Inverse problems for general second order hyperbolic equations with time-dependent coefficients

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March 4, 2015

Abstract

We study the inverse problems for the second order hyperbolic equations of general form with time-dependent coefficients assuming that the boundary data are given on a part of the boundary. The approach of this paper is a variant of the Boundary Control (BC) method developed in [E1], [E2]. We extend the results and simplify the proofs of author’s earlier works [E1], [E2], [E3], [E4] to the general case of arbitrary Lorentzian time-dependent metrics.

1 Introduction.

Consider a second order hyperbolic equation in $\mathbb{R}^{n+1}$ of the form

\begin{equation}
\sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^{n}g(x)}} \left(-i \frac{\partial}{\partial x_j} - A_j(x)\right) \sqrt{(-1)^{n}g(x)} g^{jk}(x) \left(-i \frac{\partial}{\partial x_k} - A_k(x)\right) u(x) = 0,
\end{equation}

where $x = (x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1}$, $x_0$ is the time variable. In (1.1) $g(x) = \det[g_{jk}(x)]_{j,k=0}^{n}$, where $[g_{jk}(x)]_{j,k=0}^{n} = ([g^{jk}]_{j,k=0}^{n})^{-1}$ is the metric tensor, $A(x) = \ldots$
\((A_0(x), A_1(x), ..., A_n(x))\) is the vector potential. We assume that all coefficients in (1.1) belong to \(C^\infty(\mathbb{R}^{n+1})\) and that

\[
g^{00}(x) \geq c_0 > 0, \quad \forall x \in \mathbb{R}^{n+1}.
\]

Let \((\xi_0, \xi_1, ..., \xi_n)\) be dual variables to \((x_0, x_1, ..., x_n)\). The strict hyperbolicity of (1.1) with respect to \(\xi_0\) means that the quadratic equation

\[
\sum_{j,k=0}^{n} g^{jk}(x) \xi_j \xi_k = 0
\]

has two real distinct roots \(\xi^-_0(\xi_1, ..., \xi_n) < \xi^+_0(\xi_1, ..., \xi_n)\) for all \((\xi_1, ..., \xi_n) \neq (0, ..., 0)\) and all \(x \in \mathbb{R}^{n+1}\). We have

\[
\xi^+_0(\xi_1, ..., \xi_n) = -\sum_{j=1}^{n} g^{j0}(x) \xi_j \pm \sqrt{\left(\sum_{j=1}^{n} g^{j0}(x) \xi_j\right)^2 - g^{00}(x) \sum_{j,k=1}^{n} g^{jk}(x) \xi_j \xi_k} \frac{g^{00}}{g^{00}}.
\]

The strict hyperbolicity implies that

\[
\left(\sum_{j=1}^{n} g^{j0}(x) \xi_j\right)^2 - g^{00}(x) \sum_{j,k=1}^{n} g^{jk}(x) \xi_j \xi_k > 0
\]

for all \((\xi_1, ..., \xi_n) \neq 0, \ x \in \mathbb{R}^n\).

In this paper we assume a more restrictive condition that

\[
\sum_{j,k=1}^{n} g^{jk}(x) \xi_j \xi_k \leq -c_1 \sum_{j=1}^{n} \xi_j^2,
\]

i.e. we assume that the spatial part of the equation (1.1) is elliptic for any \(x \in \mathbb{R}^{n+1}\).

Note that the quadratic form (1.3) has the signature \((+1, -1, ..., -1)\). Therefore \((-1)^n g(x) > 0\). We assume also that \(A_j(x), 0 \leq j \leq n,\) are real-valued. Thus the operator in (1.1) is formally self-adjoint.

We consider the following class of domains \(D \subset \mathbb{R}^{n+1}\). Let \(D_t = D \cap \{x_0 = t\}\) be the intersection of \(D\) with the plane \(\{t = x_0\}, t \in \mathbb{R}\). We assume that \(\partial D_t\) is a smooth closed bounded domain in \(\mathbb{R}^n\) smoothly dependent
and uniformly bounded in $t$. Let $S(x_0, x_1, ..., x_n) = 0$ be the equation of $\partial D = \bigcup_{t \in \mathbb{R}} \partial D_t$. We assume that $S$ is a time-like smooth surface in $\mathbb{R}^{n+1}$, i.e.

$$
(1.7) \quad \sum_{j,k=0}^{n} g^{jk}(x)\nu_j \nu_k < 0,
$$

where $x \in S$ and $(\nu_0, \nu_1, ..., \nu_n)$ is a normal vector to $S = 0$. The vector $(\nu_0, ..., \nu_n)$ satisfying (1.7) is called a space-like vector. Also, the surface $\Sigma$ in $\mathbb{R}^{n+1}$ is called space-like if $\sum_{j,k=0}^{n} g^{jk}(x)\nu_j(x)\nu_k(x) > 0$, where $x \in \Sigma$ and $(\nu_0(x), ..., \nu_n(x))$ is the normal vector to $\Sigma$.

Consider the initial-boundary value problem

\begin{align*}
(1.8) & \quad Lu = 0 \quad \text{in } D, \\
(1.9) & \quad u = 0 \quad \text{for } x_0 \ll 0 \quad \text{in } D, \\
(1.10) & \quad u|_S = f,
\end{align*}

where $f$ is a smooth function on $S = 0$ with compact support, $Lu = 0$ is the same as in (1.1).

It is well known that the initial-value problem (1.8), (1.9), (1.10) is well-posed (cf. [H]), assuming that (1.2) and (1.7) are satisfied.

Let $S_0 \subset S$ be a part of $S$ such that (see Fig. 1.1) $\partial D_t \cap S$ has a nonempty interior for all $t \in \mathbb{R}$. We assume also that for any $x^{(0)} \in \partial S_0$ a vector $\tau^{(1)}$, tangent to $S$ and normal to $S_0$, is not parallel to $(1,0,...,0)$.

![Diagram](image)

**Fig. 1.1.** $S_0$ is part of the boundary $S$, $S_0 \cap \partial D_t \neq \emptyset$ for $\forall t$. 3
The Dirichlet-to-Neumann operator $\Lambda$ that maps the Dirichlet data to the Neumann data on $\partial D$ is defined as

$$
\Lambda f = \sum_{j,k=0}^{k} g^{jk}(x) \left( \frac{\partial u}{\partial x_j} - i A_j(x) u \right) \nu_k(x) \left( \sum_{p,r=0}^{n} g^{pr}(x) \nu_r \nu_p \right)^{-\frac{1}{2}} \bigg|_{S},
$$

where $u(x)$ is the solution of (1.8), (1.9), (1.10) and $(\nu_0(x), \ldots, \nu_n(x))$ is the unit outward normal to $S$.

Denote by

$$
y = y(x)
$$

a proper diffeomorphism of $\mathcal{D}$ onto $\hat{D}$ such that

$$
y = x \quad \text{on} \quad S_0.
$$

We call a diffeomorphism of $\mathcal{D}$ onto $\hat{D}$ proper if for any $[t_1, t_2] \subset \mathbb{R}$ the image of $\mathcal{D} \cap \{ t_1 \leq x_0 \leq t_2 \}$ is a domain $\mathcal{D} \cap \{ S_{-}^{(t_1)}(y_1, \ldots, y_n) \leq y_0 \leq S_{+}^{(t_2)}(y_1, \ldots, y_n) \}$, where $y_0 = S_+^{(t_1)}, y_0 = S_-^{(t_2)}$ are space-like surfaces.

Let $\hat{L} \hat{u} = 0$ be the equation (1.1) in $y$-coordinates, $y \in \hat{D}$. We have

$$
\hat{L} \hat{u} \equiv \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n \hat{g}(y)}} \left( - i \frac{\partial}{\partial y_j} - \hat{A}_j(y) \right) \sqrt{(-1)^n \hat{g}(y) \hat{g}^{jk}(y)} \cdot \left( - i \frac{\partial}{\partial y_k} - \hat{A}_k(y) \right) \hat{u} = 0,
$$

where

$$
\hat{g}^{jk}(y) = \sum_{p,r=0}^{n} g^{pr}(x) \frac{\partial y_j}{\partial x_p} \frac{\partial y_k}{\partial x_r},
$$

$$
\hat{A}_j(x) = \sum_{k=0}^{n} \hat{A}_k(y) \frac{\partial y_k}{\partial x_j}, \quad 0 \leq j \leq n,
$$

Here

$$\hat{u}(y) = u(x), \quad y = y(x), \quad \hat{g}(y) = \det[\hat{g}_{jk}(y)]_{j,k=0}^{n}, \quad [\hat{g}_{jk}(y)]_{j,k=0}^{n} = \left( [\hat{g}^{jk}(y)]_{j,k=0}^{n} \right)^{-1}.$$
Note that (1.15), (1.16) are equivalent to the equalities

\begin{align}
    \sum_{k=0}^{n} A_k(x) dx_k &= \sum_{k=0}^{n} \hat{A}_k(y) dy_k, \\
    \sum_{j,k=0}^{n} \hat{g}_{jk}(y) dy_j dy_k &= \sum_{j,k=0}^{n} g_{jk}(x) dx_j dx_k,
\end{align}

where \( y \) and \( x \) are related by (1.12). Metric tensors \([g_{jk}(x)]_{j,k=0}^{n}\) and \([\hat{g}_{jk}(x)]_{j,k=0}^{n}\), related by (1.18), are called isometric.

We assume that conditions (1.2), (1.6) hold also in \( y \)-coordinates, i.e.

\begin{align}
    \hat{g}^{00}(y) &\geq C_0, \\
    \sum_{j,k=1}^{n} \hat{g}^{jk}(y) \xi_j \xi_k &\leq -C_1 \sum_{j=1}^{n} \xi_j^2.
\end{align}

Consider a subclass \( y = \hat{y}(x) \) of the maps (1.12), (1.13) of the form

\begin{align}
    y_0 = x_0, \quad y_j = \hat{y}_j(x_0, x_1, ..., x_n), \quad 1 \leq j \leq n,
\end{align}

where \( \hat{y}_j(x) \) depend smoothly on \( x_0 \).

Note that \( y = \hat{y}(x) \) maps \( \hat{D}_t \) onto \( \hat{D}_t \) for any \( t \in \mathbb{R} \). One can find \( y = \hat{y}(x) \) such that \( \hat{D}_t = \hat{D}_0 \) and \( \hat{S}_0t = \Gamma_0 \) for all \( t \in \mathbb{R} \), i.e. \( \hat{D} = \hat{D}_0 \times \mathbb{R}, \quad \hat{S}_0 = \Gamma_0 \times \mathbb{R} \), i.e. \( \hat{D} \) and \( \hat{S}_0 \) are cylindrical domains (see Fig. 1.2).

\[ \text{Fig. 1.2.} \quad \hat{D} = \hat{D}_0 \times \mathbb{R}, \quad \hat{S}_0 = \Gamma_0 \times \mathbb{R} \text{ are cylindrical domains.} \]
Here \( \hat{S}_0 \) is the image of \( S_0 \) under the map \( y = \hat{y}(x) \). Note that we can also arrange that \( \Gamma_0 \) is contained in the plane \( \hat{y}_n = 0 \) if \( \hat{\Gamma}_0 \) is small.

Let \( c(x) \in C^\infty(\overline{D}) \) be such that

\[
|c(x)| = 1, \quad x \in \overline{D},
\]

and

\[
c(x) = 1 \quad \text{on} \quad S_0.
\]

The group \( G_0(\overline{D}) \) of such \( c(x) \) is called the gauge group.

If \( Lu = 0 \) then \( u' = c^{-1}(x)u(x) \) satisfies the equation of the form \((1.1)\) with \( A_j(x) \) replaced by

\[
A'_j(x) = A_j(x) - ic^{-1}(x)\frac{\partial c}{\partial x_j}, \quad 1 \leq j \leq n,
\]

\[
A'_0(x) = A_0(x) + ic^{-1}(x)\frac{\partial c}{\partial x_0}.
\]

We shall call potentials \((A'_0, \ldots, A'_n(x))\) and \((A_0(x), \ldots, A_n(x))\) related by \((1.23)\) gauge equivalent. Note that when \( D \) is simply connected then \( c(x) = \exp i\varphi \) where \( \varphi(x) \in C^\infty(\overline{D}), \ \varphi(x) \) is real-valued and \( \varphi(x) = 0 \) on \( S_0 \).

Let \( y = y(x) \) be the change of variables, such that \( y(x) = x, \ x \in S_0 \), transforming the equation \( Lu = 0 \) in \( D \) to the equation of the form \((1.14)\) in \( \hat{D} \). We consider the initial-boundary value problem

\[
\hat{L}u = 0 \quad \text{in} \quad \hat{D},
\]

\[
\dot{u} = 0 \quad \text{for} \quad y_0 \ll 0, \ y \in \hat{D},
\]

\[
\dot{u}|_{\hat{S}} = f.
\]

Note that \( \hat{S}_0 = S_0 \), since \( \hat{y}(x) = x \) on \( S_0 \).

Since \((1.19)\) holds the initial-boundary value problem \((1.24), (1.25), (1.26)\) is also well-posed. Let \( \hat{c}(y) \in G_0(\overline{D}) \). Make the gauge transformation \( u'(y) = \hat{c}^{-1}(y)\dot{u}(y) \) and let \( L' \) be such that \( L'u' = 0 \). We have

\[
\hat{L}'u' = 0 \quad \text{in} \quad \hat{D},
\]

\[
u' = 0 \quad \text{for} \quad y_0 \ll 0, \ y \in \hat{D},
\]

\[
u'|_{\hat{S}} = f.
\]
Note that \( u' = \hat{u} \) on \( \hat{S}_0 \) since \( \hat{c}(y) = 1 \) on \( \hat{S}_0 \) and \( L'u' \) has the form

\[
(1.30) \quad L'u' = \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n \hat{g}(y)}} \left( -i \frac{\partial}{\partial y_j} A_j'(y) \right) \sqrt{(-1)^n \hat{g}(y) \hat{g}^{jk}(y) \left( -i \frac{\partial}{\partial y_k} A_k'(y) \right) } u'(y)
\]

\( = 0 \),

\( A_j'(y), \ 0 \leq j \leq n \), are potentials gauge equivalent to \( \hat{A}_j(y), \ 0 \leq j \leq n \).

Let \( \Lambda' \) be the DN operator for \((1.27), (1.28), (1.29)\)

\[
(1.31) \quad \Lambda' f = \sum_{j,k=0}^{k} \hat{g}^{jk}(y) \left( \frac{\partial u'}{\partial y_j} + i A_j'(y) u' \right) \nu_k(y) \left( \sum_{p,r=0}^{n} \hat{g}^{pr}(y) \nu_p(y) \nu_r(y) \right)^{-\frac{1}{2}} |_{\hat{S}},
\]

where \( f \) is the same as in \((1.10)\) and \((1.29)\).

It can be shown that

\[
(1.32) \quad \Lambda f \big|_{S_0} = \Lambda' f \big|_{\hat{S}_0}, \quad \forall f \in C^\infty_0(S_0),
\]

if the operator \( L' \) is obtained from \( L \) by the change of variables \((1.12), (1.13)\) and the gauge transformation \( c(y) \) such that \((1.21), (1.22)\) hold.

Therefore the inverse problem of the determination of the coefficient of \((1.1)\) can be solved only up to the changes of variables \((1.12), (1.13)\) and the gauge transformations \((1.21), (1.22)\).

We shall formulate now some conditions which will be required to solve the inverse problem.

1) Real analyticity in the time variable

One of the crucial steps in solving the inverse problem will be the use of the following unique continuation theorem of Tataru and Robbiano and Zuily (cf [T], [RZ]) that requires the analyticity in \( x_0 \):

**Theorem 1.1.** Let the coefficients of \((1.1)\) be analytic in \( x_0 \). Consider the equation \( Lu = 0 \) in a neighborhood \( U_0 \) of a point \( P_0 \). Let \( \Sigma = 0 \) be a non-characteristic surface with respect to \( L \) passing through \( P_0 \). If \( u = 0 \) in \( U_0 \cap \{ \Sigma < 0 \} \) then \( u = 0 \) in \( U_0 \cap \{ \Sigma > 0 \} \).
We shall assume not only that the coefficients of $L$ are analytic in $x_0$ but also the boundary $S(x) = 0$, the maps (1.12) and (1.20) and the gauges $c(x)$ are all analytic in $x_0$. Thus the coefficients of operators $\hat{L}, L'$ are also analytic in $y_0$.

Let $y = \varphi(x)$ be a diffeomorphism of neighborhood $U_0$ onto the neighborhood $V_0 = \varphi(U_0)$. Here $\varphi(x)$ is smooth but not analytic in any variable. It is clear that if the unique continuation property for the operator $L$ holds in $U_0$ then it holds in $V_0$ for the operator $\hat{L} = \varphi \circ L$, though the coefficients of $\hat{L}$ are not analytic. Therefore the following more general class of operators $L$ with non-analytic coefficients has the unique continuation property: For each point $x(0)$ on $D$ there is a neighborhood $U_0$ and the diffeomorphism $\Psi(x)$ of $U_0$ onto $V_0 = \psi(U_0)$ such that the coefficients of the operators $\psi \circ L$ in $V_0$ are analytic in $x_0$. Thus, the unique continuation property holds for $L$ in $U_0$.

2) The Bardos-Lebeau-Rauch condition

Consider the initial-boundary problem

$$Lu = 0, \quad u = 0 \quad \text{for} \quad x_0 \ll 0, \quad u|_{\partial D_0 \times \mathbb{R}} = f$$

in the cylindrical domain $D_0 \times \mathbb{R}$, $f$ has a compact support in $\Gamma_0 \times \mathbb{R}$, $\Gamma_0 \subset \partial D_0$. We say that BLR condition holds on $[t_0, T_{t_0}]$ if the bounded map from $f \in H_1(\Gamma_0 \times (t_0, T_{t_0}))$ to $(u|_{x_0=t_{t_0}}, \frac{\partial u}{\partial x_0}|_{x_0=t_{t_0}}) \in H_1(D_0) \times L_2(D_0)$, is onto in $H_1(D_0) \times L_2(D_0)$, where $u = 0$ for $x_0 < t_0$, $f = 0$ for $x_0 < t_0$.

Note that BLR condition obviously holds on $[t_0, T]$ for any $T > T_{t_0}$ if it holds on $[t_0, T_{t_0}]$.

Let $x = x(s), \xi = \xi(s) \in T^*_0(D_0 \times [t_0, T_{t_0}]),$ where

$$\frac{dx_j}{ds} = \frac{L_0(x(s), \xi(s))}{\partial \xi_j}, \quad x_j(0) = y_j, \quad 0 \leq j \leq n,$$

$$\frac{d\xi_j}{ds} = -\frac{L_0(x(s), \xi(s))}{\partial x_j}, \quad \xi_j(0) = \eta_j, \quad 0 \leq j \leq n,$$

be the equations of null-characteristics. Here $L_0(x, \xi) = \sum_{j,k=0}^n g^{jk}(x)\xi_j\xi_k$, $L_0(y, \eta) = 0$.

Let $t_0$ be arbitrary. It is shown in [BLR] that the BLR condition holds on $[t_0, T_{t_0}]$ for some $T_{t_0}$ if any null bicharacteristic in $T^*_0(\overline{D_0} \times [t_0, T])$ intersects $T^*_0(\Gamma_0 \times [t_0, T])$ when $T \geq T_{t_0}$.
2) Domains of influence

Let \( G(x, \xi) = \sum_{j,k=0}^{n} g_{jk}(x) \xi_j \xi_k \), \( [g_{jk}]_{j,k=0}^{n} = ([g_{jk}]_{j,k=0}^{n})^{-1} \). We say that \( x = x(\tau) \) is a forward time-like ray in \( D_0 \times \mathbb{R} \) if \( x = x(\tau) \) is piece-wise smooth, \( G(x(\tau), \frac{dx(\tau)}{d\tau}) > 0 \) and \( \frac{dx_0}{d\tau} > 0, 0 \leq \tau \). If \( G(x(\tau), \frac{dx(\tau)}{d\tau}) > 0 \) and \( \frac{dx_0}{d\tau} < 0 \) the ray \( x = x(\tau) \) is called the backward time-like ray.

One can show (cf [CH]) that the forward domain of influence \( D_+(F) \) of a closed set \( F \subset D_0 \times \mathbb{R} \) is the closure of the union of all piece-wise smooth forward time-like rays in \( D_0 \times \mathbb{R} \) starting on \( F \).

Analogously, the backward domain of influence \( D_-(F) \) of the closed set \( F \subset D_0 \times \mathbb{R} \) is the closure of the union of all backward time-like piece-wise smooth rays in \( D_0 \times \mathbb{R} \) starting at \( F \). The domain of dependence of \( F \) is the intersection \( D_+(F) \cap D_-(F) \).

Let \( \Gamma \subset \partial D_0 \) and let \( Lu = 0 \) in \( D_0 \times \mathbb{R} \). A consequence of the unique continuation property is that \( u \bigg|_{\Gamma \times [t_1, t_2]} = \frac{\partial u}{\partial \nu} \bigg|_{\Gamma \times [t_1, t_2]} \) implies \( u = 0 \) in the domain of dependence of \( \Gamma \times [t_1, t_2] \) (cf [KKL1]). Here \( \frac{\partial}{\partial \nu} \) is the normal derivative to \( \Gamma \).

In particular, we shall use the following fact:

Consider \( \Gamma \times [t_1, t_2], \Gamma \subset \partial D_0 \). Suppose \( [t_1, t_2] \) is arbitrary large. Then the domain of dependence of \( \Gamma \times [t_1, t_2] \) contains \( D_0 \times [t_1 + \delta, t_2 - \delta] \) for some \( \delta > 0 \) dependent of the metric and the domain.

In this paper we will not attempt to estimate \( \delta > 0 \) since \( [t_0 + \delta, t_2 - \delta] \) is also arbitrary large if \( [t_1, t_2] \) is arbitrary large.

Now we shall state the main result of this paper.

Making the change of variables of the form (1.20) we can, without loss of generality, assume that the domain \( D \) is cylindrical (cf Fig. 1.2), i.e. \( D = D_0 \times (-\infty, +\infty), S_0 = \Gamma_0 \times (-\infty, +\infty) \).

We consider the initial-boundary value problem

\begin{align}
(1.34) & \quad Lu = 0 \quad \text{in} \quad D_0 \times \mathbb{R}, \\
(1.35) & \quad u = 0 \quad \text{when} \quad x_0 \ll 0, \\
(1.36) & \quad u \bigg|_{\partial D_0 \times \mathbb{R}} = f,
\end{align}

where \( L \) has the form (1.1) and \( f \) has a compact support in \( \Gamma_0 \times \mathbb{R} \). Consider
another initial-boundary problem

\begin{align}
L'u' &= 0 \quad \text{in } \hat{D}_0 \times \mathbb{R}, \\
u' &= 0 \quad \text{when } y_0 \ll 0, \\
u'|_{\partial \hat{D}_0 \times \mathbb{R}} &= f',
\end{align}

where \( L'u' \) has the form (1.30), \( f' \) has a compact support in \( \Gamma_0 \times \mathbb{R} \).

**Theorem 1.2.** Consider two initial-boundary value problems (1.34), (1.35), (1.36) and (1.37), (1.38), (1.39) in domains \( D_0 \times \mathbb{R} \) and \( \hat{D}_0 \times \mathbb{R} \), respectively. Suppose \( A_j(x), A'_j(x), 0 \leq j \leq n \), are real-valued. Assume that \( \Gamma_0 \subset \partial D_0 \cap \partial \hat{D}_0 \) is nonempty and open. Let \( \Lambda \) and \( \Lambda' \) be the corresponding DN operators for \( L \) and \( L' \). Assume that \( \Lambda f|_{\Gamma_0 \times \mathbb{R}} = \Lambda' f|_{\Gamma_0 \times \mathbb{R}} \) for all smooth \( f \) with compact support in \( \Gamma_0 \times \mathbb{R} \). Suppose the conditions (1.2), (1.3) hold for \( L \) and \( L' \). Assume that the coefficients of \( L \) and \( L' \) are analytic in \( x_0 \) and \( y_0 \), respectively. Suppose also that BLR condition holds for (1.34), (1.35), (1.36) on \([t_0, T] \) for each \( t_0 \in \mathbb{R} \). Then there exists a proper map \( y = y(x) \) of \( D_0 \times \mathbb{R} \) onto \( \hat{D}_0 \times \mathbb{R} \), \( y = x \) on \( \Gamma_0 \times \mathbb{R} \), and there exists a gauge transformation with the gauge \( c'(y) \in C_0(\hat{D}_0 \times \mathbb{R}) \), \( c'(y) = 1 \) on \( \Gamma_0 \times \mathbb{R} \) such that \( L' = c' \circ y^* L \). Here \( y^* \circ L \) is the operator with \([\hat{g}^{jk}(y)]_{j,k=0}^n \) and \( \hat{A}_k(y) \), \( 0 \leq k \leq n \) as in (1.15), (1.16), \( c' \circ y^* \circ L \) is the operator with potentials \( A'_j(y) \), \( 0 \leq j \leq n \), gauge equivalent to \( \hat{A}_k(y) \), \( 0 \leq k \leq n \).

We end the introduction with the outline of the previous work and a short description of the content of the paper.

The first result on inverse hyperbolic problems with the data on the part of the boundary was obtained by Isakov in [I1]. The powerful Boundary Control (BC) method was discovered by Belishev [B1] and was further developed by Belishev [B2], [B3], [B4], Belishev and Kurylev [BK], Kurylev and Lassas [KL1], [KL2] and others (see [KKL1], [KKL2]). In [E1], [E2] the author proposed a new approach to hyperbolic inverse problems that uses substantially the idea of BC method. This approach was extended in [E3] to a class of time-dependent hyperbolic problems and in [E4] to the case of hyperbolic equations of general form with time-independent coefficients. The generalization to the case of Yang-Mills potentials was considered in [E7]. The inverse problems for the D’Alambert equation with the time-dependent scalar potentials were considered earlier by Stefanov [S] and Ramm and Sjostrand [RS] (see also Isakov [I2]). The case of the D’Alambert equation with time-dependent vector potentials was studied by Salazar [S1], [S2]:
The following observation of the importance of studying hyperbolic equations with time-dependent coefficients was made by Bardos-Lebeau-Rauch in [BLR]:

The linearizations of basic nonlinear evolution equations of mathematical physics are linear hyperbolic equations with time-dependent coefficients.

In this paper we study the inverse problem for general second order hyperbolic equations with time-dependent coefficients. The proofs are an extension and simplification of corresponding proofs in [E1], [E2], [E3], [E4] for a more general case.

The main step in the proof is the local step of solving the inverse problem in a small neighborhood near the boundary. This is done in §§2-6. In the last §7 we consider the global step leading to the proof of Theorem 1.2.

2 The Goursat coordinates

We shall prove first the Theorem 1.2 in the small neighborhood of the boundary $\partial D$.

Let $x^{(0)} \in S_0$ and let $U_0 \subset \mathbb{R}^{n+1}$ be a small neighborhood of $x^{(0)}$.

Suppose that we already did the change of variables to make $\partial D$ and $S_0$ cylindrical, i.e. $\partial D = \partial D_0 \times \mathbb{R}$ and $S_0 = \Gamma_0 \times \mathbb{R}$. We assume that we have chosen the coordinates $(x_0, x', x_n)$, $x' = (x_1, \ldots, x_{n-1})$ in $U_0$ such that $x_n = 0$ is the equation of $U_0 \cap \partial D$ and $U_0 \cap D$ is contained in the half-space $x_n > 0$. Let $(x_0^{(0)}, x_1^{(0)}, \ldots, x_{n-1}^{(0)}, 0)$ be the coordinates of the point $x^{(0)}$. Let $T_1 < x_0^{(0)} < T_2, T_2 - T_1$ is small.

Consider the initial-boundary value problem in $U_0 \cap D$:

\begin{equation}
Lu = 0, \quad x_n > 0, \quad T_1 < x_0 < T_2, \tag{2.1}
\end{equation}

\begin{equation}
|_{x_0=T_1} u = 0, \quad |_{x_0=T_1} \frac{\partial u}{\partial x_0} = 0, \tag{2.2}
\end{equation}

\begin{equation}
|_{x_n=0} u = g(x_0, x'). \tag{2.3}
\end{equation}

We assume that $L$ has the form (1.1). For the simplicity, we shall not change the notations when choosing the local coordinates such that the equation of $U_0 \cap S_0$ is $x_n = 0$. Assume that $\text{supp} \ g \subset U_0 \cap (\Gamma_0 \times [T_1, T_2])$, $g = 0$ for $x_0 < T_1$. Note that $\text{supp} \ u(x_0, x', x_n) \cap [T_1, T_2] \subset U_0 \cap [T_1, T_2]$ for $x_n > 0$ if $T_2 - T_1$ is small.
We introduce new coordinates to simplify the operator \( L \) (cf. [E4], pages 327-329) that we called the Goursat coordinates.

Denote by \( \psi^\pm(x) = (x_0, x', x_n) \) the solutions of the eikonal equations

\[
\sum_{j,k=0}^{n} g^j_k(x_0, x', x_n) \psi^\pm_{x_j}(x) \psi^\pm_{x_k}(x) = 0, \quad x_n > 0,
\]

with initial conditions

\[
\psi^+|_{x_n=0} = x_0 - T_1, \quad \psi^-|_{x_n=0} = T_2 - x_0.
\]

Since (2.4) is a quadratic equation in \( \psi^\pm_{x_n} \) one has to specify the sign of the square root. We have

\[
g^{nn}(\psi^\pm_{x_n})^2 + 2 \sum_{j=0}^{n-1} g^{nj}\psi^\pm_{x_j}\psi^\pm_{x_n} + \sum_{j,k=0}^{n-1} g^{jk}\psi^\pm_{x_j}\psi^\pm_{x_k} = 0.
\]

We will need \( \psi^+_n + \psi^-_n < 0 \) for \( x_n > 0 \) (cf. (2.16) below). So we choose the plus sign of the square root:

\[
\psi^\pm_{x_n} = -\sum_{j=0}^{n-1} g^{nj}\psi^\pm_{x_j} + \sqrt{\left(\sum_{j=0}^{n-1} g^{nj}\psi^\pm_{x_j}\right)^2 - g^{nn}\left(\sum_{j,k=0}^{n-1} g^{jk}\psi^\pm_{x_j}\psi^\pm_{x_k}\right)}}
\]

Note that \( g^{nn}(x) < 0, \psi^\pm_{x_0}|_{x_n=0} = \pm 1 \). Therefore \( \psi^\pm_{x_n}|_{x_n=0} = \frac{\mp g^{00} + \sqrt{(g^{00})^2 - g^{nn}g^{00}}}{g^{nn}} \).

The solutions \( \psi^\pm(x) \) exists for \( 0 < x_n < \delta, \delta \) is small. We assume that \( \delta \) is such that surfaces \( \psi^+ = 0 \) and \( \psi^- = 0 \) intersect when \( x_n < \delta \) and are inside \( U_0 \) when \( x_n < \delta \).

Let \( \varphi_j(x_0, x', x_n), 1 \leq j \leq n - 1 \), be solutions of the linear equation

\[
\sum_{p,k=0}^{n} g^{pk}(x_0, x', x_n) \psi^-_{x_p} \varphi_{jx_k} = 0, \quad x_n > 0,
\]

with initial condition

\[
\varphi_j(x_0, x', 0) = x_j, \quad 1 \leq j \leq n - 1.
\]

Make the following change of variables in \( U_0 \cap [T_1, T_2] \):

\[
s = \psi^+(x_0, x', x_n),
\]

\[
\tau = \psi^-(x_0, x', x_n),
\]

\[
y_j = \varphi_j(x_0, x', x_n), 1 \leq j \leq n - 1.
\]
Equation (1.1) has the following form in \((s, \tau, y')\) coordinates where \(y' = (y_1, ..., y_{n-1})\)

\[
\hat{L} u \overset{\text{def}}{=} -\frac{2}{\sqrt{|g|}} \left( \frac{\partial}{\partial s} + i \hat{A}_+(s, \tau, y') \right) \sqrt{| \hat{g} |} \hat{g}^{+,-}(s, \tau, y') \left( \frac{\partial}{\partial \tau} + i \hat{A}_+ \right) \hat{u} \\
- \frac{2}{\sqrt{|g|}} \left( \frac{\partial}{\partial \tau} + i \hat{A}_-(s, \tau, y') \right) \sqrt{| \hat{g} |} \hat{g}^{+,-}(s, \tau, y') \left( \frac{\partial}{\partial s} + i \hat{A}_- \right) \hat{u} \\
+ \sum_{j=1}^{n-1} \frac{2}{\sqrt{|g|}} \left( \frac{\partial}{\partial y_j} - i \hat{A}_j(s, \tau, y') \right) \sqrt{| \hat{g} |} \hat{g}^{+,-}(s, \tau, y') \left( \frac{\partial}{\partial \tau} - i \hat{A}_j \right) \hat{u} \\
+ \sum_{j=1}^{n-1} \frac{2}{\sqrt{|g|}} \left( \frac{\partial}{\partial s} + i \hat{A}_j(s, \tau, y') \right) \sqrt{| \hat{g} |} \hat{g}^{+,-}(s, \tau, y') \left( \frac{\partial}{\partial y_j} + i \hat{A}_j \right) \hat{u} \\
+ \sum_{j,k=1}^{n-1} \frac{1}{\sqrt{|g|}} \left( \frac{\partial}{\partial y_j} - i \hat{A}_j(s, \tau, y') \right) \sqrt{| \hat{g} |} \hat{g}^{+,-}(s, \tau, y') \left( \frac{\partial}{\partial y_k} - i \hat{A}_k \right) \hat{u},
\]

where

\[
\hat{g} = -(4 \hat{g}^{+,-})^{-2} \left( \det[ \hat{g}^{j,k}_{j,k=1} ]^{n-1} \right)^{-1}.
\]

Note that terms containing \(\frac{\partial^2}{\partial s^2}, \frac{\partial^2}{\partial \tau^2}, \frac{\partial^2}{\partial y_j \partial \tau}\) vanished because of (2.4), (2.7),

\[
2 \hat{g}^{+,-} = \sum_{j,k=0}^{n} g^{jk} \psi_+^{x_j} \psi_-^{x_k},
\]

\[
2 \hat{g}^{+,j} = \sum_{p,r=0}^{n} g^{pr} \psi_+^{x_p} \varphi_{jx_r}, \quad 1 \leq j \leq n - 1,
\]

\[
\hat{g}^{ik} = \sum_{p,r=0}^{n} g^{pr} \varphi_{jx_p} \varphi_{kx_r}, \quad 1 \leq j, k \leq n - 1,
\]

In (2.10) \(\hat{u}(s, \tau, y') = u(x_0, x', x_n)\),

\[
A_k(x) = \sum_{j=1}^{n-1} \hat{A}_j(s, \tau, y') \varphi_{jx_k} - \hat{A}_+ \psi_+^{x_k} - \hat{A}_- \psi_-^{x_k}, \quad 0 \leq k \leq n.
\]
Now we shall introduce a new system of coordinates (cf. [E4])

\[ y_0 = \frac{s - \tau + T_2 + T_1}{2} = \frac{\psi^+ - \psi^- + T_2 + T_1}{2}, \]

\[ y_j = \varphi_j(x), \quad 1 \leq j \leq n - 1, \]

\[ y_n = \frac{T_2 - T_1 - s - \tau}{2} = \frac{T_2 - T_1 - \psi^+(x) - \psi^-(x)}{2}, \]

where \( \psi^+, \psi^-, \varphi_j, \; 1 \leq j \leq n - 1, \) are the same as in (2.4), (2.7).

Note that

\[ y_0 \big|_{x_n=0} = \frac{x_0 - T_1 - (T_2 - x_0) + T_2 + T_1}{2} = x_0, \]

\[ y_j \big|_{x_n=0} = x_j, \quad 1 \leq j \leq n - 1, \]

\[ y_n \big|_{x_n=0} = \frac{T_2 - T_1 - s - \tau}{2} = \frac{T_2 - T_1 - \psi^+(x) - \psi^-(x)}{2} = 0, \]

Therefore \( y = \varphi(x) = (\varphi_0(x_1), \varphi_1(x), ..., \varphi_n(x)) \) is the identity on \( x_n = 0 \):

\[ \varphi(x) = I \quad \text{when} \quad x_n = 0. \]

Here

\[ \varphi_0 = \frac{\psi^+(x) - \psi^-(x) + T_2 + T_1}{2}, \quad \varphi_n = \frac{T_2 - T_1 - \psi^+ - \psi^-}{2}. \]

Note that \( y_n = \varphi_n(x) > 0 \) when \( x_n > 0 \) since \( \psi^+_n + \psi^-_n < 0 \) (cf. (2.6)),

\[ u_\tau = \frac{1}{2} (u_{y_0} - u_{y_n}), \quad u_\sigma = -\frac{1}{2} (u_{y_0} + u_{y_n}). \]

Thus one can easily rewrite (2.10) in \( (y_0, y', y_n) \) coordinates .

We shall further simplify (2.10) by making a gauge transformation

\[ u' = e^{-id(s,\tau)u} \hat{u}. \]

Then \( u' \) satisfies the equation

\[ L' u' = 0, \]
where $L'$ is the same as $\hat{L}$ with $\hat{A}_j, \hat{A}_+, \hat{A}_-$ replaced by $A'_j, A'_+, A'_-$, $1 \leq j \leq n - 1$, where

\begin{align}
A'_j &= \hat{A}_j - \frac{\partial d}{\partial y_j}, \quad 1 \leq j \leq n - 1, \\
A'_+ &= \hat{A}_+ - \frac{\partial d}{\partial s}, \quad A'_- = \hat{A}_- - \frac{\partial d}{\partial \tau}.
\end{align}

We choose $d(s, \tau, y')$ such that

\begin{align}
A'_+ &= -\frac{\partial d}{\partial s} + \hat{A}_+ = 0 \quad \text{for } y_n > 0, \\
d|_{y_n=0} &= 0.
\end{align}

Let

\begin{equation}
g_1 = \left| \det [\hat{g}^{jk}]_{j,k=1}^{n-1} \right|^{-1}, \quad A = \ln(g_1)^{\frac{1}{4}}.
\end{equation}

Note that

\begin{align}
\frac{\partial A}{\partial y_j} = \frac{g_{1y_j}}{4g_1} &= \frac{1}{2} \frac{1}{\sqrt{g_1}} \left( \frac{\partial}{\partial y_j} \sqrt{g_1} \right), \quad 1 \leq j \leq n - 1, \\
\frac{\partial A}{\partial s} = \frac{g_{1s}}{4g_1} &= \frac{1}{2} \frac{1}{\sqrt{g_1}} \left( \frac{\partial}{\partial s} \sqrt{g_1} \right), \\
\frac{\partial A}{\partial \tau} = \frac{g_{1\tau}}{4g_1} &= \frac{1}{2} \frac{1}{\sqrt{g_1}} \left( \frac{\partial}{\partial \tau} \sqrt{g_1} \right).
\end{align}

Since $|\hat{g}| = (4\hat{g}^{+,+})^{-2}g_1$ (cf (2.11)) we can rewrite $L'u' = 0$ in the form (cf. [E1]):
(2.24) \[ L'u' = -4\hat{g}^{+-}\left(\frac{\partial}{\partial s} + \frac{\partial A}{\partial s}\right)\left(\frac{\partial}{\partial \tau} + iA'_- + \frac{\partial A}{\partial \tau}\right)u'
- 4\hat{g}^{+-}\left(\frac{\partial}{\partial \tau} + iA'_- + \frac{\partial A}{\partial \tau}\right)\left(\frac{\partial}{\partial s} + \frac{\partial A}{\partial s}\right)u'
+ 4\hat{g}^{+-}\sum_{k=1}^{n-1}\left(\frac{\partial}{\partial s} + \frac{\partial A}{\partial s}\right)\hat{g}^{+k}_k\left(\frac{\partial}{\partial y_k} - iA'_k + \frac{\partial A}{\partial y_k}\right)u'
+ 4\hat{g}^{+-}\sum_{k=1}^{n-1}\left(\frac{\partial}{\partial y_k} - iA'_k + \frac{\partial A}{\partial y_k}\right)\hat{g}^{+k}_k\left(\frac{\partial}{\partial s} + \frac{\partial A}{\partial s}\right)u'
+ 2\sum_{j,k=1}^{n-1}\hat{g}^{+-}\left(\frac{\partial}{\partial y_j} - iA'_j + \frac{\partial A}{\partial y_j}\right)\hat{g}^{jk}_{jk}\left(\frac{\partial}{\partial y_k} - iA'_k + \frac{\partial A}{\partial y_k}\right)u'
+ 2\hat{g}^{+-}V_1 u' = 0,\]

where

(2.25) \[ V_1 = -\sum_{j,k=1}^{n-1} \left(\frac{\hat{g}^{jk}}{\hat{g}^{+-}}\frac{\partial A}{\partial y_j} + \frac{\partial A}{\partial y_k}\right)\left(\frac{\hat{g}^{jk}}{\hat{g}^{+-}}\frac{\partial A}{\partial y_j}\right)\]
+ 4\frac{\partial^2 A}{\partial s\partial \tau} + 4\frac{\partial A}{\partial s}\frac{\partial A}{\partial \tau} - 4\sum_{j=1}^{n-1}\hat{g}^{+-}\frac{\partial A}{\partial s}\frac{\partial A}{\partial y_j}
- 2\sum_{j=1}^{n-1}\left(\frac{\partial}{\partial s}\left(\frac{\hat{g}^{+-}}{\hat{g}^{+j}}\frac{\partial A}{\partial y_j}\right) + \frac{\partial A}{\partial y_j}\left(\frac{\hat{g}^{+-}}{\hat{g}^{+j}}\frac{\partial A}{\partial s}\right)\right).\]

Make the change of unknown function

(2.26) \[ u_1 = g_1^{\frac{1}{2}} u',\]

where \(g_1 = |\text{det}[\hat{g}^{jk}]_{j,k=1}^{n-1}|^{-1}\) (cf. (2.11)). Then dividing \(L'u' = 0\) by \(2\hat{g}^{+-}\) we get (cf. [E4])

\[ L_1 u_1 = 0,\]
where $L_1u_1 = 0$ has the form (cf. (2.24))

$$L_1u_1 = -2 \frac{\partial}{\partial s} \left( \frac{\partial}{\partial \tau} + iA'_- \right) u_1 - 2 \left( \frac{\partial}{\partial \tau} + iA'_- \right) \frac{\partial}{\partial s} u_1$$

$$+ 2 \sum_{j=1}^{n-1} \frac{\partial}{\partial s} \left( g_0^{+j} \left( \frac{\partial}{\partial y_j} - iA'_j \right) u_1 \right)$$

$$+ \sum_{j=1}^{n-1} 2 \left( \frac{\partial}{\partial y_j} - iA'_j \right) g_0^{+j} \frac{\partial u_1}{\partial s}$$

$$+ \sum_{j,k=1}^{n-1} \left( \frac{\partial}{\partial y_j} - iA'_j \right) g_{jk}^{+} \left( \frac{\partial}{\partial y_k} - iA'_k \right) u_1 + V_1u_1 = 0,$$

where

$$g_{jk}^{+} = \hat{g}_{jk}^{+} + \hat{g}_{jk}^{-}, \quad g_0^{+j} = \hat{g}_0^{+j} + \hat{g}_0^{-},$$

and $V_1$ is the same as in (2.25). Using that $\frac{\partial}{\partial \tau} + iA'_- = \frac{1}{2} \left( -\frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_n} \right) + iA'_- = -\frac{1}{2} \left( \frac{\partial}{\partial y_0} - iA'_0 \right) + \left( \frac{\partial}{\partial y_n} - iA'_n \right)$ we can rewrite $L_1u_1$ in $(y_0, y', y_n)$ coordinates:

$$L_1u_1 = \left( \frac{\partial}{\partial y_0} - iA'_0 \right)^2 u_1 - \left( \frac{\partial}{\partial y_n} - iA'_n \right)^2 u_1$$

$$+ \sum_{j=1}^{n-1} \left( \frac{\partial}{\partial y_0} - iA'_0 \right) g_0^{+j} \left( \frac{\partial}{\partial y_j} - iA'_j \right) u_1 + \sum_{j=1}^{n-1} \left( \frac{\partial}{\partial y_j} - iA'_j \right) g_0^{+j} \left( \frac{\partial}{\partial y_0} - iA'_0 \right) u_1$$

$$- \sum_{j=1}^{n-1} \left( \frac{\partial}{\partial y_n} - iA'_n \right) g_0^{+j} \left( \frac{\partial}{\partial y_j} - iA'_j \right) u_1 - \sum_{j=1}^{n-1} \left( \frac{\partial}{\partial y_j} - iA'_j \right) g_0^{+j} \left( \frac{\partial}{\partial y_n} - iA'_n \right) u_1$$

$$+ \sum_{j,k=1}^{n} \left( \frac{\partial}{\partial y_j} - iA'_j \right) g_{jk}^{+} \left( \frac{\partial}{\partial y_k} - iA'_k \right) u_1 + V_1u_1 = 0.$$
Rewrite \( L'u' = 0 \) in \((y_0, y', y_n)\) coordinates using (2.17).

Denote \( \hat{g}^{00} = -\hat{g}^{nn} = \hat{g}^{+,-}, \hat{g}^{nj} = \hat{g}^{jn} = \hat{g}^{+,-}, \) \(1 \leq j \leq n - 1\). We assume that \( \hat{g}^{+,-} > 0 \).

The DN operator for \( L'u' = 0 \) has the following form in \((y_0, y', y_n)\) coordinates:

\[
(2.29) \quad \Lambda' g = (\hat{g}^{+,-})^{\frac{1}{2}} \left[ (i \frac{\partial u'}{\partial y_n} - iA'_- u') - \sum_{j=1}^{n-1} \frac{\hat{g}^{nj}}{\hat{g}^{+,-}} \left( \frac{\partial u'}{\partial y_j} - iA'_j u' \right) \right] \bigg|_{y_n=0}.
\]

Since \( L' \) is obtained from (1.1) by the change of variables (2.14) and the gauge transformation (2.18) and since (2.15), (2.21) hold, we have \( \Lambda g = \Lambda' g \) on \( \{y_n = 0\} \cap U_0 \) for all \( g \) with \( \text{supp} \ g \in (\Gamma_0 \times [T_1, T_2]) \cap U_0 \). Using the form of \( L_1 u_1 = 0 \) in \((y_0, y', y_n)\) coordinates (see (2.28)) we get that DN operator \( \Lambda_1 g \) has the form

\[
(2.30) \quad \Lambda_1 g = \left( \frac{\partial u_1}{\partial y_n} - iA'_- u_1' - \sum_{j=0}^{n-1} \frac{\hat{g}^{+,-}}{\hat{g}^{+,-}} \left( \frac{\partial u_1}{\partial y_j} - iA'_j u_1 \right) \right) \bigg|_{y_n=0},
\]

where \( \hat{g}^{+,-} \).

We shall show that the DN operators \( \Lambda' \) determines the DN operator \( \Lambda_1 \) in \( U_0 \cap \Gamma_0 \).

The following lemma is well known, especially in the elliptic case (cf. [LU], [E5], §57). For the hyperbolic case see [E1], Remark 2.2.

**Lemma 2.1.** The DN operator \( \Lambda' \) determines

\[
(2.31) \quad \left. \hat{g}^{+,-} \right|_{y_n=0}, \quad \left. \hat{g}^{nj} \right|_{y_n=0}, \quad \left. \hat{g}^{jk} \right|_{y_n=0}, \quad 1 \leq j \leq n - 1, \ 1 \leq k \leq n - 1,
\]

and the derivatives of \((2.31)\) in \( y_n \) at \( y_n = 0 \).

**Proof:** The principal symbol of operator \( L' \) has the form \( \hat{g}^{+,-} p(y, \eta) \), where (cf. (2.10) in \( y \)-coordinates)

\[
(2.32) \quad p(y, \eta) = \eta_0^2 - \eta_n^2 + 2 \sum_{j=1}^{n-1} g_0^{+,-}(\eta_0 - \eta_n)\eta_j + \sum_{j,k=1}^{n-1} g_0^{jk}\eta_j\eta_k,
\]

where

\[
(2.33) \quad g_0^{+,-} = \frac{\hat{g}^{+,-}}{\hat{g}^{+,-}}, \quad g_0^{jk} = \frac{\hat{g}^{jk}}{\hat{g}^{+,-}}.
\]
Note that in the region $\Sigma = \{ \eta_0^2 + (\sum_{j=1}^{n-1} g_0^{+j} \eta_j)^2 - \varepsilon \sum_{j=1}^{n-1} \eta_j^2 < 0 \}$ of the cotangent space $T^* = U_0 \times (\mathbb{R}^{n+1} \setminus \{0\})$ the operator $p(y, \eta)$ is elliptic. We shall call $\Sigma$ the elliptic region.

There is a parametrix of the Dirichlet problem in the elliptic region and DN operator microlocally in $\Sigma$ is a pseudodifferential operator on $y_n = 0$. We shall find the principal symbol of this operator in $\Sigma$. Let $\lambda_{\pm}$ be the roots in $\eta_n$ of $p(y, \eta_0, \eta', \eta_n) = 0$:

$$\lambda_{\pm} = -\sum_{j=1}^{n-1} g_0^{+j} \eta_j \pm \sqrt{\left( \sum_{j=1}^{n-1} g_0^{+j} \eta_j \right)^2 + \left( \eta_0^2 + 2 \sum_{j=1}^{n-1} g_0^{+j} \eta_j \eta_0 + \sum_{j,k=1}^{n-1} g_{0j}^{+k} \eta_j \eta_k \right)} = -\sum_{j=1}^{n-1} g_0^{+j} \eta_j \pm \sqrt{Q},$$

where $\Im \lambda_{\pm} > 0$ in $\Sigma$. Therefore the symbol of DN in $\Sigma$ is (cf. [E5], §57):

$$\left( \hat{g}^{+,-} \right)^{\frac{1}{2}} \sqrt{Q}.$$  

Knowing $\Lambda'$ we know the symbol (2.35) for all $\eta_0, \eta'$. In particular, we can find (2.31). Computing the next term of the parametrix (cf. [E], §57) we can find the normal derivatives of (2.31).

We have

$$\Lambda_1 f = \left( \frac{\partial u_1}{\partial y_n} - iA'_- u_1 - \sum_{j=1}^{n-1} g_0^{+j} \left( \frac{\partial u_1}{\partial y_j} - iA'_j u_1 \right) \right) \bigg|_{y_n=0},$$

where $u_1 \big|_{y_n=0} = f$, $u_1 = g_1^\frac{1}{2} u'$. Note that $\frac{\partial}{\partial y_k} u_1 = g_1^\frac{1}{2} \frac{\partial}{\partial y_k} u' + (\frac{\partial}{\partial y_k} g_1^\frac{1}{2}) u'$. Therefore

$$\Lambda_1 f = g_1^\frac{1}{2} (g^{+,-})^{-\frac{1}{2}} \Lambda'(g_1^{-\frac{1}{2}} f) - \left( \frac{\partial g_1^\frac{1}{2}}{\partial y_n} - \sum_{j=1}^{n-1} g_0^{+j} \frac{\partial g_1^\frac{1}{2}}{\partial y_j} \right) g_1^{-\frac{1}{2}} f \bigg|_{y_n=0}. $$

It follows from the Lemma 2.1 that $g_1, \frac{\partial g_1}{\partial y_n}, \hat{g}^{+,-}, g_0^{+j}$ are known on $y_n = 0$ if $\Lambda'$ is known. Therefore knowing $\Lambda'$ we can determine $\Lambda_1 f$ for any $f$. 

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3 The Green’s formula

First, we introduce some notations.

Let \( \Gamma_1 \subset U_0 \cap \Gamma_0 \). Denote by \( D_{1T_1} \) the forward domain of influence of \( \Gamma_1 \times [T_1, T_2] \) in the half-space \( y_n \geq 0 \). We shall define \( \Gamma_2 \) as the intersection \( D_{1T_1} \cap \{y_n = 0\} \cap \{y_0 = T_2\} \). Analogously, let \( \Gamma_3 = D_{2T_1} \cap \{y_n = 0\} \cap \{y_0 = T_2\} \), where \( D_{2T_1} \) is the forward domain of influence of \( \Gamma_2 \times [T_1, T_2] \). We assume that \( \Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset (\Gamma_0 \cap U_0) \).

Let \( D_{jT_1} \cap \{y_n = 0\} \subset \Gamma_0 \cap U_0 \) and that \( D_{jT_1} \) does not intersect \( \Gamma_0 \times [T_1, T_2] \) outside of \( y_n = 0 \).

![Fig. 3.1.](image)

**Fig. 3.1.** \( Y_{jT_1} \) is the intersection of the plane \( \tau = 0 \) with \( D_{jT_1} \), \( \Gamma_{j+1} \) is the intersection of \( Y_{jT_1} \) with the plane \( y_n = 0 \).

Let \( D_{j\tau_0} \) be the forward domain of influence of \( \Gamma_j \times [s_0, T_2], 1 \leq j \leq 3 \), where \( T_1 \leq s_0 \leq T_2 \). Denote by \( Y_{j\tau_0} \) the intersection of \( D_{j\tau_0} \) with the plane \( \tau = T_2 - y_n - y_0 = 0 \) (cf. Fig. 3.1). Let \( X_{j\tau_0} \) be the part of \( D_{j\tau_0} \) below \( Y_{j\tau_0} \) and let \( Z_{j\tau_0} = \partial X_{j\tau_0} \setminus (Y_{j\tau_0} \cup \{y_n = 0\}) \).

We assume also that \( D_{3T_1} \cap \{y_n = 0\} \subset \Gamma_0 \cap U_0 \) and that \( D_{3T_1} \) does not intersect \( \Gamma_0 \times [T_1, T_2] \) outside of \( y_n = 0 \).
Consider the following initial-boundary value problem:

\begin{align}
L_1 u^f &= 0 \\
\left. u^f \right|_{y_n=0} &= f, \quad \text{for } y_0 = T_1, y_n > 0,
\end{align}

where $\text{supp } f \subset \Gamma_3 \times [T_1, T_2]$. Also let $v^g$ be such that

\begin{align}
L_1 v^g &= 0 \quad \text{for } y_n > 0, \\
v^g &= v^g_{y_0} = 0 \quad \text{for } y_0 = T_1, y_n > 0, \\
\left. v^g \right|_{y_n=0} &= g, \quad \text{where } \text{supp } g \subset \Gamma_3 \times [T_1, T_2].
\end{align}

Note that $L^*_1 = L_1$, i.e. $L_1$ is formally self-adjoint.

Let $(u, v)$ be the $L_2$ inner product $\int_{x_{3T_1}} u(y) \overline{v(y)} dy$. We have

\begin{align}
(L_1 u^f, v^f) - (u^f, L_1 v^g) &= 0,
\end{align}

since $L_1 u^f = 0, L_1 v^g = 0$. The Jacobian $\frac{\partial(y_0, y_n)}{\partial(s, \tau)}$ is equal to $\frac{1}{2}$. Thus $dy_0 dy_n = \frac{1}{2} ds d\tau$. Integrating by parts in $s$ we get

\begin{align}
- \int_{x_{3T_1}} \frac{\partial}{\partial s} \left( \frac{\partial}{\partial \tau} + iA'_- \right) u^f \overline{v^g} ds d\tau \\
= \int_{x_{3T_1}} \left( \frac{\partial}{\partial \tau} + iA'_- \right) u^f \frac{\partial v^g}{\partial s} ds d\tau - \int_{y_n=0} \left( \frac{\partial}{\partial \tau} + iA'_- \right) u^f \overline{v^g} d\tau.
\end{align}

Integrating by parts in $\tau$ we get

\begin{align}
- \int_{x_{3T_1}} \frac{\partial^2}{\partial \tau \partial s} u^f \overline{v^g} ds d\tau \\
= \int_{x_{3T_1}} \frac{\partial u^f}{\partial s} \frac{\partial \overline{v^g}}{\partial \tau} ds d\tau - \int_{y_n=0} \frac{\partial u^f}{\partial s} \overline{v^g} ds + \int_{\tau=0} \frac{\partial u^f}{\partial s} \overline{v^g} ds.
\end{align}

We used in (3.4), (3.5) that $u^f, v^g$ are equal to zero on $Z_{3T_1}$. Note that $s = y_0 - T_1, \tau = T_2 - y_0$ on $y_n = 0$, and $\frac{\partial}{\partial s} = \frac{1}{2} \left( \frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_n} \right), \frac{\partial}{\partial \tau} = -\frac{1}{2} \left( \frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_n} \right)$. 

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Therefore, making changes of variable $\tau = T_2 - y_0$ in the first integral and $s = y_0 - T_1$ in the second, we get

\begin{equation}
(3.6) \quad - \int_{y_n=0}^{y_n} \left( \frac{\partial}{\partial \tau} + iA'_\tau \right) u^f \overline{v^g} d\tau - \int_{y_n=0}^{y_n} \frac{\partial u^f}{\partial s} \overline{v^g} ds = \int_{y_n=0}^{y_n} \left( \frac{\partial}{\partial y_n} - iA'_n \right) u^f \overline{v^g} dy_0
\end{equation}

Analogously, integrating by parts other terms of $\int_{X \times T_1} (L_1 u^f) \overline{v^g} dsd\tau$ we get (cf. [E3], p.316)

\begin{equation}
(3.7) \quad 0 = (L_1 u^f, v^g) - (u^f, L_1 v^g)
= \int_{Y_{3T_1}} \left( \frac{\partial u^f}{\partial s} v^g - u^f \frac{\partial v^g}{\partial s} \right) dy'ds + \int_{\Gamma_3 \times [T_1,T_2]} (\Lambda_1 f \overline{g} - f \overline{\Lambda_1 g}) dy'dy_0,
\end{equation}

where $\Lambda_1$ has the form (2.30)

\begin{equation}
(3.8) \quad \Lambda_1 f = \left( \frac{\partial u^f}{\partial y_n} - iA'_n u^f \right) - \sum_{j=1}^{n-1} g_0^{+j} \left( \frac{\partial u^f}{\partial y_j} - iA'_j u^f \right) \bigg|_{y_n=0}.
\end{equation}

It follows from (3.7) that

\begin{equation}
(3.9) \quad \int_{Y_{3T_1}} (u^f \overline{v^g} - u^f \overline{v^g}) dsdy'
\end{equation}

is determined by the boundary data, i.e. by the DN operator on $\Gamma^{(3)} \times (T_1, T_2)$.

We shall denote the $L_2$ inner product in $Y_{3T_1}$ by $(u, v)_{Y_{3T_1}}$, or simply $(u, v)$ when it is clear what is the domain of integration.

Let $D_j^-$ be the backward domain of influence of $\overline{\Gamma}_j \times [T_1, T_2]$. Thus $D_j T_1 \cap D_j^-$ is the domain of dependence of $\overline{\Gamma}_j \times [T_1, T_2]$. Denote by $Q_j$ the intersection of $D_j^-$ with $\tau = 0$. Let $R_{jso} = Y_{jso} \cap Q_j$ be the rectangle $\{s_0 - T_1 \leq s \leq T_2 - T_1, \tau = 0, y' \in \overline{\Gamma}_j \}$. Note that $R_{jso}$ belongs to the domain of dependence of $\overline{\Gamma}_j \times [s_0, T_2]$. Let $H_0^1(R_{jso})$ be the subspace of the Sobolev space $H^1(R_{jso})$ consisting of $w \in H^1(R_{jso})$ such that $w = 0$ on $\partial R_{j0} \setminus \{y_n = 0\}$. Analogously, let $H_0^1(Y_{jso})$ be the subspace of $H^1(Y_{jso})$ consisting of $v \in H^1(Y_{jso})$ such that $v = 0$ on $\partial Y_{jso} \setminus \{y_n = 0\}$. Note that $R_{jso} \subset Y_{jso} \subset R_{j+1,s0}$ (cf. Fig. 3.2).
Fig. 3.2. The rectangle $R_{js_0} = \{s_0 - T_1 \leq s \leq T_2 - T_1, \tau = 0, y' \in \Gamma_j\}$, $Y_{js_0}$ is the intersection of the domain of influence of $[s_0, T_2] \times \Gamma_j$ with the plane $\tau = 0$. Note that $R_{js_0} \subset Y_{js_0} \subset R_{j+1,s_0}$.

Note that $H_0^1(R_{js_0})$ is a subspace of $H_0^1(Y_{js_0})$.

**Lemma 3.1 (Density lemma).** For any $w \in H_0^1(R_{js_0})$ there exists a sequence $\{u^{f_n}\}$ where $u^{f_n}$ are solutions of the initial-boundary value problem (3.1), $f_n(y_0, y') \in H_0^1(\Gamma_j \times [s_0, T_2])$, such that $\|w - u^{f_n}\|_{1,Y_{js_0}} \to 0$ when $n \to \infty$, $j = 1, 2, 3$.

Here $\|w\|_{1,Y_{js_0}}$ is the norm in $H_0^1(Y_{js_0})$ and $f \in H_0^1(\Gamma_j \times [s_0, T_2])$, i.e. $f = 0$ on $\partial(\Gamma_j \times [s_0, T_2]) \setminus (\Gamma_j \times \{y_0 = T_2\})$.

**Proof:** The proof of Lemma 3.1 is a simplification of the proof of Lemmas 2.2 and 3.2 in [E3]. We shall prove Lemma 3.1 for the case $s_0 = T_1$. The proof for the case $T_1 < s_0 < T_2$ is identical.

Fig. 3.3. The domain $\Delta'_2$ is bounded by $\Gamma'_1$ and $\Gamma'_2$.  

\[ s = T_2 - T_1 \]
\[ s = s_0 - T_1 \]
\[ \Gamma_j \]
\[ \Gamma_{j+1} \]
\[ R_{js_0} \]
\[ Y_{js_0} \]
\[ \Delta'_2 \]
Denote by $\Delta'_2$ the domain bounded by the half-planes $\Gamma'_1 = \{ y_n = 0, y_0 < T_2, y' \in \mathbb{R}^{n-1} \}$ and $\Gamma'_2 = \{ \tau = T_2 - y_n - y_0 = 0, s < T_2, y' \in \mathbb{R}^{n-1} \}$ (cf. Fig. 3.3). Let $\Gamma'_\infty$ be the plane $\tau = 0$. Denote by $H^{-1}_0(\Gamma'_2)$ the Sobolev space of $h \in H^{-1}(\Gamma'_\infty)$ such that supp $h \subset \Gamma'_2$, i.e. $h(s, y') = 0$ when $s > T_2$. Note that $H^{-1}_0(\Gamma'_2)$ is dual to $H^1(\Gamma'_2)$ with respect to the extension of the $L_2$ inner product on $\Gamma'_\infty$ (cf. [E5]).

**Lemma 3.2.** For any $h(s, y') \in H^{-1}_0(\Gamma'_2)$ there exists a distribution $u(s, \tau, y')$ such that

\begin{align}
(3.10) \quad & L_1 u = 0 \quad \text{in} \quad \Delta'_2, \\
(3.11) \quad & \frac{\partial u}{\partial s} \bigg|_{\Gamma'_2} = h, \\
(3.12) \quad & u \big|_{y_n=0} = 0.
\end{align}

**Proof:** Since $h(s, y') = 0$ for $s > T_2$, there exists $v(s, y') = 0$ for $s > T_2$, $v(s, y')$ belongs to $L_2$ in $s$ and to $H^{-1}$ in $y'$ and such that $\frac{\partial v}{\partial s} = h$ in $\Gamma'_\infty$. We can define $v(s, y')$ by the formula

\[
v(s, y') = \lim_{\varepsilon \to 0} e^{\varepsilon (s-T_2)} F^{-1} \frac{\tilde{h}(z_0, \xi_1, \ldots, \xi_n)}{z_0 + i\varepsilon},
\]

where $F^{-1}$ is the inverse Fourier transform, $z_0$ is the dual variable to $s$.

The distribution $\theta(-\tau) u$ satisfies the equation

\[
(3.13) \quad L_1 (\theta(-\tau) u) = 4h \delta(-\tau)
\]

in the half-space $y_n > 0$ with the boundary condition

\[
(3.14) \quad \theta(-\tau) u \big|_{y_n=0} = 0.
\]

We look for $\theta(-\tau) u$ in the form

\[
(3.15) \quad \theta(-\tau) u = \theta(-\tau) v + w,
\]

where $w$ satisfies

\begin{align}
(3.16) \quad & L_1 w = \varphi, \\
(3.17) \quad & w \big|_{y_n=0} = -\theta(-\tau) v \big|_{y_n=0}, \\
(3.18) \quad & \varphi = L_{11} (\theta(-\tau) v),
\end{align}

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where $L_{11} = L_1 + \frac{\partial^2}{\partial s \partial \tau}$. Note that $L_{11}$ is a differential operator in $\frac{\partial}{\partial s}, \frac{\partial}{\partial y_k}, 1 \leq k \leq n - 1$.

We impose the zero initial conditions on $w$ requiring that

$$w = 0 \quad \text{for} \quad y_0 > T_2.$$  

(3.19) 

Therefore $w$ is the solution of the hyperbolic equation $L_1w = \varphi$ in the half-space $y_n > 0$ with the boundary condition (3.17) and the zero initial conditions (3.19). It follows from ([H] and [E6]) that initial-boundary value problem has a unique solution in appropriate Sobolev space of negative order.

Since $\varphi$ belongs to $L^2$ in $\tau$ and to Sobolev spaces of negative order in $s$ and $y'$, we get that $w$ belongs to $H^1$ in $\tau$. Therefore $w|_{\tau=\tau_0}$ is a continuous function of $\tau_0$ with the values in Sobolev’s spaces of negative order in $(s, y')$. Therefore $u = v(s) + w(s, \tau, y')$ is the distribution solution of (3.10), (3.11), (3.12) in $\Delta'_2$.

Note that the restrictions of any distribution solution of $L_1u = 0$ to $y_n = 0$ exists since $y_n = 0$ is not a characteristic surface for $L_1$. This property is called the partial hypoellipticity (cf., for example, [E5]).

Now using Lemma 3.2 we can prove Lemma 3.1. If \{$u^f, f \in H^0_0(\Gamma_j T_1), \Gamma_j T_1 = \Gamma_j \times [T_1, T_2]$\} is not dense in $H^1_0(R_j T_1)$ then there exists nonzero $h \in H^{-1}_0(R_j T_1)$ such that $(u^f, h) = 0, \forall u^f, f \in H^1_0(\Gamma_j T_1)$. Let $v_0$ be such that $\frac{\partial v_0}{\partial s} = h$. Then $(u^f, \frac{\partial v_0}{\partial s})_{Y_j T_1} = 0, \forall f \in H^1_0(\Gamma_j T_1)$. Let $v$ be the same as in Lemma 3.2 i.e. $L_1v = 0$ in $\Delta'_2, v_{|_{Y_j T_1}} = v_0, v_{|_{y_n = 0}} = 0$. Then

$$-2(u^f, \frac{\partial v_0}{\partial s})_{Y_j T_1} = \int_{\Gamma_j T_1} f \frac{\partial v}{\partial y_n} dy' dy_0, \forall f \in H^1_0(\Gamma_j T_1).$$

(3.20)

Note that $(u^f, h)_{Y_j T_1}$ is understood as the extension of $L_2$ inner product in $(u_1, h)$ in $\Gamma'_{\infty}$, where $u_1$ is an arbitrary extension of $u^f$ for $s > T_2$. Analogously for the right hand side or (3.20). (Note that $v = 0$ for $y_0 > T_2$).

To justify (3.20) we take a sequence $h_j \in C^\infty(\Gamma'_2), h_j \rightarrow h$ in $H^{-1}_0(\Gamma'_2)$. By Lemma 3.2 there exists smooth $v_j$ such that $L_1v_j = 0$ in $\Delta'_2, v_j_{|_{y_n = 0}} = 0$,
\[
\frac{\partial v_j}{\partial s} \big|_{\tau=0} = h_j.
\]
Applying the Green’s formula (3.7) to \(u^f\) and \(v_j\) we get
\[
\int_{Y_3T_1} \left( \frac{\partial u^f}{\partial s} \nu_j - u^f \frac{\partial v_j}{\partial s} \right) dy' ds = \int_{\Gamma_3T_1} f \frac{\partial v_j}{\partial y_n} dy' dy_0,
\]
since \(v_j \big|_{y_n=0} = 0\). Integrating by parts we get
\[
\int_{Y_3T_1} \frac{\partial u^f}{\partial s} \nu_j ds dy' = - \int_{Y_3T_1} u^f \frac{\partial v_j}{\partial s} ds dy' + w^f v_j \big|_{y_n=0, y_0=T_2}.
\]
Since \(v_j = 0\) for \(s > T_2\) we have that \(v_j \big|_{y_n=0, y_0=T_2} = 0\). Therefore, taking the limit when \(j \to \infty\) we get (3.20).

Since \(f\) is arbitrary and \(\left( u^f, \frac{\partial u^f}{\partial y_n} \right) = 0\) we get that \(\frac{\partial u^f}{\partial y_n} = 0\) on \(\Gamma_j T_1\). Therefore \(L_1 v = 0\) in \(\Delta_j^0\) and \(v\) has zero Cauchy data on \(\Gamma_{2T_1}\). Then \(v = 0\) in the domain of dependence \(D_j^{-} \cap D_j T_1\), in particular, \(v = 0\) on \(R_j T_1\). Thus \(v_0 = 0\) on \(R_j T_1\) and this contradicts the assumption that \(h \neq 0\). \(\square\)

We shall prove two more theorems in this section that will be used in §4.

We shall need some known results on the initial-boundary hyperbolic problem. The following theorem holds:

**Lemma 3.3.** Let \(L_1 u = F\) in \(\mathbb{R}^n_+ \times (-\infty, T_2)\) where \(F \in H^s_+ (\mathbb{R}^n \times (-\infty, T_2), \mathbb{R}^n_+) = \{y_n > 0, y' \in \mathbb{R}^{n-1}\}\). Let \(u \big|_{y_n=0} = f\), where \(f \in H^{s+1}_+ (\mathbb{R}^{n-1} \times (-\infty, T_2))\).

Then for any \(s \geq 0\) and any \(f \in H^{s+1}_+ (\mathbb{R}^{n-1} \times (-\infty, T_2))\) and \(F \in H^s_+ (\mathbb{R}^n \times (-\infty, T_2))\) there exists a unique \(u \in H^{s+1}_+ (\mathbb{R}^n_+ \times (-\infty, T_2))\) such that
\[
\|u\|_{s+1} \leq C([f]_{s+1} + \|F\|_s).
\]
Moreover,
\[
\left[ \frac{\partial w(y_0, y', 0)}{\partial y_n} \right]_s \leq C(\|F\|_s + [f]_{s+1}).
\]

Here \(H^s_+ (\mathbb{R}^n_+ \times (-\infty, T_2))\) is the Sobolev’s space \(H^s (\mathbb{R}^n_+ \times (-\infty, T_2))\) with norm \(\|u\|_s\) consisting of \(u(y)\) with the support in \(y_0 \geq T_1\), \([f]_s\) is the norm in \(H^s_+ (\mathbb{R}^{n-1} \times (-\infty, T_2))\).

We assume that \(F(y)\) and \(f(y_0, y')\) have compact supports in \(y'\). Note that \(f = 0, \ F = 0, \ u = 0\ for \ y_0 < T_1\).

Then \(u(y_0, y', y_n)\) has also a compact support in \(y'\).
The proof of Lemma 3.3 in the case of time-dependent coefficients is given in [H] and [E6].

Note that Lemma 3.3 holds also in the case when \( \mathbb{R}^n_+ \) is replaced by an arbitrary smooth domain \( \Omega \subset \mathbb{R}^n \).

The following lemma follows from Lemma 3.3.

**Lemma 3.4.** Let, for the simplicity, \( F = 0 \). For any \( f \in H^1(\mathbb{R}^{n+1} \times (-\infty, T_2]) \), \( f = 0 \) for \( y_0 \leq T_1 \), there exists \( u \in C(H^1(\mathbb{R}^n_+), [T_1, T_2]) \cap C^1(L_2(\mathbb{R}^n_+), [T_1, T_2]) \) such that \( L_1 u = 0 \) in \( \mathbb{R}^n_+ \times (-\infty, T_2] \), \( u = 0 \) for \( y_0 < T_1 \),

\[
\max_{T_1 \leq y_0 \leq T_2} \|u(y_0, y', y_n)\|_1^2 + \max_{T_1 \leq y_0 \leq T_2} \|\frac{\partial u}{\partial y_0}(y_0, y', y_n)\|_0^2 \leq C[f]_1^2.
\]

Here \( C(H^1(\mathbb{R}^n_+), [T_1, T_2]) \cap C^1(L_2(\mathbb{R}^n_+), (T_1, T_2)) \) means that \( u(y), \frac{\partial u(y)}{\partial y_0} \) are continuous functions of \( y_0 \) with values in \( H^1(\mathbb{R}^n_+), L_2(\mathbb{R}^n_+) \), respectively.

**Proof:** Take \( s > \frac{3}{2} \). Consider the equation \( L_1 u = 0 \) in \( \mathbb{R}^n_+ \times (-\infty, T_2] \), \( u = 0 \) for \( y_0 < T_1 \), using \((y_0, y', y_n)\) coordinates (cf. (2.28)). We have

\[
\max_{T_1 \leq y_0 \leq T_2} \|u(y_0, y', y_n)\|_1^2 + \max_{T_1 \leq y_0 \leq T_2} \|\frac{\partial u}{\partial y_0}(y_0, y', y_n)\|_0^2 \leq C[f]_1^2.
\]

Let \((u, v)_{T'}\) be the \( L_2\)-inner product in \( \mathbb{R}^{n+1}_+ \times (T_1, T') \), \( T' \leq T_2 \). Integrating by parts the identity

\[
0 = (L_1 u, u_{y_0})_{T'} + (u_{y_0} L_1 u)_{T'},
\]

we get (cf. [E4])

\[
E_{T'}(u, u) + \Lambda_0(f, f) + I_1 = 0,
\]

where

\[
E_{T'}(u, u) = \int_{\mathbb{R}^n_+} \left( |u_{y_0} - i A_0' u|^2 - \sum_{j,k=1}^n g_0^{jk} (u_{y_j} - i A_j' u)(u_{y_k} - i A_k' u)' + V_1 |u|^2 \right) dy' dy_n |_{y_0 = T'},
\]

\[
\Lambda_0(f, f) = \int_{T_1} \int_{\mathbb{R}^{n-1}} \left( (\Lambda_1 f)_{y_0} + f_{y_0} \Lambda_1 f \right) dy' dy_0,
\]
\( \Lambda_1 f \) is the same as in (2.36),

\[
(3.28) \quad |I_1| \leq C \int_{\mathbb{R}^n \times [T_1, T']} \sum_{k=0}^{n} |u_{y_k}|^2 dy_0 dy' dy_n.
\]

Note that \( I = 0 \) when the coefficients of \( L_1 \) do not depend on \( y_0 \).

Let \( \|u\|_{s,T'} \) be the norm in \( H^s(\mathbb{R}_n^+) \) when \( y_0 = T' \). We have

\[
(3.29) \quad |I| \leq C \int_{T_1}^{T'} (\|u\|_{1,t}^2 + \|u_{y_0}\|_{0,t}^2) dt \leq C \int_{T_1}^{T'} |[u]|_t^2 dt
\]

where

\[
(3.30) \quad |[u]|_t^2 = \|u\|_{1,t}^2 + \|u_{y_0}\|_{0,t}^2.
\]

We have

\[
(3.31) \quad E_{T'}(u, u) \geq C|[u]|_{1,T'}^2.
\]

if \( T_2 - T_1 \) is small.

Since \( T_2 - T_1 \) is small, (3.25) implies

\[
(3.32) \quad \max_{T_1 \leq T' \leq T_2} |[u]|_{1,T'}^2 \leq C(T_2 - T_1)( \max_{T_1 \leq T' \leq T_2} |[u]|_{1,T'}^2 + |\Lambda_0(f, f)|)
\]

Note that

\[
(3.33) \quad |\Lambda_0(f, f)| \leq C \left( |f|_1^2 + \left[ \frac{\partial u(y_0, y', 0)}{\partial y_n} \right]_0 \right).
\]

Therefore (3.22), (3.25), (3.31), (3.32), (3.33) imply (3.23).

Since \( H^s_+ \) is dense in \( H^1_+ \) when \( s > 1 \) we can approximate \( f \in H^1_+ (\mathbb{R}^{n-1} \times (-\infty, T_2)) \) by functions from \( H^s_+ (\mathbb{R}^{n-1} \times (-\infty, T_2)) \), \( s > \frac{3}{2} \) and therefore the inequality (3.23) holds for \( f \in H^1_+ \).

We shall study the Goursat problem (see Fig. 3.4):
Lemma 3.5. For any \( v_0 \in H^1(\Gamma_4), v_1 \in L^2(\Gamma_4) \) there exists \( u \in H^1(\Delta_1) \) such that \( L_1 u = 0 \) in \( \Delta_1 \), \( u|_{\Gamma_4} = v_0, u|_{\Gamma_4} = v_1 \). Moreover, the traces \( \varphi = u|_{\Gamma_4}, \psi = u|_{\Gamma_3} \) exists and belongs to \( H^1(\Gamma_2), H^1(\Gamma_3) \), respectively. The following estimate holds:

\[
\|u|_{\Gamma_2}\|_{1,\Gamma_2}^2 + \|u|_{\Gamma_3}\|_{1,\Gamma_3}^2 \leq C \left( \|u|_{\Gamma_4}\|_{1,\Gamma_4}^2 + \|u|_{\Gamma_0}\|_{0,\Gamma_0}^2 \right) .
\]

Vice versa, for any \( \varphi \in H^1(\Gamma_2), \psi \in H^1(\Gamma_3), \varphi = \psi \) at \( y_0 = \frac{T_2 - T_1}{2} \), there exists \( u \in H^1(\Delta_1), L_1 u = 0 \) in \( \Delta_1 \) such that \( u|_{\Gamma_2} = \varphi, u|_{\Gamma_3} = \psi \) and the following estimate holds:

\[
\|u|_{\Gamma_2}\|_{1,\Gamma_2}^2 + \|u|_{\Gamma_0}\|_{0,\Gamma_0}^2 \leq C \left( \|u|_{\Gamma_2}\|_{1,\Gamma_2}^2 + \|u|_{\Gamma_3}\|_{1,\Gamma_3}^2 \right) .
\]

Proof: Let \( \Delta_{1,T'} \) be the domain bounded by \( \Gamma_2, \Gamma_3 \) and \( \Gamma_{4,T'} \), where \( \Gamma_{4,T'} \) is the plane \( y_0 = T', \frac{T_2 + T_1}{2} \leq T' \leq T_2 \). Denote by \((u,v)_{\Delta_{1,T'}}\) the \( L_2 \)-inner
product in $\Delta_{1,T'}$. Integrating by parts the identity

\[(L_1u, u_{x_0})_{\Delta_{1,T'}} + (u_{x_0}, L_1u)_{\Delta_{1,T'}} = 0\]

we get, as in \[E4\]:

\[(3.36) \quad E_{T'}(u, u) + Q_{T'}(u, u) + Q_{T'}^{(1)}(u, u) = I_2,\]

where $E_{T'}(u, u)$ is the same as in (3.26),

\[(3.37) \quad Q_{T'}(u, u) = \frac{1}{2} \int_{\Gamma_{2,T'}} \left[|4u_s|^2 - \sum_{j,k=1}^{n-1} g_0^{jk} \left( \frac{\partial u}{\partial y_j} - iA_j' u \right) \left( \frac{\partial u}{\partial y_k} - iA_k' u \right) \right.
\]

\[-2 \sum_{j=1}^{n-1} \left( g_0^{0j} \left( \frac{\partial u}{\partial s} + iA_j' u \right) \left( \frac{\partial u}{\partial y_j} - iA_j' u \right) + g_0^{0j} \left( \frac{\partial u}{\partial y_j} - iA_j' u \right) \left( \frac{\partial u}{\partial s} + iA_j' u \right) \right)
\]

\[+ V_1 |u|^2 \right] dy'ds,
\]

(cf. (3.22) in \[E4\]),

\[(3.38) \quad Q_{T'}^{(1)}(u, u) = \frac{1}{2} \int_{\Gamma_{3,T'}} \left( |u_{\parallel} + iA_{\perp} u|^2 - \sum_{j,k=1}^{n-1} g_0^{jk} (u_{y_j} - iA_j' u)(u_{y_k} - iA_k' u) + V_1 |u|^2 \right) dy'd\tau,
\]

\[(3.39) \quad |I_2| \leq C \int_{\Delta_{1,T'}} \sum_{j=0}^{n} \left| \frac{\partial u}{\partial y_j} \right|^2 dy'dy'dn.
\]

Here $\Gamma_{2,T'}, \Gamma_{3,T'}$ are parts of $\Gamma_2, \Gamma_3$ for $\frac{T_1 + T_2}{2} \leq T'$. When $T_2 - T_1$ is small, $Q_{T'}(u, u)$ is positive definite (cf. \[E4\], (3.23)). Therefore

\[(3.40) \quad C_1 \|u\|^2_{1,\Gamma_{2,T'}} \leq Q_{T'}(u, u) \leq C_2 \|u\|^2_{1,\Gamma_{2,T'}}.
\]

Analogously,

\[(3.41) \quad C_1' \|u\|^2_{1,\Gamma_{3,T'}} \leq Q_{T'}^{(1)}(u, u) \leq C_2' \|u\|^2_{1,\Gamma_{3,T'}}.
\]

Having (3.31), (3.32), (3.40), (3.41) we can complete the proof of Lemma 3.5 exactly as the proof of Lemma 3.1 in \[E3\].

Combining Lemmas 3.4 and 3.5 we can prove the following lemma:
Lemma 3.6. The map $f \to u^f$ is a bounded operator from $H_0^1(\Gamma_j \times [s_0, T_2])$ to $H_0^1(Y_{j,s_0}):$

(3.42) \quad \|u^f\|_{1,Y_{j,s_0}} \leq C[f]_1.$

Proof: It follows from Lemma 3.4 that

(3.43) \quad \|u^f\|_{1,\Gamma_4} + \|u^f_{y_0}\|_{0,\Gamma_4} \leq C[f]_1^2.

Then (3.34) gives

(3.44) \quad \|u^f\|_{1,\Gamma_2}^2 \leq C(\|u^f\|_{1,\Gamma_4} + \|u^f_{y_0}\|_{0,\Gamma_4}).

Combining (3.42) and (3.43) and taking into account that $\text{supp } u^f|_{\Gamma_2} = Y_{j,s_0},$ we get (3.42).

4 The Main formula

Let $L_i^{(i)}, i = 1, 2,$ be two operators of the form (2.27) such that the corresponding DN operators $\Lambda_i^{(1)}$ and $\Lambda_i^{(2)}$ are equal on $U_0 \cap \{y_n = 0\}.$ We choose $\Gamma_i^{(1)} = \Gamma_i^{(2)} = \Gamma_1$ in a neighborhood of $x^{(0)}$ in $U_0 \cap \{y_n = 0\}.$ Let $\Gamma_j^{(i)}, j = 2, 3, i = 1, 2,$ be defined as before (see Fig. 3.1) for $i = 1, 2,$ respectively.

Lemma 4.1. We have $\Gamma_j^{(1)} = \Gamma_j^{(2)}, j = 2, 3$ (cf. [E1]).

Proof: Let $\Delta_i^{(2)}_{\Gamma_1}$ be the intersection of the domain of influence $D_i^{(2)}_{\Gamma_1}, i = 1, 2,$ with the plane $y_n = 0.$ Note that $\Delta_i^{(2)}_{\Gamma_1}$ is the intersection of $y_n = 0$ with the closure of the union $\bigcup \text{supp } u_i^f$ where the union is taken over all $f \in H_0^1(\Gamma_1 \times [T_1, T_2]), L_i^{(i)} u_i^f = 0.$

Let $\tilde{\Delta}_i^{(2)}_{\Gamma_1}$ be the closure of the union $\bigcup \text{supp } \Lambda_i^{(i)} f,$ where the union is taken also over all $f \in H_0^1(\Gamma_1 \times [T_1, T_2]).$ We shall show that $\tilde{\Delta}_i^{(2)}_{\Gamma_1} = \Delta_i^{(2)}_{\Gamma_1}.$

If $x^{(0)} \not\in \Delta_i^{(2)}_{\Gamma_1}$ then $u_i^f = 0,$ $\forall f,$ in some neighborhood of $x^{(0)}$ in $U_0.$ Then $\Lambda_i^{(i)} f = 0$ in a neighborhood of $x^{(0)},$ $\forall f.$ Thus $x^{(0)} \not\in \tilde{\Delta}_i^{(2)}_{\Gamma_1},$ i.e. $\tilde{\Delta}_i^{(2)}_{\Gamma_1} \subset \Delta_i^{(2)}_{\Gamma_1}.$ Let now $x_i^{(0)} \not\in \Delta_i^{(2)}_{\Gamma_1}.$ Then $\Lambda_i^{(i)} f = 0$ in a neighborhood of $x_i^{(0)}$ for any $f \in H_0^1(\Gamma_1 \times [T_1, T_2])$ and also $f = 0$ in a neighborhood of $x'_i.$ Then by the uniqueness of the Cauchy problem (see [T], [RZ]) we have that all $u^f = 0$ in a
neighborhood of \(x'_0\) in \(\mathbb{R}^{n+1}\). Therefore \(x'_0 \not\in \Delta^{(i)}_{2T_1}\). Thus \(\Delta^{(1)}_{2T_1} = \Delta^{(1)}\). Since \(\Lambda^{(1)}_1 = \Lambda^{(2)}_1\), we have \(\Delta^{(1)}_{2T_1} = \Delta^{(2)}_{2T_1}\). Therefore \(\Delta^{(1)}_{2T_1} = \Delta^{(2)}_{2T_1}\), i.e. \(\Gamma^{(1)}_2 = \Gamma^{(2)}_2\).

Analogously one shows that \(\Gamma^{(1)}_j = \Gamma^{(2)}_j\). Since \(\Lambda^{(1)}_1 = \Lambda^{(2)}_1\), we shall write \(\Gamma^{(1)}_j, 1 \leq j \leq 3\), instead of \(\Gamma^{(i)}_j\). It follows from (3.7) that (3.9) is determined by the boundary data. Integrating by part we have

\[
(4.1) \quad \int_{Y_{3T_1}} (u^f_s v^g - u^f v^g_s) dy'
\]

\[
= 2 \int_{Y_{3T_1}} u^f_s v^g dy' - \int_{\partial Y_{3T_1} \cap \{y_n = 0\}} u^f(T_2 - T_1, 0, y') v^g(T_2 - T_1, 0, y') dy'.
\]

Since \(u^f(T_2 - T_1, 0, y') = f(T_2, y'), v^g(T_2 - T_1, 0, y') = g(T_2, y')\), we have that

\[
(4.2) \quad (u^f_s, v^g) = \int_{Y_{3T_1}} u^f_s v^g dy'
\]

is also determined by the boundary data.

**Lemma 4.2.** Let \(f \in H^1_0(\Gamma_1 \times [T_1, T_2])\). For any \(s_0 \in [T_1, T_2]\) there exists \(u_0 \in H^1_0(R_{2s_0})\) such that

\[
(4.3) \quad (u^f_s, v') = (u_0, v')
\]

for any \(v' \in H^1_0(Y_{3s_0})\). Note that \(R_{2s_0} = \{ \tau = 0, s_0 - T_1 \leq s \leq T_2 - T_1, y' \in \Gamma_2 \}\) (cf. Fig. 4.1).

**Proof:** Note that \(Y_{1T_1} \cap \{s_0 - T_1 \leq s \leq T_2 - T_1\} \subset R_{2s_0}\). Let \(w_1\) be such that \(w_{1s} = 0\) in \(R_{2s_0}\), \(w_1 = u^f\) when \(s = s_0 - T_1, y' \in \Gamma_2\). Then \(u_0 = u^f - w_1\) for \(s \geq s_0 - T_1\), \(u_0 = 0\) for \(s \leq s_0 - T_1\), belongs to \(H^1_0(R_{2s_0})\) and solves (4.3). \(\square\)
Let $L^{(i)}, i = 1, 2$, be two operators of the form (2.27). Let $f$ be in $H^1_0(\Gamma_1 \times [T_1, T_2])$ and let $u^{(i)}_0$ be the same as in (4.3) for $i = 1, 2$. Then

$$\langle u^{(1)}_0, v^{(1)}_1 \rangle_{Y^{(1)}_{2s_0}} = \langle u^{(2)}_0, v^{(2)}_2 \rangle_{Y^{(2)}_{2s_0}}$$

for all $g \in H^1_0(\Gamma_2 \times [T_1, T_2])$.

Here $u^{(1)}_i, v^{(1)}_i$ are the same as in (3.1), (3.2) for $i = 1, 2$, respectively. Operators $L^{(i)}$ and, consequently, $L^{(i)}_1$ are formally self-adjoint, $(u^{(1)}_0, v^{(1)}_1)_{Y^{(1)}_{2s_0}}$ is the $L_2$-inner product over $Y^{(i)}_{2s_0}, i = 1, 2$.

To prove Theorem 4.3 we will need the Density Lemma 3.1 and the following lemma that uses the BLR condition:

**Lemma 4.4.** Let $L^{(1)}$ and $L^{(2)}$ be two operators in $D \cap [t_0, T_2]$ having the same DN operator on $\Gamma_0 \times [t_0, T_2]$. Suppose $L^{(1)}$ satisfies the BLR condition on $[t_0, T_2]$. 

Let $L_1^{(i)}, u_1^f, X_{2s_0}$ be the same as in (3.1), $i = 1, 2, f \in H^1_0(\Gamma_{2s_0})$, where $\Gamma_{2s_0} = \Gamma_2 \times [s_0, T_2]$. Then

\begin{equation}
\|u_2^f\|_{1, \gamma_{2s_0}} \leq C_2 \|u_1^f\|_{1, \gamma_{2s_0}}. \tag{4.5}
\end{equation}

**Proof of Lemma 4.4** (cf. Lemma 2.3 in [E3]):

Suppose that BLR condition (see [BLR]) is satisfied for $L^{(1)}$ on $[t_0, T_{t_0}]$ and $t_0 < T_1$, $T_2 \geq T_{t_0}$. The BLR condition implies that the map $f \mapsto (u_1^f(x)|_{D_{T_2}}, \frac{\partial u_1^f(x)}{\partial x_0}|_{D_{T_2}})$ of $H^1_+((0, t_0) \times (T_0, T_2))$ to $H^1(D_{T_2}) \times L_2(D_{T_2})$ is onto, where $D_{T_2} = D_0^{(1)} \times \{x_0 = T_2\}$. It follows from [H] (cf. also Lemma 3.6) that

\begin{equation}
\|u_1^f(x)\|_{1, D_{T_2}}^2 + \left\| \frac{\partial u_1^f(x)}{\partial x_0} \right\|_{0, D_{T_2}}^2 \leq C_0 \left[ f \right]_{1, \Gamma_0 \times (t_0, T_2)}. \tag{4.6}
\end{equation}

By the closed graph theorem we have

\begin{equation}
\inf_{\mathcal{F}} [f']_{1, \Gamma_0 \times (t_0, T_2)} \leq C_1 \left( \|u_1^f\|_{1, D_{T_2}}^2 + \left\| \frac{\partial u_1^f}{\partial x_0} \right\|_{0, D_{T_2}}^2 \right), \tag{4.7}
\end{equation}

where $\mathcal{F} \subset H^1_+(\Gamma_0 \times (t_0, T_2))$ is the set of $f'$ such that

\begin{equation}
u_1^f(x)|_{D_{T_2}} = u_1^f(x)|_{D_{T_2}}, \quad \frac{\partial u_1^f(x)}{\partial x_0}|_{D_{T_2}} = \frac{\partial u_1^f(x)}{\partial x_0}|_{D_{T_2}}. \tag{4.8}
\end{equation}

It follows from (4.7) that there exists $f_0 \in H^1_+(\Gamma_0 \times (t_0, T_2))$, $f_0 = 0$ for $x_0 < t_0$ such that

\begin{equation}
[f_0]_{1, \Gamma_0 \times (t_0, T_2)} \leq C \left( \|u_1^f\|_{1, D_{T_2}}^2 + \left\| \frac{\partial u_1^f}{\partial x_0} \right\|_{0, D_{T_2}}^2 \right), \tag{4.9}
\end{equation}

Note that

\begin{equation}
u_1^{f_0}(x)|_{D_{T_2}} = u_1^f(x)|_{D_{T_2}}, \quad \frac{\partial u_1^{f_0}(x)}{\partial x_0}|_{D_{T_2}} = \frac{\partial u_1^f(x)}{\partial x_0}|_{D_{T_2}}. \tag{4.10}
\end{equation}

Let $\Lambda^{(i)}$ be the DN operator corresponding to $L^{(i)}, i = 1, 2$. We can introduce the Goursat coordinates for $L^{(i)}$ globally in $[t_0, T_2] \times \Gamma_0 \times (0, \varepsilon)$, where $\varepsilon$ is small. Let $L_1^{(i)}$ be the operator $L^{(i)}$ in these coordinates and let $\Lambda_1^{(i)}$ be the DN operator corresponding to $L_1^{(i)}$. The same arguments as the end of §2 show that $\Lambda_1^{(1)} = \Lambda_1^{(2)}$ on $[t_0, T_2] \times \Gamma_0$ imply that $\Lambda_1^{(1)} = \Lambda_1^{(2)}$ on $[t_0, T_2] \times \Gamma_0$. 

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Consider the identity
\[(4.11) \quad (L_1^{(i)} u_i^{f_0}, v_i^g) |_{\mathcal{I}_{j}^{(i)}} - (u_i^{f_0}, C_1^{(i)} v_i^g) |_{\mathcal{I}_{j}^{(i)}} = 0,
\]
where \(v_i^g\) is the same as in \((3.2)\). Since \(\text{supp} v_i^g \subset D(\Gamma_{s_0})\), where \(D(\Gamma_{s_0})\) is the domain of influence of \(\Gamma_{s_0}\) for \(y_0 \leq T_2\), we have that \(v_i^g = 0\) on \(Z_{s_0}\). Therefore integrating by parts in \((4.11)\) we get as in \((3.7)\):
\[(u_i^{f_0}, v_i^g) |_{\mathcal{I}_{j}^{(i)}} = -(\Lambda_1^{(1)} f_0, g) |_{\Gamma_{s_0}} + (f_0, \Lambda_1^{(1)} g) |_{\Gamma_{s_0}}.
\]
Analogously, we have for \(L^{(2)} u_2^{f_0} = 0, \ L^{(2)} v_2^g = 0:\n\[(u_2^{f_0}, v_2^g) |_{\mathcal{I}_{j}^{(2)}} = -(\Lambda_1^{(2)} f_0, g) |_{\Gamma_{s_0}} + (f_0, \Lambda_1^{(2)} g) |_{\Gamma_{s_0}}.
\]
We have that \(\Lambda_1^{(1)} f_0 = \Lambda_1^{(2)} f_0\) and \(\Lambda_1^{(1)} g = \Lambda_1^{(2)} g\) on \(\Gamma_0 \times [t_0, T_2]\). Integrating by parts we get
\[-(u_i f_0, v_i^g) = (u_i f_0, v_i^g) - \int_{R^{n-1}} (u_i f_0 v_i^g |_{s = T_2 - T_1} - u_i f_0 v_i^g |_{s = 0}) dy'.\]
Note that \(v_i^g |_{s = 0} = 0\) and \(u_i f_0 v_i^g |_{s = T_2 - T_1} = f_0(T_2, y') g(T_2, y')\). Therefore
\[(4.12) \quad (u_{1s}^{f_0}, v_1^g) = (u_{2s}^{f_0}, v_2^g)
\]
for all \(g \in H_0^1(\Gamma_{3s_0})\).

Let \(\Gamma_2, \Gamma_3, \Gamma_4\) be the same as in Lemma \[3.3\]. It was proven there that
\[(4.13) \quad \|u_i f_0\|_{1,\Gamma_4}^2 + \|u_{i, y_0} f_0\|_{0,\Gamma_4}^2 \leq C (\|u_i f_0\|_{1,\Gamma_2}^2 + \|u_i f_0\|_{1,\Gamma_3}^2),
\]
\[(4.14) \quad \|u_i f_0\|_{1,\Gamma_2}^2 + \|u_i f_0\|_{1,\Gamma_3}^2 \leq C (\|u_i f_0\|_{1,\Gamma_4}^2 + \|u_i f_{1, y_0} f_0\|_{0,\Gamma_4}^2).
\]
It follows from \(u_i f_0 |_{\Gamma_4} = u_i f_0 |_{\Gamma_4}, \ u_{i, y_0} f_0 |_{\Gamma_4} = u_{i, y_0} f_0 |_{\Gamma_4}\) that
\[(4.15) \quad u_i f_0 |_{\Gamma_2} = u_i f_0 |_{\Gamma_2}
\]
by the domain of dependence argument. Comparing \((4.12)\) with \((u_{1s}^{f_0}, v_1^g) = (u_{2s}^{f_0}, v_2^g)\) and taking into account \((4.15)\) we get
\[(4.16) \quad (u_{2s}^{f_0}, v_2^g) |_{\mathcal{I}_{2s_0}^{(2)}} = (u_{2s}^{f_0}, v_2^g) |_{\mathcal{I}_{2s_0}^{(2)}}, \quad \forall g \in H_0^1(\Gamma_3 \times (s_0, T_2)).
\]
By Lemma 3.1 \( \{ v_2^0 \} \) are dense in \( H^1_0(R_{2s_0}) \). Since \( Y_{2s_0}^{(2)} \subset R_{2s_0}^{(2)} \) we get that \( \{ v_2^0 \} \) are dense in \( H^1_0(Y_{2s_0}^{(2)}) \) and therefore \( u_{2s}^0 = u_{2s}^f \) in \( Y_{2s_0}^{(2)} \). Since \( u_2^f|_{s=T_2-T_1} = f(T_2, y') = u_1^f(T_2, y', 0) \), \( u_2^0|_{s=T_2-T_1} = f_0(T_2, y') = u_1^f(T_2, y', 0) \) and since \( u_1^f(T_2, y', 0) = u_1^f(T_2, y', 0) \) we get that \( u_2^f|_{s=T_2-T_1} = u_2^0|_{s=T_2-T_1} \). Thus

\[
(4.17) \quad u_2^0 = u_2^f \text{ on } Y_{2s_0}^{(2)}.
\]

It follows from (4.13) that

\[
(4.18) \quad \| u_2^f \|_{1, T^4}^2 + \| u_{1y_0}^f \|_{0, T_4}^2 \leq C \| u_1^f \|_{1, T_2}^2,
\]

since that \( u_1^f = 0 \) on \( \Gamma_3 \) by the domain of dependence argument.

Since \( Y_{2s_0} \) belongs to the domain of dependence of \( D_{T_2} \) we get, similarly to (4.14), that

\[
(4.19) \quad \| u_2^f \|_{1, Y_{2s_0}^{(2)}}^2 \leq C_1 (\| u_2^0 \|_{1, D_{T_2}}^2 + \| u_{2y_0}^0 \|_{0, D_{T_2}}^2),
\]

where \( D_{T_2} = D^{(2)} \cap \{ y_0 = T_2 \} \).

We also have (cf. Lemma 3.6)

\[
(4.20) \quad \| u_2^0 \|_{1, D_{T_2}}^2 + \| \frac{\partial u_2^0}{\partial x_0} \|_{0, D_{T_2}}^2 \leq C [f_0]_{1, \Gamma_0 \times [0, T_2]}^2,
\]

Combining (4.18), (4.9) with (4.19), (4.20) and taking into account (4.17), we get

\[
(4.21) \quad \| u_2^f \|_{1, Y_{2s_0}^{(2)}} \leq C \| u_1^f \|_{1, Y_{2s_0}^{(1)}}.
\]

Now we shall prove Theorem 4.3

**Proof of Theorem 4.3** Since \( u_0^{(1)} \in H^1_0(R_{2s_0}^{(1)}) \) we get, using the Density Lemma 3.1 that there exists \( u_{1n}, f_n \in H^1_0(\Gamma_2 \times [s_0, T_2]) \) such that \( \| u_0^{(1)} - u_{1n} \|_{1, Y_{2s_0}^{(1)}} \rightarrow 0 \). By Lemma 4.3 \( \{ u_2^{f_n} \} \) also converges in \( H^1_0(Y_{2s_0}^{(2)}) \) to some function \( w \in H^1_0(Y_{2s_0}^{(2)}) \). Passing to the limit in

\[
(4.22) \quad (u_{1s}^{f_n}, v_1^g) = (u_{2s}^{f_n}, v_2^g),
\]

we get

\[
(4.23) \quad (u_0^{(1)}, v_1^g) = (w_s, v_2^g) \quad \text{for any } g \in H^1_0(\Gamma_{3T_1}),
\]

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where $\Gamma_{3T_1} = \Gamma_3 \times [T_1, T_2]$ Note that (4.22) and therefore (4.23) hold also for any $g' \in H_0^1(\Gamma_{3s_0})$, i.e.

$$\tag{4.24} \left( u_{0s}^{(1)}, v_1^{g'} \right) = \left( w_s, v_2^{g'} \right).$$

For such $g'$ the equality (4.3) holds, i.e.

$$\tag{4.25} \left( u_{0s}^{(1)}, v_1^{g'} \right) = \left( u_{0s}^{(2)}, v_2^{g'} \right).$$

Comparing (4.24) and (4.25) we get

$$\tag{4.26} \left( u_{0s}^{(2)}, v_2^{g'} \right) = \left( w_s, v_2^{g'} \right),$$

Since $v_2^{g'} \in H_0^1(Y_{3s_0}^{(2)})$ are dense in $H_0^1(R_{3s_0}^{(2)})$ and $w \in H_0^1(Y_{2s_0}^{(2)}) \subset H_0^1(Y_{3s_0}^{(2)})$, we have that $u_{0s}^{(2)} = w_s$. Since $u_{0s}^{(2)}$ and $w$ are zero on $\partial Y_{3s_0}^{(2)} \setminus \{y_n = 0\}$ we get that

$$\tag{4.27} u_{0s}^{(2)} = w \text{ in } Y_{2s_0}^{(2)}.$$

Therefore (4.23) and (4.27) gives

$$\tag{4.28} \left( u_{0s}^{(1)}, v_1^{g} \right) = \left( u_{0s}^{(2)}, v_2^{g} \right)$$

for all $g \in H_0^1(\Gamma_{3T_1})$, i.e. (4.4) holds.

The following formula will be the main tool in solving the inverse problem.

**Theorem 4.5.** For any $T_1 \leq s_0 \leq T_2$ the integral

$$\tag{4.29} \int_{Y_{jT_1} \cap \{0 \leq s \leq s_0 - T_1\}} \frac{\partial u_f^{(j)}}{\partial s} \nu^sdys' \quad \forall f \in H_0^1(\Gamma_{jT_1}), \ Vg \in H_0^1(\Gamma_{jT_1}), \ j = 1, 2,$$

is determined by the DN operator on $\Gamma_{jT_1} = \Gamma_j \times [T_1, T_2]$.

**Proof:** Since $u_{0s}^{(i)} = \frac{\partial u_f^{(i)}}{\partial s}$ for $s \geq s_0 - T_1$, $u_{0s} = 0$ for $s \leq s_0 - T_1$, formula (4.28) gives that $\int_{Y_{jT_1} \cap \{s > s_0 - T_1\}} \frac{\partial u_f^{(j)}}{\partial s} \nu^sdys'$ is determined by the DN operator on $\Gamma_{jT_1}$. The integral (4.29) is the difference $(u_s^{g'}, v^g) - (u_{0s}, v^g)$ thus (4.29) is determined by DN operator.
Remark 4.1. When the coefficients of $L^{(i)}_1$, $i = 1, 2$, do not depend on $y_0$, we can obtain the estimate (4.5) without assuming the BLR condition. In this case we can derive, in addition to (3.3), another Green’s formula (cf. (3.18) in [E3]):

Consider the identity

$$0 = (L_1 u, v_{y_0}) + (u_{y_0}, L_1 v).$$

Integrating by parts as in [E3] and using that $\frac{\partial}{\partial y_0}$ and $L_1$ are commute, we get (cf. (3.20) in [E3])

$$\tilde{Q}(u, v) = -\tilde{\Lambda}_0(f, g),$$

where

$$\tilde{Q}(u, v) = \frac{1}{2} \int_{Y_0 - 2} \left[ 2(u_s + iA'_u)v_{\bar{s}} + 2u_s(v_s + iA'_v) \right.$$  

$$- 2 \sum_{j=1}^{n-1} \left( g^{0j}_0 \left( \frac{\partial u}{\partial s} + iA'_u \right) \left( \frac{\partial v}{\partial y_j} - iA'_v \right) + g^{0j}_0 \left( \frac{\partial u}{\partial y_j} - iA'_u \right) \left( \frac{\partial v}{\partial s} + iA'_v \right) \right)$$  

$$- \sum_{j,k=1}^{n} g^{jk}_0 \left( \frac{\partial}{\partial y_j} - iA'_j \right) u \left( \frac{\partial}{\partial y_k} - iA'_k \right) v + V_1 u v \bigg] dsdy'$$

and

$$\Lambda_0(f, g) = \int_{\Gamma^{(2)} \times [0, T]} \left( \Lambda_1 f \bar{y}_{y_0} + f_{y_0} \bar{\Lambda}_1 g \right) dy' dy_0.$$

As in [E3] (cf. (3.23) in [E3]) we have that $\tilde{Q}(u, u)$ is a positive definite form when $T_2 - T_1$ is small and

$$C_2 \|u\|_{1, Y_{2y_0}}^2 \leq \tilde{Q}(u, u) \leq C_1 \|u\|_{1, Y_{2y_0}}^2.$$

Let $u_i^f, i = 1, 2$, be such that $L_1^{(i)} u_i^f = 0$ in $X_{2y_0}^{(i)}$, $u_i^f |_{y_0 = 0} = f$, $u_i^f = 0$ for $y_0 < T_1$, supp $f$ is contained in $\Gamma_2 \times (T_1, T_2]$. We assume that $\Lambda_1^{(1)} = \Lambda_1^{(2)}$ on $\Gamma_{2T_1} = \Gamma_2 \times (T_1, T_2)$. It follows from (4.13), (4.31), (4.33) that

$$\tilde{Q}_1(u_1^f, u_1^f) = \tilde{Q}_2(u_2^f, u_2^f).$$
where \( \tilde{Q}_i \) corresponds to \( L_{1,i}^{(j)}, i = 1, 2 \). Thus, (4.34) implies that

\[
C_1 \| u_1^f \|_{1,Y_{2,0}^{(1)}} \leq \| u_2^f \|_{1,Y_{2,0}^{(2)}} \leq C_2 \| u_1^f \|_{1,Y_{2,0}^{(1)}},
\]

i.e. the estimate (4.5) is proven.

5 The geometric optics construction

It follows from Theorem 4.5 that the DN operator allows to determine

\[
\int_{Y_{2,0}^{(1)} \cap \{ s \leq s_0 - T_1 \}} u_s^f \overline{v^g} dsdy' \text{ for all } f \in H^1_o(\Gamma_{jT_1}), g \in H^1(\Gamma_{jT_1}), j = 1, 2,
\]

i.e. if \( u_i^f, v_i^g \) satisfy (3.1), (3.2), \( i = 1, 2 \), then

\[
\int_{Y_{2,0}^{(1)} \cap \{ s \leq s_0 - T_1 \}} u_1^f \overline{v_1^g} dsdy' = \int_{Y_{2,0}^{(2)} \cap \{ s \leq s_0 - T_1 \}} u_2^f \overline{v_2^g} dsdy'.
\]

Let \( u_i \) be the solution of \( L_{1,i}^i u_i = 0 \) such that

\[
u_{N}^{(i)} u_{i}^{(i)} = u_{N}^{(i)} + u_{i}^{(N+1)}, \quad u_{i}^{(i)} = \sum_{p=0}^{N} \frac{a_{p}^{(i)}(s, \tau, y')}{(ik)^{p}} e^{ik(s-s_0)}, \quad s_0 = s_0 - T_1,
\]

\( u_{i}^{(N+1)} \) will be chosen below, \( k \) is a large parameter. We have the following equations for \( a_{p}^{(i)}, 0 \leq p \leq N \), (cf. [E4]):

\[
-4i \left( \frac{\partial}{\partial \tau} + iA_{-}^{(i)} \right) a_{0}^{(i)} + 2i \sum_{j=1}^{n-1} g_{0}^{i j} \left( \frac{\partial}{\partial y_{j}} - iA_{j}^{(i)} \right) a_{0}^{(i)}
+ 2i \sum_{j=1}^{n-1} \left( \frac{\partial}{\partial y_{j}} - iA_{j}^{(i)} \right) (g_{0}^{i j} a_{0}^{(i)}) = 0,
\]

\[
-4i \left( \frac{\partial}{\partial \tau} + iA_{-}^{(i)} \right) a_{p}^{(i)} + 2i \sum_{j=1}^{n-1} g_{0}^{i j} \left( \frac{\partial}{\partial y_{j}} - iA_{j}^{(i)} \right) a_{p}^{(i)}
+ 2i \sum_{j=1}^{n-1} \left( \frac{\partial}{\partial y_{j}} - iA_{j}^{(i)} \right) (g_{0}^{i j} a_{0}^{(i)}) = -L_{1}^{(i)} a_{p-1}^{(i)}, \quad p \geq 1,
\]
with the initial conditions

\[ a_0^{(i)}(s, \tau, y') \big|_{\tau = \tau_0} = \chi_1(s) \chi_2(y'), \quad \tau_0 = T_2 - T_1 - s, \]

\[ a_p^{(i)}(s, \tau, y') \big|_{\tau = \tau_0} = 0, \quad p \geq 1, \]

where \( \chi_1(s) = 0 \) for \( |s - s_0'| > 2\delta \), \( \chi_1(s) = 1 \) for \( |s - s_0'| \leq \delta \), \( \chi_2(y') \in C_0^\infty(\Gamma_2) \), \( \chi_2(y') \neq 0 \) when \( |y' - y_0'| < \delta \), \( y_0' \in \Gamma_2 \) is arbitrary, \( g_{i0} \) corresponds to \( L_1^{(i)}, i = 1, 2 \). Note that \( y_n = \frac{T_2 - T_1 - s - \tau}{T_2 - T_1} = 0 \) when \( \tau = \tau_0 \).

Let \( u_i^{(N+1)} \) be such that

\[ L_1^{(i)} u_i^{(N+1)} = -\frac{1}{(ik)^N} \left( L_1^{(i)} a_N^{(i)} \right) e^{ik(s-s_0')}, \quad y_n > 0, \quad y_0 < T_2, \]

\[ u_i^{(N+1)} = u_{i0}^{(N+1)} = 0 \quad \text{when} \quad y_0 = T_1, \quad y_n > 0, \quad i = 1, 2, \quad u_i^{(N+1)} \big|_{y_n = 0} = 0, \quad y_n \leq T_2. \]

Such \( u_i^{(N+1)} \) exists (cf. [H]) and \( L_1^{(i)} (u_i^{(i)} + u_i^{(N+1)}) = 0 \).

Since \( \text{supp} u_N^{(i)} \) is contained in a small neighborhood of the line \( \{s = s_0 - T_1, y' = y_0'\} \), we have that \( \text{supp} (u_N^{(i)} + u_i^{(N+1)}) \subset D(\Gamma_2 \times [T_1, T_2]) \) when \( s_0 - T_1 > 0 \).

Let \( \beta^{(i)}(s, \tau, y') = (\beta_1^{(i)}, \beta_2^{(i)}, \ldots, \beta_{n-1}^{(i)}) \) be the solution of the system (cf. [E4])

\[ \frac{\partial \beta_j^{(i)}(s, \tau, y')}{\partial \tau} = -g_{i0}^{(j)}(s, \tau, \beta^{(i)}(s, \tau, y')), \quad 1 \leq j \leq n - 1, \quad y_n > 0, \]

\[ \beta^{(i)}(s, \tau, y') \big|_{\tau = \tau_0} = y_i', \quad i = 1, 2, \quad \tau_0 = T_2 - T_1 - s, \]

where \( y' = (\hat{y}_1, \ldots, \hat{y}_{n-1}) \in \Gamma_2 \), \( s \) is a parameter in \( (5.7) \).

Let

\[ \hat{s} = s, \quad \hat{\tau} = \tau, \quad \hat{y}' = \alpha^{(i)}(s, \tau, y'), \quad \alpha^{(i)} = (\alpha_1^{(i)}, \ldots, \alpha_{n-1}^{(i)}) \]

be the inverse to the map

\[ s = \hat{s}, \quad \tau = \hat{\tau}, \quad y' = \beta^{(i)}(\hat{s}, \hat{\tau}, \hat{y}'), \]

i.e.

\[ \alpha_j^{(i)}(s, \tau, \beta^{(i)}(s, \tau, y')) = y_j, \quad 1 \leq j \leq n - 1. \]
Note that \( \alpha_j^{(i)}(s, \tau, y'), 1 \leq j \leq n - 1, \) satisfy the equation
\[
\frac{\partial \alpha_j^{(i)}(s, \tau, y')}{\partial \tau} - \sum_{k=0}^{n-1} g_{j0}^{(i)}(s, \tau, y') \frac{\partial \alpha_j^{(i)}}{\partial y_k} = 0,
\]
\( \alpha_j^{(i)}|_{\tau=\tau_0} = y_j, \ 1 \leq j \leq n - 1. \)

Let \( \dot{a}_0^{(i)}(s, \tau, y') = a_0^{(i)}(s, \tau, y'), \) where \( y' = \beta^{(i)}(s, \tau, y'). \) Then using (5.11) and (5.2) we get
\[
(5.13) \quad \frac{\partial \dot{a}_0^{(i)}}{\partial \tau} = \frac{\partial a_0^{(i)}}{\partial \tau} + \frac{\partial a_0^{(i)}}{\partial y_j} \frac{\partial \beta_j^{(i)}}{\partial \tau} = \frac{\partial a_0^{(i)}}{\partial \tau} - \sum_{j=1}^{n} g_{0j}^{(i)} \frac{\partial a_0^{(i)}}{\partial y_j} = \dot{B}^{(i)}(s, \tau, y') \dot{a}_0^{(i)}(s, \tau, y'),
\]

where \( B^{(i)}(s, \tau, y') = -iA'_- - i \sum_{j=1}^{n-1} g_{j0}^{(i)} A_j' + \frac{1}{2} \sum_{j=1}^{n-1} \frac{\partial g_{0j}^{(i)}}{\partial y_j}, \)
\( \dot{B}^{(i)}(s, \tau, y') = B^{(i)}(s, \tau, \beta^{(i)}(s, \tau, y')), \dot{a}_0^{(i)}(s, \tau, y')|_{\tau=\tau_0} = \chi_1(s)\chi_2(y'). \)

Therefore
\[
(5.14) \quad a_0^{(i)}(s, \tau, y') = \chi_1(s)\chi_2\alpha (s, \tau, y') e^{\theta^{(i)}(s, \tau, \alpha^{(i)})}
\]

where \( \theta^{(i)}(s, \tau, y') = \int_{\tau_0}^{\tau} \dot{B}^{(i)}(s, \tilde{\tau}, y') d\tilde{\tau}. \) Substituting \( u = u^{(i)}_{N} + u^{(N+1)}_i \) into (4.29) instead of \( u^f_i, \) integrating by parts in \( s \) and taking the limit when \( k \rightarrow \infty, \) we get
\[
(5.15) \quad \int_{\mathbb{R}^{n-1}} e^{\theta^{(1)}(s_0', 0, \alpha^{(1)})} \chi_2(\alpha^{(1)}(s_0', 0, y')) v_1^{(i)}(s_0', 0, y') dy' = \int_{\mathbb{R}^{n-1}} e^{\theta^{(2)}(s_0', 0, \alpha^{(2)})} \chi_2(\alpha^{(2)}(s_0', 0, y')) v_2^{(i)}(s_0', 0, y') dy'.
\]

Note that \( \tau = 0 \) on \( Y_{2T_1}^{(i)}, i = 1, 2. \) In (5.15) \( s_0 \in (T_1, T_2] \) is arbitrary, \( s_0' = s_0 - T_1. \)

Denote by \( Y_{2T_1}^{(i)}(\tau') \) the intersection of the plane \( \tau = \tau' \) with \( X_{2T_1}^{(i)}. \) Let \( R_{2T_1}^{(i)}(\tau') \subset Y_{2T_1}^{(i)}(\tau') \) be the rectangle \( \{ \tau = \tau', 0 \leq s \leq T_2 - T_1 - \tau', y' \in \Gamma_2 \}. \)

Note that \( Y_{2T_1}^{(i)}(0) = Y_{2T_1}^{(i)} \) and \( R_{2T_1}^{(i)}(0) = R_{2T_1}^{(i)}. \)
Repeating the proof of Theorem 4.5 with \(Y_{2T_1}^{(2)}, R_{2T_1}^{(2)}\) replaced by \(Y_{2T_1}^{(2)}(\tau'), R_{2T_1}^{(2)}(\tau'), 0 \leq \tau' \leq T_2 - T_1\), we get again, using the geometric optics construction (5.11), that (5.15) holds for any \((s, \tau, y) \in \Sigma\), where \(\Sigma = \{(s, \tau), s \geq 0, \tau \geq 0, s + \tau \leq T_2 - T_1\}\). Thus, we have

\[
\int_{\mathbb{R}^{n-1}} e^{b^{(1)}} \chi_2(\alpha^{(1)}(s, \tau, y')) \overline{v_1^q(s, \tau, y')} dy' \quad = \quad \int_{\mathbb{R}^{n-1}} e^{b^{(2)}} \chi_2(\alpha^{(2)}(s, \tau, y')) \overline{v_2^q(s, \tau, y')} dy'.
\]

for \((s, \tau, y') \in X_{2T_1}^{(i)}\).

Let \(\beta^{(i)}(\Sigma \times \Gamma_2)\) be the image of \(\Sigma \times \Gamma_2\) under the map (5.10). Note that the support of geometric optics solution \(u_{N}^{(i)} + u_{i}^{(N+1)}\) is contained in \(D(\Gamma_2 \times [T_1, T_2])\). Also we have that the curve \(y' = \beta^{(i)}(s, \tau, y')\) for \(\tau_0 \leq \hat{\tau} \leq \tau\), is contained in \(X_{2T_1}^{(i)}\). Therefore \(\beta^{(i)}(\Sigma \times \Gamma_2) \subset X_{2T_1}^{(i)}\). Denote by \(X_{\Gamma_2}^{(i)}\) the intersection of \(\beta^{(i)}(\Sigma \times \Gamma_2)\) with \(\Sigma \times \Gamma_2\). Note that \(\Sigma \times \Gamma_2 = \bigcup_{0 < \tau' \leq T_2 - T_1} R_{2T_1}^{(i)}(\tau')\).

Finally, denote by \(\tilde{X}_{\Gamma_2}^{(i)}\) the image of \(X_{\Gamma_2}^{(i)}\) under the inverse map (5.9).

Note that \(\tilde{X}_{\Gamma_2}^{(i)} \subset \Sigma \times \Gamma_2\).

Making the change of variables (5.10) in (5.16) we get

\[
\int_{\Gamma_2} e^{b^{(1)}(s, \tau, y')} \chi_1(y') \overline{v_1^q(s, \tau, \beta^{(1)}(s, \tau, y'))} J_1 (s, \tau, y') dy' \\
= \quad \int_{\Gamma_2} e^{b^{(2)}(s, \tau, y')} \chi_2(y') \overline{v_2^q(s, \tau, \beta^{(2)}(s, \tau, y'))} J_2 (s, \tau, y') dy',
\]

where \(J_i\) is the Jacobian of the map (5.10), \((s, \tau, y') \in \Sigma \times \Gamma_2\).

Let \(b^{(i)} = b_1^{(i)} + ib_2^{(i)}\), where \(b_1^{(i)}, b_2^{(i)}\) are real.

Since \(\chi_2(y') \in C_0^\infty(\Gamma_2)\) is arbitrary, we have

\[
e^{b^{(1)} - ib_2^{(1)}} v_1^q (s, \tau, \beta^{(1)}(s, \tau, y')) J_1 = e^{b^{(2)} - ib_2^{(2)}} v_2^q (s, \tau, \beta^{(2)}(s, \tau, y')) J_2.
\]

Let

\[
w_1^q (s, \tau, y') = v_1^q (s, \tau, \beta^{(i)}(s, \tau, y')), \quad \hat{y}' \in \Gamma_2,
\]

\[
w_2^q (s, \tau, y') = w_2^q (s, \tau, \hat{y}') e^{-b^{(i)}(s, \tau, y')},
\]

\[42\]
Our strategy will be to show that \( w_1^g(s, \tau, \dot{y}') = w_2^g(s, \tau, \dot{y}') \) in \( \bar{X}^{(1)}_1 \) and then to show that the equations \( \bar{L}_1^{(1)} w_1^g = 0 \) and \( \bar{L}_1^{(2)} w_1^g = 0 \) have the same coefficients in \( \bar{X}^{(1)}_1 \). Here \( \bar{L}_1^{(i)} \) is obtained from \( L_1^{(i)} \) by the change of variables (5.10), \( i = 1, 2 \).

We shall show first that \( e^{2b_1^{(1)}} J_1(s, \tau, \dot{y}') = e^{2b_2^{(2)}} J_2(s, \tau, \dot{y}') \). Consider the geometric optics solutions \( v_{i,k}^g \) of the form (5.2), where \( g = \chi_1(s) \chi_3(y'), \chi_3(y') \in C_0^\infty(\Gamma_2) \) is arbitrary. Substituting \( v_{i,k}^g \) into (5.16), integrating by parts and passing to the limit when \( k \to \infty \), we get

\[
\int e^{2b_1^{(1)}} \chi_2(\alpha^{(1)}(s_0', \tau, y')) \chi_3(\alpha^{(1)}(s_0', \tau, y')) dy' = \int e^{2b_2^{(2)}} \chi_2(\alpha^{(2)}(s_0', \tau, y')) \chi_3(\alpha^{(2)}(s_0', \tau, y')) dy',
\]

where \( s_0' = s_0 - T_1 \).

Note that \( e^{b^{(i)}} e^{b^{(i)}} = e^{2b_i^{(i)}} \).

Making the change of variables \( y' = \beta^{(i)}(s_0, \tau, \dot{y}') \) and using that \( \chi_2 \) and \( \chi_3 \) are arbitrary we get

\[
e^{2b_1^{(1)}} J_1(s_0', \tau, \dot{y}') = e^{2b_2^{(2)}} J_2(s_0', \tau, \dot{y}').
\]

Therefore, (5.18) and (5.21) imply

\[
e^{-b^{(1)}(s, \tau, \dot{y}')} v_1^g(s, \tau, \beta^{(1)}(s, \tau, \dot{y}')) = e^{-b^{(2)}(s, \tau, \dot{y}')} v_2^g(s, \tau, \beta^{(2)}(s, \tau, \dot{y}')) \quad \text{in} \quad \Sigma \times \Gamma_2,
\]

i.e. \( \tilde{w}_1^g(s, \tau, \dot{y}') = \tilde{w}_2^g(s, \tau, \dot{y}') \).

As in (4.12) the integration by parts gives

\[
\int_{\Sigma_T} (u_1^g \overline{v_1^g} - u_1^g \overline{v_2^g}) ds dy' = -2 \int_{\Sigma_T} u_1^g \overline{v_2^g} ds dy' + \int_{\partial \Sigma_T \cap \{y_n = 0\}} u_1^g \mathbf{v}_{y_n = 0} \mathbf{v}_{y_n = 0}^\top dy'.
\]

Therefore \( \int_{\Sigma_T} u_1^g \overline{v_2^g} ds dy' \) is determined by the boundary data since \( u_1^g \big|_{y_n = 0} = f(T_2, \dot{y}'), \overline{\mathbb{v}}_{y_n = 0} = \overline{\mathbb{f}}(T_2, \dot{y}') \), i.e. the roles of \( u_1^g \) and \( u_2^g \) are reversed in comparison with (4.12). Therefore we get, as in (4.28),

\[
\int_{Y_{2y_n}^{(1)}} u_1^g \overline{v_1^g} ds dy' = \int_{Y_{2y_n}^{(2)}} u_2^g \overline{v_2^g} ds dy'.
\]
Substituting in (5.23) the geometric optics solution (5.2), integrating by parts in $s$, multiplying by $ik$ and, finally, taking the limit when $k \to \infty$, we get (5.16) with $v_i^g$ replaced by $v_i^{gs}$. Note that we assumed that $v_i^g \in H_0^2(\tilde{\Gamma}_{T_1})$ when integrating by parts in (5.23). This can be achieved by requiring that $g \in H_0^2(\partial \tilde{\Gamma}_{T_1})$ (cf. [H], [E6]). Therefore we get (5.18), with $v_i^g$ replaced by $v_i^{gs}$:

$$e^{b_1^{(1)} - ib_2^{(1)}} v_{1s}^{g}(s, \tau, \beta^{(1)}(s, \tau, \hat{y}')) J_1 = e^{b_1^{(2)} - ib_2^{(2)}} v_{2s}^{g}(s, \tau, \beta^{(2)}(s, \tau, \hat{y}')) J_2.$$  

Using (5.21) we get

$$e^{-b^{(1)}(s, \tau, \beta^{(1)})} v_{1s}^{g}(s, \tau, \beta^{(1)}) = e^{-b^{(2)}(s, \tau, \beta^{(2)})} v_{2s}^{g}(s, \tau, \beta^{(2)}).$$

We shall need the following lemma:

**Lemma 5.1.** The equalities

$$\alpha_{js}^{(1)}(s, \tau, \beta^{(1)}(s, \tau, \hat{y}')) = \alpha_{js}^{(2)}(s, \tau, \beta^{(2)}(s, \tau, \hat{y}')), \quad 1 \leq j \leq n - 1,$$

$$b^{(1)}(s, \tau, \hat{y}') = b^{(2)}(s, \tau, \hat{y}')$$

hold on $\tilde{X}_{T_1}^{(1)}$.

**Proof:** Making the change of variables $\hat{y}' = \alpha^{(i)}(s, \tau, y')$ in (5.19), we get

$$e^{-b^{(i)}(s, \tau, \alpha^{(i)}(s, \tau, y'))} v_i^g(s, \tau, y') = \tilde{w}_i^g(s, \tau, \alpha^{(i)}(s, \tau, y')).$$

Differentiating in $s$ we have

$$\left( - \frac{d}{ds} b^{(i)}(s, \tau, \alpha^{(i)}(s, \tau, y')) \right) e^{-b^{(i)}(s, \tau, y')} v_i^g(s, \tau, y') = e^{-b^{(i)}(s, \tau, y')} v_i^g(s, \tau, y')$$

$$\frac{\partial \tilde{w}_i^g(s, \tau, \alpha^{(i)})}{\partial s} + \sum_{j=1}^{n-1} \frac{\partial \tilde{w}_i^g(s, \tau, \alpha^{(i)})}{\partial \hat{y}_j} \alpha_{js}^{(i)}(s, \tau, \hat{y}').$$

Returning back in (5.28) to $y' = \beta^{(1)}(s, \tau, \hat{y}')$ coordinates we get

$$\frac{\partial \tilde{w}_i^g(s, \tau, \hat{y}')}{\partial s} + \sum_{j=1}^{n-1} \frac{\partial \tilde{w}_i^g(s, \tau, \hat{y}')}{\partial \hat{y}_j} \alpha_{js}^{(i)}(s, \tau, \beta^{(i)}(s, \tau, \hat{y}'))$$

$$= e^{-b^{(i)}(s, \tau, y')} v_{is}^g(s, \tau, \beta^{(i)}(s, \tau, \hat{y}')) - \frac{d}{ds} b^{(i)}(s, \tau, \alpha(s, \tau, y')) \bigg|_{y' = \beta^{(i)}(s, \tau, \hat{y}')}.$$
Subtracting (5.29) for $i = 1$ from (5.29) for $i = 2$ and taking into account (5.24) and (5.22) we get

\begin{equation}
(5.30) \\
n \sum_{j=1}^{n-1} \left( \alpha^{(1)}_{js}(s, \tau, \beta^{(1)}(s, \tau, \tilde{y}')) - \alpha^{(2)}_{js}(s, \tau, \beta^{(2)}(s, \tau, \tilde{y}')) \right) \frac{\partial \tilde{w}_i^q(s, \tau, \tilde{y})}{\partial \gamma_j} \\
+ \left( \frac{d}{ds} b^{(1)}(s, \tau, \alpha^{(1)}(s, \tau, y')) \bigg|_{y' = \beta^{(1)}} - \frac{d}{ds} b^{(2)}(s, \tau, \alpha^{(2)}(s, \tau, y')) \bigg|_{y' = \beta^{(2)}} \right) \tilde{w}_i^q(s, \tau, \tilde{y}) = 0
\end{equation}

for all $\tilde{w}_i^q(s, \tau, \tilde{y})$ where $(s, \tau, \tilde{y}) \in \Sigma \times \Gamma_2$.

Fix $\tau = \tau', 0 \leq \tau' < T_0 - T_1$. By the Density Lemma 3.1 \{v_i^q(s, \tau', y')\} are dense in $H^1_0(\tilde{R}_2^{(i)}(\tau'))$, where $g \in H^1_0(\Gamma_2 \times \{T_1 \leq y_0 \leq T_2 - \tau'\})$.

Let $\tilde{R}_2^{(i)}(\tau')$ be the image of $R_2^{(i)}(\tau') \cap \beta^{(i)}(\Sigma \times \Sigma)$ under the map (5.9).

Since $\tilde{w}_i^q = e^{-\beta^{(i)}} v_i^q(s, \tau, \beta^{(i)}(s, \tau, \tilde{y}'))$ we have that $\tilde{w}_i^q(s, \tau', \tilde{y})$ are dense in $H^1_0(\tilde{R}_2^{(i)}(\tau'))$.

The following lemma is similar to arguments in [E3], pp 1749-1750.

**Lemma 5.2.** Since $\{w_i^q(s, \tau', \tilde{y}), g \in H^1_0(\tilde{R}_2^{(i)}(\tau'))\}$ are dense in $\tilde{R}_2^{(i)}(\tau')$ we have

\begin{equation}
(5.31) \\
\alpha^{(1)}_{js}(s, \tau, \beta^{(1)}(s, \tau, \tilde{y}')) = \alpha^{(2)}_{js}(s, \tau, \beta^{(2)}(s, \tau, \tilde{y}')) \text{ on } \tilde{R}_2^{(i)}(\tau'),
\end{equation}

\begin{equation}
(5.32) \\
\frac{d}{ds} b^{(1)}(s, \tau, \alpha^{(1)}(s, \tau, y')) \bigg|_{y' = \beta^{(1)}} = \frac{d}{ds} b^{(2)}(s, \tau, \alpha^{(2)}(s, \tau, y')) \bigg|_{y' = \beta^{(2)}} \text{ on } \tilde{R}_2^{(i)}(\tau').
\end{equation}

**Proof:** Let $\gamma(s, \tau', \tilde{y}) \in C_0^{\infty}(\tilde{R}_2^{(i)}(\tau'))$. There exists a sequence $\tilde{w}_i^{qn}(s, \tau', \tilde{y}')$ convergent to $\gamma(s, \tau', \tilde{y}')$ in $H^1_0(\tilde{R}_2^{(i)}(\tau'))$. Therefore $\tilde{w}_i^{qn}$ converges weakly to $\gamma(s, \tau', \tilde{y}')$. Passing in (5.30) to the limit when $n \to \infty$ we get

\begin{equation}
(5.33) \\
n \sum_{j=1}^{n-1} \left( \alpha^{(1)}_{js}(s, \tau, \beta^{(1)}(s, \tau, \tilde{y}')) - \alpha^{(2)}_{js}(s, \tau, \beta^{(2)}(s, \tau, \tilde{y}')) \right) \frac{\partial \gamma}{\partial \gamma_j} \\
+ \left( \frac{d}{ds} b^{(1)}(s, \tau, \alpha^{(1)}(s, \tau, y')) \bigg|_{y' = \beta^{(1)}} - \frac{d}{ds} b^{(2)}(s, \tau, \alpha^{(2)}(s, \tau, y')) \bigg|_{y' = \beta^{(2)}} \right) \gamma(s, \tau', \tilde{y}') = 0.
\end{equation}
For any point \((s, \hat{y}') \in \tilde{R}_{2T_1}^{(1)}(\tau')\) we can find \(n\) \(C_0^\infty(\tilde{R}_{2T_1}^{(1)})\) functions \(\gamma_1(s, \hat{y}'), \ldots, \gamma_n(s, \hat{y}')\) such that the determinant of \(n \times n\) matrix

\[
\begin{bmatrix}
\frac{\partial \gamma_1}{\partial \hat{y}_1} & \cdots & \frac{\partial \gamma_1}{\partial \hat{y}_{n-1}} & \gamma_1 \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial \gamma_n}{\partial \hat{y}_1} & \cdots & \frac{\partial \gamma_n}{\partial \hat{y}_{n-1}} & \gamma_n 
\end{bmatrix}
\]

is not equal to zero at the point \((s, \hat{y}')\). Therefore (5.31), (5.32) hold.

Repeating the same arguments for any \(0 \leq \tau' \leq T_2 - T_1\) we get that (5.31), (5.32) hold for any \(\tau'\), i.e. it hold on \(\tilde{X}_2^{(1)} = \bigcup_{0 \leq \tau' \leq T_2 - T_1} \tilde{R}_{2T_1}^{(1)}(\tau')\), since

\[
\bigcup_{0 \leq \tau' \leq T_2 - T_1} \tilde{R}_{2T_1}^{(1)}(\tau') = \Sigma \times \Gamma_2
\]

under the map (5.9). This proves (5.25). To prove (5.26) we note that

\[
\left. \frac{d}{ds} b^{(i)}(s, \tau, \alpha^{(i)}(s, \tau, \hat{y}')) \right|_{\hat{y}' = \beta^{(i)}} = \frac{\partial b^{(i)}}{\partial s} + \sum_{j=1}^{n-1} \frac{\partial b^{(i)}}{\partial \hat{y}_j} \alpha^{(i)}_{\alpha^{(i)}}(s, \tau, \beta^{(i)}(s, \tau, \hat{y}')).
\]

Since (5.31), (5.32) hold, we have

\[
(5.34) \quad \frac{\partial}{\partial s} (b^{(1)} - b^{(2)}) + \sum_{j=1}^{n-1} \frac{\partial}{\partial \hat{y}_j} (b^{(1)} - b^{(2)}) \alpha^{(i)}_{\alpha^{(i)}}(s, \tau, \beta^{(i)}(s, \tau, \hat{y}')) = 0.
\]

Equation (5.34) is a linear homogeneous equation for \(b^{(1)}(s, \tau, \hat{y}') - b^{(2)}(s, \tau, \hat{y}')\) on \(\tilde{X}_2^{(1)}\). Since \(b^{(1)} = b^{(2)} = 0\) when \(y_n = 0\), we get

\[
(5.35) \quad b^{(1)}(s, \tau, \hat{y}') = b^{(2)}(s, \tau, \hat{y}') \quad \text{on} \quad \tilde{X}_2^{(1)}.
\]

It follows from (5.35) and (5.22) that

\[
(5.36) \quad w^q_i(s, \tau, \hat{y}') = w^q_i(s, \tau, \beta^{(i)}(s, \tau, \hat{y}')) \quad \text{on} \quad \tilde{X}_2^{(1)}
\]

where \(w^q_i = v^q_i(s, \tau, \beta^{(i)}(s, \tau, \hat{y}')).\)

### 6 The conclusion of the local step

We shall prove the following theorem:
Theorem 6.1. Let $L_1^{(i)} v_i^g = 0$, $i = 1, 2$. Make change of variables

\begin{equation}
\hat{s} = s, \quad \hat{\tau} = \tau, \quad \hat{y}' = \alpha^{(i)}(s, \tau, y'), \quad i = 1, 2.
\end{equation}

Let $\bar{L}_1^{(i)} w_i^g = 0$ be the operator $L_1^{(i)}$ in the new coordinates. Then the coefficients of $\bar{L}_1^{(1)}$ and $\bar{L}_1^{(2)}$ are equal on $\tilde{X}_1^{(1)}$.

Proof: Equations $L_1^{(i)} v_i^g = 0$ have the following form in $(s, \tau, y')$ coordinates (cf. (2.27)):

\begin{equation}
L_1^{(i)} v_i^g = -2 \frac{\partial}{\partial s} \left( \frac{\partial}{\partial \tau} + i A_{-}^{(i)} \right) v_i^g - 2 \frac{\partial}{\partial \tau} + i A_{-}^{(i)} \frac{\partial}{\partial s} v_i^g
+ \sum_{j=1}^{n-1} 2 \left( \frac{\partial}{\partial y_j} - i A_{j}^{(i)} \right) g_{j0}^{+j} \frac{\partial}{\partial s} v_i^g + \sum_{j=1}^{n-1} 2 \frac{\partial}{\partial s} \left( g_{j0}^{+j} \left( \frac{\partial}{\partial y_j} - i A_{j}^{(i)} \right) \right) v_i^g
+ \sum_{j,k=1}^{n-1} \left( \frac{\partial}{\partial y_j} - i A_{j}^{(i)} \right) g_{j0}^{jk} \left( \frac{\partial}{\partial y_k} - i A_{k}^{(i)} \right) v_i^g + V_1^{(i)} v_i^g = 0,
\end{equation}

where $i = 1, 2$, $g_{j0}^{+j} = g_{j0}^{0j}$, $V_1^{(i)}$ is the same as in (2.27).

Making the change of variables (6.1) in (6.2) we get:

\begin{equation}
\bar{L}_1^{(i)} w_i^g(s, \tau, y') = -2J_i^{-1}(s, \tau, y') \left( \frac{\partial}{\partial s} + i \tilde{A}_+^{(i)} \right) J_i \left( \frac{\partial}{\partial \tau} + i \tilde{A}_+^{(i)} \right) w_i^g
- 2J_i^{-1} \left( \frac{\partial}{\partial \tau} + i \tilde{A}_-^{(i)} \right) J_i \left( \frac{\partial}{\partial s} + i \tilde{A}_+^{(i)} \right) w_i^g
- \sum_{j=1}^{n-1} 2J_i^{-1} \left( \frac{\partial}{\partial \tau} + i \tilde{A}_-^{(i)} \right) J_i \alpha_{js}^{(i)}(s, \tau, \beta^{(i)}) \left( \frac{\partial}{\partial y_j} - i \tilde{A}_j^{(i)} \right) w_i^g
- \sum_{j=1}^{n-1} 2J_i^{-1} \left( \frac{\partial}{\partial y_j} - i \tilde{A}_j^{(i)} \right) J_i \alpha_{js}^{(i)}(s, \tau, \beta^{(i)}) \left( \frac{\partial}{\partial \tau} + i \tilde{A}_-^{(i)} \right) w_i^g
+ \sum_{j,k=1}^{n-1} J_i^{-1} \left( \frac{\partial}{\partial y_j} - i \tilde{A}_j^{(i)} \right) J_i \tilde{g}_{j0}^{jk} \left( \frac{\partial}{\partial y_k} - i \tilde{A}_k^{(i)} \right) w_i^g
+ V_1^{(i)}(s, \tau, \beta^{(i)}(s, \tau, y')) w_i^g(s, \tau, y') = 0,
\end{equation}
where \( w^g_i(s, \tau, \dot{y}') = v^g_i(s, \tau, \beta^{(i)}(s, \tau, \dot{y}')) \),

\[
\tag{6.4} \bar{g}^{jk}_{i0}(s, \tau, \dot{y}') = \sum_{p,r=1}^{n-1} g^{pr}_{i0}(s, \tau, \beta^{(i)}) A^{(i)}_{jk} \alpha^{(i)}_{ky}(s, \tau, \beta^{(i)}) \]

\[
- 2\alpha^{(i)}_{js} \alpha^{(i)}_{k\tau} - 2\alpha^{(i)}_{j\tau} \alpha^{(i)}_{ks} + 2 \sum_{p=1}^{n-1} g^{jk}_{i0} \alpha^{(i)}_{ky} (\alpha_{ks} \alpha^{(i)}_{k\tau} + \alpha_{j\tau} \alpha^{(i)}_{ks}).
\]

We used in (6.3) that (see (5.12))

\[
\tag{6.5} \bar{g}^{+j}_{i0}(s, \tau, \dot{y}') = \sum_{k=1}^{n-1} g^{+k}_{i0}(s, \tau, \beta^{(i)}(s, \tau', \dot{y}')) A^{(i)}_{jk} \alpha^{(i)}_{ky}(s, \tau, \beta^{(i)}) - \alpha^{(i)}_{j\tau}(s, \tau, \beta^{(i)}) = 0.
\]

Also we have

\[
\tag{6.6} \bar{g}^{-j}_{i0}(s, \tau, \dot{y}') = \sum_{k=1}^{n-1} g^{-k}_{i0}(s, \tau, \beta^{(i)}(s, \tau, \dot{y}')) A^{(i)}_{jk} \alpha^{(i)}_{ky}(s, \tau, \beta^{(i)}) - \alpha^{(i)}_{j\tau}(s, \tau, \beta^{(i)})
\]

\[
= -\alpha^{(i)}_{js}(s, \tau, \beta^{(i)}),
\]

since \( g^{-k}_{i0} = 0 \) (cf. (6.2)).

Note that \( A^{(i)}_+, A^{(i)}_j, A^{(i)}_\tau \) and \( \bar{A}^{(i)}_+, \bar{A}^{(i)}_j, \bar{A}^{(i)}_\tau \) are related by the equality

\[
\tag{6.7} A^{(i)}_+ ds + A^{(i)}_\tau d\tau - \sum_{j=1}^{n-1} A^{(i)}_j dy_j = \bar{A}^{(i)}_+ d\bar{s} + \bar{A}^{(i)}_\tau d\bar{\tau} - \sum_{j=1}^{n-1} \bar{A}^{(i)}_j d\bar{y}_j,
\]

where \( A^{(i)}_\tau = 0, i = 1, 2, s = \bar{s}, \tau = \bar{\tau}, y_j = \beta_j(s, \tau, \dot{y}'). \)

Note that (5.21), (5.26) imply

\[
\tag{6.8} J_1(s, \tau, \dot{y}') = J_2(s, \tau, \dot{y}') \quad \text{in} \quad \bar{X}^{(i)}_{\Gamma_2}.
\]

The first order term containing \( \frac{\partial}{\partial \tau} \) in (6.3) is equal to

\[
\tag{6.9} -2iA^{(i)}_+(\frac{\partial}{\partial \tau}) w^g_i - 2iJ^{-1}_i \left( \frac{\partial}{\partial \tau} \right) J_i \bar{A}^{(i)}_+ w^g_i + 2i \sum_{j=1}^{n-1} \bar{A}^{(i)}_j \alpha^{(i)}_{js} \frac{\partial}{\partial \tau} w^g_i
\]

\[
+ i \sum_{j=1}^{n-1} 2J^{-1}_i \left( \frac{\partial}{\partial \tau} \right) J_i \alpha^{(i)}_{js} \bar{A}^{(i)}_j w^g_i.
\]
It follows from (6.7) that

\[
A_+^{(i)} = \tilde{A}_+^{(i)} - \sum_{j=1}^{n-1} \tilde{A}_j^{(i)} \alpha_{js}(s, \tau, \beta^{(i)}(s, \tau, \tilde{y}')).
\]

Since \(A_+^{(i)} = 0\) we have that (6.10) implies that (6.9) is equal to zero.

Taking into account that \(\alpha_{js}^{(1)}(s, \tau, \beta^{(1)}) = \alpha_{js}^{(2)}(s, \tau, \beta^{(2)})\), 1 ≤ \(j\) ≤ \(n - 1\), \(J_1 = J_2\) and \(w_1^{q}(s, \tau, \tilde{y}') = w_2^{q}(s, \tau, \tilde{y}')\) we get that \(\tilde{L}_1^{(1)} - \tilde{L}_1^{(2)}\) is a differential operator in \(\frac{\partial}{\partial s}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\). We have

\[
(\tilde{L}_1^{(1)} - \tilde{L}_1^{(2)})w_1^{q} = 0.
\]

Since \(\{w_1^{q}, g \in C_0^\infty(\Gamma^{(1)} \times [T_1, T_2 - \tau'])\}\) are dense in \(H_0^1(\tilde{R}_{1T_1}(\tau'))\) we get as in Lemma 5.2 (cf. [E3]) that all coefficients of \(\tilde{L}_1^{(1)}\) and \(\tilde{L}_1^{(2)}\) are equal in \(\tilde{R}_1^{(1)}(\tau')\). Since \(\tau' \in [0, T_2 - T_1]\) is arbitrary, we get that on \(\tilde{X}_T^{(1)}\):

\[
\tilde{L}_1^{(1)} = \tilde{L}_1^{(2)} \text{ in } \tilde{X}_T^{(1)}.
\]

This completes the proof of Theorem 6.1. Let \(L'_iw_i^{q} = 0\) be the equation of the form (2.24). Making the change of variables (5.10) we get the equation \(\tilde{L}_1^{(1)}w_i^{q} = 0, i = 1, 2, \text{ on } \tilde{X}_T^{(1)}\).

Note that \(w_i^{q} = w_2^{q}\) on \(\tilde{X}_T^{(1)}\). We shall prove that \(\tilde{L}_1^{(1)} = \tilde{L}_1^{(2)}\) on \(\tilde{X}_T^{(1)}\).

Let \(\tilde{g}^{i+j}, \tilde{g}^{i-j}, \tilde{g}^{jk}, 1 ≤ j ≤ n - 1, 1 ≤ k ≤ n - 1\), be the inverse metric tensor of \(L'_i\). Note that for \(L'_i^{(1)}\) we have (cf. (2.27))

\[
g^{i+j}_{i0} = \frac{\tilde{g}^{i+j}}{\tilde{g}^{i-j}}, \quad g^{jk}_{i0} = \frac{\tilde{g}^{jk}}{\tilde{g}^{i-j}}, \quad i = 1, 2.
\]

Therefore the equation \(L'_iw_i^{q} = 0\) has the inverse metric tensor with elements (cf. (6.4), (6.5), (6.6))

\[
\tilde{g}^{jk}_i = \tilde{g}^{i-j+k}_i, \quad 1 ≤ j, k ≤ n - 1,
\]

\[
\tilde{g}^{i-k}_i = -\tilde{g}^{i-j}_i \alpha_{ks}^{(i)}(s, \tau, \beta^{(i)}), \quad \tilde{g}^{i-k}_i = 0, \quad 1 ≤ k ≤ n - 1, \quad i = 1, 2.
\]
Since $\alpha^{(1)}_{ks} = \alpha^{(2)}_{ks}$ and $\tilde{g}_{10}^{jk} = \tilde{g}_{20}^{jk}$ (see (6.12)), we get that the metric tensors of $\tilde{L}_1'$ and $\tilde{L}_2'$ are equal if we can prove that

\begin{equation}
\tilde{g}_1^{+-}(s, \tau, \beta^{(1)}(s, \tau, \tilde{y}')) = \tilde{g}_2^{+-}(s, \tau, \beta^{(2)}(s, \tau, \tilde{y}')).
\end{equation}

(6.16)  

We shall prove first that

\begin{equation}
g_1^{(1)}(s, \tau, \beta^{(1)}) = g_1^{(2)}(s, \tau, \beta^{(2)}),
\end{equation}

(6.17)  

where $g_i^{(1)} = \det[\tilde{g}_i^{jk}(n-1)]^{-1}$ (see (2.22)).

Note that $V_1^{(i)}(s, \tau, \tilde{y}')$ has the form (2.25) for $i = 1, 2$, where $A^{(i)} = \ln(g_1^{(i)})^{1/2}$. Making the change of variables (6.1) we get (cf. (6.14))

\begin{equation}
V_1^{(1)}(s, \tau, \beta^{(1)}(s, \tau, \tilde{y}')) - V_2^{(2)}(s, \tau, \beta^{(2)}(s, \tau, \tilde{y}')) = 0.
\end{equation}

(6.18)  

Note that the metric tensors for $\tilde{L}_1'$ and $\tilde{L}_2'$ are equal on $\tilde{X}_{\Gamma_2}$. Let $\tilde{A}^{(i)}(s, \tau, \tilde{y}') = A^{(i)}(s, \tau, \beta^{(i)}(s, \tau, \tilde{y}'))$.

Using the equality

\[
\tilde{A}_{y_j}^{(1)}(s, \tau, \tilde{y}') - \tilde{A}_{y_j}^{(2)}(s, \tau, \tilde{y}') = (\tilde{A}_{y_j}^{(1)} - \tilde{A}_{y_j}^{(2)})A_{y_k}^{(1)} + (\tilde{A}_{y_k}^{(1)} - \tilde{A}_{y_k}^{(2)})\tilde{A}_{y_j}^{(2)}
\]

and similar equality involving derivatives in $s$ and $\tau$ we can represent (6.18) as a homogeneous second order hyperbolic equation in $\tilde{A}^{(1)} - \tilde{A}^{(2)}$ with the coefficients depending on $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$. Since the Cauchy data for $\tilde{A}^{(1)} - \tilde{A}^{(2)} = 0$ at $y_n = 0$ (cf. Lemma 2.1) we get, by the uniqueness of the Cauchy problem (cf. [T], [J-R]), that $\tilde{A}^{(1)} = \tilde{A}^{(2)}$ in $\tilde{X}_{\Gamma_2}^{(1)}$. Therefore (6.17) holds.

Note that $\tilde{g}_i^{jk} = \tilde{g}_i^{+-} - \tilde{g}_0^{jk}$. Therefore

\[
g_1^{(i)} = \det[\tilde{g}_i^{jk}(n-1)]_{j,k=1}^{-1} = (\tilde{g}_1^{+-})^{n-1} \det[\tilde{g}_0^{jk}(n-1)].
\]

(6.17)

Since $\tilde{g}_1^{jk} = \tilde{g}_2^{jk}$ and (6.17) holds, we get

\begin{equation}
(\tilde{g}_1^{+-}(s, \tau, \beta^{(1)}))^{n-1} = (\tilde{g}_2^{+-}(s, \tau, \beta^{(1)}))^{n-1},
\end{equation}

and this proves (6.16), since we assumed that $n > 1$. Therefore metric tensors of $\tilde{L}_1'$ and $\tilde{L}_2'$ are equal. Combining this with (6.13), (6.14) we get $\tilde{L}_1' = \tilde{L}_2'$ on $\tilde{X}_{\Gamma_2}^{(1)} \supset \Gamma_1$.

**Remark 6.1.** Change $\Gamma_2$ to $\Gamma_1$. We have $\beta^{(i)}(\Sigma \times \overline{\Gamma}_1) \subset \beta^{(i)}(\Sigma \times \overline{\Gamma}_2)$. Since $\beta^{(i)}(\Sigma \times \overline{\Gamma}_1) \subset X_{1\Gamma_1}^{(i)}$ and $X_{1\Gamma_1}^{(i)} \subset (\Sigma \times \overline{\Gamma}_2)$, we get $\beta^{(i)}(\Sigma \times \overline{\Gamma}_1) \subset (\Sigma \times \overline{\Gamma}_2)$. 

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Therefore, $\beta(i)(\Sigma \times \Gamma_1) \subset (\Sigma \times \Gamma_2) \cap \beta(i)(\Sigma \times \Gamma_2) = X_{\Gamma_2}(i)$. Applying the map (5.9) to $\beta(i)(\Sigma \times \Gamma_1) \subset X_{\Gamma_2}(i)$ we get $\Sigma \times \Gamma_1 \subset \tilde{X}_{\Gamma_2}(i)$. Therefore, $\tilde{L}_1 = \tilde{L}_2$ on $\Sigma \times \Gamma_1$.

We shall summarize the results of §§2-6.

**Theorem 6.2** (Local step). Consider two initial boundary value problems

\begin{align}
L(i)u_i &= 0 \quad \text{in} \; D_0^{(i)} \times \mathbb{R}, \\
u_i &= 0 \quad \text{for} \; x_0 \ll 0, \\
u_i|_{\partial D_0^{(i)}} &= f, \; i = 1, 2,
\end{align}

where $L(i)$ have the form (1.1). Suppose $\Gamma_0 \subset \partial D_0^{(1)} \cap \partial D_0^{(2)}$ and suppose the BLR condition holds for $L^{(1)}$ on $[t_0, T_{t_0}]$. Suppose the corresponding DN operators $\Lambda^{(i)}$ are equal on $\Gamma_0 \times (t_0, T_2)$, $T_2 \geq T_{t_0}$, i.e. $\Lambda^{(1)}f = \Lambda^{(2)}f$ on $\Gamma_0 \times (t_0, T_2)$ for all $f$ with support in $\Gamma_0 \times [t_0, T_2]$. Let $T_2 - T_1$ be small. Suppose coefficients of $L^{(1)}$ and $L^{(2)}$ are analytic in $x_0$.

Let $\varphi^{(i)}$ be the changes of variables (2.14) for $i = 1, 2$ and let $\beta^{(i)}, i = 1, 2,$ be the changes of variables (5.10). Let $c_i$ be the gauge transformation (2.20), (2.21) for $i = 1, 2$. Then

\begin{align}
\beta^{(1)} \circ c_1 \circ \varphi^{(1)} \circ L^{(1)} &= \beta^{(2)} \circ c_2 \circ \varphi^{(2)} \circ L^{(2)} \quad \text{on} \; \Sigma \times \Gamma_1,
\end{align}

where

$\Sigma = \{(s, \tau), s \geq 0, \tau \geq 0, s + \tau \leq T_2 - T_1\} = \{(y_0, y_n) : 0 \leq y_n \leq \frac{T_2 - T_1}{2}, T_1 + y_n < y_0 < T_2 - y_n\}$.

**7 The global step**

Let $L_iu_i = 0$ in $D_i = D_0^{(i)} \times \mathbb{R}, i = 1, 2$, $u_i = 0$ for $x_0 \ll 0$, $\partial D_0^{(i)} \cap \partial D_0^{(2)} \supset \Gamma_0$ and $u_i|_{\partial D_0^{(i)} \times \mathbb{R}} = f, i = 1, 2$, $f$ has a compact support in $\Gamma_0 \times \mathbb{R}$.

First we extend the Theorem 6.2 for a larger time interval.

Let $[t_1, t_2]$ be an arbitrary time interval. Let $[t_0, T_{t_0}]$ be such that $T_{t_0} \leq t_1$ and the BLR condition holds on $[t_0, T_{t_0}]$. Thus the BLR condition is satisfied on $[t_0, t]$ for any $t \in [t_1, t_2]$. Let $\Gamma_1$ be arbitrary connected part of $\Gamma_0$, $\Gamma_1 \subset \Gamma_0$. Note that we do not require $\Gamma_1$ to be small.
Let \( \psi_{0i}^\pm(x_0, x', x_n), i = 1, 2, \) be the solution of the form (2.4) in \([t_0 - 1, t_2 + 1] \times \overline{\Gamma} \times [0, \varepsilon_{\pm}]\) where \( \Gamma_1 \subset \Gamma' \subset \Gamma_0.\)

We impose the following initial conditions on \( \psi_{0i}^\pm, i = 1, 2, \)

\[
(7.1) \quad \psi_{0i}^+|_{x_n=0} = x_0, \quad \psi_{0i}^-|_{x_n=0} = -x_0.
\]

Such solutions exist in \([t_0 - 1, t_2 + 1] \times \overline{\Gamma} \times [0, \varepsilon_0] \subset D_0^{(i)} \times \mathbb{R},\) when \( \varepsilon_0 \) is small. We choose \( \psi_{0i}^\pm \) such that (2.6) is satisfied and we choose \( \varepsilon_1 > 0 \) such that \( \varepsilon_1 \leq \varepsilon_0 \) and \( \{0 < x_n < \varepsilon_1, x' \in \overline{\Gamma}, x_0 \in [t_0 - 1, t_2 + 1]\} \) do not intersect \( \partial D_0^{(i)} \times \mathbb{R}.\)

Let \( \varphi_{ji}(x_0', x', x_n), 1 \leq j \leq n - 1, \) be the solutions of the linear equations (cf. (2.7))

\[
(7.2) \quad \sum_{p,k=0}^{n} g_{ij}^{pk}(x) \psi_{0ixp}' \varphi_{jix_k} = 0 \quad \text{in} \quad [t_0 - 1, t_2 + 1] \times \overline{\Gamma} \times [0, \varepsilon_1]
\]

with initial conditions

\[
(7.3) \quad \varphi_{ji} |_{x_n=0} = x_j, \quad 1 \leq j \leq n - 1.
\]

Similarly to (2.14) consider the map \((y_0^{(i)}(x), y_j^{(i)}(x), y_n^{(i)}(x)) = (\varphi_0^{(i)}, \varphi_j', \varphi_n^{(i)}), x \in [t_0 - 1, t_2 + 1] \times \overline{\Gamma} \times [0, \varepsilon_1],\) where

\[
(7.4) \quad y_0^{(i)}(x) = \frac{\psi_{0ix}^+ - \psi_{0ix}^-}{2}, \quad y_j^{(i)}(x) = \varphi_{ji}(x), \quad y_n^{(i)}(x) = -\frac{\psi_{0ix}^+ + \psi_{0ix}^-}{2}.
\]

As in (2.15) we have that the map \((x_0, x', x_n) \rightarrow (y_0, y', y_n)\) is the identity when \( x_n = 0:\)

\[
(7.5) \quad y_0^{(i)} |_{x_n=0} = x_0, \quad y_j^{(i)} |_{x_n=0} = x_j, \quad 1 \leq j \leq n - 1, \quad y_n^{(i)} |_{x_n=0} = 0.
\]

Let \( u_s = \frac{1}{2}(u_{y_0} - u_{y_n}), u_\tau = -\frac{1}{2}(u_{y_0} + u_{y_n}).\) Making the change of variables (7.4) in \( L^1 u_i = 0,\) the gauge transformation (2.18), (2.21) and the change of unknown function (2.26), we get in \( t_0 \leq y_0 \leq t_2, 0 \leq y_n \leq T_0, y' \in \overline{\Gamma},\) \( T_0 \) is small, the equation of the form

\[
L^1_i u_i^{(i)} = 0, \quad y \in \Omega_0,
\]

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where \( L^{(i)}_1 \) has the form (2.28). Here

\[
\Gamma_1 \subset \Gamma' \subset \Gamma_0, \quad \tilde{\Omega}_0 \overset{\text{def}}{=} [t_0, t_2] \times \Gamma' \times [0, T_0].
\]

We assume that \( u_1^{(i)} \) satisfy the zero initial conditions

\[
u_1^{(i)} = \frac{\partial u_1^{(i)}}{\partial y_0} = 0 \quad \text{when} \quad y_0 = t_0
\]

and

\[
u_1^{(i)} \big|_{y_n=0} = f, \quad i = 1, 2.
\]

We also assume that DN operators for \( L^i \) and subsequently for \( L_1^{(i)} \) are equal on \([t_0, t_2] \times \tilde{\Gamma} \).

Note that the change of variables

\[
\hat{y}_n = y_n, \quad \hat{y}_0 = y_0, \quad \hat{y}'_j = \alpha_j^{(i)}(y_0, y', y_n), \quad 1 \leq j \leq n - 1,
\]

where \( \alpha_j^{(i)} \) are the same as in (5.9), (5.11), are also defined globally on \( \tilde{\Omega}_0 \).

Let \([T_1, T_2] \subset [t_1, t_2] \) be arbitrary such that \( T_2 - T_1 = 2T_0 \).

Applying Theorem 6.1 to the interval \([T_1, T_2] \) we get that the coefficients of \( \tilde{L}_1^{(1)} \) and \( \tilde{L}_1^{(2)} \) and the coefficients of \( L_1' \) and \( L_2' \) are equal on \( \Sigma_{T_1T_2} \times \Gamma_1 \)

where \( \Sigma_{T_1T_2} = \{0 \leq y_n \leq T_0, T_1 + y_n \leq y_0 \leq T_2 - y_n\} \). We assume that \( \Gamma' \supset \Gamma_1 \) is such that \( \Gamma_2 \subset \Gamma_3 \subset \Gamma' \) for all \([T_1, T_2] \subset [t_1, t_2] \). Here \( \Gamma_2, \Gamma_3 \) are defined as in §3. Note that \( \Gamma_2, \Gamma_3 \) may depend on \([T_1, T_2] \).

If two intervals \([T_1, T_2] \) and \([T_1', T_2'] \) intersect, then the coefficients of \( \tilde{L}_1^{(1)} \) and \( \tilde{L}_1^{(2)} \) coincide in \( (\Sigma_{T_1T_2} \cup \Sigma_{T_1'T_2'}) \times \Gamma_1 \).

Therefore coefficients of \( \tilde{L}_1^{(1)} \) and \( \tilde{L}_1^{(2)} \) and consequently the coefficients of \( L_1' \) and \( L_2' \) (cf. (6.2), (6.3)) coincide for \( 0 \leq y_n \leq T_0, \quad y' \in \Gamma_1, \; t_1 + T_0 < y_0 < t_2 - T_0 \).

Therefore we proved

**Lemma 7.1.** Suppose \([t_1, t_2] \) is arbitrary large, \( T_0 > 0 \) is small, \( t_0 \) is such that the BLR condition is satisfied on \([t_0, t_1] \).

Let \( \Omega_0 = \{y_0 \in [t_0 + T_0, t_2 - T_0], \quad y' \in \Gamma_1, \; y_n \in [0, T_0], \} \). Assume that the coefficients of \( L^{(i)} \) are analytic in \( x_0, i = 1, 2 \). Then

\[
\beta^{(1)} \circ \varphi_1 \circ \varphi^{(1)} \circ L_1^{(1)} = \beta^{(2)} \circ \varphi_2 \circ \varphi^{(2)} \circ L_2^{(2)} \quad \text{on} \quad \Omega_0.
\]
Let $\Omega_i = (\beta^{(i)} \varphi^{(i)})^{-1} \Omega_0$, $i = 1, 2$. Note that $\Omega_i \subset D_0^{(i)} \times [t_0 - 1, t_2 + 1]$ since $T_0$ is small. We have that $\Phi_2 = (\beta^{(1)} \varphi^{(1)})^{-1} \beta_2 \varphi^{(2)}$ maps $\Omega_2$ onto $\Omega_1$. Note that $\partial \Omega_1 \cap \partial \Omega_2 \supset \Gamma_1 \times [t_0, t_2]$ and $\Phi_2 = I$ on $\Gamma_1 \times [t_0 + T_0, t_2 - T_0]$. Note also that $\beta^{(i)} \circ c_i$ can be represented as $c_i' \circ \beta^{(i)}$ where $c_i'$ is the gauge transformation in $(y_0, \dot{y}', y_n)$ coordinates. Analogously, $(\beta^{(1)} \circ c_1 \circ \varphi^{(1)})^{-1} \beta^{(2)} \circ c_2 \circ \varphi = c_3 \circ \Phi_2$, where $c_3$ is the gauge transformation. Therefore

$$c_3 \circ \Phi_2 \circ L^{(2)} = L^{(1)} \text{ in } \Omega_1.$$ 

Let $B$ be a smooth domain in $D_0^{(1)}$ such that $\partial B \cap \partial D_0^{(1)} = \gamma_1 \subset \Gamma_0$. Suppose $B$ is small and such that $B \times [t_1 + 1, t_2 - 1] \subset \Omega_1$.

Let $S_2 = \Phi_2^{-1}(B \times [t_1 + 1, t_2 - 1]) \subset D_0^{(2)} \times \mathbb{R}$ and let $S_2^+ = \Phi_2^{-1}(B \times \{x_0 = t_2 - 1\})$, $S_2^- = \Phi_2^{-1}(B \times \{x_0 = t_1 + 1\})$. Let $\tilde{S}_2^+$ be space-like surfaces in $D_0^{(2)} \times \mathbb{R}$ such that $\tilde{S}_2^+$ is the extension of $S_2^+$ and $\tilde{S}_2^-$ is the extension of $S_2^-$. We assume that the projections of $\tilde{S}_2^+$ on $D_0^{(2)}$ is $D_0^{(2)}$. Let $D_1^{(2)}$ be the domain in $D_0^{(2)} \times \mathbb{R}$ bounded by $\tilde{S}_2^+$ and $\tilde{S}_2^-$ (cf. Fig. 7.1).

It follows from [Hi], Chapter 8, that there exists an extension $\tilde{\Phi}_2$ of $\Phi_2$ from $S_2 \subset D_1^{(2)}$ to $D_1^{(2)}$ such that $\tilde{\Phi}_2|_{\Gamma_0 \times [t_1 + 1, t_2 - 1]} = I$.

Define $\overline{D}_1^{(3)} = \Phi_2(\overline{D}_1^{(2)})$. There exists also an extension $\tilde{c}_3$ of the gauge $c_3$ from $S_2$ to $D_1^{(2)}$ such that $\tilde{c}_3 = 1$ on $\Gamma_0 \times [t_1 + 1, t_2 - 1]$. Let $L^{(3)} = \tilde{c}_3 \circ \tilde{\Phi}_2 \circ L^{(2)}$, $L^{(3)}$ is defined on $D_1^{(3)}$. Thus $L^{(3)} = L^{(1)}$ on $B \times [t_1 + 1, t_2 - 1]$.

Note that $D_1^{(3)} \cap (D_0^{(1)} \times [t_1 + 1, t_2 - 1]) \supset B \times [t_1 + 1, t_2 - 1]$, $\partial D_1^{(3)} \cap (\partial D_0^{(1)} \times [t_1 + 1, t_2 - 1]) \supset \Gamma_0 \times [t_1 + 1, t_2 - 1]$. We denote by $\partial' D_1^{(3)}$ the lateral (time-like) part of $\partial D_1^{(3)}$ and by $\partial \pm D_1^{(3)}$ the top and the bottom space-like parts of $\partial D_1^{(3)}$, i.e. $\partial D_1^{(3)} = \partial' D_1^{(3)} \cup \partial_+ D_1^{(3)} \cup \partial_- D_1^{(3)}$.

Fig. 7.1. The almost cylindrical domain $D_1^{(2)}$ is the part of $D_0^{(2)} \times \mathbb{R}$ bounded from above and from below by space-like surfaces $\tilde{S}_2^+$ and $\tilde{S}_2^-$. 

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The following lemma is the key lemma of this section. It allows to reduce the solution of the inverse problem to an inverse problem in a smaller domain.

**Lemma 7.2.** Consider two initial-boundary value problem $L^{(1)} u_1 = 0$ in $D_0^{(1)} \times [t_1, t_2]$ and $L^{(3)} u_3 = 0$ in $D_1^{(3)}$,

$$u_1|_{x_0=t_1} = \frac{\partial u_1}{\partial x_0}|_{x_0=t_1} = 0, \quad x \in D_0^{(1)},$$

$$u_3|_{\partial D_1^{(3)}} = \frac{\partial u_3}{\partial x_0}|_{\partial D_1^{(3)}} = 0, \quad u_1|_{\partial D_{0}^{(1)} \times [t_1, t_2]} = f_1, \quad u_3|_{\partial D_1^{(3)}} = f_3.$$

We assume that $(\partial D_0^{(1)} \times [t_1, t_2]) \cap \partial D_1^{(3)} \supset \Gamma_0 \times [t_1, t_2]$. Assume that $L^{(1)} = L^{(3)}$ in a smooth domain $B \times [t_1, t_2]$ where $B \times [t_1, t_2] \subset (D_0^{(1)} \times [t_1, t_2]) \cap D_1^{(3)}$, $\gamma_1 = \partial D_0^{(1)} \setminus \Gamma_0$, $\Gamma_3 = \partial D_1^{(3)} \setminus (\Gamma_0 \times (t_1, t_2))$, $\partial B = \gamma_0 \cup \gamma'_0$, where $\gamma_0, \gamma'_0$ are smooth, $\gamma_0 \subset \Gamma_0$, (cf. Fig. 7.2).

Suppose $\Lambda_1 = \Lambda_3$ on $\Gamma_0 \times [t_1, t_2]$.

Consider $L^{(1)} u_1 = 0$ and $L^{(3)} u_3 = 0$ in smaller domains $(D_0^{(1)} \setminus B) \times (t_1 + \delta, t_2 - \delta)$ and $(D_1^{(3)} \cap (t_1 + \delta, t_2 - \delta)) \setminus (B \times (t_1 + \delta, t_2 - \delta))$. Note that $\partial (D_0^{(1)} \setminus B) \supset (\Gamma_0 \setminus \gamma_0) \cup \gamma'_0$. Then $\Lambda_1', \Lambda_3'$ are equal on $((\Gamma_0 \setminus \gamma_0) \cup \gamma'_0) \times (t_1 + \delta, t_2 - \delta)$ for some $\delta > 0$. Here $\Lambda_1', \Lambda_3'$ are DN operators for the initial-boundary value problem

$$L^{(1)} u_1' = 0 \text{ in } (D_0^{(1)} \setminus B) \times (t_1 + \delta, t_2 - \delta),$$

$$L^{(3)} u_3' = 0 \text{ in } (D_1^{(3)} \cap (t_1 + \delta, t_2 - \delta)) \setminus (B \times (t_1 + \delta, t_2 - \delta)),$$

$$u_1'|_{x_0=t_1+\delta} = \frac{\partial u_1'}{\partial x_0}|_{x_0=t_2+\delta} = 0,$$

$$u_3'|_{\partial (D_1^{(3)} \cap (t_1 + \delta, t_2 - \delta))} = \frac{\partial u_3'}{\partial x_0}|_{\partial (D_1^{(3)} \cap (t_1 + \delta, t_2 - \delta))} = 0,$$

$$u_1'|_{((\Gamma_0 \setminus \gamma_0) \cup \gamma'_0) \times (t_1 + \delta, t_2 - \delta)} = f, \quad u_1'|_{(\partial D_0^{(1)} \setminus \Gamma_0) \times (t_1 + \delta, t_2 - \delta)} = 0,$$

$$u_3'|_{((\Gamma_0 \setminus \gamma_0) \cup \gamma'_0) \times (t_1 + \delta, t_2 - \delta)} = f, \quad u_3'|_{(\partial D_1^{(3)} \cap (t_1 + \delta, t_2 - \delta)) \setminus (\Gamma_0 \times (t_1 + \delta, t_2 - \delta))} = 0,$$

$\text{supp } f \subset (((\Gamma_0 \setminus \gamma_0) \cup \gamma'_0) \times (t_1 + \delta, t_2 - \delta)).$
Fig 7.2. The boundary of $B$ is $\gamma_0 \cup \gamma'_0$.

The boundary of $D_\varepsilon$ is $\gamma_\varepsilon \cup \Gamma_0$, $\partial D_0^{(1)} = \Gamma_0 \cup \gamma_1$.

To prove Lemma 7.2 we will need the following version of the Runge theorem about the approximation of solutions of the equation in a smaller domain by solutions of the same equation in a larger domain.

**Lemma 7.3.** Denote by $D_\varepsilon$ the domain bounded by $\Gamma_0$ and $\gamma_\varepsilon$ such that $\gamma_\varepsilon \cup \gamma_1$ is smooth. Let $W_0$ be the space of $v \in H_s((D_0^{(1)} \setminus B) \times (t_1, t_2))$, $s \geq 1$, such that

\begin{equation}
|v|_{\gamma_1} = 0, \quad v|_{x_0 = t_1} = \frac{\partial v}{\partial x_0}|_{x_0 = t_1} = 0, \quad x \in (D_0^{(1)} \setminus B),
\end{equation}

\begin{equation}
L^{(1)} v = 0 \text{ in } (D_0^{(1)} \setminus B) \times (t_1, t_2),
\end{equation}

where $\gamma_1 = \partial D_0^{(1)} \setminus \Gamma_0$.

Denote by $K$ the closure of $W_0$ in $L_2((D_0^{(1)} \setminus B) \times (t_1, t_2))$. Consider the space $W$ of $u(x) \in H_s((D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2))$, $s \geq 1$ such that

\begin{equation}
L^{(1)} u = 0 \text{ in } (D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2), \quad u|_{(\gamma_1 \cup \gamma_\varepsilon) \times (t_1, t_2)} = 0,
\end{equation}

\begin{equation}
u|_{x_0 = t_1} = \frac{\partial u}{\partial x_0}|_{x_0 = t_1} = 0, \quad x \in D_0^{(1)} \cup D_\varepsilon.
\end{equation}

Then the closure of the restrictions of $W$ to $L_2((D_0^{(1)} \setminus B) \times (t_1, t_2))$ is also equal to $K$. Thus any function $v \in W_0$ in $(D_0^{(1)} \setminus B) \times (t_1, t_2)$ can be approximated in $L_2((D_0^{(1)} \setminus B) \times (t_1, t_2))$ norm by the functions in $W$. 

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Proof: Let $K^\perp$ be the orthogonal complement of $K$ in $L_2((D_0^{(1)} \setminus B) \times (t_1, t_2))$. Take any $g \in K^\perp$ and denote by $g_0$ the extension of $g$ by zero outside $(D_0^{(1)} \setminus B) \times (t_1, t_2)$. Let $w$ be the solution of the initial-boundary value problem

$$
L^*_1 w = g_0, \ x \in (D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2),
$$

$$
\left. w \right|_{x_0 = t_2} = \frac{\partial w}{\partial x_0} \bigg|_{x_0 = t_2} = 0, \ x \in D_0^{(1)} \cup D_\varepsilon,
$$

$$
\left. w \right|_{\partial(D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2)} = 0,
$$

where $L^*_1$ is the formally adjoint to $L^{(1)}$. Note that $\partial(D_0^{(1)} \cup D_{\varepsilon_0}) = \gamma_1 \cup \gamma_\varepsilon$ (see Fig.7.2).

By [H] and [E6] (see also Lemma 3.3) such $w(x)$ exists and belongs to $H_1((D_0^{(1)} \cup D_\varepsilon)) \times (t_1, t_2))$. We shall show that $w = 0$ in $(B \cup D_\varepsilon) \times (t_1, t_2)$. Let $\varphi \in C_0^\infty((B \cup D_\varepsilon) \times (t_1, t_2))$ and let $u(x)$ be the solution of

$$
L^{(1)} u = \varphi, \ x \in (D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2)
$$

$$
\left. u \right|_{x_0 = t_1} = \frac{\partial u}{\partial x_0} \bigg|_{x_0 = t_1} = 0, \ u \bigg|_{\partial(D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2)} = 0,
$$

(cf. [H], [E6] and Lemma 3.3), i.e. $u \in W_0$ since $\varphi = 0$ in $(D_0^{(1)} \setminus B) \times (t_1, t_2)$.

Consider the $L_2$ inner product $(\varphi, w)$ in $(D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2)$. Since $\varphi = L^{(1)} u$ we get $(\varphi, w) = (L^{(1)} u, w)$. Integrating by parts we have $(L_1 u, w) = (u, L^*_1 w) = (u, g_0) = 0$ since $u \in W_0, g_0 \in K^\perp$. Therefore $(\varphi, w) = 0, \ \forall \varphi$. Thus $w = 0$ in $(B \cup D_\varepsilon) \times (t_1, t_2)$.

Let now $\bar{w}$ be any function in $W$. We have $(\bar{w}, g_0)_0 = (\bar{w}, L^*_1 w)_0$, where $(\ ,)_0$ means that we integrate over $(D_0^{(1)} \setminus B) \times (t_1, t_2)$. Since $w = 0$ in $(B \cup D_\varepsilon) \times (t_1, t_2)$, we have that

$$
\left. w \right|_{(\Gamma_0 \setminus \gamma_0) \cup \gamma'_0} \times (t_1, t_2) = \frac{\partial w}{\partial \nu} \bigg|_{(\Gamma_0 \setminus \gamma_0) \cup \gamma'_0} \times (t_1, t_2) = 0,
$$

where $\frac{\partial}{\partial \nu}$ is the normal derivative.

Note that $(\Gamma_0 \setminus \gamma_0) \cup \gamma'_0 = \partial(D_\varepsilon \cup B) \setminus \gamma_\varepsilon$. Since $w$ satisfies the homogenous equation $L^*_1 w = 0$ in $D_\varepsilon \cup B$ the restrictions of $w$ and all derivatives on $\partial(D_\varepsilon \cup B)$ exists by the partial hypoellipticity (see, for example, [E5]). Note that $\bar{w}$ and $w$ have zero values on $\gamma_1$. Therefore, integrating by parts, we have

$$(\bar{w}, L^*_1 w)_0 = (L^{(1)} \bar{w}, w)_0 = 0,$$
since \( L^{(1)} \tilde{w} = 0 \) in \((D^{(1)}_0 \setminus B) \times (t_1, t_2)\). Therefore \((\tilde{w}, g_0)_0 = 0, \ \forall g_0 \in K^\perp\), i.e. \( \tilde{w} \in \overline{K} \).

To make the integration by parts rigorous we approximate \( \gamma_0' \cup (\Gamma_0 \setminus \gamma_0) \) by \( \gamma_{\varepsilon'_1} \), similar to \( \gamma_{\varepsilon} \), \( \gamma_{\varepsilon'_1} \subset D_{\varepsilon} \cup B \). Note that \( w = 0 \) in \( D_{\varepsilon} \cup B \). Therefore integrating by parts over domain bounded by \( \gamma_1 \cup \gamma_{\varepsilon'_1} \), and taking the limit when \( \gamma_{\varepsilon'_1} \to \gamma_0' \cup (\Gamma_0 \setminus \gamma_0) \) we get \((\tilde{w}, g_0) = 0 \).

Now we shall proof Lemma \ref{1.2}. Let supp \( f \subset \Gamma'_0 \times (t_1, t_2) \), \( \Gamma'_0 = (\Gamma_0 \setminus \gamma_0) \cup \gamma'_0 \). Let \( v_1 \) be the solutions of

\[
L^{(1)} v_1 = 0, \ x \in (D^{(1)}_0 \setminus B) \times (t_1, t_2),
\]

\[
v_1|_{x_0=t_1} = \frac{\partial v_1}{\partial x_0}|_{x_0=t_0} = 0,
\]

\[
v_1|_{\partial (D^{(1)}_0 \setminus B) \times (t_1, t_2)} = f_1,
\]

where \( \partial (D^{(1)}_0 \setminus B) = \Gamma'_0 \cup \gamma_1 \), \( f_1 = 0 \) on \( \gamma_1 \times (t_1, t_2) \), \( f_1 = f \) on \( \Gamma'_0 \times (t_1, t_2) \).

Let \( v_3 \) be solution of \( L^{(3)} v_3 = 0 \) in \( D^{(3)}_1 \setminus (B \times (t_1, t_2)) \)

\[
v_3|_{\partial_D^{(3)}_{\delta}} = \frac{\partial v_3}{\partial x_0}|_{\partial_D^{(3)}_{\delta}} = 0, \quad v_3|_{(\partial D^{(3)}_1 \setminus (\Gamma_0 \times (t_1, t_2)))} = 0, \quad v_3|_{\Gamma'_0 \times (t_1, t_2)} = f.
\]

Let \( \Lambda'_1 \) be the DN operator for \((7.14)\) and \( \Lambda'_3 \) be the DN operator for \((7.15)\). Assuming that \( \Lambda_1 = \Lambda_2 \) on \( \Gamma_0 \times (t_1, t_2) \) we shall prove that

\[
\Lambda'_1 f|_{\Gamma'_0 \times (t_1+\delta, t_2-\delta)} = \Lambda'_2 f|_{\Gamma'_0 \times (t_1+\delta, t_2-\delta)}
\]

for all \( f \) with supports in \( \Gamma'_0 \times (t_1 + \delta, t_2 - \delta) \). By Lemma \ref{3.3} there exists a sequence of smooth solutions \( w_{n1} \in W_0 \) such that

\[
\|v_1 - w_{n1}\|_0 \to 0, \ n \to \infty,
\]

where \( \|v_1\|_0 \) is the norm in \( L_2((D^{(1)}_0 \setminus B) \times (t_1, t_2)) \). Note that \( L^{(1)} w_{n1} = 0 \) in \( D^{(1)}_0 \times (t_1, t_2) \), \( w_{n1}|_{\gamma_1 \times (t_1, t_2)} = 0, \ w_{n1}|_{x_0=t_1} = \frac{\partial w_{n1}}{\partial x_0}|_{x_0=t_1} = 0 \), where \( \gamma_1 = \partial D^{(1)}_0 \setminus \Gamma_0 \). Let \( f_n = w_{n1}|_{\Gamma_0 \times (t_1, t_2)} \). Denote by \( w_{n3} \) the solution of

\[
L^{(3)} w_{n3} = 0 \ in \ D^{(3)}_1, \quad w_{n3}|_{\partial D^{(3)}_1 \setminus (\Gamma_0 \times (t_1, t_2))} = 0, \quad w_{n3}|_{\Gamma_0 \times (t_1, t_2)} = f_n,
\]

\[
w_{n3}|_{\partial D^{(3)}_1} = \frac{\partial w_{n3}}{\partial x_0}|_{\partial D^{(3)}_1} = 0.
\]

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Since $\Lambda_1 = \Lambda_2$ on $\Gamma_0 \times (t_1, t_2)$, we have

\begin{equation}
\frac{\partial w_{n1}}{\partial \nu} \bigg|_{\Gamma_0 \times (t_1, t_2)} = \frac{\partial w_{n3}}{\partial \nu} \bigg|_{\Gamma_0 \times (t_1, t_2)}.
\end{equation}

(7.17)

Since $\gamma_0 \subset \Gamma_0$, the equality (7.17) implies

\[ w_{n1}\big|_{\gamma_0 \times (t_1, t_2)} = w_{n3}\big|_{\gamma_0 \times (t_1, t_2)}, \quad \frac{\partial w_{n1}}{\partial \nu} \bigg|_{\gamma_0 \times (t_1, t_2)} = \frac{\partial w_{n3}}{\partial \nu} \bigg|_{\gamma_0 \times (t_1, t_2)}. \]

We have $L^{(1)} = L^{(3)}$ on $B \times (t_1, t_2)$. Using the uniqueness theorem of [RZ] and [T], we get

\begin{equation}
\frac{\partial w_{n1}}{\partial \nu} \bigg|_{\Gamma_0 \times (t_1, t_2)} = \frac{\partial w_{n3}}{\partial \nu} \bigg|_{\Gamma_0 \times (t_1, t_2)}.
\end{equation}

(7.18)

where $\delta > 0$ is determined by the metric and by the domaine $B$ (cf. Fig.7.2). In particular,

\begin{equation}
\frac{\partial w_{n1}}{\partial \nu} \bigg|_{\Gamma_0 \times (t_1 + \delta, t_2 - \delta)} = \frac{\partial w_{n3}}{\partial \nu} \bigg|_{\Gamma_0 \times (t_1 + \delta, t_2 - \delta)}.
\end{equation}

(7.19)

Therefore

\begin{equation}
\frac{\partial w_{n1}}{\partial \nu} \bigg|_{\Gamma_0' \times (t_1 + \delta, t_2 - \delta)} = \frac{\partial w_{n3}}{\partial \nu} \bigg|_{\Gamma_0' \times (t_1 + \delta, t_2 - \delta)},
\end{equation}

(7.20)

where $\Gamma_0' = (\Gamma_0 \setminus \gamma_0) \cup \gamma_0'$, i.e. $\Lambda_1 f'_n = \Lambda_2 f'_n$ on $\Gamma_0' \times (t_1 + \delta, t_2 - \delta)$, where $f'_n = w_{n1}\big|_{\Gamma_0' \times (t_1 + \delta, t_2 - \delta)} = w_{n3}\big|_{\Gamma_0' \times (t_1 + \delta, t_2 - \delta)}$. We have

\begin{equation}
\|f - f'_n\|_{-\frac{1}{4}, \Gamma_0' \times (t_1 + \delta, t_2 - \delta)} = \|f - f'_n\|_{-\frac{1}{4}, \partial(D_0^{(1)} \setminus B) \times (t_1 + \delta, t_2 - \delta)} \leq \|v_1 - w_{n1}\|_{0, (D_0^{(1)} \setminus B) \times (t_1 + \delta, t_2 - \delta)},
\end{equation}

(7.21)

since $\partial(D_0^{(1)} \setminus B) = \Gamma_0' \cup \gamma_1$ and $f = f'_n = 0$ on $\gamma_1 \times (t_1 + \delta, t_2 - \delta)$.

In (7.21) we again use the partial hypoellipticity property that restrictions of solutions of $L^{(1)}u = 0$ to the noncharacteristic boundary exists for any Sobolev’s space $H_s$ (cf. [E5]). The same is true for all normal derivatives of
u_1, and the same estimates hold as in the case of positive s > 0 (cf [E5] and [E1]).

By Lemma 3.3 (see [H], [E6]) we have

\[(7.22) \quad \left\| \frac{\partial v_1}{\partial \nu} - \frac{\partial w_n}{\partial \nu} \right\|_{\frac{3}{2} \partial (D_0^{(1)} \setminus B) \times (t_1 + \delta, t_2 - \delta)} \leq \left\| f - f_n \right\|_{\frac{3}{2} \partial (D_0^{(1)} \setminus B) \times (t_1 + \delta, t_2 - \delta)} \]

Analogously we have

\[(7.23) \quad \left\| \frac{\partial v_3}{\partial \nu} - \frac{\partial w_n}{\partial \nu} \right\|_{\frac{3}{2} \partial (D_1^{(1)} \cap (t_1 + \delta, t_2 - \delta)) \setminus (B \times (t_1 + \delta, t_2 - \delta))} \leq \left\| f - f_n \right\|_{\frac{3}{2} \partial (D_1^{(1)} \cap (t_1 + \delta, t_2 - \delta)) \setminus (B \times (t_1 + \delta, t_2 - \delta))}

Note that

\[\left\| f - f_n \right\|_{\frac{3}{2} \partial (D_0^{(1)} \setminus B) \times (t_1 + \delta, t_2 - \delta)} = \left\| f - f_n \right\|_{\frac{3}{2} \partial (D_1^{(1)} \cap (t_1 + \delta, t_2 - \delta)) \setminus (B \times (t_1 + \delta, t_2 - \delta))} \]

Therefore, taking the limit as n → ∞, we get, using (7.20), that

\[(7.24) \quad \frac{\partial v_1}{\partial \nu} \big|_{\Gamma_0' \times (t_1 + \delta, t_2 - \delta)} = \frac{\partial v_3}{\partial \nu} \big|_{\Gamma_0' \times (t_1 + \delta, t_2 - \delta)}.\]

Thus we proved that

\[\Lambda_1' f = \Lambda_2' f \quad \text{on} \quad \Gamma_0' \times (t_1 + \delta, t_2 - \delta)\]

for any f with \(\text{supp } f \subseteq \Gamma_0' \times (t_1 + \delta, t_2 - \delta)\).

Using Lemma 7.2 we reduce the inverse problem in \(D_0^{(1)} \times (t_1, t_2)\) to the inverse problem in smaller domains \((D_0^{(1)} \setminus B) \times (t_1 + \delta, t_2 - \delta)\) and we can continue this process.

In all lemmas below we assume that DN operators for \(L^{(1)}\) and \(L^{(2)}\) are equal on \(\Gamma_0 \times [t_1, t_2]\) and that the time interval \([t_1, t_2]\) is large enough. We shall continue to call the coordinates \((y_0, \hat{y}', y_n)\), given by the map \(\beta^{(i)} \varphi^{(i)}\), the Goursat coordinates for \(\tilde{L}_1^{(i)}, i = 1, 2\).

**Lemma 7.4.** Let \(\Gamma_1 \subset \Gamma_0\) and let \(\hat{\Gamma}_1 \subset \Gamma_0\) be such that \(\hat{\Gamma}_1 \subset \Gamma_1\). We assume that \(\hat{\Gamma}_1 \subset \Gamma_2 \subset \Gamma_3 \subset \Gamma_0\) where \(\hat{\Gamma}_j, 1 \leq j \leq 3\), are the same as \(\Gamma_j, 1 \leq j \leq 3\), in §3. Suppose the Goursat coordinates for \(L^{(1)}\) exists in \(\Omega_1 = (t_1, t_2) \times \hat{\Gamma}_3 \times [0, \varepsilon_0]\), i.e. \(L^{(1)}\) has the form \(\tilde{L}_1^{(1)}\) in these coordinates (we include the gauge transformation \((2.27)\) in \(\tilde{L}_1^{(1)}\)). Suppose the Goursat coordinates for
Let \( L^{(1)} \) exist in \((t_1, t_2) \times (\overline{\Gamma}_1 \setminus \Gamma_1) \times [0, \varepsilon_0] \). Let \( \tilde{\Omega}_2 = (t_1, t_2) \times (\overline{\Gamma}_1 \setminus \Gamma_1) \times [0, \varepsilon_0] \) and suppose \( \tilde{L}_1^{(2)} = \tilde{L}_1^{(1)} \) in \( \tilde{\Omega}_2 \). Then \( L^{(2)} \) has also Goursat coordinates in \( \Omega_3 = (t_1, t_2) \times \Gamma_1 \times [0, \varepsilon_0] \), and \( \tilde{L}_1^{(1)} = \tilde{L}_1^{(2)} \) in \( \Omega_3 \).

**Proof:** Let \( \gamma_i^{(i)}(x) \) be the transformation to the Goursat coordinates, and let \( \frac{D\psi^{(i)}(x)}{Dx} \) be the Jacobi matrix of this transformation. We have

\[
[g_i^{j,k}(y)] = \frac{D\psi^{(i)}(x)}{Dx} [g_i^{j,k}(x)] \left( \frac{D\psi^{(i)}}{Dx} \right)^T, \quad i = 1, 2,
\]

where \([g_i^{j,k}]^{-1}\) is the metric tensor in the Goursat coordinates. The Goursat coordinates degenerate at point \( y_i^{(0)} = \psi^{(i)}(x^{(0)}) \), when \( \det \frac{D\psi^{(i)}(x)}{Dx} \to \infty \) for \( y \to y_i^{(0)} \) (or \( x \to x^{(0)} \)) (cf. [E2]). We call such point a focal point. We shall prove that there is no focal points for \( L^{(2)} \) in \( \Omega_3 \).

We have

\[
\det[g_i^{j,k}(y)] = \det[g_i^{j,k}(x)] \left( \det \frac{D\psi^{(i)}}{Dx} \right)^2.
\]

Suppose there exists the focal point \( y_i^{(0)} = (y_0^{(0)}, y_0^{(0)}, y_n^{(0)}) \), \( y_0^{(0)} < \varepsilon_0, y_0' \in \Gamma_1 \) such that there is no focal points for \( L^{(2)} \) when \( y_n < y_i^{(0)} \) for all \( y_0 \in [t_1, t_2], y' \in \Gamma_1 \).

Since \( L^{(1)} \) and \( L^{(2)} \) have Goursat coordinates for \( y_n < y_i^{(0)} \) we get, by Lemma 7.1 that \( \tilde{L}_1^{(1)} = \tilde{L}_1^{(2)} \) in \((t_1, t_2) \times \overline{\Gamma}_1 \times [0, y_n^{(0)} - \varepsilon] \), \( \forall \varepsilon > 0 \), and hence \([g_1^{j,k}] = [g_2^{j,k}]\) for \( \varepsilon > 0 \). Since \( \det[g_2^{j,k}] = \det[g_1^{j,k}] \) for \( y_n < y_i^{(0)} - \varepsilon \), we have that \( \left( \frac{\det[g_2^{j,k}]}{\det[g_1^{j,k}]} \right)^2 = \frac{\det[g_1^{j,k}]}{\det[g_2^{j,k}]} \) is bounded when \( \varepsilon \to 0 \). Therefore \( y_i^{(0)} = (y_0^{(0)}, y_0', y_n^{(0)}) \) is not a focal point for \( L^{(2)} \). Thus \( L^{(2)} \) has no focal points in \( \Omega_3 \) and then, by Lemma 7.1 we have \( \tilde{L}_1^{(1)} = \tilde{L}_1^{(2)} \) in \( \Omega_3 \) (cf. [E2]).

**Lemma 7.5.** Assume that DN operators for \( L^{(1)} \) and \( L^{(2)} \) are equal on \( \Gamma_0 \times [t_1, t_2] \). Let \( \Gamma_1 \subset \Gamma_0 \). Assume that the Goursat coordinates for \( L^{(1)} \) exists in \((t_1, t_2) \times \overline{\Gamma}_1 \times [0, \frac{y_n}{2}] \). Then the Goursat coordinates for \( L^{(2)} \) also exists in \( \Omega_1 = (t_1 + \delta, t_2 - \delta) \times \overline{\Gamma}_1 \times [0, \frac{y_n}{2}] \) for some \( \delta > 0 \) and \( \tilde{L}_1^{(1)} = \tilde{L}_1^{(2)} \) in \( \Omega_1 \), where \( \tilde{L}_1^{(i)} \) are the operators \( L^{(i)} \) in the Goursat coordinates.

**Proof:** Let \( \overline{\Gamma}_1 \subset \tilde{\Gamma}_1, \tilde{\Gamma}_1 \subset \Gamma_0 \). If \( 0 \leq y_n \leq \varepsilon \), where \( \varepsilon > 0 \) is small enough, then \( \tilde{\Gamma}_1 \subset \tilde{\Gamma}_2 \subset \tilde{\Gamma}_3 \subset \Gamma_0 \), where \( \tilde{\Gamma}_j, j = 1, 2, 3 \), are the same as in Lemma 7.1. Applying Lemma 7.1 we get that the Goursat coordinates for \( \tilde{L}_1^{(1)} \) and \( \tilde{L}_1^{(2)} \) exist in \( \Omega_{1\varepsilon} = [t_1, t_2] \times \tilde{\Gamma}_1 \times [0, \varepsilon] \) and \( \tilde{L}_1^{(1)} = \tilde{L}_1^{(2)} \) in \( \Omega_{1\varepsilon} \).
Let $\Sigma_1$ be the surface in $(y', y_n)$ space such that $y_n = 0$ on $\Gamma_1 \setminus \Gamma_1, 0 \leq y_n \leq \varepsilon$ on $\partial \Gamma_1, y_n = \varepsilon$ on $\Gamma_1$. Note that $\Sigma_1$ is not smooth since it has edges when $y_n = 0, y' \in \partial \Gamma_1$ and when $y_n = \varepsilon, y' \in \partial \Gamma_1$. We shall smooth $\Sigma_1$ by replacing it by smooth surface $\tilde{\Sigma}_1$, where $\tilde{\Sigma}_1$ differs from $\Sigma_1$ in a neighborhood of edges having the size $O(\varepsilon)$. Let $\Sigma_2$ be the surface, where $y_n = \varepsilon$ when $y' \in \Gamma_1 \setminus \Gamma_1(\varepsilon)$, $\Gamma_1(\varepsilon)$ is the $\varepsilon$-neighborhood of $\Gamma_1, \varepsilon \leq y_n \leq 2\varepsilon$, when $y' \in \partial \Gamma_1(\varepsilon)$, $y_n = 2\varepsilon$ when $y' \in \Gamma_1(\varepsilon)$ (cf. Fig. 7.3).

**Fig. 7.3.**  
$\Sigma'_2$ is the surface $\{y_n = 0$ when $y' \in \tilde{\Gamma}' \setminus \Gamma_1, 0 \leq y_n \leq 2\varepsilon$ when $y' \in \partial \Gamma_1, y_n = 2\varepsilon$ when $y' \in \Gamma_1\}$; 
$\Sigma_3$ is the surface $\{y_n = \varepsilon$ when $y' \in \tilde{\Gamma} \setminus \Gamma_1(\varepsilon), \varepsilon \leq y_n \leq 3\varepsilon$ when $y' \in \partial \Gamma_1(\varepsilon), y_n = 3\varepsilon$ when $y' \in \Gamma_1(\varepsilon)\}$; 
$S_2$ is the region between $\Sigma_3$ and $\Sigma'_2$.

Let $\tilde{\Sigma}_2$ be the smoothing of $\Sigma_2$. Denote by $\tilde{\Sigma}_1$ the domain between $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ when $y' \in \tilde{\Gamma}_1$. Since $\tilde{L}^{(1)}_1 = \tilde{L}^{(2)}_1$ for $0 \leq y_n \leq \varepsilon$, we have, by Lemma 7.2, that DN operators for $L^{(1)}$ and $L^{(2)}$ are equal on $\tilde{\Sigma}_1 \times (t_1 + \delta_1, t_2 - \delta_1)$ for some $\delta_1 > 0$.

Suppose $\varepsilon > 0$ is small and such that we can introduce the Goursat coordinates for $L^{(1)}$ in $\tilde{S}_1 \times [t_1 + \delta_1, t_2 - \delta_1]$.

Note that $\varepsilon$ and $\delta_1$ are determined by $L^{(1)}$ only and are independent of $L^{(2)}$. It follows from Lemma 7.4 that Goursat coordinates for $L^{(2)}$ also hold on $\tilde{S}_1 \times (t_1 + \delta_1, t_2 - \delta_1)$ and $\tilde{L}^{(1)}_1 = \tilde{L}^{(2)}_1$ in $\tilde{S}_1 \times (t_1 + \delta_1, t_2 - \delta_1)$.  

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By Lemma [7.2] DN operators for $L^{(1)}$ and $L^{(2)}$ are equal on $\tilde{\Sigma}_2 \times (t_2 + \delta_1, t_2 - \delta_1)$.

Let $\Sigma'_2$ be the surface in $(y', y_n)$ space such that $y_n = 0$ on $\Gamma_1 \setminus [1, 0 \leq y_n \leq 2\varepsilon$ on $\partial \Gamma_1$, $y_n = 2\varepsilon$ on $\Gamma_1$ and let $\Sigma_3$ be the surface where $y_n = \varepsilon$ on $\Gamma_1 \setminus [1, \varepsilon \leq y_n \leq 3\varepsilon$ on $\partial \Gamma_1(\varepsilon)$, $y_n = 3\varepsilon$ on $\Gamma_1(\varepsilon)$. Let $\Sigma'_2$ and $\Sigma_3$ be the smoothing of $\Sigma'_2$, $\Sigma_3$ and let $\tilde{S}_2$ be the domain between $\Sigma'_2$ and $\Sigma_3$ when $y' \in \Gamma_1$. Since DN operators for $L^{(1)}$ and $L^{(2)}$ are equal on $\Sigma'_2 \times (t_1 + \delta_1, t_2 - \delta_1)$ and since the Goursat coordinates hold for $L^{(1)}$ hold on $\tilde{S}_2 \times (t_1 + \delta_1, t_2 - \delta_1)$, Lemma [7.3] implies that the Goursat coordinates hold for $L^{(2)}$ in $S_2 \times (t_1 + \delta_1, t_2 - \delta_1)$, $L_1^{(1)} = L_1^{(2)}$ in $S_2 \times (t_1 + \delta_1, t_2 - \delta_1)$ and since $\Sigma_3$ are equal on $\tilde{\Sigma}$, $\tilde{\Sigma}_2 \times (t_1 + \delta_1, t_2 - \delta_1)$. Analogously, for $k > 2$ denote by $\Sigma'_k$ the surface such that $y_n = 0$ on $\Gamma_1 \setminus [1, 0 \leq y_n \leq k\varepsilon$ on $\partial \Gamma_1$ and $y_n = k\varepsilon$ on $\Gamma_1$. Let $\Sigma_{k+1}$ be the surface, where $y_n = \varepsilon$ for $\Gamma_1 \setminus [1, \varepsilon \leq y_n \leq (k + 1)\varepsilon$ on $\partial \Gamma_1(\varepsilon)$ and $y_n = (k + 1)\varepsilon$ on $\Gamma_1(\varepsilon)$.

Denote by $\Sigma'_k$ and $\Sigma_{k+1}$ the smoothing of $\Sigma'_k$, $\Sigma_{k+1}$. Let $\tilde{S}_k$ be the domain between $\Sigma'_k$ and $\Sigma_{k+1}$. Applying successively the same arguments for $k = 3, \ldots, m$, we prove as above that $\tilde{L}_1^{(2)} = \tilde{L}_1^{(1)}$ in $\tilde{S}_k \times (t_1 + \sum_{j=1}^{k} \delta_j, t_2 - \sum_{j=1}^{k} \delta_j), k = 3, \ldots, m$.

Let $m$ be such that $(m + 1)\varepsilon \geq \frac{T_0}{2}$. Then we get that $\tilde{L}_1^{(2)} = \tilde{L}_1^{(1)}$ in $(t_1 - \delta, t_2 + \delta) \times \Gamma_1 \times [0, \frac{T_0}{2}]$, where $\delta = \sum_{j=1}^{m} \delta_j$. Note that we assume that $[t_1, t_2]$ is large. Thus $t_2 - t_1 \gg \delta$.

Suppose that after several applications of Lemma [7.2] we have
\[
L^{(1)}u_1 = 0 \quad \text{in} \quad D_0^{(1)} \times [t_1, t_2],
\]
\[
L^{(m)}u_2 = 0 \quad \text{in} \quad D_1^{(m)},
\]
where we are considering the interval $[t_1, t_2]$ instead of $[t_1 + \delta, t_2 - \delta]$ for the simplicity of notations. We assume that
\[
(\partial _{D_1^{(m)}} u_2) |_{x_0 = t_1} = 0, \quad L^{(m)}u_2 = 0 \quad \text{in} \quad D_1^{(m)},
\]
We assume that $\Gamma_0 \times (t_1, t_2) \subset \partial D_1^{(m)} \cap (\partial D_0^{(1)} \times (t_1, t_2))$ and $\Omega_1 \times (t_1, t_2) \subset (D_0^{(1)} \times (t_1, t_2)) \cap D_1^{(m)}$ (cf. Fig. 7.4).

We also assume that $L^{(1)} = L^{(m)}$ in $\Omega_1 \times (t_1, t_2)$ and that DN operators $\Lambda_1$ and $\Lambda_2$ are equal on $\partial \Omega_1 \times (t_1, t_2)$ and $\Gamma_0 \times (t_1, t_2)$. Here, as above, $\partial D_1^{(m)}$ means the time-like part of the boundary.
Consider the case when $\partial \Omega_1$ and $\partial D_0^{(1)}$ are close, i.e. there is a narrow domain $\sigma_1 \subset D_0^{(1)} \setminus \Omega_1$ such that $\gamma_1 \subset \partial D_0^{(1)}$, $\gamma_0 \subset \partial \Omega_1$ and the distance between $\gamma_0$ and $\gamma_1$ is small (cf. Fig. 7.4).

![Fig. 7.4. $\gamma_1 \subset \partial D_0^{(1)}$, $\gamma_0 \subset \Omega_1$, $\gamma_1$ and $\gamma_0$ are close.](image)

Introduce Goursat coordinates for $L^{(1)}$ and $L^{(m)}$ near $\gamma_0 \times (t_1, t_2)$. We assume that operators $L^{(1)}$ and $L^{(m)}$ are defined in domains slightly larger than $D_0^{(1)} \times (t_1, t_2)$ and $D_1^{(m)}$. Let $L_1^{(1)}$ and $L_1^{(m)}$ be the operators $L^{(1)}$ and $L^{(m)}$ in corresponding Goursat coordinates. Let $y = \varphi_1(x)$ be the transformation to the Goursat coordinates for $L^{(1)}$. Let $\sigma_0 = (t_1, t_2) \times \gamma_0 \times [0, \varepsilon_0]$ be the domain where the Goursat coordinates for $L^{(1)}$ hold. We assume that $\sigma_1$ is so small that $\varphi_1(\sigma_1 \times (t_1, t_2)) \subset \sigma_0$. Let $\tau_0 = \varphi_1(\gamma_1 \times (t_1, t_2))$, i.e. $\tau_0$ is the image of part of the boundary $\partial D_0^{(1)} \times (t_1, t_2)$ in Goursat coordinates. Denote by $\sigma_0^+$ the part of $\sigma_0$ between $\gamma_0 \times (t_1, t_2)$ and $\tau_0$, i.e. $\sigma_0^+$ is the image of $\sigma_1 \times (t_1, t_2)$ in Goursat coordinates.

We assume that the Goursat coordinates for $L^{(m)}$ also hold in $(t_1, t_2) \times \gamma_0 \times [0, \varepsilon_0]$. Moreover, applying Lemmas 7.4, 7.5 repeatedly we get that $\tilde{L}^{(m)}_1 = \tilde{L}_1^{(1)}$ in $\sigma_0^+$, where $\tilde{\sigma}_0^+ = \sigma_0^+ \cap (t_1 + \delta, t_2 - \delta)$. Here $\tilde{L}_1^{(1)}$, $\tilde{L}_1^{(m)}$ are operators $L^{(1)}$, $L^{(m)}$ in Goursat coordinates.

Consider the initial-boundary value problem for $\tilde{L}_1^{(1)}$ in Goursat coordi-
nates

\[(7.25)\]
\[
\tilde{L}^{(1)} \tilde{u}_1 = 0 \quad \text{in} \quad \tilde{\sigma}_0^+,
\]
\[
\tilde{u}_1 \big|_{x_0 = t_1 + \delta} = \frac{\partial \tilde{u}_1}{\partial x_0} \big|_{x_0 = t_1 + \delta} = 0,
\]
\[
\tilde{u}_1 \big|_{(t_1 + \delta, t_2 - \delta) \times \gamma_0} = f, \quad \tilde{u}_1 \big|_{\partial \tilde{\sigma}_0^+} = 0,
\]
where \( \text{supp} \ f \subset (t_1 + \delta, t_2 - \delta) \times \gamma_0 \).

Consider now the equation \( \tilde{L}^{(m)} \tilde{u}_2 = 0 \) in \( \tilde{\sigma}_0^+ \).

\[
\tilde{L}^{(m)} \tilde{u}_2 = 0 \quad \text{in} \quad \tilde{\sigma}_0^+,
\]
\[
\tilde{u}_2 \big|_{x_0 = t_1 + \delta} = \frac{\partial \tilde{u}_2}{\partial x_0} \big|_{x_0 = t_1 + \delta} = 0,
\]
\[
\tilde{u}_2 \big|_{(t_1 + \delta, t_2 - \delta) \times \gamma_0} = f,
\]
where \( f \) is the same as in (7.25).

Since \( \tilde{L}^{(1)} = \tilde{L}^{(m)} \) in \( \tilde{\sigma}_0^+ \) we have that \( \tilde{u}_1 = \tilde{u}_2 \) in \( \tilde{\sigma}_0^+ \). Therefore by the continuity \( \tilde{u}_2 \big|_{\partial \tilde{\sigma}_0^+} = 0 \).

Let \( \sigma_2^+ = \varphi_2^{-1}(\tilde{\sigma}_0^+) \), \( \tau_2 = \varphi_2^{-1}(\partial \tilde{\sigma}_0^+) \), where \( y = \varphi_2(x) \) is the transformation to the Goursat coordinates for \( \tilde{L}^{(m)} \).

We shall show that \( \tau_2 \) is a part of the boundary of \( D^{(m)} \).

Construct the geometric optic solution \( v_1(y) \) for \( \tilde{L}^{(1)} u_1 = 0 \) in Goursat coordinates as in (5.1). Since \( \tau_0 \) is the boundary of the domain \( \sigma_0^+ \) and since the zero Dirichlet boundary condition holds on \( \tau_0 = \tau_0 \cap (t_1 + \delta, t_2 - \delta) \) this solution must reflect at \( \tilde{\tau}_0 \) (cf. [E7]).

Consider now the geometric optics solution \( v_2(y) \) for \( \tilde{L}^{(m)} u_2 = 0 \) with the same initial condition. Since \( \tilde{L}^{(1)} v_1 = \tilde{L}^{(m)} v_2 \) in \( \tilde{\sigma}_0^+ \) we have that \( v_1(y) = v_2(y) \) before the reflection at \( \tilde{\tau}_0 \). If \( \tau_2 = \varphi_2^{-1}(\tilde{\tau}_0) \) is not a part of the boundary of \( \partial D^{(m)} \) there will be no reflection for \( v_2(y) \) at \( \tilde{\tau}_0 \). Thus, the solutions \( v_1(y) \) and \( v_2(y) \) will be different in \( \sigma_0^+ \) near \( \tilde{\tau}_0 \). This contradicts the fact that \( v_1(y) = v_2(y) \) in \( \tilde{\sigma}_0^+ \).

Therefore \( \varphi_m = \varphi_2^{-1} \varphi_1 \) maps the boundary \( \gamma_1 \times (t_1, t_2) \) of \( \partial D^{(1)} \times (t_1, t_2) \) on the part of boundary of \( \partial D^{(m)} \). Let \( \sigma_2 \subset D^{(m)} \) be the image of \( \varphi_m = \varphi_2^{-1} \varphi_1(\sigma_1 \times (t_1 + \delta, t_2 - \delta)) \). Let \( \partial_+(\Omega_1 \cup \Omega_2) \) be the space-like parts of the boundary of \( \Omega_1 \cup \Omega_2 \). Extend \( \partial_+(\Omega_1 \cup \Omega_2) \) and \( \partial_-,(\Omega_1 \cup \Omega_2) \) as space-like surfaces \( S_m^+ \) and \( S_m^- \) to the whole domain \( D^{(m)} \). Let \( D^{(m)}_1 \) be part of \( D^{(m)} \) bounded
from below and above by \( S_m^- \) and \( S_m^+ \), respectively. Note that \( \partial^j D^{(m)} \supset \Gamma_0 \times (t_1 + \delta, t_2 - \delta) \). Define \( \tilde{\varphi}_m = I \) on \( \Omega_1 \times (t_1 + \delta, t_2 - \delta) \), \( \tilde{\varphi}_m = I \) on \( \Gamma_0 \times (t_1 + \delta, t_2 - \delta) \), \( \varphi_m = \varphi_m \) on \( \sigma_1 \times (t_1 + \delta, t_2 - \delta) \). Let \( \Phi_{m+1} \) be the extension of \( \tilde{\varphi}_m \) (cf. [Hi]) to the whole domain \( \tilde{D}_1^{(m)} \). Denote by \( \tilde{D}_1^{(m+1)} \) the image of \( \tilde{D}_1^{(m+1)} \) under the map \( \Phi_{m+1} \). If \( c_m \) is a gauge transformation on \( \sigma_1 \times [t_1 + \delta, t_2 - \delta] \) we denote by \( \tilde{c}_m+1 \) the extension of \( c_m \) to \( \tilde{D}_1^{(m+1)} \) such that \( \tilde{c}_m+1 = 1 \) on \( \Omega_1 \times [t_1 + \delta, t_2 - \delta] \), \( \tilde{c}_m+1 = 1 \) on \( \Gamma_0 \times (t_1 + \delta, t_2 - \delta) \).

We just proved the following lemma:

**Lemma 7.6.** Let \( L^{(1)} \) and \( L^{(m+1)} = \tilde{c}_{m+1} \circ \Phi_{m+1} \circ L^{(m)} \) be operators in \( D_0^{(1)} \times [t_1 + \delta, t_2 - \delta] \) and \( D_1^{(m+1)} \), respectively. Then \( \Omega_2 \times (t_1 + \delta, t_2 - \delta) \subset D_1^{(m+1)} \cap (D_0^{(1)} \times (t_1 + \delta, t_2 - \delta)) \), where \( \Omega_2 = \Omega_1 \cup \sigma_1 \) and \( L^{(1)} = L^{(m+1)} \) in \( \Omega_2 \times (t_1 + \delta, t_2 - \delta) \).

We shall proceed with the enlargement of the domain \( \Omega_2 \) using Lemmas 7.2 and 7.6. It follows from Lemmas 7.4 and 7.5 that this enlargement depends only on \( L^{(1)} \) and does not depend on \( L^{(2)} \). Therefore after finite number of steps (cf. [E2]) we get a domain \( D_1^{(N)} \), operator \( L^{(N)} \) on \( D_1^{(N)} \) and the map \( \Phi_N \) of \( D_1^{(N)} \) onto \( D_0^{(1)} \times (t_1 + \delta_N, t_2 - \delta_N) \) such that \( c_N \circ \Phi_N \circ L^{(N)} = L^{(1)} \) in \( D_0^{(1)} \times (t_1 + \delta_N, t_2 - \delta_N) \) for some \( \delta_N > 0 \). Here \( c_N \) is the gauge transformation. Remind that \( \tilde{\Phi}_2 \) is the diffeomorphism of \( D_1^{(2)} \) onto \( D_1^{(3)} \), \( \Phi_3 \) is the diffeomorphism of \( D_1^{(3)} \) onto \( D_1^{(4)} \), etc. ... \( \tilde{\Phi}_{N-1} \) is the map of \( D_1^{(N-1)} \) onto \( D_1^{(N)} \) onto \( D_0^{(1)} \times (t_1 + \delta_N, t_2 - \delta_N) \).

Therefore, the diffeomorphism \( \Phi^{-1} = \Phi_1^{-1} \Phi_3^{-1} \ldots \Phi_N^{-1} \) maps \( D_0^{(1)} \times [t_0 + \delta_N, t_2 - \delta_N] \) onto \( D_1^{(2)} \). Thus \( \Phi \) maps \( D_1^{(2)} \) onto \( D_0^{(1)} \times (t_0 + \delta_N, t_2 - \delta_N) \).

Note that \( D_1^{(2)} \) is an almost cylindrical domain in \( D_0^{(2)} \times \mathbb{R} \), i.e. \( D_1^{(2)} = D_0^{(2)} \times \{ S^- (x_1, ..., x_n) \leq x_0 \leq S^+ (x_1, ..., x_n) \} \), where \( x_0 = S^+ (x_1, ..., x_n) \) are space-like surfaces, \( (x_1, ..., x_n) \in D_0^{(2)} \).

Note that \( [t_1, t_2] \) is arbitrary large and therefore \( [t_1', t_2'] = [t_1 + \delta, t_2 - \delta] \) is also arbitrary large. Therefore we obtained the following theorem:

**Theorem 7.7.** Let \( L^{(1)} \) and \( L^{(2)} \) be two operators in \( D_0^{(1)} \times \mathbb{R} \) and \( D_0^{(2)} \times \mathbb{R} \), respectively. Suppose \( \Gamma_0 \subset \partial D_0^{(1)} \cap \partial D_0^{(2)} \) and the DN operators, corresponding to \( L^{(i)} \), are equal on \( \Gamma_0 \times \mathbb{R} \) for all \( f \) that have a compact support in \( \Gamma_0 \times \mathbb{R} \). Suppose that the conditions \( L.2 \), \( L.3 \) hold for \( L^{(i)} \), \( i = 1, 2 \), and the coefficients of \( L^{(1)} \) and \( L^{(2)} \) are analytic in \( x_0 \) in \( D_0^{(i)} \times \mathbb{R} \), \( i = 1, 2 \). Suppose
for each \( t_0 \in R \) there exists \( T_0 \) such that the BLR condition is satisfied for \( L^{(1)} \) on \([t_0, T_0]\). Let \([t_1', t_2']\) be an arbitrary sufficiently large time interval. Then there exists a diffeomorphism \( \Phi^{-1} \) of \( \overline{D}_0^{(1)} \times [t_1, t_2] \) on an almost cylindrical domain \( \overline{D}_1^{(2)} \subset \overline{D}_0^{(2)} \times R \), \( \Phi = I \) on \( \Gamma_0 \times [t_1', t_2'] \) and there exists a gauge transformation \( c(y) \) on \( D_1^{(2)} \), \( |c(y)| = 1 \) on \( D_1^{(2)} \), \( c(y) = 1 \) on \( \Gamma_0 \times [t_1', t_2'] \) such that

\[ c \circ \Phi^{-1} \circ L^{(2)} = L^{(1)} \text{ on } D_0^{(1)} \times [t_1', t_2']. \]

Now we shall use Theorem 7.7 to prove Theorem 1.2.

**Proof of Theorem 1.2** Let \( L^{(i)} \) be two operators in \( D_0^{(i)} \times R, i = 1, 2 \), \( \Gamma_0 \subset \partial D_0^{(1)} \cap \partial D_0^{(2)} \) and all conditions of Theorem 7.7 are satisfied.

Let \( (t_{j1}, t_{j2}) \) be an interval as in Theorem 7.7 and \( \bigcup_{j = -\infty}^{\infty} \overline{D}_0^{(1)} \times [t_{j1}, t_{j2}] = \overline{R} \). We have \( \overline{D}_0^{(1)} \times R \subset \bigcup_{j = -\infty}^{\infty} \overline{D}_0^{(1)} \times [t_{j1}, t_{j2}] \). It follows from Theorem 7.7 that for each \( j \in Z \) there exists a diffeomorphism \( \Phi_j \) on \( D_0^{(1)} \times [t_{j1}, t_{j2}] \) and a gauge transformation \( c_j \) such that \( \Phi_j = I \) and \( c_j = 1 \) on \( \Gamma_0 \times [t_{j1}, t_{j2}] \), and

\[ (7.26) \quad c_j \circ \Phi_j^{-1} \circ L^{(2)} = L^{(1)} \text{ in } D_0^{(1)} \times [t_{j1}, t_{j2}]. \]

In (7.26) \( \Phi_j \) is a diffeomorphism of \( \overline{D}_0^{(1)} \times [t_{j1}, t_{j2}] \) onto an almost cylindrical domain \( \overline{D}_0^{(2)} \times \{ S_j^- (x_1, \ldots, x_n) \leq x_0 \leq S_j^+ (x_1, \ldots, x_n) \} \), where \( x_0 = S_j^\pm (x_1, \ldots, x_n) \) are space-like surfaces, \( \Phi_j = I \) on \( \Gamma_0 \times [t_{j1}, t_{j2}] \), \( |c_j(x)| = 1 \) for all \( x \in \overline{D}_0^{(1)} \times [t_{j1}, t_{j2}] \), \( c_j = 1 \) on \( \Gamma_0 \times [t_{j1}, t_{j2}] \).

We shall show that

\[ \Phi_j = \Phi_{j+1}, \quad c_j = c_{j+1} \]

on \( \overline{D}_0^{(1)} \times [t_{j+1,1}, t_{j2}] \) where \( [t_{j+1,1}, t_{j2}] \) is the intersection of \([t_{j1}, t_{j2}] \) and \([t_{j+1,1}, t_{j+1,2}]. \)

Let \( \Phi_j = (\Phi_{j0}, \ldots, \Phi_{jn}) \). Note that \( y = \Phi_j (x) \) satisfies the equation (cf. (1.15))

\[ [g^p_k (\Phi_j (x))]_{p,k=0} \overset{n}{=} \frac{D\Phi_j (x)}{Dx} [g^p_k (x)]_{p,k=0} \overset{n}{=} \left( \frac{D\Phi_j (x)}{Dx} \right)^T, \]

where \([g^p_k (x)]^{-1}, [g^p_k (y)]^{-1} \) are metric tensors for \( L^{(1)}, L^{(2)} \), respectively. Therefore,

\[ (7.27) \quad g^{im}_2 (\Phi_j (x)) = \sum_{p,k=0}^{n} g^p_k (x) \frac{\partial \phi_{ji}}{\partial x_p} \frac{\partial \phi_{jm}}{\partial x_k}, \quad 0 \leq i, m \leq n. \]
Note that

\(\Phi_j|_{\Gamma_0 \times [t_j, t_{j+1}]} = I, \; c_j|_{\Gamma_0 \times [t_j, t_{j+1}]} = 1.\)

By Theorem 7.7 there exists a smooth solution \(\Phi_j\) of (7.27), (7.28) in \(D_0^{(1)} \times [t_j, t_{j+1}]\).

Now we shall prove the uniqueness.

Denote the right hand side of (7.27) by \(H\left(\frac{\partial \phi_{ji}}{\partial x}, \frac{\partial \phi_{j1,m}}{\partial x}\right)\). Subtract (7.27) for \(\Phi_{j+1}\) from (7.27) for \(\Phi_j\):

\[
(7.29) \quad H\left(\frac{\partial \phi_{ji}}{\partial x}, \frac{\partial \phi_{j1,m}}{\partial x}\right) - H\left(\frac{\partial \phi_{j+1,i}}{\partial x}, \frac{\partial \phi_{j+1,m}}{\partial x}\right) = g^{im}_2(\Phi_j(x)) - g^{im}_2(\Phi_{j+1}(x)).
\]

We have

\[
(7.30) \quad H\left(\frac{\partial \phi_{ji}}{\partial x}, \frac{\partial \phi_{j1,m}}{\partial x}\right) - H\left(\frac{\partial \phi_{j+1,i}}{\partial x}, \frac{\partial \phi_{j+1,m}}{\partial x}\right) = H\left(\frac{\partial \phi_{ji}}{\partial x} - \frac{\partial \phi_{j+1,i}}{\partial x}, \frac{\partial \phi_{j1,m}}{\partial x}\right) + H\left(\frac{\partial \phi_{j1,m}}{\partial x} - \frac{\partial \phi_{j+1,m}}{\partial x}, \frac{\partial \phi_{j+1,i}}{\partial x}\right).
\]

By the mean value theorem we get

\[
(7.31) \quad g^{im}_2(\Phi_j(x)) - g^{im}_2(\Phi_{j+1}(x)) = \sum_{k=0}^{n} b_{imk,j}(x)(\varphi_{jk}(x) - \varphi_{j+1,k}(x)),
\]

where \(b_{imk,j} \in C^\infty\). Therefore, using (7.30) and (7.31) we get an overdetermined system of \(\frac{(n+2)(n+1)}{2}\) first order linear differential equations with respect to \(\varphi_{ji} - \varphi_{j+1,i}\), with zero boundary condition

\[
(\varphi_{ji} - \varphi_{j+1,i})|_{\Gamma_0 \times [t_j, t_{j+1}, t_{j2}]} = 0.
\]

The uniqueness theorem for such equation gives

\[
\varphi_{ji} = \varphi_{j+1,i} \text{ in } D_0^{(1)} \times [t_{j+1,1}, t_{j2}], \quad i = 0, 1, ..., n.
\]

Having proven that \(\Phi_i = \Phi_{i+1}\) we get that \(c_i(x)\) and \(c_{i+1}(x)\) satisfy the same equation (1.23). Since \(c_i = c_{i+1}\) on \(\Gamma_0 \times [t_{i+1,1}, t_{i2}]\), we get that \(c_i = c_{i+1}\) on \(D_0^{(1)} \times [t_{i+1,1}, t_{i2}]\). This concludes the proof of Theorem 1.2.
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