One-Loop Matching of the Heavy-Light $A_0$ and $V_0$ Currents with NRQCD Heavy and Improved Naive Light Quarks

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Abstract

One-loop matching of heavy-light currents is carried out for a highly improved lattice action, including the effects of mixings with dimension 4 $O(1/M)$ and $O(a)$ operators. We use the NRQCD action for heavy quarks, the Asqtad improved naive action for light quarks, and the Symanzik improved glue action. As part of the matching procedure we also present results for the NRQCD self energy and for massless Asqtad quark wavefunction renormalization with improved glue.

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I. INTRODUCTION

The importance of approaching the chiral limit in a controlled way in lattice simulations of heavy-light systems has become increasingly evident in recent years. Progress in reducing statistical and discretization errors has resulted in chiral extrapolation uncertainties, together with operator matching errors, dominating the final total error in heavy-light meson decay constant and form factor calculations. This has motivated the HPQCD collaboration to initiate studies of heavy-light systems using improved staggered/naive light quarks, taking advantage of the good chiral properties of such actions which allow one to go down to much smaller quark masses than has been possible in the past [1, 2, 3, 4]. Heavy-light simulations are now being carried out with light quark masses as low as $m_{\text{strange}}/8$, and contact has been established with chiral perturbation theory predictions.

An important ingredient in all heavy meson decay constant and form factor calculations is the matching of the heavy-light currents used in the simulations to their continuum QCD counterparts. The highly improved heavy-light actions introduced above necessitate a new round of matching calculations. In this article we report on the one-loop perturbative matching of the temporal component of the heavy-light axial current, $A_0$, with NRQCD heavy quarks and massless Asqtad naive quarks. These matching coefficients are directly relevant for our ongoing $f_B$, $f_{B_s}$ and $f_{D_s}$ calculations on the MILC dynamical configurations and have already been applied in the results quoted in [1, 2, 3]. Due to the chiral symmetry of naive quarks, matching coefficients for the vector current are identical to those of the axial current; thus the results presented here can also be applied to form factor calculations [5]. Our matching calculations include contributions from $1/M$ current corrections and an $\mathcal{O}(a \alpha_s)$ discretization correction. The matched heavy-light current is then correct through $\mathcal{O}(\alpha_s)$, $\mathcal{O}(a \alpha_s)$, $\mathcal{O}(\alpha_s/(aM))$ and $\mathcal{O}(\alpha_s \Lambda_{\text{QCD}}/M)$. Further corrections would come in at $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\Lambda_{\text{QCD}}^2/M^2)$ and $\mathcal{O}(a^2 \alpha_s)$.

In the next section we list the quark and glue actions employed and give a brief discussion of our calculational methods. Two independent strategies were adopted and the results tested against each other. The wavefunction renormalization constant $Z_q$ for massless Asqtad fermions, which is used in later sections on matching coefficients, is presented in section 3. Section 4 describes one-loop self-energy corrections for NRQCD heavy quarks. Both the
heavy quark wavefunction renormalization $Z_Q$ and the mass renormalization $Z_M$ enter into the current matchings. Section 5 presents the full mixing and matching calculations for the NRQCD/Asqtad heavy-light currents. We give tables of results for a range of heavy quark masses. We add several appendices with calculational details, such as a list of Feynman rules.

II. THE LATTICE ACTIONS AND CALCULATIONAL STRATEGIES

A. The Glue Action

The tree-level tadpole and $O(a^2)$ improved glue action is given by [6],

$$S_G = -\beta \sum_{x, \mu, \nu > 0} \left\{ \frac{5}{3} \frac{P_{\mu \nu}}{u_0^4} - \frac{1}{12} \frac{R_{\mu \nu}}{u_0^8} - \frac{1}{12} \frac{R_{\nu \mu}}{u_0^8} \right\},$$  \hspace{1cm} (1)

with

$$P_{\mu \nu} = \frac{1}{N_c} \text{Re} \left( \text{Tr} \{ U_\mu(x) U_\nu(x + a_\mu) U_\mu^\dagger(x + a_\nu) U_\nu^\dagger(x) \} \right),$$  \hspace{1cm} (2)

$$R_{\mu \nu} = \frac{1}{N_c} \text{Re} \left( \text{Tr} \{ U_\mu(x) U_\mu(x + a_\mu) U_\nu(x + 2a_\mu) U_\mu^\dagger(x + a_\mu + a_\nu) U_\nu^\dagger(x + a_\nu) U_\nu^\dagger(x) \} \right).$$  \hspace{1cm} (3)

$\beta = 2N_c/g^2$ and $u_0$ is the tadpole improvement factor [7]. We carry out perturbative calculations in general covariant gauge and add a gauge fixing term,

$$S_{gf} = \frac{1}{2\xi} \sum_x \left[ \sum_{\mu} \partial_\mu (a A_\mu) \right]^2,$$  \hspace{1cm} (4)

with $\partial_\mu A_\mu(x) \equiv A_\mu(x + a_\mu/2) - A_\mu(x - a_\mu/2)$ and $\xi$ the gauge parameter. We work in both the Feynman and the Landau gauges and verify that gauge invariant quantities are independent of $\xi$. In one-loop matching calculations of current operators, the only ingredient required from the gauge action is the tree-level gauge propagator. The form of the free gauge propagator for improved gauge actions has been discussed in several articles [6, 8, 9]. We follow reference [9] and invert the momentum space free improved gauge action once and for all using Mathematica. One ends up with a closed form analytic expression for the gauge propagator which is a $4 \times 4$ matrix in Lorentz space. The interested reader is referred to Appendix A of [9] for more details.
B. The Light Quark Action

The “Asqtad” action is the most successful quark action to date for simulating light dynamical and valence quarks [10, 11, 12]. The MILC and HPQCD collaborations are using the MILC dynamical configurations to study light-light, heavy-light and heavy-heavy physics [13, 14, 15, 16, 17][2, 3, 4]. This action has been designed to drastically reduce the taste-symmetry breaking effects of conventional staggered fermions and also has all other $O(a^2)$ discretization errors removed. As discussed in references [11] [1] there are two equivalent ways to write down the Asqtad action, either in terms of four component “naive” fermions or in terms of one component “staggered” fields. We prefer to use the naive fermion approach. We will use Feynman rules for four component fields and our heavy-light currents will be constructed out of improved naive light quark fields.

Three new ingredients are necessary to go from the unimproved staggered/naive action to the Asqtad action, fattening of links, the so-called “Lepage correction” and the Naik term.

One starts from the unimproved naive action,

$$S_0 = a^4 \sum_{x} \left\{ \bar{\Psi}(x) \left[ \sum_{\mu} \gamma_{\mu} \frac{1}{a} \nabla_{\mu} + m \right] \Psi(x) \right\}, \quad (5)$$

with

$$\nabla_{\mu} \Psi(x) = \frac{1}{2 u_0} \left[ U_{\mu}(x) \Psi(x + a_{\mu}) - U_{\mu}^+(x - a_{\mu}) \Psi(x - a_{\mu}) \right], \quad (6)$$

and Hermitian Euclidean $\gamma$-matrices obeying $\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu}$. The link fattening and the Lepage terms are incorporated by replacing the link variable $U_{\mu}$ in the covariant derivative by an updated variable $V'_{\mu}$ (the corresponding covariant derivative will be denoted $\nabla'_{\mu}$).

$$V'_{\mu}(x) \equiv V_{\mu}(x) - \sum_{\rho \neq \mu} \frac{(\nabla_{\rho}^2)^2}{4} U_{\mu}(x) \quad (7)$$

$$V_{\mu}(x) \equiv \prod_{\rho \neq \mu} \left( 1 + \frac{\nabla^{\ell,(2)}_{\rho}}{4} \right)_{\text{symmetrized}} U_{\mu}(x) \quad (8)$$

$V_{\mu}(x)$ is the fattened (taste violation suppressing) link and the second term in eq.(7) is the Lepage term that removes a low momentum $O(a^2)$ error. The derivatives, $\nabla_{\mu}^{\ell}$ and $\nabla_{\nu}^{\ell,(2)}$, that act on link matrix variables are defined as $(\mu \neq \nu)$,

$$\frac{1}{u_0} \nabla_{\mu}^{\ell} U_{\nu}(x) = \frac{1}{2 u_0^3} \left[ U_{\mu}(x) U_{\nu}(x + a_{\mu}) U_{\mu}^+(x + a_{\nu}) \right]$$
\[
\frac{1}{u_0} \nabla^{\mu,2}_{\mu} U_\nu(x) = \frac{1}{u_0^3} \left[ U_\mu(x) U_\nu(x + a_\mu) U^\dagger_\mu(x + a_\nu) \right.
\]
\[
+ U^\dagger_\mu(x - a_\mu) U_\nu(x - a_\mu) U_\mu(x - a_\mu + a_\nu) \bigg] 
- \frac{2}{u_0} U_\nu(x) .
\] (10)

The Naik correction adds a three-link term to the modified covariant derivative so that the final form of the Asqtad action takes on the form

\[
S_{Asqtad} = a^4 \sum_x \left\{ \bar{\Psi}(x) \left[ \sum_\mu \gamma_\mu \frac{1}{a} \left( \nabla'_{\mu} - \frac{1}{6} \nabla^{3-link}_{\mu} \right) + m \right] \Psi(x) \right\} ,
\] (11)

with

\[
\nabla^{3-link}_{\mu} \Psi(x) = \left. \left( \nabla_{\mu} \right)^3 \right|_{\text{tadpole improved}} \Psi(x)
= \frac{1}{8} \left\{ \frac{1}{u_0^3} \left[ UUU \Psi(x + 3a_\mu) - U^\dagger U^\dagger U^\dagger \Psi(x - 3a_\mu) \right] 
- \frac{3}{u_0} \left[ U \Psi(x + a_\mu) - U^\dagger \Psi(x - a_\mu) \right] \right\} .
\] (12)

Feynman rules for the action (11) have been written down in concise form by Q. Mason [18]. We verified the rules for the one-gluon emission vertices and for the subset of two-gluon emission vertices relevant to the present calculations, namely the subset symmetric in the two gluons. These Feynman rules are summarized in the appendix.

**C. The Heavy Quark Action**

We use the NRQCD action improved through \( \mathcal{O}(1/M^2) \) and \( \mathcal{O}(a^2) \), and which also includes the leading relativistic \( \mathcal{O}(1/M^3) \) correction [19]. This action is currently being used in simulations of heavy-heavy and heavy-light systems on the MILC dynamical configurations. It has been discussed in many previous publications, and hence we will be brief here. In terms of the two-component Pauli spinor \( \phi \) one has,

\[
S_{NRQCD} = \sum_x \left\{ \phi_t^\dagger \phi_t - \phi_t^\dagger \left( 1 - \frac{a \delta H}{2} \right)_t \left( 1 - \frac{a H_0}{2n} \right)_t^n \right. 
\]
\[
\times \frac{1}{u_0} U^\dagger_t (t - 1) \left( 1 - \frac{a H_0}{2n} \right)_{t-1} \left( 1 - \frac{a \delta H}{2} \right)_{t-1} \phi_{t-1} \right\} .
\] (13)
$H_0$ is the nonrelativistic kinetic energy operator,
\[ aH_0 = -\frac{\Delta^{(2)}}{2(aM_0)^2}, \]  
and $\delta H$ includes relativistic and finite-lattice-spacing corrections,
\[ a\delta H = -c_1 \frac{(\Delta^{(2)})^2}{8(aM_0)^3} + c_2 \frac{i}{8(aM_0)^2} \left( \nabla \cdot \tilde{E} - \tilde{E} \cdot \nabla \right) 
- c_3 \frac{1}{8(aM_0)^2} \sigma \cdot (\tilde{\nabla} \times \tilde{E} - \tilde{E} \times \tilde{\nabla}) 
- c_4 \frac{1}{2(aM_0)} \sigma \cdot \tilde{B} + c_5 \frac{\Delta^{(4)}}{24(aM_0)} - c_6 \frac{(\Delta^{(2)})^2}{16n(aM_0)^2}. \]  
All derivatives are tadpole improved and,
\[ \Delta^{(2)} = \sum_{j=1}^{3} \nabla_j^{(2)}, \quad \Delta^{(4)} = \sum_{j=1}^{3} \nabla_j^{(4)} \]  
Precise definitions of the $\nabla_j^{(i)}$, $i = 2, 3, 4$, and of the improved $\tilde{E}$ and $\tilde{B}$ field operators are given, for instance, most recently in Appendix B of [1] (we note, however, a factor of 2 error in eq.(B4) of this reference, which is corrected above in eq.(9)). Feynman rules for simpler versions of NRQCD actions have appeared in [20, 21, 22]. We have generalized them to rules for the more highly improved action (13) - (15) studied here. The new Feynman rules are given in an appendix.

### D. Calculational Strategies

We have employed two independent approaches to carrying out the one-loop matching calculations with the above actions and used them as checks against each other. In one method we make extensive use of Mathematica to multiply propagators and vertices, to carry out the Dirac algebra, and, where necessary, to take derivatives with respect to external momenta. The resulting expressions are then converted into FORTRAN and fed into a FORTRAN VEGAS code for the numerical integration over internal momenta.

In many instances the Mathematica expressions can be kept quite simple and general, since details of Feynman rules are only needed at the VEGAS stage. For instance, in calculations
that do not involve any derivatives with respect to external momenta, the only information the Mathematica code requires is the formal structure of the vertices in the Asqtad and NRQCD actions. For the emission of a gluon with polarization $\mu$, the Asqtad action has the general vertex,

$$V(A_\mu) = \sum_\nu w_{\mu,\nu} \gamma_\nu$$

and the NRQCD action leads to

$$V_H(A_\mu) = wh_{\mu,0} + \sum_j wh_{\mu,j} \sigma_j$$

$\sigma_j =$ Pauli matrices (see end of Appendix A.1 for a 4 component version of (19)). The dependence of the “$w$”s on the incoming and outgoing fermion momenta is specified only in the VEGAS code through subroutines that code up the Feynman rules.

Mathematica is also very useful when one needs to take derivatives with respect to external momenta. One inserts the Feynman rule expression for the $w_{\mu,\nu}$ or $wh_{\mu,\nu}$ of interest and lets Mathematica take derivatives before setting external momenta to zero.

Our second method is based on a C++ code developed by C. Morningstar for matching calculations with NRQCD/clover quark actions and which was generously made available to us. We modified the original code to replace clover by the Asqtad light quark action and also to accommodate improved glue with non-diagonal (in Lorentz space) gauge propagators. The C++ code handles derivatives via “automatic differentiation.” A C++ class is defined which carries information not just about a function, but also about its first couple of derivatives through a Taylor series expansion. When two such class instances are multiplied, for example, derivatives of the product are calculated automatically via the chain rule as part of the definition of an overloaded multiplication operation. We believe our two methods are sufficiently independent of each other so that they provide good checks of our results.

### III. $Z_q$ FOR MASSLESS ASQTAD QUARKS

Perturbative calculations with the one-component (improved staggered) version of the Asqtad action and with unimproved Wilson glue have been performed recently in [23] for the matching of light-light bilinear and four-fermion operators. Calculations using the four-component (improved naive fermion) Asqtad action and improved glue have been carried out
for bilinear and four-fermion operators [24] and for mass renormalization [25]. For the matching of heavy-light currents the light quark wavefunction renormalization \( Z_q \) is required. This quantity has been derived already with improved glue using “twisted” boundary conditions [26]. We have repeated the calculation with a gluon mass IR regulator, the IR regulator we use throughout this article. \( Z_q \) is relevant for many other light-light and heavy-light perturbative calculations. We list our results here for the case of massless light quarks.

\[
Z_q = 1 + \alpha_s \left[ C_{IR}^q + C_q \right] + \mathcal{O}(\alpha_s^2) \tag{20}
\]

\[
C_{IR}^q = \frac{1}{3\pi} \left[ 1 + (\xi - 1) \right] \ln(a^2\lambda^2) \tag{21}
\]

where \( \lambda \) is the gluon mass. The gluon mass dependence cancels, of course, upon taking the difference between continuum and lattice one-loop coefficients. The IR finite \( C_q \) has contributions from the regular rainbow diagram, the tadpole diagram, and from the \( u_0 \) tadpole improvement procedure. We write

\[
C_q = C_{q}^{\text{reg}} + C_{q}^{\text{tad}} + C_{q}^{u_0} \tag{22}
\]

and list results in Table I for both Feynman and Landau gauges. The \( u_0 \) correction contribution is given by

\[
C_{q}^{u_0} = - \left[ 4 - \frac{1}{4} - \frac{3}{2} \right] u_0^{(2)} = - \frac{9}{4} u_0^{(2)} , \tag{23}
\]

with \( u_0^{(2)} \) defined through,

\[
u_0 \equiv 1 - \alpha_s u_0^{(2)} + \mathcal{O}(\alpha_s^2) . \tag{24}\]

The three contributions in (23) come from the fat-link, Lepage, and Naik terms respectively. The numerical value of \( u_0^{(2)} \) depends on how one defines \( u_0 \) and on the gauge action. In the case of improved glue one has \( u_0^{(2)} = 0.767 \) for the plaquette definition of \( u_0 \) and \( u_0^{(2)} = 0.750 \) for the Landau link definition. The MILC dynamical configurations were created using the plaquette \( u_0 \). It then makes sense to use plaquette \( u_0 \) in nonperturbative valence light quark propagators and in perturbative light quark self energy calculations. For the NRQCD heavy quark parts of heavy-light simulations we have adopted the Landau link \( u_0 \) and this is reflected in the heavy quark parts of the perturbative matching given below. There is no contradiction in using different definitions of \( u_0 \) in different parts of a calculation. The important thing is that the perturbation theory match the choices made in the numerical simulations. We give explicit formulas for the tadpole improvement contributions (such as
FIG. 1: Self-energy diagrams which generate $C_Q$ (or $C_q$ if a light quark line is substituted).

(23) and several others below) so readers should be able to adjust the numbers in our Tables to their $u_0$ choices.

IV. HEAVY QUARK SELF ENERGY

One-loop self-energy calculations have appeared already in many articles for a variety of NRQCD actions on the lattice. Refs. [20, 22] present results for simpler NRQCD actions with unimproved glue, Ref. [27] deals with highly improved NRQCD actions and Wilson glue, and Ref. [28] works with Symanzik improved glue but at low order in NRQCD. Here we summarize results for the NRQCD action of eqns. (13) - (15) and improved glue. We are interested in wavefunction renormalization $Z_Q$, mass renormalization $Z_M$, and (for completeness) the energy shift $E_0$.

$$Z_Q = 1 + \alpha_s \left[ C_{IR}^Q + C_Q \right] + \mathcal{O}(\alpha_s^2)$$
$$Z_M = 1 + \alpha_s C_M + \mathcal{O}(\alpha_s^2)$$
$$\alpha E_0 = \alpha_s C_{E0} + \mathcal{O}(\alpha_s^2)$$

where,

$$C_{IR}^Q = \frac{1}{3\pi} \left[ -2 + (\xi - 1) \right] \ln(a^2\lambda^2).$$

Just as for light quarks, the one-loop heavy quark self energy $\alpha_s \Sigma(p)$ gets contributions from the two diagrams in Fig. 1 and from tadpole improvement. If one defines

$$\Omega_0 = \Sigma(0)$$
$$\Omega_1 = -i \left. \frac{\partial \Sigma}{\partial p_0} \right|_{p_\mu = 0}$$
$$\Omega_2 = 2(aM) \left. \frac{\partial \Sigma}{\partial p^2} \right|_{p_\mu = 0} = (aM) \left. \frac{\partial^2 \Sigma}{\partial p_x \partial p_x} \right|_{p_\mu = 0}$$

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where \( p_\mu \) stands for the heavy quark external momentum measured in units of the inverse lattice spacing (\( \Sigma \) is also taken to be dimensionless) and \( M \) is the heavy quark pole mass, then,

\[
C_{E0} = -\Omega_0 \\
C_Q = \Omega_0 + \Omega_1 \bigg|_{\text{IR finite}} \\
C_M = \Omega_2 - \Omega_1.
\]

Results for the one-loop coefficients \( C_{XX} \) are tabulated in Table II for several values of \((aM_0)\) and \(n\). To the order we are working there is no need to distinguish between the pole mass \( M \) and the bare mass \( M_0 \) in one-loop coefficients. Since simulations are carried out at fixed \((aM_0)\) it is much more convenient to express everything in terms of the bare mass. The results in Table II were obtained using the Landau link \( u_0 \), with the explicit formulas given by,

\[
C^{u0}_{E0} = \left[ -1 - \frac{3}{(aM_0)} - \frac{c_5}{2(aM_0)} + \frac{3}{2} \left( \frac{c_1}{(aM_0)^3} + \frac{c_6}{2n(aM_0)^2} \right) \right] u_0^{(2)} \tag{35}
\]

\[
C^{u0}_M = \left[ -1 + \frac{3}{2n(aM_0)} + \frac{c_5}{3} - 3(aM_0) \left( \frac{c_1}{(aM_0)^3} + \frac{c_6}{2n(aM_0)^2} \right) \right] u_0^{(2)}. \tag{36}
\]

There are no tadpole improvement corrections to \( C_Q \) since such corrections cancel between \( \Omega_0 \) and \( \Omega_1 \), as do the contributions from the tadpole diagram. Wherever possible we give analytic formulas as functions of the coefficients \( c_i \) in the NRQCD action eq.(15), however numerical results such as in Table II are always given with \( c_i = 1 \).

\( Z_M \) and \( E_0 \) are gauge invariant, however \( Z_Q \) is both gauge variant and IR divergent. We discuss in an appendix how we handle IR divergent terms such as in (25) and isolate the IR finite contribution.

V. MATCHING OF THE HEAVY-LIGHT CURRENT

The basic formalism for perturbative matching of heavy-light currents has been developed already for NRQCD/clover currents [21] and can be taken over without any modification for the case of improved naive light quarks. Again we will be brief and refer the reader to the earlier articles for details. The following three currents are needed in the NRQCD effective theory to match the temporal component of the vector or axial vector currents to full QCD.
through $O(\alpha_s/M)$.

$$J_0^{(0)}(x) = \bar{q}(x) \Gamma_0 Q(x), \quad (37)$$
$$J_0^{(1)}(x) = -\frac{1}{2(aM_0)} \bar{q}(x) \Gamma_0 \gamma \cdot \nabla Q(x), \quad (38)$$
$$J_0^{(2)}(x) = -\frac{1}{2(aM_0)} \bar{q}(x) \gamma \cdot \nabla_0 \gamma_0 \Gamma_0 Q(x). \quad (39)$$

The $Q$ fields are four component Dirac spinors with upper two components given by the NRQCD $\phi$ field and vanishing lower components, and $\Gamma_0 = \gamma_0$ or $\gamma_5 \gamma_0$. The $\nabla$ in $J_0^{(1)}$ and $J_0^{(2)}$ is the same as in (6). The relation between matrix element of $A_0$ in full QCD to those of the effective theory currents can be written as,

$$\langle A_0 \rangle = (1 + \alpha_s \tilde{\rho}_0) \langle J_0^{(0)} \rangle +$$
$$(1 + \alpha_s \rho_1) \langle J_0^{(1),sub} \rangle$$
$$+ \alpha_s \rho_2 \langle J_0^{(2),sub} \rangle + O(\alpha_s^2, \Lambda_{QCD}^2/M^2, a^2 \alpha_s), \quad (40)$$

with

$$J_0^{(i),sub} = J_0^{(i)} - \alpha_s \zeta_0 \langle J_0^{(0)} \rangle \quad (41)$$

for $i = 1, 2$. We prefer to express things in terms of the more physical matrix elements $\langle J_0^{(i),sub} \rangle$ which have had $O(\alpha_s/(aM_0))$ power law contributions subtracted out [29]. The coefficients $\tilde{\rho}_0$, $\rho_1$ and $\rho_2$ are then given by,

$$\tilde{\rho}_0 = B_0 - \frac{1}{2}(C_q + C_Q) - \zeta_0$$
$$\rho_1 = B_1 - \frac{1}{2}(C_q + C_Q) - C_M - \zeta_0 - \zeta_1$$
$$\rho_2 = B_2 - \zeta_0 - \zeta_1 \quad (42)$$

with

$$B_0 = \frac{1}{\pi} \left[ \ln(aM_0) - \frac{1}{4} \right]$$
$$B_1 = \frac{1}{\pi} \left[ \ln(aM_0) - \frac{19}{12} \right]$$
$$B_2 = \frac{4}{\pi} \quad (43)$$

$C_q$, $C_Q$ and $C_M$ are the one-loop coefficients of self-energy corrections discussed in the two previous sections. $C_M$ enters into $\rho_1$ because we have written the $1/M$ currents in (38) and
(39) using $M_0$ rather than the pole mass. Originally these currents are defined in terms of the pole mass, the mass that is common to both full QCD and the effective lattice theory. One then, for convenience, replaces $M$ by $Z_M M_0$ and notes that to one-loop order one can set $Z_M = 1$ everywhere except in the tree-level contribution from $J_0^{(1)}$, where $Z_M = 1 + \alpha_s C_M$ must be used.

The $\zeta_{ij}$ in (41) and (42) come from the mixing between the three currents. They are calculated at one-loop from diagrams shown in Fig. 2 where one puts $J_0^{(i)}$ at the vertex and projects out the tree-level expression $\langle J_0^{(j)} \rangle_{\text{tree}}$. For instance, in the case of the $\zeta_{i0}$ that appear in the power law subtraction (41), one is taking the matrix element of $J_0^{(i)}$ and asking how much of it projects back onto $J_0^{(0)}$. Since we match at zero external momentum one has $\zeta_{10} = \zeta_{20}$. As explained in references [21, 30] $\rho_2$, or more specifically $\zeta_{02}$, includes a term that removes an $O(a \alpha_s)$ discretization error from $J_0^{(0)}$. The matching procedure is such that $O(\alpha_s/M)$ and $O(a \alpha_s)$ corrections are (and must be) made at the same time.

We mention that some care is required in calculating $\zeta_{i2}$, $i = 0, 1$. There is another dimension 4 current operator involving a time derivative acting on the light quark field $\bar{q}(x)$ which is equivalent to $J_0^{(2)}$ via equations of motion.

$$J_0^{(2)}(x) = \frac{1}{2 (a M_0)} \bar{q}(x) \nabla_0 \Gamma_0 Q(x). \quad (44)$$

When calculating $\zeta_{i2}$ one must also include contributions from $\langle J_0^{(2)} \rangle_{\text{tree}}$.

Our results for $\tilde{\rho}_0$, $\rho_1$, $\rho_2$ and $\zeta_{10}$ are given in Table III. Only $\rho_1$ has a $u_0$ correction, coming from the contribution to $\zeta_{11}$ from Fig. 3:

$$\zeta_{11}^{u_0} = u_0^{(2)}. \quad (45)$$

We use the Landau link $u_0$ in calculating $\rho_1$. Table III can be used in several ways. Should one choose not to include matrix elements of the $1/M$ currents $J_0^{(1)}$ and $J_0^{(2)}$ in one’s simulation, then matching should be done with just the first term in eq.(40). In other words, one
should not include the $-\alpha_s \zeta_{10} \langle J_0^{(0)} \rangle$ term which we have absorbed into $\langle J_0^{(1),sub} \rangle$. This way one avoids unnecessarily introducing a lattice artifact $\mathcal{O}(\alpha_s/(aM_0))$ power law term that can only be canceled by the matrix element $\langle J_0^{(1)} \rangle$. Using just the first term in (40) leaves us with $\mathcal{O}(\Lambda_{QCD}/M)$ and $\mathcal{O}(a\alpha_s)$ errors. The next step in improvement would be to use $\langle A_0 \rangle = (1 + \alpha_s \tilde{\rho}_0) \langle J_0^{(0)} \rangle + \langle J_0^{(1),sub} \rangle$. Again there are no $\mathcal{O}(\alpha_s/(aM_0))$ power law contributions and errors from the heavy-light current operators come in now at $\mathcal{O}(\alpha_s \Lambda_{QCD}/M)$ and $\mathcal{O}(a\alpha_s)$. The latter two errors are removed by going to the full expression in (40).

VI. SUMMARY

We have performed a one-loop matching of the lattice NRQCD/Asqtad heavy-light current (temporal component) to its continuum QCD counterpart, correct through $\mathcal{O}(\alpha_s/M)$ for a range in heavy quark mass. One-loop results for the NRQCD heavy quark self energy corrections and the massless Asqtad quark wavefunction renormalization are also presented. We find that all the perturbative coefficients are well behaved and none of them are particularly large.

The matching coefficients are important ingredients in lattice investigations of $B$ and $D$ meson leptonic and semileptonic decays. The $\mathcal{O}(\alpha_s^2)$ errors that remain after incorporating the one-loop results of this article will, at some point, become the dominant error in decay constant and form factor determinations from the lattice (this is already happening for $f_{B_s}$ and $f_{D_s}$ calculations [2]). Consequently, there is a need to push on to two-loop matching calculations. Members of the HPQCD collaboration are already engaged in higher order perturbative calculations with improved lattice actions [31, 32, 33]. It should be possible to extend those calculations to higher order perturbative matching of heavy-light currents as well.
Acknowledgments

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APPENDIX A: FEYNMAN RULES

In deriving Feynman rules for fermionic actions we rely heavily on methods developed by Colin Morningstar in reference [27]. The basic quantity of interest is the “ξ-function”,

\[ ξ_A^{(r)} = ξ_A^{(r)}(k', k; (q_1, ν_1, b_1), (q_2, ν_2, b_2), \ldots, (q_r, ν_r, b_r)) \] (A1)

which gives the vertex due to operator \( \hat{A} \) for a fermion to emit \( r \) gluons of polarization \( ν_1, ν_2, \ldots, ν_r \), color index \( b_1, b_2, \ldots, b_r \) and with momenta \( q_1, q_2, \ldots, q_r \). \( k' \) is the momentum of the outgoing fermion and \( k \) the momentum of the incoming fermion, \( k = k' + \sum_{i=1}^r q_i \).

Given such vertices for two operators \( \hat{A} \) and \( \hat{B} \), one can derive the emission vertices for the product operator \( \hat{A} \hat{B} \) using the convolution theorem of Fourier transforms. The general expression is given in eq.(40) of [27]. Here we reproduce only the special cases of zero-, one- and two-gluon emission vertices, which are the only ones needed in the one-loop calculations of this article.

\[ ξ_{AB}^{(0)}(k', k) = δ_{k', k} \ ξ_A^{(0)}(k, k) \ ξ_B^{(0)}(k, k) \] (A2)

\[ ξ_{AB}^{(1)}(k', k; (q, ν, b)) = ξ_A^{(1)}(k', k; (q, ν, b)) \ ξ_B^{(0)}(k, k) + \ ξ_A^{(0)}(k', k') \ ξ_B^{(1)}(k', k; (q, ν, b)) \] (A3)

\[ ξ_{AB}^{(2)}(k', k; (q_1, ν_1, b_1), (q_2, ν_2, b_2)) = ξ_A^{(2)}(k', k; (q_1, ν_1, b_1), (q_2, ν_2, b_2)) \ ξ_B^{(0)}(k, k) + \ ξ_A^{(0)}(k', k') \ ξ_B^{(2)}(k', k; (q_1, ν_1, b_1), (q_2, ν_2, b_2)) + \ ξ_A^{(1)}(k', k' + q_1; (q_1, ν_1, b_1)) \ ξ_B^{(1)}(k - q_2, k; (q_2, ν_2, b_2)) \] (A4)

These equations are extremely useful and also very general. The operators \( \hat{A} \) or \( \hat{B} \) can be simple operators, such as the link variable \( U_μ \) or \( \nabla_μ \). They can also stand for more
complicated combinations such as \( \tilde{F}_{\mu\nu} \) all the way to terms like \( (1 - \frac{a \delta H}{2}) \) or \( (1 - \frac{a H_0}{2m})^n U_t^\dagger (t - 1)^n \) that appear in the NRQCD action.

Reference [27] lists the \( \xi_{\hat{A}}^{(r)} \) for \( r = 0, 1, 2, 3 \), for a wide range of elementary operators, \( U_\mu \), \( \nabla_\mu \), \( \Delta^{(2)} \), \( F_{\mu\nu} \) and for improved versions of these operators. Using the convolution equations (A2) - (A4), one can then build up the emission vertices for composite operators such as \( \nabla \cdot \mathbf{E} \) or \( (\Delta^{(2)})^2 \) or a string of \( U \) matrices. This is done, for instance, in the \( C^{++} \) code (method 2 of section IID). We have also used these equations to obtain Feynman rules once and for all by hand. They are summarized in the following two subsections.

1. Feynman Rules for the NRQCD Action

In this subsection we give the Feynman rules for the NRQCD action (13) - (15). We will do so in the two component language which is appropriate for self-energy and heavy-heavy current matching calculations. At the end of this subsection we mention the trivial generalization to a four component language which is more useful for heavy-light current matchings.

In order to streamline the notation and also to introduce spin information, we re-express the relevant \( \xi_{\hat{A}}^{(r)} \) as follows.

\[
\xi_{\hat{A}}^{(r)}(k', k; (q_1, \nu_1, b_1), (q_2, \nu_2, b_2), \ldots, (q_r, \nu_r, b_r)) \rightarrow \left[ \hat{A} \right]^{(r)\nu_1\ldots\nu_r}_{s} (k', k) T_{b_1} \ldots T_{b_r} \tag{A5}
\]

The gluon momenta \( q_1, \ldots, q_r \) are left implicit in this notation. The labels \( s = 0, 1, 2, 3 \) refer to vertices with spin structure \( (I_2, \sigma_x, \sigma_y, \sigma_z) \) respectively. We also rewrite the NRQCD action in terms of the following operators:

\[
O_1 = \left( 1 - \frac{a \delta H}{2} \right) \tag{A6}
\]
\[
O_2 = \left( 1 - \frac{a H_0}{2m} \right)^n \tag{A7}
\]
\[
O_{2U_2} = O_2 U_t^\dagger O_2 \tag{A8}
\]

and schematically one has,

\[
\mathcal{L}_{NRQCD} = \phi^\dagger (1 - O_1 O_{2U_2} O_1) \phi. \tag{A9}
\]

Feynman rules are obtained by determining,

\[
[O_1 O_{2U_2} O_1]^{(r)\nu_1\ldots\nu_r}_{s} (k', k) \tag{A10}
\]
where we are using the notation of (A5).

**Tree-level:**

One has

\[
[O_{2}^{(0)}]_{s}(k', k) = \delta_{s,0} \delta_{k',k} F(k)^n
\]

(A11)

\[
[O_{1}^{(0)}]_{s}(k', k) = \delta_{s,0} \delta_{k',k} F_{1}(k)
\]

(A12)

with

\[
F(k) = \left[1 - \frac{1}{n(aM_{0})} \sum_{j=1}^{3} \sin^{2}\left(\frac{k_{j}}{2}\right)\right]
\]

(A13)

\[
F_{1}(k) = 1 - \frac{1}{3(aM_{0})} \sum_{j=1}^{3} \sin^{4}\left(\frac{k_{j}}{2}\right) + \left[\frac{c_{1}}{(aM_{0})^{3}} + \frac{c_{6}}{2n(aM_{0})^{2}}\right] \left[\sum_{j=1}^{3} \sin^{2}\left(\frac{k_{j}}{2}\right)\right]^{2}
\]

(A14)

and

\[
[O_{2W2}]^{(0)}_{s}(k', k) = \delta_{s,0} \delta_{k',k} e^{-ik_{0}} F(k)^{2n}
\]

(A15)

The free heavy quark propagator is given by,

\[
G_{H}^{(0)}(k) = \left\{1 - [O_{1}O_{2W2}O_{1}]^{(0)}_{s=0}(k, k)\right\}^{-1}
= \left\{1 - e^{-ik_{0}} F(k)^{2n} F_{1}(k)^{2}\right\}^{-1}.
\]

(A16)

**One-gluon emission:**

Using the notation of this appendix, the coefficients \(wh_{\mu,0}\) of eq.(19) are defined (up to a color T-matrix which we omit) as,

\[
wh_{\mu,0} = [O_{1}O_{2W2}O_{1}]^{(1)\mu}_{s=0}(k', k), \quad wh_{\mu,j} \sigma_{j} = [O_{1}O_{2W2}O_{1}]^{(1)\mu}_{s=j}(k', k)
\]

(A17)

Repeated use of (A3) leads to,

\[
[O_{1}O_{2W2}O_{1}]^{(1)\mu}_{s}(k', k) = [O_{1}]^{(1)\mu}_{s}(k', k) \left\{[O_{1}O_{2W2}]^{(0)}_{s=0}(k', k') + [O_{2W2}O_{1}]^{(0)}_{s=0}(k, k)\right\} +

[O_{1}]^{(0)}_{s=0}(k', k') [O_{2W2}]^{(1)\mu}_{s}(k', k) [O_{1}]^{(0)}_{s=0}(k, k)
\]

(A18)

\[
= [O_{1}]^{(1)\mu}_{s}(k', k) \left(e^{-ik_{0}} F(k)^{2n} F_{1}(k') + e^{-ik_{0}} F(k)^{2n} F_{1}(k)\right) +

F_{1}(k') F_{1}(k) [O_{2W2}]^{(1)\mu}_{s}(k', k).
\]

(A19)

Going further down the chain one has,

\[
[O_{2W2}]^{(1)\mu=0}_{s}(k', k) = \delta_{s,0} F(k')^{n} F(k)^{n} \left(-i e^{-i(k_{0}+k_{0})/2}\right)
\]

(A20)

\[
[O_{2W2}]^{(1)\mu=j}_{s}(k', k) = [O_{2}]^{(1)j}_{s}(k', k) \left(e^{-ik_{0}} F(k')^{n} + e^{-ik_{0}} F(k)^{n}\right)
\]

(A21)
with \( k^\pm \equiv \frac{k+k'}{2} \)

\[
\begin{align*}
[O_2]^{(1)j}_s (k', k) &= \delta_{s,0} \frac{-1}{2n(aM_0)} \sin(k^+_j) F_2(k', k, n) \quad (A22) \\
F_2(k', k, n) &= \begin{cases} 
1 & n = 1 \\
\sum_{l=0}^{n-1} F(k')^{n-l-1} F(k)^l & n > 1
\end{cases} \quad (A23)
\end{align*}
\]

and

\[
[O_1]^{(1)\mu}_s (k', k) = -\frac{a}{2} \sum_{i=1}^{6} [\delta H^{(1)\mu}_i]_s (k', k). \quad (A24)
\]

The \([\delta H^{(1)\mu}_i]_s (k', k)\) can be built up from the Fourier transforms listed in [27] using again (A3). One finds,

\[
-\frac{a}{2} \left[ \delta H_1 + \delta H_6 \right]^{(1)j}_s (k', k) = \delta_{s,0} \left( \frac{c_1}{2(aM_0)^3} + \frac{c_6}{4n(aM_0)^2} \right) \sin(k^+_j) \left[ \sum_{l=1}^{3} \sin^2\left(\frac{k_{i}^l}{2}\right) + \sum_{l=1}^{3} \sin^2\left(\frac{k'_{i}^l}{2}\right) \right] \quad (A25)
\]

\[
-\frac{a}{2} \left[ \delta H_2 \right]^{(1)0}_s (k', k) = \delta_{s,0} \frac{-ic_2}{16(aM_0)^2} \sum_{i=1}^{3} \left\{ \sin(2k_{i}^{-}) \cos(k_{0}^{-}) \left[ \sin(k_{i}^+) - \sin(k_{i}) \right] \eta_{0} \right\} \quad (A26)
\]

\[
-\frac{a}{2} \left[ \delta H_2 \right]^{(1)j}_s (k', k) = \delta_{s,0} \frac{ic_2}{16(aM_0)^2} \left\{ \sin(2k_{0}^{-}) \cos(k_{j}^{-}) \left[ \sin(k_{i}^+) - \sin(k_{i}) \right] \eta_{j0} \right\} \quad (A27)
\]

\[
-\frac{a}{2} \left[ \delta H_3 \right]^{(1)0}_{s=i} (k', k) = \frac{c_3}{16(aM_0)^2} \sigma_i \sum_{j,l=1}^{3} \epsilon_{ijl} \cos(k_{i}^{-}) \left[ ss(k_{j}^+) + ss(k_{j}) \right] \sin(2k_{i}^{-}) \eta_{0} \quad (A28)
\]

\[
-\frac{a}{2} \left[ \delta H_3 \right]^{(1)j}_{s=i} (k', k) = \frac{c_3}{16(aM_0)^2} \sigma_i \sum_{l=1}^{3} \epsilon_{ijl} \cos(k_{j}^{-}) \left[ ss(k_{i}^+) + ss(k_{i}) \right] \sin(2k_{0}^{-}) \eta_{j0} \quad (A29)
\]

\[
-\frac{a}{2} \left[ \delta H_4 \right]^{(1)j}_{s=i} (k', k) = \frac{-ic_4}{4(aM_0)} \sigma_i \sum_{l=1}^{3} \epsilon_{ijl} \cos(k_{j}^{-}) \sin(2k_{i}^{-}) \eta_{jl} \quad (A30)
\]
\[
- \frac{a}{2} \left[ \delta H_3 \right]^{(1)}_s (k', k) = \delta_{s,0} \frac{c_5}{6(aM_0)} \left[ -\sin(k_j^+) + \frac{1}{2} \sin(2k_j^+) \cos(k_j^-) \right]
\]

where \( \eta_{\mu\nu} = 1 \) for unimproved \( \mathbf{E} \) and \( \mathbf{B} \) fields and \( \eta_{\mu\nu} = \frac{1}{3} [5 - \cos(2k^-_\mu) - \cos(2k^-_\nu)] \) for improved fields. The function \( ss(k_j) = \sin(k_j) \) for unimproved \( \nabla_j \) and \( ss(k_j) = \frac{1}{3} \sin(k_j) - \frac{1}{6} \sin(2k_j) \) for the improved \( \tilde{\nabla}_j \).

We have coded up the one-gluon emission vertices, \( [O_{2\nu}]^{(1)}_s (k', k), [O_1]^{(1)}_s (k', k) \), etc. as functions of arbitrary incoming and outgoing momenta, \( k \) and \( k' \). These routines are called repeatedly by the VEGAS code when calculating the integrand of the one-loop diagrams. Similar expressions are also fed into Mathematica routines, whenever one needs to take derivatives with respect to external momenta.

**Two-gluon emission:**

We will restrict the discussion to two-gluon emission relevant for the tadpole diagram, for which only a special combination of momenta are required: \( (k', k, q_1, q_2) \to (k, k, q, -q) \).

One is also interested only in the spin singlet contribution. Applying (A4) to this case one has,

\[
[O_1 O_{2\nu} O_1]^{(2)\mu,\nu}_{s=0} (k, k) = [O_1]^{(2)\mu,\nu}_{s=0} (k, k) 2 e^{-ik_0} F(k)^{2n} F_1(k) + [O_{2\nu}]^{(2)\mu,\nu}_{s=0} (k, k) F_1(k)^2 +
F_1(k) [O_{2\nu}]^{(1)}_{s=0} (k, k + q) [O_1]^{(1)\nu}_{s=0} (k + q, k) + 
[O_1]^{(1)\mu}_{s=0} (k, k + q) [O_{2\nu}]^{(1)\nu}_{s=0} (k + q, k) F_1(k) + 
\left[ [O_1]^{(1)\mu}_{s=1} (k, k + q) [O_1]^{(1)\nu}_{s=2} (k + q, k) \right]_{s=0} e^{-i(k_0+q_0)} F(k + q)^{2n} .
\]

(A32)

The two new ingredients are \( [O_1]^{(2)\mu,\nu}_{s=0} (k, k, q, -q) \) and \( [O_{2\nu}]^{(2)\mu,\nu}_{s=0} (k, k, q, -q) \).

\[
[O_1]^{(2)\mu,\nu}_{s=0} (k, k, q, -q) = \left( \frac{c_1}{2(aM_0)^3} + \frac{c_6}{4n(aM_0)^2} \right) \delta_{\mu,i} \delta_{\nu,j} 
\left[ \delta_{ij} \cos(k_j) \sum_{i=1}^{3} \sin^2(k_i) + \frac{1}{2} \sin(k_i + \frac{q_i}{2}) \sin(k_j + \frac{q_j}{2}) \right]
+ \frac{ic_2}{16(aM_0)^2} \left[ (\delta_{\mu,j} \delta_{\nu,0} + \delta_{\mu,0} \delta_{\nu,j}) \cos(k_j + \frac{q_j}{2}) \sin(q_0) \cos(\frac{q_j}{2}) \eta_{0j} 
- \delta_{\mu,j} \delta_{\nu,j} 2 \cos(k_j + \frac{q_j}{2}) \sin(q_0) \cos(\frac{q_j}{2}) \eta_{0j} \right]
+ \frac{-c_5}{12(aM_0)} \delta_{\mu,j} \delta_{\nu,j} \left[ \cos(k_j) - \cos(2k_j) \cos^2(\frac{q_j}{2}) \right]
\]

(A33)
where \( \eta_{j0} \) is the same as the \( \eta_{\mu\nu} \) defined after eq.(A31) with \( 2k_\mu^- \to q_\mu. \)

\[
[O_{2\nu}]^{(2)\mu\nu}_{s=0}(k, k, q, -q) = 2 e^{-ik_0} F(k)^n [O_{2\nu}]^{(2)\mu\nu}_{s=0}(k, k, q, -q) - \eta_{\mu0} \delta_{\nu0} \frac{1}{2} e^{-ik_0} F(k)^{2n}
- i F(k)^n e^{-i(k_0 + \frac{q_0}{2})} \left[ \delta_{\mu0} [O_{2\nu}]^{(1)\mu}_{s=0}(k + q, k) + \delta_{\nu0} [O_{2\nu}]^{(1)\mu}_{s=0}(k, k + q) \right]
+ e^{-i(k_0 + q_0)} [O_{2\nu}]^{(1)\mu}_{s=0}(k, k + q) [O_{2\nu}]^{(1)\nu}_{s=0}(k + q, k)
\]

(A34)

where

\[
[O_{2\nu}]^{(2)\mu\nu}_{s=0}(k, k, q, -q) = \delta_{\mu,i} \delta_{\nu,j} \left[ \delta_{ij} n F(k)^{n-1} \frac{-1}{4n(aM_0)} \cos(k_j) + \right.
\left. \frac{1}{2n(aM_0)} \right]^2 \sin(k_i + \frac{q_i}{2}) \sin(k_j + \frac{q_j}{2}) F_3(k, k + q, n)
\]

(A35)

with

\[
F_3(k, k + q, n) = \begin{cases} 0 & n = 1 \\ \sum_{l=1}^{n-1} l F(k)^{l-1} F(k + q)^{n-l-1} & n > 1 \end{cases}
\]

(A36)

4 component formulas :

So far we have written down Feynman rules corresponding to the NRQCD action (13) which is given in terms of 2 component Pauli spinors. One can equivalently write rules appropriate for the \( Q \) fields of (37) - (39). One needs to make the replacement

\[
(I_2, \sigma) \to (I_4, \Sigma)
\]

(A37)

with \( \Sigma_i = \text{diag}(\sigma_i, \sigma_i) \) denoting \( 4 \times 4 \) matrices. Furthermore, the above gluon emission vertices should be multiplied by \( \frac{I + 2aM_0}{2} \).

2. Feynman Rules for the Asqtad Action

As mentioned already in the main text, Feynman rules for the Asqtad action for one-gluon emission and the most general two-gluon emission vertices have been written down by Q. Mason [18]. In this appendix we list just those rules used in our calculations.

Tree-level :

The free Asqtad propagator is given by,

\[
G_q^{(0)}(k) = \left\{ i \sum_\mu \gamma_\mu \sin(k_\mu) \left[ 1 + \frac{1}{6} \sin^2(k_\mu) \right] + (am) \right\}^{-1}.
\]

(A38)
One-gluon emission:
Carrying out the derivatives and symmetrizations of eq.(8) one ends up with a fattened link consisting of [1-link, 3-link, nonplanar 5-link, nonplanar 7-link] staples with weights [1/8, 1/16, 1/64, 1/384] respectively [12]. The Lepage term leads to a 1-link contribution to \( V'_\mu \) with weight 3/8 and a planar 5-link contribution with weight -1/16. Adding to this the Naik term one ends up with,

\[
[\text{Asqtad}]^{(1)\mu} = [\text{fat link}]^{(1)\mu} + [\text{Lepage}]^{(1)\mu} + [\text{Naik}]^{(1)\mu}
\]  
(A39)

with (we again define \( k^\pm = \frac{k+q'}{2} \))

\[
[\text{fat link}]^{(1)\mu}(k', k) = -i \left\{ \gamma_\mu \cos(k^+_\mu) \cos^2(k^-_\nu) \cos^2(k^-_\rho) \cos^2(k^-_\sigma) + \sum_{\nu \neq \mu} \gamma_\nu \cos(k^+_\nu) \sin(k^-_\nu) \sin(k^-_\nu) \left[ \frac{1}{3} + \frac{1}{6} \left( \cos^2(k^-_\sigma) + \cos^2(k^-_\rho) \right) \right] \right\}
\]  
(A40)

\[
[\text{Lepage}]^{(1)\mu}(k', k) = -i \left\{ \gamma_\mu \cos(k^+_\mu) \frac{1}{4} \sum_{\nu \neq \mu} \sin^2(2k^-_\nu) - \sum_{\nu \neq \mu} \gamma_\nu \cos(k^+_\nu) \sin(k^-_\nu) \sin(k^-_\nu) \cos^2(k^-_\mu) \right\}
\]  
(A41)

\[
[\text{Naik}]^{(1)\mu}(k', k) = -i \left\{ \gamma_\mu \cos(k^+_\mu) \frac{1}{6} \left[ \cos^2(k^+_\mu) - \cos^2(k^-_\mu) \left( 1 - 4 \sin^2(k^+_\mu) \right) \right] \right\}.
\]  
(A42)

In the above formulas, the Lorentz indices \( \mu, \nu, \rho, \sigma \) are all taken to be different from each other. One can read off from these equations the coefficients \( w_{\mu,\nu} \) of eq.(18).

Two-gluon emission:
We will again present only those two-gluon emission vertices specific to the tadpole diagram where \((k', k, q_1, q_2) \rightarrow (k, k, q, -q)\). One has

\[
[\text{Asqtad}]^{(2)\mu,\nu} = i \left\{ \gamma_\mu \sin(k^-_\mu) \left[ 1 + \frac{1}{24} \left( [1 - 4 \cos^2(k^-_\mu)][1 - 4 \cos^2(k^-_\mu)] + 3 \right) \right] + \sum_{\nu \neq \mu} \gamma_\nu \sin(k^-_\nu) \sin^2\left(\frac{q^-_\nu}{2}\right) 2 \sin^2\left(\frac{q^-_\mu}{2}\right) \right\}
\]  
(A43)

\[
[\text{Asqtad}]^{(2)\mu \neq \nu} = i \left\{ \gamma_\mu \sin(k^-_\mu) \left[ -\frac{1}{6} \sin\left(\frac{q^-_\mu}{2}\right) \sin\left(\frac{q^-_\nu}{2}\right) \left( 2 + \cos^2\left(\frac{q^-_\rho}{2}\right) + \cos^2\left(\frac{q^-_\sigma}{2}\right) \right) + 2 \cos\left(\frac{q^-_\rho}{2}\right) \cos\left(\frac{q^-_\sigma}{2}\right) + \frac{1}{2} \sin\left(\frac{q^-_\mu}{2}\right) \sin\left(\frac{q^-_\nu}{2}\right) \sin(q^-_\nu) \right] \right\}
\]
\[ - \sum_{\rho \neq \mu, \nu} \gamma_\rho \sin(k_\rho) \sin^2\left(\frac{q_\rho}{2}\right) \frac{1}{6} \sin\left(\frac{q_\mu}{2}\right) \sin\left(\frac{q_\nu}{2}\right) \left(2 + \cos^2\left(\frac{q_\sigma}{2}\right)\right) \]
\[ + \left\{ \mu \rightleftharpoons \nu \right\} \]  

(A44)

**APPENDIX B: INFRARED SUBTRACTIONS**

At intermediate stages of the matching procedure several of the lattice one-loop integrals are IR divergent. We use a gluon mass \( \lambda \) to regulate those integrals and extract the IR finite contributions in the following way (similar approaches can be found, for instance, in [34][9]). If \( F_L(k, \lambda, M_0) \) is the relevant lattice integrand, one can write,

\[ \int_k F_L(k, \lambda, M_0) = \int_k [F_L(k, \lambda, M_0) - F_{\text{sub}}(k, \Lambda, \lambda, M_0)] + R(\Lambda, \lambda, M_0), \]  

with

\[ R(\Lambda, \lambda, M_0) = \int_k F_{\text{sub}}(k, \Lambda, \lambda, M_0) \]  

(B2)

and

\[ \int_k \equiv g^2 \frac{A}{3} \int \frac{d^4k}{(2\pi)^4}. \]  

(B3)

To be consistent with the previous Appendix, the momentum \( k \) is always measured in units of the inverse lattice spacing. \( \Lambda, \lambda \) and \( M_0 \) however, have dimensions of energy. \( F_{\text{sub}}(k, \Lambda, \lambda, M_0) \) is constructed to have the same IR behavior as the lattice integrand of interest, so that the first integral on the RHS of (B1) is IR finite. \( \Lambda \) is a (ultraviolet) cutoff imposed on the subtraction term so that \( F_{\text{sub}}(k, \Lambda, \lambda, M_0) = 0 \) for \( k^2 > a^2 \Lambda^2 \). We take \( \Lambda \leq \frac{\pi}{a} \) and vary \( \Lambda \) to check that the dependence on this cutoff cancels between the two terms in (B1). Another criterion for choosing a suitable \( F_{\text{sub}} \) is that the integral in (B2) be easy to do and that one be able to extract the IR divergent piece, \( R^{IR} \), analytically:

\[ R(\Lambda, \lambda, M_0) = R^{\text{finite}}(\Lambda, M_0) + R^{IR}(\Lambda, \lambda, M_0). \]  

(B4)

Otherwise there is a lot of freedom in choosing \( F_{\text{sub}} \). We give one example for the logarithmic IR divergence in the NRQCD wavefunction renormalization \( Z_Q \)

The IR divergence in \( Z_Q \) comes from \( \Omega_1 \) defined in (30). A convenient choice for \( F_{\text{sub}} \) in this case is the corresponding quantity in continuum NRQCD, which must have the same IR
behavior as the lattice theory. One can easily convince oneself that the full continuum $\Sigma$ is actually not required. The IR divergence resides in the contribution to $\Sigma$ from a temporal gluon in the one-loop rainbow diagram of Fig. 1, the only contribution that survives into the static limit. Setting $\mu = \nu = 0$ in the gluon propagator one finds,

$$F_{sub}^{Q1}(k, \Lambda, \lambda, M_0) = \begin{cases} \frac{w^2 - k^2}{[k_0^2 + w^2]^2} \left( \frac{1}{k^2 + (a\lambda)^2} \right) \left[ 1 + (\xi - 1) \frac{k_0^2}{k^2} \right] & k^2 \leq a^2\Lambda^2 \\ 0 & k^2 > a^2\Lambda^2 \end{cases}$$

(B5)

with,

$$w \equiv \frac{k_0^2}{2(aM_0)}.$$  

(B6)

Using 4D spherical coordinates the angular integrals in (B2) can be carried out analytically (we use MAPLE for this) and one is left with a 1D radial integral,

$$R(\Lambda, \lambda, M_0) = \frac{g^2}{4\pi} \frac{1}{3\pi} \int_0^{a^2\Lambda^2} \frac{dy}{(y + (a\lambda)^2)} \left\{ 2 \left[ \frac{2}{2\sqrt{y}b + 1} \right]^{3/2} + \frac{2(yb + 3\sqrt{y}b + 1 - [2\sqrt{y}b + 1]^{3/2})}{yb \left[ 2\sqrt{y}b + 1 \right]^{3/2}} \right\}$$

(B7)

where,

$$b \equiv \frac{1}{4(aM_0)^2}.$$  

(B8)

The IR divergent contribution to $R(\Lambda, \lambda, M_0)$ can be isolated straightforwardly and one has,

$$R_{finite}^{finite} = \frac{\alpha_s}{3\pi} \left( \int_0^{a^2\Lambda^2} \frac{dy}{(y + (a\lambda)^2)} \left\{ \left[ \frac{2}{2\sqrt{y}b + 1} \right]^{3/2} - 2 \right\} + (\xi - 1) \left\{ \frac{2yb + 3\sqrt{y}b + 1 - [2\sqrt{y}b + 1]^{3/2}}{yb \left[ 2\sqrt{y}b + 1 \right]^{3/2}} + 1 \right\} \right)_{a\lambda \to 0}$$

(B9)

$$R^{IR} = \frac{\alpha_s}{3\pi} \int_0^{a^2\Lambda^2} \frac{dy}{(y + (a\lambda)^2)} [2 - (\xi - 1)]$$

$$= \frac{\alpha_s}{3\pi} [2 - (\xi - 1)] \ln \frac{\Lambda^2}{\lambda^2}.$$  

(B10)

The $\Lambda$ dependence is canceled by the first term on the RHS of (B1). The gluon mass dependence is the same as in continuum QCD.
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TABLE I: Coefficient of the one-loop contribution to $Z_q$ for massless Asqtad quarks. The plaquette definition of $u_0$ is used to implement tadpole improvement. $\xi$ is the gauge parameter. Where no errors are indicated, they are of order one or less in the last digit.

| $\xi$ | $C_{q}^{\text{reg}}$ | $C_{q}^{\text{tad}}$ | $C_{q}^{u_0}$ | $C_{q}$ |
|-------|----------------------|----------------------|--------------|--------|
| 1     | -0.798(3)            | 1.600                | -1.726       | -0.924(3) |
| 0     | 0.000(3)             | 1.310                | -1.726       | -0.416(3) |

TABLE II: Coefficient of one-loop contributions to the heavy quark self-energy. The Landau link definition of $u_0$ is used to implement tadpole improvement. $\xi$ is the gauge parameter. Where no errors are indicated, they are of order one or less in the last digit.

| $\alpha M_0$ | $n$ | $C_Q(\xi = 1)$ | $C_Q(\xi = 0)$ | $C_M$  | $C_{E0}$ |
|--------------|----|----------------|----------------|--------|---------|
| 5.40         | 1  | -0.016(3)      | 0.493(3)       | -0.026(4) | 0.917   |
| 4.00         | 2  | -0.142(3)      | 0.368(3)       | 0.082(4)  | 0.850   |
| 2.80         | 2  | -0.338(3)      | 0.171(3)       | 0.235(4)  | 0.767   |
| 2.10         | 4  | -0.539(3)      | -0.030(3)      | 0.387(4)  | 0.666   |
| 1.95         | 2  | -0.611(3)      | -0.105(3)      | 0.421(4)  | 0.689   |
| 1.95         | 4  | -0.603(3)      | -0.094(3)      | 0.428(4)  | 0.646   |
| 1.60         | 4  | -0.797(3)      | -0.288(3)      | 0.543(4)  | 0.609   |
| 1.20         | 6  | -1.184(3)      | -0.678(3)      | 0.732(4)  | 0.609   |
| 1.00         | 6  | -1.545(3)      | -1.037(3)      | 0.859(4)  | 0.758   |

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TABLE III: Matching Coefficients for the temporal components of the heavy-light axial and vector currents. The Landau link $u_0$ was used to evaluate $\rho_1$. Where no errors are indicated, they are of order one or less in the last digit.

| $aM_0$ | $n$ | $\tilde{\rho}_0$ | $\rho_1$ | $\rho_2$ | $\zeta_{10}$ |
|--------|-----|-----------------|----------|----------|--------------|
| 5.40   | 1   | 0.240(2)        | 0.509(4) | −0.155(6)| −0.095       |
| 4.00   | 2   | 0.149(2)        | 0.445(4) | −0.134(6)| −0.123       |
| 2.80   | 2   | 0.043(2)        | 0.382(4) | −0.072(6)| −0.166       |
| 2.10   | 4   | −0.044(2)       | 0.346(4) | 0.002(6) | −0.210       |
| 1.95   | 2   | −0.058(3)       | 0.343(4) | 0.008(6) | −0.222       |
| 1.95   | 4   | −0.065(3)       | 0.342(4) | 0.018(6) | −0.219       |
| 1.60   | 4   | −0.117(2)       | 0.343(4) | 0.061(6) | −0.259       |
| 1.20   | 6   | −0.175(2)       | 0.385(4) | 0.107(6) | −0.329       |
| 1.00   | 6   | −0.186(2)       | 0.463(4) | 0.119(6) | −0.378       |