QUASI-ISOMETRIC EMBEDDINGS FROM THE GENERALISED 
THOMPSON’S GROUP TO THOMPSON’S GROUPS $T$

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Abstract. Brown has defined the generalised Thompson’s group $F_n$, $T_n$, $V_n$ where $n$ is an integer at least 2 and Thompson’s groups $F = F_2$, $T = T_2$, $V = V_2$ in the 80’s. Burillo, Cleary and Stein have found that there is a quasi-isometric embedding from $F_n$ to $F_m$ where $n$ and $m$ are positive integers at least 2. We show that there is a quasi-isometric embedding from $T_n$ to $T_2$ for any $n \geq 2$ and no embeddings from $T_2$ to $T_n$ for $n \geq 3$.

1. Introduction

Thompson’s groups, $F$, $T$ and $V$ are some of the most mysterious groups being investigated in the last half century, and were first introduced by Richard Thompson for finding out finitely presented groups with unsolvable word problem. These groups were later found to have connections to string rewrite systems, combinatorics, homotopy theory and large scale geometry, etc [4, 5, 9]. The groups can be considered as subgroups of the automorphism group of the Cantor set. There are purely algebraic interpretations of $F$ and $T$ as finitely presented groups [7],

\begin{equation}
F = \langle A, B \mid [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2] \rangle
\end{equation}

where $[x, y] = xyx^{-1}y^{-1}$, and

\begin{equation}
T = \langle A, B, C \mid [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2], C^{-1}B(A^{-1}CB),
\quad ((A^{-1}CB)(A^{-1}BA))^{-1}B(A^{-2}CB^{-2}), (CA)^{-1}(A^{-1}CB)^2, C^3 \rangle.
\end{equation}

The groups have been generalised by Brown [6] in the late 80’s to the families $F_n$, $T_n$ and $V_n$, where $n$ is an integer greater than or equal to 2, which are the groups we are interested in, and where $F = F_2$, $T = T_2$ . Stein later on further generalised the groups into Thompson-Stein groups [7,10]. A comprehensive introduction is given in the introductory notes by Cannon, Floyd and Parry [7].

Burillo [2] first studied undistorted subgroups of $F$. Burillo, Cleary and Stein [3] give a detailed construction of embeddings from $F_n$ into $F_m$ for any natural numbers $n, m \geq 2$. An estimate on the word length of the elements in the generalised Thompson’s group $F_n$ is given through its unique normal form and it serves as an ingredient for proving that the embedding from $F_n$ into $F_m$ is quasi-isometric.

Liousse [10] has investigated the rotation numbers for Thompson-Stein groups and raised the question: Is it possible to describe all Thompson-Stein groups up to isomorphism, up to quasi-isomorphism? She proves that selected Thompson-Stein groups are not isomorphic to each other.
In this paper, we focus on Thompson’s group $T_2$ and their generalisation $T_n$ \[6\]. We prove that there is a quasi-isometric embedding from $T_n$ to $T_2$ for any integer $n \geq 2$ and there is no embedding from $T_2$ to $T_n$ where $n \geq 3$.

2. Background

2.1. Thompson’s group $F$ and its generalisations $F_n$. We inherit the interpretations from \[5\] and \[7\]. Consider the interval $[0, 1]$ with finitely many distinguished points at dyadic rationals and we have the following.

**Definition 2.1.** Let the interval $[0, 1]$ be divided so that the subintervals are only in the form $[\frac{\ell}{2^k}, \frac{\ell+1}{2^k}]$ where $\ell + 1 < 2^k$ and $\ell, k \in \mathbb{N} \cup \{0\}$. When the interval $[0, 1]$ is divided in such a way and the number of subintervals are finite, then the division is called a dyadic subdivision and the interval $[0, 1]$ with a dyadic subdivision is called a dyadically subdivided interval.

**Definition 2.2** (\[7\]). $F$ is defined as the group of orientation-preserving piecewise-linear homeomorphisms from the unit interval $[0, 1]$ to itself which map dyadically subdivided intervals to dyadically subdivided intervals.

The following piecewise-linear functions are given as a finite list of generators of $F$

\[
A(t) = \begin{cases} 
\frac{1}{2}t & \text{if } 0 \leq t \leq \frac{1}{2} \\
\frac{1}{2}t - \frac{1}{4} & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\
2t - 1 & \text{if } \frac{3}{4} \leq t \leq 1 
\end{cases} \quad B(t) = \begin{cases} 
\frac{1}{2}t + \frac{1}{4} & \text{if } 0 \leq t \leq \frac{1}{2} \\
t - \frac{1}{4} & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\
2t - 1 & \text{if } \frac{3}{4} \leq t \leq 1 
\end{cases}
\]

A generalised version of $F$ can be defined as follows,

**Definition 2.3.** $F_n$ is defined as the group of orientation-preserving piecewise-linear homeomorphisms from the $n$-adically subdivided intervals to the $n$-adically subdivided intervals.

2.1.1. An alternative view on $F$. To see the elements of $F$ from another point of view, we first consider binary trees. Define a 2-caret to be a vertex with two edges attached to it.

Then, we can relate a dyadically subdivided interval $[0, 1]$ to a binary tree by associating a subinterval with a breakpoint in the middle to a 2-caret. A binary tree can be expanded simply by attaching 2-carets at some leaves and reduced by deleting 2-carets.

For two binary trees with the same number of leaves, we label the leaves of each tree from the left to right by $0, 1, 2, 3, \cdots, n$, and denote these two trees by $\mathcal{U}$ and $\mathcal{V}$, respectively, and denote the pair by $\langle \mathcal{U}, \mathcal{V} \rangle$. All such pairs form a set $\mathcal{S}$, i.e.

\[\mathcal{S} = \{ \langle \mathcal{U}, \mathcal{V} \rangle \mid \mathcal{U}, \mathcal{V} \text{ are trees which have the same number of leaves} \} \]

A tree pair as described above can be associated with some piecewise-linear functions mapping $[0, 1]$ to itself and we call the former tree the “source tree” and the later tree the “target tree”.

For a tree pair $\langle \mathcal{U}, \mathcal{V} \rangle$, when we attach the roots of two 2-carets to each of the leaves on $\mathcal{U}$ and $\mathcal{V}$ with the same numeric labeling, the corresponding piecewise-linear function is preserved and we call this procedure “simple expansion”. Similarly, when we delete two 2-carets with the same
labeling at its leaves on each tree, the resulting tree pair still represents the same piecewise-linear function, “simple contraction”.

Note that “simple expansion” of a tree pair here is different from the expansion of a single tree that we described above. A tree pair is reduced, if one can apply no more simple contractions on it.

We define an equivalence relation ∼ on the set of tree pairs \( \mathcal{T} \) as follows. Let \((T, S)\) and \((T', S')\) ∈ \( \mathcal{T} \), we say that \((T, S) \sim (T', S')\), if one can be obtained from the other by applying finite number of simple expansions and contractions. We hence denote the equivalence class of \((T, S)\) by \([T, S]\). It is proved in [3] that every element of \( F_n \) is represented by a unique reduced tree pair.

2.1.2. Binary operation. Let \((T, S)\) and \((U, V)\) be two tree pairs in \( \mathcal{T} \). By applying simple expansions (or simple contractions) to \((T, S)\) and \((U, V)\) so that the trees \( S \) and \( U \) become identical trees, say \( R \), we obtain the product of the two tree pairs, the pair \((W, Y)\), as the result.

\((W, Y)\) is again a tree pair with the same number of leaves. This operation induces the multiplication \(*\) on the equivalence class \( \mathcal{T} / \sim \). One can check that \( \mathcal{T} / \sim \) with \(*\) forms a group.

Similarly, the elements in \( F_n \) can be represented as equivalence classes of some \( n \)-ary tree pairs whose source and target tree have the same number of leaves.

2.2. Thompson’s group \( T \) and its generalisation \( T_n \). \( T \), known as a subgroup of the group homeomorphisms of the unit interval with two end points identified \( ([0, 1] / \sim = S^1) \) is defined as follows.

**Definition 2.4.** We call a dyadically subdivided interval \([0, 1]\) with the end point identified, a dyadically subdivided circle \( S^1 \), or a circle with dyadic subdivision.

**Definition 2.5.** \( T \) is defined as the group of orientation-preserving piecewise-linear homeomorphisms of \( S^1 \) which take a dyadically subdivided \( S^1 \) to a dyadically subdivided \( S^1 \) using affine maps on each subinterval.

The element expressed as a piecewise-linear function below is a torsion element in \( T \). Together with \( A(t) \) and \( B(t) \) in \( F \), (Section 2), they generate \( T \) [7].

\[
C(t) = \begin{cases} 
  t + \frac{1}{2} & \text{if } 0 \leq t \leq \frac{1}{2} \\
  t - \frac{1}{2} & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

Since the elements in \( T \) are homeomorphisms of \( S^1 \) rather than the unit interval, we express these elements as binary tree pairs with “cyclic” labeling.

We explain what “cyclic” labeling means below. For a binary tree we pick some leaf and label it by 0, then the leaves following it on the right are labeled by non negative integers \( 1, \cdots, k \) in order until the rightmost one. If all the leaves are labelled, then we are done. The label is exactly as in a tree of the tree pairs representing \( F \). If there are still leaves which are not labelled, these leaves are the ones on the left of the leaf labelled 0. We label these leaves from the leftmost by \( k + 1 \) onwards until reaching the leaf on the left of the leaf labeled by 0.
Consider the set of binary tree pairs, the binary tree pairs \((U, V)\) being labelled cyclically forms a set. By defining the equivalence relation and binary operation as in the case of \(F\), we obtain a group isomorphic to \(T\).

Every element of \(T\) as a tree pair also has a unique reduced tree pair. The uniqueness of tree pair representing elements in \(F\) first proved in [7] and the analogue in \(T\) is explained in details in [4].

Analogously, the generalised groups \(T_n\) can be defined as follows,

**Definition 2.6.** \(T_n\) is defined as the group of orientation-preserving piecewise-linear homeomorphisms of \(S^1\) which take an \(n\)-adically subdivided \(S^1\) to an \(n\)-adically subdivided \(S^1\) using affine maps on each subinterval.

In the case of \(T_n\), we have elements which map subintervals on the \(n\)-adic subdivision of \(S^1\) to subintervals on the other \(n\)-adic subdivision of \(S^1\) with the same number of breakpoints, hence, the trees in tree pairs are \(n\)-ary trees. The uniqueness of the reduced tree pairs in \(T_n\) can be proved as in the case of \(T\).

2.3. **Known results.** In [3], Burillo, Cleary and Stein have proved that for \(F_p\) and \(F_q\) where \(p\) and \(q\) are positive integers at least 2, there is a quasi-isometric embedding from one to the other defined by the “caret-replacement map”. Here \(q\)-carets in \(q\)-ary tree pairs representing elements in \(F_q\) are replaced by a possible arrangement of \(p\)-carets to form \(p\)-ary tree pairs representing elements in \(F_p\). For the later argument, we give a proof of the special case which resembles the one of [8, Subsection 5.3].

**Lemma 2.1.** There is an embedding from \(F_n\) to \(F_2 = F\).

**Proof.** Define a map \(\psi : \{n\text{-ary tree pairs}\} \rightarrow \{\text{binary tree pairs}\}\) as follows. For every element in \(F_n\) represented as an \(n\)-ary tree pair, we replace every \(n\)-caret by an all-right binary tree (all carets except for the top caret are attached to the right most leaf of the previous caret) with \(n\) leaves. By this replacement, we can always obtain a binary tree pair from an \(n\)-ary one.

To each simple expansion on a \(n\)-ary tree pair \((U, V)\), we apply simple expansions \(n - 1\) times on \(\psi((U, V))\) by adding an all-right binary tree with \(n\) leaves to the corresponding leaf. To each simple contraction, we apply the converse operation. These expanding or contracting operations obviously commute with the map \(\psi\), and hence, \(\psi\) induces a group homomorphism \(\psi_* : F_n \rightarrow F_2\).

It remains to show that \(\psi_*\) is injective. We consider a non identity element in \(F_n\) represented by a reduced \(n\)-ary tree pair which consists of two different trees. The image of this tree pair by \(\psi\) representing an element in \(F_2\) must consist of two different trees. Therefore, \(\psi_*\) is injective. \(\square\)

3. **Quasi-isometric embeddings**

3.1. **Elements in \(T_n\) represented as tree pairs.** From this subsection on, we will be focusing on the tree pair representation of the elements in \(T_n\). When we talk about number of leaves or the number carets, we are talking about the number of those in either tree of a tree pair.

3.1.1. **Torsion elements in \(T_n\).** \(T_n\) obviously contains torsion elements as \(T_2\) does, and Burillo, Cleary, Stein and Taback found a convenient tree pair representative of a torsion element as follows.
Theorem 3.1 (\cite{4} Prop.6.1). If $f \in T_2$ is a torsion element, then it can be represented by a (labelled) tree pair with the same source and target trees.

Proposition 3.1.1. If $f \in T_n$ is a torsion element, then it can be represented by a (labelled) tree pair with the same source and target trees.

Proof. Let $m$ be the order of $f$. Following the notation in \cite{4}, we denote $f = (S, T)$ for simplicity and we write $f = (S, f(S))$ for the convenience of the later argument. Expansion here simply means the expansion of a single tree.

We first construct the tree pair for $f^k = (S_k, T_k)$ for $k < m$ and then we could prove our hypothesis: $T_{k+1}$ is an expansion of $T_k$, by induction.

- The base case: we have $f^2 = (S, T)(S, T) = (f^{-1}(E_1), E_1)(E_1, f(E_1))$, where $E_1$ is the minimal joint expansion of $T$ and $S$. Then define $f^2 = (S_2, T_2)$ to be $(f^{-1}(E_1), f(E_1))$. Then $T_2 = f(E_1) \supset T$.

- The inductive step: we assume the hypothesis is true for the case $k < m$. We have the following,

  $$(S_{k-1}, T_{k-1})(S, T) = (f^{-(k-1)}(E_{k-1}), E_{k-1})(E_{k-1}, f(E_{k-1})) = (S_k, T_k),$$

  where $E_{k-1}$ is the minimal joint expansion $T_{k-1}$ and $S$.

  $$(S_k, T_k)(S, T) = (f^{-k}(E_k), E_k)(E_k, f(E_k)) = (S_{k+1}, T_{k+1})$$

  where $E_k$ is the minimal expansion of $T_k$ and $S$. Since $E_k$ is an expansion of both $T_k$ and $S$, and in particular so does $T_{k-1}$ by the induction hypothesis. Also since $E_{k-1}$ is a minimal expansion of $T_{k-1}$ and $S$, $E_k$ is an expansion of $E_{k-1}$. Thus, $T_{k+1} = f(E_k) \supset f(E_{k-1}) = T_k$

By this inductive construction, we have $id = f^m = (S_m, T_m)$, and thus $S_m = T_m$. $(S_m, T_m)$ is represented by

$$(S_m, T_m) = (f^{-(m-1)}(E_{m-1}), E_{m-1})(E_{m-1}, f(E_{m-1})) = (S_{m-1}, T_{m-1})(S, T).$$

Since $T_m = f(E_{m-1})$ is an expansion of $T_{m-1}$, and $E_{m-1}$ is the minimal joint expansion of both $T_{m-1}$ and $S$, $f(E_{m-1})$ is an expansion $E_{m-1}$. Also since $f(E_{m-1})$ and $E_{m-1}$ are the target and source trees of $f^{m-1}$, which indicates that they have the same number of caret. Hence $E_{m-1} = f(E_{m-1})$ and they the identical trees in one tree pair.

Having the above result, we could obtain upper bounds for the order of torsion elements in $T_n$ simply by counting the number leaves of a tree. The following proposition we obtain is a result of this argument with \cite{10} Theorem 1.

Proposition 3.1.2. The order of a torsion element in $T_n$ is a divisor of $l(n-1) + 1$ for some integer $l \geq 0$.

Proof. Following Proposition 3.1.1 we know that a torsion element always shifts the labeling of leaves. Let a tree pair $(T, T)$ represent a torsion element in $T_n$. The number of leaves in $T$ is $l(n-1) + 1$, where $l$ is the number of caret. Hence the order of an element is a divisor of the number of leaves.
3.1.2. The tree pair representatives. Recall the presentation of $F_2$ in (1), define $x_0 = A(t)^{-1}$, $x_1 = B(t)^{-1}$ and 
$$x_i = x_0^{-1} x_{i-1} x_0$$
for integer $i \geq 2$ recursively. By the definition above, $x_i$'s are also maps of dyadically subdivided unit intervals and can be represented as binary tree pairs. These tree pairs satisfy the infinite presentation of $F_2$ as follows (see for instance [7]),

$$F_2 = \langle x_i, i \geq 0 \mid x_i^{-1} x_j x_i = x_{j+1} \text{ for } i < j \rangle. \tag{3}$$

$F_n$ has a quite similar infinite presentation. Start with a finite set $\{x_0, x_1, \ldots, x_{n-1}\}$ represented by $n$-ary tree pairs in Figure 1 and define 
$$x_i = x_0^{-1} x_{i-(n-1)} x_0$$
for integer $i > n - 1$ recursively. Then, it is shown in [3] that $F_n$ admits an infinite presentation as follows,

$$F_n = \langle x_i, i \geq 0 \mid x_i^{-1} x_j x_i = x_{j+n-1} \text{ for } i < j \rangle. \tag{4}$$

Let us denote the finite generating set $\{x_i \in F_n \mid 0 \leq i < n\}$ by $\Sigma_n$ and the infinite generating set $\{x_i \in F_n \mid i \geq 0\}$ by $\Sigma'_n$.

An infinite family of torsion elements denoted by $c_{k-1}$ for positive integers $k$ in $T_n$ can be defined as a pair of all right trees with $k$ cares in $T_n$ (Figure 2), where $c_{k-1}$ is labeled so that the labelling the target tree is shifted to the left by one the source tree. $c_1$ can be identified with $C(t)$.

It is known by [4] that

$$\Sigma = \{x_0, x_1, \ldots, x_{n-1}, c_0\},$$

$$\Sigma' = \Sigma \cup \{c_k \mid 1 \leq k\} = \{x_0, x_1, \ldots, x_{n-1}, c_0, c_1, \ldots\},$$
both form a generating set of $T_n$. We will use both for the later argument.

A finite presentation of $T_2$ is given in [7], from which Burillo, Cleary, Stein and Taback [4] have deduced what they call the pumping lemma by which they can reduce $c_k$ by a word in $c_{k-1}$ and $x_i$’s. We have found an analogue of pumping relations for $T_n$ through the computation of the tree pairs representing elements in $T_n$.

Lemma 3.2. If $\ell < (k-1)(n-1)+1 = \text{ord} c_{k-1}$, then

\[
\begin{align*}
\ell c_k &= \ell_{k-1} x_{k(n-1) - \ell}, \\
(c_{k-1}^{-1})^\ell &= x_{k(n-1) - \ell} (c_{k-1}^{-1})^\ell.
\end{align*}
\]

Proof. Both identities are proved through computation using standard computation. □

3.2. A construction of an embedding from $T_n$ to $T_2$.

Theorem 3.3. There is an embedding from $T_n$ to $T_2 = T$.

Proof. Define a map $\phi : \{\text{labelled } n\text{-ary tree pairs}\} \to \{\text{labelled binary tree pairs}\}$ as follows. For every element in $T_n$ represented as an $n$-ary tree pair, we replace every $n$-caret by an all-right binary tree with $n$ leaves. Hence, we can always obtain a binary tree pair from an $n$-ary one. $\phi$ induces a group homomorphism $\phi_* : T_n \to T_2$ by the fact that simple expansion (contraction) is compatible with the operations in groups. Injectivity is proved similar as in the case of $F_n \to F$. □

Remark. This argument also works for the generalised group in the Thompson’s group $V$ case, however, we are not going to discuss here.

Despite of the embedding result of $F_2$ to $F_n$ in [3], we cannot go other way around.

Theorem 3.4. There is no injective homomorphism from $T_2$ to $T_n$ where $n > 2$.

Proof. By Proposition 3.1.2 as well as [10] Consequence 3 in Theorem1 we know that $T_n$ does not contain torsion elements of order $n - 1$. However, $T_2$ does contain torsion elements of order $n - 1$. □

Corollary 3.4.1. There is no injective homomorphism from $T_p$ to $T_q$ when $p, q$ are two consecutive numbers.
3.3. An estimate of the word length of elements in $T_n$.

3.3.1. Word length of elements in $T_n$. For estimating the word length of elements in $T_n$, we extend the argument in [4].

Extending the definition in [4, Page 9], we start with the unique reduced tree pair representing an element in $T_n$ and represent it by the concatenation of three words in $\Sigma'$. The formal definition is in the following.

**Definition 3.1 (pcq factorization).** Let the reduced labelled tree pair $(T_-, T_+)$ represent an element in $T_n(V_n)$ and $T_-$ and $T_+$ each have $k$ carets. Let $R$ be the all right tree with $k$ carets.

We write the element as a product $p\sigma q$, where:

1. $p$, a positive word in the infinite generating set $\Sigma_n$ of $F_n$ with the form $p = x_{i_1}^{r_1}x_{i_2}^{r_2} \cdots x_{i_y}^{r_y}$ where $i_1 < i_2 < \cdots < i_y$ and all $r_k$’s are positive. It is represented by $(R, T_+)$.

2. For $\omega$ representing an element in $T_n$, $\sigma = c_k^{\ell}$, where $\ell$ appearing in the exponent of $c_k$ satisfies $0 \leq \ell < k(n - 1) + 1 = \text{ord } c_k$. This element can be represented by $(R, R)$ with appropriate labeling.

3. $q$, a negative word in $\Sigma_n$ of $F_n$ with the form $x_{j_z}^{-s_z} \cdots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$ where $j_1 < j_2 < \cdots < j_z$ and all $s_k$’s are positive. It is represented by $(T_-, R)$.

We call such a product a $\text{pcq}$ factorization.

A $\text{pcq}$ factorization for an element in $T_n$ can be found by seeking a obvious path from $T_{\pm}$ to $R$ starting from the reduced pair $(T_-, T_+)$ (Figure 3). Notice that the tree pairs representing $p$ and $q$ are not necessarily reduced though $(T_-, T_+)$ is. Moreover, when the tree pair representing some element in $F_n$ is reduced, the torsion part of the $\text{pcq}$ factorization represents just an identity element.

![Figure 3. pcq factorisation](image)

For the estimation, we first consider the word length of a word only consisting of $c_k$ with respect to the finite generating set $\Sigma = \{x_0, x_1, \cdots, x_{n-1}, c_0\}$. We let $|w|_{\Sigma}$ denote the word length of $w \in T_n$ with respect to $\Sigma$.

**Proposition 3.4.1.** If $0 < \ell < k(n - 1) + 1 = \text{ord } c_{k-1}$, then $|c_k^{\ell}|_{\Sigma} < 3k + n$. 

Proof. By the division theorem, $\ell - 1$ can be presented uniquely as $q(n-1)+r$ with the remainder $0 \leq r < n-1$, and the quotient $0 \leq q \leq k$ by the assumption for $\ell$. We use the identities in Lemma 3.2 to deduce $c_k$ to the expression in $c_0$ and $x_i$'s. The computation below contains a turning point, to use either the first identity or the second one, when $\ell$ becomes greater than the order of the next finite order element. The following manipulation clarifies how we apply the identities in Lemma 3.2

\[
c_k^\ell = c_k^\ell x_{k(n-1) - \ell} = c_q^{(q+1)(n-1) - \ell} \cdots x_{k(n-1) - \ell} = (c_q^{(q+1)(n-1) - \ell} x_{k(n-1) - \ell} \cdots x_{k(n-1) - \ell})^r = x_{q(n-1) - r}^{(q+1)(n-1) - \ell} \cdots x_{k(n-1) - \ell}
\]

Then, to obtain the word expression in terms of letters in $\Sigma^k_n$, we replace the term $x_\alpha$ not in $\Sigma$ by the relation $x_\alpha = x_0^{-\gamma} x_\delta x_0^{-\gamma}$, where $\alpha = \gamma(n-1) + \delta$ such that $\gamma \geq 0$ and $0 \leq \delta < n-1$. The resulting expression has a cancelling pair between two conjugates such as $x_{\gamma(n-1)} x_{\alpha} = (x_0^{-\gamma} x_\delta x_0^{-\gamma})(x_0^{-\gamma} x_\delta x_0^{-\gamma}) = x_0^{-\gamma} x_\delta x_0^{-\gamma} x_\delta x_0^{-\gamma}$. The upper bound follows immediately. \hfill $\square$

Now we estimate the word length of an element of $T_n$ with respect to the finite generating set $\Sigma = \{x_0, x_1, \cdots, x_{n-1}, c_0\}$.

**Theorem 3.5.** The following inequalities hold,

\[
\frac{N_{\Sigma}(\omega)}{3} \leq |\omega|_{\Sigma} \leq 15(n-1)N_{\Sigma}(\omega) + 3n,
\]

where $N_{\Sigma}(\omega)$ denotes the number of carets of a reduced tree pair representing $\omega$.

**Proof.** Since the $N_{\Sigma}(x_i)$ is at most 3 for $x_i \in \Sigma$,

$N_{\Sigma}(\omega) \leq 3|\omega|_{\Sigma}$.

To see the upper bound, suppose $\omega \in T_n$ has a $\text{pcq}$ factorization,

$\omega = x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_y}^{r_y} c_k^{s_k} x_{j_1}^{s_1} \cdots x_{j_{k}}^{s_k}$,

Then, we define $D_n(\omega) = \sum_{i=1}^{y} r_i + \sum_{j=1}^{k} s_j + i_1 + j_2 + k + n$, and we first prove $|\omega|_{\Sigma} \leq 3D_n(\omega)$. Since

\[
|\omega|_{\Sigma} = |x_{i_1}^{r_1} \cdots x_{i_y}^{r_y} c_k^{s_k} x_{j_1}^{s_1} \cdots x_{j_{k}}^{s_k}|_{\Sigma} \leq |x_{i_1}^{r_1} \cdots x_{i_y}^{r_y}|_{\Sigma} + |c_k^{s_k} x_{j_1}^{s_1} \cdots x_{j_{k}}^{s_k}|_{\Sigma},
\]

replacing $x_{i_\alpha}$ in the positive word by the relation $x_{i_\alpha} = x_0^{-\gamma} x_\delta x_0^{-\gamma}$ where $i_\alpha = \gamma(n-1) + \delta$ such that $\gamma_\alpha$ is nonnegative and $0 \leq \delta_\alpha < n-1$, we can estimate its word length as follows. Notice that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_y$ because $i_1 < i_2 < \cdots < i_y$.

\[
|x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_y}^{r_y}|_{\Sigma} = |x_0^{-\gamma_1} x_{\delta_1}^{s_1} x_0^{-\gamma_1} x_0^{-\gamma_2} x_{\delta_2}^{s_2} x_0^{-\gamma_2} \cdots (x_0^{-\gamma_y} x_{\delta_y}^{s_y} x_0^{-\gamma_y})|_{\Sigma} = |x_0^{-\gamma_1} x_{\delta_1}^{s_1} x_0^{-\gamma_2} x_{\delta_2}^{s_2} x_0^{-\gamma_2} \cdots x_0^{-\gamma_{y-1}} x_{\delta_y}^{s_y} x_0^{-\gamma_{y-1}}|_{\Sigma} \leq r_1 + r_2 + \cdots + r_y + 2k_y.
\]
Similarly,
\[ |x_{j_1}^{-s_1} \cdots x_{j_z}^{-s_z} x_{j_1}^{-s_1}| \leq s_1 + s_2 + \cdots + s_z + 2j_z. \]
Summing up these two with the estimate of Proposition 3.1, we obtain \[ |\omega|_{\Sigma} \leq 3D_n(\omega). \]
Now, \[ |\omega|_{\Sigma} \] can be found by looking at the multiplication of two elements as tree pairs. Recall that the tree pairs representing \( p \) and \( q \) in this setting might be not reduced. Thus \( N_{\Sigma}(\omega) \geq N_{\Sigma}(p), \) \( N_{\Sigma}(\omega) \geq N_{\Sigma}(q). \) Also since we have started with the reduced tree pair for \( \omega, N_{\Sigma}(\omega) = N_{\Sigma}(c_k^l) = k + 1. \) We would like to estimate \( N_{\Sigma}(\omega) \) from below by the average of some estimates for \( N_{\Sigma}(p), N_{\Sigma}(q) \) and \( N_{\Sigma}(c_k^l). \)
Combining with the results in [3, Theorem 5], we have the following,
\[ N_{\Sigma}(p) \geq r_1 + r_2 + \cdots + r_y \geq \frac{1}{n-1}(r_1 + r_2 + \cdots + r_y), \]
\[ N_{\Sigma}(p) \geq \left\lfloor \frac{i_y}{n-1} \right\rfloor + 1 \geq \frac{i_y}{n-1}, \]
Similarly, \( N_{\Sigma}(q) \geq \left\lfloor \frac{j_z}{n-1} \right\rfloor + 1 \geq \frac{j_z}{n-1} \) and \( N_{\Sigma}(c_k^l) = k + 1 > \frac{k}{n-1}. \)
Taking the average of these inequalities, we have
\[ 5N_{\Sigma}(\omega) \geq \frac{1}{n-1}(D_n(\omega) - n). \]
Thus
\[ |\omega|_{\Sigma} \leq 3D_n(\omega) \leq 15(n-1)N_{\Sigma}(\omega) + 3n \]
and we are done.

3.3.2. Word length of elements in \( F_n \subset T_n. \) We then look at the word length of elements in \( F_n \subset T_n, \) and we obtain a similar result of [4, Theorem 5.3], by comparing the word length of elements in \( F_n \) and \( T_n. \)

Lemma 3.6. If \( \omega \in F_n, \) then we have the following,
\[ \frac{|\omega|_{\Sigma_n}}{36(n-1)} \leq |\omega|_{\Sigma} \leq 45(n-1)^2|\omega|_{\Sigma_n} - 1, \]
where \( \Sigma_n = \{x_0, x_1 \ldots x_{n-1}\} \) for \( F_n \) and \( \Sigma = \{x_0, x_1 \ldots x_{n-1}, c_0\} \) for \( T_n. \)

Proof. An element in \( F_n \) can be represented by a word in the \( \textbf{pcq} \) factorization corresponding to the unique reduced tree pair. As mentioned in Definition 3.1, the torsion part of the \( \textbf{pcq} \) factorization of this word is identity, which means that the \( \textbf{pcq} \) factorization of this word the same as the normal form in \( F_n \) defined in [3, Theorem 1.1]. Hence \( N_{\Sigma_n}(\omega) = N_{\Sigma}(\omega). \) By [3, Theorem 1.5] and Theorem 3.3, we have the above inequality.

\( \Box \)
3.4. **Quasi-isometric embeddings.** With the ingredients ready we obtain quasi-isometric embeddings.

**Theorem 3.7.** $F_n$ is embedded in $T_n$ without distortion.

In other words, the inclusion of $F_n$ in $T_n$ is a quasi-isometry. This follows directly from Lemma 3.6.

**Theorem 3.8.** The embedding $\phi_* : T_n \hookrightarrow T_2$ is quasi-isometric.

**Proof.** As defined in Theorem 3.3, $\phi_*$ is a label-preserving map from $T_n$ to $T_2$. We replace the $n$-carets in an $n$-ary tree pair by binary trees. Hence, we have $(n-1)N(\omega) = N(x_0,x_1,c_0)(\phi_*(\omega))$. Then by Theorem 3.5,

$$\frac{n-1}{45} |\omega|_{\Sigma} - \frac{(n-1)(n+3)}{15} \leq |\phi_*(\omega)|_{(x_0,x_1,c_0)} \leq 45(n-1)|\omega|_{\Sigma} + 15,$$

which completes the proof. \(\square\)

**Remark.** This Theorem may also be proved using Thm 3.5 and the independent work by Genevois [1].

By a similar construction we have,

**Corollary 3.8.1.** The embedding $\phi_* : T_{l((m-1)+1} \hookrightarrow T_m$ is quasi-isometric, for any positive integer $m$ and $l$. The embedding $\psi_* : T_p \hookrightarrow T_q$ is quasi-isometric, when $(q-1)$ is divisible by $(p-1)$.

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**Acknowledgments**

First, I owe a great debt of gratitude to Professor Sadayoshi Kojima without whom this work would not be possible. I am also very thankful for Dr Collin Bleak, for carefully reviewing my paper and for precious comments and suggestions. Finally, I would like to thank my supervisor Professor Sakasai for the helpful comments and conversations.

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