On a Unified Theory of Generalized Branes Coupled to Gauge Fields, Including the Gravitational and Kalb-Ramond Fields

M. Pavšič
Jožef Stefan Institute, Jamova 39, SI-1000 Ljubljana, Slovenia; E-mail: matej.pavsic@ijs.si

ABSTRACT

We investigate a theory in which fundamental objects are branes described in terms of higher grade coordinates $X^{\mu_1...\mu_n}$ encoding both the motion of a brane as a whole, and its volume evolution. We thus formulate a dynamics which generalizes the dynamics of the usual branes. Geometrically, coordinates $X^{\mu_1...\mu_n}$ and associated coordinate frame fields $\{\gamma_{\mu_1...\mu_n}\}$ extend the notion of geometry from spacetime to that of an enlarged space, called Clifford space or $C$-space. If we start from 4-dimensional spacetime, then the dimension of $C$-space is 16. The fact that $C$-space has more than four dimensions suggests that it could serve as a realization of Kaluza-Klein idea. The “extra dimensions” are not just the ordinary extra dimensions, they are related to the volume degrees of freedom, therefore they are physical, and need not be compactified. Gauge fields are due to the metric of Clifford space. It turns out that amongst the latter gauge fields there also exist higher grade, antisymmetric fields of the Kalb-Ramond type, and their non-Abelian generalization. All those fields are naturally coupled to the generalized branes, whose dynamics is given by a generalized Howe-Tucker action in curved $C$-space.
1 Introduction

Point particle is an idealization never found in nature. Physical objects are extended and possess in principle infinitely many degrees of freedom. It is now widely accepted that even at the “fundamental” level objects are extended. Relativistic strings and higher dimensional extended objects, branes, have attracted much attention during last three decades [1, 2, 3, 4].

An extended object, such as a brane, during its motion sweeps a worldsheet, whose points form an $n$-dimensional manifold $V_n$ embedded in a target space(time) $V_N$. Worldsheet is usually considered as being formed by a set of points, that is, with a worldsheet we associate a manifold of points, $V_n$. Alternatively, we can consider a worldsheet as being formed by a set of closed $(n-1)$-branes (that we shall call “loops”). For instance, a string world sheet $V_2$ can be considered as being formed by a set of 1-loops. In particular such a 1-loop can be just a closed string which in the course of its evolution sweeps a worldsheet $V_2$ which, in this case, has the form of a world tube. But in general, this need not be the case. A set of 1-loops on $V_2$ need not coincide with a family of strings for various values a time-like parameter. Thus even a worldsheet swept by an open string can be considered as a set of closed loops. The ideas that we pursue here are motivated and based to certain extent by those developed in refs. [6]–[8]. We shall employ the very powerful geometric language based on Clifford algebra [9, 10], which has turned out to be very suitable for an elegant formulation of $p$-brane theory and its generalization [11]–[16].

We thus describe branes by means of higher grade coordinates $x^{\mu_1...\mu_R}$, $R = 0, 1, 2,..., N$, corresponding to an oriented $R$-area associated with a brane, where $N$ is the dimension of the spacetime $V_N$ we started from. The latter coordinates are collective coordinates, [13, 14], analogous to the center of mass coordinates [11]. They do not provide a full description of an extended object, they merely sample it. Nevertheless, if

\footnote{In the literature, the name ‘worldsheet’ is often reserved for a 2-dimensional surface swept by 1-dimensional string. Here we use ‘worldsheet’ for a surface of any dimension $n$ swept by an $(n-1)$-dimensional brane. By symbols $V_n$ and $V_N$ we denote manifolds (we adopt an old practice), and not vectors spaces.}
higher grade coordinates $x^{\mu_1 \cdots \mu_R}$ are given, then certainly we have more information about an extended object than in the case when only its center of mass coordinates are given. By higher grade coordinates we no longer approximate an extended object with a point like object; we take into account its extra structure.

We associate all those higher grade coordinates with points of a $2^N$-dimensional space, called Clifford space, shortly $C$-space, denoted $C_{V_N}$. Every point of $C_{V_N}$ represents a possible extended event, associated with a generalized brane.

In order to consider an object’s dynamics, one has to introduce a continuous parameter, say $\tau$, and consider a mapping $\tau \rightarrow x^{\mu_1 \cdots \mu_R} = X^{\mu_1 \cdots \mu_R}(\tau)$. So functions $X^{\mu_1 \cdots \mu_R}(\tau)$ describe a curve in an $2^N$-dimensional space $C_{V_N}$. This generalizes the concept of worldline $X^{\mu}(\tau)$ in spacetime $V_N$. The action principle is given by the minimal length action in $C_{V_N}$. That the objects, sampled by $X^{\mu_1 \cdots \mu_R}$ satisfy such dynamics is our postulate \[11, 12, 16, 15\], we do not derive it.

The intersection of a $C$-space worldline $X^{\mu_1 \cdots \mu_R}(\tau)$ with an underlying spacetime $V_N$ (which is a subspace of $C_{V_N}$) gives, in general an extended event \[2\]. Therefore, what we observe in spacetime are “instantonic” extended objects that are localized both in space-like and time-like directions \[3\]. According to this generalized dynamics, worldlines are infinitely extended in $C_{V_N}$, but in general, their intersections with subspace $V_N$ are finite. In spacetime $V_N$ we observe finite objects whose time like extension may increase with evolution, and so after a while they mimic the worldlines of the usual relativity theory. This has been investigated in refs. \[11, 15\]. We have also found that such $C$-space theory includes the Stueckelberg theory \[17\]–\[24\] as a particular case, and also has implications for the resolution of the long standing problem of time in quantum gravity \[25, 12, 26\].

Objects described by coordinates $x^{\mu_1 \cdots \mu_R}$ are points in Clifford space $C_{V_N}$, also called extended events. Objects given by functions $X^{\mu_1 \cdots \mu_R}(\tau)$ are worldlines in $C_{V_N}$. A further possibility is to consider, e.g., continuous sets of extended events, described by functions $X^{\mu_1 \cdots \mu_R}(\xi^A)$, $A = 1, 2, ..., 2^n$, $n < N$, where $\xi^A \equiv \xi^{a_1 \cdots a_r}$, $r = 0, 1, 2, ..., n < N$, are $2^n$ higher grade coordinates denoting oriented $r$-areas in the parameter space $\mathbb{R}^n$. Functions $X^{\mu_1 \cdots \mu_R}(\xi^A)$ describe a $2^n$-dimensional surface in $C_{V_N}$. This generalizes the concept of worldsheet or world manifold $X^{\mu}(\xi^a)$, $a = 1, 2, ..., n$, i.e., the surface that an evolving brane sweeps in the embedding spacetime $V_N$.

A $C$-space worldline $X^{\mu_1 \cdots \mu_R}(\tau)$ does not provide a “full” description of an extended object, because not “all” degrees of freedom are taken into account; $X^{\mu_1 \cdots \mu_R}(\tau)$ only provides certain “collective” degrees of freedom that sample an extended object. On the contrary, a $C$-space worldsheet $X^{\mu_1 \cdots \mu_R}(\xi^A)$ provides much more detailed description, be-

\[2\] Analogously, in spacetime the intersection of an ordinary worldline with a 3-dimensional slice gives a point.

\[3\] They are the analog of $p = -1$ branes (instantons) that are important in string theories.
cause of the presence of $2^n$ continuous parameters $\xi^A$, on which the generalized coordinate functions $X^{\mu_1...\mu_R}(\xi^A)$ depend. In particular, the latter functions can be such that they describe just an ordinary worldsheet, swept by an ordinary brane. But in general, they describe more complicated extended objects, with an extra structure.

We equip our manifold $C_{V_N}$ with metric, connection and curvature. In the case of vanishing curvature, we can proceed as follows. We choose in $C_{V_N}$ an origin $E_0$ with coordinates $X^{\mu_1...\mu_R}(E_0) = 0$. This enables us to describe points $E$ of $C_{V_N}$ with vectors pointing from $E_0$ to $E$. Since those vectors are Clifford numbers, we call them polyvectors. So points of our flat space $C_{V_N}$ (i.e., with vanishing curvature) are described by polyvectors $x^{\mu_1...\mu_R} \gamma^{\mu_1...\mu_R}$, where $\gamma^{\mu_1...\mu_R}$ are basis Clifford numbers, that span a Clifford algebra $C_N$.

So our extended objects, the events $E$ in $C_{V_N}$, are described by Clifford numbers. This actually brings spinors into the description, since, as is well known, the elements of left (right) ideals of a Clifford algebra represent spinors [27]. So one does not need to postulate spinorial variables separately, as is usually done in string and brane theories. Our model is an alternative to the theory of spinning branes and supersymmetric branes, including spinning strings and superstrings [1]. In refs. [16] it was shown that the 16-dimensional Clifford space provides a framework for a consistent string theory. One does not need to postulate extra dimensions of spacetime. One can start from 4-dimensional spacetime, and finds that the corresponding Clifford space provides enough degrees of freedom for a string, so that the Virasoro algebra has no central charges. According to this theory all 16 dimensions of Clifford space are physical and thus observable [11] [28] [29], because they are related to the extended nature of objects. Therefore, there is no need for compactification of the extra dimensions of Clifford space.

As a next step it was proposed [28] [29] that curved 16-dimensional Clifford space can provide a realization of Kaluza-Klein theory. Gravitational as well as other gauge interactions can be unified within such a framework. In ref. [28] [29] we considered Yang-Mills gauge field potentials as components of the $C$-space connection, and Yang-Mills gauge field strengths as components of the curvature of that connection. It was also shown [29] that in a curved $C$-space which admits $K$ isometries, Yang-Mills gauge potentials occur not only in the connection, but also in the metric, or equivalently, in the vielbein. In this paper we concentrate on the latter property, and further investigate it by studying the brane action in curved background $C$-space. So we obtain the minimal coupling terms in the classical generalized brane action, and we show that the latter coupling terms contain the ordinary 4-dimensional gravitational fields, Yang-Mills gauge fields $A_\mu^\alpha$, $\alpha = 1, 2, ..., K$, and also the higher grade, in general non-Abelian, gauge fields $A^\alpha_{\mu_1...\mu_R}$ of the Kalb-Ramond type. We thus formulate an elegant, unified theory for the classical generalized branes coupled to all those various fields.
2 On the description of extended objects

2.1 Worldsheet described by a set of point events

As an example of a relativistic extended object, let us first consider the string. An evolving string sweeps a worldsheet, a physical object in the embedding spacetime. Worldsheet can be considered as being formed by a set of point events. So with a worldsheet we can associate a 2-dimensional manifold $V^2$, called world manifold, embedded in an $N$-dimensional target space $V_N$. With every point on $V_2$ we associate two parameters (coordinates) $\xi^a$, $a = 0, 1$ which are arbitrary (like “house numbers”). The embedding of $V_2$ into $V_N$ is described by the mapping

$$\xi^a \to x^\mu = X^\mu(\xi^a), \quad \xi^a \in R^n \subset R^n$$

where $x^\mu$, $\mu = 0, 1, 2, \ldots, N-1$ are coordinates describing position in $V_N$, whilst $X^\mu(\xi^a)$ are embedding functions (Fig.1), defined over a domain $R^n$ within a set $R^n$ of real numbers.

![Figure 1: A worldsheet can be considered as being formed by a set of point events associated with points of a world manifold $V_2$ embedded in $V_N$, described by the mapping $\xi^a \to x^\mu = X^\mu(\xi^a)$.](image)

With our worldsheet we thus associate a 2-parameter set of points described by functions $X^\mu(\xi^0, \xi^1)$. We distinguish here the physical object, the worldsheet, from the corresponding mathematical object, the manifold (world manifold) $V_2$. Strictly, we should use two different symbols for those different objects. In practice, we will not be so rigorous, and we will simply denote worldsheet by the symbol $V_2$ (in general $V_N$).
2.2 Worldsheet described by a set of loops

In previous section a worldsheet was described by a 2-parameter set of points described by functions $X^\mu(\xi^0, \xi^1)$. Alternatively, instead of points we can consider closed lines, loops, each being described by functions $X^\mu(s)$, where $s \in [s_1, s_2] \subset \mathbb{R}$ is a parameter along a loop. A 1-parameter family of such loops $X^\mu(s, \alpha)$, $\alpha \in [\alpha_1, \alpha_2] \subset \mathbb{R}$ sweeps a worldsheet $V_2$. This holds regardless of whether such worldsheet is open or closed. However, in the case of an open worldsheet, the loops are just kinematically possible objects, and they cannot be associated with physical closed strings. In the case of a closed worldsheet, a world tube, we can consider it as being swept by an evolving closed string.

We will now demonstrate, how with every loop one can associate an oriented area, whose projections onto the coordinate planes are $X^{\mu \nu}$. The latter quantities are functionals of a loop $X^\mu(s)$. If we consider not a single loop, but a family of loops $X^\mu(s, \alpha)$, $\alpha \in [\alpha_1, \alpha_2]$, then $X^{\mu \nu}$ are functions of parameter $\alpha$, besides being functionals of a loop. So we obtain a 1-parameter family of oriented areas described by functions $X^{\mu \nu}(\alpha)$. Let us stress again that for every fixed $\alpha$, it holds, of course, that $X^{\mu \nu}$ are functionals of $X^\mu(s, \alpha)$.

If we choose a loop $B$ on $V_2$, i.e., a loop from a given family $\{X^\mu(s, \alpha), \alpha \in [\alpha_1, \alpha_2]\}$, then we obtain the corresponding components $X^{\mu \nu}$ of the oriented area by performing the integration of infinitesimal oriented area elements over a chosen surface whose boundary is our loop $B$. Given a boundary loop $B$, it does not matter which surface we choose. In the following, for simplicity, we will choose just our worldsheet $V_2$ for the surface.

Let us now consider a surface element on $V_2$. Let $d\xi_1 = d\xi_1^a e_a$ and $d\xi_2 = d\xi_2^a e_a$ be two infinitesimal vectors on $V_2$, expanded in terms of basis vectors $e_a$, $a = 0, 1$. An infinitesimal oriented area is given by the wedge product

$$d\Sigma = d\xi_1 \wedge d\xi_2 = d\xi_1^a d\xi_2^b e_a \wedge e_b = \frac{1}{2} d\xi^{ab} e_a \wedge e_b$$

where

$$d\xi^{ab} = d\xi_1^a d\xi_2^b - d\xi_2^a d\xi_1^b$$

At every point $\xi \in V_2$ basis vectors $e_a$ span a 2-dimensional linear vector space, a tangent space $T_{\xi}(V_2)$. Following an old tradition (see, e.g., [30, 31]) we use symbol $V_n$ for an $n$-dimensional surface embedded in an $N$-dimensional space $V_N$. Thus $V_n$, and in particular $V_2$, denotes a manifold, and not a vector space. In order to simplify our wording, an expression like “vectors $e_a$ on $V_2$” will mean “tangent vectors $e_a$ at a point $\xi \in V_2$”. So whenever we talk about vectors, or whatever geometric objects, on a manifold (or in a manifold) we just mean that to a given point of the manifold we attach a geometric object (see, e.g., [32]). The latter object, of course, is not an element of our manifold, but of the tangent space. Basis vectors on $V_2$ can be considered as being induced from the target...
space basis vectors $\gamma_\mu$:

$$e_a = \partial_a X^\mu \gamma_\mu$$ \hfill (4)

In the following we will adopt the geometric calculus in which basis vectors are *Clifford numbers* satisfying

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu}$$ \hfill (5)

where $g_{\mu\nu}$ is the metric of $V_N$. Eq. (5) defines the *inner product* of two vectors as the symmetric part of the Clifford product $\gamma_\mu \gamma_\nu$. The antisymmetric part of $\gamma_\mu \gamma_\nu$ is identified with the *wedge* or *outer product*

$$\gamma_\mu \wedge \gamma_\nu \equiv \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$$ \hfill (6)

Analogous relations we have for the worldsheet basis vectors $e_a$:

$$e_a \cdot e_b \equiv \frac{1}{2} (e_a e_b + e_b e_a) = \gamma_{ab}$$ \hfill (7)

$$e_a \wedge e_b \equiv \frac{1}{2} (e_a e_b - e_b e_a)$$ \hfill (8)

where $\gamma_{ab}$ is the metric on $V_2$ which, according to eq. (4), can be considered as being induced from the target space.

If we insert the relation (4) into eq. (2) we have

$$d\Sigma = \frac{1}{2} d\xi^{ab} \partial_a X^\mu \partial_b X^\nu \gamma_\mu \wedge \gamma_\nu$$ \hfill (9)

This is an infinitesimal *bivector* or *2-vector* in the target space $V_N$.

A finite 2-vector is obtained upon integration over a finite surface $\Sigma_B$ enclosed by a loop $B$:

$$\int_{\Sigma_B} d\Sigma \equiv \frac{1}{2} X^{\mu\nu} \gamma_\mu \wedge \gamma_\nu = \frac{1}{2} \int_{\Sigma_B} d\xi^{ab} \partial_a X^\mu \partial_b X^\nu (\gamma_\mu \wedge \gamma_\nu)$$

$$\quad = \frac{1}{2} \int_{\Sigma_B} d\xi^{ab} \left( \frac{1}{2} (\partial_a X^\mu \partial_b X^\nu - \partial_a X^\nu \partial_b X^\mu) \gamma_\mu \wedge \gamma_\nu \right)$$ \hfill (10)

From eq. (10) we read

$$X^{\mu\nu}[B] = \frac{1}{2} \int_{\Sigma_B} d\xi^{ab} (\partial_a X^\mu \partial_b X^\nu - \partial_a X^\nu \partial_b X^\mu)$$ \hfill (11)

By Stokes theorem this is equal to

$$X^{\mu\nu}[B] = \frac{1}{2} \oint_B ds \left( X^\mu \frac{\partial X^\nu}{\partial s} - X^\nu \frac{\partial X^\mu}{\partial s} \right)$$ \hfill (12)

Such integration poses no problem in flat $V_N$. In curved space we may still use the same expression (11) which then defines such integral that all vectors are carried together by means of parallel transport along geodesics into a chosen point of $V_N$ where the integration is actually performed [33].
where $X^\mu(s)$ describes a boundary loop $B$, $s$ being a parameter along the loop.

Eq. (12) demonstrates that $X^{\mu\nu}$ are components of the bivector, determining an oriented area, associated with a surface enclosed by a loop $X^\mu(s)$ on the worldsheet $V_2$. Hence there is a close correspondence between surfaces and the boundary loops. The components $X^{\mu\nu}$ can be therefore be considered as bivector coordinates of a loop. These are collective coordinates, since the detailed shape (configuration) of the loop is not determined by $X^{\mu\nu}$. Only the oriented area associated with a surface enclosed by the loop is determined by $X^{\mu\nu}$. Therefore $X^{\mu\nu}$ refers to a class of loops, from which we may choose a representative loop, and say that $X^{\mu\nu}$ are its coordinates. From now on, ‘loop’ we will be often a short hand expression for a representative loop in the sense above.

By means of eqs. (10)–(12) we have performed a mapping from an infinite dimensional space of loops $X^\mu(s)$ into a finite dimensional space of oriented areas $X^{\mu\nu}$. Instead of describing loops by infinite dimensional objects $X^\mu(s)$, we can describe them by finite dimensional objects, oriented areas, with bivector coordinates $X^{\mu\nu}$. We have thus arrived at a finite dimensional description of loops (in particular, closed strings), the so called quenched minisuperspace description suggested by Aurilia et al. [13].

When we consider not a single loop $X^\mu(s)$, but a 1-parameter family of loops $X^\mu(s, \alpha)$, we have a worldsheet, considered as being formed by a set of loops. By means of eqs. (10)–(12), with every loop within such a family, i.e., for a fixed $\alpha$, we can associate bivector coordinates $X^{\mu\nu}$.

In the following we will consider two particular choices for parameter $\alpha$.

In eq. (10) we have the expression for an oriented area associated with a loop $B$. It has been obtained upon the integration of the infinitesimal oriented surface elements (2). Besides the oriented area we can associate with our loop on $V_2$ also a scalar quantity, namely the scalar area $A$ which we obtain according to

$$A = \int_{\Sigma_B} \sqrt{d\Sigma^\dagger \cdot d\Sigma}$$

(13)

Here `\dagger` denotes reversion, that is the operation which reverse the order of vectors in a product. Using the relation $e_a \wedge e_b = e e_{ab}$, where $e = e_1 \wedge e_2$ is the pseudoscalar in 2-dimensional space $V_2$ such that $e^\dagger \cdot e = (e_2 \wedge e_1) \cdot (e_1 \wedge e_2) = \gamma_{11} \gamma_{22} - \gamma_{21} \gamma_{12} = \det \gamma_{ab} \equiv \gamma$, we find

$$d\Sigma^\dagger \cdot d\Sigma = \frac{1}{4} \gamma (d\xi^{ab} e_{ab})^2$$

(14)

i.e.

$$A = \int \sqrt{d\Sigma^\dagger \cdot d\Sigma} = \frac{1}{2} \int \sqrt{\gamma} |d\xi^{ab} e_{ab}| = \int \sqrt{\gamma} |d\xi|^{12}$$

(15)

If we choose $d\xi^a_1 = (d\xi^1, 0)$, $d\xi^a_2 = (0, d\xi^2)$, then $d\xi^{12} = d\xi^1 d\xi^2$. 

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We imagine that our surface $V_2$ is covered by a family of loops $X^\mu(\alpha, s)$ (Fig. 2), such that the totality of points of all those loops is in one-to-one correspondence with the points of the manifold $V_2$, and that for parameter $\alpha$ we take the scalar area $A$. To every loop there belong bivector coordinates $X^{\mu\nu}$ (calculated according to eq. (10)) and a scalar parameter $A$ (calculated according to eq. (15)), determining the the scalar area. Dependence of $X^{\mu\nu}$ on $A$ is characteristic for a given class of surfaces $V_2$. If we consider a different class of surface $V_2$, then functions $X^{\mu\nu}(A)$ are in general different (Fig. 3).

![Figure 2: We consider a loop on $V_2$. It determines an oriented area whose extrinsic 2-vector coordinates are $X^{\mu\nu}$. The scalar area of the surface element enclosed by the loop is $A$. Given an initial loop, functions $X^{\mu\nu}(A)$ are characteristic for a class of the surface $V_2$.](image)

Figure 3: (a) Examples of two different surfaces belonging to a class of surfaces that all satisfy equation $X^{12}(A) = k$. Constant $k$ differs from zero, if a cillindric surface has only one boundary loop, so that, e.g., the upper part is open, whilst the lower part is closed. (b) Example of surfaces belonging to a class of surfaces that satisfy equation $X^{12}(A) = kA$.

Instead of starting with a given surface $V_2$ on which we determine a family of loops and calculate the functions $X^{\mu\nu}(A)$, we can start from the other end. We may assume that all what is known are functions $X^{\mu\nu}(A)$. From those functions we do not know what
the surface (worldsheet) \( V_2 \) exactly is, but we have some partial information (see Fig. 3), up to a class of surfaces (worldsheets). Functions \( X^{\mu\nu}(A) \) provide a means of describing a surface \( V_2 \), although not in all details.

Instead of the scalar area \( A \) we may take as the parameter any other parameter\(^5\). We may take, for instance, just the integral\(^6\) in the parameter space \( \mathbb{R}^2 \)

\[
\xi^{ab}(B) = \int_{\Sigma_B} d\xi^{ab}
\]

taken over a domain corresponding to a surface \( \Sigma_B \) with boundary \( B \).

So we have a mapping \( B \rightarrow \xi^{ab}(B) \), such that to any boundary loop \( B \) of our family there correspond parameters \( \xi^{ab} \). Because of the property \( \xi^{ab} = -\xi^{ba} \), (where \( a, b = 0, 1 \) if \( V_2 \) is time like), there is in fact a single parameter \( \xi^{12} \). The extrinsic 2-vector coordinates \( X^{\mu\nu} \) are functions of \( \xi^{ab} \). The mapping

\[
\xi^{ab} \rightarrow x^{\mu\nu} = X^{\mu\nu}(\xi^{ab})
\]
determines a class of surfaces \( V_2 \), embedded in \( V_N \), which are all in accordance with eq. (17). Knowing the functions \( X^{\mu\nu}(\xi^{ab}) \) means knowing a class \( \{V_2\} \), but not a particular \( V_2 \in \{V_2\} \).

The mapping \( \tau \rightarrow x^\mu = X^\mu(\tau) \), involving vector coordinates, describes a curve (a "worldline") in the space spanned by vectors \( \gamma_\mu \). The derivative of \( X^\mu \) with respect to \( \tau \), i.e., \( \dot{X}^\mu = dX^\mu/d\tau \), is the tangent vector to the worldline, or velocity.

Similarly, the mapping (17), involving bivector coordinates, describes a curve (a "worldline") in the space spanned by the bivectors \( \gamma_\mu \wedge \gamma_\nu \), and we can calculate the derivative

\[
\partial_{ab} X^{\mu\nu} = \frac{\partial X^{\mu\nu}}{\partial \xi^{ab}}
\]

which generalizes the concept of velocity.

### 2.3 Generalization to arbitrary dimensions

Let us now consider extended objects associated with manifolds \( V_n \) that have arbitrary dimension \( n \) and are embedded in a target space \( V_N \) of dimension \( N \). An infinitesimal infinitesimal oriented area element on \( V_n \) is an \( n \)-vector

\[
d\Sigma = d\xi^1 \wedge d\xi^2 \wedge \ldots \wedge d\xi^n = d\xi^{a_1} d\xi^{a_2} \ldots d\xi^{a_n} e_{a_1} \wedge e_{a_2} \wedge \ldots \wedge e_{a_n} = \frac{1}{n!} d\xi^{a_1 \ldots a_n} e_{a_1} \wedge \ldots \wedge e_{a_n}
\]

\(^5\)In a suitable choice of parameters \( \xi^a \) the determinant of the metric \( \gamma_{ab} \) on \( V_2 \) can be constant, that is \( \partial_a |\gamma| = 0 \). Choosing \(|\gamma| = 1 \), the scalar area \( A \) is given just by the integral (16).

\(^6\)Similarly, in describing a world line \( X^\mu(\tau) \), we may take any parameter \( \tau \) which, in particular, can be the length of the worldline. The analog of eq. (16) is \( \tau = \int_0^t d\tau' \).
where
\[ d\xi^{a_1...a_n} = d\xi_1^{a_1} d\xi_2^{a_2} ... d\xi_n^{a_n} \] (20)
If we consider the basis vectors \( e_a \) on \( V_n \) as being induced from the basis vectors \( \gamma_\mu \) of the embedding space \( V_N \) according to the relation (4), we have
\[ d\Sigma = \frac{1}{n!} dX^{\mu_1...\mu_n} \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_n} = \frac{1}{n!} d\xi^{a_1...a_n} \partial_{a_1} X^{\mu_1} ... \partial_{a_n} X^{\mu_n} \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_n} \] (21)
After the integration over a finite \( n \)-surface \( \Sigma_B \) with boundary \( B \) we obtain a finite \( n \)-vector
\[ \int_{\Sigma_B} d\Sigma = \frac{1}{n!} \int_{\Sigma_B} d\xi^{a_1...a_n} \frac{1}{n!} \partial_{[a_1} X^{\mu_1} ... \partial_{a_n]} X^{\mu_n} \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_n} \] (22)
Its \( n \)-vector components are
\[ X^{\mu_1...\mu_n}[B] = \int_{\Sigma_B} d\xi^{a_1...a_n} \frac{1}{n!} \partial_{[a_1} X^{\mu_1} ... \partial_{a_n]} X^{\mu_n} \] (23)
They describe an oriented \( n \)-area associated with \( \Sigma_B \), whose boundary \( B \) will be called \((n-1)\)-loop, and \( X^{\mu_1...\mu_n}[B] \) are its extrinsic coordinates. With the same \((n-1)\)-loop we can associate intrinsic coordinates (parameters), in analogy to eqs. (13)–(16), according to
\[ \xi^{a_1...a_n}(B) = \int_{\Sigma_B} d\xi^{a_1...a_n} \] (24)
With a particular choice of coordinates \( \xi^a \), such that \( \det \gamma_{ab} = 1 \), the quantities \( \xi^{a_1...a_n} \) in the above equation determine the intrinsic (scalar) \( n \)-area of the \( n \)-surface bounded by the \((n-1)\)-loop.

As in the case of \( V_2 \) we assume that on our \( n \)-dimensional worldsheet \( V_n \) there exists a family of \((n-1)\)-loops \( B \), described by functions \( X^{\mu}(s^a, \alpha) \), \( a = 1, 2, ..., n-1 \),

Instead of a manifold \( V_n \) of points we thus consider a family of loops. With every \((n-1)\)-loop of the family we can associate arbitrary parameters \( \xi^{a_1...a_n} \) (coordinates are like “house numbers”). Because of the property
\[ \xi^{a_1...a_j a_k...a_n} = -\xi^{a_1...a_k a_j...a_n} \] (25)
there is actually a single parameter. This is a particular choice for parameter \( \alpha \) of our family of loops \( X^{\mu}(s^a, \alpha) \). By means of a mapping
\[ \xi^{a_1...a_n} \rightarrow x^{\mu_1...\mu_n} = X^{\mu_1...\mu_n}(\xi^{a_1...a_n}) \] (26)
we obtain a quenched minisuperspace description of a family of \((n-1)\)-loops, i.e., a description in terms of the target space multivector coordinate functions \( x^{\mu_1...\mu_n} \). The
family is such that the totality of the points of the \((n - 1)\)-loops belonging to the family is in one-to-one correspondence with the points of the worldsheet \(V_n\). In other words, by mapping (26) we have a quenched minisuperspace description of worldsheet.

We started from a brane described by the embedding functions \(X^\mu(\xi^a)\), and derived the expression (23) and functions (26). Once we have \(X^{\mu_1...\mu_n}(\xi^{a_1...a_n})\) as functions of a parameter \(\xi^{a_1...a_n}\), we may forget about the embedding \(x^\mu = X^\mu(\xi^a)\) that we started from. We may assume that all the information available to us are just functions \(X^{\mu_1...\mu_n}(\xi^{a_1...a_n})\) given by mapping (26). Then we do not have knowledge of a particular worldsheet’s manifold \(V_n\), but of a class \(\{V_n\}\) of worldsheet’s manifolds that all satisfy eq. (26) for given functions \(X^{\mu_1...\mu_n}(\xi^{a_1...a_n})\). So we calculate the derivative

\[
\partial_{a_1...a_n} X^{\mu_1...\mu_n} \equiv \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi^{a_1...a_n}}
\]

which generalizes the notion of velocity.

In the above considerations one has to bear in mind that many loop configuration may cast the same holographic projections onto the coordinate planes as a single loop configuration. Therefore, a given set of polyvector coordinates \(X^{\mu_1...\mu_n}(\xi^{a_1...a_n})\) may describe a single or many loop configuration. Not only the details of a single loop configuration (its infinite dimensionality), but also the number of loops is undetermined in this quenched description of loops.

3 The dynamics of extended objects

3.1 Objects described in terms of \(X^\mu(\xi^a)\)

The extended objects described by the mapping (11) obey the dynamical law that is incorporated in the Dirac-Nambu-Goto minimal surface action. An equivalent action that was considered in ref. [12] is a functional of the embedding functions \(X^\mu(\xi^a)\) and the coordinate basis vector fields \(e^a(\xi)\) having the role of Lagrange multipliers:

\[
I[X^\mu, e^a] = \frac{\kappa}{2} \int d^n \xi \ |e| \left( e^a \partial_a X^\mu e^b \partial_b X_\mu + 2 - n \right)
\]

where \(|e| \equiv \sqrt{|\gamma|}\) is the determinant of \(\gamma_{ab} = e_a \cdot e_b\).

Expanding the coordinate vector fields \(e^a(\xi)\), \(a = 1, 2, ..., n\) in terms of orthonormal vector fields \(e^a, \ a = 1, 2, ..., n\), by means of a tetrad \(e^a(\xi)\) according to

\[
e^a(\xi) = e^a(\xi) e^a
\]

\(^7\)Their inner products \(e^a \cdot e^b = \eta^{ab}\) gives the Minkowski metric.
we find the following relations:

\[ \frac{\partial e^a}{\partial e^b} \equiv e^c \frac{\partial e^a}{\partial e^c_b} = n\delta^a_b \] (30)

\[ \frac{\partial |e|}{\partial e^a} = \frac{\partial |e|}{\partial \gamma^{cd}} \frac{\partial \gamma^{cd}}{\partial e^a} = -n |e| e_a \] (31)

where \( n \) comes from the contraction \( e^a e_a = n \).

Using (30),(31) we find that the variation of the action (28) with respect to \( e^a \) gives

\[ -\frac{1}{2} e_c (e^a \partial_a X^\mu \partial^\nu X_\mu + 2 - n) + \partial_c X^\mu \partial_d X_\mu e^d = 0 \] (32)

Performing the inner product with \( e_c \) and using \( e_c \cdot e_c = n \) we find

\[ e^a \partial_a X^\mu e^b \partial_b X_\mu = n \] (33)

and eq. (32) becomes

\[ e_c = \partial_c X^\mu \partial_d X_\mu e^d \] (34)

This is the equation of “motion” for the Lagrange multipliers \( e_a \). In order to understand better the meaning of eq. (34) let us perform the inner product with \( e_c \):

\[ e_c \cdot e_a = \partial_c X^\mu \partial_d X_\mu e^d \cdot e_a \] (35)

Since \( e_c \cdot e_a = \gamma_{ca} \) and \( e^d \cdot e_a = \delta^d_a \) we obtain after renaming the indices

\[ \gamma_{ab} = \partial_a X^\mu \partial_b X_\mu \] (36)

This is the relation for the induced metric on the worldsheet. On the other hand, eq. (35) can be written as

\[ e_a \cdot e_b = (\partial_a X^\mu \gamma_{\mu}) \cdot (\partial_b X^\nu \gamma_{\nu}) \] (37)

from which we have that basis vectors \( e_a \) on the worldsheet \( V_n \) are expressed in terms of the embedding space basis vectors \( \gamma_{\mu} \):

\[ e_a = \partial_a X^\mu \gamma_{\mu} \] (38)

With our procedure we have thus derived eq. (4) as a solution to our dynamical system.

Using eq. (36) we find that the action (28) is equivalent to the well known Howe–Tucker action which is a functional of \( X^\mu(\xi) \) and \( \gamma^{ab} \):

\[ I[X^\mu, \gamma_{ab}] = \frac{\kappa}{2} \int d^n\xi \sqrt{|\gamma|} \left( \gamma^{ab} \partial_a X^\mu \partial_b X_\mu + 2 - n \right) \] (39)

\[ \text{Eq.(30)} \text{ also comes directly from the relation for a derivative with respect to a vector}\]
3.2 Objects described in terms of $X^{\mu_1...\mu_n}(\xi^{a_1...a_n})$

In Sec. 2.3 we have seen that an alternative description of extended objects, up to a class in which all objects have the same coordinates $X^{\mu_1...\mu_n}$, is given by the mapping \(\text{(26)}\). Let us assume that such objects are described by the following action

\[ I[X^{\mu_1...\mu_n}, e] = \frac{\kappa}{2} \int d^n \xi |e| \left[ \frac{1}{n!} \left( \frac{1}{n!} e^{a_1}_1 \wedge ... \wedge e^{a_n}_n \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi^{a_1...a_n}} \right) \right] \]

\[ \times \left( \frac{1}{n!} e^{b_1}_1 \wedge ... \wedge e^{b_n}_n \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi^{b_1...b_n}} \right) + 1 \] \(\text{(40)}\)

Factor $1/n!$ inside the bracket comes from the definition of the worldsheet $n$-vector \((1/n!)e^{a_1}_1 \wedge ... \wedge e^{a_n}_n \partial X^{\mu_1...\mu_n}/\partial \xi^{a_1...a_n}\). The extra factor $1/n!$ in front of the bracket comes from the square of the target space $n$-vector \((1/n!)\partial X^{\mu_1...\mu_n}/\partial \xi^{a_1...a_n}\)\(\gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_n}\). The operation $\hat{\times}$ reverses the order of vectors.

Let us take into account the following relations:

\[ e_{a_1} \wedge ... \wedge e_{a_n} = e e_{a_1...a_n} \] \(\text{(41)}\)

\[ e^{a_1} \wedge ... \wedge e^{a_n} = e^{-1} e^{a_1...a_n} \] \(\text{(42)}\)

\[ e^{-1} = \frac{e}{|e|^2}, \quad |e| \equiv \sqrt{e^\dagger \cdot e} = \sqrt{\gamma} \equiv \lambda \] \(\text{(43)}\)

\[ \gamma = \det \gamma_{ab}, \quad \gamma_{ab} = e_a \cdot e_b \] \(\text{(44)}\)

Instead of the intrinsic parameters $\xi^{a_1...a_n}$, let us introduce the dual parameter

\[ \tilde{\xi} = \frac{1}{n!} e_{a_1...a_n} \xi^{a_1...a_n} \] \(\text{(45)}\)

and rewrite the $n$-area velocity according to

\[ \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi^{c_1...c_n}} = \frac{\partial X^{\mu_1...\mu_n}}{\partial \tilde{\xi}} \frac{\partial \tilde{\xi}}{\partial \xi^{c_1...c_n}} = \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi} e^{c_1...c_n} \] \(\text{(46)}\)

where we have used

\[ \frac{\partial \xi^{a_1...a_n}}{\partial \xi^{c_1...c_n}} = \delta^{a_1...a_n}_{c_1...c_n} \] \(\text{(47)}\)

and where the generalized Kronecker symbol is given by the antisymmetrized sum of products of ordinary deltas. From eq.\(\text{(46)}\) we have

\[ \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi} = \frac{1}{n!} e^{c_1...c_n} \partial X^{\mu_1...\mu_n}/\partial \xi^{c_1...c_n} \] \(\text{(48)}\)

By using eqs.\(\text{(33) and (35)}\) and \(\text{(48)}\) we can rewrite the action \(\text{(40)}\) as

\[ I[X^{\mu_1...\mu_n}, \lambda] = \frac{\kappa}{2} \int d\tilde{\xi} \left( \frac{1}{\lambda n!} \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi} \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi} + \lambda \right) \] \(\text{(49)}\)
where
\[
d\tilde{\xi} = \frac{1}{n!} \varepsilon_{a_1...a_n} d\xi^{a_1...a_n} = \frac{1}{n!} \varepsilon_{a_1...a_n} d\xi^{a_1}...d\xi^{a_n} = \varepsilon^{a_1...a_n} d\xi_1 d\xi_2...d\xi_n = d^n\xi
\]  
(50)

The last step in eq.(50) holds in a coordinates system in which \( d\xi_1 = d\xi^1, d\xi_2 = d\xi^2, \ldots, d\xi_n = d\xi^n \).

The action (49) is a functional of the \( n \)-area variables \( \xi^\mu_1...\mu_n(\tilde{\xi}) \) and a Lagrange multiplier \( \lambda \), defined in eq.(43). Variation of eq. (49) with respect to \( \lambda \) and \( \xi^\mu_1...\mu_n \), respectively, gives
\[
\delta\lambda : \frac{1}{n!} \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi} \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi} - \lambda^2 = 0
\]  
(51)
\[
\delta X^{\mu_1...\mu_n} : \frac{d}{d\tilde{\xi}} \left( \frac{1}{\lambda} \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi} \right) = 0
\]  
(52)

These are the equations of motion for the \( n \)-area variables.

Inserting eq.(51) into (49) we obtain the action which is a functional of \( X^{\mu_1...\mu_n} \) solely:
\[
I[X^{\mu_1...\mu_n}] = \kappa \int d\tilde{\xi} \left( \frac{1}{n!} \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi} \frac{\partial X^{\mu_1...\mu_n}}{\partial \xi} \right)^{1/2}
\]  
(53)

The latter action has the same form as the action for a worldline of a relativistic particle. Factor \( 1/n! \) in the latter action disappears, if we order indices according to \( \mu_1 < \mu_2 < \ldots < \mu_n \).

3.3 More general objects (generalization to \( C \)-space)

So far we have considered branes described by coordinate functions of the type \( X^\mu(\xi) \), \( X^{\mu_1\mu_2}(\xi^{a_1a_2}), \ldots \), or \( X^{\mu_1...\mu_n}(\xi^{a_1...a_n}) \) that represent a mapping of a worldsheet loop into a target space loop of the same dimensionality.: 

\( (n-1) \)-loop on \( V_n \) \( \rightarrow \) \( (n-1) \)-loop in \( V_N \).

Let us now extend the theory and consider "mixed" mappings

\( (r-1) \)-loop on \( V_n \) \( \rightarrow \) \( (R-1) \)-loop in \( V_N \).

where \( r \), in general, is different from \( R \). So we arrive at a more general extended object which is described by a mapping \([12, 15]\)
\[
\xi^{a_1...a_r} \rightarrow x^{\mu_1...\mu_R} = X^{\mu_1...\mu_R}(\xi, \xi^{a_1}, \ldots, \xi^{a_1...a_s}, \ldots, \xi^{a_1...a_n}) , \quad 0 \leq R \leq N, \quad 0 \leq r \leq n \leq N
\]  
(54)
In the compact notation we set
\[ X^M \equiv X^{\mu_1 \ldots \mu_R}, \quad \mu_1 < \mu_2 < \ldots < \mu_R \]
\[ \xi^A \equiv \xi^{a_1 \ldots a_r}, \quad a_1 < a_2 < \ldots < a_r \]
and write the mapping (54) as
\[ \xi^A \rightarrow x^M = X^M(\xi^A) \tag{55} \]
This is the parametric equation of our generalized extended object. Such object lives in a target space which is now generalized to Clifford space (shortly C-space). The worldsheet associated with the extended object is also generalized to a Clifford space. In the following we will explain this in more detail.

In eq. (54) or (55) we have a generalization of the usual relation
\[ \xi^a \rightarrow x^\mu = X^\mu(\xi^a), \quad a = 1, 2, \ldots, n; \quad \mu = 0, 1, 2, \ldots, N - 1 \tag{56} \]
that describes an \( n \)-dimensional surface, called worldsheet or world manifold, \( V_n \), embedded in \( N \)-dimensional target space \( V_N \). In eq. (56) the space \( \mathbb{R}^n \) of parameters \( \xi^a \) is isomorphic to an \( n \)-dimensional vector space \( V_n \), spanned by an orthonormal basis \( \{h_a\} \).

The vector space \( V_n \) should not be confused with the worldsheet \( V_n \), which is a manifold (embedded in a higher dimensional manifold \( V_N \)).

Instead of \( V_n \) we can consider the corresponding Clifford algebra \( C_n \) which is itself a vector space. Amongst its elements are \( r \)-vectors associated with \((r - 1)\)-loops, \( r = 0, 1, 2, \ldots, n \). A generic object is a superposition of \( r \)-vectors for different grades, and it is described by a Clifford number, a polyvector, \( \xi^{a_1 \ldots a_r} h_{a_1} \wedge \ldots \wedge h_{a_r} \in C_n \).

Our objects are now extended events \( \mathcal{E} \) \cite{29}, superpositions of \((r - 1)\)-loops, to which we assign a set of \( 2^n \) parameters (coordinates) \( \xi^A \equiv \xi^{a_1 \ldots a_r}, \quad r = 0, 1, 2, \ldots, n \) according to the mapping
\[ \mathcal{E} \rightarrow \xi^A(\mathcal{E}) \tag{57} \]
The assignment is arbitrary. We may choose an object \( \mathcal{E}_0 \) to which we assign coordinates \( \xi^A(\mathcal{E}) = 0 \). This is a coordinate origin. Choosing an origin \( \mathcal{E}_0 \), the polyvectors \( \xi^A h_A \) pointing from \( \mathcal{E}_0 \) to any \( \mathcal{E} \) are in one-to-one correspondence with extended events \( \mathcal{E} \). The space of extended events is then isomorphic to Clifford algebra \( C_n \), and the latter algebra, in turn, is isomorphic to the space of parameter \( \{\xi^A\} = \mathbb{R}^{2^n} \). Therefore we will speak about \( C_n \) as the parametric Clifford algebra or parametric polyvector space.

The parametric space \( C_n \) is by definition a (poly)vector space, spanned by a basis \( h_{a_1} \wedge \ldots \wedge h_{a_r}, \quad r = 0, 1, 2, \ldots, n \), formed by the orthonormal basis \( \{h_a\}, \quad a = 1, 2, \ldots, n \). This implies that \( C_n \) is a metric space, but its metric is just formal, without any physical content. Now let us consider the mapping (55) from \( C_n \) into a Clifford space \( C_{V_N} \) generated by a
basis $\gamma_\mu$, $\mu = 0, 1, 2, \ldots, N - 1$, with $N > n$. So we obtain a generalized, $2^n$-dimensional surface $C_{V_n}$ embedded in a target Clifford space $C_{V_N}$. The surface $C_{V_n}$ generalizes the notion of worldsheet $V_n$ to Clifford space, i.e., a manifold such that any of its tangents spaces is a Clifford algebra. If we consider only the intrinsics properties of $C_{V_n}$ (i.e., if we “forget” about its embedding into a higher dimensional Clifford space $C_{V_N}$), then we can simply denote it as $C_n$.

Below we summarize our notation of various spaces:

$V_n$ *Parametric vector space*, with an orthonormal basis $\{h_a\}$, $a = 1, 2, \ldots, n$ and elements $\xi^a h_a \in V_n$. It is isomorphic to $\mathbb{R}^n$, the space of parameters $\xi^a$.

$V_n$ *Manifold*, either flat or curved. It is a space of points (events) $P$. With every point $P \in V_n$ we associate a set of $n$ parameters (coordinates) $\xi^a(P) \equiv \xi^a \in \mathbb{R}^n$. Coordinate basis vectors are $e^a$, whilst orthonormal basis vectors are $e_a$.

$C_n$ *Parametric Clifford algebra* of $V_n$, called also *parametric polyvector space*, with basis $\{h_A\} \equiv \{h_{a_1} \wedge \ldots \wedge h_{a_r}\}$, $r = 0, 1, 2, \ldots, n$ and elements $\xi^A h_A \in C_n$, called *polyvectors*. It is isomorphic to $\mathbb{R}^{2^n}$, the space of parameters $\xi^A$.

$C_n$ *Clifford manifold*, or *Clifford space*, either flat or curved. It is a space of points that are perceived in a subspace $V_n$ as extended events $E$. With every $E \in C_n$ we associate a set of $2^n$ parameters $\xi^A(E) \equiv \xi^a_{\mu_1\ldots\mu_R} \in \mathbb{R}^{2^n}$, $\mu_1, \ldots, \mu_R = 0, 1, 2, \ldots, N$. Coordinate basis elements are $e_A \equiv e_{a_1\ldots a_r}$; orthonormal basis elements are $e_A \equiv e_{a_1\ldots a_n}$, $r = 0, 1, 2, \ldots, n$. In particular $C_n$ can be considered as being embedded in a higher dimensional Clifford space; then it is denoted as $C_{V_n}$.

$C_{V_n}$ *Generalized worldsheet*, a Clifford space embedded in a target Clifford space. Its coordinate basis elements are $e^A \equiv e_{a_1\ldots a_r}$; orthonormal basis elements are $e_A \equiv e_{a_1\ldots a_n}$, $r = 0, 1, 2, \ldots, n$.

$C_{V_N}$ *Target Clifford manifold*, or *target Clifford space*, either flat or curved. It is a space of points that are perceived in a subspace $V_N$ as extended events $E$. With every $E \in C_{V_N}$ we associate a set of $2^N$ coordinates $x^M(E) \equiv x^M \equiv x^{\mu_1\ldots\mu_R}$, $R = 0, 1, 2, \ldots N$. Coordinate basis elements are $\gamma_M \equiv \gamma_{\mu_1\ldots\mu_R}$; orthonormal basis elements are $\gamma_M \equiv \gamma_{\mu_1\ldots\mu_R}$. Instead of $C_{V_N}$ we can use simply notation $C_N$.

We follow the rule that bold symbols are used for vector spaces, whilst light symbols are used for manifolds. Since Clifford algebras also are vector spaces, they are denoted by bold symbols, whereas the corresponding Clifford spaces (manifolds of points representing extended events) are denoted by light symbols. By such notation we have attempted to simplify distinction among all those various spaces that occur in our theory of generalized
branes. For these purely physical reasons we have thus, to certain extent, deviated from
the standard notation used in mathematics.

An infinitesimal (polyvector) \( dX \in \mathcal{C}_{V_n} \), joining two points on the surface \( \mathcal{C}_{V_0} \) can be
written as
\[
 dX = dX^M \gamma_M = d\xi^A \partial_A X^M \gamma_M = d\xi^A e_A
\]
where
\[
e_A = \partial_A X^M \gamma_M
\]
These are induced basis tangent (polyvectors) on \( \mathcal{C}_{V_n} \).

At every point of the flat target \( \mathcal{C}_{V_0} \) there exists a basis
\[
 \{ \gamma^M \} = \{ 1, \gamma^\mu, \gamma^{\mu_1\mu_2}, ..., \gamma^{\mu_1...\mu_N} \}
\]
given in terms of \( 2^N \) Clifford numbers
\[
 \gamma^M \equiv \gamma^{\mu_1...\mu_r} \equiv \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge ... \wedge \gamma^{\mu_R} , \quad \mu_1 < \mu_2 < ... \mu_R , \quad R = 0, 1, 2, ...N
\]

At every point \( \xi \in \mathcal{C}_{V_n} \) on the brane’s worldsheet \( \mathcal{C}_{V_0} \), which in general is
curved, there exist a basis given in terms of \( 2^n \) Clifford numbers
\[
 \{ e^A \} = \{ e, e^a, e^{a_1a_2}, ..., e^{a_1...a_n} \}
\]
that span a tangent space \( T_\xi(\mathcal{C}_{V_n}) \). At a particular point \( \xi_0 \in \mathcal{C}_{V_n} \) it may hold
\[
 e^A \equiv e^{a_1...a_r} \equiv e^{a_1} \wedge e^{a_2} \wedge ... \wedge e^{a_r} , \quad a_1 < a_2 < ... < a_r , \quad r = 0, 1, 2, ..., n
\]
That is, at that particular point the basis polyvectors on \( \mathcal{C}_{V_n} \) are given as wedge products
of basis vectors \( e^a \). The above property cannot hold globally on a curved \( \mathcal{C}_{V_n} \). At
points different from \( \xi_0 \), basis polyvectors are in general superpositions of \( e^{a_1} \wedge e^{a_2} \wedge ... \wedge e^{a_s} \),
\( s = 0, 1, 2, ..., n \).

To sum up, every Clifford number in the target \( \mathcal{C}_{V_0} \) can be expanded in terms of
\( \gamma^M \), and every Clifford number on the worldsheet \( \mathcal{C}_{V_0} \) can be expanded in terms of
\( e^A \). Such numbers are also called Clifford aggregates or polyvectors. They are superpositions
\( r \)-vectors, the objects of definite grade that we call multivectors. This description
automatically contains spinors which are just members of the left of right ideal of Clifford
algebra \[27\].

---

\[^9\] We will use ‘basis’ and ‘frame’ as synonyms. In order to simplify notation and wording we will be
sloppy in distinguishing objects from the corresponding fields, e.g., (poly)vectors from (poly)vector fields,
frames from frame fields, etc. From the context it should not be difficult to understand which ones we
have in mind.
Metric of $C_{VN}$ is $G_{MN} = \gamma^*_M \gamma_N$, whilst metric of $C_V$ is $G_{AB} = \gamma^*_A \gamma_B$. Here ‘$*$’ denotes the scalar product of two Clifford numbers $A$ and $B$

$$A * B = \langle AB \rangle_0 \quad (64)$$

Let us now define the object $V$ which is a polyvector in target space and on the worldsheet:

$$V = e^A \frac{\partial X^M}{\partial \xi^A} \gamma_M = \sum_{r=0}^{N} \sum_{s=0}^{n} \frac{1}{r!s!} \epsilon^{a_1...a_s} \frac{\partial X^{\mu_1...\mu_r}}{\partial \xi^{a_1...a_s}} \gamma_{\mu_1...\mu_r} \quad (65)$$

In the right hand side expression we impose no restriction on the indices $\mu_1, \mu_2, ..., \mu_r$ and $a_1, a_2, ..., a_s$, therefore, in order to prevent multiple counting of equivalent terms, factors $1/r!$ and $1/s!$ are introduced.

It is illustrative to express eq. (65) in a more explicit form by employing the notation (60)–(63) and by writing

$$X^M = (\Omega, X^\mu, X^{\mu_1\mu_2}, ..., X^{\mu_1...\mu_N}) \quad (66)$$

$$\xi^A = (\xi, \xi^a, \xi^{a_1a_2}, ..., \xi^{a_1...a_n}) \quad (67)$$

We obtain

$$V = \left( \frac{\partial \Omega}{\partial \xi} + e^a \frac{\partial \Omega}{\partial \xi^a} + \frac{1}{2!} e^{a_1a_2} \frac{\partial \Omega}{\partial \xi^{a_1a_2}} + ... + \frac{1}{n!} e^{a_1...a_n} \frac{\partial \Omega}{\partial \xi^{a_1...a_n}} \right) 1 \right. \right.$$

$$+ \left( \frac{\partial X^\mu}{\partial \xi} + e^a \frac{\partial X^\mu}{\partial \xi^a} + \frac{1}{2!} e^{a_1a_2} \frac{\partial X^\mu}{\partial \xi^{a_1a_2}} + ... + \frac{1}{n!} e^{a_1...a_n} \frac{\partial X^\mu}{\partial \xi^{a_1...a_n}} \right) \gamma_{\mu} \right.$$}

$$+ \frac{1}{2!} \left( \frac{\partial X^{\mu_1\mu_2}}{\partial \xi} + e^a \frac{\partial X^{\mu_1\mu_2}}{\partial \xi^a} + \frac{1}{2!} e^{a_1a_2} \frac{\partial X^{\mu_1\mu_2}}{\partial \xi^{a_1a_2}} + ... + \frac{1}{n!} e^{a_1...a_n} \frac{\partial X^{\mu_1\mu_2}}{\partial \xi^{a_1...a_n}} \right) \gamma_{\mu_1\mu_2} \right.$$}

$$+ \quad \vdots \quad$$

$$+ \frac{1}{N!} \left( \frac{\partial X^{\mu_1...\mu_N}}{\partial \xi} + e^a \frac{\partial X^{\mu_1...\mu_N}}{\partial \xi^a} + \frac{1}{2!} e^{a_1a_2} \frac{\partial X^{\mu_1...\mu_N}}{\partial \xi^{a_1a_2}} + ... + \frac{1}{n!} e^{a_1...a_n} \frac{\partial X^{\mu_1...\mu_N}}{\partial \xi^{a_1...a_n}} \right) \gamma_{\mu_1...\mu_N}$$

One might ask how such a generalized extended object, described by eq. (55), that sweep a Clifford worldsheet $C_{VN}$, embedded in a Clifford space $C_{VN}$, looks like. Here our perceptive system again shows its shortcomings, like in the case of figuring out how higher dimensional objects look like. We are able to draw pictures of projections of an object onto 3 or 2-dimensional Euclidean space, and that is, more or less, all. But on the other hand, we are able to do algebra, and the algebra is interpreted as geometric algebra. So we have to content us by our ability to control the situation algebraically, and assume that there is a mapping between algebraic and geometric objects. The latter objects are associated with physical objects, such as, e.g., the generalized extended objects, that incorporate branes of various dimensionalities.
Let the action describing the dynamics of a generalized extended object, shortly, a
generalized brane, be described by the embedding functions \( X^M(\xi^A) \) be

\[
I[X^M, e^A] = \frac{T}{2} \int d^{dc}\xi |E| \left[ \left( e^A \frac{\partial X^M}{\partial \xi^A} \gamma_M \right)^\dagger \ast \left( e^B \frac{\partial X^N}{\partial \xi^B} \gamma_N \right) + 2 - d_C \right]
\]  

(69)

The latter action has a similar form as the action (28). But, since the indices \( M, N, \) and \( A, B \) run over the full basis (60) and (62) of the corresponding \( C \)-spaces, the action (69) is more general than (28).

Here the measure \( d^{dc}\xi |E| \) is the volume element in the worldsheet \( C \)-space \( C_{V_n} \) (whose dimension is \( 2^{dc} \)). It is equal to the product

\[
d^{dc}\xi |E| = |E| \prod_{A_i} d\xi^{A_i} = d\xi \prod_{a_1} d\xi^a \prod_{a_1<a_2} d\xi^{a_1a_2} \cdots \prod_{a_1<\ldots<a_n} d\xi^{a_1\ldots a_n} |E| \quad \text{(70)}
\]

By \( |E| \) we denote the square root of the determinant of the worldsheet \( C \)-space metric which is given by the scalar product

\[
G^{AB} = (e^A)^\dagger e^B = \langle (e^A)^\dagger e^B \rangle_0 \quad \text{(71)}
\]

where \( \langle \rangle_0 \) denotes the scalar part. Explicitly,

\[
|E| = \sqrt{|G|}, \quad G \equiv \det G_{AB} = \frac{1}{d_C!} \epsilon^{A_1\ldots A_{dc}} \epsilon^{B_1\ldots B_{dc}} G_{A_1B_1\ldots A_{dc}B_{dc}} \quad \text{(72)}
\]

The action (69) is a functional of \( X^M \) and \( e^A \), and is a \( C \)-space generalization of the action (28) which is a functional of the worldsheet embedding functions \( X^\mu \) and basis vectors \( e^a \).

An action which is classically equivalent to (69) is a functional of \( X^M \) and \( G^{AB} \):

\[
I[X^M, G^{AB}] = \frac{T}{2} \int d^{dc}\xi \sqrt{|G|} \left( G^{AB} \partial_A X^M \partial_B X_M + 2 - d_C \right) \quad \text{(73)}
\]

where \( \partial_B X_M = G_{MN} \partial_X X^N \), \( G_{MN} = \gamma_M^\dagger \gamma_N \). In eq. (73) we have a \( C \)-space generalization of the well known Howe–Tucker action [34].

Variation of the action (73) with respect to \( X^M \) and \( G^{AB} \) gives the equations of motion of our \( C \)-space extended object:

\[
\delta X^M : \quad \frac{1}{\sqrt{|G|}} \partial_A (\sqrt{|G|} \partial^A X^M ) = 0 \quad \text{(74)}
\]

\[
\delta G^{AB} : \quad G_{AB} = \partial_A X^M \partial_B X^M \quad \text{(75)}
\]

Taking into account that \( G_{AB}G^{AB} = d_C = 2^n \) and inserting eq. (75) into eq. (73) we obtain the action integral

\[
I[X^M] = T \int d^{dc}\xi \sqrt{\det \partial_A X^M \partial_B X^M} \quad \text{(76)}
\]
which is the volume of the C-space worldsheet. The latter action contains the usual p-branes, including point particles, as special cases.

Our action (73), or equivalently (76), is invariant under reparametrizations of coordinates $\xi^A$, $A = 1, 2, \ldots, 2^n$. As a consequence, there are $2^n$ primary constraints. So we are free to choose $2^n$ relations among our dynamical variables $X^M(\xi^A)$, and thus fix a gauge (a parametrization). For one of those relations we can choose, for instance,

$$\partial_{a_1 \ldots a_r} X^{\mu_1 \ldots \mu_r} = \partial_{[a_1} X^{\mu_1} \ldots \partial_{a_r]} X^{\mu_r}$$

That is, the above relation is just one of gauge fixing relations. It will be used in Sec. 5.3, where we will consider the coupling of our generalized branes to external fields.

4 On the relativity in Clifford space

The discussion of previous sections has led us to the conclusion that the space in which our extended objects live is Clifford space, shortly C-space, denoted $C_V N$ or $C_N$. A point in C-space is described by the coordinates $x^M = (\Omega, x^\mu, x^{\mu_1 \mu_2}, \ldots, x^{\mu_1 \ldots \mu_N})$ which together with the basis elements $\gamma^M = (1, \gamma^\mu, \gamma^{\mu_1 \mu_2}, \ldots, \gamma^{\mu_1 \ldots \mu_N})$ form a coordinate polyvector

$$X = x^M \gamma_M = \frac{1}{r!} \sum_{r=0}^{N} x^{\mu_1 \ldots \mu_r} \gamma_{\mu_1 \ldots \mu_r} (78)$$

From the point of view of $2^N$-dimensional C-space, $x^\mu_1 \ldots \mu_r$, $r = 1, \ldots, N$, are coordinates of a point, whilst from the point of view of the underlying Minkowski space $V_N$, these are the $r$-area ($r$-volume) variables.

The infinitesimal polyvector connecting two infinitesimally separated points of C-space is

$$dX = dx^M \gamma_M = \frac{1}{r!} \sum_{r=0}^{N} dx^{\mu_1 \ldots \mu_r} \gamma_{\mu_1 \ldots \mu_r} \equiv dx^M \gamma_M$$

The square of the distance between these points is given by the scalar product

$$|dX|^2 \equiv dX^* \cdot dX = dx^M dx^N G_{MN} = dx^M dx_N$$

where $G_{MN}$ is the metric of C-space:

$$G_{MN} = \gamma^+_M \cdot \gamma_N = \gamma^+_\mu_1 \ldots \mu_r \cdot \gamma_{\nu_1 \ldots \nu_s} (81)$$

\[10\] In flat C-space it makes sense to consider a polyvector joining the coordinate origin $E_0$ with coordinates $x^M(E_0) = 0$ and a point $E$ with coordinates $x^M(E) = x^M$, where $E_0$, $E \in C_V N$.

\[11\] A reason why we define the C-space metric by employing the reversion is in the consistency between $G_{MN}$, its inverse $G^{MN}$, and the relations (22), (23) (in which the indices $\mu_i$, $\nu_j$, ... are lowered and raised by the ordinary 4-dimensional metric $g_{\mu_i \nu_j}$ and its inverse $g^{\mu_i \nu_j}$). For more details see ref. [10].
In particular, $C$-space $C_V$ can be flat. This means that the curvature of the connection on flat $C_V$ vanishes (see sec. 5.1). In such a case one can find a coordinates system on $C_V$, such that the metric $G_{MN}$ is diagonal.

Explicitly, for different choices of the indices $M$ and $N$ we have:

$$G_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu = g_{\mu\nu} = \eta_{\mu\nu}$$

$$G_{[\mu_1\mu_2][\nu_1\nu_2]} = \gamma_{[\mu_1\mu_2} \cdot \gamma_{\nu_1\nu_2]} = (\gamma_{\mu_2} \wedge \gamma_{\mu_1}) \cdot (\gamma_{\nu_1} \wedge \gamma_{\nu_2}) = \left| \begin{array}{cc} g_{\mu_1\nu_1} & g_{\mu_1\nu_2} \\ g_{\nu_2\mu_1} & g_{\nu_2\mu_2} \end{array} \right|$$

$$G_{[\mu_1\mu_2\mu_3][\nu_1\nu_2\nu_3]} = \gamma_{[\mu_1\mu_2\mu_3} \cdot \gamma_{\nu_1\nu_2\nu_3]} = (\gamma_{\mu_3} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_1}) \cdot (\gamma_{\nu_1} \wedge \gamma_{\nu_2} \wedge \gamma_{\nu_3}) = \left| \begin{array}{ccc} g_{\mu_1\nu_1} & g_{\mu_1\nu_2} & g_{\mu_1\nu_3} \\ g_{\mu_2\nu_1} & g_{\mu_2\nu_2} & g_{\mu_2\nu_3} \\ g_{\mu_3\nu_1} & g_{\mu_3\nu_2} & g_{\mu_3\nu_3} \end{array} \right|$$

$$G_{\mu[\nu_1\nu_2]} = \gamma_\mu \gamma_{\nu_1\nu_2} = (\gamma_\mu (\gamma_{\nu_1} \wedge \gamma_{\nu_2}) = 0$$

In general we have

$$G_{[\mu_1...\mu_r][\nu_1...\nu_r]} = (\gamma_{\mu_r} \wedge ... \wedge \gamma_{\mu_1}) \cdot (\gamma_{\nu_1} \wedge ... \wedge \gamma_{\nu_r}) = \det g_{\mu_i\nu_j}, \quad i, j = 1, 2, ..., r \quad (83)$$

when $r = s$, and

$$G_{[\mu_1...\mu_r][\nu_1...\nu_s]} = 0 \quad (84)$$

when $r \neq s$.

Taking into account the explicit expressions for metric (82)–(84), the quadratic form (80) reads

$$|dX|^2 = \frac{1}{r!} \sum_{r=0}^{N} dx^{\mu_1...\mu_r} dx_{\mu_1...\mu_r}$$

$$= d\Omega^2 + dx^\mu dx_\mu + \frac{1}{2!} dx^{\mu_1\mu_2} dx_{\mu_1\mu_2} + ... + \frac{1}{N!} dx^{\mu_1...\mu_N} dx_{\mu_1...\mu_N} \quad (85)$$

In the latter expression only the factor $1/r!$ remains, since the other factor is canceled by $r!$ coming from the determinant (83). Indices $\mu_1, \mu_2, ...$ are lowered and raised by Minkowski metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$.

In the 16-dimensional Clifford space $C_4$ of the 4-dimensional Minkowski spacetime the polyvector (79) and its square (80) can be rewritten as

$$dX = d\Omega + dx^\mu \gamma_\mu + \frac{1}{2} dx^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + d\bar{x}^\mu \gamma_5 \gamma_\mu + d\bar{\Omega} \gamma_5$$

$$|dX|^2 = d\Omega^2 + dx^\mu dx_\mu + \frac{1}{2} dx^{\mu\nu} dx_{\mu\nu} - d\bar{x}^\mu \bar{x}_\mu - d\bar{s}^2$$

where

$$d\bar{x}^\mu \equiv \frac{1}{3!} dx^{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma}^\mu$$

$$22$$
\begin{equation}
\text{d}\tilde{\Omega} \equiv \frac{1}{4!} \text{d}x^{\alpha\beta\rho\sigma} \epsilon_{\alpha\beta\rho\sigma} \tag{89}
\end{equation}

The minus sign in the last two terms of the above quadratic form occurs because in 4-dimensional spacetime with signature \((++--)\) we have \(\gamma_5^2 = (\gamma_0\gamma_1\gamma_2\gamma_3)(\gamma_0\gamma_1\gamma_2\gamma_3) = -1\), and also \(\gamma_5^\dagger\gamma_5 = (\gamma_3\gamma_2\gamma_1\gamma_0)(\gamma_0\gamma_1\gamma_2\gamma_3) = -1\).

It is illustrative to write the quadratic form (line element) explicitly:

\begin{equation}
|dX|^2 = d\Omega^2 + (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\
- (dx^{01})^2 - (dx^{02})^2 - (dx^{03})^2 + (dx^{12})^2 + (dx^{13})^2 + (dx^{23})^2 \\
- (d\tilde{x}^0)^2 + (d\tilde{x}^1)^2 + (d\tilde{x}^2)^2 + (d\tilde{x}^3)^2 - d\tilde{\Omega}^2 \tag{90}
\end{equation}

Here \(\tilde{x}^0 = x^{123}\), \(\tilde{x}^1 = x^{023}\), \(\tilde{x}^2 = x^{013}\), \(\tilde{x}^3 = x^{012}\), \(\tilde{\Omega} = x^{0123}\). The factor \(1/2\) has disappeared from the term \((1/2)dx^{\mu\nu}dx_{\mu\nu}\), since \((1/2)(dx^{01}x_{01} + dx^{10}x_{10}) = dx^{01}dx_{01} = -(dx^{01})^2\), etc..

By inspecting the quadratic form \((87)\) we see that it has 8 terms with \textit{plus} and 8 terms with \textit{minus} sign. The group of transformations that leave the quadratic form \((87)\) invariant is SO(8,8). These are pseudorotations in C-space and have the role of generalized Lorentz transformations:

\begin{equation}
x'^{M} = L^M_N x^N \tag{91}
\end{equation}

The transformations matrix satisfies the relation \(L^M_J L^N_K G_{MN} = G_{JK}\).

If interpreted actively \((91)\) are the transformations that transform a point of C-space with coordinates \(x^M\) into another point with coordinates \(x'^{M}\). In the passive interpretation the point remains the same, but its coordinates \(x^M\) with respect to a reference frame \(S\) are transformed into coordinates \(x'^{M}\) with respect to a reference frame \(S'\).

From the point of view of C-space, \(x^M\) are coordinates of a \textit{point}. But from the point of view of the underlying Minkowski space, \(x^M\) are the \((p+1)\)-vector (holographic) coordinates of \(p\)-loops associated with an extended object \([11]\). In C-space, \(p\)-loops of different dimensionalities \(p\) (i.e., points, closed lines, closed 2-surfaces, closed 3-volumes) are all described on the same footing \([36, 37, 26, 11, 14]\), and can be transformed into each other by transformations \((91)\). Pseudo rotations in C-space have thus the role of polydimensional rotations in \(M_4\). A point can be transformed into a line, and in general an \((R-1)\)-loop into an \((R'-1)\)-loop. So the very dimensionality of a loop can change under a transformation. This means that, when observed in a given reference frame \(S\), a loop’s dimensionality can change from \((R-1)\) to \((R'-1)\). Alternatively, dimensionality of the same loop, when observed from different reference frames \(S\) and \(S'\), can look different. In short, dimensionality of a loop depends on the observer (associated with a given reference frame). Such relativity of dimensionality of a loop also explains why in the mapping \((54),(55)\) the dimensionality of a loop in the parameter space \(\{\xi^a\} \equiv \mathbb{R}^n\) is in general
different from the dimensionality of the same loop with respect to a reference frame in the target space $V_N$.

$C$-space in essence encodes the zero modes of $p$-loop configurations, since $p$-loops space is infinite dimensional whereas $C$-space is finite dimensional. As already mentioned before, a $p$-loop configuration, in general, can consist of many loops.

The construction with $C$-space coordinates, SO(8,8) symmetry and the brane equations of motion (74) reminds us of the constructions considered in refs. [5], where extra coordinates were introduced in order to make manifest the SO(n,n) symmetry of the duality transformations for strings and branes.

5 Curved Clifford space

5.1 General considerations

In general, the worldsheet $V_n$ swept by a brane is curved. In Sec. 3.2 we have considered a concept of a more complicated, generalized brane, whose (generalized) worldsheet $C_{V_n}$ was curved Clifford space. The latter worldsheet $C_{V_n}$ was embedded in a target space which was a flat Clifford space $C_{V_N}$, with the metric properties given in Sec. 4.

A next step is to consider curved target Clifford space $C_{V_N}$ (see refs. [28, 29]). At every point $E \in C_{V_N}$ we have a flat tangent Clifford space $T_E(C_{V_N})$ and an orthonormal basis of $2^N$ Clifford numbers

$$\{\gamma_M\} = \{1, \gamma_{\mu_1}, \gamma_{\mu_1\mu_2}, \ldots, \gamma_{\mu_1\mu_2\ldots\mu_N}\}$$

(92)

where

$$\gamma_{\mu_1\mu_2\ldots\mu_r} = \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \ldots \wedge \gamma_{\mu_r}$$

(93)

From an orthonormal basis $\{\gamma_M\}$ we can switch to a coordinate basis

$$\{\gamma_M\} = \{\gamma, \gamma_{\mu_1}, \gamma_{\mu_1\mu_2}, \ldots, \gamma_{\mu_1\mu_2\ldots\mu_N}\}$$

(94)

by means of the relation [28, 29]

$$\gamma_M = e_M^M \gamma_M$$

(95)

in which we have introduced a vielbein of curved Clifford space $C_{V_N}$, given by the scalar product $\gamma_M^\dagger \gamma^M$. Notice a distinction between bold and normal indices, used for two different kinds of basis. The coordinate basis Clifford numbers $\gamma_M = \gamma_{\mu_1\mu_2\ldots\mu_r}$ in general are not defined as a wedge product $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \ldots \wedge \gamma_{\mu_r}$. In particular, $\gamma_M$ can be multivectors of definite grade, i.e., defined as a wedge product, but such property can hold only locally at a given point $E \in C_{V_N}$, and cannot be preserved globally at all point $E$ of our curved Clifford space. The relations for $\gamma_M$ and the metric $G_{MN}$, discussed in sec. 4, refer to flat
C-space and are no longer generally valid in curved C-space. At a fixed point \( E \in C_V \)
we can choose a coordinate system such that \( \gamma_M = \gamma_M^E \), and then the relations of Sec. 4 refer to flat C-space, spanned by \( \gamma_M \), i.e., the tangent space \( T_E(C_V) \).

The set \( \{ \gamma_M \} \) of \( 2^n \) linearly independent coordinate basis fields (which depend on coordinates \( x^M \)) will be called a coordinate frame field in C-space.

The set \( \{ \gamma_M \} \) of \( 2^n \) linearly independent orthonormal basis fields (which also in general depend on \( x^M \)) will be called orthonormal frame field in C-space.

Corresponding to each field \( \gamma_M^E \) we define a differential operator—which we call \( \text{derivative}—\partial_M^E \), whose action depends on the quantity it acts on\(^\text{12}\).

(i) \( \partial_M \) maps scalars \( \phi \) into scalars

\[
\partial_M \phi = \frac{\partial \phi}{\partial x^M}
\]

Then \( \partial_M \) is just the ordinary \textit{partial derivative}.

(ii) \( \partial_M \) maps Clifford numbers into Clifford numbers. In particular, it maps a coordinate basis Clifford number \( \gamma_N^E \) into another Clifford number which can, of course, be expressed as a linear combination of \( \gamma_J^E \):

\[
\partial_M \gamma_N^E = \Gamma^J_M N \gamma_J^E \quad (97)
\]

The above relation defines the connection \( \Gamma^J_M N \) for the coordinate frame field \( \{ \gamma_M \} \).

An analogous relation we have for the orthonormal frame field:

\[
\partial_M \gamma_M^E = -\Omega_M^N \gamma_N^E \quad (98)
\]

where \( \Omega_M^N M \gamma_N \) is the connection for the orthonormal frame field \( \{ \gamma_M \} \).

When the derivative \( \partial_M \) acts on a polyvector valued field \( A = A^N \gamma_N \) we obtain

\[
\partial_M (A^N \gamma_N) = \partial_M A^N \gamma_N + A^N \partial_M \gamma_N = (\partial_M A^N + \Gamma^N_M K A^K_M) \gamma_N \equiv D_M A^N \gamma_N \quad (99)
\]

where \( D_M A^N \equiv \partial_M A^N + \Gamma^N_M K A^K_M \) are components of the covariant derivative in the coordinate basis, i.e., the ‘covariant derivative’ of the tensor analysis.

Here \( A^N \) are \textit{scalar} components of \( A \), and \( \partial_M A^N \) is just the ordinary partial derivative with respect to \( x^M \):

\[
\partial_M \equiv \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots \frac{\partial}{\partial x^n} \right) \quad (100)
\]

The derivative \( \partial_M \) behaves as a \textit{partial derivative} when acting on scalars, and it defines a \textit{connection} when acting on a basis \( \{ \gamma_M \} \). It has turned out very practical\(^\text{13}\) to use the

\(^{12}\) This operator is a generalization to curved C-space of the derivative \( \partial_\mu \), which acts in an \( n \)-dimensional space \( V_n \), and was defined by Hestenes \[9\] (who used a different symbol, namely \( \square_\mu \)).

\(^{13}\) Especially when doing long calculation (which is usually the job of a theoretical physicist) it is much easier and quicker to write \( \partial_M \) than \( \square_M, \nabla_M, D_M, \nabla_{\gamma_M} \) which all are symbols used in the literature.
easily writable symbol $\partial_M$ which —when acting on a polyvector— cannot be confused with partial derivative. In ref. [29] we provided some arguments, why also conceptually it is more appropriate to retain the same symbol $\partial_M$, regardless of whether it acts on scalar, vector, or generic polyvector fields.

The derivative $\partial_M$ is defined with respect to a coordinate frame field $\{\gamma_M\}$ in $C$-space. We can define a more fundamental derivative $\partial$ by

$$\partial = \gamma^M \partial_M$$

(101)

This is the gradient in $C$-space and it generalizes the ordinary gradient $\gamma^\mu \partial_\mu$, $\mu = 0, 1, 2, \ldots, n - 1$, discussed by Hestenes [9].

Besides the basis elements $\gamma_M$ and $\gamma_M$, we can define the reciprocal elements $\gamma^M$, $\gamma^M$ by the relations

$$(\gamma^M)\dagger \gamma_N = \delta^M_N, \quad (\gamma^M)\dagger \gamma_N = \delta^M_N$$

(102)

**Curvature.** We define the curvature of $C$-space in the analogous way as in the ordinary spacetime, namely by employing the commutator of the derivatives [9, 12, 35]. Using eq. (97) we have

$$[\partial_M, \partial_N] \gamma_J = R_{MNJK} \gamma_K$$

(103)

where

$$R_{MNJK} = \partial_M \Gamma^K_{NJ} - \partial_N \Gamma^K_{MJ} + \Gamma^K_{NJ} \Gamma^K_{MR} - \Gamma^K_{MJ} \Gamma^K_{NR}$$

(104)

is the curvature of $C$-space. Using (103) we can express the curvature according to

$$(\gamma^K)\dagger \star ([\partial_M, \partial_N] \gamma_J) = R_{MNJK}$$

(105)

An analogous relation we have if the commutator of the derivatives operates on an orthonormal basis elements and use eq. (98).

**5.2 On the dynamical curved $C$-space with sources**

We will assume that physics takes place in curved $C$-space. The latter space is a generalization of curved spacetime. As in the ordinary general relativity the metric $g_{\mu\nu}$ is a dynamical quantity, so in the generalized general relativity the $C$-space metric $G_{MN}$ is a dynamical quantity. Instead of the usual point particles and branes the role of the sources is now assumed by the generalized branes (which include the generalized point particles) described by one of the classically equivalent actions (69), (73) or (76), which we will denote by a common symbol $I_{\text{Brane}}$.

The action thus contains a term which describes a $C$-space brane and a kinetic term which describes the dynamics of the $C$-space itself:

$$I = I_{\text{Brane}} + \frac{1}{16\pi G_C} \int \! d^2 x \, \sqrt{|G|} \, R$$

(106)
Here $G_C$ is a constant (having the role of the “gravitational” constant in $C$-space), $G \equiv \det G_{MN}$ the determinant of the $C$-space metric, and $R = R_{MNJK}G^{MJ}G^{NK}$ the curvature scalar of $C$-space.

More details about how to proceed with and further generalize the theory based on the action (106) are to be found in [12]. The idea that the curved $C$-space can provide a realization of Kaluza-Klein theory has been considered in refs. [38, 41, 28, 29]. In this paper we will explore the brane part $I_{\text{Brane}}$ which contains the coupling of the $C$-space brane’s embedding functions $X^M(\xi^A)$ to the $C$-space metric $G_{MN}$.

5.3 The generalized branes in curved $C$-space

We will consider a generalized brane (a $C$-space brane) moving in a fixed curved $C$-space background. We will assume that the action is given by eq. (73) in which there occurs the metric $G_{MN}$ of the curved target $C$-space into which our generalized brane is embedded.

Suppose that in the target $C$-space there exist isometries described by $K$ Clifford numbers $k^\alpha = k^\alpha_M \gamma^M$, $\alpha = 1, 2, ..., K$; $M = 1, 2, ..., 16$, (generalized Killing “vectors”), whose components satisfy the conditions

$$D_M k^\alpha_N + D_N k^\alpha_M = 0 \quad (107)$$

where the covariant derivative $D_M A_N$ of components $A_M$ of an arbitrary polyvector $A$ is given in eq. (99).

Splitting the coordinate basis and orthonormal basis indices according to

(i) coordinate basis indices: $M = (\mu, \bar{M}), \mu = 0, 1, 2, 3; \bar{M} = 4, 5, ..., 16$

(ii) orthonormal basis indices: $\mathcal{M} = (\mu, \bar{M}), \mu = 0, 1, 2, 3; \bar{M} = 4, 5, ..., 16$

the metric and vielbein can be written as

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\mu\bar{M}} \\ G_{\bar{M}\nu} & G_{\bar{M}\bar{M}} \end{pmatrix}, \quad e^\mathcal{M}_M = \begin{pmatrix} e^\mu_{\mu} & e^\mu_{\bar{M}} \\ e_{\mu}^\mu & e_{\bar{M}}^\bar{M} \end{pmatrix} \quad (108)$$

Let us recall that the vielbein according to eq. (84) can be written as the scalar product of the $C$-space coordinate and orthonormal basis elements:

$$e_M^\mathcal{M} = \gamma^\mathcal{M}_M * \gamma^M \quad (109)$$

We can now assume that the orthonormal basis $\{\gamma_M\}$ is chosen so that $\gamma_{\mu}, \mu = 0, 1, 2, 3$, are tangent vectors to the spacetime (a subspace of $C$-space), whilst the remaining basis elements $\gamma_{\bar{M}}, \bar{M} = 4, 5, ..., 16$ are tangent to the “internal” part of $C$-space. Since the basis is orthogonal, we have

$$\gamma^\dagger_\mu * \gamma^M = 0 \quad (110)$$

Next we assume that the coordinate basis $\{\gamma_M\}$ is chosen so that $\gamma_\mu, \mu = 0, 1, 2, 3$ in general is not tangent to $V_4$, whilst the remaining coordinate basis elements $\gamma_{\bar{M}}, \bar{M} = 4, 5, ..., 16$
4, 5, ..., 16, are tangent to the “internal” part of C-space. This means that $\gamma_{\mu} \ast \gamma_{\bar{M}} \neq 0$, whilst

$$\gamma_{\mu} ^{\dagger} \ast \gamma_{\bar{M}} = 0 \tag{111}$$

The latter equation can be written as $\eta_{\mu\nu} \gamma^{\nu} \ast \gamma_{\bar{M}} = \eta_{\mu\nu} e^{\nu} \bar{M} = 0$. Since $\eta_{\mu\nu}$ is diagonal, it follows that

$$e^{\mu} \bar{M} = 0 \tag{112}$$

Taking a coordinate system in which $k_{\alpha\mu} = 0$, $k_{\alpha} \bar{M} \neq 0 \tag{113}$

the components $e_{\bar{M}} \mu$ can be written in terms of the Killing vectors and gauge fields $A_{\mu}^{\alpha}(x^{\mu})$:

$$e_{\bar{M}} \mu = e_{\bar{M}} \bar{M} k_{\alpha} \bar{M} A_{\mu}^{\alpha}; \quad \partial_{\bar{M}} A_{\mu}^{\alpha} = 0 \tag{114}$$

For the “mixed” components of the inverse vielbein we find

$$e^{\mu} \bar{M} = 0 \tag{115}$$

This follows directly from

$$\gamma_{\mu} ^{\dagger} \ast \gamma_{\bar{M}} = 0 = \gamma_{\mu} ^{\dagger} (e_{\bar{M}} \bar{M} \gamma_{\bar{M}}) = \gamma_{\mu} ^{\dagger} (e^{\mu} \bar{M} \gamma_{\mu} + e^{\bar{M}} \bar{M} \gamma_{\bar{M}}) = e^{\mu} \bar{M} e^{\bar{M}} \bar{M} \tag{116}$$

where we have used eq. (112). Since in general $e^{\bar{M}} \bar{M} \neq 0$, it follows that $e^{\mu} \bar{M}$ is equal to zero.

Using (115), (112) and (115) and the above choice of coordinates and local orthonormal frame we find that the quadratic form can be written as the sum of the 4-dimensional quadratic form and the part due to the “extra dimensions”:

$$dX_{\dagger} \ast dX = (dx^{\bar{M}} \gamma_{\bar{M}}) ^{\dagger} \ast (dx^{N} \gamma_{N}) = g_{\mu\nu} dx^{\mu} dx^{\nu} + \phi^{\bar{M} \bar{N}} dx_{\bar{M}} dx_{\bar{N}} \tag{117}$$

where

$$g_{\mu\nu} = e_{\mu}^{\nu} e_{\nu}^{\gamma} \gamma_{\nu} = G_{\mu\nu} - \phi^{\bar{M} \bar{N}} k_{\alpha} \bar{M} k_{\beta} \bar{N} A_{\mu}^{\alpha} A_{\nu}^{\beta} \quad \text{and} \quad \phi^{\bar{M} \bar{N}} \equiv e^{\bar{M}} \bar{M} e^{\bar{N}} \bar{N} \eta^{\bar{M} \bar{N}} \tag{118}$$

Here $G_{\bar{M} \bar{N}} \equiv \phi_{\bar{M} \bar{N}} = e^{\bar{M}} \bar{M} e^{\bar{N}} \bar{N} \eta_{\bar{M} \bar{N}}$, and $\phi^{\bar{M} \bar{N}}$ is the inverse of $\phi_{\bar{M} \bar{N}}$ in the “internal” space. Notice the validity of eqs. (112) and (115).

Let us now use the splitting (117) in the brane action (73). We obtain

$$\partial_{A} X^{M} \partial_{B} X^{N} G_{MN} = \partial_{A} X^{\mu} \partial_{B} X^{\nu} g_{\mu\nu} + \partial_{A} X_{\bar{M}} \partial_{B} X_{\bar{N}} \phi^{\bar{M} \bar{N}} \tag{119}$$

The auxiliary variables $G_{AB}$ and the induced metric on the (generalized, i.e., C-space) brane are related according to eq. (73). Let us introduce new auxiliary variables $G'_{AB}$ and new brane tension $T'$ according to

$$G_{AB} = G'_{AB} + \partial_{A} X_{\bar{M}} \partial_{B} X_{\bar{N}} \phi^{\bar{M} \bar{N}} \tag{120}$$
\[ T \sqrt{|G|} G^{AB} = T' \sqrt{|G'|} G'^{AB} \]  

(121)

where \(G^{AB}\) and \(G'^{AB}\) are the inverse matrices of \(G_{AB}\) and \(G'_{AB}\), respectively.

Using eqs. (120), (121) we have

\[ T \sqrt{|G|} d_C = T \sqrt{|G|} G^{AB} G_{AB} = T' \sqrt{|G'|} G'^{AB} (G'_{AB} + \partial_A X_M \partial_B X_N \Phi^{MN}) \]  

(122)

\[ T \sqrt{|G|} = \frac{T \sqrt{|G|} d_C}{d_C} = \frac{T \sqrt{|G|} G^{AB} G_{AB}}{d_C} = \frac{T' \sqrt{|G'|} G'^{AB} (G'_{AB} + \partial_A X_M \partial_B X_N \Phi^{MN})}{d_C} \]  

(123)

From eqs. (122), (123) we find that the extra term in the brane action (123) can be written

\[ T \sqrt{|G|} (2 - d_C) = T' \sqrt{|G'|} (2 - d_C) + P^A \bar{M} \partial_A X_M \Phi^{MN} \left( \frac{2}{d_C} - 1 \right) \]  

(124)

where we have written

\[ T \sqrt{|G|} G^{AB} \partial_A X_M = T' \sqrt{|G'|} G'^{AB} \partial_A X_M = P^A \bar{M} \]  

(125)

Here \(P^A \bar{M}\) are extra components of the brane canonical momentum \(P^A \bar{M} = \partial \mathcal{L} / \partial \partial_B X^M = T \sqrt{|G|} G^{AB} \partial_A X_M\).

Inserting eqs. (120), (121) and (124) into the brane action (123) we obtain

\[ I_{\text{Brane}}[X^\mu, G^{AB}] = \frac{1}{2} \int d^d \xi T' \sqrt{|G'|} \left( G'^{AB} \partial_A X^\mu \partial_B X^\nu g_{\mu\nu} + 2 - d_C \right) \]  

\[ + \frac{1}{d_C} \int d^d \xi P^A \bar{M} \partial_A X_M \Phi^{\bar{M} \bar{N}} \]  

(126)

Variation of the latter action with respect to \(G'^{AB}\) gives

\[ G'^{AB} = \partial_A X^\mu \partial_B X^\nu g_{\mu\nu} \]  

(127)

which is consistent with eqs. (75), (119), (120). Since the index \(\mu\) runs over the embedding spacetime coordinates \(\mu = 0, 1, 2, ..., N-1\) and \(A, B = 1, 2, ..., d_C\) run over the coordinates of the brane’s \(C\)-space worldsheet \(C_{V^d}\) of dimension \(d_C = 2^n\), we have that the system of equations (127) is determined, if \(2^n \leq N\). In the case \(N = 4\) we have that \(2^n \leq 4\), which is satisfied for \(n \leq 2\). In 4-dimensional spacetime the splitting described above works for point particles \((n = 1)\) and strings \((n = 2)\). Higher dimensional branes, that do not satisfy \(2^n \leq N\), of course, can also exist, but \(G'^{AB}\) then cannot have the role of auxiliary variables, because the system of equations (127) is then underdetermined. In order to obtain a determined system for auxiliary variables, a splitting of the indices \(A, B,\) analogous to that of \(M, N\) is then needed as well.

In eq. (126) we have an action for a generalized \(C\)-space brane worldsheet \(C_{V^d}\) embedded in a higher dimensional curved manifold (which in our case is the embedding \(C\)-space.
In particular, if we have just a usual point particle sweeping a 1-dimensional ‘worldsheet’, i.e., a worldline, embedded in a higher dimensional curved space, which can be either a $C$-space or just simply a spacetime with extra dimensions, then we have to take $d_C = 1$, and the action (126) becomes

$$I[X^\mu, \Lambda] = \frac{M}{2} \int d\tau \Lambda \left( \frac{\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}}{\Lambda^2} + 1 \right) + \int d\tau P^\bar{M}_\bar{N} \dot{X}_N \Phi^{\bar{M} \bar{N}}$$ \hspace{1cm} (128)

Above we have denoted

$$G^\prime_{AB} = G^\prime_{11} \equiv \Lambda^2, \quad G'^{AB} = G'^{11} \equiv \frac{1}{\Lambda^2}, \quad T' \equiv M$$

$$\partial_A X^M = \partial_1 X^M \equiv \frac{dX^M}{d\tau} \equiv \dot{X}^M$$

$$P^A_{\bar{M}} = P^1_{\bar{M}} \equiv P_{\bar{M}}$$ \hspace{1cm} (129)

The action (128) can also be obtained by splitting the following point particle action:

$$I[X^M, \lambda] = \frac{m}{2} \int d\tau \lambda \left( \frac{\dot{X}^M \dot{X}^N G_{MN}}{\lambda^2} + 1 \right)$$ \hspace{1cm} (130)

Variation of the latter action with respect to $\lambda$ gives

$$\lambda^2 = \dot{X}^M \dot{X}_M$$ \hspace{1cm} (131)

which can be split according to

$$\dot{X}^M \dot{X}_M = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \dot{X}_\bar{M} \dot{X}_{\bar{N}} \Phi^{\bar{M} \bar{N}}$$ \hspace{1cm} (132)

Introducing a new auxiliary variable $\Lambda$ and new mass $M$ (a 4-dimensional mass) according to

$$\lambda^2 = \Lambda^2 + \dot{X}_\bar{M} \dot{X}_{\bar{N}} \Phi^{\bar{M} \bar{N}}$$ \hspace{1cm} (133)

$$\frac{m}{\lambda} = \frac{M}{\Lambda}$$ \hspace{1cm} (134)

and inserting eqs. (132)–(134) into the point particle action (130) we obtain the split point particle action (128), where

$$P_{\bar{M}} = \frac{m}{\lambda} \dot{X}_{\bar{M}} = \frac{M}{\Lambda} \dot{X}_{\bar{M}}$$ \hspace{1cm} (135)

So we have verified that the split generalized brane action (126) includes a correct point particle action. Relations (132)–(135) above are point particle analog of the brane relations (119)–(121), (125).
Let us now return to the generalized brane action (126). The extra term can be written as

$$P^A_M \partial_A X_N \Phi^{MN} = P^A_M \Phi^{MN} G_N J \partial_A X^J = P^A_M \Phi^{MN} A^\alpha J k_{N\alpha} \partial_A X^J$$

$$= J^A_\alpha A^\alpha J \partial_A X^J = J^A_\alpha (A^\alpha \mu \partial_A X^\mu + A^\alpha_M \partial_A X^M)$$

where

$$J^A_\alpha \equiv P^A_M \Phi^{MN} k_{N\alpha} = P^A_M k^M_\alpha = P^A_M k^M_\alpha$$

are current densities which are conserved due to the presence of isometries. The last step in eq. (137) holds because of eq. (113). In eq. (136) we have written the metric components in terms of gauge potentials and Killing “vectors”

$$G_{\tilde{N}J} = A^\alpha J k_{N\alpha}$$

In particular,

$$G_{\tilde{N}\mu} = A^\alpha _\mu k_{N\alpha} \quad \text{if} \quad J = \mu$$

$$G_{\tilde{N}\tilde{M}} = A^\alpha _{\tilde{M}} k_{N\alpha} \quad \text{if} \quad J = \tilde{M}$$

In eq. (139) we have the well known relation between gauge potentials, Killing “vectors” and “mixed” components of the metric, the relation that was derived (see e.g., [39, 40]) within the context of an ordinary higher dimensional spacetime manifold. In eq. (140) we have rewritten the “internal” space metric components in terms of gauge potentials and Killing “vectors”.

Inserting eq. (136) into eq. (126) we obtain

$$I_{\text{Brane}}[X^\mu, G^{AB}] = \frac{1}{2} \int d^d \xi T^\nu \sqrt{|G^\gamma|} \left( G^{AB} \partial_A X^\mu \partial_B X^\nu g_{\mu\nu} + 2 - d_C \right) + \frac{1}{d_C} \int d^d \xi J^A_\alpha A^\alpha_M \partial_A X^M$$

The last term is just the interactive term between the conserved current densities $J^A_\alpha$ and gauge potentials $A^\alpha_M$ coupled to

$$\partial_A X^M \equiv \partial_{a_1...a_r} X^{\mu_1...\mu_R}, \quad r = 0, 1, 2, ..., n; \quad R = 0, 1, 2, ..., N$$

The latter potentials include the ordinary nonabelian potentials $A^\alpha _\mu$, $\alpha = 1, 2, ..., K$, coupled to $\partial_A X^\mu$, and the extra potentials $A^\alpha _M$, coupled to $\partial_A X^M$. In sec. we argued that in a particular parametrization of $\xi^A = (\xi, \xi^a, \xi a, \xi_{a_1}, ...)$, if we take $r = R = 0, 1, 2, ..., n$, we can set

$$\partial_{a_1...a_r} X^{\mu_1...\mu_r} = \partial_{[a_1} X^{\mu_1} \partial_{a_{r-1}|} X^{\mu_r}$$

If $r = 1$, the above relation is automatically true, because we have just $\partial_{a_1} X^{\mu_1}$ on both sides of the equation. If $r = 0$, then we have derivative with respect to scalar parameter
ξ, and eq. (143) is automatically satisfied as well. But for higher grades, \( r = 2, 3, \ldots, n \), eq. (143) must be imposed as an extra condition, namely a condition that fixes a gauge.

If we assume the validity of relation (143), we find that the interactive term in eq. (141), namely

\[
I_{\text{int}} = \frac{1}{dC} \int d^d \xi J^A_{\alpha} A^\alpha_M \partial_A X^M = \frac{1}{dC} \int d^d \xi J^A_{\alpha} (A^\alpha_\mu \partial_A X^\mu + A^\alpha_{\bar{M}} \partial_A X^{\bar{M}})
\]

contains the coupling of the antisymmetric gauge potentials to the antisymmetric current density. In the case of a single Killing “vector” field, \( \alpha = 1 \), the gauge fields \( A^\mu_{\mu_1 \ldots \mu_R} \) are abelian, and for \( R = r = n \), eq. (144) becomes the interactive term for the well known Kalb-Ramond fields [42]. The latter fields have an important role in string theories and brane theories [4]. Here we have demonstrated a possible broader theoretical framework for generalized branes, coupled to generalized gauge fields, which includes strings and Kalb-Ramond fields. An alternative approach to generalized gauge (Maxwell) fields in C-space has been considered in refs. [43].

To sum up, the action (141) contains the coupling of generalized gauge fields with the charge current density. Besides the ordinary gauge fields \( A^\alpha_\mu \) there also occur higher grade, Kalb-Ramond fields \( A^\alpha_{\bar{M}} \equiv A^\alpha_{\mu_1 \ldots \mu_R} \) and the zero grade, scalar, field \( A^\alpha_0 \). All those fields are included in the compact coupling term (144), where \( R = 0 \) stands for the scalar component, \( R = 1 \) for vector and \( R = 2, 3, \ldots, N \) for higher grade components, and analogously for \( r = 0, 1, 2, \ldots, n \). Eq. (144) contains the non abelian generalization of the well known coupling term for Kalb-Ramond fields considered in refs. [42, 44, 8]. The latter coupling occurs as a special case of eq. (144), if we take the terms with the grade \( r = R = n \) only, i.e., the terms with the maximal grade of the worldsheet multivectors. However, more general coupling terms with \( r \neq R \neq n \) also exists in our theory. Thus the case for \( r = n, R = 1 \) includes the well known electrically charged closed membrane considered by Dirac [45]. In the following subsection we will discuss more explicitly some particular cases of our general theory, that were previously considered in the literature as separate subjects.

5.4 Comparison with previous theories of charged branes

The electromagnetic potential \( A_\mu \) couples to the time like tangent element of the worldline \( X^\mu(\tau) \) of a charged particle:

\[
I_{\text{int}}^{\text{particle}} = q \int d\tau \frac{\partial X^\mu}{\partial \tau} A_\mu
\]

where \( q \) is the particle’s electric charge.
For extended objects the coupling involves the worldsheet tangent elements (velocities) and generalized Maxwell potentials \( A_{\mu_1...\mu_n} \). The corresponding equation for a string is

\[
I_{\text{string int}} = \frac{q^{12}}{2!} \int d^2\xi \partial_1 X^\mu \partial_2 X^\nu A_{\mu\nu}
\]  

(146)

where the two-vector charge \( q^{12} \) is coupled to a two-vector potential \( A_{\mu\nu} \). For a generic brane the coupling reads

\[
I_{\text{brane int}} = \frac{1}{n!} q^{12...n} \int d^n\xi \partial_{[1} X^{\mu_1} ... \partial_{n]} X^\nu A_{\mu_1...\mu_n}
\]  

(147)

Eq. (147) can be rewritten as

\[
I_{\text{brane int}} = \left( \frac{1}{n!} \right)^2 q^{a_1 a_2...a_n} \int d^n\xi \partial_{[a_1} X^{\mu_1} ... \partial_{a_n]} X^\nu A_{\mu_1...\mu_n}
\]  

(148)

Here \( q, q^{a_1 a_2,..., q^{a_1 a_2...a_n} } \) are the coupling strengths.

If we introduce \( q_n \equiv \frac{1}{n!} q^{a_1...a_n} \epsilon_{a_1...a_n} \) and use eqs. (46), (50) we have

\[
I_{\text{brane int}} = \frac{1}{n!} q_n \int d\tilde{\xi} \frac{\partial X^{\mu_1...\mu_n}}{\partial \tilde{\xi}} A_{\mu_1...\mu_n}
\]  

(150)

In the case considered above the vector potential \( A_\mu \) couples to the vector tangent element of the particle’s world line, the 2-vector potential \( A_{\mu\nu} \) couples to the 2-vector tangent element of the string’s world sheet, and in general, the \( n \)-vector potential \( A_{\mu_1...\mu_n} \) couples to the \( n \)-vector tangent element of the brane’s worldsheet. By this the possible couplings are not exhausted. Long time ago Dirac considered a relativistic charged closed membrane coupled to the ordinary Maxwell field given in terms of the vector potential \( A_\mu \). Later, this theory has been generalized \[46, 47\] to closed branes of any dimension.

It is well known that for open strings and membranes the electric charge \( q \) —due to its repulsive character— can only be concentrated on the boundary (e.g., at the string’s ends). But for closed strings and, in general, closed \( p \)-branes, \( q \) can be distributed over such extended objects. So instead of the total charge \( q \) we have to introduce the charge density. But since the brane moves, we have the charge current density on the brane. In a covariant description we introduce the charge current density \( j^a \) on the brane’s worldsheet \( V_n \). In ref. \[47\] the following action was considered:

\[
I[X^\mu, A_\mu] = \int d^n\xi \left( \kappa \sqrt{\left| \det \partial_a X^\mu \partial_b X_\mu \right| + j^a \partial_a X^\mu A_\mu} \right) + \frac{1}{4\pi} \int d^N x \sqrt{|g|} F_{\mu\nu} F^{\mu\nu}
\]  

(151)

The first term in eq. (151) is just the minimal surface Dirac-Nambu-Goto action for a \( p \)-brane \( (n = p + 1) \), whilst the second term represents the minimal coupling of the brane’s
electric charge current density $j^a$ with the electromagnetic field potential $A_\mu$. The last term is the kinetic term for the electromagnetic field $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The action (151) has the following transformation property:

$$I[X^\mu, A'_\mu] = I[X^\mu, A_\mu] + \int d^n \xi \, j^a \partial_a X^\mu \partial_\mu \varphi = I[X^\mu, A_\mu] + \int d^n \xi \, \partial_a (j^a \varphi)$$

where

$$A'_\mu = A_\mu + \partial_\mu \varphi$$

and

$$\partial_a j^a = 0$$

Eq.(4.8) is the gauge transformation of $A_\mu$, whilst eq.(154) expresses the charge conservation. So we have

$$\int d^n \xi \, \partial_a j^a = \oint d\Sigma_a j^a = q(\Sigma_2) - q(\Sigma_1) = 0$$

where $q = \int d\Sigma_a j^a$ is the total charge of the closed brane and $d\Sigma_a$ the hypersurface element. Here $j^a$ is a worldsheet vector density (not vector), so that (153) is covariant under reparametrizations of $V_n$.

Here $j^a$ is the intrinsic current density, a vector density on $V_n$. By the relation

$$j^\mu = \int d^n \xi \, \delta(x - X(\xi)) j^a \partial_a X^\mu$$

we obtain the extrinsic current density, i.e., the current density in the target space $V_N$. The minimal coupling Lagrangian in (151) then reads

$$\int d^n \xi \, j^a \partial_a X^\mu A_\mu = \int d^N x \, j^\mu A_\mu$$

It is straightforward to show that the conservation law (154) implies

$$\partial_\mu j^\mu = 0$$

The transformed action (which is a functional of $X^\mu$ and $A'_\mu$) differs from the “original” action $I[X^\mu, A_\mu]$ by a term with the total divergence which has no influence on the equations of motion. The canonical momentum density is

$$p_\mu^a = \frac{\mathcal{L}}{\partial_{\partial_a X^\mu}} = \kappa \sqrt{|f|} \partial^a X_\mu + j^a A_\mu$$

and it obviously is not invariant under gauge transformations (153). Therefore it is convenient to introduce the kinetic momentum density

$$\pi_\mu^a = p_\mu^a - j^a A_\mu$$
which is gauge invariant. By employing (160) we can write the phase space action

\begin{equation}
I_{\text{Brane}}[X^\mu, p_{\mu}^a, \gamma_{ab}] = \int d^n \xi \left[ p_{\mu}^a \partial_a X^\mu - \frac{1}{2} \frac{\gamma_{ab}}{\kappa \sqrt{\gamma}} (\pi_{\mu}^a \pi_\nu^b g^{\mu\nu} - \kappa^2 |\gamma| \gamma^{ab}) + \kappa \sqrt{|\gamma|} (1 - n) \right]
\end{equation}

(161)

After eliminating \( p_{\mu}^a \) from its equations of motion

\begin{equation}
p_{\nu}^c - j_c A^\nu = \kappa \sqrt{|\gamma|} \partial_c X^\nu
\end{equation}

(162)

we obtain \([47, 12]\) the Howe-Tucker action in the presence of electromagnetic field:

\begin{equation}
I_{\text{Brane}}[X^\mu, \gamma_{ab}] = \int d^n \xi \left[ \frac{\kappa \sqrt{|\gamma|}}{2} (\gamma_{ab} \partial_a X^\mu \partial_b X^\mu + 2 - n) + j^a \partial_a X^\mu A_\mu \right]
\end{equation}

(163)

The total action action is thus the sum of the brane action \( I_{\text{Brane}} \) and the kinetic term for the electromagnetic field \( I_{\text{EM}} \) (given by the last term in \(151\)).

A closed brane can thus possess two different kinds of charges coupled to two different kinds of gauge fields:

(i) Kalb-Ramond charge \( q^{12...n} \) coupled to Kalb-Ramond gauge field potentials \( A_{\mu_1...\mu_n} \) according to eq.(147);

(ii) the ordinary electric charge \( q \) coupled to the ordinary Maxwell potential \( A_\mu \) according to eq.(151) (or equivalently, eqs. (161) and (163)).

By inspecting the interactive action (147) we observe that it is just a coupling term that can be aded to the action (40). On the other hand, the action (163) is just a generalization of the action (28) to which we added the minimal coupling term for the Maxwell field \( A_\mu \). Both actions, (28) and (40), were generalized to \( C \)-space, and so we have obtained the action (69) (which is equivalent to the action (73)). As we have seen in Sec. 5.3, the two kinds of couplings, namely (i) and (ii), can be unified by employing the \( C \)-space description, in which they arise from the metric of the target \( C \)-space \( C_{V_N} \), equipped with connection, whose curvature was assumed to be in general different from zero.

6 Discussion

Spacetime as a continuum of points is just a start, from which we can arrive at a continuum of oriented areas, also called oriented volumes. Such enlarged continuum, called Clifford space or \( C \)-space provides a framework for generalizing the concepts of event, point particle, string and, in general, brane. As an ordinary brane is an object that extends in
spacetime, so a generalized brane is an object that extends in $C$-space. An important feature of $C$-space is that oriented areas (volumes) of different grades, associated with branes of different dimensionalities, can be transformed into each other. A generic geometric object has mixed grade, it is represented by a Clifford number, also called Clifford aggregate or polyvector, and it is associated with a generalized brane. Having set such a kinematics, one can construct a dynamics which employs the well known Howe-Tucker brane action, which is now generalized to $C$-space. The latter space can be curved, and this gives rise, à la Kaluza-Klein, to gauge fields as components of the $C$-space metric. The $C$-space Howe-Tucker action, minimally coupled to the $C$-space metric, contains, in particular, the well known terms for the coupling of a $p$-brane (including a point particle) to gauge fields. Amongst the latter gauge fields there also occur the Kalb-Ramond antisymmetric fields, and their non Abelian generalizations [48]. In this paper we focused our attention to the dynamics of the branes, and left aside the fact that in a more complete treatment [28, 29] $C$-space itself becomes dynamical, as in general relativity.

In the present paper we presented only a piece of the story, namely, how the classical theory of branes in a fixed background could be generalized to Clifford space. Since with the points of a flat Clifford space one can associate Clifford numbers (polyvectors), that are elements of Clifford algebra $C_N$, this automatically brings spinors (as members of left or right ideals of $C_N$) into our description. A polyvector $X^{\mu_1 \cdots \mu_R \gamma \mu_1 \cdots \mu_R}$, since it can be rewritten, e.g., in terms of a basis spanning all independent left ideals, thus contains spinorial degrees of freedom [28, 29, 16]. This means that by describing our branes in terms of the $C$-space embedding functions $X^M \equiv X^{\mu_1 \cdots \mu_R}$ we have already included spinorial degrees of freedom. We do not need to postulate them separately, as is done in ordinary string and brane theories, where besides Grassmann even (“bosonic”) variables $X^\mu$, there occur also Grassmann odd (“fermionic”) or spinorial variables. In this formulation we have a possible clue to the resolution of a big open problem, namely, what exactly is string theory. I believe that further research into that direction would provide very fruitful results and insight. An important insight is already in recognizing that 16-dimensional Clifford space provides a consistent description of quantized string theory [38, 16]. The underlying spacetime can remain 4-dimensional, there is no need for a 26-dimensional or a 10-dimensional spacetime. The extra degrees of freedom required for consistency of string theory, described in terms of variables $X^M(\tau, \sigma)$, are due to extra dimensions of $C$-space, and they need not be compactified; they are due to volume (area) evolution, and are thus all physical. But since a generic component $X^M(\tau, \sigma) \equiv X^{\mu_1 \cdots \mu_R}(\tau, \sigma)$ denotes an oriented $R$-volume, associated with an $(R-1)$-brane (i.e., a $p$-brane for $p = R - 1$), we have that string itself (i.e., 1-brane) is not enough for consistency. Higher branes are automatically present in the description with functions $X^{\mu_1 \cdots \mu_R}(\tau, \sigma)$, although they are not described in full detail, but only up to the knowledge of oriented $R$-volume. Because
of the presence of two parameters $\tau$, $\sigma$, we keep on talking about evolving strings, not in 26 or 10-dimensional spacetime, but in 16-dimensional Clifford space. In general, the number of parameters can be arbitrary, but less then 16, and so we have a brane in $C$-space, i.e., a generalized brane discussed in this paper.

The $C$-space approach to branes is possibly related to the approach with extra coordinates, considered by Duff and Lu [5] in order to describe brane dualities. Such interesting relation needs to be investigated in future research.

Clifford algebras in infinite dimensions and in continuous dimensions are still an uncharted territory worth exploring. A pioneering step into that direction has been done in ref. [12], where a theory of generalized branes was formulated in terms of the generators of the infinite dimensional Clifford algebra.

Another possible direction that remains to be further explored is related to the fact that p-brane actions can be recast as non Abelian gauge theories of volume preserving diffeomorphisms [8, 49]. It would be interesting to extend such approach to the case of the generalized branes in $C$-space considered in this paper. Also there is a number of works on polyvector generalized supersymmetries [50], on the implications to M theory [51], on generalized Yang-Mills theories, Poly-particles and duality in C-spaces [52, 48], and on quantum mechanics in C-spaces and non-commutativity [53]. A possible connection of C-space to twistors (see, e.g., [54]) would also be worth investigating.

Clifford algebra, without recourse to $C$-space and generalized branes, has been considered in numerous works attempting to explain the standard model [55]. There are diverse approaches, but common to all of them is a feeling that Clifford algebra might be a clue to the unification of fundamental forces. It is reasonable to expect that further research will bring useful results, and that it will be crucial to take into account the concepts of $C$-space and branes.

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