GROUP ACTIONS OF $A_5$ ON CONTRACTIBLE 2-COMPLEXES

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Abstract. We prove that every action of $A_5$ on a finite 2-dimensional contractible complex has a fixed point.

1. Introduction

A well-known result of Jean-Pierre Serre [Ser80] states that every action of a finite group on a contractible 1-complex (i.e. a tree) has a fixed point. By Smith theory, every action of a $p$-group on the disk $D^n$ has a fixed point. The group $A_5$ acts simplicially and fixed point freely on the barycentric subdivision $X$ of the 2-skeleton of the Poincaré homology sphere which is an acyclic 2-complex. By considering the join $X * A_5$, Edwin E. Floyd and Roger W. Richardson [FR59] proved that $A_5$ acts simplicially and fixed point freely on a contractible 3-complex. Moreover, by embedding $X * A_5$ in $\mathbb{R}^{81}$ and taking a regular neighbourhood they proved that $A_5$ acts simplicially and fixed point freely on a triangulation of the disk $D^{81}$. This was the only example known of this kind until Bob Oliver obtained a complete classification of the groups that act fixed point freely on a disk $D^n$ [Oli75]. Floyd and Richardson’s example makes clear that Serre’s result does not hold in dimension 3, but does it hold for 2-complexes? Carles Casacuberta and Warren Dicks [CD92] made the following conjecture (without requiring $X$ to be finite) which was also posed by Michael Aschbacher and Yoav Segev as a question [AS93, Question 3] in the finite case.

Conjecture 1.1. Let $G$ be a finite group. If $X$ is a 2-dimensional finite contractible $G$-complex then $X^G \neq \emptyset$.

In [CD92] the conjecture is proved for solvable groups. Independently, Segev [Seg93] studied the question of which groups act without fixed points on an acyclic 2-complex and proved Conjecture 1.1 for solvable groups and the alternating groups $A_n$ for $n \geq 6$. In [Seg94], Segev proved the conjecture for collapsible 2-complexes. Using the classification of the finite simple groups, Aschbacher and Segev proved that for many groups any action on a finite 2-dimensional acyclic complex has a fixed point [AS93]. Later, Oliver and Segev [OS02] gave a complete classification of the groups that act without fixed points on a finite acyclic 2-complex. Before [OS02], $A_5$ was the only group known to act without fixed points on an acyclic 2-complex. An excellent source to read more on this topic is Alejandro Adem’s exposition at the Séminaire Bourbaki [Ade03]. In [Cor01], J.M. Corson proved that Conjecture 1.1 holds for diagrammatically reducible complexes (in particular it holds for collapsible complexes).

The smallest group for which Conjecture 1.1 remained open is the alternating group $A_5$. The main result of this paper is the following.

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Theorem 7.2. Every action of $A_5 \cong \text{PSL}_2(2^2)$ on a finite, contractible 2-complex has a fixed point.

From our proof we also deduce the following.

Theorem 7.3. Let $X$ be a fixed point free 2-dimensional finite, acyclic $A_5$-complex and let $\pi = \pi_1(X)$. Then $\pi$ is infinite or there is an epimorphism $\pi \to A_5$.

The paper is organized as follows. In Section 3 we prove Corollary 3.14 which says that to prove Theorem 7.2 it is enough to inspect the acyclic complexes of the type considered by Oliver and Segev in [OS02]. In Section 3 we also prove Theorem 3.11 which, together with Corollary 3.14, says that, assuming a special case of the Kervaire–Laudenbach–Howie conjecture, if Conjecture 1.1 is false then there is a counterexample of a very special form. The necessary results from [OS02] are recalled in Section 2.

In Section 4 we establish the connection between Theorem 7.2 and the following group theoretic statement, using a result of Kenneth S. Brown [Bro84] in Bass–Serre theory.

Theorem 7.1. There is no presentation of $A_5$ of the form

$$\langle a, b, c, d, x_0, \ldots, x_k \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x_0ax_0^{-1} = d, w_0, \ldots, w_k \rangle$$

with $w_0, \ldots, w_k \in \ker(\phi)$, where $\phi : F(a, b, c, d, x_0, \ldots, x_k) \to A_5$ is given by $a \mapsto (2, 5)(3, 4)$, $b \mapsto (3, 5, 4)$, $c \mapsto (1, 2)(3, 5)$, $d \mapsto (2, 5)(3, 4)$ and $x_i \mapsto 1$ for each $i = 0, \ldots, k$.

In order to prove Theorem 7.1, in Section 5 we introduce a moduli of representations of the group

$$\Gamma_k = \langle a, b, c, d, x_0, \ldots, x_k \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x_0ax_0^{-1} = d \rangle$$

in $\text{SO}(3)$. In Section 6 we view these rotations in $S^3 \subset \mathbb{H}$, enabling us to do a degree argument which completes the proof of Theorem 7.1. This proof is inspired by James Howie’s proof of the Scott–Wiegold conjecture [How02]. Finally, in Section 7 we put everything together to complete the proof of Theorem 7.2.

Note. Some of the results presented here appeared originally in the author’s thesis [SC19].

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2. Fixed point free actions on acyclic 2-complexes

In this section we review the results obtained by Bob Oliver and Yoav Segev in their article [OS02] that are used later.

Throughout the paper, by $G$-complex we mean a $G$-CW complex. That is, a CW complex with a continuous $G$-action that is admissible (i.e. the action permutes the open cells of $X$, and maps a cell to itself only via the identity). For more details see [OS02, Appendix A]. We will frequently assume that the 2-cells in a $G$-complex are attached along closed edge paths, this will make no difference for the questions that we study. A graph is a 1-dimensional CW complex. By $G$-graph we always mean a 1-dimensional $G$-complex.

Definition 2.1 ([OS02]). A $G$-space $X$ is essential if there is no normal subgroup $1 \neq N \lhd G$ such that for each $H \subseteq G$, the inclusion $X^{HN} \rightarrow X^H$ induces an isomorphism on integral homology.

The main results of [OS02] are the following two theorems.

Theorem 2.2 ([OS02, Theorem A]). For any finite group $G$, there is an essential fixed point free 2-dimensional (finite) acyclic $G$-complex if and only if $G$ is isomorphic to one of the simple groups $\text{PSL}_2(2^k)$ for $k \geq 2$, $\text{PSL}_2(q)$ for $q \equiv \pm 3 \pmod{8}$ and $q \geq 5$, or $\text{Sz}(2^k)$ for odd $k \geq 3$.

Furthermore, the isotropy subgroups of any such $G$-complex are all solvable.

Theorem 2.3 ([OS02, Theorem B]). Let $G$ be any finite group, and let $X$ be any 2-dimensional acyclic $G$-complex. Let $N$ be the subgroup generated by all normal subgroups $N' \lhd G$ such that $X^{N'} \neq \emptyset$. Then $X^N$ is acyclic; $X$ is essential if and only if $N = 1$; and the action of $G/N$ on $X^N$ is essential.

The following fundamental result of Segev [Seg93, Theorem 3.4] will be used frequently, sometimes implicitly. We state the version given in [OS02].

Theorem 2.4 ([OS02, Theorem 4.1]). Let $X$ be any 2-dimensional acyclic $G$-complex (not necessarily finite). Then $X^G$ is acyclic or empty, and is acyclic if $G$ is solvable.

We denote the set of subgroups of $G$ by $\mathcal{S}(G)$.

Definition 2.5 ([OS02]). By a family of subgroups of $G$ we mean any subset $\mathcal{F} \subseteq \mathcal{S}(G)$ which is closed under conjugation. A nonempty family is said to be separating if it has the following three properties: (a) $G \notin \mathcal{F}$; (b) if $H' \subseteq H$ and $H \in \mathcal{F}$ then $H' \in \mathcal{F}$; (c) for any $H \lhd K \subseteq G$ with $K/H$ solvable, $K \in \mathcal{F}$ if $H \in \mathcal{F}$.

For any family $\mathcal{F}$ of subgroups of $G$, a $(G, \mathcal{F})$-complex will mean a $G$-complex all of whose isotropy subgroups lie in $\mathcal{F}$. A $(G, \mathcal{F})$-complex is universal (resp. $H$-universal) if the fixed point set of each $H \in \mathcal{F}$ is contractible (resp. acyclic).

If $G$ is not solvable, the separating family of solvable subgroups of $G$ is denoted by $\mathcal{SLV}$. If $G$ is perfect, then the family of proper subgroups of $G$ is denoted by $\mathcal{MA} \mathcal{X}$. 
Lemma 2.6 ([OS02, Lemma 1.2]). Let \( X \) be any 2-dimensional acyclic \( G \)-complex without fixed points. Let \( \mathcal{F} \) be the set of subgroups \( H \subseteq G \) such that \( X^H \neq \emptyset \). Then \( \mathcal{F} \) is a separating family of subgroups of \( G \), and \( X \) is an \( H \)-universal \((G, \mathcal{F})\)-complex.

Proposition 2.7 ([OS02, Proposition 6.4]). Assume that \( L \) is one of the simple groups \( \text{PSL}_2(q) \) or \( \text{Sz}(q) \), where \( q = p^k \) and \( p \) is prime (\( p = 2 \) in the second case). Let \( G \subseteq \text{Aut}(L) \) be any subgroup containing \( L \), and let \( \mathcal{F} \) be a separating family for \( G \). Then there is a 2-dimensional acyclic \((G, \mathcal{F})\)-complex if and only if \( G = L \), \( \mathcal{F} = \text{SL}_V \), and \( q \) is a power of 2 or \( q \equiv \pm 3 \) (mod 8).

If \( X \) is a poset, \( K(X) \) denotes the order complex of \( X \), that is the simplicial complex with simplices the finite nonempty totally ordered subsets of \( X \) (the complex \( K(X) \) is also known as the nerve of \( X \)).

Definition 2.8 ([OS02, Definition 2.1]). For any family \( \mathcal{F} \) of subgroups of \( G \) define

\[
i_{\mathcal{F}}(H) = \frac{1}{[\mathcal{N}_G(H) : H]}(1 - \chi(\mathcal{F}\cdot H))).
\]

Recall that if \( G \curvearrowright X \), the orbit \( G \cdot x \) is said to be of type \( G/H \) if the stabilizer \( G_x \) is conjugate to \( H \) in \( G \). In other words, if the action of \( G \) on \( G \cdot x \) is the same as the action of \( G \) on \( G/H \).

Lemma 2.9 ([OS02, Lemma 2.3]). Fix a separating family \( \mathcal{F} \), a finite \( H \)-universal \((G, \mathcal{F})\)-complex \( X \), and a subgroup \( H \subseteq G \). For each \( n \), let \( c_n(H) \) denote the number of orbits of \( n \)-cells of type \( G/H \) in \( X \). Then \( i_{\mathcal{F}}(H) = \sum_{n \geq 0} (-1)^n c_n(H) \).

Proposition 2.10 ([OS02, Tables 2,3,4]). Let \( G \) be one of the simple groups \( \text{PSL}_2(2^k) \) for \( k \geq 2 \), \( \text{PSL}_2(q) \) for \( q \equiv \pm 3 \) (mod 8) and \( q \geq 5 \), or \( \text{Sz}(2^k) \) for odd \( k \geq 3 \). Then \( i_{\text{SL}_V}(1) = 1 \).

For each family of groups appearing in Theorem 2.2 Oliver and Segev describe an example. In what follows \( D_{2m} \) is a dihedral group of order \( 2m \) and \( C_m \) is a cyclic group of order \( m \).

Proposition 2.11 ([OS02, Example 3.4]). Set \( G = \text{PSL}_2(q) \), where \( q = 2^k \) and \( k \geq 2 \). Then there is a 2-dimensional acyclic fixed point free \( G \)-complex \( X \), all of whose isotropy subgroups are solvable. More precisely, \( X \) can be constructed to have three orbits of vertices with isotropy subgroups isomorphic to \( B = \mathbb{F}_q \rtimes C_{q-1} \), \( D_{2(q-1)} \), and \( D_{2(q+1)} \); three orbits of edges with isotropy subgroups isomorphic to \( C_{q-1}, C_2 \) and \( C_2 \); and one free orbit of 2-cells.

We have \( A_5 = \text{PSL}_2(2^2) \). The barycentric subdivision of the 2-skeleton of the Poincaré dodecahedral space is an \( A_5 \)-complex of the type given in Proposition 2.11 with fundamental group the binary icosahedral group \( A_5 \cong \text{SL}(2,5) \) which has order 120. The Poincaré dodecahedral space appears in many other natural ways, for more information see [KS79].

Proposition 2.12 ([OS02, Example 3.5]). Assume that \( G = \text{PSL}_2(q) \), where \( q = p^k \geq 5 \) and \( q \equiv \pm 3 \) (mod 8). Then there is a 2-dimensional acyclic fixed point free \( G \)-complex \( X \), all of whose isotropy subgroups are solvable. More precisely, \( X \) can be constructed to have four orbits of vertices with isotropy subgroups isomorphic to \( B = \mathbb{F}_q \rtimes C_{(q-1)/2}, D_{q-1}, D_{q+1}, \) and \( A_4 \); four orbits of edges with isotropy subgroups isomorphic to \( C_{(q-1)/2}, C_2^2, C_3 \) and \( C_2 \); and one free orbit of 2-cells.

Proposition 2.13 ([OS02, Example 3.7]). Set \( q = 2^{2k+1} \) for any \( k \geq 1 \). Then there is a 2-dimensional acyclic fixed point free \( \text{Sz}(q) \)-complex \( X \), all of whose isotropy subgroups are solvable. More precisely, \( X \) can be constructed to have four orbits of vertices with isotropy subgroups
isomorphic to $M(q, \theta) \oplus D_3 \oplus C_3 \times C_3 \times C_3$. There are four orbits of edges with isotropy subgroups isomorphic to $C_3 \times C_3 \times C_3$; four orbits of edges with isotropy subgroups isomorphic to $C_3 \rtimes C_3 \rtimes C_3$; and one free orbit of 2-cells.

We also have $A_5 \cong \text{PSL}_2(5)$, so this group is addressed in both Proposition 2.11 and Proposition 2.12. There is no other such exception.

**Definition 2.14.** If $G$ is one of the groups in Theorem 2.2, the Oliver–Segev $G$-graph $\Gamma_{0S}(G)$ is the 1-skeleton of any 2-dimensional fixed point free acyclic $G$-complex without free orbits of 1-cells of the type constructed in Propositions 2.11 to 2.13. For this definition, we regard $A_5$ as $\text{PSL}_2(5)$ rather than $\text{PSL}_2(2^2)$.

Generally, there is more than one possible choice for the $G$-graph $\Gamma_{0S}(G)$. Even for $G = A_5$, thought of as $\text{PSL}_2(2^2)$, the quotient graph $\Gamma_{0S}(G)/G$ is not unique. However in Proposition 3.16 we show that $\Gamma_{0S}(G)$ is unique up to $G$-homotopy equivalence. Moreover, Corollary 3.17 shows the particular choice of $\Gamma_{0S}(G)$ is irrelevant for our purposes.

**Definition 2.15 (Our choice of $\Gamma_{0S}(A_5)$).** Here we give a construction of $\Gamma_{0S}(A_5)$ and we fix some notation in regard to this graph. Consider the subgroups

$$H_1 = \langle (2, 5)(3, 4), (3, 5, 4) \rangle \cong A_4,$$

$$H_2 = \langle (3, 5, 4), (1, 2)(3, 5) \rangle \cong D_6,$$

$$H_3 = \langle (1, 2)(3, 5), (2, 5)(3, 4) \rangle \cong D_{10}.$$

of $A_5$. The graph $\Gamma_{0S}(A_5)$ has three orbits of vertices whose representatives $v_1$, $v_2$, $v_3$ have stabilizers $H_1$, $H_2$, $H_3$ respectively. In addition, $\Gamma_{0S}(A_5)$ has three orbits of edges whose representatives $v_1 \overset{e_{12}}{\rightarrow} v_2$, $v_3 \overset{e_{23}}{\rightarrow} v_1$ and $v_2 \overset{e_{31}}{\rightarrow} v_3$ have stabilizers $H_1 \cap H_2 \cong \mathbb{Z}_3$, $H_1 \cap H_3 \cong \mathbb{Z}_2$, and $H_2 \cap H_3 \cong \mathbb{Z}_2$.

Attaching a free orbit of 2-cells to $\Gamma_{0S}(A_5)$ along the orbit of the closed edge path $(e_{12}, e_{23}, e_{31})$ we obtain an acyclic 2-dimensional fixed point free $A_5$-complex of the type given in Proposition 2.11. This complex is in fact the barycentric subdivision of the 2-skeleton of the Poincaré dodecahedral space (a simplicial complex having 21 = 5 + 10 + 6 vertices, 80 = 20 + 30 + 30 edges and 60 faces). A concrete isomorphism can be produced by mapping $v_3$ to the barycentre of a pentagonal 2-cell $ABCDE$, $v_1$ to $A$ and $v_2$ to the barycentre (midpoint) of $AB$. For more details on this see [OS02, pp. 20-21].

Recall that the coset complex of a tuple of subgroups $(H_1, \ldots, H_k)$ of a group $G$ is the simplicial complex with vertex set $G/H_1 \times G/H_2 \times \cdots \times G/H_k$ having a simplex for every subset of vertices with nonempty intersection. In [OS02, p. 21] (see also [Ade03, Section 5]) it is explained that, for $G = \text{PSL}_2(2^k)$, the graph $\Gamma_{0S}(G)$ can be taken as the 1-skeleton of the coset complex of $(B, D_{2(q-1)}, D_{2(q+1)})$. Nevertheless, the coset complex itself is not acyclic in general (see [AS93]). In Figure 1 we see a picture of $\Gamma_{0S}(G)/G$ for this particular choice.

A key property of the $G$-graph $\Gamma_{0S}(G)$ is that $H_1(\Gamma_{0S}(G))$ is a free $\mathbb{Z}[G]$-module of rank 1. From [OS02, Proposition 1.7] we deduce

**Proposition 2.16.** Let $G$ be one of the groups in Theorem 2.2. A $G$-graph $\Gamma$ is a suitable choice for $\Gamma_{0S}(G)$ if and only if the following conditions hold:

(i) The orbits of $\Gamma$ have the types prescribed by Propositions 2.11 to 2.13.

(ii) $\Gamma$ is connected.

(iii) For each $1 \neq H \leq G$, $\Gamma^H$ is acyclic or empty and is acyclic if $H$ has prime power order.
This proposition is useful for testing if a $G$-graph is a suitable choice for $\Gamma_{OS}(G)$ without having an explicit attaching map for the orbit of 2-cells.

3. A reduction

In this section we rely on the results of Oliver and Segev to prove Theorem 3.12, which allows us to reduce the proof of Theorem 7.2 to the study of acyclic complexes of the type considered in [OS02]. We also prove Theorem 3.11 which roughly says that, assuming a special case of the Kervaire–Laudenbach–Howie conjecture (see Appendix A), if Conjecture 1.1 is false, then there is a counterexample of the type constructed in [OS02]. The special case we need is the following.

**Conjecture 3.1.** Let $X$ be a finite contractible 2-complex. If $A \subset X$ is an acyclic subcomplex, then $A$ is contractible.

In [Seg94] the conjecture is proved when $X$ is collapsible. By the work of Gerstenhaber-Rothaus [GR62], we know that Conjecture 3.1 holds under the hypothesis that $\pi_1(A)$ is locally residually finite. If the fundamental group of $A$ is hyperlinear then Conjecture 3.1 is known to hold (see [NT18, Theorem 1.2], see also [Tho12], [Pes08, Section 10]). Thus the following implies Conjecture 3.1.

**Conjecture 3.2** (Connes’ embedding conjecture for groups [Pes08]). *Every group is hyperlinear.*

We first prove some results which will be used to do equivariant modifications to our complexes.

**Definition 3.3.** If $X, Y$ are $G$-spaces, a $G$-homotopy is an equivariant map $H : X \times I \to Y$. We say that $f_0(x) = H(x, 0)$ and $f_1(x) = H(x, 1)$ are $G$-homotopic and we denote this by $f_0 \simeq_G f_1$. An equivariant map $f : X \to Y$ is a $G$-homotopy equivalence if there is a map $g : Y \to X$ such that $fg \simeq_G 1_Y$ and $gf \simeq_G 1_X$. A $G$-invariant subspace $A$ of $X$ is a strong $G$-deformation retract of $X$ if there is a retraction $r : X \to A$ such that there is a $G$-homotopy $H : ir \simeq 1_X$ relative to $A$, where $i : A \to X$ is the inclusion.

**Remark 3.4.** An equivariant map $f : X \to Y$ is a $G$-homotopy equivalence if and only if $f^H : X^H \to Y^H$ is a homotopy equivalence for each subgroup $H \leq G$ (see [tD08, (2.7) Proposition]). Thus, if $f : X \to Y$ is a $G$-homotopy equivalence, the action $G \curvearrowright X$ is fixed point free (resp. essential) if and only if the action $G \curvearrowright Y$ is fixed point free (resp. essential).

From the equivariant homotopy extension property for pairs of $G$-complexes (see [Bre67, Chapter I, Section 1]) we deduce the following.
Theorem 3.5. If $A$ is a $G$-subcomplex of a $G$-complex $X$ and the inclusion $A \hookrightarrow X$ is a $G$-homotopy equivalence, then $A$ is a strong $G$-deformation retract of $X$.

Lemma 3.6. Let $X$ be an acyclic 2-dimensional $G$-complex. Let $H \leq G$ and $x_0, x_1 \in X^{(0)} \cap X^H$. Then there is a $G$-complex $Y \supset X$, such that $X$ is a strong $G$-deformation retract of $Y$ and $Y$ is obtained from $X$ by attaching an orbit of 1-cells of type $G/H$ with endpoints $\{x_0, x_1\}$ and an orbit of 2-cells of type $G/H$.

Proof. We attach an orbit of 1-cells of type $G/H$ to $X$ using the attaching map $\varphi: G/H \times S^0 \to X^{(0)}$ defined by $(gH, 1) \mapsto g \cdot x_0$, $(gH, -1) \mapsto g \cdot x_1$. Let $e$ be the 1-cell of this new orbit corresponding to the coset $H$. Since $X$ is acyclic, by Theorem 2.4 $X^H$ is also acyclic. Let $\gamma$ be an edge path in $X^H$ starting at $x_1$ and ending at $x_0$. Then we attach an orbit of 2-cells of type $G/H$ in such a way that the 2-cell corresponding to the coset $H$ is attached along the closed edge path given by $e$ and $\gamma$. It is clear that $X$ is a strong $G$-deformation retract of $Y$. □

Remark 3.7. In the situation of Lemma 3.6, we say that $Y$ is obtained from $X$ by an equivariant elementary expansion of dimension 2 and type $G/H$ or that $X$ is obtained from $Y$ by an equivariant elementary collapse of dimension 2 and type $G/H$.

The following definitions appear in [KLV01, Section 2].

Definition 3.8. A forest is a graph with trivial first homology. If a subcomplex $\Gamma$ of a CW complex $X$ is a forest, there is a CW complex $Y$ obtained from $X$ by shrinking each connected component of $\Gamma$ to a point. The quotient map $q: X \to Y$ is a homotopy equivalence and we say $Y$ is obtained from $X$ by a forest collapse.

If $X$ is a $G$-complex and $\Gamma \subset X$ is a forest which is $G$-invariant, the quotient map $q$ is a $G$-homotopy equivalence and we say the $G$-complex $Y$ is obtained from $X$ by a $G$-forest collapse. We say that a $G$-graph is reduced if it has no edge $e$ such that $G \cdot e$ is a forest.

Lemma 3.9. Let $X$ be a 2-dimensional acyclic $G$-complex. If $X^{(1)}$ is a reduced $G$-graph then stabilizers of different vertices are not comparable.

Proof. Let $\mathcal{F} = \{G_x : x \in X^{(0)}\}$ and let $M = \{v \in X^{(0)} : G_v \text{ is maximal in } \mathcal{F}\}$. We first prove, by contradiction, that $X^{(0)} = M$. Consider $v \in X^{(0)} \setminus M$ such that $G_v$ is maximal in $\{G_x : x \in X^{(0)} \setminus M\}$. Then since $X^{G_v}$ contains $v$, by Theorem 2.4 it must be acyclic. Since $v \notin M$, there is a vertex $w \in X^{G_v} \cap M$. By connectivity there is an edge $e \in X^{G_e}$ whose endpoints $v'$ and $w'$ satisfy $v' \notin M$ and $w' \in M$. Since $G_{v'} \geq G_v$ and $v' \notin M$, by our choice of $v$ we have $G_v = G_{v'}$. Since $e \in X^{G_{v'}}$ we have $G_v \leq G_e$ and since $v'$ is an endpoint of $e$ we have $G_{v'} \leq G_e$. Thus $G_e = G_{v'}$. Then the degree of $v'$ in the graph $G \cdot e$ (which has vertex set $G \cdot w' \coprod G \cdot v'$) is 1. Thus $G \cdot e$ is a forest, contradiction. Therefore we must have $M = X^{(0)}$. To conclude we have to prove that different vertices $u, v \in M$ have different stabilizers. Suppose $G_u = G_v$ to get a contradiction. Since $u, v$ are vertices of $X^{G_u}$ which is connected, there is an edge $e \in X^{G_u}$ and by maximality we must have $G_e = G_u$. If $u', v'$ are the endpoints of $e$, we have $G_{u'} = G_{v'}$. We have two cases and in any case we obtain a contradiction. If $G \cdot u' \neq G \cdot v'$ then $G \cdot e$ is a forest consisting of $|G/G_e|$ parallel edges, contradiction. Otherwise, there is a nontrivial element $g \in G$ such that $g \cdot u' = v'$ and we have $G_{u'} = G_{v'} = gG_{u'}g^{-1}$. Thus $g \in N_G(G_{u'})$. Consider the action of $\langle g \rangle$ on $X^{G_{u'}}$, which is acyclic and thus has a fixed point by the Lefschetz fixed point theorem. But this cannot happen, since this would imply that $|G_{u'} : G_{u'}| \geq |G_{u'}|$ fixes a point of $X$, which is a contradiction since $u' \in M$. □
Corollary 3.10. If \(X\) is a 2-dimensional acyclic \(G\)-complex and \(X^G\) is nonempty then there is a \(G\)-invariant maximal tree.

Proof. We define a sequence of \(G\)-complexes \(X_0, \ldots, X_k\) such that \(X_i^G \neq \emptyset\). Let \(X_0 = X\). If \(X_i\) is defined and \(X_i^{(0)} \neq \ast\) then by Lemma 3.9 there is an edge \(e_{i+1}\) of \(X_i\) such that \(G \cdot e_{i+1}\) is a forest. Then \(X_{i+1}\) is obtained from \(X_i\) by collapsing the \(G\)-forest \(G \cdot e_{i+1}\). Then \(G \cdot \{e_1, \ldots, e_k\}\) is a \(G\)-invariant spanning tree for \(X\).

Now we prove the main results of the section.

Theorem 3.11. Assume Conjecture 3.1 holds. If Conjecture 1.1 is false, then there is a 2-dimensional essential, fixed point free and contractible \(G\)-complex \(X\), where \(G\) is one of the following groups:

(i) \(\text{PSL}_2(2^p)\) for \(p\) prime.
(ii) \(\text{PSL}_2(3^p)\) for an odd prime \(p\).
(iii) \(\text{PSL}_2(q)\) for a prime \(q > 3\) such that \(q \equiv \pm 3 \mod 5\) and \(q \equiv \pm 3 \mod 8\).
(iv) \(\text{Sz}(2^p)\) for \(p\) an odd prime.

Proof. Suppose \(X\) is a counterexample for Conjecture 1.1. We may assume that \(|G|\) is minimal. Since we are assuming Conjecture 3.1, by Theorem 2.3 we have that \(X\) is essential. Then \(G\) must be one of the groups in Theorem 2.2. By minimality of \(|G|\), we have that \(X^H \neq \emptyset\) for every \(H \leq G\). Then by Lemma 2.6 \(X\) is an H-universal \((G, \mathcal{MAAX})\)-complex. By Proposition 2.7, we must have \(\mathcal{MAAX} = \mathcal{SLV}\). Then every proper subgroup of \(G\) is solvable. By [OS02, Proposition 3.3], if every proper subgroup of a group \(\text{PSL}_2(2^k)\) \((k \geq 2)\) is solvable then \(k\) is a prime (note that when \(k = 2\) the group is \(A_5\)). Also by [OS02, Proposition 3.3], if every proper subgroup of a group \(\text{PSL}_2(q)\) \((q = \pm 3 \mod 8, q > 5)\) is solvable then either \(q = 3^p\) for \(p\) an odd prime or \(q\) is prime and \(q \equiv \pm 3 \mod 5\) (since otherwise \(A_5\) is a subgroup). Finally by [OS02, Proposition 3.6], if every proper subgroup of a group \(\text{Sz}(2^k)\) is solvable then \(k\) is an odd prime. Thus \(G\) is one of the groups in the statement of Theorem 3.11.

Theorem 3.12. Let \(G\) be one of the following groups

(i) \(\text{PSL}_2(2^p)\) for \(p\) prime.
(ii) \(\text{PSL}_2(3^p)\) for an odd prime \(p\).
(iii) \(\text{PSL}_2(q)\) for a prime \(q > 3\) such that \(q \equiv \pm 3 \mod 5\) and \(q \equiv \pm 3 \mod 8\).
(iv) \(\text{Sz}(2^p)\) for \(p\) an odd prime.

Let \(X\) be a fixed point free 2-dimensional finite acyclic \(G\)-complex. Then there is a fixed point free 2-dimensional finite acyclic \(G\)-complex \(X'\) obtained from the \(G\)-graph \(\Gamma_{OS}(G)\) by attaching \(k \geq 0\) free orbits of 1-cells and \(k + 1\) free orbits of 2-cells and an epimorphism \(\pi_1(X) \to \pi_1(X')\).

Proof. Note that \(G\) satisfies \(\mathcal{SLV} = \mathcal{MAAX}\). By doing enough \(G\)-forest collapses we can assume that \(X^{(1)}\) is a reduced \(G\)-graph. The stabilizers of the vertices of \(\Gamma_{OS}(G)\) are precisely the maximal subgroups of \(G\). Therefore, since every proper subgroup of \(G\) fixes a point of \(X\), by Lemma 3.9, we have \(X^{(0)} = \Gamma_{OS}(G)^{[0]}\). Applying Lemma 3.6 enough times to modify \(X\), we may further assume \(\Gamma_{OS}(G)\) is a subcomplex of \(X\).

Finally we will modify \(X\) so that for every subgroup \(1 \neq H \leq G\), we have \(X^H = \Gamma_{OS}(G)^H\). We do this by reverse induction on \(|H|\). Assume that we have \(X\) such that it holds for every subgroup \(K\) with \(H \leq K \leq G\). Since \(\Gamma_{OS}(G)^H\) is a tree (it is acyclic and 1-dimensional) and \(X^H\) is acyclic by Theorem 2.4, the inclusion \(\Gamma_{OS}(G)^H \hookrightarrow X^H\) is a \(N_G(H)\)-homology equivalence. Now since \(\Gamma_{OS}(G)^H\) is a tree we can define a \(N_G(H)\)-retraction \(r_H : X^H \to \Gamma_{OS}(G)^H\). Then \(r_H\) is a
homology equivalence. Moreover, the stabilizer of the cells in $X^H - \Gamma_{OS}(G)^H$ is $H$ (the stabilizer cannot be bigger by the induction hypothesis). We define retractions $r_{H^g} : X^{H^g} \to \Gamma_{OS}(G)^{H^g}$ by $r_{H^g}(gx) = g \cdot r_H(x)$ which glue to give a homology equivalence

$$r : \Gamma_{OS}(G) \bigcup_{g \in G} X^{H^g} \to \Gamma_{OS}(G).$$

We may replace $X$ by the pushout $\tilde{X}$ given by the following diagram

$$\begin{array}{ccc}
\Gamma_{OS}(G) \bigcup_{g \in G} X^{H^g} & \xrightarrow{r} & \Gamma_{OS}(G) \\
\downarrow & & \downarrow \\
X & \xrightarrow{r} & \tilde{X}
\end{array}$$

It follows that $\tilde{r}$ is a homology equivalence, so the resulting $G$-complex $\tilde{X}$ is acyclic. Moreover since $\tilde{X}^{(1)}$ is a subcomplex of $X^{(1)}$ and the restriction $\tilde{r} : X^{(1)} \to \tilde{X}^{(1)}$ is a retraction, $\tilde{r}$ induces an epimorphism on $\pi_1$. This procedure removes the excessive orbits of cells of type $G/H$. By induction we obtain a complex $X'$ such that $X'^{(1)}$ coincides with $\Gamma_{OS}(G)$ up to $k \geq 0$ free orbits of 1-cells. By Lemma 2.6 $X'$ is an $H$-universal $(G, SV)$-complex. Now by Lemma 2.9 we conclude that every orbit of 2-cells of $X'$ is free and by Proposition 2.10 there are exactly $k + 1$ orbits of 2-cells.

**Remark 3.13.** If we are willing to assume Conjecture 3.1, from Theorem 3.5 it follows that there is a $G$-homotopy equivalence $X \to X'$.

**Corollary 3.14.** Let $G$ be one of the groups in Theorem 3.12. If Conjecture 1.1 is false for the group $G$, then there is a counterexample obtained from the $G$-graph $\Gamma_{OS}(G)$ by attaching $k \geq 0$ free orbits of 1-cells and $k + 1$ free orbits of 2-cells.

In particular we have the following:

**Corollary 3.15.** Assuming Conjecture 3.1, if Conjecture 1.1 is false, then there is a counterexample where every orbit of 2-cells is free.

The following explains why our particular choice of $\Gamma_{OS}(G)$ and the way the free orbits of 1-cells are attached is not relevant.

**Proposition 3.16.** Any two choices for $\Gamma_{OS}(G)$ are $G$-homotopy equivalent. Moreover, attaching $k \geq 0$ free orbits of 1-cells to any two choices for $\Gamma_{OS}(G)$ produces $G$-homotopy equivalent graphs.

**Proof.** Since any choice of $\Gamma_{OS}(G)$ is a universal $(G, SV - \{1\})$-complex, the first part follows from [OS02, Proposition A.6]. The second part follows easily from the first and [Bro06, 7.5.7].

**Corollary 3.17.** Let $\Gamma$ be a graph obtained from $\Gamma_{OS}(G)$ by attaching $k \geq 0$ free orbits of 1-cells. The set of $G$-homotopy equivalence classes of 2-dimensional acyclic fixed point free $G$-complexes with 1-skeleton $\Gamma$ does not depend on the particular choice of $\Gamma_{OS}(G)$ or the way the $k$ free orbits of 1-cells are attached. In particular, the set of isomorphism classes of groups that occur as the fundamental group of such spaces does not depend on such choices.

**Proof.** Again, this is an easy application of [Bro06, 7.5.7].
4. BROWN’S SHORT EXACT SEQUENCE

Using Bass-Serre theory, K.S. Brown gave a method to produce a presentation for a group $G$ acting on a simply connected complex $X$ [Bro84, Theorem 1]. When $X$ is not simply connected, Brown describes a presentation for an extension $\widetilde{G}_X$ of $G$ by $\pi_1(X)$ [Bro84, Theorem 2]. The group $\widetilde{G}_X$ has a description as a quotient of the fundamental group of a graph of groups. A similar result in the simply connected case was given by Corson [Cor92, Theorem 5.1] in terms of complexes of groups (higher dimensional analogues of graphs of groups).

Using Brown’s result we translate the $A_5$ case of Conjecture 1.1 into a nice looking problem in combinatorial group theory. This translation can be done in general, but to obtain similar results for the rest of the groups $G$ that appear in Theorem 3.12 we need a choice of $\Gamma_{OS}(G)$ and presentations for the stabilizers of its vertices.

In Brown’s original formulation, the result deals with actions that need not to be admissible (Brown uses the term $G$–$CW$-complex in a different way than us). Since the actions we are interested in are admissible, we state Brown’s result only in that case.

Let $X$ be a connected $G$-complex. By admissibility of the action, the group $G$ acts on the set of oriented edges. The group $\widetilde{G}_X$ depends on a number of choices that we now specify. If $e$ is an oriented edge, the same 1-cell with the opposite orientation is denoted by $e^{-1}$. Each oriented edge $e$ has a source and target, denoted by $s(e)$ and $t(e)$ and for every $g \in G$ we have $g \cdot s(e) = s(g \cdot e)$ and $g \cdot t(e) = t(g \cdot e)$. For each 1-cell of $X$ we choose a preferred orientation in such a way that these orientations are preserved by $G$. This determines a set $P$ of oriented edges. We choose a tree of representatives for $X/G$. That is, a tree $T \subset X$ such that the vertex set $V$ of $T$ is a set of representatives of $X^{(0)}/G$. Such tree always exists and the 1-cells of $T$ are inequivalent modulo $G$. We give an orientation to the 1-cells of $T$ so that they are elements of $P$. We also choose a set of representatives $E$ of $P/G$ in such a way that $s(e) \in V$ for every $e \in E$ and such that each oriented edge of $T$ is in $E$. If $e$ is an oriented edge, the unique element of $V$ that is equivalent to $t(e)$ modulo $G$ will be denoted by $w(e)$. For every $e \in E$ we choose an element $g_e \in G$ such that $t(e) = g_e \cdot w(e)$. If $e \in T$, we specifically choose $g_e = 1$. For each orbit of 2-cells we choose a closed edge path $\tau$ based at a vertex of $T$ and representing the attaching map for this orbit of 2-cells. Let $F$ be the set given by these closed edge paths.

The group $\widetilde{G}_X$ is defined as a quotient of

$$\bigast_{v \in V} G_v \ast \bigast_{e \in E} \mathbb{Z}$$

by certain relations. In order to define these relations we introduce some notation. If $v \in V$ and $g \in G_v$ we denote the copy of $g$ in the free factor $G_v$ by $g_v$. The generator of the copy of $\mathbb{Z}$ that corresponds to $e$ is denoted by $x_e$. The relations are the following:

(i) $x_e = 1$ if $e \in T$.
(ii) $x_e^{-1} g_{s(e)} x_e = (g_e^{-1} g g_e) w(e)$ for every $e \in E$ and $g \in G_v$.
(iii) $r_{\tau} = 1$ for every $\tau \in F$.

We state Brown’s theorem before giving the definition of the element $r_\omega$ associated to a closed edge path $\omega$.

**Theorem 4.1** (Brown, [Bro84, Theorems 1 and 2]). The group

$$\widetilde{G}_X = \bigast_{v \in V} G_v \ast \bigast_{e \in E} \mathbb{Z} / \langle R \rangle$$
where $R$ consists of relations (i)-(iii) is an extension

$$1 \to \pi_1(X, x_0) \xrightarrow{i} \tilde{G}_X \xrightarrow{\bar{\phi}} G \to 1.$$ 

The map $\bar{\phi}$ is defined passing to the quotient the coproduct $\phi$ of the inclusions $G_e \to G$ and the mappings $\mathbb{Z} \to G$ given by $x_e \mapsto g_e$. The map $i$ sends a closed edge path $\omega$ based at $x_0 \in V$ to $r_\omega$.

The group $\tilde{G}_X$ can be described as the quotient of the fundamental group of certain graphs of groups by relations of type (iii). Now we explain how to obtain the elements $r_\tau$. If $\alpha$ is an oriented edge, we define

$$\varepsilon(\alpha) = \begin{cases} 1 & \alpha \in P \\ -1 & \alpha \notin P \end{cases}$$

and we can always take $e \in E$ and $g \in G$ such that $\alpha = g\varepsilon(\alpha)$. Note that $e$ is unique but $g$ is not. Moreover, if $\alpha$ starts at $v \in V$, we can write

$$\alpha = \begin{cases} h \varepsilon(\alpha) & \text{with } h \in G_{s(\alpha)}, \text{ if } \alpha \in P \\ h g_i^{-1} e^{-1} & \text{with } h \in G_{w(\alpha)}, \text{ if } \alpha \notin P \end{cases}$$

Again, $h$ is not unique.

Now if $\tau = (\alpha_1, \ldots, \alpha_n)$ is a closed edge path starting at a vertex $v_0 \in V$ we define an element $r_\tau \in \bigast_{e \in E} G_e \ast \mathbb{Z}$. Recursively, we define some sequences. Since the oriented edge $\alpha_1$ starts at $v_0 \in V$, we can obtain an oriented edge $e_1$ and an element $h_1 \in G_{v_0}$ as above. We set $\varepsilon_1 = \varepsilon(\alpha_1)$ and $g_1 = h_1 g_{e_1}^{-1}$. Set $v_1 = v(\varepsilon_1)$ if $\alpha_1 \in P$ and otherwise $v_1 = v(s(\varepsilon_1))$. Now suppose we have defined $e_1, \ldots, e_k, h_1, \ldots, h_k, \varepsilon_1, \ldots, \varepsilon_k, g_1, \ldots, g_k$ and $v_1, \ldots, v_k$. The oriented edge $(g_1 g_2 \cdots g_k)^{-1} \varepsilon_{k+1}$ starts at $v_k \in V$, so we can obtain an oriented edge $e_{k+1}$ and an element $h_{k+1} \in G_{v_k}$ as before. We set $\varepsilon_{k+1} = \varepsilon(\alpha_{k+1})$ and $g_{k+1} = h_{k+1} g_{e_{k+1}}$. Set $v_{k+1} = w(\varepsilon_{k+1})$ if $\alpha_{k+1} \in P$ and otherwise $v_{k+1} = s(\varepsilon_{k+1})$. When we conclude, we can insert an element $g_1 g_2 \cdots g_n \in G_{v_0}$. Finally the relation associated to $\tau$ is given by

$$r_\tau = (h_1)_{v_0} x_{e_1}^{\varepsilon_1} (h_2)_{v_1} x_{e_2}^{\varepsilon_2} \cdots (h_n)_{v_{n-1}} x_{e_n}^{\varepsilon_n} (g_1 g_2 \cdots g_n)_{v_0}^{-1}.$$ 

The description of the inclusion $i$ along with the exactness at the middle in Brown’s short exact sequence say that for any word in $w \in \ker(\phi)$ we can find a closed edge path $\omega$ for a 2-cell such that $w = r_\omega$. We give a hands-on proof of this fact.

**Proposition 4.2.** Let $\Gamma$ be a $G$-graph and let $w \in \tilde{G}_\Gamma$. If $\bar{\phi}(w) = 1$, then there is a closed edge path $\omega$ such that $w = r_\omega$.

**Proof.** Consider a word in $\bigast_{e \in E} G_e \ast \mathbb{Z}$ representing $w$. If we insert letters $x_e$ with $e \in T$ and $1_{G_e}$ with $v \in V$ this word still represents $w$. Using these two moves we can assume the word has the form

$$(h_1)_{v_0} x_{e_1}^{\varepsilon_1} (h_2)_{v_1} x_{e_2}^{\varepsilon_2} \cdots (h_n)_{v_{n-1}} x_{e_n}^{\varepsilon_n}$$

and that we have $v_i = t(e_i^{\varepsilon_i}) = s(e_{i+1}^{\varepsilon_{i+1}}) \mod G$ for $i = 1, \ldots, n - 1$ and $v_0 = t(e_n^{\varepsilon_n}) = s(e_1^{\varepsilon_1}) \mod G$. Let $g_i = h_i g_{e_i}^{\varepsilon_i}$. Then setting

$$\alpha_i = \begin{cases} g_1 \cdots g_{i-1} h_i e_i & \text{if } \varepsilon_i = 1 \\ g_1 \cdots g_{i-1} h_i g_{e_i}^{\varepsilon_i} e_i & \text{if } \varepsilon_i = -1 \end{cases}$$

we have that $\omega = (\alpha_1, \ldots, \alpha_n)$ is a closed edge path. Moreover, we have $r_\omega = w$. 

\qed
A closed edge path $\omega$ in $X$ determines a conjugacy class $[\omega]$ of $\pi_1(X)$. The following describes the conjugation action of $\tilde{G}_X$ on $\pi_1(X)$.

**Proposition 4.3** ([Bro84, Proposition 1]). Let $\omega$ be a closed edge path in $X$ and $g \in G$. Then the conjugacy classes $[\omega]$ and $[g\omega]$ of $\pi_1(X)$ are contained in the same $\tilde{G}_X$-conjugacy class. Moreover for any element $\tilde{g} \in \tilde{\phi}^{-1}(g)$ we have $[\omega]^{\tilde{g}} = [g\omega]$.

The following proposition summarizes many ideas of this section.

**Proposition 4.4.** Let $\Gamma$ be a $G$-graph and let $w_1, \ldots, w_k \in \ker(\tilde{\phi} : \tilde{G}_{\Gamma} \to G)$. Then there is a 2-complex $X$ obtained by attaching orbits of 2-cells to $\Gamma$ along closed edge paths $\omega_1, \ldots, \omega_k$ such that $r_{\omega_i} = w_i$ and we have the following diagram with exact rows and columns.

\[
\begin{array}{ccccccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & \langle \langle \omega_i \rangle \rangle^{\tilde{G}_\Gamma} & \langle \langle w_i \rangle \rangle^{\tilde{G}_\Gamma} & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \pi_1(\Gamma) & \tilde{G}_\Gamma & \tilde{\phi} & G & 1 \\
\downarrow & i & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \pi_1(X) & \tilde{G}_X & \tilde{\phi} & G & 1 \\
\end{array}
\]

**Remark 4.5.** If $X$ is a connected $G$-complex, the group $\tilde{G}_X$ is isomorphic to the group formed by the lifts $\tilde{g}$ of elements $g : X \to X$ to the universal cover $\tilde{X}$ of $X$ (see [Bro84]). Suppose $Y$ is another $G$-complex and $h : X \to Y$ is equivariant and a homotopy equivalence. Let $\tilde{h} : \tilde{X} \to \tilde{Y}$ be a lift of $h$ to the universal covers. Then if $g \in G$, for each lift $\tilde{g}_X : \tilde{X} \to \tilde{X}$ of $g : X \to X$ there is a unique lift $\tilde{g}_Y : \tilde{Y} \to \tilde{Y}$ of $g : Y \to Y$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{h}} & \tilde{Y} \\
\downarrow {\tilde{g}_X} & & \downarrow {\tilde{g}_Y} \\
\tilde{X} & \xrightarrow{\tilde{h}} & \tilde{Y}
\end{array}
\]

Then it is easy to check that there is an isomorphism $\tilde{G}_X \to \tilde{G}_Y$ given by $\tilde{g}_X \mapsto \tilde{g}_Y$. In particular, the isomorphism type of $\tilde{G}_{\Gamma_{OS}(G)}$ does not depend on any choice.

We now apply Brown’s result Consider the following subgroups of $G = A_5$.

\[
\begin{align*}
H_1 &= \langle (2, 5)(3, 4), (3, 5, 4) \rangle \cong A_4 \\
H_2 &= \langle (3, 5, 4), (1, 2)(3, 5) \rangle \cong D_6 \\
H_3 &= \langle (1, 2)(3, 5), (2, 5)(3, 4) \rangle \cong D_{10}.
\end{align*}
\]

Recall that we can take $\Gamma_{OS}(A_5)$ to be the 1-skeleton of the coset complex of $(H_1, H_2, H_3)$. Suppose that we have an acyclic 2-complex $X$ obtained from $\Gamma_{OS}(A_5)$ by attaching a free $A_5$-orbit of 2-cells. We want to apply Brown’s method to obtain a presentation for the extension $\tilde{G}_X$. We consider the vertices $v_1 = H_1, v_2 = H_2$ and $v_3 = H_3$ of $\Gamma_{OS}(A_5)$. Then the stabilizers
of the oriented edges $e_{12} = (v_1 \rightarrow v_2)$, $e_{23} = (v_2 \rightarrow v_3)$, $e_{31} = (v_3 \rightarrow v_1)$ are

$$H_{12} = H_1 \cap H_2 = (3,5,4)$$

$$H_{23} = H_2 \cap H_3 = (1,2)(3,5)$$

$$H_{13} = H_1 \cap H_3 = ((2,5)(3,4)).$$

We take $T = \{e_{12}, e_{23}\}$. Thus $V = \{v_1, v_2, v_3\}$. We take $E = \{e_{12}, e_{23}, e_{31}\}$. Note that we have $w(e) = t(e)$ for every $e \in E$. We can take $g_e = 1$ for every $e \in E$.

Then Brown’s result gives

$$\tilde{G}_X = \frac{(H_1 *_{H_{12}} H_2 *_{H_{23}} H_3) *_{H_{13}} \langle \langle w \rangle \rangle}{\langle \langle w \rangle \rangle}$$

We explain this. First we amalgamate the groups $H_1$, $H_2$, $H_3$ identifying the copy of $H_{12}$ in $H_1$ with the copy of $H_{12}$ in $H_2$ and the copy of $H_{23}$ in $H_2$ with the copy of $H_{23}$ in $H_3$. This comes from the relations of type (ii) for $e \in T$. Then we form an HNN extension with stable letter $x = x_{e_{31}}$ that corresponds to the relation of type (ii) coming from $e_{31}$. The associated subgroups of this HNN extension are the copies of $H_{13}$ in $H_1$ and $H_3$. The quotient by the word $w$ comes from the only relation of type (iii).

Now we obtain an explicit presentation for $\tilde{G}_X$. We have $A_4 \cong \langle a, b \mid a^2, b^3, (ab)^3 \rangle$ via $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$. We have $D_6 \cong \langle b, c \mid b^3, c^2, (bc)^2 \rangle$ via $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$. Finally $D_{10} \cong \langle c, d \mid c^2, d^2, (cd)^5 \rangle$ via $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$. Thus we have a presentation

$$\tilde{G}_X = \langle a, b, c, d, x \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, xax^{-1} = d, w \rangle$$

where the word $w$ depends on the attaching map. The mapping $\bar{\phi}: \tilde{G}_X \rightarrow A_5$ is given by $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ and $x \mapsto 1$. Now if we also take into account $k$ additional free orbits of 1 and 2 cells and we recall Corollary 3.14, we obtain the following.

**Theorem 4.6.** The following are equivalent.

(i) Every finite, 2-dimensional contractible $A_5$-complex has a fixed point.

(ii) There is no presentation of $A_5$ of the form

$$\langle a, b, c, d, x_0, \ldots, x_k \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x_0ax_0^{-1} = d, w_0, \ldots, w_k \rangle$$

with $w_0, \ldots, w_k \in \ker(\phi)$, where $\phi: F(a, b, c, d, x_0, \ldots, x_k) \rightarrow A_5$ is given by $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ and $x_i \mapsto 1$.

5. A Moduli of Representations

In order to prove Theorem 7.1 we define a moduli of representations of the group

$$\Gamma_k = \langle a, b, c, d, x_0, \ldots, x_k \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x_0ax_0^{-1} = d \rangle$$

in $SO(3)$. Our argument is inspired by James Howie’s proof of the Scott–Wiegold conjecture [How02]. If $x, y \in \mathbb{C}$ we consider the matrix

$$\rho(x, y) = \begin{pmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which lies in $SO(3, \mathbb{C})$ whenever $x^2 + y^2 = 1$. Recall that $SO(n, \mathbb{C})$ is the group of matrices $M \in M_n(\mathbb{C})$ such that $M \cdot M^T = I$ and $\det(M) = 1$. 
**Theorem 5.1.** If \((x_1, y_1, x_2, y_2, x_3, y_3, X_1, \ldots, X_k) \in \mathbb{C}^6 \times \text{SO}(3, \mathbb{C})^k\) satisfies \(x_i^2 + y_i^2 = 1\) for \(i = 1, 2, 3\) then there is a group representation

\[ \Gamma_k \to \text{SO}(3, \mathbb{C}) \]

defined by the following matrices

\[
A = \begin{pmatrix}
-1 & 0 & 0 \\
0 & \frac{1}{3} & -\frac{2}{3} \sqrt{2} \\
0 & -\frac{2}{3} \sqrt{2} & -\frac{2}{3}
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
-\frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2} & 0 \\
\frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
\[
C = \rho(x_1, y_1) S_0 \rho(x_1, y_1)^T,
\]
\[
D = \rho(x_1, y_1) S_1 \rho(x_2, y_2) S_2 \rho(x_2, y_2)^T S_1^T \rho(x_1, y_1)^T,
\]
\[
X_0 = \rho(x_1, y_1) S_1 \rho(x_2, y_2) S_3 \rho(x_3, y_3) S_4.
\]

**Remark 5.2.** The coefficients of \(S_2\) and \(S_3\) are algebraic numbers. We will also view the matrices \(A, B, C, D, X_0\) as elements of \(M_3(\mathbb{C}[x_1, y_1, x_2, y_2, x_3, y_3])\).

**Remark 5.3.** This family of representations was obtained in the following way. We first obtained a single representation of the group \(\Gamma_0\) in \(\text{SO}(3, \mathbb{R})\) by choosing reflections \(r_1, r_2, r_3, r_4, r_5\) with axes forming the appropriate angles so that \(a \mapsto r_1 r_2, b \mapsto r_2 r_3, c \mapsto r_3 r_4\) and \(d \mapsto r_4 r_5\) defines a representation of the (alternating Coxeter) group generated by \(a, b, c,\) and \(d\). Since \(r_1 r_2\) and \(r_4 r_5\) are rotations of the same angle, they are conjugate, so it is possible to extend this to a representation of \(\Gamma_0\) by mapping \(x_0 \to r_6\). Then we twisted this representation in the following way to obtain three degrees of freedom. If \(\theta_1, \theta_2, \text{and} \theta_3\) are rotations commuting with \(r_1 r_2, r_2 r_3, \text{and} r_3 r_4\) respectively then \(a \mapsto \theta_1 r_1 r_2, b \mapsto \theta_1 r_2 r_3 \theta_1^{-1}, c \mapsto \theta_1 r_3 r_4 \theta_2^{-1} \theta_1^{-1}, d \mapsto \theta_1 \theta_2 \theta_3 r_4 r_5 \theta_3^{-1} \theta_2^{-1} \theta_1^{-1}\) and \(x_0 \to \theta_1 \theta_2 \theta_3 r_6\) gives a representation of \(\Gamma_0\). After tidying up these computations we obtain the moduli in Theorem 5.1.

Let \(\tilde{\phi}: \Gamma_k \to A_5\) be the homomorphism induced by \(\phi: F(a, b, c, d, x_0, \ldots, x_k) \to A_5\).

**Lemma 5.4.** We have \(\ker(\tilde{\phi}) = \langle x_0, \ldots, x_k, (bac)^3 \rangle\).

**Proof.** Since \(\Gamma_k = \Gamma_0 \ast F(x_1, \ldots, x_k)\) this reduces to the case \(k = 0\) which is proved by a GAP [GAP19] computation. The code appears in Appendix C. \(\square\)
Proposition 5.5. Let $w_0, \ldots, w_k \in \ker(\phi)$. If the group $\Gamma_k$ admits a representation $\rho$ such that 
(i) $\rho(w_i) = 1$ for each $i = 0, \ldots, k$ and 
(ii) there exists $r \in \{x_0, \ldots, x_k, (bac)^3\}$ such that $\rho(r) \neq 1$
then $\Gamma_k/\langle \langle w_0, \ldots, w_k \rangle \rangle \xrightarrow{\phi} A_5$ is not an isomorphism.

Proof. This follows from Lemma 5.4. \hfill \square

Remark 5.6. Note that in some cases (for example when $k = 0$ and $w_0 = x_0$) a representation
of $\Gamma_k \to SO(3, \mathbb{C})$ with image isomorphic to $A_5$ may suffice to conclude that $\Gamma_k/\langle \langle w_0, \ldots, w_k \rangle \rangle$
is not $A_5$. This may seem counterintuitive.

Remark 5.7. Given a family $\{w_i\}_{i \in I}$ of words in $F(a, b, c, d, x_0, \ldots, x_k)$, the set of points in
$\mathbb{C}^6 \times SO(3, \mathbb{C})^k \subseteq \mathbb{C}^{6+9k}$ such that $\rho(w_i) = 1$ for all $i \in I$ is an affine algebraic variety that we
denote $Z(\{w_i : i \in I\})$. For $k = 0$ the variety $Z(w_0)$ can be described with only 6 equations.
More generally, if we allow $X_1, \ldots, X_k$ to take values in $O(3, \mathbb{C})$ the variety $Z(\{w_0, \ldots, w_k\})$ can
be described using $6 + 9k$ equations. This suggests that it may be possible to use a result such
as Bézout’s theorem to count points. We could not finish this approach so we took a different
one.

Proposition 5.8. There is exactly one choice of

$$(x_1, y_1, x_2, y_2, x_3, y_3, X_1, \ldots, X_k) \in \mathbb{C}^6 \times SO(3, \mathbb{C})^k$$

with $x_i^2 + y_i^2 = 1$ for $i = 1, 2, 3$ such that the matrices in Theorem 5.1 satisfy

$$X_0 = X_1 = \ldots = X_k = (BAC)^3 = 1.$$

The unique solution

$$z_u = (x_1^u, y_1^u, x_2^u, y_2^u, x_3^u, y_3^u, 1, \ldots, 1)$$

is real and its exact value is given by

$$x_1^u = -\frac{1}{4} \sqrt{3 \sqrt{5} + 9} \quad x_2^u = -\sqrt{-\frac{2}{15} \sqrt{5} + \frac{1}{3}} \quad x_3^u = -\sqrt{-\frac{1}{5} \sqrt{5} + \frac{1}{2}}$$

$$y_1^u = \frac{1}{4} \sqrt{-3 \sqrt{5} + 7} \quad y_2^u = \sqrt{\frac{2}{15} \sqrt{5} + \frac{2}{3}} \quad y_3^u = \sqrt{\frac{1}{5} \sqrt{5} + \frac{1}{2}}$$

Proof. Again this reduces to the case $k = 0$ which can be proved with a SAGE computation.
Note that SAGE computes exactly over the algebraic numbers so there is no numerical error.
The proof of this proposition is given by the function find_universal_representations in
Appendix D. \hfill \square

Remark 5.9. We say that $z_u$ is universal in the following sense: if $\{w_i\}_{i \in I} \subseteq \ker(\phi)$ then
$z_u \in Z(\{w_i : i \in I\})$.

The following result is proved in Section 6.

Theorem 5.10. Let $w_0, \ldots, w_k \in \ker(\phi)$ and assume that $x_0, \ldots, x_k$ belong to the normal closure
of $\{w_0, \ldots, w_k\}$ in $\Gamma_k$. Then the variety $Z(w_0, \ldots, w_k)$ has at least two different points.
6. Quaternions

To prove Theorem 5.10 we will study the real part of the moduli, working with quaternions instead of rotation matrices. This is useful because representing a rotation as a quaternion allows to find the axis easily.

Recall that \( S^3 = \{ q \in \mathbb{H} : |q| = 1 \} \) acts on \( S^2 = \{ bi + cj + dk : b^2 + c^2 + d^2 = 1 \} \) by conjugation. Recall that any element of \( S^3 \setminus \{-1,1\} \) can be written as \( \cos(\theta/2) + \sin(\theta/2)q \) with \( \theta \in [0,2\pi] \) and \( q = bi + cj + dk \in S^2 \). Then there is a homomorphism \( p: S^3 \to SO(3,\mathbb{R}) \) which sends \( \cos(\theta) + \sin(\theta)\overline{(bi + cj + dk)} \) to the rotation matrix with angle \( \theta \) and axis \((b,c,d)\). We have \( \ker(p) = \{1, -1\} \).

Let \( \psi: \mathbb{H} \to \mathbb{R}^3 \) be given by \( a + bi + cj + dk \mapsto (b, c, d) \). Note that \( \tilde{\rho}(t) = \cos(\frac{t}{2}) + k\sin(\frac{t}{2}) \) is a lift of \( \rho(\cos(t), \sin(t)) \) by \( p \). Recall that if \( q \in S^3 \) and \( v \) is a pure quaternion we have \( \psi(qvq^{-1}) = p(q) \cdot \psi(v) \). Let \( \mathbb{D}^3 \subset \mathbb{R}^3 \) be the unit disk. Let \( \varphi: \mathbb{D}^3 \to \mathbb{H} \) be given by \((b, c, d) \mapsto \sqrt{1 - b^2 - c^2 - d^2} + bi + cj + dk \). We denote the coordinates of \([0, 2\pi]^3 \times (\mathbb{D}^3)^k \) by \( t_1, t_2, t_3, \ldots, t_{3(k+1)} \).

**Definition 6.1.** Let \( \widetilde{A}, \widetilde{B}, \widetilde{S}_0, \widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3, \widetilde{S}_4 \), be preimages by \( p \) of the matrices \( A, B, S_0, S_1, S_2, S_3, S_4 \) which appear in the statement of Theorem 5.1. We also define functions \( \widetilde{C}, \widetilde{D}, \widetilde{X}_0: [0, 2\pi]^3 \times (\mathbb{D}^3)^k \to \mathbb{H} \) by

\[
\begin{align*}
\widetilde{C}(t) &= \tilde{\rho}(t_1)\tilde{S}_0\tilde{\rho}(t_1)^{-1}, \\
\widetilde{D}(t) &= \tilde{\rho}(t_1)\tilde{S}_1\tilde{\rho}(t_2)\tilde{S}_2\tilde{\rho}(t_2)^{-1}\tilde{S}_1^{-1}\tilde{\rho}(t_1)^{-1}, \\
\widetilde{X}_0(t) &= \tilde{\rho}(t_1)\tilde{S}_1\tilde{\rho}(t_2)\tilde{S}_3\tilde{\rho}(t_3)^{-1}\tilde{S}_4.
\end{align*}
\]

For \( i = 1, \ldots, k \) we define \( \widetilde{X}_i(t) = \varphi(t_{3i+1}, t_{3i+2}, t_{3i+3}) \).

Let \( t_1^u, t_2^u, t_3^u \in [0, 2\pi]^3 \) be the unique numbers such that \( \cos(t_1^u) = x_1^u \) and \( \sin(t_1^u) = y_1^u \). Let \( t_u = (t_1^u, t_2^u, t_3^u, 0, \ldots, 0) \in [0, 2\pi]^3 \times (\mathbb{D}^3)^k \). Note that we can arrange the signs of these preimages so that \( \left( B\widetilde{A}\widetilde{C} \right)(t_u) = 1 \) and \( \widetilde{X}_0(t_u) = 1 \).

If \( w \in \ker(\phi) \) there is an induced map \( \widetilde{W}: [0, 2\pi]^3 \times (\mathbb{D}^3)^k \to S^3 \). Note that words \( w, w' \in \ker(\phi) \) which are equal in \( I_k \) induce maps \( \widetilde{W}, \widetilde{W}' \) which are equal or differ on a sign. If \( w_0, \ldots, w_k \in \ker(\phi) \) we can consider \( \widetilde{W} = (\widetilde{W}_0, \ldots, \widetilde{W}_k): [0, 2\pi]^3 \times (\mathbb{D}^3)^k \to (S^3)^{k+1} \) which can be composed with

\[
\Psi = (\psi, \ldots, \psi): \mathbb{H}^{k+1} \to \mathbb{R}^{3(k+1)}
\]

to obtain a map

\[
\Psi \widetilde{W}: [0, 2\pi]^3 \times (\mathbb{D}^3)^k \to (\mathbb{D}^3)^{k+1}.
\]

The plan is to assume \( t_u \) is the only zero in order to do a degree argument. We will get a contradiction by computing the degree in two different ways. We need some basic differentiation properties for quaternion valued analytic functions analogous to the usual ones (see Appendix B).

**Lemma 6.2.** Let \( I = [-1,1] \) and let \( \mathbb{D}^3 \subset \mathbb{R}^3 \) be the unit disk. Let

\[
(f_0, \ldots, f_k): I^3 \times (\mathbb{D}^3)^k \to (\mathbb{D}^3)^{k+1}
\]

be a continuous map such that:
For \( t_i, t_2, t_3 \in I, \ x_1, \ldots, x_k \in \mathbb{D}^3 \) we have
\[
(f_0, f_1, \ldots, f_k)((-1, t_2, t_3), x_1, \ldots, x_k) = (f_0, f_1, \ldots, f_k)((1, t_2, t_3), x_1, \ldots, x_k)
\]
\[
(f_0, f_1, \ldots, f_k)((1, -1, t_3), x_1, \ldots, x_k) = (f_0, f_1, \ldots, f_k)((1, 1, t_3), x_1, \ldots, x_k)
\]
\[
(f_0, f_1, \ldots, f_k)((t_1, t_2, -1), x_1, \ldots, x_k) = (f_0, f_1, \ldots, f_k)((1, t_1, 2), x_1, \ldots, x_k).
\]

For each \( 1 \leq i \leq k \) and for every \( (x_0, \ldots, x_k) \in I^3 \times (\mathbb{D}^3)^k \) with \( x_i \in \partial \mathbb{D}^3 \) we have
\[
(f_0, f_1, \ldots, f_k)(x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_k) = (f_0, \ldots, f_{i-1}, -f_i, f_{i+1}, \ldots, f_k)(x_0, \ldots, x_k).
\]

Suppose \( F = (f_0, \ldots, f_k) \) is nonzero on the boundary of \( I^3 \times (\mathbb{D}^3)^k \). Then the degree of the restriction \( F: \partial(I^3 \times (\mathbb{D}^3)^k) \to (\mathbb{D}^3)^{k+1} - \{0\} \) is even.

**Proof.** We fix cellular structures. For \( I \) we take the structure with two 0-cells and one 1-cell. For the cube \( I^3 \) we take the product cellular structure. For \( \mathbb{D}^3 \) we take the cell structure with two 0-cells, two 1-cells, two 2-cells and one 3-cell (the antipodal map interchanges the \( i \)-cells in each pair for \( 0 \leq i \leq 2 \)). We take the product cellular structure for \( I^3 \times (\mathbb{D}^3)^k \) and \( (\mathbb{D}^3)^{k+1} \). Let \( S = \partial(I^3 \times (\mathbb{D}^3)^k) \). Note that the \((3k + 2)\)-cells of \( S \) can be divided into \( 3 + k \) pairs of opposite cells in a natural way. Note that it is easy to define a cellular map \( h_0: I^3 \to \partial \mathbb{D}^3 \) which satisfies
\[
h_0(-1, t_2, t_3) = -h_0(1, t_2, t_3)
\]
\[
h_0(t_1, -1, t_3) = -h_0(t_1, 1, t_3)
\]
\[
h_0(t_1, t_2, -1) = -h_0(t_1, t_2, 1).
\]

Let \( h_i: \mathbb{D}^3 \to \mathbb{D}^3 \) be the identity for \( 1 \leq i \leq k \). Now we can define a homotopy between \( F|_S \) and a map \( G: S \to \partial(\mathbb{D}^3)^{k+1} \) satisfying the same condition satisfied by \( F \) and coinciding with \( H = (h_0, \ldots, h_k) \) on the \((3k + 1)\)-skeleton of \( S \). This is done by a cellular map using that \( \partial(\mathbb{D}^3)^{k+1} \) is \((3k + 1)\)-connected. For each pair of opposite \((3k + 2)\)-cells we can extend the homotopy so that the condition is also satisfied by \( G \). Clearly the degrees of \( F|_S \) and \( G \) are equal. Now note that if \( e, e' \) is a pair of opposite \((3k + 2)\)-cells then \( G_*(e), H_*(e) \in C_{3k+2}(\partial(\mathbb{D}^3)^{k+1}) \) differ on an element of \( H_{3k+2}(\partial(\mathbb{D}^3)^{k+1}) \). Moreover by our condition \( G_*(e') \) and \( H_*(e') \) differ on the same element. Thus the degree of \( H|_S \) and the degree of \( G \) are equal modulo 2. We only have to compute the degree of \( H|_S \). Note that the degree of \( h_0: \partial I^3 \to \partial \mathbb{D}^3 \) is 0. Then \( \deg(H|_S) = \deg(h_0) \cdot \deg(h_1) \cdot \ldots \cdot \deg(h_k) = 0 \) and we are done.

Now from Definition 6.1 we obtain:

**Corollary 6.3.** Let \( w_0, \ldots, w_k \in F(a, b, c, d, x_0, \ldots, x_k) \) be words and assume the total exponent of \( x_i \) in \( w_j \) is \( \delta_{i,j} \). If \( \Psi \hat{W} \) is nonzero on the boundary of \([0, 2\pi]^3 \times (\mathbb{D}^3)^k\), then the degree of the restriction \( \hat{\Psi}W: \partial ([0, 2\pi]^3 \times (\mathbb{D}^3)^k) \to \mathbb{R}^{3(k+1)} - \{0\} \) is even.

**Proof.** Since the total exponent of \( x_i \) in \( w_j \) is \( \delta_{i,j} \), by looking at Definition 6.1 we see the condition needed to apply Lemma 6.2 is satisfied.

Recall that the degree can be computed in the following way

**Lemma 6.4.** Let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be smooth and assume \( f(0) = 0 \). If \( \det(Df_0) \neq 0 \) then 0 is an isolated zero and the degree of \( f \) around 0 is given by \( \deg(f, 0) = \text{sg}(\det(Df_0)) \).

Note that \( \tilde{\rho}(t) = \cos \left( \frac{t}{2} \right) + k \sin \left( \frac{t}{2} \right) = 1 + \frac{t}{2} k + O(t^2) \).

**Lemma 6.5.** Let \( \hat{X} = (\hat{X}_0, \ldots, \hat{X}_k) \). Then \( D \left( \hat{\Psi} \hat{X} \right)_{\hat{t}_u} \) is invertible.
Proof. Again this reduces to the case \( k = 0 \) by noting that

\[
D \begin{pmatrix} \Psi \tilde{X} \end{pmatrix}_{\mathbf{t}_u} = \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}
\]

where \( M \) is the \( 3 \times 3 \) matrix we obtain in the \( k = 0 \) case. We now prove \( M \) is invertible. Recall that \( \tilde{X}_0(t_u) = 1 \). Then

\[
\tilde{X}_0(t_u + t) = \tilde{\rho}(t_i^u) \tilde{\rho}(t_1) \tilde{S}_1 \tilde{\rho}(t_2) \tilde{S}_3 \tilde{\rho}(t_3) \tilde{S}_4 + O(t^2)
\]

Now a straightforward SAGE computation shows that \( \tilde{\rho}(t_i^u) \tilde{\rho}(t_j^u) \tilde{\rho}(t_k^u) \tilde{\rho}(t_3) \tilde{S}_4 + O(t^2) \)

Now recalling that \( q_k q^{-1} = (i, j, k) \cdot p(q) \cdot (0, 0, 1) \) for any \( q \in S^3 \) we see that the columns of \( M \) are given by

\[
\begin{align*}
\frac{1}{2} \rho(x_i^u, y_i^u) \\
\frac{1}{2} \rho(x_i^u, y_i^u) S_1 \rho(x_i^u, y_i^u) \\
\frac{1}{2} S_1^{-1}
\end{align*}
\]

Now a straightforward SAGE computation shows that \( M \) is invertible. This computation is done by the function \texttt{M\_is\_invertible} in Appendix D. \( \square \)

Lemma 6.6. Let \( w \in \ker(\phi) \). Then \( \frac{\partial \tilde{W}}{\partial t_i}(t_u) \) is a pure quaternion for \( i = 1, \ldots, 3(k+1) \).

Proof. Since \( w \) belongs to \( \ker(\phi) \), in \( \Gamma_k \) it equals a product of conjugates of the \( x_i \), \( (bac)^3 \) and their inverses. Recall that \( S^2 \) is invariant by the action of \( S^3 \). By Proposition B.1, it is enough to prove that \( \frac{\partial \tilde{X}_i}{\partial t_i}(t_u) \) and \( \frac{\partial (BA\tilde{C})^3}{\partial t_i}(t_u) \) are pure quaternions.

For \( i = 0 \) the first claim follows from the computation in the proof of Lemma 6.5 and is easy to verify for \( i > 0 \). The second claim follows similarly by noting that \( (BA\tilde{C})^3(t_u) = 1 \) and writing

\[
(\tilde{B} \tilde{A} \tilde{C})^3(t_u + t) = (\tilde{B} \tilde{A} (1 + \frac{1}{2} t_1 k) \tilde{S}_0 (1 - \frac{1}{2} t_1 k))^3 + O(t^2).
\]

\( \square \)

Lemma 6.7. Let \( w_0, \ldots, w_k \in \ker(\phi) \) be words such that \( x_0, \ldots, x_k \in \langle w_0, \ldots, w_k \rangle \) (the normal closure in \( \Gamma_k \)). Then \( D \begin{pmatrix} \Psi \tilde{W} \end{pmatrix}_{\mathbf{t}_u} \) is invertible.

Proof. For each \( j \) we can write \( x_j = \prod_{s=1}^{t_j} p_{j,s} w_{j,s}^{x_j} p_{j,s}^{-1} \) in \( \Gamma_k \). We may assume without loss of generality that \( \tilde{W}_j(t_u) = 1 \) for all \( j \). Then we have

\[
\tilde{X}_j(t_u + t) = \left( \prod_{s=1}^{t_j} \tilde{P}_{j,s} \tilde{W}_{j,s}^{x_j} \tilde{P}_{j,s}^{-1} \right)(t_u + t).
\]
Therefore using Proposition B.1 we obtain
\[
\frac{\partial \tilde{X}_j}{\partial t_i}(t_u) = \sum_{s=1}^{\ell_j} \tilde{P}_{j,s}(t_u) \frac{\partial \tilde{W}_{\alpha_j,s}}{\partial t_i}(t_u) \tilde{P}_{j,s}^{-1}(t_u)
\]
\[
= \sum_{s=1}^{\ell_j} \epsilon_{j,s} \tilde{P}_{j,s}(t_u) \frac{\partial \tilde{W}_{\alpha_{j,s}}}{\partial t_i}(t_u) \tilde{P}_{j,s}^{-1}(t_u)
\]
By Lemma 6.6, \(D(\tilde{\Psi} \tilde{W})\) is invertible if and only if
\[
\left\{ \left( \frac{\partial \tilde{W}_0}{\partial t_i}(t_u), \ldots, \frac{\partial \tilde{W}_k}{\partial t_i}(t_u) \right) : 1 \leq i \leq 3(k+1) \right\}
\]
is linearly independent over \(\mathbb{R}\). If \(\beta_i \in \mathbb{R}\) satisfy
\[
\sum_{i=1}^{3(k+1)} \beta_i \frac{\partial \tilde{W}}{\partial t_i}(t_u) = 0
\]
it follows that
\[
\sum_{i=1}^{3(k+1)} \beta_i \frac{\partial \tilde{X}}{\partial t_i}(t_u) = 0.
\]
Since \(D(\tilde{\Psi} \tilde{X})\) is invertible (Lemma 6.5) again by Lemma 6.6 the set
\[
\left\{ \left( \frac{\partial \tilde{X}_0}{\partial t_i}(t_u), \ldots, \frac{\partial \tilde{X}_k}{\partial t_i}(t_u) \right) : 1 \leq i \leq 3(k+1) \right\}
\]
is linearly independent over \(\mathbb{R}\). Thus \(\beta_1 = \ldots = \beta_{3(k+1)} = 0\) and we are done. \(\square\)

Proof of Theorem 5.10. We can assume that the total exponent of \(x_i\) in \(w_j\) is \(\delta_{i,j}\). To prove this, consider the abelianization and note that it is possible to achieve this by using the following operations:

- replacing \(w_i\) by \(w_iw_j\) (if \(i \neq j\)),
- replacing \(w_i\) by \(w_i^{-1}\), and
- interchanging \(w_i\) and \(w_j\).

By Lemma 6.7 and Lemma 6.4, the degree of \(\tilde{\Psi} \tilde{W}\) near \(t_u\) is \(\pm 1\). If \(\tilde{\Psi} \tilde{W}\) has a zero on \(\partial([0,2\pi]^3 \times (\mathbb{D}^3)^k)\) we are done. Otherwise, by Corollary 6.3, the degree of \(\tilde{\Psi} \tilde{W}\) restricted to the boundary of \([0,2\pi]^3 \times (\mathbb{D}^3)^k\) is even. It follows that there must be a point \(t \neq t_u\) such that \(\tilde{\Psi} \tilde{W}(t) = 0\). This gives a second point in \(Z(w_0, \ldots, w_k)\). \(\square\)

7. Group Actions of \(A_5\) on Contractible 2-complexes

We can now prove the following.

Theorem 7.1. There is no presentation of \(A_5\) of the form
\[
\langle a, b, c, d, x_0, \ldots, x_k \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x_0ax_0^{-1} = d, w_0, \ldots, w_k \rangle
\]
with \(w_0, \ldots, w_k \in \ker(\phi)\), where \(\phi: F(a, b, c, d, x_0, \ldots, x_k) \to A_5\) is given by \(a \mapsto (2, 5)(3, 4), b \mapsto (3, 5, 4), c \mapsto (1, 2)(3, 5), d \mapsto (2, 5)(3, 4)\) and \(x_i \mapsto 1\) for each \(i = 0, \ldots, k\).

Proof. This follows from Theorem 5.10, Proposition 5.8 and Proposition 5.5. \(\square\)

Now from Theorem 7.1 and Theorem 4.6 we deduce.
Theorem 7.2. Every action of $A_5 \cong \text{PSL}_2(2^2)$ on a finite, contractible 2-complex has a fixed point.

Looking more carefully at the proof we can deduce the following.

Theorem 7.3. Let $X$ be a fixed point free 2-dimensional finite, acyclic $A_5$-complex and let $\pi = \pi_1(X)$. Then $\pi$ is infinite or there is an epimorphism $\pi \to A_5$.

Proof. By Theorem 3.12 we see that $\pi$ surjects onto the fundamental group of an acyclic 2-dimensional $A_5$ complex $X'$ which is obtained from $\Gamma_{\text{OS}}(A_5)$ by attaching $k \geq 0$ free orbits of 1-cells and $k + 1$ free orbits of 2-cells. Now note that the representation constructed to prove Theorem 7.1 restricted to $\pi_1(X')$ gives a nontrivial morphism into $\text{SO}(3, \mathbb{R})$. If the image of this morphism is finite, then it has to be isomorphic to $A_5$, since it is the only nontrivial finite perfect subgroup of $\text{SO}(3)$. This completes the proof. \qed

Recall that $N = \ker(\bar{\phi})$ is a free group of rank $60(k + 1)$. We can restate Theorem 7.1 in the following way which highlights the connection with the relation gap problem (see [Har18, Har15]).

Corollary 7.4. The extension

$$1 \to N \to \Gamma_k \xrightarrow{\bar{\phi}} A_5 \to 1$$

has a relation gap. That is, the $A_5$-module $N/[N, N]$ is free of rank $k + 1$. However $N$ cannot be generated by $k + 1$ elements as a $\Gamma_k$-group.

Note that since $\Gamma_k$ is not free this is not an example of a presentation with a relation gap.

Appendix A. Equations over groups

Let $G$ be a group. An equation over $G$ in the variables $x_1, \ldots, x_n$ is an element $w \in G * F(x_1, \ldots, x_n)$. We say that a system of equations

$$w_1(x_1, \ldots, x_n) = 1$$

$$w_2(x_1, \ldots, x_n) = 1$$

$$\cdots$$

$$w_m(x_1, \ldots, x_n) = 1$$

has a solution in an overgroup of $G$ if the map $G \to G*F(x_1, \ldots, x_m)/\langle\langle w_1, \ldots, w_m \rangle\rangle$ is injective. Such a system of equations determines an $(m \times n)$-matrix $M$ where $M_{i,j}$ is given by the total exponent of the letter $x_j$ in the word $w_i$. A system is said to be independent if the rank of $M$ is $m$.

One of the most important open problems in the theory of equations over groups is the Kervaire–Laudenbach–Howie conjecture [How81, Conjecture].

Conjecture A.1 (Kervaire–Laudenbach–Howie). An independent system of equations over $G$ has a solution in an overgroup of $G$.

Now we explain why Conjecture 3.1 follows from the Kervaire–Laudenbach–Howie conjecture for perfect groups which admit a balanced presentation. Let $A$ be an acyclic subcomplex of a contractible 2-complex $X$. Take a maximal tree $T$ for $A$ and consider a maximal tree $\overline{T}$ of $X$ containing $T$. Then $A/T \simeq A$ is an acyclic subcomplex of the contractible 2-complex $X/\overline{T}$. Then the group $G = \pi_1(A/T)$ is perfect. As usual, from $A/T$ we can read a presentation for $G$ which is balanced since $A/T$ is acyclic. Now we consider a variable $x_i$ for each 1-cell of $X/T$. 

which is not in \(A/T\) and we read words from the attaching maps for the 2-cells of \(X/T\) which are not part of \(A/T\). In this way we obtain equations in these variables with coefficients in \(A\). Since \(X/T\) is acyclic, there is an equal number of variables and equations and the determinant of the exponent matrix is 1. Thus if the Kervaire–Laudenbach–Howie conjecture holds for perfect groups which admit a balanced presentation, \(\pi_1(A)\) injects into \(\pi_1(X)\) and if \(X\) is contractible then \(A\) is contractible too.

**Appendix B. Quaternion valued analytic functions**

A *quaternion valued analytic function* is a function \(f: U \to \mathbb{H}\) where \(U \subset \mathbb{R}^n\) is open, such that its components are analytic, that is a function that can be written as

\[
f = f_1 + f_i \mathbf{i} + f_j \mathbf{j} + f_k \mathbf{k}
\]

with \(f_1, f_i, f_j, f_k: U \to \mathbb{R}\) are analytic. For \(i = 1, \ldots, n\) we can define the partial derivative

\[
\frac{\partial f}{\partial t_i} = \frac{\partial f_1}{\partial t_i} + \frac{\partial f_i}{\partial t_i} \mathbf{i} + \frac{\partial f_j}{\partial t_i} \mathbf{j} + \frac{\partial f_k}{\partial t_i} \mathbf{k}.
\]

We define

\[
Df_t = \left( \frac{\partial f}{\partial t_1}(t), \ldots, \frac{\partial f}{\partial t_n}(t) \right).
\]

The usual properties hold in this context. We need the following

**Proposition B.1.** Let \(f, g: U \to \mathbb{H}\) be analytic. Then

(i) We have the product rule

\[
\frac{\partial f \cdot g}{\partial t_i}(t) = \frac{\partial f}{\partial t_i}(t)g(t) + f(t)\frac{\partial g}{\partial t_i}(t).
\]

(ii) Suppose \(f\) is nowhere zero and \(g(t_0) \in \mathbb{R}\) then

\[
\frac{\partial f \cdot g \cdot \frac{1}{f}(t_0)}{\partial t_i} = f(t_0)\frac{\partial g}{\partial t_i}(t_0)f(t_0)^{-1}.
\]

(iii) Suppose \(f(t_0) = \pm 1\) then \(\frac{\partial f}{\partial t_i}(t_0) = -\frac{\partial f}{\partial t_i}(t_0)\).

**Proof.** (i) is a straightforward computation. (ii) and (iii) follow from (i). \(\square\)

As usual we have the Taylor series

\[
f(t_0 + t) = f(t_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial t_i}(t_0) t_i + O(t^2).
\]

From the product rule we see that we can multiply the Taylor series of two functions to obtain the Taylor series of the product.

If each coordinate of \(F = (f_1, \ldots, f_m): U \to \mathbb{H}^n\) is analytic then we use the notation

\[
\frac{\partial F}{\partial t_i} = \left( \frac{\partial f_1}{\partial t_i}, \ldots, \frac{\partial f_m}{\partial t_i} \right).
\]
APPENDIX C. GAP CODE

The following GAP computation proves Lemma 5.4.

gap> F:=FreeGroup("a","b","c","d","x0");;
gap> AssignGeneratorVariables(F);
#I Assigned the global variables [ a, b, c, d, x0 ]
gap> phi:=GroupHomomorphismByImages(>
    F,AlternatingGroup(5),>
    GeneratorsOfGroup(F),[(2,5)(3,4), (3,5,4),(1,2)(3,5),(2,5)(3,4),()]);
gap> x0^phi; # x0 in ker(phi)
()
gap> ((b*a*c)^3)^phi; # (bac)^3 in ker(phi)
()
gap> R:=[a^2,b^3,c^2,d^2,(a*b)^3,(b*c)^2,(c*d)^5,x0*a*x0^-1*d^-1,x0,(b*a*c)^3];
gap> Order(F/R); # F/R is A5. Thus ker(phi) = << x0, (bac)^3 >>.
60

APPENDIX D. SAGE CODE

The following SAGE code is used in the proofs of Theorem 5.1, Proposition 5.8 and Lemma 6.5.

The function check_rep, used in the proof of Theorem 5.1, shows A, B, C, D, X₀ satisfy the defining relations for Γ₀ in

\[ M_3(\mathbb{C}[x_1, y_1, x_2, y_2, x_3, y_3]/(x_1^2 + y_1^2 - 1, x_2^2 + y_2^2 - 1, x_3^2 + y_3^2 - 1)). \]

The function find_universal_representations, used in Proposition 5.8 to prove there is only solution to \( X₀ = (BAC)^3 = 1 \), solves the corresponding system of polynomial equations over the algebraic closure of \( \mathbb{Q} \). This function also gives the exact value of the unique universal representation \( z_u \).

The function M_is_invertible, used in the proof of Lemma 6.5, computes the matrix \( M \) and checks its determinant is nonzero. Again this computation is done over the algebraic numbers so there is no numerical error.

def rho(x,y):
    return matrix([[
        (x,y,0,0),
        (-y,x,0,0),
        (0,0,1,0),
    ]);[0]

A = matrix([[-1,0,0,0],
            (0,1/3,-2/3*sqrt(2),0),
            (0,-2/3*sqrt(2),-1/3),
            (0,0,0,0)]);[0]

B = matrix([[(-1/2,-sqrt(3)/2,0),
             (0,0,0,0),
             (0,0,0,0),
             (0,0,0,0),
             (0,0,0,0),
             (0,0,0,0)]);[0]
(sqrt(3)/2, -1/2, 0),
(0,0,1),
])

S0 = matrix([ 
(-1,0,0),
(0,1,0),
(0,0,-1)
])

S1 = matrix([ 
(-1,0,0),
(0,0,-1),
(0,-1,0),
])

S2 = matrix([ 
(-sin(9/10*pi),0,cos(9/10*pi)),
(0,-1,0),
(cos(9/10*pi),0,sin(9/10*pi)),
])

S3 = matrix([ 
(0,sin(3/10*pi),cos(3/10*pi)),
(1,0,0),
(0,cos(3/10*pi),-sin(3/10*pi)),
])

S4= matrix([ 
(0, -sqrt(3)/3, -sqrt(6)/3),
(1,0,0),
(0,-sqrt(6)/3,sqrt(3)/3),
])

def rep(x1,y1,x2,y2,x3,y3):
    C = rho(x1,y1) * S0 * rho(x1,y1).T;
    D = rho(x1,y1) * S1 * rho(x2,y2) * S2 * rho(x2,y2).T * S1.T * rho(x1,y1).T;
    X0 = rho(x1,y1) * S1 * rho(x2,y2) * S3 * rho(x3,y3) * S4;
    return (A,B,C,D,X0);

def check_rep():
    R.<x1,y1,x2,y2,x3,y3> = QQbar[];
    A,B,C,D,X0 = rep(x1,y1,x2,y2,x3,y3);
    J = R.ideal([x1^2+y1^2-1, x2^2+y2^2-1, x3^2+y3^2-1]);
    S = R.quotient(J);
    f = S.cover(); # f: R -> S
M3R = MatrixSpace(R,3,3);
M3S = MatrixSpace(S,3,3);
M3f = M3R.hom(f,M3S); # M3f: M3R -> M3S
A,B,C,D,X0 = M3f(A), M3f(B), M3f(C), M3f(D), M3f(X0);
I = matrix.identity(3);
for M in [A,B,C,D,X0]:
    assert(M*transpose(M)==I);
    assert(M.det()==1);
relations = [ A**2, (A*B)**3, B**3, (B*C)**2, C**2,
              (C*D)**5, D**2, X0*A*transpose(X0)*transpose(D) ];
for r in relations:
    assert(r==I);
print("The construction defines a representation of Gamma.");

def delta(i,j):
    if i==j:
        return 1;
    return 0;

def find_universal_representations(verbos=True):
    R.<x1,y1,x2,y2,x3,y3> = QQbar[
    A,B,C,D,X0 = rep(x1,y1,x2,y2,x3,y3);
equations = [ x1^2+y1^2-1,
              x2^2+y2^2-1,
              x3^2+y3^2-1
              ] + [ M[i][j]-delta(i,j) for i in range(3)
              for j in range(3)
              for M in [X0,(B*A*C)**3]
              ];
    I = R.ideal(equations);
dim_Z = I.dimension()
assert(dim_Z==0)
if verbose:
    print("The variety of universal representations has dimension "+str(dim_Z));
Z = I.variety();
assert(len(Z)==1);
if verbose:
    print("The number of universal representations is "+str(len(Z)));
z_u = Z[0];
if verbose:
    print("The universal representation is given by:");
    for v in [x1,y1,x2,y2,x3,y3]:
        print(v, z_u[v].radical_expression(), z_u[v], z_u[v].minpoly());
return z_u;
def M_is_invertible():
    z_u = find_universal_representations(verbose=False);
    e3 = vector([0,0,1]);
    v1 = rho(z_u["x1"],z_u["y1"]) * e3;
    v2 = rho(z_u["x1"],z_u["y1"]) * S1 * rho(z_u["x2"],z_u["y2"]) * e3;
    v3 = S4^-1 * e3;
    M = matrix([v1,v2,v3])
    return not bool(M.det()==0);

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