Abstract

We study the long time behavior of an underdamped mean-field Langevin (MFL) equation, and provide a general convergence as well as an exponential convergence rate result under different conditions. The results on the MFL equation can be applied to study the convergence of the Hamiltonian gradient descent algorithm for the overparametrized optimization. We then provide some numerical examples of the algorithm to train a generative adversarial network (GAN).

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Key words: Underdamped mean-field Langevin dynamics, ergodicity, coupling, GAN.

1 Introduction

Let $\mathcal{P}_2(\mathbb{R}^n)$ denote the space of all probability measures on $\mathbb{R}^n$ with finite second order moment, and let $F: \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^+$ be a potential function with intrinsic derivative denoted by $D_m F: \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$ (see Section 2.1 below for its definition). We study in this paper the ergodicity of the following underdamped mean-field Langevin (MFL) equation:

$$dX_t = V_t dt, \quad dV_t = -(D_m F(L(X_t), X_t)) + \gamma V_t dt + \sigma dW_t, \quad (1.1)$$

where $\gamma > 0$, $\sigma \neq 0$ are two real constants, $L(X_t)$ denotes the law of $X_t$, and $W$ is an $n$-dimensional standard Brownian motion. Let us denote the marginal distribution by $m_t := L(X_t, V_t)$, then it is easy to verify by the Itô’s formula that $m_t = (m_t)_{t \geq 0}$ is a solution (in the sense of distribution) to the nonlinear kinetic Vlasov-Fokker-Planck equation:

$$\partial_t m = -v \cdot \nabla_x m + \nabla_v \cdot ((D_m F(m^X, x) + \gamma v) m) + \frac{1}{2} \sigma^2 \Delta_v m, \quad (1.2)$$

where $m^X_t \in \mathcal{P}_2(\mathbb{R}^n)$ denotes the pushforward measure of $m_t \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)$ under the map $(x, v) \mapsto x$.

This consists in an extension of the classical underdamped Langevin equation (without the mean-field term $L(X_t)$ in (1.1)). More precisely, let $\phi: \mathbb{R}^n \to \mathbb{R}^+$ be a differentiable potential function, and

$$F(m) = \int_{\mathbb{R}^n} \phi(x) m(dx), \text{ so that } D_m F(m, x) = \nabla \phi(x), \text{ for all } (m, x) \in \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n. \quad (1.3)$$
Then (1.1) turns to be the classical underdamped Langevin equation, which has been introduced in statistical physics to describe the motion of a particle with position \( X \) and velocity \( V \) in a potential field \( \nabla_x \phi \) subject to damping and random collisions, see e.g. [33, 51, 63], and has been largely investigated in computational statistical physics, see e.g. [11, 67]. It is well known that (see e.g. [64, Proposition 6.1]), under mild conditions, the classical Langevin dynamics (i.e. Equation (1.1) with \( F \) being given by (1.3)) has a (unique) invariant measure on \( \mathbb{R}^n \times \mathbb{R}^n \) with the density:

\[
m_\infty(x, v) = C e^{-\frac{2\gamma}{\sigma^2}(\phi(x)+\frac{1}{2}|v|^2)},
\]

for some normalization constant \( C > 0 \). Moreover, in this setting, an important subject is to study the ergodicity of Langevin dynamic, i.e. the convergence as well as the convergence rate of the marginal distributions \( m_t \) to the invariant measure \( m_\infty \).

To prove the convergence result, a classical approach is the free energy method, see e.g. Bonilla, Carrillo and Soler [8]. Let us define the free energy function along the marginals \((m_t)_{t \geq 0}\) of the Langevin dynamic by

\[
\Phi(t) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( \phi(x) + \frac{1}{2} |v|^2 \right) m_t(dx, dv) + \frac{\sigma^2}{2\gamma} H(m_t), \quad t \geq 0,
\]

where \( H : \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \rightarrow \mathbb{R} \cup \{\infty\} \) is the entropy function (see Section 2.1 below for a precise definition). By computing the derivative \( \Phi'(t) \), one then proves the decay of the free energy, i.e. that \( \Phi'(t) \rightarrow 0 \). Under further technical conditions, this can then be used to deduce a general convergence result \( m_t \rightarrow m_\infty \).

As for the convergence rate, many works have been devoted to the subject and different approaches have been introduced. In [72, 73], Villani popularized the concept “hypocoercivity” and proved the exponential convergence of \( m_t \) in \( H^1_{m_\infty} \). He generalized the idea already presented in the computations performed in Talay [70, Section 3]. A more direct approach was later developed in Hérau [44] and Dolbeault, Mouhot and Schmeiser [24, 25], and it led to various results on kinetic equations. Notice that both Villani’s and Dolbeault, Mouhot and Schmeiser’s results on the exponential convergence rate highly depend on the dimension, and (therefore) do not apply to the case with mean-field interaction. It is noteworthy that in the recent paper by Cao, Lu and Wang [12], the authors developed a new estimate on the convergence rate based on the variational method proposed by Armstrong and Mourrat [2]. Let us also mention some work studying the ergodicity of underdamped Langevin dynamics using more probabilistic arguments, see e.g. Ben Arous, Cranston and Kendall [5], Wu [75], Mattingly, Stuart, and Higham [57], Rey-Bellet and Thomas [65], Talay [70], Bakry, Cattiaux and Guillin [3], Duong and Tugaut [27]. These works are mostly based on Lyapunov conditions and the rates they obtain also depend on the dimension. In the recent work by Guillin, Liu, Wu and Zhang [40], it was shown for the first time that a mean-field underdamped Langevin equation with non-convex potential is exponentially ergodic in \( H^1_{m_\infty} \). The authors’ argument combines Villani’s hypocoercivity with a certain functional inequality and Lyapunov conditions. In Monmarché [61], Guillin and Monmarché [41], the authors obtain an exponential convergence rate for the kinetic mean-field Langevin dynamics using a uniform propagation in chaos argument. To complete the brief literature review, we would draw special attention to the coupling argument applied in Bolley, Guillin and Malrieu [7] and Eberle, Guillin and Zimmer [31], which found transparent convergence rates in sense of Wasserstein-type distance.

The main objective of this paper is to study the ergodicity of the underdamped Langevin dynamic in the mean-field setting of (1.1). We will first develop the free energy approach in our setting to obtain a general convergence result of the marginal distribution \( L(X_t, V_t) \) to the invariant measure. Such an approach has already been initiated in Duong and Tugaut [27] in a
similar setting, but in a much more formal way. In a second part, we will apply the reflection-synchronous coupling technique that was initiated in Eberle, Guillin and Zimmer [31,32] to obtain an exponential contraction result in some particular cases.

For the first approach, we follow the main procedures as in Mei, Montanari and Nguyen [58], Hu, Ren, Šiška and Szpruch [46], in particular the latter, for the overdamped mean-field Langevin equation. First, we consider the follow new free energy function \( \mathfrak{F} : \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R} \cup \{\infty\} \)

\[
\mathfrak{F}(m) := F(m^X) + \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{1}{2}|v|^2 m(dx, dv) + \frac{\sigma^2}{2\gamma} H(m),
\]  

(1.5)

where \( m^X \in \mathcal{P}_2(\mathbb{R}^n) \) denotes the pushforward measure of \( m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \) under the map \( (x, v) \mapsto x \). We next deduce that the free energy is decreasing along the dynamics of the MFL, together with an explicit expression of the derivative \( \frac{d\mathfrak{F}(m)}{dt} \). This is enough to show that, if \( m^\ast \) is an accumulation points of \( (m_t)_{t \geq 0} \) and it has a density function, then it satisfies

\[
v + \frac{\sigma^2}{2} \nabla_v \log (m^\ast(x, v)) = 0.
\]

Finally, by applying LaSalle’s invariance principle for the dynamic system, we show that \( m^\ast \) must satisfy the first order condition

\[
D_m F(m^\ast, x) + \frac{\sigma^2}{2\gamma} \nabla_x \log (m^\ast(x, v)) = 0 \quad \text{and} \quad v + \frac{\sigma^2}{2\gamma} \nabla_v \log (m^\ast(x, v)) = 0.
\]  

(1.6)

When the potential function \( F : \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}_+ \) is convex so that \( \mathfrak{F} : \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R} \cup \{\infty\} \) is strictly convex, the first order condition (1.6) is sufficient to identify \( m^\ast \) as the unique minimizer of \( \mathfrak{F} \), which is also the unique invariant measure of (1.1). In this way, we are able to prove the uniqueness of the accumulation points of \( (m_t)_{t \geq 0} \), which implies \( m_t \to m^\ast \). Due to the degeneracy and the mean-field interaction of the underdamped MFL process, the proof for the claim is non-trivial. In particular, we apply the time reversal technique for the SDE in Föllmer [34] to obtain some integrability properties of the marginal densities.

For the second approach, we consider a potential functional \( F : \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R} \) which could be non-convex but is essentially with small (nonlinear) dependence on \( m \), and aims at obtaining an exponential contraction result. We mainly borrow the tools developed in Eberle, Guillin and Zimmer [31,32], where they initiate the reflection-synchronous coupling technique, further validate it in the study of the Langevin dynamics with a general non-convex potential, and make the point that the technique offers significant flexibility for additional development. But note that [31] is not concerned with mean-field interaction and the rate found there is dimension dependent. In our context, we design a new metric involving a quadratic form (see Section 4.4.2) to obtain the contraction when the coupled particles are far away, and as a result obtain a dimension-free convergence rate. The construction of the quadratic form shares some flavor with the argument in Bolley, Guillin and Malrieu [7]. Notably, our construction helps to capture the optimal rate in the area of interest (see Remark 4.16), so may be more intrinsic. Notice that most of the articles concerning the ergodicity of underdamped Langevin dynamics obtain the convergence rates depending on the dimension, and in particular very few allow both non-convex potential and the mean-field interaction. Some exceptions would be Guillin, Liu, Wu and Zhang [40], Monmarché [61] and Guillin and Monmarché [41], but they focus on a particular convolution-type interaction. When finishing our paper, we learned that independently Bolley, Guillin, Le Bris and Monmarché [6] are working out an exponential convergence result for the mean-field kinetic system through a similar approach.

More related works  

Langevin dynamics have two specific limiting regimes: the Hamiltonian limit as \( \gamma \to 0 \), and the overdamped limit as \( \gamma \to +\infty \) (in conjunction with a rescaling of time).
There are various works in the literature which carefully study the scaling of the convergence rate in terms of the friction parameter $\gamma$, and obtain lower bounds $c \min(\gamma, \gamma^{-1})$, see for instance, Dolbeault, Klar, Mouhot, Schmeiser [23], Grothaus, Stilgenbauer [38] and Iacobucci, Olla, Stoltz [47] for rather general potentials in addition to results for specific systems, see e.g. Metafune, Pallara, Priola [59] and Kozlov [50]. In our paper, as a corollary of the main convergence result, we shall prove that the underdamped MFL dynamics also converges to its overdamped limit as $\gamma \to \infty$.

Based on the ergodicity of the underdamped Langevin dynamic, and by considering various discrete time versions, it has been developed the Hamiltonian Monte Carlo methods, where the objective is to sample according to the distributions in form of (1.4), see e.g. Lelièvre, Rousset and Stoltz [53], Neal [62], Bou-Rabee, Eberle and Zimmer [9], Bou-Rabee and Schuh [10]. Nowadays this interest resurges in the community of machine learning. Notably, the underdamped Langevin dynamics has been empirically observed to converge more quickly to the invariant measure compared to the overdamped Langevin dynamics (of which the related MCMC was studied in e.g. Dalalyan [22], Durmus and Moulines [29], and it was theoretically justified by Cheng, Chatterji, Bartlett and Jordan in [19] for some particular choice of coefficients.

Ergodicity of the underdamped MFL dynamics can also be used to solve optimization problems. As we shall see in Remark 2.4, the optimization problem

$$\inf_{m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)} \mathcal{F}(m)$$

is a regularized version of problem

$$\inf_{m^X \in \mathcal{P}_2(\mathbb{R}^n)} F(m^X),$$

while the latter appears naturally in the context of neural network with plenty of neurons (see e.g. [46] as well as Section 3). By identifying the minimizer $m^x = \arg \min_{m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)} \mathcal{F}(m)$ as the limit of the marginal distributions of the underdamped MFL dynamics, we justify the underdamped MFL dynamics as an efficient numerical algorithm to solve the mean-field optimization problem.

The rest of the paper is organized as follows. In Section 2 we announce the main results. Before entering the detailed proofs, we study a numerical example concerning the so-called generative adversarial networks (GAN). The main theorems in Section 2 guide us to propose a theoretical convergent algorithm for the GAN, and the numerical test in Section 3 shows a satisfactory result. Finally, we report the proofs in Section 4.

2 Ergodicity of the mean-field Langevin dynamics

2.1 Preliminaries

Let us denote by $\mathcal{P}(\mathbb{R}^n)$ the space of all probability measures on $\mathbb{R}^n$, and by $\mathcal{P}_p(\mathbb{R}^n)$ the space of all $m \in \mathcal{P}(\mathbb{R}^n)$ with finite $p$-th moment, for all $p \geq 1$. Without further specification, in this paper the continuity on $\mathcal{P}_p(\mathbb{R}^n)$ is in the sense of $\mathcal{W}_p$ ($p$-Wasserstein) distance, i.e.

$$\mathcal{W}_p(\mu, \nu) := \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^p \pi(dx, dy) \right)^{1/p},$$

where $\Pi(\mu, \nu)$ denotes the collection of all joint distribution on $\mathbb{R}^n \times \mathbb{R}^n$ with marginal distribution $\mu$ and $\nu$ on the first and second marginal space $\mathbb{R}^n$. The spaces $\mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$, and $\mathcal{P}_p(\mathbb{R}^n \times \mathbb{R}^n)$ as well as the corresponding $p$-Wasserstein distance $\mathcal{W}_p$ are defined similarly.

A function $F : \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}$ is said to be convex if

$$F(\lambda m + (1 - \lambda)m') \leq \lambda F(m) + (1 - \lambda)F(m'), \quad \text{for all } \lambda \in [0, 1], \: m, m' \in \mathcal{P}_2(\mathbb{R}^n).$$

We say $F$ is strictly convex if the above inequality is strict whenever $m \neq m'$ and $\lambda \in (0, 1)$. Next, we say $F \in \mathcal{C}^1$, if there exists a continuous function $\frac{\partial F}{\partial m} : \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}$ satisfying
\[ \frac{\delta F}{\delta m}(m, x) \leq C(1 + |x|^2) \] for some constant \( C > 0 \), and such that, for all \( m, m' \in \mathcal{P}(\mathbb{R}^n) \), one has
\[
F(m') - F(m) = \int_0^1 \int_{\mathbb{R}^n} \frac{\delta F}{\delta m}((1 - u)m + um', x) \ (m' - m)(dx)du.
\]

When \( \frac{\delta F}{\delta m}(m, x) \) is continuously differentiable in \( x \), we define \( D_mF : \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) by
\[
D_mF(m, x) := \nabla_x \frac{\delta F}{\delta m}(m, x),
\]
which is the so-called intrinsic derivative of \( F \) introduced by Lions \([54]\). We also refer to \([13]\) Proposition 5.1.4, Proposition 5.1.5 and \([14]\) Section 2.2 for more properties and interpretations of this notion of derivative. We say function \( F \in C^\infty \) if, for all \( k \in \mathbb{N}, \ i_1, \cdots, i_k \in \mathbb{N} \), the derivatives
\[
\partial_{i_1}^{\dagger} \cdots \partial_{i_k}^{\dagger} D_mF(m, x_1, \cdots, x_k) \text{ exist and are continuous.}
\]

**Example 2.1.** Let us provide some simple examples of \( F : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \) with its derivatives.

1. In case that \( F \) is linear, namely, \( F(m) = \int \phi(x)m(dx) \), where \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) is continuously differentiable and has quadratic growth. Then
\[
\frac{\delta F}{\delta m}(m, x) = \phi(x), \text{ and hence } D_mF(m, x) = \nabla_x \phi(x).
\]

2. In case that \( F(m) = \int \int \phi(x,y)m(dx)m(dy) \). It is easy to check that
\[
\frac{\delta F}{\delta m}(m, x) = \int (\phi(x,y) + \phi(y,x))m(dy).
\]

3. In case that \( F(m) = g\left(\int \phi(x)m(dx)\right) \) with \( g, \phi \) continuously differentiable, we may apply the chain rule to obtain that
\[
\frac{\delta F}{\delta m}(m, x) = g' \left(\int \phi(y)m(dy)\right) \phi(x) \text{ and thus } D_mF(m, x) = g' \left(\int \phi(y)m(dy)\right) \nabla_x \phi(x).
\]

Recall that, for \( m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \), we denote by \( m^X \in \mathcal{P}_2(\mathbb{R}^n) \) the pushforward measure of \( m \) under the map \((x, v) \mapsto x\). Denote by \( H(m) \) the relative entropy of the measure \( m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \) with respect to the Lebesgue measure, that is,
\[
H(m) := \mathbb{E}^m\left[ \log (m(X, V)) \right] = \int_{\mathbb{R}^n \times \mathbb{R}^n} \log (m(x, v))m(x, v)dxdv.
\]

Notice that \( H(m) \) is well defined for all \( m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \). Indeed, let \( C > 0 \) be the renormalization constant such that \( Ce^{-|x|^2-|v|^2} \) is a density function for a probability measure \( \nu \), then
\[
\int_{\mathbb{R}^{2n}} \log (m(x, v))m(x, v)dxdv
= \int_{\mathbb{R}^{2n}} \log \left( \frac{m(x, v)}{Ce^{-|x|^2-|v|^2}} \right)m(x, v)dxdv - \int_{\mathbb{R}^{2n}} (|x|^2 + |v|^2)m(x, v)dxdv + \log(C)
\]
is well defined as the first term at the r.h.s. is the relative entropy between \( m \) and \( \nu \). In above, we use \( m(x, v) \) to denote the density function of the probability measure \( m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \) by abuse of notation, and let \( H(m) := \infty \) if \( m \) does not has a density. In particular we recall that \( H : \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \rightarrow \mathbb{R} \cup \{\infty\} \) is strictly convex, see e.g. \([28]\) Lemma 1.4.3].
2.2 The mean-field Langevin dynamics and free energy function

Throughout the paper we consider a potential function $F : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \cup \{\infty\}$ in the form of

$$F(m^X) = F_o(m^X) + \int_{\mathbb{R}^n} f(x)m^X(dx), \quad (2.2)$$

where $F_o : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}^+$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ satisfy $F_o \in C^1$ and $f \in C^1$. We slightly abuse the notation and denote

$$\frac{\delta F}{\delta m}(m^X, x) := \frac{\delta F_o}{\delta m}(m^X, x) + f(x), \quad \text{and} \quad D_m F(m^X, x) := D_m F_o(m^X, x) + \nabla f(x).$$

Fixing a positive constant $\gamma > 0$, a constant $\sigma \neq 0$, we introduce the underdamped mean-field Langevin (MFL) dynamics:

$$dX_t = V_t dt, \quad dV_t = -(D_m F(\mathcal{L}(X_t), X_t) + \gamma V_t) dt + \sigma dW_t. \quad (2.3)$$

We next consider the following free energy function minimization problem:

$$\inf_{m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)} \mathfrak{F}(m), \quad \text{with} \quad \mathfrak{F}(m) := F(m^X) + \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v|^2}{2} m(dx, dv) + \frac{\sigma^2}{2\gamma} H(m), \quad (2.4)$$

where $\mathfrak{F} : \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \rightarrow \mathbb{R} \cup \{\infty\}$ is the so-called free energy function. Throughout the paper we also assume that $\mathfrak{F}$ is nontrivial, that is, there exists $m_o \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\mathfrak{F}(m_o)$ is finite. The above free energy function is closely related to the Langevin dynamics (2.3). To see that, let us recall the first order condition for such an optimization problem from Hu, Ren, Šiška and Szpruch [46, Proposition 2.5].

**Lemma 2.2.** Let $F : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}^+$ be a potential function in the form of (2.2). If $m \in \arg\min_{\mu \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)} \mathfrak{F}(\mu)$, then $m$ has a density and there exists a constant $C \in \mathbb{R}$ such that

$$\frac{\delta F}{\delta m}(m^X, x) + \frac{|v|^2}{2} + \frac{\sigma^2}{2\gamma} \log \left(m(x,v)\right) = C, \quad \text{for all} \ (x, v) \in \mathbb{R}^n \times \mathbb{R}^n, \quad (2.5)$$

or equivalently, for some $C' > 0$,

$$m(x, v) = C' \exp \left(- \frac{2\gamma}{\sigma^2} \left(\frac{\delta F}{\delta m}(m^X, x) + \frac{|v|^2}{2}\right)\right), \quad \text{for all} \ (x, v) \in \mathbb{R}^n \times \mathbb{R}^n. \quad (2.6)$$

Conversely, if $m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)$ has the density function given by (2.6), and in addition $F$ is convex, then $m$ is a solution to (2.4).

**Remark 2.3.** By direct computation, when $m$ in (2.6) is smooth enough, one can check directly that $m$ is a stationary solution to the Fokker-Planck equation (1.2) in the sense that

$$-v \cdot \nabla_x m + \nabla_v \cdot ((D_m F(m^X, x) + \gamma v) m) + \frac{1}{2} \sigma^2 \Delta_v m = 0.$$

Besides, the marginal distribution of the underdamped MFL dynamics (2.3) is a weak solution to the Fokker-Planck equation (1.2). If (1.2) has a unique weak solution (with a given initial condition), then $m$ is an invariant measure to the dynamics (2.3). In fact, we will prove in our context that $m$ is indeed the unique invariant (probability) measure of the underdamped MFL dynamics (2.3).
Remark 2.4. (i) With the decomposition \( F(m^X) = F_0(m^X) + \int_{\mathbb{R}^n} f(x)m^X(dx) \), it is equivalent to define the free energy functional by
\[
\mathfrak{F}(m) = F_0(m^X) + \frac{\sigma^2}{2\gamma} H(m|\mu),
\]
where \( \mu \) is the probability measure with density \( Ce^{-\frac{x^2}{2\sigma^2}(f(x)+\frac{1}{2}|v|^2)} \) for some normalization constant \( C > 0 \), and \( H(m|\mu) \) is the relative entropy of \( m \) with respect to \( \mu \). This is in fact the formulation of the free energy function in Hu, Ren, Šiška and Szpruch [46]. Besides, it is assumed in [46, Proposition 2.5] that \( F_0 \) is convex and that \( f(x) \geq \lambda|x|^2 \) for some \( \lambda > 0 \). However, the convexity of \( F_0 \) is only used to prove that \((2.5)\) or \((2.6)\) is the sufficient condition for the optimality of \( m \) in their proof. The growth condition on \( f \) is only used to ensure the existence of a minimizer of \( \mathfrak{F}(\mu) \), which is not stated in the above lemma.

(ii) Notice that the entropy \( H \) is strictly convex, so that the free energy function \( \mathfrak{F} \) is also strictly convex whenever \( F \) is convex. Consequently, the optimization problem \((2.4)\) has at most one minimizer, which must be the unique solution to the first order equation \((2.5)\).

(iii) In particular, given a probability measure \( m \) satisfying \((2.6)\), one has
\[
\frac{\delta F}{\delta m}(m^X, x) + \frac{\sigma^2}{2\gamma} \log(m^X(x)) = C.
\]
For convex \( F \), this implies (again thanks to [46, Proposition 2.5]) that the marginal distribution \( m^X \) is the minimizer of \( m^X \mapsto F(m^X) + \frac{\sigma^2}{2\gamma} H(m^X) \). Moreover, it follows from [46, Theorem 2.11] that such \( m^X \) is also the unique invariant measure of the overdamped MFL dynamics
\[
dX_t = -D_m F(\mathcal{L}(X_t), X_t)dt + \frac{\sigma}{\sqrt{\gamma}}dW_t.
\]

Remark 2.5. To intuitively understand how the linear derivative \( \frac{\delta F}{\delta m} \) characterizes the minimizer as in the first order condition \((2.5)\), we may first ignore the terms \( \frac{\beta}{2}\mathbb{E}^m[V^2] \) and \( \frac{\sigma^2}{2\gamma} H(m) \) in the free energy \( \mathfrak{F}(m) \) and consider a convex potential function \( F : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \). Let \( m \in \mathcal{P}_2(\mathbb{R}^n) \) be such that \( \frac{\delta F}{\delta m}(m, x) = C \) for all \( x \in \mathbb{R}^n \), \( m' \in \mathcal{P}_2(\mathbb{R}^n) \) be arbitrary and \( m^\varepsilon := (1-\varepsilon)m + \varepsilon m' \) for \( \varepsilon \in [0,1] \). Then, by the convexity of \( F \), one has
\[
F(m') - F(m) \geq \frac{1}{\varepsilon} \left( F(m^\varepsilon) - F(m) \right)
= \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\mathbb{R}^n} \frac{\delta F}{\delta m}((1-u)m + um', x) \, (m' - m)(dx)du
\rightarrow \int_{\mathbb{R}^n} \frac{\delta F}{\delta m}(m, x)(m' - m)(dx) = 0, \quad \text{as} \ \varepsilon \rightarrow 0.
\]
Therefore, for arbitrary \( m' \in \mathcal{P}_2(\mathbb{R}^n) \), one has
\[
F(m') - F(m) \geq 0.
\]
In other words, when \( F \) is convex and smooth enough, the condition \( \frac{\delta F}{\delta m}(m, x) = C \) for all \( x \in \mathbb{R}^n \) is sufficient for \( m \) being a minimizer of \( F \).

2.3 Decay of the free energy and ergodicity of the Langevin dynamics

We will provide a first ergodicity result of the MFL dynamic \((2.3)\) based on the free energy approach. Let us first formulate some technical conditions.
Assumption 2.6. (i) The potential function $F : \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}_+$ is in the form of (2.2), where $F_0 : \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}_+$ and $f : \mathbb{R}^n \to \mathbb{R}_+$ satisfy $F_0 \in C^\infty_0$, $f \in C^\infty$ and

$$f(x) \geq \lambda|\mathbf{x}|^2, \quad \text{for all } x \in \mathbb{R}^n.$$  

Moreover, for each $k \geq 2$, the derivatives $D^k_m F$ is bounded, and $D_m F(m^X, x)$ is Lipschitz, i.e.

$$|D_m F(m^X_1, x_1) - D_m F(m^X_2, x_2)| \leq C(W_1(m^X_1, m^X_2) + |x_1 - x_2|).$$

(ii) For all $p \geq 1$, one has $\mathbb{E}[|X_0|^p + |V_0|^p] < \infty$, as well as $H(m_0) < \infty$.

Example 2.7. Take the third example in Example 2.1, that is, $F_0(m^X) := g(\int \phi(x)m^X(dx))$ with $g : \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$. The intrinsic derivative reads

$$D_m F_0(m^X, x) = g' \left(\int \phi(y)m^X(dy)\right)\nabla_x \phi(x).$$

Then Assumption 2.6 holds true, provided that $f(x) = \lambda|\mathbf{x}|^2$ and $g' \in C^\infty_b(\mathbb{R})$, $\phi \in C^\infty_b(\mathbb{R}^n)$. In particular, it covers the case where $g$ is the identity function, so that

$$F_0(m^X) := \int \phi(x)m^X(dx) \quad \text{and} \quad D_m F_0(m^X, x) = \nabla_x \phi(x).$$

as discussed in Example 2.1.

Remark 2.8 (Lower bound of the free energy function). Let $\mu \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)$ denote the probability measure with the density $Ce^{-\frac{g^2}{2\gamma}(|\mathbf{x}|^2 + |\mathbf{v}|^2)}$, where $C > 0$ is the normalization constant. Then the free energy function can be rewritten as

$$\mathfrak{F}(m) = F_0(m^X) + \int \left(f(x) - |x| + \frac{1}{2}|\mathbf{v}|^2 - |\mathbf{v}|\right) m(dx, dv) + \frac{\sigma^2}{2\gamma}H(m|\mu) + \frac{\sigma^2}{2\gamma} \log C.$$

Note that

$$f(x) - |x| + \frac{1}{2}|\mathbf{v}|^2 - |\mathbf{v}| + \frac{\sigma^2}{2\gamma} \log C \geq \frac{\lambda}{2}|\mathbf{x}|^2 + \frac{1}{4}|\mathbf{v}|^2 - \tilde{C},$$

where

$$\tilde{C} := \frac{1}{2\lambda} + 1 + \frac{\sigma^2}{2\gamma} \log C.$$

Further, since $F_0(m^X) \geq 0$ and $H(m|\mu) \geq 0$, we obtain a lower bound for the free energy function:

$$\mathfrak{F}(m) \geq \int \left(\frac{\lambda}{2}|\mathbf{x}|^2 + \frac{1}{4}|\mathbf{v}|^2\right) m(dx, dv) - \tilde{C}. \quad (2.8)$$

Under Assumption 2.6, it is well known that the MFL equation (2.3) admits a unique strong solution $(X_t, V_t)_{t \geq 0}$, see e.g. the proof of Sznitman [69, Theorem 1.1] or Carmona [15, Theorem 1.7]. We first prove that the function $\mathfrak{F}$ defined in (2.4) decays along the marginal $m_t := \mathcal{L}(X_t, V_t)$ of the MFL dynamics (2.3) $(X_t, V_t)_{t \geq 0}$.

Theorem 2.9. Let Assumption 2.6 hold true. Then, for all $t > 0$, $m_t$ has a smooth and strictly positive density function, which is again denoted by $m_t(\cdot)$ by abuse of notation. Moreover, for all $t > s > 0$, one has

$$\mathfrak{F}(m_t) - \mathfrak{F}(m_s) = -\int_s^t \gamma \mathbb{E} \left[|V_r + \frac{\sigma^2}{2\gamma} \nabla_x \log (m_r(X_r, V_r))|^2\right] dr.$$
Remark 2.10. The time derivative of the free energy function has been calculated in the context of the classical underdamped Langevin dynamics in Bonilla et al. [3]. Also, an informal analytical computation has been developed for some mean-field potentials in the paper of Dong and Tugaut [27].

With the help of the free energy function \( \mathfrak{F} \), we may prove the convergence of the marginal laws of (2.3) towards the minimizer \( m := \arg\min_{m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)} \mathfrak{F}(m) \), provided that the function \( F \) is convex.

**Theorem 2.11.** Let Assumption 2.6 hold true. Suppose in addition that the function \( F \) (or equivalently \( F_0 \)) is convex (see (2.1)). Then the underdamped MFL dynamics (2.3) has a unique invariant measure, which is also the unique minimizer \( F \) of (2.4). Moreover, one has

\[
\lim_{t \to \infty} W_1(m_t, m) = 0.
\]

**Remark 2.12.** The ergodicity of diffusions with mean-field interaction is a long-standing problem. Theorem 2.11 shows that, for non-degenerate confinement potentials in the sense of (2.7), \( F \) being convex on the space of probability measures (with 2nd order moment) is sufficient for the underdamped MFL dynamics to be ergodic. It is an analogue of Theorem 2.11 in Hu, Ren, Šiška and Szpruch [46], where it has been proved that the convexity of the potential function ensures the ergodicity of the overdamped MFL dynamics.

We finally provide an analogue of the classical convergence result of the underdamped Langevin dynamic to the overdamped Langevin dynamic when \( \gamma \to \infty \). To this end, we set the scaling \( \sigma = \sigma_0 \sqrt{\gamma} \) for some fixed constant \( \sigma_0 > 0 \), and denote by \((X_\gamma, V_\gamma)\) the solution of the underdamped MFL dynamics (2.3), with the same initial distribution, i.e. \( \mathcal{L}(X_0, V_0) = m_0 \).

**Corollary 2.13.** Let Assumption 2.6 hold true, and suppose that the common initial distribution \( m_0 = \mathcal{L}(X_0, V_0) \) (for all \( \gamma > 0 \)) satisfies \( \mathfrak{F}(m_0) < \infty \). Then, for all \( t \geq 0 \), one has \( X_\gamma_{\gamma t} \to Y_t \) in distribution as \( \gamma \to \infty \), where \( Y \) is the overdamped MFL dynamic defined by

\[
dY_t = -D_m F(\mathcal{L}(Y_t), Y_t) dt + \sigma_0 dW_t, \quad \text{for all } t > 0, \quad \text{and } \mathcal{L}(Y_0) = m_0. \tag{2.9}
\]

### 2.4 Exponential ergodicity given small mean-field dependence

We now study the MFL dynamics (2.3) under another set of technical conditions, where in particular \( F \) is possibly non-convex but with small mean-field dependence. We shall obtain an exponential convergence rate if the invariant measure exists.

**Assumption 2.14.** The potential function \( F : \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R} \) is given by

\[
F(m^X) = F_0(m^X) + \frac{\lambda}{2} \int_{\mathbb{R}^n} |x|^2 m^X(dx),
\]

where \( F_0 : \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R} \) belongs to \( \mathcal{C}^1 \) and \( D_m F_0 \) exists and is Lipschitz continuous. Moreover, for any \( \varepsilon > 0 \), there exists \( K > 0 \) such that for all \((m_1^X, x_1), (m_2^X, x_2) \in \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \), one has

\[
|D_m F_0(m_1^X, x_1) - D_m F_0(m_2^X, x_2)| \leq \varepsilon |x_1 - x_2|, \quad \text{whenever } |x_1 - x_2| \geq K. \tag{2.10}
\]

**Remark 2.15.** Under Assumption 2.14, one has

\[
D_m F(m^X, x) = D_m F_0(m^X, x) + \lambda x.
\]

We highlight that the condition (2.10) involves two (possibly different) arguments \( m_1^X, m_2^X \in \mathcal{P}_2(\mathbb{R}^n) \). On the other hand, (2.10) only need to hold true for \( |x_1 - x_2| \) big enough, in particular, \( D_m F_0 \) is not necessarily a small perturbation of the linear term \( \lambda x \), see Example 2.17 below.
Define the function $\psi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ by
\[
\psi(z, w) := (1 + \beta G(z, w + \gamma z)) h(\eta|w + \gamma z| + |z|),
\]
where the positive constants $\beta, \eta, \gamma$, the non-negative quadratic form $G$ and the non-decreasing concave function $h: \mathbb{R}_+ \to \mathbb{R}_+$ will be determined later, see (4.34) and (4.41). Given $(x, v), (x', v') \in \mathbb{R}^n \times \mathbb{R}^n$, we denote
\[
p := v - v' + \gamma(x - x'), \quad r := |x - x'|, \quad u := |p|,
\]
and therefore
\[
\psi(x - x', v - v') = (1 + \beta G(x - x', p)) h(\eta u + r).
\]
Notice that $\psi(x - x', v - v')$ is a semi-metric between $(x, v)$ and $(x', v')$, we then define the semi-metric (see also discussions in Remark 2.19 below):
\[
\mathcal{W}_\psi(m, m') = \inf \left\{ \int \psi(x - x', v - v')d\pi(x, v, x', v') : \pi \text{ coupling of } m, m' \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \right\}.
\]
Notice that there exists a constant $C > 0$ such that $\psi(x - x', v - v') \geq C|(x - x', v - v')|$. Therefore, the semi-metric dominates the Wasserstein-1 distance in the sense that
\[
\mathcal{W}_\psi(m, m') \geq C\mathcal{W}_1(m, m'), \quad \text{for } m, m' \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n).
\]

**Theorem 2.16 (Exponential convergence under small mean-field condition).** Let Assumption 2.14 hold true. Assume in addition that, for all $m_1^X, m_2^X \in \mathcal{P}_2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, one has
\[
|D_m F_\psi(m_1^X, x) - D_m F_\psi(m_2^X, x)| \leq \epsilon \mathcal{W}_1(m_1^X, m_2^X).
\]
Then, for $\epsilon > 0$ small enough (satisfying the quantitative condition (4.46) below), we have
\[
\mathcal{W}_\psi(m_t, m_t') \leq e^{-t} \mathcal{W}_\psi(m_0, m_0'),
\]
where $c > 0$ is a constant defined below in (4.45) (see also Remark 4.19 for a lower bound of the constant $c > 0$ in a specific example). In particular, the rate $c > 0$ does not depend on the dimension $n$, and for some $C > 0$, it satisfies that
\[
c \leq C \gamma, \quad \text{where } \gamma := \begin{cases} \gamma - \sqrt{\gamma^2 - 4\lambda}, & \text{if } \gamma^2 > 4\lambda, \\ \gamma, & \text{if } \gamma^2 < 4\lambda. \end{cases}
\]

**Example 2.17.** (i) Let $F_\psi(m^X) := \int g(x)m^X(dx)$ for some smooth function $g: \mathbb{R}^n \to \mathbb{R}$ such that
\[
\lim_{|x| \to \infty} |\nabla^2 g(x)| = 0.
\]
Then, $D_m F_\psi(m^X, x) = \nabla g(x)$ satisfies (2.10) and it has no mean-field dependence.

(ii) Let
\[
F_\psi(m^X) := g \left( \int \phi(x)m^X(dx) \right)
\]
with $g, \phi$ continuously differentiable, so that
\[
D_m F_\psi(m^X, x) = g' \left( \int \phi(y)m^X(dy) \right) \nabla_x \phi(x).
\]
Assume that $g \geq 0$, and $g', \nabla_x \phi$ are bounded, then $D_m F_\psi$ is bounded and hence it satisfies (2.10). In addition, if $g'$ is $L$-Lipschitz, then
\[
|D_m F_\psi(m_1^X, x) - D_m F_\psi(m_2^X, x)| \leq L \|\nabla_x \phi\|_\infty \mathcal{W}_1(m_1^X, m_2^X).
\]
It follows that the conditions of Theorem 2.16 are satisfied when $L$ is small enough.
Remark 2.18 (Small mean-field condition). As we shall see in the proof, it is important to first understand the exponential convergence for the Markov diffusion (without mean-field dependence). Then, the small mean-field dependence, viewed as a small perturbation from the Markov diffusion, is a "sufficient" condition for inheriting the desired exponential convergence. On the other hand, from the following example, we shall see that sometimes the small mean-field dependence is also necessary for the convergence result. Consider the potential function

\[ F(m^X) := \frac{1}{2} \left( \int_{\mathbb{R}^n} |x|^2 m^X(dx) - \alpha \left( \int_{\mathbb{R}^n} |x|m^X(dx) \right)^2 \right). \]

Notice that \( F \) is non-convex (indeed concave). The corresponding underdamped MFL dynamics reads:

\[
\begin{align*}
    dX_t &= V_t dt \\
    dV_t &= -(X_t - \alpha \mathbb{E}[X_t] + \gamma V_t)dt + \sigma dW_t.
\end{align*}
\]

It is not hard to show that if \( \alpha > 1 \), then \( \mathbb{E}[X_t] \) diverges. In other words, in this case one cannot expect the convergence of marginal distribution when the mean-field dependence is large.

Remark 2.19 (Semi-metric \( \mathcal{W}_\psi \)). We observe that \( \psi \) is not a function of the norm \( |(x - x', v - v')| \). Furthermore, the quadratic form \( G(x - x', p) \) is convex in \( (x - x', v - v') \), while \( h \) is concave. Thus, \( \psi(x - x', v - v') \) defines a semi-metric (rather than a metric) on \( \mathbb{R}^n \times \mathbb{R}^n \), and \( \mathcal{W}_\psi \) is a semi-metric on the space of measures. Consequently, the contraction result established above does not guarantee the existence of the invariant measure, but only characterizes the convergence rate under \( \mathcal{W}_\psi \), assuming that the invariant measure exists (e.g. when \( F \) is convex). Finally, considering the domination of \( \mathcal{W}_1 \) by \( \mathcal{W}_\psi \), the contraction result also implies exponential convergence to the invariant measure in \( \mathcal{W}_1 \).

Remark 2.20 (Comparison with recent results on coupling of kinetic Langevin dynamics). The proof of Theorem 2.16 relies on the reflection-synchronous coupling technique, which was developed by Eberle, Guillin, and Zimmer in [31]. In their work, they establish a contraction result under a semi-metric \( \mathcal{W}_\psi \) with

\[
\hat{\psi}((x,v),(x',v')) := (1 + \varepsilon \mathcal{V}(x,v) + \varepsilon \mathcal{V}(x',v')) h(|v - v' + \gamma (x - x')| + |x - x'|),
\]

(2.13)

where \( h \) is a constructed non-negative increasing concave function and \( \mathcal{V} \) is a constructed Lyapunov function. Notably, their contraction result holds under more general conditions than our Assumption 2.14. Specifically, they allow for a more general confinement function than \( \frac{\lambda}{2} |x|^2 \), but their contraction rate is dimension-dependent, and the kinetic dynamics do not permit mean-field dependence. To overcome these limitations, we introduce a new semi-metric that enables us to obtain exponential ergodicity in scenarios with small mean-field dependence. Note that our contraction result has an advantage over the one presented in [31], even in cases without mean-field dependence. For example, consider the MFL dynamics (2.3) \( (X_t, V_t) \) (resp. \( (X', V') \)) starting from \( (x,v) \) (resp. \( (x',v') \)). Suppose we have a Lipschitz continuous function \( w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \), and define

\[ \eta(x,v) := \mathbb{E} \left[ \int_0^\infty w(X_t, V_t)dt \right]. \]

Our contraction result guarantees that

\[ |\eta(x,v) - \eta(x',v')| \leq C \int_0^\infty \mathcal{W}_1(m_t, m'_t)dt \leq C \int_0^\infty \mathcal{W}_\psi(m_t, m'_t)dt \leq C(1 + \beta G(x - x', v - v' + \gamma (x - x')))((x,v) - (x',v')). \]
In particular, \( \limsup_{(x', v') \to (x, v)} \frac{|\eta(x, v) - \eta(x', v')|}{||(x, v) - (x', v')||} \leq C \). Hence, \( \eta \) is globally Lipschitz continuous. On the other hand, using the contraction result from [31] and a similar estimate, we obtain

\[
|\eta(x, v) - \eta(x', v')| \leq C \left( 1 + \varepsilon \mathcal{V}(x, v) + \varepsilon \mathcal{V}(x', v') \right) ||(x, v) - (x', v')||,
\]

which implies that \( \eta \) is only locally Lipschitz.

Since the initial version of this paper, there have been recent advancements in the coupling method for the kinetic Langevin dynamics. Notably, Guillin, Le Bris, and Monmarché [39] and Schuh [68] have both made important contributions using the reflection-synchronous coupling technique introduced in [31]. In [39], the authors establish a contraction result under a semi-metric \( \mathcal{W}_\psi \) with a function \( \hat{\psi} \) of the same form as in [31] (see (2.13)), but for more general cases where the kinetic process has small mean-field dependence and the confinement function can exhibit growth larger than quadratic. In [68], the author assumes a confinement function with at most quadratic growth and additionally requires that the function \( x \mapsto D_m F^0(m, x) \) to be \( L_g \)-Lipschitz continuous with a small enough constant \( L_g \). The author successfully proves a contraction result under a metric instead of a semi-metric.

It is also worth mentioning that Guillin, Liu, Wu, and Zhang provided a proof of exponential ergodicity in [40] for underdamped Langevin dynamics with convolution-type interactions, using an entirely different approach based on Villani’s hypocoercivity and functional inequality.

**Remark 2.21 (Mean-field Hamiltonian Monte Carlo method in [10]).** In their very recent work, Rou-Rabee and Schuh [10] provide a quantitative rate for the convergence of unadjusted Hamiltonian Monte Carlo method for some mean-field models. More precisely, at each round \( k \) of their Monte Carlo algorithm, they simulate the McKean-Vlasov process on \([0, T]\):

\[
\begin{align*}
\left\{ \begin{array}{ll}
dX^k_t &= V^k_t dt, \\
dV^k_t &= \left( -\nabla_x f(X^k_t) - \varepsilon \mathbb{E} \left[ \nabla_x w(X^k_t - \tilde{X}^k_t) - \nabla_x w(\tilde{X}^k_t - X^k_t) \right] \right) dt, \\
X^0_0 &= X^{-1}_T,
\end{array} \right. \\
V^k_0 &\sim \mathcal{N}(0, \frac{\sigma^2}{2}),
\end{align*}
\]

where \( \tilde{X}^k \) is an independent copy of \( X^k \), and the expectation \( \mathbb{E} \) is taken with respect to \( \tilde{X}^k \). Their main result, Theorem 3, shows that for small enough \( \varepsilon \), the distribution \( L(X^k_T) \) converges exponentially as \( k \to \infty \). Recall the second example in Example 2.1 and note that

\[
\nabla_x f(X^k_t) + \varepsilon \mathbb{E} \left[ \nabla_x w(X^k_t - \tilde{X}^k_t) - \nabla_x w(\tilde{X}^k_t - X^k_t) \right] = D_m F(L(X_t), X_t),
\]

where

\[
F(m) := \int f(x) m(dx) + \varepsilon \int w(x - y) m(dx) m(dy),
\]

so the potential function in their study is a specific example of the one in our paper, although their argument may be adaptable to more general cases. The simulated McKean-Vlasov process above closely resembles the underdamped MFL dynamics (2.3), except that it replaces the damping term \(-\gamma V dt\) and the Brownian noise \( \sigma\sqrt{\gamma} \) at each iteration.

Based on the results, both papers require small mean-field dependence to ensure exponential convergence, even though they differ in the assumptions on the potential function \( F \). Both papers use the component-wise coupling technique. In our paper, we apply the coupling to the Brownian noise \( dW_t \), while in [10], the authors apply a similar coupling to the Gaussian variable \( V^k_0 \). A significant difference is that we need to control both the couplings of \( X \) and \( V \) to prove the contraction for the underdamped MFL dynamics (2.3), while in [10], the authors only need to consider the coupling of \( X \) (as the law of \( V^k_0 \) is fixed). Consequently, our function to measure the coupling distance \( \psi \) in (2.11) is more complicated than the counterpart in [10], and the calculus in Section 4.4.3 is more elaborate.
3 Application to GAN

3.1 A mathematical model of GAN

The mean-field Langevin dynamics draws increasing attention among the attempts to rigorously prove the trainability of neural networks, in particular the two-layer networks (with one hidden layer). It becomes popular (see e.g. Chizat and Bach [20], Mei, Montanari and Nguyen [58], Rotskoff and Vanden-Eijnden [66], Hu, Ren, Šiška and Szpruch [46]) to rewrite the two-layer network training problem as an optimization problem over the space of probability measures. Namely, a two-layer network training problem can be formulated as

$$\inf_{c,a,b} \int \left| y - \sum_i c_i \varphi(a_i z + b_i) \right|^2 \mu(dy, dz),$$

with the distribution $\mu$ of the data $z$ and the label $y$. By considering the law $m$ of the random variable $X := (C, A, B)$ in $\mathbb{R}^n$, one can reformulate it as

$$\inf_{m \in \mathcal{P}_2(\mathbb{R}^n)} F(m),$$

where $F(m) := \int \left| y - \mathbb{E}^m[\Phi(X, z)] \right|^2 \mu(dy, dz)$ and $\Phi(X, z) := C \varphi(A z + B)$. (3.1)

In Mei, Montanari and Nguyen [58] and Hu, Ren, Šiška and Szpruch [46] the authors further add an entropic regularization to the minimization:

$$\inf_{m \in \mathcal{P}_2(\mathbb{R}^n)} \left\{ F(m) + \frac{\sigma^2}{2} H(m) \right\}. \quad (3.2)$$

It is due to [46, Proposition 2.5] that an optimal solution $m^*$ to (3.2) admits a density and satisfies the first order necessary condition

$$D_m F(m^*, x) + \frac{\sigma^2}{2} \nabla_x \log (m^*(x)) = 0,$$

where the intrinsic derivative $D_m F(m, x)$ reads, with $x = (c, a, b),$

$$D_m F(m, x) = \int 2 \left( \mathbb{E}^m[C \varphi(A z + B)] - y \right) \left( \begin{array}{c} \varphi(az + b) \\ c \varphi(az + b) \end{array} \right) \mu(dy, dz).$$

Moreover, since $F$ defined above is convex and $H$ is strictly convex, this is also a sufficient condition for $m^*$ being the unique minimizer. It has been proved in [46, Theorem 2.11] that such $m^*$ can be characterized as the invariant measure of the overdamped mean-field Langevin dynamics:

$$dX_t = -D_m F(\mathcal{L}(X_t), X_t) dt + \sigma dW_t.$$ 

Also it has been shown that the marginal laws $m_t$ converge towards $m^*$ in Wasserstein metric. Notably, the (stochastic) gradient descent algorithm used in training the neural networks can be viewed as a numerical discretization scheme for the overdamped MFL dynamics, see [46, Section 3.2] for more details. It is noteworthy that the optimization of the weights of the deep neural network (containing more than one hidden layer) can also be formulated as a mean-field optimization problem, which however is not convex. As a result, a general theory for the deep learning via mean-field optimization is still absent, though some partial results and analogs have been developed, see e.g. Hu, Kazeykina and Ren [45], Jabir, Šiška and Szpruch [48], Conforti, Kazeykina and Ren [21], Domingo-Enrich, Jelassi and Mensch [26], Šiška and Szpruch [74], Lu, Ma, Lu, Lu and Ying [55].
Recently, there is a strong interest in generating samplings according to a distribution only empirically known using the so-called generative adversarial networks (GAN), see e.g. the pioneering work [37]. From a mathematical perspective, the GAN can be viewed as a (zero-sum) game between two players: the generator and the discriminator, and can be trained through an overdamped Langevin process, see e.g. Conforti, Kazeykina and Ren [21], Domingo-Enrich, Jelassi, Mensch, Rotkoff and Bruna [26]. On the other hand, it has been empirically observed and theoretically proved in some cases in Cheng, Chatterji, Bartlett and Jordan in [19] as well as in Cheng, Chatterji, Abbasi-Yadkori, Bartlett and Jordan [18] that the simulation of the underdamped Langevin process converges more quickly than that of the overdamped Langevin dynamics. Therefore, in this section we shall implement an algorithm to train the GAN through the underdamped mean-field Langevin dynamics.

We first recall the mathematical model of GAN in [21]. The task of the discriminator is to compute the Nash equilibrium of the zero-sum game:

\[
\begin{align*}
D(\mu, \hat{\mu}) & := \sup_{m^X \in \mathcal{P}_2(\mathbb{R}^n)} \left\{ -F_\phi(m^X, \mu) \right\} := \sup_{m^X \in \mathcal{P}_2(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\Phi(X, z)| (\mu - \hat{\mu})(dz),
\end{align*}
\]

where \( z \mapsto \mathbb{E}[\Phi(X, z)] \) is the output of the two-layer network with an activation function \( \varphi \) as in (3.1). Indeed, the functional \( D(\cdot, \cdot) \), resembling the dual form of the Wasserstein-1 metric, can be viewed as a distance between two probability measures, and hence our model is similar to the mathematical model of the once popular Wasserstein-GAN [1]. On the other hand, the generator aims at sampling a probability measure \( \mu \) so as to minimize the distance \( D(\mu, \hat{\mu}) \).

The solution to this minimization problem is the required distribution \( \hat{\mu} \).

Let us add the velocity variable \( V \) and the regularizers to the potential \( F_\phi \):

\[
\mathcal{F}(m, \mu) := -F_\phi(m^X, \mu) - \frac{\eta}{2} \mathbb{E}[|V|^2] + \frac{\lambda_0}{2} \int |z|^2 \mu(dz) - \frac{\lambda_1}{2} \mathbb{E}[|X|^2] + \frac{\sigma_0^2}{2} H(\mu) - \frac{\eta \sigma_1^2}{2 \gamma} H(m).
\]

The regularized GAN aims at computing the Nash equilibrium of the zero-sum game:

\[
\begin{align*}
\text{generator} : & \inf_{\mu \in \mathcal{P}_2(\mathbb{R}^n)} \mathcal{F}(m, \mu) \\
\text{discriminator} : & \sup_{m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)} \mathcal{F}(m, \mu).
\end{align*}
\]  

(3.3)

**Remark 3.1.** A zero-sum game is a game in which the two players aim at minimizing and maximizing the same objective function. In particular, let \( (m^*, \mu^*) \) be a Nash equilibrium of the zero game (3.3), i.e.

\[
\begin{align*}
\mu^* & = \arg \min_{\mu \in \mathcal{P}_2(\mathbb{R}^n)} \mathcal{F}(m^*, \mu) \\
m^* & = \arg \max_{m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)} \mathcal{F}(m, \mu^*)
\end{align*}
\]

(since \( \mu \mapsto \mathcal{F}(m^*, \mu) \) and \( m \mapsto -\mathcal{F}(m, \mu^*) \) are both strictly convex, the minimizer and the maximizer above are both unique). It is well-known that the couple \( (m^*, \mu^*) \) solves the min-max problem:

\[
\mathcal{F}(m^*, \mu^*) = \inf_{\mu \in \mathcal{P}_2(\mathbb{R}^n)} \sup_{m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)} \mathcal{F}(m, \mu) = \sup_{m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)} \inf_{\mu \in \mathcal{P}_2(\mathbb{R}^n)} \mathcal{F}(m, \mu).
\]

**Remark 3.2.** Here we briefly report the relation between the regularized game (3.3) and the non-regularized one. Take a sequence \( (\lambda_0^n, \lambda_1^n, \sigma_0^n, \sigma_1^n) \to 0 \) as \( n \to \infty \), denote the corresponding regularized function by \( \mathcal{F}^n \), and denote by \( (m^n, \mu^n) \) the corresponding Nash equilibria. Let \( (m^*, \mu^*) \) be one of the \( \mathcal{W}_2 \)-accumulation point of \( (m^n, \mu^n) \), namely it is a Wasserstein-2 limit of a subsequence, still denoted by \( (m^n, \mu^n) \).
Note that the regularizers for the generator and the discriminator, namely $\frac{\lambda_0}{2} \int |z|^2 \mu(dz) + \frac{\lambda_1}{2} \mathbb{E}^m ||X||^2 + \frac{\eta}{2} \mathbb{E}^m [V]^2$, both $\Gamma$-converge (with respect to the $W_2$-distance) to 0 as $n \to \infty$, see e.g. the proof of [36, Proposition 2.3]. As a result, we have

$$
limit_{n \to \infty} \sup \mathcal{F}_n(m^n, \mu^n) = \inf_{\mu \in \mathcal{P}_2} \mathcal{F}(m^*, \mu) \quad \text{and} \quad \lim \inf_{n \to \infty} \mathcal{F}_n(m^n, \mu^n) = \sup_{m \in \mathcal{P}_2} \mathcal{F}(m, \mu^*) \quad \text{(3.4)},$$

where

$$\mathcal{F}(m, \mu) := -F_X(mX, \mu) - \frac{\eta}{2} \mathbb{E}^m [V]^2.$$  

It follows from (3.4) that

$$\mathcal{F}(m^*, \mu^*) \geq \inf_{\mu \in \mathcal{P}_2} \mathcal{F}(m^*, \mu) \geq \sup_{m \in \mathcal{P}_2} \mathcal{F}(m, \mu^*) \geq \mathcal{F}(m^*, \mu^*),$$

in other word, $(m^*, \mu^*)$ is a Nash equilibrium of the limit game. In particular, we have

$$D(m^*, \mu) = \sup_{m \in \mathcal{P}_2} \left\{ -F_X(mX, \mu^*) \right\} = \sup_{m \in \mathcal{P}_2} \left\{ -F_X(mX, \mu^*) - \frac{\eta}{2} \mathbb{E}^m [V]^2 \right\} = \mathcal{F}(m^*, \mu^*) = \inf_{\mu \in \mathcal{P}_2} \sup_{m \in \mathcal{P}_2} \mathcal{F}(m, \mu) = \inf_{\mu \in \mathcal{P}_2} \sup_{m \in \mathcal{P}_2} \left\{ -F_X(mX, \mu) - \frac{\eta}{2} \mathbb{E}^m [V]^2 \right\} = \inf_{\mu \in \mathcal{P}_2} D(m, \mu) = 0,$$

where the third and forth equalities are due to the fact that $(m^*, \mu^*)$ is a Nash equilibrium. Finally we conclude that $\mu^* = \hat{\mu}$, and that the initial sequence (not only the subsequence) $\mu^n \to \hat{\mu}$ in $W_2$. Therefore, when $\lambda_0^*, \lambda_1^*, \sigma_0^*, \sigma_1^*$ are small, the generator should eventually generate the distribution $\mu^n$ close to the target distribution $\hat{\mu}$.

In order to compute the equilibrium of the game (3.3), we observe as in Conforti, Kazeykina and Ren [21] that given the choice $m$ of the discriminator, the optimal response $\mu^*[m]$ of the generative must satisfy the first order condition

$$\frac{\delta F}{\delta \mu}(m, \mu^*[m], z) + \frac{\sigma_0^2}{2} \log \mu^*[m](z) = \text{Constant}.$$  

Therefore it has the density

$$\mu^*[m](z) = C(m) e^{-\frac{\delta}{2} \left( e^{mX} [\Phi(X, z)] + \lambda_1^* |z|^2 \right)}, \quad \text{(3.5)}$$

where $C(m)$ is the normalization constant depending on $m$. Then computing the value of the zero-sum game becomes an optimization over $m$:

$$\sup_{m \in \mathcal{P}_2} \inf_{\mu \in \mathcal{P}_2} \mathcal{F}(m, \mu) = \sup_{m \in \mathcal{P}_2} \mathcal{F}(m, \mu^*[m]).$$
Thanks to Theorem 2.11, the optimizer of the problem above can be characterized by the invariant measure of the underdamped MFL dynamics
\[
dX_t = \eta V_t dt, \quad dV_t = -(D_m F(L(X_t), X_t) + \gamma V_t) dt + \sigma_1 dW_t, \tag{3.6}
\]
with the potential function:
\[
F(m) := F_0(m^X, \mu^*[m]) + \frac{\lambda_1}{2} E_m[|X|^2] - \frac{\sigma_0^2}{2} H(\mu^*[m]).
\]
Together with (3.5), we may calculate and obtain
\[
D_m F(m, x) = \int \nabla_x \Phi(x, z)(\hat{\mu} - \mu^*[m])(dz) + \lambda_1 x. \tag{3.7}
\]

**Remark 3.3.** Instead of reporting the rigorous but tedious computation to obtain (3.7), here is a quick way to intuitively realize the form of $D_m F$. Observe the following formal calculus:
\[
D_m F(m, x) = -D_m \left( \int_{\mathbb{R}^n} E^{mX}_x[\Phi(X, z)](\mu^*[m] - \hat{\mu})(dz) + \frac{\sigma_0^2}{2} H(\mu^*[m]) \right) + \lambda_1 x
\]
\[
= \int \nabla_x \Phi(x, z)(\hat{\mu} - \mu^*[m])(dz) + \lambda_1 x
\]
\[
+ \frac{\partial}{\partial \mu^*[m]} \left( \int_{\mathbb{R}^n} E^{mX}_x[\Phi(X, z)](\mu^*[m] - \hat{\mu})(dz) + \frac{\sigma_0^2}{2} H(\mu^*[m]) \right) D_m \mu^*[m].
\]

Since by its definition $\mu^*[m] \in \arg\min_{\mu \in \mathcal{P}_2(\mathbb{R}^n)} \left( \int_{\mathbb{R}^n} E^{mX}_x[\Phi(X, z)](\mu - \hat{\mu})(dz) + \frac{\sigma_0^2}{2} H(\mu) \right)$, the formal “partial derivative” term above should be equal to 0, and this leads to the result (3.7).

### 3.2 Numerical test

We shall illustrate the theoretical result with a simple numerical example. Set $\hat{\mu}$ as the empirical law of $\hat{M} = 2000$ samples, denoted by $(\hat{z}_k)_{k \leq \hat{M}}$, of the distribution $\frac{1}{2} \mathcal{N}(-1, 1) + \frac{1}{2} \mathcal{N}(4, 1)$. We choose the activation function of our two-layer neural network as
\[
\varphi(\omega) = \max \{-10, \min\{10, \omega\}\},
\]
and the coefficients of the model as:
\[
\eta = 0.3, \quad \sigma_0 = 0.5, \quad \gamma = 3, \quad \lambda_0 = \lambda_1 = 0.02. \tag{3.8}
\]
We shall approximate the MFL dynamics (3.6) using the following Euler scheme of the $N$-particle system: for all $1 \leq i \leq N$, $t \in \mathbb{N}$,
\[
\begin{cases}
X^i_{t+1} - X^i_t = \eta V^i_t \Delta t \\
V^i_{t+1} - V^i_t = - \left( D_m F\left( \frac{1}{N} \sum_{j=1}^N \delta_{X^j_t}, X^i_t \right) + \gamma V^i_t \right) \Delta t + \sigma_1 \sqrt{\Delta t} \mathcal{N}^i_t,
\end{cases} \tag{3.9}
\]
where $(\mathcal{N}^i_t)_{t, i}$ are independent normal Gaussian random variables. In our numerical test we set
\[
N = 3000, \quad \Delta t = 0.005,
\]
and set the initial values of $(X^i, V^i)_i$ according to the following Gaussian distributions:
\[
X \sim \mathcal{N}(0, 1) \ast 12, \quad V \sim \mathcal{N}(0, 1) \ast 0.2.
\]
Recall that in order to evaluate $D_m F(\frac{1}{N} \sum_{j=1}^{N} \delta X_j^i, X^i_t)$ in the MFL dynamics, defined in (3.7), one need to sample the optimal response of the generator $\mu^*[\frac{1}{N} \sum_{j=1}^{N} \delta X_j^i]$, defined in (3.5). Given $M$ such samples, denoted by $(z^k_t)_{k \leq M}$, we may evaluate

$$D_m F(\frac{1}{N} \sum_{j=1}^{N} \delta X_j^i, X^i_t) \approx \frac{1}{M} \sum_{k=1}^{M} \nabla_x \Phi(X^i_t, \hat{z}_k) - \frac{1}{M} \sum_{k=1}^{M} \nabla_x \Phi(X^i_t, z^k_t) + \lambda_1 X^i_t.$$ 

In the numerical test, we use the Gaussian random walk Metropolis Hasting algorithm, with the optimal scaling proposed in Gelman, Roberts and Gilks [36], to generate $M = 2000$ samples of the distribution $\mu^*[\frac{1}{N} \sum_{j=1}^{N} \delta X_j^i]$. Further, on the basis of the Euler scheme (3.9), we shall apply the well-known OBABO splitting procedure for the underdamped Langevin process, see [52, Chapter 7.3.1]. Along the simulation, we record the potential energy

$$\frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{M} \sum_{k=1}^{M} \Phi(X^i_t, \hat{z}_k) - \frac{1}{M} \sum_{k=1}^{M} \Phi(X^i_t, z^k_t) \right),$$

as well as the kinetic one

$$\frac{\eta}{2} \frac{1}{N} \sum_{i=1}^{N} |\dot{X}^i_t|^2,$$

and we run 1000 iterations. As a reference, we also compute the Wasserstein-1 distance between the target distribution and the result of the generator, namely,

$$\mathcal{W}_1 \left( \bar{\mu}, \mu^* \left[ \frac{1}{N} \sum_{j=1}^{N} \delta X_j^i \right] \right) \approx \mathcal{W}_1 \left( \frac{1}{M} \sum_{k=1}^{M} \delta z_k, \frac{1}{M} \sum_{k=1}^{M} \delta z_k^i \right).$$

We have done the numerical tests for $\sigma_1$ equal to 0.5 and 1.5, respectively. In the following Figure 1, where we show the potential energy (3.10) as well as the sum of the potential energy (3.10) and the kinetic one (3.11) along the trainings. As a reference we also compute and show the $\mathcal{W}_1$-distance to the target distribution. Note that throughout the training process, the potential exhibits oscillatory behavior while the sum of the potential and kinetic energy decreases almost monotonically. Both the energy and the Wasserstein-1 distance between the target distribution and the generated one become eventually small and their histograms match satisfactorily. Note that the sum of the potential and the kinetic energy is not yet monotonously decreasing, because the entropy has not been counted in the total energy. When comparing the results for different $\sigma_1$, we observe that the energy drops more quickly with bigger $\sigma_1$ but eventually bears a bigger bias. This observation aligns with the contraction bound established in Theorem 2.16, which is highlighted in Remark 4.19.

As a comparison, we also train the GAN using the overdamped MFL dynamics as in [21]. More precisely, we simulate the diffusion following the Euler scheme:

$$X^i_{t+1} - X^i_t = -D_m F \left( \frac{1}{N} \sum_{j=1}^{N} \delta X_j^i, X^i_t \right) \Delta t + \sigma_1 \sqrt{\Delta t} N^i_t, \quad \text{for all } 1 \leq i \leq N, \ t \in \mathbb{N},$$

where again $(N^i_t)_{t \leq T}$ are independent normal Gaussian random variables. We set the same parameters as in (3.8), and choose

$$N = 3000, \ \Delta t = 0.005, \ \sigma_1 = 0.1.$$
In order to evaluate $D_m F\left(\frac{1}{N} \sum_{j=1}^N \delta_{X_j^t}, X_i^t\right)$, the samples of $\mu^*\left[\frac{1}{N} \sum_{j=1}^N \delta_{X_j^t}\right]$ are still generated by the same Gaussian random walk Metropolis Hasting algorithm as above. We run the simulation of the overdamped MFL dynamics, and compare the result to that of the underdamped MFL dynamics with the volatility $\sigma_1 = 0.1$ in Figure 2. As we can see, in both cases the Wasserstein-1 distances between the target distribution and the generated one evolve in a similar pattern as the potential energies. In the underdamped case they decrease with oscillation, whereas in the overdamped case they exhibit a predominantly monotonically decreasing trend. At the end of the training, the Wasserstein-1 distances between the target distribution and the generated one become comparably small.

4 Proofs

In this section we prove the main results stated in Section 2. In Section 4.1 we prepare some preliminary results on the properties of the marginal distributions of the underdamped MFL dynamics, in particular, the integrability result in Lemma 4.4 and the regularity result in Proposition 4.6. Using them we prove the main Theorem 2.9 (decay of free energy function) in Sec-
4.1 Some fine properties of the marginal distributions of the SDE

Let $(\Omega, \mathcal{F}, P)$ be an abstract probability space, equipped with an $n$-dimensional standard Brownian motion $W$. Let $T > 0$, $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function such that, for some constant $C > 0$,

$$|b(t, x, v) - b(t, x', v')| \leq C(|x - x'| + |v - v'|),$$

for all $(t, x, v, x', v') \in [0, T] \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, and $\sigma > 0$ be a positive constant, we study the stochastic differential equation (SDE):

$$dX_t = V_t dt, \quad dV_t = b(t, X_t, V_t)dt + \sigma dW_t,$$  \hspace{1cm} (4.1)

where the initial condition $(X_0, V_0)$ satisfies

$$\mathbb{E}[|X_0|^2 + |V_0|^2] < \infty.$$

**Remark 4.1.** (i) Under the Lipschitz condition on the drift function $b$, it is well-known that SDE (4.1) has a unique strong solution $(X, V)$, and the marginal distribution $\rho_t := \mathcal{L}(X_t, V_t)$ satisfies (in the sense of distribution) the corresponding Fokker-Planck equation:

$$\partial_t \rho + v \cdot \nabla_x \rho + v \cdot (b \rho) - \frac{1}{2} \sigma^2 \Delta_x \rho = 0.$$  \hspace{1cm} (4.2)

(ii) We will first consider the SDE with general drift function $b$ and deduce some fine properties of the density function $\rho_t(x, v)$ of $\rho_t = \mathcal{L}(X_t, V_t)$. In a second step, we apply these results to the MFL dynamic (2.3) whose marginal distribution is denoted by $m_t$. 
Existence of strict positive and smooth density function. Let us fix a time horizon $T > 0$. Let $C([0, T], \mathbb{R}^n)$ be the space of all $\mathbb{R}^n$-valued continuous paths on $[0, T]$. Denote by $\Omega := C([0, T], \mathbb{R}^n) \times C([0, T], \mathbb{R}^n)$ the canonical space, with canonical process $(X, V) = (X_t, V_t)_{0 \leq t \leq T}$ and canonical filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}$ defined by $\mathcal{F}_t := \sigma(X_s, V_s) : s \leq t)$. Let $\mathbb{P}$ be a (Borel) probability measure on $\Omega$, under which

$$\bar{X}_t = X_0 + \int_0^t V_s ds, \quad \text{a.s. and } (\sigma^{-1}(V_t - V_0))_{0 \leq t \leq T} \text{ is a Brownian motion,} \quad \text{(4.3)}$$

and $\mathbb{P} \circ (X_0, V_0)^{-1} = \mathbb{P} \circ (X_0, V_0)^{-1}$.

Then under the measure $\mathbb{P}$, $(\bar{X}_0, V_0)$ is independent of $(\bar{X}_t - X_0 - \bar{V}_0 t, \bar{V}_t - V_0)$, and the latter follows a Gaussian distribution with mean value 0 and $2n \times 2n$ variance matrix

$$\Sigma := \sigma^2 \begin{pmatrix} t^2 I_n/3 & t^2 I_n/2 \\ t^2 I_n/2 & t I_n \end{pmatrix}. \quad \text{(4.4)}$$

Let $\mathbb{Q} := \mathbb{P} \circ (X, V)^{-1}$ be the image measure of the solution $(X, V)$ to the SDE (4.1), so that

$$d\bar{X}_t = \bar{V}_t dt, \quad d\bar{V}_t = b(t, \bar{X}_t, \bar{V}_t)dt + \sigma d\bar{W}_t, \quad \mathbb{Q}-\text{a.s.,} \quad \text{(4.5)}$$

with a $\mathbb{Q}$-Brownian motion $\bar{W}$.

Similar to the time homogeneous context (i.e. $b(t, x, v)$ is independent of $t$) in Talay [70], we provide a result on the existence of strictly positive (smooth) density function of the marginal distribution of solution $(X, V)$ to (4.1).

**Lemma 4.2.** (i) The probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and, with $\bar{b}_s := b(s, \bar{X}_s, \bar{V}_s)$,

$$d\mathbb{Q} d\mathbb{P} \bigg|_{\mathcal{F}_T} = Z_T, \quad \text{with } Z_T := \exp \left( \int_0^t \sigma^{-2} \bar{b}_s \cdot d\bar{V}_s - \frac{1}{2} \int_0^t \sigma^{-1} \bar{b}_s|^2 ds \right). \quad \text{(4.6)}$$

(ii) Consequently, for the solution $(X, V)$ of (4.1), its marginal distribution $\mathcal{L}(X_t, V_t)$ has strictly positive density function, denoted by $\rho_t(x, v)$ for $t > 0$.

(iii) Assume in addition that $b \in C^\infty((0, T) \times \mathbb{R}^{2n})$ with all derivatives of order $k$ bounded for all $k \geq 1$. Then the function $(t, x, v) \mapsto \rho_t(x, v)$ belongs to $C^\infty((0, T) \times \mathbb{R}^{2n})$.

**Proof.** (i). Notice that $\bar{X}_t = \int_0^t \bar{V}_s ds$, $\mathbb{Q}$-a.s., we can then apply e.g. Üstünel and Zakaï [71, Theorem 2.4.2] to obtain that $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, and that $d\mathbb{Q} d\mathbb{P} \bigg|_{\mathcal{F}_T} = Z_T$.

(ii). We observe that under $\mathbb{P}$, $(X, V)$ can be written as the sum of a square integrable random variable and an independent Gaussian random variable with variance (1.4), then $\mathbb{P} \circ (X_t, V_t)^{-1}$ has strictly positive and smooth density function. Besides, $\mathbb{Q}$ is equivalent to $\mathbb{P}$, with strictly positive density $d\mathbb{Q}/d\mathbb{P} = Z_T$, it follows that $\mathbb{P} \circ (X_t, V_t)^{-1} = \mathbb{Q} \circ (X_t, V_t)^{-1}$ has also a strictly positive density function.

(iii). Under the additional regularity conditions on $b$, it is easy to check that the coefficients of SDE (4.1) satisfies the Hörmander’s conditions, and hence the density function $\rho \in C^\infty((0, T) \times \mathbb{R}^{2n})$, see e.g. Cattiaux and Mesnager [17, (1.5), P. 2] or Bally [4, Theorem 5.1, Remark 5.2].

**Estimates on the densities** We next provide an estimate on $\nabla_v \log (\rho_t(x, v))$, which is crucial for proving Theorem 2.9.

**Lemma 4.3** (Moment estimate). Suppose that $E[|X_0|^{2p} + |V_0|^{2p}] < \infty$ for some $p \geq 1$, then

$$E \left[ \sup_{0 \leq t \leq T} (|X_t|^{2p} + |V_t|^{2p}) \right] < \infty. \quad \text{(4.7)}$$
Consequently, the relative entropy between $\Omega$ and $\mathbb{P}$ is finite, i.e.

$$H(\Omega|\mathbb{P}) := \mathbb{E}^\mathbb{Q}\left[\log \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] = \mathbb{E}\left[\frac{1}{2} \int_0^T |\sigma^{-1}b(t, X_t, V_t)|^2 dt\right] < \infty. \quad (4.8)$$

**Proof.** Let us first consider $[4.7]$ under the condition $\mathbb{E}[|X_0|^2 + |V_0|^2] < \infty$. As $b$ is of linear growth in $(x, v)$, it is standard to apply Itô formula on $|X_t|^{2p} + |V_t|^{2p}$, and use BDG inequality and then Grönewall lemma to obtain $[4.7]$.

Next, by $[4.6]$, one has

$$H(\Omega|\mathbb{P}) = \mathbb{E}^\mathbb{Q}\left[\frac{1}{2} \int_0^T |\sigma^{-1}b(t, X_t, V_t)|^2 dt\right].$$

Since $\Omega = \mathbb{P} \circ (X, V)^{-1}$ (see $[4.1]$ and $[4.5]$), then it follows by the linear growth of $b$ together with $[4.7]$ that

$$H(\Omega|\mathbb{P}) = \mathbb{E}\left[\frac{1}{2} \int_0^T |\sigma^{-1}b(t, X_t, V_t)|^2 dt\right] < \infty.$$

\hfill $\square$

Let us introduce the time reverse process $(\tilde{X}, \tilde{V})$ and time reverse probability measures $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{Q}}$ on the canonical space $\Omega$ by

$$\tilde{X}_t := X_{T-t}, \quad \tilde{V}_t := V_{T-t}, \quad t \in [0, T], \quad \text{and} \quad \tilde{\mathbb{P}} := \mathbb{P} \circ (\tilde{X}, \tilde{V})^{-1}, \quad \tilde{\mathbb{Q}} := \Omega \circ (\tilde{X}, \tilde{V})^{-1}.$$

**Lemma 4.4.** The density function $\rho_t(x, v)$ is absolutely continuous in $v$, and it holds that

$$\mathbb{E}\left[\int_0^T |\nabla_v \log(b_s(x_s, V_s))|^2 ds\right] < \infty, \quad \text{for all} \quad t > 0. \quad (4.9)$$

**Proof.** This proof is largely based on the time-reversal argument in Föllmer [34, Lemma 3.1 and Theorem 3.10], where the author sought a similar estimate for a non-degenerate diffusion. For simplicity of notations, let us assume $\sigma = 1$.

**Step 1.** We first prove that, $(\tilde{X}, \tilde{V})$ is an Itô process under $\tilde{\mathbb{Q}}$, and there exists a $\tilde{\mathbb{F}}$-predictable process $\tilde{b} = (\tilde{b}_s)_s \in [0, T]$ such that, with a $(\tilde{\mathbb{F}}, \tilde{\mathbb{Q}})$-Brownian motion $\tilde{W}$,

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\int_0^T |\tilde{b}_s|^2 ds\right] < \infty, \quad \text{and} \quad \tilde{V}_t = \tilde{V}_0 + \int_0^t \tilde{b}_s ds + \tilde{W}_t, \quad \text{for all} \quad t \in [0, T]. \quad (4.10)$$

Let $\mathbb{P}_{x_0,v_0}(x_0,v_0)_{x_0,v_0} \in \mathbb{R}^n \times \mathbb{R}^n$ be a family of regular conditional probability distribution (r.c.p.d.) of $\mathbb{P}$ knowing $\sigma(X_0, V_0)$, such that $\mathbb{P}_{x_0,v_0}[X_0 = x_0, V_0 = v_0] = 1$ and Conditions (4.3) holds still true under $\mathbb{P}_{x_0,v_0}$ for every $(x_0, v_0) \in \mathbb{R}^n \times \mathbb{R}^n$. Let

$$\tilde{\mathbb{P}}_{x_0,v_0} := \mathbb{P}_{x_0,v_0} \circ (\tilde{X}, \tilde{V})^{-1}.$$

Recall the dynamic of $(\tilde{X}, \tilde{V})$ under $\tilde{\mathbb{P}}$ in (4.3) and notice that the marginal distribution of $(\tilde{X}_t, \tilde{V}_t)$ under $\tilde{\mathbb{P}}_{x_0,v_0}$ is Gaussian with density function

$$\rho_{x_0,v_0}^\tilde{X}_t(x, v) = \frac{1}{\sqrt{2\pi n |\Sigma|}} \exp\left(-\frac{1}{2} \left(\frac{x - (x_0 + v_0 t)}{v - v_0}\right)^\top \Sigma^{-1} \left(\frac{x - (x_0 + v_0 t)}{v - v_0}\right)\right),$$

where $\Sigma$ is defined by (4.4), so that

$$\Sigma^{-1} = 2\sigma^{-2} \begin{pmatrix} 6t^{-3}I_n & -3t^{-2}I_n \\ -3t^{-2}I_n & 2t^{-1}I_n \end{pmatrix}.$$
By direct computation, one obtains
\[
\tilde{c}_s(x_0, v_0, x, v) := \nabla_v \log (\rho_T^{x,v_0}(x, v)) = \frac{6(x_0 + (T-s)v_0 - x)}{(T-s)^2} + \frac{4(v_0 - v)}{T-s}, \quad s \in [0, T).
\]

It follows from Theorem 2.1 of Haussmann and Pardoux [42] (or Theorem 2.3 of Millet, Nualart and Sanz [60]) that \( \mathbf{V} \) is still a diffusion process w.r.t. \((\mathbb{F}, \mathbb{P}_{x_0, v_0})\), and
\[
\mathbf{V}_t - \mathbf{V}_0 - \int_0^t \nabla_v \log (\rho_T^{x,v_0}(X_s, V_s)) \, ds \quad \text{is a } (\mathbb{F}, \mathbb{P}_{x_0, v_0})-\text{Brownian motion on } [0, T).
\]

Notice that, by its definition, \((\mathbb{F}_{x_0, v_0})(x, v)) \in \mathbb{R}^n \times \mathbb{R}^n \) is a family of conditional probability of \( \mathbb{P} \)
knowing \((X_T, V_T)\), or equivalently \( \mathbb{P}_{x_0, v_0} : [\mathcal{F}] = \mathbb{P} \cdot [X_T, V_T] = (x_0, v_0) \), it follows that
\[
\mathbf{W}_t := \mathbf{V}_t - \mathbf{V}_0 - \int_0^t \tilde{c}_s(X_T, V_T, X_s, V_s) \, ds \quad \text{is a } (\mathbb{F}, \mathbb{P})-\text{Brownian motion on } [0, T),
\]
where the enlarged filtration \( \mathbb{F}^* = (\mathbb{F}_t^*)_{0 \leq t \leq T} \) is defined by
\[
\mathbb{F}^*_t := \sigma(X_T, V_T, X_s, V_s : s \in [0, t]).
\]

By the moment estimate (4.7), we have
\[
\mathbb{E}^\mathbb{Q}\left[ \int_0^t |\tilde{c}_s(X_T, V_T, X_s, V_s)|^2 \, ds \right] = \mathbb{E}^\mathbb{Q}\left[ \int_{T-t}^T |\tilde{c}_s(X_0, V_0, X_s, V_s)|^2 \, ds \right] < \infty, \quad \forall t \in [0, T).
\]

Next notice that the relative entropy satisfies
\[
H(\mathbb{Q}|\mathbb{P}) = H(\mathbb{Q}|\mathbb{P}) < \infty.
\]

Therefore, there exists a \( \mathbb{F}^* \)-predictable process \( \tilde{a} \) such that
\[
\mathbb{E}^\mathbb{Q}\left[ \int_0^T |\tilde{a}_t|^2 \, dt \right] = H(\mathbb{Q}|\mathbb{P}) < \infty,
\]
and
\[
\mathbf{W}_t := \mathbf{W}_t^1 - \int_0^t \tilde{a}_s \, ds = \mathbf{V}_t - \mathbf{V}_0 - \int_0^t (\tilde{a}_s + \tilde{c}_s(X_T, V_T, X_s, V_s)) \, ds, \quad t \in [0, T),
\]
is a \( (\mathbb{F}, \mathbb{Q}) \)-Brownian motion. Finally, by letting \((\tilde{b}_s) \in [0, T)\) be the predictable projection of the process \((\tilde{a}_s + \tilde{c}_s(X_T, V_T, X_s, V_s)) \in [0, T)\) w.r.t. \((\mathbb{F}, \mathbb{Q})\), we concludes the proof of Claim (4.10).

**Step 2.** Let \( \varphi : \Omega \rightarrow \Omega \) be the reverse operator defined by \( R(\omega) = (\tilde{\omega}_{T-t})_{0 \leq t \leq T} \). Then for every fixed \( t < T \) and \( \varphi \in C_c(\mathbb{R}^{2n}) \), one has
\[
\mathbb{E}^\mathbb{Q}\left[ (\tilde{b}_{T-t} \circ R) \varphi(X_t, V_t) \right] = - \lim_{h \searrow 0} \frac{1}{h} \mathbb{E}^\mathbb{Q}\left[ (\mathbf{V}_t - \mathbf{V}_{t-h}) \varphi(X_t, V_t) \right].
\]

Recall the dynamic of \((X, V)\) under \( \mathbb{Q} \) in (3.5), and thus
\[
\varphi(X_t, V_t) = \varphi(X_{t-h}, V_{t-h}) + \int_{t-h}^t \nabla_x \varphi(X_s, V_s) \cdot V_s \, ds
\]
\[
+ \int_{t-h}^t \nabla_v \varphi(X_s, V_s) \cdot dV_s + \frac{1}{2} \int_{t-h}^t \Delta_v \varphi(X_s, V_s) \, ds, \quad \mathbb{Q}\text{-a.s.}
\]
Therefore,
\[
\mathbb{E}^\overline{Q}\left[ (\nabla_t - \nabla_{t-h}) \varphi(\overline{X}_t, \overline{V}_t) \right] = \mathbb{E}^\overline{Q}\left[ (\nabla_t - \nabla_{t-h}) \varphi(\overline{X}_{t-h}, \overline{V}_{t-h}) \right] + \mathbb{E}^\overline{Q}\left[ (\nabla_t - \nabla_{t-h})(\varphi(\overline{X}_t, \overline{V}_t) - \varphi(\overline{X}_{t-h}, \overline{V}_{t-h})) \right] = \mathbb{E}^\overline{Q}\left[ \varphi(\overline{X}_{t-h}, \nabla_{t-h}) \int_{t-h}^t b(s, \overline{X}_s, \nabla s) ds \right] + \mathbb{E}^\overline{Q}\left[ \int_{t-h}^t \nabla_v \varphi(\overline{X}_s, \overline{V}_s) ds \right].
\]

Denoting
\[
\bar{b}_t := b(t, \overline{X}_t, \overline{V}_t),
\]
which clearly satisfies that \( \mathbb{E}^\overline{Q}\left[ \int_0^T |\bar{b}_t|^2 dt \right] < \infty, \quad (4.11) \)

it follows that
\[
\mathbb{E}^\overline{Q}\left[ (\bar{b}_{T-t} \circ R) \varphi(\overline{X}_t, \overline{V}_t) \right] = -\mathbb{E}^\overline{Q}\left[ \bar{b}_t \varphi(\overline{X}_t, \overline{V}_t) \right] - \mathbb{E}^\overline{Q}\left[ \nabla \varphi(\overline{X}_t, \overline{V}_t) \right].
\]

Therefore, denoting by \( \nabla_v \rho_t(x, v) \) the weak derivative of \( \rho \) in the sense of distribution, one has
\[
\int_{\mathbb{R}^2n} \nabla_v \rho_t(x, v) \varphi(x, v) dxdv = -\mathbb{E}^\overline{Q}\left[ \nabla \varphi(\overline{X}_s, \overline{V}_s) \right] = \mathbb{E}^\overline{Q}\left[ (\bar{b}_{T-t} \circ R + \bar{b}_t) \varphi(\overline{X}_t, \overline{V}_t) \right].
\]

As \( \varphi \in C_c(\mathbb{R}^2n) \) is arbitrary, this implies that, for a.e. \( (x, v) \),
\[
\nabla_v \rho_t(x, v) = \rho_t(x, v) \mathbb{E}^\overline{Q}\left[ (\bar{b}_{T-t} \circ R + \bar{b}_t) \right] \overline{X}_t = x, \overline{V}_t = v.
\]

Finally, it follows from the moment estimates in (4.10) and (4.11) that
\[
\mathbb{E}^\overline{Q}\left[ \int_{t_0}^{t_1} |\nabla_v \log (\rho_t(\overline{X}_t, \overline{V}_t))|^2 dt \right] = \mathbb{E}^\overline{Q}\left[ \int_{t_0}^{t_1} |\nabla_v \rho_t(\overline{X}_t, \overline{V}_t)|^2 dt \right] < \infty.
\]

We hence conclude the proof by the fact that \( \mathbb{P} \circ (X, V)^{-1} = \overline{Q} \circ (X, V)^{-1} \).

From (4.10), we already know that \( \overline{V} \) is a diffusion process w.r.t. \((\overline{F}, \overline{Q})\). With the integrability result (4.9), we can say more on its dynamics.

**Lemma 4.5.** The reverse process \((\overline{X}, \overline{V})\) is a diffusion process under \( \overline{Q} \), or equivalently, the canonical process \((\bar{X}, \bar{V})\) is a diffusion process under the reverse probability \( \overline{Q} \). Moreover, \( \overline{Q} \) is a weak solution to the SDE:
\[
\begin{align*}
\overline{d}X_t &= -\overline{V}_tdt, \\
\overline{d}V_t &= \left( -b(t, \overline{X}_t, \overline{V}_t) + \sigma^2 \nabla_v \log (\rho_{T-t}(\overline{X}_t, \overline{V}_t)) \right) dt + \sigma \overline{d}W_t, \quad \overline{Q}-a.s., \\
\end{align*}
\]

where \( \overline{W} \) is a \((\overline{F}, \overline{Q})\)-Brownian motion.

**Proof.** It follows from the Cauchy-Schwarz inequality and (4.9) that
\[
\int_t^T \int_{\mathbb{R}^2n} |\nabla_v \rho_t(x, v)| dxdv \leq \left( \int_t^T \int_{\mathbb{R}^2n} |\nabla_v \rho_t(x, v)|^2 dxdv \right)^{1/2} \rho_t(x, v) \leq \left( \int_t^T \int_{\mathbb{R}^2n} \rho_t(x, v) dxdv \right)^{1/2} < \infty,
\]

for all \( T > t > 0 \). Together with the Lipschitz assumption on the coefficient \( b(t, x, v) \), the desired result is a direct consequence of Haussmann and Pardoux [42 Theorem 2.1], or Millet, Nualart and Sanz [60 Theorem 2.3].

\( \square \)
Application to the MFL equation (2.3) We will apply the above technical results to the MFL equation (2.3). First, we know that the MFL SDE (2.3) has a unique strong solution \((X, V)\). We take \(m^X_t := \mathcal{L}(X_t)\) as an input and define

\[
b(t, x, v) := D_m F(m^X_t, x) + \gamma v = D_m F(m^X_t, x) + \nabla_x f(x) + \gamma v.
\]

(4.13)

Then \((X, V)\) is also the unique solution of SDE (4.1) with drift function \(b\) defined above.

**Proposition 4.6.** (i) Let Assumption 2.6(i) hold true. Then the function \(b(t, x, v)\) defined by (4.13) is a continuous function, uniformly Lipschitz in \((x, v)\).

(ii) Suppose in addition that Assumption 2.6(ii) holds true. Then \(b \in C^\infty((0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n)\) and all derivatives of order \(k\), for each \(k \geq 1\), are bounded on \((0, T] \times \mathbb{R}^n \times \mathbb{R}^n\) for any \(T > 0\).

**Proof.** (i). For a diffusion process \((X, V)\), it is clear that \(t \mapsto m^X_t := \mathcal{L}(X_t)\) is continuous under the weak convergence topology, then \((t, x, v) \mapsto b(t, x, v) := D_m F(m^X_t, x) + \gamma v\) is continuous. Moreover, it is clear that \(b\) is globally Lipschitz in \((x, v)\) under Assumption 2.6(i).

(ii). Let us denote

\[
b_o(t_0, x_0) := D_m F(m^X_{t_0}, x_0),
\]

so that Claim (4.14) is true for \(k = 0\).

Then it is enough to check the differentiability of \(b_o\). We claim that, for all \(k \geq 1\), one has

\[
\partial^k b_o(t_0, x_0) = E \left[ \sum_{i=0}^{k} \sum_{j=0}^{k-i} \phi^k_{i,j}(m^X_{t_0}, X_{t_0}, V_{t_0}, x_0) X^i_t V^j_t \right],
\]

(4.14)

where \(\phi^k_{i,j}(m^X, x, v, x_0)\) are bounded functions.

Further, it follows by Lemma 4.3 that, under additional conditions in Assumption 2.6(ii), one has \(E[\sup_{0 \leq t \leq T}(|X^i_t| + |V^j_t|)] < \infty\) for all \(T > 0\) and \(p \geq 1\). By the dominated convergence theorem, one has \(b_o \in C^\infty((0, \infty) \times \mathbb{R}^n)\) and hence \(b \in C^\infty((0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n)\), and in particular all its derivatives of order \(k\), for each \(k \geq 1\), are bounded on \((0, T] \times \mathbb{R}^n \times \mathbb{R}^n\) for any \(T > 0\).

Then it is enough to prove (4.14). Recall (see e.g. Carmona and Delarue [16 Proposition 5.102]) that for a smooth function \(\varphi: \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\), one has the Itô’s formula

\[
d \varphi(m^X_t, X_t, V_t) = \int_{\mathbb{R}^n} D_m \varphi(m^X_t, x, X_t, V_t) \cdot v m_t(dx, dv) dt + \nabla_x \varphi(m^X_t, X_t, V_t) \cdot V_t dt
\]

\[- \nabla_v \varphi(m^X_t, X_t, V_t) (D_m F(m^X_t, X_t) + \gamma V_t) dt
\]

\[+ \frac{1}{2} \sigma^2 \Delta_v \varphi(m^X_t, X_t, V_t) dt + \nabla_v^2 \varphi(m^X_t, X_t, V_t) \cdot \sigma dW_t.\]

(4.15)

First, for fixed \((t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n\), we set \(\varphi^0(m^X, x_0) := D_m F(m^X_t, x_0)\). Recall that the derivative \(D^k_m F\) of any order \(k \geq 2\) is bounded, then the derivative \(D^k_{m_o} \varphi^0\) of any order \(k \geq 1\) is bounded.

Applying Itô formula (4.15) on \(\varphi^0(m^X_t, x_0)\), we obtain that

\[
\partial b_o(t_0, x_0) = \frac{d \varphi^0(m^X_t, x_0)}{dt} \bigg|_{t=t_0} = E \left[ D_m \varphi^0(m^X_{t_0}, X_{t_0}, x_0, V_{t_0}) \right] = E \left[ \varphi^1(m^X_{t_0}, X_{t_0}, V_{t_0}, x_0) \right],
\]

with

\[
\varphi^1(m^X, x, v, x_0) := D_m \varphi^0(m^X, x, x_0) \cdot v,
\]

so that Claim (4.14) is true for \(k = 1\).

Next, applying Itô formula on \(\varphi^1(m^X_t, X_t, V_t, x_0)\), we obtain that

\[
\partial^2 b_o(t_0, x_0) = \frac{d}{dt} E \left[ \varphi^1(m^X_t, X_t, V_t, x_0) \right] \bigg|_{t=t_0} = E \left[ \varphi^2(m^X_{t_0}, X_{t_0}, V_{t_0}, x_0) \right],
\]
with
\[
\varphi^2(m^X, x, v, x_0) := D_m \varphi^1(m^X, x, v, x_0) \cdot v + \nabla_x \varphi^1(m^X, x, v) \cdot v
\]
\[
- \nabla_v \varphi^1(m^X, x, v) \cdot (D_m F(m^X, x) + \gamma v) + \frac{1}{2} \sigma^2 \Delta_v \varphi^1(m^X, x, v),
\]
so that Claim (4.14) is true for \( k = 2 \).

By repeating the same arguments with induction, it is easy to deduce that Claim (4.14) is true for all \( k \geq 1 \).

4.2 Proofs for Theorem 2.9 and Corollary 2.13

Proof of Theorem 2.9. In the context of (2.3), where the drift function \( b \) is given by (4.13), we use \( m_t(x, v) \) (rather than \( \rho_t(x, v) \)) to denote the density function of the marginal distribution of \((X_t, V_t)\).

Let us fix \( T > 0 \), and consider the reverse probability \( \tilde{Q} \) given before Lemma 4.4 with coefficient function \( b \) in (4.13). Recall also the dynamics of \((X, V)\) under \( \tilde{Q} \) in (4.12). Applying Itô’s formula on \( \log (m_{T-t}(X_t, V_t)) \), and then using the Fokker-Planck equation (1.2), it follows that

\[
d \log (m_{T-t}(X_t, V_t)) = \left\{ -\frac{\partial m_{T-t}}{m_{T-t}}(X_t, V_t) - \nabla_x \log (m_{T-t}(X_t, V_t)) \cdot \nabla_t + \frac{1}{2} \sigma^2 \Delta_v \log (m_{T-t}(X_t, V_t)) \\
+ \nabla_v \log (m_{T-t}(X_t, V_t)) \cdot (-b(t, X_t, V_t) + \sigma^2 \nabla_v \log (m_{T-t}(X_t, V_t))) \right\} dt \\
+ \nabla_v \log (m_{T-t}(X_t, V_t)) \cdot \sigma dW_t \\
= \left( -n \gamma + \frac{1}{2} \left| \frac{\sigma \nabla_v m_{T-t}}{m_{T-t}}(X_t, V_t) \right|^2 \right) dt + \nabla_v \log (m_{T-t}(X_t, V_t)) \cdot \sigma d\tilde{W}_t, \quad \tilde{Q}\text{-a.s.}
\]

Notice that \( m_t = \mathcal{L}(X_t, V_t) = \mathcal{L}^\tilde{Q}(X_{T-t}, V_{T-t}) \), then it follows by (4.9) that, for all \( t \in (0, T) \),

\[
dH(m_t) = d\mathbb{E}^\tilde{Q}\left[ \log (m_t(X_{T-t}, V_{T-t})) \right] \\
= \left( n \gamma - \frac{1}{2} \mathbb{E}\left[ |\sigma \nabla_v \log (m_t(X_t, X_t))|^2 \right] \right) dt.
\]

On the other hand, recall that

\[
F(m) = F_\gamma(m) + \mathbb{E}^m[f(X)], \quad \text{and} \quad D_m F(\mathcal{L}(X_t)) = D_m F_\gamma(\mathcal{L}(X_t)) + \nabla f.
\]

By a direct computation, one has

\[
dF_\gamma(\mathcal{L}(X_t)) = \mathbb{E}\left[ D_m F_\gamma(\mathcal{L}(X_t), X_t) \cdot V_t \right] dt.
\]

By Itô formula and (4.18), one has

\[
d\left( f(X_t) + \frac{1}{2} |V_t|^2 \right) \\
= \left( \nabla f(X_t) \cdot V_t - V_t \cdot (D_m F(\mathcal{L}(X_t), X_t) + \gamma V_t) + \frac{1}{2} \sigma^2 n \right) dt + V_t \cdot \sigma dW_t \\
= \left( -D_m F_\gamma(\mathcal{L}(X_t), X_t) \cdot V_t - \gamma |V_t|^2 + \frac{1}{2} \sigma^2 n \right) dt + V_t \cdot \sigma dW_t.
\]
Combining (4.17), (4.19) and (4.20), we obtain
\[ d\mathcal{F}(m_t) = d\left( F(\mathcal{L}(X_t)) + \frac{1}{2} \mathbb{E}[|V_t|^2] + \frac{\sigma^2}{2\gamma} H(m_t) \right) \]
\[ = \mathbb{E} \left[ -\gamma |V_t|^2 + \sigma^2 - \frac{\sigma^4}{4\gamma} \nabla_v \log \left( m_t(X_t, V_t) \right) \right] dt. \tag{4.21} \]

Further, by Lemmas 4.3 and 4.4 it is clear that \( \mathbb{E}[|\nabla_v \log \left( m_t(X_t, V_t) \right) \cdot V_t|] < \infty \) and by integration by parts we have
\[ \mathbb{E}[\nabla_v \log \left( m_t(X_t, V_t) \right) \cdot V_t] = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \nabla_v m_t(x, v) \cdot \nabla_v |v|^2 \right) dxdv \]
\[ = -\frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( m_t(x, v) \Delta_v |v|^2 \right) dxdv = -n. \]

Together with (4.21), it follows
\[ d\mathcal{F}(m_t) = -\gamma \mathbb{E} \left[ |V_t + \frac{\sigma^2}{2\gamma} \nabla_v \log \left( m_t(X_t, V_t) \right) |^2 \right] dt. \]

Remark 4.7 (Time reversal argument). (i) In case that \( m \mapsto F_0(m) \) is linear, Theorem 2.9 is a specific case of Fontbona and Jourdain [35, Corollary 1.6]. Indeed, when \( m \mapsto F_0(m) \) is linear, i.e. \( F_0(m) = \int f_0(x)m(dx) \), the free energy function can be viewed as a relative entropy \( m \mapsto H(m\mu) \) for some reference measure \( \mu \), see Remark 2.5. In particular, \( \mu \) shall be the invariant measure of the (classical) Langevin dynamics. Define \( \tilde{Q}^* \) to be the law of the time-reverse Langevin diffusion starting from the invariant distribution \( \mu \). In [35], the authors observed that the likelihood process
\[ \ell_t(X_t, V_t) := \frac{m_{T-t}(X_t, V_t)}{\mu}(X_t, V_t) \]
is a \( \tilde{Q}^* \)-martingale. Together with the dynamics of \((X_t, V_t)\) under \( \tilde{Q} \) and the Itô’s formula, one obtains that \( d\ell_t(X_t, V_t) = \sigma \nabla_v \ell_t(X_t, V_t) dW_t^* \) with a \( \tilde{Q}^* \)-Brownian motion \( W_t^* \) and further that
\[ d\ell_t \log \ell_t(X_t, V_t) = \frac{\sigma^2}{2} |\nabla_v \log \ell_t|^2 \ell_t(X_t, V_t) dt + \sigma (\log \ell_t + 1) \nabla_v \ell_t(X_t, V_t) dW_t^*. \]
Finally note that
\[ dH(m_t|\mu) = -d\mathbb{E}^{\tilde{Q}^*} \left[ \ell_{T-t} \log \ell_{T-t}(X_{T-t}, V_{T-t}) \right] \]
\[ = -\int \frac{\sigma^2}{2} |\nabla_v \log m_t|^2 m_t(x, v) dxdv dt, \]
which gives the result of Theorem 2.9.

(ii) In the general case where \( m \mapsto F_0(m) \) is nonlinear, we still share a similar backward pathwise calculus for the likelihood in (4.16) as in the proof of [35, Theorem 1.4]. More naturally, one might try to apply a forward pathwise calculus on \( \log(m_t(X_t, V_t)) \) instead of the backward one in (4.16). However, by doing so, we shall face an extra term involving \( \Delta_v m_t(X_t, V_t) \). Of course, one might further expect to cancel this term under the expectation by applying the integration by part
\[ \mathbb{E} \left[ \frac{\Delta_v m_t(X_t, V_t)}{m_t(X_t, V_t)} \right] = \int \Delta_v m_t(x, v) dxdv = 0. \]
Nevertheless, in order to make it rigorous, one needs some integrability property of the \( \Delta_v m_t \) term, which seems nontrivial to us. The backward pathwise calculus in (4.16) avoids this technical difficulty.
Let us define two processes \( W \) and \( P \) such that

\[
W_t = \left( X_t^\gamma, Y_t^\gamma \right), \quad P_t = (X_t^0, Y_t^0),
\]

where, by Lipschitz property of \( D \), \( \sqrt{\gamma} \) satisfies

\[
\mathbb{E}\left[ |K_\gamma| \right] \leq C \mathbb{E}\left[ |Y_t^\gamma - X_t^\gamma| \right] = C \mathbb{E}\left[ |V_t^\gamma| \right] / \gamma.
\]

As \( \sup_{\gamma > 0} \sup_{t \geq 0} \mathbb{E}\left[ |V_t^\gamma|^2 \right] < \infty \), one has for all \( t \geq 0 \),

\[
\mathbb{E}\left[ \int_0^t |K_\gamma| ds \right] \to 0, \quad \text{as } \gamma \to \infty.
\]

Then one can easily apply (with some trivial adaptation) the standard stability result of SDEs (see e.g. Jacod and Mémin [49, Section 3]) to obtain that \( Y^\gamma \) converges weakly on \( \mathcal{P}(C([0, \infty); \mathbb{R}^n)) \) to the overdamped MFL dynamics \( Y \) defined in (2.9), as \( \gamma \to \infty \).

Finally, as \( \sup_{\gamma > 0} \sup_{t \geq 0} \mathbb{E}\left[ |V_t^\gamma|^2 \right] < \infty \), it follows that, for all \( t \geq 0 \), \( X_t^\gamma \to Y_t \) weakly, as \( \gamma \to \infty \).

4.3 Proof of Theorem 2.11

Let \( (m_t)_{t \geq 0} \) be the flow of marginal laws of the solution to (2.3), given an initial law \( m_0 \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \). Define \( S(t)[m_0] := m_t \). We shall consider the so-called \( w \)-limit set:

\[
w(m_0) := \left\{ \mu \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) : \text{ there exist } t_k \to \infty \text{ such that } W_1 \left( S(t_k)[m_0], \mu \right) \to 0 \right\}.
\]

Let us recall LaSalle’s invariance principle for dynamical system.

**Lemma 4.8.** Let Assumption 2.6 hold true. Then \( (S(t))_{t \geq 0} \) is a dynamical system on \( \mathcal{W}_1 \) space, i.e.

(i). \( S(0) \) is identity;

(ii). \( S(t)(S(t')[\mu]) = S(t + t')[\mu] \) for all \( \mu \) and \( t, t' \geq 0 \);

\[27\]
(iii). for each $\mu$, $t \mapsto S(t)[\mu]$ is continuous (w.r.t. the weak convergence topology);
(iv). for each $t \geq 0$, $\mu \mapsto S(t)[\mu]$ is $\mathcal{W}_1$-continuous.

Proof. The properties (i), (ii) are trivial. The continuity in (iii) follows from the standard stability result for the McKean-Vlasov SDE. Here we prove the property (iv). Under the upholding assumptions, it follows from Lemma 4.3 that

$$\sup_{t \leq T} \int (|x|^2 + |v|^2) \, m_t(dx, dv) < \infty.$$ 

Together with the fact that $t \mapsto m_t$ is continuous w.r.t. the weak convergence topology, we deduce that $t \mapsto S(t)$ is continuous with respect to the $\mathcal{W}_1$-topology. \hfill \Box

**Proposition 4.9.** [Invariance Principle for dynamical system] Let Assumption 2.6 hold true. Then the set $w(m_0)$ is nonempty, $\mathcal{W}_1$-compact and invariant, that is,

1. for any $\mu \in w(m_0)$, we have $S(t)[\mu] \in w(m_0)$ for all $t \geq 0$;
2. for any $\mu \in w(m_0)$ and all $t \geq 0$, there exists $\mu' \in w(m_0)$ such that $S(t)[\mu'] = \mu$.

Proof. Due to Theorem 2.9 we know that $\{Z(m_t)\}_{t \geq 0}$ is bounded. Together with the lower bound of the energy function (2.8), we obtain

$$C := \sup_{t \geq 0} \mathbb{E}[|X_t|^2 + |V_t|^2] < \infty.$$ 

Therefore $(S(t)[m_0])_{t \geq 0} = (m_t)_{t \geq 0}$ lies in the $\mathcal{W}_1$-compact set

$$\left\{ m \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) : \int (|x|^2 + |v|^2) \, m(dx, dv) \leq C \right\}.$$ 

Then the desired invariance result follows from 43. Theorem 4.3.3. \hfill \Box

**Lemma 4.10.** Let Assumption 2.6 hold true. Then, every $m^* \in w(m_0)$ has a density and we have

$$v + \frac{\sigma^2}{2\gamma} \nabla_v \log (m^*(x,v)) = 0, \quad \text{Leb}^{2n} - a.e.$$ 

(4.23)

Proof. Let $m^* \in w(m_0)$ and denote by $(m_{tk})_{k \in \mathbb{N}}$ the subsequence converging to $m^*$ in $\mathcal{W}_1$.

**Step 1.** We first prove that there exists a sequence $\delta_i \to 0$ such that

$$\liminf_{k \to \infty} \mathbb{E} \left[ |V_{tk+\delta_i} + \frac{\sigma^2}{2\gamma} \nabla_v \log (m_{tk+\delta_i}(X_{tk+\delta_i}, V_{tk+\delta_i}))|^2 \right] = 0, \quad \text{for all } i \in \mathbb{N}.$$ 

(4.24)

Suppose the contrary. Then we would have for some $\delta > 0$

$$0 < \int_0^\delta \liminf_{k \to \infty} \mathbb{E} \left[ |V_{tk+s} + \frac{\sigma^2}{2\gamma} \nabla_v \log (m_{tk+s}(X_{tk+s}, V_{tk+s}))|^2 \right] ds \leq \liminf_{k \to \infty} \int_0^\delta \mathbb{E} \left[ |V_{tk+s} + \frac{\sigma^2}{2\gamma} \nabla_v \log (m_{tk+s}(X_{tk+s}, V_{tk+s}))|^2 \right] ds,$$

where the last inequality is due to Fatou’s lemma. This is a contradiction against Theorem 2.9 and the fact that $\mathbb{F}$ is bounded from below.
Step 2. Denote by \( t_k^i := t_k + \delta_i \) and \( m_t^* := S(t)[m^*] \). Note that

\[
\lim_{k \to \infty} \mathcal{W}_1 \left( m_t^k, m^* \right) = 0
\]

\[
\implies \lim_{k \to \infty} \mathcal{W}_1 \left( m_t^k, m^*_k \right) = \lim_{k \to \infty} \mathcal{W}_1 \left( S(\delta_i)[m_t^k], S(\delta_i)[m^*] \right) = 0.
\]

Now fix \( i \in \mathbb{N} \). Due to Theorem 2.9 and the fact that \( \{H(m_t^k) + \frac{1}{2} \mathbb{E} |V_t|^2 \}_{t \geq 0} \) is bounded from below, the set \( \{H(m_t^k)\}_{k \in \mathbb{N}} \) is uniformly bounded. Therefore the densities \( (m_t^k)_{k \in \mathbb{N}} \) are uniformly integrable with respect to Lebesgue measure, and thus \( m^* \) has a density. Note that

\[
0 = \lim_{k \to \infty} \mathbb{E} \left[ V_{t_k^i} + \frac{\sigma^2}{2\gamma} \nabla_v \log \left( m_t^k (X_{t_k^i}, V_{t_k^i}) \right) \right]^2
\]

\[
\geq C \limsup_{k \to \infty} \int \left( h_k^i \log (h_k^i) \right) dx.
\]

It is noteworthy that \( \int_{\mathbb{R}^n} \frac{\nabla_v (m_t^k (x,v) e^{\frac{\sigma^2}{2\gamma} |v|})^2}{m_t^k (x,v) e^{\frac{\sigma^2}{2\gamma} |v|^2}} e^{-\frac{\sigma^2}{2\gamma} |v|^2} dv \) is the relative Fisher information of the law \( m_t^k (x, \cdot) \) with respect to the Gaussian distribution \( \mu_v := \mathcal{N}(0, \sigma^2 I_n) \). Define the function

\[
h_k^i (x,v) := m_t^k (x,v) e^{\frac{\sigma^2}{2\gamma} |v|^2}.
\]

By logarithmic Sobolev inequality for the Gaussian distribution we obtain

\[
\int \left( \left( h_k^i \log (h_k^i) \right) dx \right) \leq C \int \frac{\nabla_v h_k^i}{h_k^i} \cdot \mu_v dx.
\]

Together with (4.24) we obtain

\[
0 = \lim_{k \to \infty} \mathbb{E} \left[ V_{t_k^i} + \frac{\sigma^2}{2\gamma} \nabla_v \log \left( m_t^k (X_{t_k^i}, V_{t_k^i}) \right) \right]^2
\]

\[
\geq C \limsup_{k \to \infty} \int \left( \left( h_k^i \log (h_k^i) \right) dx \right).
\]

Since \( \int h_k^i \mu_v^* = \int m_t^k dv = m_t^X \), we further have

\[
0 \geq C \limsup_{k \to \infty} \int \left( m_t^k \log (h_k^i) - m_t^k \log (m_t^X) \right) dv dx
\]

\[
= C \limsup_{k \to \infty} \int \left( m_t^k \log \frac{m_t^X}{m_t^k e^{-\frac{\sigma^2}{2\gamma} |v|^2}} \right) dv dx
\]

\[
= C \limsup_{k \to \infty} H \left( m_t^X \bigg| \mathcal{N} \left( 0, \frac{\sigma^2}{2\gamma} I_n \right) \right)
\]

\[
\geq CH \left( m^* \bigg| \mathcal{N} \left( 0, \frac{\sigma^2}{2\gamma} I_n \right) \right).
\]

The last inequality is due to the lower semi-continuity of the relative entropy in weak topology. Finally, since \( \lim_{t \to \infty} \mathcal{W}_1 (m_t^k, m^*) = 0 \), we get

\[
H \left( m^* \bigg| \mathcal{N} \left( 0, \frac{\sigma^2}{2\gamma} I_n \right) \right) = 0
\]

and thus

\[
m^* = \mathcal{N} \left( 0, \frac{\sigma^2}{2\gamma} I_n \right).
\]

This immediately implies (4.23).
Lemma 4.11. Let Assumption 2.6 hold true. Then, each \( m^* \in w(m_0) \) is equivalent to Lebesgue measure.

**Proof.** By the invariant principle we may find a probability measure \( m^* \in w(m_0) \) such that \( m^* = S(t)[m^*] \) for a fixed \( t > 0 \). Then the desired result follows from Lemma 4.2.

Note that the necessary condition \((4.23)\) for \( m^* \in w(m_0) \) is not enough to identify \( m^* \) as the invariant measure as required in Theorem 2.11. We are going to trigger the invariance principle to complete the proof.

**Proof of Theorem 2.11.** Since \( F \) is convex, so that \( \mathcal{F} \) is strictly convex, so that the optimization problem \( \inf_{m \in \mathcal{P}_{\mu}(\mathbb{R}^n)} \mathcal{F}(m) \) has a unique minimizer \( m^* \), which is given by \((2.5)\). Therefore, to conclude the proof, it is enough to check that any \( m^* \in w(m_0) \) satisfies \((2.6)\).

Let \( m^* \in w(m_0) \) and define \( m^*_t := S(t)[m^*] \) for all \( t \geq 0 \). Denote by \((X^*_t, V^*_t)_{t \geq 0}\) the solution to the MFL equation \((2.3)\) with initial distribution \( m^* \). Take a test function \( h \in C^1(\mathbb{R}^n) \) with compact support. It follows from Itô’s formula that

\[
dV^*_t h(X^*_t) = \left( -h(X^*_t)(D_mF(m^*_t, X^*_t) + \gamma V^*_t) + (\nabla_x h(X^*_t) \cdot V^*_t)V^*_t \right) dt + \sigma h(X^*_t)dW_t,
\]

where \( m^*_t \) denotes the pushforward measure of \( m^*_t \) under the map \((x, v) \mapsto x\). By the invariance principle, we have \( m^*_t \in w(m_0) \) for all \( t \geq 0 \), and by Lemma 4.10 we have

\[
v + \frac{\sigma^2}{2\gamma} \nabla_v \log \left( m^*_t(x, v) \right) = 0, \quad \text{Leb}^2 - \text{a.e.}
\]

So there exists a measurable function \((t, x) \mapsto \hat{m}_t(x)\) such that \( m^*_t(x, v) = e^{-\frac{\sigma^2}{2\gamma} v^2} \hat{m}_t(x) \). In particular, we observe that for each \( t \geq 0 \), the random variables \( X^*_t, V^*_t \) are independent and \( V^*_t \) follows the Gaussian distribution \( \mathcal{N}(0, \frac{\sigma^2}{2\gamma} I_n) \). Taking expectation on both sides of \((4.26)\), we obtain

\[
0 = \mathbb{E} \left[ -h(X^*_t)(D_mF(m^*_t, X^*_t) + \gamma V^*_t) + (\nabla_x h(X^*_t) \cdot V^*_t)V^*_t \right]
\]

\[
= \mathbb{E} \left[ -h(X^*_t)D_mF(m^*_t, X^*_t) + \frac{\sigma^2}{2\gamma} \nabla_x h(X^*_t) \right], \quad \text{for a.e. } t > 0.
\]

Observe that

\[
\mathbb{E} [\nabla_x h(X^*_t)] = C_t \int_{\mathbb{R}^n} \nabla_x h(x) \hat{m}_t(x) dx = -C_t \int_{\mathbb{R}^n} h(x) \nabla_x \hat{m}_t(x) dx,
\]

where \( C_t \) is the normalization constant such that \( C_t \hat{m}_t \) is a density function, and \( \nabla_x \hat{m}_t \) is the weak derivative in sense of distribution. Together with \((4.27)\) we have

\[
\int_{\mathbb{R}^n} h(x) \left( -D_mF(m^*_t, x) \hat{m}_t(x) - \frac{\sigma^2}{2\gamma} \nabla_x \hat{m}_t(x) \right) dx = 0, \quad \text{for a.e. } t > 0.
\]

Notice that for all \( N > 0 \), the Hilbert space \( L^2([-N, N]^n) \) has a countable (smooth functional) basis, this allows us to consider arbitrary \( h \) in a countable space to obtain that

\[
D_mF(m^*_t, x) + \frac{\sigma^2}{2\gamma} \nabla_x \log (\hat{m}_t(x)) = 0, \quad m^*_t - \text{a.s. for a.e. } t > 0.
\]
By Lemma 4.11, \( m_t^* \) is equivalent to Lebesgue measure, and thus we have

\[
\begin{align*}
D_m F(m_t^X, x) + \frac{\sigma^2}{2} \nabla_x \log (m_t^*(x, v)) &= 0, \\
v + \frac{\sigma^2}{2} \nabla_v \log (m_t^*(x, v)) &= 0,
\end{align*}
\]

for all \((x, v) \in \mathbb{R}^+ \times \mathbb{R}^{2n}\), for a.e. \( t > 0 \).

Therefore, by Lemma 2.2 one has \( m_t^* = m \). Taking into account that \( \lim_{t \to 0} W_1(m_t^*, m^*) = 0 \), we obtain \( m^* = m \). This is enough conclude that \( w(m_0) = \{m\} \), and thus \( \lim_{t \to \infty} W_1(m_t, m) = 0 \).

4.4 Exponential ergodicity given small mean-field dependence

Under Assumption 2.6 and Assumption 2.14, we consider the following equation:

\[
\begin{align*}
dX_t &= V_t dt, \\
dV_t &= -b(m_t^X, X_t) + \lambda X_t + \gamma V_t dt + \sigma dW_t,
\end{align*}
\]

(4.28)

where \( b : \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R} \) is Lipschitz in the variable \( x \)

\[
|b(m, x) - b(m, x')| \leq L|x - x'|,
\]

and for any \( \varepsilon > 0 \) there exists \( M > 0 \) such that for any \( m, m' \in \mathcal{P}_2(\mathbb{R}^n) \)

\[
|b(m, x) - b(m', x')| \leq \varepsilon|x - x'|, \quad \text{whenever } |x - x'| \geq M,
\]

and for each \( x \in \mathbb{R}^n \)

\[
|b(m, x) - b(m', x)| \leq \iota W_1(m, m'),
\]

(4.29)

where the constant \( \iota > 0 \) is small enough, satisfying the quantitative condition (4.46) below. In the sequel, for the simplicity of notation, we write \( b \) instead \( D_m F \). Besides, we notice that for this part the special form \( D_m F \), the intrinsic derivative of a function, is not necessary.

4.4.1 Reflection-Synchronous Coupling

We are going to show the contraction result in Theorem 2.16 via the coupling technique. Let \((X, V)\) and \((X', V')\) be the two solutions of (4.28) driven by the Brownian motions \( W \) and \( W' \), respectively. Define \( \delta X = X - X' \) and \( \delta V = V - V' \). We introduce the change of variable

\[
P_t := \delta V_t + \gamma \delta X_t.
\]

Then, the processes \( \delta X \) and \( P \) satisfy the following stochastic differential equations

\[
\begin{align*}
d\delta X_t &= (P_t - \gamma \delta X_t) dt, \\
\delta P_t &= -(\delta b_t + \lambda \delta X_t) dt + \sigma d\delta W_t,
\end{align*}
\]

where \( \delta W = W - W' \) and \( \delta b_t := b(m_t^X, X_t) - b(m_t^{X'}, X'_t) \).

Remark 4.12. We shall apply the reflection-synchronous coupling following the blueprint in Eberle, Guillin and Zimmer [31], of which the main idea is to separate the space \( \mathbb{R}^n \times \mathbb{R}^n \) into two parts:

(i). \( (\delta X_t, P_t) \) locates in a compact set;

(ii). \( |\delta X_t| + \eta|P_t| \) is big enough, where the constant \( \eta \) is to be determined.
As in [31] we are going to apply the reflection coupling on the area (i) and the synchronous coupling on the area (ii). However, note that in [31] the argument for the contraction on the area (ii) relies on a Lyapunov function, which can no longer play its role in the mean-field context. Therefore, we are going to construct another function (the function $G$ in (2.11)) which decays exponentially on the area (ii).

Recall the definitions
\[ r_t := |\delta X_t|, \quad u_t := |P_t|, \quad z_t := \delta X_t \cdot P_t. \]

Let $\xi > 0$. For technical reason we shall also apply the synchronous coupling on the area $u_t < \xi$, and eventually we will let $\xi \downarrow 0$. In order to couple the two processes $(X, V), (X', V')$, we consider two Lipschitz continuous functions $r_c, s_c : \mathbb{R}^{2n} \rightarrow [0, 1]$ such that $r_c^2 + s_c^2 \equiv 1$,

\[ r_c(\delta X_t, P_t) = \begin{cases} 0, & \text{when } u_t = 0 \text{ or } r_t + \eta u_t \geq 2M + \xi, \\ 1, & \text{when } u_t \geq \xi \text{ and } r_t + \eta u_t \leq 2M. \end{cases} \]

The values of the constants $\eta, M \in (0, \infty)$ will be determined later. Define
\[ e_t^x := \begin{cases} \frac{\delta X_t}{|\delta X_t|}, & \text{if } \delta X_t \neq 0, \\ 0, & \text{if } \delta X_t = 0, \end{cases} \quad \text{and} \quad e_t^p := \begin{cases} \frac{P_t}{|P_t|}, & \text{if } P_t \neq 0, \\ 0, & \text{if } P_t = 0. \end{cases} \]

With two independent Brownian motions $W^{rc}$ and $W^{sc}$ we consider the following coupling
\[
\begin{align*}
    dW_t &= r_c(\delta X_t, P_t)dW_t^{rc} + s_c(\delta X_t, P_t)dW_t^{sc}, \\
    dW'_t &= r_c(\delta X_t, P_t)(I_n - 2e_t^p(e_t^p)^\top)dW_t^{rc} + s_c(\delta X_t, P_t)dW_t^{sc},
\end{align*}
\]
in particular we have $d\delta W_t = 2r_c(\delta X_t, P_t)e_t^p(e_t^p)^\top dW_t^{rc}$. By Lévy characterization, the process $B_t := (e_t^p)^\top W_t^{rc}$ is a one-dimensional Brownian motion. For the sake of simplicity, denote $r_c := r_c(\delta X_t, P_t)$. We notice that the Lipschitz continuity of the functions $r_c, s_c$ ensures the existence and uniqueness of the coupling process.

To conclude, with the reflection-synchronous coupling, the processes $\delta X$ and $P$ satisfy the following stochastic differential equations
\[
\begin{align*}
    d\delta X_t &= (P_t - \gamma \delta X_t)dt, \\
    dP_t &= - (\delta b_t + \lambda \delta X_t)dt + 2\sigma r_c e_t^p dB_t. \tag{4.30}
\end{align*}
\]

### 4.4.2 The auxiliary function

As reported in Remark 4.12, the main novelty of our contraction result is to construct a function exponentially decaying along the process (4.30) when $r + \eta u$ is sufficiently large. In this subsection we are going to construct the auxiliary function according to the different settings.

First, it follows from (4.30) and Itô’s formula that
\[
\begin{align*}
    dr_t^2 &= 2\delta X_t \cdot d\delta X_t = 2\delta X_t \cdot 2(\delta X_t \cdot P_t - \gamma |\delta X_t|^2)dt = 2(z_t - \gamma r_t^2)dt, \\
    du_t^2 &= 2P_t \cdot dP_t + I_n \cdot d\langle P \rangle = -2(\delta b_t \cdot P_t + \lambda z_t)dt + 4\sigma r_c u_t dB_t + 4\sigma^2 r_c^2 r_t dt, \\
    dz_t &= \delta X_t \cdot dP_t + P_t \cdot d\delta X_t = (-\delta b_t \cdot \delta X_t + \lambda r_t^2 + u_t^2 - \gamma z_t)dt + 2\sigma r_c \frac{z_t}{u_t} dB_t, \\
    dr_t &= \frac{1}{|\delta X_t|} \delta X_t \cdot d\delta X_t = (e_t^x \cdot P_t - \gamma r_t)dt, \\
    du_t &= \frac{1}{2} \frac{1}{u_t} du_t^2 + \frac{1}{2} \left( -\frac{1}{4} \frac{1}{u_t^3} \right) d\langle u^2 \rangle_t = -(\delta b_t \cdot e_t^p + \lambda e_t^p \cdot \delta X_t)dt + 2\sigma r_c dB_t.
\end{align*}
\]
We write the dynamics of \((z, r^2, u^2)\) in the following way

\[
d \begin{pmatrix} z_t \\ r_t^2 \\ u_t^2 \end{pmatrix} = A \begin{pmatrix} z_t \\ r_t^2 \\ u_t^2 \end{pmatrix} dt + \begin{pmatrix} -\delta b_t \cdot \delta X_t \\ 0 \\ -2\delta b_t \cdot P_t + 4rc_t^2 \sigma^2 \end{pmatrix} dt + \begin{pmatrix} \frac{2z_t}{u_t} \\ 0 \\ \frac{4r_t \sigma}{4u_t} \end{pmatrix} \sigma r_t dB_t
\]

(4.31)

with the matrix

\[
A := \begin{pmatrix}
-\gamma & -\lambda & 1 \\
2 & -2\gamma & 0 \\
-2\lambda & 0 & 0
\end{pmatrix}.
\]

**Remark 4.13.** As we will show later, the value of \(\delta b_t\) is small whereas \(r_t + \eta u_t\) is big enough. Therefore, the coupling system is nearly linear and its contraction rate mainly depends on the matrix \(A\).

The eigenvalues of \(A\) solve the equation:

\[
0 = (\zeta + \gamma)(\zeta + \gamma + 2\lambda(\zeta + \gamma) + 2\lambda\zeta = (\zeta + \gamma)(\zeta^2 + 2\gamma \zeta + 4\lambda).
\]

We divide the discussion into two cases, based on the different values of \(\lambda\) and \(\gamma\).

(a) If \(\lambda < \frac{\gamma^2}{4}\), the matrix has three different negative eigenvalues

\[
\zeta = -\gamma, \quad \zeta = -\gamma + \sqrt{\gamma^2 - 4\lambda}, \quad \zeta = -\gamma - \sqrt{\gamma^2 - 4\lambda},
\]

in particular, it can be diagonalized. More precisely, we have \(QA = \Lambda Q\) with the transformation matrix

\[
Q := \begin{pmatrix}
-\gamma & \lambda & 1 \\
-\gamma + \sqrt{\gamma^2 - 4\lambda} & \frac{1}{2}(\gamma^2 - 2\lambda - \gamma \sqrt{\gamma^2 - 4\lambda}) & 1 \\
-\gamma - \sqrt{\gamma^2 - 4\lambda} & \frac{1}{2}(\gamma^2 - 2\lambda + \gamma \sqrt{\gamma^2 - 4\lambda}) & 1
\end{pmatrix}
\]

and the diagonal matrix \(\Lambda = \text{diag}(-\gamma, -\gamma + \sqrt{\gamma^2 - 4\lambda}, -\gamma - \sqrt{\gamma^2 - 4\lambda})\). Multiply \(Q\) on both sides of (4.31) and obtain

\[
dQ \begin{pmatrix} z_t \\ r_t^2 \\ u_t^2 \end{pmatrix} = \Lambda Q \begin{pmatrix} z_t \\ r_t^2 \\ u_t^2 \end{pmatrix} dt + Q \begin{pmatrix} -\delta b_t \cdot \delta X_t \\ 0 \\ -2\delta b_t \cdot P_t + 4rc_t^2 \sigma^2 \end{pmatrix} dt + Q \begin{pmatrix} \frac{2z_t}{u_t} \\ 0 \\ \frac{4r_t \sigma}{4u_t} \end{pmatrix} \sigma r_t dB_t.
\]

(4.32)

Further note that

\[
\left( -\gamma + \sqrt{\gamma^2 - 4\lambda} \right) z_t + \frac{1}{2} \left( \gamma^2 - 2\lambda - \gamma \sqrt{\gamma^2 - 4\lambda} \right) r_t^2 + u_t^2
\]

\[
= \left| \frac{\gamma - \sqrt{\gamma^2 - 4\lambda}}{2} \delta X_t - P_t \right|^2,
\]

(4.33)

\[
\left( -\gamma - \sqrt{\gamma^2 - 4\lambda} \right) z_t + \frac{1}{2} \left( \gamma^2 - 2\lambda + \gamma \sqrt{\gamma^2 - 4\lambda} \right) r_t^2 + u_t^2
\]

\[
= \left| \frac{\gamma + \sqrt{\gamma^2 - 4\lambda}}{2} \delta X_t - P_t \right|^2.
\]

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Now define the function $G$:

$$G(x, p) := \left| \frac{\gamma - \sqrt{\gamma^2 - 4\lambda}}{2} x - p \right|^2 + \left| \frac{\gamma + \sqrt{\gamma^2 - 4\lambda}}{2} x - p \right|^2.$$  

(4.34)

Denote by $G_t := G(\delta X_t, P_t)$. Together with (4.32), (4.33), we obtain

$$\begin{align*}
    dG_t &\leq -\left( \gamma - \sqrt{\gamma^2 - 4\lambda} \right) G_t dt \\
    &\quad + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} Q \left\{ \begin{pmatrix} -\delta b_t \cdot \delta X_t \\ -2\delta b_t \cdot P_t + 4rc_t^2 \sigma^2 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 2\gamma \sigma_t \\ 0 \\ 4u_t \end{pmatrix} \sigma rc_t dB_t \right\}.
\end{align*}$$

(b) If $\lambda > \frac{\gamma^2}{4}$, the eigenvalues of $A$ are

$$\zeta = -\gamma, \quad \zeta = -\gamma + i\sqrt{4\lambda - \gamma^2}, \quad \zeta = -\gamma - i\sqrt{4\lambda - \gamma^2}.$$ 

We have $QA = \Lambda Q$ with the transformation matrix

$$Q := \begin{pmatrix} -\gamma & \lambda & 1 \\ 4\lambda & -\lambda\gamma & -\gamma \\ 0 & \lambda\sqrt{4\lambda - \gamma^2} & -\sqrt{4\lambda - \gamma^2} \end{pmatrix}$$

and the standard form

$$\Lambda := \begin{pmatrix} -\gamma & 0 & 0 \\ 0 & -\gamma & -\sqrt{4\lambda - \gamma^2} \\ 0 & \sqrt{4\lambda - \gamma^2} & -\gamma \end{pmatrix}.$$ 

Multiplying $Q$ on both sides of (4.31), we again obtain (4.32). Now note that

$$-\gamma z_t + \lambda r_t^2 + u_t^2 = \frac{\gamma}{2} |\delta X_t - P_t|^2 + \left( \lambda - \frac{\gamma^2}{4} \right) |\delta X_t|^2 =: G(\delta X_t, P_t).$$

Together with (4.32), we obtain

$$\begin{align*}
    dG_t &= -\gamma G_t dt + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^\top Q \left\{ \begin{pmatrix} -\delta b_t \cdot \delta X_t \\ -2\delta b_t \cdot P_t + 4rc_t^2 \sigma^2 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 2\gamma \sigma_t \\ 0 \\ 4u_t \end{pmatrix} \sigma rc_t dB_t \right\}.
\end{align*}$$

By defining

$$\gamma := \begin{cases} 
\gamma - \sqrt{\gamma^2 - 4\lambda}, & \text{if } \gamma^2 > 4\lambda \\
\gamma, & \text{if } \gamma^2 < 4\lambda
\end{cases}, \quad \overline{Q} := \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \overline{Q} := \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{if } \gamma^2 > 4\lambda, \quad \text{if } \gamma^2 < 4\lambda$$

we have

$$\begin{align*}
    dG_t &\leq -\gamma G_t dt + \overline{Q} \left\{ \begin{pmatrix} -\delta b_t \cdot \delta X_t \\ -2\delta b_t \cdot P_t + 4rc_t^2 \sigma^2 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 2\gamma \sigma_t \\ 0 \\ 4u_t \end{pmatrix} \sigma rc_t dB_t \right\}.
\end{align*}$$

(4.35)

Finally, notice that in each case the function $G$ is a quadratic form and is coercive, that is,

**Lemma 4.14.** There exist $C_G, \lambda_G > 0$ such that

$$\lambda_G (r_t^2 + u_t^2) \leq G_t \leq C_G (r_t^2 + u_t^2).$$

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Proof. In both cases, the functions $G$ can be written in the form:

$$G_t = \left| \Sigma \left( \begin{array}{c} \delta X_t \\ P_t \end{array} \right) \right|^2,$$

where the matrices $\Sigma$ are of full rank in both cases. Denote by $\lambda_G$ the smallest eigenvalue of the matrix $\Sigma^T \Sigma$. Clearly $\lambda_G > 0$. Then we have $G_t \geq \lambda_G (r_t^2 + u_t^2)$. Taking $C_G := \|\Sigma\|^2$ implies the second inequality.

**Remark 4.15.** Careful readers have noticed that we did not discuss the case $\lambda = \frac{\gamma^2}{4}$. Indeed, in this case one may extract $\varepsilon > 0$ from $\lambda$ and define the new $\tilde{\lambda} := \lambda - \varepsilon < \frac{\gamma^2}{4}$. Provided that $\varepsilon$ is small enough, it will not cause trouble to the following analysis.

**Remark 4.16.** In case $b = 0$, the contraction result can directly follow from the synchronous coupling, i.e. $r_c \equiv 0$. Since by assumption, $\varepsilon > 0$.

Proof. In both cases, the functions $G$ can be written in the form:

$$G_t = \left| \Sigma \left( \begin{array}{c} \delta X_t \\ P_t \end{array} \right) \right|^2,$$

where the matrices $\Sigma$ are of full rank in both cases. Denote by $\lambda_G$ the smallest eigenvalue of the matrix $\Sigma^T \Sigma$. Clearly $\lambda_G > 0$. Then we have $G_t \geq \lambda_G (r_t^2 + u_t^2)$. Taking $C_G := \|\Sigma\|^2$ implies the second inequality.

**Remark 4.15.** Careful readers have noticed that we did not discuss the case $\lambda = \frac{\gamma^2}{4}$. Indeed, in this case one may extract $\varepsilon > 0$ from $\lambda$ and define the new $\tilde{\lambda} := \lambda - \varepsilon < \frac{\gamma^2}{4}$. Provided that $\varepsilon$ is small enough, it will not cause trouble to the following analysis.

**4.4.3 Proof of contraction**

**Lemma 4.17.** Let $c \in \mathbb{R}$, $\eta, \beta \in (0, \infty)$, and suppose that $h : [0, \infty) \rightarrow [0, \infty)$ is continuous, non-decreasing, concave, and $C^2$ except for finitely many points. Define

$$\psi_t := \psi(X_t - X_t', V_t - V_t') = (1 + \beta G_t)h(\ell_t), \quad \text{with} \quad \ell_t := r_t + \eta u_t.$$

Then,

$$e^{ct}\psi_t \leq \psi_0 + \int_0^t e^{cs}K_x ds + M_t, \quad t \geq 0,$$

(4.36)

where $M$ is a continuous local martingale, and

$$K_t = (1 + \beta G_t) h'(\ell_t) \left\{ \eta |\delta b_t + u_t + (\eta \lambda - \gamma) r_t | + (1 + \beta G_t)2h''(\ell_t)\eta^2 \sigma^2 \gamma^2 r_t \right. \\
+ 4\beta \eta^2 \sigma^2 \gamma^2 h''(\ell_t) \| \mathcal{Q} \|(r_t + 2u_t) + c\psi_t - \gamma \beta G_t h(\ell_t) \\
\left. + \beta h(\ell_t) \mathcal{Q} \left( \begin{array}{c} -\delta b_t \cdot \delta X_t \\
0 \\
-2\delta b_t \cdot P_t + 4r_t^2 \sigma^2 \gamma^2 \end{array} \right) \right\}.$$ (4.37)

Proof. Since by assumption, $h$ is concave and piecewise $C^2$, we can now apply the Itô-Tanaka formula to $h(\ell_t)$. Denote by $h'$ and $h''$ the left-sided first derivative and the almost everywhere defined second derivative. The generalized second derivative of $h$ is a signed measure $\mu_h$ such that $\mu_h(d\ell) \leq h''(\ell)d\ell$. We obtain

$$dh(\ell_t) = h'(\ell_t)(dr_t + \eta du_t) + \frac{1}{2} h''(\ell_t)d(\eta u_t)_t$$

$$= h'(\ell_t)\left\{ e^\mu \cdot P_t - \gamma r_t - \eta \delta b_t \cdot e^\mu - \eta \lambda e^\mu \cdot \delta X_t \right\} dt + 2h''(\ell_t)\eta^2 \sigma^2 \gamma^2 r_t^2 dt$$

$$+ 2h'(\ell_t)\eta \sigma r_c dB_t$$

$$\leq h'(\ell_t)\left\{ \eta |\delta b_t | + u_t + (\eta \lambda - \gamma) r_t \right\} dt + 2h''(\ell_t)\eta^2 \sigma^2 \gamma^2 r_t^2 dt + 2h'(\ell_t)\eta \sigma r_c dB_t.$$
Calculate the quadratic variation
\[ d\langle h(\ell), G\rangle_t = 2h'(\ell_t)\eta\sigma c_t \bar{Q} \left( \begin{array}{cc} 2u_t^2 & 0 \\ 0 & 4u_t \end{array} \right) \sigma c_t dt \leq 4\eta^2 \sigma^2 h'(\ell_t) \| \bar{Q} \| (u_t + 2u_t) dt. \]

Finally, again by Itô’s formula, we obtain
\[ d(e^{ct} \psi_t) = e^{ct} ((1 + \beta G_t)dh(\ell_t) + \beta h(\ell_t)dG_t + \beta d\langle h(\ell), G\rangle_t + c\psi dt) \]
\[ \leq e^{ct} (K_t dt + d\tilde{M}_t), \]
with
\[ d\tilde{M}_t = (1 + \beta G_t)2h'(\ell_t)\eta\sigma c_t dB_t + \beta h(\ell_t) \bar{Q} \left( \begin{array}{cc} 2u_t^2 & 0 \\ 0 & 4u_t \end{array} \right) \sigma c_t dB_t \]
and the process \( K \) defined in (4.37). The assertion follows by taking \( M_t = \int_0^t e^{cs} \tilde{M}_s. \]

In order to make \( \psi_t \) a contraction under expectation, it remains to choose the coefficients \( \eta, \beta, h \) so that \( E[K_t] \leq 0. \)

**Choice of coefficients** Recall \( \lambda_G \) in Lemma 4.14. We fix a constant
\[ \varepsilon_0 := \frac{\gamma \lambda_G}{14\|Q\|} \wedge \frac{\gamma}{2} < \gamma. \] \hspace{1cm} (4.38)
Recall that there exists \( M > 0 \) such that for all \( m, m' \in P_2(\mathbb{R}^n) \)
\[ |b(m, x) - b(m, x')| \leq \varepsilon_0 |x - x'| \quad \text{whenever} \quad |x - x'| \geq M. \] \hspace{1cm} (4.39)
Using \( \varepsilon_0, M \) above, we define
\[ \eta := \frac{\varepsilon_0}{Lx + \lambda + 4\|Q\|\sigma^2} \wedge \frac{M\sqrt{\varepsilon_0}}{\sigma} \quad \text{and} \quad \theta := \frac{1}{\eta} + 8\|Q\|\sigma^2, \] \hspace{1cm} (4.40)
where \( Lx \) is the Lipschitz constant of the function \( b \) in \( x \). Now we are ready to introduce the function
\[ h(\ell) = \int_0^{2M\ell} \varphi(s)g(s) ds, \] \hspace{1cm} (4.41)
with
\[ \varphi(s) = \exp \left( -\frac{\theta}{\eta^2 \sigma^2} s^2 \right), \quad g(s) = 1 - \frac{1}{2} \int_0^s \frac{\Phi(r)}{\varphi(r)} dr, \quad \Phi(r) = \int_r^0 \varphi(x) dx. \]

**Remark 4.18.** The function \( h \) and its similar variations are repeatedly used in Eberle [30], Eberle, Guillin and Zimmer [31,32], Luo and Wang [56] to measure the contraction under the reflection coupling. In particular, the functions \( \varphi, g \) and \( h \) have the following properties:

- \( \varphi \) is decreasing,
  \[ \varphi_{\min} := \min_{0 \leq s \leq 2M} \varphi(s) = \exp \left( -\frac{\theta}{\eta^2 \sigma^2} M^2 \right). \]
- \( g \) is decreasing, \( g(0) = 1 \) and \( g(s) \geq g(2M) = \frac{1}{2} \) for \( r \in [0, 2M] \).
• $h$ is non-decreasing, concave, $h(0) = 0$, $h'(0) = 1$,

$$h'(2M) = \varphi(2M) g(2M) = \frac{\varphi_{\min}}{2} > 0$$

and $h$ is constant on $[2M, \infty)$

$$h(\ell) \leq \ell, \quad \frac{\Phi(\ell)}{2} \leq h(\ell) \leq \Phi(\ell), \quad \ell \leq 2M,$$

and

$$\theta \ell h'(\ell) + 2\eta^2 \sigma^2 h''(\ell) \leq -\kappa_M h(\ell), \quad \ell \leq 2M, \quad \text{with} \quad \kappa_M := \frac{\eta^2 \sigma^2}{\int_0^{2M} \frac{\Phi(r) dr}{\Phi(r)} d\ell}.$$ (4.42)

For the later use we further define a constant $\kappa_M > 0$ such that

$$\kappa_M := \kappa_M \wedge \frac{\varphi_{\min}}{2} \left( \gamma - \eta \left( L^2 + \lambda + 4 \|Q\| \sigma^2 \right) \right).$$ (4.43)

Note that by the definition of $\eta$ in (4.40) we have $\eta \left( L^2 + \lambda + 4 \|Q\| \sigma^2 \right) < \gamma$. Next introduce the constants

$$C_1 := 4 \|Q\| L^2 M^2 \left( 1 + \frac{\gamma}{\eta} \right) + 4 \|Q\| \sigma^2,$$

and choose the coefficient $\beta \in (0,1]$ such that

$$\beta < \frac{\kappa_M}{C_1} \wedge 1,$$ (4.44)

e.g., define

$$\beta := \frac{\kappa_M}{2C_1} \wedge 1.$$

Finally we may find a constant $C_0$ such that $r + u \leq C_0 \psi$ and thus

$$W_1 \leq C_0 W_\psi.$$

For the later use, define

$$C_2 := 2 \|Q\| M \left( 1 + \frac{\gamma}{\eta} \right) C_0, \quad C_M := 4 M^2 C_G \left( 1 + \frac{1}{\eta} \right),$$

as well as

$$c := \min \left\{ \kappa_M - C_1 \beta - \left( 1 + \beta C_M \right) \eta C_0 + \beta h(2M) C_2 \right\} \ell,$$

$$\left( \gamma - \frac{7 \|Q\| \varepsilon_0}{\lambda_G} \right) \frac{2 \beta \lambda_G M^2}{1 + 2 \beta \lambda_G M^2} \right\}.$$ (4.45)

with a constant $\ell > 0$ such that

$$\kappa_M - C_1 \beta - \left( 1 + \beta C_M \right) \eta C_0 + \beta h(2M) C_2 \ell > 0.$$ (4.46)

**Remark 4.19.** The constant $c$ defined in (4.45) represents the contraction rate in Theorem 2.16.

To enhance the understanding of this quantity, here we provide a lower bound of $c$ for some specific case. First we may define the new variables

$$X' = \frac{\gamma^{3/2}}{\sigma} X, \quad V' = \frac{\gamma^{1/2}}{\sigma} V, \quad t' = \gamma^{-1} t.$$
Then $X_t', V_t'$ satisfy
\begin{equation}
\begin{aligned}
\frac{dX_t'}{dt} &= V_t' dt, \\
\frac{dV_t'}{dt} &= (-D_m F'(\mathcal{L}(X_t'), X_t') - V_t') dt + dW_t',
\end{aligned}
\end{equation}
where $W_t' = \gamma^{1/2} W_t$ is a standard Brownian motion, and $F' := \left( \frac{1}{\gamma^{1/2}} \right)^{1/2} F$. Therefore, without loss of generality we may assume $\sigma = \gamma = 1$. In this case, by a direct computation we obtain the following lower bound of $c$ whenever $\lambda < \frac{1}{4} \gamma = \frac{1}{4}$:
\[ c \geq \frac{1 - \sqrt{1 - 4\lambda}^2}{14336L^2(L^2 + \lambda + 8)^2} R_m. \]
This lower bound indicates that the rate we obtain through the coupling method is indeed small, despite the fact that we prove the contraction result in Theorem 2.16. Moreover, if we reduce the value of $\sigma$, the corresponding Lipschitz constant $L_x$ of $F'$ becomes bigger and the lower bound of $c$ above becomes smaller.

**Lemma 4.20.** With the choice of the coefficients $\eta, \beta, h, c$ above, there exists $C \geq 0$ such that $E[K_t] \leq C \xi$.

**Proof.** We divide $(r, u) \in \mathbb{R}_+ \times \mathbb{R}_+$ into two regions:

(i) $\ell_t = r_t + \eta u_t \leq 2M$: It follows by Lemma 4.14 and due to $r_t + \eta u_t \leq 2M$ that
\[ G_t \leq C_G (r_t^2 + u_t^2) \leq 4M^2 C_G \left( 1 + \frac{1}{\sigma^2} \right) = C_M. \]
It is due to the Lipschitz assumption 4.29 and the fact $W_t \leq C_0 W_\psi$ that
\[
\begin{align*}
\left| Q \begin{pmatrix}
-\delta b_t \cdot \delta X_t \\
0 \\
-2\delta b_t \cdot P_t + 4\sigma^2 r_c^2
\end{pmatrix} \right| &\leq \|Q\| \left( |\delta b_t| r_t + 2|\delta b_t| u_t + 4\sigma^2 r_c^2 \right) \\
&\leq \|Q\| \left( |C_0 t W_\psi(m_t', m'_t) + L_x r_t| (r_t + 2u_t) + 4\sigma^2 \right) \\
&\leq 2\|Q\| M \left( 1 + \frac{3}{\sigma^2} \right) C_0 t W_\psi(m_t, m'_t) \\
&\quad + 4\|Q\| L_x^2 M^2 \left( 1 + \frac{2}{\sigma^2} \right) + 4\|Q\| \sigma^2 \\
&= C_2 t W_\psi(m_t, m'_t) + C_1.
\end{align*}
\]
Together with (4.37) we obtain
\[
\begin{align*}
K_t &\leq \left( 1 + \beta G_t \right) h'(\ell_t) \left\{ \eta C_0 W_\psi(m_t, m'_t) + (\eta(L^x + \lambda) - \gamma) r_t + u_t \right\} \\
&\quad + (1 + \beta G_t) 2h''(\ell_t) \eta^2 \sigma^2 r_c^2 + 4\beta\|Q\| \eta \sigma^2 r_c^2 h'(\ell_t) (r_t + 2u_t) \\
&\quad + c\psi + h(\ell_t)\left( C_2 t W_\psi(m_t, m'_t) + C_1 \right) \\
&\leq \left( (1 + \beta C_M) \eta C_0 + \beta h(2M) C_2 \right) W_\psi(m_t, m'_t) + C_1 \beta h(\ell_t) + c\psi \\
&\quad + (1 + \beta G_t) h'(\ell_t) \left\{ \eta \left( L^x + \lambda + \frac{4\beta\|Q\| \sigma^2 r_c^2}{1 + \beta G_t} \right) - \gamma \right\} r_t + I_t
\end{align*}
\]
with $I_t := (1 + \beta G_t) h'(\ell_t) \left( 1 + \frac{8\beta\|Q\| \sigma^2 r_c^2}{1 + \beta G_t} \right) u_t + (1 + \beta G_t) 2h''(\ell_t) \eta \sigma^2 r_c^2$.

Recall $\theta$ defined in (4.40). Since $\beta < 1$, we have
\[
\frac{1}{\eta} + \frac{8\beta\|Q\| \sigma^2 r_c^2}{1 + \beta G_t} \leq \frac{1}{\eta} + 8\|Q\| \sigma^2 = \theta.
\]
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Therefore, we obtain

\[ I_t \leq (1 + \beta G_t) \theta \eta \omega h'(\ell_t) + (1 + \beta G_t) 2 \eta h''(\ell_t) \eta^2 \sigma^2 r_t^2 \]
\[ \leq (1 + \beta G_t) \left( \theta \ell_t h'(\ell_t) + 2 \eta \sigma^2 \eta h''(\ell_t) \right) 1_{\{u_t \geq \xi\}} + (1 + \beta C_M) \theta \eta \xi 1_{\{u_t \leq \xi\}} \]
\[ \leq -(1 + \beta G_t) \kappa_\lambda h(\ell_t) + (1 + \beta G_t) \kappa_M h(\ell_t) 1_{\{u_t \leq \xi\}} + (1 + \beta C_M) \theta \eta \xi \]
\[ \leq -(1 + \beta G_t) \kappa_\lambda h(\ell_t) + (1 + \beta G_t) \kappa_M r_t + (1 + \beta C_M)(\kappa_M + \theta) \eta \xi. \]

Hence,

\[ K_t \leq \left( (1 + \beta C_M) \eta C_0 + \beta h(2M) C_2 \right) \ell W_\psi(m_t, m'_t) + C_1 \beta h(\ell_t) + c \psi_t \]
\[ - (1 + \beta G_t) \kappa_\lambda h(\ell_t) + (1 + \beta G_t) \kappa_M r_t + (1 + \beta C_M)(\kappa_M + \theta) \eta \xi \]
\[ + (1 + \beta G_t) h'(\ell_t) \left\{ \eta \left( L^x + \lambda + 4 \beta \| \mathcal{Q} \| \sigma^2 r_t^2 \right) - \gamma \right\} r_t \]
\[ \leq \left( (1 + \beta C_M) \eta C_0 + \beta h(2M) C_2 \right) \ell W_\psi(m_t, m'_t) + C_1 \beta h(\ell_t) + c \psi_t - \kappa_M \psi_t \]
\[ + (1 + \beta G_t) h'(\ell_t) \left\{ \kappa_M \ell h'(\ell_t) + \eta \left( L^x + \lambda + 4 \| \mathcal{Q} \| \sigma^2 \right) - \gamma \right\} r_t \]
\[ + (1 + \beta C_M)(\kappa_M + \theta) \eta \xi. \]

Due to the choice of \( \eta \) in (4.40) and \( \kappa_M \) in (4.43), the factor of \( r_t \) above is non-positive, i.e.

\[ \frac{\kappa_M}{h'(\ell_t)} + \eta \left( L^x + \lambda + 4 \| \mathcal{Q} \| \sigma^2 \right) - \gamma \leq \frac{2 \kappa_M}{\varphi_{\min}} + \eta \left( L^x + \lambda + 4 \| \mathcal{Q} \| \sigma^2 \right) - \gamma \leq 0. \]

Therefore, we obtain

\[ K_t \leq \left( (1 + \beta C_M) \eta C_0 + \beta h(2M) C_2 \right) \ell W_\psi(m_t, m'_t) + (C_1 \beta + c - \kappa_M) \psi_t \]
\[ + (1 + \beta C_M)(\kappa_M + \theta) \eta \xi. \]

Since \( W_\psi(m_t, m'_t) \leq \mathbb{E}[\psi_t] \) and taking expectation on both sides we obtain that

\[ \mathbb{E}[K_t] \leq \left( (1 + \beta C_M) \eta C_0 + \beta h(2M) C_2 \right) \ell + (C_1 \beta + c - \kappa_M) \mathbb{E}[\psi_t] \]
\[ + (1 + \beta C_M)(\kappa_M + \theta) \mathbb{E}[\eta \xi] \]
\[ \leq (1 + \beta C_M)(\kappa_M + \theta) \mathbb{E}[\eta \xi] =: C \xi, \]

where the last inequality is due to the definition of \( c \) in (4.45).

(ii). \( \ell_t = r_t + \eta u_t \geq 2M \): In this region, \( h(\ell_t) \) is constant, \( h'(\ell_t) = h''(\ell_t) = 0 \). Therefore,

\[ K_t = c \psi_t - \gamma \beta G_t h(2M) + \beta h(2M) \left| \mathcal{Q} \left( \begin{array}{c} -\delta b_t \cdot \delta X_t \\ 0 \\ -2 \delta b_t \cdot P_t + C_0 \sigma^2 r_t^2 \end{array} \right) \right|. \] (4.48)

Further we can divide this region into two parts:

\[ \{ (r, u) : \eta u + r \geq 2M \} \subseteq \{ r \geq M \} \cup \{ u \geq \eta^{-1}(r \vee M) \}. \]

Recall that by the choice of \( \eta \) in (4.40) we have \( \sigma^2 \leq \varepsilon_0 M^2 (1 + \frac{1}{\eta})^2 \) and \( \varepsilon_0 \geq L^x \eta \). Together with (4.39) we obtain

\[ |\delta b_t| \leq \varepsilon_0 r_t, \quad r_t^2 \sigma^2 \leq \varepsilon_0 r_t^2, \quad \text{on} \quad \{ r \geq M \}. \]

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as well as
\[ |\delta b_t| \leq L' \eta u_t \leq \varepsilon_0 u_t, \quad r^2 \sigma^2 \leq \varepsilon_0 u^2_t, \quad \text{on} \quad \{ u \geq \eta^{-1}(r \lor M) \}.
\]

Combining the two estimates above, we get
\[ |\delta b_t| \leq \varepsilon_0 (r_t \lor u_t). \]

and therefore
\[ \left| Q \left( \begin{array}{c} -\delta b_t \cdot \delta X \\ 0 \\ -2\delta b_t \cdot P_t + 4r \sigma^2 \end{array} \right) \right| \leq \frac{7\|Q\|\varepsilon_0}{\lambda_G} G_t, \quad \text{(4.49)} \]

where for the last inequality we use the coercivity in Lemma 4.14. Also due to \( r_t + \eta u_t \geq 2M \) and \( \eta \leq 1 \) we have
\[ \frac{\beta G_t}{1 + \beta G_t} \geq \frac{2\beta \lambda_G M^2}{1 + 2\beta \lambda_G M^2}. \]

Together with (4.48) and (4.49) we obtain
\[ K_t \leq c\psi_t - \tau \beta G_t h(2M) + \beta h(2M) \frac{7\|Q\|\varepsilon_0}{\lambda_G} G_t \]
\[ = c\psi_t - \tau \beta G_t \psi_t + \frac{7\|Q\|\varepsilon_0}{\lambda_G} G_t \beta G_t \psi_t \]
\[ = \psi_t \left( c - \left( \tau - \frac{7\|Q\|\varepsilon_0}{\lambda_G} \right) \frac{\beta G_t}{1 + \beta G_t} \right) \]
\[ \leq \psi_t \left( c - \left( \tau - \frac{7\|Q\|\varepsilon_0}{\lambda_G} \right) \frac{2\beta \lambda_G M^2}{1 + 2\beta \lambda_G M^2} \right) \leq 0, \]

where the second last inequality is due to the choice of \( \varepsilon_0 \) in (4.38) and the last one is due to \( c \) defined in (4.45). \( \square \)

**Proof of Theorem 2.16.** Let \( \Gamma \) be a coupling of two probability measures \( m_0 \) and \( m_0' \) on \( \mathbb{R}^{2n} \) such that \( \mathcal{W}_\psi(m_0, m_0') = \int \psi \, d\Gamma \). We consider the coupling process \( ((X, V), (X', V')) \) introduced above with initial law \( ((X_0, V_0), (X_0', V_0')) \sim \Gamma \).

By taking expectation on both sides of (4.36), evaluated at localizing stopping times \( \tau_n \to t \) and applying Fatou’s lemma as \( n \to \infty \), we obtain
\[ \mathbb{E}[e^{ct}\psi_t] \leq \mathbb{E}[\psi_0] + \int_0^t e^{cs} \mathbb{E}[K_s] \, ds \leq \mathbb{E}[\psi_0] + Ce^{-1}(e^{ct} - 1) \xi, \]

for any \( \xi > 0 \) and \( t \geq 0 \). Note that \( \mathbb{E}[\psi_0] = \int \psi \, d\Gamma = \mathcal{W}_\psi(m_0, m_0') \). Therefore
\[ \mathcal{W}_\psi(m_t, m_t') \leq \mathbb{E}[\psi_t] \leq e^{-ct} \mathcal{W}_\psi(m_0, m_0') + Ce^{-1}(1 - e^{-ct}) \xi \to e^{-ct} \mathcal{W}_\psi(m_0, m_0'), \]
as \( \xi \to 0 \). Finally note that by the choice of \( \beta \) in (4.44), we have \( c > 0 \) according to (4.45) provided that \( \iota \) is small enough. \( \square \)
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