Riesz bases and Jordan form of the translation operator in semi-infinite periodic waveguides

T. Hohage\textsuperscript{b}, S. Soussi\textsuperscript{a,1,}\textsuperscript{*}

\textsuperscript{a}Department of Mathematics and Statistics, University of Limerick, Limerick, Ireland.  
\textsuperscript{b}Institut für Numerische und Angewandte Mathematik, University of Göttingen, Göttingen, Germany.

Abstract

We study the propagation of time-harmonic acoustic or transverse magnetic (TM) polarized electromagnetic waves in a periodic waveguide lying in the half-strip \((0, \infty) \times (0, L)\). It is shown that there exists a Riesz basis of the space of solutions to the time-harmonic wave equation such that the translation operator shifting a function by one periodicity length to the left is represented by an infinite Jordan matrix which contains at most a finite number of Jordan blocks of size \(>1\). Moreover, the Dirichlet-, Neumann- and mixed traces of this Riesz basis on the left boundary also form a Riesz basis. Both the cases of frequencies in a band gap and frequencies in the spectrum and a variety of boundary conditions on the top and bottom are considered.

Keywords: photonic crystal, photonic band gap, periodic dielectric medium, Floquet theory, analytic theory of operators

2000 MSC: 35P30, 78A40, 35Q60, 35B27, 35B30

1. Introduction

Periodic media have received much attention in recent years since they can prohibit the propagation of electromagnetic and acoustic waves in some frequency ranges \([1]\). This localization property is a consequence of band structure of the spectrum of the underlying differential operator and of the presence of band gaps in this spectrum. Waves with frequencies in a band gap will decrease exponentially inside such media as a consequence of the exponential decay of the Green’s kernel, see \([2]\). The band structure of the spectrum is explained by Floquet theory, see \([3, 4, 5]\). We refer the reader to \([6]\) for a mathematical introduction to photonic crystals and to \([2, 3]\) for a physical introduction.

\textsuperscript{*}Corresponding author

Email addresses: hohage@math.uni-goettingen.de (T. Hohage), sofiane.soussi@gmail.com (S. Soussi)

\textsuperscript{1}Most of this research was carried out while the second author was working in Göttingen supported by the DFG grant RTG 1023.

Preprint submitted to Elsevier

May 8, 2012
The strong localization property can be used to construct devices that mould the flow of light or sound at a very small length scale. The simulation of such devices requires the numerical solution of differential equations in locally perturbed periodic media, which is a challenging task. The proof of existence and numerical computation of defect modes has been studied in [9, 10, 11, 2, 12, 13, 14]. Problems in locally perturbed infinite periodic media with a source term considered as part of the problem have been studied more recently in [15, 16, 17]. Here a natural approach consists in solving a boundary value problem on a compact set enclosing the perturbation and using a Dirichlet-to-Neumann or a related operator on the artificial boundary. Hence, the problem is decomposed into an interior and an exterior problem. The purpose of this paper is to contribute to the understanding of exterior boundary value problems in semi-infinite periodic waveguides. A summary of our results has already been given in the abstract, for a precise formulation we refer to the next section.

The study of wave propagation in doubly periodic half planes can be reduced via the Floquet transform to wave propagation in semi-infinite periodic waveguides with quasi-periodic boundary conditions on the lateral boundaries. Moreover, using a clever trick proposed by Fliss & Joly [15], boundary value problems in the doubly periodic exterior of a square can be reduced in some sense to boundary value problems in a doubly-periodic half plane. Therefore, the analysis of this paper is also relevant for these problems.

Our results prove two open conjectures formulated in connection with a numerical method to compute the Neumann-to-Dirichlet map proposed by Fliss, Joly and Li (see the discussion of Theorem 2.2). Actually, this study arose from attempts to justify an alternative numerical approach, which will be published elsewhere. Moreover, our results explain with a new approach the exponential decay of waves in periodic media with frequency in the band gap studied in [11, 12, 13, 14] and even provides the optimal decay rate which corresponds to the one of the slowest decaying Floquet mode, which may provide guidance for photonic crystal optimization by focusing on this Floquet mode and trying to make its decay as fast as possible.

The paper is outlined as follows: in §2 we introduce the problem and present the main theorem. Some known prerequisites are collected in §3 in particular a generalization of Rouché’s theorem shown in [18], which will serve as an essential tool in the following analysis. The remaining part of the paper is dedicated to the proof of the main theorem and will be summarized at the end of section 3. There are two appendices on radiation conditions and on uniqueness results.

2. Statement of the problem and the main results

The propagation of time-harmonic acoustic or transverse magnetic (TM) polarized electromagnetic waves in a 2-D waveguide lying in the half-strip $S^+ := \mathbb{R}^+ \times (0, L)$ is described by the differential equation

$$\Delta v + \omega^2 \varepsilon_p v = 0, \quad \text{in} \ S^+. \tag{1a}$$
We assume that \( \varepsilon_p \in L^\infty(S) \) with \( S := \mathbb{R} \times (0, L) \) is periodic with period length 1 in the first variable and bounded away from 0, i.e.

\[
\varepsilon_p(x_1 + 1, x_2) = \varepsilon_p(x_1, x_2), \quad \text{for all } (x_1, x_2) \in S,
0 < \text{essinf} \varepsilon_p \leq \varepsilon_p \leq \bar{\varepsilon} \text{ a.e. in } S
\]

for some \( \bar{\varepsilon} > 0 \). On the top and bottom of \( S^+ \) we consider a boundary condition

\[
\gamma v(x_1, \cdot) = (0, 0)^T, \quad x_1 > 0
\]

with one of the following boundary value operators \( \gamma : H^2((0, L)) \to \mathbb{R}^2 \):

- **Dirichlet**
  \( \gamma_Dv := (v(0), v(L))^T \)

- **Neumann**
  \( \gamma_Nv := (v'(0), v'(L))^T \)

- **mixed**
  \( \gamma_DNv := (v(0), v'(L))^T \)

- **\( \beta \)-quasi-periodic**
  \( \gamma_\beta v := (e^{i\beta}v(0) - v(L), e^{i\beta}v'(0) - v'(L))^T, \quad \beta \in [0, 2\pi) \).

To treat \( \beta \)-quasi-periodic boundary conditions with \( \beta = 0 \) and \( \beta = \pi \) we have to impose the symmetry condition \( \varepsilon_p(x_1, x_2) = \varepsilon_p(x_1, L - x_2), \) \( x \in (0, 1) \times (0, L) \) for reasons explained in §6. Furthermore, we imposed a boundary condition on the left:

\[
\tau v = f \quad \text{with} \quad \tau v := \theta_Dv(0, \cdot) + \theta_N\frac{\partial v}{\partial x_1}(0, \cdot). \tag{1c}
\]

Here \( \theta_D, \theta_N \in \mathbb{C} \) with \( |\theta_D| + |\theta_N| > 0 \), and we consider \( \tau \) as an operator with values in a space \( H^2_\gamma \) which depends on \( \theta_N \) and \( \gamma \) and will be defined in §4.

To describe our condition on the behavior of the solution as \( x_1 \to \infty \), which will complete the formulation of the boundary value problem, we need the following definition:

**Definition 2.1.** A Floquet mode is a nontrivial solution to (1a), (1b) of the form

\[
\exp(i\xi x_1) \sum_{j=0}^m x_1^{m-j} u_j(x_1, x_2)
\]

with \( \xi \in \mathbb{C} \) and functions \( u_j \) satisfying \( u_j(x_1 + 1, x_2) = u_j(x_1, x_2) \) for all \( (x_1, x_2) \in S^+ \). We call \( \xi \) the quasi momentum and \( m \) the order of the Floquet mode, assuming that \( u^{(m)} \neq 0 \).
Note that the quasi momentum is only determined up to an additive integer multiple of $2\pi$. Typically we will choose $R\xi \in [-\pi, \pi)$. We complete the formulation of the boundary value problem by the radiation condition

$$v \in H^{1,\gamma}_{\gamma}(S^+) \quad \text{where} \quad H^{1,\gamma}_{\gamma}(S^+) := H^1_{\gamma}(S^+) \oplus \text{span}\{v_1^+, \ldots, v_{\pi}^+\},$$

(1d)

where $v_1^+, \ldots, v_{\pi}^+$ are Floquet modes with real quasi momentum, the choice of which is described in the following:

It is known (see [13, Corollary 5.1.5] or Corollary 4.6) that the waveguide supports a finite even number $2\pi$ of linearly independent Floquet modes with real quasi momentum. To formulate a radiation condition we introduce the sesquilinear form

$$q_{x_1}(v, w) := \int_0^L \left( \frac{\partial v}{\partial x_1}(x)w(x) - v(x)\frac{\partial w}{\partial x_1}(x) \right) dx, \quad x_1 \geq 0$$

(2)

for solutions $v, w$ to (1a) and (1b). Since $q_{x_1}$ is actually independent of $x_1$ as a consequence of Green’s theorem, we omit the index $x_1$ in the following. If the time dependence is given by $\exp(-i\omega t)$ with $\omega > 0$, then $\Im q(v, v)$ is proportional (with positive constant) to the energy flux through a cross section $\{x_1\} \times (0, L)$ (see e.g. [13, § 5.6.3] where the sign convention $\exp(i\omega t)$ is used).

From a physical mode we expect that energy is transported to the right, so $\Im q(v, v) > 0$. It can be shown (see [13, Theorem 5.3.2]) that there exists a basis $\{v_1^+, \ldots, v_{\pi}^+, v_1^-, \ldots, v_{\pi}^-\}$ of the span of all Floquet modes with real quasi momentum, which consists of Floquet modes satisfying the orthogonality and normalization conditions

$$q(v_j^+, v_k^-) = 0, \quad \text{for } j, k \in \{1, \ldots, \pi\},$$

(3a)

$$q(v_j^+, v_k^+) = i\delta_{j,k}, \quad q(v_j^-, v_k^-) = -i\delta_{j,k}, \quad \text{for } j, k \in \{1, \ldots, \pi\}. \quad \text{(3b)}$$

(As discussed in Appendix A, span$\{v_1^+, \ldots, v_{\pi}^+, v_1^-, \ldots, v_{\pi}^-\}$ is not necessarily uniquely determined by this condition for all $\omega$, but the following results hold for any choice of the $v_n^\pm$.)

Each element $v \in H^{1,\gamma}_{\gamma}(S^+)$ has a unique representation of the form $v = \tilde{v} + \sum_{n=1}^{\pi} \alpha_n v_n^+$ with $\tilde{v} \in H^1_{\gamma}(S^+)$ and $\alpha_n \in \mathbb{C}$, and we introduce a norm on $H^{1,\gamma}_{\gamma}(S^+)$ by $\|v\|_{H^{1,\gamma}_{\gamma}(S^+)}^2 = \|\tilde{v}\|^2_{H^1_{\gamma}(S^+)} + \sum_{n=1}^{\pi} |\alpha_n|^2$.

Let us introduce the translation operator

$$(Tv)(x) := v(x_1 + 1, x_2)$$

and assume that the Floquet modes $v_n^+$ have been chosen such that $H^{1,\gamma}_{\gamma}(S^+)$ is invariant under $T$. (This is always the case if $v_1^+, \ldots, v_{\pi}^+$ are all of order $0$.) Let $V \subset H^{1,\gamma}_{\gamma}(S^+)$ be the linear subspace of all weak solutions to (1a) and (1b). If the boundary value problem (1b) is well posed, i.e. the operator $\tau_{|V} : V \to H^\gamma_{\gamma}$ has a bounded inverse (see Prop. 6.2 and Appendix B for sufficient conditions), we can define the monodromy operator

$${\mathcal R} := \tau T (\tau_{|V})^{-1} : H^\gamma_{\gamma} \to H^\gamma_{\gamma},$$

(4)
which maps the trace of a solution to (1) to its trace at one periodicity length to the right. We are now in a position to formulate our main result:

**Theorem 2.2.**

1. There exist Floquet modes $v_n^+, n \in \mathbb{N}$ of which are defined as described above, which form a Riesz basis of $V$. This basis can be chosen such that $T : V \to V$ is represented by an infinite Jordan matrix, which contains at most a finite number of Jordan blocks of size greater than 1 all of which are of finite size.

2. If problem (1) is well posed for some choice of $\tau$, then $\{\tau v_n^+ : n \in \mathbb{N}\}$ is a Riesz basis of $H_\tau^\gamma$, and with respect to this basis the operator $R$ is represented by the same Jordan matrix as $T$ in part 1.

The second part of this theorem proves the conjectures in [17, Remark 5.1] and [16, Conjecture 3.2.52]. (In the latter case, to prove that $R |_{\text{span} \{\tau v_n^+: n > n\}}$ has spectral radius < 1, we also have to take into account Proposition 4.3.)

We point out that the well-posedness assumption in the second part of Theorem 2.2 is always satisfied for certain types of Robin trace operators (see Prop. 6.2 and B.2). We will use this fact in the proof of the first part of the theorem. For other trace operators $\tau$ non-uniqueness may occur at certain frequencies $\omega$.

3. Preliminaries and outline of the proof

3.1. Sobolev spaces on the cross section

For the boundary value operators $\gamma \in \{\gamma_D, \gamma_N, \gamma_{DN}, \gamma_\beta\}$ defined in the introduction, we define the second derivative operators

$$D_\gamma : \mathcal{D}(D_\gamma) \to L^2((0, L)), \quad w \mapsto -w''$$

with $\mathcal{D}(D_\gamma) := \{w \in H^2((0, L)) : \gamma w = (0, 0)^T\}$.

It is well known that these operators are positive and self-adjoint with compact resolvents. Moreover, complete orthonormal systems of eigepairs $\{(\tilde{\psi}_k^{\gamma}, (\tilde{\kappa}_k^{\gamma})^2) : k \in \mathbb{I}\}$ are known explicitly:

| boundary condition | $\tilde{\psi}_k^{\gamma}(t)$ | $\tilde{\kappa}_k^{\gamma}$ | $\mathbb{I}$ |
|--------------------|-----------------------------|-------------------------|----------|
| Dirichlet          | $\sqrt{\frac{2}{L}} \sin \left(\frac{\pi k}{L} t\right)$ | $\frac{\pi k}{L}$ | $\mathbb{N}$ |
| Neumann            | $\sqrt{\frac{2}{L}} \cos \left(\frac{\pi k}{L} t\right)$ | $\frac{\pi k}{L}$ | $\mathbb{N} \cup \{0\}$ |
| mixed              | $\sqrt{\frac{2}{L}} \sin \left(\frac{\pi (2k-1)}{2L} t\right)$ | $\frac{\pi (2k-1)}{2L}$ | $\mathbb{N}$ |
| $\beta$-quasi-periodic | $\sqrt{\frac{1}{L}} \exp \left(i \frac{\beta + 2\pi k}{L} t\right)$ | $\frac{\beta + 2\pi k}{L}$ | $\mathbb{Z}$ |

To simplify our notation we will often suppress the dependence of $\tilde{\psi}_k^{\gamma}$ and $\tilde{\kappa}_k^{\gamma}$ on $\gamma$ in the following. Sobolev spaces corresponding a boundary value operator $\gamma \in \{\gamma_D, \gamma_N, \gamma_{DN}, \gamma_\beta\}$ can then be defined by $H_\gamma^s((0, L)) = \mathcal{D}((I + D_\gamma)^{s/2})$ with norm $\|w\|_{H_\gamma^s} := \|(I + D_\gamma)^{s/2} w\|_{L^2}$ for $s \geq 0$, and for $s < 0$ the space...
More explicitly,
\[
\|w\|^2_{H_\gamma^s} = \sum_{k \in \mathbb{Z}} (1 + \kappa_k^2)^s |(w, \tilde{\psi}_k)|^2
\]  

It will be convenient to renumber the square roots of the $\kappa_l$’s in increasing order such that
\[
\sigma(D_\gamma) = \{ \kappa_k^2 : k \in \mathbb{I} \} = \{ \kappa_l^2 : l \in \mathbb{N} \}
\]
and $0 \leq \kappa_1 \leq \kappa_2 \leq \ldots$. This defines a bijective mapping $\mathbb{N} \to \mathbb{I}$, $k \mapsto l(k)$. Note that
\[
\kappa_{2l+1} = \kappa_1 + \frac{2\pi}{L} l \quad \text{and} \quad \kappa_{2l+2} = \kappa_2 + \frac{2\pi}{L} l
\]
for $l = 1, \ldots$. Moreover, we will write $\psi_k = \tilde{\psi}_{l(k)}$ for $k \in \mathbb{N}$.

3.2. Sobolev spaces on the strip

We introduce the self-adjoint negative Laplace operators
\[
\hat{D}_\gamma : D(\hat{D}_\gamma) \to L^2(S), \quad v \mapsto -\Delta v,
\]
with $D(\hat{D}_\gamma) := \{ v \in H^2(S) : \forall x_1 \in \mathbb{R} \gamma v(x_1, \cdot) = (0, 0)^T \}$ for $\gamma \in \{ \gamma_D, \gamma_N, \gamma_{DN}, \gamma_{\beta} \}$. Then we define
\[
H^s_\gamma(S) := D \left( (1 + \hat{D}_\gamma)^{s/2} \right), \quad s \geq 0,
\]
and $\|v\|_{H^s_\gamma} := \|(1 + \hat{D}_\gamma)^{s/2} v\|_{L^2}$. For $s < 0$, $H^s_\gamma(S)$ is defined as completion of $L^2(S)$ under the norm $\| \cdot \|_{H^s_\gamma}$. It can be shown that $H^1_{\gamma_D}(S) = \mathcal{H}_0^1(S)$, $H^1_{\gamma_N}(S) = \mathcal{H}_0^1(S)$, and $H^1_{\gamma_{DN}}(S) = \{ v \in \mathcal{H}_0^1(S) : e^{i\beta} v(\cdot, 0) = v(\cdot, L) \}$, and the norm $\| \cdot \|_{H^1_\gamma}$ is equivalent to $\| \cdot \|_{H^1}$ given by $\|v\|_{H^1_\gamma} = \int_S (|v|^2 + |\nabla v|^2) \, dx$. $H^s_\gamma(S^+)$ can be defined as the set of all restrictions to $S^+$ of functions in $H^s_\gamma(S)$ with $\|v\|_{H^s_\gamma(S^+)} := \inf \{ \|\tilde{v}\|_{H^s_\gamma(S)} : \tilde{v} |_{S^+} = v \}$. Moreover, the trace operators
\[
\tau_D : H^1_\gamma(S^+) \to H^{-1/2}_\gamma((0, L)), \quad v \mapsto v(0, \cdot),
\]
\[
\tau_N : H^1_\gamma(S^+; \Delta) \to H^{-1/2}_\gamma((0, L)), \quad v \mapsto \frac{\partial v}{\partial x_1}(0, \cdot),
\]
are well defined, continuous, and surjective, and have bounded right-inverses (see [20]). (Here $\|v\|_{H^1_\gamma(S^+; \Delta)} := \|v\|^2_{H^1_\gamma(S^+)} + \|\Delta v\|^2_{L^2(S^+)}$.) We choose $H^s : H^{-1/2}_\gamma((0, L))$ if $\theta_\Delta = 0$ and $H^s : H^{-1/2}_\gamma((0, L))$ if $\theta_\Delta \neq 0$.

3.3. Floquet transform

The Floquet transform is defined by
\[
\mathcal{F} : L^2(S) \to L^2((-\pi, \pi), L^2(\Omega))
\]
\[
\mathcal{F} v(\alpha, x) := \frac{1}{\sqrt{2\pi}} \sum_{l \in \mathbb{Z}} v(x_1 + l, x_2) e^{-i\alpha(x_1 + l)}
\]
where $\Omega = \mathbb{R}/\mathbb{Z} \times (0, L)$. It is isometric and its inverse is given by $v(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} Fv(\alpha, x) e^{i\alpha x_1} \, d\alpha$, $x \in S$ (3).

We will frequently use the orthonormal bases $\{\varphi_{m,n}^{(\gamma)} : m \in \mathbb{Z}, n \in \mathbb{N}\}$ of $L^2(\Omega)$ defined by

$$\varphi_{m,n}^{(\gamma)}(x) := \exp(2\pi imx_1)\varphi_{l_2}^{(\gamma)}(x_2), \quad x \in \Omega.$$  

(6)

If $v \in H^s_0(S)$, then for all $\alpha \in [-\pi, \pi]$ the function $Fv(\alpha, \cdot)$ belongs to the Sobolev space $H^s_0(\Omega)$ defined by $H^s_0(\Omega) := \{u \in L^2(\Omega) : \|u\|_{H^s(\Omega)} < \infty\}$ with

$$\|u\|_{H^s(\Omega)} := \left( \sum_{l \in \mathbb{Z} \times \mathbb{N}} (1 + |l|^2)^s |\langle \varphi_l^{(\gamma)}, u \rangle|^2 \right)^{1/2}$$

for $s \geq 0$ and as completion of $L^2(\Omega)$ under this norm for $s < 0$ (3). For $\alpha \in [-\pi, \pi]$ the operator $\Delta_\alpha^{(\gamma)} : H^2_0(\Omega) \rightarrow L^2(\Omega)$ uniquely defined by the property

$$\Delta_\alpha(Fv(\alpha, \cdot)) = (\mathcal{F} \Delta v)(\alpha, \cdot)$$

is given explicitly by

$$\Delta_\alpha = e^{-i\alpha x_1} \Delta e^{i\alpha x_1} = (\partial_{x_1} + i\alpha)^2 + \partial_{x_2}^2 = \Delta + 2i\alpha \partial_{x_1} - \alpha^2.$$  

(7)

3.4. Floquet modes and characteristic values of $(B_\xi)$

Due to (6) the mapping $\alpha \mapsto \Delta_\alpha$ is a polynomial with coefficients in $\mathcal{L}(H^s(\Omega), H^{s-2}(\Omega))$ for all $s$, so in particular, it has a holomorphic extension denoted by

$$\Delta_\xi := \Delta + 2i\xi \partial_{x_1} - \xi^2, \quad \xi \in \mathbb{C}.$$  

(8)

Moreover, let us introduce the operators

$$B_\xi : H^s_0(\Omega) \rightarrow H^{s-2}_0(\Omega), \quad B_\xi u := \Delta_\xi u + \omega^2 \xi u$$

for $\xi \in \mathbb{C}$, which are well defined for any $s \in [0, 2]$. For a review of the properties of holomorphic extensions of Floquet transformed periodic differential operators in a much greater generality we refer to [5].

$\xi_0 \in \mathbb{C}$ is called a characteristic value of $\xi \mapsto B_\xi$ if $B_{\xi_0}$ is not injective. If $\xi_0$ is a characteristic value of $B_\xi$ for some choice of $s \in [0, 2]$, and $B_{\xi_0}u_0 = 0$ for $u_0 \in H^s_0(\Omega) \setminus \{0\}$, then $u_0 \in H^2_0(\Omega)$ by elliptic regularity results, and $\xi_0$ is a characteristic value of $B_\xi$ for any parameter $s \in [0, 2]$. Therefore, the set of characteristic values does not depend on the parameter $s$, and for studying this set we may choose $s$ at our convenience. The operators $B_\xi$ are defined such that (with $s = 2$):

**Remark 3.1.** The following statements are equivalent for $\xi_0 \in \mathbb{C}$:

- $\xi_0$ is a characteristic value of $(B_\xi)$.
- There exists a Floquet mode $v(x) = e^{i\xi_0 x_1}u(x)$ with $u \in H^2_0(\Omega)$.  

7
It follows from the second statement that:

**Remark 3.2.** If \( \xi \in \mathbb{C} \) is a characteristic value of \( \xi \mapsto B_\xi \), then \( \xi + 2\pi l \) is a characteristic value for all \( l \in \mathbb{Z} \).

To formulate some results by Gohberg and Sigal \cite{18} in Theorem \cite{21} below, which will serve as an essential tool in the following analysis, we first have to recall the definition of the multiplicity of characteristic values. Note that for all \( \xi \) the operator \( B_\xi \) is a compact perturbation of the operator \( \Delta - 1 \) in \( \mathcal{L}(H^2_0(\Omega), L^2(\Omega)) \), and that \( \Delta - 1 \) has a bounded inverse. Therefore, by analytic Fredholm theory \cite{21}, the set of characteristic values of \( B_\xi \) is discrete. For the special case at hand the definitions in \cite{18} simplify as follows:

**Definition 3.3.** Let \( B_1 \) and \( B_2 \) be Banach spaces and \( \xi \mapsto B_\xi \) a holomorphic mapping defined on a domain in \( \mathbb{C} \) with values in \( \mathcal{L}(B_1, B_2) \). Moreover, let \( B_\xi = B + K_\xi \) where \( B \) has a bounded inverse in \( \mathcal{L}(B_2, B_1) \) and \( K_\xi \) is compact for all \( \xi \).

The point \( \xi_0 \in \mathbb{C} \) is called a characteristic value of \( (B_\xi) \) if there exists a holomorphic function \( \xi \mapsto u_\xi \) (called root function) with values in \( B_1 \) such that \( u_{\xi_0} \neq 0 \) and \( B_{\xi_0} u_{\xi_0} = 0 \). (Note that such root functions, which may be chosen constant, exist if and only if \( B_{\xi_0} \) is not invertible.) The multiplicity of a root function \( (u_\xi) \) is the order of \( \xi_0 \) as a root of \( \xi \mapsto B_{\xi} u_\xi \). \( \pi \in B_1 \) is called an eigenvector of \( (B_\xi) \) corresponding to \( \xi_0 \) if \( \pi = u_{\xi_0} \) for some root function \( (u_\xi) \) of \( (B_\xi) \) corresponding to \( \xi_0 \). The rank \( \text{rank}(\pi) \) of an eigenvector \( \pi \) is defined as the maximum of all the multiplicities of root functions \( (u_\xi) \) with \( \pi = u_{\xi_0} \).

(If the given assumptions the geometric multiplicity \( \alpha = \dim \ker B_{\xi_0} \) of \( \xi_0 \) and the ranks of all eigenvectors \( \pi \in \ker B_{\xi_0} \) are finite, see \cite{18} Lemma 2.1.)

A canonical system of eigenvectors of \( (B_\xi) \) corresponding to \( \xi_0 \) is defined as a basis \( \{u^{(1)},...,u^{(\alpha)}\} \) of \( \ker B_{\xi_0} \) with the following properties: \( \text{rank} u^{(1)} \) is the maximum of the ranks of all eigenvectors corresponding to \( \xi_0 \), and \( \text{rank} u^{(j)} \) for \( j = 2,...,\alpha \) is the maximum of the ranks of all eigenvectors in some direct complement of \( \text{span}\{u^{(1)},...,u^{(j-1)}\} \) in \( \ker B_{\xi_0} \). The numbers \( r_j = \text{rank} u^{(j)} \) \( (j = 1,...,\alpha) \) are called the partial null multiplicities of the characteristic value \( \xi_0 \), and \( \mathfrak{n}(\pi(B_\xi)\xi_0) = (r_1, r_2, ..., r_\alpha) \) the \( \alpha \)-tuple of partial null multiplicities. We call \( \mathfrak{n}(\pi(B_\xi)\xi_0) = r_1 + r_2 + ... + r_\alpha \) the (total) null multiplicity of the characteristic value \( \xi_0 \) of \( (B_\xi) \).

Finally, we call canonical Jordan chains associated to the characteristic value \( \xi_0 \) any sets \( \{u^{(j)}_0, u^{(j)}_1, ..., u^{(j)}_{r_j-1}\} \), \( 1 \leq j \leq \alpha \), of vectors in \( B_1 \) such that \( \{u^{(1)}_0, ..., u^{(\alpha)}_0\} \) is a canonical system of eigenvectors and \( \xi_0 \) is a root of order \( r_j \) of \( \xi \mapsto B_\xi (\sum_{k=0}^{j-1}(\xi - \xi_0)^k u^{(j)}_k) \) for any \( 1 \leq j \leq \alpha \).

Let \( \Gamma \) be a simple, closed, rectifiable contour contained in the domain of analyticity of \( (B_\xi) \) and of \( (B_\xi)^{-1} \) and let \( \xi_1, \xi_2, ..., \xi_n \) be the characteristic values of \( (B_\xi) \) enclosed by \( \Gamma \). (Recall that simple means that \( \Gamma \) has a continuous, bijective parametrization \( \gamma : [0,\pi) \rightarrow \Gamma \).) Then we set

\[
\mathfrak{n}(\pi(B_\xi)\Gamma) := \sum_{j=1}^{n} \mathfrak{n}(\pi(B_\xi)\xi_j).
\]
We cite the following generalization of Rouché’s Theorem:

**Theorem 3.4** (18). Assume that \((A_\xi)\) satisfies the assumptions of Definition 3.3 and let \(\Gamma\) be a simple, closed, rectifiable contour in the domain of analyticity of \((A_\xi)\) and of \((A_\xi)^{-1}\). If \(\xi \mapsto S_\xi\) is another holomorphic function defined on the same domain as \((A_\xi)\) with values in \(L(B_1, B_2)\) and if

\[||A_\xi^{-1}S_\xi|| < 1 \quad \text{for all } \xi \in \Gamma,\]

then \((A_\xi + S_\xi)^{-1}\) is analytic in some neighborhood of \(\Gamma\) and

\[\mathcal{N}((A_\xi + S_\xi); \Gamma) = \mathcal{N}((A_\xi); \Gamma). \quad (10)\]

We can now state the following generalization of Remark 3.1.

**Proposition 3.5.** Suppose that \(\xi_0\) is a characteristic value of \((B_\xi)\) with partial null multiplicities \(n(B_\xi; \xi_0) = (r_1, r_2, \ldots, r_\alpha)\). Then, the vector space \(\mathcal{V}_{\xi_0}\) of all Floquet modes with quasi momentum \(\xi_0\) is a direct sum of \(T\)-invariant subspaces \(\mathcal{V}_{\xi_0,j}\) of dimensions \(r_j \in \mathbb{N}, j = 1, \ldots, \alpha\). Moreover, there exists a basis \(\{v_{j_0,0}, \ldots, v_{j_r,1}\}\) of \(\mathcal{V}_{\xi_0,j}\) with respect to which \(\mathcal{T}: \mathcal{V}_{\xi_0,j} \to \mathcal{V}_{\xi_0,j}\) is represented by a single \(r_j \times r_j\) Jordan block, and \(v_{j,k}\) are Floquet modes of order \(k\).

Moreover, in the expansion

\[v_{j,k}(x) = e^{i\xi_0 x_1} \sum_{l=0}^{k} \frac{(ix_1)^{k-l}}{(k-l)!} u_i^{(j,k)}(x), \quad k = 0, \ldots, r_j - 1 \quad (11)\]

we have \(u_0^{(j,0)} = \cdots = u_0^{(j,r_j-1)} = \tilde{u}_0^{(j)}\), and \(\{\tilde{u}_0^{(1)}, \ldots, \tilde{u}_0^{(\alpha)}\}\) is a basis of \(\ker(B_{\xi_0})\).

**Proof.** From [18, § 3.4.3] we know that there exist canonical Jordan chains \(\{\tilde{v}_0^{(j)}, \ldots, \tilde{v}_r^{(j)}\}, 1 \leq j \leq \alpha\) associated to \(\xi_0\), and \(\mathcal{V}_{\xi_0} = \oplus_{j=1}^{\alpha} \mathcal{V}_{\xi_0,j}\) where \(\mathcal{V}_{\xi_0,j} = \text{span}\{\tilde{v}_{j,k}: 0 \leq k \leq r_j - 1\}\) and

\[\tilde{v}_{j,k} = e^{i\xi_0 x_1} \sum_{l=0}^{k} \frac{(ix_1)^{k-l}}{(k-l)!} \tilde{u}_l^{(j)}. \quad (12)\]

Since

\[(\mathcal{T}\tilde{v}_{j,k})(x) = e^{i\xi_0 (x_1 + 1)} \sum_{l=0}^{k} \frac{(ix_1 + i)^{k-l}}{(k-l)!} \tilde{u}_l^{(j)}(x) = e^{i\xi_0} e^{i\xi_0 x_1} \sum_{l=0}^{k} \sum_{m=0}^{k-l} \frac{(ix_1)^{k-l-m}}{m! (k-l-m)!} \tilde{u}_l^{(j)}(x) = e^{i\xi_0} \sum_{m=0}^{k} \frac{(ix_1)^{k-m-l}}{(k-m-l)!} \tilde{u}_l^{(j)}(x) = e^{i\xi_0} \sum_{m=0}^{k} \frac{(ix_1)^{k-m}}{(k-m)!} \tilde{v}_{j,m}(x),\]

we have

\[v_{j,k}(x) = e^{i\xi_0 x_1} \sum_{l=0}^{k} \frac{(ix_1)^{k-l}}{(k-l)!} u_i^{(j,k)}(x), \quad k = 0, \ldots, r_j - 1. \quad (11)\]
the subspaces $V_{\xi_0,j}$ are $T$-invariant and $T$ is represented by an upper triangular Toeplitz matrix $M_{\xi_0,j}$ with respect to the basis $(\tilde{v}_{j,k})_{0\leq k \leq r_j-1}$ of $V_{\xi_0,j}$. Since the entries of the first upper diagonal of $M_{\xi_0,j}$ do not vanish, we have $(M_{\xi_0,j} - e^{i\xi_0 I})^l = 0$ if and only if $l \geq r_j$, so $M_{\xi_0,j}$ is similar to a Jordan block $J_j$ of size $r_j$ with diagonal values $\exp(i\xi_0)$. It is easy to see by induction in $r_j$ that this similarity transform can be achieved by an upper triangular matrix $D$ with 1’s on the diagonal, i.e. $D = M_{\xi_0,j}$ is similar to $J_j$. Then $v_{j,k} := \tilde{v}_{j,k} + \sum_{l=0}^{k-1} D_{l,k} \tilde{v}_{j,l}$ is a Floquet mode of order $k$ of the form (11) with $u_{j,k}^{(j)} = \tilde{u}_{j,k}^{(j)}$, and $T$ is represented by $J_j$ on the subspace $V_{\xi_0,j}$ with respect to the basis $\{v_j,0,\ldots,v_j,r_j-1\}$ of $V_{\xi_0,j}$.

3.5. Outline of the proof of Theorem 2.2

We amend the finite number of right propagating Floquet modes $v_1^+,\ldots,v_n^+$ which have been described in section 2 by an infinite number of decaying Floquet modes $v_n^+, n = \pi + 1, 2\pi + 2, \ldots$, which are chosen according to Proposition 3.5 for each characteristic value of $(B_\xi)$ with positive imaginary part. These $v_n^+$ are arranged in increasing order of the imaginary part of the corresponding characteristic values and will be properly normalized. The set $\{v_n^+ : n \in \mathbb{N}\}$ is a Riesz basis of $V$ if and only if the operator

$$T : l^2(\mathbb{N}) \to H_1^{1,\infty}(S^+), \quad (a_n) \mapsto \sum_{n=1}^{\infty} a_n v_n^+$$

is a norm isomorphism from $l^2(\mathbb{N})$ to $V$. Our strategy is to compare $T$ with a reference operator $T_r$ corresponding to the case $\omega = 0$, which we will refer to as the unperturbed case. In the unperturbed case all interesting quantities can easily be computed analytically. We will proceed as follows:

- In section 4 we establish the existence of a countable number of characteristic values and derive precise estimates on the difference of characteristic values with large imaginary parts in the perturbed and the unperturbed case using Theorem 3.4.

- In section 5 we show that $T - T_r$ is compact by estimating the perturbation of eigenvectors of $(B_\xi)$ for $\xi$ with large imaginary parts. Moreover, we show that $T_r$ is a norm isomorphism from $l^2(\mathbb{N})$ to its range.

- In section 6 we show for trace operators $\tau$, which satisfy the well-posedness assumption in Theorem 2.2 that $\tau T_r : l^2(\mathbb{N}) \to H_\gamma^\tau$ is a norm isomorphism. Moreover, we show injectivity of $T$. Together with Riesz theory and the well-posedness assumption this readily implies that both $\tau T : l^2(\mathbb{N}) \to H_\gamma^\tau$ and $T : l^2(\mathbb{N}) \to V$ are norm isomorphisms. Moreover, by construction (see Proposition 3.5) both the operators $\mathcal{R}$ and $\mathcal{T}$ are represented by Jordan matrices.
4. Estimates of characteristic values

4.1. Characteristic values for $\omega = 0$

Recall from §3.1 that $\kappa_n^2$ are the eigenvalues of the negative Laplacian on the cross section with boundary conditions $\gamma$ and set

$$\xi_{m,n} := -2\pi m + i\kappa_n, \quad m \in \mathbb{Z}, \ n \in \mathbb{N}. $$

**Lemma 4.1.** The characteristic values of $(\Delta \xi)$ are precisely the numbers $\xi_{m,n}$ and $\bar{\xi}_{m,n}$ (counted with their total multiplicities if these numbers are not distinct) with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $\kappa_n > 0$ all partial null multiplicities are 1 and if $\kappa_n = 0$ the partial null multiplicity is 2. An eigenvector corresponding to both $\xi_{m,n}$ and $\bar{\xi}_{m,n}$ is given by the function $\varphi_{m,n}$ defined in (6).

**Proof.** Note that $-\Delta \varphi_{m,n} = ((2\pi m)^2 + \kappa_n^2)\varphi_{m,n} = |\xi_{m,n}|^2 \varphi_{m,n}$ for $(m,n) \in \mathbb{Z} \times \mathbb{N}$ and define $\hat{u}(m,n) := \langle u, \varphi_{m,n} \rangle$. Due to (8) we have

$$-\hat{(\Delta \xi)}(\xi u)(l) = -(\Delta + 2i\xi(2\pi m) - \xi^2)\hat{u}(l) = (|\xi|^2 + 2\xi \bar{\xi} + \xi^2)\hat{u}(l), \quad l = (m,n) \in \mathbb{Z} \times \mathbb{N}. $$

Since $\{\varphi_l : l \in \mathbb{Z} \times \mathbb{N}\}$ is an orthonormal basis of $L^2(\Omega)$, this shows that the problem separates into a countable set of scalar problems indexed by $l = (m,n)$, each of which has the two simple characteristic values $\xi_l$ and $\bar{\xi}_l$ if $\kappa_n > 0$ and the characteristic value 0 of multiplicity 2 if $\kappa_n = 0$. \hfill \square

Since the characteristic values are always $2\pi$-periodic, also for $\omega > 0$ (see Remark 3.2), it suffices to study the characteristic values in the strip $\{z \in \mathbb{C} : -\pi \leq \Re z < \pi\}$. Therefore we define

$$\xi_n^r = i\kappa_n = \xi_{0,n}, \quad n \in \mathbb{N}$$

as “reference characteristic values”. Note that $\xi_n^r$ corresponds to the physical exponentially decaying Floquet mode $\exp(i\xi_n^r x_1) = \exp(-\kappa_n x_1)$ if $\kappa_n > 0$ whereas $\bar{\xi}_n^r$ corresponds to the unphysical exponentially growing Floquet mode $\exp(\bar{\xi}_n^r x_1) = \exp(\kappa_n x_1)$.

4.2. Estimates of “large” characteristic values for Dirichlet boundary conditions

For the sake of clarity we first prove estimates on the location of characteristic values for the case of Dirichlet boundary conditions $\gamma = \gamma_\mathbb{D}$ where $\kappa_n = \frac{\pi n}{L}$ before treating general boundary conditions. The main tool will be Theorem 3.4.

We define open discs $\mathcal{D}_n$ and closed rectangles $\mathcal{R}_n$ by

$$\mathcal{D}_n := \left\{ z \in \mathbb{C} : |z - \xi_n^r| < \frac{2\omega^2 \varepsilon}{\kappa_n + \kappa_{n-1}} \right\}, \quad (15)$$

$$\mathcal{R}_n := \left\{ z \in \mathbb{C} : |\Re z| \leq \pi, |\Im z - \kappa_n| \leq \frac{\pi}{2L} \right\}. \quad (16)$$
The radii of these discs are chosen such that \( \| \Delta^{-1}_\xi \omega^2 \varepsilon_p \|_{L(L^2(\Omega))} < 1 \) for all \( \xi \in \partial D_n \) as shown later. Moreover, we choose \( N \in \mathbb{N} \) such that all disks \( D_n \) with \( n \geq N \) are mutually disjoint and contained in the strip \( \{ z \in \mathbb{C} : |\Re z| < \pi \} \), i.e.

\[
N \geq \min \left\{ n \in \mathbb{N} : \frac{\kappa_n + \kappa_{n-1}}{2} > \frac{\omega^2 \varepsilon \max(1, L)}{\pi} \right\}.
\] (17)

**Proposition 4.2.** For Dirichlet boundary conditions we have

\[
\mathcal{N}( (B_\xi), \partial D_n ) = 1 = \mathcal{N}( (B_\xi), \partial R_n ) \quad \text{for all } n \geq N.
\]

In particular, all characteristic values of \( (B_\xi) \) in the half-strip \( \{ z \in \mathbb{C} : -\pi < \Re z, \Im z > \kappa_n - \pi/(2L) \} \) are simple and contained in one of the disks \( D_n \).

**Proof.** For each \( n \geq N \) we apply Theorem 3.4 with \( A_{\xi} = \Delta_{\xi} \in L( L_2(\Omega), H^{-2}(\Omega) ) \) and \( S_{\xi} = B_\xi - \Delta_{\xi} = \omega^2 \varepsilon_p \). We will show that

\[
\| \Delta_{\xi}^{-1} \omega^2 \varepsilon_p \|_{L(L^2(\Omega))} < 1 \quad \xi \in R_n \setminus D_n.
\] (18)

Then Theorem 3.4 is applicable with \( \Gamma = \partial D_n \) and yields \( \mathcal{N}( (B_\xi), \partial D_n ) = \mathcal{N}( (\Delta_{\xi}), \partial D_n ) = 1 \) (with the help of Lemma 4.1), and it follows from the Neumann series or another application of Theorem 3.4 with \( \Gamma = \partial R_n \) that there are no characteristic values of \( (B_\xi) \) in \( R_n \setminus D_n \). This proves all claims.

Since \( \| \omega^2 \varepsilon_p \|_{L(L^2(\Omega))} = \omega^2 \varepsilon \), eq. (18) holds true if we can show that

\[
\| \Delta_{\xi}^{-1} \|_{L(L^2(\Omega))} < \frac{1}{\omega^2 \varepsilon} \quad \xi \in R_n \setminus D_n.
\]

By virtue of (14) this is equivalent to

\[
\sup \left\{ \frac{1}{|\xi - \xi_l| : l \in \mathbb{Z} \times \mathbb{N}} \right\} < \frac{1}{\omega^2 \varepsilon}, \quad \xi \in R_n \setminus D_n.
\]

Note that

\[
R_n = \{ \xi \in \mathbb{C} : \forall l \in \mathbb{Z} \times \mathbb{N} \ |\xi - \xi_{0,n}| \leq |\xi - \xi_l| \},
\] (19)

so \( |\xi - \xi_l| \geq |\xi - \xi_{0,n}| \geq \frac{2\omega^2 \varepsilon}{\kappa_n + \kappa_{n-1}} \) for all \( \xi \in R_n \setminus D_n \). Together with the inequality \( |\xi - \xi_l| \geq |3\xi - \Im \xi_l| > 3\xi \geq \frac{1}{2}(\kappa_n + \kappa_{n-1}) \) we obtain the desired estimate

\[
|\xi - \xi_l| |\xi - \xi_l| > \omega^2 \varepsilon \quad \text{for all } l \in \mathbb{Z} \times \mathbb{N}
\]

(see Fig. 2(a)). Since the supremum over \( l \) above is obviously attained, we get a strict inequality. \( \square \)
4.3. Estimates of “large” characteristic values for general boundary conditions

For general lateral boundary conditions, in particular quasi-periodic boundary conditions, the eigenvalues \( \kappa_n \) may come in pairs which are arbitrarily close together or coincide. Recall, however, that we have \( \kappa_{n+2} = \kappa_n + \frac{2\pi}{T} \) for all \( n \in \mathbb{N} \) and for all trace operators \( \gamma \). We assign to each index \( n \) its group \( G(n) \) as follows: \( G(n) := \{ n \} \) if \( \kappa_{n+1} - \kappa_n = \frac{2\pi}{T} \), i.e. if the \( \kappa_n \) are equidistant, which is the case for \( \gamma \in \{ \gamma_D, \gamma_N, \gamma_DN \} \). If \( \kappa_{n+1} - \kappa_n < \frac{2\pi}{T} \), then \( G(n) := \{ n, n+1 \} \) and if \( \kappa_{n+1} - \kappa_n > \frac{2\pi}{T} \), then \( G(n) := \{ n, \max(n-1,1) \} \).

To each group we assign a set

\[
\mathcal{R}_{G(n)} := \{ \xi \in \mathbb{C} : \forall l \in \mathbb{Z} \times \mathbb{N} \min_{n \in G(n)} |\xi - \xi_n^l| \leq |\xi - \xi_l| \},
\]

which is a closed rectangle since the \( \xi_l \) form a rectangular grid (see Fig. 2[b]). As remarked in (19), these rectangles coincide with the ones defined in the previous subsection if \( \gamma = \gamma_D \). Moreover, we introduce open disks, which again coincide with those of the previous subsection for \( \gamma = \gamma_D \):

\[
D_n := \left\{ z \in \mathbb{C} : |z - \xi_n^l| < \frac{\omega^2 \varepsilon}{\inf_{z \in \mathcal{R}_{G(n)}} \Im z} \right\}
\]

Proposition 4.3. Define the set

\[
\mathcal{S} := \bigcup_{n \geq N} D_n \quad \text{with} \quad N \geq \min \left\{ n \in \mathbb{N} : \inf_{z \in \mathcal{R}_{G(n)}} \Im z > \frac{\omega^2 \varepsilon \max(1,L)}{\pi} \right\}
\]

(see Fig. 2[b]). Then all characteristic values of \( (B_{\xi}) \) in \( \{ z \in \mathbb{C} : \Re z \in [-\pi,\pi], \Im z \geq \inf_{z \in \mathcal{R}_{G(n)}} \Im z \} \) are contained in \( \mathcal{S} \) and for each connected component \( \tilde{S} \) of \( \mathcal{S} \) we have

\[
\mathfrak{M}( (B_{\xi}) ; \partial \tilde{S} ) = \mathfrak{M}( (\Delta_{\xi}) ; \partial \tilde{S} ) = \# \{ n \in \mathbb{N} : \xi_n^l \in \tilde{S} \}.
\]

Proof. The proof is analogous to that of Proposition 4.2. Note that \( N \) is chosen such that \( \mathcal{S} \subset \{ z \in \mathbb{C} : |\Im z| < \pi \} \) and \( \bigcup_{n \in G(n)} D_n \subset \mathcal{R}_{G(n)} \) for all \( n \geq N \). By the same arguments we can show that \( \| \Delta_{\xi}^{-1} \omega^2 \varepsilon \|_{L^2(\Omega)} < 1 \) for all \( \xi \in \mathcal{R}_{G(n)} \setminus \tilde{S} \) if \( \tilde{S} \subset \bigcup_{n \in G(n)} D_n \).

4.4. Number of “small” characteristic values

Only for very small \( \omega \) the argument of Proposition 4.3 can work for all \( n \) since for large \( \omega \) some of the discs \( D_n \) overlap with their neighbors. In the following we will determine the number of characteristic values with “small” imaginary parts. First we cite the following symmetry result for the set of characteristic values:

Proposition 4.4 ([18, Theorem 5.3]). For real-valued \( \varepsilon \) the set of characteristic values of \( (B_{\xi}) \) is symmetric with respect to the real axis, and moreover \( \mathfrak{N}( (B_{\xi}) ; \xi_0 ) = \mathfrak{N}( (B_{\xi}) ; \xi_0^* ) \) for all \( \xi_0 \in \mathbb{C} \).
Figure 2: Panel (a): An important ingredient of the proof dictating our definition of the geometry are uniform lower bounds on products of the distances indicated by fat double arrows.
Panel (b): The shaded area shows the set $S$ defined in (21) in the case of quasi-periodic boundary conditions. The crosses indicate the characteristic values of $(B_\xi)$, and the dots indicate some of the reference characteristic values $(\Delta_\xi)$. The dashed lines show the boundaries of rectangles $R_{G(n)}$ defined in (20).

To count the number of these “small” characteristic values in the strip $\{z \in \mathbb{C} : -\pi \leq \Re(z) < \pi\}$ we surround them by a contour containing none of them. By Proposition 4.3 the segment $[z_N, z_N + 2\pi]$ with $z_N := -\pi + i \inf_{z \in R_{G(n)}} \Im z$ contains no characteristic value.

**Proposition 4.5.** Choose $z_N$ as above and any complex path $P$ from $z_N$ to $\bar{z}_N$ such that $\Gamma := [z_N + 2\pi, z_N] \cup P \cup [\bar{z}_N, \bar{z}_N + 2\pi] \cup -(2\pi + P)$ encloses precisely all characteristic values of $(B_\xi)$ in the rectangle $\mathcal{R} := \{z \in \mathbb{C} : \Re z \in [-\pi, \pi), |\Im z| < |\Im z_N|\}$ (see Fig. 3(a)). Then

$$\mathfrak{M}((B_\xi); \Gamma) = 2N.$$  

**Proof.** Let us define $B(\xi, \mu) := \Delta_\xi + \mu \varepsilon_p$ for $\mu \in [0, \omega^2]$. From Proposition 4.3 we can deduce that the segments $[z_N, z_N + 2\pi]$ and $[\bar{z}_N + 2\pi, \bar{z}_N]$ contain no characteristic values of $\xi \mapsto B(\xi, \mu)$ for any $\mu \in [0, \omega^2]$. For a given $\mu \in [0, \omega^2]$ it follows from the discreteness of the set of characteristic values of $\xi \mapsto B(\xi, \mu)$ that the segment $[z_N, \bar{z}_N]$ contains at most a finite number of characteristic values, and there exists $\delta > 0$ such that $\delta$-balls around these characteristic values contain no further characteristic values. We deform the straight line $[z_N, \bar{z}_N]$ in $\delta$-neighborhoods of the characteristic values to left semi-circles as shown in Fig. 3(a) to obtain a contour $P_\mu$. Because of the $2\pi$ periodicity of
the characteristic values, the imaginary parts of the characteristic values on \([z_N, \overline{z}_N]\) and on \([2\pi + [z_N, z_N]\) coincide, and replace the segment \([2\pi + [z_N, z_N]\) by \([- (2\pi + P_\mu)].\) This yields a contour \(\Gamma_\mu := [z_N + 2\pi, z_N] \cup P_\mu \cup [\xi_N, \xi_N + 2\pi] \cup -(2\pi + P_\mu)\) which contains precisely all characteristic values of \(B(\cdot, \mu)\) in the rectangle \(R.\) Moreover, \(\Gamma_\omega^2\) coincides with the contour \(\Gamma\) in the proposition. With this construction the function \(\overline{N} : [0, \omega^2] \to \{0, 1, 2, \ldots\},\)

\[\overline{N}(\mu) := N((B_\xi); \Gamma_\mu)\]

is well-defined, and in particular it does not depend on the choice of \(\Gamma_\mu.\) Moreover, it follows from Theorem 3.4 with \(A_\xi = B(\xi, \mu)\) and \(S_\xi = \tilde{\delta}\varepsilon_\mu\) such that \(|\tilde{\delta}| < 1/(\pi \max_{\xi \in \Gamma_\mu} \|A_\xi^{-1}\|_{L^2(\Omega)})\) that \(\overline{N}\) is constant in a neighborhood of

![Figure 3: Panel (a): The dashed line indicated the integration contour in the proof of Proposition 4.5. Crosses indicate characteristic values of \((B_\xi)),\) circles mark four of the “reference” characteristic values of \((\Delta_\xi)).\)

Panel (b): The proof of Lemma 5.2 involves lower bounds of the distances of points \(\xi\) in a connected component \(\tilde{S}\) of the set \(S\) (the dark shaded region) to all “reference” characteristic values \(\xi_{m,n}\) not contained in \(\tilde{S},\) see (28a).
each $\mu \in [0, \omega^2]$. Hence, $\overline{N}$ is constant on whole interval $[0, \omega^2]$ and
\[
\mathcal{N}((B_\xi); \Gamma) = \overline{N}(\omega^2) = \overline{N}(0) = \mathcal{N}((\Delta_\xi); \Gamma_0) = 2N.
\]

We mention that Proposition 4.5 yields an independent proof of the following well-known result (see [19, Corollary 5.1.5]):

**Corollary 4.6.** The dimension of the space $V_p$ spanned by Floquet modes with real quasi momentum is finite and even.

**Proof.** Due to Proposition 4.4 the contour $\Gamma$ in Proposition 4.5 encloses the same number $\overline{m}$ of characteristic values with positive and with negative imaginary parts. Together with Proposition 3.5 we find that $\dim V_p = 2N - 2\overline{m}$ is finite and even. \qed

4.5. Summary

Let us summarize our results on the characteristic values of $(B_\xi)$ in the strip $\{z \in \mathbb{C} : -\pi \leq \Re z < \pi\}$ and introduce some notation for the following sections:

- The set of characteristic values of $(B_\xi)$ in the strip $\{z \in \mathbb{C} : -\pi \leq \Re z < \pi\}$ is countable and symmetric with respect to the real axis.

- An even number $2\overline{m}$ of these characteristic values (counted with their total null multiplicity) lies on the real axis. The corresponding Floquet modes are propagating. $\overline{m}$ linearly independent “outgoing” Floquet modes $v^+_1, \ldots, v^+_\overline{m}$ are selected as described in section 2 or Appendix A and the corresponding characteristic values (or quasi-momenta) are denoted by $\xi^+_1, \ldots, \xi^+_{\overline{m}}$.

- We arrange the characteristic values with positive imaginary parts (counted with their total null multiplicities) in non-decreasing order of their imaginary parts and labeled them $\xi^+_{\overline{m}+1}, \xi^+_{\overline{m}+2}, \ldots$. Corresponding exponentially decaying Floquet modes are chosen as in Proposition 3.5 and denoted by $v^+_{\overline{m}+1}, v^+_{\overline{m}+2}, \ldots$. For $n > \overline{m}$ the characteristic values $\xi^+_n$ are close to the reference values $\xi^+_n$ and belong to the set $S$ sketched in Fig. 2(b).

- For Dirichlet and Neumann lateral boundary conditions the characteristic values $\xi^+_n$ for $n > \overline{m}$ are simple. For other boundary conditions the total null multiplicity of $\xi^+_n$ with $n > \overline{m}$ is always $\leq 2$, and except for quasi-periodic boundary conditions with $\beta \in \{0, \pi\}$ at most a finite number of them is not simple.
5. Properties of the operator $T$

5.1. Reference operator $T_i$

To study the operator $T : (a_n) \mapsto \sum_{n=1}^{\infty} a_n v_n^1$ in $H^1_\gamma(S^+)$, we introduce a reference operator $T_i$ using the Floquet modes

$$v_n^i(x) := \left(\kappa_n + \frac{1}{2\kappa_n}\right)^{-1/2} e^{-\kappa_n x_1} \psi_n(x_2), \quad n \in \mathbb{N} \text{ if } \kappa_n > 0$$

(23)

corresponding to $\omega = 0$. The normalization constant has been chosen such that $\|v_n^i\|_{H^1_\gamma(S^+)} = 1$. In the special case $\kappa_n = 0$, which can only occur for $n = 1$, we set $v_1^i(x) := e^{-\tau_1^n} \psi_1(x_2)$. In this case $v_1^i$ is not a solution to (18) with $\omega = 0$, but we will not need this property. $T_i$ is defined by

$$T_i : l^2(\mathbb{N}) \to H^1_\gamma(S^+), \quad (a_n) \mapsto \sum_{n=1}^{\infty} a_n v_n^i,$$

and has the following properties:

**Lemma 5.1.** 1. $T_i$ is well defined and isometric with $T_i(l^2(\mathbb{N})) \subset H^1_\gamma(S^+)$, i.e. $\|T_i(a_n)\|_{H^1_\gamma} = \|(a_n)\|_2$ for all $(a_n) \in l^2(\mathbb{N})$.

2. $\tau T_i : l^2(\mathbb{N}) \to H^1_\gamma$ is a Fredholm operator with index 0.

**Proof.** Part 1: The assertion follows from the fact that $\{v_n^i : n \in \mathbb{N}\}$ is an orthonormal system in $H^1_\gamma(S^+)$. Part 2: Define $\psi_n := \left(\kappa_n + \frac{1}{2\kappa_n}\right)^{-1/2} \psi_n$ if $\kappa_n > 0$ and $\psi_n := \psi_n$ else. Then $(\tau T_i)(a_n) = \sum_{n=1}^{\infty} (\theta_D - \kappa_n \theta_N) a_n \psi_n$. If $\theta_N = 0$, the assertion follows from the fact that $\{\psi_n : n \in \mathbb{N}\}$ is a complete orthogonal system in $H^1_\gamma = H^1_\gamma((0, L))$ and $\sup_{n \in \mathbb{N}} \|\psi_n\|_{H^{1/2}} / \inf_{n \in \mathbb{N}} \|\psi_n\|_{H^{1/2}} < \infty$. To treat the case $\theta_N \neq 0$ note that $\{\psi_n : n \in \mathbb{N}\}$ is also a complete orthogonal system in $H^1_\gamma = H^1_\gamma((0, L))$ and that $\sup_{n \in \mathbb{N}} ((1 + \kappa_n)^{1/2} \|\psi_n\|_{H^{1/2}}) / \inf_{n \in \mathbb{N}} ((1 + \kappa_n)^{1/2} \|\psi_n\|_{H^{1/2}}) < \infty$.

5.2. Estimates on perturbation of eigenvectors

In the following estimates on the perturbation of eigenvectors we will make all constants explicit. To do so, we first have to introduce for each connected component $\tilde{S}$ of $S$ in (21) an $L^2$-orthogonal projection operator

$$P_{\tilde{S}} u := \sum_{n \in I_{\tilde{S}}} \langle u, \varphi_{0,n} \rangle \varphi_{0,n}, \quad I_{\tilde{S}} := \{ n \in \mathbb{N} : \xi_{n}^\perp \in \tilde{S} \}$$

with $\tilde{S}$ := \{ n \in \mathbb{N} : \xi_{n}^\perp \in \tilde{S} \} (24)
and define the quantity
\[ \delta_\gamma := \frac{1}{2} \inf \{|\kappa_n - \kappa_{n'}| : n, n' \in \mathbb{N}, \kappa_n \neq \kappa_{n'}\}. \]

It is easy to see from the explicit values of the \( \kappa_n \) given in §3.1 that \( \delta_\gamma = \pi/(2L) \) for \( \gamma \in \{\gamma_D, \gamma_N, \gamma_DN\} \), \( \delta_{\gamma_0} = \min_{\xi} |\pi - \beta|/L \) if \( \beta \notin \{0, \pi\} \), and \( \delta_{\gamma_\pi} = \pi/L \) for \( \beta \in \{0, \pi\} \). Note that for a connected component \( \tilde{S} \) of the set \( S \) defined in Proposition 3.3 we have
\[ |3\xi - 3\xi_{0,n}| > \delta_\gamma \quad \text{for all } \xi \in \tilde{S}, \xi_{0,n} \notin \tilde{S} \quad (25) \]
(see Fig. 3(b)).

**Lemma 5.2 (estimates of eigenvectors).** For a connected component \( \tilde{S} \) of \( S \) in \[21\] define \( \kappa_{\tilde{S}} := \inf_{\xi \in \tilde{S}} \Re \xi \) and recall the notation \[21\]. Then all eigenvectors \( u^* \) corresponding to characteristic values in \( \tilde{S} \) satisfy the estimates
\[
\|u^* - P_{\tilde{S}}u^*\|_{L^2(\Omega)} \leq \frac{\omega^2\varepsilon}{\min(\delta_\gamma, \pi)\kappa_{\tilde{S}}} \|u^*\|_{L^2(\Omega)}, \quad (26a)
\]
\[
\|\partial_{x_1}u^*\|_{L^2(\Omega)} \leq \frac{2\omega^2\varepsilon}{\kappa_{\tilde{S}}} \|u^*\|_{L^2(\Omega)}, \quad (26b)
\]
\[
\|(u^* - P_{\tilde{S}}u^*)(0, \cdot)\|_{L^2((0, L))} \leq C \frac{\omega^2\varepsilon}{\kappa_{\tilde{S}}} \|u^*\|_{L^2(\Omega)}, \quad (26c)
\]
\[
\left\| \frac{\partial u^*}{\partial x_1}(0, \cdot) \right\|_{L^2((0, L))} \leq \frac{2\omega^2\varepsilon}{\sqrt{3}} \|u^*\|_{L^2(\Omega)}, \quad (26d)
\]
with \( C := \left( \sum_{m \in \mathbb{Z}} \frac{1}{\pi^2m^2 + \delta_\gamma^2} \right)^{1/2} \).

**Proof.** Let \( \xi \in \tilde{S} \) be a characteristic value of \((B_\xi)\) with eigenvector \( u^* \). It follows from \[14\] that \((\xi - \xi_\ell)(\xi - \xi_i)u^* (l) = \omega^2(\varepsilon_\ell u^*) (l)\), or
\[
\widehat{u^*} (l) = \frac{\omega^2(\varepsilon_\ell u^*) (l)}{(\xi - \xi_i)(\xi - \xi_\ell)}, \quad l = (m, n) \in \mathbb{Z} \times \mathbb{N}. \quad (27)
\]
This identity will be used extensively. Moreover, we need the lower bounds
\[
|\xi - \xi_\ell|^2 = (\Re \xi + 2\pi m)^2 + (3\xi - \kappa_n)^2 \geq \begin{cases} (\pi m)^2, & n \in I_{\tilde{S}} \\ (\pi m)^2 + \delta_\gamma^2, & n \notin I_{\tilde{S}} \end{cases} \quad (28a)
\]
\[
|\xi - \xi_i|^2 = (\Re \xi + 2\pi m)^2 + (3\xi + \kappa_n)^2 > (\pi m)^2 + (\kappa_{\tilde{S}} + \kappa_n)^2 \quad (28b)
\]
which hold for all \( \xi \in \tilde{S} \) and \( l \in \mathbb{Z} \times \mathbb{N} \setminus \{0\} \times I_{\tilde{S}} \) and follow by separate estimation of real and imaginary parts using \[27\]. \mid \Re \xi \mid < \pi, \kappa_n \geq 0 \) (see Fig. 3(b)).
Using the Cauchy-Schwarz inequality, (27), and the lower bounds
\[ \sum_{l \in \mathbb{Z} \cap \{0\} \times I_S} \mid \mathbf{e}_p \mathbf{u}^*(l) \mid^2 \leq \left( \frac{\omega^2 \varepsilon}{\min(\delta_\gamma, \pi) \kappa_S} \right)^2 \| \mathbf{u}^* \|_{L^2}^2. \]

To prove (26) we use that \( |\xi - \xi_l| \leq \min(\delta_\gamma, \pi) \kappa_S |m| \) and \( \partial_{x_1} \phi(0, n) = 0 \) to obtain
\[ \| \partial_{x_1} \mathbf{u}^* \|_{L^2}^2 = \left\| \sum_{l \in \mathbb{Z} \cap \{0\} \times N} \mathbf{e}_p (l) \partial_{x_1} \phi_l \right\|_{L^2}^2 = \sum_{l \in \mathbb{Z} \cap \{0\} \times N} \left| \frac{(2\pi m)\omega^2 \varepsilon \mathbf{e}_p \mathbf{u}^*(l)}{(\xi - \xi_l)(\xi - \xi_l)} \right|^2 \leq \left( \frac{2\omega^2 \varepsilon}{\kappa_S} \right)^2 \| \mathbf{u}^* \|_{L^2}^2. \]

Since
\[ \mathbf{u}^*(0, \cdot) = \sum_{l \in \mathbb{Z} \times N} \mathbf{e}_p(l) \phi_l(0, \cdot) = \sum_{n \in \mathbb{N}} \left( \sum_{m \in \mathbb{Z}} \mathbf{u}^*(m, n) \right) \psi_n, \]
we have
\[ \| (\mathbf{u}^* - P_S \mathbf{u}^*)(0, \cdot) \|_{L^2(0, L_1)}^2 = \sum_{n \in \mathbb{N} \setminus I_S} \left( \sum_{m \in \mathbb{Z}} \mathbf{u}^*(m, n) \right)^2 + \sum_{n \in I_S} \left( \sum_{m \in \mathbb{Z} \setminus \{0\}} \mathbf{u}^*(m, n) \right)^2. \]

Using the Cauchy-Schwarz inequality, (27), and the lower bounds \( |\xi - \xi_l| \geq \pi^2 m^2 + \delta_\gamma \) and \( | \xi - \xi_l | \geq \kappa_S \) (see (28)), the first term in (29) can be bounded by
\[ \sum_{n \in \mathbb{N} \setminus I_S} \left( \sum_{m \in \mathbb{Z}} \mathbf{u}^*(m, n) \right)^2 \leq \left( \sum_{m \in \mathbb{Z}} \frac{1}{\pi^2 m^2 + \delta_\gamma^2} \right) \sum_{m \in \mathbb{Z} \cap \{0\} \setminus \{0\} \times I_S} \left( \pi^2 m^2 + \delta_\gamma^2 \right) | \mathbf{u}^*(m, n) |^2 \leq C^2 \sum_{m \in \mathbb{Z} \cap \{0\} \setminus \{0\} \times I_S} \frac{\omega^2 \varepsilon | \mathbf{e}_p \mathbf{u}^*(m, n) |^2}{| \xi - \xi_l |^2} \leq C^2 \frac{\omega^2 \varepsilon}{\kappa_S} \| \mathbf{u}^* \|_{L^2}^2. \]

Using (26) the second term in (29) can be estimated by
\[ \sum_{n \in I_S} \left( \sum_{m \in \mathbb{Z} \setminus \{0\}} \mathbf{u}^*(m, n) \right)^2 \leq \left( \sum_{m \in \mathbb{Z} \setminus \{0\} \times I_S} \frac{1}{(2\pi m)^2} \right) \left( \sum_{m \in \mathbb{Z} \setminus \{0\}, n \in I_S} (2\pi m)^2 | \mathbf{u}^*(m, n) |^2 \right) \leq C^2 \| \partial_{x_1} \mathbf{u}^* \|_{L^2}^2 \leq C^2 \frac{\omega^2 \varepsilon}{\kappa_S} \| \mathbf{u}^* \|_{L^2}^2. \]
completing the proof of (26c).

To prove (26d), first note that

$$\partial_{x_1} u^*(0, \cdot) = \sum_{l \in \mathbb{Z} \times \mathbb{N}} \hat{u}^*(l) \partial_{x_1} \varphi_l(0, \cdot) = \sum_{n \in \mathbb{N}} \left( \sum_{m \in \mathbb{Z} \setminus \{0\}} (2\pi im) \hat{u}^*(m, n) \right) \psi_n.$$  

It follows from (27), the Cauchy-Schwarz inequality, the estimate $|\xi - \xi_l| |\xi - \xi_l'| \geq (\pi m)^2$ (see (28)), and the identity $\sum_{m=1}^{\infty} m^{-2} = \frac{\pi^2}{6}$ that

$$\|\partial_{x_1} u^*(0, \cdot)\|_{L^2}^2 = \sum_{n \in \mathbb{N}} \left| \sum_{m \in \mathbb{Z} \setminus \{0\}} (2\pi im)^2 \hat{u}^*(m, n) \int_{\xi_l}^{\xi_l'} \psi_n(x) \right|^2 \leq \frac{4}{3} \omega^2 \varepsilon^2 \|u^*\|_{L^2}^2,$$

which proves (26d).

We can still fix a complex scaling constant in the Floquet modes $v^+_n$ defined in (4.5). For our purposes it will be sufficient to fix this constant for all but a finite number of these $v^+_n$. We will assume in the following that

$$\gamma \notin \{\gamma_0, \gamma_\pi\}.$$  

The special case $\gamma \in \{\gamma_0, \gamma_\pi\}$ will be discussed in Appendix B. If (30) holds true, it follows from the explicit form of $\kappa_n$ given in §4 that there exists $\tilde{N} \in \mathbb{N}$ such that for all $n > \tilde{N}$ the connected component $S_n$ of $S$ containing $\xi_n'$ is a disk containing precisely one characteristic value with multiplicity. Therefore, the function

$$u_n(x) := e^{-i\kappa_n x_1 v^+_n(x), \quad n > \tilde{N}}$$  

is periodic in $x_1$, $P_{S_n} u_n$ is constant in the first variable, and $(P_{S_n} u_n)(0, \cdot)$ is a multiple of $\psi_n$. Therefore, we can choose the free complex scaling constants of $v^+$ and $u_n$ such that

$$(P_{S_n} u_n)(0, x_2) = \left( \kappa_n + \frac{1}{2\kappa_n} \right)^{-1/2} \psi_n(x_2).$$  

If $\omega = 0$ this scaling yields $v^+_n = v^+_n$.

**Corollary 5.3.** There exist a constant $C > 0$ depending only on $\omega^2 \varepsilon$ and $\gamma$ such that for all $n > \tilde{N}$

$$\kappa_n^{3/2} \|u_n - P_{S_n} u_n\|_{L^2(\Omega)} \leq C,$$

$$\kappa_n^{3/2} \|(u_n - P_{S_n} u_n)(0, \cdot)\|_{L^2(0, L)} \leq C,$$

$$\kappa_n^{1/2} \|\partial_{x_1} u_n|_{x_1=0}\|_{L^2(0, L)} \leq C.$$  

(33a) (33b) (33c)
Proof. It follows from Lemma 5.2 that \( \|u_n\|_{L^2(\Omega)} \leq 2\|P_{S_n}u_n\|_{L^2(\Omega)} \) for sufficiently large \( n \). Moreover, due to (32) we have \( P_{S_n}u_n \) \( \|L^2(\Omega) \) \( n = \left( \kappa_n + \frac{1}{\kappa_n} \right)^{-1/2} \). Now the assertions follow from Lemma 5.2. \( \square \)

5.3. Properties of the operator \( T \)

**Proposition 5.4** (properties of \( T \)).

1. The operator \( T \) is well-defined and bounded, and \( T - T_\gamma \) is compact.

2. \( T(a_n) \) satisfies (1a) and (1b) for all \( (a_n) \in l^2(\mathbb{N}) \).

**Proof.** Part 1: In the following \( C \) denotes a generic constant depending only on \( \omega^2 \) and \( \gamma \). Note that \( v'_n(x) = \exp(-\kappa_n x) \psi_n(x) \) with \( \psi_n(x) := (P_{S_n}u_n)(0, x) \) for \( n > \tilde{N} \). Inserting the term \( \pm e^{i\xi_n x_1} \tilde{\psi}_n(x_2) \) in \( \|v^+_n - v^-_n\|_{L^2} \) and applying the triangle inequality yields the estimate

\[
\|v^+_n - v^-_n\|_{L^2(S^+)} \leq \left\| e^{i\xi_n x_1} (u_n - P_{S_n}u_n) \right\|_{L^2(S^+)} + \left\| e^{i\xi_n x_1} - e^{-\kappa_n x_1} \right\|_{L^2} \|\tilde{\psi}_n\|_{L^2} \\
\leq \frac{\|u_n - P_{S_n}u_n\|_{L^2(\Omega)}}{1 - e^{-3\xi_{\tilde{N}}}} + \left( \int_0^{+\infty} \left| e^{i\xi_n x_1} - e^{-\kappa_n x_1} \right|^2 \|\tilde{\psi}_n\|_{L^2} \right)^{1/2} \\
\leq \frac{C}{\kappa_{\tilde{N}}^{3/2}}, \quad n > \tilde{N}.
\]

\( (34) \)

In the second line we have used that \( 0 < \Re \xi_{\tilde{N}} \leq \Re \xi_n \) for all \( n > \tilde{N} \) and \( \sum_{k=0}^{+\infty} e^{-k\Re \xi_{\tilde{N}}} = (1 - e^{-\Re \xi_{\tilde{N}}})^{-1} \leq (1 - e^{-\Re \xi_n})^{-1} \), and in the third line \( (33a) \) was applied together with the identity

\[
\int_0^{+\infty} \left| e^{i\xi_n x_1} - e^{-\kappa_n x_1} \right|^2 dx_1 = \frac{\xi_n - i\kappa_n}{2} \left( \frac{1}{\Re \xi_n - i\kappa_n} + \frac{1}{\kappa_n(\Re \xi_n + i\kappa_n)} \right),
\]

and Proposition 4.3.

To obtain an identity for the \( L^2 \)-distance of the gradients, we apply Green's first theorem in \( (0, l) \times (0, L) \), use the identity \( \Delta (v^+_n - v^-_n) = -\omega^2 \varepsilon_p v^+_n \), and let \( l \to \infty \):

\[
\|\nabla (v^+_n - v^-_n)\|_{L^2(S^+)} \omega^2 \int_{S^+} \varepsilon_p v^+_n \delta_{x_1} (v^+_n - v^-_n) dx \\
- \int_0^{+\infty} \frac{\partial (v^+_n - v^-_n)}{\partial x_1} (0, x_2) (v^+_n - v^-_n)(0, x_2) dx_2
\]

Since \( \frac{\partial (v^+_n - v^-_n)}{\partial x_1}(0, x_2) = i\xi_n u_n(0, x_2) + \kappa_n \tilde{\psi}_n + \frac{\partial u_n}{\partial x_1}(0, x_2) \), it follows after adding \( \pm i\xi_n \tilde{\psi}_n \) and using \( (33b) \), \( (33a) \), and Proposition 4.3 that

\[
\|\partial_{x_1} (v^+_n - v^-_n)(0, \cdot)\|_{L^2} \leq C/\sqrt{\kappa_n}. \quad \text{Using Cauchy's inequality,} \quad (34) \text{, and} \quad (33b) \text{ yields}
\]

\[
\|\nabla (v^+_n - v^-_n)\|_{L^2(S^+)} \leq \frac{C}{\kappa_n^{3/2}}, \quad n > \tilde{N}.
\]

\( (35) \)
Define $K_j : V \rightarrow H_\gamma^1(S^+)$ by $K_j(a_n) := \sum_{n=1}^{j} a_n(v_n^+ - v_n^-)$. Combining 34 and 35 and using Cauchy’s inequality, we deduce that 
\[
\| (K_{m_2} - K_{m_1})(a_n) \|_{H_\gamma^1(S^+)}^2 \leq C \left( \sum_{n=m_1+1}^{m_2} \frac{1}{\kappa_n^2} + \sum_{n=m_1+1}^{m_2} \frac{1}{\kappa_n^2} \right) \| (a_n) \|_2^2,
\]
which implies together with 31 that $(K_j)$ is a Cauchy sequence with respect to the operator norm. Therefore, $K = \lim_{j \rightarrow \infty} K_j$ is well defined, and since the range of the operators $K_j$ is finite dimensional, $K$ is compact. Moreover, $T = T_i + K$ is well-defined and bounded.

**Part 2:** Since the differential operator $\Delta + \omega^2 \epsilon_p$ is continuous from $H_\gamma^1(S^+)$ to $H_\gamma^{-1}(S^+)$, we can interchange its application with summation to show that $w := T(a_n)$ satisfies 33. Analogously, it follows from the continuity of the trace operators that $w$ satisfies 34.

\[\square\]

6. **Proof of the Main Theorem**

**Proposition 6.1.** The operator $T$ is injective.

**Proof.** Let $v := \sum_{n=1}^{\infty} a_n v_n^+$ for some sequence $(a_n) \in l^2(\mathbb{N})$ and assume that $v \equiv 0$. We have to show that $a_n = 0$ for all $n \in \mathbb{N}$. Let \{\nu_1, \nu_2, \ldots\} = \{3\xi_n : n \in \mathbb{N}\}$ with $0 \leq \nu_1 < \nu_2 < \cdots$. We show by induction in $l \in \mathbb{N}$ that $a_n = 0$ for all $n \in \mathbb{N}$ satisfying $3\xi_n \leq \nu_l$; Formally adding $\nu_0 := -1$, the induction base is trivial. For the induction step assume that the statement holds true for $l - 1$ and that there are precisely $M$ distinct characteristic values $\xi_1, \ldots, \xi_M$ with $3\xi_m = \nu_l$ and $\Re \xi_m \in [-\pi, \pi]$. Pecularities for the case $l = 1$ and $\nu_l = 0$ will be discussed at the end of the proof, so assume for the moment that $\nu_l > 0$.

Suppose that $n((B\xi), \tilde{\xi}_m) = \{r_1^{(m)}, \ldots, r_a^{(m)}\}$. Relabelling the Floquet modes $v_n^+$ and the coefficients $a_n$ corresponding to $\xi_1, \ldots, \xi_M$ by $v_{j,k}^{(n)}$ and $a_{j,k}^{(n)}$ we obtain from the induction hypothesis that for any $\epsilon > 0$ and all $x_2 \in [0, L]$ 
\[
\exp(\nu_l x_1) \sum_{m=1}^{M} \sum_{j=1}^{a_m} \sum_{k=0}^{r_j^{(m) - 1}} a_{j,k}^{(m)} v_{j,k}^{(n)}(x) = O(e^{-(\nu_l + 1 - \nu_l - \epsilon) x_1}), \quad x_1 \rightarrow \infty \quad (36)
\]
and have to show that all coefficients $a_{j,k}^{(m)}$ vanish. For simplicity, let us first assume that all partial null multiplicities $r_j^{(m)}$ are equal to 1. With the notation of Proposition 3.3 (and an additional index $m$) the function $\exp(-i\tilde{\xi}_m x_1) \sum_{j=1}^{a_m} \exp(i \tilde{\xi}_m p) \sum_{j=1}^{a_m} a_{j,0}^{(m)} u_{0,m}(x) = \sum_{j=1}^{a_m} \exp(i \tilde{\xi}_m (x_1 + p)) u_{0,m}(x)$ is 1-periodic in $x_1$ for each $m$. Pick some $x = (x_1, x_2) \in \Omega$ and set $g_m := \sum_{j=1}^{a_m} a_{j,0}^{(m)} u_{0,m}(x)$. Then it follows from 36 that 
\[
\sum_{m=1}^{M} \exp(i\Re \tilde{\xi}_m p) g_m = \exp(\nu_l p) \sum_{m=1}^{M} \exp(i\tilde{\xi}_m p) \sum_{j=1}^{a_m} a_{j,0}^{(m)} u_{0,m}(x) \quad (37)
\]
\[
= \exp(\nu_l p) \sum_{m=1}^{M} \sum_{j=1}^{a_m} a_{j,0}^{(m)} u_{0,m}(x_1 + p, x_2) \rightarrow 0, \quad \text{as } p \rightarrow \infty, p \in \mathbb{N}.
\]
If the right hand side would vanish exactly for $M$ consecutive values of $p$, say $p \in \{q+1, \ldots, q+M\}$, we could conclude immediately that $g := (g_1, \ldots, g_M)^T \in \mathbb{C}^M$ is zero since the matrix $A^{(q)} \in \mathbb{C}^{M \times M}$ defined by $A^{(q)}_{lm} := \exp(i\Re \xi_m) \cdot \text{e}^{l+q}$, $l, m = 1, \ldots, M$ is regular as shown below. However, (37) only implies that $\lim_{q \to \infty} A^{(q)} g = 0$. Therefore, we have to control $\|A^{(q)}^{-1}\|$ uniformly in $q$. For this end note that $A^{(q)}$ has a factorization

$$A^{(q)} = A^{(0)} \cdot \text{diag} \left( \exp(i\Re \xi_1)^q, \ldots, \exp(i\Re \xi_M)^q \right)$$

and $A^{(0)}$ is a Vandermonde matrix. It follows that $A^{(q)}$ is invertible and $\|A^{(q)}\|^{-1}$ is independent of $q$. Therefore, $\lim_{q \to \infty} A^{(q)} g = 0$ implies $g = 0$, i.e. $\sum_{j=1}^M g_j^{(m)} a_{0,m} = 0$ for each $m = 1, \ldots, M$. Since we know from Proposition 3.5 that $\tilde{r}_1^{(m)}$, $\ldots$, $\tilde{r}_j^{(m)}$ are linearly independent, it follows that $a_j^{(m)} = \cdots = a_{\alpha_j^{(m)}}^{(m)} = 0$.

Now assume that $\tilde{\tau} := \max\{\tilde{r}_j^{(m)} : m = 1, \ldots, M, j = 1, \ldots, \alpha_j^{(m)}\}$ is greater than 1. Multiplying (36) by $x_1^{\tilde{\tau}+1}$ and using (11) it follows that (37) holds with

$g_m := \sum_{j: r_j^{(m)} = \tau} a_j^{(m)} \cdot \tilde{a}_j^{(m)}(x)$

(with the convention that an empty sum is 0), and it follows as above that $g_1 = \cdots = g_M = 0$ for all $x \in \Omega$. Using again the linear independence of the functions $\tilde{a}_j^{(m)}$, we find that $a_j^{(m)} = 0$ for all $(j, m)$ such that $r_j^{(m)} = \tilde{\tau}$. In a second step we show analogously that $a_j^{(m)} = 0$ for all $(j, m)$ such that $r_j^{(m)} \geq \tilde{\tau} - 1$, and so on. Finally, all coefficients $a_j^{(m)}$ have to vanish.

It remains to discuss the case of propagating modes, i.e. the induction step for $\nu_1 = 0$ since not all elements of the eigenspaces are considered here. However, we can use the same technique as above to prove the stronger result that $\sum_{n=1}^\infty a_n v_n + \sum_{n=1}^\infty a_n \overline{v}_n = 0$ implies $0 = a_1 = \cdots = a_\infty = a_1^\infty = a_2^\infty = \cdots$.\qed

The well posedness result in the following proposition follows from general results in [19], but we obtain an independent proof as a side product of our analysis:

**Proposition 6.2** (well-posedness of problem (1)). Assume that the only solution $v \in H^{1,+}_{\gamma}(S^+)$ to (1) with $f = 0$ is $v = 0$. Then $F := \tau T$ is bounded and boundedly invertible, and problem (1) is well posed in the sense that for all $f \in H^{1,+}_{\gamma}$ there exists a unique solution $v \in H^{1,+}_{\gamma}(S^+)$ to (1), and $v$ depends continuously on $f$.

**Proof.** $F$ is a Fredholm operator with index 0 since $F = \tau T - \tau(T - T_1)$, and $\tau T_1$ is Fredholm with index 0 by Lemma 5.1 and $\tau(T - T_1)$ is compact by Proposition 5.4. Assume that $F(a_n) = 0$ and set $v := T(a_n)$. Then $v$ is a solution to (1) with $f = 0$, and hence $v = 0$ by our assumption. Using Proposition 6.1 we conclude that $(a_n) = 0$, i.e. $F$ is injective. This implies that $F$ has a bounded inverse. Using Proposition 5.4 it follows that $v := TF^{-1} f$ is a solution of (1), which depends continuously on $f$.\qed
The proof of our main theorem is now simple:

**Proof of Theorem 2.2.** We start with part 2 of the theorem: Since $F := \tau T$ is bounded and boundedly invertible by Proposition 6.2, the set $\{\tau v_n : n \in \mathbb{N}\}$ is a Riesz basis of $H_\gamma^2$. It follows from Propositions 6.2 and B.2 that (1) is well posed for $\tau v = v(0, \cdot) + \frac{i}{\omega p} \partial v / \partial x_1(0, \cdot)$. For this choice of $\tau$, the operator $T$ has the bounded left inverse $F^{-1} \tau$, and hence $\{v_n : n \in \mathbb{N}\}$ is a Riesz basis of $\text{ran}(T) = V$.

Since the functions $v_n$ are chosen as in Proposition 3.5, the matrix representing $T$ consists of Jordan blocks. Because of Proposition 4.3 at most a finite number of these Jordan blocks have size $> 1$. It is straightforward to see that $R$ is represented by the same matrix.

**Appendix A. On the radiation condition**

The study of solutions to the Helmholtz equation $\Delta v + \omega^2 \varepsilon_p v = 0$ in $S$ with boundary conditions $\gamma v(x_1, \cdot) = (0, 0)^T$ for $x_1 \in \mathbb{R}$ amounts to the study of spectral properties of the operator $A^{(\gamma)} := -\frac{1}{\varepsilon_p} \Delta : H_\gamma^2(S) \to L^2(S)$. Due to the isometry of the Floquet transform, the spectrum of $A^{(\gamma)}$ is the union of the spectra of the operators defined by

$$A^{(\gamma)}_\alpha := -\frac{1}{\varepsilon_p} \Delta : H_\gamma^2(\Omega) \to L^2(\Omega), \quad \alpha \in \mathbb{R}.$$ 

Since these operators are positive and self-adjoint in the weighted Hilbert space $L^2(\Omega, \varepsilon_p)$ with a compact resolvent, their spectra consist of a countable number of positive eigenvalues with finite multiplicities accumulating only at $\infty$:

$$\sigma(A^{(\gamma)}_\alpha) = \{\tilde{\lambda}_m(\alpha) : m \in \mathbb{N}\}, \quad \alpha \in [-\pi, \pi).$$

We assume that the $\tilde{\lambda}_m$ are arranged in increasing order. The functions $\tilde{\lambda}_m$ are smooth, except at points where two or more of them cross. Alternatively, the eigenvalues of $A^{(\gamma)}_\alpha$ can be arranged such that they are analytic functions $\lambda_m$ of $\alpha$, which have holomorphic extensions to a complex neighborhood $U$ of $[-\pi, \pi)$ (22). Furthermore, there exists a holomorphic family of eigenfunctions $U \to L^2(\Omega), \xi \mapsto w_{n,\xi}$:

$$\Delta_{\xi} w_{m,\xi} + \lambda_m(\xi) \varepsilon_p w_{m,\xi} = 0, \quad \xi \in U, m \in \mathbb{N}. \quad (A.1)$$

Note that if $\xi^* \in [-\pi, \pi)$ is a characteristic value of $(B_{\xi})$, then

$$\ker B_{\xi^*} = \ker(A_{\xi^*}^{(\gamma)} - \omega^2 I) = \text{span}\{w_{m,\xi^*} : \exists m \in \mathbb{N} \lambda_m(\xi^*) = \omega^2\}. \quad (A.2)$$

**Proposition A.1.** $\xi^* \in [-\pi, \pi)$ is a characteristic value of $(B_{\xi})$ with a partial null multiplicity greater than 1 if and only if there exists $m \in \mathbb{N}$ such that $\lambda_m(\xi^*) = \omega^2$ and $\lambda_m'(\xi^*) = 0$. 

24
Proof. Suppose that \( \omega^2 = \lambda_m(\xi^*) \) and \( \lambda'_m(\xi^*) = 0 \). Taking the derivative of (A.1) with respect to \( \xi \), which will be indicated by a prime in the rest of this proof, yields

\[
(\Delta \xi + \lambda_m(\xi) \varepsilon_p) w'_m,\xi + 2i(\partial_{x_1} + i\xi) w_m,\xi + \lambda'_m(\xi) \varepsilon_p w_m,\xi = 0. \tag{A.3}
\]

Since \( \lambda'_m(\xi^*) = 0 \), we have

\[
\frac{dB_\xi w_m,\xi}{d\xi}|_{\xi = \xi^*} = 0 \quad \text{and} \quad B_{\xi^*} w_m,\xi^* = 0.
\]

Therefore, \( (w_m,\xi) \) is a root function of \( (B_\xi) \) corresponding to \( \xi^* \) with a partial null multiplicity greater than 1.

Conversely, assume that \( \xi^* \) is a characteristic value of \( B_\xi \) with a partial null multiplicity greater than 1. Then, there exists a root function \( (u_\xi) \) such that

\[
B_{\xi^*} u_\xi = 0 \quad \text{and} \quad B_{\xi^*} u'_\xi = 0. \tag{A.4}
\]

Let \( \Xi(\omega^2,\xi^*) := \{m \in \mathbb{N} : \lambda_m(\xi^*) = \omega^2 \} \). Due to (A.2) there exist coefficients \( \nu_m \in \mathbb{C} \) such that

\[
w_{\xi^*} = u_{\xi^*} \quad \text{with} \quad w_{\xi} := \sum_{m \in \Xi(\omega^2,\xi^*)} \nu_m w_m,\xi.
\]

Taking a linear combination of the equations (A.3) and subtracting (A.4), we get

\[
B_{\xi^*} \left( w'_{\xi^*} - u'_{\xi^*} \right) = -\sum_{m \in \Xi(\omega^2,\xi^*)} \lambda'_m(\xi^*) \nu_m \varepsilon_p w_m,\xi^*. \tag{A.5}
\]

The right hand side of (A.5) belongs to \( \text{ran}(B_{\xi^*}) \). Since \( \xi^* \in \mathbb{R} \), \( B_{\xi^*} \) is self-adjoint in \( L^2(\Omega) \), and hence \( \ker(B_{\xi^*}) = \text{ran}(B_{\xi^*})^\perp \). As

\[
\sum_{m \in \Xi(\omega^2,\xi^*)} \lambda'_m(\xi^*) \nu_m w_m,\xi^* \in \ker(B_{\xi^*}),
\]

we have

\[
\int_\Omega \varepsilon_p \left| \sum_{m \in \Xi(\omega^2,\xi^*)} \lambda'_m(\xi^*) \nu_m w_m,\xi^* \right|^2 dx = 0.
\]

As the functions \( w_m,\xi^* \) are linearly independent, it follows that \( \lambda'_m(\xi^*) \nu_m = 0 \) for all \( m \in \Xi(\omega^2,\xi^*) \). Since not all \( \nu_m \) vanish, we obtain that \( \lambda'_m(\xi^*) = 0 \) for some \( m \).

Recall that the group velocity of a Floquet mode \( w_m,\xi(x) e^{i\xi x_1} \) is given by

\[
\frac{d\omega}{d\xi} = \frac{d\sqrt{\lambda_m(\xi)}}{d\xi} = \frac{\lambda'_m(\xi)}{2\sqrt{\lambda_m(\xi)}}. \tag{A.6}
\]

**Proposition A.2.** For a Floquet mode of the form \( v(x) = w_m,\xi(x) e^{i\xi x_1}, \lambda_m(\xi^*) = \omega^2 \) with non vanishing group velocity, i.e. \( \lambda'_m(\xi^*) \neq 0 \) the following statements are equivalent:
1. \( v \) has positive energy flux, i.e. \( \Im q(v,v) > 0 \).
2. \( v \) has positive group velocity, i.e. \( \lambda'_m(\xi^*) > 0 \).

Moreover, if \( \tilde{v}(x) = w_{n,\xi^*}(x)e^{i\xi^* x_1} \) is another Floquet mode with \( \lambda_{n}(\xi^*) = \omega^2 \) and \( m \neq n \), then \( q(v,\tilde{v}) = 0 \).

**Proof.** Since \( q_{x_1}(v,\tilde{v}) = q(v,\tilde{v}) \) is independent of \( x_1 \), it follows that

\[
q(v,\tilde{v}) = \int_0^L \left[ \frac{\partial v}{\partial x_1}(x) \bar{\tilde{v}}(x) - v(x) \frac{\partial \tilde{v}}{\partial x_1}(x) \right] dx_2 \\
= \int_0^1 \int_0^L \left[ \frac{\partial v}{\partial x_1}(x) \bar{\tilde{v}}(x) - v(x) \frac{\partial \tilde{v}}{\partial x_1}(x) \right] dx_2 dx_1 \\
= \int_\Omega \left[ (\partial_{x_1} + i\xi^*)w_{m,\xi^*}(x)w_{n,\xi^*}(x) - w_{m,\xi^*}(x)(\partial_{x_1} + i\xi^*)w_{n,\xi^*}(x) \right] dx.
\]

Taking the \( L^2 \) inner product with \( w_{n,\xi^*} \) in (A.3) and using \( \langle B_{\xi^*} w_{m,\xi^*}, w_{n,\xi^*} \rangle = \langle w_{m,\xi^*}, B_{\xi^*} w_{n,\xi^*} \rangle = 0 \) and analogously with the roles of \( w_{m,\xi^*} \) and \( w_{n,\xi^*} \) interchanged we obtain that

\[
q(v,\tilde{v}) = \frac{i}{2}(\lambda'_m(\xi^*) + \lambda'_n(\xi^*)) \int_\Omega \varepsilon_p w_{m,\xi^*} \overline{w_{n,\xi^*}} \ dx.
\]

Since \( w_{m,\xi} \) and \( w_{n,\xi} \) are orthogonal with respect to the inner product in \( L^2(\Omega,\varepsilon_p) \) for all \( \xi \in \mathbb{R} \) as eigenfunctions of the self-adjoint operators \( A_\xi^* \) in this space, we obtain the last statement. Choosing \( \tilde{v} = v \) shows the equivalence result.

**Remark A.3.** Two further important items could be added to the list of equivalent statements in Proposition A.2, which we do not want to define in detail here: It has been shown by Fliss [16, Theorem 3.2.57] that a Floquet mode has positive group velocity if and only if it satisfies the limit absorption principle. Moreover, the equivalence of the principles of limit absorption and limit amplitude has been shown in the thesis of Radosz [23]. These are very strong indications that solutions with positive group velocity are “physical solutions”, and Proposition A.2 shows that for frequencies \( \omega \) for which all real characteristic values have total multiplicity 1 (cf. Proposition A.1 for a characterization), the conditions (3) are satisfied precisely for the physical Floquet modes (up to normalization).

However, if two (or more) bands \( \lambda_m \) and \( \lambda_n \) cross at \( \omega^2 \), i.e. \( \lambda_{m}(\xi^*) = \omega^2 \) for some \( \xi^* \in [-\pi, \pi] \) and if \( \lambda'_m(\xi^*)\lambda'_n(\xi^*) < 0 \), then a system of Floquet modes satisfying (3) does not necessarily satisfy the limiting absorption principle. E.g., the Floquet modes \( v_a := \sqrt{2}v + \tilde{v} \), \( v_b := \sqrt{2}v - v \) satisfy (3), but \( v_a \) does not satisfy the limiting absorption principle. This shows in particular that Conjecture 4.2 in [24] is false in general, but true if all real characteristic values have total multiplicity 1.

For Floquet modes with group velocity 0 we do not know which are the physical solutions. A system of Floquet modes satisfying (3) is constructed in [19], but we are not aware of indications that they correspond to physical solutions.

26
Appendix B. Uniqueness results

In this appendix we discuss conditions under which solutions to the boundary value problem \( \Box \) are unique. In this case problem \( \Box \) is well posed (Proposition 6.2), and the second part of our main theorem 2.2 holds true. Proposition 13.2 is used in the proof of Theorem 2.2.

**Lemma B.1.** Assume that either \( \theta_N = 0 \) (Dirichlet condition) or \( \theta_N = 1 \) and \( 3\theta_D \geq 0 \). Moreover, assume that the only solution \( v \in H^1_\gamma(S^+) \) to the boundary value problem \( \Box \) with \( f = 0 \) is \( v = 0 \). Then the only solution \( v \in H^1_\gamma(S^+) \) to \( \Box \) with \( f = 0 \) is \( v = 0 \).

**Proof.** Let \( v = \sum_{n=1}^\infty c_n v_n^+ + w \in H^1_\gamma(S^+) \) be a solution to \( \Box \) with \( f = 0 \). Then it follows from [19, Theorem 5.1.4] that there exists \( \delta > 0 \) such that \( \exp(\delta x_1)w \in H^1_\gamma(S^+) \). Therefore, letting \( x_1 \) tend to \( \infty \) in the definition of \( q(w, v_n^+) \), it follows that \( q(w, v_n^+) = 0 \) for all \( j \), and analogously \( q(w, w) = 0 \). For the case \( \theta_N = 1 \) and \( 3\theta_D \geq 0 \) we obtain from \( \frac{\partial v}{\partial x_1}(0, x_2) = -\theta_D v(0, x_2) \) and the orthogonality and normalization conditions for the Floquet modes \( v_n^+ \) that

\[
0 \geq -2(3\theta_D) \int_0^L |v(0, x_2)|^2 \, dx_2 = 23 \int_0^L \frac{\partial v}{\partial x_1}(0, x_2)\bar{v}(0, x_2) \, dx_2 = 3q(v, v) = \sum_{j=1}^\infty |c_j|^2.
\]

This implies \( c_1 = \cdots = c_\infty = 0 \) and hence \( v = w \). For the case \( \theta_N = 0 \) it can be shown analogously that \( v = w \). Now the assumption of the lemma implies \( v = 0 \). \( \square \)

**Proposition B.2.** Let \( \tau v = \frac{\partial v}{\partial x_1}(0, \cdot) + ikv(0, \cdot) \) with \( \kappa > 0 \). Then the only solution \( v \in H^1_\gamma(S^+) \) to \( \Box \) with \( f = 0 \) is \( v = 0 \).

**Proof.** According to Lemma B.1 it suffices to consider solutions \( v \) to \( \Box \) with \( f = 0 \) in \( H^1_\gamma(S^+) \). Then there exists a sequence \( R_k \) tending to \( \infty \) such that \( \lim_{k \to \infty} \int_0^L \frac{\partial v}{\partial x_1}(R_k, x_2)\bar{v}(R_k, x_2) \, dx_2 = 0 \), and hence \( q(v, v) = 0 \). Therefore,

\[
0 = 3q(v, v) = 23 \int_0^L \frac{\partial v}{\partial x_1}(0, x_2)\bar{v}(0, x_2) \, dx_2 = 2k \int_0^L |v(0, x_2)|^2 \, dx_2.
\]

Hence, \( v(0, \cdot) = 0 \), and \( \frac{\partial v}{\partial x_1}(0, \cdot) = ikv(0, \cdot) = 0 \). Now a unique continuation principle in two dimensions (see [25, Corollary 7.4.2]) implies that \( v = 0 \). \( \square \)

**Proposition B.3.** Assume that \( \tau \) is either the Dirichlet or the Neumann trace and that \( \varepsilon_p \) satisfies the symmetry condition \( \varepsilon_p(1 - x_1, x_2) = \varepsilon_p(x_1, x_2) \). Then the only solution \( v \in H^1_\gamma(S^+) \) to \( \Box \) with \( f = 0 \) is \( v = 0 \).
\textbf{Proof.} In both cases Lemma \ref{lem:boundary_condition} applies, so it suffices to show that a solution \( v \in H^1_x(S^+) \) to \ref{eq:wave_equation} with \( f = 0 \) vanishes. For the Dirichlet condition \( v(0, \cdot) = 0 \) (i.e. \( \theta_D = 1 \) and \( \theta_N = 0 \)) we consider the odd extension \( v(-x_l, x_2) := -v(x_l, x_2) \) for \( x_1 > 0, x_2 \in (0, L) \), and for the Neumann condition \( \frac{\partial v}{\partial x_1}(0, \cdot) = 0 \) (i.e. \( \theta_N = 1 \) and \( \theta_D = 0 \)) the even extension \( v(-x_l, x_2) := v(x_l, x_2) \). In both cases the extended function \( v \) satisfies the differential equation \ref{eq:wave_equation} in the full strip, and hence \( B_\alpha(\mathcal{F}v)(\cdot, \alpha) = 0 \) for all \( \alpha \). Since \( B_\xi \) has at most a finite number of characteristic values on the real axis, it follows that \( \mathcal{F}v(\cdot, \alpha) = 0 \) for almost all \( \alpha \). As \( v \in L^2(S) \), an application of the inverse Floquet transform yields \( v \equiv 0 \). \hfill \Box

\textbf{Appendix C. Quasi-periodic boundary conditions with } \beta \in \{0, \pi\} \\

If \( \gamma \in \{\gamma_0, \gamma_\pi\} \), there are infinitely many connected components of \( S \) containing two characteristic values (with multiplicities). Here we set \( \bar{N} := N \). Then each characteristic value \( \xi_n^\pm \) with \( n \geq N \) shares a connected component \( S_\eta \) of \( S \) with precisely one other characteristic value, w.l.o.g. \( \xi_{n+1}^\pm \). For \( \gamma \in \{\gamma_0, \gamma_\pi\} \) we cannot exclude the possibility of infinitely many characteristic values with partial null multiplicity \( 2 \) in general. Treating this case would require an extension of Lemma \ref{lem:boundary_condition} and considerable additional work. To get along with Lemma \ref{lem:boundary_condition} here, we have imposed the additional assumption \( \varepsilon_p(x_1, x_2) = \varepsilon_p(x_1, L - x_2) \) in the case \( \gamma \in \{\gamma_0, \gamma_\pi\} \) above. Then by a symmetry argument we can split the problem into two problems for wave guides of width \( L/2 \): For \( \beta = 0 \) we either impose Dirichlet conditions at \( \{x : x_2 \in \{0, L/2, L\}\} \) or Neumann conditions at \( \{x : x_2 \in \{0, L/2, L\}\} \), and for \( \beta = \pi \) we impose Dirichlet conditions at \( x_2 = 0, x_2 = L \) and Neumann conditions at \( x_2 = L/2 \), or Neumann conditions at \( x_2 = 0, x_2 = L \) and Dirichlet conditions at \( x_2 = L/2 \). For each of the subproblems for waveguides of width \( L/2 \) we can exclude that possibility of infinitely many characteristic values with partial null multiplicities \( \geq 2 \) as above. Therefore, \( u_n \) defined by \ref{eq:wave_equation} is again periodic in \( x_1 \). However, \( P_{S_n}^{} u_n \) is not necessarily a multiple of \( \psi_n \), but we only have \( P_{S_n}^{} u_n \in \text{span}\{\psi_n, \psi_{n \pm 1}\} \). Therefore, we replace \ref{eq:wave_equation} by the normalization condition

\[ \|P_{S_n}^{} u_n\|_{L^2(\Omega)} = \left( \kappa_n + \frac{1}{2\kappa_n} \right)^{-1/2}, \quad n > \tilde{N}, \quad (C.1) \]

which leaves a free phase factor. Moreover, we replace \ref{eq:wave_equation} by

\[ v_n^+ (x) := e^{-\kappa_n x_1} (P_{S_n}^{} u_n)(x), \quad n > \tilde{N}. \]

It is easy to see that Lemma \ref{lem:dispersion} and Corollary \ref{cor:dispersion} remain valid. The proofs of other results are not affected.

\textbf{Acknowledgement}

The authors would like to thank two anonymous referees for their careful reading of our paper and for many very helpful suggestions. Moreover, they thank Giovanni Alessandrini for pointing out reference \ref{ref:Alessandrini} to them.
References

[1] E. Yablonovitch, Inhibited spontaneous emission in solid-state physic and electronics, Phys. Rev. Lett. 58 (20) (1987) 2059–2062.

[2] A. Figotin, A. Klein, Localized classical waves created by defects, J. Statist. Phys. 86 (1-2) (1997) 165–177.

[3] A. Figotin, P. Kuchment, Band-gap structure of spectra of periodic dielectric and acoustic media. I. Scalar model, SIAM J. Appl. Math. 56 (1) (1996) 68–88.

[4] A. Figotin, P. Kuchment, Band-gap structure of spectra of periodic dielectric and acoustic media. II. Two-dimensional photonic crystals, SIAM J. Appl. Math. 56 (6) (1996) 1561–1620.

[5] P. Kuchment, Floquet theory for partial differential equations, Vol. 60 of Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 1993.

[6] P. Kuchment, The mathematics of photonic crystals, in: Mathematical modeling in optical science, Vol. 22 of Frontiers Appl. Math., SIAM, Philadelphia, PA, 2001, pp. 207–272.

[7] S. G. Johnson, J. D. Joannopoulos, Photonic Crystals: The Road from Theory to Practice, Kluwer Academic Publishers, Boston, 2002.

[8] K. Sakoda, Optical properties of photonic crystals, Vol. 80 of Springer Series in Optical Sciences, Springer-Verlag, Berlin, 2005.

[9] H. Ammari, F. Santosa, Guided waves in a photonic bandgap structure with a line defect, SIAM J. Appl. Math. 64 (6) (2004) 2018–2033 (electronic).

[10] A. Figotin, A. Klein, Localization of classical waves. I. Acoustic waves, Comm. Math. Phys. 180 (2) (1996) 439–482.

[11] A. Figotin, A. Klein, Localization of classical waves. II. Electromagnetic waves, Comm. Math. Phys. 184 (2) (1997) 411–441.

[12] A. Figotin, A. Klein, Localization of light in lossless inhomogeneous dielectrics, J. Optical Society of America 15 (5) (1998) 1423–1435.

[13] A. Figotin, A. Klein, Midgap defect modes in dielectric and acoustic media, SIAM J. Appl. Math. 58 (6) (1998) 1748–1773 (electronic).

[14] S. Soussi, Modeling photonic crystal fibers, Adv. in Appl. Math. 36 (3) (2006) 288–317.

[15] S. Fliss, P. Joly, Exact boundary conditions for time-harmonic wave propagation in locally perturbed periodic media, Applied Numerical Mathematics 59 (2008) 2155–2178. doi:10.1016/j.apnum.2008.12.013
[16] S. Fliss, Étude mathématique et numérique de la propagation des ondes dans un milieu périodique présentant un défaut, Ph.D. thesis, École Doctorale de l’École Polytechnique (2009).

[17] S. Fliss, P. Joly, R. J. Li, Exact boundary conditions for periodic waveguides containing a local perturbation, Communications in Computational Physics 1 (6) (2006) 945–973.

[18] I. C. Gohberg, E. I. Sigal, An operator generalization of the logarithmic residue theorem and Rouché’s theorem, Mat. Sb. (N.S.) 84(126) (1971) 607–629.

[19] S. A. Nazarov, B. A. Plamenevsky, Elliptic problems in domains with piecewise smooth boundaries, Vol. 13 of de Gruyter Expositions in Mathematics, Walter de Gruyter & Co., Berlin, 1994.

[20] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, Vol. 1, Travaux et Recherches Mathématiques, No. 17, Dunod, Paris, 1968.

[21] S. Steinberg, Meromorphic families of compact operators, Arch. Rational Mech. Anal. 31 (1968/1969) 372–379.

[22] A. V. Sobolev, J. Walthoe, Absolute continuity in periodic waveguides, Proc. London Math. Soc. (3) 85 (3) (2002) 717–741.

[23] M. Radosz, The principles of limit absorption and limit amplitude for periodic operators, Ph.D. thesis, Karlsruhe Institute of Technology (2010). URL http://digbib.ubka.uni-karlsruhe.de/volltexte/1000022443

[24] M. Elhrhardt, J. Sun, C. Zheng, Evaluation of scattering operators for semi-infinite periodic arrays, Commun. Math. Sci. 7 (2) (2009) 347–364.

[25] F. Schulz, Regularity theory for quasilinear elliptic systems and Monge-Ampère equations in two dimensions, Vol. 1445 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1990.