Certifying Numerical Decompositions of Compact Group Representations

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We present a performant and rigorous algorithm for certifying that a matrix is close to being a projection onto an irreducible subspace of a given group representation. This addresses a problem arising when one seeks solutions to semi-definite programs (SDPs) with a group symmetry. Indeed, in this context, the dimension of the SDP can be significantly reduced if the irreducible representations of the group action are explicitly known. Rigorous numerical algorithms for decomposing a given group representation into irreps are known, but fairly expensive. To avoid this performance problem, existing software packages — e.g. RepLAB, which motivated the present work — use randomized heuristics. While these seem to work well in practice, the problem of to which extent the results can be trusted arises. Here, we provide rigorous guarantees applicable to finite and compact groups, as well as a software implementation that can interface with RepLAB. Under natural assumptions, a commonly used previous method due to Babai and Friedl runs in time \( O(n^5) \) for \( n \)-dimensional representations. In our approach, the complexity of running both the heuristic decomposition and the certification step is \( O(\max\{n^3 \log n, D^2 \log d\}) \), where \( d \) is the maximum dimension of an irreducible subrepresentation, and \( D \) is the time required to multiply elements of the group. A reference implementation interfacing with RepLAB is provided.

I. INTRODUCTION

Semi-definite programming is a widely used numerical tool in science and engineering. Unfortunately, runtime and memory use of SDP solvers scale poorly with the dimension of the problem. To alleviate this issue, symmetries can often be exploited to significantly reduce the dimension \( \mathcal{O} \) (see [8] for a review). This requires finding a common block-diagonalization of the matrices representing the symmetry group action. A large number of numerical methods for this task have been developed [10–23]. These algorithms must be compared along a number of different dimensions:

1. What is their runtime as a function of the relevant parameters? The most important parameters are the dimension \( n \) of the input matrices, the dimension of the algebra \( A \) they span, and the dimension \( d \) of the largest irreducible component?

2. Are they probabilistic or deterministic?

3. Do they assume a group structure, or do they work for algebras more generally?

4. Can they handle a situation where only noisy versions of the matrices representing the symmetry are available?

5. Which aspects are covered by rigorous performance guarantees?

While a detailed review of the extensive literature is beyond the scope of this paper, we summarize the performance of the approaches that come closest to the methods described here.

References [20–23] give algorithms for finding a block decomposition for general \( * \)-algebras and come with rigorous guarantees. Refs. [21, 22] require one to solve a polynomial optimization problem of degree 4 on \( \mathbb{C}^{n \times n} \). While this might work in practice, there is no general polynomial-time algorithm for this class of problems. The procedure of [20] requires one to diagonalize “super-operators”, i.e. linear maps acting on \( n \times n \)-matrices. This implies a runtime of \( O(n^6) \).

The method of [23] exhibits a runtime of \( O(\max\{n^2 \dim^2 A, n^3 \dim A\}) \). In this scaling, the first term comes from finding an orthogonal basis for \( A \) and the second term arises from using this basis to project onto the commutant and to diagonalize\( \mathcal{O} \). Meanwhile the method comes with a guarantee that the output decomposition is close to invariant, it does not guarantee that the components will be irreducible in the presence of noise. The runtime is particularly competitive for “small” algebras: If \( \alpha \in [0, 2] \) is such that \( \dim A = O(n^\alpha) \), the scaling becomes \( O(n^{3+\alpha}) \) for the case \( \alpha < 1 \). On the other hand, in the regime \( \alpha > 1 \), the runtime \( O(n^{2+2\alpha}) \) is worse than other methods discussed below.

This scaling refers to Alg. B from that reference. There, the scaling of the second term is presented as \( O(n^4 \dim A) \). Upon a closer inspection of their algorithm we found that its runtime is slightly better than claimed. It seems that the origin of the difference, in their language, is that Alg. B — as opposed to Alg. A — does not require to use the subroutine Split. Instead, Alg. B projects a single random matrix onto the commutant of \( A \), using \( O(n^3 \dim A) \) operations.
Reference [24] works on finite group representations, rather than general $*$-algebras. It generalizes Dixon’s method [25] to handle noise in the group representation. This algorithm produces a guaranteed full decomposition, however, for this it must project a full matrix basis onto the commutant of the representation and diagonalize each projection. This means that its runtime scales quite steeply, as $O(n^5)$, with $O(n^2)$ matrix diagonalizations.

Here, we suggest to split the problem of decomposing a unitary group representation $\rho$ on $\mathbb{C}^n$ into three steps:

1. Use a fast heuristic to obtain a candidate decomposition $\mathbb{C}^n \simeq R_1 \oplus R_2 \oplus \ldots$. One particular randomized algorithm running in time $O(n^3)$ has been analyzed [19, 26] and implemented as part of the RepLAB [27] software package by some of the present authors. While this algorithm seems to give accurate results in practice, this is not underpinned by a formal guarantee.

2. Certify that each of the candidate spaces $R_i$ is within a pre-determined distance $\epsilon$ of a subspace $K_i$ that is invariant under the group.

3. Certify that the invariant spaces $K_i$ are irreducible.

With the first step already covered in Ref. [19, 26], the present paper focuses on the two certification steps. Thus, we are faced with the situation that a heuristically obtained $n \times n$ matrix $\pi$ is provided, which may or may not be close to a projection onto an invariant and irreducible space. We provide a probabilistic algorithm for this decision problem. More precisely, our main result is this:

Result 1. Let $G$ be a compact group. Assume that:

1. There exists a representation $g \mapsto \rho(g)$ in terms of unitary $n \times n$ matrices.

2. In time $O(n^2)$, one can draw an element $g \in G$ according to the Haar measure, and compute an approximation $\hat{\rho}$ such that $\max_g \max_{ij} |\rho_{ij}(g) - \hat{\rho}_{ij}(g)| = o\left(\frac{1}{n^{1+\log n}}\right)$.

Then there exists an algorithm that takes as input an $n \times n$ matrix $\pi$ as well as numbers $\epsilon, p_{\text{thr}}$, and returning true or false such that:

1. [False positive rate] The probability that the algorithm returns true even though $\pi$ is not $\epsilon$-close in Frobenius norm to a projection onto an invariant and irreducible $\rho$-space is upper-bounded by $p_{\text{thr}}$.

2. [False negative rate] The probability that the algorithm returns false even though $\pi$ is $(\epsilon/2)$-close in Frobenius norm to a projection onto an invariant and irreducible $\rho$-space is approximately $2p_{\text{thr}}$.

3. [Runtime] As long as $\epsilon = o\left(\frac{1}{n^{1+\log n}}\right)$, the algorithm terminates in time

$$O\left(\left(n^3 \log n + D \text{ tr}(\pi)^2 \log \text{ tr } \pi\right) \log \frac{1}{p_{\text{thr}}} \right),$$

where $D$ is time required to multiply two elements of $G$.

This algorithm has been implemented in Python and is available in [28].

There is an asymmetry in the way we treat false positives rates (which are bounded rigorously) and false negative rates (which are only approximated). This reflects the different roles these two parameters play in practice. Indeed, if the certification algorithm returns false, the symmetry reduction has failed, no further processing will take place, and thus no further guarantees are needed. In contrast, if the algorithm returns true, the user must be able to quantify their confidence in the result – hence the necessity to have a rigorous upper bound on the false positive rate.

In the main text, we introduce an additional parameter $\delta$, which can be used to tune the false negative rate independently of the false positive rate $p_{\text{thr}}$. The interpretation is that $\delta$ is a rigorous upper bound on the false negative rate in the limiting case where $\epsilon = 0$ and the approximation $\hat{\rho}$ is in fact exact. We have chosen $\delta = 2p_{\text{thr}}$ in the displayed result, which turns out to simplify the formula for the runtime.

In practice, one can find appropriate values for $\delta$ numerically: In an exploratory phase, one can run the algorithm for increasing values of $\delta$, until it reliably identifies valid inputs as such. One would then certify a subspace by running the procedure once with the $\delta$ previously obtained.

The paper is organized as follows. In Sec. [I] we review the mathematical setting of the paper. In Sec. [II] and Sec. [IV] we present the algorithms to certify invariance and irreducibility respectively. Finally, in Sec. [V] we discuss the runtime of the algorithms.
II. MATHEMATICAL SETTING

Let $G$ be a compact group, and $(\mathbb{C}^n, \rho)$ be a unitary representation of $G$. A subset $S \subset G$ generates the group if $\langle S \rangle$ is dense in $G$, and it is symmetric if $S = S^{-1}$.

We assume that the user can evaluate a function $\tilde{\rho} : G \to \mathbb{C}^{n \times n}$ satisfying

$$\max_{ij} |\rho(g)_{ij} - \tilde{\rho}(g)_{ij}| \leq \epsilon_0, \quad \forall g \in G.$$  

If $R \subset \mathbb{C}^n$ is the subspace to be certified and $\pi_R$ projects onto it, we use $\tilde{\pi}_R$ to denote an approximation to $\pi_R$:

$$\max_{ij} |(\pi_R)_{ij} - (\tilde{\pi}_R)_{ij}| \leq \epsilon_0.$$  

We require that $\epsilon_0 < \frac{1}{2n}$, however in practice $\epsilon_0$ is typically of the order of machine precision.

In the context of our algorithms, the user has obtained $\tilde{\pi}_R$ as an output of their numerical procedure to decompose $\rho$. Using this operator as an input, the goal is to certify two statements. The first is that there exists some invariant subspace $K \subset \mathbb{C}^n$ with associated projector $\pi_K$ satisfying that

$$\|\pi_R - \pi_K\|_F \leq \epsilon,$$  

where $\| \cdot \|_F$ is the Frobenius norm and the precision parameter $\epsilon < 1/2$ is an input. We call this procedure certifying invariance. The second is that the subspace $K$ is an irreducible $G$ representation.

For this task, we assume that one knows an upper bound $r_G$ on the number of generators of $G$, and 2. can sample from the Haar measure and evaluate $\tilde{\rho}$ on the sample. In an appendix, we show how to relax the second condition and instead assume only that the user can evaluate $\tilde{\rho}$ on a well-behaved fixed generator set. The algorithms are probabilistic. A bound $p_{\text{thr}}$, on the false positive rate – i.e. the probability that an input is certified even though it is not close to the projection onto an irreducible representation – is an explicit parameter.

Bounds $r_G$ on the number of generators of $G$ are known for a wide variety of groups. For example it is known that $r_G \leq 2$ when $G$ is a finite dimensional connected compact group [29]. For a wide variety of finite simple groups, furthermore, $r_G \leq 7$ (see [30] for a review).

III. THE INVARIANCE CERTIFICATE

Here we present our algorithm for the first task, that is, certifying the approximate invariance of $R$. Section IIIA treats a closely related problem: deciding whether an operator is close to the commutant

$$\{Y \in \mathbb{C}^{n \times n} | [\rho(g), Y] = 0 \ \forall g \in G\}$$  

of $\rho$. In that section we also work in the idealized case where $\epsilon_0 = 0$. The general algorithm deciding invariance is presented in Section IIIB.

A. Estimating closeness to the commutant in the ideal case

As mentioned, in this section we assume $\epsilon_0 = 0$ – i.e. that the representation $\rho$ can be evaluated exactly – in order to bring out the key components of the argument.

Consider an $n \times n$ matrix $X$ (later, we will take $X$ to be the approximate projection $\tilde{\pi}_R$ onto a candidate subspace). The randomized Algorithm III.1 tests whether

$$\|X - P_{\text{Haar}}(X)\|_\infty \leq \epsilon.$$  

There, $\| \cdot \|_\infty$ is the spectral norm and $P_{\text{Haar}}$ is the Hilbert-Schmidt projection onto the commutant

$$P_{\text{Haar}}(X) := \mathbb{E}_g [\rho(g)X\rho^\dagger(g)],$$  

where the expectation value is with respect to the Haar distribution.
Algorithm III.1 Closeness to Commutant

Input:
- $X \in \mathbb{C}^{n \times n}$,
- $p_{\text{thr.}} \in (0, 1)$, $\epsilon \in (0, 1/2)$.

1: Set $r = 8 \left\lceil \left( \log(1/p_{\text{thr.}}) + \log(2n) \right) \right\rceil$
2: Sample $r$ group elements $g_1, \ldots, g_r \in G$ Haar-randomly
3: Compute $c = \left\| \frac{1}{r} \sum_i \rho(g_i) X \rho^\dagger(g_i) - X \right\|_\infty$
4: if $2c \leq \epsilon$ then
5: Return: True
6: end if
7: Return: False

Proposition 1. Let $X \in \mathbb{C}^{n \times n}$ satisfy $\|X - P_{\text{Haar}}(X)\|_\infty > \epsilon$. Then, the probability that Alg. III.1 returns True is at most $p_{\text{thr.}}$.

Proof. Consider the following matrix-valued random variable with mean equal to zero,

$$Z_g := \frac{1}{r} \left( \rho(g) X \rho^\dagger(g) - P_{\text{Haar}}(X) \right), \quad g \in G \text{ Haar random}.$$

Using $R := \text{Id} - P_{\text{Haar}}$ (the projector onto the orthocomplement of the commutant of $\rho$), we find $Z_g = \frac{1}{r} \rho(g) R(X) \rho^\dagger(g)$, and so,

$$\|Z_g Z_g^\dagger\|_\infty = \frac{1}{r^2} \|R(X) R(X)^\dagger\|_\infty = \frac{1}{r^2} \|R(X)\|_\infty^2, \quad \forall g \in G.$$

This way, by the matrix Hoeffding bound \[31\],

$$\text{Prob} \left[ \left\| \sum_i Z_{g_i} \right\|_\infty \geq z \|R(X)\|_\infty \right] \leq 2n \exp \left( \frac{-r z^2}{2} \right)$$

where $\{g_i\}$ are the samples in line 2 of Alg. III.1. Taking $z = 1/2$, the right-hand side above is $\leq p_{\text{thr.}}$, and so with probability at least $1 - p_{\text{thr.}}$ it holds that

$$c = \left\| \frac{1}{r} \sum_i \rho(g_i) X \rho^\dagger(g_i) - X \right\|_\infty = \left\| \sum_i Z_{g_i} - R(X) \right\|_\infty \geq \|R(X)\|_\infty - \left\| \sum_i Z_{g_i} \right\|_\infty \geq \frac{1}{2} \|R(X)\|_\infty > \epsilon/2.$$

We now show a converse result, namely, that Alg. III.1 always “detects” matrices which are close enough to the commutant.

Proposition 2. Let $X$ satisfy $\|X - P_{\text{Haar}}(X)\|_\infty \leq \epsilon/2$ for some $\epsilon < 1$. Then Alg. III.1 deterministically returns True upon the input $X$, $\epsilon$.

Proof. For any $g \in G$ it holds that

$$\|\rho(g), X\|_\infty = \|\rho(g), X - P_{\text{Haar}}(X)\|_\infty \leq 2 \|X - P_{\text{Haar}}(X)\|_\infty \leq \epsilon.$$

Therefore, using standard norm relations we obtain

$$c = \left\| \frac{1}{r} \sum_i (\rho(g_i) X \rho^\dagger(g_i) - X) \right\|_\infty \leq \frac{1}{r} \sum_i \|\rho(g_i), X\|_\infty \leq \epsilon.$$
B. The full certificate

Here, we will go beyond Section III.A in two ways: First, we allow for non-zero errors $\epsilon_0$. Second, we show that a projection that is close to being invariant is close to a projection onto an invariant subspace. The goal is, given $\bar{\pi}_R$ as an input, to certify that there is an invariant subspace $K$ with

$$\|\pi_K - \pi_R\|_F \leq \epsilon.$$  

The procedure is given in Alg. III.2

**Algorithm III.2 Invariance certificate**

**Input:**
- $\bar{\pi}_R \in \mathbb{C}^{n \times n}$
- $p_{\text{thr.}} \in (0, 1)$
- $\epsilon \in (0, 1/2)$

**Output:** True/False

1: Set $r = 8[(\log(1/p_{\text{thr.}}) + \log(2n))]$, $f_{\text{err}} = 8n\epsilon_0 + 6n^2\epsilon_0^2 + 2n^3\epsilon_0^3$, and $\epsilon' = \epsilon/2\sqrt{2 \dim R}$
2: Sample $r$ group elements $g_1, \ldots, g_r \in G$ Haar-randomly
3: Compute $\hat{c} = \|\frac{1}{r} \sum_i \bar{\rho}(g_i)\bar{\pi}_R \rho^\dagger(g_i) - \bar{\pi}_R\|_\infty$
4: if $2\hat{c} + f_{\text{err}} \leq \epsilon'$ then
5: Return: True
6: end if
7: Return: False

As before, line 4 of Alg. III.2 simply takes $k$ close to the minimum of $f_k(c)$ and does not affect the probability of falsely certifying $R$. Our main result in this section is the following guarantee on the invariance certificate.

**Theorem 1.** Assume that for all invariant subspaces $K \subset \mathbb{C}^n$,

$$\|\pi_K - \pi_R\|_F > \epsilon.$$  

(2)

Then, the probability that Alg. III.2 returns True is upper bounded by $p_{\text{thr.}}$.

To prove Thm. 1, we will first show that if $\pi_R$ is close to the commutant, then it is close to an invariant projector $\pi_K$ as in eq. (1). After that, our argument will closely follow Sec. III.A

**Proposition 3.** Assume that $\pi_R$ satisfies $2\sqrt{2 \dim R} \|P_{\text{Haar}}(\pi_R) - \pi_R\|_\infty \leq \epsilon$ for some $\epsilon < 1$. Then there exists an invariant subspace $K$ with projector $\pi_K$ satisfying $\|\pi_R - \pi_K\|_F \leq \epsilon$.

**Proof.** Let $\lambda^\dagger(M)$ be the vector of eigenvalues of a Hermitian matrix $M \in \mathbb{C}^{n \times n}$ in decreasing order. By Weyl’s perturbation theorem (see e.g. [32, Chap. VI]),

$$\|\lambda^\dagger(P_{\text{Haar}}(\pi_R)) - \lambda^\dagger(\pi_R)\|_\infty \leq \frac{\epsilon}{2\sqrt{2 \dim R}} = \epsilon'.$$

This way, the eigenvalues of $P_{\text{Haar}}(\pi_R)$ lie in $[-\epsilon', \epsilon'] \cup [1 - \epsilon', 1 + \epsilon']$, where $\epsilon' < 1/2$. Let $\pi_K$ be the projector onto all eigenspaces corresponding to eigenvalues in $1 \pm \epsilon'$. The projector $\pi_K$ is invariant and satisfies $\|\pi_K - P_{\text{Haar}}(\pi_R)\|_\infty \leq \epsilon'$. We therefore see that,

$$\|\pi_K - \pi_R\|_F \leq \sqrt{2 \dim R} \|\pi_K - \pi_R\|_\infty$$

$$\leq \sqrt{2 \dim R} \left(\|\pi_K - P_{\text{Haar}}(\pi_R)\|_\infty + \|P_{\text{Haar}}(\pi_R) - \pi_R\|_\infty\right)$$

$$\leq 2\epsilon' \sqrt{2 \dim R} = \epsilon,$$

where we used that rank$(\pi_K - \pi_R) \leq \dim K + \dim R = 2 \dim R$ in the first step.

From the proof above it becomes clear that certifying that $R$ is approximately invariant is, ultimately, just certifying that $\pi_R$ is close enough to the commutant.
Proof of Thm. 7. By Prop. 3 we may take
\[ \frac{\epsilon}{2\sqrt{2} \dim R} < \| P_{\text{Haar}}(\pi_R) - \pi_R \|_\infty. \]

Let
\[ A := \frac{1}{r} \sum_i (\rho(g_i)\pi_R \rho^\dagger(g_i) - \hat{\rho}(g_i)\hat{\pi}_R \hat{\rho}^\dagger(g_i)), \quad \Delta_R := \pi_R - \hat{\pi}_R, \]
then,
\[ \left\| \frac{1}{r} \sum_i \rho(g_i)\pi_R \rho^\dagger(g_i) - \pi_R \right\|_\infty \leq \left\| \Delta_R \right\|_\infty + \left\| A \right\|_\infty + \left\| \frac{1}{r} \sum_i \hat{\rho}(g_i)\hat{\pi}_R \hat{\rho}^\dagger(g_i) - \pi_R \right\|_\infty = n\epsilon_0 + \| A \|_\infty + \hat{c}. \]

Then, by Prop. 4 with probability at least 1 - \( p_{\text{thr.}} \) it holds that
\[ \frac{\epsilon}{2\sqrt{2} \dim R} < 2(n\epsilon_0 + \| A \|_\infty + \hat{c}). \]

We now provide an upper bound on \( \| A \|_\infty \). Let \( \Delta(g) := \rho(g) - \hat{\rho}(g) \), then
\[ \| A \|_\infty \leq \mathbb{E}_i \left[ \left\| \Delta(g_i)\pi_R \rho^\dagger(g_i) \right\|_\infty + \left\| \rho(g_i)\Delta_R \rho^\dagger(g_i) \right\|_\infty + \left\| \rho(g_i)\pi_R \Delta^\dagger(g_i) \right\|_\infty + \left\| \rho(g_i)\Delta_R \Delta^\dagger(g_i) \right\|_\infty + \left\| \Delta(g_i)\Delta_R \Delta^\dagger(g_i) \right\|_\infty \right]. \]
Submultipliciativity, together with \( \max \{ \| \Delta_R \|_\infty, \| \Delta(g) \|_\infty \} \leq n\epsilon_0 \) for all \( g \in G \), gives
\[ \| A \|_\infty \leq 3(n\epsilon_0 + n^2\epsilon_0^2) + n^3\epsilon_0^3. \]

\[ \square \]

IV. IRREDUCIBILITY CERTIFICATE

In this section we present an algorithm that certifies irreducibility. Given \( \hat{\pi}_R \) as an input, where \( R \) holds an invariance certificate, the goal is to certify that the minimizer of
\[ \min_{K \subset C^n} \| \pi_R - \pi_K \|_F \]
(3)
is irreducible. We first present the idea of the algorithm in an idealized setting, and then come back to the noisy scenario.

A. The ideal case

Let \((C^n, \rho_K)\) be a unitary representation of \( G \) and suppose that we have access to the same primitives as in Sec. III A. Namely, we can sample Haar-randomly from \( G \) and evaluate \( \rho_K \) on any sample. Our task is to certify if \( \rho_K \) is irreducible. The following algorithm uses random walks to achieve this.
For any Proposition 4. connection to the dimension of the commutant is made by the following statement.

Proof. Unitarity ensures that of the commutant of much below 2. otherwise it holds that \( \text{tr} > \) large. We will bound this rate at the end of this subsection. if \( \text{tr} \) chosen too small, the algorithm could fail to recognize irreducible representations —its false negative rate would be \( S = \theta \) for any \( g \).

Theorem 2. Let \( \rho_K \) be reducible, then the probability that Alg. [IV.1] returns True upon this input is at most \( p_{thr} \).

Our proof of Thm. [2] will work for any value of \( t \), i.e. it does not rely on using \( t = 2 + \log n_K \). However, if \( t \) is chosen too small, the algorithm could fail to recognize irreducible representations —its false negative rate would be large. We will bound this rate at the end of this subsection.

The key for the proof of Thm. [2] is Schur’s lemma —if \( \rho_K \) were irreducible it would hold that \( \text{tr} P_{\text{Haar}} = 1 \) and otherwise it holds that \( \text{tr} P_{\text{Haar}} \geq 2 \). What the algorithm does estimate a quantity which is larger than the dimension of the commutant of \( \rho_K \). As we will see, if \( \rho_K \) is reducible then it is exceedingly unlikely for this estimator to fall too much below 2.

The quantity being estimated is, in fact, \( \text{tr} P_S^{2t} \), where \( P_S \) is the random walk operator associated to \( \rho_K \). The connection to the dimension of the commutant is made by the following statement.

Proposition 4. For any \( t \) it holds that \( \text{tr} P_{\text{Haar}} \leq \text{tr} P_S^{2t} \).

Proof. Unitarity ensures that \( \|P_S\|_\infty = 1 \). Because \( r \geq r_G \), the probability that \( S \) generates \( G \) is one. Together with \( S = S^{-1} \), this ensures that \( P_S \) is self-adjoint and that the +1 eigenspace corresponds exactly to the commutant of \( \rho_K \).

Let \( \{\lambda_i\} \) be all the eigenvalues of \( P_S \) different from one. The statement follows from

\[
\text{tr} P_S^{2t} = \text{tr} P_{\text{Haar}} + \sum_i \lambda_i^{2t} \geq \text{tr} P_{\text{Haar}}.
\]

Proof of Thm. [2] It is clear that \( E_m \) is an estimator for \( \text{tr} P_S^{2t} \). Since \( \rho_K \) is unitary, furthermore, \( |\text{tr} \rho_K(g)|^2 \leq n_K^2 \) for any \( g \), and so by Chernoff’s bound,

\[
\Pr \left[ E_m \leq (1 - \theta) \text{tr} P_S^{2t} \right] \leq \exp \left( -\frac{\theta^2 m \text{tr} P_S^{2t}}{2n_K^2} \right),
\]

for any \( \theta \in (0, 1) \). But by the assumption on \( m \) we may use \( \theta = \theta_m \) in the equation above. Then, using Prop. [4] and \( \text{tr} P_{\text{Haar}} \geq 2 \),

\[
\Pr \left[ E_m \leq 2(1 - \theta_m) \right] \leq \Pr \left[ E_m \leq (1 - \theta_m) \text{tr} P_S^{2t} \right] \leq \exp \left( -\frac{\theta^2 m \text{tr} P_S^{2t}}{2n_K^2} \right) \leq \exp \left( -\frac{\theta^2 \text{tr} P_S^{2t}}{n_K^2} \right) < p_{thr}.
\]

As mentioned, the proof above doesn’t rely on the particular choice of \( t \) in line 3 of Alg. [IV.1] It also only uses the bound \( m \geq 2n_K^2 \log(1/p_{thr}) \) on the number of samples (cf. line 2). In Prop. [6] we use \( t > 2 + \log n \) and \( m > 16n_K^2 \log_2(1/(p_{thr}, - p_{thr})) \) to bound the false negative rate of the algorithm. To prove it, it’s convenient to show the following intermediate result first.
Proposition 5. Let $S$ be sampled as in Alg. IV.1. The probability that $\|P_{\text{Haar}} - P_S\|_\infty > 1/2$ is strictly less than

$$2n^2 \exp\left(\frac{-r}{8}\right) \leq p_{\text{thr}}.$$

Proof. Let $\sigma$ be the representation of $G$ acting by conjugation on $\mathbb{C}^{n \times n}$. For a group element $g \in G$ sampled Haar-randomly, the operator

$$V_g := \frac{1}{r}\left(\frac{1}{2}(\sigma(g) + \sigma^\dagger(g)) - P_{\text{Haar}}\right)$$

is a Hermitian random variable with zero mean. Furthermore, by unitarity of $\rho$ and because $\sigma(g)$ and $P_{\text{Haar}}$ are simultaneously diagonalizable, we have that

$$\|V_g\|_\infty \leq \frac{1}{r}, \quad \|V_g^2\|_\infty \leq \frac{1}{r^2}.$$ 

But then, writing $S = \{g_i\}_{i=1}^r \cup \{g_i^{-1}\}_{i=1}^r$, we see that

$$P_S - P_{\text{Haar}} = \sum_{i=1}^r V_{g_i},$$

where the operators $V_{g_i}$ are independent random variables satisfying the conditions above. Then, by the matrix Hoeffding bound [31],

$$\text{Prob} (\lambda_{\text{max}}(P_S - P_{\text{Haar}}) > x) < n^2 e^{-\frac{x^2}{4}},$$

where $\lambda_{\text{max}}$ is the maximum eigenvalue. Finally, repeating the statement above for $\lambda_{\text{max}}(P_{\text{Haar}} - P_S)$ and using the union bound, we conclude that

$$\text{Prob} (\|P_{\text{Haar}} - P_S\|_\infty > x) < 2n^2 e^{-\frac{x^2}{4}}.$$ 

Using $x = 1/2$ and the fact that $r \geq 8[(\log(1/p_{\text{thr.}}) + 2\log(n))$ we recover the claimed statement.

Proposition 6. Let $\rho_K$ be irreducible, then the probability that Alg. IV.1 returns False upon this input is at most $p'_{\text{thr.}}$.

Proof. By Prop. [3] with probability at least $1 - p_{\text{thr.}}$ it holds that

$$\|P_S^{2t} - P_{\text{Haar}}\|_\infty \leq 2^{-2t},$$

where we used $P_S^{2t} - P_{\text{Haar}} = (P_S - P_{\text{Haar}})^{2t}$ because $P_S$ and $P_{\text{Haar}}$ commute. This way,

$$\text{tr} \, P_S^{2t} \leq \text{tr} \, P_{\text{Haar}} + n_K^2 2^{-2t} \leq \text{tr} \, P_{\text{Haar}} + \frac{1}{16} = \frac{17}{16}.$$ 

Furthermore, by our assumption in $m$, we have $2(1 - \theta_m) \geq 3/2$. But then, the Chernoff bound says that the probability that $E_m \geq 3/2$ is at most

$$\exp\left(\frac{-m}{n_K^2 \cdot 3 \times 256}\right) \leq \exp\left(\frac{-m}{16n_K^2}\right) \leq p'_{\text{thr.}} - p_{\text{thr.}}.$$ 

A false positive can occur if either eq. (4) does not hold, or if conditioned on it holding, $E_m \geq 3/2$. By the union bound, this probability is at most $p_{\text{thr.}} + (1 - p_{\text{thr.}})(p'_{\text{thr.}} - p_{\text{thr.}}) < p'_{\text{thr.}}$. 

B. The noisy case

In this section we adapt the idea presented above to the noisy scenario. Suppose we have certified that a subspace $R \subset \mathbb{C}^n$ is invariant (with precision $\epsilon$). We now wish to certify that the minimizer $K$ of (3) is irreducible.
Our approach uses $Q$ irreducibility certificate. Assume that the minimizer $K$ is irreducible. In fact, this parameter is used in the same way that $\epsilon$, $\epsilon_0$ reduces to Alg. IV.1 in the limit of $\epsilon$, $\epsilon_0 \to 0$, we expect that the false negative rate is well approximated by $\delta_{\text{conf.}}$ when $\epsilon$ and $\epsilon_0$ are small enough. Since the runtime of the algorithm scales with $\max\log(1/p_{\text{thr.}})$, $\log(1/(\delta_{\text{conf.}} - p_{\text{thr.}}))$, a reasonable choice for the confidence parameter is $\delta_{\text{conf.}} = 2p_{\text{thr.}}$.

Within Alg. IV.2 and throughout this section we use the following conventions:

$$c_1 := 2(\epsilon + n\epsilon_0)(1 + \epsilon + n\epsilon_0) + n\epsilon_0(1 + \epsilon + n\epsilon_0)^2,$$
$$c_2 := 2c_1(1 + c_1),$$
$$h_t(x) := (1 + x)^t - 1,$$
$$d_t := h_t(c_2),$$
$$e_t := d_{2t}(\text{int}(\text{tr} \pi_R)^2 + d_{2t}).$$

For the sake of clarity, we have shifted the proofs of several propositions in this subsection to App. A.

### Algorithm IV.2 Irreducibility certificate

**Input:**
- $\pi_R \in \mathbb{C}^{n \times n}$, $\epsilon \in (0, 1/2)$
- $p_{\text{thr.}} \in (0, 1)$
- $\delta_{\text{conf.}}$

**Output:** True/False.

1. if $e_t \geq 2$ then
   2. return False
3. end if
4. Set $r = \max\{r_G, 12(\log(2/p_{\text{thr.}}) + 2\log(n))\}$
5. Set $m = 2\left[\frac{\text{int}(\text{tr} \pi_R)^2 + d_{2t}}{2 - e_t}, \max\{\log(p_{\text{thr.}}^{-1}), 8\log((\delta_{\text{conf.}} - p_{\text{thr.}})^{-1})\}\right]$ $\triangleright G$ generated by $\leq r_G$ elements.
6. Set $t = 2 + \lceil\log_2\text{int}(\text{tr} \pi_R)\rceil$ $\triangleright m$ random walks
7. Sample $r$ elements $g_i \in G$, set $S = \{g_i\} \cup \{g_i^{-1}\}$ $\triangleright 2t$ random walk length
8. Sample $m$ words $s_i \in S^{2t}$ uniformly
9. Compute $E = e_t + \frac{1}{m} \sum_i |\text{tr} \tilde{\rho}_R(s_i)|^2$
10. Set $\theta_m = \sqrt{2\log(1/p_{\text{thr.}})}(\text{int}(\text{tr} \pi_R)^2 + d_{2t})/m(2 - e_t)$
11. if $E < 2(1 - \theta_m)$ then
   12. return True
13. end if
14. return False

**Theorem 3.** Assume that the minimizer $K$ of eq. (3) is reducible. Then the probability that Alg. IV.2 outputs True is at most $p_{\text{thr.}}$.

Similar to the ideal case, the proof of this theorem relies on characterizing the approximate random walk operator $Q_S^R$ given by

$$Q_S^R(\cdot) := \frac{1}{|S|} \sum_{s \in S} \pi_R \tilde{\rho}(s)^\dagger \pi_R(\cdot) \tilde{\rho}(s)^\dagger \pi_R.$$

Our approach uses $Q_S^R$ to upper-bound the dimension of the commutant of $\rho$ restricted to $K$, that is $\text{tr} P_{\text{Haar}}^K$, where

$$P_{\text{Haar}}^K(\cdot) := \int_G d\mu_{\text{Haar}}(g) (\pi_K \rho(g) \pi_K)(\cdot) \pi_K \rho^\dagger(g) \pi_K.$$
An important object in our proof is the restricted random walk operator,

\[ P^K_S(\cdot) := \frac{1}{|S|} \sum_{s \in S} \pi_K \rho(s) \pi_K(\cdot) \pi_K \rho(s) \pi_K. \]

Notice that \( Q^R_S \) is a small perturbation of \( P^K_S \).

**Proposition 7.** Use the notation above, let \( Q_\epsilon := P^K_S - Q^R_S \) and \( \gamma \) be such that \( \|Q_\epsilon\|_\infty \leq \gamma \). Then, for all \( t \) it holds that

\[ \text{tr} P^K_{\text{Haar}} \leq \text{tr}((Q^R_S + \gamma I)^{2t}). \]

**Proof.** Let \( \{r_i\} \) be the eigenvalues of \( P^K_S \). By Weyl’s perturbation theorem, for each \( r_i \), there is some eigenvalue \( q_i \) of \( Q^R_S \) satisfying \( q_i \in r_i \pm \gamma \). In particular, \( Q^R_S + \gamma I \) has \( P^K_{\text{Haar}} \)-many eigenvalues in the range \([1, 1 + 2\gamma]\). Then,

\[ \text{tr}((Q^R_S + \gamma I)^{2t}) \geq \text{tr} P^K_{\text{Haar}} + \sum_{i,s,t} (q_i + \gamma)^{2t} \geq \text{tr} P^K_{\text{Haar}}. \]

\[ \square \]

We will show that \( \|Q_\epsilon\|_\infty \leq c_2 \) in Prop. 11 from App. A and so we use \( \gamma = c_2 \) henceforth. Then, if for any \( t \) it holds that

\[ \text{tr}((Q^R_S + c_2 I)^{2t}) < 2, \]

\( K \) is irreducible. We may expand

\[ \text{tr}((Q^R_S + c_2 I)^{2t}) = \sum_{k=0}^{2t} \binom{2t}{k} c_2^{2t-k} \text{tr}((Q^R_S)^k) \]

\[ = \sum_{k=0}^{2t} \binom{2t}{k} c_2^{2t-k} \frac{1}{|S|^k} \sum_{s \in S^k} |\text{tr} \hat{\rho}_R(s)|^2, \]

where we used,

\[ \hat{\rho}_R(s) := \pi_K \hat{\rho}(s) \pi_R. \]

Our approach is to bound the norm of all terms with \( k < 2t \) and estimate the one with \( k = 2t \). This is because in the regime of interest \( c_2 \) is small, and so terms with non-trivial powers of \( c_2 \) are of subleading order. The following proposition will be used to bound the size of subleading terms.

**Proposition 8.** Let \( R \) hold an invariance certificate with precision \( \epsilon < 1/2 \) and let \( K \) be the minimizer in eq. 3. Then, for any \( s \in S^k \), it holds that

\[ |\text{tr} \hat{\rho}_R(s)|^2 \leq \dim^2 K + d_k. \]

The following proposition uses the previous result to bound the size of the subleading contributions to eq. 6.

**Proposition 9.** Let \( R, K \) and \( \epsilon \) be as in Prop. 8 and let \( n\epsilon_0 < 1/2 \). Then,

\[ \left| \sum_{k=0}^{2t-1} \binom{2t}{k} c_2^{2t-k} \text{tr}((Q^R_S)^k) \right| \leq \epsilon t. \]

We therefore obtain

\[ \text{tr} P^K_{\text{Haar}} \leq \epsilon t + \text{tr}((Q^R_S)^{2t}) = \epsilon t + \frac{1}{|S|^{2t}} \sum_{s \in S^{2t}} |\text{tr} \hat{\rho}_R(s)|^2. \]

All that is left to be shown is that the estimator for the second term used by Alg. IV.2 concentrates sharply around its mean. For this we will use the following proposition, a simple consequence of the Chernoff bound.
Proposition 10. Let $R$, $K$ and $\epsilon$ be as in Prop. 8 and assume that $K$ is reducible. Let $\{s_i\}$ be $m$ uniformly random samples from $S^{2r}$. Then, for any $\theta \in (0, 1)$, it holds that

$$\Pr \left[ \frac{1}{m} \sum_{i=1}^{m} |\tr \hat{\rho}_R(s_i)|^2 \leq (1 - \theta) \tr((Q_S^R)^{2r}) \right] < \exp \left( \frac{-\theta^2 m(2 - \epsilon_t)}{2(\dim^2 K + d_{2r})} \right).$$

We may now prove the first main result of this subsection.

Proof of Thm. 10 By our assumption on $m$, it holds that $\theta_m < 1$. But then using Prop. 10 with $\theta = \theta_m$,

$$\Pr \left[ \frac{1}{m} \sum_{i} |\tr \hat{\rho}_R(s_i)|^2 + \epsilon_t \leq 2(1 - \theta_m) \right] \leq \Pr \left[ \frac{1}{m} \sum_{i} |\tr \hat{\rho}_R(s_i)|^2 + \epsilon_t \leq (1 - \theta_m) \tr((Q_S^R)^{2r}) + \epsilon_t \right]$$

$$\leq \Pr \left[ \frac{1}{m} \sum_{i} |\tr \hat{\rho}_R(s_i)|^2 \leq (1 - \theta_m) \tr((Q_S^R)^{2r}) \right]$$

$$< \exp \left( \frac{-\theta^2 m(2 - \epsilon_t)}{2(\dim^2 K + d_{2r})} \right) < p_{thr}.$$

\[ \square \]

V. TIME COMPLEXITY

Here we analyse the runtime of the certification procedures proposed and discuss several ways to optimize it.

Alg. 11 runs in $O(n^3 \log n)$ steps: the main sources of complexity are the $r = O(\log n)$ matrix products and the spectral norm appearing in line 3. The latter has complexity at most $O(n^3)$ through the singular value decomposition.

In practice, this last step is significantly cheaper. Ref. 33 estimates the spectral norm in time $O(n^2 \log n)$. Note that the method of 33 is probabilistic and so it raises the false positive rate, albeit in a controllable way. Alternatively, the spectral norm can be bounded by the Frobenius norm in $O(n^2)$ operations.

To compute the runtime of Alg. 12 we assume that $\epsilon_t$ and $\epsilon$ are small enough that $d_{(2 + \log_2 d)}$ and $d_{2 + \log_2 d}$ are non-increasing functions of $d := \dim R$ and $n$. Here, $d_t$ and $\epsilon_t$ are defined as in the top of Sec. IV.B and we use $t = 2 + \log d$. For this it is sufficient to take

$$\epsilon < \frac{1}{48(d^2 + 1)(2 + \log_2 d)}, \quad \epsilon_0 < \frac{1}{120n(d^2 + 1)(2 + \log_2 d)}.$$

In this regime the runtime of the algorithm, as it is written in the main text, is

$$O \left( n^3 d^2 \log d \left( \log \frac{1}{p_{thr}} + \log \frac{1}{\delta_{\text{conf.}} - p_{thr}} \right) \right).$$

Because the false negative rate is of secondary importance for our certificate, a convenient choice is $\delta_{\text{conf.}} = 2p_{thr}$, where both terms above have the same scaling.

The main bottleneck of 13 is the $n^3$ factor, coming from the fact that the algorithm evaluates matrix products on $C^{n \times n}$. This can be significantly reduced by either: 1. taking products in the group and then obtaining the image, or 2. restricting matrices $\hat{\rho}_R(s)$ to the subspace $R$ first, and taking products in this smaller space. Letting $D$ denote the runtime of whichever of these two is faster, the runtime becomes $O(Dd^2 \log d \log p_{thr}^{-1})$.

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Appendix A: Proofs

Proposition 11. Let $Q_c$ be as in Prop. A.9 and $c_2$ be as in the beginning of Sec. IV.B. Then $\|Q_c\|_\infty \leq c_2$.

Proof. Let $\rho_K(s) := \pi_K \rho(s) \pi_K$ and $D(s) := \hat{\rho}_R(s) - \rho_K(s)$. Using subadditivity, we bound
\[
\|Q_c\|_\infty \leq \max_s \|D(s) \otimes \hat{\rho}_K(s) + \rho_K(s) \otimes \hat{D}(s) + D(s) \otimes \hat{D}(s)\|_\infty \leq \max_s (2\|D(s)\|_\infty + \|D(s)\|_\infty^2).
\]

Further writing $\Delta := \hat{\pi}_R - \pi_K$ and $\Delta(s) := \hat{\rho}(s) - \rho(s)$, we observe that
\[
D(s) = \Delta \rho(s) (\pi_K + \Delta)^\dagger + (\pi_K + \Delta) \rho(s) \Delta^\dagger + (\pi_K + \Delta) \Delta(s)(\pi_K + \Delta)^\dagger,
\]
and so,
\[
\|D(s)\|_\infty \leq 2\|\Delta\|_\infty (1 + \|\Delta\|_\infty) + \|\Delta(s)\|_\infty (1 + \|\Delta\|_\infty)^2.
\]

We can directly bound $\|\Delta(s)\|_\infty \leq n\epsilon_0$. Then, because $R$ holds an invariance certificate with precision $\epsilon$, we deduce
\[
\|\Delta\|_\infty \leq n\epsilon_0 + \epsilon.
\]

It follows that $\|D(s)\|_\infty \leq c_1$, where $c_1$ is defined as in the top of Sec. IV.B and the claim follows. \hfill \Box

Proof of Prop. A.8. As in the proof of Prop. A.11 let $D(s) := \hat{\rho}_R(s) - \rho_K(s)$. For the sake of convenience, let us introduce the following notation: $B_1(s) = D(s)$, $B_0(s) = \rho_K(s)$, and for any bit string $v \in \mathbb{F}_2^k$ and $s \in S^k$, $B_v(s) = B_{v_1}(s_1) B_{v_2}(s_2) \cdots B_{v_k}(s_k)$.

Then, using submultiplicativity, subadditivity and unitary invariance we find that
\[
|\text{tr}(\hat{\rho}_R(s))|^2 \leq \sum_{v \in \mathbb{F}_2^k} |\text{tr} B_v(s)|^2
\]
\[
\leq \sum_{v \in \mathbb{F}_2^k} \|B_v(s)\|^2_{\mathcal{F}}
\]
\[
\leq \dim^2 K + \sum_{v \neq 0} \max_s \|D(s)\|^w_{\mathcal{F}}
\]
\[
\leq \dim^2 K + \sum_{w=1}^k \binom{k}{w} \max_s \|D(s)\|^w_{\mathcal{F}}
\]
\[
\leq \dim^2 K + (1 + \max_s \|D(s)\|_{\mathcal{F}})^k - 1,
\]
where $w(v)$ denotes the Hamming weight of $v$. Then, because $R$ holds an invariance certificate with precision $\epsilon$, we may use an argument analogous to the proof of Prop. A.11 to bound $\max_s \|D(s)\|_{\mathcal{F}}$ by $c_1$ (defined in the top of Sec. IV.B). This finalizes the proof. \hfill \Box

Proof of Prop. A.9. We begin by observing that $d_k \leq d_{2t}$ for all $k \leq 2t$, and so Prop. A.8 implies
\[
\left| \sum_{k=0}^{2t-1} \binom{2t}{k} c_2^{2t-k} \text{tr}((Q_c^R)^k) \right| \leq [(1 + c_2)^{2t} - 1](\dim^2 K + d_{2t}).
\]

Since $\epsilon < 1/2$, $\dim K = \dim R$. Finally, $n\epsilon_0 < 1/2$ implies that $\text{int}(\pi_R \hat{\pi}_R) = \text{tr} \pi_R = \dim R$. \hfill \Box
Proof of Prop. 10 By Prop. 8 \( |\text{tr } \hat{\rho}_R(s_i)|^2 / (\dim^2 K + d_{2t}) \) is a random variable in \([0, 1]\), so Chernoff’s bound gives

\[
\Pr \left[ \frac{1}{|I|} \sum_i |\text{tr } \hat{\rho}_R(s_i)|^2 \leq (1 - \theta) \text{tr}((Q^g_S)^{2t}) \right] < \exp \left( -\frac{\theta^2 m \text{tr}((Q^g_S)^{2t})}{2(\dim^2 K + d_{2t})} \right)
\]

But by Prop. 7 \( \text{tr}((Q^g_S + c_2 \mathbb{I})^{2t}) \geq \text{tr}P^K_{\text{Haar}} \geq 2 \), and by Prop. 9 \( \text{tr}((Q^g_S)^{2t}) \geq 2 - \epsilon_t \), which finishes the proof. \( \square \)

Appendix B: Extension to a weaker scenario

Here we show how to modify our algorithms to a setting in which the user has considerably less control over the group than is assumed in the main text. To keep the the line of argument clean, we provide only short proof sketches for the claimed statements, and include these at the end of the appendix. In the following, the Lie algebra \( \mathfrak{g} \) of \( G \) is endowed with a \( G \)-invariant inner product \( \langle \cdot, \cdot \rangle_{\mathfrak{g}} \) and a corresponding 2-norm \( \| \cdot \|_{\mathfrak{g}} \).

In the current setting, the user is assumed to know \( \hat{\rho} \) evaluated on a fixed symmetric generator set \( S \). The set \( S \) and the representation \( \rho \) must also satisfy two requirements.

The first is that \( S \) is not too ‘ill-conditioned’: We say that \( S \) is \((\delta, k)\)-dense, if for any \( g \in G \) there exists a word \( s_1 \cdots s_k \) of length \( k \) in \( S \) for which

\[
\| \log g^{-1} s_1 \cdots s_k \|_{\mathfrak{g}} \leq \delta.
\]

The second requirement is that the \( \rho \)-images of close-by group elements are also close-by. That is, we say that \( \rho \) is \( q \)-bounded if it holds that

\[
\|d\rho(X)\|_F \leq q \|X\|_{\mathfrak{g}}, \quad \forall X \in \mathfrak{g},
\]

where \( d\rho \) is the representation of \( \mathfrak{g} \) corresponding to \( \rho \). In summary, we assume that the user knows some numbers \((\delta, k, q)\) such that \( S \) is \((\delta, k)\)-dense and \( \rho \) is \( q \)-bounded (we say that \((G, S, \rho)\) is \((\delta, k, q)\)-well conditioned).

In the case \( G \) is finite, one may take \( k \) to be the Cayley diameter and \( q = \delta = 0 \). When \( G \) is continuous, to the best of our knowledge there are no explicit generator sets \( S \) known to be \((\delta, k)\)-dense. For special unitary groups, the Solovay-Kitaev theorem provides an asymptotic result: certain generator sets are \((\delta, O(\log^4 \delta^{-1}))\)-dense. In the case of \( SU(2) \), some progress towards an explicit scaling for the Solovay-Kitaev theorem has been made in [34].

Remark 1. One can modify the algorithms presented here to use a bound on the spectral gap \( \|P_S - P_{\text{Haar}}\|_\infty \) as an input instead of \((\delta, k, q)\). However, such a bound is rarely known without diagonalizing \( P_S \). While results stating the existence of a gap exist for a variety of compact groups, these do not quantify how large it is (e.g. [35–37]). Because of this, we do not present such a modification.

1. Invariance certificate

The invariance certificate in this scenario is given by Alg. B.1, where we use

\[
f(x) = 2\sqrt{2} \dim R(xk + 2kne_0(n\epsilon_0 + 1) + 2q\delta \exp(q\delta) + 2n\epsilon_0).
\]

Algorithm B.1 Modified invariance certificate

**Input:**
- \( \{\hat{\rho}(s) : s \in S\} \subset \mathbb{C}^{n \times n}, \)
- \( \delta \in (0, 1), k \in \mathbb{N}, q \in \mathbb{R}_+, \)
- \( \tilde{\pi}_R \in \mathbb{C}^{n \times n}, \)
- \( \epsilon \in (0, 1/2). \)

**Output:** True/False

1. Let \( f \) be defined as in eq. (B1)
2. if \( f(\max_{s \in S} \|\hat{\rho}(s), \tilde{\pi}_R\|_F) \leq \epsilon \) then
3. Return: True
4. end if
5. Return: False
As in the main text, the key quantity to be bounded is \( \| P_{\text{Haar}}(\pi_R) - \pi_R \|_F \). This is achieved by the following two propositions.

**Proposition 12.** Let \((G, S, \rho)\) be \((\delta, k, q)\)-well conditioned and assume that
\[
\| \tilde{\rho}(s) \tilde{\pi}_R \|_F \leq c_3, \quad \forall s \in S.
\]

Then, for all \(g \in G\) we have that
\[
\| P_{\text{Haar}}(\pi_R) - \pi_R \|_F \leq kc_3 + 2kn\epsilon_0(n\epsilon_0 + 1) + 2q\delta \exp(q\delta) =: c_4(c_3).
\]

Putting this together with Prop. 3 shows that if Alg. B.1 returns True, then \(R\) is approximately invariant up to precision \(\epsilon\).

### 2. Irreducibility certificate

We now move on to the irreducibility certificate. For simplicity we only present the procedure in the ideal case, given by Alg. B.2. The certificate is in essence the same as Alg. IV.1 with the prominent difference that \(S\) is not sampled at the start. The proof of Thm. 2 carries over exactly to the current case showing that this algorithm’s false positive rate is at most \(p_{\text{thr.}}\).

Alg. B.2 furthermore includes the parameter \(t\) as an input (compare line 3 of Alg. IV.1). This choice is made for the sake of performance. Specifically, in Prop. 13 we bound the false negative rate whenever \(t\) is large enough —this is in the same spirit as Prop 6. Here, though, the bound on \(t\) is too large to be useful in many practical settings.

Rather than using Prop. 13 to choose \(t\), we have instead tested the performance of the algorithm for different values of \(t\) (see [38]). There it is found that, for a variety of finite group representations, taking \(t \gg k\) is sufficient to bring the empirical false negative rate down to zero.

**Algorithm B.2 Modified ideal irreducibility certificate**

**Input:**
- \(\{\rho_K(s) : s \in S\} \subset \mathbb{C}^{n_K \times n_K}\),
- \(p_{\text{thr.}} \in (0, 1)\),
- \(t \in \mathbb{N}\).

**Output:** True/False.

1: Set \(m = 3[n_K^2 \log(1/p_{\text{thr.}})] + 1\)
2: Set \(\theta_m = n_K \sqrt{\frac{2 \log(1/p_{\text{thr.}})}{m}}\)
3: Compute \(E_m = \frac{1}{m} \sum_{i=1}^{m} |\text{tr} \rho_K(s_i)^2|\), with \(s_i \in S^{2t}\) sampled uniformly
4: if \(E_m < 2(1 - \theta_m)\) then
5: \(\text{return True}\)
6: end if
7: \(\text{return False}\)

We thus conclude by analysing the false negative rate of Alg. B.2. This probability is intimately related to the spectral gap of \(P_K\), —the mixing time of random walks in \(S\). Here, we show how to obtain a bound on this spectral gap from the parameters \((\delta, k, q)\). This result follows from Ref. [39, Lemma 5] up to some minor technical detail.

**Proposition 13.** There exists a constant \(c_0\) such that for any compact group \(G\), generator set \(S \subset G\) and irreducible representation \(\rho_K\) the following holds. If \((G, S, \rho_K)\) is \((\delta, k, q)\)-well conditioned with \(\delta \leq (c_0 q)^{-c_0}\), then for any \(t \geq \frac{1}{2} \log u - 1\), it holds that the probability that Alg. B.2 returns False upon this input is at most
\[
\exp\left(\frac{-m}{3 \dim^2 K} \left(\frac{2 - \theta_m}{1 + (n - 1)(1 - k^{-2}\epsilon)^{2t} - 1}\right)^2\right).
\]
Our approach is the following. Ref. [39, Lemma 5] gives a bound on this spectral gap as a function of $\delta$, $k$ and a third parameter, the maximal weight length defined by

$$\max \left\{ \|\omega\|_{g^*}^2 \mid \omega \text{ weight in } \rho_K \right\}.$$ 

The following proposition relates this quantity to our parameter $q$, which in turn allows us to obtain a bound on the mixing time in terms of $(\delta, k, q)$.

**Proposition 14.** Let $(K, \rho_K)$ be a unitary representation of $G$ with maximal weight length equal to $w$. Then

a) $\rho_K$ is $\sqrt{w \dim K}$-bounded,

b) if $\rho_K$ is $q$-bounded, then $q$ must satisfy $q \geq w$.

### 3. Proofs

**Proof of Prop. [12]** We directly compute that for all $s \in S$

$$\|\left[ \rho(s), \pi_R \right] \|_F \leq c_3 + 4n\epsilon_0 + 2n^2\epsilon_0^2 = c_5.$$ 

Similarly, for any $s \in S^k$,

$$\| [\rho(s), \pi_R] \|_F \leq kc_5,$$

where we used the identity $[AB, C] = A[BC] + [A, C]B$ iteratively.

Now, let $g \in G$ be arbitrary. By assumption, there exists a word $g_s := s_1 \cdots s_k$ in $S$, together with an element $g_X := \exp(X)$ for which

$$g = g_sg_X,$$

$$\|X\|_g \leq \delta.$$

Subadditivity and submultiplicativity imply that

$$\| \rho(g) - \rho(g_s) \|_F = \| \exp d\rho(X) - I \|_F \leq \| d\rho(X) \|_F \| \exp(\| d\rho(X) \|_F) \|_F \leq q\delta \exp(q\delta),$$

and so

$$\| [\rho(g), \pi] \|_F \leq 2q\delta \exp(q\delta) + kc_5 = c_4, \quad \forall g \in G.$$ 

Finally, we may use the unitarity of $\rho$ to obtain

$$\| P_{\text{Haar}}(\pi_R) - \pi_R \|_F \leq \mathbb{E}_{g \sim G} \left[ \| [\rho(g), \pi_R] \|_F \right],$$

which proves the claim. 

**Proof of Prop. [14]** Let $\{\omega_i\}$ be the set of weights appearing in $\rho_K$, let $\omega_0$ be a weight in that set with maximal length (so $\|\omega_0\|^2_{g^*} = w$) and let $t$ be the Lie algebra of the maximal torus in $G$. We begin by noting that because $\|\cdot\|_g$ is invariant under the adjoint $G$-action, we know that

$$\sup_{X \in t} \frac{\|d\rho_K(X)\|_F^2}{\|X\|_g^2} = \sup_{X \in t} \frac{\|d\rho_K(X)\|_F^2}{\|X\|_g^2}.$$ 

For any $X \in t$,

$$\|d\rho_K(X)\|_F^2 = \sum_i |\omega_i(X)|^2 = \sum_i |\omega_i^*, X|_g|^2,$$  

(B2)
where $\omega_i^*$ is the dual of $\omega_i$ with respect to the invariant inner product. Using Cauchy-Schwarz on eq. (B2) we obtain

$$
\|d\rho_K(X)\|_F^2 \leq \|X\|_g^2 \sum_i \|\omega_i^*\|_g^2 \leq (w \dim K) \|X\|_g^2,
$$

which proves the first statement.

For the second statement, let us choose $X = \omega_0^*/\|\omega_0^*\|_g$ in eq. (B2). We obtain

$$
\|d\rho_K(X)\|_F^2 = \sum_i \frac{|\langle \omega_i^*, \omega_0^* \rangle|_g^2}{\|\omega_0^*\|_g^2} \geq \|\omega_0^*\|_g^2 = w.
$$

But $\|X\|_g = 1$ so any $q \leq w$ would be inconsistent with the equation above.

Proof of Prop. 13. By Prop. 14, the maximal weight-length $r$ of $\rho_K$ can be at most $q$. Consider the random walk operator $P_S$ associated to $\rho_K$ and let $\lambda$ be the spectral norm of the restriction of $P_S$ to the traceless subspace, —by the assumption that $\rho_K$ is irreducible, we know that $\lambda < 1$.

Ref. [39, Lemma 5] implies that there exists a universal constant $c_0 > 0$ such that if $\delta \leq (c_0 q)^{-c_0}$, then

$$
1 - \lambda \geq \frac{1}{|S|k^2}.
$$

Hence,

$$
\text{tr} P_S^{2t} \leq 1 + (n - 1) \left(1 - \frac{1}{|S|k^2}\right)^{2t}.
$$

(B3)

Then, for any $x \leq 1$, the right-hand side is smaller than $2 - x$ if and only if

$$
t \geq \frac{1}{2} \log \frac{n - 1}{x} =: t_x.
$$

Equivalently, for any $t$ given as in the assumption of the theorem, the right-hand side of (B3) is at most $2 - x_t$, where

$$
x_t := 1 - (n - 1)(1 - 1/|S|k^2)^{2t}.
$$

The Chernoff bound implies that for any $\alpha > 0$, if $\{s_i\}$ are $m$ uniform samples from $S^{2t}$, then

$$
\text{Prob} \left[ \frac{1}{m} \sum_i |\text{tr} \rho_K(s_i)| \geq (2 - x_t)(1 + \alpha) \right] \leq \exp(-\alpha^2 m/3 \dim^2 K).
$$

(B4)

Consider the choice

$$
\alpha = \frac{2 - \theta_m}{2 - x_t} - 1,
$$

where $\theta_m$ is as in Line 1 of Alg. [IV.1]. Then, eq. (B4) becomes

$$
\text{Prob} \left[ \frac{1}{m} \sum_i |\text{tr} \rho_K(s_i)| \geq (2 - x_t)(1 + \alpha) \right] \leq \exp\left(\frac{m}{3 \dim^2 K} \left(\frac{2 - \theta_m}{2 - x_t} - 1\right)^2\right).
$$

(B5)