Remark about string field for general configuration of $N$ D-instantons

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Abstract: In this paper we would like to suggest matrix form of the string field for any configuration of $N$ D-instantons in bosonic string field theory.

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1. Introduction

Renewed attention has been paid to Witten’s cubic bosonic open string field theory [1], following Sen’s conjectures that this formalism can be used to give analytic description of D25-brane decay in bosonic string theory [3] (For review and extensive list of references, see [2].) It seems to be possible that string field theory could give very interesting information about nonperturbative nature of string theory and consequently about M theory. For that reason it seems to be interesting to study the relation between string field theory and matrix theory [4], which is the most successful nonperturbative definition of M theory. For example, recent progress in the vacuum string field theory [3, 4, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 19, 20, 21] suggests that the string field theory could be useful for better understanding of the basic fabric of the string theory.

In order to find relation between matrix theory and string field theory, it would be perhaps useful to study the nonabelian extension of string field theory as well. As was stressed in the original paper [1], nonabelian extension of string field theory can be very easily implemented into its formalism by introducing Chan-Paton factors for various fields in the string field theory action and including the trace over these indices. In modern language this configuration corresponds to $N$ coincident D25-branes.

In the previous paper [22], we have proposed a generalised form of the string field theory action that was suitable for the description of general configuration of D-instantons. We have formulated this theory in a pure abstract form following the seminal paper [1]. In order to support further our proposal, we think that it would be desirable to have an alternative formulation of the generalised matrix string field theory action which would allow us to perform more detailed calculation.
In particular, it would be nice to have a matrix generalisation of the action written in the conformal field theory (CFT) language [23, 24]. In this paper we suggest a possible form of the matrix valued string fields that will be building blocks for the matrix CFT formulation of the string field action. We present a compact form of this matrix valued string field. We will study its operator product expansion (OPE) with the open string stress energy tensor and we will show that in order that any general component of the string field to have a well defined conformal dimension the background configuration of \(N\) D-instantons (In this paper we will discuss D-instantons only, the extension to Dp-branes of any dimension is trivial.) must obey one particular condition that can be interpreted as a requirement that the background configuration of D-instantons is a solution of the equation of motion arising from the low energy action for the D-instanton matrix model. In our opinion this situation is similar with the fact that consistent string field theory should be formulated using conformal field theory that forces the background field to obey the equation of motion.

Then we extend our analysis to the case of infinitely many D-instantons and we show that well known nonabelian configuration can be very easily included in our formalism. In particular, we find such a matrix form of the string field and hence vertex operators that precisely corresponds to the string field theory formulated around a noncommutative D-brane background [25, 26].

In conclusion we outline our results and suggest extension of this work. In particular, it will be clear from this paper that the extension of our approach to the supersymmetric case can be very easily performed.

2. String field theory in the CFT formalism

In this section we review basic facts about bosonic string field theory, following mainly [3]. Gauge invariant string field theory is described with the full Hilbert space of the first quantized open string including \(b, c\) ghost fields subject to the condition that the states must carry ghost number one, where \(b\) has ghost number \(-1\), \(c\) has ghost number 1 and \(SL(2, C)\) invariant vacuum \(\langle 0\rangle\) carries ghost number 0. We denote \(\mathcal{H}\) the subspace of the full Hilbert space carrying ghost number 1. Any state in \(\mathcal{H}\) will be denoted as \(|\Phi\rangle\) and corresponding vertex operator \(\Phi(x)\) is the vertex operator that creates state \(|\Phi\rangle\) out of the vacuum state \(|0\rangle\)

\[
|\Phi\rangle = \Phi(x) |0\rangle .
\]  

Since we are dealing with open string theory, the vertex operators should be put on the boundary of the world-sheet.

The open string field theory action has a form

\[
S = \frac{1}{g_0^2} \left( \frac{1}{2\alpha'} \langle I \circ \Phi(0) Q_B \Phi(0) \rangle + \frac{1}{3} \langle f_1 \circ \Phi(0) f_2 \circ \Phi(0) f_3 \circ \Phi(0) \rangle \right) ,
\]  

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where $g_0$ is open string coupling constant, $Q_B$ is BRST operator and $<$ denote correlation function in the combined matter ghost conformal field theory. $I, f_1, f_2, f_3$ are conformal mapping exact form of which is reviewed in [4] and $f_i \circ \Phi(0)$ denotes the conformal transformation of $\Phi(0)$ by $f_i$. For example, for $\Phi$ a primary field of dimension $h$, then $f_i \circ \Phi(0) = (f'_i(0))^h \Phi(f_i(0))$.

We can expand any state $|\Phi\rangle \in \mathcal{H}$ as

$$|\Phi\rangle = \Phi(0) |0\rangle = (\phi(y) + A_\mu(y)\alpha_\mu^{\nu-1} + B_{\mu\nu}(y)\alpha_\mu^{\nu-1}\alpha^{\nu}_{\nu-1} + \ldots)c_1 |0\rangle = \sum_\alpha \phi^\alpha(y) \Phi_\alpha(0) |0\rangle,$$

(2.3)

where coefficients $\phi^\alpha(y)$ in front of basis states $|\Phi_\alpha\rangle$ of $\mathcal{H}$ depend on the centre-of-mass state coordinate $y$ and where index $\alpha$ labels all possible vertex operators of ghost number one. As we think of the coefficient functions $\phi_\alpha(y)$ as space-time particle fields, we call $|\Phi\rangle$ as a string field [2]. The vertex operator $\Phi(z)$ defined above is also called as a string field.

The previous action describes string field theory living on one single D25-brane. In order to describe a configuration of $N$ coincident D25-branes we equip the open string with Chan-Paton degrees of freedom so that coefficient functions become matrix valued and so $|\Phi\rangle$. In the following we restrict to the case of $N$ D-instantons where strings obey Dirichlet boundary conditions in all dimensions and where coefficient functions are $N \times N$ matrices without any dependence on $y$. We will write such a string field as $|\hat{\Phi}\rangle$ and call it mostly in the text as a "Matrix valued string field" keeping in mind that this is $N \times N$ matrix where each particular component $|\hat{\Phi}_{ij}\rangle$ corresponds to the string field that describes the state of the string connecting the i-th D-instanton with the j-th D-instanton.

As we claimed in the introduction, it would be interesting to have a formulation of the string field action for any configuration of D-instantons. While some progress in this direction has been made in [22], we would like to find such a formulation of the action based on the CFT description. As the first step in searching such a string field theory action we propose generalised matrix valued vertex operators carrying CP factors that describe any configuration of D-instantons. We will discuss this approach in the next section.

3. String fields for $N$ D-instantons

We propose the form of the matrix valued string field which in our opinion provides description of the general configuration of $N$ D-instantons in the bosonic string field theory. D-instantons are characterised by the strings having Dirichlet boundary conditions in all dimensions $y^I, I = 1, \ldots, 26$. Let us consider the situation with $N$ D-instantons placed in general positions. This configuration is described with the
matrices

\[
Y^I = \begin{pmatrix}
  y^I_1 & 0 & \ldots & 0 \\
  0 & y^I_2 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \ldots & 0 & y^I_N
\end{pmatrix}, \quad I = 1, \ldots, 26,
\]

(3.1)

where \( y^I_i \) labels coordinate of i-th D-instanton. Moreover, the configuration (3.1) corresponds to the solution of the equation of motion of the low energy matrix model effective action and as we will see, some consistency requirements that will be posed on the matrix valued string fields also imply that the background configuration of \( N \) D-instantons (3.1) should have this form.

It is well known that the string stretching from the i-th D-instanton to the j-th D-instanton has an energy proportional to the distance between these two branes. More precisely, vertex operator describing the ground state of the string going from the i-th D-instanton to the j-th D-instanton is given by

\[
|ij⟩ \equiv u_{ij}(z = 0)c(0) |0⟩ \equiv c(0) \exp \left( \frac{i(y^I_i - y^I_j)}{2\pi\alpha'} g_{IJ}X^J(0) \right) |0⟩, \quad i, j = 1, \ldots, N,
\]

(3.2)

with \(|0⟩\) being the \( SL(2, C) \) invariant vacuum state. In the previous expression \( g_{IJ} \) is a flat closed string metric \( g_{IJ} = \delta_{IJ} \) with signature \((+, \ldots, +)\). Let us consider some state of ghost number one from the first quantized Hilbert space of the open string that does not depend on the zero mode of \( X^I(z) \) which means that \( \Phi_\alpha = \Phi_\alpha(\partial X, c, b) \) commutes with \( u_{ij} \) given above. The index \( \alpha \) labels all possible vertex operators of ghost number one. Then any string field corresponding to the string going from the i-th D-instanton to the j-th D-instanton can be written in the similar form as in (2.3)

\[
|\hat{\Phi}⟩_{ij} = \sum_\alpha A^\alpha_{ij} \Phi(0)_{\alpha} u_{ij}(0) |0⟩,
\]

(3.3)

where \((A)_{ij}^\alpha\) is analogue of \( \phi^\alpha(y) \) in (2.3). Roughly speaking, matrix \( A^\alpha \in U(N) \) contains information which string from the collection of all possible \( N^2 \) strings of the system of \( N \) D-instantons (or more precisely, with which amplitude of probability) is excited in given state characterised by the world-sheet operator \( \Phi_\alpha(z) \). In the following we restrict ourselves to one particular CFT operator \( \Phi_\alpha(z) \) and its corresponding \( A^\alpha \). For that reason we omit the index \( \alpha \) in our formulas. In spite of this fact we will still call \( \hat{\Phi} \) a string field since it describes the whole system.

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1Because we implicitly presume that \( U \) is normal ordered we will not write symbol of the normal ordering \( : \). For simplicity, we will also consider the dependence of the world-sheet fields \( X^I(z) \) on the holomorphic coordinate \( z \) only.
From the previous analysis it is clear that any string field is $N \times N$ matrix that in more detailed description has a form

$$\hat{\Phi}(0) = \begin{pmatrix}
A_{11}u_{11}(0) & A_{12}u_{12}(0) & \ldots & A_{1N}u_{1N}(0) \\
A_{21}u_{21}(0) & A_{22}u_{22}(0) & \ldots & A_{2N}u_{2N}(0) \\
\vdots & \vdots & \ddots & \vdots \\
A_{N1}u_{N1}(0) & \ldots & A_{N,N-1}u_{N,N-1}(0) & A_{NN}u_{NN}(0)
\end{pmatrix} \times \Phi(0). \quad (3.4)$$

We would like to argue that this expression can be written in more symmetric form. Let us define $N \times N$ matrix-operator that is a function of the matrices $Y^I$ and the world-sheet fields $X^I(z)$

$$U(z)(\cdot \ ) = \exp\left(\frac{i}{2\pi\alpha'} [Y^I, \cdot \ ]g_{IJ}X^J(z)\right), \quad (3.5)$$

where definition of its action on any $N \times N$ matrix will be given below. Our proposal is that the generalised matrix valued string field can be written as

$$\hat{\Phi}(z) = U(z)(A)\Phi(z). \quad (3.6)$$

We will show that for the background given (3.1) the operator (3.6) reduces to (3.4). To see this, we must firstly explain the meaning of the expression $U(z)(A)$.

This expression simply corresponds to the expansion of the exponential function and successive acting of the commutators on matrix $A$. More precisely

$$U(A)_{ij} = A_{ij} + \frac{i}{2\pi\alpha'} [Y^I, A]_{ij}g_{IJ}X^J(z) + \frac{1}{2} \left(\frac{i}{2\pi\alpha'}\right)^2 [Y^I, [Y^K, A]]_{ij}g_{IJ}g_{KL}X^J(z)X^L(z) + \ldots. \quad (3.7)$$

The second term in (3.7) for $Y^I$ given in (3.1) is equal to

$$\frac{i}{2\pi\alpha'} [Y^I, A]_{ij}g_{IJ}X^J(z) = \frac{i}{2\pi\alpha'} [y^I_ikA_{kj} - A_{ik}y^I_k\delta_{kj}]g_{IJ}X^J(z) = \frac{i}{2\pi\alpha'} [y^I_i - y^I_j]A_{ij}g_{IJ}X^J(z), \quad (3.8)$$

where there is no summation over $i, j$. In the same way the third term in (3.7) gives

$$\frac{1}{2} \left(\frac{i}{2\pi\alpha'}\right)^2 [Y^I, [Y^K, A]]_{ij}g_{IJ}g_{KL}X^J(z)X^L(z) =$$

$$= \frac{1}{2} \left(\frac{i}{2\pi\alpha'}\right)^2 [y^I_{im} - y^I_m]A_{mj}g_{KL}X^J(z)X^L(z) =$$

$$= \frac{1}{2} \left(\frac{i}{2\pi\alpha'}\right)^2 \left(y^I_m\delta_{im} - y^I_jA_{mj} - (y^I_m - y^I_jA_{im})\delta_{mj}y^I_j\right)g_{KL}X^J(z)X^L(z) =$$
\[
\left(\frac{i}{2\pi \alpha'}\right)^2 (y_i^I - y_j^I) A_{ij} = \left(\frac{i}{2\pi \alpha'}\right)^2 (y_i^K - y_j^K) y_j^I A_{ij} = \frac{1}{2} A_{ij} \left(\frac{i}{2\pi \alpha'}\right)^2 (y_i^I - y_j^I) g_{IJ} X^J(z) g_{KL} X^L(z) = \frac{1}{2} A_{ij} \left(\frac{i}{2\pi \alpha'}(y_i^I - y_j^I) g_{IJ} X^J(z)\right)^2,
\]

where from the fourth row there is no summation over \(i, j\). To show that (3.6) really corresponds to (3.4) for \(Y^I\) given in (3.1) we use the proof by the mathematical induction. Let us presume that the following relation is valid for any \(P\)

\[
\left(\frac{i}{2\pi \alpha'}\right)^P [Y^{I_1}, [Y^{I_2}, \ldots, [Y^{I_P}, A]]]_{ij} g_{I_1 J_1} g_{I_2 J_2} \ldots g_{I_P J_P} X^{J_1}(z) X^{J_2}(z) \ldots X^{J_P}(z) = A_{ij} \left(\frac{i}{2\pi \alpha'}(y_i^I - y_j^I) g_{IJ} X^J(z)\right)^P.
\]

Then for \(P + 1\) we have

\[
\left(\frac{i}{2\pi \alpha'}\right)^{P+1} [Y^{I_1}, [Y^{I_2}, \ldots, [Y^{I_{P+1}}, A]]]_{ij} g_{I_1 J_1} g_{I_2 J_2} \ldots g_{I_{P+1} J_{P+1}} X^{J_1}(z) X^{J_2}(z) \ldots X^{J_{P+1}}(z) = A_{ij} \left(\frac{i}{2\pi \alpha'}(y_i^I - y_j^I) g_{IJ} X^J(z)\right)^P -
\]

\[
\left(\frac{i}{2\pi \alpha'}\right)^{P+1} \left(\frac{i}{2\pi \alpha'}(y_i^I - y_j^I) g_{IJ} X^J(z)\right)^P \left(\frac{i}{2\pi \alpha'}(y_i^K - y_j^K) g_{KL} X^L(z)\right) = A_{ij} \left(\frac{i}{2\pi \alpha'}(y_i^I - y_j^I) g_{IJ} X^J(z)\right)^P -
\]

\[
-y_j^K A_{ij} \left(\frac{i}{2\pi \alpha'}(y_i^I - y_j^I) g_{IJ} X^J(z)\right)^P g_{KL} X^L(z) = A_{ij} \left(\frac{i}{2\pi \alpha'}(y_i^I - y_j^I) g_{IJ} X^J(z)\right)^{P+1},
\]

where again there is no summation over \(i, j\) from the fourth row. Using the previous result we obtain the expression

\[
U(A)_{ij}(z) = A_{ij} \exp \left(\frac{i}{2\pi \alpha'}(y_i^I - y_j^I) g_{IJ} X^J(z)\right)
\]

without summation over \(i, j\). We then see that (3.8) has a correct form of the matrix valued string field for the description of the string configuration in the background (3.1) of \(N\) D-instantons.

Before we turn to the next example, we must certainly find some consistency conditions which these generalised conformal operators should obey. We will proceed
as follows. Let us start with general configuration of $N$ D-instantons described with any $U(N)$ valued matrices $Y_I$. Then we require that the matrix valued string field $\Phi$ should obey linearised equation of motion of the string field theory action. In abelian case this leads to the requirement that given state is annihilated by the BRST operator $Q_B$. It is reasonable to presume that this holds in nonabelian case as well so we obtain the condition

$$Q_B \, \Phi = 0 , \quad \Phi = \Phi(0) U(A)(0) \, |0\rangle \ .$$

(3.13)

We will study consequence of this equation. In order to do that we must find an operator product expansion (OPE) between various matrix valued operators and stress energy tensor of the open string theory

$$T(z) = -\frac{1}{\alpha'} \partial_z X^I(z) \partial_z X^J(z) g_{IJ}, \quad X^I(z) X^J(w) = -\frac{1}{2} \alpha' \ln(z-w) g^{IJ}, \quad I, J = 1, \ldots, 26 .$$

(3.14)

Using (3.14) we can easily calculate OPE between $T(z)$ and $U(0)$. For example, let us consider the OPE between stress energy tensor (3.14) and the first two terms in the expansion of $U(0)$ acting on any $A$ corresponding to any CFT operator $\Phi$

$$T(z) \frac{i}{2\pi \alpha'} [Y^I, A] g_{IJ} X^J(0) \sim \frac{1}{z} \frac{i}{2\pi \alpha'} [Y^I, A] g_{IJ} \partial_z X^J(0) ,$$

$$T(z) \frac{1}{2} \left( \frac{i}{2\pi \alpha'} \right)^2 [Y^{I_1}, [Y^{I_2}, A]] g_{I_1J_1I_2J_2} X^{J_1}(0) X^{J_2}(0) \sim$$

$$\sim \frac{1}{2z} \left( \frac{i}{2\pi \alpha'} \right)^2 [Y^{I_1}, [Y^{I_2}, A]] g_{I_1J_1I_2J_2} \partial_z (X^{J_1}(0) X^{J_2}(0)) -$$

$$- \frac{\alpha'}{4z^2} \left( \frac{i}{2\pi \alpha'} \right)^2 [Y^I, [Y^J, A]] g_{IJ} .$$

(3.15)

Generally we have

$$T(z) \frac{1}{P!} \left( \frac{i}{2\pi \alpha'} \right)^P [Y^{I_1}, [Y^{I_2}, \ldots, [Y^{I_P}, A]]] g_{I_1J_1} \ldots g_{I_PJ_P} \times$$

$$\times X^{J_1}(0) \ldots X^{J_P}(0) \sim \frac{1}{z} \frac{1}{P!} \left( \frac{i}{2\pi \alpha'} \right)^P \sum_{k=1}^P [Y^{I_1}, [Y^{I_2}, \ldots, [Y^{I_k}, \ldots, [Y^{I_P}, A]]] \times$$

$$\times g_{I_1J_1} \ldots g_{I_kJ_k} \ldots g_{I_PJ_P} X^{J_1}(0) \ldots X^{J_{k-1}}(0) \partial_z X^{J_k}(0) X^{J_{k+1}}(0) \ldots X^{J_P}(0) -$$

$$- \frac{1}{z^2 4P!} \left( \frac{i}{2\pi \alpha'} \right)^P \sum_{m=1, n=2, m \neq n}^P g_{I_1J_m} g_{I_mJ_n} [Y^{I_1}, [Y^{I_2}, \ldots, [Y^{I_m}, \ldots, [Y^{I_n}, \ldots, [Y^{I_P}, A]]] \times$$

$$\times g_{I_1J_1} \ldots g_{I_PJ_P} X^{J_1}(0) \ldots X^{J_{m-1}}(0) X^{J_{m+1}} \ldots X^{J_{n-1}}(0) X^{J_{n+1}} \ldots X^{J_P}(0) .$$

(3.16)

From the previous expression we can deduce that generally there is not well defined OPE between stress energy tensor and matrix valued string field. As a consequence
of this fact we cannot define how such a matrix valued string field transforms under conformal transformations and hence we cannot define string field action. In fact, in analogy with the abelian case we would like to have an OPE in the form

$$T(z)\Phi(0) \sim \frac{1}{z^2} h(\Phi)(0) + \frac{1}{z} \partial_z \Phi(0) ,$$  

(3.17)

where $h(\Phi)$ is a conformal dimension of given field $\Phi$. In order to obtain OPE in similar form we demand that the background configuration of D-instantons obeys following rule

$$[Y^I, [Y^J, B]] - [Y^J, [Y^I, B]] = 0, \forall B , I, J = 1, \ldots, 26 ,$$  

(3.18)

or equivalently

$$[[Y^I, Y^J], B] = 0 \Rightarrow [Y^I, Y^J] = i\theta^{IJ} 1_{N \times N} ,$$  

(3.19)

that holds of course only in case of infinite dimensional $U(N)$ matrices $Y^I$. We can expect that this configuration describes higher dimensional D-brane with noncommutative world-volume. In case of finite dimensional matrices the only possible solution is

$$[Y^I, Y^J] = 0 .$$  

(3.20)

Conditions (3.19),(3.20) are precisely solutions of the equation of motion arising from the low energy action for $N$ D-instantons. We have seen the similar result in our previous paper [22] where the requirement of the nilpotence of the matrix valued BRST operator leads to the conclusion that the background configuration of D-instantons must obey equation (3.19), (3.20).

Using (3.18) we can move $Y^I_m$ and then $Y^I_o$ on the left hand side of the second expression in (3.16). Since we have $P$ possible $I$ and $P - 1$ $J$ and all appear in the expression in the symmetric way, the summation in the second expression in (3.16) gives the factor $P(P - 1)$ so that the second term in (3.16) gives

$$-\frac{\alpha' P(P - 1)}{4P! z^2} \left( \frac{i}{2\pi \alpha'} \right)^P g_{IJ}[Y^I, [Y^J, [Y^{I_1}, \ldots, [Y^{I_{P-2}}, A]]] \times$$

$$\times g_{I_1 J_1} \cdots g_{I_{P-2} J_{P-2}} X^{J_1}(0) \cdots X^{J_{P-2}}(0) =$$

$$= -\frac{\alpha'}{4z^2(P - 2)!} \left( \frac{i}{2\pi \alpha'} \right)^2 g_{IJ}[Y^I, [Y^J, \left( \frac{i}{2\pi \alpha'} [Y^K, \cdot] g_{KL} X^L(0) \right)^P - A]$$

(3.21)

and the first equation in (3.16) gives

$$\frac{1}{z P!} \left( \frac{i}{2\pi \alpha'} \right)^P \sum_{k=1}^{P} [Y^{I_1}, \ldots, [Y^{I_k}, \ldots, [Y^{I_P}, A]]] \times$$

$$\times g_{I_1 J_1} \cdots g_{I_k J_k} \cdots g_{I_P J_P} X^{J_1}(0) \cdots X^{J_{k-1}}(0) \partial_z X^{J_k}(0) X^{J_{k+1}}(0) \cdots X^{J_P}(0) =$$

$$= \frac{1}{z P!} \partial_z \left( \left( \frac{i}{2\pi \alpha'} \right)^P [Y^I, \cdot] g_{IJ} X^J(0) \right)^P (A) .$$

(3.22)
Collecting all previous results we obtain the following operator product expansion

\[ T(z)\Phi(0) \sim \frac{1}{z^2} \left( \frac{1}{16\pi^2\alpha'} g_{IJ}[Y^I, [Y^J, \Phi(0)]] + h_\Phi \Phi(0) \right) + \frac{1}{z} \partial_z \Phi(0) , \]  

(3.23)

where \( h_\Phi \) is the conformal dimension of the operator \( \Phi(0) \) in (3.6). We see that "the conformal dimension" of \( \hat{\Phi} \) is now matrix valued and depends on the configuration of various D-instantons. Since the ghost sector does not depend on the background configuration of D-instantons, the acting of the BRST operator on \( \Phi \) is the same as in the abelian case. When we also use the gauge

\[ b_0 |\Phi\rangle = 0 \]  

(3.24)

we see that the linearised equation of motion (3.13) leads to condition

\[ \frac{1}{16\pi^2\alpha'} g_{IJ}[Y^I, [Y^J, \Phi(0)]] + h_\Phi \Phi(0) = 0 , \]  

(3.25)

where (in gauge \( b_0 |\Phi\rangle = 0 \))

\[ Q_B |\Phi\rangle = h_\Phi |\Phi\rangle . \]  

(3.26)

Condition (3.25) expresses the fact that each component \( \hat{\Phi}_{ij} \) of the matrix valued string field describes state of the open string connecting the i-th D-instanton with the j-th D-instanton which is on the mass shell. For example, for the diagonal background (3.1) the first term in the bracket in (3.23) gives

\[ \frac{1}{16\pi^2\alpha'} g_{IJ}[Y^I, [Y^J, \Phi(0)]]_{ij} = \]

\[ = \frac{1}{16\pi^2\alpha'} g_{IJ} [y_i^I - y_j^I] [y_j^J - y_i^J] \hat{\Phi}_{ij}(0) \]

(3.27)

(again no summation over \( i, j \)). Then we have a natural result that the conformal dimension of each component \( \hat{\Phi}_{ij} \) is proportional to the distance between the i-th D-instanton and the j-th D-instanton.

The OPE between generalised matrix valued string field and the stress energy tensor has also an important consequence for the conformal transformation of given string field and hence for the generalised form of the string field action. Recall, that a primary vertex operator of conformal dimension \( h \) transforms under \( z' = f(z) \) as

\[ \mathcal{O}(z') = \left( \frac{\partial f}{\partial z} \right)^{-h} \mathcal{O}(z) = \exp \{ (-h \ln(f'(z)) \} \mathcal{O}(z) . \]

(3.28)

From the second form of this description and from the fact that for the matrix valued string field the first term in (3.23) acts as a matrix on given string field we can anticipate following generalised matrix valued conformal transformation

\[ \hat{\Phi}'(z') = \exp \left( - \ln(f(z)) \left[ \frac{1}{16\pi^2\alpha'} g_{IJ}[Y^I, [Y^J, \cdot]] + h_\Phi \right] \right) (\hat{\Phi})(z) , \]

(3.29)
where the exponential function should be understood as a matrix valued function and its action on \( \hat{\Phi}(z) \) in form of the Taylor expansion and where \( h_{\Phi} \) is the conformal dimension of the operator \( \Phi(z) \). In particular, for the background (3.1) we have

\[
\hat{\Phi}(z)_{ij} = A_{ij} \exp \left( \frac{i}{2\pi\alpha'} (y^I_i - y^I_j) g_{IJ} X^J(z) \right) .
\]  

(3.30)

In order to determine the behaviour of this field under conformal transformation, we must expand exponential function in (3.29) and let it to act on \( \hat{\Phi} \). For example, for the background (3.1) we obtain the result that the field \( \hat{\Phi}_{ij} \) that describes the state of the string going from the i-th D-instanton to the j-th D-instanton transforms under the general conformal transformation according to the usual rule

\[
\hat{\Phi}'(z')_{ij} = \left( \frac{df(z)}{dz} \right)_{16\pi\alpha'(y_i - y_j)^2 - h_{\Phi}} \hat{\Phi}(z)_{ij}.
\]  

(3.31)

Now we turn to the second example which is the background configuration of \( N \) D-instantons (in this case \( N \to \infty \)) in the form

\[
[Y^a, Y^b] = i\theta^{ab}, \quad a, b = 1, \ldots, 2p, \quad Y^m = 0, \quad m = 2p + 1, \ldots, 26 .
\]  

(3.32)

As in the previous case we begin with (3.33) where the second term is proportional to

\[
i[Y^I, A]g_{IJ} X^J(0) .
\]  

(3.33)

Following [27, 28, 29] we introduce the set of matrices

\[
O_k = e^{i\theta^{ij} k_i p_j}, \quad p_b = \theta_{bc} Y^c, \quad \theta^{ac} \theta^{cb} = \delta^b_a .
\]  

(3.34)

Then we can write any matrix as follows

\[
A = \int d^2p \exp[i\theta^{ab} k_a p_b] A(k) ,
\]  

(3.35)

where \( A(k) \) is an ordinary function. Then it is easy to see [28, 29]

\[
[p_i, O_k] = k_i O_k, \quad [p_i, p_j] = -i\theta_{ij}
\]  

(3.36)

and consequently

\[
\frac{2\pi}{4\pi^2\alpha'} [Y^a, A]g_{ab}X^b(0) = [p_a, A]G^{ab}\tilde{X}_b(0) = \int d^2p k_a G^{ab} \tilde{X}_b(0) A(k) O_k ,
\]  

\[
G^{ab} = -\frac{1}{4\pi\alpha'} g_{cd} \theta^{ab} , \quad \tilde{X}_b(0) = 2\pi\alpha' \theta_{bc} X^c(0),
\]  

\[
\left( \frac{i}{2\pi\alpha'} \right)^P [Y^{I_1}, [Y^{I_2}, \ldots, [Y^{I_P}, A]]]g_{I_1 J_1} g_{I_2 J_2} \ldots g_{I_P J_P} X^{J_1}(z) X^{J_2}(z) \ldots X^{J_P}(z) =
\]  

\[
i^P \int d^2p k_a G^{a_1 b_1} \tilde{X}_{b_1}(z) \ldots k_{a_P} G^{a_P b_P} \tilde{X}_{b_P}(z) A(k) O_k .
\]  

(3.37)
Then it follows
\[ U(A)(z) = \int d^{2p} k \exp \left( i k \bar{X}(z) \right) A(k) O_k, \quad k \bar{X} = k_a G^{ab} \bar{X}_b \] (3.38)
and consequently for any conformal field theory operator \( \Phi(z) \) (corresponding to some particular \( A \)) we get the matrix valued string field \( \hat{\Phi} \)
\[ \hat{\Phi} = U(A) \Phi(z) = \int d^{2p} k \Phi(z) \exp \left( i k \bar{X}(z) \right) A(k) O_k . \] (3.39)

We define generalised matrix valued vertex operators the form of which we can deduce from (3.39)
\[ V(k, \Phi(z)) = \Phi(z) \exp \left( i k \bar{X}(z) \right) O_k \] (3.40)
It is important to include the matrix \( O_k \) into the definition of the matrix valued vertex operator \( V \) in order to stress its matrix nature since any correlation function of these operators contains the trace over matrix indices. Let us consider two such matrix valued vertex operators
\[ V(k_1, \Phi)(z) = O_{k_1} \Phi(z) e^{ik_1 \bar{X}(z)} , \quad V(k_2, \Psi)(z) = O_{k_2} \Psi(z) e^{ik_2 \bar{X}(z)} , \] (3.41)
which should appear in the calculation of the correlation function and in particular in the string field theory action. Let us calculate the generalised OPE of these two operators where we include the matrix multiplication. In fact, the calculation of the OPE is an easy task. Matrix multiplication affects only the parts containing \( O_{k_1}, O_{k_2} \) that gives
\[ O_{k_1} O_{k_2} = \exp \left( i \theta^{ij} (k_1 + k_2)_i p_j - \frac{1}{2} i \theta^{ab} k_a k_b \right) = e^{-\frac{i}{2} \theta^{ab} k_a k_b} O_{k_1 + k_2} \] (3.42)
using
\[ [i \theta^{ab} k_a p_b, i \theta^{cd} k_c p_d] = -\theta^{ab} \theta^{cd} k_a k_c (-i \theta_{bd}) = -i \theta^{ab} k_a k_b \] (3.43)
and also using the relation
\[ e^A e^B = e^{A+B+\frac{i}{2}[A,B]} \] (3.44)
that is valid for operators whose commutator is a pure number. Then we have
\[ V(k_1, \Phi)(z) V(k_2, \Psi)(w) = O_{k_1} \Phi(z) e^{ik_1 \bar{X}(z)} O_{k_2} \Psi(w) e^{ik_2 \bar{X}(w)} \sim e^{-\frac{i}{2} \theta^{ab} k_a k_b} O_{k_1+k_2} \times \]
\[ \times \left( z - w \right)^{\frac{i}{2} k_{1a} G^{ab} k_{2b}} \exp \left( i (k_1 + k_2) \bar{X}(w) \right) \Phi(w) \Psi(w) + \ldots \right) \] (3.45)
where dots mean other possible singular terms arising from the expansion of \( e^{ik \bar{X}(z)} \) and from the OPE between \( \Phi(z) \) and \( \Psi(w) \). We see that the previous OPE has the same form as the OPE of the vertex operators in the presence of the background field \( B_{ab} = \left( \frac{i}{2} \right)_{ab} \) as is well known from the seminal paper [25]. It is also important to
stress that thanks to the redefinition \( X^c(z) = \frac{1}{2\pi\alpha'} \theta^{cd} \tilde{X}_d(z) \) the stress energy tensor looks like
\[
T(z) = -\frac{1}{\alpha'} \partial X(z)^j \partial X(z)^j g_{11} = \frac{1}{\alpha'} \partial X^i(z) \partial X^j(z) g_{ij} = \frac{1}{\alpha'} \partial \tilde{X}_a(z) \partial \tilde{X}_b(z) G^{ab} - \frac{1}{\alpha'} \partial X^i(z) \partial X^j(z) g_{ij}.
\]  
(3.46)

In other words, the world-sheet stress energy tensor is expressed in terms of the open string metric \( G_{ab} \) in dimensions labelled with \( \tilde{X}_a, \tilde{X}_b, \ldots \) hence the OPE between the part of the stress energy tensor depending on the open string metric and any matrix valued vertex operator is a function of the open string quantities only again with agreement with [25].

We should also study the generalised conformal transformation (3.29)
\[
V'(k, \Phi, z') = \exp \left( -\ln(f(z)) \left[ \frac{1}{16\pi^2\alpha'} g_{11} [Y^I, [Y^J, \cdot]] + h_\Phi \right] \right) V(k, \Phi, z).
\]  
(3.47)

In fact, \( h_\Phi \) is given solely by the conformal dimension of \( \Phi(z) \) and the matrix multiplication defined in the exponential function in (3.47) acts on \( O_k \) only. Then we can expand the exponential function and use (3.36). It is easy to see that we obtain the standard conformal transformation of the vertex operator with the momentum \( k_a \)
\[
V'(k, \Phi, z') = \exp \left( -\ln(f(z)) \left[ \frac{\alpha'}{4} k^2 + h_\Phi \right] \right) V(k, \Phi, z) = \left( \frac{df(z)}{dz} \right)^{-\frac{\alpha'k^2}{4} - h_\Phi} V(k, \Phi, z)
\]  
(3.48)

with \( k^2 = k_a G^{ab} k_b \). From this expression we see that \( V(k, \Phi, z) \) has the conformal dimension equal to \( \alpha' k^2/4 + h_\Phi \) as we could expect. We can also calculate the OPE between the stress energy tensor and matrix valued vertex operators. Since the OPE between vertex operator and stress energy tensor determines the conformal dimension of given operator and this is known from (3.48), we do not need work out this OPE and can determine its form directly from (3.48).

**More general example**

Let us consider the background configuration of D-instantons in the form
\[
Y^a = 1_{N \times N} \otimes y^a, \ Y^i = \begin{pmatrix}
y^1_1 \otimes 1 & 0 & \ldots & 0 \\
0 & y^1_2 \otimes 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & y^i_N \otimes 1
\end{pmatrix}, i = 2p + 1, \ldots, 26,
\]  
(3.49)

where
\[
[y^a, y^b] = i \theta^{ab}, \ a, b = 1, \ldots, 2p
\]  
(3.50)
are infinite dimensional matrices. It is easy to see that this configuration obeys (3.19) and hence corresponds to the consistent background configuration. Now we will write any matrix $A$ as follows

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1N} \\ A_{21} & A_{22} & \ldots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & \ldots & \ldots & A_{NN} \end{pmatrix},$$

(3.51)

where $A_{xy}$, $x, y = 1, \ldots, N$ are infinite dimensional matrices. Let us write any $A_{xy}$ in the form

$$A_{xy} = \int d^2p k_{xy} A_{xy}(k_{xy}) \exp \left( i \theta^{ab} k_{axy} p_b \right).$$

(3.52)

Now we can write

$$\frac{i}{2\pi \alpha'} [Y^I, A]_{xy} g_{IJ} X^J(z) = \frac{i}{2\pi \alpha'} [\delta_{xz} \otimes y^a, A]_{xy} g_{ab} X^b(z) + \frac{i}{2\pi \alpha'} [y_i^a \delta_{xz} \otimes 1, A]_{xy} g_{ij} X^j(z) = [p_a, A]_{xy} g^{ab} \bar{X}_b(z) + \frac{i}{2\pi \alpha'} (y_i^a - y_i^b) g_{ij} X^j(z) A_{xy} =$$

$$= i \int d^2p k_{xy} \left( k_{xya} g^{ab} \bar{X}_b(z) + \frac{1}{2\pi \alpha'} (y_i^a - y_i^b) g_{ij} X^j(z) \right) A_{xy}(k_{xy}) O_{k_{xy}}.$$

(3.53)

The second term in (3.4) gives

$$\left( \frac{i}{2\pi \alpha'} \right)^2 [Y^I, [Y^J, A]] g_{LK} g_{KL} X^K(z) X^L(z) =$$

$$= i^2 \int d^2p k_{xy} k_{xya} G^{ab}_{xy} \bar{X}_b(z) k_{xya} G^{ab}_{xy} \bar{X}_b(z) A_{xy}(k_{xy}) O_{k_{xy}} +$$

$$+ i^2 \int d^2p k_{xy} A(k_{xy})_{xy} \left( \frac{1}{2\pi \alpha'} (y_i^a - y_i^b) g_{ij} X^j(z) \right)^2 +$$

$$+ 2i^2 \int d^2p k_{xy} O_{k_{xy}} k_{xya} G^{ab} \bar{X}_b(z) \frac{1}{2\pi \alpha'} (y_i^a - y_i^b) g_{ij} X^j(z) =$$

$$= i^2 \int d^2p k_{xy} O_{k_{xy}} A(k_{xy})_{xy} \left( k_{xya} G^{ab} \bar{X}_b(z) + \frac{1}{2\pi \alpha'} (y_i^a - y_i^b) g_{ij} X^j(z) \right)^2.$$

(3.54)

Generally, we have

$$\left( \frac{i}{2\pi \alpha'} \right)^P [Y^{I_1}, [Y^{I_2}, \ldots, [Y^{I_P}, A]]_{xy} g_{I_1J_1} g_{I_2J_2} \ldots g_{I_PJ_P} X^{I_1}(z) X^{J_2}(z) \ldots X^{I_P}(z) =$$

$$= i^P \int d^2p k_{xy} \left( k_{xya} G^{ab} \bar{X}_b(z) + \frac{1}{2\pi \alpha'} (y_i^a - y_i^b) g_{ij} X^j(z) \right)^P A(k_{xy})_{xy} O_{k_{xy}}.$$

(3.55)
Using these results we can write the generalised matrix valued string field (3.6) in the form

\[ \hat{\Phi}_{xy}(z) = \int d^2p k_{xy} A(k_{xy})_{xy} \exp \left( ik_{xy} \tilde{X}(z) + \frac{i}{2\pi \alpha'} (y^i_x - y^i_y) g_{ij} X^j(z) \right) O_{k_{xy}}. \]  

(3.56)

In summary, we have obtained the matrix valued string field for configuration of \(N\) D2p-branes with the noncommutative world-volume in dimensions labelled with \(x^a, a = 1, \ldots, 2p\), that are placed in the different transverse positions labelled with \(y^i_x, i = 2p + 1, \ldots, 26, x = 1, \ldots, N\).

We hope that three examples given above sufficiently support our proposed form of the generalised matrix string field (3.6). Then we propose that the string field theory action for any configuration of \(N\) D-instantons obeying (3.19) has a form

\[ S = \frac{1}{g_0^2} \text{Tr} \left( \frac{1}{2\alpha'} \left( I \circ \hat{\Phi}(0) Q_B \hat{\Phi}(0) \right) + \frac{1}{3} \left( f_1 \circ \hat{\Phi}(0) f_2 \circ \hat{\Phi}(0) f_3 \circ \hat{\Phi}(0) \right) \right), \]  

(3.57)

where now conformal transformations \(I \circ \hat{\Phi}(0), f_i \circ \hat{\Phi}(0), i = 1, 2, 3\) are defined by (3.29). The precise study of this action, its particular solutions will be performed in the forthcoming work.

4. Conclusion

In this paper we have proposed the matrix valued form of the string field that could be useful for description of D-instanton configuration using the string field theory action written in the conformal field theory language [23, 24]. We have calculated OPE of these matrix valued string fields with the stress energy tensor. We have seen that the condition that the OPE is well defined leads to the requirement that the background configuration of D-instantons should obey the equation that can be interpreted as the equation of motion arising from the low energy matrix theory action. We have also proposed the generalised conformal transformation of the matrix valued string fields.

As a next step of our research we will study the proposed matrix valued string field theory action (3.57). We will also extend this approach to the supersymmetric case.

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