Aspherical completions and rationally inert elements

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Abstract

Let $X$ be a connected space. An element $[f] \in \pi_n(X)$ is called rationally inert if $\pi_*(X) \otimes \mathbb{Q} \to \pi_*(X \cup_f D^{n+1}) \otimes \mathbb{Q}$ is surjective. We extend the results of [16] and prove in particular that if $X \cup_f D^{n+1}$ is a Poincaré duality complex and the algebra $H(X)$ requires at least two generators then $[f] \in \pi_n(X)$ is rationally inert. On the other hand, if $X$ is rationally a wedge of at least two spheres and $f$ is rationally non trivial, then $f$ is rationally inert. Finally if $f$ is rationally inert then the rational homotopy of the homotopy fibre of the injection $X \to X \cup_f D^{n+1}$ is the completion of a free Lie algebra.

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In [2] and [16] the authors define and establish the properties of rationally inert elements in the homotopy groups of simply connected CW complexes $X$ of finite type: $[f] \in \pi_n(X)$ is rationally inert if

$$\pi_*(X) \otimes \mathbb{Q} \to \pi_*(X \cup_f D^{n+1}) \otimes \mathbb{Q}$$

is surjective. Our objective here is to use Sullivan completions $X \to X_{\mathbb{Q}}$ to extend the definitions to $[f] \in \pi_n(X)$, $n \geq 1$, where $X$ is any connected CW complex, and then to extend the principal results of [16] to this more general setting and establish several applications. For details about Sullivan completions the reader is referred to [14].

Inverse homotopy equivalences between the homotopy categories of connected CW complexes, $X$, and connected simplicial sets, $S$, are provided by $X \mapsto \text{Sing } X$, the singular simplices in $X$, and by $S \mapsto |S|$, its Milnor realization. These identify a map $X \to |S|$ with a morphism $\text{Sing } X \to S$. For simplicity we denote both by $X \to S$,

and refer to either a connected CW complex or a connected simplicial set simply as a connected space.

Additionally, for simplicity, we adopt the

Convention. Our base field is $\mathbb{Q}$. When the meaning is clear, we will suppress the differentials from the notation. For simplicity, we will also write

$$(-)^\vee := \text{Hom}(-, \mathbb{Q}), \quad \text{and } H(-) := H^*(-; \mathbb{Q}),$$

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for singular cohomology. Moreover, where there is no ambiguity we suppress the differential from the notation for a complex, and write $A$ instead of $(A, d)$.

As detailed in §1 below, a Sullivan completion $X_Q$ appears naturally as a simplicial set. Sullivan models and Sullivan completions are reviewed in §1. In particular, if $X$ is simply connected and of finite type, then [8, Theorem 15.11] its Sullivan completion induces an isomorphism $\pi_\ast(X) \otimes \mathbb{Q} \cong \pi_\ast(X_Q)$. Thus we extend the definition of rationally inert elements as follows:

**Definition.** If $X$ is a connected space then $[f] \in \pi_n(X)$, some $n \geq 1$, is rationally inert if the inclusion $i : X \to X \cup_f D^{n+1}$ induces a surjection,

$$\pi_\ast(i_Q) : \pi_\ast(X_Q) \to \pi_\ast((X \cup_f D^{n+1})_Q).$$

This condition can be characterized in terms of the homotopy type of the fibre $F(f)$ of $i_Q$ (Theorem 1). Applications are then provided in Theorems 2, 3 and 4. To state Theorem 1 we need the

**Definition.** A connected space $Y$ is rationally wedge-like if for some non-void linearly ordered set $S = \{\sigma\}$, and integers $n_\sigma > 0$, there is a homotopy equivalence,

$$Y \xrightarrow{\cong} \lim_{\sigma_1 < \cdots < \sigma_r} (S^{n_{\sigma_1}} \vee \cdots \vee S^{n_{\sigma_r}})_Q,$$

where the inverse system is defined by the projections of $S^{n_{\sigma_1}} \vee \cdots \vee S^{n_{\sigma_r}}$ on the sub wedges.

**Remark:** Note that in general $(X \vee Y)_Q$ is different from $X_Q \vee Y_Q$!

**Theorem 1.** For any connected space $X$, a homotopy class, $[f] \in \pi_n(X)$, some $n \geq 1$, is rationally inert if and only if the homotopy fibre $F(f)$ of $X_Q \to (X \cup_f D^{n+1})_Q$ is rationally wedge-like.

Applications are then provided in Theorems 2, 3 and 4. To state Theorem 1 we need the

Theorem 3.3 in [16] is a special case of Theorem 1 since in that case the homotopy fibre of $X \to X \cup_f D^{n+1}$ is rationally a wedge of spheres if and only if its rationalization is rationally wedge-like.

An example of rationally inert elements is provided by the following theorem, established for simply connected spaces in ([16, Theorem 5.1]).

**Theorem 2.** If $X \cup_f D^{n+1}$ is a Poincaré duality complex and the algebra $H(X)$ requires at least two generators then $[f] \in \pi_n(X)$ is rationally inert.
(14, §4), for any minimal Sullivan algebra, $\wedge Z, \pi_*\Omega(\wedge Z)$ is naturally a graded Lie algebra, complete with respect to a natural filtration. Its Lie bracket is given explicitly in terms of the Whitehead products in $\pi_*\langle \wedge Z \rangle$. We generalize ([16, Theorem 3.3 (I)]) in

**Theorem 3.** Suppose $X$ is a connected space and $[f] \in \pi_n(X)$, some $n \geq 1$, is rationally inert. Then $\pi_*(\Omega F(f))$ is the completion of a free sub Lie algebra, freely generated by a subspace $S \cong H_*(\Omega(X \cup_f D^{n+1})$.

A general question asks what conditions on a group $G$ imply that $(BG)_\mathbb{Q}$ is aspherical; i.e., a $K(\pi,1)$. This is true when $G$ is a finitely generated free group, when $G$ is the fundamental group of a Riemann surface or when $G$ is a right-angled Artin group ([19], [7]). We consider here the one-relator groups, $\pi_1(X \cup_f D^2)$, obtained by adding a 2-cell to a wedge of circles along a continuous map $f : S^1 \to X$. The well known Lyndon theorem ([18], [20], [6]) states that if $f$ is not a proper power, then $X \cup_f D^2$ is aspherical. In general it may happen that a connected space $X$ is aspherical, but $X_{\mathbb{Q}}$ is not. However, the spaces considered by Lyndon remain aspherical when rationalized:

**Theorem 4.** If $X$ is a wedge of at least two circles then any non zero $[f] \in \pi_1(X)$ is rationally inert; equivalently, $(X \cup_f D^2)_\mathbb{Q}$ is aspherical.

**Remark.** Note that even if $f$ is a proper power, where Lyndon’s theorem does not apply, it is true that $(X \cup_f D^2)_\mathbb{Q}$ is aspherical.

Finally recall a famous unsolved problem of JHC Whitehead [21]: is a subcomplex of an aspherical two-dimensional CW complex aspherical? As observed by Anick [1] it is sufficient to consider the case that both subcomplexes share the same 1-skeleton and base point. The problem then reduces to the question: If $X$ is a finite 2-dimensional connected CW complex and $X \cup (\coprod_{k=1}^p D^2_k)$ is aspherical, is $X$ aspherical?

In [1] Anick provides a positive answer to an analogous question for simply connected rational spaces. Here we have a positive answer for Sullivan completions of connected spaces.

**Theorem 5.** If $X$ is a connected space and $(X \cup \coprod_{k=1}^p D^2_k)_\mathbb{Q}$ is aspherical, then $X_{\mathbb{Q}}$ is aspherical.

1 Sullivan models and Sullivan completions

We review briefly the basic facts and notation from Sullivan’s theory. For details the reader is referred to [14]. A $\Lambda$-algebra is a commutative differential graded algebra (cdga) of the form $(\wedge V, d)$, where $V = V^{\geq 0}$ is a graded vector space and $\wedge V$ is the free graded commutative algebra generated by $V$. Moreover the differential is required to satisfy the Sullivan condition: $V = \cup_{n \geq 0} V(n)$, where

$$V(0) = V \cap \ker d \quad \text{and} \quad V(n+1) = V \cap d^{-1}(\wedge V(n)).$$

Here $V$ is a generating vector space for $\wedge V$. If $V = V^{\geq 1}$ then $V$ is a Sullivan algebra.
Moreover, $\wedge V = \oplus_{p \geq 0} \wedge^p V$, where $\wedge^p V$ denotes the linear span of the monomials in $V$ of length $p$; $p$ is called the wedge degree. In particular, a $\Lambda$-algebra is minimal if $d : V \to \wedge^2 V$ and quadratic if $d : V \to \wedge^2 V$. Thus associated with a minimal $\Lambda$-algebra $(\wedge V, d)$ is the quadratic $\Lambda$-algebra $(\langle \wedge V, d_1 \rangle)$ defined by: $d_1 v$ is the component of $dv$ in $\wedge^2 V$.

Note that if $V = V^g$ then the inclusion of a subspace $W \subset \wedge^2 V$ extends to an isomorphism $\wedge W \cong \wedge V$ if and only if $W \oplus \wedge^2 V = \wedge V$. In this case $\wedge W$ satisfies the same condition as $\wedge V$: the definition of a Sullivan algebra does not depend on the choice of generating vector space. Observe as well that if $V = V^g$ then the natural map

$$\wedge V \cong \prod_p \wedge^p V$$

is an isomorphism.

With each connected space $Y$ is associated a cdga $A_{PL}(Y)$ and a unique isomorphism class of minimal Sullivan algebras $(\wedge V, d)$ characterized by the existence of a quasi-isomorphism $(\langle \wedge V, d \rangle) \cong A_{PL}(Y)$. By definition $(\wedge V, d)$ is the minimal Sullivan model of $Y$. Among their properties are the natural isomorphisms $H(\langle \wedge V, d \rangle) \cong H(Y)$ of graded algebras. Moreover, any map, $f : X \to Y$ determines a "homotopy class" of morphisms, $\varphi : \wedge V \to \wedge W$, from the minimal Sullivan model of $Y$ to that of $X$; $\varphi$ is a Sullivan representative of $f$.

On the other hand, the construction of Sullivan completions is accomplished by a functor associating to a $\Lambda$-algebra, $\wedge W$, a simplicial set $\langle \wedge W \rangle$, with the property that $< >$ converts direct limits to inverse limits. In particular, if $\wedge W$ is a minimal Sullivan model of a connected space $X$ then this determines a based homotopy class of maps

$$X \to X_\mathbb{Q} := \langle \wedge W \rangle,$$

the Sullivan completion of $X$. In particular, if $\varphi : \wedge V \to \wedge W$ is a Sullivan representative of $f : X \to Y$ then

$$f_\mathbb{Q} = \langle \varphi \rangle : \langle \wedge W \rangle \to \langle \wedge V \rangle.$$

Moreover, ([9, Theorem 1.3]) for any minimal Sullivan algebra, $\wedge W$, there is a natural bijection $\pi_* (\langle \wedge W \rangle) \cong \text{Hom}(\wedge^1 W/ \wedge^2 W)$, and the isomorphism $W \cong \wedge^1 W/ \wedge^2 W$ then induces a bijection

$$\pi_* (\langle \wedge W \rangle) \cong W^\vee.$$

Therefore, for any morphism $\varphi : \wedge V \to \wedge W$ of minimal Sullivan algebras, it follows that $\pi_* (\langle \varphi \rangle)$ is surjective if and only if $\varphi : \wedge^1 V/ \wedge^2 V \to \wedge^1 W/ \wedge^2 W$ is injective, or equivalently, if the generating vector space $W \subset \wedge W$ can be chosen so that $\varphi : V \to W$ is the inclusion of a subspace. In this case

$$\pi_* (\langle \varphi \rangle) = \varphi^\vee : W^\vee \to V^\vee.$$

Now a general morphism $\varphi : \wedge V \to \wedge W$ of Sullivan algebras factors ([9, Theorem 3.1]) as

$$\wedge V \xrightarrow{\eta} \wedge V \otimes \wedge Z \xrightarrow{\gamma} \wedge W,$$

in which (i) $\eta(v) = v \otimes 1$, (ii) $\gamma$ is a quasi-isomorphism, (iii) $Z = Z^0$, (iv) $Z = \cup_n Z(n)$ satisfying

$$Z(0) = Z \cap d^{-1}(\wedge V) \quad \text{and} \quad Z(n+1) = Z \cap d^{-1}(\wedge V \otimes \wedge Z(n)),$$

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and (v) the quotient \( (\wedge Z, \overline{d}) = \mathbb{Q} \otimes_{\wedge V} (\wedge V \otimes \wedge Z) \) is a minimal \( \Lambda \)-algebra. Here \( \wedge V \otimes \wedge Z \) is a minimal \( \Lambda \)-extension of \( \wedge V \).

**Remark.** If \( \pi_* \langle \varphi \rangle \) is surjective we take \( \eta = \varphi \) to be an inclusion \( V \to W \) and \( \wedge W = \wedge V \otimes \wedge Z \).

In particular, with each minimal Sullivan algebra \( (\wedge V, d) \) is associated a unique isomorphism class of \( \Lambda \)-extensions, \( (\wedge V \otimes \wedge U, d) \), its acyclic closures. These are characterized by the following two properties: (i) the augmentation \( \wedge V \to \mathbb{Q} \) extends to a quasi-isomorphism \( \wedge V \otimes \wedge U \cong \mathbb{Q} \) with \( U \to 0 \), and (ii) the quotient differential in \( \wedge U = \mathbb{Q} \otimes_{\wedge V} (\wedge V \otimes \wedge U) \) is zero.

Finally, a minimal Sullivan algebra \( \wedge V \) determines the graded homotopy Lie algebra \( L_V = (L_V)_{\geq 0} \) given by

\[
s(L_V)_p = \text{Hom}(V^{p+1})
\]

and

\[
< v, s[x, y] > = (-1)^{1+\text{deg} y} < d_1v, sx, sy >, \quad v \in V, x, y \in L_V.
\]

(Here \( s \) is the degree 1 suspension isomorphism.) Thus

\[
s(L_V) = \pi_* (\wedge V).
\]

### 2 Rationally wedge-like spaces

**Lemma 1.** The following two conditions on a minimal Sullivan algebra, \( \wedge Z \), are equivalent:

(i) The generating vector space \( Z \subset \wedge Z \) can be chosen so that

\[
Z \cap \ker d \xrightarrow{\cong} H^{\geq 1}(\wedge Z).
\]

(ii) \( \wedge Z \) is the minimal Sullivan model of a cdga \( A = \mathbb{Q} \oplus A_{\geq 1} \) in which the differential and products in \( A_{\geq 1} \) are zero.

*If these hold then \( Z \) can be chosen so that \( Z \cap \ker d \xrightarrow{\cong} H^{\geq 1}(\wedge Z) \) and \( (\wedge Z, d) \) is quadratic.*

**proof:** If (i) holds let \( A \) be the quotient of \( \wedge Z \) by \( \wedge \geq 2 Z \) and by a direct summand of the image of \( \ker d \) in \( Z \). If (ii) holds set \( V_0 = A_{\geq 1} \) and define a quadratic Sullivan algebra \( \wedge V \) by setting \( V(k) = \oplus_{0 \leq k} V_k \), with \( d : V_{k+1} \to \wedge^2 V(k) \cap \ker d \) inducing an isomorphism in homology. Then \( (\wedge V, d) \) has zero homology in wedge degree 2, and it follows that \( \wedge V \) has zero homology in wedge degrees \( \geq 2 \). Hence \( \wedge V \) is a quadratic Sullivan model for \( A \). Thus \( \wedge V \cong \wedge Z \), and so \( Z \) can be chosen so that \( d : Z \to \wedge^2 Z \). Thus the final assertion is part of ([14, Proposition 6]).

**Example:** Finite wedges of spheres: \( S = S^{\sigma_1} \vee \cdots \vee S^{\sigma_k} \).

The quasi-isomorphism \( A_{PL}(S) \to \oplus_{\mathbb{Q}} A_{PL}(S^{\sigma_i}) \cong \oplus_{\mathbb{Q}} H(S^{\sigma_i}) \) identifies the minimal Sullivan model of \( S \) as a minimal Sullivan algebra \( \wedge Z \) satisfying the conditions of Lemma 1. Here \( Z \cap \ker d \) has a basis \( z_1, \ldots, z_k \) representing orientation classes of \( S^{\sigma_1}, \ldots, S^{\sigma_k} \).
Now choose elements $x_i$ in the homotopy Lie algebra $L_S$ of $S$ so that $\langle z_i, sx_j \rangle = \delta_{ij}$. The $x_j$ then freely generate a free sub Lie algebra $E \subset L_S$. In fact, the rescaling argument in [9, p.230] generalizes to reduce to the case $S = S_{\geq 2}^n$, in which case the result is established in [8, §23, Example 2]. Moreover, it follows from [9, Chap. 2] that

$$L_S = \lim_{n} L_S/L_S^n$$

where $L_S^n$ is the ideal spanned by the iterated commutators in $L_S$ of length $n$. According to [9, Chapter 2], the $x_{\sigma_i}$ map to a basis of $L_S/L_S^2$ and hence the inclusion $E \rightarrow L_S$ induces isomorphisms $E/E_n \cong L_S/L_S^n$.

**Proposition 1.** A connected space $F$ is rationally wedge-like if and only if it has the form $F = \langle \wedge Z \rangle$, where $\wedge Z$ satisfies the equivalent conditions of Lemma 1.

**proof:** Suppose first that $F = \langle \wedge Z \rangle$, where $\wedge Z$ satisfies the conditions of Lemma 1, and pick a linearly ordered basis of $Z \cap \ker d$. Then each finite subset $z_{\sigma_1} < \cdots < z_{\sigma_k}$ determines an inclusion

$$\wedge Z(\sigma_1, \ldots, \sigma_k) \hookrightarrow \wedge Z$$

of quadratic Sullivan algebras with $Z(\sigma_1, \ldots, \sigma_k) \subset Z$, and for which $\{z_{\sigma_i}\}$ is a basis of $H^{\geq 1}(\wedge Z(\sigma_1, \ldots, \sigma_p))$, and $\wedge Z(\sigma_1, \ldots, \sigma_k)$ is a Sullivan model for $S^{\sigma_1 \vee \cdots \vee S^{\sigma_k}}$. Moreover, the inclusions $\wedge Z(\sigma_1, \ldots, \sigma_k) \hookrightarrow \wedge Z(\sigma_1, \ldots, \sigma_k)$ are Sullivan representatives for the projections $S^{\sigma_1 \vee \cdots \vee S^{\sigma_k}} \rightarrow S^{\sigma_1} \vee \cdots \vee S^{\sigma_k}$.

Now

$$\wedge Z = \lim_{\sigma_1 \cdots < \sigma_k} \wedge Z(\sigma_1, \ldots, \sigma_k)$$

and so

$$\langle \wedge Z \rangle = \lim_{\sigma_1 \cdots < \sigma_k} \langle \wedge Z(\sigma_1, \ldots, \sigma_k) \rangle = \lim_{\sigma_1 \cdots < \sigma_k} (S^{\sigma_1} \vee \cdots \vee S^{\sigma_k})_Q.$$

In the reverse direction, suppose $F$ is rationally wedge like, so that

$$F = \lim_{\sigma_1 \cdots < \sigma_k} (S^{n_{\sigma_1}} \vee \cdots \vee S^{n_{\sigma_k}})_Q.$$

Then let $\wedge Z$ be a Sullivan algebra satisfying the conditions of Lemma 1 in which $Z \cap \ker d$ has a basis $\{z_{\sigma_i}\}$ of degrees $n_{\sigma_i}$. Thus any subset $\sigma_1 < \cdots < \sigma_k$ determines a sub Sullivan algebra $Z(\sigma_1 \ldots \sigma_k)$ by the requirement that $Z(\sigma_1 \ldots \sigma_k) = \bigoplus_n Z(\sigma_1 \ldots \sigma_k; n)$ in which

$$Z(\sigma_1 \ldots \sigma_k; 0) = \bigoplus_n Qz_{\sigma_i}$$

and

$$Z(\sigma_1 \ldots \sigma_k; n+1) = Z \cap d^{-1}(\wedge Z(\sigma_1 \ldots \sigma_k; (n))).$$

This gives as above that

$$\langle \wedge Z \rangle = \lim_{\sigma_1 \cdots < \sigma_k} \langle \wedge Z(\sigma_1 \ldots \sigma_k) \rangle = \lim_{\sigma_1 \cdots < \sigma_k} (S^{n_{\sigma_1}} \vee \cdots \vee S^{n_{\sigma_k}})_Q = F.$$
Corollary 1. If \( X = \vee_{\sigma} S^{n_{\sigma}} \) is a wedge of spheres, then \( X_\mathbb{Q} \) is rationally wedge-like. If all the spheres are circles then \( X_\mathbb{Q} \) is aspherical.

Corollary 2. If \( \langle \wedge Z \rangle \) is rationally wedge-like and \( \dim H^{\geq 1}(\wedge Z) > 1 \), then the sum of the solvable ideals in \( L_Z \) is zero.

proof. It follows from Lemma 1 that \( \text{cat}(\wedge Z) = 1 \), and so from \([11]\), Sdepth \( L_Z < \infty \). Now \([12]\) Theorem 1] asserts that the sum, \( \text{rad} L_Z \), of the solvable ideals in \( L_Z \) is finite dimensional, and that \( L_Z \) acts nilpotently in \( \text{rad} L_Z \). In particular, if \( \text{rad} L_Z \neq 0 \) then the center of \( L_Z \) is non-zero. Let \( x \in L_Z \) be an element in the center.

Since \( \langle \wedge Z \rangle \) is rationally wedge-like, \( \wedge Z = \varprojlim Z(\sigma_1, \ldots, \sigma_k) \) where \( \wedge Z(\sigma_1, \ldots, \sigma_k) \) is the minimal Sullivan model of a wedge of \( k \) spheres, and \( S \) has by hypothesis at least two elements. Then \( L_Z = \varprojlim L_Z(\sigma_1, \ldots, \sigma_k) \), and the maps \( L_Z \to L_Z(\sigma_1, \ldots, \sigma_k) \) are surjective. Thus if \( x \neq 0 \) it maps to a non-zero element in some \( L_Z(\sigma_1, \ldots, \sigma_k) \) with \( k > 1 \). This would contradict the Example above.

Remark. Rationally wedge-like spaces provide examples of minimal Sullivan algebras \( \wedge Z \) for which \( \langle \wedge Z \rangle \) is not the Sullivan completion of a space. For example, suppose \( Z = Z^3 \) has a countably infinite basis, so that \( \pi_*(\wedge Z) = \pi_3(\wedge Z) = (Z^3)^\vee \).

Thus for any minimal Sullivan algebra \( \wedge V \), the condition \( \langle \wedge V \rangle = \langle \wedge Z \rangle \) would imply that \( V = V^3 \) and \( (V^3)^\vee \cong (Z^3)^\vee \). But if \( \wedge V \) were the minimal model of a space \( X \) then we would have \( V^3 \cong H^3(X) = H_3(X)^\vee \) and so either \( \dim V^3 < \infty \) or \( \text{card} (V^3) \geq \text{card} \mathbb{R} \).

In the second case, \( \text{card} ((V^3)^\vee) > \text{card} \mathbb{R} \) and so \( (V^3)^\vee \) and \( (Z^3)^\vee \) are not isomorphic.

Proposition 2. Suppose \( X \) and \( Y \) are connected spaces, one of which has rational homology of finite type. Then

(i) The homotopy fibre, \( F \), of the natural map

\[ i_\mathbb{Q} : (X \vee Y)_\mathbb{Q} \to (X \times Y)_\mathbb{Q} \]

is rationally wedge-like.

(ii) If \( X_\mathbb{Q} \) and \( Y_\mathbb{Q} \) are aspherical then so are \( F \) and \( (X \vee Y)_\mathbb{Q} \).

This result is analogous to the fact that the usual fibre of the injection \( X \vee Y \to X \times Y \) is the join of \( \Omega X \) and \( \Omega Y \) and thus a suspension. (But note that \( (X \vee Y)_\mathbb{Q} \) may be different from \( X_\mathbb{Q} \vee Y_\mathbb{Q} \).)

Proposition 2 follows easily from a result about Sullivan algebras (Proposition 3, below). For this, consider minimal Sullivan algebras, \( \wedge W \) and \( \wedge Q \). The natural surjection \( \wedge W \otimes \wedge Q \to \wedge W \times Q \wedge Q \) is surjective in homology, and so extends to a minimal Sullivan model

\[ \varphi : \wedge T := \wedge W \otimes \wedge Q \otimes \wedge R \to \wedge W \times Q \wedge Q. \]

Filtering by wedge degree then yields a morphism

\[ \varphi_1 : (\wedge T, d_1) \to (\wedge W, d_1) \times Q (\wedge Q, d_1) \]

between the associated bigraded cdga’s. (Here \( (\wedge -, d_1) \) is the associated quadratic Sullivan algebra.)

Proposition 3. With the hypotheses and notation above,
(i) \( \langle \wedge R \rangle \) is rationally wedge-like.

(ii) \( \varphi_1 \) is a quasi-isomorphism.

proof: (i) Let \( \wedge W \otimes \wedge U_W \) and \( \wedge Q \otimes \wedge U_Q \) denote the respective acyclic closures. Then \( \wedge R \) is quasiferomorphic to

\[
\wedge T \otimes \wedge W \otimes \wedge Q \otimes \wedge U_Q \simeq A := (\wedge W \oplus \wedge Q) \otimes \wedge U_W \otimes \wedge U_Q.
\]

Dividing \( A \) by the ideal generated by \( W \) yields the short exact sequence

\[
0 \to \wedge^{\geq 1} W \otimes \wedge U_W \otimes \wedge U_Q \to A \to \wedge Q \otimes \wedge U_W \otimes \wedge U_Q \to 0.
\]

Decompose the differential in \( \wedge W \otimes \wedge U_W \) in the form \( d = d_1 + d' \) with \( d_1(W) \subset \wedge^2 W \), \( d_1(U_W) \subset W \otimes \wedge U_W \), \( d'(W) \subset \wedge^{\geq 3} W \) and \( d'(U_W) \subset \wedge^{\geq 2} W \otimes \wedge U_W \). Then \( d_1 \) is a differential and \( (\wedge W \otimes \wedge U_W, d_1) \) is the acyclic closure of \( (\wedge W, d_1) \). Choose a direct summand, \( S \), of \( d_1(\wedge^{\geq 1} U_W) \) in \( W \otimes \wedge U_W \). Then \( I = (\wedge^{\geq 2} W \otimes \wedge U_W) \oplus S \) is acyclic for the differential \( d_1 \) and therefore also for the differential \( d \). Thus \( J = I \otimes \wedge U_Q \) is an acyclic ideal in \( A \) and \( A \xrightarrow{\sim} A/J \).

Now consider the short exact sequence

\[
0 \to (\wedge^{\geq 1} W \otimes \wedge U_W \otimes \wedge U_Q)/J \to A/J \to \wedge Q \otimes \wedge U_W \otimes \wedge U_Q \to 0.
\]

The inclusion of \( \wedge U_W \) in the right hand term is a quasi-isomorphism. This yields a quasi-isomorphism

\[
d_1(\wedge^{\geq 1} U_W) \otimes \wedge^{\geq 1} U_Q \simeq A/J \simeq A.
\]

Since the differential and the multiplication in \( d_1(\wedge^{\geq 1} U_W) \otimes \wedge^{\geq 1} U_Q \) are zero, it follows from Proposition 1 that \( \langle \wedge R \rangle \) is rationally wedge-like.

(ii) The surjection \( (\wedge W \otimes \wedge Q, d_1) \to (\wedge W \times_Q \wedge Q, d_1) \) extends to a quasi-isomorphism

\[
\hat{\varphi} : \wedge \hat{T} := (\wedge W \otimes \wedge Q \otimes \wedge \hat{R}, \delta) \to (\wedge W \times_Q \wedge Q, d_1)
\]

from a minimal Sullivan algebra. We first show that \( \hat{R} \) can be chosen so that \( (\wedge \hat{T}, \delta) \) is quadratic. Then we extend \( \delta \) to a differential \( \hat{d} = \sum_{i \geq 1} \hat{d}_i \) in which \( \hat{d}_1 = \delta \) and

\[
\hat{d}_i : \wedge \hat{T} \to \wedge^{i+1} \hat{T} \quad \text{and} \quad \hat{\varphi} \circ \hat{d} = d \circ \varphi.
\]

It is automatic that \( (\wedge \hat{T}, \hat{d}) \) will be a minimal Sullivan algebra. Moreover, filtering by wedge degree shows that \( \hat{\varphi} \) is a quasi-isomorphism and so \( \wedge \hat{T} \) is a minimal Sullivan model for \( \wedge W \times_Q \wedge Q \). In particular this identifies \( \hat{T} \) with \( T \), \( R \) with \( \hat{R} \) and \( \hat{\varphi} \) with \( \varphi \), thereby establishing (ii).

To accomplish the first step, define \( d_1 : U_W \to W \otimes \wedge U_W \) and \( d_1 : U_Q \to Q \otimes \wedge U_Q \) as in (i). Assign \( \wedge W \) and \( \wedge Q \) wedge degree as a second degree and assign \( U_W \) and \( U_Q \) second degree 0. Then \( (\wedge W \otimes \wedge U_W, d_1) \) and \( (\wedge Q \otimes \wedge U_Q, d_1) \) are the respective acyclic closures of \( (\wedge W, d_1) \) and \( (\wedge Q, d_1) \), and \( d_1 \) increases the second degree by 1. Now \( \hat{\varphi} \) and \( \hat{T} \) may be constructed so that \( \hat{R} \) is equipped with a second gradation for which \( \delta \) increases the second degree by one and \( \hat{\varphi} \) is bihomogeneous of degree zero.
The argument in the proof of (i) now yields a sequence of bihomogeneous quasi-isomorphisms connecting

\[ \mathbb{Q} \oplus (d_1(\wedge^+W) \otimes \wedge^+U) \simeq \wedge R. \]

Thus \( H^{\geq 1}(\wedge R) \) is concentrated in second degree 1. Therefore \( \wedge R \) satisfies condition (i) of Proposition 1, and it follows that we may choose \( \hat{R} \) so that the quotient cdga \( \wedge R \) is quadratic and \( H^{\geq 1}(\wedge R) \) embeds in \( \hat{R} \). This implies that \( \hat{R} \) is concentrated in second degree 1 and that \( \delta : \hat{R} \to \wedge^2(W \oplus Q \oplus T) \).

In particular, \((\wedge \hat{T}, \delta)\) is a quadratic Sullivan algebra.

The construction of \( \hat{d} \) proceeds as follows. Write the differential in \( \wedge W \times \mathbb{Q} \wedge Q \) as \( d = \sum_{r \geq 1} d_r \) in which \( d_r \) is a derivation raising wedge degree by \( r + 1 \). Thus for each \( r \), \( \sum_{i+j=r} d_id_j = 0 \). Now we construct by induction a sequence of derivations \( \hat{d}_1 = \delta, \ldots, \hat{d}_r \ldots \), in \( \wedge \hat{T} \) in which \( \hat{d}_r \) increases the wedge degree by \( r + 1 \), and

\[ \sum_{i+j=r} \hat{d}_i \hat{d}_j = 0 \quad \text{and} \quad \hat{\varphi} \hat{d}_i = d_i \hat{\varphi}. \]

Thus, in view of (i), \( \hat{d} := \sum \hat{d}_i \) will define a differential in \( \wedge \hat{T} \), \((\wedge \hat{T}, \hat{d})\) will be a Sullivan algebra, and \( \varphi : (\wedge \hat{T}, \hat{d}) \to (\wedge W \times \mathbb{Q} \wedge Q, d) \) will be a cdga morphism. Filtering by wedge degree shows that \( \hat{\varphi} \) is a quasi-isomorphism.

It remains to construct the \( \hat{d}_i, i \geq 2 \). For this, set \( \hat{T}(k) = W \oplus Q \oplus \hat{R}^{<k} \). Since \((\wedge \hat{T}, \delta)\) is a Sullivan algebra it follows that each \( \hat{R}^k \) is the union of an increasing family of subspaces \( F^p(\hat{R}^k) \) such that

\[ \delta : F^0(\hat{R}^k) \to \wedge \hat{T}(k-1) \quad \text{and} \quad \delta : F^{p+1}(\hat{R}^k) \to \wedge \hat{T}(k-1) \otimes F^p(\hat{R}^k). \]

Set \( \hat{d}_1 = \delta \) and assume by induction that \( \hat{d}_1, \ldots, \hat{d}_r \) have been constructed, and that \( \hat{d}_{r+1} \) has been constructed in \( \hat{R}^{<r} \otimes F^p(\hat{R}^k) \).

Let \( y_i \) be a basis for a direct summand of \( F^p(\hat{R}^k) \) in \( F^{p+1}(\hat{R}^k) \). Then

\[ \hat{\varphi}(\hat{d}_{r+1} \hat{d}_1 y_i) = \hat{d}_{r+1} \hat{d}_1 \psi y_i = -\hat{d}_1 \hat{d}_{r+1} \varphi y_i - \sum_{j=2}^r (\hat{d}_j \hat{d}_{r+2-j}) \varphi y_i = -\hat{d}_1 \hat{d}_{r+1} \varphi y_i - \sum_{j=2}^r \hat{\varphi}(\hat{d}_j \hat{d}_{r+2-j}) y_i. \]

It follows that

\[ \hat{d}_1 \hat{\varphi}(\hat{d}_{r+1} + \sum_{j=2}^r \hat{d}_j \hat{d}_{r+2-j}) y_i = 0. \]

Since \( \hat{\varphi} \) is a surjective quasi-isomorphism with respect to \( \hat{d}_1 \) and \( d_1 \), this implies that

\[ (\hat{d}_{r+1} \hat{d}_1 + \sum_{j=2}^r \hat{d}_j \hat{d}_{r+2-j}) y_i = \hat{d}_1 \Phi_i \]
with \( \hat{\varphi}_i = -\hat{d}_{r+1} \hat{y}_i \). Extend \( \hat{d}_{r+1} \) to \( F^{p+1} (\hat{R}^k) \) by setting \( \hat{d}_{r+1} = -\Phi_i \).

\[ \square \]

**proof of Proposition 2:** (i) Let \( \wedge W \) and \( \wedge Q \) be the minimal Sullivan models of \( X \) and \( Y \). A Sullivan representative of the inclusion \( i : X \vee Y \to X \times Y \) is then the inclusion

\[ \wedge W \otimes \wedge Q \to \wedge T := \wedge W \otimes \wedge Q \otimes \wedge R. \]

It follows that \( i_Q \) is the surjection

\[ \langle \wedge W \otimes \wedge Q \rangle \to \langle \wedge W \otimes \wedge Q \otimes \wedge R \rangle. \]

But this surjection is a fibration ([S Proposition 17.9]) with fibre \( \langle \wedge R \rangle \), which is a rationally wedge-like by Proposition 3.

(ii) When \( X_Q \) and \( Y_Q \) are aspherical, then \( U_W \) and \( U_Q \) are concentrated in degree 0 and \( W \) is concentrated in degree 1. This shows that \( F \) is aspherical. Since one of \( X, Y \) has rational homology of finite type, \( (X \times Y)_Q = X_Q \times Y_Q \) is aspherical. We deduce then from the homotopy sequence of the fibration \( F \to (X \vee Y)_Q \to (X \times Y)_Q \) that \( (X \vee Y)_Q \) is also aspherical.

\[ \square \]

3 Cell attachments and Theorem 1

Before undertaking the proof of Theorem 1 we set up the basic framework that translates the topology of a cell attachment to Sullivan’s theory, and establish two preliminary Propositions.

Suppose \( f : S^n \to X \) is the map of Theorem 1, and denote by \( (\wedge W, d) \) the Sullivan minimal model of \( X \). A Sullivan representative of \( f \) is a morphism from \( \wedge W \) to the minimal model of \( S^n \). Composing with the quasi-isomorphism from that model to \( H(S^n) \) gives a morphism \( \psi : \wedge W \to H(S^n) \). Now define a linear map of degree \(-n\),

\[ \varepsilon : \wedge W \to \mathbb{Q}, \]

by setting \( \varepsilon(1) = 0 \) and \( \psi(\Phi) = \varepsilon(\Phi) \cdot [S^n] \), \( \Phi \in \wedge \geq 1 W \), where \([S^n]\) denotes an orientation class in \( S^n \). In particular, \( \varepsilon \circ d = 0 \) and \( \varepsilon(\wedge \geq 2 W) = 0 \).

Now define a cdga \( (\wedge W \oplus \mathbb{Q} a, D) \) as follows: \( \deg a = n + 1, \ a^2 = a \cdot \wedge^+ W = 0, \) and

\[ Da = 0 \quad \text{and} \quad D\Phi = d\Phi + \varepsilon(\Phi)a, \quad \Phi \in \wedge W. \]

By [S (13)b and (13)d], division by \( a \) yields the commutative diagram,

\[ \begin{array}{ccc}
\mathbb{Q} a & \xrightarrow{\tau} & (\wedge W \oplus \mathbb{Q} a, D) \\
& & \xrightarrow{\lambda} (\wedge V, d) \\
& & \downarrow \approx \\
& & (\wedge W, d)
\end{array} \]

in which \( (\wedge V, d) \) is a minimal Sullivan model for \( X \cup_f D^{n+1} \), and \( \lambda \) is a Sullivan representative for the inclusion \( i : X \to X \cup_f D^{n+1} \). In particular, \( i_Q : X_Q \to (X \cup_f D^{n+1})_Q \) is identified with \( \langle \lambda \rangle : \langle \wedge W \rangle \to \langle \wedge V \rangle \).
As described in §1, \( \lambda \) factors as

\[
\begin{align*}
\lambda : (\land V, d) & \xrightarrow{\eta} (\land V \otimes \land Z, d) \xrightarrow{\gamma} (\land W, d),
\end{align*}
\]

in which \( \land V \otimes \land Z \) is a \( \Lambda \)-extension of \( \land V \), \( \gamma \) is a quasi-isomorphism, and the quotient

\[
(\land Z, \delta) := Q \otimes_{\land V} (\land V \otimes \land Z, d)
\]
is a minimal \( \Lambda \)-algebra. Since \( H^1(i) \) is injective, it follows that \( \lambda : V^1 \to W^1 \) is injective. Therefore \( Z = Z^{\geq 1} \) and \( \land Z \) is a minimal Sullivan algebra.

Further, because \( \gamma \) is a quasi-isomorphism of Sullivan algebras, \( \langle \gamma \rangle \) is a homotopy equivalence, which (up to homotopy) identifies \( \langle \eta \rangle \) with \( \langle \lambda \rangle \). But \( \langle \eta \rangle \) is the projection of a Serre fibration with fibre \( \langle \land Z \rangle \). Thus \( \langle \land Z \rangle \), the homotopy fibre of \( \langle \lambda \rangle \), and the homotopy fibre \( F(f) \) of \( i_Q \), all have the same homotopy type:

\[
\langle \land Z \rangle \simeq F(f). \tag{4}
\]

On the other hand, we have

**Proposition 4.** With the hypotheses and notation of (3), let \( \land V \otimes \land U \) be the acyclic closure of \( \land V \). Then there is a degree 1 isomorphism,

\[
H^{\geq 1}(\land Z, \delta) \xrightarrow{\simeq} Qa \otimes \land U,
\]

and \( H^{\geq 1}(\land Z, \delta) \cdot H^{\geq 1}(\land Z, \delta) = 0 \).

**proof:** First observe that in diagram (3), \( \tau \Phi = \lambda \Phi + \alpha(\Phi) a \). Thus \( \tau \) must coincide with \( \lambda \) in \( \land^{\geq 2} V \), and that also \( D \circ \tau = D \circ \lambda \). Thus for \( \Phi \in \land V \),

\[
d(\lambda \Phi) + \varepsilon(\lambda \Phi)a = D(\lambda \Phi) = D(\tau \Phi) = \tau d \Phi = \lambda(d \Phi) = d(\lambda \Phi).
\]

Hence

\[
\varepsilon \circ \lambda = 0. \tag{5}
\]

Now let \( \land V \otimes \land U \) be the acyclic closure of \( \land V \). Apply \( - \otimes_{\land V} \land V \otimes \land U \) to diagram (3) to obtain a short exact sequence of complexes,

\[
0 \to Qa \otimes \land U \to (\land W \oplus Qa) \otimes_{\land V} (\land V \otimes \land U) \to \land W \otimes \land U \to 0, \tag{6}
\]
in which the differential in \( Qa \otimes \land U \) is zero and the homology of the central complex is \( Q1 \) in positive degrees. It follows that \( H^0(\land W \otimes \land U) = Q1 \) and that the connecting homomorphism is an isomorphism of degree 1. By (5), \( \varepsilon \) vanishes on \( \land V \), and hence \( (\varepsilon \otimes \text{id}) \circ (\lambda \otimes \text{id}) = 0 \) in \( \land V \otimes \land U \). Now a straightforward calculation shows that the connecting homomorphism is given explicitly by

\[
H(\varepsilon \otimes \text{id}) : H^{\geq 1}(\land W \otimes \land U) \xrightarrow{\simeq} Qa \otimes \land U. \tag{7}
\]

On the other hand, applying \( - \otimes_{\land V} \land V \otimes \land U \) to the quasi-isomorphism \( \gamma \) yields quasi-isomorphisms

\[
(\land Z, \delta) \xrightarrow{\simeq} \land V \otimes \land U \otimes \land Z \xrightarrow{\gamma} \land W \otimes \land U,
\]
so that
we have a degree 1 isomorphism $H^{\ge 1}(\wedge Z, \overline{d}) \xrightarrow{\cong} \mathbb{Q}a \otimes U$. It is immediate that $H(\varepsilon \otimes id)$ vanishes on products, which gives the second assertion.

\[ \square \]

Theorem 1 is now contained in

**Theorem 1'.** Suppose $X$ is a connected CW complex, and $[f] \in \pi_n(X)$, some $n \geq 1$. Then in the factorization (3)

\[ \lambda : (\wedge V, d) \xrightarrow{\eta} (\wedge V \otimes \wedge Z, d) \xrightarrow{\gamma} (\wedge W, d), \]

the following conditions are equivalent:

(i) $[f]$ is rationally inert.

(ii) The generating space $Z$ can be chosen so that

\[ \overline{d} : Z \rightarrow \wedge^2 Z \quad \text{and} \quad H(\wedge Z) = \mathbb{Q} \oplus (Z \cap \ker \overline{d}). \]

(iii) The homotopy fibre of $F(f)$ of $i_Q : X_Q \rightarrow (X \cup_f D^{n+1})_Q$ is rationally wedge-like.

**proof:** (i) $\Rightarrow$ (ii): Since $\langle \lambda \rangle$ is identified with $i_Q$, $[f] \in \pi_n(X)$ is rationally inert if and only if the generating space $W$ can be chosen so that $\lambda$ restricts to an inclusion $V \rightarrow W$. In this case, $\wedge W$ decomposes as a Sullivan extension $\wedge V \rightarrow \wedge V \otimes \wedge Z = \wedge W$. Thus we may take $\eta = \lambda$ and $\gamma = id_{\wedge W}$. Note that if $\wedge V \otimes \wedge U$ is the acyclic closure of $\wedge V$, then the augmentation $\wedge V \otimes \wedge U \xrightarrow{\cong} \mathbb{Q}$ defines a quasi-isomorphism $\wedge W \otimes \wedge U = \wedge V \otimes \wedge Z \otimes \wedge U \xrightarrow{\cong} \wedge Z$.

If $\dim H^{\ge 1}(\wedge Z) = 1$, then necessarily $\wedge Z$ is the minimal Sullivan model of a sphere $S^k$ and $\langle \wedge Z \rangle = S^k_Q$. If $\dim H^{\ge 1}(\wedge Z) \geq 2$, let $\sigma : \wedge Z \rightarrow \wedge W \otimes \wedge U$ be a right inverse to the quasi-isomorphism $\wedge W \otimes \wedge U \xrightarrow{\cong} \wedge Z$ above. Since $\wedge V \otimes \wedge Z$ is a minimal Sullivan algebra, it will follow that

\[ \sigma : Z \rightarrow \wedge^{\ge 1} W \otimes \wedge U. \]

But this will imply that $\sigma : \wedge^{\ge 2} Z \rightarrow \wedge^{\ge 2} W \otimes \wedge U$. Now a simple calculation shows that the connecting homomorphism vanishes on any $\wedge$-cycle in $\wedge^{\ge 2} Z$. Since the connecting homomorphism is an isomorphism it follows that division by $\wedge^{\ge 2} Z$ induces an injection $H^{\ge 1}(\wedge Z) \rightarrow Z$, and (ii) follows from Lemma 1.

To complete this direction of the proof we need to establish (iii). For this write $Z = \cup_k Z(k)$ in which $Z(0) = Z \cap \ker \overline{d}$ and $Z(k+1) = Z \cap \overline{d}^{-1}(\wedge Z(k))$. Assuming by induction that $\sigma : Z(k) \rightarrow \wedge^{\ge 1} W \otimes \wedge U$ we obtain that for $z \in Z(k+1)$, $d\sigma(z) = \sigma(\overline{d} z) \in \wedge^{\ge 2} W \otimes \wedge U$. Now let $\Phi$ be the component of $\sigma(z)$ in $1 \otimes \wedge U$. Since $\wedge W$ is minimal it follows that $d : \wedge^{\ge 1} W \otimes \wedge U \rightarrow \wedge^{\ge 2} W \otimes \wedge U$. But if $\Phi \neq 0$ then $d(1 \otimes \Phi)$ has a non-zero component in $V \otimes \wedge U$. Therefore $\Phi = 0$ and (iii) follows by induction on $k$.

(ii) $\Rightarrow$ (iii): Since $\langle \wedge Z \rangle \simeq F(f)$ it follows from Proposition 1 that $F(f)$ is rationally wedge-like.
(iii) ⇒ (i): First suppose that $F(f)$ is a rational sphere $S^k_\mathbb{Q}$. Then $\wedge Z$ is the minimal Sullivan model of a sphere, and so $\dim Z \cap \ker d = 1$. Thus it follows from Proposition 4 that $U = 0 = V$. Since $\wedge V$ is the minimal Sullivan model for $X \cup_f D^{n+1}$ this implies that $\pi_*(X \cup_f D^{n+1})_\mathbb{Q} = 0$ and $[f]$ is rationally inert.

Otherwise $F(f)$ is the inverse limit of rational wedges of at least two spheres. If $[f]$ is not inert then in the sequence

$$
\pi_*(\Omega(X \cup_f D^{n+1})_\mathbb{Q}) \to \pi_{*+1}(F(f)) \to \pi_{*+1}(X_\mathbb{Q})
$$

the image of $\pi_*(\Omega(X \cup_f D^{n+1})_\mathbb{Q})$ contains a non-zero class $\omega \in \pi_{*+1}(F(f))$. Because $\Omega(X \cup_f D^{n+1})_\mathbb{Q}$ acts on $F(f)$, it follows that the Whitehead product $\omega \bullet \beta$ of $\omega$ and any $\beta \in \pi_*(F(f))$ is zero.

Then, because $\pi_*(F(f)) = \lim \pi_*(S^{\sigma_1} \vee \cdots \vee S^{\sigma_k})$ it follows that for some $r \geq 2$, the image $\overline{\omega}$ of $\omega$ in some $\pi_*(S^{\sigma_1} \vee \cdots \vee S^{\sigma_k})_\mathbb{Q}$ is non-zero, and that

$$
\overline{\omega} \bullet \beta = 0, \quad \beta \in \pi_*(S^{\sigma_1} \vee \cdots \vee S^{\sigma_k})_\mathbb{Q}.
$$

As observed in [2], $\pi_*(S^{\sigma_1} \vee \cdots \vee S^{\sigma_r})_\mathbb{Q}$ is the suspension of its homotopy Lie algebra $L$, and it follows from [9, Chapter 2] that $\overline{\omega}$ determines a non-zero element in the center of $L$. But the center of $L$ is zero, and therefore $[f]$ is rationally inert. \hfill \Box

4 Poincaré duality complexes

We say a CW complex $Y = X \cup_f D^{n+1}$ is a rational Poincaré duality complex if $H(Y)$ is a Poincaré duality algebra and the top class is in the image of $H(Y,X)$. In this case it follows that $H^{\leq n}(X) \cong H(X)$. Poincaré duality complexes are rational Poincaré duality complexes, and so Theorem 2 follows from

**Theorem 2’.** If $Y = X \cup_f D^{n+1}$ is a rational Poincaré duality complex and the algebra $H(Y)$ requires at least two generators, then $[f] \in \pi_n(X)$ is rationally inert.

Before undertaking the proof we establish some notation. Let $\wedge V$ be the minimal Sullivan model of $Y$, and let $S$ be a direct summand in $(\wedge V)^{n+1}$ of $(\wedge V)^{n+1} \cap \ker d$. Then division by $S$ and by $(\wedge V)^{n+1}$ defines a surjective quasi-isomorphism $\wedge V \cong A$, and

$$
A^{n+1} = A^{n+1} \cap \Im d \oplus \mathbb{Q} \omega,
$$

where $\omega$ is a cycle representing the top cohomology class of $Y$. As shown in ([16, §5]), a cdga model of the inclusion $X \hookrightarrow Y$ is then provided by the inclusion

$$
(j : (A,d) \to (A \otimes \mathbb{Q}t,d),
$$

where $\deg t = n$, $t \cdot A^+ = 0$, and $dt = \omega$.

Thus if $A \otimes \wedge U$ is the acyclic closure of $A$, then a cdga model for the homotopy fibre of $j$ is given by

$$
(A \otimes \mathbb{Q}t) \otimes_A (A \otimes \wedge U) = (A \otimes \mathbb{Q}t) \otimes \wedge U.
$$
Thus from the short exact sequence
\[ 0 \to A \otimes \wedge U \to (A \oplus \mathbb{Q}t) \otimes \wedge U \to \mathbb{Q}t \otimes \wedge U \to 0 \]
we deduce that
\[ H^{\geq 1}((A \oplus \mathbb{Q}t) \otimes \wedge U) \xrightarrow{\partial} \mathbb{Q}t \otimes \wedge U \]
is an isomorphism of graded vector spaces.

For the proof of Theorem 2 we first eliminate two special cases. First if \( V^1 = 0 \) the argument of ([16, §5]) shows that \((A \oplus \mathbb{Q}t) \otimes \wedge U\) is a cdga model of a wedge of spheres, and so \([f]\) is rationally inert. (Note that in [16] it is assumed that \( X \) is simply connected; however the proof of this assertion relies only on the fact that \( V^1 = 0 \).) Secondly, if \( n = 1 \) then \( X \simeq \mathbb{Q}S^1 \vee \cdots \vee \mathbb{Q}S^2 \) and so \( Y \) is rationally equivalent to an oriented Riemann surface. In this case Theorem 2' is established in [13].

Thus to prove Theorem 2' we may assume that \( n \geq 2 \) and that \( A^1 \) contains a non-zero cycle \( x \). Since \( H(A) \) is a Poincaré duality algebra there is a cycle \( w \in A^n \) such that \( wx = \omega \). The first step for the proof is then

**Lemma 2.** With the hypotheses and notation above, \( A^{n+1} \otimes \wedge U \subset d(A^n \otimes \wedge U) \).

**proof:** Choose \( \pi \in U^0 \) so that \( d\pi = x \). Since \( \wedge V \) is a minimal Sullivan algebra, \( V \) is the union of an increasing sequence of subspaces \( V(0) \subset \cdots \subset V(q) \subset \cdots \) in which \( V(0) = \mathbb{Q}x \) and \( d : V(q+1) \to \wedge V(q) \). It follows that \( U \) is the union of an increasing sequence of subspaces \( U(0) \subset \cdots \subset U(q) \subset \cdots \) in which \( U(0) = \mathbb{Q}\pi \) and
\[ d : U(q+1) \to A^{\geq 1} \otimes \wedge U(q). \]

We show by induction on \( q \) that
\[ A^{n+1} \otimes \wedge U(q) \subset d(A^n \otimes \wedge U(q)) \tag{10} \]

First note that any \( z \in A^{n+1} \) has the form \( z = dy + \lambda wx \), some \( \lambda \in \mathbb{Q} \). Thus
\[ z \otimes 1 = d(y \otimes 1) \pm d(\lambda w \pi) \in d(A^n \otimes \wedge U(0)). \]

Then for \( r \geq 1 \),
\[ z \otimes \pi^r = d(y \otimes \pi^r) \pm \frac{1}{r+1} w \otimes \pi^{r+1}) + ry \otimes \pi^{r-1}. \]

It follows by induction on \( r \) that \( A^{n+1} \otimes \wedge U(q) \subset d(A^n \otimes \wedge U(q)) \).

Now fix a direct summand, \( T \), of \( U(q) \) in \( U(q+1) \), and assume by induction that for some \( s \),
\[ A^{n+1} \otimes \wedge U(q) \otimes \wedge^{\leq s} T \subset d(A^n \otimes \wedge U(q) \otimes \wedge^{\leq s} T). \]

Then write \( \Phi \in A^{n+1} \otimes \wedge U(q) \otimes \wedge^{\leq s+1} T \) as \( \Phi = \sum \Phi_i \otimes \Psi_i \) with \( \Phi_i \in A^{n+1} \otimes \wedge U(q) \) and \( \Psi_i \in \wedge^{\leq s+1} T \). By the hypothesis \( \Phi_i = d\Omega_i \) with \( \Omega_i \in A^n \otimes \wedge U(q) \). Therefore
\[ \sum \Phi_i \otimes \Psi_i = d(\sum \Omega_i \otimes \Psi_i) \pm \sum \Omega_i \wedge d\Psi_i. \]

The first term is in \( d(A^n \otimes \wedge U(q) \otimes \wedge^{\leq s+1} T) \). On the other hand, \( d\Psi_i \in A^{\geq 1} \otimes \wedge U(q) \otimes \wedge^{\leq s} T \) and so the second term is in \( A^{n+1} \otimes \wedge U(q) \otimes \wedge^{\leq s} T \). By hypothesis, the second term is contained in \( d(A^n \otimes \wedge U(q) \otimes \wedge^{\leq s} T) \). This closes the induction. \( \square \)
**proof of Theorem 2':** Let \( \Phi \in \wedge U \). Then

\[
t - (-1)^n w \xi \in (A \oplus \mathbb{Q}t) \otimes \wedge U
\]

is a cycle, and

\[
d((t - (-1)^n w \xi)\Phi) = -w \xi \ d\Phi \in A^{n+1} \otimes \wedge U.
\]

By Lemma 2, \( w \xi d\Phi = d\Psi \) for some \( \Psi \in A^n \otimes \wedge U \). Thus \( (t - (-1)^n)w \xi)\Phi + \Psi \) is a cycle projecting to \( t \otimes \Phi \) in \( \mathbb{Q}t \otimes \wedge U \). Then such cycles map to a basis of \( \mathbb{Q}t \otimes \wedge U \). But because \( n \geq 2 \), \( 2n > n + 1 \) and so the product of any two of those cycles is zero. Therefore this defines a cdga quasi-isomorphism from the cohomology of a wedge of spheres to \( (A \oplus \mathbb{Q}t) \otimes \wedge U \). Lemma 1 and Theorem 1' together then imply that \( [f] \) is rationally inert. 

\( \square \)

5 **The structure of \( L_Z \) and Theorem 3**

Any minimal Sullivan algebra \( \wedge V \) equips \( L_V \) with a natural additional structure ([14 §3]), defined as follows. Associated with \( \wedge V \) is the set, directed by inclusion, of the finite dimensional subspaces \( V_\alpha \subset V \) for which \( \wedge V_\alpha \) is preserved by \( d \). For convenience we denote this set by \( J_V = \{ \alpha \} \). In particular,

\[
L_V = \varprojlim \alpha \in J_V L_\alpha, \quad L_\alpha \text{ the homotopy Lie algebra of } \wedge V_\alpha.
\]

That structure permits the explicit description of the Whitehead products in \( \pi_\ast \langle \wedge V \rangle \) in terms of the Lie brackets in \( L_V \) ([14, Formula (11)]).

Moreover, for any augmented graded algebra, \( A \), the **classical completion** is defined by \( \hat{A} = \varprojlim_n A/I^n \), \( I^n \) denoting the \( n \)th power of the augmentation ideal. The Sullivan completion of \( UL_V \) is then the inverse limit,

\[
UL_V = \varprojlim \alpha \in J_V UL_\alpha.
\]

Further, by ([10, Proposition 3.3]), there are natural isomorphisms \( \hat{H}_\ast(\Omega \langle \wedge V_\alpha \rangle; \mathbb{Q}) \xrightarrow{\cong} \hat{UL}_\alpha \). Passing to inverse limits then yields the isomorphism of the **Sullivan completions**, \( \hat{H}_\ast(\Omega \langle \wedge V \rangle; \mathbb{Q}) \xrightarrow{\cong} UL_V \). \hspace{1cm} (11)

Similarly, the **Sullivan central series** is the filtration of \( L_V \) given by

\[
L_V^{(r)} = \varprojlim \alpha \in J_V L_\alpha^{(r)},
\]

where \( L_\alpha^{(r)} \) is the ideal spanned by iterated commutators of length \( r \). It satisfies ([14 §5])

\[
L_V/L_V^{(r)} = \varprojlim \alpha \in J_V L_\alpha/L_\alpha^{(r)} \quad \text{and} \quad L_V = \varprojlim_r L_V/L_V^{(r)}.
\]

In the case that \( \langle \wedge V \rangle \) is the homotopy fibre of \( i_Q : X_Q \to (X \cup_f D^{n+1})_Q \) when \( [f] \in \pi_n(X) \) is rationally inert, this additional structure has the striking properties provided in Theorem 3' below.

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Suppose next that $\wedge W = \wedge V \otimes \wedge Z$ is the decomposition of a minimal Sullivan algebra determined by an inclusion $\wedge V \to \wedge W$ with $V \subset W$, and denote $Q \otimes_{\wedge V} \wedge W = (\wedge Z, \overline{d})$. Then the short exact sequence $V \to W \to Z$ dualizes to the short exact sequence

$$0 \leftarrow L_V \leftarrow L_W \leftarrow L_Z \leftarrow 0$$

of Lie algebra morphisms, which identifies $L_Z$ as an ideal in $L_W$. The holonomy representation $\overline{\vartheta}$ of $L_V$ in $H(\wedge Z)$, ([9, Chapter 4]), then extends ([14, §7]) to a holonomy representation of $UL_V$ in $H(\wedge Z)$.

On the other hand, the right adjoint representation of $L_W$ in $L_Z$ extends to the right adjoint representation of $UL_W$ in $L_Z$, which further factors to give a right representation of $UL_V$ in $L_Z/L_Z^{(2)}$ ([14, Proposition 7]).

Now suppose $(\wedge Z, \overline{d})$ is a quadratic Sullivan algebra. The surjection $\wedge^{\geq 1} Z \to Z$ with kernel $\wedge^{\geq 2} Z$ induces a surjection $H^{\geq 1}(\wedge Z) \to Z \cap \ker \overline{d}$ of $UL_V$-modules. This in turn dualizes to an inclusion

$$(Z \cap \ker \overline{d})^\vee \to H^{\geq 1}(\wedge Z)^\vee$$

of right $UL_V$-modules. Moreover, according to ([14, Propositions 6 and 7]) the pairing $Z \times sL_Z \to Q$ induces an isomorphism

$$L_Z/L_Z^{(2)} \cong (Z \cap \ker \overline{d})^\vee$$

of right $UL_V$-modules.

For the rest of this section we fix a map to a connected CW complex,

$$f : S^n \to X,$$

some $n \geq 1$, for which $[f]$ is rationally inert.

As observed in the Remark in §1, a Sullivan representative $\wedge V \to \wedge W$ for the inclusion $X \to X \cup_f D^{n+1}$ has the form

$$\wedge V \to \wedge V \otimes \wedge Z = \wedge W,$$

and as above we denote the quotient differential in $\wedge Z$ by $(\wedge Z, \overline{d})$. It follows from Theorem 1’ that $(\wedge Z, \overline{d})$ is a quadratic Sullivan algebra and that $H^{\geq 1}(\wedge Z, \overline{d}) = Z \cap \ker \overline{d}$.

Now recall from §2 the linear map

$$\varepsilon : \wedge W \to Q$$

of degree $-n$. Since $\varepsilon(V) = 0$, $\varepsilon$ factors to give

$$\hat{\varepsilon} \in (Z^n)^\vee = (L_Z)_{n-1}.$$

Thus, in view of ([11], Theorem 3 is contained in

**Theorem 3’**. With the hypotheses and notation above, let $\overline{\tau} \in L_Z/L_Z^{(2)}$ denote the image of $\hat{\varepsilon}$. Then
(i) Both $L_Z/L_Z^{(2)}$ and $H^{\geq 1}(\wedge Z)^{\vee}$ are free $UL_V$-modules, respectively generated by $\varepsilon$ and $\hat{\varepsilon}$.

(ii) The map $\Phi \mapsto \varepsilon \cdot \Phi, \Phi \in UL_W,$ is a surjection

\[ \tau : UL_W \to L_Z, \]

of $UL_W$-modules.

(iii) Any subspace $S \subset L_Z$ with $S \cong L_Z/L_Z^{(2)}$ freely generates a free sub Lie algebra, $E \subset L_Z$, and

\[ \lim_{\leftarrow} E/E \cap L_Z^{(r)} \cong L_Z. \]

**Remark.** When $X$ is simply connected with finite Betti numbers and $n \geq 2$, then Theorem 3' is established in \cite[Theorem 3.3]{16}.

Before undertaking the proof of Theorem 3’ we establish a preliminary Proposition. For this, denote by $\varepsilon_W : \wedge V \otimes \wedge U \cong Q$ the augmentation in the acyclic closure of $\wedge V$ defined by $\varepsilon_W(U) = 0$. Since the quotient differential in $\wedge U$ is zero, the holonomy representation of $UL_V^{\vee}$ is a representation in $\wedge U$. On the other hand, the holonomy representation of $UL_V$ in $H^{\geq 1}(\wedge Z)$ is a representation in $Z \cap \text{Ker} \cdot d$. Now we strengthen Proposition 4 with

**Proposition 5.** With the hypotheses and notation above, there is a commutative diagram

\[
\begin{array}{ccc}
\wedge U & \xrightarrow{\psi} & Z \cap \text{Ker} \cdot d \\
\varepsilon_W \downarrow & & \downarrow \varepsilon \\
Q & & \varepsilon
\end{array}
\]

in which $\psi$ is an isomorphism of $UL_V$-modules of degree $n + 1$.

**proof.** Implicit in the isomorphism $\wedge W = \wedge V \otimes \wedge Z$ is the choice of a left inverse, $\wedge Z \to \wedge W$, of graded algebras for the surjection $\wedge W \to \wedge Z = Q \otimes_{\wedge V} \wedge W$. This, with $id_{\wedge V}$, defines an isomorphism $\wedge V \otimes \wedge Z \cong \wedge W$, and identifies $id \otimes \varepsilon$ with $\varepsilon$. A simple and standard argument using Proposition 1 shows that this left inverse can be chosen so that the image of $\wedge V \otimes ((Z \cap \text{Ker} \cdot d) \oplus \mathbb{Q})$ is preserved by $d$. It is then immediate that the inclusion of this subcomplex in $(\wedge V \otimes \wedge Z)$ is a quasi-isomorphism. Thus from the commutative diagram (3) we obtain the row exact sequence

\[
0 \to \mathbb{Q} a \to \wedge V \otimes (Z \cap \text{Ker} \cdot d \oplus \mathbb{Q}) \oplus \mathbb{Q} a \to \wedge V \otimes (Z \cap \text{Ker} \cdot d \oplus \mathbb{Q}) \to 0
\]

Since $\varepsilon(\wedge V) = 0$, $\wedge V$ is a subcomplex. Division by this subcomplex yields the row exact sequence of complexes,

\[
0 \to \mathbb{Q} a \to \wedge V \otimes (Z \cap \text{Ker} \cdot d) \oplus \mathbb{Q} a \to \wedge V \otimes (Z \cap \text{Ker} \cdot d) \to 0
\]
in which the middle complex has zero homology. It is immediate that the connecting quasi-isomorphism $\delta$, is then given by

$$\Phi \otimes z \mapsto \begin{cases} \hat{\varepsilon}(z) a & \text{if } \Phi = 1 \\ 0 & \text{if } \Phi \in \land^{\geq 1}V. \end{cases}$$

With a shift of degrees, regard $\varepsilon_W$ as a quasi-isomorphism $\land V \otimes \land U \xrightarrow{\sim} \mathbb{Q}a$, sending $1 \mapsto a$. Then, since $\land V \otimes \land U$ is $\land V$-semifree, in the diagram,

$$\land V \otimes \land U \xrightarrow{\varepsilon_W} \land V \otimes (Z \cap \text{Ker } d) \xrightarrow{\chi} \land V \otimes Z \cap \text{Ker } d \xrightarrow{\psi} Z \cap \text{Ker } d,$$

we may lift $\varepsilon_W$ through $\delta$ to obtain the quasi-isomorphism, $\chi$, of $\land V$-modules. But $\land V \otimes (Z \cap \text{Ker } d)$ is also $\land V$-semifree. Therefore applying $\mathbb{Q} \otimes_{\land V} -$ yields a quasi-isomorphism $\psi : \land U \xrightarrow{\sim} Z \cap \text{Ker } d$.

Now the differentials in $\land U$ and in $Z \cap \text{Ker } d$ are zero, and so $\psi$ is an isomorphism. Moreover, $\mathbb{Q} \otimes_{\land V} -$ converts morphisms between $\land V$-semifree modules to morphisms of $L_V$-modules. In this case $\psi$ is then automatically a morphism of $UL_V$-modules. Finally, it is also immediate that the diagram of the Proposition commutes. □

**proof of Theorem 2 (i).** Here we rely consistently on the notation and conventions of §2.

First, observe that the dual of a $UL_V$-module inherits a right $UL_V$-module structure in the standard way. Thus replacing $\psi$ by $\psi^{-1}$ in the diagram of Proposition 5 and then dualizing yields the commutative diagram

$$\begin{array}{c}
(\land U)^\vee \\
\downarrow \sim \\
\mathbb{Q} \\
\downarrow \\
(Z \cap \text{Ker } d)^\vee
\end{array}$$

in which $1 \in \mathbb{Q}$ maps to $\varepsilon_W \in (\land U)^\vee$ and to $\hat{\varepsilon} \in (Z \cap \text{Ker } d)^\vee$. By ([14, Proposition 8]) $(\land U)^\vee$ is a free right $UL_V$-module, freely generated by $\varepsilon_W$. Since $H^{\geq 1}(\land Z) = (Z \cap \text{Ker } d)$, it follows from ([13]) that $H^{\geq 1}(\land Z)^\vee$ is a free right $UL_V$-module freely generated by $\hat{\varepsilon}$.

(ii) To establish that the map

$$\tau : UL_W \to L_Z$$

is surjective, note that if $\beta \geq \alpha \in J$ and $s \geq r$, then since $Z_\beta \supset Z_\alpha$,

$$L_{Z_\beta}/L_{Z_\beta}^s \to L_{Z_\alpha}/L_{Z_\alpha}^r$$

is a surjection of finite dimensional spaces. Thus it is sufficient to show that the composites

$$UL_W \to L_Z \to L_{Z_\alpha}/L_{Z_\alpha}^{r+1}$$

are all surjective.
When \( r = 1 \), this is immediate from part (i) of the Theorem. Moreover, it follows from the construction of \( \tau \) that its image is an ideal in \( L_Z \). This, together with the surjectivity of (14) when \( r = 1 \) implies via the obvious induction that (14) is surjective for all \( r \).

(iii) To show that \( E \) is free it is sufficient to show that any linearly independent elements \( x_1, \ldots, x_k \in S \) generate a free sub Lie algebra \( F \). But by (ii) the restriction of \( S \) to \( Z \cap \ker d \) is an isomorphism \( sS \xrightarrow{\cong} (Z \cap \ker d)^\vee \). It follows that there are \( z_1, \ldots, z_k \in Z \cap \ker d \) such that

\[
\langle z_i, sx_j \rangle = \delta_{ij}.
\]

Let \( T \) be the linear span of the \( z_i \), so that \( \mathbb{Q} \oplus T \subset \mathbb{Q} \oplus (Z \cap \ker d) \) is a sub cdga, with minimal Sullivan model \( \wedge Z_T \subset \wedge Z \) satisfying \( T = Z_T \cap \ker d \), and with homotopy Lie algebra \( L_T \). The surjection \( L_Z \to L_T \) maps the generating set \( \{x_i\} \) of \( F \) bijectively to a dual basis for \( T \). As shown in the Example in §2, it follows that \( F \) is free.

Finally, let \( S_\alpha \) be the image of \( S \) in \( L_{Z_\alpha} \). Since \( L_{Z_\alpha}^{(2)} \to L_{Z_\alpha}^{(2)} \) is surjective, it follows that \( S_\alpha + L_{Z_\alpha}^2 = L_{Z_\alpha} \). Therefore, because \( L_{Z_\alpha} \) is nilpotent, the induced maps \( E \to L_{Z_\alpha} \) are surjective. Hence, these induce surjections \( E/E \cap L_Z^{(r)} \to L_{Z_\alpha}/L_{Z_\alpha}^{(r)} \).

Since each \( L_{Z_\alpha}/L_{Z_\alpha}^r \) is finite dimensional, it follows that passing to inverse limits yields surjections

\[
E/E \cap L_Z^{(r)} \to L_Z/L_Z^{(r)}.
\]

It is immediate from this that \( \lim_{\to \infty} E/E \cap L_Z^{(r)} \xrightarrow{\cong} L_Z \).

\[\square\]

6 One-relator groups

Our objective here is the proof of

**Theorem 4** If \( X \) is a wedge of at least two circles then any non-zero \([f] \in \pi_1(X)\) is rationally inert or, equivalently, \((X \cup_f D^2)_Q\) is aspherical.

**proof:** First observe that in fact

\[
[f] \text{ is rationally inert } \iff (X \cup_f D^2)_Q \text{ is aspherical.} \tag{15}
\]

In fact, the same argument as in the Example in §2 shows that the minimal Sullivan model of \( X \) is cdga equivalent to \( \mathbb{Q} \oplus H^1(X_Q) \). It follows that the homotopy Lie algebra, \( L \), is concentrated in degree 0 and since \( \pi_*(X_Q) = sL \), \( X_Q \) is aspherical. Thus if \([f] \) is rationally inert then \((X \cup_f D^2)_Q \) is aspherical. On the other hand, a Sullivan representative for the inclusion \( i : X \to X \cup_f D^2 \) is a morphism \( \gamma : \wedge V \to \wedge W \) of minimal Sullivan algebras. Since \( \pi_1(i) \) is injective, \( H^1(i) \) is surjective and it follows that \( \gamma : V^1 \to W^1 \) is injective. But if \((X \cup_f D^2)_Q \) is aspherical, then \( V = V^1 \), \( \gamma \) is injective, and by definition \([f] \) is rationally inert.

Next note that it is sufficient to prove the Theorem when \( X \) is a finite wedge of circles. Simply write \( X = Y \vee Y' \) in which \( Y \) is a finite wedge of circles, \( Y' \) is a wedge of circles, and \( f : S^1 \to Y \). Then, as just observed, \( Y'_Q \) is aspherical. It follows from Proposition 2 that if \((Y \cup_f D^2)_Q \) is aspherical, then so is \((X \cup_f D^2)_Q = [(Y \cup_f D^2) \vee Y']_Q \). Thus by (15), \([f] \in \pi_1(Y \cup_f D^2)_Q \) is rationally inert if and only if \([f] \in \pi_1(X_Q) \) is rationally inert.
In summary, we may and do assume henceforth that

\[ X = S^1 \lor \cdots \lor S^1. \]

On the other hand, we observe that

\[ [f] \neq 0 \implies \text{a Sullivan representative of } f \text{ is non-zero.} \quad (16) \]

In fact, denote \( G = \pi_1(X) \), so that \( G_Q = \pi_1(X_Q) \). According to \([9, \text{Theorem 7.5}]\), \( G^n/G^{n+1} \otimes \mathbb{Q} \xrightarrow{\cong} G^n_Q/G^{n+1}_Q \). But by \([15]\), \( G^n/G^{n+1} \) is a free abelian group, and hence \( G^n/G^{n+1} \rightarrow G^n_Q/G^{n+1}_Q \) is injective. Since \( G \) is a free group, \( G \rightarrow \lim_n (G/G^n)_{\mathbb{Q}} \) is injective and the image of \([f] \) in \( G_Q \) is non-zero. In particular, a Sullivan representative of \( f \) is non-zero.

Next recall from the Example in §2 and Lemma 1 that \( S^1 \lor \cdots \lor S^1 \lor S^2 \) has a quadratic minimal Sullivan model, \((\wedge W, d_1)\) in which \( W \cap \ker d_1 = H^\geq (S^1 \lor \cdots \lor S^1 \lor S^2)\). In particular, \( W^1 \cap \ker d_1 = H^1(S^1 \lor \cdots \lor S^1) \). Moreover, \( W^{>1} \cap \ker d_1 = W^2 \cap \ker d_1 = Qa \), where \( a \) represents the orientation class of \( S^2 \). It follows that

\[ W = W^1 \oplus Qa \oplus R, \]

and that the identity in \( \wedge W^1 \) extends to a quasi-isomorphism

\[ \varphi : (\wedge W, d_1) \xrightarrow{\cong} (\wedge W^1 \oplus Qa, d_1) \]

with \( \varphi(a) = a \) and \( \varphi(R) = 0 \).

**Note:** In comparing with the general situation described in §3, observe that the \( \wedge W^1 \) here corresponds to the \( \wedge W \) in §3, and that the \( \wedge W \) here has no analogue in §3.

In particular \( \varphi \) preserves wedge degrees when \( a \) is assigned wedge degree 1. Thus not only is \( H(\ker \varphi) = 0 \), but in fact for cycles \( \Phi \in \wedge W \),

\[ \Phi \in \wedge^k W \cap \ker \varphi \implies \Phi = d_1 \Psi \quad \text{for some } \Psi \in \wedge^{k-1} W \cap \ker \varphi. \quad (17) \]

The proof of Theorem 3 is now accomplished in the following steps:

**Step One:** Construction of a linear map of degree 1, \( d_0 : W \rightarrow W \), whose extension, also denoted \( d_0 \), to a derivation in \( \wedge W \) provides a cdga \( (\wedge W, d_1 + d_0) \) connected by cdga quasi-isomorphisms to \( A_{PL}(X \cup_f D^2) \).

**Step Two:** \((\wedge W, d_0 + d_1)\) is a Sullivan algebra, and hence a Sullivan model for \( X \cup_f D^2 \).

**Step Three:** The minimal Sullivan model of \((\wedge W, d_1 + d_0)\) has the form \((\wedge V^1, D)\), and so \((X \cup_f D^2)_Q\) is aspherical, and \([f] \) is rationally inert.

**Step One:** Construction of \( d_0 : W \rightarrow W \) whose extension to a derivation (also denoted by \( d_0 \)) provides a cdga \( (\wedge W, d_0 + d_1) \) connected by cdga quasi-isomorphisms to \( A_{PL}(X \cup_f D^2) \).

For this, fix a Sullivan representative \( \psi : (\wedge W^1, d) \rightarrow (\wedge V, 0) \) for \( f \) and, as at the start of §3, define \( \varepsilon : \wedge W^1 \rightarrow \mathbb{Q} \) by

\[ \varepsilon(1) = \varepsilon(\wedge^{\geq 2} W^1) = 0 \quad \text{and} \quad \psi(w) = \varepsilon(w)v, \quad w \in W^1. \]
Then define a derivation $\delta$ in $\wedge W^1 \oplus \mathbb{Q}a$ by setting

$$\delta(w) = \varepsilon(w)a \quad \text{and} \quad \delta(\wedge W^2 \oplus \mathbb{Q}a) = 0.$$  

Then $d_1 \delta = 0 = \delta d_1$ and $\delta^2 = 0$, so that $(\wedge W^1 \oplus \mathbb{Q}a, d_1 + \delta)$ is a cdga. As observed at the start of §3, this cdga is connected by cdga quasi-isomorphisms to $A_{PL}(X \cup_f \mathcal{D}^2)$.

Next, we construct a linear map $d_0 : W \to W$ of degree 1 such that $d_0 d_1 + d_1 d_0 = 0$ and $\varphi \circ d_0 = \delta \circ \varphi$.

For this, recall that $W = \cup_n W(n)$ with $W(0) = W \cap \ker d_1$ and $W(n + 1) = W \cap d_1^{-1}(\wedge W(n))$. By convention, $W(-1) = 0$. We assume by induction that $d_0$ is constructed in $W(n - 1)$, and write $W(n) = W(n - 1) \oplus S$. If $w \in S$, then

$$d_1 d_0 d_1 w = -d_0 d_1^2 w = 0,$$

and so $d_0 d_1 w$ is a cycle in $(\wedge W^1, d_1)$.

Suppose first that $w \in W^1$. Then $d_1 w \in \wedge^2 W^1(n - 1)$ and

$$\varphi(d_0 d_1 w) = \delta \varphi(d_1 w) = 0.$$

Thus by (17), for some $u \in \ker \varphi \cap W^2$,

$$d_0 d_1 w = d_1 u.$$

Moreover, $\delta : W^1 \to \mathbb{Q}a$, and so we may regard $\delta w$ as an element of $W^2$ for which $d_1 \delta w = 0$ in $\wedge W$. Set $d_0 w = \delta w - u$. Then

$$d_1 d_0 w = -d_1 u = -d_0 d_1 w$$

and, since $\varphi u = 0$,

$$\varphi(d_0 w) = \varphi(\delta w) = \delta w = \delta(\varphi w).$$

On the other hand suppose $w \in W^k$, some $k \geq 2$. Then $d_0 d_1 w \in (\wedge^2 W)^{k+2}$ and so $d_0 d_1 w \in R \wedge W \oplus \mathbb{Q}a^2$. Thus $\varphi(d_0 d_1 w) = 0$ and again by (17) $d_0 d_1 w = d_1 u$ for some $u \in W^2 \subseteq R$. Set $d_0 w = -u$, so that again

$$d_1 d_0 w = -d_1 u = -d_0 d_1 w.$$

Then, since $u \in R$, $\varphi u = 0$ while $\varphi w \in \mathbb{Q}a$ and so $\delta \varphi w = 0$ as well. This completes the construction of $d_0$. By construction,

$$\varphi \circ (d_1 + d_0) = (d_1 + \delta) \circ \varphi.$$

Finally we show that $d_0^2 = 0$ so that $d_1 + d_0$ is a differential, and that

$$\varphi : (\wedge W, d_1 + d_0) \xrightarrow{\sim} (\wedge W^1 \oplus \mathbb{Q}a, d_1 + \delta). \quad (18)$$

In fact $d_1 d_0^2 = d_0^2 d_1$. Assume by induction that $d_0^2 = 0$ in $W(n - 1)$. Then for $w \in S$, $d_0^2 w$ is a $d_1$-cycle and $\varphi(d_0^2 w) = \delta^2 \varphi w = 0$. Thus by (17), $d_0^2 w$ is a $d_1$-boundary, and hence $d_0^2 w = 0$. Thus $(\wedge W, d_1 + d_0)$ is a cdga and $\varphi$ is a morphism of cdga’s with respect to $d_1 + d_0$ and $d_1 + \delta$. Filter both sides by the difference between degree and wedge degree.
The map induced by $\varphi$ in the $0^{th}$ term of the spectral sequence is the quasi-isomorphism $\varphi : (\wedge W, d_1) \iso (\wedge W^1 \oplus \mathbb{Q} a, d_1)$. This establishes (15).

Note that by (16), the Sullivan representative $\psi$ is non-zero, and so for some $w \in W^1$, $\delta w = a$, and $d_0 w \neq 0$.

**Step Two:** $(\wedge W, d_1 + d_0)$ is a Sullivan algebra, and hence is a Sullivan model for $X \cup_f D^2$.

Here we prove a more general result: if $(\wedge V, d)$ is any minimal Sullivan algebra and $d_0 : V \to V$ is a linear map of degree 1 such that $d_0^2 = dd_0 + d_0 d = 0$, then $(\wedge V, d + d_0)$ is a Sullivan algebra.

For this, fix an increasing filtration $0 = V(0) \subset \cdots \subset V(n) \subset \cdots$ such that $V = \cup_n V(n)$ and $d : V(n + 1) \to \wedge \geq 2 V(n)$. Then, as follows, define by induction a sequence of subspaces of $V$ of the form

$$Q(0) \subset P(0) \subset \cdots \subset Q(n) \subset P(n) \subset \cdots$$

so that

- $d$ and $d_0 : Q(n + 1) \to \wedge P(n)$,
- $d$ and $d_0 : P(n + 1) \to \wedge Q(n + 1)$,

and

$$P(n) \supset V(n).$$

First, we set $Q(0) = P(0) = 0$. Then suppose $Q(k)$, and $P(k)$ are constructed for $k \leq n$. Write

$$V(n + 1) = V(n + 1) \cap P(n) \oplus S(n + 1),$$

and set

$$Q(n + 1) = P(n) + d_0(S(n + 1)) \quad \text{and} \quad P(n + 1) = Q(n + 1) + S(n + 1).$$

It is immediate that

$$P(n + 1) \supset P(n) + S(n + 1) \supset V(n) + S(n + 1) = V(n + 1).$$

Moreover, if $x \in S(n + 1)$ then

$$dd_0 x = -d_0 dx \in d_0 d(S(n + 1)) \subset d_0(\wedge \geq 2 V(n))$$

$$\subset d_0(\wedge \geq 2 P(n)) \subset \wedge \geq 2 P(n).$$

In particular, $d : Q(n + 1) \to \wedge \geq 2 P(n)$. Further $d_0^2(S(n + 1)) = 0$ and so $d_0(Q(n + 1)) = d_0(P(n)) \subset P(n)$.

On the other hand, if $x \in S(n + 1)$ then $d_0 x \in Q(n + 1)$ by construction, while $dx \in \wedge \geq 2 V(n) \subset \wedge P(n)$. This closes the induction and exhibits $(\wedge V, d + d_0)$ as a Sullivan algebra.

**Step Three:** The minimal Sullivan model of $(\wedge W, d_1 + d_0)$ has the form $(\wedge V^1, D)$, and so $(X \cup_f D^2)_Q$ is aspherical.
Recall from the Example in §2 that the homotopy Lie algebra of $(\wedge W, d_1)$ is the completion, $\hat{L}$ of the free Lie algebra $L(x_1, \ldots, x_r, y)$ generated by vectors $x_i$ dual to the orientation classes of the circles, and by $y$ dual to the orientation class of $S^2$. By construction, $W^{\geq 2} = \mathbb{Q}a \oplus R$, and we may choose $y$ so that

$$\langle a, sy \rangle = 1 \quad \text{and} \quad \langle R, sy \rangle = 0.$$ 

Now dualize $d_0 : W \to W$ to $d : \hat{L} \to \hat{L}$. Since $\deg d = -1$ it follows that $d : \hat{L}(x_1, \ldots, x_r) \to 0$ and $dy \in \hat{L}(x_1, \ldots, x_r)$. Moreover, because $d_0$ is a derivation satisfying $d_0d_1 + d_1d_0 = 0 = d_0^2$, it follows that $d$ is a derivation in the Lie algebra $\hat{L}$ and that $d^2 = 0$.

Moreover, if $(\wedge V, D)$ is the minimal Sullivan model of $(\wedge W, d_1 + d_0)$ then $V \cong H(W, d_0)$. Therefore $H(\hat{L}, d) = (H(W, d_0))^\vee$, and so it is sufficient to prove that

$$H_{\geq 1}(\hat{L}, d) = 0.$$

Recall also from Step One that a Sullivan representative for $f$ determines a linear map $\varepsilon : W^1 \to \mathbb{Q}$. Thus $\varepsilon$ desuspends to $\alpha \in L_W^1 = \hat{L}(x_1, \ldots, x_r)$. We show now that

$$dy = \alpha,$$ 

so that $dy \neq 0$.

For this, recall from Step One that if $w \in W^1$ then $d_0w = \varepsilon(w)a - u$, where $u \in W^2 \cap \ker \varphi = \mathbb{Z}$. It follows that

$$\langle w, sy \rangle = -\langle d_0w, sy \rangle = -\langle \varepsilon(w)a - u, sy \rangle = \langle w, s\alpha \rangle,$$

which establishes (19).

Denote by $L_q(x_i)$ the linear span of the commutators of length $q$ in the $x_i$. Write $dy$ as a series

$$dy = \sum_{q \geq n} \alpha_q,$$

where $\alpha_q \in L_q(x_i)$ and $\alpha_n \neq 0$. Then form the differential graded Lie algebra $(L(x_i, y), \partial)$ with $\partial(x_i) = 0$ and $\partial(y) = \alpha_n$. Since $\alpha_q$ belongs to $L_n(x_i)$ we can modify the degrees in $L(x_i)$ by assigning deg 2 to the $x_i$, without changing the homology with respect to $\partial$. Thus it follows from [16] Theorem 3.12 that $H_q(L(x_i, y), \partial) = 0$ for $q > 0$.

Now let $\omega = \sum_{q \geq p} \omega_q$ be a $d$-cycle in degree $r > 0$ in $\hat{L}(x_i, y)$, with $\omega_q \in L_q(x_i, y)$. Then $\omega_p$ is a $\partial$-cycle, and so a $\partial$-boundary. Choose $\beta_{p-n+1} \in L_{p-n+1}(x_i, y)$ with $\partial(\beta_{p-n+1}) = \omega_p$. Write $\omega(1) = \omega - d(\beta_{p-n+1})$, then $\omega(1)$ is a sum $\sum_{s \geq p+1} \omega(1)_s$. One again $\omega(1)_{p+1}$ is a $\partial$-cycle. This determines $\beta_{p-n+2}$. Continue in this way to obtain at the and an element

$$\beta = \sum_{s \geq p-n+1} \beta_s$$

with $d\beta = \omega$. 

\[\square\]

**Corollary.** With the notation of Theorem 3, set $V = W^1 \cap \ker d_0$. Then $(\wedge V, d_1) \to (\wedge W^1, d_1 + d_0)$ is the minimal Sullivan model of $X \cup_f D^2$. 

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proof: First note that any element in $\wedge^2 W^1$ can be written as $\Phi = \sum_{i=1}^n w_i \wedge w'_i$ in which $w_1, \ldots, w_n, w'_1, \ldots, w'_n$ are all linearly independent. Thus if $d_0 \Phi = 0$ then each $d_0 w_i = d_0 w'_i = 0$. But $d_1 : V \to \wedge^2 W^1 \cap \ker d_0$, and so $\wedge V$ is preserved by $d_1$. It is immediate from Step Three that $V \cong H(W, d_0)$, and it follows that $(\wedge V, d_1)$ is the minimal Sullivan model of $X \cup_f D^2$.

\[ \square \]

7 Whitehead’s problem and Theorem 5

Theorem 5. If $X$ is a connected CW complex and $(X \cup \bigvee_{k=1}^p D^2)_Q$ is aspherical then $X_Q$ is aspherical.

proof: The obvious induction reduces the statement to the case $p = 1$. Then, since $\pi_*(X \cup_f D^2)_Q \cong V^V$ as sets where $\wedge V$ is the minimal Sullivan model of $X \cup_f D^2$, our hypothesis simply implies that $V = V^1$. Let $\varphi : (\wedge V, d) \to (\wedge W, d)$ be a Sullivan representative for the inclusion $i : X \to X \cup_f D^2$. Since $H^1(X \cup_f D^2) \to H^1(X)$ is injective, it follows that $\varphi$ is injective and so $\wedge W$ decomposes as $\wedge V \otimes \wedge Z$, with $Z = Z^{2,1}$. In particular $|f|$ is rationally inert. Moreover, it follows from Proposition 1 that

$$H^{\geq 1}(\wedge Z, \overline{d}) \cong \mathbb{Q}b \otimes \wedge U,$$

where $\deg b = 1$ and $\wedge V \otimes \wedge U$ is the acyclic closure of $\wedge V$. Since $V = V^1$, $U = U^0$ and $H^{\geq 1}(\wedge Z, \overline{d}) = H^1(\wedge Z, \overline{d})$. This in turn implies $Z = Z^1$ and $X_Q$ is aspherical.

\[ \square \]

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