Stochastic perturbation of integrable systems: a window to weakly chaotic systems.

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May 21, 2013

Abstract

Integrable non-linear Hamiltonian systems perturbed by additive noise develop a Lyapunov instability, and are hence chaotic, for any amplitude of the perturbation. This phenomenon is related, but distinct, from Taylor’s diffusion in hydrodynamics. We develop expressions for the Lyapunov exponents for the cases of white and colored noise. The situation described here being ‘multi-resonance’ – by nature well beyond the Kolmogorov-Arnold-Moser regime, it offers an analytic glimpse on the regime in which many near-integrable systems, such as some planetary systems, find themselves in practice. We show with the aid of a simple example, how one may model in some cases weakly chaotic deterministic systems by a stochastically perturbed one, with good qualitative results.

1 Introduction

The problem

Lyapunov exponents measure the average rate of expansion of volumes advected by the trajectory of a dynamical system. When a dynamical system is chaotic, some of its Lyapunov exponents are positive, a small difference in the initial conditions is amplified exponentially with time. An integrable system with \( N \) degrees of freedom, having \( N \) constants of motion, has all its Lyapunov equal to zero. The motion is restricted to an \( N \)-dimensional torus in \( 2N \)-dimensional phase-space.

Consider one such integrable Hamiltonian dynamics, but now perturbed by a weak additive noise:

\[
\dot{q}_i = \frac{\partial H}{\partial p_i} \\
\dot{p}_i = -\frac{\partial H}{\partial q_i} + \epsilon^{1/2} \xi_i(t)
\]  

In this paper we shall mostly consider the case in which the \( \xi(t) \) are independent Gaussian white noises:

\[
\langle \xi_i(t) \rangle = 0 \quad \text{and} \quad \langle \xi_i(t) \xi_i(t') \rangle = 2 \delta_{ij} \delta(t-t').
\]

Such a system diffuses slowly from one torus to another, but we shall consider times short enough that this diffusion is small.

It may come as a surprise that for every \( \epsilon > 0 \) the system (1) generically develops a Lyapunov instability: two trajectories starting at nearby points and subjected to the same noise \( \xi \) diverge exponentially (mostly, as we shall see, on the surface of the torus): the system acquires \( N \) positive Lyapunov exponents. Because the underlying Hamiltonian system is, by assumption, integrable, the exponents vanish in the limit of zero noise amplitude – as \( \epsilon^{1/3} \), as we show below [1]. In what follows we shall derive expressions for this, and more general situations.

Motivation

Before launching into rather long calculations, let us discuss our motivation. Systems that are integrable and subjected to a small non-integrable perturbation are quite common in physics: the example
of planetary systems, where the perturbation is the interaction between different planets, immediately
comes to mind. Another family of problems of this kind arises when one considers a system with many
interacting degrees of freedom $N$, such that in some initial condition the average interaction is integrable
in the $N \to \infty$ limit. This is the case of stars belonging to a (to a first approximation) homogeneous,
spherical stellar cloud: each star perceives the rest as a spherical integrable potential, although the system
is most definitely not integrable when one takes into account the inhomogeneities of mass distribution.
Finally, one should remark that even numerical roundoff errors themselves may induce a Lyapunov in-
stability in a system that has none, at least when the Lyapunov exponents are calculated on the basis
of the tangent dynamics associated with a single trajectory.

Small perturbations of integrable systems evoke the Kolmogorov-Arnold-Moser (KAM) theorem,
which states that under certain conditions, once perturbation is turned on, regularity is not totally lost,
and there remain some regions where trajectories belong to tori and have zero Lyapunov exponents.
Remarkable as it is, the KAM result is very often irrelevant as soon as one considers systems with a few
degrees of freedom. Indeed, planetary systems such as the solar system are known to be chaotic [11, 25].

As mentioned above, the stochastic perturbation which is our main concern here, drives the system
out of regularity even for arbitrarily small amplitudes. To understand that this does not contradict
the KAM theorem, we argue as follows: the stochastic equation (1), a Langevin process with infinite
temperature, may be derived by considering the system coupled with a bath composed of an infinite
number of oscillators, with a continuum spread of frequencies [26]. We are hence in a situation as
described above: we may think of (1) as a system with infinitely many degrees of freedom, those of the
original system plus those of the bath.

The purpose of this paper is then to understand in better detail this regime, as a basis for treating
systems in which the perturbation is not stochastic, but is due to the effect of the rest of the system
with a particular degree of freedom.

The Lyapunov and the Taylor regimes

Before concluding this introduction, let us write Equations (1) in a more flexible and general way.
Considering a bath of oscillators coupled in a generic way to a Hamiltonian system, one may write
the most general Markovian Langevin equation (here restricted to the infinite temperature limit) in a
canonically invariant way [3]. Denoting $G_k$ phase-space functions which specify the coupling between
system an bath, and the phase-space variables $x = (q_1, \ldots, p_1, \ldots)$, one has:

\begin{equation}
(S) \quad \dot{x}_i = \{x_i, H\} + \epsilon^{1/2} \sum_k \{x_i, G_k\} \xi_k(t)
\end{equation}

\begin{equation}
(I) \quad \dot{x}_i = \{x_i, H\} + \epsilon^{1/2} \sum_k \{x_i, G_k\} \xi_k(t) + \epsilon \sum_k \{G_k, \{G_k, x_i\}\}
\end{equation}

Here $\{A, B\}$ are the Poisson brackets. The first is the equation in the Stratonovitch, and the second in
the Ito convention. One can now check that the usual Langevin equations (1) are obtained for $G_k = -q_k$.
Equation (3) leads to the evolution for the phase-space probability distribution $P(x)$:

\begin{equation}
\frac{\partial P}{\partial t} + \{P, H\} = \epsilon \sum_k \left[ \{G_k, \{G_k, H\} P\} + \{G_k, \{G_k, P\}\} \right]
\end{equation}

We have made explicit the amplitude $\epsilon$ of the noise, which we shall assume throughout to be small.
The advantage of this canonically invariant representation is that we may take advantage of the integrable nature of the Hamiltonian dynamics: we may now write everything in terms of the angle $\theta_i$ and action $I_i$ variables. The Hamiltonian is then a function of the $I_i$, and the equations read, for example in the Stratonovitch convention:

$$\dot{I}_i = \varepsilon^{1/2} \sum_k \{I_i, G_k\} \xi_k(t)$$
$$\dot{\theta}_i = \omega_i + \varepsilon^{1/2} \sum_k \{\theta_i, G_k\} \xi_k(t)$$

Here, $\omega_i = \partial H / \partial I_i$ is the angular frequency of $\theta_i$. The $G_k$ have to be expressed in terms of the action and angle variables, $\{\theta_i, G_k\} = \partial G_k / \partial \theta_i$ and $\{I_i, G_k\} = -\partial G_k / \partial I_i$.

In the absence of noise, the system remains confined to a torus labeled by the value of the constants of motion $I_i$, and spanned by the $\theta_i$. The effect of the noise is to add some diffusion, within and away from the torus. Because the amplitude of the noise is by assumption small (of order $\varepsilon^{1/2}$) and random, the typical time for this diffusion is $t_{\text{diff}} \sim \varepsilon^{-1}$. Consider now two trajectories starting in nearby points on the same torus, under the effect of the same noise: apart from their common diffusion, there is an exponential separation of trajectories that, as we shall see, has a characteristic time $\tau \sim \varepsilon^{-1/3}$. Once two trajectories have diverged substantially (of $O(1)$), the fact that their noise is the same becomes irrelevant, and each follows its own diffusion. In the small $\varepsilon$ limit, there is a large range of timescales $\varepsilon^{-1/3} \ll \tau \ll \varepsilon^{-1}$ where the diffusive drift away from a torus is still very small, but the Lyapunov instability is well defined. We shall in what follows concentrate on such times.

The interplay of noise and regular dynamics has a history in the hydrodynamics of laminar flows: the enhancement in diffusion due to the interplay with regular advection goes under the name of Taylor instability. In this paper we shall be mostly concerned with the initial exponential separation regime, with copies subjected to the same noise.

The equations of motion read:

$$H = I - \frac{1}{6} I^3 \quad ; \quad \omega(I) = 1 - \frac{1}{2} I^2,$$

and let us choose:

$$G = \sqrt{2} \cos \theta.$$

As we shall see below, because $\sin \theta$ is multiplying a white noise of small amplitude in (7), it may without loss of generality be replaced by its root-mean-square average. If we now make the identification of $I$ as the transverse and $\theta$ the longitudinal direction, our example corresponds then precisely to a Poiseuille flow on a two-dimensional channel [8], with transverse diffusion — the textbook example of Taylor diffusion:

$$\dot{x} = 1 - y^2$$
$$\dot{y} = -\varepsilon \xi(t)$$

The results may be seen in figure [1]. When the two particles have independent realizations of noise, their separation evolves in a purely diffusive manner, as $\sqrt{2\varepsilon t}$, until the diffusion reaches the walls, when the distribution becomes stationary. A surprising phenomenon occurs then: the copies perform an essentially longitudinal diffusion with an enhanced effective coefficient $\varepsilon_{\text{eff}}$. This is the Taylor-Aris dispersion [20,2]. The origin of this enhancement is simple: particles behave like cars which advance deterministically along a highway with lanes having different speeds, but diffuse laterally. As they diffuse back to their original lane, they do so with a fluctuation in the longitudinal direction that is the result of the stochastic excursion along faster and slower lanes. In Figure [1] one may fit $\varepsilon_{\text{eff}} = 0.0025 / \varepsilon$, which is in agreement with the expressions [20,2] for a case initially without diffusion along the channel.

Consider now our case, when the two particles are subjected to the same noise. The separation is initially exponential $e^{\lambda_{\infty} t}$, where $\lambda_{\infty}$ is by definition the Lyapunov exponent. Just as in the case of independent noise realizations, at long times the system crosses over to a (predominately longitudinal) Taylor dispersion regime. In this paper we shall be mostly concerned with the initial exponential separation regime, with copies subjected to the same noise.
Figure 1: (a) Evolution of the distance between two points at $I = 0$ initially separated by a distance $\sqrt{\delta I^2 + \delta \theta^2} = 10^{-6}$. We let the systems evolve with Eqs. (7), with diffusion coefficient $\varepsilon = 10^{-3}$. The results are averaged over 1024 realizations. Dotted line and full line correspond to different and same noise for the two realizations. For different noise realizations there is an initial diffusion with coefficient $\varepsilon$, followed by a faster Taylor diffusion with $\varepsilon_{\text{eff}} = 0.0025/\varepsilon$. For the case of equal noise there is an initial exponential separation, followed by the Taylor diffusion regime with $\varepsilon_{\text{eff}}$. (b) Displacement along $I$ for the same problem.
2 Evolution of the tangent vectors

We now turn to the evolution of two nearby trajectories $x_i(t)$ and $x_j(t)$, and the tangent vector $u_i(t) = \frac{x_i(t) - x_j(t)}{\|x_i(t) - x_j(t)\|}$. One has to be careful about the prescription. One obtains in the Stratonovitch convention:

$$\dot{x}_i = \{x_i, H\} + \varepsilon^{1/2} \sum_k \{x_i, G_k\} \xi_k(t)$$

$$\dot{u}_i = \sum_j \frac{\partial}{\partial x_j} \left[ \{x_i, H\} + \varepsilon^{1/2} \sum_k \{x_i, G_k\} \xi_k(t) \right] u_j$$

(9)

Note that the evolution of the $u_i$ is ‘slaved’ to that of the $x_i$. For small $\varepsilon$ and times smaller than $t_{\text{diff}} \sim \varepsilon^{-1}$, we may neglect the effect of noise on the evolution of the $x_i$, so that the original variables move on a torus. Further progress is made by writing Eqs (9) in angle-action variables $(\theta_i, I_i)$. We have:

$$\dot{\theta}_i = \{\theta_i, H\} = \omega_i(t)$$

$$\dot{I}_i = 0$$

(10)

Denoting the set $\{u_1, \ldots, u_N\} = \{(u_{i_1}, \ldots, u_{i_N}), (u_{\theta_1}, \ldots, u_{\theta_N})\}$, the evolution in the tangent space becomes, in the Stratonovitch convention:

$$\begin{align*}
\dot{u}_{\theta_i} &= \sum_j \varepsilon^{1/2} \sum_k \frac{\partial \theta_i}{\partial \theta_j} \frac{\partial I_i}{\partial I_j} \xi_k(t) u_{\theta_j} + \sum_j \varepsilon^{1/2} \sum_k \frac{\partial I_i}{\partial I_j} \xi_k(t) u_{I_j} \\
\dot{u}_{I_i} &= \sum_j \varepsilon^{1/2} \sum_k \frac{\partial I_i}{\partial \theta_j} \xi_k(t) u_{\theta_j} + \sum_j \varepsilon^{1/2} \sum_k \frac{\partial I_i}{\partial I_j} \xi_k(t) u_{I_j}
\end{align*}$$

(11)

where we may assume (for the times that concern us here) that the phase-space variables are unperturbed by the noise and are given by Eqs (10).

In order to compare the terms in the limit of small $\varepsilon$, we propose a rescaling of the $u_j$ and time, as follows:

$$t \rightarrow \varepsilon^{-\alpha} t \quad ; \quad u_{\theta_i} \rightarrow \varepsilon^-\beta u_{\theta_i} \quad ; \quad u_{I_i} \rightarrow u_{I_i}$$

(12)

We shall assume and check that $\alpha > 0$ and $\beta > 0$. Equations (11) become:

$$\begin{align*}
\dot{u}_{\theta_i} &= \sum_j \varepsilon^{1/2} - \alpha \sum_k \frac{\partial \theta_i}{\partial \theta_j} \frac{\partial I_i}{\partial I_j} \xi_k(\varepsilon^{-\alpha} t) u_{\theta_j} \\
&\quad + \sum_j \varepsilon^{1/2 - \alpha + \beta} \sum_k \frac{\partial I_i}{\partial I_j} \xi_k(\varepsilon^{-\alpha} t) u_{I_j} \\
\dot{u}_{I_i} &= \sum_j \varepsilon^{1/2 - \alpha} \sum_k \frac{\partial I_i}{\partial \theta_j} \xi_k(\varepsilon^{-\alpha} t) u_{\theta_j} + \sum_j \varepsilon^{1/2} \sum_k \frac{\partial I_i}{\partial I_j} \xi_k(\varepsilon^{-\alpha} t) u_{I_j}
\end{align*}$$

(13)

Comparing the first terms of (13) and (14), we conclude that we may neglect the former; while comparing the first term of (13) and the third of (14), that we may neglect the latter. Also, comparing the first term and second terms of (14), we see that we may neglect the latter. We are left with:

$$\dot{u}_{\theta_i} = \varepsilon^{-\alpha + \beta} \sum_j \frac{\partial^2 H}{\partial I_i \partial I_j} u_{I_j}$$

$$\dot{u}_{I_i} = \varepsilon^{1/2 - \alpha} \sum_j \sum_k \frac{\partial^2 G_k}{\partial \theta_i \partial \theta_j} (\varepsilon^{-\alpha} t) \xi_k(\varepsilon^{-\alpha} t) u_{\theta_j}$$

(15)

(16)

Here the $\frac{\partial^2 H}{\partial I_i \partial I_j}$ are constants, and the $\frac{\partial^2 G_k}{\partial \theta_i \partial \theta_j}$ quantities that are depend on time through the angles $\theta_i(t)$, defined by the torus and given by (10).
This is as far as we can go for a general perturbation. If we now we consider the case of white noise, we have that: $\xi_k(\varepsilon^{-\alpha}t) = \varepsilon^{\frac{\alpha}{2}} \xi_k(t)$. We may proceed as follows: we choose $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{4}$:

$$u_{\theta_i} = \sum_j \frac{\partial^2 H}{\partial I_j \partial I_j} u_{I_j}$$
(17)

$$\dot{u}_{I_i} = \sum_{j,k} \frac{\partial^2 G_k}{\partial \theta_j \partial \theta_j} (\varepsilon^{-\alpha} t) \xi_k(t) u_{\theta_j}$$
(18)

This is not yet the final product. We have to note now that the $\frac{\partial^2 G_k}{\partial \theta_j \partial \theta_j} (\varepsilon^{-\alpha} t)$ are rapidly oscillating functions of (rescaled) time.

We now use the fact that in the limit of high frequency (in our case $\varepsilon \to 0$), one may replace the oscillating terms by their root mean square average. To see that this is generically the case, consider a stochastic process with generator $L(\omega t)$, a periodic function of time. The generating function over one period is given by the time-ordered exponential $T e^{-\int L(\omega t) dt} = e^{-L_{av} t}$, where the averaged generator may be developed using the Magnus expansion [19]:

$$L_{av} = \frac{\omega}{2\pi} \left[ \int_0^{2\pi} dt L(\omega t) dt + \int_0^{2\pi} dt \int_0^t dt' [L(\omega t), L(\omega t')] + ... \right]$$
(19)

Rescaling times, one finds that the second term is of order $\omega^{-1}$, the subsequent one $\omega^{-2}$, and so on. Averaging over time the generator means, going back to the equation [18] which is in Langevin form, that we substitute the noises terms by white, correlated Gaussian noises $\rho_{ij}$ with correlations:

$$L_{ijkl} = \frac{1}{\tilde{t}} \int_0^t dt \int_0^{t'} dt' \frac{\partial^2 G_k}{\partial \theta_i \partial \theta_j} (\varepsilon^{-\alpha} t') \frac{\partial^2 G_k}{\partial \theta_i \partial \theta_j} (\varepsilon^{-\alpha} t') \xi_i(t) \xi_j(t')$$
(20)

where $\tilde{t}$ is a time that is long enough that it encompasses an almost integer number of cycles of the variables, but is short with respect to $\varepsilon^{-1/3}$. We finally obtain:

$$\dot{u}_{\theta_i} = \sum_j \frac{\partial^2 H}{\partial I_j \partial I_j} u_{I_j}$$
(21)

$$\dot{u}_{I_i} = \sum_j \rho_{ij}(t) u_{\theta_j}$$
(22)

with $\rho_{ij}(t) \rho_{kl}(t') = 2L_{ijkl} \delta(t-t')$. We are now in a position to write the equation for the evolution equation of the probability distribution $P(u_{\theta_i}, u_{I_i})$ of the $u_i$:

$$\frac{\partial P}{\partial t} = \sum_{ij} \left( \frac{\partial^2 H}{\partial I_j \partial I_j} u_{I_j} \frac{\partial}{\partial u_{\theta_i}} + \sum_{ijlm} L_{ijkl} \frac{\partial^2}{\partial u_{I_i} \partial u_{I_l}} u_{\theta_i} u_{\theta_m} \right) P$$
(23)

Note that in [21], [22] and [23] time here has been rescaled as $t \to \varepsilon^{-1/3} t$ (cfr Eq [12]). In equations [21], [22] Ito and Stratonovich conventions coincide, because to leading order in $\varepsilon$ the terms multiplying the stochastic noise depend on the $u_{\theta_i}$, and are absent in the equation for $\dot{u}_{\theta_i}$. Had we started in the Ito convention, we would have carried along the ‘drift terms’ in the second of [3] and the corresponding terms in the equation for $\dot{u}_i, u_0$: one may check that, consistently, their effect is subleading in $\varepsilon$.

### 3 A single degree of freedom

Let us now specialize to a single degree of freedom. The equations [21] and [22] read, in this case:

$$\dot{u}_\theta = \frac{d^2 H}{dI^2} u_I$$
(24)

$$\dot{u}_I = \rho(t) u_\theta$$
(25)
with $t(t') = \delta(t - t')\Lambda_{1100}$. The root mean square geometric factor for the amplitude of the noise reads:

$$
\Lambda_{1100} = \frac{1}{\ell} \int_0^\ell dt \left[ \frac{d^2G_k}{d\theta^2}(t) \right]^2 \equiv \left( \frac{d^2G}{d\theta^2} \right)^2 = \left( \frac{G}{\omega(I)^2} \right)^2
$$

(26)

where we have used the fact that $\frac{d}{dm} = \frac{dt}{dm} = \frac{1}{\omega(I)} \frac{d}{dt}$. The corresponding Fokker-Planck equation is:

$$
\frac{\partial P}{\partial t} = \left[ \left( \frac{d^2H}{d\theta^2} \right) u \frac{\partial}{\partial u} + \Lambda_{1100} \frac{\partial^2}{\partial u_0^2} u_0^2 \right] P
$$

(27)

In the one degree of freedom case, we may now perform a further rescaling of $t$ and $u_0$, and obtain an adimensional equation for the evolution of the probability $\tilde{P}$ of the rescaled variables:

$$
\frac{\partial \tilde{P}}{\partial t} = \left[ -\frac{\partial}{\partial u_0} u_I + \frac{\partial^2}{\partial u_0^2} u_0^2 \right] \tilde{P}(\tilde{u}_0, u_I, t).
$$

(28)

This equation appears frequently in the theory of one-dimensional localization, and in the related problem of the harmonic oscillator with randomly diffusing frequency (see References [9], [11], [23] and [13], whose approaches we shall follow).

The rescaled time $\ell t$ is expressed, with respect to the original time $t$ as: $\ell = \frac{\ell}{\pi}$ where $\tau$ is the characteristic time

$$
\tau = \frac{1}{\ell} \left[ \frac{d^2G}{d\theta^2} \right]^2 \left( \frac{d\omega}{dI} \right)^2 = \frac{1}{\ell} \left( \frac{d\omega}{dH} \right)^2
$$

(29)

where we have used that $\frac{d\omega}{dH} = \frac{dt}{dH} = \frac{dH}{d\theta^2}$. The factor $\left( \frac{d\omega}{dH} \right)^2$ appearing in the characteristic time $\tau$ is a measure of the difference is period of neighboring orbits, and we shall hence call it isochronicity parameter. It is zero for a harmonic oscillator. Denoting $t_P = 2\pi/\omega$ the period of oscillations, we may also write

$$
\left( \frac{1}{\ell} \frac{d\omega}{dH} \right)^2 = \left( \frac{1}{t_P} \frac{dt_P}{dH} \right)^2.
$$

(30)

Starting from an initial length $\|u(0)\| = 1$, we define the (quenched) Lyapunov exponent as the average of the logarithmic separation:

$$
\lambda(t) = \frac{1}{\ell} \ln \|u(t)\|.
$$

(31)

An annealed estimate may be also defined as:

$$
\lambda^{(2)}(t) = \frac{1}{2\ell} \ln \|u(t)\|^2.
$$

(32)

where averages are taken over the stochastic noise realizations. Because all the dependence on the problem is through the timescale $\tau$, we have that both exponents are proportional to $\tau^{-1}$, with different dimensionless proportionality constants of order one.

**Annealed Lyapunov exponent $\lambda^{(2)}$**

The annealed Lyapunov exponent is easy to calculate using the property [13] that the moments of order two

$$
\langle u_0 u_0 \rangle = \int u_0 u_0 P(u_0, u_I, t) du_0 du_I.
$$

(33)

evolve through a closed system of equations. Using equation (27) one may easily see that, to leading order in $\varepsilon$:

$$
\frac{d}{dt} \begin{pmatrix} \langle u_0 u_0 \rangle \\ \langle u_0 u_I \rangle \\ \langle u_I u_I \rangle \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{\Lambda_{1100}}{\tau} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \langle u_0 u_0 \rangle \\ \langle u_0 u_I \rangle \\ \langle u_I u_I \rangle \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \langle u_0 u_0 \rangle \\ \langle u_0 u_I \rangle \\ \langle u_I u_I \rangle \end{pmatrix}
$$

(34)
The largest eigenvalue $\mu_M$ of the matrix in the right hand side yields the annealed Lyapunov exponent $\lambda^{(2)}$. The eigenvalue equation is easy to derive:

$$\mu_M^3 = \frac{1}{2} \varepsilon \Lambda_{1100}[\omega'(I)]^2 = \frac{1}{2} \tau^{-3}$$

and we get:

$$2\lambda^{(2)} = \mu_M = \frac{2^{-1/3}}{\tau}$$

(36)

We easily check that $\langle u_\theta u_I \rangle \propto \varepsilon^{1/3}(u_\theta^3)$ et $\langle u_\theta^2 \rangle \propto \varepsilon^{2/3}(u_\theta^2)$; which means that the if the Lyapunov vector has a component of order one along the $\theta$ direction (tangent to the torus), it has a component of order $\varepsilon^{1/3}$ along the $I$ direction (i.e. transverse to the torus).

**‘Quenched’ Lyapunov exponent**

In order to have a more complete description, it is useful to introduce the Riccati variable [9, 4]:

$$z = \frac{u_I}{u_\theta} = \left( \frac{d\omega}{dH} \right)^{-1} \left( \frac{u_\theta}{u_\theta} \right)$$

(37)

Clearly, the average Lyapunov exponent is given by:

$$\lambda = \omega'(I) z^2 + \varepsilon^{1/2} \xi(t).$$

(39)

with $\xi(t)\xi(t') = \Lambda_{1100} \delta(t-t')$ From this, or directly from (27), we obtain the Fokker-Planck version:

$$\frac{\partial P}{\partial t} = \left[ \omega'(I) \frac{\partial}{\partial z} z^2 + \varepsilon \Lambda_{1100} \frac{\partial^2}{\partial z^2} \right] P(z, t).$$

(40)

Again, we may rescale out all physical constants:

$$\hat{t} = t/\tau$$

$$\hat{z} = z/h$$

(41)

(42)

where $\tau$ given in (29) and

$$h = \left[ \tau \omega'(I) \right]^{-1} = \left( \tau \omega \frac{d\omega}{dH} \right)^{-1}.$$ 

(43)

We get:

$$\frac{\partial \hat{P}}{\partial \hat{t}} = \frac{\partial}{\partial \hat{z}} \left[ \hat{z}^2 + \frac{\partial}{\partial \hat{z}} \right] \hat{P}(\hat{z}, \hat{t})$$

(44)

(which we could have obtained directly from (28), and):

$$\hat{z} = -\omega'(I) \hat{z}^2 + \hat{\xi}(t)$$

(45)

where $\hat{\xi}(t)$ is a Gaussian white noise of variance 2.

In order to calculate the Lyapunov exponent, we need the expectation value of $\hat{z}$, computed with the stationary solution of Equation (44) satisfying $\partial P_\infty / \partial \hat{t} = 0$. Note that we are trying to solve for the stationary solution of a particle in an unbounded (cubic) potential. This is in fact impossible unless we re-inject at $-\infty$ particles that have reached $+\infty$: the stationary state has a constant current. This is not as strange as it seems, because as we shall see below, $z$ has the interpretation of the tangent of an angle which grows monotonically. The solution we find is then:

$$\hat{P}_\infty(\hat{z}) = \frac{1}{N} \exp \left( -\frac{\hat{z}^3}{3} \right) \left[ C + \int_0^{\hat{z}} \exp \left( \frac{\bar{y}^3}{3} \right) d\bar{y} \right] = \frac{1}{N} \int_{-\infty}^{\hat{z}} \exp \left( \frac{\bar{y}^3 - \hat{z}^3}{3} \right) d\bar{y}$$

(46)
where we have put
\[ C = \int_{-\infty}^{0} \exp\left(\frac{\tilde{y}^3}{3}\right) \, d\tilde{y} \] (47)
in order to assure normalizability and positivity. The normalization constant is given by [16]:
\[ N = \int_{-\infty}^{\infty} \int_{-\infty}^{\tilde{z}} \exp\left(\frac{\tilde{y}^3 - \tilde{z}^3}{3}\right) \, d\tilde{y} \, d\tilde{z} = \pi^2 \left[ \text{Ai}^2(0) + \text{Bi}^2(0) \right] \left(\frac{2}{3}\right)^{1/3} \frac{\sqrt{\pi} \Gamma\left(\frac{1}{6}\right)}{\sqrt{3}} \simeq 4.97605 \] (48)
where Ai et Bi are the Airy functions. The average \( \langle \tilde{z} \rangle \) is readily obtained as:
\[ \langle \tilde{z} \rangle = \int d\tilde{z} \, \tilde{P}_\infty(\tilde{z}) \, \tilde{z} = \left(\frac{3}{2}\right)^{1/3} \frac{\sqrt{\pi} \Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{2}\right)} \simeq 0.364506. \] (49)

The Lyapunov exponent is then:
\[ \lambda = \omega'(I)z = \frac{\langle \beta \rangle}{\tau} = \left(\frac{3}{2}\right)^{1/3} \frac{\sqrt{\pi} \Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{2}\right)} \left[ \epsilon \left\langle \frac{1}{H} \right\rangle \left(\frac{1}{\omega \, dH} \right)^2 \right]^{1/3}. \] (50)

4 Lyapunov jumps and Lyapunov-vector phase-slips

Evolution of the direction of the Lyapunov vector

Let us introduce the angle of the Lyapunov vector as:
\[ \alpha = \arctan z = \arctan \left(\frac{u_I}{u_\theta}\right) \] (51)
The evolution of the probability distribution \( P_\alpha(\alpha, t) \) may be obtained directly by changing variables in
Figure 3: The stationary distribution of the angle $\alpha$ for different values of parameters.

The Fokker-Planck equation \[ \partial P_{\alpha} / \partial t = \partial / \partial \alpha \left[ dV / d\alpha + (\tau \omega')^{-2} \partial / \partial \alpha \cos^4 \alpha \right] P_{\alpha} \] to get:

$$\frac{\partial P_{\alpha}}{\partial t} = \frac{\partial}{\partial \alpha} \left[ \frac{dV}{d\alpha} + (\tau \omega')^{-2} \frac{\partial}{\partial \alpha} \cos^4 \alpha \right] P_{\alpha} \tag{52}$$

where

$$V(\alpha) = -\frac{\omega'}{2} \left( \alpha - \frac{1}{2} \sin 2\alpha - (\tau \omega')^{-3} \cos^4 \alpha \right). \tag{53}$$

Equivalently, we find that the angle $\alpha$ follows a Langevin equation:

$$\dot{\alpha} = -\frac{dV}{d\alpha} + (\tau \omega')^{-1} \cos^2 \alpha \xi(t) \approx -\frac{dV}{d\alpha} + (\tau \omega')^{-1} \xi(t) \tag{54}$$

where we have replaced the cosine by one, the value it takes at the only times when the noise is non-negligible. The system is a marginal washboard potential (Fig. 4) with very small corrections and small noise. Away from saddles, the angle evolves monotonically and almost deterministically: these are the ‘phase slips’. This deterministic motion by itself would leave it trapped in the saddles: here the effect of noise – or in general any form of perturbation – is crucial, because it allows the system to traverse the saddle and start a new phase slip. Because the noise is weak, most of the time is spent around saddles where $\alpha = 0 \mod 2\pi$, and for those times the Lyapunov vector stays tangent to the torus.

Let us see what happens during a phase slip. During those times, we may neglect the noise in the equation for $z$. Solving the deterministic equation $\ddot{z} = -\dot{z}^2$ with some initial condition $u_0(0), u_I(0)$ we obtain: $u_I(t) = -|u_I(0)|$ and $u_\theta(t) = |u_\theta(0)| - |u_I(0)|t$. The norm of the vector evolves smoothly until the slip starts, then dips to a minimum of $\sqrt{u_I(t)^2 + u_\theta(t)^2} \sim |u_I(0)|$ which is achieved at half-slip $\alpha = \frac{\pi}{2}$, and then quickly recovers in the next half-slip what it had lost during the first.

The average time elapsed between phase slips is proportional to the Lyapunov time. In order to compute this we calculate the flux of $z$ defined from the Fokker-Planck equation $\partial P / \partial t = -\partial \tilde{j} / \partial \tilde{z}$ at stationarity.

$$\tilde{j}(\tilde{z}) = -\frac{1}{N\tau}. \tag{55}$$

The average time between slips is simply given by $1 / |j|:

$$\langle t_{\text{slip}} \rangle = \frac{1}{|j|} = N\tau = \frac{\pi}{\sqrt{3}} \lambda^{-1} \simeq 1.8138 \lambda^{-1}. \tag{56}$$
Let us see how this comes about in a simple example, the dynamics with \( \omega' = \Lambda_{1100} = 1 \).

\[
\begin{align*}
(I) \quad \dot{u}_\theta &= u_I \\
\dot{u}_I &= \varepsilon^{1/2} \xi(t) \ u_\theta
\end{align*}
\]

(57)

and \( \varepsilon = 10^{-3} \). Equations are of the form (24) and (25). We start with a random vector \( u(0) \) with unit norm \( \|u(0)\| = 1 \) and random orientation \( \alpha(t = 0) \).

Figure 5 shows the evolution of \( \alpha \) (which should be considered only modulo \( 2\pi \)): the phase slips are clearly visible. Whenever there is phase-slip, the finite time Lyapunov exponent shows a dip. These general features are clearly visible in the computations of Lyapunov exponents of planetary motion [17]. Although the Lyapunov vector is unfortunately not generally quoted in those cases, one expects that phase slips are the cause of the dips also for planets.

5 The role of separatrices: the example of the simple pendulum

As one would expect, the instability of trajectories is larger in and around separatrices. In order to see this, consider the example of the simple pendulum \( H = \frac{1}{2} p^2 + 1 - \cos q \). The frequency in terms of the energy is shown in Fig 6.

Small oscillations correspond to the linear regime, for which one has [24] \( \frac{dT}{dH} \big|_{H=0} = \frac{\tau_0}{\sin q} \neq 0 \) so that even a small amplitude trajectory will develop an instability in the presence of noise. The neighborhood of a separatrix \( H = 2 \) is also interesting. For \( \delta \equiv |H - 2| \), one may compute

\[
\omega(\delta \to 0) \simeq \frac{\pi}{|\ln \delta|} \to 0
\]

(58)

from which

\[
\frac{1}{\omega} \frac{d\omega}{dH} \sim \frac{1}{\delta |\ln \delta|} \to \infty.
\]

(59)

Because \( G^2 = \langle \dot{q} \rangle^2 \) is of order \( \omega^4 \) we find that the Lyapunov exponent scales as: \( \varepsilon^{1/3}|\ln \delta|^2 \delta^{-2/3} \to \infty \), which means that just on the separatrix it scales differently with \( \varepsilon \).
Figure 5: Evolution of the Lyapunov angle $\alpha$ and norm for $\varepsilon = 10^{-3}$. The characteristic time is $\tau = 10$. 

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The behavior of the pendulum is quite generic for nondegenerate fixed points. Consider the dynamics around a fixed point, say \((q_0, p_0) = (0, 0)\). We may assume that \(H_0 = H(q_0, p_0) = 0\), and generically to lowest order \(H\) reads:

\[
H(q, p) = \frac{p^2}{2m} + \alpha qp + \frac{k}{2} q^2
\]

where the constants \(m, k, \alpha\) may have any magnitude or sign. Hamilton’s equations read:

\[
\ddot{q} + \left(\frac{k}{m} - \alpha^2\right) q = 0.
\]

There are two possible cases: either

- \(\frac{k}{m} - \alpha^2 > 0\), the system is locally a harmonic oscillator \(\omega_0^2 = \frac{k}{m} - \alpha^2\), and the fixed point is elliptic. The development of \(\omega\) is to first order \(\omega = \omega_0 + \Delta \omega(H - H_0)\) so that \((\frac{d\omega}{dH})^2\) is a minimum at the center of the ellipse. For a fixed value of the noise, in this point the Lyapunov exponent is minimal.

- \(\frac{k}{m} - \alpha^2 < 0\). The fixed point is hyperbolic, the trajectory is a separatrix. Consider a trajectory starting close to this point, of energy \(H = \delta \ll 1\). The dynamics in \((q, p)\) starts along the unstable direction and the distance grows as \(\sim \sqrt{\delta} e^{\omega_0 t}\) or \(\omega_0 = |\frac{k}{m} - \alpha^2|\), becoming of order one \(\sqrt{\delta} e^{\omega_0 t} \sim 1\), after a time of order \(\frac{1}{\omega_0} |\ln \delta| \gg 1\). Once the system is away from the critical point, its subsequent evolution takes a time of order one. We hence conclude that the frequency close to an elliptic points goes as:

\[
\omega \sim \frac{\omega_0}{|\ln \delta|} \ll 1.
\]

We thus find that the behavior near minima and separatrix of the pendulum is generic for nondegenerate \((\omega_0 \neq 0)\) situations.

## 6 Analogies

In this section we discuss the close physical analogy between our problem and Anderson Localization. It will become clear that the situation we are dealing with is critical: in our language it is in the limit...
between a regime with exponentially rare (in terms of $\varepsilon$) phase slips, and a regime with frequent ($O(1)$) slips. This criticality shows up in the localization language in that the system corresponds to a band edge.

Consider equations (17) and (18) and eliminate the $u_L$. We get:

$$\ddot{u}_\theta_j + \sum_j \left\{ -\sum_{kl} \frac{\partial^2 H}{\partial I_i \partial I_k} \frac{\partial^2 G_k}{\partial \theta_j \partial \theta_i} (\varepsilon^{-\alpha} t) \xi_k(t) \right\} u_{\theta_i} = \ddot{u}_\theta_j + \sum_j \dot{H}_{ij} u_{\theta_j}$$

(63)

which defines $\dot{H}_{ij}(t)$ as the term in the bracket. If we now make the correspondence $u_{\theta_i} \rightarrow \psi_i$ and $t \rightarrow x$, we may write the Shroedinger eigenvalue equation

$$\nabla^2 \psi + \dot{H} \psi = e\psi$$

(64)

where $\psi$ is an $N$ component wavefunction of $x$. Our problem concerns what happens around ‘energy’ $e = 0$. Lyapunov exponents are related to the decay of $\psi$ for large $x$, and this is indeed a question of localization of wavefunctions in the presence of a potential $\hat{H}$. This relation has been long understood, and indeed we have used several results originally thought for localization problems (cfr refs. [9, 4]).

Consider the problem in one dimension. The Lyapunov exponent is related to the exponent in the decay of a localized function. On the other hand, phase slips are related to the nodes and indeed we have used several results originally thought for localization problems (cfr refs. [9, 4]).

In conclusion, we should emphasize two points:

- Our problem is a one dimensional (the time) localization situation, in the presence of weak noise. It is hence marginal, and we are in a band edge situation.
- Our potential is random if the perturbation is random, but we may still think of cases for which the perturbation is deterministic: the problem of a planet perturbed by the small interaction with others is the classical example. In the language of localization one may ask the question as to whether a deterministic (but complicated) potential might or not be represented by a random one. This has a long tradition in solid state physics: although there are no definite universal answers, such identification gave useful insights, perhaps the most spectacular being the explanation of Fishman, Grempel and Prange [5] of energy localization in ‘kicked’ quantum systems in terms of Anderson localization.

7 More general types of perturbation

7.1 Non Gaussian noise

One expects that any Markovian noise with a non-Gaussian distribution will give the same results as the Gaussian with the corresponding variance. The reason is that the noise is weak, so what matters is its cumulative effect over time, and this is in fact Gaussian by a central limit theorem property. Formally, this may be seen at the level of equation (39), writing it in Martin-Siggia-Rose form:

$$1 = \int D[\xi] P[\xi] \int D[z] \delta \left[ \dot{z} + \omega'(I) z^2 - \varepsilon^{1/2} \xi(t) \right]$$

$$= \int D[\xi] P[\xi] \int D[z] \int D[\dot{z}] \exp \left\{ i \int dt \dot{z} \left[ \dot{z} + \omega'(I) z^2 - \varepsilon^{1/2} \xi(t) \right] \right\}$$

$$= \int D[z] \int D[\dot{z}] \exp \left\{ i \int dt \dot{z} \left[ \dot{z} + \omega'(I) z^2 \right] - \mathcal{F}(\varepsilon^{1/2} \dot{z}) \right\}$$

where we have introduced the noise probability $P$ and the corresponding cumulant generator $e^{\mathcal{F}[\nu]} = \int D[\xi] P[\xi] \exp \left\{ -i \int dt \dot{z} \nu(t) \right\}$. Expanding $\mathcal{F}[\varepsilon^{1/2}]$ in powers of $\varepsilon$, to second order, we recover a Gaussian case.
7.2 Noise with long time correlations

Ornstein–Uhlenbeck process  A simple way of introducing long range correlations is to consider $\zeta(t)$ evolving as:

$$
(I) \quad \dot{\zeta} = -\frac{1}{\tau_\ast} \zeta + \frac{1}{\tau_\ast} \xi(t)
$$

where $\tau_\ast$ is the time scale of the process and $\xi(t)$ is a white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t) \xi(t') \rangle = 2\varepsilon \delta(t-t')$. This is an Ornstein–Uhlenbeck process [15]. The autocorrelation reads

$$
\langle \zeta(t) \zeta(t') \rangle = \frac{\varepsilon}{\tau_\ast} e^{\frac{|t-t'|}{\tau_\ast}}.
$$

In particular, $\langle \zeta(t)^2 \rangle = \frac{\varepsilon}{\tau_\ast}$. In the limit $\tau_\ast \to 0$, $\zeta(t)$ becomes delta-correlated.

Because we now consider noise that is not white, we are not always justified in replacing $\frac{\partial^2 G}{\partial \theta^2}$ by its root mean squared value as we did in above. Let us write an equation for the Riccati variable without averaging over the angle variables:

$$
(I) \quad \dot{\zeta} = -\omega'(I) z^2 + \frac{\partial^2 G}{\partial \theta^2} \zeta
$$

$$
\dot{\zeta} = \frac{1}{\tau_\ast} \zeta + \frac{1}{\tau_\ast} \xi(t)
$$

In the limit $\tau_\ast = 0$, the Lyapunov exponent $\lambda_0$, given by (50), inversely proportional to $\tau_0$ given by (29).

If $\tau_\ast \neq 0$, $\tau_\ast$ is a time scale in the problem, in addition to $2\pi / \omega(I)$ and $\tau_0$. We expect that the Lyapunov exponent $\lambda$ is given by a form

$$
\frac{\lambda}{\lambda_0} = f \left( \lambda_0 \tau_\ast, \omega \tau_\ast, \varepsilon \right)
$$

where $f$ is adimensional. Let shall analyze the case in which $\frac{\partial^2 G}{\partial \theta^2}$ has a nonzero and a zero time average, respectively.

Non-zero time average  We assume that $\frac{\partial^2 G}{\partial \theta^2} = \frac{\partial^2 G}{\partial \theta^2}^{\text{non}} + \text{(zero average term)}$, and for definiteness that the time average is positive.

Assume first that $\lambda_0 \tau_\ast \gg 1$. In this case, $z(t)$ is a fast variable with respect to $\zeta$. During the times when

- $\zeta(t) < 0$. $\dot{z} < 0$ and the tangent vector turns rapidly. Note that $\zeta(t)$ stays negative for times longer than many ‘phase slips’. These events may be seen as short steps in figure [7]. During such times, $z(t)$ is zero on average, and there is no contribution to the Lyapunov exponent.

- $\zeta(t) > 0$. During such periods: $\zeta \sim \oint_0^\infty \zeta p(\zeta) d\zeta \sim \sqrt{\varepsilon / \tau_\ast}$. and $z$ follows adiabatically the equilibrium configuration for each $\zeta$, i.e. $\dot{z} \sim 0$ in (67), so that

$$
z(t)^2 \sim \frac{1}{\omega'(I)} \frac{\partial^2 G}{\partial \theta^2} \zeta(t) \sim \frac{1}{\omega'(I)} \frac{\partial^2 G}{\partial \theta^2} \sqrt{\frac{\varepsilon}{\tau_\ast}}
$$

from which one obtains the typical value of $z(t)$ during such times.

The two regimes are equally probable, so that

$$
\langle z \rangle = \frac{1}{2} \langle z |_{\zeta < 0} \rangle + \frac{1}{2} \langle z |_{\zeta > 0} \rangle \sim \left[ \frac{1}{\omega'(I)^2} \left( \frac{\partial^2 G}{\partial \theta^2} \right)^2 \frac{\varepsilon}{\tau_\ast} \right]^{1/4}
$$

and we obtain

$$
\lambda = \omega'(I) \langle z \rangle \sim \left[ \frac{\varepsilon}{\tau_\ast} \omega'(I)^2 \left( \frac{\partial^2 G}{\partial \theta^2} \right)^2 \right]^{1/4} = \lambda_0 (\lambda_0 \tau_\ast)^{-1/4}.
$$

Figure [8] shows the values of Lyapunov exponents in terms of the parameters. We have $\lambda / \lambda_0 \to 1$ for the regime $\lambda_0 \tau_\ast \ll 1$ (just as the white noise case), and $\lambda / \lambda_0 \propto (\lambda_0 \tau_\ast)^{-1/4}$ for the case $\lambda_0 \tau_\ast \gg 1$. All the dependence in $\varepsilon$ is through $\lambda_0$.
Figure 7: A realization of the dynamics (67) with $\varepsilon = 10^{-3}$ and $\tau_\ast = 200$, in the case $\frac{dG}{d\theta} > 0$. 
Figure 8: Lyapunov exponent $\lambda$ in terms of the correlation time $\tau_*$, both adimensionalized by $\lambda_0$. $\frac{\partial^2 G}{\partial \theta^2} > 0$, for $\varepsilon = 1, 10^{-1}, 10^{-3}, 10^{-5}$. The straight lines correspond to exponents 0 and $-1/4$.

Zero average: $\frac{\partial^2 G}{\partial \theta^2} = 0$. Here $\frac{\partial^2 G}{\partial \theta^2}$ oscillates with frequency $\omega$.

Clearly, in this case $\frac{\partial^2 G}{\partial \theta^2}$ changes sign periodically over a short timescale $2\pi/\omega$. and we cannot apply the arguments above.

Let us consider $\lambda_0 \tau_* \gg 1$ and $\omega \tau_* \gg 1$. Because of timescale separation, we may consider instead of $\dot{z}$, $z^2$ and $\frac{\partial^2 G}{\partial \theta^2} \zeta(t)$, their averages over $2\pi/\omega$, which we shall denote: $\overline{\dot{z}}, \overline{z^2}$ et $\overline{\frac{\partial^2 G}{\partial \theta^2} \zeta(t)}$. In particular, $\overline{\cos \omega t \zeta(t)}$ is typically of the order of the variation of $\zeta(t)$ in a short period, $\zeta/\omega$. We may thus make the same argument as before, but considering, instead of the sign of $\zeta(t)$, the sign of $\overline{\dot{z}}$.

In the regime $\overline{\dot{z}} > 0$, the slow variable $\overline{z} \sim 0$ equilibrates, so that Equation (67) averaged over time reads:

$$0 \sim -\omega'(I) \overline{z^2} + \Gamma \cos \omega(t) \overline{\zeta(t)}$$

(73)

from which:

$$\overline{z^2} \sim \frac{\Gamma}{\omega'(I)} \frac{\overline{\zeta}}{\omega} \sim \frac{\Gamma}{\omega'(I)} \frac{1}{\omega \tau_*} \sqrt{\frac{\varepsilon}{\tau_*}}$$

(74)

Again, the two regimes $\overline{\zeta} < 0$ and $\overline{\dot{z}} > 0$ are equiprobable, and

$$\lambda = \omega'(I) \overline{z} \sim \omega'(I) \overline{z}^{1/2} \sim \left(\frac{\varepsilon}{\tau_*} \Gamma^2 \omega'(I)^2\right)^{1/4} \left(\omega \tau_*\right)^{-1/2} = \lambda_0 (\lambda_0 \tau_*)^{-1/4} (\omega \tau_*)^{-1/2}.$$  

(75)

Figure 9 shows a plot of $\lambda$. We find that $\lambda/\lambda_0 \rightarrow 1$ when $\tau_* \rightarrow 0$, as in the Markovian case. For weak noise, and $\tau_*$ large enough (75) is well reproduced.
Figure 9: Lyapunov exponent $\lambda$ in terms of the correlation time $\tau_*$, for $\varepsilon = 1 - 10^{-4}$. Both variables are made dimensionless using $\lambda_0$ Here $\frac{\partial^2 G}{\partial \theta^2} = 0$. The straight line is a power law with exponent $-1$.

8 Many degrees of freedom

Largest exponent (annealed)

We start from equations (21) and (25). Putting $\hat{u}_{ii} = \sum_j \frac{\partial^2 H}{\partial I_i \partial I_j} u_{ij}$ we obtain:

$$\dot{u}_{\theta i} = \hat{u}_{Ii}$$  (76)
$$\dot{u}_{Ii} = \sum_j \hat{\rho}_{ij}(t) u_{\theta j}$$  (77)

with $\hat{\rho}_{ij}(t) = \hat{\rho}_{ij}(t') = \delta(t - t') \hat{\Lambda}_{ijkl}$, where:

$$\hat{\Lambda}_{ijkl} = \sum_{i'm'} \frac{\partial^2 H}{\partial I_i \partial I_{i'm'}} \frac{\partial^2 H}{\partial I_{i'm'} \partial I_j} \Lambda_{i'm'kl}$$  (78)

Equation (23) becomes:

$$\frac{\partial P}{\partial t} = \left[ \dot{u}_{Ii} \frac{\partial}{\partial u_{\theta i}} + \hat{\Lambda}_{ijkl} \frac{\partial^2}{\partial u_{\theta i} \partial u_{\theta j}} u_{\theta i} u_{\theta j} \right] P$$  (79)

We may generalize the calculation of the largest (annealed) Lyapunov exponent by considering the $3N(N-1)/2$-dimensional vector: $\{ \langle u_{\theta i} u_{\theta j} \rangle, \langle u_{Ii} u_{\theta j} + \langle u_{Ij} u_{\theta i} \rangle/2, \langle u_{Ii} u_{Ij} \rangle \}$

$$\frac{d}{dt} \left( \frac{\langle u_{\theta i} u_{\theta j} \rangle}{\langle u_{Ii} u_{Ij} \rangle} \right) = 2 \left[ \begin{array}{c} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{array} \right] + \varepsilon \left[ \begin{array}{c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hat{\Lambda} & 0 & 0 \end{array} \right] \left( \frac{\langle u_{\theta i} u_{\theta j} \rangle}{\langle u_{Ii} u_{Ij} \rangle} \right)$$  (80)

Just as in the one dimensional case, it is easy to see just by writing the eigenvalue equation, that all eigenvalues satisfy:

$$\frac{1}{2} \varepsilon \hat{\Lambda}_{ijkl} V_{kl} = \mu^3 V_{ij}$$  (81)
The annealed version of the largest Lyapunov exponent is given by

\[ 2\lambda^{(2)}_{\text{max}} = \mu_M \]  

Kolmogorov-Sinai entropy

In order to obtain a Riccati form for (76) and (77), we start by writing them as:

\[ \ddot{u}_{\theta i} = \sum_j \hat{\rho}_{ij}(t)u_{\theta j} \]  

We now apply this to \( N \) independent vectors \( u_l^{\theta i} \), which we shall denote as an \( N \times N \) matrix \( \Theta \):

\[ \ddot{\Theta}_{il} = \sum_j \hat{\rho}_{ij} \Theta_{jl} \]  

Defining the matrix Riccati variable \( Z = \dot{\Theta} \Theta - \frac{1}{2} \), we get the equation:

\[ \dot{Z}_{ij} - [Z^2]_{ij} = \hat{\rho}_{ij} \]  

The Kolmogorov-Sinai entropy is given by the rate of \((N\text{-dimensional})\) volume expansion in the \( \theta_i \) space:

\[ h_{KS} = \langle \frac{d}{dt} \text{Tr} \ln \Theta \rangle = \text{Tr} \langle Z \rangle \]  

where we have used the identity \( \frac{d}{dt} \text{Tr} \ln \Theta = \text{Tr} \{ \dot{\Theta} \Theta - \frac{1}{2} \} \).

9 An example: Foucault’s pendulum

Consider Foucault’s pendulum. The one in the Musee des Arts et Metiers in Paris has a mass of \( m = 25 \text{ kg} \) with radius \( R = 0.09 \text{ m} \) hanging at the end of a 18 m thread. The frequency of small oscillations is \( \omega_0 = \sqrt{g/l} \) where \( g = 9.81 \text{ m/s}^2 \). The pendulum describes small oscillations of amplitude \( q_0 \sim 10^{-2} \) radians. As we saw above, for a simple pendulum at small oscillations \( \frac{d}{dt} |_{H=0} = \frac{V}{\text{sm}g} \). We neglect friction, because we assume that some mechanism compensates it. On the other hand, we assume the stochastic element of the force fluctuations may be considered to be Markovian. We are thus led to a situation where \( G = -q \), so that \( (G)^2 \sim \omega_0^4 q_0^2 \). The diffusion constant in air is related to the viscosity via the Stokes-Einstein relation:

\[ D = \frac{k_BT}{6\pi \eta R} \sim 10^{-16} \text{ m}^2/\text{s}. \]  

The intensity of noise is then \( D/(l^2 \omega_0) \). The Lyapunov time is given estimated by \( 29 \)

\[ \tau \sim \left( \frac{D}{l^2 \omega_0 (\omega_0 q_0^2)} \right)^{1/3} \sim \left( \frac{q_0^2 \omega_0^2 D}{64 \omega_0^2} \right)^{-1/3} \]  

We find \( \tau \sim 5 \) years. For a pendulum of length in the order of centimeters, and a mass of radius in the order of millimeters, the Lyapunov time turns out to be in the order of days.

10 Stochastic treatment to model weakly nonintegrable systems beyond KAM regime

The main motivation of this paper is the perspective of treating weakly nonintegrable systems beyond the KAM regime, by substituting the integrability-breaking interactions by random noise. For example, one might hope to obtain an estimate of the Lyapunov exponents of a planet by treating the perturbation due to the other planets as stochastic. The restriction of being beyond the KAM regime is easily satisfied by systems with larger numbers of degrees of freedom \( N \), and in general one might expect that the stochastic
approximation of the weak interactions be better in that limit. However, as we shall see below, for large $N$ one may have effects of ‘crowding’ of degrees of freedom that become hence more correlated: this is clear in the example of a planetary system of given size and large number of planets.

**Many-body Lyapunov exponents and passive approximation**

In order to fix ideas, consider a weakly interacting system such as a planetary system with $N$ planets. In order to test its stability properties of the orbits of planet $A$ one may proceed in different ways:

- **i)** Compute two trajectories starting with slightly different positions $r_A(t)$ and $r_A(t) + \delta r_A(t)$ of planet $A$, using two copies of the full $6N$-dimensional dynamics $r_i(t)$ and $r_i(t) + \delta r_i(t)$, and then measuring the evolution of the distance $\delta r_A(t)$ between the two copies of planet $A$.

- **ii)** Compute two nearby trajectories $r_A(t)$ and $r_A(t) + \delta r_A(t)$ of planet $A$, but treating planet $A$ using the same trajectory of all other planets in the two copies (i.e. $\delta r_i(t) = 0$ for $i \neq A$). Planet $A$ is passive in the sense that the change in its initial conditions and subsequent trajectory does not reflect in a change in the trajectory of all others.

- **iii)** Even more extreme, one may neglect all interactions except those that the other planets exert on $A$: planet $A$ is then completely passive.

Procedure (i) gives, for any $A$ and at long times, the largest Lyapunov exponent of the whole system, even if separation between trajectories is measured only for planet $A$, although finite-time effects may be large and long lasting [22]. The reason is easy to understand: the Lyapunov vector has a norm that grows exponentially with time, and, unless its projection with any particular direction vanishes exponentially with time, its time dependence will follow that of the norm.

Procedures (ii) and (iii) give different values for each planet, and these are approximations that for weakly interacting systems might give a very good estimate of the finite-time sensitivity to initial conditions of a single planet. One may also conjecture that in that limit the exponents so obtained, treating by turns each planet as passive, might give a good approximation of the entire set of $6N$ Lyapunov exponents $\lambda_1, ..., \lambda_{6N}$.

For a general system with weak interaction:

$$H = H(I_1, ..., I_N) + \epsilon H_{\text{int}}(\theta_1, ..., \theta_N, I_1, ..., I_N)$$

(89)

with equations of motion

$$\dot{\theta}_i = \omega_i(I_1, ..., I_N) + \epsilon \frac{\partial H_{\text{int}}}{\partial I_i}$$

$$\dot{I}_i = -\epsilon \frac{\partial H_{\text{int}}}{\partial \theta_i}$$

(90)

procedure (iii) for a single degree of freedom amounts to calculating the Lyapunov exponent of the following system:

$$\dot{\theta}_A = \omega_A(I_1, ..., I_N) + \epsilon \frac{\partial H_{\text{int}}}{\partial I_A}$$

$$\dot{I}_A = -\epsilon \frac{\partial H_{\text{int}}}{\partial \theta_A}$$

(91)

where $H_{\text{int}}$ is taken as a function of $\theta_A, I_A$ and all other values $i \neq A$ are fixed as $\theta_i = \theta_i(0) + \omega_i t$, and $I_i(t) = I_i(0)$.

**Froeschl´e model**

Let us consider a toy model, which turns out to be quite instructive. We study the $N$ degree of freedom version of the Hamiltonian introduced in [9]:

$$H_F = \sum_{i=1}^{N} \frac{I_i^2}{2} + I_0 + \frac{\epsilon(N+2)}{1 + \frac{1}{N+2} \sum_{i=0}^{N} \cos \theta_i}$$

(92)
We have scaled coefficients so that both terms are extensive and $\epsilon$ intensive. When $\epsilon = 0$, the system is integrable, and the $\theta_i$ turn with angular speed $I_i$, except for $\theta_0$, which has unit angular frequency. When $\epsilon > 0$, $H$ is no longer integrable. The equations of motion read, with $i \geq 1$,

$$\dot{\theta}_0 = 1,$$
$$\dot{\theta}_i = I_i,$$
$$\dot{I}_i = -\epsilon \sin \theta_i - \epsilon \sin \theta_i \xi(t),$$

with

$$\xi(t) = \left(1 + \frac{1}{N+2} \sum_i \cos \theta_i \right)^{-2} - 1. \quad (94)$$

If $\epsilon > 0$ is small enough, the KAM theorem applies and some invariant tori survive. The values of $\epsilon$ for this to be the case are expected to be exponentially small in $N$, and become extremely small already for $N = 6$ \cite{13}. The first regions of phase space where tori break, are the places where $\sum_i n_i \omega_i = 0$, where the $n_i$ are integers. This scenario was observed in Ref. \cite{10} for $N = 2$.

We shall consider a torus given by $I_i(= \omega_i)$ chosen from a Gaussian distribution with zero mean and variance $\beta^{-1}$. For $I_i$ incommensurate, the quantity $\xi(t)$ is a sum of projections of incommensurate angles, and one expects it to behave as a pseudo-random number generator, at least for $N$ large enough. The question as to if and when such signals may be taken as random, and the more refined one of the recurrences in their autocorrelations, has received enormous attention both in mathematics and physics. (The reader will find a discussion and references in Zwanzig’s book \cite{28}).

**Statistical properties of $\xi(t)$**

If we assume that the angles are random enough that $\sum_{i=0}^{N} \cos \theta_i = O(\sqrt{n})$, we may develop for large $N$:

$$\xi(t) = -\frac{2}{N+2} \sum_{i=0}^{N} \cos \theta_i + \frac{3}{(N+2)^2} \left( \sum_{i=0}^{N} \cos \theta_i \right)^2 + O(N^{-3}). \quad (95)$$

The assumption that the $\theta_i$ are decorrelated angles requires at the very least that we are not on a resonance. This amounts to treating $\xi(t)$ as deriving from $\theta_i$ that are independent, random, and uniformly distributed in $[0, 2\pi]$. To lowest order, we have:

$$\langle \xi \rangle = \frac{3}{2} \frac{N + 1}{(N + 2)^2} \approx \frac{3}{2N}, \quad (96)$$
$$\langle \xi^2 \rangle = \frac{2}{N + 2} \frac{N + 1}{(N + 2)^2} \approx \frac{2}{N} \quad (97)$$

and

$$\sigma^2 = \langle \xi^2 \rangle - \langle \xi \rangle^2 \approx \frac{2}{N}. \quad (98)$$

Let us now calculate the autocorrelation of $\xi(t)$. We consider constant $\omega_i (= I_i)$, as we shall be interested in the dynamics before the system leaves the vicinity of a torus. We hence put $\theta_i(t) = \theta_i(0) + \omega_i t$. For large values of $N$,

$$C(t) = \frac{1}{\sigma^2} \langle \xi(0) \xi(t) \rangle = \frac{4}{\sigma^2(N+2)^2} \sum_{i,j} \langle \cos \theta_i(0) \cos \theta_j(0) + \omega_i \omega_j \rangle = \frac{1}{N+1} \sum_i \cos \omega_i t. \quad (99)$$

We have averaged over initial conditions: $\theta_i(0)$ drawn from a uniform distribution in $[0, 2\pi]$. We wish to estimate the time $\tau_\epsilon$ of decay of the correlation. We have $C(0) = 1$ by construction, and as $t \gg \tau_\epsilon$, if the $\omega_i$ are not commensurable, $C(t)$ is a fluctuating quantity with variance $1/N$.

The value of $\tau_\epsilon$ depends on the distribution of the $\omega_i$. For large $N$, we have

$$C(t) = \frac{1}{\sqrt{2\pi \beta^{-1}}} \int \cos \omega t \, e^{-\beta \omega^2/2} d\omega \approx \exp \left( -\frac{1}{2} \beta^{-1} t^2 \right), \quad (100)$$

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so that

$$\tau_* = \beta^{1/2}$$  \quad (101)$$

may be interpreted as the autocorrelation time. It is of the order of the average period of oscillation, and is independent of $N$. The autocorrelation is shown in Fig.10.

In principle, one could make some effort to calculate better estimates of $\sigma^2$ and $\tau_*$. However, the power law with exponent $\frac{1}{3}$ in the scaling law (103) tells us that the Lyapunov exponent only depends little on these parameters, so that very precise estimates are not needed.

**Testing validity of stochastic treatment.**

Let us first consider an extreme form of ‘passive’ approximation: we shall see how $\xi(t)$ acts on a single degree of freedom that has no feedback on the rest of the variables:

$$\dot{u}_\theta = u_I$$

$$\dot{u}_I = \varepsilon^{1/2} \xi(t) u_\theta$$  \quad (102)$$

If $\xi(t)$ is a real noise with zero average, variance $\langle \xi^2(t) \rangle = \sigma^2$ and correlation time $\tau_*$, we know from the results of section 3 that the Lyapunov exponent of this dynamics should scale as

$$\lambda \propto (\varepsilon \sigma^2 \tau_*)^{1/3}.$$  \quad (103)$$

Note that $\varepsilon$ is a parameter (not to be confused with the perturbation parameter $\epsilon$) that can be freely varied, providing some flexibility in testing the validity of our stochastic treatment.

We now substitute the gaussian noise by the one generated in a true Froeschl´e model setting (artificially) the parameter $\varepsilon$ to one. We used different realisations of $\xi(t)$ for $\beta = 0.8$, $N$ between 4 and 8192 and $\epsilon$ of order $10^{-3}$ (see Fig 11). We measured the Lyapunov exponent by estimating the exponential rate of growing of $(u_\theta, u_I)$.

Figure 11 shows the Lyapunov exponent obtained in this way, compared to the analytical expression, with the autocorrelation time $\tau_*$ estimated from (101). The agreement is very good for weak values of noise, over several decades of noise intensity.

The problem of an unusually unstable degree of freedom: the Modified Froeschl´e model

Next step is to go back to the true Froeschl´e model and compute its true Lyapunov exponent, and compare it with the largest one obtained by procedure (iii) above, that is, by treating the effect on a given degree of freedom by mimicking the perturbation produced by all other degrees of freedom with a random correlated noise. Because the effective perturbation has amplitude $O(N^{-\frac{1}{2}})$, one would expect the Lyapunov exponent to be proportional to $N^{-\frac{1}{3}}$. We have tried and we have checked numerically that this is not so, the largest Lyapunov exponent is very weakly dependent of $N$, if at all.

The explanation of this surprising fact is instructive. In our procedure we are choosing $N$ values of $I_i$ in an interval $\beta$ of order one. Each degree of freedom performs a motion following equation 93, corresponding to a pendulum of amplitude with ‘energy’ $I_i^2/2$ and ‘gravity field’ $\sim \epsilon$ (because of the term
−ε sin θᵢ in the last equation). Some of these degrees of freedom are close to the separatrix, the distance being

$$δ_i = \left| \frac{I_i^2}{2} - ε \right|$$

(104)

One may estimate from the Gaussian distribution of the $I - i$ that the smallest $δ_i$ scale as $δ_{min} \sim \frac{1}{N^2}$, while the noise scales as $ε_{eff} \sim N^{-\frac{1}{2}}$. Recall now the discussion of section 5: the Lyapunov exponent of a degree of freedom scales as $ε_{eff} |ln δ|^2 δ^{-\frac{3}{2}}$, which actually increases with $N$. Because the global Lyapunov exponent, whichever projection we measure, will be dominated by the largest, we conclude that the exponent is much larger than $\sim N^\frac{1}{3}$. In a word, the "crowding" of many degrees of freedom as $N \to \infty$ has produced interactions that become large, and in fact grow with $N$.

On way to minimize this problem is consider a model with an extra term:

$$H_{mF} = \sum_{i=1}^{N} \left( \frac{I_i^2}{2} + I_0 + \frac{ε(N+2)}{1 + \frac{1}{N+2}} \right) + \sum_{i=0}^{N} \frac{cos θ_i}{1 - \frac{1}{N+2} \sum_{i=0}^{N} cos θ_i}$$

(105)

so that the equations of motion read, with $i \geq 1$,

$$\dot{θ}_0 = 1,$$

$$\dot{θ}_i = I_i,$$

$$\dot{I}_i = -ε sin θ_i ξ(t),$$

(106-108)

We have measured the lyapunov exponent of one passive degree of freedom of this model, and obtained a good agreement even for relatively low values of $N$ (see Fig 12).
\[ \eta = 10^{-3} \quad \beta = 0.8 \]

**Figure 12:** Measured Lyapunov exponent of *one degree of freedom* (see text) of the modified Froeschle model \( \varepsilon = 1 \) and \( \beta = 0.8 \).

## 11 Conclusions

We have derived expressions for the Lyapunov exponents of an integrable system perturbed by additive stochastic noise. The motivation is to use this knowledge to estimate the effect of deterministic perturbations on an almost integrable system which is, however, far from the KAM and Nekhoroshev regimes – as will be the case as soon as one considers systems with many degrees of freedom and reasonable strong perturbations. The field of application of such an approach could be widespread: we have mentioned already planets and stellar clusters, but even a sound wave traveling in a liquid is an example of a near integrable system interacting with many (microscopic) degrees of freedom. We suppose that such an approximation must be implicitly present in one way or another in the literature (see e.g. [21]), but a systematic and general study seems to be missing. We hope this paper may offer a step in that direction.

### Acknowledgments

We wish to thank A. Politi, S. Ruffo, S. Tremaine and A. deWijn for clarifying remarks and suggestions.

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