Energy of the Bardeen model using an approximate symmetry method

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Abstract
In this paper, we investigate the energy problem in general relativity using approximate Lie symmetry methods for differential equations. This procedure is applied to the Bardeen model (the regular black-hole solution). Here, we are forced to evaluate the third-order approximate symmetries of the orbital and geodesic equations. It is shown that energy must be re-scaled by some factor in the third-order approximation. We discuss the insights into this re-scaling factor.

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1. Introduction
Energy–momentum is an important conserved quantity in any physical theory and its definition has been under investigation for a long time from the viewpoint of general relativity. Unfortunately, there is still no generally accepted definition of energy–momentum for gravitational fields. The main problem is with the expression defining the gravitational field energy part. To obtain a meaningful expression for the energy, momentum and angular momentum for a general relativistic system, Einstein proposed an expression called Einstein’s energy–momentum complex [1]. After that, many complexes have been found, for instance by Landau–Lifshitz [2], Tolman [3], Papapetrou [4], Möller [5], Weinberg [6] and Bergmann and Thompson [7]. Some of these definitions are coordinate dependent, whereas others are not. Also, some of these expressions cannot be used to define angular momentum.

Virbhadra [8] gave marvelous information about coincidence of some complexes, which attracted many researchers. Aguirreagabiria et al [9] proved that Einstein, Landau–Lifshitz, Papapetrou and Weinberg prescriptions yielded the same distribution of energy for any metric of the Kerr–Schild class if calculations are performed in the Kerr–Schild coordinates. Vagenas [10] and Sharif [11] proved that these four prescriptions coincide for the (2+1)-dimensional rotating BTZ black hole (named after Banados, Teitelboim and Zanelli) and the regular black-hole solution, respectively. Xulu [12] extended this investigation and found the same energy distribution for the dyadosphere of the Reissner–Nordström (RN) black hole. However, all these prescriptions have their own drawbacks.

The concept of approximate symmetry theories comes from the combination of Lie group theory and perturbations. Using this idea, two different so-called ‘approximate symmetry’ theories have been constructed. The first theory was proposed by Baikov et al [13] and the second theory by Fushchich and Shtelen [14]. It should be mentioned here that a manifold having no exact symmetry can possess approximate symmetries. As compared to exact symmetry, one can acquire more interesting information from a slightly broken (approximate) symmetry. This idea of symmetries would be more accurate as it does not depend on pseudo-tensors and hence agrees with the principles of general relativity. In order to discuss the approach of approximate symmetries, we use the concept of Lie symmetries that is substantiated in spacetime isometries as well as in Noether symmetries.

Kara et al [15] used the approximate Lie symmetry method [13] to discuss the conservation laws of the Schwarzschild spacetime. Later, using the same procedure, in the second-order approximation, the lost conservation laws for RN [16], Kerr–Newmann [17] and Kerr–Newmann AdS [18] spacetimes were recovered and some energy re-scaling factors were obtained. In a recent paper [19], we have discussed re-scaling of energy in the stringy charged black-hole solutions using approximate symmetries. In going from the Minkowski spacetime to the non-flat regular black-hole solution, the recovered conservation laws (recovered in first-order approximation) are lost. It is expected that by minimizing the charge, one should recover all lost...
conservation laws. In this paper, we use the same procedure to formulate the energy re-scaling factor for the regular black-hole solution.

The format of the paper is as follows. In the next section, we review the basic material about approximate symmetry methods for the solution of differential equations (DEs) and also give exact symmetries of the Minkowski spacetime and the first-order approximate symmetries of the Schwarzschild spacetime. In section 3, we study approximate symmetries of the regular black-hole solution. Finally, we summarize and discuss the results.

2. Mathematical formalism

Symmetry is a point transformation (a transformation that maps one point \((x, y)\) into another \((x', y')\)) under which the form of DE does not change. These transformations form a group known as point transformation group. Symmetries are important for their direct connection with the conservation laws through the Noether theorem [20]. It has been mentioned that if, for a given system of DEs, there is a variational principle, then a continuous symmetry invariant under the action of the given functional provides the corresponding conservation law [21, 22].

A system of \(p\) ODEs, each of \(n\)-th order [23, 24]

\[
E_\alpha(s; x(s), x'(s), x''(s), \ldots, x^{(n)}(s)) = 0 \quad (\alpha = 1, 2, 3, \ldots, p)
\]

(here \(x\) is a dependent variable, \(s\) is an independent variable and \(x', x'', \ldots, x^{(n)}\) denote the first, second and so on \(n\)-th order derivatives of \(x\) w.r.t. \(s\)) under the point transformation \((s, x) \rightarrow (\xi(s, x), \eta(s, x))\) can admit a symmetry generator

\[
X = \xi(s, x) \frac{\partial}{\partial s} + \eta(s, x) \frac{\partial}{\partial x}
\]

if and only if, on the solution of the ODEs \(E_\alpha = 0\), the following symmetry condition,

\[
X^{(n)}(E_\alpha)|_{E_\alpha=0} = 0,
\]

is satisfied. Here \(n\)-th order extension of the symmetry generator given by equation (2) can be written as

\[
X^{(n)} = \xi(s, x) \frac{\partial}{\partial s} + \eta(s, x) \frac{\partial}{\partial x} + \eta_1(s, x, x') \frac{\partial}{\partial x'} + \cdots + \eta_{n}(s, x, x', \ldots, x^{(n)}) \frac{\partial}{\partial x^{(n)}}
\]

and the corresponding prolongation coefficients are

\[
\eta_1 = \frac{d\eta}{ds}, \quad \eta_2 = \frac{d\eta_1}{ds} - x^{(0)} \frac{d\xi}{ds}, \quad n \geq 2.
\]

The \(k\)-th order approximate symmetries of a perturbed ODE

\[
E = E_0 + \epsilon E_1 + \epsilon^2 E_2 + \cdots + \epsilon^k E_k + O(\epsilon^{k+1}); \quad \epsilon \in \mathbb{R}^+
\]

are given by the generator

\[
X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots + \epsilon^k X_k
\]

if the following symmetry condition holds [25]:

\[
XE = [(X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots + \epsilon^k X_k)](E = E_0 + \epsilon E_1 + \epsilon^2 E_2 + \cdots + \epsilon^k E_k) \mid_{E_0 + \epsilon E_1 + \epsilon^2 E_2 + \cdots + \epsilon^k E_k} = O(\epsilon^{k+1}).
\]

Here \(E_0\) is the exact part of ODE, \(E_1\) and \(E_2\) are, respectively, the first- and second-order approximate parts of the perturbed ODE and so on. \(X_0\) represents the exact symmetry generator and \(X_1, X_2\) denote, respectively, the first- and second-order approximate parts of the symmetry generator and so on. For the \(k\)-th order approximate symmetry generator, the terms involving \(\epsilon^{k+1}\) and its higher powers can be substituted to zero so that the rhs of equation (8) becomes zero. As compared to exact symmetries, these approximate symmetries do not necessarily form a Lie algebra, rather than do form the so-called ‘approximate Lie algebra’ (up to a specified degree of precision) [26]. The perturbed equation always admits an approximate symmetry \(\epsilon X_0\) called a trivial symmetry. If a symmetry generator \(X = X_0 + \epsilon X_1\) exists with \(X_0 \neq 0\) and \(X_1 \neq kX_0\), where \(k\) is an arbitrary constant, then it is called nontrivial symmetry [27].

It has been mentioned [28] that ten Killing vectors (appendix A) of Minkowski spacetime provide conservation laws for energy and linear momentum as well as for angular and spin angular momentum. The algebra calculated from the geodesic equations contains some additional symmetries which provide no conservation law [29]. The Schwarzschild spacetime has four isometries \(Y_0, Y_1, Y_2, Y_3\), which correspond to conservation laws of energy and angular momentum only. The symmetry algebra of the Schwarzschild spacetime (calculated through the geodesic equations) consists of four isometries and the dilation algebra. In the limit of small mass of a point gravitating source, \(\epsilon = 2M [15]\), all lost conservation laws would be recovered as first order trivial approximate symmetries. For the orbital equation, the first-order approximate symmetries are given in appendix B.

3. The Bardeen model

The RN solution is a static, spherically symmetric and asymptotically flat solution of the Einstein–Maxwell field equations having a singularity structure. There arise some problems when one applies the laws of physics at the singularity point (origin of the radial coordinate) of the RN solution. To resolve this problem, Bardeen constructed a singularity-free solution [30, 31] using the energy–momentum tensor of nonlinear electrodynamics as the source of the field equations. The Bardeen model is known as a regular black-hole solution because for a particular ratio of mass to charge, this represents a black hole and a singularity-free structure. After that many regular black-hole solutions have been discussed by various authors, for example [32]. These metrics, being analogues to the RN metric with magnetic charge, have quite a similar global but regular structure as that for the RN spacetime.
3.1. Isometry algebra and orbital equation of motion

The line element representing the Bardeen model is given by [31]

\[ ds^2 = \left[ 1 - \frac{2mr^2}{(r^2 + e^2)^{3/2}} \right] dt^2 - \left[ 1 - \frac{2mr^2}{(r^2 + e^2)^{3/2}} \right]^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

(9)

The asymptotic behavior of the solution can be observed by the simple expansion of the term \( g_{tt} \), as follows:

\[ g_{tt} = 1 - \frac{2m}{r} + \frac{3me^2}{r^3} + O \left( \frac{1}{r^5} \right), \]

where \( m \) is the mass of the configuration. When \( r \to \infty \), the parameter \( e \) does not vanish as the cylindrical term \( \frac{r}{r^2} \) but vanishes as the term \( \frac{1}{r} \). Therefore, one cannot recognize the parameter \( e \) as the Coulomb charge \( Q \) of the RN spacetime, for which \( g_{tt} \) is given by

\[ g_{tt} = 1 - \frac{2m}{r} + Q^2 r^2. \]

Indeed, in the Bardeen model, the parameter \( e \) represents the magnetic charge of the nonlinear self-gravitating monopole [33], while we have taken the gravitational units, i.e. \( G = c = 1 \). For \( e = 0 \), it turns out to be the Schwarzschild spacetime, while for \( m = 0 = e \), it becomes the Minkowski spacetime. The simultaneous solution of the Killing equations for the Bardeen model is given by

\[
k^0 = c_0, \quad k_1 = 0, \quad k^2 = c_1 \sin \phi - c_2 \cos \phi, \\
k^3 = \cot \theta (c_1 \cos \phi + c_2 \sin \phi) + c_3.
\]

This solution corresponds to the generators \( Y_0, Y_1, Y_2 \) and \( Y_3 \) that form an algebra \( so(3) \oplus \mathbb{R} \) implying the conservation laws for energy and angular momentum only, while the linear and spin angular momentum conservation laws are lost. We will check whether one can recover these lost conservation laws in the limit of small gravitational mass and charge or not.

The set of geodesic equations for the spacetime (9) can be written as

\[
\ddot{t} + 2 \left[ \frac{m(r^2 + e^2)^{1/2}}{[1 - \frac{2mr}{(r^2 + e^2)^{3/2}}]} \right] \left[ \frac{(r^2 - 2re^2)}{(r^2 + e^2)^{3/2}} \right] i \dot{r} = 0, \tag{10}
\]

\[
\ddot{r} + \left[ \frac{mr^2 - 2e^2}{(r^2 + e^2)^{3/2}} \right] \left[ 1 - \frac{2mr^2}{(r^2 + e^2)^{3/2}} \right] i \dot{t} = 0,
\]

\[
- \left[ \frac{mr^2 - 2e^2}{(r^2 + e^2)^{3/2}} + mr^2 - 2e^2 \right] \left[ \frac{(r^2 - 3e^2)}{(r^2 + e^2)^{3/2}} \right]^{-1} r^2 = 0,
\]

\[
- r \left[ \frac{2mr^2}{(r^2 + e^2)^{3/2}} \right] (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = 0, \tag{11}
\]

\[
\ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0, \tag{12}
\]

\[
\ddot{\phi} + \frac{2}{r} \dot{r} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0. \tag{13}
\]

Using the spacetime, four geodesic equations and the symmetries of the equatorial plane, we obtain the following relativistic equation of motion (orbital equation of motion):

\[
\frac{d^2u}{dt^2} + u = \frac{2mu^2}{(1 + 2e^2u^2)^{3/2}} + \frac{m(1 - 2e^2u^2)}{h^2(1 + e^2u^2)^{3/2}} + \frac{m^2}{h^2(1 + e^2u^2)^{3/2}}, \tag{14}
\]

where \( h \) is the classical angular momentum per unit mass and \( u = \frac{1}{r} \).

3.2. Perturbation parameters and approximate symmetries of the orbital and geodesic equations

The Bardeen model is a singularity-free solution and represents the black hole for the following inequality:

\[
27e^2 \leq 16m^2. \tag{15}
\]

Since the vanishing of mass and charge provides the Minkowski spacetime, the perturbation parameters are to be defined in terms of mass and charge of the black hole to recover the lost conservation laws. The required perturbation parameters are

\[
\epsilon = 2m, \quad \epsilon^2 = ke^2, \quad 0 < k \leq \frac{4}{27}.
\]

(15)

(15)

(15)

Introducing these parameters in the Bardeen model, we obtain the following third-order perturbed spacetime:

\[
ds^2 = \left[ 1 - \frac{e}{r} + \frac{3ke^3}{2r^3} \right] dt^2 - \left[ 1 + \frac{e}{r} + \frac{e^2}{r^2} - \frac{3ke^3}{2r^4} \right] dr^2
\]

\[ - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

Retaining only the terms of first and second order in the above equation and neglecting \( O(\epsilon^3) \), it reduces to the second-order perturbed spacetime for the Schwarzschild solution. Also, it reduces to the Minkowski spacetime in the limit \( \epsilon = 0 \). The third-order perturbed orbital equation of motion can be written as

\[
\frac{d^2u}{dt^2} + u = \epsilon \left( \frac{1}{2h^2} + \frac{3}{2u^2} \right)^3 - \epsilon^3 \left( \frac{15ku^4}{4} + \frac{9ku^2}{4h^2} \right); \tag{16}
\]

For this equation of motion, the exact and first-order parts of the symmetry generator are the same as for the Schwarzschild spacetime given by equations (B.1)-(B.7). Since there is no term quadratic in \( \epsilon \), its second-order approximate symmetries are zero. Now we calculate its third-order approximate symmetries. For this purpose, we apply the operator

\[
X^{[2]} = \xi(\phi, u) \frac{\partial}{\partial \phi} + \eta(\phi, u) \frac{\partial}{\partial u} + \eta(\phi, u, u') \frac{\partial}{\partial u'} \\
+ \eta(\phi, u, u', u'') \frac{\partial}{\partial u''}
\]

to equation (16) and then substitute the values of the prolongation coefficients and retain only the terms involving
We have seen that out of four constants corresponding to exact symmetry generators, two have appeared and canceled out, while the six constants corresponding to first- and second-order parts of the symmetry generators appear. All these six constants turn out to be zero when we solve this system of equations simultaneously and it results in the system of DEs for the Minkowski spacetime. Its solution provides ten Killing vectors and the generators corresponding to the dilation algebra $d_0$. This leads to no nontrivial symmetry in the third-order approximation. For this model, the constants corresponding to the dilation algebra $\xi_0(s) = c_0s + c_1$ do not cancel automatically but collect some expression for cancellation. This expression is given as follows:

$$1 - \frac{9k}{2} = 1 - \frac{9e^2}{8m^2}, \quad (22)$$

where we have used the value of $k$. Since $\xi(s)$ is the coefficient of $s$, its specific value does not matter.

4. Summary and discussion

This paper is devoted to the study of the energy of the Bardeen model (regular black-hole solution) by using approximate symmetries. These solutions have the isometry algebra $so(3) \oplus R$, while the system of geodesic equations has the algebra $so(3) \oplus R \oplus d_z$. The strength of the approximate symmetries (i.e. perturbation parameters) provides information about the amount of energy contained in the spacetime field. The second-order approximation to the symmetry generator yields only the lost conservation laws, and hence no information regarding the energy re-scaling factor is obtained. For the energy re-scaling factor, we have to evaluate the third-order approximate symmetries.

Firstly, we have evaluated the third-order approximate symmetries of the orbital equation of motion. The exact and the first-order parts of the symmetry generators are the same as those for the Schwarzschild spacetime. However, the second-order approximate part of the symmetry generator is zero. In the third-order approximation, one can obtain the trivial symmetries only (i.e. of Minkowski spacetime) given by equations (B.1)–(B.5).

Secondly, we have calculated the third-order approximate symmetries of the perturbed geodesic equations. Here the exact, first- and second-order approximate parts of the symmetry generator turn out to be the same as those for the Schwarzschild spacetime. The third-order approximate symmetries of the third-order perturbed geodesic equations provide non trivial symmetry generator. However, we have obtained the energy re-scaling factor given by equation (22). The re-scaling factor is of great interest as it shows the reduction in energy by the ratio of the magnetic self-energy of the source to its gravitational self-energy. For $e = 0$, it vanishes and coincides with the Schwarzschild spacetime.

When a particle is placed in the field produced by a charge gravitating source, some force is exerted on it that should be position dependent. However, it is not clear whether the modification in mass (energy) in the presence of charge.
should be position dependent. The expression for re-scaling of the force calculated through the pseudo-Newtonian formalism [34] is given by
\[
m = \frac{m}{r^2(r^2 + e^2)^{3/2}(r^3 - 2re^2)}.
\]

It is position dependent and provides the reduction in force by a factor depending on the ratio of the magnetic potential energy at a distance \(r\) to the rest energy of the gravitational source. However, it is not a convenient expression for relating the magnetic self-energy of the configuration to its gravitational self-energy. It should be position independent where the radial parameter \(r\) would be canceled out. Our obtained re-scaling factor is clearly position independent and corresponds to change in mass (energy) due to the presence of magnetic charge. Therefore, it provides a physically more significant expression for relating these self-energies as compared to the results found through different approaches [11].

We would like to mention here that the Bardeen model with magnetic charge is the analogue of the RN spacetime with electric charge for which, in the second-order approximation, an energy re-scaling factor
\[
1 - 2k = 1 - \frac{Q^2}{2M^2}
\]

is obtained. However, in contrast to this, we are forced to calculate its third-order approximate symmetries because the second-order approximation provides no information about the energy re-scaling factor. Clearly both these scaling factors are position independent providing more significant results. For the orbital equation of motion, the second-order approximate symmetries are zero (while for the RN spacetime, there exist nonzero and trivial second-order symmetries). In the literature [15, 16], a difference between the conservation laws for the full system of geodesic equations and single orbital equation of motion is noted. This difference also holds for the Bardeen solution. It would be worthwhile to investigate the approximate symmetry generators and the concept of energy re-scaling for the rotating stringy black-hole solutions that contain either electric or magnetic charge or both charges.

**Appendix A**

The symmetry generators of the Minkowski spacetime are given as
\[
Y_0 = \frac{\partial}{\partial t}, \quad Y_1 = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \\
Y_2 = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}, \\
Y_3 = \frac{\partial}{\partial \phi}, \\
Y_4 = \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} - \csc \theta \cos \phi \frac{\partial}{\partial \phi}.
\]

Using these symmetry generators, we can find the energy re-scaling factor as
\[
Y_5 = \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} - \csc \theta \cos \phi \frac{\partial}{\partial \phi},
\]
\[
Y_6 = \cos \theta \frac{\partial}{\partial \phi} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta},
\]
\[
Y_7 = r \sin \theta \cos \phi \frac{\partial}{\partial t} + t \left( \sin \theta \frac{\partial}{\partial \phi} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \csc \theta \sin \phi \frac{\partial}{\partial \phi} \right),
\]
\[
Y_8 = r \sin \theta \sin \phi \frac{\partial}{\partial t} + t \left( \sin \theta \frac{\partial}{\partial \phi} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \csc \theta \cos \phi \frac{\partial}{\partial \phi} \right),
\]
\[
Y_9 = r \cos \theta \frac{\partial}{\partial \phi} + t \left( \cos \theta \frac{\partial}{\partial \phi} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right).
\]

**Appendix B**

For the orbital equation, the first-order approximate symmetries of the Schwarzschild spacetime include
\[
Y_0 = u \frac{\partial}{\partial u}, \quad Y_1 = \cos \phi \frac{\partial}{\partial \theta}, \quad Y_2 = \sin \phi \frac{\partial}{\partial \phi}, \quad (B.1)
\]
\[
Y_3 = \frac{\partial}{\partial \phi}, \quad Y_4 = 2\cos \phi \frac{\partial}{\partial \phi} - u \sin 2\phi \frac{\partial}{\partial \phi}, \quad (B.2)
\]
\[
Y_5 = 2\cos \phi \frac{\partial}{\partial \phi} + u \cos 2\phi \frac{\partial}{\partial \phi}, \quad (B.3)
\]
\[
Y_6 = u \cos \phi \frac{\partial}{\partial \phi} - u^2 \sin \phi \frac{\partial}{\partial \phi}, \quad (B.4)
\]
\[
Y_7 = u \sin \phi \frac{\partial}{\partial \phi} + u^2 \cos \phi \frac{\partial}{\partial \phi}, \quad (B.5)
\]

and two nontrivial stable approximate symmetry generators are
\[
Y_{a1} = \sin \phi \frac{\partial}{\partial \phi} + \epsilon \left( 2\cos \phi \frac{\partial}{\partial \phi} - u \cos \phi \frac{\partial}{\partial \phi} \right), \quad (B.6)
\]
\[
Y_{a2} = \cos \phi \frac{\partial}{\partial \phi} - \epsilon \left( 2\cos \phi \frac{\partial}{\partial \phi} - u \sin \phi \frac{\partial}{\partial \phi} \right). \quad (B.7)
\]

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