Abstract. In this study, we define discrete fractional Sturm-Liouville (DFSL) operators within Riemann-Liouville and Grünwald-Letnikov fractional operators with both delta and nabla operators. We show self-adjointness of the DFSL operator for the first time and prove some spectral properties, like orthogonality of distinct eigenfunctions, reality of eigenvalues, parallely in integer and fractional order differential operator counterparts.

Keywords: Sturm-Liouville, discrete fractional, self-adjointness, eigenvalue, eigenfunction.

AMS Subject Classification: 34B24, 39A70, 34A08.

1. Introduction

Fractional calculus is rather attractive subject due to having wide-ranging application areas of theoretical and applied sciences. Although there is a large number of worthwhile mathematical works on the fractional differential calculus, there is no noteworthy parallel improvement of fractional difference calculus up to lately. This improvement has shown that discrete fractional calculus has certain unforeseen hardship.

Fractional sums and differences were obtained firstly in Diaz-Osler [17], Miller and Ross [15] and Gray and Zhang [16] and they found discrete types of fractional integrals and derivatives. Later, several authors started to touch upon discrete fractional calculus [1, 14, 27, 22, 28]. Nevertheless, discrete fractional calculus is a rather novel area. The first studies has been done by Atici et al. [2 – 5], Anastassiou [6, 7], Abdeljawad et al. [9 – 13], and Cheng [21], and so forth.

Self-adjoint operators have an important place in differential operators. Especially, various spectral properties of Sturm-Liouville differential operators, like orthogonality of distinct eigenfunctions, reality of eigenvalues, could be analyzed by the help of the self-adjointness. Self-adjointness of difference operators enabled to be analyzed integer order Sturm-Liouville difference operators parallely to differential counterpart.

Self-adjointness of fractional Sturm-Liouville differential operators have been shown by [23 – 26]. However, although self-adjointness of DFSL operators has been mentioned, it has not been proved yet. In this study, we define DFSL operators, which is defined differently from [18], and prove the self-adjointness for the first time, which means we have mentioned the important self-adjointness property. Moreover, we study some spectral properties of the operator parallely in integer and fractional order differential operator counterparts.
distinct eigenfunctions, reality of eigenvalues of DFSL operator and we prove these properties by using self-adjointness.

In this study, differently we analyze DFSL equation within Riemann-Liouville and Grünwald-Letnikov fractional operators with both delta and nabla operators. The aim of this paper is to contribute to the theory of DFSL operator.

In this paper, we discuss DFSL equation in two different ways;

i) (nabla right and left) Riemann-Liouville (R-L) fractional operator,

\[ L_1 x(t) = \nabla_0^\mu (p(t) \nabla^\mu x(t)) + q(t) x(t) = \lambda r(t) x(t), \]

ii) (delta left and right) Grünwald-Letnikov (G-L) fractional operator,

\[ L_2 x(t) = \Delta_-^\mu (p(t) \Delta_+^\mu x(t)) + q(t) x(t) = \lambda r(t) x(t). \]

2. Preliminaries

Definition 1. [1] Delta and nabla difference operators are defined by respectively

\[ \Delta x(t) = x(t+1) - x(t), \quad \nabla x(t) = x(t) - x(t-1). \]

Definition 2. [8] Falling function is defined by, \( \alpha \in \mathbb{R} \),

\[ t^\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - n)}, \]

where \( \Gamma \) is the gamma function.

Definition 3. [8] Rising function is defined by, \( \alpha \in \mathbb{R} \),

\[ t^{\bar{\alpha}} = \frac{\Gamma(t + \alpha)}{\Gamma(t)}. \]

Gamma function must be well-defined in the above two definitions.

Remark 1. Delta and nabla operators have the following properties

\[ \Delta t^\alpha = \alpha t^{\alpha-1}, \]
\[ \nabla t^{\bar{\alpha}} = \alpha t^{\bar{\alpha}-1}. \]

Definition 4. [2] [8] [15] Fractional sum operators are defined by,

(i) The nabla left fractional sum of order \( \mu > 0 \) is defined

\[ \nabla_{\alpha}^{-\mu} x(t) = \frac{1}{\Gamma(\mu)} \sum_{s=a+1}^{t} (t - \rho(s))^{\mu-1} x(s), \quad t \in \mathbb{N}_{a+1}, \]

(ii) The nabla right fractional sum of order \( \mu > 0 \) is defined
where $\rho(t) = t - 1$ is called backward jump operators, $\mathbb{N}_a = \{a, a + 1, \ldots\}$, $\mathbb{bN} = \{b, b - 1, \ldots\}$.

**Definition 5.** [10] [12] Fractional difference operators are defined by,

(i) The nabla left fractional difference of order $\mu > 0$ is defined

$$\nabla_a^\mu x(t) = \nabla_a^n \nabla_a^{-(n-\mu)} x(t) = \frac{\nabla^n}{\Gamma(n-\mu)} \sum_{s=a+1}^{t} (t - \rho(s))^{n-\mu-1} x(s), \ t \in \mathbb{N}_{a+1}, \quad (7)$$

(ii) The nabla right fractional difference of order $\mu > 0$ is defined

$$\nabla_b^\mu x(t) = (-1)^n \nabla_b^n \nabla_a^{-(n-\mu)} x(t) = \frac{(-1)^n \nabla^n}{\Gamma(n-\mu)} \sum_{s=a+1}^{t} (s - \rho(t))^{n-\mu-1} x(s), \ t \in \mathbb{bN} \quad (8)$$

Fractional differences in (7–8) are called the **Riemann–Liouville (R-L)** definition of the $\mu$-th order nabla fractional difference.

**Definition 6.** [18–20] Fractional difference operators are defined by,

(i) The delta left fractional difference of order $\mu$, $0 < \mu \leq 1$, is defined

$$\Delta_-^\mu x(t) = \frac{1}{h^\mu} \sum_{s=0}^{t} (-1)^s \frac{\mu(\mu - 1) \ldots (\mu - s + 1)}{s!} x(t-s), \ t = 1, \ldots, N. \quad (9)$$

(ii) The delta right fractional difference of order $\mu$, $0 < \mu \leq 1$, is defined

$$\Delta_+^\mu x(t) = \frac{1}{h^\mu} \sum_{s=0}^{N-t} (-1)^s \frac{\mu(\mu - 1) \ldots (\mu - s + 1)}{s!} x(t+s), \ t = 0, \ldots, N - 1, \quad (10)$$

Fractional differences in (9–10) are called the **Grünewald–Letnikov (G-L)** definition of the $\mu$-th order delta fractional difference.

**Definition 7.** [12] We define the integration by parts formula for R-L nabla fractional difference operator, $u$ is defined on $\mathbb{bN}$ and $v$ is defined on $\mathbb{N}_a$, then

$$\sum_{s=a+1}^{b-1} u(s) \nabla_a^\mu v(s) = \sum_{s=a+1}^{b-1} v(s) \nabla_a^\mu u(s). \quad (11)$$

**Definition 8.** [18] [19] We define the integration by parts formula for G-L delta fractional difference operator, $u, v$ is defined on $\{0, 1, \ldots, n\}$, then

$$\sum_{s=0}^{n} u(s) \Delta_-^\mu v(s) = \sum_{s=0}^{n} v(s) \Delta_+^\mu u(s). \quad (12)$$

3. Main Results

3.1. Discrete Fractional Sturm-Liouville Equations

We consider DFSLE equations in two different ways;
where \( p(t) > 0, r(t) > 0, q(t) \) is defined and real valued, \( \lambda \) is the spectral parameter, \( x(t) \in l^2[a + 1, b - 1]. \)

ii) (delta left and right) \( G-L \) fractional operator is defined by,

\[
L_2 x(t) = \Delta^\mu a (p(t) \Delta^\mu b x(t)) + q(t) x(t) = \lambda r(t) x(t),
\]

where \( p, q, r, \lambda \) is as defined above, \( x(t) \in l^2[0, n]. \)

Firstly, let’s consider the equation (13) and give the following theorems and proofs,

**Theorem 9.** DFSL operator \( L_1 \), denoted by the equation (13), is self-adjoint.

**Proof.**

\[
\begin{align*}
\langle u(t) L_1 v(t) \rangle &= u(t) \nabla^\mu_a (p(t) \nabla^\mu_b v(t)) + u(t) q(t) v(t), \quad (15) \\
v(t) L_1 u(t) &= v(t) \nabla^\mu_a (p(t) \nabla^\mu_b u(t)) + v(t) q(t) u(t). \quad (16)
\end{align*}
\]

If (15–16) is subtracted from each other

\[
u(t) L_1 v(t) - v(t) L_1 u(t) = u(t) \nabla^\mu_a (p(t) \nabla^\mu_b v(t)) - v(t) \nabla^\mu_a (p(t) \nabla^\mu_b u(t))
\]

and definite sum operator to the both side of the last equality is applied, we have

\[
\sum_{s=a+1}^{b-1} (u(s) L_1 v(s) - v(s) L_1 u(s)) = \sum_{s=a+1}^{b-1} u(s) \nabla^\mu_a (p(s) \nabla^\mu_b v(s)) - \sum_{s=a+1}^{b-1} v(s) \nabla^\mu_a (p(s) \nabla^\mu_b u(s)). \quad (17)
\]

If we apply the integration by parts formula in (11) to right hand side of (17), we have

\[
\sum_{s=a+1}^{b-1} (u(s) L_1 v(s) - v(s) L_1 u(s)) = \sum_{s=a+1}^{b-1} p(s) \nabla^\mu_b v(s) \nabla^\mu_a u(s) - \sum_{s=a+1}^{b-1} p(s) \nabla^\mu_a u(s) \nabla^\mu_b v(s) = 0,
\]

\[
\langle L_1 u, v \rangle = \langle u, L_1 v \rangle.
\]

The proof completes.

**Theorem 10.** Eigenfunctions, corresponding to distinct eigenvalues, of the equation (14) are orthogonal.

**Proof.** Let \( \lambda_\alpha \) and \( \lambda_\beta \) are two different eigenvalues corresponds to eigenfunctions \( u(n) \) and \( v(n) \) respectively for the the equation (13),

\[
\begin{align*}
\nabla^\mu_a (p(t) \nabla^\mu_b u(t)) + q(t) u(t) - \lambda_\alpha r(t) u(t) &= 0, \\
\nabla^\mu_a (p(t) \nabla^\mu_b v(t)) + q(t) v(t) - \lambda_\beta r(t) v(t) &= 0,
\end{align*}
\]

If we multiply last two equations to \( v(n) \) and \( u(n) \) respectively, subtract from each other and apply definite sum operator, since the self-adjointness of the operator \( L_1 \), we get,
since \( \lambda_\alpha \neq \lambda_\beta \),

\[
\sum_{s=a+1}^{b-1} r(s) u(s) v(s) = 0
\]

\[
\langle u(t), v(t) \rangle = 0
\]

The proof completes.

**Theorem 11.** All eigenvalues of the equation (13) are real.

**Proof.** Let \( \lambda = \alpha + i\beta \), since the self-adjointness of the operator \( L_1 \), we have

\[
\langle L_1 u, u \rangle = \langle u, L_1 u \rangle,
\]

\[
\langle \lambda ru, u \rangle = \langle u, \lambda ru \rangle,
\]

\[
(\lambda - \bar{\lambda}) \langle u, u \rangle_r = 0
\]

Since \( \langle u, u \rangle_r \neq 0 \),

\[
\lambda = \bar{\lambda}
\]

and hence \( \beta = 0 \). The proof completes.

Secondly, let’s consider the equation (13) and give the following theorems and proofs,

**Theorem 12.** DFSL operator \( L_2 \), denoted by the equation (14), is self-adjoint.

**Proof.**

\[
u(t) L_2 v(t) = u(t) \Delta_\mu^- (p(t) \Delta_\mu^v v(t)) + u(t) q(t) v(t),\]

\[
v(t) L_2 u(t) = v(t) \Delta_\mu^- (p(t) \Delta_\mu^u u(t)) + v(t) q(t) u(t).
\]

If (18 – 19) is subtracted from each other

\[
u(t) L_2 v(t) - v(t) L_2 u(t) = u(t) \Delta_\mu^- (p(t) \Delta_\mu^v v(t)) - v(t) \Delta_\mu^- (p(t) \Delta_\mu^u u(t))
\]

and definite sum operator to the both side of the last equality is applied, we have

\[
\sum_{s=0}^{n} (u(s) L_1 v(s) - v(s) L_2 u(s)) = \sum_{s=0}^{n} u(s) \Delta_\mu^- (p(s) \Delta_\mu^v v(s)) - \sum_{s=0}^{n} v(s) \Delta_\mu^- (p(s) \Delta_\mu^u u(s)).
\]

If we apply the integration by parts formula in (12) to right hand side of (20), we have

\[
\sum_{s=0}^{n} (u(s) L_2 v(s) - v(s) L_2 u(s)) = \sum_{s=0}^{n} p(s) \Delta_\mu^v v(s) \Delta_\mu^u u(s)
\]

\[
- \sum_{s=0}^{n} p(s) \Delta_\mu^u u(s) \Delta_\mu^v v(s)
\]

\[
= 0
\]
Theorem 13. Eigenfunctions, corresponding to distinct eigenvalues, of the equation (14) are orthogonal.

Proof. Let \( \lambda_\alpha \) and \( \lambda_\beta \) are two different eigenvalues corresponds to eigenfunctions \( u(n) \) and \( v(n) \) respectively for the the equation (14),

\[
\Delta^\mu \left( p(t) \Delta^\mu u(t) \right) + q(t) u(t) - \lambda_\alpha r(t) u(t) = 0, \\
\Delta^\mu \left( p(t) \Delta^\mu v(t) \right) + q(t) v(t) - \lambda_\beta r(t) v(t) = 0.
\]

If we multiply last two equations to \( v(n) \) and \( u(n) \) respectively, subtract from each other and apply definite sum operator, since the self-adjointness of the operator \( L_2 \), we get

\[
(\lambda_\alpha - \lambda_\beta) \sum_{s=0}^{n} r(s) u(s) v(s) = 0,
\]

since \( \lambda_\alpha \neq \lambda_\beta \),

\[
\sum_{s=0}^{n} r(s) u(s) v(s) = 0, \\
\langle u(t), v(t) \rangle = 0.
\]

So, the eigenfunctions are orthogonal. The proof completes.

Theorem 14. All eigenvalues of the equation (14) are real.

Proof Let \( \lambda = \alpha + i\beta \), since the self-adjointness of the operator \( L_2 \)

\[
\langle L_2 u, u \rangle = \langle u, L_2 u \rangle, \\
\langle \lambda r u, u \rangle = \langle u, \lambda r u \rangle,
\]

\[
(\lambda - \overline{\lambda}) \langle u, u \rangle_r = 0
\]

Since \( \langle u, u \rangle_r \neq 0 \),

\[
\lambda = \overline{\lambda},
\]

and hence \( \beta = 0 \). The proof completes.

4. Conclusion

In this study, differently we analyze DFSL equation within Riemann-Liouville and Grünwald-Letnikov fractional operators with both delta and nabla operators. We define DFSL operators, which is defined differently from \[13\], and prove the self-adjointness for the first time. Hereupon, we prove some spectral properties of the operator paralelly in integer and fractional order differential counterparts, like orthogonality of distinct eigenfunctions, reality of eigenvalues. The aim of this paper is to contribute to the theory of Sturm-Liouville fractional difference operator.
References

[1] Goodrich, Christopher, and Allan C. Peterson. Discrete fractional calculus. Berlin: Springer, 2015.

[2] Atıcı, Ferhan M., and Paul W. Eloe. "Discrete fractional calculus with the nabla operator." Electronic Journal of Qualitative Theory of Differential Equations 2009.3 (2009): 1-12.

[3] Acar, Nihan, and Ferhan M. Atıcı. "Exponential functions of discrete fractional calculus." Applicable Analysis and Discrete Mathematics (2013): 343-353.

[4] Atici, Ferhan, and Paul Eloe. "Initial value problems in discrete fractional calculus." Proceedings of the American Mathematical Society 137.3 (2009): 981-989.

[5] Atici, Ferhan M., and PAUL W. Eloe. "Linear forward fractional difference equations." Commun. Appl. Anal 19 (2015): 31-42.

[6] Anastassiou, George A. "Nabla discrete fractional calculus and nabla inequalities." Mathematical and Computer Modelling 51.5 (2010): 562-571.

[7] Anastassiou, George A. "Right nabla discrete fractional calculus." Int. J. Difference Equ 6.2 (2011): 91-104.

[8] Bohner, Martin, and Allan C. Peterson, eds. Advances in dynamic equations on time scales. Springer Science & Business Media, 2002.

[9] Abdeljawad, Thabet. "Dual identities in fractional difference calculus within Riemann." Advances in Difference Equations 2013.1 (2013): 36.

[10] Abdeljawad, Thabet, and Dumitru Baleanu. "Fractional Differences and Integration by Parts." Journal of Computational Analysis & Applications 13.3 (2011).

[11] Abdeljawad, Thabet. "On Riemann and Caputo fractional differences." Computers & Mathematics with Applications 62.3 (2011): 1602-1611.

[12] Abdeljawad, Thabet, and Ferhan M. Atıcı. "On the definitions of nabla fractional operators." Abstract and Applied Analysis. Vol. 2012. Hindawi Publishing Corporation, 2012.

[13] Abdeljawad, Thabet. "On delta and nabla Caputo fractional differences and dual identities." Discrete Dynamics in Nature and Society 2013 (2013).

[14] Baleanu, Dumitru, Shahram Rezapour, and Saeid Salehi. "On some self-adjoint fractional finite difference equations." Journal of Computational Analysis & Applications 19.1 (2015).

[15] Miller, Kenneth S., and Bertram Ross. "Fractional difference calculus." Proceedings of the international symposium on univalent functions, fractional calculus and their applications. 1988.
[17] Diaz, J. B., and T. J. Osler. "Differences of fractional order." Mathematics of Computation 28.125 (1974): 185-202.

[18] Almeida, Ricardo, et al. "Variational methods for the solution of fractional discrete/continuous Sturm–Liouville problems." Journal of Mechanics of Materials and Structures 12.1 (2016): 3-21.

[19] Bourdin, Loïc, et al. "Variational integrator for fractional Euler–Lagrange equations." Applied Numerical Mathematics 71 (2013): 14-23.

[20] Dubois, François, Ana Cristina Galucio, and Nelly Point. "Introduction à la dérivation fractionnaire. Théorie et applications." Techniques de l'ingénieur (2010).

[21] Cheng, Jin-Fa, and Yu-Ming Chu. "Fractional difference equations with real variable." Abstract and Applied Analysis. Vol. 2012. Hindawi Publishing Corporation, 2012.

[22] Mohan, J. Jagan, and G. V. S. R. Deekshitulu. "Fractional order difference equations." International Journal of Differential Equations 2012 (2012).

[23] Bas, Erdal, and Funda Metin. "Fractional singular Sturm-Liouville operator for Coulomb potential." Advances in Difference Equations 2013.1 (2013): 300.

[24] Bas, Erdal. "Fundamental spectral theory of fractional singular Sturm-Liouville operator." Journal of Function Spaces and Applications 2013 (2013).

[25] Klimek, Malgorzata, and Om Prakash Agrawal. "Fractional Sturm–Liouville problem." Computers & Mathematics with Applications 66.5 (2013): 795-812.

[26] Klimek, Malgorzata, and Om P. Agrawal. "On a regular fractional Sturm-Liouville problem with derivatives of order in (0, 1)." Carpathian Control Conference (ICCC), 2012 13th International. IEEE, 2012.

[27] Lizama, Carlos, Fractional Difference Equations and Applications, http://www.iumpa.upv.es/wp-content/uploads/2016/05/Fractional-Diference-Equations-and-Applications.pdf

[28] Ahrendt, Kevin - Rolling, Tim - Dewolf, Lydia - Mazurowski, Liam - Mitchell, Kelsey & Veconi, Dominic, Initial and Boundary Value Problems for the Caputo Fractional Self-Adjoint Difference Equations, Enlightenment of Pure and Applied Mathematics, Volume 2 (2016), Issue 1, Pages 105-141.