Curing fermion mass gauge variance in QED$^{2+1}$

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Abstract

The problem of the gauge dependence of the fermion mass in the Maxwell-Chern-Simons QED$_{2+1}$ is revisited. Using Proca mass term as an intermediate infrared regulator we are demonstrating gauge-invariance of the fermion mass shell in QED$_{2+1}$ in all orders of the perturbation theory.

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1 Introduction

Quantum field theories in 3 space-time dimensions provide fertile ground for studying high-temperature asymptotics of 4-dimensional models [1] and different effects in planar systems in condensed matter [2]. Comparing with ordinary 4-dimensional theories the lower dimension looks as an apparent simplification. However, lower space-time dimension brings infrared singularities that have grave consequences. In one of the first papers on 3-dimensional models [3] it was indicated that such a singularity leads to the gauge dependence of the one-loop fermion mass shell. On the one-loop level a simple solution of the problem was proposed [3]: only the transversal (Landau) gauge $\alpha = 0$ is valid, since it provides the same result as the Coulomb gauge (which is actually ill-defined), but in higher orders of the perturbation theory the situation is uncertain. Moreover, gauge variance of the fermion mass shell may indicate presence of an anomaly even in the transversal gauge.

In this letter we shall discuss application of the Proca regularization to the topologically massive QED$_{2+1}$. We shall demonstrate that in the Proca-Maxwell-Chern-Simons QED$_{2+1}$ (Proca model) in a relativistic gauge the Ward identities imply gauge-invariance of the fermion mass shell. The regularization may be taken off and in the limit of zero Proca mass the model reduces to the topologically massive QED$_{2+1}$ in the transversal gauge. We shall demonstrate the infrared finiteness of the topologically massive QED$_{2+1}$ in the transversal gauge in all orders of the perturbation theory and prove the unitarity of the model.

The Lagrangian of the Proca model is the following:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{4} \varepsilon^{\mu\nu\alpha} F_{\mu\nu} A^\alpha + \frac{m^2}{2} A^\mu A^\mu - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \bar{\psi}(i\gamma^\mu + eA - M)\psi.$$  (1)

We are using standard notations, $\theta$ is the Chern-Simons mass and $m$ is the Proca mass. The corresponding gauge field propagator is

$$D^{Proca}_{\mu\nu}(p) = -i \frac{p^2 - m^2}{(p^2 - m^2)^2 - \theta^2 p^2} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + i\theta\varepsilon_{\mu\nu\alpha} \frac{p^\alpha}{p^2 - m^2} - i\frac{p_\mu p_\nu}{p^2} \frac{\alpha}{p^2 - \alpha m^2}.$$  (2)

(at $m \rightarrow 0$ the latter is a propagator of the topologically massive QED$_{2+1}$ in a relativistic gauge $\alpha$).

The Ward identities in the Proca model have the standard form:

$$\partial_\mu \frac{\delta \bar{\Gamma}}{\delta A^\mu(x)} = ie\psi(x) \frac{\delta \bar{\Gamma}}{\delta \bar{\psi}(x)} - ie\bar{\psi}(x) \frac{\delta \bar{\Gamma}}{\delta \psi(x)},$$  (3)

$$\Gamma = \bar{\Gamma} - \frac{1}{2\alpha} (\partial_\mu A^\mu)(\partial_\nu A^\nu) + \frac{m^2}{2} A^\mu A^\mu = W - IA - \eta\psi - \bar{\eta}\bar{\psi},$$  (4)

$$e^{iW(I,\eta,\bar{\eta})} = \int D\bar{A}D\bar{\psi}D\bar{\psi}e^{i\int d^4x(\mathcal{L} + IA + \eta\bar{\psi} + \bar{\eta}\psi)}.$$  (5)

Taking Eq. (3) into account one may obtain the derivative of the generating functional of the vertex functions $\bar{\Gamma}$,
\[
\frac{\delta \Gamma}{\delta \alpha} = \frac{e}{2\alpha} \int dx dy \left[ \partial_\mu \frac{\delta^2 W}{\delta I^\mu(x) \delta \eta(y)} D(x - y) \frac{\delta \Gamma}{\delta \psi(y)} - \partial_\mu \frac{\delta^2 W}{\delta I^\mu(x) \delta \bar{\eta}(y)} D(x - y) \frac{\delta \Gamma}{\delta \bar{\psi}(y)} \right],
\]

\[
D(x) = \int \frac{d\phi}{(2\pi)^3} \frac{e^{-ipx}}{p^2 - \alpha m^2}.
\]

Calculating variations of \(\delta \Gamma / \delta \alpha\) with respect to \(A, \psi, \bar{\psi}\) one may obtain derivatives of the corresponding vertex functions with respect to the gauge parameter \(\alpha\). For instance, the derivative of the fermion mass operator \(\Sigma\) is the following:

\[
\frac{\partial \Sigma(x - y)}{\partial \alpha} = -i e^2 \int dz \left[ \int dz_1 dz_2 D^\mu(z - z_1) \Gamma^\mu_\alpha(z_1, x, z_2) S(z_2 - z) \right] S^{-1}(z - y) - e^2 \int dz S^{-1}(x - z) \left[ \int dz_1 dz_2 D^\mu(z - z_2) S(z - z_1) \Gamma^\mu_\alpha(z_1, z_2, y) \right],
\]

\[
\Gamma^\mu(x, y, z) = \frac{\partial}{\partial A_\mu(x)} \frac{\partial}{\partial \delta \psi(y)} \frac{\partial}{\partial \delta \bar{\psi}(z)} \frac{\partial \Gamma}{\partial \psi(y, \bar{\psi}(z))} \bigg|_{\mu=\psi=\bar{\psi}=0},
\]

\(S\) is an exact fermion propagator, the mass operator \(\Sigma\) is defined as \(S^{-1} = S_0^{-1} + i \Sigma\), \(S_0\) is the bare fermion propagator, and the function \(D^\mu\) is the following:

\[
D^\mu_\alpha(x) = \int \frac{d\phi}{(2\pi)^3} \frac{e^{-ipx}}{p^2 - \alpha m^2} \frac{p^\mu}{(p^2 - \alpha m^2)^2}.
\]

In the one-loop approximation

\[
\frac{\partial \Sigma_1(x - y)}{\partial \alpha} = e^2 (\hat{p} - M) \Sigma'_1(p),
\]

\[
\Sigma'_1(p) = -\frac{i}{(2\pi)^3} \int dk \frac{(\hat{p} - \hat{k} + M)\hat{k}}{(k^2 - \alpha m^2)^2((p - k)^2 - M^2)}. \tag{13}
\]

On the fermion mass shell the quantity \(\Sigma'_1\) is nonsingular, \(\Sigma'_1(\hat{p} = M) = \frac{1}{8\pi m\sqrt{\alpha}}\) and Eq. \(\tag{12}\) implies independence of the one-loop fermion mass of the gauge parameter in the Proca model. Therefore, the one-loop fermion mass renormalization does not depend on the gauge parameter and in the \(m \to 0\) limit it coincides with the mass renormalization in the topologically massive QED \(2+1\).

\(^1\)In the topologically massive QED \(2+1\) the corresponding expression for \(\Sigma'_1\) is \(\Sigma'_1(p) = \frac{1}{8\pi} \cdot \frac{1}{p - M} + \ldots\), where \(\ldots\) denotes terms logarithmically divergent and finite on the mass shell, and the gauge invariance of the one-loop fermion mass does not follow from the Ward identities.
QED_{2+1} in $\alpha = 0$ gauge. We shall show below that in the Proca model all the vertex functions on the fermion mass shell are independent of the gauge parameter, and transversal in the photon momenta.

Transversality of vertex functions with respect to external photon momenta follows from Eq. (3) providing that the corresponding diagram is nonsingular on the fermion mass shell. The gauge invariance of the vertex functions follows from Eq. (6). To prove the gauge invariance of the model, which is equivalent to establishing the fact that Eq. (6) vanishes on the mass shell, we shall use the approach elaborated in Ref. [4]. The right-hand side of Eq. (6) has the following structure:

\[
\int dp \psi_{\text{in}}(p)K(p) \int dk \frac{k_\mu}{(k^2 - \theta^2)k^2} M_\mu(k),
\]

where the dotted line denotes the propagator $\frac{k_\mu}{k^2 - \theta^2}$, $K(p) = \hat{p} - M$, $\psi_{\text{in}}(p)$ is a solution of the free equation of motion, $K(p)\psi_{\text{in}}(p) = 0$, $\psi_{\text{in}}(p) \sim \delta(p^2 - M^2)$.

Eq. (14) vanishes if $K(p)$ acts on a function that has no $\frac{1}{p^2}$ singularities. If the diagram in the right-hand side of Eq. (14) is one-particle irreducible, the latter equation may be rewritten as

\[
\int dk \frac{(\hat{p} - \hat{k} + M)k_\mu}{(k^2 - 2pk + \delta)k^2(k^2 - \theta^2)} M_\mu(k), \quad \delta = p^2 - M^2,
\]

and it vanishes indeed on the fermion mass shell.

If the diagram Eq. (14) is a reducible one,

\[
\int dk \frac{k_\mu}{(k^2 - 2pk + \delta)k^2} \Gamma, \quad \Gamma = p - k,
\]

the equation similar to Eq. (13) is nonzero. However, its contribution will be cancelled identically after the fermion wave function renormalization [4] (note that in the topologically massive QED_{2+1} in an arbitrary $\alpha$-gauge the dotted line in Eqs. (14),(13) would be $k_\mu/k^4$, that makes Eq. (14) singular, even if the vertex function exists). Transversality of the vertex functions may be proven along the same lines.

Thus we have shown that the fermion mass shell in the Proca model is gauge-invariant. Now, if we could take the regularization off, we would obtain a limit corresponding to the topologically massive QED_{2+1} in $\alpha = 0$ gauge, which would imply the unitarity of the regularized topologically massive QED_{2+1}.

Let us show that $m \to 0$ limit of the Proca model is regular. The gauge field in the Proca model has the following eigenvectors decomposition:
\[
A_{\mu}^{\text{Proc\-a}}(x) = \sum_{i=1}^{3} \int \frac{d\mathbf{p}}{2\pi\sqrt{2\omega_{i}}} (e^{-i\mathbf{p}\cdot x} u_{\mu}^{(i)} a_{i}(\mathbf{p}) + \text{h.c.}) , \quad p_{\mu}^{(i)} = (\omega_{i}, \mathbf{p}) , \quad (17)
\]

\[
u_{\mu}^{(1,2)} = \frac{\omega_{1,2}}{|\mathbf{p}|\sqrt{\omega_{1,2}^{2} - \mathbf{p}^{2} + m^{2}}} (p_{\mu}^{(1,2)} - g_{\mu 0} \frac{\omega_{1,2}^{2} - \mathbf{p}^{2}}{\omega_{1,2}} + i \frac{\omega_{1,2}^{2} - \mathbf{p}^{2} - m^{2}}{\theta \omega_{1,2}} \varepsilon_{\mu 0} p^{(1,2)\alpha}) , \quad (18)
\]

\[
\omega_{1,2} = \sqrt{\mathbf{p}^{2} + m_{1,2}^{2}} , \quad m_{1,2}^{2} = m^{2} + \frac{\theta^{2}}{4} , \quad (19)
\]

\[
u_{\mu}^{(3)} = \frac{1}{m} p_{\mu}^{(3)} , \quad \omega_{3} = \sqrt{\mathbf{p}^{2} + m_{3}^{2}} , \quad m_{3}^{2} = \alpha m^{2} . \quad (20)
\]

Excitations \(u_{\mu}^{(1)}\) and \(u_{\mu}^{(2)}\) are the physical modes, and the excitation \(u_{\mu}^{(3)}\) is a nonphysical longitudinal mode. Due to the transversality of the vertex functions all the scattering amplitudes wherein photons with the polarization \(u_{\mu}^{(3)}\) are involved do not contribute to the scattering matrix.

On the other hand, in the topologically massive QED\(_{2+1}\) just one excitation is physical, namely \(e_{\mu}^{(1)}\):

\[
A_{\mu}^{\text{QED}}(x) = \int \frac{d\mathbf{p}}{2\pi\sqrt{2\omega}} (e^{-i\mathbf{p}\cdot x} e_{\mu}^{(1)} a_{\text{ph}}(\mathbf{p}) + \text{h.c.}) + \int \frac{d\mathbf{p}}{2\pi\sqrt{2|\mathbf{p}|}} (e^{-i\mathbf{p}\cdot x} (e_{\mu}^{(2)} a_{2}(\mathbf{p}) + e_{\mu}^{(3)} a_{3}(\mathbf{p})) + \text{h.c.}) , \quad (21)
\]

\[
e_{\mu}^{(1)} = \frac{\omega}{\theta |\mathbf{p}|} (p_{\mu}^{(1)} - g_{\mu 0} \frac{\theta^{2}}{\omega} + i \frac{\theta}{\omega} \varepsilon_{\mu 0} p^{(1)\alpha}) , \quad \omega = p_{\mu}^{(1)} = \sqrt{\theta^{2} + \mathbf{p}^{2}} . \quad (22)
\]

\[
e_{\mu}^{(2)} = \frac{1}{\theta |\mathbf{p}|} \frac{p_{\mu}^{(2)}}{\sqrt{\theta |\mathbf{p}|}} , \quad p_{0}^{(2)} = p_{0}^{(3)} = |\mathbf{p}| \quad (23)
\]

\[
e_{\mu}^{(3)} = -\frac{1}{2} \sqrt{\frac{\theta}{|\mathbf{p}|}} \left( \alpha g_{\mu 0} + \frac{2i}{\theta} \varepsilon_{\mu 0} p^{(3)\beta} + \left( \frac{|\mathbf{p}|}{\theta} - \frac{\alpha}{2 |\mathbf{p}|} - i\alpha x_{0} \right) p_{\mu}^{(3)} \right) . \quad (24)
\]

\(e_{\mu}^{(2)}\) and \(e_{\mu}^{(3)}\) are massless nonphysical excitations.

In \(m \to 0\) limit \(u_{\mu}^{(1)}\) turns into \(e_{\mu}^{(1)}\), and \(u_{\mu}^{(2)}\) may be rewritten as follows:

\[
u_{\mu}^{(2)}|_{m \to 0} = \frac{1}{m} p_{\mu} + ml_{\mu}(p) \quad (25)
\]

the vector \(l_{\mu}\) having a finite limit at \(m \to 0\). After taking the regularization off the contribution of the photons of polarization \(u_{\mu}^{(2)}\) to the scattering amplitudes should vanish. Due to the transversality of vertices the contribution of the longitudinal part of \(u_{\mu}^{(2)}\) vanishes and the remainder is proportional to \(m\). Therefore, emitting of \(n\) "soft" photons (of the polarization \(u_{\mu}^{(2)}\)) gives rise to the coefficient \(m^{n}\). Thus, if diagrams with \(n\) \(u_{\mu}^{(2)}\) photons have infrared singularities lower than \(m^{n}\), the corresponding scattering matrix elements vanish in \(m \to 0\) limit.
Fortunately, in the transversal gauge all diagrams are infrared finite. The proof is straightforward. In $\alpha = 0$ gauge the infrared-dominant term of the gauge field propagator is proportional to $\varepsilon_{\mu\nu\alpha}k^\alpha/k^2$. Consider a closed fermion loop with $n$ outgoing photons, $\Pi_{\mu_1...\mu_n}(p_1,...,p_n)$. Due to the transversality, $p^\mu \Pi_{\mu_1...\mu_n}(p_1,...,p_n) = 0$, the diagram $\Pi_{\mu_1...\mu_n}(p_1,...,p_n)$ for $n > 2$ is proportional to a momentum of each photon, thus suppressing (possible) infrared singularities of photon legs. The two-point photon vertex has the following structure:

$$\Pi_{\mu\nu} = (p^2 g_{\mu\nu} - p_{\mu}p_{\nu})A + \varepsilon_{\mu\nu\lambda}p^\lambda B$$

with $A$ and $B$ having finite limits at $p \rightarrow 0$. Therefore, dressed photon lines have the same linear infrared singularity as the bare one, and any insertion of fermion loops is infrared safe.

Nonclosed internal fermion lines end with external fermion legs. These (dressed) internal fermion lines may be tied together by photon lines through (dressed) vertices. Let us choose the momenta of internal photon lines as the integration momenta in the diagram loops. Suppose the loop has no vertices with external photon legs. In this case, since the photon propagator brings $1/k$ singularity, infrared divergency in the loop integral may arise when the loop contains two other lines with $1/k$ singularities. The additional $1/k$ singularities may come only from the denominators of neighboring fermion propagators when other terms in these denominators are cancelled due to the mass shell condition. Therefore, potentially logarithmically divergent diagrams have the following structure:

$$\int \frac{dk}{p_1, \alpha_1 \quad p_1 + k \quad p_2, \alpha_2 \quad p_2 - k}$$

The infrared-dangerous part of Eq. (27) is equal to

$$\int \frac{dk}{(p_1 + k)^2 - M^2)((p_2 - k)^2 - M^2)k^2}T_{\beta_1\beta_2}$$

In the vicinity of the fermion mass shell, i.e. when $\hat{p} - M = \delta/2M$, Eq. (28) is proportional to $\delta \ln \delta$ and vanishes on the fermion mass shell, $\delta \rightarrow 0$. Naively, the dressing of the fermion lines in Eq. (27) with external soft photons would worsen the diagram. However, additional fermion propagators are infrared-regular. For instance, let a photon with momentum $p$ be emitted from the upper fermion line, Eq. (27). The denominator of the additional fermion propagator is equal to

$$(p_1 + k - p)^2 - M^2 = k^2 - 2pk + p^2 + \delta - 2p_1p + 2kp_1$$

and it is nonzero even at $p^2 = 0$.

Thus we have demonstrated that the anomaly is absent from the topologically massive QED$_{2+1}$ and calculations may be performed in any relativistic gauge $\alpha$ provided that proper regularization is done. Detailed analysis of the problem of the fermion mass shell gauge invariance will be presented elsewhere [5].
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