We discuss a novel method to calculate $f_B$ on the lattice, introduced in [1], based on the study of the dependence of finite size effects upon the heavy quark mass of flavoured mesons and on a non–perturbative recursive finite size technique. This method avoids the systematic errors related to extrapolations from the static limit or to the tuning of the coefficients of effective Lagrangian and the results admit an extrapolation to the continuum limit. We show the results of a first estimate at finite lattice spacing, but close to the continuum limit, giving $f_B = 170(11)(5)(22)$ MeV. We also obtain $f_{B_s} = 192(9)(5)(24)$ MeV. The first error is statistical, the second is our estimate of the systematic error from the method and the third the systematic error from the specific approximations adopted in this first exploratory calculation. The method can be generalized to two–scale problems in lattice QCD.

1. Introduction

Lattice QCD evaluations of quantities characterised by two scales with a large hierarchy require in general a very high lattice resolution and a sizeable total physical volume to correctly account the dynamics of the small distance scale and to dispose of the finite size effects related to the large distance scale. A good example is provided by the pseudoscalar $B$ meson decay constant [2], where the small distance scale is represented by the inverse of the bottom quark mass and the large distance scale by the radius of the $B$ meson, related in turn to the inverse of the light quark mass. A straight evaluation of the decay constant would require lattices with $N = 80^4$ points or more, exceeding the present generation computers capabilities, and, in the case of unquenched simulations, the ones of the next generation. One resorts to approximate calculations based on extrapolations from the static limit or on non–relativistic formulations of standard QCD. All the available methods introduce systematic errors related to extrapolation fits and/or to the use of effective Lagrangians.

We discuss a novel approach based on the study of the dependence upon the heavy quark mass of the pseudoscalar decay constant of heavy flavoured mesons (see [1] for details). The basic assumption is that the finite size effects are mainly related to the light quark mass and rather insensitive to the one of a sufficiently heavy quark. We discuss the general features of the method assuming the continuum limit has been taken. The relevant quantity is the ratio $\sigma = f_B(2L)/f_B(L)$ of the pseudoscalar constants at different volumes, where $f_B(L)$ is the value of the decay constant on a volume with linear size $L$. The dimensionless $\sigma$ depends on general grounds upon three dimensionless variables: $m_L$, $m_hL$, and $\Lambda_{QCD} L$. For a sufficiently large heavy quark mass $m_h$, the dependence is basically dominated by the light quark and the expansion for large $m_h$ takes the form

$$\sigma = \sigma(m_L, \Lambda_{QCD} L) + \frac{C(m_L, \Lambda_{QCD} L)}{m_h L}.$$  

A simple phenomenological ansatz for $\sigma$ can be made based on the concept of a reduced mass constructed out of the heavy and light quark masses

$$\sigma = \sigma(m_{red} L, \Lambda_{QCD} L),$$

where $m_{red} = (\mu_1 \mu_2)/(\mu_1 + \mu_2)$. The quantity $\mu_i$ is a function of the quark mass, but not only: indeed, for very light masses, finite size effects are regulated by the physical meson size, which is expected to remain finite when the light quark mass approaches zero.
A crucial question is the threshold value of the quark mass on a given volume where the large $m_h$ expansion becomes reliable. As we will show, this value falls in a mass range of the order of a couple of GeV in the renormalization invariant mass scheme, where the calculation on a single lattice is affordable. Under these circumstances, the strategy to obtain $f_B$ is the following. One first performs a calculation on a lattice where the resolution is suitable for $b$ quark propagation, but the total volume is unavoidably a small one. This sets $f_B$ on a finite volume. In order to connect to the large volume results, one needs the step scaling function $\sigma$ for values of heavy quark masses generally lower than those of the simulation where the finite size value of $f_B$ was obtained. The possibility of extrapolating $\sigma$ to heavier masses depends upon the validity of the asymptotic expansion: in a favourable case, as is the real one, one can evaluate the finite size effects in a reliable way, connecting, by a repeated iteration of the procedure, small volume values of $f_B$ to the ones on large volumes,

$$f_B^{\text{phys}} = f_B(L_0)\sigma(L_0)\sigma(2L_0)\ldots,$$

and the recursion stops on a volume where $\sigma \simeq 1$ within a required precision. The continuum limit is obtained by extrapolating to zero lattice spacing the step scaling function obtained at fixed physical quantities. We use the lattice scale in terms of $r_0$ [3,4]. Meson masses, on different volumes, are tuned by fixing the physical value of the Renormalization Group Invariant quark masses [3,4].

2. Results and Discussion

The results are obtained at finite lattice spacing. The size of the smallest volume follows from the decision of making our estimate for the finite size $f_B$ on a $48 \times 24^3$ lattice with a cutoff of about $a_0^{-1} \simeq 12$ GeV. The value of the bare coupling for this lattice spacing has been obtained from a fit in ref. [5]. The procedure fixes $\beta(a_0) = 7.3$ and the physical volume $L_0 = 0.4$ fm. On this lattice, we simulate heavy quark masses up to 0.3 in lattice units, corresponding to bare physical masses slightly above 4 GeV. Indeed, as a general caution against large lattice artifacts, at all $\beta$ values we take the maximum heavy quark mass in lattice units of the order of 0.3. The first $\Sigma$ (we distinguish between the continuum step function $\sigma$ and the one at finite lattice spacing $\Sigma$) goes from the volume of 0.4 fm to the one of 0.8 fm. In terms of lattice points, we go from 12 to 24, and we have to match the starting volume of 0.4 fm with a resolution which is half of the one used for a correct estimate of the bottom quark propagation. According to our caveat, it follows that the maximum bare quark mass that we can achieve is correspondingly halved, i.e. of about a couple of GeV at a bare coupling $\beta = 6.737$. We make a further iteration with a second $\Sigma$ going from 0.8 fm to 1.6 fm, where our investigation of heavy quark masses stops at the order of the charm quark mass. The corresponding bare coupling is $\beta = 6.211$. The finite volume effects for this second evolution step are small enough to make the neglection of the residual volume effects a safe assumption, that however can be tested explicitly.

The plots in Figs. 1 and 2 show the dependence of $\Sigma$ upon the heavy RGI quark mass $M_{RGI}^h$ for the two volume jumps and provide evidence for a plateau of insensitivity to heavy quark masses: the data have been obtained from a linear extrapolation in $M_{RGI}^h$ to the down and strange RGI quark masses reported in [3]. These figures, with
the figures of ref. [1], support the procedure proposed. The statistical errors are computed by a jacknife method (see ref. [1] for details). The finite size value of \( f_B \) is obtained by a calculation on the highest resolution lattice and the RGI bottom quark mass is obtained from the equation

\[
M = \hat{Z}_M(g_0)m_{W\Gamma}(g_0)
\]

In order to obtain the renormalisation constant \( Z_M(g_0) \) at \( \beta = 7.3 \) and \( \beta = 6.737 \), we have used a safe interpolation of the pseudoscalar renormalisation constant \( Z_P(g_0, \mu) \) at a value of \( \mu \) three times the reference value used in eq. (6.8) of ref. [8]. The value for \( f_B \) that we obtain is

\[
f_B(0.4\text{fm}) = 483(4)\text{MeV}
\]

By using the values of \( \Sigma \) for the \( b \) quark at constant RGI mass

\[
\Sigma_{0.4-0.8}^{bd} = 0.401(4), \quad \Sigma_{0.8-1.6}^{bd} = 0.88(4)
\]

we obtain our estimate of \( f_B \) on the large volume:

\[
f_B^{\text{phys}} = 170(11)\text{MeV},
\]

where the error quoted in the previous equation is statistical only. In this the written version of the talk we include our estimate of the systematic errors that can be partly ascribed to specific approximations used in the present computation that can be eventually removed, and partly to the uncertainty in the extrapolation in the heavy quark mass of finite size effects, inherent to the method proposed. The overall error on the number \( f_B \) coming from the removable systematic uncertainties is of about 13% and of at most 2 - 3% from the ones deriving from the unavoidable extrapolation in the heavy quark mass, leading to a global uncertainty of about 25 MeV of which about 20 are removable while 5 stay with the method:

\[
f_B^{\text{phys}} = 170(11)(5)(22)\text{MeV}.
\]

In the same way we obtain

\[
f^{\text{phys}}_{B_s} = 192(9)(5)(24)\text{MeV}.
\]

We quote also the results of the ratios

\[
\frac{f_{B_s}^{\text{phys}}}{f_B^{\text{phys}}} = 1.13(2)(1), \quad \frac{f_{D_s}^{\text{phys}}}{f_D^{\text{phys}}} = 1.10(1)(1)
\]

that were asked at the end of the talk.

The method proposed can be generalized to problems characterised by two very different mass scales, if the decoupling of the large mass scale from the low scales of non-perturbative QCD dynamics holds true. This appears to be the case in the example discussed and is somehow supported by the wide success of the predictions of perturbative QCD calculations for hard processes that are insensitive to the dressing mechanism of quarks and gluons into standard hadronic final states.

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