Relationship between MP and DPP for stochastic recursive optimal control problem under volatility uncertainty

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Abstract. In this paper, we study the relationship between maximum principle (MP) and dynamic programming principle (DPP) for stochastic recursive optimal control problem driven by $G$-Brownian motion. Under the smooth assumption for the value function, we obtain the connection between MP and DPP under a reference probability $P^*_{t,x}$. Within the framework of viscosity solution, we establish the relation between the first-order super-jet, sub-jet of the value function and the solution to the adjoint equation respectively.

Key words. Stochastic recursive optimal control, Maximum principle, Dynamic programming principle, $G$-expectation

AMS subject classifications. 93E20, 60H10, 35K15

1 Introduction

Motivated by the study of volatility uncertainty in finance, Peng [18,19] established the theory of $G$-expectation which is a consistent sublinear expectation. The representation of $G$-expectation as the supremum of expectations over a set of nondominated probability measures $\mathcal{P}$ was obtained in [3,15]. Epstein and Ji [8,9] studied the volatility uncertainty in economic and financial problems by using $G$-expectation as a tool. Hu et al. [12,13] obtained the existence and uniqueness theorem and other properties for backward stochastic differential equation driven by $G$-Brownian motion ($G$-BSDE), which is completely different from the classical BSDE due to the set of nondominated probability measures $\mathcal{P}$ representing $G$-expectation. In addition, Soner et al. [23] studied a new type of fully nonlinear BSDE, called 2BSDE, by different formulation and method, and obtained the deep result of the existence and uniqueness theorem for 2BSDE.

Hu and Ji [10,11] studied the maximum principle (MP) and dynamic programming principle (DPP) for the following stochastic recursive optimal control problem driven by $G$-Brownian motion:

\[
\begin{align*}
    dX_{s}^{t,x,u} &= b(s, X_{s}^{t,x,u}, u_s) ds + h_{ij}(s, X_{s}^{t,x,u}, u_s) d\langle B^i, B^j \rangle_s + \sigma(s, X_{s}^{t,x,u}, u_s) dB_s, \\
    X_{t}^{t,x,u} &= x, \quad s \in [t, T],
\end{align*}
\]

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and the MP in [10] holds, where $P$ explains why the MP holds under a reference probability $P$ obtain that, for some $V$ function

In Example 3.7, we prove that the relational expression (1.4) does not hold for any framework by different formulation and method. Equation. Let us mention that Biagini et al. [2] and Xu [25] also studied the MP under $x$ is of polynomial growth in $s$.

The relationship between MP and DPP for different types of stochastic optimal control problems was established in [14, 17, 21, 27]. Bahlali et al. [1] obtained the stochastic optimal control problems. The relationship between MP and DPP for different types of stochastic optimal control problems is studied in many literatures, we refer the readers to [1, 14, 17, 21, 26–28] and to our knowledge, there is no result on this topic. If the value function

It is well-known that MP and DPP are two important methods to study control problems. The relationship between MP and DPP for the control problem (1.1)-(1.3) was established in [14, 17, 21, 27]. Bahlali et al. [1] obtained the MP for the control problem (1.1)-(1.3) under a reference probability $P_{t,x}^* \in \mathcal{P}_{t,x} \subset \mathcal{P}$ (see (3.1) for definition), and proved that this MP is also a sufficient condition under some convex assumptions. In particular, the adjoint equation (2.2) is introduced under $P_{t,x}^*$ and has a unique solution $(p_x, q_x, N_x) \in [t, T]$. By introducing a new implied partition approach, Hu and Ji [11] obtained the DPP and the related HJB equation. Let us mention that Biagini et al. [2] and Xu [25] also studied the MP under $G$-expectation framework by different formulation and method.

In this paper, we study the relationship between MP and DPP for the control problem (1.1)-(1.3). Up to our knowledge, there is no result on this topic. If the value function $V(\cdot) \in C^{1,2}([t, T] \times \mathbb{R})$ and $\partial_x^2 V(\cdot)$ is of polynomial growth in $x$, then, for any $P_{t,x}^* \in \mathcal{P}_{t,x}^* \subset \mathcal{P}_{t,x}$ (see (3.4) for definition), we have

In Example 3.7, we prove that the relational expression (1.4) does not hold for any $P \notin \mathcal{P}_{t,x}^*$. If the value function $V(\cdot) \in C^{1,2}([t, T] \times \mathbb{R})$, $\partial_x^2 V(\cdot)$ is of polynomial growth in $x$ and $\partial^2_x V(\cdot)$ is continuous, then we obtain that, for some $P_{t,x}^* \in \mathcal{P}_{t,x}^*$,

and the MP in [10] holds, where $P_{t,x}^*$ satisfies the condition (3.7). Thus, the relational expression (1.4) explains why the MP holds under a reference probability $P_{t,x}^*$. Furthermore, we study the relationship
between MP and DPP in the viscosity sense and obtain
\[
D^{1+}_{x} V(t, x) \subseteq [\tilde{p}_l, \tilde{p}_r], \quad \tilde{p}_l, \tilde{p}_r \in D^{1+}_{x} V(t, x) \text{ if } \tilde{p}_l = \tilde{p}_r,
\]
where \(\tilde{p}_l\) and \(\tilde{p}_r\) are defined in Theorem 3.4. In particular, \(D^{1+}_{x} V(t, x)\) may be empty in Example 3.3. The relational expression (1.5) is completely new and different from the classical case.

This paper is organized as follows. We recall some basic results of \(G\)-expectation, the MP and DPP for stochastic recursive optimal control problem driven by \(G\)-Brownian motion in Section 2. In Section 3, we obtain the relationship between MP and DPP for stochastic recursive optimal control problem driven by \(G\)-Brownian motion.

2 Preliminaries

2.1 \(G\)-expectation

In this subsection, we recall some basic notions and results of \(G\)-expectation. The readers may refer to [20] for more details.

Let \(T > 0\) be fixed and let \(\Omega_T = C_0([0, T]; \mathbb{R}^d)\) be the space of \(\mathbb{R}^d\)-valued continuous functions on \([0, T]\) with \(\omega_0 = 0\). The canonical process \(B_t(\omega) := \omega_t\), for \(\omega \in \Omega_T\) and \(t \in [0, T]\). For each given \(t \in [0, T]\), set
\[
\text{Lip}(\Omega_t) := \{\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) : N \geq 1, t_1 \leq \cdots \leq t_N \leq t, \varphi \in C_b, \text{Lip}(\mathbb{R}^{d \times N})\},
\]
where \(C_b, \text{Lip}(\mathbb{R}^{d \times N})\) denotes the space of bounded Lipschitz functions on \(\mathbb{R}^{d \times N}\).

Let \(G : S_d \to \mathbb{R}\) be a given monotonic and sublinear function, where \(S_d\) denotes the set of \(d \times d\) symmetric matrices. Then there exists a bounded and convex set \(\Sigma \subseteq S^+_d\) such that
\[
G(A) = \frac{1}{2} \sup_{\gamma \in \Sigma} \text{tr}[\gamma A] \quad \text{for } A \in S_d,
\]
where \(S^+_d\) denotes the set of \(d \times d\) nonnegative matrices. In this paper, we consider the non-degenerate \(G\), i.e., there exists a \(\alpha > 0\) such that \(\gamma \geq \alpha I_d\) for any \(\gamma \in \Sigma\). Specially, if \(d = 1\), then \(G(a) = \frac{1}{2}\alpha^2 (a^2 - a^2)\) for \(a \in \mathbb{R}\) with \(\alpha \geq \sigma > 0\).

Peng [18, 19] constructed the \(G\)-expectation \(\hat{E} : \text{Lip}(\Omega_T) \to \mathbb{R}\) and the conditional \(G\)-expectation \(\hat{E}_t : \text{Lip}(\Omega_T) \to \text{Lip}(\Omega_t)\) as follows:

(i) For \(s < t \leq T\), define \(\hat{E}_t[\varphi(B_t - B_s)] = u(t - s, 0)\), where \(u\) is the viscosity solution (see [4]) of the following \(G\)-heat equation:
\[
\partial_t u - G(\partial^2_{xx} u) = 0, \quad u(0, x) = \varphi(x).
\]

(ii) For \(X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) \in \text{Lip}(\Omega_T)\), define
\[
\hat{E}_t[X] = \varphi_i(B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}) \text{ for } i = N - 1, \ldots, 1 \text{ and } \hat{E}[X] = \hat{E}[\varphi_1(B_{t_1})],
\]
where \(\varphi_{N-1}(x_1, \ldots, x_{N-1}) := \hat{E}[\varphi(x_1, \ldots, x_{N-1}, B_{t_N} - B_{t_{N-1}})]\) and
\[
\varphi_i(x_1, \ldots, x_i) := \hat{E}[\varphi_{i+1}(x_1, \ldots, x_{i}, B_{t_{i+1}} - B_{t_i})] \text{ for } i = N - 2, \ldots, 1.
\]
Theorem 2.1 \([5,15]\) There exists a unique convex and weakly compact set of probability measures \(\mathcal{P}\) on \((\Omega_T, \mathcal{B}(\Omega_T))\) such that
\[
\hat{E}[X] = \sup_{P \in \mathcal{P}} P[X] \text{ for all } X \in L^1_G(\Omega_T).
\]
\(\mathcal{P}\) is called a set that represents \(\hat{E}\).

For this \(\mathcal{P}\), we define capacity
\[
c(A) := \sup_{P \in \mathcal{P}} P(A) \text{ for } A \in \mathcal{B}(\Omega_T).
\]
A set \(A \in \mathcal{B}(\Omega_T)\) is polar if \(c(A) = 0\). A property holds “quasi-surely” (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables \(X\) and \(Y\) if \(X = Y\) q.s.

Definition 2.2 Let \(M^0_G(0,T)\) be the space of simple processes in the following form: for each \(N \in \mathbb{N}\) and \(0 = t_0 < \cdots < t_N = T\),
\[
\eta = \sum_{i=0}^{N-1} \xi_i I_{(t_i, t_{i+1})}(t),
\]
where \(\xi_i \in \text{Lip}(\Omega_t)\) for \(i = 0, 1, \ldots, N - 1\).

Denote by \(M^2_G(0,T)\) the completion of \(M^0_G(0,T)\) under the norm \(\|\eta\|_{M^2_G} := \left(\hat{E}[I^T_0 |\eta|^2 dt]\right)^{1/2}\) for \(p \geq 1\).

For each \(\eta \in M^2_G(0,T)\), \(i = 1, \ldots, d\), denote \(\eta = (\eta^1, \ldots, \eta^d) \in M^2_G(0,T; \mathbb{R}^d)\), the G-Itô integral \(\int_0^T \eta^i_t dB_t\) is well defined.

2.2 MP and DPP under volatility uncertainty

For each given \(t \leq s \leq T\), set
\[
\text{Lip}(\Omega_s^t) := \{ \varphi(B_t - B_s, \ldots, B_{t_n} - B_t) : N \geq 1, t \leq t_1 \leq \cdots \leq t_N \leq s, \varphi \in C_{Lip}(\mathbb{R}^{d \times N}) \},
\]
\[
M^{0,t}_G(t,T) = \left\{ \eta_t = \sum_{i=0}^{N-1} \xi_i I_{(t_i, t_{i+1})}(s) : N \geq 1, t = t_0 \leq \cdots \leq t_N = T, \xi_i \in \text{Lip}(\Omega_s^t) \right\}.
\]

Denote by \(M^{0,t}_G(t,T)\) (resp. \(L^p_G(\Omega_s^t)\)) the completion of \(M^{0,t}_G(t,T)\) (resp. \(\text{Lip}(\Omega_s^t)\)) under the norm \(\|\eta\|_{M^2_G} := \left(\hat{E}[\int_0^T |\eta|^2 dt]\right)^{1/2}\) (resp. \(\|X\|_{L^p_G} := \left(\hat{E}[|X|^p]\right)^{1/p}\)) for \(p \geq 1\).

For simplicity of presentation, we only consider the relationship between MP and DPP under 1-dimensional case studied in \([10,11]\). The results still hold for the general case \([12,13]\). More precisely, let \(U \subset \mathbb{R}^m\).
be a given nonempty convex and compact set, consider the following 1-dimensional forward and backward SDEs driven by 1-dimensional $G$-Brownian motion $B$ for each $(t, x) \in [0, T] \times \mathbb{R}$:

$$
\begin{aligned}
    dX_{t,x}^{s,u} &= h(s, X_{s,x}^{s,u}, u_s) d(B)_s + \sigma(s, X_{s,x}^{s,u}, u_s) dB_s, \\
    dY_{t,x}^{s,u} &= -g(s, X_{s,x}^{s,u}, Y_{s,x}^{t,x,u}, Z_{s,x}^{t,x,u}, u_s) d(B)_s + Z_{s,x}^{t,x,u} dB_s + dK_{s,x}^{t,x,u}, \\
    X_{t,x}^{t,x,u} &= x, \quad Y_{T,x}^{t,x,u} = \Phi(X_{T,x}^{t,x,u}), \quad s \in [t, T],
\end{aligned}
$$

(2.1)

where $\langle B \rangle$ is the quadratic variation of $B$, $h, \sigma, \Phi$ in $(x, y, z, u)$ are continuous in $(s, x, y, z, u)$.

**H1** The derivatives of $h, \sigma, \Phi$ in $(x, y, z, u)$ are continuous in $(s, x, y, z, u)$.

**H2** There exists a constant $L > 0$ such that for any $(s, x, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} 	imes U$,

$$
|h_x(s, x, u)| + |h_u(s, x, u)| + |\sigma_x(s, x, u)| + |\sigma_u(s, x, u)| + |g_x(s, x, y, z, u)| + |g_u(s, x, y, z, u)| & \leq L,

|g_x(s, x, y, z, u)| + |g_u(s, x, y, z, u)| + |\Phi'(x)| \leq L(1 + |x| + |u|).
$$

**Remark 2.3** Here $G(a) = \frac{1}{2} (\sigma^2 a^+ - \sigma^2 a^-)$ for $a \in \mathbb{R}$ with $\sigma \geq \sigma > 0$. By Corollary 3.5.5 in [20], we know $d(B)_s = \gamma_s ds$ with $\sigma^2 \leq \gamma_s \leq \sigma^2$.

For each $t \in [0, T]$, we denote by

$$
\mathcal{U}^t[t, T] := \{u(\cdot) : u(\cdot) \in M^2_{\mathbb{R}}(t, T; \mathbb{R}^m) \text{ with values in } U \} \quad \text{(2.2)}
$$

the set of admissible controls on $[t, T]$. For each $u(\cdot) \in \mathcal{U}^t[t, T]$, the equation (2.1) has a unique solution $(X_t^{t,x,u}, Y_t^{t,x,u}, Z_t^{t,x,u}, K_t^{t,x,u})$ (see [12]) such that

$$
\mathbb{E} \left[ \sup_{s \in [t, T]} \left( |X_s^{t,x,u}|^p + |Y_s^{t,x,u}|^p + \left( \int_t^T |Z_s^{t,x,u}|^2 ds \right)^{p/2} + |K_T^{t,x,u}|^p \right) \right] < \infty \text{ for each } p \geq 2 \quad \text{(2.3)}
$$

and

$$
X_s^{t,x,u}, Y_s^{t,x,u}, K_s^{t,x,u} \in L^p_G(\Omega^t_t), \quad K_t^{t,x,u} \text{ is a non-increasing } G\text{-martingale with } K_0^{t,x,u} = 0.
$$

Then $Y_t^{t,x,u} \in L^p_G(\Omega^t_t) = \mathbb{R}$. For each fixed $(t, x) \in [0, T] \times \mathbb{R}$, we define the cost functional

$$
J(t, x; u(\cdot)) = Y_t^{t,x,u} \quad \text{(2.4)}
$$

and the value function

$$
V(t, x) = \inf_{u(\cdot) \in \mathcal{U}^t[t, T]} J(t, x; u(\cdot)). \quad \text{(2.5)}
$$

Let $\bar{u}(\cdot) \in \mathcal{U}^t[t, T]$ be an optimal control. Then $V(t, x) = J(t, x; \bar{u}(\cdot))$. The corresponding solution $(X_t^{t,x,\bar{u}}, Y_t^{t,x,\bar{u}}, Z_t^{t,x,\bar{u}}, K_t^{t,x,\bar{u}})$ to equation (2.1) is called the optimal trajectory. Set

$$
\mathcal{P}^*_{t,x} = \{P \in \mathcal{P} : E_P[K_T^{t,x,\bar{u}}] = 0 \}. \quad \text{(2.6)}
$$
Remark 2.4  Note that $K_{T}^{t,x,u} \leq 0$, then $E_{P}[K_{T}^{t,x,u}] = 0$ iff $K_{T}^{t,x,u} = 0$ under $P$.

For each fixed $P_{t,x}^{*} \in \mathcal{P}_{t,x}^{*}$, the following adjoint equation

$$
\begin{cases}
    dp_{s} = -\{ [h_{x}(s) + g_{y}(s) + g_{z}(s)\sigma_{x}(s)] p_{s} + [g_{x}(s) + \sigma_{x}(s)] q_{s} + g_{z}(s) \} d(B)_{s} \\
    + q_{s} dB_{s} + dN_{s}, \\
    p_{T} = \Phi'(X^{t,x,u}_{T}), \ s \in [t, T],
\end{cases}
$$

(2.7)

has a unique solution $(p(\cdot), q(\cdot), N(\cdot)) \in M_{P_{t,x}^{*}}^{2}(t, T) \times M_{P_{t,x}^{*}}^{2}(t, T) \times M_{P_{t,x}^{*}}^{2}(t, T)$ under $P_{t,x}^{*}$ (see [3, 4]), where $h_{x}(s) = h_{x}(s, X^{t,x,u}_{t,x}, \bar{u}_{s})$, similar for $\sigma_{x}(s)$, $g_{x}(s), g_{y}(s)$ and $g_{z}(s)$, $F_{s}^{x} = \sigma(B_{r} - B_{t} : t \leq r \leq s)$,

$$
M_{P_{t,x}^{*}}^{2}(t, T) = \left\{ (\eta_{s})_{s \in [t, T]} : \eta_{s} \in F_{s}^{2}, \text{ and } E_{P_{t,x}^{*}} \left[ \int_{t}^{T} |\eta_{s}|^{2} ds \right] < \infty \right\},
$$

$$
M_{P_{t,x}^{*}}^{2}(t, T) = \left\{ (N_{s})_{s \in [t, T]} : N_{t} = 0, N_{s} \in F_{s}^{2}, \text{ N is a square integrable martingale that is orthogonal to } B \right\}.
$$

Define the Hamiltonian $H : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U \times U \times \mathbb{R} \times \mathbb{R} \times [t, T] \to \mathbb{R}$ as follows:

$$
H(x, y, z, u, v, p, q, s) = h(s, x, u)p + \sigma(s, x, u)q + g_{z}(s, x, y, z, v)\sigma(s, x, u)p + g(s, x, y, z, u).
$$

(2.8)

Hu and Ji [10] obtained the following MP.

Theorem 2.5 ([10]) Suppose that Assumptions (H1) and (H2) hold. Let $\bar{u}(\cdot) \in U^{t}[t, T]_{a.s.}$ be an optimal control for the control problem (2.3) and let $(X^{t,x,u_{t}}_{s}, Y^{t,x,u_{t}}_{s}, Z^{t,x,u_{t}}_{s}, K^{t,x,u_{t}}_{s})$ be the corresponding trajectory. Then there exists a $P_{t,x}^{*} \in \mathcal{P}_{t,x}^{*}$ such that the following maximum principle holds:

$$(H_{u}(X^{t,x,u_{t}}_{s}, Y^{t,x,u_{t}}_{s}, Z^{t,x,u_{t}}_{s}, \bar{u}_{s}, \bar{u}_{t}, p_{s}, q_{s}, s), u - \bar{u}_{s}) \geq 0, \ \forall u \in U, \ a.e. \ s \in [t, T], \ P_{t,x}^{*}-a.s.,$$

where $(p(\cdot), q(\cdot), N(\cdot))$ is the solution of the adjoint equation (2.7) under $P_{t,x}^{*}$, $H(\cdot)$ is defined in (2.8).

Hu and Ji [11] obtained that the value function $V(\cdot)$ satisfies the DPP. Based on DPP, they obtained the following theorem.

Theorem 2.6 ([11]) Suppose that Assumptions (H1) and (H2) hold. Let $V(\cdot)$ be the value function defined in (2.3). Then $V(\cdot)$ is the unique viscosity solution of the following second-order partial differential equation:

$$
\partial_{t} V(t, x) + \inf_{u \in U} G(F(t, x, V(t, x), \partial_{x} V(t, x), \partial_{x}^{2} V(t, x), u)) = 0, \ V(T, x) = \Phi(x),
$$

(2.9)

where $F : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}$ defined by

$$
F(t, x, a_{1}, a_{2}, a_{3}, u) = \sigma^{2}(t, x, u)a_{3} + 2h(t, x, u)a_{2} + 2g(t, x, a_{1}, \sigma(t, x, u)a_{2}, u).
$$

6
3 Main results

In the following, we first study the relationship between MP and DPP under the smooth case.

**Theorem 3.1** Suppose that Assumptions (H1) and (H2) hold. Let \( \bar{u}(\cdot) \in \mathcal{U}^t[t,T] \) be an optimal control for the control problem and let \( (X^{t,x,\bar{u}}, Y^{t,x,\bar{u}}, Z^{t,x,\bar{u}}, K^{t,x,\bar{u}}) \) be the corresponding trajectory.

(1) If the value function \( V(\cdot) \in C^{1,2}([t,T] \times \mathbb{R}) \) and \( \partial^2_{xx} V(\cdot) \) is of polynomial growth in \( x \), then

\[
\tilde{P}^{*}_{t,x} = \{ P \in \mathcal{P} : E_P[\tilde{K}_T] = 0 \} \tag{3.1}
\]

is nonempty and belongs to \( \mathcal{P}^*_{t,x} \), where \( \mathcal{P}^*_{t,x} \) is defined in (3.2),

\[
\begin{align*}
\tilde{K}_s &= \frac{1}{2} \int^s_t F(r)d\langle B \rangle_r - \int^s_t G(F(r))dr, s \in [t,T], \\
F(s) &= F(s, X^{t,x,\bar{u}}_s, V(s, X^{t,x,\bar{u}}_s), \partial_x V(s, X^{t,x,\bar{u}}_s), \partial^2_{xx} V(s, X^{t,x,\bar{u}}_s), \bar{u}_s).
\end{align*}
\]

Moreover for any \( P^*_{t,x} \in \tilde{P}^*_{t,x} \), we have, \( P^*_{t,x} \)-a.s., for a.e. \( s \in [t,T] \),

\[
\begin{align*}
Y^{t,x,\bar{u}}_s &= V(s, X^{t,x,\bar{u}}_s), \\
Z^{t,x,\bar{u}}_s &= \sigma(s, X^{t,x,\bar{u}}_s, \bar{u}_s) \partial_x V(s, X^{t,x,\bar{u}}_s),
\end{align*}
\]

and

\[
\begin{align*}
-\partial_x V(s, X^{t,x,\bar{u}}_s) &= G(F(s, X^{t,x,\bar{u}}_s, V(s, X^{t,x,\bar{u}}_s), \partial_x V(s, X^{t,x,\bar{u}}_s), \partial^2_{xx} V(s, X^{t,x,\bar{u}}_s), \bar{u}_s)) \\
&= \min_{u \in \mathcal{U}} G(F(s, X^{t,x,\bar{u}}_s, V(s, X^{t,x,\bar{u}}_s), \partial_x V(s, X^{t,x,\bar{u}}_s), \partial^2_{xx} V(s, X^{t,x,\bar{u}}_s), u)).
\end{align*}
\]

(2) If the value function \( V(\cdot) \in C^{1,2}([t,T] \times \mathbb{R}) \), \( \partial^2_{xx} V(\cdot) \) is of polynomial growth in \( x \) and \( \partial^2_{xx} V(\cdot) \) is continuous, then for any \( P^*_{t,x} \in \tilde{P}^*_{t,x} \), we have, \( P^*_{t,x} \)-a.s., for a.e. \( s \in [t,T] \),

\[
\begin{align*}
\partial^2_{xx} V(s, X^{t,x,\bar{u}}_s) &= -\frac{1}{2} \gamma_s \partial_x F(s) \text{ if } F(s) \neq 0, \\
\partial^2_{xx} V(s, X^{t,x,\bar{u}}_s) &= -\frac{1}{2} v_s \partial_x F(s) \text{ if } F(s) = 0,
\end{align*}
\]

where \( d\langle B \rangle_s = \gamma_s ds, v_s = -2 \partial^2_{xx} V(s, X^{t,x,\bar{u}}_s) (\partial_x F(s))^{-1} I_{(\partial_x F(s) \neq 0)} + \gamma_s I_{(\partial_x F(s) = 0)} \in [\mathbb{R}^2, \sigma^2], P^*_{t,x} \)-a.s., and

\[
\begin{align*}
\partial_x F(s) &= \partial_x F(s, X^{t,x,\bar{u}}_s, V(s, X^{t,x,\bar{u}}_s), \partial_x V(s, X^{t,x,\bar{u}}_s), \partial^2_{xx} V(s, X^{t,x,\bar{u}}_s), \bar{u}_s).
\end{align*}
\]

(3) If the value function \( V(\cdot) \in C^{1,3}([t,T] \times \mathbb{R}) \), \( \partial^2_{xx} V(\cdot) \) is of polynomial growth in \( x \), \( \partial^2_{xx} V(\cdot) \) is continuous and there exists a \( P^*_{t,x} \in \tilde{P}^*_{t,x} \) such that, \( P^*_{t,x} \)-a.s., for a.e. \( s \in [t,T] \),

\[
\begin{align*}
\partial^2_{xx} V(s, X^{t,x,\bar{u}}_s) &= -\frac{1}{2} \gamma_s \partial_x F(s) \text{ if } F(s) \neq 0, \\
\partial^2_{xx} V(s, X^{t,x,\bar{u}}_s) &= -\frac{1}{2} v_s \partial_x F(s) \text{ if } F(s) = 0,
\end{align*}
\]
then we have, \( P_{t,x}^* \)-a.s., for a.e. \( s \in [t, T] \),

\[
\begin{align*}
p_s &= \partial_s V(s, X_s^{t,x,u}), \\
q_s &= \sigma(s, X_s^{t,x,u}, \bar{u}_s)\partial_{xx}^2 V(s, X_s^{t,x,u}), \\
N_s &= 0,
\end{align*}
\]

and

\[
(H_u(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, \bar{u}_s, \bar{u}_s, p_s, q_s, s), u - \bar{u}_s) \geq 0, \quad \forall u \in U,
\]

where \((p(\cdot), q(\cdot), N(\cdot))\) is the solution of the adjoint equation \( \text{(2.17)} \) under \( P_{t,x}^* \), \( H(\cdot) \) is defined in \( \text{(2.8)} \).

**Proof.** (1) Applying Ito’s formula to \( V(s, X_s^{t,x,u}) \), we have

\[
\begin{align*}
d\tilde{Y}_s &= \Pi_s ds - g(s, X_s^{t,x,u}, \tilde{Y}_s, \tilde{Z}_s, \bar{u}_s) d\langle B \rangle_s + \tilde{Z}_s dB_s + d\tilde{K}_s, \\
\tilde{Y}_T &= \Phi(X_T^{t,x,u}), \quad s \in [t, T],
\end{align*}
\]

where \( \tilde{K}_s \) and \( F(s) \) are defined in \( \text{(3.2)} \).

\( \tilde{Y}_s = V(s, X_s^{t,x,u}), \quad \tilde{Z}_s = \sigma(s, X_s^{t,x,u}, \bar{u}_s)\partial_s V(s, X_s^{t,x,u}), \quad \Pi_s = \partial_s V(s, X_s^{t,x,u}) + G(F(s)) \).

By \( \text{(2.3)} \) and the assumptions of \( V(\cdot) \), we can obtain

\[
\hat{\mathbb{E}} \left[ \sup_{s \in [t,T]} |\tilde{Y}_s|^2 + \int_t^T (|\tilde{Z}_s|^2 + |F(s)|^2) ds \right] < \infty.
\]

It follows from Proposition 4.1.4 in \( \text{(2.16)} \) that \( (\tilde{K}_s)_{s \in [t,T]} \) is a non-increasing \( \hat{G} \)-martingale with \( \tilde{K}_t = 0 \) and \( \hat{\mathbb{E}} \left[ |\tilde{K}_T|^2 \right] < \infty \). Noting that \( V(\cdot) \) is a solution to PDE \( \text{(2.9)} \), we obtain \( \Pi_s \geq 0 \). On the other hand, \( (Y_s^{t,x,u}, Z_s^{t,x,u}, K_s^{t,x,u}) \) satisfies

\[
\begin{align*}
dY_s^{t,x,u} &= -g(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, \bar{u}_s) d\langle B \rangle_s + Z_s dB_s + dK_s^{t,x,u}, \\
Y_T^{t,x,u} &= \Phi(X_T^{t,x,u}), \quad s \in [t, T].
\end{align*}
\]

By \( \hat{\mathbb{E}} \left[ K_T \right] = 0 \) and Theorem \( \text{(2.1)} \) it is easy to get that \( \tilde{P}_{t,x}^* \) defined in \( \text{(3.1)} \) is nonempty. For any given \( P_{t,x}^* \in \tilde{P}_{t,x}^* \), since

\[
dK_s^{t,x,u} - d\Pi_s ds = dK_s^{t,x,u} - \Pi_s ds \leq 0 \text{ for } s \in [t, T], \quad P_{t,x}^* \text{-a.s.},
\]

and \( \tilde{Y}_t = V(t, x) = Y_t^{t,x,u} \), by the strict comparison theorem for BSDEs \( \text{(3.10)} \) and \( \text{(3.11)} \) under \( P_{t,x}^* \) (see Theorem 2.2 in \( \text{(2.2)} \)), we obtain

\[
dK_s^{t,x,u} - \Pi_s ds = 0 \text{ for } s \in [t, T], \quad P_{t,x}^* \text{-a.s.},
\]

which implies \( K_T^{t,x,u} = 0 \) and \( \Pi_s = 0 \) for \( s \in [t, T] \), \( P_{t,x}^* \)-a.s. Thus \( P_{t,x}^* \in \tilde{P}_{t,x}^* \) and \( \text{(3.3)} \), \( \text{(3.4)} \) hold under \( P_{t,x}^* \).
Thus we obtain
\[ \partial_t V(s, x) + G(F(s, x, V(s, x), \partial_x V(s, x), \partial_{xx}^2 V(s, x), \bar{u}_s)) \geq 0. \] (3.12)

For any given \( P_{t,x}^* \in \bar{P}_{t,x} \), by (3.4) we know, \( P_{t,x}^* \)-a.s., for a.e. \( s \in [t,T] \),
\[ \partial_s V(s, X_t^{s,x,u}) + G(F(s)) = 0. \] (3.13)

Under the case \( F(s) > 0 \), noting that \( F(s, x, V(s, x), \partial_x V(s, x), \partial_{xx}^2 V(s, x), \bar{u}_s) \) is continuous in \( x \), we obtain that, for \( x \rightarrow X_t^{s,x,u} \),
\[ G(F(s, x, V(s, x), \partial_x V(s, x), \partial_{xx}^2 V(s, x), \bar{u}_s)) - G(F(s)) = \frac{1}{2} \sigma^2 [F(s, x, V(s, x), \partial_x V(s, x), \partial_{xx}^2 V(s, x), \bar{u}_s) - F(s)], \]
which implies \( \partial_{xx}^2 V(s, X_t^{s,x,u}) + \frac{1}{2} \sigma^2 \partial_x F(s) = 0 \). Since
\[ \bar{K}_T = \frac{1}{2} \int_t^T F(s)d(B)_s - \int_t^T G(F(s))ds \]
and \( \sigma^2 \leq \gamma_s \leq \sigma^2 \), we get \( \gamma_s = \sigma^2 \) if \( F(s) > 0 \). Thus we have \( \partial_{xx}^2 V(s, X_t^{s,x,u}) = -\frac{1}{2} \gamma_s \partial_x F(s) \) under the case \( F(s) > 0 \). By the same method, we can get \( \partial_{xx}^2 V(s, X_t^{s,x,u}) = -\frac{1}{2} \gamma_s \partial_x F(s) \) under the case \( F(s) < 0 \).

Under the case \( F(s) = 0 \) and \( \partial_x F(s) = 0 \), it is clear that \( \partial_x |F(s)| = 0 \). Since \( |G(u)| \leq \frac{1}{2} \sigma^2 |u| \) for \( u \in \mathbb{R} \), we have \( \partial_x G(F(s)) = 0 \). Thus \( \partial_{xx}^2 V(s, X_t^{s,x,u}) = 0 \) by (3.12) and (3.13), which implies \( \partial_{xx}^2 V(s, X_t^{s,x,u}) = -\frac{1}{2} \gamma_s \partial_x F(s) \). Under the case \( F(s) = 0 \) and \( \partial_x F(s) > 0 \), it is obvious that
\[ F(s, x, V(s, x), \partial_x V(s, x), \partial_{xx}^2 V(s, x), \bar{u}_s) > 0 \] for \( x \uparrow X_t^{s,x,u} \)
and
\[ F(s, x, V(s, x), \partial_x V(s, x), \partial_{xx}^2 V(s, x), \bar{u}_s) < 0 \] for \( x \downarrow X_t^{s,x,u} \).

Thus we obtain \( \partial_{xx}^2 V(s, X_t^{s,x,u}) + \frac{1}{2} \sigma^2 \partial_x F(s) \geq 0 \) and \( \partial_{xx}^2 V(s, X_t^{s,x,u}) + \frac{1}{2} \sigma^2 \partial_x F(s) \leq 0 \) by (3.12) and (3.13), which implies \( \partial_{xx}^2 V(s, X_t^{s,x,u}) = -\frac{1}{2} \gamma_s \partial_x F(s) \) for some \( \mu \in [\sigma^2, \sigma^2] \). By the same analysis, we can get
\[ \partial_{xx}^2 V(s, X_t^{s,x,u}) = -\frac{1}{2} \gamma_s \partial_x F(s) \] under the case \( F(s) = 0 \) and \( \partial_x F(s) < 0 \).

(3) Applying Ito’s formula to \( \partial_x V(s, X_t^{s,x,u}) \) under \( P_{t,x}^* \), we have
\[
\begin{cases}
\begin{aligned}
d \partial_x V(s, X_t^{s,x,u}) &= \left[ \partial_{xx}^2 V(s, X_t^{s,x,u})h(s, X_t^{s,x,u}, u_s) + \frac{1}{2} \partial_{xxx}^3 V(s, X_t^{s,x,u})\sigma^2(s, X_t^{s,x,u}, u_s) \right] d(B)_s \\
&\quad + \partial_{xx}^2 V(s, X_t^{s,x,u}) ds + \partial_{xx}^2 V(s, X_t^{s,x,u}) \sigma(s, X_t^{s,x,u}, u_s) dB_s \\
\partial_x V(T, X_T^{s,x,u}) &= \Phi(X_T^{s,x,u}), \quad s \in [t,T].
\end{aligned}
\end{cases}
\]

By (3.8) and (3.7), we know \( \partial_{xx}^2 V(s, X_t^{s,x,u}) ds = -\frac{1}{2} \partial_x F(s)d(B)_s \) \( P_{t,x}^* \)-a.s. Then, by (3.8), it is easy to verify that
\[ (p_s, q_s, N_s) = (\partial_x V(s, X_t^{s,x,u}), \sigma(s, X_t^{s,x,u}, u_s)\partial_{xx}^2 V(s, X_t^{s,x,u}), 0) \]
satisfies the adjoint equation (2.7) under $P^*_{t,x}$, which implies (3.5).

It follows from (3.3) that, $P^*_{t,x}$-a.s., for a.e. $s \in [t,T]$,

$$G(F(s, X^s_{t,x,u}, V(s, X^s_{t,x,u}), \partial_x V(s, X^s_{t,x,u}), \partial_{xx}^2 V(s, X^s_{t,x,u}, u))) \geq G(F(s)).$$

Under the case $F(s) > 0$, for any given $u \in U$, we know $u^\varepsilon = \bar{u}_s + \varepsilon(u - \bar{u}_s) \in U$ with $\varepsilon \in [0, 1]$ and

$$G(F(s, X^s_{t,x,u}, V(s, X^s_{t,x,u}), \partial_x V(s, X^s_{t,x,u}), \partial_{xx}^2 V(s, X^s_{t,x,u}, u^\varepsilon))) - G(F(s)) = \frac{1}{2} \sigma^2 \big[ F(s, X^s_{t,x,u}, V(s, X^s_{t,x,u}), \partial_x V(s, X^s_{t,x,u}), \partial_{xx}^2 V(s, X^s_{t,x,u}, u^\varepsilon)) - F(s) \big] \geq 0 \text{ for } \varepsilon \downarrow 0.$$

From this, it is easy to deduce that

$$\langle \partial_x F(s, X^s_{t,x,u}, V(s, X^s_{t,x,u}), \partial_x V(s, X^s_{t,x,u}), \partial_{xx}^2 V(s, X^s_{t,x,u}, u_s)), u - u_s \rangle \geq 0, \forall u \in U. \quad (3.14)$$

By the same method, it is easy to check that (3.14) still holds under the cases $F(s) < 0$ and $F(s) = 0$. Thus it is easy to get (3.3) by (3.14).

**Remark 3.2** Although we obtain (3.3), we do not know whether there exists a $P^*_{t,x} \in \tilde{P}^*_{t,x}$ such that (3.7) holds (see Remark 2.3 in [23]).

The following proposition gives a sufficient condition for the assumption (3.7) to hold.

**Proposition 3.3** Suppose that Assumptions (H1) and (H2) hold. Let $\bar{u}(\cdot) \in U[t,T]$ be an optimal control for the control problem (3.9) and let $(X^{t,x,u}, Y^{t,x,u}, Z^{t,x,u}, K^{t,x,u})$ be the corresponding trajectory. The value function $V(\cdot) \in C^{1,2}([t,T] \times \mathbb{R})$, $\partial_{xx}^2 V(\cdot)$ is of polynomial growth in $x$ and $\partial_{xx}^2 V(\cdot)$ is continuous. If $h(\cdot)$, $\sigma(\cdot)$ are bounded functions and there exists a $\beta > 0$ such that $\sigma^2(\cdot) \geq \beta$, then the assumption (3.7) holds for any $P^*_{t,x} \in \tilde{P}^*_{t,x}$, where $\tilde{P}^*_{t,x}$ is defined in (3.7).

**Proof.** We only need to show that, for a.e. $s \in [t,T]$,

$$\partial_{xx}^2 V(s, X^s_{t,x,u}) = -\frac{1}{2} \gamma_0 \partial_x F(s) P^*_{t,x} \text{ a.s.}$$

Set

$$A_s = \{ x \in \mathbb{R} : \partial_x V(s, x) = 0, \partial_{xx}^2 V(s, x) \neq 0 \}.$$

For each $x \in A_s$, it is easy to find a $\delta > 0$ such that $\partial_x V(s, x') \neq 0$ for $x' \in (x - \delta, x) \cup (x, x + \delta)$, which implies that $A_s$ is countable. By (3.20), it is easy to deduce that

$$\left\{ \partial_{xx}^2 V(s, X^s_{t,x,u}) \neq -\frac{1}{2} \gamma_0 \partial_x F(s) \right\} \subseteq \left\{ X^s_{t,x,u} \in A_s \right\}.$$

By Theorem 3.7 in [10], we know that $c(\{ X^s_{t,x,u} \in A_s \}) = 0$, which implies $P^*_{t,x}(\{ X^s_{t,x,u} \in A_s \}) = 0$. Thus we obtain (3.7) for any $P^*_{t,x} \in \tilde{P}^*_{t,x}$. \qed

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For simplicity, the constant $C > 0$ will change from line to line in the following, and we only consider $D^1_x^+ V(t, x)$ and $D^1_x^- V(t, x)$.

**Theorem 3.4** Suppose that Assumptions (H1) and (H2) hold. Let $\bar{u}(\cdot) \in \mathcal{U}[t, T]$ be an optimal control for the control problem (2.5) and let $(X^{t,x,\bar{u}}, Y^{t,x,\bar{u}}, Z^{t,x,\bar{u}}, K^{t,x,\bar{u}})$ be the corresponding trajectory. Then we have

$$D^1_x^+ V(t, x) \subseteq \bar{p}_t,$$

where $\bar{p}_t = \sup_{P_x \in \mathcal{P}_T^x, P^x_t} \bar{p}_x$, $\bar{p}_t = \inf_{P_x \in \mathcal{P}_T^x, P^x_t}$, $P^x_t$ is defined in (2.6), $(P^x_T(\cdot), q^{P^x}(\cdot), N^{P^x}(\cdot))$ is the solution of the adjoint equation (2.7) under $P^x$. Moreover, if $\bar{p}_t = \bar{p}_t$, then

$$\{\bar{p}_t\} \subseteq D^1_x^+ V(t, x).$$

**Proof.** For each $x' \in \mathbb{R}$ and $s \in [t, T]$, set

$$\tilde{X}_s = X^{t,x',\bar{u}} - X^{t,x,\bar{u}}, \quad \tilde{Y}_s = Y^{t,x',\bar{u}} - Y^{t,x,\bar{u}},$$
$$\tilde{Z}_s = Z^{t,x',\bar{u}} - Z^{t,x,\bar{u}}, \quad \tilde{K}_s = K^{t,x',\bar{u}} - K^{t,x,\bar{u}}.$$

For simplicity, let $x$ be a fixed constant and $x' \in [x - 1, x + 1]$. The proof is divided into six steps.

**Step 1:** Estimates for $\tilde{X}, \tilde{Y}, \tilde{Z}$ and $\tilde{K}$ under $\mathbb{E}[^\cdot]$.

By the estimates of G-SDEs (see [13, 20]), we have, for each $p \geq 2$,

$$\mathbb{E}\left[ \sup_{s \in [t, T]} |\tilde{X}_s|^p \right] \leq C |x' - x|^p \quad \text{and} \quad \mathbb{E}\left[ \sup_{s \in [t, T]} |X^{t,x,\bar{u}}|^p \right] \leq C,$$

where the constant $C > 0$ depends on $T, \sigma^2, L$ and $p$. It follows from Proposition 5.1 in [12] that, for each $p \geq 2$,

$$|\tilde{Y}_s|^p \leq C \mathbb{E}_s \left[ \Phi(X^{t,x,\bar{u}}) - \Phi(X^{t,x,\bar{u}}) \right]^p + \left( \int_t^T |\tilde{g}_r| dr \right)^p,$$

where $C > 0$ depends on $T, \sigma^2, L$ and $p$. $\tilde{g}_s = g(r, X^{t,x,\bar{u}}, Y^{t,x,\bar{u}}, Z^{t,x,\bar{u}}, \bar{u}_r) - g(r, X^{t,x,\bar{u}}, Y^{t,x,\bar{u}}, Z^{t,x,\bar{u}}, \bar{u}_r).$

By Doob’s inequality under $\mathbb{E}[\cdot]$ (see [22, 24]), we have

$$\mathbb{E}\left[ \sup_{s \in [t, T]} |\tilde{Y}_s|^p \right] \leq C \left( \mathbb{E}\left[ \Phi(X^{t,x,\bar{u}}) - \Phi(X^{t,x,\bar{u}}) \right]^{p+1} + \left( \int_t^T |\tilde{g}_r| dr \right)^{p+1} \right)^{p/(p+1)}.$$

By (H2), (3.15) and Hölder’s inequality, we can easily obtain

$$\mathbb{E}\left[ \sup_{s \in [t, T]} |\tilde{Y}_s|^p \right] \leq C |x' - x|^p,$$

(3.16)
where $C > 0$ depends on $T$, $\bar{\sigma}^2$, $L$ and $p$. By Proposition 3.8 in \cite{12}, we deduce

$$E \left( \left( \int_t^T |\tilde{Z}_s|^2 ds \right)^{p/2} \right) \leq C|x' - x|^{p/2} \text{ for } p \geq 2,$$

(3.17)

where $C > 0$ depends on $T$, $\bar{\sigma}^2$, $\sigma^2$, $L$ and $p$.

Step 2: Estimates for $\tilde{Z}$ and $\tilde{K}$ under $P \in \mathcal{P}$.

Set

$$\tilde{P}_{t,x}^* = \{ P \in \mathcal{P} : E_P[K_{t,x}^{t,x}] = 0 \}.$$

For each fixed $P^{x'} \in \tilde{P}_{t,x}^*$, applying Ito’s formula to $|\tilde{Y}_s|^2$ under $P^{x'}$, we obtain

$$|\tilde{Y}_t|^2 + \int_t^T |\tilde{Z}_s|^2 d(B)_s = |\tilde{Y}_T|^2 + 2 \int_t^T \tilde{Y}_s \tilde{g}_s d(B)_s - 2 \int_t^T \tilde{Y}_s \tilde{Z}_s dB_s + 2 \int_t^T \tilde{Y}_s dK_{t,x}^{t,x},$$

where $\tilde{g}_s = g(s, X_t^{t,x'}, \dot{Y}_s^{t,x'}, Z_t^{t,x'}, \tilde{u}_s) - g(s, X_t^{t,x}, \dot{Y}_s^{t,x}, Z_t^{t,x}, \bar{u}_s)$. By the Burkholder-Davis-Gundy inequality, (3.15) and (3.16), we obtain

$$E_{P^{x'}} \left( \left( \int_t^T |\tilde{Z}_s|^2 ds \right)^{p/2} \right) \leq C|x - x'|^p + CE_{P^{x'}} \left( \left( \int_t^T |\tilde{Y}_s|^2 ds \right)^{p/2} \right),$$

(3.18)

where $C > 0$ depends on $T$, $\bar{\sigma}^2$, $\sigma^2$, $L$ and $p$. Since

$$-K_{t,x}^{t,x} = \tilde{Y}_T - \tilde{Y}_t + \int_t^T \tilde{g}_s d(B)_s - \int_t^T \tilde{Z}_s dB_s \text{ } P^{x'}-\text{a.s.,}$$

we get

$$E_{P^{x'}} \left( |K_{t,x}^{t,x}|^p \right) \leq C|x - x'|^p + E_{P^{x'}} \left( \left( \int_t^T |\tilde{Z}_s|^2 ds \right)^{p/2} \right),$$

(3.19)

where $C > 0$ depends on $T$, $\bar{\sigma}^2$, $\sigma^2$, $L$ and $p$. Combining (3.18) and (3.19), we get

$$E_{P^{x'}} \left( \left( \int_t^T |\tilde{Z}_s|^2 ds \right)^{p/2} + |K_{t,x}^{t,x}|^p \right) \leq C|x' - x|^p,$$

(3.20)

where $C > 0$ depends on $T$, $\bar{\sigma}^2$, $\sigma^2$, $L$ and $p$. By similar method, we can obtain that, for each fixed $P^x \in \mathcal{P}_{t,x}^*$ and $p \geq 2,$

$$E_{P^x} \left( \left( \int_t^T |\tilde{Z}_s|^2 ds \right)^{p/2} + |K_{t,x}^{t,x}|^p \right) \leq C|x' - x|^p,$$

(3.21)

where $C > 0$ depends on $T$, $\bar{\sigma}^2$, $\sigma^2$, $L$ and $p$.

Step 3: Variation of $\tilde{X}$ and $\tilde{Y}$.

Rewrite the equation of $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{K})$ as follows:

$$\begin{cases}
  d\tilde{X}_s &= [h_x(s) + \varepsilon_1(s)]\tilde{X}_s d(B)_s + [\sigma_x(s) + \varepsilon_2(s)]\tilde{X}_s dB_s, \\
  d\tilde{Y}_s &= -([g_x(s) + \varepsilon_3(s)]\tilde{X}_s + [g_y(s) + \varepsilon_4(s)]\tilde{Y}_s + [g_z(s) + \varepsilon_5(s)]\tilde{Z}_s) d(B)_s \\
  &+ \tilde{Z}_s dB_s + dK_{t,x}^{t,x} - dK_{t,x}^{t,x}, \\
  \tilde{X}_t &= x' - x, \quad \tilde{Y}_T = [\Phi(X_T^{t,x}) + \varepsilon_6(T)]\tilde{X}_T, \quad s \in [t,T],
\end{cases}$$

(3.22)
where
\[
\begin{align*}
\varepsilon_1(s) &= \int_0^1 [h_x(s, X_s^x, u) + \alpha \dot{X}_s, \bar{u}_s) - h_x(s)]da, \\
\varepsilon_2(s) &= \int_0^1 [\sigma_x(s, X_s^x, u) + \alpha \dot{X}_s, \bar{u}_s) - \sigma_x(s)]da, \\
\varepsilon_3(s) &= \int_0^1 [g_x(s, X_s^x, u) + \alpha \dot{X}_s, Y_s^x, u, \dot{Y}_s, Z_s^x, u) - g_x(s)]da, \\
\varepsilon_4(s) &= \int_0^1 [g_y(s, X_s^x, u) + \alpha \dot{X}_s, Y_s^x, u, \dot{Y}_s, Z_s^x, u) - g_y(s)]da, \\
\varepsilon_5(s) &= \int_0^1 [g_z(s, X_s^x, u) + \alpha \dot{X}_s, Y_s^x, u, \dot{Y}_s, Z_s^x, u, \dot{Z}_s, \bar{u}_s) - g_z(s)]da, \\
\varepsilon_6(t) &= \int_0^1 [\Phi'(X_s^x) + \alpha \dot{X}_s) - \Phi'(X_s^x)]da.
\end{align*}
\]

Let \((l_s)_{s \in [t, T]}\) be the solution of the following G-SDE:
\[
dl_s = g_y(s)l_s d(B)_s + g_z(s)l_s dB_s, \quad l_t = 1.
\]

For each given \(P^{o'} \in \tilde{P}_{t,x}\), the solution of the adjoint equation (24.7) is denoted by \((p^{o'}(\cdot), q^{o'}(\cdot), N^{o'}(\cdot))\).

Applying Itô’s formula to \(l_s(\dot{Y}_s - \dot{p}^{o'}(\cdot) \dot{X}_s)\) under \(P^{o'}\) and noting that \(dK_s^{t,x,\alpha} = 0\), we obtain
\[
\dot{Y}_t - \dot{p}^{o'}_t \dot{X}_t = E_{P^{o'}} \left[ \varepsilon_6(T) \right] l_T \dot{Y}_T + \int_t^T l_s dK_s^{t,x,\alpha} + \int_t^T \left\{ \varepsilon_4(s) l_s \dot{Y}_s + \varepsilon_5(s) l_s \dot{Z}_s \\
+ \left[ \varepsilon_1(s)p_s^{o'} + \varepsilon_2(s)q_s^{o'} + \varepsilon_3(s)g_s(s)p_s^{o'} + \varepsilon_3(s) \right] l_s \dot{X}_s \right\} d(B)_s.
\]

Step 4: Estimates of every terms in the right side of (3.23).

By \(l_s \geq 0\) and \(dK_s^{t,x,\alpha} \leq 0\), we get \(\int_t^T l_s dK_s^{t,x,\alpha} \leq 0\). It follows from Hölder’s inequality, \(d(B)_s \leq \sigma^2 ds\) and (4.20) that
\[
\left| E_{P^{o'}} \left[ \int_t^T \varepsilon_5(s) l_s \dot{Z}_s d(B)_s \right] \right| \\
\leq C \left( E_{P^{o'}} \left[ \int_t^T |\dot{Z}_s|^2 ds \right] \right)^{1/2} \left( E_{P^{o'}} \left[ \int_t^T |\varepsilon_5(s)|^4 ds \right] \right)^{1/4} \left( E_{P^{o'}} \left[ \int_t^T |l_s|^4 ds \right] \right)^{1/4}
\]
\[
\leq C \left( \mathbb{E} \left[ \int_t^T |\varepsilon_5(s)|^4 ds \right] \right)^{1/4} \left( \mathbb{E} \left[ \sup_{s \in [t, T]} |l_s|^4 \right] \right)^{1/4} x - x',
\]
where \(C > 0\) depends on \(T, \sigma^2, \mathbb{P}^2\) and \(L\). Similar to (3.15), we know
\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |l_s|^4 \right] \leq C,
\]
where \(C > 0\) depends on \(T, \sigma^2\) and \(L\). Now, we prove
\[
\mathbb{E} \left[ \int_t^T |\varepsilon_5(s)|^4 ds \right] \to 0 \text{ as } x' \to x.
\]

For each given \(N > 0\) and \(\delta \in [0, 1]\), set
\[
\omega_N(\delta) = \sup \{ g_z(s, x + x', y + y', z + z', u) - g_z(s, x, y, z, u) : |x| \leq N, \\
|y| \leq N, |z| \leq N, u \in U, |x'| \leq \delta, |y'| \leq \delta, |z'| \leq \delta \}.
\]

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By (H1), we have \( \bar{\omega}_N(\delta) \to 0 \) as \( \delta \to 0 \) for each fixed \( N > 0 \). Thus

\[
|\varepsilon_5(s)| \leq \bar{\omega}_N(\delta) + \frac{2L}{N} \left( \sqrt{|X_s| + |Y_s|} + \sqrt{|Z_s|} \right) + \frac{2L}{N} \left( \sqrt{|X_s^t,x,u|} + \sqrt{|Y_s^t,x,u|} + \sqrt{|Z_s^t,x,u|} \right).
\]

By (3.15), (3.16) and (3.17), we obtain

\[
\hat{Y}_t = \int_t^T \varepsilon_5(s) ds \leq C \left( |\bar{\omega}_N(\delta)|^4 + \frac{1}{\Delta^2} |x' - x| + \frac{1}{N^2} \right),
\]

where \( C > \delta \) which implies (3.24) by letting \( \delta \to 0 \) and then \( N \to \infty \). Then we obtain

\[
E_{P^{x'}} \left[ \int_t^T \varepsilon_5(s) ds \right] = o(|x' - x|).
\]

(3.25)

Similar to the proof of (3.25), we can prove that the other terms in the right side of (3.23) are \( o(|x' - x|) \). Note that \( \bar{X}_t = x' - x \), then we get

\[
\hat{Y}_t \leq p_t^{x'}(x' - x) + o(|x' - x|) \text{ for } P^{x'} \in \bar{\mathcal{P}}_{t,x}'.
\]

(3.26)

Similar to the proof of (3.25), we can deduce

\[
\hat{Y}_t \geq p_t^{x'}(x' - x) + o(|x' - x|) \text{ for } P^x \in \mathcal{P}_{t,x}'.
\]

(3.27)

Step 5: First order derivative for \( \hat{Y}_t \).

For \( x' > x \), we get by (3.27) that

\[
\liminf_{x' \downarrow x} \frac{\hat{Y}_t}{x' - x} = \liminf_{x' \downarrow x} \frac{Y_t^{t,x',u} - Y_t^{t,x,u}}{x' - x} \geq \sup_{P^{x'} \in \bar{\mathcal{P}}_{t,x'}} p_t^{x'} = \bar{p}_t.
\]

(3.28)

It follows from (3.15) that \( \bar{Y}_t(x' - x)^{-1} \) is bounded. Thus we can choose \( x^n > x, n \geq 1 \), such that \( x^n \downarrow x \) and

\[
\limsup_{x' \downarrow x} \frac{Y_t^{t,x',u} - Y_t^{t,x,u}}{x' - x} = \lim_{n \to \infty} \frac{Y_t^{t,x^n,u} - Y_t^{t,x,u}}{x^n - x} < \infty.
\]

(3.29)

Since \( P^{x^n} \in \bar{\mathcal{P}}_{t,x,n} \subset \mathcal{P} \) and \( \mathcal{P} \) is weakly compact, there exists a subsequence of \( \{x^n : n \geq 1\} \), denoted by \( \{x^n : i \geq 1\} \), such that \( P^{x^n_i} \to P^* \in \mathcal{P} \) weakly as \( i \to \infty \). By Lemma 29 in [3], we have

\[
E_{P^{x^n_i}} \left[ K_{T}^{t,x,u} \right] \to E_{P^*} \left[ K_{T}^{t,x,u} \right] \text{ as } i \to \infty.
\]

(3.30)

Noting that \( K_{T}^{t,x^n_i,u} = 0 \) \( P^{x^n_i} \)-a.s., we get

\[
- E_{P^{x^n_i}} \left[ K_{T}^{t,x,u} \right] = E_{P^{x^n_i}} \left[ K_{T}^{t,x^n_i,u} - K_{T}^{t,x,u} \right] \leq \hat{E} \left[ |K_{T}^{t,x^n_i,u} - K_{T}^{t,x,u}| \right].
\]

(3.31)
By (3.17), (3.30) and (3.31), we deduce \( E_{P_t^i} [R_{T_t}^{i,x,u}] = 0 \), which implies \( P^* \in \mathcal{P}^*_t \). Since the adjoint equation (2.7) is a linear BSDE, by Proposition 2.2 in [7], we obtain
\[
p_t^{P^*_t} = E_{P_t^i} \left[ \lambda_T \Phi(X_T^{t,x,u}) + \int_t^T \lambda_s g_x(s) d(B)_s \right],
\]
where
\[
\lambda_s = \exp \left( \int_t^s \beta_r dB_r + \int_t^s (\alpha_r - \frac{1}{2} |\beta_r|^2) d(B)_r \right),
\]
\[
\alpha_r = h_x(r) + g_y(r) + g_z(r) \sigma_x(r), \quad \beta_r = g_z(r) + \sigma_x(r).
\]
Since \( \lambda_T \Phi(X_T^{t,x,u}) + \int_T^T \lambda_s g_x(s) d(B)_s \in L^1_G(\Omega_T) \), we get
\[
\lim_{i \to \infty} p_t^{P^*_t} \to p_t^{P^*_t} \quad \text{as} \quad i \to \infty
\]
by Lemma 29 in [3]. By (3.25), (3.26) and (3.27), we obtain
\[
\lim_{x' \to x} \frac{Y_t^{t',x',u} - Y_t^{t,x,u}}{x' - x} = \bar{p}_t = \sup_{P^* \in \mathcal{P}^*_t} p_t^{P^*},
\]
which implies
\[
Y_t^{t',x',u} - Y_t^{t,x,u} = \bar{p}_t (x' - x) + o(|x' - x|) \quad \text{for} \quad x' > x.
\]
Similar to (3.32), we can obtain
\[
Y_t^{t',x',u} - Y_t^{t,x,u} = \bar{p}_t (x' - x) + o(|x' - x|) \quad \text{for} \quad x' < x,
\]
where \( \bar{p}_t = \inf_{P^* \in \mathcal{P}^*_t} p_t^{P^*} \).

Step 6: \( D_{x}^{1,-} V(t,x) \) and \( D_{x}^{1,+} V(t,x) \).

Noting that \( V(t,x) = Y_t^{t,x,u} \) and \( V(t,x') \leq Y_t^{t',x',u} \), we have
\[
V(t,x') - V(t,x) \leq Y_t^{t',x',u} - Y_t^{t,x,u}.
\]
For any given \( a \in D_{x}^{1,-} V(t,x) \), by definition of \( D_{x}^{1,-} V(t,x) \), we get
\[
a(x' - x) + o(|x' - x|) \leq V(t,x') - V(t,x).
\]
By (3.32)-(3.39), we deduce \( a \leq \bar{p}_t \) if \( x' > x \) and \( a \geq \bar{p}_t \) if \( x' < x \). Thus \( a \in [\bar{p}_t, \bar{p}_t] \), which implies \( D_{x}^{1,-} V(t,x) \subseteq [\bar{p}_t, \bar{p}_t] \). If \( \bar{p}_t = \bar{p}_t \), by (3.33)-(3.35) and the definition of \( D_{x}^{1,+} V(t,x) \), we have \( \bar{p}_t \in D_{x}^{1,+} V(t,x) \).

The first example shows that \( D_{x}^{1,+} V(t,x) \) may be empty if \( \bar{p}_t \neq \bar{p}_t \).

**Example 3.5** Consider the following simple control system:

\[
\begin{align*}
\left\{ \begin{array}{l}
dX_t^{t,x,u} &= u_x d(B)_s, \quad X_t^{t,x,u} = x, \quad s \in [t,T], \\
\quad dY_s^{t,x,u} &= Z_s^{t,x,u} dB_s + dR_s^{t,x,u}, \quad Y_T^{t,x,u} = (X_T^{t,x,u})^2,
\end{array} \right.
\end{align*}
\]
where $U = \{1\}$, $\bar{v}^2 = 1$ and $\bar{\sigma}^2 = 0.2$. Obviously, $\bar{u} = 1$ and the value function

$$V(t,x) = \mathbb{E}\left[\left(x + \langle B \rangle_T - \langle B \rangle_t\right)^2\right], \ (t,x) \in [0,T] \times \mathbb{R}.$$  

By Theorem 3.5.4 in [20], we have

$$\mathbb{E}\left[\left(x + \langle B \rangle_T - \langle B \rangle_t\right)^2\right] = \sup_{\bar{\sigma}^2 \leq \sigma^2} \left(x + v(T-t)\right)^2. \quad (3.37)$$

Thus we obtain

$$V(t,x) = \begin{cases} 
(x + (T-t))^2, & x \geq -0.6(T-t), \\
(x + 0.2(T-t))^2, & x < -0.6(T-t).
\end{cases}$$

On the point $(t^*, x^*) = (t^*, -0.6(T-t^*))$ for some $t^* < T$, it is easy to verify that the maximum value in (3.37) is obtained at $v = \bar{\sigma}^2$ or $v = \sigma^2$. Thus $\{P^v : v = \bar{\sigma} \text{ or } v = \sigma\} \subseteq \mathcal{P}^{t*,x*}_t$, where $P^v$ is a probability measure on $(\Omega_T, \mathcal{B}(\Omega_T))$ such that $\langle B \rangle_s = v^2 \sigma^2 \text{ a.s.}$ Under this case,

$$p^{P^v}_{t^*} = 2E_{P^v}[x^* + \langle B \rangle_T - \langle B \rangle_{t^*}] = 0.8(T-t^*), \quad p^{P^v}_{t^*} = 2E_{P^v}[x^* + \langle B \rangle_T - \langle B \rangle_{t^*}] = -0.8(T-t^*),$$

which implies $\bar{p}^v \neq \bar{p}^v$. It is easy to check that

$$\lim_{x \downarrow x^*} \frac{V(t^*, x) - V(t^*, x^*)}{x - x^*} = 0.8(T-t^*), \quad \lim_{x \uparrow x^*} \frac{V(t^*, x) - V(t^*, x^*)}{x - x^*} = -0.8(T-t^*),$$

which implies $D_{t^*}^1 V(t^*, x^*) = \emptyset$ by the definition of $D_{t^*}^1 V(t^*, x^*)$.

The second example is a linear quadratic optimal control problem under $\mathbb{E}[\cdot]$ studied in [10].

Example 3.6 Consider the following control system:

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\frac{dX^t_{t,x,u}}{dt} &= \left(4X^t_{t,x,u} + u_s\right)d(B) + \left(X^t_{t,x,u} + u_s\right)dB,
\frac{dY^t_{t,x,u}}{dt} &= -\frac{1}{2}(X^t_{t,x,u})^2 + (u_s)^2d(B) + Z^t_{t,x,u}dB + dK^t_{t,x,u},
X^t_{t,x,u} &= x, Y^t_{t,x,u} = \frac{1}{2}(X^t_{t,x,u})^2, \ s \in [t,T],
\end{array}
\right.
\end{align*}
\]

where $x \neq 0$, $U = \mathbb{R}$, $\bar{\sigma}^2 = 1$ and $\bar{\sigma}^2 = 0.2$. Let $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ be an optimal control. By Theorem 3.5 there exists a $P^*_{t,x} \in \mathcal{P}^{t,x}_T$ such that $E_{P^*_{t,x}}[K^t_{t,x,u}] = 0$ and

$$p_s + q_s + \bar{u}_s = 0, \ a.e. \ s \in [t,T], \ P^*_{t,x}-a.s., \quad (3.38)$$

where $(p(\cdot), q(\cdot), N(\cdot))$ satisfies the following BSDE under $P^*_{t,x}$:

\[
\begin{align*}
\left\{ 
\begin{array}{l}
dp_s &= -[4p_s + q_s + X^t_{s,x,u}]d(B) + q_s dB + dN_s, \\
p_T &= X^t_{T,x,u}, \ s \in [t,T].
\end{array}
\right.
\end{align*}
\]
Suppose that \( p = P_sX^{t,x,\bar{u}}, P^*_t - \text{a.s.} \)
and \( dP_s = \Lambda_s d\langle B \rangle_s, \ P_T = 1, \ P^*_t - \text{a.s.} \).

Applying Itô’s formula to \( P_sX^{t,x,\bar{u}} \) under \( P^*_t \), we get
\[
\begin{cases}
\Lambda_s X^{t,x,\bar{u}} = -9P_sX^{t,x,\bar{u}} - X^{t,x,\bar{u}} - 2P_s \bar{u}, \\
q_s = P_sX^{t,x,\bar{u}} + P_s \bar{u}, \ N_s = 0.
\end{cases}
\tag{3.39}
\]

By (3.38) and (3.39), we obtain
\[
\bar{u}_s = -\frac{2P_s}{1 + P_s}X^{t,x,\bar{u}}, \ \Lambda_s = -\frac{5P^2_s + 10P_s + 1}{1 + P_s}.
\]

Suppose that \( d\langle B \rangle_s = \sigma^2 ds = ds \) under \( P^*_t \). Then we obtain
\[
dP_s = -\frac{5P^2_s + 10P_s + 1}{1 + P_s} ds, \ P_T = 1, \tag{3.40}
\]
which implies
\[
d(1 + P_s)^2 = [-10(1 + P_s)^2 + 8] ds, \ P_T = 1.
\]

The solution of (3.40) is
\[
P_s = \sqrt{\frac{16}{5}(10(T-s)) + \frac{1}{5} - 1}, \ s \in [0,T]. \tag{3.41}
\]

In the following, we prove that \( d\langle B \rangle_s = ds \) under \( P^*_t \). Let \((P_s)_{s \in [0,T]}\) be the solution of (3.41) and
\[
\bar{u}_s = -\frac{2P_s}{1 + P_s}X^{t,x,\bar{u}}. \tag{3.42}
\]

Applying Itô’s formula to \( \frac{1}{2}P_s(X^{t,x,\bar{u}})^2 \), it is easy to verify that
\[
Y^{t,x,\bar{u}} = \frac{1}{2}P_s(X^{t,x,\bar{u}})^2, \ Z^{t,x,\bar{u}} = P_s(X^{t,x,\bar{u}})^2 + P_sX^{t,x,\bar{u}} \bar{u}, \ K^{t,x,\bar{u}} = \int_t^s \frac{5P^2_s + 10P_s + 1}{2(1 + P_s)} (X^{t,x,\bar{u}})^2 d\langle B \rangle_r - dr.
\]

Since \( P_r > 0 \) and \((X^{t,x,\bar{u}})^2 > 0 \), we obtain that \( P_t \) contains only one element \( P^*_t \) such that \( d\langle B \rangle_r = dr \).

By Theorem 5.4 in [10], \( \bar{u} \) defined in (3.42) is an optimal control.

By Theorem 3.7, the value function \( V(\cdot) \) satisfies the following HJB equation:
\[
\begin{cases}
\partial_t V(t,x) + \inf_{v \in \mathbb{R}} G((x + v)^2 \partial^2_{xx} V(t,x) + 2(4x + v) \partial_x V(t,x) + x^2 + v^2) = 0, \\
V(T,x) = \frac{1}{2}x^2.
\end{cases}
\]

It is easy to verify that \( V(t,x) = \frac{1}{2}P_t x^2 \) is the unique solution of the above HJB equation, where \((P_s)_{s \in [0,T]}\) is the solution of (3.40). Thus one can easily check that the relations (3.35) and (3.38) in Theorem 3.1 hold.

The third example shows that the relation (3.38) in Theorem 3.1 holds only for \( P^*_t \) in \( \mathbb{P}^*_t \) defined in (3.1).
Example 3.7 Consider the following control system:

\[
\begin{align*}
\begin{cases}
    dX_{t,x,u}^s &= u_s d(B)_s, \\
    dY_{t,x,u}^s &= Z_{t,x,u}^s dB_s + dK_{t,x,u}^s.
\end{cases}
\end{align*}
\]

where \( T > 1, U = [1, 2], \sigma^2 = 1 \) and \( \sigma^2 = 0.5 \). Under this case, the value function

\[
V(t,x) = \inf_{u \in \mathcal{U}([t,T])} Y_{t,x,u}^T = \inf_{u \in \mathcal{U}([t,T])} \mathbb{E} \left[ - \left( x + \int_t^T u_s d(B)_s - 1 \right)^2 \right].
\]

(3.43)

By Theorem 2.6, the value function \( V(\cdot) \) satisfies the following HJB equation:

\[
\partial_t V(t,x) + \inf_{v \in \mathcal{U}} G(2v \partial_x V(t,x)) = 0, \quad V(T,x) = -(x-1)^2.
\]

It is easy to check that

\[
V(t,x) = -(x + T - t - 1)^2
\]

(3.44)
satisfies the above HJB equation.

Let \((t,x) = (T - 1, 0)\) be fixed. From (3.43) and (3.44), we obtain that \( \bar{u}_s = v, s \in [t,T], \) is an optimal control for any fixed \( v \in U \). Take \( \bar{u}_s = v \) with \( v \in (1,2) \), it is easy to check that

\[
Y_{t,x,u}^s = \phi(s, v((B)_s - (B)_t) - 1), \quad Z_{t,x,u}^s = 0,
\]

\[
K_{t,x,u}^s = \int_t^s \partial_s \phi(r, v((B)_r - (B)_t) - 1)vd(B)_r + \int_t^s \partial_v \phi(r, v((B)_r - (B)_t) - 1)dr,
\]

where

\[
\phi(s, x') = \begin{cases} 
-(x' + \frac{v}{2}(T-s))^2, & x' > -\frac{v}{2}(T-s), \\
0, & x' \in [-v(T-s), -\frac{v}{2}(T-s)], \\
-(x' + v(T-s))^2, & x' < -v(T-s).
\end{cases}
\]

For each \( P_{t,x}^* \in \mathcal{P}_{t,x}^* \) defined in (3.7), we know that for, \( s \in [t,T], \)

\[
\int_t^s [\partial_v \phi(r, v \int_t^s \gamma d\theta - 1) v + \partial_v \phi(r, v \int_t^s \gamma d\theta - 1)]dr = 0, \quad P_{t,x}^* - \text{a.s.},
\]

where \( d(B)_r = \gamma_r dr \). Then we obtain

\[
\partial_v \phi(s, v \int_t^s \gamma d\theta - 1) v + \partial_v \phi(s, v \int_t^s \gamma d\theta - 1) = 0, \quad \text{a.e.} \ s \in [t,T], \quad P_{t,x}^* - \text{a.s.}
\]

By simple calculation of \( \partial_v \phi \) and \( \partial_s \phi \), we can easily obtain that there exists a \( \Omega_0 \subset \Omega \) with \( P_{t,x}^*(\Omega_0) = 1 \) such that, for \( \omega \in \Omega_0, \) a.e. \( s \in [t,T], \)

\[
\gamma_\omega(s) = \begin{cases}
0.5, & v \int_t^s \gamma d\theta - 1 > -\frac{v}{2}(T-s), \\
1, & v \int_t^s \gamma d\theta - 1 < -v(T-s).
\end{cases}
\]

(3.45)
If \( v \int_t^{s*} \gamma_{\theta}(\omega)d\theta - 1 > -\frac{v}{2}(T-s) \) for some \( s_0 \in (t,T] \), then \( s^* := \sup\{s < s_0 : v \int_t^{s} \gamma_{\theta}(\omega)d\theta - 1 = -\frac{v}{2}(T-s)\} \) such that

\[
v \int_t^{s^*} \gamma_{\theta}(\omega)d\theta - 1 = -\frac{v}{2}(T-s^*),
\]

\[
v \int_t^{s} \gamma_{\theta}(\omega)d\theta - 1 > -\frac{v}{2}(T-s) \text{ for } s \in (s^*, s_0].
\]

By (3.45), we get \( \gamma_0(\omega) = 0.5 \) for \( s \in (s^*, s_0) \) a.e. From this, we have

\[
v \int_t^{s} \gamma_{\theta}(\omega)d\theta - 1 = v \int_t^{s^*} \gamma_{\theta}(\omega)d\theta - 1 + \frac{v}{2}(s-s^*)
\]

\[
= -\frac{v}{2}(T-s^*) + \frac{v}{2}(s-s^*)
\]

\[
= -\frac{v}{2}(T-s), \text{ for } s \in (s^*, s_0],
\]

which is a contradiction. Thus we obtain \( v \int_t^{s} \gamma_{\theta}(\omega)d\theta - 1 \leq -\frac{v}{2}(T-s) \) for \( s \in [t,T] \). Similarly, we can get \( v \int_t^{s} \gamma_{\theta}(\omega)d\theta - 1 \geq -v(T-s) \) for \( s \in [t,T] \). Taking \( s = T \), we obtain

\[
v \int_t^{T} \gamma_{\theta}(\omega)d\theta - 1 = 0, \quad P_{t,x}^{*}\text{-a.s.}
\]

From this we get

\[
v \int_t^{s} \gamma_{\theta}(\omega)d\theta - 1 = -v \int_t^{s} \gamma_{\theta}(\omega)d\theta \in [-v(T-s), -\frac{v}{2}(T-s)].
\]

Thus

\[
P_{t,x}^{*} = \{ P \in \mathcal{P} : B_1 = v^{-1}, \text{ P\text{-a.s.}} \}. \tag{3.46}
\]

For each \( P_{t,x}^{*} \in \mathcal{P}_{t,x}^{*} \), it is clear that \( Y_{s,t,x}^{*,\omega} = 0 \). But \( V(s, X_{s,t,x}^{*,\omega}) = -\left(\int_t^{s} (v_{\gamma_{\theta}} - 1) d\theta\right)^2 \) may not be equal to 0. Thus the relational expression (3.33) in Theorem 3.3 does not hold for each \( P_{t,x}^{*} \in \mathcal{P}_{t,x}^{*} \). Similar to the analysis of (3.45), we can obtain that \( \mathcal{P}_{t,x}^{*} \) contains only one element \( P_{t,x}^{*} \) such that \( v_{\gamma_{\theta}} = 1 \), a.e. \( s \in [t,T] \), \( P_{t,x}^{*}\text{-a.s.} \). Thus the relational expression (3.3) in Theorem 3.3 holds for \( P_{t,x}^{*} \in \mathcal{P}_{t,x}^{*} \). It is easy to verify that the relational expressions (3.7) and (3.8) in Theorem 3.3 hold for \( P_{t,x}^{*} \in \mathcal{P}_{t,x}^{*} \).

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