On the normal bundle of Levi-flat real hypersurfaces

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Abstract

Let $X$ be a connected complex manifold of dimension $\geq 3$ and $M$ be a smooth compact Levi-flat real hypersurface in $X$. We show that the normal bundle to the Levi foliation does not admit a Hermitian metric with positive curvature along the leaves. This generalizes a result obtained by Brunella.

1 Introduction

A real hypersurface $M$ (of class at least $C^2$) in a complex manifold is called Levi-flat if its Levi-form vanishes identically or, equivalently, if it admits a foliation by complex hypersurfaces. Another equivalent formulation is that $M$ is locally pseudoconvex from both sides.

Given a Levi-flat real hypersurface $M$ in a complex manifold $X$ of dimension $n$, we call $N^{1,0}_M = (T^{1,0}_X)_M/T^{1,0}M$ the holomorphic normal bundle of $M$. The restriction of $N^{1,0}_M$ to each $(n-1)$-dimensional complex submanifold of $M$ has a structure of a holomorphic line bundle induced from that of $T^{1,0}_X$.

In this paper we prove the following

Theorem 1.1

Let $X$ be a complex manifold of dimension $n \geq 3$. Then there does not exist a smooth compact Levi-flat real hypersurface $M$ in $X$ such that the normal bundle to the Levi foliation admits a Hermitian metric with positive curvature along the leaves.

Classical nontrivial examples of Levi-flat hypersurfaces were described by Grauert as tubular neighborhoods of the zero section of a generically chosen line bundle over a non-rational Riemann surface [G2]. In these examples, the Levi-flat hypersurfaces arise as the boundary of a pseudoconvex domain admitting only constant holomorphic functions. On the other hand,
there are also examples of compact Levi-flat real hypersurfaces bounding Stein domains. For example, the product of an annulus and the punctured plane is biholomorphically equivalent to a domain in $\mathbb{P}^1 \times \{ \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \}$ with Levi-flat boundary [O1]. Further examples of complex surfaces that can be cut into two Stein domains along smooth Levi-flat real hypersurface can be found in [N]. From [A1] one even obtains examples of Levi-flat hypersurfaces in complex surfaces having hyperconvex complement.

These examples above show that Levi-flat hypersurfaces can be of quite different nature and therefore explain a certain interest in the classification of compact Levi-flat real hypersurfaces. Let us also mention that some of these constructions can be extended to higher dimensions.

On the other hand, the study of Levi-flat real hypersurfaces is related to basic questions in dynamical systems and foliation theory: Levi-flats arise as stable sets of holomorphic foliations, and a real-analytic Levi-flat real hypersurface extends to a holomorphic foliation leaving $M$ invariant. Relating to this, a famous open problem is whether or not $\mathbb{CP}^2$ contains a smooth Levi-flat real hypersurface. This problem arose as part of a conjecture that, for any codimension one holomorphic foliation on $\mathbb{CP}^2$ (with singularities), any leaf accumulates to a singular point of the foliation [C-L-S]. This problem is still open. It is only known that if $\mathbb{CP}^2$ admits a smooth Levi-flat real hypersurface, then it has to satisfy a restrictive curvature condition [A-B].

For $n \geq 3$, however, it is known that there does not exist any smooth real Levi-flat hypersurface $M$ in $\mathbb{CP}^n$. This was first proved by LinsNeto in [LN] for real-analytic $M$ and by Siu in [S] for $C^{12}$-smooth $M$. For further improvements concerning the regularity, we refer the reader to [L-M] and [C-S].

The proofs of the above-mentioned results essentially exploited the positivity of $T^{1,0}\mathbb{CP}^n$. Brunella’s main observation [B] was that the positivity of the normal bundle itself is enough to ensure that the complement of $M$ is pseudoconvex. If $X = \mathbb{CP}^n$, or if $X$ admits a hermitian metric of positive curvature, then the normal bundle $N^1_1_M$ is automatically positive (it is a quotient of $T^{1,0}X$, and therefore more positive than $T^{1,0}X$).

This led Brunella to prove that if $X$ is a compact Kähler manifold with $\dim X \geq 3$, and if $M$ is a smooth Levi-flat real hypersurface such that there exists a holomorphic foliation on a neighborhood of $M$ leaving $M$ invariant, then the normal bundle of this foliation does not admit any fiber metric with positive curvature.

Ohsawa generalized this in [O3] to a nonexistence result for smooth Levi-
flat real hypersurfaces admitting a fiber metric whose curvature form is semi-positive of rank \( \geq 2 \) along the leaves of \( M \) (in any compact Kähler manifold).

Our Theorem 1.1 is a generalized version of Brunella’s result in the sense that we are able to drop the compact Kähler assumption on the ambient \( X \). This was conjectured in [O4, Conjecture 5.1].

The following example from [Br, Example 4.2] and [O4, Theorem 5.1] shows that Theorem 1.1 cannot hold for \( n = 2 \), even for \( X \) compact Kähler:

Let \( \Sigma \) be a compact Riemann surface of genus \( g \geq 2 \). Let \( D \) be the open unit disc, and let \( \Gamma \) be a discrete subgroup of \( \text{Aut} D \subset \text{Aut} \mathbb{C}P^1 \) such that \( \Sigma \simeq D/\Gamma \). Then \( \Gamma \) also acts on \( D \times \mathbb{C}P^1 \) by

\[(z, w) \mapsto (\gamma(z), \gamma(w)), \quad \gamma \in \Gamma.\]

The quotient \( X = (D \times \mathbb{C}P^1)/\Gamma \) is a compact complex surface, ruled over \( \Sigma \) (and hence projective). From the horizontal foliation on \( D \times \mathbb{C}P^1 \), we get a holomorphic foliation on \( X \), leaving invariant a real analytic Levi-flat hypersurface \( M \) induced from the \( \Gamma \)-invariant \( D \times S^1 \). The Bergman metric induces a metric with positive curvature on the normal bundle of \( M \) (see [O4] for more details).

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2 Sketch of the proof

Let us begin by recalling the essential steps of Brunella’s proof in [Br]: Assume that \( X \) is a connected compact Kähler manifold of dimension \( n \geq 3 \), and let \( M \) be a smooth Levi-flat real hypersurface such that there exists a holomorphic foliation on a neighborhood of \( M \) leaving \( M \) invariant. Under the assumption that the normal bundle of this foliation admits a fiber metric with positive curvature, Brunella shows that \( X \setminus M \) is strongly pseudoconvex.

Then the argument is as follows: Since the normal bundle of the foliation is
topologically trivial, its curvature form $\theta$ is $d$-exact on a tubular neighborhood $U$ of $M$. Thus $\theta = d\beta$ on $U$, where the primitive $\beta = \beta^{1,0} + \beta^{0,1}$ can be chosen of real type ($\beta^{1,0} = \beta^{0,1}$) and one has $\bar{\partial}\beta^{0,1} = 0$. Since $\dim X \geq 3$, the vanishing theorem of Gauert and Riemenschneider combined with Serre’s duality implies that the $\overline{\partial}$-cohomology with compact support $H^0_{\overline{\partial}}(X \setminus M)$ is zero. This means that one can extend $\beta^{0,1}$ $\overline{\partial}$-closed to $X$. Hodge symmetry on the Kähler manifold $X$ means $H^{0,1}(X) \simeq \overline{H^{1,0}(X)}$. Hence $\beta^{0,1} = \eta + \partial\alpha$, with $\partial\eta = 0$. But then $\partial\beta^{0,1} = \partial\overline{\partial}\alpha$. Therefore, setting $\phi = i(\overline{\alpha} - \alpha)$, one obtains $\theta = i\partial\overline{\partial}\phi$. The existence of a potential for the positive curvature form is, however, a contradiction to the maximum principle on the leaves of the foliation.

Our proof follows this general idea. We assume by contradiction that there exists a smooth compact Levi-flat real hypersurface $M$ in $X$ such that the normal bundle to the Levi foliation admits a Hermitian metric with positive curvature along the leaves. However, since our $M$ is not embedded in a compact Kähler manifold, we have to make several important modifications. Since $M$ has a tubular neighborhood which is pseudoconcave (of dimension $\geq 3$), this tubular neighborhood can be compactified to a compact manifold $\tilde{X}$. Then $\tilde{X} \setminus M$ is a strongly pseudoconvex manifold, and we can even arrange that it carries a complete Kähler metric (section 4 and 5). By means of $L^2$-estimates on $\tilde{X} \setminus M$, we will then extend the normal bundle to $M$ to a holomorphic line bundle over $\tilde{X}$ (section 6). We also show that $CR$ sections of high tensor powers of the normal bundle extend to holomorphic sections over $\tilde{X}$ (section 7), again by means of solving some Cauchy-problem for the $\overline{\partial}$-equation using $L^2$-estimates. This permits us to find sufficiently many sections that provide a holomorphic embedding of a tubular neighborhood of $M$ into a compact Kähler manifold (section 8). This proves the nonexistence of such $M$ as before.

3 Preliminaries

Let $Y$ be a complex manifold of dimension $n$ endowed with a Hermitian metric $\omega$, and let $E$ be a holomorphic vector bundle on $Y$ with a Hermitian metric $h$. For integers $0 \leq p, q \leq n$, we use the following notations:

- $\mathcal{C}^p,q_c(Y,E)$ denotes the space of smooth, compactly supported $E$-valued $(p,q)$-forms on $Y$.
- $L^2_{p,q}(Y,E,\omega,h)$ denotes the Hilbert space obtained by completing $\mathcal{C}^p,q_c(Y,E)$ with respect to the $L^2$-norm $\| \cdot \|_{\omega,h}$ induced by $\omega$ and $h$.

If $\text{rk}E = 1$, and the metric on $E$ is given by $h = e^{-\varphi}$, we write $L^2_{p,q}(Y,E,\omega,\varphi)$ instead of $L^2_{p,q}(Y,E,\omega,h)$.
As usual, the differential operator $\overline{\partial}$ is extended as a densely defined closed linear operator on $L^2_{p,q}(Y,E,\omega,h)$, whose domain of definition is
\[
\text{Dom} \overline{\partial} = \{ f \in L^2_{p,q}(Y,E,\omega,h) \mid \overline{\partial} f \in L^2_{p,q+1}(Y,E,\omega,h) \},
\]
where $\overline{\partial} f$ is computed in the sense of distributions. The Hilbert space adjoint of $\overline{\partial}$ will be denoted by $\overline{\partial}^*$ ($= \overline{\partial}^*_\omega,h$).

We also define the space of harmonic forms,
\[
\mathcal{H}^{p,q}(Y,E,\omega,h) = L^2_{p,q}(Y,E,\omega,h) \cap \text{Ker} \overline{\partial} \cap \text{Ker} \overline{\partial}^*_\omega,h,
\]
and the $L^2$-Dolbeault cohomology groups of $Y$,
\[
H^{p,q}_{L^2}(Y,E,\omega,h) = L^2_{p,q}(Y,E,\omega,h) \cap \text{Ker} \overline{\partial} / L^2_{p,q}(Y,E,\omega,h) \cap \text{Im} \overline{\partial}.
\]
Whenever we feel that it is clear from the context, we will omit the dependency of the $L^2$-spaces, norms, operators etc. on the hermitian metric $h$ of the vector bundles under considerations.

For the reader’s convenience, we also recall the well-known Bochner-Kodaira-Nakano inequality, which is the starting point for all $L^2$-estimates for $\overline{\partial}$. If $\omega$ is a Kähler metric, then for every $u \in C^{p,q}_c(Y,E)$ we have the following a priori estimate (see [D Lemme 4.4]):
\[
\|\overline{\partial} u\|_{\omega,h}^2 + \|\overline{\partial}^*_\omega,h u\|^2 \leq |i\Theta_h(E),\Lambda_\omega| u,u \gg_{\omega,h} (3.1)
\]
Here $i\Theta_h(E)$ is the curvature of the bundle $E$, and $\Lambda_\omega$ is the adjoint of multiplication by $\omega$. It is important to note that if the metric $\omega$ is complete, then the inequality (3.1) extends to all forms $u \in L^2_{p,q}(Y,E,\omega,h) \cap \text{Dom} \overline{\partial} \cap \text{Dom} \overline{\partial}^*_\omega,h$. For metrics that are not Kähler, there is an additional curvature term (see [O4]).

Moreover, if $E$ is a line bundle, and if $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of $i\Theta(E)$ with respect to $\omega$, then we have
\[
\langle i\Theta_h(E),\Lambda_\omega \rangle u,u \omega,h \geq (\lambda_1 + \ldots + \lambda_q - \lambda_{p+1} - \ldots - \lambda_n)|u|_{\omega,h}^2 \quad (3.2)
\]
if $u$ is of bidegree $(p,q)$ (see [D2 (13.6)]).

In section 7, we shall also use the following variant of the $\overline{\partial}$-operator: by $\overline{\partial}_c$ we denote the strong minimal realization of $\overline{\partial}$ on $L^2_{p,q}(Y,E,\omega,h)$. This means that $u \in \text{Dom} \overline{\partial}_c \subset L^2_{p,q}(Y,E,\omega,h)$ if there exists $f \in L^2_{p,q+1}(Y,E,\omega,h)$
and a sequence \((u_\nu)_{\nu \in \mathbb{N}} \subset D^p\bar{q}(Y,E)\) such that \(u_\nu \rightharpoonup u\) and \(\bar{\partial}u_\nu \rightharpoonup f = \bar{\partial}_c u\) in \(L^{2}_{p,q+1}(Y,E,\omega,h)\).

The Hilbert space adjoint of \(\bar{\partial}_c\) will be denoted by \(\vartheta\); it is the weak maximal realization of the formal adjoint of \(\bar{\partial}\) on \(L^{2}_{p,q}(Y,E,\omega,h)\).

4 Convexity properties

Let \(M\) be a smooth Levi-flat real hypersurface in a Hermitian manifold \(X\). By considering a double covering, we may assume that \(M\) is orientable and that the complement of \(M\) in \(X\) divides \(X\) into two connected components (shrinking \(X\) if necessary), see also [Br]. So we may assume that \(X\) is sufficiently small so that there exists a smooth real-valued function \(\rho\) on \(X\) such that

\[ M = \{ z \in X \mid \rho(z) = 0 \} \]

and \(d\rho \neq 0\) on \(M\). We further fix a Hermitian metric \(\omega_o\) on \(X\).

We now assume by contradiction that the normal bundle of the Levi foliation admits a Hermitian metric of positive curvature along the leaves. As in [Br], this implies that the complement of \(M\) is strongly pseudoconvex. The following proposition was proved in [O3], we include the proof for the sake of completeness.

**Proposition 4.1**

*Let \(M\) be a compact smooth Levi-flat real hypersurface in a Hermitian manifold \(X\) of dimension \(\geq 2\), such that the normal bundle \(N^{1,0}_{M}\) of the Levi foliation admits a smooth Hermitian metric of leafwise positive curvature. Then, after possibly shrinking \(X\), there exists a \(C^2\)-smooth nonnegative function \(v\) on \(X\), smooth on \(X \setminus M\), and a positive constant \(c > 0\) such that

\[ i\bar{\partial}(\log v) \geq c\omega_o \quad \text{on} \quad X \setminus M. \tag{4.1} \]

Moreover, we have that \(v = O(\rho^2)\).*

**Proof.** As in [Br] one can find a finite family of local coordinate neighborhoods \(U_\gamma\) in \(X\) covering \(M\) such that \(U_\gamma \cap M = \{ z \in U_\gamma \mid \text{Im}f_\gamma = 0 \}\), where \(\bar{\partial}f_\gamma\) vanishes to infinite order on \(M\), and that \(T^{1,0}_M = \text{Ker}f_\gamma\) on \(M \cap U_\gamma\).

Let \(\varpi = \{ \varpi_\gamma \}\) be a 1-form on \(M\) defining its Levi-foliation. We may assume that \(\varpi_\gamma\) is defined on \(U_\gamma\). Let \(h = \{ h_\gamma \}\) be the fibre metric of \(N^{1,0}_{M}\) such that

\[ h_\delta = |\varpi_\gamma/\varpi_\delta|^2 h_\gamma \tag{4.2} \]
on $U_\gamma \cap U_\delta \cap M$. By assumption on the curvature of $N_{M,0}^{1,0}$, we may assume that $-\log h_\gamma$ is strictly plurisubharmonic on the leaves of $M$.

We have $\varpi_\gamma = e_\gamma df_\gamma$ for some smooth function $e_\gamma$ which is nowhere vanishing on $U_\gamma$ and holomorphic along the leaves of $M$. From (4.2) it follows that we have

$$h_\gamma |e_\gamma|^2 (\operatorname{Im} f_\gamma)^2 - h_\delta |e_\delta|^2 (\operatorname{Im} f_\delta)^2 = O(\rho^3) \text{ on } U_\gamma \cap U_\delta$$

Therefore, invoking Whitney’s extension theorem, there exists a $C^2$ function $v$ defined in a tubular neighborhood of $M$, smooth away from $M$, such that $v = h_\gamma |e_\gamma|^2 (\operatorname{Im} f_\gamma)^2 + O(\rho^3)$ on $U_\gamma$.

To see that $-\log v$ is strictly plurisubharmonic in a tubular neighborhood of $M$, it suffices to estimate the Levi-form of $-\log(h_\gamma |e_\gamma|^2 (\operatorname{Im} f_\gamma)^2)$. Indeed, let $V \in T^{1,0}X$ be a unitary vector that we decompose orthogonally into $V = V_t + V_n$, with $V_t \in \ker \partial \rho$. Then the strict plurisubharmonicity of $-\log h_\gamma$ and the holomorphicity of $e_\gamma$ along the leaves of $M$ imply that there exists $c > 0$ such that

$$-i\partial \overline{\partial} \log(h_\gamma |e_\gamma|^2) (V_t, \overline{V_t}) \geq (c - \epsilon) \omega_o(V_t, \overline{V_t}),$$

where $\epsilon$ can be made as small as we wish by shrinking $X$. On the other hand, since $h_\gamma$ and $|e_\gamma|^2$ do not vanish,

$$-i\partial \overline{\partial} \log(h_\gamma |e_\gamma|^2) (V_n, \overline{V_n}) \geq -C \omega_o(V_n, \overline{V_n}).$$

The mixed terms in $(V_t, \overline{V_n})$ can be handled as follows:

$$-i\partial \overline{\partial} \log(h_\gamma |e_\gamma|^2) (V_t, \overline{V_n}) \geq -\epsilon \omega_o(V_t, \overline{V_t}) - \frac{C}{\epsilon} \omega_o(V_n, \overline{V_n}).$$

Moreover, since $\overline{\partial} f_\gamma$ vanishes to infinite order on $M$, we have

$$-i\partial \overline{\partial} \log((\operatorname{Im} f_\gamma)^2) (V, \overline{V}) = (2i \operatorname{Im} f_\gamma)^2 + \frac{2i \partial \overline{\operatorname{Im} f_\gamma} \wedge \overline{\operatorname{Im} f_\gamma}}{(\operatorname{Im} f_\gamma)^2} (V, \overline{V})$$

$$\geq -\epsilon \omega_o(V, \overline{V}) + 2i \frac{\partial \overline{\operatorname{Im} f_\gamma} \wedge \overline{\operatorname{Im} f_\gamma}}{(\operatorname{Im} f_\gamma)^2} (V_n, \overline{V_n})$$

Again, $\epsilon$ can be made as small as we wish by shrinking $X$. Combining the above estimates permits to conclude by taking $\epsilon$ sufficiently small. □

Remark. In [A2, Proposition 3.3], a converse statement is proved: If there exists a boundary distance function of $X \setminus M$ with positive Diederich-Fornaess exponent, then the normal bundle $N_{M}^{1,0}$ is positive along the leaves.
5 A first compactification

For sufficiently large $\alpha \in \mathbb{R}^+$, Proposition 4.1 implies that the set
\[ \{ z \in X \mid -\log v(z) > \alpha \} \]
is a pseudoconcave manifold (of complex dimension $\geq 3$). By a theorem of Rossi [R] and Andreotti-Siu [A-S] it can be compactified. Hence we may assume that $M$ is embedded as a real hypersurface in a compact complex manifold $X'$ of dimension $n$, and $X' \setminus M$ is a strongly pseudoconvex manifold (or a 1-convex manifold, using a different terminology): $X' \setminus M$ admits a smooth exhaustion function (given by Proposition 4.1), which is strictly plurisubharmonic outside a compact subset.

Before continuing with the proof, we will make some standard modifications of $X'$ in order to facilitate the following arguments.

By [G1] there exists a compact analytic subset $A \subset X' \setminus M$ and a proper holomorphic map $\pi'$ from $X' \setminus M$ onto a Stein space $S$ such that $\pi'$ is a biholomorphic mapping from $X' \setminus (M \cup A)$ to $S \setminus \pi'(A)$.

By Hironaka's method, there is a complex manifold $\tilde{S}$ obtained from $S$ by blowing up $S$ along smooth centers, several times if necessary, such that the induced bimeromorphic map $\tilde{\pi}: \tilde{S} \to S$ is holomorphic. Moreover, following [G1], it is possible to choose $\tilde{S}$ such that

- $(\pi')^{-1} \circ \tilde{\pi}$ is biholomorphic on $\tilde{S} \setminus \tilde{\pi}^{-1}(\pi'(A))$,
- $\tilde{\pi}^{-1}(\pi'(A))$ is a divisor with normal crossings whose irreducible components $\tilde{A}_j$ are nonsingular, and
- there exists positive integers $p_1, \ldots, p_{\nu}$ such that the line bundle $\mathcal{O}(\tilde{A})$ induced by the divisor $\tilde{A} = \sum_{j=1}^{\nu} p_j \tilde{A}_j$ is negative.

This modification permits us to prove the following proposition.

**Proposition 5.1**

$M$ is a real hypersurface in a compact complex manifold $\tilde{X}$ of dimension $n$ such that

(i) $\tilde{X} \setminus M$ is strongly pseudoconvex, and moreover there exists a smooth exhaustion function $\varphi$ on $\tilde{X} \setminus M$, plurisubharmonic on $\tilde{X} \setminus M$, strictly plurisubharmonic outside a compact of $\tilde{X} \setminus M$, such that $\varphi \sim -2\log |\rho|$ outside a compact of $K$ of $\tilde{X} \setminus M$.

(ii) $\tilde{X} \setminus M$ admits a complete Kähler metric $\tilde{\omega}$ such that $\tilde{\omega} = i\partial \bar{\partial} \varphi$ outside a compact of $\tilde{X} \setminus M$. 

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There exists a line bundle $L$ over $\widetilde{X}$, defined by a divisor $\widetilde{A} = \sum_{j=1}^{n} p_j \widetilde{A}_j$, with compact support in $\widetilde{X} \setminus M$, such that $L$ is negative on an open neighborhood of $K$ and holomorphically trivial outside a compact of $\widetilde{X} \setminus M$.

Remark. In the following sections, we implicitly assume that $L$ is endowed with a flat metric outside of $K$ (where $L$ is holomorphically trivial).

Proof. Gluing a pseudoconcave tubular neighborhood of $M$ in $X$ to a suitable relatively compact domain with strictly pseudoconvex boundary in $\tilde{S}$ we obtain a compact complex manifold $\tilde{X}$ of dimension $n$, containing the Levi-flat real hypersurface $M$, that has all the required properties. Indeed, (i) and (iii) follow easily from the discussion preceding the proposition.

To see that $\tilde{X} \setminus M$ is complete Kähler, we consider the Kähler metric $\omega_{\tilde{S}} = i\Theta(\mathcal{L}^* = i\Theta(\mathcal{O}(-\tilde{A}))$ on $\tilde{S}$. On $\tilde{S} \setminus \tilde{A}$, the line bundle $\mathcal{O}(-\tilde{A})$ is holomorphically trivial, hence there exists a smooth function $\psi$ such that $i\partial\bar{\partial}\psi = i\mathcal{O}(-\tilde{A})$ on $\tilde{S} \setminus \tilde{A}$. Now we choose a smooth cut-off function $\chi$ such that $\chi \equiv 1$ on a sufficiently large compact of $\tilde{S}$ containing $\tilde{A}$ and such that the support of $\chi$ is compact in $\widetilde{X} \setminus M$. Then, for $\varepsilon > 0$ small enough,

$$\tilde{\omega} = \begin{cases} \varepsilon i\partial\bar{\partial}(\chi\psi) + i\partial\bar{\partial}\varphi \\ \varepsilon i\Theta(\mathcal{O}(-\tilde{A})) + i\partial\bar{\partial}\varphi \end{cases}$$

defines a Kähler metric on $\tilde{X} \setminus M$. Moreover, $\tilde{\omega}$ is complete on $\tilde{X} \setminus M$. Indeed, it follows from (4.1) as in [O-S1] that there exists $0 < \eta \leq 1$ such that $i(\partial\bar{\partial})(\log v) \geq \varepsilon \partial\bar{\partial}(\log v)$. This implies that $i\partial\bar{\partial}(\log v) \geq \eta \partial\bar{\partial}(\log v)$, showing that $\tilde{\omega}$ is complete. $\square$

6 Holomorphic extension of the normal bundle

The aim of this section is to prove that the holomorphic normal bundle of $M$ extends to a holomorphic line bundle over the compact manifold $\widetilde{X}$. The main ingredient needed for the extension is the following $L^2$-vanishing result:

Proposition 6.1
For every $N \in \mathbb{N}$ the following holds: Assume $0 \leq q \leq n - 1$, and let $f \in L^2_{\partial\bar{\partial}}(\tilde{X} \setminus M, \tilde{\omega}, -N\varphi) \cap \text{Ker}\,\partial\bar{\partial}$. Then there exists a $(0,q-1)$-form $g \in L^2_{\partial\bar{\partial}}(\tilde{X} \setminus M, \tilde{\omega}, -N\varphi)$ satisfying $\partial\bar{\partial}g = f$.

Proof. The proof is similar to the one of Theorem 2.1 in [O3]. The metric $\tilde{\omega}$ is Kähler, and, since $\varphi$ is plurisubharmonic on $\tilde{X} \setminus M$, we have $-i\partial\bar{\partial}\varphi \leq 0$.
on $\tilde{X} \setminus M$ and $-i\partial\bar{\partial}\varphi = \tilde{\omega}$ outside a compact $K$ of $\tilde{X} \setminus M$. From (3.1) and (3.2) we then obtain

$$N \int_{\tilde{X} \setminus (M \cup K)} |u|^2 e^{N\varphi} dV_{\tilde{\omega}} \leq \|\nabla u\|_{\tilde{\omega}, -N\varphi}^2 + \|\nabla_{\tilde{\omega}, -N\varphi} u\|_{\tilde{\omega}, -N\varphi}^2$$

(6.1)

for every $u \in L^2_{0,q}(\tilde{X} \setminus M, \tilde{\omega}, -N\varphi) \cap \text{Dom} \bar{\partial} \cap \text{Dom} \nabla_{\tilde{\omega}, -N\varphi}$, $0 \leq q \leq n - 1$, $N \in \mathbb{N}$.

It is well-known that (6.1) implies that $\text{Im} \nabla$ is closed in $L^2_{0,q}(\tilde{X} \setminus M, \tilde{\omega}, -N\varphi)$ and that $H^0,q_{L^2}(\tilde{X} \setminus M, \tilde{\omega}, -N\varphi)$ is finite dimensional. This entails

$$H^0,q_{L^2}(\tilde{X} \setminus M, \tilde{\omega}, -N\varphi) \simeq H^0,q(\tilde{X} \setminus M, \tilde{\omega}, -N\varphi).$$

By (6.1) it then follows that every element of $H^0,q_{L^2}(\tilde{X} \setminus M, \tilde{\omega}, -N\varphi)$ is zero outside of $K$, so that it vanishes identically by Aronszajn’s unique continuation theorem for elliptic operators. Hence $H^0,q_{L^2}(\tilde{X} \setminus M, \tilde{\omega}, -N\varphi) = \{0\}$. □

Since close to $M$, the weight $e^{N\varphi}$ (up to a bounded function) equals $\rho^{-2N}$ (where $\rho$ is a defining function for $M$, see section 4), Proposition 6.1 enables us to extend $CR$ objects on $M$ to holomorphic objects on $\tilde{X}$ by solving $\bar{\partial}$-equations with zero Cauchy data along $M$. In particular we can prove the following

**Proposition 6.2**

There exists a holomorphic line bundle $\tilde{N} \to \tilde{X}$ such that $\tilde{N}|_M = N^{1,0}_M$ ($\tilde{N}$ extends the $CR$ line bundle $N^{1,0}_M$).

**Proof.** The $CR$ line bundle $N^{1,0}_M$ is topologically trivial over $M$. Therefore it is in the image of the exponential map

$$\exp : H^1(M, \mathcal{O}_M) \to H^1(M, \mathcal{O}^*_M),$$

where $\mathcal{O}_M$, resp $\mathcal{O}^*_M$ denotes the sheaf of germs of smooth $CR$ functions on $M$, resp. nonvanishing $CR$ functions on $M$. Let us therefore choose $\xi \in H^1(M, \mathcal{O}_M)$, and identify it with a smooth $\partial_M$-closed $(0,1)$-form $g$ on $M$. Now $g$ admits a $\bar{\partial}$-closed extension to $\tilde{X}$. Indeed, consider a smooth extension $\tilde{g}$ to a neighborhood of $M$ such that $\bar{\partial}\tilde{g}$ vanishes to infinite order along $M$. Multiplying $\tilde{g}$ by a cut-off function whose support is contained in an arbitrary small tubular neighborhood of $M$, we can arrange that $\bar{\partial}\tilde{g}$ vanishes outside a small tubular neighborhood of $M$. This means that the $(0,2)$-form $f = \bar{\partial}\tilde{g}$ satisfies

$$\int_{\tilde{X} \setminus M} |f|^2 e^{N\varphi} dV_{\tilde{\omega}} < +\infty$$
for any $N \in \mathbb{N}$. Since $2 \leq n - 1$ by assumption on $X$, we may apply Proposition 6.1 and obtain a smooth $(0,1)$-form $u$ satisfying $\bar{\partial}u = \bar{\partial}\tilde{g}$ on $\tilde{X} \setminus M$ and

$$\int_{\tilde{X} \setminus M} |u|^2 e^{N\varphi} dV_\tilde{\omega} < +\infty.$$  

The solution that is minimal with respect to this $L^2$-norm moreover satisfies an elliptic equation. Using the regularity result obtained in [B, Theorem 2.1], we may therefore assume that $u$ vanishes to sufficiently high order along $M$ (by taking $N$ sufficiently large). But then $\tilde{g} - u$ is $\bar{\partial}$-closed on $\tilde{X}$ and coincides with $g$ on $M$. Therefore the holomorphic line bundle defined by $\tilde{N} = \exp(\tilde{g} - u)$ extends the CR line bundle $N_{\tilde{M}}^{1,0}$. It is topologically trivial, since it is in the image of the exponential map above. □

7 Holomorphic extension of sections

The key result of this section is Proposition 7.2, the extension of CR sections of the normal bundle to holomorphic sections of $\tilde{N}$ over $\tilde{X}$. This will enable us to holomorphically embed a tubular neighborhood of $M$ into some complex projective space in the next section. The proof of this holomorphic extension property needs several steps; it is actually the technically most demanding part of this paper.

In order to extend CR sections of a line bundle over $M$ to holomorphic sections over $\tilde{X}$, the following vanishing result is very useful; it is in the same spirit as Proposition 6.1, but less precise. Remember that the holomorphic line bundle $L \to \tilde{X}$ is given by Proposition 5.1.

**Proposition 7.1**

Let $L \to \tilde{X}$ be a holomorphic line bundle. Then there exist $k \in \mathbb{N}$ and $N \in \mathbb{N}$ such that the following holds: Assume $0 \leq q \leq n - 1$, and let $f \in L^2_{0,q}(\tilde{X} \setminus M, E \otimes L^k, \tilde{\omega}, -N\varphi) \cap \text{Ker} \bar{\partial}$. Then there exists a $(0, q - 1)$-form $g \in L^2_{0,q-1}(\tilde{X} \setminus M, E \otimes L^k, \tilde{\omega}, -N\varphi)$ satisfying $\bar{\partial}g = f$ and $\|g\|_{\tilde{\omega}, -N\varphi} \leq \|f\|_{\tilde{\omega}, -N\varphi}$.

**Proof.** Given the holomorphic line bundle $E$, we first choose $k$ big enough such that $E \otimes L^k$ is negative on the compact where $\varphi$ is only weakly plurisubharmonic. Then we choose $N$ big enough such that $i\Theta(E \otimes L^k) - Ni\bar{\partial}\partial\varphi \leq -\tilde{\omega}$ on $\tilde{X}$. We may then conclude from (3.1) and (3.2) as in the proof of Proposition 6.1. □

In the next section, we want to holomorphically extend CR sections of some high tensor power of the normal bundle of $M$. Since the normal bundle...
$N_M^{1,0}$ is positive, also $\tilde{N}$ can be equipped with a metric of positive curvature near $M$.

On the other hand, we can multiply the metric of $\tilde{N}$ by $e^{\nu}\varphi$. This adds $-Ni\partial\overline{\partial}\varphi$ to the curvature, so the curvature of $\tilde{N}$ can be made negative near $M$ by taking $N$ sufficiently large. This modification, however, would require the $CR$ sections that we wish to extend to be sufficiently regular. In [A1] it was shown that even $CR$ sections given an embedding into projective space are not necessarily $C^\infty$-smooth if $\text{dim } X = 2$. If $\text{dim } X \geq 3$, however, we can use some approximation arguments, reducing the involved $\overline{\partial}$-equation to compactly supported forms. As a result we can prove

**Proposition 7.2**

Let $\ell \in \mathbb{N}$ be sufficiently large, and assume that $s$ is a $CR$-section of $(N_M^{1,0})^\ell$ of class at least $C^4$. Then there exists a holomorphic section $\tilde{s}$ of $\tilde{N}^{\ell}$ on a tubular neighborhood of $M$ such that $\tilde{s}|_M = s$.

**Proof.** First we choose a $C^4$-extension $s_o$ of $s$ to $\hat{X}$ such that $\overline{\partial}s_o$ vanishes to the third order along $M$, i.e. $|\overline{\partial}s_o|_{\omega_o} = O(|\rho|^3)$.

Now we consider an exhaustion of $\hat{X}$ by strictly pseudoconvex domains $\Omega_\varepsilon = \{ z \in \hat{X} \mid \rho^2(z) > \varepsilon^2 \}$. Moreover, we define the annular domains $D_j = \Omega_{\varepsilon} \setminus \overline{\Omega_{\varepsilon}}$.

Then we choose a sequence of smooth cut-off functions $\chi_j$ with compact support in $\Omega_{\varepsilon}$ such that $\chi_j \equiv 1$ on $\overline{\Omega_{\varepsilon}}$ and $|d\chi_j|^2 \leq 1$ (this is possible since $\hat{w}$ is complete on $\hat{X} \setminus M$). Then

$$f_j := \overline{\partial}(\chi_j \overline{\partial}s_o) \in L^2_{0,2}(\hat{X} \setminus M, \hat{N}^{\ell}, \hat{\omega}) \cap \text{Ker } \overline{\partial}$$

is compactly supported in $D_j$.

Applying Lemma 7.3 yields $u_j \in L^2_{0,1}(\hat{X} \setminus M, \hat{N}^{\ell}, \hat{\omega})$ supported in $D_j$ satisfying $\overline{\partial}u_j = f_j$ and

$$\|u_j\|_{L^2}^2 \leq C^2 j^2 \|f_j\|_{L^2}^2 \lesssim j^2 \|\overline{\partial}\chi_j \wedge \overline{\partial}s_o\|_{L^2}^2$$

$$\leq j^2 \int_{D_j} |\overline{\partial}s_o|_{\omega_o}^2 dV_{\omega} \leq j^2 \int_{D_j} \rho^{-2} |\overline{\partial}s_o|_{\omega_o}^2 dV_{\omega} \lesssim j^{-2}$$

Now $g_j = \chi_j \overline{\partial}s_o - u_j$ is $\overline{\partial}$-closed and supported in $\Omega_{\varepsilon}$, hence compactly supported in $\hat{X} \setminus M$. Note that we may view $g_j$ as forms with values in $\hat{N}^{\ell} \otimes L^k$ (since $\hat{L}$ is holomorphically trivial outside a compact $K$ of $X \setminus M$).

By Proposition 7.1 there exist $k, N \in \mathbb{N}$ such that we can find solutions $h_j \in L^2_{0,0}(\hat{X} \setminus M, \hat{N}^{\ell} \otimes L^k, \hat{\omega}, -\nu \varphi)$ satisfying $\overline{\partial}h_j = g_j$. Hence $g_j \in L^2_{0,1}(\hat{X} \setminus M, \hat{N}^{\ell} \otimes L^k, \hat{\omega}, -\varphi) \cap \text{Im } \overline{\partial}$. By Lemma 7.3 we can therefore find $h_j \in$
\(L^2_{0,0}(\tilde{X} \setminus M, \tilde{N}^\ell \otimes L^k, \tilde{\omega}, -\varphi)\) satisfying \(\partial h_j = g_j\) and \(\|h_j\|_{\tilde{\omega}, -\varphi} \leq C_o \|g_j\|_{\tilde{\omega}, -\varphi}\). But

\[
\|g_j\|_{\tilde{\omega}, -\varphi}^2 \lesssim \|\chi_j \partial s_{0}\|_{\tilde{\omega}, -\varphi}^2 + \|u_j\|_{\tilde{\omega}, -\varphi}^2 \\
\lesssim \int_{\Omega_{\frac{1}{2}}} |\partial s_o|_{\tilde{\omega}}^2 \rho^{-2} dV_{\tilde{\omega}} + \int_{D_j} |u_j|_{\tilde{\omega}}^2 \rho^{-2} dV_{\tilde{\omega}} \\
\lesssim \int_{\Omega_{\frac{1}{2}}} \rho^6 \rho^{-4} dV_{s_o} + j^2 \|u_j\|_{\tilde{\omega}}^2 \lesssim 1
\]

Therefore the sequence \((h_j)\) is bounded in \(L^2_{0,0}(\tilde{X} \setminus M, \tilde{N}^\ell \otimes L^k, \tilde{\omega}, -\varphi)\), hence has a subsequence that weakly converges to \(h_o \in L^2_{0,0}(\tilde{X} \setminus M, \tilde{N}^\ell \otimes L^k, \tilde{\omega}, -\varphi)\). Since \(\partial h_j = \partial s_o\) on \(\Omega_{\frac{1}{2}}\), we must therefore have \(\partial h_o = \partial s_o\) in \(\tilde{X} \setminus M\). Moreover, since \(h_o \in L^2_{0,0}(\tilde{X} \setminus M, \tilde{N}^\ell \otimes L^k, \tilde{\omega}, -\varphi)\), we have \(\int_{\tilde{X} \setminus M} |h_o|^2 \rho^{-2} dV_{\tilde{\omega}} < +\infty\). This clearly implies that the trivial extension of \(h_o\) to \(\tilde{X}\) satisfies \(\partial h_o = \partial s_o\) as distributions on \(\tilde{X}\) (not only on \(\tilde{X} \setminus M\)). Hence \(h_o\) is of class at least \(C^4\) by the hypoellipticity of \(\partial\), and must therefore vanish on \(M\).

Thus \(\tilde{s} = s_o - h_o\) is a holomorphic section of \(\tilde{N}^\ell\) in a tubular neighborhood of \(M\) (where the line bundle \(L\) is holomorphically trivial) extending \(s\).

\[\square\]

**Lemma 7.3**

Let \(\ell \in \mathbb{N}\) be sufficiently large and \(f_j\) be defined by (7.4). For some constant \(C > 0\), independent of \(j \in \mathbb{N}\), there exists \(u_j \in L^2_{0,1}(\tilde{X} \setminus M, \tilde{N}^\ell, \tilde{\omega})\), supported in \(D_j\), such that \(\partial u_j = f_j\) and

\[
\|u_j\|_{\tilde{\omega}} \leq C_j \|f_j\|_{\tilde{\omega}}.
\]

**Proof.** We will fix a hermitian metric \(\tilde{h}\) on \(\tilde{N}\). By assumption on \(N^1_{M}\), we may choose \(\tilde{h}\) such that \((\tilde{N}, \tilde{h})\) is positive near \(M\).

Replacing \(\tilde{N}^\ell\) by \(\tilde{N}^\ell \otimes K^*_X =: F = F(\ell)\) (which is still positive for \(\ell\) sufficiently large), we may also assume that \(f_j\) is an \((n, 2)\)-form rather than a \((0, 2)\)-form.

Note that the boundary of \(D_j\) consists of two parts: a strictly pseudo-convex part \(\partial \Omega_{\frac{1}{2}}\) and a strictly pseudoconcave part \(-\partial \Omega_{\frac{1}{2}}\). Since \(n \geq 3\), this implies that \(D_j\) satisfies condition \(Z(n-2)\) (see [FK]), hence the \(\partial\)-Neumann problem satisfies subelliptic estimates in degree \((p, n-2)\) for all \(0 \leq p \leq n\).
Now we use a duality argument from [Ch-S]; let \( \partial_c \) be the strong minimal realization of \( \partial \) on \( L^2_{n,1}(D_j, F, \omega_o) \). Then by Theorem 3 of [Ch-S] the range of \( \partial_c \) is closed in \( L^2_{n,2}(D_j, F, \omega_o) \), and \( \partial_c \)-exact forms \( f \in L^2_{n,2}(D_j, F, \omega_o) \) are characterized by the usual orthogonality condition:

\[
\int_{D_j} f \wedge \eta = 0 \quad \forall \eta \in L^2_{0,n-2}(D_j, F^*, \omega_o) \cap \text{Ker} \partial
\]

But, using Stokes’ theorem, we get for \( \eta \in C^\infty_0(n-2)(D_j, F^*, \omega_o) \cap \text{Ker} \partial \)

\[
\int_{D_j} \partial(x_j \partial s_o) \wedge \eta = \int_{\partial D_j} (x_j \partial s_o) \wedge \eta = -\int_{\partial \Omega_2^c} \partial s_o \wedge \eta = -\int_{\partial \Omega_2^c} \partial(s_o \wedge \eta) = 0,
\]

and this also holds for \( \eta \in L^2_{0,n-2}(D_j, F^*, \omega_o) \cap \text{Ker} \partial \) using a standard approximation argument and the subelliptic estimates in degree \((0, n-2)\).

Hence \( f_j = \partial(x_j \partial s_o) \) belongs to the image of \( \partial_c \), i.e. there exists \( u_j \in L^2_{n,1}(D_j, F, \omega_o) \) satisfying \( \partial_c u_j = f_j \). As usual, we assume that \( u_j \) is the minimal \( L^2 \)-solution i.e. \( u_j \in L^2_{n,1}(D_j, F, \omega_o) \cap (\text{Ker} \partial_c)^\perp \). In particular, \( u_j \) is smooth on \( D_j \), and the trivial extension of \( u_j \) by zero outside \( \overline{D_j} \) (which we still denote by \( u_j \)), satisfies \( \partial u_j = f_j \) as distributions on \( X \setminus M \) (by definition of the strong minimal realization \( \partial_c \)). It remains to estimate \( \|u_j\|_\omega \).

To do so, we may assume that \( u_j = \partial \alpha_j \) for some \( \alpha_j \in L^2_{n,2}(D_j, F, \omega_o) \cap \text{Dom} \partial \cap \text{Dom} \partial_c \) satisfying \( \partial_c \alpha_j = 0 \), i.e. \( (\partial_c \partial + \partial \partial_c) \alpha_j = f_j \). By the subelliptic estimates, \( \alpha_j \) is also sufficiently smooth on \( \overline{D_j} \) (\( f_j \) is of class \( C^3 \) and vanishes outside a compact of \( D_j \), so \( \alpha_j \) is at least in the Sobolev space \( W^3 \) and smooth up to the boundary outside the support of \( f_j \)).

We will now estimate \( \alpha_j \) by using a priori estimates from [Gri] for negative line bundles over the strictly pseudoconcave domains \( W_j = X \setminus \overline{\Omega_2} \).

From [Gri] Theorem VI and Theorem 7.4 it follows that there exists \( \lambda > 0 \) such that

\[
\|v\|_{\omega_o, W_j}^2 \leq \frac{\lambda}{\ell} (\|\partial v\|_{\omega_o, W_j}^2 + \|\partial v\|_{\omega_o, W_j}^2)
\]

for all \( v \in L^2_{0,q}(W_j, F^*, \omega_o) \cap \text{Dom} \partial \cap \text{Dom} \partial_c \), \( 0 \leq q \leq n-2 \). From this we infer by Serre duality as in [Ch-S] that

\[
\|v\|_{\omega_o, W_j}^2 \leq \frac{\lambda}{\ell} (\|\partial_c v\|_{\omega_o, W_j}^2 + \|\partial v\|_{\omega_o, W_j}^2)
\]

(7.2)

for all \( v \in L^2_{n,q}(W_j, F, \omega_o) \cap \text{Dom} \partial_c \cap \text{Dom} \partial, q \geq 2 \).
Note that $D_j$ has two connected components $D_j^\pm$. We now choose an extension $\tilde{\alpha}_j^\pm$ of $\alpha_j|_{D_j^\pm}$ to $W_j$ such that $\tilde{\alpha}_j^\pm \in \text{Dom}\partial_c \cap \text{Dom} \theta$ (on $W_j$!) and such that

$$\|\partial \tilde{\alpha}_j^\pm\|_{\omega_o, W_j}^2 + \|\overline{\partial} \tilde{\alpha}_j^\pm\|_{\omega_o, W_j}^2 \leq b(\|\partial \alpha_j\|_{\omega_o, D_j^\pm}^2 + \|\overline{\partial} \alpha_j\|_{\omega_o, D_j^\pm}^2 + \|\alpha_j\|_{\omega_o, D_j^\pm}^2)$$

for some constant $b$ not depending on $\alpha_j$ nor on $j$. This is possible for $j$ sufficiently large by general Sobolev extension methods (locally we flatten the boundary $\partial \Omega_j$ and extend the sufficiently smooth $\alpha_j$ componentwise across $\partial \Omega_j$ by first order reflection, then we use a partition of unity, cf e.g. \[\text{[E]}\].

Applying (7.2) with $\tilde{\alpha}_j^\pm$ yields

$$\|\tilde{\alpha}_j^\pm\|_{\omega_o, W_j}^2 \leq \frac{\lambda}{\ell} (\|\overline{\partial_c} \tilde{\alpha}_j^\pm\|_{\omega_o, W_j}^2 + \|\partial \tilde{\alpha}_j^\pm\|_{\omega_o, W_j}^2) \leq \frac{\lambda}{\ell} b(\|\partial \alpha_j\|_{\omega_o, D_j^\pm}^2 + \|\overline{\partial} \alpha_j\|_{\omega_o, D_j^\pm}^2 + \|\alpha_j\|_{\omega_o, D_j^\pm}^2) = \frac{\lambda}{\ell} b(\|\partial \alpha_j\|_{\omega_o, D_j^\pm}^2 + \|\alpha_j\|_{\omega_o, D_j^\pm}^2)$$

For $\ell$ sufficiently large we thus obtain

$$\|\alpha_j\|_{\omega_o}^2 \leq \|\partial \alpha_j\|_{\omega_o}^2 \ll \overline{\partial}_c \partial \alpha_j, \alpha_j \gg \omega_o$$

$$= \ll f_j, \alpha_j \gg \omega_o \ll \|f_j\|_{\omega_o} \|\alpha_j\|_{\omega_o},$$

which implies

$$\|\alpha_j\|_{\omega_o} \leq \|f_j\|_{\omega_o}.$$ 

Thus

$$\|u_j\|_{\omega_o}^2 = \ll \partial \alpha_j, \alpha_j \gg \omega_o = \ll \overline{\partial}_c \partial \alpha_j, \alpha_j \gg \omega_o = \ll f_j, \alpha_j \gg \omega_o \ll \|f_j\|_{\omega_o}^2.$$ 

It remains to compare the norms $\|u_j\|_{\omega_o}$ and $\|u_j\|_{\omega_o}$. To do so, we re-identify $u_j$ and $f_j$ with $N^\ell$-valued (0, 1) and (0, 2)-forms again. Since $M$ is Levi-flat, we have $dV_\omega \sim \rho^{-2} dV_{\omega_o}$. Using the Levi-flatness of $M$ again, we also have $|f_j|_{\omega_o}^2 = |\overline{\partial} \chi_j \wedge \theta|_{\omega_o}^2 \lesssim \rho^{-2} |f_j|_{\omega}^2$. On the other hand, we have $\omega \gg \omega_o$, which implies $|u_j|_{\omega_o} \gtrsim |u_j|_{\omega}$. Since $u_j$ is supported in $D_j$, we thus have

$$\|u_j\|_{\omega_o}^2 \lesssim \|u_j\|_{\omega_o}^2 \lesssim \|u_j\|_{\omega}^2 \lesssim j^2 \|f_j\|_{\omega_o}^2 \lesssim j^2 \|f_j\|_{\omega}^2,$$

which proves the desired estimate. \qed

The point of the following lemma is that even though $\ell \in \mathbb{N}$ can be arbitrary big, the weight function $-\varphi$ stays the same (it does not have to be multiplied by a large integer as $\ell$ increases!).
Lemma 7.4
Let $\ell, k \in \mathbb{N}$ be arbitrary. Then $\mathrm{Im} \overline{\Theta}$ is closed in $L^2_{0,1}(\tilde{X} \setminus M, \tilde{\omega}^{\ell} \otimes L^k, \tilde{\omega}, -\varphi)$. This implies that there exists a constant $C_o$ such that every $f \in L^2_{0,1}(\tilde{X} \setminus M, \tilde{\omega}^{\ell} \otimes L^k, \tilde{\omega}, -\varphi) \cap \mathrm{Im} \overline{\Theta}$ has a solution $u \in L^2_{0,0}(\tilde{X} \setminus M, \tilde{\omega}^{\ell} \otimes L^k, \tilde{\omega}, -\varphi)$ satisfying $\overline{\partial} u = f$ and $\|u\|_{\tilde{\omega}, -\varphi} \leq C_o \|f\|_{\tilde{\omega}, -\varphi}$.

Proof. We will show that for $\ell, k \in \mathbb{N}$ arbitrary, $\mathrm{Im} \overline{\Theta}$ is closed in $L^2_{n,n}(\tilde{X} \setminus M, (\tilde{\omega}^{\ast})^{\ell} \otimes (L^*)^k, \tilde{\omega}, \varphi)$; then we argue by duality.

First we consider a smooth extension of the the hermitian metric $\omega_o$ on $X$ to $\tilde{X}$. Recall that in degree $(n, n)$, the curvature term in (3.2) is given by the trace of the curvature form with respect to the metric under consideration. We now modify the metric in $(\tilde{\omega}^{\ast})^{\ell} \otimes (L^*)^k$ by a bounded factor $\exp(-m \rho^2)$. This adds to the curvature a term which is $m \partial \overline{\partial} \rho^2 = 2m \partial \overline{\partial} \rho + 2m \partial \overline{\partial} \rho \wedge \overline{\partial} \rho$. Taking $m$ sufficiently large, we may therefore assume that $\text{Trace}_{\omega_o}(i\Theta((\tilde{\omega}^{\ast})^{\ell} \otimes (L^*)^k))$ is positive outside a compact of $\tilde{X} \setminus M$. Also, by a theorem of Greene and Wu, $\tilde{X} \setminus M$ admits a strongly subharmonic exhaustion function with respect to the metric $\omega_o$ (since it is non compact). Pasting a multiple of this exhaustion function together with $\varphi$ (and still calling this modified exhaustion function $\varphi$), we may therefore assume that

$$\text{Trace}_{\omega_o}(i\Theta((\tilde{\omega}^{\ast})^{\ell} \otimes (L^*)^k) + i\partial \overline{\partial} \varphi) \gtrsim \rho^{-2}$$
on $\tilde{X} \setminus M$. If necessary, we can also modify the metric $\omega_o$ such that its torsion can be absorbed by the right-hand side of the above inequality. But then the above estimate implies that for $f \in L^2_{n,n}(\tilde{X} \setminus M, (\tilde{\omega}^{\ast})^{\ell} \otimes (L^*)^k, \omega_o, \varphi, \text{loc})$, there exists $u \in L^2_{n,n-1}(\tilde{X} \setminus M, (\tilde{\omega}^{\ast})^{\ell} \otimes (L^*)^k, \omega_o, \varphi)$ such that

$$\int_{\tilde{X} \setminus M} |u|^2_{\omega_o} e^{-\varphi} dV_{\omega_o} \lesssim \int_{\tilde{X} \setminus M} \rho^2 |f|^2_{\omega_o} e^{-\varphi} dV_{\omega_o},$$

provided the latter integral is finite. Since $u$ is of bidegree $(n, n-1)$ and $\tilde{\omega} \gtrsim \omega_o$, we get

$$\int_{\tilde{X} \setminus M} |u|^2_{\omega_o} e^{-\varphi} dV_{\tilde{\omega}} \lesssim \int_{\tilde{X} \setminus M} |u|^2_{\omega_o} e^{-\varphi} dV_{\omega_o}.$$

On the other hand, the Levi-flatness of $M$ implies $dV_{\tilde{\omega}} \lesssim \rho^{-2} dV_{\omega_o}$. Since $f$ is of top degree, this means that

$$\int_{\tilde{X} \setminus M} \rho^2 |f|^2_{\omega_o} e^{-\varphi} dV_{\omega_o} \lesssim \int_{\tilde{X} \setminus M} |f|^2_{\omega_o} e^{-\varphi} dV_{\tilde{\omega}}.$$

So we get

$$\|u\|_{\tilde{\omega}, -\varphi} \lesssim \|f\|_{\tilde{\omega}, -\varphi}.$$
This proves that $\text{Im} \partial$ is closed in $L^2_{n,m}(\tilde{X} \setminus M, \tilde{N}^{(-\ell)} \otimes L^{(-k)}, \tilde{\omega}, \varphi)$.

Now Serre duality permits to conclude that $\text{Im} \partial$ is closed in $L^2_{0,1}(\tilde{X} \setminus M, \tilde{N}^{\ell} \otimes L^{k}, \tilde{\omega}, -\varphi)$. □

8 Realization as a hypersurface of a compact Kähler manifold

The final step in the proof of Theorem 1.1 is to show the following Proposition.

Proposition 8.1

A tubular neighborhood of $M$ in $X$ can be holomorphically embedded into a compact Kähler manifold of dimension $n$.

Proof. In [O-S2] (see also [O2]) and [H-M]), Kodaira’s embedding theorem was generalized to the setting of compact Levi-flat CR manifolds, and it was shown that sufficiently high tensor powers of a positive CR line bundle over a smooth, compact Levi-flat hypersurface $M$ admit enough CR sections $s_0, \ldots, s_m$, so that the CR map $[s_0 : \ldots : s_m]$ provides a CR-embedding of $M$ into $\mathbb{CP}^m$. This applies to our situation, since $N^{1,0}_M = \tilde{N}|_M$ is assumed to be positive.

In particular, it was proved in [H-M] that if $\ell$ is big enough, then the $C^4$-smooth CR-sections of $\tilde{N}^{\otimes \ell}|_M$ separate the points on $M$ and give local coordinates on $M$. Using Proposition 7.2, the CR-sections of $\tilde{N}^{\otimes \ell}|_M$ can be extended to holomorphic sections of $\tilde{N}^{\otimes \ell}$ over a tubular neighborhood of $M$ in $\tilde{X}$.

Arguing by continuity, it is not difficult to see that if $\ell$ is big enough, then, after possibly shrinking $X$, the holomorphic sections of $\tilde{N}^{\otimes \ell}$ separate points and give local coordinates on $X$. Hence we have a holomorphic embedding $\Psi : X \hookrightarrow \mathbb{CP}^m$.

Let $\omega_{FS}$ denote the Fubini-Study metric on $\mathbb{CP}^m$. We will show that $\Psi^* \omega_{FS}$ extends to a Kähler metric on a divisorial blow-up of $\tilde{X}$.

First we extend $\Psi$ to a meromorphic map $\tilde{\Psi} : \tilde{X} \longrightarrow \mathbb{CP}^m$: The embedding $\tilde{\Psi}$ is obtained from holomorphic sections $s_j$, $j = 0, \ldots, m$ of the line bundle $\tilde{N}^{\otimes \ell}$ over $X$ for some large $\ell$. Each of these holomorphic sections $s_j$ can be extended to a holomorphic section $\tilde{s}_j$ of the line bundle $\tilde{N}^{\otimes \ell} \otimes L^{\otimes k}$.
over \(\hat{X}\) for some large \(k\) (use Proposition 7.1 and see the proof of Proposition 7.2; also note that \(L\) is trivial over \(X\)). Then \([\tilde{s}_0 : \ldots : \tilde{s}_m]\) gives a meromorphic extension \(\tilde{\Psi}\) of \(\Psi\).

By Hironaka’s method we may blow up \(\tilde{X}\) along smooth centers, several times if necessary, to obtain a smooth complex manifold \(\hat{X}\) of dimension \(n\), together with a holomorphic map \(p : \hat{X} \to \tilde{X}\), such that the induced map \(\hat{\Psi} = \tilde{\Psi} \circ p : \hat{X} \to \mathbb{C}\mathbb{P}^m\) is holomorphic. Let \(Z\) denote the exceptional divisor of \(p\).

We have \(\hat{\Psi}^*\omega_{FS} \geq 0\) on \(\hat{X}\), and \(\hat{\Psi}^*\omega_{FS} > 0\) on \(\{z \in \hat{X} \mid \text{Jac}\hat{\Psi}(z) \neq 0\}\). But since \(\Psi\) gives an embedding of \(X\), the analytic set \(\{z \in \hat{X} \mid \text{Jac}\hat{\Psi}(z) = 0\}\) is compact in the strongly pseudoconvex manifold \(\hat{X} \setminus M\). But this means that \(\{z \in \hat{X} \mid \text{Jac}\hat{\Psi}(z) = 0\}\) \(\subset Z\).

We choose a relatively compact open subset \(V\) of \(\hat{X} \setminus M\) that contains \(Z\).

According to Grauert \([G1], \S 3, \text{Satz}\ 1\), the line bundle \(O(Z)\) associated to the divisor \(Z\) is negative. The curvature form \(\Omega = i\Theta(O(-Z))\) defines a positive Kähler form on \(V\). Since \(O(-Z)\) is trivial over \(V \setminus Z\), there exists a smooth function \(\psi\) such that \(i\partial\bar{\partial}\psi = \Omega\) on \(V \setminus Z\). Now we choose a smooth cut-off function \(\lambda\) with compact support in \(V\) such that \(\lambda \equiv 1\) in a neighborhood of \(Z\). For some sufficiently small \(\tau > 0\), the form \(\omega = \begin{cases} \hat{\Psi}^*\omega_{FS} + \tau i\partial\bar{\partial}(\lambda\psi) & \text{on } \hat{X} \setminus Z \\ \hat{\Psi}^*\omega_{FS} + \tau\Omega & \text{near } Z \end{cases}\)
is then a Kähler metric on \(\hat{X}\).

\(\square\)

End of the proof of Theorem 1.1. The last step in the proof of the theorem is as in [Br] or [O3] (see also section 2). By Proposition 8.1 \(M\) can be realized as a smooth real hypersurface in a compact Kähler manifold \(\hat{X}\). Repeating the arguments from section 6, the holomorphic normal bundle \(N^1_{M}\) (which is topologically trivial) extends to a topologically trivial holomorphic line bundle over \(\hat{X}\). The Hermitian metric on \(N^1_{M}\) can be extended to a Hermitian metric of this holomorphic line bundle. Since the holomorphic line bundle is topologically trivial, its curvature form is \(d\)-exact. Applying the \(\partial\bar{\partial}\)-lemma on Kähler manifolds, we may thus conclude that the curvature form admits a potential, i.e. there exists a smooth function on \(M\) which is strictly plurisubharmonic along the leaves of the Levi foliation of \(M\). This contradicts the maximum principle. Therefore such \(M\) cannot exist. \(\square\)
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