), by choosing \( \nu \leq \epsilon \)

\[
\kappa_{\sigma,\eta} = O\left(\log(n)n^{(1+\nu/2)(1-\epsilon)}\right) = O(n^{1-\epsilon^2}).
\]

**Proof.** For each \( \ell < \kappa_{\sigma,\eta} \) we consider the forest \( \tilde{W}_{x,\sigma}^{\mathcal{L}}[\ell] \) constructed as in Definition 18, but if an edge \((e_{\ell'}, f_{\ell'})\) for some \( \ell' < \ell \) is not open, we attach a fictitious leaf (with no future children) to \( e_{\ell'} \), to which we assign the weight \( \tilde{w}_{\sigma,\eta}(e_{\ell'}) \). This construction ensures that for every \( \ell \) both the graph \( G_{x,\sigma}^{\mathcal{L}}[\ell] \) and the forest \( \tilde{W}_{x,\sigma}^{\mathcal{L}}[\ell] \) have exactly \( \ell \) edges. Call \( F_{\ell} \) the set of leaves of \( \tilde{W}_{x,\sigma}^{\mathcal{L}}[\ell] \). By construction, for all \( v \in F_{\ell} \) there is a unique \( z \in \mathcal{L}_s^{x,\sigma} \) and a unique path \( p(v) : z \rightarrow v \) of length at most \( r \) in \( \tilde{W}_{x,\sigma}^{\mathcal{L}}[\ell] \). The weight of such a path
is given by \( \hat{w}_{\sigma,\eta}(e_v) \) where \( e_v \) is any tail in \( E_u^\ell \) if \( (u, v) \) is the last edge in the path \( p(v) \). It follows that

\[
\sum_{z \in L^\ell_*} \sum_{v \in F_z} \sum_{p \rightarrow v} \hat{w}_{\sigma,\eta}(e_v) \leq 1.
\]

Since all \( v \in F_\ell \) are such that \( \hat{w}_{\sigma,\eta}(e_v) \geq \hat{w}_{\sigma,\eta}(e_\ell) \), we obtain

\[
|F_\ell| \hat{w}_{\sigma,\eta}(e_\ell) \leq 1.
\]

By the absence of cycles in \( \tilde{W}^\sigma_\eta[e] \) we also have that

\[
\ell \leq r|F_\ell|.
\]

In conclusion

\[
\hat{w}_{\sigma,\eta}(e_\ell) \leq \frac{1}{|F_\ell|} \leq \frac{r}{\ell}.
\]

If we replace \( \ell = \kappa_{\sigma,\eta} - 1 \) we get

\[
\kappa_{\sigma,\eta} - 1 \leq \frac{r}{\hat{w}_{\sigma,\eta}(e_{\kappa_{\sigma,\eta}-1})} \leq \frac{r}{\hat{w}_{\min}}.
\]

Next, we define the set of nice paths.

**Definition 20.** We call nice a path \( p = (v_0, \ldots, v_t) \) s.t.

1. \( w(p) \leq e^{-(1-\varepsilon)/2} m_t \).
2. \( p \) belongs to \( \mathcal{T}_0^\sigma(s) \) up to time \( s \).
3. \( p \) belongs to \( \mathcal{W}^\sigma_{\eta}(r) \) from time \( s \) to time \( t - h = r + s \).
4. \((v_{t-h}, \ldots, v_t)\) is the unique path of length at most \( h \) in the graph \( G(\eta) \) from \( v_{t-h} \) to \( v_t \).

For every \( x, y \in [n] \), we write \( \mathcal{N}_{x,y} \) for the set of nice paths \( (v_0, \ldots, v_t) \) with \( v_0 = x \) and \( v_t = y \).

We now focus on proving (4.23), which will be a consequence of the law of large numbers in Lemma 15 together with the forthcoming Lemma 21. The latter shows, via a martingale argument, that w.h.p. the walk will not exit the forest \( W_{\sigma,\eta} \). We follow [9, 10], where a very similar statement was proved for the walk on a single environment. For simplicity we write \( \hat{w} \) instead of \( \hat{w}_{\sigma,\eta} \). We compute the first two conditional moments of the increment \( M_{\ell+1} - M_\ell \):

\[
\mathbb{E} [M_{\ell+1} - M_\ell \mid S_\ell] \leq 1_{\ell+1 < \kappa_{\sigma,\eta}} \frac{\hat{w}(e_{\ell+1}) \Delta |\mathcal{G}_{\sigma,\eta}[\ell]|}{m - \ell},
\]

\[
\mathbb{E} [(M_{\ell+1} - M^2_\ell) \mid S_\ell] \leq 1_{\ell+1 < \kappa_{\sigma,\eta}} \frac{\hat{w}(e_{\ell+1})^2 \Delta |\mathcal{G}_{\sigma,\eta}[\ell]|}{m - \ell}.
\]

Fix any \( \ell = \Theta(\log n) \) and observe that since \( |\mathcal{G}_{\sigma,\eta}[\ell]| \leq \ell, \) \( \hat{w}(e_\ell) \leq \frac{r}{\ell}, \) we have

\[
\hat{w}(e_{\ell+1}) \Delta |\mathcal{G}_{\sigma,\eta}[\ell]| = O(\log n), \quad \sum_{\ell \geq \ell} \hat{w}(e_\ell) = O(\log^2(n)).
\]
Set

\[ a := \sum_{\ell \geq \bar{\ell}} E [M_{\ell+1} - M_{\ell} \mid S_{\ell}] = O \left( \log(n)n^{-\epsilon^2} \right) = o(1), \]

\[ b := \sum_{\ell \geq \bar{\ell}} E \left[ (M_{\ell+1} - M_{\ell})^2 \mid S_{\ell} \right] = O \left( \log^3(n)n^{-1} \right). \]

Fix any \( \varepsilon \in (0, 1) \) and define

\[ Z_{\ell+1} = \frac{4}{\varepsilon} \left( M_{\ell+1} - M_{\ell} - E [M_{\ell+1} - M_{\ell} \mid S_{\ell}] \right). \]

We observe that \( E[Z_{\ell+1} \mid S_{\ell}] = 0 \) and that \( |Z_{\ell+1}| \leq 1 \), since if \( \ell \geq \bar{\ell} = \omega(1) \), then \( \hat{w}(e_{\ell+1}) \to 0 \), and in particular \( M_{\ell+1} - M_{\ell} \leq \varepsilon \).

Consider the martingale

\[ W_u = \sum_{\ell=\bar{\ell}+1}^u Z_{\ell}, \quad \forall u > \bar{\ell}. \]

Notice that

\[ W_{\kappa,\eta}(\frac{\varepsilon}{4} M_{\kappa,\eta} - M_{\ell} - a) \quad \text{and} \quad b' := \sum_{\ell \geq \bar{\ell}} \text{Var}(Z_{\ell} \mid S_{\ell}) \leq \frac{16}{\varepsilon^2} b. \]

A martingale version of Bennett’s inequality introduced in [20, Theorem 1.6] ensures that, for \( c > 0 \),

\[ \mathbb{P}_{\sigma,x}(\exists u \geq \bar{\ell} \text{ s.t. } W_u \geq c) \leq e^{c\left(\frac{b'}{c + b'}\right)^{c+b'}}. \]

In particular,

\[ \mathbb{P}_{\sigma,x}(M_{\kappa,\eta} - M_{\ell} \geq \varepsilon) = \mathbb{P}_{\sigma,x}(\frac{\varepsilon}{4} W_{\kappa,\eta} + a \geq \varepsilon) \leq \mathbb{P}_{\sigma,x}(\frac{\varepsilon}{4} W_{\kappa,\eta} \geq \frac{\varepsilon}{2}) = \mathbb{P}_{\sigma,x}(W_{\kappa,\eta} \geq 2) = o(n^{-1}). \]

We are left to show that for every \( \varepsilon > 0 \),

\[ \mathbb{P}_{\sigma,x}(M_{\ell} \leq \varepsilon) = 1 - o(n^{-1}). \]

The number of non-open edges in the first \( \bar{\ell} \) steps in the process from Definition 18 is stochastically dominated by a binomial with parameters \( \ell \) and

\[ p = \Delta \bar{\ell} / (m - \bar{\ell}) = O(n^{-1} \log n). \]

Therefore, the probability of having 2 or more edges that are not open in the first \( \bar{\ell} \) steps is \( o(n^{-1}) \). Combined with the fact that for every \( \ell \geq 0 \), \( \hat{w}(e_{\ell}) \leq 2^{-s} \), this implies

\[ \mathbb{P}_{\sigma,x}(M_{\ell} \geq 2^{-s+1}) = o(n^{-1}), \]

which is enough to derive the desired conclusions. \( \square \)

The proof of (4.23) is achieved by collecting the results obtained so far.

**Proposition 22.** For every \( \varepsilon > 0 \):

\[ \lim_{n \to \infty} \mathbb{P} \left( \min_{x \in S_n^\sigma} Q_{x,y}^\sigma \left( (X_0, \ldots, X_t) \in \bigcup_{y \in [n]} N_{x,y} \right) > 1 - \varepsilon \right) = 1. \]

**Proof.** We check the conditions in Definition 20 one by one:

1. follows from Lemma 15;
2. this is the content of [10, Proposition 13] and [9, Proposition 10];
We rewrite $Z$ for every concentration inequality. Following lemma, which is based on the constructions in Definitions 17 and 18 and on a boundary of the in-neighborhood of $24$.

Proof. Fix $\sigma, \eta$ of $W$ edges of $y$ of field generated by this construction. Clearly, in the construction of the in-neighborhood constructed in the usual breadth first way, see e.g. [11, Section 3.3]. Let $S$ denote the $\sigma$-field generated by this construction. Clearly, in the construction of the in-neighborhood of $y$ we cannot reveal more than $\Delta_h = o(n)$ edges. Therefore, by Lemma 19 at most $o(n)$ edges of $\eta$ have been revealed up to this point. Let $\mathcal{E}_y$ denote the tails of the leaves of $\mathcal{W}^{x,(\eta)}_s(r)$ at distance $r$ from $L^{x,\sigma}_s$ and call $\mathcal{F}_y$ the set of heads of the vertices $v$ in the boundary of the in-neighborhood of $y$ such that there is a unique path of length at most $h - 1$ to $y$ in the configuration $\eta$. Both $\mathcal{E}_y$ and $\mathcal{F}_y$ are $\mathcal{S}$-measurable, and

$$\mathbb{E}[1_{\eta(e) = f} | S] = \frac{1}{m}(1 + o(1)), \quad (4.26)$$

for any $e \in \mathcal{E}_y$, $f \in \mathcal{F}_y$. Associate to each head $f \in \mathcal{F}_y$ the weight

$$\hat{w}'(f) = P_{\eta}^{h-1}(v(f), y).$$

At this point we notice that by definition of nice paths,

$$\hat{Q}^t_s(x, y) = \sum_{e \in \mathcal{E}_y} \hat{w}_{\sigma, \eta}(e) \sum_{f \in \mathcal{F}_y} \hat{w}'(f) 1_{\hat{w}(e) \hat{w}'(f) \leq e^{-(1-v/2)Ht}} 1_{\eta(e) = f}.$$

We remark that

$$\frac{1}{m} \sum_{f \in \mathcal{F}_y} \hat{w}'(f) \leq \mu_{\eta} P_{\eta}^{h}(y), \quad \sum_{e \in \mathcal{E}_y} \hat{w}_{\sigma, \eta}(e) \leq 1. \quad (4.27)$$

Since a matching $\eta(e) = f$ of $e \in \mathcal{E}_y$ and $f \in \mathcal{F}_y$ can only occur after the generation of $\sigma, G^{x,(\eta)}_s(r), \mathcal{F}_y$, (4.26) and (4.27) show that

$$\mathbb{E}[\hat{Q}^t_s(x, y) | S] \leq \mu_{\eta} P_{\eta}^{h}(y).$$

We rewrite $Z := \hat{Q}^t_s(x, y) = \sum_{e \in \mathcal{E}_y} c(e, \eta(e))$, where

$$c(e, f) = \hat{w}_{\sigma, \eta}(e) \hat{w}'(f) 1_{\hat{w}(e) \hat{w}'(f) \leq e^{-(1-v/2)Ht}}.$$

Here we observe that we can take $v$ such that $(1 - v/2)(1 + v) \geq 1 + v/3$ and therefore

$$\|c\|_{\infty} \leq \max_{e,f} c(e, f) \leq e^{-(1-v/2)Ht} \leq n^{-1-v/3}. \quad (4.28)$$

We are now left with showing the validity of (4.21). Such a result is achieved by the following lemma, which is based on the constructions in Definitions 17 and 18 and on a concentration inequality.

Lemma 23. For every $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}(\forall x, y \in [n], \hat{Q}^t_s(x, y) \leq (1 + \varepsilon) \mu_{\eta} P_{\eta}^{h}(y) + \varepsilon / n) = 1.$$

Proof. Fix $x, y \in [n]$. Generate, in this order, the configuration $\sigma$, the graph $G^{x,(\eta)}_s(r)$ as in Definition 18, and the in-neighborhood of $y$ up to distance $h - 1$. The latter can be constructed in the usual breadth first way, see e.g. [11, Section 3.3]. Let $\mathcal{S}$ denote the $\sigma$-field generated by this construction. Clearly, in the construction of the in-neighborhood of $y$ we cannot reveal more than $\Delta_h = o(n)$ edges. Therefore, by Lemma 19 at most $o(n)$ edges of $\eta$ have been revealed up to this point. Let $\mathcal{E}_y$ denote the tails of the leaves of $\mathcal{W}^{x,(\eta)}_s(r)$ at distance $r$ from $L^{x,\sigma}_s$ and call $\mathcal{F}_y$ the set of heads of the vertices $v$ in the boundary of the in-neighborhood of $y$ such that there is a unique path of length at most $h - 1$ to $y$ in the configuration $\eta$. Both $\mathcal{E}_y$ and $\mathcal{F}_y$ are $\mathcal{S}$-measurable, and

$$\mathbb{E}[1_{\eta(e) = f} | S] = \frac{1}{m}(1 + o(1)), \quad (4.26)$$

for any $e \in \mathcal{E}_y$, $f \in \mathcal{F}_y$. Associate to each head $f \in \mathcal{F}_y$ the weight

$$\hat{w}'(f) = P_{\eta}^{h-1}(v(f), y).$$

At this point we notice that by definition of nice paths,

$$\hat{Q}^t_s(x, y) = \sum_{e \in \mathcal{E}_y} \hat{w}_{\sigma, \eta}(e) \sum_{f \in \mathcal{F}_y} \hat{w}'(f) 1_{\hat{w}(e) \hat{w}'(f) \leq e^{-(1-v/2)Ht}} 1_{\eta(e) = f}.$$

We remark that

$$\frac{1}{m} \sum_{f \in \mathcal{F}_y} \hat{w}'(f) \leq \mu_{\eta} P_{\eta}^{h}(y), \quad \sum_{e \in \mathcal{E}_y} \hat{w}_{\sigma, \eta}(e) \leq 1. \quad (4.27)$$

Since a matching $\eta(e) = f$ of $e \in \mathcal{E}_y$ and $f \in \mathcal{F}_y$ can only occur after the generation of $\sigma, G^{x,(\eta)}_s(r), \mathcal{F}_y$, (4.26) and (4.27) show that

$$\mathbb{E}[\hat{Q}^t_s(x, y) | S] \leq \mu_{\eta} P_{\eta}^{h}(y).$$

We rewrite $Z := \hat{Q}^t_s(x, y) = \sum_{e \in \mathcal{E}_y} c(e, \eta(e))$, where

$$c(e, f) = \hat{w}_{\sigma, \eta}(e) \hat{w}'(f) 1_{\hat{w}(e) \hat{w}'(f) \leq e^{-(1-v/2)Ht}}.$$

Here we observe that we can take $v$ such that $(1 - v/2)(1 + v) \geq 1 + v/3$ and therefore

$$\|c\|_{\infty} \leq \max_{e,f} c(e, f) \leq e^{-(1-v/2)Ht} \leq n^{-1-v/3}. \quad (4.28)$$
We can now invoke the concentration inequality (see [13, Proposition 1.1] and [9, Section 6.2])

\[ \mathbb{P}(Z - \mathbb{E}[Z|S] \geq a | S) \leq \exp \left( -\frac{a^2}{2\|c\|_\infty (2\mathbb{E}[Z|S] + a)} \right). \]

Choosing \( a := \frac{\varepsilon}{2} \mathbb{E}[Z|S] + \frac{\varepsilon}{2n} \), (4.28) shows that this probability is bounded by \( o(n^{-2}) \) for every fixed choice of \( x, y \). Taking a union bound we conclude the desired result. \( \square \)

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# Dipartimento di Matematica e Fisica, Università di Roma Tre, Largo S. Leonardo Murti-aldo 1, 00146 Roma, Italy.

*E-mail address:* caputo@mat.uniroma3.it

3 Dipartimento di Matematica e Fisica, Università di Roma Tre, Largo S. Leonardo Murti-aldo 1, 00146 Roma, Italy.

*E-mail address:* matteo.quattropani@uniroma3.it