ABSTRACT. We prove that the tree-like Deligne-Mumford operad is a homotopical model for the trivialization of the circle in the higher-genus framed little discs operad. Our proof is based on a geometric argument involving nodal annuli. We use Riemann surfaces with analytically parametrized boundary as a model for higher-genus framed little discs.

CONTENTS

1. Introduction 2
2. Operads and topology 7
  2.1. A brief reminder on operads 7
  2.2. Free operad 8
  2.3. Pushout of operads 12
3. Operads based on Riemann surfaces with boundary 12
  3.1. The operad of framed surfaces 12
  3.2. The monoid of framed annuli 15
  3.3. Framed nodal annuli 18
  3.4. Tree-like nodal surfaces 20
  3.5. Split Surfaces and the Geometric Pushout Theorem 22
4. Model Categories and Homotopy (Co)limits 29
  4.1. Model category theory 29
  4.2. The homotopy category 31
  4.3. Some important model categories 32
  4.4. Quillen adjunction 33
  4.5. Homotopy (co)limits 34
5. The Berger-Moerdijk Model Structure for Operads 35
  5.1. Existence of model structure 35
  5.2. W-construction and cofibrant replacement 35

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1. INTRODUCTION

The following result was proven by Drummond-Cole [11]. Let

\[ FLD \]

be the operad of framed little disks and let

\[ \overline{M}_{0,*} \]

be the genus zero Deligne-Mumford operad with \( k \)-to-one operations indexed by points in \( \overline{M}_{0,k+1} \), the moduli space of genus zero nodal surfaces with \( k + 1 \) marked points labeled by the \( k \) inputs and one output. Let

\[ FLD_{1,1} \]

be framed little disks with one input and one output (with only a space of \( 1 \to 1 \) operations, which is up to homotopy the group \( S^1 \)), and \( * \) the operad with only one identity \( 1 \to 1 \) operation. Then in any model structure on operads with weak equivalences spanned by maps of topological operads which are level-wise weak equivalences, we have the following result.

**Theorem** (Drummond-Cole [11]). The homotopy colimit of the diagram

\[ \text{pt} \leftarrow FLD_{1,1} \to FLD \]

is related by a canonical sequence of weak equivalences to \( \overline{M}_{0,*} \).

In this paper we give a higher-genus generalization of this result, which provides in particular a more geometric (and indeed motivic, as seen

\[ ^{\text{In fact, the result can be formulated in the } \infty \text{-category associated to the model structure, which only depends on the weak equivalences.}} \]
in [31]) interpretation of Drummond-Cole’s result in genus zero. To this end we define in [31] the operad

$$\text{Fr}_\partial$$

of \textit{framed surfaces} with spaces of operations given by the moduli spaces of complex, i.e. conformal, surfaces with analytically parametrized boundary, and with composition given by gluing boundary components along compatible parametrizations.

The operad $FLD$ embeds in $\text{Fr}_\partial$ by viewing the boundary of the “large” disk in which the framed little disks embed as the outgoing boundary of a conformal surface of genus zero, and the boundaries of the interior disks as incoming boundaries. This embedding establishes a homotopy equivalence between $FLD$ and the suboperad $\text{Fr}_{\partial,g=0}$ of framed surfaces of genus zero, and so $\text{Fr}_\partial$ is a natural higher-genus generalization of $FLD$ in the category of topological operads. Let

$$\text{Ann}$$

be the suboperad of $\text{Fr}_\partial$ consisting of annuli, i.e. genus zero framed surfaces with one incoming and one outgoing boundary components. This operad is homotopy equivalent to $FLD_{1,1}$ and to $S^1$, and each annulus is a homotopy unit for $\text{Fr}_\partial$. It is convenient to enlarge $\text{Ann}$ and $\text{Fr}_\partial$ to strictly unital operads

$$\widetilde{\text{Ann}}, \quad \widetilde{\text{Fr}}_\partial$$

by including infinitely thin annuli (see [3.2]).

The main result of the present paper is the following theorem. As before, suppose we are working with a model structure with weak equivalences spanned by maps of topological operads which are level-wise weak equivalences. Let

$$\text{DM}^{\text{tree}}$$

be the operad of “tree-like” nodal surfaces of arbitrary genus, whose spaces of operations are the partial compactifications of the moduli spaces $\mathcal{M}_{g,k+1} = \{M_{g,k+1} : g \geq 0, k \geq 0\}$ of closed Riemann surfaces of genus $g$ with $k + 1$ marked points by boundary components consisting of nodal curves whose dual graph is a tree.

**Theorem 1.1.** The homotopy colimit of the diagram

$$\text{pt} \leftarrow \widetilde{\text{Ann}} \rightarrow \widetilde{\text{Fr}}_\partial$$

of unital operads is related by a canonical sequence of weak equivalences to the Deligne-Mumford operad $\text{DM}^{\text{tree}}$.

The same statement holds for the homotopy colimit of the diagram of nonunital operads

$$\text{pt} \leftarrow \text{Ann} \rightarrow \text{Fr}_\partial.$$
Note that a priori the Deligne-Mumford operad is an operad of (topological) orbifolds. There is a canonical way to resolve it in the homotopy category, which we discuss in Appendix A. If one is only interested in the operad as an object of a rational homotopy category (e.g., by considering its chains over a field of characteristic zero), the orbifold structure can be ignored without changing the homotopy type and the homotopy colimit result holds on the level of coarse moduli spaces (note that in genus zero this distinction is irrelevant as there are no stabilizers).

This implies the following corollary, motivated by mirror symmetry considerations (see the discussion below). Let $\text{DM}_{\text{tree coarse}}$ be the operad built out of the underlying coarse moduli spaces of the orbifold-valued operad $\text{DM}_{\text{tree}}$. Let $k$ be a field of characteristic zero. In this context, the operads of chains with coefficients in $k$ on $\text{DM}_{\text{tree}}$ and $\text{DM}_{\text{tree coarse}}$ are equivalent (as the homology of finite groups is trivial in characteristic zero). From Theorem 1.1 we deduce the following result.

**Corollary 1.2.** Let $k$ be a field of characteristic 0. The data of an algebra over the operad of chains $C_\ast(\text{DM}_{\text{tree}})$ is equivalent to the data of a dg algebra $A$ over $C_\ast(\text{Fr}_D)$ together with a derived $S^1$-trivialization, i.e., a chain of quasiisomorphisms of $C_\ast(S^1)$-modules $\tau : A \cong V$, with $V$ a complex of $k$-modules carrying a trivial $S^1$-action.

This corollary follows from Theorem 1.1, essentially by the universal property of colimits. For a rigorous proof, see Appendix B. □

Corollary 1.2 is a higher-genus generalization of a result of Drummond-Cole and Vallette [12, Theorem 7.8]. The derived $S^1$-trivialization above is equivalent to a Hodge-to-de Rham degeneration data (in the sense of [12]). The characteristic zero condition above can be removed, at the cost of working with $\text{DM}_{\text{tree}}$ instead of $\text{DM}_{\text{tree coarse}}$, and considering algebras over an appropriate model-theoretic replacement of the operads involved.

**Motivation and sketch of proof.** The homological mirror symmetry conjecture of Kontsevich [22] has opened up the perspective of understanding Gromov-Witten invariants in terms of the Fukaya category. The archetypal relationship goes via considering the so-called closed-open map which conjecturally induces for (sufficiently nice) symplectic manifolds an isomorphism between symplectic cohomology and Hochschild cohomology of the Fukaya category, see [1, 14, 22]. The Hochschild cohomology carries the structure of a BV-algebra, reflected at chain level as a structure of an algebra over the operad of chains on the framed little discs. This includes an $S^1$-action, corresponding (conjecturally) to the $S^1$-action on symplectic cohomology. When the symplectic manifold is closed, the symplectic cohomology is canonically identified (via the fixed point map) with quantum cohomology and the
$S^1$-action is canonically trivial. On the other hand, quantum cohomology carries the structure of an algebra over the homology of the genus 0 Deligne-Mumford-Knudsen (DMK) operad. In 2003, Kontsevich formulated the conjecture that the framed little discs operad with a trivialization of the circle should be equivalent to the DMK operad. This was proved in various settings by Drummond-Cole [11], Drummond-Cole and Vallette [12], Khoroshkin, Markarian, Shadrin [21], see also Dotsenko, Shadrin, Vallette [10] for the relation between this picture and the Givental group action, and Costello [9] for a point of view inspired by field theory.

Our operad $\text{Fr}_\partial$ of framed surfaces is a higher genus analogue of the operad of framed little discs. Our Main Theorem 1.1 extends the equivalence of operads proved by Drummond-Cole [11] to higher genus, and Corollary 1.2 extends to higher genus the algebraic formulations from [10, 21]. Modulo some necessary model-category and $\infty$-category digressions, our proof is geometric and makes use of certain explicit and canonical degenerations of Riemann surfaces. Remarkably, our use of Riemann surfaces with analytically parametrized boundary, which makes the gluing operation well-defined, has a motivic counterpart discussed by the second author in [31].

The key technique in our proof consists in replacing the diagram

$$\text{pt} \leftarrow \text{Ann} \rightarrow \text{Fr}_\partial$$

by the homotopy equivalent, but much more geometrically meaningful diagram (cf. Theorem 3.9)

$$\text{NodAnn} \leftarrow \text{Ann} \rightarrow \text{Fr}_\partial.$$  

Here

$$\text{NodAnn}$$

is the operad of (stable) nodal annuli, with only $1 \to 1$ operations consisting of a compactification of Ann by allowing the modulus to tend to $\infty$, which we geometrically interpret as the annulus developing a node (disjoint from either parametrized boundary component). The resulting operad turns out to be contractible (Lemma 3.6), hence gives rise to a diagram whose homotopy colimit is equivalent to the homotopy colimit $\text{pt} \leftarrow \text{Ann} \rightarrow \text{Fr}_\partial$ of the theorem above. In fact, in this formulation the homotopy colimit result is visible geometrically, as the Geometric Pushout Theorem 3.9 from §3.4. The proof of the Main Theorem 1.1 relies in fact on a mild homotopy enhancement of the proof of Theorem 3.9.

**Two technical remarks.**

1. We use the Berger-Moerdijk model category structure on topological operads throughout, induced by a given model category structure on topological spaces. Note that the topological spaces we work with
are not CW complexes. This makes it inconvenient to use the standard (Quillen) model category structure on the category of topological spaces, and we replace it by the so-called mixed model category structure due to Cole [8]. On the other hand, all spaces we work with are homotopy equivalent (and not just weak homotopy equivalent) to CW complexes, which implies that our results will also hold in the Strøm model category structure [30], where only homotopy equivalences are inverted. This formalism is explained in §5.

2. Two of the operads that we will be working with are operads valued in topological orbifolds, which do not strictly speaking fit into the Berger-Moerdijk model category formalism used in the body of the paper. Nevertheless, as the difference is just one of semantics, we allow ourselves to freely write down and study comparison maps between topological and orbifold-valued operads. An explanation of how to interpret these orbifold-valued operads and the maps between them rigorously is given in Appendix A. We use the formalism of ∞-operads, which is equivalent to the one of Berger-Moerdijk by the recent work of Cisinski-Moerdijk [6,7], which builds on Barwick [2] and Heuts-Hinich-Moerdijk [10].

Structure of paper. Taking advantage of the analogous nature of the proofs of the homotopical Main Theorem 1.1 and of the Geometric Pushout Theorem 3.9 we first give in §2 and §3 a self-contained statement and proof of Theorem 3.9, along with a brief introduction to topological operads and their pushouts. A reader interested in the flavor of our proof without the topological technicalities can read those sections only. In §4, §5 we introduce the formalism of model categories and the Berger-Moerdijk model category structure on topological operads, which we will be working with. Appendix A contains a brief exposition of ∞-operads following Lurie [24], which is necessary in order to work rigorously with orbifold-valued operads such as DMtree. We prove the Main Theorem 1.1 in §6.

Note that analogues of our Geometric Pushout Theorem 3.9 hold in the category of cyclic and modular operads. We expect homotopy-theoretic results akin to Theorem 1.1 to hold as well, but are unable to prove them at the moment lacking knowledge of a sufficiently well developed model category theory in these contexts.

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2. Operads and topology

2.1. A brief reminder on operads. Operads were initially defined by May [25] in a topological context.

An operad $O$ is a construction that specifies a class of composable operations with multiple inputs. An operad in $\text{sets}$ is a collection of sets $O_n$, $n \geq 0$ of operations “with $n$ inputs and one output”, or operations “of arity $n$”, together with composition rules

$$\gamma : O_k \times O_{n_1} \times O_{n_2} \times \cdots \times O_{n_k} \to O_{n_1 + \cdots + n_k}$$

and permutation rules, consisting of right actions of the symmetric groups $\mathfrak{S}_n$ on $O_n$, $n \geq 0$, where $\mathfrak{S}_0 = 1$ by convention,

$$O_n \times \mathfrak{S}_n \to O_n, \quad (o, \sigma) \mapsto o\sigma.$$ 

The composition rules and the permutation rules are required to satisfy certain tautological relations which essentially encode the fact that they behave like composition and permutation of inputs. Generally, operads are also required to have a unit, $1 \in O_1$, with the property that composing $o \in O_k$, $k \geq 1$ by 1 on the left or with the tuple $(1, 1, \ldots, 1)$ on the right does not change $o$. By default, when we use the word “operad” we will mean unital operad.

A representation of an operad $O$ (in sets), or an algebra over $O$, is a set $(S, \rho)$ with a collection of maps $\rho_o : S^n \to S$ indexed by $o \in O_n$, $n \geq 0$. By convention $S^0$ consists of a single point and therefore we interpret the collection of maps $\rho_o$, $o \in O_0$ as a distinguished collection of elements in $S$. The case in which $O_0$ consists of a single element is historically important, see May [25], but the operads that we will construct in this paper will have naturally richer spaces $O_0$. The collections of maps $\rho_o$, $o \in O_n$, $n \geq 1$ are interpreted as spaces of operations with $n$ inputs and one output in $S$. We require these maps to satisfy the permutation rule

$$\rho_{o\sigma}(s_1, \ldots, s_n) = \rho_o(s_{\sigma(1)}, \ldots, s_{\sigma(n)})$$

for $\sigma \in \mathfrak{S}_n$ a permutation, and also the associativity rule

$$\rho_o \circ (\rho_{o_1} \times \cdots \times \rho_{o_k}) = \rho_{\gamma(o(o_1, \ldots, o_k))} : S^{m_1} \times \cdots \times S^{m_k} \to S.$$ 

Note that the only property needed in order to define algebras over operads in this context is that the category $\text{Sets}$ has a symmetric monoidal structure with respect to the cartesian product and the permutation action $\mathfrak{S}_n \times S^n \to S^n$, $\sigma(s_1, \ldots, s_n) = (s_{\sigma(1)}, \ldots, s_{\sigma(n)})$. In particular,
we can define the notion of algebra over an operad in any symmetric monoidal category \((\mathcal{C}, \otimes)\) with choice of unit object.

For convenience, we shall impose a slightly stronger condition: namely, that the symmetric monoidal category \(\mathcal{C}\) we work with be \emph{closed} (see \[3, \S 2\]), which in particular implies it has (small) colimits, the colimits distribute over pushouts and there is an internal Hom functor. The cases of most interest to us are the categories Top of (Hausdorff) topological spaces, Vect of vector spaces and Vect\(_{dg}\) of differential graded vector spaces. Given a functor between monoidal categories \(\mathcal{C} \to \mathcal{D}\), we get a functor of associated operad categories \(\mathbf{Op}_\mathcal{C} \to \mathbf{Op}_\mathcal{D}\).

Operads can be equivalently defined by specifying a smaller set of composition rules, the so-called \emph{partial compositions}. More precisely, given a unital operad \(O\) one defines the partial compositions

\[-\circ_i : O_k \times O_\ell \to O_{k+\ell-1}, \quad 1 \leq i \leq k\]

as \(u \circ_i v = \gamma(u; 1, \ldots, 1, v, 1, \ldots, 1)\) for \(1 \leq i \leq k\). These obey the tautological relations \(u \circ_i (v \circ_j w) = (u \circ_i v) \circ_{i+1} w\) for \(i, j \geq 1\) and \((u \circ_j w) \circ_i v = (u \circ_i v) \circ_{j+1} w\) for \(j > i \geq 1\) and \(v \in O_\ell\), called respectively \emph{sequential composition} and \emph{parallel composition}. These relations determine uniquely all the other composition and permutation rules for the operad \(O\), allowing for an equivalent definition of the operad structure.

2.2. \textbf{Free operad.} This section is based on \[3, \S 5.8\] and \[4, \S 3\].

Many constructions in algebra canonically output graded objects, i.e. objects of the form \(\bigsqcup X_i\) for \(\bigsqcup\) the coproduct operation (\(\oplus\) for vector spaces) and \(i \in I\) running over some indexing set. For example the free unital monoid on a set \(\Gamma\) (resp., a vector space \(V\)) is \(\text{Free}(\Gamma) = \bigsqcup_{n \in \mathbb{N}} \Gamma^n\) (resp., the tensor algebra \(\text{Free}(V) = \bigoplus_{n \in \mathbb{N}} V \otimes^n\)), indexed by the integers \(\mathbb{N}\). Note that the monoid structure on \(\text{Free}(\Gamma)\) “lives over” the standard additive monoid structure on \(\mathbb{N}\). Many constructions in algebra begin by resolving a monoid \(M\), or an associative algebra \(A\), by free ones using a simplicial object (or chain complex) based on the free algebra construction of the bar complex. The analogue of the bar complex in the theory of operads is indexed not by the monoid of natural numbers but by the operad of trees, whose \(n \to 1\) operations are given by certain trees with \(n\) distinguished “input edges” and one distinguished “output edge”. Note that some trees have automorphisms, which interact with the \(S_n\)-actions on spaces of operations, so properly speaking the free construction is indexed by the \emph{groupoid} of trees. Our take here is to resolve the groupoid of trees by the \emph{set} of planar trees (which have no automorphisms). These form an associative operad, without \(S_n\)-actions: the \(S_n\)-actions have to then be re-introduced by hand. We now explain this construction starting...
Let $C$ be any monoidal symmetric category and $\text{Op}_C$ be the category of operads in $C$. Define the category of $S$-collections on $C$, written $S_{-}\text{gr}\ C$, to be the category of sequences $X_* := (X_0, X_1, X_2, \ldots)$ with $X_n \in C$ a $S_n$-module for $n \geq 0$, and with morphisms given by equivariant sequences of morphisms in $C$.

We have a canonical forgetful functor
$$\text{forg} : \text{Op}_C \to S_{-}\text{gr}\ C$$
which associates to an operad $O$ the sequence $(O_0, O_1, O_2, \ldots)$ of its spaces of operations. This functor has a left adjoint
$$\text{Free} : S_{-}\text{gr}\ C \to \text{Op}_C$$
called the free operad functor. The adjunction relation reads
$$\text{Hom}_{\text{Op}_C}(\text{Free}(X_*), O) \cong \text{Hom}_{S_{-}\text{gr}\ C}(X_*, \text{forg}(O)).$$

We describe below the situation for $C = \text{Top}$.

2.2.1. The operad of labeled rooted trees. We describe first an important operad based on trees. The heuristic idea is the following: an operation with $n$ inputs is represented by a rooted tree with $n$ distinguished leaves labeled by the set $\{1, \ldots, n\}$, up to isomorphism. Composition of operations is represented by grafting such trees one upon another. The implementation of this idea is slightly delicate because of the need to keep track of the action of the symmetric groups on labelings.

A graph with half-edges is a graph $\Gamma$ with a set of vertices $\text{Vert}\_\Gamma$, a set of oriented edges $\text{Edge}\_\Gamma$, each having one tail and one head vertex, and an additional set of oriented half-edges $\text{Half}\_\Gamma$ with only one end (either head or tail). We denote $\text{Half}\_\Gamma^+ \Gamma$ the set of incoming half-edges. We denote $\overline{\Gamma}$ the oriented graph of full edges. We say that a graph with half-edges $\Gamma$ is a tree of operations if $\Gamma$ is a planar tree, $\overline{\Gamma}$ is a (planar) rooted tree and $\Gamma$ has exactly one outgoing half-edge which is attached to the root of $\overline{\Gamma}$. The planar structure of $\Gamma$ is understood to be part of the data. This definition allows for an arbitrary number (including 0) of incoming half-edges for $\Gamma$, it allows for some (or all) of the leaves of $\overline{\Gamma}$ to have no incoming half-edge attached to them, and it allows for the incoming half-edges of $\Gamma$ to be attached at any vertex of $\overline{\Gamma}$. See Figure 1. Note that each interior vertex of $\overline{\Gamma}$ has a unique outgoing edge attached to it. In addition to the above, we introduce the trivial tree $|$, consisting of a unique edge and no vertex.

A labeling of a tree of operations with $n \geq 1$ incoming half-edges is the bijective assignment of an element in $\{1, \ldots, n\}$ to each incoming half-edge. A labeled tree of operations is a pair $(\tau, \lambda)$ consisting of a tree of operations $\tau$ and a labeling $\lambda : \{1, \ldots, n\} \to \text{Half}^+_\tau$. Two
labeled trees \((\tau, \lambda)\) and \((\tau', \lambda')\) are equivalent if there exists a non-planar isomorphism \(\phi : \tau \xrightarrow{\sim} \tau'\) such that \(\lambda' = \phi^+ \lambda\), where \(\phi^+ : \text{Half}_+^{\tau} \xrightarrow{\sim} \text{Half}_+^{\tau'}\) is the induced bijection on the set of incoming half-edges.

Write

\[
\text{PlanarTree}_n
\]

for the set of all labeled trees of operations with \(n \geq 0\) incoming half-edges, and write

\[
\text{Tree}_n
\]

for the equivalence classes under the above equivalence relation. This is a \(S_n\)-equivariant groupoid with respect to the right action of \(S_n\) on labelings.

The \(S_n\)-groupoids \(\text{Tree}_n\), \(n \geq 0\) form an operad in the following way:

- Given an equivalence class of a labeled tree \([\tau, \lambda] \in \text{Tree}_k\) and a collection \([\tau_i, \lambda_i] \in \text{Tree}_{n_i}, 1 \leq i \leq k\) we define the composition \(\gamma([\tau, \lambda], ([\tau_1, \lambda_1], \ldots, [\tau_k, \lambda_k]))\) as follows. We choose representatives for each of the previous equivalence classes, we build a tree \(T\) by gluing for each \(i \in \{1, \ldots, k\}\) the outgoing half-edge of \(\tau_i\) to the \(i\)-th incoming half-edge of \(\tau\) as distinguished by the labeling \(\lambda\), producing thus for each \(i\) a new interior edge whose tail vertex is the root of \(\tau_i\) and whose head vertex is the same as that of the \(i\)-th incoming edge of \(\tau\). (If \(\tau_i\) is the trivial tree, the gluing is innocuous.) We define a labeling \(\ell\) of the tree \(T\) by concatenating the labelings \(\lambda_1, \ldots, \lambda_k\). The result of the composition is the equivalence class of the labeled tree \((T, \ell)\).

- The unit is provided by the trivial tree \(\mid\) with its unique labeling.

The resulting operad, denoted

\[
\text{Tree},
\]

is the operad of labeled rooted trees.

**Remark 2.1.** In the above description of the operad \(\text{Tree}\) we have used planar trees. This means that an ordering of the incoming edges and half-edges at each vertex is implicit, induced by a given orientation of the plane. Without loss of generality we could have used non-planar
trees without specifying such an ordering, and the resulting operad would have been the same: any tree (with half-edges) admits a planar representative, and indeed a choice of ordering of the incoming edges (and half-edges) at each vertex determines a planar embedding of the tree.

2.2.2. The free operad functor. Given a \( \mathcal{S} \)-space \( X_* \), the heuristic idea for the construction of the free operad \( \text{Free}(X_*) \) is the following: the space of operations in arity \( n \geq 0 \) consists of elements of \( \text{Tree}_n \), decorated at each vertex \( v \) by an element of \( X_{|v|} \), where \( |v| \geq 0 \) is the number of incoming edges and half-edges at \( v \), also called the valency of \( v \). The composition of operations is inherited from the composition of trees. As in the previous section, the implementation of this idea is subtle because the elements of \( \text{Tree}_n \) are equivalence classes. The construction of \( \text{Free}(X_*) \) mixes the \( \mathcal{S} \)-structure on \( X_* \) with the action of isomorphisms of trees on the sets of incoming edges and half-edges at corresponding vertices.

Let \( \tau \in \text{PlanarTree}_n \) be a tree of operations. For \( v \in \text{Vert}_\tau \) denote by \( \text{in}(v) \) the set of incoming edges and half-edges at \( v \) and let \( |v| = |\text{in}(v)| \geq 0 \) be the valency of \( v \). Given a \( \mathcal{S} \)-space \( X_* \), define

\[
X^\tau = \prod_{v \in \text{Vert}_\tau} X_{|v|}.
\]

A labeled rooted tree with vertices coloured by elements of \( X_* \) and with \( n \geq 0 \) incoming half-edges is a triple \( (\tau, \mathbf{x}, \lambda) \) with \( \tau \in \text{PlanarTree}_n \), \( \mathbf{x} \in X^\tau \), and \( \lambda : \{1, \ldots, n\} \sim \rightarrow \text{Half}_{|v|}^+ \) a labeling of the incoming half-edges of \( \tau \). Two such triples \( (\tau, \mathbf{x}, \lambda) \) and \( (\tau', \mathbf{x}', \lambda') \) are equivalent if there exists a non-planar isomorphism \( \phi : \tau \sim \rightarrow \tau' \) such that \( \lambda' = \phi^+ \lambda \) as above and such that the following relation holds for the colours \( \mathbf{x} = (x_v)_{v \in \text{Vert}_\tau} \) and \( \mathbf{x}' = (x'_w)_{w \in \text{Vert}_{\tau'}} \): for each vertex \( v \) of \( \tau \) we have \( x'_{\phi(v)} = x_v \sigma_\phi \), where \( \sigma_\phi \in \mathcal{S}_{|v|} \) is the permutation determined by \( \phi \) after identifying both sets \( \text{in}(v) \) and \( \text{in}(\phi(v)) \) with \( \{1, \ldots, |v|\} \) according to the planar ordering. We denote \( \text{PlanarTree}_n(X_*) \) the space of labeled rooted trees with vertices coloured by elements of \( X_* \), and with \( n \geq 0 \) incoming half-edges. This carries a natural topology and splits as a disjoint union of topological spaces indexed by the elements of \( \text{PlanarTree}_n \). We denote \( \text{Tree}_n(X_*) \) the space of equivalence classes under the above equivalence relation, which again carries a natural topology and splits as a disjoint union of topological spaces indexed by the elements of \( \text{Tree}_n \). This is naturally a \( \mathcal{S}_n \)-space under the action of the permutation group on labelings.
Definition 2.2. The spaces of operations in the free operad $\text{Free}(X_\ast)$ are

$$\text{Free}(X_\ast)_n = \text{Tree}_n(X_\ast), \quad n \geq 0.$$ These form a topological operad with compositions, unit, and $S$-structure inherited from the operad $\text{Tree}$.

2.3. Pushout of operads. Suppose that

$$P \leftarrow A \rightarrow Q$$

is a diagram of topological operads. We define the amalgamated product, or pushout

$$P \ast_A Q$$

to be – as is conventional – the colimit of the diagram in topological operads. Explicitly, this is a quotient (interpreted as a colimit) of the free operad $\text{Free}(P_\ast \sqcup Q_\ast)$ and is defined as follows. Consider the counit of the free-forgetful adjunction: this is the natural transformation between the functors $\text{Free} \circ \text{forg}$ and $\text{Id}_{\text{Op}}$ which associates to each operad $O$ the “product” morphism of operads $\prod : \text{Free}(O_\ast) \to O$ obtained by applying composition maps in $O$ to a tree of elements in $O$ recursively until the tree has a single vertex (this is independent of the order by the associativity of operations in operads). Now $P \ast_A Q$ is the quotient of $\text{Free}(P_\ast \sqcup Q_\ast)$ by the equivalence relation generated by the relations

$$\sim_1 \sqcup \sim_2$$

described as follows.

- $(\sim_1)$ If $o_{\text{free}} \in \text{Free}(P_\ast \sqcup Q_\ast)$ is a free element over a tree $\tau$ and $\tau$ has a sub-tree $\tau_0$ all of whose vertices are labeled by elements of $P$ (resp., of $Q$) then $o_{\text{free}}$ is equivalent to $o'_{\text{free}}$ with all vertices and all full edges of $\tau_0$ contracted to a point, and with the product $\prod(o_{\text{free}}|_{\tau_0})$ written at that point.

- $(\sim_2)$ Denote the two operad maps $i : A \to P$ and $j : A \to Q$. If $o_{\text{free}} \in \text{Free}(P_\ast \sqcup Q_\ast)$ is a free element over a tree $\tau$ which on some vertex $v \in \tau$ has a label which is equal to $i(a)$ for some $a \in A$, we set $o_{\text{free}} \sim o'_{\text{free}}$ where $o'_{\text{free}}$ has the label on $v$ replaced by $j(a)$.

The amalgamated product can be defined more generally for operads in categories which do not live over the category of sets. The above relations should then be understood as equalizer conditions in the underlying category.

3. OPERADS BASED ON RIEMANN SURFACES WITH BOUNDARY

3.1. The operad of framed surfaces.
Definition 3.1. A framed surface is a compact Riemann surface $\Sigma$ with boundary $\partial \Sigma$ locally analytically modelled on the upper half plane $\{z \in \mathbb{C} : \text{Im} \, z \geq 0\}$, together with an analytic parametrization $\varphi_i : S^1 \to C_i$ for each boundary component $C_i \subset \partial \Sigma$.

A component $C_i \subset \partial \Sigma$ is called an input or an output if the orientation induced by the parametrization coincides, respectively is opposite to the boundary orientation of $C_i$.

Write $\text{Fr}_{m,n}^\partial$ for the moduli space of framed surfaces with $m$ incoming and $n$ outgoing boundary components.

The space of oriented analytic diffeomorphisms $S^1 \to S^1$ which preserve a basepoint $1 \in S^1$ is contractible. Indeed, this set is identified with the space of analytic functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions $f(0) = 0$, $f(x+1) = f(x)+1$ for all $x \in \mathbb{R}$, and $f' > 0$, which is convex. (The function $f(x) = x$ can be taken as a basepoint.) As such, once an orientation of each boundary component has been specified (which is the same as a labelling of the components as inputs or outputs), an analytic parametrization $\varphi_i : S^1 \to C_i$ is determined up to homotopy by the choice of a basepoint $p_i = \varphi_i(1) \in C_i$.

Framed surfaces can be glued at inputs and outputs because of the following phenomenon.

A framed surface is canonically isomorphic in the neighborhood of each of its boundary components to a closed annulus $A_\epsilon = \{z \in \mathbb{C} : 1 - \epsilon \leq |z| \leq 1\}$ for some $\epsilon > 0$. Indeed, given a component $C_i$ with analytic parametrization $\varphi_i$, the latter locally extends uniquely, and these local extensions coincide on the overlaps by uniqueness of holomorphic continuation. The original parametrization $\varphi_i$ corresponds then to the restriction of the extended parametrization to the circle $\{|z| = 1\}$ if $C_i$ is an output, respectively to the restriction to the circle $\{|z| = 1 - \epsilon\}$ if $C_i$ is an input. As a consequence, any two framed surfaces are uniquely locally isomorphic in the neighborhood of any of their incoming, respectively outgoing boundary components.

Given two annuli $A_\epsilon, A_{\epsilon'}$ (viewed as complex manifolds with canonically parametrized boundary) the incoming boundary of the first can be glued to the outgoing boundary of the second (to produce an annulus with modulus $\ln 1/(1 - \epsilon) + \ln 1/(1 - \epsilon')$, cf. §3.2 below). Since every framed surface is isomorphic in a neighborhood of each of its boundary components to such an annulus, this gives us the local data necessary for gluing two framed surfaces along boundary components of opposite orientation,

$$(\Sigma, \gamma), (\Sigma, \gamma') \mapsto \Sigma_{\gamma, \gamma'} \Sigma'.$$
Note that this also makes sense if $\Sigma, \Sigma'$ are disconnected and also if $\gamma, \gamma'$ are boundary components consisting of multiple circles, as long as the orientations are compatible.

In particular, the moduli spaces $F_{\partial}^{m,n}$ form a topological PROP, and the moduli spaces $F_{\partial}^{m,1}$ with one output form a topological operad (see [May]). We denote this latter topological operad by $F_{\partial}$.

Note that the moduli space $F_{\partial}^{m,n}$ is a priori a stacky object, as a surface can have automorphisms. However, this can only happen when both $m$ and $n$ are equal to zero, as no nontrivial automorphism of a connected complex surface can fix an embedded curve or boundary component pointwise. In particular, since in this paper we will only be interested in the operad $F_{\partial}$ (which involves the moduli spaces $F_{\partial}^{m,n}$ with $n = 1$), we will never encounter any stacky phenomena involving framed surfaces.

**Remark 3.2.** It is understood here that the elements of $F_{\partial}$ are labeled framed Riemann surfaces, meaning that, for each framed Riemann surface $\Sigma \in F_{\partial}^{m,1}$, we are given a bijection $\lambda$ between $\{1, \ldots, m\}$ and the set of incoming boundary components of $\Sigma$. The bijection $\lambda$ is called a labeling, and there are of course $m!$ such choices of labelings. The labeling is necessary in order to define composition by gluing and hence the operad structure on $F_{\partial}$. This additional presence of labelings is standard for operads constructed out of Riemann surfaces, similarly to the case of the Deligne-Mumford spaces $\overline{M}_{g,n}$ where the $n$ marked points are also labeled. The symmetric group $S_m$ acts on the right on the set of labelings of a framed Riemann surface $\Sigma$ by composition at the source $(\lambda, \sigma) \mapsto \lambda \sigma, \sigma \in S_m$. For readability we will henceforth not mention explicitly the labelings of surfaces, but whenever we will write “framed surface” we will mean “labeled framed surface”.

**Remark 3.3.** Since we will be interested in $F_{\partial}$ as a topological operad, a few more words need to be said in order to specify the topology on the moduli spaces involved. Given any point of $F_{\partial}^{m,1}$ corresponding to a surface $S$, we can glue in disks (with standard parametrization of boundary) to all the inputs and outputs of $S$ to obtain a closed Riemann surface $\tilde{S}$. This gives an identification of $F_{\partial}^{m,1}$ with the moduli space of Riemann surfaces with $m + 1$ parametrized loops bounding disks isomorphic to the standard disk $D \subset \mathbb{C}$ and with standard induced parametrization on $\mathbb{C}$. In particular, $F_{\partial}$ is a subspace of the space of tuples $(X, \gamma_1, \ldots, \gamma_{m+1})$ with $X$ a closed Riemann surface (corresponding to a point of some $\mathcal{M}_{g,m+1}$) and the $\gamma_i, i = 1, \ldots, m + 1$ pairwise nonintersecting contractible embedded analytic loops in $X$. This is a bundle over $\mathcal{M}_{g,m+1}$. We topologize $F_{\partial}^{m,1}$ as a locally closed subset of this bundle of tuples.
This presents $\frf_{\partial}$ as a complex infinite-dimensional manifold. Its local model at a framed Riemann surface of genus $g$ with $m + 1$ boundary components is the total space of a fibration over a neighborhood of the corresponding element in $\mathcal{M}_{g,m+1}$ with fiber given by $m + 1$-tuples embeddings of the disc in $\mathbb{C}$ close to the standard one. The fact that the corresponding element in $\mathcal{M}_{g,m+1}$ may be an orbifold point is irrelevant here.

3.2. The monoid of framed annuli. The genus 0 and arity 1 part of $\frf_{\partial}$ forms a topological monoid which we denote

$$\text{Ann}$$

and call the monoid of framed annuli. A framed annulus is a genus 0 Riemann surface $A$ with two boundary components $\partial A = \partial^+ A \sqcup \partial^- A$ labeled as input and output, together with analytic parametrizations $f_+$ of the input $\partial^+ A$ and $f_-$ of the output $\partial^- A$. Ignoring the parametrizations of the boundary components, such an annulus is conformally determined by its modulus $\alpha \in (0, \infty)$ (Schottky’s theorem, [28]). This is the logarithm of the ratio of the radii

$$\alpha = \ln \frac{R}{r}$$

of a standard annulus $A_{R,r} = \{z \in \mathbb{C} : r \leq |z| \leq R\}$, $r < R$ which is conformally equivalent to $A$, where the outer circle $|z| = R$ is labeled as input and the inner circle $|z| = r$ is labeled as output. The group of conformal automorphisms of the underlying Riemann surface $A$ is canonically isomorphic to $S^1$: up to replacing $A$ with a conformally equivalent standard annulus, its group of automorphisms is represented by the rotations of $\mathbb{C}$ which fix the origin. As such, the pair $(f_-, f_+)$ is considered modulo global rotations $\theta \cdot (f_-, f_+) = (\theta + f_-, \theta + f_+), \theta \in S^1$. With this understood, we write $[(A, f_-, f_+, \alpha)]$ for the equivalence class of a framed annulus $(A, f_-, f_+, \alpha)$.

Remark. The modulus behaves additively under gluing of standard annuli. However, it does not behave additively under gluing of general framed annuli. This can be seen explicitly by studying configurations of nested circles in $\mathbb{C}$.

The topological monoid $\text{Ann}$ is not unital. In order to achieve unitality, it is convenient to enlarge it to the topological monoid of possibly degenerate framed annuli, denoted $\widetilde{\text{Ann}}$, by including the moduli space of framed annuli of modulus 0, denoted $\text{Ann}^0$.

A framed annulus of modulus 0 is a triple $(C, f_-, f_+)$ consisting of a connected closed analytic 1-dimensional manifold $C$ together with analytic diffeomorphisms $f_{\pm} : S^1 \to C$. We will also refer to $(C, f_-, f_+)$
as being a framed annulus of thickness zero, or as being a degenerate framed annulus. Two such framed annuli \((C, f_-, f_+)\) and \((D, g_-, g_+)\) are equivalent if there exists an analytic diffeomorphism \(\psi : C \to D\) such that \(g_\pm = \psi f_\pm\). As such, the framed annulus \((C, f_-, f_+)\) is equivalent to \((S^1, \text{id}, f_-^{-1}f_+)\) and also to \((S^1, f_+^{-1}f_-, \text{id})\). We choose the first expression to realize a bijection

\[
\text{Ann}^0 \xrightarrow{\sim} \text{Aut}(S^1), \quad [(C, f_-, f_+)] \mapsto f_-^{-1}f_+.
\]

The composition of the equivalence classes of two framed annuli of modulus 0 is defined by

\[
[(C, f_-, f_+)] \circ [(D, g_-, g_+)] = [(C, f_-f_+^{-1}g_+, g_+)] = [(D, g_-^{-1}f_-, g_+)].
\]

This makes \(\text{Ann}^0\) into a group. The neutral element is the class \([(S^1, \text{id}, \text{id})]\), consisting of degenerate annuli \((C, f_-, f_+)\) with \(f_- = f_+\). The inverse of \([(C, f_-, f_+)]\) is \([(C, f_+, f_-)]\). As such the above bijection

\[
\text{Ann}^0 \xrightarrow{\sim} \text{Aut}(S^1)
\]

is a group isomorphism. (Had we chosen to associate to the class of an annulus \([(C, f_-, f_+)\]) the element \(f_-^{-1}f_+ \in \text{Aut}(S^1)\), suggested by choosing as a representative the degenerate annulus \((S^1, f_+^{-1}f_-, \text{id})\), we would have obtained a bijective group anti-homomorphism.)

The topological monoid \(\text{Ann}\) is a trivial fiber bundle over \((0, \infty)\), which is the space of moduli of unframed annuli, with fiber \(\text{Aut}(S^1) \times _{S^1} \text{Aut}(S^1)\), where \(\text{Aut}(S^1)\) stands for the group of analytic automorphisms of the circle and \(S^1\) acts diagonally on \(\text{Aut}(S^1) \times \text{Aut}(S^1)\) by translations in the target. We topologize \(\text{Ann}\) by extending this trivial fiber bundle to a trivial fiber bundle over \([0, \infty)\) and collapsing the fiber at 0 via the diagonal action of \(\text{Aut}(S^1)\) given by \(\varphi \cdot (f_-, f_+) = (\varphi f_-, \varphi f_+)\). We identify the quotient with \(\text{Aut}(S^1)\) via \((f_-, f_+) \mapsto f_-^{-1}f_+\) as above.

We extend the monoid structure from \(\text{Ann}\) to \(\widetilde{\text{Ann}}\) as described above for two elements in \(\text{Ann}^0\) and by defining

\[
[(A, f_-, f_+, \alpha)] \circ [(C, g_-, g_+, \alpha)] = [(A, f_-f_+g_-^{-1}g_+, \alpha)],
\]

and

\[
[(D, h_-, h_+, \alpha)] \circ [(A, f_-, f_+, \alpha)] = [(A, f_-h_+^{-1}h_+, f_+, \alpha)]
\]

for \([(A, f_-, f_+, \alpha)] \in \text{Ann}\) and \([(C, g_-, g_+)], [(D, h_-, h_+)]) \in \text{Ann}^0\).

We claim that this monoid structure is compatible with the above topology, i.e. \(\widetilde{\text{Ann}}\) is a topological monoid. To prove the claim, let us consider sequences \([(A^\nu, f^\nu_-, f^\nu_+, \alpha^\nu)]\) and \([(B^\nu, g^\nu_-, g^\nu_+, \beta^\nu)]\), \(\nu \geq 1\) with \(\alpha^\nu, \beta^\nu > 1\), and such that, for \(\nu \to \infty\), we have \(\alpha^\nu \to \alpha, \beta^\nu \to \beta\) with \(\alpha\) or \(\beta\) equal to 1. We can assume without loss of generality that \(A^\nu\) and \(B^\nu\) are standard annuli whose inner radius is equal to 1 and whose outer radius is equal to \(\alpha^\nu\), respectively \(\beta^\nu\), and also that

\[
\text{Ann}^0 \xrightarrow{\sim} \text{Aut}(S^1), \quad [(C, f_-, f_+)] \mapsto f_-^{-1}f_+.
\]
\( f_\pm^\nu \to f_\pm, \ g_\pm^\nu \to g_\pm, \) the limits being analytic parametrizations of the standard circles of corresponding radii 1, \( \alpha \) and \( \beta \).

We prove the claim in the case \( \alpha > 1 \) and \( \beta = 1 \). The glued annulus \( A^\nu \# B^\nu \) has input given by the boundary component \( \partial^+ B^\nu \) with parametrization \( g_+^\nu \), and output given by the boundary component \( \partial^- A^\nu \) with parametrization \( f_-^\nu \). See Figure 2. As \( \nu \to \infty \), the input \( \partial^+ B^\nu \) of \( B^\nu \) — which is the standard circle of radius \( \beta^\nu \) in \( \mathbb{C} \) — converges pointwise with respect to the standard parametrization to the standard circle of radius 1 with its standard parametrization, viewed as \( \partial^- B^\nu \) for all \( \nu \). The latter is identified with \( \partial^+ A^\nu \) via \( f_-^\nu g_-^\nu \). As such, the limit of the composition \( A^\nu \# B^\nu \) is canonically identified with the limit \( A \) of the sequence \( A^\nu \), and this identification is given by \( f_- + g_- \) along the input boundary component. The input boundary component of the limit inherits the parametrization \( g_+ \), and via this identification the latter corresponds to the parametrization \( f_- g_-^{-1} g_+ \) of the input boundary component of \( A \). As far as the output boundary component of the limit is concerned, it is canonically identified with the output boundary component of \( A \) and inherits as such the parametrization \( f_- \).

This shows that
\[
\lim_{\nu \to \infty} [(A^\nu, f_-^\nu, f_+^\nu, \alpha^\nu)] \circ [(B^\nu, g_-^\nu, g_+^\nu, \beta^\nu)] = [(A, f_-, f_+ g_-^{-1} g_+, \alpha)]
\]
\[
= [(A, f_-, f_+, \alpha)] \circ [(S^1, g_-, g_+)]
\]
\[
= \lim_{\nu \to \infty} [(A^\nu, f_-^\nu, f_+^\nu, \alpha^\nu)] \circ \lim_{\nu \to \infty} [(B^\nu, g_-^\nu, g_+^\nu, \beta^\nu)].
\]

The proof of the claim in the cases \( \alpha = 1, \beta > 1 \) and \( \alpha = \beta = 1 \) is analogous and we omit it.

**Figure 2.** We depict a (framed) annulus as a horizontal cylinder of finite length, with its input boundary component to the right and its output boundary component to the left. The composition \( A \circ B \) of two framed annuli is depicted by drawing \( A \) to the left of \( B \).
Definition 3.4. We define $\tilde{Fr}_\partial$ to be the extension of $Fr_\partial$ by possibly degenerate framed annuli,

$$\tilde{Fr}_\partial = Fr_\partial \sqcup_{Ann} \tilde{Ann}.$$ 

The definition makes sense since $Ann$ is open in $Fr_\partial$. A straightforward generalization of the arguments showing that $Ann$ is a topological monoid shows that $\tilde{Fr}_\partial$ is a topological operad, i.e. that composition with degenerate annuli is continuous over $Fr_\partial$. We omit the details.

3.3. Framed nodal annuli. Ordinary annuli have modulus parameter $\alpha \in (0, \infty)$. By introducing degenerate annuli, we have extended the possible parameters to $[0, \infty)$. In this section we will further extend the possible modulus parameters from $[0, \infty)$ to $[0, \infty]$. We do this by adding a new class of annuli, called nodal annuli, which have modulus parameter $\infty$. While introducing degenerate moduli did not change the homotopy type of the topological monoid $Ann$, adding in nodal annuli has a strong destructive effect: it makes the monoid contractible.

Definition 3.5. We say that a complex surface with analytically parametrized boundary is a framed nodal annulus if it has two boundary components, genus zero, and at most nodal singularities. (In order to shorten notation, the term “nodal annuli” includes ordinary annuli with no nodes.)

We say that a framed nodal annulus is unstable if it has an irreducible component which contains no boundary components (equivalently, if it has a component of genus zero and infinite automorphism group), and stable otherwise. See Figure 3. Note that all stable framed nodal annuli either have one irreducible component containing both boundary circles (i.e. they are ordinary framed annuli), or two irreducible components of which one contains the incoming circle and the other contains the outgoing circle. The stabilization of an unstable nodal annulus is obtained by contracting all irreducible components which have no boundary. We will be interested in the moduli space of stable framed nodal annuli, viewed as quotients of possibly unstable framed nodal annuli by the equivalence relation induced by stabilization. We write $\text{NodAnn}$ for the moduli space of stable framed nodal annuli. We topologize this space similarly to our moduli space of surfaces with boundary above. Namely, given a stable framed nodal annulus, we get a point of $\overline{M}_{0,4}$ by gluing in disks along both parametrized boundary components, and marking the images of $\pm 1 \subset S^1$ in both boundary components in the resulting genus zero curve. In this way, we can view $\text{NodAnn}$ as a subspace in the bundle over $\overline{M}_{0,4}$ whose fiber consists of
pairs of parametrized disjoint embedded analytic closed curves whose parametrizations map $\pm 1 \in S^1$ to the marked points.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{nodal_annuli.png}
\caption{Unstable/stable framed nodal annuli.}
\end{figure}

For the next Lemma, recall that we denote $\text{Aut}(S^1)$ the group of analytic automorphisms of $S^1$ with analytic inverse, and $\text{Aut}_0(S^1) \subset \text{Aut}(S^1)$ denotes the subgroup of automorphisms which fix $1 \in S^1$.

**Lemma 3.6.** The moduli space of stable framed nodal annuli is homeomorphic to $$(\text{Aut}_0(S^1) \times \text{Aut}_0(S^1)) \times \mathbb{C}.$$ In particular, it is contractible.

**Proof.** Consider the action of $S^1$ on $\text{Aut}(S^1)$ by translations in the target. The moduli space of stable framed nodal annuli containing a node is identified with $\text{Aut}(S^1)/S^1 \times \text{Aut}(S^1)/S^1$. Indeed, each of the two irreducible components of the underlying Riemann surface is equivalent to a disk with a marked point at the origin. The group of automorphisms of the latter is $S^1$, given by rotations, and it acts on the analytic parametrizations of its boundary by translations in the target. Writing $f(\text{mod } S^1)$ for the class of an element of $\text{Aut}(S^1)$ modulo the action of $S^1$, an arbitrary element of this moduli space can thus be written $(f^-, f^+ \text{mod } S^1)$.

With this understood, the topology on $\text{NodAnn}$ can be alternatively described as follows. Let $[(A^\nu, f^\nu_{\pm})], \nu \geq 1$ be a sequence in $\text{Ann}$ with moduli $\alpha^\nu \to \infty, \nu \to \infty$. Choose representatives $A^\nu = [-\alpha^\nu/2, \alpha^\nu/2] \times S^1$ and $(f^\nu_-, f^\nu_+ \in \text{Aut}(S^1) \times_{S^1} \text{Aut}(S^1)$ and assume that $(f^\nu_-, f^\nu_+) \to (f^-, f^+)$ as $\nu \to \infty$. We then have by definition

$$[(A^\nu, f^\nu_{\pm})] \to (f^-(\text{mod } S^1), f^+(\text{mod } S^1)), \quad \nu \to \infty.$$ 

By marking the point $1 \in S^1$ we obtain homeomorphisms

$$\text{Aut}(S^1)/S^1 \simeq \text{Aut}_0(S^1), \quad \text{Aut}(S^1) \simeq \text{Aut}_0(S^1) \times S^1,$$

and

$$\text{Aut}(S^1) \times_{S^1} \text{Aut}(S^1) \simeq \text{Aut}_0(S^1) \times \text{Aut}_0(S^1) \times S^1.$$ 

(None of these identifications preserves any group structure, see also Remark 3.7 below.)

We have already seen that the moduli space $\text{Ann}$ of framed annuli is a trivial bundle over $(0, \infty)$ with fiber $\text{Aut}(S^1) \times_{S^1} \text{Aut}(S^1)$, where
$S^1$ acts diagonally. In view of the isomorphism $S^1 \times (0, \infty) \simeq \mathbb{C}^\times$, after choosing a trivialization of the bundle $\text{Ann} \to (0, \infty)$ we obtain a homeomorphism

$$\text{Ann} \simeq \text{Aut}_0(S^1) \times \text{Aut}_0(S^1) \times \mathbb{C}^\times.$$ 

With respect to this identification, the projection $\text{Ann} \to (0, \infty)$ corresponds to the projection $\mathbb{C}^\times \to (0, \infty)$, $z \mapsto |z|$. Also, with respect to the identification of the moduli space of stable framed nodal annuli containing a node with $\text{Aut}_0(S^1) \times \text{Aut}_0(S^1)$, the definition of convergence for a sequence $(f_\nu^-, f_\nu^+, z_\nu) \in \text{Ann} \simeq \text{Aut}_0(S^1) \times \text{Aut}_0(S^1) \times \mathbb{C}^\times$ such that $|z_\nu| \to \infty$ and $(f_\nu^-, f_\nu^+) \to (f_-, f_+)$ as $\nu \to \infty$ translates into $(f_\nu^-, f_\nu^+, z_\nu) \to (f_-, f_+)$. In other words, we have a homeomorphism

$$\text{NodAnn} \simeq \text{Aut}_0(S^1) \times \text{Aut}_0(S^1) \times (\mathbb{C}^\times \cup \{\infty\}).$$

Up to an inversion on the factor $\mathbb{C}^\times$, this is the statement of the Lemma.

**Remark 3.7.** Consider the group homomorphism with kernel $\text{Aut}_0(S^1)$ given by the evaluation $\text{Aut}(S^1) \to S^1$, $f \mapsto f(1)$. This admits a section which associates to each element of $S^1$ the corresponding translation, and thus exhibits $\text{Aut}(S^1)$ as a semi-direct product $\text{Aut}(S^1) \simeq \text{Aut}_0(S^1) \rtimes S^1$. Although the action of $S^1$ on $\text{Aut}_0(S^1)$ by conjugation is nontrivial, we do nevertheless have a homeomorphism at the level of the underlying topological spaces $\text{Aut}(S^1) \simeq \text{Aut}_0(S^1) \times S^1$. On the other hand, there is of course no canonical group structure on the quotient $\text{Aut}(S^1)/S^1$.

Nodal annuli provide a partial compactification of the space of annuli “in the modulus $\infty$ limit”, whereas in the previous section we gave a compactification of the space of annuli “in the modulus 0 limit”. In particular, these two compactifications can be combined into a new separable topological space of possibly degenerate stable framed nodal annuli,

$$\tilde{\text{NodAnn}} = \tilde{\text{Ann}} \sqcup_{\text{Ann}} \text{NodAnn}.$$ 

Given two possibly degenerate nodal annuli we can glue them to produce a new possibly degenerate nodal annulus. Note that if both annuli have modulus $\infty$ (i.e. have two irreducible components), the resulting glued space will be unstable. Under our convention, we identify the resulting space with its stabilization. It is immediate to check that the resulting composition operation is associative; it is continuous by an argument analogous to the one used in the previous section for the continuity of the multiplication operation on $\text{Ann}$.

### 3.4. Tree-like nodal surfaces.

We recall that all our framed surfaces are labeled, see Remark 3.2.
Definition 3.8. Define \( \text{NodFr}_{\partial}^{\text{tree}} \) to be the moduli space of stable nodal Riemann surfaces with non-nodal analytically parametrized boundary, with the restriction that the dual graph of irreducible components is a tree. Further define

\[
\text{NodFr}_{\partial}^{\text{tree}} = \text{NodFr}_{\partial}^{\text{tree}} \sqcup \text{Ann} \text{Ann}.
\]

Note that \( \text{NodFr}_{\partial}^{\text{tree}} \) can have (stable) interior components which carry no boundary parametrizations, and these can have discrete automorphism groups, so \( \text{NodFr}_{\partial}^{\text{tree}} \) is an orbifold. The glued space \( \text{NodFr}_{\partial}^{\text{tree}} \) makes sense because \( \text{Ann} \subset \text{NodFr}_{\partial}^{\text{tree}} \) is open. More precisely, we have \( \text{NodAnn} \subset \text{NodFr}_{\partial}^{\text{tree}} \) and \( \text{NodFr}_{\partial}^{\text{tree}} \) differs from \( \text{NodFr}_{\partial}^{\text{tree}} \) in that it contains degenerate annuli. We topologize \( \text{NodFr}_{\partial}^{\text{tree}} \) as before, by viewing it as embedded in a bundle over the (tree-like) moduli space of closed nodal Riemann surfaces (possibly with some marked points). Glueing along the boundary and possibly collapsing determines an operad structure on \( \text{NodFr}_{\partial}^{\text{tree}} \) (see Appendix A for details on operad structures for orbifolds). We call it the operad of possibly degenerate tree-like framed nodal surfaces. We also define

\[
\text{NodFr}_{\partial}^{\text{tree}}, \text{coarse} = \text{NodFr}_{\partial} \sqcup \text{Ann} \text{Ann}.
\]

For further reference we denote by

\[
\text{NodFr}_{\partial}
\]

the moduli space of stable nodal Riemann surfaces with non-nodal analytically parametrized boundary, without any restriction on the dual graph, and also

\[
\text{NodFr}_{\partial} = \text{NodFr}_{\partial} \sqcup \text{Ann} \text{Ann}.
\]

Theorem 3.9 (Geometric Pushout Theorem). The operad \( \text{NodFr}_{\partial}^{\text{tree}, \text{coarse}} \) of possibly degenerate tree-like framed nodal surfaces is canonically isomorphic to the pushout of the following diagram, in which both arrows are inclusions:

\[
\text{NodAnn} \leftarrow \text{Ann} \rightarrow \text{Fr}_{\partial}.
\]

The heuristic idea of the proof is that a nodal surface can be described, though not uniquely, by a successive gluing of framed non-nodal surfaces and nodal annuli. See Figure 4. When the dual graph of irreducible components is a tree, this data is equivalently encoded in the pushout construction. The equivalence relations defining the pushout
construction precisely eliminate the ambiguity, i.e. non-uniqueness, of this description. The equivalence relations underlie pushouts in the topological category, and in particular (as we are not taking homotopy pushouts yet), self-equivalences are ignored, and this explains the "coarse" nature of the resulting pushout.

![Figure 4. One possible presentation of a nodal surface by gluing.](image)

As such the proof of Theorem 3.9 on the level of operads in sets is quite straightforward, and we will make it explicit in the next section. Also in the next section, by re-interpreting the free-forgetful adjunction on the operad of framed surfaces and its relatives, we give a proof of this theorem which also accounts for the topology on the two sides. While the theorem does not imply the homotopy-theoretic pushout result (in order to get a correct model for the homotopy pushout, the diagram of operads must be replaced by a suitable resolution), it is a good intuitive approximation for it. Indeed, the eventual homotopical proof will be based on a topologically enhanced version of exactly the argument presented in the next section.

3.5. **Split Surfaces and the Geometric Pushout Theorem.** The objects of interest in this section will be various moduli spaces of framed surfaces with “seams” at embedded curves, which we call “split surfaces”. We again recall that all our framed surfaces are labeled, see Remark 3.2.

**Definition 3.10.** A split framed surface with $k$ interior seams is a pair $(\Sigma, S)$ consisting of a framed surface $\Sigma$ with boundary, together with an analytic embedding $S : (S^1)^\sqcup_k \hookrightarrow \Sigma$ mapping into the interior of $\Sigma$.

By definition, the seams are parametrized curves: the interior seams are the components $S_i : S^1 \to \Sigma$ of the embedding $S = \sqcup_{i=1}^k S_i : (S^1)^\sqcup_k \hookrightarrow \Sigma$; the parametrized boundary components of $\Sigma$ are called exterior seams. In the definition we allow $k = 0$, i.e. no interior seams.

Given a framed surface $\Sigma$, write

$$\text{Split}^k_\Sigma$$

for the moduli space of all split surface structures on $\Sigma$ with $k$ unordered interior seams. Equivalently, $\text{Split}^k_\Sigma$ is the space of analytic embeddings $(S^1)^\sqcup_k \hookrightarrow \Sigma$ endowed with the compact-open topology. Write

$$\text{Split}^k = \sqcup_{\Sigma \in \text{Fr}_\partial} \text{Split}^k_\Sigma$$
for the moduli space of all split framed surfaces with \( k \) unordered interior seams, and write

\[
\Split\quad (= \bigsqcup_{k \geq 0} \Split^k) 
\]

for the moduli space of all split framed surfaces with an arbitrary number of unordered interior seams, and

\[
\Split_{\Sigma} \quad (= \bigsqcup_{k \geq 0} \Split^k_{\Sigma})
\]

for the moduli space of all split surface structures on \( \Sigma \) with an arbitrary number of unordered interior seams.

To every split surface \((\Sigma, S)\) is associated a “dual graph”

\[
\Gamma_{\Sigma, S}
\]

with \( k \) interior edges, which is a directed graph with half-edges. Vertices are indexed by the connected components of \( \Sigma \setminus S \), internal edges are indexed by interior seams (which separate locally and thus inherit an orientation determined by the orientation of \( \Sigma \)) and half-edges are indexed by external seams (each of these belongs to the closure of a single connected component of \( \Sigma \setminus S \)). In particular, since the incoming external seams of \( \Sigma \) are labeled by definition, the dual graph inherits a labeling of its incoming half-edges. Note that two split surfaces in the same connected component of \( \Split \) have the same dual graph, so given a labeled graph \( \Gamma \) we can write

\[
\Split_{\Gamma}
\]

for the union of connected components of \( \Split \) with dual graph \( \Gamma \). The following observation is straightforward.

**Lemma 3.11.** Let \( \Sigma \) be connected. The dual graph \( \Gamma_{\Sigma, S} \) associated to a split surface \((\Sigma, S)\) is a tree if and only if the image of each interior seam is separating, i.e. its complement is disconnected. \(\square\)

Split framed surfaces are a convenient model for the free operad on the \( \mathcal{S} \)-graded space underlying \( \Fr_\partial \) (the source of the free-forgetful adjunction map), as we now explain. Write

\[
\Split^{k, \text{tree}}_{\Sigma} \subset \Split^k_{\Sigma}
\]

for the moduli space of all split surface structures \( S \) on \( \Sigma \) with \( k \) unordered interior seams such that the dual graph \( \Gamma_{\Sigma, S} \) is a tree. Further denote

\[
\Split^{k, \text{tree}} = \bigsqcup_{\Sigma \in \Fr_\partial} \Split^{k, \text{tree}}_{\Sigma} \subset \Split^k, \\
\Split^{\text{tree}}_{\Sigma} = \bigsqcup_{k \geq 0} \Split^{k, \text{tree}}_{\Sigma} \subset \Split_{\Sigma}, \\
\Split^{\text{tree}} = \bigsqcup_{k \geq 0} \Split^{k, \text{tree}} \subset \Split,
\]

and

\[
\Split^{\text{tree}}_{\Gamma} = \Split_{\Gamma}
\]
for any labeled tree $\Gamma$ without planar structure. We call these moduli spaces of split surface structures tree-like.

**Remark 3.12.** In the notation $\text{Split}^\text{tree}_\Gamma$ we think of $\Gamma$ as being a tree with no distinguished planar structure. Equivalently, we can upgrade $\Gamma$ by endowing it with a planar structure and then quotient out by the corresponding equivalence relation as in §2.2.2. See also Remark 2.2.1.

Let $\tau$ be a labeled tree of operations from which we discard the planar structure. We denote by $[\tau]$ its isomorphism class with respect to the isomorphism relation described in §2.2.1. Recall that, for any operad $O$, the free operad $\text{Free}(O)$ has components $\text{Free}_{[\tau]}(O)$ indexed by such isomorphism classes.

**Lemma 3.13.** Let $\tau$ be a labeled tree of operations from which we discard the planar structure. We have a canonical homeomorphism

$$G : \text{Free}_{[\tau]}(\text{Fr}_\partial) \xrightarrow{\simeq} \text{Split}^\text{tree}_\tau.$$ 

The fact which underlies the proof of Lemma 3.13 is that, given a framed surface $\Sigma''$, the data of an interior seam whose image is separating is equivalent to the data of a decomposition of $\Sigma''$ as a gluing of two framed surfaces along one boundary component. Obviously, such a seam determines such a decomposition of $\Sigma''$. Conversely, given two framed surfaces $\Sigma, \Sigma'$ and a choice of boundary components $\gamma \subset \Sigma$ which is incoming and $\gamma' \subset \Sigma'$ which is outgoing, with corresponding framings $f : S^1 \to \gamma$ and $f' : S^1 \to \gamma'$, the glued surface $\Sigma'' = \Sigma^\sharp_{\gamma, \gamma'} \Sigma'$ inherits a seam, i.e., a distinguished analytic embedding of $S^1$ into its interior, given with respect to the canonical inclusions $\Sigma, \Sigma' \hookrightarrow \Sigma''$ by either of the equal compositions

$$S^1 \xrightarrow{f} \gamma \hookrightarrow \Sigma \hookrightarrow \Sigma''$$

or

$$S^1 \xrightarrow{f'} \gamma' \hookrightarrow \Sigma' \hookrightarrow \Sigma''.$$ 

**Proof of Lemma 3.13.** Let $n = |\text{Half}^+(\tau)|$ be the number of incoming half-edges of $\tau$. Recall from §2.2.2 that $\text{PlanarTree}_n(\text{Fr}_\partial)$ denotes the space of labeled rooted trees with vertices coloured by elements of $\text{Fr}_\partial$ and with $n$ incoming half-edges. Denote $\text{PlanarTree}_{[\tau]}(\text{Fr}_\partial) \subset \text{PlanarTree}_n(\text{Fr}_\partial)$ the subset consisting of those elements whose underlying labeled rooted tree is isomorphic to $\tau$. We have a canonical “gluing” map

$$G : \text{PlanarTree}_{[\tau]}(\text{Fr}_\partial) \to \text{Split}^\text{tree}_\tau$$

given by gluing framed surfaces according to the underlying labeled rooted tree. Indeed, the planar structure of any labeled tree $\tau'$ equivalent to $\tau$ induces a labeling of the incoming edges and half-edges at each of its vertices. Since the incoming boundary components of the
framed surface $\Sigma_v$ attached to a vertex $v$ are also labeled by definition, this prescribes uniquely the order in which the framed surfaces which correspond to vertices adjacent to $v$ have to be attached to $\Sigma_v$ along their outgoing boundary components. By definition, the resulting split surface belongs to $\text{Split}^\text{tree}_v$.

The map is clearly continuous, surjective, and the fiber over each element of $\text{Split}^\text{tree}_v$ is canonically identified with an equivalence class as described in §2.2.2. As such, it descends to a homeomorphism

$$G : \text{Free}_{[\tau]}(\text{Fr}_0) \rightarrow \text{Split}^\text{tree}_v,$$

where $\text{Free}_{[\tau]}(\text{Fr}_0) = T\text{ree}_{[\tau]}(\text{Fr}_0)$ is the quotient of $\text{PlanarTree}_{[\tau]}(\text{Fr}_0)$ under the equivalence relation described in §2.2.2. □

To extend the above result to $\tilde{\text{Fr}}_0$, we compactify $\text{Split}$ by allowing interior components of thickness zero:

**Definition 3.14.** Let

$$\text{Split}$$

be the partial compactification of $\text{Split}$ which allows two seams (internal or external) $S^1 \to \Sigma$ to intersect if and only if they have the same image with the same orientation, and which also allows $\Sigma$ to be a framed degenerate annulus.

We have corresponding partial compactifications

$$\text{Split}^k, \text{Split}_\Sigma,$$

of the moduli spaces $\text{Split}^k, \text{Split}_\Sigma$ respectively, and also for their tree-like and labeled tree-like counterparts, with similar notations $\text{Split}^{\text{tree}}$ etc.

Points of $\text{Split}_\Sigma$ over a fixed surface $\Sigma$ are indexed by maps $S : (S^1)^{\cup k} \to \Sigma$ which allow seams with compatible orientation to coincide as above, with the additional data of an ordering of all copies of $S^1$ mapping to a given closed oriented curve. The notion of dual graph $\Gamma = \Gamma(\Sigma, S)$ for such an element $(\Sigma, S)$ is defined as follows. The vertices of $\Gamma$ are of two kinds: they correspond either to the connected components of $\Sigma \setminus S$, or to pairs of interior seams which have the same image and which are immediate successors for the given ordering. The edges correspond to interior seams. One sees that the ordering of the copies of $S^1$ mapping to a given closed oriented curve precisely resolves the ambiguity in the dual graph by specifying a “composition order” of the thickness-zero annuli they “bound”.

Given a labeled tree $\Gamma$ (again with no distinguished planar structure), we have corresponding moduli spaces $\text{Split}_\Gamma = \text{Split}^{\text{tree}}_\Gamma$. The proof of the following Lemma is in all points similar to that of Lemma 3.13 hence we omit it.
Lemma 3.15. Let \( \tau \) be a labeled tree of operations from which we discard the planar structure. We have a canonical homeomorphism

\[
G : \text{Free}_\tau(\hat{\mathbb{F}}_{\partial}) \xrightarrow{\simeq} \hat{\text{Split}}_\tau.
\]

\( \square \)

In order to extend the result to \( \hat{\text{NodFr}}_{\partial} \) we need to further define moduli spaces of framed nodal surfaces with seams.

Definition 3.16. Let \( \text{NodSplit} \) be the space of framed nodal surfaces \( \Sigma \) endowed with an embedding \( (S^1)^{\cdot k} \to \hat{\Sigma}_{\text{smooth}} \) of a finite number \( k \geq 0 \) of parametrized seams in the open smooth locus. The elements of \( \text{NodSplit} \) are called split framed nodal surfaces.

The notion of dual graph for a split framed nodal surface \( (\Sigma, S) \) is defined as follows: its vertices are the connected components (not the irreducible components) of \( \Sigma \setminus S \), and in particular the dual graph in this context ignores nodes. The edges correspond to interior seams as before. We can further define moduli spaces \( \text{NodSplit}^{\text{tree}}, \text{NodSplit}^{\text{tree, coarse}} \) etc. as above.

It is again convenient for unitality purposes to extend the setup by including degenerate annuli.

Definition 3.17. Let \( \hat{\text{NodSplit}} \) be the partial compactification of \( \text{NodSplit} \) obtained by allowing \( S \) to include coinciding circles bounding thickness-zero annuli, as in Definition 3.14.

Similarly to the non-nodal case, we consider as part of the data an ordering of the interior seams which have the same oriented image. We have the same notion of dual graph, and we can further define moduli spaces \( \hat{\text{NodSplit}}^{\text{tree}}, \hat{\text{NodSplit}}^{\text{tree, coarse}} \) etc. as above.

The next Lemma is the counterpart of Lemmas 3.13 and 3.15.

Lemma 3.18. Let \( \tau \) be a labeled tree of operations from which we discard the planar structure. We have a canonical homeomorphism

\[
G : \text{Free}_\tau(\hat{\text{NodFr}}_{\partial}) \xrightarrow{\simeq} \hat{\text{NodSplit}}_\tau,
\]

and similarly for the coarse moduli spaces. \( \square \)

In the proof of the Geometric Pushout Theorem 3.9 we will encounter the following new kind of moduli space. We single out the definition before the proof, for the convenience of the reader.
Definition 3.19. Define

\[ \widetilde{\text{NodSplit}}^{\text{tree}}_{\text{protected}} \]

to be the moduli space of split nodal surfaces with dual graph a tree (with half-edges) and such that every nodal component is a nodal annulus. We call such surfaces tree-like and protected.

The idea of the definition is that every node has to be “protected” on two sides by a pair of seams.

\[ \equiv \]

Figure 5. Tree-like split structure on a nodal surface, together with its dual graph. It becomes protected by adding one seam around the node \( N \) on the trivalent component.

Proof of the Geometric Pushout Theorem 3.14. Consider the diagram

\[ \widetilde{\text{NodAnn}} \leftarrow \widetilde{\text{Ann}} \rightarrow \tilde{\text{Fr}}_{\partial} \]

Recall from §2.3 the definition of its pushout

\[ P \simeq \text{Free}(\tilde{\text{Fr}}_{\partial} \sqcup \widetilde{\text{NodAnn}}) / \sim, \]

where \( \sim \) is the equivalence relation generated by relations \( \sim_1 \sqcup \sim_2 \).

As in Lemma 3.13 there is a tautological gluing map

\[ \text{Free}(\tilde{\text{Fr}}_{\partial} \sqcup \widetilde{\text{NodAnn}}) \rightarrow \widetilde{\text{NodSplit}}^{\text{tree,coarse}}. \]

The preimage of this map over a given split surface \( (\Sigma, S) \) consists of all possible choices of decorating the vertices of the dual graph of \( (\Sigma, S) \) by the corresponding point of \( \text{NodAnn} \) or \( \tilde{\text{Fr}}_{\partial} \). Thus, in order for a split surface \( (\Sigma, S) \) to have nonempty preimage, connected components of
$\Sigma \setminus S$ must be either smooth surfaces or nodal annuli. Components indexed by non-nodal annuli can be labeled either way and contribute to the ambiguity of the lifting. It is precisely this ambiguity that is resolved by the relation $\sim_2$, in a way that is obviously compatible with the topology as it identifies connected components in their entirety. Thus the map to $\text{NodSplit}_{\text{tree, coarse}}$ above factors as

\[
\begin{array}{ccc}
\text{Free}(\text{Fr}_\partial \sqcup \text{NodAnn}) & \sim_2 & \text{NodSplit}_{\text{tree, coarse}} \\
\downarrow \cong & & \downarrow \cong \\
\text{NodSplit}_{\text{protected}}_{\text{tree}} & & \\
\end{array}
\]

where it maps homeomorphically the partial quotient to the target $\text{NodSplit}_{\text{protected}}_{\text{tree}}$, which is in turn a union of connected components of $\text{NodSplit}_{\text{tree}}$ consisting of split nodal curves with dual graph a tree, and such that each nodal component is a nodal annulus. (Note that as protected split curves are glued out of smooth framed curves and nodal annuli, neither of which have automorphisms, there is no need to take the coarse space here.)

Consider now the map

\[
\text{NodSplit}_{\text{protected}}_{\text{tree}} \to \text{NodFr}_\partial
\]

defined by erasing the seams. Note that erasing a seam which is a common boundary component of two framed surfaces in $\text{Fr}_\partial$ corresponds precisely to gluing, i.e. composition in the operad $\text{Fr}_\partial$. Similarly, erasing a seam which is a common boundary component of two nodal annuli creates an unstable component which must be further discarded, and this corresponds again to gluing, i.e. composition in the operad $\text{NodAnn}$. It thus follows that the above map is constant along the equivalence classes defined by relation $\sim_1$, which identifies pairs of points inside $\text{NodSplit}_{\text{protected}}_{\text{tree}}$ which are related by removing a single seam (note that such a seam must either be between two nodal annuli or between two smooth framed surfaces). On the level of sets, it is clear that $\sim_1$ identifies any two points in $\text{NodSplit}_{\text{protected}}_{\text{tree}}$ which correspond to splittings of the same nodal curve. We turn this intuition into a precise topological colimit argument as follows.

Given a tree-like nodal surface $\Sigma$ with $\nu$ nodes, choose a piecewise analytic (with analytic boundaries) Hermitian metric on the normalization of $\Sigma$ (this can be done in a suitably consistent manner on a
small neighborhood of $\Sigma$ in the moduli space of tree-like nodal surfaces). Write $\text{Split}_\Sigma^\epsilon \subset \text{Split}_\Sigma^{\text{tree}}$ for those split surfaces that have no seams at a distance $\leq \epsilon$ from any node. Since seams are not allowed to pass through nodes, these spaces filter $\text{Split}_\Sigma^{\text{tree}}$ as $\epsilon \to 0$. Now (for $\epsilon$ sufficiently small), write $(\Sigma, S_\epsilon) \in \text{Split}_\Sigma^{\text{tree}}$ for the splitting given by the collection of $2\nu$ circles consisting of all points at radius $\epsilon$ from the $2\nu$ preimages of the nodes in a normalization (and parametrized in some analytic fashion). Then every element in $\text{Split}_\Sigma^\epsilon$ is identified (in a way consistent with the topology) with $(\Sigma, S_\epsilon)$ via $\sim_1$. Further, for $\epsilon' < \epsilon$ we have $(\Sigma, S_\epsilon) \sim (\Sigma, S_{\epsilon'})$: indeed, by $\sim_1$ used for $\tilde{\text{NodAnn}}$ they are both equivalent to the split surface $(\Sigma, S_\epsilon \sqcup S_{\epsilon'})$.

We have thus proved that the fiber of the map (2) is given by the equivalence classes with respect to $\sim_1$. By choosing metrics consistently in a neighborhood of $\Sigma$ in the moduli space of nodal curves, we get a homeomorphism

$$\tilde{\text{NodSplit}}_{\text{tree}}^{\text{protected}} \sim_1 \tilde{\text{NodFr}}_{\partial, \text{coarse}}.$$ 

Together with the homeomorphism (1), we obtain a homeomorphism

$$\tilde{\text{Fr}}_{\partial} *_{\text{Ann}} \tilde{\text{NodAnn}} \cong \tilde{\text{NodFr}}_{\partial, \text{coarse}}.$$ 

\[\square\]

4. Model Categories and Homotopy (Co)limits

Our references for this section are Lurie [23, Appendix A.2], May-Ponto [26], Hovey [19] and Ginot [15].

4.1. Model category theory. Suppose that $C$ is a category and $I$ is a class of morphisms in $C$ “to be inverted”. We say that $I$ is a class of weak equivalences if the following conditions are satisfied:

- (Category structure). The objects of $C$ with the morphisms in $I$ form a subcategory.
- (2 out of 3). Given any commutative diagram

$$\begin{array}{ccc}
A & \rightarrow & \downarrow \\
\downarrow & & \\
B & \rightarrow & C
\end{array}$$

with two of the three morphisms in $I$, the third is also in $I$.

Note that the first axiom is sometimes replaced by an identity axiom, as composition compatibility is part of the 2 out of 3 axiom. Now given a class of weak equivalences, one would like to produce a “localized”
category in which these are inverted, i.e. a category $C_I$ with a functor $C \to C_I$ such that the image of any morphism in $I$ is invertible, and which is initial—up to taking care of set-theoretic issues—among such categories. Modulo some set-theoretic difficulties such a $C_I$ can be proven to exist. In fact, when $C$ is an ordinary category, the localization $C_I$ comes naturally as the set of connected components of morphism spaces in a simplicial category, which should be considered in the context of $\infty$-category theory.

The problem is that for a general class $I$ of weak equivalences, the localization $C_I$ (whether as a category or a simplicial category) is incredibly difficult to access. In particular, it is hopeless to calculate $\text{Hom}_{C_I}(X, Y)$ for two objects $X, Y$ of $C$. In order to turn $C_I$ into a manageable object, it is necessary to endow $C$ with some supplementary data. One remarkably elegant and versatile such additional datum is a so-called “model category structure”. A model category structure consists in endowing $C$ with two new classes of morphisms called fibrations, $P$, and cofibrations, $Q$, such that the objects of $C$ with either $P$ or $Q$ form subcategories of $C$. The category $C$ together with the classes $I, P, Q$ need to satisfy a collection of conditions among themselves, for which we refer the reader to [17, §3]. Some conditions that we will use here are as follows.

1. The category $C$ has an initial object, $\emptyset$, a final object, $\text{pt}$, and all finite limits and colimits.
2. For any morphism $X \xrightarrow{f} Y$ of objects, there is a “fibrant replacement” $X \xrightarrow{i} X' \xrightarrow{f'} Y$ such that $i \in I \cap Q$ is a cofibrant weak equivalence and $f' \in P$ is a fibration.
3. Similarly, for any morphism $X \xrightarrow{f} Y$ of objects, there is a “cofibrant replacement” $X \xrightarrow{j} X' \xrightarrow{j'} Y$ such that $f' \in Q$ is a cofibration and $j \in F \cap I$ is a fibrant weak equivalence.
4. All three categories $P, Q, I$ are closed with respect to taking retracts of morphisms.
5. Given the subcategories $I$ of weak equivalences and $Q$ of cofibrations (resp., the category $P$ of fibrations), the subcategory $P$ of fibrations (respectively, $Q$ of cofibrations) is uniquely characterized by a lifting property.

Note that neither cofibrant nor fibrant replacement is required to be functorial, though there often is a functorial choice (in fact, there is a sense in which the choice is unique up to homotopy). If a map $X \xrightarrow{f} Y$ is a fibration we write shorthand $X \xrightarrow{f} Y$. 
and similarly if \( X \to^f Y \) is a cofibration we write \( X \leftarrow^f Y \).

If \( X \to^f Y \) is an equivalence we write \( X \leftarrow^f \sim Y \), with evident compound meanings for \( X \leftarrow^f \sim Y \) ("acyclic cofibrations", or cofibrant weak equivalences) and \( X \to^f \sim Y \) ("acyclic fibrations", or fibrant weak equivalences).

4.2. The homotopy category. Suppose that \( C \) is a category with weak equivalences \( I \) and model structure \( P,Q \). We say that an object \( X \) is fibrant if the map \( X \to pt \) to the terminal object is fibrant (in \( P \)), and cofibrant if the map \( \emptyset \to X \) is cofibrant (in \( Q \)). Note that by applying a suitable factorization axiom to the map \( \emptyset \to X \) or \( X \to pt \), we see that every object \( X \) admits a (cofibrant) weak equivalence (in general non-canonically) to a fibrant object, \( X \leftarrow^f \sim X \), and a (fibrant) weak equivalence from a cofibrant one, \( \emptyset \leftarrow^f X \to pt \).

Let \( X \sqcup X \leftarrow^i_0 \sqcup \leftarrow^i_1 C_X \) be the codiagonal map, and \( X \sqcup X \leftarrow^i C_X \sim X \) a factorization. Any such object \( C_X \) is called a cylinder object for \( X \).

It admits a fibrant equivalence \( C_X \leftarrow^f \sim X \) and two cofibrant maps \( X \leftarrow^i_{0,1} C_X \), which are cofibrant equivalences by the two out of three axiom. Similarly, we can factorize the diagonal map \( \Delta : X \to X \times X \) as \( X \leftarrow^i P_X \to X \times X \); such a \( P_X \) is called a path object. It admits a cofibrant equivalence \( X \leftarrow^i P_X \) and two fibrant maps \( P_X \leftarrow^p_{0,1} X \), which are fibrant equivalences also by the two out of three axiom.

**Definition 4.1.** Write \( C,P,C_Q,CP \) for the full subcategories of \( C \) consisting of fibrant, cofibrant, and fibrant-cofibrant objects, respectively.

**Definition 4.2.** Suppose that \( f,g : X \to Y \) is a pair of maps, and choose a cylinder object \( C_X \) and a path object \( P_Y \). We say that \( f \) and \( g \) are left homotopic if \( f \sqcup g : X \sqcup X \to Y \) factors through \( C_X \) as \( X \sqcup X \leftarrow^i_{0,1} C_X \leftarrow^h Y \) for some choice of map ("homotopy") \( h \).

We say that \( f \) and \( g \) are right homotopic if the map \( X \to^f \times Y \) factors through \( P_Y \) as \( X \leftarrow^k P_Y \to X \times Y \) for some choice of map ("cohomotopy") \( k \).

**Lemma 4.3** ([27], [19, Proposition 1.2.5]). If \( X \) is cofibrant (and \( Y \) arbitrary), the relation \( \sim_L \) of left homotopy equivalence on \( \text{Hom}(X,Y) \) is an equivalence relation, and does not depend on the choice of cylinder object \( C_X \). Similarly, if \( Y \) is fibrant, the relation \( \sim_R \) of right homotopy equivalence is an equivalence relation and does not depend on choice of path object \( P_Y \). If \( X \) is fibrant and \( Y \) is cofibrant, then the two equivalence relations \( \sim_L \) and \( \sim_R \) on \( \text{Hom}(X,Y) \) are the same.
Definition 4.4. The category $\text{Ho}_C$ is the category with objects $C_{QP}$ and morphisms $\text{Hom}_{\text{Ho}_C}(X,Y)$ defined as the quotient of $\text{Hom}_C(X,Y)$ by left (or equivalently, right) homotopy equivalence.

Theorem 4.5 ([27], [19, Theorem 1.2.10]). The homotopy category $\text{Ho}_C$ is canonically equivalent to the localized category $C[I^{-1}]$.

Remark 4.6. Recall that given a ring $A$ with a localizing set of elements $I$, there is a condition on $I$ called the left (resp., right) Ore condition which allows one to write down the localization $A[I^{-1}]$ as the ring of fractions $i^{-1}f$ (resp., $fi^{-1}$) for $i \in I$. Similarly, given a category $C$ there is a notion of left (resp., right) Ore condition, which is part of a so-called calculus of fractions on $C$ [20, A.2.1.11(h)]. If the left Ore condition is satisfied then the category $C[I^{-1}]$ can be expressed as the category of objects of $C$ with morphisms $X \rightarrow Y$ represented by “roofs” $X \xrightarrow{f} Z \xleftarrow{g} Y$, with $Z$ arbitrary and $g$ a weak equivalence, subject to a straightforward equivalence relation determined by diagrams of maps commuting with a weak equivalence $Z' \rightarrow Z$. If $C$ is a model category then the category of cofibrant objects and maps up to left homotopy satisfies the left Ore condition with quotient $\text{Ho}_C$ and the category of fibrant objects and maps up to right homotopy satisfies the right Ore condition with quotient $\text{Ho}_C$.

4.3. Some important model categories. We will give a few examples of model category structures on topological spaces and differential complexes that will be important to us. Recall that in order to define a model structure, it suffices to specify just two classes of morphisms: weak equivalences and fibrations or weak equivalences and cofibrations. The third class is then determined by a lifting property.

4.3.1. Model category structures on topological spaces. Let $\text{Top}$ be the category of Hausdorff topological spaces. Recall that a map $f : X \rightarrow Y$ is a homotopy equivalence if it admits a homotopy inverse and a weak homotopy equivalence if it is a bijection on path-connected components and for any $x \in X$, the map $\pi_n(X,x) \rightarrow \pi_n(Y,f(x))$ is an isomorphism for each $n$. Any homotopy equivalence $X \rightarrow Y$ is a weak homotopy equivalence, and the converse is true provided $X,Y$ are CW complexes, but not true in general. Both homotopy equivalences and weak homotopy equivalences evidently satisfy the conditions required to define a class of weak equivalences. We denote the class of weak homotopy equivalences $\text{WE}$.

Proposition 4.7 (Quillen model structure [27, II.2, Theorem 1]). There is a model structure on the category $\text{Top}$ with weak equivalences given by $\text{WE}$ and fibrations given by Serre fibrations. This model category structure is called the Quillen model structure. A space $X$ is
cofibrant in this model structure if and only if it is a retract of a CW complex. Any space is fibrant.

Proposition 4.8 (Strøm model structure [30]). There is a model structure on the category Top, called the Strøm model structure ([30]) with weak equivalences given by homotopy equivalences and with fibrations given by Hurewicz fibrations. The cofibrations are retracts of Hurewicz cofibrations with closed image. Any topological space is cofibrant in the Strøm model structure.

Proposition 4.9 (Mixed model structure [8]). There is a model structure called the mixed model structure on topological spaces with weak equivalences WE and fibrations given by Hurewicz fibrations. A space is cofibrant in the intermediate model category structure if it is homotopy equivalent to a CW complex. This structure was constructed by Cole [8], see also [26, §17.3-4].

4.3.2. Chain complexes. Let \( k \) be a ring (e.g. \( k = \mathbb{Z} \) or \( k = \mathbb{Q} \)). Then the categories \( C(k) \) (resp. \( C_+(k) \)) of chain complexes of \( k \)-modules (resp. supported in non-negative degrees) have model structures with weak equivalences given by quasi-isomorphisms and fibrations given by maps of complexes which are term-wise surjective (resp. in all positive degrees). Cofibrant objects are then term-wise projective complexes of \( k \)-modules. All objects are fibrant. This is called the standard or projective model structure on the category of complexes [15, §2.3], [26, §18.4-5].

4.4. Quillen adjunction. It is a natural question to ask when a functor of model categories induces a functor of homotopy categories, and when this functor is a weak equivalence. (The functor most interesting for us will be the chains functor from topological operads up to weak homotopy equivalence to dg operads up to quasi-isomorphism.) A convenient condition on a functor \( F : \mathcal{C} \to \mathcal{D} \) of model categories that guarantees (in a functorial way) a functor on homotopy categories is the notion of so-called Quillen adjunction.

Definition 4.10. A functor \( F : \mathcal{C} \to \mathcal{D} \) between model categories is a left Quillen functor if it admits a right adjoint \( G : \mathcal{D} \to \mathcal{C} \) such that \( F \) preserves cofibrations and acyclic cofibrations and \( G \) preserves fibrations and acyclic fibrations. In this situation we call \( G \) a right Quillen functor, and the adjunction \( (F,G) \) is called a Quillen adjunction.

Quillen adjunctions induce pairs of adjoint (in the conventional sense) functors between homotopy categories: one gets \( \text{ho}_F : \text{Ho}_\mathcal{C} \rightleftarrows \text{Ho}_\mathcal{D} : \text{ho}_G \), defined by applying \( F \), resp. \( G \) to fibrant-cofibrant representatives (in fact, it is sufficient to apply \( F \) to a fibrant representative and \( G \) to a cofibrant representative to get the correct functor on homotopy
categories). These should be thought of as the left (resp. right) derived functors of $F$ (resp. $G$). A Quillen adjunction is called a Quillen equivalence if $ho_F$ (equivalently, $ho_G$) is an equivalence on homotopy categories.

The primordial Quillen adjunction, and one that will be important in this paper, is the adjunction $C_* : \text{Top} \leftrightarrow C_+(\mathbb{Z}) : |\cdot|$. Here we define $C_*(X)$ for $X \in \text{Top}$ to be the complex of singular chains on $X$. Its adjoint $|\cdot|$ is given by taking the geometric realization of an associated simplicial set. This adjunction can be written as a composition

$$\text{Top} \leftarrow \text{SSet} \leftrightarrow \text{SAb} \leftrightarrow C_+(\mathbb{Z}).$$

The first one associates to a topological space its singular simplicial set, and to a simplicial set its geometric realization, and is in fact a Quillen equivalence. The second one is the free-forgetful adjunction between simplicial sets and simplicial Abelian groups, which is not a Quillen equivalence. The third one is the Dold-Kan correspondence, and is an equivalence of categories. See [27, §II.3], [15, Cor. 3.2.15, Th. 3.4.4] and [33, §8.4].

4.5. Homotopy (co)limits. Our sources for this section are the articles by Stephan [29], Dwyer and Spalinski [13] and Chapter 13 from the book of Hirschhorn [18].

Suppose $\mathcal{C}$ is a model category and $J$ is a small “diagram” category which we are interested in mapping to $\mathcal{C}$. Then the functor category $\mathcal{C}^J := \text{Fun}(J, \mathcal{C})$ inherits a natural notion of weak equivalence: we say that a natural transformation $F \to G$ of functors $F, G : J \to \mathcal{C}$ is a weak equivalence if $F(j) \to G(j)$ is such for each object $j \in J$. There are several natural model structures on the diagram category, one of which is the injective model structure, with cofibrations determined objectwise on a map of diagrams. If $X$ is an object of $\mathcal{C}$, there is a constant diagram $\underline{X}$ with every object of $J$ sent to $X$ and every arrow sent to the identity morphism of $X$. This determines a functor $\text{const} : \mathcal{C} \to \mathcal{C}^J$. Its left adjoint is by definition the colimit functor (and its right adjoint is the limit functor, when $J$-indexed limits exist):

$$\text{colim} : \mathcal{C}^J \rightleftarrows \mathcal{C} : \text{const}.$$ 

This determines a Quillen adjunction, and thus induces a functor of associated homotopy categories, called the homotopy colimit functor, written

$$\text{hocolim} : Ho_{\mathcal{C}^J} \to Ho_{\mathcal{C}}.$$ 

The homotopy colimit is well-defined up to equivalence, but giving an explicit model depends on the choice of cofibrant resolution of a diagram. (In fact when passing from the homotopy category to the
richer ∞-category language, the category of such choices is contractible, and thus homotopy colimits are unique up to homotopy in a strong sense.) In the homotopy category there is a canonical map from the ordinary colimit of a diagram to its homotopy colimit. If this map is a weak equivalence, the diagram is called \textit{colimit exact}.

\textbf{Definition 4.11 (\cite{18} Definition 13.1.1).} A model category \( C \) is left proper if the pushout by a cofibration preserves weak equivalences.

A category all of whose objects are cofibrant is left proper \cite[Corollary 13.1.3]{18}.

\textbf{Theorem 4.12 (\cite{18} Proposition 13.3.8, Proposition 7.1.9).} If \( C \) is a left proper category then a diagram \( Y \leftarrow X \rightarrow X' \) is colimit exact if \( X \rightarrow X' \) is a cofibration.

5. The Berger-Moerdijk Model Structure for Operads

5.1. \textbf{Existence of model structure.} Suppose that \( C \) is a symmetric monoidal category with weak equivalences. Then we say that a map of operads \( O \rightarrow O' \) in \( C \) is a weak equivalence if it is so objectwise, i.e. if \( O_n \rightarrow O'_n \) is a weak equivalence for each \( n \). Berger and Moerdijk \cite{3} show that if \( C \) is a model category satisfying certain additional conditions, then this notion of weak equivalence is part of a model category structure on operads in \( C \), for which the fibrations \( O \rightarrow O' \) are object-wise fibrations. In particular, they prove the following result.

\textbf{Theorem 5.1 (\cite{3} Theorem 3.2).} If \( C \) is a cartesian+ closed symmetric monoidal model category, then the category of operads in \( C \) has a model structure with weak equivalences and fibrations determined levelwise.

We have not spelled out the meaning of “cartesian+”. This is a shorthand notation for cartesian category satisfying some additional properties (cofibrantly generated with cofibrant terminal object and admitting symmetric monoidal fibrant replacement functor), cf. the assumptions of Theorem 3.2 in \cite{3}. For our purposes it suffices to record that this holds for all three model structures that we consider on Top.

5.2. \textbf{W-construction and cofibrant replacement.}

The \textit{W-construction} for operads plays the role of the familiar bar resolution for algebras. Our references here are Vogt \cite{32} and Berger-Moerdijk \cite{4}. We refer to §2.2 for notation concerning the definition of the free operad associated to a graded object.

Given a topological operad \( O \), recall that we denote \( O_* \) the graded topological space \( O_* = (O_1, O_2, \ldots) \). We define a new operad \( W(O) \)
out of $\text{Free}(O_*)$ as follows. For each $n \geq 1$ we define

$$W(O)_n = \coprod_{[\tau] \in \text{Tree}_n} O^{[\tau]} \times [0,1]^{\text{Edge}_{[\tau]}} / \sim_W,$$

for a certain equivalence relation $\sim_W$. Here $O^{[\tau]} \times [0,1]^{\text{Edge}_{[\tau]}}$ is a notation for the quotient of $\bigcup_{\tau \in [\tau]} O^\tau \times [0,1]^{\text{Edge}^\tau}$, where $\tau \in \text{PlanarTree}_n$ ranges over the elements of the equivalence class $[\tau] \in \text{Tree}_n$, by the equivalence relation given by non-planar isomorphisms of labeled trees, which act on the first factor as in §2.2.2 and which act on the second factor via their action on the sets of edges of trees. Thus $O^{[\tau]} \times [0,1]^{\text{Edge}_{[\tau]}}$ should be interpreted as the $[\tau]$-component of $\text{Free}(O_*)$, which consists of all possible labelings of the vertices $v$ of a tree $\tau$ by elements of $O_{\text{Child}(v)}$, with the additional data of a length in $[0,1]$ for each internal edge. The equivalence relation $\sim_W$ consists simply in identifying two vertices $v, w$ which are connected by an edge of length 0, and replacing their corresponding labels, which are elements of $O_{\text{Child}(v)}$ and $O_{\text{Child}(w)}$, by their composition in $O$ which is an element of $O_{\text{Child}(v)+\text{Child}(w)}-1$.

Loosely speaking, $W(O)$ is obtained from $\text{Free}(O_*)$ by giving lengths to internal edges of trees and merging vertices according to the composition rules in $O$ when the connecting edges acquire length zero. The composition rule in $W(O)$ is inherited from that of $\text{Free}(O_*)$, with the convention that each new internal edge which results from a composition by gluing two half-edges is attributed length 1.

Given a point $o \in W(O)_n$, we obtain a point of $O_n$ by composing the operations in the corresponding tree. This results in a functorial map of operads $W(O) \to O$ which is (essentially by construction) a homotopy equivalence, see [4, Theorem 5.1].

The $W$ construction is useful for replacing maps of operads by cofibrations. Namely, we have the following theorem.

**Theorem 5.2** ([4 Proposition 6.6]). If $O \to O'$ is a map of operads which is a cofibration on the level of $\mathfrak{S}$-equivariant graded spaces, then $W(O) \to W(O')$ is a cofibration.

In particular, the $W$-construction is a functorial cofibrant replacement in the Strom model category structure. Moreover, if the spaces $O_n$ are homotopy equivalent to CW complexes (as is the case for the operads we are interested in), the $W$-construction is a cofibrant replacement in the mixed model category structure.

### 5.3. Model category structures on algebras.

The main use of operads comes via the study of their “representations”, i.e. through the association to each operad $O$ of a notion of algebra object over $O$. For the sake of completeness, we include here a brief discussion of model
category structure on the category of algebras over a dg operad (see also Appendix B).

In this section \( O \) is a connective dg operad, i.e. an operad all of whose homology groups vanish in negative degrees. There is a notion of a strict dg algebra \( V_* \) over \( O \), consisting of a complex \( V_* \) together with dg maps \( V_* \otimes O_n \to V_* \), which are strictly associative and \( S_n \)-equivariant with respect to the \( S_n \)-action. There is also a notion of weak dg algebra which is a dg vector space \( V_* \) together with dg maps \( V_* \otimes O_n \to V_* \), which are only associative and equivariant up to coherent homotopy. We say that a map of strict (resp., weak) algebras \( V_* \to V'_* \) is an equivalence if it is a quasi-isomorphism forgetting the operad action.

**Definition 5.3.** Write \( \text{Alg}(O) \), resp., \( \text{Alg}_{\text{weak}}(O) \) for the categories of strong, respectively, weak algebras over \( O \).

In general, it is the notion of weak algebra category that gives the “correct” category of “\( O \)-algebras up to equivalence”, as the category of strict \( O \)-algebras up to equivalence might depend on the homotopy type of \( O \), while the category of weak algebras does not. However, this is a moot point if the \( S_n \)-action on \( O_n \) is free. Namely, we have the following result.

**Lemma 5.4 (\cite{B} Corollary 4.5).** If \( O \) is an operad and \( O_n \) has free \( S_n \)-action, or if \( O \) is any operad and \( k \) has characteristic 0, then the functor of categories \( \text{Alg}_{\text{weak}}(O) \to \text{Alg}_{\text{strong}}(O) \) is a Quillen equivalence for a certain natural choice of model category structures on both sides.

In particular, we obtain the following

**Corollary 5.5.** The categories \( \text{Alg}_{\text{weak}}(O) \) and \( \text{Alg}_{\text{strong}}(O) \) have canonically equivalent homotopy categories (both as ordinary and as \((\infty, 1)\)-categories).

## 6. Proof of the main theorem

### 6.1. Homotopy colimits

Let us consider the map of \( \mathcal{G} \)-equivariant spaces \( \widetilde{\text{Ann}} \to \widetilde{\text{Fr}}_\partial \). This is a map of free \( \mathcal{G} \)-spaces which at the level of the underlying topological spaces is an embedding of a connected component. Moreover, the remaining connected components of \( \widetilde{\text{Fr}}_\partial \) are homotopy equivalent to CW-complexes, hence are cofibrant in the mixed model structure. As a consequence of Theorem 5.2, the map

\[
W(\widetilde{\text{Ann}}) \to W(\widetilde{\text{Fr}}_\partial)
\]

is a cofibration. By Theorem 4.12, the homotopy colimit

\[
\text{hocolim}(\widetilde{\text{NodAnn}} \leftarrow \widetilde{\text{Ann}} \to \widetilde{\text{Fr}}_\partial)
\]
is computed by the operad colimit

$$\text{colim}(W(\tilde{\text{NodAnn}}) \leftarrow W(\tilde{\text{Ann}}) \rightarrow W(\tilde{\text{Fr}})).$$

Given a labeled tree of operations $\tau$ with $n$ inputs, recall from Definition 3.14 the moduli space $\tilde{\text{Split}}_\tau$ of framed split surfaces with possibly coinciding seams and dual graph $\tau$. Let $\text{Edge}_\tau$ be the set of edges of $\tau$. We glue together the spaces $\tilde{\text{Split}}_\tau \times [0, 1]^{\text{Edge}_\tau}$, where we view the coordinate $[0, 1]^{e}$ for $e$ an edge as being attached to the seam $S_e$, into the Humpty-Dumpty space

$$\text{HD} = \bigsqcup_{\tau} \tilde{\text{Split}}_\tau \times [0, 1]^{\text{Edge}_\tau},$$

where $\sim$ is the equivalence relation

$$(\Sigma, (S_1, \ldots, S_E), (t_1, \ldots, t_\text{E}) \sim (\Sigma, (S_1, \ldots, \hat{S}_e, \ldots, S_E), (t_1, \ldots, \hat{t}_e, \ldots t_\text{E})).$$

As a direct consequence of Lemma 3.15 we have the following

**Lemma 6.1.** There is a homeomorphism $\text{HD} \simeq W(\tilde{\text{Fr}})).$ □

Similarly, we define the nodal Humpty-Dumpty space as

$$\text{NodHD} = \bigsqcup_{\tau} \tilde{\text{NodSplit}}_\tau \times [0, 1]^{\text{Edge}_\tau},$$

where $\sim$ is the same equivalence relation as above. As a consequence of Lemma 3.18 we have the following

**Lemma 6.2.** There is a homeomorphism

$$\text{NodHD} \simeq W(\tilde{\text{NodFr}}_\text{tree}).$$ □

We define

$$W(\tilde{\text{NodFr}}_\text{tree})_{\text{protected}}$$

to be the space of tuples $(\Sigma, \{S_e\}, \{t_e\})$ such that each node of $\Sigma$ is surrounded on both sides by curves which can be contracted to the node.

The next statement is a direct consequence of the definition. The proof is very similar to that of the Geometric Pushout Theorem 3.9 and we omit it.

**Lemma 6.3.** We have a canonical isomorphism

$$\text{colim}(W(\tilde{\text{NodAnn}}) \leftarrow W(\tilde{\text{Ann}}) \rightarrow W(\tilde{\text{Fr}})) \cong W(\tilde{\text{NodFr}}_\text{tree})_{\text{protected}}.$$ □
6.2. **Recovering the Deligne-Mumford operad.** Recall that we denote by $\text{DM}^{\text{tree}}$ the tree component of the Deligne-Mumford operad.

**Lemma 6.4.** We have a homotopy equivalence of operads
\[ \text{DM}^{\text{tree}} \xrightarrow{\simeq} \tilde{\text{NodFr}}_{\partial}^{\text{tree}}. \]

**Proof.** Let $\mathbb{D}$ be the standard unit disk, framed with the standard boundary parametrization $\theta \mapsto \exp(2\pi i\theta)$ for $\theta \in \mathbb{R}/\mathbb{Z}$ and let $0 \in \mathbb{D}$ be the origin. Let $\overline{\mathbb{D}}$ be the disk framed with the reverse boundary parametrization $\theta \mapsto \exp(-2\pi i\theta)$. Let $A_\alpha$ be the annulus of modulus $\alpha \in (0, \infty)$ framed with the standard boundary parametrizations. We then have
\[ \lim_{\alpha \to \infty} A_\alpha = \mathbb{D} \cup_0 \overline{\mathbb{D}} \]
in NodAnn, compatibly with boundary parametrizations.

Given a marked nodal surface $X \in \text{DM}^{\text{tree}}$, write
\[ Fr(X) \]
for the framed nodal surface obtained by gluing (at $0 \in \mathbb{D}$) a copy of $\mathbb{D}$ at every input marked point of $X$ and a copy of $\overline{\mathbb{D}}$ at the output marked point of $X$. See Figure 6.

![Figure 6](image)

**Figure 6.** By attaching disks at marked points one turns a surface with marked points into a framed nodal surface.

By a simple stabilization argument we see that
\[ Fr : \text{DM}^{\text{tree}} \to \tilde{\text{NodFr}}_{\partial}^{\text{tree}} \]
is a map of topological operads. On the other hand we see that, as a map of spaces, $Fr$ is the embedding of a homotopy retract. Indeed, let $\text{cap} : \tilde{\text{NodFr}}_{\partial}^{\text{tree}} \to \text{DM}^{\text{tree}}$ be the map (now of $\mathfrak{S}$–graded spaces, not operads) which assigns to a nodal surface with boundary the surface with marked points obtained
by gluing a copy of $\overline{D}$ at each input and a copy of $D$ at the output, and marking all images of $0 \in \overline{D}$, respectively $D$. See Figure 7.

![Figure 7](image)

**Figure 7.** By attaching caps along the boundary and stabilizing one turns a framed nodal surface into a nodal surface with marked points.

Then it is clear that 
\[ \text{cap} \circ Fr = \mathbb{I}_{DM}. \]

On the other hand, consider the maps 
\[ \text{stretch}_\alpha : \text{NodFr}_\partial \rightarrow \text{NodFr}_\partial, \quad \alpha \in [0, \infty] \]
defined by gluing $A_\alpha$ at every input and output of a framed nodal surface. This defines a homotopy equal to the identity map at $\alpha = 0$ and equal to $Fr \circ \text{cap}$ at $\alpha = +\infty$, which proves the homotopy retract property.

The following result will complete the proof of our main theorem.

**Lemma 6.5.** There is a homotopy equivalence of operads 
\[ W(\text{NodFr}_\partial)^{\text{tree}}_{\text{protected}} \simeq_\pi \text{NodFr}_\partial^{\text{tree}}. \]

**Proof.** We essentially repeat the proof of Theorem 3.9 in §3.5. Note that a natural map $\pi$ is the forgetful map which forgets all internal seams. If we work in the nondegenerate setting, i.e., if we exclude from the operad $\text{NodFr}_\partial^{\text{tree}}$ the space of thickness zero annuli $\text{Ann}^0$, we get a map 
\[ \pi_{\text{nondeg}} : W(\text{NodFr}_\partial^{\text{tree}})_{\text{protected}} \simeq_{\pi_{\text{nondeg}}} \text{NodFr}_\partial^{\text{tree}}. \]

This map has the same homotopy type as $\pi$ and has homotopic fibers. Now $\pi_{\text{nondeg}}$ is a Serre fibration, so it suffices to prove that the fibers of $\pi_{\text{nondeg}}$ are contractible.

Let $\Sigma \in \text{NodFr}_\partial^{\text{tree}}$ be a surface, and let $W_\Sigma := \pi^{-1}(\Sigma)$. Choose a real analytic metric on (the normalization of) $\Sigma$. For $\epsilon > 0$ let 
\[ W_\Sigma^{\epsilon} \subset W_\Sigma. \]
be the subspace that contains no seams at a distance \( \leq \epsilon \) from a node. Then the spaces \( W^I_\Sigma, \epsilon > 0 \) filter \( W_\Sigma \) by opens. In particular, write

\[
W^I_\Sigma := \bigcup_{\epsilon \in (0; \epsilon_0]} W^\epsilon_\Sigma \subset I \times W_\Sigma
\]

for some sufficiently small \( \epsilon_0 \) (depending on \( \Sigma \)), with \( I = (0, \epsilon_0] \). The map

\[
W^I_\Sigma \to W_\Sigma
\]

is a homotopy equivalence because all its fibers are nonempty intervals (with one open endpoint 0). We are thus left to prove that \( W^I_\Sigma \) is contractible.

Consider the canonical map

\[
\text{gap} : W^I_\Sigma \to W^I_\Sigma
\]

given by the tautological inclusions \( W^\epsilon_\Sigma \hookrightarrow W^{\epsilon/2}_\Sigma, \epsilon \in (0, \epsilon_0], \) i.e.

\[
(w, \epsilon) \mapsto (w, \epsilon/2), \quad w \in W^\epsilon_\Sigma.
\]

This map is well-defined because the filtration is decreasing, and it is obviously a homotopy equivalence.

Choose a continuous map

\[
\text{protect} : I \to W^I_\Sigma
\]

which takes \( \epsilon \) to a protected weighted split surface \( \Sigma_\epsilon = (\Sigma, S_\epsilon) \) with seams given by geodesic parametrizations of the circles of radius \( \epsilon \) on both sides of each node (one chooses \( \epsilon_0 \) small enough such that all these collections of circles are embedded).

We claim that \( \text{gap} : W^I_\Sigma \to W^I_\Sigma \) is homotopic to the map \( \tau : (w, \epsilon) \mapsto (\text{protect}(3\epsilon/4), \epsilon/2) \). The homotopy is constructed in two steps: we first put in the circles corresponding to \( S_{3\epsilon/4} \) with weight continuously changing from 0 to 1 (this is allowed because there are no other seams at distance \( \leq \epsilon \) from a node), and then continuously reduce all the other weights to zero (this is allowed because the presence of the \( S_{3\epsilon/4} \) (with weight 1) guarantees that the nodes remain protected).

Since \( \tau \) factors through an interval, it is homotopic to a constant. Since \( \text{gap} \) is a homotopy equivalence we infer that the space \( W^I_\Sigma \) is contractible, which finishes the proof of the Lemma.

\[ \square \]

Proof of the Main Theorem 1.1. Since the map of equivariant spaces \( \tilde{\text{Ann}} \to \tilde{\text{Fr}}_\partial \) is an embedding of a connected component and we are considering free \( \mathcal{S} \)-spaces, we infer that the map

\[
W(\tilde{\text{Ann}}) \to W(\tilde{\text{Fr}}_\partial)
\]
is a cofibration. This holds in the mixed model structure on Top. Thus the homotopy colimit
\[
\text{hocolim}(\widetilde{\text{NodAnn}} \leftarrow \widetilde{\text{Ann}} \rightarrow \widetilde{\text{Fr}}_{\partial})
\]
is computed by the operad colimit
\[
\text{colim}(W(\widetilde{\text{NodAnn}}) \leftarrow W(\widetilde{\text{Ann}}) \rightarrow W(\widetilde{\text{Fr}}_{\partial})).
\]
In view of Lemmas 6.3, 6.4 and 6.5 we obtain the sequence of equivalences
\[
\text{colim}(W(\widetilde{\text{NodAnn}}) \leftarrow W(\widetilde{\text{Ann}}) \rightarrow W(\widetilde{\text{Fr}}_{\partial})) \\
\cong W(\widetilde{\text{NodFr}}_{\partial}^{\text{tree}})_{\text{protected}} \widetilde{\rightarrow} \text{NodFr}_{\partial}^{\text{tree}} \leftarrow \text{DM}_{\text{tree}}.
\]
The proof carries over verbatim for the non-unital versions of our operads Ann and Fr_{\partial}, as in the statement of Theorem 1.1. Alternatively, one can use the fact that the forgetful functor from operads to non-unital operads commutes with homotopy push-outs.

7. Motivic extensions

Note that the extra work involving infinite-dimensional manifolds could be simplified somewhat, if in a less elegant way, by using the Kimura-Stasheff-Voronov (KSV) operad, a finite-dimensional model for \(\widetilde{\text{Fr}}_{\partial}\). However, this would lose the key message of the paper, which is contained in Theorems 1.1 and 3.9. In particular, the results of the present paper imply the following result. Let \(|\widetilde{\text{Ann}}|, |\widetilde{\text{NodAnn}}|, |\widetilde{\text{Fr}}_{\partial}|\) be the sets underlying the operads we defined above. Let \(|\text{NodFr}_{\partial}|\) be the groupoid underlying this orbifold, viewed as a discrete orbifold. Then we have the following "discrete" homotopy pushout result:

**Theorem 7.1.** The homotopy pushout in the category of topological groupoids of the discrete groupoids
\[
|\widetilde{\text{Ann}}| \leftarrow |\widetilde{\text{Fr}}_{\partial}| \rightarrow |\widetilde{\text{NodAnn}}|
\]
is homotopy equivalent (in the category of operads of orbifolds with our notion of weak equivalence) to the orbifold \(|\widetilde{\text{NodFr}}_{\partial}^{\text{tree}}|\).

Here it is essential that we are using \(|\widetilde{\text{NodFr}}_{\partial}^{\text{tree}}|\) rather than \(|\text{DM}_{\text{tree}}^{\text{tree}}|\) (which is no longer homotopy equivalent, as the fact that \(\text{DM}_{\text{tree}}^{\text{tree}} \rightarrow \text{NodFr}_{\partial}^{\text{tree}}\) is a homotopy equivalence uses non-discrete structure). As the second author showed in [31], analogues of both Theorems 3.9 and 7.1 hold in a much wider context: they admit a \(p\)-adic analogue (where the moduli spaces in question classify rigid analytic objects) and a motivic analogue, which takes into account the compatibility of complex manifold structures between Fr_{\partial} and NodFr_{\partial} (note that here
we need to get rid of unitality, as the complex manifold structure does not extend to the boundary of $\tilde{\mathcal{F}}_q$).

**Appendix A. Orbifolds, Operads and $\infty$-Categories**

In this paper we consider the objects $DM^{\text{tree}}$ and $\tilde{\text{NodFr}}^{\text{tree}}$ as operads. The operad structure is defined quite intuitively as a glueing of curves, but as the moduli spaces corresponding to spaces of operations of these operads are topological stacks (in fact, topological orbifolds) rather than topological spaces, some care is needed in working with them and comparing them to homotopy colimits in the model category of operads of topological spaces.

There are two standard ways to make rigorous the relevant notion of stacky operad. One, of a more classical flavor, would be to “enlarge” the spaces of operations in a homotopy invariant way that does not change stabilizers. One concrete such construction would be, for example, to keep track of the data of an additional $C^\infty$ function on each curve which vanishes along with its derivatives on the boundary and is required to be non-invariant under automorphisms. The space of possible functions on a fixed curve is an infinite-dimensional vector space, and the condition of asymmetry removes a subspace of infinite codimension, keeping the space of choices finite. In fact, our space $W(\tilde{\text{NodFr}}^{\text{tree}})^{\text{protected}}$ from [10] can be used as just such a solution (at every number of inputs, it-classifies the same data as $\tilde{\text{NodFr}}^{\text{tree}}$ together with contractible additional data). However, it is desirable to have a more direct answer to the question of what the operad $\tilde{\text{NodFr}}^{\text{tree}}$ really is (and similarly for $DM^{\text{tree}}$). A better point of view is that it is an operad valued in stacks, i.e. rather than being an operad valued in topological spaces, it is a functor from topological spaces to operads valued in groupoids. Indeed, given a “test” topological space $S$, one should imagine an operad in groupoids $DM(S)$ with $n \to 1$ operations given by the category whose objects are families of continuously varying algebraic tree-like nodal curves over $S$ with $n+1$ marked points ($n$ inputs and one output) and whose morphisms are isomorphisms of such data. Composition is then given by glueing of curves. This point of view almost works (indeed, it is sufficient for defining the individual spaces of operations), but runs into problems related to groupoids naturally forming a two-category rather than a category. The easiest and most elegant language to circumvent these issues is the language of $\infty$-operads, which is sufficiently universal to naturally incorporate topological and categorical inputs. It is now known (via results of [16], [6], [7], [2]) that $\infty$-operads form a model category equivalent to Berger and Moerdijk’s...
model category of topological operads, and in particular we lose no information by passing to this level.

A.1. From topological operads to $\infty$-operads. An $\infty$-operad structure (introduced by Lurie [24]) is a simplicial set that is fibered over a certain fixed simplicial set, whose simplices (as we shall see) are indexed by forests with some additional data. A topological operad can be converted to such a datum in three steps: first, the operad structure gets interpreted as a simplicial topological space over the simplicial operad of forests. Then each space involved in the structure gets replaced by its simplicial chains (the simplicial set built out of the sets $\text{Map}(\Delta^n, X)$, whose topological realization is weakly homotopy equivalent to $X$) to produce a bisimplicial set, and finally the bisimplicial set gets totalized as a simplicial set (all three pictures turn out to be models for equivalent $\infty$-categorical objects).

Now for an operad valued in stacks, a very similar construction works, except that there is a minor difference that gives an additional simplicial enhancement: namely, the functor of chains $\text{Map}(\Delta^n, X)$ for $X$ a topological stack returns not a simplicial set but a simplicial category, and so from a stacky operad (appropriately defined) one obtains a simplicial category enhanced in simplicial sets, fibered over forests. Taking the nerve of the category direction, one obtains a trisimplicial category fibered over forests, and totalizing produces an ordinary $\infty$-category, and therefore an $\infty$-operad.

To make this precise, we use the $\infty$-categorical language freely, and to avoid set-theoretic issues, implicitly replace all relevant categories (Sets, Top, etc.) by $\kappa$-small categories for a suitably large cardinal $\kappa$ (alternatively, we work in a Grothendieck universe).

Let $\text{Fin}_+$ be the category of finite sets with partially defined maps, i.e. with morphisms $\text{Hom}(\Gamma_1, \Gamma_2)$ given by a pair $(\Gamma'_1, f)$ with $\Gamma'_1 \subset \Gamma_1$ and $f : \Gamma'_1 \to \Gamma_2$. We view this category as a small category (for concreteness, we can take our sets to be subsets of $\mathbb{N}$). This category is naturally equivalent to the category of pointed finite sets. Let $N\text{Fin}_+$ be the nerve of the category $\text{Fin}_+$, an $\infty$-category (i.e., a simplicial set with a certain horn-filling property). We write partially defined morphisms of sets as $f : \Gamma \to \Gamma'$. For a set $\Gamma$ with subset $\Gamma'$, let $\chi_{\Gamma'} : \Gamma \to \Gamma'$ be the characteristic morphism of the subset, which sends $x \mapsto x$ for $x \in \Gamma'$ and is undefined at all other elements.

We say that a map $\Gamma \to \Sigma$ is inert if it is isomorphic to a characteristic map of the form $\chi_{\Gamma'} : \Gamma \to \Gamma'$. Recall that a marked simplicial set is a simplicial set with a distinguished collection of “marked” one-simplices which includes all degenerate one-simplices. We view $N\text{Fin}_+$ as a marked category by marking the one-simplices corresponding to inert partial maps of sets.
Definition A.1 ([24, Definition 2.1.4.1]). Define an \(\infty\)-preoperad to be a marked simplicial set \(O^\otimes\) together with a map of marked simplicial sets \(O^\otimes \to NFin_+\).

There is a model category structure on \(\infty\)-preoperads. A map of \(\infty\)-preoperads \((O^\otimes, p)\) and \((P^\otimes, p')\) is a commutative diagram

\[
\begin{array}{ccc}
O^\otimes & \rightarrow & P^\otimes \\
\downarrow p & & \downarrow p' \\
NFin_+ & \rightarrow & 
\end{array}
\]

compatible with markings. Fibrant objects in this model category structure are called (coloured) \(\infty\)-operads. An \(\infty\)-preoperad \((O^\otimes, p)\) is an \(\infty\)-operad if

Definition A.2 ([24, Definition 2.1.1.10]). A (coloured) \(\infty\)-operad \(O^\otimes\) is a pair \((O^\otimes, p)\) with \(O^\otimes\) an \(\infty\)-category and \(p : O^\otimes \to NFin_+\) an inner fibration satisfying the following conditions. Write \(O^\otimes(\Gamma)\) for the fiber category of \(p\) over the object \(\Gamma \in N(Fin_+)\). We make the following requirements.

1. All marked (i.e., inert) morphisms in \(NFin_+\) admit coCartesian lifts, and all such lifts are marked in \(O^\otimes\).
2. For \(x \in \Gamma\) an element, let \(F^x : O^\otimes(\Gamma) \to O^\otimes(x)\) be the functor representing the correspondence between fiber categories defined by the edge \(\chi^x\) (representable because coCartesian liftable). Require the product functor \(\prod_{\Gamma} O^\otimes(\Gamma) \to \prod_{x \in \Gamma} O^\otimes(x)\) to be an equivalence of categories.
3. Similarly, for \(f : \Gamma^- \to \Gamma'\) another morphism, require the natural functor of fiber categories over one-simplices \(\Gamma(f) \to \prod_{x \in \Gamma} \Gamma(\chi^x \circ f)\) to be an equivalence.

If a map \(f : (O^\otimes, p) \to (P^\otimes, p')\) of \(\infty\)-preoperads is an equivalence of \(\infty\)-categories between \(O^\otimes\) and \(P^\otimes\), this map is a weak equivalence, and this is an if and only if when \(O^\otimes, P^\otimes\) are \(\infty\)-operads.

If \(O = \{O_n, \circ, \rho_n : \mathfrak{S}_n \to Aut(O_n)\}\) is a classical (or a simplicial) operad, one defines the associated \(\infty\)-operad \((O^\otimes, p)\) fiberwise over \(Fin_+\) by taking the nerve of a (simplicial) category fibered over \(Fin_+\), with fiber categories \(O^\otimes(\Gamma) := pt\) (for all finite sets \(\Gamma\)) and with fiber over the arrow \(f : \Gamma^- \to \Gamma'\) given by \(O^\otimes(f) = \prod_{x \in \Gamma'} O_{[f^{-1}(x)]}\) : the symmetry group action now ensures the coCartesian liftability of the invertible arrows \(\sigma \in Aut_{Fin_+}(\langle n \rangle)\) and operad composition provides a way of composing arrows. Note in particular that for a morphism of the type \(\chi_{\Gamma'}\) for \(\Gamma' \subset \Gamma\) a subset, a morphism over \(\chi\) is a collection of one-to-one operations in \(O\) indexed by \(\Gamma'\). A natural coCartesian lift for \(\chi_{\Gamma'}\) is the
arrow $\mathbb{I}_{\Gamma,\Gamma'}$ corresponding to choosing the identity, $\mathbb{I}$, for each of the $\Gamma'$ one-to-one operations, since for any map $f : \Gamma' \to \Sigma$, any arrow over the composition $f \circ \chi_{\Gamma'}$ factors naturally through $\mathbb{I}_{\Gamma,\Gamma'}$.

A similar construction works for coloured operads, where the fiber over $\langle n \rangle$ is the set of $n$-tuples of colours. Conversely, if $O = (O^\otimes, p)$ is a coloured $\infty$-operad and $C$ is a set of objects of $O^\otimes(1)$ which contains each object up to equivalence, then for $t_n : \langle n \rangle \to \langle 1 \rangle$ the fully defined map to a point, the sets $O_n(c_1, c_2) := \pi_0(O^\otimes(c_1, c_2)_{t_n})$ (morphisms between two objects of $O^\otimes$ living over the map $t_n$ in $Fin_+$) naturally acquire a coloured operad structure with colours $c_i$. The full operad structure encodes the higher compatibilities involved in deriving this data. Note that this notion of operad only uses “invariant up to homotopy equivalence” notions of functors of $\infty$-categories, namely, Cartesian liftability of arrows and the requirement that certain universal diagrams are equivalences. In this sense, it is independent of the specific combinatorial model for $\infty$-categories we use. In particular, if we use the model for $\infty$-categories as topological categories, there is an obvious functor from topological operads in the sense of Berger-Moerdijk to $\infty$-operads. We will use a more combinatorial comparison below.

First, it will be convenient for us to give an alternative interpretation of the simplicial sets $NFin_+$ in terms of forests with extra data.

**Definition A.3.** A graded forest is a graph $\Gamma$ together with a grading of vertices $V(\Gamma)$ and edges $E(\Gamma)$ by “levels” $V(\Gamma) = \sqcup V_i(\Gamma)$, $E(\Gamma) = \sqcup E_i(\Gamma)$ such that $\Gamma$ is a forest and every edge in $E(\Gamma_i)$ is oriented and connects a vertex of level $i - 1$ to a vertex of level $i$, with at most one edge going out of each vertex (i.e., such that the root of each component tree is its highest-graded vertex). We call an edge between vertices of weight $n - 1$, $n$ a vertex of weight $n$, and label vertices and edges of each level by a set. These sets are allowed to repeat between levels.

Define $For_n$ to be the set (via our convention of using small models) of “$n$-bounded” forests, consisting of all graded forests with vertices of indices in $0, \ldots, n$.

**Definition A.4.**

- For an $n$-bounded graded forest $\Gamma \in For_n$ and $1 \leq i \leq n$, define $d_i(\Gamma)$ to be the $n + 1$-graded forest given by putting a vertex of weight $i$ in the middle of every edge of level $i$, and shifting the weight of all edges of weight $\geq i$ in $\Gamma$ up by 1. (The $i$-graded vertices of $s_i(\Gamma)$ are labeled by the same set as the $i$-labeled edges of $\Gamma$.)

- For an $n$-bounded graded forest $\Gamma \in For_n$ and $1 \leq i \leq n$, define $s_i(\Gamma)$ to be the $n - 1$-graded forest given by contracting each edge of level $i$, and reducing the level of each vertex of level $\geq i + 1$ by one. The level $i - 1$ vertices of $d_i(\Gamma)$ are labeled by the same
set as level \(i-1\) vertices of \(\Gamma\), and level \(i\) vertices, resp., level \(i\) edges of \(d_i(\Gamma)\) are labeled by the same set as level \(i+1\) vertices, resp., level \(i+1\) edges of \(\Gamma\).

**Lemma A.5.** The simplicial set \((\text{For}_n, d_i, s_i)\) with \(n\)-simplices \(\text{For}_n\) and edge, degeneracy maps \((d_i, s_i)\) as above is isomorphic to the simplicial nerve \(N(\text{Fin}_+^n)\).

**Proof.** As there is at most one edge coming out of each vertex, to a forest as above we can associate the sequence of partial maps \(V(\Gamma) \to V(\Gamma)\) of level \(i\), resp., level \(i+1\) edges of \(\Gamma\) and edge and degeneracy maps are in agreement. \(\square\)

**Definition A.6.** Given a topological operad \(O = (O_n, o_i, \rho_n : \mathfrak{S}_n \to \text{Aut}(O_n))\), we define a simplicial space \(O^\otimes\) over \(\text{For}_n\) as follows.

Choose an ordering of each finite set in our (small) category \(\text{Fin}_+^n\), i.e., an isomorphism of each object of \(\text{Fin}_+^n\) with the set \(\langle n \rangle = \{1, \ldots, n\}\).

To a forest \(\Gamma\), associate the graph with half-edges \(\Gamma^+\) obtained by removing all “leftmost” vertices at level 0 and adding a half-edge to each “root” (i.e., vertex with no children). Now define \(O^\otimes\) to be the product of free operad spaces \(O^\tau\) (a point of \(O^\tau\) is a choice of point of \(O|_v|\) for each vertex \(v\) of \(\Gamma^+\) of degree \(|v|+1\)). The simplicial structure is given by operad composition, after identifying the set of parents of each vertex \(v \in \Gamma^+\) with \(\langle |v| \rangle\) according to our choices of ordering.

The resulting simplicial topological space can be understood as a “topologically enriched \(\infty\)-operad”. Instead of working in the \(\infty\)-category of such, we observe (\cite[Example 2.1.1.21]{24}) that if \(O^\otimes\) is a discrete operad, the result is a simplicial set over \(N\text{Fin}_+\), which is an \(\infty\)-operad (in fact, it is the nerve of a certain 1-category over \(\text{Fin}_+\) associated with the classical operad \(O\)). It follows by functoriality that for any test topological space \(T\), the simplicial set \(O^\otimes(T)\) with \(O^\otimes(T)_n := \text{Hom}(T, O^\otimes_n)\) is a topological operad, and so the bisimplicial set \(O^\otimes(\Delta^*)_{\text{bisimp}}\) given by the diagram of simplicial sets \(O^\otimes(\Delta^0) \leftrightarrow O^\otimes(\Delta^1) \leftrightarrow \cdots\) is a simplicial object in the category of simplicial sets over \(N\text{Fin}_+\) which are \(\infty\)-operads. In particular, its totalization

\[
O^\otimes(\Delta^*) := \text{Tot}(O^\otimes(\Delta^*)_{\text{bisimp}})
\]

is a simplicial object which (as it is a totalization of a simplicial fibered \(\infty\)-category) is an \(\infty\)-category fibered over \(\text{Fin}_+\), and as levelwise weak equivalences of bisimplicial sets imply equivalences of totalizations, all of the conditions in our definition \(1\) follow from the corresponding conditions on each individual \(O^\otimes(\Delta^i)\).

At the end of the day, we have produced out of a topological operad an \(\infty\)-operad. Our construction coincides with the one in \cite{7}, thus giving
an equivalence of ∞-categories between the localized ∞-category underlying the Berger-Moerdijk model structure (the ∞-categorical localization of topological operads by weak equivalences) and the category of ∞-operads defined in [24] with a “single colour” condition (i.e. the requirement that the fiber of $O^\otimes$ over the one-element set is connected and pointed).

A.2. The stacky operads $\text{DM}^\text{tree}$ and $\widetilde{\text{NodFr}}^\text{tree}$ as ∞-operads.

Stacky operads such as $\text{DM}^\text{tree}$ and $\widetilde{\text{NodFr}}^\text{tree}$ fit into this picture, with a small modification. Suppose $O^\otimes$ is a simplicial groupoid (i.e. a simplicial object in the category of one-groupoids with functors) with map $p : O^\otimes \to \text{NFin}_+$, where the groupoid structure on simplices of $\text{NFin}_+$ is discrete. Then we can define a new simplicial object $|O^\otimes|$ over $\text{NFin}_+$ as the totalization of the bisimplicial complex $NO^\otimes$ given by applying the nerve construction to the groupoid $O_n$ for each $n$ (resulting in a new simplicial grading).

Define a topological prestack to be a functor from topological spaces to (small) groupoids. To each topological prestack $X$ we associate a functor $NX : \text{Top} \to \text{SSet}$ by sending each topological space $S$ to the nerve $N(X(S))$. We define the underlying simplicial set of $X$, denoted $|X|$, to be the simplicial set underlying the bisimplicial set $|[X]|$ with $|[X]|_{ij} := NX_i(\Delta^j)$ (corresponding to applying $NX$ to the standard cosimplicial topological space $\Delta^*\otimes\Delta^*$). It is clear that an equivalence of stacks $X \to Y$ induces a homotopy equivalence on underlying simplicial sets, and more generally if $X \to Y$ is a fibration of stacks with contractible fibers, the corresponding map on simplicial sets will be a homotopy equivalence.

Now we define objects $(\text{DM}^\text{tree})^\otimes_{\text{stack}}$ and $(\widetilde{\text{NodFr}}^\text{tree})^\otimes_{\text{stack}}$ as simplicial topological stacks over $\text{NFin}_+$. Namely, for each test topological space $S$ and each forest $\tau$, we define $(\text{DM}^\text{tree})^\otimes_{\text{stack}}(S)_\tau$ to be the category whose objects are continuously varying disconnected nodal curves $X$ with markings $M \subset X$ together with continuous maps $X \to \text{Vert}(|\Gamma|)$ and $M \to \text{Edge}_{1/2}(|\Gamma|)$ (with discrete target) where $\text{Edge}_{1/2}(|\Gamma|)$ is the set of pairs $(v, e)$ where $v$ is a vertex and $e$ is an edge (or half-edge) containing $v$. We require that (over every point of the topological base $S$) the component over each vertex is connected and tree-like and markings map bijectively to the elements of $\text{Edge}_{1/2}$ which include $v$.

There is a map

$$(\text{DM}^\text{tree})^\otimes(S)_\tau \to \prod_{v \in \tau} (\bigcup_{g \geq 0} \overline{\mathcal{M}}^\text{tree}_{g, |\text{Child}(v)| + 1}(S))$$

to the groupoids of collections of nodal curves over $S$ labeled by vertices of $\tau$ with appropriate numbers of marked points. This is evidently an equivalence of groupoids; the difference is essentially semantic, in that
we consider collections of marked curves which are pieces of a larger “disjoint union” curve. These canonically form a simplicial groupoid over the simplicial set $For$, with boundary maps given by gluing and degeneracy maps given by including extra copies of the identity operation (the unique point in $\overline{M}_{0,2}$). We define the simplicial groupoid $(\widetilde{\text{NodFr}}\theta)_{\text{tree},\text{stack}}$ similarly over $For$, with simplices over a tree $\tau$ given by the category of appropriate disjoint tree-like nodal curves with parametrized boundary components over each vertex of $\tau$. Finally, the $\infty$-operads corresponding to the topological operads $\tilde{\text{Fr}}\theta$, $W(\tilde{\text{Fr}}\theta)$, and $W(\tilde{\text{Fr}}\theta)_{\text{protected}}$ are all equivalent to the $\infty$-preoperads given by considering the corresponding objects as (stabilizer-free) moduli stacks in the evident way. Write $(\tilde{\text{Fr}}\theta)_{\text{stack}}^{\otimes}$, etc., for the corresponding simplicial topological stacks.

We define $(\text{DM}^{\text{tree}})^{\otimes}$ to be the object of the $\infty$-category of $\infty$-operads corresponding to the preoperad $|(\text{DM}^{\text{tree}})_{\text{stack}}^{\otimes}|$ (defined above, as the simplicial set underlying the bisimplicial set $|(\text{DM}^{\text{tree}})^{\otimes,\text{stack}}|$.) The map in Theorem 6.4 (written there on the level of stacks of operations) now extends in an obvious way to a map $(\text{DM}^{\text{tree}})^{\otimes,\text{stack}} \to (\widetilde{\text{NodFr}}\theta)^{\otimes,\text{stack}}$ of simplicial topological stacks fibered over $For$, and the homotopy equivalence property on fibers implies that the map is an equivalence on the level of stacks of simplices, hence the corresponding map of $\infty$-preoperads gives an equivalence. Similarly, the map in the proof of Lemma 6.5 should be interpreted in this language as a map of simplicial topological stacks

$$(W(\tilde{\text{Fr}}\theta)_{\text{protected}})^{\otimes,\text{stack}} \to (\widetilde{\text{NodFr}}\theta)^{\otimes,\text{stack}},$$

which is (by an obvious extension to the case of disconnected curves of the homotopy-trivialization-of-fibers argument in the proof of this lemma) an equivalence of topological stacks on the level of simplices. Now $W(\tilde{\text{Fr}}\theta)_{\text{protected}}$ is canonically equivalent in the Berger-Moerdijk category (which is equivalent as an $\infty$-category to the category of $\infty$-operads) to the homotopy pushout of the diagram of topological operads $\text{pt} \leftarrow S^1 \to \tilde{\text{Fr}}\theta$. The diagram of equivalences of simplicial topological stacks

$$
\begin{array}{ccc}
(W(\tilde{\text{Fr}}\theta)_{\text{protected}})^{\otimes} & \xleftarrow{\sim} & (W(\tilde{\text{Fr}}\theta)_{\text{protected}})^{\otimes,\text{stack}} \\
& & \xrightarrow{\sim} (\widetilde{\text{NodFr}}\theta)^{\otimes,\text{stack}} \\
& & \downarrow \\
\end{array}
$$

induces a canonical equivalence of $\infty$-operads between $(\text{DM}^{\text{tree}})^{\otimes}$ and the desired homotopy pushout.
Here we give a short proof of Corollary 1.2. Let $O$ be an operad in a symmetric monoidal category $C$. Then the structure of an $O$-algebra on an object $V \in C$ is a map of operads $O \to V^\otimes$, where $V^\otimes$ is the operad “spanned by $V$”, with $n$-ary operations $\text{Hom}(V^\otimes n, V)$. Let

\[
\begin{array}{ccc}
O' & \longrightarrow & O'' \\
\downarrow & & \downarrow \\
O & \longrightarrow & P
\end{array}
\]

be a pushout diagram of operads. Then the colimit property implies that a structure of $P$-algebra on $V$ is uniquely determined by a diagram

\[
\begin{array}{ccc}
O' & \longrightarrow & O'' \\
\downarrow & & \downarrow \\
O & \longrightarrow & V^\otimes.
\end{array}
\]

It follows from [5, Theorem 2.1] that for $C$ the dg category of complexes over a field $k$ of characteristic 0 (which is a symmetric monoidal model category with a “commutative interval”) the category of algebras over a dg operad $O$ in $C$ is invariant under quasiisomorphism of operads (a famous special case is that, in characteristic 0, the category of CDGA’s up to equivalence is equivalent to the category of $E_\infty$-algebras up to equivalence). In other words, in the category of operads in $C$, any map $\tilde{O} \to V^\otimes$ for $\tilde{O}$ a cofibrant replacement of $O$ is equivalent to a map $O \to V^\otimes$, and this map is unique up to unique equivalence (more generally, this is true on an $\infty$-categorical level).

Now the chains functor $C_\ast(-, k) : \text{Top} \to C$ commutes with colimits and is symmetric monoidal, and this implies that it commutes with colimits when viewed as a functor between the corresponding categories of operads. Corollary 1.2 follows by setting $O = C_\ast \tilde{F}_{\partial}, O' = C_\ast S^1$ and $O'' = C_\ast pt$, using the fact that in any model category, given a triple

\[
\begin{array}{ccc}
O' & \longrightarrow & O'' \\
\downarrow & & \downarrow \\
O & \longrightarrow & \text{of maps }
\end{array}
\]

with cofibrant homotopy pushout $P$ and $O, O', O''$ cofibrant, the data of a map $P \to X$ up to equivalence (for $X$ any object) is equivalent to a pair of maps $\alpha : O \to X$ and $\alpha'' : O'' \to X'$ together with a chain of equivalences between $\alpha |_{O'}$ and $\alpha'' |_{O'}$ in the model category of maps out of $O'$. 

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