Relating Gauge Theories via Gauge/Bethe Correspondence

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Abstract

In this note, we use techniques from integrable systems to study relations between gauge theories. The Gauge/Bethe correspondence, introduced by Nekrasov and Shatashvili, identifies the supersymmetric ground states of an $\mathcal{N} = (2, 2)$ supersymmetric gauge theory in two dimensions with the Bethe states of a quantum integrable system. We make use of this correspondence to relate different quiver gauge theories which correspond to different formulations of the Bethe equations of the xxx and the $tJ$ models.
1 Introduction

In this note, we use techniques from integrable systems to study relations between gauge theories. There have been many examples of the interplay between gauge theories (often with string theory interpretations) and integrable systems [1–12]. In [13, 14], a beautiful dictionary between quantum integrable systems such as spin chains, and \( \mathcal{N} = (2, 2) \) supersymmetric gauge theories in two dimensions was introduced. In short, the supersymmetric ground states of the gauge theory are mapped directly to the Bethe spectrum of the integrable model. The effective twisted superpotential of the gauge theory in the Coulomb branch is identified with the Yang–Yang counting function which serves as a potential for the Bethe equations, whose solutions are the spectrum of the integrable system. In [15], this correspondence was extended to four dimensional gauge theories which correspond to Toda or Calogero–Moser models. A brane construction in the topological string A–model for the theories described in [15] was provided in [16]. The Gauge/Bethe correspondence is thought to encompass also the AGT–correspondence [17] which was explained from the point of view of matrix models in [18].
Here we will concentrate on the simpler correspondence between spin chain–type integrable models and supersymmetric gauge theories in two dimensions. We use the Gauge/Bethe correspondence to make a statement about the supersymmetric ground states of seemingly unrelated gauge theories. We use the fact that integrable models can give rise to several different systems of Bethe equations, which nonetheless lead to the same spectrum. Since the Gauge/Bethe correspondence relates the Yang–Yang function to the effective twisted superpotential, different Bethe equations correspond to different gauge theories. This means that the Gauge/Bethe correspondence can be used to relate the low energy properties of different gauge theories. By the correspondence, we know that gauge theories which can be traced back to equivalent Bethe equations have the same supersymmetric ground states.

The simplest instance of this phenomenon is found in the XXX spin chain with twisted boundary conditions, where configurations with \( N \) magnons can be mapped into configurations with \( L - N \) magnons which describe the same state. Another, richer, example is the so–called \( tJ \) model. The fact that the \( tJ \) spin chain has the supergroup \( sl(1|2) \) as symmetry group leads to three different, but ultimately equivalent sets of Bethe equations, corresponding to the different choices of the Cartan matrix of \( sl(1|2) \).

On the other side of the correspondence, we have three quiver–type gauge theories with two different gauge groups each. Their supersymmetric ground states are thus shown to be equivalent. Note that our statement is stronger than a mere counting of the vacua of the three theories. The three sets of Bethe equations are actually different ways of writing the same non–linear conditions. It is reasonable to expect the correspondence to go beyond the ground states and to relate also the solitons which interpolate between them. To give more weight to our claim, we show that at least in the case of zero twisted masses, it is possible to embed the three quiver gauge theories into string theory in terms of brane cartoons of \( D_2 \), \( D_4 \) and NS5–branes, and that it is possible to relate them via brane motions. While we exemplify our approach using the \( tJ \) model, we believe that it can be used in a wider context. We believe that new insights into supersymmetric gauge theories in two dimensions can be obtained by expressing them as quantum integrable systems.

The plan of this note is the following. In Section 2, we briefly review \( \mathcal{N} = (2, 2) \) supersymmetric gauge theories in two dimensions. We introduce the field content and action (§ 2.1), the low energy effective action (§ 2.2), and finally the three quiver gauge theories we set out to relate in this note (§ 2.4). In Section 3, we introduce integrable spin chains (§ 3.1), spell out the necessary knowledge of the algebraic Bethe ansatz (§ 3.2), state the Gauge/Bethe correspondence as described by Nekrasov and Shatashvili (§ 3.3), and introduce the \( tJ \) model (§ 3.5). The explicit matching is done in § 3.6. As an alternative justification, brane cartoons for the massless cases of our quiver gauge theories are introduced in Section 4, where also their relation via brane transitions is detailed. In Section 5, we conclude with a view on more general applications of the approach taken in this note. The basics of the superalgebra \( sl(1|2) \) are collected in Appendix A, while the proof of the equivalence of the three Bethe equations underlying the gauge theories is given in Appendix B.
\section{\( \mathcal{N} = (2,2) \) supersymmetric gauge theories in two dimensions}

In this section, we will introduce the necessary notation for dealing with \( \mathcal{N} = (2,2) \) gauge theories in two dimensions, discuss their low energy action after integrating out massive fields, and finally present three seemingly unrelated quiver gauge theories, which will be our explicit examples throughout this note.

\subsection{Field content and action}

Let us quickly review the basics of \( \mathcal{N} = (2,2) \) theories. We will use the notation of \cite{19}; for greater detail, we refer our readers to \cite{19-21}.

\( \mathcal{N} = (2,2) \) theories are field theories in \( 1 + 1 \) dimensions with two (real) positive and two (real) negative chirality supercharges. Superspace is described by the two bosonic coordinates \( x^0, x^1 \), and the four fermionic coordinates \( \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^- \). We define the differential operators

\[
D_\pm = \frac{\partial}{\partial \theta^\pm} - i \bar{\theta}^\pm \partial_\pm, \quad \overline{D}_\pm = - \frac{\partial}{\partial \bar{\theta}^\pm} + i \theta^\pm \partial_\pm. \tag{2.1}
\]

The \( \theta \)–expansion of the vector superfield in Wess–Zumino gauge is given by

\[
V = \theta^+ \bar{\theta}^-(A_0 - A_1) + \theta^+ \bar{\theta}^+(A_0 + A_1) - \theta^- \bar{\theta}^+ \sigma - \theta^+ \bar{\theta}^- \sigma \\
+ i \theta^+ \bar{\theta}^- (\bar{\theta}^- \lambda_- + \bar{\theta}^+ \lambda_+) + i \bar{\theta}^+ \theta^- (\theta^- \lambda_- + \theta^+ \lambda_+) + \theta^- \theta^+ \bar{\theta}^- D, \tag{2.2}
\]

where \( A_\mu \) is a vector field, \( \lambda_\pm, \bar{\lambda}_\pm \) are Dirac fermions which are conjugate to each other, \( \sigma \) is a complex scalar, and \( D \) is a real auxiliary field. With this, we can now define the gauge covariant derivative

\[
D_\pm = e^{-V} D_\pm e^V, \quad \overline{D}_\pm = e^V \overline{D}_\pm e^{-V}. \tag{2.3}
\]

A chiral superfield satisfies \( \overline{D}_\pm \Phi = 0 \). The \( \theta \)–expansion of the chiral superfield is given by

\[
\Phi = \phi(y^\pm) + \theta^a \psi_a(y^\pm) + \theta^+ \theta^- F(y^\pm), \tag{2.4}
\]

where \( \phi \) is a complex scalar field, \( \psi_a \) a Dirac fermion, \( F \) a complex auxiliary field, \( y^\pm = x^\pm - i \theta^\pm \bar{\theta}^\pm \), and \( x^\pm = x_0 \pm x_1 \). A twisted chiral superfield satisfies \( \overline{D}_+ \Sigma = D_- \Sigma = 0 \). The super field strength \( \Sigma = \frac{1}{2} \{ \overline{D}_+, D_- \} \) is a twisted chiral superfield and its \( \theta \)–expansion is given by

\[
\Sigma = \sigma(y^\pm) + i \theta^+ \bar{\lambda}_+(y^\pm) - i \bar{\theta}^- \lambda_-(y^\pm) + \theta^+ \bar{\theta}^- [D(y^\pm) - i A_{01} (y^\pm)] + \ldots, \tag{2.5}
\]

where \( \bar{\theta}^\pm = x^\pm \mp i \theta^\pm \bar{\theta}^\pm \), and \( A_{01} = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] \).

The \( \mathcal{N} = (2,2) \) field theory in two dimensions can be understood as the dimensional reduction of \( \mathcal{N} = 1 \) supersymmetric gauge theory in four dimensions. The scalar \( \sigma \)
results from the $x^{2,3}$ components of the vector field $A_\mu$ in four dimensions.

In the supersymmetric action, there are three kinds of couplings:

- the $D$–term: $\int d^2 x \ d^4 \theta \ K$, where $K$ is an arbitrary (real) differential function of the superfields,
- the $F$–term (plus its Hermitian conjugate): $\int d^2 x \ d\theta^+ \ d\theta^- W |_{\theta^+ = 0} + \text{h.c.}$, where the superpotential $W$ is a holomorphic function of the chiral multiplets,
- the twisted $F$–term (plus its Hermitian conjugate): $\int d^2 x \ d\bar{\theta}^- \ d\theta^+ \ \tilde{W} |_{\theta^+ = \theta^- = 0} + \text{h.c.}$, where the twisted superpotential $\tilde{W}$ is a holomorphic function of the twisted superfields.

The supersymmetric structure implies some decoupling and non–renormalization theorems: in particular neither the $F$–term nor the twisted $F$–term get renormalized. Moreover, in the effective action, the $F$–term and twisted $F$–term cannot mix.

Let us consider a gauge theory with gauge group $G$ and chiral matter multiplets $X_k$ (we denote them by $Q$ if they are in the fundamental, $\bar{Q}$ if they are in the anti–fundamental, $B$ if they are in the bifundamental, and $\Phi$ if they are in the adjoint representation). The kinetic term of the Lagrangian is given by

$$L_{\text{kin}} = \int d^4 \theta \left( \sum_k X_k^+ e^V X_k - \frac{1}{2e^2} \text{Tr}(\Sigma^+ \Sigma) \right), \tag{2.6}$$

where $e$ is the gauge field strength. We can consider the following additional terms:

- Fayet–Iliopoulos (FI) and theta–term: $L_{\text{FI,} \theta} = -\frac{i}{2} \tau \int d\bar{\theta}^- d\theta^+ \text{Tr} \Sigma + \text{h.c.}$, where $\tau = i r + \theta/2\pi$. Such a term can be turned on for every $U(1)$–factor in the center of $G$.
- The complex mass: $L_{\text{mass}} = \sum_{k,l} \int d^2 \theta \ m_k^l \bar{X}_l X_k + \text{h.c.}$, where $m_k^l$ are complex parameters.
- The twisted masses: $L_{\text{tw}} = \int d^4 \theta \ (X^t e^\theta \bar{\theta}^+ \bar{m}_X + \text{h.c.})$, where $e^\theta \bar{\theta}^+ \bar{m}_X$ are matrices in the same representation as $X$ of the maximal torus of the global symmetry group.\footnote{It is possible to obtain the same theory by dimensional reduction from six dimensions. In this case, the $x^{4,5}$ components of the vector field turn into the complex scalar $\phi$ of the chiral multiplet in the adjoint representation.}

While the complex masses are already present in the four dimensional theory, this is not the case for the twisted masses. The twisted masses are deformations of the theory which are related to the global symmetry group of the theory. They can be obtained by first gauging the global symmetry group, giving a vev to the scalar component of the vector superfield, and in the end making the fields vanish \cite{21}. The global

\footnote{In the rest of this note, all the twisted masses are defined up to a scale factor $u$ that we set equal to 1.}
symmetry group $H$ of an $\mathcal{N} = (2, 2)$ gauge theory receives a factor of $U(L_i)$ for each set of $L_i$ fundamental or anti–fundamental multiplets $Q^i$. For each adjoint multiplet $\Phi^i$ and each bifundamental multiplet $B_{ij}^i$, $H$ receives a further factor of $U(1)$. Moreover, there are two $U(1)$ R–symmetries which are usually denoted by $U(1)_V$ and $U(1)_A$. Turning on twisted masses will break $H$ down to its maximal torus. Also turning on a superpotential will in general break the global symmetry group to some extent. This implies that a general superpotential will be incompatible with general twisted masses. However, special choices of the superpotential and twisted mass parameters are possible which allow both deformations to coexist.

Once the twisted masses are turned on, the matter fields become massive and can be integrated out to obtain a low energy effective action.

2.2 Low energy effective action

In this section, we describe the Coulomb branch of the theory. We therefore consider the low energy effective theory obtained for slowly varying $\sigma$ fields after integrating out the massive matter fields. In this way, we obtain an effective twisted superpotential $\tilde{W}_{\text{eff}}(\Sigma)$; the vacua of the theory $[16]$ are the solutions of the equation

$$\exp\left[2\pi \frac{\partial \tilde{W}_{\text{eff}}(\sigma)}{\partial \sigma_i}\right] = 1.$$ (2.7)

Consider the case of a $U(1)$ gauge theory with one chiral superfield $Q$ of charge 1 and twisted mass $\tilde{m}_Q$. The most general supersymmetric action containing terms with at most four fermions and two derivatives is given by

$$S_{\text{eff}}(\Sigma) = -\int d^4 \theta K_{\text{eff}}(\Sigma, \Sigma) + \frac{1}{2} \int d^2 \theta \tilde{W}_{\text{eff}}(\Sigma) + \text{h.c.}.$$ (2.8)

In the absence of an $F$–term, the action $S(\Sigma, Q)$ is quadratic in $Q$, and the effective action can be evaluated exactly via a one–loop calculation:

$$e^{S_{\text{eff}}(\Sigma)} = \int DQ e^{S(\Sigma, Q)}.$$ (2.9)

The bosonic determinant equals

$$\int \frac{d^2 k}{(2\pi)^2} \log(k^2 + |\sigma - \tilde{m}_Q|^2 + D),$$ (2.10)

and expanding in powers of $D$,

$$\log(k^2 + |\sigma - \tilde{m}_Q|^2 + D) = \log(k^2 + |\sigma - \tilde{m}_Q|^2) + \frac{D}{k^2 + |\sigma - \tilde{m}_Q|^2} + \ldots$$ (2.11)

The zeroth order term is cancelled by the fermionic determinant, while the first order
term leads, after integrating over the momenta, to the effective twisted superpotential

$$\tilde{W}_{\text{eff}}(\Sigma) = \frac{1}{2\pi} (\Sigma - \tilde{m}_Q) \left( \log(\Sigma - \tilde{m}_Q) - 1 \right) - i\tau \Sigma, \quad (2.12)$$

where we also added the contribution of the Fayet–Iliopoulos term. In the general case, an $F$–term is possible but, thanks to the decoupling theorem, it would not change the expression of the effective twisted superpotential.

In the following, we will be mainly interested in quiver gauge theories. In this case, the gauge group $G$ and the flavor group $F$ are direct products:

$$G = \prod_{a=1}^{r} U(N_a), \quad F = \prod_{a=1}^{r} U(L_a). \quad (2.13)$$

These theories can be represented via quiver diagrams. Each factor $U(N_a)$ corresponds to a node, a bifundamental field in the representation $N_a \otimes N_b$ is denoted by an arrow going from node $a$ to node $b$, and an adjoint field is an arrow starting and ending on the same node. Each component $U(L_a)$ of the flavor group is represented by an extra node, joined by a dotted arrow to the relevant component of the gauge group (see Figure 1 for some examples). The evaluation of the effective twisted superpotential in the non–Abelian case is very similar to the $U(1)$ calculation once one observes that the classical vacuum equations require $\sigma$ to be diagonalizable. If we assume that $\sigma_i \neq \sigma_j$ for $i \neq j$, which breaks the gauge group $U(N)$ to its maximal torus $U(1)^N$, we can perform exactly the same Gaussian integration of the chiral fields as above. We thus obtain the following contributions to the effective twisted superpotential:

- For each fundamental field $Q_k$ with twisted mass $\tilde{m}_k^f$:
  $$\tilde{W}_{\text{eff}}^f = \frac{1}{2\pi} \sum_{i=1}^{N} \left( \sigma_i - \tilde{m}_k^f \right) \left( \log(\sigma_i - \tilde{m}_k^f) - 1 \right). \quad (2.14)$$

- For each anti–fundamental field $\overline{Q}_k$ with twisted mass $\tilde{m}_k^\overline{f}$:
  $$\tilde{W}_{\text{eff}}^{\overline{f}} = \frac{1}{2\pi} \sum_{i=1}^{N} \left( -\sigma_i - \tilde{m}_k^{\overline{f}} \right) \left( \log(-\sigma_i - \tilde{m}_k^{\overline{f}}) - 1 \right). \quad (2.15)$$

- For each adjoint field $\Phi$ with twisted mass $\tilde{m}_k^{\text{adj}}$:
  $$\tilde{W}_{\text{eff}}^{\text{adj}} = \frac{1}{2\pi} \sum_{i,j=1}^{N} \left( \sigma_i - \sigma_j - \tilde{m}_k^{\text{adj}} \right) \left( \log(\sigma_i - \sigma_j - \tilde{m}_k^{\text{adj}}) - 1 \right). \quad (2.16)$$
For each bifundamental $B^{12}$ in the representation $N_1 \otimes N_2$ and twisted mass $\tilde{m}^b$:

$$\tilde{W}_{\text{eff}}^{b} = \frac{1}{2\pi} \sum_{i=1}^{N_1} \sum_{p=1}^{N_2} \left( -\sigma_i^{(1)} + \sigma_p^{(2)} - \tilde{m}^b \right) \left( \log(-\sigma_i^{(1)} + \sigma_p^{(2)} - \tilde{m}^b) - 1 \right),$$  \hspace{1cm} (2.17)

where the $\sigma_i^{(a)}$ are the scalar components of the vector multiplet for the group $U(N_a)$.

### 2.3 Example: two gauge theories

Consider the $N = (2, 2)$ theory with gauge group $U(N)$ and the following matter content \[13\]:

- an adjoint field $\Phi$ with twisted mass $\tilde{t}$,
- $L$ fundamentals and anti–fundamentals $Q_k, \bar{Q}_k$ with twisted mass $-\tilde{t}/2$.

The effective twisted superpotential is given by:

$$\tilde{W}_{\text{eff}}^N(\sigma) = \frac{L}{2\pi} \sum_{i=1}^{N} \left[ (\sigma_i + \frac{\tilde{t}}{2}) \left( \log(\sigma_i + \frac{\tilde{t}}{2}) - 1 \right) - (\sigma_i - \frac{\tilde{t}}{2}) \left( \log(-\sigma_i + \frac{\tilde{t}}{2}) - 1 \right) \right]$$

$$+ \frac{1}{2\pi} \sum_{i \neq j}^N (\sigma_i - \sigma_j - \tilde{t}) \left( \log(\sigma_i - \sigma_j - \tilde{t}) - 1 \right) - \tau \sum_{i=1}^{N} \sigma_i. \hspace{1cm} (2.18)$$

We intend to compare it to a system with gauge group $U(L - N)$, $L$ fundamental and antifundamentals and opposite Fayet–Iliopoulos term, which admits the following effective twisted superpotential:

$$\tilde{W}_{\text{eff}}^{L-N}(\sigma) = \frac{L}{2\pi} \sum_{i=1}^{L-N} \left[ (\sigma_i + \frac{\tilde{t}}{2}) \left( \log(\sigma_i + \frac{\tilde{t}}{2}) - 1 \right) - (\sigma_i - \frac{\tilde{t}}{2}) \left( \log(-\sigma_i + \frac{\tilde{t}}{2}) - 1 \right) \right]$$

$$+ \frac{1}{2\pi} \sum_{i \neq j}^{L-N} (\sigma_i - \sigma_j - \tilde{t}) \left( \log(\sigma_i - \sigma_j - \tilde{t}) - 1 \right) + \tau \sum_{i=1}^{N} \sigma_i. \hspace{1cm} (2.19)$$

### 2.4 Example: three quiver gauge theories

The main claim of this note is that the three quiver gauge theories described below, which have different gauge groups and different matter contents, share the same chiral ring (and therefore have the same supersymmetric vacua). The three theories are given as follows:

**Case A**  A quiver gauge theory with gauge groups $U(N_h + N_i)$ and $U(N_i)$ (the reason for the names of the parameters $N_h$ and $N_i$ will become clear in the following), with the following matter content:
a bifundamental $B^{12}$ in the representation $\overline{N_h} \otimes (N_h + N_L)$ with twisted mass $i/2$,

- a bifundamental $B^{21}$ in the representation $(\overline{N_h} + N_L) \otimes N_h$ with twisted mass $i/2$,

- an adjoint $\Phi^2$ for $U(N_h + N_L)$ with twisted mass $i$,

- $L$ fundamentals and anti–fundamentals $(Q^2_k, \overline{Q^2_k})$ for $U(N_h + N_L)$ with twisted mass $-i/2$.

The global symmetry group $H$ (which is broken down to its maximal torus by the twisted masses) is $U(L)_Q \times U(L)_{\overline{Q}} \times U(1)_B \times U(1)_{\overline{B}} \times U(1)_{\Phi}$. The quiver diagram is represented in Figure 1(a) Using the results above, we find the following effective twisted superpotential

$$W^{\text{eff}}_{ih}(\sigma) = \frac{L}{2\pi} \sum_{p=1}^{N_h+N_L} \left[ \left( \sigma_p^{(2)} + \frac{i}{2} \right) \left( \log(\sigma_p^{(2)} + \frac{i}{2}) - 1 \right) - \left( \sigma_p^{(1)} - \frac{i}{2} \right) \left( \log(-\sigma_p^{(1)} + \frac{i}{2}) - 1 \right) 
+ \frac{1}{2\pi} \sum_{p=1}^{N_h+N_L} \left( \sigma_i^{(1)} - \sigma_p^{(2)} - \frac{i}{2} \right) \left( \log(\sigma_i^{(1)} - \sigma_p^{(2)} + \frac{i}{2}) - 1 \right) - \left( \sigma_i^{(1)} - \sigma_p^{(2)} + \frac{i}{2} \right) \left( \log(-\sigma_i^{(1)} + \sigma_p^{(2)} - \frac{i}{2}) - 1 \right) 
+ \frac{1}{2\pi} \sum_{p,q=1}^{N_h+N_L} \left( \sigma_p^{(2)} - \sigma_q^{(2)} - i \right) \left( \log(\sigma_p^{(2)} - \sigma_q^{(2)} - i) - 1 \right) - ir_1 \sum_{i=1}^{N_h} \sigma_i^{(1)} - ir_2 \sum_{p=1}^{N_h+N_L} \sigma_p^{(2)} \right]. \quad (2.20)$$

The twisted masses are compatible with a superpotential of the type

$$W_A(Q^2, \overline{Q^2}, \Phi^2, B^{12}, B^{21}) = \sum_k \left[ a Q^2_k \Phi^2 Q^2_{\overline{k}} + b Q^2_k B^{21} Q^2_{\overline{k}} \right], \quad (2.21)$$

where $a$ and $b$ are parameters.

**Case B**  A quiver gauge theory with gauge groups $U(N_h + N_L)$ and $U(N_L)$, with the following matter content:

- a bifundamental $B^{12}$ in the representation $\overline{N_L} \otimes (N_h + N_L)$ and twisted mass $i/2$,

- a bifundamental $B^{21}$ in the representation $(\overline{N_h} + \overline{N_L}) \otimes N_L$ and twisted mass $i/2$,
• $L$ fundamentals and anti–fundamentals $(Q^2_k, \overline{Q}^2_k)$ for $U(N_h + N_i)$ with twisted mass $-i/2$.

The global symmetry group $H$ (which is broken down to its maximal torus by the twisted masses) is $U(L)_Q \times U(L)_{\overline{Q}} \times U(1)_B \times U(1)_{\overline{B}}$. The quiver diagram is shown in Figure 1(b). In this case, the effective twisted superpotential is given by

$$
\hat{W}^{\text{eff}}_{\overline{B}}(\sigma) = \frac{L}{2\pi} \sum_{p=1}^{N_i + N_i} \left[ \left( \sigma_p^{(2)} + \frac{i}{2} \right) \left( \log(\sigma_p^{(2)} + \frac{i}{2}) - 1 \right) - \left( \sigma_p^{(2)} - \frac{i}{2} \right) \left( \log(-\sigma_p^{(2)} + \frac{i}{2}) - 1 \right) \right]
+ \frac{1}{2\pi} \sum_{i=1}^{N_i} \sum_{p=1}^{N_i} \left[ \left( \sigma_i^{(1)} - \sigma_p^{(2)} - \frac{i}{2} \right) \left( \log(\sigma_i^{(1)} - \sigma_p^{(2)} - \frac{i}{2}) - 1 \right) \right]
- \left( \sigma_i^{(1)} - \sigma_p^{(2)} + \frac{i}{2} \right) \left( \log(-\sigma_i^{(1)} + \sigma_p^{(2)} - \frac{i}{2}) - 1 \right) - i\tau_1 \sum_{i=1}^{N_i} \sigma_i^{(1)} - i\tau_2 \sum_{p=1}^{N_i} \sigma_p^{(2)}. \quad (2.22)
$$

The twisted masses are compatible with a superpotential of the type

$$
W_B(Q^2, \overline{Q}^2, B^{12}, B^{21}) = a \sum_{k} \left[ Q^2_k B^{21} B^{12} \overline{Q}^2_k \right]. \quad (2.23)
$$

**Case C** A quiver gauge theory with gauge groups $U(N_c)$ and $U(N_i)$, with the following matter content:

• a bifundamental field $B^{12}$ in the representation $\overline{N}_c \otimes N_e$ with twisted mass $i/2$,
• a bifundamental field $B^{21}$ in the representation $\overline{N}_e \otimes N_i$ with twisted mass $i/2$,
• an adjoint field $\Phi^1$ for $U(N_i)$ with mass $i$,
• $L$ fundamental and anti–fundamentals fields $(Q^2_k, \overline{Q}^2_k)$ for $U(N_e)$ with mass $-i/2$.

The global symmetry group $H$ (which is broken down to its maximal torus by the twisted masses) is $U(L)_Q \times U(L)_{\overline{Q}} \times U(1)_B \times U(1)_{\overline{B}} \times U(1)_{\Phi}$. The quiver diagram is given in Figure 1(c). The effective twisted superpotential reads

$$
\hat{W}^{\text{eff}}_{\Phi}(\sigma) = \frac{L}{2\pi} \sum_{p=1}^{N_i} \left[ \left( \sigma_p^{(2)} + \frac{i}{2} \right) \left( \log(\sigma_p^{(2)} + \frac{i}{2}) - 1 \right) - \left( \sigma_p^{(2)} - \frac{i}{2} \right) \left( \log(-\sigma_p^{(2)} + \frac{i}{2}) - 1 \right) \right]
+ \frac{1}{2\pi} \sum_{i=1}^{N_i} \sum_{p=1}^{N_i} \left[ \left( \sigma_i^{(1)} - \sigma_p^{(2)} - \frac{i}{2} \right) \left( \log(\sigma_i^{(1)} - \sigma_p^{(2)} - \frac{i}{2}) - 1 \right) \right]
- \left( \sigma_i^{(1)} - \sigma_p^{(2)} + \frac{i}{2} \right) \left( \log(-\sigma_i^{(1)} + \sigma_p^{(2)} - \frac{i}{2}) - 1 \right) - i\tau_1 \sum_{i=1}^{N_i} \sigma_i^{(1)} - i\tau_2 \sum_{p=1}^{N_i} \sigma_p^{(2)}. \quad (2.24)
$$
The twisted masses are compatible with a superpotential of the type

\[ W_C(Q^2, \overline{Q}^2, \Phi^1, B^{12}, B^{21}) = \sum_k \left[ a B^{21} \Phi^1 B^{12} + b Q^2_k B^{21} B^{12} \overline{Q}^2_k \right]. \quad (2.25) \]

Even though these three theories have different gauge groups and field content, we will show with the help of the Gauge/Bethe correspondence that their supersymmetric ground states are the same.

### 3 Gauge/Bethe correspondence

The Gauge/Bethe correspondence, as detailed in [13, 14], relates two dimensional \( \mathcal{N} = (2, 2) \) supersymmetric gauge theories to quantum integrable systems. The supersymmetric vacua of the gauge theories form a representation of the chiral ring, which is a distinguished class of operators which are annihilated by one chirality of the supercharges \( Q \). The commuting Hamiltonians of the quantum integrable system are identified with the generators of the chiral ring. The space of states of the quantum integrable system, i.e. the spectrum of the commuting Hamiltonians, is thus mapped to the supersymmetric vacua of the gauge theory. Arguably, this correspondence holds true for all integrable systems, in the sense that to any spin chain solvable by the Bethe ansatz, we can associate a corresponding \( \mathcal{N} = (2, 2) \) gauge theory.

#### 3.1 Parameters of a general spin chain

In this section, we collect the possible parameters of the quantum integrable systems which we will need to match to the parameters of the \( \mathcal{N} = (2, 2) \) gauge theories. We will be very brief; for more detail, we refer the reader to the original work [13]. We are only considering integrable systems which correspond to two–dimensional gauge theories, therefore we only look at spin chains without anisotropy\(^3\).

Quantum integrable systems in \( 1 + 1 \) dimensions usually correspond to spin chain–type systems. Such a system lives on a one–dimensional lattice of length \( L \). To each point \( k \) we associate a representation \( \Lambda \) of the symmetry group \( K \) and call the corresponding Hilbert space \( \mathcal{H}_k \). The dynamics is described by the Hamiltonian

\[ \mathcal{H} = -\sum_{k=1}^{L-1} [\Pi_{k,k+1} - 1], \quad (3.1) \]

where \( \Pi_{k,k+1} \) is the permutation operator between the points \( k \) and \( k + 1 \). Moreover, one needs to specify boundary conditions via an operator \( \mathcal{K} \in \text{End}(\mathcal{H}) \). For a closed spin chain, \( \mathcal{K} \) depends on \( r = \text{rank}(K) \) twist parameters \( \{ \theta_a \}_{a=1}^r \).

Since \( \mathcal{H} \) commutes with the maximal torus \( T \subset K \) (summed over the chain), we can

\(^3\)By the term anisotropy we refer to the spin interactions in the Hamiltonian not being the same in the \( x, y \) and \( z \) directions, as is the case in the \( xxz \) and \( xyz \) models.
decompose the Hilbert space of states into a direct sum

\[ \mathcal{H} = \bigotimes_{k=1}^{L} \mathcal{H}_{k} = \bigoplus_{a=1}^{r} \bigoplus_{N_a=0}^{L_k} \mathcal{H}^{(a)}_{N_a}. \] (3.2)

An element \( \Psi \in \mathcal{H}^{(a)}_{N_a} \) is a magnon describing a state with \( N_a \) particles of species \( a \). The magnon \( \Psi \) depends on the rapidities (quasi–momenta) \( \{ \lambda_i^{(a)} \}_{i=1}^{N_a} \). There can be different effective lengths \( L_a \) for each species. For a general spin chain, each point \( k = 1, \ldots, L \) can carry a different representation \( \Lambda_k = [\Lambda_k^1, \ldots, \Lambda_k^r] \) of the symmetry group, and furthermore, one can turn on inhomogeneities \( v_k^{(a)} \) (which can be understood as displacements) in each position of the chain. Special cases are the xxx spin chain, where each point carries the fundamental representation of \( su(2) \), and the \( tJ \) model where each point carries the fundamental representation of \( sl(1|2) \).

### 3.2 Algebraic Bethe ansatz

In this section, we introduce some necessary definitions from the theory of integrable models. In particular, we show how to construct a system of commuting Hamiltonians starting from the Yang–Baxter relations for a graded (supersymmetric) vector space \( C^{(m|n)} \). In the special case \( C^{(1|2)} \), the construction provides the Bethe ansatz for the \( tJ \) model, which is related via the Gauge/Bethe correspondence to the three quiver gauge theories introduced in Sec. 2.4. A pedagogical introduction to the algebraic Bethe ansatz can be found in [22].

Consider a homogeneous chain of length \( L \) where each position carries the fundamental representation of the algebra \( sl(m|n) \), i.e. we have \( m \) bosonic and \( n \) fermionic degrees of freedom. Associate to each point a copy of the Hilbert space \( \mathcal{H} = C^{(m|n)} \), where \( C^{(m|n)} \) is a \( \mathbb{Z}_2 \)–graded vector space \( C^{(m|n)} = C^m \oplus C^n \) with parity

\[ |x| = \begin{cases} 0 & \text{if } x \in C^m \\ 1 & \text{if } x \in C^n. \end{cases} \] (3.3)

Introduce now the matrix (linear operator on \( C^{(m|n)} \otimes C^{(m|n)} \)) \( R(\lambda) \), depending on the spectral parameter \( \lambda \). The matrix \( R \) satisfies the Yang–Baxter equation (YBE) if the following identity (on \( C^{(m|n)} \otimes C^{(m|n)} \otimes C^{(m|n)} \)) holds:

\[ (1 \otimes R(\lambda - \mu)) (R(\lambda) \otimes 1) (1 \otimes R(\mu)) = (R(\mu) \otimes 1) (1 \otimes R(\lambda)) (R(\lambda - \mu) \otimes 1). \] (3.4)

The solution to the YBE can be written in terms of the identity and the permutation operator

\[ R(\lambda) = \frac{t}{\lambda + t} \mathbb{1} + \frac{\lambda}{\lambda + t} \Pi. \] (3.5)
Moreover, the ybe can be rewritten in the form

\[ R_{12}(\lambda - \mu) \left( \Pi_{13} R_{13}(\lambda) \otimes \Pi_{23} R_{23}(\mu) \right) = \left( \Pi_{13} R_{13}(\mu) \otimes \Pi_{23} R_{23}(\lambda) \right) R_{12}(\lambda - \mu), \quad (3.6) \]

where the indices 1, 2, 3 indicate on which of the three \( C^{(m|n)} \) spaces the operator is acting. In this form, we have singled out the third space. Now we can choose to consider it differently from the other two and interpret it as the Hilbert space \( \mathcal{H}_k \) living on a point of the chain while the others take the role of auxiliary spaces. In turn, we can require a ybe to be satisfied at each point. It is convenient to introduce the Lax operator \( L_k \) at the point \( k \) as follows:

\[ L_k(\lambda) = \Pi R(\lambda) = \frac{1}{\lambda + i} \Pi + \frac{\lambda}{\lambda + i} \Pi. \quad (3.7) \]

Since \( \mathcal{H} \) is an inner quantum space, \( L_k(\lambda) \) can now be seen as an \((m + n) \times (m + n)\) matrix, whose entries are quantum operators. The ybe at point \( k \) is written as

\[ R(\lambda - \mu) \left( L_k(\lambda) \otimes L_k(\mu) \right) = \left( L_k(\mu) \otimes L_k(\lambda) \right) R(\lambda - \mu). \quad (3.8) \]

In physical terms, we can interpret the Lax operator as a connection along the chain, in the sense that \( L_k \) defines the transport between the points \( k \) and \( k + 1 \), via the Lax equation:

\[ \Psi_{k+1} = L_k \Psi_k, \quad (3.9) \]

where the vector \( \Psi \) has \( m + n \) entries in \( \mathcal{H} \). It is thus natural to define the monodromy matrix \( T(\lambda) \) as the ordered product of \( L_k(\lambda) \) along the chain:

\[ T(\lambda) = L_L(\lambda) L_{L-1}(\lambda) \cdots L_1(\lambda). \quad (3.10) \]

One can show by induction that the monodromy matrix satisfies the same ybe:

\[ R(\lambda - \mu) \left( T(\lambda) \otimes T(\mu) \right) = \left( T(\mu) \otimes T(\lambda) \right) R(\lambda - \mu). \quad (3.11) \]

Taking the (super) trace of \( T(\lambda) \) in the auxiliary space gives the transfer matrix \( t(\lambda) \):

\[ t(\lambda) = \text{Tr}[T(\lambda)], \quad (3.12) \]

the ybe implies that transfer matrices at different values of the spectral parameter commute:

\[ [t(\lambda), t(\mu)] = 0. \quad (3.13) \]

This is the fundamental property that turns \( t(\lambda) \) into the generating object for the \( L - 1 \) integrals of motion that make the system integrable. It is customary to define the \( L - 1 \) Hamiltonians \( \mathcal{H}_{(l)} \) as the coefficients of the development

\[ \log[t(\lambda)t(0)^{-1}] = \sum_{l=1}^{L} \frac{l}{l!} \mathcal{H}_{(l)} \quad (3.14) \]
In particular, the logarithmic derivative of \( t(\lambda) \) in \( \lambda = 0 \) is the Hamiltonian for the spin chain that we introduced in Eq. (3.1):

\[
\mathcal{H} = \mathcal{H}_{(1)} = -i \left. \frac{d}{d\lambda} \log t(\lambda) \right|_{\lambda=0} = -\sum_{k=1}^{L-1} [\Pi_{k,k+1} - 1].
\] (3.15)

Eigenvectors for the transfer matrix are at the same time eigenvectors for all the commuting Hamiltonians. In order to construct them, consider the simplest case \( n = 2, m = 0 \), corresponding to the xxx spin chain. The monodromy can be written as an operator–valued \( 2 \times 2 \) matrix:

\[
T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.
\] (3.16)

Introduce a reference state \( \Omega \) such that

\[
\Omega = \bigotimes_{k=1}^{L} \omega_k, \quad L_k \omega_k = \begin{pmatrix} \alpha(\lambda) & * \\ 0 & \delta(\lambda) \end{pmatrix} \omega_k \Rightarrow T(\lambda) \Omega = \begin{pmatrix} \alpha(\lambda)^L & * \\ 0 & \delta(\lambda)^L \end{pmatrix} \Omega,
\] (3.17)

where \( \alpha(\lambda) = \lambda + i/2 \) and \( \delta(\lambda) = \lambda - i/2 \), then by construction \( \Omega \) is an eigenvector for \( t(\lambda) \):

\[
t(\lambda) \Omega = \left( \alpha(\lambda)^L + \delta(\lambda)^L \right) \Omega.
\] (3.18)

The algebraic Bethe ansatz consists in looking for eigenvectors of \( t(\lambda) \) of the form

\[
\Phi(\{\lambda\}) = B(\lambda_1) B(\lambda_2) \cdots B(\lambda_N) \Omega.
\] (3.19)

Imposing the ybe for the monodromy matrix, one finds that \( \Phi(\{\lambda\}) \) is an eigenvector if and only if the parameters \( \{\lambda\} \) satisfy the Bethe equations:

\[
\left( \frac{\lambda_i + \frac{i}{2}}{\lambda_i - \frac{i}{2}} \right)^L = \prod_{j=1}^{N} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}, \quad i = 1, 2, \ldots, N.
\] (3.20)

The case of a higher rank symmetry group is analyzed with a recursive procedure known as the nested Bethe ansatz (see [23, 24]). For \( n + m > 2 \), applying the construction above produces, together with a set of Bethe equations for variables \( \{\lambda^{(1)}\} \), a new Lax operator with rank \( n + m - 1 \). The construction can now be repeated, and at each step one obtains a system of equations for a new set of variables \( \{\lambda^{(a)}\} \) and a new Lax operator of rank \( n + m - a \). After \( n + m - 2 \) steps, one arrives at the final set of equations. The overall result can be put into a very elegant form in which the Bethe equations only depend on the root space decomposition of the symmetry group [25].

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Explicitly,

\[
\left( \frac{\lambda_i^{(a)} + \frac{1}{2} \Lambda^a}{\lambda_i^{(a)} - \frac{1}{2} \Lambda^a} \right)_i^L = \prod_{(b,j) = (1,1)}^{r N_a} \lambda_i^{(a)} - \lambda_j^{(b)} + \frac{i}{2} C^{ab}, \quad a = 1, 2, \ldots, r, \quad i = 1, 2, \ldots, N_a,
\]

(3.21)

where \( r = \text{rank}(sl(m|n)) = n + m - 1 \), \( C^{ab} \) is Cartan matrix, and \( [\Lambda^1, \ldots, \Lambda^r] \) is the highest weight of the representation.\(^4\) Two remarks are of importance:

1. Even though we started by considering the fundamental representation for \( sl(m|n) \), the Bethe ansatz equations (3.21) are more general and valid for an arbitrary representation \( \Lambda \) of the symmetry group \( K \).

2. The result depends on the choice of the Cartan matrix \( C^{ab} \) of the symmetry group. While this is unique (up to conjugation) for \( sl(n) \), this is not the case for \( sl(m|n) \), where there are \( \binom{m+n}{m} \) conjugacy classes of Borel subalgebras. This means that for a given spin chain (and for a given ring of commuting Hamiltonians), there are \( \binom{m+n}{m} \) sets of Bethe ansatz equations which are by construction equivalent.

We would like to stress that for a given spin chain with supergroup symmetry, there is a unique ring of commuting Hamiltonians, but the choice of Borel subalgebra can lead to different–looking Bethe equations. This is the property that we will use in the following to prove that different quiver gauge theories (one for each choice of Bethe equations) have the same ground states (which are ultimately identified by the ring of Hamiltonians).

A very non–trivial statement is that the Bethe equations describe the critical points of a potential, which was first introduced in \(^{26}\) by Yang and Yang. The Yang–Yang function corresponding to the nested Bethe ansatz equations in Eq. (3.21) reads:

\[
Y(\lambda) = \frac{L}{2\pi} \sum_{a=1}^{r} \Lambda^a \sum_{i=1}^{N_a} \hat{x}
\]

\[
\left( \frac{2\lambda^{(a)}_i}{\Lambda^a} \right) - \frac{1}{4\pi} \sum_{a,b=1}^{r} C^{ab} \sum_{(i,j) = (1,1)}^{(N_a, N_b)} \hat{x}
\]

\[
\left( \frac{2\lambda^{(a)}_i - 2\lambda^{(b)}_j}{e^{2\pi \hat{x}}} \right) + \sum_{a=1}^{r} \sum_{i=1}^{N_a} n^{(a)}_i \lambda^{(a)}_i,
\]

(3.22)

where the \( n^{(a)}_i \) are integers and \( \hat{x} \) is the function

\[
\hat{x}(\lambda) = \lambda \arctan(\lambda^{-1}) + \frac{1}{2} \log(1 + \lambda^2),
\]

(3.23)

which satisfies

\[
e^{2\pi \hat{x}(\lambda)} = \frac{\lambda + i/a}{\lambda - i/a}.
\]

(3.24)

It follows that the system of Bethe equations in Eq. (3.21) can be written as

\[
e^{2\pi \omega_i^{(a)}(\lambda)} = 1, \quad a = 1, 2, \ldots, r, \quad i = 1, 2, \ldots, N_a,
\]

(3.25)

\(^4\)In this notation, the spin \( \frac{1}{2} \) representation of \( su(2) \) of the “standard” xxx model has weight \( [\Lambda^1] = [1] \).
where \( \omega_i^{(a)}(\lambda) \) are the components of the closed one–form \( \omega(\lambda) = dY(\lambda) \),

\[
\omega(\lambda) = \sum_{a=1}^{r} \sum_{i=1}^{N_a} \omega(\lambda)_i^{(a)} d\lambda_i^{(a)} = \sum_{a=1}^{r} \sum_{i=1}^{N_a} \frac{\partial Y(\lambda)}{\partial \lambda_i^{(a)}} d\lambda_i^{(a)} = dY(\lambda). \tag{3.26}
\]

### 3.3 The Dictionary

The main statement of \[13, 14\] is that the effective twisted superpotential \( \tilde{W}_{\text{eff}}(\sigma) \) can be identified with the Yang–Yang counting function \( Y(\lambda) \), once the parameters of both theories are properly matched. In this section, finally, we give the precise dictionary between the quantities of the \( \mathcal{N} = (2,2) \) gauge theory and the integrable systems we have introduced.

The first observation is that the equation (2.7) for the vacua of the gauge theory and the Bethe ansatz equation (3.25) for the rapidities have the same form. Most properties of the gauge theory are determined by the symmetry group \( K \) of the integrable system. The sector with particle numbers \( \{N_a\}_{a=1}^{r} \) for each species leads to a product gauge group of the form \( \prod_{a=1}^{r} U(N_a) \). This results in a quiver gauge theory with \( r \) nodes, where the node \( a \) carries the gauge group \( U(N_a) \). Each effective length \( L_a \) gives rise to \( L_a \) fundamentals and \( L_a \) anti–fundamental fields being attached to node \( a \). The twisted masses of the bifundamental and adjoint fields can be read off from the Cartan matrix of \( K \). In the quiver diagram, we only draw those lines between nodes \( a, b \) which correspond to a non–zero entry \( C_{ab} \) (i.e. non–zero twisted mass). We are thus lead to a quiver diagram of the type shown in Figure 2. The twisted masses of the \( k \)–th fundamental and anti–fundamental field at node \( a \) are given by the weight of the representation of the symmetry group \( K \) that the position \( k \) in the chain is carrying, plus the possible inhomogeneity at position \( k \). The boundary conditions for closed spin chains, which are encoded in the \( \hat{\theta}^a \), enter the FI terms of the gauge theory.\footnote{Periodic spin chains give rise to \( U(N) \) gauge groups, while open chains result in \( SO(N) \) or \( Sp(N) \) gauge groups, depending on the boundary condition. The boundary conditions for open spin chains are not described by \( \hat{\theta}^a \)–parameters, which corresponds to the fact that the \( SO(N) \) and \( Sp(N) \) groups do not have a central \( U(1) \)–factor and thus have no FI–terms.}

The Coulomb branch only depends on the effective twisted superpotential and is not affected by the presence of an \( F \)–term. Nevertheless, in general the superpotential...
Table 1: Dictionary in the Gauge/Bethe correspondence.

| gauge theory | integrable model |
|--------------|------------------|
| number of nodes in the quiver | $r$ | $r$ |
| gauge group at $a$–th node | $U(N_a)$ | $N_a$ |
| effective twisted superpotential | $\tilde{W}_{\text{eff}}(\sigma)$ | $Y(\lambda)$ |
| equation for the vacua | $e^{2\pi i d\tilde{W}_{\text{eff}}} = 1$ | $e^{2\pi i dY} = 1$ |
| flavor group at node $a$ | $U(L_a)$ | $L_a$ |
| lowest component of the twisted chiral superfield | $\sigma_i^{(a)}$ | $\lambda_i^{(a)}$ |
| twisted mass of the fundamental field | $\tilde{m}_k^{(a)}$ | $\frac{i}{2}a_k^{(a)} + \nu_k^{(a)}$ |
| twisted mass of the anti–fundamental field | $\tilde{m}_k^{(a)}$ | $\frac{i}{2}a_k^{(a)} - \nu_k^{(a)}$ |
| twisted mass of the adjoint field | $\tilde{m}_{\text{adj}}^{(a)}$ | $\frac{i}{2}C^{aa}$ |
| twisted mass of the bifundamental field | $\tilde{m}_{(ab)}^{(a)}$ | $\frac{i}{2}C^{ab}$ |
| FI–term for $U(1)$–factor of gauge group $U(N_a)$ | $\tau_a$ | $\hat{\theta}^a$ |

will break (part of) the global symmetries which results in constraints on the possible values of the twisted masses. These constraints are to be compared with those that come from the theory of representations of the symmetry group $K$ on the integrable model side (e.g. the Cartan matrix containing only integer entries, or the allowed values for the highest weights).

All the relevant parameters and their matching are collected in Table 1.

### 3.4 Example: XXX spin chain

The xxx spin chain is one of the best studied integrable models. It describes a system of electrons on a lattice with spin exchange interactions. Each site can be either occupied by a spin up ($\uparrow$) or down ($\downarrow$). The Hilbert space at each point is

$$\mathcal{H}_k = \mathbb{C}^2,$$

(3.27)
which corresponds to the fundamental representation of $sl(2)$. Using the results of the previous section and choosing as a reference state

$$\Omega = \bigotimes_{i=1}^{L} e_i,$$

(3.28)

one finds that the rapidities $\lambda$ satisfy the Bethe Ansatz equations

$$\left( \frac{\lambda_i + \frac{1}{2}}{\lambda_i - \frac{1}{2}} \right)^L e^{i\hat{\theta}} = \prod_{\substack{j=1\ j\neq i}}^{N} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}, \quad i = 1, \ldots, N,$$

(3.29)

where $L$ is the length of the chain, $N$ is the number of magnons and $\hat{\theta}$ is the boundary twist parameter.

It is a known fact (for a modern discussion see [27]) that there is a completely equivalent set of equations obtained by choosing the opposite reference state

$$\Omega = \bigotimes_{i=1}^{L} e_i,$$

(3.30)

and considering $L - N$ dual magnons:

$$\left( \frac{\lambda_i + \frac{1}{2}}{\lambda_i - \frac{1}{2}} \right)^L e^{-i\hat{\theta}} = \prod_{\substack{j=1\ j\neq i}}^{L-N} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}, \quad i = 1, \ldots, N.$$  

(3.31)

Comparing the Bethe Ansatz equations and their corresponding Yang–Yang functions to the effective twisted superpotentials in Eq. (2.18) and Eq. (2.19) we find that the two gauge systems described in Sec. 2.3 admit the same $2^L$ supersymmetric ground states.

### 3.5 Example: $tJ$ model

The $tJ$ model [28] describes a system of electrons on a lattice with a Hamiltonian that describes nearest-neighbor hopping (with coupling $t$) and spin interactions (with coupling $J$). Consider a lattice of length $L$ with periodic boundary conditions. Each site can be either free ($\circ$) or occupied by a spin up ($\uparrow$) or down ($\downarrow$) electron. Excluding double occupancy, the Hilbert space at each point $k$ is:

$$\mathcal{H}_k = \mathbb{C}^{(1/2)},$$

(3.32)

which corresponds to the fundamental representation of $sl(1/2)$. It is convenient to introduce anticommuting creation–annihilation pairs $c_{k,s}^\dagger, c_{k,s}, s = \{ \uparrow, \downarrow \}$ at each site,
acting as

$$|s\rangle_k = c_{k,s}^\dagger |\circ\rangle_k$$

for $s = \{\uparrow, \downarrow\}$,

$$|\circ\rangle_k$$

is the vacuum, annihilated by $c_{k,s}$. Let $n_{k,s} = c_{k,s}^\dagger c_{k,s}$ be the number of $s$ electrons at position $k$ and $n_k = n_{k,\uparrow} + n_{k,\downarrow}$. We can further introduce $sl(2)$ spin operators at each site:

$$S^-_k = c_{k,\uparrow}^\dagger c_{k,\downarrow}, \quad S^+_k = c_{k,\downarrow}^\dagger c_{k,\uparrow}, \quad S^z_k = \frac{1}{2}(n_{k,\uparrow} - n_{k,\downarrow}).$$

With these ingredients, we can write down the Hamiltonian

$$H = \sum_{k=1}^{L-1} \left[ -t P \sum_{s=\uparrow,\downarrow} (c_{k,s}^\dagger c_{k+1,s} + \text{h.c.}) P + \sum_{s=\uparrow,\downarrow} J \left( S_k \cdot S_{k+1} - \frac{1}{4} n_k n_{k+1} + 2 n_k - \frac{1}{2} \right) \right],$$

where $P$ projects out double occupation. It is convenient to introduce the number of holes $N_h$, of spins up $N_\uparrow$, of spins down $N_\downarrow$, and electrons $N_e$:

$$N_h = \sum_{k=1}^{L} (1 - n_k), \quad N_\uparrow = \sum_{k=1}^{L} n_{k,\uparrow}, \quad N_\downarrow = \sum_{k=1}^{L} n_{k,\downarrow}, \quad N_e = N_\uparrow + N_\downarrow.$$

Single occupancy implies

$$L = N_h + N_\uparrow + N_\downarrow.$$

The Hamiltonian is remarkable for being supersymmetric for the choice $J = 2t = 2$, in the sense that it is invariant under the action of the superalgebra $sl(1|2)$ (see Appendix A for an explicit realization of the algebra in terms of creation/annihilation operators and for some basic properties). In the supersymmetric case, $H$ can also be conveniently expressed as

$$H = -\sum_{k=1}^{L-1} [\Pi_{k,k+1} - 1],$$

where $\Pi_{k,k+1}$ interchanges the configurations at sites $k$ and $k+1$, with an extra minus sign if they are both fermionic:

$$\Pi_{k,k+1} |\circ\rangle_k \otimes |\circ\rangle_{k+1} = |\circ\rangle_k \otimes |\circ\rangle_{k+1},$$

$$\Pi_{k,k+1} |\circ\rangle_k \otimes |s\rangle_{k+1} = |s\rangle_k \otimes |\circ\rangle_{k+1}, \quad s = \{\uparrow, \downarrow\},$$

$$\Pi_{k,k+1} |s_1\rangle_k \otimes |s_2\rangle_{k+1} = -|s_2\rangle_k \otimes |s_1\rangle_{k+1}, \quad s_1, s_2 = \{\uparrow, \downarrow\}.$$

This is precisely the same structure that we introduced in the previous section.

Having recognized the $tJ$ model Hamiltonian as an example of a spin chain solvable by the algebraic Bethe ansatz, we can use the results of Sec. 3.2. A fundamental remark is in order. This spin chain is $sl(1|2)$–invariant, and this supergroup admits different inequivalent choices of the Cartan matrix (as shown in Appendix A). This means that
the same physical system of electrons and holes is described by three different Bethe ansatz equations, as explained in [29]. By construction, the three choices must be equivalent, as was shown explicitly in [29, 30] (see also Appendix B). We will examine all three of them here.

**Case A** The first case corresponds the Kac–Dynkin\(^6\) diagram \(\circlearrowleft\circlearrowright\). This leads to

\[
C^{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \quad \Lambda = [0 \ 1], \quad N_1 = N_h, \quad N_2 = N_h + N_\downarrow. \tag{3.42}
\]

The nested Bethe equations are given by

\[
\left( \frac{\lambda_p^{(2)} + i \frac{1}{2}}{\lambda_p^{(2)} - i \frac{1}{2}} \right)^L = \prod_{q=1 \atop q \neq p}^{N_h+N_i} \lambda_p^{(2)} - \lambda_q^{(2)} + i \prod_{i=1}^{N_h} \lambda_p^{(2)} - \lambda_i^{(1)} - i \frac{1}{2}, \quad p = 1, \ldots, N_h + N_\downarrow, \tag{3.43a}
\]

\[
1 = \prod_{p=1}^{N_h+N_i} \lambda_p^{(2)} - \lambda_i^{(1)} - i \frac{1}{2}, \quad i = 1, \ldots, N_h. \tag{3.43b}
\]

The Yang–Yang function (3.22) reads:

\[
Y_A(\lambda) = \frac{L}{2\pi} \sum_{p=1}^{N_h+N_i} (2\lambda_p^{(2)}) - \frac{1}{2\pi} \sum_{p,q=1 \atop p \neq q}^{N_h+N_i} (2\lambda_p^{(2)} - 2\lambda_q^{(2)}) + \frac{1}{2\pi} \sum_{p=1}^{N_h+N_i} \sum_{i=1}^{N_h} (2\lambda_p^{(2)} - 2\lambda_i^{(1)}) + \sum_{i=1}^{N_h} n_i^{(1)} \lambda_i^{(1)} + \sum_{p=1}^{N_h+N_i} n_p^{(2)} \lambda_p^{(2)}. \tag{3.44}
\]

**Case B** The second case corresponds to the Kac–Dynkin diagram \(\circlearrowleft\circlearrowright\). This leads to

\[
C^{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \Lambda = [0 \ 1], \quad N_1 = N_\downarrow, \quad N_2 = N_h + N_\downarrow. \tag{3.45}
\]

\(^6\)See Appendix A.
The nested Bethe equations are given by

\[
\left( \frac{\lambda_p^{(2)} + i}{\lambda_p^{(2)} - i} \right)^L = \prod_{i=1}^{N_i} \frac{\lambda_p^{(1)} - \lambda_p^{(2)} - i}{\lambda_p^{(1)} - \lambda_p^{(2)} + i}, \quad p = 1, \ldots, N_h + N_i,
\]

\[
1 = \prod_{p=1}^{N_h + N_i} \frac{\lambda_i^{(1)} - \lambda_p^{(2)} - i}{\lambda_i^{(1)} - \lambda_p^{(2)} + i}, \quad i = 1, \ldots, N_i.
\]

The Yang–Yang function (3.22) reads:

\[
Y_B(\lambda) = \frac{L}{2\pi} \sum_{p=1}^{N_h + N_i} \hat{x}(2\lambda_p^{(2)}) + \frac{1}{2\pi} \sum_{p=1}^{N_h + N_i} \sum_{i=1}^{N_i} \hat{x}(2\lambda_i^{(1)} - 2\lambda_p^{(2)}) + \sum_{i=1}^{N_i} n_i^{(1)} \lambda_i^{(1)} + \sum_{p=1}^{N_h + N_i} n_p^{(2)} \lambda_p^{(2)}.
\]

**Case C** The third case corresponds to Kac–Dynkin diagram \( \bigcirc \longrightarrow \bigcirc \). This leads to

\[
C^{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, \quad \Lambda = [0 \ 1], \quad N_1 = N_1, \quad N_2 = N_1 + N_1.
\]

The nested Bethe equations are given by

\[
\left( \frac{\lambda_p^{(2)} - i}{\lambda_p^{(2)} + i} \right)^L = \prod_{i=1}^{N_i} \frac{\lambda_p^{(1)} - \lambda_p^{(2)} - i}{\lambda_p^{(1)} - \lambda_p^{(2)} + i}, \quad p = 1, \ldots, N_h,
\]

\[
\prod_{p=1}^{N_h} \frac{\lambda_p^{(2)} - \lambda_i^{(1)} - i}{\lambda_p^{(2)} - \lambda_i^{(1)} + i} = \prod_{j=1}^{N_i} \frac{\lambda_j^{(1)} - \lambda_i^{(1)} - i}{\lambda_j^{(1)} - \lambda_i^{(1)} + i}, \quad i = 1, \ldots, N_i.
\]

The resulting Yang–Yang function (3.22) is given by

\[
Y_C(\lambda) = \frac{L}{2\pi} \sum_{p=1}^{N_h} \hat{x}(2\lambda_p^{(2)}) - \frac{1}{2\pi} \sum_{p=1}^{N_h} \sum_{i=1}^{N_i} \hat{x}(2\lambda_p^{(2)} - 2\lambda_i^{(1)}) + \frac{1}{2\pi} \sum_{i=1}^{N_i} \sum_{j=1}^{N_i} \hat{x}^2(\lambda_i^{(1)} - \lambda_j^{(1)})

+ \sum_{i=1}^{N_i} n_i^{(1)} \lambda_i^{(1)} + \sum_{p=1}^{N_h} n_p^{(2)} \lambda_p^{(2)}.
\]

The three systems of equations admit a number of solutions \( Z(N_h, N_\uparrow, N_\downarrow) \) that can be written explicitly as follows [31]:

\[
Z(N_h, N_\uparrow, N_\downarrow) = \sum_{q=0}^{N_h + N_\uparrow} \frac{N_\uparrow - N_\downarrow + 1}{N_h + N_\uparrow + 1} \binom{q - 1}{q} \binom{N_h + N_\uparrow + 1}{N_h} \binom{N_h - N_\downarrow - 1}{q - 1}.
\]
Table 2: Comparing quiver diagrams for the three supersymmetric theories and Dynkin–Kac diagrams for the fundamental representation. For each node in the Dynkin diagram, there is a gauge group. For each white node, there is an adjoint field. A flavor group is attached to the nodes with non-zero label.

### 3.6 Gauge/Bethe correspondence for the $tJ$ model

After having collected the relevant quantities both for our quiver gauge theories and the $tJ$ model, we are ready to identify the effective twisted superpotentials given in Sec. 2.4 with the Yang–Yang functions derived in the previous section. Observing that

$$\frac{2\lambda}{a} f(a\lambda) = \left(\lambda + \frac{1}{a}\right) \left(\log(\lambda + \frac{1}{a}) - 1\right) - \left(\lambda - \frac{1}{a}\right) \left(\log(\lambda - \frac{1}{a}) - 1\right) + \text{const.},\quad (3.52)$$

we are now in a position to identify the gauge theories whose effective twisted superpotentials reproduce the Yang–Yang functions above:

- The Yang–Yang function in Eq. (3.44) corresponds to a quiver gauge theory with the effective twisted superpotential given in Eq. (2.20) with $\theta$–angles $\theta_1 = (N_h + N_i) \pi$ and $\theta_2 = (N_h + N_i + 1) \pi$.

- The Yang–Yang function in Eq. (3.47) corresponds to a quiver gauge theory with the effective twisted superpotential given in Eq. (2.22) with $\theta_1 = (N_h + N_i) \pi$ and $\theta_2 = (N_h + N_i) \pi$.

- The Yang–Yang function in Eq. (3.50) corresponds to a quiver gauge theory with effective twisted superpotential given in Eq. (2.24) with $\theta_1 = (N_i + 1) \pi$ and $\theta_2 = (N_h + N_i) \pi$.

In Table 2 the Kac–Dynkin diagrams and the quiver diagrams for the corresponding gauge theories are shown.

We would like to stress once more the logic behind our construction. The $tJ$ model admits three sets of Bethe ansatz equations corresponding to the same ring of commuting Hamiltonians. To each of these, we associate a quiver gauge theory, according to the dictionary in Table 1. Since the commuting Hamiltonians are the same, also the three gauge theories have the same chiral ring and, equivalently, the same supersymmetric ground states.
Having considered a supergroup symmetry, we are in the position to slightly extend the dictionary in Section 3.3. The quiver diagrams for the supersymmetric gauge theories are to be compared to the Kac–Dynkin diagrams of the superalgebra. For each node in the Dynkin diagram, there is a gauge group. Furthermore, each white node carries an adjoint field. A flavor group is attached to the nodes with non–zero label.

We would like to end this section with an observation concerning the constraints on the mass parameters coming from the two sides of the correspondence. Consider for simplicity the case of the distinguished Borel subalgebra of \( sl(m|n) \), whose Dynkin diagram has \( m - 1 \) white nodes, followed by a grey node and \( n - 1 \) white nodes (for \( sl(1|2) \) this is the choice corresponding to case C), see Eq. (A.13). According to the dictionary, we have adjoint fields \( \Phi^a \) for every white node, and fundamentals at each node. This means that for each white node, we can introduce a superpotential of the type

\[
W = Q_k^a (\Phi^a)^{\Lambda^a} \bar{Q}_k^a, \quad a \neq m,
\]

which conserves a \( U(1) \)–symmetry, thus imposing a constraint on the twisted masses:

\[
\Lambda^a \bar{m}^{\text{adj}(a)} + \bar{m}^{\text{f}(a)} + m_k^{\text{f}(a)} = 0, \quad a \neq m.
\]

Requiring the superpotential to be a polynomial in the fields translates to the conditions

\[
\Lambda^a \in \mathbb{N}, \quad a \neq m; \quad \Lambda^m \in \mathbb{R}.
\]

This reproduces exactly the conditions that the representation \( \Lambda \) has to satisfy in order to be finite–dimensional (see Appendix A). In this case, the two sides of the correspondence lead to the same constraints.

4 Embedding in type IIA string theory

We have shown that the three quiver gauge theories introduced in Section 2.4 have the same supersymmetric ground states. It is reasonable to expect that this connection also manifests itself in other ways. Here we show that they can also be related to each other using a string theory embedding. In this section, we propose a possible mechanism based on brane transitions that faithfully reproduces the matter content of our three quiver gauge theories. In the present setup, the construction corresponds to vanishing twisted masses. A complete type IIA embedding that reproduces the twisted masses and an M–theory description of the transition are currently under investigation.

Brane constructions such as the ones in \([21, 32, 33]\) are likely candidates for relating the three quiver gauge theories. Our setup (in type IIA string theory) is the following. We consider two parallel NS5 branes \( \text{NS5}_i, i = 1, 2 \) which are extended in the 012345–directions, and another NS5–brane \( \text{NS5}_5 \) extended in the 012389–directions. There are

\[\text{We thank Kentaro Hori for suggesting this construction.}\]
Let $D_2$–branes (extended in the 016–directions) stretching between the $N_5$–branes. Furthermore, we have $D_4$–branes extended in the 01789–directions. The setup is summarized in Table 4. This configuration preserves 4 of the 32 supercharges of type $\text{IIA}$ string theory. Note the invariance under the rotations in the (01), (23), (45) and (89) planes: these appear as Lorentz invariance and as global symmetries in the field theory.

|       | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|---|
| $N_5^{1,2}$ | × | × | × | × | × | × | × | × | × | × |
| $N_5'$ | × | × | × | × | × | × | × | × | × | × |
| $D_2$ | × | × | × | × | × | × | × | × | × | × |
| $D_4$ | × | × | × | × | × | × | × | × | × | × |

Table 3: Brane setup for the type $\text{IIA}$ embedding.

Open strings stretching between $D_2$–branes located between parallel $N_5$–branes correspond to the adjoint fields in the $\mathcal{N} = 2$ sector, while open strings stretching between two separate stacks of $D_2$–branes correspond to bifundamental fields. Open strings stretching between the $D_2$ branes and the stack of $D_4$–branes correspond to the fundamental and anti–fundamental fields of the flavor group $U(L)$. The parameters of the field theory are encoded in the positions of the $N_5$ and $D_4$ branes. Here we restrict ourselves to the case of all matter fields being massless.

The transitions between the three different brane configurations work as follows:

1. We start with the brane cartoon in Figure 3(a) corresponding to the quiver of case A (Figure 1(a)). There are $N_h$ $D_2$ branes between $N_5'$ and $N_5^{1}$, $N_h + N_i$ $D_2$ branes between $N_5^{1}$ and $N_5^{2}$, and the $D_4$ branes are located between the two parallel $N_5$ branes.

2. Moving the $N_5'$ brane past $N_5^{1}$, we obtain the brane cartoon in Figure 3(b) which corresponds to the setup of case B (Figure 1(b)). Note that now, there are no $D_2$ branes stretching between parallel $N_5$ branes, which explains why there is no adjoint matter.

3. To reach the last configuration of case C (figure 1(c)), we need to go through two intermediate steps. First, we move the $D_4$–branes to the right past $N_5^2$, thus generating $L$ $D_2$ branes (see Figure 3(c)).

4. Then we move also the $N_5'$–brane to the right, past $N_5^2$ (see Figure 3(d)). This creates an $\mathcal{N} = 2$ sector on the left hand side of the cartoon (with the corresponding adjoint field).

5. In the last step, we make the $N_5'$–brane coincide with the $D_4$–branes. The resulting brane cartoon (Figure 3(e)) corresponds to the quiver diagram of case C.

---

*Similar setups have been discussed in [21, 33].*
Figure 3: Brane transitions connecting the quiver gauge theories of cases A, B, C

With this, the three theories have been related by a mechanism completely different from the Gauge/Bethe correspondence, which can also be used as a further alley of investigation.

5 Conclusions

In this note, we have used the Gauge/Bethe correspondence by Nekrasov and Shatashvili to relate different supersymmetric quiver gauge theories in two dimensions. These theories, despite having different gauge groups and matter content, turn out to have the same chiral ring and therefore the same supersymmetric ground states. We have thus used quantum integrable systems as a tool to make statements about gauge theories:

- in the xxx case, we showed that a theory with $N$ colors and $L$ flavors has the same supersymmetric ground states as a theory with $L - N$ colors and $L$ flavors;
- in the $tJ$ model case, we used in particular the fact that integrable systems with supergroup symmetry give rise to several sets of Bethe equations, which
correspond to different quiver gauge theories. In particular, in the $\text{sl}(m|n)$ case, there are $\binom{m+n}{m}$ equivalent quiver gauge theories with $m + n - 1$ nodes.

It is little surprising that the gauge theories under consideration can also be related via a string theory construction using brane movements.

While the translation of two–dimensional supersymmetric gauge theories into integrable systems is less straight–forward than going in the opposite direction, we suggest to follow this path in order to gain knowledge about gauge theories via quantum integrable systems. The parameters of the quantum integrable models translate into precise values for the twisted masses of the supersymmetric gauge theories, which can be rather restrictive for the allowed values of twisted masses. For integrable systems with supergroup symmetry, the range of possible values is quite large, though: in the case of the distinguished Borel subalgebra, one finds that nodes carrying adjoint fields (white nodes) admit any non–negative integer weights, and the weights for nodes without adjoint (grey nodes) are even continuous parameters. While the explicit examples we used were based on the xxx and $t/J$ models, we believe that our approach can be applied in a wider context. It is conceivable to tune the twisted masses of the gauge theories under consideration to values compatible with a spin chain embedding to check whether their supersymmetric ground states can be matched. Once this relation is established, other means of investigation such as the realization of the systems via brane cartoons can be used to study the gauge theories at different values for the twisted masses.

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A The superalgebra $sl(1|2)$

In this appendix, we collect some facts about Lie superalgebras. For details see [34, 35].

Superalgebra and spin operators. A superalgebra $\mathfrak{g}$ can be decomposed into an even and an odd part, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The even part $\mathfrak{g}_0 = gl(1) \oplus sl(2)$ of the superalgebra $\text{sl}(1|2)$ is generated by the operators $S^\pm, S^z, Z$ with commutation relations

\[
[S^Z, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^z, \quad [Z, S^\pm] = 0, \quad [Z, S^z] = 0.
\]  (A.1)
There are two additional fermionic multiplets $Q^\pm_s$, $s = \{\uparrow, \downarrow\}$ which transform as $(\pm \frac{1}{2}, \frac{1}{2})$ with respect to $g_0$. Explicitly:

$$[S^z, Q^\pm_s] = \pm \frac{1}{2} Q_s, \quad [S^\pm, Q^\pm_s] = 0, \quad [Z, Q^\pm_s] = \frac{1}{2} Q^\pm_s, \quad [Z, Q^\pm_s] = -\frac{1}{2} Q^\pm_s. \quad (A.2)$$

The fermionic generators satisfy the following anticommutation relations:

$$\{Q^+_s, Q^-_s\} = 0, \quad \{Q^\pm_s, Q^\pm_s\} = S^\pm, \quad \{Q^\pm_s, Q^\pm_s\} = Z \pm S^z. \quad (A.3)$$

The $tJ$ model admits a natural representation of the $sl(1|2)$ algebra. At each point $k$ of the lattice, the generators can be represented in terms of creation–annihilation operators as

$$S^-_k = c^\dagger_{k\uparrow} c_{k\downarrow}, \quad S^+_k = c^\dagger_{k\downarrow} c_{k\uparrow}, \quad S^z_k = \frac{1}{2} (n_{k\uparrow} - n_{k\downarrow}), \quad (A.4)$$

$$Q^-_{k\uparrow} = (1 - n_{k\uparrow}) c^\dagger_{k\uparrow}, \quad Q^+_{k\downarrow} = (1 - n_{k\downarrow}) c^\dagger_{k\downarrow}, \quad Q^-_{k\downarrow} = (1 - n_{k\uparrow}) c_{k\uparrow}, \quad (A.5)$$

$$Q^+_{k\uparrow} = (1 - n_{k\downarrow}) c_{k\downarrow}, \quad Z_k = 1 - \frac{1}{2} n_k. \quad (A.6)$$

**Root decomposition.** Let $\{\delta_1, \ldots, \delta_m, \epsilon_1, \ldots, \epsilon_n\}$ be a basis for $\mathbb{C}^{(m|n)}$ with inner product

$$(\delta_i, \delta_j) = \delta_{ij}, \quad (\epsilon_i, \epsilon_j) = -\delta_{ij}, \quad (\delta_i, \epsilon_j) = 0. \quad (A.7)$$

The superalgebra $sl(m|n)$ admits a root space decomposition

$$sl(m|n) = \mathfrak{h} \oplus \bigoplus_{a \in \Phi} \mathfrak{g}_a, \quad (A.8)$$

where $\mathfrak{h}$ is the Cartan subalgebra (diagonal matrices) and $\Phi$ is the root system:

$$\Phi = \{\delta_i - \delta_j\} \cup \{\epsilon_i - \epsilon_j\} \cup \{\pm (\delta_i - \epsilon_j)\} \quad i, j = 1, \ldots, m; \quad i, j = 1, \ldots, n. \quad (A.9)$$

The **standard set** of positive roots (corresponding to the distinguished Borel subalgebra) is

$$\Phi^+ = \{\delta_i - \delta_j \mid i < j\} \cup \{\epsilon_i - \epsilon_j \mid \bar{i} < \bar{j}\} \cup \{\delta_i - \epsilon_j\} \quad i, j = 1, \ldots, m; \quad \bar{i}, \bar{j} = 1, \ldots, n, \quad (A.10)$$

where $\{\delta_i - \epsilon_j\}$ are odd roots, i.e.

$$(\delta_i - \epsilon_j, \delta_i - \epsilon_j) = 0. \quad (A.11)$$

For Lie algebras, all possible sets of positive roots can be obtained by reflection, and the corresponding Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$ are conjugate. This is not the case for superalgebras, since the reflection of an odd root $\delta_i - \epsilon_j \rightarrow \epsilon_i - \delta_i$ produces a new system of positive roots whose associated Borel subalgebra is not conjugate to the initial one. For $sl(1|2)$, there are six possible choices of positive roots, which are
organized into three conjugacy classes under reflection. It is convenient to represent
the positive roots by using the Cartan matrix or, equivalently, a Dynkin diagram. Now
we need to distinguish between even roots (white nodes \(\bigcirc\)) and odd roots (grey nodes
\(\otimes\)). The standard Dynkin diagrams and the other two obtained by reflection of odd
roots are represented in Table 4. In the general \(sl(m|n)\) case, there are \(\binom{m+n}{m}\) conjugacy
classes (and Dynkin diagrams), one for each sequence of \(m\) repetitions of the symbol \(\delta\)
and \(n\) repetitions of the symbol \(\epsilon\).

**Kac–Dynkin diagrams.** Representations of \(sl(m|n)\) are labelled uniquely by so-called
Kac–Dynkin diagrams. These are Dynkin diagrams in which a number \(\Lambda^a\) is associated
to each node. For example, the fundamental representation for \(sl(1|2)\) can be associated
to three non–equivalent Kac–Dynkin diagrams:

\[
\begin{array}{ccc}
\bigotimes & \bigcirc \\
0 & 1 \\
\end{array} &
\begin{array}{ccc}
\bigotimes & \bigotimes \\
0 & 1 \\
\end{array} &
\begin{array}{ccc}
\bigcirc & \bigotimes \\
0 & 1 \\
\end{array}.
\]

If we choose the distinguished Borel subalgebra, a representation

\[
\Lambda = \Lambda^1 \circ \Lambda^2 \circ \cdots \bigotimes \Lambda^{m-1} \bigotimes \Lambda^m \bigotimes \Lambda^{m+1} \cdots \bigotimes \Lambda^{m+n-1}
\]

is finite dimensional if and only if the labels of the white nodes are non–negative
integers and the label of the grey node \(\Lambda^m\) is a real number.

**B Equivalence of \(t|J\) Bethe Equations**

We want to show the equivalence of the three Bethe ansatz equations described in
Section 3.5 explicitly by using an argument originally introduced in [30].
Consider the Bethe ansatz equations in Eq. (3.46):

\[
\left( \frac{\lambda_p^{(2)} + \frac{i}{2}}{\lambda_p^{(2)} - \frac{i}{2}} \right)^L = \prod_{i=1}^{N_{\lambda}} \frac{\lambda_i^{(1)} - \lambda_p^{(2)} - \frac{i}{2}}{\lambda_i^{(1)} - \lambda_p^{(2)} + \frac{i}{2}}, \quad p = 1, \ldots, N_h + N_i 
\]

\[
1 = \prod_{p=1}^{N_{\lambda} + N_i} \frac{\lambda_i^{(1)} - \lambda_p^{(2)} - \frac{i}{2}}{\lambda_i^{(1)} - \lambda_p^{(2)} + \frac{i}{2}}, \quad i = 1, \ldots, N_i. 
\]

The second set (the unknowns are \( \lambda_i^{(1)} \)) can be written as a polynomial equation:

\[
1 = \prod_{p=1}^{N_{\lambda} + N_i} \frac{\lambda_i^{(1)} - \lambda_p^{(2)} - \frac{i}{2}}{\lambda_i^{(1)} - \lambda_p^{(2)} + \frac{i}{2}} \quad \Leftrightarrow \quad p(w) = \prod_{p=1}^{N_{\lambda} + N_i} \left( w - \lambda_p^{(2)} - \frac{i}{2} \right) - \prod_{p=1}^{N_{\lambda} + N_i} \left( w - \lambda_p^{(2)} + \frac{i}{2} \right) = 0. 
\]

The polynomial \( p(w) \) has degree \( N_i + N_{\lambda} \). We identify the variables \( \lambda_i^{(1)} \) with the first \( N_i \) solutions \( w_i \) of \( p(w) = 0 \). We call the other \( N_{\lambda} \) solutions \( \bar{w}_j, j = 1, \ldots, N_{\lambda} \).

Using the residue theorem, we can write the rhs of the first set of BEA (Eq. (3.46)(a)) as

\[
\sum_{i=1}^{N_i} \log \left[ \frac{\lambda_p^{(2)} - w_i + \frac{i}{2}}{\lambda_p^{(2)} - w_i - \frac{i}{2}} \right] = \sum_{i=1}^{N_i} \frac{1}{2\pi i} \oint_{C_i} \log \left[ \frac{\lambda_p^{(2)} - z + \frac{i}{2}}{\lambda_p^{(2)} - z - \frac{i}{2}} \right] \frac{dz}{dz} \log(p(z)),
\]

where \( C_i \) is a contour around \( w_i \) (see Figure 4(a)). The logarithm has a branch cut from \( (\lambda_p^{(2)} + i/2) \) to \( (\lambda_p^{(2)} - i/2) \). We can change the contour, picking residues from the other \( N_{\lambda} \) poles of \( p(z) \), plus the contributions of the branch cut (see Figure 4(b)):

\[
\sum_{i=1}^{N_i} \log \left[ \frac{\lambda_p^{(2)} - w_i + \frac{i}{2}}{\lambda_p^{(2)} - w_i - \frac{i}{2}} \right] = -\sum_{j=1}^{N_{\lambda}} \frac{1}{2\pi i} \oint_{C_j} \log \left[ \frac{\lambda_p^{(2)} - z + \frac{i}{2}}{\lambda_p^{(2)} - z - \frac{i}{2}} \right] \frac{dz}{dz} \log(p(z)) = \log \left[ \frac{p(\lambda_p^{(2)} + i/2)}{p(\lambda_p^{(2)} - i/2)} \right].
\]

Writing \( p(z) \) explicitly:

\[
p(\lambda_p^{(2)} + i/2) = -\prod_{q=1}^{N_{\lambda} + N_i} \left( \lambda_p^{(2)} - \lambda_q^{(2)} + i \right), \quad p(\lambda_p^{(2)} - i/2) = \prod_{q=1}^{N_{\lambda} + N_i} \left( \lambda_p^{(2)} - \lambda_q^{(2)} - i \right),
\]

\( (B.5) \)
we can exponentiate:

\[
\prod_{i=1}^{N_h} \frac{\lambda_p^{(2)}(2) - w_i + \frac{i}{2}}{\lambda_p^{(2)} - w_i - \frac{i}{2}} = \prod_{j=1}^{N_h} \frac{\lambda_p^{(2)} - \bar{w}_j - \frac{i}{2}}{\lambda_p^{(2)} - \lambda_q^{(2)}(2) + i} \prod_{q=1 \atop q \neq p}^{N_h+N_\downarrow} \lambda_p^{(2)} - \lambda_q^{(2)} - i,
\]

(B.6)

and using the set in Eq. (3.46a) on the LHS, we find

\[
\left(\frac{\lambda_p^{(2)}(2) + i}{\lambda_p^{(2)} - i} \right)^L = \prod_{q=1 \atop q \neq p}^{N_h+N_\downarrow} \frac{\lambda_p^{(2)} - \lambda_q^{(2)} + i}{\lambda_p^{(2)} - \lambda_q^{(2)} - i} \prod_{i=1}^{N_h} \frac{\lambda_p^{(2)} - \bar{w}_i - \frac{i}{2}}{\lambda_p^{(2)} - \lambda_i^{(1)} + \frac{i}{2}} \quad p = 1, \ldots, N_h + N_\downarrow,
\]

(B.7)

which coincides with the equation of case A Eq. (3.43a) if we identify \(\bar{w}_i = \lambda^{(1)}_i\):

\[
\left(\frac{\lambda_p^{(2)} + i}{\lambda_p^{(2)} - i} \right)^L = \prod_{q=1 \atop q \neq p}^{N_h+N_\downarrow} \frac{\lambda_p^{(2)} - \lambda_q^{(2)} + i}{\lambda_p^{(2)} - \lambda_q^{(2)} - i} \prod_{i=1}^{N_h} \frac{\lambda_p^{(2)} - \lambda_i^{(1)} - \frac{i}{2}}{\lambda_p^{(2)} - \lambda_i^{(1)} + \frac{i}{2}} \quad p = 1, \ldots, N_h + N_\downarrow,
\]

(B.8a)

\[
1 = \prod_{p=1}^{N_h+N_\downarrow} \frac{\lambda_p^{(2)} - \lambda_i^{(1)} - \frac{i}{2}}{\lambda_p^{(2)} - \lambda_i^{(1)} + \frac{i}{2}} \quad i = 1, \ldots, N_h.
\]

(B.8b)

The identification reproduces also the other equations (3.43b) since these can be put into the very same polynomial form \(p(w) = 0\), and the \(\bar{w}_j\) are by construction solutions.
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