Deterministic Entanglement Transmission on Series-Parallel Quantum Networks

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The performance of the entanglement transmission task, i.e., distributing entanglement between two arbitrarily distant nodes in a large-scale quantum network (QN), is notably benchmarked by the classical entanglement percolation (CEP) scheme. Improvement of entanglement transmission beyond CEP can only be achieved, with great loss of generality, by nonscalable strategies and/or on restricted QN topologies. The possibility of finding a better scheme than CEP that remains scalable and adaptable arises eventually from a new concurrency percolation theory (ConPT), essentially an alternative and more effective mapping of QN than classical percolation, that roots not in probability but in concurrence—a fundamental measure of bipartite entanglement. ConPT suggests using not generically probabilistic but deterministic protocols to produce a single highly entangled state between two nodes. In light of the ConPT mapping, here we formalize a specific deterministic entanglement transmission (DET) scheme that is fully analogous to resistor circuit analysis, implementing the corresponding series and parallel rules by deterministic entanglement swapping and purification protocols, respectively. The DET is designed for general d-dimensional information carriers and is scalable and adaptable for any series-parallel QN, as it reduces the entanglement transmission task to a purely topological problem of calculating path connectivity. Unlike CEP, different levels of optimality in terms of k-concurrences can be observed in DET, depending on specific QN topologies. More interestingly, our results suggest that the nesting feature of the well-known repeater protocol for qubits is not helpful for optimizing the final concurrence, the proof of which necessarily relies on a special reverse arithmetic mean–geometric mean (AM–GM) inequality.

I. INTRODUCTION

Many quantum communication technologies find their roots in entanglement, one of the most vital quantum resources [1]. However, the nonlocality of entanglement facilitating quantum communication becomes unrealistic at large space scales where faithful quantum information is submerged in thermal incoherent noise [2], making it very inefficient to directly establish point-to-point long-distance entanglement. To overcome this limit of classical locality and indirectly distribute entanglement among arbitrary parties requires utilizing some intermediate parties as “relays” and applying to them a variety of fundamental communication protocols, including entanglement swapping [3, 4] and entanglement purification [5, 6], as well as composite protocols that are usually recursive and more complex, such as the nested purification protocol using quantum repeaters [7–9], etc.

Since all parties are physically separated from one another so that locality holds, only local operations and classical communication (LOCC) can be applied to the information carriers (e.g., qubits) of different parties. Such consideration allows us to treat each party as a node that consists of a collection of information carriers, and treat each bipartite (resp. multipartite) entangled state formed by inter-node information carriers as a link (resp. hyperlink), essentially giving rise to a complex network representation of the structure of locality on entanglement resources, widely known as the quantum network (QN) [10]. A number of QN-based protocols have since been introduced, e.g., q-swapping [11] and path routing [12], combining communication protocols with network science [13]. In practice, basic communication protocols on small-scale QNs have been experimentally demonstrated using diamond nitrogen-vacancy centers [14, 16] and ion traps [17, 18]. It is believed that practical QNs are easier to scale than universal quantum computing platforms, since the nonlocality of information carriers only needs to be maintained within each node but not across the global QN platform. The future of a worldwide large-scale QN, namely, a “quantum Internet” [19] is hence foreseeable. This, on the other hand, urgently demands a more efficient design of QN-based communication protocols that should be scalable with network size and adaptable to different network topology, in accordance with the well-developed methods of complex network analysis [20].

The particular task of entanglement distribution [21] between two arbitrary nodes—which we coin as entanglement transmission in the QN context—is of special interest. The discovery of an exact mapping between entanglement transmission and classical bond percolation theory [22] gives rise to a straightforward entanglement transmission scheme, called classical entanglement percolation (CEP) [10], that greatly reduces the task of distributing a singlet (i.e., two maximally entangled qubits) between two nodes to a pure percolation problem. The CEP scheme is naturally scalable and adaptable for arbi-
trary QN. However, immediately after it was introduced, the CEP was proven not to produce the optimal probability of obtaining such a singlet [10]. The problem of calculating the optimal probability was later proven to be extremely complicated even for a one-dimensional (1D) chain [23, 24]. Even just improving this probability on large-scale QN can only be systematically done for special network topology with a few other limitations [23], making it difficult to find a better entanglement transmission scheme than CEP that is scalable and adaptable in the same satisfactory way.

This difficulty eventually leads to the questioning of the exclusivity of the classical-percolation-theory-based mapping itself: it was only recently realized that another mapping from entanglement transmission to a new statistical theory called concurrence percolation theory (ConPT) [25] may exist, bearing its name due to its analogy to classical percolation yet rooting not in probability but in concurrence—a fundamental measure of bipartite entanglement [26, 27]. The ConPT predicts a lower entanglement transmission threshold than the classical percolation threshold, exhibiting a “quantum advantage” that is purely structural, independent of nontopological details. To achieve the ConPT threshold, it has been suggested that deterministic communication protocols that only produce outcomes with certainty (up to unitary equivalence) should be used, so that the protocols can be applied recursively without becoming more and more involved with probabilistic distributions [25].

In light of this, here we formalize a specific deterministic entanglement transmission (DET) scheme for arbitrary two nodes in QN, provided that the network topology between the two nodes is series-parallel, i.e., can be decomposed into only two connectivity rules—series and parallel rules, in accordance with resistor circuit analysis. This DET scheme is built by explicitly expressing the series and parallel rules as two functions—a swapping function $S$ and a purification function $P$, which can be implemented by two fundamental communication protocols—entanglement swapping and purification, respectively. Such an implementation is possible since both protocols can be tuned to be deterministic. The formulation of our scheme has two important features:

1. The DET scheme is fully scalable and adaptable for arbitrary series-parallel QN, as it reduces the entanglement transmission task to a problem of calculating path connectivity between arbitrary two nodes, fully analogous to calculating the total resistance in a series-parallel resistor network.

2. The swapping function $S$ and purification function $P$ are defined for $d$-dimensional information carriers (qutrits, ququarts, etc.) in general, making the DET an all-purpose scheme for more exotic design of quantum information devices.

In particular, we show that unlike the CEP scheme, different levels of optimality can indeed be observed in the DET, depending on specific QN topologies (Fig. 2), given in terms of a family of $d$ concurrence monotones $C_k$ (see Appendix B). Using majorization theory, we calculate inequalities for the $S$ and $P$ functions and their recursive combinations, confirming that to some degree, the deterministic outcomes of DET are still the best, even when we relax the requirement of determinacy and detune the entanglement swapping and purification protocols to be probabilistic. The results are summarized in Table I.

| Network topology       | DET optimizes…       |
|------------------------|----------------------|
| Simple series          | Average $C_d$        |
| Simple parallel        | Average $C_k$ ($1 < k \leq d$) |
| Parallel-then-series  | Average $C_d$        |
| Series-then-parallel   | Worst-case $C_d$ ($d = 2$ only) |
| Series-parallel        | Worst-case $C_d$ ($d = 2$ only) |

Interestingly, our results suggest that the well-known nested purification protocol [7] introduced for a 1D chain of parties with multiple bipartite entangled qubits shared in between—which is essentially a parallel-then-series QN—is not a good strategy for optimizing concurrence. Instead of applying entanglement swapping and purification in a nested way, we argue that the best strategy is to apply entanglement purification once for all between every two adjacent nodes, and then perform entanglement swapping along the purified links. The proof of this result has not been derived before, to our best knowledge, since it relies on a special reverse arithmetic mean–geometric mean (AM–GM) inequality which we will prove in Appendix D. Note that the unnecessity of nesting could greatly simplify communication protocol design, suggesting that DET is practically more preferable.

II. QUANTUM NETWORK

In this paper, we focus on a “minimal” version of QN where each link is a $d^2$-dimensional bipartite pure state shared by the two connected nodes [10]. Such a bipartite pure state can always be written as $\sum_{\mu,\nu} \Psi_{\mu\nu} |\mu\nu\rangle \in \mathbb{C}^{d \times d}$ which allows matrixization, i.e., to be mapped to a $d \times d$ matrix $\Psi$ with elements $\Psi_{\mu\nu}$, $\mu, \nu = 1, 2, \ldots, d$. Left or right multiplication of $\Psi$ by a unitary matrix corresponds to a local unitary transformation performed by either of the two parties, respectively. Thus, up to unitary equivalence, the state can always be locally transformed into a diagonal form $|\lambda\rangle = \sum_{j=1}^{d} \sqrt{\lambda_j} |j\rangle$ by a singular value decomposition, so that each link is exclusively represented by $d$ positive Schmidt numbers, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$. Note that $\lambda$ is always subject to the normalization constraint $\sum_{j=1}^{d} \lambda_j = 1$. In particular, when $d = 2$ (qubits), there is only one degree of freedom, and thus each link can always be represented by only one
FIG. 1: Demonstration of the deterministic entanglement transmission (DET) scheme on arbitrary series-parallel quantum network (QN) with \( d \)-dimensional information carriers. From an initial QN (i), a final state (viii) can always be deterministically produced between Alice (A) and Bob (B) using the series and parallel rules (Table II).

A. Quantum Communication Protocols

A quantum communication protocol is a set of LOCC that allows entanglement to be distributed among different parties. We briefly review two fundamental bipartite protocols which are of special interest to us:

1. Entanglement swapping

Suppose there are three parties, Alice–Relay–Bob (A–R–B), who share two bipartite states, \( |\lambda_a\rangle = \sum_{j=1}^{d} \sqrt{\lambda_{a,j}} |jj\rangle \) (shared between A–R) and \( |\lambda_b\rangle = \sum_{k=1}^{d} \sqrt{\lambda_{b,k}} |kk\rangle \) (shared between R–B), respectively. This can be represented as a QN of three nodes and two links. Although A and B do not directly share any entanglement, a swapping protocol can indeed be performed, which requires a set of quantum measurements to be conducted on R. The result, which is probabilistic, is an ensemble of different bipartite states directly shared between A–B, usually entangled. To be specific, if we matricize \( |\lambda_a\rangle \) and \( |\lambda_b\rangle \) and denote them by two diagonal matrices \( \text{diag}(\lambda_a) \) and \( \text{diag}(\lambda_b) \), respectively, then after R performs the measurements, the (unnormalized) result state between A and B can always be written in the matrix form as

\[
\Psi^\alpha = \text{diag}(\lambda_a)^{1/2} X^\alpha \text{diag}(\lambda_b)^{1/2}
\]  

for outcome \( \alpha \), subject to probability \( p_\alpha = \|\Psi^\alpha\|^2 \) [29].

The quantum measurements are encoded by a set of arbitrary matrices \( \{X^\alpha\} \) constrained by the completeness relation,

\[
\sum_\alpha X_{\mu\nu\alpha}^\ast X_{\mu'\nu'\alpha} = \delta_{\mu\mu'} \delta_{\nu\nu'},
\]  

parameter w.l.o.g. [25].

This representation by a sequence of Schmidt numbers is effective, since singular values allow a built-in majorization preorder powered by matricization (see Appendix A for basic notions of the majorization theory). Obviously, this only works for bipartite states. Tripartite states, for example, although can be completely classified by five parameters including one phase and four moduli [28], are not yet compatible with such a preorder, leaving us a difficult “(quantum) three-body problem”.

Note that within a traditional network representation, each link is also a “bipartite” notation since it essentially connects two nodes. Hence, it seems a perfect match to study the statistics of a large number of bipartite states using a “bipartite” network theory, namely a “statistical theory of (quantum) two-body problems”. In contrast, a “complex” QN consisting of multipartite entangled states as “hyperlinks” is beyond the scope of the present paper, since the difficulties coming from both quantum many-body theory and “hypernetwork” theory are actually twofold.
which is sufficient for \( \{X^n\} \) to denote a positive operator-valued measure (POVM).

2. Entanglement purification

Given \( |\lambda\rangle = \sum_{j=1}^{m} \sqrt{\lambda_j} |jj\rangle \) for \( m \in \mathbb{Z}^+ \), what is the maximum probability to convert it to a new state \( |x\rangle = \sum_{j=1}^{m} \sqrt{\lambda_j} |jj\rangle \)? Such a probability is indeed well known and can be achieved by a purification protocol \[6\]. To be specific, this maximum probability is given by

\[
p_{\lambda\rightarrow x} = \min_{0 \leq k < m} \frac{1 - \sum_{j=1}^{k} \lambda_j^k}{1 - \sum_{j=1}^{k} \lambda_j^k}, \tag{3}
\]

and the LOCC implementation has been explicitly given in Ref. \[30\]. It is clear from Eq. \((3)\) that \( p_{\lambda\rightarrow x} = 1 \) when \( x \succ \lambda \), a result originally given by Nielsen’s theorem \[31\]. Also, note that it is possible to obtain a new state \( |x\rangle \) of a different dimension \( d \) if \( m \geq d \), since one can always pad \( x \) by zeros \[32\] and build a new sequence of \( m \), \( d \), \( n \).

Schmidt numbers, \( x' = x \oplus (0, 0, \cdots, 0) \), which inserted into Eq. \((3)\) yields the maximum probability \( p_{\lambda\rightarrow x'} \).

B. Entanglement Transmission Schemes

We denote a scheme by a (possibly infinite) number of quantum communication protocols that can be collectively applied to QN. One of the most important entanglement transmission schemes is the classical entanglement percolation (CEP) scheme for \( d = 2 \). The CEP scheme consists of two steps \[10\]: (i) Use the purification protocol to convert each link to a singlet, the probability of which is given by Eq. \((3)\). (ii) Find a path of links connecting two arbitrary nodes A and B that have all been successfully converted to singlets, and apply the swapping protocol to them one by one. The final state between A and B is guaranteed to be a singlet by Eq. \((1)\).

The probability of finding such a path can be considered as a measure of the entanglement transmission ability between A and B, determined by the network topology. This probability, however, is known to be nonoptimal, since for some special topology, e.g., a honeycomb with double links between every two connected nodes, adding a “preprocessing” step of entanglement swapping to the CEP scheme may change the network topology to a triangular type and potentially increase the probability of establishing a singlet between A and B \[10\].

A deterministic entanglement transmission (DET) scheme, on the other hand, does not use generic probabilistic protocols to increase the singlet conversion probability, but uses deterministic protocols to produce a single highly entangled state between A and B (demonstrated in Fig. \[1\]). Note that after performing each deterministic protocol, the new system is still an intact QN, i.e., a pure state as a whole, not a probabilistic ensemble. This makes DET easily scalable with network size.

As we will see, the specific DET scheme we introduce utilizes only deterministic bipartite protocols, which can be applied to arbitrary series-parallel QN.

III. A DETERMINISTIC ENTANGLEMENT TRANSMISSION SCHEME

A. Series-Parallel Network Topology

Whether a network is series-parallel or not depends on which two nodes of interest are chosen \[33\]. Given two nodes A and B, the network topology can be characterized into different categories (Fig. \[2\]). All topologies between A and B given in Figs. \[2(a)-2(e)\] are considered series-parallel, but the topology in Fig. \[2(f)\] is not due to the existing “bridge”. Note also that most realistic complex networks can be approximated as series-parallel, since loops are rare and can usually be ignored in infinite-dimensional systems \[34\], leaving a tree-like structure which is essentially series-parallel between any two nodes.

B. Series and Parallel Rules

The series and parallel rules of our DET scheme are given in Table \[11\] where \( \otimes \) stands for the Kronecker product, and, for simplicity, the infix notation \( \sim \) is introduced for representing binary operations, i.e., \( x \sim f \equiv f(x, y)\),

\[
\begin{align*}
(a) \text{Simple series QN.} & \\
(b) \text{Simple parallel QN.} & \\
(c) \text{Parallel-then-series QN.} & \\
(d) \text{Series-then-parallel QN.} & \\
(e) \text{Series-parallel QN.} & \\
(f) \text{General QN.}
\end{align*}
\]
which is left grouping, i.e., $x^+ f^+ y^+ f^+ z \equiv (x^+ f^+ y) f^+ z$.

The series and parallel rules are implemented by the entanglement swapping and purification protocols (Section IA), respectively, in a deterministic manner.

1. **Swapping function** $S(x, y)$

- **Definition.** We define $S : (\mathbb{R}_+^d, \mathbb{R}_+^d) \to \mathbb{R}_+^d$ by
  \[
  S(x, y) = d \times \sigma^2(\text{diag}(x^1)^{1/2}V \text{diag}(y^1)^{1/2})
  \]  
  (4)

  where $\sigma(\Psi)$ denotes the singular values of matrix $\Psi$, and $\sigma^2(\Psi)$ the entry-wise square of $\sigma(\Psi)$, both arranged in descending order so that $S(x, y) = S(\text{perm}(x), y)$. The matrix $V$ is constant and unitary, with elements $V_{\mu\nu} = d^{-1/2} \exp(-2\pi\text{i}\mu\nu/d)$, $\mu, \nu = 1, 2, \ldots, d$.

- **Properties.** $S(x, y)$ has the following properties:
  - Permutation invariance ($x \to \text{perm}(x)$):
    \[
    S(x, y) = S(\text{perm}(x), y).
    \]  
    (5)
  - Trace preserving:
    \[
    \text{tr}(S(x, y)) = \text{tr}(x)\text{tr}(y).
    \]  
    (6)
  - Isotone:
    \[
    S(x, z) \succ S(y, z), \forall z \text{ if } x \succ y.
    \]  
    (7)
  - Commutativity:
    \[
    S(x, y) = S(y, x).
    \]  
    (8)
  - Associativity:
    \[
    S(S(x, y), z) = S(x, S(y, z)) \text{ for } d \leq 3.
    \]  
    (9)

  Equations (5) and (6) hold because $\sigma(\Psi)$ is invariant under Hermitian conjugate and unitary transformation of $\Psi$. Equation (7) holds as a result of Theorem 12 (see Appendix C), given the facts that $S$ is permutation invariant [Eq. (5)] and convex (see Lemma 1 below). Equation (8) can be inferred from its validity for the special case $\text{tr}(x) = \text{tr}(y) = 1$.

  In particular, Eq. (9) is valid for $d \leq 3$ because its left-hand side (LHS) and right-hand side (RHS) are both governed by two constraints: trace preserving [Eq. (9)] and determinant preserving (see Lemma 2). Additionally, employing the duality (see Lemma 3) produces two more constraints in terms of “dual” trace preserving and “dual” determinant preserving. The latter, however, can be shown to be equivalent to the original determinant preserving constraint. Therefore, there are three independent constraints in total, and thus the LHS must be equal to the RHS when $d \leq 3$. When $d > 3$, counterexamples can be easily found.

- **Implementability by LOCC.**—It remains to be shown that $S(\lambda_a, \lambda_b)$ can be implemented by the entanglement swapping protocol [Eq. (i)] in a deterministic manner. This construction was originally given in Ref. 27. In short, given a special set of matrices $\{\lambda^\alpha\}$ each with elements $\lambda^\alpha_{\mu\nu} = d^{-1}e^{-\alpha(\mu+\nu)2\pi\text{i}/d^2 - 2\pi\text{i}\mu\nu/d}$ for $\alpha = 1, 2, \cdots, d^2$, one can verify that $\{\lambda^\alpha\}$ satisfies Eq. (2), thus denoting a POVM. Hence, performing the swapping protocol and following Eq. (1), the (unnormalized) result state $\Psi^\alpha'$ is obtained, for each outcome $\alpha$, with elements $\Psi^\alpha_{jk} = d^{-1/2} \sqrt{\lambda^\alpha_{jk} e^{-(\alpha\mu+\nu+\delta)2\pi\text{i}/d^2}}$. Now, as long as Alice and Bob are shared by the Relay the classical information of which $\alpha$ is obtained, they can always rotate their shared state $\Psi^\alpha$ by some phase accordingly and transform it locally into a new $\alpha$-independent state $\Psi'$ with elements $\Psi'_{jk} = d^{-1/2} \sqrt{\lambda^\alpha_{jk} e^{-(\alpha\mu+\nu+\delta)2\pi\text{i}/d^2}}$, the Schmidt numbers of which are then given by $S(\lambda_a, \lambda_b)$ [Eq. (4)] after normalization of $\Psi'$ [27]. The independence of $\Psi'$ on $\alpha$ confirms that the protocol is deterministic.

2. **Purification function** $P(x)$

- **Definition.**—We define $P : \mathbb{R}_+^d \to \mathbb{R}_+^d$ ($m \geq d$) by the following pseudocode:

  function $P(x)$:
  $\triangleright x^+ \leftarrow \left(x^+_1, x^+_2, \cdots, x^+_m\right)$
  $s \leftarrow \sum_{j=1}^m x_j$
  for $l \leftarrow 1, 2, \cdots, d$ do
    $\chi_l \leftarrow \max\left\{x^+_j, s/(d+1-l)\right\}$
    $s \leftarrow s - \chi_l$
  end
  return $\chi$
  $\triangleright \chi = (\chi_1, \cdots, \chi_d)$

  (10)

  The above definition implies that $P(x)$ is always given in descending order, $P(x) \equiv [P(x)]^1$, and

  \[
  [P(x)]_j \geq x^+_j, \quad j = 1, 2, \cdots, d.
  \]  
  (11)

  Moreover, if $[P(x)]_l > x^+_l$ for some $l$, then

  \[
  [P(x)]_j = [P(x)]_l \text{ and } [P(x)]_j > x^+_j
  \]  
  (12)

  for all $l \leq j \leq d$.

- **Properties.** $P(x)$ has the following properties:
Permutation invariance ($x \mapsto \text{perm}(x)$):
\[ \mathcal{P}(x) = \mathcal{P}(\text{perm}(x)). \quad (13) \]

- Trace preserving:
\[ \text{tr}(\mathcal{P}(x)) = \text{tr}(x). \quad (14) \]

- Isotone:
\[ \mathcal{P}(x) \triangleright \mathcal{P}(y) \text{ if } x \triangleright y. \quad (15) \]

- Commutativity:
\[ \mathcal{P}(x \otimes y) = \mathcal{P}(y \otimes x). \quad (16) \]

- Associativity:
\[ \mathcal{P}(\mathcal{P}(x \otimes y) \otimes z) = \mathcal{P}(x \otimes \mathcal{P}(y \otimes z)). \quad (17) \]

Equations (13), (14) and (16) all hold by definition [Eq. (10)]. And again, Eq. (15) holds as a result of Theorem 12 (see Appendix C), given the facts that $\mathcal{P}$ is permutation invariant [Eq. (13)] and convex (see Lemma 5 below).

In particular, Eq. (17) is valid because, on the one hand, by Eq. (15), we have $\mathcal{P}(\mathcal{P}(x \otimes y) \otimes z) \triangleright \mathcal{P}(x \otimes y \otimes z)$; on the other hand, since $\mathcal{P}(x \otimes y \otimes z) \triangleright (0,0,\cdots,0)$, by Lemma 4 below we have $\mathcal{P}(x \otimes y \otimes z) \triangleright \mathcal{P}(x \otimes y \otimes z)$. Thus, we must have $\mathcal{P}(x \otimes y \otimes z) = \mathcal{P}(x \otimes y \otimes z)$, since $\mathcal{P}$ is permutation invariant. But the same is also true for $\mathcal{P}(x \otimes \mathcal{P}(y \otimes z))$. Hence, together we have $\mathcal{P}(\mathcal{P}(x \otimes y) \otimes z) = \mathcal{P}(x \otimes \mathcal{P}(y \otimes z)) = \mathcal{P}(x \otimes y \otimes z)$.

**Implementability by LOCC.**—The definition [Eq. (11)] implies that
\[ \mathcal{P}(\lambda) \oplus (0,0,\cdots,0) \triangleright \lambda. \quad (18) \]
Thus, any new bipartite state of Schmidt numbers $\mathcal{P}(\lambda)$ can be produced deterministically (i.e., $p_{\lambda \rightarrow \mathcal{P}(\lambda)} = 1$) from a bipartite state of Schmidt numbers $\lambda$ by the entanglement purification protocol [Eq. (3)].

**IV. OPTIMALITY**

We have shown that both the series and parallel rules in Table I are implementable by the entanglement swapping and purification protocols (Section II A). Naturally, we are interested in how good is the DET scheme built by these rules. Here, we proceed to study the optimality of these rules. As we will see, the degree of optimality of our series and parallel rules actually varies for different series-parallel network topologies. This variety and its implications will be addressed in detail below.

When discussing the optimality of the series and parallel rules, we will compare our specific rules to generic bipartite protocols. We will show that, to some degree, the deterministic outcomes by the series and parallel rules are still the best, even when we relax the requirement of determinacy and consider general probabilistic ensembles of outcomes. Since an entanglement-efficient protocol that gives general probabilistic outcomes necessarily requires more storage and more sophisticated algorithms to keep track of every possible outcome, the complexity of the full scheme must inevitably scale with the QN size. Thus, given the optimality as well as the determinacy of our series and parallel rules, we believe that the DET scheme based on these rules is potentially the best scheme in practice for entanglement transmission on large-scale QN.

**Concurrence monotones.**—To start with, it is necessary to introduce a proper entanglement measure on the Schmidt numbers $\lambda$. Although $\lambda$ only admits a preorder, the entanglement measure as a real function admits a strict total order, which can be used to quantify the amount of entanglement. We will adopt a special family of entanglement monotones, called the concurrence monotones $C_k$, $k = 1,2,\cdots,d$. We will show that the series and parallel rules optimize the task of entanglement transmission on series-parallel QNs when measured by the average or worst-case $k$-concurrence, or $G$-concurrence (see Appendix B for definitions and details), depending on the network topology.

**A. Some Basic Lemmas**

We first prove some basic Lemmas for the swapping and purification functions, $S(x,y)$ and $\mathcal{P}(x)$:

**Lemma 1. (Convexity)** $S(x,\alpha z)$ is convex, i.e.,
\[ \sum_{\alpha} p_{\alpha} S(x_{\alpha}, z) \triangleright S\left(\sum_{\alpha} p_{\alpha} x_{\alpha}, z\right), \quad p_{\alpha} \in \mathbb{R}_+, \quad z \in \mathbb{R}_+^d. \]

**Proof.** Equation (4) can be rewritten as
\[ S(x,y) = d \times \text{eig}(\text{diag}(x^4)V^\dagger \text{diag}(y^4)V) \quad (19) \]
where “eig” denotes the eigenvalues, arranged in descending order. Thus,
\[ S\left(\sum_{\alpha} p_{\alpha} x_{\alpha}, z\right) \triangleright d \sum_{\alpha} p_{\alpha} \text{eig}(\text{diag}(x_{\alpha}^4)V^\dagger \text{diag}(z^4)V) \]
by Lidskii’s theorem on eigenvalues (Theorem III.4.1 of Ref. [35]).

**Lemma 2. (Determinant preserving)**
\[ \text{det}(S(x,y)) = d^d \text{det}(x) \text{det}(y). \]
where \( \det(x) \equiv \det(\text{diag}(x)) = x_1 x_2 \cdots x_d \).

**Proof.** By Eq. (19), we directly have \( \det(S) = d^d \det(x) |\det(V)|^2 \det(y) = d^d \det(x) \det(y) \).

\[
L_{\alpha}. \text{(Duality)} \text{ Let } z = S(x, y). \text{ Then, } \quad \text{adj}(z)^\dagger = d^{-2} S(\text{adj}(x), \text{adj}(y)).
\]

where \( \text{adj}(x) \equiv (\det(x)/x_1, \det(x)/x_2, \cdots, \det(x)/x_d) \).

**Proof.** By Eq. (19),
\[
S(\text{adj}(x), \text{adj}(y)) = d \times \text{eig}(\text{diag}(\text{adj}(x)^{\dagger}))V^{\dagger}\text{diag}(\text{adj}(y^{\dagger}))V
\]
\[
= d \times \text{eig}(\text{adj}(x^{\dagger}))\text{adj}(V)\text{adj}(\text{diag}(y^{\dagger}))\text{adj}(V^{\dagger})
\]
\[
= d \times \text{adj}(V^{\dagger}\text{diag}(y^{\dagger})\text{diag}(x^{\dagger}))
\]
\[
= d \times \text{adj}(S(x, y)/d)^\dagger = d^{2-d} \text{adj}(z)^\dagger,
\]
where we have used several properties of the adjugate of a matrix \( \Psi \), e.g., \( \text{adj}(\Psi) = \det(\Psi)\Psi^{-1} \) and \( \text{adj}(c\Psi) = c^{d-1}\text{adj}(\Psi) \) for \( c \in \mathbb{R} \).

\[
L_{\alpha}. \text{(Majorization extremity)} \text{ Let } z' \in \mathbb{R}^d_+. \text{ If } \quad (z')^\dagger = (0,0, \cdots ,0) \succ x, \text{ then } z' \succ P(x).
\]

**Proof.** We will prove it by induction, i.e., suppose there exists \( |P(x)|_j > x_j^l \) for some \( l \), and
\[
\sum_{j=1}^k z_j^l \geq \sum_{j=1}^l |P(x)|_j, \quad \forall k = 1, \cdots , l - 1,
\]
we would like to prove
\[
\sum_{j=1}^k z_j^l \geq \sum_{j=1}^l |P(x)|_j.
\]

Indeed,
\[
\sum_{j=1}^k z_j^l = \sum_{j=1}^{l-1} z_j^l + \sum_{j=1}^{d} z_j^l - \sum_{j=1}^{l-1} z_j^l \geq \sum_{j=1}^{l-1} z_j^l + \sum_{j=1}^{d} z_j^l - \sum_{j=1}^{l-1} z_j^l = \sum_{j=1}^{l-1} |P(x)|_j + (d + 1 - l)^{-1} \sum_{j=t}^d |P(x)|_j
\]
\[
\geq \sum_{j=1}^{l-1} |P(x)|_j + (d + 1 - l)^{-1} \sum_{j=t}^d |P(x)|_j
\]
\[
= \sum_{j=1}^{l-1} |P(x)|_j + |P(x)|_j = \sum_{j=1}^l |P(x)|_j
\]

where the first inequality holds because the maximum is never less than the mean, and the second because \( \text{tr}(P(x)) = \text{tr}(z') \) and \( 1 - (d + 1 - l)^{-1} \geq 0 \). Hence Eq. (21) is proved.

Finally, choose the minimum \( l \) that satisfies \( |P(x)|_j > x_j^l \). In other words, \( |P(x)|_j = x_j^l \) for \( j < l \). Hence,
\[
\sum_{j=1}^k |P(x)|_j = \sum_{j=1}^k x_j^l \leq \sum_{j=1}^k z_j^l, \quad \forall k = 1, \cdots , l - 1,
\]
which completes the induction.

\[
L_{\alpha}. \text{(Convexity)} \text{ } P(x, y) \text{ is convex, i.e., } \quad \sum_{\alpha} p_{\alpha} P(x, y) \succ P(\sum_{\alpha} p_{\alpha} x_{\alpha}), \quad p_{\alpha} \in \mathbb{R}_+.
\]

**Proof.** Similar to Lemma 4, we will prove it by induction, i.e., suppose there exists \( |P(\sum_{\alpha} p_{\alpha} x_{\alpha})|_j > |\sum_{\alpha} p_{\alpha} x_{\alpha}|^l_j \) for some \( l \), and
\[
\sum_{j=1}^k \left[ \sum_{\alpha} p_{\alpha} P(x_{\alpha}) \right]^l_j \geq \sum_{j=1}^k \left[ P(\sum_{\alpha} p_{\alpha} x_{\alpha}) \right]^l_j,
\]
\[
\forall k = 1, \cdots , l - 1, \text{ we would like to prove}
\]
\[
\sum_{j=1}^k \left[ \sum_{\alpha} p_{\alpha} P(x_{\alpha}) \right]^l_j \geq \sum_{j=1}^k \left[ P(\sum_{\alpha} p_{\alpha} x_{\alpha}) \right]^l_j.
\]

But this is true and can be proved in the same way as in Eq. (22). Finally, choose the minimum \( l \) that satisfies \( |P(\sum_{\alpha} p_{\alpha} x_{\alpha})|_j > |\sum_{\alpha} p_{\alpha} x_{\alpha}|^l_j \). In other words, \( |P(\sum_{\alpha} p_{\alpha} x_{\alpha})|_j = |\sum_{\alpha} p_{\alpha} x_{\alpha}|^l_j \) for \( j < l \). Hence,
\[
\sum_{j=1}^k \left[ \sum_{\alpha} p_{\alpha} x_{\alpha} \right]^l_j \leq \sum_{j=1}^k \left[ \sum_{\alpha} p_{\alpha} P(x_{\alpha}) \right]^l_j,
\]
\[
\forall k = 1, \cdots , l - 1, \text{ which completes the induction.}
\]

\[
L_{\alpha}. \text{(A sum-product mixing majorization inequality)} \text{ Let } x, y, z \in \mathbb{R}^d_+. \text{ If } \quad \ln x^y + \ln y^z \succ_w \ln z^w,
\]

then
\[
\ln P(x) + \ln P(y) \succ_w \ln P(z).
\]

**Proof.** To begin with, w.l.o.g. we will just assume \( m = d + 1 \), which means that \( P(x) \) is only one dimension less than \( x \). Indeed, for any \( m > d \), the purification process [Eq. 10] can be equally constructed by applying such a one-dimension-less \( P(x) \) to \( x \) recursively \( m - d \) times. Thus, if our statement holds for \( m = d + 1 \), then it holds for general \( m \) too.
First, we would like to prove the following inequality,
\[
\left( \prod_{j=1}^{d} [\mathcal{P}(x)]_{j} \right) \left( \prod_{j=1}^{d} [\mathcal{P}(y)]_{j} \right) \geq \prod_{j=1}^{d} [\mathcal{P}(z)]_{j}. \tag{25}
\]
By the definition of \( \mathcal{P}(x) \) [Eq. (12)], we have
\[
\prod_{j=1}^{d} [\mathcal{P}(z)]_{j} = \left( \prod_{j=1}^{l-1} z_{j}^{\downarrow} \right) \left[ \sum_{j=l}^{d+1} z_{j}^{\downarrow} / (d + 1 - l) \right]^{d+1-l},
\]
where \( l \) separates \( \mathcal{P}(z) \) and satisfies \( [\mathcal{P}(z)]_{j} = z_{j}^{\downarrow} \) for \( j < l \) and \( [\mathcal{P}(z)]_{j} > z_{j}^{\downarrow} \) for \( l \leq j \leq d \).

Now we take advantage of a special reverse AM–GM inequality which we will introduce and prove in Appendix D. Let \( \Delta = z_{d+1}^{\downarrow}, \varepsilon_{j} = z_{d+1-j}^{\downarrow}, \) and \( E_{k} = \Delta + \sum_{j=1}^{d} \varepsilon_{j}, k = 1, 2, \ldots, d + 1 = l \). We can see that \( \varepsilon_{k} \leq \varepsilon_{k+1} \leq E_{k} / k, \forall k \) [implied by Eq. (10)]. Thus, using Corollary 16,
\[
\left( \sum_{j=1}^{d+1} z_{j}^{\downarrow} / (d + 1 - l) \right)^{d+1-l} \leq \exp\left( z_{d+1}^{\downarrow} / z_{d}^{\downarrow} \right) \prod_{j=1}^{d} z_{j}^{\downarrow} \leq \left( 1 + \sqrt{z_{d+1}^{\downarrow} / z_{d}^{\downarrow}} \right)^{2} \frac{d}{l} z_{d}^{\downarrow},
\]
where the second inequality holds since \( z_{d+1}^{\downarrow} \leq z_{d}^{\downarrow} \). Thus,
\[
\prod_{j=1}^{d} [\mathcal{P}(z)]_{j} \leq \left( 1 + \sqrt{z_{d+1}^{\downarrow} / z_{d}^{\downarrow}} \right)^{2} \prod_{j=1}^{d} z_{j}^{\downarrow}
\leq \left( x_{d}^{\downarrow} + x_{d+1}^{\downarrow} + 2 \sqrt{x_{d}^{\downarrow} y_{d+1}^{\downarrow}} \right) \prod_{j=1}^{d-1} z_{j}^{\downarrow}
\leq \left( x_{d}^{\downarrow} + x_{d+1}^{\downarrow} \right) \left( y_{d}^{\downarrow} + y_{d+1}^{\downarrow} \right) \prod_{j=1}^{d-1} z_{j}^{\downarrow}
\leq \left( \prod_{j=1}^{d} [\mathcal{P}(x)]_{j} \right) \left( \prod_{j=1}^{d} [\mathcal{P}(y)]_{j} \right),
\]
where in the third step we use Theorem 13 (see Appendix C), in the fourth step the usual AM–GM inequality, and in the last step the definition of the purification process [Eq. (10)]. Hence Eq. (25) is proved.

Now we will complete the proof by induction, i.e., suppose there exists \( [\mathcal{P}(z)]_{j} > z_{j}^{\downarrow} \) for some \( l \), and
\[
\sum_{j=1}^{k} \ln [\mathcal{P}(x)]_{j} + \sum_{j=1}^{k} \ln [\mathcal{P}(y)]_{j} \geq \sum_{j=1}^{k} \ln [\mathcal{P}(z)]_{j}, \tag{26}
\]
\[\forall k = 1, \ldots, l - 1, \] we would like to prove
\[
\sum_{j=1}^{l} \ln [\mathcal{P}(x)]_{j} + \sum_{j=1}^{l} \ln [\mathcal{P}(y)]_{j} \geq \sum_{j=1}^{l} \ln [\mathcal{P}(z)]_{j}. \tag{27}
\]
The proof structure is the same as in Eq. (22), while the only difference is that, for the second inequality in Eq. (22) to hold, we need to show that \( \text{tr}(\ln \mathcal{P}(x)) + \text{tr}(\ln \mathcal{P}(y)) \geq \text{tr}(\ln \mathcal{P}(z)) \). But this has been proved by Eq. (25). Hence Eq. (27) is proved.

Finally, choose the minimum \( l \) that satisfies \( [\mathcal{P}(z)]_{j} > z_{j}^{\downarrow} \). In other words, \( [\mathcal{P}(z)]_{j} = z_{j}^{\downarrow} \) for \( j < l \). Hence,
\[
\sum_{j=1}^{k} \ln [\mathcal{P}(x)]_{j} + \sum_{j=1}^{k} \ln [\mathcal{P}(y)]_{j} \leq \sum_{j=1}^{k} \ln x_{j}^{\downarrow} + \sum_{j=1}^{k} \ln y_{j}^{\downarrow}
\]
\[\leq \sum_{j=1}^{k} \ln [\mathcal{P}(x)]_{j} + \sum_{j=1}^{k} \ln [\mathcal{P}(y)]_{j}, \forall k = 1, \ldots, l - 1 \]
which completes the induction. \(\square\)

### B. Single Link

First, we ask: if we relax the requirement of determinacy, would a better strategy exist if we use LOCC to convert a link into a probabilistic ensemble of links before applying the protocols? The answer is negative, as confirmed by the following theorem.

**Theorem 7.** Given an arbitrary series-parallel QN, converting an inbetween link into a probabilistic ensemble cannot increase the average \( k \)-concurrence produced by the series and parallel rules (Table II) between \( A \) and \( B \) for \( k = 1, 2, \ldots, d \).

**Proof.** Given a probabilistic ensemble derived from link \( \lambda \) by LOCC, let \( \lambda_{\alpha} \) be the probabilistic outcome \( \alpha \), subject to probability \( p_{\alpha} \). Then,
\[
\sum_{\alpha} p_{\alpha} \lambda_{\alpha} > \lambda. \tag{28}
\]
Equation (28) represents a fundamental limit of locality of LOCC. This result can be obtained by applying Lidskii’s theorem (Theorem III.4.1 of Ref. 35) to the ensemble of reduced density matrices produced by LOCC [32].

Now, by Lemmas 1 and 4
\[
\sum_{\alpha} p_{\alpha} S(\lambda_{\alpha}, z) > S(\sum_{\alpha} p_{\alpha} \lambda_{\alpha}, z),
\]
\[
\sum_{\alpha} p_{\alpha} \mathcal{P}(\lambda_{\alpha}) > \mathcal{P}(\sum_{\alpha} p_{\alpha} \lambda_{\alpha}), \tag{29}
\]
and by Eqs. (B4) and (B5) in Appendix B
\[
\sum_{\alpha} p_{\alpha} C_{k}(\lambda_{\alpha}) \leq C_{k}(\sum_{\alpha} p_{\alpha} \lambda_{\alpha}) \leq C_{k}(\lambda). \tag{30}
\]
Together, Eqs. \([29]\) and \([30]\) imply that the average \(k\)-
concurrence will never be greater than what could be
obtained from \(\lambda\) itself. Hence, it never helps to convert
\(\lambda\) into a probabilistic ensemble.

\(\Box\)

**C. Simple Series**

For simple series network topology [Fig. 2(a)], the
swapping protocol is responsible for entanglement
transmission. What is the maximum of the final average \(k\)-
concurrence [from Eq. \([1]\)]
\[
\sum_{\alpha} p_{\alpha} C_{d}^{\alpha} = \sum_{\alpha} \|\Psi^{\alpha}\|_{2}^{2} C_{k}(\sigma(\|\Psi^{\alpha}\|_{2}^{-1}\Psi^{\alpha}))
\]

between \(A\) and \(B\) that can be obtained by the swapping
protocol? And how does it compare with the deterministic
series rule? This is answered by the following theorem.

**Theorem 8.** Given a simple series QN, compared
with generic entanglement swapping protocols of probabilistic
outcomes, the series rule (Table II) produces the optimal
average \(G\)-concurrence between \(A\) and \(B\).

**Proof.** It suffices to prove the theorem for two links \(\lambda_{a}\)
and \(\lambda_{b}\) in series. The nice properties of \(S(\lambda_{a}, \lambda_{b})\)
guarantee that the proof can be generalized to \(n\) links.

Let \(\{X^{\alpha}\}\) denote a set of quantum measurements as
used in the entanglement swapping protocol [Eq. \([1]\)].
In general, given probability \(p_{\alpha}\) and the corresponding
\(k\)-concurrence \(C_{k}^{\alpha}\) of outcome \(\alpha\), the maximum
average \(k\)-concurrence \(\max\{\|X^{\alpha}\|\sum_{\alpha} p_{\alpha} C_{k}^{\alpha}\}\) is intractable, and
the corresponding optimal \(\{X^{\alpha}\}\) should implicitly depend
on \(\lambda_{a}\) and \(\lambda_{b}\). However, this is not the case for \(G\)-
concurrence. To see this, let \(k = d\), then
\[
\sum_{\alpha} p_{\alpha} C_{d}^{\alpha} = d \sum_{\alpha} |\det(\Psi^{\alpha})|^{2/d} = d \sum_{\alpha} \det(\lambda_{a})^{1/d} |\det(X^{\alpha})|^{2/d} \det(\lambda_{b})^{1/d} = d^{-1} C_{d}(\lambda_{a}) C_{d}(\lambda_{b}) \sum_{\alpha} |\det(X^{\alpha})|^{2/d} \leq d^{-2} C_{d}(\lambda_{a}) C_{d}(\lambda_{b}) \sum_{\alpha} \operatorname{tr}(X^{\alpha}X^{\alpha}) = C_{d}(\lambda_{a}) C_{d}(\lambda_{b}).
\]
(31)

where the AM–GM inequality is used in the second last
step and Eq. \([2]\) in the last step. Equation \([31]\) indicates
that the final average \(G\)-concurrence will never be greater
than the product of the \(G\)-concurrences of \(\lambda_{a}\) and \(\lambda_{b}\).
This proof was originally given in Ref. \([27]\). As we can see,
this is a very special result that comes from the unique
feature of the multiplicity of determinants, which is not
applicable to other \(k\)-concurrence monotones.

It remains to be shown that the equality in Eq. \([31]\)
holds for \(S(\lambda_{a}, \lambda_{b})\). By Lemma \([2]\)
\[
C_{d}(S(\lambda_{a}, \lambda_{b})) = d \det(S(\lambda_{a}, \lambda_{b}))^{1/d} = d^{2} \det(\lambda_{a})^{1/d} \det(\lambda_{b})^{1/d} = C_{d}(\lambda_{a}) C_{d}(\lambda_{b}).
\]
(32)

Thus,
\[
\sum_{\alpha} p_{\alpha} C_{d}^{\alpha} \leq C_{d}(S(\lambda_{a}, \lambda_{b}))
\]
which completes the proof.

\(\Box\)

**Remark.**—Note that Eq. \([33]\) says nothing about
the majorization preorder. In fact, usually there is
\[
\sum_{\alpha} p_{\alpha} \sigma^{2}(\Psi^{\alpha}/\|\Psi^{\alpha}\|_{2})^{\dagger} = \sum_{\alpha} \sigma^{2}(\Psi^{\alpha})^{\dagger} \not\succ S(\lambda_{a}, \lambda_{b}).
\]
(34)

Would the inequality in Eq. \([34]\) hold, one could prove
Eq. \([33]\) for not only \(k = d\) but \(k < d\) as well. Unfortunately
this is not true.

**D. Simple Parallel**

For simple parallel network topology [Fig. 2(b)], the
purification protocol is responsible for entanglement
transmission. Similarly, we ask: what is the maximum
of outcome \(k\)-concurrences of \(\lambda_{a}\) and \(\lambda_{b}\) that can be obtained by
the purification protocol? And how does it compare with the deterministic parallel
rule? The answer is given by the following theorem.

**Theorem 9.** Given a simple parallel QN, compared
with generic entanglement purification protocols of probabilistic
outcomes, the parallel rule (Table II) produces the optimal
average \(k\)-concurrence between \(A\) and \(B\) for
\(k = 1, 2, \cdots, d\).

**Proof.** Again, it suffices to prove the theorem for two
links \(\lambda_{a}\) and \(\lambda_{b}\) in parallel. The nice properties of \(P(\lambda)\)
guarantee that the proof can be generalized to \(n\) links.

For simple parallel network topology [Fig. 2(b)], the
locality of LOCC [Eq. \([28]\)], if we can obtain
an ensemble of outcomes \(\lambda_{a}', \lambda_{b}'\), each with probability \(p_{\alpha}\),
from \(\lambda_{a} \otimes \lambda_{b}\), then
\[
\left(\sum_{\alpha} p_{\alpha} \lambda_{a}'^{\alpha} \otimes (0, 0, \cdots, 0)\right) \succ \lambda_{a} \otimes \lambda_{b}.
\]

Hence, by Lemma \([4]\)
\[
\sum_{\alpha} p_{\alpha} \lambda_{a}'^{\alpha} \succ P(\lambda_{a} \otimes \lambda_{b}),
\]
(35)

and thus
\[
\sum_{\alpha} p_{\alpha} C_{k}(\lambda_{a}') \leq C_{k}(\sum_{\alpha} p_{\alpha} \lambda_{a}'^{\alpha}) \leq C_{k}(P(\lambda_{a} \otimes \lambda_{b}))
\]
for \(k = 1, 2, \cdots, d\).

\(\Box\)

**Remark.**—Note that unlike Eq. \([34]\) for the series rule,
Eq. \([35]\) does hold. Therefore, the optimality of the parallel
rule is for all \(k\)-concurrence, not just \(G\)-concurrence.
E. Parallel-then-Series

For parallel-then-series network topology [Fig. 2(c)], both the swapping and purification protocols are responsible for entanglement transmission. We prove the following theorem:

**Theorem 10.** Given a parallel-then-series QN, compared with generic entanglement swapping and purification protocols of probabilistic outcomes, the series and parallel rules (Table II) produce the optimal average G-concurrence between A and B.

**Proof.** It suffices to prove the theorem for four links: \( \lambda_a \) and \( \lambda_b \) in parallel, \( \lambda_c \) and \( \lambda_d \) in parallel, and then the two parallel groups are connected in series. Now, let \( \Psi^\alpha = \text{diag}(\lambda_a \otimes \lambda_b)^{1/2}X^\alpha\text{diag}(\lambda_c \otimes \lambda_d)^{1/2} \), as used in the entanglement swapping protocol [Eq. (1)]. The difference, however, is that \( \Psi^\alpha \) is not a \( d \times d \) matrix, but a \( d^2 \times d^2 \) matrix. This is because we are considering a generic swapping protocol, which must be applied to the full \( d^2 \) dimensions. Then, the final \( d \)-dimensional outcome \( \alpha \) will be given by \( \mathcal{P}(\sigma^2(\Psi^\alpha)) \).

By the Gel’Jand-Naimark theorem on singular values (Theorem III.4.5 of Ref. [35]), we have

\[
\ln \sigma^2(\Psi^\alpha) < \ln (\lambda_a \otimes \lambda_b)^{1/2} + \ln (\lambda_c \otimes \lambda_d)^{1/2} + \ln (X^\alpha).
\]

Thus, by Lemma 6, we have, in particular,

\[
\text{det}(\mathcal{P}(\sigma^2(\Psi^\alpha))) \leq \text{det}(\mathcal{P}(\lambda_a \otimes \lambda_b))\text{det}(\mathcal{P}(\lambda_c \otimes \lambda_d))\text{det}(\mathcal{P}(\lambda_e \otimes \lambda_f)).
\]

Hence, similar to Eq. (31),

\[
\sum \alpha p_\alpha C^\alpha d = d \sum \alpha \text{det}(\mathcal{P}(\sigma^2(\Psi^\alpha)))^{1/d} \leq d^{-2} C_d(\mathcal{P}(\lambda_a \otimes \lambda_b))C_d(\mathcal{P}(\lambda_c \otimes \lambda_d)) \sum \alpha \text{tr}(\mathcal{P}(\sigma^2(\Psi^\alpha)))
\]

\[
= d^{-2} C_d(\mathcal{P}(\lambda_a \otimes \lambda_b))C_d(\mathcal{P}(\lambda_c \otimes \lambda_d)) \sum \alpha \text{tr}(X^{\alpha})
\]

\[
= C_d(\mathcal{P}(\lambda_a \otimes \lambda_b))C_d(\mathcal{P}(\lambda_c \otimes \lambda_d)).
\]

(37)

where the AM–GM inequality is used in the second step, Eq. (14) in the third step, and Eq. (2) in the last step. Again, Eq. (37) indicates that the final average G-concurrence will never be greater than the product of the G-concurrences of \( \mathcal{P}(\lambda_a \otimes \lambda_b) \) and \( \mathcal{P}(\lambda_c \otimes \lambda_d) \), which is equal to \( C_d(S(\mathcal{P}(\lambda_a \otimes \lambda_b)), \mathcal{P}(\lambda_c \otimes \lambda_d)) \) [Eq. (32)]. \( \square \)

**Remark.**—At first glimpse, it seems that we would have more degrees of freedom to design a swapping protocol directly on the full \( d^2 \)-dimensional \( \lambda_a \otimes \lambda_b \) and \( \lambda_c \otimes \lambda_d \), and we would get a better result in terms of G-concurrence. Interestingly though, Theorem 10 says that the best strategy is to first reduce them to \( d \)-dimensional \( \mathcal{P}(\lambda_a \otimes \lambda_b) \) and \( \mathcal{P}(\lambda_c \otimes \lambda_d) \) and then apply a swapping protocol on them. Our result implies a strong practical convenience in entanglement transmission, that it is in fact unnecessary to keep all \( d^m \) dimensions for \( m \) parallel links. We can safely purify and produce a \( d \)-dimensional link from the \( m \) parallel links and then swap it with other links connected in series. The final G-concurrence is guaranteed to be optimal.

Note particularly that the nested purification protocol [7] is designed to swap \( \lambda_a \otimes \lambda_b \) and \( \lambda_c \otimes \lambda_d \) using two independent swapping protocols on \( a, c \) and \( b, d \), respectively, before applying purification protocols on the outcomes. But the tensor product of the two swapping protocols is just a special case of a \( d^2 \)-dimensional swapping protocol. Thus, the nested purification protocol does not optimize the final average G-concurrence.

F. Series-then-Parallel

For series-then-parallel network topology [Fig. 2(d)], unfortunately, the final average \( k \)-concurrence between A and B will not be maximized by the series and parallel rules but instead by some generic nondeterministic swapping and purification protocols.

A counterexample can be constructed. Let \( d = 2 \) and \( \lambda_a = \lambda_b = \alpha = (0.9, 0.1) \), \( \lambda_c \) and \( \lambda_d \) are connected in series and then with \( \lambda_c \) in parallel. The series and parallel rules (Table II) yield a deterministic worst-case \( G \)-concurrence of probabilistic outcomes, the series and parallel rules (Table II) yield a deterministic final outcome (the ZZ basis [23]) on \( \lambda_a \) and \( \lambda_b \), which is guaranteed to be optimal. However, applying a special nondeterministic swapping protocol (the ZZ basis [24]) on \( \lambda_a \) and \( \lambda_b \), we can get four probabilistic outcomes, \( \lambda_1 = (1/2, 1/2) \) with probabilities \( p_1 = p_2 = 0.41 \) and \( p_3 = p_4 = 0.09 \), respectively. The final average G-concurrence is 2\( p_1 C_d(\mathcal{P}(\lambda_a \otimes \lambda_1)) + 2 p_2 C_d(\mathcal{P}(\lambda_c \otimes \lambda_3)) \approx 0.695 \). Thus, the series and parallel rules are not optimal. The deeper reason is that the inequality in Eq. (34) does not hold in general. For this example, we clearly have

\[
p_1 \lambda_1^2 + p_2 \lambda_2^2 + p_3 \lambda_3^2 + p_4 \lambda_4^2 = (0.819, 0.181) \neq S(\lambda_a, \lambda_b) = (0.966, 0.034).
\]

G. Series-Parallel

For arbitrary series-parallel network topology [Fig. 2(c)], we already know from the previous discussion that no optimality can be observed for the final average \( k \)-concurrence. That being said, when \( d = 2 \), a special theorem of optimality can indeed be made in terms of the final worst-case G-concurrence (see Appendix B). We prove the following theorem:

**Theorem 11.** (For \( d = 2 \) only.) Given a series-parallel QN, compared with generic entanglement swapping and purification protocols of probabilistic outcomes, the series and parallel rules (Table II) produce the optimal worst-case G-concurrence between A and B.

**Proof.** From Theorems 8 and 9 we know that, for simple series and simple parallel network topologies, given a generic ensemble of probabilistic outcomes \( \alpha \), there must
be at least one \( \lambda_n \) of which the \( G \)-concurrence is smaller than (at most equal to) the \( G \)-concurrence which the series and parallel rules would produce, respectively. Since \( d = 2 \), this \( \lambda_n \) majorizes what the series and parallel rules would produce. Then, given that the series and parallel rules are isotone [Eqs. (7) and (15)], the final probabilistic outcome related to \( \lambda \) rules are isotone \[ Eqs. (7) \text{ and } (15) \], the final probabilistic outcome related to \( \lambda \) will remain as the worst case when the series and parallel rules are applied over the full series-parallel network.

Remark.—This special optimality on the worst-case \( G \)-concurrence for arbitrary series-parallel network topology with \( d = 2 \) was first noticed in Ref. \[25\]. When \( d > 2 \), Theorem \[11\] does not hold in general. This is because other deterministic series or parallel rules different than Table \[11\] may exist. As long as the other deterministic rules can produce a \( \lambda \) with better \( G \)-concurrence (which is only possible for series-then-parallel network topology), then the worst-case \( G \)-concurrence is automatically larger than what Table \[11\] can produce.

V. DISCUSSION

What truly makes the formulation of deterministic entanglement transmission (DET) scalable with network size and adaptable to different (at least series-parallel) network topologies is the use of deterministic quantum communication protocols \[25\]. One may wonder if the specific DET scheme based on the series and parallel rules (Table \[11\]) that we have introduced is the only possible DET scheme using deterministic protocols. The answer is negative, because different series and parallel rules may exist as well:

For example, the recipe of \( S(\lambda_n, \lambda_n) \) [Eq. (14)] may still be implementable by LOCC if the matrix \( V \) is not a constant but depends explicitly on \( \lambda_n \) and \( \lambda_n \). When properly chosen, this new swapping function can even produce a better \( k \)-concurrence \((k \neq d)\) than our swapping function \[27\]. Moreover, if quantum catalysts \[36\] are allowed, then it is also possible to have a new purification function \( P(\lambda) \) that is implementable by LOCC without satisfying the majorization relation [Eq. (15)]. These new functions still correspond to deterministic protocols and therefore can be used to build a different set of series and parallel rules.

What remains is the question of whether we can also find deterministic protocols that are applicable to non-series-parallel QN. The complexity of answering this comes from the fact that multipartite protocols \[27\] and multilink-based QN routing \[12\] may have to be used for producing deterministic outcomes. The existence of such deterministic protocols can certainly help us study entanglement transmission on a more general QN.

Given the well-behaved scalability and adaptability of DET, we find it particularly interesting to apply our scheme to an infinite series-parallel QN that has an infinite number of nodes. We expect that percolation-like criticality in terms of \( \lambda \) of each link can be observed for entanglement transmission in the thermodynamic limit. However, note that to each QN link assigned is not a single number but a sequence of Schmidt numbers which have more than one degree of freedom when \( d > 2 \). This prohibits us from establishing an exact one-to-one mapping from \( \lambda \) to a real parameter, e.g., temperature. It is unknown how criticality behaves in such case.

Appendix A: Basic Notations

We briefly recall some basic notions in matrix analysis and majorization theory \[35\]:

Let \( x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ \). Note that some notations for matrices can also be defined for \( x \). Given the diagonal matrix of \( x \),

\[
\text{diag}(x) \equiv \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix},
\]

we can simply define the trace of \( x \) as \( \text{tr}(x) \equiv \text{tr}(\text{diag}(x)) = \sum_{j=1}^n x_j \), the determinant of \( x \) as \( \text{det}(x) \equiv \text{det}(\text{diag}(x)) = x_1 x_2 \cdots x_n \), and the adjugate of \( x \) as \( \text{adj}(x) \equiv (\text{det}(x)/x_1, \text{det}(x)/x_2, \cdots, \text{det}(x)/x_n) \).

Let \( x^\downarrow \) and \( x^\uparrow \) be the sequences given by rearranging the coordinates of \( x \) in decreasing order and increasing order, respectively. In other words, \( x_1^\downarrow, x_2^\downarrow, \cdots, x_n^\downarrow \) as the coordinates of \( x^\downarrow \) satisfy \( x_1^\downarrow \geq x_2^\downarrow \geq \cdots \geq x_n^\downarrow \). Similarly, for \( x^\uparrow \) there is \( x_1^\uparrow \leq x_2^\uparrow \leq \cdots \leq x_n^\uparrow \).

Let \( x, y \in \mathbb{R}^n_+ \). We say that \( x \) is weakly submajorized by \( y \), or \( x \prec_w y \) \[35\], if

\[
\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow, \quad \forall k = 1, \cdots, n. \tag{A1}
\]

In particular, if \( x \) is weakly submajorized by \( y \) and

\[
\sum_{j=1}^n x_j^\downarrow = \sum_{j=1}^n y_j^\downarrow, \quad \text{or, } \text{tr}(x) = \text{tr}(y), \tag{A2}
\]

then we say that \( x \) is majorized by \( y \), or \( x \prec y \) \[35\].

For example, when \( x \in \mathbb{R}^n_+ \) and \( \text{tr}(x) = 1 \), we always have \((1/n, \cdots, 1/n) \prec x \prec (1, 0, \cdots, 0)\). On the other hand, given two arbitrary \( x, y \in \mathbb{R}^n_+ \) with \( \text{tr}(x) = \text{tr}(y) = 1 \), we may have \( x \not\prec y \) and \( y \not\prec x \) both.

Thus, the majorization relation is not a total order. It is not a partial order either, because \( x \succ y \) and \( y \succ x \) do not necessarily imply \( x = y \), since they may differ by a permutation \[35\]. The majorization relation is only a preorder on \( \mathbb{R}^n_+ \).

In this work, our attention is mostly focused on \( d^2 \)-dimensional normalized states with \( d \) Schmidt numbers \( \lambda \) satisfying \( \text{tr}(\lambda) \equiv 1 \). We exclusively use the symbol \( \lambda \) whenever we implicitly know that it has a trace of unity.
and $d - 1$ degrees of freedom. For general vectors, we use the symbols $x, y, z, \ldots \in \mathbb{R}_+^{k}$ instead.

Appendix B: A Family of Concurrence Monotones: $k$-Concurrence

Given a $d \times d$-dimensional bipartite pure state, up to unitary equivalence, $|\lambda\rangle = \sum_{j=1}^{d} \sqrt{x_j} |jj\rangle$, there are $d$ concurrence monotones, as developed in Ref. [27],

$$C_k(\lambda) \equiv \left[ \sum_{i=1}^{d} \lambda_i \right]^{1/k}, \quad \text{tr}(\lambda) \equiv 1,$$

$$k = 1, 2, \ldots, d,$$ where $S_k(\lambda)$ is the $k$-th elementary symmetric polynomial [35], e.g.,

$$S_0(\lambda) = 1,$$

$$S_1(\lambda) = \sum_{i=1}^{d} \lambda_i,$$

$$S_2(\lambda) = \sum_{i<j}^{d} \lambda_i \lambda_j,$$

$$\ldots$$

$$S_d(\lambda) = \lambda_1 \lambda_2 \cdots \lambda_d.$$ Note that $C_k$ is named $k$-concurrence since the whole monotone family is nothing but a generalization of $C_2(\lambda)$, the “concurrence” which is more commonly referred to in the context of quantum information theory. Another special concurrence of the family is $C_d$ (i.e., $k = d$), the $G$-concurrence, as it stands for the geometric mean of the Schmidt numbers [27]. The importance of $G$-concurrence can be understood from the main text.

All $k$-concurrences have the following properties [27]:

- **Permutation invariance ($\lambda \rightarrow \text{perm}(\lambda)$):**
  $$C_k(\lambda) = C_k(\text{perm}(\lambda)).$$

- **Unit measure:**
  $$0 \leq C_k(\lambda) \leq 1,$$ and
  $$C_k(\lambda) = 1 \text{ if } \lambda = \left(\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d}\right).$$

- **Isotone:**
  $$C_k(\lambda_a) \leq C_k(\lambda_b) \text{ if } \lambda_a > \lambda_b.$$

- **Concavity:**
  $$C_k(\sum_{\alpha} p_{\alpha} \lambda_{\alpha}) \geq \sum_{\alpha} p_{\alpha} C_k(\lambda_{\alpha}), \quad p_{\alpha} \in \mathbb{R}_+.$$ (B5)

In particular, if $\sum_{\alpha} p_{\alpha} = 1$, then $p_{\alpha}$ and $\lambda_{\alpha}$ can be explained as a probabilistic ensemble. The average $k$-concurrence is then defined as the RHS of Eq. [B5], $\sum_{\alpha} p_{\alpha} C_k(\lambda_{\alpha})$, and the worst-case $k$-concurrence is defined as $\min_{\alpha} C_k(\lambda_{\alpha})$, accordingly.

Appendix C: Some Useful Theorems

**Theorem 12.** Let $F : \mathbb{R}_+^{n} \to \mathbb{R}_+$ be convex, i.e.,

$$\sum_{\alpha} p_{\alpha} F(x_{\alpha}) \geq F(\sum_{\alpha} p_{\alpha} x_{\alpha}), \quad p_{\alpha} \in \mathbb{R}_+.$$ If, for all $x$,

$$F(\text{perm}(x)) = F(x),$$

then $F$ is isotone, i.e., $F(x) \geq F(y)$ if $x \succeq y$.

**Proof.** Let $x \succeq y$ in $\mathbb{R}_+^{n}$. By Theorem II.1.10 of Ref. [36] there exist a set of $m \times m$-dimensional permutation matrices $P_1, P_2, \ldots$ and a set of $p_{1}, p_{2}, \ldots \in \mathbb{R}_+$ with $\sum_{\alpha} p_{\alpha} = 1$ such that $y = \sum_{\alpha} p_{\alpha} P_{\alpha} x$. Thus,

$$F(y) = F(\sum_{\alpha} p_{\alpha} P_{\alpha} x) \geq \sum_{\alpha} p_{\alpha} F(P_{\alpha} x) = \sum_{\alpha} p_{\alpha} F(x),$$

i.e., $F(x) \geq F(y)$. □

**Theorem 13.** Let $x, y \in \mathbb{R}_+^{n}$. If $\ln x \succ_{w} \ln y$, then

$$\left(\prod_{j=1}^{l-1} x_j \right)^{\sum_{j=1}^{k} \left(\frac{1}{l}\right)} \geq \left(\prod_{j=1}^{l-1} y_j \right)^{\sum_{j=1}^{k} \left(\frac{1}{l}\right)}.$$

∀$k = 1, 2, \ldots, n$ and $\forall l = 1, 2, \ldots, k$, given that $0 \leq s \leq k - l + 1$.

**Proof.** Let $c = \prod_{j=1}^{l-1} \left(\frac{y_j}{x_j}\right) \leq 1$. Then,

$$\prod_{j=l}^{k} x_j \geq c \prod_{j=l}^{k} y_j \geq \prod_{j=l}^{k} \left(\frac{c^{l+1}}{s}\right) = \prod_{j=l}^{k} \left(\frac{1}{s}\right) y_j,$$

which proves Eq. [C3]. Note that Eq. [C5] holds because $f(e^t) = (e^t)^s$ is convex and monotone increasing in $t$. □

Appendix D: A Reverse AM–GM Inequality

We will prove the following inequality:

**Theorem 14.** Let $\varepsilon \in \mathbb{R}_+^{n}$ and $\Delta \geq 0$. Let $E_k = \Delta + \sum_{j=1}^{k} \varepsilon_j$, $k = 1, 2, \ldots, n$. If $\varepsilon_k \leq \varepsilon_{k+1} \leq E_{k}/k$, $\forall k$, then

$$\left(\frac{E_n}{n}\right)^{n} \leq \varepsilon_n \varepsilon_{n-1} \cdots \varepsilon_1 \left(1 + \frac{\Delta}{\varepsilon_1}\right)^{n}$$

with equality if and only if $\varepsilon_n = \varepsilon_{n-1} = \cdots = \varepsilon_1$. □
Proof. To start with, note that $E_k/k = \varepsilon_k/k + E_{k-1}/k \leq E_{k-1}/(k-1)$. Therefore, the following inequality holds:

$$\frac{E_n}{n} \leq \frac{E_{n-1}}{n-1} \leq \cdots \leq \frac{E_1}{1}. \quad (D2)$$

Now we will prove Eq. \([D1]\) by induction, i.e., suppose

$$\left( \frac{E_{j-1}}{j-1} \right)^{j-1} \leq \varepsilon_j - \varepsilon_{j-2} \cdots \varepsilon_1 \left( 1 + \frac{\Delta/\varepsilon_j}{j-1} \right)^{j-1}, \quad (D3)$$

we would like to prove

$$\left( \frac{E_j}{j} \right)^{j} \leq \varepsilon_j \varepsilon_{j-1} \cdots \varepsilon_1 \left( 1 + \frac{\Delta/\varepsilon_j}{j} \right)^{j}. \quad (D4)$$

To do so, define $f_j(\varepsilon_j, \varepsilon_{j-1}, \cdots, \varepsilon_1)$ as the right-hand side minus the left-hand side of Eq. \([D4]\). Taking the derivative of $f_j$ w.r.t. $\varepsilon_k$ for $k = 2, \cdots, j$ yields

$$\frac{\partial f_j}{\partial \varepsilon_k} = \frac{\varepsilon_j \varepsilon_{j-1} \cdots \varepsilon_1}{\varepsilon_k} \left( 1 + \frac{\Delta/\varepsilon_j}{j} \right)^{j-1} - \left( \frac{E_j}{j} \right)^{j-1}$$

$$\geq \varepsilon_j \varepsilon_{j-1} \cdots \varepsilon_1 \left( 1 + \frac{\Delta/\varepsilon_j}{j} \right)^{j-1} - \left( \frac{E_j}{j} \right)^{j-1}$$

$$\geq \varepsilon_j \varepsilon_{j-1} \cdots \varepsilon_1 \left( 1 + \frac{\Delta/\varepsilon_j}{j-1} \right)^{j-1} - \left( \frac{E_j}{j-1} \right)^{j-1}$$

$$\geq f_{j-1} \geq 0 \quad (D5)$$

with equality if and only if $\varepsilon_j = \varepsilon_{j-1} = \cdots = \varepsilon_1$. Thus, fixing $\varepsilon_1$ and noticing that $\varepsilon_2$ is constrained by $\varepsilon_1$ only, we conclude that, since $\partial f_j/\partial \varepsilon_2 \geq 0$, $f_j$ takes the minimum if and only if $\varepsilon_2 \geq \varepsilon_1$ actually takes the equality; next, fixing $\varepsilon_1$ and $\varepsilon_2 = \varepsilon_1$ and noticing that $\varepsilon_3$ is constrained by $\varepsilon_1$ and $\varepsilon_2$, we conclude that $f_j$ takes the minimum if and only if $\varepsilon_3 \geq \varepsilon_2$ actually takes the equality; …

Taken together, we conclude that $f_j$ takes the minimum if and only if $\varepsilon_j = \varepsilon_{j-1} = \cdots = \varepsilon_1$, which put back into $f_j$ yields $f_j(\varepsilon_1, \varepsilon_1, \cdots, \varepsilon_1) = 0$. Hence $f_j \geq 0$. The induction is thus completed given $f_1 = 0$. \(\square\)

We note that, despite being a reversed inequality of the AM–GM type, Theorem \([14]\) is tight. This is because $\Delta$ can take any nonnegative value. The two independent constraints $\varepsilon_{k+1} \geq \varepsilon_k$ and $\varepsilon_{k+1} \leq E_k/k$ in Theorem \([14]\) together imply that the deviations between the coordinates of $\varepsilon$ cannot be too large, which are controlled by $\Delta$. In particular, if $\Delta = 0$, then Eq. \([D1]\) just becomes the reverse of the usual AM–GM inequality. But $\Delta = 0$ also requires $\varepsilon_{k+1} = \varepsilon_k = \cdots = \varepsilon_1$ given the constraints. Therefore Eq. \([D1]\) will not be violated since it can only take the equality.

**Corollary 15.** (Same prerequisites as in Theorem \([14]\))

$$\left( \frac{E_j}{j} \right)^{j} \leq \varepsilon_j \varepsilon_{j-1} \cdots \varepsilon_1 \left( 1 + \frac{\Delta/\varepsilon_j}{j} \right)^{j}, \quad j = 1, 2, \cdots, n \quad (D6)$$

with equality if and only if $\varepsilon_j = \varepsilon_{j-1} = \cdots = \varepsilon_1$. \(\square\)

**Proof.** This has been proved in the proof of Theorem \([14]\), which is a special case $(j = n)$ of this Corollary.

**Corollary 16.** (Same prerequisites as in Theorem \([14]\))

$$\left( \frac{E_j}{j} \right)^{j} \leq \varepsilon_j \varepsilon_{j-1} \cdots \varepsilon_1 e^{\Delta/\varepsilon_1}, \quad j = 1, 2, \cdots, n \quad (D7)$$

with equality if and only if $\Delta = 0$.

**Proof.** This is because

$$\left( 1 + \frac{\Delta/\varepsilon_1}{j} \right)^{j} \leq e^{\Delta/\varepsilon_1}, \quad j > 0 \quad (D8)$$

with equality if and only if $\Delta = 0$. \(\square\)

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