COMPLETE MONOTONICITY OF A FAMILY OF FUNCTIONS INVOLVING THE TRI- AND TETRA-GAMMA FUNCTIONS

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Abstract. The psi function \( \psi(x) \) is defined by \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \) and \( \psi^{(i)}(x) \) for \( i \in \mathbb{N} \) denote polygamma functions, where \( \Gamma(x) \) is the gamma function. In this paper, we prove that the function

\[
|\psi'(x)|^2 + \psi''(x) - \frac{x^2 + \lambda x + 12}{12x^4(x+1)^2}
\]

is completely monotonic on \( (0, \infty) \) if and only if \( \lambda \leq 0 \), and so is its negative if and only if \( \lambda \geq 4 \). From this, some inequalities are refined and sharpened.

1. Introduction

We recall from [8, Chapter XIII] and [16, Chapter IV] that a function \( f \) is said to be completely monotonic on an interval \( I \) if \( f \) has derivatives of all orders on \( I \) and

\[
0 \leq (-1)^nf^{(n)}(x) < \infty \quad (1)
\]

for \( x \in I \) and \( n \geq 0 \). The class of completely monotonic functions may be characterized by the famous Bernstein-Widder Theorem [16, p. 161, Theorem 12b] which reads that a necessary and sufficient condition that \( f(x) \) should be completely monotonic for \( 0 < x < \infty \) is that

\[
f(x) = \int_0^\infty e^{-xt} \, d\alpha(t), \quad (2)
\]

where \( \alpha(t) \) is non-decreasing and the integral converges for \( 0 < x < \infty \).

We also recall that the classical Euler’s gamma function \( \Gamma(x) \) is defined by

\[
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt, \quad x > 0. \quad (3)
\]

The logarithmic derivative of \( \Gamma(x) \), denoted by \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \), is called the psi or di-gamma function, and the derivatives \( \psi^{(i)}(x) \) for \( i \in \mathbb{N} \) are respectively called the polygamma functions. In particular, the functions \( \psi'(x) \) and \( \psi''(x) \) are called the tri- and tetra-gamma functions.

In [2, p. 208, (4.39)], it was established that the one-sided inequality

\[
|\psi'(x)|^2 + \psi''(x) > \frac{p(x)}{900x^4(x+1)^{10}} \quad (4)
\]

holds for \( x > 0 \), where

\[
p(x) = 75x^{10} + 900x^9 + 4840x^8 + 15370x^7 + 31865x^6 + 45050x^5
+ 44101x^4 + 29700x^3 + 13290x^2 + 3600x + 450. \quad (5)
\]
By the same technique as in [2, p. 208, (4.39)], we can easily obtain
\[
[\psi'(x)]^2 + \psi''(x) < \frac{36 + 180x + 408x^2 + 504x^3 + 352x^4 + 132x^5 + 21x^6}{36x^4(1 + x)^6} \tag{6}
\]
for \(x > 0\).

In [17], the function
\[
[\psi'(x)]^2 + \psi''(x) - \frac{p(x)}{900x^4(x + 1)^{10}} \tag{7}
\]
was proved to be completely monotonic on \((0, \infty)\).

In [18], the following results were obtained:

1. The two-sided inequality
\[
\frac{x^2 + 12}{12x^4(x + 1)^2} < [\psi'(x)]^2 + \psi''(x) < \frac{x + 12}{12x^4(x + 1)} \tag{8}
\]
holds on \((0, \infty)\).
2. The functions
\[
f(x) = [\psi'(x)]^2 + \psi''(x) - \frac{x^2 + 12}{12x^4(x + 1)^2} \tag{9}
\]
and
\[
g(x) = \frac{x + 12}{12x^4(x + 1)} - \left\{[\psi'(x)]^2 + \psi''(x)\right\} \tag{10}
\]
are completely monotonic on \((0, \infty)\).

The inequalities (4) and (6) and the ones in (8) are not included each other.

For more information on results related to the function \([\psi'(x)]^2 + \psi''(x)\), please refer to [3, 4, 5, 6, 11, 12, 13, 15, 17], the expository and survey article [10, 14] and the literature listed therein.

The aim of this paper is to establish necessary and sufficient conditions on \(\lambda \in \mathbb{R}\) for the function
\[
f_\lambda(x) = [\psi'(x)]^2 + \psi''(x) - \frac{x^2 + \lambda x + 12}{12x^4(x + 1)^2} \tag{11}
\]
to be completely monotonic on \((0, \infty)\).

Our main results may be stated as the following theorem.

**Theorem 1.** Let \(\lambda \in \mathbb{R}\).

1. The function \(f_\lambda(x)\) defined by (11) is completely monotonic on \((0, \infty)\) if and only if \(\lambda \leq 0\);
2. The function \(-f_\lambda(x)\) is completely monotonic on \((0, \infty)\) if and only if \(\lambda \geq 4\);
3. The double inequality
\[
\frac{x^2 + \mu x + 12}{12x^4(x + 1)^2} < [\psi'(x)]^2 + \psi''(x) < \frac{x^2 + \nu x + 12}{12x^4(x + 1)^2} \tag{12}
\]
holds on \((0, \infty)\) if and only if \(\mu \leq 0\) and \(\nu \geq 4\).

In next section we supply several proofs for Theorem 1. In the final section we derive some corollaries and pose a double inequality of \([\psi'(x)]^2 + \psi''(x)\) on \((0, \infty)\).
2. Proofs of Theorem 1

In this section we provide several proofs for Theorem 1 by different approaches.

First proof of Theorem 1. By the recursion formula

\[ \psi^{(n-1)}(x+1) = \psi^{(n-1)}(x) + \frac{(-1)^{n-1}(n-1)!}{x^n} \]  \hspace{1cm} (13)

for \( x > 0 \) and \( n \in \mathbb{N} \), see [1, pp. 258 and 260, 6.3.5 and 6.4.6], we have

\[
f_{\lambda}(x) - f_{\lambda}(x + 1) = \left[ \psi'(x) - \psi'(x + 1) \right] \left[ \psi'(x) + \psi'(x + 1) \right]
\]

\[ + \left[ \psi''(x) - \psi''(x + 1) \right] - \left[ \frac{x^2 + \lambda x + 12}{12x^4(x+1)^2} - \frac{(x+1)^2 + \lambda(x+1) + 12}{12(x+1)^4(x+2)^2} \right]. \]

\[ = \frac{1}{x^2} \left[ 2\psi'(x) - \frac{1}{x^2} \right] - \frac{2}{x^3} \left[ \frac{x^2 + \lambda x + 12}{12x^4(x+1)^2} - \frac{(x+1)^2 + \lambda(x+1) + 12}{12(x+1)^4(x+2)^2} \right]. \]

\[ = \frac{2}{x^2} \left\{ \psi'(x) - \frac{1}{2x^2} - \frac{1}{x} + \frac{x^2 + \lambda x + 12}{12x^4(x+1)^2} + \frac{x^2(x+1)^2 + \lambda(x+1) + 12}{12(x+1)^4(x+2)^2} \right\}. \]

\[ = \frac{2}{x^2} \left[ \psi'(x) - \frac{1}{2x^2} + \frac{13 - \lambda}{24x} + \frac{48}{24x} \frac{1}{24x} + \frac{1}{2x^4} \right]. \]

\[ \triangleq \frac{2}{x^2} h_{\lambda}(x). \]

Using the formula

\[ \frac{1}{x^n}\Gamma(r) \int_0^\infty t^{r-1} e^{-xt} dt \]  \hspace{1cm} (15)

for \( r > 0 \) and \( x > 0 \), see [1, p. 255, 6.1.1], and the integral representations

\[ \psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1-e^{-xt}} e^{-xt} dt \]  \hspace{1cm} (16)

for \( n \in \mathbb{N} \) and \( x \in (0, \infty) \), see [1, p. 260, 6.4.1], yields

\[ h_{\lambda}(x) = \int_0^\infty \left( \frac{t}{1-e^{-t}} - t - \frac{\lambda}{24} + \frac{37 - 2\lambda e^{-t}}{6} + \frac{13 - \lambda}{6} t e^{-2t} \right) dt \]

\[ + \frac{9\lambda - 172}{24} e^{-t} + \frac{28 - \lambda}{48} t e^{-t} + \frac{\lambda - 48}{48} t e^{-t} + \frac{1}{12} t^3 e^{-t} \right) e^{-xt} dt \]

\[ = \frac{1}{48} \int_0^\infty \left[ \frac{4P(t)}{Q(t)} - \lambda \right] Q(t)e^{-(x+2)t} dt, \]

where

\[ P(t) = \frac{e^{2t}(t^3 - 12t^2 + 54t - 86) - e^t(t^3 - 12t^2 + 16t - 160) - 26t - 74}{e^t - 1} \]

and

\[ Q(t) = 2e^{2t} - e^t(t^2 - 6t + 18) + 8(t + 2) \]

on \((0, \infty)\).

By expanding the function \( Q(t) \) into power series at \( t = 0 \), we have

\[ Q(t) = \sum_{k=3}^{\infty} \frac{2^{k+1} - (k - 9)(k + 2)}{k!} t^k, \hspace{1cm} t > 0. \]

Direct differentiation gives

\[ \frac{d}{dt} \left[ \frac{P(t)}{Q(t)} \right] = \frac{\theta(t)}{(e^t - 1)^2 Q(t)^2}. \]
where

\[ \theta(t) = 2e^{5t}(t^3 - 15t^2 + 78t - 140) + e^{4t}(t^4 - 16t^3 + 132t^2 - 420t + 1236) - 2e^{3t}(5t^4 - 26t^3 + 125t^2 - 132t + 1018) + e^{2t}(17t^4 - 88t^3 + 356t^2 - 228t + 1600) - e^t(8t^4 - 38t^3 + 40t^2 - 228t + 756) + 176 \]

for \( t > 0 \). Straightforward differentiating leads to

\[ \theta'(t) = 2e^t \left[ e^{4t}(5t^3 - 72t^2 + 360t - 622) + 2e^{3t}(1131 - 354t + 120t^3 - 15t^4 + t^5) - e^{2t}(2922 - 164t^2 - 297t^3 - 58t^4 + 15t^5) + e^t(1674 + 224t^2 - 54t^3 + 17t^4) - 264 + 74t + 37t^2 + 3t^3 - 4t^4 \right] \]

\[ \theta''(t) = e^{3t}(20t^3 - 273t^2 + 1296t - 2128) + 2e^{2t}(3039 - 822t + 315t^2 + 41t^3 + 3t^4) - 2e^{t}(2549 + 151t + 210t^2 + 28t^3 + 15t^4) + e^t(1674 + 376t + 62t^2 + 14t^3 + 17t^4) + 74 + 74t + 9t^2 - 16t^3, \]

\[ \theta'''(t) = 2e^{4t}(40t^4 - 516t^3 + 2319t^2 - 3607) + 6e^{3t}(2765 - 612t + 274t^2 - 37t^3 + 3t^4) - 2e^{2t}(5849 + 722t + 336t^2 + 4t^3 + 30t^4) + e^t(2250 + 700t + 104t^2 + 82t^3 + 17t^4) + 74 + 18t - 48t^2, \]

\[ \theta_1^{(3)}(t) = 2e^{4t}(160t^3 - 1944t^2 + 8244t - 12113) + 6e^{3t}(7683 - 1288t + 711t^2 - 99t^3 + 9t^4) - 8e^{2t}(3105 + 529t + 171t^2 + 32t^3 + 15t^4) + e^t(2950 + 908t + 350t^2 + 150t^3 + 17t^4) + 18 - 96t, \]

\[ \theta_1^{(4)}(t) = 32e^{4t}(40t^3 - 456t^2 + 1818t - 2513) + 6e^{3t}(21761 - 2442t + 1836t^2 - 261t^3 + 27t^4) - 8e^{2t}(6739 + 1400t + 438t^2 + 124t^3 + 30t^4) + e^t(3858 + 1608t + 800t^2 + 218t^3 + 17t^4) - 96, \]

\[ \theta_1^{(5)}(t) = e^t \left[ 64e^{3t}(80t^3 - 852t^2 + 3180t - 4117) + 18e^{2t}(20947 - 1218t + 1575t^2 - 225t^3 + 27t^4) - 16e^t(7439 + 1838t + 624t^2 + 184t^3 + 30t^4) + 5466 + 3208t + 1454t^2 + 286t^3 + 17t^4 \right] \]

\[ \triangleq e^t\theta_2(t), \]
In the light of Descartes' Sign Rule, the function

\[ u(t) = 883 - 1932t + 252t^2 + 720t^3 \]
has at most two zeros on $[0, \infty)$. Since $u(0) = 883$, $u(1) = -77$ and $u(2) = 3787$, these two zeros are all less than 2, which implies that the function $u(t)$ is positive on $[2, \infty)$, and so $\theta_3^{(4)}(t) > 0$ on $[2, \infty)$.

In [7, p. 269, 3.6.6] and [9], it was listed that
\[
e^x \leq \frac{2 + x}{2 - x}, \quad 0 \leq x < 2. \tag{17}
\]

Hence, for $t \in (0, 2)$, we have
\[
\theta_3^{(4)}(t) \geq e^{2t}\left[\left(\frac{2 - t}{2 + t}\right)^2(101476 + 129284t + 34131t^2 + 7317t^3
\right.
\]
\[
+ 1008t^4 + 54t^5) + 96(883 - 1932t + 252t^2 + 720t^3)
\]
\[
= \frac{e^{2t}}{(t + 2)^2}(54t^7 + 792t^6 + 72621t^5 + 309567t^4
\]
\[
+ 209804t^3 - 839488t^2 - 291584t + 744976)
\]
\[
= \frac{e^{2t}}{(t + 2)^2}(54t^7 + 792t^6 + 72621t^5
\]
\[
+ (309567t^2 + 828938t + 508821)(t - 1)^2 - 102880t + 236155]
\]
\[
> \frac{236155 - 102880t}{(t + 2)^2}e^{2t}
\]
\[
> 0.
\]

In conclusion, the function $\theta_3^{(4)}(t)$ is positive on $(0, \infty)$.

A direct calculation yields
\[
\begin{align*}
\theta(0) &= 0, & \theta'(0) &= 0, & \theta''(0) &= 0, \\
\theta_1''(0) &= 0, & \theta_1^{(3)}(0) &= 0, & \theta_1^{(4)}(0) &= 0, \\
\theta_1^{(5)}(0) &= 0, & \theta_2(0) &= 0, & \theta_2'(0) &= 0, \\
\theta_2''(0) &= 22960, & \theta_2^{(3)}(0) &= 216160, & \theta_2^{(4)}(0) &= 1188248, \\
\theta_2^{(5)}(0) &= 5009904, & \theta_3(0) &= 313119, & \theta_3'(0) &= 809098, \\
\theta_3''(0) &= 1651644, & \theta_3^{(3)}(0) &= 2711964, & \theta_3^{(4)}(0) &= 3352392.
\end{align*}
\]

This implies that
\[
\begin{align*}
\theta(t) &> 0, & \theta'(t) &> 0, & \theta''(t) &> 0, \\
\theta_1''(t) &> 0, & \theta_1^{(3)}(t) &> 0, & \theta_1^{(4)}(t) &> 0, \\
\theta_1^{(5)}(t) &> 0, & \theta_2(t) &> 0, & \theta_2'(t) &> 0, \\
\theta_2''(t) &> 22960, & \theta_2^{(3)}(t) &> 216160, & \theta_2^{(4)}(t) &> 1188248, \\
\theta_2^{(5)}(t) &> 5009904, & \theta_3(t) &> 313119, & \theta_3'(t) &> 809098, \\
\theta_3''(t) &> 1651644, & \theta_3^{(3)}(t) &> 2711964, & \theta_3^{(4)}(t) &> 3352392.
\end{align*}
\]
on $(0, \infty)$. As a result, the function $\frac{P(t)}{Q(t)}$ is decreasing on $(0, \infty)$, with
\[
\lim_{t \to 0} \frac{P(t)}{Q(t)} = 1 \quad \text{and} \quad \lim_{t \to \infty} \frac{P(t)}{Q(t)} = 0.
\]

Hence,

(1) when $\lambda \leq 0$, the function $h_\lambda(x)$ is completely monotonic on $(0, \infty)$;
(2) when \( \lambda \geq 4 \), the negative of \( h_\lambda(x) \) is completely monotonic on \( (0, \infty) \).

Since \( \frac{2}{x^2} \) is completely monotonic on \( (0, \infty) \) and the product of finitely many completely monotonic functions is also completely monotonic,

(1) when \( \lambda \leq 0 \), the difference \( f_\lambda(x) - f_\lambda(x + 1) \) is completely monotonic on \( (0, \infty) \), that is,

\[
0 \leq (-1)^i[f_\lambda(x) - f_\lambda(x + 1)]^{(i)} = (-1)^i[f_\lambda(x)]^{(i)} - (-1)^i[f_\lambda(x + 1)]^{(i)}, \quad i \geq 0.
\]

By virtue of induction, we obtain

\[
(-1)^i[f_\lambda(x)]^{(i)} \geq (-1)^i[f_\lambda(x + 1)]^{(i)} \geq \cdots \geq (-1)^i[f_\lambda(x + k)]^{(i)} \geq \lim_{k \to \infty} (-1)^i[f_\lambda(x + k)]^{(i)} = 0
\]

for \( i \geq 0 \). So the function \( f_\lambda(x) \) for \( \lambda \leq 0 \) is completely monotonic on \( (0, \infty) \).

(2) when \( \lambda \geq 4 \), a similar argument leads to the complete monotonicity of \( f_\lambda(x) \) on \( (0, \infty) \).

The sufficiency is proved.

Multiplying by \( x^n \) on both sides of (13) yields

\[
\lim_{x \to 0^+} [x^n \psi^{(n-1)}(x)] = (-1)^n(n-1)!, \quad n \in \mathbb{N}.
\]  

Using L'Hôpital's rule, the limit (18), and the formula (13), we have

\[
\begin{align*}
\lim_{x \to 0^+} & \frac{12x^4(x + 1)^2 \left\{ [\psi'(x)]^2 + \psi''(x) \right\} - x^2 - 12}{x} \\
= & \lim_{x \to 0^+} \left( 24(x + 1)x^4 \left\{ [\psi'(x)]^2 + \psi''(x) \right\} \\
& + 12(x + 1)^2x^4 \left\{ 2\psi'(x)\psi''(x) + \psi^{(3)}(x) \right\} \\
& + 48(x + 1)^2x^3 \left\{ [\psi'(x)]^2 + \psi''(x) \right\} - 2x \right)
\end{align*}
\]

\[
= 24 \lim_{x \to 0^+} [x^2 \psi'(x)]^2 + 12 \lim_{x \to 0^+} [x^4 \psi^{(3)}(x)] + 48 \lim_{x \to 0^+} [x^3 \psi''(x)]
\]

\[
+ 24 \lim_{x \to 0^+} [x^4 \psi''(x)] + 24 \lim_{x \to 0^+} \left\{ x^4 \psi'(x)\psi''(x) + 2x^3 \psi'(x) \right\}
\]

\[
= 24 \lim_{x \to 0^+} \left\{ x^2 \psi'(x) + 2x \psi'(x) \right\}
\]

\[
= 24 \lim_{x \to 0^+} \left\{ x^2 \psi'(x) + x^2 \psi''(x) \right\}
\]

\[
= 24 \lim_{x \to 0^+} \left\{ x^2 \psi''(x) + \left( \frac{-1)^{3-1}(3-1)!}{x^3} \right) \right\}
\]

\[
+ 2x \left[ \psi'(x) + \frac{-1)^{2-1}(2-1)!}{x^2} \right]
\]

\[
= 24 \lim_{x \to 0^+} \left\{ x^2 \psi''(x) + x^2 \psi'(x) + 2x \psi'(x) \right\}
\]

\[
= 0.
\]

In [1, p. 260, 6.4.12 and 6.4.13], it was listed that

\[
\psi'(z) \sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^4} + \frac{1}{42z^5} - \frac{1}{30z^6} + \cdots
\]

(19)

and

\[
\psi''(z) \sim -\frac{1}{z^2} - \frac{1}{2z^3} - \frac{1}{2z^4} + \frac{1}{6z^5} - \frac{1}{6z^6} + \frac{3}{10z^{10}} - \frac{5}{6z^{12}} + \cdots
\]

(20)

as \( z \to \infty \) in \( |\arg z| < \pi \). Therefore, we have

\[
\frac{12x^4(x + 1)^2 \left\{ [\psi'(x)]^2 + \psi''(x) \right\} - x^2 - 12}{x} \sim 4 - \frac{82}{15x} + \frac{14}{3x^2} - \frac{29}{35x^3} + \cdots
\]
as \( x \to \infty \). Hence
\[
\lim_{x \to \infty} 12x^4(x + 1)^2 \left\{ \left[ \psi'(x) \right]^2 + \psi''(x) \right\} - x^2 - 12 = 4.
\]
If the function \( f_\lambda(x) \) is completely monotonic on \((0, \infty)\), then
\[
\left[ \psi'(x) \right]^2 + \psi''(x) - \frac{x^2 + \lambda x + 12}{12x^4(x + 1)^2} > 0
\]
which can be rearranged as
\[
\lambda < \frac{12x^4(x + 1)^2 \left\{ \left[ \psi'(x) \right]^2 + \psi''(x) \right\} - x^2 - 12}{x} \to 0^+
\]
as \( x \to 0^+ \), which means that \( \lambda \leq 0 \). If \( -f_\lambda(x) \) is completely monotonic on \((0, \infty)\), then
\[
\lambda > \frac{12x^4(x + 1)^2 \left\{ \left[ \psi'(x) \right]^2 + \psi''(x) \right\} - x^2 - 12}{x} \to 4
\]
as \( x \to \infty \), so \( \lambda \geq 4 \). The necessity is proved.

The double inequality (12) and its best possibility follow from the necessary and sufficient conditions for the function \( f_\lambda(x) \) to be completely monotonic on \((0, \infty)\). The proof of Theorem 1 is complete. \( \square \)

Second proof for the first part of Theorem 1. It is easy to see that
\[
f_\lambda(x) = f(x) - \lambda h(x), \tag{21}
\]
where \( f(x) \) is defined by (9) and
\[
h(x) = \frac{1}{12x^3(x + 1)^2}. \tag{22}
\]
From [18] it is known that the function \( f(x) \) is completely monotonic on \((0, \infty)\), as mentioned on page 2. Since \( h(x) \) is completely monotonic on \((0, \infty)\), it follows that the function \( f_\lambda(x) \) is completely monotonic on \((0, \infty)\) when \( \lambda \leq 0 \). Utilizing (13) for \( n = 1 \) and \( n = 2 \) yields that
\[
f_\lambda(x) = \left[ \psi'(x + 1) + \frac{1}{x^2} \right]^2 + \left[ \psi''(x + 1) - \frac{2}{x^2} \right]^2 - \frac{x^2 + \lambda x + 12}{12x^4(x + 1)^2}
\]
\[
= -\lambda \left[ \frac{1}{12} - \frac{2}{x^2} + \frac{3}{x} - \frac{4 + 3x}{(1 + x)^2} \right] - \frac{37}{12x^2} + \frac{25}{6x} - \frac{63 + 50x}{12(1 + x)^2}
\]
\[
+ \frac{2}{x^2} \psi'(x + 1) + \left[ \psi'(x + 1) \right]^2 + \psi''(x + 1).
\]
Therefore, if \( \lambda > 0 \) then \( \lim_{x \to 0^+} f_\lambda(x) = -\infty \). This implies that the function \( f_\lambda(x) \) is completely monotonic on \((0, \infty)\) if and only if \( \lambda \leq 0 \). \( \square \)

Second proof for the necessity of the second part of Theorem 1. From the asymptotic expansions in (19) and (20) it is not difficult to obtain that
\[
f_\lambda(x) = \frac{4 - \lambda}{12x^5} + O\left( \frac{1}{x^6} \right)
\]
as \( x \to \infty \). This implies that the function \( f_\lambda(x) \) is positive for small positive number \( x \) when \( \lambda < 4 \). So the necessary condition in the second part of Theorem 1 follows. \( \square \)

Second proof for a part of the sufficiency in the second part of Theorem 1. It is clear that
\[
-f_\lambda(x) = g(x) + \frac{\lambda - 13}{12x^3(1 + x)^2},
\]
where \( g(x) \) is defined by (10). From the complete monotonicity of the function \( g(x) \) mentioned on page 2, it follows that if \( \lambda \geq 13 \) the function \( -f_\lambda(x) \) is completely monotonic on \((0, \infty)\). This gives an alternative proof for a part of the sufficiency in the second part of Theorem 1.

\[ \square \]

Remark 1. The first proof of Theorem 1 is direct and independent, but other partial proofs are based on the main result in [18].

3. More remarks

In this section we derive some corollaries from the first proof of Theorem 1 and pose a double inequality for bounding the function \( [\psi'(x)]^2 + \psi''(x) \) on \((0, \infty)\).

Remark 2. It is easy to see that the double inequality (12) for \( \mu = 0 \) and \( \nu = 4 \) recovers the left-hand side inequality and refines the right-hand side inequalities in (6) and (8).

It is clear that the bounds in (8) and (12) are simpler than (4) and (6).

Remark 3. From the proof of Theorem 1, it is easy to deduce that the function

\[
\psi'(x) - \frac{\lambda(4 + 8x + 5x^2)}{24x(1 + x)^3(2 + x)^2} - \frac{24 + 120x + 288x^2 + 399x^3 + 345x^4 + 181x^5 + 51x^6 + 6x^7}{6x^2(1 + x)^3(2 + x)^2}
\]

is completely monotonic on \((0, \infty)\) if and only if \( \lambda \leq 0 \), and so is its negative if and only if \( \lambda \geq 4 \).

Remark 4. Integrating the function (23) and considering its positivity, it is immediate to see that the double inequality

\[
\xi \left[ \ln x - 9 \ln(1 + x) + 8 \ln(2 + x) - 18 + 33x + 14x^2 \right] \quad \frac{24}{48(1 + x)^2(2 + x)}
\]

\[
< \psi(x) - \frac{28x^4 + 87x^3 + 73x^2 + 3x - 12}{6x(x + 1)^3(x + 2)} - \frac{43 \ln(x + 1) - 37 \ln(x + 2)}{6}
\]

\[
< \eta \left[ \ln x - 9 \ln(1 + x) + 8 \ln(2 + x) - 18 + 33x + 14x^2 \right] \quad \frac{24}{48(1 + x)^2(2 + x)}
\]

holds on \((0, \infty)\) if and only if \( \xi \geq 4 \) and \( \eta \leq 0 \).

Remark 5. We note that the function

\[
G_\lambda(x) = \ln \Gamma(x) + x + \frac{1}{12(1 + x)^2} + \frac{\lambda - 48}{48(1 + x)} + \frac{H_\lambda(x)}{24}
\]

(1) is completely monotonic on \((0, \infty)\) if and only if \( \lambda \leq 0 \);
(2) satisfies \((-1)^{k+1}[G_\lambda(x)]^{(k)} > 0\) for \( k \in \mathbb{N} \) on \((0, \infty)\) if and only if \( \lambda \geq 4 \);
(3) has limits

\[
\lim_{x \to 0^+} G_\lambda(x) = \frac{488 \ln 2 - 44 + \lambda(1 - 24 \ln 2)}{48}
\]

and

\[
\lim_{x \to \infty} G_\lambda(x) = \frac{124 + 12 \ln(2\pi) - 7\lambda}{24},
\]

where

\[
H_\lambda(x) = (24 - \lambda x) \ln x + [(9\lambda - 172)x + 12\lambda - 256] \ln(x + 1) + 4(37 - 2\lambda)x - 3\lambda + 61] \ln(x + 2).
\]
In particular, we have
\[
G_0(x) = \ln \Gamma(x) + \ln x + x + \frac{1}{12(1 + x)^2} - \frac{1}{1 + x} 
\frac{(61 + 37x) \ln (2 + x) - (64 + 43x) \ln (1 + x)}{6} 
\tag{29}
\]
and
\[
G_4(x) = \ln \Gamma(x) + \ln x + x + \frac{1}{12(1 + x)^2} - \frac{11}{12(1 + x)} 
\frac{(49 + 29x) \ln (2 + x) - 2(26 + 17x) \ln (1 + x) - x \ln x}{6} 
\tag{30}
\]
on \((0, \infty)\).

Remark 6. We conjecture that the double inequality
\[
\frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(x + 1)^2} \right] ^\alpha < [\psi'(x)]^2 + \psi''(x) < \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(x + 1)^2} \right] ^\beta 
\tag{31}
\]
holds on \((0, \infty)\) if and only if \(\alpha \geq \frac{6}{5}\) and \(\beta \leq 1\).

References

[1] M. Abramowitz and I. A. Stegun (Eds), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, 9th printing, Washington, 1970.

[2] H. Alzer, Sharp inequalities for the digamma and polygamma functions, Forum Math. 16 (2004), no. 2, 181–221; Available online at http://dx.doi.org/10.1515/form.2004.009.

[3] B.-N. Guo and F. Qi, A class of completely monotonic functions involving divided differences of the psi and tri-gamma functions and some applications, J. Korean Math. Soc. 48 (2011), no. 3, 655–667; Available online at http://dx.doi.org/10.4134/JKMS.2011.48.3.655.

[4] B.-N. Guo and F. Qi, A completely monotonic function involving the tri-gamma function and with degree one, Appl. Math. Comput. 218 (2012), no. 19, 9890–9897; Available online at http://dx.doi.org/10.1016/j.amc.2012.03.075.

[5] B.-N. Guo and F. Qi, Refinements of lower bounds for polygamma functions, Proc. Amer. Math. Soc. (2012), in press; Available online at http://dx.doi.org/10.1090/S0002-9939-2012-11387-5.

[6] B.-N. Guo, F. Qi, and H. M. Srivastava, Some uniqueness results for the non-trivially complete monotonicity of a class of functions involving the polygamma and related functions, Integral Transforms Spec. Funct. 21 (2010), no. 11, 103–111; Available online at http://dx.doi.org/10.1080/10652461003748112.

[7] D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, 1970.

[8] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[9] F. Qi, A method of constructing inequalities about \(e^x\), Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 8 (1997), 16–23.

[10] F. Qi, Bounds for the ratio of two gamma functions, J. Inequal. Appl. 2010 (2010), Article ID 493058, 84 pages; Available online at http://dx.doi.org/10.1155/2010/493058.

[11] F. Qi, P. Cerone, and S. S. Dragomir, Complete monotonicity of a function involving the divided difference of psi functions, Bull. Aust. Math. Soc. (2013), in press; Available online at http://dx.doi.org/10.1017/S0004972712001025.

[12] F. Qi and B.-N. Guo, Completely monotonic functions involving divided differences of the \(n\)- and \((n-1)\)-gamma functions and some applications, Commun. Pure Appl. Anal. 8 (2009), no. 6, 1975–1989; Available online at http://dx.doi.org/10.3934/cpaa.2009.8.1975.

[13] F. Qi and B.-N. Guo, Necessary and sufficient conditions for functions involving the \((n+1)\)- and \((n+2)\)-gamma functions to be completely monotonic, Adv. Appl. Math. 44 (2010), no. 1, 71–83; Available online at http://dx.doi.org/10.1016/j.aam.2009.03.003.

[14] F. Qi and Q.-M. Luo, Bounds for the ratio of two gamma functions—From Wendel’s and related inequalities to logarithmically completely monotonic functions, Banach J. Math. Anal. 6 (2012), no. 2, 132–158.

[15] F. Qi, Q.-M. Luo, and B.-N. Guo, Complete monotonicity of a function involving the divided difference of digamma functions, Sci. China Math. (2013), in press.
[16] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.

[17] J.-L. Zhao, B.-N. Guo, and F. Qi, *A completely monotonic function involving the tri- and tetra-gamma functions*, Math. Slovaca 63 (2013), no. 1-2, in press.

[18] J.-L. Zhao, B.-N. Guo, and F. Qi, *Complete monotonicity of two functions involving the tri- and tetra-gamma functions*, Period. Math. Hungar. 65 (2012), no. 1, 147–155; Available online at http://dx.doi.org/10.1007/s10998-012-9562-x.

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