SOLVABILITY OF THE HANKEL DETERMINANT PROBLEM FOR REAL SEQUENCES

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ABSTRACT. To each nonzero sequence \( s := \{s_n\}_{n \geq 0} \) of real numbers we associate the Hankel determinants \( D_n = \det \mathcal{H}_n \) of the Hankel matrices \( \mathcal{H}_n := (s_{i+j})_{i,j=0}^n, \ n \geq 0 \), and the nonempty set \( \mathbb{N}_s := \{n \geq 1 \mid D_{n-1} \neq 0\} \). We also define the Hankel determinant polynomials \( P_0 := 1 \), and \( P_n, \ n \geq 1 \) as the determinant of the Hankel matrix \( \mathcal{H}_n \), modified by replacing the last row by the monomials \( 1, x, \ldots, x^n \). Clearly \( P_n \) is a polynomial of degree at most \( n \) and of degree \( n \) if and only if \( n \in \mathbb{N}_s \). Kronecker established in 1881 that if \( \mathbb{N}_s \) is finite then rank \( \mathcal{H}_n = r \) for each \( n \geq r - 1 \), where \( r := \max \mathbb{N}_s \). By using an approach suggested by I.S. Iohvidov in 1969 we give a short proof of this result and a transparent proof of the conditions on a real sequence \( \{t_n\}_{n \geq 0} \) to be of the form \( t_n = D_n, \ n \geq 0 \) for a real sequence \( \{s_n\}_{n \geq 0} \). This is the Hankel determinant problem. We derive from the Kronecker identities that each Hankel determinant polynomial \( P_n \) satisfying \( \deg P_n = n \geq 1 \) is preceded by a nonzero polynomial \( P_{n-1} \) whose degree can be strictly less than \( n - 1 \) and which has no common zeros with \( P_n \). As an application of our results we obtain a new proof of a recent theorem by Berg and Szwarc about positive semidefiniteness of all Hankel matrices provided that \( D_0 > 0, \ldots, D_{r-1} > 0 \) and \( D_n = 0 \) for all \( n \geq r \).

1. Introduction

We use the notation \( \mathbb{N} := \{1, 2, \ldots\} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). To a sequence \( s := \{s_n\}_{n \geq 0} \) of real numbers we associate the Hankel matrices \( \mathcal{H}_n := (s_{i+j})_{i,j=0}^n, \ n \geq 0 \) and the determinants \( D_n = D_n(s) := \det \mathcal{H}_n, \ n \geq 0 \). In this way we get a mapping \( D : \{s_n\}_{n \geq 0} \rightarrow \{D_n(s)\}_{n \geq 0} \) in the space \( \mathbb{R}^{\mathbb{N}_0} \) of sequences of real numbers. We call this mapping the Hankel determinant transform. It was introduced and studied by Layman in [13], who emphasized that such a transform is far from being injective by proving that a sequence \( s \) and its binomial transform \( \beta(s) \) defined by

\[
\beta(s)_n := \sum_{k=0}^{n} \binom{n}{k} s_k, \quad n \geq 0,
\]

have the same image under this mapping. Concerning the missing injectivity let us here just point out that the Hankel determinant transform of all the sequences \( \{a^n\}_{n \geq 0}, \ a \in \mathbb{R} \) is \( \{1, 0, 0, \ldots\} \).

Several authors have been concerned with the sign pattern of the sequence \( D(s) \) in order to use this for the determination of the rank and signature of the Hankel matrices. This is given in rules of e.g. Jacobi, Gundelfinger and Frobenius. See [8], [10] for a treatment of these questions, which become quite technical when zeros occur in the sequence \( D(s) \).

The Hankel determinant problem for real sequences is to characterize the image \( D(\mathbb{R}^{\mathbb{N}_0}) \) in \( \mathbb{R}^{\mathbb{N}_0} \), i.e., to find a necessary and sufficient condition for a sequence \( t \in \mathbb{R}^{\mathbb{N}_0} \) to be of the form

\[
\begin{pmatrix}
  s_0 & s_1 & \cdots & s_n \\
  s_1 & s_2 & \cdots & s_{n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  s_n & s_{n+1} & \cdots & s_{2n}
\end{pmatrix} = t_n, \quad n \geq 0.
\]

with some sequence \( s \) of real numbers. It turns out that such conditions are similar to those that were obtained by G. Frobenius [6, p.207] in 1894 for all possible signs of the numbers \( \{t_n\}_{n \geq 0} \). His arguments were simplified by F. Gantmacher [8, p.348] in 1959 and by I.S. Iohvidov [10, (12.8), p.83] in 1982 in an essential way. The purpose of the present paper is to obtain a further simplification of the Frobenius reasoning by giving in Theorem 3 a new
Proposition 1. Let \( t := \{ t_n \}_{n \geq 0} \) be a sequence of real numbers and \( Z_t := \{ n \geq 0 \mid t_n \neq 0 \} \).

If \( Z_t = \emptyset \) then the equation (1.1) is satisfied if and only if \( s_n = 0 \) for all \( n \geq 0 \). If \( Z_t \neq \emptyset \) consists of \( 1 \leq m \leq \infty \) distinct elements \( \{ n_k \}_{0 \leq k < m} \) arranged in increasing order then the equation (1.1) is solvable if and only if the following Frobenius conditions (see [8, p.348]) hold

\[
(-1)^{n_0 + 1} t_{n_0} > 0, \quad \text{if} \quad n_0 + 1 \in 2N,
\]

\[
(-1)^{\frac{n_k - 1 - n_k}{2}} t_{n_k + 1} t_{n_k} > 0, \quad \text{if} \quad n_k + 1 - n_k \in 2N, \quad 0 \leq k < m - 1, \quad 2 \leq m \leq \infty.
\]

It follows from Theorem 1 that (1.1) is solvable if \( t_n \neq 0 \) for all \( n \geq 0 \), and not solvable if \( t = \{0, 1, 0, 0, \ldots\} \). Furthermore, the condition \((-1)^{\frac{n(n+1)}{2}} t_n \geq 0 \) for all \( n \geq 0 \) is sufficient for the existence of at least one solution of (1.1).

Let us formulate an elementary result about existence and uniqueness of solutions to (1.1) and which is independent of Theorem 1. For this we need the following notation. For a \( n \times n \) determinant \( A \), we denote by \( A^{k,m} \), \( 1 \leq k, m \leq n \), the \( (n-1) \times (n-1) \) determinant obtained by deleting the \( k \)’th row and \( m \)’th column of \( A \). For Hankel determinants we follow Frobenius [6, p.212] in writing \( D_{n+1}' = D_{n+1}^{n+2n+1} \), \( n \geq 0 \), i.e.,

\[
D_1' = s_1, \quad D_2' = \begin{vmatrix} s_0 & s_2 \\ s_1 & s_3 \end{vmatrix}, \quad D_n' = \begin{vmatrix} s_0 & s_1 & \ldots & s_{n-2} & s_{n-1} & s_{n+1} \\ s_1 & s_2 & \ldots & s_{n-1} & s_n & s_{n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_{n-1} & s_n & \ldots & s_{2n-3} & s_{2n-2} & s_{2n} \\ s_n & s_{n+1} & \ldots & s_{2n-2} & s_{2n-1} & s_{2n+1} \end{vmatrix}, \quad n \geq 2. \quad (1.2)
\]

Proposition 1. Given two sequences \( t, t' \) of real numbers such that \( t_n \neq 0 \) for all \( n \geq 0 \), there exists a unique sequence \( s \) of real numbers such that

\[
D_n = t_n \in \mathbb{R} \setminus \{0\}, \quad D_{n+1}' = t_{n}' \in \mathbb{R}, \quad n \geq 0.
\]

To see this we use the Laplace expansion of \( D_n \) and \( D_{n+1}' \) along the last column and note that \((D_{n+1}')^{n-k,n+1} = D_{n-k,n+1} \). This gives the following recurrence formulas

\[
s_0 = D_0, \quad s_1 = D_1'; \quad s_2 D_0 = D_1 + s_1^2, \quad s_3 D_0 = D_2' + s_1 s_2; \\
s_2 n D_{n-1} = D_{n} + \sum_{k=0}^{n-1} (-1)^k s_{2n-1-k} D_{n-k,n+1}^{n-k,n+1} =: D_{n} + F_n(s_0, \ldots, s_{2n-1}), \\
s_2 n+1 D_{n-1}' = D_{n+1}' + \sum_{k=0}^{n-1} (-1)^k s_{2n-k} D_{n-k,n+1}^{n-k,n+1} =: D_{n+1}' + G_n(s_0, \ldots, s_{2n}), \quad n \geq 1;
\]

where \( D_{n-k,n+1}^{n-k,n+1} \), \( 0 \leq k \leq n - 1 \), depend only on \( s_j \), \( 0 \leq j \leq 2n - 1 \), \( F_n \) is a function of \( s_0, \ldots, s_{2n-1} \) and \( G_n \) a function of \( s_0, \ldots, s_{2n} \). If \( t''_n = D_{n+1}', t_n = D_n \neq 0, n \geq 0 \) are assumed to be given, these relations determine the sequence \( s \) uniquely, and the assertion follows.

A complete description of all solutions of (1.1), when some of the numbers \( t_n \) vanish, can be derived from the Frobenius results in [6], but this is of no relevance in the present context.

Let \( \mathbb{P}[\mathbb{R}] \) denote the set of all algebraic polynomials with real coefficients. Given a sequence \( \{s_n\}_{n \geq 0} \) of real numbers we introduce two sequences of polynomials in \( \mathbb{P}[\mathbb{R}] \):

\[
P_0(x) := 1, \quad P_1(x) := \begin{vmatrix} s_0 & s_1 \\ 1 & x \end{vmatrix}, \quad P_n(x) := \begin{vmatrix} s_0 & s_1 & s_2 & \ldots & s_n \\ s_1 & s_2 & s_3 & \ldots & s_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & s_{n+1} & \ldots & s_{2n-1} \\ 1 & x & x^2 & \ldots & x^n \end{vmatrix}.
\]
Let $D_{n+k} = D_{n+k-1-k} = D_{n+k-1} = D_{n+k-1-k}$. Note that $D_{n+1,n+1-k} = D_{n+1-k,n+1}$, so $k \leq n$, $n \geq 1$ and

$$P_1(x) = D_0 x - D_1', P_n(x) = D_{n-1} x^n - D_n x^{n-1} + \sum_{k=1}^{n-1} (-1)^{k-1} x^{n-1-k} D_{n-k}^{n-k,n+1}, \quad n \geq 2.$$  

The polynomials $\{P_n\}_{n \geq 0}$ are called Hankel determinant polynomials with respect to the sequence $\{s_n\}_{n \geq 0}$. Let $L : P[\mathbb{R}] \rightarrow \mathbb{R}$ denote the linear functional determined by

$$L(x^n) = s_n, \quad n \geq 0.$$  

Then

$$L(x^k P_n(x)) = 0, \quad 0 \leq k \leq n - 1, \quad n \geq 1,$$  

and also

$$L\left(P_n(x)^2\right) = D_n D_{n-1}, \quad n \geq 0, \quad D_{-1} := 1.$$  

Already Stieltjes considered this kind of functional, see [18, p. 25]. It is also used in [3, Definition 2.1, p.6]).

In the classical case where all the Hankel determinants $D_n > 0$, these polynomials are proportional to the classical orthonormal polynomials (see [3, p.10; p.15; Exercise 3.1(a), p.17])

$$p_n(x) := \frac{P_n(x)}{\sqrt{D_n D_{n-1}}}, \quad n \geq 0, \quad D_{-1} := 1,$$  

and those of the second kind.

In the general case of an arbitrary sequence $\{s_n\}_{n \geq 0}$ of real numbers Frobenius [6, (5), p.212] obtained in 1894 a recurrent relation for the polynomials $\{P_n\}_{n \geq 0}$ in the following determinant form

$$D_{n-1} D_n x P_n(x) = D_{n-1}^2 P_{n+1}(x) + (D_{n-1} D'_n + D_n D'_n) P_n(x) + D_n^2 P_{n-1}(x),$$  

where $n \geq 0$, $P_{-1}(x) := 0$, $P_0(x) = 1$ and $D_{-1} := 1$, $D'_0 := 0$. If $D_n \neq 0$ for all $n \geq 0$, then the functional $L$ is called quasi-definite (see [3, Definition 3.2, p.16]) and the monic polynomials

$$p_n(x) := P_n(x)/D_n, \quad n \geq 0,$$  

are usually considered for which the recurrence (1.9) is written in the Jacobi form (see [3, Theorem 4.1, p.18])

$$p_{n+1}(x) = (x - a_n) p_n(x) - b_n p_{n-1}(x), \quad n \geq 0, \quad p_0(x) = 1, \quad p_{-1}(x) = 0,$$  

$$a_n = D_{n+1}^2 / D_n D_{n-1}, \quad n \geq 0, \quad b_n = D_n D_{n-2} / D_{n-1}^2, \quad n \geq 1, \quad a_0 = D_1^2 / D_0, \quad b_0 = D_0,$$  

where the relations (1.12) are invertible (cp. [3, Theorem 4.2, p.19])

$$D_n = \prod_{k=0}^{n} b_k^{n+1-k}, \quad D_{n+1} = \left(\sum_{k=0}^{n} a_k\right) \prod_{k=0}^{n} b_k^{n+1-k}, \quad n \geq 0.$$  

and $b_n \neq 0$ for all $n \geq 0$. Conversely, given the recurrence formula (1.11) for monic polynomials $\{p_n\}_{n \geq 0}$ with two arbitrary real sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ satisfying $b_n \neq 0$ for all $n \geq 0$, we determine by (1.13) and Proposition 1 a quasi-definite functional $L$ such that $L(p_n(t) p_m(t)) = 0$ and $L(p_n(t)^2) \neq 0$ for all $n, m \geq 0, n \neq m$, by virtue of (1.5), (1.10), (1.6) and (1.7). This fact is known as the generalized Favard theorem for quasi-definite functionals (see [3, Theorem 4.4, p.21]).

By Theorem 1, if $\{s_n\}_{n \geq 0}$ is assumed nonzero, i.e., $s_n \neq 0$ for at least one $n \geq 0$, there exists $r \geq 1$ such that $D_{r-1} \neq 0$, and then $P_r$ is a polynomial of degree $r$. In Theorem 2 we
derive from the Kronecker identities (2.7) a simple result about zeros of the polynomials \( P_r \) and \( Q_r \).

In 1881 Kronecker [12] also characterized all those nonzero sequences \( \{s_n\}_{n \geq 0} \) of real numbers whose Hankel matrices \( \{s_{i+j}\}_{i,j=0}^{\infty} \) are of finite rank, see Theorem A of Section 2.

In Corollary 1 we provide a new interpretation of this result based on Theorem 3.

The results obtained by Frobenius [6] in 1894 are formulated in Theorem D of Section 3.

In Subsection 5.3 we use Theorem 3 to derive a recent theorem of Berg and Szwarc [1], see Theorem E.

2. Kronecker’s results from 1881

Let \( 1 \leq m \leq n \) and \( A_n = \{a_{i,j}\}_{i,j=1}^{n} \) be a nonzero square matrix of order \( n \), where \( A_n \) being nonzero means that it has at least one nonzero element. For arbitrary \( 1 \leq i_1 < i_2 < \ldots < i_m \leq n \) and \( 1 \leq j_1 < j_2 < \ldots < j_m \leq n \) the determinant \( \det(a_{i_k,j_k})_{k=1}^{m} \) is called a minor of \( A_n \) of order \( m \). The largest order of the nonzero minors of \( A_n \) is called the rank of the matrix \( A_n \) and is denoted by \( \text{rank} A_n \) (see [8, p.2]). The rank of an infinite matrix \( A_\infty = \{a_{i,j}\}_{i,j=1}^{\infty} \) is defined by \( \text{rank} A_\infty := \sup_{n \geq 1} \text{rank} A_n \in \mathbb{N} \cup \{\infty\} \), where \( \mathbb{N} := \{1, 2, \ldots\} \) (see [9, p.205]).

Theorem A (Kronecker (1881)). Let \( \{s_n\}_{n \geq 0} \) be a nonzero sequence of real numbers, \( H_n := (s_{i+j})_{i,j=0}^{n} \), \( D_n := \det H_n \), \( n \geq 0 \), and \( H_\infty := (s_{i+j})_{i,j=0}^{\infty} \). Then a necessary and sufficient condition for \( H_\infty \) to have a finite rank \( r \in \mathbb{N} \) is that

\[
D_{r-1} \neq 0 \quad , \quad D_n = 0 \quad , \quad n \geq r \, .
\]

The necessity of the condition is formulated in Kronecker [12, p.560], Frobenius [6 p. 204], Gantmacher [9, p.206] and Iohvidov [10, p. 74], while the sufficiency, proved by Kronecker [12, p.563], is less known and can be found in Iohvidov [10, item 11, p.79].

It has been proved by Kronecker in [12, (G\(^r\))], (G\(r\)), (p.567)] that if \( (s_{i+j})_{i,j=0}^{\infty} \) is of finite rank \( r \) then

\[
Q_r(x)/P_r(x) = \sum_{k \geq 0} s_k x^{-k-1} \, .
\]

Since (1.3) yields \( x^r P_r(1/x) |_{x=0} = D_{r-1} \neq 0 \), the change of variable \( x \to 1/z \) in (2.2) shows that it is equivalent to the Taylor expansion at the origin

\[
\psi^r(z) := \frac{z^{r-1} Q_r(1/z)}{z^r P_r(1/z)} = \sum_{k \geq 0} s_k z^k
\]

of the analytic function \( \psi^r \) on the open disk \( |z| < 1/\rho_r \) where \( \rho_r := \max \{ |z| : |P_r(z) = 0\} \).

Therefore the series in the right-hand side of (2.2) converges absolutely for every \( |x| > \rho_r \), and (2.3) below holds by the Cauchy-Hadamard formula (see [16, (2), p.200]).

Conversely, Kronecker proved in [12, p.568] that if the numbers \( \{s_n\}_{n \geq 0} \) are the coefficients in the expansion (2.5) of \( q/p \) for \( p, q \in \mathbb{P}[\mathbb{R}] \), \( \deg p = r \in \mathbb{N} \) and \( \deg q < r \) then \( H_\infty \) has the rank \( r \), provided \( D_{r-1} \neq 0 \) (see [14, Section 45, p.198], [9, Theorem 8, p.207]). Thus, the following characterization of the Hankel matrices of finite rank holds.

Theorem B. Let \( \{s_n\}_{n \geq 0} \) be a nonzero sequence of real numbers and \( H_\infty := (s_{i+j})_{i,j=0}^{\infty} \).

(a) If \( H_\infty \) has a finite rank \( r \in \mathbb{N} \) then \( \deg P_r = r \),

\[
\lim_{k \to \infty} \sqrt[k]{|s_k|} = \max \{ |z| : |z| \in \mathbb{C} , \ P_r(z) = 0 \} \, ,
\]

and

\[
\sum_{k \geq 0} \frac{s_k}{x^{k+1}} = \frac{Q_r(x)}{P_r(x)} \, ,
\]

where the series is absolutely convergent for every \( |x| > \max \{ |z| : |z| \in \mathbb{C} , \ P_r(z) = 0 \} \).
(b) If \( R := \lim_{k \to \infty} \sqrt[n]{s_k} < +\infty \) and there exist \( p, q \in \mathbb{P}[\mathbb{R}] \), \( p \) of degree \( r \in \mathbb{N} \) and \( q \) of degree at most \( r - 1 \) such that
\[
\sum_{k \geq 0} \frac{s_k}{z^{k+1}} = \frac{q(z)}{p(z)}, \quad |z| > R,
\] (2.5)
then \( \mathcal{H}_\infty \leq r \), where the equality is attained if \( p \) and \( q \) have no common roots.

The following theorem of Kronecker [12, pp.560, 561, 571] clarifies the structure of the sequences satisfying \( \mathcal{H}_\infty < \infty \) (see also [9, Theorem 7, p.205] and 9, p.234]).

**Theorem C.** Let \( \{s_n\}_{n \geq 0} \) be a nonzero sequence of real numbers and \( \mathcal{H}_\infty := (s_{n+j})_{j=0}^\infty \).

(a) \( \mathcal{H}_\infty \) has a finite rank \( r \in \mathbb{N} \) if and only if \( D_{r-1} \neq 0 \) and there exist \( r \) numbers \( d_0, d_1, \ldots, d_{r-1} \) such that
\[
\sum_{k=0}^{r-1} d_k s_{k+m} = s_{r+m}, \quad m \geq 0.
\] (2.6)

(b) If \( \mathcal{H}_\infty \) has a finite rank \( r \in \mathbb{N} \) then for every \( n \geq 0 \) there exist \( r \) numbers \( d_{n,0}, d_{n,1}, \ldots, d_{n,r-1} \) such that
\[
\sum_{k=0}^{r-1} d_{n,k} s_{k+m} = s_{r+n+m}, \quad m \geq 0,
\]
where \( d_{n,k} \) is equal to \( d_k \) from (2.6) for each \( 0 \leq k \leq r - 1 \).

(c) If \( \mathcal{H}_\infty \) has a finite rank \( r \in \mathbb{N} \) then the sequence \( \{s_n\}_{n \geq 0} \) is uniquely determined by the values of \( s_0, s_1, \ldots, s_{2r-1} \).

Finally, we note that the equality (2.4) proved by Kronecker in [12, (G\(^{(m)}\)), (G\(^*\)), p.567]) asserts implicitly that the polynomials \( P_r \) and \( Q_r \) have no common roots provided that \( D_{r-1} \neq 0 \). Furthermore, this fact also follows from the identity
\[
P_{r-1}(x)Q_r(x) - P_r(x)Q_{r-1}(x) = D_{r-1}^2,
\] (2.7)
written by Kronecker in [12, (F), p.564] for arbitrary \( r \geq 1 \) (see also [6, (14), p.220]).

Observe that (2.7) can easily be proved when \( D_n \neq 0 \) for all \( n \geq 0 \) (see 7, III.15, p.48], [2, Theorem 2.12, p.54]). These restrictions can be removed by the so-called perturbation technique. More precisely, by using the Hilbert matrix \( M_n^i := ((i+j+1)^{-1})_{i,j=0}^n \), for every \( \varepsilon > 0 \) we introduce the perturbed sequence
\[
\{s^n_{\varepsilon k}\}_{n \geq 0}, \quad s^n_{\varepsilon k} := s_n + \frac{\varepsilon^{n+1}}{n+1}, \quad M_n^i := \left( \frac{\varepsilon^{i+j+1}}{i+j+1} \right)^{n}_{i,j=0}, \quad n \geq 0,
\]
whose Hankel determinant \( D_{\varepsilon}^n = \det(M_n + M_n^i) = m_n\varepsilon^{(n+1)^2} + \ldots \) for every \( n \geq 0 \) is a polynomial of degree \( (n+1)^2 \) in the variable \( \varepsilon \) with positive leading coefficient \( m_n := \det M_n^i > 0 \) (see 15, 3, p.92]. Since the zeros of all polynomials \( D_\varepsilon^n \), \( n \geq 0 \), form an at most countable set, there exists a sequence \( \{\varepsilon_k\}_{k \geq 0} \) of positive numbers \( \varepsilon_k \) tending to zero as \( k \to \infty \) such that
\[
\det \{s^n_{\varepsilon_k}+i+j \}_{i,j=0}^n \neq 0, \quad n, k \geq 0.
\]

With (2.7) in hand for \( \{s^n_{\varepsilon_k}\}_{n \geq 0}, \varepsilon_k \geq 0 \), we conclude by the continuous dependence of (2.7) on \( s^n_{\varepsilon_k} \), \( 0 \leq m \leq 2r - 1 \), that (2.7) holds for \( \{s_n\}_{n \geq 0} \).

It also follows from (2.7) that \( P_r \) and \( P_{r-1} \) have no common roots, provided that \( D_{r-1} \neq 0 \). We have therefore proved the following property (cf. [2, Theorem 2.14, p.57])

**Theorem 2.** Let \( \{s_n\}_{n \geq 0} \) be an arbitrary nonzero sequence of real numbers and \( r \) be a positive integer satisfying \( D_{r-1} \neq 0 \). Then \( \deg P_r = r, P_{r-1} \neq 0, Q_r \neq 0 \) and the polynomial \( P_r \) has no common zeros with the polynomials \( P_{r-1} \) and \( Q_r \).

Observe, that Theorem 2 can also be easily deduced from [5, Theorem 1.9, p.80; Theorem 1.3(ii), p.44]. We will in the sequel use the following notion.
Definition 1. Let \( \{s_n\}_{n \geq 0} \) be a nonzero sequence of real numbers. The rank of the infinite Hankel matrix \((s_{i+j})_{i,j=0}^{\infty}\) is called the Hankel rank of \( \{s_n\}_{n \geq 0} \).

Since \( \text{rank}(s_{i+j})_{i,j=0}^{\infty} \in \mathbb{N} \cup \{\infty\} \), the Hankel rank of a real nonzero sequence can be equal to any positive integer or infinity.

### 3. Frobenius’ theorem from 1894

Let \( s := \{s_n\}_{n \geq 0} \) be an arbitrary nonzero sequence of real numbers and

\[
N_s := \{ r \in \mathbb{N} \mid D_{r-1} \neq 0 \}.
\]

Theorem 1 yields \( N_s \neq \emptyset \). Suppose that \( N_s \) consists of \( m \) \((1 \leq m \leq \infty)\) distinct elements \( \{n_k\}_{1 \leq k < m+1} \) arranged in increasing order and \( n_0 := 0 \), where it is assumed that \( a+\infty = \infty \) for arbitrary \( a \in \mathbb{R} \). Then

\[
\{0\} \cup N_s = \{n_k\}_{0 \leq k < m+1}, \quad 1 \leq m \leq \infty, \quad 0 = n_0 < n_1 < \ldots.
\]

We say that the Hankel determinant polynomial \( P_n \) defined by (1.3) is of full degree if \( \deg P_n = n \). It follows from (1.3), (1.4), (3.1) and (3.2) that \( P_n \) is of full degree if and only if \( n = n_k \) for some \( 0 \leq k < m + 1 \), i.e.,

\[
\{ P_n \mid \deg P_n = n, \ n \geq 0 \} = \{ P_{n_k} \}_{0 \leq k < m+1} = \{ P_{n_0} = 1, P_{n_1}, \ldots, P_{n_k}, P_{n_{k+1}}, \ldots \},
\]

\[
\deg P_{n_k} = n_k, \ 0 \leq k < m + 1.
\]

Theorem 2 states that the identities (2.7) proved by Kronecker in 1881 imply that for each \( 0 \leq k < m \) the polynomial \( P_{n_{k+1}} \) is preceded by a nonzero polynomial \( P_{n_{k+1}-1} \) which has no common zeros with \( P_{n_{k+1}} \) and whose degree can be strictly less than \( n_{k+1} - 1 \).

In 1894 Frobenius established \( 6, (10), p.210 \) that \( P_{n_{k+1}-1} \) for such \( k \) is proportional with a nonzero real constant of proportionality to the previous polynomial \( P_{n_k} \) of full degree provided that \( \deg P_{n_{k+1}-1} < n_{k+1} - 1 \), i.e., there exists \( \gamma_k \in \mathbb{R} \setminus \{0\} \) such that

\[
P_{n_{k+1}-1}(x) = \gamma_k P_{n_k}(x),
\]

if \( n_{k+1} - n_k \geq 2 \) and \( 0 \leq k < m \) (see also \( 5, \text{Theorem 1.3(ii), p.44} \)). Furthermore, he proved in \( 6, (8), p.214 \) that for \( m \geq 2 \) the recurrence relations

\[
p_{n_{k+1}}(x) = a_k(x)p_{n_k}(x) - \beta_k p_{n_{k-1}}(x), \quad 1 \leq k < m,
\]

hold between the monic polynomials

\[
p_{n_k}(x) := P_{n_k}(x)/D_{n_k-1}, \quad 0 \leq k < m + 1, \quad D_{-1} := 1,
\]

corresponding to the polynomials \( P_n \) of full degree, where \( \{\beta_k\}_{1 \leq k < m} \) are nonzero real numbers and \( a_k(x) \in \mathbb{P}[\mathbb{R}] \) is a monic polynomial of degree \( n_{k+1} - n_k \) for every \( 1 \leq k < m \) (see also \( 5, \text{Remark 1.2, p.71} \)). It is also proved in \( 6, (9), p.210 \) that

\[
P_{n_{k+1}}(x) \equiv \ldots \equiv P_{n_{k+1}-2}(x) \equiv 0
\]

provided that \( n_{k+1} - n_k \geq 3 \) and \( 0 \leq k < m \) (see also \( 5, \text{Theorem 1.3, p.44} \)). Thus, the following theorem was proved by Frobenius \( 6 \) in 1894.

Theorem D. Let \( \{s_n\}_{n \geq 0} \) be an arbitrary nonzero sequence of real numbers and \( N_s, m \) and \( \{n_k\}_{0 \leq k < m+1} \) be defined as in (3.1) and (3.2).

For the Hankel determinant polynomials \( \{P_n\}_{n \geq 0} \) defined by (1.3) the following assertions hold.

(a) If \( n_1 \geq 2 \) then there exists \( \gamma_0 \in \mathbb{R} \setminus \{0\} \) such that

\[
P_0 \equiv 1, \quad P_1 \equiv \gamma_0, \quad \deg P_{n_1} = n_1 = 2,
\]

when \( n_1 = 2 \) and

\[
P_0 \equiv 1, \quad P_1 \equiv 0, \quad \ldots, \quad P_{n_1-2} \equiv 0, \quad P_{n_1-1} \equiv \gamma_0, \quad \deg P_{n_1} = n_1,
\]

when \( n_1 \geq 3 \).
(b) If \( m \geq 2, 1 \leq k < m \) and \( n_{k+1} - n_k \geq 2 \) then there exists \( \gamma_k \neq 0 \) such that
\[
\deg P_{n_k} = n_k, \quad P_{n_{k+1}-1} = \gamma_k P_{n_k}, \quad \deg P_{n_{k+1}} = n_{k+1},
\]
when \( n_{k+1} - n_k = 2 \) and
\[
\deg P_{n_k} = n_k, \quad P_{n_k+1} = 0, \quad \ldots, \quad P_{n_{k+1}-2} = 0, \quad P_{n_{k+1}-1} = \gamma_k P_{n_k}, \quad \deg P_{n_{k+1}} = n_{k+1},
\]
when \( n_{k+1} - n_k \geq 3 \).

(c) If \( m \geq 2 \) then for the monic polynomials
\[
p_0(x) = 1, \quad p_k(x) := \frac{P_{n_k}(x)}{D_{n_k-1}}, \quad 0 \leq k < m + 1,
\]
there exist monic polynomials in \( \mathbb{P}[\mathbb{R}] \)
\[
a_k(x), \quad \deg a_k(x) = n_{k+1} - n_k \geq 1, \quad 0 \leq k < m,
\]
and nonzero real numbers \( \{\beta_k\}_{0 \leq k < m} \) such that
\[
p_{n_{k+1}}(x) = a_k(x)p_{n_k}(x) - \beta_k p_{n_k-1}(x), \quad 0 \leq k < m, \quad p_{n-1} := 0. \quad (3.4)
\]

(d) If \( m < \infty \) then \( n_m = \max \mathbb{N}_s \) and \( P_n \equiv 0 \) for all \( n \geq n_m + 1 \).

It should be noted that Theorem D (d) follows directly from Theorem A and Theorem C (a). Indeed, the conditions of Theorem D (d) imply the validity of (2.1) for \( r = n_m \) and in view of Theorem A we obtain that \( H_n \) has a finite rank \( n_m \). But for arbitrary \( n \geq n_m + 1 \) the \( (n_m + 1) \)-th row of the determinant for \( P_n \) in (1.3) is the linear combination of the first \( n_m \) rows by virtue of (2.6). Hence, \( P_n \equiv 0 \) and the desired result is proved.

Theorem D shows that except of polynomials of full degree and proportional to them the sequence \( \{P_n\}_{n \geq 0} \) defined in (1.3) contains no other nonzero polynomials. Furthermore, if \( n \geq 1 \) then it follows from \( P_n \equiv 0 \) that \( \deg P_{n+1} < n+1 \) while \( P_n \neq 0 \) and \( \deg P_n < n \) imply \( \deg P_{n+1} = n+1 \). Observe that Theorem D (c) was essentially generalized by A. Draux [5, Theorem 6.2, p.477] in 1983.

4. Iohvidov’s approach from 1969

Throughout this section we fix an arbitrary nonzero sequence \( s := \{s_n\}_{n \geq 0} \) of real numbers and use the set \( \mathbb{N}_s \) defined as in (3.1). The analysis below will not use the statements from the previous Sections 2 and 3.

In 1969 Iohvidov [11, see also 10] suggested a new technique for dealing with Hankel matrices. For every \( r \in \mathbb{N}_s \) he proposed to use the approximating sequence \( s^{(r)} \) defined as follows.

We first put
\[
s_n^{(r)} = s_n, \quad 0 \leq n \leq 2r - 1. \quad (4.1)
\]
Since the first \( 2r - 1 \) numbers \( s_0, s_1, s_2, \ldots s_{2r-2} \) of the sequence \( s \) satisfy
\[
D_{r-1} = \begin{bmatrix}
0 & s_0 & s_1 & \cdots & s_{r-2} & s_{r-1} \\
1 & s_1 & s_2 & \cdots & s_{r-3} & s_r \\
. & . & . & \cdots & . & . \\
s_{r-2} & s_{r-1} & s_{r-2} & \cdots & s_{2r-4} & s_{2r-3} \\
s_{r-1} & s_r & s_{r-1} & \cdots & s_{2r-3} & s_{2r-2}
\end{bmatrix} \neq 0, \quad (4.2)
\]
Theorem 3. It is possible to determine uniquely all \( r \) numbers \( d^{(r)}_0, d^{(r)}_1, \ldots, d^{(r)}_{r-1} \) from the system

\[
\begin{pmatrix}
  s_0 & s_1 & \ldots & s_{r-2} & s_{r-1} \\
  s_1 & s_2 & \ldots & s_{r-1} & s_r \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  s_{r-2} & s_{r-1} & \ldots & s_{2r-4} & s_{2r-3} \\
  s_{r-1} & s_r & \ldots & s_{2r-3} & s_{2r-2}
\end{pmatrix}
\begin{pmatrix}
  d^{(r)}_0 \\
  d^{(r)}_1 \\
  \vdots \\
  d^{(r)}_{r-2} \\
  d^{(r)}_{r-1}
\end{pmatrix}
= \begin{pmatrix}
  s_r \\
  s_{r+1} \\
  \vdots \\
  s_{2r-2} \\
  s_{2r-1}
\end{pmatrix},
\tag{4.3}
\]

and then to define recursively the numbers \( s^{(r)}_{2r}, s^{(r)}_{2r+1}, \ldots \) by the formulas

\[
s^{(r)}_{2r} = s_r d^{(r)}_0 + s_{r+1} d^{(r)}_1 + \ldots + s_{2r-2} d^{(r)}_{r-2} + s_{2r-1} d^{(r)}_{r-1}, \quad s^{(r)}_{2r+m} = \sum_{k=0}^{r-1} s^{(r)}_{2r+m+k} d^{(r)}_k, \quad m \geq 1. \tag{4.4}
\]

We obtain the sequence

\[
s^{(r)} = \{ s^{(r)}_n \}_{n \geq 0}, \quad s^{(r)} = \{ s_0, s_1, s_2, \ldots, s_{2r-1}, s_{2r}, s^{(r)}_{2r+1}, \ldots \},
\tag{4.5}
\]

whose terms satisfy a homogeneous linear recurrence relation with constant coefficients of the form

\[
\sum_{k=0}^{r-1} d^{(r)}_k s^{(r)}_{k+m} = s^{(r)}_{m}, \quad m \geq 0,
\tag{4.6}
\]

which is equivalent to the simultaneous validity of (4.4) and (4.3) provided that (4.1) holds.

The statements of Theorem 3 (d) and (e) below follow easily from the main results of Iohvidov in Chapter II of [10], while their proofs given in Subsections 6.3 and 6.4 are based on a somewhat different approach than the one used in [10].

**Theorem 3.** Let \( s := \{ s_n \}_{n \geq 0} \) be a nonzero sequence of real numbers and \( \mathcal{H}_n := (s_{i+j})_{i,j=0}^n \), \( D_n := \det \mathcal{H}_n, n \geq 0 \).

(a) For arbitrary \( n \in \mathbb{N} \) the property

\[
s_0 = s_1 = s_2 = \ldots = s_{n-1} = 0, \quad s_n \neq 0,
\tag{4.7}
\]

is equivalent to

\[
D_0 = D_1 = \ldots = D_{n-1} = 0, \quad D_n \neq 0.
\]

If \( n \in \mathbb{N} \) and \( D_0 = D_1 = \ldots = D_{n-1} = 0 \) then

\[
D_n = (-1)^{\frac{n(n+1)}{2}} s_n^{n+1}.
\]

(b) The set

\[
\mathbb{N}_s := \{ r \geq 1 \mid D_{r-1} \neq 0 \}
\tag{4.8}
\]

is nonempty and for every \( r \in \mathbb{N}_s \) the formulas (4.1), (4.3) and (4.4) produce the sequence

\[
s^{(r)} := \{ s^{(r)}_n \}_{n \geq 0}, \quad s^{(r)} = \{ s_0, s_1, s_2, \ldots, s_{2r-1}, s_{2r}, s^{(r)}_{2r+1}, \ldots \},
\]

such that

\[
\text{rank} \left( s^{(r)}_{i+j} \right)_{i,j=0}^\infty = r.
\tag{4.9}
\]

(c) For every \( r \in \mathbb{N}_s \) we have

\[
\lim_{k \to \infty} \sqrt[k]{|s^{(r)}_k|} = \max \{ |z| \mid z \in \mathbb{C}, \ P_r(z) = 0 \},
\tag{4.10}
\]

and

\[
\frac{s_0}{x} + \frac{s_1}{x^2} + \ldots + \frac{s_{2r-1}}{x^{2r}} + \sum_{k \geq 2r} \frac{s_k}{x^{k+1}} = \frac{Q_r(x)}{P_r(x)},
\tag{4.11}
\]
where the series is absolutely convergent for every $|x| > \max\{ |z| \mid z \in \mathbb{C} , \ P_r(z) = 0 \}$.

(d) For arbitrary $d \in \mathbb{N}$ and $r \in \mathbb{N}_s$ the following statements hold:

$$D_r = \left(s_{2r} - s_{2r}(r)\right) D_{r-1} : \quad D'_{r+1} = \left(s_{2r+1} - s_{2r+1}(r)\right) D_{r-1} - \left(s_{2r} - s_{2r}(r)\right) D'_r ; \quad (4.12)$$

$$D_r = \ldots = D_{r+d-1} = 0 \iff s_{2r} = s_{2r}(r) , s_{2r+1} = s_{2r+1}(r) , \ldots s_{2r+d-1} = s_{2r+d-1}(r) ; \quad (4.13)$$

$$D_r = \ldots = D_{r+d-1} = 0 \Rightarrow D_{r+d} = (-1)^{\frac{d(d+1)}{2}} \left(s_{2r+d} - s_{2r+d}(r)\right) D_{r-1} . \quad (4.14)$$

(e) For every $r \in \mathbb{N}_s$ the Ioibzidov characteristic function

$$d_r := \inf \left\{ m \geq 0 \mid s_{2r+m} \neq s_{2r+m}(r) \right\} \in \{0,1,2,...,\} \cup \{\infty\} , \quad (4.15)$$

where it is assumed that $\inf \emptyset := \infty$, possesses the following property

$$d_r = \inf \left\{ m \geq 0 \mid D_{r+m} \neq 0 \right\} . \quad (4.16)$$

In particular, for arbitrary $r \in \mathbb{N}_s$ we have

$$d_r = \infty \iff s = s^{(r)} \iff D_{r-1} \neq 0 , \ D_r = D_{r+1} = \ldots = 0 . \quad (4.17)$$

The case of Theorem 3(a) was first considered by Frobenius [6, p.206] in 1894. In 1969 Ioibzidov [11, (5), p.244] introduced the characteristic function $d_r$ and established in [11, (7), p.246] the equalities (4.14) (see also [10, (10.5), p.62; (11.2), p.70]). A formula similar to (4.14) has been established in [1, Lemma 2.3]. However, the setting of Theorem 3 (d) and (e) differs from that of [10, Chapter 2] because only the finite Hankel matrices $(s_{i+j})_{i,j=0}^n$ are considered there.

5. Consequences of Theorem 3

5.1. Approximating sequence. The first immediate consequence of Theorem 3 is that the sequence $s^{(r)}$ for every $r \in \mathbb{N}_s$ can equivalently be defined by the expansion (4.11) which in view of $x' P_r(1/x) |_{x=0} = D_{r-1} \neq 0$ can be considered as the Taylor expansion at the origin of the rational function

$$\frac{z^{r-1} Q_r(1/z)}{z^r P_r(1/z)} = \sum_{k \geq 0} \frac{s_{r}^{(r)} z^k}{z^k} , \quad |z| < \min\{ |\zeta| \mid \zeta \in \mathbb{C} \setminus \{0\} , \ P_r(1/\zeta) = 0 \} .$$

5.2. Finiteness of rank. Assume now that for a given nonzero sequence $\{s_n\}_{n \geq 0}$ of real numbers the infinite Hankel matrix $(s_{i+j})_{i,j=0}^\infty$ has a finite rank. Then the set $\mathbb{N}_s$ defined as in (4.8) is finite because $D_{r-1} \neq 0$ means that the first $r$ rows and the first $r$ columns of the matrix $(s_{i+j})_{i,j=0}^\infty$ are linearly independent. There exists therefore a maximal element $r_* := \max \mathbb{N}_s \geq 1$ of $\mathbb{N}_s$ for which we have $d_{r_*} = \infty$ by virtue of (4.16). Then (4.15) yields $s = s^{(r_*)}$ and (4.9) implies $\operatorname{rank}(s_{i+j})_{i,j=0}^\infty = r_*$. Conversely, if $s = s^{(r)}$ for a certain $r \in \mathbb{N}_s$, then we have the validity of (4.9) in view of Theorem 3 (b). Thus, $\operatorname{rank}(s_{i+j})_{i,j=0}^\infty = r \iff D_{r-1} \neq 0$ and $s = s^{(r)}$. Combining this assertion with (4.11), (4.17) and with (4.1), (4.6), (6.15) we obtain the validity of the following corollary which contains the statements of Theorems A, B and Theorem C (a), while Theorem C (b) follows from (6.3) below.

Corollary 1. Let $s := \{s_n\}_{n \geq 0}$ be a nonzero sequence of real numbers, $D_n := \det(s_{i+j})_{i,j=0}^n$, $n \geq 0$, and let the sequence $s^{(m)} := \{s^{(m)}_n\}_{n \geq 0}$ for every $m \geq 1$ satisfying $D_{m-1} \neq 0$ be defined by the expansion

$$\frac{Q_m(x)}{P_m(x)} = \sum_{k \geq 0} \frac{s_{k}^{(m)} x^k}{x^{k+1}} , \quad |x| > \max\{ |z| \mid P_m(z) = 0 \} .$$
Then \( s_n^{(m)} = s_n \), \( 0 \leq n \leq 2m - 1 \) for every such \( m \), and the infinite Hankel matrix
\( \mathcal{H}_{\infty} := (s_{i+j})_{i,j=0}^{\infty} \) has a finite rank \( r \geq 1 \) if and only if \( D_{r-1} \neq 0 \) and one of the following equivalent condition holds:

(a) \( s = s^{(r)} \); 

(b) \( D_n = 0 \), \( n \geq r \); 

(c) \( \sum_{k\geq0} \frac{s_k}{x^k r!} = \frac{Q_n(x)}{P_r(x)} \), \( |x| > \max\{ |z| \mid z \in \mathbb{C} \}, \ P_r(z) = 0 \}; 

(d) \( D_{r-1}s_{r+m} + \sum_{k=0}^{r-1} p_{r,k} s_{k+m} = 0 \), \( m \geq r \), where \( P_r(x) = D_{r-1}x^r + \sum_{k=0}^{r-1} p_{r,k} x^k \). 

In other words, for arbitrary \( r \geq 1 \) satisfying \( D_{r-1} \neq 0 \) we have 

\( (a) \iff (b) \iff (c) \iff (d) \iff \text{rank } \mathcal{H}_{\infty} = r \), 

while \( \text{rank } \mathcal{H}_{\infty} = r \) implies \( D_{r-1} \neq 0 \) for arbitrary positive integer \( r \).

### 5.3. Positive semidefiniteness

The method used by Darboux in deriving formula \([4, (68), p.413]\), now called the Christoffel-Darboux summation formula, can be applied to the polynomials \( P_n \). This leads to the following formula (see \([3, \text{Theorem } 4.5, p.23]\), \([2, \text{Theorem } 2.6, p.50]\))

\[
D_{r-1}^2 \sum_{k=0}^{r-1} \frac{P_k(x)P_k(y)}{D_k D_{k-1}} = \frac{P_r(x)P_{r-1}(y) - P_r(y)P_{r-1}(x)}{x - y}, \quad D_{r-1} := 1, \ x \neq y,
\]

provided that \( D_k \neq 0 \) for all \( 0 \leq k \leq r - 1 \) and \( r \) is a positive integer. Thus,

\[
D_{r-1}^2 \sum_{k=0}^{r-1} \frac{P_k(x)^2}{D_k D_{k-1}} = P'_r(x)P_{r-1}(x) - P_r(x)P'_{r-1}(x),
\]

and

\[
D_{r-1}^2 \sum_{k=0}^{r-1} \frac{|P_k(z)|^2}{D_k D_{k-1}} = \frac{\text{Im } P_r(z)P_{r-1}(\overline{z})}{\text{Im } z}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

where \( \overline{z} \) is a complex conjugate of \( z \).

Under the conditions

\( D_0 > 0, \ D_1 > 0, \ldots, \ D_{r-1} > 0 \),

we then get

\[
\frac{\text{Im } P_r(z)P_{r-1}(\overline{z})}{\text{Im } z} = \frac{\text{Im } P_r(z)P_{r-1}(z)}{\text{Im } z} = D_{r-1}^2 \sum_{k=0}^{r-1} \frac{|P_k(z)|^2}{D_k D_{k-1}} \geq D_{r-1} D_{r-2} |P_{r-1}(z)|^2, \quad \text{Im } z \neq 0,
\]

so if \( P_r(z) = 0 \) for a \( z \) with \( \text{Im } z \neq 0 \), the last inequality above implies \( P_{r-1}(z) = 0 \), which is contradicting that \( P_r \) and \( P_{r-1} \) have no common zeros according to (2.7). Therefore all zeros of \( P_r \) are real, and if \( \lambda \) is a real zero of the polynomial \( P_r \) we have from (5.6)

\[
P'_r(\lambda)P_{r-1}(\lambda) = D_{r-1}^2 \sum_{k=0}^{r-1} \frac{P_k(\lambda)^2}{D_k D_{k-1}},
\]

while (2.7) yields

\[
P_{r-1}(\lambda)Q_r(\lambda) = D_{r-1}^2.
\]

Since (5.10) means that \( P_{r-1}(\lambda) \neq 0 \), (5.9) implies \( P'_r(\lambda) \neq 0 \) because

\[
\sum_{k=0}^{r-1} \frac{P_k(\lambda)^2}{D_k D_{k-1}} \geq \frac{P_{r-1}(\lambda)^2}{D_{r-1} D_{r-2}} > 0.
\]
Thus, all zeros \( \{\lambda_n\}_{n=1}^r \) of \( P_r \) are simple and by virtue of (5.9) and (5.10) we have

\[
\mu_n := \frac{Q_r(\lambda_n)}{P_r'(\lambda_n)} = \left( \sum_{k=0}^{r-1} \frac{P_k(\lambda_n)^2}{D_kD_{k-1}} \right)^{-1} \in (0, +\infty) , \quad 1 \leq n \leq r ,
\]

which gives the following form of the partial fraction decomposition of \( Q_r/P_r \):

\[
\frac{Q_r(x)}{P_r(x)} = \sum_{n=1}^{r} \frac{Q_r(\lambda_n)}{P_r'(\lambda_n)(x - \lambda_n)} = \sum_{m=0}^{\infty} \frac{1}{x^{m+1}} \sum_{n=1}^{r} \mu_n \lambda_n^m , \quad |x| > \max_{1 \leq n \leq r} |\lambda_n| .
\] (5.11)

Assume now that (2.1) and (5.8) hold. Then \( r \in \mathbb{N}_z \) and (5.5) implies the validity of (5.3) which in view of (5.11) yields

\[
s_m = \sum_{n=1}^{r} \mu_n \lambda_n^m , \quad m \geq 0 , \quad \mu_n = \left( \sum_{k=0}^{r-1} \frac{P_k(\lambda_n)^2}{D_kD_{k-1}} \right)^{-1} > 0 , \quad P_r(\lambda_n) = 0 , \quad 1 \leq n \leq r .
\]

We have completely proved the following assertion \(^1\)

**Theorem E (2015, [1, Theorem 1.1, p.1569]).** Let \( \{s_n\}_{n \geq 0} \) be an arbitrary sequence of real numbers and \( \mathcal{H}_n := (s_{i+j})_{i,j=0}^n , \quad n \geq 0 . \) Assume that there exists a positive integer \( n_0 \) such that

\[
D_n := \det \mathcal{H}_n > 0 , \quad 0 \leq n \leq n_0 - 1 , \quad \det \mathcal{H}_n = 0 , \quad n \geq n_0 .
\]

Then there exist \( n_0 \) distinct real numbers \( \{x_k\}_{k=1}^{n_0} \) and \( n_0 \) positive numbers \( \{\mu_k\}_{k=1}^{n_0} \) such that

\[
s_n = \int_{-\infty}^{+\infty} x^n d\mu(x) , \quad n \geq 0 , \quad \mu := \sum_{k=1}^{n_0} \mu_k \delta_{x_k} ,
\]

where \( \delta_y \) denotes the Dirac measure placed at the point \( y \in \mathbb{R} .
\]

6. Proof of Theorem 3

6.1. **Proof of Theorem 3(a).** If (4.7) holds then the first column in the matrices \( \mathcal{H}_0, \ldots, \mathcal{H}_{n-1} \) is zero, and therefore \( D_0 = D_1 = \ldots = D_{n-1} = 0 , \) while

\[
D_n = \begin{vmatrix}
0 & 0 & \ldots & 0 & s_n \\
0 & 0 & \ldots & s_n & s_{n+1} \\
0 & s_n & \ldots & s_{2n-2} & s_{2n-1} \\
0 & s_{n+1} & \ldots & s_{2n-1} & s_{2n}
\end{vmatrix} = (-1)^{(n+1)/2} s_n^{n+1} . \quad (6.1)
\]

Conversely, the identity \( D_0 = s_0 \) together with the condition \( D_0 = 0 \) imply \( s_0 = 0 . \) Then (6.1) gives \( D_1 = -s_1^2 , \) which by virtue of the condition \( D_1 = 0 \) yields \( s_1 = 0 . \) Pursuing a finite number of repetitions of this fact, we arrive at \( s_0 = s_1 = \ldots = s_{n-1} = 0 . \) In view of (6.1), \( D_n = (-1)^{(n+1)/2} s_n^{n+1} \neq 0 \) and therefore \( s_n \neq 0 . \) This concludes the proof of Theorem 3 (a).

\(^1\) During the preparation of the present paper the second author learned that the result is formulated in [17, Theorem 1.2, p.5]
6.2. Proof of Theorem 3(b). Since \( s \) is a nonzero sequence, Theorem 3(a) implies that \( N_s \) is nonempty. To prove that the Hankel rank of \( s^{(r)} \) is equal to \( r \), we observe that the relations (4.6) and (4.3) can also be written as follows

\[
\begin{pmatrix}
  s_{m+r}^{(r)} \\
  s_{m+r-1}^{(r)} \\
  \vdots \\
  s_{m+1}^{(r)} \\
  s_{m}^{(r)}
\end{pmatrix}
= 
\begin{pmatrix}
  d_{r-1}^{(r)} & d_{r-2}^{(r)} & \cdots & d_{2}^{(r)} & d_{1}^{(r)} & d_{0}^{(r)} \\
  1 & 0 & \cdots & 0 & 0 & 0 \\
  0 & 1 & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & 1 & 0 & 0 \\
  0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
  s_{m+r-1}^{(r)} \\
  s_{m+r-2}^{(r)} \\
  \vdots \\
  s_{m+2}^{(r)} \\
  s_{m+1}^{(r)} \\
  s_{m}^{(r)}
\end{pmatrix}, \quad m \geq 0.
\]

Therefore

\[
\begin{pmatrix}
  s_{m+n+r}^{(r)} \\
  s_{m+n+r-1}^{(r)} \\
  \vdots \\
  s_{m+n+2}^{(r)} \\
  s_{m+n+1}^{(r)}
\end{pmatrix}
= 
\begin{pmatrix}
  d_{r-1}^{(r)} & d_{r-2}^{(r)} & \cdots & d_{2}^{(r)} & d_{1}^{(r)} & d_{0}^{(r)} \\
  1 & 0 & \cdots & 0 & 0 & 0 \\
  0 & 1 & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & 1 & 0 & 0 \\
  0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}^{n+1}
\begin{pmatrix}
  s_{m+r-1}^{(r)} \\
  s_{m+r-2}^{(r)} \\
  \vdots \\
  s_{m+2}^{(r)} \\
  s_{m+1}^{(r)} \\
  s_{m}^{(r)}
\end{pmatrix}, \quad m, n \geq 0, \quad (6.2)
\]

and the first row of the matrix in the righthand side of (6.2) gives the existence of \( r \) numbers \( d_{n,0}^{(r)}, d_{n,1}^{(r)}, \ldots, d_{n,r-1}^{(r)} \) satisfying

\[
\sum_{k=0}^{r-1} d_{n,k}^{(r)} s_{k+m}^{(r)} = s_{r+n+m}^{(r)}, \quad m \geq 0,
\]

where \( d_{n,k}^{(r)} \) is equal to \( d_{k}^{(r)} \) from (4.6) for each \( 0 \leq k \leq r - 1 \). This means that

\[
\begin{pmatrix}
  s_{0}^{(r)} \\
  s_{1}^{(r)} \\
  \vdots \\
  s_{r}^{(r)} \\
  s_{r+1}^{(r)} \\
  \vdots \\
  s_{2r}^{(r)} \\
  s_{2r+1}^{(r)} \\
  \vdots \\
  \vdots \\
  s_{2r-1}^{(r)} \\
  s_{2r}^{(r)} \\
  \vdots \\
  \vdots \\
  s_{(r)}^{(r)} \\
  s_{(r)}^{(r)} \\
  \vdots \\
  \vdots \\
  \vdots
\end{pmatrix} = 
\begin{pmatrix}
  s_{0}^{(r)} \\
  s_{1}^{(r)} \\
  \vdots \\
  s_{r}^{(r)} \\
  s_{r+1}^{(r)} \\
  \vdots \\
  s_{2r}^{(r)} \\
  s_{2r+1}^{(r)} \\
  \vdots \\
  \vdots \\
  s_{2r-1}^{(r)} \\
  s_{2r}^{(r)} \\
  \vdots \\
  \vdots \\
  s_{(r)}^{(r)} \\
  s_{(r)}^{(r)} \\
  \vdots \\
  \vdots \\
  \vdots
\end{pmatrix} + \begin{pmatrix}
  d_{n,0}^{(r)} \\
  d_{n,1}^{(r)} \\
  \vdots \\
  d_{n,r-1}^{(r)}
\end{pmatrix} + \cdots + \begin{pmatrix}
  d_{n,0}^{(r)} \\
  d_{n,1}^{(r)} \\
  \vdots \\
  d_{n,r-1}^{(r)}
\end{pmatrix}, \quad (6.4)
\]

i.e., for arbitrary \( n \geq 0 \) the \((r+n+1)\)-th column \( (s_{n+r+j}^{(r)})_{j=0}^{\infty} \) of the infinite matrix \( (s_{i+j}^{(r)})_{i,j=0}^{\infty} \) is a linear combination of the first \( r \) columns which are linearly independent by virtue of (4.2). Thus,

\[
\text{rank } (s_{i+j}^{(r)})_{i,j=0}^{\infty} = r,
\]

and we conclude that the Hankel rank of \( s^{(r)} \) is equal to \( r \). Theorem 3(b) is proved.

6.3. Proof of Theorem 3(d). For the sequence \( \hat{s}^{(r)} = \{\hat{s}_{n}^{(r)}\}_{n \geq 0} \) defined by

\[
\hat{s}_{n}^{(r)} := s_{n} - s_{n}^{(r)}, \quad n \geq 0, \quad \hat{s}^{(r)} = s - s^{(r)},
\]

we have, by virtue of (4.5),

\[
\hat{s}^{(r)} = \{s_{n} - s_{n}^{(r)}\}_{n \geq 0} = \begin{pmatrix}
  0, 0, 0, \ldots, 0, \hat{s}_{2r}^{(r)}, \hat{s}_{2r+1}^{(r)}, \ldots
\end{pmatrix}.
\]

It is appropriate at this point to recall (see [8, Definition 8, p.61]) that two square matrices \( A \) and \( B \) are called equivalent if there exist two square matrices \( P \) and \( Q \) with nonzero
It is evident that for two 1-equivalent square matrices \( B = PAQ \). If \( \det P = \det Q = 1 \) we say that \( A \) and \( B \) are 1-equivalent and write

\[
A \overset{1}{\sim} B .
\]

It is evident that for two 1-equivalent square matrices \( A \) and \( B \) we have \( \det A = \det B \), and also \( \operatorname{rank} A = \operatorname{rank} B \) in view of [8, Theorem 2, p.62].

For arbitrary \( r \in \mathbb{N}_5 \) consider the matrix

\[
\mathcal{H}_r = \begin{pmatrix}
0 & s_1 & \cdots & s_{r-2} & s_{r-1} & s_r \\
1 & s_2 & \cdots & s_{r-1} & s_r & s_{r+1} \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
s_{r-1} & s_r & \cdots & s_{2r-3} & s_{2r-2} & s_{2r-1} \\
s_r & s_{r+1} & \cdots & s_{2r-2} & s_{2r-1} & s_{2r}
\end{pmatrix} = \begin{pmatrix}
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r)
\end{pmatrix}.
\]

Subtracting from the last column a linear combination of the first \( r \) columns with the coefficients from (6.4) with \( n = 0 \), we conclude that

\[
\mathcal{H}_r \overset{1}{\sim} \begin{pmatrix}
0 & s_1 & \cdots & s_{r-1} & 0 \\
0 & s_2 & \cdots & s_r & 0 \\
& \cdots & \cdots & \cdots & \cdots \\
0 & s_{r-1} & s_r & \cdots & s_{2r-2} \\
0 & s_r & s_{r+1} & \cdots & s_{2r-1} \\
\hat{s}_{2r}
\end{pmatrix}.
\]

(6.5)

The Laplace expansion of the determinant of the righthand side of (6.5) by minors along column \( r + 1 \) leads to the validity of the lefthand equality in (4.12),

\[
D_r = D_{r-1} \cdot \begin{pmatrix}
s_{2r} - s_{2r} \\
r \in \mathbb{N}_5
\end{pmatrix}
\]

Thus, \( D_r = 0 \) if and only if \( s_{2r} = s_{2r}^{(r)} \), which proves (4.13) for \( d = 1 \).

To prove (4.13) for \( d > 1 \) assume that

\[
s_{2r} = s_{2r}^{(r)} , \ s_{2r+1} = s_{2r+1}^{(r)} , \ldots , s_{2r+d-1} = s_{2r+d-1}^{(r)}.
\]

(6.7)

Consider the matrix

\[
\mathcal{H}_{r+d} = \begin{pmatrix}
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r)
\end{pmatrix}.
\]

For every \( 0 \leq n \leq d \) we subtract from the \((r + n + 1)\)-th column a linear combination of the first \( r \) columns with the coefficients from the equality (6.4), and we obtain

\[
\mathcal{H}_{r+d} \overset{1}{\sim} \begin{pmatrix}
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r) \\
(r) & (r) & \cdots & (r) & (r) & (r)
\end{pmatrix}.
\]

(6.8)
Expanding the determinant of the matrix in the righthand side of (6.8) after the last row $d + 1$ times or by using that the matrix is quasi-triangular (see [8, p.43]) and using the formulas [8, (67), p.43], (6.7) and (4.1), we get

$$D_{r+d} = (-1)^{d(d+1)/2} \left( s_{2r+d} - s_{2r+d}^{(r)} \right) D_{r-1}.$$  

(6.9)

Furthermore, it follows from (6.8) that for all $0 \leq n \leq d - 1$ the matrix $H_{r+n}$ is 1-equivalent to the matrix with zero $(r + 1)$-th column. Therefore

$$D_r = \ldots = D_{r+d-1} = 0,$$

(6.10)

which proves the implication \(\Leftarrow\) in (4.13) for $d \geq 1$.

To prove the inverse implication in (4.13) for such $d$ assume that (6.10) holds for some $1 \leq d < \infty$. Then $D_r = 0$ implies $s_{2r} = s_{2r}^{(r)}$ by virtue of (6.6). Therefore (6.7) holds for $d = 1$, and we can use the expression (6.9) for $D_{r+1}$ to give $s_{2r+1} = s_{2r+1}^{(r)}$ if $D_{r+1} = 0$. Pursuing a finite number of repetitions of this trick, we get at last $s_{2r+d-1} = s_{2r+d-1}^{(r)}$ which completes the proof of (4.13).

Finally, the equivalence (4.13) and the implication (6.7) \(\Rightarrow\) (6.9) just deduced give the validity of (4.14).

To prove the righthand equality in (4.12) we take $r \in \mathbb{N}_s$ and consider

$$D'_{r+1} := \begin{vmatrix} s_0 & s_1 & \ldots & s_{r-2} & s_{r-1} & s_{r+1} \\ s_1 & s_2 & \ldots & s_{r-1} & s_r & s_{r+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_{r-2} & s_{r-1} & \ldots & s_{2r-4} & s_{2r-3} & s_{2r-1} \\ s_{r-1} & s_r & \ldots & s_{2r-3} & s_{2r-2} & s_{2r} \\ s_r & s_{r+1} & \ldots & s_{2r-2} & s_{2r-1} & s_{2r+1} \end{vmatrix}.$$

Subtracting from the last column a linear combination of the first $r$ columns with the coefficients from (6.4) with $n = 0$, we obtain

$$D'_{r+1} = \begin{vmatrix} s_0^{(r)} & s_1^{(r)} & \ldots & s_{r-2}^{(r)} & s_{r-1}^{(r)} & 0 \\ s_1^{(r)} & s_2^{(r)} & \ldots & s_{r-1}^{(r)} & s_r^{(r)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_{r-2}^{(r)} & s_{r-1}^{(r)} & \ldots & s_{2r-4}^{(r)} & s_{2r-3}^{(r)} & 0 \\ s_{r-1}^{(r)} & s_r^{(r)} & \ldots & s_{2r-3}^{(r)} & s_{2r-2}^{(r)} & s_{2r} - s_r^{(r)} \\ s_r^{(r)} & s_{r+1}^{(r)} & \ldots & s_{2r-2}^{(r)} & s_{2r-1}^{(r)} & s_{2r+1} - s_{2r+1}^{(r)} \end{vmatrix}.$$  

(6.11)

The Laplace expansion of the determinant in the righthand side of (6.11) by minors along the column $r + 1$ and (4.1) lead to the validity of the righthand equality in (4.12). This finishes the proof of Theorem 3(d).

6.4. Proof of Theorem 3(e). The formula (6.6) proves that $D_r \neq 0$ if and only if $s_{2r} \neq s_{2r}^{(r)}$. Thus, the definitions of $d_r$ given in (4.15) and in (4.16) are the same in the case where $d_r = 0$.

Assume now that the number $d_r$ defined by (4.15) is finite and $1 \leq d_r < \infty$. Then (6.7) holds for $d = d_r$ and $s_{2r+d} \neq s_{2r+d}^{(r)}$, which by (6.9) means that $D_{r+d} \neq 0$. Thus, $d_r$ coincides with the infimum in the righthand side of (4.16).

Conversely, if $d_r$ is defined by (4.16) and $1 \leq d_r < \infty$ then $D_{r+d_r} \neq 0$ and (6.10) holds for $d = d_r$ as well as (6.7) by virtue of (4.13). We can therefore apply the formula (6.9) for $d = d_r$ to conclude that $D_{r+d_r} \neq 0$ yields $s_{2r+d} \neq s_{2r+d}^{(r)}$. This means that $d_r$ coincides with the infimum in the righthand side of (4.15). Thus, two definitions of $d_r$ given in (4.15) and in (4.16) also coincide when $1 \leq d_r < \infty$.

Finally, (4.17) follows directly from (4.13) applied for every positive integer $d$. This completes the proof of Theorem 3(e).
6.5. Proof of Theorem 3(c). If \( p(x, t) = \sum_{k=0}^{n} \sum_{j=0}^{m} p_{k,j} x^k t^j \) is an algebraic polynomial with real coefficients of two variables \( x \) and \( t \), we use the linear functional \( L \) from (1.5) with respect to the \( t \)-variable to get

\[
L_t(p(x, t)) = \sum_{k=0}^{n} \sum_{j=0}^{m} s_j p_{k,j} x^k = \sum_{k=0}^{n} \left( \sum_{j=0}^{m} s_j p_{k,j} \right) x^k \in \mathcal{P}[\mathbb{R}].
\]

For example,

\[
L_t \left( \frac{1-1}{x-t} \right) = 0, \quad L_t \left( \frac{x-t}{x-t} \right) = s_0, \quad L_t \left( \frac{x^2-t^2}{x-t} \right) = s_0 x + s_1, \quad L_t \left( \frac{x^n-t^n}{x-t} \right) = \sum_{j=0}^{n-1} s_j x^{n-1-j}, \quad n \geq 1.
\]

Comparing these equalities with (1.3) we see that

\[
Q_n(x) := L_t \left( \frac{P_n(x) - P_n(t)}{x-t} \right), \quad n \geq 0,
\]

and if \( P_n(x) := \sum_{k=0}^{n} p_{k,n} x^k \), \( Q_n(x) := \sum_{k=0}^{n-1} q_{n,k} x^k \), \( n \geq 1 \), we obtain

\[
P_0(x) = 1, \quad P_1(x) = s_0 x - s_1, \quad P_n(x) = D_{n-1} x^n + \sum_{k=0}^{n-1} p_{n,k} x^k, \quad p_{n,n} = D_{n-1},
\]

\[
P_n(x) - P_n(t) = \sum_{m=1}^{n} p_{m,n} \frac{x^m - t^m}{x-t} = \sum_{m=1}^{n-1} p_{m,n+1} \sum_{k=0}^{m} x^k t^{m-k} = \sum_{k=0}^{n-1} \left( \sum_{m=k}^{n-1} p_{m,n+1} t^{m-k} \right) x^k,
\]

\[
Q_1(x) := s_0^2, \quad Q_n(x) = L_t \left( \frac{P_n(x) - P_n(t)}{x-t} \right) = \sum_{k=0}^{n-1} \left( \sum_{m=k}^{n-1} p_{n,m+1} s_{m-k} \right) x^k,
\]

\[
q_{n,m} = \sum_{k=m}^{n-1} p_{n,k+1} s_{k-m} = \sum_{k=0}^{n-m-1} p_{n,k+m+1} s_k, \quad 0 \leq m \leq n-1, \quad n \geq 1,
\]

where the latter equalities can be written in the following form:

\[
q_{n,n-1-m} = \sum_{k=0}^{n-m} p_{n,n-(m-k)} s_k, \quad 0 \leq m \leq n-1, \quad n \geq 1.
\]  \( \text{(6.12)} \)

Let \( r \in \mathbb{N}_s \), i.e., \( D_{r-1} \neq 0 \). According to (1.3) we have \( L(t^m P_r(t)) = 0 \) for every \( 0 \leq m \leq r-1 \) and therefore

\[
\sum_{k=0}^{r} p_{r,k} s_{k+m} = 0, \quad 0 \leq m \leq r-1,
\]  \( \text{(6.13)} \)

which gives

\[
\sum_{k=0}^{r-1} (-p_{r,k}) s_{k+m} = D_{r-1} s_{r+m}, \quad 0 \leq m \leq r-1,
\]

or

\[
\begin{pmatrix}
  s_0 & s_1 & \ldots & s_{r-2} & s_{r-1} \\
  s_1 & s_2 & \ldots & s_{r-1} & s_r \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  s_{r-2} & s_{r-1} & \ldots & s_{2r-4} & s_{2r-3} \\
  s_{r-1} & s_r & \ldots & s_{2r-3} & s_{2r-2}
\end{pmatrix}
\begin{pmatrix}
  -p_{r,0}/D_{r-1} \\
  -p_{r,1}/D_{r-1} \\
  \ldots \\
  -p_{r,r-2}/D_{r-1} \\
  -p_{r,r-1}/D_{r-1}
\end{pmatrix}
= \begin{pmatrix}
  s_r \\
  s_{r+1} \\
  \ldots \\
  s_{2r-2} \\
  s_{2r-1}
\end{pmatrix}.
\]  \( \text{(6.14)} \)

Since \( D_{r-1} \neq 0 \) it follows from (4.3) and (6.14) that

\[
d_k^{(r)} = -\frac{p_{r,k}}{D_{r-1}}, \quad 0 \leq k \leq r-1, \quad P_r(x) = D_{r-1} x^r + \sum_{k=0}^{r-1} p_{r,k} x^k, \quad r \in \mathbb{N}_s,
\]  \( \text{(6.15)} \)

and therefore the recursive formulas (4.4) can be written in the following manner

\[
D_{r-1} s_{2r+m} + \sum_{k=0}^{r-1} p_{r,k} s_{r+m+k} = 0, \quad m \geq 0,
\]
which together with (6.13) gives
\[
\sum_{k=0}^{r} p_{r,k}s_{k+m}^{(r)} = 0, \quad m \geq 0,
\]
or
\[
D_{r-1}s_{r+m}^{(r)} + \sum_{k=0}^{r-1} p_{r,k}s_{k+m}^{(r)} = 0, \quad m \geq 0. \quad (6.16)
\]

Denote \( P_r^*(x) := x^r P_r(1/x) \) and \( Q_r^*(x) := x^{r-1} Q_r(1/x) \). Then
\[
P_r^*(x) = D_{r-1} + \sum_{k=0}^{r-1} p_{r,k} x^{r-k}, \quad P_r^*(0) = D_{r-1} \neq 0, \quad Q_r^*(x) = \sum_{k=0}^{r-1} q_{r,k} x^{r-1-k},
\]
and the function \( \psi_r(z) := Q_r^*(z)/P_r^*(z) \) is analytic on the open disk \(|z| < 1/\rho_r\) where \( \rho_r := \max\{ |z| \mid z \in \mathbb{C}, P_r(z) = 0 \} \). Let \( a_n, n \geq 0 \), be the coefficients of the Taylor expansion of \( \psi_r(z) \) at the origin,
\[
\psi_r(z) = \sum_{n=0} a_n z^n, \quad |z| < 1/\rho_r,
\]
which is obviously equivalent to the expansion of the form
\[
\frac{Q_r(x)}{P_r(x)} = \sum_{m=0} a_m x^{m+1}, \quad |x| > \rho_r. \quad (6.18)
\]
The identity
\[
\left( \sum_{m=0}^{r} p_{r,m} x^m \right) \sum_{m=0} a_m \frac{x^m}{x^{m+1}} = \sum_{m=0}^{r} \frac{p_{r,k} a_{m+k}}{x^{m+1}} + \sum_{m=0}^{r-1} x^m \sum_{k=m}^{r-1} p_{r,k+1} a_{k-m},
\]
and (6.18) imply
\[
\sum_{k=m}^{r-1} p_{r,k+1} a_{k-m} = q_{r,m}, \quad 0 \leq m \leq r - 1,
\]
\[
\sum_{k=0}^{r} p_{r,k} a_{m+k} = 0, \quad m \geq 0,
\]
which can be written as
\[
\sum_{k=0}^{m} p_{r,r-(m-k)} a_k = q_{r,r-1-m}, \quad 0 \leq m \leq r - 1, \quad (6.19)
\]
\[
D_{r-1} a_{m+r} + \sum_{k=0}^{r-1} p_{r,k} a_{m+k} = 0, \quad m \geq 0.
\]
Observe that (4.1), (6.12) for \( n = r \) and (6.16) mean that
\[
\sum_{k=0}^{m} p_{r,r-(m-k)} s_k^{(r)} = q_{r,r-1-m}, \quad 0 \leq m \leq r - 1,
\]
\[
D_{r-1} s_{m+r}^{(r)} + \sum_{k=0}^{r-1} p_{r,k} s_{m+k}^{(r)} = 0, \quad m \geq 0.
\]
and by subtracting from these equalities the corresponding equalities in (6.19) we obtain
\[
D_{r-1} \left[ s_m^{(r)} - a_m \right] + \sum_{k=0}^{m-1} p_{r,r-(m-k)} \left[ s_k^{(r)} - a_k \right] = 0, \quad 0 \leq m \leq r - 1,
\]
\[
D_{r-1} \left[ s_m^{(r)} - a_m + r \right] + \sum_{k=0}^{m-1} p_{r,k} \left[ s_{m+k}^{(r)} - a_{m+k} \right] = 0, \quad m \geq 0,
\]
where it is assumed that \( \sum_{0}^{r-1} := 0 \). From these recurrence relations we obtain \( s_m^{(r)} = a_m \) for all \( m \geq 0 \) as a consequence of \( D_{r-1} \neq 0 \). Together with (4.1) this proves (4.11). Since the radius of convergence of the Taylor series (6.17) is known we get the validity of (4.10) by virtue of the Cauchy-Hadamard formula (see [16, (2), p.200]). Theorem 3 (c) is proved.
7. Proof of Theorem 1

If \( t_n = 0 \) for all \( n \geq 0 \) then Theorem 3 (a) yields \( s_n = 0 \) for every \( n \geq 0 \) and therefore in this case (1.1) has only one solution as stated in the theorem.

To examine the case \( Z_t \neq 0 \) we introduce the notation

\[
\Delta_0 := (-1)^{\frac{n_0+1}{2}} t_{n_0}, \quad \Delta_{k+1} := (-1)^{\frac{n_{k+1}-n_k}{2}} t_{n_{k+1}} t_{n_k}, \quad 0 \leq k < m-1, \quad 2 \leq m \leq \infty .
\] (7.1)

7.1. Necessity of Theorem 1. To prove necessity, we assume that (1.1) has at least one solution \( s := \{s_n\}_{n \geq 0} \) for a given \( \{t_n\}_{n \geq 0} \). Then by Theorem 3 (a),

\[
t_{n_0} = (-1)^{\frac{n_0(n_0+1)}{2}} s_{n_0+1} .
\] (7.2)

Furthermore, in (4.8) we have

\[ N_k = \{n_k + 1\}_{0 \leq k < m} , \]

and for every \( 0 \leq k < m \) the formulas (4.1), (4.3) and (4.4) for \( r = n_k + 1 \) and the numbers \( s_0, s_1, \ldots, s_{2n_k+1} \) produce the sequence

\[ s^{(n_k+1)} = \{s_0, s_1, s_2, \ldots, s_{2n_k+1}, s_{2n_k+2}, \ldots\} , \quad 0 \leq k < m . \] (7.3)

For arbitrary \( 0 \leq k < m - 1, \ 2 \leq m \leq \infty \), it follows from Theorem 3 (d) with \( r = n_k + 1 \) and \( d = n_{k+1} - n_k - 1 \) that

\[
t_{n_{k+1}} = (-1)^{\frac{(n_{k+1}-n_k)(n_{k+1}-n_k-1)}{2}} \left( s_{n_{k+1}+n_k+1} - s_{n_{k+1}+n_k+1}^{(n_{k+1})} \right) t_{n_k} .
\] (7.4)

But in view of (7.1), (7.2) and (7.4) we have

\[
\Delta_0 = (-1)^{\frac{(n_0+1)^2}{2}} s_{n_0+1} > 0 , \quad \Delta_{k+1} = (-1)^{\frac{(n_{k+1}-n_k)^2}{2}} \left( s_{n_{k+1}+n_k+1} - s_{n_{k+1}+n_k+1}^{(n_{k+1})} \right) t_{n_k} > 0 , \quad 0 \leq k < m - 1 ,
\]

provided that \( n_0 + 1 \in 2N \) and \( n_{k+1} - n_k \in 2N \) for \( 0 \leq k < m - 1, \ 2 \leq m \leq \infty \). This proves the necessity part of Theorem 1.

7.2. Sufficiency of Theorem 1. The proof of sufficiency proceeds by induction on \( k \). Given a sequence \( \{t_n\}_{n \geq 0} \) satisfying the conditions of Theorem 1 we will determine the terms of the sequence \( s := \{s_n\}_{n \geq 0} \) such that (1.1) holds. Let \( D_n := \det(s_{i+j})_{i,j=0}^n, \ n \geq 0 \).

If \( n_0 = 0 \) we put \( s_0 = t_0 \) to obtain \( D_0 = s_0 = t_0 \). If \( n_0 = 1 \) and \( n_0 + 1 \in 2N + 1 \), we set

\[ s_0 = s_1 = \ldots = s_{n_0-1} = 0 , \quad s_{n_0} = (-1)^{\frac{n_0}{2}} \frac{1}{t_{n_0+1}} .
\]

According to Theorem 3 (a) we have

\[ D_0 = D_1 = \ldots = D_{n_0-1} = 0 , \quad D_{n_0} = (-1)^{\frac{n_0(n_0+1)}{2}} s_{n_0+1} = (-1)^{\frac{n_0(n_0+1)}{2}} t_{n_0} = t_{n_0} .
\]

Assume now that \( n_0 + 1 \in 2N \). Then, in view of (7.1),

\[
t_{n_0} = (-1)^{\frac{n_0+1}{2}} \Delta_0 , \quad \Delta_0 > 0 ,
\]

and if we put

\[ s_0 = s_1 = \ldots = s_{n_0-1} = 0 , \quad s_{n_0} = \Delta_0^{\frac{1}{n_0+1}} ,
\]

then by Theorem 3 (a) we obtain

\[ D_0 = D_1 = \ldots = D_{n_0-1} = 0 , \quad D_{n_0} = (-1)^{\frac{n_0(n_0+1)}{2}} s_{n_0+1} = (-1)^{\frac{n_0(n_0+1)}{2}} \Delta_0 = t_{n_0} .
\]

Therefore in both cases the numbers \( s_0, \ldots, s_{2n_0} \) with \( s_{n_0+1}, \ldots, s_{2n_0} \) chosen arbitrarily satisfy (1.1) for \( 0 \leq n \leq n_0 \).

Suppose that for a certain \( k \) satisfying \( 0 \leq k < m - 1 \) where \( 2 \leq m \leq \infty \), the numbers \( s_0, \ldots, s_{2n_k} \) satisfy (1.1) for \( 0 \leq n \leq n_k \). We prove that it is possible to determine the numbers \( s_{2n_k+1}, s_{2n_k+2}, \ldots, s_{2n_{k+1}} \) such that (1.1) holds for \( 0 \leq n \leq n_{k+1} \).

Choosing arbitrarily the number \( s_{2n_{k+1}} \) we construct the sequence \( s^{(n_{k+1})} \) as in (7.3).
Assume first that \( n_{k+1} = n_k + 1 \). In view of (4.12) for \( r = n_k + 1 \),

\[
D_{n_{k+1}} = \left( s_{2n_{k+1}} - s_{2n_{k+1}}^{(n_k+1)} \right) t_{n_k},
\]

and by putting

\[
s_{2n_{k+1}} = s_{2n_{k+1}}^{(n_k+1)} + \frac{t_{n_{k+1}}}{t_{n_k}},
\]

we obtain the desired equality \( D_{n_{k+1}} = t_{n_{k+1}} \).

Assume now that \( n_{k+1} - n_k \geq 2 \). Then

\[
t_{n_{k+1}} = ... = t_{n_{k+1} - 1} = 0
\]

and if we set

\[
s_{2n_k+2} = s_{2n_k+2}^{(n_k+1)} , \ s_{2n_k+3} = s_{2n_k+3}^{(n_k+1)} , \ ... , \ s_{n_k+n_k+1} = s_{n_k+n_k+1}^{(n_k+1)},
\]

we obtain, by virtue of (4.13) with \( r = n_k + 1 \) and \( d = n_k+1 - n_k - 1 \),

\[
D_{n_{k+1}} = 0 = t_{n_{k+1}} , \ ... , \ D_{n_{k+1} - 1} = 0 = t_{n_{k+1} - 1},
\]

and in view of (4.14),

\[
D_{n_{k+1}} = (-1)^{\frac{(n_k+1)-n_k-1)(n_{k+1}-n_k)}{2}} \left( s_{n_{k+1}+n_k+1} - s_{n_{k+1}+n_k+1}^{(n_k+1)} \right) t_{n_{k+1} - 1}. \tag{7.5}
\]

If \( n_{k+1} - n_k \in 2\mathbb{N} + 1 \) we can choose

\[
s_{n_{k+1}+n_k+1} = s_{n_{k+1}+n_k+1}^{(n_k+1)} + (-1)^{\frac{(n_k+1)-n_k-1}{2}} \left( \frac{t_{n_{k+1}}}{t_{n_k}} \right)^{n_{k+1}-n_k},
\]

to have from (7.5), \( D_{n_{k+1}} = t_{n_{k+1}} \).

But if \( n_{k+1} - n_k \in 2\mathbb{N} \) we set

\[
s_{n_{k+1}+n_k+1} = s_{n_{k+1}+n_k+1}^{(n_k+1)} + \left( \frac{\Delta_{k+1}}{t_{n_k}^2} \right)^{\frac{1}{n_{k+1}-n_k}},
\]

where according to the conditions of the theorem

\[
\Delta_{k+1} = (-1)^{\frac{n_{k+1}-n_k}{2}} t_{n_{k+1}} t_{n_k}, \quad \Delta_{k+1} > 0.
\]

Then (7.5) gives

\[
D_{n_{k+1}} = (-1)^{\frac{(n_{k+1}-n_k-1)(n_{k+1}-n_k)}{2}} \left( s_{n_{k+1}+n_k+1} - s_{n_{k+1}+n_k+1}^{(n_k+1)} \right) t_{n_{k+1}} - 1
\]

\[
\Delta_{k+1} \left( \frac{t_{n_{k+1}}}{t_{n_k}} \right)^{n_{k+1}-n_k} = (-1)^{\frac{(n_{k+1}-n_k)^2}{2}} t_{n_{k+1}} = t_{n_{k+1}}.
\]

Therefore in both cases the numbers \( s_0, \ ... , \ s_{n_{k+1}+n_{k+1}} \) satisfy (1.1) for \( 0 \leq n \leq n_{k+1} \) independently on the choice of \( s_{n_{k+1}+n_{k+2}}, \ ... , \ s_{2n_{k+1}} \). Choosing arbitrarily the latter numbers we obtain the desired result. This finishes the proof of Theorem 1.
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