Interference phenomena in the decay $D_s^+ \to \eta\pi^0\pi^+$ induced by the $a_0^0(980) - f_0(980)$ mixing

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Using the data on the decay $D_s^+ \to f_0(980)\pi^0\pi^+ \to K^+K^-\pi^+$, we estimate the amplitude of the process $D_s^+ \to \left[f_0(980) \to (K^+K^- + K^0\bar{K}^0) \to a_0^0(980)\right] \pi^+ \to \eta\pi^0\pi^+$, caused by the mixing of $a_0^0(980)$ and $f_0(980)$ resonances that breaks the isotopic invariance due to the $K^+$ and $K^0$ meson mass difference. Effects of the interference of this amplitude with the amplitudes of the main mechanisms responsible for the decay $D_s^+ \to \eta\pi^0\pi^+$ are analyzed. As such mechanisms, we examine the transition $D_s^+ \to \eta\pi^0\pi^+$, which is observed in experiment, and the possible transition $D_s^+ \to \left(a_0^0(980)\pi^+ + a_0^0(980)\pi^0\right) \to \eta\pi^0\pi^+$. It is shown that the rapidly varying phase of the $a_0^0(980) - f_0(980)$ transition amplitude strongly influences the interference curves.

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I. INTRODUCTION

A threshold phenomenon known as the the mixing of $a_0^0(980)$ and $f_0(980)$ resonances appreciably breaks the isotopic invariance, since the effect is proportional to $\sqrt{2(M_{K^0} - M_{K^+})/M_{K^0}} \approx 0.13$ in the modulus of the amplitude \[1\]; see also Ref. \[2\]. This effect appears as the narrow (with the width of about $2(M_{K^0} - M_{K^+}) \approx 8$ MeV) resonant peak between the $K^+K^-$ and $K^0\bar{K}^0$ thresholds owing to the transition $a_0^0(980) \to K\bar{K} \to f_0(980)$ or vice versa $f_0(980) \to K\bar{K} \to a_0^0(980)$. There are many proposals in the literature concerning both the searching of the $a_0^0(980) - f_0(980)$ mixing and estimating the effects related with this phenomenon; the detailed list of references may be found, for example, in Ref. \[8\].

Recently, this phenomenon has been discovered experimentally and studied with the help of detectors VES in Protvino in $\pi^-N$ collisions \[4, 5\] and BESIII in Beijing in $J/\psi$ decays \[6, 7\]. As a result it has become clear \[3, 9, 10\] that the similar isospin breaking effect can appear not only due to the $a_0^0(980) - f_0(980)$ mixing, but also due to any mechanism of the production of the $K\bar{K}$ pairs with the definite isospin in the $S$ wave, $X \to K\bar{K} \to f_0(980)/a_0^0(980)$ \[11\]. Thus, a new tool emerged to study the production mechanism and nature of light scalars.

In the present work, we discuss, for the first time, the possibility of the $a_0^0(980) - f_0(980)$ mixing detection in three-body hadronic decay of the $D_s^+$ mesons into $\eta\pi^0\pi^+$. We pay attention to the fact that the manifestation of the isospin-breaking amplitude $f_0(980) \to K\bar{K} \to a_0^0(980)$ can be enhanced in this decay owing to its interference with the amplitudes of other mechanisms. The sharp and large variation of the phase of the $f_0(980) - a_0^0(980)$ transition amplitude (by about 90$^\circ$ in the region between $K^+K^-$ and $K^0\bar{K}^0$ thresholds) plays an important role in the interference phenomenon. So far, this characteristic feature of the $a_0^0(980) - f_0(980)$ mixing has remained in the shadows \[12, 13\]. By our estimates, the decay $D_s^+ \to \eta\pi^0\pi^+$ has potential for the $a_0^0(980) - f_0(980)$ mixing detection.

II. THE $a_0^0(980) - f_0(980)$ MIXING IN $D_s^+ \to \eta\pi^0\pi^+$

A. The case of two mechanisms

Figure 1 shows the BaBar data \[13\] on the $S$-wave mass spectrum of the $K^+K^-$ system produced in the decay $D_s^+ \to K^+K^-\pi^+$. Its shape, as well as the shape of the $S$-wave $\pi^+\pi^-$ spectrum in $D_s^+ \to \pi^+\pi^-\pi^+$ \[14\], is approximated by the $f_0(980)$ resonance contribution (see Figs. 1 and 2, and Ref. \[17\]).

The solid curves in Figs. 1 and 2 are proportional to the modulus squared of the $f_0(980)$ resonance propagator, i.e., $|S_{K^+K^-}|^2 \sim 1/|D_{f_0}(m_{K^+K^-})|^2$, where $m_{K^+K^-}$ is the invariant mass of $K^+K^-$ in the region above the $K^+K^-$ threshold, and $|S_{\pi^+\pi^-}|^2 \sim 1/|D_{f_0}(m_{\pi^+\pi^-})|^2$, where $m_{\pi^+\pi^-}$ is the invariant mass of $\pi^+\pi^-$, respectively. Here the $f_0(980)$ propagator, $1/D_{f_0}$, was taken from Ref. \[8\] without any changes.

The Particle Data Group (PDG) gives \[18\]

$$BR(D_s^+ \to f_0(980)\pi^+ \to K^+K^-\pi^+) = (1.15 \pm 0.32)\%.$$  \hspace{1cm} (1)$$

This value and its accuracy require further careful study (see discussions of the assumptions made by BaBar \[13\] and CLEO \[19\] with the treatment of the initial data). In fact, in the original BaBar \[13\] and CLEO \[19\] analyses a possible presence of the $a_0^0(980)$ resonance has been neglected so that the number given in Eq. (1) effectively corresponds to a sum of the $f_0(980)$ and $a_0^0(980)$ contributions in the decay of the $D_s^+$ mesons. Therefore, we consider the results of our analysis as some guide and hope that the detection of the $a_0^0(980) - f_0(980)$ mixing may shed extra light on the mechanisms of the $f_0(980)$ and $a_0^0(980)$ production in $D_s^+$ decays.

Using Eq. (1), together with the values of the $f_0(980)$ and $a_0^0(980)$ resonance parameters (see Appendix), obtained in Ref. \[3\] by analyzing the BESIII data \[6\] on the intensity of the $a_0^0(980) - f_0(980)$ mixing in the decays $J/\psi \to \phi f_0(980) \to \phi\eta\pi$ and $\psi' \to \phi f_0(980) \to \phi\eta\pi$, we can estimate the contributions of the $a_0^0(980)$ and $f_0(980)$ resonances in the BaBar data. The results of our analysis are given in Table 1, where we compare the fractions of the $a_0^0(980)$ and $f_0(980)$ resonances in the final states, $f_j(980)\pi^+$, with the values of the $f_j(980)$ resonances obtained in Ref. \[3\] and Ref. \[13\].
The data corresponding to the modulus squared of the transition amplitudes of the decay $\gamma \chi^s \rightarrow \pi^0 \pi^0$ are presented just below in Eq. (9).

The available data on the decay $D_s^+ \rightarrow \eta \pi^0 \pi^+$ show that it proceeds predominantly via the $\eta \rho^+$ intermediate state:

$$BR(D_s^+ \rightarrow \eta \rho^+ \rightarrow \eta \pi^0 \pi^+) = (8.9 \pm 0.8)\%,$$

$$BR(D_s^+ \rightarrow \eta \pi^0 \pi^+) = (9.2 \pm 1.2)\%.$$ (3) (4)

Let us denote the $D_s^+ \rightarrow \eta \rho^+ \rightarrow \eta \pi^0 \pi^+$ and $D_s^+ \rightarrow [f_0(980) \rightarrow (K^+ K^- + K^0\bar{K}^0) \rightarrow \eta \pi^0 \pi^+]$ transition amplitudes as $A_{\eta \rho^+}$ and $A_{f_0 a_0^0}$, respectively. For the description of their dependence on the mass variables, we use the following expressions:

$$A_{\eta \rho^+} \equiv A_{\eta \rho^+}(m_{\eta \rho^+}^2, m_{\eta \pi^0}^2, m_{\pi^0 \pi^+}^2) \equiv A_{\eta \rho^+}(s, t, u)$$

$$= C_{D_s^+ \eta \rho^+} \frac{s - t}{D_{\rho^+}(u)} F_{\rho}(u) \sqrt{\frac{g^2_{\rho \pi \rho}}{16\pi}},$$ (5)

$$A_{f_0 a_0^0} \equiv A_{f_0 a_0^0}(m_{\eta \pi^0}^2) \equiv A_{f_0 a_0^0}(s)$$

$$= C_{D_s^+ f_0 \pi^+} \frac{\Pi_{\rho \pi \rho}^2}{D_{\rho^+}^2(s) F_{\rho}(u) \sqrt{\frac{g^2_{\rho \pi \rho}}{16\pi}}},$$ (6)

where $s = m_{\eta \rho^+}^2$, $t = m_{\eta \pi^0}^2$, and $u = m_{\pi^0 \pi^+}^2$ are the invariant masses squared of the indicated meson pairs in the decay $D_s^+ \rightarrow \eta \pi^0 \pi^+$, and here we neglect the $\pi^0$ and $\pi^+$ mass difference and put $m_\pi = 0.135$ GeV; $D_{\rho^+}(u)$, $D_{\rho^+}(s)$, $D_{\rho^+}(t)$, and $\Pi_{\rho \pi \rho}^2(s)$ are the inverse propagators of $\rho^+$, $a_0^0(980)$, $f_0(980)$ resonances and the amplitude of the $a_0^0(980) \rightarrow (K^+ K^- + K^0\bar{K}^0) \rightarrow f_0(980)$ transition, respectively, $F_{\rho}(u)$ is the centrifugal barrier penetration factor (formulas for all these quantities are presented in Appendix); $g_{\rho \pi \rho}$ and $a_0^0 m_{\eta \pi^0}$ are the coupling constants (see also the Appendix); $C_{D_s^+ \eta \rho^+}$ and $C_{D_s^+ f_0 \pi^+}$ are the invariant amplitudes of the decays $D_s^+ \rightarrow \eta \rho^+$ and $D_s^+ \rightarrow f_0(980) \pi^+$, respectively. In so doing, the effective vertices $D_s^+ \rightarrow \eta \rho^+$ and $\rho^+ \rightarrow \eta \pi^0 \pi^+$ are taken in the form

$$V_{D_s^+ \eta \rho^+} = C_{D_s^+ \eta \rho^+} (\epsilon_\rho^\ast + \epsilon_{\rho \pi^+} + \epsilon_\pi^\ast + \epsilon_{\pi \rho^+}),$$

$$V_{\rho^+ \pi^0 \pi^+} = g_{\rho \pi \rho} (\epsilon_\rho^\ast + \epsilon_{\rho \pi^+} + \epsilon_\pi^\ast + \epsilon_{\pi \rho^+}),$$

where $\epsilon_\rho$ is the polarization four-vector of the $\rho^+$ meson, $\epsilon_{\rho \pi^0}$, $\epsilon_{\rho \pi^+}$, and $\epsilon_{\pi \rho^+}$ are the four-momenta of the $D_s^+$, $\eta$, $\pi^0$, and $\pi^+$ mesons in the decay $D_s^+ \rightarrow \eta \rho^+$. Hence the kinematical factor $s - t$ in Eq. (5) is $(p_{\rho^+} + p_\eta + p_{\pi^0} - p_{\pi^+})$.

The amplitude $A_{f_0 \pi^+}$, responsible for the decay $D_s^+ \rightarrow f_0(980) \pi^+ \rightarrow K^+ K^- K^+ \pi^+$ [see Eq. (1)], is given by

$$A_{f_0 \pi^+} \equiv A_{f_0 \pi^+}(m_{K^+ K^-}) \equiv A_{f_0 \pi}(s)$$

$$= C_{D_s^+ f_0 \pi^+} \frac{1}{D_{f_0}(s)} \sqrt{\frac{g^2_{f_0 \pi \pi^+}}{16\pi}}.$$ (9)

Each invariant amplitude $C_{D_s^+ \eta \rho^+}$ and $C_{D_s^+ f_0 \pi^+}$ is represented by two real numbers, a modulus and a phase,
which are independent of the mass variables, i.e., \( C_{D^+_0\eta_{\pi^0}} = a_1 e^{i\varphi_1} \) and \( C_{D^+_0\eta_{\pi^+}} = a_2 e^{i\varphi_2} \). Such an approximation of the amplitudes of heavy quarkonium decays with the participation of light resonances in intermediate states is commonly used in the data treatments (fits to experimental distributions in the Dalitz plots), see, for example, Refs. 13, 16, 19. We use this approximation for our estimates.

Taking into account Eqs. (2) and (3), we present in Figs. 3(a) and 3(b) the \( \eta_{\pi^0} \) and \( \pi^0\pi^+ \) mass spectra in the decay \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) for the case of the incoherent sum of the contributions from the \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) and \( D^+_s \rightarrow [f_0(980) \rightarrow (K^+K^- + K^0\bar{K}^0) \rightarrow a_0^0(980)] \pi^+ \rightarrow \eta_{\pi^0}\pi^+ \) mechanisms. The sharp peak with the width of about \( 2(m_{K^0} - m_{K^+}) \approx 8 \) MeV in Fig. 3(a) in the region of \( K^+K^- \) and \( K^0\bar{K}^0 \) thresholds arises owing to the \( a_0^0(980) = f_0(980) \) mixing. Figures 3(c) and 3(d) show, as an example, the \( s-u \) and \( s-t \) Dalitz plots for approximately \( 10^8 D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) Monte Carlo events generated for the above hypothetical case of the incoherent sum of two mechanisms. As seen from Eq. (5), the \( s-u \) and \( s-t \) distributions for the \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) decay mechanism vanish on the dashed lines \( u = m^2_{D_s} + 2m^2_{\pi} + m^2_{\eta} - 2s \) and \( t = s \) shown in Figs. 3(c) and 3(d), respectively. These lines divide the \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) events into two equal parts. The events caused by the \( a_0^0(980) = f_0(980) \) mixing concentrate in the vicinity of \( s = 4m^2_{K^0} \approx m^2_{f_0} \) on the \( s-u \) and \( s-t \) Dalitz plots. They make up about one-hundredth of a half of the \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) Dalitz plots.

This large for the isospin breaking contribution which, at the first sight, could be naturally expected to have the magnitude at the level of \( \approx 10^{-2} \) for the above hypothetical case of the incoherent sum of \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) and \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) transitions. Figures 3(e) and 3(f) show four variants of the \( \eta_{\pi^0} \) mass spectra in the region of the \( K^+K^- \) and \( K^0\bar{K}^0 \) thresholds with taking into account the interference of the \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) and \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) transition amplitudes \( A_{\eta_{\pi^0}p^+} \) and the amplitude \( A_{f_0a_0^0} \) caused by the \( a_0^0(980) = f_0(980) \) mixing,

\[
\frac{dN_{\eta_{\pi^0}}}{dm_{\eta_{\pi^0}}} = \int_{a_-(s)}^{a_+(s)} |A_{\eta_{\pi^0}p^+} + A_{f_0a_0^0}|^2 2m_{\eta_{\pi^0}} dm_{\eta_{\pi^0}}^2 \ . \tag{10}\]

Here the integration is made over the physical region of the variable \( m^2_{\eta_{\pi^0}p^+} = u \) from \( a_-(s = m^2_{\eta_{\pi^0}}) \) to \( a_+(s = m^2_{\eta_{\pi^0}}) \), where

\[
a_\pm(s) = \frac{1}{2} \left( \Sigma - s - \frac{(m^2_{D^+_s} - m^2_{\pi})(m^2_{\eta} - m^2_{\pi})}{s} \right) \pm \frac{2m_{D^+_s}}{\sqrt{s}} p(s)q(s) \ . \tag{11}\]

\[
p(s) = \sqrt{m^2_{D^+_s} - 2m^2_{\pi}(s + m^2_{\pi}) + (s - m^2_{\pi})^2/(2m_{2\eta})} \ , \tag{12}\]

\[
q(s) = \sqrt{s^2 - 2s(m^2_{\eta} + m^2_{\pi}) + (m^2_{\pi} - m^2_{\eta})^2/(2\sqrt{s})} \ . \tag{13}\]

Using the data from Eqs. 11 and 13, we find \( C_{D^+_s f_0\pi^+}/C_{D^+_s \eta_{\pi^0}} = (a_0/a_1) \xi \approx (4.5 \text{ GeV}) \xi \) where \( \xi = e^{i\varphi_{12}} \) and \( \varphi_{12} = \varphi_2 - \varphi_1 \) is the relative phase of the amplitudes \( C_{D^+_s f_0\pi^+} \) and \( C_{D^+_s \eta_{\pi^0}} \). This phase is unknown and to illustrate the possible interference patterns

Figure 3: The illustration of the \( a_0^0(980) = f_0(980) \) mixing manifestation in the decay \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) against the mechanism \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \). The solid curves in (a) and (b) show, respectively, the \( \eta_{\pi^0} \) and \( \pi^0\pi^+ \) mass spectra in the decay \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) for the case of the incoherent sum of the contributions from the \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) and \( D^+_s \rightarrow [f_0(980) \rightarrow (K^+K^- + K^0\bar{K}^0) \rightarrow a_0^0(980)] \pi^+ \rightarrow \eta_{\pi^0}\pi^+ \) mechanisms. The \( s-u \) and \( s-t \) Monte Carlo Dalitz plot distributions for this case are shown in (c) and (d), respectively. Plots (e) and (f) show the \( \eta_{\pi^0} \) mass spectra in the region of the \( K^+K^- \) and \( K^0\bar{K}^0 \) thresholds for four variants of the interference between the amplitudes \( D^+_s \rightarrow \eta_{\pi^0}\pi^+ \) and \( D^+_s \rightarrow [f_0(980) \rightarrow (K^+K^- + K^0\bar{K}^0) \rightarrow a_0^0(980)] \pi^+ \rightarrow \eta_{\pi^0}\pi^+ \) in comparison with the incoherent case; the curves are described in the text.
we put $\varphi_{21} = 0^\circ$, ±90°, and 180° (respectively, $\xi = 1$, ±1, and −1). The short and long dashed curves in Fig. 3(c) show the $\eta\pi^0$ mass spectra for $\xi = 1$ and $\xi = −1$, respectively. The dotted curve in this figure shows the contribution from the amplitude $A_{0\eta\pi^0}$ only, and the solid curve corresponds to the above case of the incoherent sum of two mechanisms. The solid and dotted curves in Fig. 3(f) show the same as in Fig. 3(c), and the short and long dash curves illustrate the interference patterns corresponding to $\xi = i$ and $\xi = −i$, respectively.

Note that the interference of $A_{f0\eta\pi^0}$ with the other contributions will be practically always essential (see Figs. 3(c) and 3(f)) in consequence of the sharp change of the phase of the $a_0^0(980) − f_0(980)$ transition amplitude $\Pi_{a_0^0f_0}(s)$ by about 90° in the region between $K^+K^−$ and $K^0\bar{K}^0$ thresholds [13, 14], where the modulus of $\Pi_{a_0^0f_0}(s)$ is maximal and approximately constant (see Appendix for details).

**B. The case of three mechanisms**

In principle, the decay $D^+_s \to \eta\pi^0\pi^+$ can proceed not only via the $\eta\pi^0$ intermediate state but also via the $(a_0^0(980)\pi^0)^+\pi^+$ production. $D^+_s \to [a_0^0(980)\pi^0 + a_0^0(980)\pi^+] \to \eta\pi^0\pi^+$. However, such a transition should be expected to be small. Based on the data quoted in Eqs. (9) and (13), we put $BR(D^+_s \to (a_0^0(980)\pi^0)^+) \to \eta\pi^0\pi^+ \approx 1\%$ as a very rough upper estimate. Note that by our estimate the relevant upper limit for $BR(D^+_s \to a_0^0(980)\pi^+) \to K^+K^−\pi^+$ is $\approx 0.1\%$. This estimate consists with the initial dominance of the $f_0(980)$ resonance in the decay $D^+_s \to f_0(980)\pi^+ \to K^+K^−\pi^+$ (see Eq. (1) and the discussion after it, and also Ref. [22]).

Thus, we have three interfering mechanisms of the decay $D^+_s \to \eta\pi^0\pi^+$. The corresponding decay amplitude is

$$A_{D^+_s \to \eta\pi^0\pi^+} = A_{0\eta\pi^0} + A_{f0\eta\pi^0} + A_{0\eta\pi^+},$$

where the amplitude $A_{0\eta\pi^0}$ describes the transition $D^+_s \to (a_0^0(980)\pi^0)^+ \to \eta\pi^0\pi^+$. Like $A_{0\eta\pi^0}$ [see Eq. (9)], the amplitude $A_{0\eta\pi^0}$ has to be antisymmetric with respect to permutation of the $s$ and $t$ variables [23]. Taking this into account, we approximate the amplitude $A_{0\eta\pi^0}$ by the following expression

$$A_{0\eta\pi^0} \equiv A_{0\eta\pi}(m^2_{\eta\pi^0}, m^2_{\eta\pi^+}) \equiv A_{0\eta\pi}(s,t) = C_{D^+_s a_0^0\pi^0} \left[ \frac{1}{D^+_s(a_0^0)}(s) - \frac{1}{D^+_s(a_0^0)}(t) \right] \sqrt{\frac{2a_0^{tot\pi^+}}{16\pi}},$$

where the production amplitude $C_{D^+_s a_0^0\pi^0} = a_3^e e^{i\varphi_3}$ is assumed to be the $s$- and $t$-independent complex constant. Note that any coherent sum of the amplitudes $A_{0\eta\pi^0}$ and $A_{0\eta\pi^0}$ gives the symmetric distribution of the $\eta\pi^0\pi^+$ events in the $s−t$ Dalitz plot relative to the $t = s$ line. The isospin-breaking amplitude $A_{f0\eta\pi^0} = A_{f0\eta\pi^0}(s)$ caused by the $a_0^0(980) \to f_0(980)$ mixing depends exclusively on $s$ and therefore is responsible for the asymmetry of the distribution of the $\eta\pi^0\pi^+$ events in the $s−t$ Dalitz plot (relative to the $t = s$ line).

By our estimate $C_{D^+_s a_0^0\pi^+}/C_{D^+_s \eta\pi^+} = (a_3/a_1)\xi' \approx (1.65 \text{ GeV})\xi'$, where $\xi' = e^{i\varphi_{31}}$ and $\varphi_{31} = \varphi_3 - \varphi_1$ is an unknown relative phase of the amplitudes $C_{D^+_s a_0^0\pi^+}$ and $C_{D^+_s \eta\pi^+}$. We examined 16 variants of the interference patterns corresponding to different combinations of the relative phase values $\varphi_{31} = 0^\circ$, ±90°, ±180° or parameters $\xi = 1$, ±1, −1 and $\xi' = 1$, ±1, −1. To illustrate possible manifestations of the $a_0^0(980) \to f_0(980)$ mixing effect, we chose 4 of them with $(\xi, \xi') = (i, 1), (−1, 1), (1, i)$, and $(1, −1)$. The solid curves in Figs. 4(a) and 4(c) show the $\eta\pi^0$ mass spectra,

$$\frac{dN_{\eta\pi^0}}{dm_{\eta\pi^0}} = \int_{a_{−}(s)}^{a_{+}(s)} |A_{D^+_s \to \eta\pi^0\pi^+}|^2 m_{\eta\pi^0} dm_{\eta\pi^0}^2,$$

calculated with the use of Eqs. (5), (6), (11)–(15). The corresponding distributions of the Monte Carlo events (~ $|A_{D^+_s \to \eta\pi^0\pi^+}|^2$) in the $s$–$t$ Dalitz plots are shown in Figs. 4(b) and 4(d). The variant represented in Figs. 4(a) and 4(b) corresponds to combination $(\xi, \xi') = (i, 1)$ for which the influence of the $a_0^0(980) \to f_0(980)$ mixing seems most appreciable. The variant represented in Figs. 4(c) and 4(d) corresponds to combination $(\xi, \xi') = (−1, 1)$. In this case, the $\eta\pi^0$ mass spectrum demonstrates a small narrow peak located on the smooth background in the region of the $KK$ thresholds [see Fig. 4(c)]. Nevertheless, the asymmetry effect is clearly visible in the Dalitz plot [see Fig. 4(d)] (though it almost collapses in the $\eta\pi^0$ projection). The mass spectra $dN_{\eta\pi^0}/dm_{\eta\pi^0}$ in the $a_0^0(980)$ resonance region are presented in more detail in Fig 5 for variants with $(\xi, \xi') = (i, −1), (−1, 1), (1, i)$, and $(1, −1)$.

The dotted curves in Figs. 4(a), 4(c), and 5 correspond to the mass spectra $dN_{\eta\pi^0}/dm_{\eta\pi^0}$ without the contribution of the amplitude $A_{f0\eta\pi^0}$. Note that the asymmetry in the $s$–$t$ Dalitz plot distributions relative to the $t = s$ line (see Fig. 4) manifests itself in all considered 16 variants.

Detecting signs of the $D^+_s \to [a_0^0(980)\pi^0]^+ \to \eta\pi^0\pi^+$ decay mechanisms is one of the interesting problems both for the weak hadronic decay physics of the $D^+_s$ meson and for the physics of the light scalar $a_0^0(980)$ and $f_0(980)$ mesons. At present, intensive investigations in these lines are realized by the LHCb, BaBar, CLEO, Belle, and BESIII Collaborations (see, for example, recent reviews [18, 24, 26]).

**III. CONCLUSION AND DISCUSSION**

Light meson spectroscopy from hadronic charm meson decays (in particular, study of the $a_0^0(980)$ and $f_0(980)$ resonances) is one of the main lines of the LHCb program on charm physics [24, 25]. It is hoped that the
measurements of the $D_s^+$ meson decays with huge statistics, really reachable at LHCb, will allow us to reveal the isospin breaking effect caused by the $a_0^0(980) - f_0(980)$ mixing in the $D_s^+ \rightarrow \eta \pi^0 \pi^+$ channel and obtain new information on the production mechanisms and nature of the light scalar mesons.

Note that the investigations of the $a_0^0(980) - f_0(980)$ mixing in three-body decays of the $D^0$ meson, such as $D^0 \rightarrow K^0_S \pi^+ \pi^-$, $D^0 \rightarrow K^0_S \eta \pi^0$, $D^0 \rightarrow K^- K^+ K^+$, $D^0 \rightarrow K^+ K^- \pi^0$, and $D^0 \rightarrow \pi^+ \pi^- \pi^0$, are also promising and interesting. These decays differ appreciably from those of the $D_s^+$ meson. We hope to present detailed estimates for the case of the $D^0$ decays elsewhere in the near future.

Note also that the $a_0^0(980) - f_0(980)$ mixing in the semileptonic decays $D_s^+ \rightarrow [\pi^0 \eta, \pi \pi, \pi^0 \nu$ has been discussed recently in Ref. [27].

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APPENDIX: PROPAGATORS AND $a_0^0(980) - f_0(980)$ MIXING AMPLITUDE

The inverse propagator of the $\rho^+$ meson in Eq. (5) is

$$D_{\rho^+}(u) = m_{\rho^+}^2 - u - i \sqrt{u} \Gamma_{\rho}(u),$$

where $\Gamma_{\rho}(u) = (m_{\rho^+}^2/u) \Gamma_{\rho}(q(u)/q(m_{\rho}^2)^3 F^2(u), F^2(u) = [1 + r^2 g^2(m_{\rho}^2)]/[1 + r^2 g^2(u)], r_{\rho} = 5 \text{ GeV}^{-1}, q(u) = \sqrt{u - 4m_{\rho}^2}/2, m_{\rho} = 0.775 \text{ GeV, } \Gamma_{\rho} = 0.148 \text{ GeV, } g_{\rho\rho\pi}/(4\pi) = 2.8 \text{ GeV}.$$

The $a_0^0(980) - f_0(980)$ mixing amplitude in Eq. (6), caused by the diagrams shown in Fig. 6, has the form

$$\Pi_{a_0^0 f_0}(s) = \frac{g_0 K K - g_{f_0 K K}}{16 \pi} \left[ i \left( \rho_{K K} - \frac{s}{s} \right) \right].$$
where \( s \) (the square of the invariant virtual mass of scalar resonances) \( \geq 4m_K^2 \) and \( \rho_K \Sigma(s) = \sqrt{1 - 4m_K^2/s} \); in the region \( 0 \leq s \leq 4m_K^2 \), \( \rho_K \Sigma(s) \) should be replaced by \( i/\rho_K \Sigma(s) \). The modulus and the phase of \( \Pi_{a_0}^{ab} \) (980) mixing have been taken into account in Eq. (6) for the mixing amplitude \( A_{f_0a} \). [1,3]

\[
-D_r(s) = m_r^2 - s + \sum_{ab} |\text{Re}\Pi_r^{ab}(m_r^2) - \Pi_r^{ab}(s)|,
\]

where \( \Pi_r^{ab}(s) \) stands for the diagonal matrix element of the polarization operator of the resonance \( r \) corresponding to the contribution of the \( ab \) intermediate state [28].

At \( s > (m_a + m_b)^2 \),

\[
\Pi_r^{ab}(s) = \frac{g_{r}^2}{16\pi} \left[ \frac{m_{ab}^+ m_{ab}^-}{\pi s} \ln \frac{m_b}{m_a} + \rho_{ab}(s) \right] \times \left( i - \frac{1}{\pi} \ln \frac{s - m_{ab}^+ m_{ab}^-}{s - m_{ab}^+ m_{ab}^-} \right),
\]

where \( g_{r}^2 \) is the coupling constant of \( r \) with \( ab \),

\[
\rho_{ab}(s) = \sqrt{s - m_{ab}^+ m_{ab}^-} \frac{s - m_{ab}^+ m_{ab}^-}{s - m_{ab}^+ m_{ab}^-}, \quad m_{ab} = m_a \pm m_b,
\]

and \( m_a \geq m_b \). \( \text{Im}\Pi_r^{ab}(s) = \sqrt{s - m_{ab}^+ m_{ab}^-} \rho_{ab}(s) \). At \( m_{ab}^+ m_{ab}^- < s < m_{ab}^{-1} \),

\[
\Pi_r^{ab}(s) = \frac{g_{r}^2}{16\pi} \left[ \frac{m_{ab}^+ m_{ab}^-}{\pi s} \ln \frac{m_b}{m_a} \rho_{ab}(s) \left( 1 - \frac{2}{\pi} \arctan \sqrt{\frac{m_{ab}^+ m_{ab}^-}{s - m_{ab}^+ m_{ab}^-}} \right) \right],
\]

where \( \rho_{ab}(s) = \sqrt{m_{ab}^+ m_{ab}^-} \). At \( s \leq m_{ab}^{-1} \),

\[
\Pi_r^{ab}(s) = \frac{g_{r}^2}{16\pi} \left[ \frac{m_{ab}^+ m_{ab}^-}{\pi s} \ln \frac{m_b}{m_a} \right] + \rho_{ab}(s) \left( 1 - \frac{2}{\pi} \arctan \sqrt{\frac{m_{ab}^+ m_{ab}^-}{s - m_{ab}^+ m_{ab}^-}} \right),
\]

where \( \rho_{ab}(s) = \sqrt{m_{ab}^+ m_{ab}^-} \).

The propagators \( 1/D_{a_0}^{ab}(s) \) and \( 1/D_{f_0}^{ab}(s) \) constructed with taking into account the finite width corrections [see Eqs. (19)–(22)] satisfy the Källén-Lehman representation in the wide domain of coupling constants of the scalar mesons with two-particle states and, due to this fact, provide the normalization of the total decay probability to unity: \( \sum_{ab} BR(r \to ab) = 1 \) [29].

Here we use the numerical estimates of the coupling constants \( g_{f_0}^{2a_0}/(16\pi) \) and \( g_{a_0}^{2f_0}/(16\pi) \) obtained in Ref. [2]

\[
\frac{g_{f_0}^{2a_0}}{16\pi} = \frac{3 g_{f_0}^2}{2} \frac{g_{f_0}^2}{16\pi} = 0.098 \text{ GeV}^2,
\]

\[
\frac{g_{a_0}^{2f_0}}{16\pi} = \frac{2 g_{a_0}^2}{16\pi} = 0.4 \text{ GeV}^2,
\]

\[
\frac{g_{a_0}^{2f_0}}{16\pi} = \frac{2 g_{a_0}^2}{16\pi} = 0.5 \text{ GeV}^2.
\]

As in Ref. [2], we fix \( m_{a_0}^0 = 0.985 \text{ GeV}, m_{f_0} = 0.985 \text{ GeV} \) and set \( g_{a_0}^{2f_0}/g_{f_0}^2 = g_{a_0}^{2a_0}/g_{f_0}^2 = g_{a_0}^{2f_0}/g_{f_0}^2 \) by the \( q^2q^2 \) model, see, e.g., Refs. [2, 30].

[1] N. N. Achasov, S. A. Devyanyan, and G. N. Shestakov, Phys. Lett. B 88, 367 (1979).

[2] N. N. Achasov, S. A. Devyanyan, and G. N. Shestakov, Yad. Fiz. 33, 1337 (1981) [Sov. J. Nucl. Phys. 33, 715]
The sharp jump of the phase suggests an idea to study the amplitude between the \( K^+K^- \) and \( K^0\bar{K}^0 \) thresholds.

The points with error bars in this plot correspond to the data of the BaBar \cite{15} and CLEO \cite{19} assumption of the \( K^0\bar{K}^0 \) dominance in the decay \( D_s^+ \rightarrow K^+K^-\pi^+ \). The situation requires clarification. By our estimate, the branching ratio \( BR(D_s^+ \rightarrow f_0(980)\pi^+ \rightarrow \pi^+\pi^-\pi^+) \approx 3.4\% \) is in agreement with Eq. (1) and the curve in Fig. 2. Let us emphasize once again that the detection of the \( a^0_0(980) \) mixing can provide a unique opportunity to clarify the puzzling mechanisms of the \( f_0(980) \) and \( a^0_0(980) \) production and decay in the three-body hadronic decays of the \( D_s^+ \) meson. The branching ratio of the \( f_0(980) \) resonance decay into \( \pi^+\pi^- \) is larger than into \( K^+K^- \) because of the phase space factor of the \( K^+K^- \) system that suppresses the \( f_0(980) \) to \( K^+K^- \) decay near the threshold.

Note that the direct production of the \( (a_0^0(980)\pi^+) \) system via the \( D_s^+ \rightarrow W^+ \rightarrow (a_0^0(980)\pi^+ + a_0^0(980)\pi^+) \) transition is impossible, i.e., the matrix element of the isovector axial vector current \( \langle 0|j_{a^*}^{(1)}|a_0(980)\pi^+ \rangle = 0 \). The matter is that the isovector axial vector current has the negative \( G \) parity in Standard Model, whereas the \( G \) parity of the \( (a^0(980)\pi^+) \) system is positive. The final \( (a_0(980)\pi^+) \) system is produced in the weak \( D_s^+ \) decay in the state with isospin \( f = 1 \). This implies that the effective interaction Lagrangian responsible for the transition \( D_s^+ \rightarrow (a_0(980)\pi^+) \) is proportional to \( (a_0^0(980)\pi^+ - a_0^0(980)\pi^+) \). Thus, the amplitude \( A_{a^0\pi^+} \), unlike \( A_{a^0\pi^0} \), is antisymmetric with respect to permutation of the \( s \) and \( t \) variables.