EXPLICIT SOLUTIONS TO THE OPPENHEIM CONJECTURE FOR INDEFINITE TERNARY DIAGONAL FORMS

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ABSTRACT. We give a new proof of the Oppenheim conjecture for indefinite ternary diagonal forms of the type $x^2 + y^2 - \alpha z^2$ where $\alpha$ is an irrational number. Our method is explicit in the sense that we are able to construct a solution to the problem and we obtain an effective bound on the solution. The method is geometrical and is based on continued fractions.

1. INTRODUCTION

We are interested in the following diophantine problem, given any real number $\varepsilon > 0$ and a positive irrational number $\alpha$, is there exists a nonzero vector $(x, y, z) \in \mathbb{Z}^3$ such that

$$|x^2 + y^2 - \alpha z^2| \leq \varepsilon.$$ 

This apparently simple question found a solution only in the mid-eighties thanks to G.A. Margulis which proved that the answer is positive. In fact, Margulis showed in [Mar89] a much more general statement which encompasses all indefinite quadratic forms in $n \geq 3$ variables, provided they are not proportional to a rational one. This result was conjectured by Oppenheim in 1929 [Opp29] and remained open in full generality until Margulis’ breakthrough. The Oppenheim conjecture reduces to the three dimensional case which is strangely the most difficult case. The strategy of the proof used by Margulis, was to solve a particular case of another conjecture due to M.S. Raghunathan. The resolution of the Oppenheim conjecture is a consequence of Margulis’ proof of the Raghunathan conjecture in the case $n = 3$. Few time later, Ratner’s proved the Raghunathan conjecture in full generality for any connected Lie group [R90]. These results were the starting point of a tremendous amount of activity around which is now called homogeneous dynamics. This point of view shows to be very fruitful in order to treat various unsolved problems especially in diophantine approximation. The litterature about this conjecture and others related questions is abundant. The interested reader may find most of the main contributions on this conjecture in Margulis’ survey [Mar03] which is by far the most complete.

A natural question is whether the Oppenheim conjecture could be proved with another method, namely without using homogeneous dynamics. As far as we know the answer is negative for $n = 3$, unless for a very specific case due to Watson which we will discuss later on. The most powerful method to solving diophantine inequalities is the Circle method but it requires a large number of variables compared to the degree of the polynomial involved. In the early times of the conjecture, Davenport and Heilbronn succeeded to prove the Oppenheim conjecture for irrational diagonal forms in five variables by using a variant of the Circle method [DH46]. Their proof has the advantage to be effective. The same result was proved earlier by Chowla for $n \geq 9$ using lattice points counting in irrational ellipsoids. The barrier $n = 4$ has been breached by Oppenheim itself in its seminal paper [Opp29] using some old results of Korkine and Zolotareff on representation of definite forms [KZ72]. In the late seventies, Iwaniec [Iw77] proved the Oppenheim conjecture for some quaternary diagonal forms using sieve theory. A last attempt to prove the conjecture was due to R.K. Baker and H.P. Schlickewei who proved the conjecture in full generality for $n \geq 21$
For quadratic forms, i.e. in degree 2, it seems that the circle method can only operate if \( n \geq 5 \). Using the full power of analytic methods combined with geometry of numbers, an effective version of the Oppenheim conjecture was proved very recently for \( n \geq 5 \) by P. Buterus, F. Götze, T. Hille and G.A. Margulis \([\text{BGHM}]\). This results have been sharpened by P. Buterus, F. Götze, T. Hille in \([\text{BGH}]\) for diagonal forms extending Birch-Davenport method to dimensions at least five combined with a result of Schlickewei. The latter proofs are quite involved and very technical.

The three dimensional case.

It is noteworthy to mention the difficulty of the problem for \( n = 3 \). The case of forms \( Q_\alpha(x, y, z) = x^2 + y^2 - \alpha z^2 \) we are concerned with shows a curious behaviour. Indeed it has been remarked by Eskin Margulis and Mozes ([EMM98], Theorem 2.2.) that \( Q_\alpha(\mathbb{Z}^3) \) fails to be equidistributed for a dense set of values of \( \alpha \). This contrasts with the analog in higher dimension, in the same paper it is proved that the set \( Q(\mathbb{Z}^3) \) is equidistributed in the real line given any form \( Q \) of signature \((p, q) \neq (2, 1) \) or \((2, 2)\) which satisfy the assumptions of the Oppenheim conjecture.

For a very specific class of quadratics forms, Watson \([\text{Wat46}]\) gave an explicit proof of the Oppenheim conjecture by showing how to construct the solution and therefore providing bounds for the solution.

Watson considered quadratic forms of the type \( Q(x, y, z) = x^2 - a\alpha y^2 - \alpha^2 z^2 \) where \( a \) is a positive integer and \( \alpha \) is an irrational number with continued fraction representation \([a; a, \ldots] = [a; \overline{a}]\). When \( a \geq 2 \) such numbers are sometimes called \textit{silver means}, in analogy with the case \( a = 1 \) which is just the golden ratio. The convergents of such numbers satisfies very a simple recurrence relation, if \( c_n = p_n / q_n \) is the \( n^{th} \) convergent of \( \alpha \) then \( q_n = p_{n-1} \). For each integer \( n > 0 \), let us set

\[
    x_n = q_{n+1}, \quad y_n = q_n, \quad \text{and} \quad z_n = q_{n-1}.
\]

By means of easy manipulations Watson showed that

\[
    |x_n^2 - a\alpha y_n^2 - \alpha^2 z_n^2| \leq \frac{\alpha + \overline{\alpha}}{q_n q_{n-1} B_n B_{n-1}}
\]

where \( \overline{\alpha} \) is the algebraic conjugate of \( \alpha \) and \( B_n = |\overline{\alpha} - p_n / q_n| \). Since \( \alpha \) has bounded partial quotients, in fact all equal to \( a \), the \( B_n \)'s are bounded. Thus,

\[
    |x_n^2 - a\alpha y_n^2 - \alpha^2 z_n^2| \ll_n \frac{1}{q_n q_{n-1}}.
\]

Let us choose an arbitrary \( \varepsilon > 0 \), then if \( n \) is taken large enough in order to fullfill the inequality

\[
    \frac{1}{q_n q_{n-1}} \leq \frac{1}{q_{n-1}^2} \leq \varepsilon.
\]

This ensures that \( v_n = (q_{n+1}, q_n, q_{n-1}) \) solves the Oppenheim conjecture for \( n \) as above. Note that this gives an asymptotic sequence of solutions not only one solution.

A bound for the solution \( v_n \) depends on the least integer \( n_1 \) such that \( m_1 = q_{n_1 - 1} \geq \frac{1}{\sqrt{\varepsilon}} \). Thus for \( n \geq n_1 \)

\[
    \|v_n\|_\infty = q_{n+1} \ll \frac{1}{\sqrt{\varepsilon}}.
\]

This result is quite exceptional among the bunch of results surrounding the Oppenheim conjecture. In fact, it gives a computable solution and it is \textit{effective} in the sense that it gives a bound of the sequence \( F(N) = \min_{v \in \mathbb{Z}^3, v \neq 0, \|v\|_\infty < N} |Q(v)| \). As we have seen, by taking \( N = \varepsilon^{-1/2} \), Watson’s
result gives
\[ F(N) = \min_{v \in \mathbb{Z}^3, v \neq 0, ||v||_{\infty} < N} |Q(v)| \ll N^{-2}. \]

The problem of effectiveness in Margulis’ theorem amounts to finding optimal bounds for \( F(N) \). Although, Ratner’s theorems are not effective in general, Lindenstrauss and Margulis [EL10] succeeded to overcome this issue by giving upper bounds on \( e^{P(1/\varepsilon)} \) for some polynomial \( P \). Their deep result is valid for all indefinite forms in degree three and is based on homogeneous dynamics. Shorty after Bourgain [Brg10] gave optimal bounds for \( F(N) \) for ternary diagonal forms. The works of Ghosh- Gorodnik-Nevo [GGN20], Ghosh-Kelmer [GK18] and Athreya-Margulis [AM18] gave closely related results for generic families of quadratic forms.

The bound provided in Watson’s result is outstanding, in the sense that, as far as we know, this is the best known bound for an individual quadratic form. Indeed, one of the output of Bourgain’s result predicts that, under Lindelöf hypothesis for the Riemann Zeta function the best bound one can hope for a generic form is \( F(N) \ll N^{-1+o(1)} \). Watson’s peculiar example improves it by a factor \( N^{-1} \). Note a slight difference with Bourgain, indeed he considered forms of the type \( Q(x, y, z) = x_1^2 + \alpha_2 x_2^2 - \alpha_3 x_3^2 \) with \( \alpha_2, \alpha_3 > 0 \) whereas Watson’s example is of the form \( Q(x, y, z) = x_1^2 - \alpha_2 x_2^2 - \alpha_3 x_3^2 \) with \( \alpha_2, \alpha_3 > 0 \). Be that as it may, Watson’s result is the best result one can expect in solving diophantine inequalities of the form \( |Q(v)| \leq \varepsilon \).

**The main results.** The aim of the paper is to construct explicit solutions to the Oppenheim conjecture for ternary forms of the type \( Q_n(x, y, z) = x_1^2 + y^2 - \alpha z^2 \) where \( \alpha \notin \mathbb{Q} \). In turn, one is able to obtain effective bounds on such solutions. The proof essentially relies on diophantine properties of the irrational number \( \beta = \sqrt{\alpha} \), more precisely its measure of irrationality. The measure of irrationality of an real number \( \beta \) is defined as the least positive real number \( \mu \) such that for
\[
\mu(\beta) = \inf\{\omega : \frac{C}{q^{\omega+\sigma}} < \left| \beta - \frac{p}{q} \right| \text{ for all rational } p/q (q > 0), \text{ every } \sigma > 0 \text{ and for some constant } C > 0\}. 
\]

A deep theorem due to Roth states that \( \mu(\beta) = 2 \) whenever \( \beta \) is an algebraic number. The converse is not true, indeed \( \theta(e) = 2 \) whereas the constant \( e \approx 2.718 \) is a transcendental number. There exists transcendental numbers \( x \) for which \( \mu(x) = \infty \), these are termed Liouville numbers.

Let us fix an arbitrary small parameter \( \sigma > 0 \). For any irrational number \( \beta \) which is not a Liouville number, we define the following quantity
\[
\theta_\sigma(\beta) := \mu(\beta) - 1 + \sigma.
\]

In any case, one has \( \theta > 1 \) and the following diophantine condition holds for any irrational \( \beta \)
\[ \inf_{q \geq 1} q^\theta (q\beta) > 0 \]

(2)

| Method               | Variables(s) | Type      | Quantitative | Effective | Explicit |
|----------------------|--------------|-----------|--------------|-----------|----------|
| Homogeneous Dynamics | n \geq 3     | general   | ✓            | ✓         | ×        |
| Circle method        | n \geq 5     | general   | ✓            | ✓         | ×        |
| Geometry of numbers  | n = 4, n \geq 9 | diagonal | ✓            | ✓         | ×        |
| Sieve Theory         | n = 4        | diagonal  | ×            | ×         | ×        |
| Continued Fractions  | n = 3, 4     | diagonal  | ×            | ✓         | ✓        |

**Figure 1.** Comparison of the different proofs of the Oppenheim conjecture.
where \((x)\) denotes the distance of a real number \(x\) to the nearest integer. We are going to give an explicit proof of the Oppenheim conjecture for quadratic forms of the type \(Q_\alpha(x, y, z) = x^2 + y^2 - \alpha z^2\) where \(\alpha\) is an irrational number. The convergents of \(\beta\) are simply denoted \(c_n = p_n/q_n\), and \(\theta\) stands for \(\theta_\sigma(\beta)\).

**Theorem 1.1.** Given any real number \(\varepsilon > 0\) and a positive irrational number \(\alpha\). There exists a nonzero vector \(v = (x, y, z) \in \mathbb{Z}^3\) such that

\[
|Q_\alpha(v)| \leq \varepsilon.
\]

Moreover, if \(\beta\) is not a Liouville number then the solution satisfies

\[
||v||_\infty \ll q_{2n_1}^{2/(\theta + 1)}
\]

where \(q_{2n_1}\) is the denominator of the convergent of order \(2n_1\) of \(\beta\) with

\[
n_1(\varepsilon) = 2 + \left[\frac{\theta + 1}{\theta - 1}\right] \ln(\varepsilon)/\ln 2.
\]

We can extend the class of forms for which the conjecture is valid by considering classes of forms equivalent to the type \(Q_\alpha\) as given in Theorem 1.1. Given a subgroup \(N\) of \(GL(3, \mathbb{R})\), we say that two quadratic forms \(Q_1\) and \(Q_2\) are \(N\)-equivalent if there exists a \(g \in N\) such that

\[
Q_1(x) = Q_2(gx)\quad \text{for every } x \in \mathbb{R}^3.
\]

Any indefinite ternary form \(Q\) is \(SL(3, \mathbb{R})\)-equivalent to

\[
Q_0(x, y, z) = x^2 + y^2 - z^2.
\]

Let us denote by \(H\) the subgroup of \(GL(3, \mathbb{R})\) defined by

\[
H = \left\{ \begin{bmatrix} A & 0 \\ 0 & h_{33} \end{bmatrix} : A \in SL(3, \mathbb{Q}), h_{33} \notin \mathbb{Q} \right\}.
\]

From Theorem 1.1 we derive the following result.

**Corollary 1.2.**

(1) Suppose that \(Q\) is an indefinite quadratic form which is \(SL(3, \mathbb{Q})\)-equivalent to a \(Q_\alpha\) with \(\alpha \notin \mathbb{Q}\). Then the Oppenheim conjecture holds for \(Q\).

(2) Suppose that \(Q\) is an indefinite quadratic form which is \(H\)-equivalent to \(Q_0\). Then the Oppenheim conjecture holds for \(Q\). In particular if \(Q_1 = f(x, y) - \beta^2 z^2\) where \(f(x, y)\) is a rational binary form and \(\beta \notin \mathbb{Q}\) then the conjecture holds for \(Q_1\).

**Remarks.** (1) A great advantage of our method is that we know how to construct the solution. As a byproduct we obtain an effective bound on the solution. The quality of the bound depends on the value of \(\theta\) and the growth of the denominators \((q_n(\beta)))_{n \geq 1}\) of the convergents of \(\beta\).

(2) The idea of the proof is geometrical and relies on the following observations. The line of equation \(x = \beta z\) is a generatrix for the cone \(\{Q_\alpha = 0\}\) restricted to the plane \(y = 0\). For every \(\varepsilon > 0\), this line is inside the region \(\{-\varepsilon \leq Q_\alpha \leq \varepsilon\}\) and because \(\beta\) is irrational, this line cannot contain a nontrivial lattice point. Nevertheless, Dirichlet’s approximation theorem tells us that there exists lattice points lying arbitrarily near the line at any level of precision. Given any \(\varepsilon > 0\), one expects that such lattice point lies in the region \(\{|Q_\alpha| \leq \varepsilon\}\). Unsurprisingly we show in section §2 that Dirichlet’s theorem is not enough to prove the Oppenheim conjecture for \(Q_\alpha\). To overcome this problem we introduce a sequence of rational lines which are nearly parallel to the line passing through a lattice point \(u_n = (x_n, 0, z_n)\) given by Dirichlet’s given a certain order of approximation \(q_{2n}^{-1}\), i.e.

\[
\text{dist}(u_n, L_\beta) \ll \frac{1}{q_{2n}^{-1}}.
\]
where \( 1 \leq z_n \leq q_{2n}^{1-\eta} \) with \( \eta = \frac{\theta - 1}{\theta + 1} \). Given any \( n \), we define the line \( L_\beta^n \) by setting

\[
(L_\beta^n) : \ u_n + \mathbb{R}(c_{2n}(\beta), \frac{1}{q_{2n}(\beta)}, 1).
\]

A parametrization of this line for the downward direction is given by

\[
(L_\beta^n)^+ : v_n(t) = (x_n - tc_{2n}(\beta), -\frac{t}{q_{2n}(\beta)}, z_n - t) \quad (t \geq 0).
\]

The proposition 2.1 is going to show that for \( n \) large enough the parametrization \( v_n(t) \) of the intersection \( L_\beta^n \cap \{ |Q_\alpha| \leq \varepsilon \} \) is supported by two disjoint intervals \( I^1_n \) and \( I^2_n \). Thus, in order to have a lattice point in \( \{ |Q_\alpha| \leq \varepsilon \} \) it suffices to find a multiple of \( q_{2n} \), say \( t_n \), in the union of \( I^1_n \) and \( I^2_n \). In this case, one can clear denominators and the solution is given by \( v_n(t_n) \in \mathbb{Z}^3 \cap L_\beta^n \cap \{ |Q_\alpha| \leq \varepsilon \} \). The key lemma 3.1 says that this is possible if \( n \) is greater or equal than some integer \( n_1(\varepsilon) \) which is explicitly computable.

3) When \( \beta = \sqrt{\alpha} \) is a Liouville number, we can easily prove that the Oppenheim conjecture is satisfied in dimension \( n = 2 \) for the form \( q(x, z) = x^2 - \beta^2 z^2 \). Since we have \( Q_\alpha(x, 0, z) = q(x, z) \), then the Oppenheim conjecture is satisfied for \( Q_\alpha \).

4) In the case when \( \beta \) is not a Liouville number, it is always possible to find a real number \( \theta > 1 \) large enough such that for every integer \( q \geq 1 \),

\[
q^\theta \langle q/\beta \rangle > 0.
\]

The irrationality measure \( \mu(\beta) \) is introduced only with the aim of obtaining optimal bounds and to quantify the growth of the sequence \( q_{n+1}(\beta)/q_n(\beta) \). For instance the main theorem gives an explicit integral solution for our favorite example of irrational indefinite form \( Q(x, y, z) = x^2 + y^2 - \sqrt{2}z^2 \).

5) The corollary 1.2 shows that we can find a solution to the Oppenheim problem for quadratic forms of the type

\[
Q(x, y, z) = ax^2 + bxy + cy^2 - \alpha z^2
\]

where \( a, b, c \in \mathbb{Q} \) and \( \alpha \notin \mathbb{Q} \). This is the best we can do, and it would be interesting to find explicit solutions for general indefinite irrational forms. For the general case, one would be led to use the multidimensional version of the Dirichlet’s approximation theorem. Using the same kind of strategy applied to a product of linear forms instead of a quadratic form, the author was able to derive a set of sufficient conditions for the Littlewood conjecture to hold.

2. SEQUENCES OF RATIONAL LINES INTERSECTING \( \{ |Q_\alpha| \leq \varepsilon \} \)

We focus our attention on forms of the type

\[
Q_\alpha(x, y, z) = x^2 + y^2 - \alpha z^2
\]

where \( \alpha \in \mathbb{R}_+ \). We assume that \( \alpha \) is irrational and therefore the form \( Q_\alpha \) is an indefinite quadratic form which is not proportional to a form which rational coefficients. The output of Margulis’s result tells us that for every \( \varepsilon > 0 \), there must exist a nonzero lattice vector \( v \in \mathbb{Z}^3 \) such that

\[
0 \leq |Q_\alpha(v)| \leq \varepsilon.
\]

We are going to reprove this result by constructing an explicit solution to this problem, i.e. to find a nonzero vector in \( \mathcal{A}(\varepsilon) \cap \mathbb{Z}^3 \) where the domain \( \mathcal{A}(\varepsilon) \) is delimited by the level sets \( \{ Q_\alpha = -\varepsilon \} \).
and \( \{ Q = \varepsilon \} \). A parametrization of the cone \( \{ Q_\alpha = 0 \} \) is as follows,

\[
\begin{align*}
  x(t, \theta) &= \sqrt{\alpha} t \cos \theta \\
  y(t, \theta) &= \sqrt{\alpha} t \sin \theta \quad (0 \leq \theta < 2\pi). \\
  z(t, \theta) &= t.
\end{align*}
\]

This parametrization shows that the cone \( \{ Q_\alpha = 0 \} \) is generated by a continuous family of lines given by \( \mathcal{L}_\alpha(\theta) = \mathbb{R}(\sqrt{\alpha} \cos \theta, \sqrt{\alpha} \sin \theta, 1) \) where the angle \( \theta \) varies in \([0, 2\pi)\). The line corresponding to the intersection of the \( xz\)-plane with the cone \( \{ Q_\alpha = 0 \} \) is exactly given by \( \mathcal{L}_\beta(0) = \mathbb{R}(\sqrt{\alpha}, 0, 1) \), we denote it by \( \mathcal{L}_\beta \) where \( \beta = \sqrt{\alpha} \). An equation of this line in the \( xz\)-plane is just \( x = \beta z \). Since \( \beta^2 = \alpha \) is irrational, \( \beta \) itself is irrational too. Thus given any positive integer \( N > 1 \) we obtain from Dirichlet’s Theorem that there exists \( (p_0, q_0) \in \mathbb{N}^2 \) with \( 1 \leq q_0 \leq N \) such that

\[
|p_0 - \beta q_0| \leq \frac{1}{N}.
\]

This tells us that we can always find a lattice vector \( (p_0, 0, q_0) \) arbitrarily near the line of equation \( x = \beta z \) in the \( xy\)-plane provided \( \beta \) is irrational.

2.1. Irrationality Measures. We follow the notations of [H90], section 3.

For each real number, let \( \langle x \rangle \) denote the distance of \( x \) to the closest integer. Dirichlet’s theorem says that \( \inf_{q \geq 1} q \beta q(\beta) < 1 \), the question is to know in which extent one can improve this approximation. We can assign to \( \beta \) a number called the irrationality measure of \( \beta \) which is defined as follows,

\[
\mu(\beta) := \inf \{ \omega \in \mathbb{R}_+ : \inf_{q \geq 1} q^{\omega-1+\sigma} \beta q(\beta) > 0 \text{ for every real } \sigma > 0 \}.
\]

In other words,

\[
\mu(\beta) = \inf \{ \omega : \frac{C}{q^{\omega+\sigma}} < \left| \beta - \frac{p}{q} \right| \text{ for all rational } p/q (q > 0), \text{ every } \sigma > 0 \text{ and for some constant } C > 0 \}.
\]

An alternative definition of \( \mu \)

Suppose \( \beta \) has an infinite continued fraction expansion \( \beta = [b_0; b_1, b_2, \ldots] \), the \( n^{th} \) convergent of \( \beta \) is the rational number \( c_n(\beta) = [b_0; b_1, \ldots, b_n] \) which has reduced expression \( \frac{p_n(\beta)}{q_n(\beta)} \). Then the measure of irrationality of \( \beta \) is related to the growth of the denominators of \( c_n(\beta) \) through the following relation which can be taken as an alternative definition of \( \mu \),

\[
\mu(\beta) = 1 + \limsup_n \frac{\ln q_{n+1}(\beta)}{\ln q_n(\beta)}.
\]

Provided the existence of the limit, one has the following asymptotic behaviour

\[
q_{n+1} \asymp q_n^{\mu-1}.
\]

If we denote by \( \lambda_n \) the ratio \( q_{n+1}/q_n \), the last asymptotic estimate could be read as follows

\[
\lambda_n \asymp q_n^{\mu-2}.
\]

Its lowest value for an irrational number is \( \mu(\beta) = 2 \) and it is reached for any algebraic number of degree \( d \geq 2 \). This fact is a highly non trivial theorem due to Roth [Roth]. In the other extreme side, the value \( \mu(\beta) = \infty \) correspond to the case when \( \beta \) is the Liouville number. In general, it is extremely difficult to compute this measure in practice.

A nice consequence of Roth’s theorem is that \( x \) is a transcendental number as soon as \( \mu(x) > 2 \). Unfortunately, this criterion is not enough to characterize transcendental numbers because
the converse of Roth’s result is not true. Indeed, Adams’ proved that $\theta(e) = 2$ showing that a transcendental number could reach the same bound (see e.g. [Dav78]). More precisely, it can be proved that for all rational numbers $p/q$ ($q \geq 2$)

$$|qe - p| > c_1 \frac{\log \log q}{q \log q}.$$ 

Since the continued fraction expansion of $e$ is given $e = [2; 1, 2, 1, 1, 2n]_{n \geq 2}$, its partial quotients are unbounded. This implies that $e$ is not a badly approximable number, thus $\inf_{q \geq 1} q\langle qe \rangle = 0$. But for every $\sigma > 0$, it is not difficult to see that for every $q \geq 2$

$$\frac{\log \log q}{\log q} > \frac{1}{q^\sigma}.$$ 

This shows that for every $\sigma > 0$ and rationals $p/q$

$$|qe - p| > \frac{c}{q^{1+\sigma}}$$ 

for some constant $c$. The latter amounts to say that $\mu(e) = 2$, and it shows that some transcendental numbers are not well-approximated by rationals and behave like algebraic numbers in view of Roth’s theorem.

**The exponent theta associated to $\beta$**

By definition suppose that $\beta$ is not a Liouville number i.e. $\mu(\beta) < \infty$. Then for every $\sigma > 0$ there exists $C > 0$ such that for any $p, q$ integers with $q \geq 1$,

$$\frac{C}{q^{\mu(\beta)+\sigma}} < \left| \beta - \frac{p}{q} \right|$$ (7)

or also,

$$\frac{C}{q^{\mu(\beta)-1+\sigma}} < |q\beta - p|.$$ (8)

Let us fix a real parameter $\sigma > 0$ and introduce the following useful quantity associated with any irrational number $\beta$

$$\theta(\beta) := \mu(\beta) + \sigma - 1.$$ 

This exponent gives a lower bound for the approximation of the irrational number $\beta$ by rational numbers provided it is not a Liouville number. It particular, since $\beta$ is not a Liouville number one has that $C = \inf_{q \geq 1} q^\theta \|q\beta\|$ is positive and therefore for every $p$ and $q$ integers, $q \geq 1$

$$\frac{C}{q^{\theta+1}} < \left| \beta - \frac{p}{q} \right|.$$ (9)

2.2. *Dirichlet versus Oppenheim.* Dirichlet’s theorem does not give a very precise estimate about how close is the lattice point $u_0 = (p, 0, q)$, obtained in (5), to the line $\mathbb{R}(\beta, 0, 1)$. Let us explain why $u_0 = (p_0, 0, q_0)$ falls out $A(\varepsilon)$ for any choice of $N$. Otherwise the conjecture would be proved for $Q_\alpha$ and $u_0$ would be the solution. It is not difficult to quantify by how much Dirichlet’s fails to prove the Oppenheim conjecture for $Q_\alpha$. In particular, combining (9) with Dirichlet approximation (5) we have

$$\frac{C}{q_0^{\theta+1}} < \left| \beta - \frac{p_0}{q_0} \right| \leq \frac{1}{q_0 N}.$$
Taking the inverse if necessary, we can assume that \( u_0 \) is in the first octant with \( q_0 \leq N \) one infers that

\[
\frac{C}{q_0^\theta} < p_0 - \beta q_0 \leq \frac{1}{N},
\]

and the latter inequality gives in addition sharp bounds for \( q_0 \)

\[
(CN)^{1/\theta} < q_0 \leq N.
\]

From (10) we get

\[
\beta q_0 + \frac{C}{q_0^\theta} < p_0 < \beta q_0 + \frac{1}{N}.
\]

Using (11) one obtains

\[
\beta(CN)^{1/\theta} + \frac{C}{N^\theta} < p_0 \leq \beta N + \frac{1}{N}.
\]

Therefore

\[
2\beta(CN)^{1/\theta} + \frac{C}{N^\theta} < p_0 + \beta q_0 \leq 2\beta N + \frac{1}{N}.
\]

We finally obtain the following bounds for \( Q_\alpha(u_0) = p_0^2 - \beta^2 q_0^2 \)

\[
\frac{C}{N^\theta} \left( 2\beta(CN)^{1/\theta} + \frac{C}{N^\theta} \right) < p_0^2 - \beta^2 q_0^2 \leq \frac{1}{N} \left( 2\beta N + \frac{1}{N} \right) .
\]

In particular we can do than the inequality \( Q_\alpha(u_0) < 2\beta + \frac{1}{N^2} \). Thus Dirichlet’s theorem is unable to provide a solution to the Oppenheim conjecture whatever the choice of \( N \).

2.3. Error in the approximation by the convergents. We have a precise of the rate of error of this approximation, set \( e_n(\beta) := \beta - c_n(\beta) \), so we have (see e.g. Exercise 3.1.5. [EW])

\[
\frac{1}{2q_{n+1}(\beta)^2} \leq |e_n(\beta)| \leq \frac{1}{q_n(\beta)q_{n+1}(\beta)} < \frac{1}{q_n(\beta)^2}.
\]

The sequence \( q_n(\beta) \) is increasing and the rate of convergence is determined by the diophantine properties of \( \beta \), in particular it tends to infinity with at least exponential rate since \( 2^{(n-2)/2} \leq q_n(\beta) ([Kh], Theorem 12). \)

The convergents \( c_n(\beta) \) tends to \( \beta \) by oscillating so that the sign of \( e_n(\beta) \) is alternating. From now on, we choose even indices which implies that the error terms assume only positive values. We infer that,

\[
\frac{1}{2q_{2n+1}(\beta)^2} \leq e_{2n}(\beta) < \frac{1}{q_{2n}(\beta)^2}.
\]

2.4. Rational lines of approximation. We introduce an object which is at the core of our strategy. It is a sequence of rational lines which will cross \( A(\varepsilon) \) in a sufficiently large time in order to contain a lattice point. We have two degrees of freedom given by the integral parameters \( n \) and \( N \). We are going to reduce to merely one parameter, namely \( n \). To do this, let us first fix the real parameter

\[
\eta := \frac{\theta - 1}{\theta + 1}
\]

where

\[
\theta(\beta) = \mu(\beta) + \sigma - 1.
\]
In all cases, $\theta > 1$, and therefore
\[ 0 < \eta < 1. \]
Let us choose $N$ to be a sequence $(N_n)$ satisfying the growth condition
\[ N_n = q_{2n}^{1-\eta}. \]
For each nonnegative integer $n$, Dirichlet’s theorem tells us that there exists a two-dimensional lattice vector $(x_n, z_n)$ with $1 \leq z_n \leq N_n$ such that
\[ |\beta z_n - x_n| \leq \frac{1}{N_n}. \]
Moreover, using $\theta$ there exists a constant $C$ such that
\[ \frac{C}{N_n^\theta} \leq \frac{C}{z_n^\theta} < |\beta z_n - x_n| \leq \frac{1}{N_n}. \]
This gives the crucial bound on the denominators,
\[ (CN_n)^{1/\theta} < z_n \leq N_n \]
For each $n$, from (19) we form the three-dimensional integral vector $u_n := (x_n, 0, z_n) \in \mathbb{Z}^3$ which is close to the axis $x = \beta z$. As we have seen earlier Dirichlet’s approximation theorem is not enough in order to ensure that $u_n$ is in $\mathcal{A}(\varepsilon)$. However, we have at our disposal a sequence of lattice points $(u_n)_n$ near $\mathcal{A}(\varepsilon)$ from which we built a sequence of affine lines $L_{\beta}^n$ by setting
\[ (L_{\beta}^n) : u_n + \mathbb{R}(c_{2n}(\beta), \frac{1}{q_{2n}(\beta)}, 1). \]
The lines $(L_{\beta}^n)$ are good candidates for containing lattice points in $\mathcal{A}(\varepsilon)$. Indeed the first interesting feature is that this lines pass through lattice points, namely the $u_n$’s, and such lines are directed by rational vectors so that they can contain lattice points. Another crucial feature is geometrical, the fact that the lines are nearly parallel to the generatrix of the cone, namely the line $\mathbb{R}(\beta, 0, 1)$. This leads us to expect that $(L_{\beta}^n)$ spends a sufficient amount of time in $\mathcal{A}(\varepsilon)$ for $n$ large enough.
We will rather focus on the downward half-line parametrized as follows
\[ (L_{\beta}^n)^+: v_n(t) = (x_n - tc_{2n}(\beta), -\frac{t}{q_{2n}(\beta)}, z_n - t) \ (t \geq 0). \]
We are interested in the intersection of this half-line with the domain $\mathcal{A}(\varepsilon)$. Note that for each increment of the index $n$, the line will never remains in a same plane, in that two successive lines $(L_{\beta}^n)^+$ and $(L_{\beta}^{n+1})^+$ will never be coplanar. A geometric observation allows us to guess that this line $(L_{\beta}^n)^+$ will cut the boundary of $\mathcal{A}(\varepsilon)$, namely $\{Q_\alpha = \pm \varepsilon\}$, in at most four points. This will be made explicit in our computations. Our first task is to estimate the time spent by $(L_{\beta}^n)^+$ in $\mathcal{A}(\varepsilon)$. The answer is given in the following proposition.

**Proposition 2.1.** Let $I^n$ be the set of times at which the half-line $(L_{\beta}^n)^+ = \{v_n(t) \mid t \geq 0\}$ intersects $\mathcal{A}(\varepsilon)$. Then, for $n$ large enough, $I$ is the union of two intervals $I_1^n$ and $I_2^n$.

**Proof.** Let us fix $\varepsilon > 0$. For each positive integer $n$, the half-line $(L_{\beta}^n)^+$ lies in $\mathcal{A}(\varepsilon)$ if and only if for every $t \geq 0$
\[ v_n(t) = (x_n - tc_{2n}(\beta), -\frac{t}{q_{2n}(\beta)}, z_n - t) \in \mathcal{A}(\varepsilon). \]
This amounts to say that the time variable $t$ is constrained to satisfy the inequalities
\[ -\varepsilon \leq (x_n - tc_{2n}(\beta))^2 + (tq_{2n}(\beta)^{-1})^2 - \alpha(z_n - t)^2 \leq \varepsilon. \]
Let us define the quadratic polynomial in the real variable $t$ (the time)
\[ f_n(t) := (x_n - tc_{2n}(\beta))^2 + (tq_{2n}(\beta)^{-1})^2 - \alpha(z_n - t)^2. \]

Ordering the terms we get
\[ f_n(t) = \{c_{2n}(\beta)^2 + q_{2n}(\beta)^{-2} - \beta^2\}t^2 - 2\{c_{2n}(\beta)x_n - y_n\beta^2\}t + \{x_n^2 - \beta^2z_n^2\}. \]

Which is important to us is the intersection points of the graph of $f_n(t)$ with the two lines corresponding to $\pm \varepsilon$. Thus we are reduced to solve the two following equations (remember $\beta^2 = \alpha$) provided such solutions exists
\[ f_n(t) = \{c_{2n}(\beta)^2 + q_{2n}(\beta)^{-2} - \beta^2\}t^2 - 2\{c_{2n}(\beta)x_n - y_n\beta^2\}t + \{x_n^2 - \beta^2z_n^2 \pm \varepsilon\} = 0. \]

Set $A_n = c_{2n}(\beta)^2 + q_{2n}(\beta)^{-2} - \beta^2$, $B_n = -2\{c_{2n}(\beta)x_n - y_n\beta^2\}$ and $C_n^\pm = x_n^2 - \beta^2z_n^2 \pm \varepsilon$.

Thus one has to solve the (two) equations
\[ A_nt^2 + B_nt + C_n^\pm = 0. \]

We need to estimate the discriminants $\Delta_n^\pm(\varepsilon) = B_n^2 - 4A_nC_n^\pm$ and in fact we only need to focus on the roots and their relative distance not on their ordering nor their signs.

Since we have a nice control of the error in the approximation by the convergents, we replace $c_{2n}(\beta)$ by $\beta - e_{2n}(\beta)$. The coefficients are therefore given by,
\[
\begin{align*}
A_n & = -2\beta e_{2n}(\beta) + e_{2n}(\beta)^2 + q_{2n}(\beta)^{-2}, \\
B_n & = -2(\beta x_n - x_n e_{2n}(\beta) - z_n\beta^2) = -2(\beta(x_n - \beta z_n) - x_n e_{2n}(\beta)), \\
C_n^\pm & = (x_n - \beta z_n)(x_n + \beta z_n) \pm \varepsilon.
\end{align*}
\]

Let us set $\delta_n = x_n - \beta z_n$, $\bar{\delta}_n = x_n + \beta z_n$, note that $Q_\alpha(u_n) = \delta_n\bar{\delta}_n$, so that $C_n^\pm = Q_\alpha(u_n) \pm \varepsilon$ and $B_n = -2(\beta\delta_n - x_n e_{2n}(\beta))$. The Dirichlet lattice point $u_n = (x_n, 0, z_n)$ is exterior to $\{-\varepsilon \leq Q_\alpha \leq \varepsilon\}$, changing $u_n$ to $-u_n$ if necessary we can assume that $Q_\alpha(u_n) > \varepsilon$. Thus, $x_n^2 > \beta^2z_n^2 \pm \varepsilon$ and in particular $C_n > 0$.

We deduce from (20) and (21) the following bounds for $z_n, \delta_n$ and $x_n$.
\[
(\mathcal{C}q_{2n}^{1-\eta})^\theta = (CN_n)^{1/\theta} < z_n \leq N_n = q_{2n}^{1-\eta}.
\]

Since $x_n = \beta z_n + \delta_n$, using (23) and (24) we get
\[
\beta C^{1/\theta}q_{2n}^{(1-\eta)/\theta} < \beta z_n + C\frac{\theta}{N_n} < x_n \leq \beta z_n + \frac{1}{N_n} \leq \beta q_{2n}^{1-\eta} + \frac{1}{q_{2n}^{1-\eta}}.
\]

We define the following quantities,
\[
U_n := -\frac{B_n}{A_n} \quad \text{and} \quad V_n := \frac{A_n}{B_n^2}
\]

Thus $\Delta_n^\pm(\varepsilon) = B_n^2 (1 - 4V_n(Q_\alpha(u_n) \pm \varepsilon))$. We are going to show that $V_n$ tends to zero, this will prove that $\Delta_n^\pm(\varepsilon)$ are both positive for $n$ large enough.

For any positive $n$, the inequalities in (17) gives
\[
\frac{1}{24q_{2n+1}(\beta)^2} \leq e_{2n}(\beta) < \frac{1}{q_{2n}(\beta)^2}.
\]

Concerning $B_n = 2\{x_n e_{2n}(\beta) - \beta \delta_n\}$, with (24), (25) and (26) one has that
\[
\frac{1}{2q_{2n+1}^2} \left( \frac{C}{N_n^\theta} + \beta(CN_n)^{1/\theta} \right) - \frac{2\beta}{N_n} < xe_2 - 2\delta_n\beta \leq \frac{1}{q_{2n}^2} \left( \beta N_n + \frac{1}{N_n} \right) - \frac{2\beta}{N_n^\gamma}. \tag{27}
\]

We can rearrange the terms in order to get the following bounds for \(B_n\)

\[
\frac{1}{q_{2n}} \left( \frac{C}{\lambda_{2n} q_{2n} N_n^\theta} + \frac{\beta(CN_n)^{1/\theta}}{\lambda_{2n}^2 q_{2n}} \right) - \frac{2\beta q_{2n}}{N_n} \leq B_n \leq \frac{2}{q_{2n}^2} \left( \beta N_n + \frac{1}{N_n} \right) - \frac{4\beta}{N_n^\gamma}. \tag{28}
\]

Replacing \(N_n\) by \(q_{2n}^{1-\eta}\) in (28) we obtain a lower bound for \(B_n\)

\[
\frac{1}{q_{2n}} \left( \frac{C}{\lambda_{2n}^2 q_{2n} h_{2n} \lambda_{1+\eta} q_{2n}} + \frac{\beta(CN_n)^{1/\theta}}{\lambda_{2n}^2 q_{2n} \lambda_{1+\eta} q_{2n}} \right) - \frac{2\beta q_{2n}^\eta}{N_n} \leq B_n. \tag{29}
\]

We claim that \(V_n\) tends to zero as \(n\) goes to infinity. Indeed, one has

\[
V_n = \frac{e_{2n}^2 - 2\beta e_{2n} + 1/q_{2n}^2}{(xe_{2n} - 2\delta_n)^2}.
\]

For the numerator of \(V_n\) we have the bound

\[
A_n = e_{2n}^2 - 2\beta e_{2n} + q_{2n}^{-2} \leq \frac{1}{q_{2n}^2} - \frac{\beta}{q_{2n+1}^2} + \frac{1}{q_{2n}^2} = \frac{1}{q_{2n}^2} \left( 1 - \frac{\beta}{\lambda_{2n}^2} + \frac{1}{q_{2n}^2} \right). \tag{30}
\]

Thus, using (29) we get

\[
0 < |V_n| \ll \left( \frac{C}{\lambda_{2n}^2 q_{2n} \lambda_{1+\eta} q_{2n}} + \frac{\beta(CN_n)^{1/\theta} q_{2n}^{(1-\eta)/\theta-1}}{\lambda_{2n}^2 q_{2n} \lambda_{1+\eta} q_{2n}} - \frac{2\beta q_{2n}^\eta}{N_n} \right)^{-2}. \tag{31}
\]

Taking under consideration the fact that \(\lambda_{2n}^2 \approx q_{2n}^{2(\mu-2)}\) which follows from (6) we obtain that

\[
\lim_{n} V_n = 0.
\]

Using (15) we infer that

\[
\left| Q_\alpha(u_n) \right| \leq 2C \beta + \frac{1}{N_n} = 2C \beta + \frac{1}{q_{1-\eta}^\gamma}. \tag{32}
\]

---

**Figure 2.** The domain \(-\varepsilon \leq Q_\alpha(v_n(t)) = f_n(t) \leq \varepsilon\) is supported by two intervals.
Thus the term

\[ 1 - 4V_n(Q_\alpha(u_n) \pm \varepsilon) \]

can be made positive and less than 1 provided \( n \) is taken large enough. Hence the discriminants are always positive when \( n \) becomes larger than some positive integer \( n_0 = n_0(\varepsilon) \) depending on \( \varepsilon \). In this range the roots are given by

\[
t_{1,2,3,4}(n, \varepsilon) = \frac{1}{2} U_n \left( 1 \pm \sqrt{1 - 4V_n(Q_\alpha(u_n) \pm \varepsilon)} \right).
\]

In more details, these correspond to the hitting times, \( t_1 < t_2 < t_3 < t_4 \) given by

\[
\begin{align*}
((L^\alpha_\beta)^+ \cap \{Q_\alpha = -\varepsilon\}) & \quad \begin{cases} 
   t_1(n) = \frac{1}{2} U_n \left( 1 - \sqrt{1 - 4V_n(Q_\alpha(u_n) + \varepsilon)} \right) \\
   t_4(n) = \frac{1}{2} U_n \left( 1 + \sqrt{1 - 4V_n(Q_\alpha(u_n) + \varepsilon)} \right)
\end{cases}
\end{align*}
\]
and

$$((L^\alpha_\beta)^+ \cap \{Q_\alpha = \varepsilon\}) \left\{ \begin{aligned} t_2(n) &= \frac{1}{2} U_n \left( 1 - \sqrt{1 - 4V_n(Q_\alpha(u_n) - \varepsilon)} \right) \\ t_3(n) &= \frac{1}{2} U_n \left( 1 + \sqrt{1 - 4V_n(Q_\alpha(u_n) - \varepsilon)} \right). \end{aligned} \right.$$ 

In particular the values of $t$ for which $v_n(t) \in (L^\alpha_\beta)^+ \cap A(\varepsilon)$ is the union the two disjoint intervals $I^n_1(\varepsilon) = [t_1, t_2]$ and $I^n_2(\varepsilon) = [t_3, t_4]$ when $n \geq n_0$. The proposition 2.1 is proved.

\[ \square \]

**Remark.** The signs and the hitting times are not important for our purposes. The two intervals of the proposition 2.1, $I^n_1$ and $I^n_2$ plays a symmetric role and their size is the same. One interval should comprise negative times while the other consists of positive ones. Assume for instance, changing the order if necessary, that the roots are sorted such that $t_1 < t_2 < t_3 < t_4$ for $n \geq n_0$. Then in view of the previous proposition, for $n \geq n_0$ the intersection of the half-line $(L^n_\beta)^+$ with the two level sets $\{Q_\alpha = \pm \varepsilon\}$ behaves as follows, see figure 2.4

$$Q_\alpha(v_n(t)) < -\varepsilon \quad \text{if} \quad 0 < t < t_1 \quad \text{(out)}$$
$$-\varepsilon \leq Q_\alpha(v_n(t)) \leq \varepsilon \quad \text{if} \quad t_1 \leq t < t_2 \quad \text{(in)}$$
$$\varepsilon < Q_\alpha(v_n(t)) \quad \text{if} \quad t_2 \leq t < t_3 \quad \text{(out)}$$
$$\varepsilon < Q_\alpha(v_n(t)) \leq \varepsilon \quad \text{if} \quad t_3 \leq t < t_2 \quad \text{(in)}$$
$$Q_\alpha(v_n(t)) < -\varepsilon \quad \text{if} \quad t_4 < t \quad \text{(out)}.$$

3. **A Solution to the Oppenheim Conjecture for $Q_\alpha$**

Let $\varepsilon > 0$ be an arbitrary small real number, we are interested to finding $n$ and $t_n$ such that $v_n(t_n)$ is a nonzero vector is in $\mathbb{Z}^3 \cap A(\varepsilon)$ i.e.

$$0 < |Q_\alpha(v_n(t_n))| \leq \varepsilon.$$

In order that $v_n(t_n)$ provides the required lattice point, we necessarily need $t_n$ to be a multiple of $q_{2n}$ so that we can clear the denominators. By symmetry, we only need to focus on one interval, say $I^n_1(\varepsilon)$. The following combinatorial argument shows that it is always possible to do so for large enough values of $n$.

**Lemma 3.1.** There exists a positive integer $n_1(\varepsilon)$ such that the interval $I^n_1(\varepsilon) = [t_1, t_2]$ contains a multiple of $q_{2n}$ whenever $n \geq n_1$.

**Proof.** Let us set for each positive integer $n$, the following counting function

$$M_n := \text{Card}([t_1, t_2] \cap \mathbb{Z} q_{2n}).$$

$M_n$ is the number of multiples of $q_{2n}$ in $I^n_1 = [t_1, t_2]$. In particular we have that $M_n = \left\lfloor \frac{l(I^n_1)}{q_{2n}} \right\rfloor$ where $l(I^n_1)$ is the length of the interval $I^n_1$. The aim is to show that this quantity is at least 1 when $n$ is larger than a certain threshold $n_1$. As $n$ gets large, we have that

$$t_1 \asymp 2U_n V_n(Q_\alpha(u_n) - \varepsilon) = -\frac{2(Q_\alpha(u_n) - \varepsilon)}{B_n} \quad (33)$$

and

$$t_2 \asymp 2U_n V_n(Q_\alpha(u_n) + \varepsilon) = -\frac{2(Q_\alpha(u_n) + \varepsilon)}{B_n}. \quad (34)$$

The length of the interval $I^n_1$ is asymptotically given by
Thus, \[ M_n \approx \frac{4\varepsilon}{q_{2n}|B_n|}. \]

Concerning the denominator,
\[
q_{2n}|B_n| \leq \frac{1}{q_{2n}} \left( \beta N_n + \frac{1}{N_n} \right) - \frac{2\beta}{N_n^\eta} q_{2n}
\leq \beta \frac{N_n}{q_{2n}} + \frac{1}{q_{2n}} \frac{1}{N_n} - \frac{2\beta}{N_n^\eta} q_{2n}
\leq \frac{\beta}{q_{2n}^\eta} + \frac{1}{q_{2n}^{2-\eta}} - \frac{2\beta}{q_{2n}^{(1-\eta)\theta - 1}}.
\]

Then the choice\(^1\) of \( \eta \) gives that \((1 - \eta)\theta - 1 = \eta\), then
\[
q_{2n}|B_n| \leq \frac{\beta}{q_{2n}^\eta} + \frac{1}{q_{2n}^{2-\eta}} - \frac{2\beta}{q_{2n}^{(1-\eta)\theta - 1}}.
\]

Thus we have the upper estimate
\[
q_{2n}|B_n| \ll \frac{1}{q_{2n}^\eta}.
\]

Taking the inverse,
\[
\varepsilon q_{2n}^\eta \ll \frac{4\varepsilon}{q_{2n}|B_n|}.
\]

Finally we infer the following crucial bound
\[
\varepsilon q_{2n}^\eta \ll M_n \quad (36)
\]

In particular, since \((q_{2n})_n\) diverges there exists \(n_1(\varepsilon)\) such that for all \(n \geq n_1(\varepsilon)\)
\[
1 < \varepsilon q_{2n}^\eta.
\]

Hence (36) shows that \(M_n \geq 1\) for \(n \geq n_1\), meaning that the interval of times \(I^\eta_1(\varepsilon) = [t_1, t_2]\) contains at least one multiple of \(q_{2n}\) for \(n \geq n_1\). Let us estimate the integer \(n_1\) which depends on the choice of \(\varepsilon\) and \(\eta\), and which can be seen formally as
\[
\varphi_\beta(\varepsilon) = \min\{n \geq n_0 \mid M_n \geq 2\}.
\]

Here \(n_0 = n_0(\varepsilon)\) is the least integer which ensures that \(I^\eta_1(\varepsilon) \neq \emptyset\) coming from Proposition 2.1 whereas \(n_1(\varepsilon)\) is the least integer such that \(I^\eta_1(\varepsilon)\) contains a multiple of \(q_{2n}\). In particular, \(n_0(\varepsilon) < n_1(\varepsilon)\). The number \(n_1\) is not going to be optimal, i.e. it will be an upper estimate for \(\varphi_\beta(\varepsilon)\).

Since \(2^{n-1} \leq q_{2n}\), a sufficient condition in order the inequality \(\varepsilon q_{2n}^\eta > 1\) to hold is

---

\(^1\)This is the only moment we need that \(\theta > 1\)
Applying logarithms, we get
\[ n > 1 + \eta^{-1} \ln \left( \frac{\varepsilon}{\eta} \right) / \ln 2. \]

Thus a good choice for \( n_1 \) is
\[ n_1(\varepsilon) := 2 + \lfloor \eta^{-1} \ln (\varepsilon) / \ln 2 \rfloor. \]

This finishes the proof of the Lemma.

**Proof of Theorem 1.1.** Let \( \varepsilon > 0 \) be fixed.

**Case 1** Assume \( \beta \) is a Liouville number and let \( n \) be a positive large enough integer so that
\[ 2^{-n} \beta + 2^{-2(n+2)} \leq \varepsilon. \]

Since \( \mu(\beta) = \infty \) and given \( n \) as above we can always find a rational number \( p/q \) such that
\[ |q\beta - p/q| < 1/q^{n+2}. \]  
(38)

From this, we deduce that \( q\beta - 1/q^{n+2} < p < q\beta - 1/q^{n+2} \), thus
\[ 2q\beta - 1/q^{n+2} < p + \beta q < 2q\beta + 1/q^{n+2}. \]  
(39)

Thus,
\[ |Q_\alpha(p,0,q)| = |p^2 - \beta^2 q^2| = (p + \beta q)|p - \beta q| < \frac{1}{q^{n+2}}(2q\beta + 1/q^{n+2}). \]  
(40)

Since \( q \geq 2 \),
\[ |Q_\alpha(p,0,q)| < \frac{1}{2^{n+2}}(4\beta + 1/2^{n+2}). \]

The choice of \( n \) implies that \( v = (p,0,q) \) is a nonzero integral solution of
\[ |Q_\alpha(p,0,q)| < \varepsilon. \]

In other words, the Oppenheim conjecture holds for \( Q_\alpha \) in this case.

**Case 2** Assume \( \beta \) is a not a Liouville number.

The lemma (3.1) shows that there exists an explicit integer \( n_1(\varepsilon) > 0 \) such that \( I_1^n(\varepsilon) \) contains a multiple of \( q_{2n_1} \) say \( a_{n_1} q_{2n_1} \in [t_1,t_2] \) where \( a_{n_1} \) is a nonzero integer. The proposition (2.1) implies that \( v_{n_1}(a_{n_1} q_{2n_1}) \in A(\varepsilon) \). Moreover,
\[ v_{n_1}(a_{n_1} q_{2n_1}) = (x_{n_1} - a_{n_1} p_{2n_1}, -a_{n_1} z_{n_1} - a_{n_1} q_{2n_1}) \in \mathbb{Z}. \]

Thus, we have a nonzero integral vector \( v_1 := v_{n_1}(a_{n_1} q_{2n_1}) \) in \( A(\varepsilon) \) that is,
\[ |Q_\alpha(v_1)| \leq \varepsilon. \]

This proves the first assertion of the theorem. We give an estimate the size of the solution, set
\[ \|v_1\|_\infty = \max\{|x_{n_1} - a_{n_1} p_{2n_1}|, |z_{n_1} - a_{n_1} p_{2n_1}|, |a_{n_1}|\}. \]

A crude bound is given by
\[ \|v_1\|_\infty \leq |x_{n_1}| + |a_{n_1}| p_{2n_1}. \]

We know from (25) that
\[ |x_{n_1}| \leq \beta q_{2n_1}^{1-\eta}. \]
Also by construction we have $a_{n_1} \in [t_1/q_{2n_1}, t_2/q_{2n_1}]$, thus in view of (32), (33) and (34) one has

$$|a_{n_1}| \leq |Q_{n_1}(u_n)| + \varepsilon \leq 2C\beta + q_{2n_1}^{-1+\eta} + \varepsilon.$$ 

Thus,

$$\|v_1\|_\infty \lesssim q_{2n_1}^{1-\eta} + \frac{2C\beta + \varepsilon}{q_{2n_1}|B_{2n_1}|} c_{2n_1} + \frac{c_{2n_1}}{q_{2n_1}|B_{2n_1}|}.$$ 

or equivalently

$$\|v_1\|_\infty \lesssim q_{2n_1}^{1-\eta} + \frac{2C\beta + \varepsilon}{|B_{n_1}|} c_{2n_1} + \frac{c_{2n_1}}{|B_{n_1}|}.$$ 

Using (29)

$$\frac{1}{|B_{n_1}|} \leq \frac{1}{|C\lambda^{-2}_{2n_1} q_{2n_1}^{2-(1-\eta)\theta} + \beta C^{1/\theta} q_{2n_1}^{-2(1-\eta)/\theta-2} - 2\beta q_{2n_1}^{-1}|}. \quad (41)$$

We have the relation $1 - \eta = 2/(\theta(\theta + 1))$ and the irrationality measure $\mu$ which comes into play using (6), thus

$$\lambda^{-2}_{2n_1} q_{2n_1}^{-2(1-\eta)\theta} \asymp q_{2n_1}^{-2(\mu-2)-2(1-\eta)\theta} = q_{2n_1}^{-2\mu+2-2(\theta+1)} = q_{2n_1}^{-2(\mu-1+1/(\theta+1))}$$

and

$$\lambda^{-2}_{2n_1} q_{2n_1}^{(1-\eta)/\theta-2} \asymp q_{2n_1}^{-2(\mu-2)+(1-\eta)/\theta-2} = q_{2n_1}^{-2\mu+2+2/(\theta^2(\theta+1))} = q_{2n_1}^{-2(\mu-1-1/(\theta^2(\theta+1))).}$$

Therefore,

$$\frac{1}{|B_{n_1}|} \ll \frac{1}{|Cq_{2n_1}^{-2(\mu-1+1/(\theta+1))} + \beta C^{1/\theta} q_{2n_1}^{-2(\mu-1-1/(\theta^2(\theta+1)))} - 2\beta q_{2n_1}^{-2(\theta(\theta+1))}|}. \quad (41)$$

Let us set $\kappa = \frac{2}{\theta(\theta + 1)} = 1 - \eta,$

$$\frac{1}{|B_{n_1}|} \ll \frac{1}{|Cq_{2n_1}^{-2(\mu-1) - \kappa} + \beta C^{1/\theta} q_{2n_1}^{-2(\mu-1)+\kappa/\theta} - 2\beta q_{2n_1}^{-\kappa}|}.$$ 

$$\ll \frac{q_{2n_1}^\kappa}{|1 - Cq_{2n_1}^{-2(\mu-1)/2\beta} - C^{1/\theta} q_{2n_1}^{-2(\mu-1)+\kappa/\theta} / 2|}.$$ 

Thus, since $2(\mu - 1) \geq 2 > \kappa/\theta + \kappa$, one has

$$\frac{1}{|B_{n_1}|} \ll q_{2n_1}^{\kappa} = q_{2n_1}^{1-\eta}.$$ 

Hence we get,

$$\|v_1\|_\infty \ll q_{2n_1}^{1-\eta} + O(1).$$

In short,

$$\|v_1\|_\infty \ll q_{2n_1}^{1-\eta}.$$
By definition we have $\eta = 1 - 2/(\theta + 1)$ so that the last inequality reads

$$\|v_1\|_{\infty} \ll q_{2n_1}^{2/(\theta+1)}$$

where

$$n_1(\varepsilon) := 2 + \left\lfloor \eta^{-1} \left| \ln (\varepsilon) \right| / \ln 2 \right\rfloor.$$ 

This finishes the proof of Theorem 1.1.

Proof of Corollary 1.2. (1) By assumption there exists $\gamma \in \text{SL}(3, \mathbb{Q})$, such that $Q(x) = Q_\alpha(\gamma x)$. Let us consider an arbitrary real $\varepsilon > 0$. Let $a = \text{lcm}\{\text{den}((\gamma^{-1})_{i,j}), 1 \leq i, j \leq 3\}$ be the least common multiple of the denominator of the coefficients of $\gamma^{-1}$. Thus $a\gamma^{-1}$ is an integral matrix. Theorem 1.1 gives the existence of a nonzero integral vector $v \in \mathbb{Z}^3$ such that $|Q_\alpha(v)| \leq \varepsilon/a^2$.

Since $a\gamma^{-1} \in \text{SL}(3, \mathbb{Z})$, $v_1 = a\gamma^{-1} v$ is a nonzero integral vector such that

$$|Q(v_1)| = |Q(a\gamma^{-1} v)| = a^2 |Q(\gamma^{-1} v)| = a^2 |Q_\alpha(v)| \leq \varepsilon.$$

Hence $Q$ satisfies the Oppenheim conjecture.

(2) Let $h \in H$ such that $Q(x) = Q_0(hx)$, where for some $A \in \text{SL}(3, \mathbb{Q})$ and $h_{33} \notin \mathbb{Q}$ one has

$$h = \begin{bmatrix} A & 0 \\ 0 & h_{33} \end{bmatrix}.$$

The matrix $h$ can factorized as follows

$$h = \begin{bmatrix} I_2 & 0 \\ 0 & h_{33} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

Set $\alpha = h_{33}^2$ and $\gamma = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \in \text{SL}(3, \mathbb{Q})$, thus

$$Q(x) = Q_0(hx) = Q_\alpha(\gamma x).$$

The form $Q$ is $\text{SL}(3, \mathbb{Q})$-equivalent to the form $Q_\alpha$. Then the assertion (1) of the corollary allows us to show that $Q$ fullfills the conjecture.

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