Characterizing nonclassical correlations of tensorizing states in a bilocal scenario

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Abstract
In the present paper, we attempt to address the question of “can tensorizing states \((\rho \otimes \rho \text{ or } \rho \otimes \rho')\) have quantum advantages?”. To answer this question, we exploit the notion of measurement-induced nonlocality (MIN) and advocate a fidelity-based nonbilocal measure to capture the nonlocal effects of tensorizing states due to locally invariant von Neumann projective measurements. We show that the properties of the fidelity-based nonbilocal measures are retrieved from that of MIN. Analytically, we evaluate the nonbilocal measure for any arbitrary pure state. The upper bounds of the nonbilocal measure based on fidelity are also obtained in terms of eigenvalues of correlation matrix. As an illustration, we have computed the nonbilocality for some popular input states.

Keywords Entanglement · Quantum correlation · Projective measurements · Nonlocality

1 Introduction

The existence of nonlocal attributes in composite states continues to be a fundamental and unique feature of quantum systems which have no counterparts in the classical domain [1]. In fact, it has fueled the development of quantum technologies. A few
notable nonlocal attributes are coherence [2, 3], entanglement [4–6], steering and quantum correlations beyond entanglement [7–10]. Ever since the identification of EPR paradox [4], it is believed that the entanglement is the only manifestation of nonlocality of the quantum system. Recent studies have reinforced the fact that entanglement alone cannot capture the entire spectrum of the nonlocality and one will have to go beyond entanglement to get an estimate of the entire spectrum. The nonlocality of pure states is completely characterized by the violation of Bell inequality and entanglement [11]. On the other hand, the nonlocal aspects of mixed states are not understood and are still found to be mysterious entities in quantum information theory [11, 12]. Despite the ongoing debate on the manifestation of nonlocality, the quantification of entanglement and quantum correlations beyond entanglement besides their characterization continue to capture the attention of researchers in the domain of quantum information processing.

In a recent quantum entanglement swapping experiment with multi-measurements and multi-sources, it is shown that the independence of the sources can cause the nonlocal behaviors of probability distributions and is called nonbilocal correlations [13, 14]. These kinds of correlations are captured using nonlinear inequalities and one important class of these inequalities is the so-called binary-input-and-output bilocality inequality which is called the bilocality inequality [13, 14]. In recent times, some interesting progress has been made in this direction [15–23]. Gisin et al. [24] have shown that a pair of entangled states can violate the bilocality inequality implying that tensorizing states may possess nonlocal correlations. There is a curiosity in understanding the nonlocal behavior of quantum systems when two bipartite states with vanishing correlations are combined and whether the tensorizing state possesses nonlocal advantages or not.

The phenomena of combining two quantum systems show better quantum advantages than the individual counterparts. This is known as superactivation of nonlocality, symbolized as $0 + 0 > 0$ and it cannot occur in a purely classical world. The superactivation of nonlocality provides an answer for “can the state $\rho \otimes \rho$ be nonlocal if $\rho$ is local.” Recently, the superactivation of quantum nonlocality in the context of violating certain Bell inequalities with an entangled bound state has been demonstrated [25]. The same study has been carried out in the context of tensor networks as well [26, 27]. Further, the superactivation was also considered for arbitrary entangled states by allowing local preprocessing on the tensor product of different quantum states $(\rho \otimes \rho')$ symbolized as $1 + 0 > 1$ [28]. Since the nonlocality inequalities can detect only the nonlocal aspects in quantum systems, there is a dire necessity to look for an analytical method to quantify this quantum correlation in a bilocal scenario.

In order to characterize the nonclassical correlation of bilocal states, we exploit the property of bipartite measurement-induced nonlocality (MIN) and define a nonbilocal measure using fidelity between the states. The relation between the nonlocal and nonbilocal correlation measures is established and it is shown that nonbilocality is always greater than the nonlocal correlation. Further, the upper bounds of the nonbilocal correlation measure are also obtained for arbitrary mixed input states. To validate the properties of nonbilocal measures, we study the proposed quantity for different input states.
The present paper is structured as follows: To start with in Sect. 2, we review the concept of measurement-induced nonlocality and definition of fidelity-based MIN. In Sect. 3, we introduce fidelity-based nonbilocal measure and establish a relationship with the nonlocal measure. Section 4 quantifies the nonbilocality of arbitrary mixed states. In Sect. 5, we compute the proposed nonbilocal measure for some well-known input states and compare with the fidelity-based MIN. Finally, the conclusions are presented in Sect. 6.

2 Measurement-induced nonlocality

Measurement-induced nonlocality (MIN) is manifested in the nonlocal effects due to locally invariant eigenprojective measurements and is a faithful measure of bipartite quantum correlations. It is originally defined as the maximal Hilbert–Schmidt distance pre- and post-measurements states and is defined as [10]

$$N(\rho) = \max_{\Pi^a} \| \rho - \Pi^a(\rho) \|^2,$$

where the maximization is taken over the locally invariant projective measurements on subsystem $a$, $\Pi^a(\rho) = \sum_k (\Pi^a_k \otimes 1^b) \rho (\Pi^a_k \otimes 1^b)$ with $\Pi^a = \{ \Pi^a_k \} = \{|k\rangle \langle k|\}$ being the projective measurements on the subsystem $a$ which does not change the marginal state $\rho^a$ locally, i.e., $\Pi^a(\rho^a) = \rho^a$. Here $\|O\| = \sqrt{\text{Tr}O^T O}$ is the Hilbert–Schmidt norm of operator $O$. The MIN is in some sense dual to the geometric version of quantum discord (GD) of the given state $\rho$ and is formulated as [8, 9]

$$D(\rho) = \min_{\Pi^a} \| \rho - \Pi^a(\rho) \|^2.$$

If $\rho^a$ is nondegenerate, the optimization is not required and the above measures are equal. The Hilbert–Schmidt distance (HS) is an important quantity and its operational meaning is the informational distance between quantum states [29]. Recently, it is shown that the HS norm is easy to compute and measurable experimentally using many-particle interference techniques [30]. Further, the Hilbert–Schmidt distance has been widely used in quantum mechanics, particularly in quantifying quantum resources, such as quantum entanglement [31], quantum discord [8, 9], measurement-induced nonlocality [10, 32] and asymmetry [33]. The relations between the Hilbert–Schmidt distance and the trace distance have been studied [34]. Nevertheless, Hilbert–Schmidt distance is not a bonafide measure of quantum correlation which is shown by considering a simple map $\Gamma^\sigma : X \rightarrow X \otimes \sigma$, i.e., the map adding a noisy ancillary state to the party $b$ [35]. Under such an operation, we have

$$\|X\| \rightarrow \|\Gamma^\sigma X\| = \|X\| \sqrt{\text{Tr}\sigma^2}.$$

Due to the addition of local ancilla $\rho^c$, the MIN of resultant state is

$$N(\rho^{a:bc}) = N(\rho^{ab})\text{Tr}(\rho^c)^2.$$
implying that MIN differs arbitrarily due to local ancilla as long as $\rho^c$ is mixed. Defining the MIN in terms of any one of the contractive distance measures seems to be a natural way of rectifying the local ancilla problem. One such form of MIN based on the fidelity is given by [36]

$$N_F(\rho) = 1 - \min_{\Pi^a} F(\rho, \Pi^a(\rho)).$$  (4)

Here, the minimum is taken over the locally invariant projective measurements on subsystem $a$ and $F(\rho, \sigma) = \text{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$ is the fidelity between the states $\rho$ and $\sigma$ [37]. This measure has been explored in different contexts of quantum information processing such as cloning [38], teleportation [39], quantum chaos [40] and phase transition in physical systems [41].

The above definition of fidelity involves the square root of density operator. Hence, the computation of fidelity in higher dimensions is quite intractable. To reduce the computational complexity, we follow another version of fidelity [42]:

$$F(\rho, \sigma) = \frac{\text{Tr}(\rho\sigma)^2}{\text{Tr}(\rho^2)\text{Tr}(\sigma^2)}.$$  (5)

This measure also possesses all the properties of fidelity introduced by Josza [37] and is useful in defining MIN [43]. Based on the fidelity, MIN is defined as [43, 44]

$$N_F(\rho) = 1 - \min_{\Pi^a} F(\rho, \Pi^a(\rho)).$$  (6)

It is worth pointing out at this juncture that the fidelity-based MIN fixes the local ancilla problem of Hilbert–Schmidt MIN and satisfies all the necessary axioms of a bonafide measure of quantum correlations.

### 3 Nonbilocality measure

Before introducing the nonbilocality measure, we first review the notion of bilocal scenario. The entanglement swapping is a typical example of this scenario. Consider two separate sources $S_1$ and $S_2$ distributing the physical systems to the distant observers as shown in Fig. 1. A source $S_1$ ($S_2$) distributes the system to observer Alice (Bob) and Bob (Charles). The measurements on all the three parties Alice, Bob and Charles are labeled as $x$, $y$ and $z$ and their corresponding outcomes $a$, $b$ and $c$, respectively. In particular, Bob has two particles and may do the joint measurements. In general, the tripartite joint probability distribution is written as

$$p(a, b, c|x, y, z) = \int \int d\lambda_1 d\lambda_2 q_1(\lambda_1)q_2(\lambda_2)p(a|x, \lambda_1)p(b|y, \lambda_1, \lambda_2)p(c|z, \lambda_2)$$  (7)
where $\lambda_1$ and $\lambda_2$ are the independent shared random variables according to the densities $q_1(\lambda_1)$ and $q_2(\lambda_2)$, respectively. The above equation holds good if only if $p(a, b, c|x, y, z)$ is local.

Assume, Alice and Bob have the binary inputs $x = 0, 1$ and $z = 0, 1$ giving the binary outputs $a_x = \pm 1$ and $c_z = \pm 1$. Then, Bob has four possible outcomes, i.e., $b_0 = \pm 1$ and $b_1 = \pm 1$. Then, the bilocality inequality can be written as

$$S = \sqrt{|I|} + \sqrt{|J|} \leq 2$$  

where

$$I = \langle (a_0 + a_1)b_0(c_0 + c_1) \rangle, \quad J = \langle (a_0 - a_1)b_1(c_0 - c_1) \rangle$$

and the bracket notation $\langle \cdot \rangle$ stands for the expectation value of many experimental runs. Further, Gisin et al. [24] have brought out the closer connection between the bilocality and CHSH inequality as

$$\rho \text{ violates CHSH} \rightarrow \rho \otimes \rho \text{ violates the bilocality inequality.}$$

Next, we introduce the fidelity-based nonbilocal measure. Let us consider a bilocal quantum state (Fig. 2) shared by four parties in the separable composite finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \otimes \mathcal{H}_d$ with bipartition $\rho_{ab}$ (shared between $a$ and $b$) and $\rho_{cd}$ (shared between $c$ and $d$). We define the nonbilocal measure in terms of fidelity as

$$N_F(\rho_{ab} \otimes \rho_{cd}) = 1 - \min_{\Pi_{bc}} F(\rho_{ab} \otimes \rho_{cd}, \Pi_{bc}(\rho_{ab} \otimes \rho_{cd})),$$

where the optimization is taken over the locally invariant eigenprojective measurements $\Pi_{bc} = \{\Pi_{k}^{bc}\}$ which does not alter the marginal state $\rho_{bc} = \text{Tr}_{ad}(\rho_{ab} \otimes \rho_{cd})$ locally, $d_F(\cdot, \cdot)$ quantifies the distance between the state and its post-measurement state and is given by $\Pi_{bc}(\sqrt{\rho_{ab} \otimes \rho_{cd}}) = \sum_{k,l} (1^a \otimes \Pi_{kl}^{bc} \otimes 1^d) \sqrt{\rho_{ab} \otimes \rho_{cd}} (1^a \otimes \Pi_{kl}^{bc} \otimes 1^d)$ with $1^{a(d)}$ being a $2 \times 2$ unit matrix acting on $a(d)$. Here, $\rho^b = \sum_i \lambda_i |i_b\rangle\langle i_b|$ and $\rho^c = \sum_j \lambda_j |j_c\rangle\langle j_c|$ are the marginal states of $\rho_{bc}$. If any one of the states is nondegenerate, the measurement takes the form $\Pi_{bc} = \{\Pi^b \otimes \Pi^c\}$. 
Using the orthogonality of projectors and cyclic property of trace of matrices, we can show that \( \text{Tr}(\Pi^{bc}(\rho_{ab} \otimes \rho_{cd}))^2 = \text{Tr}(\rho_{ab} \otimes \rho_{cd} \Pi^{bc}(\rho_{ab} \otimes \rho_{cd})) \). Hence, the definition of nonbilocal measure can be recast as

\[
N_F(\rho_{ab} \otimes \rho_{cd}) = 1 - \min_\Pi^{bc} \frac{\text{Tr}(\rho_{ab} \otimes \rho_{cd} \Pi^{bc}(\rho_{ab} \otimes \rho_{cd}))}{\text{Tr}(\rho_{ab} \otimes \rho_{cd})^2}. \tag{12}
\]

The above measure quantifies the quantum correlation of any bilocal state from the perspectives of eigenprojective measurements. It is worth mentioning at this juncture that MIN is a crucial resource for bipartite quantum communication. In view of this, the nonbilocal measure is also helpful for multipartite communication protocol. In general, the bilocal states are useful in typical entanglement swapping and the fidelity-based nonbilocal measure involves swapping of quantum resources. Recently, the nonbilocal correlation measures have been established using Hellinger distance [45] and affinity [46]. Both the measures are quite hard to compute in higher dimensions due to the presence of square root of density matrix. However, the nonbilocal measure given by Eq. (12) based on fidelity is easy to compute and enjoys a variety of nice properties:

(i) \( N_F(\rho_{ab} \otimes \rho_{cd}) \geq 0 \) and the equality holds for any product input states defined by \( \rho_{ab} = \rho^a \otimes \rho^b \) and \( \rho_{cd} = \rho^c \otimes \rho^d \). Also, the nonbilocal measure vanishes for classical-quantum state \( \rho_{ab} = \sum_i \rho^a_i \otimes p_i |i_b\rangle \langle i_b| \) and \( \rho_{cd} = \sum_j \rho^c_j \otimes |j_c\rangle \langle j_c| \).

(ii) \( N_F(\rho_{ab} \otimes \rho_{cd}) \) is locally unitary invariant in the sense that

\[
N_F((U_{ab} \otimes U_{cd})\rho_{ab} \otimes \rho_{cd}(U_{ab} \otimes U_{cd})^\dagger) = N_F(\rho_{ab} \otimes \rho_{cd}), \tag{13}
\]

where \( U_{ab} = U_a \otimes U_b \) and \( U_{cd} = U_c \otimes U_d \) are the local unitary operators.
(iii) If $N_F(\rho_{ab} \otimes \rho_{cd})$ is positive, then, it implies that at least any one of the input states is entangled.

(iv) If $\rho^b$ and $\rho^c$ are both nondegenerate, then

$$N_F(\rho_{ab} \otimes \rho_{cd}) = 1 - F(\rho_{ab} \otimes \rho_{cd}, \Pi_{bc}(\rho_{ab} \otimes \rho_{cd})).$$

(v) For any pure input state, the fidelity-based nonbilocal measure is given by

$$N_F(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = 1 - \sum_{i,j} s_i^4 r_j^4$$

where $s_i$ and $r_j$ are the Schmidt coefficients of $|\Psi_{ab}\rangle$ and $|\Psi_{cd}\rangle$, respectively.

(vi) For any arbitrary pure state, the fidelity-based MIN and nonbilocal measures are related as

$$N_F(\rho_{ba} \otimes \rho_{ab}) \geq N_{\text{MIN}}(\rho).$$

(vii) Although $N_F(\rho_{ab}) = N_F(\rho_{cd}) = 0$, $N_F(\rho_{ab} \otimes \rho_{cd}) > 0$.

The properties (i)–(iv) can be easily proved. Hence, we concentrate on the detailed proof of the remaining properties.

To validate property (v), we employ the Schmidt decomposition of pure input states. Let $|\Psi_{ab}\rangle$ and $|\Psi_{cd}\rangle$ be the pure input states with the following Schmidt decomposition

$$|\Psi_{ab}\rangle = \sum_{i} s_i |i_a i_b\rangle$$

and

$$|\Psi_{cd}\rangle = \sum_{j} r_j |j_c j_d\rangle$$

and $s_i$ and $r_j$ are the respective Schmidt coefficients of input states. Further, $|i_a (b)\rangle$ and $|j_c (d)\rangle$ are the orthonormal bases of the subsystems $a(b)$ and $c(d)$, respectively.

We note that

$$\rho_{ab} \otimes \rho_{cd} = |\Psi_{ab}\rangle \langle \Psi_{ab}| \otimes |\Psi_{cd}\rangle \langle \Psi_{cd}|$$

$$= \sum_{i i' j j'} s_i s_i' r_j r_j' |i_a\rangle \langle i_a| \otimes |i_b\rangle \langle i_b| \otimes |j_c\rangle \langle j_c| \otimes |j_d\rangle \langle j_d|. \quad (14)$$

For pure input states $\text{Tr}(\rho_{ab} \otimes \rho_{cd})^2 = 1$, the nonbilocal measure takes the form

$$N_F(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = 1 - \min_{\Pi_{bc}} \text{Tr}(\rho_{ab} \otimes \rho_{cd} \Pi_{bc}(\rho_{ab} \otimes \rho_{cd})). \quad (15)$$

Further, we compute the marginal state as

$$\rho^{bc} = \text{Tr}_{ad}(|\Psi_{ab}\rangle \langle \Psi_{ab}| \otimes |\Psi_{cd}\rangle \langle \Psi_{cd}|) = \sum_{ij} s_i^2 r_j^2 |i_b j_c\rangle \langle i_b j_c| \quad (16)$$

The von Neumann projective measurement is expressed as

$$\Pi_{bc} = \{\Pi_{hk}^{bc} \equiv U |h_b k_c\rangle \langle h_b k_c| U^\dagger \} \quad (17)$$
and the marginal state can be expressed as a spectral decomposition of $\rho^{bc}$ as

$$\rho^{bc} = \sum_{hk} \langle h_b k_c | U^\dagger \rho^{bc} U | h_b k_c \rangle U | h_b k_c \rangle \langle h_b k_c | U^\dagger . \quad (18)$$

We wish to point out that $\{ U | h_b k_c \} \}$ is an orthonormal base with the eigenvalue $s_i^2 r_j^2 = \langle h_b k_c | U^\dagger \rho^{bc} U | h_b k_c \rangle$.

The post-measurement state $\Pi^{bc} (\rho_{ab} \otimes \rho_{cd})$ can be computed as

$$\Pi^{bc} (\sqrt{\rho_{ab} \otimes \rho_{cd}})$$

$$= \Pi^{bc} (\rho_{ab} \otimes \rho_{cd})$$

$$= \sum_{hk} (1^a \otimes \Pi^{bc}_{hk} \otimes 1^d) (\langle \Phi_{ab} | \otimes | \Phi_{cd} \rangle | \Phi_{cd} \rangle) (1^a \otimes \Pi^{BC}_{hk} \otimes 1^d)$$

$$= \sum_{hk} (1^a \otimes \Pi^{bc}_{hk} \otimes 1^d) \left( \sum_{ii'jj'} s_i s_{i'} r_j r_{j'} \langle i_a | \otimes | i_b \rangle \otimes | j_c \rangle \otimes | j_d \rangle \langle j_d' | \right)$$

$$= \sum_{hk} \sum_{ii'jj'} s_i s_{i'} r_j r_{j'} \langle i_a | \otimes U | h_b k_c \rangle \langle h_b k_c | U^\dagger | i_b j_c \rangle \langle i_b j_c | U | h_b k_c \rangle \langle h_b k_c | U^\dagger \otimes | j_d \rangle \langle j_d' | .$$

Consequently, we have

$$\rho_{ab} \otimes \rho_{cd} \Pi^{bc} (\rho_{ab} \otimes \rho_{cd})$$

$$= \left( \sum_{ii'jj'} s_i s_{i'} r_j r_{j'} \langle i_a | \otimes | i_b \rangle \otimes | j_c \rangle \otimes | j_d \rangle \langle j_d' | \right)$$

$$= \sum_{hh} \sum_{uu'vv'} s_u s_{u'} r_v r_{v'} \langle h_b k_c | U^\dagger | u_b v_c \rangle \langle u_b v_c | U | h_b k_c \rangle | u_a \rangle \langle u_a' | \otimes U | h_b k_c \rangle$$

$$= \sum_{ii'jj'} \sum_{hh} \sum_{uu'vv'} s_i s_{i'} r_j r_{j'} s_u s_{u'} r_v r_{v'} \langle e_{BfC} | U^\dagger | u_{BfC} \rangle \langle u_{BfC} | U | h_b k_f \rangle | i_a \rangle \langle i_a' | \otimes | i_b j_c \rangle \langle i_b j_c' | U | h_b k_c \rangle \langle h_b k_c | U^\dagger \otimes | j_d \rangle \langle j_d' | v_d \rangle \langle v_d' | .$$
Then, the fidelity between the pre- and post-measurement states is computed as
\[
\mathcal{F}(\rho_{ab} \otimes \rho_{cd}, \Pi^{bc}(\rho_{ab} \otimes \rho_{cd})) \\
= \text{Tr} \rho_{ab} \otimes \rho_{cd} \Pi^{bc}(\rho_{ab} \otimes \rho_{cd}) \\
= \sum_{iuvhk} s_i^2 s_u^2 r_i^2 r_u^2 \langle h_b k_c | U^\dagger | h_b k_c \rangle \langle h_b k_c | U \rangle \langle h_b k_c | U^\dagger | i_b j_c \rangle \\
= \sum_{hk} (\langle h_b k_c | U^\dagger \rho_{bc} U | h_b k_c \rangle)^2.
\]

The nonbilocal measure for any pure state is given by
\[
N_{\mathcal{F}}(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = \max_{\Pi^{bc}} d_{\mathcal{F}}(\rho_{ab} \otimes \rho_{cd}, \Pi^{bc}(\rho_{ab} \otimes \rho_{cd})) \\
= 1 - \min_{\Pi^{bc}} \mathcal{F}(\rho_{ab} \otimes \rho_{cd}, \Pi^{bc}(\rho_{ab} \otimes \rho_{cd})) \\
= 1 - \min_{\Pi^{bc}} \sum_{hk} (\langle h_b k_c | U^\dagger \rho_{bc} U | h_b k_c \rangle)^2,
\]

where the optimization is taken over all locally invariant eigenprojective measurements given in Eq. (17) leaving the marginal state \(\rho^{bc}\) invariant. Following the spectral decomposition given by Eq. (18), we obtain the property (v)
\[
N_{\mathcal{F}}(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = 1 - \sum_{i,j} s_i^4 r_i^4.
\]
Hence, proved.

To establish the property (vi), we first recall the definition of fidelity-based nonbilocal correlation measure as
\[
N_{\mathcal{F}}(\rho_{ba} \otimes \rho_{ab}) = \max_{\Pi^{aa}} d_{\mathcal{F}}(\rho_{ba} \otimes \rho_{ab}, \Pi^{aa}(\rho_{ba} \otimes \rho_{ab})) \\
= 1 - \min_{\Pi^{aa}} \mathcal{F}(\rho_{ba} \otimes \rho_{ab}, \Pi^{aa}(\rho_{ba} \otimes \rho_{ab})) \\
= 1 - \min_{\Pi^{bc}} \frac{\text{Tr}(\rho_{ab} \otimes \rho_{cd} \Pi^{bc}(\rho_{ab} \otimes \rho_{cd}))}{\text{Tr}(\rho_{ab} \otimes \rho_{cd})^2}.
\]
For pure input states, \(\text{Tr}(\rho_{ab} \otimes \rho_{cd})^2 = 1\) and the nonbilocal correlation measure is given by
\[
N_{\mathcal{F}}(\rho_{ba} \otimes \rho_{ab}) = 1 - \min_{\Pi^{aa}} \text{Tr}(\rho_{ab} \otimes \rho_{cd} \Pi^{bc}(\rho_{ab} \otimes \rho_{cd})) \\
\geq 1 - \min_{\Pi^{aa}} \text{Tr}(\rho_{ab} \otimes \rho_{ab}(\Pi^a \otimes \Pi^a)(\rho_{ba} \otimes \rho_{ab})). \\
= 1 - \min_{\Pi^{aa}} \text{Tr}(\rho_{ab} \Pi^a(\rho_{ab}))^2 \\
\geq 1 - \min_{\Pi^{aa}} \text{Tr}(\rho_{ab} \Pi^a(\rho_{ab})) \\
= N_{\mathcal{F}}^{MIN}(\rho),
\]
where the first inequality follows from the fact that $\Pi^a \otimes \Pi^a$ is not necessarily optimal and the second inequality is due to the square of the fidelity between pre- and post-measurement states and is either equal to or less than unity. Hence, the theorem is proved. The above relation provides a closer connection between the nonbilocal and nonlocal measures implying that the nonbilocal measure is always greater than MIN.

4 Nonbilocal correlation for mixed states

To compute the fidelity-based nonbilocal measure for any arbitrary mixed input state, we first define some basic notation in the operator Hilbert space. Let $\mathcal{L}(\mathcal{H}_\alpha)$ be the Hilbert space of linear operators on $\mathcal{H}_\alpha (\alpha = a, b, c, d)$ with the inner product $\langle X | Y \rangle = \text{Tr}(X^\dagger Y)$. An arbitrary $m \times n$-dimensional bipartite state can be written as

$$\rho_{ab} = \sum_{i,j} \lambda_{ij}^{ab} X_i \otimes Y_j,$$

where $\{X_i : i = 0, 1, \ldots, m^2 - 1\}$ and $\{Y_j : j = 0, 1, \ldots, n^2 - 1\}$ are the orthonormal operator bases of the subsystems $a$ and $b$, respectively, satisfying the relation $\text{Tr}(X_k X_l) = \delta_{kl}$ and $\Lambda_{ab}$ is the correlation matrix for the subsystem $ab$ which characterizes the correlation between the system $a$ and $b$ and its matrix elements $\lambda_{ij}^{ab} = \text{Tr}(\rho_{ab} X_i \otimes Y_j)$ are real. Similarly, one can define the orthonormal operator bases as $\{P_k : k = 0, 1, \ldots, u^2 - 1\}$ and $\{Q_l : l = 0, 1, \ldots, v^2 - 1\}$ for another input state $\rho_{cd}$ with $u$ and $v$ being the dimensions of the marginal systems $c$ and $d$, respectively. Then, the state $\rho_{cd}$ is defined as

$$\rho_{cd} = \sum_{k,l} \lambda_{kl}^{cd} P_k \otimes Q_l,$$

where $\lambda_{kl}^{cd} = \text{Tr}(\rho_{cd} P_k \otimes Q_l)$ are the matrix elements of matrix $\Lambda_{cd}$. Then, the bilocal state is written as

$$\rho_{ab} \otimes \rho_{cd} = \sum_{i,j} \sum_{k,l} \lambda_{ij}^{ab} \lambda_{kl}^{cd} X_i \otimes Y_j \otimes P_k \otimes Q_l. \quad (20)$$

**Theorem 1** For any arbitrary bilocal input states represented in Eq. (20), the upper bound of nonbilocal measure is given by

$$N_F(\rho_{ab} \otimes \rho_{cd}) \leq 1 - \frac{1}{||\Lambda||^2} \sum_{s=1}^{nu} \mu_s, \quad (21)$$

where $\mu_s$ are the eigenvalues of the matrix $\Lambda_{ab,cd} \Lambda_{ab,cd}^\dagger$ arranged in increasing order and $\Lambda_{ab,cd}^\dagger$ denotes the transpose of the matrix $\Lambda_{ab,cd}$. 

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If the measurement operators are given by \( \Pi^{bc} = \{ \1^a \otimes \Pi^{bc}_h \otimes \1^d \} \), the post-measurement state becomes

\[
\Pi^{bc}(\rho_{ab} \otimes \rho_{cd}) = \sum_h \sum_{ijkl} \lambda_{ij}^{ab} \lambda_{kl}^{cd} X_i \otimes \Pi^{bc}_h(Y_j \otimes P_k) \Pi^{bc}_h \otimes Q_l,
\]

\[
= \sum_h \sum_{ij'k'l} \lambda_{ij}^{ab} \lambda_{k'l}^{cd} \gamma_{hj'k'} \gamma_{hj} \gamma_{k'l},
\]

(22)

where \( \gamma_{hjk} = \text{Tr} \Pi^{bc}_h(Y_j \otimes P_k) \) are the elements of matrix \( \Gamma \). Next, we compute the fidelity between pre- and post-measurement states and is given by

\[
F(\rho_{ab} \otimes \rho_{cd}, \Pi^{bc}(\rho_{ab} \otimes \rho_{cd})) = \sum_h \sum_{ij'k'l} \lambda_{ij}^{ab} \lambda_{k'l}^{cd} \gamma_{hj'k'} \gamma_{hj} \gamma_{k'l},
\]

(23)

where \( \Gamma \) is an \( nu \times n^2 u^2 \)-dimensional matrix. Then,

\[
N_F(\rho_{ab} \otimes \rho_{cd}) = 1 - \min_{\Pi^{bc}} \frac{1}{\| \Lambda \|^2} \Gamma \Lambda_{ab,cd} \Lambda_{ab,cd}^t \Gamma^t \leq 1 - \frac{1}{\| \Lambda \|^2} \sum_{s=1}^{nk} \mu_s,
\]

where \( \mu_s \) are the eigenvalues of the matrix \( \Lambda_{ab,cd} \Lambda_{ab,cd}^t \) listed in increasing order. Hence the theorem is proved.

**Theorem 2** If the marginal state \( \rho^b \) is nondegenerate, the nonbilocal measure \( N_F(\rho_{ab} \otimes \rho_{cd}) \) due to the measurement \( \Pi^{bc} \) has the upper bound as

\[
N_F(\rho_{ab} \otimes \rho_{cd}) \leq 1 - \frac{1}{\| \Lambda_{cd} \|^2} F(\rho_{ab}, \Pi^b(\rho_{ab})) \times \left( \begin{array}{c} u \\ \sum_{\tau=1}^u \mu_{\tau} \end{array} \right),
\]

(24)

where \( \mu_{\tau} \) are the eigenvalues of matrix \( \Lambda_{cd} \Lambda_{cd}^t \) arranged in increasing order and \( F(\rho_{ab}, \Pi^b(\rho_{ab})) \) is the fidelity between the state \( \rho_{ab} \) and post-measurement state \( \Pi^b(\sqrt{\rho_{ab}}) \).

To prove the theorem, it is worth reiterating that if the marginal state is nondegenerate, the optimization is not required. Assuming that the state \( \rho^b \) is nondegenerate and the optimization of the measure given by Eq. (12) is taken over \( \Pi^c \) alone, the measurement operator is defined as \( \Pi^b \otimes \Pi^c = \{ \Pi^b_j \otimes \Pi^c_k \} = \{ |j_b\rangle \langle j_b| \otimes \Pi^c_k \} \). The nonbilocality measure based on the fidelity becomes

\[
N_F(\rho_{ab} \otimes \rho_{cd}) = 1 - \min_{\Pi^{bc}} F(\rho_{ab} \otimes \rho_{cd}, \Pi^{bc}(\rho_{ab} \otimes \rho_{cd}))
\]

\[
= 1 - \min_{\Pi^{bc}} \frac{\text{Tr} \rho_{ab} \otimes \rho_{cd} \cdot \Pi^{bc}(\rho_{ab} \otimes \rho_{cd})}{\text{Tr}(\rho_{ab} \otimes \rho_{cd})^2}
\]
\[ = 1 - \min_{\Pi^c} \frac{\text{Tr} \rho_{ab} \otimes \rho_{cd} \cdot (\Pi^b \otimes \Pi^c)(\rho_{ab} \otimes \rho_{cd})}{\text{Tr} \rho_{ab}^2 \cdot \text{Tr} \rho_{cd}^2} \]

\[ = 1 - \frac{1}{\| \Lambda_{cd} \|^2} F(\rho_{ab}, \Pi^b(\rho_{ab})) \min_{\Pi^c} \text{Tr} \rho_{cd} \Pi^c(\rho_{cd}). \]  

(25)

where \( F(\rho_{ab}, \Pi^b(\rho_{ab})) \) is the fidelity between the state \( \rho_{ab} \) and post-measured state \( \Pi^b(\rho_{ab}) \). Following the optimization procedure given in [43], we write the second term of the above equation as

\[ \min_{\Pi^c} \text{Tr} \rho_{cd} \Pi^c(\rho_{cd}) = \min_C \text{Tr} C \Lambda_{cd} A^t_{cd} C^t. \]  

(26)

The quantity \( \text{Tr} \rho_{ab} \Pi^b(\rho_{ab}) \) is the fidelity between the state \( \rho_{ab} \) and post-measurement state \( \Pi^b(\rho_{ab}) \). Then, the fidelity-based nonbilocality measure is given by

\[ N_F(\rho_{ab} \otimes \rho_{cd}) = 1 - F(\rho_{ab}, \Pi^b(\rho_{ab})) \min_C \text{Tr} C \Lambda_{cd} A^t_{cd} C^t \]

\[ \leq 1 - 1 - \frac{1}{\| \Lambda_{cd} \|^2} F(\rho_{ab}, \Pi^b(\rho_{ab})) \times \sum_{\tau=1}^u \mu_\tau, \]  

(27)

where \( \mu_\tau \) are the eigenvalues of matrix \( \Lambda_{cd} A^t_{cd} \) arranged in increasing order.

If the marginal states \( \rho^b \) and \( \rho^c \) are nondegenerate and the dimension of \( \rho^c \) is 2 \( (u = 2) \), then, the closed formula of nonbilocality measure is expressed as

\[ N_F(\rho_{ab} \otimes \rho_{cd}) \leq 1 - F(\rho_{ab}, \Pi^b(\rho_{ab})) \times \frac{(\mu_1 + \mu_1)}{\| \Lambda_{cd} \|^2}. \]  

(28)

### 5 Illustrations

In this section, we compute the fidelity-based measurement-induced nonbilocality for some well-known input states.

**Example 1** Let \(|\Psi_{ab}\rangle = |00\rangle\) and \(|\Psi_{cd}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}\) be the two input states.

According to property (v), the nonbilocality measure is

\[ N_F(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = 1 - \sum_{i,j} s^4_i r^4_j. \]  

(29)

The Schmidt coefficients for \(|\Psi_{ab}\rangle\) are 0 and 1. Similarly, \(|\Psi_{cd}\rangle\) has the Schmidt coefficients \(1/\sqrt{2}\) and \(1/\sqrt{2}\). Then, \(N_F(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = 0.5\). The above example validates the property (iii) of \(N_F(\rho_{ab} \otimes \rho_{cd})\).

**Example 2** The input state is \(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \otimes (|00\rangle + |11\rangle)/\sqrt{2}\).

Then,

\[ N_F(|\Psi_{ab}\rangle \otimes |\Psi_{cd}\rangle) = 1 - 4 \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{4}. \]  

(30)
Example 3 Next, we consider the isotropic state given by

$$\rho = \frac{1 - x}{m^2 - 1} \mathbb{1} + \frac{m^2 x - 1}{m^2 - 1} |\Psi\rangle\langle\Psi|; \quad x \in [0, 1].$$  \(31\)

where $\Psi = \frac{1}{\sqrt{m}} \sum_i |ii\rangle$ is a maximally entangled state and $m$ is the dimension of the state.

Here, we compare the F-MIN and nonbilocal measure. The F-MIN is zero when $x = 1/m^2$ and the corresponding state is a maximally mixed state. We have computed the fidelity-based nonlocal and nonbilocal measures and plotted them as a function of state parameter $x$ in Fig. 3. From Fig. 3, we observe that the nonbilocal measure is also zero at $x = 1/m^2$. Further, we notice that nonbilocal measure is always greater than nonlocal measure (F-MIN).

Example 4 Next, we study the nonbilocality of the $m \times m$-dimensional Werner state and is given by

$$\rho = \frac{m - x}{m^3 - m} \mathbb{1} + \frac{mx - 1}{m^3 - 3} S; \quad x \in [-1, 1].$$  \(32\)

where $S = \sum_{\mu, \nu} |\mu\rangle \langle \nu| \otimes |\nu\rangle \langle \mu|$ is an exchange operator.

We observe that the F-MIN of Werner state vanishes at $x = 1/m$. In Fig. 4, we have plotted the F-MIN and fidelity-based nonbilocal measure as a function of state parameter $x$. We clearly notice that the nonbilocal measure is also zero at $x = 1/m$ and is always greater than the F-MIN.

6 Conclusions

In this article, we have introduced a measure of nonbilocal correlations of two bipartite input states using fidelity-based measurement-induced nonlocality. We notice that a closer connection exists between the nonlocal and nonbilocal correlation measures.
For any arbitrary pure input state, we have evaluated the nonbilocal correlations analytically. The upper bounds of the fidelity-based nonbilocal measure are also obtained for mixed input states. As an illustration, the nonbilocality is computed for some well-known examples.

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