BKP tau-functions as square roots of KP tau-functions

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Abstract

It is well-known that a BKP tau-function is the square root of a certain KP tau-function, provided one puts the even KP times equal to zero. In this paper we compute for all polynomial BKP tau-function its corresponding KP "square". We also give, in the polynomial case, a representation theoretical proof of a recent result by Alexandrov, viz. that a KdV tau-function becomes a BKP tau-function when one divides all KdV times by 2.

1 Introduction

In the 1980’s Date, Jimbo, Kashiwara and Miwa, inspired by the pioneering work of Sato [19], described many soliton hierarchies of KP and KdV type [3], [4], [5], [9]. In particular they introduced the BKP hierarchy in [5], which is related to the lie algebra $b_{\infty}$. They define the level one spin module of this infinite dimensional orthogonal Lie algebra by action of certain fermionic creation and annihilation operators on a vacuum vector $|0\rangle$. The BKP hierarchy describes the $B_{\infty}$-group orbit of this highest weight vector $|0\rangle$ in this spin module. Elements in this orbit are the BKP tau-functions, which, in the polynomial case, can be describe as certain Pfaffians of vacuum expectation values. Since the Pfaffian of an anti-symmetric matrix is the square root of the determinant of this matrix, these BKP tau-functions are square roots of certain determinants and in fact Date, Jimbo, Kashiwara and Miwa show that it is the square root of a certain KP tau-function, provided one puts the even KP times equal to zero. This fact, was used by the author and A. Yu. Orlov in [16], in another realization of this $b_{\infty}$ group orbit, to give a representation theoretical proof of the fact that Pfaff Lattice tau-functions are square roots of 2D Toda lattice tau-functions. J. Harnad and A.Yu. Orlov [6]-[8] also use this observation to express KP and BKP tau-functions as sums over products of pairs of $Q$ Schur functions.

In [12], [13], V.G. Kac and the author gave explicit formulas for all KP and also BKP tau-function. In this paper, we calculate for every polynomial BKP tau-function its corresponding square, i.e. the corresponding KP tau-function.

In section [7] we give a representation theoretical explanation, at least in the polynomial case, of a recent result of A. Alexandrov [1], viz. that a KdV tau-function,
which is a KP tau-function that does not depend on the even times, becomes a BKP tau-function when one divides all KdV times by 2.

2 The fermionic formulation of KP

Consider the infinite matrix group \( GL_∞ \), consisting of all complex matrices \( G = (g_{ij})_{i,j \in \mathbb{Z}} \) which are invertible and all but a finite number of \( g_{ij} - \delta_{ij} \) are 0. We denote its Lie algebra by \( gl_∞ \) consisting of all complex matrices \( g = (g_{ij})_{i,j \in \mathbb{Z}} \) for which all but a finite number of \( g_{ij} \) are 0. Both the group and its Lie algebra act naturally on the vector space \( C^∞ = \bigoplus_{j \in \mathbb{Z}} C e_j \) (via the usual formula \( E_{ij}(e_k) = \delta_{jk} e_i \)).

The semi-infinite wedge representation \([14],[12]\). \( F = \Lambda_{-}^{1,∞} C^∞ \) is the vector space with a basis consisting of all semi-infinite monomials of the form \( e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \ldots \), where \( i_0 > i_1 > i_3 > \ldots \) and \( i_{l+1} = i_l - 1 \) for \( l >> 0 \). One defines the representation \( R \) of \( GL_∞ \) and \( r \) of \( gl_∞ \) on \( F \) by

\[
R(G)(e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \cdots) = Ge_{i_1} \wedge Ge_{i_2} \wedge Ge_{i_3} \wedge \cdots .
\]

The corresponding representation \( r \) of the Lie algebra \( gl_∞ \) of \( GL_∞ \) can be described in terms of a Clifford algebra. Define the wedging and contracting operators \( \psi^+_j \) and \( \psi^-_j \) \((j \in \mathbb{Z} + \frac{1}{2})\) on \( F \) by

\[
\psi^+_j(e_{i_0} \wedge e_{i_1} \wedge \cdots) = e_{-j+\frac{1}{2}} \wedge e_{i_0} \wedge e_{i_1} \ldots ,
\]

\[
\psi^-_j(e_{i_0} \wedge e_{i_1} \wedge \cdots) = \begin{cases} 0 & \text{if } j + \frac{1}{2} \neq i_s \text{ for all } s \\ (-1)^s e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_{s-1}} \wedge e_{i_{s+1}} \wedge \cdots & \text{if } j + \frac{1}{2} = i_s. \end{cases}
\]

These operators satisfy the relations \((i,j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -)\):

\[
\psi^\lambda_i \psi^\mu_j + \psi^\mu_j \psi^\lambda_i = \delta_{\lambda,-\mu} \delta_{i,-j},
\]

hence they generate a Clifford algebra, which we denote by \( \mathcal{C} \ell \). Introduce the following elements of \( F \) \((m \in \mathbb{Z})\):

\[
|m\rangle = e_m \wedge e_{m-1} \wedge e_{m-2} \wedge \cdots .
\]

It is clear that \( F \) is an irreducible \( \mathcal{C} \ell \)-module such that

\[
\psi^+_j |0\rangle = 0, \text{ for } j > 0.
\]

It will be convenient to define also the opposite spin module with vacuum vector \( \langle 0 \mid \), here

\[
\langle 0 | \psi^+_j = 0, \text{ for } j < 0,
\]

and for \( m > 0 \) one defines

\[
\langle \pm m \mid = \langle 0 | \psi^+_{\frac{1}{2}} \psi^+_{\frac{3}{2}} \cdots \psi^+_{m-\frac{1}{2}}.
\]
The vacuum expectation value is defined as $\langle a \rangle = \langle 0 | a | 0 \rangle$ and $\langle 0 | 1 | 0 \rangle = 1$. It is straightforward that the representation $r$ of $g\ell_\infty$ is given by the formula $r(E_{ij}) = \psi^+_{i-\frac{1}{2}} \psi^-_{j-\frac{1}{2}}$. Define the charge decomposition

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}, \quad \text{where charge}(|m\rangle) = m \text{ and charge}(w^\pm_j) = \pm 1.$$ \[ \]

The space $F^{(m)}$ is an irreducible highest weight $g\ell_\infty$-module, where $|m\rangle$ is its highest weight vector, i.e.

$$r(E_{ij})|m\rangle = 0 \text{ for } i < j, \quad r(E_{ii})|m\rangle = 0 \text{ (resp. } = |m\rangle) \text{ if } i > m \text{ (resp. if } i < m).$$ \]

Let $S$ be the following operator on $F \otimes F$

$$S = \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi^+_{i} \otimes \psi^-_{i}$$ \]

and let

$$\mathcal{O}_m = R(GL_\infty)|m\rangle \subset F^{(m)}$$ \]

be the $GL_\infty$-orbit of the highest weight vector $|m\rangle$.

**Theorem 1** ([14], Theorem 5.1) Let $M$ be an integer and let $f = \oplus_{m \in \mathbb{Z}} f_m \in \oplus_{m \in \mathbb{Z}} F^{(m)}$ be such that all $f_m \neq 0$ and $f_m = |m\rangle$ for $m < M$. Then $f \in \oplus_{m \in \mathbb{Z}} \mathcal{O}_m$ if and only if for all $k, \ell \in \mathbb{Z}$, such that $k \geq \ell$, one has

$$S(f_k \otimes f_\ell) = \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi^+_{i} f_k \otimes \psi^-_{i} f_\ell = 0. \quad (2)$$ \]

Equation (2) is called the $(k - \ell)$-th modified KP hierarchy in the fermionic picture. The 0-th modified KP is the KP hierarchy. The collection of all such equations $k, \ell \in \mathbb{Z}$ with $k \geq \ell$ is called the (full) MKP hierarchy in the fermionic picture.

### 3 The fermionic formulation of BKP

The Lie group $B_\infty$ and the corresponding Lie algebra $b_\infty$ can be defined using the following bilinear form on $\mathbb{C}_\infty$, see e.g. [10], section 7.11:

$$(e_i, e_j)_B = (-1)^i \delta_{i, -j}. \quad (3)$$ \]

Then

$$B_\infty = \{ G \in GL_\infty \mid (G(v), G(w)) = (v, w)_B \text{ for all } v, w \in \mathbb{C}_\infty \},$$

$$b_\infty = \{ g \in gl_\infty \mid (g(v), w) + (v, g(w))_B = 0 \text{ for all } v, w \in \mathbb{C}_\infty \},$$ \]

The elements $F_{jk} = E_{-j, k} - (-1)^{j+k} E_{-k, j} = -(-1)^{j+k} F_{kj}$, with $j > k$ form a basis of $b_\infty$. Note that

$$r(F_{jk}) = \psi^+_{j+\frac{1}{2}} \psi^-_{k-\frac{1}{2}} - (-1)^{j+k} \psi^+_{k+\frac{1}{2}} \psi^-_{j-\frac{1}{2}}.$$ \]
This suggests to define linear anti-involutions on the Clifford algebra $C\ell$, which respects the relations (1):

$$
\iota_B(\psi_{j+\frac{1}{2}}) = (-1)^j \psi_{j-\frac{1}{2}}, \quad \iota_B(\psi_{k-\frac{1}{2}}) = (-1)^k \psi_{k+\frac{1}{2}}.
$$

(4)

This induces via $r$ the following anti-involution on $gl_\infty$

$$
\iota_B(E_{jk}) = (-1)^{j+k} E_{-k,-j}
$$

thus

$$
b_\infty = \{ g \in gl_\infty | \iota_B(g) = -g \}.
$$

Instead of $\psi^\pm_i$, and inspired by [5] (see also [20]), we choose different operators that generate $C\ell$, viz. eigenvectors of $\iota_B$

$$
\phi_i = \frac{\psi_{i+\frac{1}{2}} + (-1)^i \psi_{i-\frac{1}{2}}}{\sqrt{2}}, \quad \hat{\phi}_i = \sqrt{-1} \frac{\psi_{i+\frac{1}{2}} - (-1)^i \psi_{i-\frac{1}{2}}}{\sqrt{2}}, \text{ for } i \in \mathbb{Z},
$$

(5)

related to $b_\infty$. These elements satisfy the following relations:

$$
\phi_i \phi_j + \phi_j \phi_i = (-1)^i \delta_{i,-j}, \quad \phi_i \hat{\phi}_j + \hat{\phi}_j \phi_i = 0, \quad \hat{\phi}_i \phi_j + \hat{\phi}_j \hat{\phi}_i = (-1)^i \delta_{i,-j}, \quad i, j \in \mathbb{Z}.
$$

(6)

Thus, we have the following symmetric bilinear form

$$
(\phi_i, \phi_j)_B = (\hat{\phi}_i, \hat{\phi}_j)_B = (-1)^i \delta_{i,-j}, \quad (\hat{\phi}_i, \phi_j)_B = 0, \text{ for } i, j \in \mathbb{Z}.
$$

(7)

We observe that

$$
r(F_{jk}) = \frac{(-1)^k}{2}(\phi_j \phi_k - \phi_k \phi_j) + \frac{(-1)^k}{2}(\hat{\phi}_j \hat{\phi}_k - \hat{\phi}_k \hat{\phi}_j), \text{ for } i, j \in \mathbb{Z},
$$

(8)

and that in both cases

$$
\phi_j |0\rangle = \hat{\phi}_j |0\rangle = 0, \text{ for } j > 0.
$$

The action of $\phi_0$ and $\hat{\phi}_0$ is special and one has

$$
\phi_0 |0\rangle = \frac{1}{\sqrt{2}} |1_B\rangle := \frac{1}{\sqrt{2}} |1\rangle - 1), \quad \hat{\phi}_0 |0\rangle = -\frac{\sqrt{-1}}{\sqrt{2}} |1_B\rangle := -\frac{\sqrt{-1}}{\sqrt{2}} |1\rangle - 1),
$$

\[
\phi_0 | - 1\rangle = \frac{1}{\sqrt{2}} |0\rangle, \quad \hat{\phi}_0 | - 1\rangle = \frac{\sqrt{-1}}{\sqrt{2}} |0\rangle
\]

and

\[
\langle 0 | \phi_0 = \frac{1}{\sqrt{2}} \langle 1_B | := \frac{1}{\sqrt{2}} \langle -1 |, \quad \langle 0 | \hat{\phi}_0 = \frac{\sqrt{-1}}{\sqrt{2}} \langle -1 | := \frac{\sqrt{-1}}{\sqrt{2}} \langle 1_B |
\]

\[
\langle -1 | \phi_0 = \frac{1}{\sqrt{2}} \langle 0 |, \quad \langle -1 | \hat{\phi}_0 = -\frac{\sqrt{-1}}{\sqrt{2}} \langle 0 |
\]

(9)
which gives that $|1_B⟩ = |↑_B⟩$, $⟨1_B| = ⟨↓_B|$ and
\[
⟨0|\hat{ϕ}_0ϕ_0|0⟩ = -⟨0|ϕ_0\hat{ϕ}_0|0⟩ = -1 = -⟨-1|\hat{ϕ}_0ϕ_0|1⟩ = \frac{-1}{2}.
\]
Note that
\[
ψ^+_i = \frac{ϕ_i - \sqrt{-1}\hat{ϕ}_i}{\sqrt{2}}, \quad ψ^-_i = \frac{(-1)^iϕ_i + \sqrt{-1}\hat{ϕ}_i}{\sqrt{2}}, \text{ for } i ∈ \mathbb{Z}, \quad (11)
\]

The $gl_∞$ level one representation $r$, when restricted to $b_∞$ gives a level two representation of this orthogonal infinite dimensional Lie algebra. The formula's $(8)$ make it possible to define the level one spin representations of this algebra in two ways on $F_B$, $\hat{F}_B$:
\[
r_B(F_{jk}) = \frac{(-1)^k}{2}(ϕ_jϕ_k - ϕ_kϕ_j) \quad \text{or} \quad \hat{r}_B(F_{jk}) = \frac{(-1)^k}{2}(\hat{ϕ}_j\hat{ϕ}_k - \hat{ϕ}_k\hat{ϕ}_j).
\]

Each module splits in to two irreducible level one representations $F_B = F_B^0 \oplus F_B^1$, $\hat{F}_B = \hat{F}_B^0 \oplus \hat{F}_B^1$ for $r_B$, $\hat{r}_B$, respectively, with highest weight vectors $|0⟩$ and $|1_B⟩$.

The elements $ϕ_{j_1}ϕ_{j_2} \cdots ϕ_{j_p}0⟩$ (resp. $\hat{ϕ}_{j_1}\hat{ϕ}_{j_2} \cdots \hat{ϕ}_{j_p}|0⟩$) with $j_1 < j_2 < \cdots < j_p \leq 0$ form a basis of $F_B$ (resp. $\hat{F}_B$).

Let $S_B, S_B$ be the following operator on $F_B \otimes F_B, \hat{F}_B \otimes \hat{F}_B$, respectively:
\[
S_B = \sum_{j∈\mathbb{Z}}(-1)^jϕ_j \otimes ϕ_{-j}, \quad S_B = \sum_{j∈\mathbb{Z}}(-1)^j\hat{ϕ}_j \otimes \hat{ϕ}_{-j}.
\]

To define the hierarchies in the B case, we assume that $τ ∈ F^ν_B$ (resp. $τ ∈ \hat{F}^ν_B$), has the form $τ = g|ν⟩$, it is called a $\tau$-function of the BKP hierarchy if,
\[
S_B(g|ν⟩ \otimes g|ν⟩) = gϕ_0|ν⟩ \otimes gϕ_0|ν⟩ \quad \text{(resp. } S_B(g|ν⟩ \otimes g|ν⟩) = g\hat{ϕ}_0|ν⟩ \otimes g\hat{ϕ}_0|ν⟩). \quad (13)
\]

In fact, see e.g. [11] or [13], equation $(13)$ describes the $B_\infty$-orbit of $|ν⟩$, where $ν = 0$ or 1.

### 4 Vertex operators

In this section, we want to realize the spin module $F$ in two different ways. An indication that these isomorphisms exist is given by the following gradation of our module $F$. Define
\[
\text{deg}(|0⟩) = 0, \quad \text{deg}(ϕ_{-i}) = \text{deg}(\hat{ϕ}_{-i}) = \text{deg}(ψ^±_{-i ± 1/2}) = i
\]
and let $F_k = \{f ∈ F| \text{deg}(f) = k\}$. The character formula $\dim_q F = \sum_{k∈\mathbb{Z}} \dim(F_k)q^k$, is clearly equal to
\[
2 \prod_{k=1}^{∞}(1 + q^k)^2.
\]
since the elements $\psi^+_1\psi^+_2 \cdots \psi^+_i \psi^-_j \psi^-_j \cdots \psi^-_m |0\rangle$, with $i_1 < i_2 < \cdots < i_m < 0$ and $j_1 < j_2 < \cdots < j_n < 0$, form a basis of $F$. We can rewrite this character formula in two different ways. The first one is

$$2 \prod_{k=1}^{\infty} (1 + q^k)^2 = 2 \prod_{k=1}^{\infty} \left( (1 + q^k) \frac{1 - q^k}{1 - q^k} \right)^2 = 2 \prod_{k=1}^{\infty} \left( \frac{1 - q^{2k}}{1 - q^k} \right)^2 = 2 \prod_{k=1}^{\infty} \left( \frac{1}{1 - q^{2k-1}} \right)^2$$

and for the second one we use the Jacobi triple product identity which gives that

$$2 \prod_{k=1}^{\infty} (1 + q^k)^2 = \sum_{j \in \mathbb{Z}} q^{j(j-1)/2} \prod_{k=1}^{\infty} \frac{1}{1 - q^k}.$$

We define two isomorphisms $\sigma$ and $\overline{\sigma}$, such that $\sigma(F) = B$ and $\overline{\sigma} = \overline{B}$, where

$$B = \mathbb{C}[q, q^{-1}, t_k | k = 1, 2, \ldots], \quad \overline{B} = \mathbb{C}[\theta, \overline{t}_k, \overline{t}_k | k = 1, 3, 5, \ldots]. \quad (14)$$

Here $\theta$ is a Grassmann variable, i.e. $\theta^2 = 0$, which commutes with all the other indeterminates.

The isomorphisms are uniquely determined by the following properties [11]. First, $\sigma(|0\rangle) = \overline{\sigma}(|0\rangle) = 1$. Second,

$$\sigma \psi^\pm(z) \sigma^{-1} = \sum_{k \in \frac{1}{2} + \mathbb{Z}} \sigma \psi^\pm_k \sigma^{-1} z^{-k - \frac{1}{2}} = q^{\pm 1} z^{\pm \overline{t}_k} \frac{\partial}{\partial \overline{t}_k} \exp \left( \pm \sum_{i=1}^{\infty} t_i z^i \right) \exp \left( \mp \sum_{i=1}^{\infty} \frac{\partial}{\partial t_i} \frac{z^i}{i} \right)$$

and

$$\overline{\sigma} \phi(z) \overline{\sigma}^{-1} = \sum_{k \in \mathbb{Z}} \overline{\sigma} \phi_k \overline{\sigma}^{-1} z^{-k} = \frac{\partial \theta}{\partial \overline{t}_k} \left( \sum_{i > 0, \text{odd}} \overline{t}_i z^i \right) \exp \left( -2 \sum_{i > 0, \text{odd}} \frac{\partial}{\partial \overline{t}_i} \frac{z^i}{i} \right),$$

$$\overline{\sigma} \phi(z) \overline{\sigma}^{-1} = \sum_{k \in \mathbb{Z}} \overline{\sigma} \phi_k \overline{\sigma}^{-1} z^{-k} = \sqrt{-1} \frac{\partial \theta}{\partial \overline{t}_k} \left( \sum_{i > 0, \text{odd}} \overline{t}_i z^i \right) \exp \left( -2 \sum_{i > 0, \text{odd}} \frac{\partial}{\partial \overline{t}_i} \frac{z^i}{i} \right).$$

Note that

$$\sigma(|m\rangle) = q^m, \quad \text{and} \quad \overline{\sigma}(|-1\rangle) = \theta.$$

Both isomorphisms make it possible to express an element $f = g|0\rangle \in F$ as function in $B$ or $\overline{B}$, viz.

$$\sigma(f) = \sum_{k \in \mathbb{Z}} g_k(t) q^k, \quad \overline{\sigma}(f) = \overline{g}_0(\overline{t}, \overline{t}) + \overline{g}_1(\overline{t}, \overline{t}) \theta.$$ 

To determine these functions it will be convenient to introduce the oscillator algebra associated to the above fermionic fields. Let : $ab := ab - (0|ab|0)$ stand for the normal ordered product of two elements. Define

$$\alpha(z) = \sum_{k \in \mathbb{Z}} \alpha_k z^{-k-1} =: \psi^+(z) \psi^-(z), \quad (17)$$
and
\[
\beta(z) = \sum_{k \in 2Z+1} \beta_k z^{-k-1} =: \phi(z) \frac{\phi(-z)}{z} ::, \quad \hat{\beta}(z) = \sum_{k \in 2Z+1} \hat{\beta}_k z^{-k-1} =: \hat{\phi}(z) \frac{\hat{\phi}(-z)}{z} ::,
\]
then
\[
\sigma(\alpha(z)) = q \frac{\partial}{\partial q} z^{-1} + \sum_{k=1}^{\infty} \left( k t_k z^{k-1} + \frac{\partial}{\partial t_k} z^{-k-1} \right)
\]
and
\[
\sigma(\beta(z)) = \sum_{0 < k \in 2Z+1} \left( k \tilde{t}_k z^{k-1} + 2 \frac{\partial}{\partial \tilde{t}_k} z^{-k-1} \right),
\]
\[
\sigma(\hat{\beta}(z)) = \sum_{0 < k \in 2Z+1} \left( k \hat{t}_k z^{k-1} + 2 \frac{\partial}{\partial \hat{t}_k} z^{-k-1} \right).
\]

We observe that
\[
\beta(z) + \hat{\beta}(z) =: \phi(z) \frac{\phi(-z)}{z} : + : \hat{\phi}(z) \frac{\hat{\phi}(-z)}{z} :
\]
\[
= \sum_{i,j \in \mathbb{Z}} z^{-i-j-1} \left( \psi^+_i \psi^-_{j-\frac{1}{2}} : + (-1)^{i+j} \psi^-_i \psi^+_{j+\frac{1}{2}} : \right)
\]
\[
= \psi^+(z) \psi^-(z) : + : \psi^+(z) \psi^-(z) :
\]
\[
= \alpha(z) + \alpha(-z)
\]
\[
= 2 \sum_{k \in 2Z+1} \alpha_k z^{-k-1}.
\]

Define
\[
H(s) = \sum_{k>0} s_k \alpha_k, \quad \overline{H}(s) = \sum_{k>0, \text{odd}} s_k \beta_k, \quad \text{and} \quad \dot{H}(s) = \sum_{k>0, \text{odd}} \frac{s_k}{2} \hat{\beta}_k,
\]
then
\[
\sigma(H(s)) = \sum_{k>0} s_k \frac{\partial}{\partial t_k}, \quad \sigma(\overline{H}(s)) = \sum_{k>0, \text{odd}} s_k \frac{\partial}{\partial \overline{t}_k}, \quad \sigma(\dot{H}(s)) = \sum_{k>0, \text{odd}} s_k \frac{\partial}{\partial \dot{t}_k}.
\]

One has
\[
H(s_1, 0, s_3, 0, s_5, 0, \ldots) = \overline{H}(s) + \dot{H}(s)
\]
and

**Lemma 2 (a)**
\[
\exp(H(s))|0\rangle = \exp(\overline{H}(s))|0\rangle = \exp(\dot{H}(s))|0\rangle = |0\rangle,
\]
(b)
\[
\exp(H(s)) \psi^\pm(z) \exp(-H(s)) = \psi^\pm(z) \exp \left( \pm \sum_{k>0} s_k z^k \right).
\]

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\[
\exp(\overline{H}(s))\phi(z)\exp(-\overline{H}(s)) = \phi(z) \exp \left( \sum_{k>0, odd} s_k z^k \right),
\]
\[
\exp(\hat{H}(s))\hat{\phi}(z)\exp(-\hat{H}(s)) = \hat{\phi}(z) \exp \left( \sum_{k>0, odd} s_k z^k \right),
\]
and
\[
\exp(\overline{H}(s))\hat{\phi}(z)\exp(-\overline{H}(s)) = \hat{\phi}(z),
\]
\[
\exp(\hat{H}(s))\phi(z)\exp(-\hat{H}(s)) = \phi(z).
\]

**Proof.** (a) follows from the fact that all \( \alpha_k|0\rangle = \beta_k|0\rangle = \hat{\beta}_k|0\rangle = 0 \) for all \( k > 0 \).

(b) (resp. (c)) follows from the fact that \( [\alpha^i_k, \psi^{\pm j}(z)] = \pm \delta_{ij} z^k \psi^{\pm j}(z) \) and \( [\frac{1}{2} \beta_k, \phi(z)] = z^k \phi(z) \) (resp. \( [\frac{1}{2} \hat{\beta}_k, \hat{\phi}(z)] = z^k \hat{\phi}(z) \)). \( \square \)

We deduce from (19) and part (b) and (c) of the above lemma that
\[
\exp(\overline{H}(s) + \hat{H}(s))\psi^{\pm}(z)\exp(-\overline{H}(s) - \hat{H}(s)) = \exp(H(s)\psi^{\pm}(z)\exp(-H(s)|_{alt \ s_{2k}=0})
\]
\[
= \psi^{\pm}(z) \exp \left( \pm \sum_{k>0, odd} s_k z^k \right).
\]

(21)

Now let \( \exp(H(s)) \) act on \( f = g|0\rangle \in F \). Since we can decompose such an element as \( f = \sum_{k \in \mathbb{Z}} f_k \), where each \( f_k \in F^{(k)} \), thus we can write
\[
f = g|0\rangle = \sigma^{-1} \left( \sum_k g_k(t) q^k \right) = \sum_k \sigma^{-1} (g_k(t)) |k\rangle.
\]

This gives
\[
\exp(\overline{H}(s)) f = \exp(\overline{H}(s)) g|0\rangle = \sum_k \sigma^{-1} (g_k(t+s)) |k\rangle.
\]

Now, let \( T_k(s) \) be the coefficient of the highest weight vector \( |k\rangle \) in the above expression, then
\[
T_k(s) = \langle k| \exp(H(s)) f = g_k(s).
\]

Thus
\[
\sigma(f) = \sigma(g|0\rangle) = \sum_{k \in \mathbb{Z}} \langle k| \exp(H(t)) g|0\rangle q^k.
\]

(22)

In a similar way one obtains:
\[
\overline{\sigma}(f) = \overline{\sigma}(g|0\rangle) = (|0\rangle + \theta(-1) \exp(\overline{H}(t) + \hat{H}(t)) g|0\rangle.
\]

(23)
5 The bosonic formulation of MKP and BKP

Under the isomorphism $\sigma$ we can rewrite (2), using (15), to obtain the MKP hierarchy:

Let $[z] = (z, z^2, z^3, \ldots)$, $y = (y_1, y_2, \ldots)$, and $\text{Res} \sum_i f_i z^i dz = f_{-1}$, then

$$\text{Res} \ z^{k-\ell} \tau_k(t - [z^{-1}]) \tau_\ell(y + [z^{-1}]) \exp \left( \sum_{i=1}^{\infty} (t_i - y_i) z^i \right) dz = 0, \quad k \geq \ell. \quad (24)$$

The equations (24) first appeared in [9], (2.4)l,l. In a similar way, but now using the isomorphism $\sigma$ and (16) we can reformulate (13), to obtain the BKP hierarchy [5],[9],[11]:

Let $[z]_{\text{odd}} = (z, z^3, \ldots)$, $y = (y_1, y_3, \ldots)$, then

$$\text{Res} \ \tau(t - 2[z^{-1}]_{\text{odd}}) \tau(y + 2[z^{-1}]_{\text{odd}}) \exp \left( \sum_{i=1}^{\infty} (t_{2i-1} - y_{2i-1}) z^{2i-1} \right) \frac{dz}{z} = \tau(t) \tau(y). \quad (25)$$

6 Polynomial tau-functions

A polynomial tau-function of the BKP hierarchy [13] corresponds to an element (cf. [13])

$$f^k = v_1 v_2 \cdots v_k |0\rangle, \quad \text{with} \quad v_i = \sum_{j \in \mathbb{Z}} (-1)^j (v_i, \phi_{-j}) \phi_j \in \mathbb{C}^\infty, \quad \text{or}$$

$$\hat{f}^k = \hat{v}_1 \hat{v}_2 \cdots \hat{v}_k |0\rangle, \quad \text{with} \quad \hat{v}_i = \sum_{j \in \mathbb{Z}} (-1)^j (\hat{v}_i, \hat{\phi}_{-j}) \hat{\phi}_j \in \hat{\mathbb{C}}^\infty, \quad (26)$$

This is obvious from the fact that $v_i \otimes v_i$ commutes with $S_B$. Indeed, using that an element $v \in \mathbb{C}^\infty$ can be written as $v = \sum_j (-1)^j (v, \phi_{-j}) \phi_j$, $(v, v) = \sum_j (-1)^j (v, \phi_{-j}) (v, \phi_j)$ and that $v^2 = \frac{(v, v)}{2}$, one finds

$$(v \otimes v) S_B = \sum_j (-1)^j v \phi_j \otimes v \phi_{-j}$$

$$= \sum_j ((v, \phi_j) - \phi_j v) \otimes ((-1)^j (v, \phi_{-j}) - (-1)^j \phi_{-j} v)$$

$$= (v, v) 1 \otimes 1 - v^2 \otimes 1 - 1 \otimes v^2 + \sum_j (-1)^j \phi_j v \otimes \phi_{-j} v$$

$$= 0 + \sum_j (-1)^j \phi_j v \otimes \phi_{-j} v$$

$$= S_B(v \otimes v).$$

Thus if $f^{k-1}$ satisfies (13) then $vf^{k-1}$, again satisfies (13).
In order to express BKP tau-functions as the square root of a certain KP tau-
function, as was shown in [5], we want to calculate

\[ g^k = v_1 v_2 \cdots v_k \hat{v}_1 \hat{v}_2 \cdots \hat{v}_k |0\), \] with \( v_i = \sum_j a_{ij} \phi_j, \) \( \hat{v}_i = \sum_j a_{ij} \hat{\phi}_j, \) (27)

where for every \( 1 \leq i \leq k \) the coefficients \( a_{ij} \) that appear in \( v_i \) and \( \hat{v}_i \) are equal. Now,

\[ g^k = (-1)^{\frac{k(k-1)}{2}} v_1 \hat{v}_1 v_2 \hat{v}_2 \cdots v_k \hat{v}_k |0\]

and

\[ v_i \hat{v}_i = \sum_j a_{ij} \phi_j \sum_{\ell} a_{i\ell} \hat{\phi}_{\ell} \]

\[ = \sqrt{-1} \sum_{j, \ell} a_{ij} a_{i\ell} \frac{\psi^+_{j+\frac{1}{2}} + (-1)^j \psi^-_{j-\frac{1}{2}} \psi^+_{\ell+\frac{1}{2}} - (-1)^{\ell} \psi^-_{\ell-\frac{1}{2}}}{\sqrt{2}} \]

\[ = - \frac{\sqrt{-1}}{2} \sum_{j, \ell} a_{ij} a_{i\ell} \left((-1)^j \psi^+_{j+\frac{1}{2}} \psi^-_{\ell-\frac{1}{2}} - (-1)^{\ell} \psi^-_{j-\frac{1}{2}} \psi^+_{\ell+\frac{1}{2}}\right). \]

Hence the element \( g^k \in F^{(0)} \). Moreover,

**Proposition 3** The element \( g^k \in F^{(0)} \) of (27) satisfies the KP hierarchy [2].

**Proof.** To prove this, it will be sufficient to show that \( v_i \hat{v}_i \otimes v_i \hat{v}_i \) commutes with \( S \).

Note that up to a constant the element \( v_i \hat{v}_i \) is of the form \( w^+ v^- - v^- w^+ \) where

\[ w^+ = \sum_j a_{ij} \psi^+_{j+\frac{1}{2}}; \quad v^- = \sum_j (-1)^j a_{ij} \psi^-_{j-\frac{1}{2}}. \]

Now, observe that \( v^- v^- = w^+ w^+ = 0 \), thus

\[ S(w^+ v^- - v^- w^+) \otimes (w^+ v^- - v^- w^+) \]

\[ = \sum_k \psi^+_k (w^+ v^- - v^- w^+) \otimes \psi^-_k (w^+ v^- - v^- w^+) \]

\[ = \sum_k (-2(\psi^+_k, v^-) w^+ + (w^+ v^- - v^- w^+)) \psi^+_k \otimes (2(\psi^-_k, w^+) v^- + (w^+ v^- - v^- w^+)) \psi^-_k) \]

\[ = -4(w^+, v^-) w^+ \otimes v^- - 2w^+ \otimes (w^+ v^- - v^- w^+) v^- + 2(w^+ v^- - v^- w^+) w^+ \otimes v^- \]

\[ + (w^+ v^- - v^- w^+) \otimes (w^+ v^- - v^- w^+) S \]

\[ = -4(w^+, v^-) w^+ \otimes v^- + 2w^+ \otimes (w^+, v^-) v^- + 2(w^+, v^-) w^+ \otimes v^- \]

\[ + (w^+ v^- - v^- w^+) \otimes (w^+ v^- - v^- w^+) S \]

\[ = (w^+ v^- - v^- w^+) \otimes (w^+ v^- - v^- w^+) S \]

Thus \( v_i \hat{v}_i \otimes v_i \hat{v}_i \) commutes with \( S \). \( \square \)
In order to calculate the corresponding tau-functions, we first calculate several vacuum expectation values.

\[
\exp\left(\overline{H}(\bar{t}) + \hat{H}(\hat{t})\right) v_i \exp\left(-\overline{H}(\bar{t}) - \hat{H}(\hat{t})\right) = \exp\left(\overline{H}(\bar{t}) + \hat{H}(\hat{t})\right) \sum_{j > -N_i} a_{ij} \phi_j \exp\left(-\overline{H}(\bar{t}) - \hat{H}(\hat{t})\right)
\]

\[
= \text{Res} \sum_{j > -N_i} a_{ij} z^j \exp\left(\overline{H}(\bar{t}) + \hat{H}(\hat{t})\right) \phi(z) \exp\left(-\overline{H}(\bar{t}) - \hat{H}(\hat{t})\right) \frac{dz}{z}
\]

\[
= \text{Res} z^{-N_i-1} \sum_{j > 0} a_{i,j-N_i} z^j \exp\left(\sum_{k > 0, \text{odd}} \overline{t}_k z^k \right) \phi(z) \frac{dz}{z}.
\]

Here we assume that \(a_{i,-N_i} \neq 0\). We then write

\[
\sum_{j > 0} a_{i,j-N_i} z^j = a_{i,-N_i} \exp\left(\sum_{i=1}^{\infty} c_{ij} z^j \right). \quad (28)
\]

Hence for \(\bar{t} = (\overline{\bar{t}}_1, 0, \overline{\bar{t}}_3, 0, \ldots)\), we find that

\[
\exp\left(\overline{H}(\bar{t}) + \hat{H}(\hat{t})\right) v_i \exp\left(-\overline{H}(\bar{t}) - \hat{H}(\hat{t})\right) = a_{i,-N_i} \text{Res} z^{-N_i} \exp\left(\sum_{k > 0} (\overline{t}_k + c_{ik}) z^k \right) \phi(z) \frac{dz}{z}
\]

Thus,

\[
\langle 0 \mid \exp\left(\overline{H}(\bar{t}) + \hat{H}(\hat{t})\right) v_i v_j \exp\left(-\overline{H}(\bar{t}) - \hat{H}(\hat{t})\right) \mid 0 \rangle = a_{i,-N_i} a_{j,-N_j} \text{Res} z^{-N_i} w^{-N_j} \exp\left(\sum_{k > 0} (\overline{t}_k + c_{ik}) z^k \right) \exp\left(\sum_{k > 0} (\overline{t}_k + c_{jk}) w^k \right) \langle 0 \mid \phi(z) \phi(w) \mid 0 \rangle \frac{dz}{z} \frac{dw}{w}.
\]

Using that

\[
\langle 0 \mid \phi(z) \phi(w) \mid 0 \rangle = (zw)^{-1} \left(1 + \sum_{i=1}^{\infty} \left(-\frac{w}{z}\right)^i\right)
\]

and

\[
\exp\left(\sum_{k > 0} t_k z^k \right) = \sum_{j = 0}^{\infty} s_j(t) z^j,
\]

we find that

\[
\langle 0 \mid \exp\left(\overline{H}(\bar{t}) + \hat{H}(\hat{t})\right) v_i v_j \exp\left(-\overline{H}(\bar{t}) - \hat{H}(\hat{t})\right) \mid 0 \rangle = a_{i,-N_i} a_{j,-N_j} \overline{\chi}_{N_i,N_j}(\bar{t} + c_i, \hat{t} + c_j),
\]

where

\[
\overline{\chi}_{N,M}(s, t) = \frac{1}{2} s_N(t) s_M(s) + \sum_{k=1}^{M} (-1)^k s_{N+k}(t) s_{M-k}(s). \quad (29)
\]

Clearly,

\[
\langle 0 \mid \exp\left(\overline{H}(\bar{t}) + \hat{H}(\hat{t})\right) \hat{v}_i \hat{v}_j \exp\left(-\overline{H}(\bar{t}) - \hat{H}(\hat{t})\right) \mid 0 \rangle = a_{i,-N_i} a_{j,-N_j} \overline{\chi}_{N_i,N_j}(\hat{t} + c_i, \hat{t} + c_j)\]
From which we deduce that

\[ \langle 0 | \exp \left( \mathcal{P}(\hat{t}) + \hat{H}(\hat{t}) \right) v_i \hat{v}_j \exp \left( -\mathcal{P}(\hat{t}) - \hat{H}(\hat{t}) \right) | 0 \rangle = -\frac{\sqrt{-1}}{2} a_{i,-N_i} a_{j,-N_j} s_{N_i}(\hat{t} + c_i) s_{N_j}(\hat{t} + c_j). \]

Next, we calculate

\[ \exp (H(t)) v_i \exp (-H(t)) = \]

\[ = \text{Res} \sum_{j=-N_i}^{\infty} \frac{a_{ij}}{\sqrt{2}} z^j \left( \exp \left( \sum_{k>0} t_k z^k \right) \psi^+(z) + z^{-1} \exp \left( \sum_{k>0} -t_k (-z)^k \right) \psi^-(z) \right) \] \[ \times \left( \exp \left( \sum_{k>0} (t_k + c_{ik}) z^k \right) \psi^+(z) + z^{-1} \exp \left( \sum_{k>0} (-1)^{k+1} t_k + c_{ik} \right) \psi^-(z) \right) \] \[ d z, \]

and analogously, we find

\[ \exp (H(t)) \hat{v}_i \exp (-H(t)) = \frac{a_{i,-N_i} \sqrt{-1}}{\sqrt{2}} \text{Res} \left( z^{-N_i} z_t \right) \]

\[ \times \left( \exp \left( \sum_{k>0} (t_k + c_{ik}) z^k \right) \psi^+(z) - z^{-1} \exp \left( \sum_{k>0} (-1)^{k+1} t_k + c_{ik} \right) \psi^-(z) \right) \] \[ d z. \]

From which we deduce that

\[ \langle 0 | \exp (H(t)) v_i v_j \exp (-H(t)) | 0 \rangle = \langle 0 | \exp (H(t)) \hat{v}_i \hat{v}_j \exp (-H(t)) | 0 \rangle = \]

\[ = \frac{a_{i,-N_i} a_{j,-N_j}}{2} \text{Res} \left( z^{-N_i} z_t \right) \left( \sum_{k=0}^{\infty} \left( -\frac{w}{z} \right)^k \exp \left( \sum_{k>0} (t_k + c_{ik}) z^k + (-1)^{k+1} t_k + c_{jk} \right) \right) \]

\[ + \sum_{k=1}^{\infty} \left( -\frac{w}{z} \right)^k \exp \left( \sum_{k>0} (t_k + c_{jk}) w^k + (-1)^{k+1} t_k + c_{ik} \right) \] \[ d z \frac{d w}{w} \]

\[ = a_{i,-N_i} a_{j,-N_j} \chi_{N_i,N_j}^+(t + c_i, t - \tilde{c}_j, t - \tilde{c}_i, t + c_j). \]

Here \( \tilde{c} = (-c_1, c_2, -c_3, c_4, \ldots) \) and

\[ \chi_{N,M}^+(s, t, u, v) = \frac{1}{2} \sum_{k=0}^{M} (-1)^N s_{N+k}(s) s_{M-k}(t) \pm \frac{1}{2} \sum_{k=1}^{M} (-1)^M s_{N+k}(-u) s_{M-k}(v). \]

Finally,

\[ \langle 0 | \exp (H(t)) v_i \hat{v}_j \exp (-H(t)) | 0 \rangle = -\frac{\sqrt{-1}}{2} a_{i,-N_i} a_{j,-N_j} \text{Res} \left( z^{-N_i} w^{-N_j} \right) \times \]

\[ \left( \sum_{k=0}^{\infty} \left( -\frac{w}{z} \right)^k \exp \left( \sum_{k>0} (t_k + c_{ik}) z^k + (-1)^{k+1} t_k + c_{jk} \right) \right) \] \[ - \sum_{k=1}^{\infty} \left( -\frac{w}{z} \right)^k \exp \left( \sum_{k>0} (t_k + c_{jk}) w^k + (-1)^{k+1} t_k + c_{ik} \right) \] \[ d z \frac{d w}{w} \]

\[ = -\sqrt{-1} a_{i,-N_i} a_{j,-N_j} \chi_{N_i,N_j}^-(t + c_i, t - \tilde{c}_j, t - \tilde{c}_i, t + c_j). \]
The Pfaffian (31) is obtained by applying Wick's theorem. Next, we calculate that, with generality, than is needed here. Denote by \( \text{matrix in a sum of determinants times Pfaffians. We first state the formula in more general form.}

\[\hat{\sigma}(\phi_{v_1v_2\cdots v_{2n}|0})) = \sum_{j>\lambda_i} a_{ij} \phi_j = \phi_{\lambda_i} + \sum_{j>\lambda_i} s_{j-\lambda_i}(c_i) \phi_j.\] (32)

The Pfaffian (31) is obtained by applying Wick's theorem. Next, we calculate \( \hat{\sigma}(\phi_{v_1v_2\cdots v_{2n}|0}) \). Using the above vacuum expectation values, we find that, with

\[
\begin{align*}
A(\bar{t}) &= \left( \bar{\chi}_{\lambda_i,\lambda_j}(\bar{t} + c_i, \bar{t} + c_j) \right)_{1 \leq i,j \leq 2n},
B(\bar{t}, \hat{t}) &= \left( -\frac{\sqrt{-1}}{2} s_{\lambda_i}(\bar{t} + c_i) s_{\lambda_j}(\hat{t} + c_j) \right)_{1 \leq i,j \leq 2n},
\end{align*}
\]

(33)

the vacuum expectation value

\[
\begin{align*}
\bar{\sigma}(\phi_{v_1v_2\cdots v_{2n}|0}) &= Pf \begin{pmatrix} A(\bar{t}) & -B(\bar{t}, \hat{t})^T \\ -B(\bar{t}, \hat{t}) & A(\bar{t}) \end{pmatrix} \\
&= Pf(\bar{A}(\bar{t})) Pf(\bar{A}(\bar{t})) \\
&= Pf(\bar{A}(\bar{t})) Pf(\bar{A}(\bar{t})) \\
&= Pf(\bar{A}(\bar{t})) Pf(\bar{A}(\bar{t})) \\
&= Pf(\bar{A}(\bar{t})) Pf(\bar{A}(\bar{t}))
\end{align*}
\]

(34)

In the second equality of (34), we use a formula, due to E. R. Caianiello [2], for a proof see [18]. This formula expresses the Pfaffian of the \( 4n \times 4n \) skew symmetric matrix in a sum of determinants times Pfaffians. We first state the formula in more generality, than is needed here. Denote by \( [m] = \{1,2,\ldots,m\} \) for a nonnegative integer \( m \). Let \( I \subset [m] \) and \( J \subset [k] \), denote by \( W(I, J) \) the matrix that we construct out of the \( m \times n \) matrix \( W \) by erasing the rows \( i \) with \( i \in [m]\setminus I \) and the columns \( j \) with \( j \in [k]\setminus J \). If \( J = \{j_1,j_2,\ldots,j_s\} \), we write \( \sum(J) = j_1 + j_2 + \ldots + j_s \). Then

**Proposition 6** Let \( m \) and \( k \) be nonnegative integers such that \( m + k \) is even. Let \( X \), respectively \( Y \), be a skew-symmetric \( m \times m \), respectively \( k \times k \) matrix, and \( W \)
Thus we have shown:

\[ Pf \left( \begin{array}{cc} X & W \\ -W^T & Y \end{array} \right) = \sum_{I,J} \varepsilon(I, J) Pf(X(I, I)) Pf(Y(J, J)) \det(W([m]\setminus I; [k]\setminus J)), \tag{35} \]

where the sum is taken over all pairs of even-element subsets \((I, J)\) such that \(I \subset [m], J \subset [n]\) and where \(m - |I| = k - |J|\). Here

\[ \varepsilon(I, J) = (-1)^{\sum(I) + \sum(J) + (m) + (k) + (m-|I|)}. \]

We use this proposition, to obtain the second equality of (34). In our case \(m = k = 2n\) and the matrix \(W = \overline{B}(\hat{t}, \hat{\tau})\) has rank 1, which means that all the terms on the right-hand side of (35) are zero, except when \(I = J = [2n]\).

To obtain the Main Theorem of this paper, we calculate

\[ \sigma(v_1v_2 \cdots v_{2n}\hat{v}_1\hat{v}_2 \cdots \hat{v}_{2n}|0)), \]

From the above vacuum expectation values we deduce using Wick’s theorem that the KP tau-function

\[ \tau_{2n,2n}^2(t) = \sigma(v_1v_2 \cdots v_{2n}\hat{v}_1\hat{v}_2 \cdots \hat{v}_{2n}|0)) = Pf \left( \begin{array}{cc} A^+(t) & -\sqrt{-1}A^-(t) \\ \sqrt{-1}A^-(t)^T & A^+(t) \end{array} \right), \tag{36} \]

where (cf. (30))

\[ A^\pm(t) = \left( \chi_{\lambda_i, \lambda_j}^\pm (t + c_i, t - \hat{c}_j, t - \hat{c}_i, t + c_j) \right)_{1 \leq i, j \leq 2n}. \tag{37} \]

Thus we have shown:

**Theorem 7** Let \(\overline{t} = (\overline{t}_1, 0, \overline{t}_3, 0, \ldots)\), then the BKP tau-function \(\tau_{2n}^2(\overline{t})\) of Theorem 3 is the square root of the KP tau-function \(\tau_{2n,2n}^2(t)\), given in (36), i.e.,

\[ \tau_{2n}^2(\overline{t}) = \sqrt{\tau_{2n,2n}^2(t)}. \]

**Remark 8** If we put all the constants equal to zero, then the above KP tau-function \(\tau_{2n,2n}^2(t)\) corresponds to the following element \(\Phi F\):

\[ \phi_{-\lambda_1} \phi_{-\lambda_2} \cdots \phi_{-\lambda_{2n}} \phi_{-\lambda_1} \phi_{-\lambda_2} \cdots \phi_{-\lambda_{2n}}|0), \]

which is up to a sign equal to

\[
\begin{align*}
\begin{cases}
\psi^+_{-\lambda_1 + \frac{1}{2}} \psi^+_{-\lambda_2 + \frac{1}{2}} \cdots \psi^+_{-\lambda_{2n} + \frac{1}{2}} \psi^-_{-\lambda_1 - \frac{1}{2}} \psi^-_{-\lambda_2 - \frac{1}{2}} \cdots \psi^-_{-\lambda_{2n} - \frac{1}{2}} |0), & \text{if } \lambda_{2n} \neq 0 \text{ and } \\
\psi^+_{-\lambda_1 + \frac{1}{2}} \psi^+_{-\lambda_2 + \frac{1}{2}} \cdots \psi^+_{-\lambda_{2n} - 1 + \frac{1}{2}} \psi^-_{-\lambda_1 - \frac{1}{2}} \psi^-_{-\lambda_2 - \frac{1}{2}} \cdots \psi^-_{-\lambda_{2n} - 1 - \frac{1}{2}} |0), & \text{if } \lambda_{2n} = 0.
\end{cases}
\end{align*}
\]

This element corresponds to \(s_{(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{2n} - 1, \lambda_1, \lambda_2, \ldots, \lambda_{2n})}(t)\), where we use the Frobenius notation for a partition, see e.g. [17]. This means that the tau-function \(\tau_{2n,2n}^2(t)\)
is the lowest element (i.e. it generates) the KP Schubert cell (cf. [12]) corresponding to the partition

\[ \begin{cases} (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{2n} - 1 | \lambda_1, \lambda_2, \ldots, \lambda_{2n} ), & \text{if } \lambda_{2n} \neq 0 \text{ and} \\ (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{2n-1} - 1 | \lambda_1, \lambda_2, \ldots, \lambda_{2n-1} ), & \text{if } \lambda_{2n} = 0. \end{cases} \]

While its "square root", the BKP tau-function \( \tau^2_B(\mathcal{I}) \), which is equal up to a multiplicative constant, to the Q-Schur function \( Q_{(\lambda_1, \lambda_2, \ldots, \lambda_{2n})} \left( \frac{1}{2} \right) \), generates the BKP Schubert cell corresponding to the strict partition

\( (\lambda_1, \lambda_2, \ldots, \lambda_{2n} ), \) respectively \( (\lambda_1, \lambda_2, \ldots, \lambda_{2n-1} ), \) if \( \lambda_{2n} = 0. \)

7 The relation KdV versus BKP

A. Alexandrov showed in a recent short publication [1], that all KdV tau-functions, i.e., KP tau-functions that are independent of the even times \( t_{2i} \), are BKP tau-functions when one replaces all times \( t_{2i+1} \) by \( \frac{t_{2i+1}}{2} \). In this section, we give a representation theoretical explanation for this.

It is clear from [12], but was proved already in the 80's in [14], that all polynomial KdV tau functions can be obtained as the following vacuum expectation value, for a certain \( k = 0, 1, 2, \ldots \), where the \( c_{2i-1} \) are arbitrary constants.

\[ \tau_k(t + c) = \langle 0 | e^{\sum_{i=1}^{\infty} (t_{2i-1} + c_{2i-1}) \alpha_{2i-1} \psi^+_{-k+i\frac{1}{2}} \psi^+_{-k+i\frac{3}{2}} \cdots \psi^+_{-k+i\frac{1}{2}} } | -k \rangle. \quad (38) \]

Note that we can obtain all tau-functions [39], by calculating the above expression with all \( c_{2i-1} = 0 \) and then substituting \( t_{2i-1} + c_{2i-1} \) for \( t_{2i-1} \). Thus from now on we will put all \( c_{2i-1} = 0 \). In fact (cf. [12] or [14]), it is not difficult to show that

\[ \tau_k(t) = s_{(k,k-1,...,2,1)}(t_1, t_2, t_3, \ldots) = s_{(k,k-1,...,2,1)}(t_1, 0, t_3, 0, \ldots), \]

the Schur function corresponding to the partition \( \lambda = (k, k - 1, \ldots, 2, 1) \), which is independent of the even times.

\[ \tau_k(t) = \langle 0 | e^{\sum_{i=1}^{\infty} t_{2i-1} \alpha_{2i-1} \psi^+_{-k+i\frac{1}{2}} \psi^+_{-k+i\frac{3}{2}} \cdots \psi^+_{-k+i\frac{1}{2}} } | -k \rangle \]

\[ = \langle 0 | e^{\sum_{i=1}^{\infty} (t_{2i-1} + c_{2i-1}) \alpha_{2i-1} \psi^+_{-k+i\frac{1}{2}} \psi^+_{-k+i\frac{3}{2}} \cdots \psi^+_{-k+i\frac{1}{2}} } | 0 \rangle \]

\[ = \pm \langle 0 | e^{\sum_{i=1}^{\infty} t_{2i-1} \frac{1}{2} \psi^+_{-k+i\frac{1}{2}} \psi^+_{-k+i\frac{3}{2}} \cdots \psi^+_{-k+i\frac{1}{2}} } | 0 \rangle \]

\[ = \pm \langle 0 | e^{\sum_{i=1}^{\infty} t_{2i-1} \frac{1}{2} \psi^+_{-k+i\frac{1}{2}} \psi^+_{-k+i\frac{3}{2}} \cdots \psi^+_{-k+i\frac{1}{2}} } | 0 \rangle \quad (39) \]

The above calculation is up to a multiplicative sign. Now using [11], we can rewrite

\[ 1 \text{One obtains the Q-Schur functions as given in Macdonald's book on symmetric functions [17] by substituting for } t_i = \sum_{j>0} x_j^i \]
(39), again up to a sign, to:

\[
\tau_k(t) = \pm \langle 0| \sum_{i=1}^\infty t_{2i-1} \alpha_{2i-1} \phi_{-k} - \sqrt{-1} \hat{\phi}_{-k} \phi_{-k+1} + \sqrt{-1} \hat{\phi}_{-k+1} \times \\
\phi_{-k+2} - \sqrt{-1} \hat{\phi}_{-k+2} \sqrt{2} \phi_{-1(-1)^k} + \sqrt{-1} \hat{\phi}_{-1(-1)^k} \sqrt{2} |0\rangle. 
\]

Instead of this expression, we focus on

\[
\tau_k(s, t) = \pm \langle 0| \sum_{i=1}^\infty s_{2i-1} \beta_{2i-1} + t_{2i-1} \beta_{2i-1} \phi_{-k} - \sqrt{-1} \hat{\phi}_{-k} \phi_{-k+1} + \sqrt{-1} \hat{\phi}_{-k+1} \times \\
\phi_{-k+2} - \sqrt{-1} \hat{\phi}_{-k+2} \sqrt{2} \phi_{-1(-1)^k} + \sqrt{-1} \hat{\phi}_{-1(-1)^k} \sqrt{2} |0\rangle. 
\]

Differentiate (31) by \( \frac{\partial}{\partial s_{2j+1}} - \frac{\partial}{\partial t_{2j+1}} \), we thus obtain

\[
\left( \frac{\partial}{\partial s_{2j+1}} - \frac{\partial}{\partial t_{2j+1}} \right) \tau_k(s, t) = \pm \langle 0| \sum_{i=1}^\infty s_{2i-1} \beta_{2i-1} + t_{2i-1} \beta_{2i-1} \left( \frac{\beta_{2j-1}}{2} - \frac{\hat{\beta}_{2j-1}}{2} \right) \phi_{-k} - \sqrt{-1} \hat{\phi}_{-k} \phi_{-k+1} + \sqrt{-1} \hat{\phi}_{-k+1} \times \\
\phi_{-k+2} - \sqrt{-1} \hat{\phi}_{-k+2} \sqrt{2} \phi_{-1(-1)^k} + \sqrt{-1} \hat{\phi}_{-1(-1)^k} \sqrt{2} |0\rangle. 
\]

However, since

\[
\left[ \frac{\beta_{2j-1}}{2} - \frac{\hat{\beta}_{2j-1}}{2}, \phi_{-k+\ell} - (-1)^\ell \sqrt{-1} \hat{\phi}_{-k+\ell} \right] = \phi_{-k+\ell+2j-1} - (-1)^{\ell+2j-1} \sqrt{-1} \hat{\phi}_{-k+\ell+2j-1},
\]

we conclude that

\[
\left( \frac{\partial}{\partial s_{2j+1}} - \frac{\partial}{\partial s_{2j+1}} \right) \tau_k(s, t) = 0.
\]

Thus \( \tau_k(s, t) \) is a function of \( s + t \). From which we deduce that, up to a multiplicative sign,

\[
\pm \tau_k \left( \frac{t}{2}, \frac{t}{2} \right) = \tau_k \left( t, (1 - \epsilon)t \right) = \tau_k(t, 0).
\]

We now calculate explicitly \( \tau_k(t, 0) \). Note first, that if \( k \) is odd,

\[
\frac{\phi_0 + \sqrt{-1} \hat{\phi}_0}{\sqrt{2}} |0\rangle = \sqrt{2} \phi_0 |0\rangle.
\]

Thus

\[
\tau_k(t, 0) = \pm \langle 0| \sum_{i=1}^\infty t_{2i-1} \phi_{-k} - \sqrt{-1} \hat{\phi}_{-k} \phi_{-k+1} + \sqrt{-1} \hat{\phi}_{-k+1} \times \\
\phi_{-k+2} - \sqrt{-1} \hat{\phi}_{-k+2} \sqrt{2} \phi_{-1(-1)^k} + \sqrt{-1} \hat{\phi}_{-1(-1)^k} \sqrt{2} |0\rangle 
\]

\[
= \pm \frac{1}{(\sqrt{2})^k} \langle 0| \sum_{i=1}^\infty t_{2i-1} \phi_{-k} \phi_{-k+1} \phi_{-k+2} \phi_{-1(-1)^k} |0\rangle. 
\]

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which clearly is a BKP tau-function.

More explicitly,

\[ \tau_k(t, 0) = \pm \frac{1}{(\sqrt{2})^k} P f \left( \chi_{i,j}(t, t) \right) \prod_{1 \leq i,j \leq k} \frac{1-(-1)^k}{2} \]

Hence, it is (up to a multiplicative constant) the Q-Schur function corresponding to the strict partition \((k, k-1, \ldots, 2, 1)\), and it is, again up to a multiplicative constant, the square root of the Schur function \(s_{(k+1)}(t_1, 0, t_3, 0, \ldots)\). We thus obtain (cf. [1]):

**Proposition 9** All polynomial KdV tau-functions \(\tau_k(t+c) = s_{(k,k-1,\ldots,2,1)}(t_1+c_1, t_3+c_3, \ldots)\) become BKP tau-functions, when one replaces \(t_i+c_i\) by \(\frac{t_i+c_i}{2}\). Moreover, up to a multiplicative constant, \(\tau_k\left(\frac{t+c}{2}\right)\) is equal to \(Q_{(k,k-1,\ldots,2,1)}\left(\frac{t+c}{2}\right)\) and to the square root of \(s_{(k+1)}(t_1+c_1, 0, t_3+c_3, 0, \ldots)\), here \((k^{k+1})\) is the partition \((k, k, \ldots, k)\), where \(k\) appears \(k+1\) times.

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