A priori error analysis for a mixed VEM discretization of the spectral problem for the Laplacian operator.

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Abstract

The aim of the present work is to derive error estimates for the Laplace eigenvalue problem in mixed form, by means of a virtual element method. With the aid of the theory for non-compact operators, we prove that the proposed method is spurious free and convergent. We prove optimal order error estimates for the eigenvalues and eigenfunctions. Finally, we report numerical tests to confirm the theoretical results together with a rigorous computational analysis of the effects of the stabilization in the computation of the spectrum.

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1. Introduction

In the recent years, the virtual element method (VEM), which is a generalization of the classic finite element method to polygonal meshes, has shown important breakthroughs in the numerical resolution of partial differential equations.

The eigenvalue problems are a subject of study where the classical numerical methods provided by the finite element method (FEM) has been plenty developed in different contexts, as for example, acoustic interactions, elastoacoustic problems, elasticity problems, vibrations of structures, fluid stability, etc. Due the importance of knowing the natural vibration frequencies of the mentioned problems, it is relevant to have numerical tools that improve the accuracy in the approximation of solutions, with reduced computational costs. Is in this sense where the VEM presents important features in comparison with FEM, that makes it attractive for mathematicians and engineers.

A priori error estimates for spectral problems implementing VEM have been developed in the past years, with important results. We mention \cite{8, 14, 17, 18, 23, 26, 28, 29, 30}, only to mention a few. Although, in the literature is possible to find several studies on the implementation of VEM methods for mixed formulations like \cite{2, 3, 7, 12, 13, 19, 20}. On the other hand, the first work related to spectral problems with mixed formulations is presented in \cite{27} where a VEM for the mixed formulation of the Laplace eigenvalue problem has been analyzed. For the analysis, the authors lie in the well developed theory of \cite{10} and take advantage of the compact solution operator in order to obtain convergence of the eigenvalues and eigenfunctions of the Laplace eigenproblem, and therefore, error estimates, using the classic theory of \cite{3}. Moreover, in this references have
analyzed as VEM for virtual spaces BDM-type, where the local spaces are defined for polynomial of degree \( k \geq 1 \), which have an additional cost compared with the VEM spaces similar to Raviart-Thomas elements (i.e., for \( k \geq 0 \)).

We are interested in mixed formulations for eigenvalue problems and as cornerstone for more challenging mixed formulations, we begin with the mixed formulation for the Laplace eigenvalue problem in two dimensions. In one hand, we present a rigorous mathematical analysis for the proposed VEM method, which is based on the general theory of non-compact operators of [15] in first place, in order to prove convergence of our method. The error estimates for the eigenfunctions and eigenvalues will be derived by adapting the results of [16] for the VEM framework and the VEM spaces that we will analyze are of the Raviart-Thomas-type, where the cost of implementation is less than the BDM-type spaces. With this choice of VEM spaces, we will prove that our method is convergent, spurious free and delivers the optimal double order of convergence for the eigenvalues.

On the other hand, it is well known from the literature that some numerical methods that depend on some particular stabilizations may introduce spurious eigenvalues for certain choices of this parameter. Recently for DG methods based in interior penalization, applied in spectral problems, this phenomena has been studied in [23, 24] and also for VEM methods in [8, 29]. Since our mixed formulation also depends on a stabilization, which is intrinsic in the VEM framework, we will also study from a numerical point of view how this stabilization affects the computation of the spectrum in different polygonal meshes in order to obtain a threshold in which our method works perfectly.

Also, we will discuss our proposed VEM for a more general Laplace eigenproblem, where the boundary can be split in two parts: a Dirichlet and Neumann boundary. This mixed boundary conditions are relevant for our purposes, since the regularity of the solution is clearly affected for this nature of the split boundary and hence, the computation of the spectrum may introduce spurious modes, which needs to be controlled by means of the stabilization term of our VEM.

The paper is organized as follows: In section 2 we present the Laplace eigenvalue problem, the mixed formulation for the problem, and we recall important properties of this problem, as the spectral characterization and regularity results. In section 3 we introduce the standard hypothesis for the mesh that the VEM framework requires, the virtual spaces, degrees of freedom and hence, the discrete bilinear forms that are considered for the discrete mixed formulation. Finally, in section 5 we report some numerical tests that illustrates the performance of the method and confirms the theoretical results obtained in the previous sections together with a computational analysis of the effects of the stabilization in the computation of the spectrum in a domain with mixed boundary conditions.

2. The spectral problem

Let \( \Omega \subset \mathbb{R}^2 \) be an open bounded domain with Lipschitz boundary \( \Gamma \). The Laplace eigenvalue problem reads as follows:

**Problem 1.** Find \( (\lambda, u) \in \mathbb{R} \times H^1(\Omega), u \neq 0 \), such that

\[
\begin{align*}
-\Delta u &= \lambda u & \text{in } \Omega, \\
n &= 0 & \text{on } \Gamma.
\end{align*}
\] (2.1)

In order to obtain a mixed variational formulation of (2.1), we introduce the additional unknown \( \sigma = \nabla u \). Then, replacing this new unknown in (2.1), multiplying with suitable test functions, integrating by parts, and using the boundary condition, we obtain the following equivalent mixed weak formulation:
Problem 2. Find \((\lambda, \sigma, u) \in \mathbb{R} \times \mathrm{H}(\text{div}, \Omega) \times L^2(\Omega), \; (\sigma, u) \neq (0, 0)\), such that

\[
\int_{\Omega} \sigma \cdot \tau + \int_{\Omega} \text{div} \tau u = 0 \quad \forall \tau \in \text{H}(\text{div}, \Omega),
\]
\[
\int_{\Omega} \text{div} \sigma v = -\lambda \int_{\Omega} uv \quad \forall v \in L^2(\Omega).
\]

We define the spaces \(V := \text{H}(\text{div}, \Omega)\) and \(Q = L^2(\Omega)\). Let us remark that these spaces will be endowed with the usual norms which we denote by \(\| \cdot \|_V\) and \(\| \cdot \|_Q\), respectively, and the product space \(V \times Q\) will be endowed with the natural norm of product spaces which we denote by \(\| \cdot \|_{V \times Q}\).

With these definitions at hand, we introduce the bilinear forms \(a : V \times V \to \mathbb{R}\) and \(b : V \times Q \to \mathbb{R}\), defined as follows

\[
a(\sigma, \tau) := \int_{\Omega} \sigma \cdot \tau, \quad \sigma, \tau \in V,
\]
\[
b(\tau, v) := \int_{\Omega} v \text{div} \tau, \quad \tau \in V, \; v \in Q.
\]

Then, if \((\cdot, \cdot)_Q\) denotes the usual \(Q\) inner-product, we rewrite Problem 2 as follows:

Problem 3. Find \((\lambda, \sigma, u) \in \mathbb{R} \times V \times Q, \; (\sigma, u) \neq (0, 0)\), such that

\[
a(\sigma, \tau) + b(\tau, u) = 0 \quad \forall \tau \in V,
\]
\[
b(\sigma, v) = -\lambda(u, v)_Q \quad \forall v \in Q.
\]

We remark that each of the previous bilinear forms are bounded and symmetric.

Let \(K\) be the kernel of bilinear form \(b(\cdot, \cdot)\) defined as follows:

\[
K := \{ \tau \in V : b(\tau, v) = 0 \quad \forall v \in Q\} = \{ \tau \in V : \text{div} \tau = 0 \text{ in } \Omega\}.
\]

It is well-known that bilinear form \(a(\cdot, \cdot)\) is elliptic in \(K\) and that \(b(\cdot, \cdot)\) satisfies the following inf-sup condition (see \[11\])

\[
\sup_{0 \neq \tau \in V} \frac{b(\tau, v)}{\|\tau\|_V} \geq \beta \|v\|_Q \quad \forall v \in Q,
\]

where \(\beta\) is a positive constant.

Remark 2.1. The eigenvalues of Problem 3 are positive. Indeed, taking \(\tau = \sigma\) and \(v = u\) in Problem 3 and adding the resulting forms, we have obtain

\[
\lambda = \frac{a(\sigma, \sigma)}{\|u\|_Q^2} \geq 0.
\]

In addition, \(\lambda = 0\) implies \((\sigma, u) = (0, 0)\).

To analyze Problem 3, we introduce the following linear solution operator \(T\)

\[
T : Q \rightarrow Q,
\]
\[
f \mapsto Tf := \tilde{u},
\]

where \((\tilde{\sigma}, \tilde{u}) \in V \times Q\) is the solution of the corresponding source problem:

\[
\left\{
\begin{array}{ll}
a(\tilde{\sigma}, \tau) + b(\tau, \tilde{u}) & = 0 \quad \forall \tau \in V, \\
b(\tilde{\sigma}, v) & = -(f, v)_Q \quad \forall v \in Q,
\end{array}
\right.
\]

(2.3)
which is the variational formulation of the following problem

\[
\begin{align*}
\tilde{\sigma} &= \nabla \tilde{u} \quad \text{in } \Omega, \\
\text{div} \tilde{\sigma} &= -f \quad \text{in } \Omega, \\
\tilde{u} &= 0 \quad \text{on } \Gamma.
\end{align*}
\]  
(2.4)

From the fact that \( a(\cdot, \cdot) \) is \( K \)-elliptic and (2.2), it is well known that problem (2.3) admits a unique solution \((\tilde{\sigma}, \tilde{u}) \in V \times Q\) and there exists a positive constant \(C\) such that

\[
\| (\tilde{\sigma}, \tilde{u}) \|_{V \times Q} \leq C \| f \|_Q.
\]  
(2.5)

As a consequence, we have that \(T\) is well defined, self-adjoint with respect to \((\cdot, \cdot)_Q\) and compact. Moreover, if \((\lambda, (\sigma, u)) \in \mathbb{R} \times V \times Q\) solves Problem (2.3) if and only if \((1/\lambda, u)\) is an eigenpair of \(T\), i.e., if

\[Tu = \mu u, \quad \text{with } \mu := \frac{1}{\lambda}.
\]

According to [1], the regularity for the solution of (2.3) is the following: there exists a constant \(r > 1/2\) depending on \(\Omega\) such that the solution \(\tilde{u} \in H^{1+r}(\Omega)\), where \(r\) is at least 1 if \(\Omega\) is convex and \(r\) is at least \(\pi/\omega - \varepsilon\), for any \(\varepsilon > 0\) for a non-convex domain, with \(\omega < 2\pi\) being the largest reentrant angle of \(\Omega\).

Hence we have the following additional regularity result for the solution of problem (2.3).

**Lemma 2.1.** There exist a positive constant \(C\) such that

\[
\| \sigma \|_{r, \Omega} + \| u \|_{1+r} \leq C \| f \|_Q.
\]

On the other hand, since \(T\) is a self-adjoint compact operator, we have the following spectral characterization result (see [3]).

**Lemma 2.2.** The spectrum of \(T\) satisfies \(\text{sp}(T) = \{0\} \cup \{\mu_n : n \in \mathbb{N}\}\), where \(\{\mu_n\}_{n \in \mathbb{N}}\) is a sequence of positive eigenvalues which converge to zero with the multiplicity of each non-zero eigenvalue being finite. In addition, the following additional regularity result holds true for eigenfunctions

\[
\| \sigma \|_{r, \Omega} + \| u \|_{1+r} \leq C \| u \|_Q,
\]

with \(\bar{r} > 1/2\) and \(C > 0\) depending on the eigenvalue.

Now we are in position to introduce our approximation scheme.

3. The virtual element method

3.1. Mesh assumptions and virtual spaces

We begin this section establishing the framework in which we will operate. The VEM method needs particular assumptions for the construction of the meshes, which are well established in [4]. Let \(\{T_h\}\) be a family of decompositions of \(\Omega\) into polygons \(K\). Let \(h_K\) denote the diameter of the element \(K\) and \(h := \max_{K \in \Omega} h_K\).

For the analysis, we make the following assumptions on the meshes as in [5, 9]: there exists a positive real number \(C_T\) such that, for every \(K \in T_h\) and for every \(T_h\),
• \(A_1\): the ratio between the shortest edge and the diameter of \(K\) is larger than \(C_T\);

• \(A_2\): \(K\) is star-shaped with respect to every point of a ball of radius \(C_T h_K\).

For any subset \(S \subseteq \mathbb{R}^2\) and any non-negative integer \(k\), we indicate by \(P_k(S)\) the space of polynomials of degree up to \(k\) defined on \(S\). To keep the notation simpler, we denote by \(n\) a generic normal unit vector; its precise definition will be clear from the context.

We consider now a polygon \(K\) and, for any fixed non-negative integer \(k\), we define the following finite dimensional space (inspired in [9, 5]):

\[
\mathcal{V}_K^h := \left\{ \tau_h \in H(\text{div}; K) : (\tau_h \cdot n) \in P_k(e) \quad \forall e \subset \partial K, \quad \text{div} \tau_h \in P_k(K), \quad \text{rot} \tau_h = 0 \text{ in } K \right\}.
\]

We define the following degrees of freedom for functions \(\tau_h\) in \(\mathcal{V}_K^h\):

\[
\left( \int_e (\tau_h \cdot n) q ds \quad \forall q \in P_k(e), \quad \forall \text{ edge } e \subset \partial K, \right) \tag{3.1}
\]
\[
\left( \int_K \tau_h \cdot \nabla q \quad \forall q \in P_k(K)/P_0(K). \right) \tag{3.2}
\]

These degrees of freedom are unisolvent, as is stated in [8, Proposition 1].

For each decomposition \(T_h\) of \(\Omega\) into polygons \(K\), we define

\[
\mathcal{V}_h := \left\{ \tau_h \in H(\text{div}, \Omega) : \tau_h|_K \in \mathcal{V}_K^h \right\}.
\]

In agreement with the local choice, we choose the following global degrees of freedom:

\[
\left( \int_e (\tau_h \cdot n) q ds \quad \forall q \in P_k(e), \quad \text{for each internal edge } e \not\subset \Gamma, \right)
\]
\[
\left( \int_K \tau_h \cdot \nabla q \quad \forall q \in P_k(K)/P_0(K), \quad \text{for each element } K \subset T_h. \right)
\]

Additionally we introduce the following finite dimensional space:

\[
\mathcal{Q}_h := \{ v_h \in \mathcal{Q} : v_h|_K \in P_k(K), \quad \forall K \subset T_h \}.
\]

As is customary in the VEM framework, the bilinear forms \(a(\cdot, \cdot)\) and \(b(\cdot, \cdot)\) are written elementwise as follows

\[
a(\sigma, \tau) = \sum_{K \in T_h} a^K(\sigma, \tau) = \sum_{K \in T_h} \int_K \sigma \cdot \tau \quad \sigma, \tau \in \mathcal{V}_h,
\]
\[
b(\sigma, v) = \sum_{K \in T_h} b^K(\sigma, v) = \sum_{K \in T_h} \int_K v \text{ div } \sigma \quad \sigma \in \mathcal{V}, \quad v \in \mathcal{Q}.
\]

Observe that with the degrees of freedom that we are operating, \(a^K(\cdot, \cdot)\) is not explicitly computable, contrary of \(b(\cdot, \cdot)\). For this reason we need to introduce a projection operator to circumvent this drawback.

First, we define for each polygon \(K\) the space

\[
\hat{\mathcal{V}}_h^K := \nabla(P_{k+1}(K)) \subset \mathcal{V}_h^K.
\]
Then, we define the \([L^2(\Omega)]^2\)-orthogonal projector \(\Pi^K_h : [L^2(\Omega)]^2 \to V_h^K\) by
\[
\int_K \Pi^K_h \tau \cdot \tilde{u}_h = \int_K \tau \cdot \tilde{u}_h \quad \forall \tilde{u}_h \in V_h^K.
\]

Let \(S^K(\cdot, \cdot)\) be any symmetric positive definite (and computable) bilinear form to be chosen as to satisfy
\[
c_0 a^K(\tau_h, \tau_h) \leq S^K(\tau_h, \tau_h) \leq c_1 a^K(\tau_h, \tau_h) \quad \forall \tau_h \in \tau^K_h,
\]
for some positive constants \(c_0\) and \(c_1\) depending only on the constant \(C_T\) from mesh assumptions \(A_1\) and \(A_2\). Then, we define on each element \(K\) the bilinear form
\[
a^K_h(\sigma_h, \tau_h) := \int_K \Pi^K_h \sigma_h \cdot \Pi^K_h \tau_h + S^K(\sigma_h - \Pi^K_h \sigma_h, \tau_h - \Pi^K_h \tau_h), \quad \sigma_h, \tau_h \in V_h^K,
\]
and, in a natural way,
\[
a_h(\sigma, \tau) := \sum_{K \in T_h} a^K_h(\sigma_h, \tau_h), \quad \sigma_h, \tau_h \in V_h.
\]

The following two properties of the bilinear form \(a^K_h(\cdot, \cdot)\) are easily derived by repeating in our case the arguments from \[\text{Proposition 4.1}].

- **Consistency:**
  \[
a^K_h(u_h, \tau_h) = \int_K u_h \cdot \tau_h \quad \forall u_h \in V^K_h, \quad \forall \tau_h \in V^K_h, \quad \forall K \in T_h.
\]

- **Stability:** There exist two positive constants \(\alpha_*\) and \(\alpha^*\), independent of \(K\), such that:
  \[
  \alpha_* \int_K \tau_h \cdot \tau_h \leq a^K_h(\tau_h, \tau_h) \leq \alpha^* \int_K \tau_h \cdot \tau_h \quad \forall \tau_h \in V^K_h, \quad \forall E \in T_h.
  \]

Now we are in a position to introduce the virtual element discretization of Problem 3.

### 3.2. The discrete eigenvalue problem

With the VEM spaces and degrees of freedom defined above, we introduce the discretization of Problem 3 as follows

**Problem 4.** Find \((\lambda_h, \sigma_h, u_h) \in \mathbb{R} \times V_h \times Q_h, (\sigma_h, u_h) \neq (0, 0)\), such that
\[
\begin{align*}
a_h(\sigma, \tau) + b(\tau, u) & = 0 \quad \forall \tau \in V_h, \\
b(\sigma, v) & = \lambda_h(u, v) \quad \forall v \in Q_h.
\end{align*}
\]

Let \(K_h\) be the discrete kernel of bilinear form \(b(\cdot, \cdot)\) defined as follows:
\[
K_h := \{ \tau_h \in V_h : b(\tau_h, v_h) = 0 \quad \forall v_h \in Q_h \}.
\]

We observe that by virtue of (3.4), the bilinear form \(a_h(\cdot, \cdot)\) is bounded. Moreover, as is shown in the following lemma, it is also uniformly elliptic.

**Lemma 3.1.** There exists a constant \(\beta > 0\), independent of \(h\), such that
\[
a_h(\tau_h, \tau_h) \geq \beta \|\tau_h\|_V^2 \quad \forall \tau_h \in K_h.
\]
Proof. Thanks to (3.4), the above inequality holds with $\alpha := \min \{\alpha_*, 1\}$. \hfill \Box

Also, the following discrete inf-sup condition holds.

**Lemma 3.2.** There exists $\hat{\beta} > 0$, independent of $h$, such that
\[
\sup_{v \neq \tau_h \in V_h} \frac{b(\tau_h, v_h)}{\|\tau_h\|_{V_h}} \geq \hat{\beta}\|v_h\|_{Q_h} \quad \forall v_h \in Q_h.
\]

**Proof.** The proof is straightforward by adapting the arguments of [12, Lemma 5.3]. \hfill \Box

The next step is to introduce the discrete version of the operator $T$:
\[
T_h : Q_h \rightarrow Q_h,
\]
\[
f_h \mapsto T_h f_h := \tilde{u}_h,
\]
where $(\tilde{\sigma}_h, \tilde{u}_h) \in V_h \times Q_h$ is the solution of the corresponding discrete source problem:
\[
\begin{aligned}
& a_h(\tilde{\sigma}_h, \tau_h) + b(\tau_h, \tilde{u}_h) = 0 \quad \forall \tau_h \in V_h, \\
& -b(\tilde{\sigma}_h, v_h) = (f_h, v_h)_{Q} \quad \forall v_h \in Q_h.
\end{aligned}
\]

In what follows, we state some auxiliary results about the approximation properties of this interpolant (see [8]). The first one concerns approximation properties of $\text{div} \psi_I$ and follows from a commuting diagram property for this interpolant, which involves the $L^2(\Omega)$-orthogonal projection $P_k : L^2(\Omega) \rightarrow \{q \in L^2(\Omega) : q|_K \in P_k(K) \quad \forall K \in T_h \}$.

For $P_k$ we have the following approximation estimate (see [8]): if $0 \leq s \leq k + 1$, it holds
\[
\|v - P_k(v)\|_{0,\Omega} \leq C h^s \|v\|_{s,\Omega} \quad \forall s \in H^s(\Omega) \cap Q.
\]

**Lemma 3.3.** Let $\tau \in \mathcal{V}$ be such that $\tau \in [H^t(\Omega)]^2$ with $t > 1/2$. Let $\tau_I \in V_h$ be its interpolant defined by (3.1)–(3.2). Then,
\[
\text{div} \psi_I = P_k(\text{div} \tau) \quad \text{in} \ \Omega.
\]
Consequently, for all $K \in T_h$, $\|\text{div} \psi_I\|_{0,K} \leq \|\text{div} \tau\|_{0,K}$ and, if $\text{div} \tau|_K \in H^r(K)$ with $r \geq 0$, then
\[
\|\text{div} \tau - \text{div} \psi_I\|_{0,K} \leq Ch_k^{\min\{r,k+1\}} |\text{div} \tau|_{r,K}.
\]

**Proof.** See [8, Appendix]. \hfill \Box

The second result concerns the $L^2(\Omega)$ approximation property of $\tau_I$.

**Lemma 3.4.** Let $\tau \in \mathcal{V}$ be such that $\tau \in [H^t(\Omega)]^2$ with $t > 1/2$. Let $\tau_I \in V_h$ be its interpolant defined by (3.1)–(3.2). Let $K \in T_h$. If $1 \leq t \leq k + 1$, then
\[
\|\tau - \tau_I\|_{0,K} \leq Ch_k^{t} |\tau|_{t,K},
\]
whereas, if $1/2 < t \leq 1$, then
\[
\|\tau - \tau_I\|_{0,K} \leq C \left( h_k^{t} |\tau|_{t,K} + h_K \|\text{div} \tau\|_{0,K} \right).
\]

**Proof.** See [8, Appendix]. \hfill \Box
The end this section by recalling the following technical result.

**Lemma 3.5.** There exists a constant $C > 0$ such that, for every $p \in H^{1+t}(\Omega)$ with $1/2 < t \leq k+1$, there holds
\[
\|\nabla p - \Pi_h(\nabla p)\|_{0,\Omega} \leq Ch^t \|\nabla p\|_{t,\Omega},
\]
where $(\Pi_h v)|_K := \Pi^K_h(v|_K)$ for all $K \in T_h$.

**Proof.** See [8, Lemma 8].

\section{Spectral approximation}

In what follows, we will prove that convergence properties for the numerical method proposed in Section 3. We begin this section by recalling some definitions of spectral theory.

Let $X$ be a generic Hilbert space and let $S$ be a linear bounded operator defined by $S : X \to X$. If $I$ represents the identity operator, the spectrum of $S$ is defined by $\text{sp}(S) := \{z \in \mathbb{C} : (zI - S) \text{ is not invertible}\}$ and the resolvent is its complement $\rho(S) := \mathbb{C} \setminus \text{sp}(S)$. For any $z \in \rho(S)$, we define the resolvent operator of $S$ corresponding to $z$ by $R_z(S) := (zI - S)^{-1} : X \to X$.

Despite to the fact that $T$ is compact, since the discrete solution operator is defined from $Q_h$ onto itself, the non-compact theory of [15] is suitable for this setting.

We introduce the following definition
\[
\|T\|_h := \sup_{0 \neq f_h \in Q_h} \frac{\|Tf_h\|_Q}{\|f_h\|_Q}.
\]

No we recall properties P1 and P2 of [15].

- **P1:** $\|T - T_h\|_h \to 0$ as $h \to 0$;
- **P2:** $\forall \tau \in Q_h, \inf_{\tau_h \in Q_h} \|\tau - \tau_h\|_Q \to 0$ as $h \to 0$.

Our task consists into prove properties P1 and P2 in order to ensure the spectral convergence. We observe that P2 is an immediate consequence from the fact that the smooth functions are dense in $Q$. Hence, only remains to prove property P1.

**Lemma 4.1.** There exists $C > 0$ such that, for all $f_h \in Q_h$, if $\tilde{u} = Tf_h$ and $\tilde{u}_h = T_hf_h$, then
\[
\|(T - T_h)f_h\|_h = \|\tilde{u} - \tilde{u}_h\|_h \leq Ch^r.
\]

**Proof.** Let $f \in Q_h$ such that $\tilde{u} = Tf_h$, $\tilde{u}_h = T_hf_h$ and $\overline{\sigma}_I \in V_h$. From triangular inequality we have,
\[
\|\overline{\sigma} - \overline{\sigma}_h\|_{0,\Omega} \leq \|\overline{\sigma} - \overline{\sigma}_I\|_{0,\Omega} + \|\overline{\sigma}_I - \overline{\sigma}_h\|_{0,\Omega}.
\]
We set $\tau_h := \overline{\sigma}_I - \overline{\sigma}_h$, thanks to Lemma 3.3, equations (2.3) and (3.5), we have $\text{div} \overline{\sigma}_I =$.
Therefore, we obtain
\[ P_k(\text{div} \, \tilde{\sigma}) = f_h = \text{div} \, \tilde{\sigma}_h, \] then \( \text{div} \, \tau_h = 0 \). Hence \( \tau_h \in K_h \subset K \). Therefore, we have
\[
\alpha \| \tau_h \|^2_{0, \Omega} = \alpha \| \tau_h \|^2_{V} \leq a_h(\tilde{\sigma}_I, \tau_h) - a_h(\tilde{\sigma}_h, \tau_h) + b(\tau_h, u_h)
\]
\[
= \sum_{K \in T_h} \left[ a^K_h(\tilde{\sigma}_I - \Pi^K_h \tilde{\sigma}, \tau_h) + a^K(\Pi^K_h \tilde{\sigma} - \tilde{\sigma}, \tau_h) \right] + a(\tilde{\sigma}, \tau_h)
\]
\[
= \sum_{K \in T_h} \left[ a^K_h(\tilde{\sigma}_I - \Pi^K_h \tilde{\sigma}, \tau_h) + a^K(\Pi^K_h \tilde{\sigma} - \tilde{\sigma}, \tau_h) \right]
\]
\[
\leq C \sum_{K \in T_h} \left( \| \tilde{\sigma} - \tilde{\sigma}_I \|_{0,K} + \| \tilde{\sigma} - \Pi^K_h \tilde{\sigma} \|_{0,K} \right) \| \tau_h \|_{0,\Omega}.
\]
Therefore, we obtain
\[
\| \tilde{\sigma} - \tilde{\sigma}_h \|_{0,\Omega} \leq C \left( \sum_{K \in T_h} \left( \| \tilde{\sigma} - \tilde{\sigma}_I \|_{0,K} + \| \tilde{\sigma} - \Pi^K_h \tilde{\sigma} \|_{0,K} \right) \right).
\]
(4.1)

The next step is to control \( \| (T - T_h) f_h \|_{0,\Omega} \). Again, using triangle inequality we obtain
\[
\| (T - T_h) f_h \|_{0,\Omega} = \| \tilde{u} - \tilde{u}_h \|_{0,\Omega} \leq \| \tilde{u} - P_k(\tilde{u}) \|_{0,\Omega} + \| P_k(\tilde{u}) - \tilde{u}_h \|_{0,\Omega}.
\]
(4.2)

Now, adapting the arguments of [12, Lemma 5.3], we deduce that there exists \( \tilde{\sigma}_h \in V_h \) such that
\[
\text{div} \, \tilde{\sigma}_h = P_k(\tilde{u}) - \tilde{u}_h \quad \text{and} \quad \| \tilde{\sigma}_h \|_V \leq c \| P_k(\tilde{u}) - \tilde{u}_h \|_{0,\Omega}.
\]
(4.3)

Hence,
\[
\| P_k(\tilde{u}) - \tilde{u}_h \|^2_{0,\Omega} = \int_\Omega (P_k(\tilde{u}) - \tilde{u}_h) \text{div} \, \tilde{\sigma}_h = \int_\Omega (\tilde{u} - \tilde{u}_h) \text{div} \, \tilde{\sigma}_h
\]
\[
= b(\tilde{\sigma}_h, \tilde{u}) - b(\tilde{\sigma}_h, \tilde{u}_h) = a_h(\tilde{\sigma}_h, \tilde{\sigma}_h) - a(\tilde{\sigma}, \tilde{\sigma}_h)
\]
\[
= \sum_{K \in T_h} \left[ a^K_h(\tilde{\sigma}_h - \Pi^K_h \tilde{\sigma}, \tilde{\sigma}_h) - a^K(\tilde{\sigma} - \Pi^K_h \tilde{\sigma}, \tilde{\sigma}_h) \right]
\]
\[
\leq C \sum_{K \in T_h} \left( \| \tilde{\sigma}_h - \tilde{\sigma} \|_{0,K} + \| \tilde{\sigma} - \Pi^K_h \tilde{\sigma} \|_{0,K} \right) \| \tilde{\sigma}_h \|_V
\]

It follows of the above estimate, (4.1), (4.3) and (4.2)
\[
\| (T - T_h) f_h \|_{0,\Omega} \leq C \left( \| \tilde{u} - P_k(\tilde{u}) \|_{0,\Omega} + \sum_{K \in T_h} \left( \| \tilde{\sigma} - \tilde{\sigma}_I \|_{0,K} + \| \tilde{\sigma} - \Pi^K_h \tilde{\sigma} \|_{0,K} \right) \right)
\]
(4.4)

Now, we need to estimate the three terms on the right-hand side above. For the first term, invoking (3.6) we obtain
\[
\| \tilde{u} - P_k(\tilde{u}) \|_{0,\Omega} \leq C h^{1+\tau}_K \| \tilde{u} \|_{1+\tau,\Omega}.
\]
(4.5)

For the second term, we using Lemma 3.3 and 3.4 we have
\[
\sum_{K \in T_h} \| \tilde{\sigma} - \tilde{\sigma}_I \|_{0,K} \leq C \left( \sum_{K \in T_h} (h^K_K |\tilde{\sigma}|_{r,K} + h_K |\text{div} \, \tilde{\sigma}|_{0,K}) \right)
\]
Finally for the third term, using \( \tilde{\sigma} = \nabla (\tilde{u}) \) (see (2.4)) and Lemma 3.5 we obtain
\[
\sum_{K \in T_h} \| \tilde{\sigma} - \Pi^K_h \tilde{\sigma} \|_{0,K} = \sum_{K \in T_h} \| \nabla (\tilde{u}) - \Pi^K_h \nabla (\tilde{u}) \|_{0,K} \leq C \sum_{K \in T_h} h^K_K |\tilde{\sigma}|_{r,K}
\]
(4.6)
Substituting (4.5)–(4.6) in (4.4) and using (2.5), we have
\[
\| (T - T_h) f_h \|_{0,\Omega} \leq C h \| \tilde{u} \|_{1+r,\Omega} + \| \tilde{\sigma} \|_{r,\Omega} + \| \text{div} \tilde{\sigma} \|_{0,\Omega} \leq C h \| f_h \|_{0,\Omega}.
\]
Hence we conclude the proof.

As a consequence of P1, we have the following results (see [15, Lemma 1 and Theorem 1]).

The first of these results establishes that the discrete resolvent is bounded.

**Lemma 4.2.** Assume that P1 hold. Let \( F \subset \rho(T) \) be closed. Then, there exist positive constants \( C \) and \( h_0 \), independent of \( h \), such that for \( h < h_0 \)
\[
\sup_{v_h \in Q_h} \| R_z(T_h)v_h \|_\Omega \leq C \| v_h \|_\Omega \quad \forall z \in F.
\]

The following results establishes that the numerical method does not introduce spurious eigenvalues.

**Theorem 4.1.** Let \( U \subset \mathbb{C} \) be an open set containing \( \text{sp}(T) \). Then, there exists \( h_0 > 0 \) such that \( \text{sp}(T_h) \subset U \) for all \( h < h_0 \).

As a consequence of the previous results is that the proposed numerical method does not introduce spurious eigenvalues. Moreover, according to [15, Section 2] we have the spectral convergence of \( T_h \) to \( T \) as \( h \) goes to zero. In fact, if \( \mu \in (0,1) \) is an isolated eigenvalue of \( T \) with multiplicity \( m \) and \( \mathcal{C} \) is an open circle on the complex plane centered at \( \mu \) with boundary \( \gamma \), we have that \( \mu \) is the only eigenvalue of \( T \) lying in \( \mathcal{C} \) and \( \gamma \cap \text{sp}(T) = \emptyset \). Also, invoking [15, Section 2], we deduce that for \( h \) small enough there exist \( m \) eigenvalues \( \mu_1^h, \ldots, \mu_m^h \) of \( T_h \) (according to their respective multiplicities) that lie in \( \mathcal{C} \) and hence, the eigenvalues \( \mu_i^h, i = 1, \ldots, m \) converge to \( \mu \) as \( h \) goes to zero.

### 4.1. Error estimates

As a direct consequence of Lemma 4.1, standard results about spectral approximation (see [22], for instance) show that isolated parts of \( \text{sp}(T) \) are approximated by isolated parts of \( \text{sp}(T_h) \). More precisely, let \( \mu \in (0,1) \) be an isolated eigenvalue of \( T \) with multiplicity \( m \) and let \( \mathcal{E} \) be its associated eigenspace. Then, there exist \( m \) eigenvalues \( \mu_1^h, \ldots, \mu_m^h \) of \( T_h \) (repeated according to their respective multiplicities) which converge to \( \mu \). Let \( \mathcal{E}_h \) be the direct sum of their corresponding associated eigenspaces.

We recall the definition of the *gap* \( \tilde{\delta} \) between two closed subspaces \( \mathcal{X} \) and \( \mathcal{Y} \) of \( L^2(\Omega) \):
\[
\tilde{\delta}(\mathcal{X}, \mathcal{Y}) := \max \{ \delta(\mathcal{X}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{X}) \}, \quad \text{where} \quad \delta(\mathcal{X}, \mathcal{Y}) := \sup_{x \in \mathcal{X}: \|x\|_\mathcal{Q} = 1} \left( \inf_{y \in \mathcal{Y}} \|x - y\|_\mathcal{Q} \right).
\]

We define
\[
B_h := T_h P_k : \mathcal{Q} \rightarrow \mathcal{Q},
\]
such that \( B_h \) and \( T_h \) have the same non-zero eigenvalues and corresponding eigenfunctions.

Let \( E : \mathcal{Q} \rightarrow \mathcal{Q} \) be the spectral projector of \( T \) corresponding to the isolated eigenvalue \( \mu \), namely
\[
E := \frac{1}{2\pi i} \int_\gamma R_z(T)dz.
\]
On the other, we define $F_h : Q \rightarrow Q$ as the spectral projector of $T_h$ corresponding to the isolated eigenvalue $\mu_h$, namely

$$F_h := \frac{1}{2\pi i} \int_\gamma R_z(B_h)dz.$$

From [15, Lemma 1] we have the following result.

**Lemma 4.3.** There exist strictly positive constants $h_0$ and $C$ such that

$$\|R_z(B_h)\| \leq C \quad \forall h < h_0, \; \forall z \in \gamma.$$

The following result will be used to prove the convergence between the continuous and discrete eigenspaces.

**Lemma 4.4.** There exist positive constants $C$ and $h_0$ such that, for all $h < h_0$, the following estimates hold

$$\|(E - F_h)\| \leq C \|(T - B_h)\| \leq C h^{\min\{\tilde{r}, k\}},$$

where $\tilde{r} > 1/2$ is such that $E \subset [H^{\tilde{r}}(\Omega)]$ (cf. Lemma 2.2).

**Proof.** The first estimate is straightforward from [16, Lemma 3] and Lemma 4.2. For the second estimate we procede as follows: Let $u \in E(Q)$. Then

$$\|(T - B_h)u\| \leq \|(T - TP_k)u\| + \|(TP_k - B_h)u\|$$

$$= \|T(I - P_k)u\| + \|(T - Th)P_ku\|$$

$$\leq \|T\||I - P_k\| + \|(T - Th)\|P_ku\|$$

$$\leq Ch^{\min\{\tilde{r}, k\}} \|u\|,$$

where we have used triangular inequality, Lemma 4.1 the fact that $T$ is bounded and that $P_k$ is the $L^2$-projection. This concludes the proof.

The following error estimates for the approximation of eigenvalues and eigenfunctions hold true.

**Theorem 4.2.** There exists a strictly positive constant $C$ such that

$$\delta(F_h(Q), E(Q)) \leq C \xi_h,$$

$$|\mu - \mu_h^{(i)}| \leq C \xi_h, \quad i = 1, \ldots, m,$$

where

$$\xi_h := \sup_{f \in E(Q): \|f\|_Q = 1} \|(T - Th)f\|_Q.$$

**Proof.** The proof runs identically as in [14, Theorem 1].

The next step is to show an optimal order estimate for this term.

**Theorem 4.3.** For all $r \in (\frac{1}{2}, r_{\Omega})$, there exists a positive constant $C$ such that

$$\|(T - Th)f\|_Q \leq C h^{\min\{\tilde{r}, k\}} \|f\|_Q \quad \forall f \in \mathcal{E}$$

and, consequently,

$$\xi_h \leq C h^{\min\{\tilde{r}, k\}}.$$
Proof. The proof is identical to that of Lemma 2.1 but using now the additional regularity from Lemma 2.1.

The error estimate for the eigenvalue \( \mu \in (0, 1) \) of \( T \) leads to an analogous estimate for the approximation of the eigenvalue \( \lambda = \frac{1}{\mu} \) of Problem 2 by means of the discrete eigenvalues \( \lambda_h^{(i)} := \frac{1}{\mu_h^{(i)}}, 1 \leq i \leq m \), of Problem 4. However, the order of convergence in Theorem 4.2 is not optimal for \( \mu \) and, hence, not optimal for \( \lambda \) either. Our next goal is to improve this order.

**Theorem 4.4.** For all \( r \in \left( \frac{1}{2}, r_\Omega \right) \), there exists a strictly positive constant \( C \) such that

\[
\left| \lambda - \lambda_h^{(i)} \right| \leq C h^{2 \min \{r, k \}}.
\]

Proof. Let \( (\sigma_h, u_h) \) be such that \( (\lambda_h, (\sigma_h, u_h)) \) is a solution of Problem 4 with \( \|u_h\|_Q = 1 \). Also, according to Theorems 4.2 and 4.3, there exists a solution of Problem 2 that satisfies

\[
\|\sigma - \sigma_h\|_V + \|u - u_h\|_Q \leq C h^{\min \{r, k \}}. \tag{4.7}
\]

Let us rewrite Problems 4 and 2 as follows:

\[
A((\sigma, u); (\tau, v)) = -\lambda (u, v)_Q \quad \forall (\tau, v) \in V \times Q,
\]

\[
A_h((\sigma_h, u_h); (\tau_h, v_h)) = -\lambda (u_h, v_h)_Q \quad \forall (\tau_h, v_h) \in V_h \times Q_h,
\]

where the bilinear forms \( A : V \times Q \rightarrow \mathbb{R} \) and \( A : V_h \times Q_h \rightarrow \mathbb{R} \) are defined by

\[
A((\sigma, u); (\tau, v)) := a(\sigma, \tau) + b(\tau, u) + b(\sigma, v),
\]

and

\[
A_h((\sigma_h, u_h); (\tau_h, v_h)) := a_h(\sigma_h, \tau_h) + b(\tau_h, u_h) + b(\sigma_h, v_h).
\]

With these definitions at hand, we have

\[
A((\sigma - \sigma_h, u - u_h); (\sigma - \sigma_h, u - u_h)) + \lambda (u - u_h, u - u_h)_Q
\]

\[
= A((\sigma_h, u_h); (\sigma_h, u_h)) + \lambda (u_h, u_h)_Q
\]

\[
= A((\sigma_h, u_h); (\sigma_h, u_h)) + \lambda (u_h, u_h)_Q + \lambda (u_h, u_h)_Q - \lambda_h (u_h, u_h)_Q
\]

\[
= A((\sigma_h, u_h); (\sigma_h, u_h)) + \lambda_h (u_h, u_h)_Q + \lambda (u_h, u_h)_Q - \lambda_h (u_h, u_h)_Q
\]

\[
= A((\sigma_h, u_h); (\sigma_h, u_h)) - A_h((\sigma_h, u_h); (\sigma_h, u_h)) + (\lambda - \lambda_h)(u_h, u_h)_Q.
\]

Then, we arrive to the following identity

\[
(\lambda - \lambda_h)(u_h, u_h)_Q = A_h((\sigma_h, u_h); (\sigma_h, u_h)) - A_h((\sigma_h, u_h); (\sigma_h, u_h)) + (\lambda - \lambda_h)(u_h, u_h)_Q.
\]

The aim now is to estimate terms I and II. For I we have

\[
|I| = \left| A((\sigma - \sigma_h, u - u_h); (\sigma - \sigma_h, u - u_h)) - \lambda (u - u_h, u - u_h)_Q \right|
\]

\[
\leq \left| A((\sigma - \sigma_h, u - u_h); (\sigma - \sigma_h, u - u_h)) \right| + \left| \lambda (u - u_h, u - u_h)_Q \right|
\]

\[
\leq \|\sigma - \sigma_h\|_V^2 + \|u - u_h\|_Q^2 \leq C h^{2 \min \{r, k \}}.
\]
Let $\sigma_h \in L^2(\Omega)$ be such that $\sigma_h|_K \in P_k(E)$, for all $K \in T_h$. From the definition of $A(\cdot, \cdot)$, $A_h(\cdot, \cdot)$ and $a_h(\cdot, \cdot)$, triangular inequality and (3.3), the term $\Pi$ is controlled as follows:

$$|\Pi| = |a_h(\sigma_h, \sigma_h) - a(\sigma_h, \sigma_h)|$$

$$= \left| \sum_{K \in T_h} a^K_h(\sigma_h, \sigma_h) - a^K(\sigma_h, \sigma_h) \right|$$

$$= \left| \sum_{K \in T_h} a^K_h(\sigma_h - \Pi^K_h \sigma, \sigma_h - \Pi^K_h \sigma) - a^K(\sigma_h - \Pi^K_h \sigma, \sigma_h - \Pi^K_h \sigma) \right|$$

$$\leq C \sum_{K \in T_h} \left\| \sigma_h - \Pi^K_h \sigma \right\|_{0,K}^2 \leq C \sum_{K \in T_h} \left( \left\| \sigma_h - \sigma \right\|_{0,K}^2 + \left\| \sigma - \Pi^K_h \sigma \right\|_{0,K}^2 \right) \leq C h^{2 \min\{p,k\}},$$

where we have used (4.7) and the properties of the projection. This concludes the proof.

5. Numerical results

In this section we report some numerical tests which allows us to assess the performance of the method. Following the ideas proposed in [6], we have implemented in a MATLAB code a lowest-order VEM ($k = 0$) on arbitrary polygonal meshes. A natural choice for $S^K(\cdot, \cdot)$ is given by

$$S^K(\sigma_h, \tau_h) := w_K \sum_{k=1}^{N_K} \left( \int_{e_k} \sigma_h \cdot n \right) \left( \int_{e_k} \tau_h \cdot n \right)$$

(5.1)

where $N_K$ represents the number of edges in the polygon $K$ and $w_K$ is the so-called stability constant which will be taken of the order of unity, see [3, Section 5] for more details.

We report in this section a couple of numerical tests which allowed us to assess the theoretical results proved above.

We begin with some numerical tests to assess the performance of the proposed virtual element method. More precisely, we are interested, first, in the computation of convergence orders to confirm the theoretical results of the analysis.

With this goal in mind, we present three scenarios in which we will prove our method: the first is to compute the eigenvalues and convergence rates in the unitary square, the second will correspond to a non-convex domain and the last one considers a square with mixed boundary conditions.

5.0.1. Test 1: unit square

In this test, the domain is the unit square $\Omega = (0, 1)^2$. Due the simplicity of this domain, the exact solutions are known. Indeed, the eigenvalues for this problem are

$$\lambda = (m^2 + n^2)\pi^2, \quad m, n \in \mathbb{N}, \ m, n \neq 0,$$

(5.2)

with the associated eigenfunctions

$$u(x, y) = \sin(m\pi x) \sin(n\pi y), \quad m, n \in \mathbb{N}, \ m, n \neq 0.$$

For our numerical tests, we have used four different families of meshes which we describe in the following list:
\[ T_1^h: \] triangular meshes;
\[ T_2^h: \] square meshes;
\[ T_3^h: \] square meshes with \( N \) nodes per side then it perturbs all nodes but the central one and the boundary ones
\[ T_4^h: \] trapezoidal meshes which consist of partitions of the domain into \( N \times N \) congruent trapezoids, all similar to the trapezoid with vertices \((0,0), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{2}{3}\right), (0, \frac{2}{3})\).

The refinement parameter \( N \), used to label each mesh, represents the number of elements intersecting each edge. In the unit square, the eigenfunctions of problem (2.1) are smooth and hence, the approximation orders for our method are optimal. In Table 1 we report the first six eigenvalues computed with our method. In the row 'Order' we present the convergence rates for each eigenvalue. These order have been computed respect to the exact eigenvalues provided by (5.2).

| \( T_1^h \) | \( N \) | \( \lambda_{h1} \) | \( \lambda_{h2} \) | \( \lambda_{h3} \) | \( \lambda_{h4} \) | \( \lambda_{h5} \) | \( \lambda_{h6} \) |
|---|---|---|---|---|---|---|---|
| 8 | 19.2126 | 46.0629 | 46.2998 | 71.0656 | 86.7093 | 86.9136 |
| 16 | 19.6065 | 48.5605 | 48.5691 | 76.9474 | 95.6175 | 95.6228 |
| 32 | 19.7047 | 49.1402 | 49.1409 | 78.4259 | 97.8534 | 97.8615 |
| 64 | 19.7309 | 49.2955 | 49.2961 | 78.8238 | 98.4859 | 98.4886 |
| Order | 1.9881 | 1.9825 | 1.9541 | 1.9372 | 1.9364 |
| Exact | 19.7392 | 49.3480 | 49.3480 | 78.9568 | 98.6960 | 98.6960 |

| \( T_2^h \) | \( N \) | \( \lambda_{h1} \) | \( \lambda_{h2} \) | \( \lambda_{h3} \) | \( \lambda_{h4} \) | \( \lambda_{h5} \) | \( \lambda_{h6} \) |
|---|---|---|---|---|---|---|---|
| 8 | 18.7724 | 42.0875 | 42.0875 | 65.4027 | 69.7660 | 69.7660 |
| 16 | 19.4886 | 47.2890 | 47.2890 | 75.0894 | 89.3259 | 89.3259 |
| 32 | 19.6760 | 48.8153 | 48.8153 | 77.9546 | 96.1565 | 96.1656 |
| 64 | 19.7234 | 49.2137 | 49.2137 | 78.7039 | 98.0505 | 98.0505 |
| Order | 1.9881 | 1.9218 | 1.9218 | 1.9180 | 1.8346 | 1.8346 |
| Exact | 19.7392 | 49.3480 | 49.3480 | 78.9568 | 98.6960 | 98.6960 |

| \( T_3^h \) | \( N \) | \( \lambda_{h1} \) | \( \lambda_{h2} \) | \( \lambda_{h3} \) | \( \lambda_{h4} \) | \( \lambda_{h5} \) | \( \lambda_{h6} \) |
|---|---|---|---|---|---|---|---|
| 8 | 18.7419 | 41.9133 | 42.0552 | 65.5365 | 69.0829 | 70.2747 |
| 16 | 19.4846 | 47.2780 | 47.2843 | 74.9461 | 89.3402 | 89.4316 |
| 32 | 19.6745 | 48.8119 | 48.8146 | 77.9310 | 96.1702 | 96.1818 |
| 64 | 19.7230 | 49.2130 | 49.2131 | 78.6981 | 98.0522 | 98.0529 |
| Order | 1.9727 | 1.8968 | 1.8866 | 1.8548 | 1.7757 | 1.7494 |
| Exact | 19.7392 | 49.3480 | 49.3480 | 78.9568 | 98.6960 | 98.6960 |

| \( T_4^h \) | \( N \) | \( \lambda_{h1} \) | \( \lambda_{h2} \) | \( \lambda_{h3} \) | \( \lambda_{h4} \) | \( \lambda_{h5} \) | \( \lambda_{h6} \) |
|---|---|---|---|---|---|---|---|
| 8 | 18.6654 | 41.8949 | 42.2913 | 64.0705 | 70.1566 | 71.6883 |
| 16 | 19.4595 | 47.2477 | 47.3714 | 74.6558 | 89.6327 | 90.2307 |
| 32 | 19.6685 | 48.8055 | 48.8384 | 77.8372 | 96.2649 | 96.4358 |
| 64 | 19.7215 | 49.2112 | 49.2196 | 78.6739 | 98.0769 | 98.1212 |
| Order | 1.9746 | 1.9256 | 1.9295 | 1.9094 | 1.8478 | 1.8567 |
| Exact | 19.7392 | 49.3480 | 49.3480 | 78.9568 | 98.6960 | 98.6960 |

It is clear from Table 1 the double order of convergence for the eigenvalues due the smoothness of the eigenfunctions. Also, no spurious eigenvalues are observed in these test, which confirms the accuracy and stability of the proposed mixed method. Also In Figure 1 we present plots for the first four eigenfunctions.
5.1. Effect of the stability constant

The aim of this test is to analyze the influence of the stability (see (5.1)) on the computed spectrum, to know whether the quality of the computations can be affected by this constant. We will consider the same geometrical configuration of the previous test. We will compute the lowest eigenvalue for different values $w_K$ using the family of meshes $\mathcal{T}_h^2$. 

Figure 1: Test 1. Plots of the first four eigenfunctions for $T_h^1$ (top left), $T_h^2$ (top right), $T_h^3$ (bottom left) and $T_h^4$ (bottom right) with $N = 32$
It can be seen from Table 2 that despite the fact that the different stabilization numbers do not introduce spurious eigenvalues in the method, the order of convergence is affected when the stabilization constant is large.

5.2. Test 2: L-shaped domain

In this stage, we consider the classic non-convex domain called L-shaped domain. The non-convexity of this geometry leads to obtain non-smooth eigenfunctions due to the singularity and hence, non-optimal order of convergence. Contrary to the square domain, for this test we do not have analytical solutions. Hence, we obtain the eigenvalues for different meshes and different refinement levels and compute the order of convergence respect to an extrapolated value, which is reported in the row 'Extrap.' of Table 3. These extrapolated values have been computed by means of a least-square fitting.

In this test we have used four different families of meshes (see Fig. 2):
- $T_h^5$: non-structured hexagonal meshes made of convex hexagons;
- $T_h^6$: triangular meshes;
- $T_h^7$: square meshes;

In Table 3 we report the first six eigenvalues of (2.1) computed with our mixed method in the L-shaped domain.

| $N$ | $w_K = 4^0$ | $w_K = 4^1$ | $w_K = 4^2$ | $w_K = 4^3$ | $w_K = 4^4$ | $w_K = 4^5$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| 8   | 0.0623      | 0.2470      | 0.9530      | 3.3404      | 8.9395      | 15.3873     |
| 16  | 0.2469      | 0.9521      | 3.3296      | 8.8625      | 15.1606     | 18.4360     |
| 32  | 0.9519      | 3.3269      | 8.8434      | 15.1048     | 18.3535     | 19.3965     |
| 64  | 3.3262      | 8.8386      | 15.0909     | 18.3330     | 19.3735     | 19.6524     |
| 128 | 8.8374      | 15.0874     | 18.3279     | 19.3678     | 19.6465     | 19.7174     |
| 256 | 15.0865     | 18.3266     | 19.3664     | 19.6450     | 19.7160     | 19.7338     |

| Order | 0.3746 | 0.7304 | 1.1463 | 1.5302 | 1.8027 | 1.9400 | 1.9885 |
|--------|--------|--------|--------|--------|--------|--------|--------|
| $\lambda_1$ | 19.7392 | 19.7392 | 19.7392 | 19.7392 | 19.7392 | 19.7392 | 19.7392 |

In Table 3 we report the first six eigenvalues $\lambda_1$ for $w_K = 0$ and $w_K = 4^{-k}$ with $-6 \leq k \leq 6$ and $T_h^2$. 

| $N$ | $w_K = 4^{-1}$ | $w_K = 4^{-2}$ | $w_K = 4^{-3}$ | $w_K = 4^{-4}$ | $w_K = 4^{-5}$ | $w_K = 4^{-6}$ | $w_K = 0$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|-------------|
| 8   | 19.8649        | 20.1582        | 20.2328        | 20.2516        | 20.2563        | 20.2575        | 20.2579     |
| 16  | 19.7708        | 19.8427        | 19.8607        | 19.8652        | 19.8664        | 19.8666        | 19.8667     |
| 32  | 19.7471        | 19.7650        | 19.7695        | 19.7706        | 19.7709        | 19.7709        | 19.7710     |
| 64  | 19.7412        | 19.7457        | 19.7468        | 19.7470        | 19.7471        | 19.7471        | 19.7471     |
| 128 | 19.7397        | 19.7408        | 19.7411        | 19.7412        | 19.7412        | 19.7412        | 19.7412     |
| 256 | 19.7393        | 19.7396        | 19.7397        | 19.7397        | 19.7397        | 19.7397        | 19.7397     |

| Order | 1.9978 | 2.0039 | 2.0049 | 2.0052 | 2.0052 | 2.0052 | 2.0052 |
|--------|--------|--------|--------|--------|--------|--------|--------|
| $\lambda_1$ | 19.7392 | 19.7392 | 19.7392 | 19.7392 | 19.7392 | 19.7392 | 19.7392 |
Table 3: Test 2. The lowest computed eigenvalues $\lambda_{hi}$, $1 \leq i \leq 6$ for different $T_h$.

| $T_h$ | $N$ | $\lambda_{h1}$ | $\lambda_{h2}$ | $\lambda_{h3}$ | $\lambda_{h4}$ | $\lambda_{h5}$ | $\lambda_{h6}$ |
|-------|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| $T_h^5$ | 18  | 37.7081        | 59.7051        | 76.6486        | 113.201        | 119.976        | 155.366        |
|       | 38  | 38.3107        | 60.5558        | 78.4827        | 117.017        | 125.714        | 163.376        |
|       | 54  | 38.4262        | 60.6812        | 78.7317        | 117.587        | 126.701        | 164.712        |
|       | 70  | 38.4735        | 60.7270        | 78.8254        | 117.797        | 127.079        | 165.205        |
|       | 90  | 38.4999        | 60.7511        | 78.8784        | 117.909        | 127.284        | 165.454        |
|       |     | Order          | 1.65           | 2.04           | 2.13           | 2.02           | 1.82           | 1.89           |
|       |     | Extrap.        | 38.5625        | 60.7943        | 78.9530        | 118.109        | 127.721        | 166.000        |
| $T_h^6$ | 10  | 35.6472        | 56.2124        | 71.8313        | 102.1495       | 105.329        | 133.136        |
|       | 20  | 37.6435        | 59.5705        | 77.0532        | 113.680        | 121.000        | 155.988        |
|       | 30  | 38.0916        | 60.2398        | 78.0999        | 116.088        | 124.487        | 161.234        |
|       | 50  | 38.3550        | 60.5895        | 78.646         | 117.359        | 126.413        | 164.123        |
|       | 60  | 38.4064        | 60.6502        | 78.7409        | 117.580        | 126.763        | 164.641        |
|       |     | Order          | 1.68           | 1.90           | 1.89           | 1.84           | 1.73           | 1.70           |
|       |     | Extrap.        | 38.5495        | 60.8048        | 78.9882        | 118.184        | 127.800        | 166.259        |
| $T_h^7$ | 20  | 37.1216        | 58.8887        | 76.4390        | 111.187        | 117.648        | 148.894        |
|       | 40  | 38.0961        | 60.2981        | 78.3125        | 116.279        | 124.824        | 161.043        |
|       | 60  | 38.3168        | 60.5690        | 78.6692        | 117.276        | 126.3007       | 163.633        |
|       | 80  | 38.4047        | 60.6648        | 78.7948        | 117.629        | 126.846        | 164.582        |
|       | 100 | 38.4497        | 60.7094        | 78.8530        | 117.793        | 127.110        | 165.034        |
|       |     | Order          | 1.66           | 1.94           | 1.96           | 1.92           | 1.83           | 1.79           |
|       |     | Extrap.        | 38.5476        | 60.7943        | 78.9613        | 118.111        | 127.635        | 166.007        |

From Table 3 we observe that the order of convergence for the first eigenvalue is not optimal. This is expectable due to the non-convexity of the chosen domain. In the other hand, the rest of the eigenvalues converge to the extrapolated ones with double order, since in these cases, the singularity does not deteriorate the smoothness of the associated eigenfunctions.

In Figure 2 we present plots for the first two eigenfunctions of the Laplace eigenproblem in the L-shaped domain, computed with different type of meshes.

![Figure 2: Test 2. Plots of the first two eigenfunctions for $T_h^5$ (left) and $T_h^6$ (right).](image)
5.3. Square with mixed boundary conditions.

Let us mention the natural extension of system (2.1) to the mixed boundary conditions case. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with Lipschitz boundary $\Gamma$. We assume that this boundary is splitted in two parts $\Gamma_D$ and $\Gamma_N$ such that $\Gamma := \Gamma_D \cup \Gamma_N$. The Laplace eigenvalue problem reads as follows:

Problem 5. Find $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$, $u \neq 0$, such that

$$
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma_D, \\
u \cdot n = 0 & \text{on } \Gamma_N,
\end{cases}
$$

(5.3)

where $n$ denotes the outward unitary vector respect to $\Gamma_N$. A mixed variational formulation of (5.3) is the following: Find $(\lambda, \sigma, u) \in \mathbb{R} \times H_0(\text{div}, \Omega) \times L^2(\Omega)$, $(\sigma, u) \neq (0, 0)$, such that

$$
\int_{\Omega} \sigma \cdot \tau + \int_{\Omega} \text{div} \tau u = 0 \quad \forall \tau \in H_0(\text{div}, \Omega),
$$

$$
\int_{\Omega} \text{div} \sigma v = -\lambda \int_{\Omega} uv \quad \forall v \in L^2(\Omega).
$$

where the condition $u \cdot n$ is imposed in the space $H(\text{div}, \Omega)$. The analysis of this variational formulations is analogous to the Dirichlet boundary condition case considered before. The main difference lies on the regularity of the solution $u$, which is affected due the mixed boundary conditions (see [21]).

If the $\Omega$-domain is convex, the analysis of convergence and error estimates holds with no major changes. However, for non-convex domains the analysis is more complex and this will be studied in a future work.

In the present test we consider as computational domain the square $\Omega := (-1,1)^2$. The boundary conditions provided in (2.1) are imposed as follows: the condition $u = 0$ (i.e., condition on $\Gamma_D$) is considered in the top and bottom of the square, meanwhile the condition on $\Gamma_N$ is considered on the other two sides of the square.

We begin with the analysis of the effects of the stabilization $w_K$ in the computation of the spectrum. Since we have mixed boundary conditions, which is obviously different compared with our already presented experiments only for the boundary condition $u = 0$, it is expectable that spurious modes can arise for certain stabilizations. The following tables report computed eigenvalues for two different meshes: one of triangles ($T^1_h$) and the other of squares ($T^2_h$). We present this two choices only for simplicity. We mention that for other polygonal meshes the results are similar.

For the experiments, we fix $T^1_h$ and $T^2_h$ in $N = 10$. In tables 4 and 5 we report the first ten eigenvalues computed with our method, considering the meshes mentioned before and for different values of $w_K$. The numbers in boxes represent spurious eigenvalues and the last column present the exact eigenvalues of the problem, which we compare with the computed ones.
Table 4: Computed lowest eigenvalues for different values of $w_K$ with $N = 10$.

| $w_K$ | $w_K = \frac{1}{10}$ | $w_K = \frac{1}{10}$ | $w_K = \frac{1}{10}$ | $w_K = \frac{1}{10}$ | $\lambda_{hi}$ |
|-------|----------------------|----------------------|----------------------|----------------------|---------------|
| 2.4718 | 2.4716 | 2.4691 | 2.4650 | 2.4582 | 2.4674 |
| 4.9502 | 4.9490 | 4.9385 | 4.9212 | 4.8925 | 4.9348 |
| 9.9277 | 9.9230 | 9.8810 | 9.8118 | 9.6984 | 9.8701 |
| 12.4172 | 12.4101 | 12.3466 | 12.2403 | 12.0572 | 12.3373 |
| 12.4407 | 12.4329 | 12.3634 | 12.2511 | 12.0779 | 12.3374 |
| 20.0183 | 19.9999 | 19.8354 | 19.5668 | 19.1338 | 19.7404 |
| 22.5181 | 22.4946 | 22.2854 | 21.9447 | 21.3974 | 22.2094 |
| 25.0231 | 24.9958 | 24.7517 | 24.3525 | 23.6593 | 24.6753 |
| 25.1170 | 25.0863 | 24.8131 | 24.3723 | 23.7235 | 24.6757 |
| 32.6591 | 32.6124 | 32.1980 | 31.5263 | 30.4107 | 32.0801 |
| $w_K = 1$ | $w_K = 1.5$ | $w_K = 2$ | $w_K = 5$ | $w_K = 10$ | $\lambda_{hi}$ |
| 2.4447 | 2.4314 | 2.4182 | 2.3416 | 2.2236 | 2.4674 |
| 4.8362 | 4.7810 | 4.7270 | 4.4256 | 3.9962 | 4.9348 |
| 9.4787 | 9.2680 | 9.0657 | 8.0046 | 6.6680 | 9.8701 |
| 11.6989 | 11.3597 | 11.0385 | 9.4235 | 7.5451 | 12.3373 |
| 11.7524 | 11.4439 | 11.1505 | 9.6523 | 7.8569 | 12.3374 |
| 18.3195 | 17.5672 | 16.8702 | 13.5654 | 10.0645 | 19.7404 |
| 20.3747 | 19.4378 | 18.5765 | 14.5876 | 10.5526 | 22.2094 |
| 22.3529 | 21.1738 | 20.1051 | 15.3520 | 10.9359 | 24.6753 |
| 22.5443 | 21.4719 | 20.4918 | 16.0154 | 11.5723 | 24.6757 |
| 28.2992 | 26.4446 | 24.8059 | 17.9568 | 12.0748 | 32.0801 |
Table 5: Test 3. Computed lowest eigenvalues for different values of $w_K$ with $N = 10$.

| $w_K = 0$ | $w_K = \frac{1}{10}$ | $w_K = \frac{1}{100}$ | $w_K = \frac{1}{1000}$ | $w_K = \frac{1}{10000}$ | $\lambda_{hi}$ |
|-----------|------------------|---------------------|---------------------|---------------------|-----------------|
| 2.5086    | 2.5073           | 2.4960              | 2.4775              | 2.4472              | 2.4674          |
| 5.0171    | 5.0146           | 4.9921              | 4.9550              | 4.8943              | 4.9348          |
| 10.5573   | 10.5350          | 10.3390             | 10.0279             | 9.5492              | 9.8701          |
| 13.0658   | 13.0423          | 12.8350             | 12.5054             | 11.9963             | 12.3373         |
| 13.0658   | 13.0423          | 12.8350             | 12.5054             | 11.9963             | 12.3374         |
| 21.1146   | 21.0701          | 20.6780             | 20.0559             | 19.0983             | 19.7404         |
| 25.9616   | 25.8275          | 24.6801             | 22.9788             | 20.6107             | 22.2094         |
| 28.4702   | 28.3348          | 27.1762             | 25.4563             | 23.0579             | 24.6753         |
| 28.4702   | 28.3348          | 27.1762             | 25.4563             | 23.0579             | 24.6757         |
| 36.5189   | 36.3626          | 35.0191             | 33.0067             | 30.1599             | 32.0801         |

We observe from tables 4 and 5 that the behaviour of the spurious eigenvalues clearly depend on the choice of $w_K$. Notice that when $w_K$ is close to zero (even for the case $w_K = 0$) the spurious vanish, meanwhile when $w_K$ increases, the spurious eigenvalues appear, for both $T^1_h$ and $T^2_h$, from $w_K = 5$ in forward. Moreover, we observe that the method do not introduce spurious for $w_K < 2$. This phenomenon is analogous for other meshes.

We remark that the appearance of spurious eigenvalues is not visible for the domain with the Dirichlet boundary condition, which leads to conclude that the mixed boundary is also a relevant factor at the moment to compute the spectrum of the Laplace eigenvalue problem.

Now we are interested to observe if the refinement of the meshes influence the appearance of spurious eigenvalues when we choose a value of $w_K$ that introduces pollution of the spectrum. To do this task, we chose $w_K = 10$ (since with this value of $w_K$ several spurious eigenvalues are observed) and refine the meshes $T^1_h$ and $T^2_h$ that we have used in the previous tests.
Table 6: Test 3. Computed lowest eigenvalues for $w_K = 10$

| $\lambda_{hi}$ | $N = 10$ | $N = 20$ | $N = 30$ | $N = 40$ |
|----------------|----------|----------|----------|----------|
| 2.4674         | 2.2236   | 2.4071   | 2.4399   | 2.4520   |
| 4.9348         | 3.9962   | 4.6921   | 4.8230   | 4.8729   |
| 9.8701         | 6.6680   | 8.9636   | 9.4461   | 9.6299   |
| 12.3373        | 7.5451   | 10.9239  | 11.6687  | 11.9544  |
| 12.3373        | 7.8569   | 10.9440  | 11.6810  | 11.9617  |
| 19.7404        | 10.0645  | 16.4216  | 18.0867  | 18.8010  |
| 22.2094        | 10.5526  | 18.0007  | 20.1514  | 21.0296  |
| 24.6753        | 10.9359  | 19.5387  | 22.0980  | 23.1650  |
| 24.6757        | 11.5723  | 19.8019  | 22.1888  | 23.2291  |
| 32.0801        | 12.0748  | 23.9208  | 27.8445  | 29.5927  |

| $\lambda_{hi}$ | $N = 10$ | $N = 20$ | $N = 30$ | $N = 40$ |
|----------------|----------|----------|----------|----------|
| 2.4674         | 1.6705   | 2.2045   | 2.3432   | 2.3960   |
| 4.9348         | 3.3409   | 4.4090   | 4.6864   | 4.7919   |
| 9.8701         | 3.3930   | 6.6819   | 8.1431   | 8.8180   |
| 12.3373        | 4.1925   | 8.8864   | 10.4863  | 11.2139  |
| 12.3373        | 4.5674   | 8.8864   | 10.4863  | 11.2139  |
| 19.7404        | 4.7619   | 10.7096  | 15.0342  | 17.5084  |
| 22.2094        | 4.8714   | 12.9141  | 16.2862  | 17.6359  |
| 24.6753        | 4.9359   | 12.9141  | 17.3774  | 19.9043  |
| 24.6757        | 4.9737   | 13.3637  | 17.3774  | 19.9043  |
| 32.0801        | 4.9937   | 13.5721  | 21.3607  | 26.3263  |

From table 6 we observe that the spurious eigenvalues disappear when the meshes are refined, which is expectable, since the accuracy of the VEM is improved when the meshes are refined. Similar results have been obtained for other numerical methods that depend on some particular stabilization parameter (for instance, the DG method [24, 23]).

In table 7 we report the first six eigenvalues computed with our method considering different polygonal meshes and $w_K = 1$. It is clear that the double order of convergence is obtained with this configuration of the geometry, as is expected.
Table 7: Test 3. The lowest computed eigenvalues $\lambda_{hi}$, $1 \leq i \leq 6$ for different $T_h$.

| $T_h$ | $N$ | $\lambda_{h1}$ | $\lambda_{h2}$ | $\lambda_{h3}$ | $\lambda_{h4}$ | $\lambda_{h5}$ | $\lambda_{h6}$ |
|-------|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $T_1^1$ | 8   | 2.4343          | 4.8005          | 9.3442          | 11.4928         | 11.5519         | 17.7203         |
|       | 16  | 2.4592          | 4.9004          | 9.7437          | 12.1144         | 12.1295         | 19.2004         |
|       | 32  | 2.4653          | 4.9263          | 9.8362          | 12.2820         | 12.2839         | 19.6013         |
|       | 64  | 2.4669          | 4.9327          | 9.8613          | 12.3237         | 12.3239         | 19.7054         |
|       |     | Order 2.0200    | 1.9600          | 2.0800          | 1.9100          | 1.9100          | 1.8900          |
|       |     | Extr. 2.4673    | 4.9351          | 9.8670          | 12.3406         | 12.3391         | 19.7470         |
| $T_2^1$ | 8   | 2.3465          | 4.6931          | 8.1753          | 10.5219         | 10.5219         | 16.3507         |
|       | 16  | 2.4361          | 4.8722          | 9.3862          | 11.8223         | 11.8223         | 18.7724         |
|       | 32  | 2.4595          | 4.9190          | 9.7443          | 12.2038         | 12.2038         | 19.4886         |
|       | 64  | 2.4654          | 4.9308          | 9.8380          | 12.3034         | 12.3034         | 19.6760         |
|       |     | Order 1.9400    | 1.9400          | 1.7900          | 1.8000          | 1.8000          | 1.7900          |
|       |     | Extr. 2.4676    | 4.9353          | 9.8822          | 12.3496         | 12.3496         | 19.7644         |
| $T_3^1$ | 8   | 2.3474          | 4.6887          | 8.1662          | 10.4683         | 10.5147         | 16.3106         |
|       | 16  | 2.4363          | 4.8705          | 9.3909          | 11.8172         | 11.8204         | 18.7349         |
|       | 32  | 2.4596          | 4.9186          | 9.7456          | 12.2024         | 12.2029         | 19.4837         |
|       | 64  | 2.4654          | 4.9308          | 9.8383          | 12.3032         | 12.3032         | 19.6746         |
|       |     | Order 1.9500    | 1.9300          | 1.8100          | 1.8300          | 1.8000          | 1.7400          |
|       |     | Extr. 2.4675    | 4.9354          | 9.8811          | 12.3473         | 12.3495         | 19.7787         |
| $T_4^1$ | 8   | 2.3575          | 4.6664          | 8.3096          | 10.4737         | 10.5728         | 16.0176         |
|       | 16  | 2.4390          | 4.8649          | 9.4302          | 11.8119         | 11.8429         | 18.6639         |
|       | 32  | 2.4603          | 4.9171          | 9.7561          | 12.2014         | 12.2069         | 19.4503         |
|       | 64  | 2.4656          | 4.9304          | 9.8410          | 12.3028         | 12.3049         | 19.6685         |
|       |     | Order 1.9500    | 1.9300          | 1.8100          | 1.8100          | 1.8200          | 1.7700          |
|       |     | Extr. 2.4676    | 4.9355          | 9.8799          | 12.3491         | 12.3477         | 19.7701         |

Finally, in figure [3] we present plots of the first four eigenfunctions for the Laplace eigenvalue problem with mixed boundary conditions, obtained with different polygonal meshes.
Figure 3: Test 3. Plots of the first four eigenfunctions for $T_h^1$ (top left), $T_h^2$ (top right), $T_h^3$ (bottom left) and $T_h^4$ (bottom right) computed with different meshes and $N = 32$.

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