Extending Dekking’s construction of an infinite binary word avoiding abelian 4-powers

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Abstract

We construct an infinite binary word with critical exponent 3 that avoids abelian 4-powers. Our method gives an algorithm to determine if certain types of morphic sequences avoid additive powers. We also show that there are $\Omega(1.172^n)$ binary words of length $n$ that avoid abelian 4-powers, which improves on previous estimates.

1 Introduction

Two words $x$ and $y$ are abelian equivalent if $x$ and $y$ are anagrams of each other. An abelian square (resp. abelian cube) is a word $xy$ (resp. $xyz$) where $x$ and $y$ (resp. $x$, $y$, and $z$) are abelian equivalent. More generally, an abelian $k$-power is a word $x_1x_2\cdots x_k$, where the $x_i$ are all abelian equivalent.

Erdős [6] asked if there was an infinite word over a finite alphabet that avoided abelian squares. Evdokimov [7] constructed such a word over a 25-letter alphabet; Keränen [8] improved this to a 4-letter alphabet. Dekking [5] showed that there is an infinite ternary word that avoids abelian cubes and an infinite binary word that avoids abelian 4-powers. This paper presents some results that strengthen or extend Dekking’s result on the binary alphabet.

The first direction we explore in trying to strengthen Dekking’s construction is to consider the simultaneous avoidance of abelian 4-powers and ordinary (i.e., non-abelian) powers. Ordinary powers are defined as follows. Let $w = w_1w_2\cdots w_n$ be a word of length $n$ and period $p$; i.e., $w_i = w_{i+p}$ for $i = 0, \ldots, n-p$. If $p$ is the smallest period of $w$, we say that the exponent of $w$ is $n/p$. We also say that $w$ is an $(n/p)$-power of order $p$. Words of exponent 2 (resp. 3) are called squares (resp. cubes). Words of exponent > 2 (resp. > 3) are called...
2+-powers (resp. 3+-powers). If $x$ is an infinite sequence, we define the critical exponent of $x$ as
\[
\sup\{e \in \mathbb{Q} : \text{there is a factor of } x \text{ with exponent } e\}.
\]
One can then ask: What is the least critical exponent among all infinite binary words that avoid abelian 4-powers?

We answer this question in this paper, but our proof involves a method for proving the avoidance of a variation on abelian powers, namely, additive powers. An additive square is a word $xy$ over an integer alphabet where $x$ and $y$ have the same length and the sum of the letters of $x$ is the same as the sum of the letters of $y$. Additive cubes and additive $k$-powers are defined analogously. Over the alphabet $\{0, 1\}$, a word is an abelian $k$-power if and only if it is an additive $k$-power, but on larger alphabets these two notions are no longer equivalent.

Three important papers on the avoidance of additive powers are: the paper of Cassaigne, Currie, Schaeffer, and Shallit [2], the paper of Rao and Rosenfeld [11], and the paper of Lietard and Rosenfeld [9].

Our first main result is the following:

**Theorem 1.** There exists an infinite binary word with critical exponent 3 that avoids additive/abelian 4-powers.

Another problem that goes beyond the existence of infinite binary words that avoid abelian 4-powers is to estimate the number of such words of length $n$. Currie [3] showed that there are $\Omega(1.044^n)$ binary words of length $n$ that avoid abelian 4-powers. We improve this to the following:

**Theorem 2.** There are $\Omega(1.172^n)$ binary words of length $n$ that avoid abelian 4-powers.

## 2 The construction for Theorem 1

Let $f : \{0, 1, 2\}^* \to \{0, 1, 2\}^*$ be the morphism defined by
\[
f(0) = 001, \quad f(1) = 012, \quad f(2) = 212
\]
and let $g : \{0, 1, 2\}^* \to \{0, 1\}^*$ be the morphism defined by
\[
g(0) = 0001001110010001100011, \quad g(1) = 0001001110011101100011, \quad g(2) = 011100111001101100011.
\]

We will prove that both $f^\omega(0)$ and $g(f^\omega(0))$ avoid additive 4-powers. We therefore introduce some notation related to additive powers. Let $u$ be a word over an integer alphabet. Then we write:

- $|u|$ for the length of $u$;
- $S(u)$ for the sum of the letters of $u$ and,
- $\sigma(u)$ for the vector $[|u|, S(u)]^T$. 

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Also, if $h : X^* \to Y^*$ is a morphism, we define
\[ W_h = \max_{a \in X} |h(a)|. \]
In particular, we have $W_f = 3$ and $W_g = 22$.

**Lemma 3.** Let $X$ and $Y$ be subsets of the integers and let $h : X^* \to Y^*$ be a morphism. Suppose that for every $x \in X$ we have $|h(x)| = a + bx$ and $S(h(x)) = c + dx$ for some integers $a, b, c, d$. Define the matrix
\[ M_h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \]
Then for any word $u \in X^*$ we have $\sigma(h(u)) = M_h \sigma(u)$. Consequently, if $M_h$ is invertible, we have $\sigma(u) = M_h^{-1} \sigma(h(u))$.

**Proof.** The result clearly holds when $u$ is the empty word, so suppose $u$ is non-empty and write $u = u'x$, where $x$ is the last letter of $u$. If the claim holds for $u'$, then we have
\[
M_h \sigma(u) = M_h \begin{bmatrix} |u'| + |h(u')| \\ S(h(u')) + S(h(x)) \end{bmatrix} = \sigma(h(u)).
\]

Note that for $x \in \{0, 1, 2\}$, we have $|f(x)| = 3 + 0x$ and $S(f(x)) = 1 + 2x$. Thus, we have
\[
M_f = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.
\]
Similarly, for $x \in \{0, 1, 2\}$, we have $|g(x)| = 22 + 0x$ and $S(g(x)) = 9 + 2x$, so we have
\[
M_g = \begin{bmatrix} 22 & 0 \\ 9 & 2 \end{bmatrix}.
\]
Clearly, both $M_f$ and $M_g$ are invertible.

We adapt the template method (see [4] for an early version of this method, or [11] for a much more powerful version of the method) to deal with additive powers. A template (for additive 4-powers) is an 8-tuple
\[ t = [a_0, a_1, a_2, a_3, a_4, d_0, d_1, d_2], \]
where \( a_i \in \{0, 1, 2\}^* \), \( |a_i| \leq 1 \), and the \( d_i \) are in \( \mathbb{Z}^2 \). A word \( w \) is an instance of \( t \) if we can write
\[
    w = a_0x_0a_1x_1a_2x_2a_3x_3a_4,
\]
such that for each \( i \), \( d_i = \sigma(x_{i+1}) - \sigma(x_i) \). Let \( h \in \{f, g\} \). For \( A \in \{0, 1, 2\}^* \), \( |A| \leq 1 \), an \( h \)-split of \( A \) is a triple \([p, a, s]\), such that \( h(A) = pas \), and \( a \in \{0, 1, 2\}^* \), \( |a| \leq 1 \).

Let
\[
    T = [A_0, A_1, A_2, A_3, A_4, D_0, D_1, D_2] \quad \text{and} \quad t = [a_0, a_1, a_2, a_3, a_4, d_0, d_1, d_2]
\]
be templates. We say that \( T \) is an \( h \)-parent of \( t \) if the \( A_i \) have \( h \)-splits \([p_i, a_i, s_i]\) such that
\[
    d_i = M_hD_i + b_i,
\]
where \( b_i = \sigma(s_{i+1}p_{i+2}) - \sigma(s_ip_{i+1}) \). We call a template \( t' \) an \( h \)-ancestor of \( t \) if \( t' \) is in the reflexive and transitive closure of the \( h \)-parent relation on \( t \).

**Lemma 4.** Let \( h \in \{f, g\} \) and let \( U \in \{0, 1, 2\}^* \). Suppose template \( T = [A_0, A_1, A_2, A_3, A_4, D_0, D_1, D_2] \) is an \( h \)-parent of template \( t = [a_0, a_1, a_2, a_3, a_4, d_0, d_1, d_2] \). If \( U \) contains an instance of \( T \), then \( h(U) \) contains an instance of \( t \).

**Proof.** Suppose that \( U \) contains an instance \( V = A_0X_0A_1X_1 \cdots X_3A_4 \) of \( T \). Since \( T \) is an \( h \)-parent of \( t \), the \( A_i \) have \( h \)-splits \([p_i, a_i, s_i]\) such that \( d_i = M_hD_i + b_i \), where \( b_i = \sigma(s_{i+1}p_{i+2}) - \sigma(s_ip_{i+1}) \). Thus \( u = h(U) \) contains a factor \( v = a_0x_0a_1x_1a_2x_2a_3x_3a_4 \), where \( x_i = s_ih(X_i)p_{i+1} \). We have
\[
    \sigma(x_{i+1}) - \sigma(x_i) = \sigma(s_{i+1}h(X_{i+1})p_{i+2}) - \sigma(s_ih(X_i)p_{i+1})
    = \sigma(h(X_{i+1})) + \sigma(s_{i+1}p_{i+2}) - \sigma(h(X_i)) - \sigma(s_ip_{i+1})
    = M_h\sigma(X_{i+1}) + \sigma(s_{i+1}p_{i+2}) - M_h\sigma(X_i) - \sigma(s_ip_{i+1})
    = M_h(\sigma(X_{i+1}) - \sigma(X_i)) + \sigma(s_{i+1}p_{i+2}) - \sigma(s_ip_{i+1})
    = M_hD_i + b_i
    = d_i
\]
Hence, the word \( v \) is an instance of \( t \).

If \( t \) is a template, we write \( d_i = [d_i^{(0)}, d_i^{(1)}] \) and define the quantities
\[
    \Delta(t) = \max \left\{ |d_0^{(0)}|, |d_1^{(0)}|, |d_2^{(0)}| \right\}
\]
and
\[
    B_h(t) = 6 + 4(W_h - 2) + 6\Delta(t).
\]

**Lemma 5.** Let \( h \in \{f, g\} \) and let \( U \in \{0, 1, 2\}^* \). If \( h(U) \) contains an instance \( v \) of template \( t \) of length at least \( B_h(t) \), then there is an \( h \)-parent \( T \) of \( t \) such that \( U \) contains an instance \( V \) of \( T \) and \( |V| < |v| \).
Proof. Suppose that \( v = a_0x_0a_1x_1a_2x_2a_3x_3a_4 \) is a factor of \( u = h(U) \) and is an instance of \( t = [a_0, a_1, a_2, a_3, a_4, d_0, d_1, d_2] \).

If \(|x_i| \geq W_h - 1\) for each \( i \), then each \( x_i \) can be written in the form \( x_i = s_i h(X_i) p_{i+1} \) for some \( X_i \), where \( s_i \) is a suffix of some \( h(A_i) \) and \( p_{i+1} \) is a prefix of some \( h(A_{i+1}) \). The analysis for Lemma 4 is thus reversible, and \( U \) contains an instance of \( T = [A_0, A_1, A_2, A_3, A_4, D_0, D_1, D_2] \) where the \( A_i \) have \( h \)-splits \([p_i, a_i, s_i]\) and

\[
D_i = M_h^{-1}(d_i - b_i).
\]

To complete the proof it suffices to show that if \(|v| \geq B_h(t)\), then for each \( i \) we have \(|x_i| \geq W_h - 1\). Suppose to the contrary that for some \( i \) we have \(|x_i| \leq W_h - 2\). Let \( \{\xi_0 \leq \cdots \leq \xi_3\} = \{|x_0|, \ldots, |x_3|\} \); i.e., the \( \xi_i \) are the lengths of the \( x_i \) arranged in non-decreasing order. Our hypothesis then is that \( \xi_0 \leq W_h - 2 \). By the definition of \( \Delta(t) \), we have \( \xi_i \leq \xi_0 + i\Delta(t) \) for \( i = 0, 1, 2, 3 \). Hence,

\[
\sum_{i=0}^{3} |x_i| = \sum_{i=0}^{3} \xi_i \leq \sum_{i=0}^{3} (\xi_0 + i\Delta(t)) = 4\xi_0 + \Delta(t) \sum_{i=0}^{3} i = 4\xi_0 + 6\Delta(t) \leq 4(W_h - 2) + 6\Delta(t).
\]

Consequently, we have

\[
|v| \leq 5 + \sum_{i=0}^{3} |x_i| \leq 5 + 4(W_h - 2) + 6\Delta(t) < B_h(t),
\]

which is a contradiction. □

Theorem 6. The infinite word \( g(f^\omega(0)) \) contains no additive 4-powers.

Proof. An additive 4-power \( v \) in \( g(f^\omega(0)) \) is an instance of the template

\[
t_0 = [\epsilon, \epsilon, \epsilon, \epsilon, [0, 0], [0, 0], [0, 0]].
\]

Applying Lemma 5 with \( h = g \) and \( t = t_0 \) (and thus \( \Delta(t) = 0 \) and \( B_g(t) = 6 + 4(22 - 2) + 6(0) = 86 \)), we see that if \(|v| \geq 86\), then the infinite word \( f^\omega(0) \) contains an instance \( V \) of some \( g \)-parent \( T \) of \( t_0 \). First, we verify by a brute force computation that \( g(f^\omega(0)) \) contains no additive 4-power \( v \) of length less than 86. Next, we use a computer to build a list \( Anc \) of all possible \( g \)-parents of \( t_0 \). We find that there are 17056 such \( g \)-parents. We then compute the set of all \( f \)-ancestors of these \( g \)-parents. For each \( g \)-parent, we compute all of its \( f \)-parents and add any new templates found to the list \( Anc \). We find 48 new templates, so our list now contains 17104 templates. We again compute \( f \)-parents of these 48 new templates, but we find that this results in no new templates. The process therefore terminates at this step with \( Anc \) containing 17104 templates. It follows that \( v \) is a factor of \( g(f(V)) \) for some instance \( V \) of a template \( T \) in \( Anc \).

Furthermore, we find that for every template \( t \) in \( Anc \), we have \( \Delta(t) \leq 2 \). Hence, applying Lemma 5 iteratively (with \( h = f \) and \( B_f(t) \leq 6 + 4(3 - 2) + 6(2) = 22 \)), we see that if \( f^\omega(0) \)
contains an instance $V$ of $T$, then it contains one where $|V| \leq 21$. Furthermore, Lemma 4 guarantees that $g(f(V))$ contains an additive 4-power.

To complete the proof, it suffices to check that for every factor $V$ of $f^\omega(0)$ of length at most 21, the word $g(f(V))$ does not contain an additive 4-power. A brute force computation shows that $f^6(0)$ contains all length-21 factors of $f^\omega(0)$. We can then examine the prefix $g(f^7(0))$ of $g(f^\omega(0))$ and verify that it contains no additive 4-powers, which establishes the claim.

**Theorem 7.** The infinite word $g(f^\omega(0))$ contains cubes but no $3^+$-powers.

**Proof.** This is proved with Walnut [10]. Here are the Walnut commands:

```wasm
morphism g "0->0001001110010001100011 1->0001001110011101100011
  2->0111001110011101100011":
morphism f "0->001 1-> 012 2->212":
promote aut f:
image SA g aut:
```

```wasm
eval containsthreeplus "?msd_3 Ei, n (n>=1) & At (t<=2*n) => SA[i+t]=SA[i+n+t]":
# checks to see if the word contains a 3+-power (answers FALSE)
```

```wasm
eval containscubes "?msd_3 Ei, n (n>=1) & At (t<2*n) => SA[i+t]=SA[i+n+t]":
# checks to see if there are cubes (answers TRUE)
```

Theorem 1 now follows from Theorems 6 and 7. It is optimal in the sense that the longest binary word avoiding both abelian 4-powers and (ordinary) 3-powers is the following word of length 39:

$$001101011011001001101100100110110101100$$

and there is only one such word (up to binary complement).

We should also point out that, in principle, the method of Rao and Rosenfeld [11] would also work to prove Theorem 6. We initially attempted to apply this method, but the computation produced a large number of templates and we were not able to complete the required computer checks.

### 3 Generalizing to additive $k$-powers

In this section we sketch how to generalize the method of the previous section to additive $k$-powers. We first need to generalize the definition of template. Given an integer alphabet $X$, an **additive $k$-template** is a $2k$-tuple

$$t = [a_0, \ldots, a_k, d_0, \ldots, d_{k-1}],$$

where $a_i \in X^*$, $|a_i| \leq 1$, and the $d_i$ are in $\mathbb{Z}^2$. A word $w$ is an **instance** of $t$ if we can write

$$w = a_0 x_0 a_1 x_1 \cdots x_{k-1} a_k,$$
such that for each $i$, $d_i = \sigma(x_{i+1}) - \sigma(x_i)$.

If $h$ is a morphism defined on an integer alphabet, we define $h$-split, $h$-parent, and $h$-ancestor as in Section 2. We need Lemma 3 and additive $k$-power analogues of Lemmas 4 and 5. We state these analogues next; their proofs are straightforward generalizations of those of Lemmas 4 and 5. Our method only applies to morphisms that satisfy the conditions of Lemma 3, so let us call any such morphism $h$ a linear morphism and let us use $M_h$ to denote the matrix of Lemma 3.

**Lemma 8.** Let $h$ be a linear morphism and let $U \in X^*$. Suppose additive $k$-template $T$ is an $h$-parent of additive $k$-template $t$. If $U$ contains an instance of $T$, then $h(U)$ contains an instance of $t$.

If $h$ is a morphism and $t$ is an additive $k$-template, we write $d_i = \begin{bmatrix} d_i(0) \\ d_i(1) \end{bmatrix}$ and define the quantities

$$\Delta(t) = \max \left\{ |d_i(0)| : i = 0, \ldots, k-2 \right\}$$

and

$$B_h(t) = k + 2 + k(W_h - 2) + \frac{(k-1)k}{2} \Delta(t).$$

**Lemma 9.** Let $h$ be a linear morphism such that $|h(a)| \geq 2$ for all $a \in X$ and such that $M_h$ is invertible. Let $U \in X^*$. If $h(U)$ contains an instance $v$ of additive $k$-template $t$ of length at least $B_h(t)$, then there is an $h$-parent $T$ of $t$ such that $U$ contains an instance $V$ of $T$ and $|V| < |v|$.

**Theorem 10.** Let $h$ be a linear morphism such that all eigenvalues of $M_h$ are larger than 1 in absolute value and let $t$ be an additive $k$-template. Then the set of $h$-ancestors of $t$ is finite.

**Proof.** If the eigenvalues of $M_h$ are larger than 1 in absolute value, then $M_h$ is invertible and its eigenvalues are smaller than 1 in absolute value. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $M^{-1}_h$ and let $M^{-1}_h = P^{-1}JP$, where $J$ is the Jordan form of $M_h$. Then either

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}. $$

Hence,

$$M_h^{-n} = P^{-1} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P \quad \text{or} \quad M_h^{-n} = P^{-1} \begin{bmatrix} \lambda_1^n & n\lambda_1^{n-1} \\ 0 & \lambda_1^n \end{bmatrix} P,$$

and so $\sum_{i \geq 0} M_h^{-i}$ converges.

Let $t = [a_0, \ldots, a_k, d_0, \ldots, d_{k-1}]$ be an additive $k$-template and let $T = [A_0, \ldots, A_k, D_0, \ldots, D_{k-1}]$
be an $h$-ancestor of $t$ obtained by applying the $h$-parent relation $\ell$ times. If $\ell = 1$, then, as in the proof of Lemma 5, we have $D_i = M_h^{-1}(d_i - b_i)$, where there are only finitely many choices for $b_i$. For larger $\ell$, by iteration, we find that $D_i$ has the form

$$D_i = c_\ell M_h^{-\ell} + c_{\ell-1} M_h^{-(\ell-1)} + \cdots + c_1 M_h^{-1} + c_0,$$

(1)

where the $c_j$ are all taken from a finite set of vectors. Since $\sum_{i \geq 0} M_h^{-i}$ converges and since $D_i \in \mathbb{Z}^2$, we see that there are only finitely many possible vectors $D_i$, and hence, only finitely many $h$-ancestors $T$ of $t$.

**Corollary 11.** Let $f : X^* \to X^*$ and $g : X^* \to Y^*$ be linear morphisms over integer alphabets $X$ and $Y$ such that $g(f(a))$ is an infinite word for some $a \in X$. If

- $|f(a)| \geq 2$ and $|g(a)| \geq 2$ for all $a \in X$,
- $M_f$ and $M_g$ are invertible, and
- the eigenvalues of $M_f$ are larger than 1 in absolute value,

then it is possible to decide if $g(f^\omega(a))$ avoids additive $k$-powers.

**Proof.** We follow the procedure from the proof of Theorem 6. An additive $k$-power $v$ in $g(f^\omega(0))$ is an instance of the additive $k$-template

$$t_0 = [\epsilon, \ldots, \epsilon, [0, 0], \ldots, [0, 0]].$$

We first check by a brute force computation that $g(f^\omega(a))$ contains no additive $k$-power of length less than $B_g(t_0) = k + 2 + k(W_g - 2)$. As in the proof of Theorem 6, we then need to compute the set of $g$-parents of $t_0$ and the set $\text{Anc}$ of all $h$-ancestors of each of the $g$-parents of $t_0$. By Theorem 10, the set $\text{Anc}$ is finite and effectively computable. By Lemmas 8 and 9, it now suffices to verify that for every $t \in \text{Anc}$, no factor of $f^\omega(a)$ of length less than $B_f(t)$ is an instance of $t$. This is a finite (computer) check. \qed

### 4 The construction for Theorem 2

A *(multi-valued)* substitution on the alphabet $\Sigma$ is a function $\theta : \Sigma^* \to 2^{\Sigma^*}$ that satisfies

$$\theta(uv) = \{UV : U \in \theta(u), V \in \theta(v)\}$$

for all $u, v \in \Sigma^*$. We extend $\theta$ to $2^{\Sigma^*}$ by defining

$$\theta(S) = \bigcup_{u \in S} \theta(u)$$

for all $S \subseteq \Sigma^*$. This extension allows us to compose substitutions.

A substitution $\theta$ on $\Sigma$ is *abelian $n$-power-free* if every word $v \in \Sigma^*$ avoids abelian $n$-powers. Currie [3] established that there are $\Omega(1.044^n)$ binary words of length $n$ that avoid abelian 4-powers by exhibiting an abelian 4-power free substitution on $\{0,1\}$. We improve this to $\Omega(1.172^n)$ by composing several abelian 4-power free substitutions on $\{0,1\}$.

We begin with a technical lemma. We say that a substitution $\theta$ on $\Sigma$ is *letter-wise uniform* if for every $a \in \Sigma$, we have $|u| = |v|$ for all $u, v \in \theta(a)$. 


Lemma 12. Suppose that there exists an abelian 4-power free word $w \in \{0, 1\}^\omega$ in which the frequency of the letter 0 exists and is equal to $\alpha$. Let $\theta : \{0, 1\}^* \rightarrow 2^{\{0,1\}^*}$ be an abelian 4-power free substitution which is letter-wise uniform. For $a \in \{0, 1\}$, let $\ell_a = |u|$ for some $u \in \theta(a)$, and let $m_a = |\theta(a)|$. Then for any fixed $\epsilon > 0$, there are $\Omega(\beta^n)$ abelian 4-power free binary words of length $n$, where

$$\beta = (m_0^{\alpha-\epsilon} m_1^{-\alpha-\epsilon})^{\frac{|\ell(0)| + 1}{|\ell(0)| + (1 - \alpha + \epsilon)|1|}}.$$  

Proof. Let $W \in \theta(w)$, and consider the prefix $V$ of $W$ of length $n$. Write $V = V'p$, where $V' \in \theta(v)$ for some prefix $v$ of $w$ and $|p| \leq \max_{a \in \{0,1\}}(\ell_a - 1)$. By the assumption that $\theta$ is abelian 4-power free, every word in $\theta(v)$ avoids abelian 4-powers.

Fix $\epsilon > 0$. For $n$ sufficiently large, we have $(\alpha - \epsilon)|v| \leq |v_0| \leq (\alpha + \epsilon)|v|$. Therefore, we have

$$n - |p| = \ell_0|v_0| + \ell_1|v_1| \leq ((\alpha + \epsilon)\ell_0 + (1 - \alpha + \epsilon)\ell_1)|v|,$$

which gives

$$|v| \geq \frac{n - |p|}{(\alpha + \epsilon)\ell_0 + (1 - \alpha + \epsilon)\ell_1}.$$  

Thus we have

$$|\theta(v)| = m_0^{|v|\ell_0} m_1^{|v|\ell_1} \geq m_0^{\alpha-\epsilon}|v| m_1^{(1-\alpha-\epsilon)|v|} \geq (m_0^{\alpha-\epsilon} m_1^{1-\alpha-\epsilon})^{\frac{|v|}{n - |p|}} \geq (m_0^{\alpha-\epsilon} m_1^{1-\alpha-\epsilon})^{\frac{1}{(\alpha + \epsilon)\ell_0 + (1 - \alpha + \epsilon)\ell_1}} = c \left[(m_0^{\alpha-\epsilon} m_1^{1-\alpha-\epsilon})^{\frac{|v|}{(\alpha + \epsilon)\ell_0 + (1 - \alpha + \epsilon)\ell_1}}\right]^n, $$

where $c = (m_0^{\alpha-\epsilon} m_1^{1-\alpha-\epsilon})^{\frac{|v|}{(\alpha + \epsilon)\ell_0 + (1 - \alpha + \epsilon)\ell_1}}$. 

Let $\theta_0, \theta_1 : \Sigma^* \rightarrow 2^{\Sigma^*}$ be defined as follows:

$$\begin{align*}
\theta_0(0) &= \{0001\} & \theta_1(0) &= \{011, 101\} \\
\theta_0(1) &= \{011, 101\} & \theta_1(1) &= \{0001\}
\end{align*}$$

Note that $\theta_1$ is obtained from $\theta_0$ by swapping the images of 0 and 1. Currie [3] proved that $\theta_0$ is abelian 4-power free by generalizing a result of Dekking [5]. It follows that $\theta_1$ is abelian 4-power free as well.

Let $x \in \{0, 1\}^*$, and write $x = x_1 x_2 \cdots x_k$, where the $x_i$ are in $\{0, 1\}$. Then we define $\theta_x = \theta_{x_k} \circ \theta_{x_{k-1}} \cdots \circ \theta_{x_1}$. Since $\theta_0$ and $\theta_1$ are letter-wise uniform and abelian 4-power free, it follows that $\theta_x$ is letter-wise uniform and abelian 4-power free. Further, for any fixed word $x \in \{0, 1\}^*$, it is straightforward to compute the constants $\ell_0$, $\ell_1$, $m_0$, and $m_1$ described in the statement of Lemma 12 for the substitution $\theta_x$.

Proof of Theorem 2. Let $h : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be the morphism defined by $h(0) = 0001$ and $h(1) = 011$. It is well-known that $h^\omega(0)$ avoids abelian 4-powers [5]. By a straightforward calculation (see [1, Theorem 8.4.7]), the frequency of the letter 0 in $h^\omega(0)$ is $\alpha = \frac{\sqrt{5} - 1}{2}$.
Table 1: The unique word $x \in \{0, 1\}^k$ that maximizes $\beta_x$. (The value of $\beta_x$ is truncated after 8 decimal places.)

| $k$ | $x$         | $\beta_x$   |
|-----|-------------|-------------|
| 1   | 1           | 1.13503537  |
| 2   | 01          | 1.15986115  |
| 3   | 101         | 1.16840698  |
| 4   | 1101        | 1.17123737  |
| 5   | 11101       | 1.17195987  |
| 6   | 111101      | 1.17220553  |
| 7   | 1111101     | 1.17226224  |
| 8   | 11111101    | 1.17228469  |
| 9   | 011111101   | 1.17228931  |
| 10  | 111111101   | 1.17229090  |
| 11  | 01111111101 | 1.17229161  |
| 12  | 001011111101| 1.17229185  |
| 13  | 0001011111101| 1.17229194 |
| 14  | 00001011111101| 1.17229198 |
| 15  | 000101011111101| 1.17229199 |

For each $k \in \{1, 2, \ldots, 15\}$ and every $x \in \{0, 1\}^k$, we used a computer to apply Lemma 12 with the word $w = h^\omega(0)$, the substitution $\theta = \theta_x$, and the number $\epsilon = 10^{-5}$ to obtain a number $\beta_x$ such that the number of abelian 4-power free binary words of length $n$ is $\Omega(\beta^n_x)$. For each $k \in \{1, 2, \ldots, 10\}$, the maximum value of $\beta_x$ was maximized by a unique word $x \in \{0, 1\}^k$. This word $x$, and the value of $\beta_x$, is shown in Table 1. In particular, this establishes Theorem 2.

When we performed the computations reported in Table 1, we didn’t just use $\theta_0$ and $\theta_1$, but we also included substitutions $\theta_2$ defined by $\theta_2(0) = \{0111\}$, $\theta_2(1) = \{001, 010\}$, and $\theta_3$ defined by swapping the images of $\theta_2(0)$ and $\theta_2(1)$, which both satisfy the criteria of Currie [3] to be abelian 4-power free. However, there was always an optimal composition that only involved $\theta_0$ and $\theta_1$.

The current best known upper bound for the number of abelian 4-power free words over a binary alphabet is $O(1.374164^n)$ due to Samsonov and Shur [12].

5 Open problems

The set of all infinite binary words avoiding abelian 4-powers is still not completely understood and there are still problems to investigate. One is the following:

Problem 1. What is the minimum possible frequency of 0’s in an infinite binary word avoiding abelian fourth powers?

Some empirical calculations suggest it might be around 1/3. The frequency of 0’s in the binary complement of the word $h^\omega(0)$ from the proof of Theorem 2 is $(3 - \sqrt{5})/2 \approx 0.381966$. 

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