On the software computation of the formulae for the $n$-th prime number

Dimitar G. Dimitrov

Faculty of Mathematics and Informatics
Sofia University
5 James Bourchier Str., Sofia, Bulgaria
e-mail: dgdimitrov@fmi.uni-sofia.bg

Received: 27 January 2019     Revised: 14 September 2019     Accepted: 16 September 2019

Abstract: Many formulae for calculating the $n$-th prime number exist. In this paper, a comparison of the computation time of different existing formulae is made.

Keywords: Prime number, Arithmetic formula, Comparison, Software computation.

2010 Mathematics Subject Classification: 11A41, 11A25, 11-04.

1 Introduction

A primary focus of number theory is the study of prime numbers. The problem of finding a formula for calculating the $n$-th prime number is very old and has always been of interest to researchers.

Following are some well-known formulae:

\[ p_n = 1 + \sum_{m=0}^{2^n} \left[ \sum_{j=1}^m \frac{\cos^2 \pi \left( \frac{(j-1)! + 1}{j} \right)}{n} \right]^{1/n} \]  \hspace{1cm} (1)
\[ p_n = 1 + \sum_{m=0}^{2^n} \left| \begin{array}{c} n \\ \sin \frac{\pi n}{j} \end{array} \right| \right|^{1/n} \]

\[ p_n = 1 + \sum_{m=0}^{2^n} \left| \begin{array}{c} n \\ 1 + \sum_{j=2}^{m} \left| \sin \frac{\pi (j-1)!}{j} \right| \end{array} \right| \right|^{1/n} \]

\[ p_n = \left| 1 - \frac{1}{\log_2 \log \left( -\frac{1}{2} + \sum_{d \mid P_{n-1}} \mu(d) \right) \right| \]

where \( \mu \) is the Möbius function and \( P_{n-1} \) is the product of all previously calculated primes \( p_1, \ldots, p_{n-1} \).

Formulae (1), (2) are defined by Willans [10], (3) is a modification which uses an alternative formula for the prime-counting function \( \pi \) introduced by J. Minač [10], and (4) is a recurrent formula by J. M. Gandhi [6].

In [2–5] K. Atanassov introduced the following 10 formulae:

\[ p_n = \sum_{i=0}^{C(n)} sg \left( n - \sum_{j=2}^{i} \mu(j-1) - \varphi(j) \right) \]

\[ p_n = \sum_{i=0}^{C(n)} sg \left( n - \sum_{j=2}^{i} \psi(j) - j - 1 \right) \]

\[ p_n = \sum_{i=0}^{C(n)} sg \left( n - \sum_{j=2}^{i} \sigma(j) - j - 1 \right) \]

\[ p_n = \sum_{i=0}^{C(n)} sg \left( n - \sum_{j=2}^{i} fr \left( \frac{j}{(j-1)!} \right) \right) \]

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\begin{align*}
p_n &= \sum_{i=0}^{C(n)} sg\left( n - \sum_{j=2}^{i} \frac{1}{\delta(j)} \right) \quad (9) \\
p_n &= \sum_{i=0}^{C(n)} sg\left( n - \sum_{j=2}^{i} \frac{1}{\tau(j)} \right) \quad (10) \\
p_n &= \sum_{i=0}^{C(n)} sg\left( n - \sum_{j=2}^{i} \right) \quad (11) \\
p_n &= \sum_{i=0}^{C(n)} sg\left( n - \sum_{j=2}^{i} \frac{1}{\Omega(j)} \right) \quad (12) \\
p_n &= \sum_{i=0}^{C(n)} sg\left( n - \pi_P(i) \right) \quad (13)
\end{align*}

where \( n = \prod_{i=1}^{k} p_i^{\alpha_i} \) is the canonical representation of a natural number \( n > 1 \),

\begin{align*}
C(n) &= \left\lfloor \frac{n^2 + 3n + 4}{4} \right\rfloor, \\
sg(x) &= \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}, \\
\overline{sg}(x) &= \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}, \\
\varphi(n) &= \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1), \\
\psi(n) &= \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 1), \\
\sigma(n) &= \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}.
\end{align*}
for \( \frac{p}{q} \)\]
\[
fr\left( \frac{p}{q} \right) = \begin{cases} 
0, & \text{if } \frac{p}{\gcd(p, q)} = 1 \\
1, & \text{else}
\end{cases}
\]
\[
\eta(n) = \sum_{i=1}^{k} \alpha_{i}p_{i},
\]
\[
\delta(n) = \prod_{i=1}^{k} \alpha_{i}p_{1}^{\alpha_{i}} \ldots p_{i-1}^{\alpha_{i-1}} p_{i}^{\alpha_{i}} \ldots p_{i+1}^{\alpha_{i+1}} \ldots p_{k}^{\alpha_{k}},
\]
\[
\tau(n) = \prod_{i=1}^{k} (1 + \alpha_{i}),
\]
\[
\overline{\omega}(n) = \begin{cases} 
\min(\alpha_{1}, \ldots, \alpha_{k})k, & \text{if } k > 1 \\
1, & \text{if } k = 1
\end{cases}
\]
\[
\Omega(n) = \sum_{i=1}^{k} \alpha_{i},
\]
\[
\pi_{p}(n) = \begin{cases} 
0, & \text{if } n < 2 \\
\pi_{p}(n - 1) + P(n), & \text{else}
\end{cases}
\]
\[
P(n) = \begin{cases} 
1, & \text{if } n \text{ is prime} \\
0, & \text{if } n \text{ is composite}
\end{cases}
\]

We will also include some recently developed formulae. S. M. Ruiz introduced the following three formulae [11, 12]:

\[
p_{n} = 1 + \sum_{k=1}^{\left\lfloor 2n\log_{n}+2 \right\rfloor} \left( 1 - \frac{1}{n} \sum_{j=2}^{k} \left( 1 - \left\lfloor \frac{1}{j} \left( 2 - \sum_{s=1}^{j} \left( \left\lfloor \frac{j}{s} \right\rfloor - \left\lfloor \frac{j-1}{s} \right\rfloor \right) \right) \right) \right),
\]
\[
p_{n} = 1 + \sum_{k=1}^{\left\lfloor 2n\log_{n}+2 \right\rfloor} \left( 1 - \frac{1}{n} \sum_{j=2}^{k} \left\lfloor \frac{\text{lcm}(1, 2, \ldots, j)}{j \cdot \text{lcm}(1, 2, \ldots, j-1)} \right\rfloor \right),
\]
\[
p_{n} = \left\lfloor n\log_{n} \right\rfloor + \sum_{k=\left\lfloor n\log_{n} \right\rfloor}^{\left\lfloor n\log_{n}+n\log(n)-0.5 \right\rfloor+3} \left( 1 - \frac{1}{n} \sum_{j=2}^{k} \left\lfloor \frac{\text{lcm}(1, 2, \ldots, j)}{j \cdot \text{lcm}(1, 2, \ldots, j-1)} \right\rfloor \right).
\]

The newest formula in the present research was developed by I. Kaddoura and S. Abdul-Nabi [7]:
\[ p_n = 3 + 2 \lfloor n \log n \rfloor + 2 \sum_{x=7}^{\lfloor 2 n \log n \rfloor + 2} \left( 4 + \frac{\sum_{j=1}^{\lfloor \frac{x}{6} \rfloor} \lfloor S(6j + 1) \rfloor + \sum_{j=1}^{\lfloor \frac{x}{6} \rfloor} \lfloor S(6j - 1) \rfloor}{n} \right), \quad (18) \]

where

\[ S(x) = -\sum_{k=1}^{\lfloor \frac{\sqrt{x}}{6} \rfloor + 1} \left( \left\lfloor \frac{x}{6k+1} \right\rfloor - \frac{x}{6k+1} \right) + \left( \left\lfloor \frac{x}{6k-1} \right\rfloor - \frac{x}{6k-1} \right) \]

\[ \frac{2}{\left\lfloor \frac{\sqrt{x}}{6} \right\rfloor + 1} \]

2 Methodology

All of the listed formulae have been implemented in C++ strictly following their definitions. No optimizations that change their definitions have been made.

The time used for calculating some of the mathematical functions such as \( \mu(d) \) greatly depends on the specific implementations chosen. A list of all implementation details follows.

Some formulae directly or indirectly check whether a given natural number is prime or composite. An efficient AKS primality test [1] has been chosen. The algorithm requires polynomial time.

The following pseudocode represents the algorithm used to calculate the Möbius function [8]:

```
function MOEBIUS(n) ▷ Where \( n \in \mathbb{N} \) and \( n > 0 \)
    if \( n = 1 \) then return 1
    end if
    if \( n = 2 \) then return -1
    end if
    \( p \leftarrow 0 \)
    for all \( i \in \{2\} \cup \{m \in \mathbb{N} | m \geq 3 \land m \mod 2 = 1 \land m^2 \leq n\} \) do
        if \( n \mod i = 0 \) then
            \( n \leftarrow n/i \)
            \( p \leftarrow p + 1 \)
            if \( n \mod i = 0 \) then return 0
        end if
    end for
    if \( p \mod 2 = 0 \) then
        return -1
    else
        return 0
    end if
```

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The following is a pseudocode for finding the prime factorization of a natural number \( n \) [9]:

```
function DECOMPOSE(n)  ▷ Where \( n \in \mathbb{N} \) and \( n > 1 \)
    \( k \leftarrow 0 \)
    \( \text{for all } i \in \{2\} \cup \{m \in \mathbb{N} | m \geq 3 \land m \mod 2 = 1 \land m^2 \leq n\} \text{ do} \)
    \( c \leftarrow 0 \)
    \( \text{while } n \mod i = 0 \text{ do} \)
        \( c \leftarrow c + 1 \)
        \( n \leftarrow n/i \)
    \( \text{end while} \)
    \( \text{if } c > 0 \text{ then} \)
        \( \text{result}[k] \leftarrow (i, c) \)
        \( k \leftarrow k + 1 \)
    \( \text{end if} \)
\( \text{end for} \)
\( \text{if } n > 2 \text{ then} \)
    \( \text{result}[k] \leftarrow (n, 1) \)
    \( k \leftarrow k + 1 \)
\( \text{end if} \)
return result
end function
```

The number \( x^n \), where \( x \in \mathbb{Z} \) and \( n \in \mathbb{N} \), is calculated using the following well known fast exponential algorithm:

\[
x^n = \begin{cases} 
1, & \text{if } n = 0 \\
x.x^{n-1}, & \text{if } n \text{ is odd} \\
(x^{2})^2, & \text{if } n \text{ is even} 
\end{cases}
\]

The greatest common divisor (gcd) of two positive natural numbers \( a \) and \( b \) is calculated using the Stein’s binary GCD algorithm [13]. It is very efficient compared to the conventional Euclidean algorithm. This algorithm is also used in the implementation of the least common multiple function used in (16) and (17).

The source code was compiled with the gcc compiler on Ubuntu 16.04 LTS operating system. Testing of the formulae was performed on a desktop computer with an Intel i7 CPU.

1024-bit integer data types from the Boost Multiprecision Library were used (\texttt{uint1024\_t} and \texttt{int1024\_t}) so the implementation can handle large intermediate values such as factorials and powers of two. For example, the largest integer for which factorial can be calculated using 32 bits is only 12, using 128 bits – 34, and so on. As a result, larger data types are needed even for relatively small values of \( n \).
Table 1. Results for very small values of $n$

| Formula | $n = 4$   | $n = 9$   | $n = 10$  | $n = 20$  | $n = 30$  |
|---------|-----------|-----------|-----------|-----------|-----------|
| (1)     | 0.000187  | 3.256814  | 27.839838 | time out  | time out  |
| (2)     | 0.000170  | 3.429308  | 28.946541 | time out  | time out  |
| (3)     | 0.000135  | 3.396208  | 28.492246 | time out  | time out  |
| (4)     | 0.000039  | 1.055240  | 23.841273 | time out  | time out  |
| (5)     | 0.000039  | 0.000388  | 0.000578  | 0.008815  | 0.044728  |
| (6)     | 0.000038  | 0.000383  | 0.000579  | 0.008401  | 0.044321  |
| (7)     | 0.000041  | 0.000492  | 0.000839  | 0.010981  | 0.055538  |
| (8)     | 0.000167  | 0.005568  | 0.008969  | 0.605112  | 7.757162  |
| (9)     | 0.000027  | 0.000333  | 0.000451  | 0.007046  | 0.036736  |
| (10)    | 0.000030  | 0.000385  | 0.000545  | 0.008474  | 0.043577  |
| (11)    | 0.000025  | 0.000285  | 0.000404  | 0.006202  | 0.032856  |
| (12)    | 0.000190  | 0.000264  | 0.000371  | 0.005800  | 0.030580  |
| (13)    | 0.000016  | 0.000229  | 0.000335  | 0.005386  | 0.027922  |
| (14)    | 0.000006  | 0.000012  | 0.000014  | 0.000067  | 0.000128  |
| (15)    | 0.000166  | 0.004771  | 0.008037  | 0.116386  | 0.578732  |
| (16)    | 0.003338  | 0.210815  | 0.418192  | 15.278138 | time out  |
| (17)    | 0.000508  | 0.026317  | 0.040497  | 1.371085  | 10.674773 |
| (18)    | 0.000039  | 0.000441  | 0.000656  | 0.004996  | 0.017033  |

Each combination of a formula and a value of $n$ was executed 10 times and then the average time in seconds was calculated.

3 Results

Table 1 shows the measured times for small values of $n$. A time out was set to 60 seconds.

As expected, formulae (1)–(4) which use sums with $2^n$ summands, as well as (8), (16) and (17) are significantly slower than the other formulae. Also for these formulae with increasing $n$, the number of digits required for intermediate numbers increases drastically and therefore calculations will be slower. For that reason, these formulae were excluded from further testing with larger values. Results can be seen in Table 2. Time out was set to 300 seconds.

It is important to note that the presented results are large because 1024-bit integer types were used. This was needed because, as already pointed, some formulae contain factorials and powers of two which require a large number of digits even for small values of $n$. If we use standard 32 or 64-bit integers, calculations will be faster. Table 3 shows the results when 64-bit data types are used.
| Formula | $n = 100$ | $n = 200$ |
|--------|-----------|-----------|
| (5)    | 7.611714  | 167.108847|
| (6)    | 7.499903  | 163.355355|
| (7)    | 9.037541  | 188.893689|
| (9)    | 6.611778  | 148.642090|
| (10)   | 7.491476  | 163.455926|
| (11)   | 6.104673  | 140.214233|
| (12)   | 5.820569  | 135.284282|
| (13)   | 5.541861  | 130.774057|
| (14)   | 0.001620  | 0.008684  |
| (15)   | 50.872570 | time out  |
| (18)   | 0.494740  | 3.383555  |

Table 2. Results for larger values of $n$

| Formula | $n = 10$  | $n = 100$ | $n = 200$ | $n = 1000$ | $n = 20000$ |
|---------|-----------|-----------|-----------|------------|-------------|
| (5)     | 0.000030  | 0.309244  | 7.135392  | time out   | time out    |
| (6)     | 0.000024  | 0.307875  | 7.115837  | time out   | time out    |
| (7)     | 0.000030  | 0.371489  | 8.190629  | time out   | time out    |
| (9)     | 0.000022  | 0.275384  | 6.708046  | time out   | time out    |
| (10)    | 0.000036  | 0.322498  | 7.475469  | time out   | time out    |
| (11)    | 0.000021  | 0.275238  | 6.627389  | time out   | time out    |
| (12)    | 0.000021  | 0.258818  | 6.370391  | time out   | time out    |
| (13)    | 0.000020  | 0.271519  | 6.572727  | time out   | time out    |
| (14)    | 0.000002  | 0.000054  | 0.000321  | 0.027147   | 113.856200  |
| (18)    | 0.000049  | 0.022168  | 0.158724  | 14.874442  | time out    |

Table 3. Results for implementation with 64-bit data types

### 4 Conclusion

The time needed for calculating each formula greatly depends on software implementation details such as data types and algorithms for formulae like $\sin$, $\log$, $\mu$, etc. The present research shows that Atanassov’s formula (14) is the fastest one at least for values of $n$ up to 20000. It inherits its efficiency as an immediate consequence of applying the AKS prime testing formula to find $\pi_P(i)$. The most recent formula (18) developed by Kaddoura and Abdul-Nabi is also fast, and it took second place.
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