GENERALIZATION OF THE GROSS-PERRY METRICS

M. Jakimowicz and J. Tafel

Institute of Theoretical Physics, University of Warsaw,
Hoża 69, 00-681 Warsaw, Poland, email: tafel@fuw.edu.pl

Abstract

A class of $SO(n+1)$ symmetric solutions of the $(N+n+1)$-dimensional Einstein equations is found. It contains 5-dimensional metrics of Gross and Perry and Millward.

1 Introduction

An extra spatial dimension was introduced by Kaluza and Klein (see e.g. [1]) in order to unify electromagnetism and gravity. Recently, due to the string theory, extra dimensions became a permanent part of the high energy theoretical physics. For instance, in brane-world models (see [2] for a review) matter fields are confined to a four-dimensional brane and gravity can propagate in higher dimensional bulk. These higher dimensional models motivate studying Einstein’s equations in $D > 4$ dimensions. Some techniques of the 4-dimensional Einstein theory were already generalized to higher dimensions. They refer mainly to the classification of the Weyl tensor, the Robinson-Trautman solutions, spacetimes with vanishing invariants and metrics with $D-2$ abelian symmetries (see e.g. [3, 4, 5, 6]).

Symmetry assumptions are one the most efficient methods of solving the Einstein equations. All well known multidimensional exact solutions like the Myers-Perry black hole [7], black ring of Emparan and Reall [8] and the Gross-Perry metrics [9] (see also [10]) admit several dimensional symmetry groups.

In [11] we proposed a construction of vacuum metrics admitting $SO(n+1)$ spherical symmetry, which was based on the symmetry reduction of $(N + n + 1)$–dimensional Einstein equations to $(N + 1)$–dimensional equations with a scalar field $\phi$. There was used an additional assumption that the field of normal vectors to surfaces $\phi = const$ is geodetic and the induced metric of the surfaces is an Einstein metric. The construction, for zero cosmological constant and timelike surfaces $\phi = const$, can be summarized as follows.

Let $\gamma_{ij}$ and $P_{ij}$ be symmetric tensors depending on coordinates $x^i, i = 0, \ldots, N - 1$. Assume that $\gamma_{ij}$ has the Lorentz signature and $P_{ij}$ satisfies the following conditions

$$P^i_{\ i} = 0,$$  
(1)

$$P^i_j P^j_i = 2c = const,$$  
(2)

$$P^k_{\ ;i;k} = 0.$$  
(3)

1
where \( P^i_j = \gamma^i_{jk} P_{kj} \) and the semicolon denotes covariant derivative related to the metric \( \gamma_{ij} dx^i dx^j \). From matrices \( \gamma = (\gamma_{ij}) \) and \( P = (P^i_j) \) we compose metric corresponding to \( \gamma e^{Pr} \), where \( \tau \) is a function of another coordinate \( s \). We assume that its Ricci tensor satisfies

\[
R^i_j (\gamma e^{Pr}) = \lambda \delta^i_j \, , \quad \lambda = \text{const} \, .
\]

Given \( c \) and \( \lambda \) we look for solutions \( \beta(s) \) and \( \phi(s) \) of the following equations

\[
(\beta \dot{\phi}) \dot{\phi} = -\beta V_{,\phi} \, ,
\]

\[
-N\lambda \beta^{-2/N} = \left(1 - \frac{1}{N}\right) \frac{\dot{\beta}^2}{\beta^2} - \frac{2c}{\beta^2} - \dot{\phi}^2 + 2V \, ,
\]

where the dot denotes the partial derivative with respect to \( s \) and \( V \) is a function of \( \phi \).

The main result of [11] is that, under conditions (1)-(4), metric

\[
\tilde{g} = -ds^2 + \tilde{g}_{ij} dx^i dx^j \, ,
\]

was

\[
\tilde{g}_{ij} = \beta^{2/N} (\gamma e^{Pr})_{ij}
\]

and \( \tau(s) \) is defined via equation

\[
\beta \dot{\tau} = 2 \, ,
\]

satisfies \((N+1)\)–dimensional Einstein equations with the scalar field \( \phi \) and potential \( V \).

Moreover, if

\[
V = -\frac{1}{2} n (n-1) e^{-2\sqrt{\frac{N}{n(n-1)}}} \phi
\]

then

\[
g = e^{-2\sqrt{\frac{n}{n(n-1)}}} \tilde{g} - e^{2\sqrt{\frac{N-1}{n(n-1)}}} \phi d\Omega_n^2
\]

is an \((N+n+1)\)–dimensional vacuum metric invariant under the group \( SO(n+1) \). Here \( d\Omega_n^2 \) is the standard metric of the \( n \)-dimensional sphere.

A particular solution of conditions (1)–(4), for any \( N > 1 \), is given by

\[
\gamma_{ij} = \text{diag}(+1, -1, -1, ..) \, , \quad P_{ij} = P_{ji} = \text{const} \, , \quad P^i_i = 0 \, , \quad \lambda = 0.
\]

For \( N = 2 \) conditions (1)–(4) can be solved in full generality. They lead either to \( \gamma^i_{j} \) or to \( c = 0 \) and to \( \gamma e^{Pr} \) equivalent to the metric

\[
(\gamma e^{Pr})_{ij} dx^i dx^j = \frac{dudv}{(1 + \frac{1}{2} uv)^2} + \tau h(u) du^2 \, ,
\]

where \( h \) is an arbitrary function of coordinate \( u \). In the next section we find solutions of equations (3), (4) and construct corresponding vacuum metrics. In section 3 we discuss properties of these metrics.
2 Multi dimensional vacuum metrics

In [11] we gave examples of vacuum metrics derived by our method. Other solutions with \( n > 1 \) can be obtained by inspection of the Gross-Perry metrics [9]. Let \( \lambda = 0 \) and \( s = s(r) \) be a function of a new coordinate \( r \). Then equations (5), (6), (9) take the form

\[
\left( \frac{\beta \phi'}{\alpha} \right)' = \alpha \beta V, \phi (14)
\]

\[
(1 - \frac{1}{N}) \frac{\beta^2}{\alpha^2} - \frac{\beta^2 \phi^2}{\alpha^2} + 2\beta^2 V = 2c , \quad (15)
\]

\[
\tau' = \frac{2\alpha}{\beta} , \quad (16)
\]

where the prime denotes the derivative with respect to \( r \) and \( \alpha = s' \). Metric (7) is given by

\[
\tilde{g} = -\alpha^2 dr^2 + \beta^2/N (\gamma e^{P r})_{i,j} dx^i dx^j . \quad (17)
\]

If \( N = n = 2 \) equations (14), (15) are satisfied by functions \( \alpha, \beta, \phi \) corresponding to the Gross-Perry metric [9]. Changing parameters in these functions leads to the following solutions for arbitrary dimensions \( N > 1 \) and \( n > 1 \)

\[
\alpha = \alpha_0 |r|^{l-1} |r - r_0|^{-p} |r + r_0|^{l+p} \quad (18)
\]

\[
\beta = \beta_0 (r^2 - r_0^2) \alpha \quad (19)
\]

\[
e^{V \frac{n(N-1)}{n-1} \phi} = (n - 1)|r\alpha| . \quad (20)
\]

Here \( l \) is a number defined by \( n \) and \( N \)

\[
l = \frac{n + N - 1}{(n-1)(N-1)} \quad (21)
\]

and \( p, \alpha_0, \beta_0 \) and \( r_0 \neq 0 \) are parameters related to the constant \( c \) via

\[
c = 2\beta_0^2 r_0^2 \left[ \frac{n}{n-1} - p^2 (n-1)(N-1)^2 \right] N(n + N - 1) . \quad (22)
\]

Integrating equation (16) yields

\[
\tau = \frac{1}{\beta_0 r_0} \ln \left| \frac{r + r_0}{r - r_0} \right| + \tau_0 . \quad (23)
\]

Due to a freedom of transformations of \( r, P \) and \( \gamma \) we can assume

\[
r_0 > 0 , \quad |\alpha_0| = \frac{1}{n-1} , \quad \beta_0 = 1 , \quad \tau_0 = 0 \quad (24)
\]

(note that a sign of \( \alpha_0 \) can be still adjusted to have \( \beta > 0 \) for \( r \neq 0, \pm r_0 \)). Thus, \( p \) and \( r_0 > 0 \) remain as free parameters.
Let $N = 2$. In the case (12) and $c > 0$ the matrix $P$ can be diagonalized by a 2-dimensional Lorentz transformation. Hence, one obtains

$$\tilde{g} = -\alpha^2 dr^2 + \beta \left( e^{\pm \sqrt{c} \tau} dt^2 - e^{\mp \sqrt{c} \tau} dy^2 \right),$$  \hspace{1cm} (25)$$

where $t$ and $y$ denote coordinates $x^i$. Substituting (18)-(24) into (25) and (11) yields the following (n+3)-dimensional vacuum metric

$$g = \left| \frac{r - r_0}{r + r_0} \right|^{p' - q} \left| \frac{r - r_0}{r + r_0} \right|^{p' + q} dt^2 - \frac{r + r_0}{|r|^2} \frac{2^{p' + 2}}{(n-1)^2} \left( \frac{dt^2}{(n-1)^2} + r^2 d\Omega_n^2 \right).$$  \hspace{1cm} (26)$$

Parameters $p'$ and $q$ are related to $p$ and $c$ by

$$p' = \frac{n - 1}{n + 1} p , \quad q = \pm \frac{\sqrt{|c|}}{r_0}. \hspace{1cm} (27)$$

Because of (22) they are constrained by

$$(n + 1)p^2 + (n - 1)q^2 = 2n. \hspace{1cm} (28)$$

For n=2 solution (26) is exactly the Gross-Perry metric [9] under the identification

$$r_0 = m, \quad p' = \frac{1}{\alpha}(\beta + 1), \quad q = \frac{1}{\alpha}(\beta - 1). \hspace{1cm} (29)$$

Here $m$, $\alpha$ and $\beta$ are parameters used by Gross and Perry, constrained by the condition $\alpha = \sqrt{\beta^2 + \beta + 1}$.

If $c < 0$ the matrix $P$ can be put into the off diagonal form. Instead of (25) one obtains

$$\tilde{g} = -\alpha^2 dr^2 + \beta \left[ \cos(\sqrt{|c|})(dt^2 - dy^2) \pm 2 \sin(\sqrt{|c|} \tau) dt dy \right].$$  \hspace{1cm} (30)$$

In this case the vacuum metric corresponding to (18)-(24) reads

$$g = \left| \frac{r - r_0}{r + r_0} \right|^{p'} \left[ \cos \left( q \ln \left| \frac{r + r_0}{r - r_0} \right| \right)(dt^2 - dy^2) + 2 \sin \left( q \ln \left| \frac{r + r_0}{r - r_0} \right| \right) dt dy \right]$$

$$- \frac{|r + r_0|^{2^{p' + 2}}}{|r|^2 |r - r_0|^{2^{p' - 2}} (n-1)^2} \left( \frac{dr^2}{r^2} + r^2 d\Omega_n^2 \right).$$  \hspace{1cm} (31)$$

Relation (27) is still valid, but now parameters $p'$, $q$ are constrained by

$$(n + 1)p^2 - (n - 1)q^2 = 2n. \hspace{1cm} (32)$$

If $N = 2$ and

$$p = \pm \frac{\sqrt{2n(n+1)}}{n - 1} \hspace{1cm} (33)$$
then it follows from (22) that \( c = 0 \) and one can merge solutions (18)-(20) with metric (13) for \( \lambda = 0 \). In this way the following vacuum metric is obtained

\[
g = \left| \frac{r - r_0}{r + r_0} \right|^{\pm \sqrt{\frac{2n}{n+1}}} \left( dudv + \ln \left| \frac{r + r_0}{r - r_0} \right| h(u)du^2 \right) \tag{34}
\]

\[
- \frac{|r + r_0|^{\frac{2}{n+1}} (\pm \sqrt{\frac{2n}{n+1} + 1})}{|r|^{\frac{2}{n+1}} |r - r_0|^{\frac{2}{n+1}} (\pm \sqrt{\frac{2n}{n+1} - 1})} \left( \frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2 \right).
\]

In the case \( n = 2 \), \( h(u) = 0 \) metric (34) with the lower sign coincides with the metric given by Millward [12] under the identification

\[
b = \frac{1}{\sqrt{3}} \ln \left| \frac{r - r_0}{r + r_0} \right|, \quad M = \frac{\sqrt{3}}{2} r_0. \tag{35}
\]

For \( N > 2 \) one can easily construct vacuum solutions based on relations (11), (12), (17)-(24). They generalize metrics (26) and (31). In this case a classification of symmetric tensors (here \( P_{ij} \)) in multidimensional Lorentzian manifolds [3] can be useful in order to distinguish nonequivalent solutions. One can also construct metrics which generalize (34) by taking \( \gamma e^{\tau u} \) corresponding to the metric

\[
dudv + \tau h(u)du^2 + \sum_{a=1}^{N-2} e^{c_a \tau} dy_a^2, \tag{36}
\]

where constants \( c_a \) are constrained by

\[
\sum_{a=1}^{N-2} c_a = 0. \tag{37}
\]

In this case we can use functions defined by (18)-(24) with constant \( c \) given by

\[
c = \frac{1}{2} \sum_{a=1}^{N-2} c_a^2. \tag{38}
\]

### 3 Discussion

In addition to \( SO(n+1) \) symmetries metrics (26) and (31) admit one timelike and one spacelike Killing vector (note that interpretation of \( \partial_t \) and \( \partial_y \) in case (31) can change depending on value of \( r \)). Metric (26) is static and metric (31) is stationary. In the limit \( r \to \infty \) they behave like

\[
dt^2 - dy^2 - r^{-2(n-2)} \left( \frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2 \right). \tag{39}
\]
Under the change $r' = r^{\frac{1}{n-1}}$ metric (39) takes the standard form of the (n+3)-dimensional Minkowski metric. Thus, metrics (26) and (31) are asymptotically flat on surfaces $y = \text{const}$.

Metric (34) has a null Killing vector field $\partial_v$ and it belongs to generalized Kundt’s class [14]. If $r \to \infty$ it tends asymptotically to the flat metric in the form

$$du dv - r^{-2 \frac{2n-2}{n-1}} \left( \frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2 \right). \tag{40}$$

Generalizing results of [13] for the Gross-Perry metric to arbitrary $n$, one can show that both metrics (26) and (31) are of algebraic type $I$. For $h \neq 0$ metric (34) is of algebraic type $II_i$ and for $h(u) = 0$ it is of type D. Aligned null vector fields for metrics (26), (31) and (34) are given in Appendix A.

All metrics (26), (31) and (34) are singular at $r = \pm r_0$ and $r = 0$. Near $r = 0$ they behave as

$$dt^2 - dy^2 - r^{-\frac{2n}{n-1}} \left( \frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2 \right). \tag{41}$$

Substituting $r' = r^{\frac{1}{n-1}}$ shows that (41) is the flat metric. Thus, $r = 0$ is a coordinate singularity. By calculating the Kretschmann invariant (see Appendix B) it can be shown, that singularity at $r = \pm r_0$ is essential for all values of parameters in the case of metrics (26) and (34). In the case of metric (31) the singularity at $r = r_0$ and $r = -r_0$ is essential when, respectively, $p' < n$ or $p' > -n$. For $p' > n$ or $p' < -n$ the geodesic distance along $\partial_r$ tends to infinity when $r \to r_0$ or $r \to -r_0$, respectively. Thus, these regions represents an infinity different from that given by $r \to \infty$. For these values of parameters the Riemann tensor (in an orthonormal basis) tends to zero when $r \to r_0$ or $r \to -r_0$, respectively. However, the asymptotic metric is not the (n+3)-dimensional Minkowski metric. Its coefficients in front of $dt$ and $dy$ tend to zero whereas the coefficient in front of $d\Omega_n^2$ tends to infinity like the geodesic distance to the power $2(p' - 1)/(p' - n)$ or $2(p' + 1)/(p' + n)$, respectively.

Since metrics (26), (31) are invariant under the nonnull field $\partial_u$ they can be interpreted in the context of the Kaluza-Klein theory. Then metric (26) is equivalent to the scalar field given by $g_{yy}$ and the asymptotically flat (n+2)-dimensional metric induced on the surface $y=\text{const}$. The case $n = 2$ (the Gross-Perry metric) was studied in this framework by Ponce de Leon [15]. In order to interpret metric (31) with $q \neq 0$ in this vein one can write it in the form

$$g = -\Phi(dy - A_0 dt)^2 + g_{n+2}. \tag{42}$$

Here

$$A_0 = \tan \left( q \ln \left| \frac{r + r_0}{r - r_0} \right| \right) \tag{43}$$

is the electromagnetic potential,

$$\Phi = \left| \frac{r - r_0}{r + r_0} \right|^{p'} \cos \left( q \ln \left| \frac{r + r_0}{r - r_0} \right| \right) \tag{44}$$
corresponds to a scalar field and
\[ g_{n+2} = \frac{|r - r_0|^{\nu'}}{|r + r_0|} \left( \frac{dt^2}{\cos(q \ln \frac{r + r_0}{r - r_0})} - \frac{|r + r_0|^{2\nu' q}}{|r|^{2n-1}} - \frac{dr^2}{(n-1)^2 + r^2 d\Omega_n^2} \right) \] (45)
defines, modulo a power of \( \Phi \), a \((n+2)\)-dimensional metric. This metric is Lorentzian and asymptotically flat for large values of \( r \) and becomes singular when \( r \) diminishes to a value satisfying condition \( q \ln \frac{r + r_0}{r - r_0} = \pm \pi/2 \).

**Acknowledgements.** This work was partially supported by the Polish Committee for Scientific Research (grant 1 PO3B 075 29).

**Appendix A**

The aligned null direction is given by
\[ \hat{l} = \left( \frac{|r - r_0|^{\nu - q}}{|r + r_0|} + \frac{1}{2} f^2 \right) dt + \left( \frac{|r - r_0|^{\nu}}{|r + r_0|} - \frac{1}{2} f^2 \frac{|r - r_0|^{q}}{|r + r_0|} \right) dy + \frac{f |r + r_0|^{\nu + 1}}{(n-1)|r|^{n-1}} |r - r_0|^{\nu - 1} dr \] (A.1)

for metric (26) and by
\[ \hat{l} = \left( \sin \left( q \ln \frac{r + r_0}{r - r_0} \right) + 1 \right) \left( 1 - \frac{1}{2} f^2 \frac{|r - r_0|^{q}}{|r + r_0|} \right) \cos \left( q \ln \frac{r + r_0}{r - r_0} \right) \frac{\cos \left( q \ln \frac{r + r_0}{r - r_0} \right)}{\left( \sin \left( q \ln \frac{r + r_0}{r - r_0} \right) + 1 \right)^2} dt + \left( \frac{1}{2} f^2 \frac{|r - r_0|^{\nu}}{|r + r_0|} - \cos \left( q \ln \frac{r + r_0}{r - r_0} \right) \right) dy + \frac{|r + r_0|^{\nu + 1}}{(n-1)|r|^{n-1}} |r - r_0|^{\nu - 1} dr \] (A.2)

for metric (31). The function \( f \) is a solution of the polynomial equation
\[ f^4 - 8 f^2 \left( \frac{p'}{q} + \frac{2 q (n - 1) rr_0}{n (r^2 + r_0^2 - 2 p' r r_0)} \right) A - 16 A^2 = 0, \] (A.3)

where
\[ A = \frac{|r - r_0|^{\nu - q}}{|r + r_0|} \] (A.4)
in the case (26) and
\[ A = \frac{|r - r_0|^{- \nu}}{|r + r_0|} \left( \sin \left( q \ln \frac{r + r_0}{r - r_0} \right) + 1 \right) \] (A.5)
in the case (31). For metric (34) the aligned null directions are defined by

\[ \hat{n} = \frac{r + r_0}{r - r_0} \pm \sqrt{\frac{2n}{n+1}} \] \quad (A.6)

and

\[ \hat{l} = \frac{1}{2} \left( \ln \frac{r - r_0}{r + r_0} + f^2 \frac{r + r_0}{r - r_0} \pm \sqrt{\frac{2n}{n+1}} \right) du + \frac{1}{2} dv + \frac{r + r_0}{r - r_0} \frac{1}{n-1} \left( \pm \sqrt{\frac{2n}{n+1}} \right) dy. \] \quad (A.7)

In this case

\[ f^2 = \left( \mp \sqrt{\frac{n+1}{2n}} + \frac{2(n^2 - 1)rr_0}{n(n+1)(r^2 + r_0^2)} \mp \sqrt{2n(n+1)rr_0} \right) h(u) \left( \pm \sqrt{\frac{2n}{n+1}} \right). \] \quad (A.8)

**Appendix B**

The Kretschmann invariant for metrics (26) and (31) has the following form

\[ R_{\mu\nu\delta\sigma} R^{\mu\nu\delta\sigma} = 16n(n - 1)r_0^2 \left| r - r_0 \right|^{2(n+1)} \left( 4(n - 2p^2) + 2p' (n(n + 1)(p^2 - 3) + 2(1 + p^2)) r_0 + (4 - 3n - 5n^2 - 2(-2 + n(n + 3))p^2 + 4(n(n + 3))p^2) r^2 r_0^2 + 2p'(n(n + 1)(p^2 - 3) + 2(1 + p^2)) r_0^3 + (n^2 + n - 2p^2) r_0^4 \right). \] \quad (B.1)

In the case of metric (31) the Kretschmann invariant is given by (B.1) with \( p' = \pm \sqrt{\frac{2n}{n+1}} \).

**References**

[1] Coquereaux R. and Jadczyk A. 1988, Riemannian geometry, fiber bundles, Kaluza-Klein theories and all that..., *World Scientific Lecture Notes in Physics* 16

[2] Maartens R. 2004, Brane-world gravity, *Living Rev. Relativity* 7, [http://www.livingreviews.org/lrr-2004-7](http://www.livingreviews.org/lrr-2004-7)

[3] Coley A., Milson R., Pravda V., Pravdova A. 2004, Classification of the Weyl Tensor in Higher Dimensions *Class. Quantum Grav.* 21 L35-L42

[4] Podolsky J. and Ortaggio M. 2006, Robinson-Trautman spacetimes in higher dimensions, *Class. Quantum Grav.*, 23 57855797

[5] Coley A., Fuster A., Hervik S., Pelavas N. 2006, Higher dimensional VSI spacetimes, *Class. Quantum Grav.*, 23 7431-7444
[6] Harmark T. 2004, Stationary and Axisymmetric Solutions of Higher-Dimensional General Relativity, *Phys. Rev.*, **D70** 124002

[7] Myers R.C. and Perry M.J. 1986, Black holes in higher dimensional space-time, *Annals Phys.*, **172** 304

[8] Emparan R. and Reall H.S. 2002, Rotating black ring in five dimensions, *Phys. Rev. Lett.*, **88** 101101

[9] Gross D.J. and Perry M.J. 1983, Magnetic monopoles in Kaluza-Klein theories, *Nucl. Phys.*, **B 226** 29

[10] Davidson A. and Owen D. 1985, Black holes as a windows to extra dimensions *Phys. Lett.*, **B 155** 247

[11] Jakimowicz M. and Tafel J. 2008, SO(n + 1) symmetric solutions of the Einstein equations in higher dimensions, *Class. Quantum Grav.*, **25** 175002

[12] Millward R.S. 2008, A five-dimensional Schwarzschild-like solution, arXiv: gr-qc/0603132

[13] Coley A. and Pelavas N. 2006, Classification of Higher Dimensional Spacetimes, *Gen. Rel. Grav.* **38** 445-461

[14] Coley A. 2008, Classification of the Weyl Tensor in Higher Dimensions and Applications, *Class. Quantum Grav.*, **25** 033001

[15] Ponce de Leon J. 2007, Exterior spacetime for stellar models in 5-dimensional Kaluza-Klein gravity, *Class. Quantum Grav.* **24** 1755