ON THE GENERIC PART OF THE COHOMOLOGY OF LOCAL
AND GLOBAL SHIMURA VARIETIES

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Abstract. Using the work of Fargues-Scholze, we prove a vanishing theorem for the generic unramified part of the cohomology of local Shimura varieties of general linear groups. This gives an alternative approach to vanishing results of Caraiani-Scholze for the cohomology of unitary Shimura varieties.

1. Introduction

1.1. Local vanishing. Let $F$ be a finite extension of $\mathbb{Q}_p$ for some $p$, with the residue field isomorphic to $\mathbb{F}_q$. Take a local Shimura datum $(G, b, \mu)$ over $F$ with $G$ being a product of general linear groups $\text{GL}_{n_i}$ over $F$ indexed by a finite set $I$. Let $E \subset F$ denote the reflex field and put $C := \hat{E}$. We will work with a hyperspecial open compact subgroup $K := \prod_{i \in I} \text{GL}_{n_i}(\mathcal{O}_F) \subset G(F)$, and let $\mathcal{M}_{(G, b, \mu), K}$ denote the corresponding local Shimura variety of level $K$ [SW20, 24.1.3]. Consider the compactly supported cohomology $R\Gamma_c(\mathcal{M}_{(G, b, \mu)}, K; \mathbb{Z}_\ell)$ for $\ell \neq p$ (taken after base change to $C$), equipped with left action of $J_b(F)$, and right action of the Hecke algebra $H_K := \mathbb{Z}_\ell[\mathcal{K} \backslash G(F)/\mathcal{K}]$. (The Weil group $\mathcal{W}_E$ also acts but this will not be relevant to us.)

Suppose $m \subset H_K$ is the maximal ideal corresponding to a character $\lambda_m : H_K \to \mathbb{F}_\ell$. We have the associated unramified $L$-parameter $\rho_m : \text{Frob}_F \to \prod_{i \in I} \text{GL}_{n_i}(\mathbb{F}_\ell)$; this is just determined by a conjugacy class of semisimple elements. Let us say that $\rho_m$ is generic if, for every $i \in I$, the eigenvalues $\alpha_1, \ldots, \alpha_{n_i}$ of $\rho_m(\text{Frob}_F)$ regarded as an element of $\text{GL}_{n_i}(\mathbb{F}_\ell)$ satisfy $\alpha_j/\alpha_j' \neq q$ for all $j \neq j'$.

Theorem 1.1. If $\rho_m$ is generic and $J_b$ is not quasi-split, then the localized cohomology $H^i_c(\mathcal{M}_{(G, b, \mu), K}; \mathbb{Z}_\ell)_m$ vanishes for every integer $i$.

The work of Fargues-Scholze [FS] is fundamental in our proof. In some sense, they prove that “Jacquet-Langlands transfer” appears in the cohomology of local Shimura varieties even for mod $\ell$ coefficients. We also use the results of [HKW] and [MS14b] to relate it to the actual Jacquet-Langlands correspondence in characteristic 0. The genericity assumption implies that $\lambda_m$ is not in the image of “Jacquet-Langlands transfer”. (The underlying idea is loosely related to some arguments in [CS17, CS], e.g., [CS17, 5.4.3].) It is possible to prove a variant for not necessarily generic representations by putting more restriction on $J_b$ instead; this
is related to the work of Boyer on the Harris-Taylor Shimura variety [Boy19], and will be discussed elsewhere.

1.2. Application to unitary Shimura varieties. We can use Theorem 1.1 to give an alternative argument for vanishing theorems of [CS17]. To be precise, our argument only uses their results on the geometry of the Hodge-Tate period map.

Let $F$ be a CM field. Let $(B, *, V, (\cdot, \cdot))$ be a PEL datum of type A with $B$ being a central simple $F$-algebra, and let $(G, X)$ denote the associated Shimura datum. Take a prime $p$ that completely splits over $F$. In particular, choosing suitable places $v_1, \ldots, v_a$ of $F$ dividing $p$, we have an isomorphism

$$G_{Q_p} \cong G_\mathbb{m} \times \prod_i G_{v_i},$$

where $G_{v_i}$ is a product of general linear groups over $F_{v_i} (\cong Q_p)$. Take a hyperspecial open compact subgroup $K_p = \mathbb{Z} \times \mathbb{Z}_p \times \prod K_{v_i}$ of $G(Q_p)$. The Hecke algebra at $p$

$$H_{K_p} := \mathbb{Z}_p[G(Q_p)/K_p]$$

acts via right on the cohomology of Shimura variety

$$H^{i}(S_K, \overline{\mathbf{F}}_\ell), \quad H'_{c}(S_K, \overline{\mathbf{F}}_\ell)$$

for sufficiently small open compact subgroup $K = K_pK^p \subset G(Q_p) \times (K^p)$. (We are considering the right action, but it is common in the literature to consider the left action obtained by composing the right action with an involution of $H_{K_p}$.) Set $d := \dim S_K$.

**Conjecture 1.2.** Let $\lambda_{m_p} : H_K \to \mathbf{F}_\ell$ denote the character corresponding to a maximal ideal $m_p \subset H_{K_p}$ such that $\rho_{m_p}$ is generic. If $H^{i}(S_K, \overline{\mathbf{F}}_\ell)_{m_p} \neq 0$, then $i \geq d$. If $H'_{c}(S_K, \overline{\mathbf{F}}_\ell)_{m_p} \neq 0$, then $i \leq d$.

Let us restrict ourselves to the settings of [CS17].

**Theorem 1.3.** Assume $B = F$, $V = F^{2n}$, and $G$ is a quasi-split similitude unitary group. Then, Conjecture 1.2 holds true.

**Theorem 1.4.** Assume $G$ is anisotropic modulo center. Then, Conjecture 1.2 holds true: if $H^{i}(S_K, \overline{\mathbf{F}}_\ell)_{m_p} \neq 0$, then $i = d$.

**Theorem 1.3** (resp. **Theorem 1.4**) is proved in [CS] (resp. [CS17]) after localizing at a maximal ideal of the global Hecke algebra under some assumptions, and their version of Theorem 1.2 plays a very important role in the application to (potential) automorphy [ACC⁺]. Note however that our statement only uses the local Hecke algebra at $p$. Our method uses neither trace formula nor detailed analysis of the cohomology of Igusa varieties and their boundaries, while we still crucially relies on the semiperversity result of Caraiani-Scholze [CS Theorem 4.6.1]. (This is the only reason we work in the setting of [CS]. Once their results on the geometry of Igusa varieties are generalized, Conjecture 1.2 can be proved by our method.)

Let us describe the strategy of our proof. We start with the following form of Mantovan’s formula (see Theorem 7.1): $R\Gamma(S_K, \overline{\mathbf{F}}_\ell)_{m_p}$ admits a $H_{K_p}$-equivariant
filtration (in the derived sense) whose graded pieces are

\[ R\Gamma(Ig^b, F_\ell)^{op} \otimes_{C_\ell(J_0(K_p))} R\Gamma_c(M(G, b, \mu), \infty, F_\ell(d_b))_{m_p}[2d_b] \cong \\
(R\Gamma_c(Ig^b, F_\ell(d_b))^* \otimes_{C_\ell(J_0(K_p))} R\Gamma_c(M(G, b, \mu), \infty, F_\ell(d_b))_{m_p} \]

for \( b \in B(G_{Q_p}, \mu^{-1}) \), where \( Ig^b \) denotes the corresponding Igusa variety, \( d_b = \dim Ig^b \), and \((-)^* \) denotes the smooth dual.

The first step is to use Theorem 1.1.

**Proposition 1.5.** If \( b \) is not ordinary,

\[ R\Gamma(Ig^b, F_\ell)^{op} \otimes_{C_\ell(J_0(Q_p))} R\Gamma_c(M(G_{Q_p}, b, \mu), \infty, F_\ell(d_b))_{m_p} \cong 0. \]

**Proof.** To apply Theorem 1.1 use that \( J_b \) is quasi-split if and only if \( b \) is ordinary in the current setting; cf. proofs of [CS17] 5.5.4, 5.5.5. Also note that the twist \( F_\ell(d_b) \) does not affect the Hecke action. \( \square \)

It remains to work with the ordinary term; let \( b_0 \) denote the ordinary element. For this, we recall the Hodge-Tate period map, which is actually hidden in Mantovan’s formula. Let \( S_{K^p} \) denote the perfectoid Shimura variety of level \( K^p \). Let \( S_{K^p}^0 \) denote the good reduction locus, which lives over the adic generic fiber of the formal completion of the integral model of level \( K \). Let \( F_\ell \) denote the flag variety of \( G_{Q_p} \) associated with \( \mu \) (or \( \mu^{-1} \), depending on the sign convention). We have the \( G(Q_p) \)-equivariant Hodge-Tate period maps [CS17] 2.1.3:

\[ \pi_{HT}: S_{K^p} \to F_\ell, \quad \pi_{HT}^0: S_{K^p}^0 \to F_\ell \]

as diamonds; the second is just the restriction of the first. We also consider their quotients by \( K_p \), and use the same notation. We now want to show that

\[ R\Gamma_c([F_\ell(Q_p)/K_p], i^{b_0*} R\pi_{HT*}^0 F_\ell)_{m_p} \]

sits in degree \( \geq d \); here, \( F_\ell(Q_p) \) is the ordinary locus and

\[ i^{b_0}: [F_\ell(Q_p)/K_p] \hookrightarrow [F_\ell/K_p] \]

is a closed immersion of v-stacks. The key result from [CS] is

**Theorem 1.6 (CS).** Let \( C \) be any complete algebraically closed nonarchimedean extension of \( Q_p \). Then, \( R\pi_{HT*} F_\ell \) belongs to \( \mathcal{O}^\perp X, F_\ell \); the precise claim involves integral models and nearby cycles.

We can deduce from this that

\[ R\Gamma_c([F_\ell(Q_p)/K_p], R^{b_0} R\pi_{HT*}^0 F_\ell) \]

lives in degree \( \geq d \). The final step is to kill the contribution from the non-ordinary loci after localizing at \( m_p \):

**Proposition 1.7.** A natural map

\[ R\Gamma_c([F_\ell(Q_p)/K_p], R^{b_0} R\pi_{HT*}^0 F_\ell)_{m_p} \to R\Gamma_c([F_\ell(Q_p)/K_p], i^{b_0*} R\pi_{HT*}^0 F_\ell)_{m_p}. \]

is an isomorphism.
This is proved again using the machinery of [FS]. There is a $H_{K_p}$-equivariant filtration on $Ri^{bh_1}_!R\pi^\circ_{HT,*}\mathbb{F}_\ell$ whose graded pieces are

$$Ri^{bh_1}_!b^*_iR\pi^\circ_{HT,*}\mathbb{F}_\ell$$

where $i^b: [\mathcal{F}^b/K_p] \hookrightarrow [\mathcal{F}/K_p]$ is a locally closed embedding. The piece from $b_0$ is exactly $i^{b_0}_*R\pi^\circ_{HT,*}\mathbb{F}_\ell$. Now, it suffices to control the composite $Ri^{bh_1}_!l^*_b$ for a non-ordinary $b$. This can be reduced to an analogous problem in the context of the moduli stack $\text{Bun}_{G_{Q_p}}$ of $G_{Q_p}$-bundles on the Fargues-Fontaine curve studied in [FS]: one can show that $Ri^{bh_1}_!l^*_bR\pi^\circ_{HT,*}$ comes from the corresponding object from $\text{Bun}_{G_{Q_p}}$ as $i^{b_0}_*R\pi^\circ_{HT,*}$ comes from the strata $\text{Bun}_{G_{Q_p}}^{b}$ corresponding to $b$. Thus, we can write the contribution from $b$ in the form of

$$(Ri^{bh_1}_!l^*_b(-))^\op \otimes_{C_\ell}(\mathcal{M}_{G_{Q_p},b_0,\mu},K_p,\mathbb{F}_\ell(d))_{\mathfrak{m}_p}$$

for $i^b: \text{Bun}_{G_{Q_p}}^{b} \hookrightarrow \text{Bun}_{G_{Q_p}}$. To show that such contribution vanishes, we look at actions of excursion operators introduced in [FS]: the action on the right term is “generic” by the assumption on $\mathfrak{m}_p$, while the action on the left term cannot be “generic” as it comes from the non-split group $J_b$.

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2. **Mod $\ell$ representations of inner forms of general linear groups**

Let $F$ be a finite extension of $Q_p$ with residue field $F_q$. Let $G$ be an inner form of $\text{GL}_n$ over $F$. Let $\ell$ be a prime different from the residue characteristic of $F$ and fix $q^{1/2} \in \mathbb{F}_\ell$. Let $\text{Rep}_{\mathbb{F}_\ell}(G)$ denote the category of smooth $\mathbb{F}_\ell$-representation of $G(F)$. We will recall some results on $\text{Rep}_{\mathbb{F}_\ell}(G)$.

First recall that an irreducible representation $\pi$ of $G(F)$ is called cuspidal (resp. supercuspidal) if $\pi$ is not a quotient (resp. subquotient) of any properly parabolically induced representation. Two notions are different in general. The pair $(M,\sigma)$ consisting of a Levi subgroup of $G$ and a supercuspidal irreducible representation $\sigma$ of $M(F)$ is the supercuspidal support of $\pi$ if $\pi$ is a subquotient of the normalized parabolic induction $i^G_M\sigma$; such $(M,\sigma)$ always exists. (For the current $G$, the pair $(M,\sigma)$ is known to be unique up to $G$-conjugacy [MS14a Théorème A].)

We need the following result on supercuspidal representations:

**Theorem 2.1 ([MS14a 3.27, 3.28]).** Any supercuspidal irreducible $\mathbb{F}_\ell$-representation admits a lift to a supercuspidal irreducible $Q_\ell$-representation.

3. **A lemma on $L$-parameters**

Using the results explained in the previous section, we prove a lemma on $L$-parameters.

Let $F$ be a finite extension of $Q_p$ with residue field $F_q$. Let $G$ be a product of general linear groups over $F$. Fix $b \in B(G)$, and let $J_b$ denote the associated algebraic group over $F$ as usual.

Fix $\ell$ different from $p$ and $q^{1/2}$ in $\mathbb{Z}_\ell$. Let $L$ be an algebraically closed field over $\mathbb{Z}_\ell$. For any irreducible smooth $L$-representation $\pi$ of $J_b(F)$, Fargues-Scholze
\[\text{(FS) IX.4.1, IX.7.1]}\] constructed the (semisimple) \(L\)-parameter well-defined up to \(\hat{J}_b(L)\)-conjugacy:
\[\varphi_\pi : W_F \to \hat{J}_b(L) \times W_F \hookrightarrow \hat{G}(L) \times W_F,\]
where the embedding is twisted as in \[\text{(FS) IX.7.1}]. (Note that \(\pi\) is known to be Schur-irreducible.)

Let us recall some properties of \(L\)-parameters.

1. The construction is compatible with products of groups \[\text{(FS) IX.6.2}]: \text{if} \ G = G_1 \times G_2, b = (b_1, b_2), \text{and} \ \pi = \pi_1 \boxtimes \pi_2, \text{then} \ \varphi_\pi = (\varphi_{\pi_1}, \varphi_{\pi_2}) \text{as homomorphisms to} \ \hat{J}_b(L) = \hat{J}_{b_1}(L) \times \hat{J}_{b_2}(L).

2. The construction is compatible with the parabolic induction \[\text{(FS) IX.7.3}]: \text{if} \ \pi \text{is a subquotient of a unnormalized parabolically induced representation} \ \text{Ind}_M^L \sigma, \ \text{then} \ \varphi_\pi \text{is obtained from} \ \varphi_\sigma \text{via the twisted embedding} \ \hat{M}(L) \hookrightarrow \hat{J}_b(L). \ \text{Equivalently, if} \ \pi \text{is a subquotient of a normalized parabolically induced representation} \ i_M^L \sigma, \ \text{then} \ \varphi_\pi \text{is obtained from} \ \varphi_\sigma \text{via the untwisted embedding} \ \hat{M}(L) \hookrightarrow \hat{J}_b(L).

3. The construction is compatible with the mod \(\ell\) reduction \[\text{(FS) IX.5.2}]: \text{if} \ \pi \text{is an irreducible} \ \mathbf{F}_\ell\text{-representation and obtained as the mod} \ \ell\text{ reduction of an irreducible} \ \overline{\mathbf{Q}}_\ell\text{-representation} \ \tilde{\pi}, \ \varphi_\pi \text{is the mod} \ \ell\text{ reduction of} \ \varphi_{\tilde{\pi}}.

4. For an irreducible \(\overline{\mathbf{Q}}_\ell\text{-representation} \ \tilde{\varphi}, \ \varphi_\pi \text{is the same as the usual semisimplified} \ L\text{-parameter (i.e., it is compatible with the one of Harris-Taylor via the Jacquet-Langlands correspondence if we ignore the monodromy operator); this is} \ [\text{HKW} 1.0.3].

Now we can prove the lemma we need:

**Lemma 3.1.** Suppose \(J_b\) is not quasi-split. If \(\pi\) is an irreducible smooth \(\mathbf{F}_\ell\text{-representation of} \ J_b(F), \text{then} \ \varphi_\pi \text{is not a generic unramified} \ L\text{-parameter}.

**Proof.** Write \(G = \prod_i \text{GL}_{n_i}, b = (b_i), J_b = J_{b_i}, \text{and} \ \pi = \mathbb{Z}_i \pi_i. \text{For some} \ i, J_{b_i} \text{is not quasi-split by the assumption. By the definition of genericity and properties of} \ L\text{-parameters, we may assume that} \ G = \text{GL}_n. \text{Assume} \ \varphi_\tilde{\pi} \text{is a generic unramified} \ L\text{-parameter. Take the supercuspidal support} \ (M, \sigma') \text{of} \ \pi, \text{and a lift} \ \tilde{\sigma} \text{of} \ \sigma \text{using Theorem \[\text{(2.1).]} \text{The} \ L\text{-parameter} \ \varphi_{\tilde{\sigma}} : W_F \to \hat{M}(\overline{\mathbf{Q}}_\ell) \hookrightarrow \text{GL}_n(\overline{\mathbf{Q}}_\ell) \)
\[
\text{with the untwisted embedding, factors through} \ \text{GL}_n(\mathbb{Z}_\ell), \text{and its reduction to} \ \text{GL}_n(\mathbf{F}_\ell) \text{recovers} \ \varphi_\tilde{\pi}. \text{By the genericity, we have a decomposition} \ \varphi_{\tilde{\sigma}} = \oplus_{i=1}^n \chi_i \text{into characters} \ \chi_1, \ldots, \chi_n \text{and} \ \chi_i/\chi_j \text{is not the cyclotomic character for any} \ i \neq j; \text{cf.} \ [\text{CSTP} 6.2.2] \text{and} \ [\text{CS} \text{ proof of 5.1.3]. Since} \ J_b \text{is not quasi-split,} \ M \text{has a factor of the form of} \ \text{GL}_{m_i}(D) \text{for some central division} \ F\text{-algebra} \ D \text{of dimension} > 1 \text{and} \ m \geq 1. \text{Thus, no such representation} \ \tilde{\sigma} \text{exists.} \]

4. **Proof of Theorem \[\text{[1.1].} \]

We shall use Hecke operators and excursion operators. Let us first recall the description of the cohomology of local Shimura varieties from \[\text{[FS] IX.3].} \text{Let} \ r_\mu \text{denote the representation of} \ \hat{G} \text{whose highest weight is conjugate to the cocharacter} \}

corresponding to \( \mu \), and set \( d = \dim \pi \). Let \( i^1 \) (resp. \( i^b \)) denote the immersion \( \text{Bun}_G^1 \hookrightarrow \text{Bun}_G \) (resp. \( \text{Bun}_G^b \hookrightarrow \text{Bun}_G \)). Then, there is an identification

\[
R\Gamma_c(\mathcal{M}(G,b,\mu), K, Z_{\ell}[q^{1/2}])[d](d/2) \cong i^b \ast T_{_{\pi}} (i^1 \text{c-Ind}_K^{G(\ell)} Z_{\ell}[q^{1/2}])
\]
as representations of \( J_b(F) \), and each \( H^r_c \) is a finitely generated smooth representation of \( J_b(\mathbb{Q}_p) \) [FS, IX.3.1]. Moreover, this identification is compatible with right actions of the Hecke algebra \( H_K \).

Given \( m \subset H_K \), it induces

\[
R\Gamma_c(\mathcal{M}(G,b,\mu), K, Z_{\ell}[q^{1/2}])[d](d/2) \cong i^b \ast T_{_{\pi}} (i^1 \text{c-Ind}_K^{G(\ell)} Z_{\ell}[q^{1/2}])_m
\]
as \( \ell \)-parameters for unramified irreducible representations as \( L \)-parameters of Fargues-Scholze agree with the usual one. Now, \( \lambda \) determines the localization

\[
Z_{\text{spec}}(G, Z_{\ell}[q^{1/2}]) \to \text{End}_{G(\ell)}(\text{c-Ind}_K^{G(\ell)} Z_{\ell}[q^{1/2}]) = H^0_K.
\]
(Beware that \( \text{c-Ind}_K^{G(\ell)} Z_{\ell}[q^{1/2}] \) is being regarded as a right module of \( H_K \).) Composing it with an involution \( KhK \mapsto Kh^{-1}K \) of \( H_K \), we obtain a map

\[
Z_{\text{spec}}(G, Z_{\ell}[q^{1/2}]) \to H_K;
\]
this is the map compatible with usual \( L \)-parameters for unramified irreducible representations as \( L \)-parameters of Fargues-Scholze agree with the usual one. Now, \( \lambda_m \) determines the localization

\[
Z_{\text{spec}}(G, Z_{\ell}[q^{1/2}]) \to Z_{\text{spec}}(G, Z_{\ell}[q^{1/2}])_m.
\]
It suffices to show that

the action of \( Z_{\text{spec}}(G, Z_{\ell}[q^{1/2}]) \) on \( i^b \ast T_{_{\pi}} (i^1 \text{c-Ind}_K^{G(\ell)} Z_{\ell}[q^{1/2}])_m \)
factors through \( Z_{\text{spec}}(G, Z_{\ell}[q^{1/2}])_m \).

As the action of \( Z_{\text{spec}}(G, Z_{\ell}[q^{1/2}]) \) commute with excursion operators [FS, IX.5.2], we need only observe that

the action of \( Z_{\text{spec}}(G, Z_{\ell}[q^{1/2}]) \) on \((\text{c-Ind}_K^{G(\ell)} Z_{\ell}[q^{1/2}])_m \) factors through \( Z_{\text{spec}}(G, Z_{\ell}[q^{1/2}])_m \).

This completes the proof of Theorem [\ref{mainthm}].
5. Shimura varieties of PEL type and the Hodge-Tate period map

We start the discussion on Shimura varieties. Let \((\mathcal{O}_B, \ast, \Lambda, (\cdot, \cdot))\) be an integral PEL datum of type A or C unramified at \(p\); cf. [CS17, 4.3]. Let \((G, X)\) denote the associated Shimura datum with the reflex field \(E \subset \mathbb{C}\). We write \(S_K\) for the Shimura variety of level \(K \subset G(\mathbb{A}_f)\). Fix a place \(p\) of \(E\) above \(p\) and an embedding \(\mathbb{Q} \to \mathbb{Q}_p\) that induces \(p\). Let \(S_{K,E_p}\) denote the adic space over \(E_p\) associated with \(S_K\). Suppose for the moment that \(K = K_p\) and \(K_p = G(\mathbb{Z}_p)\). The Shimura variety has the canonical integral model \(S_K\) over \(\mathcal{O}_E\). Let \(S_{K,C}\) denote the open immersion from the adic generic fiber of the \(p\)-adic completion \(\mathcal{S}_K\) of \(\mathcal{S}\); this is the locus of “good reduction”. For deeper \(K\), we take the inverse image to define \(S_{K,E_p}\). Consider the associated diamonds [SW20, Sch].

Lemma 5.1. Let \(C\) be a complete algebraically closed nonarchimedean field over \(\mathbb{Q}_p\), the natural map \(\Gamma(S_K, \mathbb{Q}_\ell) \to \Gamma(S_{K,C}, \mathbb{Q}_\ell)\) is an isomorphism for \(\ell \neq p\).

Proof. This follows from results of Lan-Stroh [LS18, 5.20] and Huber; see [CS, 2.6.4].

We now define perfectoid Shimura varieties. Set \(S_{K,p} := \varprojlim K_p S_{K,K_p}\p\) both are actually perfectoid spaces after base change to a perfectoid field. Let \(\mathcal{F}_\ell\) denote the flag variety of \(G(E_p)\) associated with the cocharacter \(\mu\) (or, perhaps better, its inverse) determined by \(h \in X\); cf. [CS17, 2.1]. We regard it as a diamond over \(\text{Spd}(E_p)\).

Theorem 5.2 ([CS17, 2.1.3]). There is a \(G(\mathbb{Q}_p)\)-equivariant morphism \(\pi_{HT}: \mathcal{S}_{K,p} \to \mathcal{F}_\ell\) with a specific construction. This is called the Hodge-Tate period map.

We write \(\pi_{HT}^0\) for the restriction of \(\pi_{HT}\) to \(\mathcal{S}_{K,p}^0\). The restriction \(\pi_{HT}^0\) is quasi-compact and quasiseparated.

6. Newton stratifications and Igusa varieties

It is important to study the Hodge-Tate period map using Newton stratifications. There are two types of stratifications. We fix a complete algebraically closed nonarchimedean extension \(C\) of \(\mathbb{Q}_p \supset E_p\).

The first stratification is classical, and is induced from the Newton stratification of the geometric special fiber \(\mathcal{F}_K\) over \(\mathbb{F}_p\) of the integral model: for each \(b \in B(G_{\mathbb{Q}_p, \mu^{-1}})\), the inverse images of the stratum \(\mathcal{F}_K^b\) along the specialization maps define locally closed subspaces \(S_{K,C}^{\leq b} \subset S_{K,C}, S_{K,C}^{\leq b} \subset S_{K,C}\). Note that

\[
S_{K,C}^{\geq b} := \bigcup_{b' \geq b} S_{K,C}^{b'} = S \setminus \bigcup_{b' < b} S_{K,C}^{b'}
\]
is a quasicompact open subspace of $S_{K_p,C}^{(b)}$ for every $b$. In particular, for $\mu$-ordinary $b_0$, the unique maximal element, $S_{K_p,C}^{(b_0)}$, $S_{K_p,C}^{(b)}$ are open strata.

On the other hand, Caraiani-Scholze defined the Newton stratification of $\mathcal{F}_C$ with the reverse direction [CS17, Section 3], e.g., $\mathcal{F}_C^{(b)}$ is a closed stratum. In fact, it is induced from the stratification of $\text{Bun}_{G_{Q_p}}$ of the Fargues-Fontaine curve. Let us recall this description. The flag variety $\mathcal{F}_C$ as a diamond over $\text{Spd}(C)$ has the following interpretation: it sends a perfectoid space $S$ over $C$ to the set of isomorphism classes of modification of the trivial $G_{Q_p}$-bundle $E_1$ on the relative Fargues-Fontaine curve $X_S$ bounded by $\mu$, i.e., pairs $(\mathcal{E}, i : E_1 \to \mathcal{E})$ of a $G_{Q_p}$-bundle $\mathcal{E}$ and a modification $i$ that is bounded by $\mu$ along the divisor determined the structure map $\mathcal{F}_C \to \text{Spd}(C)$. There is a natural map $\mathcal{F}_C \to \text{Bun}_{G_{Q_p}}$; $(\mathcal{E}, i) \mapsto E_1$ to the moduli stack $\text{Bun}_{G_{Q_p}}$ of $G_{Q_p}$-bundles [FS, III]. The moduli stack $\text{Bun}_{G_{Q_p}}$ is an $\ell$-cohomologically smooth Artin v-stack of dimension 0 [FS, IV.1.19]. As explained in [FS, III], each $b \in B(G_{Q_p})$ gives rise to a locally closed subfunctor $\text{Bun}_{G_{Q_p}}^{(b)} \hookrightarrow \text{Bun}_{G_{Q_p}}$ classifying $G$-bundles of type $b \in B(G_{Q_p})$. In fact, this defines a stratification of $\text{Bun}_{G_{Q_p}}$. The stratification of $\mathcal{F}_C$ is induced from that of $\text{Bun}_{G_{Q_p}}$ via the map $\mathcal{F}_C \to \text{Bun}_{G_{Q_p}}$ as is clear from the definition in [CS17, 3.5.6]. In particular, $\mathcal{F}_C^{(b)}$ is a locally closed locally spatial subdiamond of $\mathcal{F}_C$; it was merely a locally closed topological subset in [CS17].

The following lemma is useful later:

Lemma 6.1 ([Han, 2.10]). Fix $\ell \neq p$. The map $\mathcal{F}_C \to \text{Bun}_G$ factors through a map

$$q : [\mathcal{F}_C/G(E_p)] \to \text{Bun}_G,$$

and $q$ is $\ell$-cohomologically smooth. Moreover, for any open compact subgroup $K_p \subset G(Q_p)$, the composite

$$q_K : [\mathcal{F}_C/K_p] \to [\mathcal{F}_C/G(E_p)] \xrightarrow{q} \text{Bun}_G$$

does $\ell$-cohomologically smooth.

Fix $b \in B(G_{Q_p}, \mu^{-1})$. The intersection

$$S_{K_p,C}^{(b)} \cap (\pi_{HT})^{-1}(\mathcal{F}_C^{(b)}) = S_{K_p,C}^{(b)} \times_{\mathcal{F}_C} \mathcal{F}_C^{(b)}$$

defines a quasicompact and quasiseparated open immersion to $(\pi_{HT})^{-1}(\mathcal{F}_C^{(b)})$, and it contains all points of rank 1. Let

$$\pi_{HT} : S_{K_p,C}^{(b)} \times_{\mathcal{F}_C} \mathcal{F}_C^{(b)} \to \mathcal{F}_C^{(b)}$$

denote the map induced from $\pi_{HT}$, which is quasicompact and quasiseparated.

Lemma 6.2. A natural map

$$(R\pi_{HT})_* \mathcal{F}_C \to R\pi_{HT}^* \mathcal{F}_C$$

is an isomorphism in $D_{\text{et}}(\mathcal{F}_C^{(b)}, \mathcal{F}_C)$ [Sch, 14.13].
Proof: As all diamonds involved are locally spatial, we are allowed to work with $D^{+}(\mathcal{F}_{\ell,C,\text{ét}}, F_{\ell})$. It suffices to show that

$$(R\pi_{HT}^{\circ}, F_{\ell})_{\overline{\mathbb{F}}^{\text{et}}} \to (R\pi_{HT}^{\circ b}, F_{\ell})_{\overline{\mathbb{F}}^{\text{et}}}$$

is an isomorphism for all geometric points $\overline{\mathbb{F}}$: $\text{Spa}(C, (C')^{+}) \to \mathcal{F}_{\ell,C}^{b}$, because $\mathcal{F}_{\ell,C,\text{ét}}^{b}$ has enough points [Sch 14.3]. This follows from [CS17 4.4.1, 4.4.2].

We next recall the product formula that describes the structure of $\pi_{HT}^{b}$. We need some preparation.

Let $\pi_{HT}: \mathcal{M}_{(G,b,\mu),\infty} \to \mathcal{F}_{\ell,C}$ denote the Hodge-Tate period map from (the diamond associated with) the Rapoport-Zink space at infinite level, which factors through $\mathcal{F}_{\ell,C}$. [CS17 4.2.6]. Using the interpretation as the moduli of $G_{Q_{p}}$-shtukas [SW20 24.3.5], we have the following cartesian diagram of $v$-stacks

$$
\begin{array}{c}
\mathcal{M}_{(G,b,\mu),\infty} \\
\downarrow \pi_{HT}^{b} \\
\mathcal{F}_{\ell,C} \\
\downarrow \\
\text{Bun}_{C}^{b, G_{Q_{p}}}.
\end{array}
$$

Let us also recall that $\text{Bun}_{C}^{b, G_{Q_{p}}}$ has the form of $[s/J_{b}]$, where $J_{b}$ is the extension of $J_{b}(Q_{p})$ by a “unipotent” group $\tilde{J}_{b}$ of dimension $d_{b} := (2\rho, \nu_{b})$, with a canonical splitting $J_{b}(Q_{p}) \hookrightarrow \tilde{J}_{b}$. [FS III].

The product formula describes the fiber product

$$
\text{Spec}_{K_{p}, C}^{b} \times \mathcal{F}_{\ell,C} \mathcal{M}_{(G,b,\mu),\infty}
$$

using Igusa varieties. Choose a completely slope divisible $p$-divisible group $X_{b}$ over $\overline{\mathbb{F}}_{p}$ with $G_{Q_{p}}$-structure corresponding to $b \in B(G_{Q_{p}}, \mu^{-1})$. (It is known to exist.) This choice determines the (perfect) Igusa variety $Ig_{b}$ of level $K$ over $\overline{\mathbb{F}}_{p}$ [CS17 4.3.1], equipped with the action of $J_{b}(Q_{p})$. Its dimension is $d_{b}$. As $Ig_{b}$ is perfect [CS17 4.3.5], it lifts uniquely to a flat $p$-adic formal scheme $Ig_{W(\overline{\mathbb{F}}_{p})}^{b}$ over $W(\overline{\mathbb{F}}_{p})$.

**Proposition 6.3 ([CS17 4.3.19, 4.3.20]).** Choosing $(X_{b})_{C} \in \mathcal{M}(G_{Q_{p}}, b, \mu)(C)$, we have an isomorphism of diamonds

$$
\text{Spec}_{K_{p}, C}^{b} \times \mathcal{F}_{\ell,C} \mathcal{M}(G_{Q_{p}}, b, \mu) \cong \mathcal{M}(G_{Q_{p}}, b, \mu) \times C (Ig_{W(\overline{\mathbb{F}}_{p})}^{b})_{C}^{\text{ad}},
$$

where $(Ig_{W(\overline{\mathbb{F}}_{p})}^{b})_{C}^{\text{ad}}$ denotes the adic generic fiber of $Ig_{W(\overline{\mathbb{F}}_{p})}^{b}$ base changed to $C$.

7. Mantovan’s formula

We combine results explained in the previous section to prove a form of Mantovan’s formula. (For usual Mantovan’s formula, see [Man04, HK19]) Fix $\ell \neq p$ and set $C := \widehat{\mathbb{Q}}_{p}$. The Rapoport-Zink space at infinite level $\mathcal{M}(G_{b,\mu},\infty)$ will be regarded as a diamond over $\text{Spd}(C)$.

Recall that the perfect Igusa variety $Ig_{b}$ is the perfection of the limit of Igusa varieties at finite level [CS17 4.3.8]. Define $R\Gamma_{c}(Ig_{b}^{\text{ad}}, \overline{\mathbb{F}}_{\ell}(d_{b}))$ to be the colimit of compactly supported cohomology of $\overline{\mathbb{F}}_{\ell}(d_{b})$ on Igusa varieties at finite level. The Poincaré duality at finite level implies that

$$
R\Gamma(Ig_{b}^{\text{ad}}, \overline{\mathbb{F}}_{\ell})^{*} \cong R\Gamma(Ig_{b}^{\text{ad}}, \overline{\mathbb{F}}_{\ell}(d_{b}))[2d_{b}], R\Gamma_{c}(Ig_{b}^{\text{ad}}, \overline{\mathbb{F}}_{\ell}(d_{b}))^{*} \cong R\Gamma(Ig_{b}^{\text{ad}}, \overline{\mathbb{F}}_{\ell})[2d_{b}]
$$
in the derived category of smooth $\mathbf{F}_l$-representations of $J_b(\mathbb{Q}_p)$, where $(-)^*$ is the smooth dual and $\mathbf{F}_l(d_b)$ is regarded as an $J_b(\mathbb{Q}_p)$-equivariant sheaf via the functoriality of dualizing complexes; compare with the argument in [KS 7.1].

**Theorem 7.1.** There is a filtration of $R\Gamma(S_{K^p,\mathbb{Q}}, \mathbf{F}_l)$ by complexes of smooth representations of $G(\mathbb{Q}_p) \times W_{E_p}$ whose graded pieces are

$$R\Gamma(Ig^b, \mathbf{F}_l)^{op} \otimes_{\mathcal{C}_c(J_b(\mathbb{Q}_p))} R\Gamma_c(M_{(G,b,\mu),\infty}, \mathbf{F}_l(d_b))[2d_b] \cong (R\Gamma_c(Ig^b, \mathbf{F}_l(d_b))^*)^{op} \otimes_{\mathcal{C}_c(J_b(\mathbb{Q}_p))} R\Gamma_c(M_{(G,b,\mu),\infty}, \mathbf{F}_l(d_b)).$$

Here, $\mathbf{F}_l(d_b)$ on $M_{(G,b,\mu),\infty}$ is equipped with a $J_b(\mathbb{Q}_p)$-equivariant structure as in Lemma 7.4 below. An analogous claim holds at the level $K^p K_p$ by taking $K_p$-invariants.

**Remark 7.2.** The dual statement gives a filtration of $R\Gamma_c(S_{K^p,\mathbb{Q}}, \mathbf{F}_l(d))[2d]$ whose graded pieces are

$$R\text{Hom}_{J_b(\mathbb{Q}_p)}(R\Gamma_c(M_{(G,b,\mu),\infty}, \mathbf{F}_l(d_b)), R\Gamma_c(Ig^b, \mathbf{F}_l(d_b))).$$

As the twists on the source and target are cancelled out (see Lemma 7.6 below), the resulting equality in the Grothendieck group of smooth representations of $G(\mathbb{Q}_p) \times W_{E_p}$ is the $\mathbf{F}_l$-version of usual Mantovan’s formula.

It is not clear to the author if this dual statement implies Theorem 7.1 itself as $R\Gamma_c(M_{(G,b,\mu),\infty}, \mathbf{F}_l(d_b))$ is not admissible.

The rest of this section is devoted to the proof of this theorem. By Lemma 5.1 we have isomorphisms

$$R\Gamma(S_{K^p,\mathbb{Q}}, \mathbf{F}_l) \cong R\Gamma(S_{K^p,C}, \mathbf{F}_l) \cong R\Gamma(\mathcal{F}_C, R\pi_{HT^+}, \mathbf{F}_l).$$

Using excision sequences, we obtain a filtration whose graded pieces are

$$R\Gamma_c(\mathcal{F}_C, R\pi_{HT^+}, \mathbf{F}_l) \cong R\Gamma_c(\mathcal{F}_C, R\pi_{HT^+}, \mathbf{F}_l),$$

where the isomorphism comes from Lemma 7.2. Therefore, it remains to identify $R\Gamma_c(\mathcal{F}_C, R\pi_{HT^+}, \mathbf{F}_l)$. Let us factorize $\pi_{HT^+}$ as

$$\mathcal{M}_{(G,b,\mu),\infty} \xrightarrow{\pi_{b\text{unip}}} \mathcal{M}_{(G,b,\mu),\infty}/\tilde{b}_{b,C} \xrightarrow{\pi_{b}^{ss}} \mathcal{F}_C,$$

where $\pi_{b\text{unip}}$ (resp. $\pi_{b}^{ss}$) is a $\tilde{b}_{b,C}$-torsor (resp. $J_b(\mathbb{Q}_p)$-torsor) as v-sheaves.

As in [Man04 Proposition 5.12],

$$R\Gamma(Ig^b, \mathbf{F}_l)^{op} \otimes_{\mathcal{C}_c(J_b(\mathbb{Q}_p))} R\Gamma_c(M_{(G,b,\mu),\infty}, \mathbf{F}_l) \cong (R\Gamma(Ig^b, \mathbf{F}_l)^{op} \otimes_{\mathcal{F}_l} R\Gamma_c(M_{(G,b,\mu),\infty}, \mathbf{F}_l)) \otimes_{\mathcal{C}_c(J_b(\mathbb{Q}_p))} \mathbf{F}_l.$$
Moreover, the natural homomorphism
\[ R\pi^b_{unip!} R\pi^b_{unip} \to \text{id} \]
is an equivalence.

Lemma 7.5. There is an isomorphism
\[ R\Gamma_c(\mathcal{M}_{(G,b,\mu)}; \mathcal{F}_\ell \otimes^L_{E_\ell} R\pi^b_{unip!} \mathcal{F}_\ell) \]
isomorphic to
\[ R\Gamma_c(\mathcal{M}_{(G,b,\mu)}; \mathcal{F}_\ell \otimes^L_{E_\ell} R\pi^b_{unip!} \mathcal{F}_\ell) \]
compatible with actions of \( G(\mathbb{Q}_p) \), \( J_b(\mathbb{Q}_p) \), \( W_{E_p} \).

Proof of Lemma 7.5. We first claim that
\[ R\Gamma_c(\mathcal{M}_{(G,b,\mu)}; \mathcal{F}_\ell \otimes^L_{E_\ell} \mathcal{F}_\ell) \]
is an equivalence.

This completes the proof of Lemma 7.3.

Proof of Lemma 7.4. As \( \pi^b_{unip} \) is a \( \mathcal{J}^0_b \)-torsor, it follows from [FS, III.5.1] that \( \pi^b_{unip} \) is \( \ell \)-cohomologically smooth of dimension \( d_b \). The equivalence of
\[ R\pi^b_{unip!} R\pi^b_{unip} \to \text{id} \]
is shown in the proof of [FS, V.2.1]. It is also shown there that \( R\pi^b_{unip!} \) is fully faithful. To identify \( R\pi^b_{unip!} \mathcal{F}_\ell \), note that \( \mathcal{J}^0_{b,W}(\mathbb{F}_p) \) is the v-sheaf associated with the formal scheme of the form of
\[ \text{Spf}(W(\mathbb{F}_p)[x_1^{1/p\infty}, \ldots, x_{d_b}^{1/p\infty}]); \]
see [CS17, 4.2.11]. So, by [Sch] Section 27, the dualizing complex of
\[ \mathcal{J}^0_{b,W}(\mathbb{F}_p) \to \text{Spd}(W(\mathbb{F}_p)) \]
is naturally isomorphic to \( \mathcal{F}_\ell(d_b)[2d_b] \). So, the same holds for
\[ \text{Spd}(W(\mathbb{F}_p)) \to [\text{Spd}(W(\mathbb{F}_p))/\mathcal{J}^0_{b,W}(\mathbb{F}_p)] \);
which in turn gives the desired identification $R\pi_{\text{unip}}^{b!}\mathbf{F}_\ell \cong \mathbf{F}_\ell (d_b)[2d_b]$. □

Proof of Lemma 7.5. By Proposition 6.3 we have the following cartesian diagram of locally spatial diamonds:

$$
\begin{array}{c}
\mathcal{M}_{(G_{Q_p}, b, \mu), \infty} \times C ((I_{g_W}^b\mathbf{F}_p)_C)^{ad} \\
\downarrow \\
\mathcal{S}_{K_{p,C}}^b \\
\downarrow \pi_{HT}^b \\
\mathcal{M}_{(G_{Q_p}, b, \mu), \infty} \\
\end{array}
$$

Using the base change isomorphisms twice, we see that $R\pi_{HT}^b R\pi_{HT}^b \mathbf{F}_\ell$ identifies with the pullback of $R\Gamma(I_{g_W}^b\mathbf{F}_p)_C^{ad}, \mathbf{F}_\ell)$ along the structure map $\mathcal{M}_{(G_{Q_p}, b, \mu), \infty} \to \text{Spd}(C)$.

As there is an isomorphism supplied by [CS17, 4.4.3] $R\Gamma((I_{g_W}^b\mathbf{F}_p)_C^{ad}, \mathbf{F}_\ell) \cong R\Gamma(I_{g}^b, \mathbf{F}_\ell)$, we finish by the projection formula [Sch, 22.23]. □

Let us remark that the twist has no effect in the following sense:

**Lemma 7.6.** The $J_b(Q_p)$-equivariant structure on $\mathbf{F}_\ell (d_b)$ on $I_{g}^b$ and on $R\pi_{\text{unip}}^{b!}\mathbf{F}_\ell$ are given by the same smooth character $\kappa$:

$J_b(Q_p) \to \mathbf{F}_\times^\ell$.

**Proof.** The first remark is that $I_{g}^b$ is nonempty [VW13, 11.2]. It is clear that the action of $J_b(Q_p)$ on $\mathbf{F}_\ell (d_b)$ is given by some character on the $J_b(Q_p)$-orbit of a connected component of $I_{g}^b$ (which we choose); it will be denoted by $\kappa_1$. Moreover, writing $I_{g}^b$ as the perfection of the inverse limit of Igusa varieties at finite level, one sees that this character $\kappa_1$ is smooth.

It remains to study $R\pi_{\text{unip}}^{b!}\mathbf{F}_\ell$. Recall the cartesian diagram

$$
\begin{array}{c}
\mathcal{M}_{(G_{Q_p}, b, \mu), \infty} \\
\downarrow \pi_{\text{unip}}^b \\
\mathcal{M}_{(G_{Q_p}, b, \mu), \infty} / J_{p,C}^0 \\
\downarrow \\
[\text{Spd}(C) / J_{p,C}^0].
\end{array}
$$

By commutation of upper shriek with base change [Sch, 23.12], $R\pi_{\text{unip}}^{b!}\mathbf{F}_\ell$ is obtained from the pullback from the corresponding object on $\text{Spd}(C)$, on which $J_b(Q_p)$ acts by a character since it is already isomorphic to $\mathbf{F}_\ell (d_b)[2d_b]$ on $\text{Spd}(C)$ as in the proof of Lemma 7.1. Write $\kappa_2$ for this character.

To compare $\kappa_1$ and $\kappa_2$, we need only observe that $J_{p,C}^0$ acts freely on the v-sheaf $I_{g_W}^b\mathbf{F}_p$ associated with the formal scheme $I_{g_W}^b\mathbf{F}_p$. (In particular, $\kappa_1$ is independent of the choice of the connected component and $\kappa_2$ is also smooth.) □

8. Semiperversity

From now on, we work in the setting of [CS, Section 2]. So, $B = F$ is a CM field and $V = F^{2n}$, and $G$ is a quasi-split similitude unitary group. From [CS], we recall the key semiperversity result.

Let $C$ be a complete algebraically closed nonarchimedean extension of $Q_p$ with the ring of integers $O_C$ and the residue field $k$. 
Theorem 8.1 ([CS 4.6.1]). There is a cofinal system of formal models \( \mathfrak{X} \) of \( \mathcal{F}_\ell \) over \( \mathcal{O}_C \) such that the nearby cycle

\[
R\psi(R\pi^0_{HT}, \mathcal{F}_\ell)
\]

belongs to \( \pi D^{\geq d}(\mathcal{F}_k, \mathcal{F}_\ell) \).

Corollary 8.2. Let \( i^0 \) denote the the closed immersion \( \mathcal{F}_\ell(\mathbb{Q}_p) \hookrightarrow \mathcal{F}_\ell(C) \). The local cohomology

\[
R\Gamma(\mathcal{F}_\ell(C), R\pi^0_{HT\ast}, \mathcal{F}_\ell) := R\Gamma(\mathcal{F}_\ell(\mathbb{Q}_p), R^{\text{rig}}_{i^0} R\pi^0_{HT\ast}, \mathcal{F}_\ell)
\]

belongs to \( D^{\geq d}(\mathcal{F}_\ell) \). Similarly,

\[
\Gamma_c([\mathcal{F}_\ell(\mathbb{Q}_p)/K_p], R^{\text{rig}}_{i^0} R\pi^0_{HT\ast}, \mathcal{F}_\ell)
\]

belongs to \( D^{\geq d}(\mathcal{F}_\ell) \).

Proof. Set \( \mathcal{F} := R\pi^0_{HT\ast}, \mathcal{F}_\ell \). Let \( j^{bo} : \mathcal{F}_\ell(C) \setminus \mathcal{F}_\ell(\mathbb{Q}_p) \hookrightarrow \mathcal{F}_\ell \) denote the open immersion from the complement of the ordinary locus. We will work in the \( \infty \)-categorical setup. By Lemma 8.3 below, there is a canonical fiber sequence

\[
i^{bo}_* R^{\text{rig}}_{bo \ast} \mathcal{F} \rightarrow \mathcal{F} \rightarrow R j^{bo}_* j^{bo \ast} \mathcal{F}
\]

giving rise to a canonical fiber sequence

\[
R\Gamma(\mathcal{F}_\ell(C), \mathcal{F}) \rightarrow R\Gamma(\mathcal{F}_\ell(C), \mathcal{F}) \rightarrow R\Gamma(\mathcal{F}_\ell(C) \setminus \mathcal{F}_\ell(\mathbb{Q}_p), \mathcal{F}).
\]

Let us first fix a formal model \( \mathfrak{X} \) as in Theorem 8.1. The image of the ordinary locus under the specialization map is a closed set \( i_Z : Z \hookrightarrow \mathcal{F}_k \) consisting of finitely many closed points. As \( i^{bo}_Z \) is left \( t \)-exact, \( i^{bo}_Z R\psi(\mathcal{F}) \) lives in \( D^{\geq d}(Z, \mathcal{F}_\ell) \). Let \( j_Z : \mathcal{F}_k \setminus Z \hookrightarrow \mathcal{F}_k \) denote the open immersion from the complement of \( Z \). As above, there is a canonical fiber sequence

\[
i^{bo}_Z R\psi(\mathcal{F}) \rightarrow R\psi(\mathcal{F}) \rightarrow R j_Z^* j_Z^* R\psi(\mathcal{F})
\]

giving rise to a canonical fiber sequence

\[
R\Gamma_Z(\mathcal{F}_k, R\psi(\mathcal{F})) \rightarrow R\Gamma(\mathcal{F}_C, \mathcal{F}) \rightarrow R\Gamma(\mathcal{F}_C \setminus sp^{-1}(Z), \mathcal{F})
\]

with \( R\Gamma_Z(\mathcal{F}_k, R\psi(\mathcal{F})) \in D^{\geq d}(\mathcal{F}_\ell) \).

Now taking the limits by varying \( \mathfrak{X} \), we obtain a canonical fiber sequence

\[
R \lim R\Gamma_Z(\mathcal{F}_k, R\psi(\mathcal{F})) \rightarrow R\Gamma(\mathcal{F}_C, \mathcal{F}) \rightarrow R\Gamma(\mathcal{F}_C \setminus \mathcal{F}_\ell(\mathbb{Q}_p), \mathcal{F})
\]

as fibre sequences commute with limits. So, we conclude that

\[
R\Gamma(\mathcal{F}_\ell(\mathbb{Q}_p), \mathcal{F}_\ell(C), \mathcal{F}) \cong R \lim R\Gamma_Z(\mathcal{F}_k, R\psi(\mathcal{F})) \in D^{\geq d}(\mathcal{F}_\ell)
\]

as desired.

The second part follows from the first part. \( \square \)

The following lemma was used above:

Lemma 8.3. Let \( i : Z \hookrightarrow X \) be a closed immersion of locally spatial diamonds with the open complement \( j : U \hookrightarrow X \). The functor \( i_* \) admits a left adjoint \( R! \), and for any object \( \mathcal{F} \) of the \( \infty \)-category \( D_{\text{et}}(X, \mathcal{F}_\ell) \) [Sch 17.1], there is a canonical fiber sequence

\[
i_* R! \mathcal{F} \rightarrow \mathcal{F} \rightarrow R j_* j^* \mathcal{F}.
\]

In particular, \( R! \) agrees with the one in [Sch]. A similar statement holds for schemes.
Proof. One sees that the essential image of the fully faithful functor $i_*$ consists of objects $G$ with $j^*G \cong 0$. Thus, the fiber sequence characterizes the left adjoint functor $Ri^!$, and it is well-defined. A more detail is given in [GL19 2.2.5.5] for quasi-projective schemes over an algebraically closed field (this is not an essential assumption; see also [GL19 2.2.5.6]). □

9. Proof of Theorem 1.3

We only consider $H^*$; the case of $H^*_\rho$ follows from this case by the Poincaré duality as the dual of a generic unramified $L$-parameter is generic unramified. (See the proof of [CS, 1.1].)

By Proposition 1.5, we have

$$\Gamma(G_b^b, F_{\ell})^{op} \otimes L_{C_c(J_b(Q_p))} \Gamma_c(M_{(G_{Q_p}, b, \mu)}, K_p, F_{d}(d)_{m_p}) \cong 0.$$ for any non-ordinary $b$. So, by Theorem 7.1, we see that

$$\Gamma(S_K, F_{\ell})_{m_p} \cong \Gamma(G_b^b, F_{\ell})^{op} \otimes L_{C_c(J_b(Q_p))} \Gamma_c(M_{(G_{Q_p}, b, \mu)}, K_p, F_{d}(d)_{m_p}[2d],$$

where $b_0$ is the unique ordinary element. As in the proof of Theorem 7.1, we have

$$\Gamma(G_b^b, F_{\ell})^{op} \otimes L_{C_c(J_b(Q_p))} \Gamma_c(M_{(G_{Q_p}, b, \mu)}, K_p, F_{d}(d)_{m_p}[2d]) \cong \Gamma_c([-F_{\ell}(Q_p)/K_p], i^{b_0 *} R\pi^o_{HT*} F_{\ell})_{m_p},$$

where $i^{b_0} : F_{\ell}(Q_p) \hookrightarrow F_{\ell}C$ is a closed immersion from the ordinary stratum. We know, by Corollary 8.2, that

$$\Gamma_c([-F_{\ell}(Q_p)/K_p], R.i^{b_0} R\pi^o_{HT*} F_{\ell})_{m_p}$$

belongs to $D^{2d}(F_{\ell})$. We shall prove that

$$\Gamma_c([-F_{\ell}(Q_p)/K_p], R.i^{b_0} R\pi^o_{HT*} F_{\ell})_{m_p} \cong 0$$

for every $b \neq b_0$. From the proof of Theorem 7.1, one sees that $i^{b_0} R\pi^o_{HT*} F_{\ell}$ comes from some object

$$V_b \in D_{ech}(\text{Bun}^b_{G_{Q_p}}, F_{\ell}) \cong D(J_b(Q_p), F_{\ell})$$

under the pullback along $q_{K_p}^b : [\mathcal{F}^b_C/\mathcal{K}_p] \to \text{Bun}^b_{G_{Q_p}}$ (the restriction of $q_{K_p}$), where the equivalence is given by [FS, V.2.2.2]. Moreover, $V_b$ is a bounded complex of admissible representations of $J_b(Q_p)$. As $q_{K_p}$ is $\ell$-cohomologically smooth by Lemma 6.1, there is an identification

$$R.i^{b_0} i^o \gamma^{b_0} R\pi^o_{HT*} F_{\ell} \cong q_{K_p}^{b_0} (R.i^{b_0} i^o \gamma^{b_0} V_b)$$

given by smooth base change [Sch 23.16.(iii)]. Namely, $R.i^{b_0} i^o \gamma^{b_0} R\pi^o_{HT*} F_{\ell}$ comes from an object of $D(J_b(Q_p), F_{\ell})$ by regarding $R.i^{b_0} i^o \gamma^{b_0} V_b$ as such an object. As in the proof of Theorem 7.1, we see that $\Gamma_c([-F_{\ell}(Q_p)/K_p], R.i^{b_0} i^o \gamma^{b_0} R\pi^o_{HT*} F_{\ell})_{m_p}$ is given by

$$(R.i^{b_0} i^o \gamma^{b_0} V_b)^{op} \otimes L_{C_c(J_b(Q_p))} \Gamma_c(M_{(G_{Q_p}, b, \mu)}, F_{d}(d)_{m_p}[2d].$$
It suffices to show the vanishing
\[ R \text{Hom}_{J_{b_0}(Q_p)}(R\Gamma_c(M(G_{Q_p}, b_0, \mu), K_p, \mathcal{F}_\ell(d)), (Ri^{b_0,1}V_b)^*) \cong 0. \]

We need the following lemma to proceed:

**Lemma 9.1.** The Artin \( \psi \)-stack \( \text{Bun}^{b_0}_{G_{Q_p}} \) is \( \ell \)-cohomologically smooth of dimension \(-d\). Moreover, the dualizing complex \( D_{\text{Bun}^{b_0}_{G_{Q_p}}} \) on \( \text{Bun}^{b_0}_{G_{Q_p}} \) has the form of
\[
\kappa^{-1}[-2d] \in D(J_{b_0}(Q_p), \mathcal{F}_\ell) \cong D_{\text{et}}(\text{Bun}^{b_0}_{G_{Q_p}}, \mathcal{F}_\ell)
\]
for the character \( \kappa : J_{b_0}(Q_p) \to \mathcal{F}_\ell^\times \) in Lemma 7.8.

**Proof.** The first part is [FS, IV.1.22]. For the second part, let us consider maps
\[
[s/J_{b_0}(Q_p)] \to \text{Bun}^{b_0}_{G_{Q_p}} \cong [s/\mathcal{J}_{b_0}] \xrightarrow{p_{\text{unip}}} [s/J_{b_0}(Q_p)] \xrightarrow{p_{\text{ss}}} *
\]
where \( p_{\text{unip}}, p_{\text{ss}} \) are natural projections and \( s \) is a section of \( p_{\text{unip}} \). As in the proof of Lemma 7.8, \( R_s \mathcal{F}_\ell \) identifies with \( \kappa[2d] \) and \( R_{s[t]} \kappa[2d] \cong \mathcal{F}_\ell \). Since \( s \) is a section of \( p_{\text{unip}} \), we see that
\[
R^1 p_{\text{unip}}^! \kappa[2d] \cong \mathcal{F}_\ell, \quad R^1 p_{\text{unip}}^! \mathcal{F}_\ell \cong \kappa^{-1}[-2d].
\]
So, it suffices to show that \( R^1 p_{\text{unip}}^! \mathcal{F}_\ell \cong \mathcal{F}_\ell \). For this, we show that the dualizing complex \( D_{J_{b_0}^G/\mathcal{J}(Q_p)} \) on \( J_{b_0}^G/\mathcal{J}(Q_p) \) identifies with \( \mathcal{F}_\ell[2 \dim J_0] \). Consider a tower \( \{ J_{b_0}^G/\mathcal{J}_{J_0}(Q_p) \} \) étale over \( J_{b_0}^G/\mathcal{J}(Q_p) \) for open compact pro-\( p \) subgroups \( J \subset J_0(Q_p) \), with Hecke action. Fixing an \( \mathcal{F}_\ell \)-valued Haar measure of \( J_0(Q_p) \), we can identify dualizing complexes \( D_{J_{b_0}^G/\mathcal{J}(Q_p)} \) with \( \mathcal{F}_\ell[2 \dim J_0] \) via the pullback to \( D(J_{b_0}^G/\mathcal{J})(\mathcal{F}_\ell) \) by [Sch 24.2]. The resulting \( J_0(Q_p) \)-equivariant structures on \( \mathcal{F}_\ell[2 \dim J_0] \) is trivial, so this gives the desired identification of \( D_{J_{b_0}^G/\mathcal{J}(Q_p)} \) as \( D_{J_{b_0}^G/\mathcal{J}(Q_p)} \) are pullbacks of \( D_{J^G/\mathcal{J}(Q_p)} \).

By this lemma, we have
\[
R \text{Hom}_{J_{b_0}(Q_p)}(R\Gamma_c(M(G_{Q_p}, b_0, \mu), K_p, \mathcal{F}_\ell(d)), (Ri^{b_0,1}V_b)^*) \cong R \text{Hom}_{J_{b_0}(Q_p)}(R\Gamma_c(M(G_{Q_p}, b_0, \mu), K_p, \mathcal{F}_\ell), (Ri^{b_0,1}V_b)^*) \cong R \text{Hom}_{J_{b_0}(Q_p)}(R\Gamma_c(M(G_{Q_p}, b_0, \mu), K_p, \mathcal{F}_\ell), D(Ri^{b_0,1}V_b)) \cong R \text{Hom}_{J_{b_0}(Q_p)}(R\Gamma_c(M(G_{Q_p}, b_0, \mu), K_p, \mathcal{F}_\ell), D(i^{b_0,1}D(i^{b_0}V_b)))) \cong R \text{Hom}_{J_{b_0}(Q_p)}(R\Gamma_c(M(G_{Q_p}, b_0, \mu), K_p, \mathcal{F}_\ell), D(i^{b_0,1}D(i^{b_0}V_b)))) \]
where the second and fourth isomorphisms can be seen from the characterization of reflexive objects [FS, V.6.2]. To show that this vanishes, we may assume \( V_b \) is concentrated in one degree. We may also assume that \( R\Gamma_c(M(G_{Q_p}, b_0, \mu), K_p, \mathcal{F}_\ell) \) itself does not vanish, otherwise there is nothing to prove.

We now look at excursion operators. The support of the action of the spectral Bernstein center \( Z_{\text{spec}}(G_{Q_p}, \mathcal{F}_\ell) \) on \( R\Gamma_c(M(G_{Q_p}, b_0, \mu), K_p, \mathcal{F}_\ell) \) has the only closed point corresponding to \( p_m \), as in the proof of Theorem 1.1 and this L-parameter is generic by assumption. On the other hand, the action of \( Z_{\text{spec}}(G_{Q_p}, \mathcal{F}_\ell) \) on \( D(i^{b_0}V_b) \) is the same as the Verdier dual of the action on \( i^{b_0}V_b \) twisted by the involution \( D_{\text{spec}} \) of \( Z_{\text{spec}}(G_{Q_p}, \mathcal{F}_\ell) \) as follows from [FS, IX.2.2]; compare with [FS, IX.5.3]. In
particular, the support of the action on $D(i_b^tV_b)$ is contained in the support of the action on $i_b^tV_b$, up to the involution $D^{spec}$. As $J_b$ is not quasi-split, the support of the action on $i_b^tV_b$ does not contain any point corresponding to a generic unramified $L$-parameter by Lemma 3.1. As the class of generic unramified $L$-parameters is preserved under $D^{spec}$, we conclude the same for $D(i_b^tV_b)$. This implies the desired vanishing: assume that $Ext^i$ is nonzero for some $i$. We may assume $i = 0$ after replacing $V_b$. Two natural actions of $Z^{spec}(G_{Q_p},\mathbf{F}_\ell)$, via the target or the source, on 

$$\text{Hom}_{\text{spec}}(\mathbf{F}_\ell)\circ (\mathbf{H}_{Q_p}^t,\mathbf{F}_\ell)_{\mathbf{Z}_p}, i_b^t\text{D}(i_b^tV_b) \neq 0$$

are the same since the action of $Z^{spec}(G_{Q_p},\mathbf{F}_\ell)$ on representations of $J_{\text{spec}}(Q_p)$ is functorial. But, this contradicts properties of supports explained above. \hfill \square

Finally let us remark that Theorem 1.4 can be proved in the same way using [CS17] instead:

**Proof of Theorem 1.4.** In this case, Shimura varieties are compact. Using the perverse [CS17] 6.1.3, we can argue as in the proof of Theorem 1.3 to show that $H^i(S_K\mathbf{Q}_\ell)_{\mathbf{Z}_p} \neq 0$ implies $i \geq d$. Using the Poincaré duality, we also see that $i \leq d$ holds, i.e., $i = d$. \hfill \square

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