Anomalies and chiral defect fermions

by

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ABSTRACT

Chiral defect fermions in the background of an external, $2n$ dimensional gauge field are considered. Assuming first a finite extra dimension, we calculate the axial anomaly in a vector-like, gauge invariant model for arbitrary $n$, and the consistent anomaly in a gauge variant model with a chiral spectrum. For technical reasons, the latter calculation is limited to the $2 + 1$ dimensional case. We also show that the infinite lattice chiral model, when properly defined, is in fact a limiting case of the above gauge-variant model. The behaviour of this model with a dynamical gauge field is discussed.
1. Introduction

Recently, interest in the long standing problem of constructing chiral gauge theories on the lattice was revived [1-12] following the interesting proposal of D. Kaplan [1]. Kaplan starts with a Wilson action in $2n + 1$ dimensions, with the peculiar property that the Dirac mass has the shape of a domain wall, i.e. the mass term changes sign across the $2n$ dimensional surface $s = 0$. For a certain range of the Dirac mass and the Wilson parameter, the spectrum contains a single, $2n$ dimensional chiral fermion that lives on the domain wall.

Kaplan suggested that a chiral gauge theory in $2n$ dimensions may emerge in the continuum limit of his model when dynamical gauge fields are introduced. His suggestion was based on the expectation that it may be possible to decouple all but the massless chiral mode in the low energy limit.

There are several ways in which one can proceed from the free domain wall model to a full fledged interacting theory. In particular, the gauge field may be chosen to be $2n+1$ dimensional, as originally proposed by Kaplan, or it can be made $2n$ dimensional by eliminating its extra component and insisting that the remaining $2n$ components be independent of the extra coordinate [3,4]. In this case the extra coordinate becomes a sophisticated flavour space. We will restrict our attention in this paper to the latter possibility. For recent results on models with a $2n + 1$ dimensional gauge field see ref. [5].

A basic difficulty is how to define the interacting model with an infinite extra direction as the limit of models with a finite extra direction. The origin of this difficulty is as follows. If the extra coordinate $s$ has an infinite range, the chiral fermion on the domain wall is the only massless excitation. On the other hand, if the extra coordinate has a finite range there appears another massless mode with the opposite chirality. For example, choosing periodic boundary conditions necessitates the existence of an anti-domain wall, and the chiral fermion on the anti-domain wall has the opposite chirality. More examples of this phenomenon are discussed below.

The successful construction of a chiral gauge theory using Kaplan’s method therefore depends critically on one’s ability to decouple the extra, unwanted massless mode. Especially stringent constraints exist if one attempts to achieve this decoupling at the level of perturbation theory, because here the Nielsen-Ninomiya theorem [13] is directly applicable. Once the extra coordinate is made finite, and as long as gauge invariance is maintained, both massless modes couple with equal strength to the gauge field and the theory is vector-like. A sequence of such models with increasing size of the extra direction will eventually give rise to a vector-like theory as well.
In an attempt to avoid this problem from the outset, Narayanan and Neuberger have proposed to work directly with an infinite extra direction [4]. The spectrum then contains a single chiral fermion, and at least within the context for perturbation theory, one can investigate the model by writing down the corresponding Feynman rules. However, a closer look reveals that the Feynman rules of the infinite lattice model are ambiguous. Different values for the divergence of the source current are obtained if the infinite summations over $s$ at the vertices of the relevant diagram are carried out in different ways.

Narayanan and Neuberger circumvented this difficulty by showing that a chiral effective action can be defined using transfer matrix methods [6]. In this approach, the transfer matrix for a single site translation in the $s$-direction is gauge invariant, and possible breakdown of gauge invariance resides entirely in the choice of the boundary conditions. However, the relation between the transfer matrix construction and the original, infinite lattice model has not been fully clarified.

In this paper we reexamine the infinite lattice chiral model directly. The crucial property of a consistent set of Feynman rules is that, during the process of sending the range of the $s$-summations to infinity, all vertices in a given diagram should describe the coupling of the gauge field to the \textit{same} source current. (In other words, the IR regularized diagram should satisfy Bose symmetry). Enforcing this requirement uniquely determines how to perform the infinite $s$-summations. Moreover, it implies that the infinite lattice chiral model is defined as the limit of a sequence of models with a finite extra direction, in which \textit{gauge invariance is broken in a particular way}. As a check, we calculate the consistent anomaly in the three dimensional model and show that the correct result is obtained.

We begin in sect. 2 with a discussion of the axial anomaly in vector-like gauge invariant models. We do so because the use of chiral defect fermions for lattice simulations of QCD is an important subject by itself, and because understanding the relation between the the vector-like and chiral models may help in making further progress on the difficult question of lattice chiral gauge theories.

In a lattice formulation of QCD using chiral defect fermions there is a natural way to define the axial current. We calculate the axial anomaly using this definition. We show that, up to a numerical factor, this calculation can be reduced to the calculation of the Chern-Simons action. Using the result of Golterman, Jansen and Kaplan [7], we find the coefficient of the axial anomaly for any $n$.

In sect. 3, after a discussion of the ambiguities of the infinite lattice chiral model, we introduce the \textit{gauge variant} chiral model on a lattice with a finite extra direction. We show that the carefully defined Feynman rules of the infinite lattice model are
reproduced by the gauge-variant model for a sufficiently large extra dimension. We then calculate the consistent anomaly in the gauge-variant model for the three-dimensional case. As explained below, although conceptually similar, the general case is technically more complicated.

In sect. 4 we discuss the behaviour of the gauge-variant model in the presence of dynamical gauge fields. We argue that dynamical restoration of gauge invariance takes place [14], and that, very likely, it is accompanied by the appearance of a new massless charged fermion in the spectrum which makes the theory vector-like [9,10].

In most of this paper we will use the boundary fermions variant [8] of chiral defect fermions. Models with boundary fermions are simpler, yet they have the same physical content as the corresponding domain wall models. We will explain how the same results can be obtained with domain wall fermions.

Chiral fermions arise on the $2n$ dimensional boundaries of a $2n + 1$ dimensional lattice model of Wilson fermions with a constant mass $M$. They exist provided free boundary conditions are chosen in the extra direction, and $M$ is in the allowed range. (We set the Wilson parameter to $r = 1$). Models with a single boundary (semi-infinite extra direction) as well as models with two boundaries (finite extra direction) can both be discussed.

The chiral spectrum is the same as in the domain wall case for the same choice of the absolute value of the mass. Additional considerations [4] restrict the range of $M$ to $0 < M < 1$. For this choice every boundary (like every domain wall) supports a single chiral fermion.

It is important to note that the relative sign of $M$ and $r$ in the boundary fermion model is the opposite from their relative sign in the conventional Wilson action. Thus, contrary to conventional Wilsonian QCD, the sum of the $2n + 1$ dimensional Wilson term and the mass term is not a positive operator. This property has profound impact on the dynamics of the model and is crucial for the existence of stable massless modes.

In Kaplan’s definition of the domain wall model [1], the relative sign of $M$ and $r$ is the same as in the Wilson action on the negative half-space. Again, the region with unconventional relative sign – the positive half-space – is in some sense more important. This can be seen from the fact that the Chern-Simons current flows only into this half-space, while it vanishes away from the wall on the other side. One can say that the negative half-space can be discarded without affecting the dynamics. This is precisely what one does in going from the domain wall to the boundary fermions model.
2. Axial anomaly in the vector-like model

The vector-like model is defined as follows [8]. We consider a $d = 2n + 1$ dimensional lattice. The first $2n$ coordinates, labeled $x_\mu$, have an infinite range, whereas the extra coordinate $s$ takes the values $s = 1, \ldots, 2N$. We have chosen an even number of sites to minimize the amount of relabeling needed to see the correspondence between the vector-like model and the chiral model discussed later. None of our results depends on this choice. The action is

$$S = \sum_{x,y,s} \overline{\psi}(x,s)D_{x,y}^\parallel(U)\psi(y,s) + \sum_{x,s,s'} \overline{\psi}(x,s)D_{s,s'}^\perp\psi(x,s'),$$  \hspace{1cm} (1)

$$D_{x,y}^\parallel(U) = \frac{1}{2} \sum_\mu \left( (1 + \gamma_\mu)U_\mu(x)\delta_{x+\hat{\mu},y} + (1 - \gamma_\mu)U_\mu^\dagger(x - \hat{\mu})\delta_{x-\hat{\mu},y} \right) + (M - 4)\delta_{x,y}$$  \hspace{1cm} (2)

$$D_{s,s'}^\perp = \frac{1}{2} \left( (1 + \gamma_d)\delta_{s+1,s'} + \frac{1}{2} (1 - \gamma_d)\delta_{s-1,s'} - \delta_{s,s'} \right).$$  \hspace{1cm} (3)

Notice that, apart from the unconventional sign of the mass term, $D_{x,y}^\parallel(U)$ is the usual $2n$ dimensional gauge covariant Dirac operator for Wilson fermions. The spectrum contains a right-handed fermion near the boundary $s = 1$ and a left-handed fermion near the other boundary.

The action (1) is invariant under a $180^\circ$ rotation in the $(k,d)$ plane around an axis located at the hyper-plane $s = N + 1/2$. Explicitly

$$\psi(x_1, \ldots, x_{2n}, s) \rightarrow i\gamma_k\gamma_d\psi(x_1, \ldots, x_{k-1}, -x_k, x_{k+1}, \ldots, x_{2n}, 2N + 1 - s).$$  \hspace{1cm} (4)

For simplicity, we will consider below a U(1) gauge theory. The generalization to the non-abelian case is straightforward. We now want to define the various $2n$ dimensional currents. The vector current is uniquely determined by the coupling to the gauge field, and it is given by

$$J_\mu^V(x) = \sum_{s=1}^{2N} j_\mu(x,s),$$  \hspace{1cm} (5)

where $j_\mu(x,s)$ stands for the first $2n$ components of the $d$-dimensional current

$$j_\mu(x,s) = \frac{1}{2} \left( \overline{\psi}(x,s)(1 + \gamma_\mu)U_\mu(x)\psi(x + \hat{\mu}, s) - \overline{\psi}(x + \hat{\mu}, s)(1 - \gamma_\mu)U_\mu^\dagger(x)\psi(x,s) \right),$$  \hspace{1cm} (6a)

$$j_d(x,s) = \frac{1}{2} \left( \overline{\psi}(x,s)(1 + \gamma_d)\psi(x,s + 1) - \overline{\psi}(x,s + 1)(1 - \gamma_d)\psi(x,s) \right).$$  \hspace{1cm} (6b)

The $d$-dimensional current satisfies the continuity equation

$$\sum_\mu \Delta_\mu j_\mu(x,s) = \begin{cases} -j_d(x,1), & s = 1, \\ -\Delta_d j_d(x,s), & 1 < s < 2N - 1, \\ j_d(x,2N - 1), & s = 2N. \end{cases}$$  \hspace{1cm} (7)
Here $\Delta_\mu f(x,s) = f(x,s) - f(x - \hat{\mu},s)$. Notice the peculiar form of the boundary terms in the continuity equation. Eqs. (5) and (7) imply the conservation of the vector current.

Next we want to define an axial transformation, from which the axial current can be derived in the usual way. There is a lot of arbitrariness in the choice of the axial transformation. Any transformation that assigns opposite charges to the two chiral modes will reduce to the usual axial transformation in the continuum limit.

For example, we could define the axial transformation to be the usual 2n dimensional one, applied equally to the fermions on all 2n dimensional layers. The disadvantage of this definition is that the divergence of (singlet or non-singlet) axial currents will involve an axial variation of the mass and Wilson terms, summed over all $s$. It has been argued that QCD with chiral defect fermions does not require any fine tuning of the mass term in the continuum limit [8]. The verification of this statement is extremely complicated with the above axial transformations, whereas it becomes trivial (within the context of perturbation theory) with the axial transformation defined below.

The natural definition of an axial transformation in a model of chiral defect fermions takes advantage of the fact that the two chiral modes are globally separated in the $s$-direction. We first define left-handed and right-handed transformations as follows

\begin{align*}
\delta_{L,R}\psi_{x,s} &= \pm iq_{L,R}(s)\psi_{x,s}, \\
\delta_{L,R}\overline{\psi}_{x,s} &= -iq_{L,R}(s)\overline{\psi}_{x,s},
\end{align*}

(8a)

where

\begin{align*}
q_{R}(s) &= \begin{cases} 
1, & 1 \leq s \leq N, \\
0, & N < s \leq 2N,
\end{cases} & q_{L}(s) &= \begin{cases} 
0, & 1 \leq s \leq N, \\
1, & N < s \leq 2N.
\end{cases}
\end{align*}

(9)

Notice that the two transformations are related by the discrete symmetry of eq. (4). The left-handed and right-handed transformations act vectorially, but only on half of the fermions, whereas fermions in the other half-space are invariant. Thus, the right-handed massless mode at the $s = 1$ boundary transforms only under the right-handed transformation etc. The non-invariance of the action under these transformations resides entirely in the coupling between the $N$-th layer and the $(N + 1)$-st layer.

We define the axial transformation to be the product of a right-handed transformation and an inverse left-handed transformation. Thus, fermions in the two half-spaces transform with opposite charges under the axial transformation. The
corresponding currents are
\[ J^R_\mu(x) = \sum_{s=1}^{N} j_\mu(x,s), \]  
(10a)
\[ J^L_\mu(x) = \sum_{s=N+1}^{2N} j_\mu(x,s), \]  
(10b)
\[ J^A_\mu(x) = J^R_\mu(x) - J^L_\mu(x). \]  
(10c)

The divergences equations are
\[ \Delta_\mu J^R_\mu(x) = -j_d(N,x), \]  
(11a)
\[ \Delta_\mu J^L_\mu(x) = j_d(N,x), \]  
(11b)
\[ \Delta_\mu J^A_\mu(x) = -2j_d(N,x). \]  
(11c)

Feynman rules are obtained in the usual way by making the weak coupling expansion
\[ U_{x,\mu} = \exp igV_\mu(x + \mu/2). \]  
(12)

Because of lack of translation invariance in the extra coordinate, we go to momentum space only in the first 2n coordinates. The fermion propagator \( G_F(s,s';p) \) is given in ref. [8]. (When necessary, the dependence of the propagator on the size of the extra direction will be indicated by a superscript). For \( s \) and \( s' \) both near the same boundary, \( G_F(s,s';p) \) is dominated by the zero mode’s contribution, whereas for \( s \) or \( s' \) far from the boundaries, \( G_F(s,s';p) \) can be approximated up to an exponentially small error by the translationally invariant \( d \)-dimensional propagator
\[ G_F(s,s';p) \approx \int_{-\pi}^{\pi} \frac{dp_d}{2\pi} e^{ip_d(s-s')} G_0(p_\alpha), \]  
(13)
\[ G_0(p_\alpha) = \frac{i\gamma_\alpha \hat{p}_\alpha + w(p_\alpha) - M}{\hat{p}_\alpha \hat{p}_\alpha + (w(p_\alpha) - M)^2}, \]  
(14)
where \( p_\alpha = (p_\mu, p_d) \) is the \( d \)-dimensional momentum (summation over \( \alpha \) is implied), \( \hat{p}_\alpha = \sin p_\alpha \), and \( w(p_\alpha) = \sum_\alpha 1 - \cos p_\alpha \).

The single photon vertex is given by
\[ \bar{\Lambda}_\mu(s,s'; p, q, k) = i\delta_{s,s'} \delta^{2n}(k + p - q) \Lambda_\mu(p + q), \]  
(15)
Here \( p \) and \( q \) are the \( 2n \) dimensional momenta of the incoming and outgoing fermions respectively, and \( k \) is the momentum of the external gauge field. \( \Lambda_\mu \) is the usual \( 2n \) dimensional vertex
\[ \Lambda_\mu(p + q) = \gamma_\mu \cos \left( \frac{p_\mu + q_\mu}{2} \right) - i \sin \left( \frac{p_\mu + q_\mu}{2} \right). \]  
(16)
In addition, the Feynman rules include an integration over a \(2n\) dimensional Brillouin zone for every closed loop, and a summation over \(s = 1, \ldots, 2N\) at each vertex.

We are now ready to compute the divergence of the axial current in the presence of an external gauge field. For a finite lattice spacing one expects

\[
\Delta_\mu J_\mu^A = C_A \epsilon_{\mu_1 \ldots \mu_n, \nu_1 \ldots \nu_n} \left( \partial_{\mu_1} V_{\nu_1} \right) \cdots \left( \partial_{\mu_n} V_{\nu_n} \right) + O(a).
\]  

(17)

We want to calculate the coefficient \(C_A\). Using eq. (11c), this is obtained from the correlator of \(j_d(N, x)\) with \(n\) vector currents. Following ref. [7] closely, we Taylor expand this diagram. Keeping track of the various symmetry factors we find

\[
C_A = \frac{i \epsilon_{\mu_1 \ldots \mu_n, \nu_1 \ldots \nu_n}}{(2n)!} \frac{\partial \cdots \partial}{\partial (p_1)_{\mu_1} \cdots \partial (p_n)_{\mu_n}} T^A_{\nu_1 \ldots \nu_n}(p_1, \ldots, p_n) \bigg|_{p_i = 0},
\]  

(18)

\[
T^A_{\nu_1 \ldots \nu_n} = -2 \int_{-\pi}^{\pi} \frac{d^{2n} k}{(2\pi)^{2n}} \sum_{s_1 \ldots s_n} \left( T^+_{\nu_1 \ldots \nu_n} - T^-_{\nu_1 \ldots \nu_n} \right),
\]  

(19)

\[
T^\pm_{\nu_1 \ldots \nu_n} = \frac{1}{2} \text{tr} \left( 1 \pm \gamma_d \right) G_F(N \pm \eta_{\nu_1}, s_1; k) \Lambda_{\nu_1}(k + k_1) G_F(s_1, s_2; k_1) \cdots \Lambda_{\nu_n}(k_{n-1} + k_n) G_F(s_n, N \pm \eta_{\nu_n}; k_n).
\]  

(20)

Here \(\eta_+ = 1, \eta_- = 0\), and \((k_i)_\mu = k_\mu + (p_1)_\mu + \ldots + (p_i)_\mu\).

Thanks to the exponential damping of the fermionic propagator in the \(s\)-direction, only \(s\)-values in the neighbourhood of \(s = N\) are important on the r.h.s. of eq. (19). For sufficiently large \(N\) we can therefore replace \(G_F\) in eq. (20) by the translation invariant propagator \(G_0\), and extend the range of the \(s\)-summations to \(-\infty < s < \infty\). This gives rise to a considerable simplification, and allows us to express \(T^A_{\nu_1 \ldots \nu_n}\) as an integral over a \(2n + 1\) dimensional Brillouin zone. The resulting expression coincides with the corresponding one in ref. [7] up to a numerical factor. Using the result of ref. [7] for the coefficient of the Chern-Simons action we find

\[
C_A = \frac{2i (-)^{n+1}}{(2\pi)^n n!},
\]  

(21)

in agreement with known results [15].

If one works with domain wall fermions, the vector-like model is defined by imposing periodic boundary conditions. One lets \(s\) range from \(-2N\) to \(2N\) with the layers \(s = 2N\) and \(s = -2N\) identified, the domain wall is at \(s = 0\) and the anti-domain wall at \(s = 2N\). The right-handed transformation is applied to the half-space \(-N < s \leq N\), and the left-handed transformation to the other half-space. The divergence of right-handed current is \(j_d(-N, x) - j_d(N, x)\), with similar expressions for
the other currents. As shown in ref. [7], the new term makes no contribution to the anomaly.

3. Consistent anomaly in the chiral model

We now proceed to discuss the infinite lattice chiral model. We will first describe the naive Feynman rules of this model and exhibit their ambiguities. Next we will explain how a unique consistent prescription for evaluating the infinite $s$-summation is obtained. We then show that the infinite lattice chiral model is really the limit of a sequence of gauge-variant models with increasing extra direction. Finally we calculate the consistent anomaly in the three dimensional case and find the correct result.

The boundary fermion version of the chiral model is formally defined by replacing the finite range of $s$ by a semi-infinite one $s \geq 1$. With this change in the range of $s$, the action is still given by eqs. (1-3). The two dimensional source current is

$$J_\mu(x) = \sum_{s=1}^{\infty} j_\mu(x, s).$$

(22)

The continuity equation for the $2n+1$ dimensional current $j_\alpha(x, s)$ takes the standard form for all $s \geq 2$, while for $s = 1$ it is given by the first row of eq. (7). If one is not careful about the limiting procedure implicit on the r.h.s. of eq. (22), one find that the formal divergence of the source current is zero. It is our purpose to examine this question more carefully below.

The Feynman rules undergo the following modifications. In each vertex, the $s$-summation now extends over all $s \geq 1$. Also, the fermion propagator has a different form [8], which exhibits the presence of a single chiral fermion near the single boundary at $s = 1$.

In more detail, if we restrict our attention to any finite interval $1 \leq s, s' \leq N$ and take $N' \gg 1$, than the semi-infinite lattice propagator $G_F^\infty$ is well approximated by the propagator $G_F^{N+N'}$ of a lattice with a finite extra direction $1 \leq s \leq N + N'$. In fact, there is a uniform bound

$$|G_F^\infty(s, s'; p) - G_F^{N+N'}(s, s'; p)| \leq c_1 e^{-c_2 N'|p|}, \quad 1 \leq s, s' \leq N,$$

(23)

for some calculable positive constants $c_1$ and $c_2$. In eq. (23) we have explicitly shown the momentum dependence of the bound. For exponentially small momenta the bound is useless. This momentum range can however be disregarded because, for an appropriate definition of the continuum limit, it can be made exponentially small also compared to any physical scale of the interacting theory.
On the other hand, the two propagators differ radically for \( p \ll 1 \) (in lattice units) and \( s, s' \lesssim N + N' \). In this range, \( G_F^{N+N'} \) propagates a massless state whereas \( G_F^{\infty} \) does not. Another obvious difference is that \( G_F^{N+N'} \) is not even defined outside the range \( 1 \leq s, s' \leq N + N' \). This, however, is less important because \( G_F^{\infty} \) does not propagate any light states for large \( s \). Any spurious effect of the existence of infinitely heavy fields can be cancelled by the introduction of appropriate Pauli-Villars fields [4,11].

The infinite \( N \) limit is therefore singular. This singularity is reflected in the ambiguities of the naive Feynman rules of the infinite lattice model, to which we now turn.

The example we will consider is the vacuum polarization in the three dimensional model. The naive Feynman rules of the infinite lattice model give rise to the following expression

\[
\Pi_{\mu\nu} = \int_{-\pi}^\pi \frac{d^2k}{(2\pi)^2} \sum_{s_1,s_2=1}^\infty I_{\mu\nu}, \quad \text{(formally)} \tag{24}
\]

\[
I_{\mu\nu} = \Lambda_\mu(-2k-p) G_F^\infty(s_1,s_2;k) \Lambda_\nu(2k+p) G_F^\infty(s_2,s_1;k+p). \tag{25}
\]

Unfortunately, the r.h.s. of eq. (24) is ambiguous. In order to exhibit this ambiguity let us consider the IR regularized quantity

\[
\Pi_{\mu\nu}^{N_1,N_2} = \int_{-\pi}^\pi \frac{d^2k}{(2\pi)^2} \sum_{s_1=1}^{N_1} \sum_{s_2=1}^{N_2} I_{\mu\nu}. \tag{26}
\]

Notice that \( \Pi_{\mu\nu}^{N_1,N_2} \) is the correlator

\[
\Pi_{\mu\nu}^{N_1,N_2} = \langle J_\mu^{N_1} J_\nu^{N_2} \rangle, \tag{27}
\]

where the regularized current is

\[
J_\mu^{N} = \sum_{s=1}^{N} j_\mu(x,s). \tag{28}
\]

Here \( N \) stands for \( N_1 \) or \( N_2 \).

The regularized current satisfies a divergence equation analogous to eq. (11a). In momentum space we find

\[
\Pi_{\nu}^{N_1,N_2} \equiv 2 \sum_{\mu} \sin(p_\mu/2) \Pi_{\mu\nu}^{N_1,N_2} \tag{29a}
\]

\[
= -\int_{-\pi}^\pi \frac{d^2k}{(2\pi)^2} \sum_{s_2=1}^{N_2} \left( I^+_{\nu} - I^-_{\nu} \right), \tag{29b}
\]

\[
I^\pm_{\nu} = \pm \frac{1}{2} \text{tr} (1 \pm \sigma_3) G_F(N_1 + \eta_{\pm,s_2};k) \Lambda_{\nu}(2k+p) G_F(s_2,N_1 + \eta_{\pm};k+p). \tag{30}
\]

Notice that eq. (30) coincides with eq. (20) of the vector-like model for \( n = 1 \).
Let us now consider several limiting procedures and check the transversality of the resulting expression. We define

\[ \Pi_\nu = \lim_{N_1 \to \infty} \Pi_{\nu}^{N_1,N_2}, \]
\[ \Pi'_\nu = \lim_{N_2 \to \infty} \lim_{N_1 \to \infty} \Pi_{\nu}^{N_1,N_2}, \]
\[ \Pi''_\nu = \lim_{N_1 \to \infty} \lim_{N_2 \to \infty} \Pi_{\nu}^{N_1,N_2}. \]

Every one of these limiting procedures gives a different result. Using the exponential damping of the propagator in the \( s \)-direction, the reader can easily verify that \( \Pi'_\nu = 0 \). \( \Pi''_\nu \) is recognized as the divergence of the right-handed current in the vector-like theory (see eq. (19) and the following discussion), which is equal to one half of the axial anomaly.

The correct limiting procedure is the one in eq. (31a). The limiting procedures (31b) and (31c) are unacceptable because they involve a regularized vacuum polarization which does not obey Bose symmetry. A glance at eq. (27) reveals that, except for \( N_1 = N_2 \), \( \Pi_{\nu}^{N_1,N_2} \) describes the coupling of two photons to two different source currents, which is obviously incorrect. Insisting that the regularized vacuum polarization describe the coupling of two photons to the same source current implies the limiting procedure (31a). More generally, the correct prescription for evaluating any Feynman graph is the following. One first performs all the \( s \)-summations over a finite range \( 1 \leq s_1, s_2, \ldots \leq N \). Only at the end the common largest value \( N \) is sent to infinity.

We will soon extract the two dimensional consistent anomaly from \( \Pi_\nu \). But first we show that the Feynman rules of the infinite lattice chiral model, as carefully defined above, identify it with the limit of a sequence of gauge-variant lattice models with a finite extra direction.

The gauge-variant model consists of \( N \) layers of charged fermions and \( N' \) layers of neutral fermions. Gauge invariance is broken in the coupling between the last layer of charged fermions and the first layer of neutral fermions. The action is (compare eq. (1))

\[
S = \sum_{x,y} \sum_{s=1}^{N} \bar{\psi}(x,s) D_{x,y}^\parallel(U) \psi(y,s) \\
+ \sum_{x,y} \sum_{s=N+1}^{N+N'} \bar{\psi}(x,s) D_{x,y}^\parallel(1) \psi(y,s) \\
+ \sum_{x} \sum_{s,s'=1}^{N+N'} \bar{\psi}(x,s) D_{s,s'}^\perp \psi(x,s'),
\]

(32)

The tree level spectrum consists of a charged right-handed fermion at the \( s = 1 \) boundary, and a neutral left-handed fermion at the boundary \( s = N + N' \). The source
current is given by eq. (28). The $s$-summation in every vertex extends only over the limited range $1 \leq s_1, s_2, \ldots \leq N$. At this stage, the similarity to the correct Feynman rules of the infinite lattice model is already clear. To make it precise, we make use of the bound (23). This bound is applicable because we need the propagator only in the above limited range. As a result, in the limit $N' \to \infty$ one can replace the finite lattice propagator $G_{F}^{N+N'}$ by $G_{F}^{\infty}$, thus recovering the regularized form of the diagrams of the infinite lattice chiral model.

We comment that, unlike the ambiguous situation described earlier, the precise order of the limits $N \to \infty$ and $N' \to \infty$ is unimportant. The reason is that for any large value of $N'$ one avoids the dangerous region where the finite and infinite lattice propagators disagree, and this is true regardless of the value of $N$. In particular, one can choose $N = N'$. For this choice, the correlator of any number of source currents in the gauge-variant model is equal to the correlator of the same number of right-handed currents in the vector-like model provided the external gauge field is switched off.

Finally, let us calculate the consistent anomaly of the three dimensional model. Recall that at the linearized level, the consistent anomaly is given by an expression similar to the r.h.s. of eq. (17), but with the constant $C_A$ replaced with another constant $C_{cons}$. We thus have to evaluate the relevant part of $\Pi_\nu$ using eqs. (29), (30) and (31a) in the limit of large $N$. We try to apply the same reasoning as in the calculation of the axial anomaly in the vector-like model. In eq. (29b), only the region $s_2 \approx N$ contributes to the $s_2$-summation (we let $N_1 = N_2 = N$), and so we can replace the finite lattice propagator by the translation invariant propagator $G_0$. For the same reason, the lower limit of the $s_2$-summation can be extended to $-\infty$. However, the upper limit lies in the center of the region $s_2 \approx N$ and it cannot be altered. Making the substitution $s = s_2 - N$ we thus obtain

$$
\Pi_\nu = - \int_{-\pi}^{\pi} \frac{d^2 k}{(2\pi)^2} \sum_{s = -\infty}^{0} \left( \hat{I}_\nu^+ - \hat{I}_\nu^- \right),
$$

$$
\hat{I}_\nu^\pm = \pm \frac{1}{2} \text{tr} \left( 1 \pm \sigma_3 \right) G_0(\eta^\pm, s; k) \Lambda_\nu(2k + p) G_0(s, \eta^\pm; k + p).
$$

Up to this point, everything could be trivially generalized to an arbitrary dimension. Thus, we are able to express the consistent anomaly using translationally invariant propagators, but summations that extend only over a semi-infinite extra direction. One can go to momentum representation also in the extra direction by making use of the Fourier transform of the lattice $\theta$-function. However, we have not found this representation very useful in the computation of the diagram.

In the three dimensional case we proceed as follows. By making a $180^\circ$ rotation
in the $(2,3)$ plane we find $\Pi_\nu = \tilde{\Pi}_\nu$, where

$$\tilde{\Pi}_\nu = - \int_{-\pi}^{\pi} \frac{d^2 k}{(2\pi)^2} \sum_{s=1}^{\infty} (\hat{I}_\nu^+ - \hat{I}_\nu^-).$$

(35)

Notice that the difference between $\Pi_\nu$ and $\tilde{\Pi}_\nu$ is in the range of the $s$-summation. We comment that for the symmetric choice $N = N'$, this rotation is essentially the discrete symmetry eq. (4). However, the limiting expressions eqs. (33) and (35) are related through this rotation also for an asymmetric limit where $N \neq N'$.

We complete the computation by invoking the relation to the vector-like model. Specifically, $\Pi_\nu$ and $\tilde{\Pi}_\nu$ correspond to the correlators $\langle \Delta_\mu J^R_\mu J^R_\nu \rangle$ and $\langle \Delta_\mu J^R_\mu J^L_\nu \rangle$ of the vector-like model respectively. Using $J^V_\mu = J^R_\mu + J^L_\mu$, the equality of the above two correlators and eq. (11), we find that in three dimensions $C_{\text{cons}} = C_A/4$. Using eq. (21) for $n = 1$ we find $C_{\text{cons}} = i/4\pi$ as expected.

We comment that for $n > 1$, the above discrete symmetry is not sufficient to reduce the calculation of the consistent anomaly to that of the axial anomaly, (which, as we have seen, can be related to the calculation of the Chern-Simons action). This is somewhat disappointing because the Chern-Simons action in $2n + 1$ dimensions is closely related to the consistent anomaly in $2n$ dimensions. However, we stress that the difficulties we have encountered are of a technical nature, and we have every reason to believe that similar results should exist for any $n$.

4. Discussion

A model of chiral defect fermions on a lattice with a (semi)-infinite extra direction is formally both gauge invariant and chiral. An arbitrary chiral spectrum should give rise to anomalies. On the other hand, a manifestly gauge invariant lattice action implies exact current conservation. The conflict between these two properties raises the suspicion that the infinite lattice chiral model is not well defined. Indeed, we have explicitly demonstrated the ambiguities of its Feynman rules.

In a carefully defined model, either gauge invariance or the chiral character of the spectrum are lost. We have shown above that the carefully defined (semi)-infinite lattice model is actually the limit of gauge-variant models with a finite extra direction. These models consist of a charged and a neutral $2n + 1$ dimensional slabs, each with a finite extra direction. If the two slabs were decoupled, there would be one massless mode of each chirality on every slab. The direct coupling between the two slabs breaks gauge invariance and, at the same time, eliminates one charged and one neutral field from the massless spectrum.
In an attempt to construct chiral gauge theories on the lattice, giving up exact
gauge invariance at the lattice scale is a reasonable price, if one can show that the
spectrum remains chiral and gauge invariance is recovered in the continuum limit
for anomaly free theories. There are strong indications, however, that this scenario
is not realized in the gauge-variant model we have constructed. When the gauge
fields are promoted to full-fledged dynamical variables, what is likely to happen is
dynamical restoration of gauge invariance, accompanied by the appearance of new
massless charged fields which make the theory vector-like.

The action of an arbitrary lattice model with a dynamical gauge field can be
written as $S = S_{inv} + S_{non}$, where $S_{inv}$ and $S_{non}$ are the gauge invariant and gauge
variant parts of the action respectively. (We are allowing for the possibility that one
of these terms is zero). Using gauge invariance of the lattice measure, the partition
function can be rewritten as a functional integral over the original fields plus a fixed
radius Higgs field $\phi$ taking values in the gauge group [14]. The action becomes
$S = S_{inv} + S_{non}(\phi)$, where every field in $S_{non}(\phi)$ undergoes the gauge transformation
defined by the field $\phi$.

If the above procedure is applied to the action (32), the quadratic gauge variant
terms turn into gauge invariant Yukawa interactions. A model of this type has been
recently investigated in ref. [9]. If the Yukawa coupling is switched off, the charged
and neutral fermions decouple and the model is vector-like. This model could have
a chiral continuum limit if it had a strongly interacting symmetric (PMS) phase, in
which the charged and neutral massless modes that couple through the Yukawa inter-
action become a massive Dirac field, while the other charged field remains massless.
Although the results of ref. [9] are not conclusive, the cumulative evidence points
to the absence of such a phase, and hence that the vector-like spectrum persists
throughout the entire phase diagram.

More generally, we have recently derived a No-Go theorem [10] that extends the
Nielsen-Ninomiya theorem to interacting lattice theories with arbitrary couplings.
Apart from unitarity and gauge invariance, which are common to both theorems, the
key assumption of ref. [10] is that the decay rate of a certain two point function (the
retarded anti-commutator) at space-like separations satisfies a mild bound. When
this bound is satisfied, one can show that the inverse retarded propagator, considered
as an effective hamiltonian, is sufficiently differentiable to apply the Nielsen-Ninomiya
theorem.

Lattice models lacking reflection positivity are in general not unitary at the lattice
scale, yet they can have a perfectly consistent continuum limit. An examination of
the details of our theorem reveals that this kind of non-unitarity is irrelevant, because
in the proof one uses only unitarity of the low energy spectrum. In particular, the inclusion of heavy Pauli-Villars fields in a model of chiral defect fermions should not affect the validity of the No-Go theorem since they automatically decouple from low energy physics.

What makes the application of our No-Go theorem to a given lattice model less straightforward than in the case of the Nielsen-Ninomiya theorem, is the need to verify the validity of the above mentioned bound. Various arguments indicate that many (and perhaps all) short range lattice models should satisfy this bound. At the moment, however, direct evidence is limited to the free field case and to strongly interacting models amenable to certain analytical methods.

In conclusion, our purpose in the present paper was to show that the infinite lattice model of chiral defect fermions, when properly defined, is in fact a special case of the gauge invariant model proposed in ref. [3]. Further study along the lines of refs. [9] and [10] is needed before a definite conclusion can be drawn about the feasibility of a chiral continuum limit in this model.

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