Stochastic Maps, Wealth Distribution in Random Asset Exchange Models and the Marginal Utility of Relative Wealth

Sitabhra Sinha*

The Institute of Mathematical Sciences, C.I.T. Campus, Taramani, Chennai - 600 113, India

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Abstract

We look at how asset exchange models can be mapped to random iterated function systems (IFS) giving new insights into the dynamics of wealth accumulation in such models. In particular, we focus on the “yard-sale” (winner gets a random fraction of the loser's wealth) and the “theft-and-fraud” (winner gets a random fraction of the poorer players wealth) asset exchange models. Several special cases including 2-player and 3-player versions of these “gamers” allow us to connect the results with observed features in real economies, e.g., lock-in (positive feedback), etc. We then implement the realistic notion that a richer agent is less likely to be aggressive when bargaining over a small amount with a poorer player. When this simple feature is added to the yard-sale model, in addition to the accumulation of the total wealth by a single agent (“condensation”), we can see exponential and power-law distributions of wealth. Simulation results suggest that the power-law distribution occurs at the cross-over of the system from exponential phase to the condensate phase.

1. Introduction

Recently, there has been a considerable amount of work done in developing the statistical mechanics of economic activities leading to wealth accumulation and distribution in society [1]. One of the simplest classes of models which explore these mechanisms are the “asset exchange models” [2–7]. In analogy with the physics of ideal gases, economic agents can be viewed as particles which have random elastic collisions with each other, resulting in wealth circulation throughout the system. One reliable indicator of whether these models reflect economic reality is to test whether they reproduce the observed wealth distributions in various societies.

It has been known for over a century that almost all human societies tend to exhibit the same type of wealth distribution. If $P(x)$ is the probability distribution for income or wealth $x$ of individuals, then for large $x$, it follows the so-called Pareto law:

$$P(x) \sim x^{-v(1+v)},$$

i.e., a power law distribution with the exponent $v$ between 1 and 2 [8], while for small $x$, an exponential distribution is observed [9,10].

Unfortunately, the simplest asset exchange models do not show this distribution. The asymptotic states of these models exhibit either an exponential phase, or even more extremely, all the wealth condensing into the hands of a single individual; power-law distributions, if seen at all, turn out to be transient [2–5]. However, recent work on the effects of introducing random saving propensities in the asset exchange models, have shown asymptotic power law distribution similar to those observed in reality [6,7]. In this paper, we look at another possible modification of the asset exchange model: the same amount of money may have different relative values to a rich agent and a poor agent. In other words, the relative importance of making a net gain in a round of trading is dependent on the relative wealths of the agents involved. By introducing this simple principle into the model, we observe a wide range of distributions: from an exponential phase to a condensate phase, with a power-law distribution appearing in the transition region between the two phases.

We consider a simple model of a closed economic system where the total wealth (amount of money) available for exchange, $M$, and the total number of agents, $N$, trading with each other, are fixed. Wealth is neither created nor destroyed, but only change hands through trading between agents. Further, the system is observed only at discrete time intervals $t = 0, 1, 2, 3, \ldots$ Each agent $i$ has some wealth $x_i(t)$ associated with it at some time step $t$. Starting from an arbitrary initial distribution of wealth $(x_i(0), i = 1, 2, 3, \ldots)$, during each time step two randomly chosen agents $i$ and $j$ exchange a fraction of their combined wealth. Each such transaction is obeying the constraint that the combined wealth of the two agents is conserved by the trade, and that neither of the two has negative wealth after the trade (i.e., debt is not allowed). In general, one of the players will gain and the other player will lose as a result of the trade.

Different types of exchange models are defined based on the choice of the fraction of wealth that will be exchanged in a trade. If the wealth exchanged is a fraction of the wealth owned by the poorer of the two agents, then the model (in accordance with the terminology introduced in Ref. [1]) is the so-called “Yard-Sale” model (YS), whereas if the exchanged amount is a fraction of the losing agent’s wealth, it is the “Theft-and-Fraud” model (TF) [1]. These names reflect the fact that usually (e.g., in a yard sale) the richer agent is unlikely to stake its entire holdings in a trade with a poorer agent. The only circumstances during which such an event is likely to occur is when the poorer agent is dishonest, and either the exchange itself, or the wealth of the poorer agent, is unknown to the richer agent (corresponding to theft and fraud respectively).

If we consider an arbitrarily chosen pair of agents $(i,j)$ who trade at a time step $t$ resulting in a net gain of wealth by agent $i$, then the change in their wealth as a result of
trading is:

\[ x_i(t+1) = x_i(t) + \Delta x; \quad x_j(t+1) = x_j(t) - \Delta x, \]

where \( \Delta x \) is the net wealth exchanged between the two agents. In the YS model

\[ \Delta x = \alpha \min(x_i(t), x_j(t)), \]

while in the TF model

\[ \Delta x = \alpha x_i(t) \] (agent \( j \) has lost),

with \( \alpha \) as a uniformly distributed random number in the interval \([0, 1]\). Whether agent \( i \) or \( j \) will “win” in a particular trading encounter is decided by the toss of a fair coin, i.e., each has a probability 1/2 of making a net gain. A possible variant, where, the two agents randomly redistribute their total wealth can be seen as a manifestation of the TF model.

In the next section, the asset exchange models are seen as a class of random dynamical systems. This picture allows us to understand in simple terms various features of the distribution seen in the two models. Section 3 introduces the concept of diminishing bargaining efficiency as the wealth of an agent is increased. Results of 2-agent and \( N \)-agent asset exchange models are given. Finally, we conclude with a summary and discussion of possible directions for future work.

2. Asset exchange models as stochastic IFS

If we consider \( \alpha \) to be a constant in Eqs. (2) and (3), then for \( N = 2 \), the asset exchange model (YS or TF) is a system of two maps of the unit interval \([0,1]\) onto itself, with the system randomly switching between the two maps. The map selected at a particular instant depends upon which agent wins in that particular trading round. Such stochastic dynamical systems are called Iterated Function Systems (IFS) [11]. The 2 map IFS corresponding to the YS and TF models are shown in Fig. 1 for \( \alpha = 0.5 \).

2.1. 2-Agent models

Let us consider the YS model with \( N = 2 \), and total wealth \( M = \sum_{i=1}^{N} x_i = 1 \) (i.e., normalized). Then, the state of such an economy at any given time \( t \), is completely specified by the wealth of any one of the agents, \( x(t) \) (since the other agent’s wealth is \( 1 - x(t) \)). For constant \( \alpha \), the corresponding IFS is given by

\[
\text{Map 1} : x(t+1) = (1 + \alpha) x(t), \quad \text{if } x(t) < 0.5, \\
= x(t) + \alpha (1 - x(t)), \quad \text{otherwise},
\]

and,

\[
\text{Map 2} : x(t+1) = (1 - \alpha) x(t), \quad \text{if } x(t) < 0.5, \\
= x(t) - \alpha (1 - x(t)), \quad \text{otherwise}.
\]

For \( \alpha = 0 \), the initial distribution is unchanged by the IFS, but for any \( \alpha > 0 \), the final distribution corresponds to two delta function peaks at \( x = 0 \) and \( x = 1 \). In other words, the entire wealth eventually ends up in the hands of one of the two agents through a process of gradual wealth condensation. The transition to this condensate phase from an arbitrary initial distribution takes longer and longer time as \( \alpha \to 0 \), in a process analogous to critical slowing down.

In the TF model with \( N = 2 \), the wealth dynamics is given by the IFS:

\[
\text{Map 1} : x(t+1) = (1 - \alpha) x(t) + \alpha, \\
\text{and}
\]

\[
\text{Map 2} : x(t+1) = (1 - \alpha) x(t).
\]

The asymptotic state for a constant value of \( \alpha(>0.5) \) is a Cantor set fractal distribution. In particular, for \( \alpha = 2/3 \), the final distribution is the middle-third Cantor set, generated by successively dividing intervals into three equal parts and then removing the central section. Therefore, the wealth possessed by an agent at any given time form a discontinuous range of values.

For randomly varying \( \alpha \), the asymptotic distribution of the TF model is composed of an infinite number of Cantor sets generated by the different values of \( \alpha \). This turns out to be a power law distribution with an exponent \( \simeq 0.45 \). The occurrence of a power-law or scale-free distribution can be understood as follows. Each of the Cantor set distributions generated for a fixed \( \alpha \) has a length scale associated with it, corresponding to the fraction of an interval removed recursively during its generation. For random \( \alpha \), since all Cantor sets are represented, this implies that all length scales are present in the asymptotic distribution. This results in a scale-free distribution for randomly varying \( \alpha \).

To understand why YS and TF models lead to very different asymptotic distributions, we look at the effect of a sequence of unfavorable outcomes (i.e., losses) on the wealth of the richer agent (Fig. 2). Let us suppose that in both models, the rich agent (\( A \), say) owns a significant fraction of the total wealth (\( x = 0.95 \), say). If we now look at the result of a series of losses, we find that agent \( A \) is much less affected in the YS model than in the TF model, and the larger the initial wealth of \( A \), the greater is the

\[
\text{Fig. 1. The IFS corresponding to the 2-agent (left) YS model and (right) TF model with } \alpha = 0.5. \text{ The system stochastically switches from one map to the other depending on which of the two agents (A, B) wins a particular round of trading.}
\]
increasing the balance in favor of the other agent becomes increasingly unlikely with the occurrence of the number of unfavorable outcomes needed to shift the total wealth; this imbalance will be consolidated in subsequent trading, the YS model, if initially one of the agents acquires a significant fraction of other hand, it takes six successive unfavorable outcomes needed (and therefore, acquiring the agent, the larger is the number of stable fixed points distribution over a wide range as the order of the return maps for each of the two systems (Fig. 3). For the YS model, all the higher iterate maps have stable fixed points only at \( x = 0 \) and \( x = 1 \). The TF model on the other hand has increasing number of stable fixed points distributed over a wide range as the order of the return map is increased. This immediately implies that randomly switching between maps in the TF model will lead to a fairer distribution of wealth, whereas that will not be the case in the YS model. Further, for the YS model we find that the attractor where the system will eventually end up, is decided by the first few outcomes of bargaining among the two agents. Once the system enters the basin of attraction of one of the attractors (\( x = 0 \) or \( x = 1 \)) through a series of favorable outcomes, it takes many more unfavorable outcomes for it to come out and enter the other attractor’s domain. If one of the agents, by chance, wins the first few successive bargaining rounds then it is almost bound to be the eventual winner. This is reminiscent of the phenomena of positive feedback leading to “lock-in” in real economic systems [12].

difference between the two models. This means that, if initially one of the agents acquire a significant fraction of the total wealth (by random fluctuations), then the dynamics of the YS model assures that the agent will consolidate this position. The greater the value of wealth fraction owned by A, \( x(t) \) (right) indicates that, while in the TF model, it takes only one loss to change the status of agent A from the richer (\( x \geq 0.5 \)) to the poorer agent (\( x < 0.5 \)), in the YS model, on the other hand, it takes six successive losses to change the status of A. So, in the YS model, if initially one of the agents acquire a significant fraction of the total wealth, this imbalance will be consolidated in subsequent trading, as the occurrence of the number of unfavorable outcomes needed to shift the balance in favor of the other agent becomes increasingly unlikely with increasing \( x \).

2.2. 3-Agent models

The 3-agent TF model (with the total wealth normalized to 1) can be represented in the triangular region \([0,0),(0,1), (1,0)\] in the two-dimensional plane. This is because we need to only explicitly specify the wealths \( x_1(t) \) and \( x_2(t) \) of two agents (A and B, say), the third agent (C) having wealth \( 1 - x_1(t) - x_2(t) \).

The 3-agent model is the simplest case where we can study the effects of different types of interaction coupling among agents. For example, instead of allowing all the agents to trade among themselves, we can forbid trade between agents B and C (say). Therefore, B and C can only trade with each other through a “go-between” (in this case, agent A). Our simulations showed no differences in the results for the two schemes. Figure 4(a) shows the attractors corresponding to the two types of coupling among agents. The asymptotic distribution is again a power law, with the exponent value of 0.5 ± 0.005 (Fig. 4(b)) for both types of network interaction structure.

2.3. N-Agent models

As the number of agents \( N \) is increased, the models become increasingly difficult to understand in terms of dynamical systems. However, most of the features of the 2- and 3-agent games carry over to the \( N \)-agent case. For example, the observation that all wealth condenses into the hands of a single agent, holds true in the \( N \)-agent YS model as \( N \rightarrow \infty \). In the TF model, as \( N \) increases, the asymptotic wealth distribution becomes exponential.

A few special cases can be understood completely even in the \( N \rightarrow \infty \) limit. One of these is the case when \( \alpha = 1 \). For the YS model, this corresponds to a “Double-or-nothing” scenario, where the poorer agent stands to either double its wealth (if it wins) or lose everything to the richer agent (if it loses). The richer agent, on the other hand, stands to lose

Fig. 2. Difference between the 2-agent YS model (broken lines) and TF model (solid lines) in the effect of a sequence of unfavorable outcomes on the wealth of the initially richer agent (A) with \( \alpha = 0.5 \). In both models, agent A starts out with wealth \( x_0 = 0.95 \). The return maps (left) of the two models show that the losses affect A more strongly in the TF model than in the YS model (especially when \( x \) is large). The time evolution of the wealth fraction owned by A, \( x(t) \) (right) indicates that, while in the TF model, it takes only one loss to change the status of agent A from the richer (\( x \geq 0.5 \)) to the poorer agent (\( x < 0.5 \)), in the YS model, on the other hand, it takes six successive losses to change the status of A. So, in the YS model, if initially one of the agents acquire a significant fraction of the total wealth, this imbalance will be consolidated in subsequent trading, as the occurrence of the number of unfavorable outcomes needed to shift the balance in favor of the other agent becomes increasingly unlikely with increasing \( x \).

Fig. 3. The second return map of the 2-agent (left) YS model and (right) TF model with \( \alpha = 0.5 \). Both the models have four fixed points, but while all four are stable in the TF model, only two (\( x = 0 \) and \( x = 1 \)) are stable in the YS model. Each curve is labelled by a pair of letters “XY”, indicating that it corresponds to the case where a win of agent X in the first time instant \( (t + 1) \), is followed by a win of agent Y in the next time instant \( (t + 2) \). Note that, in the YS model, for the case of both agents winning once (i.e., AB or BA), the basin of attraction for \( x = 1 \) is larger than \( x = 0 \) when A wins first, and the reverse holds true when B wins first. This indicates the strong dependence of the final state of the system on the initial states (i.e., the winner of the first few “trades” has a very high probability of emerging as the eventual winner).

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or gain only an amount of wealth equal to that owned currently by the poorer agent. The corresponding situation in the TF model gives the winner (independent of whether the richer or the poorer agent wins) all the assets of the loser. Therefore, this situation corresponds to a “winner-take-all” scenario. It can be easily seen that both cases quickly lead to the concentration of wealth into the hands of a steadily decreasing minority.

3. Diminishing bargaining efficiency with wealth

We have so far assumed that the probability that an agent will gain net wealth is independent of its wealth. However, in any real situation, it is unlikely that an agent who owns 1,000 units of wealth (say) will be as concerned about winning or losing 1 additional unit, as an agent who has only 1 unit. Therefore, the relative value of the amount of wealth won or lost by an agent is clearly a function of its wealth at that given time. This results in the increasing aggressiveness of the poorer agent in getting a favorable deal during any trade with a wealthier agent.

This is implemented in the asset exchange model by expressing the probability that agent $i$ will win when trading with agent $j$, $p(i|j)$, as a “Fermi function”:

$$p(i|j) = \frac{1}{1 + \exp \left( \frac{x_i - x_j}{\beta} \right)},$$

with $\beta$ parametrizing the significance of the relative value of wealth between trading agents. For $\beta = 0$, the original asset exchange model is retrieved, where the probability that any agent wins a round of trading is $1/2$, independent of their wealth. When $\beta > 0$, the poorer agent has a higher probability of winning, the difference from the original probability ($= 1/2$) depending on the value of $\beta$. In the special circumstance when the two agents have the same wealth, the probability of each of them winning is $1/2$, irrespective of the value of $\beta$.

As $\beta \to \infty$, it becomes certain that in any encounter the poorer agent will win. Intuitively it is clear that this will ensure a fairer distribution of wealth. In fact, when $\alpha$ is a constant, the corresponding stochastic IFS reduces to a deterministic map; when $N = 2$, it is a map of the unit interval $[0,1]$ onto itself (Fig. 5).

In the YS model, with $\alpha = 1$ (“Double-or-nothing” for the poorer agent), the wealth distribution remains in the condensate phase even as $\beta \to \infty$. This is because, although wealth changes hands frequently, the number of solvent agents who can trade steadily decreases over time. Wealth gets accumulated into the hands of a steadily diminishing number of agents, although the label of the wealthiest agent keeps changing. But for any $\alpha < 1$, the wealth distribution will tend to be fairer, i.e., the asymptotic distribution no longer corresponds to all the wealth ending up with just one agent. The chaotic map corresponding to the YS model at $\beta \to \infty$ (for a fixed value of $\alpha$) ensures a more uniform distribution of wealth among the agents.

When we implement this principle in the 2-agent YS model with randomly varying $\alpha$, we immediately find that...
varying $\beta$ completely alters the asymptotic wealth distribution. At the limit $\beta \to \infty$ the distribution function is triangular or “tent”-shaped:

$$P(x) = 4x, \quad \text{if } x \leq 0.5; \quad 4(1-x), \quad \text{otherwise,}$$

where $x$ is confined to the unit interval. As $\beta$ is decreased, we find that the central peak of the distribution (at $x = 0.5$) gradually diminishes, while the tails of the distribution (at $x = 0$ and 1) gradually start rising (Fig. 6). The distribution finally becomes identical to the delta function peak distribution of the conventional YS model as $\beta \to 0$.

In the 2-agent TF model (Fig. 5 (right)), if $\alpha$ is fixed, then at $\beta \to \infty$, the asymptotic state is a 2-cycle, with the two players alternately switching between two wealth values $x_1, x_2$ ($x_1 > 0.5 > x_2$). The exact numerical value of $x_1, x_2$ depends on $\alpha$. With randomly varying $\alpha$ we find a smooth asymptotic distribution which peaks at the center ($x = 0.5$). Unlike the TF model, here the distribution is not piecewise linear. With decreasing $\beta$, the central peak diminishes with corresponding increase at the tails of the distribution (Fig. 7).

When we implement the principle in the $N$-agent YS model, with $\alpha$ fixed to a constant value ($= 0.5$, say), we find that a power-law distribution is observed at intermediate values of $\beta$. The same result is seen when $\alpha$ is randomly varied (Fig. 8). For large value of $\beta$, the asymptotic distribution is exponential for large wealth (Fig. 8 (a)), with an initial power law increase (which disappears with decreasing $\beta$). With increasing $\beta$, the distribution shows an increasing peak at the most probable value of $x = 1$, i.e., the average wealth per agent. In other words, as the effect of the relative value of wealth becomes more pronounced, it becomes more likely that every agent will have the same amount of money (on average). As already mentioned, at intermediate values of $\beta$, a power-law distribution is observed (Fig. 8 (b)). As $\beta$ is decreased further, a condensation at very high value of wealth is observed in addition to the power law distribution. Further decrease of $\beta$ leads to the increasing condensation of the wealth in the hands of fewer and fewer agents, until, at $\beta = 0$, a single agent acquires all the wealth (the conventional YS model).

Note that if we implement increasing bargaining efficiency with increasing wealth, we have a situation

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\[ \beta = 10 \]

\[ \beta = 100 \]

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\[ \beta = 0.667 \]

\[ \beta = 8 \]

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essentially similar to the “greedy exchange” model investigated in Ref. [2]. This possibly corresponds to the condition that the wealthier agent has greater chance of dictating terms to the poorer agent during any bargaining.

4. Discussion
In this paper we have first discussed random asset exchange models as a class of stochastic dynamical systems. This allows, us to understand why different asymptotic distributions are observed in the YS and TF models. We can also connect the features observed in these models with phenomena like positive feedback (“lock-in”) seen in real economic situations. We have then introduced the realistic notion that the “value” of a certain quantity of wealth is relative to the wealth currently owned by an agent, i.e., to a very wealthy agent the gain or loss of a very small quantity is not as important as it is to a very poor agent. Implementing this principle, we observe exponential distribution of wealth among agents, in addition to the condensate phase observed in the original YS model. More interestingly, we see a power-law distribution of wealth in the region of transition between the exponential and condensate phases.

The use of a Fermi function (Eq. (8)) to express the probability of an agent to win a particular exchange with another agent, \( p(i|j) \), as a function of their relative wealth, \( x_i/x_j \), can be interpreted as an implementation of a principle of diminishing marginal utility of relative wealth. If the utility \( U \) is expressed as a function of the relative wealth \( w \), then the probability \( p(i|j) \) is related to the marginal utility \( \partial U/\partial w \). Therefore, the probability of winning, and hence, the marginal utility corresponding to an utility function which initially rises rapidly with \( w \), but for higher values of \( w \) shows very little or no growth, is represented very well by a Fermi function. In the \( \beta \to \infty \) limit, Eq. (8) implies that the utility function increases linearly up to some maximum value \( U = U_{\text{max}} \) (when the relative wealth \( w = 1 \)), and then stays constant at all higher values of \( w \). On the other hand, when \( \beta = 0 \), the utility function is a strictly linear increasing function of the relative wealth.

A possible interesting aspect not fully explored is the role of the network structure of interaction among the agents. There has been some study of small-world effects in wealth distribution through interactive multiplicative stochastic process [13]. In the conventional YS model, the introduction of small-world or regular structure, requiring agents to interact preferentially with some agents, does not change the eventual delta-function peak distribution of wealth. The only effect of a regular network structure is that multiple delta-function peaks coexist, depending on the allowed range of interaction. We have not carried out a corresponding study of the model under the condition of diminishing bargaining efficiency with wealth.

Another possibly fruitful area of future work is the connection of the models studied in this paper to mass-aggregation models allowing diffusion, aggregation and dissociation [14]. Such models show nonequilibrium phase transitions, from a state where the steady state mass distribution decays exponentially to another state where an infinite aggregate is observed in addition to a power law distribution. The principal difference with the models that we study here is that in mass aggregation all the units exchanged have the same fixed mass value, whereas here we have no limit to the smallest amount of wealth an agent can possess and exchange. However, it is striking that both types of models show similar phases.

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