Berry Phase in homogeneous Kähler manifolds with linear Hamiltonians

Luis J. Boya†, Askold M. Perelomov‡ and Mariano Santander‡

† Departamento de Física Teórica, Facultad de Ciencias,
Universidad de Zaragoza, E–50009 Zaragoza, Spain

‡ Departamento de Física Teórica, Facultad de Ciencias,
Universidad de Valladolid, E–47011 Valladolid, Spain

emails: luisjo@posta.unizar.es, perelomo@dftuz.unizar.es, santander@fta.uva.es

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Abstract

We study the total (dynamical plus geometrical (Berry)) phase of cyclic quantum motion for coherent states over homogeneous Kähler manifolds $X = G/H$, which can be considered as the phase spaces of classical systems and which are, in particular cases, coadjoint orbits of some Lie groups $G$. When the Hamiltonian is linear in the generators of a Lie group, both phases can be calculated exactly in terms of classical objects. In particular, the geometric phase is given by the symplectic area enclosed by the (purely classical) motion in the space of coherent states.

I. INTRODUCTION

Let us consider a quantum state $\psi(t)$ whose evolution follows a time–dependent Schrödinger equation. If the final state $\psi(T)$ coincides with initial one $\psi(0)$, then the representative state vectors $|\psi(0)\rangle$ and $|\psi(T)\rangle$ differ one from another just by a phase factor $\exp(i\alpha)$. This phase factor can be splitted into two parts $\alpha = \beta + \gamma$, called respectively dynamical phase and geometrical phase. Both $\beta$ and $\gamma$ are important characteristics of the evolution of the system under consideration.

In particular the geometric phase turns out to depend on the Hamiltonian in a rather indirect way, as it is determined only by the closed loop traversed by the state in the

†On leave of absence from the Institute for Theoretical and Experimental Physics, 117259 Moscow, Russia
state space. This geometrical phase associated to any quantum cyclic motion with time-dependent Hamiltonians appears, in addition to the well-known dynamical phase, due to the natural curvature of the line bundle over the projective Hilbert space of states. This was found by Berry\(^1\) for adiabatic motion, interpreted by Simon\(^2\) as above, and extended by Aharonov-Anandan in \(^3\) (see also \(^4,5\) for arbitrary cyclic motion). However, there are very few cases in which the calculation can be performed explicitly, and it would be nice to exhibit examples where the phases of a cyclic quantum motion can be calculated in closed terms.

We shall consider the important cases in which the Hamiltonian \(H(t)\) is linear in the generators of a Lie algebra \(G\) acting through some unitary irreducible representation \(T^\lambda\) in a Hilbert space \(H^\lambda\), where \(\lambda\) labels the representation. The aim of this paper is to show that in these cases, explicit expressions for both \(\beta\) and \(\gamma\) can be given in terms of a related classical dynamical system. This is achieved by using the generalized coherent state technique\(^6,7,8\), and is done in a frame general enough to cover a wide variety of examples and particular cases, including the well-known situation for evolution of a spin 1/2 in a magnetic field, an standard example which is however an oversimplified one, because its quantum state space is the Riemann sphere \(\mathbb{C}P^1\).

Therefore all information on dynamical and geometrical phases for these quantum systems can be obtained by studying the motion of a purely classical system in a suitable phase space. As we shall see these are the Kähler (and hence naturally symplectic) homogeneous spaces \(X = G/H\), with \(G\) the Lie group of the Lie algebra \(G\). Important examples of such spaces are the orbits of the coadjoint representation of compact semisimple Lie groups. For \(G = U(n)\) the generic (maximal dimension) coadjoint orbit is \(U(n)/U(1)^n\); this space is called a flag manifold, and plays an important role in many areas\(^9,10\).

The set-up of this paper is as follows: in Section 2 we present the main ideas leading to closed expressions for dynamical and geometrical phases, in terms of motion in the space \(X\) taken as a classical space. This is possible when the quantum Hamiltonian is linear in the generators of some representation of a Lie algebra \(G\) and besides \(X\) is an homogeneous Kähler manifold of the Lie group \(G\). In Section 3 we describe some homogeneous Kähler manifolds; they include: (i) coadjoint orbits of semisimple compact Lie groups, (ii) the so-called bounded symmetric domains which are not compact, and (iii) some other cases, like the Heisenberg "plane".

Finally in the Appendix A we collect explicit expressions for the kernels which determine the Kähler potential, and we give some differential and topological information on Kähler manifolds, including the Poincaré polynomials. A resumé of relevant details on coherent states, extracted from\(^8\), is also included as Appendix B.

### II. THE GROUP THEORETICAL COMPUTATION OF PHASES

Let us consider the time–dependent Schrödinger equation

\[
\frac{id}{dt}\psi(t) = \tilde{H}(t)\psi(t),
\]

with a Hamiltonian of the form:

\[
\tilde{H}(t) = \sum_j a_j(t) X_j^\lambda, \quad X_j^\lambda = T^\lambda(X_j),
\]
where $\mathcal{T}^\lambda$ is an unitary irreducible representation of the Lie algebra $\mathcal{G}$, whose generators $X_i$ are represented in $\mathcal{T}^\lambda$ by the (antihermitian) operators $iX_i^\lambda$ and $a_j(t)$ are arbitrary real functions of time. We consider here only those cases when the representation Hilbert space $\mathcal{H}^\lambda$ may be realized as a space $\mathcal{F}^\lambda$ of holomorphic functions on a complex homogeneous space $X = G/H$ which is also a Kähler one. We assume also that the initial state is a generalized coherent state $|x_0\rangle$ labeled by the point $x_0 \in X$; for details, see $^8$.

In this case, under time evolution the initial coherent state remains a coherent state

$$|x(t)\rangle = U(t,0)|x(0)\rangle$$

and then $x(t)$ is a solution of the Hamilton equation for the corresponding classical system

$$\dot{x} = \{H(t), x\}^\lambda, \quad \dot{x} = \frac{dx}{dt},$$

where $\{, \}$ is the Poisson bracket induced on $X$ by the representation $\mathcal{T}^\lambda$.

The mapping $X \to \mathcal{H}^\lambda$ which associates the point $x_0 \in X$ to the coherent state $|x_0\rangle$ allows an isomorphic identification of actual quantum “trajectories” starting from $|x_0\rangle$ and obeying the usual Schrödinger equation

$$i\frac{d}{dt}|x(t)\rangle = \hat{H}(t)|x(t)\rangle$$

to some classical motions in $X$ (taken as a classical phase space, not a configuration space), satisfying (2.4).

Under this identification, if $\Gamma$ is a closed loop in $X$ with period $T$, it is still closed in the projective Hilbert space $\overline{\mathcal{H}}^\lambda$, which should be considered as the true state space, but not necessarily in the linear Hilbert space $\mathcal{H}^\lambda$. In this cyclic motion, the state vector picks up a phase

$$|\psi(T)\rangle = \exp (i\alpha)|\psi(0)\rangle, \quad \alpha = \beta + \gamma.$$  

This can be also seen as follows $^4$. Let $\gamma$ be a closed path (loop) in the projective Hilbert space $\overline{\mathcal{H}}^\lambda = CP^\infty$ of states; let $|\psi\rangle = |\psi(t)\rangle$ be a generic point in $\Gamma$. There is a tautological line bundle, whereby each point carries its vectors; this line bundle is hermitian, by the hermitian product in $\mathcal{H}$. Let $P(t)$ be in the fibre over $|\psi(t)\rangle$. The Hamiltonian $\hat{H}$ works in $\mathcal{H}$, and by projection in $\overline{\mathcal{H}}$ also, so the time evolution carries $P(0) \to P(t)$ and projects to $U(0,t): |\psi(0)\rangle \to |\psi(t)\rangle$. As $P(T)$ is the fibre over $\psi(T)$ which coincides with the fibre over $\psi(0)$, we must have:

$$P(T) = \exp (i\alpha)P(0),$$

where $\alpha$ is the total phase for the cyclic motion. The lift of the path $\Gamma$ through the connection of the line bundle $L^\lambda$ would produce an $U(1)$ holonomy $\gamma$; this is the geometric phase, and the difference, $\beta = \alpha - \gamma$ is the dynamical phase. As explained in detail in $^2$, we have the following explicit expressions for both parts of the total phase $\alpha$:

$$\beta = \int \langle \psi(t)|\hat{H}(t)|\psi(t)\rangle dt, \quad \gamma = \int \langle \psi'(t)|\left( -i \frac{d}{dt}\right)|\psi'(t)\rangle dt,$$

where $|\psi'(t)\rangle$ is a trivializing section, i.e. there is no dynamical phase for the whole loop, see $^4, ^{11}$. The connection 1-form of this line bundle $\theta^\lambda$ is related locally to the symplectic
2-form as \( \omega^\lambda = d\theta^\lambda \), and this symplectic 2-form is in turn induced by the imaginary part of the Hermitian scalar product in \( \mathcal{H}^\lambda \).

Under the conditions stated, both phases can be computed directly in terms of the classical motion in \( X \). For the dynamical part we have:

\[
\beta = \int \langle \psi(t) | \hat{H}(t) | \psi(t) \rangle \, dt = \int \langle x(t) | \hat{H}(t) | x(t) \rangle \, dt = \int a_j(t) X_j^\lambda(t) \, dt,
\]

(2.9)

where \( X_j^\lambda(t) = \langle x(t) | X_j^\lambda | x(t) \rangle \).

The geometric phase \( \gamma \) is given as the integral along \( \Gamma \) of the connection 1-form \( \theta^\lambda \) which depends on the representation. Due to the abelian nature of the \( U(1) \) group, the Stokes theorem applies and gives:

\[
\gamma = \gamma_{cl} = \int_\Gamma \theta^\lambda = \int_\Sigma \omega^\lambda,
\]

(2.10)

where \( \omega^\lambda = d\theta^\lambda \) and \( \Sigma \) is any surface having \( \Gamma = \partial \Sigma \) as its boundary. Hence, we give the expression for the geometric phase in terms of symplectic area of any surface whose boundary is the given (classical) closed circuit in our Kähler manifold \( X \).

Thus formula (2.10) is valid for arbitrary homogeneous Kähler manifolds. Now if \( F^\lambda \) denotes the Kähler potential\(^{12,13} \), the expressions for the connection and curvature forms are (see Appendix A):

\[
\theta^\lambda = \frac{1}{i} \left( \frac{\partial F^\lambda}{\partial z_\mu} \, dz_\mu - \frac{\partial F^\lambda}{\partial \bar{z}_\mu} \, d\bar{z}_\mu \right), \quad \omega^\lambda = \frac{i}{2} \frac{\partial^2 F^\lambda}{\partial z_\mu \partial \bar{z}_\nu} \, dz_\mu \wedge d\bar{z}_\nu = d\theta^\lambda,
\]

(2.11)

and the Kähler potential itself is related to the kernel, which generalizes Bergmann’s kernel, as:

\[
F(z, \bar{z}) = \ln (K(z, \bar{w}))(w = z).
\]

(2.12)

The simplest closed loops are geodesic triangles. For them we can give explicit expressions.

Let us first consider the simplest case \( \mathcal{G} = su(2) \), where \( X = \mathbb{C}P^1 = SU(2)/U(1) \) is the Riemann sphere, and \( U(1) \rightarrow SU(2) = S^3 \rightarrow S^2 \) is the second Hopf sphere bundle. In this case there is a single complex coordinate \( z \), related with the point \( x \) on the sphere by the usual stereographic projection, and the Bergmann’s kernel is given by

\[
K(z, \bar{w}) = 1 + z\bar{w}.
\]

(2.13)

It is clear that any vertex can be carried to a prescribed point on the sphere, say the North pole, corresponding to \( z = 0 \). Let \( |x\rangle, |y\rangle \) denote the two coherent states determining the remaining triangle vertices, corresponding to points \( x, y \) on the sphere, and let us denote \( z, w \) the complex coordinates corresponding to \( x, y \). The closed expression for the geometric phase associated to this closed loop is:

\[
\gamma = \frac{1}{2i} \ln \frac{K(z, w)}{K(z, \bar{w})} = \frac{1}{2i} \ln \frac{1 + z\bar{w}}{1 + z\bar{w}} = \frac{1}{2i} \ln \frac{\langle x|y \rangle}{\langle y|x \rangle}.
\]

(2.14)

For \( \mathcal{G} = su(2) \) this result for the phase has been given already in\(^6 \) (see also\(^8 \)). In this case the symplectic area is proportional to the riemannian area for the standard riemannian
structure on \( S^2 \); this proportionality is however accidental and may be misleading because this does not hold in higher dimensions; for instance in \( \mathbb{C}P^n \) \((n > 1)\) the symplectic area of any finite triangle is not proportional to its riemannian Fubini-Study area.

Let us now consider the general case where the generators \( X_j^\lambda \) close to an unitary irreducible representation of the Lie algebra \( \mathcal{G} \). The symplectic area of any closed loop in \( \mathcal{H} \) is completely determined by the loop; this does not hold for the ‘riemannian’ area determined by a general Kähler metric, which depends essentially on the two-dimensional surface whose boundary is the prescribed loop. In this general case (with group \( \mathcal{G} \)), it suffices again to give a closed expression for the symplectic area of a triangular loop. If one vertex is carried to a prescribed point on the Kähler homogeneous manifold \( \mathcal{X} \) (say determined by the complex coordinates \( z_\mu = 0 \)), the remaining vertices \( x, y \) will correspond to the complex coordinates \( z_\mu, w_\nu \). The same argument as before leads in this case to the expression:

\[
\gamma \propto \frac{1}{2i} \ln \frac{K(z, w)}{K(z, w)}.
\] (2.15)

Appendix A contain explicit expressions for the kernels \( K(z, w) \) for Hermitian symmetric spaces, taken from \(^8\), where further details on the construction of the kernel \( K(z_\mu, w_\nu) \) for homogeneous Kähler manifolds can be found.

In the case of complex grassmannians, the usual choice for complex coordinates are called Pontrjagin coordinates and can be arranged as a complex rectangular matrix \( Z \). After substituting for the relevant kernel, the basic expression (2.15) reduces to:

\[
\gamma \propto \frac{1}{2i} \ln \frac{\det(I + ZW^{\dagger})}{\det(I + WZ^{\dagger})} \] (2.16)

and coincides with the formula for the geometric phase derived through explicit computation by Berceanu, who also points out the validity of a similar formula for any Hermitian symmetric space \(^{14}\). However, the arguments given in \(^8\) appear to hold unrestrictedly for arbitrary homogeneous Kähler manifold \( \mathcal{X} \), and not only for Hermitian symmetric spaces.

In the well understood example (see e.g. \(^{15,16}\)), of a spin 1/2 particle in a magnetic field,

\[
i \frac{d}{dt} |\psi(t)\rangle = -\mu \mathbf{B}(t) \cdot \mathbf{S} |\psi(t)\rangle \] (2.17)

the Hamiltonian is a linear combination of three operators which span a Lie algebra \( su(2) \), and quantum evolution can be thus translated into a classical motion of a point on the homogeneous space \( SU(2)/U(1) \), the Riemann sphere again. The coadjoint orbits are 2-spheres and \( x = \mathbf{x} \) is a unit vector in \( \mathbb{R}^3 \), so at any fixed time \( \hat{H}(t) \) splits into two parts:

\[
\hat{H}(t) = \hat{H}_{\parallel} + \hat{H}_{\perp},
\] (2.18)

where

\[
\hat{H}_{\perp} |\mathbf{x}(t)\rangle = 0, \quad \hat{H}_{\parallel} |\mathbf{x}(t)\rangle = E(t) |\mathbf{x}(t)\rangle.
\] (2.19)

The longitudinal part produces only a dynamical phase, as the ray of \( |\mathbf{x}\rangle \) and of \( E(t)|\mathbf{x}\rangle \) are the same. The geometrical phase comes entirely from the transverse part. In particular, if the field is constant in direction:

\[
H(t) = B(t) \sigma_z
\] (2.20)
and the initial state is \(|x\rangle = \cos(\theta/2) |+\rangle + \sin(\theta/2) |-\rangle\), the solution is readily obtained

\[ |\psi(t)\rangle = a(t) |+\rangle + b(t) |-\rangle, \quad (2.21) \]

where \(a(t) = a(0) \exp(-i \int B(t) \, dt)\), \(b(t) = b(0) \exp(i \int B(t) \, dt)\). For \(\theta = 0\) or \(\theta = \pi\) we have a purely dynamical phase, while for \(\theta = \pi/2\) the phase is purely geometrical.

For arbitrary \(B(t)\) there is also a local splitting, and the “parallel” \(H_{\parallel}\) and “perpendicular” \(H_{\perp}\) parts of the Hamiltonian carry respectively the dynamical and geometric phases.

### III HOMOGENEOUS SYMPLECTIC MANIFOLDS AND KÄHLER MANIFOLDS

Relative to the definition of a symplectic manifold, see the book

**Definition.** A symplectic manifold \((M, \omega)\) is called homogeneous if there exists on it a transitive action \(\Phi_g : M \rightarrow M\) of some Lie group \(G = \{g\}\) which acts as a group of symplectic transformations, i.e., it leaves invariant the form \(\omega\), \(\Phi_g^* \omega = \omega\).

**Theorem.** Any homogeneous symplectic manifold on which a connected Lie group \(G\) acts transitively and by symplectic transformations is locally isomorphic to an orbit of a coadjoint representation of this group \(G\) or of a central extension of \(G\) by \(\mathbb{R}\).

Thus any coadjoint orbit of the group \(G\) is an homogeneous symplectic manifold.

Among the class of all homogeneous symplectic manifolds, the main important subclass is those of coadjoint orbits of **semi-simple** Lie groups. These have an additional Kähler homogeneous structure. A Kähler manifold is defined as a complex manifold \(M\) endowed with a Kähler metric \(h\), whose imaginary part is a closed two-form. A Kähler metric is an hermitian metric \(h\) which comes from a function \(F(z, \overline{z})\) called the Kähler potential:

\[
\begin{align*}
\text{ds}^2 &= h_{\mu\nu} \text{d}z^\mu \overline{\text{d}z}^\nu, \\
h_{\mu\nu}(z, \overline{z}) &= \partial_{\mu} \partial_{\overline{\nu}} F(z, \overline{z}), \\
\partial_{\mu} &= \frac{\partial}{\partial z^\mu}, \\
\partial_{\overline{\nu}} &= \frac{\partial}{\partial \overline{z}^\nu}. 
\end{align*}
\]

(3.1)

The imaginary part of this metric is a symplectic 2-form

\[
\omega = \frac{i}{2} h_{\mu\nu}(z, \overline{z}) \text{d}z^\mu \wedge \overline{\text{d}z}^\nu, \\
d\omega = 0.
\]

(3.2)

The connection between orbits of the coadjoint representation of compact simple Lie groups and Kähler homogeneous manifolds is stated in the following important result of A. Borel:

**Theorem.** Any orbit of the coadjoint representation of a compact simple Lie group is a compact Kähler homogeneous simply-connected manifold, and any compact Kähler homogeneous simply-connected manifold is some orbit of the coadjoint representation of the some compact simple Lie group.

Orbits of the coadjoint representation of a compact Lie group are even rational manifolds. Topologically they are compact and simply-connected manifolds. Their topology is described, for example, in the review in the Appendix A we give some pertinent results.
Many examples of Kähler homogeneous manifolds with a compact group $G$ are known; these spaces are compact, even-dimensional, simply-connected and oriented. As the cohomology class $\omega \neq 0$, all the even Betti numbers are nonzero. Let us recall some simple examples.

For $G = SU(2) = Spin(3) \sim SO(3)$, the generic coadjoint orbits in $su(2) \approx \mathbb{R}^3$ are spheres $S^2$; there is an isolated orbit consisting of a single point, the origin. For each sphere the 2-form is just the area (volume) form, automatically closed by dimensionality. It is a complex (one-dimensional) manifold, the Riemann sphere.

For $G = SU(3)$, there are three types of coadjoint orbits in $su(3) \approx \mathbb{R}^8$: the origin, four-dimensional orbits isomorphic to $\mathbb{C}P^2 = \frac{SU(3)}{SU(2) \times U(1)}$, and six-dimensional maximal orbits, isomorphic to the flag manifold $\mathbb{F}^3 = \frac{SU(3)}{U(1) \times U(1)}$.

For $G = SU(n)$, the description of the orbits is essentially given by the partitions of $n$ (see 21).

The general calculation of Kähler metrics on the coadjoint orbits for any compact simple Lie group (the classical and exceptional structures of Cartan) was expelled in 21.

The main reason why these manifolds are Kähler is that the homogeneous structure is also obtained from the complex extension $G^C$ of $G$. The role of the subgroup $H$ here is played by some triangular (Borel) subgroup $B$; both $G^C$ and $B$ are analytic manifolds, and so is $G^C/B$ which turns out to be isomorphic to $G/H$. The space $X$ is also obviously simply connected, because $G$ can be taken simply connected (for any $X$) and $H$ is connected. This construction $X = G/H = G^C/B$ is also basic in the Borel-Weil-Bott theory of analytic construction of irreducible representations of $G$ as sections in some holomorphic bundles.

When $G$ is a general simple or semisimple compact Lie group, the orbits of the coadjoint representation exhaust all the compact homogeneous Kähler manifolds.

Other examples of (non-compact) Kähler manifolds are the so-called bounded symmetric domains (see 22). Recall that a bounded domain $D \subset \mathbb{C}^n$ is called symmetric if each point in $D$ is fixed by an involutive holomorphic diffeomorphism of $D$. These are characterized by the result:

Theorem [Helgason 22, p.310]. (i) Each bounded symmetric domain $D$, when equipped with the Bergmann metric, is a Hermitian symmetric space of the non-compact type. In particular, a bounded symmetric domain is necessarily simply connected.

(ii) Let $M$ be a Hermitian symmetric space of the non-compact type. Then there exists a bounded symmetric domain $D$ and a holomorphic diffeomorphism of $M$ onto $D$.

The paradigmatic example is the Lobachevsky plane. This a Kähler manifold which is non-compact, and of constant negative curvature.

A complete classification of Kähler manifolds is still lacking. Hermitian symmetric spaces, which are completely classified, are examples of Kähler manifolds, while the remaining non-hermitian symmetric spaces are not Kähler (e.g. the even dimensional spheres $S^{2n}$, $n > 1$ are homogeneous and symmetric, but obviously not Kähler).

Some nonsemisimple groups also provide other Kähler manifolds. A very basic example is that obtained from the Heisenberg-Weyl algebra $hw(1)$ generated by the usual operators $p,q,1$, by quotient by the subgroup generated by the subalgebra 1. This space is the basic "quantum" space $q,p$, whose non-compact Kähler character becomes obvious after
introduction of the complex coordinate \( z = p + iq \).

APPENDIX A. KÄHLER HOMOGENEOUS MANIFOLDS

We start by listing some examples of compact Kähler homogeneous manifolds. More details can be found in \(^{19}\) and in \(^{21}\).

1. \( G = SO(3) \sim SU(2) \) is the rotation group of a three-dimensional vector space \( \mathbb{R}^3 \). Here the sign \( \sim \) means a locally isomorphic group and \( G^* \) is the dual space to the Lie algebra \( G^* = \{ x | x = (x_1, x_2, x_3) \} \approx \mathbb{R}^3 \). There is a zero-dimensional orbit (the origin) while the remaining orbits are generic and are two-dimensional spheres \( S_r^2 = \{ x | x^2 = x_1^2 + x_2^2 + x_3^2 = r^2 \} \).

2. \( G = SU(3) \). Here we have three types of coadjoint orbits in \( su(3) \approx \mathbb{R}^8 \): First, the origin \( x = 0 \). Second, four-dimensional orbits (isomorphic to \( \mathbb{C}P^2 \))

\[
\mathcal{O} = \frac{SU(3)}{SU(2) \times U(1)},
\]

and third, six-dimensional orbits isomorphic to the complex flag space \( \mathbb{F}^3 \)

\[
\mathcal{O} = \frac{SU(3)}{U(1) \times U(1)}.
\]

3. \( G = SU(n) \). Here, in addition to the trivial zero-dimensional orbit, we have orbits isomorphic to the complex projective space \( \mathbb{C}P^{n-1} \),

\[
\mathcal{O} = \frac{SU(n)}{SU(n-1) \times U(1)} \sim \mathbb{C}P^{n-1}.
\]

There are also orbits isomorphic to the complex grassmannians \( \mathbb{C}G_{m,n} \),

\[
\mathcal{O} = \frac{SU(m+n)}{SU(m) \times SU(n) \times U(1)} \sim \mathbb{C}G_{m,n},
\]

and finally the generic maximal orbits are isomorphic to the complex flag manifold \( \mathbb{F}^n \)

\[
\mathcal{O} = \frac{SU(n)}{U(1) \times U(1) \times \cdots \times U(1)} \sim \mathbb{F}^n.
\]

4. For compact simple Lie algebras, the coadjoint orbits of minimal non-zero dimension were investigated in \(^{23}\), and are given in the following Table:
### A.1 Kernels for some Hermitian symmetric spaces

In this section we give the explicit expressions for kernels of the Hermitian symmetric spaces of classical type, either compact or non-compact (bounded symmetric domains). They belong to four families, which in the Cartan notation are $A_{III}, C_I, D_{III}$ and $BD_I(q=2)$. There are two further exceptional Hermitian symmetric spaces, $E_{III}, E_{VII}$ related to exceptional Lie algebras.

#### $A_{III}$

For the complex Grassmannians $\mathbb{C}G_{p,q}$ of $p$-planes in $\mathbb{C}^{p+q}$:

$$X = SU(p+q)/(SU(p) \otimes SU(q) \otimes U(1)), \quad p \geq q,$$

(A.6)

in terms of the $pq$ complex coordinates arranged in a rectangular $p \times q$ complex matrix $Z$:

$$K(z, \overline{v}) = \det(I^{(p)} + ZW^\dagger).$$

(A.7)

#### $A_{III}^c$

The non-compact Cartan duals of the complex Grassmannians are the spaces:

$$X = SU(p,q)/(SU(p) \otimes SU(q) \otimes U(1)), \quad p \geq q.$$  

(A.8)

which can be realized as the bounded domain $I^{(p)} - ZZ^\dagger \geq 0$ with $Z$ as above; its kernel is:

$$K(z, \overline{v}) = \det(I^{(p)} - ZW^\dagger).$$

(A.9)

#### $C_I$

For the manifold of Lagrangian $p$-spaces in $\mathbb{C}^{2p}$, which is the compact symmetric Hermitian space:

$$X = Sp(p)/U(p)$$

(A.10)

the kernel is given in terms of $p(p+1)/2$ complex coordinates arranged in a $p \times p$ complex symmetric matrix $Z$ as:

$$K(z, \overline{v}) = \det(I^{(p)} + ZW^\dagger).$$

(A.11)
\( C_{\text{I nc}} \)

The Cartan dual to the previous space:

\[
X = Sp(2p, \mathbb{R})/U(p)
\]  

(A.12)

can be realized as the bounded domain \( I^{(p)} - ZZ^\dagger \geq 0 \) in terms of the coordinate matrix \( Z \) as above; its kernel is:

\[
K(z, \overline{w}) = \det(I^{(p)} - ZW^\dagger).
\]  

(A.13)

\( D_{\text{III c}} \)

The kernel for the compact Hermitian symmetric space:

\[
X = SO(2p)/U(p)
\]  

(A.14)

is given in terms of \( p(p - 1)/2 \) complex coordinates arranged in a rectangular \( p \times p \) complex skew-symmetric matrix \( Z \) as:

\[
K(z, \overline{w}) = \det(I^{(p)} + ZW^\dagger).
\]  

(A.15)

\( D_{\text{III nc}} \)

For the non-compact Cartan dual space:

\[
X = SO^*(2p)/U(p)
\]  

(A.16)

realized as the bounded domain \( I^{(p)} - ZZ^\dagger \geq 0 \) in terms of the coordinates \( Z \) as above, the kernel is:

\[
K(z, \overline{w}) = \det(I^{(p)} - ZW^\dagger).
\]  

(A.17)

\( B D_{\text{I c}} \)

The real Grassmannian \( \mathbb{R}G_{2,p} \) of 2-planes in \( \mathbb{R}^{p+2} \):

\[
X = SO(p + 2)/(SO(p) \otimes SO(2))
\]  

(A.18)

In terms of \( p \) complex coordinates arranged as a \( p \times 1 \) row complex vector \( z \), with \( z' \) denoting the transpose \( 1 \times p \) column complex vector, then

\[
K(z, \overline{w}) = 1 + (z \cdot z')(\overline{w} \cdot \overline{w}') + 2(z \cdot \overline{w}').
\]  

(A.19)

\( B D_{\text{I nc}} \)

The non-compact dual space:

\[
X = SO(p, 2)/(SO(p) \otimes SO(2))
\]  

(A.20)

can be realized as the bounded domain

\[
|z \cdot z'| < 1, \quad 1 + |z \cdot z'|^2 - 2\overline{z} \cdot z' > 0,
\]  

(A.21)

where the \( p \) complex coordinates are arranged as a \( p \times 1 \) row complex vector \( z \), as above; the kernel is:

\[
K(z, \overline{w}) = 1 + (z \cdot z')(\overline{w} \cdot \overline{w}') - 2(z \cdot \overline{w}').
\]  

(A.22)

The two exceptional Hermitian symmetric spaces can be dealt with similarly, by using \( 3 \times 3 \) octonionic matrices, as discussed by U. Hirzebruch\textsuperscript{25}. 

10
A.2 Topology of orbits

Orbits of a coadjoint representation of compact Lie groups are compact simply-connected manifolds; this follows from the exact homotopy sequence. They have a non-trivial second homotopy group $\pi_2(X)$ because they are compact symplectic manifolds. Further information on their topology may be found, for example, in the review $^9$.

Let $P_X(t) = \sum_{j=0}^{N} b_j t^j$ be the Poincaré polynomial of manifold $X$, $b_j$ being the Betti numbers of the manifold $X$ of dimension $N$. In our case $X = G/H$, where $H$ is some compact semisimple subgroup of $G$, and $\text{rank } H = \text{rank } G = r$. In this case, the Hirsch formula (see $^9$) is valid

$$P_X(t) = \prod_{j=1}^{r} \frac{1 - t^{2n_j}}{1 - t^{2m_j}},$$

where $n_j$ and $m_j$ are the degrees of basic invariants of the Weyl group $W$ of the groups $G$ and $H$ (see $^26$). Let us give a few applications of this formula. We have

(i) For the complex projective space:

$$X = \mathbb{C} P^n, \quad P_X(t) = P_n(t) = 1 + t^2 + t^4 + \cdots + t^{2n}. \quad (A.24)$$

(ii) For the complex flag manifold $\mathbb{F}^n$:

$$X = \mathbb{F}^n = \frac{SU(n)}{U(1) \times \cdots \times U(1)}, \quad P_X(t) = P_1(t) P_2(t) \cdots P_{n-1}(t), \quad (A.25)$$

where the polynomial $P_n(t)$ was defined above.

(iii) An example of a real grassmannian $\mathbb{R} G_{3,2}$:

$$X = \frac{SO(5)}{SO(3) \times SO(2)}, \quad P_X(t) = P_3(t). \quad (A.26)$$

(iv) An example of a real “flag-like” manifold:

$$X = \frac{SO(5)}{SO(2) \times SO(2)}, \quad P_X = P_1(t) P_3(t). \quad (A.27)$$

(v) For the minimal orbits of the coadjoint representation of $G_2$,

$$X = \frac{G_2}{SU(2) \times U(1)}, \quad P_X = P_5(t). \quad (A.28)$$

(vi) For the octonionic “flag-like” coadjoint orbit of $G_2$:

$$X = \frac{G_2}{U(1) \times U(1)}, \quad P_X = \frac{(1 - t^4)(1 - t^{12})}{(1 - t^2)(1 - t^2)} = P_1(t) P_5(t). \quad (A.29)$$

(vii) For the complex Grassmann manifolds $\mathbb{C} G_{m,n}$

$$X = \mathbb{C} G_{m,n} = \frac{SU(m + n)}{SU(m) \times SU(n) \times U(1)}, \quad (A.30)$$
\[
P_X = \frac{(1 - t^4) \ldots (1 - t^{2(m+n)})}{(1 - t^2)[(1 - t^4) \ldots (1 - t^{2m})][(1 - t^4) \ldots (1 - t^{2n})]}.
\] (A.31)

For example, for the lowest dimensional complex Grassmann manifold, \( \mathbb{C}G_{2,2} \), we have
\[
X = \mathbb{C}G_{2,2} = \frac{SU(4)}{SU(2) \times SU(2) \times U(1)},
\] (A.32)
\[
P_X = \frac{(1 - t^6)(1 - t^8)}{(1 - t^2)(1 - t^4)} = (1 + t^4)(1 + t^2 + t^4) = 1 + t^2 + 2t^4 + t^6 + t^8.
\] (A.33)

(viii) For the octonionic Cayley plane,
\[
X = \mathbb{F}_{4}C_{3} \times SO(2), \quad P_X = \frac{(1 - t^{16})(1 - t^{24})}{(1 - t^2)(1 - t^8)},
\] (A.34)
\[
P_X = (1 + t^8)(1 + t^2 + t^4 + \cdots + t^{22}) = 1 + t^2 + t^4 + \cdots + 2t^8 + 2t^{10} + \cdots + 2t^{22} + t^{24} + \cdots + t^{30}.
\] (A.35)

**APPENDIX B. COHERENT STATES**

As discussed in Section 2, we consider here classical Hamiltonian systems which correspond to quantum systems of a special type for which the quantum properties are expressed simply in terms of classical ones.

Let \((X, \omega)\) be a compact simply-connected symplectic manifold on which the semi-simple compact Lie group \(G\) act transitively.

As it was shown by A. Borel, this class of manifolds coincides with the class of orbits of a coadjoint or (what is equivalent) adjoint representation of the compact semi-simple Lie group \(G\). These manifolds are Kähler homogeneous manifolds, and have even dimension. This means that they admit a Hermitian \(G\)-invariant metric, as given in (3.1), whose imaginary part is a closed two-form given in (3.2). Both are determined by a single function \(F(z, \overline{z})\), called the potential of the Kähler metric, which may be found from the Gauss decomposition of the group \(G\).

The \(G\)-invariant Hermitian metric (and the \(G\)-invariant symplectic structure) on the orbits of coadjoint actions is not uniquely determined. The most general ones are a linear combination of a number \(r\) of basic metrics or symplectic forms, the number \(r\) being equal to the rank of the manifold.

Let us recall now the construction of unitary irreducible representations of simple compact Lie groups \(G\) of rank \(r\). Such representation is characterized by an \(r\)-dimensional vector \(\lambda = (\lambda_1, \ldots, \lambda_r)\) — the so-called highest weight: \(T(g) = T^\lambda(g)\), where \(\lambda = \sum \lambda_j w_j\), \(w_j\) are the fundamental weights and \(\lambda_j\) are non-negative integers.

Correspondingly, in the representation space \(\mathcal{H}^\lambda\), there exists a vector (the highest vector) \(|\lambda\rangle\) satisfying the conditions
\[
\hat{E}_\alpha |\lambda\rangle = 0, \quad \alpha \in R_+, \quad \hat{H}_j |\lambda\rangle = \lambda_j |\lambda\rangle,
\] (B.1)
where \(\hat{E}_\alpha\) and \(\hat{H}_j\) are operators in \(\mathcal{H}^\lambda\) which represent the Chevalley basis for \(\mathcal{G}^C\).
In the space $\mathcal{H}^{\lambda}$, there exists a basis $\{|\mu\rangle\}$, where $\mu$ is a weight vector, i.e., an eigenvector of all operators $H_j$:

$$H_j |\mu\rangle = \mu_j |\mu\rangle. \quad (B.2)$$

A general representation $T^\lambda(g)$ characterized by the highest weight $\lambda = (\lambda_1, \ldots, \lambda_r)$ corresponds to a fiber bundle over $X = G/H = G^C/B_+ = X_-$, with the circle as a fiber, with connection and curvature forms:

$$\theta^\lambda = \frac{1}{2i} \left( \frac{\partial F^\lambda}{\partial \bar{z}_\mu} d\bar{z}_\mu - \frac{\partial F^\lambda}{\partial z_\mu} dz_\mu \right), \quad \omega^\lambda = \frac{1}{2i} \frac{\partial^2 F^\lambda}{\partial z_\mu \partial \bar{z}_\nu} d\bar{z}_\mu \wedge dz_\nu = d\theta^\lambda, \quad (B.3)$$

where $F = \sum_l \lambda_l F^l, \quad l = 1, 2, \ldots, r$. The representation $T^\lambda(g)$ with the highest weight $\lambda$ may be realized in the space of polynomials $F^\lambda$ over $X_-$. Namely,

$$T^\lambda(g) f(z) = \alpha_\lambda(z, g) f(z), \quad (B.4)$$

where the quantities $\alpha_\lambda(z, g)$ and $z_g$ may be found from the Gaussian decomposition

$$z_g = \zeta_1 h_1 z_1, \quad (B.5)$$

$$z_g = z_1, \quad \alpha_\lambda(z, g) = \delta_1^{\lambda_1} \cdots \delta_r^{\lambda_r}. \quad (B.6)$$

The invariant scalar product $F^\lambda$ is introduced by the formulas

$$(f_1, f_2) = d_\lambda \int F^\lambda_1(z) f_2(z) d\mu_\lambda(z), \quad (B.7)$$

where $d_\lambda$ is the dimension of the representation $T^\lambda$. In this case we have

$$T^\lambda(g) f(z) = \exp \{ i S^\lambda(z, g) \} f(z), \quad (B.8)$$

where

$$S^\lambda(z, g) = \int_0^z (\theta^\lambda - g_\ast \cdot \theta^\lambda) + S^\lambda(0, g), \quad (B.9)$$

and the Kähler potential is:

$$F^\lambda = \sum_l \lambda_l F^\lambda_l(z, \bar{z}) = -\ln(\langle \lambda | T^\lambda(zz^+) | \lambda \rangle), \quad (B.10)$$

which determines after (B.3) the connection $\theta^\lambda$ and curvature $\omega^\lambda$ forms in the fiber bundle with base $X$, a circle as a fiber, and related to the representation $T^\lambda(g)$.

A similar construction works also for degenerate representations for which the highest weight $\lambda$ is singular, i.e. $(\lambda, \alpha) = 0$ for one or several roots $\alpha$. Then the isotropy subgroup $B$ of a vector $|\psi_0\rangle$ is one of the so-called parabolic subgroups. This means that $B$ contains the Borel subgroup $\tilde{B}$, i.e. the maximal solvable subgroup. The coset space $X = G^C/B$ is the degenerate orbit of the coadjoint representation, but this space is still the homogeneous Kähler manifold$^{19}$. Hence the construction considered above is valid completely also in this case.

Following$^{6,7,8}$, let us now construct the coherent state (hereafter CS) systems for an arbitrary compact Lie group.

To this aim one has to take an initial vector $|0\rangle$ in the space $\mathcal{H}^{\lambda}$. Note first of all that the isotropy subgroup $H_\mu$ for any state $|\mu\rangle$ corresponding to weight vector $\mu$ contains the
Cartan subgroup $H = U(1) \times \cdots \times U(1) = T^r$, where $r$ is the number of $U(1)$ factors entering in $H$, and is called the rank of group $G$. For generic weight vectors subgroup $H_\mu$ coincides with $H$.

In general, the isotropy subgroup for a linear combination of weight vector is a subgroup of the Cartan subgroup. Therefore it is convenient to choose a weight vector $|\mu\rangle$ as an initial element of the CS system. In the general case, the isotropy subgroup $H_\mu$ coincides with $H$ or in another form, $H_\mu = H$. For generic weight vectors subgroup $H_\mu$ coincides with $H$, and a CS is characterized by a point of $X = G/H$.

For the degenerate representation, where the highest weight $\lambda$ is orthogonal to some root $\alpha : (\lambda, \alpha) = 0$, the isotropy subgroup $H_\mu$ may be larger than $T^r$ for some state vector $|\mu\rangle$. Then any CS $|x\rangle$ is characterized by a point of a degenerate orbit of the adjoint representation. Indeed, in all cases,

$$H_j |x\rangle = [T(g) H_j T^{-1}(g)] |x\rangle = \mu_j |x\rangle, \quad |x\rangle = T(g) |\mu\rangle.$$  \hfill (B.11)

Therefore if we take a state vector $|\mu\rangle$ as the initial vector $|0\rangle$, then the coherent state $|x\rangle$ is characterized by a point of an orbit of adjoint representation, and the orbit may be degenerate.

Now suppose that $T^\lambda(g)$ is a non-degenerate representation of the compact Lie group $G$ with the highest weight $\lambda$, i.e., $(\lambda, \alpha) \neq 0$ for any $\alpha \in R$. We take the vector with the lowest weight $| - \lambda \rangle$ as the initial vector $|0\rangle$ for the CS system. Let us consider the action on this state of operators $H_j, E_\alpha$ and $E_{-\alpha}$ $(\alpha \in R_+)$ representing the Lie algebra $G_C$. One can see that subalgebra $B_- = \{H_j, E_{-\alpha}\}, \alpha \in R_+$ is the isotropy subalgebra for the vector $|\lambda\rangle$. The corresponding group $B_-$ is a subgroup of $G_C$.

Taking the lowest weight vector $|\lambda\rangle$ as $|0\rangle$, applying operators $T^\lambda(g)$ and using the Gaussian decomposition $g = \zeta h z$, with $\zeta \in Z_+$, we obtain the CS system

$$|\zeta\rangle = N T^\lambda(\zeta) |0\rangle = N \exp \left( \sum_{\alpha \in R_+} \zeta_\alpha E_\alpha \right) |0\rangle, \quad N = \langle 0 | T^\lambda(g) |0\rangle,$$  \hfill (B.12)

or in another form,

$$|\zeta\rangle = D(\xi) |0\rangle, \quad D(\xi) = \exp \left[ \sum (\xi_\alpha E_\alpha - \bar{\xi}_\alpha E_{-\alpha}) \right].$$  \hfill (B.13)

Note that the unitary operators $D(\xi)$ do not form a group but their multiplication law is

$$D(\xi_1) D(\xi_3) = D(\xi_3) \exp \left( i \sum_j \varphi_j H_j \right).$$  \hfill (B.14)

Note also that these CS are eigenstates of operators

$$T(g) H_j T^{-1}(g) = \tilde{H}_j, \quad \tilde{H}_j |x\rangle = -\lambda_j |x\rangle.$$  \hfill (B.15)

The last equations determine the CS up to a phase factor $\exp(i\alpha)$. The constructed CS system has all properties of a general CS system. Some of the most important ones are noted below.

1. Operators $T^\lambda(g)$ transform one CS into another,

$$T^\lambda(g) |x\rangle = \exp(i \phi_\lambda(x, g)) |x \rangle,$$  \hfill (B.16)

where $\phi_\lambda(x, g)$ is a phase shift.
2. CS are not mutually orthogonal. The scalar product is

\[ \langle \zeta_1 | \zeta_2 \rangle = N_1 N_2 \langle 0 | T^+ (\zeta_1) T(\zeta_2) | 0 \rangle = N_1 N_2 \langle 0 | T(\zeta_1^+ \zeta_2) | 0 \rangle \]

\[ = K_{\lambda}(\zeta_1^+ \zeta_2) \left[ K_{\lambda}(\zeta_1^+ \zeta_1) K_{\lambda}(\zeta_2^+ \zeta_2) \right]^{-1/2}, \quad (B.17) \]

where

\[ K_{\lambda}(\zeta_1^+ \zeta_2) = \Delta_{1}^{\lambda_1}(\zeta_1^+ \zeta_2) \cdots \Delta_{r}^{\lambda_r}(\zeta_1^+ \zeta_2) \]

and quantities \( \Delta_j \) may be found from the Gaussian decomposition. For the group \( G = SU(n), G^C = SL(n, \mathbb{C}), \) the quantity \( \Delta_j \) is the lower angular minor of order \( j \) of the matrix \( \zeta_1^+ \zeta_2. \)

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