FACTORS OF SOME LACUNARY $q$-BINOMIAL SUMS

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Abstract. In this paper, we prove a divisibility result for the lacunary $q$-binomial sum

$$\sum_{k \equiv r \pmod{c}} (-1)^k q^{(k \choose l)} \left[ \frac{n}{k} \right] _q \left[ \frac{(k - r)/c}{l} \right] _q.$$ 

1. Introduction

Suppose that $p$ is a prime. A classical result of Fleck asserts that

$$\sum_{k \equiv r \pmod{p}} (-1)^k \binom{n}{k} \equiv 0 \pmod{p^{\left\lfloor \frac{n - p - 1}{p-1} \right\rfloor}}, \tag{1.1}$$

where $\left\lfloor x \right\rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$ is the floor function. In 1977, Weisman generalized Fleck’s congruence to prime power moduli in the following way:

$$\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \equiv 0 \pmod{p^{\left\lfloor n/p^\alpha - 1 - \frac{\nu_p(n!)}{p-1} \right\rfloor}). \tag{1.2}$$

In 2009, with help of $\psi$-operator in Fontaine’s theory of $(\phi, \Gamma)$-modules, Sun [6] and Wan [9] obtained a polynomial-type extension of (1.1) and (1.2):

$$\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \left( \frac{(k - r)/p^\alpha}{l} \right) \equiv 0 \pmod{p^{\left\lfloor \frac{n - p - 1 - \frac{\nu_p(n!)}{p-1} - l}{p-1} \right\rfloor}}. \tag{1.3}$$

On the other hand, motivated by the homotopy exponents of the special unitary group SU($n$), Davis and Sun [3, 8] proved another two congruences with a little different flavor:

$$e \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \left( \frac{(k - r)/p^\alpha}{l} \right)^l \equiv 0 \pmod{p^{\nu_p(n/p^\alpha)!}}, \tag{1.4}$$

$$\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \left( \frac{(k - r)/p^\alpha}{l} \right) \equiv 0 \pmod{p^{\left\lfloor \frac{n - p - 1 - l}{p-1} - \nu_p(l)! \right\rfloor}}. \tag{1.5}$$

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where \( \nu_p(x) = \max\{i \in \mathbb{N} : p^i \mid x\} \) is the \( p \)-adic order of \( x \). Notice that neither (1.4) nor (1.5) could be deduced from (1.3), though (1.4) and (1.5) are often weaker than (1.3) provided \( l \) is small.

In this paper, we shall consider the \( q \)-analogues of (1.4) and (1.5). For an integer \( n \), as usual, define the \( q \)-integer

\[
[n]_q = \frac{1 - q^n}{1 - q}.
\]

And define the \( q \)-binomial coefficient

\[
\begin{align*}
\binom{n}{k}_q &= \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q \cdot [k-1]_q \cdots [1]_q}, \\
\binom{n+1}{k}_q &= q^k \binom{n}{k}_q + \binom{n}{k-1}_q.
\end{align*}
\]

In particular, we set \([n]_0 = 1 \) and \([n]_k = 0 \) for \( k < 0 \). It is easy to see \([n]_k \) is a polynomial in \( q \) since

\[
\binom{n+1}{k}_q = q^k \binom{n}{k}_q + \binom{n}{k-1}_q.
\]

Let \( \mathbb{Z}[q] \) denote the polynomial ring in \( q \) with integral coefficients. Then we have the following \( q \)-analogue of (1.3).

**Theorem 1.1.** For \( n, c \in \mathbb{Z}^+ \) and \( r, h \in \mathbb{Z} \), the lacunary \( q \)-binomial sum

\[
\sum_{k \equiv r \pmod{c}} (-1)^k q^{(k)_q + hk} \binom{n}{k}_q \left[\frac{(k - r)/c}{l}\right]_{q^c}
\]

is divisible by

\[
\prod_{d \mid c} \Phi_d(q)^{[n/d] - [l/c/d]} \prod_{b \mid c \atop b < c} \Phi_b(q)^{[n/b] - [r/b] - ([n-r]/b)}
\]

over \( \mathbb{Z}[q] \), where \( \Phi_d \) is the \( d \)-th cyclotomic polynomial.

Since \( \Phi_p^\alpha(q) = [p]_{q^{p^\alpha-1}} \) for prime \( p \), we may get

\[
\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k q^{(k)_q + hk} \binom{n}{k}_q \left[\frac{(k - r)/p^\alpha}{l}\right]_{q^{p^\alpha}} \equiv 0 \mod \prod_{j=\alpha}^{\infty} \Phi_{p^j}(q)^{[n/p^j] - [l/p^{j-\alpha}] - ([n-r]/p^j)}.
\]

(1.6)

Note that

\[
\nu_p(n!) = \sum_{j=2}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor,
\]

(1.7)
and for $1 \leq j \leq \alpha - 1$

$$\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{r}{p^j} \right\rfloor - \left\lfloor \frac{n-r}{p^j} \right\rfloor = \left\lfloor \{r\}_{p^{\alpha-1}} + \{n-r\}_{p^{\alpha-1}} \right\rfloor - \left\lfloor \{r\}_{p^{\alpha-1}} \right\rfloor - \left\lfloor \{n-r\}_{p^{\alpha-1}} \right\rfloor,$$

where $\{r\}_{p^{\alpha-1}}$ denotes the least non-negative residue of $r$ modulo $p^{\alpha-1}$. Substituting $q = 1$ in (1.6), we can get the following stronger version of (1.5) [8, (1.1)]:

$$\nu_p \left( \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \frac{(k-r)/p^\alpha}{l} \right) \geq \nu_p \left( \left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor ! \right) - \nu_p (l!) + \tau_p (\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}),$$

(1.8)

where

$$\tau_p (a, b) = \text{ord}_p \left( \binom{a+b}{a} \right).$$

We shall prove Theorem 1.1 in the next section. For the advantage of $q$-congruences, our proof of Theorem 1.1 is even simpler than the original one of (1.5).

**Remark.** Quite recently, some Fleck type $q$-congruences also have been established by Schultz and Walker [5].

### 2. Proofs of Theorem 1.1

Let $\mathbb{Q}[q]$ denote the polynomial ring in $q$ with rational coefficients. Note that the greatest common divisor of all coefficients of $\Phi_d(q)$ is 1. By a well-known result of Gauss, if $\Phi_d(q)$ divides $F(q) \in \mathbb{Z}[q]$ over $\mathbb{Q}[q]$, then $\Phi_d(q)$ also divides $F(q)$ over $\mathbb{Z}[q]$. So below we don’t distinguish the $q$-congruences over $\mathbb{Z}[q]$ and $\mathbb{Q}[q]$.

**Lemma 2.1.** $(\zeta^r q^h; q)_n$ is divisible by $\Phi_d(q)^{[n/d]}$ for any $r, s \in \mathbb{Z}$, where $\zeta = e^{2\pi \sqrt{-1}/d}$.

**Proof.** We know that

$$\Phi_d(q) = \prod_{\substack{k=1 \\ (k,d)=1}}^d (1 - \zeta^k q).$$

For any $k$ with $(k, d) = 1$, let $0 \leq e_k < d$ be the integer such that $e_k k \equiv r \pmod{d}$. Then we have $1 - \zeta^r \zeta^{-e_k k} = 0$, i.e., $1 - \zeta^k q$ divides $1 - \zeta^r q^j$ if $j \equiv e_k \pmod{d}$. Thus

$$(\zeta^r q^h; q)_n = \prod_{j=h}^{n+h-1} (1 - \zeta^r q^j)$$

is divisible by $(1 - \zeta^k q)^{[n/d]}$. \hfill \Box
Lemma 2.2.
\[
\sum_{k \equiv r \pmod{c}} (-1)^k q^{(\frac{k}{2}) + \frac{hk}{2}} \binom{n}{k}_q \equiv 0 \pmod{\prod_{c|d} \Phi_d(q)^{\lfloor n/d \rfloor}}. \tag{2.1}
\]

Proof. In view of the \(q\)-binomial theorem (cf. [2, Corollary 10.2.2(c)]),
\[
\sum_{k=0}^{n} (-1)^k q^{\frac{k}{2}} \binom{n}{k}_q x^k = (x; q)_n.
\]
So letting \(\zeta = e^{2\pi \sqrt{-1}/c}\),
\[
\sum_{k \equiv r \pmod{c}} (-1)^k q^{\frac{k}{2}} + \frac{hk}{2} \binom{n}{k}_q = \frac{1}{c} \sum_{k=0}^{n} (-1)^k q^{\frac{k}{2}} + \frac{hk}{2} \binom{n}{k}_q \sum_{t=0}^{c-1} \zeta^{-rt} = \frac{1}{c} \sum_{t=0}^{c-1} \zeta^{-rt} (\zeta^h; q)_n.
\]
Thus (2.1) immediately follows from Lemma 2.1, since \(\zeta\) is also a \(d\)-th root of unity if \(c | d\). \(\square\)

Lemma 2.3.
\[
\binom{n}{k}_q = \prod_{1 < d \leq n} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}.
\]

Proof. Clearly
\[
[n]_q! = \prod_{j=1}^{n} [j]_q = \prod_{j=1}^{n} \prod_{d > 1 \text{gcd}(d,j)} \Phi_d(q) = \prod_{1 < d \leq n} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}. \tag{2.2}
\]
Hence
\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \prod_{1 < d \leq n} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}.
\]

Proof of Theorem 1.1. We shall prove
\[
\sum_{k \equiv r \pmod{c}} (-1)^k q^{\frac{k}{2}} \binom{n}{k}_q \frac{(k-r)/c}{l} q^c \equiv 0 \pmod{\prod_{c|d} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor l/c/d \rfloor}}. \tag{2.3}
\]
by using an induction on \(l\). The case \(l = 0\) follows from Lemma 2.2. Assume that \(l \geq 1\) and (2.3) holds for the smaller values of \(l\). Compute

\[
\sum_{k \equiv r \pmod{c}} (-1)^k q^{(k)/2+hk} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \left[ \begin{array}{c} (k-r)/c \\ l \end{array} \right]_{q^c} = \sum_{k \equiv r \pmod{c}} (-1)^k q^{(k)/2+hk} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \cdot q^{-r} \left[ \frac{[k]_q - [r]_q}{[c]_q[l]_q} \right] \left[ \begin{array}{c} (k-r)/c - 1 \\ l-1 \end{array} \right]_{q^c}
\]

\[
= \frac{q^{-r}[n]_q}{[lc]_q} \sum_{k \equiv r \pmod{c}} (-1)^k q^{(k)/2+hk} \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]_q \left[ \begin{array}{c} (k-r)/c \\ l-1 \end{array} \right]_{q^c}
\]

\[
= \frac{q^{-r}[n]_q}{[lc]_q} \sum_{k \equiv r \pmod{c}} (-1)^k q^{(k)/2+hk} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \left[ \begin{array}{c} (k-r)/c \\ l-1 \end{array} \right]_{q^c}
\]

Note that \([lc]_q\) is divisible by or prime to \(\Phi_d(q)\) according to whether \(d \mid lc\) or not. By the induction hypothesis, we obtain that

\[
\sum_{k \equiv r \pmod{c}} (-1)^k q^{(k)/2+hk} \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]_q \left[ \begin{array}{c} (k-r)/c \\ l-1 \end{array} \right]_{q^c} = 0 \pmod{\prod_{c \mid d} \Phi_d(q)^{(n-1)/d+\lfloor l(1-c/d)\rfloor-1/d\lfloor lc\rfloor}},
\]

and

\[
\sum_{k \equiv r \pmod{c}} (-1)^k q^{(k)/2+hk} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \left[ \begin{array}{c} (k-r)/c \\ l-1 \end{array} \right]_{q^c} = 0 \pmod{\prod_{c \mid d} \Phi_d(q)^{(n/d)\lfloor 1/(d/c)\rfloor-1/d\lfloor lc\rfloor}},
\]

where for an assertion \(A\) we adopt the notation \(1_A = 1\) or 0 according to whether \(A\) holds or not. Thus by noting that for arbitrary positive integers \(s\) and \(t\)

\[
1_{l|s} = \left\lfloor \frac{s}{t} \right\rfloor - \left\lfloor \frac{s-1}{t} \right\rfloor,
\]

(2.3) is concluded.

On the other hand, with help of Lemma 2.3

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q \equiv 0 \pmod{\prod_{b \mid c} \Phi_b(q)^{(n/b)\lfloor r/b\rfloor-\lfloor(n-r)/b\rfloor}}
\]

5
whenever $k \equiv r \pmod{c}$, since for any $b \mid c$

$$\left\lfloor \frac{n}{b} \right\rfloor - \left\lfloor \frac{k}{b} \right\rfloor - \left\lfloor \frac{n-k}{b} \right\rfloor = \left\lfloor \frac{n}{b} \right\rfloor - \left\lfloor \frac{r}{b} \right\rfloor - \left\lfloor \frac{n-r}{b} \right\rfloor.$$

All are done. \hfill \square

It is easy to check that

$$[l]_{q^e}[l-1]_{q^e} \cdots [2]_{q^e}[1]_{q^e} = \prod_{1<j \leq l} \Phi_j(q^e)^{[l/j]} \equiv 0 \pmod{\prod_{c \mid d, d > c} \Phi_d(q)^{\lfloor n/d \rfloor}}$$

and

$$[k]_{q^e}[k-1]_{q^e} \cdots [k-l+1]_{q^e} = q^{-(l)}[k]_{q}(\lfloor k \rfloor_{q} - [1]_{q}) \cdot ([k]_{q} - [l-1]_{q}).$$

So applying a simple induction on $l$, we can deduce the $q$-analogue of (1.4):

**Corollary 2.1.**

$$\sum_{k \equiv r \pmod{c}} (-1)^k q^{(k^2/2)} \cdot \frac{n}{k} \cdot q^{n/c} \cdot \left\lfloor \frac{n-r}{c} \right\rfloor_{q^e}$$

is divisible by

$$\Phi_c(q)^{[n/c]-l} \cdot \prod_{c \mid d, d > c} \Phi_d(q)^{[n/d]} \cdot \prod_{b \mid c, b < c} \Phi_b(q)^{[n/b]-[r/b]-\lfloor (n-r)/b \rfloor}.$$  

In particular, for prime $p$,

$$\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k q^{(k^2/2)} \cdot \frac{n}{k} \cdot q^{n/p^\alpha} \cdot \left\lfloor \frac{n-r}{p^\alpha} \right\rfloor_{q^e}$$

is divisible by

$$[p]_{q^e}^{[n/p^\alpha]-l} \prod_{j=0}^{\alpha-1} [p]_{q^e}^{[n/p^{j+1}]} \cdot \prod_{j=1}^{\alpha-1} [p]_{q^e}^{[n/p^{j}] - [r/p^{j}] - \lfloor (n-r)/p^{j} \rfloor}.$$  

Substituting $q = 1$, we obtain the improvement of (1.4) [3, Theorem 5.1]:

$$\nu_p \left( \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \left( \frac{n}{k} \cdot \left\lfloor \frac{n-r}{p^\alpha} \right\rfloor \right) \right) \geq \max \{ \nu_p([n/p^\alpha]!) - l, \nu_p([n/p^\alpha]!) + \tau_p(\{r\}_{p^\alpha-1}, \{n-r\}_{p^\alpha-1}) \}.$$  

(2.4)
3. Lucas type and Wolstenholme-Ljunggren type $q$-congruences

Let

$$T_{p^\alpha,l}(n,r) = \frac{lp^l}{[n/p^\alpha-1]!} \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \binom{(k-r)/p^\alpha}{l}.$$

In [8], Sun and Davis established the following Lucas type congruence:

$$T_{p^\alpha+1,l}(pn+s,pr+t) \equiv (-1)^t \binom{s}{t} T_{p^\alpha,l}(n,r) \pmod{p}, \quad (3.1)$$

where $p$ is a prime, $\alpha \geq 1$, $n,r \geq 0$ and $0 \leq s,t \leq p-1$. Now we may give a $q$-analogue of (3.1). For $b,c \geq 1$ with $b \mid c$, define

$$T_{c,l}(n,r; q) = \frac{[l]_q!\Phi_c(q)^l}{[n/c]_q!\Phi_c(q)^[n/c]} \sum_{k \equiv r \pmod{c}} (-1)^k \binom{k}{c} \binom{(k-r)/c}{l} q^c,$$

where $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$.

**Theorem 3.1.** Let $b \geq 2$ and $n,r,s,t \geq 0$ be integers with $0 \leq s,t \leq b-1$. Suppose that $c$ is a positive multiple of $b$. Then

$$T_{b^\alpha,l}(bn+s,br+t; q) \equiv (-1)^t \binom{s}{t} T_{c,l}(n,r; q^b) \pmod{\Phi_b(q)}. \quad (3.2)$$

**Proof.** By the $q$-Lucas congruence (cf. [4, Proposition 2.2]), we have

$$\binom{bn+s}{br+t}_q \equiv \binom{n}{r}_q \binom{s}{t}_q (\pmod{\Phi_b(q)}). \quad (3.3)$$

Since $q^b \equiv 1 (\pmod{\Phi_b(q)})$, (3.3) can be rewritten as

$$\binom{bn+s}{br+t}_q \equiv \binom{n}{r}_q \binom{s}{t}_q (\pmod{\Phi_b(q)}). \quad (3.4)$$

On the other hand, we have

$$(-1)^{bk+t} q^{(bk+t)/2} \equiv (-1)^{k+t} q^{(k+t)/2} (\pmod{\Phi_b(q)}). \quad (3.5)$$

In fact, since $q^{(bk+t)/2} = q^{bk(2k+2t-1)/2 + (k+t)/2}$, (3.5) easily follows when $b$ is odd. And if $b$ is even, then

$$q^{b/2} = \frac{1-q^b}{1-q^{b/2}} - 1 \equiv -1 (\pmod{\Phi_b(q)}).$$

Thus (3.5) is also valid for even $b$. Since $b \mid c$, it is not difficult to see that $\Phi_{bc}(q) = \Phi_c(q^b)$. Also, $[j]_{q^c} = (1-q^{jc})/(1-q^c)$ is prime to $\Phi_b(q)$ for any $j \geq 1$. 

Hence,

\[
T_{bc}(bn+s, br+t; q) = \frac{[l]_{q^c}!\Phi_{bc}(q)^l}{\left[\left[(bn + s)/c\right] \right]_{q_{bc}/k!} \sum_{bk+t \equiv br+t \pmod{bc}} (-1)^{bk+t} q^{(bk+t)/2} \left(\frac{bn + s}{bk + t}\right)_{q \left(\frac{(bk + t) - (br + t)}{bc}\right)}}_{q^{bc}}
\]

\[
= \frac{[l]_{q^c}!\Phi_{c}(q)^l}{\left[\left[nb/c\right] \right]_{q_{c}/k!} \sum_{k \equiv r \pmod{c}} (-1)^{k+t} q^{b(k+t)/2} \left(\frac{n}{k}\right)_{q^{l}} \left(\frac{k - r}{c}\right)}_{q^c}
\]

\[
= (-1)^{t} q^{(l)/2} \left[\frac{s}{t}\right]_{q} T_{c,l}(n, r; q^b) \pmod{\Phi_b(q)}.
\]

\[
\Box
\]

Furthermore, define

\[
T_{c,l}(n, r; q, z) = \frac{[l]_{q^c}!\Phi_{c}(q)^l}{\left[\left[nb/c\right] \right]_{q_{c}/k!} \sum_{k \equiv r \pmod{c}} (-1)^{k} z^k q^{k/2} \left(\frac{n}{k}\right)_{q^{l}} \left(\frac{k - r}{c}\right)}_{q^c}.
\]

Then we also have

\[
T_{bc}(bn+s, br+t; q, z) \equiv (-1)^{t} z^t q^{(l)/2} \left[\frac{s}{t}\right]_{q} T_{c,l}(n, r; q^b, z^b) \pmod{\Phi_b(q)}.
\] (3.6)

Below we consider the special case that \(s = t = 0\). We need the following \(q\)-analogue of the Wolstenholme-Ljunggren congruence.

**Lemma 3.1.**

\[
\left[\frac{bn}{bm}\right] \left[\frac{n}{m}\right]_{q^b} \equiv \left((-1)^{b-1} q^{\left(\frac{b}{2}\right)}\right)^{(n-m)m} + \frac{(b^2 - 1)nm(n-m)}{24} (1-q^b)^2 \pmod{\Phi_b(q)^3}.
\] (3.7)

**Proof.** By Andrews’ discussions in \([1]\), we have

\[
\frac{(q^{j+b}; q)_{b-1} - (-1)^{j+b-1} q^{\left(\frac{b}{2}\right)} (q; q)_{b-1}}{(1-q^{j+b})(1-q^b)} \equiv \frac{(b^2 - 1)b}{24} \pmod{\Phi_b(q)},
\] (3.8)

though he only proved (3.8) when \(b\) is prime. Noting that \((q; q)_{b-1} \equiv b \pmod{\Phi_b(q)}\) and \(1-q^b \equiv j(1-q^b) \pmod{\Phi_b(q)^2}\), (3.8) can be rewritten as

\[
\frac{(q^{j+b}; q)_{b-1}}{(q; q)_{b-1}} \equiv (-1)^{j+b-1} q^{\left(\frac{b}{2}\right)} + \frac{(b^2 - 1)j(1+1)}{24} (1-q^b)^2 \pmod{\Phi_b(q)^3},
\]
It follows that
\[
\begin{align*}
\left[\frac{[bn]}{[bm]}\right]_q/\left[\frac{m}{n}\right]_{q^b} &= \prod_{j=m-n}^{m-1} \left((q^{j+1}; q)_{b-1}/(q; q)_{b-1}\right) \\
&\equiv (-1)^{(b-1)(n-m)m} q^{\left(\frac{b}{b}\right)(n-m)} \left(1 + \frac{(b^2 - 1)mn(n-m)}{24} (1 - q^b)^2\right) \pmod{\Phi_b(q)^3}.
\end{align*}
\]

In view of \((3.5)\), we get \((3.7)\).

Thus,
\[
\begin{align*}
\left[\frac{[\ell]}{[\ell]}\right]_{q^{\ell}} &\equiv \left[\frac{[\ell]}{[\ell]}\right]_{q^{\ell}} \prod_{k=R}^{m} \left(-1\right)^{b_k} z^{b_k} \left[\frac{[bn]}{[bk]}\right] q^{\left(\frac{b}{b}\right) \left[\frac{m}{n}\right]_{q^b}} \\
&\equiv \left[\frac{[\ell]}{[\ell]}\right]_{(q^{\ell})^{\ell}} \sum_{k=R}^{m} \left(-1\right)^{b_k} z^{b_k} q^{n_k} \left[\frac{n}{k}\right] q^{\left(\frac{k-r}{c}\right) \left[\frac{m}{n}\right]_{q^b}} \\
&\equiv \frac{(b^2 - 1)n}{24} \sum_{k=R}^{m} \left(-1\right)^{b_k} z^{b_k} (1 - q^{b_k}) (1 - q^{n_k}) \left[\frac{n}{k}\right] q^{\left(\frac{k-r}{c}\right) \left[\frac{m}{n}\right]_{q^b}} \pmod{\Phi_b(q)^3}.
\end{align*}
\]

That is,

**Theorem 3.2.** Suppose that \(b, c \ge 2\) and \(b \mid c\). Then for \(n, r \ge 0\),
\[
\begin{align*}
\frac{1}{(1 - q^b)^2} \cdot (T_{bc, l}^{(b)}(bn, br; q, z) - (-1)^{(b-1)n} T_{c, l}^{(b)}(n, r; q^b, z^b q^{n(\frac{b}{b})}))) \\
&\equiv - \frac{1}{[\ell]_{q^{\ell}}!} \cdot \frac{(b^2 - 1)n^2(n - 1)}{24} \cdot z^b T_{c, l}^{(b)}(n - 2, r - 1; q^b, z^b) \pmod{\Phi_b(q)}.
\end{align*}
\]

**In particular,**
\[
\begin{align*}
T_{b, l}^{(b)}(bn, br; q, z) - (-1)^{(b-1)n} T_{b, l}^{(b)}(n, r; q^b, z^b q^{n(\frac{b}{b})}) \\
&\equiv - \frac{(b^2 - 1)n}{24} \cdot z^b (1 - q^b)^2 T_{b, l}^{(b)}(n - 2, r - 1; q^b, z^b) \pmod{\Phi_b(q)^3}.
\end{align*}
\]

**4. From q-congruences to integer congruences**

In \([3]\), Sun and Davis conjectured that
\[
\frac{p^l}{[n/p]!} \sum_{k=R}^{m} \left(-1\right)^{k} \left(\frac{pn}{pk}\right) \left(\frac{k-r}{p}\right)^l \equiv \frac{p^l}{[n/p]!} \sum_{k=R}^{m} \left(-1\right)^{k} \left(\frac{n}{k}\right) \left(\frac{k-r}{p}\right)^l \pmod{p^3} \tag{4.1}
\]


for prime $p \geq 5$. This conjecture was confirmed by Sun in [7], with help of some arithmetical properties of the Stirling numbers of the second kind.

Define

$$S_{c,l}^{(b)}(n, r; q, z) = \frac{\Phi_b(q)^l}{[(nb/c)]_{q^p}} \sum_{k \equiv r \pmod{c}} (-1)^k z^k q^{(k)} \left[\begin{array}{c} n \\ k \end{array}\right]_q [(k - r)/c]_{q^p}^l.$$

Using the similar discussions as above, we also can obtain that

\[
S_{bc,l}^{(b)}(bn + s, br + t; q, z) \equiv (-1)^t z^t q^{(t)} \left[\begin{array}{c} s \\ t \end{array}\right] q^{(t)} S_{c,l}^{(b)}(n, r; q, z) \pmod{\Phi_b(q)},
\]

and

\[
\frac{1}{1 - q^p} \cdot \left( S_{bc,l}^{(b)}(bn, br; q, z) - (-1)^{(b-1)n} S_{c,l}^{(b)}(n, r; q, z, q^n) \right) \equiv - \frac{(n - 2)b/c!}{[nb/c]!} \cdot \frac{(b^2 - 1)n^2(n - 1)}{24} \cdot z^b S_{c,l}^{(b)}(n - 2, r - 1; q^b, z^b) \pmod{\Phi_b(q)}.
\]

In particular, for prime $p \geq 5$,

\[
\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k q^{(k)} \left[\begin{array}{c} n \\ k \end{array}\right]_q [(k - r)/p^\alpha]_{q^p}^l \equiv \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k q^{nk(p)^\alpha} \left[\begin{array}{c} n \\ k \end{array}\right]_q [(k - r)/p^\alpha]_{q^p}^l + \frac{(p^2 - 1)n[p]_{q^p}^{p^\alpha}}{24[n/p^\alpha]_{q^p}^{p^\alpha}} \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k (1 - q^{kp})(1 - q^{(n-k)p}) \left[\begin{array}{c} n \\ k \end{array}\right]_q [(k - r)/p^\alpha]_{q^p}^l \pmod{[p]_q^3}.
\]

However, one might doubt whether (4.4) surely implies (4.1), since neither side of (4.4) is a polynomial in $q$. So we need to give an explanation how to deduce (4.1) from (4.4) by substituting $q = 1$.

Let $L(q)$ and $R(q)$ denote the left side and the right side of (4.4) respectively. Let

\[
F(q) = \frac{[n/p^\alpha]_{q^p}^{p^\alpha}!}{\prod_{j \geq \alpha + 1}[p]_{q^p}^{[n/p^\alpha]}!}.
\]

Then clearly $F(q) \in \mathbb{Z}[q]$ in view of (2.2). And from Corollary 2.1 we also know that $F(q) L(q), F(q) R(q) \in \mathbb{Z}[q]$. Hence there exists a polynomial $H(q) \in \mathbb{Z}[q]$ such that

\[
F(q) L(q) - F(q) R(q) = [p]_q^3 H(q).
\]
Substituting \( q = 1 \), we get
\[
F(1)L(1) \equiv F(1)R(1) \pmod{p^3}.
\]
But by (1.7),
\[
F(1) = \frac{\lfloor n/p^\alpha \rfloor!}{p^{\sum_{j \geq \alpha+1} \lfloor n/p^j \rfloor}}
\]
is not divisible by \( p \). Thus (4.1) is concluded.

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