Velocity and velocity bounds in static spherically symmetric metrics

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Abstract

We find simple expressions for velocity of massless particles in dependence of the distance $r$ in Schwarzschild coordinates. For massive particles these expressions put an upper bound for the velocity. Our results apply to static spherically symmetric metrics. We use these results to calculate the velocity for different cases: Schwarzschild, Schwarzschild-de Sitter and Reissner-Nordström with and without the cosmological constant. We emphasize the differences between the behavior of the velocity in the different metrics and find that in cases with naked singularity there exists always a region where the massless particle moves with a velocity bigger than the velocity of light in vacuum. In the case of Reissner-Nordström-de Sitter we completely characterize the radial velocity and the metric in an algebraic way. We contrast the case of classical naked singularities with naked singularities emerging from metric inspired by noncommutative geometry where the radial velocity never exceeds one. Furthermore, we solve the Einstein equations for a constant and polytropic density profile and calculate the radial velocity of a photon moving in spaces with interior metric. The polytropic case of radial velocity displays an unexpected variation bounded by a local minimum and maximum.

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I. INTRODUCTION

The class of static spherically symmetric metrics is widely used in General Relativity and Astrophysics [1, 2]. The trajectory of a test body in the gravitational field determined by this metric can be obtained in a closed form up to an integral. The latter is not always solvable in a compact form. Therefore, more direct analytical and easily accessible information on the motion of the test body is welcome. One such possible method is to cast the equation of motion into a form resembling the corresponding expression of classical mechanics with an effective potential \( U_{\text{eff}} \). This is certainly possible for exterior metrics and the power of this method for the Schwarzschild-de Sitter case (with a cosmological constant \( \Lambda \)) has been demonstrated in [3]. Here, we attempt a yet different way to gain information about the particle motion by examining the radial and angular velocity of the particle in dependence of the distance. Such an information can be derived without the necessity of solving the full equation of motion. The usefulness of the results can be then shown by applying the general result to the different cases of exterior and interior spherically symmetric metrics. For instance, in contrast to the Schwarzschild metric, where the radial velocity of a massless particle approaches asymptotically one at large distances, the same quantity in the Schwarzschild-de Sitter metric reaches a maximal value \( 1 - (3r_s r_1^{-1})^{2/3} \) at \( r = r_1 \equiv (3r_s r_A^2)^{1/3} \) \( (r_A = 1/\sqrt{\Lambda}) \) beyond which it decreases. Note that \( r_1 \) is of astrophysical order of magnitude as it is a combination of a small quantity, \( r_s = GM \), with a large one \( r_A \). Indeed, \( r_1 \) is also of the order of magnitude of the typical extension of a star cluster (if the mass \( M \) is a typical star mass), galaxy (if \( M \) is of the order of magnitude of a star cluster) and galaxy cluster (choosing \( M \) to be the mass of a typical galaxy) [3]. As we will show, this implies the interesting result of the Schwarzschild-de Sitter space-time, i.e. the velocity of a photon leaving one of the above astrophysical objects does not grow with distance (after having travelled a distance \( r_1 \)) to approach asymptotically one, but decreases monotonically. The velocity expressions which we derive allow also a deeper understanding of the special role of naked singularities. As we will demonstrate in the case of naked singularities, the radial velocity of a massless particle exceeds in the Schwarzschild coordinates in certain space regions the value one, i.e. the velocity of light in vacuum (and in the absence of gravitational fields). The naked singularities which we study in this work are of Reissner-Nordström and Reissner-Nordström-de Sitter type. In the latter case we give an algebraic characterization of the metric and the radial velocity of a massless particle deriving among other the conditions for the existence of the naked singularity and for the regions where the radial velocity is bigger than one. We give a definition of the extreme Reissner-Nordström metric in terms of the radial velocity. We also study the case of naked singularities encountered in metrics inspired by noncommutative geometry. We find that in contrast to the classical case the radial velocity of a massless particles is always smaller-equal one. We apply our velocity expressions also to the case of an interior metric. We do this first for the case of constant density and later assume the density to be determined by a polytropic equation of state and the Tolman-Oppenheimer-Volkov hydrostatic equilibrium equation. In both cases, the radial velocity of the photon will be smaller than one and approximately one if the gravitational field is weak. The interesting feature is, however, that in the polytropic case it will display a variation with some local minima and maxima present.

The concept of a velocity i.e. the change of distance with respect to the time coordinate is in General Relativity a frame dependent concept as it is not an invariant. This is not a drawback as it corresponds to the real measurements we perform in practice. Although the
Equivalence Principle assures that we can always find a local frame in which the velocity of a photon is the velocity of light (as defined in vacuum), this is not the frame which is conform with our measurement devices. Indeed, all our measurements of local (as opposed to cosmological) effects of General Relativity refer explicitly to Schwarzschild coordinates in which we establish and measure the precession of perihelia, deflection of light or radar echo delay. In these coordinates the radial velocity of light is not a constant as it is determined by the geodesic equation of motion. It is therefore of some interest to study the radial velocity of a photon in Schwarzschild coordinates.

The paper is organized as follows. In section 2 and 3 we derive the general expression for the radial velocity in Schwarzschild coordinates in dependence of the radial coordinate. In section 4 we apply the radial velocity expressions to vacuum metric and discuss the significance of the results. In section 5 we study the radial velocity of a massless particle in interior metric using constant density and a polytropic model. In section 6 we present the conclusions.

II. THE GENERAL STATIC SPHERICALLY SYMMETRIC METRIC

Let us remind the reader that the most general form of a static spherically symmetric metric can be written as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2).$$  (1)

As it is well known there exist two associated conserved quantities \[1, 2\]

$$E = A(r)\frac{dt}{d\tau},$$  (2)

and

$$r_l = r^2\frac{d\varphi}{d\tau} = r^2\dot{\varphi}\frac{E}{A(r)},$$  (3)

where \(\tau\) is the proper time, i.e. we have chosen the affine parameter to be \(\tau\) and dotted quantities signify derivative with respect to time \(t\). Another constant which emerges due to the geodesic equation taken together with the metric compatibility is

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau},$$

where \(\epsilon = 1\) for massive particles and \(\epsilon = 0\) for massless particles. Rewriting the above expression by means of the metric (1) we obtain

$$-\frac{\epsilon}{B(r)} = -\frac{A(r)}{B(r)}\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2 + \frac{r^2}{B(r)}\left(\frac{d\vartheta}{d\tau}\right)^2 + \frac{r^2\sin^2\vartheta}{B(r)}\left(\frac{d\varphi}{d\tau}\right)^2.$$

The conserved quantities given in (2) and (3) enable us to cast the above equation into the form

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{r_l^2}{2r^2B(r)} = \frac{E^2}{2A(r)B(r)} - \frac{\epsilon}{2B(r)},$$  (4)
which is essentially the equation of motion where the plane of motion is such that \( \theta = \pi/2 \).

Taking into account that for the metric (1) we have \( A(r)B(r) = 1 \), equation (4) becomes

\[
\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + U_{\text{eff}} = \frac{E^2}{2}, \quad U_{\text{eff}} = \left( \frac{r^2}{2r^2} + \frac{\epsilon}{2} \right) A(r).
\]

(5)

Instead of solving directly (4) or (5), we will pursue here a different strategy. First of all, notice that the conserved quantities (2) and (3) permit us to write equation (4) as

\[
E^2 \left( 1 - \frac{\dot{r}^2}{A^2(r)} - \frac{r^2 \dot{\phi}^2}{A(r)} \right) = \frac{\epsilon}{B(r)}.
\]

(6)

Similarly, we can express (3) in the following form

\[
\dot{r}^2 \left( \frac{1}{B^2(r)} - \dot{r}^2 - \frac{r^2 \dot{\phi}^2}{B(r)} \right) = r^4 \dot{\phi}^2 \frac{\epsilon}{B(r)}.
\]

(7)

In the subsequent sections the last two equations will be applied to different cases of exterior and interior spherically symmetric metrics.

III. VELOCITY AND VELOCITY BOUNDS

We assume first \( \epsilon = 1 \), i.e. the case of massive particles. In all examples we will discuss later the metric elements \( B(r) \) and \( A(r) \) satisfy \( B(r) \geq 0 \) and \( A(r) \geq 0 \) for \( r \in K \subset \mathbb{R}^+ \), where the explicit form of \( K \) depends on the particular metric under consideration. For instance, in the Schwarzschild case we would have \( K = [2r_s, \infty) \). Let \( v_r(r) \) and \( v_\phi(r) \) be the radial and tangential velocities of a test particle, respectively. Then, for \( \epsilon = 1 \) the r.h.s. of equation (3) must be positive definite and we obtain

\[
1 - \frac{v_r^2(r)}{A^2(r)} - \frac{v_\phi^2(r)}{A(r)} > 0.
\]

This expression gives us the velocity bounds of a massive particle. In particular, notice that if the motion is purely radial, i.e. \( \dot{\phi} = 0 \), then we obtain the following bound for the radial velocity

\[
v_r(r) < A(r), \quad r \in K \subset \mathbb{R}^+,
\]

whereas in the case of purely tangential motion, that is \( \dot{r} = 0 \), a similar bound can be found, namely

\[
v_\phi(r) < \sqrt{A(r)}, \quad r \in K \subset \mathbb{R}^+.
\]

In the massless case the inequalities we derived for massive particles become equalities and we have

\[
v_r(r) = A(r), \quad v_\phi(r) = \sqrt{A(r)}.
\]

(8)

Notice that for the Minkowski metric in spherical coordinates \( ds^2 = -dt^2 + dr^2 + r^2d\Omega^2 \) where \( A(r) = B(r) = 1 \), we obtain as expected \( v_r(r) < 1 \) for the massive case and \( v_r(r) = 1 \) for the photon’s radial velocity.
FIG. 1: Comparison of the radial velocities of photon moving in the Schwarzschild and Schwarzschild-de Sitter metric. Evidently, the photon velocity does not reach 1, not even asymptotically, in the latter case and decreases after a maximum value smaller than one.

IV. VACUUM METRICS

In this section we discuss the velocity of a test particle in dependence of the distance \( r \) for different types of metrics. The results are depicted in figures 1-4, which display the main features of the radial velocity as zeros, local extrema and regions where the velocity is bigger than one. In the figures we have not used the actual values of \( r_s = G_N M \) and \( r_A = 1/\sqrt{\Lambda} \) (where \( G_N \) is the Newtonian and \( \Lambda \) the cosmological constant) as both these scales are very far apart. We therefore decided to use arbitrary units in the figures. Notice that a common feature of the velocity profiles is that \( v_r(r) \) vanishes when evaluated at the event horizon \( r_H \) since light cannot escape from a black hole as seen from the standpoint of a distant observer in Schwarzschild coordinates.

A. Schwarzschild and Schwarzschild-de Sitter metric

In the case of Schwarzschild-de Sitter, also called Kottler metric

\[
A(r) = 1 - \frac{2r_s}{r} - \frac{1}{3} \frac{r^2}{r_A^2}
\]
the velocities are given by

\begin{align*}
v_r(r) &= 1 - \frac{2r_s}{r} - \frac{1}{3} \frac{r^2}{r_A^2}, \quad v_\phi(r) = \sqrt{1 - \frac{2r_s}{r} - \frac{1}{3} \frac{r^2}{r_A^2}},
\end{align*}

where the Schwarzschild case is recovered for \( r_A \to \infty \). The interesting feature emerging from FIG. 1 is that in the Schwarzschild-de Sitter metric the photon radial velocity reaches a maximum velocity smaller than one at a finite distance \( r_1 = (3r_s)^{1/3} \).

After which \( v_r \) decreases. Moreover, from (9) we also see that in order there exists an interval of the radial variable \( r \) such that \( A(r) > 0 \) it must be

\[ r_A > 3r_s. \]  

Under condition (10) the radial velocity will become zero at \( \tilde{r}_0 \approx 2r_s \) and \( r_c \approx \sqrt{3}r_A \). Hence, we have

\[ 0 < v_r(r) < v_r(r_1) < 1 \quad \text{for} \quad r \in (\tilde{r}_0, r_c) \quad \text{and} \quad r_A > 3r_s. \]

Note that \( r_1 \) is a combination of a small parameter \( r_s \) with a large one, i.e. \( r_A \). Therefore, its actual value is of astrophysical order of magnitude even if \( r_A \) is very large. The actual value of \( r_1 \) can be estimated by using the currently favored value \( \rho_{\text{vac}} \approx 0.7\rho_{\text{crit}} \) together with

\[ \Lambda = 3 \left( \frac{\rho_{\text{vac}}}{\rho_{\text{crit}}} \right) H_0^2 \]

where \( H_0 \) is the Hubble constant. For sun masses it comes out to be approximately the size of a globular cluster, for the mass of a globular cluster it is roughly the size of a average galaxy and finally, inserting the mass of a galaxy \( r_1 \) approaches the size of a galaxy cluster [3]. The distance \( r_1 \) has a fourfold meaning

(i) It is the maximal size of a bound orbit in Schwarzschild-de Sitter metric in the approximation of small angular momentum [3].

(ii) It is the maximal size of virialized matter in space-time with \( \Lambda \) [3].

(iii) As shown above, it is the distance at which a photon reaches its maximal velocity smaller than one. This curious features of the photon radial velocity can have an astrophysical effect, namely the radial velocity of a photon leaving the surface of a typical galaxy and after travelling roughly a distance of about the size of a galaxy cluster starts decreasing with distance.

(iv) It is the critical distance after which the radiation is blue-shifted [3].

It is also clear that the velocity of light as measured in Schwarzschild coordinates at the earth’s surface is not the velocity of light which we would measure in an empty space (Minkowski space-time). Measured in the radial direction the velocity of light on earth would be \( 1 - 1.39 \times 10^{-9} \) which has a small historical relevance. In the second half of the twenty century it was thought that a more accurate measurement of the velocity of light was hampered
by inaccurate definition of the meter \[6\]. Therefore in 1983 the meter has been redefined as the length travelled by light in vacuum during a time interval of \(1/299792458\) seconds \[9\]. This definition makes the velocity of light in vacuum an exact number: \(299792458\) m/s. The best value of directly measured velocity of light is \(299792456.2\pm1.1\) m/s \[6\]. This means that a direct measurement achieves an accuracy of about \(10^{-9}\) which is exactly of the same size as the effect of measuring the radial velocity of light at the surface of earth. This also means that such an effect is actually not (yet) measurable. One could object that our estimate is based on a constant velocity concept whereas the actual radial velocity in Schwarzschild coordinates depends on \(r\). By defining an average

\[
<v_r> = \frac{1}{r_b - r_a} \int_{r_a}^{r_b} v_r(r) dr
\]

and choosing \(r_a = R_\oplus\), i.e. the average radius of the earth and \(r_b = R_\oplus - \delta r\), we obtain

\[
<v_r> = 1 - 2\frac{r_s}{\delta r} \ln \left(1 + \frac{\delta r}{R_\oplus}\right) \simeq 1 - \frac{2r_s}{R_\oplus} + \frac{2r_s}{R_\oplus} \frac{\delta r}{R_\oplus}
\]

which shows that one can neglect the \(\delta r\) corrections near the earth surface.

### B. Noncommutative inspired Schwarzschild and Reissner-Nordström metrics

An interesting class of static, spherically symmetric metrics inspired by noncommutative geometry consists of the so called noncommutative geometry inspired Schwarzschild and Reissner-Nordström metrics \[7\]. The noncommutative geometry inspired Schwarzschild black hole has line element (in this subsection we choose \(c = G = 1\))

\[
ds^2 = g(r)dt^2 - g(r)^{-1}dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)
\]

where

\[
g(r) = 1 - \frac{2Ms(r)}{r}, \quad M_s(r) = \frac{2M}{\sqrt{\pi}} \gamma(3/2, r^2/4\vartheta), \quad \gamma(3/2, r^2/4\vartheta) = \int_0^{r^2/4\vartheta} du \frac{u^{3/2}}{e^u - 1}
\]

and \(\vartheta\) is a parameter encoding noncommutativity. Notice that for \(r/\sqrt{\vartheta} \to \infty\) the line element \(13\) becomes the classical Schwarzschild metric. For \(M < M_0 = 1.9\sqrt{\vartheta} \) \(13\) (we assume that at least formally it is possible to have such an \(M\)) does not exhibit any horizon and describes what we call a naked, self-gravitating droplet of anisotropic fluid \[7\]. Finally, for \(M > M_0\) there exist two horizons and only one horizon at \(r_0 = 3.0\sqrt{\vartheta}\) for \(M = M_0\). The noncommutative geometry inspired Schwarzschild metric has a line element of the form

\[
ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)
\]

with

\[
f(r) = 1 - \frac{4M}{r\sqrt{\pi}} \gamma(3/2, r^2/4\vartheta) + \frac{Q^2}{\pi r^2} \left[ F(r) + \frac{2}{\vartheta} r \gamma(3/2, r^2/4\vartheta) \right],
\]

\[
F(r) = \gamma^2(1/2, r^2/4\vartheta) - \frac{r}{\sqrt{2\vartheta}} \gamma(1/2, r^2/2\vartheta), \quad \gamma(a/b, x) = \int_0^x \frac{du}{u} u^{a/b} e^{-u},
\]
FIG. 2: Photon radial velocity versus \( x = r/\sqrt{\theta} \) in metrics inspired by noncommutative geometry with a naked singularity. The upper curve corresponds to the Schwarzschild noncommutative case whereas the lower one refers to noncommutative Reissner-Nordström. The choice of the parameters is: \( M/\sqrt{\theta} = 1 \), \( Q/\sqrt{\theta} = 2 \). The velocity is always smaller-equal one.

where \( M \) and \( Q \) are the mass and charge parameters, respectively. Asymptotically for \( r/\sqrt{\theta} \to \infty \) the above line element reproduces the classic Reissner-Nordström metric. Depending on the values of both \( M \) and \( Q \) the line element (14) describes a charged black hole with two horizons, an extremal charged black hole or a naked, charged, self-gravitating droplet of anisotropic fluid. We do not give the algebraic details here to define the naked singularity in the charged case, but rely on numerical results. Since the above metrics are static and spherically symmetric we can apply our formula (8) to study the radial velocity of a massless particle in the aforementioned metrics. In Figure 2 we plot the radial velocity of a photon moving in the above metrics under the condition that we have a naked droplet. It is evident that a massless particle starting with velocity one asymptotically away from the droplet will experience a deceleration as it approaches the droplet till it reaches a minimum velocity at \( r_m = 3\sqrt{\theta} \). For \( r < r_m \) the particle starts accelerating and its velocity will be one again at \( r = 0 \). Worth noticing is the fact that the velocity is everywhere smaller-equal one. This is different if the naked singularity were of classical type, a case discussed in the next subsection.

C. What is indecent about naked singularities?

We will discuss here the effect of naked singularity on the radial velocity of a photon in the Reissner-Nordström metric [10] with and without cosmological constant. In the present case the function \( A(r) \) is given by

\[
A(r) = 1 - \frac{2r_s}{r} + \frac{r_Q^2}{r^2} - \frac{1}{3} \frac{r^2}{r_A^3}.
\] (15)
There exists a region where the velocity exceeds one. This allows us to calculate the velocities as follows

\[
v_r(r) = 1 - \frac{2r_s}{r} + \frac{r_Q^2}{r^2} - \frac{1}{3} \frac{r^2}{r_A^2}, \quad v_\phi(r) = \sqrt{1 - \frac{2r_s}{r} + \frac{r_Q^2}{r^2} - \frac{1}{3} \frac{r^2}{r_A^2}}.
\]

The Reissner-Nordström case is obtained in the limit \( r_A \to \infty \). Notice that in this limit and for \( r_s < r_Q \) the radial velocity exceeds the velocity of light (represented by the value 1) below

\[
r_0 = \frac{1}{2} \frac{r_Q^2}{r_s},
\]

and displays a local minimum at

\[
r_1 = \frac{r_Q^2}{r_s}, \quad v_r(r_1) = 1 - \left( \frac{r_s}{r_Q} \right)^2,
\]

after which it approaches one. A similar behavior can be seen if a positive cosmological constant is introduced. To this purpose it is interesting to study under which conditions on the parameter \( r_s, r_Q \) and \( r_A \) the Reissner-Nordström-deSitter geometry exhibits a naked singularity. In what follows the only hypothesis we will make is that \( r_A > r_s \) and \( r_A > r_Q \). The minimal requirement for the emergence of a naked singularity is that \( (15) \) possesses two real intersections with the \( r \)-axis having opposite signs. This condition assures the only existing horizon is the cosmological one. Finally, there is a second characteristic point

\[
(16)
\]
important to define the extreme case, namely that the radial velocity has local minimum \( r_m \) and one local maximum \( r_M \). Concerning the first requirement, we have to study the roots of the equation
\[
- \frac{r^4}{3r_A^2} + r^2 - 2rsr + r_Q^2 = 0. \tag{17}
\]
According to [8] the above quartic equation will have two real roots and one conjugate pair of complex roots if its associated discriminant \( \Delta_I \) is negative, that is
\[
\Delta_I = 4I^3_I - J^2_I < 0
\]
where
\[
I_I = 1 - 4 \frac{r_Q}{r_A^2}, \quad J_I = -24 \frac{r_Q^2}{r_A^2} + 36 \frac{r_s^2}{r_A^2} - 2.
\]
Hence, the function \( v_r(r) \) will intersect the \( r \)-axis at two different locations if
\[
(r^2_Q - 4r^2_Q)^3 - r^2_A(12r^2_Q - 18r^2_s + r^2_A)^2 < 0. \tag{18}
\]
Fig. 3 shows that in the naked singularity scenario the radial velocity function may exhibit one local minimum and maximum or a saddle point. In order that \( v_r(r) \) has at least a local minimum and maximum, we have to require that \( dv_r/dr = 0 \). This leads to the quartic
\[
- \frac{r^4}{3r_A^2} + rsr - r_Q^2 = 0. \tag{19}
\]
The above equation will have two real roots and one conjugate pair of complex roots if the discriminant \( \Delta_{II} \) is negative, that is
\[
\Delta_{II} = 4I^3_{II} - J^2_{II} < 0
\]
with
\[
I_{II} = 4 \frac{r_Q}{r_A^2}, \quad J_{II} = 9 \frac{r_s^2}{r_A^2}.
\]
Therefore, the condition for the existence of a local minimum and maximum reads
\[
r_Q < r_Q^{\text{crit}}, \quad r_Q^{\text{crit}} = \left( \frac{3}{4} \right)^{2/3} (r_A r_s^2)^{1/3}. \tag{20}
\]
Moreover, the real solutions of (19) are
\[
r_{m,M} = \frac{\sqrt{2}}{4} \left( \alpha^{1/2} + \sqrt{12\sqrt{2} \frac{rsr_A}{\alpha^{1/2}} - 16 \frac{r_Q^2 r_A^2}{\beta^{1/3}} - \beta^{1/3}} \right)
\]
with
\[
\alpha = \frac{\beta^{2/3} + 16 r_Q^2 r_A^2}{\beta^{1/3}}, \quad \beta = 36 r_s^2 r_A^4 + 4 \sqrt{-256 r_Q^6 r_A^6 + 81 r_s^6 r_A^8}.
\]
Notice that (20) ensures that \( \beta > 0 \). Moreover, there are two sign changes in (19), hence Descartes sign rule implies that the maximum number of positive roots is two. On the other
hand, the polynomial equation obtained from (19) by reversing the sign of \( r \) does not exhibit any sign change and we conclude that the maximum number of negative real roots must be zero. From these considerations it follows that \( r_m > 0 \) whenever (20) is satisfied. Finally, the radial velocity evaluated at the local minimum will be always positive in virtue of (18). Furthermore, we can characterize an extreme Reissner-Nordström-de Sitter black hole by a choice of the parameters \( r_s, r_Q \) and \( r_A \) such that \( v_r(r_m) = 0 \) together with the condition that (17) has four real roots with two of them coinciding. In other words, this will ensure the existence of only one extra horizon apart from the cosmological one. For the sake of brevity, we will treat here only the special case \( r_A^2 > r_Q^3 \) implying \( r_Q \ll r_Q^{\text{crit}} \). In terms of the small parameter \( \gamma = r_Q/r_Q^{\text{crit}} \) the local minimum is located at

\[
    r_m = \left( \frac{3}{4} \right)^{4/3} (r_A^2 r_s)^{1/3} \gamma^2 \left[ 1 + \frac{27}{256} \gamma^6 + \mathcal{O}(\gamma^{12}) \right]
\]

whereas the radial velocity at \( r_m \) takes the value

\[
    v_r(r_m) = 1 - \left( \frac{4}{3} \right)^{4/3} \left( \frac{r_s}{r_A} \right)^{2/3} \frac{1}{\gamma^2} + \mathcal{O}(\gamma^4).
\]

Notice that in physical interesting cases \( r_s \ll r_A \). Hence, the quantity \( (r_s/r_A)^{2/3} \gamma^{-2} \) will be the product of a small quantity multiplying a large one. A simple computation reveals that \( (r_s/r_A)^{2/3} \gamma^{-2} \ll (r_s/r_Q)^2 \) and we conclude that \( (r_s/r_A)^{2/3} \gamma^{-2} \) will be of order one if \( r_s \approx r_Q \). Finally, we find the maximum at \( r_M \)

\[
    r_M = (3r_A^2 r_s)^{1/3} \left[ 1 - \frac{\gamma^2}{2^{8/3}} - \frac{\gamma^4}{2^{13/3}} + \mathcal{O}(\gamma^6) \right]
\]

with

\[
    v_r(r_M) = 1 - \left( \frac{3 r_s}{r_A} \right)^{2/3} + \frac{1}{2^{8/3}} \left( \frac{3 r_s}{r_A} \right)^{2/3} \gamma^2 + \mathcal{O}(\gamma^4).
\]

It is not difficult to see that at the fourth order in \( \gamma \) we always have \( v_r(r_M) < 1 \). The radial velocity becomes bigger than one below a value given approximately by \( r_0 \) in equation (16). In general, \( v_r(r) < 1 \) for those values of \( r \) such that

\[
    p(r) = -\frac{r^4}{3r_A^2} - 2r_s r + r_Q^2 > 0.
\]

It is not difficult to check that the discriminant of the quartic equation \( p(r) = 0 \) is always negative, thus implying that there will be always two real roots and one conjugate pair of complex roots. Moreover, according to the Descartes sign rule there will be one negative and one positive real root. The latter is

\[
    R_0 = \frac{1}{2} \left( -\tilde{\alpha}^{1/2} + \sqrt{12 \frac{r_s^2 r_A^2}{\tilde{\alpha}^{1/2}} + 4 \frac{r_Q^2 r_A^2}{\tilde{\beta}^{1/3}} - \tilde{\beta}^{1/3}} \right)
\]

with

\[
    \tilde{\alpha} = \frac{\tilde{\beta}^{2/3} - 4 r_Q^2 r_A^2}{\tilde{\beta}^{1/3}}, \quad \tilde{\beta} = 18 r_s^2 r_A^2 + 2 \sqrt{16 r_Q^6 r_A^6 + 81 r_s^4 r_A^4}.
\]
If \( r_A r_s^2 \gg r_Q^3 \) then the critical radius below which we encounter velocities bigger than one is

\[
R_0 = \frac{1}{2} \left( \frac{3}{4} \right)^{4/3} (r_A r_s)^{1/3} \gamma^2 \left[ 1 - \frac{27}{4096} \gamma^6 + \mathcal{O}(\gamma^{12}) \right].
\]

Notice that at the second order in \( \gamma \) we have \( R_0 \approx r_m/2 \) and in the regime \( r_Q \ll r_{\text{crit}} \) the radial velocity exceeds one whenever \( r < r_m/2 \). The question ‘what is so special about naked singularities’ can be answered, using the formulae we derived for the velocity, fast and efficiently. There exist regions where the velocity of a massless particle is bigger than the velocity of light as defined in special theory of relativity. It is known that naked singularities are gravitational singularities which are not hidden by an event horizon, and hence they might be observable. If loop quantum gravity is correct, then naked singularities could exist in nature, implying that the cosmic censorship hypothesis does not hold. Numerical calculations and some other arguments have also hinted at this possibility \([11]\). Furthermore, it cannot be excluded that closed time-like curves might arise in the vicinity of naked singularities. For example in \([12]\) the possibility of strong causality violation in the presence of a naked singularity has been analyzed. Moreover, for Kerr black holes with sufficiently high rotational parameter the space-time contains a naked singularity and also closed time-like curves. However, the connection between naked singularity and causality violation is not clear and deserves further studies \([13]\).

Two comments are in order here: we emphasize once again that as long as we have the Equivalence Principle we can find a frame in which locally the velocity of light is one. It is only in the Schwarzschild coordinates that in the case of naked singularities this velocity exceeds the value one. Should the Equivalence Principle be broken (say, by some quantum effects) such a local frame will not exist \([14]\). Secondly, the aspects of the particle motion in metrics with a cosmological constant can be considered as examples of local (as opposed to cosmological) effects of this constant. This is expected since the cosmological constant is part of the Einstein tensor and not of the energy-momentum tensor and will therefore affect, in principle, any measurable quantity (for other local effects see \([15,17]\)).

V. INTERIOR METRICS

For the interior metrics we have a density profile (e.g. a constant density or any other form), say of the general form

\[ \rho(r) = \rho_0(r), \quad r \leq R \quad \text{and} \quad \rho(r) = 0, \quad r \geq R \]

for some \( R \) which denotes the extension of the object, i.e. \( P(R) = 0 \) with \( P \) being the pressure in the perfect fluid energy momentum tensor. We demand the radial velocities of the fluid to be zero which corresponds to a hydrostatic equilibrium. This condition is equivalent to the Tolman-Oppenheimer-Volkov (TOV) equation

\[
\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2} \left[ 1 + \frac{P(r)}{M(r)} \right] \left[ 1 + \frac{4\pi r^3 P(r)}{M(r)} \right]^{-1} \left[ 1 - \frac{2GM(r)}{r} \right]^{-1}
\]

with \( M(r) = 4\pi \int_0^r d\xi \xi^2 \rho(\xi) \). After solving the TOV equation we have all necessary ingredients to write down the metric elements \( A(r) \) and \( B(r) \). They read \([18]\)

\[
A(r) = \exp \left\{ -2G \int_r^\infty \frac{d\xi}{\xi^2} \frac{M(\xi)}{1 - 2GM(\xi)/\xi} \right\}, \quad B(r) = \left[ 1 - \frac{2GM(r)}{r} \right]^{-1}.
\]
FIG. 4: The radial velocity of a massless particle in a Reissner-Nordström-de Sitter metric with a naked singularity. The only (cosmological) horizon is at $\sqrt{3}r_A - r_s$. There exists a region where the velocity exceeds one. For the explanations of $r_0 = R_0$, $r_1 = r_m$ and $r_2 = r_M$ see the text.

For $r > R$ we have the Schwarzschild vacuum metric. This implies that $\rho = P = 0$ for $r \geq R$.

A. Constant density

The case of constant density is illuminating as all the formulae can be written in closed form. Indeed, one obtains

$$A(r) = \frac{1}{4} \left[ 3 \sqrt{1 - \frac{2r_s}{R}} - \sqrt{1 - \frac{2r_s r^2}{R^3}} \right]^2 \quad \text{and} \quad B(r) = \left[ 1 - \frac{2r_s r^2}{R^3} \right]^{-1}.$$  

The radial and angular velocity of a photon (being at the same time the upper bounds for the corresponding velocities for a massive particle) are

$$v_r(r) = \frac{1}{2} \left[ 3 \sqrt{1 - \frac{2r_s}{R}} - \sqrt{1 - \frac{2r_s r^2}{R^3}} \right] \sqrt{1 - \frac{2r_s r^2}{R^3}},$$

$$v_\phi(r) = \frac{1}{2} \left[ 3 \sqrt{1 - \frac{2r_s}{R}} - \sqrt{1 - \frac{2r_s r^2}{R^3}} \right].$$
In Figure 4 we contrast the radial velocity of a photon moving in matter with constant density with the velocity of a photon moving in matter obeying a polytropic equation of state, an issue explained in detail below.

**B. Velocity of a photon moving in polytropic matter**

Let us explore a more realistic situation related to the density profile as given by a non-relativistic hydrostatic equilibrium and a polytropic equation of state. To be specific we use the following approximations

\[
\frac{P}{\rho} \ll 1, \quad \frac{4\pi r^3 P(r)}{M(r)} \ll 1, \quad \frac{2GM(r)}{r} \ll 1
\]

such that the TOV equation reduces to

\[
\frac{dP}{dr} = -G M(r) \rho(r) \frac{r^2}{r^2}.
\]

We model matter by means of the polytropic equation of state, i.e. \( P = K \rho^2 \). This setting gives us the density as

\[
\rho(x) = \rho_0 \Theta(x), \quad \Theta(x) = \sin \frac{x}{x}, \quad x = \sqrt{\frac{4\pi G}{2K}} r.
\]

The radius \( R \) is fixed by the first zero of \( \rho(x) \). This occurs at \( x_1 = \pi \) and we have \( R = \sqrt{2\pi K/4G} \). The transformation linking the coordinates \( x \) and \( r \) can be written as \( x = \pi r/R \). The initial density \( \rho_0 \) is fixed by the total mass \( M \) and the radius \( R \) through \( \rho_0 = \pi M/4R^3 \). The pressure is given through \( P = K \rho^2 \). From \( x = \sqrt{4\pi G/2K} r \) we get (for \( r = R \)) \( \pi = \sqrt{4\pi G/2K} R \) which allows to write \( K \) in terms of \( R \) as follows

\[
K = \frac{2GR^2}{\pi}.
\]

Hence, the pressure can be expressed as

\[
P(r) = \frac{GM^2 \sin^2 \left( \frac{\pi r}{R} \right)}{8\pi R^2 \left( \frac{\pi r}{R} \right)^2}
\]

and in the interval \( 0 \leq r \leq R \) the mass function can be rewritten as

\[
M(r) = \frac{M}{\pi} \left[ \sin \left( \frac{\pi r}{R} \right) - \pi \frac{r}{R} \cos \left( \frac{\pi r}{R} \right) \right].
\]

Concerning \( A(r) \) we have

\[
A(r) = \exp \left\{ -2G \int_r^\infty \frac{du}{u^2} \left[ M(u) + 4\pi u^3 P(u) \right] \right\},
\]

\[
= \exp \left\{ -2G \int_r^R \frac{du}{u^2} \left[ M(u) + 4\pi u^3 P(u) \right] \right\} \cdot \exp \left\{ -2GM \int_R^\infty \frac{dr}{r^2 \left( 1 - \frac{2GM}{r} \right)} \right\}.
\]
Since
\[ \int_{\infty}^{\infty} \frac{dr}{r^2 (1 - \frac{2GM}{r})} = -\frac{1}{2GM} \ln \left(1 - \frac{2GM}{R}\right) \]
we obtain
\[ A(r) = \left(1 - \frac{2GM}{R}\right) \exp \left\{ -2G \int_{r}^{R} \frac{B(u)}{u^2} \left[ M(u) + 4\pi u^3 P(u) \right] \right\}, \]
\[ = \left(1 - \frac{2GM}{R}\right) \exp \left\{ -2G \int_{r}^{R} \frac{B(u)M(u)}{u^2} \left[ 1 + 4\pi u^3 P(u) \right] \right\}. \]
Since we are in the regime (24) the expression for \( A(r) \) simplifies as follows
\[ A(r) \simeq \left(1 - \frac{2GM}{R}\right) \exp \left\{ -2G \int_{r}^{R} \frac{B(u)M(u)}{u^2} \right\}. \]
This allows us to write the integral in the exponential as
\[ I = \int_{r}^{R} \frac{B(u)M(u)}{u^2} = \frac{M}{\pi} \int_{r}^{R} \frac{du}{u^2} \left[ \sin \left(\frac{\pi u}{R}\right) - \frac{\pi u}{R} \cos \left(\frac{\pi u}{R}\right) \right], \]
\[ = \frac{M}{R} \int_{x}^{\pi} \frac{dv}{v^2 - \alpha v (\sin v - \cos v)} \]
with \( \alpha = 2r_s/R \). Restricting ourselves to the case \( \alpha \ll 1 \) and since the function \( \sin v - v \cos v \) is of bounded variation we can approximate the denominator with \( v^2 \) to arrive at
\[ A(r) = (1 - \alpha) \exp \left\{ -\frac{\alpha \sin x}{x} \right\}. \]
Collecting our results the radial velocity of the photon becomes
\[ v_r(x) = \sqrt{(1 - \alpha) \left[ 1 - \frac{\alpha}{\pi} (\sin x - x \cos x) \right] \exp \left\{ -\frac{\alpha \sin x}{x} \right\}}. \]
This velocity is plotted in Figure 4. It is a priori not obvious that the case of radial photon velocity displays more variation than the case of constant velocity. Note that these variations are bounded by a local minimum and maximum and the total range of the variation is close to one.

VI. CONCLUSIONS

We derived a simple expression, equation (24), for the photon radial velocity in dependence of the distance and as functional of metric elements in a general static spherically symmetric metric. Equation (24) allows a quick analytical insight into the behaviour of the photon radial velocity without the necessity of solving the geodesic equation of motion. We have applied this general result to different metrics known from the literature. The radial velocity of massless particles shows some interesting features:
FIG. 5: The radial photon’s velocity in interior metrics: the upper figure is for a constant density whereas the lower one corresponds to polytropic configuration with polytropic index 2. In both cases $\alpha = 2r_s/R = .01$.

1. In the Schwarzschild case the effect on earth changes the velocity of light to $1 - 1.39 \times 10^{-9}$ which is actually within the range of the error bars of directly measured velocity of light. Even though this makes the effect unmeasurable in laboratory, it is good to recall that in the past the error bars were attributed to a fuzzy definition of the meter.

2. The Schwarzschild-de Sitter case is particularly interesting as it might be the actual exterior space-time of spherical astrophysical objects since a positive cosmological constant offers a simple explanation of the acceleration of the Universe. In such a case the radial velocity of a photon reaches a maximum value at an astrophysical distance after which it decreases in contrast to the Schwarzschild case where the velocity approaches one.

3. If the central singularity of a Reissner-Nordström or Reissner-Nordström-de Sitter black hole is not protected by a horizon, we encounter radial velocities bigger than one. From the point of view of Schwarzschild coordinates this is what makes the naked singularities so special. We give a full algebraic characterization of Reissner-Nordström-de Sitter metric and the radial velocity of a massless particle moving in this metric. For instance we give conditions when the radial velocity has a local minimum and maximum provided the charge is smaller than a critical value. We use this to define the extreme case. We note that if the naked singularity is encountered in metrics motivated by noncommutative geometry, the radial velocity of a photon never exceeds one. We find this a curious fact which deserves attention.

4. For interior metrics we contrasted the radial velocity of a photon moving in matter of constant density with a polytropic case. Whereas the velocity function of the constant density case is monotonically decreasing, the polytropic case displays a local minimum
and maximum. We mention that in both cases the variation of the radial velocity (as compared to one) is rather small. As a future prospect we mention that it would be of interest to study the radial velocity of a photon (in Schwarzschild coordinates) moving in a spherically symmetric Dark Matter Halo.
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