The resilience of the Etherington–Hubble relation

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19 May 2022

ABSTRACT

The Etherington reciprocity theorem, or distance duality relation (DDR), relates the mutual scaling of cosmic distances in any metric theory of gravity where photons are massless and propagate on null geodesics. In this paper, we make use of the DDR to build a consistency check based on its degeneracy with the Hubble constant, \( H_0 \). We parameterise the DDR using the form \( n(z) = 1 + \varepsilon z \), thus only allowing small deviations from its standard value. We use a combination of late time observational data to provide the first joint constraints on the Hubble parameter and \( \varepsilon \) with percentage accuracy: \( H_0 = 68.6 \pm 2.5 \text{ km s}^{-1} \text{Mpc}^{-1} \) and \( \varepsilon = 0.001^{+0.023}_{-0.026} \). We build our consistency check using these constraints and compare them with the results obtained in extended cosmological models using cosmic microwave background data. We find that extensions to \( \Lambda \)CDM involving massive neutrinos and/or additional dark radiation are in perfect agreement with the DDR, while models with non-zero spatial curvature show a preference for DDR violation, i.e. \( \varepsilon \neq 0 \) at the level of \( \sim 1.5\sigma \). Most importantly, we find a mild \( 2\sigma \) discrepancy between the validity of the DDR and the latest publicly available Cepheid-calibrated SNIa constraint on \( H_0 \). We discuss the potential consequences of this for both the Etherington reciprocity theorem and the \( H_0 \) tension.

Key words: Cosmology: observations – distance scale – cosmological parameters – Cosmology: theory – gravitation

1 INTRODUCTION

Cosmology: observations – distance scale – cosmological parameters – Cosmology: theory – gravitation

The era of precision cosmology began with the launch of the COBE satellite in 1989, which measured both the near-perfect black body spectrum and the tiny temperature anisotropies of the cosmic microwave background (CMB) (Fixsen et al. 1994; Bennett et al. 1996). Since then, a vast array of surveys and experiments have probed nearly all epochs and scales of the Universe, converging on the standard \( \Lambda \)CDM model of cosmology. This model describes a Universe which is spatially flat (the curvature density \( \Omega_k \approx 0 \)), is dominated at late times by a cosmological constant \( \Lambda \) and in which the majority of matter only interacts gravitationally. We call this type of matter cold dark matter (CDM). Furthermore, we assume that general relativity is the correct description of gravity, and that on the largest scales the Universe is homogeneous and isotropic. This cosmological principle is encapsulated in the construction of the Friedmann–Lemaître–Robertson–Walker (FLRW) metric (Weinberg 1972).

The Hubble expansion rate \( H(z) \) is a fundamental quantity in any cosmological model based on a theory of gravity which uses the FLRW metric. There are a number of different ways that the value of the Hubble parameter at redshift zero, i.e. the Hubble constant \( H_0 \), can be measured, some of which depend on the cosmological model chosen. With the increasing precision of the observational data at our disposal, it has become clear that a significant statistical tension exists between some of the different measurements of \( H_0 \) (Di Valentino et al. 2015b; Verde et al. 2019; Pogosian et al. 2021; Freedman 2021; Di Valentino et al. 2021d). Specifically, when measurements are made by constructing a distance ladder to Type Ia supernovae (SNIa) using parallax distances and the period–luminosity relation of Cepheid variable stars, the Hubble constant is found to be \( H_0 = 73.2 \pm 1.3 \text{ km s}^{-1} \text{Mpc}^{-1} \) (by the SH0ES collaboration (Riess et al. 2021b), R20 hereafter), which is in tension with the Planck value stemming from measurements of the temperature anisotropies and polarisation of the cosmic microwave background, \( H_0 = 67.4 \pm 0.5 \text{ km s}^{-1} \text{Mpc}^{-1} \) (Aghanim et al. 2020c) (P18 hereafter), a cosmological-model-dependent result.

It is worth noting that while the SH0ES collaboration uses light curves of Cepheid variable stars to anchor the SNIa and hence infer \( H_0 \), alternative methods of calibration, using either Mira variable stars or stars at the tip of the red giant branch (TRBG) have measured values of \( H_0 = 73.6 \pm 3.9 \text{ km s}^{-1} \text{Mpc}^{-1} \) (Huang et al. 2018, 2020) and \( H_0 = 69.6 \pm 1.9 \text{ km s}^{-1} \text{Mpc}^{-1} \) respectively (Freedman et al. 2019; Freedman 2021). In particular, the TRGB result shows a \( \sim 2\sigma \) tension with R20 that can be linked to an inconsistency in their measurements of distances to common SNIa hosts (Efstathiou 2021). Some unaccounted-for systematics in the SNIa calibration could possibly resolve the moderate tension between these methods but a compelling answer is yet to be found (Mortsell et al. 2021; Martinelli & Tutusaus 2019).

Similar arguments can be applied to CMB observations, as either a modification of the recombination physics (by e.g. early dark energy (Poulin et al. 2019; Niedermann & Sloth 2020; Freese & Winkler 2021) or particles beyond the Standard Model (D’Eramo et al. 2018;
Arias-Aragón et al. 2020, 2021; Giarè et al. 2021b) or of the expansion rate (by e.g. interacting dark energy (Salvatelli et al. 2014; Martinelli et al. 2019), dynamical dark energy (Zhao et al. 2017; Bonilla et al. 2021) or modified gravity (Raveri 2020; Peirone et al. 2019)) can lead to significant variations in the value of \( H_0 \) inferred by Planck. There have been many efforts in this direction to find a solution to the Hubble tension, but so far these have been largely unsuccessful (see e.g. Knox & Millea 2020; Di Valentino et al. 2021a; Abdalla et al. 2022 for reviews).

Besides the Hubble expansion rate \( H(z) \), another fundamental quantity in universes described by the FLRW metric is the distance duality relation (DDR), or Etherington reciprocity theorem, which relates cosmological distances measured through the luminosity and the angular size of astrophysical objects (Etherington 1933). The validity of the DDR stems from the metricity of the gravitational theory and the masslessness and number conservation of photons. It therefore holds in any theory of gravity which assumes photons propagate on the null geodesics of the spacetime, regardless of the assumed matter–energy content. It can be measured directly with “golden” observables for which we know both the angular size and the speed of light \( c \).

A spatially flat, homogeneous and isotropic spacetime is described by the Friedmann–Lemaître–Robertson–Walker (FRLW) metric, which describes the evolution of the scale factor and the comoving distance can be derived, where the speed of light \( c \) is the cosmic scale factor, \( \chi \) is the comoving distance and \( \Omega _R = \frac{\Theta}{2} \tanh^2 \Theta d\Theta^2 \) is the metric on a 2-sphere in polar coordinates. By considering the radial propagation of massless particles in this metric, a relation between the evolution of the scale factor and the comoving distance can be derived,

\[
\frac{d\chi}{dz} = H(z)^{-1},
\]

where we have used \( a(t) = (1 + z)^{-1} \) to define the redshift \( z \) and introduced the Hubble parameter \( H(t) = \frac{d}{dt} \ln a = \frac{d}{dz} \ln a \) in terms of \( z \), with \( H(z = 0) = H_0 \) being the Hubble constant. This relation is known as Hubble’s law.

Cosmological observations cannot directly measure the comoving distance. Instead, they infer either the angular diameter distance, \( d_A \), from the angular scale of an object of known size, or the luminosity distance, \( d_L \), from the flux of a source of known intrinsic brightness. While they are both related to the comoving distance, these relations are model-dependent. However, for any metric theory of gravity where photons propagate on null geodesics and for which photon number is conserved, the mutual scaling of the luminosity and angular distances with redshift follows the DDR (Etherington 1933),

\[
\eta(z) \equiv \frac{d_L(z)}{(1 + z)^2 d_A(z)} = 1,
\]

where the luminosity distance \( d_L = (1 + z) \chi \) and the angular diameter distance \( d_A = \chi/(1 + z) \). Deviations from \( \eta(z) = 1 \) would appear if photons did not propagate on null geodesics (i.e. were not massless), if photon number was not conserved (perhaps through a decay of photons into another particle), or if the spacetime was not described by a pseudo-Riemannian manifold.\(^1\) Note that the DDR will hold regardless of the form one chooses to parameterise the expansion rate \( H(z) \) with, since it is a fundamental relation which stems from the geodesic motion of massless particles (Etherington 1933). In particular, it will hold for curved spacetimes where the three-dimensional manifold curvature is different from zero.

We hazard that it is not a particularly strong assumption to expect \( \eta(z) = 1 \). Nevertheless, it is important to consider what measurements we can make to confirm this expectation.

At first glance, it appears that we can probe deviations from the DDR directly through observations of luminosity and angular distances. However, from the definition of the FLRW metric, it is clear that these distances depend on the reference frame that the observer chooses to describe their spacetime. This means that they are not true observables.

If we have some object of known angular size (a standard ruler), we cannot measure the true (angular) distance to that object. Rather, we measure the ratio of the angular distance to the physical length of the ruler, \( \theta^{-1} = \frac{d_A(z)}{R} \), where \( R \) is the ruler length and \( \theta \) is the measured angular size of the object. If we have a set of standard rulers at different redshifts, we can measure the evolution of \( d_A(z)/R \), or alternatively \( H(z)R \), using Equation 2. The dependence on the intrinsic ruler length can be eliminated by multiplying the constraints to estimate the combination \( d_A(z)H(z) \).

Similarly, if we have some object of known intrinsic luminosity (a standard or more properly a standardisable candle), we cannot measure the true (luminosity) distance to that object. Instead, what we probe is the ratio of two fluxes, or else \( dL^2/R^2 \sim (H_0 dL)^2/D^2 \), where \( D \) is the distance we would measure to the source if it were at \( z = 0 \) and we have used Equation 2 to write \( D \) in terms of the Hubble constant.

It is therefore convenient to rewrite Equation 3 in a form that takes the peculiarities of standard rulers and candles into account,

\[
\frac{\eta(z) H_0}{H(z)} = \frac{1}{(1 + z)^2} \frac{[H_0 d_L(z)]_{candle}}{[H(z) d_A(z)]_{ruler}}.
\]

From Equation 4, it is therefore possible to derive constraints on both the Hubble constant and the DDR by employing a specific choice for \( H(z) \).

The first possibility is to use the Einstein field equations to calculate the equations of motion associated with the FLRW metric. This yields the Friedmann equation, which relates the cosmic expansion to the matter–energy content of the Universe,

\[
E^2(z) = \frac{H_0^2}{H^2} = \Omega_m (1 + z)^3 + \Omega_r (1 + z)^4 + \Omega_{DE} X(z) .
\]

\(^1\) A theory of gravity is metric if the Christoffel symbols can be expressed through the metric tensor. This implies the manifold to be pseudo-Riemannian.
where $E^2(z)$ is the dimensionless expansion rate and $\Omega_k = \frac{\rho_k}{\rho_{\text{crit}}}$ represents the fractional density of the $k^{th}$ component today, $\rho_{\text{crit}} = 3H^2/8\pi G$ being the critical energy density. The subscript DE refers to any possible deviation from the standard cosmological model in the form of dark energy or a modification of gravity. The possible time-dependence of this component is expressed through $X(z)$. When $X(z) = 1$, we have a cosmological constant and hence $E^2(z)$ describes $\Lambda$CDM. In other words, the parameter $X(z)$ is able to capture any contribution to $E^2(z)$ which does not come from matter or radiation.

This approach is manifestly model-dependent, as a specific choice of $X(z)$ is required to obtain $H(z)$. However, $H(z)$ is a measurable quantity and can be constrained through observations. To show this, we redefine $H(z)$ in the following way,

$$H(z) = \frac{\ln a(z)}{dz} = \frac{1}{(1+z)} \frac{dz}{dz}.$$  

Finally, we rewrite Equation 4 to eliminate the $H(z)$ dependency,

$$\eta(z)H_0 = \frac{1}{(1+z)^2} \left[ \frac{H_0 d_L(z)}{d_A(z)} \right]^{\text{candle}} \left[ \frac{H_0 d_A(z)}{H_0} \right]^{\text{clock + ruler}}.$$  

This empirical relation allows us to constrain $H_0$ and $\eta(z)$ at the same time, without needing to explicitly define the form of the cosmological model $E^2(z)$. We call this expression the Etherington–Hubble relation.

### 3 RECONSTRUCTING DISTANCES FROM OBSERVATIONAL DATA

In order to fully exploit the Etherington–Hubble relation presented in Equation 7, we must employ a method which does not involve explicitly defining $E^2(z)$. We follow an approach based on Gaussian processes (GPs) to interpolate cosmological data and infer the distance–redshift relation. A Gaussian process is defined as “a collection of random variables, any finite number of which have a joint Gaussian distribution” (Rasmussen & Williams 2006). A GP is therefore a generalisation of a Gaussian probability distribution, but where a probability distribution describes finite-dimensional random variables, a GP describes the properties of functions, and can be used to reconstruct a function $f(z)$ given the function values $z$, as well as the mean and covariance function of the GP, also known as the kernel. We will discuss the kernel choice when presenting the results of our analysis.

Given a set of observations from cosmological data of some generic function of redshift, $f(z)$, we build a GP interpolation of $f(z)$ tailored to that data. This allows us to obtain a continuous set of probability distribution functions (PDFs) that represent $f(z)$ at each redshift. From these PDFs we can construct any function, $F(z) = F(f(z))$, by propagating samples (random variates) of the PDFs of $f(z)$ into those of $F(z)$. The propagation $f \rightarrow F$ is done in a similar way to the evaluation of Markov chain Monte Carlo (MCMC) parameters which are not sampled during the likelihood maximisation (Renzi et al. 2021). For this reason we refer to this methodology as Gaussian Process Monte Carlo (GPMC), which was first introduced by Renzi & Silvestri (2020). We now describe the datasets we use to reconstruct the distances involved in Equation 7.

### 3.1 Standard rulers and clocks

As discussed in the previous section, standard rulers provide two different constraints that can be combined to obtain $d_A(z)H(z)$, while standard clocks provide $H(z)$. Combining rulers and clocks, we can reconstruct the distance–redshift relation in terms of the angular diameter distance. For this purpose, we employ baryon acoustic oscillation (BAO) data from the latest release of the Sloan Digital Sky Survey (SDSS) collaboration as standard rulers, and a collection of observations of $H(z)$ obtained from cosmic chronometers as standard clocks (both datasets are reported in Appendix A). To build $d_A(z)$, we start by constructing the $d_A(z)/R$ and $H(z)/R$ distributions from the SDSS BAO data, assuming they are Gaussian. The SDSS collaboration has provided seven such measurements in the range $0.3 \leq z \leq 2.3$, which we use to build $d_A(z)/R$ and $H(z)/R$. We then combine them to obtain seven effective measurements of $d_A(z)H(z)$. The second step to obtain $d_A(z)$ is to interpolate the cosmic chronometer data using a GP regression to get the value of $H(z)$ at the same redshift as the BAO data. In this way, we can combine the two datasets into measurements of $d_A(z)$ at the seven BAO redshifts. The combination of the two datasets is done by taking samples of the distributions of $d_A(z)H(z)$ and combining them algebraically into a sample of $d_A(z)$. We fix the number of samples at each redshift for all distributions to 10,000, to ensure that the samples provide a fair representation of the PDFs from which they are drawn. We discuss the systematics of the BAO data points in Appendix B.

### 3.2 Standard candles

To reconstruct the distance–redshift relation in terms of the luminosity distance we proceed in a similar way to the method used to obtain the angular diameter distance. We use a collection of 1048 B-band observations of the relative magnitudes of Type Ia supernovae, $m_B(z)$, collectively known as the Pantheon dataset (Scolnic et al. 2018) as our standard candles. The first step is to interpolate the Pantheon dataset to infer the value of $m_B(z)$ at the redshifts of the BAO. The relative supernova magnitude is related to the distance from the supernova itself by the following equation,

$$m_B(z) - M_B = 5 \log_{10} d_L(z) + 25,$$

where $M_B$ is the magnitude any supernova would have if it was at a distance of 10pc from the observer, also known as the supernova absolute magnitude. The value of $M_B$ can be easily related to the Hubble constant by taking into account that for $z \rightarrow 0$ the luminosity distance can be determined through the local Hubble law, $d_L(z) = (1 + z)H(z)/H_0$. This allows us to rewrite Equation 8, substituting $H_0$ in place of $M_B$, i.e.

$$m_B(z) = 5 \log_{10}(H_0 d_L(z)) - 5a_B,$$

where $a_B$ is the intercept of the magnitude–redshift relation, approximately given by $\log_{10} z = 0.2 m_B^0$ with $m_B^0 = m_B(z \sim 0)$. For a generic expansion and $z > 0$, the value of $a_B$ can be expressed through a cosmographic expansion, as in Riess et al. (2016),

$$e^{a_B+0.2 m_B^0} = \left[ 1 + \frac{1}{2}(q_0 - 1) z \right]^{-\frac{1}{6} (1 - q_0 + 3q_0^2 + j_0)^2 z^2 + O(z^3)}.$$  

In the following we will assume a Gaussian distribution for $a_B$ with $a_B = 0.71723 \pm 0.00176$ (Riess et al. 2016). This value is obtained by fixing $q_0 = -0.55$ and $j_0 = 1$. However, the dependency of $q_0$ on the choice of $q_0$ and $j_0$ is negligible given the redshift range in which the

2 This is a fair assumption considering the symmetry of the bounds on the BAO constraints.
We firstly present the constraints on where

These combinations are again performed in an MCMC-like fashion, and

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function varies, and the magnitude of the variations.

reconstruction which is the best fit to the data. The hyperparameters

and the Pantheon datasets as this yields the lowest

̃ and ℎ obtained using the Matérn kernel from now on. This kernel

consistent across the kernels. For this reason, we choose to focus on

the points are clearly visible.

Note that we introduce a small artificial offset in redshift so that all

measurements of the distance duality relation

4.1 The GPMC analysis

We firstly present the constraints on ℎ( ) obtained using the GPMC

method described above. We begin by checking the stability of our

results against changes in the parameterisation of ℎ( ), finding no differences. Specifically, we found no change in our results using ℎ( ) = 1 + ε, ℎ( ) = 1 + ε( / (1 + )) or ℎ( ) = 1 + ε( (1 + )) = ℎ( ) = 1 + ε. However, in cases where GP results are very sensitive to a change in kernel or the dataset is very large, a genetic algorithms or neural network approach may be more appropriate (Bernardo & Levi Said 2021; Dialeetopoulos et al. 2021).

To obtain constraints on ℎ( ), we parameterise its evolution in redshift as ℎ( ) = 1 + ε, meaning that if ε is measured to be zero, ℎ( ) = 1 and we find no violation of the DDR. This parameterisation is motivated by the simple fact that we expect the DDR, if violated, to deviate only slightly from unity, as indicated by current constraints on ℎ( ) (Li et al. 2011; Ma & Corsaniti 2018; Holanda et al. 2016, 2017; Rana et al. 2017; Zhou et al. 2020; Holanda & da Silva 2020; Xu & Huang 2020; Bora & Desai 2021). We present the constraints on ℎ( ) and ε obtained with our pipeline in Figure 2. Note that we also checked the stability of our results against changes in the parameterisation of ℎ( ), finding no differences. Specifically, we found no change in our results using ℎ( ) = 1 + ε, ℎ( ) = 1 + ε( / (1 + )), ℎ( ) = 1 + ε ln(1 + ) or ℎ( ) = 1 + ε( (1 + )) or ℎ( ) = 1 + ε( (1 + )) = ℎ( ) = 1 + ε. However, in cases where GP results are very sensitive to a change in kernel or the dataset is very large, a genetic algorithms or neural network approach may be more appropriate (Bernardo & Levi Said 2021; Dialeetopoulos et al. 2021).

It is clear from Figure 2 that there is a degeneracy between ε (or equivalently ℎ( )) and ℎ( ), and we also explicitly show how the imposition of different priors on ℎ( ) can alter the measurement of ε obtained from the same data. We reweight the marginalised posterior distributions obtained for ℎ( ) using two different Gaussian priors. We use the P18 prior of ℎ( ) = 67.4 ± 0.5 km s⁻¹ Mpc⁻¹ (Aghanim et al. 2020c) (shown in red in Figure 2) and R20 prior of ℎ( ) = 73.2 ± 1.3 km s⁻¹ Mpc⁻¹ (shown in yellow in Figure 2). This demonstrates how changing the cosmological model (here done implicitly by the choice of prior – a model-dependent value from the CMB, or a model-independent value from Cepheid-calibrated supernovae) can in principle alter the measurement of the DDR. In other words, the ℎ( ) tension can be recast as a tension in measurements made of the DDR.

While the reweighted results are still both consistent with ℎ( ) = 1 (P18 at 1σ and R20 at 2σ), depending on one’s level of tolerance, the 2σ discrepancy between the R20 measurement of ε and ε = 0 may be taken more or less seriously.

Although its statistical significance is not high enough to draw reliable conclusions, if we are uncomfortable with this level of disagreement between measurements and we trust the DDR, our result can be interpreted as a clue that the local measurements of ℎ( ) are biased towards an artificially high value. As many have previously speculated, this could be due to systematic errors in the SNIa calibration (Efstathiou 2021; Camarena & Marra 2021; Renzi & Silvestri 2020). Conversely, if we trust the SH0ES measurements, this 2σ discrepancy may be seen as a hint for new physics beyond the standard cosmological model at late times, lending weight to

Figure 1. Measurements of the distance duality relation ℎ( ) at several redshifts obtained by combining BAO, SNIa and CC data using our GPMC method with four different kernels, and assuming ℎ( ) = 70 km s⁻¹ Mpc⁻¹.

fit for aB is typically performed ( ≤ 0.2) and current experimental

uncertainties (Riess et al. 2016; Camarena & Marra 2021). Combining

the prior information on aB with the GP reconstruction of mB( ) we can finally obtain estimates of the luminosity distance at the BAO redshifts and use them to constrain the Etherington–Hubble relation. These combinations are again performed in an MCMC-like fashion, algebraically combining the PDF samples at each redshift through Equation 7.

4 RESULTS

4.1 The GPMC analysis

We firstly present the constraints on ℎ( ) obtained using the GPMC

method described above. We begin by checking the stability of our

results against a change in the GP kernel used. In Figure 1, we

show measurements of ℎ( ) when considering four different kernels

available in the Python library scikit-learn³: the Matérn, Rational

Quadratic, Dot Product and Radial Basis Function (RBF) kernels.

Note that we introduce a small artificial offset in redshift so that all

the points are clearly visible.

From this plot, we can see that the measurement of ℎ( ) is extremely

consistent across the kernels. For this reason, we choose to focus on

the results obtained using the Matérn kernel from now on. This kernel

takes the form

where and is the Euclidean distance between the two input points and , controls the shape of the kernel. We fix = 7/2 for both the CC and the Pantheon datasets as this yields the lowest value i.e. the reconstruction which is the best fit to the data. The hyperparameters and represent the length scale over which the reconstructed function varies, and the magnitude of the variations.

We fix the kernel hyperparameters by minimising the logarithm of

³ https://scikit-learn.org/
the possibility that the solution to the $H_0$ tension could involve the breakdown of the Etherington reciprocity theorem.

Leaving aside observational systematics, the possible causes of such a violation can be attributed to only a few physical effects. One possibility is that photons are interacting with some particles not in the Standard Model e.g. photon conversion into axion-like particles (Tiwari 2017; Buen-Abad et al. 2020; De Bernardis et al. 2006; Mirizzi et al. 2008; Masaki et al. 2017, 2020; Mukherjee et al. 2019; Avgoustidis et al. 2010; D’Amico & Kaloper 2015). Another possibility is that photons have some small but non-zero mass. While being a rather exotic avenue, the possibility remains open, even though strong constraints on photon mass have been derived using astrophysical observations (Schaefer 1999; Wu et al. 2016). A final possibility is to consider the breakdown of the FLRW metric by non-uniform Hubble flow (McClure & Dyer 2007; Krishnan et al. 2021b,a; Luongo et al. 2021) or by theories of gravitation which do not respect the metricity ansatz, one example being teleparallel theories (De Andrade et al. 2000; Bahamonde et al. 2021). Generally speaking, all these possibilities involve modifying the propagation of photons.

Let us close this section with the following remark: the constraint we obtain for $H_0$ is the first ever measurement that does not rely on any cosmological assumptions. All we have assumed is the existence of standardisable observables and that they are all tracers of the same cosmic expansion and therefore related to the same underlying $H(z)$. This means that our constraints on the Hubble constant are completely model-independent.

### 4.2 Changing the cosmological model

In this section, we now move to describe how the validity of the DDR can be used as a consistency test for extensions to ΛCDM.

Firstly, it is important to note that the full Planck likelihood uses a temperature scaling relation which assumes that photons propagate on null geodesics, i.e.

$$T = (1 + z) T_0,$$

where $z$ is the redshift and $T_0$ is the observed CMB temperature at $z = 0$. This relation is exactly equivalent to rewriting the DDR in terms of the temperature (Avgoustidis et al. 2012, 2016) and therefore implies $\eta(z) = 1$. As the results of a fit to the CMB data can be summarised (in terms of late time observables) with $H_0$, the value obtained for this parameter is forced to be consistent with $\epsilon = 0$. We can see from Figure 2 that this is indeed the case for the result obtained using the Planck prior on $H_0$ assuming ΛCDM expansion.

As pointed out in section 2, the validity of the distance duality relation, i.e. $\eta(z) = 1$, is a robust assumption to make. We now use the resilience of this relation and the fact that it is implicitly assumed in the CMB fit as a general consistency check for a given cosmological model.

We firstly estimate the value of $H_0$ using the Planck 2018 observations of the cosmic microwave background temperature anisotropies and polarisation. While neither the results obtained by the Gaussian Process Monte Carlo technique or the local measurement of $H_0$ provided by the SH0ES collaboration rely on the precise form of $E^2(z)$, inferring the value of the present day expansion rate from observations of the early universe (i.e. the CMB) necessarily requires a cosmological model.

We start by considering the standard ΛCDM model based on the usual six free parameters and then proceed by including different combinations of the additional parameters, such as the total neutrino mass $M_\nu \equiv \sum m_\nu$, the effective number of relativistic species $N_{\text{eff}}$, and the spatial curvature parameter $\Omega_k$. Indeed, it is well known that the constraints on $H_0$ can be significantly changed by including additional degrees of freedom in the sample (Di Valentino et al. 2016b, MNRAS 000, 1–11 (2021)).
Table 1. Measurements of $\epsilon$ considering different cosmological models, and their consistency with the DDR, R20 and the GPMC.

| Cosmological model | Inferred $\epsilon$ (P18) | Consistency with DDR ($\epsilon = 0$) | Consistency with R20 ($\epsilon = -0.031 \pm 0.016$) | Consistency with GPMC ($\epsilon = 0.001^{+0.023}_{-0.002}$) |
|-------------------|---------------------------|---------------------------------|--------------------------------|---------------------------------|
| $\Lambda$CDM      | 0.006 ± 0.016             | 0.4 $\sigma$                    | 1.6 $\sigma$                | 0.2 $\sigma$                    |
| $\Lambda$CDM + $M_{\nu}$ | 0.009 ± 0.018             | 0.5 $\sigma$                    | 1.7 $\sigma$                | 0.2 $\sigma$                    |
| $\Lambda$CDM + Neff | 0.014 ± 0.019             | 0.7 $\sigma$                    | 1.8 $\sigma$                | 0.4 $\sigma$                    |
| $\Lambda$CDM + $M_{\nu}$ + Neff | 0.016 ± 0.021             | 0.8 $\sigma$                    | 1.8 $\sigma$                | 0.4 $\sigma$                    |
| $\Lambda$CDM + $\Omega_k$ | 0.048$^{+0.026}_{-0.029}$ | 1.6 $\sigma$                    | 2.4 $\sigma$                | 1.2 $\sigma$                    |
| $\Lambda$CDM + $\Omega_k$ + $M_{\nu}$ | 0.032 ± 0.027             | 1.2 $\sigma$                    | 2.0 $\sigma$                | 0.8 $\sigma$                    |
| $\Lambda$CDM + $\Omega_k$ + Neff | 0.044 ± 0.027             | 1.6 $\sigma$                    | 2.4 $\sigma$                | 1.1 $\sigma$                    |
| $\Lambda$CDM + $\Omega_k$ + $M_{\nu}$ + Neff | 0.037 ± 0.027             | 1.4 $\sigma$                    | 2.2 $\sigma$                | 1.0 $\sigma$                    |

For instance, as robustly indicated by oscillation experiments (de Salas et al. 2021; De Salas et al. 2018), neutrinos should be regarded as massive particles, and cosmology provides a powerful (albeit indirect) means to constrain their mass (De Salas et al. 2018; Hagstotz et al. 2020; Vagnozzi 2019; Vagnozzi et al. 2018a,b, 2017; Giusarma et al. 2016; Bond et al. 1980; Capozzi et al. 2021; Di Valentino et al. 2021b; Xu et al. 2021; Green & Meyers 2021). The strong correlation between the expansion rate and the total neutrino mass can alter the final constraint on $H_0$.

Similarly, by testing departures from the reference value $N_{\text{eff}} = 3.046$ (Mangano et al. 2005; Akiti & Yamaguchi 2020; Froustey et al. 2020; Bennett et al. 2020; Schwarz 2003), one can probe and constrain several extended models both of cosmology and particle physics that predict extra dark radiation in the early Universe, including the cases of additional neutrino species and hot relics beyond the standard model of elementary particles (Melchiorri et al. 2007; Di Valentino et al. 2015a, 2016a; Archidiacono et al. 2015; D’Eramo et al. 2018; Giarè et al. 2021a,b; Green et al. 2021; D’Eramo & Yun 2021; DePorzio et al. 2021; D’Eramo et al. 2021b,a). Higher values of the effective number of relativistic species can lead to smaller values of the sound horizon at recombination, resulting in a preference for higher values of $H_0$.

Finally, the last Planck Collaboration data release (Aghanim et al. 2020a,b,c) confirmed the presence of an enhanced lensing amplitude in CMB power spectra, larger than what is expected in the standard $\Lambda$CDM model. While such a preference could be a manifestation of some unaccounted-for systematics in the CMB data (Efstathiou & Gratton 2020; Vagnozzi et al. 2020, 2021), as argued by Di Valentino et al. (2019); Handley (2021), the possibility of a closed Universe ($\Omega_k < 0$) can provide another valid physical explanation for this effect. Despite the fact that inflationary theory naturally predicts a spatially flat geometry, it is nevertheless possible to construct inflationary models with positive curvature (Linde 1995, 2003; Ratra 2017; Bonga et al. 2016; Handley 2019; Bonga et al. 2017; Ooba et al. 2018; Ellis et al. 2002; Uzan et al. 2003; Unger & Poplawski 2019; Gordon et al. 2021; Sloan et al. 2020; Motaharfar & Singh 2021). When the assumption of flatness is relaxed, the analysis of Planck CMB temperature and polarisation (TT TE EE) data show a preference for a closed Universe at more than 95% confidence level (CL hereafter) (Aghanim et al. 2020c; Di Valentino et al. 2019; Handley 2021; Di Valentino et al. 2021a,d,c; Forconi et al. 2021; Zuckerman & Anchordoqui 2021). In such models, the Hubble tension is substantially increased and a large discordance arises in most of the local cosmological observables compared to what is observed by other means, including BAO (Di Valentino et al. 2019; Handley 2021; Di Valentino et al. 2021d). Since the DDR is preserved in a closed Universe, we can maintain an agnostic perspective on the spatial geometry and examine several extended non-flat cosmologies to study if the aforementioned tension can be recast in terms of a violation of the DDR.

For each model analysed, we perform a Markov chain Monte Carlo parameter estimation using the Planck 2018 likelihood, in each case deriving a posterior distribution function for $H_0$. We adopt these posterior distributions in the likelihood for the distance duality relation and therefore infer the value of $\epsilon$ (defined as before by the relation $\eta(z) = 1 + \epsilon z$) in all the different models. This allows us to use the DDR as a consistency check for the different models and to study whether the tensions between early and late time estimations of $H_0$ are also reflected in $\epsilon$. We present our results for the different models in Table 1 and we summarise them in Figure 3.

As already discussed in subsection 4.1, considering the value of $H_0$ obtained by the Planck Collaboration within the standard $\Lambda$CDM model, we infer $\epsilon = 0.006 \pm 0.016$ at 68% CL, consistent with the DDR at 0.4 $\sigma$. This value is also in perfect agreement (0.2 $\sigma$) with the result obtained following the GPMC technique. Interestingly, the tension between the Planck estimation of $H_0$ and the local measurement of the $SH_0ES$ collaboration is reflected in a difference in the respective values of $\epsilon$ at the level of 1.6 $\sigma$.

We note that the inclusion of the total neutrino mass and the number of relativistic degrees of freedom as additional parameters of our cosmological model does not significantly change our results. In particular, the value of $\epsilon$ inferred in these models always remains consistent with the DDR and the GPMC estimation to within one standard deviation, while the difference with respect to the value of $\epsilon$.

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4 Other possible explanations to this effect involve for instance modified gravity theories (Raveri et al. 2021; Aoki et al. 2020).

5 It is worth noting that including in the analysis the lensing spectrum as measured by the Planck Collaboration and the BAO data we recover the preference for a flat Universe (Aghanim et al. 2020a; Forconi et al. 2021). However these results should be considered with caution because local measurements are in strong disagreement with Planck when the curvature parameter is free to vary (Di Valentino et al. 2019; Handley 2021; Di Valentino et al. 2021d).
was released, along with the addition of approximately 700 new SNIa when considering the total neutrino mass and/or the effective number with respect to the DDR prediction at the level of $\sigma$.

Conversely, the addition of spatial curvature significantly increases the tensions between the different datasets. While this is a fair assumption, recent analyses of the Pantheon catalogue have outlined a discrepancy between the high and low redshift bins of the datasets (Dainotti et al. 2021; Kazantzidis et al. 2021; Di Valentino et al. 2020b). If confirmed by other independent observations, this could mean that the cosmographic expansion used to fit the magnitude-Hubble intercept is incorrect. We stress however that the latest release of the SH0ES collaboration (Riess et al. 2021a) has reanalysed such assumptions, finding no inconsistency with the assumptions of Riess et al. (2016).

Although not the focus of this work, an important avenue of recent interest is the possibility that the $H_0$ tension could be resolved with ultra late-time effects, such as in the context of modified gravity (Perivolaropoulos & Skara 2021), by assuming that $H_0$ varies with redshift (Dainotti et al. 2021; Dainotti et al. 2022) or by a steep “phase transition” in the expansion history or in the SNIa absolute magnitude (Marra & Perivolaropoulos 2021; Camarena & Marra 2021). Many of these theories warrant deeper investigation. However, including these theories in a well-defined set-up which can be used to constrain the cosmic history, such as the pipeline which we have presented in this paper, is far from trivial. As we discuss in section 4, such models would alter the scaling of the CMB temperature-redshift relation possibly leading to spectral distortions in the CMB black-body spectrum Chluba (2014).

Furthermore, the analysis we perform to compare our constraints with the results from Planck would be inconsistent with these models, since the Boltzmann solver used to analyse Planck data assumes a standard temperature-scaling relation Ivanov et al. (2020). We also note that by choosing the relation between the magnitude of SNIa and $H_0$, we are implicitly assuming an FRLW metric which again assumes $H_0$ is a constant.

We therefore leave a thorough investigation of such late-time modifications of the expansion history for a future work, since, in a nutshell, the aim of this current work was to use the resilience of the Etherington–Hubble relation as a consistency check for beyond $\Lambda$CDM models, rather than explicitly investigating the $H_0$ tension or an apparent $H_0$ evolution.

6 CONCLUSION

The Etherington reciprocity theorem relates the mutual scaling of the luminosity and angular distances with redshift in any metric theory of gravity where photons propagate on null geodesics and for which photon number is conserved. Given its resilience, the Etherington reciprocity theorem, also known as the distance duality relation, can be regarded as a robust property of any Universe described by an FLRW line element where fundamental interactions obey the Standard Model of particle physics. In this work, we explicitly investigated the degeneracy of the DDR with the Hubble rate, constructing a consistency test methodology for any cosmological model based only on the assumption of the validity of the DDR.

We firstly followed a GPMC procedure to measure the DDR function $\eta(z)$ using a dataset of SNIa, BAO and CC, finding that the data is in agreement with $\eta(z) = 1$ at the level of $1\sigma$; in other words we find no violation of the DDR in our collection of late time data. We then obtained joint constraints on $H_0$ and the DDR parameter $\epsilon$, revealing the degeneracy between these two quantities. By imposing two different priors on $H_0$, one from P18 and one from R20, we showed how the constraint on $\epsilon$ changes for these two datasets.
In particular, we found the R20 prior to be in a mild statistical discrepancy (2σ) with the constraints from late time data. While our results are not accurate enough to provide a statistically significant answer to the Hubble tension, they support the argument that the present discrepancy in the value of the Hubble constant may simply arise from unaccounted-for systematics, in line with the findings of other recent studies (Efstathiou 2021; Renzi & Silvestri 2020). However, it is also interesting to consider the idea that, if the discrepancy is not due to systematics, the solution to the H0 tension could involve the breakdown of the Etherington reciprocity theorem. As we explained in section 4, this may be a signal for the need to build a theory of gravity that is not constructed using a pseudo-Riemannian manifold or to consider new physics beyond the Standard Model of elementary interactions, involving mechanisms of photon conversion into other particles or even more exotic scenarios. When considering P18 data and a ΛCDM cosmology, we found instead an almost perfect agreement with the DDR validity, as expected by the fact that the fit to CMB data assumes the standard temperature scaling relation.

In the second part of our results, we used the finding that the DDR is valid and robust as a consistency check for different cosmological models. When testing the model of ΛCDM plus a varying total neutrino mass and number of relativistic degrees of freedom, we found that the value of ε agrees with the DDR at the level of 1σ. However, when considering a model in which the spatial curvature is free to vary, we found a value of ε which disagrees with the validity of the DDR at around 1.6σ. This implies that, if we trust that the DDR is valid, a cosmological model of ΛCDM + Ω_k is in turn disfavoured at the level of 1.6σ. Given that the direction of this discrepancy is opposite (in terms of ε) to that of the R20 prior we conclude that it is a manifestation of the well-known discrepancy between Planck and late time data when a non-flat spacetime is considered (Di Valentino et al. 2019; Handley 2021; Di Valentino et al. 2021d).

In conclusion, we have demonstrated how the DDR can be used as a powerful consistency test for beyond ΛCDM models and, generally speaking, for cosmological measurements involving the Hubble constant, an endeavour which is becoming ever more pressing as we move relentlessly forward into the epoch of ultra-precision cosmology.

ACKNOWLEDGEMENTS

We thank Eleonora Di Valentino, Alessandro Melchiorri and Matteo Martinelli for useful comments and suggestions during the preparation of this work. FR acknowledges support from the NWO and the Dutch Ministry of Education, Culture and Science (OCW) (through NWO VIDI Grant No.2019/ENW/00678104 and from the D-ITP consortium). NBH is supported by a postdoctoral position funded through two “la Caixa” Foundation fellowships (ID 100010434), with fellowship codes LCF/BQ/PI19/11690015 and LCF/BQ/PI19/11690018. WG is supported by “Theoretical astroparticle Physics” (TASP), iniziativa specifica INFN.

DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author.

REFERENCES

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APPENDIX A: COSMOLOGICAL DATA

The baryon acoustic oscillation and cosmic chronometer data used in this work are reported in Table A1 and Table A2. The Pantheon dataset (Scolnic et al. 2018) is publicly available at https://github.com/dscolnic/Pantheon.

APPENDIX B: SYSTEMATICS

In this appendix, we review the main systematics involved in the cosmic chronometer and baryon acoustic oscillation data.

Cosmic Chronometers

In our approach, we have employed SNIa and BAO data to effectively constrain the dimensionless expansion rate $E^2(z)$, while CC are used to set an absolute scale which provides the value of $H(z)$ at each redshift. The systematics involved in CC measurements could therefore affect our constraints on $H_0$ and $\epsilon$.

Possible sources of systematic errors in these data come from the modelling of the stellar population in the galaxies used as standard clocks (Moresco et al. 2012a; Moresco 2015; Moresco et al. 2016, 2018, 2020). In particular, a subdominant stellar population may impact the selection of the unbiased tracer, which is required in CC measurements to determine the differential age of the Universe with redshift. Passively evolving galaxies are excellent tracers of the cosmic differential age but the simple description of these galaxies which uses a single stellar population is insufficiently accurate. Instead, the impact of using a realistic star formation history (SFH) must be assessed. The uncertainties in modelling the metallicity of the stellar population must also be considered in the error budget, as this information is used to calibrate the relative age of the population. Finally, the stellar population synthesis (SPS) model used in calibrating the relative stellar age is also a possible source of systematics.

The CC data employed in the main text (see Table A2) already include the uncertainties coming from the SFH and stellar metallicity (Moresco et al. 2012a; Moresco 2015; Moresco et al. 2016), while the contribution of subdominant stellar population is negligible for our dataset (Moresco et al. 2018).

The uncertainties associated with SPS models are instead not included. In Moresco et al. (2020), these were shown to produce an additional $\lesssim 16\%$ uncertainty in the value of $H(z)$ measured with the CC data. We include this additional error in our fit by first reconstructing the evolution of the SPS contribution with redshift through a GP fit on the values of the third column of Table 3 in Moresco et al. (2020). We then sum this with that reported in Table A2.

The effect of adding this SPS uncertainty is rather small. We found that it mainly contributes to the decision of the SPS uncertainty in the value of $H(z)$ measured with the CC data. We include this additional error in our fit by first reconstructing the evolution of the SPS contribution with redshift through a GP fit on the values of the third column of Table 3 in Moresco et al. (2020). We then sum this with that reported in Table A2.

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While the R20 discrepancy is slightly reduced, the constraints we suffer from a similar problem, since its largest redshift is there are very few SNIa data points above fit to the Pantheon data are ill-constrained at higher redshifts, since Bourboux et al. 2017). It is therefore important to assess their impact model (Aubourg et al. 2015; Font-Ribera et al. 2014; du Mas des Λ

\[ H(z) = \frac{c}{z} \frac{d_{\text{M}}}{r_s} \]

Table A1. The BAO data used in this work given in terms of \( d_{\text{M}} = (1 + z) d_A \) and \( d_H = c H(z)^{-1} \).

\[
\begin{array}{cccc}
\text{Type} & z & d_{\text{M}}/r_s & d_H/\Delta s \\
\hline
\text{BOSS galaxy–galaxy} & 0.38 & 10.27 \pm 0.15 & 24.89 \pm 0.58 & \text{Alam et al. (2017)} \\
\text{eBOSS galaxy–galaxy} & 0.51 & 13.38 \pm 0.18 & 22.43 \pm 0.48 & \text{Alam et al. (2017)} \\
& 0.70 & 17.65 \pm 0.30 & 19.78 \pm 0.46 & \text{Bautista et al. (2020); Gil-Marín et al. (2020)} \\
& 0.85 & 19.50 \pm 1.00 & 19.60 \pm 2.10 & \text{Tamone et al. (2020); de Mattia et al. (2020)} \\
& 1.48 & 30.21 \pm 0.79 & 13.23 \pm 0.47 & \text{Neveux et al. (2020); Hou et al. (2020)} \\
\text{eBOSS Ly–α–Ly–α} & 2.34 & 37.41 \pm 1.86 & 8.86 \pm 0.29 & \text{de Sainte Agathe et al. (2019)} \\
\text{eBOSS Ly–α–quasar} & 2.35 & 36.30 \pm 1.80 & 8.20 \pm 0.36 & \text{Blomqvist et al. (2019)} \\
\end{array}
\]

Table A2. The CC data used in this work.

\[
\begin{array}{ccc}
\hline
z & H(z) & \text{Reference} \\
\hline
0.07 & 69.0 \pm 19.6 & \text{Zhang et al. (2014)} \\
0.09 & 69.0 \pm 12.0 & \text{Stern et al. (2010)} \\
0.12 & 68.6 \pm 26.2 & \text{Zhang et al. (2014)} \\
0.17 & 75.0 \pm 4.0 & \text{Moresco et al. (2012b)} \\
0.19 & 75.0 \pm 5.0 & \text{Moresco et al. (2012b)} \\
0.2 & 72.9 \pm 29.6 & \text{Zhang et al. (2014)} \\
0.27 & 77.0 \pm 14.0 & \text{Stern et al. (2010)} \\
0.28 & 88.8 \pm 36.6 & \text{Zhang et al. (2014)} \\
0.352 & 83.0 \pm 14.0 & \text{Moresco et al. (2012b)} \\
0.3802 & 83.0 \pm 13.5 & \text{Moresco et al. (2016)} \\
0.4 & 95.0 \pm 17.0 & \text{Stern et al. (2010)} \\
0.4004 & 77.0 \pm 10.2 & \text{Moresco et al. (2016)} \\
0.4247 & 87.1 \pm 11.2 & \text{Moresco et al. (2016)} \\
0.44497 & 92.8 \pm 12.9 & \text{Moresco et al. (2016)} \\
0.47 & 89.0 \pm 49.6 & \text{Ratsimbazafy et al. (2017)} \\
\hline
\end{array}
\]

The resilience of the Etherington–Hubble relation

\[
H(z) = \frac{c}{z} \frac{d_{\text{M}}}{r_s} 
\]

The CC data used in this work.

the Ly–α data. The high-redshift BAO data (at \( z = 2.34 \) and \( z = 2.35 \) respectively) peak positions have been shown to have a \( 3\sigma \) discrepancy with the CMB prediction from the Planck data in a \( \Lambda CDM \) model (Aubourg et al. 2015; Font-Ribera et al. 2014; du Mas des Bourboux et al. 2017). It is therefore important to assess their impact on the results described in the main text.

An additional consideration is that the GP errors resulting from the fit to the Pantheon data are ill-constrained at higher redshifts, since there are very few SNIa data points above \( z \gtrsim 1 \). The CC dataset suffers from a similar problem, since its largest redshift \( z = 1.965 \).

Without including Ly–α data, we found a small shift in the mean value of \( H_0 \) with almost the same accuracy of the whole BAO dataset, \( H_0 = 69.2 \pm 2.6 \) kms\(^{-1}\)Mpc\(^{-1}\). The bound on \( \epsilon \) instead worsens by a factor of two, \( \epsilon = 0.004^{+0.040}_{-0.040} \). The discrepancy of the R20 results with \( \epsilon = 0 \) consequently reduces to \( 1.5\sigma \) when the R20 prior is combined with our late time data (\( \epsilon = 0.031^{+0.041}_{-0.039} \) at 95% CL). While the R20 discrepancy is slightly reduced, the constraints we draw without the Ly–α BAO do not change the conclusions outlined in the main text.

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