Optimal rate decay for weak solutions of general porous medium type equations

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Abstract

In this paper, we show that bounded weak solutions of the Cauchy problem for general degenerate parabolic equations of the form

$$u_t + \text{div} f(x,t,u) = \text{div} (|u|^{\alpha} \nabla u), \quad x \in \mathbb{R}^n, \quad t > 0,$$

where $\alpha > 0$ is constant, decrease to zero, under fairly broad conditions on the advection flux $f$. Besides that, we derive an optimal decay rate for these solutions.

Key words: Porous medium equation; rate decay.

1 Introduction

The main object of this work is to obtain an optimal rate decay for (signed) weak solutions of the problem

$$u_t + \text{div} f(x,t,u) = \text{div} (|u|^{\alpha} \nabla u), \quad x \in \mathbb{R}^n, \quad t > 0,$$

$$u(\cdot,0) = u_0 \in L^{p_0}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n), \quad (1)$$

given constants $\alpha > 0$ and $1 \leq p_0 < \infty$, and $f \in C^1$ satisfying

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x,t,u) u \geq 0 \quad \forall \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad u \in \mathbb{R}, \quad (2)$$

In the case of $f$ not depending on $x$ and $t$, we have, in particular, the problem

$$u_t + \text{div} f(u) = \text{div} (|u|^{\alpha} \nabla u), \quad x \in \mathbb{R}^n, \quad t > 0,$$

$$u(\cdot,0) = u_0 \in L^{p_0}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n), \quad (3)$$

whose solutions exhibit a lot of knowing properties of parabolic problems in a conservative way as, for example, regularity, decay in $L^1$ norm, mass conservation and comparation properties. If the condition (2) does not hold, some of these
properties are no longer valid in general, such as decay in $L^q$ norm for $q > 1$, the Total Variation Diminishing (TVD), contrativity in $L^1$, global existence, decay to zero in various norms when $t \to \infty$, in case of global existence, etc.

In fact, the problem (1) could be much more complicated when the condition (2) is violated, as we indicate below. For this, we consider, for simplicity, the one-dimensional problem

$$u_t + (f(x) | u |^k u)_x = (| u |^\alpha u_x)_x \quad x \in \mathbb{R}, \; t > 0,$$

$$u(\cdot, 0) = u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

(4)

where $J := \{ x \in \mathbb{R} : f'(x) < 0 \} \neq \emptyset$.

Rewriting the first equation as

$$u_t + (k + 1) f(x) | u |^k u_x = (| u |^\alpha u_x)_x - f'(x) | u |^k u,$$

(5)

we can see that $u(x,t)$ tends to be stimulated to grow in magnitude at the points where $x \in J$, in particular where $-f'(x) \gg 1$. On the other hand, as $\| u(\cdot,t) \|_{L^1(\mathbb{R})} \leq \| u_0 \|_{L^1(\mathbb{R})}$ for all $t$ (while the solution exists), an intense growth in a given part of $J$ results in the formation of elongated structures (as is illustrated in fig. 1), that tend to be efficiently dissipated by the diffusivity term present. The greater the growth of $| u(x,t) |$, the greater will be the effect of the term $-f'(x) | u |^k u$ on the right side of (5) in forcing the additional growth and greater will be the dissipative capacity of the diffusivity term in (5) to inhibit such growth, given the increase of the own diffusion coefficient and of the intensification of the stretching effects on the profile of $u(\cdot,t)$!

The competition between the diffusive and the forcing terms in the equation (4) can, in this way, become so intense that the final result of this interaction (explosion or not on finite time, global existence and the behavior when $t \to \infty$, etc) is very difficult to be foreseen.

Besides that, in contrast to the current literature (see e.g. [9, 16, 17]), this kind of interaction (with mass conservation or similar links) only began to be investigated mathematically very recently (in [1, 2, 7]).

In the case of globally defined solutions, we can examine other open questions also for the problem (1) with the condition (2) even in $n = 1$ dimension. For example, we don’t know general conditions about $f, u_0$ that prevent blow-up at the infinity, that is, in order to have $u(\cdot,t) \in L^\infty([0, \infty), L^\infty(\mathbb{R}^n))$, or conditions that ensure asymptotic decay
Figure 1: Representation of the solution $u(\cdot, t)$ on the instant $t = 5$ (full curve) corresponding to equation $5$ above with $f(x) = -\tanh x, \alpha = 0.5, k = 1.5$, and initial state $u_0$ indicated (dashed curve). We note the growth of $u(\cdot, t)$ due $f'(x) < 0$, with formation of elongated structures ("High frequency waves") due the mass conservation.

$[\lim_{t \to \infty} \| u(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = 0]$, or convergence to stationary states (when they exist), and so on.

These questions will not be examined in this article, with one exception: we will show that the condition $2$ above ensures the decay: given $u_0 \in L^{p_0}(\mathbb{R}^n)$, the solution $u(\cdot, t) \in C^0([0, \infty), L^{p_0}(\mathbb{R}^n))$ corresponding to the problem $1$ with the condition $2$ satisfies

$$\| u(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \leq K(n, p_0, \alpha) \| u(\cdot, 0) \|_{L^{p_0}(\mathbb{R}^n)}^{\delta_0} t^{-\gamma_0} \quad \forall \ t > 0,$$

(6)

where

$$\delta_0 = \frac{2p_0}{2p_0 + n\alpha}, \quad \gamma_0 = \frac{n}{2p_0 + n\alpha},$$

(7)

and where $K(n, p_0, \alpha) > 0$ is a constant that depends only on the parameters $n, p_0, \alpha$ (and not on $t, u, u_0$, or $f$).

There are some results about existence and some estimates of the solutions to some similar and simpler equations. For example, the porous medium equation

$$u_t - (u^m)_{xx} = 0 \quad x \in \mathbb{R}, \ t > 0,$$

$$u(\cdot, 0) = u_0$$

(8)
for $m > 1$ and continuous positive initial data $u_0$ with connected compact support was considered by Caffarelli and Friedman [3]. They proved that a classical solution of problem (8) exists up to the free boundaries for some $t$. Also in [8], some optimal regularity results were obtained.

The porous medium equation

$$u_t = \Delta u^m$$

with $m > 1$ was treated, for example, in [10, 11] in smooth bounded domains. Kim and Lee [10] showed the short time existence of a smooth solution for this equation with a positive initial condition $u(x, 0) = u_0$ in $\Omega$ and with $u(x, t) = 0$ in $\partial \Omega$. With this same conditions, the long time existence was also showed by Kim [11].

Considering external forces in the porous medium equation in $\mathbb{R}^n$, the problem becomes

$$u_t - \Delta u^m = (\text{div} f) + g, \quad x \in \mathbb{R}^n, t > 0,$$

$$u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^n \quad (9)$$

Oleĭnik and Kalašinkov-Čžou [14] and Lions [12] showed the existence of weak solutions. Also, some a priori Hölder estimates for weak solutions can be found in [13].

The existence of local in time $C^\infty$-solutions for the following equation

$$u_t = \text{div}(u^l \nabla u), \quad x \in \mathbb{R}^n, t \geq 0,$$

$$u(x, 0) = u_0(x) \in \bigcap_{k=0}^{\infty} H^k(\mathbb{R}^n), \quad (10)$$

was showed in [15] considering that $l$ is an even natural number.

If $l = m - 1$, where $m < 1$ is a constant, the equation is referred as a fast diffusion equation. Assuming that $u(x, 0) = u_0(x) \geq 0$ cm $\mathbb{R}^n$, the existence of a kind of weak solution, the uniqueness and the regularity were discussed in [18].

Let us consider the problem

$$u_t + \text{div} f(x, t, u) = \text{div}(|u|^\alpha \nabla u), \quad x \in \mathbb{R}^n, t > 0,$$

$$u(x, 0) = u_0 \in L^{p_0}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad (11)$$
where $1 \leq p_0$ and $\alpha > 0$. This paper is organized as follows. Section 2 is devoted to showing an estimate for $L^q$ norm of the solutions of this problem. An energy inequality and some estimates decay for $L^p$ and $L^q$ norm for each $p_0 \leq q \leq \infty$ and $0 \leq t \leq T$ are presented in Section 3. In both Sections, we will consider $u_0 > 0$ (or $u_0 < 0$) for all $x \in \mathbb{R}^n$. Finally, Section 4 is devoted to find an optimal decay rate for weak solutions with any $u_0 \in L^{p_0}(\mathbb{R})$.

We remark that, in this paper, we understand as smooth and weak solution to the problem (11) a function that satisfies the following definitions, respectively:

**Definition 1.1.** A smooth function $u \in L^\infty_{loc}([0,T_*), L^\infty(\mathbb{R}^n))$ is a bounded classical solution in a maximal interval of existence $[0,T_*)$, where $0 \leq T_* \leq \infty$, if it satisfies classically the first equation of (11) and, besides that, $u(\cdot,t) \to u_0$ in $L^{p_0}_{loc}(\mathbb{R}^n)$, when $t \to 0$.

**Definition 1.2.** A weak solution to the problem (11) is a function $u$ that satisfies

$$\int_0^T \int_{\mathbb{R}^n} u(x,\tau)\Psi_t(x,\tau) + (f(x,\tau,u),\Psi_t(x,\tau)) + \frac{|u(x,\tau)|\alpha+1}{\alpha+1} \Delta \Psi(x,\tau) \, dx \, d\tau = 0,$$

for any $\Psi \in C_0^\infty(\mathbb{R}^n \times (0,T))$ and $u(\cdot,t) \to u_0$ in $L^{p_0}_{loc}$, when $t \to 0$.

When $u_0 > 0$ (or $u_0 < 0$) for all $x \in \mathbb{R}^n$, the solutions of the problem (11) are strictly positive (or strictly negative), see [7]. So the first equation is uniformly parabolic, which is the reason why we consider smooth solutions in Sections 2 and 3, and weak solutions in Section 4 where the initial condition and solutions can change sign. For a more complete discussion of regularity see e.g. [4, 5, 6, 19, 20].

2 Decreasing $L^q$ norm for smooth solutions

In this section we consider $f$ satisfying the hypothesis below.

**($f_1$)** $\exists K_f(T) < \infty$ such that $\sup \left\{ \frac{\partial f}{\partial u}(x,t,u) : |u| \leq M(T) \right\} < K_f(T)$, with $M(T) = \sup_{0<\tau<T} \|u(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}$.

**($f_2$)** $\frac{\partial f_j}{\partial u}(x,t,\cdot) \in L^1_{loc}(\mathbb{R})$, for each $1 \leq j \leq n$.

Let $0 < \varepsilon \leq 1$ and $\zeta_R$ a cut-off function given by $\zeta_R = 0$ if $|x| > R$ and $\zeta_R = e^{-\varepsilon \sqrt{1+|x|^2}} - e^{-\varepsilon \sqrt{1+R^2}}$ if $|x| \leq R$. Considering $q \geq 2$, $q \geq p_0$ and $\delta > 0$,
we define $\Phi_\delta(v) := L_\delta^q(v)$, where $L_\delta \in C^2(\mathbb{R})$ by
\[
L_\delta(u) := \int_0^u S(v/\delta) \, dv, \quad u \in \mathbb{R},
\]
where $S(0) = 0$, $S(u) = -1$ if $u \leq -1$, $S(u) = 1$ if $u \geq 1$ and $S$ is smooth for $-1 \leq u \leq 1$.

This cut-off functions will be useful in this Section and in Section 3.

**Theorem 2.1.** Let $q \geq 2$ and $T > 0$. If $u(x,t)$ is a smooth and bounded solution in $\mathbb{R}^n \times [0, T]$ of (11) and $f$ satisfies (f1 - f2), then
\[
\|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq \|u(\cdot, t_0)\|_{L^q(\mathbb{R}^n)}, \quad \forall 0 \leq t_0 \leq t \leq T, \text{ for each } p_0 \leq q \leq \infty.
\]

**Proof.** Let $p_0 \leq q \leq \infty$, $M(T) := \sup \{\|u(\cdot, \tau)\|_{L^\infty(\mathbb{R}^n)} : 0 < \tau < T\}$, $K_f(T) := \sup \left\{\left|\frac{\partial}{\partial \nu} [f(x, t, v)]\right| : |v| \leq M(T)\right\}$ and $\delta > 0$. Multiplying the first equation of (11) by $\Phi_\delta'(u)\zeta_R(x)$ and integrating in $\mathbb{R}^n \times [t_0, T]$ for each $0 < t_0 < t \leq T$, we obtain
\[
\int_{t_0}^t \int_{|x| < R} \Phi_\delta'(u)\zeta_R(x) |u_t| \, dx \, d\tau + \int_{t_0}^t \int_{|x| < R} \Phi_\delta'(u) \zeta_R(x) \text{div}(f) \, dx \, d\tau
\]
\[
= \int_{t_0}^t \int_{|x| < R} \Phi_\delta'(u) \zeta_R(x) \text{div}(|u|^\alpha \nabla u) \, dx \, d\tau. \tag{13}
\]

By (f1), letting $\delta \to 0$, we have that
\[
-q \int_{t_0}^t \int_{B_R} L_\delta^{-1}(u) L_\delta'(u) \left[ \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}(x, \tau, u) + \frac{\partial f_{ij}}{\partial v_j}(x, \tau, u) u_j \right] \zeta_R(x) \, dx \, d\tau
\]
\[
\leq K_f(T) \int_{t_0}^t \int_{B_R} |u(x, t)|^q \|
abla \zeta_R(x)\| \, dx \, d\tau,
\]
Let be $G_\delta(u) := \int_0^u |u|^\alpha \Phi_\delta'(w) \, dw$. As $\Phi_\delta'(u)\zeta_R(x)|u|^\alpha \langle \nabla u, \nabla u \rangle \geq 0$, applying the Divergence Theorem in the right side of (13) we have that
\[
\int_{t_0}^t \int_{|x| < R} \Phi_\delta'(u) \zeta_R(x) \text{div}(|u|^\alpha \nabla u) \, dx \, d\tau \leq \int_{t_0}^t \int_{|x| \leq R} \Phi_\delta'(u) |u|^\alpha \langle \nabla \zeta_R(x), \nabla u \rangle \, dx \, d\tau
\]
\[
\leq - \int_{t_0}^t \int_{|x| \leq R} \langle \nabla \zeta_R(x), \nabla G_\delta(u) \rangle \, dx \, d\tau
\]
\[
\leq \int_{t_0}^t \int_{|x| \leq R} \Delta \zeta_R(x) G_\delta(u) \, dx \, d\tau
\]
\begin{align*}
+ \frac{1}{R} \int_{t_0}^t \int_{|x|=R} G_\delta(u)(\nabla \zeta_R(x), x) \, d\sigma(x) \, d\tau.
\end{align*}

Note that
\begin{align*}
G_\delta(u) \leq \int_0^{u(x,t)} M^\alpha(T) \Phi'_\delta(w) \, dw \leq M^\alpha(T) \Phi_\delta(u).
\end{align*}

This estimate and Cauchy-Schwarz inequality give us
\begin{align*}
\int_{t_0}^t \int_{|x|<R} |\nabla \zeta_R(x)| \, |u(x,\tau)|^q \, dx \, d\tau \leq M^\alpha(T) \int_{t_0}^t \int_{|x|\leq R} |\Delta \zeta_R(x)| \, |u(x,\tau)|^q \, dx \, d\tau
\end{align*}
\begin{align*}
+ M^\alpha(T) \int_{t_0}^t \int_{|x|=R} |\nabla \zeta_R(x)| \, d\sigma(x) \, d\tau.
\end{align*}

Using the previous estimates, applying the Fubini’s Theorem in the first term on the left hand side, and letting \( \delta \to 0 \), we obtain
\begin{align*}
0 &\leq \int_{|x|<R} \zeta_R(x) |u(x,t)|^q \, dx \\
&\leq \int_{|x|<R} \zeta_R(x) |u(x,t_0)|^q \, dx \\
&\quad + K_f(T) \int_{t_0}^t \int_{B_R} |\nabla \zeta_R(x)| \, |u(x,\tau)|^q \, dx \, d\tau \\
&\quad + M^\alpha(T) \int_{t_0}^t \int_{B_R} |\Delta \zeta_R(x)| \, |u(x,\tau)|^q \, dx \, d\tau \\
&\quad + M^\alpha(T) \int_{t_0}^t \int_{|x|=R} |u(x,\tau)|^q \, |\nabla \zeta_R(x)| \, d\sigma(x) \, d\tau.
\end{align*}

The triangular inequality and estimates for \( |\nabla \zeta_R(x)| \) and \( |\Delta \zeta_R(x)| \), give us
\begin{align*}
\int_{|x|<R} |u(x,t)|^q \zeta_R(x) \, dx &\leq \int_{|x|<R} |u(x,t_0)|^q \zeta_R(x) \, dx \\
&\quad + \xi K_f(T) \int_{t_0}^t \int_{B_R} |u(x,\tau)|^q \, e^{-\xi \sqrt{1+|x|^2}} \, dx \, d\tau \\
&\quad + n \xi M^\alpha(T) \int_{t_0}^t \int_{B_R} |u(x,\tau)|^q \, e^{-\xi \sqrt{1+|x|^2}} \, dx \, d\tau \\
&\quad + \xi M^\alpha(T) \int_{t_0}^t \int_{|x|=R} |u(x,\tau)|^q \, e^{-\xi \sqrt{1+R^2}} \, d\sigma(x) \, d\tau,
\end{align*}

Letting \( R \to \infty \) and applying the Gronwall Lemma, we obtain
\begin{align*}
\int_{\mathbb{R}^n} |u(x,t)|^q e^{-\xi \sqrt{1+|x|^2}} \, dx \leq \int_{\mathbb{R}^n} |u(x,t_0)|^q e^{-\xi \sqrt{1+|x|^2}} \, dx \exp(S(\xi,T,t)),
\end{align*}
where \( S(\xi, T, t) = (n\xi M^\alpha(T) + \xi K_f(T)) t \), and \( S(\xi, T, t) \to 0 \), if \( \xi \to 0 \). Then, letting \( \xi \to 0 \) and \( t_0 \to 0 \) (in this order), we obtain

\[
\|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq \|u_0\|_{L^q(\mathbb{R}^n)} < \infty.
\]

**Observation 2.2.** Note that \( u(\cdot, t) \in L^q(\mathbb{R}^n) \) for each \( p_0 \leq q \leq \infty \) and for each \( 0 \leq t \leq T \), because \( u_0 \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) implies \( u_0 \in L^q(\mathbb{R}^n) \), \( \forall p \leq q < \infty \) by Interpolation Inequality.

**Observation 2.3.** In particular, by Theorem 2.1 we have, for each \( 0 \leq t \leq T \),

\[
\int_{t_0}^t \sum_{j=1}^n \frac{\partial f_j(x, \tau, u)}{\partial x_j} u(x, \tau)|u(x, \tau)|^{q-2} u(x, \tau) \, dx \, d\tau < \infty,
\]

as for each \( 0 \leq t \leq T \), we have

\[
\int_{t_0}^t \sum_{j=1}^n \frac{\partial f_j(x, t, u)}{\partial x_j} L_{s-1}^q(u) L_{s}^q(u) \zeta(x) \, dx \, d\tau \leq K \|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} < \infty.
\]

### 3 Decay estimates for \( L^q \) and \( L^\infty \) norms

In this section we will obtain one important energy inequality. This inequality will be fundamental to obtaining the decay velocity for smooth solutions in the last section.

**Theorem 3.1.** Let \( q \geq 2 \) and \( T > 0 \). If \( u(x, t) \) is a smooth and bounded solution in \( \mathbb{R}^n \times [0, T] \) of (11) and \( f \) satisfies (f1 - f2), then

\[
(t - t_0)^\gamma \|u(\cdot, t)\|_{L^q(\mathbb{R}^n)}^q + q(q - 1) \int_{t_0}^t (\gamma - 1) \int_{\mathbb{R}^n} |u(x, \tau)|^{q-2} |\nabla u|^2 \, dx \, d\tau
\leq \gamma \int_{t_0}^t (\gamma - 1) \|u(\cdot, \tau)\|_{L^q(\mathbb{R}^n)}^q \, d\tau.
\]

**Proof.** Let \( T > 0 \), \( M(T) := \sup\{\|u(\cdot, \tau)\|_{L^\infty(\mathbb{R}^n)} \mid 0 < \tau < T\} \), \( p_0 \leq q < \infty \), with \( 2 \leq q \), \( K_f(T) := \sup\left\{ \left| \frac{\partial f}{\partial u}(v) \right| : |v| \leq M(T) \right\} \) and \( \delta > 0 \). Multiplying the first equation of the problem (11) by \( (\tau - t_0)^\gamma \Phi_{\delta}(u) \zeta_R(x) \), integrating on \( \mathbb{R}^n \times [t_0, t] \), where \( 0 < t_0 < t \leq T \), and applying Fubini’s Theorem in the first term on the left side we obtain

\[
\int_{B_R} \zeta_R(x)(t - t_0)^\gamma \Phi_{\delta}(u(x, t)) \, dx - \gamma \int_{t_0}^t \int_{B_R} \zeta_R(x)(\tau - t_0)^{\gamma-1} \Phi_{\delta}(u(x, \tau)) \, dx \, d\tau
\]
\[
\int_{t_0}^{t} \int_{|x|<R} (\tau - t_0)^\gamma \Phi_\delta'(u) \zeta_R(x) \text{div}(f) \, dx \, d\tau \\
= \int_{t_0}^{t} \int_{|x|<R} (\tau - t_0)^\gamma \Phi_\delta'(u) \zeta_R(x) \text{div}(|u|^{\alpha} \nabla u) \, dx \, d\tau. \quad (14)
\]

Note that
\[
- \int_{t_0}^{t} \int_{|x|<R} (\tau - t_0)^\gamma \Phi_\delta'(u) \zeta_R(x) \text{div}(f) \, dx \, d\tau \\
\leq - \int_{t_0}^{t} \int_{|x|<R} (\tau - t_0)^\gamma \Phi_\delta'(u) \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}(x, t, u) u_{x_j} \zeta_R(x) \, dx \, d\tau, \quad (15)
\]
as \[ -q \int_{t_0}^{t} \int_{B_R} (\tau - t_0)^\gamma \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}(x, t, u) L_\delta^{q-1}(u) L_\delta'(u) \zeta_R(x) \, dx \, d\tau \leq 0. \]

Also, writing \( G_\delta(u) := \int_{0}^{u} |w|^{\alpha} \Phi_\delta'(w) \, dw \) and applying the Divergence Theorem, we have that the term on the right hand side of \((14)\) can be written as
\[
\int_{t_0}^{t} \int_{|x|<R} (\tau - t_0)^\gamma \Phi_\delta'(u) \zeta_R(x) \text{div}(|u|^{\alpha} \nabla u) \, dx \, d\tau \\
= - \int_{t_0}^{t} \int_{|x|\leq R} (\tau - t_0)^\gamma \langle \nabla \zeta_R(x), \nabla G_\delta(u) \rangle \, dx \, d\tau \\
-q(q-1) \int_{t_0}^{t} \int_{|x|=R} (\tau - t_0)^{\gamma} L_\delta(u)^{q-2}(L_\delta'(u))^2 \zeta_R(x) |u|^{\alpha} |\nabla u|^2 \, dx \, d\tau.
\]

Applying one more time the Divergence Theorem and using Cauchy-Schwarz inequality, it holds that
\[
\int_{t_0}^{t} \int_{|x|<R} (\tau - t_0)^\gamma \Phi_\delta'(u) \zeta_R(x) \text{div}(|u|^{\alpha} \nabla u) \, dx \, d\tau \\
\leq M^\alpha(T) \int_{t_0}^{t} \int_{|x|\leq R} (\tau - t_0)^\gamma \Phi_\delta(u) \Delta \zeta_R(x) \, dx \, d\tau \\
+ M^\alpha(T) \int_{t_0}^{t} \int_{|x|=R} (\tau - t_0)^\gamma \Phi_\delta(u) |\nabla \zeta_R(x)| \, d\sigma(x) \, d\tau \\
-q(q-1) \int_{t_0}^{t} \int_{|x|=R} (\tau - t_0)^{\gamma} L_\delta(u)^{q-2}(L_\delta'(u))^2 \zeta_R(x) |u|^{\alpha} |\nabla u|^2 \, dx \, d\tau, \quad (16)
\]
as \( G_\delta(u) \leq M^\alpha(T) \Phi_\delta(u) \). Substituting \((15)\) and \((16)\) in \((14)\),
\[
\int_{B_R} \zeta_R(x)(t-t_0)^\gamma \Phi_\delta(u(x, t)) \, dx \leq \gamma \int_{t_0}^{t} \int_{B_R} \zeta_R(x)(\tau-t_0)^{\gamma-1} \Phi_\delta(u(x, \tau)) \, dx \, d\tau
\]
By the Nirenberg-Gagliardo-Sobolev’s Interpolation Inequality, such that
\[ \|u(t)\|_{L^q(R^n)} \leq \frac{\|u(t)\|_{L^\infty(R^n)}}{\delta} (t - t_0)^{-\kappa}, \]
where \( \delta = \frac{2q + n\alpha}{2q + 2n\alpha} \) and \( \kappa = \frac{n}{2q + 2n\alpha}. \)

Theorem 3.2. Let \( q \geq 2 \) and \( T > 0 \). If \( u(x,t) \) is a smooth and bounded solution in \( \mathbb{R}^n \times [0,T] \) of (11) and \( f \) satisfies (f1 - f2), then

\[ \|u(t)\|_{L^\infty(R^n)} \leq K_q(n, \alpha) \|u_0\|_{L^{n/2}(\mathbb{R}^n)} (t - t_0)^{-\kappa}, \]
for all \( (t_0,T) \), \( \forall 2p_0 \leq q \leq \infty, \)

where \( \delta = \frac{2q + n\alpha}{2q + 2n\alpha} \) and \( \kappa = \frac{n}{2q + 2n\alpha}. \)

Proof. Let \( u(x,t) \) be a smooth solution of (11). Defining \( w(x,t) := |u(x,t)|^{\frac{2q - n}{2}} \), we have \( w(t) \in L^\beta(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), where \( \beta = \frac{2q}{q + \alpha} \). By the inequality (17), it follows that

\[ (t - t_0)^\gamma \|w(t)^\beta\|_{L^\beta(\mathbb{R}^n)} + \frac{4q(q - 1)}{(q + \alpha)^2} \int_{t_0}^t (t - \tau)^\gamma \|\nabla w(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \]

\[ \leq \gamma \int_{t_0}^t (t - \tau)^{\gamma - 1} \|w(\cdot, \tau)\|_{L^\beta(\mathbb{R}^n)}^\beta d\tau, \]

By the Nirenberg-Gagliardo-Sobolev’s Interpolation Inequality, \( \exists C > 0 \) (constant) such that

\[ \|w(t)\|_{L^\beta(\mathbb{R}^n)} \leq C \|w(t)\|_{L^{n/2}(\mathbb{R}^n)}^{1 - \theta} \|\nabla w(t)\|_{L^2(\mathbb{R}^n)}^\theta. \]
where \( \frac{1}{\beta} = \theta \left( \frac{1}{2} - \frac{1}{n} \right) + (1 - \theta) \frac{2}{\beta} \). So we have \( \theta = \frac{n(q + \alpha)}{nq + 2q + 2n\alpha} \) and

\[
(t - t_0)\gamma \|w(\cdot, t)\|_{L^\beta(\mathbb{R}^n)}^{\beta} + \frac{4q(q - 1)}{(q + \alpha)^2} \int_{t_0}^{t} (\tau - t_0)\gamma \|\nabla w(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}^{2} d\tau \leq \\
\leq \gamma C^\beta \int_{t_0}^{t} (\tau - t_0)\gamma^{-1} \|w(\cdot, \tau)\|_{L^{\alpha/2}(\mathbb{R}^n)}^{(1 - \theta)\beta} \|\nabla w(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}^{\theta \beta} d\tau \\
\leq \gamma C^\beta \|u(\cdot, t_0)\|_{L^{\alpha/2}(\mathbb{R}^n)}^{(1 - \theta)\beta} \int_{t_0}^{t} (\tau - t_0)\gamma^{-1} \|\nabla w(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}^{\theta \beta} d\tau,
\]

as \( \|w(\cdot, t)\|_{L^{\beta/2}(\mathbb{R}^n)}^{\beta/2} = \|u(\cdot, t)\|_{L^{\alpha/2}(\mathbb{R}^n)}^{q/2} \leq \|u(\cdot, t_0)\|_{L^{\alpha/2}(\mathbb{R}^n)}^{q/2} \) by Theorem 2.4.

Applying Hölder’s inequality and Young’s inequality (in this order) both with \( p = \frac{2}{\theta \beta} \) and \( q = \frac{2}{2 - \theta \beta} \), we obtain

\[
(t - t_0)\gamma \|w(\cdot, t)\|_{L^\beta(\mathbb{R}^n)}^{\beta} + \frac{4q(q - 1)}{(q + \alpha)^2} \int_{t_0}^{t} (\tau - t_0)\gamma \|\nabla w(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}^{2} d\tau \leq \\
\leq \gamma C^\beta \|u(\cdot, t_0)\|_{L^{\alpha/2}(\mathbb{R}^n)}^{q(1 - \theta)\beta} (t - t_0)^{\frac{2 - \theta \beta}{q} + \frac{2q(q - 1)}{(q + \alpha)^2}} \left( \int_{t_0}^{t} (\tau - t_0)\gamma^{-\frac{\theta \beta}{q}} \|\nabla w(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}^{2} d\tau \right)^{\frac{\theta \beta}{q}} \\
\leq \left( \gamma C^\beta \|u(\cdot, t_0)\|_{L^{\alpha/2}(\mathbb{R}^n)}^{q(1 - \theta)\beta} \right)^{\frac{2 - \theta \beta}{q} + \frac{2q(q - 1)}{(q + \alpha)^2}} (t - t_0) + \\
+ \frac{2q(q - 1)}{(q + \alpha)^2} \int_{t_0}^{t} (\tau - t_0)^{\gamma^{-1}} \|\nabla w(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}^{2} d\tau,
\]

where in the last inequality we choose \( \gamma \) so that \( (\gamma - 1) \frac{2}{\theta \beta} = \gamma \), that is, \( \gamma = \frac{2}{2 - \theta \beta} \).

So \( \int_{0}^{\beta} \tau^{-1} d\tau < \infty \), as \( \gamma - 1 > -1 \). Then,

\[
(t - t_0)\gamma \|w(\cdot, t)\|_{L^\beta(\mathbb{R}^n)}^{\beta} + \frac{2q(q - 1)}{(q + \alpha)^2} \int_{t_0}^{t} (\tau - t_0)\gamma \|\nabla w(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}^{2} d\tau \leq \\
\leq \left( \gamma C^\beta \|u(\cdot, t_0)\|_{L^{\alpha/2}(\mathbb{R}^n)}^{q(1 - \theta)\beta} \right)^{\frac{2 - \theta \beta}{q} + \frac{2q(q - 1)}{(q + \alpha)^2}} (t - t_0). 
\]

Writing the previous inequality in terms of \( u \) we obtain, in particular,

\[
\|u(\cdot, t)\|_{L^\alpha(\mathbb{R}^n)}^{q} \leq \left( \gamma C^\beta \|u(\cdot, t_0)\|_{L^{\alpha/2}(\mathbb{R}^n)}^{q(1 - \theta)\beta} \right)^{\frac{2 - \theta \beta}{q} + \frac{2q(q - 1)}{(q + \alpha)^2}} (t - t_0)^{1 - \gamma},
\]

that is,
\[ \|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq (C^\beta)^{\frac{1}{2} - \frac{\theta q}{2}} \left( \frac{2 - \theta \beta}{2} \right)^{\frac{1}{2}} \frac{\theta \beta (q + \alpha)^2}{4q(q - 1)} \|u(\cdot, t_0)\|_{L^q(\mathbb{R}^n)} \left( t - t_0 \right)^{-\frac{\gamma}{q}}. \]

As \( \frac{2(1 - \theta)}{2 - \theta \beta} = \frac{2q + n\alpha}{2q + 2n\alpha} \) and \( \frac{1 - \gamma}{q} = -\frac{n}{2q + 2n\alpha} \), we get
\[ \|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq K_q \|u(\cdot, t_0)\|_{L^q(\mathbb{R}^n)} \left( t - t_0 \right)^{-\frac{\gamma}{q}}, \]
where \( K_q = K_q(n, \alpha) = (C^\beta)^{\frac{1}{2} - \frac{\theta q}{2}} \left( \frac{2 - \theta \beta}{2} \right)^{\frac{1}{2}} \frac{\theta \beta (q + \alpha)^2}{4q(q - 1)} \). \( \square \)

**Theorem 3.3.** Let \( q \geq 2 \) and \( T > 0 \). If \( u(x, t) \) is a smooth and bounded solution in \( \mathbb{R}^n \times [0, T] \) of (11) and \( f \) satisfies (f1 - f2), then
\[ \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq K_n \|u_0\|_{L^\infty(\mathbb{R}^n)} \left( t - t_0 \right)^{-\kappa}, \quad \forall t \in (t_0, T], \forall 2p_0 \leq q \leq \infty, \]
where \( \delta = \frac{2q}{2q + n\alpha}, \quad \kappa = \frac{n}{2q + n\alpha} \) and \( K_n(q, \alpha) \) is constant.

**Proof.** Let \( u(\cdot, t) \) be a smooth solution of (11). By Theorem 3.2
\[ \|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq K_q \|u(\cdot, t_0)\|_{L^q(\mathbb{R}^n)} \left( t - t_0 \right)^{-\frac{\gamma}{q}}, \]
where \( K_q(n, \alpha) = C^{\beta - \frac{\theta q}{2}} \left( \frac{nq + 2q + 2n\alpha}{q + n\alpha} \right)^{\frac{1}{2}} \frac{\theta \beta (q + \alpha)^2}{4q(q - 1)} \left( t - t_0 \right)^{-\frac{\gamma}{q}}, \)
as \( \gamma = \frac{2q}{2 - \theta \beta}, \quad \beta = \frac{2q}{q + \alpha} \) and \( \theta \beta = \frac{2nq}{q + 2q + 2n\alpha} \).

Let \( m \in \mathbb{N} \) with \( m \geq 1 \), we define \( t_j^{(m)} = \frac{2^{-m} t}{m} \) and \( t_j^{(m)} = t_0 + (1 - 2^{-j})t \) for all \( 1 \leq j \leq m \). Applying this inequality, \( m \) times, for \( q = 2^{m} q \), \( \epsilon t_0 = t_0^{(m)} \), \( q = 2^{m} - 1 \), \( \epsilon t_0 = t_0^{(m-1)} \), \( \cdots \), we obtain
\[ \|u(\cdot, t_j^{(m)})\|_{L^q(\mathbb{R}^n)} \leq K_m \|u(\cdot, t_{j-1}^{(m)})\|_{L^q(\mathbb{R}^n)} \left( t - t_{j-1}^{(m)} \right)^{-\frac{\gamma}{q}}, \]
\[ \leq K_m \|u(\cdot, t_{j-1}^{(m)})\|_{L^q(\mathbb{R}^n)} \left( t - t_{j-1}^{(m)} \right)^{-\frac{\gamma}{q}}, \]
\[ \vdots \]
\[ \leq K_m \|u(\cdot, t_{j-1}^{(m)})\|_{L^q(\mathbb{R}^n)} \left( t - t_{j-1}^{(m)} \right)^{-\frac{\gamma}{q}} \cdots \left( \frac{2^{m-1} q + n \alpha}{2^{m-1} q + 2n \alpha} \right)^{\frac{1}{2}} \frac{\theta \beta (q + \alpha)^2}{4q(q - 1)} \left( t - t_{j-1}^{(m)} \right)^{-\frac{\gamma}{q}}, \]
\[ \leq K_m \|u(\cdot, t_0^{(m)})\|_{L^q(\mathbb{R}^n)} \left( t - t_0^{(m)} \right)^{-\frac{\gamma}{q}} \cdots \left( \frac{2^{m-1} q + n \alpha}{2^{m-1} q + 2n \alpha} \right)^{\frac{1}{2}} \frac{\theta \beta (q + \alpha)^2}{4q(q - 1)} \left( t - t_0^{(m)} \right)^{-\frac{\gamma}{q}} \cdots, \]
\[ \leq K_m \|u(\cdot, t_0^{(m)})\|_{L^q(\mathbb{R}^n)} \left( t - t_0^{(m)} \right)^{-\frac{\gamma}{q}} \cdots \left( \frac{2^{m-1} q + n \alpha}{2^{m-1} q + 2n \alpha} \right)^{\frac{1}{2}} \frac{\theta \beta (q + \alpha)^2}{4q(q - 1)} \left( t - t_0^{(m)} \right)^{-\frac{\gamma}{q}} \cdots, \]
\[ \leq \cdots \cdots \]
where, for each $1 \leq j \leq m$,

$$K_j = \frac{2^j q(n+2) + 2n\alpha}{2^j q + 2na} \left( \frac{2^j q(n+4) + 4n\alpha}{2^j q + 4na} \right)^{2j - \frac{2j q(n+2) + 2n\alpha}{2^j q + 2na}},$$

$$A_m = \prod_{j=1}^{m} \frac{2^j q + na}{2^j q + 2na} = \frac{1}{2^m} \prod_{j=1}^{m} \frac{2^j q + na}{2^j q + 2na} = \frac{1}{2^m} \frac{2^m q + na}{2q + na}, \quad B_0 = 1 e$$

$$B_j = \prod_{k=0}^{j-1} \frac{2^{m-k} q + na}{2^{m-k} q + 2na} = \frac{1}{2^j} \frac{2^m q + na}{2^m q + 2na} \text{ para } 1 \leq j \leq m.$$ 

Since $t_j^{(m)} - t_{j-1}^{(m)} = 2^{-j} t$, for all $1 \leq j \leq m$, we can rewrite the previous inequality as

$$\|u(\cdot, t_m^{(m)})\|_{L^2(q, \mathbb{R}^n)} \leq \prod_{j=1}^{m} \left[ K_j^{B_{m-j}} \|u(\cdot, t_0^{(m)})\|_{L^2(q, \mathbb{R}^n)} (2^{-j} t)^{-\frac{n}{2^j q + 2na} B_{m-j}} \right].$$

Now let us estimate, separate, $\prod_{j=1}^{m} K_j^{B_{m-j}}$ and $\prod_{j=1}^{m} (2^{-j} t)^{-\frac{n}{2^j q + 2na} B_{m-j}}$.

Note that $\prod_{j=1}^{m} t^{\frac{n}{2^j q + 2na} B_{m-j}} = \prod_{j=1}^{m} t^{\frac{-n}{2^j q + 2na} B_{m-j}}$. 

First, let us observe that,

$$\prod_{j=1}^{m} (2^{-j} t)^{-\frac{n}{2^j q + 2na} B_{m-j}} = \frac{\sum_{j=1}^{m} (2^{-j} t)^{-\frac{n}{2^j q + 2na} B_{m-j}}}{\sum_{j=1}^{m} (2^{-j} t)^{-\frac{n}{2^j q + 2na} B_{m-j}}},$$

and that (changing $m - j$ for $j$), we obtain

$$\prod_{j=1}^{m} \frac{2^{m-j} q + 2n\alpha}{2^{m-j} q + 2na} B_{m-j} = \frac{1}{\sum_{j=1}^{m} \frac{2^{m-j} q + 2n\alpha}{2^{m-j} q + 2na} B_{m-j}}.$$

Defining $\hat{\alpha} = \frac{2^m q + na}{2n\alpha}$, we get

$$\prod_{j=1}^{m} \frac{-n}{2^j q + 2na} B_{m-j} = \frac{2n(2^m q + na)}{\hat{\alpha}(2na)^2} \sum_{j=0}^{m-1} \frac{1}{\hat{\alpha}^{2^{-j+1}} + 1 \hat{\alpha}^{2^{-j}}}.$$
\[-\frac{2n(2^m 2q + n\alpha)}{\alpha(2n\alpha)^2} \sum_{j=0}^{m-1} \frac{1}{\alpha 2^{-j} + 1} - \frac{1}{\alpha 2^{-j+1} + 1} = -\frac{2n(2^m 2q + n\alpha)}{\alpha(2n\alpha)^2} \left[ \frac{1}{\alpha 2^{-m+1} + 1} - \frac{1}{2\alpha + 1} \right] = -\frac{2n(2^m 2q + n\alpha)}{2m^2 q} \left[ \frac{1}{4q + 2n\alpha} - \frac{1}{2m^4 q + 2n\alpha} \right] = -\frac{2n(2q + \frac{n\alpha}{2})}{2q} \left[ \frac{1}{4q + 2n\alpha} - \frac{1}{2m^4 q + 2n\alpha} \right]. \]

Now, letting \( m \to +\infty \), we obtain

\[
\sum_{j=1}^{\infty} \frac{-n}{2^j 2q + 2n\alpha} B_{m-j} = \frac{-2n}{4q + 2n\alpha} = \frac{-n}{2q + n\alpha}, \text{ and } \lim_{m \to +\infty} A_m = \lim_{m \to +\infty} \frac{2q + \frac{n\alpha}{2}}{2q + n\alpha} = \frac{2q}{2q + n\alpha}.
\]

So, we show that, in fact, \( \gamma = \frac{2q}{2q + n\alpha} \) and \( \kappa = \frac{n}{2q + n\alpha} \). It remains to obtain a bound to \( K_n(\alpha, q) \) independent of \( m \).

Note that

\[
\prod_{j=1}^{m} \left(2^{-j}\right) \frac{-n}{2^j 2q + 2n\alpha} B_{m-j} = 2^{-\sum_{j=1}^{m} \frac{n}{2^j 2q + 2n\alpha}} B_{m-j}. \]

As \( B_j = \frac{1}{2^j} \frac{2^m 2q + n\alpha}{2^m 2q + n\alpha} = \frac{2^m 2q + n\alpha}{2^m 2q} = 1 + \frac{n\alpha}{2^m 2q}, \forall 1 \leq j \leq m \), we have

\[
\prod_{j=1}^{m} \left(2^{-j}\right) \frac{-n}{2^j 2q + 2n\alpha} B_{m-j} \leq 2^m (1 + \frac{n\alpha}{2^m 2q})^{\sum_{j=1}^{m} \frac{n}{2^j 2q + 2n\alpha}} \leq 2^{\frac{n}{2^m 2q + 2n\alpha}}(1 + \frac{n\alpha}{2^m 2q}) \sum_{j=1}^{m} \frac{1}{2^j}. \]

So, letting \( m \to \infty \), we obtain

\[
\prod_{j=1}^{\infty} \left(2^{-j}\right) \frac{-n}{2^j 2q + 2n\alpha} B_{m-j} \leq 2^{\frac{n}{2^m 2q + 2n\alpha}}(1 + \frac{n\alpha}{2^m 2q}) \sum_{j=1}^{\infty} \frac{1}{2^j} = 2^{\frac{n}{2^m 2q + 2n\alpha}}(1 + \frac{n\alpha}{2^m 2q}). \]

Finally, we estimate \( \prod_{j=1}^{m} K_j^{B_{m-j}} \). For this, note that

\[
K_j = C \frac{1}{2^j q + \alpha} \left( \frac{2^j q (n + 2) + 2n\alpha}{2^j q + n\alpha} \right)^{2^j q + 2n\alpha} \left( \frac{(2^j q + \alpha)^2}{2^j q (2^j q - 1)} \right)^{2^j q + 2n\alpha}.
\]

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\[ \leq C^{n+2 + \frac{2n\alpha}{2q}} \left( n + 2 + \frac{2n\alpha}{2q} \right)^\frac{n+1}{2} \cdot \left( \frac{(2^j q + \alpha)^2}{2^j q (2^j q - 1)} \right)^{\frac{n}{2}}. \]

Defining \( j \geq j_0 \) so that \( \frac{2n\alpha}{2^j q} < 1 \), \( \frac{\alpha}{2^j (2q - 1)} < 1 \) and \( \frac{\alpha^2}{2^{2j} q (2^j q - 1)} < 1 \), we get

\[ K_j \leq C^{\frac{n+2 + \frac{2n\alpha}{2q}}{2^j q}} (n + 3) \left( \frac{2^j q + \alpha^2}{2^j q (2^j q - 1)} \right) \frac{n}{2^j q} \]

\[ \leq C^{\frac{n+2 + \frac{2n\alpha}{2q}}{2^j q}} (n + 3) \left( \frac{q}{2(2q - 1)} + \frac{\alpha}{2^j (2q - 1)} + \frac{\alpha^2}{2^{2j} q (2^j q - 1)} \right) \frac{n}{2^j q} \]

So

\[ \prod_{j=1}^{m} K_j^{B_{m-j}} \leq \prod_{j=1}^{m} \left[ C^{\frac{n+2 + \frac{2n\alpha}{2q}}{2^j q}} (n + 3) \left( \frac{q}{2(2q - 1)} + 2 \right) \frac{n}{2^j q} \right]^{\frac{4+\alpha}{4q}} \]

\[ = \left( \prod_{j=1}^{j_0-1} K_j^{\frac{4+\alpha}{4q}} \right) \left( n + 3 \right) \frac{n^{n+\alpha} 4^q}{4^q} C^{\frac{n+2 + \frac{2n\alpha}{2q}}{2^j q}} \frac{n}{2^j q} (1-2^{-m}) + \frac{2n\alpha}{4q} \left( \frac{1}{1-2^{-m}} \right) \cdot \left( \frac{q}{2(2q - 1)} + 2 \right)^\frac{4+\alpha}{4q} \frac{n}{2^j q} (1-2^{-m}). \]

Now, letting \( m \to +\infty \), we obtain

\[ \prod_{j=1}^{m} K_j^{B_{m-j}} \leq \left( \prod_{j=1}^{j_0-1} K_j^{\frac{4+\alpha}{4q}} \right) \left( n + 3 \right) \frac{n^{n+\alpha} 4^q}{4^q} C^{\frac{n+2 + \frac{2n\alpha}{2q}}{2^j q}} \frac{n}{2^j q} (1-2^{-m}) + \frac{2n\alpha}{4q} \left( \frac{1}{1-2^{-m}} \right) \cdot \left( \frac{q}{2(2q - 1)} + 2 \right)^\frac{4+\alpha}{4q} \frac{n}{2^j q} (1-2^{-m}), \]

that is, \( \prod_{j=1}^{m} K_j^{B_{m-j}} < \infty \). \qed
4 Optimal rate for signed solutions

In this section we will obtain an optimal rate decay for weak solutions of the problem

\[ u_t + \text{div} f(x,t,u) = \text{div}(|u(x,t)|^\alpha \nabla u) \quad x \in \mathbb{R}^n, \ t > 0, \]

\[ u(\cdot,0) = u_0 \in L^{p_0}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \quad (18) \]

where \( u_0 \) is any function in \( L^{p_0}(\mathbb{R}) \).

Let us consider the auxiliary problems

\[ u_t + \text{div} f(x,t,u) = \text{div}(|u(x,t)|^\alpha \nabla u) \quad x \in \mathbb{R}^n, \ t > 0, \]

\[ u(\cdot,0) = u_0^+ + \epsilon \psi \in L^{p_0}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \quad (19) \]

and

\[ u_t + \text{div} f(x,t,u) = \text{div}(|u(x,t)|^\alpha \nabla u) \quad x \in \mathbb{R}^n, \ t > 0, \]

\[ u(\cdot,0) = -u_0^- - \epsilon \psi \in L^{p_0}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \quad (20) \]

where \( \epsilon > 0 \) and \( 0 < \psi \in L^{p_0}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \).

To prove the main result of this article, we need \( f \) to satisfy the hypothesis below

(\( f_3 \)) \( |f(x,t,u) - f(x,t,v)| \leq C_f(M,T) |u - v|, \ \forall \ x \in \mathbb{R}^n, \ 0 \leq t \leq T. \)

**Theorem 4.1.** Let \( q \geq 2 \) and \( T > 0 \). If \( u(x,t) \) is a weak and bounded solution in \( \mathbb{R}^n \times [0,T] \) of (18) and \( f \) satisfies (\( f_1 - f_3 \)), then

\[ \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq K_n(\alpha,q)\|u_0\|_{L^q(\mathbb{R}^n)}^{\delta} (t - t_0)^{-\kappa}, \ \forall \ t \in (t_0,T], \ \forall 2p_0 \leq q \leq \infty, \]

where \( \delta = \frac{2q}{2q + n\alpha}, \ \kappa = \frac{n}{2q + n\alpha} \) and \( K_n(\alpha,q) \) is constant.

**Proof.** Let \( u,v,w \) be, respectively, solutions of (18), (19) and (20), by comparison (see [7]), we have \( w(x,t) \leq u(x,t) \leq v(x,t), \ w(x,t) \leq 0 \) and \( 0 \leq v(x,t), \ \forall x \in \mathbb{R}^n \) and \( \forall t > 0. \) Now, by Theorem 3.3 the estimates obtained for smooth solutions, that do not change sign, of problems with initial data that do
not change sign, it follows that

\[ \|w(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq K_n(\alpha, n)\| - u^-_0 - \epsilon \psi \|^\delta_{L^1(\mathbb{R}^n)} t^{-\kappa}, \forall t \in (t_0, T], \forall 2 p_0 \leq q \leq \infty, \]

where \( \delta = 2q = \frac{2q + n\alpha}{2q + n\alpha}, \) \( \kappa = \frac{n}{2q + n\alpha} \) and \( K_n(\alpha, q) \) is constant. In particular, we have

\[ -K_n(\alpha, n)\| - u^-_0 - \epsilon \psi \|^\delta_{L^1(\mathbb{R}^n)} t^{-\kappa} \leq w(x, t) \leq u(x, t) \]

\[ \leq v(x, t) \leq K_n(\alpha, n)\| u^+_0 + \epsilon \psi \|^\delta_{L^1(\mathbb{R}^n)} t^{-\kappa}, \]

hence

\[ \| u(\cdot, t) \| \leq K_n(\alpha, n) \max \left\{ \| - u^-_0 - \epsilon \psi \|^\delta_{L^1(\mathbb{R}^n)}, \| u^+_0 + \epsilon \psi \|^\delta_{L^1(\mathbb{R}^n)} \right\} t^{-\kappa}, \forall \epsilon > 0. \]

As \( \max \left\{ \| - u^-_0 - \epsilon \psi \|^\delta_{L^1(\mathbb{R}^n)}, \| u^+_0 + \epsilon \psi \|^\delta_{L^1(\mathbb{R}^n)} \right\} \) decreases when \( \epsilon \to 0^+ \), we obtain

\[ \| u(\cdot, t) \| \leq K_n(\alpha, n) \max \left\{ \| u^-_0 \|^\delta_{L^1(\mathbb{R}^n)}, \| u^+_0 \|^\delta_{L^1(\mathbb{R}^n)} \right\} t^{-\kappa}. \]

\[ \square \]

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