Knots in collapsible and non-collapsible balls

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Abstract

We construct the first explicit example of a simplicial 3-ball \(B_{15,66}\) that is not collapsible. It has only 15 vertices. We exhibit a second 3-ball \(B_{12,38}\) with 12 vertices that is collapsible and evasive, but not shellable. Finally, we present the first explicit triangulation of a 3-sphere \(S_{18,125}\) (with only 18 vertices) that is not locally constructible. All these examples are based on knotted subcomplexes with only three edges; the knots are the trefoil, the double trefoil, and the triple trefoil, respectively. The more complicated the knot is, the more distant the triangulation is from being polytopal, collapsible, etc. Further consequences of our work are:

1. Unshellable 3-spheres may have vertex-decomposable barycentric subdivisions. (This shows the strictness of an implication proven by Billera and Provan.)
2. For \(d\)-balls, vertex-decomposable implies non-evasive implies collapsible, and for \(d = 3\) all implications are strict. (This answers a question by Barmak.)
3. Locally constructible 3-balls may contain a double trefoil knot as a 3-edge subcomplex. (This improves a result of Benedetti and Ziegler.)
4. Rudin’s ball is non-evasive.

1 Introduction

Collapsibility is a combinatorial property introduced by Whitehead, and somewhat stronger than contractibility. In 1964, Bing proved, using knot theory, that some triangulations of the 3-ball are not collapsible [11, 18]. Bing’s method works as follows. One starts with a finely-triangulated 3-ball embedded in the Euclidean 3-space. Then one drills a knot-shaped tubular hole inside it, stopping one step before destroying the property of being a 3-ball. The resulting 3-ball contains a knot that consists of a single interior edge plus many boundary edges. This interior edge is usually called knotted spanning. If the knot is sufficiently complicated (like a double, or a triple trefoil), Bing’s ball cannot be collapsible [11, 18]; see also [7]. In contrast, if the knot is simple enough (like a single trefoil), then the Bing ball may be collapsible [24].

Thus the existence of a short knot in the triangulation prevents a 3-ball from having a desirable combinatorial property, namely, collapsibility. This turned out to be a recurrent motive in literature. In the Eighties, several authors asked whether all 3-spheres are shellable. This was answered in 1991 by Lickorish in the negative [23]: The presence in a 3-sphere of a triple trefoil on three edges prevents it from being shellable. It remained open whether all spheres are constructible (a slightly weaker property than shellability). However, in 2000 Hachimori and Ziegler showed that the presence of any

*Supported by the Swedish Research Council, grant “Triangulerade Mångfalder, Knutteori i diskrete Morseteori” and the DFG Collaborative Research Center TRR 109, “Discretization in Geometry and Dynamics”.
†Supported by the DFG Research Group “Polyhedral Surfaces”.

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non-trivial knot on three vertices in a 3-sphere even prevents it from being constructible. Finally, in 1994 the physicists Durhuus and Jonsson asked whether all 3-spheres are locally constructible. Once again, a negative answer, based on Lickorish’s original argument, was found using knot theory; see Benedetti–Ziegler [10].

These examples represent spheres that are far away from being polytopal. Thus, they are good candidates for testing properties that are true for polytopes, but only conjectured to be true for spheres. Moreover, they represent good test instances for algorithms in computational topology, as they are complicated triangulations of relatively simple spaces.

Unfortunately, the knotted counterexamples by Lickorish and others have a defect: They are easy to explain at the blackboard, but they yield triangulations with many vertices. The purpose of this paper is to come up with analogous ‘test examples’ that are smaller in size, but still contain topological obstructions that prevent them from having nice combinatorial properties.

A first idea to save on the number of faces is to start by realizing the respective knot in 3-space, using (curved) arcs. Obviously, any knot can be realized with exactly three arcs in $\mathbb{R}^3$ (we just need to draw it and insert three vertices along the knot). If we thicken the arcs into three ‘bananas’, the resulting 3-complex $P$ is homeomorphic to a solid torus pinched three times. By inserting 2-dimensional membranes, $P$ can be made contractible, and then it can be thickened to a 3-ball (or a 3-sphere) simply by adding cones. This approach costs a lot of manual effort, but a posteriori, it allows us to obtain new insight. In fact, here comes the second idea: We can ask a computer to perform random bistellar flips to the triangulation of the ball, without modifying the subcomplex $P$. Performing the flips according to a simulated annealing strategy [12] we were able to decrease the size of the triangulation, but for sure the flips will preserve the knotted substructure and its number of arcs.

This construction was introduced by the second author in [28], who applied it to the single trefoil, thereby obtaining a knotted 3-ball $B_{12, 38}$ with 12 vertices and 38 tetrahedra. Here we apply the method
to the double trefoil and the triple trefoil. The resulting spheres turn out to be interesting in connection with some properties which we will now describe.

The notion of **Evasiveness** has appeared first in theoretical computer science, in Karp’s conjecture on monotone graph properties. Kahn, Saks and Sturtevant [22] extended the evasiveness property to simplicial complexes, showing that non-evasiveness strictly implies collapsibility. One can easily construct explicit examples of collapsible evasive 2-complexes in which none of the vertex-links is contractible [5]; see also [8]. Basically there are three known ways to prove that a certain complex $E$ is evasive:

(A) One shows that none of its vertex-links is contractible, cf. [5];
(B) one proves that the Alexander dual of $E$ is evasive, cf. [22];
(C) one shows (for example, via knot-theoretic arguments [11]) that $E$ is not even collapsible.

But are there collapsible evasive balls? And if so, how do we prove that they are evasive? Clearly, none of the approaches above would work. This was asked to us by Barmak (private communication).

Once again, we found a counterexample in the realm of knotted triangulations: specifically, Lutz’s triangulation $B_{12,38}$, which contains a single-trefoil knotted spanning edge.

**Main Theorem 1.** *The 3-ball $B_{12,38}$ is collapsible and evasive. However, it is not shellable and not locally constructible.*

To prove collapsibility, we tried, using the computer, several collapsing sequences, until we found a lucky one. To show evasiveness, we used some sort of ‘trick’: We computed the homology of what would be left from $B_{12,38}$ after deleting roughly half of its vertices. It turns out that deleting five vertices from $B_{12,38}$ (no matter which ones) yields almost always some complex with non-trivial homology. From that we were able to exclude non-evasiveness.

En passant, we also prove the non-evasiveness of other existing triangulations that were known to be collapsible, like Rudin’s ball (Theorem 6.3) or Lutz’s triangulations $B_{7,10}$ [26] and $B_{9,18}$ [25].

Main Theorem 1 can be viewed as an improvement on the result from 1972 by Lickorish–Martin [24] and Hamstrom–Jerrard [21] that a ball with a knotted spanning edge can be collapsible. Recently Benedetti–Ziegler [10] constructed a similar example with all vertices on the boundary. In contrast, our $B_{12,38}$ has exactly one interior vertex. We also mention that $B_{12,38}$ is the first example of a manifold that admits a perfect discrete Morse function, but cannot admit a perfect Fourier–Morse function in the sense of Engström [16]. In fact, a complex is non-evasive if and only if it admits a Fourier–Morse function with only one critical cell.

**Vertex-decomposability** is a strengthening of shellability, much like non-evasiveness is a strengthening of collapsibility. It was introduced by Billera and Provan in 1980, in connection with the Hirsch conjecture [30]. For 3-balls, we have the following diagram of implications:

\[
\begin{array}{ccc}
\text{vertex-decomposable} & \Rightarrow & \text{shellable} \\
\downarrow & & \downarrow \\
\text{non-evasive} & \Rightarrow & \text{collapsible}
\end{array}
\]

In addition, the barycentric subdivision of any shellable complex is vertex-decomposable [30] — and the barycentric subdivision of any collapsible complex is non-evasive [32]. What about the converse? Can an unshellable ball or sphere become vertex-decomposable after a single barycentric subdivision? The answer is positive. The barycentric subdivision of $B_{12,38}$ is, in fact, vertex-decomposable. The same holds for $S_{13,56}$, the unshellable 3-sphere obtained coning off the boundary of $B_{12,38}$; see Proposition 6.8.
Next, we turn to a concrete question from DISCRETE QUANTUM GRAVITY. Suppose that we wish to take a walk on the various triangulations of $S^3$, by starting with the boundary of the 4-simplex and performing a random sequence of bistellar flips (also known as ‘Pachner moves’). All triangulated 3-spheres can be obtained this way [29], but some may be less likely to appear than others, like the 16-vertex triangulation $S_{16,104}$ by Dougherty, Faber and Murphy [4, 13]. (In fact, any ‘Pachner walk’ from the boundary of the 4-simplex to $S_{16,104}$ must pass through spheres with more than 16 vertices.) This ‘random Pachner walk’ model is used in discrete quantum gravity, by Ambjørn, Durhuus, Jonsson and others, to estimate the total number of triangulations of $S^3$ [2, 3]. Durhuus and Jonsson have also developed the property of local constructibility, conjecturing it would hold for all 3-spheres [14]. As we said, the conjecture was negatively answered in [10], but it remained unclear how difficult it is to reach counterexamples, using a random Pachner walk. In other words: How outspread should the simulation be, before we have the chance to meet a non-locally constructible sphere?

Here we answer this question by presenting the first explicit triangulation of a non-locally constructible 3-sphere. For that, we have to adapt the construction of $B_{12,38}$ from the single trefoil to the triple trefoil. In the end, we manage to use only 18 vertices. The surprise is that via Pachner moves, the final triangulation is reachable rather straightforwardly.

**Main Theorem 2.** Some 17-vertex triangulation $B_{17,95}$ of the 3-ball contains a triple trefoil knotted spanning edge. This $B_{17,95}$ is not collapsible. Coning off the boundary of $B_{17,95}$ one obtains a knotted 3-sphere $S_{18,125}$ that is not locally constructible. Removing any tetrahedron from $S_{18,125}$ one obtains a knotted 3-ball that is neither locally constructible nor collapsible. This $S_{18,125}$ is ‘3-stellated’, in the notation of Bagchi–Datta [4]: it can be reduced to the boundary of a 4-simplex by using 94 Pachner moves that do not add further vertices.

After dealing with the single trefoil and the triple trefoil, let us turn to the intermediate case of the double trefoil. By the work of Benedetti–Ziegler, any 3-ball containing a 3-edge knot in its 1-skeleton cannot be locally constructible if the knot is the sum of three or more trefoils [10]. But is this bound best possible? In [10] it is shown with topological arguments that a collapsible 3-ball may contain a double trefoil knot on 3 edges. Recall that locally constructible 3-balls are characterized by the property of collapsing onto their boundary minus a triangle [10]. This is stronger than just being collapsible. It remained unclear whether a locally constructible 3-ball may indeed contain a double trefoil on three edges.

We answer this question affirmatively in Section 4. As before, the key consists in triangulating cleverly, so that computational approaches may succeed. On the way to this result, we produce a smaller example of a non-collapsible ball, using only 15 vertices and 66 tetrahedra.

**Main Theorem 3.** Some 15-vertex triangulation $B_{15,66}$ of the 3-ball contains a double trefoil knotted spanning edge. This $B_{15,66}$ is not collapsible. Coning off the boundary of $B_{15,66}$ one obtains a knotted 3-sphere $S_{16,92}$ that is locally constructible. Removing the tetrahedron 191415 from $S_{16,92}$ one obtains a knotted 3-ball that is collapsible and locally constructible.

Now, for each $d \geq 3$ one has the following hierarchy of combinatorial properties of triangulated $d$-spheres [10]:

\[
\{\text{vertex-decomposable}\} \subset \{\text{shellable}\} \subset \{\text{constructible}\} \subset \{\text{LC}\} \subset \{\text{all }d\text{-spheres}\}.
\]

An analogous hierarchy holds for $d$-balls ($d \geq 3$) [10]:

\[
\{\text{vertex-decomp.}\} \subset \{\text{shellable}\} \subset \{\text{constructible}\} \subset \{\text{LC}\} \subset \{\text{collapsible onto } (d-2)\text{-complex}\} \subset \{\text{all }d\text{-balls}\}.
\]

(When $d = 3$, “collapsible onto a 1-complex” is equivalent to “collapsible”.)
3-Ball

We say that the complex and link \(C\) and \(v\) of a single point, or (3) we say that the complex \(C\) is non-evasive. Here is another interesting hierarchy for balls, which can be merged with the previous one.

**Main Theorem 4.** There are the following inclusion relations between families of simplicial \(d\)-balls:

\[
\text{\{vertex-decomposable\}} \subseteq \text{\{non-evasive\}} \subseteq \text{\{collapsible\}} \subseteq \text{\{all \(d\)-balls\}}.
\]

For 2-balls all inclusions above are equalities, whereas for 3-balls all inclusions above are strict. More precisely, we have the following 'mixed' hierarchy:

\[
\text{\{vertex-decomposable\}} \subsetneq \text{\{shellable AND non-evasive\}} \subsetneq \text{\{shellable OR non-evasive\}} \subsetneq \text{\{collapsible\}} \subseteq \text{\{all 3-balls\}}.
\]

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**Table 1: List of 3-balls and 3-spheres discussed here**

| Trefoils | 3-ball \(B\) | 3-Sphere \(\partial (v + B)\) | 3-ball \(\partial (v + B) - \Sigma\) |
|----------|---------------|-----------------|-----------------|
| 0        | \(B_{7,10}\)  | \(S_{8,20}\)      | \(B_{8,19}\)      |
| 0        | \(B_{8,13}\)  | \(S_{9,25}\)      | \(B_{9,24}\)      |
| 0        | \(B_{9,18}\)  | \(S_{10,32}\)     | \(B_{10,31}\)     |
| 1        | \(B_{12,38}\) | \(S_{13,56}\)     | \(B_{13,55}\)     |
| 2        | \(B_{15,66}\) | \(S_{16,92}\)     | \(B_{16,91}\)     |
| 3        | \(B_{17,95}\) | \(S_{18,125}\)    | \(B_{18,124}\)    |

Note: \(VD = vertex-decomposable, sh. = shellable, constr. = constructible, LC = locally constructible, coll. = collapsible, NE = non-evasive\). “TREFOILS: \(t\)” means “containing a \(t\)-fold trefoil on 3 edges”.

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\]

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**2 Background**

**2.1 Combinatorial properties of triangulated spheres and balls**

A \(d\)-complex is pure if all of its top-dimensional faces (called facets) have the same dimension.

A pure \(d\)-complex \(C\) is constructible if either \(C\) is a simplex, or \(C\) is a disjoint union of points, or \(d \geq 1\) and \(C\) can be written as \(C = C_1 \cup C_2\), where \(C_1\) and \(C_2\) are constructible \(d\)-complexes and \(C_1 \cap C_2\) is a constructible \((d - 1)\)-complex.

A pure \(d\)-complex \(C\) is shellable if either (1) \(C\) is a simplex, or (2) \(C\) is a disjoint union of points, or (3) \(d \geq 1\) and \(C\) can be written as \(C = C_1 \cup C_2\), where \(C_1\) is a shellable \(d\)-complex, \(C_2\) is a \(d\)-simplex, and \(C_1 \cap C_2\) is a shellable \((d - 1)\)-complex.

A pure \(d\)-complex \(C\) is vertex-decomposable if either (1) \(C\) is a simplex, or (2) \(C\) is a disjoint union of points, or (3) \(d \geq 1\) and there is a vertex \(v\) in \(C\) (called shedding vertex) such that \(\text{del}(v, C)\) and \(\text{link}(v, C)\) are both vertex-decomposable (and \(\text{del}(v, C)\) is pure \(d\)-dimensional).

A (not necessarily pure!) \(d\)-complex \(C\) is non-evasive if either (1) \(C\) is a simplex, or (2) \(C\) is a single point, or (3) \(d \geq 1\) and there is a vertex \(v\) in \(C\) such that \(\text{del}(v, C)\) and \(\text{link}(v, C)\) are both non-evasive.

An elementary collapse is the simultaneous removal from a \(d\)-complex \(C\) of a pair of faces \((\sigma, \Sigma)\) with the prerogative that \(\Sigma\) is the only face properly containing \(\sigma\). (This condition is usually abbreviated in the expression ‘\(\sigma\) is a free face of \(\Sigma\’; some complexes have no free face). If \(C' := C - \Sigma - \sigma\), we say that the complex \(C\) collapses onto the complex \(C'\). Even if \(C\) is pure, this \(C'\) need not be pure. We say that the complex \(C\) collapses onto \(D\) if \(C\) can be reduced to \(D\) by some finite sequence of
elementary collapses. A (not necessarily pure) $d$-complex $C$ is collapsible if it collapses onto a single vertex.

A simplicial 3-ball is locally constructible (or shortly LC) if it can be collapsed onto its boundary minus a triangle. A simplicial 3-sphere is locally constructible (or shortly LC) if the removal of some tetrahedron makes it collapsible onto one of its vertices.

2.2 Perfect discrete Morse functions

A map $f : C \rightarrow \mathbb{R}$ on a simplicial complex $C$ is a discrete Morse function on $C$ if for each face $\sigma$

(i) there is at most one boundary facet $\rho$ of $\sigma$ such that $f(\rho) \geq f(\sigma)$ and

(ii) there is at most one face $\tau$ having $\sigma$ as boundary facet such that $f(\tau) \leq f(\sigma)$.

A critical face of $f$ is a face of $C$ for which

(i) there is no boundary facet $\rho$ of $\sigma$ such that $f(\rho) \geq f(\sigma)$ and

(ii) there is no face $\tau$ having $\sigma$ as boundary facet such that $f(\tau) \leq f(\sigma)$.

A collapse-pair of $f$ is a pair of faces $(\sigma, \tau)$ such that

(i) $\sigma$ is a boundary facet of $\tau$ and

(ii) $f(\sigma) \geq f(\tau)$.

Forman [17, Section 2] showed that for each discrete Morse function $f$ the collapse pairs of $f$ form a partial matching of the face poset of $C$. The unmatched faces are precisely the critical faces of $f$. Each complex $K$ endowed with a discrete Morse function is homotopy equivalent to a cell complex with exactly one cell of dimension $i$ for each critical $i$-face [17]. In particular, if we denote by $c_i(f)$ the number of critical $i$-faces of $f$, and by $\beta_i(C)$ the $i$-th Betti number of $C$, one has

$$c_i(f) \geq \beta_i(C)$$

for all discrete Morse functions $f$ on $C$. These inequalities need not be sharp. If they are sharp for all $i$, the discrete Morse function is called perfect. However, for each $k$ and for each $d \geq 3$ there is a $d$-sphere $S$ [7] such that for any discrete Morse function $f$ on $S$, one has

$$c_{d-1}(f) \geq k + \beta_{d-1}(S) = k.$$

2.3 Knots and knot-theoretic obstructions

A knot is a simple closed curve in a 3-sphere. All the knots we consider are tame, that is, realizable as 1-dimensional subcomplexes of some triangulated 3-sphere. A knot is trivial if it bounds a disc; all the knots we consider here are non-trivial. The knot group is the fundamental group of the knot complement inside the ambient sphere. For example, the knot group of the trefoil knot (and of its mirror image) is $\langle x, y \mid x^2 = y^3 \rangle$. Ambient isotopic knots have isomorphic knot groups. A connected sum of two knots is a knot obtained by cutting out a tiny arc from each and then sewing the resulting curves together along the boundary of the cutouts. For example, summing two trefoils one obtains the “granny knot”; summing a trefoil and its mirror image one obtains the so-called “square knot”. When we say “double trefoil”, we mean any of these (granny knot or square knot): From the point of view of the knot group, it does not matter. A knot is $m$-complicated if the knot group has a presentation with $m + 1$ generators, but no presentation with $m$ generators. By “at least $m$-complicated” we mean “$k$-complicated for some $k \geq m$”. There exist arbitrarily complicated knots: Goodrick [18] showed that the connected sum of $m$ trefoil knots is at least $m$-complicated.

A spanning edge of a 3-ball $B$ is an interior edge that has both endpoints on the boundary $\partial B$. An $\Sigma$-knotted spanning edge of a 3-ball $B$ is a spanning edge $xy$ such that some simple path on $\partial B$
between \(x\) and \(y\) completes the edge to a (non-trivial) knot \(\mathcal{L}\). From the simply-connectedness of 2-spheres it follows that the knot type does not depend on the boundary path chosen; in other words, the knot is determined by the edge. More generally, a spanning arc is a path of interior edges in a 3-ball \(B\), such that both extremes of the path lie on the boundary \(\partial B\). If every path on \(\partial B\) between the two endpoints of a spanning arc completes the latter to a knot \(L\), the arc is called \(\mathcal{L}\)-knotted. Note that the relative interior of the arc is allowed to intersect the boundary of the 3-ball; compare Ehrenborg–Hachimori \([15]\).

Below is a list of known results on knotted spheres and balls. As for the notation, if \(B\) is a 3-ball with a knotted spanning edge, by \(S_B\) we will mean the 3-sphere \(\partial (v \ast B)\), where \(v\) is a new vertex. By \(L_t\) we denote a connected sum of \(t\) trefoil knots.

**Theorem 2.1** (Benedetti/Ehrenborg/Hachimori/Ziegler). Any 3-ball with an \(\mathcal{L}_t\)-knotted spanning arc of \(t\) edges cannot be LC \([7]\), but it can be collapsible \([10, 24]\). An arbitrary 3-ball with an \(\mathcal{L}_1\)-knotted spanning arc of less than \(3\) edges cannot be shellable nor constructible \([20]\). In contrast, some shellable 3-balls have a \(\mathcal{L}_1\)-knotted spanning arc of \(3\) edges \([20]\).

**Theorem 2.2** (Adams et al. \([1, Theorem 7.1]\)). Any knotted 3-ball in which the knot \(\mathcal{L}_t\) is realized with \(e\) edges cannot be rectilinearly embeddable in \(\mathbb{R}^3\) if \(e \leq 2t + 3\).

**Theorem 2.3** (Benedetti/Ehrenborg/Hachimori/Shimokawa/Ziegler). A 3-sphere or a 3-ball, with a subcomplex of \(m\) edges, isotopic to the sum of \(t\) trefoil knots,

- cannot be vertex-decomposable if \(t \geq \left\lceil \frac{m}{3} \right\rceil\) \([20]\),
- cannot be constructible/shellable if \(t \geq \left\lceil \frac{m}{2} \right\rceil\) \([15, 19]\), and
- cannot be LC if \(t \geq m \([10]\).

The first two bounds are known to be sharp for \(t = 1 \([20]\); the latter bound is sharp for all \(t\), as far as spheres are concerned \([6, 10]\).

**Theorem 2.4** (Benedetti). Let \(S\) be a 3-sphere with a subcomplex of \(m\) edges, isotopic to the sum of \(t\) trefoil knots. For any discrete Morse function \(f\) on \(S\), one has

\[c_2(f) \geq t - m + 1.\]

3. **The single trefoil**

In this section, we study the 3-ball \(B_{12,38}\) introduced in \([28]\) and given by the following 38 facets:

\[
2347, \quad 23410, \quad 23710, \quad 2457, \quad 24510, \quad 25713, \quad 25810, \quad 25813, \\
26911, \quad 261113, \quad 261213, \quad 27810, \quad 27811, \quad 271113, \quad 28911, \quad 28912, \\
281213, \quad 3467, \quad 34610, \quad 35813, \quad 35911, \quad 35913, \quad 36712, \quad 361013, \\
361213, \quad 371012, \quad 38911, \quad 38912, \quad 381213, \quad 391012, \quad 391013, \quad 4567, \\
45610, \quad 5679, \quad 56911, \quad 561011, \quad 57913, \quad 6101113.
\]

The ball is contructed in a way such that the edge 23 is a knotted spanning edge for \(B_{12,38}\), the knot being a single trefoil. In particular, by Theorem 2.1, \(B_{12,38}\) is not shellable, not constructible and not LC. Here we show that:

(1) \(B_{12,38}\) is not rectilinearly-embeddable in \(\mathbb{R}^3\);
(2) \(B_{12,38}\) is evasive;
(3) \(B_{12,38}\) is collapsible;
(4) The 3-sphere \(\partial (1 \ast B_{12,38})\) minus the facet 1 269 is an LC knotted 3-ball.
Proposition 3.1. $B_{12,38}$ is not rectilinearly-embeddable in $\mathbb{R}^3$.

Proof. The boundary of $B_{12,38}$ consists of the following 18 triangles:

\[
269, \ 2612, \ 2912, \ 358, \ 3511, \ 3811, \ 5810, \ 51011, \ 679, \\
6712, \ 7810, \ 7811, \ 7913, \ 71012, \ 71113, \ 91012, \ 91013, \ 101113.
\]

In particular, the four edges 26, 67, 78 and 38 form a boundary path from the vertex 2 to the vertex 3. Together with the interior edge 23, this path closes up to a pentagonal trefoil knot. By Theorem 2.2, $B_{12,38}$ cannot be rectilinearly embedded in $\mathbb{R}^3$, because the stick number of the trefoil knot is 6. □

Proposition 3.2. $B_{12,38}$ is collapsible, but not LC.

Proof. By Theorem 2.1, $B$ is not LC; in particular, $B$ does not collapse onto its boundary minus a triangle. So, in the first phase of the collapse (the one in which the tetrahedra are collapsed away) we have to remove several boundary triangles in order to succeed. Now, finding a collapse can be difficult, but verifying the correctness of a given collapse is fast. The following is a certificate of the collapsibility of $B_{12,38}$.

First phase (pairs “triangle” → “tetrahedron”):

\[
101113 \rightarrow 6101113, \ 7913 \rightarrow 57913, \ 61011 \rightarrow 561011, \ 5611 \rightarrow 56911, \ 2612 \rightarrow 261213, \\
579 \rightarrow 5679, \ 91012 \rightarrow 391012, \ 71113 \rightarrow 271113, \ 5911 \rightarrow 35911, \ 2713 \rightarrow 25713, \\
3912 \rightarrow 38912, \ 2613 \rightarrow 261113, \ 3812 \rightarrow 381213, \ 3911 \rightarrow 38911, \ 71012 \rightarrow 371012, \\
8912 \rightarrow 28912, \ 61013 \rightarrow 361013, \ 358 \rightarrow 35813, \ 6911 \rightarrow 26911, \ 81213 \rightarrow 281213, \\
3613 \rightarrow 361213, \ 31013 \rightarrow 391013, \ 3513 \rightarrow 35913, \ 6712 \rightarrow 36712, \ 367 \rightarrow 3467, \\
567 \rightarrow 4567, \ 7811 \rightarrow 27811, \ 2911 \rightarrow 28911, \ 346 \rightarrow 34610, \ 457 \rightarrow 2457, \\
5610 \rightarrow 45610, \ 3410 \rightarrow 23410, \ 247 \rightarrow 2347, \ 237 \rightarrow 23710, \ 5810 \rightarrow 25810, \\
5813 \rightarrow 25813, \ 7810 \rightarrow 27810, \ 245 \rightarrow 24510.
\]

Second phase (pairs “edge” → “triangle”):

\[
812 \rightarrow 2812, \ 78 \rightarrow 278, \ 713 \rightarrow 5713, \ 810 \rightarrow 2810, \ 911 \rightarrow 8911, \ 79 \rightarrow 679, \\
1011 \rightarrow 51011, \ 711 \rightarrow 27111, \ 58 \rightarrow 258, \ 912 \rightarrow 2912, \ 712 \rightarrow 3712, \ 511 \rightarrow 3511, \\
35 \rightarrow 359, \ 57 \rightarrow 257, \ 1012 \rightarrow 31012, \ 311 \rightarrow 3811, \ 67 \rightarrow 467, \ 47 \rightarrow 347, \\
27 \rightarrow 2710, \ 811 \rightarrow 2811, \ 212 \rightarrow 21213, \ 1013 \rightarrow 91013, \ 34 \rightarrow 234, \ 23 \rightarrow 2310, \\
710 \rightarrow 3710, \ 910 \rightarrow 3910, \ 310 \rightarrow 3610, \ 610 \rightarrow 4610, \ 46 \rightarrow 456, \ 45 \rightarrow 4510, \\
24 \rightarrow 2410, \ 36 \rightarrow 3612, \ 210 \rightarrow 2510, \ 312 \rightarrow 31213, \ 1213 \rightarrow 61213, \ 25 \rightarrow 2513, \\
56 \rightarrow 569, \ 613 \rightarrow 61113, \ 513 \rightarrow 5913, \ 1113 \rightarrow 21113, \ 213 \rightarrow 2813, \ 913 \rightarrow 3913, \\
69 \rightarrow 269, \ 39 \rightarrow 389, \ 38 \rightarrow 3813, \ 28 \rightarrow 289, \ 611 \rightarrow 2611.
\]

Third phase (pairs “vertex” → “edge”):

\[
12 \rightarrow 612, \ 4 \rightarrow 410, \ 6 \rightarrow 26, \ 10 \rightarrow 510, \ 11 \rightarrow 211, \ 5 \rightarrow 59, \ 7 \rightarrow 37, \ 2 \rightarrow 29, \ 9 \rightarrow 89, \\
3 \rightarrow 313, \ 13 \rightarrow 813.
\]

The above collapsing sequence was found with the randomized approach of [9].

Proposition 3.3. $B_{12,38}$ is evasive.

Proof. Let us establish some notation first. We identify each vertex of $B_{12,38}$ with its label, which is an integer in $A := \{2, \ldots, 13\}$. For each subset $S$ of $A$, we denote by $C_S$ the complex obtained from $B_{12,38}$ by deleting the vertices in $S$.

Now, suppose by contradiction that $B$ is non-evasive. The vertices of $B_{12,38}$ can be reordered so that their progressive deletions and links are non-evasive. In particular, there exists a five-element subset $F$ of $A$ such that $C_F$ is non-evasive.
With the help of a computer program, we checked the homologies of all complexes obtained by deleting five vertices from $B$. Since the order of deletion does not matter, there are only $\binom{12}{5} = 792$ cases to check, so the computation is extremely fast. It turns out that these homologies are never trivial, except for the following three cases:

1. $F_1 = \{4, 5, 8, 10, 11\}$,
2. $F_2 = \{4, 5, 10, 11, 12\}$,
3. $F_3 = \{4, 6, 7, 9, 12\}$.

So, the non-evasive complex $C_F$ whose existence was postulated above must be either $C_{F_1}$, or $C_{F_2}$, or $C_{F_3}$. However, it is easy to see that the deletion of any vertex from $C_{F_1}$ yields a non-acyclic complex. The same holds for $C_{F_2}$ and $C_{F_3}$. Therefore, all three complexes $C_{F_1}$, $C_{F_2}$ and $C_{F_3}$ are evasive: A contradiction.

**Remark 3.4.** Let $S_B$ be the sphere obtained by coning off the boundary of $B_{12,38}$ with an extra vertex, labeled by 1. Let $\Sigma$ be the tetrahedron 1269 and let $\sigma$ be its facet 269. With the help of the computer, one can check that $S_B - \Sigma$ collapses onto the 2-ball $D$ consisting of the triangles 126, 129 and 169. Since $D = \partial (S_B - \Sigma) = \partial (S_B - \Sigma) - \sigma$, it follows that the knotted 3-ball $S_B - \Sigma$ is locally constructible (because it collapses onto its boundary minus the triangle $\sigma$). For a proof, see [6].

4 The double trefoil

In the following, we present the construction of a triangulated 3-sphere that contains a double trefoil knot on three edges in its 1-skeleton. In fact, there are two different ways to form the connected sum of two trefoil knots, the granny and the square knot. We base our construction on the square knot.

Let 12, 23, 13 be the three edges forming the square knot, which, for our purposes, we simply call the double trefoil knot. An embedding of the knot in $\mathbb{R}^3$ is depicted in Figure 2.
Our strategy to place the knot into the 1-skeleton of a triangulated 3-dimensional sphere is as follows. We
- start with an embedding of the knot in $\mathbb{R}^3$,
- triangulate the region around the knot to obtain a triangulated 3-ball,
- complete it to a triangulation of $S^3$ by adding the cone over its boundary.

Once the knot edges 12, 23, 13 are placed in $\mathbb{R}^3$ we need to shield off these edges to prevent unwanted identifications of distant vertices later on. We protect each of the knot edges by placing a spindle around it; see Figure 3 for images of the spindles and Table 2 for lists of nine tetrahedra each, which form the three spindles. The additional vertices on the boundaries of the spindles allow us to close the holes of the knot by gluing in (triangulated) membrane patches.

Table 2: Part I of the sphere $S_{33,192}$: The three spindles.

| 1245  | 2478 | 231011 | 3101314 | 131920 | 3161719 |
|-------|------|--------|---------|--------|---------|
| 1246  | 2458 | 231012 | 3101114 | 131921 | 3171920 |
| 1256  | 2589 | 231112 | 3111415 | 132021 | 3171820 |
|       | 2569 |        | 311215  |        | 3182021 |
|       | 2479 |        | 3101315 |        | 3161819 |
|       | 2469 |        | 3101215 |        | 3181921 |

In Figure 2, the diagonal edges on the boundaries of the spindles and also the interior edges of the spindles are not shown. All that we need at the moment are the vertices on the boundaries of the spindles. For example, if we move along the left spindle 1–2 from apex 1 to apex 2, we first meet the vertices 4, 5, 6 and then the vertices 7, 8, 9 on the spindle boundary.

The membrane patches can be read off from Table 3. The central triangle 11114 connects the left part with the right part of Figure 2 and contributes to the closure of the upper central hole. Next
Table 3: The triangles of the membranes in the sphere $S_{33,192}$.

|   |   |   |   |
|---|---|---|---|
|45|11|157|1911|
|41122|57111|179|11418|
|21122|78111|8911|11620|
|2722|81011|11|141920|
|6722|58110|179|1441620|
|4622|51012|11618|1441923|
|567|511112|11623|31423|

To the triangle 11114 on the left hand side in Figure 2 is the triangle 1911 from the third column of Table 3, followed by triangle 179 and so on. Once all the membrane triangles of Table 3 are in place in Figure 2, the resulting complex is a mixed 2- and 3-dimensional simplicial complex, consisting of spindle tetrahedra and membrane triangles. Since we closed all holes of the initial double trefoil knot, the resulting complex is contractible.

Table 4: Part II of the sphere $S_{33,192}$: Tetrahedra to be added to Part I to obtain a ball $B_{32,140}$.

|   |   |   |   |   |   |
|---|---|---|---|---|---|
|462425|17926|152627|17924|19212429|1161830|
|562425|27926|511267|191124|20212429|3161830|
|5102425|1567|1011267|891124|13202429|1162021|
|5101225|16726|8101127|8101124|13152029|1162310|
|5111225|672225|781127|581024|14152029|1621330|
|571125|272225|571127|56924|14162029|1612330|
|781125|14626|15727|58924|14161729|1192130|
|891125|462226|46924|14171829|1162301|
|56725|14526|8102728|47924|16202129|1192030|
|672225|451126|581028|16212329|14192030|
|462225|4112226|5101228|1111424|14192032|
|272225|2112226|5111228|10111424|14192032|
|27825|2101126|451128|10131424|14192032|
|28925|4112228|3171829|14192332|

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Our next aim is to thicken the intermediate mixed 2- and 3-dimensional complex to a triangulated 3-ball $B_{32,140}$. For this end we add local cones to Figure 1 with respect to the nine new vertices 24, 25, ..., 32. These cones are listed in Table 3, the positions of their apices are marked in Figure 1 by boxes containing the new vertices.

If we add together all the (spindle) tetrahedra from Table 2 (Part I of the sphere $S_{33,192}$) with all the (cone) tetrahedra from Table 4 (Part II of the sphere $S_{33,192}$), we obtain a triangulated 3-ball $B_{32,140}$ with 33 vertices and 140 tetrahedral facets. By construction, the 3-ball $B_{32,140}$ contains the double trefoil knot in its 1-skeleton and all the membrane triangles in its 2-skeleton.

In a final step, we add to the 3-ball $B_{32,140}$ the cone over its boundary with respect to the vertex 33 (Part III of the sphere $S_{33,192}$ with tetrahedra as listed in Table 5) to obtain the 3-sphere $S_{33,192}$.

| Table 6: The sphere $S_{16,92}$. |
|----------------------------------|
| 1256 | 12512 | 12612 | 1378 | 13711 | 13811 | 1456 | 14516 |
| 14612 | 141013 | 141016 | 141213 | 151213 | 151316 | 1789 | 17911 |
| 18914 | 181014 | 181015 | 181115 | 191115 | 191415 | 1101314 | 1101516 |
| 1131416 | 1141516 | 23413 | 23415 | 231315 | 231316 | 2478 | 24715 | 24816 |
| 241013 | 241016 | 25614 | 251214 | 26812 | 26816 | 26914 | 26916 |
| 2789 | 27910 | 271013 | 271315 | 28914 | 281214 | 291016 | 341213 |
| 341215 | 3567 | 35614 | 3578 | 35811 | 351114 | 36716 | 36914 |
| 36916 | 371114 | 371416 | 391213 | 391216 | 391315 | 391415 | 312156 |
| 3141516 | 4567 | 4578 | 45816 | 46715 | 461215 | 581113 | 581316 |
| 51111213 | 5111214 | 671315 | 671316 | 681215 | 681315 | 681316 | 791012 |
| 791112 | 7101214 | 7101314 | 7111214 | 7131416 | 8101214 | 8101215 | 8111315 |
| 9101216 | 91111213 | 9111315 | 10121516 |

Proposition 4.1. The 3-sphere $S_{33,192}$ consists of 192 tetrahedra and 33 vertices. It has face vector $f = (33, 225, 384, 192)$ and contains the double trefoil knot on three edges in its 1-skeleton.

The 3-sphere $S_{33,192}$ is not minimal with the property of containing the double trefoil knot in its 1-skeleton. One way of obtaining smaller triangulations is by applying bistellar flips, cf. [12], to the triangulation $S_{33,192}$. If we want to keep the knot while doing local bistellar modifications on the triangulation, we merely have to exclude the knot edges 12, 23, 13 as pivot edges in the bistellar flip program BISTELLAR [27]. The smallest triangulation we found this way is $S_{16,92}$; see Table 6 for the list of facets of $S_{16,92}$. 

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Theorem 4.2. The 3-sphere $S_{16,92}$ has 92 tetrahedra and 16 vertices. It has face vector $f = (16, 108, 184, 92)$ and contains the double trefoil knot on three edges in its 1-skeleton.

If we remove from the 3-sphere $S_{16,92}$ the facet 191415, then the resulting 3-ball is LC, although it contains a double trefoil knot as a three-edge subcomplex.

Proposition 4.3. The removal of the tetrahedron 191415 from $S_{16,92}$ yields a locally constructible 3-ball $B_{16,91}$ with 16 vertices and 91 tetrahedra.

Proof. Let $D$ be the 2-ball given by the triangles 1915, 11415 and 91415. Clearly $D$ is a subcomplex of the boundary of $B_{16,91}$; it is in fact equal to $\partial B_{16,91}$ minus the triangle 1914. Our goal is to show that $B_{16,91}$ collapses onto $D$. The following is a certificate that this is true:

First phase (pairs “triangle” → “tetrahedron”):

| 1914 | → | 18914, 8914 → 28914, 189 → 1789, 289 → 2789, |
| 178 | → | 1378, 137 → 13711, 378 → 3578, 138 → 13811, |
| 278 | → | 2478, 2814 → 281214, 1711 → 17911, 357 → 3567, |
| 279 | → | 27910, 5811 → 581113, 1811 → 181115, 4816 → 481616, |
| 1815 | → | 181015, 2914 → 26914, 81115 → 8111315, 5816 → 581316, |
| 51136 | → | 151316, 269 → 26916, 1911 → 191115, 111315 → 1111315, |
| 1810 | → | 181014, 6916 → 36916, 247 → 24715, 2415 → 23415, |
| 10114 | → | 1101314, 457 → 4578, 2916 → 291016, 3511 → 351114, |
| 1513 | → | 151213, 81014 → 8101214, 4516 → 451615, 145 → 1456, |
| 356 | → | 35614, 11314 → 1131416, 1512 → 12512, 71114 → 7111214, |
| 91315 | → | 391315, 3913 → 391213, 156 → 1256, 3714 → 371416, |
| 71416 | → | 7131416, 5614 → 25614, 234 → 23413, 146 → 14612, |
| 6713 | → | 671316, 81015 → 8101215, 467 → 46715, 1416 → 141016, |
| 51113 | → | 5111213, 4612 → 461215, 21016 → 240161, 111214 → 511214, |
| 31215 | → | 3121516, 3614 → 36914, 101215 → 10121516, 6716 → 36716, |
| 101214 | → | 7101214, 11016 → 1101516, 1612 → 12612, 71012 → 791012, |
| 71315 | → | 271315, 2816 → 26816, 91016 → 9101216, 2413 → 241013, |
| 61136 | → | 681316, 7912 → 791112, 31315 → 23313, 4123 → 14123, |
| 41013 | → | 141013, 6815 → 681215, 2812 → 26812, 3915 → 391415, |
| 31416 | → | 3141516, 11416 → 1141516, |

Second phase (pairs “edge” → “triangle”):

| 89 | → | 789, 29 → 2910, 16 → 126, 516 → 1516, 38 → 358, |
| 815 | → | 81215, 13 → 1311, 1315 → 21315, 15 → 125, 17 → 179, |
| 810 | → | 81012, 18 → 1814, 811 → 81113, 28 → 268, 35 → 3514, |
| 1113 | → | 111213, 12 → 1212, 814 → 81214, 57 → 578, 916 → 91216, |
| 613 | → | 6813, 311 → 31114, 812 → 6812, 913 → 91213, 78 → 478, |
| 111 | → | 11115, 1114 → 51114, 1115 → 91115, 48 → 458, 45 → 456, |
| 511 | → | 51112, 56 → 256, 47 → 4715, 58 → 5813, 25 → 2514, |
| 46 | → | 4615, 513 → 51213, 512 → 51214, 68 → 6816, 415 → 3415, |
| 1213 | → | 11213, 813 → 81316, 112 → 1412, 412 → 3412, 34 → 3413, |
| 313 | → | 2313, 23 → 2315, 215 → 2715, 413 → 1413, 715 → 6715, |
| 67 | → | 367, 27 → 2713, 615 → 61215, 612 → 2612, 37 → 3716, |
| 14 | → | 1410, 1215 → 121516, 716 → 71316, 213 → 21013, 212 → 21214, |
| 210 | → | 2410, 214 → 2614, 410 → 401016, 24 → 2416, 1214 → 71214, |
| 26 | → | 2616, 713 → 71314, 616 → 3616, 714 → 71014, 614 → 6914, |
| 69 | → | 369, 712 → 71112, 710 → 7910, 79 → 7911, 911 → 91112, |
| 1014 | → | 101314, 910 → 91012, 912 → 3912, 1012 → 101216, 1216 → 31216, |
| 1314 | → | 131416, 1013 → 11013, 39 → 3914, 1416 → 141516, 110 → 11015, |
| 1316 | → | 11316, 1016 → 101516, 314 → 31415, 315 → 31516, 116 → 11516, |
Third phase (pairs “vertex” \(\rightarrow\) “edge”):

\[ 13 \rightarrow 113, \ 5 \rightarrow 514, \ 6 \rightarrow 36, \ 10 \rightarrow 1015, \ 7 \rightarrow 711, \ 11 \rightarrow 1112, \ 12 \rightarrow 312, \]
\[ 2 \rightarrow 216, \ 3 \rightarrow 316, \ 4 \rightarrow 416, \ 8 \rightarrow 816, \ 16 \rightarrow 1516. \]

\[ \square \]

If we remove from the 3-sphere \(S_{16, 92}\) the entire star of the vertex 1 (one of the three knot vertices), we obtain a 3-ball \(B_{15, 66}\) with many interesting properties. In the following we will show that

1. \(B_{15, 66}\) contains a knotted spanning edge 23, where the knot is the double trefoil;
2. \(B_{15, 66}\) is not embeddable in \(\mathbb{R}^3\);
3. \(B_{15, 66}\) is not collapsible, but it admits a discrete Morse function with one critical vertex, one critical edge and one critical triangle.

**Proposition 4.4.** \(B_{15, 66}\) is not rectilinearly-embeddable in \(\mathbb{R}^3\).

**Proof.** The boundary of \(B_{15, 66}\) consists of the following 26 triangles:

\[
\begin{align*}
256, \ &2512, \ 2612, \ 378, \ 3711, \ 3811, \ 456, \ 4516, \ 4612, \ 41013, \ 41016, \ 41213, \ 51213, \ 51316, \ 789, \ 7911, \ 8914, \ 81014, \ 81015, \ 81115, \ 91115, \ 91415, \ 101314, \ 101516, \ 131416, \ 141516.
\end{align*}
\]

In particular, the five edges 25, 513, 1013, 810 and 38 form a boundary path from the vertex 2 to the vertex 3. Together with the interior edge 23, this path closes up to a hexagonal double trefoil knot. By Theorem 2.2, \(B_{15, 66}\) cannot be rectilinearly embedded in \(\mathbb{R}^3\). \[ \square \]

**Theorem 4.5.** \(B_{15, 66}\) admits a discrete Morse function with three critical faces, all of them belonging to the boundary \(\partial B_{15, 66}\).

**Proof.** We will show that there is a 2-dimensional subcomplex \(C\) of \(B_{15, 66}\) such that:

- \(B_{15, 66}\) collapses onto \(C\) and
- \(C\) minus the triangle 258 collapses onto a pentagon.

Here is the right collapsing sequence:

First phase (pairs “triangle” \(\rightarrow\) “tetrahedron”):

\[
\begin{align*}
41016 \rightarrow 241016, \ &41013 \rightarrow 241013, \ 91415 \rightarrow 391415, \ 101516 \rightarrow 10121516, \\
81115 \rightarrow 8111315, \ &3811 \rightarrow 35811, \ 81315 \rightarrow 681315, \ 131416 \rightarrow 713416, \\
4516 \rightarrow 45816, \ &6815 \rightarrow 681215, \ 456 \rightarrow 4567, \ 81015 \rightarrow 8101215, \\
8914 \rightarrow 28914, \ &2413 \rightarrow 23413, \ 141516 \rightarrow 3141516, \ 2512 \rightarrow 251214, \\
4816 \rightarrow 24816, \ &2814 \rightarrow 281214, \ 248 \rightarrow 2478, \ 81012 \rightarrow 8101214, \\
2313 \rightarrow 231315, \ &3711 \rightarrow 371114, \ 4612 \rightarrow 461215, \ 2612 \rightarrow 261214, \\
91115 \rightarrow 9111315, \ &2816 \rightarrow 26816, \ 41215 \rightarrow 341215, \ 289 \rightarrow 2789, \\
31416 \rightarrow 371416, \ &458 \rightarrow 4578, \ 567 \rightarrow 5678, \ 356 \rightarrow 35614, \\
6813 \rightarrow 681316, \ &31315 \rightarrow 391315, \ 3413 \rightarrow 341213, \ 581 \rightarrow 581313, \\
247 \rightarrow 24715, \ &51214 \rightarrow 511214, \ 357 \rightarrow 3578, \ 2616 \rightarrow 26916, \\
121516 \rightarrow 3121516, \ &2415 \rightarrow 23415, \ 61316 \rightarrow 671316, \ 2914 \rightarrow 26914, \\
2916 \rightarrow 291016, \ &2614 \rightarrow 25614, \ 51213 \rightarrow 511213, \ 31216 \rightarrow 391216, \\
71314 \rightarrow 7101314, \ &31114 \rightarrow 351114, \ 71114 \rightarrow 7111214, \ 51113 \rightarrow 581113, \\
91216 \rightarrow 9101216, \ &71013 \rightarrow 271013, \ 91012 \rightarrow 791012, \ 7910 \rightarrow 27910, \\
3614 \rightarrow 36914, \ &3916 \rightarrow 36916, \ 367 \rightarrow 36716, \ 4715 \rightarrow 46715, \\
31213 \rightarrow 391213, \ &71012 \rightarrow 7101214, \ 21315 \rightarrow 271315, \ 7912 \rightarrow 791112, \\
91112 \rightarrow 9111213, \ &6715 \rightarrow 671315.
\end{align*}
\]
Second phase (pairs “edge” → “triangle”):

\[
\begin{align*}
89 & \to 789, \quad 1416 \to 71416, \quad 45 \to 457, \quad 311 \to 3511, \quad 1015 \to 101215, \\
1415 & \to 31415, \quad 1314 \to 101314, \quad 57 \to 578, \quad 810 \to 81014, \quad 814 \to 81214, \\
410 & \to 2410, \quad 413 \to 41213, \quad 1516 \to 31516, \quad 516 \to 51316, \quad 416 \to 2416, \\
1216 & \to 101216, \quad 48 \to 478, \quad 1013 \to 21013, \quad 1012 \to 101214, \quad 24 \to 234, \\
412 & \to 3412, \quad 216 \to 21016, \quad 23 \to 2315, \quad 513 \to 5813, \quad 215 \to 2715, \\
1115 & \to 111315, \quad 1016 \to 91016, \quad 313 \to 3913, \quad 916 \to 6916, \quad 910 \to 2910, \\
1014 & \to 71014, \quad 213 \to 2713, \quad 715 \to 71315, \quad 710 \to 71610, \quad 47 \to 467, \\
815 & \to 81215, \quad 46 \to 4615, \quad 415 \to 3415, \quad 512 \to 51112.
\end{align*}
\]

Let \( C \) be the obtained 2-complex. Note that \( C \) contains the triangle 258, which belongs to \( \partial B_{15,66} \) and has not been collapsed yet. Let \( D \) be the complex obtained from \( C \) after removing the (interior of the) triangle 258. Here is a proof:

First phase (pairs “edge” → “triangle”):

\[
\begin{align*}
25 & \to 2514, \quad 214 \to 21214, \quad 56 \to 5614, \quad 614 \to 6914, \quad 914 \to 3914, \\
212 & \to 2812, \quad 812 \to 6812, \quad 612 \to 61215, \quad 615 \to 61315, \quad 1215 \to 31215, \\
315 & \to 3915, \quad 1315 \to 91315, \quad 613 \to 6713, \quad 312 \to 3912, \quad 912 \to 91213, \\
39 & \to 369, \quad 713 \to 71316, \quad 36 \to 3616, \quad 316 \to 3716, \quad 1213 \to 111213, \\
67 & \to 6716, \quad 616 \to 6816, \quad 816 \to 81316, \quad 68 \to 268, \quad 913 \to 91113, \\
26 & \to 269, \quad 813 \to 81113, \quad 28 \to 278, \quad 78 \to 378, \quad 38 \to 358, \\
37 & \to 3714, \quad 35 \to 3514, \quad 811 \to 5811, \quad 514 \to 51114, \quad 29 \to 279, \\
911 & \to 7911, \quad 711 \to 71112, \quad 1112 \to 111214, \quad 712 \to 71214.
\end{align*}
\]

Final phase (pairs “vertex” → “edge”):

\[
\begin{align*}
2 & \to 27, \quad 15 \to 915, \quad 3 \to 314, \quad 12 \to 1214, \quad 6 \to 69, \quad 8 \to 58, \quad 5 \to 511, \quad 9 \to 79.
\end{align*}
\]

At this point we are left with the pentagon \( P \) given by the five edges 714, 716, 1113, 1114, and 1316. The latter edge, 1316, belongs to the boundary of \( B_{15,66} \). Clearly, \( P \) minus this edge yields a collapsible 1-ball. Thus, \( B_{15,66} \) admits a discrete Morse function whose critical faces are the vertex 13, the edge 1366 and the triangle 258. This discrete Morse function is the best possible, since \( B_{15,66} \) cannot be collapsible (because of its knotted spanning edge 23).

\[\square\]

5 The triple trefoil

In this section, we are constructing a triangulation \( S_{44,284} \) of the 3-sphere \( S^3 \) that contains a triple trefoil knot with three edges in its 1-skeleton. We then use bistellar flips to obtain a reduced triangulation \( S_{18,125} \).

As before for the double trefoil, we place a triple trefoil knot on the three edges 12, 23, 13 in \( \mathbb{R}^3 \), as depicted in Figure 4. Each of the three knot edges is protected by a spindle; see Figure 5 for the spindles and Table 7 for the list of tetrahedra of the spindles.

To close the holes of the knot we glue in the membrane triangles of Table 8 and then add the local cones with respect to the vertices 34, 35, . . . , 43 from Table 9 to obtain a 3-ball \( B_{43,214} \).

Finally, we add to \( B_{43,214} \) the cone over its boundary with respect to the vertex 44 (as given in Table 10) to obtain the 3-sphere \( S_{44,284} \).

Proposition 5.1. The 3-sphere \( S_{44,284} \) consists of 284 tetrahedra and 44 vertices. It has face vector \( f = (44, 328, 568, 284) \) and contains the triple trefoil knot on three edges in its 1-skeleton.
Figure 4: The triple trefoil in the sphere $S_{44,284}$.

Again, the 3-sphere $S_{44,284}$ is not minimal with the property of containing the triple trefoil knot in its 1-skeleton. The smallest triangulation we found via bistellar flips is $S_{18,125}$; see Table 11 for the list of facets of $S_{18,125}$.

**Theorem 5.2.** The 3-sphere $S_{18,125}$ consists of 125 tetrahedra and 18 vertices. It has face vector $f = (18, 143, 250, 125)$ and contains the triple trefoil knot on three edges in its 1-skeleton.

Because of the knot, $S_{18,125}$ is not LC. So it cannot admit a discrete Morse with fewer than four critical cells. However, it does admit a discrete Morse function with one critical vertex, one critical edge, one critical triangle and one critical tetrahedron, as we once more found by a random search.

**Theorem 5.3.** $S_{18,125}$ admits a discrete Morse function with one critical vertex, one critical edge, one critical triangle and one critical tetrahedron.
Figure 5: The spindles of $S_{44,284}$.

Table 7: Part A of the sphere $S_{44,284}$: The three spindles.

| 1278 | 1469 | 27910 | 231617 | 2131518 | 3161819 | 132526 | 3222427 | 1252728 |
|------|------|-------|--------|----------|---------|--------|---------|--------|
| 1279 | 1479 | 291012 | 231618 | 2131618 | 3181921 | 132527 | 3222527 | 1272830 |
| 1289 | 1457 | 27811 | 231718 | 2134146 | 3161720 | 132627 | 3222325 | 1252629 |
| 1578 | 271011 | 2141617 | 3161920 | 3232526 | 1252829 |
| 1569 | 28912 | 2141518 | 3171821 | 3232427 | 1262730 |
| 1589 | 281112 | 2141718 | 3172021 | 3232627 | 1262930 |

Table 8: The triangles of the membranes in the sphere $S_{44,284}$.

| 41322 |
|-------|
| 8930  | 62830 | 13031 | 121718 | 101215 | 21232 | 212627 | 192124 | 32133 |
| 6830  | 6928 | 93031 | 121517 | 101518 | 121832 | 212426 | 192427 | 212733 |
| 468   | 92829 | 7931 | 131517 | 101118 | 161832 | 222426 | 192027 | 252733 |
| 4812  | 92930 | 1631 | 131721 | 111218 | 21532 | 222630 | 202127 | 32433 |
| 41214 | 6731 | 132123 | 151632 | 52230 | 4522 | 242533 |
| 41314 | 678 | 132223 | 151617 | 4522 | 242526 |
| 121415 | 212324 | 5630 |
Table 9: Part B of the sphere $S_{44,284}$. Tetrahedra to thicken Part A to a ball $B_{43,214}$.

| Tetrahedra | 9293031 | 11121832 | 20212733 | 1313740 | 2323841 | 3333942 |
|------------|---------|-----------|-----------|---------|---------|---------|
| 1293031    | 2111232 | 3202133   | 7313740   | 16323841| 25333942|         |
| 793134     | 16183235| 25273336  | 7103740   | 16193841| 25283942|         |
| 9293134    | 11183235| 20273336  | 10153740  | 19243841| 6283942  |         |
| 1293134    | 2113235 | 3203336   | 14153740  | 23243841| 569342   |         |
| 1282934    | 2101135 | 3192036   | 14173740  | 23263841| 589342   |         |
| 46934      | 13151835| 22242736  | 14151840  | 23242741| 56942    |         |
| 47934      | 13161835| 22252736  | 14171840  | 23262741| 58942    |         |
| 9289294    | 10111835| 19202736  | 10151840  | 19242741| 692842   |         |
| 6928234    | 10151835| 19242736  | 10111840  | 19202741| 928924   |         |
| 6283034    | 10121535| 19212436  | 11121840  | 20212741| 69293024|         |
| 1280304    | 2101235 | 3192136   | 8111240   | 17202141| 26293042|         |
| 92934      | 13183537| 22535368  | 7101140   | 16192041| 2528924  |         |
| 1012145    | 19212324| 562830    | 161340    | 2153241 | 3243342  |         |
| 791037     | 16181938| 25272839  | 14640     | 2131541 | 3222442  |         |
| 9101237    | 18192138| 27283039  | 46840     | 13151741| 22242642|         |
| 10121437   | 19212338| 5283039   | 481240    | 13172141| 22263042|         |
| 10141537   | 19232438| 562839    | 4132243   |         |         |         |
| 13141637   | 22232538| 45739     | 4132243   |         |         |         |
| 14161737   | 23252638| 57839     | 4133743   |         |         |         |
| 4131437    | 13222338| 452239    | 1337343   |         |         |         |
| 4124137    | 13212338| 523039    | 1322834   |         |         |         |
| 481237     | 13172138| 2263039   | 22283943  |         |         |         |
| 891237     | 17182138| 26273039  | 4223943   |         |         |         |
| 46837      | 13151738| 22242639  | 4373943   |         |         |         |
| 683037     | 12151738| 21242639  |           |         |         |         |
| 893037     | 12171838| 21262739  |           |         |         |         |
| 9303137    | 12183238| 21273339  |           |         |         |         |
| 1303137    | 2123238 | 3213339   |           |         |         |         |
| 793137     | 16183238| 25273339  |           |         |         |         |
| 1303437    | 2123538 | 3213639   |           |         |         |         |
| 6303437    | 12153538| 21243639  |           |         |         |         |
| 463437     | 13153538| 22243639  |           |         |         |         |
Table 10: Part C of the sphere $S_{44,284}$: Cone over the boundary of the ball $B_{43,214}$.

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 14544 | 144044 | 15644 | 163144 | 131344 | 1343744 | 1374044 |
| 2131444 | 2134144 | 2141544 | 2153244 | 2323544 | 2353844 | 2384144 |
| 3222344 | 3224244 | 3232444 | 3243344 | 3333644 | 3363944 | 3394244 |
| 4522444 | 4124444 | 4124044 | 4131444 | 4132244 | 563044 | 5223044 |
| 67844 | 6731444 | 683044 | 783944 | 731344 | 7343944 | 893044 |
| 8942444 | 8394244 | 9304244 | 12141544 | 12151744 | 12171844 | 12184044 |
| 13212344 | 13214144 | 13222344 | 15161744 | 15163244 | 16173744 | 16323544 |
| 16353744 | 17184044 | 17374044 | 21232444 | 21242644 | 21274144 | 21274144 |
| 22304244 | 24252644 | 24253344 | 25263844 | 25333644 | 25363844 | 26274144 |
| 26384144 | 34373944 | 35373844 | 36383944 | 37384344 | 3794344 | 38394344 |

Table 11: The sphere $S_{18,125}$.

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 1249 | 12415 | 12915 | 13810 | 13812 | 131012 | 14514 | 14516 |
| 14914 | 141516 | 15711 | 15714 | 151117 | 151216 | 151217 | 171112 |
| 171216 | 171416 | 181013 | 181217 | 181318 | 181718 | 191415 | 1101213 |
| 1111218 | 1111718 | 1121318 | 1141516 | 23513 | 23514 | 231314 | 24615 |
| 24617 | 24917 | 251014 | 251018 | 251318 | 261112 | 261116 | 261215 |
| 261617 | 27810 | 27811 | 271018 | 271112 | 271216 | 271618 | 281013 |
| 281116 | 281318 | 281618 | 291215 | 291216 | 291617 | 2101314 | 34812 |
| 34815 | 341012 | 341016 | 341516 | 35713 | 35714 | 36914 | 36918 |
| 361116 | 361118 | 361417 | 361617 | 37913 | 37918 | 371418 | 381015 |
| 391314 | 3101517 | 3101617 | 3111516 | 3111517 | 3111718 | 3141718 | 451014 |
| 451016 | 46815 | 46817 | 481217 | 491314 | 491317 | 4101213 | 4101314 |
| 4121317 | 5678 | 56713 | 5689 | 56918 | 561318 | 57811 | 58911 |
| 591016 | 591018 | 591115 | 591215 | 591216 | 5111517 | 5121517 | 67815 |
| 671315 | 68914 | 681417 | 6111218 | 6121315 | 6121318 | 781015 | 791017 |
| 791018 | 791317 | 7101517 | 7131517 | 7141618 | 891114 | 8111416 | 8141618 |
| 8141718 | 9101617 | 9111415 | 11141516 | 12131517 |   |   |   |
6 Non-evasiveness and vertex-decomposability

In this section, we show that all vertex-decomposable balls are non-evasive, while the converse is false already in dimension three. For example, we show that Rudin’s ball is non-evasive, but it is neither vertex-decomposable nor shellable. The following Lemma is well known.

**Lemma 6.1.** Let v be a shedding vertex of a vertex-decomposable d-ball $B$. Then v lies on the boundary of the ball. In particular,

(i) link$(v, B)$ is a vertex-decomposable $(d - 1)$-ball;
(ii) del$(v, B)$ is a vertex-decomposable d-ball.

**Proof idea:** If v is an interior vertex, then the deletion of v is d-dimensional but not $(d - 1)$-connected and therefore not vertex-decomposable.

**Theorem 6.2.** Every vertex-decomposable d-ball is non-evasive. In particular, all 2-balls are non-evasive.

**Proof.** A zero-dimensional vertex-decomposable ball is just a point, so it is indeed non-evasive. Let $B$ be a vertex-decomposable d-ball, with $d > 0$. By Lemma 6.1 there is a boundary vertex v such that del$(v, B)$ is a vertex-decomposable d-ball and link$(v, B)$ is a vertex-decomposable $(d - 1)$-ball. The deletion of v from $B$ has fewer facets than $B$, and the link of v in $B$ has smaller dimension than $B$. By double induction on the dimension and the number of facets, we may assume that both del$(v, B)$ and link$(v, B)$ are non-evasive. By definition, then, $B$ is non-evasive.

Next, we prove that the converse of Theorem 6.2 above is false.

**Theorem 6.3.** Rudin’s ball $R$, which has 14 vertices and 41 facets, is non-evasive.

**Proof.** Rudin’s ball is given by the following 41 facets $\{31\}$:

\[
13713, 13913, 15711, 15911, 171113, 191113, 24814, 241014, 26812, 261012, 281214, 2101214, 34711, 34712, 361011, 361014, \\
37123, 371114, 391213, 3101114, 45812, 45813, 471112, 481112, 481314, 4101314, 56913, 56914, 571114, 581213, 591213, 591114, \\
681112, 691314, 6101112, 6101314, 7111213, 8121314, 9111314, 10111214, 11121314.
\]

To prove non-evasiveness, we claim that the sequence $(a_1, \ldots, a_{14}) = (3, 4, 5, 12, 13, 1, 7, 9, 14, 8, 11, 10, 2, 6)$

has the following two properties:

(I) For each $i \leq 5$, link$_{a_i} \text{del}_{a_1, \ldots, a_{i-1}} R$ is a non-evasive 2-complex;

(II) del$_{3, 4, 5, 12, 13} R$ is a non-evasive 2-complex.

To prove that an arbitrary 2-complex $C$ with $n$ vertices is non-evasive, we need to find an order $a_1, \ldots, a_k, a_{k+1}, \ldots, a_n$ of its vertices so that:

(i) For each $i \leq k$, link$_{a_i} \text{del}_{a_1, \ldots, a_{i-1}} R$ is a tree;

(ii) $\text{del}_{a_1, \ldots, a_k} R$ is a tree.
All trees and all simplicial 2-balls are vertex-decomposable and non-evasive, cf. Theorem 6.2. In particular, the link of 3 in $R$ is a non-evasive 2-ball. Let us delete this vertex 3, and proceed with the proof of the claim:

- The link of 4 in $\text{del}_3 R$ is the 2-complex $C$ given by the following 8 facets
  $$2814, \ 21014, \ 5812, \ 71112, \ 81112, \ 81314, \ 101314, \ 5813.$$  
  Let us show that $C$ is non-evasive. The link of 7 in $C$ is a single edge, hence non-evasive. The deletion of 7 from $C$ yields a complex with the same triangles as $C$, except 71112. Inside this smaller complex, the link of 8 is a path, and the deletion of 8 yields the 2-complex
  $$21014, \ 512, \ 1112, \ 1314, \ 101314, \ 513.$$  
  This is a 2-ball with a 3-edge path attached, hence non-evasive. In particular, $C$ is non-evasive.

- The link of 5 in $\text{del}_{3,4} R$ is the 2-complex $D$ given by the following 8 facets
  $$1711, \ 1911, \ 6913, \ 6914, \ 71114, \ 81213, \ 91114, \ 91213.$$  
  We can delete 8 first (its link is an edge), then 9 (because its link is a 6-edge path). The resulting 2-complex,
  $$1711, \ 613, \ 614, \ 71114, \ 1213,$$
  is a 2-ball with a 3-edge path attached, hence non-evasive. So $D$ is also non-evasive.

- The link of 12 in $\text{del}_{3,4,5} R$ is the (non-pure) 2-complex $E$ given by the following 11 facets
  $$268, \ 2610, \ 2814, \ 21014, \ 6811, \ 61011, \ 71113, \ 81314, \ 913, \ 101114, \ 111314.$$  
  We can delete 9 and 7, as their links are a point and an edge (respectively); after that, we delete 13, whose link is now a path. The resulting 2-complex $E'$ has 7 facets:
  $$268, \ 2610, \ 2814, \ 21014, \ 6811, \ 61011, \ 101114.$$  
  The link of 14 inside $E'$ is a 3-edge path, and the deletion of 14 from $E'$ yields a (non-evasive) 2-ball. So, $E'$ and $E$ are non-evasive.

- The link of 13 in $\text{del}_{3,4,5,12} R$ is the 2-complex $F$ given by the following 6 facets
  $$1711, \ 1911, \ 6914, \ 61014, \ 814, \ 91114.$$  
  We can delete 8 first (its link is a point), then 7 (its link is single edge). The resulting 2-complex is a 2-ball. In particular, $F$ is non-evasive.

- Finally, let us examine the 2-complex $G := \text{del}_{3,4,5,12,13} R$. It consists of 13 facets:
  $$1711, \ 1911, \ 268, \ 2610, \ 21014, \ 2814, \ 6811, \ 6914, \ 61011, \ 61014, \ 71114, \ 91114, \ 101114.$$  
  From $G$ we can delete 1 (it has a 2-edge link), then 7 (1-edge link), and then 9 (2-edge link). The resulting 2-complex $H := \text{del}_{1,7,9} G$ consists of 8 facets:
  $$268, \ 2610, \ 21014, \ 2814, \ 6811, \ 61011, \ 61014, \ 101114.$$  
  The link of 14 inside $H$ is a 4-edge path, and the deletion from $H$ of 14 yields a 2-ball. So $H$ is non-evasive; therefore $G$ is non-evasive as well.
Corollary 6.4. Some non-evasive balls are (constructible and) not shellable.

For a more general statement on non-evasiveness of convex 3-balls see [XXX].

Proposition 6.5. Let $B_{7,10}$ be the smallest shellable 3-ball that is not vertex-decomposable [26]. This $B_{7,10}$ is non-evasive.

Proof. $B_{7,10}$ is given by the following 10 tetrahedra:

$$0126, 0134, 0136, 0235, 0256, 0356, 1245, 1246, 1346, 2456.$$  

As explained in [26], the deletion of 6 yields the (non-pure!) 3-complex $A$ given by the facets

$$012, 0134, 0235, 1245.$$  

The link of the vertex 5 in $A$ consists of two triangles with a point in common; this is non-evasive. Deleting 5 from $A$, we obtain the 3-complex $B$ with the following facets.

$$012, 0134, 023, 124.$$  

The link of the vertex 4 inside $B$ is a triangle with an edge attached, hence non-evasive. The deletion of the vertex 4 from $B$ is a 2-ball. Therefore, $B$ is non-evasive, $A$ is non-evasive, and $B_{7,10}$ is non-evasive as well. The sequence of deletions certificating its non-evasiveness is the ‘countdown sequence’ 6–5–4–3–2–1–0. 

Corollary 6.6. Some non-evasive balls are shellable but not vertex-decomposable.

Proposition 6.7. Let $B_{9,18}$ be the smallest non-shellable 3-ball, described in [25]. $B_{9,18}$ is non-evasive and constructible.

Proof. $B_{9,18}$ is given by the following 18 tetrahedra:

$$0123, 0124, 0145, 0157, 0168, 0178, 0234, 0678, 1236, 1245, 1258, 1268, 1578, 2347, 2367, 2467, 2468, 4678.$$  

Consider the 2-sphere $S$ given by the following 12 triangles:

$$023, 024, 036, 045, 057, 068, 078, 236, 245, 258, 268, 578.$$  

It is easy to see that $S$ minus the triangle 036 is the same 2-complex as the link of 1 inside $B_{9,18}$. Since a 2-sphere minus a triangle yields a 2-ball, and all 2-balls are shellable, it follows that the link of 1 inside $B_{9,18}$ is shellable. Since shellability is preserved by taking cones, the closed star $C_1$ of 1 inside $B_{9,18}$ is also shellable. Let $B_1 := C_1 \cup 0678$. Since $C_1 \cap 0678$ consists of the two triangles 068 and 078, $B_1$ is also shellable. (A shelling order for $B_1$ is the shelling order for $C_1$, plus 0678 as last facet.) Now, let $B_2$ be the shellable 3-ball with 7 vertices (labeled by 0, 2, 3, 4, 6, 7, 8) with the following 6 facets, already given in a possible shelling order:

$$0234, 2347, 2367, 2467, 2468, 4678.$$  

Clearly, $B_{9,18}$ splits as $B_1 \cup B_2$. Moreover, the intersection $B_1 \cap B_2$ is a 2-ball, given by the following 5 facets:

$$023, 024, 236, 268, 678.$$  

In particular, $B_{9,18}$ is constructible. We still have to prove that $B$ is non-evasive; we will show this by deleting the vertices 1–0–6–3–7–2–4–5–8, in this order. The link of vertex 1 in $B_{9,18}$ is the
(non-evasive, shellable) 2-ball described above. The deletion of 1 from $B_{9,18}$ yields the following 3-complex $A$:

$$0234, 0678, 2347, 2367, 2467, 2468, 4678, 045, 057, 245, 258, 578.$$  

Inside $A$, the link of the vertex 0 consists of two triangles joined by a 2-edge path. Such a 2-complex is clearly non-evasive. Deleting the vertex 0 from $A$ we obtain the 3-complex $B$ described as follows:

$$2347, 2367, 2467, 2468, 4678, 245, 258, 578.$$  

Next, we delete 6, whose link inside $B$ is a 2-ball with 4 triangles. The result is this 3-complex $C$:

$$2347, 245, 248, 258, 478, 578.$$  

From $C$ we can delete first 3 (whose link is a triangle) and then 7 (whose link is a 3-edge path). The result is a 2-ball, so $C$ is non-evasive. As a consequence, $B, A$ and $B_{9,18}$ are all non-evasive.  

Our last result highlights the positive effects of barycentric subdivisions.

**Proposition 6.8.** Let $B$ be a simplicial complex.

1. Although $B_{9,18}$ is not shellable, its barycentric subdivision is vertex-decomposable.
2. Although $S_{13,56}$ is not constructible, its barycentric subdivision is vertex-decomposable.
3. Although $B_{12,38}$ is evasive and not LC, its barycentric subdivision is LC and non-evasive.

**Proof.** Sequences of deletions that prove vertex-decomposability of $\text{sd}B_{9,18}$ and $S_{13,56}$ were found with a computer backtrack search. Since $B_{12,38}$ is collapsible, by a result of Welker $\text{sd}B_{12,38}$ is non-evasive [32]. Since $B_{12,38}$ is a collapsible 3-ball, by a result of the first author $\text{sd}B_{12,38}$ is locally constructible [7].  

**Corollary 6.9.** Some non-evasive balls are (LC and) not constructible.

**Proof.** The barycentric subdivision of $B_{12,38}$ cannot be constructible by Theorem 2.1, because it contains a knotted spanning arc of two edges.  

7 Open problems

The following questions remain open:

- Are there constructible $d$-spheres that are not shellable? The problem is open already for $d=3$.
- Are there non-evasive balls with a knotted spanning edge?
- Are there examples of non-shellable spheres that become vertex-decomposable after stacking all facets? (This would imply that a non-simplicial 4-ball can be vertex-decomposable but not shellable.)
- Are there evasive collapsible 4-balls?
- Are there non-evasive balls that are not LC? Are there LC (3-)balls that are evasive?
- Are the 3-spheres $S_{16,92}$ and $S_{18,125}$ vertex-minimal with the property of having the double trefoil and the triple trefoil knot on three edges in their 1-skeleton, respectively? What happens if we replace the square knot by the granny knot?

Acknowledgements.

Thanks to Jonathan Barmak and Alex Engström for helpful discussions.
References

[1] C. C. Adams, B. M. Brennen, D. L. Greilsheimer, and A. K. Woo, Stick numbers and composition of knots and links. J. Knot Theory Ramifications 6 (1997), 149–161.
[2] J. Ambjørn, B. Durhuus, and T. Jonsson, Quantum Geometry. Cambridge University Press, Cambridge, 1997.
[3] J. Ambjørn and S. Varsted, Three dimensional simplicial quantum gravity. Nucl. Phys. B, 373 (1992), 557–577.
[4] B. Bagchi and B. Datta, On stellated spheres, shellable balls, lower bounds and a combinatorial criterion for tightness. Preprint (2012, 46 pages) available at arxiv:1102.0856v2.
[5] J. A. Barmak and G. E. Minian, Strong homotopy types, nerves and collapses. Discr. Comput. Geometry 47 (2012), 301-328.
[6] B. Benedetti, On Locally Constructible Manifolds. PhD thesis, TU Berlin (2010). Available online at http://opus.kobv.de/tuberlin/volltexte/2010/2519/.
[7] B. Benedetti, Discrete Morse theory for manifolds with boundary. Trans. AMS 364 (2012), 6631–6670.
[8] B. Benedetti and F. H. Lutz, The dunce hat and a minimal non-extendably collapsible 3-ball. To appear in Electronic Geometry Models. Preprint (2013, 6 pages) at arxiv:0912.3723v2.
[9] B. Benedetti and F. H. Lutz, Random discrete Morse theory and a new library of triangulations. Preprint (2013, 33 pages) available at arxiv:1303.6422.
[10] B. Benedetti and G. M. Ziegler, On locally constructible spheres and balls. Acta Math. 206(2011), 205-243.
[11] R. H. Bing, Some aspects of the topology of 3-manifolds related to the Poincaré conjecture. In Lectures on Modern Mathematics, T. Saaty, ed., vol. II, Wiley (1964), 93–128.
[12] A. Björner and F. H. Lutz, Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere. Experiment. Math. 9 (2000), 275–289.
[13] R. Dougherty, V. Faber, and M. Murphy, Unflippable tetrahedral complexes. Discrete Comput. Geom. 32 (2004), 309–315.
[14] B. Durhuus and T. Jonsson, Remarks on the entropy of 3-manifolds. Nucl. Phys. B 445 (1995), 182–192.
[15] R. Ehrenborg and M. Hachimori, Non-constructible complexes and the bridge index. Europ. J. Combin. 22 (2001), 475–491.
[16] A. Engström, Discrete Morse functions from Fourier transforms. Experimental Mathematics 18 (2009), 45–53.
[17] R. Forman, Morse theory for cell complexes. Adv. in Math. 134 (1998), 90–145.
[18] R. Goodrick, Non-simplicially collapsible triangulations of $P^n$. Proc. Camb. Phil. Soc. 64 (1968), 31–36.
[19] M. Hachimori and K. Shimokawa, Tangle sum and constructible spheres. Journal of Knot Theory and Its Ramifications 13 (2004), 373–383.
[20] M. Hachimori and G. M. Ziegler, Decompositions of simplicial balls and spheres with knots consisting of few edges. Math. Z. 235 (2000), 159–171.
[21] M. E. Hamstrom and R. P. Jerrard, Collapsing a triangulation of a “knotted” cell. Proc. Amer. Math. Soc. 21 (1969), 327–331.
[22] J. Kahn, M. Saks, and D. Sturtevant, A topological approach to evasiveness. Combinatorica 4 (1984), 297–306.
[23] W. B. R. Lickorish, Unshellable triangulations of spheres, Europ. J. Combin., 12 (1991), 527–530.
[24] W. B. R. Lickorish and J. M. Martín, Triangulations of the 3-ball with knotted spanning 1-simplices and collapsible r-th derived subdivisions. Trans. Amer. Math. Soc. 170 (1972), 451–458.
[25] F. H. Lutz, A vertex-minimal non-shellable simplicial 3-ball with 9 vertices and 18 facets. In www.eg-models.de, Electronic Geometry Model No. 2003.05.004.
[26] F. H. Lutz, Vertex-minimal not vertex-decomposable balls. In www.eg-models.de, Electronic Geometry Model No. 2003.06.001.
[27] F. H. Lutz, BISTELLAR, Version 11/03. http://page.math.tu-berlin.de/~lutz/stellar/BISTELLAR, 2003.
[28] F. H. Lutz, Small examples of nonconstructible simplicial balls and spheres. SIAM J. Discrete Math. 18 (2004), 103–109.
[29] U. Pachner, P. L. homeomorphic manifolds are equivalent by elementary shelling. Europ. J. Combin. 12 (1991), 129–145.
[30] J. S. Provan and L. J. Billera, Decompositions of simplicial complexes related to diameters of convex polyhedra. Math. Operations Research 5 (1980), 576–594.
[31] M. E. Rudin, An unshellable triangulation of a tetrahedron. Bull. Amer. Math. Soc. 64 (1958), 90–91.
[32] V. Welker, Constructions preserving evasiveness and collapsibility. Discr. Math. 207 (1999), 243–255.