Group Theory in Finite Hamiltonian Systems

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Abstract. We give a short review of the methods and aims of the Óptica Matemática projects in studying discrete Hamiltonian systems with group-theoretical methods.

There are several partially overlapping definitions of Hamiltonian systems in the applied mathematical literature—and of phase space. They have been proposed to be useful in classical mechanics, geometric optics, and others for wave and quantum systems. We study discrete Hamiltonian systems, where ‘configuration’ space is a denumerable collection of points, finite or infinite. They have corresponding phase spaces that are compact or noncompact manifolds stemming from the mother Lie algebra of the system.

Three postulates
We propose three postulates to define Hamiltonian systems: the first two are the geometric and dynamic Hamilton equations, \[\{K, X\} = -iP\] and \[\{K, P\} = iF(X)\] expressed with Lie brackets of operators associated with the classical observable of position \(X\) and its tangent operator \(P\) (momentum) in the direction determined by the generator of evolution \(K\) (Hamiltonian + any constant). The third postulate is algebraic; it determines the Lie bracket \[\{X, P\} = i(\alpha 1 + \beta K)\] between the canonical position and momentum operators. When they commute and the bracket is Poisson, the Hamiltonian system is classical; and when they are self-adjoint operators or matrices subject to commutators, it is a quantum system. When we postulate that \(\alpha = h\), \(\beta = 0\), the Hamiltonian system will be mothered by the Heisenberg-Weyl algebra and group \(HW\), yielding Schrödinger quantum-mechanical systems. We can postulate instead that the key commutator close into a different Lie algebra. In one space dimension we can have the spin \(SU(2)\) group \((\beta = 1)\) \([1, 2]\), the Lorentz group \((\beta = -1)\) \(SU(1, 1)\) \([3]\), the group of Euclidean motions \(ISO(2)\) \((\beta = 0)\) \([4]\), or any three-parameter algebra, or \(q\)-algebra \([5, 6, 7, 8]\), which contracts to \(HW\). This postulate thus entails the correspondence principle: when the number and density of points increase without limit in a discrete Hamiltonian system, the limit should be some well-known continuous Hamiltonian system \([9, 10]\).

We have found particularly fruitful the group \(SU(2)\), which is compact, and hence the spectrum of the position, momentum, and Hamiltonian (displaced energy) is finite: \(\{x_m\}\) for \(m \in \{-j, -j+1, \ldots, j\}\), comprising \(N = 2j + 1\) points. Since the rotation of a sphere projects its points to harmonic motion, the system will be a discrete harmonic oscillator. Its ‘states’ (or

1 We apologize that most of the references below are to our own work, where further sources and differences can be found; we have tried to give a condensed overview of results published during the past few years.
$N$-point signals, or wavefunctions, or kets) are the overlap between the position and Hamiltonian bases of the complex Hilbert space $\mathbb{C}^N$. The special functions required are Wigner little-$d$ functions, widely known in quantum angular momentum theory, with their expression in terms of binomial distributions and Kravchuk orthogonal discrete polynomials. With the position generator compact, as said above we have infinite discrete Hamiltonian systems: the free system mothered by $ISO(2)$ and the repulsive oscillator mothered by $SU(1,1)$. Finally, Hamiltonian systems mothered by $q$- and super- algebras yield discrete oscillators whose wavefunctions include $q$-Kravchuk [7], $p$-Kravchuk, Meixner, Pollaczek [11] and Hahn [12, 13] polynomials, all of which contract to the common quantum oscillator.

We should underline that the group action can also be bilateral: on ket-bra matrices that stand for density distributions of pure or entangled states. Hence, discrete Hamiltonian systems are actually quantum systems.

Phase space

The phase space of discrete Hamiltonian systems is a symplectic manifold embedded in the space of coadjoint orbits of the mother algebra [14, 15], which we may call meta-phase space. Clearly, if the spectra of the generators of the mother Lie algebra are discrete and their interval finite, phase space should be a compact manifold —a sphere in the case of $SU(2)$. The traditional Fourier approach to finite systems considers discrete tori; yet, there one cannot rotate around a point (as a harmonic oscillator would) without tearing the torus somewhere [16, 17, 18]. In the ring of the generated Lie group we find a Wigner operator that is covariant under that group: recall that under HW and $Sp(2,\mathbb{R})$ canonical transforms [19], the common HW-Wigner function transforms its phase space plane covariantly [20]. For compact groups such as $SU(2)$, the Wigner operator is an $N \times N$ matrix, a covariant function on this group, and reducible to one of the coset spaces. Its expectation value in a state yields a quasiprobability distribution that is the $SU(2)$-covariant Wigner function. We can thus see the music sheet of finite signals on a sphere (best projected on a rectangle), where we can classify the $N^2$ unitary $U(N)$ transformations that linearly map and aberrate signals [21], in correspondence with geometric optical models. In the contraction limit to the HW-Wigner function, unitarity somehow becomes canonicity.

Two-dimensional systems

Two-dimensional discrete Hamiltonian systems are useful for the analysis of finite pixellated images, following cartesian or polar coordinates [22], and maintaining close relation with the continuous geometric-optical counterpart model. The $N^2$ complex pixel values of an $N \times N$-pixel screen form an irreducible representation space under the direct product group $SU(2)_x \times SU(2)_y = SO(4)$ [23, 24]. Through symmetry importation, the images can be unitarily transformed under the four-parameter $U(2)$-Fourier group of geometric optics that includes isotropic and anisotropic Fourier-Kravchuk transforms, as well as rotations and gyrations [25]. The latter are the discrete counterparts of the transformation between Hermite-Gauss (HG) and Laguerre-Gauss (LG) wave-optical beams [26]. Finally, we also examine the relative advantages of using sampled values of HG and LG functions versus our discrete Kravchuk functions to analyze images into their mode [27] and angular momentum [28] components, and synthesize the original image faithfully.

We are assembling these results in an orderly fashion for a volume in order to show the conceptual economy that group theory brings into the analysis of discrete Hamiltonian systems, revealing their symmetry and dynamics, classical as well as quantum.

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