Renormalizability of the gradient flow in the 2D $O(N)$ non-linear sigma model

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It is known that the gauge field and its composite operators evolved by the Yang–Mills gradient flow are ultraviolet (UV) finite without any multiplicative wave function renormalization. In this paper, we prove that the gradient flow in the 2D $O(N)$ non-linear sigma model possesses a similar property: The flowed $N$-vector field and its composite operators are UV finite without multiplicative wave function renormalization. Our proof in all orders of perturbation theory uses a $(2 + 1)$-dimensional field theoretical representation of the gradient flow, which possesses local gauge invariance without gauge field. As an application of the UV finiteness of the gradient flow, we construct the energy–momentum tensor in the lattice formulation of the $O(N)$ non-linear sigma model that automatically restores the correct normalization and the conservation law in the continuum limit.

Subject Index B31, B32, B34, B38

1. Introduction and summary

The Yang–Mills gradient flow or the Wilson flow [1] has attracted much attention in recent years in the context of lattice gauge theory. Its known applications include: scale setting [1,2], definition of the topological charge [1,3], definition of non-perturbative gauge coupling [4,5], chiral condensation [6], improvement of step scaling [7], etc. Even its application to supersymmetric theory [8] and to the operator product expansion [9] is considered. Reference [10] is a review of this notion and further related works can be found in a review [11] and in a most recent paper on the non-perturbative beta function [12].

A crucial property of the Yang–Mills gradient flow, underlying the above applications, is its “ultraviolet (UV) finiteness” [1,13]. The gradient flow is a one-parameter (called the flow-time) evolution of the gauge field, according to a “heat diffusion equation” (called the flow equation). A remarkable fact that can be rigorously proven [13] in all orders of perturbation theory is that any correlation function of the evolved (or flowed) gauge field becomes UV finite without the wave function renormalization, as long as the parameters of the theory are properly renormalized. Moreover, any local product of the flowed gauge field remains UV finite without further (multiplicative as well as subtractive) renormalization. This remarkable property of the gradient flow facilitates, in particular, the construction of renormalized composite operators of the gauge field. That is, any simple product of the flowed (bare) gauge field as it stands is a renormalized (i.e., UV-finite) quantity.

In Ref. [14], as a possible application of the gradient flow, one of us (H.S.) considered the construction of the energy–momentum tensor in lattice gauge theory. This application of the gradient...
flow to the energy–momentum tensor was further developed from a somewhat different perspective in Ref. [15]. The construction was then generalized to gauge theories including the fermion field [16]. The genuine energy–momentum tensor cannot be defined on the lattice because the lattice structure breaks the translational invariance explicitly. Even the construction of a lattice operator that reduces to the correctly normalized conserved energy–momentum tensor in the continuum limit is quite non-trivial, as investigated in Refs. [17,18]. Reference [19] is a pioneering work on this issue.

The basic idea of Refs. [14,16], which uses the UV finiteness of the gradient flow in an essential way, is recapitulated in Sect. 6 of the present paper. The aim of Refs. [14,16] is to construct a lattice operator that automatically reduces to the correctly normalized conserved energy–momentum tensor in the continuum limit. Theoretically, there is only little room for doubt on the reasoning in Refs. [14, 16]. Practically, however, it is not a priori clear whether presently available lattice parameters are sufficient to extract physical information by using the construction. On this issue, the promising result in Ref. [20] for thermodynamical quantities in quenched QCD is quite encouraging. Still, it is indispensable to numerically demonstrate the conservation law of the energy–momentum tensor by using lattice Monte Carlo simulations.

Under these situations, it seems useful to study a simpler system that would allow a similar construction of the lattice energy–momentum tensor using the gradient flow. One of the basic assumptions in Refs. [14,16] is that the theory is asymptotically free. Not so many field theories exhibit asymptotic freedom, however. This was our original motivation for the present study on the gradient flow in the 2D $O(N)$ non-linear sigma model [21–23]. It is well known [24] that the physics of this systems possesses many similarities with the 4D non-Abelian gauge theory. These include asymptotic freedom, dynamical generation of the mass gap, and, for $N = 3$, the topological term and associated $\theta$-parameter. See also Refs. [25,26]. This system is also advantageous from a computational perspective (and thus from our original motivation), because there exists a very efficient cluster simulation algorithm [27,28]. The state of the art in non-perturbative lattice study of the 2D $O(N)$ non-linear sigma model can be found in Ref. [29].

In the present paper, we will show that there exists another surprising similarity between the 2D $O(N)$ non-linear sigma model and the 4D gauge theory: Any correlation function of the flowed $N$-vector field in the former becomes UV finite without the wave function renormalization, as long as the parameters of the theory are renormalized. This UV finiteness also persists for any local product of the flowed $N$-vector field. This similarity is surprising, because the UV finiteness of the flowed gauge field is a non-trivial consequence [13] of the gauge BRS symmetry that acts non-linearly on the gauge field. In fact, matter fields such as the fermion field transform linearly under the gauge BRS symmetry and they do require wave function renormalization even after the flow [6]. In the 2D $O(N)$ non-linear sigma model, however, it is not clear at first glance what plays the same role as this gauge BRS symmetry in the 4D gauge theory. Our proof clarifies this point. On the other hand, happily, because of the UV finiteness of the gradient flow in the 2D $O(N)$ non-linear sigma model, we can repeat the construction of the lattice energy–momentum tensor in Refs. [14,16].

The following describes the organization of the present paper and gives a summary of the contents of each section.

In Sect. 2, we introduce the flow equation in the 2D $O(N)$ non-linear sigma model. If one considers the application in lattice numerical simulations, this is the equation that should be solved numerically in conjunction with the conventional Monte Carlo simulations. We then formulate the perturbative expansion for the system defined by the combination of the 2D $O(N)$ non-linear sigma model and the flow equation (the flowed system).
In Sect. 3, on the basis of the perturbative expansion developed in Sect. 2, we explicitly compute the two-point function of the flowed bare $N$-vector field to the one-loop order. This explicit calculation shows that the two-point function is made UV finite by the conventional parameter renormalization in the non-linear sigma model [30], but without the wave function renormalization. We carry out the computation in dimensional regularization and in lattice regularization and arrive at the same conclusion. Although this computation is only in the one-loop level, it strongly indicates that the gradient flow in the non-linear sigma model has a similar UV property as the gauge theory.

As the proof for the 4D gauge theory in Ref. [13] and the renormalizability proof in the stochastic quantization [31,32], our proof in all orders of perturbation theory uses a local field theory with one spacetime dimension higher: We use a $(2 + 1)$-dimensional field theoretical representation of the flowed system. In Sect. 4, we define this $(2 + 1)$-dimensional local field theory. Then we show that the system defined through the flow equation in Sect. 2 and the $(2 + 1)$-dimensional field theory have equivalent perturbative expansions. It is easy to see the rough equivalence. However, a closer look reveals that there are discrepancies between the two systems; the measure term in the former is missing in the latter, while the former does not have the flow-line loop diagrams of the latter. Presumably, the step to show that these two apparently different elements are actually equivalent (Sect. 4.4) is the hardest part in our argument. We will find that, to address this very subtle problem in a convincing manner, it is necessary to first discretize the flow-time derivative and then take the continuum limit for this discretization; this necessity of discretization is also counterintuitive.

Once having obtained a local field theory that is (perturbatively) equivalent to the flowed system, a possible way to proceed is to write down a Ward–Takahashi relation or a Zinn-Justin equation [33] (see, e.g., Ref. [34]) for the 1PI generating functional,\(^1\) which restricts the possible form of counterterms, on the basis of a certain symmetry in the $(2 + 1)$-dimensional system. This is the content of Sect. 5. Here, we encounter another surprise: The $(2 + 1)$-dimensional field theory possesses local gauge symmetries, although it does not contain any gauge field. Note that the unique internal symmetry in the original 2D $O(N)$ non-linear sigma model is the global $O(N)$ symmetry. Because of these gauge symmetries, we have to fix the gauge. Even under the gauge fixing, there still remains a residual symmetry that acts non-linearly on various fields. We will find that the Zinn-Justin equation associated with this non-linear symmetry does the job. Then, by listing possible counterterms (by borrowing the information obtained in Sect. 4.4) and examining the restriction implied by the Zinn-Justin equation, we finally show that the only counterterms required are those of the original 2D $O(N)$ non-linear sigma model. In particular, the flowed $N$-vector field (and its composite operators) is not renormalized. This completes our proof for the UV finiteness of the gradient flow.

In Sect. 6, on the basis of the UV finiteness established in Sect. 5, we construct the energy–momentum tensor in a lattice formulation of the non-linear sigma model, following the line of reasoning of Refs. [14,16].

In summary, we have found another example in which the gradient flow exhibits a remarkable UV finiteness: in the 2D $O(N)$ non-linear sigma model, any correlation function of the flowed $N$-vector field and its composite operators is UV finite without multiplicative (as well as subtractive) renormalization. Our proof in the present paper also clarifies subtle but very interesting technical issues arising in the theoretical analysis of the gradient flow, such as the necessity of the discretization\(^2\).

\(^1\) In this aspect, our approach is more conventional than the approach in Ref. [13].
of the flow-time derivative and the emergence of gauge and/or non-linear symmetries in the corresponding local field theory with one dimension higher. The knowledge obtained here will be useful in considering the application of the gradient flow to a wider range of systems.

Also, going back to our original motivation, we hope to numerically test the idea of Refs. [14,16] by using the energy–momentum tensor constructed in Sect. 6 in the near future.

2. Gradient flow in the 2D $O(N)$ non-linear sigma model

2.1. 2D $O(N)$ non-linear sigma model and the flow equation

The 2D $O(N)$ non-linear sigma model is a field theory of an $N$ component vector with the unit length. Its partition function is given by

$$Z_{O(N)} = \int \prod_{i=1}^{N} Dn^i \left( \prod_{x} \delta(n(x)^2 - 1) \right) \exp \left( -\frac{1}{2g_0^2} \int d^Dx \sum_{i=1}^{N} \partial_{\mu}n^i(x)\partial_{\mu}n^i(x) \right) , \quad (2.1)$$

where $n(x)^2 \equiv \sum_{i=1}^{N} n^i(x)n^i(x)$ and $g_0$ is the bare coupling constant. Although the spacetime dimension $D$ is 2 for our target theory, expressions for generic $D$ are useful because we will extensively use dimensional regularization in what follows.

In the present paper, as an analogue of the Yang–Mills gradient flow [1], we consider the following $t$-evolution of the $N$-vector field (the flow equation):

$$\partial_t n^i(t, x) = P^{ij}(t, x)\partial_\mu \partial_\mu n^j(t, x), \quad (2.2)$$

where the initial value is given by the $N$-vector field in the $O(N)$ non-linear sigma model,

$$n^i(t = 0, x) = n^i(x), \quad (2.3)$$

which is subject to the functional integral (2.1). The projection operator $P^{ij}(t, x)$ in the right-hand side of the flow equation (2.2) is defined by

$$P^{ij}(t, x) \equiv \delta^{ij} - n^i(t, x)n^i(t, x) \quad (2.4)$$

(in Eq. (2.2) and in what follows, the sum over the repeated index is understood). The projection operator is introduced so that the flow is consistent with the constraint $n(t, x)^2 = 1$, where $n(t, x)^2 \equiv \sum_{i=1}^{N} n^i(t, x)n^i(t, x)$, i.e., $\partial_t n(t, x)^2 = 0$. The latter would be a natural requirement for the flow equation for the $O(N)$ non-linear sigma model. In fact, a flow equation identical to Eq. (2.2) has also been advocated in Appendix B of Ref. [8] from the perspective of the symmetry of the present system.\footnote{Throughout the present paper, the symbol $D$ is used for the functional integral over functions on the $D$-dimensional spacetime.}

\footnote{It is legitimate to call Eq. (2.2) the “gradient” flow, because the right-hand side of Eq. (2.2) can also be obtained as the equation of motion (i.e., the gradient in the functional space) in the system (2.1).}
2.2. Perturbative expansion

As usual, for the perturbative treatment of the $O(N)$ non-linear sigma model, we parametrize the constraint $n(x)^2 = 1$ in Eq. (2.1) in terms of $N - 1$ independent components (the $\pi$-field) as

\[ n^k(x) = \pi^k(x), \quad \text{for } k = 1, \ldots, N - 1, \]

\[ n^N(x) = \sqrt{1 - \pi(x)^2}, \quad \pi(x)^2 \equiv \sum_{k=1}^{N-1} \pi^k(x)\pi^k(x), \]

and then expand expressions regarding $\pi(x)$ as a small fluctuation. In this perturbative treatment, the partition function becomes

\[ Z_{O(N)} = \int \prod_{k=1}^{N-1} D\pi^k \left[ \prod_x \frac{1}{\sqrt{1 - \pi(x)^2}} \right] \times \exp \left( -\frac{1}{2g_0^2} \int d^Dx \left\{ \left[ \partial_\mu \pi(x) \right]^2 + \left[ \partial_\mu \sqrt{1 - \pi(x)^2} \right]^2 \right\} \right). \]

The above arbitrary choice of the perturbative branch, Eq. (2.6) with small $\pi(x)$, however, induces infrared (IR) divergences in the perturbative expansion of $O(N)$ non-invariant quantities [35]. To regularize the IR divergences, we introduce the mass term

\[ S_{\text{mass}} = -\frac{m_0^2}{g_0^2} \int d^Dx \left[ n^N(x) - 1 \right] = \frac{m_0^2}{g_0^2} \int d^Dx \left\{ \frac{1}{2} \pi(x)^2 + \frac{1}{8} \left[ \pi(x)^2 \right]^2 \right\} , \]

and take the massless limit $m_0 \to 0$ in the very end of the calculation. With this mass term, the particular perturbative branch (2.6) is favored for a weak coupling.

Also for the flowed field $n^i(t, x)$, since $n(t, x)^2 = 1$ holds along the flow evolution, we set

\[ n^k(t, x) = \pi^k(t, x), \quad \text{for } k = 1, \ldots, N - 1, \]

\[ n^N(t, x) = \sqrt{1 - \pi(t, x)^2}, \quad \pi(t, x)^2 \equiv \sum_{k=1}^{N-1} \pi^k(t, x)\pi^k(t, x). \]

Then the perturbative expansion of the flow equation (2.2) is obtained from the integral representation,

\[ \pi^k(t, x) = \int d^Dy \left[ K_t(x-y)\pi^k(y) + \int_0^t ds K_{t-s}(x-y)R^k(s, y) \right], \]

where $K_t(x)$ is the heat kernel,\(^4\)

\[ K_t(x) = \int_p e^{ipx} e^{-tp^2}. \]

\(^4\)Throughout the present paper, we use the abbreviation

\[ \int_p \equiv \int \frac{d^Dp}{(2\pi)^D}. \]
Fig. 1. A double wavy line represents the heat kernel (2.13).

Fig. 2. A single wavy line represents the free propagator (2.15).

Fig. 3. Diagram 01: A one-loop diagram that gives rise to the contribution (3.2) to the two-point function.

\begin{equation}
R^k(t,x) \equiv -\pi^k(t,x) \left[ \pi^I(t,x) \partial_\mu \partial_\mu \pi^I(t,x) + \sqrt{1 - \pi(t,x)^2} \partial_\mu \partial_\mu \sqrt{1 - \pi(t,x)^2} \right].
\end{equation}

Noting that \((\partial_t - \partial_\mu \partial_\mu)K_t(x) = 0\) and \(K_{t=0}(x) = \delta^D(x)\), we see that Eq. (2.11) solves Eq. (2.2) with the initial condition (2.3). By iteratively solving Eq. (2.11) in terms of the initial value \(\pi^k(y)\), therefore, we have a perturbative solution of the flow equation. This expansion can be represented diagrammatically (the flow Feynman diagram [13]) and, throughout this paper, we represent the heat kernel (2.13) by a double wavy line in Fig. 1. This line is also called the “flow-line propagator” or simply the “flow line”.

On the other hand, the combination \(R^k\) in Eq. (2.14) represents the effect of non-linear terms in the flow equation and, in what follows, this interaction will be denoted by an open circle (the flow vertex); see Fig. 4 for an example.

The initial value of the flow, \(\pi^k(y)\) in Eq. (2.11), is a quantum field subject to the functional integral (2.7). From Eq. (2.11) and Eq. (2.7) (with the mass term (2.8)), one then sees that the quantum free propagator of the flowed field is given by

\begin{equation}
\langle \pi^k(t,x)\pi^l(s,y) \rangle_0 = g_0^2 \delta^{kl} \int_p e^{ip(x-y)} \frac{e^{-p^2/2m_0^2}}{p^2 + m_0^2}.
\end{equation}

Note that, in this propagator, the flow times at the end points appear in the sum (not the difference). Throughout this paper, this free propagator will be denoted by a single wavy line (Fig. 2).

Finally, the functional integral (2.7) generates interaction vertices among the \(\pi^k(x)\). The interaction vertices in the action integral will be denoted by a filled circle (see Fig. 3 for an example). On the other hand, the interaction vertices arising from the functional measure in Eq. (2.7), the “measure term”,

\begin{equation}
\prod_x \frac{1}{\sqrt{1 - \pi(x)^2}} = \exp \left\{-\frac{1}{2} \delta^D(0) \int d^Dx \ln \left[1 - \pi(x)^2\right] \right\},
\end{equation}

will be represented by a cross as in Fig. 5.

3. One-loop calculation of correlation functions of the flowed field

An explicit one-loop calculation of the correlation functions of the flowed field is quite instructive, because it shows a remarkable UV property of the gradient flow. As the UV regularization, we first
Fig. 4. Diagram 02: A one-loop diagram that gives rise to the contribution (3.3) to the two-point function.

We adopt dimensional regularization, setting

\[ D = 2 - \epsilon. \]  

Let us compute the two-point function of the flowed \( \pi \)-field. The lowest-order (tree-level) two-point function is given by the free propagator (2.15) in Fig. 2.

In the one-loop level, diagram 01 in Fig. 3, which contains the interaction vertex in the original non-linear sigma model only, gives

\[
\left\{ \pi^k(t, x) \pi^l(s, y) \right\} = \frac{g_0^2}{4\pi} \left[ -2 + \ln \left( \frac{e^{\gamma_E} m_0^2}{4\pi} \right) \right] \int_p e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2 + m_0^2} \delta^{kl} \int_p e^{ip(x-y)} e^{-(t+s)p^2} \frac{m_0^2}{(p^2 + m_0^2)^2},
\]

where \( \gamma_E \) is Euler’s constant.

On the other hand, the contribution of another one-loop diagram, diagram 02 in Fig. 4, that contains the flow vertex is

\[
\left\{ \pi^k(t, x) \pi^l(s, y) \right\} = \frac{g_0^2}{4\pi} (N - 1) \left[ -2 + \frac{1}{2} \ln(8\pi t) + \frac{1}{2} \ln(8\pi s) + m_0^2 t \ln \left( 2e^{\gamma_E - 1} m_0^2 t \right) \\
+ m_0^2 s \ln \left( 2e^{\gamma_E - 1} m_0^2 s \right) \right] \times \frac{g_0^2}{4\pi} \int_p e^{ip(x-y)} e^{-(t+s)p^2} \frac{m_0^2}{p^2 + m_0^2}. \]

We note that the measure term in Eq. (2.16) vanishes identically in dimensional regularization with which \( \delta^D(0) \equiv 0 \). Thus, in total, we have

\[
\left\{ \pi^k(t, x) \pi^l(s, y) \right\} = \left\{ \pi^k(t, x) \pi^l(s, y) \right\} \left[ 1 + \frac{g_0^2}{4\pi} \left( N - 2 \right) \left[ -2 + \frac{1}{2} \ln(8\pi t) + \frac{1}{2} \ln(8\pi s) \\
+ (N - 1) m_0^2 t \ln \left( 2e^{\gamma_E - 1} m_0^2 t \right) \\
+ (N - 1) m_0^2 s \ln \left( 2e^{\gamma_E - 1} m_0^2 s \right) \right] \right\} \times \frac{g_0^2}{4\pi} \int_p e^{ip(x-y)} e^{-(t+s)p^2} \frac{m_0^2}{p^2 + m_0^2} \delta^{kl} \int_p e^{ip(x-y)} e^{-(t+s)p^2} \frac{m_0^2}{(p^2 + m_0^2)^2} + O \left( \frac{4}{g_0^4} \right). \]

(3.4)
Now, the parameter renormalization in the original $O(N)$ non-linear sigma model (2.7) with the mass term (2.8) is known to be (in the minimal subtraction (MS) scheme)

$$g_0^2 \equiv \mu \epsilon g^2 Z, \quad Z = 1 - \frac{g^2}{4\pi} 2(N-2)\frac{1}{\epsilon} + O(g^4),$$

(3.5)

and

$$m_0^2 = \frac{Z}{Z_3^{1/2}} m^2 = \left[ 1 - \frac{g^2}{4\pi} (N-3)\frac{1}{\epsilon} + O(g^4) \right] m^2, \quad Z_3 = 1 - \frac{g^2}{4\pi} 2(N-1)\frac{1}{\epsilon} + O(g^4).$$

(3.6)

where $Z_3$ is the wave function renormalization factor for the unflowed $\pi$-field, $\pi^k(x) = Z_3^{1/3} \pi_R^k(x)$. If we make these substitutions in Eqs. (3.4), we obtain the following completely UV-finite expression:

$$\langle \pi^k(t,x) \pi^l(s,y) \rangle = \left\{ 1 + \frac{g^2}{4\pi} \left[ \ln \left( \frac{e^{\gamma_E} m^2}{4\pi \mu^2} \right) + \frac{1}{2} (N-1) \ln \left( 8\pi \mu^2 t \right) + \frac{1}{2} (N-1) \ln \left( 8\pi \mu^2 s \right) \\
+ (N-1)m^2 t \ln \left( 2e^{\gamma_E-1} m^2 t \right) + (N-1)m^2 s \ln \left( 2e^{\gamma_E-1} m^2 s \right) \right] \right\}
\times g^2 \delta^{kl} \int_p e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2 + z_m m^2} + O(g^4),$$

(3.7)

where

$$z_m = 1 - \frac{g^2}{4\pi} \frac{1}{2} (N-3) \ln \left( \frac{e^{\gamma_E} m^2}{4\pi \mu^2} \right).$$

(3.8)

Remarkably, when expressed in terms of renormalized parameters, the two-point function of the flowed $\pi$-field is UV finite without multiplicative wave function renormalization. This UV finiteness of the flowed field is similar to that of the 4D gauge field flowed by the Yang–Mills gradient flow, a property first observed in Ref. [1] in lower-order perturbative computations and then proven in all orders of perturbation theory in Ref. [13]. The above result indicates that, by a similar mechanism to the 4D gauge theory, the $N$-vector field flowed to positive flow times is UV finite only with parameter renormalization.

It is also instructive to repeat the above calculation by using lattice regularization instead of dimensional regularization. We adopt the prescription that in Eq. (2.7) $\int d^D x \rightarrow a^2 \sum_x$, where $a$ denotes the lattice spacing, and the derivative $\partial_\mu$ is replaced by the forward difference operator. The Laplacian in the flow equation (2.2) is replaced by $\partial_\mu \partial_\mu \rightarrow \partial_\mu^* \partial_\mu^*$, where $\partial_\mu$ and $\partial_\mu^*$ are the forward

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5 In Sect. 5, as a byproduct of our analysis, we will have a proof for these renormalization rules.

6 Kengo Kikuchi and his collaborators independently observed this UV finiteness (private communication).
and backward difference operators, respectively. Then the contribution of Fig. 3 is

\[
\langle \pi^k(t, x) \pi^l(s, y) \rangle = \frac{g_0^2}{4\pi} \left[ \ln (am_0)^2 - 5 \ln 2 + \pi \right] g_0^2 \delta_{kl} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2 + m_0^2},
\]

\[
+ \frac{g_0^2}{4\pi} \left\{ \frac{N-3}{2} \left[ \ln (am_0)^2 - 5 \ln 2 \right] - \pi \right\} g_0^2 \delta_{kl} \times \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2 + m_0^2},
\]

\[
+ \frac{g_0^2}{4\pi} \left( \frac{4\pi}{a^2} \right) g_0^2 \delta_{kl} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2 + m_0^2},
\]

which is quadratically divergent. The quadratic divergence in the last term is canceled by the measure term (2.16) with \( \delta_D(0) \rightarrow 1/a^2 \) for lattice regularization. In fact, the contribution of the measure term to the two-point function (Fig. 5) is

\[
\langle \pi^k(t, x) \pi^l(s, y) \rangle = \frac{g_0^2}{4\pi} \left( \frac{4\pi}{a^2} \right) g_0^2 \delta_{kl} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2 + m_0^2}.
\]

On the other hand, the contribution of Fig. 4 is

\[
\langle \pi^k(t, x) \pi^l(s, y) \rangle = \frac{g_0^2}{4\pi} (N-1) \left[ - \ln(am_0)^2 + 5 \ln 2 + \frac{1}{2} \ln \left( 2e^{\gamma_E}m_0^2t \right) + \frac{1}{2} \ln \left( 2e^{\gamma_E}m_0^2s \right) \right] \times \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2 + m_0^2}.
\]

Thus, we have in total

\[
\langle \pi^k(t, x) \pi^l(s, y) \rangle = \left\{ 1 + \frac{g_0^2}{4\pi} \left[ -(N-2) \left[ \ln (am_0)^2 - 5 \ln 2 \right] + \pi + \frac{1}{2} (N-1) \ln \left( 2e^{\gamma_E}m_0^2t \right) \right] \right\} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2 + m_0^2}
\]

\[
+ \frac{g_0^2}{4\pi} \left\{ \frac{N-3}{2} \left[ \ln (am_0)^2 - 5 \ln 2 \right] - \pi \right\} g_0^2 \delta_{kl} \times \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2 + m_0^2} + O\left( g_0^4 \right).
\]

It is obvious that all UV divergences are removed by the parameter renormalization (3.5) and (3.6) with the replacement \( 1/\epsilon \rightarrow -\ln a \); again, remarkably, no wave function renormalization is required.

Although the two-point function (3.7) is UV finite, it contains IR divergences (i.e., it diverges for \( m \rightarrow 0 \)) because it is not an \( O(N) \) invariant “physical” quantity [35]. As a simple example of an
IR-finite $O(N)$-invariant observable, we can consider the “energy density”, defined by

$$E(t, x) \equiv \frac{1}{2} \partial_\mu n^i(t, x) \partial_\mu n^i(t, x),$$

(3.13)

which is analogous to the energy density introduced in Ref. [1] for the gauge theory.

For the vacuum expectation value,

$$\langle E(t, x) \rangle = \frac{1}{2} \left\{ \left[ \partial_\mu \pi(t, x) \right]^2 + \left[ \partial_\mu \sqrt{1 - \pi(t, x)}^2 \right]^2 \right\},$$

(3.14)

there are four flow Feynman diagrams to the next-to-leading order, as depicted in Figs. 6–9 (the cross denotes the operator $E(t, x)$). A straightforward calculation using dimensional regularization yields

$$\langle E(t, x) \rangle = \frac{g_0^2}{4\pi} (N - 1) \frac{1}{4t} (8\pi t)^{\epsilon/2} \left[ 1 + \frac{g_0^2}{4\pi} 2(N - 2) \frac{1}{\epsilon} (8\pi t)^{\epsilon/2} + O\left(\frac{g_0^4}{\epsilon}\right) \right]$$

$$= \frac{g^2}{4\pi} (N - 1) \frac{1}{4t} \left[ 1 + \frac{g^2}{4\pi} (N - 2) \ln \left(8\pi \mu^2 t\right) + O\left(g^4\right) \right].$$

(3.15)

This is IR finite as expected and UV finite in terms of the renormalized coupling constant, again indicating the UV finiteness of the flowed field. If this UV finiteness persists to all orders (we will prove this in a later section), the result (3.15) shows that the combination $t \langle E(t, x) \rangle$ provides a possible non-perturbative definition of a renormalized coupling as the gradient flow scheme in the 4D gauge theory (see, e.g., Refs. [4,5]). That is, we can set

$$g_R^2 \left(1/\sqrt{8t}\right) = \frac{16\pi}{N - 1} t \langle E(t, x) \rangle = g_0^2 + \cdots.$$  (3.16)

Then it must be interesting to investigate the running of this non-perturbative coupling in numerical lattice simulations, in view of the expected conformal and walking behaviors of the $O(3)$ non-linear sigma model with non-zero $\theta$-parameters [26].

4. $(D + 1)$-dimensional field theoretical representation of the gradient flow

In the next section, we reveal the renormalization structure of the flowed system defined in Sect. 2. We prove in particular that the flowed $N$-vector field does not require the wave function renormalization.
Our strategy is identical to the case of the 4D gauge theory [13]; we seek a \((D + 1)\)-dimensional local field theory that reproduces the flow Feynman rules in the preceding sections and use this to show the renormalizability. We neglect the IR-regulating mass term (2.8) in this section, because it complicates the argument destroying the \(O(N)\) symmetry. We will consider the effect of the mass term at the very end of the next section.

4.1. Partition function

As in Refs. [6,13], we consider a \((D + 1)\)-dimensional \((D = 2\) for our target theory\) field theory defined in the half space, \((t, x) \in [0, \infty) \times \mathbb{R}^D\), that (at least perturbatively) is equivalent to the gradient flow in the 2D \(O(N)\) non-linear sigma model. We will find that, to resolve subtleties associated with the measure term and the flow-line loop (see Sect. 4.4), it is necessary to specify a prescription for the flow-time derivative. We will use the forward difference prescription (with the discretization length \(\epsilon\)) for this.

The regularization for the \(D\)-dimensional “spacetime” direction is, on the other hand, arbitrary and we may assume, for instance, dimensional regularization or lattice regularization.

The partition function of the \((D + 1)\)-dimensional field theory that we consider is defined by

\[
Z \equiv \int \left[ \prod_{i=1}^{N} D\xi^i(x) \right] \left[ \prod_{i=1}^{N} Dn^i(t) \right] \left[ \prod_{x} \delta(n(x)^2 - 1) \right] \\
\times \left[ \prod_{t=0}^{\infty} \prod_{i=1}^{N} D\lambda^i(t) \right] \left[ \prod_{t=0}^{\infty} \prod_{i=1}^{N} Dn^i(t) \right] \left[ \prod_{t=0}^{\infty} \prod_{x} \delta(n(t, x)^2 - 1) \right] \sqrt{1 - n_\perp(t + \epsilon, x)^2} e^{-S},
\]

(4.1)

where \(t = 0, \epsilon, 2\epsilon, \ldots\), and

\[
S \equiv \frac{1}{2g_0^2} \int d^Dx \partial_\mu n^i(x) \partial_\mu n^i(x) \\
- i\epsilon \sum_{t=0}^{\infty} \int d^Dx \lambda^i(t, x) P^{ij}(t, x) \left\{ \frac{1}{\epsilon} \left[ n^j(t + \epsilon, x) - n^j(t, x) \right] - \partial_\mu \partial_\mu n^i(t, x) \right\} \\
- i \int d^Dx \xi^i(x) \left[ n^i(0, x) - n^i(x) \right].
\]

(4.2)

In these expressions, \(n^i(x)\) corresponds to the \(N\)-vector field in the \(D\)-dimensional \(O(N)\) non-linear sigma model (2.1) and \(n^i(t, x)\) corresponds to the \(N\)-vector field evolved by the flow equation (2.2). The basic idea is that the functional integral over the Lagrange multiplier \(\lambda^i(t, x)\) imposes the flow

---

\(7\) Our renormalization proof uses a \((D + 1)\)-dimensional system that assumes a particular forward difference for the flow-time derivative. We do not mean, however, that the time evolution in the gradient flow must be defined by the forward time difference; any sound discretization of the flow-time derivative can be used to implement the flow equation (2.2) in numerical simulations. The \((D + 1)\)-dimensional system below is merely an intermediate tool for the renormalization proof and, in our present context, is not an object to be simulated.
equation (2.2) with the discretized flow time. Note that the left-hand side of Eq. (2.2) can equivalently be written as $P^{ij}(t, x) \partial_t n^j(t, x)$ with the projection operator $P^{ij}(t, x)$ in Eq. (2.4). The integration over another Lagrange multiplier $\xi^i(x)$ in Eq. (4.1), on the other hand, imposes the initial condition (2.3).

In Eq. (4.1), $n_\perp(t + \epsilon, x)^2 = \sum_{i=1}^{N} n^i_\perp(t + \epsilon, x)n^i_\perp(t + \epsilon, x)$, and

$$n^i_\perp(t + \epsilon, x) = \epsilon P^{ij}(t, x) \partial_t n^j(t, x).$$

(4.3)

It can be shown that, with the factor $\sqrt{1 - n_\perp(t + \epsilon, x)^2}$ in the integration measure, the partition function $Z$ (4.1) can be obtained from the original partition function $Z_{O(N)}$ (2.1) by inserting unity (up to infinite gauge volume; see below). However, since $\sqrt{1 - n_\perp(t + \epsilon, x)^2} = 1 + O(\epsilon^2) \to 1$ for $\epsilon \to 0$, this factor can be neglected in the $\epsilon \to 0$ limit and we do not explicitly include this factor in what follows.

4.2. Symmetries and the gauge fixing

The above $(D + 1)$-dimensional system possesses the following symmetries. One is the global $O(N)$ symmetry that is inherited from the original $O(N)$ non-linear sigma model:

$$\delta n^i(x) = \epsilon^{ij} n^j(x), \quad \delta \xi^i(x) = \epsilon^{ij} \xi^j(x),$$

$$\delta n^i(t, x) = \epsilon^{ij} n^j(t, x), \quad \delta \lambda^i(t, x) = \epsilon^{ij} \lambda^j(t, x),$$

(4.4)

where $\epsilon^{ij} = -\omega^{ij}$ are infinitesimal constant parameters.

Other, somewhat unexpected ones are local gauge symmetries:

$$\delta n^i(x) = 0, \quad \delta \xi^i(x) = g(x) \bar{n}^i(x),$$

$$\delta n^i(t, x) = 0, \quad \delta \lambda^i(t, x) = h(t, x) n^i(t, x),$$

(4.5)

where

$$\bar{n}^i(x) \equiv \frac{n^i(0, x) + n^i(x)}{2},$$

(4.6)

and $g(x)$ and $h(t, x)$ are local parameters that can depend on their arguments. These local symmetries, which exist even with the discretized flow-time and $D$-dimensional regularization, follow from the constraints $n(x)^2 = n(t, x)^2 = 1$ in the functional integral and the property $n^i(t, x) P^{ij}(t, x) = 0$. Because of these gauge symmetries, the partition function (4.1) itself is infinite. This is not a problem in our present context, because what we need at this moment is a generating functional of the perturbative expansion of the flowed system.

To formulate perturbation theory in the above $(D + 1)$-dimensional field theory, we thus have to first fix the gauge symmetries (4.5). For this, we adopt the following gauge fixing conditions:

$$\xi^N(x) = 0, \quad \lambda^N(t, x) = 0,$$

(4.7)

and follow the Faddeev–Popov procedure. Thus we insert unity

$$\int Dg \left[ \prod_x \delta \left( \xi^N(x) - g(x) \bar{n}^N(x) \right) \left| \bar{n}^N(x) \right| \right] \times \left[ \prod_{t=0}^{\infty} Dh(t) \left[ \prod_x \prod_{i=0}^{\infty} \delta \left( \lambda^N(t, x) - h(t, x) n^N(t, x) \right) \left| n^N(t, x) \right| \right] = 1$$

(4.8)

into the functional integral (4.1). Then, using the invariance of the action and the functional measure under the transformations (4.5), we can factor out the gauge volume.
\[
\int Dg \left[ \prod_{t=0}^{\infty} Dh(t) \right]
\]

from the partition function (4.1).

We further solve the constraints \( n(x)^2 = n(t, x)^2 = 1 \) in terms of \( N - 1 \) independent components, as Eqs. (2.5) and (2.6) and Eqs. (2.9) and (2.10). Then, after the gauge volume is factored out, the partition function is given by

\[
Z' = \int \left[ \prod_{k=1}^{N-1} D\xi^k \right] \left[ \prod_{k=1}^{N-1} D\pi^k \right] \times \left[ \prod_{t=0}^{\infty} \prod_{k=1}^{N-1} D\lambda^k(t) \right] \left[ \prod_{t=0}^{\infty} \prod_{k=1}^{N-1} D\pi^k(t) \right] \prod_x \sqrt{1 - \pi(x)^2} e^{-S},
\]

where

\[
\sqrt{1 - \pi(x)^2} \equiv \sqrt{1 - \pi(0, x)^2 + \sqrt{1 - \pi(x)^2}}
\]

and

\[
S = \frac{1}{2g_0^2} \int d^Dx \left\{ \left[ \partial_\mu \pi(x) \right]^2 + \left[ \partial_\mu \sqrt{1 - \pi(x)^2} \right]^2 \right\} - i e \sum_{t=0}^{\infty} \int d^Dx \lambda^k(t, x) \left( \frac{1}{\epsilon} \left[ \pi^k(t + \epsilon, x) - \pi^k(t, x) \right] - \partial_\mu \partial_\mu \pi^k(t, x) - R^k(t, x) \right) + \mathcal{E}
\]

\[
- i \int d^Dx \xi^k(x) \left[ \pi^k(0, x) - \pi^k(x) \right],
\]

where the combination \( R^k(t, x) \) is defined by Eq. (2.14) and

\[
\mathcal{E} = i e \sum_{t=0}^{\infty} \int d^Dx \lambda^k(t, x) \pi^k(t, x) \left\{ \pi^l(t, x) \frac{1}{\epsilon} \left[ \pi^l(t + \epsilon, x) - \pi^l(t, x) \right] + \sqrt{1 - \pi(t, x)^2} \frac{1}{\epsilon} \left[ \sqrt{1 - \pi(t + \epsilon, x)^2} - \sqrt{1 - \pi(t, x)^2} \right] \right\}.
\]

### 4.3. Feynman rules in the \((D + 1)\)-dimensional system

Next we derive the Feynman rules in the above system (4.10)–(4.13). To write down the free propagator, we introduce the heat kernel with the discretized flow time, by

\[
K^\epsilon_f(x) \equiv \int_\rho e^{i px} \left( 1 - \epsilon p^2 \right)^{1/\epsilon},
\]

which fulfills

\[
\frac{1}{\epsilon} \left[ K^\epsilon_{f + \epsilon}(x) - K^\epsilon_f(x) \right] - \partial_\mu \partial_\mu K^\epsilon_f(x) = 0, \quad K^\epsilon_0(x) = \delta^D(x).
\]

Clearly, \( K^\epsilon_f(x) \) reduces to the heat kernel (2.13) in the continuum flow-time limit, \( K^\epsilon_f(x) \xrightarrow{\epsilon \to 0} K_f(x) \).

By using this object, we change the integration variables from \( \pi^k(t, x) \) to \( p^k(t, x) \) as [13]

\[
\pi^k(t, x) = \int d^Dy K^\epsilon_f(x - y) \pi^k(y) + p^k(t, x).
\]
Note that \( \vartheta(\text{implicitly implied in Eq. (2.11) through the retarded time-ordering, being cubic or higher in fields.}) \). The first line of Eq. (4.12) of course reproduces the interaction terms in the action of the \( \pi\pi \)-propagator (4.22) reproduces the flow-line propagator This completes our derivation of free propagators. In the continuum flow-time limit \( \epsilon \to 0 \), abbreviated terms are cubic or higher in fields. It is then straightforward to find free propagators and the result is

\[
\left\langle \pi^k(x)\pi^l(y) \right\rangle_0 = g_0^2 \delta^{kl} \int_p e^{ip(x-y)} \frac{1}{p^2},
\]

(4.18)

\[
\left\langle p^k(t,x)\lambda^l(y) \right\rangle_0 = i\delta^{kl} \vartheta(t-s) K_{t-s}(x-y),
\]

(4.19)

\[
\left\langle p^k(t,x)\xi^l(y) \right\rangle_0 = i\delta^{kl} \vartheta(t+s) K_{t+s}(x-y),
\]

(4.20)

where \( \vartheta(t) \) is a “regularized” step function,

\[
\vartheta(t) = \begin{cases} 
1, & \text{for } t > 0, \\
0, & \text{for } t = 0, \\
0, & \text{for } t < 0.
\end{cases}
\]

(4.21)

Note that \( \vartheta(0) = 0 \) (not, e.g., 1/2). Since other free propagators among \( \pi^k(x), p^k(t,x), \lambda^k(t,x), \) and \( \xi^k(x) \) vanish, Eqs. (4.18)–(4.20) in conjunction with Eq. (4.16) show

\[
\left\langle \pi^k(t,x)\pi^l(s,y) \right\rangle_0 = g_0^2 \delta^{kl} \int_p e^{ip(x-y)} \frac{(1 - \epsilon p^2)^{(t+s)/\epsilon}}{p^2},
\]

(4.22)

\[
\left\langle \pi^k(t,x)\lambda^l(s,y) \right\rangle_0 = i\delta^{kl} \vartheta(t-s) K_{t-s}^\epsilon(x-y),
\]

(4.23)

\[
\left\langle \pi^k(t,x)\xi^l(y) \right\rangle_0 = i\delta^{kl} \vartheta(t+s) K_{t+s}^\epsilon(x-y).
\]

(4.24)

In passing, we note

\[
\left\langle \pi^k(t+\epsilon,x)\lambda^l(t,y) \right\rangle_0 = i\delta^{kl} \delta^D(x-y), \quad \left\langle \pi^k(t,x)\lambda^l(t,y) \right\rangle_0 = 0,
\]

(4.25)

and

\[
\left\langle \pi^k(0,x)\xi^l(y) \right\rangle_0 = i\delta^{kl} \delta^D(x-y).
\]

(4.26)

This completes our derivation of free propagators. In the continuum flow-time limit \( \epsilon \to 0 \), the \( \pi\pi\)-propagator (4.22) reproduces the \( \pi\pi\)-propagator in Eq. (2.15) and the \( \pi\lambda\)-propagator (4.23) reproduces the flow-line propagator \( K_{t-s}(x-y) \) in Eq. (2.11); the step function \( \vartheta(t-s) \) is implicitly implied in Eq. (2.11) through the retarded time-ordering, \( t > s \).

The interaction terms in the present \( (D+1) \)-dimensional system are given by terms in Eq. (4.12) being cubic or higher in fields. The first line of Eq. (4.12) of course reproduces the interaction terms in the action of the \( O(N) \) non-linear sigma model, Eq. (2.7). On the other hand, the term \( i\epsilon \sum_{t=0}^{\infty} \int d^Dx \lambda^k(t,x) R^k(t,x) \) in the limit \( \epsilon \to 0 \), combined with the above \( \pi\lambda\)-propagator, precisely reproduces the last term of the integral equation (2.11) (i.e., the flow vertex).
Thus, we have observed that our present \((D + 1)\)-dimensional system basically reproduces the perturbative expansion of the flowed system defined in Sect. 2; they seem to be basically equivalent. Nevertheless, we should note that the equivalence appears not quite complete. The measure term \((2.16)\) is missing in Eqs. \((4.10)-(4.13)\) (the factor \(\prod_x \sqrt{1 - \pi(x)^2}/\sqrt{1 - \pi(x)^2}\) becomes unity under the integration over \(\xi\) and this is not the measure term). Although the measure term \((2.16)\) identically vanishes when one uses dimensional regularization, it plays an important role in other regularizations, such as lattice regularization. If the equivalence including the measure term does not hold, then the renormalizability proof in the next section, which is based on the present \((D + 1)\)-dimensional field theory, does not apply to the gradient flow with, e.g., lattice regularization. Then, the UV finiteness of the gradient flow with lattice regularization, which we observed through an explicit calculation in Sect. 3, is not explained by the proof.

We will find that, rather surprisingly, the measure term is generated from naively \(O(\epsilon)\) terms in the action \((4.12)\). The aim of the next subsection is to clarify this point and to establish the perturbative equivalence between the above \((D + 1)\)-dimensional system and the flowed system in Sect. 2.

4.4. Equivalence with the perturbative expansion of the gradient flow

We first integrate over the Lagrange multiplier \(\xi^k(x)\) in the partition function \((4.10)\). Then \(\pi^k(0, x)\) is identified with \(\pi^k(x)\) and we have \(\prod_x \sqrt{1 - \pi(x)^2}/\sqrt{1 - \pi(x)^2} = 1\) in Eq. \((4.10)\).

Next we note that the perturbative expansion of Eq. \((4.10)\) generates loop diagrams consisting solely of the flow-line propagator \((4.23)\). Such “flow-line loop diagrams” are depicted in Figs. 10 and 11.\(^8\)

It is now very important to recognize that there is no counterpart to the above flow-line loop diagrams in the perturbative expansion of the flowed system in Sect. 2. This is a consequence of the retarded nature of the flow equation and one can confirm this by drawing flow-line diagrams starting from Eq. \((2.11)\). Thus, there appears some (apparent; see below) discrepancy between the perturbative expansions of the above two systems.

Let us begin our investigation from the flow-line loop diagram in Fig. 10, which starts and ends at the same flow vertex. First note that the \(\pi \lambda\)-propagator \((4.23)\) vanishes when the flow time of \(\lambda\) is

\(^8\)The flow-line loops cannot become higher than one-loop, because the flow vertex is linear in \(\lambda\).
greater or equal to the flow time of \( \pi \). Therefore, in Eq. (4.12), the genuine flow vertex containing the non-linear term \( R^k(t, x) \) does not contribute to the flow-line loop diagram in Fig. 10. What contributes is the self-contraction in the combination \( \mathcal{E} \) (4.13). The self-contraction of \( -\mathcal{E} \) yields

\[
\epsilon \sum_{t=0}^{\infty} \delta^D(0) \int d^Dx \frac{1}{\epsilon} \left\{ \pi^k(t, x) \left[ \pi^k(t, x) - \frac{1 - \pi(t, x)}{1 - \pi(t + \epsilon, x)} \pi^k(t + \epsilon, x) \right] \right\}.
\]

(4.27)

If we Taylor expand \( \pi^k(t + \epsilon, x) \) in this expression with respect to \( \epsilon \), we find

\[
\int_0^\infty dt \frac{1}{2} \delta^D(0) \int d^Dx \partial_t \ln \left[ 1 - \pi(t, x)^2 \right] + O(\epsilon).
\]

(4.28)

Since this is a total derivative, only the boundary field \( \pi^k(t = 0, x) = \pi^k(x) \) is contained. Then, remarkably, Eq. (4.28) coincides with the measure term (2.16) for \( \epsilon \to 0 \).

The above result (4.28) can be obtained in a somewhat different manner. We first Taylor expand \(-\mathcal{E}\), which yields

\[
i \int_0^\infty dt \int d^Dx \left( \frac{1}{2} \lambda^k(t, x) \pi^k(t, x) \left[ \partial_t \pi^l(t, x) \right]^2 + \left[ \frac{\partial^l(t, x) \partial_t \pi^l(t, x)}{1 - \pi(t, x)^2} \right]^2 \right) \epsilon + O(\epsilon^2),
\]

(4.29)

which is \( O(\epsilon) \). This \( O(\epsilon) \) term becomes \( O(1) \) under the self-contraction, because in the \( \epsilon \to 0 \) limit of the \( \pi \lambda \)-propagator (4.23) behaves as

\[
\left\{ \partial_t \pi^k(t, x) \lambda^l(s, y) \right\}_0 = i \delta^{kl} \delta(t - s) \delta^D(x - y) + i \delta^{kl} \theta(t - s) \partial_t K_{t-s}(x - y),
\]

(4.30)

and the delta function at the equal flow-time is interpreted as \( \delta(0) = 1/\epsilon \). The self-contraction in Eq. (4.29) thus cancels the factor \( \epsilon \) and leaves the \( O(1) \) result, Eq. (4.28).

Next, we see that a flow-line loop diagram that contains a plurality of flow vertices, such as the diagram in Fig. 11, vanishes as \( \epsilon \to 0 \). A little thought shows that all vertices in such a flow-line loop diagram must be the vertex arising from \( \mathcal{E} \) (4.13). This is again because the \( \pi \lambda \)-propagator (4.23) vanishes when the flow time of \( \lambda \) is greater than or equal to that of \( \pi \). The vertex is \( O(\epsilon) \) as in Eq. (4.29).

The integration of the flow time of each vertex eliminates one delta function and finally one is left with an overall integration and \( \delta(0) = 1/\epsilon \). In the present case of a plurality of flow vertices, however, the power of \( \epsilon \) coming from the vertices is always greater than or equal to two; thus the flow-line loop diagram vanishes for \( \epsilon \to 0 \). The conclusion is that Eq. (4.28) is the unique contribution of the flow-line loop diagrams for \( \epsilon \to 0 \).

By similar reasoning, it can be confirmed that Eq. (4.28) is the unique place in which an apparent \( O(\epsilon) \) term in the action contributes in the \( \epsilon \to 0 \) limit. The integration of the flow time of each vertex eliminates one delta function and the singularity \( \delta(0) = 1/\epsilon \) can arise only from the flow-line loop diagrams, the case already considered above. This observation justifies the Taylor expansion with respect to \( \epsilon \) and the neglect of the \( O(\epsilon) \) terms besides that particular term in Eq. (4.29).

Thus, we have observed that the perturbative expansions in the above two systems are equivalent by a remarkable mechanism: A flow-line loop diagram in the \((D + 1)\)-dimensional system, which is not generated in the perturbative expansion of the original flow equation, reproduces the measure term (2.16), which is absent in the original partition function of the \((D + 1)\)-dimensional system, Eq. (4.10). The mechanism is remarkable, because an apparent \( O(\epsilon) \) term in the action, i.e., \( \mathcal{E} \) (4.13), plays the crucial role through the flow-line loop.
Now, having established the equivalence between the \((D + 1)\)-dimensional field theory (4.10)–(4.13) and the flowed system in Sect. 2, we are ready to prove the UV finiteness of the gradient flow in Sect. 2.

5. Proof of the renormalizability of the gradient flow

In this section, on the basis of the \((D + 1)\)-dimensional field theory in the preceding section, we show that any correlation function of the flowed \(N\)-vector field in terms of the renormalized coupling is UV finite, without the wave function renormalization.

5.1. Residual non-linear symmetry

We first note that, even with the gauge fixing (4.7), there remains a residual symmetry that is a particular combination of the global \(O(N)\) symmetry (4.4) and the local symmetries (4.5). It is given by the requirement that it does not affect the gauge fixing conditions. That is,

\[
\delta \xi^N(x) = \epsilon^{Nk} \xi^k(x) + g(x) \overline{m}^N(x) = 0, \tag{5.1}
\]

\[
\delta \lambda^N(t, x) = \epsilon^{Nk} \lambda^k(t, x) + h(t, x) n^N(t, x) = 0. \tag{5.2}
\]

From these, we have

\[
g(x) = \frac{\epsilon^{kN} \xi^k(x)}{\overline{m}^N(x)}, \quad h(t, x) = \frac{\epsilon^{kN} \lambda^k(t, x)}{n^N(t, x)}. \tag{5.3}
\]

Under this residual symmetry, other field components transform as

\[
\delta n^i(x) = \epsilon^{ij} n^j(x), \quad \delta \xi^k(x) = \epsilon^{kl} \xi^l(x) + \epsilon^{1N} \xi^1(x) \overline{m}^k(x) \overline{m}^N(x),
\]

\[
\delta n^i(t, x) = \epsilon^{ij} n^j(t, x), \quad \delta \lambda^k(t, x) = \epsilon^{kl} \lambda^l(t, x) + \epsilon^{1N} \lambda^1(t, x) n^k(t, x) n^N(t, x), \tag{5.4}
\]

where indices \(k\) and \(l\) run over only from 1 to \(N - 1\).

The interesting part in the above residual symmetry is the \(O(N)/O(N - 1)\) part corresponding to the choice of parameters \(\epsilon^{kl} = 0\). Writing \(\epsilon^k \equiv \epsilon^{kN}\), it induces the following non-linear transformations:

\[
\delta \pi^k(x) = \epsilon^k \sqrt{1 - \pi(x)^2}, \quad \delta \xi^k(x) = \epsilon^l \xi^l(x) \frac{\overline{\pi}^k(x)}{\sqrt{1 - \pi(x)^2}},
\]

\[
\delta \pi^k(t, x) = \epsilon^k \sqrt{1 - \pi(t, x)^2}, \quad \delta \lambda^k(t, x) = \epsilon^l \lambda^l(t, x) \frac{\pi^k(t, x)}{\sqrt{1 - \pi(t, x)^2}}. \tag{5.5}
\]

It can be directly confirmed that the integration measure and the action in Eqs. (4.10)–(4.13) are invariant under this non-linear transformation; this is expected, because the original partition function with the discrete flow time, Eq. (4.1) with Eq. (4.2), is invariant under Eqs. (4.4) and (4.5).

5.2. Ward–Takahashi relation or the Zinn-Justin equation

We can express the invariance of the system under the non-linear transformation (5.5) as an identity for the generating functional of 1PI correlation functions. First, we introduce the source terms for

\[
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\]

\[\text{Downloaded from https://academic.oup.com/ptep/article-abstract/2015/3/033B08/1587088 by guest on 15 March 2020}\]
elementary fields,

\[ S_J = - \int d^D x \left[ J^k_P(x) \pi^k(x) \right] - \epsilon \sum_{i=0}^{\infty} \int d^D x \left[ J^k_i(t, x) \pi^k(t, x) + J^k_{\lambda}(t, x) \lambda^k(t, x) \right], \quad (5.6) \]

except for the Lagrange multiplier field \( \xi^k(x) \). It turns out that this omission of the \( \xi \)-source greatly simplifies the discussion of the renormalization. This implies that we omit correlation functions including \( \xi^k(x) \) from our consideration. However, since the \( \xi \)-field appears only in the quadratic (i.e., free) part of the action \( S \) only linearly, if 1PI correlation functions of other elementary fields turn out to be UV finite after renormalization, any correlation functions including the elementary \( \xi \)-field are also UV finite. Hence nothing is lost by the omission of the \( \xi \)-source for our present purpose.

To write down the Ward–Takahashi relation associated with the symmetry \((5.5)\), we also supplement additional terms to the action, as

\[ S_{\text{tot}} = S + S_H + S_K, \quad (5.7) \]

where

\[ S_H = - \int d^D x \left[ H(x) \sqrt{1 - \pi(x)^2} \right] - \epsilon \sum_{i=0}^{\infty} \int d^D x \left[ H(t, x) \sqrt{1 - \pi(t, x)^2} \right], \quad (5.8) \]

\[ S_K = - \epsilon \sum_{i=0}^{\infty} \int d^D x \sum_n \sum_{l_1 \ldots l_n} K^{k,l_1 \ldots l_n}(t, x) O^{k,l_1 \ldots l_n}(t, x), \quad (5.9) \]

and

\[ O^{k,l_1 \ldots l_n}(t, x) \equiv \lambda^k(t, x) \frac{\pi^{l_1}(t, x)}{\sqrt{1 - \pi(t, x)^2}} \ldots \frac{\pi^{l_n}(t, x)}{\sqrt{1 - \pi(t, x)^2}}, \quad (5.10) \]

where the source \( K^{k,l_1 \ldots l_n}(t, x) \) is symmetric in indices \((l_1, \ldots, l_n)\) by definition.

We now consider the variation of integration variables of the form of Eq. \((5.5)\) in the partition function:

\[ Z'' = \int \left[ \prod_{k=1}^{N-1} D\xi^k \right] \left[ \prod_{k=1}^{N-1} D\pi^k \right] \times \left[ \prod_{t=0}^{\infty} \prod_{k=1}^{N-1} D\lambda^k(t) \right] \left[ \prod_{t=0}^{\infty} \prod_{k=1}^{N-1} D\pi^k(t) \right] \prod_x \frac{1 - \pi(x)^2}{\sqrt{1 - \pi(x)^2}} e^{-S_{\text{tot}} - S_J}. \quad (5.11) \]

We note

\[ \delta \sqrt{1 - \pi(x)^2} = -\epsilon^m \pi^m(x), \quad \delta \sqrt{1 - \pi(t, x)^2} = -\epsilon^m \pi^m(t, x), \quad (5.12) \]

and

\[ \delta O^{k,l_1 \ldots l_n}(t, x) = \epsilon^m \left[ O^{m,k,l_1 \ldots l_n}(t, x) + nO^{k,m,l_1 \ldots l_n}(t, x) \right] + \sum_{i=1}^{n} \epsilon^l O^{k,l_i \ldots l_n \ldots l_n}(t, x). \quad (5.13) \]

Then, by the standard argument, the invariance of the integration measure and of \( S \) imply that the generating functional of 1PI functions, defined by the Legendre transformation,

\[ \Gamma = -\ln Z'' + \int d^D x \left[ J^k_P(x) \pi^k(x) \right] + \epsilon \sum_{i=0}^{\infty} \int d^D x \left[ J^k_i(t, x) \pi^k(t, x) + J^k_{\lambda}(t, x) \lambda^k(t, x) \right], \quad (5.14) \]
where $\pi^k(x)$, $\pi^k(t, x)$, and $\lambda^k(t, x)$ denote expectation values of elementary fields, follows an identity

\[
\int d^D x \frac{\delta \Gamma}{\delta \pi^m(x)} \frac{\delta H(x)}{\delta (x)} + \int_0^\infty dt \int d^D x \left[ \frac{\delta \Gamma}{\delta \pi^m(t, x)} \frac{\delta H(t, x)}{\delta (t, x)} + \frac{\delta \Gamma}{\delta \lambda^k(t, x)} \frac{\delta H(t, x)}{\delta K^m,k(t, x)} \right] \\
+ \int d^D x H(x) \pi^m(x) + \int_0^\infty dt \int d^D x H(t, x) \pi^m(t, x) \\
+ \int_0^\infty dt \int d^D x K^{k,l}(t, x) \left[ \frac{\delta \Gamma}{\delta K^{m,kl}(t, x)} + \frac{\delta \Gamma}{\delta K^{m,ml}(t, x)} - \delta^{lm} \lambda^k(t, x) \right] \\
+ \int_0^\infty dt \int d^D x \sum_{n=2}^\infty K^{k,l_1...l_n}(t, x) \\
\times \left[ \frac{\delta \Gamma}{\delta K^{m,kl_1...l_n}(t, x)} + \frac{\delta \Gamma}{\delta K^{m,ml_1...l_n}(t, x)} + \sum_{i=1}^n \frac{\delta}{\delta K^{k,l_1...l_{i-1}l_i...l_n}} \frac{\delta \Gamma}{\delta K^{k,l_1...l_{i-1}l_i...l_n}(t, x)} \right] = 0. \quad (5.15)
\]

In writing down this identity, we have taken the continuum flow-time limit $\epsilon \to 0$. This is justified because we have observed that the symmetry (5.5) is preserved by the flow-time discretization. Also, we have observed that the $(D + 1)$-dimensional system Eqs. (4.10)–(4.13) with $\epsilon \to 0$ reproduces the perturbative expansion of the flow equation. Thus we can study the renormalizability of the flowed system in Sect. 2 by using the identity (5.15).

### 5.3. Structure of the renormalization

Our statement of the renormalizability is that the 1PI generating functional $\Gamma$ can be made UV finite in terms of renormalized quantities, by appropriately choosing the constants $Z$ and $Z_3$ in

\[
g_0^2 = \mu^\varepsilon g^2 Z, \quad \pi^k(x) = Z_3^{1/2} \pi^k_R(x), \quad H(x) = Z_3^{-1/2} H_R(x) \quad (5.16)
\]

order by order in perturbation theory. In particular, we claim that the flowed or “bulk” fields, $\pi^k(t, x)$ and $\lambda^k(t, x)$, do not require multiplicative renormalization.

Our argument proceeds by mathematical induction based on the loop expansion. We set

\[
\Gamma = \sum_{\ell=0}^\infty \Gamma^{(\ell)}, \quad (5.17)
\]

where $\Gamma^{(\ell)}$ is the generating functional in the $\ell$th loop order. The above assertion is certainly true for $\ell = 0$ (tree-level approximation) for which $Z = Z_3 = 1$ is sufficient. Then suppose that, in perturbation theory with renormalization quantities fixed, the constants $Z$ and $Z_3$ in Eq. (5.16) can be chosen so that $\Gamma^{(0)}, \ldots, \Gamma^{(\ell)}$, are UV finite in terms of renormalized quantities. Then consider the $(\ell + 1)$th loop order calculation on the basis of the above chosen $Z$ and $Z_3$. Since $Z$ and $Z_3$ have already been chosen so that $\Gamma^{(0)}, \ldots, \Gamma^{(\ell)}$ are finite, by considering the UV-divergent part of the identity (5.15) in the $(\ell + 1)$th loop order, we have

\[
\Gamma^{(0)} \ast \Gamma^{(\ell + 1)\text{div}} = 0, \quad (5.18)
\]
where $\Gamma^{(\ell+1)\text{div}}$ denotes the UV-divergent part of $\Gamma^{(\ell+1)}$ and

$$
\Gamma^{(0)} \equiv \int d^D x \left[ \frac{\delta \Gamma^{(0)}}{\delta \pi_R^m(x)} \frac{\delta}{\delta H_R(x)} + \frac{\delta \Gamma^{(0)}}{\delta H_R(x)} \frac{\delta}{\delta \pi_R^m(x)} \right]
+ \int_0^\infty dt \int d^D x \left[ \frac{\delta \Gamma^{(0)}}{\delta \pi^m(t,x)} \frac{\delta}{\delta H(t,x)} + \frac{\delta \Gamma^{(0)}}{\delta H(t,x)} \frac{\delta}{\delta \pi^m(t,x)} \right]
+ \int_0^\infty dt \int d^D x \left[ \frac{\delta \Gamma^{(0)}}{\delta \pi^m(t,x)} \frac{\delta}{\delta K^{m,k}(t,x)} + \frac{\delta \Gamma^{(0)}}{\delta K^{m,k}(t,x)} \frac{\delta}{\delta \pi^m(t,x)} \right]
+ \int_0^\infty dt \int d^D x \sum_{n=2}^\infty K^{k,l_1\ldots l_n}(t,x)

\times \left[ \frac{\delta}{\delta K^{m,k,l_1\ldots l_n}(t,x)} + n \frac{\delta}{\delta K^{m,l_1\ldots l_n}(t,x)} + \sum_{i=1}^n \frac{\delta}{\delta K^{k,l_1\ldots l_i\ldots l_n}(t,x)} \right],
$$

(5.19)

and

$$
\Gamma^{(0)} \equiv S_{\text{tot}}|_{Z=Z_3=1}.
$$

(5.20)

We next study the most general form of the divergent part $\Gamma^{(\ell+1)\text{div}}$. First of all, by a general theorem, the divergent part must be an integral of a local polynomial of fields and their derivatives. We then note that there is no divergence corresponding to a local term in the “bulk” $t > 0$, a term that is written as an $\int_0^\infty dt \int d^D x$ integral of a local polynomial of fields and their derivatives: As we explained in detail in Sect. 4.4, there is no loop diagram consisting solely of the “flow-line” $\pi \lambda$-propagator (4.23), other than the diagram in Fig. 10, which reduces to the measure term at the boundary $t = 0$, Eq. (2.16). Then, since the $\pi \pi$-propagator (4.22) possesses the Gaussian damping factor $e^{-\epsilon(t+s)}p^2$ (for $\epsilon \to 0$), any loop diagram in which the flow times of the vertices (they must be the same for the divergent part) are positive is UV finite. Therefore, there is no divergence that is written as the bulk integral.

Any divergent part is thus written as the integral on the boundary $t = 0$. Noting that, for $D = 2$, the fields $\pi_R^m(x)$ and $\pi^k(t,x)$ possess the mass dimension 0, $H_R(x)$, $\lambda^k(t,x)$, and $K^{k,l_1\ldots l_n}(t,x)$ possess 2, and $H(t,x)$ possesses 4, the most general possible form of the divergent part is

$$
\Gamma^{(\ell+1)\text{div}} = \int d^D x \left[ B(\pi_R(x), \partial_\mu \pi_R(x)) + H_R(x) C(\pi_R(x)) + \lambda^k(0,x) D^k(\pi_R(x)) + \sum_{n=1}^\infty K^{k,l_1\ldots l_n}(0,x) E^{k,l_1\ldots l_n}(\pi_R(x)) \right],
$$

(5.21)

where $B$ contains at most two derivatives and $E^{k,l_1\ldots l_n}$ is symmetric in indices $(l_1, \ldots, l_n)$. Note that we have not included the flow field at zero flow time, $\pi^k(0,x)$, in the possible form of the divergent part (5.21). The redundancy to use this field variable in addition to $\pi_R^m(x)$ follows from the relation

$$
\pi^k(0,x) = \pi^k(x),
$$

(5.22)

i.e., the expectation value of the variation of the action with respect to the $\xi$-field. Note that here the field variables denote the expectation values in the presence of source fields and not the integration.
variables in the functional integral. This identity shows that, as the arguments of the 1PI generating functional, the variables \( \pi^k(0, x) \) and \( \pi^k(x) \) cannot be independent, because they cannot take different values for any configuration of the source fields.

We note also that the combination

\[
\int d^Dx \frac{\partial}{\partial \pi^m_R(x)} \delta \pi^m_R(x) F^k(\pi_R(x))
\]

(5.23)
does not appear in Eq. (5.21): An external \( \pi^k(t, x) \) line in a 1PI diagram can arise only from a flow vertex that inevitably contains the Lagrange multiplier field \( \lambda^k(t, x) \). Since there is no flow-line loop (other than the diagram in Fig. 10, which reduces to a boundary term), the flow-line propagator starting from \( \lambda^k(t, x) \) can end only at another flow vertex that contains another \( \lambda^k(s, x) \). This shows that any 1PI diagram containing \( \pi^k(t, x) \) must accomplish at least one \( \lambda \). The combination (5.23) does not match this rule.

Now, having obtained the general form of the divergent part, Eq. (5.21), we examine the implication of the identity (5.18) with Eq. (5.19).

First of all, examining the coefficient of \( \partial \pi^k(0, x) \) in Eq. (5.18) that arises from \( \delta \Gamma^{(0)} / \delta \lambda^k(0, x) \) in Eq. (5.19), we have

\[
E^{m,k} = 0.
\]

(5.24)

Then, from various terms in Eq. (5.18), we have

\[
\frac{\partial C}{\partial \pi^m_R(x)} = \frac{\pi^m_R(x)}{1 - \pi_R(x)^2} C,
\]

(5.25)

\[
\int d^Dx \sqrt{1 - \pi_R(x)^2} \frac{\delta}{\delta \pi^m_R(x)} \int d^Dx B
\]

\[
= \int d^Dx \left[ \frac{1}{\mu^e g^2} \left[ -\partial_\mu \partial_\mu \pi^m_R(x) + \frac{\pi^m_R(x)}{\sqrt{1 - \pi_R(x)^2}} \partial_\mu \partial_\mu \sqrt{1 - \pi_R(x)^2} \right] C, \right.
\]

(5.26)

\[
\frac{\partial D^k}{\partial \pi^m_R(x)} + \delta^{mk} \frac{\pi^l_R(x)}{1 - \pi_R(x)^2} D^l = 0,
\]

(5.27)

and

\[
E^{m,kl} + E^{k,ml} \equiv \sqrt{1 - \pi_R(x)^2} \frac{\partial E^{k,l}}{\partial \pi^m_R(x)},
\]

(5.28)

\[
E^{m,kl_1...l_n} + nE^{k,ml_1...l_n} = -\sum_{i=1}^n \delta^{l_i m} E^{k,l_1...l_{i-1}l_{i+1}...l_n} + \sqrt{1 - \pi_R(x)^2} \frac{\partial E^{k,l_1...l_n}}{\partial \pi^m_R(x)}, \quad n \geq 2.
\]

(5.29)

The above conditions for \( B \) and \( C \), Eqs. (5.25) and (5.26), are completely identical to the conditions on the divergent part in the original 2D \( O(N) \) non-linear sigma model [30]. The general solution to these is given by [30]

\[
C = -\frac{1}{2} \delta Z_3 \frac{1}{\sqrt{1 - \pi_R(x)^2}},
\]

(5.30)

\[
B = \delta Z \frac{1}{2 \mu^e g^2} \left\{ \left[ \partial_\mu \pi_R(x) \right]^2 + \left[ \partial_\mu \sqrt{1 - \pi_R(x)^2} \right]^2 \right\}
\]

\[
- \delta Z_3 \frac{1}{2 \mu^e g^2} \left\{ \left[ \partial_\mu \pi_R(x) \right]^2 - \partial_\mu \sqrt{1 - \pi_R(x)^2} \partial_\mu \frac{\pi_R(x)^2}{\sqrt{1 - \pi_R(x)^2}} \right\},
\]

(5.31)

where \( \delta Z \) and \( \delta Z_3 \) are constants.
Next, from the linearly realized $O(N - 1)$ symmetry (that is preserved in our all steps), one can set $D^k = \pi_R^k(x) d(\pi_R(x)^2)$. Then Eq. (5.27) immediately shows that $d = 0$ and

$$D^k = 0. \quad (5.32)$$

Next, from Eqs. (5.24) and (5.28), and the fact that $E^{m,kl}$ is symmetric under the exchange $k \leftrightarrow l$, we have

$$E^{m,kl} = -E^{k,ml} = -E^{k,lm} = +E^{l,km} = +E^{l,mk} = -E^{m,lk} = -E^{m,kl} = 0. \quad (5.33)$$

Finally, we note that, if the right-hand side of Eq. (5.29) vanishes, then

$$E^{m,k\ldots l_n} = -n E^{k,m\ldots l_n} = +n^2 E^{m,k\ldots l_n}, \quad (5.34)$$

and thus

$$E^{m,k\ldots l_n} = 0, \quad n \geq 2. \quad (5.35)$$

This is actually the case by mathematical induction because the right-hand side of Eq. (5.29) vanishes for $n = 2$ from Eqs. (5.24) and (5.33) and then for $n = 3$ again from Eq. (5.33); we see that $E^{k,l\ldots l_n} = 0$ for all $n \geq 1$.

In summary, we observed that the possible divergent part in the present system is given by Eq. (5.21) with Eqs. (5.30) and (5.31) and $D^k = E^{k,l\ldots l_n} = 0$. This is identical to the divergent part in the 2D $O(N)$ non-linear sigma model. One can see that the divergent part (5.21) with Eqs. (5.30) and (5.31) is canceled by the variation of the total action $S_{\text{tot}}$ (5.7) under the change of the renormalization constants in Eq. (5.16) by $(\ell + 1)$th loop order quantities:

$$Z \rightarrow Z + \delta Z, \quad Z_3 \rightarrow Z_3 + \delta Z_3. \quad (5.36)$$

The 1PI generating functional in the $(\ell + 1)$th loop order, $\Gamma^{(\ell+1)}$, is thus made UV finite. This completes the mathematical induction for the renormalizability. In particular, we showed that there is no need of the wave function renormalization for the flowed fields, $\pi^k(t, x)$ and $\lambda^k(t, x)$.

We have shown that any correlation function of the flowed fields is UV finite under the conventional parameter renormalization, without multiplicative wave function renormalization. Then, it is easy to see that, because of Gaussian damping factors in propagators, this UV finiteness holds even when some spacetime coordinates of the correlation function coincide, i.e., even in the equal-point limit, as long as all flow-time coordinates of flowed fields are strictly positive.\(^9\) The local product of any number of flowed fields does not contain UV divergences. This robust UV finiteness, that makes the construction of renormalized composite operators straightforward, is the key property in application of the gradient flow in lattice field theory.

In renormalized perturbation theory, one uses the propagators and the vertices in terms of renormalized parameters and renormalized fields. This renormalized Feynman rule is obtained by making

\(^9\) Again, the absence of the flow-line loop diagram is crucial for this finiteness.
the substitution (5.16) in the action (4.12). The part including the $\xi$-field becomes

$$-i \int d^D x \xi^k(x) \left[ \pi^k(0, x) - \pi^k_R(x) \right] + i \int d^D x \left( Z_3^{1/2} - 1 \right) \xi^k(x) \pi^k_R(x), \tag{5.37}$$

and the second term is regarded as the perturbation. In this renormalized perturbation theory, from the first term, the free propagator is given by

$$\langle \pi^k(t, x) \pi^l(s, y) \rangle_0 = \mu \epsilon g^2 \delta^{kl} \int_p e^{i p(x-y)} e^{-(t+s)p^2/p^2}, \tag{5.38}$$

while the second term is regarded as a counterterm. In this way, we can also use Eq. (5.38) for $\pi^k_R(x)$ by identifying $\pi^k_R(x) = \pi^k(0, x)$. As the $\pi \xi$-propagator (4.24) shows, the second term in Eq. (5.37) acts as a two-point vertex at the boundary $t = 0$ that connects between $\pi^k(t, x)$ and $\pi^l_R(y)$. This counterterm thus plays the same role as the boundary counterterm $\Delta S_{bc}$ in the gauge theory (Sect. 7.1 of Ref. [13]).

Finally, the IR-regulating mass term (2.8) can readily be incorporated in the above argument by the substitution

$$H(x) \to H(x) + \frac{m_0^2}{g_0^2}. \tag{5.39}$$

In particular, from Eq. (5.16), we see that the generating functional becomes UV finite in terms of

$$Z_3^{1/2} \left[ H(x) + \frac{m_0^2}{g_0^2} \right] = H_R(x) + \frac{1}{\mu \epsilon g^2} \frac{Z_3^{1/2} m_0^2}{Z}. \tag{5.40}$$

This shows that the mass parameter is renormalized as $m_0^2 = \left( \frac{Z}{Z_3^{1/2}} \right) m^2$, as we already noted in Eq. (3.6).

### 6. Lattice energy–momentum tensor

In the preceding section, we have shown that any local product (the composite operator) of the bare flowed $N$-vector field becomes UV finite under the conventional parameter renormalization in the 2D $O(N)$ non-linear sigma model. As application of this fact, in the present section, we consider the construction of the energy–momentum tensor, the Noether current associated with the translational invariance, in a lattice formulation of the non-linear sigma model. The idea is the same as that in Refs. [14] and [16]: Since lattice regularization explicitly breaks the translational invariance, the construction of the energy–momentum tensor is awkward. Instead of considering this construction directly, we construct a composite operator of the flowed field, which, under dimensional regularization, becomes the energy–momentum tensor. Since dimensional regularization preserves the translational invariance, the description of the energy–momentum tensor that fulfills the correct Ward–Takahashi relation is straightforward. On the other hand, since the composite operator of the flowed field is UV finite under the parameter renormalization, it must become independent of the regularization in the limit that the regulator is removed (after the renormalization, as long as the same renormalization conditions are adopted). In this way, low-energy correlation functions of the energy–momentum tensor may be computed by using lattice regularization. This construction in Ref. [14] has been applied to the thermodynamics of quenched QCD in Ref. [20] and promising results have been obtained.
6.1. Energy–momentum tensor with dimensional regularization

The energy–momentum tensor $T_{\mu\nu}(x)$ can be obtained from the variation of the action (2.1) under the infinitesimal translation with a localized parameter,

$$\delta n^i(x) = \xi_\mu(x) \partial_\mu n^i(x),$$  \hspace{1cm} (6.1)

as

$$\delta S = - \int d^D x \xi_\nu(x) \partial_\mu T_{\mu\nu}(x).$$  \hspace{1cm} (6.2)

The explicit form is given by

$$T_{\mu\nu}(x) = \frac{1}{g^2} \left[ \partial_\mu n^i(x) \partial_\nu n^i(x) - \frac{1}{2} \delta_{\mu\nu} \partial_\rho n^i(x) \partial_\rho n^i(x) \right].$$  \hspace{1cm} (6.3)

Assuming that we are using dimensional regularization, which preserves the translational invariance, the above classical expression as it stands fulfills the correct Ward–Takahashi relation associated with the translational invariance:

$$\left\langle O_{\text{ext}} \int_\mathcal{D} d^D x \partial_\mu \{ T_{\mu\nu} \} \right\rangle_R(x) O_{\text{int}} = - \left\langle O_{\text{ext}} \partial_\nu O_{\text{int}} \right\rangle. \hspace{1cm} (6.4)$$

In this expression, $\mathcal{D}$ is a bounded integration region, $O_{\text{ext}}$ is an operator outside the region $\mathcal{D}$, and $O_{\text{int}}$ is an operator inside the region. We defined the renormalized energy–momentum tensor by subtracting the vacuum expectation value,

$$\{ T_{\mu\nu} \} \equiv T_{\mu\nu}(x) - \langle T_{\mu\nu}(x) \rangle.$$

The Ward–Takahashi relation ensures that the bare quantity $T_{\mu\nu}(x)$ is not multiplicatively renormalized.

Although naively the energy–momentum tensor (6.3) is traceless for $D = 2$, UV divergences in the composite operator $(1/g^2) \partial_\rho n^i(x) \partial_\rho n^i(x)$ being proportional to $1/\epsilon$ makes this expectation invalid even for $\epsilon \to 0$. Instead, we have the the trace anomaly,

$$\delta_{\mu\nu} \{ T_{\mu\nu} \} \equiv \frac{\beta}{g^3} \left\{ \partial_\rho n^i \partial_\rho n^i \right\}_R(x), \hspace{1cm} (6.5)$$

where the MS scheme is assumed in the renormalized composite operator in the right-hand side and the coefficient is given by the $\beta$ function,

$$\beta \equiv \left( \mu \frac{\partial}{\partial \mu} \right) g = \frac{\epsilon}{2} g - g^3 \sum_{k=0}^{\infty} b_k g^{2k} \hspace{1cm} (6.6)$$

(here, the derivative with respect to the renormalization scale $\mu$ is taken while bare quantities are kept fixed), and [36–38]

$$b_0 = \frac{1}{4\pi} (N - 2), \quad b_1 = \frac{1}{(4\pi)^2} 2(N - 2), \quad b_2 = \frac{1}{(4\pi)^3} (N - 2)(N + 2), \hspace{1cm} (6.7)$$

and

$$b_3 = \frac{1}{(4\pi)^4} (N - 2) \left[ -\frac{2}{3} (N^2 - 22N + 34) + 12(N - 3) \zeta(3) \right]. \hspace{1cm} (6.8)$$

6.2. Small flow-time expansion and the energy–momentum tensor

We construct a composite operator of the flowed field, which reduces to the 2D composite operator (6.3) by using the small flow-time expansion introduced in Ref. [13]. For this, we take an
$O(N)$-invariant dimension-2 second-rank composite operator of the flowed field:
\[
\partial_\mu n^i(t, x) \partial_\nu n^i(t, x) = \partial_\mu \pi^k(t, x) \partial_\nu \pi^k(t, x) + \partial_\mu \sqrt{1 - \pi(t, x)^2} \partial_\nu \sqrt{1 - \pi(t, x)^2}.
\] (6.9)

According to the argument in Ref. [13], for $t \to 0$, this composite operator of the flowed field can be expressed as a series of 2D local operators with increasing mass dimensions, as
\[
\partial_\mu n^i(t, x) \partial_\nu n^i(t, x) = \left[\partial_\mu n^i(t, x) \partial_\nu n^i(t, x)\right] + \zeta_{11}(t) \left[\partial_\mu n^i(x) \partial_\nu n^i(x) - \left(\partial_\mu \partial_\nu n^i(x) \partial_\nu n^i(x)\right)\right]
+ \zeta_{12}(t) \left[\partial_\mu \partial_\nu n^i(x) \partial_\nu n^i(x) - \left(\partial_\mu \partial_\nu n^i(x) \partial_\nu n^i(x)\right)\right] + O(t).
\] (6.10)

Similarly, we have
\[
\partial_\rho n^i(t, x) \partial_\rho n^i(t, x) = \left[\partial_\rho n^i(t, x) \partial_\rho n^i(t, x)\right]
+ \zeta_{22}(t) \left[\partial_\rho n^i(x) \partial_\rho n^i(x) - \left(\partial_\rho \partial_\rho n^i(x) \partial_\rho n^i(x)\right)\right] + O(t).
\] (6.11)

Inverting these relations with respect to the 2D operators and substituting them into Eq. (6.3), we have
\[
\{ T_{\mu\nu} \}_R(x) \equiv T_{\mu\nu}(x) - \{ T_{\mu\nu}(x) \}
= c_1(t) \left[\partial_\mu n^i(t, x) \partial_\nu n^i(t, x) - \frac{1}{2} \delta_{\mu\nu} \partial_\rho n^i(t, x) \partial_\rho n^i(t, x)\right]
+ c_2(t) \left[\frac{1}{2} \delta_{\mu\nu} \partial_\rho n^i(t, x) \partial_\rho n^i(t, x) - \left(\frac{1}{2} \delta_{\mu\nu} \partial_\rho n^i(t, x) \partial_\rho n^i(t, x)\right)\right] + O(t),
\] (6.12)

where
\[
c_1(t) = \frac{1}{g_0^2} \zeta_{11}(t)^{-1},
\] (6.13)
\[
c_2(t) = \frac{1}{g_0^2} \left[\left(-2 \zeta_{11}(t)^{-1}\right) \zeta_{12}(t) - \left(\zeta_{12}(t)^{-1} - 1\right)\right].
\] (6.14)

Hence, if we know the $t \to 0$ behavior of the coefficients $\zeta_{IJ}(t)$ in Eqs. (6.10) and (6.11), then the energy–momentum tensor (6.3) can be obtained by the $t \to 0$ limit of the right-hand side of Eq. (6.12).

Thus, we are interested in the $t \to 0$ behavior of the coefficients $\zeta_{IJ}(t)$. Since all the composite operators in the above expansions are bare ones, by the standard renormalization group argument, the expansion coefficients are independent of the renormalization scale $q$, if they are expressed in terms of the running parameter $\bar{g}(q)$. In particular, we may take $q = 1/\sqrt{8t}$. Then, because of the asymptotic freedom, the running coupling behaves as $\bar{g}(1/\sqrt{8t}) \to 0$ for $t \to 0$ and $\zeta_{IJ}(t)$ for $t \to 0$ can be evaluated by perturbation theory.

To find the coefficients $\zeta_{IJ}(t)$ in Eq. (6.10), we consider correlation functions,
\[
\left\langle \partial_\mu n^i(t, x) \partial_\nu n^i(t, x) \pi^k(y) \pi^l(z) \right\rangle \equiv g_0^4 \delta^{kl} \int_{p, q} \frac{e^{ip(x-y)} e^{iq(x-z)}}{p^2 + m_0^2 q^2 + m_0^2} \mathcal{M}_{\mu\nu}(p, q; t)
\] (6.15)

and
\[
\left\langle \partial_\mu n^i(x) \partial_\nu n^i(x) \pi^k(y) \pi^l(z) \right\rangle \equiv g_0^4 \delta^{kl} \int_{p, q} \frac{e^{ip(x-y)} e^{iq(x-z)}}{p^2 + m_0^2 q^2 + m_0^2} \mathcal{M}_{\mu\nu}(p, q).
\] (6.16)
and compute the coefficients of combinations,

\[ ip_\mu iq_\nu + iq_\mu ip_\nu, \quad 2\delta_{\mu\nu}ip_\rho iq_\rho, \]

(6.17)
in tensors \( M_{\mu\nu} (p, q; t) \) and \( M_{\mu\nu} (p, q) \). Then, we determine \( \zeta_{IJ} (t) \) so that the relation (6.10) holds in correlation functions in view of the combinations (6.17).

Set \( \zeta_{IJ} (t) = \zeta_{IJ}^{(0)} (t) + \zeta_{IJ}^{(1)} (t) + \cdots \), where superscripts denote the number of loops. In the tree level, \( \zeta_{IJ}^{(0)} (t) = \delta_{IJ} \). From this, it follows that the one-loop correction \( \zeta_{11}^{(1)} (t) (\zeta_{12}^{(1)} (t)) \) is given by the difference of one-loop coefficients of the former (the latter) combination in Eq. (6.17) between Eqs. (6.15) and (6.16). In the one-loop level, for Eq. (6.15), we have four diagrams in Figs. 12–15. For Eq. (6.16), we have only two diagrams in Figs. 12 and 15. We use dimensional regularization to regularize the UV divergences and the mass term (2.8) to regularize the IR divergences.

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\[ ^{10} \text{We neglect standard one-particle irreducible diagrams because they give rise to the same contributions to Eq. (6.15) and to Eq. (6.16).} \]
appearing in Eq. (6.12) are renormalized ones. This must be so, because all composite operators are finite in terms of the renormalized parameter. By taking the trace of Eq. (6.3) and comparing it with Eq. (6.5), we find

\[ g_0 \]

and from these

\[ \zeta_{11}(t) = 1 + \frac{g_0^2}{4\pi} (N - 2) \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln(8\pi t) \right] + O\left(g_0^4\right), \]

\[ (6.18) \]

and these

\[ \zeta_{12}(t) = \frac{g_0^2}{4\pi} (-1)(N - 2) \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln(8\pi t) \right] + O\left(g_0^4\right), \]

\[ (6.19) \]

\[ \zeta_{22}(t) = \zeta_{11}(t) + (2 - \epsilon)\zeta_{12}(t) = 1 + \frac{g_0^2}{4\pi} (N - 2) + O\left(g_0^4\right). \]

\[ (6.20) \]

Note that IR divergences are canceled out in the coefficients. Using these results in Eqs. (6.13) and (6.14), we have

\[ c_1(t) = \frac{1}{g^2} - \frac{1}{4\pi} (N - 2) \ln \left( 8\pi \mu^2 \right) + O\left(g^2\right), \]

\[ (6.21) \]

\[ c_2(t) = \frac{1}{4\pi} (N - 2) + O\left(g^2\right) = b_0 + O\left(g^2\right), \]

\[ (6.22) \]

where \( g \) is the renormalized coupling in the MS scheme (3.5). Note that these coefficients are UV finite in terms of the renormalized parameter. This must be so, because all composite operators appearing in Eq. (6.12) are renormalized ones.

For \( c_2(t) \) (6.22), one may proceed one step further [14] by requiring that Eq. (6.12) reproduces the trace anomaly (6.5) to the two-loop order. By taking the trace of Eq. (6.3) and comparing it with Eq. (6.5), we find

\[ \left\{ \partial_\rho n_i^i \partial_\rho n_i^i \right\}_R(x) = \left[ 1 + O\left(g^4\right) \right] \left[ \partial_\rho n_i^i(x)\partial_\rho n_i^i(x) - \left\{ \partial_\rho n_i^i(x)\partial_\rho n_i^i(x) \right\} \right]. \]

\[ (6.23) \]
Using Eq. (6.11) with Eq. (6.20), we find that, for Eq. (6.12) to reproduce the trace anomaly (6.5) to the two-loop order,

\[ c_2(t) = b_0 + \left( b_1 - b_0^2 \right) g^2 + O\left( g^4 \right) \]

\[ = \frac{1}{4\pi} (N - 2) - \frac{1}{(4\pi)^2} (N - 2)(N - 4) g^2 + O\left( g^4 \right). \]  

(6.24)

The expression for the energy–momentum tensor that is usable with lattice regularization is thus given by the \( t \to 0 \) limit of Eq. (6.12) with the coefficients in Eqs. (6.21) and (6.24). As noted above, we can replace the renormalization constant \( g \) and the renormalization scale \( \mu \) in Eqs. (6.21) and (6.24) by the running coupling \( \bar{g}(q) \) with the renormalization scale \( q \) and set \( q = 1/\sqrt{8t} \). We may use, e.g., the four-loop running coupling \[ 39, \]

\[ \bar{g}(q)^2 = \frac{1}{b_0\ell} \left[ 1 - \frac{b_1}{b_0^2} \frac{\ln \ell}{\ell} + \frac{b_1^2 \left( \ln^2 \ell - \ln \ell - 1 \right) + b_0 b_2}{b_0^4 \ell^2} \right. \]

\[ - \left. \frac{b_3^2 \left( \ln^3 \ell - \frac{5}{2} \ln^2 \ell - 2 \ln \ell + \frac{1}{2} \right) + 3b_0 b_1 b_2 \ln \ell - \frac{1}{2} b_0^2 b_3}{b_0^6 \ell^3} \right] \], \quad \ell \equiv \ln \left( \frac{q^2}{\Lambda^2} \right). \]  

(6.25)

in actual numerical simulations.

6.3. A facile computational method for \( \zeta_{IJ}(t) \)

In the above calculation of the matching coefficients \( \zeta_{IJ}(t) \), we have regularized IR divergences by introducing the bare mass \( m_0 \) for the \( N \)-vector field. The required computation is, as a result, somewhat troublesome. In this final subsection, we point out that, at least in the one-loop level, there exists a “facile method” that avoids the introduction of the IR-regularizing mass \[ 40 \]. This method has been particularly useful for gauge theories \[ 14,16 \] because one can regularize IR divergences without introducing a gauge-breaking mass parameter; IR divergences are regularized by “dimensional regularization”.

We first note that, for a Feynman diagram that contributes to Eq. (6.16), e.g., the diagram in Fig. 15, there always exists a corresponding flow Feynman diagram that contributes to Eq. (6.15). The topology of both diagrams is identical (Fig. 15 for the present example) but in the latter the propagators carry the Gaussian damping factor \( e^{-t\ell^2} \), where \( \ell \) is the loop momentum, as in Eq. (2.15). As we have observed above, what is relevant to \( \zeta_{IJ}(t) \) is the difference of the values of these two diagrams, which, by dimensional counting, has the structure

\[ \int \frac{e^{-2t\ell^2}}{\ell^2 + m_0^2} - \int \frac{1}{\ell^2 + m_0^2} = \int \frac{e^{-2t\ell^2} - 1}{\ell^2 + m_0^2}. \]  

(6.26)

In this combination, IR divergences are canceled out and thus we may set \( m_0 \to 0 \) in this combination.\[ 11 \] On the other hand, this integral contains UV divergences and we use the complex dimension

\[ \text{11 A flow Feynman diagram that does not have its counterpart in the 2D field theory, such as the diagrams in Figs. 13 or 14, is IR convergent; dimensional counting shows that the loop integral has the structure } \int_0^1 ds \int_0^{2\pi} e^{-it\ell} \text{ without the denominator.} \]
For $m_0 \to 0$, the result is given by

$$
\int_{\ell} e^{-2t\ell^2} \ell^2 - \frac{1}{(4\pi)^{D/2}} - \frac{1}{D/2 + 1} (2t)^{-D/2+1},
$$

as the analytic continuation from $\text{Re}(D) < 2$. This computation corresponds to the computational method in the preceding subsection.

On the other hand, if we forget to include the contribution corresponding to Eq. (6.16), we will have only the first term of Eq. (6.26):

$$
\int_{\ell} e^{-2t\ell^2} \ell^2 + m_0^2.
$$

This is UV convergent, but contains IR divergences for $m_0 \to 0$. Thus, we set $m_0 \to 0$ and instead use the complex dimension $D$ to regularize IR divergences. The result is given by

$$
\int_{\ell} e^{-2t\ell^2} \ell^2 = \frac{1}{(4\pi)^{D/2}} - \frac{1}{D/2 + 1} (2t)^{-D/2+1},
$$

as the analytic continuation from $\text{Re}(D) > 2$.

Now, interestingly, the right-hand side of Eq. (6.27) and that of Eq. (6.29) are identical as a function of $D$. Thus, we may use the latter method instead of the former. The latter is computationally much simpler because only flow Feynman diagrams have to be computed and the IR regulator $m_0$ is not necessary. We have tabulated the result of this facile method in Table 3. It can be confirmed that each entry coincides with the difference between corresponding entries of Tables 1 and 2, as must be the case; the resulting matching coefficients $\zeta_{IJ}(t)$ obtained directly from Table 3 are, of course, identical to the previous ones, Eqs. (6.18)–(6.20).

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**Note added in proof**

In recent papers [41,42], the solution to the flow equation (2.2) in the $1/N$ expansion is studied. In the former work, the expectation value of Eq. (6.12) at finite temperature is computed for the large-$N$ limit, and it is shown that the expectation value correctly reproduces thermodynamic quantities in the presence of a non-perturbative mass gap.

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### Table 3. The result of the facile method, in units of $g_0^2/(4\pi)$.

| Diagram | $i p_{\mu} q_{\nu} + i q_{\mu} p_{\nu}$ | $2\delta_{\mu\nu} i p_{\rho} i q_{\rho}$ |
|---------|---------------------------------|---------------------------------|
| 07      | $-2 \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln(8\pi t) \right]$ | 0 |
| 08      | $(2N - 2) \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln(8\pi t) \right]$ | 0 |
| 09      | $\frac{3}{4} \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln(8\pi t) \right]$ | $\frac{1}{2} N - \frac{5}{8}$ |
| 10      | $-\frac{3}{4} \left[ (N - 2) \frac{1}{\epsilon} + \frac{1}{2} \ln(8\pi t) \right]$ | $-\frac{1}{2} N + \frac{5}{8}$ |
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