A $q$-ANALOG OF SCHUR’S $Q$-FUNCTIONS

GEANINA TUDOSE AND MICHAEL ZABROCKI

Abstract. We present a family of analogs of the Hall-Littlewood symmetric functions in the $Q$-function algebra. The change of basis coefficients between this family and Schur’s $Q$-functions are $q$-analog of numbers of marked shifted tableaux. These coefficients exhibit many parallel properties to the Kostka-Foulkes polynomials.

1. Introduction

The space of $Q$-functions, $\Gamma$, is defined to be the algebra generated by the odd power sum elements $\{p_1, p_3, p_5, \ldots\}$ as a subalgebra of the space of symmetric functions, $\Lambda$. $\Gamma$ is associated to the representation theory of the spin group and is also related to the projective representation theory of the symmetric and alternating groups. The fundamental basis for this space are Schur’s $Q$-functions, $Q_\lambda[X]$, which are indexed by strict partitions $\lambda$. These functions hold the place that the Schur $S$-functions, $s_\mu[X]$ for $\mu$ a partition, represent in the algebra of the symmetric functions.

The space of symmetric functions contains an important basis, $H_\mu[X; q]$, the Hall-Littlewood symmetric functions \cite{6}. Through specializations of the parameter $q$ these functions interpolate several well studied bases of the symmetric functions and generalize features of these bases. They have elegant properties and may be seen in many different contexts of combinatorics, algebra, representation theory, geometry and mathematical physics.

It is natural to ask the question of what the analog of the Hall-Littlewood symmetric functions in the $Q$-function algebra should be. In this paper we introduce a family of functions $G_\lambda[X; q] \in \Gamma$ that answers this question since we observe that this family shares many of the combinatorial and algebraic properties of the $H_\mu[X; q]$ functions in the space of symmetric functions. We expect that these functions will also be interesting from the perspective of other fields as well.

From the combinatorial standpoint, we note that the coefficient of $s_\lambda[X]$ in the symmetric function $H_\mu[X; q]$ is the well known Kostka-Foulkes polynomial. This family of coefficients are known to be polynomials in the parameter $q$ with non-negative integer coefficients and at $q = 1$ represent the number of column strict tableaux of shape $\lambda$ and content $\mu$. The combinatorial tools of jeu de taquin and the plactic monoid were in part developed to explain the connections between the Kostka-Foulkes polynomials and the column strict tableaux \cite{14}.

By comparison the coefficient of $Q_\lambda[X]$ in the function $G_\mu[X; q]$ is also a polynomial in $q$ and we conjecture (and prove in certain cases) that it also has coefficients that are non-negative integers. At $q = 1$ we know that these coefficients are the number of marked shifted tableaux with shape $\lambda$ and content $\mu$. A version of the RSK-algorithm was developed by Sagan, Worley and others \cite{5}, \cite{18}, \cite{23}, and
used to develop the theory of marked shifted tableaux. We hope that this theory can be extended to help answer the question of a combinatorial interpretation for these coefficients.

Our definition for the functions $G_{\lambda}[X; q]$ is motivated by viewing the symmetric functions $s_{\mu}[X]$ and $H_{\mu}[X; q]$ as compositions of operators. In the case of the Schur functions, the Bernstein operator $S_m \in \text{End}(\Lambda)$ (\cite{16} p. 96) has the property for $m \geq \mu_1$,

$$S_m(s_{\mu}[X]) = s_{(m, \mu_1, \ldots, \mu_{\ell(\mu)})}[X].$$

That is, this formula is a recursive definition for the Schur functions of degree $n + m$ as an algebraic relation that raises the degree of a symmetric function by $m$ acting on a Schur function of degree $n$. For the Hall-Littlewood symmetric functions, the operator $H_m \in \text{End}(\Lambda)$ with

$$H_m(H_{\mu}[X; q]) = H_{(m, \mu_1, \ldots, \mu_{\ell(\mu)})}[X; q]$$

for $m \geq \mu_1$ is due to Jing \cite{8}. In \cite{26}, it was noticed that these operators (as well as many others) are related by a simple algebraic $q$-twisting, $S_m = H_m$ (the definition of $\tilde{q}$ is stated precisely in equation (11) below).

The Schur’s $Q$-functions, $Q_{\lambda}[X]$ may also be seen from this perspective (\cite{7}, \cite{13}, \cite{16} p. 262-3). That is, there exists an operator $Q_m \in \text{End}(\Gamma)$ such that for $m > \lambda_1$,

$$Q_m(Q_{\lambda}[X]) = Q_{(m, \lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})}[X].$$

Since the definition of the $q$-twisting $\tilde{q}$ extends to $\text{End}(\Gamma)$, a natural definition for an analog to of the Hall-Littlewood symmetric functions in $\Gamma$ is to define $G_m := \tilde{Q}_m$ and then for $m > \lambda_1$, we set

$$G_{(m, \lambda_1, \ldots, \lambda_{\ell(\lambda)})}[X; q] := G_m(G_{\lambda}[X; q]).$$

This framework provides us only with a possible definition for the Hall-Littlewood analogs in $\Gamma$. It remains to show that these functions share properties similar to those of the Hall-Littlewood functions. In this case we find some striking similarities that say we have indeed found the correct analog.

This work is inspired by the results of the Hall-Littlewood functions and the desire to find analogous structure in the $Q$-function algebra. In addition, part of the motivation of defining these functions and identifying their properties is to find what features of the Hall-Littlewood symmetric functions are not unique to the symmetric function algebra and should hold in a more general setting. A goal of this research is to possibly identify what the $q$-twisting of equation (11) represents on a combinatorial, geometric or representation theoretical level and to show that the $G_{\lambda}[X; q]$ are another example of a structure that seems to exist in a more general context.

The remainder of this paper is divided into three sections and two appendices. The first section is simply a exposition of definitions and and notation related to the symmetric functions and $Q$-function algebra. We develop in some detail the perspective that bases of the symmetric functions and $Q$-functions can be seen as compositions of operators that have simple algebraic definitions. In the next section we introduce the $G_{\lambda}[X; q]$ functions and derive recurrences and some properties that are analogous to those that exist for the Hall-Littlewood symmetric functions. In a final section we discuss a generalization of the functions $G_{\lambda}[X; q]$ that are indexed by a sequence of strict partitions $(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)})$ and the motivation
for this generalization. These functions correspond to a $q$-analog of the product $Q_{\mu^1}[X]Q_{\mu^2}[X] \cdots Q_{\mu^k}[X]$ and the coefficients of these functions correspond to analogs of the generalized or parabolic Kostka polynomials of \cite{10, 12, 19, 21} and \cite{22}.

Finally, in the first appendix we include tables of transition coefficients between the $G_\lambda[X;q]$ basis and the $Q_\lambda[X]$ basis for degrees 3 through 9. These tables are evidence of a very strong conjecture that these coefficients are polynomials in $q$ with non-negative integer coefficients and represent a $q$ analog of the number of marked shifted tableaux. This suggests that the marked shifted tableaux should have a poset structure similar to the charge poset for the column strict tableaux. In the second appendix we include a conjectured diagram for a rank function on the marked shifted tableaux of content $(4,3,2)$. This diagram suggests that the structure for the statistic on marked shifted tableaux is somewhat different than that of the charge poset even though we conjecture that these statistics should share many of the same properties.

2. Notation and Definitions

2.1. Symmetric functions, partitions and columns strict tableaux. Consider $\Lambda^X$ the ring of series of finite degree in the variables $x_1, x_2, x_3, \ldots$ which are invariant under all permutations of the variables. This ring is algebraically generated by the set of elements $\{p_k[X] = \sum_i x_i^k\} \subset \Lambda^X$ and hence $\Lambda^X$ is isomorphic to the ring $\Lambda = \mathbb{C}[p_1, p_2, p_3, \ldots]$ with $\text{deg}(p_k) = k$. We will refer to both $\Lambda$ and $\Lambda^X$ as the ring of symmetric functions.

$\Lambda$ is a graded ring and basis for the component of degree $n$ is given by the monomials $p_{\lambda} := p_{\lambda_1}p_{\lambda_2} \cdots p_{\lambda_\ell}$. The entries of $\lambda$ are called the parts of the partition. The number of parts that are of size $i$ in $\lambda$ will be represented by $m_i(\lambda)$ and the total number of non-zero parts is represented by $\ell(\lambda) := \sum_i m_i(\lambda)$ and the size by $|\lambda| := \sum_k km_k(\lambda) = \sum_i \lambda_i$. A common statistic associated to partitions is $n(\lambda) := \sum_i (i - 1)\lambda_i$.

The partial order on partitions, $\lambda \leq \mu$ if and only if $\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i$ for all $1 \leq k \leq \ell(\lambda)$, is called the dominance order. We call the operators $R_{ij}\lambda = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots, \lambda_{\ell(\lambda)})$ for $1 \leq i < j \leq \ell(\lambda)$ ‘raising operators’ and they have the property that if $R_{ij}\lambda$ is a partition, then $R_{ij}\lambda \geq \lambda$.

We will consider three additional bases of $\Lambda$ here. Following the notation of \cite{10}, we define the homogeneous (complete) symmetric functions as $h_\lambda := h_{\lambda_1}h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$, where $h_\lambda = \sum_{\lambda \vdash n} p_\lambda / z_\lambda$ and $z_\lambda = \prod_{i=1}^{\ell(\lambda)} \lambda^{m_i(\lambda)} m_i(\lambda)!$. The elementary symmetric functions are $e_\lambda := e_{\lambda_1}e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$ where $e_\lambda = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} p_\lambda / z_\lambda$. By convention we set $p_0 = h_0 = e_0 = 1$ and $p_{-k} = h_{-k} = e_{-k} = 0$ for $k > 0$. The Schur functions are given by $s_\lambda = \text{det} [ h_{\lambda_i + j - 1} ]_{1 \leq i, j \leq \ell(\lambda)}$. The sets $\{p_\lambda\}_{\lambda \vdash n}$, $\{h_\lambda\}_{\lambda \vdash n}$, $\{e_\lambda\}_{\lambda \vdash n}$ and $\{s_\lambda\}_{\lambda \vdash n}$ all form bases for the symmetric functions of degree $n$.

We will consider the elements of $\Lambda$ as functors on the space $\Lambda^X$. If $E$ is an element of $\Lambda^X$ then we let $p_k[E] = E$ with $x_i$ replaced with $x_i^k$, so that $p_k : \Lambda^X \rightarrow \Lambda^X$. We then extend this relation algebraically, $p_\lambda[E]$ will represent the expression $p_{\lambda_1}[E]p_{\lambda_2}[E] \cdots p_{\lambda_{\ell(\lambda)}}[E]$. In particular, if we take $E = X = x_1 + x_2 + x_3 + \cdots$ then
$p_k[X] = \sum_i x_i^k$ and the map that sends $p_k \mapsto p_k[X]$ is a ring isomorphism $\Lambda \to \Lambda^X$ since $p_k[X] = x_1^k + x_2^k + x_3^k + \cdots$.

Identities which hold in the ring $\Lambda$ specialize as well to the ring of symmetric polynomials in a finite number of variables. We will use the notation $X_n$, to represent a series of variables (e.g. $X = \sum_i x_i$ or $Y = \sum_i y_i$), while capital letters indexed by a number will represent a polynomial sum of variables (e.g. $X_n = \sum_{i=1}^n x_i$ or $Z_n = \sum_{i=1}^n z_i$).

We will need to adjoin to each of the rings $\Lambda, \Lambda^X$ and $\Lambda^{X_n}$ a special element $q$ (or many special elements, if necessary) which acts much like a variable in this ring, however $q$ will specialize to values in the field. $q$ has the special property that $p_k[qX] = q^k p_k[X]$ and hence is not an element of our base field since for $c \in \mathbb{C}$, we have that $p_k[e^X] = c p_k[X]$.

Notice that by definition we have in general $p_k[a X + b Y] = ap_k[X] + bp_k[Y]$ for $a, b \in \mathbb{C}$. This implies that $f[-X]$ does not represent the symmetric series $f[X]$ with $x_i$ replaced by $-x_i$ since $p_k[X]|_{x_i \to -x_i} = (-1)^k p_k[X]$, while $p_k[-X] = -p_k[X]$. To this end we introduce the notation $f[\varepsilon X] = f[qX]|_{q=-1}$. In the case of the power sums we have that $p_k[\varepsilon X] = (-1)^k p_k[X]$ and hence $f[\varepsilon X] = f[X]|_{x_i \to -x_i}$.

Consider the series $\Omega = \sum_{n \geq 0} h_n$ which is not an element of the ring $\Lambda$, but lies in the completion of this ring. We will use the morphism $p_k \mapsto p_k[X]$ on this element as well and manipulations of this notation allow us to derive the following identities, which we will use repeatedly in our calculations.

\begin{align}
(1) \quad & \Omega[X] = \prod_i \frac{1}{1-x_i} = \sum_{n \geq 0} h_n[X] \\
(2) \quad & \Omega[-X] = \prod_i 1 - x_i = 1/\Omega[X] \\
(3) \quad & \Omega[X + Y] = \Omega[X] \Omega[Y] \\
(4) \quad & \Omega[-\varepsilon X] = \prod_i 1 + x_i = \sum_{n \geq 0} e_n[X]
\end{align}

Define a generating function of operators $S(z) = \sum_m S_m z^m$ where for an arbitrary symmetric function $P[X]$, $S(z) P[X] = P[X - 1/z] \Omega[zX]$. In this manner $S_m$ acts on any symmetric function raising the degree of the function by $m$ and has the action $S_m P[X] = S(z) P[X]|_{z^m}$. A composition of the operators $S(z_i)$ produces the expression

\begin{align}
(5) \quad & S(z_1) S(z_2) \cdots S(z_k) \mathbf{1} = \Omega[Z_k X] \prod_{1 \leq i < j \leq n} (1 - z_j / z_i),
\end{align}

where $Z_k$ represents the sum $z_1 + z_2 + \cdots + z_k$. Since the coefficient of $z^\lambda$ in $\Omega[Z_k X]$ then it must be that the coefficient of $z^\lambda$ in the right hand side of (5) is given by $\prod_{1 \leq i < j \leq n} (1 - R_{ij}) h_{\lambda}^\lambda[X]$, where $R_{ij} h_{\lambda}^\lambda[X] = h_{R_{ij} \lambda}^\lambda[X]$ (considering the $h$-functions indexed by sequences of numbers). This is an expression for the Schur function $s_\lambda[X]$, hence it follows that $S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_k} \mathbf{1} = s_\lambda[X]$. These operators are due to
Bernstein [16] p.96. It follows that \( S_m(s_\lambda[X]) = s_{(m,\lambda)}[X] \) where \((m, \lambda)\) denotes \((m, \lambda_1, \ldots, \lambda_{\ell(\lambda)})\) and \( s_{(a_1, \ldots, a_\ell)}[X] = \det[h_{a_i,i+j}] \).

**Remark 1:** We follow [16] in the use of raising operators for our definitions, however we are being imprecise since our raising operators are not associative or commutative as defined. We will consider a symmetric function as a composition of operators (for example in equation (6)) and the operators \( R_{ij} \) serve to raise or lower the indexing integer of the operator in the \( i \) and \( j^{th} \) positions respectively.

By acting on an arbitrary symmetric function using these operations and the relations in equations (1) (2) and (3) commutation relations of the operators follow very nicely. By expanding the left and right side of the following expression verifies that

\[
S(z)S(u)P[X] = -\frac{u}{z}S(u)S(z)P[X].
\]

By taking the coefficient of \( u^m z^n \) in both sides of the equation, we find that \( S_n S_m = -S_{m-1}S_{n+1} \) which also implies that \( S_m S_{m+1} = 0 \). Many of the calculations of commutation relations are of a similar sort of manipulation.

A Young diagram for a partition will be a collection of cells of the integer grid lying in the first quadrant. For a partition \( \lambda \), \( Y(\lambda) = \{(i,j) : 0 \leq j < \ell(\lambda) \text{ and } 0 \leq i \leq \lambda_j \} \). The reason why we consider empty cells rather than say, points, is because we wish to consider fillings of these cells. A tableau is a map from the set \( Y(\lambda) \) to \( \mathbb{N} \), this may be represented on a Young diagram by writing integers within the cells of a graphical representation of a Young diagram (see figure 1). The shape of the tableau is the partition \( \lambda \). We say that a tableau \( T \) is column strict if \( T(i,j) \leq T(i+1,j) \) and \( T(i,j) < T(i,j+1) \) whenever the points \((i+1,j)\) or \((i,j+1)\) are in \( Y(\lambda) \). Let \( m_k(T) \) represent the number of points \( p \) in \( Y(\lambda) \) such that \( T(p) = k \). The vector \((m_1(T), m_2(T), \ldots)\) is the content of the tableau \( T \).

The Pieri rule describes a combinatorial method for computing the product of \( h_m[X] \) and \( s_\mu[X] \) expanded in the Schur basis. We will use the notation \( \lambda/\mu \in \mathcal{H}_m \) to represent that \(|\lambda| - |\mu| = m\) and for \( 1 \leq i \leq \ell(\lambda), \mu_i \leq \lambda_i \) and \( \mu_i \geq \lambda_{i+1} \). It may be easily shown that

\[
h_m[X]s_\mu[X] = \sum_{\lambda/\mu \in \mathcal{H}_m} s_\lambda[X].
\]  

This gives a method for computing the expansion of the \( h_\mu[X] \) basis in terms of the Schur functions. Consider the coefficients \( K_{\lambda \mu} \) defined by the expression

\[
h_\mu[X] = \sum_{\lambda \vdash |\mu|} K_{\lambda \mu} s_\lambda[X].
\]

\( K_{\lambda \mu} \) are called the Kostka numbers and are equal to the number of column strict tableaux of shape \( \lambda \) and content \( \mu \).

### 2.2. Kostka polynomials and Hall-Littlewood symmetric functions

Define the following symmetric functions

\[
H_\lambda[X;q] = \prod_{i \leq j} \frac{1 - R_{ij}}{1 - q R_{ij}} h_\lambda[X]
\]

\[
= \prod_{1 \leq i < j \leq n} (1 + (q-1)R_{ij} + (q^2 - q)R_{ij}^2 + \cdots) h_\lambda[X].
\]
They will be referred to as Hall-Littlewood symmetric functions as they are transformations of the symmetric polynomials defined by Hall [8] (see [10] for a modern account where \( Q_\mu(x; q) \) in their notation is \( H_\mu[X(1 - q); q] \) in ours). The coefficient of \( s_\lambda[X] \) in \( H_\mu[X; q] \) is known as the Koskta-Foulkes polynomial \( K_{\lambda\mu}(q) \). That is, we have the expansion

\[
(9) \quad H_\mu[X; q] = \sum_\lambda K_{\lambda\mu}(q)s_\lambda[X].
\]

We will present some of the properties of the Kostka-Foulkes polynomials and the Hall-Littlewood symmetric functions below. First, it will be important to establish some identities for manipulating these functions.

Let \( H(z) = \sum_ m H_m z^m \) be defined as the operation \( H(z)P[X] = P[X - (1 - q)/z]\Omega[zX] \). Taking the coefficient of \( H_m = H(z) \) has the effect of raising the degree of the symmetric function it is acting on by \( m \). A composition of these operators has the expression

\[
(10) \quad H(z_1)H(z_2) \cdots H(z_k) = \Omega[Z_kX] \prod_{1 \leq i < j \leq k} \frac{1 - z_j/z_i}{1 - qz_j/z_i}.
\]

Now since \( h_{\mu, \lambda} = \frac{\delta_{\lambda\nu}}{z_1^\lambda} \Omega[Z_kX] \), it is clear that the coefficient of \( z_\lambda \) in the right hand side of (10) is exactly the right hand side of (8) and hence \( H_\lambda H_{\lambda_2} \cdots H_{\lambda_\ell(\lambda)} = H_\lambda[X; q] \). This operator also satisfies the relations \( H_{m-1}H_m = qH_mH_{m-1} \) and \( H_{m-1}H_m - qH_mH_{m-1} = qH_mH_{m-1} - H_{m-1}H_m \). This relation can be derived as we did for the Schur function operators by demonstrating \((z - u)H(z)H(u) = (qz - u)H(u)H(z)\) on an arbitrary symmetric function \( P[X] \).

This family of operators \( H_m \) is due to Jing [8] and they are sometimes referred to as 'vertex operators' for the Hall-Littlewood symmetric functions.

For an element \( V \in Hom(\Lambda, \Lambda) \), define

\[
(11) \quad \tilde{V}^q P[X] = V^q P[qX + (1 - q)Y] \bigg|_{Y=X},
\]

where \( V^Y \) denotes that as an operation on symmetric functions in the \( Y \) variables only and \( Y = X \) represents setting the \( Y \) variables equal to the \( X \) variables after the operation is completed. This is a \( q \)-analog of the operator \( V \) and we remark that \( S(z)^q = H(z) \). This follows by calculating

\[
\tilde{S}(z)^q P[X] = S^Y(z)P[qX + (1 - q)Y] \bigg|_{Y=X} = P[qX + (1 - q)(Y - 1/z)]\Omega[zY] \bigg|_{Y=X} = P[X - (1 - q)/z]\Omega[zX] = H(z)P[X],
\]

This relationship between \( S(z) \) and \( H(z) \) is the motivation for our definition of the \( q \)-analog of Schur's \( Q \)-functions.

The functions \( H_\lambda[X; q] \) interpolate between the functions \( s_\lambda[X] = H_\lambda[X; 0] \) and \( h_\lambda[X] = H_\lambda[X; 1] \). The Kostka-Foulkes polynomials are defined as the \( q \)-polynomial coefficient of \( s_\lambda[X] \) in \( H_\mu[X; q] \) and hence we have the expansion analogous to (7).

\[
(12) \quad H_\mu[X; q] = \sum_{\lambda \preceq \mu} K_{\lambda\mu}(q)s_\lambda[X].
\]
The coefficients \( K_{\lambda \mu}(q) \) are clearly polynomials in \( q \), but it is surprising to find that the coefficients of the polynomials are non-negative integers.

A defining recurrence can be derived \( K_{\lambda \mu}(q) \) in terms of the Kostka-Foulkes polynomials indexed by partitions of size \( |\mu| - |\lambda| \) using the formula for \( H_m \). This recurrence is often referred to as the ‘Morris recurrence’ for the Kostka-Foulkes polynomials \([17]\). The action of \( H_m \) on the Schur functions is given by

\[
H_m(s_\mu[X]) = \sum_{\lambda \in \mathcal{H}^m} \sum_{i \geq 0} q^i s_{(m+i, \lambda)}[X].
\]

It is not immediately obvious that at \( q = 1 \), the previous equation reduces to the Pieri rule and at \( q = 0 \) the formula is simply \( S_m(s_\mu[X]) = s_{(m, \mu)}[X] \). Using \([13]\) and equating coefficients of \( s_\lambda \) on both sides of the equation \( H_m(H_\mu[X; q]) = \sum_\lambda K_{\lambda \mu}(q)H_m(s_\lambda[X]) \), we arrive at the Morris recurrence

\[
K_{\alpha,(m, \mu)}(q) = \sum_{s=1}^{t: \alpha_s \geq m} (-1)^{s-1} q^{\alpha_s - m} \sum_{\lambda; \lambda / \alpha^{(s)} \in H(\alpha_s - m)} K_{\lambda \mu}(q),
\]

where \( m > \mu_1 \) and \( \alpha^{(s)} \) is \( \alpha \) with part \( \alpha_s \) removed.

The Kostka-Foulkes polynomials and the generating functions \( H_\mu[X; q] \) have the following important properties which we simply list here so that we may draw a connection to analogous formulae. For a more detailed reference of these sorts of properties we refer the interested reader to the excellent survey article \([6]\).

1. the degree in \( q \) of \( K_{\lambda \mu}(q) \) is \( n(\mu) - n(\lambda) \).
2. \( K_{\lambda \mu}(0) = \delta_{\lambda \mu} \) which implies \( H_\mu[X; 0] = s_\mu[X] \), \( K_{\lambda \mu}(1) = K_{\lambda \mu}, \) so that \( H_\mu[X; 1] = h_\mu[X], \) \( K_{\lambda \lambda}(q) = 1 \) and \( K_{(\mu)|\mu}(q) = q^{n(\mu)}. \) We also have that \( K_{\lambda \mu}(q) = 0 \) if \( \lambda < \mu \).
3. \( K_{\lambda \mu}(q) = \sum_T q^{c(T)} \), where the sum is over all column strict tableaux of shape \( \lambda \) and content \( \mu \) and \( c(T) \) denotes the charge of a tableau \( T \) (see \([13]\) and hence is a polynomial with non-negative integer coefficients.
4. A combinatorial interpretation for these coefficients exists in terms of objects known as rigged configurations \([11]\).
5. \( H_{(1^n)}[X; q] = e_n \left( \frac{X}{1-q} \right)(g; q)_n \) where \( (g; q)_n = \prod_{i=1}^n (1-q^i) \).
6. If \( \zeta \) is the \( k^{th} \) root of unity, \( H_\mu[X; \zeta] \) factors into a product of symmetric functions.
7. Set \( K'_\mu(q) := q^{n(\lambda) - n(\mu)} K_{\mu \lambda}(1/q), \) then \( K'_{\mu \lambda}(q) \geq K_{\mu \lambda}(q) \) for \( \lambda \leq \nu \).
8. \( K_{\lambda+\mu}(q) \geq K_{\lambda \mu}(q) \), where \( \lambda + \mu \) represents the partition \( \lambda \) with a part of size \( a \) inserted into it.
9. \( K_{\lambda \mu}(q) = \sum_{a \in S_n} \text{sign}(w) P_q(w(\lambda + \rho) - (\mu + \rho)) \) where \( P_q(\alpha) \) is the coefficient of \( x^\alpha \) in \( \prod_{1 \leq i < j \leq n} (1-qx_i/x_j)^{-1} \), a \( q \) analog of the Kostant partition function
and \( \rho = (\ell(\mu) - 1, \ell(\mu) - 2, \ldots, 1, 0) \).
10. \( H_\mu[X; q] H_\lambda[X; q] = \sum_{\nu} d_{\nu \mu}(q) H_\nu[X; q] \), for some coefficients \( d_{\nu \mu}(q) \) with the property that if the Littlewood-Richardson coefficient \( c_{\nu \mu}(q) = 0 \) then \( d_{\nu \mu}(q) = 0 \). These coefficients are a transformation of the Hall algebra structure coefficients.
11. For the scalar product \( \langle s_\lambda[X], s_\mu[X] \rangle = \delta_{\lambda \mu} \), the basis \( H_\mu[X(1-q); q] \) is orthogonal with respect to \( H_\lambda[X; q] \), that is \( \langle H_\lambda[X; q], H_\mu[X(1-q); q] \rangle = 0 \) if \( \lambda \neq \mu \).
2.3. Schur’s $Q$-functions, strict partitions, and marked shifted tableaux.

The $Q$-function algebra is a sub-algebra of the symmetric functions $\Gamma = \mathbb{C}[p_1, p_3, p_5, \ldots]$.

A typical monomial in this algebra will be $p_\lambda$, where $\lambda$ is a partition and $\lambda_i$ is odd. A partition $\lambda$ is strict if $\lambda_i > \lambda_{i+1}$ for all $1 \leq i \leq \ell(\lambda) - 1$ and a partition $\lambda$ is odd if $\lambda_i$ is odd for $1 \leq i \leq \ell(\lambda)$. We will use the notation $\lambda \vdash n$ (respectively $\lambda \vdash n$) to denote that $\lambda$ is a partition of size $n$ that is strict (respectively odd). Note that the number of strict partitions of size $n$ and the number of odd partitions of size $n$ is the same (proof: write out a generating function for each sequence).

The analog of the homogeneous and elementary symmetric functions in $\Gamma$ are the $Q$-functions $q_\lambda := q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda(\lambda)}$, where $q_n = \sum_{\lambda \vdash n} \frac{2^{\ell(\lambda)}}{z_{\lambda}}$. Define an algebra morphism $\theta : \Lambda \rightarrow \Gamma$ by the action on the $p_n$ generators as $\theta(p_n) = (1 - (-1)^n)p_n$. That is $\theta(p_n) = 2p_n$ if $n$ is odd and $\theta(p_n) = 0$ for $n$ even. $\theta$ has the property that $\theta(h_n) = \theta(e_n) = q_n$ and may be represented in our notation as $\theta(p_n[X]) = p_n(1 - z)X$. Under this morphism, our Cauchy element may also be considered a generating function for the $q_n$ elements since

$$\Omega[(1 - \epsilon)X] = \sum_{n \geq 0} q_n[X] = \prod_i \frac{1 + x_i}{1 - x_i}. \quad (15)$$

It follows that $\{p_\lambda\}_{\lambda \vdash n}$, $\{q_\lambda\}_{\lambda \vdash n}$, $\{q_\lambda\}_{\lambda \vdash n}$ are all bases for the subspace of $Q$-functions of degree $n$. Another fundamental basis for this space are the Schur’s $Q$-functions $Q_\lambda[X] = \theta(H_\lambda[X]; -1)$. These functions hold a similar place in the $Q$-function algebra that the Schur functions hold in $\Lambda$. In particular, $\{Q_\lambda[X]\}_{\lambda \vdash n}$ is a basis for the $Q$-functions of degree $n$.

In analogy with the Schur functions, $Q_\lambda[X]$ may also be defined with a raising operator formula by setting $g = -1$ and applying the $\theta$ homomorphism to equation (8). We arrive at the formula:

$$Q_\lambda[X] = \prod_{i < j} \frac{1 - R_{ij}}{1 + R_{ij}} q_\lambda[X] = \prod_{i < j} (1 - 2R_{ij} + 2R_{ij}^2 - \cdots)q_\lambda[X], \quad (16)$$

where the operators now act as $R_{ij} q_\lambda[X] = q_{R_{ij} \lambda}[X]$. Furthermore, they have a formula as the coefficient in a generating function:

$$Q_\lambda[X] = \Omega[(1 - \epsilon)Z_n X] \prod_{1 \leq i < j \leq n} \frac{1 - z_i z_j}{1 + z_i z_j} \Omega[1 -$$

As with Schur functions and the Hall-Littlewood functions, the raising operator formula leads us to an operator definition. By setting $Q(z)P[X] = P [X - \frac{z}{2}]\Omega[1-$

\[Q(z)P[X] = P [X - \frac{z}{2}]\Omega[1-\]
\( Q(z_1)Q(z_2) \cdots Q(z_n)1 = \Omega[(1-\epsilon)Z_nX] \prod_{1 \leq i < j \leq n} \frac{1-z_j/z_i}{1+z_j/z_i}, \)

and hence if we set \( Q_mP[X] = Q(z)P[X] \bigg|_{z^m} \) then \( Q_m(Q_\lambda[X]) = Q_{(m,\lambda)}[X] \) as long as \( m > \lambda_1 \). The commutation relations for the \( Q_m \) are

\[
Q_mQ_n = -Q_nQ_m \text{ for } m \neq -n, \tag{18}
\]

\[
Q_mQ_{-m} = 2(-1)^m - Q_{-m}Q_m \text{ if } m \neq 0, \tag{19}
\]

\[
Q_m^2 = 0 \text{ if } m \neq 0 \text{ and } Q_0^2 = 1. \tag{20}
\]

These formulas allow us to straighten the \( Q_\mu[X] \) functions when they are not indexed by a strict partition.

The \( Q \)-function algebra is endowed with a natural scalar product. If we set \( \langle Q_\lambda, Q_\mu \rangle_T = 2^\ell(\lambda)\delta_{\lambda\mu}z_{\lambda} \) for \( \lambda, \mu \vdash n \), then it may be shown that we also have

\[
\langle Q_\lambda[X], Q_\mu[X] \rangle_T = 2^\ell(\lambda)\delta_{\lambda\mu}. \tag{21}
\]

A shifted Young diagram for a partition will again be a collection of cells lying in the first quadrant. For a strict partition \( \lambda \), let \( YS(\lambda) = \{(i,j) : 0 \leq j \leq \ell(\lambda) \text{ and } j-1 \leq i \leq \lambda_j + j - 1 \} \). A marked shifted tableau \( T \) of shape \( \lambda \) is a map from \( YS(\lambda) \) to the set of marked integers \( \{1 < 2 < 2' < \ldots \} \) that satisfy the following conditions

- \( T(i,j) \leq T(i+1,j) \) and \( T(i,j) \leq T(i,j+1) \)
- If \( T(i,j) = k \) for some integer \( k \) (i.e. has an unmarked label) then \( T(i,j+1) \neq k \)
- If \( T(i,j) = k' \) for some marked label \( k' \) then \( T(i+1,j) \neq k' \)

We may represent these objects graphically with a Young diagram representing \( \lambda \) and the cells filled with the marked integer alphabet. If \( T \) is a marked shifted tableau, then we will set \( m_i(T) \) as the number of occurrences of \( i \) and \( i' \) in \( T \). The sequence \( (m_1(T), m_2(T), m_3(T), \ldots) \) is the content of \( T \).

The combinatorial definition of the marked shifted tableaux is defined so that it reflects the change of basis coefficients between the \( q_\mu \) and \( Q_\lambda \) basis. The rule for computing the product of \( q_\mu[X] \) and \( Q_\lambda[X] \) when expanded in the Schur \( Q \)-functions is the analog of the Pieri rule for the \( \Gamma \) space. If \( \lambda/\mu \in H_m \) then \( a(\lambda/\mu) \) will represent \( 1+ \) the number of \( 1 < j \leq \ell(\lambda) \) such that \( \lambda_j > \mu_j \) and \( \mu_{j+1} > \lambda_j \).

We may show that

\[
q_m[X]Q_\mu[X] = \sum_{\lambda/\mu \in H_m} 2^{a(\lambda/\mu)-\ell(\lambda)+\ell(\mu)}Q_\lambda[X]. \tag{22}
\]

Denote by \( L_{\lambda\mu} \) the number of marked shifted tableaux \( T \) of shape \( \lambda \) and content \( \mu \) (where \( \lambda \) is a strict partition) such that \( T(i,i) \) is not a marked integer. We may expand the function \( q_\mu[X] \) in terms of the \( Q \)-functions using (22) to show

\[
q_\mu[X] = \sum_{\lambda \vdash |\mu|} L_{\lambda\mu}Q_\lambda[X]. \tag{23}
\]
3. The $Q$-Hall-Littlewood basis $G_{\lambda}(x; q)$ for the algebra $\Gamma$

In this section we define a new family of functions $G_{\lambda}[X, q]$ which seems to play the same role as the Hall-Littlewood functions $H_{\lambda}[X, q]$ in the $Q$-functions algebra. These functions are introduced via a raising operator formula similar to (8). This definition permits an equivalent interpretation via a corresponding vertex operator $G_m$ whose properties are analogues to both the Hall-Littlewood vertex operator $H_m$ and $Q_m$.

Note: From here, unless otherwise stated, all partitions are considered strict.

3.1. Raising operator formula. We define the following analog of the Hall-Littlewood functions in the subalgebra $\Gamma$

\begin{equation}
G_{\lambda}[X; q] := \prod_{1 \leq i < j \leq n} \left( \frac{1 + qR_{ij}}{1 - qR_{ij}} \right) \left( \frac{1 - R_{ij}}{1 - R_{ij}} \right) q_{\lambda}[X] = \prod_{1 \leq i < j \leq n} \left( \frac{1 + qR_{ij}}{1 - qR_{ij}} \right) Q_{\lambda}[X].
\end{equation}

We call the functions $G_{\lambda} \in \Gamma \otimes \mathbb{C}(q)$ the $Q$-Hall-Littlewood functions.

In $\Gamma \otimes \mathbb{C}(q)$ this family can be expressed in the basis of $Q$-functions as

\begin{equation}
G_{\mu}[X; q] = \sum_{\lambda} L_{\lambda \mu}(q) Q_{\lambda}[X],
\end{equation}

which can be viewed as a $q$-analogue of (23). We call the coefficients $L_{\lambda \mu}(q)$ the $Q$-Kostka polynomials. We shall see that this family of polynomials shares many of the same properties with the classical Kostka-Foulkes polynomials. Tables of these coefficients are given in an Appendix. It follows from (24) that $L_{\lambda \mu}(q)$ have integer coefficients and $L_{\lambda \mu}(q) = 0$ if $\lambda < \mu$. This shows

**Proposition 1.** The $G_{\lambda}$, $\lambda$ strict, form a $\mathbb{Z}$-basis for $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}(q)$.

The basis $G_{\lambda}$ interpolates between the Schur’s $Q$-functions and the functions $q_{\mu}$ because $G_{\lambda}[X; 0] = Q_{\lambda}[X]$ and $G_{\lambda}[X; 1] = q_{\lambda}[X]$ as is clear from (24).

Since the coefficient of $z^h$ in $\Omega[(1 - \epsilon)Z_n X]$ is $q_{\lambda}[X]$ equation (24) implies

\begin{equation}
G_{\lambda}[X; q] = \prod_{1 \leq i < j \leq n} \left( \frac{1 - z_j / z_i}{1 + z_j / z_i} \right) \left( \frac{1 + qz_j / z_i}{1 - qz_j / z_i} \right) \Omega[(1 - \epsilon)Z_n X]_{\lambda}.\end{equation}

Define the operator $G(z)$ acting on an arbitrary symmetric function $P[X]$ as

\begin{equation}
G(z)P[X] = P \left[ X - \frac{1 - q}{z} \right] \Omega[(1 - \epsilon)zX].
\end{equation}

The operator $G(z)$ defines a family of operators as $G(z) = \sum_{m \in \mathbb{Z}} G_m z^m$ and hence

$G_m P[X] = G(z)P[X]_{z^{-m}}$.

If we consider a composition of these operators acting on the symmetric function 1 we obtain the following

\begin{equation}
G(z_1)G(z_2) \cdots G(z_n)1 = \prod_{1 \leq i < j \leq n} \left( \frac{1 - z_j / z_i}{1 + z_j / z_i} \right) \left( \frac{1 + qz_j / z_i}{1 - qz_j / z_i} \right) \Omega[(1 - \epsilon)Z_n X],
\end{equation}

which together with relation (26) gives

$$G_{\lambda}[X; q] = G_{\lambda_1} \cdots G_{\ell(\lambda)}(1).$$
Next we investigate some properties of this operator. First, $G_m$ satisfies the following commutation relation.

**Proposition 2.** For all $r, s \in \mathbb{Z}$ we have

$$
(1 - q^2)(G_rG_s + G_sG_r) + q(G_{r-1}G_{s+1} - G_{s+1}G_{r-1}) + G_{s-1}G_{r+1} - G_{r+1}G_{s-1}) = 2(-1)^r(1 - q)^2\delta_{r,-s}.
$$

**Proof** We will prove this relation in a few steps. Consider $G(u)$ and $G(z)$ the operator $G$ defined above on the variable $u$ and $z$ respectively. We are looking at the composition of these two operators.

**Step 1.** We may write

$$
G(u)G(z) = G(z)uF(z/u)F(-z/u),
$$

where $G(z, u)$ is an operator symmetric in $z$ and $u$ defined by

$$
G(z)uP[X] := P \left[ X - \frac{1 - q}{u} - \frac{1 - q}{z} \right] \Omega[(1 - \epsilon)(z + u)X]
$$

and $F(t) := \frac{1 - t}{1 + \ell}\cdot$ This is easily seen from

$$
G(u)G(z)P[X] = G(u)P \left[ X - \frac{1 - q}{z} \right] \Omega[(1 - \epsilon)zX] =
$$

$$
P \left[ X - (1 - q) \left( \frac{1}{z} + \frac{1}{u} \right) \right] \Omega[(1 - \epsilon)(z + u)X] \Omega[(1 - \epsilon)(q - 1)]z(u)\sum_{r \in \mathbb{Z}}2(-1)^r\ell^r.
$$

Note that first two factors are exactly $G(u, z)$ and $\Omega \left[ (1 - \epsilon)(q - 1)z(u)\sum_{r \in \mathbb{Z}}2(-1)^r\ell^r \right]$ is equal to $F(z/u)F(-z/u)$. Denote by $\alpha(t) := F(t) + F(t^{-1}) = \sum_{r \in \mathbb{Z}}2(-1)^r\ell^r$.

**Step 2.** Let $q_i^+$ be the adjoint of multiplication by $q_i[X]$ in the algebra $\Gamma$ under the scalar product $\langle, \rangle$. Consider $Q(t)$ its generating function, i.e., $Q(t) = \sum q_i^+t^i$.

It is not difficult to see that $P[X + t] = Q^+(t)P[X]$. In fact it suffices to show this for a suitable basis element in $\Gamma$, namely the power sum $p_\lambda[X]$, where $\lambda$ is an odd partition. We show that $q_i^+p_\lambda[X] = p_\lambda[X]t^{\mu_i}$.

$$
p_{\mu}[X + t] \mid_{t}^{\mu_i} = (p_{\mu_1}[X] + t^{\mu_1}) \ldots (p_{\mu_\ell}[X] + t^{\mu_\ell}) \mid_{t}^{\mu_i}.
$$

At the same time we also know that $p_{\lambda}p_\mu = 2^{-\ell(\lambda)}z_{\nu}^{-1}p_\nu$ if $\nu \looparrowright \lambda = \mu$ since

$$
p_{\lambda}p_\mu \mid_{p_\nu} = \left( p_{\lambda}p_\mu, \frac{2^{\ell(\nu)}z_{\nu}^{-1}}{z_\nu} \right)_{\Gamma} = \frac{2^{\ell(\nu)}z_{\nu}^{-1}}{2^{\ell(\mu)}z_\nu} \delta_{\mu, \lambda\nu}.
$$

Now because $q_i = \sum_{\lambda\nu, \mu}2^{\ell(\lambda)}p_\lambda/z_i$, then $q_i^+p_\nu[X] = \sum_{\lambda\nu, \mu}z_{\nu}^{-1}p_\nu$ where $\lambda\vdash i$ and this is exactly the right hand side of $[3]$. Also recall that $\Omega[(1 - \epsilon)\mu] = Q(t)$, where $Q$ is the generating functions for $q_i$. If we replace these in the expression of $G(z, u)$ we obtain

$$
G(z, u) = Q(z)Q(u)Q^+(-u^{-1})Q^+(-z^{-1})Q^+(u^{-1})Q^+(z^{-1})Q((1 - \epsilon)z).
$$

We know that $Q(u)Q(z)\alpha(u/z) = \alpha(u/z)$ (see Chap III. 8, p. 263), and hence we have as well $Q^+u)Q^+(z)\alpha(u/z) = \alpha(u/z)$. Therefore

$$
G(z, u)\alpha(u/z) = \alpha(u/z).
$$
Step 3. Consider the following expression.
\[
(G(u)G(z)(1 - qz/u)(1 + qu/z) + G(z)G(u)(1 - qu/z)(1 + qz/u)) = \\
= (1 + qz/u)(1 + qu/z)G(u, z)\alpha(u/z) \\
= (1 + qz/u + qu/z + q^2)\alpha(u/z),
\]
and so
\[
(G(u)G(z)(1 - qz/u + qu/z - q^2) + G(u)G(z)(1 - qu/z + qz/u - q^2)) \\
= (1 + qz/u + qu/z + q^2)\alpha(u/z) = (1 - q^2)\alpha(u/z).
\]
since we have as a series \(\alpha(t) = -\alpha(t)\). By taking the coefficient of \(u^r z^s\) in the left hand side when expanded as series and equating with the corresponding coefficient in the right hand side we obtain the desired relation.

For \(q = 0\) in the relation above we recover the commutation relations of the operator \(Q\) given in equations (18), (19) and (20). At \(q = 1\), \(G_m\) becomes multiplication by \(q_m\) and hence is commutative.

Formula (26) may be used to derive the action of the operator \(G_m\) on the basis of Schur’s \(Q\)-functions.

Proposition 3. For \(m > 0\),
\[
G_m(Q_\lambda[X]) = \sum_{i \geq 0} q^i \sum_{\mu : \lambda / \mu \in H_i} 2^{a(\lambda / \mu)}(-1)^{s(m+i, \mu)}Q_{\mu + (m+i)}[X],
\]
where \(\mu + (k)\) denotes the partition formed by adding a part of size \(k\) to the partition \(\mu\), and \(s(k, \mu) + 1\) represents which part \(k\) becomes in \(\mu + (k)\) (\(Q_{\mu + (k)}[X] = 0\) if \(\mu\) contains a partition of size \(k\)). For \(m \leq 0\) a similar statement can be made using the commutation relations (18), (19) and (20).

Proof From (26) the action of \(G_m\) on a function \(P[X] \in \Gamma\) can be written as
\[
G_m P[X] = P[X - (1 - q)/z] \Omega[(1 - \epsilon)zX] \big|_{z^m} \\
= \sum_{i \geq 0} q^i z^{-i}(q_i q_i^{-1}) P[X - 1/z] \Omega[(1 - \epsilon)zX] \big|_{z^m} \\
= \sum_{i \geq 0} q^i (q_i q_i^{-1}) P[X - 1/z] \Omega[(1 - \epsilon)zX] \big|_{z^{m+i}},
\]

since \(P[X + t] = \sum_{i \geq 0} q_i t^i P[X] t^i\). Thus
\[
G_m Q_\lambda[X] = \sum_{i \geq 0} q_i Q_{m+i}(q_i^{-1}Q_\lambda[X]),
\]
where \(q_i^{-1}\) applied to \(Q_\lambda\) is
\[
q_i^{-1}Q_\lambda[X] = \sum_{\mu : \lambda / \mu \in H_i} 2^{a(\lambda / \mu)}Q_\mu[X],
\]
If \(m > 0\), equation (26) follows from (18) and (20). If \(m \leq 0\) we may need to use the commutation relation (13) for \(Q_\alpha\) to straighten the index to a strict partition. In general we have \(Q_{m+i}(Q_\mu[X]) = Q_{(m+i, \mu)}[X]\), where we prepend the part \((m + i)\) (possibly negative) to the partition \(\mu\). \(\square\)
Example 1. We compute \( G_{(3,2,1)}[X; q] \) using the proposition above. We have
\[
G_{(3,2,1)}[X; q] = G_3(G_2(Q_{(1)}[X]))
\]
\[
= G_3 \left( \sum_{i \geq 0} \sum_{(1)/\mu \in H_i} 2^{\alpha((1)/\mu)} (-1)^{\epsilon(2+i,\mu)} Q_{\mu+(2+i)}[X] \right)
\]
\[
= G_3(Q_{(2,1)}[X]) + 2qG_3(Q_{(3)}[X])
\]
\[
= \sum_{i \geq 0} \sum_{(2,1)/\mu \in H_i} 2^{\alpha((2,1)/\mu)} (-1)^{\epsilon(3+i,\mu)} Q_{\mu+(3+i)}[X] +
\]
\[
+ 2q \sum_{i \geq 0} \sum_{(3)/\nu \in H_i} 2^{\alpha((3)/\nu)} (-1)^{\epsilon(3+i,\nu)} Q_{\nu+(3+i)}[X]
\]
\[
= (q^0 q^0 Q_{(3,2,1)} + q^1 2^1 Q_{(4,2)} + q^2 2^1 Q_{(5,1)})
\]
\[
+ 2q(q^1 2^1 Q_{(4,2)} + q^2 2^1 Q_{(5,1)} + q^3 2^1 Q_{(6)})
\]
\[
= Q_{(3,2,1)} + (2q + 4q^2)Q_{(4,2)} + (2q^2 + 4q^3)Q_{(5,1)} + 4q^4 Q_{(6)}.
\]

3.2. Properties of the polynomials \( L_{\lambda\mu}(q) \). The \( Q \)-Kostka polynomials introduced here have a number of remarkable properties that are very similar to those of Kostka-Foulkes polynomials listed in the previous section. We have already seen the analog of Property 2 holds for \( Q \)-Kostka polynomials. In what follows we will consider some of the other remaining properties.

An important consequence of equation (32) is a Morris-like recurrence which expresses the \( Q \)-Kostka polynomials \( L_{\lambda\mu}(q) \) in terms of smaller ones.

Proposition 4. We have the following recurrence

\[
(33) \quad L_{\alpha_s(n,\mu)}(q) = \sum_{(3)/\nu \in H_i} (-1)^{\epsilon \alpha_s - n} 2^{\alpha(\lambda/\alpha^{(s)})} L_{\lambda\mu}(q),
\]

where \( n > \mu_1 \) and \( \alpha^{(s)} \) is \( \alpha \) with part \( \alpha_s \) removed.

Proof If \( n > \mu_1 \) we have that
\[
G_n G_{\mu}[X; q] = G_{(n,\mu)}[X; q] = \sum_{\alpha} L_{\alpha(n,\mu)}(q)Q_{\alpha}[X].
\]
On the other hand \( G_{\mu}[X; q] = \sum_{\lambda} L_{\lambda\mu}(q)Q_{\lambda}[X] \) and so
\[
G_n \left( \sum_{\lambda} L_{\lambda\mu}(q)Q_{\lambda}[X] \right) = \sum_{\mu} L_{\lambda\mu}(q)G_n(Q_{\lambda}[X]).
\]
Using the action in (32) we have
\[
(35) \quad G_n G_{\mu}[X; q] = \sum_{\lambda} L_{\lambda\mu}(q) \sum_{i \geq 0} q^i \sum_{\nu,\lambda/\nu \in H_i} 2^{\alpha(\lambda/\nu)} (-1)^{\epsilon(n+i,\nu)} Q_{\nu+(n+i)}[X].
\]
For \( \alpha = \nu + (n + i) \), equating the coefficients of \( Q_{\alpha} \) in (34) and (35) we get
\[
L_{\alpha_s(n,\mu)}(q) = \sum_{\lambda} \sum_{i \geq 0} q^i \frac{1}{2} \alpha(\lambda/\alpha^{(s)} - (n+i)) (-1)^{\epsilon(n+i,\alpha^{(s)} - (n+i))} L_{\lambda\mu}(q).
\]
By reindexing \( i := \alpha_s - n \) for \( \alpha_s - n \geq 0 \) we obtain the desired recurrence (33). \( \square \)
Example 2. Let $n = 5$ and $L_{(6,2),(5,2,1)}(q) = 2q + 4q^2$. Using the recurrence we have one $s$ such that $\alpha_s \geq 5$, i.e. $\alpha_1 = 6$. So

$$L_{(6,2),(5,2,1)}(q) = q^{6-5} \sum_{\lambda/\mu \in H_1} 2^{a(\lambda/\mu)} L_{\lambda}(q)$$

$$= q(2L_{(21),(21)}(q) + 2L_{(3),(21)}(q)) = q(2 + 2 \cdot 2q) = 2q + 4q^2.$$  

As a consequence of the Morris-like recurrence we have the following

Corollary 5. Let $\mu \leq \lambda$ in dominance order.
1. If $n > \lambda_1$ then $L_{(n,\lambda_1),(n,\mu)}(q) = L_{\lambda\mu}(q)$.
2. $L_{\lambda\lambda}(q) = 1$ and $L_{(3),(\lambda)}(q) = 2^{\ell(\lambda)-1}q^{n(\lambda)}$.
3. $2^{\ell(\mu)-\ell(\lambda)}$ divides $L_{\lambda\mu}(q)$.

Proof 1. There is only one term in the recurrence (33) in this case which is exactly $L_{\lambda\mu}(q)$.

2. The first is a consequence of (1). For the second, we have that the only term on the right hand side is $q^{\ell(\lambda)-1} \cdot 2L_{(|\lambda|-\ell(\lambda))}(q)$ which by induction is $q^{\ell(\lambda)-1} \cdot 2^{\ell(\lambda)-2} = 2^{\ell(\lambda)-1}q^{n(\lambda)}$. This is the analog of Property 2 for the Kostka-Foulkes polynomials.

3. We use induction on $\ell(\mu)$ and the Morris recurrence to derive this property. If $\ell(\mu) = 1$ we know that $G_{(m)}[X; q] = Q_{(m)}[X]$ and thus $\lambda$ can be only $(m)$ and the assertion holds.

For the induction step we use equation (33). We need to show that $2^{\ell(\mu)-\ell(\lambda)+1}$ divides $L_{\alpha,n\mu}(q)$. From the induction hypothesis we know that $2^{\ell(\mu)-\ell(\lambda)}$ divides $L_{\lambda\mu}(q)$ for every such $\lambda$ on the left hand side of equation (33).

The partitions $\lambda$ have $\ell(\lambda) \in \{\ell(\alpha), \ell(\alpha) - 1\}$. If $\ell(\lambda) = \ell(\alpha)$ then we are done. If $\ell(\lambda) = \ell(\alpha) - 1$ and $\alpha \neq \lambda$, since $\alpha \neq \lambda$ has length one less than $\alpha$, and thus $a(\lambda/\alpha) \geq 1$. This implies that $2^{\ell(\mu)-\ell(\lambda)+1}$ divides $2^{a(\lambda/\alpha)} L_{\lambda\mu}(q)$ and thus $L_{\alpha,n\mu}(q)$.\qed

Using the Morris recurrence we are also able to obtain a formula for the degree of $L_{\lambda\mu}(q)$ similar to Property 1 for the Kostka-Foulkes polynomials.

Proposition 6. If $\mu \leq \lambda$ in dominance order, we have

$$\deg_q L_{\lambda\mu}(q) = n(\mu) - n(\lambda).$$

Proof We prove this assertion by induction on $\ell(\mu)$. For $\ell(\mu) = 1$, the equality is obvious.

For the induction step we use the recurrence (33). Fix now an index $s$ on the right hand side of the equation (33). Denote by $\mu(s) = (\alpha_1 + \alpha_2 - n, \alpha_2, \ldots, \alpha_s, \ldots, \alpha_{s+1}, \ldots)$. We claim that $n(\mu(s)) < n(\lambda)$ for any other $\lambda$ such that $\lambda/\alpha(s) \in H(\alpha_n)$. We have that

$$n(\mu(s)) = \sum_{i=2}^{s-1} (i-1)\alpha_i + \sum_{i=2}^{s-1} (s + i - 1)\alpha_{s+i+1}$$

while

$$n(\lambda) = \sum_{i=2}^{s-1} (i-1)(\alpha_i + \epsilon_i) + \sum_{i=0}^{s} (s + i - 1)(\alpha_{s+i+1} + \epsilon_{s+i}).$$
where \( \lambda = (\alpha_1 + \epsilon_1, \ldots, \alpha_s + 1 + \epsilon_{s-1}, \alpha_{s+1} + \epsilon_s, \ldots) \) and \( \sum \epsilon_i = \alpha_s - n \). Moreover if \( \lambda \neq \mu^{(s)}_{(0)} \) there exists at least one \( \epsilon_i \) with \( i \geq 2 \) such that \( \epsilon_i \neq 0 \).

Therefore \( n(\mu^{(s)}_{(0)}) < n(\lambda) \). We thus have proved that among the polynomials \( L_{\lambda \mu}(q) \) in the second sum of (33), the polynomial \( L_{\mu^{(s)}_{(0)}, \mu}(q) \) has the highest degree, namely \( n(\mu) - n(\mu^{(s)}_{(0)}) \).

Next we show that in the first sum, the highest degree is obtained for \( s = 1 \). That is to say \( \deg_q \left( q^{\alpha_1 - n} L_{\mu^{(1)}_{(0)}, \mu}(q) \right) > \deg_q \left( q^{\alpha_1 - n} L_{\mu^{(s)}_{(0)}, \mu}(q) \right) \), hence

\[
\alpha_1 - n + n(\mu) - n(\mu^{(1)}_{(0)}) > \alpha_s - n + n(\mu) - n(\mu^{(s)}_{(0)}),
\]

\[
\alpha_1 + \sum_{i=2}^{s-1} (i-1)\alpha_i + \sum_{j \geq s+1} (j-2)\alpha_j > \alpha_s + \sum_{i=3}^{s-1} (i-2)\alpha_i + \sum_{j \geq s} (j-2)\alpha_j
\]

which is

\[
\alpha_1 + \alpha_2 + \sum_{i=3}^{s-1} (i-1)\alpha_i > \sum_{i=3}^{s-1} (i-2)\alpha_i + (s-1)\alpha_s = (\alpha_3 + \alpha_s) + (2\alpha_4 + \alpha_s) + \cdots + [(s-3)\alpha_{s-1} + \alpha_s] + 2\alpha_s.
\]

The last inequality is true as \((i-1)\alpha_i > (i-2)\alpha_i + s\) for \( i = 3, \ldots, s-1 \) and \( \alpha_1 + \alpha_2 > 2\alpha_s \).

Thus we have \( \deg_q L_{\alpha, (n, \mu)}(q) = (\alpha_1 - n) + n(\mu) - n(\mu^{(1)}_{(0)}) \). Finally we need to show this is in fact \( n((n, \mu)) - n(\alpha) \). That is

\[
\alpha_1 - n + \sum_{i \geq 2} (i-1)\mu_i - \sum_{i \geq 3} (i-2)\alpha_i = \sum_{i \geq 0} \mu_i - \sum_{i \geq 2} (i-1)\alpha_i,
\]

and by simplifying we obtain \( \sum \alpha_i - n = \sum \mu_i \), which is obviously true.

Hence \( \deg_q \left( L_{\alpha, (n, \mu)}(q) \right) = n((n, \mu)) - n(\alpha) \) and the proof is complete. \( \square \)

The property that is most suggestive that these polynomials are analogs of the Kostka-Foulkes polynomials is

**Conjecture 7.** The Q-Kostka polynomials \( L_{\lambda \mu}(q) \) have non-negative coefficients.

We will prove this conjecture for some particular cases. In general we believe that there should exist a similar combinatorial interpretation as for the Kostka-Foulkes polynomials. More precisely there should exist a statistic function \( d \) on the set of marked shifted tableaux, similar to the charge function on column strict tableaux, such that

\[
L_{\lambda \mu}(q) = \sum_T q^{d(T)}
\]

summed over marked shifted tableaux of shifted shape \( \lambda \) and content \( \mu \) with diagonal entries unmarked.

In addition, we conjecture that this function must have the property that if \( T \) and \( S \) are two marked shifted tableaux such that by erasing the marks the two resulting tableaux coincide, then \( d(T) = d(S) \).

For some of the polynomials \( L_{\lambda \mu}(q) \), this observation determines completely the statistic on the tableaux. For instance there are two marked shifted tableaux classes of shape \((5,3,1)\) and content \((4,3,2)\) and \( L_{(5,3,1),(4,3,2)}(q) = 2q + 4q^2 \). Clearly the tableau with a 3 in the first row must have statistic 1 and with 3 in the second row
has statistic 2. On the other hand, \( L_{\circ(1), (4,3,2)}(q) = 4q^5 + 4q^6 \). This polynomial does not uniquely determine which of the two tableaux have statistic 5 and 6. We have used the function \( G_{(4,3,2)}[X; q] \) to draw a conjectured tableau poset (similar to the case of column strict tableau) for the marked shifted tableaux with unmarked diagonals of content \((4,3,2)\) in an appendix.

Another intriguing property of this statistic function \( d \) is that the values it takes are not too different than the charge function. It seems that in general we have that for given \( \lambda \) and \( \mu \) the set of \( \{ d(T), T \} \) in the summation of \( L_{\lambda\mu}(q) \) is a subset of \( \{ c(T), T \} \) column strict tableaux of shape \( \lambda \) and content \( \mu \), where \( c \) is the usual charge. This suggests that there should be a relationship between these two statistics; however, we have so far not been able to establish what that link might be.

**Proposition 8.** (1) For a two-row partition and \( \lambda > \mu \) we have \( L_{\lambda\mu}(q) = 2q^{n(\mu) - n(\lambda)} \).

(2) If \( \mu \) has the property \( \mu_i \geq \sum_{j=i+1} \mu_j \), Conjecture 3 is true.

**Proof** (1). Let us consider \( G_{\mu}[X; q] \) and let \( \mu = (n, m) \). We have that \( G_{(n,m)}[X; q] = G_n(Q_{(m)}, X) \) which by Proposition 3 is

\[
G_n(Q_{(m)}[X]) = \sum_{i \geq 0} q^i \sum_{\mu: (m)/\mu \in \mathcal{H}_i} 2^{n((m)/\mu)} (-1)^{\ell(n, i, \mu)} Q_{\mu + (n+i)}[X].
\]

From this we deduce that \( i = 0, 1, \ldots, m \) and \( \mu = (m - i) \). Thus

\[
G_n(Q_{(m)}[X]) = Q_{(n,m)}[X] + \sum_{i=1}^m 2^i Q_{(n+i,m-i)}[X]
\]

and the proof is complete.

(2). In this case we prove it by induction on \( \ell(\mu) \) and using the Morris recurrence 3.

The case \( \ell(\mu) = 1 \) is clear as \( L_{\lambda\mu}(q) = \delta_{\lambda\mu} \). For the induction step consider \( L_{\alpha(n,m)}(q) \) as in the right hand side of (33). Under our assumption there is just one index in the first sum, i.e., only \( \alpha_1 \) can be greater than \( n \). This is true since \( |\alpha| = n + |\mu| \), \( \alpha \geq (n, \mu) \) in dominance order and \( n \geq |\mu| \). Thus the right hand side does not contain negative signs and by induction it is non-negative. Hence \( L_{\alpha(n,m)}(q) \) has non-negative coefficients.

We also note that monotonicity properties, similar to Property 7 and 8, hold for the \( Q \)-Kostka polynomials.

**Conjecture 9.** Let \( L'_{\lambda\mu}(q) := q^{n(\mu) - n(\lambda)} L_{\lambda\mu}(q^{-1}) \). We have

\[
L'_{\lambda\mu}(q) \geq 2^{\ell(\nu) - \ell(\mu)} L'_{\lambda\nu}(q), \quad \text{for } \mu \leq \nu \text{ in dominance order}.
\]

We can prove this fact by using induction and the recurrence (33) for the case \( \mu_1 = \nu_1 \).

**Example 3.** Let \( \lambda = (6, 2), \mu = (4, 3, 1), \nu = (5, 2, 1) \). We have \( n(\lambda) = 2, n(\mu) = 5 \), and \( n(\nu) = 4 \). The \( L' \) polynomials are

\[
L'_{\lambda\mu} = q^{5-2}(4q^2 + 4q^3) = 4 + 4q, \quad L'_{\lambda\nu} = q^{4-2}(2q + 4/q^2) = 4 + 2q,
\]

and thus \( L'_{\lambda\mu}(q) \geq 2^{3-3} L'_{\lambda\nu}(q) \).
Another property of the Kostka-Foulkes polynomials case that seems to hold in our case refers to the growth of the polynomials $L$. For the Kostka-Foulkes polynomials the conjecture belongs to Gupta (see [3] and references therein).

**Conjecture 10.** If $r$ is an integer that is not a part in either partitions $\lambda$ or $\mu$, then

$$L_{\lambda+(r),\mu+(r)}(q) \geq L_{\lambda\mu}(q).$$

The case where $r > \lambda_1$ (which also ensures that $r > \mu_1$) is obviously true since $L_{(r,\lambda),(r,\mu)}(q) = L_{\lambda\mu}(q)$ (see Corollary 3).

**Example 4.** Let $\lambda = (5,3), \mu = (4,3,1)$ and $a = 2$. We have

$$L_{(5,3,2),(4,3,2,1)}(q) - L_{(5,3),(4,3,1)}(q) = 2q + 4q^2 + 8q^3 - (2q + 4q^2) = 8q^3.$$

3.3. **Another expression for $L_{\lambda\mu}(q)$**. The polynomials $L_{\lambda\mu}(q)$ have a similar interpretation to property [3] using an analog of the $q$-Kostant partition function. We follow the construction in [3]. In order to write equation (16) as

$$L_{\lambda}\left(\frac{1}{1-R_{ij}}\right)^{-1} Q_{\lambda}[X].$$

we will use linear maps from the group algebra $\mathbb{Z}[\mathbb{Z}^n]$ to the algebra $\Gamma$. A basis of $\mathbb{Z}[\mathbb{Z}^n]$ will consist of formal exponentials $\{e^\alpha\}_{\alpha \in \mathbb{Z}}$ which satisfy relations $e^\alpha e^\beta = e^\alpha + e^\beta$. In fact we identify the ring with the ring of Laurent polynomials in $x_1, \ldots, x_n$ and set $e^\alpha = x^\alpha$. With this in mind we are viewing all our polynomials in $\Gamma$ (or $\Lambda_n$) as linear homomorphisms from $\mathbb{Z}[\mathbb{Z}^n]$ to $\Gamma$ i.e.

$$Q : e^\lambda \rightarrow Q(e^\lambda) = Q_{\lambda} \quad q : e^\lambda \rightarrow q(e^\lambda) = q_{\lambda}.$$

If we now set $\zeta_n := \prod_{1 \leq i < j \leq n} \left(1 + \frac{x_i}{x_j}\right)$, we have that $\zeta_n = \sum_{\alpha \in \mathbb{Z}^n} R(\alpha)e^\alpha$ where $R(\alpha) = \sum_i a_i\alpha^2$ and $a_i$ counts the number of ways the vector $\alpha$ can be written as a sum of positive roots of type $\Lambda_n$, $t$ of which are distinct. The positive roots in the root lattice of $A_{n-1}$ are $\{e_i - e_j\}_{1 \leq i < j \leq n}$, where $e_i = (0, \ldots, 1, \ldots) = 0$ is the canonical basis of $\mathbb{Z}^n$.

Since $q(e^\lambda) = Q(\zeta_n e^\lambda)$ we have that

$$q_{\lambda}[X] = q(e^\lambda) = Q(\sum_{\alpha \in \mathbb{Z}^n} R(\alpha)e^\alpha e^\lambda) = \sum_{\alpha \in \mathbb{Z}^n} R(\alpha)Q_{\lambda + \alpha}[X].$$

If we consider the same argument for $G_{\lambda}[X, q] = \prod_{1 \leq i < j \leq n} \left(1 + qR_{ij}\right) Q_{\lambda}[X]$ we need to define the $q$-analog of $\zeta_n$ as

$$\zeta_n(q) := \prod_{1 \leq i < j \leq n} \left(1 + qx_i/x_j\right),$$

and thus $\zeta_n(q) = \sum_{\alpha \in \mathbb{Z}^n} R_q(\alpha)e^\alpha$, where $R_q(\alpha) = \sum a_{t,k} 2^t q^k$ and $a_{t,k}$ counts the number of ways the vector $\alpha$ can be written as a sum of $k$ positive roots, $t$ of which
are distinct. Hence

\[ G_\lambda [X, q] = \sum_{\alpha \in \mathbb{Z}^n} R_q(\alpha) Q_{\lambda + \alpha} [X]. \]

This yields another expression for the \(Q\)-Kostka polynomials in terms of \(R_q(\alpha)\) as

\[ L_\lambda(\mu)(q) = \sum_{\alpha : \mu + \lambda = \pm 2^r \lambda} \pm 2^r R_q(\alpha). \]

The index of the sum reflects the straightening of a \(Q\)-function indexed by an integer sequence and it is a consequence of the commutation relations (19). It is possible to express the equation above using the action of the symmetric group on Schur’s \(Q\)-functions, yielding an alternating sum similar to Property 9. Unfortunately the action of the symmetric group on Schur’s \(Q\)-functions indexed by a general integer vector is not as elegant as for Schur functions.

**Remark:** Most of the properties of the \(Q\)-Kostka polynomials \(L_\lambda(\mu)(q)\) are analogous to the Kostka-Foulkes polynomials, but a few properties do not seem to generalize.

1. The analog of Property 6 does not seem to hold since computations of \(G_\lambda [X; q]\) where \(q\) is set to a root of unity do not factor.
2. There does not seem to exist an elegant relationship between \(G_\lambda [X; q]\) and its dual basis (Property 11).
3. A property similar to that of Property 10 does not seem to hold. We do not know if there is a relationship between \(G_\lambda [X; q]\) and a Hall-like algebra.
4. We do not know if an analog of the Macdonald symmetric functions should exist. A family of functions which mimic the formulas for the Macdonald symmetric functions in [26] may easily be defined, but the specializations of the variables indicate that the same sort of positivity and symmetry properties of the coefficients cannot hold through this definition.

### 4. Generalized (parabolic) \(Q\)-Kostka Polynomials

There exists in the literature a few generalizations of the Kostka-Foulkes polynomials that correspond to \(q\)-analogs of multiplicities of irreducibles in tensor products of irreducible representations (Littlewood-Richardson coefficients). In [22] formulas were introduced for realizing ‘generalized’ or ‘parabolic’ Kostka coefficients [21] as coefficients appearing in families of symmetric functions defined as compositions of operators. This construction may also be extended to the \(Q\)-function algebra providing a means of defining a generalization of the \(Q\)-Kostka polynomials that corresponds to a \(q\)-analogue of coefficients in products of \(Q_\mu [X]\).

Let \(\mu^* = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)})\) be a sequence of partitions and \(\bar{\mu}^*\) the concatenation of all those partitions. Take \(\eta = (\ell(\mu^{(1)}), \ell(\mu^{(2)}), \ldots, \ell(\mu^{(k)}))\) and \(n = \sum \eta_i\), then set \(\text{Roots}_\eta = \{(i, j) : 1 \leq i \leq \eta_1 + \cdots + \eta_r < j \leq n \text{ for some } r\}\).

Now set

\[ (37) \quad H_{\mu^*}[X; q] = \prod_{(i, j) \in \text{Roots}_\eta} \frac{1}{1 - qR_{ij}} s_{\mu^*} [X]. \]

The parabolic Kostka polynomials are defined as the coefficients of the Schur basis in these symmetric functions. That is, the polynomials \(K_{\lambda, \mu^*}(q)\) are defined
by the coefficients in the expansion

\begin{equation}
H_{\mu^*}(X; q) = \sum_{\lambda \vdash |\mu^*|} K_{\lambda; \mu^*}(q)s_\lambda[X].
\end{equation}

The functions \( H_{\mu^*}(X; q) \) and the parabolic Kostka coefficients have the following properties.

- If \( \bar{\mu} \) is a partition then it is conjectured that \( K_{\lambda; \mu^*}(q) \) has non-negative integer coefficients (in certain cases this is known).
- \( H_{\mu^*}(X; 0) = s_{\bar{\mu}^*}[X] \). \( \mu^* \) need not be a partition, but this is consistent with the definition of \( s_\lambda \) in section 2.1.
- \( H_{\mu^*}(X; 1) = s_{\mu^{(1)}}[X]s_{\mu^{(2)}}[X] \cdots s_{\mu^{(k)}}[X] \) and in this sense the coefficients \( K_{\lambda; \mu^*}(q) \) are \( q \)-analogos of the Littlewood-Richardson coefficients.
- If \( \mu^* = ((\gamma_1), (\gamma_2), \ldots, (\gamma_{\ell(\gamma)})) \) where \( \gamma \) is a partition, then \( H_{\gamma}(X; q) = H_{\gamma}[X; q]. \)
- If \( \bar{\mu} \) is a partition then \( H_{\mu^*}(X; q) = s_{\bar{\mu}^*}[X] + \sum_{\lambda > \mu^*} K_{\lambda; \mu^*}(q)s_\lambda[X]. \)
- There exists an operator \( \mathbf{H} \), such that \( \mathbf{H}_\gamma(H_{\mu^*}(X; q)) = H_{(\gamma^{(1)}, \ldots, \gamma^{(k)})}[X; q] \) (see 22).

In addition, analogs of most properties of the Hall-Littlewood functions and the Kostka-Foulkes polynomials also seem to hold (see for instance 22).

We should also mention that there is an analog of several other formulas for the Hall-Littlewood functions. It follows from the definition of the functions \( H_{\mu^*}(X; q) \) that

\begin{equation}
H_{\mu^*}(X; q) = \Omega[Z_nX] \prod_{1 \leq i < j \leq n} (1 - z_j/z_i) \prod_{(i, j) \in \text{Roots}_n} \frac{1}{1 - qz_j/z_i} |z_{\mu^*}^\gamma|,
\end{equation}

For \( k > 0 \), if we define the operation,

\begin{equation}
\mathbf{H}(Z^k)P[X] = P[X - (1 - q)Z^\ast]\Omega[ZX] \prod_{1 \leq i < j \leq k} (1 - z_j/z_i),
\end{equation}

where \( Z^\ast = \sum_{i=1}^{k} \frac{1}{z_i} \), then

\begin{equation}
\mathbf{H}(Z^n)\mathbf{H}(Z^{n^2}) \cdots \mathbf{H}(Z^{n^{(n)}})1 = \Omega[Z_nX] \prod_{1 \leq i < j \leq n} (1 - z_j/z_i) \prod_{(i, j) \in \text{Roots}_n} \frac{1}{1 - qz_j/z_i}
\end{equation}

and therefore \( \mathbf{H}(Z^k)H_{\mu^*}(X; q) \bigg|_{z^\gamma} = H_{(\gamma, \mu^*)}[X; q]. \)

This construction exists in complete analogy within the Q-function algebra. We will create a family of functions in \( \Gamma \) which are indexed by a sequence of strict partitions. Let \( \mu^* = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}) \) where \( \mu^{(i)} \) is a strict partition and set \( \eta = (\ell(\mu^{(1)}), \ell(\mu^{(2)}), \ldots, \ell(\mu^{(k)})) \). Define \( \text{Roots}_n^\eta \) as before and then define the function

\begin{equation}
G_{\mu^*}(X; q) = \prod_{(i, j) \in \text{Roots}_n^\eta} \frac{1 + qR_{ij}}{1 - qR_{ij}} Q_{\bar{\mu}^*}[X].
\end{equation}
We may also view these elements of $\Gamma$ as the result of a family of operators acting on 1. Consider the composition of the operators
\[ Q_\lambda P[X] := Q_{\lambda_1}Q_{\lambda_2} \cdots Q_{\lambda_k}P[X] \]
\[ = P[X - Z_k^\lambda]\Omega[(1 - \epsilon)Z_kX] \prod_{1 \leq i < j \leq k} \frac{1 - z_j/z_i}{1 + z_j/z_i} z^\lambda \]
and then set $G_\lambda := \bar{Q}_\lambda q^q$, that is
\[ G_\lambda P[X] = P[X + (q - 1)Z_k^\lambda]\Omega[(1 - \epsilon)Z_kX] \prod_{1 \leq i < j \leq k} \frac{1 - z_j/z_i}{1 + z_j/z_i} z^\lambda. \]

As with the other operators of this sort, it is easily shown that a composition of $G_{\mu(1)}$ acting on 1 is equivalent to the defining relation (12) of the functions $G_{\mu^*}[X; q]$ and hence that
\[ G_{\mu^*}[X; q] = G_{(\gamma_1, \mu(1), \ldots, \mu(k))}[X; q]. \]

The $G_{\mu^*}[X; q]$ functions seem to share many of the same properties of the $H_{\mu^*}[X; q]$ and $G_{\gamma}[X; q]$ analogs. Define the polynomials $L_{\lambda, \mu^*}(q)$ by the expansion
\[ G_{\mu^*}[X; q] = \sum_{\lambda} L_{\lambda, \mu^*}(q)Q_\lambda[X]. \]

- $G_{\mu^*}[X; 0] = Q_{\mu^*}[X]$. $\mu^*$ need not be a strict partition, however if it is not then the straightening relations (18), (19) and (20) may be applied to reduce the expression.
- $G_{\mu^*}[X; 1] = Q_{\mu(1)}[X]Q_{\mu(2)}[X] \cdots Q_{\mu(k)}[X]$ and hence $L_{\lambda, \mu^*}(1)$ is equal to the coefficient of $Q_\lambda[X]$ in the product $Q_{\mu(1)}[X]Q_{\mu(2)}[X] \cdots Q_{\mu(k)}[X]$.
- If $\mu^* = ((\gamma_1), (\gamma_2), \ldots, (\gamma_{t(\gamma)}))$ where $\gamma$ is a partition, then $G_{\mu^*}[X; q] = G_{\gamma}[X; q]$.
- If $\mu^*$ is a strict partition then $G_{\mu^*}[X; q] = Q_{\mu^*}[X] + \sum_{\lambda > \mu^*} L_{\lambda, \mu^*}(q)Q_\lambda[X]$.

Computing these coefficients suggests the following remarkable conjecture and indicates that these coefficients are an important $q$-analog of the structure coefficients of the $Q_\lambda[X]$ functions in the same way that the $K_{\lambda, \mu^*}(q)$ polynomials are $q$-analogs of the Littlewood-Richardson coefficients.

**Conjecture 11.** For a sequence of partitions $\mu^*$, if $\mu^*$ is a partition then $L_{\lambda, \mu^*}(q)$ is a polynomial in $q$ with non-negative integer coefficients.

If this conjecture is true then the polynomials $L_{\lambda, \mu^*}(q)$ are a $q$ analog of the coefficient of $Q_\lambda[X]$ in the product $Q_{\mu(1)}[X]Q_{\mu(2)}[X] \cdots Q_{\mu(k)}[X]$. A combinatorial description for these coefficients was given in [23] and hence we are looking for an additional statistic on the set of objects counted by them which includes as a special case the coefficients $L_{\lambda, \mu}(q)$.

This conjecture suggests that the $L_{\lambda, \mu^*}(q)$ should also share many of the properties that are held by the $K_{\lambda, \mu^*}(q)$ and that generalize the case of the Kostka-Foulkes polynomials.

We remark that the parabolic Kostka polynomials indexed by a sequence of partitions $\mu^*$ where each $\mu^{(i)}$ is a rectangle (i.e. each $\mu^{(i)} = (a_i, a_i, \ldots, a_i)$ for some $a_i$) is a special subfamily of these polynomials. In this case, explicit combinatorial formulas are known for the coefficients (see for example [19], [24] or [12]) which...
imply that the coefficients $K_{\lambda,\mu^*}(q)$ are positive. By contrast, for the generalizations of the $Q$-Kostka polynomials we know that if $\mu^{(i)}$ has two equal parts for any $i$ then $G_{\mu^*}[X;q] = 0$, hence this special case is not of interest in this setting.

5. Appendix: Tables of $2^{\ell(\lambda)-\ell(\mu)} L_{\lambda\mu}(q)$ for $n = 4, 5, 6, 7, 8, 9$

\[
\begin{bmatrix}
(3, 1) & (4) \\
1 & q \\
0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
(3, 2) & (4, 1) & (5) \\
1 & 2q & q^2 \\
0 & 1 & q \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
(3, 2, 1) & (4, 2) & (5, 1) & (5, 1) & (6) \\
1 & 2q^2 + q & 2q^3 + q^2 & q^4 \\
0 & 1 & 2q & q^2 \\
0 & 0 & 1 & q \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
(4, 2, 1) & (4, 3) & (5, 2) & (6, 1) & (6, 1) & (7) \\
1 & q & 2q^2 + q & 2q^3 + q^2 & q^4 \\
0 & 1 & 2q & 2q^2 & q^3 \\
0 & 0 & 1 & 2q & q^2 \\
0 & 0 & 0 & 1 & q \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
(4, 3, 1) & (5, 2, 1) & (5, 3) & (6, 2) & (6, 2) & (7, 1) & (7, 1) & (8) \\
1 & 2q & 2q^2 + q & 2q^2 + 2q^3 & q^5 + 2q^4 & q^5 \\
0 & 1 & q & 2q^2 + q & 2q^3 + q^2 & q^4 \\
0 & 0 & 1 & 2q & 2q^2 & q^3 \\
0 & 0 & 0 & 1 & 2q & q^2 \\
0 & 0 & 0 & 0 & 1 & q \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
(4, 3, 2) & (5, 3, 1) & (5, 4) & (6, 2, 1) & (6, 3) & (7, 2) & (8, 1) & (9) \\
1 & 2q + 4q^2 & 2q^3 + q^2 & 2q^2 + 4q^3 & q^2 + 2q^4 + 4q^5 & 4q^4 + q^3 + 2q^5 & 2q^6 + 2q^5 & q^7 \\
0 & 1 & q & 2q & 2q^2 + q & 2q^2 + 2q^3 & q^3 + 2q^4 & q^5 \\
0 & 0 & 1 & 0 & 2q & 2q^2 & 2q^3 & q^4 \\
0 & 0 & 0 & 1 & q & 2q^2 + q & 2q^3 + q^2 & q^4 \\
0 & 0 & 0 & 0 & 1 & 2q & 2q^2 & q^3 \\
0 & 0 & 0 & 0 & 0 & 1 & 2q & q^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & q \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
6. **Appendix: Example of Conjectured Tableaux Poset of Content**

\((4, 3, 2)\)

![Diagram of tableaux poset]

**Figure 2.** The cells marked with a \(k^*\) can be labeled with either \(k\) or \(k'\), we conjecture that the statistic is independent of these markings. The value of \(G_{(4,3,2)}[X;q]\) determines the position of each of the shifted tableaux here except for the two of shape \((8,1)\), however the statistics in smaller polynomials (e.g. \(G_{(4,3,1)}[X;q]\)) suggest this rank function. The covering relation is unknown, but the rank function indicates that it is not the same as the charge statistic.

**Acknowledgment:** Thank you to Nantel Bergeron for many helpful suggestions on this research.

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E-mail address: tudose@math.umn.edu, zabrocki@mathstat.yorku.ca

School of Mathematics, University of Minnesota, Minneapolis, Minnesota, 55455 and Department of Mathematics and Statistics, York University, Toronto, Ontario, M3J 1P3