Schwarzschild-Tangherlini Metric from Scattering Amplitudes

Gustav Ulhø Jakobsen
Niels Bohr International Academy, The Niels Bohr Institute, University of Copenhagen Blegdamsvej 17, DK-2100 Copenhagen, Denmark

Dated: June 3, 2020

We show in detail how the Schwarzschild-Tangherlini metric of a point particle in arbitrary dimensions can be derived from a scattering amplitude to all orders in $G_N$ in covariant gauge (i.e. $R_{\xi}$-gauge) with a generalized de Donder-type gauge function, $G_{\sigma}$. The metric is independent of the covariant gauge parameter $\xi$ and obeys the classical gauge condition $G_{\sigma} = 0$. We explicitly compute the metric to second order in $G_N$ where gravitational self-interactions become important. Interestingly, after generalizing to arbitrary dimension, a logarithmic dependence on the radial coordinate appears in space-time dimension $D = 5$.

I. INTRODUCTION

The classical limit of effective quantum gravity is a successful description of general relativity. Here, quantum field theoretic methods are used to derive results in classical general relativity $[1–9]$. In this approach gravitational interactions are mediated by spin-2 gravitons and general relativity is recast in the language of quantum field theory $[10]$

The field theoretic description of gravity is easily generalized to arbitrary space-time dimensions, $D$ $[11–14]$. Already when working with Einstein gravity in $D = 4$, if the dimensional regularization scheme is used, it is to some extent necessary to work with an arbitrary dimension $D$ when pursuing the field-theoretic framework. A classic result is the Schwarzschild-Tangherlini metric which describes the gravitational field of a neutral non-rotating point particle.

The quantum field description of gravity has given new insights into the gauge theory of gravity. A well known example is the double-copy nature of gravity in terms of Yang-Mills gauge theory $[15]$. Interest in the gauge freedom of gravity has led to the study of new perturbative gauges and field redefinitions which e.g. can be used to reduce the complexity of the Feynman rules or make apparent the double copy nature of gravity $[16–17]$. In general, these studies give hope that a thorough understanding, and exploitation, of the gauge freedom of gravity will result in simplifications of the complicated tensor structure of quantum gravity and possibly offer an improved starting point from which to continue investigations into quantum corrections.

In this paper, we analyze the quantum field theoretic expansion of the Schwarzschild-Tangherlini metric from a series of Feynman diagrams with an ever-increasing number of loops. Such an all-order expansion was suggested in $[2]$ where it was shown how the loop integrals can be reduced in the classical limit. Already, such expansions have been done to second $[12–15]$ and third $[3]$ order in the gravitational constant $G_N$.

Our analysis uses a novel generalized gauge fixing function which combines harmonic gauge, $g^\mu\nu T^\sigma_{\mu\nu} = 0$, and the linearized version de Donder gauge, $\partial_\sigma h_\sigma^\mu = \frac{1}{2} \partial_\mu h$. Working all the time in arbitrary dimensions $D$ we use covariant gauge (i.e. $R_{\xi}$-gauge) so that our analysis depends on the arbitrary parameter $\xi$. This approach clearly demonstrates how the classical limit depends on the quantum gauge fixing procedure.

The standard coordinates of the Schwarzschild-Tangherlini metric are spherical and not of the perturbative kind used in effective quantum gravity. Perhaps the most well-known perturbative gauge is harmonic gauge. In space-time dimension $D = 4$ analytic results in harmonic gauge to all orders in $G_N$ are known $[13–19]$. However in dimensions $D \neq 4$ and in de Donder gauge analytic results are rare. In $[3]$ we find the metric in de Donder gauge to third order in $G_N$ and in $[12]$ we find it, also, in de Donder gauge in arbitrary dimensions $D$ to second order in $G_N$.

After presenting general formulas relating the Schwarzschild-Tangherlini metric to scattering amplitudes we explicitly compute the metric to second order in $G_N$. This gives a new general result for the perturbative expansion of the Schwarzschild-Tangherlini metric in the generalized gauge including both de Donder and harmonic gauge in arbitrary dimensions. As a consistency check, in appendix $[13]$ we compare the amplitude approach with a derivation using only methods from classical general relativity.

In space-time dimension $D = 5$ we find the curious appearance of a logarithmic dependence on the radial variable at second order in $G_N$. This is analogous to the case in $[3]$ in $D = 4$ at third order in $G_N$. We explain how the arbitrary scale thus introduced corresponds to a coordinate transformation which is allowed because of redundant gauge freedom. From this explanation it is expected that the appearance of logarithmic dependence is limited to $D = 5$ at second and higher orders in $G_N$ and $D = 4$ at third and higher orders in $G_N$.

The paper is organized as follows. In Sec. $[1]$ we discuss the gauge-fixed action and the generalized de Donder-type gauge function in detail. We consider the Feynman rules and present the graviton propagator in covariant gauge. Then, in the first part of Sec. $[11]$ we consider gen-
eral ideas of the all order expansion of the Schwarzschild-Tangherlini metric in terms of scattering amplitudes. In the subsections 11A and 11B we compute the first and second order contribution to the metric, respectively. In Sec. IV we discuss the appearance of logarithms in the metric. There are two appendices. First, in appendix A we present the relevant Feynman rules for our computations. Second, in appendix B we go through the alternative derivation of the expansion of the Schwarzschild-Tangherlini metric.

II. COVARIANT AND GENERALIZED GAUGE FIXING

We work with the Einstein-Hilbert action minimally coupled to a scalar field together with the covariant gauge fixing term:

\[ S = \int d^D x \sqrt{-g} \left( \frac{2R}{\kappa^2} + \mathcal{L}_\phi \right) + \int d^D x \frac{\eta^\mu \nu G_\mu G_\nu}{\kappa^2 \xi} . \]  

(1)

Here \( \kappa^2 = 32\pi G_N \) and \( \mathcal{L}_\phi = \frac{1}{2} (g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2) \) and we use the mostly minuses metric. Also, \( \xi \) is the covariant gauge parameter. Additionally, from the path integral gauge fixing procedure, there would be a ghost term which we, however, will not consider since it does not contribute in the classical limit.

We choose a de Donder-type family of gauge functions \( G_\sigma \) which depend on the arbitrary parameter \( \alpha \):

\[ G_\sigma = (1 - \alpha) \partial_\mu (h_\sigma^\mu - \frac{1}{2} \eta_\mu \nu h_\sigma^\nu) + \alpha g^{\mu \nu} \Gamma_{\sigma \mu \nu} . \]  

(2)

Here \( h_\mu \nu = g_\mu \nu - \eta_\mu \nu \) and indices on \( h_\mu \nu \) are raised and lowered with the flat space metric. The index on \( \Gamma_{\sigma \mu \nu} \) was lowered with \( g_\mu \nu \).

Note the details of this gauge function. When \( \alpha = 0 \) we have de Donder gauge, \( \partial_\mu (h_\sigma^\mu - \frac{1}{2} \eta_\mu \nu h_\sigma^\nu) = 0 \) and when \( \alpha = 1 \) we have harmonic gauge \( g^{\mu \nu} \Gamma_{\sigma \mu \nu} = 0 \). Here we have used similar terminology as [10]. Any choice of \( \alpha \), however, results in a valid gauge choice of the same generalized type as discussed in [10]. When \( G_\sigma \) is expanded in \( h_\mu \nu \) the linear term is independent of \( \alpha \) while the nonlinear terms are linear in \( \alpha \). Thus, the gauge parameter \( \alpha \) scales all the non-linear terms of \( G_\sigma \). For the 1-loop computation we need only the linear and quadratic terms which we find to be:

\[ G_\sigma \approx h_\sigma^\mu - \frac{1}{2} h_\mu. + - \alpha \left( h^{\mu \nu} h_{\sigma \mu \nu} - \frac{1}{2} h^{\mu \nu} h_{\nu \sigma \mu} \right) . \]  

(3)

Here, and later, we use the comma notation for partial derivatives.

The classical equations of motion \( \delta S = 0 \) depend on both gauge parameters. First, we will focus on the dependence on the covariant parameter. We get the equations of motion

\[ G^{\mu \nu} + \frac{1}{\xi} H^{\mu \nu} = - \frac{\kappa^2}{4} T^{\mu \nu} , \]  

(4a)

\[ \sqrt{-g} H^{\mu \nu} = \alpha G^\rho \Gamma_{\rho \sigma \beta} g^{\mu \rho} g^{\nu \beta} + \left( I_{\mu \nu}^{\rho \sigma} - \frac{1}{2} g^{\mu \nu} g^{\rho \sigma} \right) \]  

\[ \times \partial_\sigma \left( G^\rho (\eta^\beta - \alpha (g^{\alpha \beta} - \eta^{\alpha \beta})) \right) , \]  

(4b)

where in Eq. (4b) indices on \( G_\sigma \) are raised with the flat space metric. We use the standard notation \( I_{\mu \nu}^{\rho \sigma} = \frac{1}{2} \delta_{\mu}^{( \rho} \delta_{\nu}^{\sigma)} + \frac{1}{2} \delta_{\mu}^{\sigma} \delta_{\nu}^{\rho} \) and, later, \( P_{\mu \nu}^{\rho \sigma} = I_{\mu \nu}^{\rho \sigma} + \frac{1}{2} \eta^{\mu \nu} \eta_{\rho \sigma} \) from [10] [20]. In Eq. (4a), \( G^{\mu \nu} \) is the Einstein tensor, \( T^{\mu \nu} \) is the energy-momentum tensor of matter, and \( H^{\mu \nu} \) is a Lorentz covariant tensor which depends on the gauge fixing function \( G_\sigma \) in such a way that \( G_\sigma = 0 \) implies \( H^{\mu \nu} = 0 \). The tensor \( H^{\mu \nu} \) is independent of \( \xi \) and hence all dependence on \( \xi \) is explicit in Eq. (4a). Like \( G_\sigma \), the tensor \( H^{\mu \nu} \) is not general covariant and it breaks the general covariance of the Einstein equations.

It is not clear that the classical limit described by Eq. (4a) is independent of \( \xi \). However, taking the covariant derivative on both sides gives some indication in this direction. Since both \( D_\mu G^{\mu \nu} = 0 \) and \( D_\mu T^{\mu \nu} = 0 \) we get as a consequence of Eq. (4a) that

\[ D_\mu H^{\mu \nu} = 0 . \]  

(5)

which we interpret as the classical gauge condition.

It is clear that the equation \( G_\sigma = 0 \) implies Eq. (5). From a perturbative expansion of Eq. (5) in \( G_\sigma \) we make the opposite conclusion as well, that \( G_\sigma = 0 \) is implied by Eq. (5). At each order in the perturbative expansion we find that \( \partial^2 G_\sigma^{(n)} \) is given by earlier terms in \( G_\sigma \). Starting from the exact equation \( \partial^2 G_\sigma^{(1)} = 0 \), we conclude that \( G_\sigma^{(1)} = 0 \) and by induction that \( \partial^2 G_\sigma^{(n)} = 0 \). Finally we conclude that \( G_\sigma = 0 \).

From \( G_\sigma = 0 \) it follows that \( H^{\mu \nu} = 0 \). Clearly solutions to Eq. (4a) are then independent of \( \xi \). This conclusion should not be surprising since if we at all hope to find a metric it should satisfy the Einstein equations and then \( H^{\mu \nu} \) is forced to vanish as well. The conclusion is that Eq. (4a) is equivalent to classical general relativity with the gauge coordinate-condition \( G_\sigma = 0 \). In this paper we explicitly verify that the metric is independent of \( \xi \) and obeys the condition \( G_\sigma = 0 \) to second order in \( G_\sigma \).

The two gauge parameters \( \xi \) and \( \alpha \) play very different roles. The covariant gauge parameter \( \alpha \) appears only during intermediate steps and the classical metric is independent of \( \xi \). During the calculation, however, it is convenient to separate quantities into parts according to their dependence on \( \xi \). The parameter \( \alpha \) is introduced to describe an entire family of classical gauge choices. The classical limit then depends on \( \alpha \) since the gauge condition \( G_\sigma = 0 \) does.

To derive the Feynman rules we expand the action around flat space-time in \( h_\mu \nu \). Since the linear term of the gauge function \( G_\sigma \) is independent of \( \alpha \), the quadratic
term in the action $S$ will also be independent of $\alpha$. From the quadratic term in $S$ we derive the graviton propagator in covariant de Donder gauge in momentum space:

$$\frac{iG_{\alpha\beta}^{\mu\nu}}{q^2+i\epsilon} = \frac{i}{q^2+i\epsilon} \left( \mathcal{P}^{-1\mu\nu}_{\alpha\beta} - 2(1-\xi)I^{\mu\nu}_{\alpha\beta} \frac{q^2}{q^2} I^{\rho\sigma}_{\alpha\beta} \right).$$  \hspace{1cm} (6)

Here $\mathcal{P}^{-1\mu\nu}_{\alpha\beta}$ is the inverse operator to $\mathcal{P}^{\mu\nu}_{\alpha\beta}$ which is the well known de Donder propagator

$$\mathcal{P}^{-1\mu\nu}_{\alpha\beta} = I^{\mu\nu}_{\alpha\beta} - \frac{1}{D-2} \eta^{\mu\nu} \eta_{\alpha\beta},$$  \hspace{1cm} (7)

to which the covariant propagator reduces for $\xi = 1$. For other values of $\xi$ a new momentum-dependent term appears in the propagator. Later, it will be convenient to separate the propagator into two terms, one independent of $\xi$ and the other linear in $\xi$.

Expanding the action in $h_{\mu\nu}$ generates terms with an arbitrary number of gravitons. For the 1-loop calculation only the $\phi^2 h$ and $h^3$ vertices are necessary. These are included in appendix [4]. We note, however, how the vertices in general depend on the gauge parameters $\xi$ and $\alpha$. The coupling of $h$ to $\phi$ is independent of the gauge fixing and hence the vertices $\phi^2 h^\alpha$ as well. The graviton self-interaction vertices can conveniently be separated into two terms, one independent of $\xi$ and one linear in $\frac{1}{\xi}$. As for the terms linear in $\frac{1}{\xi}$ these can then be divided into terms linear or quadratic in $\alpha$.

III. DIAGRAM EXPANSION OF THE SCHWARZSCHILD-TANGHERLINI METRIC

It is an exciting idea that the Schwarzschild-Tangherlini metric can be computed from scattering amplitudes and Feynman diagrams [11, 12]. Since the metric is not a gauge-invariant object, the relevant diagrams cannot be gauge-invariant either and they will include an external graviton. In this section we derive the Schwarzschild-Tangherlini metric from the exact vertex function of a massive scalar emitting a graviton. This amplitude is shown in figure 1.

[Diagram of a massive scalar emitting a graviton]

In the classical limit, diagrams with an arbitrary number of loops still contribute and loops correspond to orders in $G_N$. We put the incoming scalar on-shell so that $k^2 = m^2$. The amplitude is then multiplied together with a $\delta$-function $\delta(kq)$ which, in the classical limit, puts the outgoing scalar on-shell as well. The graviton is not on-shell, although in the classical limit where $\hbar \to 0$ we have that $q^2 \to 0$. In this limit, when $\hbar$ is reintroduced as a dimensional parameter and sent to zero, it becomes relevant to distinguish wavelengths from classical momenta. The scalar momenta are taken as classical momenta while the graviton momenta should be considered as wavelengths. This makes them scale with $\hbar$ so that the momentum of the external graviton $q^\mu$ goes to zero while the scalar momentum $h^\mu$ stays finite. This analysis applies to intermediate particles as well where graviton loop momenta $l^\mu \to 0$. See [14] for detailed discussions of the classical limit.

The Lorentz covariance of the perturbative quantum field theoretic framework invites us to work in an arbitrary inertial frame. It will be convenient then to introduce a notation which separates tensors into parallel and orthogonal parts with respect to the point particle momentum $k^\mu$. We introduce the following projection operators

$$\eta_{\mu\nu} = \frac{k_\mu k_\nu}{m^2},$$

$$\eta_{\mu\nu}^\perp = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2},$$  \hspace{1cm} (8a, b)

and use similar symbols to signify projection with respect to these. This notation is similar to that in [12]. These operators are particularly simple in the inertial frame of $k^\mu$ where they are diagonal and represent the time- and space components of $\eta_{\mu\nu}$ respectively. Our definition of the Fourier transform between position and momentum space will be that of relativistic quantum field theory.

In the classical limit we can interpret the amplitude in figure 1 as the source of the metric, $h_{\mu\nu}$, generated from the point particle and the surrounding gravitational field. The source of $h_{\mu\nu}$, that is $\mathcal{M}^{\mu\nu}_{\text{vertex}}$, is combined of an energy-momentum pseudo-tensor and a gauge fixing term. In the classical limit, we get

$$2\pi \delta(kq) \mathcal{M}^{\mu\nu}_{\text{vertex}} = -\kappa \tilde{\tau}^{\mu\nu}(q) + \frac{1}{\xi} 2\pi \delta(kq) \mathcal{M}^{\mu\nu}_{gf},$$  \hspace{1cm} (9)

where $\tilde{\tau}^{\mu\nu}(q)$ and $\mathcal{M}^{\mu\nu}_{gf}$ are independent of $\xi$. As explained, the $\delta$-function $\delta(kq)$ puts $(k-q)^\mu$ on-shell and enables us to relate $\mathcal{M}^{\mu\nu}_{\text{vertex}}$ to $\tilde{\tau}^{\mu\nu}(q)$ which is the total energy-momentum tensor of matter and gravitation in momentum space. This tensor is e.g. discussed in [13, 14]. It is locally conserved and therefore obeys $q_\mu \tilde{\tau}^{\mu\nu}(q) = 0$. To zeroth order, it is given by the point particle energy-momentum tensor of special relativity, and loop-corrections describe energy-momentum from the surrounding, self-interacting gravitational field.

It is not clear that, in the classical limit, the dependence on $\xi$ of $\mathcal{M}^{\mu\nu}_{\text{vertex}}$ can be reduced to that of the simple expression in Eq. (9), since each graviton self-interaction
vertex includes a factor $\frac{1}{2}$ and each graviton propagator a factor $\xi$. However, due to several cancellations, all powers of $\xi$ different from $\frac{1}{2}$ disappear in the classical limit. Looking at the classical equation of motion, $\delta S = 0$, Eq. (4) we see that $\mathcal{M}_{\nu\alpha}^{\mu}$ corresponds to the non-linear part of the gauge-breaking term $H^{\mu\nu}$ in momentum space:

$$2\pi \delta(kq) \mathcal{M}_{\nu\alpha}^{\mu} = -\frac{4}{\kappa^2} H^{\mu\nu}_{\text{non-linear}}.$$  

(10)

To get the metric, we solve the classical equation $\delta S = 0$ by contracting $\mathcal{M}_{\nu\alpha}^{\mu}$ with the graviton propagator Eq. (6). Finally we can go to position space with a Fourier transform to get the Schwarzschild-Tangherlini metric:

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{\kappa}{2} \int\frac{d^D q}{(2\pi)^{D-1}} \frac{\delta(kq)}{q^2} G_{\mu\alpha\beta}^{\nu\gamma} M_{\alpha\beta}^{\gamma\nu}.$$  

(11)

This exciting equation relates the metric from classical general relativity to the scattering amplitude, $\mathcal{M}_{\nu\alpha}^{\mu}$, to all orders in $G_N$.

Although both the vertex function and the graviton propagator depend on $\xi$, the metric does not. This is due to the Einstein equations combined with the gauge condition $G_N = 0$ both of which are exact in the classical limit. If we separate the graviton propagator into two parts according to $\xi$, $G_{\mu\alpha\beta}^{\nu\gamma} = (G^{\nu\gamma} + \xi G_{\nu\gamma}^{\mu})_{\mu\alpha\beta}$ as in Eqs. (A6) we find that the following combinations vanish, $G^{\beta\gamma} \tau = 0$ and $G^{\nu\gamma} M_{\alpha\beta}^{\gamma\nu} = 0$, where we have omitted indices. These two equations correspond to the Einstein equations and the gauge condition, respectively, and secure that $\xi$ disappears from the metric. Using these relations, we get an expression for $h_{\mu\nu}$ independent of $\xi$ in momentum space:

$$\hat{h}_{\mu\nu} = \frac{\mathcal{P}_{\mu\alpha}^{\mu\beta}}{q^2} \left(\frac{\kappa^2}{2} \tau_{\alpha\beta} + 2 H_{\alpha\beta}^{\gamma\nu}_{\text{non-linear}}\right).$$  

(12)

In this equation and in Eq. (11) indices on the propagator was lowered with the flat space metric.

Let us compare Eq. (12) with the approach in [15]. There, loop-correctons to $\tau_{\mu\nu}$ was calculated in $D = 4$ with the background field method and the metric was obtained in harmonic gauge by solving the classical Einstein equations with the non-linear harmonic gauge condition $\Gamma_{\mu\nu}^{\gamma\nu} g^{\mu\nu} = 0$ which meant that a gauge-dependent term was added to the energy-momentum tensor. In our approach the gauge-dependent term is already included in the amplitude in the form of $H_{\mu\nu}^{\gamma\nu}_{\text{non-linear}}$ and this tensor exactly corresponds to their gauge-dependent correction to $\tau_{\mu\nu}$.

Eq. (12) is particularly simple in de Donder gauge where $\alpha = 0$. In this gauge $\sqrt{-g} H_{\mu\nu}$ is linear in $h_{\mu\nu}$ which implies that $H_{\mu\nu}^{\gamma\nu}_{\text{non-linear}} = 0$ so that the second term on the right hand side disappears. Thus in De Donder gauge, the graviton $h_{\mu\nu}$ couples directly to the local energy-momentum tensor $\tau_{\mu\nu}$. In general the linear gauge of $\alpha = 0$ is special since then, the $\xi$-dependence of the graviton self-interaction vertices disappear. In this case “Landau gauge” $\xi \to 0$ is possible.

As an example we will first compute the tree-level contribution to $\mathcal{M}_{\nu\alpha}^{\mu}$ from which we derive the first order correction to the metric. Afterwards we will focus on the 1-loop contribution, where gravitational self-interactions first appear, which gives the $(G_N)^2$ metric contribution.

### A. Tree Level: Newton Potential in Arbitrary Dimensions

As a simple example we compute the first order Newton correction to the Schwarzschild-Tangherlini metric. This comes from the tree diagram where a single graviton is connected to the scalar line. We get, in the classical limit, $i\mathcal{M}_{\nu\alpha}^{\mu} = -ikk^\mu k^\nu$, where we have used the same labeling of momenta as in figure 1 and the $\hbar c^2$ vertex rule from appendix A and neglected factors of $q^t$. This amplitude is independent of the gauge-parameters and for $\tau_{\mu\nu}$ we find to zeroth order that $\tau_{\mu\nu} \approx 2\pi \delta(kq)k^\mu k^\nu$. This is indeed conserved $q_{\mu} \tau_{\nu\mu} = 0$ and reproduces, in position space, the simple energy-momentum tensor of an inertial point particle. Using Eq. (11) we propagate the tree-amplitude and go to position space to get the Newton potential in arbitrary dimensions:

$$h_{\nu\mu}^{(1)} = -\frac{\mu}{\sqrt{-x}} \left(\eta_{\mu\nu} - \frac{1}{D-3} \eta^{\mu}_{\nu\mu}\right).$$  

(13)

We use the Lorentz covariant notation of Eqs. (8). The Schwarzschild-Tangherlini parameter $\mu$ is

$$\mu = \frac{16\pi G_N m}{(D-2)\Omega_{D-2}}.$$  

(14)

where $\Omega_{d-1}$ is the surface area of a sphere in $d$-dimensional space and $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d-1}{2})}$. The first order metric in Eq. (13) agrees with the results in [12] [14]. It is independent of both $\xi$, as expected, and $\alpha$ since $\alpha$ only enters in the self-interaction vertices.

### B. One-Loop Contribution to the Metric

The $(G_N)^2$ contribution to the metric comes from the triangle diagram in figure 2. Other 1-loop diagrams do not contribute with non-analytic classical terms [2]. This diagram depends on both gauge parameters although $\xi$ disappears in the metric.

The triangle loop-integrals relevant for this amplitude have been treated in detail in [2] [11]. In the classical limit, they can be reduced to a very simple convolution integral. To do this, we first integrate away the massive propagator with the time-like component $l_3^\sigma$. As a result, the single scalar line is effectively split into two independent scalar lines, that is, two tree-diagrams are
These expressions are Lorentz covariant and valid in any dimension. However, the factor \( \cos(\frac{\pi}{2}D) \) in the denominator makes them diverge in odd dimensions. It is not a problem, however, because the divergent term is analytic since it is proportional to an integer power of \( q^2 \). Such analytic terms describe local corrections to the metric. When the divergent term is neglected a finite logarithmic dependence on \( q^2 \) remains. In the end, this only has significance in \( D = 5 \) where a logarithm appears in position space. We will discuss the divergence in detail in Sec. IV.

Clearly, \( \tilde{\tau}_{1\text{-loop}}^{\mu
u} \) is locally conserved which implies that \( G_{\mu\nu} \tilde{\tau}_{1\text{-loop}} \) vanishes as expected from the discussion above Eq. (12). It is a straightforward check that \( G_{\mu\nu} \tilde{\tau}_{1\text{-loop}} \) disappears as well. This verifies that the metric is independent of \( \xi \) to second order in \( G_N \). At one-loop order \( H^{\mu\nu} \) is linear in \( \alpha \) while \( \tau^{\mu\nu} \) is independent of \( \alpha \). Going to higher orders in \( G_N \) we would expect \( \alpha \) to appear to any integer power in both \( H^{\mu\nu} \) and \( \tau^{\mu\nu} \).

When we go to position space we distinguish the two cases \( D \neq 5 \) and \( D = 5 \). In the final part of this section we treat \( D \neq 5 \) and in Sec. IV we focus on \( D = 5 \). We use Eqs. (11) and (12) to transform to position space and get:

\[
h^{(2)}_{\mu\nu} = \frac{\mu^2}{(2D-3)} \left( \frac{1}{2} \eta^{\parallel\mu\nu} - \left( \frac{4\alpha - 3}{4(D-5)} \right) x^\mu x^\nu - \frac{2(1-\alpha)}{4(D-3)^2(D-5)} \eta^{\perp\mu\nu} \right),
\]

Here, \( r^2 = -x^2 \). The pole in \( D = 5 \) makes it evident that, in this dimension, the amplitude was not regularized correctly before the Fourier transform. This metric satisfies \( G_\sigma = 0 \) to second order in \( G_N \) as expected (see Eq. (15)) where \( G_\sigma \) is expanded to second order in \( G_N \). A generally useful formula to perform the relevant Fourier transforms is

\[
\int \frac{d^D q}{(2\pi)^D} e^{-ix^\perp q^\perp} (-q^2)^{\frac{D}{2}} = \frac{2^n \Gamma(\frac{D+n}{2})}{\sqrt{\pi^D} \Gamma(-\frac{D}{2})} (-x^2)^{\frac{D+n}{2}},
\]

which is also found in [12].

In de Donder gauge where \( \alpha = 0 \) we find agreement of Eq. (16) with [12] in any dimension [12]. For harmonic gauge \( \alpha = 1 \) we know only of any comparison in \( D = 4 \) e.g. [19]. For general \( \alpha \) we have made an independent derivation in appendix B with methods from classical general relativity and we find agreement.

We can choose any value for \( \alpha \) and we can e.g. use this freedom to remove the coefficient of \( \eta^{\perp\mu\nu} \). The special choice of \( \alpha = \frac{5}{6} \) removes the pole in \( D = 5 \), which will explained in the next section.

IV. APPEARANCE OF LOGARITHMS IN THE PERTURBATIVE EXPANSION

In this section we will focus on the divergences of Eqs. (15) and how this leads to a logarithmic term in the metric in \( D = 5 \). We will explain why this term appears and learn that besides \( D = 5 \) logarithmic terms are only expected in \( D = 4 \).

The divergence in Eqs. (15) comes from the factor \( \cos(\frac{\pi}{2}D) \) in the denominator in odd dimensions. To analyze these divergences we use the dimensional regularization scheme and take the limit where the dimension

1 Note, that there is a misprint in the fourth line of their Eq. (5.34) where \( (D - p - 3)^2 \) should be replaced by \( (D - p - 3) \). We thank Paolo Di Vecchia for confirming this.
would not expect logarithms to appear in $D = 5 + 2n + 2\epsilon$ where $n$ is an integer and $\epsilon$ is infinitesimal. As explained, the pole in $\epsilon$ is an analytic function of $q^2$, namely $(-q^2)^n$ (after multiplying Eqs. (15) by the propagator), which describes local corrections to the metric. However, the finite term includes a non-analytic, logarithmic dependence on $q^2$ which describe long-range classical physics. This results in the following prescription to remove the divergence in odd dimensions

$$\frac{\sqrt{-q^2}^{D-3}}{\cos(\frac{\pi}{2} D)} \to (-1)^{\frac{D-3}{2}} (-q^2)^{\frac{D-3}{2}} \ln(-q^2)q^2, \quad (18)$$

which is valid only in $D \neq 5$. We will now compute this integral in $D = 5$ by using the replacement rule in Eq. (18).

The logarithmic dependence on $q^2$ in Eq. (18) can be rewritten in terms of powers of $q^2$ with

$$\ln(-q^2) = \frac{1}{\epsilon} \left((q^2)^\epsilon - 1\right), \quad (20)$$

where $\epsilon$ is infinitesimal. The Fourier integral Eq. (17) can now be used. Using these tools, we get that in $D = 5$ the integral corresponding to Eq. (19) becomes:

$$\int \frac{d^D q \delta(k_{\perp}^2)}{(2\pi)^D} e^{-iqx} \frac{\ln(-r_0^2 q^2)}{q^2} \frac{\eta^\mu_{\nu} q^\mu q^\nu}{q^2} = \left(\frac{2}{\Omega_{D-2}(D-3)\sqrt{-x_\perp^{D-3}}}ight)^2 \frac{1}{D-5} \left(\frac{1}{x_\perp^2} - 2(D-3)\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}\right), \quad (19)$$

with $r_0$ is an arbitrary scale which is introduced from the dimensional dependence of $G_N$. With this replacement Eqs. (19) are finite in all dimensions.

In principle it would be necessary to separate Eqs. (15) into two expressions for even/odd dimensions before going to position space. However, all cases but $D = 5$ can be treated simultaneously because the analytic functions can be neglected with dimensionally regularized integrals such as Eq. (17) which gives the Fourier transform of $(-q^2)^n/2$. In this equation it is seen that when $\frac{n}{2}$ is an integer so that we are transforming an analytic function, the result is zero.

In Eq. (16) the pole in $D = 5$ comes from the integral

$$\int d^D q \delta(k_{\perp}^2) - \frac{1}{2} \left(\frac{\ln(-r_0^2 q^2)}{q^2} \frac{\eta^\mu_{\nu} q^\mu q^\nu}{q^2}\right),$$

have compared Eq. (22) with an independent derivation in appendix 3 and find agreement.

Note that we can make the logarithm disappear with the special choice $\alpha = \frac{5}{2}$. The arbitrary scale, however, would in principle still be there. In analogy, in $D = 4$ we know that for $\alpha = 1$ in harmonic gauge there is no logarithms.

The arbitrary scale corresponds to a redundant gauge freedom. It is well know from linearized gravity that even after choosing de Donder gauge, we can still translate the coordinates with $e^\sigma$ as long as it is a harmonic function, $\partial^2 e^\sigma = 0$. In our situation the relevant coordinate transformation is

$$x^\mu \to x^\mu + \beta \frac{\mu^2}{r^4} x^\mu + \ldots$$

which does not change our gauge since $(x_\perp^2/r^4)$ is a harmonic function (when $r \neq 0$). At higher orders in $G_N$ this coordinate transformation gets corrected which is indicated by the ellipsis. Choosing $\beta$ in Eq. (23) appropriately makes the coordinate transformation equivalent to a scaling of the arbitrary parameter $r_0 \to \gamma r_0$ in Eq. (22). Thus, the arbitrary parameter in the metric in $D = 5$ is unproblematic and is related to the coordinate system not being completely specified yet.

In arbitrary space-time dimensions the equivalent transformation would be $x_\perp^2/r^{D-1}$ which is a harmonic function. In any dimension we would be able to introduce an arbitrary parameter with such a transformation. However, only in $D = 4$ or $D = 5$ does this lead to the appearance of logarithms in the metric. This is due to the fact that only in these dimensions, such a transformation would be possible using the dimensions of $\mu$. In $D = 5$ it is accompanied by $\mu^2$ while in $D = 4$ we get $\mu^3$ so that the logarithms appear respectively at second and third order. In other dimensions the transformation would have to be scaled by fractional powers of $\mu$. 

\[ \]
V. CONCLUDING REMARKS

We have analyzed the problem of deriving the Schwarzschild-Tangherlini metric from scattering amplitudes in detail. We calculated the metric to second order in $G_N$ and explicitly verified our general conclusions to this order. These include that the metric is independent of the covariant parameter $\xi$, that it obeys the classical gauge condition $G_\sigma = 0$, and that it simply is the Fourier transform of the exact three-point vertex of a scalar emitting a graviton after propagation by the graviton propagator.

In $D = 5$ a logarithmic dependence appeared in position space analogous to the case in $D = 4$ at third order in $G_N$ [4]. We analyzed this curious phenomenon in terms of redundant gauge freedom and coordinate transformations. This freedom makes it possible to introduce an arbitrary parameter in any dimension, though only in the two cases $D = 4$ and $D = 5$ does it lead to logarithmic terms in the metric.

The full all-order expansion of the Schwarzschild-Tangherlini metric from scattering amplitudes is still to be performed explicitly. This requires an inductive relation between the loop amplitudes at different orders. Already several exciting simplifications are known [2]. A logical continuation is to analyze the analogous problem for particles with spin and eventually look at quantum corrections [9, 18]. Also, it would be interesting to continue investigations into solutions in classical general relativity is to analyze the analogous problem between the loop amplitudes at different orders.

We derive the graviton propagator by inverting the quadratic operator in the $h^2$ term of the action. In momentum space, this term reads

$$ S_{h^2} = \frac{1}{2} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2} \Delta_{\alpha\beta}^{\mu\nu} h_{\alpha\beta}^{\mu\nu}, $$

where

$$ \Delta_{\alpha\beta}^{\mu\nu} = P_{\alpha\beta}^{\mu\nu} - \frac{2(1 - \frac{1}{\xi})}{\xi} \frac{l^2}{l^2} P_{\alpha\beta}^{\sigma\sigma}. $$

is a tensor depending on both the momentum $l^\mu$ and the covariant gauge parameter $\xi$. We invert the tensor and for $(\Delta_{\alpha\beta}^{\mu\nu})^{-1}$, we find

$$ G_{\alpha\beta}^{\mu\nu} = P_{\alpha\beta}^{\mu\nu} - 2(1 - \frac{1}{\xi}) \frac{l^2}{l^2} P_{\alpha\beta}^{\sigma\sigma}, $$

so that $G_{\alpha\beta}^{\mu\nu} \Delta_{\alpha\beta}^{\mu\nu} = I_{\alpha\beta}^{\mu\nu}$. Here, $P^{-1}$ is the inverse operator to $P$ defined in Eq. 7. To understand the structure of these operators better it is advantageous to separate them into parts according to the dependence on $\xi$, so that $G = G_c + \xi G_f$ and $\Delta = \Delta_c + \frac{1}{\xi} \Delta_{gf}$. The parts turn out to be:

$$ G_{\alpha\beta}^{\mu\nu} = P_{\alpha\beta}^{\mu\nu} - 2l^2 \frac{l^2}{l^2} I_{\alpha\beta}^{\sigma\sigma}, $$

$$ \Delta_{\alpha\beta}^{\mu\nu} = P_{\alpha\beta}^{\mu\nu} - 2l^2 \frac{l^2}{l^2} P_{\alpha\beta}^{\sigma\sigma}, $$

$$ G_{\alpha\beta}^{\mu\nu} = 2l^2 \frac{l^2}{l^2} I_{\alpha\beta}^{\sigma\sigma}, $$

$$ \Delta_{\alpha\beta}^{\mu\nu} = 2P_{\alpha\beta}^{\mu\nu} \frac{l^2}{l^2} P_{\alpha\beta}^{\sigma\sigma}. $$

ACKNOWLEDGMENTS

I want to thank Poul Henrik Damgaard and Emil Bjerrum-Bohr for helpful discussions and comments.

Appendix A: Feynman Rules

The Feynman rules in covariant de Donder-type gauge are derived in similar fashion as other gauge choices such as the background field method in $D = 4$ [10, 20] and supergravity in de Donder gauge in arbitrary dimensions [12].

We use the path integral method and expand the metric around flat space-time $g_{\mu\nu} = \eta_{\mu\nu} + k h_{\mu\nu}$. We raise and lower all indices in this section with $\eta^{\mu\nu} = \eta_{\mu\nu}$ and use the standard notations $I_{\alpha\beta}^{\mu\nu} = \frac{1}{2} (\eta_{\mu}^{\alpha} \eta_{\nu}^{\beta} + \eta_{\nu}^{\alpha} \eta_{\mu}^{\beta})$ and $P_{\alpha\beta}^{\mu\nu} = I_{\alpha\beta}^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \eta_{\alpha\beta}$. The terms in the action which are relevant for the one-loop computation are $h^2, \phi^2, h\phi^2$ and $h^3$. Expanding the action from Eq. 1 in $h_{\mu\nu}$, we get

$$ S \approx \frac{1}{2} \int d^D x \left( \eta_{\mu\nu} \left( \delta_{\alpha\beta}^{\mu\nu} - 2(1 - \frac{1}{\xi}) \frac{l^2}{l^2} \frac{l^2}{l^2} \right) h_{\alpha\beta}^{\mu\nu} \right) + \frac{1}{2} \int d^D x \left( \partial^\mu \phi \partial_\mu \phi - m^2 \phi^2 \right) $$

$$ - \frac{\kappa}{2} \int d^D x \left( \eta_{\mu\nu} \partial^\mu \partial_\nu \phi - \frac{1}{2} h^0 (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) \right) + \kappa \int d^D x U^{\alpha\beta} \phi^{\gamma\delta} h_{\mu\nu} h_{\alpha\beta,\rho} h_{\gamma\delta,\sigma} , $$

where we have introduced the tensor $U^{\alpha\beta} \phi^{\gamma\delta}$ to describe the cubic graviton vertex. This tensor can be read off from the expression

$$ U^{\alpha\beta} \phi^{\gamma\delta} h_{\mu\nu} h_{\alpha\beta,\rho} h_{\gamma\delta,\sigma} = -h_{\mu\nu} h_{\rho,\sigma} h_{\alpha\beta}^{\gamma\delta} + h_{\mu\nu} h_{\rho,\mu} (h_{\rho,\sigma} - h_{\rho,\sigma}^{\gamma\delta}) + h_{\mu\nu} h_{\rho,\rho} (h_{\sigma,\delta} - h_{\sigma,\delta}^{\gamma\delta}) $
And they obey simple identities
\[
\Delta_c G_{gf} = \Delta_g G_c = 0, \quad (A7a)
\]
\[
\Delta G = \Delta_c G_c + \Delta_g G_{gf} = I, \quad (A7b)
\]
in which we have left out indices, but matrix multiplication is understood. The graviton propagator is then
\[
\frac{i}{l^2 + i\varepsilon} G_{\alpha\beta}, \quad (A8)
\]
and the scalar propagator \( \frac{i}{r - m + i\varepsilon} \).

The \( h\phi^2 \) vertex is relatively simple:
\[
\begin{align*}
  p - k & \ 

\end{align*}
\]

And the \( h^3 \) vertex:
\[
\begin{align*}
  &

\end{align*}
\]

Appendix B: Classical derivation of the Schwarzschild-Tangherlini metric in de Donder-type coordinates

We perform an independent calculation of the Schwarzschild-Tangherlini metric in de Donder-type coordinates that satisfy \( G_c = 0 \). We change coordinates from the standard Schwarzschild-Tangherlini metric in spherical coordinates to new cartesian-like coordinates which we determine perturbatively to obey the de Donder-type gauge condition. The method is analogous to that in Weinberg [3] for harmonic gauge in \( D = 4 \).

In standard coordinates, the Schwarzschild-Tangherlini metric is given by (see e.g. [14]):
\[
d\tau^2 = (1 - \frac{\mu}{R^n})dt^2 - \frac{1}{1 - \frac{\mu}{R^n}}dR^2 - R^2d\Omega_{D-2}^2. \quad (B1)
\]

Here \( n = D - 3 \) and \( \mu \) is the Schwarzschild-Tangherlini parameter from Eq. (14). The Schwarzschild-Tangherlini metric solves the Einstein equations in arbitrary dimensions \( D \).

We change the radial coordinate \( R \) into a new one \( r \) and then go to cartesian coordinates with respect to \( r \). We determine the relationship between \( r \) and \( R(r) \) perturbatively so that the cartesian-like coordinates obey the gauge condition \( G_c = 0 \). The metric in terms of the new coordinates in the inertial frame of the point particle is
\[
d\tau^2 = B dt^2 - \frac{1}{B} \frac{dR^2}{dr^2} \left( \frac{\bar{x}\bar{d}\bar{x}}{r} \right)^2 - \frac{R^2}{r^2} \left( d\bar{x}^2 - \left( \frac{\bar{x}\bar{d}\bar{x}}{r} \right)^2 \right), \quad (B2)
\]
where \( B = 1 - \frac{\mu}{R^n} \) and \( r^2 = |\bar{x}|^2 \). We generalize to the covariant notation of Eqs. (8) and get
\[
g_{\mu\nu} = B \eta_{\mu\nu} + \frac{R^2}{r^2} \eta_{\mu\nu} + \left( \frac{1}{B} \frac{dR^2}{dr^2} - \frac{R^2}{r^2} \right) \frac{x^\mu x^\nu}{x_\bot^2}, \quad (B3)
\]
where now \( r^2 = -x_\bot^2 \). We expand the Schwarzschild-Tangherlini radial coordinate \( R \) in terms of the new coordinate \( r \) in powers of \( \mu \):
\[
R = r \left( 1 + a \frac{\mu}{r^n} + b \left( a \frac{\mu}{r^n} \right)^2 + ... \right). \quad (B4)
\]

Where \( a \) and \( b \) are to be determined by the condition \( G_c = 0 \). As we will see, this expansion is not sufficient in \( D = 5 \) and the coefficient \( b \) has to changed \( b \rightarrow b_0 + b_1 \ln \frac{r}{r_0} \). For now we will ignore \( D = 5 \) and continue. Inserting our expansion of \( R \) into the metric Eq. (B3) we get an expansion of the metric depending on the coefficients \( a \) and \( b \):
\[
g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + ... \quad (B5)
\]

The gauge fixing function is expanded similarly
\[
G_\sigma \approx h^{(1)}_{\sigma,\mu} - \frac{1}{2} h^{(1)}_{\mu,\sigma} \quad (B6)
\]

\[
+ h^{(2)}_{\sigma,\mu} - \frac{1}{2} h^{(2)}_{\mu,\sigma} - \alpha \left( h^{(1)}_{\mu,\sigma} h^{(1)}_{\sigma,\nu} - \frac{1}{2} h^{(1)}_{\mu,\nu} h^{(1)}_{\sigma,\sigma} \right),
\]

where the first line is the first order term and the second line the second order term. Perturbatively \( G_\sigma = 0 \) means that each line vanishes by itself. The first order term of \( G_\sigma \) determines the coefficient \( a \). We compute \( h^{(1)}_{\mu\nu} \) in terms of \( a \)
\[
h^{(1)}_{\mu\nu} = \frac{\mu}{r^n} \left( - \eta_{\mu\nu}^{\parallel} + 2a\eta_{\mu\nu}^{\perp} - 2na - 1 \right) \frac{x^{\mu}_{\perp} x^{\nu}_{\perp}}{x^{\perp}_{\perp}}, \quad (B7)
\]
from which we find the first order gauge condition
\[
h^{(1)}_{\mu,\nu} - \frac{1}{2} h^{(1)}_{\mu,\sigma} = \frac{\mu}{r^{n+1}} (2na - 1) \frac{x^{\perp}_{\sigma}}{x^{\perp}_{\perp}}, \quad (B8)
\]
so that \( G_\sigma = 0 \) means \( a = \frac{1}{2n} \). Eq. (B7) then agrees with our tree-level result.

Going to second order we find an expression for \( h^{(2)}_{\mu\nu} \) in terms of \( b \):
\[
h^{(2)}_{\mu\nu} = \frac{\mu^2}{r^{2n}} \left( \frac{1}{2} \eta_{\mu\nu}^{\parallel} + \frac{2b}{4n^2} \eta_{\mu\nu}^{\perp} - \frac{4b}{4n} \frac{x^{\mu}_{\perp} x^{\nu}_{\perp}}{x^{\perp}_{\perp}} \right).
\]

The second order gauge condition reads:
\[
h^{(2)}_{\sigma,\mu} - \frac{1}{2} h^{(2)}_{\mu,\sigma} = \alpha \left( h^{(1)}_{\sigma,\nu} h^{(1)}_{\mu,\nu} - \frac{1}{2} h^{(1)}_{\mu,\nu} h^{(1)}_{\sigma,\sigma} \right). \quad (B10a)
\]
For the right hand side we find
\[ h_{\sigma\mu}^{(1)} h_{\mu\nu}^{(1)} = \frac{1}{2} \rho^{(1)\mu \nu} = -\frac{\rho^2}{2} \frac{n + 1}{r^{2n+1}} \frac{x_\sigma^+}{r} , \quad \text{(B10b)} \]
and the left hand side:
\[ h_{\sigma\mu}^{(2)} h_{\mu\nu}^{(2)} = -\frac{\rho^2}{2} \frac{n^2 + 1 + (n-2)b}{r^{2n+1}} \frac{x_\sigma^+}{r} . \quad \text{(B10c)} \]
Combining Eqs. (B10) we determine \( b \) to be:
\[ b = \frac{-1 - \alpha}{n + 1} \text{.} \quad \text{(B11)} \]
We see that \( b \) diverges in \( D = 5 \) which means our choice of expansion of \( R(r) \) must be changed in \( D = 5 \). Inserting \( b \) into \( h_{\mu\nu}^{(2)} \) in Eq. (B9) produces the same result as our one-loop computation for \( D \neq 5 \).

1. Appearance of a Logarithm in \( D = 5 \)

In \( D = 5 \) it is necessary to generalize the expansion of \( R \) in terms of \( r \). We change Eq. (B4) into
\[ R = r \left( \frac{\mu}{r} + (b_0 + b_1 \ln \frac{r}{r_0}) \left( \frac{\mu}{r} \right)^2 + \ldots \right) , \quad \text{(B12)} \]
where we have let \( b \to b_0 + b_1 \ln \frac{r}{r_0} \). We repeat the analogous steps as above with the new expansion of \( R \). For example Eq. (B10c) changes into
\[ h_{\sigma\mu}^{(2)} h_{\mu\nu}^{(2)} = -\frac{\rho^2}{2} \frac{x_\sigma^+}{r} \left( \frac{n^2 + 1 + (n-2)b}{2n} \right) \frac{3n - 2}{4n^2} b_1 \right) , \quad \text{(B13)} \]
where \( b = b_0 + b_1 \ln \frac{r}{r_0} \). The second order gauge condition Eq. (B10a) becomes:
\[ (n-2)b \left( 3n - 2 \right) b = an(n+1) - n^2 - 1 \quad \text{(B14)} \]
This equation is identical to the one that determined \( b \) above in Eqs. (B10) only that the term with \( b_1 \) is new and that \( b \) now includes a logarithmic term. For \( D \neq 5 \) we are forced to remove the logarithmic dependence in \( b \) so that \( b_1 = 0 \). However, in \( D = 5 \) we find that \( b_1 = 5 - 6\alpha \neq 0 \) while both \( b_0 \) and \( r_0 \) are arbitrary. We compute \( h_{\mu\nu}^{(2)} \) in terms of \( b_0 \) and \( b_1 \) for \( D = 5 \)
\[ h_{\mu\nu}^{(2)} = \frac{\rho^2}{r^2} \left( \frac{1}{2} \eta_{\mu\nu} + \frac{2b}{16} \eta_{\mu\nu} - \frac{4b - b_1}{8} x_\mu^+ x_\nu^+ \right) , \quad \text{(B15)} \]
where again \( b = b_0 + b_1 \ln \frac{r}{r_0} \). Inserting \( b_1 = 5 - 6\alpha \) and \( b_0, r_0 \) arbitrary produces the same result as our 1-loop calculation for \( D = 5 \).

[1] M. Duff, Quantum Tree Graphs and the Schwarzschild Solution, Phys. Rev. D 7, 2317 (1973).
[2] N. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Plant, and P. Vanhove, General Relativity from Scattering Amplitudes, Phys. Rev. Lett. 121, 171601 (2018).
[3] W. D. Goldberger and I. Z. Rothstein, An Effective field theory of gravity for extended objects, Phys. Rev. D 73, 104029 (2006).
[4] D. A. Kosower, B. Maybee, and D. O’Connell, Amplitudes, Observables, and Classical Scattering, JHEP 02, 137 (2018).
[5] C. Cheung, I. Z. Rothstein, and M. P. Solon, From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion, Phys. Rev. Lett. 121, 251101 (2018).
[6] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order, Phys. Rev. Lett. 122, 201603 (2019).
[7] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Black Hole Binary Dynamics from the Double Copy and Effective Theory, JHEP 10, 206 (2019).
[8] A. Cristofoli, N. Bjerrum-Bohr, P. H. Damgaard, and P. Vanhove, Post-Minkowskian Hamiltonians in general relativity, Phys. Rev. D 100, 084040 (2019).
[9] M.-Z. Chung, Y.-T. Huang, and J.-W. Kim, Kerr-Newman stress-tensor from minimal coupling to all spin, (2019).
[10] J. F. Donoghue, Introduction to the effective field theory description of gravity, in Advanced School on Effective Theories (1995) arXiv:gr-qc/9512024.
[11] A. Cristofoli, P. H. Damgaard, P. Di Vecchia, and C. Heissenberg, Second-order Post-Minkowskian scattering in arbitrary dimensions, (2020).
[12] A. K. Collado, P. Di Vecchia, R. Russo, and S. Thomas, The subleading eikonal in supergravity theories, JHEP 10, 038 (2018).
[13] A. Petrov and J. B. Pitts, The Field-Theoretic Approach in General Relativity and Other Metric Theories. A Review, (2020).
[14] R. Emparan and H. S. Reall, Black Holes in Higher Dimensions, Living Rev. Rel. 11, 6 (2008).
[15] Z. Bern, J. J. M. Carrasco, and H. Johansson, Perturbative Quantum Gravity as a Double Copy of Gauge Theory, Phys. Rev. Lett. 105, 061602 (2010).
[16] C. Cheung and M. P. Solon, Classical Gravitational Scattering at O(G^4) from Feynman Diagrams, (2020).
[17] C. Cheung and G. N. Remmen, Twofold Symmetries of the Pure Gravity Action, JHEP 01, 104 (2019).
[18] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, Quantum corrections to the Schwarzschild and Kerr metrics, Phys. Rev. D 68, 084005 (2003) [Erratum: Phys.Rev.D 71, 069904 (2005)], arXiv:hep-th/0211071.

[19] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley and Sons, New York, 1972).

[20] N. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, Quantum gravitational corrections to the nonrelativistic scattering potential of two masses, Phys. Rev. D 67, 084033 (2003) [Erratum: Phys.Rev.D 71, 069903 (2005)], arXiv:hep-th/0211072.