Izergin-Korepin approach to symmetric functions

Kohei Motegi\textsuperscript{1} and Kazumitsu Sakai\textsuperscript{2}

\textsuperscript{1} Faculty of Marine Technology, Tokyo University of Marine Science and Technology, Etchujima 2-1-6, Koto-Ku, Tokyo, 135-8533, Japan
\textsuperscript{2} Department of Physics, Tokyo University of Science, Kagurazaka 1-3, Shinjuku-ku, Tokyo, 162-8601, Japan

E-mail: \textsuperscript{1} kmoteg0@kaiyodai.ac.jp \textsuperscript{2} k.sakai@rs.tus.ac.jp

Abstract. Recently, the Izergin-Korepin technique, which was originally a method to analyze the domain wall boundary partition functions initiated by Korepin and Izergin, was extended to the wavefunctions of integrable six-vertex models. We illustrate for the case of the rational integrable models.

1. Introduction

Quantum inverse scattering method [1, 2, 3] is one of the traditional methods to study quantum integrable models. In the early days of the birth of the quantum inverse scattering method, Korepin [4] introduced the domain wall boundary partition functions of the six-vertex model, and at the same time he introduced a method to extract the properties which uniquely define the polynomials representing the partition functions. Later, Izergin [5] found the explicit determinant form (Izergin-Korepin determinant) satisfying the properties. The Izergin-Korepin determinant has found applications to the problem of the enumeration of the alternating sign matrices [6, 7]. Also, the Izergin-Korepin technique has been extended to the variations of the domain wall boundary partition functions [7, 8], scalar products [9], and to the Andrews-Baxter-Forrester elliptic integrable model [10] in [11, 12, 13]. The domain wall boundary partition functions for other class of integrable models such as the Perk-Schultz model, the Felderhof model and their elliptic analogues (Okado-Deguchi-Fujii-Martin model, Foda-Wheeler-Zuparic model) [14, 15, 16, 17, 18, 19, 20, 24] have also been investigated [21, 22, 23, 24]. And we recently found a way to extend to the wavefunctions [25, 26, 27, 28, 56]. Today, the wavefunctions are getting paid attention due to their connections with mathematics, in particular with combinatorics and representation theory. This is because the explicit representations of the wavefunctions are expressed as symmetric functions, and the symmetric functions which appear are not only the celebrated ones such as the Schur, Hall-Littlewood, Grothendieck polynomials but also their quantum group deformations and elliptic analogues, and one can study the symmetric functions by using this correspondence. We gave a detailed study for the Izergin-Korepin approach to the wavefunctions for several trigonometric and elliptic models and boundary conditions in [25]. In this article, we illustrate the case for the rational integrable models which were not treated in the paper. Note that there are several other types of techniques of the quantum inverse scattering method developed to analyze the wavefunctions. Today there are extensive studies on this subject. See [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51], and references therein for example.
2. Rational \( L \)-operator

Let us first introduce two types of two-dimensional vector spaces \( W_a \) and \( F_j \), and denote the orthonormal basis of \( W_a \) and its dual as \( \{ 0 \}_{a}, \{ 1 \}_{a} \) and \( \{ 0 \}_{a}, \{ 1 \}_{a} \), the orthonormal basis of \( F_j \) and its dual as \( \{ 0 \}_{j}, \{ 1 \}_{j} \) and \( \{ 0 \}_{j}, \{ 1 \}_{j} \).

We also introduce the following Pauli spin operators \( \sigma^+ \) and \( \sigma^- \) as operators acting on the (dual) orthonormal basis as

\[
\sigma^+|0\rangle = |1\rangle, \quad \sigma^+|1\rangle = 0, \quad \langle 0|\sigma^+ = \langle 1|, \quad \langle 1|\sigma^- = \langle 0|, \quad \langle 0|\sigma^- = 0.
\]

We now introduce the following rational \( L \)-operator

\[
L_{a_{j}}(z, w_{j}, a_{j}, b_{j}, c_{j}, d_{j}, e_{j}, f_{j}) = \begin{pmatrix}
\frac{a_{j}(z-w_{j})+b_{j}}{h-1} + a_{j} & 0 & 0 & 0 \\
0 & \frac{a_{j}(z-w_{j})+b_{j}}{h-1} & c_{j} & 0 \\
0 & d_{j} & e_{j}(z-w_{j}) + f_{j} - e_{j} & 0 \\
0 & 0 & 0 & e_{j}(z-w_{j}) + f_{j} \frac{h}{h-1}
\end{pmatrix},
\]

which acts on the tensor product space \( W_a \otimes F_j \), and \( a_{j}, b_{j}, c_{j}, d_{j}, e_{j}, f_{j} \) and \( h \) are constant parameters satisfying the following relations

\[
a_{j}e_{j} - c_{j}d_{j}h = 0, \quad b_{j}e_{j} - a_{j}f_{j} + h(h-1)c_{j}d_{j} = 0.
\]

The \( L \)-operator satisfies the RLL relation

\[
R_{ab}(z_{1} - z_{2})L_{a_{j}}(z_{1}, w_{j})L_{b_{j}}(z_{2}, w_{j}) = L_{b_{j}}(z_{2}, w_{j})L_{a_{j}}(z_{1}, w_{j})R_{ab}(z_{1} - z_{2}),
\]

where \( R_{ab}(z) \) is the following rational \( R \)-matrix

\[
R_{ab}(z) = \begin{pmatrix}
\frac{z}{\bar{h}} + 1 - \frac{1}{\bar{h}} & 0 & 0 & 0 \\
0 & \frac{z}{\bar{h}} & 1 - \frac{1}{\bar{h}} & 0 \\
0 & 1 - \frac{1}{\bar{h}} & \bar{h} & 0 \\
0 & 0 & 0 & \frac{z}{\bar{h}} + 1 - \frac{1}{\bar{h}}
\end{pmatrix},
\]

acting on the tensor product space \( W_a \otimes W_b \).

Note that by setting \( a_{j} = c_{j} = d_{j} = 1 - 1/h, \ b_{j} = w_{j} = 0, \ e_{j} = h - 1, \ f_{j} = (h - 1)^2 \), the rational \( L \)-operator \( L_{a_{j}}(z, w_{j}, a_{j}, b_{j}, c_{j}, d_{j}, e_{j}, f_{j}) \) becomes the rational \( R \)-matrix \( R_{a_{j}}(z) \). Note also the following furthermore limit \( h \to \infty \) of the \( R \)-matrix from the six-vertex model to the five-vertex model

\[
R_{a_{j}}(z)_{h \to \infty} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & z & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

3. Wavefunctions

The wavefunctions are constructed from the \( L \)-operator as follows. First, we introduce the \( B \)-operator

\[
B(z|w_{1}, \ldots, w_{M}) = a \langle 0|L_{a_{M}}(z, w_{M}, a_{M}, b_{M}, c_{M}, d_{M}, e_{M}, f_{M}) \cdots L_{a_{1}}(z, w_{1}, a_{1}, b_{1}, c_{1}, d_{1}, e_{1}, f_{1})|1\rangle_{a}.
\]
The $B$-operators commute with each other

$$[B(z_1|w_1, \ldots, w_M), B(z_j|w_1, \ldots, w_M)] = 0,$$

which follows from the $RLL$ relation (5). The wavefunctions are defined as the following matrix elements of the product of $B$-operators:

$$W_{M,N}(z_1, \ldots, z_N|w_1, \ldots, w_M|x_1, \ldots, x_N) = \langle x_1 \cdots x_N|B(z_1|w_1, \ldots, w_M) \cdots B(z_N|w_1, \ldots, w_M)|0^M \rangle.$$

Here, $|0^M \rangle := |0\rangle_1 \otimes \cdots \otimes |0\rangle_M \in F_1 \otimes \cdots \otimes F_M$, $|0^M \rangle := 1|0\rangle_1 \otimes \cdots \otimes |0\rangle_M \in F_1^* \otimes \cdots \otimes F_M^*$ are the vacuum vector and its dual, and $\langle x_1 \cdots x_N| (1 \leq x_1 < x_2 < \cdots < x_N \leq M)$ are simple states defined as

$$\langle x_1 \cdots x_N| = \langle 0^M | \prod_{j=1}^N \sigma_{x_j}^z \in F_1^* \otimes \cdots \otimes F_M^*.$$

One can show the following properties of the wavefunctions constructed from the rational $L$-operator.

**Proposition 1.** The wavefunctions $W_{M,N}(z_1, \ldots, z_N|w_1, \ldots, w_M|x_1, \ldots, x_N)$ satisfy the following properties.

1. $W_{M,N}(z_1, \ldots, z_N|w_1, \ldots, w_M|x_1, \ldots, x_N)$ are polynomials of degree $N - 1$ in $w_M$ if $x_N = M$.
2. $W_{M,N}(z_1, \ldots, z_N|w_1, \ldots, w_M|x_1, \ldots, x_N)$ are symmetric with respect to $z_j$, $j = 1, \ldots, N$.
3. The following recursive relations between the wavefunctions hold if $x_N = M$:

$$W_{M,N}(z_1, \ldots, z_N|w_1, \ldots, w_M|x_1, \ldots, x_N)|w_N = z_N + h - b_M/a_M = c_M \prod_{j=1}^{N-1} \frac{a_M(z_j - w_N + 1 - h)}{h(h - 1)} \prod_{j=1}^{M-1} \left( \frac{e_j(z_N - w_j) + f_j}{h - 1} - e_j \right) \times W_{M-1,N-1}(z_1, \ldots, z_{N-1}|w_1, \ldots, w_{M-1}|x_1, \ldots, x_{N-1}).$$

If $x_N \neq M$, the following factorizations hold for the wavefunctions:

$$W_{M,N}(z_1, \ldots, z_N|w_1, \ldots, w_M|x_1, \ldots, x_N) = \prod_{k=1}^{N} \left( \frac{a_M(z_k - w_M) + b_M}{h - 1} + a_M \right) W_{M-1,N}(z_1, \ldots, z_N|w_1, \ldots, w_{M-1}|x_1, \ldots, x_N).$$

4. The following expression holds for the case $N = 1$, $x_1 = M$

$$W_{M,1}(z|w_1, \ldots, w_M|M) = c_M \prod_{k=1}^{M-1} \left( \frac{e_k(z - w_k) + f_k}{h - 1} - e_k \right).$$

Proposition 1 is an extension of the celebrated Korepin’s Lemma for the domain wall boundary partition functions to the wavefunctions. The point is that when we deal with the wavefunctions, one has to deal with two cases $x_N = M$ and $x_N \neq M$, and the smaller wavefunctions which are connected with the wavefunctions of the original size are different between the two cases $x_N = M$ and $x_N \neq M$. 


4. Symmetric functions

We introduce the following symmetric functions.

**Definition 2.** We define the following symmetric functions

\[ S_{M,N}(z_1, \ldots, z_n | w_1, \ldots, w_M | x_1, \ldots, x_N) \]

depending on the symmetric variables \( z_1, \ldots, z_n \), complex parameters \( w_1, \ldots, w_M \), and integers \( x_1, \ldots, x_N \) satisfying \( 1 \leq x_1 < \cdots < x_N \leq M \),

\[
S_{M,N}(z_1, \ldots, z_n | w_1, \ldots, w_M | x_1, \ldots, x_N) = \sum_{\sigma \in S_N} \prod_{j=1}^{N} \prod_{k=x_j+1}^{M} \left( \frac{a_k(z_{\sigma(j)} - w_k) + b_k}{h - 1} + a_k \right) \prod_{1 \leq j < k \leq N} \frac{z_{\sigma(j)} - z_{\sigma(k)} + 1 - h}{h(z_{\sigma(j)} - z_{\sigma(k)})} \\
\times \prod_{j=1}^{N} \prod_{k=1}^{x_j-1} \left( \frac{e_k(z_{\sigma(j)} - w_k) + f_k}{h - 1} - e_k \right) \prod_{j=1}^{N} c_{x_j}.
\]  

We make some remarks. By setting \( a_j = c_j = d_j = 1 - 1/h, \) \( b_j = w_j = 0, \) \( e_j = h - 1, \) \( 15 \) becomes

\[
S_{M,N}(z_1, \ldots, z_n | w_1, \ldots, w_M | x_1, \ldots, x_N) = \left( 1 - \frac{1}{h} \right)^N \sum_{\sigma \in S_N} \prod_{1 \leq j < k \leq N} \frac{z_{\sigma(j)} - z_{\sigma(k)} + 1 - h}{h(z_{\sigma(j)} - z_{\sigma(k)})} \\
\times \prod_{j=1}^{N} \prod_{k=x_j+1}^{M} \left( \frac{z_{\sigma(j)} - w_k}{h} + 1 - \frac{1}{h} \right) \prod_{j=1}^{N} \prod_{k=1}^{x_j-1} (z_{\sigma(j)} - w_k).
\]  

It is easy to take the limit \( h \to \infty \) of (16):

\[
S_{M,N}(z_1, \ldots, z_n | w_1, \ldots, w_M | x_1, \ldots, x_N)|_{h \to \infty} = \sum_{\sigma \in S_N} \prod_{1 \leq j < k \leq N} \frac{1}{z_{\sigma(k)} - z_{\sigma(j)}} \prod_{j=1}^{N} \prod_{k=1}^{x_j-1} (z_{\sigma(j)} - w_k) \\
= \left( -1 \right)^{\sigma} \sum_{\sigma \in S_N} \prod_{j=1}^{N} \prod_{k=1}^{x_j-1} (z_{\sigma(j)} - w_k).
\]  

Using the Young diagram \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \) \( (M - N \geq \lambda_1 \geq \cdots \geq \lambda_N \geq 0) \) in the right hand side of (17) instead of \( x_1, \ldots, x_N \) by the transformation rule \( \lambda_j = x_{N-j+1} - N + j - 1, \) \( j = 1, \ldots, N, \) we find limit of \( S_{M,N}(z_1, \ldots, z_n | w_1, \ldots, w_M | x_1, \ldots, x_N) \) is nothing but the factorial Schur functions

\[
S_{M,N}(z_1, \ldots, z_n | w_1, \ldots, w_M | x_1, \ldots, x_N)|_{h \to \infty} = \det_N \left( \frac{(z_k | w)^{j+N-j}}{(z_j - z_k)} \right)_{1 \leq j < k \leq N},
\]  

where

\[
(z|w)^m = (z - w_1) \cdots (z - w_m).
\]  

The following correspondence between the wavefunctions and the symmetric functions hold.
Theorem 3. The wavefunctions $W_{M,N}(z_1,\ldots,z_N|w_1,\ldots,w_N|x_1,\ldots,x_N)$ are explicitly expressed as the symmetric functions $S_{M,N}(z_1,\ldots,z_N|w_1,\ldots,w_N|x_1,\ldots,x_N)$

$$W_{M,N}(z_1,\ldots,z_N|w_1,\ldots,w_M|x_1,\ldots,x_N) = S_{M,N}(z_1,\ldots,z_N|w_1,\ldots,w_M|x_1,\ldots,x_N). \quad (20)$$

Theorem 3 can be proved by showing that the rational symmetric functions $S_{M,N}(z_1,\ldots,z_N|w_1,\ldots,w_N|x_1,\ldots,x_N)$ satisfy all the properties listed in Proposition 1.

By setting $a_j = c_j = d_j = 1 - 1/h$, $b_j = w_j = 0$, $e_j = h - 1$ and taking the limit $h \to \infty$, we get the correspondence between the wavefunctions of the five-vertex model and the factorial Schur functions

$$W_{M,N}(z_1,\ldots,z_N|w_1,\ldots,w_M|x_1,\ldots,x_N)|_{h \to \infty} = \frac{\det_N (z_k^j w_{j+N-j})}{\prod_{1 \leq j < k \leq N} (z_j - z_k)}, \quad (21)$$

5. Discussion

We illustrated the Izergin-Korepin analysis on the wavefunctions constructed from the rational $L$-operator. One can extend the analysis to the elliptic models by using the notion of elliptic polynomials [52, 12]. The notion of the elliptic polynomials was applied to compute the explicit form of the domain wall boundary partition functions of the Andrews-Baxter-Forrester model [10] in [11, 12], for example. The Izergin-Korepin analysis can be extended to the wavefunctions of the elliptic integrable models as well [25, 27, 28]. The Izergin-Korepin technique can also be applied to other boundary conditions such as the reflecting boundary conditions, and we find generalizations of symplectic Schur functions [53, 54] and Bump-Friedberg-Hoffstein Whittaker functions [55] appear. See [56] for example.

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