Gleason’s Theorem on Self-Dual Codes and Its Generalizations

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TO EIICHI BANNAI, ON THE OCCASION OF HIS 60TH BIRTHDAY.

Abstract

One of the most remarkable theorems in coding theory is Gleason’s 1970 theorem about the weight enumerators of self-dual codes. In the past 36 years there have been hundreds of papers written about generalizations and applications of this theorem to different types of codes, always on a case-by-case basis. In this talk I will state the theorem and then describe the far-reaching generalization that Gabriele Nebe, Eric Rains and I have developed which includes all the earlier generalizations at once. The full proof has just appeared in our book *Self-Dual Codes and Invariant Theory* (Springer, 2006).

This paper is based on my talk at the conference on Algebraic Combinatorics in honor of Eiichi Bannai, held in Sendai, Japan, June 26–30, 2006.

1. Motivation

Self-dual codes are important because they intersect with

- communications
- combinatorics
- block designs, spherical designs
- group theory
- number theory
- sphere packing
- quantum codes
- conformal field theory, string theory
2. Introduction

In classical coding theory (as for example in MacWilliams and Sloane [13]), a code $C$ of length $N$ over a field $F$ is a subspace of $F^N$. The dual code is

$$C^\perp := \{ u \in F^N : u \cdot c = 0, \forall c \in C \}.$$

**Example:** $C = \{000, 111\}$, $C^\perp = \{000, 011, 101, 110\}$ with $F = F_2$. The weight enumerators of these two codes are

$$W_C(x, y) = x^3 + y^3, \quad W_{C^\perp}(x, y) = x^3 + 3xy^2.$$

A code is self-dual if $C = C^\perp$. For example, the binary code $i_2 := \{00, 11\}$ is self-dual, with weight enumerator

$$W_{i_2}(x, y) = x^2 + y^2. \quad (1)$$

**Example:** The Hamming code $h_8$ of length 8 is self-dual. This is the binary code with generator matrix:

$$\begin{matrix}
\infty & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{matrix}$$

The second row of the matrix has 1’s under the quadratic residues 0, 1, 2 and 4 mod 7. The remaining rows are obtained by fixing the infinity coordinate and cycling the other coordinates. This code has weight enumerator

$$W_{h_8}(x, y) = x^8 + 14x^4y^4 + y^8. \quad (2)$$

As can be seen from the generator matrix, this code is closely related to the incidence matrix of the projective plane of order 2.

If we replace the prime 7 in this construction by 23, we get the binary Golay self-dual code
$g_{24}$ of length 24, with generator matrix as follows:

| 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 |
| 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1 |
| 0, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0 |
| 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1 |
| 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0 |
| 0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1 |
| 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 1, 1 |
| 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1 |
| 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 0, 1 |

This has weight enumerator

$$W_{g_{24}}(x, y) = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}. \tag{3}$$

3. MacWilliams’ Theorem, 1962

In her Ph.D. thesis at Harvard in 1962, Jessie MacWilliams \cite{macwilliams} showed that the weight enumerator of the dual of a linear code is determined by the weight enumerator of the code:

**Theorem 1.** For a code $C$ over $\mathbb{F}_q$,

$$W_{C^\perp}(x, y) = \frac{1}{|C|} W_C(x + (q - 1)y, x - y) . \tag{4}$$

The proof uses the Poisson summation formula, in the form that says that the sum of a function $f$ over a vector space is equal to the average of the appropriate Fourier transform of $f$ over the dual vector space.

**Corollary.** If $C$ is self-dual, $W_C(x, y)$ is fixed under the “MacWilliams” transformation

$$\left( \begin{array}{c} x \\ y \end{array} \right) \mapsto \frac{1}{\sqrt{q}} \left( \begin{array}{cc} 1 & q - 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) . \tag{5}$$

4. First there were four types

As can be seen from \cite{glicksberg} and \cite{ihara}, for some binary self-dual codes the Hamming weights of all the codewords (the powers of $y$) are multiples of 4. In other cases (as in \cite{macwilliams}) the weights may only be even. Gleason and Pierce showed that there are essentially only four possibilities for this phenomenon to occur with self-dual codes over fields:
Theorem 2. (Gleason-Pierce (1967), see [1], [18]). If $C$ is a self-dual code over $\mathbb{F}_q$ with Hamming weights divisible by $m$, then one of the following holds:

I) $q = 2 \ (\Rightarrow m = 2)$

II) $q = 2$ and $m = 4$

III) $q = 3 \ (\Rightarrow m = 3)$

IV) $q = 4$ and Hermitian ($\Rightarrow m = 2$) or $q = 4$ and Euclidean,

or else $c = 2$, $q$ is arbitrary, $N$ is even and $W(x, y) = (x^2 + (q - 1)y^2)^{N/2}$.

Because of this theorem, self-dual codes falling into one of those four classes came to be known as codes of Types I, II, III and IV, respectively.

Incidentally, the codes with $c = 2$ mentioned in the final clause of the theorem still have not been fully classified (see [18]).

5. Gleason’s Theorem (1970, Nice)

At the International Congress of Mathematicians in Nice, 1970, Gleason established the following result.

Theorem 3. (Gleason [8]). If $C$ is a self-dual code of one of the four types mentioned in Theorem 2 then the weight enumerator of $C$ belongs to the polynomial ring $\mathbb{C}[f, g]$, where:

| Type | $f$ | $g$ |
|------|-----|-----|
| I    | $x^2 + y^2$ | $x^2y^2(x^2 - y^2)^2$ |
|      | $i_2$ | Hamming code $h_8$ |
| II   | $x^8 + 14x^4y^4 + y^8$ | $x^4y^4(x^4 - y^4)^4$ |
|      | Hamming code $h_8$ | binary Golay code $g_{24}$ |
| III  | $x^4 + 8xy^3$ | $y^3(x^3 - y^3)^3$ |
|      | tetracode | ternary Golay code |
| IV*  | $x^2 + 3y^2$ | $y^2(x^2 - y^2)^2$ |
|      | $i_2 \otimes \mathbb{F}_4$ | hexacode |

*In fact Gleason omitted this case, which was first given in [12].

Under each polynomial we have written the name of a code whose weight enumerator leads to that polynomial. For example, the theorem states that the weight enumerator of a Type I
A self-dual code belongs to the ring generated by the weight enumerators of the codes $i_2$ and $h_8$, that is, by $f = W_{i_2} = x^2 + y^2$ (see (1)) and $W_{h_8}$ (see (2)). It is simpler to replace $W_{h_8}$ as a generator of this ring by

$$g := \frac{1}{4} (f^4 - W_{h_8}) = x^2 y^2 (x^2 - y^2),$$

as in the first row of the table.

In the following years many generalizations of Gleason’s theorem were published, for example to self-dual codes over other fields ($\mathbb{F}_5$, . . .), to biweight enumerators, split weight enumerators, codes containing the all-ones vector, etc.

The main applications of these theorems are in the classification of self-dual codes of moderate lengths, and in the determination of the optimal (or extremal) codes of the various Types. The book [15] contains an extensive bibliography.

6. $\mathbb{Z}/4\mathbb{Z}$ appears!

In the early 1990’s, coding theory changed forever when it was discovered that certain infamous nonlinear binary codes were really linear (and in some cases self-dual) codes over the ring $\mathbb{Z}/4\mathbb{Z}$ of integers mod 4. For example, the Nordstrom-Robinson code is a famous nonlinear binary code of length 16 that contains 256 codewords and has minimal distance 6, more than is possible with any linear code of the same length and the same number of codewords (cf. [13]). Although nonlinear, its weight enumerator behaves like that of a linear binary code — it is fixed under (5) (with $q = 2$). In 1992, Forney, Trott and I [7] showed that this code is really a linear self-dual code over $\mathbb{Z}/4\mathbb{Z}$, a code already known as the octacode (cf. [5], [6]).

This work was extended by Hammons, Kumar, Calderbank, Solé and me in [9] to cover many other families of binary nonlinear codes, and this in turn was followed by numerous other papers that studied self-dual codes over the rings $\mathbb{Z}/m\mathbb{Z}$ for integers $m \geq 4$ (again see [15] for references).

7. And then there were nine!

In 1998, Eric Rains and I wrote a 120-page survey for the Handbook of Coding Theory [17] in which we distinguished nine types of self-dual codes, extending the original four types to include such families as linear codes over $\mathbb{Z}/4\mathbb{Z}$, linear codes over $\mathbb{Z}/m\mathbb{Z}$ for $m \geq 4$, additive codes over $\mathbb{F}_4$, etc. Again each version of Gleason’s theorem was treated separately.
8. Higher-genus weight enumerators

In the mid-1990’s there was a major breakthrough. As the result of a really amazing coincidence, we were led to investigate a certain family of “Clifford groups” \( C(m) \) with structure \( 2^{1+2m} \cdot O_{2m}(2) \). The story of this astonishing coincidence can be found in [4] and [15], so I will not repeat it here. Studying these “Clifford” groups led to breakthroughs in quantum codes [4] and to generalizations of Gleason’s theorem to higher-genus or multiple weight enumerators [14].

9. The new book

After writing [14], we realized that the arguments used to handle the invariants of the Clifford groups could be extended to handle other classes of self-dual codes. The result is a far-reaching generalization of Gleason’s theorem which defines the “Type” of a self-dual code in such a way that the weight enumerator of any code of that Type belongs to the invariant ring of a certain “Clifford-Weil” group associated with the Type, and furthermore that this invariant ring is spanned by weight enumerators of codes of that Type.

These are the two properties that previously had to be proved for each Type on a case-by-case basis. Now we know that this is automatically true, provided the codes fall into one of certain very general classes.

The proof of the general theorem is not easy, and occupies perhaps 150 pages of the new 400-page book, “Self-Dual Codes and Invariant Theory” [15].

For me, the book represents the culmination of thirty-five years of work.

In the rest of this talk I will give an outline of our approach, omitting all the technical details (and the category theory).
10. Notation: codes over rings

We will use the following notation:

\[
R = \text{ground ring = ring with unit} \\
V = \text{left } R\text{-module = alphabet} \\
\text{(usually we assume } R \text{ and } V \text{ are finite)} \\
C = \text{code of length } N \\
= R\text{-submodule of } V^N \\
c \in C, \ r \in R \Rightarrow rc \in C.
\]

Dual code:

\[
\beta = \text{nonsingular bilinear form, } \beta : V \times V \to \mathbb{Q}/\mathbb{Z} \\
\text{E.g. } \beta(u, v) = \frac{1}{2}uv \text{ (in the binary case)} \\
C^\perp := \left\{ u \in V^N : \sum_{i=1}^{N} \beta(u_i, c_i) = 0, \forall c \in C \right\}
\]

11. Weight enumerators

Let \( C \subseteq V^N \) be a code, where the alphabet \( V = \{v_0, v_1, \ldots\} \). The complete weight enumerator of \( C \) is:

\[
cwe(C) := \sum_{c \in C} \prod_{i=1}^{N} x_{c_i} \in \mathbb{C}[x_{v_0}, x_{v_1}, \ldots].
\]

Example: \( \mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\} \),

\[
codeword c = 0011 \omega \bar{\omega} \Rightarrow x_0^2 x_1^2 x_\omega x_{\bar{\omega}}.
\]

The symmetrized weight enumerator is obtained by identifying \( x_v \) and \( x_w \) in \( \text{cwe}(C) \) if we do not need to distinguish \( v \) and \( w \). E.g. we usually set \( x_\omega = x_{\bar{\omega}} \) for codes over \( \mathbb{F}_4 \). The Hamming weight enumerator is obtained from \( \text{cwe}(C) \) by setting \( x_0 = x \) and all other \( x_v = y \).

12. Biweight or genus-2 weight enumerator

Take an ordered pair of codewords \( b, c \in C \) in all possible ways and write one above the other:

\[
\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & \omega & \bar{\omega} & \cdots \\ 0 & 1 & 0 & 1 & \omega & \bar{\omega} & \cdots \end{bmatrix}.
\]
Then the biweight or genus-2 weight enumerator of $C$ is

$$cwe_2(C) := \sum_{(b,c) \in C \times C} \prod_{i=1}^{N} x \binom{b_i}{c_i}.$$ 

**Remark:**

$$cwe_2(C) = cwe(C \otimes R^2).$$

For we have

$$C \otimes R^2 \leq V^N \otimes R^2 \cong V^{2N} \cong (V^2)^N.$$ 

Note that the ground ring for $C \otimes R^2$ is

$$\text{Mat}_2(R),$$

the ring of $2 \times 2$ matrices over $R$. So, even in the case of classical binary codes, we need to use noncommutative rings when we consider higher-genus weight enumerators!

### 13. Extra conditions

Often one wants to consider self-dual codes with certain additional properties, for example that the weights are divisible by 4, or the code contains the all-ones vector. Some of these properties can be included in the new notion of Type, provided they can be described in terms of “quadratic mappings”. Oversimplifying (see [15, Chapter 1] for the precise definition), a quadratic mapping is a map from $V$ to $Q/\mathbb{Z}$ which is the sum of a quadratic part and a linear part. If $\Phi$ is a collection of quadratic mappings then we say that a code $C$ is *isotropic* with respect to $\Phi$ if

$$\sum_{i=1}^{N} \phi(c_i) = 0, \quad \forall c \in C, \forall \phi \in \Phi.$$ 

**Examples:**

- $\phi(x) = \frac{1}{4}x^2$ (to get weights divisible by 4 in the binary case)
- $\phi(x) = \frac{1}{p}x$, $p$ odd (to ensure that $1 \in C$)
- $\phi(x) = \beta(x, x)$ (specialization of $\beta$, always present)
14. The new definition of Type

We say that a code \( C \leq V^N \) has

\[
\text{Type } \rho := (R, V, \beta, \Phi)
\]

if \( C \) is self-dual with respect to \( \beta \) and isotropic with respect to \( \Phi \).

**Memo:** Many details have been concealed here. See [15] for further information.

We call \( (R, V, \beta, \Phi) \) a form ring, adapting a term from algebraic \( K \)-theory (cf. Bak [2]).

15. Symmetric idempotents

A symmetric idempotent \( \iota \in R^* \) satisfies \( \iota^2 = \iota \) together with certain extra conditions (see [15]), and has the property that there are “left” and “right” elements \( l_\iota \) and \( r_\iota \) associated with it such that

\[
\iota = l_\iota r_\iota
\]

**Examples:**

- \( R = \mathbb{Z}/6\mathbb{Z}: \iota = 3 = 3 \cdot 3 \) or \( \iota = 4 = 4 \cdot 4 \)
- \( R = \text{Mat}_m(R') : \iota = \text{diag}\{1, 0, 0, \ldots, 0\} \)

16. The Clifford-Weil group \( \mathcal{C}(\rho) \)

We associate with the form ring \( \rho = (R, V, \beta, \Phi) \) a certain finite subgroup of \( GL_{|V|}(\mathbb{C}) \) that we call the Clifford-Weil group \( \mathcal{C}(\rho) \). This generalizes the familiar group of order 192 generated by

\[
\begin{bmatrix}
1 & 0 \\
0 & i
\end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\]

that arises from Gleason’s theorem for Type II (or doubly-even) binary codes, and also generalizes the Clifford groups \( \mathcal{C}(m) \) mentioned above. The generators for \( \mathcal{C}(\rho) \) are:

\[
\rho(r) : x_v \mapsto x_{r v}, \quad \forall r \in R^* \quad (\text{because } C \text{ is a code})
\]

\[
\rho(\phi) : x_v \mapsto e^{2\pi i \phi(v)} x_v, \quad \forall \phi \in \Phi \quad (\text{because } C \text{ is isotropic})
\]

and the “MacWilliams” transformations (generalizing \([3]\)) for every symmetric idempotent \( \iota = l_\iota r_\iota \) the associated MacWilliams transformation is

\[
h_{\iota, r} : x_v \mapsto \frac{1}{\sqrt{|V|}} \sum_{w \in V} e^{2\pi i \beta(w, r_v) v} x_{w + (1-\iota)v}.
\]
Example of $h_{ι,r}$:

\[ R = V = \mathbb{Z}/6\mathbb{Z}, \]

two symmetric idempotents $3 = 3 \cdot 3, 4 = 4 \cdot 4$

For $ι = 3, r_ι = 3, |3V| = 2$:

\[
h_{3,r_3} = \frac{1}{\sqrt{2}} \begin{bmatrix}
+ & 0 & 0 & + & 0 & 0 \\
0 & - & 0 & 0 & + & 0 \\
0 & 0 & + & 0 & 0 & + \\
+ & 0 & 0 & - & 0 & 0 \\
0 & + & 0 & 0 & + & 0 \\
0 & 0 & + & 0 & 0 & -
\end{bmatrix},
\]

where $+$ stands for $+1$ and $-$ for $-1$.

Our reasons for calling $C(\rho)$ the Clifford-Weil group are that (i) when the groups $C(m)$ mentioned in §8 act on the Barnes-Wall lattices, they act as the full orthogonal group $O_{2m}^+(2)$ on the Clifford algebra of the quadratic form, and (ii) in some situations $C(\rho)$ coincides with the groups studied by Weil in his famous paper “Sur certaines groups d’opérateurs unitaires” [20].

17. Quasi-chain rings

Our main theorems will cover self-dual codes over a very large class of rings.

A chain ring is one in which the left ideals are linearly ordered by inclusion.

A quasi-chain ring is a direct product of matrix rings over chain rings.

Examples of quasi-chain rings:

- matrix rings over finite fields
- matrix rings over $\mathbb{Z}/m\mathbb{Z}$
- matrix rings over Galois rings

But not all rings are covered by the present theory. Examples of rings that are not (yet) covered:

- the group ring $\mathbb{F}_3 Sym(3)$
- the matrix ring

\[
\begin{bmatrix}
\mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} \\
2\mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z}
\end{bmatrix}
\]
18. The main theorems

**Theorem.** Let $R$ be a finite chain ring or quasi-chain ring, and let $\rho$ be the form ring 

\[ \rho = (R, V, \beta, \Phi). \]

Consider codes $C \leq V^N$ of Type $\rho$. Then (i) $\text{cwe}(C)$ belongs to the invariant ring $\text{Inv}(C(\rho))$, and (ii) $\text{Inv}(C(\rho))$ is spanned by the $\text{cwe}(C)$, where $C$ runs through codes of Type $\rho$.

The proof, as already mentioned, uses category theory and is long and hard, and takes up a good part of the book [15].

We believe, but have not been able to prove, that the theorem should hold without the restriction to quasi-chain rings. We state this as the:

**Weight Enumerator Conjecture:** The theorem should hold for any finite ring $R$.

19. An application

In their 1999 paper “Type II codes, even unimodular lattices and invariant rings” [3], Bannai, Dougherty, Harada and Oura consider codes of (in our new notation) Type $4_{II}^Z$. The corresponding Clifford-Weil group has order 1536, and the ring to which the complete weight enumerators belong has Molien series

\[ \frac{1 + t^8 + 2t^{16} + 2t^{24} + t^{32} + t^{40}}{(1 - t^8)^3(1 - t^{24})}. \] (6)

They remark that “it is not known if the invariant ring is generated by the complete weight enumerators of codes”. This now follows immediately from our main theorem.

Incidentally, the nonzero coefficients of the Molien series in (6) form sequence A051462 in [19], where the reader will find references to both [3] and [15]. A great many Molien series arise in studying self-dual codes [4], and [19] provides a convenient way to keep track of them.\footnote{The index to [15] lists the sequence numbers for over 100 such Molien series.}
20. Example: Hermitian self-dual codes over $\mathbb{F}_9$

The form ring for Hermitian self-dual codes over $\mathbb{F}_9$ is $\rho = (R = V = \mathbb{F}_9, \beta, \Phi)$, where

$$\beta(v, w) = \frac{1}{3} \text{Tr}_{\mathbb{F}_9/\mathbb{F}_3}(vw)$$

$$\Phi = \{\beta(av, v) : a \in \mathbb{F}_9\}$$

$$\mathbb{F}_9 = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^7\},$$

with $\alpha^2 + \alpha = 1$, $\alpha^4 = -1$.

Generators for $C(\rho)$ are:

$$\rho(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} =: M_1.$$  

The isotropic conditions are:

$$\Phi = \{\phi(a) : a \in \mathbb{F}_9\},$$

$$\phi(a)(v) = \frac{1}{3} \text{Tr}(av\bar{v}) = \frac{1}{3} \text{Tr}(av^4)$$

$$= \frac{1}{3}(av^4 + a^3v^4) = \frac{1}{3}(a + a^3)v^4.$$  

Take $a = \alpha$, $\alpha + \alpha^3 = -1$. Then

$$\rho(\phi(\alpha)) : x_v \mapsto e^{2\pi i \frac{4}{3}} x_v$$

giving the matrix

$$M_2 := \text{diag}\{1, \omega^2, \omega, \omega^2, \omega, \omega^2, \omega, \omega^2, \omega\},$$

where $\omega = e^{2\pi i / 3}$.

The MacWilliams transformation:

idempotent $\iota = 1$
\[ x_v \mapsto \frac{1}{\sqrt{9}} \sum_{w \in \mathbb{F}_9} e^{2\pi i \frac{1}{3} \text{Tr}(\alpha \bar{w})} x_w \]

giving the matrix

\[
M_3 := \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \bar{\omega} & \omega & 1 & \omega & \bar{\omega} & 1 & \bar{\omega} & 1 \\
1 & \omega & \omega & \bar{\omega} & 1 & \bar{\omega} & \omega & \omega & 1 \\
1 & 1 & \bar{\omega} & \omega & \bar{\omega} & 1 & \omega & \bar{\omega} & \omega \\
1 & \omega & 1 & \omega & \omega & \bar{\omega} & 1 & \omega & \bar{\omega} \\
1 & \omega & \omega & \bar{\omega} & 1 & \bar{\omega} & \omega & \omega & 1 \\
1 & \bar{\omega} & \bar{\omega} & \omega & 1 & \omega & \omega & \omega & 1 \\
1 & 1 & \omega & \omega & \bar{\omega} & 1 & \bar{\omega} & \bar{\omega} & \omega \\
1 & \bar{\omega} & 1 & \bar{\omega} & \bar{\omega} & \omega & 1 & \omega & \omega
\end{bmatrix}.
\]

Then the Clifford-Weil group is

\[ \mathcal{C}(\rho) = \langle M_1, M_2, M_3 \rangle, \]

a nine-dimensional group of order 192.

The Molien series for this group is

\[
\frac{1 + 3t^4 + 24t^6 + 74t^8 + 156t^{10} + \cdots + 989t^{20} + \cdots + t^{38}}{(1 - t^2)^2(1 - t^4)^2(1 - t^6)^3(1 - t^8)(1 - t^{12})}
\]

Remarks

- The coefficients of the Taylor series expansion form sequence A092354 in [19].

- There are at least 6912 secondary invariants (set \( t = 1 \) in numerator).

- This complexity is typical of most groups — see Huffman and Sloane [10].

- This ring is spanned by cwe’s of codes, by our main theorem.

- It would be hopeless to try to find a corresponding set of codes!

- To get the Hamming weight enumerator theorem, we cannot simply identify \( x_1 = x_\alpha = \cdots = x_{\alpha^7} = x \) (this fails because \( M_2 \) does not act nicely, and if we ignore the generator \( M_2 \) the resulting ring has Molien series \( 1/(1 - t^2)^2 \), which is wrong) — this is what we call an “illegal symmetrization”.

- The correct way to obtain the Hamming weight enumerator theorem is first to divide up the elements of \( \mathbb{F}_9 \) into three orbits \( \{0\}, \{1, \alpha^2, \alpha^4, \alpha^6\} \) (which square to 1) and
\{\alpha, \alpha^3, \alpha^5, \alpha^7\} (which square to \(-1\)). The generators now collapse nicely, to

\[\tilde{M}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tilde{M}_2 = \begin{bmatrix} 1 & 0 \\ 0 & \omega \\ 0 & \omega^2 \end{bmatrix}, \quad \tilde{M}_3 = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 4 & -2 \\ 4 & -2 \end{bmatrix},\]

generating a group of order 48 with Molien series

\[\frac{1}{(1-t^2)(1-t^4)(1-t^5)}.\]

Codes that correspond to the terms in the denominator can be taken to be:

\[[1 \alpha], \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & \alpha & 2\alpha \end{bmatrix}].\]

If their Hamming weight enumerators are denoted by \(f_2, f_4, f_6\) respectively, then the ring of Hamming weight enumerators is

\[\mathbb{C}[f_2, f_4] \oplus f_6 \mathbb{C}[f_2, f_4].\]

This is not the ring of invariants of any finite group of \(2 \times 2\) matrices.

### 21. Higher-genus weight enumerators

To handle higher-genus or multiple weight enumerators we use tensor products, as mentioned in §12, and Morita theory. The form ring for genus-\(m\) weight enumerators is

\[\rho \otimes R^m = \text{Mat}_m(\rho) := (\text{Mat}_m(R), V \otimes R^m, \beta^{(m)}, \Phi_m).\]

**Theorem.** (1) The space of homogeneous invariants of degree \(N\) of the corresponding Clifford-Weil group \(C_m(\rho)\) is spanned by the genus-\(m\) weight enumerators \(\text{cwe}_m(C)\), where \(C\) ranges over a set of permutation representatives of codes of Type \(\rho\) and length \(N\). (2) If every length \(N\) code of Type \(\rho\) is generated by at most \(m\) elements, then these genus-\(m\) weight enumerators are a basis for the space of homogeneous invariants of degree \(N\).

**Corollary.** The Molien series of \(C_m(\rho)\), \(\text{Mol}_{C_m(\rho)}(t)\), converges monotonically as \(m\) increases:

\[\lim_{m \to \infty} \text{Mol}_{C_m(\rho)}(t) = \sum_{N=0}^{\infty} \nu_N t^N,\]

where \(\nu_N\) is the number of permutation-equivalence classes of codes of Type \(\rho\) and length \(N\).
Example: Binary self-dual (or Type 2I) codes. The order of $C_m(\rho)$ and the Molien series for genera 1 to 4 are as follows:

Genus 1: $|C_1| = 16$ (Gleason [8]):

$$\frac{1}{(1 - t^2)(1 - t^8)}$$

Genus 2: $|C_2| = 2304$ (see [12]):

$$\frac{1 + t^{18}}{(1 - t^2)(1 - t^8)(1 - t^{12})(1 - t^{24})}$$

Genus 3: $|C_3| = 5160960$ (see [14]):

$$\frac{\text{degree } 154}{(1 - t^2)(1 - t^{12}) \cdots (1 - t^{40})}$$

— there are at least 720 secondary invariants

Genus 4: $|C_4| = 178362777600$ (see Oura [16])

$$\frac{\text{degree } 504}{(1 - t^2) \cdots (1 - t^{120})}$$

— there are over $10^{10}$ secondary invariants

The convergence of the Molien series mentioned in the above Corollary can be seen in the following table, which gives the initial terms of the expansion of the Molien series for genera 1–5:

| $m$ \ $N$ | 0  | 2  | 4  | 6  | 8  | 10 | 12 | 14 | 16 | $\cdots$ |
|-----------|----|----|----|----|----|----|----|----|----|---------|
| 1         | 1  | 1  | 1  | 1  | 2  | 2  | 2  | 2  | 3    |         |
| 2         | 1  | 1  | 1  | 1  | 2  | 2  | $3^*$| 3  | 4    |         |
| 3         | 1  | 1  | 1  | 1  | 2  | 2  | 3  | 4  | 6    |         |
| 4         | 1  | 1  | 1  | 1  | 2  | 2  | 3  | 4  | 7    |         |
| 5         | 1  | 1  | 1  | 1  | 2  | 2  | 3  | 4  | 7    |         |

*This entry 3 corresponds to the fact that the biweight enumerators of the three codes $i_2^k$, $h_8i_2^k$ and $d_{12}^k$ are linearly independent.

Incidentally, the 8-dimensional group $C_3(\rho)$ of order 5160960 is the group whose magical emergence from the computer — leading to the astonishing coincidence mentioned in §8 — indirectly led to our writing the book.

22. There is no time to mention:

- Our new construction for the Barnes-Wall lattices as lattices over $\mathbb{Z}[\sqrt{2}]$ whose automorphism groups are the Clifford-Weil groups $C_m(2I)$ (Chapter 6).
The theorem that the automorphism group of the genus-$m$ weight enumerator of any Type 2$_I$ code that is not generated by codewords of weight 2 is the Clifford-Weil group $C_m(2_I)$. There is an analogous assertion for doubly-even or Type 2$_{II}$ codes. (Chapter 6)

- The generalizations to maximal self-orthogonal codes (Chapter 10).
- Quantum codes (Chapter 13).
- The extensive tables giving the classification of all codes and of extremal codes of modest lengths (Chapters 11, 12).
- Applications to spherical designs (Chapters 5, 6).
- “Closed codes”: What definition of duality guarantees that $C^{\perp\perp} = C$?
- Our attempts at generalizing the theory to handle self-dual lattices.
23. Finally, the new list of Types

Chapter 2 of the book ends with a list of the principal Types and the sections in which they are discussed. To entice the reader, but without giving any further details, here is that list:

| Type | Description |
|------|-------------|
| 2I (The old Type I) | 4Z/nZ |
| 2II (The old Type II) | mZ |
| 2S | mZ |
| 2lin, 2lin | mZ |
| 2lin', 2lin | mZ |
| 4E (The old Type IV) | mZ |
| 4E | mZ |
| qE (even) | GR(p^e, f)^E |
| qE | GR(p^e, f)_I^E |
| 3 (The old Type III) | GR(p^e, f)_2^E |
| qE (odd) | GR(p^e, f)_2^E |
| qE (odd) | GR(p^e, f)_2^E |
| 4H (The old Type IV) | GR(p^e, f)^H |
| qH | GR(p^e, f)^H |
| qH | GR(p^e, f)^H |
| 4H+ | GR(p^e, f)^H+ |
| 4H+ | GR(p^e, f)^H+ |
| qH+ (even) | Z_p (Codes over the p-adic integers) |
| qH+ (even) | F_{q^2} + F_{q^2}u |
| qH+ (even) | |
| qH+ (even) | |
| qH+ (odd) | |
| qH+ (odd) | |
| qH+ (odd) | |
| qlin, qlin | |
| qlin, qlin | |
| qlin', qlin' | |
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