An eigenfunction expansion formula for one-dimensional two-state quantum walks

Tatsuya Tate

Received: 21 September 2021 / Accepted: 24 July 2022
© The Author(s) 2022

Abstract
The purpose of this paper is to give a direct proof of an eigenfunction expansion formula for one-dimensional two-state quantum walks, which is an analog of that for Sturm–Liouville operators due to Weyl, Stone, Titchmarsh, and Kodaira. In the context of the theory of CMV matrices, it had been already established by Gesztesy–Zinchenko. Our approach is restricted to the class of quantum walks mentioned above, whereas it is direct and it gives some important properties of Green functions. The properties given here enable us to give a concrete formula for a positive-matrix-valued measure, which gives directly the spectral measure, in a simplest case of the so-called two-phase model.

Keywords Quantum walks · Eigenfunction expansion · QW-Fourier transform · Green functions

Mathematics Subject Classification 47A70 · 47A10 · 34B20

1 Introduction

The quantum walks are certain unitary operators, and they are sometimes regarded as quantum counterparts of the classical random walks. The homogeneous two-state quantum walks (in one dimension with a constant coin matrix) are well understood fundamental models (see, for example, [6, 9, 25]), and recently the scattering-theoretical aspects, as a perturbation of homogeneous walks, are intensively investigated (see [15–17, 20, 21]). The Schrödinger operators in one dimension are often called the Sturm–Liouville operators and they are well...
studied. Thus, it would be rather natural to understand resemblance between one-
dimensional quantum walks and Sturm–Liouville operators. The purpose of the
present paper is to give a proof of an eigenfunction expansion formula for one-
dimensional two-state quantum walks which is analogous to classical formulas of
Weyl [27, 28], Stone [24], Titchmarsh [26], and Kodaira [10] for Sturm–Liouville
operators. The theory of eigenfunction expansions for Sturm–Liouville operators
are discussed, for example, in [12, 14, 18, 19] and a short review can be found in
[23]. Probabilistic aspects of one-dimensional quantum walks are also intensively
investigated. The notion of transfer matrix is introduced in [11] to construct sta-
tionary measures from eigenfunctions for quantum walks, and it is suitable for
our analysis. Then, our basic idea in this paper is to use the transfer matrix to
develop a theory analogous to that for Sturm–Liouville operators.

Before going to explain our setting-up, we should mention about the work by
Gesztesy–Zinchenko [8] on Weyl–Titchmarsh theory for CMV matrices with Ver-
blunsky coefficients in the unit disk. The notion of CMV matrices has been intro-
duced by Cantero–Moral–Velázquez [2] and has further developed and deepened
by Simon [22, 23]. The one-dimensional two-state quantum walks are special
CMV matrices and the theory of CMV matrices applied to this class of quan-
tum walks in [1, 5] and other works. Therefore, many of our results in this paper
are essentially contained in [8]. However, since our presentations and proofs are
direct without using the theory of CMV matrices, and formulas are given in usual
representations of unitary evolutions for quantum walks. Although our approach
only works for the class of quantum walks mentioned above, the setting of our
presentation could have advantageous aspect when the quantum walks are applied
and used in areas different from pure mathematics such as information science
or quantum physics. Furthermore, it seems that a property of the Green function,
Theorem 1.4 below, is new, and it can be used to give a concrete formula, Theo-
rem 5.7, to compute the positive-matrix-valued measure, which gives directly the
spectral resolution, for a certain special simplest case of the so-called two-phase
model [3].

Now, let us prepare notation to mention some of results in the paper. All the inner
products in the paper are complex linear in the first variable and anti-complex linear
in the second. We denote the standard Hermitian inner product of the two-dimen-
sional complex vector space $\mathbb{C}^2$ by $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ and the standard orthonormal basis of $\mathbb{C}^2$
by $\{e_L, e_R\}$

$$e_L = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_R = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (1)$$

The orthogonal projection onto the one-dimensional subspaces, $C e_L$, $C e_R$, are
denoted by $\pi_L, \pi_R$. In general, the set of maps from a set $X$ to another set $Y$ is denoted
by $\text{Map}(X, Y)$. We fix $C \in \text{Map}(\mathbb{Z}, U(2))$, where $U(2)$ is the group of unitary $2 \times 2$
matrices, and define a linear map

$$U(C) : \text{Map}(\mathbb{Z}, \mathbb{C}^2) \to \text{Map}(\mathbb{Z}, \mathbb{C}^2)$$

by the following formula:
An eigenfunction expansion formula for one-dimensional…

\[ U(\mathbb{C})\Psi(n) = \pi_+ C(n+1)\Psi(n+1) + \pi_- C(n-1)\Psi(n-1), \]

(2)

where \( \Psi \in \text{Map}(\mathbb{Z}, \mathbb{C}) \), \( n \in \mathbb{Z} \). Let \( \ell^2(\mathbb{Z}, \mathbb{C}^2) \) be the Hilbert space of \( \ell^2 \)-functions whose inner product is given by

\[ \langle f, g \rangle = \sum_{n \in \mathbb{Z}} \langle f(n), g(n) \rangle_{\ell^2} \quad (f, g \in \ell^2(\mathbb{Z}, \mathbb{C}^2)). \]

The linear map \( U(\mathbb{C}) \) defined in (2) becomes a unitary operator on \( \ell^2(\mathbb{Z}, \mathbb{C}^2) \) when it is restricted to \( \ell^2(\mathbb{Z}, \mathbb{C}^2) \), and it preserves the space \( C_0(\mathbb{Z}, \mathbb{C}^2) \) of finitely supported \( \mathbb{C}^2 \)-valued functions. In this paper, we call the linear map defined in (1), (2) the quantum walk with the coin matrix \( C : \mathbb{Z} \rightarrow U(2) \). We write

\[ C(n) = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}, \quad \Delta_n = \det C(n) = a_n d_n - b_n c_n \quad (n \in \mathbb{Z}). \]

(3)

Throughout this paper, we assume the following:

\[ a_n \neq 0 \quad \text{for all } n \in \mathbb{Z}. \]

(4)

Under the assumption (4), the unitarity of the matrix \( C(n) \) causes \( d_n \neq 0 \) for any \( n \in \mathbb{Z} \).

**Theorem 1.1** [11, 13, 17] Suppose that the coin matrix \( C \) satisfies the assumption (4). For any \( n \in \mathbb{Z} \) and any \( \lambda \in \mathbb{C} \setminus \{0\} \), we define a \( 2 \times 2 \) matrix \( T_\lambda(n) \) by

\[ T_\lambda(n) = \begin{bmatrix} \frac{1}{a_{n+1}}(\lambda - \lambda^{-1}c_n b_{n+1}) - \lambda^{-1}d_n b_{n+1}a_n \\ \lambda^{-1}c_n \end{bmatrix}, \]

(5)

and a \( 2 \times 2 \) matrix \( F_\lambda(n) \) by

\[ F_\lambda(n) = \begin{cases} I & (n = 0), \\ T_\lambda(n-1)T_\lambda(n-2) \cdots T_\lambda(1)T_\lambda(0) & (n \geq 1), \\ T_\lambda(n)^{-1}T_\lambda(n+1)^{-1} \cdots T_\lambda(-2)^{-1}T_\lambda(-1)^{-1} & (n \leq -1). \end{cases} \]

(6)

For any \( u \in \mathbb{C}^2 \), we define \( \Phi_\lambda(u) \in \text{Map}(\mathbb{Z}, \mathbb{C}^2) \) by

\[ \Phi_\lambda(u)(n) = F_\lambda(n)u \quad (n \in \mathbb{Z}). \]

(7)

Then, the map \( \Phi_\lambda : \mathbb{C}^2 \rightarrow \text{Map}(\mathbb{Z}, \mathbb{C}^2) \) is injective and the eigenspace \( \mathcal{M}_\lambda \) of \( U(\mathbb{C}) \) in \( \text{Map}(\mathbb{Z}, \mathbb{C}^2) \) with an eigenvalue \( \lambda \in \mathbb{C} \setminus \{0\} \) coincides with the image of \( \Phi_\lambda \). Hence, \( \dim \mathcal{M}_\lambda = 2 \) for each such \( \lambda \). Furthermore, we have \( \dim \mathcal{M}_\lambda \cap \ell^2(\mathbb{Z}, \mathbb{C}^2) \leq 1 \) for any \( \lambda \in S^1 \).
The matrix $T_\lambda(n)$ is called the transfer matrix, and it is useful to describe various functions and quantities related to the quantum walk $U(C)$. For example, the Green function can be expressed in terms of the transfer matrix. To be more precise, let

$$R(\lambda) = (U - \lambda)^{-1} \quad (\lambda \in \mathbb{C}\setminus \sigma(U)) \quad (8)$$

be the resolvent of the restriction $U$ of $U(C)$ to $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, where $\sigma(U)$ denotes the spectrum of the operator $U$. For any $u \in \mathbb{C}^2$ and $k \in \mathbb{Z}$, we define a function $\delta_k \otimes u \in C_0(\mathbb{Z}, \mathbb{C}^2)$ by

$$(\delta_k \otimes u)(n) = \begin{cases} u & (n = k), \\ 0 & (n \neq k). \end{cases} \quad (9)$$

The Green function $R(\lambda) \in \text{Map}(\mathbb{Z}^2, M_2(\mathbb{C}))$, where $M_2(\mathbb{C})$ denotes the space of complex $2 \times 2$ matrices, is defined as

$$R(\lambda)(n, m)u = [R(\lambda)(\delta_m \otimes u)](n) \quad ((n, m) \in \mathbb{Z}^2, u \in \mathbb{C}^2, \lambda \in \mathbb{C}\setminus \sigma(U)). \quad (10)$$

Then, the Green function $R(\lambda)(n, m)$ is expressed in terms of the matrix $F_\lambda(n)$ as in the following.

**Theorem 1.2** We define $z_L, z_R \in \text{Map}(\mathbb{Z}, M_2(\mathbb{C}))$ by

$$z_L(n) = \begin{bmatrix} 1 & 0 \\ -c_n/d_n & 0 \end{bmatrix}, \quad z_R(n) = \begin{bmatrix} 0 & -b_n/a_n \\ 0 & 1 \end{bmatrix}. \quad (11)$$

We set $x_0(\lambda) = R(\lambda)(0, 0)$. Then, we have

$$R(\lambda)(n, m) = \begin{cases} F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(m)^* & (m > n), \\ F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_L(0)]F_{1/\lambda}(m)^* & (m < n), \end{cases} \quad (12)$$

and for each $n \in \mathbb{Z}$, we have

$$R(\lambda)(n, n) = F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_L(0)]F_{1/\lambda}(n)^* - \lambda^{-1}z_L(n)$$

$$= F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(n)^* - \lambda^{-1}z_R(n). \quad (13)$$

As in Theorem 1.2, the matrix-valued holomorphic function $x_0(\lambda) = R(\lambda)(0, 0)$ plays one of the central roles in the present paper. Therefore, it is important to develop methods to compute $x_0$, one of which is given by the following two theorems.

**Theorem 1.3** Let $\lambda \in \mathbb{C}\setminus \{0 \cup S^1\}$. Let $A_1(\lambda)$ (resp. $A_2(\lambda)$) be the vector subspace in $\mathbb{C}^2$ consisting of all vectors $w \in \mathbb{C}^2$ satisfying

---

1 The $2 \times 2$ matrix-valued function $T_\lambda(n)$ is introduced in [11]. There is another matrix called the transfer matrix used in [13]. See Appendix A in this paper.

© Birkhäuser
\[
\sum_{n \geq 1} \|F_\lambda(n)w\|^2_{\mathbb{C}^2} < +\infty \quad \text{resp.} \quad \sum_{n \leq -1} \|F_\lambda(n)w\|^2_{\mathbb{C}^2} < +\infty.
\]

Then, we have \( \dim A_L(\lambda) = \dim A_R(\lambda) = 1 \). In particular, we have

\[
\text{rank } [x_0(\lambda) + \lambda^{-1}z_L(0)] = \text{rank } [x_0(\lambda) + \lambda^{-1}z_R(0)] = \text{rank } R_\lambda(n, m) = 1
\]

for any \( \lambda \in \mathbb{C}\setminus\{0 \} \) and \( n, m \in \mathbb{Z} \) with \( n \neq m \).

For \( \lambda \in \mathbb{C}\setminus\{0 \} \), let \( v_\pm(\lambda) \) be unit vectors satisfying

\[
\sum_{n \geq 1} \|F_\lambda(n)v_+(\lambda)\|^2_{\mathbb{C}^2} < +\infty, \quad \sum_{n \leq -1} \|F_\lambda(n)v_-(\lambda)\|^2_{\mathbb{C}^2} < +\infty. \tag{14}
\]

The existence of these unit vectors is assured by Theorem 1.3.

**Theorem 1.4** For \( \lambda \in \mathbb{C}\setminus\{0 \} \) the unit vectors \( v_+(\lambda), v_-(\lambda) \) are linearly independent. The matrix-valued holomorphic function \( x_0(\lambda) = R_\lambda(0, 0) \) is given by

\[
x_0(\lambda)e_L = -\lambda^{-1} \left( z_L(0)e_L, v_+(\lambda)\right)_{\mathbb{C}^2} \left( v_-(\lambda), v_+(\lambda)\right)_{\mathbb{C}^2},
\]

\[
x_0(\lambda)e_R = -\lambda^{-1} \left( z_R(0)e_R, v_-(\lambda)\right)_{\mathbb{C}^2} \left( v_+(\lambda), v_-(\lambda)\right)_{\mathbb{C}^2}, \tag{15}
\]

where \( v_\pm(\lambda)^\perp \) denotes any unit vector in \( \mathbb{C}^2 \) perpendicular to \( v_\pm(\lambda) \), respectively.

The eigenfunction expansion theorem due to Weyl [27, 28], Stone [24], Titchmarsh [26], and Kodaira [10]² is regarded as an inversion formula for a generalized Fourier transform defined by eigenfunctions for Sturm–Liouville operators. Let us state an eigenfunction expansion formula for the quantum walk \( U(\mathcal{C}) \) defined by the coin matrix \( \mathcal{C} \) satisfying (4). For any \( f \in C_0(\mathbb{Z}, \mathbb{C}^2) \), we define a function \( F_\mathcal{C}[f] \) on \( \mathbb{C}\setminus\{0 \} \) by

\[
F_\mathcal{C}[f](\lambda) = \hat{\mathcal{F}}^\mathcal{C}(\lambda) := \sum_{n \in \mathbb{Z}} F_{1/\lambda}(n)^* f(n), \quad \lambda \in \mathbb{C}\setminus\{0 \}. \tag{16}
\]

The sum in (16) is finite, because the support of \( f \) is finite. Therefore, the function \( F_\mathcal{C}[f](\lambda) \) is a Laurent polynomial in \( \lambda \in \mathbb{C}\setminus\{0 \} \). We call \( F_\mathcal{C}[f] = \hat{\mathcal{F}}^\mathcal{C} \) the QW-Fourier transform of \( f \).

**Theorem 1.5** There exists a positive-matrix-valued measure \( \Sigma \) on \( S^1 \), such that we have the following.

(1) The resolvent \( R(\lambda) \) is written as

² Some historical notes can be found in [10, 18].
The positive-matrix-valued measure $\Sigma$ satisfying (17) is unique.

(2) For any $f, g \in C_0(\mathbb{Z}, \mathbb{C}^2)$, we have

$$\langle f, g \rangle = \int_{S^1} \langle d\Sigma(\zeta)\mathcal{F}_c[f](\zeta), \mathcal{F}_c[g](\zeta) \rangle_{\mathbb{C}^2}. \quad (18)$$

(3) Let

$$U = \int_{S^1} \lambda \, dE(\lambda)$$

be the spectral resolution of the unitary operator $U$, where $E$ is a projection-valued measure on $S^1$. Then, for each Borel set $A$ in $S^1$, the projection $E(A)$ on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ is written as

$$[E(A)f](n) = \int_A F_\zeta(n) d\Sigma(\zeta)\mathcal{F}_c[f](\zeta), \quad f \in C_0(\mathbb{Z}, \mathbb{C}^2). \quad (19)$$

In particular, the following inversion formula holds for $f \in C_0(\mathbb{Z}, \mathbb{C}^2)$:

$$f(n) = \int_{S^1} F_\zeta(n) d\Sigma(\zeta)\mathcal{F}_c[f](\zeta). \quad (20)$$

**Corollary 1.6** The following holds.

1. The spectrum $\sigma(U)$ coincides with the support of $\Sigma$.
2. $\lambda \in S^1$ is an eigenvalue of $U$ if and only if $\Sigma(\{\lambda\}) \neq 0$. When $\lambda$ is an eigenvalue of $U$, the projection $E(\{\lambda\})$ onto the eigenspace of $\lambda$ is given by

$$[E(\{\lambda\})f](n) = F_\zeta(n)\Sigma(\{\lambda\})\mathcal{F}_c[f](\lambda).$$

We refer the readers to [7, 8] for properties of positive-matrix-valued measures. We note that the matrix-valued function $x_0(\lambda)$ is not an $m$-Carathéodory function in the sense of [22], because our operator is unitary. Instead, we use the matrix

$$x(\lambda) = I + 2\lambda x_0(\lambda), \quad (21)$$

which is indeed an $m$-Carathéodory function. The positive-matrix-valued measure $\Sigma$ is then a boundary value of the function $x(\lambda)$ in the sense that $\Sigma$ satisfies

$$x(\lambda) = \int_{S^1} \frac{\zeta + \lambda}{\zeta - \lambda} \, d\Sigma(\zeta) \quad (|\lambda| < 1), \quad (22)$$

and $\Sigma$ is characterized as
An eigenfunction expansion formula for one-dimensional…

Let \( C(S^1, \mathbb{C}^2) \) be the space of continuous \( \mathbb{C}^2 \)-valued functions on \( S^1 \). For any \( k, l \in C(S^1, \mathbb{C}^2) \), we define

\[
\langle k, l \rangle_{\Sigma} = \int_{S^1} \langle d\Sigma(\zeta)k(\zeta), l(\zeta) \rangle_{\mathbb{C}^2}.
\]

This is a positive semi-definite Hermitian sesquilinear form on \( C(S^1, \mathbb{C}^2) \), and hence, it defines an inner product on the quotient space \( C(S^1, \mathbb{C}^2) / N \) of \( C(S^1, \mathbb{C}^2) \) by the subspace \( N = \{ k \in C(S^1, \mathbb{C}^2) \mid \|k\|_{\Sigma} = 0 \} \), where we set

\[
\|k\|_{\Sigma} = \sqrt{\langle k, k \rangle_{\Sigma}} \quad (k \in C(S^1, \mathbb{C}^2)).
\]

We denote \( L^2(S^1, \mathbb{C}^2)_{\Sigma} \) the completion of \( C(S^1, \mathbb{C}^2) / N \) by the norm on \( C(S^1, \mathbb{C}^2) / N \) naturally induced by (25). The QW-Fourier transform (16) induces a map from \( C_0(\mathbb{Z}, \mathbb{C}^2) \) to \( L^2(S^1, \mathbb{C}^2)_{\Sigma} \) which we denote \( \mathcal{F}_C \).

**Theorem 1.7** The map \( \mathcal{F}_C \) extends to a unitary operator from \( \ell^2(\mathbb{Z}, \mathbb{C}^2) \) to \( L^2(S^1, \mathbb{C}^2)_{\Sigma} \). The quantum walk \( U \) on \( \ell^2(\mathbb{Z}, \mathbb{C}^2) \) is unitarily equivalent to the operator defined by the multiplication by \( \lambda \in S^1 \) on \( L^2(S^1, \mathbb{C}^2)_{\Sigma} \), namely, we have

\[
\mathcal{F}_C[U(C)f](\lambda) = \lambda \mathcal{F}_C[f](\lambda)
\]

for any \( f \in \ell^2(\mathbb{Z}, \mathbb{C}^2) \).

The organization of the paper is as follows. In Sect. 2, we solve two equations, an inhomogeneous eigenvalue equation and its conjugate. The definition of the QW-Fourier transform (16) comes from the fact that it gives a defect of the left-inverse of \( U(C) - \lambda \) obtained by solving a conjugate equation to an inhomogeneous eigenvalue equation to be the right-inverse. See Theorem 2.6 for a precise statement. The solutions to these equations are used to prove Theorem 1.2 in Sect. 3. Some of the properties of the Green functions, such as Theorem 1.3, are proved also in Sect. 3. In Sect. 4, we give proofs of Theorems 1.5 and 1.7. We calculate the positive-matrix-valued measure \( \Sigma \) in two examples, homogeneous quantum walks, and a simplest case of the two-phase models in Sect. 5.

### 2 Inhomogeneous eigenvalue equations and its conjugate

Let \( f \in \text{Map}(\mathbb{Z}, \mathbb{C}^2) \), \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( w \in \mathbb{C}^2 \). We consider the following initial value problem:

\[
(U(C) - \lambda)\Psi = f, \quad \Psi(0) = w.
\]

\[
d\Sigma(\zeta) = w * -\lim_{r \uparrow 1} \text{Re} x(r\zeta) = w * -\lim_{r \uparrow 1} [r\zeta x_0(r\zeta) - r^{-1}\zeta x_0(r^{-1}\zeta)].
\]
Any $\Psi(a)$ with a fixed integer $a \in \mathbb{Z}$ can be chosen for an initial value, but we have chosen $a = 0$ for simplicity of notation. To prove Theorem 1.2, it is important to construct solutions to the problem (27) and its conjugate problem

$$(U(C)^* - \tilde{\lambda})\Psi = f, \quad \Psi(0) = w, \quad (28)$$

where the map

$$U(C)^* : \text{Map}(\mathbb{Z}, \mathbb{C}^2) \to \text{Map}(\mathbb{Z}, \mathbb{C}^2)$$

(29)
is the extension of the formal adjoint operator (on $C_0(\mathbb{Z}, \mathbb{C}^2)$) of $U(C)$ given by

$$(U(C)^*\Psi)(n) = C(n)\pi_L\Psi(n-1) + C(n)\pi_R\Psi(n+1) \quad (n \in \mathbb{Z}). \quad (30)$$

A brief account on the formal adjoint operator for a linear map $A : C_0(\mathbb{Z}, \mathbb{C}^2) \to \text{Map}(\mathbb{Z}, \mathbb{C}^2)$ is given in Appendix B. It is well known that $U(C)$ defines a unitary operator on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$. This property comes from the following lemma.

**Lemma 2.1** As linear maps on $\text{Map}(\mathbb{Z}, \mathbb{C}^2)$, we have $U(C)^* = U(C)^{-1}$.

**Proof** Let $f \in \text{Map}(\mathbb{Z}, \mathbb{C}^2)$. Then, we have

$$[U(C)U(C)^*f](n)$$

$$= \pi_L C(n+1)[U(C)^*f](n+1) + \pi_R C(n-1)[U(C)^*f](n-1)$$

$$= \pi_L C(n+1)[C(n+1)^*\pi_L f(n) + C(n+1)^*\pi_R f(n+2)]$$

$$+ \pi_R C(n-1)[C(n-1)^*\pi_L f(n-2) + C(n-1)^*\pi_R f(n)]$$

$$= \pi_L C(n+1)^*\pi_L f(n) + \pi_L C(n+1)^*\pi_R f(n+2) + \pi_R C(n-1)^*\pi_L f(n-2) + \pi_R C(n-1)^*\pi_R f(n) = f(n),$$

$$[U(C)^*U(C)f](n)$$

$$= C(n)^*\pi_L C(n-1)[U(C)f](n-1) + C(n)^*\pi_R C(n+1)[U(C)f](n+1)$$

$$= C(n)^*\pi_L C(n)f(n) + C(n)^*\pi_R C(n-2)f(n-2)$$

$$+ C(n)^*\pi_R C(n+2)f(n+2) + C(n)^*\pi_R C(n)f(n)$$

$$= C(n)^*\pi_L C(n)f(n) + C(n)^*\pi_R C(n)f(n)$$

$$= C(n)^*[\pi_L + \pi_R]C(n)f(n) = C(n)^*C(n)f(n) = f(n),$$

which shows the assertion. \qed

In Proposition 2.3 below, we give formulas for the solutions to the initial value problems (27) and (28). Before proceed to Proposition 2.3 and its proof, we give recurrence equations equivalent to the equations in (27) and (28).

**Lemma 2.2** The initial value problem (27) is equivalent to the equation
\[ \Psi(n + 1) = T_\lambda(n)\Psi(n) + \frac{1}{a_{n+1}}\pi_L f(n) - \frac{1}{d_n} T_\lambda(n)\pi_R f(n + 1) \] (31)

with \( \Psi(0) = w. \) The initial value problem (28) is equivalent to the equation

\[ \Psi(n + 1) = T_{1/\lambda}(n)\Psi(n) - \frac{\lambda}{a_{n+1}}\pi_L C(n + 1)f(n + 1) \]

\[+ \frac{\lambda}{a_{n+1}} \left( T_{1/\lambda}(n) - \frac{\lambda}{a_{n+1}}\pi_L \right) f(n) \] (32)

with \( \Psi(0) = w. \)

**Proof** We suppose that \( \Psi \in \text{Map}(\mathbb{Z}, \mathbb{C}^2) \) is a solution to Eq. (27) with \( \Psi(0) = w \in \mathbb{C}^2. \) We set

\[ \Psi(n) = \begin{bmatrix} \psi_L(n) \\ \psi_R(n) \end{bmatrix}, \quad f(n) = \begin{bmatrix} f_L(n) \\ f_R(n) \end{bmatrix}, \] (33)

so that Eq. (27) is written as

\[
\begin{cases}
  a_{n+1}\psi_L(n + 1) + b_{n+1}\psi_R(n + 1) = \lambda \psi_L(n) + f_L(n), \\
  c_{n-1}\psi_L(n - 1) + d_{n-1}\psi_R(n - 1) = \lambda \psi_R(n) + f_R(n).
\end{cases}
\] (34)

By the assumption (4), we can rewrite (34) as

\[
\begin{cases}
  \psi_L(n + 1) = -\frac{b_{n+1}}{a_{n+1}}\psi_R(n + 1) + \frac{\lambda}{a_{n+1}}\psi_L(n) + \frac{1}{a_{n+1}}f_L(n), \\
  \psi_R(n - 1) = -\frac{c_{n-1}}{d_{n-1}}\psi_L(n - 1) + \frac{\lambda}{d_{n-1}}\psi_R(n) + \frac{1}{d_{n-1}}f_R(n).
\end{cases}
\] (35)

Shifting the variable \( n \) in (34), we have

\[
\begin{cases}
  a_n\psi_L(n) + b_n\psi_R(n) = \lambda \psi_L(n - 1) + f_L(n - 1), \\
  c_n\psi_L(n) + d_n\psi_R(n) = \lambda \psi_R(n + 1) + f_R(n + 1).
\end{cases}
\] (36)

Solving the second equation of (36) in \( \psi_R(n + 1) \), we have

\[ \psi_R(n + 1) = \lambda^{-1}c_n \psi_L(n) + \lambda^{-1}d_n \psi_R(n) - \lambda^{-1}f_R(n + 1). \] (37)

Substituting (37) into the first equation of (35), we see

\[
\begin{align*}
\psi_L(n + 1) &= \frac{1}{a_{n+1}} (\lambda - \lambda^{-1}b_{n+1}c_n) \psi_L(n) - \lambda^{-1}b_{n+1}d_n \psi_R(n) \\
&\quad + \lambda^{-1}b_{n+1}f_R(n + 1) + \frac{1}{d_n} f_L(n).
\end{align*}
\] (38)
Equations (37) and (38) give Eq. (31). Next, we suppose that $\Psi \in \text{Map} (\mathbb{Z}, \mathbb{C}^2)$ satisfies (31) and $\Psi(0) = w$. We write $\Psi$ as in (33). Then, $\psi_L, \psi_R$ satisfy (37) and (38). Solving (37) in $f_R(n+1)$ and substituting the result into (38), we obtain the first equation in (35). Shifting the variable $n$ to $n + 1$ in (37), we have the second equation in (35). Therefore, $\Psi$ satisfies (35). Since (35) is equivalent to the equation in (27), $\Psi$ solves the initial value problem (27).

Next, we assume that $\Psi \in \text{Map} (\mathbb{Z}, \mathbb{C}^2)$ solves the initial value problem (28), and we write $\Psi$ as in (33). Equation (28) is equivalent to the equation

$$\pi_L \Psi(n) + \pi_R \Psi(n + 1) = C(n) \left[ \lambda \Psi(n) + f(n) \right]$$

with $\Psi(0) = w$, which is written as

$$\left\{ \begin{array}{l}
\psi_L(n) = \frac{\lambda}{a_{n+1}} \psi_L(n + 1) + \frac{1}{\lambda} b_{n+1} \psi_R(n + 1) + a_{n+1} f_L(n + 1), \\
\psi_R(n) = \frac{\lambda}{c_{n-1}} \psi_L(n - 1) + \frac{1}{\lambda} d_{n-1} \psi_R(n - 1) + c_{n-1} f_L(n - 1). 
\end{array} \right. \quad (39)$$

Shifting the variable $n$ in (39), we see

$$\left\{ \begin{array}{l}
\psi_L(n) = \frac{\lambda}{a_{n+1}} \psi_L(n + 1) + \frac{1}{\lambda} b_{n+1} \psi_R(n + 1) + a_{n+1} f_L(n + 1), \\
\psi_R(n) = \frac{\lambda}{c_{n-1}} \psi_L(n - 1) + \frac{1}{\lambda} d_{n-1} \psi_R(n - 1) + c_{n-1} f_L(n - 1). 
\end{array} \right. \quad (40)$$

Solving the first equation of (40) in $\psi_L(n + 1)$, we see

$$\psi_L(n + 1) = \frac{\lambda}{a_{n+1}} \psi_L(n) - \frac{b_{n+1}}{a_{n+1}} \psi_R(n + 1)$$

$$- \frac{1}{\lambda} a_{n+1} f_L(n + 1) - \frac{1}{\lambda} b_{n+1} f_R(n + 1). \quad (41)$$

Substituting the second equation of (39) into (41) shows

$$\psi_L(n + 1) = \frac{1}{a_{n+1}} \left( \frac{\lambda}{\lambda} - \frac{\lambda b_{n+1} c_{n}}{a_{n+1}} \right) \psi_L(n) - \frac{b_{n+1} d_{n}}{a_{n+1}} \psi_R(n) - \frac{b_{n+1} c_{n}}{a_{n+1}} f_L(n)$$

$$- \frac{b_{n+1} d_{n}}{a_{n+1}} f_R(n) - \frac{1}{\lambda} a_{n+1} f_L(n + 1) - \frac{1}{\lambda} b_{n+1} f_R(n + 1). \quad (42)$$

Since the combination of the two equations, (42) and the second line of (39), is equivalent to Eq. (32), $\Psi$ solves (32). Conversely, we suppose that $\Psi \in \text{Map} (\mathbb{Z}, \mathbb{C}^2)$ satisfies (32) with $\Psi(0) = w$. Thus, $\Psi$ satisfies (42) and the second line of (39). From these two equations, we have (41). Shifting the variable $n$ in (41) to $n - 1$ and solving it in $\psi_L(n - 1)$, we have the first line of (39). Therefore, $\Psi$ solves Eq. (28).
Lemma 2.2 is used to deduce the concrete formulas of the functions $v_\lambda(n, m)$, $w_\lambda^0(n, m)$ defined in (44), (47) below which give solutions to the problems (27), (28). Indeed, if we give an initial value $\Psi(0) = w$, then the function $\Psi$ on the set of non-negative integers satisfying Eq. (32) is automatically determined. Shifting the variable $n$ to $n - 1$ in (32) and solving it in $\Psi(n - 1)$, we see

$$
\Psi(n - 1) = T_1/\lambda(n - 1)^{-1}\Psi(n) + \frac{1}{a_n}T_1/\lambda(n - 1)^{-1}\pi_L C(n)f(n)
$$

Thus, once we give $\Psi(0) = w$, the function $\Psi$ on the set of non-negative integers satisfying Eq. (32) is automatically determined. Therefore, in principle, we can solve the Eq. (28). Similar discussion is also applicable for (27) using (31). The formulas of the solutions given in Proposition 2.3 are deduced from the concrete form of $\Psi(n)$ obtained using Eqs. (31) and (32) for several small $n \in \mathbb{Z}$ in the absolute value.

**Proposition 2.3**

(1) We define a function $v_\lambda \in \text{Map}(\mathbb{Z}^2, M_2(\mathbb{C}))$ by the following formula:

$$
v_\lambda(n, 0) = \begin{cases} 
\lambda^{-1} F_\lambda(n) z_L(0) & (n \geq 1), \\
\lambda^{-1} F_\lambda(n) z_R(0) & (n \leq -1), 
\end{cases}
$$

$$
v_\lambda(n, m) = \begin{cases} 
\lambda^{-1} F_\lambda(n) F_\lambda(m)^{-1}(z_L(m) - z_R(m)) & (1 \leq m \leq n - 1), \\
-\lambda^{-1} z_R(n) & (1 \leq m = n), \\
\lambda^{-1} F_\lambda(m) F_\lambda(m)^{-1}(z_R(m) - z_L(m)) & (n + 1 \leq m \leq -1), \\
-\lambda^{-1} z_L(n) & (n = m \leq -1), \\
0 & (\text{otherwise}).
\end{cases}
$$

(44)

Then, for each $f \in \text{Map}(\mathbb{Z}, \mathbb{C}^2)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}^2$, Eq. (27) has a unique solution given by

$$
\Psi = \Phi_\lambda(w) + V_\lambda f,
$$

(45)

where $\Phi_\lambda(w)$ is defined in (7) and $V_\lambda f$ is defined by

$$
V_\lambda f(n) = \sum_{m \in \mathbb{Z}} v_\lambda(n, m)f(m).
$$

(46)

(2) We define a function $w_\lambda^0 \in \text{Map}(\mathbb{Z}^2, M_2(\mathbb{C}^2))$ by the following formula:
\[ w_\lambda''(n, 0) = \begin{cases} \frac{1}{\lambda} F_{1/\lambda}(n)z_R(0)^* & (n \geq 1), \\ \frac{1}{\lambda} F_{1/\lambda}(n)z_L(0)^* & (n \leq -1), \end{cases} \]

\[ w_\lambda''(n, m) = \begin{cases} \frac{1}{\lambda} F_{1/\lambda}(n)F_{1/\lambda}(m)^{-1}(z_R(m)^* - z_L(m)^*) & (1 \leq m \leq n - 1), \\ -\frac{1}{\lambda} z_L(n)^* & (1 \leq m = n), \\ \frac{1}{\lambda} F_{1/\lambda}(n)F_{1/\lambda}(m)^{-1}(z_L(m)^* - z_R(m)^*) & (n + 1 \leq m \leq -1), \\ -\frac{1}{\lambda} z_R(n)^* & (n = m \leq -1), \\ 0 & \text{(otherwise)}. \end{cases} \] (47)

Then, for each \( f \in \text{Map}(\mathbb{Z}, \mathbb{C}^2), \) \( \lambda \in \mathbb{C}\backslash\{0\} \) and \( w \in \mathbb{C}^2, \) Eq. (28) has a unique solution given by

\[ \Psi = \Phi_{1/\lambda}(w) + W'_{\lambda}f, \] (48)

where \( W'_{\lambda}f \) is defined by

\[ W'_{\lambda}f(n) = \sum_{m \in \mathbb{Z}} w_\lambda''(n, m)f(m). \] (49)

We need some of the following formulas to prove Proposition 2.3.

**Lemma 2.4** For any \( \lambda \in \mathbb{C}\backslash\{0\}, n \in \mathbb{Z}, \) we have the following:

1. \( F_{1/\lambda}(n + 1) = T_\lambda(n)F_{1/\lambda}(n), \)
2. \( \pi_L \mathcal{C}(n)T_\lambda(n - 1) = \lambda \pi_L, \)
3. \( \pi_R \mathcal{C}(n)T_\lambda(n)^{-1} = \lambda \pi_R, \)
4. \( z_L(n) = \lambda \frac{d_n}{d_{n+1}} T_\lambda(n)^{-1} \pi_L = \frac{\Delta_n}{d_n} \mathcal{C}(n)^* \pi_L = \frac{1}{d_n} \mathcal{C}(n)^* \pi_L, \)
5. \( z_R(n) = \frac{\lambda}{d_{n-1}} T_\lambda(n - 1) \pi_R = \frac{\Delta_n}{d_n} \mathcal{C}(n)^* \pi_R = \frac{1}{d_n} \mathcal{C}(n)^* \pi_R, \)
6. \( z_L(n)^* + z_R(n) = z_L(n) + z_R(n)^* = I, \)
7. \( a_n z_L(n)^* + d_n z_R(n)^* = \pi_L \mathcal{C}(n)z_L(n)^* + \pi_R \mathcal{C}(n)z_R(n)^* = \mathcal{C}(n), \)
8. \( T_\lambda(n)[z_L(n) - z_R(n)]T_{1/\lambda}(n)^* = [z_L(n + 1) - z_R(n + 1)]. \)

**Proof** (1) follows from (6), (2) and the first two equalities in (5) follow from (5), (11) and the unitarity of \( \mathcal{C}(n). \) The inverse \( T_{1/\lambda}(n)^{-1} \) of the matrix \( T_\lambda(n) \) is given by

\[ T_{1/\lambda}(n)^{-1} = \begin{bmatrix} c_n d_n a_n & d_n \lambda^{-1} b_{n+1} \\ -\frac{1}{d_n} (\lambda - \lambda^{-1} b_{n+1} c_n) \end{bmatrix}. \] (50)

From this and the unitarity of \( \mathcal{C}(n), \) (3) and the first two equalities in (4) follow. Since \( \mathcal{C}(n) \) is unitary, we have
\[ \triangle_n a_n = d_n, \quad \triangle_n b_n = -c_n, \quad \triangle_n c_n = -b_n, \quad \triangle_n d_n = a_n. \] (51)

This and (11) show (6). By a direct computation using (11) and (51), we see

\[
a_n z_L(n)^* = \pi_L \mathcal{C}(n) z_L(n)^* = \begin{bmatrix} a_n & b_n \\ 0 & 0 \end{bmatrix},
\]

\[
d_n z_R(n)^* = \pi_R \mathcal{C}(n) z_R(n)^* = \begin{bmatrix} 0 & 0 \\ c_n & d_n \end{bmatrix},
\]

from which the item (7) follows. To prove (8), we first note that, by (51), the matrix \( T_{1/\lambda}(n)^* \) is written as

\[
T_{1/\lambda}(n)^* = \begin{bmatrix} \frac{\triangle_{n+1}}{d_{n+1}} \left( \lambda^{-1} - \frac{\lambda b_n c_{n+1}}{\triangle_n \triangle_{n+1}} \right) & -\frac{\lambda}{\triangle_n} b_n \\ \lambda a_n c_{n+1} & \lambda a_n \end{bmatrix}.
\] (52)

From this and the item (4), we have

\[
T_\lambda(n) z_L(n) T_{1/\lambda}(n)^* = \frac{\lambda}{a_{n+1}} \pi_L T_{1/\lambda}(n)^*
\]

\[
= \begin{bmatrix} \frac{\triangle_{n+1}}{a_{n+1} d_{n+1}} - \lambda^2 b_n c_{n+1} / a_{n+1} d_{n+1} & -\lambda^2 b_n \\ 0 & \lambda^2 b_n / a_{n+1} d_{n+1} \end{bmatrix}.
\] (53)

Using the concrete form for \( T_{1/\lambda}(n)^* \) mentioned above, we also have

\[
z_R(n) T_{1/\lambda}(n)^* = \frac{\lambda}{\triangle_n d_{n+1}} \begin{bmatrix} -b_n c_{n+1} / a_{n+1} & -b_n \\ a_{n+1} c_{n+1} & a_{n+1} d_{n+1} \end{bmatrix}.
\]

By a direct computation using the definition of \( T_\lambda(n) \) and the above formula for \( z_R(n) T_{1/\lambda}(n)^* \), we have

\[
T_\lambda(n) z_R(n) T_{1/\lambda}(n)^*
\]

\[
= \begin{bmatrix} -\lambda^2 b_n c_{n+1} / a_{n+1} d_{n+1} & -\lambda^2 b_n c_{n+1} / a_{n+1} \\ \lambda^2 b_n / a_{n+1} d_{n+1} & \lambda^2 b_n / a_{n+1} \end{bmatrix}.
\] (54)

Subtracting (54) from (53), we conclude (8).

**Proof of Proposition 2.3** According to (11), the matrices \( z_L(n), z_R(n), z_L(n) - z_R(n) \), and their adjoints are all nonzero for any \( n \in \mathbb{Z} \). Since the matrix \( F_\lambda(n) \) is nonsingular for any \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( n \in \mathbb{Z} \), \( v_\lambda(n, m) \) and \( w_\lambda(n, m) \) are nonzero if and only if \( n \neq 0 \) and \( m \) lies between 0 and \( n \). In particular, the sums in (46) and (49) are finite. Thus, \( V_\lambda f \) and \( W_\lambda^2 f \) are well defined as elements in \( \text{Map}(\mathbb{Z}, \mathbb{C}^2) \) for any \( f \in \text{Map}(\mathbb{Z}, \mathbb{C}^2) \).

First, we show the uniqueness of the solution to each of the initial value problems (27) and (28). Suppose that \( \Psi_1, \Psi_2 \in \text{Map}(\mathbb{Z}, \mathbb{C}^2) \) solve the initial value problem (27). Then, we have \( \Psi_1(0) = \Psi_2(0) = w \in \mathbb{C}^2 \). We set \( \Psi = \Psi_1 - \Psi_2 \). Then, \( \Psi \) satisfies
(U(\mathcal{C}) - \lambda) \Psi = 0 \text{ with } \Psi(0) = 0. \text{ According to Theorem 1.1, } \Psi \text{ is in the image of the map } \Phi_\lambda : \mathbb{C}^2 \to \text{Map}(\mathbb{Z}, \mathbb{C}^2) \text{ defined in (7). Thus, we can write } \Psi = \Phi_\lambda(u) \text{ with some } u \in \mathbb{C}^2. \text{ Since } F_\lambda(0) = 1, \text{ we see } 0 = \Psi(0) = \Phi_\lambda(u)(0) = F_\lambda(0) u = u, \text{ and hence, } \Psi = \Phi_\lambda(0) = 0. \text{ This shows that the solution to the initial value problem (27) is unique. Next, suppose that } \Psi_1, \Psi_2 \text{ are solutions to the initial value problem (28). Then, we have } \Psi_1(0) = \Psi_2(0) = w. \text{ We set } \Psi = \Psi_1 - \Psi_2. \text{ Then, } \Psi \text{ satisfies } (U(\mathcal{C}) - \bar{\lambda}) \Psi = 0 \text{ with } \Psi(0) = 0. \text{ Applying } U(\mathcal{C}) \text{ to the equation } U(\mathcal{C})^* \Psi = \bar{\lambda} \Psi \text{ and using Lemma 2.1, we have } \Psi = \bar{\lambda} U(\mathcal{C}) \Psi \text{ or, what is the same to say, } U(\mathcal{C}) \Psi = \bar{\lambda}^{-1} \Psi, \text{ because } \bar{\lambda} \neq 0 \text{ as assumed. Therefore, by Theorem 1.1, } \Psi \text{ is in the image of the map } \Phi_{1/\bar{\lambda}} : \mathbb{C}^2 \to \text{Map}(\mathbb{Z}, \mathbb{C}^2), \text{ and hence, we can write } \Psi = \Phi_{1/\bar{\lambda}}(u) \text{ with some } u \in \mathbb{C}^2. \text{ Then, we have } 0 = \Psi(0) = \Phi_{1/\bar{\lambda}}(u)(0) = u \text{ by exactly the same discussion as before. Therefore, we see } \Psi = \Phi_{1/\bar{\lambda}}(0) = 0, \text{ which shows that the solution to (28) is unique.}

Next, we check that the function defined by (45) (resp. (48)) solves the initial value problem (27) (resp. (28)). Since, by Theorem 1.1, we have \(U(\mathcal{C}) - \lambda) \Phi_\lambda(w) = (U(\mathcal{C}) - \bar{\lambda}) \Phi_{1/\bar{\lambda}}(w) = 0\) and \(\Phi_\lambda(w)(0) = \Phi_{1/\bar{\lambda}}(w)(0) = w\), it is enough to check that the function \(\Psi = V_\lambda f\) (resp. \(\Psi = W_\lambda f\)) solves the initial value problem (27) (resp. (28)) with the initial value \(w = 0\). The definitions (44) and (47) show that \(v_\lambda(0, m) = w_\lambda(0, m) = 0\) for any \(m \in \mathbb{Z}\), and hence we see \(V_\lambda f(0) = W_\lambda f(0) = 0\). Thus, by Lemma 2.2, we need only to show that \(\Psi = V_\lambda f\) (resp. \(\Psi = W_\lambda f\)) satisfies the recurrence equation (31) (resp. (32)). Before proceeding to the proof that \(V_\lambda f\) satisfies (31), it is useful to give concrete formulas for \(T_\lambda v_\lambda(n, m)\). Namely, we have

\[
T_\lambda(n)v_\lambda(n, 0) = \begin{cases} 
\nu_\lambda(n + 1, 0) & (n \geq 1), \\
0 & (n = 0), \\
\lambda^{-1}z_R(0) & (n = -1), \\
\nu_\lambda(n + 1, 0) & (n \leq -2), 
\end{cases}
\]

\[
T_\lambda(n)v_\lambda(n, m) = \begin{cases} 
v_\lambda(n + 1, m) & (1 \leq m \leq n - 1), \\
-\lambda^{-1}T_\lambda(n)z_R(n) & (1 \leq m = n), \\
\lambda^{-1}(z_R(n + 1) - z_L(n + 1)) & (n + 1 = m \leq -1), \\
v_\lambda(n + 1, m) & (n + 2 \leq m \leq -1), \\
-\frac{1}{a_{m+1}} \pi_L & (n = m \leq -1), \\
0 & (\text{otherwise}).
\end{cases}
\]

These can be obtained directly from the definition (44) and the items (1), (4) in Lemma 2.4. (The item (4) in Lemma 2.4 is used only to obtain the formula when \(n = m \leq -1\).) In what follows, we use the notation, for example, ‘\(1/\lambda\)’ to indicate that the item (1) in Lemma 2.4 is used to show the equality. We now show that \(\Psi = V_\lambda f\) satisfies (31). We have
An eigenfunction expansion formula for one-dimensional…

To prove (31)

we consider the case

which shows that the function

which shows that

with

the right-hand side of (31) using (55) as follows:

for

satisfies (31) when

Next,

we calculate

Next,

Next,

which coincides with

and hence, \( V_\lambda f \) satisfies (31) when \( n = -1 \). Next, we consider the case \( n \geq 1 \). We have

which shows that the function \( V_\lambda f \) satisfies (31) for \( n \geq 1 \). For \( n \leq -3 \), we calculate the right-hand side of (31) using (55) as follows:
\[T_\lambda(n) \Psi = (\neq)\]

\[
T_\lambda(n) \Psi = \left(\frac{1}{a_{n+1}} + \frac{1}{a_{n+1}} \pi_L f(n) - \frac{1}{d_n} T_\lambda(n) \pi_R f(n + 1)\right) + T_\lambda(n) v_\lambda(n, n + 1) f(n + 1) + \sum_{m = n + 2}^0 T_\lambda(n) v_\lambda(n, m) f(m)
\]

\[\text{(2.29)}\]

\[
T_\lambda(n) \pi_R f(n + 1) + \lambda^{-1} (z_R(n + 1) - z_L(n + 1)) f(n + 1) + \sum_{m = n + 2}^0 v_\lambda(n + 1, m) f(m) = \frac{1}{d_n} T_\lambda(n) \pi_R f(n + 1) + \lambda^{-1} z_R(n + 1) f(n + 1) + \sum_{m = n + 1}^0 v_\lambda(n + 1, m) f(m)
\]

\[\text{(5)}\]

For \(n = -2\), also by (55), we see

\[
T_\lambda(-2) V_\lambda f(-2) + \frac{1}{a_{-1}} \pi_L f(-2) - \frac{1}{d_{-2}} T_\lambda(-2) \pi_R f(-1) = \sum_{m = -2}^0 T_\lambda(-2) v_\lambda(-2, m) f(m) + \frac{1}{a_{-1}} \pi_L f(-2) - \frac{1}{d_{-2}} T_\lambda(-2) \pi_R f(-1)
\]

\[\text{(2.29)}\]

\[
v_\lambda(-1, 0) f(0) + \lambda^{-1} (z_R(-1) - z_L(-1)) f(-1) = \frac{1}{d_{-2}} T_\lambda(-2) \pi_R f(-1)
\]

\[\text{(5)}\]

\[
v_\lambda(-1, 0) f(0) + v_\lambda(-1, -1) f(-1) = V_\lambda f(-1).
\]

This shows that \(V_\lambda f\) satisfies (31) for all \(n \in \mathbb{Z}\). Next, to show that the function \(\Psi = W_\lambda^o f\) satisfies (32), we prepare the formulas for \(T_{1/\lambda}^o w_\lambda^o(n, m)\) as follows:

\[
T_{1/\lambda}^o(n) w_\lambda^o(n, m) = \begin{cases} 
  w_\lambda^o(n + 1, 0) & (n \geq 1), \\
  0 & (n = 0), \\
  \frac{1}{\lambda} z_L(0)^* & (n = -1), \\
  w_\lambda^o(n + 1, 0) & (n \leq -2).
\end{cases}
\]

\[\text{(56)}\]
For \( n = 0 \), we have

\[
W_\alpha^n f(1) = w_\alpha^n(1, 1)f(1) + w_\alpha^n(1, 0)f(0)
\]

\[
= -\frac{\lambda}{a_0} z_L(1)^*f(1) + \frac{\lambda}{a_1} T_{1/\lambda}(1)z_R(0)^*f(0)
\]

\[ (4),(1) \]

\[
= -\frac{\lambda}{a_0} \pi_L C(1)f(1) + \frac{\lambda}{a_1} T_{1/\lambda}(0)z_R(0)^*f(0)
\]

\[ (6) \]

\[
= -\frac{\lambda}{a_0} \pi_L C(1)f(1) + \frac{\lambda}{a_1} \left( T_{1/\lambda}(0) - \frac{\lambda}{a_1} \pi_L \right) f(0),
\]

which shows that \( W_\alpha^n f \) satisfies (32) for \( n = 0 \). For \( n = -1 \), we calculate the right-hand side of (32) with \( n = -1 \), by (56), as

\[
T_{1/\lambda}(-1)W_\alpha^n f(-1) = \frac{\lambda}{a_0} \pi_L C(0) f(0) + \frac{\lambda}{a_1} \left( T_{1/\lambda}(-1) - \frac{\lambda}{a_1} \pi_L \right) f(-1)
\]

\[
= T_{1/\lambda}(-1)(w_\lambda^n(-1, -1)f(-1) + w_\lambda^n(-1, 0)f(0))
\]

\[
- \frac{\lambda}{a_0} \pi_L C(0) f(0) + \frac{\lambda}{a_1} \left( T_{1/\lambda}(-1) - \frac{\lambda}{a_1} \pi_L \right) f(-1)
\]

\[ (2,30) \]

\[
= -\lambda \frac{1}{a_0} \pi_L C(0) f(0) + \lambda \left( T_{1/\lambda}(-1) - \frac{\lambda}{a_1} \pi_L \right) f(-1)
\]

\[ (57) \]

\[
= \lambda \left( T_{1/\lambda}(-1)(I - z_R(-1)^*) - \frac{\lambda}{a_1} \pi_L \right) f(-1)
\]

\[
+ \lambda \left( z_L(0)^* - \frac{1}{a_0} \pi_L C(0) \right) f(0).
\]

By the item (4) in Lemma 2.4, we have \( z_L(0)^* = \frac{1}{a_0} \pi_L C(0) \). By the item (6) in Lemma 2.4, we have \( I - z_R(-1)^* = z_L(-1) \). Substituting these formulas into (57), we obtain

\[
T_{1/\lambda}(-1)W_\alpha^n f(-1) = \frac{\lambda}{a_0} \pi_L C(0) f(0) + \lambda \left( T_{1/\lambda}(-1) - \frac{\lambda}{a_1} \pi_L \right) f(-1)
\]

\[
= \lambda \left( T_{1/\lambda}(-1)z_L(-1) - \frac{\lambda}{a_1} \pi_L \right) f(-1) \quad \text{(4)} = 0.
\]

This shows that \( W_\alpha^n f \) satisfies (32) for \( n = -1 \), because \( W_\alpha^n f(0) = 0 \). Let \( n \geq 1 \). We have
\[ W^\omega_{\lambda} f(n + 1) = \sum_{m=0}^{n-1} w^\omega_{\lambda}(n + 1, m) f(m) + w^\omega_{\lambda}(n + 1, n) f(n) + w^\omega_{\lambda}(n + 1, n + 1) f(n + 1) \]

\[
(2.30)_1 \begin{array}{c}
\sum_{m=0}^{n-1} T_{1/\lambda}(n) w^\omega_{\lambda}(n, m) f(m) \\
+ \lambda^{-1} T_{1/\lambda}(n) (z_R(n) - z_L(n)) f(n) - \lambda^{-1} z_L(n + 1)^* f(n + 1)
\end{array}
\]

\[
(2.21) \begin{array}{c}
T_{1/\lambda}(n) W^\omega_{\lambda} f(n) + \lambda^{-1} T_{1/\lambda}(n) z_R(n)^* f(n) - \lambda^{-1} z_L(n + 1)^* f(n + 1)
\end{array}
\]

\[
(6), (4) \begin{array}{c}
T_{1/\lambda}(n) W^\omega_{\lambda} f(n) + \lambda^{-1} \left( T_{1/\lambda}(n) - \frac{\lambda^{-1}}{a_{n+1}} \pi_L \right) f(n) - \frac{\lambda^{-1}}{a_{n+1}} \pi_L C(n + 1) f(n + 1)
\end{array}
\]

which shows that \( W^\omega_{\lambda} f \) satisfies \((32)\) for \( n \geq 1 \). When \( n \leq -2 \), we calculate the right-hand side of \((32)\) for \( \Psi = W^\omega_{\lambda} f \) as

\[
T_{1/\lambda}(n) W^\omega_{\lambda} f(n) - \frac{\lambda^{-1}}{a_{n+1}} \pi_L C(n + 1) f(n + 1) + \lambda^{-1} \left( T_{1/\lambda}(n) - \frac{\lambda^{-1}}{a_{n+1}} \pi_L \right) f(n)
\]

\[
+ \sum_{m=n}^{0} T_{1/\lambda}(n) w^\omega_{\lambda}(n, m) f(m)
\]

\[
(2.30) \begin{array}{c}
\lambda^{-1} \left( T_{1/\lambda}(n) - \frac{\lambda^{-1}}{a_{n+1}} \pi_L \right) f(n) - \frac{\lambda^{-1}}{a_{n+1}} \pi_L C(n + 1) f(n + 1)
\end{array}
\]

\[
+ \sum_{m=n+2}^{0} w^\omega_{\lambda}(n + 1, m) f(m) - \lambda^{-1} T_{1/\lambda}(n) z_R(n)^* f(n)
\]

\[
+ \lambda^{-1} (z_L(n + 1)^* - z_R(n + 1)^*) f(n + 1)
\]

\[
(2.21), (6) \begin{array}{c}
\lambda^{-1} \left( T_{1/\lambda}(n) - \frac{\lambda^{-1}}{a_{n+1}} \pi_L \right) f(n) - \lambda^{-1} T_{1/\lambda}(n) (I - z_L(n)) f(n)
\end{array}
\]

\[
+ \sum_{m=n+1}^{0} w^\omega_{\lambda}(n + 1, m) f(m)
\]

\[
- \frac{\lambda^{-1}}{a_{n+1}} \pi_L C(n + 1) f(n + 1) + \lambda^{-1} z_L(n + 1)^* f(n + 1)
\]

\[
(4) \begin{array}{c}
W^\omega_{\lambda} f(n + 1)
\end{array}
\]
This shows that $W_\lambda^o f$ satisfies (32) for any $n \in \mathbb{Z}$. \hfill \blacksquare

**Corollary 2.5** For $\lambda \in \mathbb{C}\setminus\{0\}$, we define $w_\lambda \in \text{Map} (\mathbb{Z}^2, M_2(\mathbb{C}))$ by

$$w_\lambda(n, m) = w_\lambda^o(m, n)^* \quad (n, m \in \mathbb{Z}).$$

(58)

Let $W_\lambda : C_0(\mathbb{Z}, \mathbb{C}^2) \rightarrow \text{Map} (\mathbb{Z}, \mathbb{C}^2)$ be a map defined by

$$(W_\lambda f)(n) = \sum_{m \in \mathbb{Z}} w_\lambda(n, m) f(m) \quad (f \in C_0(\mathbb{Z}, \mathbb{C}^2)).$$

(59)

Then, $W_\lambda f$ satisfies

$$W_\lambda (U(\mathbb{C}) - \lambda)f = f$$

(60)

for any $f \in C_0(\mathbb{Z}, \mathbb{C}^2)$.

**Proof** We first note that $W_\lambda$ defined in (59) is the formal adjoint operator of the linear map $W_\lambda^o : C_0(\mathbb{Z}, \mathbb{C}^2) \rightarrow \text{Map} (\mathbb{Z}, \mathbb{C}^2)$.\footnote{We take $f, g \in C_0(\mathbb{Z}, \mathbb{C}^2)$.} Then, $(U(\mathbb{C}) - \lambda)f \in C_0(\mathbb{Z}, \mathbb{C}^2)$. By Proposition 2.3, (2) and taking $w = 0$ in (48), we have $(U(\mathbb{C})^* - \lambda W_\lambda^o g = g$. Since $W_\lambda$ is the formal adjoint operator of $W_\lambda^o : C_0(\mathbb{Z}, \mathbb{C}^2) \rightarrow \text{Map} (\mathbb{Z}, \mathbb{C}^2)$, we have

$$\langle W_\lambda^o (U(\mathbb{C}) - \lambda)f, g \rangle = \langle (U(\mathbb{C}) - \lambda)f, W_\lambda^o g \rangle = \langle f, (U(\mathbb{C})^* - \lambda) W_\lambda^o g \rangle = \langle f, g \rangle.$$

Since $f, g \in C_0(\mathbb{Z}, \mathbb{C}^2)$ is arbitrary, we have (60). \hfill \blacksquare

The function $w_\lambda$ is given explicitly by the following:

$$w_\lambda(0, m) = \begin{cases} \lambda^{-1} z_R(0) F_{1/\lambda}(m)^* & (m \geq 1), \\ -\lambda^{-1} z_L(0) F_{1/\lambda}(m)^* & (m \leq -1), \\ \end{cases}$$

$$w_\lambda(n, m) = \begin{cases} \lambda^{-1} (z_R(n) - z_L(n)) F_{1/\lambda}(n)^* F_{1/\lambda}(m)^* & (1 \leq n \leq m - 1), \\ -\lambda^{-1} z_L(n) & (1 \leq n = m), \\ \lambda^{-1} (z_L(n) - z_R(n)) F_{1/\lambda}(n)^* F_{1/\lambda}(m)^* & (m + 1 \leq n \leq -1), \\ -\lambda^{-1} z_R(n) & (m = n \leq -1), \\ 0 & \text{otherwise}. \end{cases}$$

(61)

One of the most important properties of the operator $W_\lambda$ is the following.

**Theorem 2.6** For any $f \in C_0(\mathbb{Z}, \mathbb{C}^2)$, we have $W_\lambda f \in C_0(\mathbb{Z}, \mathbb{C}^2)$ and

$$(U - \lambda) W_\lambda f(n) = f(n) - (\delta_0 \otimes \widehat{f}(\lambda))(n) \quad (n \in \mathbb{Z}),$$

(62)

where the QW-Fourier transform $\widehat{f}(\lambda)$ of $f$ is defined in (16), and the function $\delta_m \otimes u \in C_0(\mathbb{Z}, \mathbb{C}^2)$ with $m \in \mathbb{Z}, u \in \mathbb{C}^2$ is defined in (9).

\footnote{See Appendix B for the notion of formal adjoint operators and their construction.}
\textbf{Proof} For fixed \( m \in \mathbb{Z} \), the function \( w_\lambda(n, m) \) in \( n \) can be nonzero only when \( n \) lies between 0 and \( m \). Therefore, we have \( W_\lambda f \in C_0(\mathbb{Z}, \mathbb{C}^2) \) for any \( f \in C_0(\mathbb{Z}, \mathbb{C}^2) \). Let \( f \in C_0(\mathbb{Z}, \mathbb{C}^2) \) and \( n \geq 2 \). By (4) in Lemma 2.4 and (61), we have

\[
\pi_R C(n - 1) w_\lambda (n - 1, n - 1) = \lambda^{-1} \pi_R C(n - 1) z_L (n - 1),
\]

by (2), (4), (5) in Lemma 2.4, we also have for \( m \geq n \)

\[
\pi_R C(n - 1) w_\lambda (n - 1, m) = \lambda^{-1} \pi_R C(n - 1) (z_R(n - 1) - z_L(n - 1)) F_{\frac{n}{\lambda}}(n - 1)^{s-1} F_{\frac{1}{\lambda}}(m)^s,
\]

and for \( m \geq n + 2 \)

\[
\pi_L C(n + 1) w_\lambda (n + 1, m) = \lambda^{-1} \pi_L C(n + 1) (z_R(n + 1) - z_L(n + 1)) \times F_{\frac{n}{\lambda}}(n + 1)^{s-1} F_{\frac{1}{\lambda}}(m)^s,
\]

Thus, the formula for \( m = n + 1 \) in (63) can be regarded as a special case of (64) and (65). We have

\[
\pi_L C(n + 1) W_\lambda f(n + 1) = -\lambda^{-1} \pi_L T_{\frac{n}{\lambda}}(n)^{s-1} F_{\frac{n}{\lambda}}(n)^{s-1} \sum_{m \geq n + 1} F_{\frac{1}{\lambda}}(m)^s f(m),
\]

\[
\pi_R C(n - 1) W_\lambda f(n - 1) = \lambda^{-1} \pi_R T_{\frac{n}{\lambda}}(n - 1)^{s-1} F_{\frac{n}{\lambda}}(n - 1)^{s-1} \sum_{m \geq n} F_{\frac{1}{\lambda}}(m)^s f(m).
\]
where we have used (1) in Lemma 2.4. Thus, \(U(C)W_\lambda f(n)\) can be calculated as follows:
\[
U(C)W_\lambda f(n) = \pi_L C(n + 1)W_\lambda f(n + 1) + \pi_R C(n - 1)W_\lambda f(n - 1)
\]
\[
= -\lambda^{-1} \frac{\Delta n+1}{d_{n+1}} \pi_L T_{1/\lambda}(n)(n)^{s-1} F_{1/\lambda}(n)^{s-1} \sum_{m \geq n+1} F_{1/\lambda}(m)^{s} f(m)
\]
\[
+ \lambda^{-1} \frac{\Delta n-1}{a_{n-1}} \pi_R T_{1/\lambda}(n-1)(n-1)^{s} F_{1/\lambda}(n-1)^{s} \sum_{m \geq n} F_{1/\lambda}(m)^{s} f(m)
\]
\[
= \lambda^{-1} \frac{\Delta n-1}{a_{n-1}} \pi_R T_{1/\lambda}(n-1)(n-1)^{s} f(n)
\]
\[
+ \lambda^{-1} \left( \frac{\Delta n-1}{a_{n-1}} \pi_R T_{1/\lambda}(n-1)^{s} - \frac{\Delta n+1}{d_{n+1}} \pi_L T_{1/\lambda}(n)^{s-1} \right)
\]
\[
\times F_{1/\lambda}(n)^{s-1} \sum_{m \geq n+1} F_{1/\lambda}(m)^{s} f(m).
\]

We calculate this expression further. By Lemma 2.4, we see
\[
\lambda^{-1} \frac{\Delta n+1}{d_{n+1}} \pi_L T_{1/\lambda}(n)(n)^{s-1} \equiv \frac{1}{a_n} \pi_L C(n) = z_L(n)^{s},
\]
\[
\lambda^{-1} \frac{\Delta n-1}{a_{n-1}} \pi_R T_{1/\lambda}(n-1)(n-1)^{s} \equiv \frac{1}{d_n} \pi_R C(n) = z_R(n)^{s},
\]
for all \(n \in \mathbb{Z}\). Using these formulas for \(n \geq 2\), (66) can be further calculated as
\[
U(C)W_\lambda f(n) = \lambda^{-1} \frac{\Delta n-1}{a_{n-1}} \pi_R T_{1/\lambda}(n-1)(n-1)^{s} f(n)
\]
\[
+ \lambda^{-1} \left( \frac{\Delta n-1}{a_{n-1}} \pi_R T_{1/\lambda}(n-1)^{s} - \frac{\Delta n+1}{d_{n+1}} \pi_L T_{1/\lambda}(n)^{s-1} \right)
\]
\[
\times F_{1/\lambda}(n)^{s-1} \sum_{m \geq n+1} F_{1/\lambda}(m)^{s} f(m)
\]
\[
\equiv z_R(n)^{s} f(n) + (z_R(n)^{s} - z_L(n)^{s}) F_{1/\lambda}(n)^{s-1} \sum_{m \geq n+1} F_{1/\lambda}(m)^{s} f(m)
\]
\[
\equiv f(n) - z_L(n)f(n) + (z_R(n) - z_L(n))F_{1/\lambda}(n)^{s-1} \sum_{m \geq n+1} F_{1/\lambda}(m)^{s} f(m)
\]
\[
\equiv f(n) + \lambda w_\lambda(n,n)f(n) + \lambda \sum_{m \geq n+1} w_\lambda(n,m)f(m)
\]
\[
= f(n) + \lambda W_\lambda f(n),
\]
which shows (62) for \( n \geq 2 \). Next, we consider the case \( n \leq -2 \). By (5) in Lemma 2.4, we have
\[
\pi_L C(n+1)w_{\lambda}(n+1, n+1)
\]
\[
= (2.35) -\lambda^{-1} \pi_L C(n+1)z_R(n+1) \overset{(5)}{=} -\lambda^{-1} \frac{\Delta n+1}{a_{n+1}} \pi_L \pi_R = 0,
\]
\[
\pi_R C(n-1)w_{\lambda}(n-1, n-1)
\]
\[
= (2.35) -\lambda^{-1} \pi_R C(n-1)z_R(n-1) \overset{(5)}{=} -\lambda^{-1} \frac{\Delta n-1}{a_{n-1}} \pi_R.
\]

By (1), (4), (5) in Lemma 2.4, we also have for \( m \leq n \)
\[
\pi_L C(n+1)w_{\lambda}(n+1, m)
\]
\[
= (2.35) -\lambda^{-1} \pi_L C(n+1)\left( z_L(n+1) - z_R(n+1) \right) F_{1/\lambda}(n+1) s^{-1} F_{1/\lambda}(m)^*
\]
\[
\overset{(4),(5)}{=} -\lambda^{-1} \pi_L \left( \frac{\Delta n+1}{d_{n+1}} \pi_L - \frac{\Delta n+1}{a_{n+1}} \pi_R \right) F_{1/\lambda}(n+1) s^{-1} F_{1/\lambda}(m)^*
\]
\[
\overset{(1)}{=} -\lambda^{-1} \frac{\Delta n+1}{d_{n+1}} \pi_L T_{1/\lambda}^{-}(n) s^{-1} F_{1/\lambda}(n)^* F_{1/\lambda}(m)^*,
\]
and for \( m \leq n - 2 \)
\[
\pi_R C(n-1)w_{\lambda}(n-1, m)
\]
\[
= (2.35) -\lambda^{-1} \pi_R C(n-1)\left( z_L(n-1) - z_R(n-1) \right) F_{1/\lambda}(n-1) s^{-1} F_{1/\lambda}(m)^*
\]
\[
\overset{(4),(5)}{=} -\lambda^{-1} \pi_R \left( \frac{\Delta n-1}{d_{n-1}} \pi_L - \frac{\Delta n-1}{a_{n-1}} \pi_R \right) F_{1/\lambda}(n-1) s^{-1} F_{1/\lambda}(m)^*
\]
\[
\overset{(1)}{=} -\lambda^{-1} \frac{\Delta n-1}{a_{n-1}} \pi_R T_{1/\lambda}^{-}(n-1) s^{-1} F_{1/\lambda}(n)^* F_{1/\lambda}(m)^*.
\]

Thus, the second line in (68) is a special case of the formula for \( \pi_R C(n-1)w_{\lambda}(n-1, m) \) in the case \( m \leq n - 2 \) given above, and we have
\[
\pi_L C(n+1)W_{\lambda}f(n+1)
\]
\[
= -\lambda^{-1} \frac{\Delta n+1}{d_{n+1}} \pi_L T_{1/\lambda}^{-}(n) s^{-1} F_{1/\lambda}(n)^* \sum_{m \leq n} F_{1/\lambda}(m)^* f(m),
\]
\[
\pi_R C(n-1)W_{\lambda}f(n-1)
\]
\[
= -\lambda^{-1} \frac{\Delta n-1}{a_{n-1}} \pi_R T_{1/\lambda}^{-}(n-1) s^{-1} F_{1/\lambda}(n)^* \sum_{m \leq n-1} F_{1/\lambda}(m)^* f(m).
\]

Therefore, \( U(\mathcal{C})W_{\lambda}f \) can be calculated as
An eigenfunction expansion formula for one-dimensional…

\[ U(C)f(n) = \pi_L C(n+1)W_\lambda f(n+1) + \pi_R C(n-1)W_\lambda f(n-1) \]

\[ = \lambda^{-1} \frac{\Delta^{n+1}}{d_{n+1}} \pi_L T_{1/\lambda} (n) f(n) \]

\[ + \lambda^{-1} \left( \frac{\Delta^{n+1}}{d_{n+1}} \pi_L T_{1/\lambda} (n) f(n) - \frac{\Delta^{n-1}}{a_{n-1}} \pi_R T_{1/\lambda} (n-1) f(n) \right) \]

\[ \times F_{1/\lambda}^{*} (n) \sum_{m \leq n-1} F_{1/\lambda} (m) f(m) \]

(2.41) \[ = z_L (n)^* f(n) \] (69)

\[ + (z_L (n)^* - z_R (n)^*) F_{1/\lambda} (n)^* \sum_{m \leq n-1} F_{1/\lambda} (m) f(m) \]

(6) \[ = f(n) - z_R (n) f(n) \]

\[ + (z_L (n) - z_R (n)) F_{1/\lambda} (n)^* \sum_{m \leq n-1} F_{1/\lambda} (m) f(m) \]

(2.35) \[ = f(n) + \lambda W_\lambda f(n) , \]

which shows (62) for \( n \leq -2 \). Since \( w_\lambda (0, 0) = 0 \), we have

\[ U(C)W_\lambda f(1) = \sum_{m \geq 2} \pi_L C(2) w_\lambda (2, m) f(m) + \sum_{m \geq 1} \pi_R C(0) w_\lambda (0, m) f(m) \]

(2.35) \[ = -\lambda^{-1} \pi_L C(2) z_L (2) f(2) \]

\[ + \lambda^{-1} \pi_L C(2) (z_R (2) - z_L (2)) F_{1/\lambda} (2)^* \sum_{m \geq 3} F_{1/\lambda} (m) f(m) \]

\[ + \lambda^{-1} \pi_R C(0) z_R (0) \sum_{m \geq 1} F_{1/\lambda} (m) f(m) \]

(1),(4),(5) \[ = -\lambda^{-1} \frac{\Delta^2}{d_2} \pi_L F_{1/\lambda} (2)^* \sum_{m \geq 2} F_{1/\lambda} (m) f(m) \]

\[ + \lambda^{-1} \frac{\Delta^0}{a_0} \pi_R T_{1/\lambda} (0)^* F_{1/\lambda} (1)^* \sum_{m \geq 1} F_{1/\lambda} (m) f(m) \]

(2.41) \[ = -z_L (1)^* F_{1/\lambda} (1)^* \sum_{m \geq 2} F_{1/\lambda} (m) f(m) \]

\[ + z_R (1)^* F_{1/\lambda} (1)^* \sum_{m \geq 2} F_{1/\lambda} (m) f(m) \]

\[ = z_R (1)^* f(1) + (z_R (1)^* - z_L (1)^*) F_{1/\lambda} (1)^* \sum_{m \geq 2} F_{1/\lambda} (m) f(m) \]

(6),(2.35) \[ = f(1) - z_L (1) f(1) + \lambda \sum_{m \geq 2} w_\lambda (1, m) f(m) \]

(2.35) \[ = f(1) + \lambda W_\lambda f(1) . \]
For \( n = -1 \), we have
\[
U(C)W_\lambda f(-1) = \sum_{m \leq -1} \pi_L C(0) w_\lambda(0,m) + \sum_{m \leq -2} \pi_R C(-2) w_\lambda(-2,m)f(m)
\]
\[
= \lambda^{-1} \sum_{m \leq -1} \pi_L C(0) z_L(0) F_{1/\lambda}(m)^* f(m) - \lambda^{-1} \pi_R C(-2) z_R(-2)f(-2)
\]
\[
+ \sum_{m \leq -3} \pi_R C(-2)(z_L(-2) - z_R(-2)) F_{1/\lambda}(m)^* f(m)
\]
\[
\overset{(2.35)}{=} \lambda^{-1} \sum_{m \leq -1} \pi_L C(0) z_L(0) F_{1/\lambda}(m)^* f(m) - \lambda^{-1} \pi_R C(-2) z_R(-2)f(-2)
\]
\[
+ \sum_{m \leq -3} \pi_R C(-2)(z_L(-2) - z_R(-2)) F_{1/\lambda}(m)^* f(m)
\]
\[
\overset{(4),(5)}{=} \lambda^{-1} \Delta_0 \pi_L F_{1/\lambda}(-1)^* f(-1) + \lambda^{-1} \Delta_0 \pi_L \sum_{m \leq -2} F_{1/\lambda}(m)^* f(m)
\]
\[
- \lambda^{-1} \Delta_{a-2} \pi_R F_{1/\lambda}(-2)^* \sum_{m \leq -2} F_{1/\lambda}(m)^* f(m)
\]
\[
\overset{(1)}{=} \lambda^{-1} \Delta_0 \pi_L T_{1/\lambda}(-1)^* f(-1)
\]
\[
+ \lambda^{-1} \left( \Delta_0 \pi_L T_{1/\lambda}(-1)^* - \Delta_{a-2} \pi_R T_{1/\lambda}(-2)^* \right) \times F_{1/\lambda}(-1)^* \sum_{m \leq -2} F_{1/\lambda}(m)^* f(m)
\]
\[
\overset{(2.41)}{=} z_L(-1)^* f(-1) + (z_L(-1)^* - z_R(-1)^*) \times F_{1/\lambda}(-1)^* \sum_{m \leq -2} F_{1/\lambda}(m)^* f(m)
\]
\[
\overset{(6)}{=} f(-1) - z_R(-1)f(-1) + (z_L(-1) - z_R(-1)) \times F_{1/\lambda}(-1)^* \sum_{m \leq -2} F_{1/\lambda}(m)^* f(m)
\]
\[
\overset{(2.35)}{=} f(-1) + \lambda W_\lambda f(-1).
\]

These calculations show that (62) holds also for \( n = \pm 1 \). Finally, we calculate \( U(C)W_\lambda f(0) \). By Lemma 2.4 again, we see for \( m \geq 1 \)
\[
\pi_L C(1) w_\lambda(1,m) = -z_L(0)^* F_{1/\lambda}(m)^* = (-I + z_R(0)) F_{1/\lambda}(m)^* = -F_{1/\lambda}(m)^* + \lambda w_\lambda(0,m),
\]
and for \( m \leq -1 \)
\[
\pi_R C(-1) w_\lambda(-1,m) = -z_R(0)^* F_{1/\lambda}(m)^* = (-I + z_L(0)) F_{1/\lambda}(m)^* = -F_{1/\lambda}(m)^* + \lambda w_\lambda(0,m).
\]

From this, we conclude
An eigenfunction expansion formula for one-dimensional…

\[ U(C)W_\lambda f(0) = \sum_{m \geq 1} \pi_L C(1)w_\lambda(1, m)f(m) + \sum_{m \leq -1} \pi_R C(-1)w_\lambda(-1, m)f(m) \]

\[ = - \sum_{m \neq 0} F_{1/\lambda}(m)^*f(m) + \sum_{m \neq 0} \lambda w_\lambda(0, m)f(m) \]

\[ = \lambda W_\lambda f(0) - \tilde{f}^C(\lambda) + f(0), \]

which shows that (62) holds for all \( n \in \mathbb{Z} \).

Proof of (26) for \( f \in C_0(\mathbb{Z}, \mathbb{C}^2) \). The concrete form of the matrix \( T_{1/\lambda}(n)^* \) is given in (52) and that of the matrix \( T_{1/\lambda}(n)^{*-1} \) is given as

\[ T_{1/\lambda}(n)^{*-1} = \begin{bmatrix} \lambda \frac{d_{n+1}}{\Delta_{n+1}} & \frac{\Delta_n b_n}{a_n \Delta_{n+1}} \\ -\lambda \frac{d_n b_n}{\Delta_n} & \lambda^{-1} - \frac{\Delta_n c_n}{\Delta_{n+1} \Delta_{n+1}} \end{bmatrix}. \]  

Using (52) and (70), it can be shown directly that

\[ T_{1/\lambda}(m - 1)^{*-1} \pi_L C(m) = \lambda C(m)^* \pi_L C(m) = \frac{\lambda}{\Delta_m} \begin{bmatrix} a_m d_m & b_m c_m \\ -a_m c_m & -b_m d_m \end{bmatrix}, \]

\[ T_{1/\lambda}(m)^* \pi_R C(m) = \lambda C(m)^* \pi_R C(m) = \frac{\lambda}{\Delta_m} \begin{bmatrix} -b_m c_m & -b_m d_m \\ a_m c_m & a_m d_m \end{bmatrix}, \]

for all \( m \in \mathbb{Z} \). From these formulas and (1) in Lemma 2.4, we have for \( f \in C_0(\mathbb{Z}, \mathbb{C}^2) \)

\[ \mathcal{F}_C[U(C)f](\lambda) = \sum_{m \in \mathbb{Z}} F_{1/\lambda}(m)(U(C)f)(m) \]

\[ = \sum_{m \in \mathbb{Z}} F_{1/\lambda}(m)[\pi_L C(m + 1)f(m + 1) + \pi_R C(m - 1)f(m - 1)] \]

\[ = \sum_{m \in \mathbb{Z}} F_{1/\lambda}(m - 1)^{*-1} \pi_L C(m)f(m) + \sum_{m \in \mathbb{Z}} F_{1/\lambda}(m + 1)^* \pi_R C(m)f(m) \]

\[ = \frac{1}{\lambda} \sum_{m \in \mathbb{Z}} F_{1/\lambda}(m)^* T_{1/\lambda}(m - 1)^{*-1} \pi_L C(m)f(m) \]

\[ + \sum_{m \in \mathbb{Z}} F_{1/\lambda}(m)^* T_{1/\lambda}(m)^* \pi_R C(m)f(m) \]

\[ = \lambda \sum_{m \in \mathbb{Z}} F_{1/\lambda}(m)^* \left[ C(m)^* \pi_L C(m) + C(m)^* \pi_R C(m) \right] f(m) \]

\[ = \lambda \sum_{m \in \mathbb{Z}} F_{1/\lambda}(m)^* f(m) = \lambda \mathcal{F}_C[f], \]

which shows (26).

\[ \square \]
3 Green function and its properties

In this section, we investigate properties of the Green function (10) and give proofs of Theorems 1.2, 1.3 and 1.4. By the definition (10) of the Green function and (188), (189) in Appendix B, we see

$$R(\lambda) f(n) = \sum_{m \in \mathbb{Z}} R_{\lambda}(n, m) f(m) \quad (f \in C_0(\mathbb{Z}, \mathbb{C}^2)).$$

It is well known that the resolvent $R(\lambda)$ is holomorphic as a bounded operator-valued function in $\lambda \in \mathbb{C} \setminus \sigma(U)$, and thus, $R_{\lambda}(n, m)$ is holomorphic as an $M_2(\mathbb{C})$-valued function in $\lambda \in \mathbb{C} \setminus \sigma(U)$ for each fixed $m, n \in \mathbb{Z}$. For $\lambda \in \mathbb{C} \setminus \sigma(U)$, we set

$$x_0(\lambda) = R_{\lambda}(0, 0).$$ (72)

Using the concrete formula (30) of $R(0) = U^*$, we have

$$x_0(0) u = U^* (\delta_0 \otimes u)(0) = \mathbb{C}^0 \pi_L (\delta_0 \otimes u)(-1) + \mathbb{C}^0 \pi_R (\delta_0 \otimes u)(1) = 0$$

for $u \in \mathbb{C}^2$, and hence, $x_0(0) = 0$. For any $f \in C_0(\mathbb{Z}, \mathbb{C}^2)$ and $\lambda \in \mathbb{C} \setminus \sigma(U)$, the difference $R(\lambda)f - V_{\lambda}f \in \text{Map}(\mathbb{Z}, \mathbb{C}^2)$, where $V_{\lambda}f$ is defined in (46), satisfies

$$(U(\mathbb{C}) - \lambda)(R(\lambda)f - V_{\lambda}f) = (U - \lambda)R(\lambda)f - (U(\mathbb{C}) - \lambda)V_{\lambda}f = 0$$

by Proposition 2.3. Therefore, by Theorem 1.1, there exists a unique vector $r_{\lambda}(f) \in \mathbb{C}^2$, such that

$$R(\lambda)f(n) = V_{\lambda}f(n) + \Phi_{\lambda}(r_{\lambda}(f))(n) \quad (n \in \mathbb{Z}).$$ (73)

Since $V_{\lambda}f(0) = 0$ and $\Phi_{\lambda}(r_{\lambda}(f))(0) = r_{\lambda}(f)$, we have

$$R(\lambda)f(0) = V_{\lambda}f(0) + \Phi_{\lambda}(r_{\lambda}(f))(0) = r_{\lambda}(f),$$

and for any $u \in \mathbb{C}^2$

$$x_0(\lambda) u = R_{\lambda}(0, 0) u = R(\lambda)(\delta_0 \otimes u)(0) = r_{\lambda}(\delta_0 \otimes u).$$ (74)

More generally, we have

$$r_{\lambda}(\delta_m \otimes u) = R(\lambda)(\delta_m \otimes u)(0) = R_{\lambda}(0, m) u \quad (u \in \mathbb{C}^2).$$ (75)

To prove Theorem 1.2, we need the following.

**Lemma 3.1** For any $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, we have

$$F_{\lambda}(n)[z_L(0) - z_R(0)] F_{1/\lambda}(n)^* = [z_L(n) - z_R(n)].$$ (76)

**Proof** Since $F_\lambda(0) = I$ for any $\lambda \in \mathbb{C} \setminus \{0\}$, (76) for $n = 0$ is obvious. By the item (8) in Lemma 2.4, and the definition (6) of the matrix $F_\lambda(n)$, we see
Let $k \geq 1$ and suppose that (76) holds for $n = k$. By the items (1), (8) in Lemma 2.4, we see

$$F_{\lambda}(k + 1)[z_{L}(0) - z_{R}(0)]F_{1/\lambda}(k + 1)^*$$

$$= T_{\lambda}(k)[z_{L}(0) - z_{R}(0)]T_{1/\lambda}(k + 1)^*$$

$$= T_{\lambda}(k)[z_{L}(k) - z_{R}(k)]T_{1/\lambda}(k)^*$$

$$= [z_{L}(k + 1) - z_{R}(k + 1)];$$

hence, (76) holds for $k + 1$. By induction, (76) holds for all $n \geq 1$. By the item (8) in Lemma 2.4, we see

$$[z_{L}(0) - z_{R}(0)] = T_{\lambda}(-1)[z_{L}(-1) - z_{R}(-1)]T_{1/\lambda}(-1)^*,$$

and hence, we have

$$F_{\lambda}(-1)[z_{L}(0) - z_{R}(0)]F_{1/\lambda}(-1)^*$$

$$= T_{\lambda}(-1)^{-1}[z_{L}(0) - z_{R}(0)]T_{1/\lambda}(-1)^{*-1}$$

$$= [z_{L}(-1) - z_{R}(-1)].$$

This shows that (76) holds for $n = -1$. Let $k \leq -1$ and suppose that (76) holds for $n = k$. Then, we have

$$F_{\lambda}(k - 1)[z_{L}(0) - z_{R}(0)]F_{1/\lambda}(k - 1)^*$$

$$= T_{\lambda}(k - 1)^{-1}F_{\lambda}(k)[z_{L}(0) - z_{R}(0)]F_{1/\lambda}(k)^*T_{1/\lambda}(k - 1)^{*-1}$$

$$= T_{\lambda}(k - 1)^{-1}[z_{L}(k) - z_{R}(k)]T_{1/\lambda}(k - 1)^{*-1}$$

$$= [z_{L}(k - 1) - z_{R}(k - 1)],$$

which shows that (76) holds also for $n = k - 1$. By induction, (76) holds for all $n \leq -1$ and, hence, for all $n \in \mathbb{Z}$. 

\textbf{Proof of Theorem 1.2} By the definition (46) of the operator $V_{\lambda}$, we have

$$v_{\lambda}(n, m)u = V_{\lambda}(\delta_m \otimes u)(n)$$

for any $n, m \in \mathbb{Z}$ and $u \in \mathbb{C}^2$. Thus, by setting $f = \delta_m \otimes u$ with $m \in \mathbb{Z}$, $u \in \mathbb{C}^2$ in (73), we have
\[ R_\lambda(n, m)u = R(\lambda)(\delta_m \otimes u)(n) \]
\[ = V_\lambda(\delta_m \otimes u)(n) + \Phi_\lambda(r_\lambda(\delta_m \otimes u))(n) \]
\[ = v_\lambda(n, m)u + F_\lambda(n)r_\lambda(\delta_m \otimes u) \]
\[ (77) \]

Applying \( R(\lambda) \) to Eq. (62) in Theorem 2.6, we have
\[ R(\lambda)f(n) = W_\lambda f(n) + R(\lambda)(\delta_0 \otimes \tilde{f}(\lambda))(n) = W_\lambda f(n) + R_\lambda(n, 0)\tilde{f}(\lambda) \]
\[ (78) \]
for \( f \in C_0(\mathbb{Z}, \mathbb{C}^2) \) and \( n \in \mathbb{Z} \). We set \( f = \delta_m \otimes u \) in the above. By the definition (16) of \( F_\lambda \), we have \( F_\lambda(\delta_m \otimes u)(\lambda) = F_{1/\lambda}(m)^*u \). By the definition (59) of \( W_\lambda \), we see
\[ W_\lambda(\delta_m \otimes u)(n) = w_\lambda(n, m)u \quad (u \in \mathbb{C}^2). \]

Hence, substituting \( f = \delta_m \otimes u \) into (78), we obtain
\[ R_\lambda(n, m)u = w_\lambda(n, m)u + R_\lambda(n, 0)F_{1/\lambda}(m)^*u. \]
\[ (79) \]
Setting \( m = 0 \) in (77) gives
\[ R_\lambda(n, 0) = v_\lambda(n, 0) + F_\lambda(n)x_0(\lambda), \]
and substituting this into (79) yields
\[ R_\lambda(n, m) = w_\lambda(n, m) + v_\lambda(n, 0)F_{1/\lambda}(m)^* + F_\lambda(n)x_0(\lambda)F_{1/\lambda}(m)^* \]
\[ = \begin{cases} 
  w_\lambda(n, m) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_L(0)]F_{1/\lambda}(m)^* & (n \geq 1), \\
  w_\lambda(n, m) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(m)^* & (n \leq -1), \\
  w_\lambda(0, m) + x_0(\lambda)F_{1/\lambda}(m)^* & (n = 0), 
\end{cases} \]
\[ (80) \]
where we used the definition (44) of \( v_\lambda \). For \( n = 0 \), it follows from (80) and (61) that:
\[ R_\lambda(0, m) = \begin{cases} 
  [x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(m)^* & (m \geq 1), \\
  [x_0(\lambda) + \lambda^{-1}z_L(0)]F_{1/\lambda}(m)^* & (m \leq -1). 
\end{cases} \]
\[ (81) \]
From (61), we have \( w_\lambda(n, m) = 0 \) for the two cases \((n \leq -1, m \geq n + 1)\) and \((n \geq 1, m \leq n - 1)\). Hence, we have
\[ R_\lambda(n, m) = \begin{cases} 
  F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(m)^* & (n \leq -1, m \geq n + 1), \\
  F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_L(0)]F_{1/\lambda}(m)^* & (n \geq 1, m \leq n - 1). 
\end{cases} \]
\[ (82) \]
To consider the other cases, we note that by Lemma 3.1
\[ w_\lambda(n, m) = \begin{cases} \lambda^{-1}F_\lambda(n)[z_R(0) - z_L(0)]F_{1/\lambda}(m^*) & (1 \leq n \leq m - 1), \\ \lambda^{-1}F_\lambda(n)[z_L(0) - z_R(0)]F_{1/\lambda}(m^*) & (m + 1 \leq n \leq -1). \end{cases} \] (83)

Therefore, (80) gives

\[ R_\lambda(n, m) = \begin{cases} F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(m^*) & (1 \leq n \leq m - 1), \\ F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_L(0)]F_{1/\lambda}(m^*) & (m + 1 \leq n \leq -1). \end{cases} \] (84)

The formula (12) in Theorem 1.2 is a rewritten form of (81), (82), and (84). For \( n = m \geq 1 \), the formulas (80), (61) and Lemma 3.1 show

\[
R_\lambda(n, n) = w_\lambda(n, n) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_L(0)]F_{1/\lambda}(n^*)
\]
\[
= -\lambda^{-1}z_L(n) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_L(0)]F_{1/\lambda}(n^*)
\]
\[
= -\lambda^{-1}z_L(n) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0) + \lambda^{-1}(z_L(0) - z_R(0))]F_{1/\lambda}(n^*)
\]
\[
= -\lambda^{-1}z_L(n) + \lambda^{-1}(z_L(n) - z_R(n)) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(n^*)
\]
\[
= -\lambda^{-1}z_R(n) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(n^*),
\]
which implies (13) for \( n = m \geq 1 \). Similarly, for \( n \leq -1 \), we have

\[
R_\lambda(n, n) = w_\lambda(n, n) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(n^*)
\]
\[
= -\lambda^{-1}z_R(n) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(n^*)
\]
\[
= -\lambda^{-1}z_R(n) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_L(0) + \lambda^{-1}(z_R(0) - z_L(0))]F_{1/\lambda}(n^*)
\]
\[
= -\lambda^{-1}z_R(n) + \lambda^{-1}(z_R(n) - z_L(n)) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_L(0)]F_{1/\lambda}(n^*)
\]
\[
= -\lambda^{-1}z_L(n) + F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]F_{1/\lambda}(n^*),
\]
which implies (13) for \( n = m \leq -1 \) and, hence, for all \( n \in \mathbb{Z} \). \( \square \)

The Green function \( R_\lambda(n, m) \) is, as above, expressed in terms of the products \( F_\lambda(n) \) of the transfer matrices \( T_\lambda(n) \) and the special value \( x_0(\lambda) = R_\lambda(0, 0) \) of the Green function. Therefore, we will face a computation of the matrix-valued function \( x_0(\lambda) \) when we apply results in this paper. Theorems 1.3 and 1.4 give us one of the methods to calculate \( x_0(\lambda) \) whose proof is given in the rest of this section. The following is one of the most important facts in the proof of Theorem 1.3.
Lemma 3.2 Let \( \lambda \in \mathbb{C} \setminus \{0 \cup S^1\} \). Then, we have
\[
\sum_{n \geq 1} \|F_\lambda(n)\|^2_{HS} = +\infty, \quad \sum_{n \leq -1} \|F_\lambda(n)\|^2_{HS} = +\infty. \tag{85}
\]

In (85), the norm \( \|A\|_{HS} \) of a 2 \times 2 matrix
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
is the Hilbert–Schmidt norm \( \|A\|_{HS} \) defined by
\[
\|A\|^2_{HS} = |a|^2 + |b|^2 + |c|^2 + |d|^2 = \|Ae_L\|^2_{\mathbb{C}^2} + \|Ae_R\|^2_{\mathbb{C}^2}.
\]

Lemma 3.2 could be proved by a method in [17]. However, we give a proof different from it.

**Proof** Let \( \lambda \in \mathbb{C} \setminus \{0 \cup S^1\} \). First, we show the following formula:
\[
C(n)F_\lambda(n) = \lambda \pi_L F_\lambda(n - 1) + \lambda \pi_R F_\lambda(n + 1) \quad (n \in \mathbb{Z}). \tag{86}
\]
Indeed, by (1), (2), (3) in Lemma 2.4, we have
\[
C(n)F_\lambda(n) = \pi_L C(n)F_\lambda(n) + \pi_R C(n)F_\lambda(n)
\]
\[
= \pi_L C(n)T_\lambda(n - 1)F_\lambda(n - 1) + \pi_R C(n)T_\lambda(n)^{-1}F_\lambda(n + 1) \tag{1}
\]
\[
= \lambda \pi_L F_\lambda(n - 1) + \lambda \pi_R F_\lambda(n + 1), \tag{2,3}
\]
which gives (86). To prove that
\[
D := \sum_{n \geq 1} \|F_\lambda(n)\|^2_{HS}
\]
is infinity, we assume \( D < +\infty \) and deduce a contradiction. We note that \( D > 0 \). In general, for \( C \in U(2) \) and \( A, B \in M_2(\mathbb{C}) \), we have
\[
\|CA\|_{HS} = \|A\|_{HS}, \quad \|\pi_L A + \pi_R B\|^2_{HS} = \|\pi_L A\|^2_{HS} + \|\pi_R B\|^2_{HS}. \tag{87}
\]
Equation (86) and the identities in (87) imply
\[
\|F_\lambda(n)\|^2_{HS} = \|C(n)F_\lambda(n)\|^2_{HS}
\]
\[
= |\lambda|^2 \|\pi_L F_\lambda(n - 1)\|^2_{HS} + |\lambda|^2 \|\pi_R F_\lambda(n + 1)\|^2_{HS}.
\]
Therefore, we have
\[ D = \sum_{n \geq 1} \| C(x) F_{\lambda}(n) \|_{HS}^2 = \sum_{n \geq 1} \| \lambda \pi_L F_{\lambda}(n - 1) + \lambda \pi_R F_{\lambda}(n + 1) \|_{HS}^2 \]
\[ = | \lambda |^2 \sum_{n \geq 1} \| \pi_L F_{\lambda}(n - 1) \|_{HS}^2 + | \lambda |^2 \sum_{n \geq 1} \| \pi_R F_{\lambda}(n + 1) \|_{HS}^2 \]
\[ = | \lambda |^2 \sum_{n \geq 1} \| \pi_L F_{\lambda}(n) \|_{HS}^2 + | \lambda |^2 \sum_{n \geq 1} \| \pi_R F_{\lambda}(n) \|_{HS}^2 \]
\[ = | \lambda |^2 \| \pi_L F_{\lambda}(0) \|_{HS}^2 - | \lambda |^2 \| \pi_R F_{\lambda}(1) \|_{HS}^2 \]
\[ + | \lambda |^2 \sum_{n \geq 1} \left( \| \pi_L F_{\lambda}(n) \|_{HS}^2 + \| \pi_R F_{\lambda}(n) \|_{HS}^2 \right) \]
\[ = | \lambda |^2 \| \pi_L F_{\lambda}(0) \|_{HS}^2 - | \lambda |^2 \| \pi_R F_{\lambda}(1) \|_{HS}^2 + | \lambda |^2 D. \]

By definition, we have \( F_{\lambda}(0) = I \) and \( F_{\lambda}(1) = T_{\lambda}(0) \). These and the definition of \( T_{\lambda}(n) \) show
\[ \| \pi_L F_{\lambda}(0) \|_{HS}^2 = 1, \quad \| \pi_R F_{\lambda}(1) \|_{HS}^2 = | \lambda |^{-2} (|c_0|^2 + |d_0|^2) = | \lambda |^{-2}. \]

By substituting these into the above equation and by the assumption \( | \lambda | \neq 1 \), we conclude
\[ D = | \lambda |^2 - 1 + | \lambda |^2 D, \quad \text{hence} \quad D = -1, \]
which is a contradiction. Therefore, \( D = +\infty \). Next, suppose that
\[ E := \sum_{n \leq -1} \| F_{\lambda}(n) \|_{HS}^2 < +\infty. \]

We note that \( E > 0 \). We have
\[ E = \sum_{n \leq -1} \| C(n) F_{\lambda}(n) \|_{HS}^2 = \sum_{n \leq -1} \| \lambda \pi_L F_{\lambda}(n - 1) + \lambda \pi_R F_{\lambda}(n + 1) \|_{HS}^2 \]
\[ = \sum_{n \leq -1} | \lambda |^2 \| \pi_L F_{\lambda}(n - 1) \|_{HS}^2 + \sum_{n \leq -1} | \lambda |^2 \| \pi_R F_{\lambda}(n + 1) \|_{HS}^2 \]
\[ = \sum_{n \leq -2} | \lambda |^2 \| \pi_L F_{\lambda}(n) \|_{HS}^2 + \sum_{n \leq 0} | \lambda |^2 \| \pi_R F_{\lambda}(n) \|_{HS}^2 \]
\[ = -| \lambda |^2 \| \pi_L F_{\lambda}(-1) \|_{HS}^2 + | \lambda |^2 \| \pi_R F_{\lambda}(0) \|_{HS}^2 \]
\[ + | \lambda |^2 \sum_{n \leq -1} \left( \| \pi_L F_{\lambda}(n) \|_{HS}^2 + \| \pi_R F_{\lambda}(n) \|_{HS}^2 \right) \]
\[ = -| \lambda |^2 \| \pi_L F_{\lambda}(-1) \|_{HS}^2 + | \lambda |^2 \| \pi_R F_{\lambda}(0) \|_{HS}^2 + | \lambda |^2 E. \]

Since \( F_{\lambda}(0) = I \), we have \( \| \pi_R F_{\lambda}(0) \|_{HS}^2 = 1 \). By the definition of \( F_{\lambda}(-1) = T_{\lambda}(-1)^{-1} \) and its concrete form (50), we see
\[ \| \pi_L F_{\lambda}(-1) \|_{HS}^2 = \| \pi_L T_{\lambda}(-1)^{-1} \|_{HS}^2 = | \lambda |^{-2} (|a_0|^2 + |b_0|^2) = | \lambda |^{-2}. \]

Hence, we have
which is a contradiction. Therefore, we conclude $E = +\infty$. \qed

**Lemma 3.3** Let $\lambda \in \mathbb{C}\setminus\{0\} \cup S^1)$. Then, the dimensions of the subspaces $A_L(\lambda)$ and $A_R(\lambda)$ in Theorem 1.3 are less than or equal to 1.

**Proof** Suppose that $\dim A_L(\lambda) = 2$. Then, we have
\[
\sum_{n \geq 1} \|F_\lambda(n)e_L\|_{C^2}^2 < +\infty, \quad \sum_{n \geq 1} \|F_\lambda(n)e_R\|_{C^2}^2 < +\infty.
\]
Summing these two sums gives $\sum_{n \geq 1} \|F_\lambda(n)\|_{HS}^2 < +\infty$, which contradicts Lemma 3.2. Thus, we conclude $\dim A_L(\lambda) \leq 1$. Next, suppose that $\dim A_R(\lambda) = 2$. Then, we see
\[
\sum_{n \leq -1} \|F_\lambda(n)e_L\|_{C^2}^2 < +\infty, \quad \sum_{n \leq -1} \|F_\lambda(n)e_R\|_{C^2}^2 < +\infty.
\]
Summing these two sums gives $\sum_{n \leq -1} \|F_\lambda(n)\|_{HS}^2 < +\infty$, which contradicts Lemma 3.2. Thus, we conclude $\dim A_R(\lambda) \leq 1$. \qed

The function $R(\lambda)(\delta_0 \otimes u)$ is in $\ell^2(Z, C^2)$ for any $u \in C^2$ and we have by Theorem 1.2
\[
\|R(\lambda)(\delta_0 \otimes u)\|^2 = \sum_{n \in Z} \|R(\lambda)(\delta_0 \otimes u)(n)\|_{C^2}^2 = \sum_{n \in Z} \|R_\lambda(n, 0)u\|_{C^2}^2
\]
\[
= \|x_0(\lambda)u\|_{C^2}^2 + \sum_{n \geq 1} \|F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_L(0)]u\|_{C^2}^2
\]
\[
+ \sum_{n \leq -1} \|F_\lambda(n)[x_0(\lambda) + \lambda^{-1}z_R(0)]u\|_{C^2}^2 < +\infty.
\]
(88)
Since $\dim A_L \leq 1$ and $\dim A_R \leq 1$ by Lemma 3.3, we see
\[
\text{rank}[x_0(\lambda) + \lambda^{-1}z_L(0)] \leq 1, \quad \text{rank}[x_0(\lambda) + \lambda^{-1}z_R(0)] \leq 1.
\]

**Proof of Theorem 1.3** It is left to prove that, for $\lambda \in \mathbb{C}\setminus\{0\} \cup S^1)$, the following holds:
\[
\text{rank}[x_0(\lambda) + \lambda^{-1}z_L(0)] = \text{rank}[x_0(\lambda) + \lambda^{-1}z_R(0)] = 1.
\]
(89)
To prove (89), suppose that $\text{rank}[x_0(\lambda) + \lambda^{-1}z_L(0)] = 0$. Then, $x_0(\lambda) = -\lambda^{-1}z_L(0)$. By (88), we see
\[
\sum_{n \leq -1} \| F_{\lambda}(n)[z_L(0) - z_R(0)]u \|^2_{C^2} < +\infty
\]

for any \( u \in C^2 \). However, since \( \det(z_L(0) - z_R(0)) = \frac{\Delta_0}{\alpha_0d_0} \), the matrix \( z_L(0) - z_R(0) \) is non-singular. Thus, for any \( w \in C^2 \), there exists \( u \), such that \( (z_L(0) - z_R(0))u = w \). Therefore, we have \( \sum_{n \leq -1} \| F_{\lambda}(n)w \|^2_{C^2} < +\infty \) for any \( w \in C^2 \). This contradicts the fact that \( \dim A_R(\lambda) = 1 \) in Lemma 3.3. Hence, \( \text{rank}[x_0(\lambda) + \lambda^{-1}z_L(0)] = 1 \). Next, suppose that \( \text{rank}[x_0(\lambda) + \lambda^{-1}z_R(0)] = 0 \). Then, \( x_0(\lambda) = -\lambda^{-1}z_R(0) \). By (88), we see

\[
\sum_{n \geq 1} \| F_{\lambda}(n)[z_L(0) - z_R(0)]u \|^2_{C^2} < +\infty
\]

for any \( u \in C^2 \). As above, for any \( w \in C^2 \), there exists \( u \), such that \( (z_L(0) - z_R(0))u = w \). Thus, we have \( \sum_{n \geq 1} \| F_{\lambda}(n)w \|^2_{C^2} < +\infty \) for any \( w \in C^2 \). This contradicts the fact that \( \dim A_L(\lambda) = 1 \) in Lemma 3.3. Hence, \( \text{rank}[x_0(\lambda) + \lambda^{-1}z_R(0)] = 1 \). \( \square \)

To prove Theorem 1.4, we use the following lemma.

**Lemma 3.4** Let \( \lambda \in \mathbb{C}\setminus\{0\} \cup S^1 \) and \( u \in C^2 \) be arbitrary. Then, a vector \( w \in C^2 \) satisfies

\[
\sum_{n \geq 1} \| F_{\lambda}(n)[w + \lambda^{-1}z_L(0)u] \|^2_{C^2} < +\infty,
\]

and

\[
\sum_{n \leq -1} \| F_{\lambda}(n)[w + \lambda^{-1}z_R(0)u] \|^2_{C^2} < +\infty \tag{90}
\]

if and only if \( w = x_0(\lambda)u \).

**Proof** Let \( \lambda \in \mathbb{C}\setminus\{0\} \cup S^1 \) and \( u, w \in C^2 \). We define \( \Psi \in \text{Map}(Z, C^2) \) by

\[
\Psi(n) = \begin{cases}
  w & (n = 0), \\
  F_{\lambda}(n)[w + \lambda^{-1}z_L(0)u] & (n \geq 1), \\
  F_{\lambda}(n)[w + \lambda^{-1}z_R(0)u] & (n \leq -1).
\end{cases} \tag{91}
\]

By Proposition 2.3, we have \( \Psi = V_{\lambda}(\delta_0 \otimes u) + \Phi_{\lambda}(w) \). Therefore, again by Proposition 2.3, we have

\[
(U(\mathcal{C}) - \lambda)\Psi = \delta_0 \otimes u. \tag{92}
\]

Now, suppose that \( w \) satisfies (90). Then, \( \Psi \) defined above is in \( \ell^2(Z, C^2) \). Therefore, by applying \( R(\lambda) \) to (92), we have

\[
\Psi(n) = R(\lambda)(\delta_0 \otimes u)(n) = R_{\lambda}(n, 0)u.
\]
In particular, we have \( w = \Psi(0) = R_\lambda(0,0)u = x_0(\lambda)u \) by the definition (72) of \( x_0(\lambda) \). Conversely, if \( w = x_0(\lambda)u \), we have \( \Psi(n) = R_\lambda(n,0)u = R_\lambda(0)\delta_0 \otimes u(n) \) by Theorem 1.2 which is in \( \ell^2(\mathbb{Z}, \mathbb{C}) \). Hence, \( w = x_0(\lambda)u \) satisfies (90).

**Proof of Theorem 1.4** Let \( \lambda \in \mathbb{C}\setminus\{0\} \cup S^1 \). We first show that the unit vectors \( v_+(\lambda), v_-(\lambda) \) form a basis in \( \mathbb{C}^2 \). Indeed, suppose that \( av_+(\lambda) + bv_-(\lambda) = 0 \) with \( a, b \in \mathbb{C} \) not simultaneously zero. If \( a = 0 \), then \( bv_-(\lambda) = 0 \), and hence \( b = 0 \), because \( v_-(\lambda) \) is a nonzero vector in \( \mathbb{C}^2 \). Thus, \( a \neq 0 \), and thus, we can write \( v_+(\lambda) = cv_-(\lambda) \) with \( c = -b/a \). Then, by the property (14), the function \( \Phi_\lambda(v_+(\lambda)) \) is in \( \ell^2(\mathbb{Z}, \mathbb{C}^2) \), and hence, \( \lambda \) is in the spectrum \( \sigma(U) \), an eigenvalue, of the unitary operator \( U = U(\mathbb{C})|_{\ell^2(\mathbb{Z}, \mathbb{C}^2)} \). This is a contradiction, since we have assumed \( \lambda \not\in S^1 \).

Thus, \( v_\pm(\lambda) \) forms a basis of \( \mathbb{C}^2 \), and hence, the matrix

\[
\begin{bmatrix}
1 & -\overline{m(\lambda)} \\
m(\lambda) & -1
\end{bmatrix},
\]

is non-singular. We define \( a_L(\lambda), b_L(\lambda), a_R(\lambda), b_R(\lambda) \in \mathbb{C} \) by

\[
\begin{bmatrix}
a_L(\lambda) \\ b_L(\lambda)
\end{bmatrix} = \lambda^{-1} \begin{bmatrix}
1 & -\overline{m(\lambda)} \\
m(\lambda) & -1
\end{bmatrix}^{-1} \begin{bmatrix}
\langle z_L(0)e_L, v_+(\lambda) \rangle_{\mathbb{C}^2} \\ \langle z_L(0)e_L, v_-(\lambda) \rangle_{\mathbb{C}^2}
\end{bmatrix},
\]

\[
\begin{bmatrix}
a_R(\lambda) \\ b_R(\lambda)
\end{bmatrix} = -\lambda^{-1} \begin{bmatrix}
1 & -\overline{m(\lambda)} \\
m(\lambda) & -1
\end{bmatrix}^{-1} \begin{bmatrix}
\langle z_R(0)e_R, v_+(\lambda) \rangle_{\mathbb{C}^2} \\ \langle z_R(0)e_R, v_-(\lambda) \rangle_{\mathbb{C}^2}
\end{bmatrix}. \tag{93}
\]

Then, we claim that

\[
x_0(\lambda)e_L = b_L(\lambda)v_-(\lambda) = -\lambda^{-1}z_L(0)e_L + a_L(\lambda)v_+(\lambda),
\]

\[
x_0(\lambda)e_R = a_R(\lambda)v_+(\lambda) = -\lambda^{-1}z_R(0)e_R + b_R(\lambda)v_-(\lambda). \tag{94}
\]

From the definition (93) and that \( \|v_\pm(\lambda)\|_{\mathbb{C}^2} = 1 \), we have
\[
\begin{align*}
\langle z_L(0)e_L, v_+(\lambda) \rangle_{C^2} & = \lambda \left[ \begin{array}{c} -m(\lambda) \\ m(\lambda) \end{array} \right] \\
\langle z_L(0)e_L, v_-\lambda) \rangle_{C^2} & = \lambda \left[ \begin{array}{c} a_L(\lambda) \\ b_L(\lambda) \end{array} \right] \\
& = \left[ \lambda a_L(\lambda) - \lambda \langle v_-(\lambda), v_+(\lambda) \rangle_{C^2} b_L(\lambda) \right] \\
& = \left[ \lambda a_L(\lambda) v_+(\lambda) - \lambda b_L(\lambda) v_-(\lambda), v_+(\lambda) \rangle_{C^2} \right] \\
& = \left[ \langle -\lambda a_R(\lambda) v_+(\lambda) + \lambda b_R(\lambda) v_-(\lambda), v_+(\lambda) \rangle_{C^2} \right]. \\
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
\langle \lambda a_L(\lambda) v_+(\lambda) - \lambda b_L(\lambda) v_-(\lambda) - z_L(0)e_L, v_+(\lambda) \rangle_{C^2} & = 0, \\
\langle -\lambda a_R(\lambda) v_+(\lambda) + \lambda b_R(\lambda) v_-(\lambda) - z_R(0)e_R, v_+(\lambda) \rangle_{C^2} & = 0.
\end{align*}
\]

Since \( \{ v_+(\lambda), v_-(\lambda) \} \) is a basis of \( C^2 \), we obtain

\[
\begin{align*}
b_L(\lambda)v_-(\lambda) + \lambda^{-1} z_L(0)e_L &= a_L(\lambda)v_+(\lambda), \\
a_R(\lambda)v_+(\lambda) + \lambda^{-1} z_R(0)e_R &= b_R(\lambda)v_-(\lambda). \\
\end{align*}
\]

Therefore, by (96), the vector \( w = b_L(\lambda)v_-(\lambda) \) satisfies

\[
\sum_{n \geq 1} \| F_\lambda(n)[w + \lambda^{-1} z_L(0)e_L] \|^2_{C^2} < +\infty,
\]

\[
\sum_{n \leq -1} \| F_\lambda(n)[w + \lambda^{-1} z_R(0)e_L] \|^2_{C^2} < +\infty,
\]

since we have \( z_R(0)e_L = 0 \) and the vector \( v_-(\lambda) \) satisfies (14). By Lemma 3.4, we have \( b_L(\lambda)v_-(\lambda) = x_0(\lambda)e_L \). This and (96) show the first line of (94). Similarly, by (96), the vector \( w = a_R(\lambda)v_+(\lambda) \) satisfies
\[
\sum_{n \geq 1} \| F_\lambda(n) [w + \lambda^{-1} z_L(0) e_R] \|^2_{C^2} < +\infty,
\]

\[
\sum_{n \leq -1} \| F_\lambda(n) [w + \lambda^{-1} z_R(0) e_R] \|^2_{C^2} < +\infty,
\]
since we have \( z_L(0) e_R = 0 \) and the vector \( v_+ (\lambda) \) satisfies (14). This and (96) show the second line of (94). For a unit vector \( u \in \mathbb{C}^2 \), we take any unit vector \( u^\perp \) perpendicular to \( u \). Since \( \{ v_+ (\lambda), v_+ (\lambda)^\perp \}, \{ v_-(\lambda), v_-(\lambda)^\perp \} \) are orthonormal bases of \( \mathbb{C}^2 \), we see

\[
1 = \| v_-(\lambda) \|^2_{C^2} = | \langle v_-(\lambda), v_+(\lambda) \rangle_{C^2} |^2 + | \langle v_-(\lambda), v_+(\lambda)^\perp \rangle_{C^2} |^2
\]

and hence

\[
1 - | m(\lambda) |^2 = | \langle v_-(\lambda), v_+(\lambda)^\perp \rangle_{C^2} |^2 = | \langle v_+(\lambda), v_-(\lambda)^\perp \rangle_{C^2} |^2.
\]

Using this and (93), we have

\[
\lambda b_L(\lambda) = - \frac{1}{| \langle v_-(\lambda), v_+(\lambda)^\perp \rangle_{C^2} |^2}
\]

\[
\times \left[ | \langle z_L(0) e_L, v_-(\lambda) \rangle_{C^2} - \langle v_+(\lambda), v_-(\lambda) \rangle_{C^2} \langle z_L(0) e_L, v_+(\lambda) \rangle_{C^2} |^2 \right]
\]

\[
= - \frac{1}{| \langle v_-(\lambda), v_+(\lambda)^\perp \rangle_{C^2} |^2}
\]

\[
\times | \langle z_L(0) e_L - \langle z_L(0) e_L, v_+(\lambda) \rangle_{C^2} v_+(\lambda), v_-(\lambda) \rangle_{C^2} |^2
\]

\[
= - \frac{1}{| \langle v_-(\lambda), v_+(\lambda)^\perp \rangle_{C^2} |^2}
\]

\[
\times \left( z_L(0) e_L, v_+(\lambda)^\perp \right)_{C^2} \langle v_+(\lambda)^\perp, v_-(\lambda) \rangle_{C^2}
\]

\[
= - \left( z_L(0) e_L, v_+(\lambda)^\perp \right)_{C^2}.
\]

Similarly, we have

\[
\lambda a_R(\lambda) = \frac{1}{| \langle v_+(\lambda), v_-(\lambda)^\perp \rangle_{C^2} |^2}
\]

\[
\times \left[ | \langle z_R(0) e_R, v_-(\lambda) \rangle_{C^2} - \langle z_R(0) e_R, v_+(\lambda) \rangle_{C^2} |^2 \right]
\]

\[
= - \frac{1}{| \langle v_+(\lambda), v_-(\lambda)^\perp \rangle_{C^2} |^2}
\]

\[
\langle z_R(0) e_R - \langle z_R(0) e_R, v_+(\lambda) \rangle_{C^2} v_+(\lambda), v_-(\lambda) \rangle_{C^2}
\]

\[
= - \frac{1}{| \langle v_+(\lambda), v_-(\lambda)^\perp \rangle_{C^2} |^2}
\]

\[
\times \left( z_R(0) e_R, v_-(\lambda)^\perp \right)_{C^2} \langle v_-(\lambda)^\perp, v_+(\lambda) \rangle_{C^2}
\]

\[
= - \left( z_R(0) e_R, v_-(\lambda)^\perp \right)_{C^2}.
\]
Equations (94), (97) and (98) complete the proof of (15) in Theorem 1.4.

4 Integral representation for the Green function

As is seen in the previous sections, the function $x_0 : \mathbb{C} \setminus \sigma(U) \to M_2(\mathbb{C})$ introduced in (72) plays one of the central roles in the series of our results. However, since it does not seem that the real part $[x_0(\lambda) + x_0(\lambda)^*]/2$ has nice properties as the $m$-Carathéodory functions have. The definition of $m$-Carathéodory functions is given in the statement of Lemma 4.1 below. Instead of using $x_0$, we use the following function:

$$x(\lambda) = I + 2\lambda x_0(\lambda).$$

By (99), we have

$$\text{Re } x(\lambda) := \frac{1}{2} [x(\lambda) + x(\lambda)^*] = \lambda x_0(\lambda) - \overline{\lambda} x_0(1/\lambda),$$

which shows the second equality in (23).

Lemma 4.1 The function $x : \mathbb{C} \setminus \sigma(U) \to M_2(\mathbb{C})$ is an $m$-Carathéodory function in the sense that it is a holomorphic function on the unit disc with a positive real part and $x(0) = I$.

Proof Since $x_0$ is holomorphic on $\mathbb{C} \setminus \sigma(U)$, $x$ is also holomorphic on $\mathbb{C} \setminus \sigma(U)$, and in particular, on the unit disc. By definition, we see $x(0) = I$. We set

$$K(\lambda) = (U - \lambda)^{-1}(U + \lambda) = R(\lambda)(U - \lambda + 2\lambda) = I + 2\lambda R(\lambda) \quad (\lambda \in \mathbb{C} \setminus \sigma(U)).$$

(102)

For $\lambda \in \mathbb{C} \setminus \sigma(U)$ and $u, v \in \mathbb{C}^2$, we have

$$\langle x_0(0)u, v \rangle_{\mathbb{C}^2} = \langle R(0, 0)u, v \rangle_{\mathbb{C}^2} = \langle R(\lambda)(\delta_0 \otimes u), \delta_0 \otimes v \rangle,$$

and hence

$$\langle x(\lambda)u, v \rangle_{\mathbb{C}^2} = \langle [I + 2\lambda R(\lambda)]\delta_0 \otimes u, \delta_0 \otimes v \rangle$$

$$= \langle K(\lambda)\delta_0 \otimes u, \delta_0 \otimes v \rangle.$$

(103)

This shows that
\[ \langle x(\bar{\lambda})^* u, v \rangle_{C^2} = \langle u, x(\bar{\lambda}) v \rangle_{C^2} = \overline{\langle x(\bar{\lambda}) v, u \rangle_{C^2}} \]
\[ = \langle K(\bar{\lambda}) \delta_0 \otimes v, \delta_0 \otimes u \rangle = \langle \delta_0 \otimes u, K(\bar{\lambda}) \delta_0 \otimes v \rangle \]
\[ = \langle K(\bar{\lambda})^*(\delta_0 \otimes u), \delta_0 \otimes v \rangle \]
\[ = \langle [(U^* + \bar{\lambda})(U^* - \bar{\lambda})^{-1}] \delta_0 \otimes u, \delta_0 \otimes v \rangle. \]

We note that
\[
\frac{1}{2}(K(\lambda) + K(\lambda)^*)
= \frac{1}{2}[(U - \lambda)^{-1}(U + \lambda) + (U^* + \bar{\lambda})(U^* - \bar{\lambda})^{-1}]
= \frac{1}{2}(U - \lambda)^{-1}[(U + \lambda)(U^* - \bar{\lambda}) + (U - \lambda)(U^* + \bar{\lambda})](U^* - \bar{\lambda})^{-1}
= (1 - |\lambda|^2)(U - \lambda)^{-1}(U^* - \bar{\lambda})^{-1} = (1 - |\lambda|^2)R(\lambda)^*R(\lambda). \tag{104} \]

Therefore, the real part Re \(x(\lambda)\) of \(x(\lambda)\) satisfies
\[
\text{Re} \langle x(\bar{\lambda}) u, u \rangle_{C^2} = \langle [(K(\lambda) + K(\lambda)^*)/2](\delta_0 \otimes u), \delta_0 \otimes u \rangle
= (1 - |\lambda|^2)(R(\lambda)^*R(\lambda)\delta_0 \otimes u, \delta_0 \otimes u)
= (1 - |\lambda|^2)||R(\lambda)(\delta_0 \otimes u)||^2 > 0 \]
if \(|\lambda| < 1\) and \(u \neq 0\). This completes the proof. \(\square\)

Therefore, there exists a unique positive \(2 \times 2\)-matrix-valued measure
\[
\Sigma = \begin{bmatrix} \mu_L & \bar{\alpha} \\ \alpha & \mu_R \end{bmatrix}, \tag{105} \]
on \(S^1\), where \(\mu_L\) and \(\mu_R\) are probability measures and \(\alpha\) is a complex measure on \(S^1\), such that we have the following Herglotz representation:
\[
x(\lambda) = \int_{S^1} \frac{\zeta + \bar{\lambda}}{\zeta - \lambda} \, d\Sigma(\zeta), \quad \Sigma(S^1) = I. \tag{106} \]

We refer the readers to [8, 12] for the proof of the above fact. We denote \(C(X)\) the space of complex-valued continuous functions on a compact topological space \(X\) with the norm defined by \(\|f\|_{C(X)} = \sup_{x \in X} |f(x)|\). The positive-matrix-valued measure \(\Sigma\) is characterized by
\[
\int_{S^1} h(\zeta) \langle \, d\Sigma(\zeta)u, u \rangle_{C^2} = \lim_{r \uparrow 1} \int_{S^1} h(\zeta) \langle [\text{Re} \, x(r\zeta)]u, u \rangle_{C^2} \, d\ell(\zeta) \tag{107} \]
for any \(h \in C(S^1)\) and \(u \in C^2\), where \(d\ell(\zeta)\) denotes the Lebesgue measure on \(S^1\) with the unit total mass. In what follows, we choose the counterclockwise orientation on the unit circle \(S^1\). Then, we can write
\[ \int_{S^1} h(\zeta) \, d\ell(\zeta) = \frac{1}{2\pi i} \int_{S^1} h(\zeta) \frac{d\zeta}{\zeta} = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{i\theta}) \, d\theta. \]

To prove Theorem 1.5, (1), we consider, for a fixed \( f \in C_0(\mathbb{Z}, \mathbb{C}^2) \), the following function:

\[ H_f(\lambda) = \langle K(\lambda)f, f \rangle \quad (\lambda \in \mathbb{C}^2, |\lambda| < 1). \]  

(108)

The function \( H_f \) is holomorphic on the unit disc and, by (104), its real part is

\[ \text{Re} H_f(\lambda) = (1 - |\lambda|^2)\|R(\lambda)f\|^2. \]  

(109)

Hence, \( H_f \) is a scalar Carathéodory function with \( H_f(0) = \|f\|^2 \). Therefore, there exists a unique positive measure \( \nu_f \) on \( S^1 \), such that

\[ H_f(\lambda) = \int_{S^1} \frac{\zeta + \lambda}{\zeta - \lambda} \, d\nu_f(\zeta), \quad \nu_f(S^1) = \|f\|^2, \]  

(110)

and the measure \( \nu_f \) is the unique measure satisfying

\[ \int_{S^1} h(\zeta) \, d\nu_f(\zeta) = \lim_{r \downarrow 1} \int_{S^1} h(\zeta) \text{Re} H_f(r\zeta) \, d\ell(\zeta) \]  

(111)

for any \( h \in C(S^1) \). In what follows, we prove the formula (17) with the matrix-valued measure appeared in (105) and (106). We need the following.

**Lemma 4.2** The matrix-valued functions \( F_\lambda(n) \), \( F_{1/\lambda}(n)^* \) are holomorphic in \( \lambda \in \mathbb{C}\setminus\{0\} \) for any fixed \( n \in \mathbb{Z} \).

**Proof** By (5) and (52), the matrix-valued functions \( T_\lambda(n) \), \( T_{1/\lambda}(n)^* \) are non-singular and holomorphic in \( \lambda \in \mathbb{C}\setminus\{0\} \) for any \( n \in \mathbb{Z} \). Therefore, their inverses are also holomorphic. By the definition (6), \( F_\lambda(n) \) is defined as a product of some \( T_\lambda(m) \)'s or their inverses. Thus, \( F_\lambda(n) \) is holomorphic in \( \lambda \in \mathbb{C}\setminus\{0\} \). Similarly, \( F_{1/\lambda}(n)^* \) is defined as a product of some \( T_{1/\lambda}(m)^* \)'s or their inverses. Therefore, \( F_{1/\lambda}(n)^* \) is also holomorphic. \( \square \)

**Lemma 4.3** Let \( f \in C_0(\mathbb{Z}, \mathbb{C}^2) \). We set

\[ h_f(\lambda) := H_f(\lambda) - 2\langle x_0(\lambda)f, f(1/\lambda) \rangle_{\mathbb{C}^2}. \]

Then, \( h_f(\lambda) \) is holomorphic on \( \mathbb{C}\setminus\{0\} \), and we have \( \text{Re} h_f(\zeta) = ||\hat{f}(\zeta)||^2_{\mathbb{C}^2} \) for \( \zeta \in S^1 \).

**Proof** We first note that the statement on the analyticity of the function \( h_f(\lambda) \) is non-trivial, because it involves, at first glance, the matrix-valued function \( x_0(\lambda) \) which is holomorphic only on \( \mathbb{C}\setminus\sigma(U) \). By the identity \( K(\lambda) = I + 2\lambda R(\lambda) \) and the definition (16) of the QW-Fourier transform \( \hat{f}^\mathcal{C} \), we see
\[
\frac{1}{2} h_f(\lambda) = \frac{1}{2} \|f\|^2 + \lambda \sum_{m,n \in \mathbb{Z}} \langle R_\lambda(n,m)f(m), f(n) \rangle_{\mathbb{C}^2}
\]

\[
- \lambda \sum_{m,n \in \mathbb{Z}} \langle x_0(\lambda)F_{1/\lambda}(m)^*, F_\lambda(n)f(n) \rangle_{\mathbb{C}^2}
\]

\[
= \frac{1}{2} \|f\|^2 + \sum_{m,n \in \mathbb{Z}} \langle \lambda[R_\lambda(n,m) - F_\lambda(n)x_0(\lambda)F_{1/\lambda}(m)^*]f(m), f(n) \rangle_{\mathbb{C}^2}.
\]

The sums appeared in (112) are all finite sums, because \(f\) has a finite support. By Theorem 1.2, we have

\[
\lambda[R_\lambda(n,m) - F_\lambda(n)x_0(\lambda)F_{1/\lambda}(m)^*] = \begin{cases} 
F_\lambda(n)z_L(0)F_{1/\lambda}(m)^* & (m \leq n - 1), \\
F_\lambda(n)z_R(0)F_{1/\lambda}(m)^* & (m - 1 \geq n), \\
F_\lambda(n)z_L(0)F_{1/\lambda}(n)^* - z_L(n) & (n = m).
\end{cases}
\]

Substituting these formulas into (112), we have

\[
\frac{1}{2} h_f(\lambda) = \sum_{n \in \mathbb{Z}} \sum_{m \leq n - 1} \langle F_\lambda(n)z_L(0)F_{1/\lambda}(m)^*f(m), f(n) \rangle_{\mathbb{C}^2}
\]

\[
+ \sum_{n \in \mathbb{Z}} \sum_{m \geq n + 1} \langle F_\lambda(n)z_R(0)F_{1/\lambda}(m)^*f(m), f(n) \rangle_{\mathbb{C}^2}
\]

\[
+ \sum_{n \in \mathbb{Z}} \langle F_\lambda(n)z_L(0)F_{1/\lambda}(n)^*f(n), f(n) \rangle_{\mathbb{C}^2}
\]

\[
- \sum_{n \in \mathbb{Z}} \langle z_L(n)f(n), f(n) \rangle_{\mathbb{C}^2} + \frac{1}{2} \|f\|^2.
\]

From (113) and Lemma 4.2, the function \(h_f(\lambda)\) is holomorphic in \(\lambda \in \mathbb{C} \setminus \{0\}\). By taking the complex conjugate of (113), we see

\[
\frac{1}{2} h_f^*(\lambda) = \sum_{n \in \mathbb{Z}} \sum_{m \leq n - 1} \langle F_{1/\lambda}(m)z_L(0)^*F_\lambda(n)^*f(n), f(m) \rangle_{\mathbb{C}^2}
\]

\[
+ \sum_{n \in \mathbb{Z}} \sum_{m \geq n + 1} \langle F_{1/\lambda}(m)z_R(0)^*F_\lambda(n)^*f(n), f(m) \rangle_{\mathbb{C}^2}
\]

\[
+ \sum_{n \in \mathbb{Z}} \langle F_{1/\lambda}(n)z_L(0)^*F_\lambda(n)^*f(n), f(n) \rangle_{\mathbb{C}^2}
\]

\[
- \sum_{n \in \mathbb{Z}} \langle z_L(n)^*f(n), f(n) \rangle_{\mathbb{C}^2} + \frac{1}{2} \|f\|^2.
\]
By the item (6) in Lemma 2.4, we have $z_L(0)^* = I - z_R(0)$, $z_R(0)^* = I - z_L(0)$, and $z_L(n)^* = I - z_R(n)$. Substituting these identities into (114), we obtain

$$
\frac{1}{2} h_f(\lambda) = \sum_{n \in \mathbb{Z}} \sum_{m \leq n-1} \langle F_\lambda(n)^* f(n), F_{1/\lambda}(m)^* f(m) \rangle_{\mathbb{C}^2} \\
- \sum_{n \in \mathbb{Z}} \sum_{m \geq n+1} \langle F_\lambda(n)^* z_\lambda(0) F_\lambda(n)^* f(n), f(m) \rangle_{\mathbb{C}^2} \\
+ \sum_{n \in \mathbb{Z}} \sum_{m \geq n+1} \langle F_\lambda(n)^* f(n), F_{1/\lambda}(m)^* f(m) \rangle_{\mathbb{C}^2} \\
- \sum_{n \in \mathbb{Z}} \sum_{m \geq n+1} \langle F_{1/\lambda}(m) z_\lambda(0) F_\lambda(n)^* f(n), f(m) \rangle_{\mathbb{C}^2} \\
+ \sum_{n \in \mathbb{Z}} \langle (F_\lambda(n)^* f(n), f(n) \rangle_{\mathbb{C}^2} - \frac{1}{2} \|f\|^2. 
$$

(115)

Gathering the first, third, and fifth lines in (115) and using (16), we obtain

$$
\frac{1}{2} h_f(\lambda) = \langle \hat{f}^\ast (1/\lambda), \hat{f}^\ast (\lambda) \rangle_{\mathbb{C}} - \frac{1}{2} \|f\|^2 + \sum_{n \in \mathbb{Z}} \langle z_R(n) f(n), f(n) \rangle_{\mathbb{C}^2} \\
- \sum_{n \in \mathbb{Z}} \sum_{m \leq n-1} \langle F_{1/\lambda}(m) z_\lambda(0) F_\lambda(n)^* f(n), f(m) \rangle_{\mathbb{C}^2} \\
- \sum_{n \in \mathbb{Z}} \sum_{m \geq n+1} \langle F_{1/\lambda}(m) z_\lambda(0) F_\lambda(n)^* f(n), f(m) \rangle_{\mathbb{C}^2} \\
- \sum_{n \in \mathbb{Z}} \langle F_{1/\lambda}(n) z_\lambda(0) F_\lambda(n)^* f(n), f(n) \rangle_{\mathbb{C}^2}. 
$$

(116)

In the second and the third lines in (116), we change the order of the summations in $n$ and $m$, and exchange the role of the letters $m$, $n$; we see

$$
\frac{1}{2} h_f(\lambda) = \langle \hat{f}^\ast (1/\lambda), \hat{f}^\ast (\lambda) \rangle_{\mathbb{C}} - \frac{1}{2} \|f\|^2 + \sum_{n \in \mathbb{Z}} \langle z_R(n) f(n), f(n) \rangle_{\mathbb{C}^2} \\
- \sum_{n \in \mathbb{Z}} \sum_{m \geq n+1} \langle F_{1/\lambda}(n) z_\lambda(0) F_\lambda(m)^* f(m), f(n) \rangle_{\mathbb{C}^2} \\
- \sum_{n \in \mathbb{Z}} \sum_{m \leq n-1} \langle F_{1/\lambda}(n) z_\lambda(0) F_\lambda(m)^* f(m), f(n) \rangle_{\mathbb{C}^2} \\
- \sum_{n \in \mathbb{Z}} \langle F_{1/\lambda}(n) z_\lambda(0) F_\lambda(n)^* f(n), f(n) \rangle_{\mathbb{C}^2}. 
$$

(117)

In the last line of (117), we use Lemma 3.1, namely the formula

$$
F_{1/\lambda}(n) z_\lambda(0) F_\lambda(n)^* = F_{1/\lambda}(n) z_\lambda(0) F_\lambda(n)^* + z_R(n) - z_L(n).
$$
and we finally obtain
\[
\frac{1}{2} \mathcal{H}(\lambda) = - \sum_{n \in \mathbb{Z}} \sum_{m \leq n-1} \langle F_{1/\lambda}(n) z_L(0) F_{\lambda}(m)^* f(m), f(n) \rangle_{C^2} \\
- \sum_{n \in \mathbb{Z}} \sum_{m \geq n+1} \langle F_{1/\lambda}(n) z_L(0) F_{\lambda}(m)^* f(m), f(n) \rangle_{C^2} \\
- \sum_{n \in \mathbb{Z}} \langle F_{1/\lambda}(n) z_L(0) F_{\lambda}(n)^* f(n), f(n) \rangle_{C^2} \\
+ \langle \hat{\mathcal{C}}(1/\lambda), \hat{\mathcal{C}}(\lambda) \rangle - \frac{1}{2} \|f\|^2 \\
+ \sum_{n \in \mathbb{Z}} \langle z_L(n) f(n), f(n) \rangle_{C^2}.
\]

(118)

Now, we take \( \lambda = \zeta \in S^1 \). Since \( 1/\zeta = \zeta \), all of the terms in (113) and (118) are canceled each other after taking the sum of them except the term
\[
\langle \hat{\mathcal{C}}(1/\zeta), \hat{\mathcal{C}}(\zeta) \rangle_{C^2} = \|\hat{\mathcal{C}}(\zeta)\|_{C^2}^2
\]
in (118), and this shows the assertion. \( \square \)

We prepare an estimate of an integral which will be used to prove Proposition 4.5 below.

**Lemma 4.4** Let \( u \in \mathbb{C}^2 \). Then, there exists a positive constant \( C_u \), such that, for any real number \( r \) satisfying \( 2/3 \leq r < 1 \), we have
\[
\int_{S^1} \|x(r\zeta)u\|_{C^2} d\zeta \leq C_u(1 - r)^{-1/2}.
\]

**Proof** We take \( \zeta \in S^1 \) and \( r \in [2/3, 1) \). By definition, we have \( x(r\zeta) = I + 2r\zeta x_0(r\zeta) \). This shows that
\[
\|x(r\zeta)u\|_{C^2} \leq \|u\|_{C^2} + 2r\|x_0(r\zeta)u\|_{C^2} \leq \|u\|_{C^2} + 2\|x_0(r\zeta)u\|_{C^2}
\]
for any \( u \in \mathbb{C}^2 \). Taking \( f = \delta_0 \otimes u \) in (109), we see
\[
\|R(r\zeta)(\delta_0 \otimes u)\| = (1 - r^2)^{-1/2} \left[ \text{Re} \, H_{\delta_0} (r\zeta) \right]^{1/2} \\
= (1 + r)^{-1/2} (1 - r)^{-1/2} \left[ \text{Re} \, H_{\delta_0} (r\zeta) \right]^{1/2} \\
\leq (1 - r)^{-1/2} \left[ \text{Re} \, H_{\delta_0} (r\zeta) \right]^{1/2}.
\]

(120)

It follows from (119), (120) and the definition of \( x_0(\lambda) \) in (72) that:
Integrating this inequality over $S^1$ with respect to the normalized Lebesgue measure $d\zeta$, and using the Cauchy–Schwarz inequality, we see

$$\int_{S^1} \|x(r\zeta)u\|_{C^2} \, d\zeta \leq \|u\|_{C^2} + 2(1 - r)^{-1/2} \left[ \text{Re} H_{\delta_0 \otimes u}(r\zeta) \right]^{1/2}.$$  

Now, from (111), we have

$$\langle \nu_{\delta_0 \otimes u}(S^1) = \lim_{r \to 1} \int_{S^1} \text{Re} H_{\delta_0 \otimes u}(r\zeta) \, d\zeta, \rangle$$

which shows that the integral $\int_{S^1} \text{Re} H_{\delta_0 \otimes u}(r\zeta) \, d\zeta$ is continuous in $r \in [3/2, 1]$. Hence, we can take a positive constant $A_u$, such that

$$\int_{S^1} \text{Re} H_{\delta_0 \otimes u}(r\zeta) \, d\zeta \leq A_u^2$$

for any $r \in [2/3, 1]$. Therefore, we have

$$\int_{S^1} \|x(r\zeta)u\|_{C^2} \, d\zeta \leq [(1 - r)^{1/2} \|u\|_{C^2} + 2A_u](1 - r)^{-1/2} \leq \|u\|_{C^2}/\sqrt{3} + 3A_u(1 - r)^{-1/2}.$$  

The assertion follows by setting $C_u = \|u\|_{C^2}/\sqrt{3} + 3A_u$.

The following proposition plays a central role in the proof of Theorem 1.5.

**Proposition 4.5** For any $f \in C_0(\mathbb{Z}, \mathbb{C})$, the measure $\nu_f$ on $S^1$ given in (110), (111) is written as

$$dv_f(\zeta) = \langle d\Sigma(\zeta) \tilde{f}(\zeta), \tilde{f}(\zeta) \rangle_{C^2},$$  

where $d\Sigma$ is the positive-matrix-valued measure given in (106), (107).

**Proof** We use the characterization (111) of the measure $dv_f$. By Lemma 4.3, the function $H_f(\lambda)$ can be written as
\[
H_f(\lambda) = 2\lambda \langle x_0(\lambda) \hat{f}^c(\lambda), \hat{f}^c(1/\lambda) \rangle_{C^2} + h_f(\lambda)
\]
\[
= \langle x(\lambda) \hat{f}^c(\lambda), \hat{f}^c(1/\lambda) \rangle_{C^2} - \langle \hat{f}^c(\lambda), \hat{f}^c(1/\lambda) \rangle_{C^2} + h_f(\lambda),
\]
where \( h_f \) is defined in Lemma 4.3. Taking the real part, we see
\[
\text{Re} \ H_f(\lambda) = \text{Re} \langle x(\lambda) \hat{f}^c(\lambda), \hat{f}^c(1/\lambda) \rangle_{C^2}
\]
\[
- \text{Re} \langle \hat{f}^c(\lambda), \hat{f}^c(1/\lambda) \rangle_{C^2} + \text{Re} h_f(\lambda).
\]
By Lemmas 4.2 and 4.3, the functions \( \hat{f}^c(\lambda), h_f(\lambda) \) are holomorphic on \( \mathbb{C} \setminus \{0\} \). Thus, by Lemma 4.3, we see
\[
\lim_{r \to 1} \left[ \text{Re} \langle \hat{f}^c(r\xi), \hat{f}^c(r^{-1}\xi) \rangle_{C^2} - \text{Re} h_f(r\xi) \right] = \|\hat{f}^c(\xi)\|_{C^2}^2 - \text{Re} h_f(\xi) = 0.
\]
Therefore, (111) shows that
\[
dv_f(\zeta) = w \ast -\lim_{r \to 1} \text{Re} \langle x(r\xi) \hat{f}^c(r\xi), \hat{f}^c(r^{-1}\xi) \rangle_{C^2} \, \text{d}e'(\zeta). \tag{122}
\]
We calculate the right-hand side of (122) using (107) as follows. For simplicity of notation, we set
\[
F_L(\lambda) = \langle \hat{f}^c(\lambda), e_L \rangle_{C^2}, \quad F_R(\lambda) = \langle \hat{f}^c(\lambda), e_R \rangle_{C^2},
\]
which are holomorphic on \( \mathbb{C} \setminus \{0\} \). We write
\[
\langle x(\lambda) \hat{f}^c(\lambda), \hat{f}^c(1/\lambda) \rangle_{C^2}
\]
\[
= F_L(\lambda)F_L(1/\lambda) \langle x(\lambda)e_L, e_L \rangle_{C^2} + F_L(\lambda)F_R(1/\lambda) \langle x(\lambda)e_L, e_R \rangle_{C^2}
\]
\[
+ F_R(\lambda)F_L(1/\lambda) \langle x(\lambda)e_R, e_L \rangle_{C^2} + F_R(\lambda)F_R(1/\lambda) \langle x(\lambda)e_R, e_R \rangle_{C^2},
\]
and hence
\[
2\text{Re} \langle x(r\xi) \hat{f}^c(r\xi), \hat{f}^c(r^{-1}\xi) \rangle = g_1(r\xi) + g_2(r\xi) + g_3(r\xi) + g_4(r\xi), \tag{123}
\]
where we set
\[
g_1(\lambda) = F_L(\lambda)F_L(1/\lambda) \langle x(\lambda)e_L, e_L \rangle_{C^2} + F_L(1/\lambda)F_L(\lambda) \langle x(\lambda)e_L, e_L \rangle_{C^2},
\]
\[
g_2(\lambda) = F_R(\lambda)F_L(1/\lambda) \langle x(\lambda)e_R, e_L \rangle_{C^2} + F_R(1/\lambda)F_L(\lambda) \langle x(\lambda)e_R, e_L \rangle_{C^2},
\]
\[
g_3(\lambda) = F_L(\lambda)F_R(1/\lambda) \langle x(\lambda)e_L, e_R \rangle_{C^2} + F_L(1/\lambda)F_R(\lambda) \langle x(\lambda)e_L, e_R \rangle_{C^2},
\]
\[
g_4(\lambda) = F_R(\lambda)F_R(1/\lambda) \langle x(\lambda)e_R, e_R \rangle_{C^2} + F_R(1/\lambda)F_R(\lambda) \langle x(\lambda)e_R, e_R \rangle_{C^2}.
\]
For \( 0 < r < 1 \) and \( \zeta \in S^1 \), we set
\[
k_L(r, \zeta) = F_L(r^{-1}\zeta) - F_L(r\zeta), \quad k_R(r, \zeta) = F_R(r^{-1}\zeta) - F_R(r\zeta),
\]
so that

\[
\begin{align*}
g_1(r\zeta) &= F_L(r\zeta)k_L(r, \zeta)(x(r\zeta)e_L, e_L)_{C^2} \\
&+ F_L(r\zeta)k_L(r, \zeta)(x(r\zeta)^*e_L, e_L)_{C^2} \\
&+ 2|F_L(r\zeta)|^2\langle |\text{Re } x(r\zeta)|e_L, e_L \rangle_{C^2}, \\
g_2(r\zeta) &= F_R(r\zeta)k_L(r, \zeta)(x(r\zeta)e_R, e_L)_{C^2} \\
&+ F_L(r\zeta)k_R(r, \zeta)(x(r\zeta)^*e_R, e_L)_{C^2} \\
&+ 2F_R(r\zeta)|F_L(r\zeta)|\langle |\text{Re } x(r\zeta)|e_R, e_L \rangle_{C^2}, \\
g_3(r\zeta) &= F_L(r\zeta)k_R(r, \zeta)(x(r\zeta)e_L, e_R)_{C^2} \\
&+ F_R(r\zeta)k_L(r, \zeta)(x(r\zeta)^*e_L, e_R)_{C^2} \\
&+ 2F_L(r\zeta)|F_R(r\zeta)|\langle |\text{Re } x(r\zeta)|e_L, e_R \rangle_{C^2}, \\
g_4(r\zeta) &= F_R(r\zeta)k_R(r, \zeta)(x(r\zeta)e_R, e_R)_{C^2} \\
&+ F_R(r\zeta)k_R(r, \zeta)(x(r\zeta)^*e_R, e_R)_{C^2} \\
&+ 2|F_R(r\zeta)|^2\langle |\text{Re } x(r\zeta)|e_R, e_R \rangle_{C^2}. \\
\end{align*}
\]

(124)

Since the functions \(F_L, F_R\) are holomorphic on \(\mathbb{C}\setminus\{0\}\), we have, for \(2/3 \leq r < 1\)

\[
|k_L(r, \zeta)| = |F_L(r^{-1}\zeta) - F_L(r\zeta)| \\
\leq \|F_L'\|_{C(A)}|r^{-1} - r| \leq 3C(1-r), \quad (125)
\]

\[
|k_R(r, \zeta)| = |F_R(r^{-1}\zeta) - F_R(r\zeta)| \\
\leq \|F_R'\|_{C(A)}|r^{-1} - r| \leq 3C(1-r),
\]

where we set

\[
A = \{ \lambda \in \mathbb{C} \mid 2/3 \leq |\lambda| \leq 3/2\}, \quad C = \max\{\|F_L'\|_{C(A)}, \|F_R'\|_{C(A)}\}.
\]

For simplicity, we denote \(k(r, \zeta)\) one of \(k_L(r, \zeta), k_L(r, \zeta), k_R(r, \zeta)\) and \(k_R(r, \zeta)\), and we denote \(F(\lambda)\) one of \(F_L(\lambda), F_L(\lambda), F_R(\lambda)\) and \(F_R(\lambda)\). Then, for any \(h \in C(S^1)\), any \(u, v \in \mathbb{C}^2\), and any \(r \in [2/3, 1)\), Lemma 4.4 and the estimate (125) show

\[
\left| \int_{S^1} h(\zeta)F(r\zeta)k(r, \zeta)(x(r\zeta)u, v)_{C^2} \, d\zeta(\zeta) \right| \\
\leq \int_{S^1} |h(\zeta)||F(r\zeta)||k(r, \zeta)||x(r\zeta)u||_{C^2}\|v\|_{C^2} \, d\zeta(\zeta) \\
\leq 3C\|v\|_{C^2}\|h\|_{C(S^1)}\|F\|_{C(A)}(1-r) \int_{S^1} \|x(r\zeta)u\|_{C^2} \, d\zeta(\zeta) \\
\leq 3CC_u\|v\|_{C^2}\|h\|_{C(S^1)}\|F\|_{C(A)}(1-r)^{1/2},
\]

\(\Birkhäuser\)
which tends to zero as \( r \uparrow 1 \). This shows that, for any \( h \in C(S^1) \), in the integral

\[
\int_{S^1} h(\zeta)g_i(r\zeta) \, d\zeta(i = 1, 2, 3, 4),
\]

the contributions from the first and the second term of \( g_i(r\zeta) \) in (124) tend to zero as \( r \uparrow 1 \). To handle the contributions from the last term involving \( \text{Re} \, x(r\zeta) \) of \( g_i(r\zeta) \) in (124), let \( G(\lambda) \) be a smooth function on \( A \) and we take \( u, v \in \mathbb{C}^2 \). We write \( G(\lambda) = G(x, y) \) for \( \lambda = x + iy \). Then, for \( \zeta \in S^1 \), we have

\[
|G(\zeta) - G(r\zeta)| \leq B(1 - r), \quad B = \|G_x\|_{C(A)} + \|G_y\|_{C(A)},
\]

where \( G_x \) (resp. \( G_y \)) is the partial derivative of \( G \) with respect to \( x \) (resp. \( y \)). Hence, for any \( h \in C(S^1) \), we see by Lemma 4.4

\[
\left| \int_{S^1} h(\zeta)(G(r\zeta) - G(\zeta))\langle [\text{Re} \, x(r\zeta)]u, v \rangle_{C^2} \, d\zeta(\zeta) \right|
\leq \|h\|_{C(S^1)}B(1 - r) \int_{S^1} |\langle [\text{Re} \, x(r\zeta)]u, v \rangle_{C^2} | \, d\zeta(\zeta)
\leq \|h\|_{C(S^1)}B(1 - r) \int_{S^1} (\|x(r\zeta)u\|_{C^2}\|v\|_{C^2} + \|x(r\zeta)v\|_{C^2}\|u\|_{C^2}) \, d\zeta(\zeta)
\leq \|h\|_{C(S^1)}B(1 - r)(C_u\|v\|_{C^2} + C_v\|u\|_{C^2})(1 - r)^{-1/2}
= \|h\|_{C(S^1)}B(C_u\|v\|_{C^2} + C_v\|u\|_{C^2})(1 - r)^{1/2},
\]

which tends to zero as \( r \uparrow 1 \). This and (107) show that

\[
\lim_{r \uparrow 1} \int_{S^1} h(\zeta)G(r\zeta)\langle [\text{Re} \, x(r\zeta)]u, v \rangle_{C^2} \, d\zeta(\zeta)
= \lim_{r \uparrow 1} \int_{S^1} h(\zeta)G(\zeta)\langle [\text{Re} \, x(r\zeta)]u, v \rangle_{C^2} \, d\zeta(\zeta)
+ \lim_{r \uparrow 1} \int_{S^1} h(\zeta)(G(r\zeta) - G(\zeta))\langle [\text{Re} \, x(r\zeta)]u, v \rangle_{C^2} \, d\zeta(\zeta)
= \lim_{r \uparrow 1} \int_{S^1} h(\zeta)G(\zeta)\langle [\text{Re} \, x(r\zeta)]u, v \rangle_{C^2} \, d\zeta(\zeta)
\]

\[
= \int_{S^1} h(\zeta)G(\zeta) \, d\Sigma(\zeta)u, v \rangle_{C^2}.
\]

Therefore, by taking \( G(\lambda) \) as one of functions \(|F_L(\lambda)|^2, F_R(\lambda)F_L(\lambda), F_L(\lambda)\overline{F_R(\lambda)}, |F_R(\lambda)|^2 \), we obtain
An eigenfunction expansion formula for one-dimensional…

$$\lim_{r \uparrow 1} \int_{S^1} h(\zeta) g_1(r\zeta) \, d\zeta(\zeta) = 2 \int_{S^1} h(\zeta) |F_L(\zeta)|^2 \langle d\Sigma(\zeta)e_L, e_L \rangle_{C^2},$$

$$\lim_{r \uparrow 1} \int_{S^1} h(\zeta) g_2(r\zeta) \, d\zeta(\zeta) = 2 \int_{S^1} h(\zeta) F_R(\zeta) \overline{F_L(\zeta)} \langle d\Sigma(\zeta)e_R, e_L \rangle_{C^2},$$

$$\lim_{r \uparrow 1} \int_{S^1} h(\zeta) g_3(r\zeta) \, d\zeta(\zeta) = 2 \int_{S^1} h(\zeta) F_L(\zeta) \overline{F_R(\zeta)} \langle d\Sigma(\zeta)e_L, e_R \rangle_{C^2},$$

$$\lim_{r \uparrow 1} \int_{S^1} h(\zeta) g_4(r\zeta) \, d\zeta(\zeta) = 2 \int_{S^1} h(\zeta) |F_R(r\zeta)|^2 \langle d\Sigma(\zeta)e_R, e_R \rangle_{C^2}.$$ 

Therefore, by (122), (123), we conclude

$$\int_{S^1} h(\zeta) \, dv_J(\zeta)$$

$$\quad = \lim_{r \uparrow 1} \int_{S^1} h(\zeta) \text{Re} \langle x(r\zeta)^{\hat{\mathcal{C}}}(r\zeta), \hat{\mathcal{F}}(r^{-1}\zeta) \rangle_{C^2} \, d\zeta(\zeta)$$

$$\quad = \lim_{r \uparrow 1} \frac{1}{2} \int_{S^1} h(\zeta) [g_1(r\zeta) + g_4(r\zeta) + g_3(r\zeta) + g_4(r\zeta)] \, d\zeta(\zeta)$$

$$\quad = \int_{S^1} h(\zeta) |F_L(\zeta)|^2 \langle d\Sigma(\zeta)e_L, e_L \rangle_{C^2}$$

$$\quad + \int_{S^1} h(\zeta) F_R(\zeta) \overline{F_L(\zeta)} \langle d\Sigma(\zeta)e_R, e_L \rangle_{C^2}$$

$$\quad + \int_{S^1} h(\zeta) F_L(\zeta) \overline{F_R(\zeta)} \langle d\Sigma(\zeta)e_L, e_R \rangle_{C^2}$$

$$\quad + \int_{S^1} h(\zeta) |F_R(r\zeta)|^2 \langle d\Sigma(\zeta)e_R, e_R \rangle_{C^2}$$

$$\quad = \int_{S^1} h(\zeta) \langle d\Sigma(\zeta)^{\hat{\mathcal{C}}}(\zeta), \hat{\mathcal{F}}(\zeta) \rangle_{C^2},$$

which completes the proof.

Proof of Theorem 1.5 The formula (121) in Proposition 4.5 is equivalent to the formula

$$\langle K(\lambda)f, f \rangle = \int_{S^1} \frac{\zeta + \lambda}{\zeta - \lambda} \langle d\Sigma(\zeta)^{\hat{\mathcal{C}}}(\zeta), \hat{\mathcal{F}}(\zeta) \rangle_{C^2}. \quad (127)$$

Since $H_f(\lambda) = \langle K(\lambda)f, f \rangle$, by setting $\lambda = 0$ in (127), we obtain

$$\langle f, f \rangle = \int_{S^1} \langle d\Sigma(\zeta)^{\hat{\mathcal{C}}}(\zeta), \hat{\mathcal{F}}(\zeta) \rangle_{C^2}, \quad (128)$$

which shows (18) for $f = g \in C_0(\mathbb{Z}, C^2)$. Since $K(\lambda) = I + 2\lambda R(\lambda)$, we see
\[
\langle R(\lambda)f, f \rangle = \frac{1}{2\lambda} \langle K(\lambda)f, f \rangle - \frac{1}{2\lambda} \langle f, f \rangle \\
= \int_{S^1} \left( \frac{1}{2\lambda} \frac{\zeta + \lambda}{\zeta - \lambda} - \frac{1}{2\lambda} \right) \langle d\Sigma(\zeta) \hat{f}^c(\zeta), \hat{f}^c(\zeta) \rangle_{C^2} \\
= \int_{S^1} \frac{1}{\zeta - \lambda} \langle d\Sigma(\zeta) \hat{f}^c(\zeta), \hat{f}^c(\zeta) \rangle_{C^2},
\]
which shows (17) for \( f = g \in C_0(\mathbb{Z}, C^2) \). The polarization identity
\[
\langle f, g \rangle = \frac{1}{4} \sum_{k=0}^{3} \langle f + i^k g, f + i^k g \rangle
\]
shows (17) and (18) for any \( f, g \in C_0(\mathbb{Z}, C^2) \). Hence, the assertion (2) in Theorem 1.5 has been proved. To prove the assertion (1) in Theorem 1.5, suppose that the positive-matrix-valued measure \( \Omega \) also satisfies the identity
\[
\langle R(\lambda)f, g \rangle = \int_{S^1} \frac{1}{\zeta - \lambda} \langle d\Omega(\zeta) \hat{f}^c(\zeta), \hat{g}^c(\zeta) \rangle_{C^2}
\]
(130)
for any \( f, g \in C_0(\mathbb{Z}, C^2) \). We note that \( Uf \in C_0(\mathbb{Z}, C^2) \) for \( f \in C_0(\mathbb{Z}, C^2) \). The formula (26) shows that for \( \lambda, \mu \in \mathbb{C}\backslash\{0\} \)
\[
\mathcal{F}_C[(U - \lambda)f](\mu) = (\mu - \lambda) \hat{f}^c(\mu).
\]
Thus, replacing \( f \) by \( (U - \lambda)f \) in (130) shows
\[
\langle f, g \rangle = \langle R(\lambda)(U - \lambda)f, g \rangle \\
= \int_{S^1} \frac{1}{\zeta - \lambda} \langle d\Omega(\zeta) \mathcal{F}_C[(U - \lambda)f](\zeta), \hat{g}^c(\zeta) \rangle_{C^2} \\
= \int_{S^1} \frac{1}{\zeta - \lambda} \langle d\Omega(\zeta) (\zeta - \lambda) \hat{f}^c(\zeta), \hat{g}^c(\zeta) \rangle_{C^2} \\
= \int_{S^1} \langle d\Omega(\zeta) \hat{f}^c(\zeta), \hat{g}^c(\zeta) \rangle_{C^2}.
\]
Then, using the identity \( K(\lambda) = I + 2\lambda R(\lambda) \) again, we have
\[
\langle K(\lambda)f, g \rangle = \int_{S^1} \langle d\Omega(\zeta) \hat{f}^c(\zeta), \hat{g}^c(\zeta) \rangle_{C^2} \\
+ 2\lambda \int_{S^1} \frac{1}{\zeta - \lambda} \langle d\Omega(\zeta) \hat{f}^c(\zeta), \hat{g}^c(\zeta) \rangle_{C^2} \\
= \int_{S^1} \frac{\zeta - \lambda + 2\lambda}{\zeta - \lambda} \langle d\Omega(\zeta) \hat{f}^c(\zeta), \hat{g}^c(\zeta) \rangle_{C^2} \\
= \int_{S^1} \frac{\zeta + \lambda}{\zeta - \lambda} \langle d\Omega(\zeta) \hat{f}^c(\zeta), \hat{g}^c(\zeta) \rangle_{C^2}.
\]
From this, (103), and the identity \( \mathcal{F}_c[\delta_0 \otimes u](\lambda) = u \) for any \( u \in \mathbb{C}^2 \), we obtain the following Herglotz representation of the matrix \( x(\lambda) : \)

\[
\langle x(\lambda)u, v \rangle_{\mathbb{C}^2} = \int_{S^1} \frac{\xi + \lambda}{\xi - \lambda} \langle d\Omega(\xi)u, v \rangle_{\mathbb{C}^2} \quad (u, v \in \mathbb{C}^2),
\]

which shows \( \Sigma = \Omega \). Thus, the uniqueness of the positive-matrix-valued measure in (17) and the assertion (1) in Theorem 1.5 have been proved. We remark that

\[
\langle h(U)f, g \rangle = \int_{S^1} h(\xi) \langle d\Sigma(\xi)\hat{f}(\xi), \hat{g}(\xi) \rangle_{\mathbb{C}^2}
\]

(131) for any \( h \in C(S^1) \) and any \( f, g \in C_0(\mathbb{Z}, \mathbb{C}^2) \). Let us prove (131) before proceeding to the proof of the assertion (3) in Theorem 1.5. By the formula (26), we have

\[
\hat{f}(\lambda) = \mathcal{F}_c[UU^{-1}f](\lambda) = \lambda \mathcal{F}_c[U^{-1}f](\lambda), \quad \mathcal{F}_c[U^{-1}f](\lambda) = \lambda^{-1}\hat{f}(\lambda)
\]

for any \( f \in C_0(\mathbb{Z}, \mathbb{C}^2) \). Iterated use of this and the formula (26) show that, for any integer \( n \), positive or negative

\[
\mathcal{F}_c[U^n f](\lambda) = \lambda^n \hat{f}(\lambda).
\]

By this formula and the item (2) in Theorem 1.5, we have

\[
\langle U^n f, g \rangle = \int_{S^1} \langle d\Sigma(\xi)\mathcal{F}_c[U^n f](\xi), \hat{g}(\xi) \rangle_{\mathbb{C}^2}
\]

\[= \int_{S^1} \xi^n \langle d\Sigma(\xi)\hat{f}(\xi), \hat{g}(\xi) \rangle_{\mathbb{C}^2}.
\]

Therefore, for any Laurent polynomial \( p \) on \( \mathbb{C}\{0\} \), we see

\[
\langle p(U)f, g \rangle = \int_{S^1} p(\xi) \langle d\Sigma(\xi)\hat{f}(\xi), \hat{g}(\xi) \rangle_{\mathbb{C}^2}.
\]

Since the set of Laurent polynomials in \( \xi \in S^1 \) is dense in \( C(S^1) \) with respect to the norm \( \| \cdot \|_{C(S^1)} \), and since \( \|h(\xi)\|_{op} \leq \|h(\xi)\|_{C(S^1)} \), where \( \|h(\xi)\|_{op} \) denotes the operator norm of \( h(\xi) \) on \( \ell^2(\mathbb{Z}, \mathbb{C}^2) \), we have (131). We now proceed to the proof of the assertion (3) in Theorem 1.5. For any \( h \in C(S^1) \) and \( f \in C_0(\mathbb{Z}, \mathbb{C}^2) \), we have

\[
\langle h(U)f, f \rangle = \int_{S^1} h(\xi) d\|E(\xi)f\|^2 = \int_{S^1} h(\xi) \langle d\Sigma(\xi)\hat{f}(\xi), \hat{f}(\xi) \rangle_{\mathbb{C}^2}.
\]

Since the spectral measure \( \|E(\xi)f\|^2 \) is the unique Borel measure on \( S^1 \) satisfying the above with the total mass \( \|f\|^2 \), we see

\[
\|E(A)f\|^2 = \langle E(A)f, f \rangle = \int_A \langle d\Sigma(\xi)\hat{f}(\xi), \hat{f}(\xi) \rangle_{\mathbb{C}^2}
\]

for any Borel set \( A \) in \( S^1 \), and hence, by the polarization identity (129), we see
\[ \langle E(A)f, g \rangle = \int_A \langle d\Sigma(\zeta)\hat{\mathcal{C}}(\zeta), \hat{g}^\mathcal{C}(\zeta) \rangle_{C^2}. \]

Taking \( g = \delta_n \otimes v \) with \( n \in \mathbb{Z} \) and \( v \in \mathbb{C}^2 \), and noting \( \mathcal{F}_c[\delta_n \otimes v](\lambda) = F_{1/\lambda}(n)^*v \), we have
\[
\langle [E(A)f](n), v \rangle_{C^2} = \langle E(A)f, \delta_n \otimes v \rangle = \int_A \langle d\Sigma(\zeta)\hat{\mathcal{C}}(\zeta), F_\zeta(n)^*v \rangle_{C^2}
= \int_A \langle F_\zeta(n)d\Sigma(\zeta)\hat{\mathcal{C}}(\zeta), v \rangle_{C^2}. \tag{132}
\]

Since (132) holds for any \( v \in \mathbb{C}^2 \), we have (19). By setting \( A = S^1 \) in (19), we conclude (20).

**Proof of Corollary 1.6** By taking \( f = \delta_0 \otimes u \) and \( n = 0 \) in (132), we have
\[
\langle [E(A)\delta_0 \otimes u](0), v \rangle_{C^2} = \int_A \langle d\Sigma(\zeta)u, v \rangle_{C^2} = \langle \Sigma(A)u, v \rangle_{C^2}. \tag{133}
\]

We remark that the support, \( \text{supp}(\Sigma) \), of the positive-matrix-valued measure \( \Sigma \) on \( S^1 \) is defined as the complement of the open set
\[
\bigcup_{U: \text{open in } S^1, \Sigma(U) = 0} U.
\]

Now, suppose that \( \zeta_o \in S^1 \) is not contained in \( \sigma(U) \). Then, since \( \text{supp}(E) = \sigma(U) \), there exists an open neighborhood \( U \) of \( \zeta_o \) in \( S^1 \), such that \( E(U) = 0 \). The identity (133) shows \( \Sigma(U) = 0 \). Therefore, \( \zeta_o \notin \text{supp}(\Sigma) \). Conversely, suppose that \( \zeta_o \in S^1 \) is not contained in \( \text{supp}(\Sigma) \), and let \( U \) be an open neighborhood of \( \zeta_o \) satisfying \( \Sigma(U) = 0 \). This means that each of the entry of \( \Sigma \) is zero on \( U \). Thus, the right-hand side of (132) with \( A = U \) is zero for any \( f \in C_0(\mathbb{Z}, \mathbb{C}^2), n \in \mathbb{Z} \) and \( v \in \mathbb{C}^2 \). This means that \( E(U) = 0 \). Thus, we conclude that \( \zeta_o \notin \text{supp}(E) \), which proves the assertion (1) in Corollary 1.6. Next, to prove the assertion (2) in Corollary 1.6, suppose that \( \lambda \in S^1 \) satisfies \( \Sigma(\{ \lambda \}) \neq 0 \). Then, by (133), we see \( \langle [E(\{ \lambda \})(\delta_0 \otimes u)](0), v \rangle_{C^2} \neq 0 \) for some \( u, u \in \mathbb{C}^2 \). Hence, \( E(\{ \lambda \}) \neq 0 \) and \( \lambda \) is an eigenvalue of \( U \). Conversely, suppose that \( \lambda \in S^1 \) is an eigenvalue of \( U \). Let \( f \in \ell^2(\mathbb{Z}, \mathbb{C}^2) \) be an eigenfunction of \( U \) with the eigenvalue \( \lambda \). Since \( E(\{ \lambda \}) \) is the projection onto the eigenspace of \( U \) with the eigenvalue \( \lambda \), we see \( E(\{ \lambda \})f = f \). By (132), we have
\[
\|f(n)\|^2_{C^2} = \langle [E(\{ \lambda \})f](n), f(n) \rangle_{C^2} = \langle F_\lambda(n)\Sigma(\{ \lambda \})\hat{\mathcal{C}}(\lambda), f(n) \rangle_{C^2}
\]
for any \( n \in \mathbb{Z} \). The left-hand side of the above expression is nonzero for some \( n \in \mathbb{Z} \), because \( f \) is not identically zero. Therefore, \( \Sigma(\{ \lambda \}) \neq 0 \), which proves the assertion (2) in Corollary 1.6. \( \square \)
Before proceeding to the proof of Theorem 1.7, we give, based on [7], some accounts on the sesquilinear form $\langle \cdot , \cdot \rangle_{\Sigma}$ defined in (24). In particular, we show that the positivity of the measure $\langle d\Sigma(\zeta)k(\zeta), k(\zeta) \rangle_{\mathbb{C}^2}$ for any $k \in C(S^1, \mathbb{C}^2)$. This fact can be proved using the characterization (107) of $\Sigma$ and the non-negativity of $\text{Re} x(\lambda)$. However, we give here an alternate proof.

We recall that, about the entries of positive-matrix-valued measure $\Sigma$, the diagonals $\mu_L$ and $\mu_R$ are Borel probability measures, while $\alpha = \langle \Sigma e_L, e_R \rangle$ is a complex Borel measure satisfying $|\alpha(A)|^2 \leq \mu_L(A)\mu_R(A)$ for any Borel set $A$, because $\det \Sigma(A) \geq 0$. From this, it follows that the total variation measure $|\alpha|$ of $\alpha$ also satisfies the inequality $|\alpha|(A)^2 \leq \mu_L(A)\mu_R(A)$ for any Borel set $A$ in $S^1$. This shows that $\alpha, |\alpha|$ is absolutely continuous with respect to both of $\mu_L$ and $\mu_R$. We define a probability measure $\mu$ by $\mu(A) = (\mu_L(A) + \mu_R(A))/2$, that is $\mu = \text{Tr}(\Sigma)/2$. Then, we have $|\alpha|(A) \leq \mu(A)$, and thus, $\alpha, \mu_L$ and $\mu_R$ are all absolutely continuous with respect to the measure $\mu = \text{Tr}(\Sigma)/2$. We can write $\alpha = \rho \mu, \mu_L = \rho_L \mu, \mu_R = \rho_R \mu$ with $\rho_L, \rho_R, \rho \in L^1(S^1, \mu)$, where $\rho_L$ and $\rho_R$ are non-negative $\mu$-a.e. Then, we have $|\alpha| = |\rho| \mu$. For any $\xi \in S^1$ and $\varepsilon > 0$, we denote $A(\xi, \varepsilon)$ the arc in $S^1$ centered at $\xi$ and length $\varepsilon$. Then, by the Lebesgue differentiation theorem (see, for example, [4], 2.9.8 Theorem), we have

$$0 \leq \frac{1}{\mu(A(\xi, \varepsilon))²} \left( \mu_L(A(\xi, \varepsilon))\mu_R(A(\xi, \varepsilon)) - |\alpha|(A(\xi, \varepsilon))^2 \right)$$

$$= \frac{1}{A(\xi, \varepsilon)} \int_{A(\xi, \varepsilon)} \rho_L \, d\mu \times \frac{1}{A(\xi, \varepsilon)} \int_{A(\xi, \varepsilon)} \rho_R \, d\mu - \left( \int_{A(\xi, \varepsilon)} |\rho| \, d\mu \right)^2$$

$$\rightarrow \rho_L(\xi)\rho_R(\xi) - |\rho(\xi)|²$$

as $\varepsilon \downarrow 0$ for $\mu$-a.e $\xi \in S^1$. Let $X$ be a Borel set, such that $\mu(X) = 1$ and the above holds for any $\xi \in X$. Then, $|\rho|^2 \leq \rho_L\rho_R$ on $X$. Let $k \in C(S^1, \mathbb{C}^2)$ and we write $k = (k_L, k_R)$ with $k_L, k_R \in C(S^1)$. Then, for any Borel set $A$ on $S^1$, we have

$$\int_A \langle d\Sigma(\zeta)k(\zeta), k(\zeta) \rangle_{\mathbb{C}^2}$$

$$= \int_A \left( |k_L|^² \, d\mu_L + k_L k_R \, d\alpha + \overline{k_L} k_R \, d\overline{\alpha} + |k_R|^² \, d\mu_R \right)$$

$$= \int_A \left( |k_L|^² \rho_L + k_L k_R \rho + \overline{k_L} k_R \overline{\rho} + |k_R|^² \rho_R \right) \, d\mu.$$

Since the integrand in (134) can be written as

$$\rho_L \left| k_L + \frac{k_R \overline{\rho}}{\rho_L} \right|^² + |k_R|^² \left( \rho_R - \frac{|\rho|^²}{\rho_L} \right),$$

it is non-negative at points where $\rho_L > 0$ in $X$. At a point $\xi \in X$ satisfying $\rho_L(\xi) = 0$, the integrand in (134) becomes $|k_R(\xi)|² \rho_R(\xi) \geq 0$. Thus, the integral in (134) is non-negative. Hence, the measure $\langle d\Sigma(\zeta)k(\zeta), k(\zeta) \rangle_{\mathbb{C}^2}$ is indeed a positive measure.
Proof of Theorem 1.7 For $k \in C(S^1, C^2)$, we define $\|k\|_\infty$ by the formula $\|k\|_\infty = \sup_{\zeta \in S^1} \|k(\zeta)\|_{C^2}$. We first show the inequality
\[
\|p\|^2_\Sigma \leq 4\|p\|^2_\infty. \tag{135}
\]
Indeed, using (134) for $A = S^1$, we see
\[
\|k\|^2_\Sigma = \int_{S^1} \left( |k_L|^2 \rho_L + k_R k_R \rho + |k_R|^2 \rho_R \right) d\mu \\
= \int_{S^1} \|k\|^2_\Sigma \mu_L(S^1) + 2 \int_{S^1} |k_L||k_R| |\rho| d\mu + \|k\|^2_\infty \mu_R(S^1) \\
\leq 2\|k\|^2_\Sigma + 2\|k\|^2_\infty |\alpha(S^1)| \leq 4\|k\|^2_\infty,
\]
which shows (135). Let $\pi$ be the natural projection from $C(S^1, C^2)$ to $C(S^1, C^2)/N$. Then, the inner product $\langle \cdot, \cdot \rangle_\Sigma$ on $C(S^1, C^2)/N$ is defined by $\langle \pi(k), \pi(l) \rangle_\Sigma = \langle k, l \rangle_\Sigma$ for $k, l \in C(S^1, C^2)$, and $L^2(S^1, C^2)_\Sigma$ is the completion of $C(S^1, C^2)/N$ by the norm $\|\pi(k)\|_\Sigma = \sqrt{\langle \pi(k), \pi(k) \rangle_\Sigma}$. The inner product $\langle \cdot, \cdot \rangle_\Sigma$ and its norm $\| \cdot \|_\Sigma$ is defined on the whole of $L^2(S^1, C^2)_\Sigma$ in a standard manner. The map
\[
\mathfrak{F}_C : C_0(\mathbb{Z}, C^2) \rightarrow L^2(S^1, C^2)_\Sigma
\]
is defined by the composition of $F_C, \pi$ and the inclusion $C(S^1, C^2)/N \hookrightarrow L^2(S^1, C^2)_\Sigma$. For any $u \in C^2$ and $n \in \mathbb{Z}$, we have, by (26), $F_C[U^n(\delta_0 \otimes u)](\lambda) = \lambda^n u$. Therefore, the subspace $F_C[C_0(\mathbb{Z}, C^2)]$ of $C(S^1, C^2)$ contains the space of $C^2$-valued Laurent polynomials, and hence, it is dense in $C(S^1, C^2)$ with respect to the supremum norm $\| \cdot \|_\infty$. We take $s \in L^2(S^1, C^2)_\Sigma$ and $\epsilon > 0$. Then, we can take a function $k \in C(S^1, C^2)$, such that $\|\pi(k) - s\|_\Sigma < \epsilon$. Since $F_C[C_0(\mathbb{Z}, C^2)]$ is dense in $C(S^1, C^2)$, we can take a function $f \in C_0(\mathbb{Z}, C^2)$, such that $\|F_C(f) - k\|_\infty < \epsilon$. Therefore
\[
\|\mathfrak{F}_C(f) - s\|_\Sigma \leq \|\pi \circ F_C(f) - \pi(k)\|_\Sigma + \|\pi(k) - s\|_\Sigma \\
= \|F_C(f) - k\|_\Sigma + \|\pi(k) - s\|_\Sigma \\
\leq 4\|F_C(f) - k\|_\infty + \epsilon \leq 5\epsilon.
\]
This shows that $\mathfrak{F}_C[C_0(\mathbb{Z}, C^2)]$ is dense in $L^2(S^1, C^2)_\Sigma$. For any $f \in C_0(\mathbb{Z}, C^2)$, (18) shows that $\|\mathfrak{F}_C(f)\|_\Sigma = \|f\|_\Sigma$ for $f \in C_0(\mathbb{Z}, C^2)$. To extend the map $\mathfrak{F}_C$ to the whole space $\ell^2(\mathbb{Z}, C^2)$, we take $f \in \ell^2(\mathbb{Z}, C^2)$. Noting that $C_0(\mathbb{Z}, C^2)$ is dense in $\ell^2(\mathbb{Z}, C^2)$, we take a sequence $\{f_n\}_{n=1}^\infty$ of functions $f_n \in C_0(\mathbb{Z}, C^2)$, such that $\|f - f_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Then, we have $\|\mathfrak{F}_C(f_n) - \mathfrak{F}_C(f_m)\|_\Sigma = \|f_n - f_m\|_\infty \rightarrow 0$ as $n, m \rightarrow 0$, and thus, the limit $s = \lim_{n \rightarrow \infty} \mathfrak{F}_C(f_n)$ exists in $L^2(S^1, C^2)_\Sigma$. We have
\[
\|s\|_\Sigma = \lim_{n \rightarrow \infty} \|\mathfrak{F}_C(f_n)\|_\Sigma = \lim_{n \rightarrow \infty} \|f_n\| = \|f\|. \tag{137}
\]
We take another sequence $\{g_n\}_{n=1}^\infty$ of functions $g_n \in C_0(\mathbb{Z}, C^2)$ converging $f$. Then $\|\mathfrak{F}_C(g_n) - \mathfrak{F}_C(f_n)\|_\Sigma = \|g_n - f_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Hence
\[
\|\mathfrak{F}_C(g_n) - s\|_\Sigma \leq \|\mathfrak{F}_C(g_n) - \mathfrak{F}_C(f_n)\|_\Sigma + \|\mathfrak{F}_C(f_n) - s\|_\Sigma \rightarrow 0
\]
as \( n \to \infty \). This shows that \( s = \lim_{n \to \infty} \mathfrak{F}_c(f_n) \) does not depend on the choice of sequences of functions in \( C_0(\mathbb{Z}, \mathbb{C}^2) \) converging \( f \). Thus, we can define \( \mathfrak{F}_C(f) = \lim_{n \to \infty} \mathfrak{F}_c(f_n) \). We prove that the map \( \mathfrak{F}_c : l^2(\mathbb{Z}, \mathbb{C}^2) \to L^2(S^1, \mathbb{C}^2)\Sigma \) is unitary. That the map \( \mathfrak{F}_c \) preserves the norm has proved in (137). To prove the surjectivity, we take \( s \in L^2(S^1, \mathbb{C}^2)\Sigma \). For each positive integer \( n \), we take \( k_n \in C(S^1, \mathbb{C}^2) \) such that \( \| \pi(k_n) - s \|_{\Sigma} < 1/n \). Since \( FC[ C_0(\mathbb{Z}, \mathbb{C}^2) ] \) is dense in \( C(S^1, \mathbb{C}^2) \), we can take \( f_n \in C_0(\mathbb{Z}, \mathbb{C}^2) \), such that \( \| FC(f_n) - k_n \|_{\infty} < 1/n \). Then, we have

\[
\| \mathfrak{F}_c(f_n) - s \|_{\Sigma} \leq \| \mathfrak{F}_c(f_n) - \pi(k_n) \|_{\Sigma} + \| \pi(k_n) - s \|_{\Sigma} < 5/n, \\
\| f_n - f_m \| = \| \mathfrak{F}_c(f_n) - \mathfrak{F}_c(f_m) \|_{\Sigma} \leq \| \mathfrak{F}_c(f_n) - s \|_{\Sigma} + \| s - \mathfrak{F}_c(f_m) \|_{\Sigma} < 5/n + 5/m.
\]

This shows that the sequence \( \{f_n\} \) converges to a function \( f \in l^2(\mathbb{Z}, \mathbb{C}^2) \) and \( \mathfrak{F}_c(f) = \lim_{n \to \infty} \mathfrak{F}_c(f_n) = s \). This shows that \( \mathfrak{F}_c \) is surjective. Therefore, \( \mathfrak{F}_c : l^2(\mathbb{Z}, \mathbb{C}^2) \to L^2(S^1, \mathbb{C}^2)\Sigma \) is a unitary operator. This completes the proof of Theorem 1.7.

\[ \square \]

5 Examples

In this section, we consider two examples, homogeneous quantum walks and certain quantum walks with non-constant coin matrix. The quantum walk with a non-constant coin matrix considered here is a special case of the so-called two phase models discussed originally in [3]. In these examples, we use Theorem 1.4 to compute concretely the matrix-valued function \( x_0(\lambda) \) and the positive-matrix-valued measure \( \Sigma \).

5.1 Homogeneous quantum walks

First of all, let us consider the fundamental example, namely the case where the coin matrix \( C \) is constant, say \( C(n) = C \in U(2) \) for any \( n \in \mathbb{Z} \). For simplicity of notation, we assume

\[ C \in SU(2), \]

so that \( C \) can be written as

\[
C = \begin{bmatrix}
\alpha & \beta \\
-\bar{\beta} & \bar{\alpha}
\end{bmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1, \quad \alpha \neq 0.
\]

The transfer matrix \( T_{\lambda}(n) \) does not depend on \( n \in \mathbb{Z} \) and we denote it by \( T_C(\lambda) \), which and whose inverse are given by

\[ \text{Birkhäuser} \]
The matrix $F_{\lambda}(n)$ is then given by

$$F_{\lambda}(n) = T_C(\lambda)^n \quad (n \in \mathbb{Z}).$$

When $\beta = 0$, we have $|\alpha| = 1$ and the transfer matrix $T_C(\lambda)$ is a diagonal matrix, and the matrices $F_{\lambda}(n), F_{1/\lambda}(n)^* \varepsilon$ are given by

$$F_{\lambda}(n) = T_C(\lambda)^n = \alpha^{-n} \begin{bmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{bmatrix},$$

$$F_{1/\lambda}(n)^* = \alpha^n \begin{bmatrix} \lambda^{-n} & 0 \\ 0 & \lambda^n \end{bmatrix} \quad (138).$$

In this case, we can take

$$\left\{ \begin{array}{ll}
v_+(\lambda) = e_L, & v_-(\lambda) = e_R \quad (|\lambda| < 1), \\
v_+(\lambda) = e_R, & v_-(\lambda) = e_L \quad (|\lambda| > 1).
\end{array} \right.$$  

for the unit vectors $v_+(\lambda), v_-(\lambda)$ in Theorem 1.4. By the definition (11) of $z_L, z_R$, we have $z_L(n) = \pi_L, z_R(n) = \pi_R$. By Theorem 1.4 and (21), we have

$$x_0(\lambda) = \begin{cases} 0 & (|\lambda| < 1), \\ -\lambda^{-1}I & (|\lambda| > 1), \end{cases} \quad x(\lambda) = \begin{cases} I & (|\lambda| < 1), \\ -I & (|\lambda| > 1). \end{cases}$$

Therefore, using the characterization (107) of $\Sigma$, it is concluded that the positive-matrix-valued measure $\Sigma$ is given by $d\Sigma(\zeta) = Id\epsilon(\zeta)$, the identity matrix times the normalized Lebesgue measure $d\epsilon(\zeta)$. The corresponding QW-Fourier transform $F_C$ is given by

$$F_C[f](\lambda) = \sum_{n \in \mathbb{Z}} \alpha^n \begin{bmatrix} \lambda^{-n}f_L(n) \\ \lambda^n f_R(n) \end{bmatrix}, \quad f(n) = \begin{bmatrix} f_L(n) \\ f_R(n) \end{bmatrix} \in C_0(\mathbb{Z}, \mathbb{C}^2).$$

This is basically a usual Fourier series expansion. To be more precise, we define the Fourier series $F[f]$ with $f \in C_0(\mathbb{Z}, \mathbb{C}^2)$ by

$$F[f](\lambda) = \sum_{n \in \mathbb{Z}} \lambda^n f(n) \quad (\lambda \in \mathbb{C} \setminus \{0\}) \quad (139).$$

The function $F[f]$ is a $\mathbb{C}^2$-valued Laurent polynomial. We introduce a map $\mathcal{F} : C(S^1, \mathbb{C}^2) \rightarrow C(S^1, \mathbb{C}^2)$ by
(\mathcal{J}k)(\zeta) = \begin{bmatrix} k_L(\zeta^{-1}) \\ k_R(\zeta) \end{bmatrix}, \quad k(\zeta) = \begin{bmatrix} k_L(\zeta) \\ k_R(\zeta) \end{bmatrix} \in C(S^1, \mathbb{C}^2).

Then, we have
\[ \mathcal{J}F_C = F. \]

Next, we consider the case \( \beta \neq 0 \). In this case, we have \( 0 < |a| < 1 \). The characteristic equation for \( T_C(\lambda) \) is
\[ \frac{\lambda}{2} J(\lambda) z + \frac{\alpha}{\lambda} = 0, \quad J(\lambda) = \frac{\lambda + \lambda^{-1}}{2}. \] (140)

Let \( S, \ T \) be subsets of \( S^1 \) defined by
\[ S = \{ \zeta \in S^1 \mid |\text{Re}(\zeta)| \leq |a| \}, \quad T = \{ \zeta \in S^1 \mid |\text{Re}(\zeta)| \geq |a| \}. \] (141)

To calculate \( \Sigma \), we prepare the following lemma.

**Lemma 5.1** The following holds.

1. The matrix \( T_C(\lambda) \) has an eigenvalue in \( S^1 \) if and only if \( \lambda \in S \).
2. \( T_C(\lambda) \) has an eigenvalue with multiplicity two if and only if \( \lambda \) is one of the following four points:
   \[ |a| + i|\beta|, \quad -|a| + i|\beta|, \quad -|a| - i|\beta|, \quad |a| - i|\beta|. \]

**Proof** Suppose that \( T_C(\lambda) \) has an eigenvalue \( z_o \) satisfying \( |z_o| = 1 \). Then, by (140), we have
\[ J(\lambda) = \frac{\alpha z_o + \overline{\alpha z_o}}{2}. \]

This shows that \( J(\lambda) \) is real and \( |J(\lambda)| \leq |a| \). The parameter \( \lambda \) satisfies the equation
\[ \lambda^2 - 2r\lambda + 1 = 0, \quad r = J(\lambda). \]

Since \( r \) is real and \( |r| \leq |a| < 1 \), we have \( \lambda \in S^1 \) and \( \text{Re}(\lambda) = J(\lambda) \). Therefore, \( |\text{Re}(\lambda)| \leq |a| \). Conversely, suppose that \( \lambda \in S^1 \) and \( |\text{Re}(\lambda)| \leq |a| \). Since \( |\lambda| = 1 \), we see \( J(\lambda) = \text{Re}(\lambda) \) and \( |J(\lambda)| \leq |a| < 1 \). We write \( \lambda = e^{i\theta} \) with \( \theta \in (0, \pi) \). Then, \( \cos \theta = J(\lambda) \) and the solution to Eq. (140) is
\[ z = \frac{\cos \theta \pm i\sqrt{|a|^2 - \cos^2 \theta}}{\alpha}. \] (142)

Then, a direct calculation shows \( |z| = 1 \), and which proves (1). Next, suppose that \( \lambda \) is one of the four points in (2). Then, \( J(\lambda) = \pm |a| \), and the discriminant of (140) is zero. Hence, \( T_C(\lambda) \) has an eigenvalue with multiplicity two. Conversely, suppose that \( T_C(\lambda) \) has an eigenvalue \( z_o \) with multiplicity two. Then, by (140), we have
\( J(\lambda)^2 = |a|^2 \). Therefore, we see \( \lambda = J(\lambda) \pm i \sqrt{1 - |a|^2} \) which coincides with one of the four points in the statement. \( \square \)

Thus, for \( \lambda \in \mathbb{C}\setminus \{0\} \cup S^1 \), the matrix \( T_C(\lambda) \) has two mutually different eigenvalues, and, by (140), the absolute value of one of them is less than one and that of another is greater than one. We remark that, since \( T_C(\lambda) \) is holomorphic and its eigenvalues are simple for \( \lambda \in \mathbb{C}\setminus \{0\} \cup S^1 \), the eigenvalues can be labelled, so that they are holomorphic there. Let \( z_+(\lambda) \) be the eigenvalues of \( T_C(\lambda) \) satisfying \( |z_+(\lambda)| < 1 < |z_-(\lambda)| \) for \( \lambda \in \mathbb{C}\setminus \{0\} \cup S^1 \) and holomorphic there.

**Lemma 5.2** The function \( z_+(\lambda) \) is given by

\[
\alpha z_+ (\lambda) = \begin{cases} 
J(\lambda) - \sqrt{J(\lambda)^2 - |a|^2} & (|\lambda| < 1, \operatorname{Re} \lambda > 0) \text{ or } (|\lambda| > 1, \operatorname{Re} \lambda > 0), \\
J(\lambda) + \sqrt{J(\lambda)^2 - |a|^2} & (|\lambda| < 1, \operatorname{Re} \lambda < 0) \text{ or } (|\lambda| > 1, \operatorname{Re} \lambda < 0), \\
J(\lambda) + i \sqrt{|a|^2 - J(\lambda)^2} & (|\lambda| < 1, \operatorname{Im} \lambda > 0) \text{ or } (|\lambda| > 1, \operatorname{Im} \lambda < 0), \\
J(\lambda) - i \sqrt{|a|^2 - J(\lambda)^2} & (|\lambda| < 1, \operatorname{Im} \lambda < 0) \text{ or } (|\lambda| > 1, \operatorname{Im} \lambda > 0),
\end{cases}
\]

where we denote \( \sqrt{z} \) for \( z \in \mathbb{C}\setminus (-\infty, 0] \) the branch of the square root whose values are positive for positive real numbers.

**Proof** These expressions are solutions to Eq. (140). We also remark that the function \( \sqrt{J(\lambda)^2 - |a|^2} \) is holomorphic on \( \mathbb{C}\setminus (S \cup i\mathbb{R}) \) and the function \( \sqrt{|a|^2 - J(\lambda)^2} \) is holomorphic on \( \mathbb{C}\setminus (T \cup \mathbb{R}) \). For a nonzero real number \( \lambda \), the function \( z_+ (\lambda) \) is given by

\[
\alpha z_+ (\lambda) = \begin{cases} 
J(\lambda) - \sqrt{J(\lambda)^2 - |a|^2} & (0 < \lambda), \\
J(\lambda) + \sqrt{J(\lambda)^2 - |a|^2} & (0 > \lambda),
\end{cases}
\]

which can be checked by the requirement \( |z_+ (\lambda)| < 1 \). The function in the first line of (144) is holomorphic on \( \{ \lambda \in \mathbb{C}\setminus \{0\} \mid \operatorname{Re} (\lambda) > 0 \} \setminus S \), and that in the second line is holomorphic on \( \{ \lambda \in \mathbb{C}\setminus \{0\} \mid \operatorname{Re} (\lambda) < 0 \} \setminus S \). Thus, these expressions of \( z_+ (\lambda) \) still hold on the respective regions. We note that

\[
\sqrt{z} = \begin{cases} 
i \sqrt{-z} & (\operatorname{Im} z > 0), \\
-i \sqrt{-z} & (\operatorname{Im} z < 0).
\end{cases}
\]

A direct calculation shows

\[
\operatorname{Im} (J(\lambda)^2 - |a|^2) = -\frac{1}{2|\lambda|^4} (1 - |\lambda|^4) \operatorname{Re} (\lambda) \operatorname{Im} (\lambda).
\]

Thus, we see
An eigenfunction expansion formula for one-dimensional…

\[ \sqrt{J(\lambda)^2 - |\beta|^2} \]
\[ = \begin{cases} 
-\i \sqrt{|\alpha|^2 - J(\lambda)^2} & (0 < |\lambda| < 1, \Re(\lambda) > 0, \Im(\lambda) > 0), \\
\i \sqrt{|\alpha|^2 - J(\lambda)^2} & (0 < |\lambda| < 1, \Re(\lambda) < 0, \Im(\lambda) > 0),
\end{cases} \]

and hence, we have

\[ \alpha z_+(\lambda) = J(\lambda) + \i \sqrt{|\alpha|^2 - J(\lambda)^2} \hspace{1cm} (0 < |\lambda| < 1, \Im(\lambda) > 0). \]

Other expressions of \( z_+(\lambda) \) are obtained in the same way. \( \square \)

**Remark 5.3** As in the proof of Lemma 5.2, the function \( z_+(\lambda) \) is analytically continued through the relative interior of the subset \( T \) in \( S^1 \).

For the case of the constant coin, it is rather easy to use the usual Fourier series (139) to calculate the matrix \( x_0(\lambda) \). Indeed, we have

\[ R_j(n, m) = \int_{S^1} z^{m-n} (\hat{U}(z) - \lambda)^{-1} \, d\ell(z), \quad (145) \]

where \( \hat{U}(z) \) is the matrix-valued function given by

\[ \hat{U}(z) = \begin{bmatrix} \alpha z^{-1} & \beta z^{-1} \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}. \]

Calculating the integral (145) for \( n = m = 0 \) using the residue formula and the fact that \( z_\pm(\lambda) \) are the solutions to Eq. (140), we see

\[ x_0(\lambda) = \frac{1}{\lambda \alpha(z_+(\lambda) - z_-(\lambda))} \begin{bmatrix} \lambda - \alpha z_+(\lambda) & \beta z_+(\lambda) \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}. \quad (146) \]

The formula (146) can also be obtained using Corollary 1.4, although it needs somehow complicated calculation. We give an outline of a calculation. We take the vectors \( v_+(\lambda), v_-(\lambda) \) as eigenvectors corresponding to the eigenvalues \( z_\pm(\lambda) \). Explicitly, we set

\[ v_+(\lambda) = \frac{1}{\sqrt{|\lambda z_+(\lambda) - \bar{\alpha}|^2 + |\beta|^2}} \begin{bmatrix} \lambda z_+(\lambda) - \bar{\alpha} \\ -\bar{\beta} \end{bmatrix}, \]
\[ v_-(\lambda) = \frac{1}{\sqrt{|\lambda z_-(\lambda) - \bar{\alpha}|^2 + |\beta|^2}} \begin{bmatrix} \lambda z_-(\lambda) - \bar{\alpha} \\ -\bar{\beta} \end{bmatrix}. \quad (147) \]

Suppose that \( 0 < \lambda < 1 \). Then, \( \alpha z_\pm(\lambda) \) is real. Using the fact that \( z_\pm(\lambda) \) are the roots of (140), we find that the vectors \( v_+(\lambda) \) and \( v_-(\lambda) \) form an orthonormal basis in \( \mathbb{C}^2 \) for \( \lambda > 0 \). By Theorem 1.4, we see
That the two formulas (148) and (146) are identical for $0 < \lambda < 1$ can be verified by the formula

\[ \frac{\lambda z_-(\lambda) - 1}{\lambda \alpha (|\lambda z_-(\lambda) - \bar{\alpha}|^2 + |\beta|^2)} \left[ \begin{array}{c} \lambda z_-(\lambda) - \bar{\alpha} \\ -\beta \end{array} \right], \]

\[ \frac{\beta z_+(\lambda)}{-\alpha (|\lambda z_+(\lambda) - \bar{\alpha}|^2 + |\beta|^2)} \left[ \begin{array}{c} \lambda z_+(\lambda) - \bar{\alpha} \\ -\beta \end{array} \right]. \]

That the two formulas (148) and (146) are identical for $0 < \lambda < 1$ can be verified by the formula

\[ |\lambda z_+(\lambda) - \bar{\alpha}|^2 + |\beta|^2 = \frac{\lambda \alpha}{\bar{\alpha}} (z_+ - z_-)(\lambda z_+ - \bar{\alpha}), \]

\[ |\lambda z_-(-\lambda) - \bar{\alpha}|^2 + |\beta|^2 = \frac{\lambda \alpha}{\bar{\alpha}} (z_+ - z_-)(\bar{\alpha} - \lambda z_-), \]

which hold for $0 < \lambda < 1$. Then, one can use the coincidence theorem for holomorphic function $x_0(\lambda)$ to show that (146) holds also on the region $\text{Re}(\lambda) > 0$, $0 < |\lambda| < 1$. The same discussion works well for other region in $\mathbb{C} \setminus \{0\} \cup S^1$. We note that from the formula (145), the Green function must satisfy

\[ R_\lambda(n, n) = R_\lambda(0, 0) = x_0(\lambda), \]

which is, according to (13) in Theorem 1.2, equivalent to

\[ T_C(\lambda)[x_0(\lambda) + z_L(0)]T_C(1/\bar{\lambda})^* = [x_0(\lambda) + z_L(0)], \] (149)

because $z_L(n) = z_L(0)$. Equation (149) can be verified directly by (146) and the definition of $T_C(\lambda)$. A straightforward calculation shows

\[ x(\lambda) = I + 2\lambda x_0(\lambda) \]

\[ = \frac{2}{\alpha(z_+(\mu) - z_-(\lambda))} \left[ \begin{array}{c} K(\lambda) \frac{\beta}{\alpha} z_+(\lambda) \\ -\frac{\beta}{\alpha} a z_+(\lambda) \end{array} \right], \]

\[ K(\lambda) = \frac{\lambda - \lambda^{-1}}{2}. \] (150)

The positive-matrix-valued measure $\Sigma$ is described as follows.

**Theorem 5.4** The positive-matrix-valued measure $d\Sigma(\zeta)$ is given by

\[ d\Sigma(\zeta) = \frac{1}{\sqrt{|\alpha|^2 - \text{Re} \zeta^2}} \left[ \begin{array}{c} \text{Im}(\zeta) \\ -i \frac{\beta}{\alpha} \text{Re} \zeta \end{array} \right] \chi_{\sigma(U)}(\zeta), \]

where $\chi_{\sigma(U)}$ is the characteristic function of the spectrum $\sigma(U) = \{ \zeta \in S^1 \mid \text{Re} \zeta \leq |\alpha| \}$ of $U$.

**Proof of Theorem 5.4** Let $u \in \mathbb{C}^2$ and write it as
An eigenfunction expansion formula for one-dimensional $\ldots$

$$u = \begin{bmatrix} u_L \\ u_R \end{bmatrix}, \quad u_L, u_R \in \mathbb{C}.$$ 

We need to calculate the limit

$$\lim_{r \uparrow 1} \int_{S^1} h(\zeta) \langle [\text{Re} x(r\zeta)]u, u \rangle_{\mathbb{C}^2} \, d\ell(\zeta) \quad (151)$$

for $h \in C(S^1)$. We have the expression

$$\text{Re} x(\lambda) = \begin{bmatrix} 2\text{Re} \left( \frac{K(\lambda)}{a(z_+(\lambda)-z_-(\lambda))} \right) & 2i\frac{\beta}{\alpha} \text{Im} \left( \frac{z_+(\lambda)}{z_+(\lambda)-z_-(\lambda)} \right) \\ -2i\frac{\beta}{\alpha} \text{Im} \left( \frac{z_+(\lambda)}{z_+(\lambda)-z_-(\lambda)} \right) & 2\text{Re} \left( \frac{K(\lambda)}{a(z_+(\lambda)-z_-(\lambda))} \right) \end{bmatrix},$$

and hence

$$\langle [\text{Re} x(\lambda)]u, u \rangle_{\mathbb{C}^2} = 2||u||_{\mathbb{C}^2}^2 \text{Re} \left[ \frac{K(\lambda)}{a(z_+(\lambda)-z_-(\lambda))} \right]$$

$$- 4\text{Im} \left( \frac{\beta}{\alpha} u_L u_R \right) \text{Im} \left[ \frac{z_+(\lambda)}{z_+(\lambda)-z_-(\lambda)} \right]. \quad (152)$$

Thus, we need to calculate the limits of the integrals

$$\int_{S^1} h(\zeta) \text{Re} \left[ \frac{K(r\zeta)}{a(z_+(r\zeta)-z_-(r\zeta))} \right] \, d\ell(\zeta),$$

$$\int_{S^1} h(\zeta) \text{Im} \left[ \frac{z_+(r\zeta)}{z_+(r\zeta)-z_-(r\zeta)} \right] \, d\ell(\zeta)$$

as $r \uparrow 1$. We only consider the integral on the positive orthant $\Delta = \{ \zeta \in S^1 \mid \text{Re} (\zeta) \geq 0, \text{Im} (\zeta) \geq 0 \}$, because the other parts of the integral can be handled similarly. Thus, we set

$$I_r = \int_{\Delta} h(\zeta) \text{Re} \left[ \frac{K(r\zeta)}{a(z_+(r\zeta)-z_-(r\zeta))} \right] \, d\ell(\zeta)$$

$$= \frac{1}{2\pi} \int_0^{\pi/2} h(e^{it}) \text{Re} \left[ \frac{K(re^{it})}{a(z_+(re^{it})-z_-(re^{it}))} \right] \, dr,$$

$$J_r = \int_{\Delta} h(\zeta) \text{Im} \left[ \frac{z_+(r\zeta)}{z_+(r\zeta)-z_-(r\zeta)} \right] \, d\ell(\zeta)$$

$$= \frac{1}{2\pi} \int_0^{\pi/2} h(e^{it}) \text{Im} \left[ \frac{z_+(re^{it})}{z_+(re^{it})-z_-(re^{it})} \right] \, dr.$$

Let $\psi \in (0, \pi/2)$ be the real number satisfying $\cos \psi = |\alpha|$, and we divide the arc $\Delta$ into two parts, $D_1 = \{ 0 \leq t \leq \psi \}$ and $D_2 = \{ \psi \leq t \leq \pi/2 \}$. From Lemma 5.2, we have
\[
\frac{z_+(\lambda)}{z_+(\lambda) - z_-(\lambda)} = \frac{J(\lambda)}{2i\sqrt{|\alpha|^2 - J(\lambda)^2}} + \frac{1}{2} \quad (0 < |\lambda| < 1, \ \text{Im}(\lambda) > 0).
\]

By a direct calculation, we see
\[
\left| |\alpha|^2 - J(re^{it})^2 \right|^2 = D(t)^2 + K(r)^2 [J(r)^2 \sin^2(2t) - 2D(t) \cos(2t)]
+ K(r)^4 \cos^2(2t),
\]
where we set
\[
D(t) = |\alpha|^2 - \cos^2 t.
\]

Since \(J(r) \geq 1, \sin(2t) \neq 0 \) and \(D(\psi) = 0\), we can take a sufficiently small \(\varepsilon > 0\), such that
\[
J(r)^2 \sin^2(2t) - 2D(t) \cos 2t > 0 \quad (t \in [\psi, \psi + \varepsilon], \ r \in [2/3, 1]).
\]

Then, we see
\[
\left| |\alpha|^2 - J(re^{it})^2 \right|^2 > D(t)^2, \quad \text{hence}
\frac{1}{\sqrt{|\alpha|^2 - J(re^{it})^2}} \leq \frac{1}{\sqrt{D(t)}} = \frac{1}{\sqrt{|\alpha|^2 - \cos^2 t}}
\]
for \(t \in [\psi, \psi + \varepsilon]\) and \(r \in [2/3, 1]\). For \(t \in (\psi, \pi/2]\), we have
\[
\lim_{r \uparrow 1} \Im \left[ \frac{z_+(re^{it})}{z_+(re^{it}) - z_-(re^{it})} \right] = -\frac{\cos t}{2D(t)^{1/2}} = -\frac{\Re(\zeta)}{2\sqrt{|\alpha|^2 - \Re(\zeta)^2}}.
\]

Since \(1/D(t)^{1/2}\) is integrable on \([\psi, \pi/2]\), the Lebesgue convergence theorem shows
\[
\lim_{r \uparrow 1} \int_{\psi}^{\psi + \varepsilon} h(e^{it}) \Im \left[ \frac{z_+(re^{it})}{z_+(re^{it}) - z_-(re^{it})} \right] \, dt
\]
\[
= -\int_{\psi}^{\psi + \varepsilon} h(e^{it}) \frac{\cos t}{2\sqrt{|\alpha|^2 - \cos^2 t}} \, dt.
\]

On the interval \([\psi + \varepsilon, \pi/2]\), \(D(t)\) is bounded, and thus, the Lebesgue convergence theorem is also applicable. Therefore, we see
\[
\lim_{r \uparrow 1} \int_{D_2} h(\zeta) \Im \left[ \frac{z_+(r\zeta)}{z_+(r\zeta) - z_-(r\zeta)} \right] \, d\zeta(\zeta)
\]
\[
= -\int_{D_2} h(\zeta) \frac{\Re(\zeta)}{2\sqrt{|\alpha|^2 - \Re(\zeta)^2}} \, d\zeta(\zeta).
\]

On the arc \(D_1\), we can use the first expression of \(z_+(\lambda)\) in (143). We can still use the estimate (153) around \(t = \psi\). However, this time, the imaginary part of
\( z_+(r\zeta)/(z_+(r\zeta) - z_-(r\zeta)) \) tends to zero as \( r \uparrow 1 \). Therefore, again, the Lebesgue convergence theorem shows

\[
\lim_{r \uparrow 1} \int_{D_1} h(\zeta) \text{Im} \left[ \frac{z_+(r\zeta)}{z_+(r\zeta) - z_-(r\zeta)} \right] d\zeta(\zeta) = 0.
\]

Thus, we obtain

\[
\lim_{r \uparrow 1} I_r = - \int_{D_2} h(\zeta) \frac{\text{Re} (\zeta)}{2\sqrt{|\alpha|^2 - \text{Re} (\zeta)^2}} d\zeta(\zeta).
\]

(154)

The limit of the integral also handled in the same way. This time, on the arc \( D_1 \), the real part of \( K(\zeta) \) equals zero. On the arc \( D_2 \), the real part of \( K(r\zeta)/\alpha(z_+(r\zeta) - z_-(r\zeta)) \) converges, as \( r \uparrow 1 \), to \( \text{Im} (\zeta)/2\sqrt{|\alpha|^2 - \text{Re} (\zeta)^2} \). Then, we obtain

\[
\lim_{r \uparrow 1} I_r = \int_{D_2} h(\zeta) \frac{\text{Im} (\zeta)}{2\sqrt{|\alpha|^2 - \text{Re} (\zeta)^2}} d\zeta(\zeta).
\]

(155)

The same calculations on other orthants combined with (152) show the assertion.

\( \square \)

### 5.2 Simplest two-phase model

Let \( C_0, C_\pm \) be three \( 2 \times 2 \) special unitary matrices, and we write them as

\[
C_0 = \begin{bmatrix} \alpha_0 & \beta_0 \\ -\beta_0 & \alpha_0 \end{bmatrix}, \quad C_\pm = \begin{bmatrix} \alpha_\pm & \beta_\pm \\ -\beta_\pm & \alpha_\pm \end{bmatrix},
\]

\[
|\alpha_0|^2 + |\beta_0|^2 = |\alpha_\pm|^2 + |\beta_\pm|^2 = 1.
\]

Let \( C : \mathbb{Z} \to U(2) \) be a coin matrix defined as

\[
C(n) = \begin{cases} C_+ & (n \geq 1), \\
C_0 & (n = 0), \\
C_- & (n \leq -1). \end{cases}
\]

The quantum walk \( U(C) \) defined by the coin matrix \( C \) is called a two-phase model with one defect. Since the calculation is a bit complicated, we impose the following strong assumptions:

\[
0 < \rho := |\alpha_+| = |\alpha_-| < 1, \quad \beta := \beta_+ = \beta_- , \quad |\text{Re} (\beta_0\beta)| < |\beta|^2.
\]

(156)

Therefore, \( C_+ \) and \( C_- \) differs only by the phase of the diagonals. We note that the first two assumptions are for simplifying the calculation, but the last assumption is imposed for \( U \) to have eigenvalues. Eigenvalues of much general two-phase models are studied in [13]. In this case, the transfer matrix \( T_\lambda(n) \) satisfies
\[ T_{\lambda}(n) = \begin{cases} T_{C_+}(\lambda) & (n \geq 1), \\ T_{C_-}(\lambda) & (n \leq -2), \end{cases} \]

and the matrix-valued function \( F_{\lambda}(n) \) satisfies

\[ F_{\lambda}(n) = \begin{cases} T_{C_+}(\lambda)^n T_{\lambda}(0) & (n \geq 1), \\ T_{C_-}(\lambda)^{n+1} T_{\lambda}(-1)^{-1} & (n \leq -1). \end{cases} \]  \quad (157)

As in the case of constant coins, we denote \( z_{\pm}(C_\pm, \lambda) \) the eigenvalues of \( T_{C_\pm}(\lambda) \) satisfying

\[ |z_+(C_\pm, \lambda)| < 1 < |z_-(C_\pm, \lambda)|, \quad \lambda \in \mathbb{C}\setminus\{0\} \cup S^1, \]  \quad (158)

and we define vectors \( w_{\pm}(C_\pm, \lambda) \) as

\[
\begin{align*}
w_{\pm}(C_+, \lambda) &= \left[ \begin{array}{c} \alpha_0(\lambda z_{\pm}(C_+, \lambda) - \alpha_+) \\ -\alpha_0\beta \end{array} \right], \\
w_{\pm}(C_-, \lambda) &= \left[ \begin{array}{c} \alpha_0\alpha_-(z_{\pm}(C_-, \lambda) - \alpha_-\lambda^{-1}) \\ -\alpha_0\alpha_-\beta\lambda^{-1} \end{array} \right].
\end{align*}
\]  \quad (159)

The vectors \( w_{\pm}(C_\pm, \lambda) \) are eigenvectors of \( T_{C_\pm}(\lambda) \) with the eigenvalues \( z_{\pm}(C_\pm, \lambda) \), respectively. We define unit vectors \( v_{\pm}(\lambda) \) by

\[
\begin{align*}
v_+(\lambda) &= \frac{1}{\| T_{\lambda}(0)^{-1}w_+(C_+, \lambda) \|_{C^2}} T_{\lambda}(0)^{-1}w_+(C_+, \lambda), \\
v_-(\lambda) &= \frac{1}{\| T_{\lambda}(-1)^{-1}w_-(C_-, \lambda) \|_{C^2}} T_{\lambda}(-1)^{-1}w_-(C_-, \lambda).
\end{align*}
\]  \quad (160)

These vectors satisfy

\[
\begin{align*}
F_{\lambda}(n)v_+(\lambda) &= z_+(C_+, \lambda)^n v_+(\lambda) \quad (n \geq 1), \\
F_{\lambda}(n)v_-(\lambda) &= z_-(C_-, \lambda)^{n+1} v_-(\lambda) \quad (n \leq -1).
\end{align*}
\]  \quad (161)

From (161), it follows that we can use the unit vectors \( v_+(\lambda), v_-(\lambda) \) to apply Theorem 1.4. We set

\[ S = \{ \zeta \in S^1 \mid |\text{Re}(\zeta)| \leq \rho \}. \]  \quad (162)

Then, by Lemma 5.1, for \( \lambda \in \mathbb{C}\setminus\{0\} \), we have \( |z_\pm(C_\pm, \lambda)| = 1 \) if and only if \( \lambda \in S \).

**Lemma 5.5** There are no eigenvalues of \( U \) in \( S \).

**Proof** This is proved in [17], but we give a proof for completeness. Suppose contrary that \( U \) has an eigenvalue \( \lambda \) in \( S \). Then, there exists a nonzero vector \( u \in \mathbb{C}^2 \), such that the function \( \Phi_{\lambda}(u) \) defined in (7) is in \( \ell^2(\mathbb{Z}, \mathbb{C}^2) \). Since \( \lambda \in S \), all the eigenvalues of \( T_{C_\pm}(\lambda) \) are in \( S^1 \) by Lemma 5.1. By an argument using the Jordan normal form of \( T_{C_\pm}(\lambda) \), we can find a positive constant \( c \), such that
An eigenfunction expansion formula for one-dimensional…

\[ \| T_{\mathcal{E}}(\lambda)^n w \|_{C^2} \geq c \| w \|_{C^2} \]

for any \( w \in C^2 \) and any positive integer \( n \). By (157), we have

\[ \| \Phi_\delta(u)(n) \|_{C^2}^2 \geq c^2 \| T_\delta(0)u \|_{C^2}^2, \quad \| \Phi_\delta(u)(-n) \|_{C^2}^2 \geq c^2 \| T_\delta(-1)^{-1}u \|_{C^2}^2. \]

Since \( \Phi_\delta(u) \in \ell^2(\mathbb{Z}, C^2) \), we must have \( T_\delta(0)u = T_\delta(-1)^{-1}u = 0 \). This implies that \( u = 0 \) which is a contradiction. \( \square \)

The matrices \( T_\delta(0), T_\delta(-1) \) and their inverses are given by

\[
T_\delta(0) = \begin{bmatrix}
\frac{1}{\alpha_0}(\lambda + \lambda^{-1}\beta_0\beta) - \lambda^{-1}\overline{\alpha_0}\beta
& -\lambda^{-1}\overline{\beta_0}

-\lambda^{-1}\beta_0
& \lambda^{-1}\alpha_0
\end{bmatrix},
\]

\[
T_\delta(0)^{-1} = \begin{bmatrix}
\lambda^{-1}\alpha_+ & \lambda^{-1}\beta

\lambda^{-1}\beta_+ & \frac{1}{\alpha_0}(\lambda + \lambda^{-1}\beta_0\beta)
\end{bmatrix},
\]

\[
T_\delta(-1) = \begin{bmatrix}
\frac{1}{\alpha_0}(\lambda + \lambda^{-1}\overline{\beta_0}\beta) - \lambda^{-1}\overline{\beta_0}\alpha_0
& -\lambda^{-1}\overline{\beta}

-\lambda^{-1}\overline{\beta}
& \lambda^{-1}\alpha_-
\end{bmatrix},
\]

\[
T_\delta(-1)^{-1} = \begin{bmatrix}
\lambda^{-1}\alpha_0 & \lambda^{-1}\beta_0

\lambda^{-1}\beta_0 & \frac{1}{\alpha_0}(\lambda + \lambda^{-1}\overline{\beta_0}\beta)
\end{bmatrix}.
\]

Then, the concrete form of the vectors \( v_\pm(\lambda) \) is given by

\[
v_+(\lambda) = \frac{1}{D_+(\lambda)} \left[ \frac{\overline{\alpha_0}(Z_+(\lambda) - \lambda^{-1})}{\overline{\beta_0}(Z_+(\lambda) - \lambda^{-1}) - \beta \lambda} \right],
\]

\[
v_-(\lambda) = \frac{1}{D_-(\lambda)} \left[ \frac{(\lambda + \overline{\beta_0}\lambda^{-1})Z_-(\lambda) - \rho^2}{-\alpha_0\lambda^{-1}Z_-(\lambda)} \right],
\]

where \( D_\pm(\lambda) \) are normalization terms and \( Z_\pm(\lambda) \) are given by

\[
Z_+(\lambda) = \alpha_+z_+(C_+, \lambda), \quad Z_-(\lambda) = \alpha_-z_-(C_-, \lambda). \tag{163}
\]

By the assumption (156), \( z = Z_\pm(\lambda) \) are the solutions to the equation

\[
z^2 - 2J(\lambda)z + \rho^2 = 0. \tag{164}
\]

In the case of constant coins, the denominators \( \langle v_+(\lambda), v_-(\lambda) \rangle_{C^2} \) and \( \langle v_-(\lambda), v_+(\lambda) \rangle_{C^2} \) in (15) are constant functions in \( \lambda \) and they do not contribute the asymptotic behavior of \( x_0(\lambda) \) as \( |\lambda| \to 1 \). However, in the two-phase model, there are points \( \zeta \in S^1 \) where these denominators are asymptotically zero as \( \lambda \to \zeta \in S^1 \). We need to describe these points and analyze the behavior of \( x_0(\lambda) \) near these points. We set

\[
s = \text{Re}(\beta_0\overline{\beta}), \quad t = \text{Im}(\beta_0\overline{\beta}). \tag{165}
\]
In what follows, we use the unit vector $u^\perp$ perpendicular to a unit vector $u$ defined by

$$u^\perp = \begin{bmatrix} -\bar{b} \\ \bar{a} \end{bmatrix} \quad \text{when} \quad u = \begin{bmatrix} a \\ b \end{bmatrix}. \quad (166)$$

**Lemma 5.6** The denominators are given by

$$\langle v_+(\lambda), v_-^\perp(\lambda) \rangle_{C^2} = -\langle v_-(\lambda), v_+(\lambda) \rangle_{C^2} = \frac{2\bar{\beta}}{D_+(\lambda)D_-(\lambda)}Z_-(\lambda)(1 - s - J(\lambda)Z_-(\lambda)).$$

**Proof** Since $Z_+(\lambda)$ are the solution to Eq. (164), they satisfy $Z_+(\lambda) - \lambda^{-1} = Z_-(\lambda) - \lambda$ and $\rho^2 = 2J(\lambda)Z_-(\lambda) - Z_-(\lambda)^2$. Thus, we see

$$v_+(\lambda) = \frac{1}{D_+(\lambda)} \begin{bmatrix} \bar{a}_0(\lambda - Z_-(\lambda)) \\ \bar{b}_0(\lambda - Z_-(\lambda)) - \bar{\beta}\lambda \end{bmatrix},$$

$$v_-(\lambda) = \frac{Z_-(\lambda)}{D_-(\lambda)} \begin{bmatrix} (\bar{b}_0\bar{\beta} - 1)\lambda^{-1} + Z_-(\lambda) \\ -\alpha_0\bar{\beta}\lambda^{-1} \end{bmatrix}. \quad (167)$$

Therefore, a direct calculation using the property $|\alpha_0|^2 + |\beta_0|^2 = 1$ shows

$$\langle v_+(\lambda), v_-^\perp(\lambda) \rangle_{C^2} = \frac{Z_-(\lambda)}{D_+(\lambda)D_-(\lambda)} \times \left(2\bar{\beta} - 2J(\lambda)\bar{\beta}Z_-(\lambda) + 2J(\lambda)\bar{\beta}_0Z_-(\lambda) - \bar{\beta}_0Z_-(\lambda)^2 - \bar{\beta}_0 - \beta_0\bar{\beta}^2 \right).$$

Substituting $2J(\lambda)Z_-(\lambda) - Z_-(\lambda)^2 = \rho^2 = 1 - \beta\bar{\beta}$ into the above, we see

$$\langle v_+(\lambda), v_-^\perp(\lambda) \rangle_{C^2} = \frac{Z_-(\lambda)}{D_+(\lambda)D_-(\lambda)}(1 - s - J(\lambda)Z_-(\lambda)).$$

By the choice of the vector $v_\pm(\lambda)^\perp$ perpendicular to $v_\pm(\lambda)$ described in (166), we have

$$\langle v_+(\lambda), v_-^\perp(\lambda) \rangle_{C^2} = -\langle v_-(\lambda), v_+(\lambda) \rangle_{C^2},$$

which completes the proof. \[\square\]

In what follows, we sometimes write

$$G(\zeta_{(-)}) = \lim_{r \downarrow 1} G(r\zeta), \quad G(\zeta_{(+)}) = \lim_{r \downarrow 1} G(r^{-1}\zeta) \quad (\zeta \in S^1)$$

for a function $G$ on $\mathbb{C}\setminus S^1$, if the limit exists. We write $\zeta = z + iy \in S^1$. We define a function $f(\lambda)$ by
\[ f(\lambda) = 1 - s - J(\lambda)Z_-(\lambda). \] (168)

Then, by Lemma 5.2, for \( \zeta \not\in S \), we have

\[ f(\zeta) := f(\zeta_{(+)})) = f(\zeta_{(-)}) = 1 - s - x(x + \text{sgn}(x)\sqrt{x^2 - \rho^2}), \]

where

\[ \text{sgn}(x) = \begin{cases} +1 & (x > 0), \\ -1 & (x < 0). \end{cases} \]

Thus, \( f(\zeta) = 0 \) if and only if \( \zeta \) is one of the following four points:

\[ \zeta_* = e^{i\theta_*} := x_* + iy_*, \quad -\bar{\zeta}_*, \quad -\zeta_*, \quad \bar{\zeta}_*. \] (169)

where positive real numbers \( x_*, y_* \) are given by

\[ x_* = \frac{1 - s}{\sqrt{2 - 2s - \rho^2}}, \quad y_* = \sqrt{1 - x_*^2} = \frac{\sqrt{1 - \rho^2 - s^2}}{2 - 2s - \rho^2}. \] (170)

It is well known (see [13]) that these points are actually the eigenvalues of \( U \), which will also be shown as point masses of \( \Sigma \) in the following theorem.

**Theorem 5.7** The positive-matrix-valued measure \( \Sigma \) for the two-phase model \( U(C) \) with the assumption (156) is given by the following:

\[
\begin{align*}
d\Sigma(\zeta) = & \frac{\sqrt{\rho^2 - x^2}}{(1 - s)^2 - (2 - 2s - \rho^2)x^2} \\
& \times \left[ (1 - s)|y| + \text{sgn}(y)tx - \text{sgn}(y)ia_0\beta x \\
& + \frac{x_*\sqrt{x_*^2 - \rho^2}}{2y_*(1 - s)^2} \left[ [(1 - s)y_* + tx_*] - ia_0\beta x_* \\
& + \frac{x_*\sqrt{x_*^2 - \rho^2}}{2y_*(1 - s)^2} \left[ [(1 - s)y_* - tx_*] + ia_0\beta x_* \right] \right] \right] \chi_S(\zeta) \, d\ell'(\zeta) \\
& + \delta_{\zeta_+} + \delta_{\zeta_-} \right) \\
& + \delta_{\bar{\zeta}_+} + \delta_{\bar{\zeta}_-},
\end{align*}
\]

where we write \( \zeta = x + iy \in S^1 \) with \( x, y \in \mathbb{R} \), and \( \zeta_* = x_* + iy_* \) is defined in (170).

**Proof of Theorem 5.7** By (167), we have

\[ \langle Z_L(0)e_L, v_+(\lambda) \rangle_{C^2} = \frac{\bar{\beta}\lambda}{D_+}, \quad \langle Z_R(0)e_R, v_-(\lambda) \rangle_{C^2} = \frac{Z_-}{D_-}(Z_--\lambda^{-1}). \]

Therefore, by Theorem 1.4, we have
\[ \lambda x_0(\lambda) = \frac{1}{2f(\lambda)} \left[ \beta_0 \beta - 1 + \lambda Z_-(\lambda) \right] - \alpha_0 \beta \right] \bar{\beta}_0 \beta - 1 + \lambda Z_-(\lambda), \]  

(171)

where the holomorphic function \( f(\lambda) \) is given in (168). We first calculate the matrix-valued measure \( \Sigma \) on \( S \). We only consider the upper part \( S_+ = \{ \zeta \in S \mid \text{Im}(\zeta) > 0 \} \) of \( S \). The measure \( \Sigma \) on the lower part of \( S \) can be calculated in the same way. For \( \zeta \in S_+ \), by Lemma 5.2, we have

\[ Z_-(r\zeta) = J(r\zeta) - i\sqrt{\rho^2 - J(r\zeta)}, \]

\[ Z_-(r^{-1}\zeta) = J(r^{-1}\zeta) + i\sqrt{\rho^2 - J(r^{-1}\zeta)}. \]

Therefore, writing \( \zeta = x + iy \) as before

\[ f(\zeta_{(-)}) = 1 - s - x^2 + ixw, \quad f(\zeta_{(+)}) = 1 - s - x^2 - ixw, \quad w = \sqrt{\rho^2 - x^2}. \]

By the assumption (156), we have \(|s| < 1 - \rho^2\), and hence, \( f(\zeta_{(\pm)}) \) are nonzero. Since both of \( \pm \sqrt{\rho^2 - J(\lambda)^2} \) are continuous on \( S \), the Lebesgue convergence theorem shows

\[
\begin{align*}
\mathcal{W} & = \lim_{r \to 1} [r\zeta x_0(r\zeta) - r^{-1}\zeta x_0(r^{-1}\zeta)] \chi_S(\zeta) \\
& = \frac{1}{2f(\zeta_{(-)})} \left[ \beta_0 \beta - 1 + \zeta(x - i\sqrt{\rho^2 - x^2}) \right] - \alpha_0 \beta \bar{\beta}_0 \beta - 1 + \zeta(x + i\sqrt{\rho^2 - x^2}) \right] \chi_S(\zeta) \text{d}\mathcal{E}(\zeta) \\
& - \frac{1}{2f(\zeta_{(+)})} \left[ \beta_0 \beta - 1 + \zeta(x + i\sqrt{\rho^2 - x^2}) \right] - \alpha_0 \beta \bar{\beta}_0 \beta - 1 + \zeta(x - i\sqrt{\rho^2 - x^2}) \right] \chi_S(\zeta) \text{d}\mathcal{E}(\zeta) \\
& = \frac{w}{(1 - s)^2 - [2(1 - s) - \rho^2]^2} \left[ (1 - s)y + tx \right] - \frac{i\alpha_0 \beta x}{(1 - s)y - tx} \chi_S(\zeta) \text{d}\mathcal{E}(\zeta).
\end{align*}
\]

(172)

Next, we calculate the matrix-valued measure \( \Sigma \) on \( S^1 \setminus S \). We have

\[ \lim_{r \to 1} (Z_-(r\zeta) - Z_-(r^{-1}\zeta)) = 0 \quad (\zeta \in S^1 \setminus S \cup \{ \pm \xi_+, \pm \xi_- \}). \]

(173)

Therefore, we only need to calculate the behavior of \( x_0(\lambda) \) near \( \{ \pm \xi_+, \pm \xi_- \} \). On the domains

\[ \Omega_+ := \{ \lambda \in \mathbb{C} \mid \text{Re}(\lambda) > 0 \} \setminus (S \cup i\mathbb{R}), \quad \Omega_- := \{ \lambda \in \mathbb{C} \mid \text{Re}(\lambda) < 0 \} \setminus (S \cup i\mathbb{R}), \]

the function \( Z_-(\lambda) \) is given as
\[ Z_-(\lambda) = \begin{cases} J(\lambda) + \sqrt{J(\lambda)^2 - \rho^2} & (\lambda \in \Omega_+), \\
J(\lambda) - \sqrt{J(\lambda)^2 - \rho^2} & (\lambda \in \Omega_-). \end{cases} \]

We still use the notation \( f(\lambda) = 1 - s - J(\lambda)Z_-(\lambda) \) on the domains \( \Omega_\pm \). We denote \( \zeta_o = x_o + iy_o \) one of the points \( \{ \pm \zeta_o, \pm \overline{\zeta_o} \} \) and write \( e^{i\theta_o} = \zeta_o \). For any small \( \epsilon > 0 \) and \( r \in (2/3, 1) \) close to 1, we define

\[ J_{r,\varepsilon} = \{ re^{i\theta} \mid r \leq t \leq r^{-1}, |\theta - \theta_o| \leq \epsilon \}, \]

\[ A_\varepsilon = J_{r,\varepsilon} \cap S^1 = \{ e^{i\theta} \mid |\theta - \theta_o| \leq \epsilon \}, \]

where \( \varepsilon > 0 \) is chosen, so that \( J_{r,\varepsilon} \) is contained in one of \( \Omega_\pm \) containing \( \zeta_o \) and \( J_{r,\varepsilon} \) contains only \( \zeta_o \) among \( \{ \pm \zeta_o, \pm \overline{\zeta_o} \} \). We note that, by Lemma 5.6, \( f(\lambda) \) is nonzero for \( |\lambda| \neq 1 \) since, for \( |\lambda| \neq 1 \), \( \{ v_+(\lambda), v_-(\lambda) \} \) is a basis of \( \mathbb{C}^2 \). Then, \( \zeta_o \) is a zero of \( f(\lambda) \) of order 1 and there are no zeros of \( f(\lambda) \) on a neighborhood of \( J_{r,\varepsilon} \) where the function \( f(\lambda) \) is defined in (168) which is holomorphic on \( \Omega_\pm \). Thus, for any holomorphic function \( h(\lambda) \) near \( J_{r,\varepsilon} \), Cauchy’s integral formula gives

\[ \frac{1}{2\pi i} \int_{\partial J_{r,\varepsilon}} \frac{h(\lambda)}{f(\lambda)} \frac{d\lambda}{\lambda} = \frac{h(\zeta_o)}{\zeta_o f'(-\zeta_o)}, \quad (174) \]

where \( \partial J_{r,\varepsilon} \) is the boundary of \( J_{r,\varepsilon} \) with the counterclockwise direction. Since \( Z_-(\lambda) \) satisfies the Eq. (164), we have

\[ f(\lambda) = 1 - s - J(\lambda)Z_-(\lambda) = 1 - s - \frac{1}{2}(Z_-(\lambda)^2 + \rho^2). \]

Thus, we have \( f'(\lambda) = -Z_-(\lambda)Z'_-(\lambda) \). Substituting this into (174) yields

\[ \frac{1}{2\pi i} \int_{\partial J_{r,\varepsilon}} \frac{h(\lambda)}{f(\lambda)} \frac{d\lambda}{\lambda} = \frac{h(\zeta_o)}{\zeta_o f'(-\zeta_o)} = - \frac{h(\zeta_o)}{\zeta_o Z_-(\zeta_o)Z'_-(\zeta_o)}. \quad (175) \]

By calculating the contour integral in the left-hand side of (175), we see

\[ \lim_{r \downarrow 1} \left[ \int_{A_\varepsilon} \frac{h(\zeta)}{f(\zeta)} \frac{d\zeta}{\zeta} - \int_{A_\varepsilon} \frac{h(\zeta)}{f(\zeta)} \frac{d\zeta}{\zeta} \right] = \frac{h(\zeta_o)}{\zeta_o Z_-(\zeta_o)Z'_-(\zeta_o)}. \quad (176) \]

The set of all Laurent polynomials is dense in the space of continuous functions on \( A_\varepsilon \). Thus, the formula (176) still holds for any continuous function \( h \) on \( A_\varepsilon \). Let \( u = [u_L, u_R] \in \mathbb{C}^2 \) and \( h \in C(A_\varepsilon) \). According to the formula (171), we set

\[ G(\lambda) = (\beta_0 \overline{\beta} - 1 + \lambda Z_-(\lambda))|u_L|^2 + \alpha_0 \overline{\alpha} \overline{u_L}u_R - \alpha_0 \overline{\beta} u_L \overline{u_R} + (\beta_0 \overline{\beta} - 1 + \lambda Z_-(\lambda))|u_R|^2, \]

so that

\[ \langle [r\zeta x_0(r\zeta) - r^{-1}\zeta x_0(r^{-1}\zeta)]u, u \rangle_{\mathbb{C}^2} = \frac{G(r\zeta)}{2f(r\zeta)} - \frac{G(r^{-1}\zeta)}{2f(r^{-1}\zeta)}. \]
From (171) and (176), we have

\[
\lim_{r \to 1} \int_{\Delta_x} h(\zeta) \left[ r^{-1} \xi \xi_0(r \xi) - r^{-1} \xi \xi_0(r^{-1} \xi) \right] u, u \right]_{\mathbb{C}} \, d\ell(\zeta)
\]

\[
= \lim_{r \to 1} \left( \int_{\Delta_x} \frac{h(\zeta) G(r \xi)}{2f(r \xi)} \, d\ell(\zeta) - \int_{\Delta_x} \frac{h(\zeta) G(r^{-1} \xi)}{2f(r^{-1} \xi)} \, d\ell(\zeta) \right)
\]

\[
= \frac{h(\xi_0) G(\xi_0)}{2 \xi_0 Z_-(\xi_0) Z'_-(\xi_0)}.
\]  

(177)

A direct calculation shows

\[
\lambda Z'_-(\lambda) = \lambda \frac{J'(\lambda)}{\sqrt{J(\lambda)^2 - \rho^2}} \left( J(\lambda) + \sqrt{J(\lambda)^2 - \rho^2} \right) = \frac{K(\lambda)}{\sqrt{J(\lambda)^2 - \rho^2}} Z_-(\lambda),
\]

where, as before, \(K(\lambda) = (\lambda - \lambda^{-1})/2\). Since \(0 = f(\xi_0) = 1 - s - \xi_0 Z_-(\xi_0)\), when \(\xi_0 = \text{Re}(\xi_0) > 0\) (that is, \(\xi_0 = \xi_x\)), we have

\[
\xi_0 Z_-(\xi_0) Z'_-(\xi_0) = \frac{K(\xi_0)}{\sqrt{\xi_0^2 - \rho^2}} \frac{Z_-(\xi_0)^2}{\xi_0^2 \sqrt{\xi_0^2 - \rho^2}} = \frac{i \alpha_0 (1 - s)^2}{\sqrt{\xi_0^2 - \rho^2}}.
\]

Then, we also have

\[
\beta_0 \overline{\beta} - 1 + \xi_0 Z_-(\xi_0) = s + it - 1 + \frac{1 - s}{\xi_0} (x_0 + iy_0) = \frac{i}{\xi_0} ((1 - s)y_0 + tx_0).
\]

Therefore, from (177), we obtain, when \(\xi_0 = \text{Re}(\xi_0) > 0\)

\[
\Sigma(\{\xi_0\}) = \frac{1}{2 \xi_0 Z_-(\xi_0) Z'_-(\xi_0)} \times \left[ \begin{array}{c} \beta_0 \overline{\beta} - 1 + \xi_0 Z_-(\xi_0) \delta_{\xi_0} \\ - \alpha_0 \overline{\beta} \delta_{\xi_0} \\ \beta_0 \overline{\beta} - 1 + \xi_0 Z_-(\xi_0) \delta_{\xi_0} \end{array} \right]
\]

\[
= \frac{x_0 \sqrt{\xi_0^2 - \rho^2}}{2y_0 (1 - s)^2} \left[ \begin{array}{c} (1 - s)y_0 + tx_0 \delta_{\xi_0} \\ i \alpha_0 \overline{\beta} x_0 \delta_{\xi_0} \\ (1 - s)y_0 - tx_0 \delta_{\xi_0} \end{array} \right],
\]

(178)

where \(\delta_{\xi_0}\) is the delta measure at the point \(\xi_0\), and for \(\text{Re}(\xi_0) < 0\)

\[
\Sigma(\{\xi_0\}) = -\frac{x_0 \sqrt{\xi_0^2 - \rho^2}}{2y_0 (1 - s)^2} \times \left[ \begin{array}{c} (1 - s)y_0 + tx_0 \delta_{\xi_0} \\ i \alpha_0 \overline{\beta} x_0 \delta_{\xi_0} \\ (1 - s)y_0 - tx_0 \delta_{\xi_0} \end{array} \right],
\]

(179)

From this, we conclude the assertion in Theorem 5.7. \(\square\)
Appendix A: Proof of Theorem 1.1

Theorem 1.1 is proved in [11, 13, 17]. However, the way of presentation here is somehow different from them. Thus, we give its proof for completeness. Suppose that $f \in \text{Map}(\mathbb{Z}, \mathbb{C}^2)$ satisfies $U(C)f = \lambda f$ with $\lambda \in \mathbb{C} \setminus \{0\}$. We write

$$f(n) = \begin{bmatrix} f_L(n) \\ f_R(n) \end{bmatrix} \quad (n \in \mathbb{Z}, \ f_L(n), f_R(n) \in \mathbb{C}).$$

Then, by the definition of $U(C)$, we have

$$\begin{cases} a_n f_L(n + 1) + b_n f_R(n + 1) = \lambda f_L(n), \\ c_n f_L(n - 1) + d_n f_R(n - 1) = \lambda f_R(n). \end{cases} \quad (180)$$

Shifting the variable $n$, we get

$$\begin{cases} a_n f_L(n) + b_n f_R(n) = \lambda f_L(n - 1), \\ c_n f_L(n) + d_n f_R(n) = \lambda f_R(n + 1). \end{cases} \quad (181)$$

The second equation of (181) gives

$$f_R(n + 1) = \frac{1}{c_n} c_n f_L(n) + \frac{1}{d_n} c_n f_L(n) \quad (182)$$

Substituting this into the first equation of (180) shows

$$f_L(n + 1) = \frac{1}{a_n} (\lambda - \frac{1}{a_n} b_n c_n f_L(n) - \frac{1}{d_n} d_n f_R(n)). \quad (183)$$

Equations (182) and (183) give

$$f(n + 1) = T_\lambda(n) f(n) \quad (n \in \mathbb{Z}), \quad (184)$$

where $T_\lambda(n)$ is defined in (5). From (184) and the definition (6) of the matrix $F_\lambda(n)$, we have $f(n) = F_\lambda(n) f(0) = \Phi_\lambda(f(0))(n)$, and hence, $f \in \Phi_\lambda(\mathbb{C}^2)$. Conversely, we consider the function $f(n) : = \Phi_\lambda(u)(n)$ defined in (7) with a nonzero vector $u \in \mathbb{C}^2$. Using Lemma 2.4, we see

$$[U(C)f](n) = \pi_L C(n + 1) F_\lambda(n + 1) u + \pi_R C(n - 1) F_\lambda(n - 1) u$$

$$\overset{(1)}{=} \pi_L C(n + 1) T_\lambda(n) f(n) + \pi_R C(n - 1) T_\lambda(n - 1)^{-1} f(n)$$

$$\overset{(1), (2)}{=} \lambda \pi_L f(n) + \lambda \pi_R f(n) = \lambda f(n),$$

which shows that $f \in \mathcal{M}^\lambda$. Therefore, we have $\mathcal{M}^\lambda = \Phi_\lambda(\mathbb{C}^2)$. Since $F_\lambda(n)$ is an invertible matrix for each $\lambda \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{Z}$, the map $\Phi_\lambda : \mathbb{C}^2 \to \text{Map}(\mathbb{Z}, \mathbb{C}^2)$ is injective. This shows that $\dim \mathcal{M}^\lambda = \dim \text{Im} \Phi_\lambda = 2$. Next, we introduce a map

$$J : \text{Map}(\mathbb{Z}, \mathbb{C}^2) \to \text{Map}(\mathbb{Z}, \mathbb{C}^2)$$
defined as

\[(Jf)(n) = \begin{bmatrix} f_L(n-1) \\ f_R(n) \end{bmatrix}, \quad \text{where } f(n) = \begin{bmatrix} f_L(n) \\ f_R(n) \end{bmatrix}.\]

Let \( f = \Phi_x(w) \) with \( w \in \mathbb{C}^2 \). Using (181) we see

\[
\begin{align*}
  f_L(n) &= \frac{\lambda}{a_n}f_L(n-1) - \frac{b_n}{a_n}f_R(n), \\
  f_R(n+1) &= \frac{c_n}{a_n}f_L(n) + \frac{d_n}{a_n}f_R(n) \\
  &= \lambda^{-1}c_n f_L(n) + \frac{\lambda}{a_n}f_L(n-1) - \frac{b_n}{a_n}f_R(n) + \lambda^{-1}d_n f_R(n) \\
  &= \frac{c_n}{a_n}f_L(n-1) + \frac{\lambda^{-1}\triangle_n}{a_n}f_R(n).
\end{align*}
\]

This is equivalent to

\[
J\Phi_x(w)(n+1) = S_x(n)J\Phi_x(w)(n), \quad S_x(n) = \frac{1}{a_n} \begin{bmatrix} \lambda & -b_n \\ c_n & \lambda^{-1}\triangle_n \end{bmatrix} \quad (185)
\]

for all \( w \in \mathbb{C}^2 \). The matrix \( S_x(n) \) is also called the transfer matrix [13, 17]. Compared with \( T_x(n) \) which we used throughout the paper, \( S_x(n) \) has a nice property: \( \det S_x(n) = d_n/a_n \), and hence, \( |\det S_x(n)| = 1 \), whereas \( \det T_x(n) = d_n/a_{n+1} \) whose absolute value is, in general, not equal to 1. We take \( v, w \in \mathbb{C}^2 \) and suppose that \( \Phi_x(v) \in \ell^2(\mathbb{Z}, \mathbb{C}^2) \) and \( \|\Phi_x(w)(n)\|_{\ell^2} \) is bounded. By the definition of \( J \), \( \|(J\Phi_x(w))(n)\|_{\ell^2} \) is bounded. Since \( J \) is unitary, \( J\Phi_x(v) \in \ell^2(\mathbb{Z}, \mathbb{C}) \). Thus, we have \( \|(J\Phi_x(v))(n)\|_{\ell^2} \to 0 \) as \( n \to \infty \). We define a function \( W : \mathbb{Z} \to \mathbb{R} \) by the formula

\[
W(n) = |\det[(J\Phi_x(v))(n), (J\Phi_x(w))(n)]|.
\]

By (185), we see that

\[
W(n+1) = |\det[(J\Phi_x(v))(n+1), (J\Phi_x(w))(n+1)]| \\
= |\det[S_x(n)(J\Phi_x(v))(n), S_x(n)(J\Phi_x(w))(n)]| \\
= |\det(S_x(n))| \det[(J\Phi_x(v))(n), (J\Phi_x(w))(n)] = W(n)
\]

for each \( n \in \mathbb{Z} \). Therefore, \( W \) is a constant function. Since \( \|(J\Phi_x(v))(n)\|_{\ell^2} \to 0 \), we have \( W(n) \to 0 \) as \( n \to \infty \). Hence, \( W(n) \) is identically zero. This shows that \( J\Phi_x(v)(n) \) and \( J\Phi_x(w)(n) \) are linearly dependent for each \( n \in \mathbb{Z} \). We take \( s_0, t_0 \in \mathbb{C} \) which is not simultaneously zero satisfying

\[
s_0J\Phi_x(v)(0) + t_0J\Phi_x(w)(0) = 0.
\]

Then, by (185), we have, when \( n \geq 1 \)
An eigenfunction expansion formula for one-dimensional…

\[ s_0 J \Phi_A(v)(n) + t_0 J \Phi_A(w)(n) \]
\[ = s_0 S_A(n-1)J \Phi_A(v)(n-1) + t_0 S_A(n-1)J \Phi_A(w)(n-1) \]
\[ = S_A(n-1)[s_0 J \Phi_A(v)(n-1) + t_0 J \Phi_A(w)(n-1)] \]
\[ = \ldots \]
\[ = S_A(n-1) \ldots S_A(0)[s_0 J \Phi_A(v)(0) + t_0 J \Phi_A(w)(0)] = 0, \]
\[ s_0 J \Phi_A(v)(-n) + t_0 J \Phi_A(w)(-n) \]
\[ = S_A(-n)^{-1}[s_0 J \Phi_A(v)(-n+1) + t_0 J \Phi_A(w)(-n+1)] \]
\[ = \ldots \]
\[ = S_A(-n)^{-1} \ldots S_A(-1)^{-1}[s_0 J \Phi_A(v)(0) + t_0 J \Phi_A(w)(0)] = 0. \]

Therefore, we have \( s_0 J \Phi_A(v) + t_0 J \Phi_A(w) = 0 \) on \( \mathbb{Z} \), and this shows \( \dim \mathcal{M}^2 \cap \ell^2(\mathbb{Z}, \mathbb{C}^2) \leq 1 \).

\[ \square \]

Appendix B: Formal adjoint operators

In this section, we prove the following.

Lemma B.1 Let \( A : C_0(\mathbb{Z}, \mathbb{C}^2) \rightarrow \text{Map}(\mathbb{Z}, \mathbb{C}^2) \) be a linear map. Then, there exists a unique linear operator \( A^* : C_0(\mathbb{Z}, \mathbb{C}^2) \rightarrow \text{Map}(\mathbb{Z}, \mathbb{C}^2) \) satisfying

\[ \langle A^* f, g \rangle = \langle f, A g \rangle \quad (f, g \in C_0(\mathbb{Z}, \mathbb{C}^2)). \quad (186) \]

We call the operator \( A^* \) the formal adjoint of \( A \).

Before proceeding to the proof of Lemma B.1, let us prepare some notation and simple properties of a linear map \( A : C_0(\mathbb{Z}, \mathbb{C}^2) \rightarrow \text{Map}(\mathbb{Z}, \mathbb{C}^2) \). We recall that, for \( m \in \mathbb{Z} \) and \( u \in \mathbb{C}^2 \), the function \( \delta_m \otimes u \in C_0(\mathbb{Z}, \mathbb{C}^2) \) is defined as in (9). Then, any function \( g \in C_0(\mathbb{Z}, \mathbb{C}^2) \) can be written as

\[ g = \sum_{m \in \mathbb{Z}} \delta_m \otimes g(m). \quad (187) \]

Let \( A : C_0(\mathbb{Z}, \mathbb{C}^2) \rightarrow \text{Map}(\mathbb{Z}, \mathbb{C}^2) \) be a linear map. For any \( m, n \in \mathbb{Z} \), we define a linear map \( k_A(n, m) : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) by

\[ k_A(n, m)u = A(\delta_m \otimes u)(n) \quad (u \in \mathbb{C}^2). \quad (188) \]

The \( 2 \times 2 \) matrix-valued function \( k_A(m, n) \) is called the kernel function of \( A \). The sum in (187) is finite for \( g \in C_0(\mathbb{Z}, \mathbb{C}^2) \), and hence, we have

\[ (Ag)(n) = \sum_{m \in \mathbb{Z}} A(\delta_m \otimes g(m))(n) = \sum_{m \in \mathbb{Z}} k_A(n, m)g(m). \quad (189) \]
Proof of Lemma B.1 First, we show that the operator satisfying (186) is unique. Indeed, suppose that there are two linear maps \( B, C \) from \( C_0(\mathbb{Z}, C^2) \) to \( \text{Map}(\mathbb{Z}, C^2) \) satisfying

\[
\langle Bf, g \rangle = \langle Cf, g \rangle = \langle f, Ag \rangle \quad (f, g \in C_0(\mathbb{Z}, C^2)).
\]

We set \( X = B - C \). Then, we have \( \langle Xf, g \rangle = 0 \) for any \( f, g \in C_0(\mathbb{Z}, C^2) \). For any \( n \in \mathbb{Z}, u \in C^2 \) and \( f \in C_0(\mathbb{Z}, C^2) \), we see

\[
0 = \langle Xf, \delta_n \otimes u \rangle = \sum_{m \in \mathbb{Z}} \langle (Xf)(m), (\delta_n \otimes u)(m) \rangle_{C^2} = \langle (Xf)(n), u \rangle_{C^2}.
\]

This shows that \( \langle (Xf)(n), u \rangle_{C^2} = 0 \) for any \( u \in C^2 \), and hence \( (Xf)(n) = 0 \). Since \( n \in \mathbb{Z} \) is arbitrary, we see \( Xf = 0 \). Since \( f \in C_0(\mathbb{Z}, C^2) \) is arbitrary, we see \( X = 0 \). Namely, \( B = C \), which shows that an operator \( A^* \) satisfying (186) is unique if it exists. Next, we construct an operator \( A^* \) satisfying (186). For \( f \in C_0(\mathbb{Z}, C^2) \) and \( n \in \mathbb{Z} \), we define \( (A^*f)(n) \in C^2 \) by

\[
(A^*f)(n) = \sum_{m \in \mathbb{Z}} k_A(m, n)^* f(m),
\]

where \( k_A(m, n)^* \) is the adjoint matrix of the matrix \( k_A(m, n) \). The sum in (190) is finite, since \( f \in C_0(\mathbb{Z}, C^2) \). Thus, the formula (190) defines a vector \( (A^*f)(n) \in C^2 \). Therefore, we have a linear map \( A^* : C_0(\mathbb{Z}, C^2) \rightarrow \text{Map}(\mathbb{Z}, C^2) \). For \( g \in C_0(\mathbb{Z}, C^2) \), the pairing \( \langle A^*f, g \rangle \) is defined and

\[
\langle A^*f, g \rangle = \sum_{n \in \mathbb{Z}} \langle (A^*f)(n), g(n) \rangle_{C^2} = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle k_A(m, n)^* f(m), g(n) \rangle_{C^2}
\]

\[
= \sum_{m \in \mathbb{Z}} \langle f(m), k_A(m, n) g(n) \rangle_{C^2} = \sum_{m \in \mathbb{Z}} \langle f(m), (Ag)(n) \rangle_{C^2}
\]

\[
= \langle f, Ag \rangle,
\]

which shows that \( A^* \) defined by (190) satisfies (186). \( \square \)

For example, the kernel function \( U(n, m) (n, m \in \mathbb{Z}) \) of the quantum walk \( U(\mathcal{C}) \) defined in (2) is given as

\[
U(n, m) = \delta(n - m + 1)\pi_L \mathcal{C}(m) + \delta(n - m - 1)\pi_R \mathcal{C}(m),
\]

where \( \delta : \mathbb{Z} \rightarrow \mathbb{C} \) is defined by

\[
\delta(n) = \begin{cases} 1 & (n = 0), \\ 0 & (n \neq 0). \end{cases}
\]

The kernel function \( U^*(n, m) \) of the formal adjoint \( U(\mathcal{C})^* \) is given as

\[
U^*(n, m) = U(m, n)^* = \delta(m - n + 1)\mathcal{C}(n)^*\pi_L + \delta(m - n - 1)\mathcal{C}(n)^*\pi_R.
\]

From this, we have the formula (30).
Acknowledgements The author would like to thank the anonymous referees for their comments and suggestions that help improve the presentation of the paper. The author is partially supported by JSPS KAKENHI Grant No. 18K03267, 17H06465.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Cantero, M.J., Grünbaum, F.A., Moral, L., Velázquez, L.: Matrix-valued Szegö polynomials and quantum random walks. Commun. Pure Appl. Math. 63(4), 464–507 (2010)
2. Cantero, M.J., Moral, L., Velázquez, L.: Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle. Linear Algebra Appl. 362, 29–56 (2003)
3. Endo, T., Konno, N., Segawa, E., Takei, M.: Limit theorems of a two-phase quantum walk with one defect. Quantum Inf. Comput. 15, 1373–1396 (2015)
4. Federer, H.: Geometric Measure Theory. Classics in Mathematics (Reprint of the 1969 Ed.). Springer, Berlin (1996)
5. Fillman, J., Ong, D.C.: Purely singular continuous spectrum for limit-periodic CMV operators with applications to quantum walks. J. Funct. Anal. 272(12), 5107–5143 (2017)
6. Gesztesy, F., Janson, S., Scudo, P.: Weak limits for quantum random walks. Phys. Rev. E 69, 026119 (2004)
7. Grimmett, G., Janson, S., Scudo, P.: Weak limits for quantum random walks. Phys. Rev. E 69, 026119 (2004)
8. Gesztesy, F., Tsekanovskii, E.: On matrix-valued Herglotz functions. Math. Nachr. 218(1), 61–138 (2000)
9. Gesztesy, F., Zinchenko, M.: Weyl–Titchmarsh theory for CMV operators associated with orthogonal polynomials on the unit circle. J. Approx. Theory 139(1–2), 172–213 (2006)
10. Konno, N.: A new type of limit theorems for the one-dimensional quantum random walk. J. Math. Soc. Jpn. 57, 1179–1195 (2005)
11. Kodaira, K.: The eigenvalue problem for ordinary differential equations of the second order and Heisenberg’s theory of S-matrices. Am. J. Math. 71(4), 921–945 (1949)
12. Kawai, H., Komatsu, T., Konno, N.: Stationary measure for two-state space-inhomogeneous quantum walk in one dimension. Yokohama Math. J. 63, 59–74 (2017)
13. Kotani, K., Matano, H.: Differential Equations and Eigenfunction Expansion. Iwanami Shoten, Tokyo (2006) (in Japanese)
14. Kiuchi, C., Saito, K.: Eigenvalues of two-phase quantum walks with one defect in one dimension. Quantum Inf. Process. 20, 11 (2021) (Article no. 171)
15. Marchenko, V.A.: Sturm–Liouville Operators and Applications. Birkhäuser, Basel (1986)
16. Morioka, H.: Generalized eigenfunctions and scattering matrices for position-dependent quantum walks. Rev. Math. Phys. 31(7), 1950019 (2019)
17. Morioka, H., Segawa, E.: Detection of edge defects by embedded eigenvalues of quantum walks. Quantum Inf. Process. 18(2), 18 (2019) (Article no. 283)
18. Maeda, M., Sasaki, H., Segawa, S., Suzuki, S., Suzuki, K.: Dispersive estimates for quantum walks on 1D lattice. J. Math. Soc. Japan 74(1), 217–246 (2022)
19. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness. Academic Press, Inc., San Diego (1975)
20. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, III: Scattering Theory. Academic Press, Inc., San Diego (1979)
20. Richard, A., Suzuki, R., de Aldecoa, T.: Quantum walks with an anisotropic coin, I: spectral theory. Lett. Math. Phys. **108**, 331–357 (2018)

21. Richard, A., Suzuki, R., de Aldecoa, T.: Quantum walks with an anisotropic coin, II: scattering theory. Lett. Math. Phys. **109**, 61–88 (2019)

22. Simon, B.: Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, Colloquium Publications, Part 1, vol. 54. American Mathematical Society, Providence (2005)

23. Simon, B.: Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory. American Mathematical Society Colloquium Publications, Part 2, vol. 54. American Mathematical Society, Providence (2005)

24. Stone, M.H.: Linear Transformations in Hilbert Space. American Mathematical Society Colloquium Publications, vol. 15. American Mathematical Society, Providence (1932)

25. Sunada, T., Tate, T.: Asymptotic behavior of quantum walks on the line. J. Funct. Anal. **262**(6), 2608–2645 (2012)

26. Titchmarsh, E.C.: Eigenfunction Expansion, Part I. Oxford University Press, Oxford (1962)

27. Weyl, H.: Über gewöhnliche Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen. Göttinger Nachrichten 230–254 (1935)

28. Weyl, H.: Über gewöhnliche Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen. Göttinger Nachrichten 442–467 (1910)