Landau Level Mixing and Levitation of Extended States in Two Dimensions

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We study the effects of mixing of different Landau levels on the energies of one-body states, in the presence of a strong uniform magnetic field and a random potential in two dimensions. We use a perturbative approach and develop a systematic expansion in both the strength and smoothness of the random potential. We find the energies of the extended states shift upward, and the amount of levitation is proportional to \((n + 1/2)/B^3\) for strong magnetic field, where \(B\) is the magnetic field strength and \(n\) is the Landau level index.

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The behavior of extended electronic states of non-interacting electrons in a uniform magnetic field \(B\) with a random substrate potential \(V(\mathbf{r})\), is of central importance to the understanding of the integer quantum Hall effect \([1]\). In this Letter, we report a new and rather simple calculation that exposes the microscopic origin of the so-called “levitation” of extended states \([2,3]\) in the large-\(B\) limit, which has been the subject of recent interest.

On the one hand, it is now widely accepted that, in the limit \(B = 0\), there are no extended (delocalized) single-electron states at any finite energy \([1]\). While in the strong field limit, there exist discrete energies near the center of each disorder-broadened Landau level, at which states are extended. An appealing (but heuristic) scenario, known as the “levitation” of extended states, has been proposed to explain how the interpolation between these limiting behaviors might occur \([2,3]\). This holds that one-electron states are localized at all energies except at a discrete set \(E_n(B) = (n + 1/2)\hbar\omega_c + \epsilon_n(B)\), \(n \geq 0\), where \(\omega_c = |eB|/m\). The energies \(\epsilon_n(B) \to \epsilon_c\), a constant \([1]\), as \(|B| \to \infty\), and increase monotonically as \(|B|\) decreases, in such a way that \(E_n(B) < E_{n+1}(B) < \infty\) for \(|B| > 0\), finally diverging as \(B \to 0\).

As the Fermi level is raised through \(E_c(B)\), the \(T = 0\) Hall conductance jumps from \(ne^2/h\) to \((n + 1)e^2/h\). As the Fermi energy approaches the critical energy from above or below, the localization length of single-electron states at the Fermi level diverges, and a zero-temperature quantum critical point is approached. This has been widely studied in numerical simulations \([3,4]\). Using the notion of a law of corresponding states related by a Chern-Simons effective field theory, this picture has been suitably reinterpreted and extended to interacting electrons and the fractional quantum Hall effect \([10]\).

While the levitation scenario is appealing, it has apparently not yet been derived from microscopic considerations, and recently there has been considerable interest in testing it experimentally and numerically \([11,20]\), and in identifying its microscopic origin. The effect must be associated with Landau-level mixing, which gives rise to a level-repulsion effect, which would tend to lower rather than raise the energy levels. (This is clear for the case \(n = 0\), but is generally true, as the level repulsion due to mixing with higher Landau levels is always stronger than that from lower ones.)

In this Letter, we resolve this paradox, and provide what appears to be a rather simple derivation of the initial appearance of levitation associated with Landau-level mixing at large but finite fields, namely the \(O(B^{-3})\) levitation

\[
\epsilon_n(B) = \epsilon_c + (n + 1/2)\hbar\omega_c \left( \frac{(\ell/\xi)^2}{\omega_c \tau} \right)^2 + O(B^{-4}),
\]

where \(\ell = \sqrt{\hbar/|eB|}\) is the “magnetic length” (the classical radius of the ground-state cyclotron orbit). Here \(\hbar/\tau\) is the energy scale of Landau-level broadening in the high-field limit (essentially the variance of the fluctuations of \(V(\mathbf{r})\)), and \(\xi\) is a characteristic length scale over which the potential varies by this amount. This result is derived in the limit \(\ell/\xi << 1\) and \(\omega_c \tau >> 1\), which is always achieved at sufficiently high magnetic fields provided the potential is bounded, local, and smoothly-varying.

In the large-\(B\) limit, when Landau quantization becomes exact, the dynamics of cyclotron and “guiding center” motion of electrons decouple, and the latter can be treated semiclassically: the electrons move adiabatically along equipotentials of the potential \(V(\mathbf{r})\), with the local drift-velocity, \(\mathbf{v}_d = \hat{z} \times \nabla V(\mathbf{r})/eB\), where \(\hat{z}\) is the direction of the magnetic field normal to the two-dimensional surface.

If the topology of the region \(V(\mathbf{r}) < \epsilon\) is considered as a function of \(\epsilon\), Trugman \([21]\) noted that there is a classical percolation transition between a picture of disconnected “lakes” in a dry “continent” for \(\epsilon < \epsilon_c\), and a picture of a continuous “ocean” with isolated dry “islands” for \(\epsilon > \epsilon_c\). Thus as the Fermi energy is raised, there is a transition in which the regions where the Landau level is locally filled join up to form a continuously-connected region, giving rise to a quantum Hall effect.

Corrections to this semiclassical behavior will occur...
when the equipotential line on which a particle is moving comes close to a saddle-point of \( V(\mathbf{r}) \), and tunneling to a nearby equipotential line at the same energy can occur \(^{22}\). This breakdown is believed to control the quantum critical behavior when \( \epsilon \) is close to \( \epsilon_c \). The Chalker-Coddington “Network model” \(^{1} \) attempts to model this by replacing the Hamiltonian by an effective model, representing it as a network of saddle-points with energies close to \( \epsilon_c \), connected by essentially inert directional leads (the equipotential lines). A random scattering matrix describes the transition amplitudes between the two incoming and two outgoing leads at each saddle-point.

The extended states are thus clearly identified with saddle-point energies. In the limit of the strong-field limit of a finite system, a single critical saddle-point will control the energy at which transmission across the system first occurs. This picture allows the identification

\[
\epsilon_c = V(\mathbf{r}_c),
\tag{2}
\]

where \( \mathbf{r}_c \) is the location of the critical saddle-point. If the thermodynamic limit is first taken, one may anticipate that the distribution of saddle-point energies is singular at \( \epsilon_c \), and there will be many critical saddle-points satisfying (2). In either case, the energy of delocalized states in the strong-field limit must be associated with saddle-point energies. Our result follows from a systematic analysis of mixing between Landau levels, computed perturbatively in the strong-field limit as before, but now replacing the leading effect of mixing is accounted for by working in a network of saddle-points with energies close to \( \epsilon_c \), connected by essentially inert directional leads (the equipotential lines). A random scattering matrix describes the transition amplitudes between the two incoming and two outgoing leads at each saddle-point.

The next order correction is given by

\[
V^{(n)}_{3}(\mathbf{r}) = \frac{3}{\pi}(n + \frac{1}{2}) \left( \frac{\ell^4}{\hbar \omega_c} \right) u(\mathbf{r}),
\tag{5}
\]

where

\[
u(\mathbf{r}) = (\nabla^2 V(\mathbf{r}))^2 - \det |\nabla_i \nabla_j V(\mathbf{r})|,
\]

\[\equiv (\nabla_x^2 V - \nabla_y^2 V)^2 + (2\nabla_x \nabla_y V)^2 \geq 0. \tag{6}\]

At a saddle-point \( u(\mathbf{r}_c) > 0 \), as the determinant of second derivatives is negative. (Note that \( u(\mathbf{r}) \) only vanishes if the matrix of second derivatives is rotationally-invariant, which is not true at a saddle-point). Thus, in contrast to a generic point where the corrections to the effective potential due to Landau-level mixing are negative, the leading correction at saddle-points, which control the energies of extended states, is positive, giving rise to the levitation effect. The result \(^{11}\) follows from an estimate of \( u(\mathbf{r}_c) \) as being of order (\( \hbar / \tau \ell^2 \)). Our results are schematically illustrated in Fig. \( \text{(1)} \).

We now sketch the technical derivation of (3)-(6). We write the substrate potential \( V(\mathbf{r}) \) in terms of its Fourier components \( \tilde{V}(\mathbf{q}) \)

\[
V(\mathbf{r}) = \frac{1}{A} \sum_{\mathbf{q}} \tilde{V}(\mathbf{q}) e^{i \mathbf{q} \cdot \mathbf{r}}
\tag{7}
\]

where for convenience we have imposed (quasi-)periodic boundary conditions on an area \( A \) that contains an integral number of magnetic flux quanta. We now write

\[
e^{i \mathbf{q} \cdot \mathbf{r}} = e^{i \mathbf{q} \cdot \mathbf{R}} U(\mathbf{q}), \quad U(\mathbf{q}) = e^{i \mathbf{q} \cdot (\mathbf{r} - \mathbf{R})}, \tag{8}\]

where \( \mathbf{R} \) is the “guiding center” of the cyclotron orbit \(^{23}\), which obeys the algebra \(^{24,23}\)

\[
e^{i \mathbf{q} \cdot \mathbf{R}} e^{i \mathbf{q} \cdot \mathbf{R}'} = \exp(\frac{i}{2} (\mathbf{q} \times \mathbf{q'})^2) e^{i (\mathbf{q} + \mathbf{q'}) \cdot \mathbf{R}}. \tag{9}\]

(Here \( \mathbf{q} \times \mathbf{q'} \equiv q_x q_y' - q_y q_x' \).) The unitary operator \( U(\mathbf{q}) \) acts entirely on the cyclotron orbit (Landau level) variables, and commutes with the guiding center. In the strong-field limit, the potential term projected into the Landau level \( n \) becomes

\[
\frac{1}{A} \sum_{\mathbf{q}} \tilde{V}(\mathbf{q}) e^{i \mathbf{q} \cdot \mathbf{R}} U(\mathbf{q})_{nn}, \tag{10}\]

where \( U(\mathbf{q})_{nn'} \equiv \langle n | U(\mathbf{q}) | n' \rangle \) (\( n \) and \( n' \) are Landau-level indices): for \( n \geq n' \), \( U(\mathbf{q}) \) is given by

\[
\left( \frac{q_x + i q_y}{\sqrt{2}} \right)^{n-n'} L^{n-n'}_{n'} \left( \frac{2 q^2 \ell^2}{4} \right) \exp\left( -\frac{1}{4} q^2 \ell^2 \right), \tag{11}\]

where \( L^{n}_{n'}(x) \) is a Laguerre polynomial.

The problem in the high-field limit is to diagonalize the projected potential (10), in the subspace of a given Landau level. When the field strength is strong but finite, states in different Landau levels are still well separated in
energy. Nevertheless, electrons in a given Landau level may be scattered into other Landau levels by the random potential, and will eventually come back due to energy conservation. The effect of such (virtual) processes is to renormalize the effective potential seen by the electrons in this Landau level [see Fig. (2)], which we calculate below.

The trick we will use to characterize the renormalization is to develop a perturbative expansion in $V/\hbar\omega_c$, and rewrite the effective Hamiltonian in the form (10), but with a renormalized $\hat{V}_{\text{eff}}^{(n)}(\mathbf{r})$, which can then be expanded in powers of $\ell$ as well as in $1/\hbar\omega_c$, to give a true $1/B$ expansion. We then carry out the Fourier transform to find the renormalized $\hat{V}_{\text{eff}}^{(n)}(\mathbf{r})$ that this corresponds to.

Using standard perturbative renormalization formalism (2), we find the leading $O(V^2/\hbar\omega_c)$ term in the effective Hamiltonian is

$$
\frac{1}{A^2} \sum_{\mathbf{q} \mathbf{ q}'} \frac{\hat{V}(\mathbf{q})\hat{V}(\mathbf{q}') e^{i\mathbf{q} \cdot \mathbf{R} - i\mathbf{q}' \cdot \mathbf{R}}}{\hbar\omega_c} \sum_{n'} U_{n'n}(\mathbf{q}) U_{n'n}(\mathbf{q}') (n - n') \tag{12}
$$

The primed sum means that the singular term $n' = n$ is excluded. We must now express this term in the form (10), using the contraction (8). The general $O(V^m/(\hbar\omega_c)^{m-1})$ contribution to $\hat{V}_{\text{eff}}^{(n)}(\mathbf{r})$ may be written (for $m > 1$) in the form

$$
\frac{\hbar\omega_c}{A^m} \sum_{\mathbf{q}_1 \ldots \mathbf{q}_m} \left( \prod_{i=1}^{m} \frac{\hat{V}(\mathbf{q}_i) e^{i\mathbf{q}_i \cdot \mathbf{r}}}{\hbar\omega_c} \right) f_m^{(n)}(\mathbf{q}_1, \ldots, \mathbf{q}_m), \tag{13}
$$

where $f_m^{(n)}(\mathbf{q}_1, \ldots, \mathbf{q}_m)$ is a symmetric and analytic function of the $\{\mathbf{q}_i, \ell\}$ (it is derived from the $U_{n'\mathbf{q}}(\mathbf{q})$, which are analytic). It is also rotationally-invariant, and must vanish as any of the $\mathbf{q}_i \to 0$, as addition of a spatially-constant term ($\mathbf{q} = 0$ Fourier component) to the potential cannot affect the non-linear terms in $\hat{V}_{\text{eff}}^{(n)}(\mathbf{r})$. The term $f_2^{(n)}(\mathbf{q}_1, \mathbf{q}_2)$ is the symmetric part of

$$
\frac{e^{i\mathbf{q}_1 \cdot \mathbf{q}_2 \ell^2/2}}{U_{n'\mathbf{q}}(\mathbf{q}_1 + \mathbf{q}_2)} \sum_{n'} U_{n'\mathbf{q}}(\mathbf{q}_1) U_{n'\mathbf{n}}(\mathbf{q}_2) (n - n'). \tag{14}
$$

It is straightforward to expand $f_2^{(n)}(\mathbf{q}_1, \mathbf{q}_2)$ in powers of $\ell$, using (12); we find that, up to terms of order $\ell^4$, it is given by

$$
\frac{1}{2}(\mathbf{q}_1 \cdot \mathbf{q}_2) \ell^2 + \frac{3}{8}(n + \frac{1}{2})( (\mathbf{q}_1 \cdot \mathbf{q}_2)^2 - (\mathbf{q}_1 \times \mathbf{q}_2)^2) \ell^4. \tag{15}
$$

This corresponds to a gradient expansion of the effective potential in real space, and gives the leading terms of $O(B^{-2})$ and $O(B^{-3})$ in (8).

We find that the leading term in the gradient expansion of the term of order $O(V^3/(\hbar\omega_c)^2)$ is of order $\ell^4$ (this in fact follows directly from the general properties of $f_2^{(n)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ mentioned above). This means that its leading contribution to the effective potential is $O(B^{-4})$, and it does not contribute to the leading terms. Higher-order terms in $V/\hbar\omega_c$ vanish even faster at large $B$.

In the following we discuss the experimental implications of our results, and the relation between these results and existing work on this subject. The results above have interesting implications for attempts to experimentally detect levitation based on the relative motion of the extended state energy and the mean energy of the broadened Landau level (as defined using the density of states) at large $B$.

Our result shows that the leading effect in this limit is an $O(B^{-2})$ downwards motion of the mean energy of the Landau level, while the extended state is static to this order, and only levitates to $O(B^{-3})$. In this limit at least, experimental evidence (2) that the extended state rises relative to the mean energy of the Landau level would be demonstrating not levitation of extended states, but the lowering of localized state energies due to level-repulsion between Landau levels. We also note that evidence of levitation of extended states has been found in previous numerical work, in both the continuum system (11), and the tight binding model (5), although there is controversy in the latter case (12).

Recently Shabazyan and Raikh (5) (see also Ref. (16)) used an extension of the network model (6) to simulate the continuum system in the presence of a smooth random potential. They considered the effects of strongly-localized orbitals of different Landau levels with energies close to the saddle-point energies of a particular Landau level of interest, and find that resonant tunneling into such orbitals results on average in a reduction of the transmission rate through the saddle-points, implying an upward shift of the energy of extended states, which does not depend on the Landau level index $n$. We note that in order for this effect to be important, there must be significant overlap in the density of states (DOS) of different Landau levels; while it is clear from our results that levitation occurs even if there is no overlap in the DOS of different Landau levels (which is the case when $B$ is large).

Later, Gramada and Raikh (20) studied the effects of a short-range impurity potential on the transmission rate through a nearby saddle-point, and again find a reduction of the transmission rate on average. They estimate the upward shift of the extended state energy due to this effect to be of order $B^{-4}$ for large $B$. We believe the $O(1/B^3)$ levitation we identify here is the dominant one, at large $B$.

There are recent observations (13, 26) of apparently-direct transitions from quantum Hall states with large $\nu$ to insulating states at very weak magnetic field, which appears to be inconsistent with the conventional one-electron extended-state-levitation picture and the global
phase diagram [10]. We note at very weak magnetic field electron-electron interactions may become important and the one-body picture may not be sufficient. However, a quantitative validation of the levitation scenario for non-interacting electrons in the $B \to 0$ limit clearly urgently needs to be attempted.

To summarize: we have used a perturbative approach to study the effects of mixing between Landau levels in a two-dimensional non-interacting electron system, due to a random substrate potential. In high magnetic fields, we find that although most of the states in the Landau level with index $n$ are pushed to lower energy by such mixing, the energy of extended states shifts upward, and the amount of this shift is proportional to $(n + 1/2)/B^3$.

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[1] For a review, see B. Huckestein, Rev. Mod. Phys. 67, 357 (1995), and references therein.
[2] D. E. Khmelnitskii, Phys. Lett. A 106, 182 (1984).
[3] R. B. Laughlin, Phys. Rev. Lett. 52, 2304 (1984).
[4] E. Abrahams, P. W. Anderson, D. C. Licciardello and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).
[5] If $V(r)$ has particle-hole symmetry, as often assumed in theoretical studies, we have $\epsilon_c = 0$.
[6] J.T. Chalker and P.D. Coddington, J. Phys. C 21, 2665 (1988).
[7] B. Huckestein and B. Kramer, Phys. Rev. Lett. 64, 1437 (1990).
[8] Y. Huo and R.N. Bhatt, Phys. Rev. Lett. 68, 1375 (1992).
[9] D. Liu and S. Das Sarma, Phys. Rev. B 49, 2677 (1994).
[10] S. Kivelson, D.H. Lee and S.-C. Zhang, Phys. Rev. B 46, 2223 (1992).
[11] T. Ando, J. Phys. Soc. Jpn. 53, 3126 (1984).
[12] I. Glozman, C. E. Johnson and H. W. Jiang, Phys. Rev. Lett. 74, 594 (1995).
[13] S.V. Kravchenko et al., Phys. Rev. Lett. 75, 910 (1995).
[14] J.E. Furneaux et al., Phys. Rev. B 51, 17227 (1995).
[15] T. V. Shahbazyan and M. E. Raikh, Phys. Rev. Lett. 75, 304 (1995).
[16] V. Kagalovsky, B. Horovitz and Y. Avishai, Phys. Rev. B 52, R17044 (1995).
[17] D. Z. Liu, X. C. Xie and Q. Niu, Phys. Rev. Lett. 76, 975 (1996); X. C. Xie and D. Z. Liu, preprint cond-mat/9512015.
[18] Kun Yang and R.N. Bhatt, Phys. Rev. Lett. 76, 1316 (1996).
[19] D. N. Sheng, Z. Y. Weng and Q. Gao, preprint cond-mat/9512008.
[20] A. Gramada and M. E. Raikh, preprint cond-mat/9601023.
[21] S. A. Trugman, Phys. Rev. B 27, 7539 (1983).
[22] H. A. Fertig and B. I. Halperin, Phys. Rev. B 36, 7969 (1987).
[23] F.D.M. Haldane, in The Quantum Hall Effect, edited by R. E. Prange and S. M. Girvin (Springer, New York, 1990).
[24] S. M. Girvin and T. Jach, Phys. Rev. B 29, 5617 (1984); S. M. Girvin, A. H. MacDonald and P. M. Platzman, Phys. Rev. B 33, 2481 (1986).
[25] See, for example, P. W. Anderson, J. Phys. C 3, 2436 (1970).
[26] S. Song et al., unpublished.

FIG. 1. Density of states and energy of extended states in a given Landau level before (dashed lines) and after (solid lines) Landau level mixing is taken into account.

FIG. 2. Schematic perturbative expansion of the effective potential seen by electrons in the $n$th Landau level.
Haldane, Yang PRL Fig. 1

\[ E \]

\[ \varepsilon_c \]
Haldane, Yang PRL Fig. 2

\[ V_n^{\text{eff}} = V + \ldots \]