Polynomial Recursion Equations in Form Factors of ADE-Toda Field Theories

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Abstract

It is shown that the problem of calculating form factors in ADE affine Toda field theories can be reduced to the nonperturbative recursive calculation of polynomials symmetric in each sort of variables. We determine these recursion equations explicitly for the ADE series and characterize the polynomial solutions by an interplay between the weight space of the underlying Lie algebra and representations of the symmetric group.
1.

The knowledge of form factors, i.e. matrix elements of a local operator in a quantum field theory, gives a deep insight into the quantum structure of a given Lagrangian field theory. Once one has calculated all form factors it is possible to classify the operator content of the quantum theory or to calculate correlation functions.

In the class two-dimensional field theories with factorizable scattering it is known \[1\] that form factors do satisfy a system of four axioms which makes the problem of calculating them somehow feasible.

Form factors have been studied for several diagonal scattering theories in recent years, see \[2, 3, 4, 5, 6, 7, 8\] and references therein. In this paper we shall be interested in affine Toda field theories which is a class of two-dimensional models with factorizable diagonal scattering for which the classical S-matrices are known \[9, 10\].

Form factors for the \(A_1^{(1)}\) and for the \(A_2^{(2)}\) Toda field theories have been studied in \[1, 4\] and \[3\] respectively. The theories are simple in the sense that only one type of particle is present and that the S-matrix contains only poles of first order. In fact only these first order poles are covered by the axioms \[1\] mentioned above. Recently some form factors have been calculated for \(A_n^{(1)}\) theories in \[11\] and \(D_{2n}^{(1)}\) in \[12\].

In this paper we show that the axioms of \[1\] can be applied with slight modifications to situations with many different kinds of particles and S-matrices with higher order poles. We prove for the case of ADE Toda field theories that, once the minimal form factors are known, the only thing to be done is to solve a polynomial recursion equation. These equations are exact and do not rely on any perturbative treatment! Even though it might be difficult to find the solutions explicitly we succeed in characterizing them by means of the representation theory of the symmetric group.

2.

Let us define the Lagrangian of affine Toda field theory in the following way (cf. \[13, 14\]). Let \(g\) be a rank \(r\) Lie algebra of type A, D, or E \[15\]. The set of simple roots of \(g\) is given by \(\{\alpha_i\} \) with \(i \in I = \{1, 2, \ldots, r\}\) and we set \(\alpha_0\) to be the negative of the highest root. Let \(I = \hat{I} \cup \{0\}\).

The fundamental weights \(\lambda_i\) are defined using the standard euclidean form \((,\,\,)\) on \(g\) by the condition \((\lambda_i, \alpha_j) = \delta_{ij}\). Notice in the definition that we restrict our discussion to ADE Lie algebras and therefore we do not have to care about long and short roots.

The Cartan matrix of \(g\) is given by \(C_{ij} = (\alpha_i, \alpha_j)\). Because we will make use of it below we mention that the inverse of \(C\) is given by \(C^{-1}_{ij} = (\lambda_i, \lambda_j)\).
Define integers $k_i$ with $i \in \hat{I}$ by the condition $k_0 = 1$ and $\sum_{i \in \hat{I}} k_i \alpha_i = 0$. It is of course known that the Coxeter number is given by $h = \sum_{i \in \hat{I}} k_i$. We then set $X^\pm = \sum_{i \in \hat{I}} \sqrt{k_i} X_i^\pm$, where the $X_i^\pm$ are the generators of $\mathbf{g}$ corresponding to the simple root $\alpha_i$. $X^+$ and $X^-$ commute and are elements of the Cartan subalgebra of $\mathbf{g}$.

Let $\Phi$ be a scalar field taking values in the Cartan subalgebra of $\mathbf{g}$. The Lagrangian of the affine Toda field theories is the given by the following expression \[14, 13\] which is defined in a two-dimensional Minkowski space.

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi, \partial^\mu \Phi) - \frac{m^2}{\beta} \left( e^{ad(\Phi)} X^+ , X^- \right),$$  

where $ad$ denotes the adjoint action. The parameters $m$ and $\beta$, which we take to be real throughout this paper, are the mass scale and the coupling respectively. It turns out that for the purposes of our study the effective coupling \[16\]

$$B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1 + \beta^2/4\pi}$$

will be of fundamental importance.

3.

As implicitly stated above, the fields or particles in an affine Toda field theory are in one-to-one correspondence to the nodes in the Dynkin diagram of the Lie algebra $\mathbf{g}$. More precisely \[10\], a particle of species $a$ in a given theory might be identified with the fundamental weight $\lambda_a$, for $a \in I$.

According to the symmetries of the Dynkin diagram, particle $a$ has a charge conjugate partner $\bar{a}$ which might be identical with $a$. We remark that the theories coming from $\mathbf{g} \in \{D_{2n}, E_7, E_8\}$ contain only self-conjugate particles. For details see \[9\].

Using this correspondence it can be shown that fusing, i.e. non-vanishing three point couplings, of particles $a$, $b$, and $c$ occurs if there exist integers $r_1$, $r_2$, and $r_3$ such that the following equation holds.

$$w^{r_1} \lambda_a + w^{r_2} \lambda_b + w^{r_3} \lambda_c = 0. \tag{3}$$

$w$ denotes the Coxeter element of the Weyl group of $\mathbf{g}$. This equation is known as Dorey’s rule \[10\], see also \[13\].

If we denote the rapidities of a particle by $\theta$ the fusing $a + b \rightarrow \bar{c}$ occurs at $i\theta_{ab}^c$. We will refer to $\theta_{ab}$ as well as to $w_{ab} = \theta_{ab} h/\pi$ and $\bar{u} = h - u$ as fusing angles for the particular
fusing. Explicit formulae for these angles can be found in [9, 10]. It can be shown that there exists a correspondence between these angles and the integers occurring in (3).

The classical S-matrices for ADE affine Toda field theories have been constructed as a solution of the bootstrap equation

$$S_{dc}(\theta) = S_{da}(\theta - i\bar{\theta}^0_{bc})S_{db}(\theta + i\theta^0_{bc}).$$

(4)

It was shown in [10] that these S-matrices can be nicely written in the following way.

$$S_{ab}(\theta) = \frac{h}{\prod_{p=0}^{h-1} (\langle 2p + 1 + \varepsilon_{ab}\rangle_{+}^{(\lambda_a,w^{-p}\phi_b)})}\xi_{ab}(\theta).$$

(5)

We have been using the following notation. For $i \in I$ we define $\phi_i = (1 - w^{-1})\lambda_i$. (These are just linear combinations of fundamental root vectors with the property that $w\phi_i$ is a negative root [15]). The functions $\langle r \rangle_{+}(\theta)$ are defined by

$$\langle r \rangle_{+}(\theta) = \frac{(r - 1)_{+}(\theta)(r + 1)_{+}(\theta)}{(r - 1 + B)_{+}(\theta)(r - 1 + B)_{+}(\theta)}; \quad (r)_{+}(\theta) = \sinh((\theta + \frac{i\pi r}{h})/2).$$

(6)

Using the standard [11, 13] two-coloring of the nodes of the Dynkin diagram we set $c(\lambda_a) = 1$ if the node is of color white and $c(\lambda_a) = -1$ if the color is black. With this convention we set $\varepsilon_{ab} = (c(\lambda_a) - c(\lambda_b))/2$.

For later purposes the following form of the S-matrix, completely equivalent to (5), is useful.

$$S_{ab}(\theta) = \frac{\xi_{ab}(-\theta)}{\xi_{ab}(\theta)}, \quad \xi_{ab}(\theta) = \prod_{p=0}^{h-1} (\langle p \rangle_{+}(\theta))^{m_{ab}(p)}.$$  

(7)

Comparing this form with (3) it is clear that the integers $m_{ab} \geq 0$ can be easily related to the exponent in (3). We list a few properties of this quantity.

$$m_{ab}(1) = \delta_{ab}, \quad m_{ab}(-p) = -m_{ab}(p), \quad m_{ab}(h-p) = m_{ab}(p), \quad m_{ad}(p) = m_{ad}(p - \bar{u}^b_{ac}) + m_{bd}(p + \bar{u}^a_{bc}).$$

(8)

The last equation is of course a descendant of the bootstrap equation (4).

4.

The object of study in this paper are the form factors in an ADE Toda field theory. This is a matrix element of a local operator $O(x)$. Without loss of generality we can take the
local operator at the origin of our space-time and allow the matrix element to be taken for several incoming particles and the vacuum.

\[ F_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_n) := \langle 0 | \mathcal{O}(0) | \theta_1, \ldots, \theta_n \rangle. \]  

(9)

In our conventions the indices \( a_i \) denote the species of a particle of rapidity \( \theta_i \).

It is known that the form factor in the class of theories we are considering is subject to four axioms [1]. The first two of them are known as Watson’s equations. One of them reflects the fact that particles \( a_i \) and \( a_{i+1} \) can be interchanged by the S-matrix \( S_{a_i a_{i+1}}(\theta_i - \theta_{i+1}) \) while the other one is just a monodromy property of \( F \).

The solutions of Watson’s equations can be constructed by the following procedure. Since \( F \) is a meromorphic function in several variables we can split it locally into a part, which we will denote by \( K \), containing poles and zeros in a given region, times another part which is analytic without zeroes there. We perform this split in the strip \( 0 < \text{Im} \theta < 2\pi \).

\[ F_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_n) = K_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_n) \prod_{i<j} F_{a_i a_j}^{\min}(\theta_i - \theta_j), \]  

(10)

\( F_{a_i a_j}^{\min}(\theta_i - \theta_j) \) is known as the minimal form factor. It can be shown that once one assumes some obvious monodromy properties for \( K \) (mainly periodicity), this object drops out of Watson’s equations and \( F_{a_i a_j}^{\min}(\theta_i - \theta_j) \) then satisfies a simple two-particle equation. The minimal solutions in the ADE-case are known and can be given either in an integral form [17] or in a product expansion of \( \Gamma \)-functions [11].

Using these solutions one can show that the two remaining axioms lead to equations for \( K \) only. One of these equations arises from the fact that we have particles and their conjugate partners in our theory, and reflects a zero angle scattering of a given particle with all the other fields approaching the local operator, for details see the appendix of [1].

We then obtain the following kinematical residue equation.

\[ -i \text{ res}_{\theta' = \theta + i\pi} K_{d_{a_1} \ldots d_n}(\theta', \theta, \theta_1, \ldots, \theta_n) = \]  

\[ = K_{d_1 \ldots d_n}(\theta_1, \ldots, \theta_n) \left( \prod_{j=1}^n \xi_{adj}(\theta - \theta_j) - \prod_{j=1}^n \xi_{adj}(\theta_j - \theta) \right) / F_{a_1 a_2}^{\min}(i\pi). \]  

(11)

The last condition to be imposed is a consequence of the presence of intermediate bound states, i.e. fusions. In [1] this condition was established only for fusions arising from first order poles of the S-matrix. We will show below that with some minor modifications this last axiom remains valid in the presence of any odd order forward channel pole of the S-matrix. For a related statement see [8]. The bound state residue equation for \( K \) is then:
Based on the results in \[11, 12\] we will now propose a general ansatz for the \( K \)-part of the form factor. We denote by \( N_k \), \( k \in I \), the number of particles of type \( k \) in the expression \( \\[14\] \). Rather than to work with the rapidities \( \theta_i \) itself it is possible and useful to take \( x_i = e^{\theta_i} \) as basic variables. All these \( x_i \)'s of a given species will be grouped together in the vector
\[
x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_{N_k}^{(k)}) , \quad k \in I .
\] (15)
The singularities leading to the kinematical residue are parametrized by expressions of the form \( (x_i + x_j)^{-1} \) while the ones corresponding to fusings are contained in functions of the form \( \\[11\] \)
\[
W_{kl}(x_1^{(k)}, x_j^{(l)}) = \prod_{p=0}^{h-2} (x_1^{(k)} - \Omega^{p+1} x_j^{(l)}) \Omega^{-p-1} x_j^{(l)} - \Omega^{p+1} x_1^{(k)} , \quad \Omega = e^{i\pi/h} .
\] (16)
Notice that the product only goes up to \( h - 2 \) to ensure consistency of this ansatz \( \\[11\] \). With this notation the parametrization of \( K \) is the following,
\[
K_{[N_1, \ldots, N_r]}(x^{(1)}, \ldots, x^{(r)}) = Q_{[N_1, \ldots, N_r]}(x^{(1)}, \ldots, x^{(r)}) \prod_{k \in J} \prod_{i, j} \frac{1}{W_{kl}(x_i^{(k)}, x_j^{(l)})} \times 
\prod_{k=1}^{r} \prod_{i < j} \frac{1}{W_{kl}(x_i^{(k)}, x_j^{(k)})} \times \prod_{k=1}^{r-1} \prod_{l=k+1}^{r} \prod_{i=1}^{N_k} \prod_{j=1}^{N_l} \frac{1}{W_{kl}(x_i^{(k)}, x_j^{(l)})} .
\] (17)
For the kinematical pole part we have been introducing the index set \( J \) which contains the indices of all self-conjugate particles, and the indices of the non-self-conjugate ones, such that only the particles and not their conjugates are being counted.

The ansatz (17) is chosen in such a way that it contains in an explicit way all informations about the poles necessary for the residue equations (11) and (12). In other words we have reduced the problem of calculating the form factor to the problem of calculating the \( Q's \) in (17).

We are going to prove that \( Q_{N_1,\ldots,N_r}(x^{(1)}, \ldots, x^{(r)}) \) is a polynomial, which is symmetric in at least all the components of the vectors \( x^{(k)} \).

Let \( A_{ab} \) be the set of integers \( p \) in the expression for the S-matrix (7) which have \( m_{ab}(p) > 0 \), such that \( p \) occurs \( m_{ab}(p) \) times in \( A_{ab} \). We denote the number of elements in \( A_{ab} \) by \( \#A_{ab} \). Since this number will be frequently used in the sequel we will show the following.

**Proposition 1:** Let \( C_{ki}^{-1} \) as in section 2 denote an entry in the inverse of the Cartan matrix of the Lie algebra \( g \). We then have:

\[
\#A_{ab} = C_{ab}^{-1} + C_{\bar{a}b}^{-1}.
\]  

**Proof:** Considering the exponent in (3) we can see that \( \sum_{p=0}^{h-1} (\lambda_a, w^p \phi_b) = \sum_{p=0}^{h-1} (\lambda_a, (w^p - w^{-p-1}) \lambda_b) = 0 \) because of the cyclicity of the Coxeter element. Having a closer look at (4) and (5) one can see that for any \( p \in \{1, 2, \ldots, h-1\} \) we have with an entry \( \langle p \rangle \) always an entry of the form \( \langle 2h-p \rangle^{-1} = \langle -p \rangle^{-1} \). These two facts show that in order to compute \( \#A_{ab} \) we only have to consider the “positive” \( \langle p \rangle \)-entries, hence to evaluate the sum

\[
\#A_{ab} = \sum_{p=0}^{s} (\lambda_a, (w^p - w^{-p-1}) \lambda_b) = (\lambda_a, \lambda_b) - (\lambda_a, w^{-s-1} \lambda_b).
\]  

Due to (3) \( s \) is given by \( s = h/2 - 1 \) for ADE Lie algebras apart from \( A_{2n} \) while for \( A_{2n} \) which has an odd Coxeter number we have \( s = h/2 - 1 - \varepsilon_{ab} \).

In the Weyl group of a Lie algebra there is a longest element \( w_0 \) with the property \( w_0 \lambda_a = -\lambda_a \), for \( a \in I \). This special element can be related to the Coxeter element in a standard way \([13]\). We are using this result in a weak sense which leads in all cases to

\[
(\lambda_a, w^{-s-1} \lambda_b) = (\lambda_a, w_0 \lambda_b) = -(\lambda_a, \lambda_b).
\]  

Using this relation in (19) together with the fact mentioned in section 2 that the inner product of two fundamental weights is just an entry in the inverse of the Cartan matrix establishes the Proposition.
We are now in a position to show the main result of this section, which is, of course, strongly supported by the results of [7, 8, 11, 12].

**Theorem 1:** The object \( Q_{[N_1,\ldots,N_r]}(x^{(1)},\ldots,x^{(r)}) \) appearing in the ansatz (17) is a polynomial for all ADE affine Toda field theories.

**Proof:** In order to verify the Theorem we have to show that by inserting the ansatz for \( K \) in (17) into the kinematical (11) and the bound state (12) residue equations we get equations for the \( Q \)'s which are polynomial.

The first observation to make is that due to the structure of the polynomial in the denominator of \( K \) (17) it is possible to separate there the variables involved in the limiting process for the residues in (11) and (12) from those being only spectators in the process. This means that by this procedure we split the denominator polynomial in all \( N + 2 \) variables into a polynomial in \( N \) variables not containing the variables involved in taking the residue and another polynomial in \( N + 2 \) variables.

The actual proof of our result basically consists of a tedious calculation which makes several times use of the identities (8) and the explicit form of the S-matrix (5) and (7).

In order to write down the result of the calculation as transparent as possible we again have to introduce some notation. The residues in (11) and (12) are taken in the vicinity of the rapidity \( \theta \) which in our exponential language of (15) corresponds to the variable \( x = e^\theta \). Evaluating the residue for \( W_{kl} \) in (16) suggests to introduce the symbol

\[
[p]_i^{(k)} = x - \Omega^p x_i^{(k)}, \quad k \in I, \quad i \in \{1,\ldots,N_k\}.
\]

From this symbol we can construct the following compound in a form in which it will appear in the equations for the \( Q \)'s.

\[
\{(p)\}_i^{(k)} = \left[\left(h - p - 1\right)_{i}^{(k)}\left(h - p + 1\right)_{i}^{(k)}\right]^{m_{ab}(h-p)} \left([-p-1+B]_{i}^{(k)}\left[-p+1-B\right]_{i}^{(k)}\right)^{m_{ab}(p)}. \tag{22}
\]

\( B \) is the effective coupling of the Toda field theory as defined in (2). From the kinematical equation (11) we derive:

\[
Q_{a[a_1,\ldots,a_r]}(-x, x; x^{(1)},\ldots,x^{(r)}) = -i(-1)^{N_a} x^{2(C_{a}^{-1}+C_{a}^{-1})-1} \prod_{i=1}^{N_a} \frac{\prod_{k=1}^{h-2}(2\cos\left(\frac{\pi}{2}+\frac{k-1}{2}\right))^2}{\mathcal{D}_{a}^{(ir)}} \times
\]

\[
\prod_{i=1}^{N_a} \left[\prod_{i=1}^{N_a} \frac{1}{[0]^{(a)}} \times \prod_{i=1}^{N_a} \frac{1}{[h]^{(a)}} \times \left( \prod_{k=1}^{r} \prod_{i=1}^{N_k} \left[\left(p\right)_{i}^{(k)}\right]^{\mu_{ab}(p)} - \prod_{k=1}^{r} \prod_{i=1}^{N_k} \left[\left(-p\right)_{i}^{(k)}\right]^{\mu_{ab}(p)} \right) \right) \right] \times
\]

\[
Q_{[N_1,\ldots,N_r]}(x^{(1)},\ldots,x^{(r)}).
\]

(23)
This is a polynomial recursion equation. The factors $1/[0]^{(a)}$ and $1/[h]^{(a)}$ are cancelled by corresponding expressions in the bracket which contains the difference.

It remains to state the equation for $Q$ arising from the bound state residue equation (12).

\[
Q_{ab[N_1,...,N_r]}(x\Omega^{\tilde{u}_{ac}}_{bc}, x^{(1)}, \ldots x^{(r)}) = i x^{2(C_{ab}^{-1}+C_{ab}^{-1})-\delta_{ab}} H_{abc} \frac{\Gamma_{ab}^c}{F_{ab}^\min (i\theta_{ab}^c)} \prod_{i=1}^{N_a} \frac{1}{[u_{ab}]^{(a)}(\Omega_{ab})^i} \times
\]
\[
\prod_{i=1}^{N_b} \frac{1}{[-u_{bc}]^{(b)}(\Omega_{bc})^i} \times \prod_{k=1}^{r} \prod_{i=1}^{N_c} \left( \prod_{p=1}^{\tilde{u}_{ab}} (\{\Omega_{ac} - \tilde{u}_{ac}\}_i) \mu_{ab}(p) \times \prod_{p=1}^{\tilde{u}_{bc}} (\{\Omega_{bc} - \tilde{u}_{bc}\}_i) \mu_{bc}(p) \right)
\]
\[
Q_{c[N_1,...,N_r]}(x; x^{(1)}, \ldots x^{(r)})
\]

(24)

To be complete we give the explicit form of the complicated factor:

\[
H_{abc} = \Omega^{\tilde{u}_{ac}}_{bc}(N_a+2N+m_{ab}(u_{ab}^{-1})) \Omega^{-\tilde{u}_{bc}}_{bc}(N_b+2N) \left( \Omega^{\tilde{u}_{ac}}_{ac} + \Omega^{-\tilde{u}_{bc}}_{bc} \right) \delta_{ab} \times
\]
\[
\prod_{p=1}^{h-2} \left( \Omega^{\tilde{u}_{ac}}_{ac} - \Omega^{-\tilde{u}_{bc}+p-1} \right) m_{ab}(p) \times \prod_{p=1}^{h-2} \left( \Omega^{\tilde{u}_{bc}}_{bc} - \Omega^{-\tilde{u}_{bc}+p-1} \right) m_{bc}(p)
\]

(25)

where $N = \sum_{i=1}^{r} N_i$.

We have thus found that even the bound state recursion equation (12) leads to a polynomial equation for $Q$. The negative powers in (24) are actually cancelled, as can be easily seen.

As a result we have established the theorem.

**Remark:** It is known that the S-matrices of ADE Toda field theories might have poles which correspond to forward channel fusings [9, 10] in order greater than one. Initially we have been using the axiom of [1] for our bound states, which in [1] had been established for first order poles only. However, it turns out that that this axiom can be applied to higher order poles as well provided one replaces the term “res” in (12) by simply taking the coefficient in the leading order singularity at the position of the pole.

It is moreover remarkable that our result in the case of higher order poles is nonperturbative. It does not at all make use of the separate Feynman diagrams which contribute to a singularity of the S-matrix [4]. This fact reflects the bootstrap property of the S-matrix on the level of form factors [12], see also [3].

Since our calculation by (8) and (13) basically made use only of the fact that the S-matrix has the bootstrap property [4], we conjecture that in other diagonal scattering theories with bootstrap property it should also be possible to reduce the problem of calculating form factors to one of evaluating recursive equations for polynomials.
6.

We are now going to determine the structure of the polynomials \( Q_{[N_1, \ldots, N_r]}(x^{(1)}, \ldots, x^{(r)}) \). The first property is its total degree. This quantity is independent of the equations (23) and (24) and can be calculated by the following consideration.

A Lorentz transformation shifts the rapidities simply by a parameter. We can therefore determine the effect of a Lorentz transformation on the denominator in (17). The full form factor as defined in (9) is required to be Lorentz invariant. Hence, the object \( Q_{[N_1, \ldots, N_r]}(x^{(1)}, \ldots, x^{(r)}) \) must have the same behaviour under a Lorentz transformation as the denominator of \( K \). A direct calculation using Proposition 1 and (8) leads to:

**Proposition 2:** Let \( M_{sc} \) and \( M_{nsc} \) the set of all self-conjugate and of all non-self-conjugate particles respectively in an ADE Toda field theory. The total degree of the polynomial \( Q_{[N_1, \ldots, N_r]}(x^{(1)}, \ldots, x^{(r)}) \) is given by

\[
\text{deg}Q_{[N_1, \ldots, N_r]}(x^{(1)}, \ldots, x^{(r)}) = \sum_{k=1}^{r} N_k(N_k - 1)(C_{kk}^{-1} + C_{kk}^{-1} - \delta_{kk}) + 2 \sum_{k=1}^{r-1} \sum_{l=k+1}^{r} N_kN_l(C_{kl}^{-1} + C_{kl}^{-1} - \delta_{kl}) + \frac{1}{2} \sum_{k \in M_{sc}} N_k(N_k - 1) + \frac{1}{2} \sum_{k \in M_{nsc}} N_kN_k.
\]

Since the coefficients in the recursion equations (23) and (24) are invariant under the transformation \( B - 1 \rightarrow 1 - B \) we find by induction in the numbers of particles:

**Proposition 3:** The polynomial \( Q_{[N_1, \ldots, N_r]}(x^{(1)}, \ldots, x^{(r)}) \) is invariant under the weak-strong coupling duality \( B - 1 \rightarrow 1 - B \).

Let us introduce the elementary symmetric polynomials \( e_i^n \) in \( n \) variables \( x_1, x_2, \ldots, x_n \) by (28)

\[
\prod_{i=1}^{n}(1 + tx_i) = \sum_{l=0}^{n} e_i^n t^l.
\]

Comparing this expression with the bracket symbol \( [p] \) introduced in (21) we recognize that the recursion coefficients in (23) and (24) entirely consist of elementary symmetric polynomials.

Given a partition \( \pi = (\pi_1, \pi_2, \ldots, \pi_p) \), with \( \pi_1 \geq \pi_2 \geq \ldots \geq \pi_p \), we can define an elementary symmetric polynomial corresponding to this partition by

\[
E^n_\pi = e^n_{\pi_1} e^n_{\pi_2} \cdots e^n_{\pi_p}.
\]
Characteristic quantities associated to a partition are its length $l(\pi)$, which is the number of nonzero entries of $\pi$, and its weight $|\pi| = \sum_{i=1}^{p} \pi_i$.

Since we are dealing with polynomials in $r$ different types of variables (15) we introduce the following polynomial which is symmetric only in each species of variables.

$$E_{\Pi}(x^{(1)}, \ldots, x^{(r)}) = E_{(\pi^{(1)}|\pi^{(2)}|\ldots|\pi^{(r)})}(x^{(1)}, \ldots, x^{(r)}) = \prod_{k=1}^{r} E_{\pi^{(k)}}(x^{(k)}). \quad (29)$$

Having in mind that the recursion coefficients in (23) and (24) consist just of polynomials of this kind, we find by induction in the number of particles:

**Proposition 4:**

$$Q_{[N_1, \ldots, N_r]}(x^{(1)}, \ldots, x^{(r)}) = \sum_{|\Pi| = \text{deg} Q} c_{\Pi} E_{\Pi}(x^{(1)}, \ldots, x^{(r)}), \quad (30)$$

where $c_{\Pi}$ are constants to be determined by the recursive equations (23) and (24).

We are now turning to the question which partitions out of the many possible ones in (30) do actually contribute with a nonvanishing $c_{\Pi}$.

The quantity we would like to compute is the partial degree of the $Q_{[N_1, \ldots, N_r]}(x^{(1)}, \ldots, x^{(r)})$.

This is the highest power to which any $x_i^{(k)}$ with $i \in \{1, 2, \ldots, N_k\}$ appears in the polynomial in question. Using Proposition 4 and (28), (29) it is clear that this partial degree of the variables of species $k$ is nothing else than the longest partition $\pi^{(k)}$ in $\Pi$. This means that for a given $N$-particle polynomial $Q_{[N_1, \ldots, N_r]}(x^{(1)}, \ldots, x^{(r)})$ we have to find the upper bounds for $l(\pi^{(k)})$ for all $k \in I$.

This upper bound can be calculated by investigating how the lengths of the partitions evolve when we add one variable to the polynomial $Q$, i.e. if we consider the recursion equation (24) which links an $N + 2$-particle polynomial to an $N + 1$-particle polynomial.

In order to do this we first determine the number $\nu_{abcd}^{c}$ of $\langle p \rangle^{-1}$ entries in $\lambda_{abcd}^{c}(\theta)$, introduced in (13). We take (14) and count the number of $\langle p \rangle^{-1}$ entries on both sides of the equation. Using Proposition 1 we find

$$\nu_{ca,d}^{b} + \nu_{abcd}^{c} = (\lambda_a, \lambda_d) + (\lambda_a, \lambda_d). \quad (31)$$

Considering all possible permutations of $a$, $b$, and $c$ and the exchange of $d$ and $\bar{d}$ in this equation we find that
\[ \nu_{ab,cd} = (\lambda_a + \lambda_b - \lambda_c, \lambda_d). \] (32)

The main entity of the recursion coefficient in (24) is the bracket symbol \( \{p\}_i^{(k)} \) introduced in (22). The structure of this symbol is quite close to the one of \( \lambda_{ab,cd}(\theta) \). Taking the product \( \prod_{i=1}^{N_i} \{p\}_i^{(k)} \) we see using (21) and (27) that it can be expanded into elementary symmetric polynomials. The maximum length of the partitions in the \( k \) component in the recursion coefficient in (24) can the be determined to be

\[ 2 (\lambda_a + \lambda_b - \lambda_c + \lambda_{\bar{a}} + \lambda_{\bar{b}} - \lambda_{\bar{c}}, \lambda_k) + \delta_{ck} - \delta_{bk} - \delta_{ak}. \] (33)

This means that from one step in the recursion procedure, \( l(\pi^{(k)}) \) in the \( N + 2 \)-particle polynomial can be at most the maximal length of the partitions in the \( N + 1 \)-particle polynomial plus the expression in (33).

Suppose we are given a particle vector \( \mathbf{N} = [N_1, \ldots, N_r] \) which by a series of successive fusings, with \( \mathbf{N}_i \) being the intermediate vectors, terminates in a vector \( \mathbf{N}' \). Hence we have \( \mathbf{N}' < \mathbf{N}_1 < \mathbf{N}_2 < \ldots < \mathbf{N}_n < \mathbf{N} \), where the ordering is given by the absolute number of particles in a specific vector.

We know that a particle in a Toda field theory corresponds to a fundamental weight vector. Moreover, we have seen in (32) that a fusing \( a + b \to \bar{c} \) corresponds to a weight vector (conveniently to be taken in Dynkin components) \( \lambda_a + \lambda_b - \lambda_{\bar{c}} \).

Using this interpretation it is clear that the particle vectors introduced above are nothing else than vectors in the positive weight lattice of the Lie algebra \( \mathfrak{g} \). Any difference \( \mathbf{N}_i - \mathbf{N}_j \) is in the positive weight lattice as long as \( i > j \).

Let us denote \( \Lambda = \mathbf{N} - \mathbf{N}' \). Rewriting the \( \delta \)'s in (33) by expressions of the form \( (\lambda_{\bar{c}}, \alpha_k) \) we get the following:

**Lemma 1:** Suppose we know, in the above notation, that the polynomial \( Q_{\mathbf{N}} \) can be obtained by a polynomial \( Q_{\mathbf{N}'} \) by a process of successive fusings. Then the maximum length of the partitions belonging to the variables of type \( k \) for any \( k \in I \) are related by

\[
l(\pi^{(k)})|_{\mathbf{N}} = l(\pi^{(k)})|_{\mathbf{N}'} + \sum_i \left( 2(\lambda_{a_i} + \lambda_{b_i} - \lambda_{c_i}, \lambda_k) + 2(\lambda_{\bar{a}_i} + \lambda_{\bar{b}_i} - \lambda_{\bar{c}_i}, \lambda_k) - (\lambda_{\bar{a}_i} + \lambda_{\bar{b}_i} - \lambda_{\bar{c}_i}) \right)
\]

\[
= l(\pi^{(k)})|_{\mathbf{N}'} + 2(\Lambda + \bar{\Lambda}, \lambda_k) - (\bar{\Lambda}, \alpha_k).
\] (34)

This result is interesting because it gives a link between the representations of the symmetric group acting on the polynomials \( Q_{[N_1, \ldots, N_r]} \) and the weight space of the Lie algebra.
g. Namely, the quantity $l(\pi^{(k)})$ restricts the representations of the symmetric group which do appear in the expression (30).

We would like to rewrite the result (34) in more direct terms using the correspondence $N = \sum_{i=1}^{r} \lambda_i N_i$. By formula (26) we know that the degree of $Q$-polynomial is zero if exactly one particle is present. Combining (34) and (26) keeping in mind that we exclude to talk about a polynomial without any variables in our context we get:

**Corollary:** The maximum length of the partitions belonging to particles of type $k$ for any $k \in I$ in the polynomial $Q[N_1, \ldots, N_r]$ is given by:

$$l(\pi^{(k)})|_{[N_1, N_2, \ldots, N_r]} = 2 \sum_{i=1}^{r} N_i(C_{ik}^{-1} + C_{ik}^{-1}) - N_k \delta_{kk} - 2(C_{kk}^{-1} + C_{kk}^{-1} - \frac{1}{2} \delta_{kk}).$$  \hspace{1cm} (35)

We would like to close this paper with a conjecture on the minimal length of the partitions contributing nonvanishingly to (30). We rewrite (17) as $K[N_1, \ldots, N_r] = Q[N_1, \ldots, N_r]/D[N_1, \ldots, N_r]$. By construction we have as in (30)

$$D[N_1, \ldots, N_r](x^{(1)}, \ldots, x^{(r)}) = \sum_{|\Xi|=\deg Q[N_1, \ldots, N_r]} d_{\Xi} E_{\Xi}(x^{(1)}, \ldots, x^{(r)}),$$  \hspace{1cm} (36)

where all constants $d_{\Xi}$ are of course fixed by the explicit form of $D[N_1, \ldots, N_r]$ in (17).

By construction it is clear that the polynomial $D[N_1, \ldots, N_r]$ is just the kernel of the recursive equations (23) and (24) and has therefore to be added to any particular solution to these equations in order to have the most general solution for any $Q[N_1, \ldots, N_r]$. This, of course, introduces a free parameter at each stage of the recursion process.

Let $l_{\min}(\xi^{(k)})$ be the minimum length of of the partitions $\xi^{(k)}$, belonging to variables of species $k$, for any $k \in I$, contributing nonvanishingly to the sum (36).

Based on calculating solutions to the equations (23) and (24) we would like to state the following:

**Conjecture:** The partitions $\pi^{(k)}$ contributing nonvanishingly to the a general solution $Q[N_1, \ldots, N_r]$ in the form of (30) are at least equally long as $l_{\min}(\xi^{(k)})$, for any $k \in I$. We would like to put it even stronger. The partitions $\Pi$ in (30) do contribute nonvanishingly to a general solution $Q[N_1, \ldots, N_r]$ if and only if the length of its subpartitions $\pi^{(k)}$ is between this lower bound and the upper bound (35) for $k \in I$.

It is possible to compute $l_{\min}(\xi^{(k)})$ explicitly:

$$l_{\min}(\xi^{(k)}) \geq \frac{1}{2} (N_k - 1) + \delta_{kk}(C_{kk}^{-1} + C_{kk}^{-1} - 1)(N_k - 1),$$  \hspace{1cm} (37)
where \(1/2(N_k - 1)\] means that if this expression is not an integer we have to take the next integer.

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