Fake Supersymmetry and Extremal Black Holes

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Abstract

We derive the BPS type of first order differential equations for the rotating black hole solutions in the three-dimensional Einstein gravity coupled minimally with a self-interacting scalar field, using fake supersymmetry formalism. It turns out that the formalism is not complete and should be augmented by an additional equation to imply the full equations of motion. We identify this additional equation as a constraint by using an effective action method. By computing the renormalized boundary stress tensor, we obtain the mass and angular momentum of the black hole solutions of these first order equations and confirm that they saturate the BPS bound.

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1 Introduction

Supersymmetry, if confirmed experimentally, has a profound significance in our nature. It would give us various predictions and new perspectives for particle phenomenology and cosmology. Apart from these implications, it explains systematically many interesting analytic results which may be ad-hoc or difficult to understand otherwise. This analytic nature is more or less related to the so-called Bogomol’nyi-Prasad-Sommerfield (BPS) states in extended supersymmetric field theories, which preserve supersymmetry partially. One interesting and nice aspect of these BPS states is that they admit the Killing spinors which satisfy the Killing spinor equations (KSE). These KSE are usually lower order differential equations than the original equations of motion and therefore are easier to solve. Explicitly, in the usual two derivative theory, the bosonic equations of motion are given by second order differential equations, while BPS states can be described by first order equations.

Typically these BPS states exist even in the model which contains only bosonic sector of the supersymmetric theory. In this reduced model they usually correspond to the states which minimize the energy, from which, once again, the lower order equations can be obtained. Inspired by this, fake supersymmetry method has been developed to obtain these BPS states which satisfy lower order equations of motion for non-supersymmetric, i.e. purely bosonic, model [1][2][3]. The basic idea is simple: One may consider a ‘fake’ supersymmetric extension of the bosonic model and introduce a spinor which satisfies fake KSE in the corresponding supersymmetric model. Since the EOM of original bosonic model are the same as the bosonic EOM of the supersymmetric model for vanishing fermions, these reduced order ‘fake’ KSE would almost imply the full EOM as in genuine supersymmetric theory.

Along this line, various interesting BPS solutions of gravity with a minimally coupled scalar field have been found. They include domain wall solutions [1] and black hole solutions with a scalar hair [2]. They were found by considering some reduced EOM which are consistent with the full EOM. In the case of domain wall solutions and some static black hole solutions, those reduced EOM have been obtained by using fake supersymmetry formalism [3].

In this paper we would like to establish a systematic method to obtain these reduced order EOM by using fake supersymmetry formalism. Specifically, we consider the three-dimensional Einstein gravity with a minimally coupled and self-interacting scalar field. It would be considered as a bosonic sector of some fake supergravity. It was found that the model admits asymptotically anti-de Sitter black hole solutions with a scalar hair [2][7][8] as well as Banados-Teitelboim-Zanelli (BTZ) black holes [9]. We use the KSE of the fake supergravity to find the lower order EOM. It turns out that the KSE are not enough to uniquely determine the solutions. We find that it is due to the fact that the Killing vector associated with the fake Killing spinor is null-like. We identify the missing equation and argue that this corresponds to the constraint equation rather than the dynamical EOM. In order to support the claim, we consider the effective action formalism. The resultant solutions are shown to correspond to quarter BPS solutions in the supersymmetric counterpart.
Since the solutions are asymptotically anti-de Sitter, these can be studied in the context of AdS/CFT correspondence. We determine the counter terms for the scalar and the metric fields and compute renormalized boundary stress tensor. From this we obtain the mass and angular momentum of the solutions and confirm that they really saturate the BPS bound.

\section{Einstein gravity with an interacting scalar field}

The action of three-dimensional Einstein gravity with a minimally coupled scalar field is given by

\[ S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \tag{1} \]

where we have taken the convention of the metric as mostly plus signs and the curvature tensors as \( [\nabla_\mu \nabla_\nu] V_\rho = R_{\mu\nu\rho\sigma} V^\sigma \) and \( R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} \).

The EOM are composed of scalar field equation and the metric field equations as follows:

\[ 0 = E_\phi \equiv \nabla^2 \phi - \frac{\partial V}{\partial \phi}, \quad 0 = E_{\mu\nu} \equiv G_{\mu\nu} - T_{\mu\nu}, \tag{2} \]

where \( G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \), \( T_{\mu\nu} \equiv \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left[ \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + V(\phi) \right] \).

As usual, the trace part of \( E_{\mu\nu} \) can be used to rewrite EOM as

\[ 0 = E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} V, \tag{3} \]

which is the relevant form for our study in next sections.

We are interested in the asymptotically AdS black holes with a scalar hair, which would be deformations of BTZ black holes. Our metric ansatz for rotating AdS black holes with axial symmetry in AdS-Schwarzschild-like coordinates is taken as

\[ ds^2 = L^2 \left[ - e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + r^2 \left( d\theta + e^{C(r)} dt \right)^2 \right], \tag{4} \]

where \( L \) denotes the radius of asymptotic AdS space. Accordingly, the scalar field \( \phi \) is taken as a function only of the radial coordinate \( r \). The asymptotic conditions on the metric functions \( A(r), B(r), C(r) \) are taken as

\[ e^{A(r)} \bigg|_{r \to \infty} \to r + O\left( \frac{1}{r} \right), \quad e^{B(r)} \bigg|_{r \to \infty} \to \frac{1}{r} + O\left( \frac{1}{r^2} \right), \quad e^{C(r)} \bigg|_{r \to \infty} \to \text{const.} + O\left( \frac{1}{r^2} \right). \tag{5} \]

The boundary condition for the scalar field consistent with this metric ansatz is given by

\[ \phi(r) \bigg|_{r \to \infty} = \text{const.} + O\left( \frac{1}{r} \right). \tag{6} \]
We have taken our fall-off boundary conditions for the metric as the standard Brown-Henneaux type which allow us to obtain the central charge by Brown-Henneaux method \cite{10}. One may note that the above metric admits a time-like Killing vector $\frac{\partial}{\partial t}$ and a rotational Killing vector $\frac{\partial}{\partial \theta}$, which would generate the full isometry group in the generic case as was shown in the rotating BTZ case \cite{11}.

Explicitly, EOM for the above ansatz are given by

\begin{align}
0 &= E_\phi = \frac{1}{L^2} e^{-2B} \left[ \left( A' - B' + \frac{1}{r} \right) \phi' + \phi'' \right] - \frac{\partial V}{\partial \phi}, \\
0 &= -E_{rr} = L^2 e^{2B} V + \frac{1}{2} \phi'^2 + A'' + A'^2 - A'B' - \frac{1}{r} B' - \frac{r^2}{2} C'^2 e^{2C - 2A}, \\
0 &= -\frac{1}{r^2} e^{2B} E_{\theta \theta} = L^2 e^{2B} V + \frac{1}{r} (A' - B') + \frac{r^2}{2} C'^2 e^{2C - 2A}, \\
0 &= -\frac{1}{r^2} e^{2B - C} E_{\theta t} = L^2 e^{2B} V + \frac{1}{2} \left[ C'' + C'^2 + r^2 C'^2 e^{2C - 2A} - (A' + B')C' \right. \\
&\quad \left. + \frac{2}{r} (A' - B') + \frac{3}{r} C' \right], \\
0 &= -e^{2B} E_{tt} = L^2 (r^2 e^{2C} - e^{2A}) e^{2B} V - e^{2A} \left[ A'' + A'^2 - A'B' + \frac{1}{r} A' \right] \\
&\quad + r^2 e^{2C} \left[ C'' + \frac{3}{2} C'^2 + \frac{r^2}{2} C'^2 e^{2C - 2A} - (A' + B')C' + \frac{1}{r} (A' - B' + 3C') \right],
\end{align}

where $'$ denotes the differentiation with respect to the radial coordinate $r$. These equations are called the full EOM in the following.

In Ref. \cite{7} extremally rotating black hole solutions with a scalar hair were found as solutions of the above EOM. It has been known that extremal BTZ black hole solutions preserve partial supersymmetry in the context of supergravity. Since the extremal black hole solutions with scalar hair can be considered as a deformation of extremal BTZ, it is natural to expect that the supersymmetry-like argument might play some roles to the solutions.

### 3 Fake Supersymmetry and Effective Action

In this section, by using the, so-called, fake supersymmetry technique, we obtain Bogomol’nyi type of first order differential equations which solve the full EOM. This can be considered as the generalization of the domain wall case to the extremally rotating $AdS_3$ black holes. This turns out to be the systematic derivation of the first order equations for extremal black holes \cite{7}. It turns out that fake Killing spinor equations are not sufficient to obtain all of the first order equations. As in the case of genuine supersymmetric theory with null Killing spinors, the fake Killing spinors turn out to be null-like and should be augmented by a certain component of EOMs. In our case, by using effective action method, we show that this component of EOMs
becomes effectively a first order equation and, in fact, it corresponds to a certain constraint not the dynamical equation.

3.1 Fake supersymmetry

Our convention for Γ-matrices is taken such as \( \{\Gamma^\alpha, \Gamma^\beta\} = 2\eta^\alpha{}^\beta \). Explicitly, 1 + 2 dimensional (lower indices) Γ-matrices may be taken as real and symmetric ones:

\[
\Gamma^a_{\alpha\beta} = (-1, \sigma^1, \sigma^3),
\]

where \( \sigma^a \)'s are Pauli matrices. Note that \( \epsilon^\alpha{}^\beta \Gamma^a_{\alpha\beta} = 0 \). Spinor indices are raised or lowered by rank two \( \epsilon \)-tensor as

\[
\Gamma^\alpha{}^\beta = \epsilon^\beta{}^\gamma \Gamma^\alpha{}^\gamma.
\]

Then, Clifford algebra is realized as

\[
\{\Gamma^a, \Gamma^b\}_{\alpha\beta} = (\Gamma^a)_{\rho\alpha} (\Gamma^b)_{\beta\rho} - (\Gamma^b)_{\rho\alpha} (\Gamma^a)_{\beta\rho} = 2\eta^a{}^b \delta^\beta{}^\alpha.
\]

We also take \( \epsilon^{\tilde{a}\tilde{b}} = 1 \) such that

\[
\Gamma^{\tilde{a}\tilde{b}} \equiv \frac{1}{2}[\Gamma^{\tilde{a}}, \Gamma^{\tilde{b}}] = \epsilon^{\tilde{a}\tilde{b}} \Gamma_{\tilde{c}}, \quad \Gamma^{\tilde{a}\tilde{b}} = 1.
\]

Though there is another inequivalent irreducible representation of Γ-matrices in three dimensions, one may deal with the inequivalent ones simply by taking \( \tilde{\Gamma}^{a} \equiv -\Gamma^{a} \).

In our case, the fake Killing spinors under ‘fake’ supersymmetry are determined by two equations, one of which corresponds to the (fake) dilatino variation and the other to the (fake) gravitino variation as

\[
\left( \Gamma^\mu \partial_\mu \phi + \frac{1}{4L} \partial_\phi \right) \epsilon = 0, \quad \left( D_\mu - \frac{1}{4L} \mathcal{W} \Gamma_\mu \right) \epsilon = 0,
\]

where \( \mathcal{W} = \mathcal{W}(\phi) \), the so-called superpotential, denotes a certain function of the scalar field \( \phi \) and the curved index Γ-matrices are defined as \( \Gamma^\mu \equiv e^a_\mu \Gamma^a \). The covariant derivatives in the above fake Killing spinor equations (KSE) are defined by

\[
D_\mu \epsilon \equiv \left( \partial_\mu + \frac{1}{4} \omega_{\mu}^{\tilde{a}\tilde{b}} \Gamma_{\tilde{a}\tilde{b}} \right) \epsilon,
\]

where \( \omega_{\mu}^{\tilde{a}\tilde{b}} \) denotes the spin connection.

The integrability conditions of the above fake KSE, after the contraction with a Γ-matrix, lead to the following conditions

\[
0 = \Gamma^\nu \left[ D_\nu - \frac{1}{4L} \mathcal{W}_\nu, \ D_\mu - \frac{1}{4L} \mathcal{W}_\mu \right] \epsilon = \frac{1}{2} E_{\mu\nu} \Gamma^\nu \epsilon, \quad (9)
\]

\[
0 = \Gamma^\mu \left[ D_\mu - \frac{1}{4L} \mathcal{W}_\mu, \ \Gamma^\nu \partial_\nu \phi + \frac{1}{L} \partial_\phi \mathcal{W} \right] \epsilon = E_\phi \epsilon,
\]

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where $\partial_\phi$ denotes the differentiation with respect to the scalar field, $\phi$, and the scalar potential $V(\phi)$ should be taken in the form of

$$V(\phi) = \frac{1}{2L^2}(\partial_\phi W)^2 - \frac{1}{2L^2}W^2.$$  

(10)

The above contracted integrability conditions show us that EOMs for metric and scalar fields are almost satisfied. However, as in the case of genuine Killing spinors, the fake KSE or their integrability conditions may not imply the full EOM. According to the nature of fake Killing spinors, one may need an additional condition to imply the full EOM as will be shown in the following.

Now, let us solve the fake KSE explicitly. For our metric ansatz, dreibeins can be taken as

$$e^t = Le^A(r) dt, \quad e^r = Le^B(r) dr, \quad e^\theta = Lr(d\theta + e^C(r) dt).$$  

(11)

The spin connection one forms, $\omega^{\hat{a}\hat{b}} = \omega^a_{\mu} dx^\mu$, for these dreibeins are given by

$$\begin{align*}
\omega^{\hat{t}\hat{r}} &= \left(A'e^{A-B} - \frac{1}{r^2}C'e^{2C-A-B} \right) dt - \frac{1}{2}r^2 C'e^{C-A-B} d\theta, \\
\omega^{\hat{t}\hat{\theta}} &= \frac{1}{2} e^{A/2} e^B \partial_\phi W \\
\omega^{\hat{r}\hat{\theta}} &= \frac{1}{2}(2 + r C')e^{C-B} dt - e^{-B} d\theta.
\end{align*}$$  

(12)

Firstly, let us solve the fake dilatino equation. Since the scalar field depends only on the radial coordinate $r$ in our case, one can see that

$$\left(e^{-B} \phi' \Gamma^{\hat{r}} + \partial_\phi W \right) \epsilon = 0,$$

which leads to

$$\Gamma^{\hat{r}} \epsilon = \pm \epsilon, \quad \phi' = \mp e^B \partial_\phi W.$$  

(13)

For definiteness, let us take $\Gamma^{\hat{r}} \epsilon = \epsilon$ case, which may be regarded as a projection. By solving directly the KSE corresponding to the fake gravitino variation, it turns out that the fake Killing spinor is a function only of the radial coordinate $r$ and given in terms of the metric function $A(r)$ as

$$\epsilon_\alpha = e^{A/2} \epsilon_0, \quad \epsilon_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$  

(14)

Furthermore, it turns out that metric functions and the scalar field $\phi$ should be related through first order differential equations as

$$(e^C)' = \frac{1}{r}e^A \left(e^B W - \frac{2}{r} \right) = \left(\frac{1}{r} e^A \right)'.$$  

(15)

It has been known that the KSE imply the full bosonic EOM if the Killing vector formed by genuine Killing spinors is time-like, while it doesn’t if the corresponding Killing vector is null-like. It is natural to expect the same behavior for the fake KSE. We show that in our case the
Killing vector constructed from the fake Killing spinors is null-like and therefore the KSE are insufficient to satisfy the full EOM.

Through the standard procedure, one can construct the one-form dual to Killing vector by the bilinear of the fake Killing spinors as

$$\xi \equiv \xi_\mu dx^\mu = (\bar{\epsilon} \Gamma_\mu \epsilon) \, dx^\mu. \quad (16)$$

It is straightforward to check that $\xi_\mu$ satisfies $\nabla_\mu (\xi_\nu) = 0$ by using fake KSE, which tells us that $\xi^\mu$ is a Killing vector. Using the Fierz identity of three-dimensional $\Gamma$-matrices, it is also straightforward to see that

$$\xi^\mu \xi_\mu = -3(\bar{\epsilon} \epsilon)^2 = 0, \quad (17)$$

which shows us that the Killing vector is null-like and the fake KSE is insufficient to imply full EOM. This manifests from three equations in Eq.(13) and (15) from KSE for four unknown variables.

Following the standard way in the genuine KSE, let us identify the missing equation for KSE to imply the full EOM in our case. To achieve this, it is convenient to introduce null coordinates adapted to the above Killing vector as

$$\xi = f \, e^+, \quad (18)$$

where $f$ is a certain normalization function. By direct computation from the fake Killing spinor expression given in Eq.(14), one can take $e^+$ (with $f \sim e^A$) as

$$e^+ \equiv \frac{L}{\sqrt{2}} [(r e^C - e^A) dt + rd\theta]. \quad (19)$$

Then, our metric can be written as

$$ds^2 = 2e^+ e^- + e^\phi e^\phi, \quad (20)$$

where

$$e^- \equiv \frac{L}{\sqrt{2}} [(r e^C + e^A) dt + rd\theta].$$

It is interesting to note that the projection condition, $\Gamma^\phi \epsilon = \epsilon$, for fake Killing spinor implies $\Gamma^+ \epsilon = 0$.

Now, let us identify the missing equation. The following procedure is a direct adaptation of the genuine Killing spinor case \cite{12,13} to the fake one. By the spinor contraction of $\bar{\epsilon}$ with the contracted integrability condition, $0 = E_{\mu\nu} \Gamma^\nu \epsilon$, one obtains

$$E_{\mu\nu} \xi^\nu = 0. \quad (21)$$

\footnote{The Dirac conjugate of spinor is defined as $\bar{\epsilon} \equiv \epsilon^\dagger \Gamma^i$.}
Since $\xi^-$ is the only non-vanishing component of a Killing vector $\xi = \xi^+ e^+$, the above condition implies that all the components $E_{-\mu}$ should vanish. By multiplying $E_{\mu\sigma} \Gamma^\sigma$ to the contracted integrability condition, $0 = E_{\mu} \Gamma^\mu \epsilon$, and symmetrizing the free indices, one also obtains

$$E_{\mu\nu} E_{\nu\rho} g^{\rho\sigma} = 0.$$  \hspace{1cm} (22)

Using this condition (or its flat space index form), one can see that all the components of EOM are implied by fake KSE except $0 = E_{\hat{+} \hat{+}}$. Therefore, to imply full EOM, fake KSE should be augmented by the equation $0 = E_{\hat{+} \hat{+}}$, which can be written in our case as

$$0 = E_{\hat{+} \hat{+}} = \frac{-2A}{2L^2} \left[ E_{\hat{u}} - \frac{2}{r} (r e^C + e^A) E_{\hat{u}} + \frac{1}{r^2} (r e^C + e^A)^2 E_{\hat{u}} \right].$$

Using the conditions from KSE or the automatically vanishing components of bosonic equations, the necessary condition to imply the full EOM is given by

$$0 = E_{\hat{+} \hat{+}} = \frac{2}{rL^2} \left[ A' + B' - \frac{r}{2} \phi'^2 \right].$$  \hspace{1cm} (23)

In the following we will show that this missing equation can be identified as a certain constraint not a dynamical equation in the effective action formulation.

Collecting the previous results for fake Killing spinors given in Eq. (13) and Eq. (15) with the condition Eq. (23), one obtains the following first order differential equations, which satisfy the full EOM,

$$\phi' = -e^B \partial_\phi W, \quad A' + \frac{1}{r} = e^B W, \quad (e^C)' = \left( \frac{1}{r^2} e^{2A} \right)' , \quad A' + B' = \frac{r}{2} \phi'^2.$$  \hspace{1cm} (24)

These differential equations, called reduced EOM, were obtained by some educated guess in Ref. \[7\].

Some comments are in order. If we choose the other inequivalent representation for $\Gamma$-matrices, $\hat{\Gamma}$, and take the projection choice of fake Killing spinor as $\hat{\Gamma}^r \epsilon = \epsilon$, we obtain the same equations in Eq. (24) except for the third one which changes into

$$(e^C)' = - \left( \frac{1}{r} e^A \right)'.$$  \hspace{1cm} (25)

Since the above equations in Eq. (24) was derived by solving KSE for the fixed representation of $\Gamma$-matrices with definite projection $\Gamma^r \epsilon = \epsilon$, one may say that solutions of these reduced EOM preserves 1/4 fake supersymmetries just like extremal rotating BTZ black holes.\footnote{In Ref. \[14\] fake supersymmetry is anticipated to play some roles even for three-dimensional supergravity, which might be related to our case.} Note that the third equation in Eq. (24) can be integrated as

$$e^C = C_+ + \frac{1}{r^2} e^A ,$$  \hspace{1cm} (26)
where the integration constants $C_+$ can take any value consistently with the asymptotic boundary conditions. One of the convenient choices may be to take the integration constant as $C_+ = 0$, so that the metric function $C(r)$ is simply given by

$$e^C = \frac{1}{r} e^A .$$  \hspace{1cm} (27)

Note that the standard choice, $C_+ = -1$, for instance for BTZ black holes in AdS-Schwarzschild coordinates, can be recovered by a simple coordinate transformation, $\theta \to \theta + C_+ t$. One advantage of this choice is the fact that one of the null coordinates can be identified with $\theta$ coordinate.

One can see that the Killing one-form, $\xi$, from fake Killing spinor becomes identified with $r d\theta$ as can be shown from Eq.(19). This explains partially the result that the equation, $E_{\theta \theta} = 0$, can be taken instead of the missing equation, $E_{\hat{T} \hat{T}} = 0$.

### 3.2 Effective Action

In order to clarify the nature of the missing equation in the fake Killing spinor formalism, we consider the effective action. By inserting the metric ansatz into the action (1), one obtains the effective action as

$$S_{eff} = -\frac{1}{16\pi G} \int d^3 x \ L_r \ e^{A-B} \left[ 2A'' + 2A'^2 - 2A'B' - \frac{r^2}{2} C'^2 e^{2C-2A} + \frac{2}{r} (A' - B') + \frac{1}{2} \phi'^2 + L^2 e^{2B} V(\phi) \right]$$

$$= -\frac{1}{16\pi G} \int d^3 x \ L_r \ e^{A-B} \left[ - \frac{2}{r} A' - \frac{r^2}{2} C'^2 e^{2C-2A} + \frac{1}{2} \phi'^2 + L^2 e^{2B} V(\phi) \right] + \text{total deriv.}$$

whose EOM can be obtained, after rearranging results from the variation of the action with respect to $A, B, C, \phi$, as

$$0 = L^2 e^{2B} V + \frac{1}{r} (A' - B') + \frac{r^2}{2} C'^2 e^{2C-2A} ,$$

$$0 = A' + B' - \frac{r}{2} \phi'^2 ,$$

$$0 = (r^3 C' e^{-A-B+C})' ,$$

$$0 = L^2 re^{A+B} \partial_\phi V - (re^{A-B} \phi')' .$$

One can verify that these equations are equivalent to the full EOM in Eq. (7). First of all, one may notice that there are just four equations rather than five compared with the original full EOM. However, one can see that one of the five equations in the full EOM is redundant as follows. Basically, the redundant equation corresponds to the one containing $A''$ term, for instance $0 = E_{rr}$ in Eq. (7). Let us derive this equation from the above four equations. By differentiating the first equation with respect to the radial coordinate $r$, one can obtain an equation containing $A''$ term. Though this equation also has $V'$ term, this term can be eliminated though the equation obtained by multiplying the last equation by $\phi'$. By combining the resultant
equation with the second and third equations in the above, one can derive a differential equation containing $A''$ term which can be shown to be equivalent to $0 = E_{rr}$.

Up to total derivative, the effective action can be rewritten as

$$S_{\text{eff}} = \frac{1}{16\pi G} \int d^3x \, L \left[ \frac{1}{2} \partial^2 \phi + \frac{1}{2} e^{A+B} \left( (\partial_\phi W)^2 - W^2 \right) + e^{A-B} \left( \frac{1}{r} (A' + B') - \frac{1}{2} \phi'^2 \right) \right],$$

where it is clear that $e^{A-B}$ becomes a Lagrange multiplier and thus a variation with respect to this gives us a constraint equation, $A' + B' - r\phi'^2/2 = 0$. This equation is nothing but the missing equation obtained in Eq. (23). In appendix A we present preliminary study on the canonical formulation of our model to investigate the origin of this constraint.

Let us try to extremize the above effective action by a complete square to obtain BPS like first order equations. By squaring the Lagrangian successively, one obtains

$$S_{\text{eff}} = -\frac{1}{16\pi G} \int d^3x \, \frac{Lr}{2} \left[ e^{A-B} \left\{ (\phi' + e^B \partial_\phi W)^2 - \left( A' + \frac{1}{r} - e^B W \right)^2 \right\} - e^{-(A+B)} r^2 \left( (e^C)' + \frac{1}{r} e^A \right) \left( (e^C)' - \frac{1}{r} e^A \right) \right] + \text{total deriv.}$$

One can see that the following conditions extremize the effective action partially

$$\phi' = -e^B \partial_\phi W, \quad A' + \frac{1}{r} = e^B W,$$

which should be augmented by the constraint from Lagrange multiplier $e^{A-B}$. After inserting the above first order equations $^{(32)}$ in the effective action with the constraint$^{(3)},$ the effective action can be further reduced as

$$S_{\text{eff}} = \frac{1}{16\pi G} \int d^3x \, \frac{Lr}{2} e^{-(A+B)} \left[ \left( r(e^C)' + e^A (e^B W - \frac{2}{r}) \right) \left( r(e^C)' - e^A (e^B W - \frac{2}{r}) \right) \right] + \text{total deriv.}$$

This reduced effective action can be extremized by the following first order equation

$$(e^C)' = \pm \left( \frac{1}{r} e^A \right)'.$$  

These seem to suggest that the effective action formalism may reproduce the first order equations obtained from fake SUSY formalism.

### 4 Boundary Stress Tensor and Conserved Charges

It was shown that first order differential equations derived in the previous section describe extremally rotating AdS black holes by near horizon analysis and, moreover, some of analytic

\footnote{The necessity of a certain constraint was noticed in a different context $^{[15]}$}
solutions for these first order equations, called reduced EOM, were also presented in Ref. [7]. In this section we obtain renormalized boundary stress tensor on the AdS black hole solutions for these reduced EOM, which is interpreted as the stress tensor of dual CFT on the asymptotic boundary by the standard AdS/CFT dictionary [16]. We also confirm the extremality of these black hole solutions by obtaining mass and angular momentum through renormalized boundary stress tensor. It is interesting to note that mass and angular momentum from the renormalized boundary stress tensor have contribution from both metric and scalar fields, while two contributions are obtained in one stroke through metric in the so-called ADT formalism [17] [18] [19].

The (holographically) renormalized boundary stress tensor is given by the subtraction of an appropriate counter term from quasi-local stress tensor introduced by Brown and York [20] [16]. This boundary stress tensor becomes finite after the subtraction and can be identified with the (renormalized) stress tensor in the dual field theory according to the AdS/CFT correspondence. Using these renormalized boundary stress tensor, one can compute conserved charges in dual field theory which can also be identified with those in the bulk gravity. In the following, we obtain the renormalized boundary stress tensor for our model and also verify the previous expressions of conserved charges. The aim of this section is two-fold. On the one hand we would like to obtain the contribution of a scalar hair to the boundary stress tensor and on the other we verify the conserved charge expression of our concerned black hole solutions in another way and confirm the extremality of those black holes.

Solving reduced EOM in Eq. (24) perturbatively at the asymptotic infinity, one can see that the asymptotic fall-off behaviors of AdS black hole solutions are given by

\[ e^A(r) = r + \frac{a_1}{r} + \cdots, \quad e^B(r) = \frac{1}{r^3} + \frac{b_1}{r^3} + \cdots, \quad e^C(r) = -1 + \frac{1}{r} e^A, \]  
\[ \phi(r) = \phi_\infty + \frac{\phi_1}{r} + \cdots, \quad W(\phi) = 2 + \frac{1}{2} (\phi - \phi_\infty)^2 + \cdots, \]  

where constants \( a_1, b_1 \) and \( \phi_1 \) are related as \( a_1 + b_1 = -\phi_1^2/4 \). Note that the integration constant are taken as \( C_+ = -1 \), which is more appropriate to obtain conserved charges correctly. For the superpotential \( W(\phi) \) which is an even function of \( (\phi - \phi_\infty) \), one can show that the asymptotic form of the scalar field \( \phi \) is given by

\[ \phi(r) = \phi_\infty + \frac{\phi_1}{r} + \mathcal{O}\left(\frac{1}{r^3}\right). \]

In fact, by using reduced EOM one can show that the coefficients in the next leading term is given by [7]

\[ a_1 = -\frac{1}{2} \Delta_0, \quad b_1 = -\frac{1}{4} \phi_1^2 + \frac{1}{2} \Delta_0, \]  

where \( \Delta_0 \) is a constant related to the horizon value of the superpotential as \( \Delta_0 = r_H^2 W(\phi_H) \). As was mentioned in the previous section, these asymptotic boundary conditions for metric functions satisfy the so-called Brown-Henneaux boundary conditions [10]. Together with this
metric fall-off boundary condition, the scalar field should satisfy the similar fall-off boundary condition to be consistent with the EOM. As an explicit example, by turning off the scalar field, that is to say, setting $\phi = \phi_\infty$, one obtains the extremal BTZ black holes, of which solutions are given in the above coordinates as

$$
e^A(r) = e^{-B(r)} = r - \frac{r_H^2}{r}, \quad e^C = -\frac{r_H^2}{r^2}, \quad \phi = \phi_\infty = \phi_H, \quad \mathcal{W} = 2. \tag{37}$$

For the boundary stress tensor computation it is very convenient to consider the metric foliated in the radial direction with the further decomposition of the boundary metric in the ADM form. Note that our metric ansatz is already in such a form. Explicitly, our metric ansatz can be written as

$$ds^2 = N^2 dr^2 + \gamma_{ij} dx^i dx^j, \quad N \equiv L e^B,$$

where

$$\gamma_{ij} dx^i dx^j = -L^2 e^{2A} dt^2 + \sigma (d\theta + e^C dt)^2, \quad \sigma \equiv L^2 r^2.$$

As is clear from the definition of the boundary stress tensor or its unregularized Brown-York tensor form, there are two contributions to the boundary stress tensor. One contribution comes from metric fields and the other from the scalar field. The metric contribution to the renormalized boundary stress tensor is well-known [16] and given in our case by the following form

$$T_G^{ij} = \frac{1}{16\pi G} \left( K^{ij} - K^{ji} - \frac{1}{L} \gamma^{ij} \right), \tag{38}$$

where $K^{ij}$ denotes the extrinsic curvature and $K$ is its trace, $K \equiv \gamma^{ij} K_{ij}$. Our convention for the extrinsic curvature $K_{ij}$ is

$$K_{ij} \equiv \frac{1}{2N} \left[ \partial_r \gamma_{ij} - \nabla_i N_j - \nabla_j N_i \right],$$

where $\nabla_i$ denotes the covariant derivative with respect to the metric $\gamma_{ij}$. Therefore, we focus only on the scalar part in the action, in the following. Fortunately for our purpose, the scalar field contribution to the boundary stress tensor was already determined for a specific scalar potential in Ref. [21]. However, the fall-off boundary conditions and the scalar potential are different in our case from that. Therefore, we need to rederive the scalar contribution which is appropriate in our case.

According to the standard construction of counter terms, they are chosen to cancel the unwanted divergent part of the on-shell action. To apply this procedure, let us consider the variation of the scalar part in our action. After inserting the bulk EOM in the variation of the action, one obtains

$$\delta S = -\frac{1}{16\pi G} \int d^2 x \sqrt{-\gamma} n^\mu \partial_\mu \phi \delta \phi, \tag{39}$$

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where \( n^n \) denotes the unit outward normal to the hypersurface or the boundary surface.\(^4\) To cancel this term, one needs to introduce the variation of counter term for the scalar field as

\[
\delta S_{ct} = \frac{1}{16\pi G} \int_{\partial M} d^2x \sqrt{-\gamma} n^r \partial_r \phi \delta \phi \tag{40}
\]

In the second equality, we have expanded the integrand in powers of \( 1/r \) according to the fall-off boundary conditions.

Now, let us take the integrated version of the above variational form of counter term as

\[
S_{ct} = \frac{1}{16\pi G L} \int_{\partial M} d^2x \sqrt{-\gamma} \left[ \alpha L n^r \partial_r \phi - \beta \phi^2 \right], \tag{41}
\]

where \( \alpha \) and \( \beta \) are certain constant and will be determined in the following. The variation of the above counter term leads to

\[
\delta S_{ct} = -\frac{1}{16\pi G} \int_{\partial M} d^2x \sqrt{-\gamma} \frac{1}{L r^2} \left[ 2(\alpha + \beta) \phi_1 + \mathcal{O}\left(\frac{1}{r^2}\right) \right] \delta \phi_1,
\]

which should be matched to the above variational form of the counter term. This condition determines only the combination of \( \alpha \) and \( \beta \) as

\[
\alpha + \beta = \frac{1}{2}, \tag{42}
\]

which means that the counter term may not be unique. This is not so strange since this ambiguity does not affect the conserved charges, as will be shown in below. It is also useful to recall that counter terms in higher curvature gravity, which has additional degrees of freedom through higher curvature terms, have such ambiguity \([22][23]\).

One can verify that conserved charges are independent of the ambiguity explicitly as follows. First, one may note that the renormalized boundary stress tensor is given by the sum of metric and scalar contributions as follows:

\[
T^{ij}_B = T^{ij}_G + T^{ij}_\phi = \frac{1}{8\pi G} \left( K^{ij} - K^j^i - \frac{1}{L} \gamma^{ij} \right) + \frac{1}{16\pi G L} \gamma^{ij} \left[ \alpha L n^r \partial_r \phi - \beta \phi^2 \right], \tag{43}
\]

with the condition \( \alpha + \beta = 1/2 \). Note that the scalar contribution solely comes from the counter term in \([41]\). Then, the conserved charges can be computed by

\[
Q_\xi = \frac{1}{8\pi G} \int d\theta \sqrt{\sigma} \ u_i \xi_j T^{ij}_B, \tag{44}
\]

where \( u^i \) and \( \xi^j \), defined on the boundary, denote the time-like unit vector normal to the hypersurface and a Killing vector for the conserved charge, respectively.

\( ^4\)In our case its nonvanishing component is just \( n^r = e^{-B}/L \) or its dual one form is given by \( n = Nd\).
To obtain the mass of our black hole solutions, one can take the time-like Killing vector as $\xi = e^A u$ with unit one form $u = Le^A dt$. Then, one can see that the metric and scalar contributions are given, respectively, by

$$M_G = \frac{1}{8\pi G} \int d\theta \sqrt{\sigma} u_i \xi_j T^{ij}_G = \frac{2\Delta_0 - \phi_1^2}{16G}, \quad M_\phi = \frac{1}{8\pi G} \int d\theta \sqrt{\sigma} u_i \xi_j T^{ij}_\phi = \frac{\phi_1^2}{16G},$$

in which $\alpha$ and $\beta$ appear only through a combination, $\alpha + \beta = 1/2$. The total mass of the black hole solutions is given by

$$M = M_G + M_\phi = \frac{\Delta_0}{8G}. \quad (46)$$

By taking the space-like Killing vector for angular momentum as $\xi = Lr \nu$ with unit one form $\nu = Lr (d\theta + e^C dt)$, one obtains metric and scalar contribution to angular momentum as

$$J_G = L \frac{\Delta_0}{8G}, \quad J_\phi = 0,$$

which leads to the total angular momentum as

$$J = J_G + J_\phi = L \frac{\Delta_0}{8G}. \quad (48)$$

The above results on mass and angular momentum show us that the ambiguity in the counter term is harmless. Furthermore, the expressions of conserved charges confirm the extremality of the considered black holes $ML = J$, which were shown independently by the so-called ADT formalism \[7\] \[24\]. As alluded in the above, it is crucial for the correct conserved charge that one should keep the appropriate coordinates or the appropriate integration constant $C_+ = -1$, which was also the case in the ADT formalism.

Though the ambiguity in counter term is not physical, one may determine the counter term completely by considering more generic fall-off boundary condition for the scalar field as

$$\phi(r) - \phi_\infty = \frac{\phi_1 r}{r} + \zeta \frac{\phi_2}{r^2} + \cdots,$$

where $\zeta$ is a constant. Under this generalized fall-off condition for the scalar field, the required counter term variation becomes

$$\delta S_{ct} = -\frac{1}{16\pi G} \int_{\partial M} d^2x \sqrt{-\gamma} \frac{1}{rL^2} \left[ \phi_1 + \zeta \frac{4}{r^2} \phi_1^2 + \cdots \right] \delta \phi_1.$$

The variation of integration ansatz is given by

$$\delta S_{ct} = -\frac{1}{16\pi G} \int_{\partial M} d^2x \sqrt{-\gamma} \frac{1}{Lr^2} \left[ 2(\alpha + \beta) \phi_1 + \zeta \frac{3}{r^3} (3\alpha + 2\beta) \phi_1^2 + \cdots \right] \delta \phi_1,$$

where we have retained $\zeta$ as a constant during the variation. Comparing the above two expressions for the variation of the counter terms, one obtains

$$\alpha = \frac{1}{3}, \quad \beta = \frac{1}{6}. \quad (49)$$

Then the counter term in our case can be chosen uniquely as the limit of such a counter term by taking $\zeta \rightarrow 0$. This phenomenon such as less ambiguity for additional fall-off tail has also analogy in higher curvature gravity, where the more general fall-off solutions determine the counter terms with less ambiguity \[22\].
5 Conclusion

In this paper we have considered fake supersymmetry to derive first order differential equations for the rotating black hole solutions in the three-dimensional Einstein gravity with a minimally coupled self-interacting scalar field. It turns out that the fake Killing spinor is null in the sense that it leads to the null Killing vector, so that the fake KSE should be augmented by one of EOM, $0 = E_{\dot{+}+}$ in our convention, to imply the full EOM. We have also shown that this additional equation can be regarded as a certain constraint by using the effective action method.

We also computed the renormalized boundary stress tensor from which we determined the mass and the angular momentum of our black hole solutions with a scalar hair. They saturate the mass bound for the angular momentum just like the usual extremally rotating BTZ black holes.

It is somewhat unclear how to obtain all the first order equations in the effective action formalism while the fake supersymmetry formalism may not be complete in the case of null Killing spinor. It would be very interesting to investigate further the nature of the missing equation in the generic context of the fake supersymmetry formalism. Our investigation suggests that it may correspond to a constraint equation. In this context it would be nice if one can identify the missing equation through the canonical approach with light-cone foliation.

The fake supersymmetry formalism has been a powerful tool to study the BPS states in gravity models. Since the theory itself is not supersymmetric, the solutions of fake KSE are not guaranteed to be stable. It would be an separate issue to determine the stability of those solutions. It would also be very interesting to extend the fake supersymmetry formalism to the higher derivative gravity with scalar fields.

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Appendix

A Canonical Formalism

In this appendix, we describe the canonical formalism of our model. Since the canonical formalism for the scalar field is trivial, we focus on the formalism for the metric. The aim of this section is to indicate that the missing equation in the fake supersymmetry formalism may be connected with the Hamiltonian and momentum constraints. Here, we adopt the standard notation in the canonical formulation with time-like foliation, which will be used only in this appendix.

Through the ADM decomposition of the metric as
\[ ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N_i dt)(dx^j + N_j dt), \] (A.1)
one can apply the canonical formalism to gravity. In this formulation, \( \gamma_{ij} \)'s are taken as canonical variables and their conjugate momentums are given in terms of the extrinsic curvature \( K_{ij} \) by
\[ \pi^{ij} \equiv \sqrt{\gamma} \left[ K^{ij} - \gamma^{ij} K \right]. \] (A.2)

In our convention the extrinsic curvature \( K_{ij} \) is defined by
\[ K_{ij} \equiv \frac{1}{2N} \left[ \partial_t \gamma_{ij} - \nabla_i N_j - \nabla_j N_i \right], \quad K \equiv \gamma^{ij} K_{ij} \]
where \( \nabla_i \) denotes the covariant derivative with respect to the metric \( \gamma_{ij} \).

By diffeomorphism invariance, one obtains constraints which are called as Hamiltonian and momentum constraints. These constraints can be written in our case respectively as
\[ 0 = \mathcal{H} = -\sqrt{\gamma} \left[ (2)^R - \frac{1}{2} \partial^i \phi \partial_i \phi - V \right] + \frac{1}{\sqrt{\gamma}} \left[ \pi^{ij} \pi_{ij} - \pi^2 + \frac{1}{2} \pi_{\phi}^2 \right], \] (A.3)
\[ 0 = \mathcal{P}_i = -2\sqrt{\gamma} \nabla_j \left( \frac{1}{\sqrt{\gamma}} \pi^j_i \right) + \pi_{\phi} \partial_i \phi, \]
where \((2)^R\) denotes the curvature scalar in two dimensions for \((r, \theta)\) and
\[ \pi_{\phi} \equiv -\frac{1}{N} \sqrt{\gamma} \left( \partial_t \phi - N^i \partial_i \phi \right) \]
denotes the conjugate momentum for the scalar field \( \phi \). Using our ansatz for the black hole metric, one can see that these constraints lead to
\[ 0 = L^2 e^{2B} V + \frac{1}{2} \phi'^2 - \frac{2}{r} B' + \frac{r^2}{2} (e^C)'^2 e^{-2A}, \] (A.4)
\[ 0 = \left( r^3 e^{-(A+B)} (e^C)' \right)'. \]

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The canonical Hamiltonian is given by

\[ H = \int d^2 x \sqrt{\gamma} \left[ \mathcal{N} \mathcal{H} + \mathcal{N}^i \mathcal{P}_i \right] + \text{surface term}, \]  

(A.5)

and dynamical equations in the canonical formalism are given by

\[
\frac{\delta H}{\delta \pi_{ij}} = \partial_t \gamma_{ij}, \quad \frac{\delta H}{\delta \gamma_{ij}} = -\partial_t \pi_{ij},
\]

where the first equation is nothing but the condition determining the extrinsic curvature by \( \partial_t \gamma_{ij} \). In terms of the extrinsic curvature \( K_{ij} \), the second dynamical equation can be written as

\[
\partial_t K_{ij} = N^k \nabla_k K_{ij} + K_{ik} \nabla_j N^k + K_{jk} \nabla_i K^k + \nabla_i \nabla_j N \tag{A.6}
\]

\[
- N \left[ (2)R_{ij} - 2K^k_k K_{kj} + K K_{ij} - \frac{1}{2} \partial_i \phi \partial_j \phi - \gamma_{ij} V(\phi) \right].
\]

As is clear from this expression, this equation leads to two equations among EOM for the metric field as follows

\[
0 = L^2 e^{2B} V + \frac{1}{2} \phi'^2 + A'' + A'^2 - A'B' - \frac{1}{r} B' - \frac{r^2}{2} C'^2 e^{2C-2A},
\]

\[
0 = L^2 e^{2B} + \frac{1}{r} (A' - B') + \frac{r^2}{2} C'^2 e^{2C-2A}.
\]

These equations correspond to \( 0 = E_{rr} \) and \( 0 = E_{\theta\theta} \) in the full EOM. If one use the first order equations obtained from fake supersymmetry formalism, the combination of those constraints and the equation \( 0 = E_{\theta\theta} \) becomes the missing equation. This indicates that the constraint by the Lagrange multiplier \( e^{A-B} \) in the effective action may appear in the light-cone foliation.
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