Corrigendum: The luminosity distance–redshift relation up to second order in the Poisson gauge with anisotropic stress (2015 Class. Quantum Grav. 32 045004)

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Starting from equation (3.6) and until the end of the manuscript the angular variables should always be \( \theta^a \) and not \( \theta^i \). This misinterpretation propagated along the rest of the paper and is responsible for the major part of the following typos and errors.

- Starting from equation (3.12) and until the end of the manuscript \( \nu^a_{Li} \) should be always replaced by \( -\nu^a_{Li} \).

- At the end of the line after equation (3.12) the text ‘along the line of sight’ should be replaced by ‘of the source position’.

- In equations (3.15) and (4.25) the term

\[
\int_{\eta_s}^{\eta_0} d\eta' \partial_0 (\partial_0 \psi (\eta')) \int_{\eta_s}^{\eta_0} d\eta' \gamma_0^{ab} \partial_b \int_{\eta_s}^{\eta_0} d\eta' \psi (\eta'),
\]

should be replaced by

\[
+4 \int_{\eta_s}^{\eta_0} d\eta' \partial_0 (\partial_0 \psi (\eta')) \int_{\eta_s}^{\eta_0} d\eta' \gamma_0^{ab} \partial_b \int_{\eta_s}^{\eta_0} d\eta' \psi (\eta'),
\]

- In equation (3.31) the terms

\[
+ \left( 1 - \frac{1}{H_s \Delta \eta} \right) \left\{ 4 \int_{\eta_s}^{\eta_0} d\eta' \partial_0 (\partial_0 \psi (\eta')) \int_{\eta_s}^{\eta_0} d\eta' \gamma_0^{ab} \partial_b \int_{\eta_s}^{\eta_0} d\eta' \psi (\eta') \right. \\
\left. - \frac{4}{\Delta \eta} \left[ \int_{\eta_s}^{\eta_0} d\eta' \partial_0 (\partial_0 \psi (\eta')) \int_{\eta_s}^{\eta_0} d\eta' \gamma_0^{ab} \partial_b \int_{\eta_s}^{\eta_0} d\eta' \psi (\eta') \right] \right. \\
\left. \left. - \partial_0 \left( \int_{\eta_s}^{\eta_0} d\eta' \gamma_0^{ab} \partial_b \int_{\eta_s}^{\eta_0} d\eta' \psi (\eta') \right) \right) \partial_0 \left( \int_{\eta_s}^{\eta_0} d\eta' \gamma_0^{ac} \partial_c \int_{\eta_s}^{\eta_0} d\eta' \psi (\eta') \right) \right.
\]

should be replaced by

\[
\int_{\eta_s}^{\eta_0} d\eta' \partial_0 (\partial_0 \psi (\eta')) \int_{\eta_s}^{\eta_0} d\eta' \gamma_0^{ab} \partial_b \int_{\eta_s}^{\eta_0} d\eta' \psi (\eta'),
\]
\[ -2 \int_{n_c}^{n_a} \frac{1}{(\eta_n - \eta_c)^2} \Delta_2 \left[ \int_{n_\eta}^{n_{\eta_a}} d\eta \gamma^{ab}_{\eta} \partial_a \int_{\eta_c}^{\eta_a} d\eta' \psi^I(\eta') \right] \partial_b \int_{n_\eta}^{n_{\eta_a}} d\eta' \psi^I(\eta') \\
+ \frac{2}{\eta_n - \eta_c} \int_{n_c}^{n_\eta} \frac{d\eta'}{(\eta_n - \eta_c)^2} \left[ \left( \partial_b \psi^I(\eta') \right) \int_{n_\eta}^{n_{\eta_a}} d\eta \gamma^{ab}_{\eta} \partial_a \int_{\eta_c}^{\eta_a} d\eta' \psi^I(\eta') \right] \\
- 2 \int_{n_c}^{n_\eta} \frac{d\eta'}{(\eta_n - \eta_c)^2} \left[ \left( \int_{n_\eta}^{n_{\eta_a}} d\eta \gamma^{ab}_{\eta} \partial_a \int_{\eta_c}^{\eta_a} d\eta' \psi^I(\eta') \right) \partial_b \int_{n_\eta}^{n_{\eta_a}} d\eta' \psi^I(\eta') \right] \\
- 4 \cot \frac{\theta}{(\sin \theta)^2} \int_{n_c}^{n_\eta} \frac{d\eta'}{(\eta_n - \eta_c)^2} \partial_\eta \left[ \left( \int_{n_\eta}^{n_{\eta_a}} d\eta \gamma^{ab}_{\eta} \partial_a \int_{\eta_c}^{\eta_a} d\eta' \psi^I(\eta') \right) \partial_b \int_{n_\eta}^{n_{\eta_a}} d\eta' \psi^I(\eta') \right] \\
- \frac{1}{\eta_n - \eta_c} \left[ \int_{n_c}^{n_\eta} d\eta' \frac{\eta_n - \eta_c}{\eta_n - \eta'} \partial_b \psi^I(\eta') \right]^2 \\
\]

should be replaced by

\[ + \left( 1 - \frac{1}{\mathcal{H}, \eta} \right) \left\{ + 4 \int_{n_c}^{n_\eta} d\eta' \partial_a \left( \partial_b \psi^I(\eta') \right) \int_{n_\eta}^{n_{\eta_a}} d\eta \gamma^{ab}_{\eta} \partial_a \int_{\eta_c}^{\eta_a} d\eta' \psi^I(\eta') \right\} \\
- 4 \int_{n_c}^{n_\eta} d\eta' \partial_a \left( \partial_b \psi^I(\eta') \right) \int_{n_\eta}^{n_{\eta_a}} d\eta \gamma^{ab}_{\eta} \partial_a \int_{\eta_c}^{\eta_a} d\eta' \psi^I(\eta') \right\} \\
+ \partial_\eta \left( \int_{n_c}^{n_\eta} d\eta' \psi^I(\eta') \right) \partial_b \left( \int_{n_\eta}^{n_{\eta_a}} d\eta \gamma^{ab}_{\eta} \partial_a \int_{\eta_c}^{\eta_a} d\eta' \psi^I(\eta') \right) \\
+ \frac{2}{\eta_n - \eta_c} \int_{n_c}^{n_\eta} d\eta' \frac{\eta_n - \eta_c}{\eta_n - \eta'} \partial_b \left[ \Delta_2 \psi^I(\eta') \right] \int_{n_\eta}^{n_{\eta_a}} d\eta \gamma^{ab}_{\eta} \partial_a \int_{\eta_c}^{\eta_a} d\eta' \psi^I(\eta') \right\} \\
+ \frac{1}{\eta_n - \eta_c} \left[ \int_{n_c}^{n_\eta} d\eta' \frac{\eta_n - \eta_c}{\eta_n - \eta'} \partial_b \psi^I(\eta') \right]^2 \]

- In the second line of the first paragraph of section IV, the text ‘basically agree’ should be replaced by ‘agree’.
- All the text and equations starting from after equation (4.9) and up the paragraph after equation (4.14) included should be erased and replaced by the following sentence ‘Using equations (4.4)–(4.9) to evaluate equations (4.1)–(4.3) one can prove that these are perfectly equivalent to equations (3.28), (3.29) and (3.31) with \( \psi^I = \psi \) and \( \psi^A = 0 \).’
- In the paragraph after equation (4.25), the text ‘One can then note that while the terms: \( \delta^{(2)} zS, \delta^{(2)} zSW, \delta^{(2)} zSWXISW, \delta^{(2)} zDop, \delta^{(2)} zDopL, \) and \( \delta^{(2)} zDop[ISW] \) coincide with the corresponding terms of [11], the terms: \( \delta^{(2)} zDop[ISW], \delta^{(2)} zSWXDop \) and \( \delta^{(2)} zISW \) seem to differ from the corresponding terms of [11]’ should be replaced by ‘One can then note that while the terms: \( \delta^{(2)} zS, \delta^{(2)} zSW, \delta^{(2)} zSWXISW, \delta^{(2)} zDop, \delta^{(2)} zDopL, \delta^{(2)} zDop[ISW], \delta^{(2)} zDop[ISW], \delta^{(2)} zDop[ISW] \) coincide with the corresponding terms of [11], the terms: \( \delta^{(2)} zSWXDop \) and \( \delta^{(2)} zISW \) seem to differ from the corresponding terms of [11].’
• In the second paragraph after equation (4.25) the text ‘along the line-of-sight ($\theta^\eta(\eta) \neq \theta^\eta_0$ for $\eta \neq \eta_0$, see equation (3.13))’ should be replaced by ‘at the source ($\theta^\eta_i \neq \theta^\eta_0$ for $\eta_i \neq \eta_0$, see equation (3.13))’.

• In the fourth paragraph of the section V, the text ‘almost reduce to’ should be replaced by ‘reduce to’.
The luminosity distance–redshift relation up to second order in the Poisson gauge with anisotropic stress

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Abstract
We present the generalization of previously published results, about the perturbed redshift and the luminosity–redshift relation up to second order in perturbation theory, for the case of the Poisson gauge and in the presence of anisotropic stress. The results are therefore valid for general dark energy models and (most) modified gravity models. We use an innovative approach based on the recently proposed ‘geodesic light-cone’ gauge. We then compare our findings with other results, which recently appeared in the literature, for the particular case of vanishing anisotropic stress. Arriving at a common accepted expression for the nonlinear and relativistic corrections to the redshift and distance–redshift relation is of fundamental importance in view of future cosmological surveys. Thanks to these surveys the Universe will be further probed with high precision and at very different scales, where nonlinear and relativistic effects can play a key role.

Keywords: cosmological perturbations theory, luminosity distance–redshift relation, anisotropic stress

1. Introduction
In the near future cosmology will enter a new era in which the use of Newtonian gravity will no longer be sufficient in studying large scale structure (LSS). In fact, the next generation of LSS surveys will probe the Universe with high precision and at very different scales, where nonlinear and relativistic effects can play a key role. Therefore, it is of fundamental importance to have a reliable description of the observables which describe the physical information carried by light-like signals traveling along our past light-cone, at least up to second order in perturbation theory. Among these observables the redshift $z$ and the luminosity distance $d_L$.
occupy important positions. In fact, following the pioneering work of [1], $d_\Delta$ has been computed to first order in the longitudinal gauge (for a CDM model in [2], CDM and $\Lambda$CDM in [3]), and to second order in the synchronous gauge, but only for CDM, in [4].

Here we generalize the results presented in [5, 6] (and used in [7–9]), where the perturbed redshift and luminosity distance–redshift relation were obtained up to second order in the Poisson gauge (PG) and for a general dark energy model but with vanishing anisotropic stress, to the case where anisotropic stress is present.

The evaluation of LSS observables in the presence of anisotropic stress is one of the major issues to be addressed in view of the next generation of LSS surveys. In fact the anisotropic stress, which vanishes for the $\Lambda$CDM model, frequently appears in other dark energy and/or modified gravity models. The main point is that the presence of anisotropic stress can induce deviations from the standard observational predictions based on $\Lambda$CDM. Therefore, if we are able to isolate and measure the effect of the anisotropic stress in future LSS cosmological observations we will be able to conclude that the Universe is not described by a $\Lambda$CDM model.

In [5, 6] the perturbed redshift and luminosity–redshift relation were derived for the first time up to second order in the PG, and for a general dark energy model, starting from the recently proposed ‘geodesic light-cone’ (GLC) gauge [10] and using an innovative approach. On the other hand, the final results of [5, 6] are valid only for the case with vanishing anisotropic stress and are only partially written using a formalism which is simply related to the one already used at first order (see, for example, [2]). Here we fill this gap.

For problems associated with the observation of light sources lying on the past light-cone of a given observer, the GLC gauge, an adapted system of coordinates, is extremely helpful. In this system several quantities simplify greatly [10] and the so-called Jacobi Map can be obtained exactly, non-perturbatively, [6], while keeping all the required degrees of freedom for applications to general geometries. As a consequence, starting from the GLC gauge one can express light-cone observables in any gauge by computing a coordinate transformation that connects the GLC to the chosen gauge (see [5, 6] for details). This new procedure considerably simplifies the task of writing LSS observables (like redshift and luminosity distance) to a given order in perturbation theory. In practice, one can start from a given non-perturbative exact expression for the observable in question in the GLC gauge, and go to its perturbative counterpart, e.g. in the PG, using a coordinate transformation valid at the desired order in the perturbative theory.

In the second part of the paper we also attempt a comparison of the results of [5, 6], and of the ones here presented, with other results, most notably [11] (see also [12]), for the case of vanishing anisotropic stress. As we shall see, even after translation the comparison is not straight-forward due to the length of the expressions and the possibility of transforming them by integrations by parts. The result is simply that the expressions derived here and in [5, 6] do not agree with the ones derived in [11]. Further work will be needed to resolve the discrepancies. In order to encourage colleagues to look further into this comparison, we believe it is useful to include here this first attempt.

The paper is organized as follows. In section 2 we recall the definition and special properties of the GLC gauge. We also specify the PG up to second order in perturbation theory for the case with anisotropic stress, and find the connection between the two gauges up to second order. In section 3, we first give the result for the redshift up to second order in perturbation theory in the PG in terms of the observer’s angular coordinates, and for a generic dark energy (modified gravity) model with anisotropic stress, using the standard formalism. We then move to the luminosity distance as a function of the observed redshift and of the observer’s angular coordinates, also up to second order in perturbation theory in the PG and for a generic dark energy (modified gravity) model with anisotropic stress. In section 4 we...
consider the particular case of vanishing anisotropic stress and compare our results (and the ones of [5, 6]) with the results of [11, 12]. In section 5 we summarize our results and draw some conclusions.

2. From the GLC to the PG

Following [5, 6] we give in this paper the expression for the redshift and the luminosity distance–redshift relation in a generic homogeneous FLRW Universe with perturbation, up to second order and in the PG with anisotropic stress. We start with the so-called GLC coordinate defined in [10]. GLC coordinates consist of a timelike coordinate \( \tau \) (which can always be identified with the proper time of the synchronous gauge and, therefore, describes a geodesic observer static in this gauge [13]), of a null coordinate \( w \) and of two angular coordinates \( \tilde{\theta}^a \) \((a = 1, 2)\).

The line-element of the GLC metric takes the form:

\[
ds^2 = Y^2 dw^2 - 2Y dw d\tau + \gamma_{ab} \left( d\tilde{\theta}^a - U^a dw \right) \left( d\tilde{\theta}^b - U^b dw \right), \quad a, b = 1, 2, \tag{2.1}
\]

and depends on six arbitrary functions \((Y, U^a \text{ and } \gamma_{ab} = \gamma_{wb})\). In matrix form:

\[
g_{\mu\nu} = \begin{pmatrix} 0 & -Y & \tilde{\theta} \\ -Y & Y^2 + U^2 & -U_b \\ \tilde{\theta}^T & -U_a^T & \gamma_{ab} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1 & -Y^{-1} & -U^b/Y \\ -Y^{-1} & 0 & \tilde{\theta} \\ -U^a/Y^T & \tilde{\theta}^T & \gamma^{ab} \end{pmatrix}, \tag{2.2}
\]

where \(\gamma_{ab}\) and its inverse \(\gamma^{ab}\) lower and raise the two-dimensional indices\(^1\).

The condition \(w = \text{constant}\) defines a null hypersurface \(\partial w = 0\), corresponding to the past light-cone of the given observer, hereafter chosen to be the geodesic one. The vector \(u_\mu = -\partial_\tau\) is the 4-velocity of this geodesic observer, \((\partial^\mu \tau) V_\mu (\partial_\tau \tau) = 0\). Let us also recall that, in the GLC gauge, the null geodesics connecting sources and observer are characterized simply by the tangent vector \(k^\mu = -\omega g^{\mu\nu} \partial_\nu w = -\omega Y^{-1} \delta^\mu_\nu\) (where \(\omega\) is an arbitrary normalization constant), meaning that photons travel at constant values of \(w\) and \(\tilde{\theta}^a\). This renders the calculation of the redshift particularly simple in this gauge.

We now determine the redshift and the cosmological distances, as the luminosity and angular distance, exactly, non-perturbatively in the GLC gauge. Let us denote by subscripts ‘o’ and ‘s’, respectively, quantities evaluated at the observer and source space–time position, and let us consider a light ray emitted by a static geodesic source lying on the past light-cone of a static geodesic observer (defined by \(w = w_o\)) and on the spatial hypersurface \(\tau = \tau_s\). The light ray will be received by the static geodesic observer at \(\tau = \tau_o > \tau_s\). The exact non-perturbative expression of the redshift \(z_o\) associated with this light ray is then simply given by [10]

\[
(1 + z_o) = \frac{(k u_\mu)_{o}}{k(u_\mu)_{o}} = \frac{(\partial^\mu w \partial_\mu \tau)_{o}}{(\partial^\mu w \partial_\mu \tau)_{o}} = \frac{Y(\tau_o, \tilde{\theta}^a)}{Y(\tau_o, \tilde{\theta}^a)} \tag{2.3}
\]

On the other hand, in [6] an exact expression for the so-called Jacobi Map [14] is derived in the GLC gauge and the following non-perturbative solution for the luminosity (area) distance is obtained:

\(^1\) However, in analogy with the synchronous gauge, the GLC gauge also has some residual gauge freedom [6].
\[
d_L^2 = (1 + z_s)^4 \frac{4\sqrt{\tau}}{[\text{det}(\gamma^{-1} \partial \gamma_{ab})]^{3/2}}.
\]

(2.4)

where \(\gamma\) denotes the determinant of the two-dimensional matrix \(\gamma_{ab}\).

Let us now define the gauge in which we want to express the redshift and the luminosity distance, given by equations (2.3) and (2.4). Neglecting vector and tensor contributions, the PG metric [15] (sometimes denoted at first order ‘Newtonian gauge’ or ‘longitudinal gauge’) takes the form

\[
d\delta T^2 = a^2(\eta)\left[-(1+2\Phi)d\eta^2 + (1 - 2\Psi)d\chi^2\right],
\]

(2.5)

where the (generalized) Bardeen potentials \(\Phi\) and \(\Psi\) are defined, up to second order, as follows:

\[
\Phi \equiv \phi + \frac{1}{2}\phi^{(2)}, \quad \Psi \equiv \psi + \frac{1}{2}\psi^{(2)},
\]

(2.6)

and we make no assumption on the anisotropic stress, so that \(\Psi\) and \(\Phi\) can be different also at first order.

In order to compute the redshift and the luminosity distance given in equations (2.3) and (2.4) in terms of standard PG variables we have to transform the GLC gauge quantities to quantities in PG. This generalizes what is done in [5, 6], because we consider here the general case with anisotropic stress up to second order. Starting from the following suitable boundary conditions: (i) the transformation is non-singular around \(r = 0\), and (ii) the two-dimensional spatial section \(r = \text{const}\) are locally parametrized at the observer position by standard spherical coordinates \((\theta, \phi)\), the coordinate transformation to second order is then given by

\[
\tau = \tau^{(0)} + \tau^{(1)} + \tau^{(2)}
\]

with\[
\tau^{(0)} = \left(\int_{\eta_n}^{\eta} d\eta' a(\eta')\right), \quad \tau^{(1)} = a(\eta) \mathcal{P}(\eta, r, \theta^a),
\]

\[
\tau^{(2)} = \int_{\eta_n}^{\eta} d\eta' \frac{a(\eta')}{2} \left[\phi^{(2)} - \phi^2 + \left(\partial_\rho \mathcal{P}\right)^2 + \gamma_{0}^{ab} \partial_\rho \mathcal{P} \partial_b \mathcal{P}\right](\eta', r, \theta^a),
\]

(2.7)

\[
w = w^{(0)} + w^{(1)} + w^{(2)}
\]

with \(w^{(0)} = \eta_n^+, \quad w^{(1)} = \mathcal{Q}(\eta_n^+, \eta_n^-, \theta^a), \quad w^{(2)} = \frac{1}{2} \int_{\eta_n}^{\eta} \text{dx} \left[\psi^{(2)} + \phi^{(2)} + 2(\psi^2 - \phi^2) + 2(\psi + \phi) \partial_\chi \mathcal{Q} + \gamma_0^{ab} \partial_\rho \mathcal{Q} \partial_b \mathcal{Q}\right](\eta_n^+, x, \theta^a),
\]

(2.8)

\[
\hat{\theta}^a = \hat{\theta}^{a(0)} + \hat{\theta}^{a(1)} + \hat{\theta}^{a(2)}
\]

with \(\hat{\theta}^{a(0)} = \theta^a, \quad \hat{\theta}^{a(1)} = \frac{1}{2} \int_{\eta_n}^{\eta} \text{dx} \left[\gamma_0^{ab} \partial_\rho \mathcal{Q}\right](\eta_n^+, x, \theta^a)
\]

4
\[ \tilde{\gamma}^{a(2)} = \int_{\eta_0}^{\eta} dx \left[ \frac{1}{2} \tilde{\gamma}^{ac}_{\theta} \partial_c \tilde{\gamma}^{(2)}_{\theta} + \tilde{\gamma}^{ac}_{\theta} \partial_c \tilde{\gamma}^{(1)}_{\theta} + \frac{1}{2} \tilde{\gamma}^{dc}_{\theta} \partial_d \tilde{\gamma}^{(1)}_{a} \right] + \frac{1}{2} (\psi + \phi) \partial_\phi \tilde{\gamma}^{(1)} + (\phi - \psi) \partial_\phi \tilde{\gamma}^{(1)} - \partial_\phi \tilde{\gamma}^{(1)}_d \partial_d \tilde{\gamma}^{(1)}_a (\eta_0, x, \theta^a). \] (2.9)

where \((\gamma_0^{ab}) = \text{diag}(r^{-2}, r^{-2} \sin^2 \theta)\), and \(\eta_0\) represents an early enough time when the perturbations (or better their integrands) were negligible. We have also introduced the zeroth-order light-cone variables \(\eta_\pm = \eta \pm r\), with corresponding partial derivatives:

\[ \partial_\eta = \partial_+ + \partial_- \quad \partial_r = \partial_+ - \partial_- \quad \partial_\pm = \frac{\partial}{\partial \eta_\pm} = \frac{1}{2} (\partial_\eta \pm \partial_r), \] (2.10)

and defined

\[ P(\eta, r, \theta^a) = \int_{\eta_0}^{\eta} d\eta \frac{a(\eta')}{a(\eta)} \phi(\eta', r, \theta^a), \]

\[ Q(\eta_+, \eta_-, \theta^a) = \int_{\eta_0}^{\eta} dx \frac{1}{2} (\psi + \phi) (\eta_+, x, \theta^a). \] (2.11)

With this we can then compute the non-trivial entries of the GLC metric of equation (2.2) in terms of the variable \((\eta, r, \theta^a)\):

\[ Y^{-1} = \frac{1}{a(\eta)} \left[ 1 + \partial_+ \mathcal{Q} - \partial_r P + \frac{1}{2} (\psi - \phi) + \partial_\theta w^{(2)} + \frac{1}{a} (\partial_\theta - \partial_r) \mathcal{Q}^{(2)} - \phi^{(2)} + 2 \phi - \frac{1}{2} \phi (\psi + \phi) - \phi \partial_+ \mathcal{Q} + \partial_r P \left( \frac{1}{2} \phi - \frac{3}{2} \psi \right) - \partial_\theta P \partial_+ \mathcal{Q} - \gamma_0^{ab} \partial_a \partial_b \mathcal{Q} \right], \] (2.12)

\[ U^a = \partial_\eta \tilde{\gamma}^{a(1)} - \frac{1}{a} \gamma_0^{ab} \partial_b \tau_1 + \partial_\phi \tilde{\gamma}^{a(2)} + \frac{1}{a} \gamma_0^{ab} \partial_b \tau_2 - \frac{1}{a} \partial_\phi \tau_1 \partial_\phi \tilde{\gamma}^{a(1)} \]

\[ - \phi \partial_\tau \tilde{\gamma}^{a(1)} - \frac{1}{a} \psi \gamma_0^{ab} \partial_b \tau_1 - \frac{1}{a} \gamma_0^{cd} \partial_\tau \tau_1 \partial_\phi \tilde{\gamma}^{a(1)} + \left( \partial_\phi \mathcal{Q} - \partial_\phi P + \frac{1}{2} (\psi - \phi) \right) \left( - \partial_\phi \tilde{\gamma}^{a(1)} + \frac{1}{a} \gamma_0^{ab} \partial_\phi \tau_1 \right). \] (2.13)

\[ \gamma^{ab} = a^{-2} \left[ \gamma_0^{ab} (1 + 2 \psi) + \left[ \gamma^{ac}_{\theta} \partial_c \tilde{\gamma}^{(1)}_{\theta} + (a \leftrightarrow b) \right] + \gamma^{ac}_{\theta} \left( \psi^{(2)} + 4 \psi^2 \right) - \partial_\phi \tilde{\gamma}^{(1)}_{\theta} \partial_\phi \tilde{\gamma}^{(1)}_{\theta} \right. \]

\[ + \partial_\phi \tilde{\gamma}^{(1)}_{\theta} \partial_\phi \tilde{\gamma}^{(1)}_{\theta} + 2 \psi \left[ \gamma^{ac}_{\theta} \partial_c \tilde{\gamma}^{(1)}_{\theta} + (a \leftrightarrow b) \right] + \gamma^{cd}_{\theta} \partial_\theta \tilde{\gamma}^{(1)}_{\theta} \partial_\phi \tilde{\gamma}^{(1)}_{\theta} \left. \right] + \gamma^{ac}_{\theta} \partial_c \tilde{\gamma}^{(2)}_{\theta} + (a \leftrightarrow b) \right]. \] (2.14)

### 3. Redshift and luminosity distance–redshift relation: going from GLC to PG

#### 3.1. Redshift

Let us begin with the redshift. Starting from the non-perturbative solution (2.3) and using the coordinate transformation defined above (in particular equation (2.12)), we can obtain its second order perturbative expression in the PG for a general dark energy model with
anisotropic stress. This was first done in [5] for the case with vanishing anisotropic case, but the final expression was not explicitly given. Here we present the final result in standard form, using the conformal time as affine parameter of our line-of-sight. To this aim we underline that since \( \tau \) plays the role of the effective gauge-invariant velocity potential (see [6]), we can define in polar coordinates the spatial components of the perturbed velocity \( v_\mu \) of the PG (geodesic) observer as:

\[
v_\mu = \left( v_r, v_{r\perp} + v_{r\perp}^{(2)} \right) \quad \text{with} \quad v_r = -\partial_r \tau^{(1)}, \quad v_r^{(2)} = -\partial_r \tau^{(2)},
\]

\[
v_{r\perp} = -\partial_{\theta} \tau^{(1)}, \quad v_{r\perp}^{(2)} = -\partial_{\theta} \tau^{(2)},
\]

(3.1)

where \( \tau^{(1)} \) and \( \tau^{(2)} \) are the first- and second-order part of the coordinate transformation \( \tau = \tau(\eta, r, \theta^a) \) between the PG and GLC gauge (see equation (2.7)). The unit vector \( n_\mu \) along the direction connecting the source to the observer can then be expanded, in polar coordinates and to first order (which is enough for our purpose), as:

\[
n_\mu = \left( 0, -\frac{1}{a} (1 + \psi), 0, 0 \right), \quad n_\mu = (0, -a (1 - \psi), 0, 0).
\]

(3.2)

Taking then its scalar product with the spatial component of the perturbed velocity we have

\[
\bar{v} \cdot \hat{n} = v_{r\perp} + v_{r\perp}^{(2)} = \partial_r P + \psi \partial_\theta P + \frac{1}{a(\eta)} \int_{\eta_0}^{\eta} \frac{a(\eta')}{a(\eta)} \partial_\eta \left( \phi^{(2)} - \phi^2 + (\partial_\eta P)^2 \right)
\]

\[+ \gamma_0^{ab} \partial_{\eta b} P \partial_{\theta a} P(\eta', r, \theta^a)
\]

(3.3)

and we also have that

\[
v_{r\perp} a = \gamma_0^{ab} \partial_{\eta b} P \partial_{\theta a} P.
\]

(3.4)

Les us now define the following useful variables

\[
\psi^I = \frac{\psi + \phi}{2}, \quad \psi^A = \frac{\psi - \phi}{2},
\]

(3.5)

which define the isotropic and anisotropic part of the Bardeen potential. Hereafter we will use such variables to express our perturbed quantities.

Finally, also with the help of the following results

\[
Q_s = -2 \int_{\eta_0}^{\eta} d\eta' \psi^I \left( \eta', \eta_0 - \eta', \theta^a \right),
\]

(3.6)

\[
\partial_+ Q_s = \psi^I_\prime - \psi^I_s - 2 \int_{\eta_0}^{\eta} d\eta' \partial_\eta' \psi^I \left( \eta', \eta_0 - \eta', \theta^a \right).
\]

(3.7)

\[
\partial_+ \omega_s^{(2)} = \frac{1}{4} \left[ \left( \phi^{(2)}_\prime + \psi^{(2)}_\prime \right) - \left( \psi^{(2)}_s + \psi^{(2)}_s \right) \right] - \psi^I_s \left[ \psi^I_\prime - \psi^I_s \right]
\]

\[-2 \int_{\eta_0}^{\eta} d\eta' \partial_\eta' \psi^I \left( \eta', \eta_0 - \eta', \theta^a \right) + 2 \left( \psi^I_{s} \psi^A_{s} - \psi^I_{s} \psi^A_{s} \right)
\]
we obtain the redshift up to second order in perturbation theory (hereafter we only use the conformal time $\eta$ as argument inside the integral over the line-of-sight, instead of extended arguments like $(\eta, \eta_0 - \eta^*, \theta^\alpha)$):

$$1 + z_s = \frac{a(\eta)}{a(\eta_0)} \left[ 1 + \delta^{(1)}z + \delta^{(2)}z \right]$$

(3.10)

with

$$\delta^{(1)}z = \nu_0^\perp - \nu_I^\perp + \left( \psi_o^I - \psi_s^A \right) - \left( \psi_o^I - \psi_s^A \right) - 2 \int_{\eta}^{\eta_0} d\eta' \partial_\eta \psi^I(\eta'),$$

(3.11)

and

$$\delta^{(2)}z = \nu_0^{(2)} - \nu_I^{(2)} + \frac{1}{2} \left( \phi_o^{(2)} - \phi_s^{(2)} \right) - \frac{1}{2} \int_{\eta}^{\eta_0} d\eta' \partial_\eta \left[ \phi_o^{(2)}(\eta') + \psi^{(2)}(\eta') \right]$$

$$+ \frac{1}{2} \left( \nu_0^\perp - \nu_I^\perp \right)^2 + \frac{1}{2} \left( \psi_o^I - \psi_s^A \right)^2 + \nu_0^\perp - \nu_I^\perp - \nu_0^\parallel - \psi_s^I$$

$$\times \left( \psi_o^I - \psi_s^I - 2 \int_{\eta}^{\eta_0} d\eta' \partial_\eta \psi^I(\eta') \right) + \frac{1}{2} \left( \nu_0^\parallel \nu_I^\perp - \nu_I^0 \nu_0^\perp \right).$$
Let us underline how the results above are still written in terms of the angles along the line-of-sight.

On the other hand, the perturbed redshift should be written as a function of the observer’s angular coordinates. Using the properties of the GLC gauge, this corresponds to writing the redshift as a function of the GLC angular coordinates \( \tilde{\theta}^a \). In fact, as recalled in the previous section, \( \tilde{\theta}^a \) are equivalent to the standard angular coordinate at the observer position and are constant along the line-of-sight. Therefore the perturbed redshift \( 1 + z_1 \) is given by Taylor-expanding \( 1 + z_1 \), around \( \tilde{\theta}^a \) (we use a bar to denote that the redshift is now expressed in terms of \( \tilde{\theta}^a \)). To this purpose it is enough to invert equation (2.9) to first order since the background redshift is independent from the angles, therefore we need the expansion

\[
\delta \bar{z} = \delta \tilde{z} \left| \theta = \tilde{\theta}^a \right. ,
\]

(3.14)

Then the redshift as a function of the observer’s angular coordinates \( \tilde{\theta}^a \) will be given by equations (3.11), (3.12), (3.14) and (3.15) with \( \phi^a \) replaced by \( \tilde{\phi}^a \) plus a further term given by Taylor-expanding \( \delta \bar{z} \) around \( \tilde{\theta}^a \). Namely we have

\[
\delta \bar{z} = \delta \tilde{z} \left| \theta = \tilde{\theta}^a \right. ,
\]

(3.14)

Let us point out that, in particular, the perturbed redshift, given in equations (3.11), (3.12), (3.14) and (3.15), is uniquely defined after we go to the fully gauge fixed PG, even if we start from the GLC gauge where some residual gauge freedom is still present [6].

3.2. Luminosity distance

Let us now move to the luminosity distance \( d_L \). We first give the basic steps to arrive at the final expression for the luminosity distance–redshift relation to a given order in perturbation theory, in the case under consideration the second, without going into full details (see [5, 6]). The first step consists in writing \( d_L \) up to second order in perturbation theory in PG using the exact expression in equation (2.4) and the second order coordinate transformation given in
equations (2.7)–(2.14). On the other hand, as mentioned, we want to write $d_L$ as a function of the observed redshift and with respect to the angles at the observer position. We can first write $d_L$ as function of the observed redshift defining a fiducial model with coordinates $(\eta_0^{(0)}, r_s^{(0)}, \theta_0^2)$ for which the observed redshift and the past light-cone of our observer are given by

$$1 + z_s = \frac{a(\eta_0)}{a(\eta_0^{(0)})}, \quad w = w_0 = \eta_0 = \eta_0^{(0)} + r_s^{(0)}.$$  \hspace{1cm} (3.16)

We then expand conformal time and radial PG coordinates around the coordinates of the fiducial model as $\eta = \eta_0 + \eta_1 + \eta_2$ and $r_s = r_s^{(0)} + r_s^{(1)} + r_s^{(2)}$, and obtain the terms of these expansions by perturbatively solving the following system of equations:

$$1 + z_s = \frac{a(\eta_0)}{a(\eta_0^{(0)})} = \frac{a(\eta_0)}{a(\eta_0^{(0)})} \left[ 1 + \delta^{(1)}_z + \delta^{(2)}_z \right],$$  \hspace{1cm} (3.17)

$$w = \eta_0 = w_0^{(0)} + w_0^{(1)} + w_0^{(2)},$$  \hspace{1cm} (3.18)

where we have to use equations (2.8), (3.11) and (3.12). In particular, for the case under consideration we have the following solution for our fiducial model

$$\eta_1^{(1)} = \frac{\delta^{(1)}_z}{H_s},$$  \hspace{1cm} (3.19)

$$\eta_2^{(2)} = \frac{1}{H_s} \left[ \delta^{(2)}_z + \delta^{(1)}_z - \frac{1}{2} \left( 1 + \frac{H'_s}{H_s} \right) \left( \delta^{(1)}_z \right)^2 \right],$$  \hspace{1cm} (3.20)

and

$$r_s^{(0)} = \eta_0 - \eta_s^{(0)} = \Delta \eta, \quad r_s^{(1)} = -\eta_s^{(1)} + 2 \int_{\eta_0}^{\eta_0^{(0)}} d\eta' \psi L' (\eta'),$$  \hspace{1cm} (3.21)

$$r_s^{(2)} = -\eta_s^{(2)} - w_0^{(2)} - w_0^{(1-2)},$$  \hspace{1cm} (3.22)

where $H_s = a'(\eta_0^{(0)}) / a(\eta_0^{(0)})$ is the comoving Hubble parameter of the fiducial model, and the quantities $\delta^{(1)}_z$ and $w_0^{(1-2)}$ stand for the second order contribution coming from Taylor expanding $\delta^{(1)}_z$ and $w_0^{(1)}$ around the background source position of our fiducial model. These are given by

$$\delta^{(1-2)}_z = \eta_0^{(1)} \left( \partial_s \psi_L' + \partial_s \psi_A' + H_0 \chi_s \eta_s + \partial_s \chi_s \right) + \left[ \frac{3}{2} \partial_s \psi_L' + \frac{1}{2} \partial_s \psi_A' - 2 \partial_s \psi_L' \right] \left( 2 \int_{\eta_0}^{\eta_0^{(0)}} d\eta' \partial_s^2 \psi L' (\eta') \right),$$  \hspace{1cm} (3.23)

$$w^{(1-2)} = \left[ \psi_L' - \psi_s' - 2 \int_{\eta_0}^{\eta_0^{(0)}} d\eta' \partial_s \psi L' (\eta') \right] \left( 2 \int_{\eta_0}^{\eta_0^{(0)}} d\eta' \psi L' (\eta') \right) + \psi_s' \left[ 2 \eta_0^{(1)} - 2 \int_{\eta_0}^{\eta_0^{(0)}} d\eta' \psi L' (\eta') \right],$$  \hspace{1cm} (3.24)

where, we underline, all the quantities above are now expressed with respect to the background conformal time of our fiducial model.
Once we have \( \eta_r(0), \eta_r(1), \eta_r(2) \) we can obtain the luminosity distance–redshift relation \( d_L(z_s, \theta^a) \) by Taylor expanding the second order solution for \( d_L \), previously found as function of the PG coordinates, around the fiducial values \( \eta_r(0), \eta_r(1), \eta_r(2) \).

This yields \( d_L \) as function of \( \eta_r(0) \), which determines the observed redshift. The last step is to write it as a function of the observer’s angular coordinates. We proceed as done for the observed redshift, using the properties of the GLC gauge for which \( \theta^\alpha \) are equivalent to the standard angular coordinate at the observer position and are constant along the line-of-sight. Therefore the luminosity distance \( d_L(z_s, \theta^a) \) is given by Taylor expanding \( d_L(z_s, \theta^a) \) around \( \theta^a \) (we use a bar to denote that the luminosity distance is now expressed in terms of \( \theta^\alpha \)). To this purpose it is enough to use equation (3.13), as for the redshift the background value \( d_L(0) \) is independent from the angles.

In [5] the procedure above is followed in full detail for the case with vanishing anisotropic stress. Here, apart from the above results, we give only the final results, skipping the major part of the technical details. Therefore, following the procedure summarized above, we obtain

\[
\frac{\tilde{d}_L(z_s, \theta^a)}{(1 + z_s)\Delta \eta} = \frac{d_L^{\text{FLRW}}(z_s)}{d_L^{\text{FLRW}}(z_s)} = 1 + \tilde{\delta}^{(1)}_S(z_s, \theta^a) + \tilde{\delta}^{(2)}_S(z_s, \theta^a),
\]

where the first order luminosity distance is given by

\[
\tilde{\delta}^{(1)}_S(z_s, \theta^a) = -\left(1 - \frac{1}{\mathcal{H}_s\Delta \eta}\right)\psi_s - \frac{1}{\mathcal{H}_s\Delta \eta}\psi_s^t - \left(\psi^t_s + \psi^t_s\right) + \left(1 - \frac{1}{\mathcal{H}_s\Delta \eta}\right)
\times \left[\left(\psi^t_s - \psi^t_s\right) - \left(\psi^t_s - \psi^t_s\right) - 2\int_{\eta_s(0)}^{\eta_s(1)} d\eta' \psi^t_s(\eta')\right] + \frac{2}{\Delta \eta}\int_{\eta_s(0)}^{\eta_s(1)} d\eta' \psi^t_s(\eta') - \frac{1}{\Delta \eta}\int_{\eta_s(0)}^{\eta_s(1)} d\eta' \frac{\eta' - \eta_s(0)}{\eta_s(0) - \eta'} \psi^t_s(\eta')
\]

(3.26)

with \( \Delta_2 = \tilde{\delta}^2_{\psi} + \cot \theta \partial_\theta + 1/(\sin \theta)^2 \tilde{\delta}^2_{\theta} \) the two-dimensional angular Laplacian. This is in full agreement with the previous results of [2, 3] and with the ones of [6] for the case of vanishing anisotropic case.

The second order result is much more involved. We split it into three different parts:

\[
\tilde{\delta}^{(2)}_S(z_s, \theta^a) = \tilde{\delta}^{(2)}_{\text{path}} + \tilde{\delta}^{(2)}_{\text{pos}} + \tilde{\delta}^{(2)}_{\text{mixed}},
\]

(3.27)

where \( \tilde{\delta}^{(2)}_{\text{path}} \) denotes terms connected to the photon path and to the boundary terms; \( \tilde{\delta}^{(2)}_{\text{pos}} \) is for the terms manifestly generated by the source and observer peculiar velocity. Finally, \( \tilde{\delta}^{(2)}_{\text{mixed}} \) mixes peculiar velocity effects with all others. We then obtain the following final result for \( \tilde{\delta}^{(2)}_{\text{pos}} \)

\[
\tilde{\delta}^{(2)}_{\text{pos}} = \left(1 - \frac{1}{\mathcal{H}_s\Delta \eta}\right)\left[\frac{1}{2} \psi^t_\theta \psi^t_\theta - \frac{1}{\mathcal{H}_s}\left(\psi_\theta - \psi_\theta\right)\partial_\theta \psi_\theta - \psi_\theta^2\right]
+ \frac{1}{\mathcal{H}_s\Delta \eta} \left(\frac{1}{2} \psi^t_\theta \psi^t_\theta - \psi_\theta^2\right) - \frac{1}{2} \psi_\theta^2 + \psi_\theta \psi_\theta^t + \frac{1}{2} \frac{\mathcal{H}'_s}{\mathcal{H}_s} \left(\psi_\theta - \psi_\theta^t\right)^2.
\]

(3.28)

For the parts \( \tilde{\delta}^{(2)}_{\text{mixed}} \) and \( \tilde{\delta}^{(2)}_{\text{path}} \) we perform a further split: \( \tilde{\delta}^{(2)}_{\text{mixed}} \) and \( \tilde{\delta}^{(2)}_{\text{path},\alpha} \) contain terms which explicitly depend only on \( \psi^t \) (and, in case, from the genuine second order
variables), while \( \delta_{\text{mixed}, b}^{(2)} \) and \( \delta_{\text{path}, b}^{(2)} \) contain the rest of the terms which depend also on \( \psi^A \).

We then have

\[
\bar{\delta}_{\text{mixed}, o}^{(2)} = \left( 1 - \frac{1}{\mathcal{H}_s \Delta \eta} \right) \left[ + \psi^l_o \eta \eta^l_o + \psi^l_o \frac{1}{2 \Delta \eta} \int_{s}^{\eta^l_o} \mathrm{d} \eta' \frac{\eta' - \eta^l_o}{\eta^l_o - \eta} \Delta_2 \psi^l (\eta') \right. \\
+ 2 \alpha_l s \delta_o \int_{s}^{\eta^l_o} \mathrm{d} \eta' \psi^l (\eta') + \frac{1}{\mathcal{H}_s} \left( \psi^l_o - \psi^l_s - 2 \int_{s}^{\eta^l_o} \mathrm{d} \eta' \partial \eta \psi^l (\eta') \right) \\
- 2 \mathcal{H}_s \int_{s}^{\eta^l_o} \mathrm{d} \eta' \psi^l (\eta') \delta_s \eta \eta^l_o + 2 \partial \delta \psi^l \eta^l_o \int_{s}^{\eta^l_o} \mathrm{d} \eta' \partial_l \psi^l (\eta') \right] \\
+ \psi^l_s \frac{1}{\mathcal{H}_s \Delta \eta} \left[ \psi^l_s + \psi^l_o \int_{s}^{\eta^l_o} \mathrm{d} \eta' \frac{\eta' - \eta^l_o}{\eta^l_o - \eta} \Delta_2 \psi^l (\eta') \right] \\
- \psi^l_s + \frac{2 \mathcal{H}_s \int_{s}^{\eta^l_o} \mathrm{d} \eta' \psi^l (\eta') + (\psi^l_s - \psi^l_o) \left[ \frac{1}{\mathcal{H}_s \Delta \eta} - \frac{1}{\mathcal{H}_s \Delta \eta} \right] \psi^l_s - 2 \\
\times \int_{s}^{\eta^l_o} \mathrm{d} \eta' \psi^l_s (\eta') + \frac{1}{\mathcal{H}_s \Delta \eta} \psi^l_s + \frac{1}{\mathcal{H}_s \Delta \eta} \partial \psi^l_s \\
- \frac{1}{\mathcal{H}_s} \partial \psi^l_s \left( \psi^l_s - \psi^l_o \right) \int_{s}^{\eta^l_o} \mathrm{d} \eta' \partial \psi^l (\eta') \right) - \frac{1}{\mathcal{H}_s} \partial \psi^l_s \left( \psi^l_s - \psi^l_o \right) \int_{s}^{\eta^l_o} \mathrm{d} \eta' \partial \psi^l (\eta') \right] \\
\left( \frac{1}{\mathcal{H}_s \Delta \eta} - \frac{1}{\mathcal{H}_s \Delta \eta} \right) \left( \psi^l_s - \psi^l_o \right) \right) \\
\left( \frac{1}{\mathcal{H}_s \Delta \eta} - \frac{1}{\mathcal{H}_s \Delta \eta} \right) \left( \psi^l_s - \psi^l_o \right) \right) \\
\left( \frac{1}{\mathcal{H}_s \Delta \eta} - \frac{1}{\mathcal{H}_s \Delta \eta} \right) \left( \psi^l_s - \psi^l_o \right) \right) \\
(3.29)
\]

\[
\delta_{\text{mixed}, b}^{(2)} = \left( 1 - \frac{1}{\mathcal{H}_s \Delta \eta} \right) \left[ + \psi^l_s \psi^l_o s \int_{s}^{\eta^l_o} \mathrm{d} \eta' \psi^l_s (\eta') + \int_{s}^{\eta^l_o} \mathrm{d} \eta' \psi^l_s (\eta') \right] - \frac{1}{\mathcal{H}_s \Delta \eta} \mathcal{H}^s_s \eta^l_o \\
\times \left( \psi^l_s - \psi^l_o \right) + \frac{1}{\mathcal{H}_s \Delta \eta} \left( 2 \psi^l_s - \psi^l_o \right) \eta^l_o \\
+ \left( \frac{1}{\mathcal{H}_s \Delta \eta} \partial \psi^l_s \psi^l_s - \partial \psi^l_s \psi^l_s \right) \frac{1}{\mathcal{H}_s} \left( \psi^l_s - \psi^l_o \right) \\
(3.30)
\]

and

\[
\bar{\delta}_{\text{path}, o}^{(2)} = \left( 1 - \frac{1}{\mathcal{H}_s \Delta \eta} \right) \left\{ \frac{1}{2} \left( \phi^l_s (\eta^l_o) - \phi^l_s (\eta^l_o) \right) - \frac{1}{2} \int_{s}^{\eta^l_o} \mathrm{d} \eta' \partial \eta \left( \psi^l_s (\eta') + \phi^l_s (\eta') \right) \right\} \\
- \frac{1}{2} \psi^l_s (\eta^l_o) - \frac{1}{4} \int_{s}^{\eta^l_o} \mathrm{d} \eta' \eta^l_s - \eta^l_o \Delta_2 \left( \psi^l_s (\eta') + \phi^l_s (\eta') \right) \\
+ \frac{1}{2} \int_{s}^{\eta^l_o} \mathrm{d} \eta' \left( \psi^l_s (\eta') + \phi^l_s (\eta') \right) + \left( 1 - \frac{1}{\mathcal{H}_s \Delta \eta} \right) \\
\times \left\{ \left[ - \frac{1}{2} \partial \psi^l_s + \frac{3}{2} \partial \psi^l_s + \partial \psi^l_s + 2 \partial \psi^l_s + 2 \int_{s}^{\eta^l_o} \mathrm{d} \eta' \partial \psi^l (\eta') \right] \right\} \\
\right.
\]
\[ \times \left( -2 \int_{n_1}^{n_0} \, \text{d} \eta \, \psi^I (\eta^I) \right) - \left( \psi_o^I - \psi_s^I - \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) \right) \]

\[ \times \frac{1}{\Delta \eta} \int_{n_0}^{n_0} \, \text{d} \eta \, \frac{\eta^I - \eta_o}{\eta^I - \eta_s} \Delta_2 \psi^I (\eta^I) + \frac{1}{2} (\psi_s^I)^2 - \frac{1}{2} (\psi_o^I)^2 \]

\[ - 2 \psi_o^I \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) - 4 \int_{n_0}^{n_0} \, \text{d} \eta \left[ -\psi^I (\eta^I) \partial_{\eta^I} \psi^I (\eta^I) - \partial_{\eta^I} \psi^I (\eta^I) \right] \]

\[ \times \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) - \psi^I (\eta^I) \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) \]

\[ + \gamma_0^{ab} \partial_a \left( \int_{n_0}^{n_0} \, \text{d} \eta \, \gamma^{\eta^I} \psi^J (\eta^J) \partial_b \left( \int_{n_0}^{n_0} \, \text{d} \eta \, \gamma^{\eta^I} \psi^J (\eta^J) \right) \right) \]

\[ + 2 \partial_{\psi^I} \int_{n_0}^{n_0} \, \text{d} \eta \, \gamma^{\eta^I} \partial_b \partial_b \int_{n_0}^{n_0} \, \text{d} \eta \, \gamma^{\eta^I} \psi^J (\eta^J) + 4 \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) \]

\[ \times \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) + 4 \psi_s^I \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) \]

\[ + \frac{3}{2} (\psi_s^I)^2 - 2 \psi_s^I \psi_o^I + \frac{1}{H_s} \left( \partial_{\psi^I} - \frac{1}{H_s \Delta \eta} \partial_{\psi^I} \right) \]

\[ \times \psi_o^I - \psi_s^I - 2 \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) \]

\[ - 2 \partial_{\psi^I} \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) \]

\[ \times \left( \frac{1}{2} \left( \frac{1}{H_s \Delta \eta} \right) \left[ (\psi_s^I)^2 + 2 (\psi_s^I - \psi_o^I) \right] \right) \]

\[ + \frac{2}{\Delta \eta} \int_{n_0}^{n_0} \, \text{d} \eta \left[ \psi^I (\eta^I) \left( \psi_o^I - \psi_s^I + 2 \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) \right) \right] \]

\[ + \gamma_0^{ab} \partial_a \left( \int_{n_0}^{n_0} \, \text{d} \eta \, \gamma^{\eta^I} \psi^J (\eta^J) \partial_b \left( \int_{n_0}^{n_0} \, \text{d} \eta \, \gamma^{\eta^I} \psi^J (\eta^J) \right) \right) \]

\[ \left( \psi^I - \frac{2}{\Delta \eta} \int_{n_0}^{n_0} \, \text{d} \eta \, \gamma^{\eta^I} \psi^J (\eta^J) \right) \frac{1}{\Delta \eta} \int_{n_0}^{n_0} \, \text{d} \eta \, \gamma^{\eta^I} \psi^J (\eta^J) \]

\[ + 2 \left( \frac{1}{\Delta \eta} \int_{n_0}^{n_0} \, \text{d} \eta \, \gamma^{\eta^I} \psi^J (\eta^J) \right) \frac{1}{\Delta \eta} \int_{n_0}^{n_0} \, \text{d} \eta \, \gamma^{\eta^I} \psi^J (\eta^J) \]

\[ \Delta_2 \psi^I (\eta^I) \]

\[ + \left[ \frac{1}{H_s \Delta \eta} (\psi_o^I - \psi_s^I) \right] \]

\[ - 2 \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) \]

\[ - \frac{1}{\Delta \eta} \int_{n_0}^{n_0} \, \text{d} \eta \, \partial_{\eta^I} \psi^I (\eta^I) \]

\[ \times \left( \int_{n_0}^{n_0} \, \text{d} \eta \, \frac{1}{(\eta_s - \eta_o)^2} (\psi^I (\eta^I) - \psi_o^I) \right) \]
\[ + 2 \int_{n_i}^{n_0} \mathrm{d} \eta' \frac{1}{(\eta_0 - \eta')} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) \int_{n_i}^{n_0} \mathrm{d} \eta' \Delta_2 \psi \left( \eta' \right) \]

\[ + 2 \partial_n \psi \left( \eta' \right) \int_{n_i}^{n_0} \mathrm{d} \eta' \frac{1}{(\eta_0 - \eta')} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) \int_{n_i}^{n_0} \mathrm{d} \eta' \Delta_2 \psi \left( \eta' \right) - \frac{4}{\Delta \eta} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) \]

\[ \times \int_{n_i}^{n_0} \mathrm{d} \eta' \frac{1}{(\eta_0 - \eta')} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) \]

\[ \times \int_{n_i}^{n_0} \mathrm{d} \eta' \frac{1}{(\eta_0 - \eta')} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) + \left( \ddot{\partial}_a \right) \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) \]

\[ \times \left[ 4 \int_{n_i}^{n_0} \mathrm{d} \eta' \frac{1}{(\eta_0 - \eta')} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) - 3 \int_{n_i}^{n_0} \mathrm{d} \eta' \gamma_{ab}^{\psi} \partial_{a} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{b} \psi \left( \eta' \right) \right] \]

\[ - 6 \int_{n_i}^{n_0} \mathrm{d} \eta' \gamma_{ab}^{\psi} \partial_{a} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{b} \psi \left( \eta' \right) \]

\[ - \partial_{a} \left( \int_{n_i}^{n_0} \mathrm{d} \eta' \gamma_{ab}^{\psi} \partial_{a} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{b} \psi \left( \eta' \right) \right) - 3 \int_{n_i}^{n_0} \mathrm{d} \eta' \gamma_{ab}^{\psi} \partial_{a} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{b} \psi \left( \eta' \right) \]

\[ - 2 \left( \int_{n_i}^{n_0} \mathrm{d} \eta' \psi \left( \eta' \right) \right) \int_{n_i}^{n_0} \mathrm{d} \eta' \left[ \frac{1}{(\eta_0 - \eta')} \int_{n_i}^{n_0} \mathrm{d} \eta' \Delta_2 \psi \left( \eta' \right) \right] \]

\[ + \frac{1}{(\eta_0 - \eta')} \left( \frac{1}{2} \Delta_2 \psi \left( \eta' \right) + \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) \right) \int_{n_i}^{n_0} \mathrm{d} \eta' \Delta_2 \psi \left( \eta' \right) \]

\[ + \partial_{a} \left( \frac{1}{\mathcal{H}} \left[ - \psi \left( \eta' \right) - 2 \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) \right] \right) \gamma_{ab}^{\psi} \]

\[ \times \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{\psi} \psi \left( \eta' \right) - 2 \int_{n_i}^{n_0} \mathrm{d} \eta' \frac{1}{(\eta_0 - \eta')} \Delta_2 \left[ \int_{n_i}^{n_0} \mathrm{d} \eta' \gamma_{ab}^{\psi} \partial_{b} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{a} \psi \left( \eta' \right) \right] \]

\[ - \partial_{a} \left( \int_{n_i}^{n_0} \mathrm{d} \eta' \frac{\eta' - n_i}{(\eta_0 - \eta')} \left[ \left( \partial_{a} \psi \left( \eta' \right) \right) \right] \int_{n_i}^{n_0} \mathrm{d} \eta' \gamma_{ab}^{\psi} \partial_{a} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{b} \psi \left( \eta' \right) \right] \]

\[ - 2 \int_{n_i}^{n_0} \mathrm{d} \eta' \gamma_{ab}^{\psi} \partial_{a} \left[ \int_{n_i}^{n_0} \mathrm{d} \eta' \gamma_{ab}^{\psi} \partial_{a} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{b} \psi \left( \eta' \right) \right] \]

\[ \times \partial_{a} \partial \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{b} \psi \left( \eta' \right) - 4 \frac{\cot \theta}{(\sin \theta)^2} \int_{n_i}^{n_0} \mathrm{d} \eta' \frac{1}{(\eta_0 - \eta')} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{a} \partial \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{b} \psi \left( \eta' \right) \]

\[ \times \partial_{a} \partial \left[ \left( \int_{n_i}^{n_0} \mathrm{d} \eta' \frac{\eta' - n_i}{(\eta_0 - \eta')} \partial_{a} \psi \left( \eta' \right) \right) \partial_{a} \int_{n_i}^{n_0} \mathrm{d} \eta' \partial_{b} \psi \left( \eta' \right) \right] \]
- \frac{1}{(\sin \bar{\theta})^2} \left\{ \frac{1}{\Delta \eta} \int_{\eta}^{\eta'} d\eta \frac{\eta' - \eta}{\eta - \eta'} \partial_{\eta} \psi^t (\eta') \right\}^2 - \frac{1}{\Delta \eta} \int_{\eta}^{\eta'} d\eta \frac{\eta' - \eta}{\eta - \eta'} \\
\times \Delta_2 \left[ \psi^t (\eta') \left( \psi^t_{\eta} - \psi^t (\eta') \right) - 2 \int_{\eta'}^{\eta} d\eta \partial_{\eta} \psi^t (\eta') \right] + \gamma_0^{ab} \\
\times \partial_{\eta} \left( \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \right) \partial_{\bar{\eta}} \left( \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \right) \\
- \int_{\eta}^{\eta'} d\eta \left\{ \psi^t (\eta') \lim_{\bar{\eta} \to \eta} \left( \frac{1}{(\eta - \bar{\eta})^2} \right) \int_{\eta}^{\eta'} d\eta \Delta_2 \psi^t (\eta') \right\} - 2 \psi^t (\eta') \frac{1}{\eta - \eta'} \\
\times \int_{\eta}^{\eta'} d\eta \frac{\eta' - \eta}{\eta - \eta'} \Delta_2 \partial_{\eta} \psi^t (\eta') + 2 \gamma_0^{ab} \partial_{\eta} \left( \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \right) \\
\times \frac{1}{\eta - \eta'} \int_{\eta}^{\eta'} d\eta \frac{\eta' - \eta}{\eta - \eta'} \partial_{\eta} \Delta_2 \psi^t (\eta') - \left( \psi^t_{\eta} - 2 \psi^t (\eta') \right) \\
- 2 \int_{\eta}^{\eta'} d\eta \psi^t \partial_{\eta} \psi^t (\eta') \frac{1}{(\eta - \eta')^2} \int_{\eta}^{\eta'} d\eta \Delta_2 \psi^t (\eta') + \partial_{\eta} \psi^t (\eta') \\
\times \left[ \lim_{\bar{\eta} \to \eta} \left( \gamma_0^{ab} \partial_{\eta} \left( \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \right) \right) - 2 \int_{\eta}^{\eta'} d\eta \gamma_0^{ab} \right] \\
\times \partial_{\eta} \left( \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \right) + 2 \partial_{\eta} \gamma_0^{ab} \partial_{\eta} \left( \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \right) \\
\times \int_{\eta}^{\eta'} d\eta \partial_{\eta} \partial_{\eta} \left( \gamma_0^{ab} \partial_{\eta} \left( \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \right) \right) + 2 \gamma_0^{ab} \partial_{\eta} \left( \psi^t (\eta') \right) \\
+ \int_{\eta}^{\eta'} d\eta \partial_{\eta} \partial_{\eta} \psi^t (\eta') \partial_{\eta} \left( \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \right) \right\}. \tag{3.31}

\delta^{(2)}_{\text{path},b} = \left\{ 1 - \frac{1}{\mathcal{H}_s \Delta \eta} \left\{ - \left( \psi^A_{\eta} - \psi^A_{\eta'} \right) \frac{1}{\Delta \eta} \int_{\eta}^{\eta'} d\eta \frac{\eta' - \eta}{\eta - \eta'} \Delta_2 \psi^t (\eta') - \psi^A_{\eta} \psi^A_{\eta'} \\
+ \left( \psi^A_{\eta} \right)^2 + \psi^A_{\eta} \psi^A_{\eta} - 2 \psi^A_{\eta} \psi^A_{\eta} \right\} \right\} \\
- \frac{1}{\mathcal{H}_s \Delta \eta} \partial_{\eta} \psi^A_{\eta} \left( \psi^A_{\eta} - \psi^A_{\eta'} \right) - \frac{2}{\mathcal{H}_s \Delta \eta} \partial_{\eta} \psi^A \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \\
- \frac{1}{\mathcal{H}_s \Delta \eta} \partial_{\eta} \psi^A_{\eta} - \partial_{\eta} \psi^A_{\eta} \left( \psi^A_{\eta} - \psi^A_{\eta'} \right) - \frac{2}{\mathcal{H}_s \Delta \eta} \partial_{\eta} \psi^A \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \\
- 2 \psi^A_{\eta} \frac{1}{\Delta \eta} \int_{\eta}^{\eta'} d\eta \psi^t (\eta') \right\} + \psi^A_{\eta} \frac{1}{\Delta \eta} \int_{\eta}^{\eta'} d\eta \frac{\eta' - \eta}{\eta - \eta'} \Delta_2 \psi^t (\eta')
+ \frac{1}{H_s \Delta \eta} \left( \psi^A_s - \psi^A_o \right) \right) \frac{1}{\Delta \eta} \int_{\eta_o}^{\eta} d\eta' \Delta_2 \psi (\eta') + \left[ \frac{1}{H_s \Delta \eta} (-\psi^A_s + 2 \psi^A_o) \right) \\
+ \frac{1}{H_s \Delta \eta} \left( \psi^A_s - \psi^A_o \right) \right) \left( \frac{-2 \int_{\eta_o}^{\eta} d\eta' \partial_\psi \psi (\eta') \right) \\
- \frac{1}{2} \left( \psi^A_s \right)^2 + \psi^A_s \psi^A_o - \left( \psi^A_o \right)^2 + \psi^A_o \psi^A_o - \psi^A_s \psi^A_s - \psi^A_s \psi^A_s - 2 \psi^A_o \psi^A_s + \frac{1}{H_s \Delta \eta} \right) \\
\times \left[ \frac{1}{2} \left( \psi^A_s - \psi^A_o \right)^2 - \psi^A_s \psi^A_o + \psi^A_o \psi^A_o - \psi^A_s \psi^A_s - \psi^A_s \psi^A_s \right] + \frac{2}{H_s \Delta \eta} \partial_\psi \psi^A_s \\
\times \int_{\eta_o}^{\eta} d\eta' \gamma^a_0 \delta \int_{\eta_o}^{\eta} d\eta' \psi^A (\eta') + \frac{1}{H_s} \partial_\psi \psi^A \rho_{0a} \partial_\psi \int_{\eta_o}^{\eta} d\eta' \psi^A (\eta') \\
+ \frac{4}{\Delta \eta} \int_{\eta_o}^{\eta} d\eta' \left( \psi^A \psi^A (\eta') \right) - \frac{2}{\Delta \eta} \int_{\eta_o}^{\eta} d\eta' \left. \psi^A \psi^A (\eta') \right|_{\eta_o}^{\eta} - \Delta_2 \left( \psi^A \psi^A (\eta') \right). 
\tag{3.32}

We can note how several new terms appear when we consider an anisotropic stress. In particular, we have a new genuine second order lensing (see last term of equation (3.32)), which is non-zero only when we consider models of dark energy with anisotropic stress (or modified gravity models). As a consequence, this could be used to test these models.

4. Comparison with previous results: vanishing anisotropic stress

Let us now consider the particular case with vanishing anisotropic stress. We begin by showing that the results for the luminosity distance–redshift relation reported in [6] basically agree, for this particular case, with the above results. In the case of vanishing anisotropic stress we have $\psi^A = \psi^A = 0$, $\delta_l$ can be then easily obtained using equations (3.26)–(3.32). In particular we have that $\delta_{\text{mixed}, b}^{(2)} = \delta_{\text{path}, b}^{(2)} = 0$, the form of $\delta_{\text{pos}}^{(2)}$ does not change, and $\delta_{\text{mixed}, a}^{(2)}$ and $\delta_{\text{path}, a}^{(2)}$ are obtained just substituting $\psi^A$ with $\psi^A$.

Let us start from the explicit expressions reported in [6]:

$$
\delta_{\text{pos}}^{(2)} = \frac{1}{2} \left\{ 1 - \frac{1}{H_s \Delta \eta} \right\} \left\{ \left( \partial_\psi P_s \right)^2 + \left( \gamma^a_0 \right) \partial_\psi P_s \partial_\psi P_s \right\} \\
- \frac{2}{H_s} \left( \partial_\psi P_s - \partial_\psi P_s \right) \left( \partial_\psi \psi^A + \partial_\psi \psi^A \right) - \int_{\eta_o}^{\eta} d\eta' \frac{\alpha (\eta')}{\alpha (\eta_o)} \right\} \\
\times \partial_\psi \left[ \phi^{(2)} - \psi^2 + \left( \partial_\psi P_s \right)^2 + \chi_0^{ab} \partial_\psi P_s \partial_\psi P_s \right] \left( \eta', \Delta \eta, \vec{\partial} \right) \\
+ \frac{1}{2H_s \Delta \eta} \left\{ \left( \partial_\psi P_s \right)^2 + \lim_{r \to 0} \left[ \chi_0^{ab} \partial_\psi P_s \partial_\psi P_s \right] \\
- \int_{\eta_o}^{\eta} d\eta' \frac{\alpha (\eta')}{\alpha (\eta_o)} \partial_\psi \left[ \phi^{(2)} - \psi^2 + \left( \partial_\psi P_s \right)^2 + \chi_0^{ab} \partial_\psi P_s \partial_\psi P_s \right] \right\} \\
- \frac{1}{2H_s \Delta \eta} \left( 1 - \frac{H_s}{H_s^2} \right) \left( \partial_\psi P_s - \partial_\psi P_s \right)^2.
\tag{4.1}
$$

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\[ \delta_{\text{mixed}}^{(2)} = \left( 1 - \frac{1}{\mathcal{H}_s \Delta \eta} \right) \left\{ \partial_s P_s J_s^{(1)} - \left( \partial_s P_s - \partial_s P_o \right) \frac{1}{\mathcal{H}_s} \partial_s \psi - \left( \gamma_{0b}^{ab} \right) s Q_s \partial_s P_s + \frac{1}{\mathcal{H}_s} \partial_s Q_s \partial_s^2 P_s + Q_s \partial_s^2 P_s \\
+ \frac{1}{2} \partial_s \left( \partial_s P_s - \partial_s P_o \right) \left( \int_{\eta_0}^{\eta_s} d\eta \left[ \gamma_{0b}^{ab} \partial_b Q \left( \eta_s^{(0)+} , x , \tilde{\theta}^a \right) \right] \right) \right\} \\
- \frac{1}{\mathcal{H}_s \Delta \eta} \left( \psi_s - \psi_s - J_s^{(1)} \right) \partial_s P_o + \frac{Q_s}{\mathcal{H}_s} \partial_s P_s \\
+ \frac{1}{\Delta \eta} \left( \partial_s P_s - \partial_s P_o \right) \left( \frac{1}{\mathcal{H}_s} \left( 1 - \frac{\mathcal{H}_s}{\mathcal{H}_s^2} \right) \partial_s Q_s + \frac{2}{\mathcal{H}_s} \psi_s \right) \\
+ \frac{1}{\mathcal{H}_s} \left( \partial_s P_s - \partial_s P_o \right) \left( \partial_b \psi_s - \partial_s \psi_s - \frac{1}{\delta_s^2} \right) \right. \\
\times \int_{\eta_0}^{\eta_s} d\eta \Delta_2 \psi \left( \eta_s , \eta_s , - \eta_s , \tilde{\theta}^a \right) \right\} , \tag{4.2} \]

\[ \delta_{\text{path}}^{(2)} = \left( 1 - \frac{1}{\mathcal{H}_s \Delta \eta} \right) \left\{ - \frac{1}{4} \left( \phi_s^{(2)} - \phi_o^{(2)} \right) + \frac{1}{4} \left( \psi_s^{(2)} - \psi_o^{(2)} \right) + \frac{1}{2} \psi_s^2 - \frac{1}{2} \psi_o^2 \\
- \left( \psi_s + J_s^{(1)} \right) \partial_s Q_s + \frac{1}{4} \left( \gamma_{0b}^{ab} \right) \partial_s Q_s \partial_s Q_s + Q_s \left( - \partial_s^2 Q_s + \partial_s \psi_s \right) + \frac{1}{\mathcal{H}_s} \partial_s Q_s \partial_s \psi_s \\
+ \frac{1}{4} \int_{\eta_0}^{\eta_s} d\eta \partial_s \left[ \phi_s^{(2)} + \psi_s^{(2)} + 4 \psi_s \partial_s Q + \gamma_{0b}^{ab} \partial_b Q \partial_b Q \left( \eta_s^{(0)+} , x , \tilde{\theta}^a \right) \right] \right. \\
- \frac{1}{2} \partial_s \left( \partial_s Q_s \right) \left( \int_{\eta_0}^{\eta_s} d\eta \left[ \gamma_{0b}^{ab} \partial_b Q \left( \eta_s^{(0)+} , x , \tilde{\theta}^a \right) \right] \right) \right\} - \frac{1}{2} \psi_s^{(2)} \\
- \frac{1}{2} \psi_s^2 - K_2 + \psi_s J_s^{(1)} + \frac{1}{2} \left( J_s^{(1)} \right)^2 + J_s^{(1)} \frac{Q_s}{\Delta \eta} - \frac{1}{\mathcal{H}_s \Delta \eta} \\
\times \left( 1 - \frac{\mathcal{H}_s}{\mathcal{H}_s^2} \right) \frac{1}{2} \left( \partial_s Q_s \right)^2 - \frac{2}{\mathcal{H}_s \Delta \eta} \psi_s \partial_s Q_s \\
+ \frac{1}{2} \partial_s \left( \psi_s + J_s^{(1)} + \frac{Q_s}{\Delta \eta} \right) \left( \int_{\eta_0}^{\eta_s} d\eta \left[ \gamma_{0b}^{ab} \partial_b Q \left( \eta_s^{(0)+} , x , \tilde{\theta}^a \right) \right] \right) \\
+ \frac{1}{4} \partial_s Q_s \partial_s \left( \int_{\eta_0}^{\eta_s} d\eta \left[ \gamma_{0b}^{ab} \partial_b Q \left( \eta_s^{(0)+} , x , \tilde{\theta}^a \right) \right] \right) \\
+ \frac{1}{16} \partial_s \left( \int_{\eta_0}^{\eta_s} d\eta \left[ \gamma_{0c}^{bc} \partial_c Q \left( \eta_s^{(0)+} , x , \tilde{\theta}^a \right) \right] \partial_b \right) \\
\times \left( \int_{\eta_0}^{\eta_s} d\eta \left[ \gamma_{0d}^{ad} \partial_b Q \left( \eta_s^{(0)+} , \tilde{x} , \tilde{\theta}^a \right) \right] - \frac{1}{4 \Delta \eta} \int_{\eta_0}^{\eta_s} d\eta \left[ \phi_s^{(2)} + \psi_s^{(2)} + 4 \psi_s \partial_s Q \right] \right) . \]
\[ + \gamma_{0}^{ab} \partial_{a} Q \partial_{b} \mathcal{Q} \left( \eta_{s}^{(0)+}, x, \hat{\vartheta}^{a} \right) \]
\[ + \frac{1}{H_{s}} \partial_{s} Q_{s} \left\{ -\partial_{\eta} \psi_{s} + \partial_{x} \psi_{s} + \frac{1}{\Delta \eta_{s}^{2}} \int_{\eta_{s}^{(0)-}}^{\eta_{s}^{(0)+}} d\eta \Delta_{2} \psi \left( \eta', \eta_{o} - \eta', \hat{\vartheta}^{a} \right) \right\} \]
\[ + Q_{s} \left\{ \partial_{x} \psi_{s} + \partial_{\eta} \int_{u_{s}}^{\eta_{s}^{(0)-}} dx \frac{1}{(\eta_{s}^{(0)+} - x)^{2}} \int_{y_{s}}^{\eta_{s}^{(0)+}} dy \Delta_{2} \psi \left( \eta_{s}^{(0)+}, y, \hat{\vartheta}^{a} \right) \right\} \]
\[ + \frac{1}{2 \Delta \eta_{s}^{2}} \int_{\eta_{s}^{(0)-}}^{\eta_{s}^{(0)+}} d\eta \Delta_{2} \psi \left( \eta', \eta_{o} - \eta', \hat{\vartheta}^{a} \right) \]
\[ + \frac{1}{16 \sin^{2} \theta} \left( \int_{u_{s}}^{\eta_{s}^{(0)-}} d\eta \left[ \gamma_{0}^{ab} \partial_{a} Q \left( \eta_{s}^{(0)+}, x, \hat{\vartheta}^{a} \right) \right] \right)^{2}. \] (4.3)

To compare these terms with the results in equations (3.28), (3.29) and (3.31) (with \( \psi = \psi \)) we have to express them in a more familiar form. To this aim we use the results of equations (2.11) and (3.6)–(3.9) (with \( \psi = \psi \)), together with the following relations evaluated for the case of vanishing anisotropic stress:
\[ \partial_{x}^{2} Q_{s} = \frac{3}{2} \left( \partial_{\eta} \psi_{s} - \partial_{x} \psi_{s} \right) + \frac{1}{2} \left( \partial_{x} \psi_{s} - \partial_{y} \psi_{s} \right) - 2 \int_{u_{s}}^{\eta_{s}^{(0)-}} d\eta' \partial_{\eta'}^{2} \psi (\eta'), \] (4.4)
\[ \partial_{x} \int_{u_{s}}^{\eta_{s}^{(0)-}} dx \frac{1}{(\eta_{s}^{(0)+} - x)^{2}} \int_{y_{s}}^{\eta_{s}^{(0)+}} dy \Delta_{2} \psi \left( \eta_{s}^{(0)+}, y, \hat{\vartheta}^{a} \right) \]
\[ = \int_{u_{s}}^{\eta_{s}^{(0)-}} d\eta' \left[ -\frac{1}{(\eta_{o} - \eta')^{2}} \int_{\eta'}^{\eta_{s}^{(0)+}} d\eta'' \Delta_{2} \psi (\eta'') \right. \]
\[ + \left. \frac{1}{(\eta_{o} - \eta')^{2}} \left( \frac{1}{2} \Delta\psi (\eta') + \int_{\eta'}^{\eta_{s}^{(0)+}} d\eta'' \partial_{\eta''} \Delta_{2} \psi (\eta'') \right) \right]. \] (4.5)
\[ \frac{1}{4} \partial_{x} Q_{s} \partial_{x} \left( \int_{u_{s}}^{\eta_{s}^{(0)-}} dx \left[ \gamma_{0}^{ab} \partial_{a} Q \left( \eta_{s}^{(0)+}, x, \hat{\vartheta}^{a} \right) \right] \right) \]
\[ = \partial_{x} \left( \int_{u_{s}}^{\eta_{s}^{(0)-}} d\eta' \psi (\eta') \right) \int_{\eta_{s}^{(0)-}}^{\eta_{s}^{(0)+}} d\eta' \left[ \frac{2}{(\eta_{o} - \eta')} \gamma_{0}^{ab} \partial_{a} \int_{u_{s}}^{\eta_{s}^{(0)-}} d\eta'' \psi (\eta'') \right. \]
\[ - \gamma_{0}^{ab} \partial_{b} \left( \psi (\eta') + 2 \int_{\eta'}^{\eta_{s}^{(0)+}} d\eta'' \partial_{\eta''} \psi (\eta'') \right). \] (4.6)

Furthermore, \( J_{2}^{(1)} \) is the first order lensing term and is given by
\[ J_{2}^{(1)} = \frac{1}{\Delta \eta} \int_{\eta_{s}^{(0)-}}^{\eta_{s}^{(0)+}} d\eta' \frac{\eta'}{\eta_{o} - \eta'} \Delta_{2} \psi \left( \eta', \eta_{o} - \eta', \hat{\vartheta}^{a} \right). \] (4.7)
This corresponds to the lowest order contribution obtained from

$$J_2 = \frac{1}{2} \left[ \cot \theta \theta^{(1)} + \partial_a \theta^{(2)} \right].$$

(4.8)

We then have also a genuine second order lensing term given by

$$K_2 = \frac{1}{2} \left[ \cot \theta \theta^{(2)} + \partial_a \theta^{(2)} \right] = \frac{1}{2} V_\alpha \theta^{(2)}$$

$$= \frac{1}{4 \Delta \eta} \int \eta^a \frac{\eta^a - \eta^a_{o}}{\eta^a_{o} - \eta^a} \Delta_2 \left[ \phi^{(2)}(\eta^a) + \psi^{(2)}(\eta^a) + 4 \psi(\eta^a)(\psi_a - \psi(\eta^a)) \right]$$

$$- 2 \int \eta^a \frac{\eta^a - \eta^a_{o}}{\eta^a_{o} - \eta^a} \Delta_2 \psi(\eta^a) + 4 \gamma_0^{ab} \partial_b \left( \int \eta^a \frac{\eta^a - \eta^a_{o}}{\eta^a_{o} - \eta^a} \Delta_2 \psi(\eta^a) \right)$$

$$+ \int \eta^a \frac{1}{\eta^a_{o} - \eta^a} \Delta_2 \psi(\eta^a) - 2 \psi(\eta^a) \int \eta^a \frac{\eta^a - \eta^a_{o}}{\eta^a_{o} - \eta^a} \Delta_2 \psi(\eta^a)$$

$$\times \int \eta^a \gamma_0^{ab} \partial_b \left( \int \eta^a \frac{\eta^a - \eta^a_{o}}{\eta^a_{o} - \eta^a} \Delta_2 \psi(\eta^a) \right)$$

$$\times \int \eta^a \gamma_0^{ab} \partial_b \left( \int \eta^a \frac{\eta^a - \eta^a_{o}}{\eta^a_{o} - \eta^a} \Delta_2 \psi(\eta^a) \right)$$

$$\times \int \eta^a \gamma_0^{ab} \partial_b \left( \int \eta^a \frac{\eta^a - \eta^a_{o}}{\eta^a_{o} - \eta^a} \Delta_2 \psi(\eta^a) \right)$$

$$\times \int \eta^a \gamma_0^{ab} \partial_b \left( \int \eta^a \frac{\eta^a - \eta^a_{o}}{\eta^a_{o} - \eta^a} \Delta_2 \psi(\eta^a) \right)$$

$$= 2 \gamma_0^{ab} \partial_b \left( \int \eta^a \frac{\eta^a - \eta^a_{o}}{\eta^a_{o} - \eta^a} \Delta_2 \psi(\eta^a) \right) - 2 \psi(\eta^a) \int \eta^a \frac{\eta^a - \eta^a_{o}}{\eta^a_{o} - \eta^a} \Delta_2 \psi(\eta^a)$$

(4.9)

Using equations (4.4)–(4.9) to evaluate equations (4.1), (4.2) and (4.3) one can prove that these are almost equivalent to equations (3.28), (3.29) and (3.31) with $\psi^l = \psi$ and $\psi^A = 0$. The only sources of disagreement derive from the expansion $J_2^{(1-2)}$, i.e. the expansion around the background fiducial model of the term $J_2$, and from the second order terms that one obtains from

$$\left( 1 - \frac{1}{H_0 \Delta \eta} \right) \left[ -2 \int \eta^a \partial_a \psi(\eta^a) \right] + \frac{2}{\Delta \eta} \int \eta^a \partial_a \psi(\eta^a) - J_2$$

(4.10)

when one expands the angular coordinates around the angular coordinates at the observer position. The evaluation of these terms was not fully addressed in [5, 6]. In particular, when we restrict ourselves to our past light-cone the PG coordinates are no longer a set of independent coordinates and angular partial derivatives do not commute with the integrals along the line-of-sight. Taking this in consideration we obtain that
and, expanding around the angular coordinates at the observer position, that

\[ J_{2}^{(1-2)} = -\frac{1}{H_{\xi}} \left( v_{\eta_{1}} - v_{\eta_{2}} + \psi_{0} - \psi_{s} - 2 \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\eta} \psi (\eta') \right) \]

\[ - \frac{1}{\Delta \eta} \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\eta} \psi (\eta') \int_{\eta_{1}}^{\eta_{2}} d\eta' \Delta_{2} \psi (\eta') + 2 \left( \int_{\eta_{1}}^{\eta_{2}} d\eta' \psi (\eta') \right) \]

\[ \times \int_{\eta_{1}}^{\eta_{2}} d\eta' \left[ - \frac{1}{(\eta_{0} - \eta')} \int_{\eta_{1}}^{\eta_{2}} d\eta' \Delta_{2} \psi (\eta') + \frac{1}{(\eta_{0} - \eta')}^{2} \right] \]

\[ \times \left( \frac{1}{2} \Delta_{2} \psi (\eta') + \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\eta} \left( \Delta_{2} \psi (\eta') \right) \right) \]

\[ + \frac{1}{\Delta \eta} \int_{\eta_{1}}^{\eta_{2}} d\eta' \left( \frac{1}{(\eta_{0} - \eta')} \int_{\eta_{1}}^{\eta_{2}} d\eta' \Delta_{2} \psi (\eta') + \frac{1}{(\eta_{0} - \eta')}^{2} \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\eta} \psi (\eta') \right) \]

\[ \times \int_{\eta_{1}}^{\eta_{2}} d\eta' \gamma_{0}^{ab} \left[ - \frac{1}{2} \left( \eta_{0} - \eta' \right) \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\beta} \psi (\eta') + \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\beta} \partial_{\eta} \psi (\eta') \right] \]

\[ + \frac{1}{2} \partial_{\beta} \psi (\eta') \right] - \partial_{\beta} \left\{ \frac{1}{H_{\xi}} \left[ v_{\eta_{1}} - v_{\eta_{2}} + \psi_{0} - \psi_{s} \right] \right. \]

\[ - 2 \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\eta} \psi (\eta') \int_{\eta_{1}}^{\eta_{2}} d\eta' \Delta_{2} \psi (\eta') \right\} \gamma_{0}^{ab} \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\beta} \psi (\eta') \]  \( (4.11) \)

and, expanding around the angular coordinates at the observer position, that

\[ \left( 1 - \frac{1}{H_{\xi} \Delta \eta} \right) \left[ -2 \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\eta} \psi (\eta') \right] \]

\[ = \left( 1 - \frac{1}{H_{\xi} \Delta \eta} \right) \left[ -2 \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\eta} \psi (\eta', \eta' - \eta_{0}, \tilde{\theta}^{a}) \right] \]

\[ + 4 \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\beta} \left( \partial_{\eta} \psi (\eta') \right) \int_{\eta_{1}}^{\eta_{2}} d\eta' \gamma_{0}^{ab} \partial_{\beta} \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\eta} \psi (\eta') \]  \( (4.12) \)

\[ \frac{2}{\Delta \eta} \int_{\eta_{1}}^{\eta_{2}} d\eta' \psi (\eta') = \frac{2}{\Delta \eta} \int_{\eta_{1}}^{\eta_{2}} d\eta' \psi (\eta', \eta' - \eta_{0}, \tilde{\theta}^{a}) - \frac{4}{\Delta \eta} \]

\[ \times \left[ \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\beta} \psi (\eta') \int_{\eta_{1}}^{\eta_{2}} d\eta' \gamma_{0}^{ab} \partial_{\beta} \int_{\eta_{1}}^{\eta_{2}} d\eta' \partial_{\eta} \psi (\eta') \right] \]  \( (4.13) \)
These results do not agree with the correspondent ones in [5, 6] (see equations (3.26) and (3.31) of [5]). On the other hand, the sources of disagreement are only angular total derivatives and subleading terms with respect to the leading lensing and Doppler contributions. Therefore, these new terms affect in a totally negligible way the backreaction effect calculated in [7–9].

In [11] (see also [12]) the results for the perturbed redshift and luminosity distance was also derived, but in a different way, for the particular case of vanishing anisotropic stress. In [11] the authors work mainly in a perturbed Minkowski space–time and conformally transform their results to the original FLRW space–time at the end. Therefore, the comparison of these two independent results, those of [5, 6] here further generalized and those of [11], is of fundamental importance. Considering only scalar perturbations2 and the different formalism used, we find that the results of equations (3.11), (3.12), (3.14) and (3.15), for \( \psi_\perp = \psi \) and \( \psi_\parallel = 0 \), do not exactly coincide with the results of [11]. To be more precise, the first order coincides, while splitting the second order contribution of equation (3.15) in a similar way as proposed in [11] we find some discrepancies. Namely, we can write:

\[
\delta^{(2)} \zeta = \delta^{(2)} \zeta_S + \delta^{(2)} \zeta_{SW} + \delta^{(2)} \zeta_{SW\times ISW} + \delta^{(2)} \zeta_{SW\times Dop} + \delta^{(2)} \zeta_{Dop} \\
+ \delta^{(2)} \zeta_{Dop,L} + \delta^{(2)} \zeta_{Dop,ISW} + \delta^{(2)} \zeta_{Dop,ISW_L} + \delta^{(2)} \zeta_{ISW} + \delta^{(2)} \zeta_{\varphi} 
\] (4.15)

2 In [5] the vector and tensor perturbations were also considered. The results of [5] are in agreement [16] with the ones given in [17].
with
\[\delta^{(2)}z_S = v^{(2)} - \psi^{(2)} + \frac{1}{2}(\phi_o^{(2)} - \phi_s^{(2)}) - \frac{1}{2}\int_{\eta}^{\eta_o} d\eta' \partial_{\eta'} \left[ (\phi^{(2)}(\eta') + \psi^{(2)}(\eta')) \right],\]  
(4.16)
\[\delta^{(2)}z_{SW} = \frac{3}{2}\psi_0^2 - \frac{1}{2}\psi_o^2 - \psi_0 \psi_o,\]  
(4.17)
\[\delta^{(2)}z_{SWxISW} = 2(\psi_0 - \psi_o) \int_{\eta}^{\eta_o} d\eta' \partial_{\eta'} \psi(\eta'),\]  
(4.18)
\[\delta^{(2)}z_{SWxDep ||} = (v^{(2)} - \eta^{(2)})(\psi_0 - \psi_o) - \left( \psi_0 v^{(2)} + \psi_o v^{(2)} \right) - \left( \psi_0 v^{(2)} + \psi_o v^{(2)} \right),\]  
(4.19)
\[\delta^{(2)}z_{Dep ||} = \frac{1}{2}(v^{(2)} - \eta^{(2)})^2,\]  
(4.20)
\[\delta^{(2)}z_{Dep \perp} = \frac{1}{2}(v^{(2)} - \eta^{(2)}),\]  
(4.21)
\[\delta^{(2)}z_{Dep ||xISW} = -2(\eta^{(2)} - \eta^{(2)}) \int_{\eta}^{\eta_o} d\eta' \partial_{\eta'} \psi(\eta'),\]  
(4.22)
\[\delta^{(2)}z_{Dep \perp xISW} = 2a \int_{\eta}^{\eta_o} d\eta' \partial_{\eta'} \psi(\eta'),\]  
(4.23)
\[\delta^{(2)}z_{ISW} = 4 \int_{\eta}^{\eta_o} d\eta' \psi(\eta') \partial_{\eta'} \psi(\eta') + \partial_{\eta'} \psi(\eta') \int_{\eta}^{\eta_o} d\eta' \partial_{\eta'} \psi(\eta') + \psi(\eta') + \psi(\eta') \times \int_{\eta}^{\eta_o} d\eta' \partial_{\eta'}^2 \psi(\eta') - \chi_0^{ab} \partial_{\eta'} \left( \int_{\eta}^{\eta_o} d\eta' \psi(\eta') \right) \partial_{\eta'} \left( \int_{\eta}^{\eta_o} d\eta' \partial_{\eta'} \psi(\eta') \right),\]  
(4.24)
\[\delta^{(2)}z_{\theta^c} = +2 \partial_{\eta'}(\eta^{(2)} + \eta^{(2)}) \int_{\eta}^{\eta_o} d\eta' \partial_{\eta'} \partial_{\eta'} \psi(\eta') + \partial_{\eta'} \psi(\eta') \int_{\eta}^{\eta_o} d\eta' \partial_{\eta'} \partial_{\eta'} \psi(\eta') + \partial_{\eta'} \psi(\eta') \int_{\eta}^{\eta_o} d\eta' \partial_{\eta'} \psi(\eta').\]  
(4.25)

Here \(\delta^{(2)}z_S\) stands for the genuine second order terms and the other names are given in terms of the physical effects connected with the relative term (Doppler effect, Sachs–Wolfe effect, integrated and double integrated Sachs–Wolfe effect). One can then note that while the terms: \(\delta^{(2)}z_S, \delta^{(2)}z_{SW}, \delta^{(2)}z_{SWxISW}, \delta^{(2)}z_{Dep ||}, \delta^{(2)}z_{Dep \perp}\) and \(\delta^{(2)}z_{Dep ||xISW}\) coincide with the corresponding terms of [11], the terms: \(\delta^{(2)}z_{Dep \perp xISW}, \delta^{(2)}z_{SWxDep ||}\) and \(\delta^{(2)}z_{ISW}\) seem to differ from the corresponding terms of [11]. Just to give an example, in the term \(\delta^{(2)}z_{ISW}\) a double partial derivative with respect to the conformal time is present, while this is not present in the correspondent term of [11].

The term \(\delta^{(2)}z_{\theta^c}\) comes from the fact that we want to write the perturbed redshift as a function of the observer’s angular coordinates (see section 3.1), which differ from the angular coordinates along the line-of-sight (\(\theta^c(\eta) \neq \theta^c_o\) for \(\eta \neq \eta_o\), see equation (3.13)). This term appears to be absent in [11], suggesting that in [11] the observed redshift is not written as a function of the direction of observation.

Moving to the luminosity distance–redshift relation, the comparison of the results in equations (3.28), (3.29) and (3.31) (with \(\psi^l = \psi^l\) and \(\psi^A = 0\) with the ones of [11] is very involved, in fact results that are equal can look different by using a simple integration by
parts. On the other hand, we can try to compare terms with the same physical meaning and which can be isolated easily from the other terms. As an example, considering only the term \( \sim v_{ls}^2 \), in \( \delta_{\text{pos}}^{(2)} \), we have
\[
\frac{1}{2} \left[ \frac{H^2}{\Delta \eta} \right] v_{ls}^2 \quad \text{while if we look at equation (140) of [11] we have}
\]
\[
\left( \frac{H}{\Delta \eta} \right)^2 v_{ls}^2.
\]
Clearly the quantities appearing in equations (4.26) and (4.27) are different. Therefore, from this simple check, and from the fact that also the second order perturbed redshift of [11] differs from our expression of equation (3.15), we conclude that the recent results in [11] do not agree with the results of [5, 6] and with the ones here presented.

5. Conclusions

Let us summarize the results of this work and make some further comment. The main results of the paper are the perturbed redshift and luminosity distance–redshift relation up to second order in perturbation theory in PG, obtained by generalizing the results in [5, 6] to include an anisotropic stress. These results, presented in equations (3.11), (3.12), (3.14) and (3.15), and in equations (3.26)–(3.32), are therefore valid for general dark energy models and (most) modified gravity models, for which a non-vanishing anisotropic stress frequently appears. The results are presented using a standard formalism close to the one introduced in [2].

The evaluation of LSS observables in models with anisotropic stress is of fundamental importance for the understanding of the dark energy problem. In fact, a direct detection of an observational signature of anisotropic stress will rule out \( \Lambda \) CDM. Taking into consideration the anisotropic stress several new terms appear in equations (3.26)–(3.32). In particular, in equation (3.32) a new genuine second order lensing term (the last one in (3.32)) appears. This is non-zero only for models of dark energy with anisotropic stress (or for modified gravity models), and, as a consequence, could be used to test alternative models of dark energy.

Furthermore, we have seen that the results obtained in equations (3.26)–(3.32) almost reduce to the ones given in [5, 6] for the particular case of vanishing anisotropic stress.

To this aim we use a series of results (equations (2.11), (3.6)–(3.9) and (4.4)–(4.9)) which constitute a dictionary to translate the quantities that one usually obtains when going from the GLC gauge to the PG, to more standard variables.

We have then summarized, and in part illustrated, the innovative approach used in [5, 6] to write LSS observables (like redshift and luminosity distance) to a given order in perturbation theory starting from the GLC gauge. This innovative approach enormously simplifies this task and can be used also to obtain other useful observables like the galaxy number counts at second order [18] (see [19, 20] for the first order case, and [21] for other recent results at second order).

Finally, we have partially compared our results about the perturbed redshift and luminosity–redshift relation, with those of [11] (see also [12]) for the case of vanishing anisotropic stress, showing that there is still some disagreement with the results presented here and in [5, 6]. We stress that arriving at a commonly accepted expression for the perturbed
redshift and luminosity–redshift relation is of fundamental importance in view of the future cosmological surveys. In fact, any uncertainty in the theoretical description of the observed redshift and luminosity distance will impact on the reliability of the interpretation of our present and future cosmological observations. We do hope that this work will stimulate further investigation in this direction.

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