Decay of Quantum Correlations on a Lattice by Heat Kernel Methods

Laurent Amour, Claudy Cancelier, Pierre Lévy-Bruhl, and Jean Nourrigat

Abstract. We prove some estimates of the correlation of two local observables in quantum lattice models at high temperature. For that, we describe the heat kernel of the Hamiltonian for finite subsets of the lattice, which may tend to the whole lattice. (In other words, we study the heat kernel of a Schrödinger in large dimension.)

1. Introduction

In the last twenty years, many works were devoted to the estimates, or asymptotics, of the correlation of two local observables, (or Ursell functions of $n$ local observables), for classical lattice models, at high, or at low temperature. In these works, an interaction is a given function $V_{\Lambda}$ on $(\mathbb{R}^p)^\Lambda$ for each finite subset $\Lambda$ of a lattice $L$, (where $p$ is the number of degrees of freedom at each site). Then, for each $\beta > 0$ (the inverse of the temperature), the mean value of the local observable $f$, (i.e., of a function $f$ on $(\mathbb{R}^p)^\Lambda$), is:

$$E_{\Lambda,\beta}(f) = Z_{\Lambda}(\beta)^{-1} \int_{(\mathbb{R}^p)^\Lambda} e^{-\beta V_{\Lambda}(x)} f(x) dx$$

The bilinear analogue (correlation of two local observables), or the multilinear analogue (Ursell function of $n$ local observables), are defined in a standard way, (see, for example, D. Ruelle [23], B. Simon [24] or R. A. Minlos [18]). We say that $f$ is supported in a subset $E$ of $\Lambda$ if $f$ depends only on the variables $x_{\lambda}$ corresponding to sites $\lambda$ which are in $E$. A classical problem is to estimate the decay of the correlation of two local observables $f$ and $g$ with disjoint supports $E$ and $F$ when the distance of $E$ and $F$ tends to $+\infty$. When $\beta$ is small enough (high temperature), this is a classical result of L. Gross [10]. For the study of Ursell functions at high temperature in classical mechanics, (tree decay), see, for instance, Bertini, Cirillo and Oliveri [8]. At low temperature, the problem is more

$$Z_{\Lambda}(\beta) = \int_{(\mathbb{R}^p)^\Lambda} e^{-\beta V_{\Lambda}(x)} dx.$$
complicated and more hypotheses are needed: see the works of Helffer, Sjöstrand, V. Bach, T. Jeccko, J.S. Möller, O. Matte, . . . [6,7,11] to [17,26,27].

Like R. A. Minlos, A. Verbeure and V. A. Zagrebnov in [20], we are interested in similar results in quantum statistical mechanics. Let us consider a quantum d-dimensional lattice of particles, each of them moving in \( \mathbb{R}^p \). To each finite subset \( \Lambda \) of the lattice \( L = \mathbb{Z}^d \), we shall define below, with more details, a potential \( V_\Lambda \). We denote by \( H_\Lambda \) the following differential operator in \( (\mathbb{R}^p)^\Lambda \), depending on the Planck’s constant \( h \):

\[
H_\Lambda = -\frac{h^2}{2} \sum_{\lambda \in \Lambda} \Delta_x + V_\Lambda(x)
\]  

(1.1)

where \( x = (x_\lambda)_{\lambda \in \Lambda} \) denotes the variable of \( (\mathbb{R}^p)^\Lambda \), each variable \( x_\lambda \) being in \( \mathbb{R}^p \).

With suitable hypotheses (see below), the exponential \( e^{-\beta H_\Lambda} \) will be defined for \( \beta > 0 \), and will be of trace class. Then, a local observable is no more a function, but a bounded operator \( A \) on the Hilbert space \( \mathcal{H}_\Lambda = L^2((\mathbb{R}^p)^\Lambda) \). The mean value \( E_{\Lambda,\beta}(A) \) is classically defined by:

\[
E_{\Lambda,\beta}(A) = Z_{\Lambda}(\beta)^{-1} \text{Tr}(e^{-\beta H_\Lambda}A), \quad Z_{\Lambda}(\beta) = \text{Tr}(e^{-\beta H_\Lambda}).
\]  

(1.2)

If \( E \) is a subset of \( \Lambda \), we say that \( A \) is supported in \( E \) if \( A \) can be seen as an operator on the Hilbert space \( \mathcal{H}_E \). If \( A \) and \( B \) are two local observables, supported in two disjoint subsets \( E_1 \) and \( E_2 \) of \( \Lambda \), a natural definition for the correlation is:

\[
\text{Cov}_{\Lambda,\beta}(A, B) = E_{\Lambda,\beta}(AB) - E_{\Lambda,\beta}(A)E_{\Lambda,\beta}(B).
\]  

(1.3)

One of the goals of this work, like in [20], is to give an analogue of the result of L. Gross in this situation, and to estimate the decay of \( \text{Cov}_{\Lambda,\beta}(A, B) \) when the distance of the supports \( E_1 \) and \( E_2 \) of \( A \) and \( B \) tends to +\( \infty \) (see Theorem 1.3 below), assuming that \( \beta \) is small enough. Such an estimate is obtained in [20] under suitable assumptions on the interaction (see later), using Feynman integral techniques. Our aim is to give, with a different hypothesis on the interaction, and by a non probabilistic method, another estimate.

Let us give more details on the interaction, i.e., on the family of functions \((V_\Lambda)_{\Lambda \subset \mathbb{Z}^d}\). We consider a function \( A \in C^\infty(\mathbb{R}^p, \mathbb{R}) \) and, for each pair of sites \( \lambda \) and \( \mu \) in the lattice, a function \( B_{\lambda,\mu} \in C^\infty(\mathbb{R}^{2p}, \mathbb{R}) \). For each finite subset \( \Lambda \) of \( \mathbb{Z}^d \), we denote by \( V_\Lambda \) the following potential in \( (\mathbb{R}^p)^\Lambda \):

\[
V_\Lambda(x) = \sum_{\lambda \in \Lambda} A(x_\lambda) + \sum_{\lambda,\mu \in \Lambda, \lambda \neq \mu} B_{\lambda,\mu}(x_\lambda, x_\mu) \quad x = (x_\lambda)_{\lambda \in \Lambda}.
\]  

(1.4)

For the sake of simplicity, we assume that \( B_{\mu,\lambda} = B_{\lambda,\mu} \). When translation invariance is needed, \( B_{\lambda,\mu} \) will depend only on \( \lambda - \mu \). We shall assume that \( B_{\lambda,\mu} \) is small when \( |\lambda - \mu| \) is large. More precisely, we assume that, for some \( \varepsilon \in ]0, 1[ \), the following hypothesis is satisfied:
(H_ε) For each \( \alpha \geq 0 \) and \( \beta \geq 0 \), there exists \( C_{\alpha, \beta}(\varepsilon) > 0 \) such that

\[
\sup_{\lambda \in \mathbb{Z}^d} \sum_{\mu \in \mathbb{Z}^d} \frac{\| \nabla_{x,\lambda} \nabla_{x,\mu} B_{\lambda,\mu} \|}{\varepsilon^{1 - |\lambda - \mu|}} \leq C_{\alpha, \beta}(\varepsilon). \tag{1.5}
\]

The function \( A \) is bounded from below. All the derivatives of order \( \geq 1 \) of \( A \) are bounded. For each \( m > 0 \) and \( \beta > 0 \), we have:

\[
\sup_{x \in \mathbb{R}^p} (1 + |x|^m) e^{-\beta A(x)} = C_m(\beta) < +\infty. \tag{1.6}
\]

The hypotheses (1.4) and (H_ε) are the same as in Sjöstrand [25]. With these hypotheses, the one site potential has less than quadratic growth at infinity. For the case of polynomial growth at infinity for \( A \), with quadratic nearest neighbor interaction, or for the light mass asymptotic, see [1,19–21].

Under the above hypotheses, and suitable conditions on \( \beta \) and \( h \), our main result is an upper bound for the correlation of two local observables supported in two disjoint sets \( E_1 \) and \( E_2 \). If (H_ε) is satisfied and if \( \varepsilon < \delta < 1 \), we shall see in Theorem 1.3 that this correlation is \( O(\delta \text{dist}(E_1, E_2)) \). If \( E_1 \) and \( E_2 \) are single points, we don’t need the hypothesis (H_ε) for all derivatives, but for a finite number. The same remark is valid if the two local observables are multiplications by bounded functions.

Our analysis relies on two main ingredients. First, (in Theorem 1.1), we give a description and some precise estimates for the integral kernel of the operator \( e^{-\beta H_\Lambda} \). These estimates are uniform with respect to \( |\Lambda| \), which may be large. We also estimate all the derivatives of the function \( \psi_\Lambda \) appearing there, using suitable norms for the higher differentials, like in [25]. Second, we use a cluster type decomposition of the heat kernel (Theorem 1.2 and its consequences in Section 7).

Throughout this paper, we denote by \( \| \| \) the \( L^\infty \) norm of a function, and the parameter \( \beta \) will be denoted by \( t \), since we shall use evolution equations. If (H_ε) is satisfied for some \( \varepsilon \in ]0,1[ \), there exist \( M_1(\varepsilon) \) and \( M_2(\varepsilon) \) (independent of \( \Lambda \)), such that, for each finite set \( \Lambda \), and for each point \( \lambda \in \Lambda \):

\[
\sup_{\lambda \in \Lambda} \| \nabla_{x,\lambda} V_\Lambda \| \leq M_1(\varepsilon) \sup_{\lambda \in \Lambda} \sum_{\mu \in \Lambda} \frac{\| \nabla_{x,\lambda} \nabla_{x,\mu} V_\Lambda \|}{\varepsilon^{1 - |\lambda - \mu|}} \leq M_2(\varepsilon). \tag{1.7}
\]

All the results of this work, excepted the bounds for the first order derivatives, will be valid under the following condition:

\[
h t = h \beta \leq T_0 \quad T_0 = M_2(\varepsilon)^{-1/2}. \tag{1.8}
\]

We denote by \( \nabla_\Lambda \) the differential, with respect to \((x_\lambda, y_\lambda) \in \mathbb{R}^{2p} \ (\lambda \in \Lambda)\) of a \( C^\infty \) function \( f \) on \((\mathbb{R}^{2p})^\Lambda\). The norm of \( \nabla_\Lambda f(x) \) is its norm in \((\mathbb{R}^{2p})^\Lambda\). We denote by \( \text{diam}(A) \) the diameter of a subset \( A \) of \( \mathbb{Z}^d \). The norm in \( \mathbb{Z}^d \) will be the \( \ell^\infty \) norm. The hypothesis (1.6) is not needed for Theorems 1.1, devoted to the description of the integral kernel of \( e^{-tH_\Lambda} \).
Theorem 1.1. Under the previous hypotheses, the integral kernel $U_{\Lambda}(x,y,t)$ of $e^{-tH_{\Lambda}}$ can be written in the form

$$U_{\Lambda}(x,y,t) = (2\pi \hbar)^{-p|\Lambda|/2} e^{-\frac{|x-y|^2}{2\hbar}} e^{-\psi_{\Lambda}(x,y,t)},$$

where $\psi_{\Lambda}$ is a $C^{\infty}$ function in $(\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda} \times [0, +\infty]$, depending on the parameter $\hbar > 0$. Moreover, if $(H_{\varepsilon})$ is satisfied, for each finite subset $\Lambda$ of $\mathbb{Z}^d$, we have, for all $t > 0$

$$\sup_{\lambda \in \Lambda} \|\nabla_{\lambda} \psi_{\Lambda}(\cdot, t)\| \leq t M_1(\varepsilon).$$

(1.10)

For each $m \geq 2$, for all points $\lambda^{(1)}, \ldots, \lambda^{(m-1)}$ in $\Lambda$, we can write, if $\hbar t \leq T_0$ (the constant of (1.4)):

$$\sup_{(x,y) \in (\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda}} \sum_{\mu \in \Lambda} |\nabla_{\lambda^{(1)}} \cdots \nabla_{\lambda^{(m-1)}} \nabla_{\mu} \psi_{\Lambda}(x,y,t)| e^{diam(\lambda^{(1)}, \ldots, \lambda^{(m-1)}, \mu)} \leq t K_m(\varepsilon),$$

(1.11)

where $K_m(\varepsilon)$ is independent of $\Lambda$.

A result of Sjöstrand [25] asserts that, near the diagonal, the integral kernel $U_{\Lambda}$ can be written in the form (1.9), and gives estimates for an approximation modulo $O(h^{\infty})$ of the function $\psi_{\Lambda}$. Here, we study the function $\psi_{\Lambda}$ itself, not an approximation, and the estimation is global.

Now, we prepare the cluster type decomposition of the heat kernel. The first step will be a decomposition of $\psi_{\Lambda}$. It will be used in Section 7 to construct a Mayer decomposition of the heat kernel. In (1.4), the potential is written as a sum, where each term corresponds to a point $\lambda$ or to a couple of points $(\lambda, \mu)$. The function $\psi_{\Lambda}$ will have a somewhat similar decomposition, but the terms $T_{Q} \psi_{\Lambda}$ of the sum will be indexed by the boxes $Q \subseteq \Lambda$. When it is restricted to the diagonal, the function associated to the box $Q$ will depend only on the variables $x_{\lambda}$ ($\lambda \in Q$). The function will decrease almost like $e^{diam(Q)}$ when $Q$ is large.

The next theorem makes more precise these ideas. A box of $\mathbb{Z}^d$ is a set of the following form:

$$\Lambda = \prod_{j=1}^d [a_j, b_j]$$

(1.12)

where $a_j$ and $b_j$ are in $\mathbb{Z}$ ($a_j \leq b_j$).

Theorem 1.2. We can define, for each box $\Lambda$ of $\mathbb{Z}^d$, and for each box $Q \subseteq \Lambda$, $(Q$ may be a single point), a function $(T_Q \psi_{\Lambda})(x,y,t)$ (where $\psi_{\Lambda}$ is the function of Theorem 1.1), such that:

1. The function $(T_Q \psi_{\Lambda})(x,y,t)$ is $C^{\infty}$ and depends only on the variables $x_{\lambda}$ and $y_{\lambda}$ such that $\lambda \in Q$, and on the variables $x_{\lambda} - y_{\lambda}$ such that $\lambda \notin Q$ (restricted to the diagonal, this function is supported in $Q$).

2. We have:

$$\psi_{\Lambda}(x,y,t) = \psi_{\Lambda}(0,y - x,t) + \sum_{Q \subseteq \Lambda} (T_Q \psi_{\Lambda})(x,y,t)$$

(1.13)

where the sum is taken over all boxes $Q$ contained in $\Lambda$, including the points.
3. If \( (H_2) \) is satisfied \((0 < \varepsilon < 1)\), if \( ht \leq T_0 \) (defined in (1.8) and (1.7)), for each integer \( m \geq 1 \), for each points \( \lambda^{(1)}, \ldots, \lambda^{(m)} \) in \( \Lambda \), we have, for some constant \( K_m(\varepsilon) > 0 \) independent of \( \Lambda \):

\[
\| \nabla \lambda^{(1)} \ldots \nabla \lambda^{(m)} (T_{Q, \psi_{\Lambda}})(., t, h) \| \leq K_m(\varepsilon) t e^{\text{diam}(Q \cup \{ \lambda^{(1)}, \ldots, \lambda^{(m)} \})} (Q)^{2d}.
\]

where\( (Q) = 1 + \text{diam}(Q) \).

\textbf{If} \( m = 0 \), this result is valid for boxes \( Q \) not reduced to single points.

4. If \( m = 0 \) and \( Q \) is a single point \( \lambda \), we can write:

\[
|T_{\lambda} \psi_{\Lambda}(x, y, t) - t \tilde{A}(x_{\lambda}, y_{\lambda}) + t \tilde{A}(0, y_{\lambda} - x_{\lambda})| \leq K(\varepsilon)(t + h^2 t^2). \tag{1.15}
\]

where \( \tilde{A} \) is the function appearing in (1.4), and:

\[
\tilde{A}(x_{\lambda}, y_{\lambda}) = \int_{0}^{1} A(y_{\lambda} + \theta(x_{\lambda} - y_{\lambda})) d\theta. \tag{1.16}
\]

The decomposition of Theorem 1.2 will be used in Section 7 to construct an analogue of the Mayer decomposition (see B. Simon [24]), and the later will be used in the study of correlations.

Let us denote by \( K_{\text{op}, \text{op}}(E_1, E_2, t, h) \), \( (E_1 \text{ and } E_2 \text{ are disjoint subsets of } \mathbb{Z}^d) \), the smallest positive constant such that, for each \( A \in \mathcal{L}(H_{E_1}) \) and \( B \in \mathcal{L}(H_{E_2}) \), for each box \( \Lambda \) containing \( E_1 \) and \( E_2 \), we have:

\[
|\text{COV}_{\Lambda, \tau}(A, B)| \leq K_{\text{op}, \text{op}}(E_1, E_2, t, h) \| A \| \| B \|. \tag{1.17}
\]

Let \( K_{\text{fc}, \text{fc}}(E_1, E_2, t, h) \) be the smallest positive constant such that, for every continuous, bounded functions \( f \) and \( g \), supported in \( E_1 \) and \( E_2 \), and for each box \( \Lambda \) containing \( E_1 \) and \( E_2 \), we have:

\[
|\text{COV}_{\Lambda, \tau}(M_f, M_g)| \leq K_{\text{fc}, \text{fc}}(E_1, E_2, t, h) \| f \| \| g \|. \tag{1.18}
\]

We define in the same way \( K_{\text{fc}, \text{op}}(E_1, E_2, t, h) \) and \( K_{\text{op}, \text{fc}}(E_1, E_2, t, h) \).

**Theorem 1.3.** Let \( (V_{\lambda})_{\lambda \in \mathbb{Z}^d} \) be an interaction satisfying \( (H_2) \), \((0 < \varepsilon < 1)\). Then, for each \( \delta \) such that \( 0 < \varepsilon < \delta < 1 \), there exists \( t_1(\varepsilon, \delta) \) and functions \( M(1, t, h, \varepsilon, \delta) \) and \( N(1, \varepsilon, \delta) \) with the following property. If \( ht \leq T_0 \) (defined in (1.8)), and if \( t \leq t_1(\varepsilon, \delta) \), for every finite disjoint sets \( E_1 \) and \( E_2 \), we have, for some functions \( M_1, M_2, M_3, M_4 \) depending only on the written variables:

a) \( K_{\text{op}, \text{op}}(E_1, E_2, t, h) \leq M_1(|E_1 \cup E_2|, t, h, \varepsilon, \delta) \delta^{\text{dist}(E_1, E_2)} \).

b) \( K_{\text{fc}, \text{fc}}(E_1, E_2, t, h) \leq t \inf(|E_1|, |E_2|) M_2(\varepsilon, \delta) \delta^{\text{dist}(E_1, E_2)} \).

c) \( K_{\text{op}, \text{fc}}(E_1, E_2, t, h) \leq M_3(|E_1|, t, h, \varepsilon, \delta) \delta^{\text{dist}(E_1, E_2)} \).

\text{If} \( f \) has bounded derivatives, and \( g \) is bounded, we can write:

\[
|\text{COV}_{\Lambda, \tau}(M_f, M_g)| \leq \| \nabla f \|_{\infty} \| g \| M_4(|E_1|, t, h, \varepsilon, \delta) \delta^{\text{dist}(E_1, E_2)}.
\]

The functions denoted by \( M \), as functions of \( t \) and \( h \), for fixed \( \varepsilon \) and \( \delta \), are bounded on each compact of the set \( \{(t, h), h > 0, 0 < t < t_1(\varepsilon, \delta), ht < T_0\} \). The constant
$t_1(\varepsilon, \delta)$, limiting the validity of the result, is independent of the three sets $E_1$, $E_2$ and $\Lambda$.

Now, we turn our attention to thermodynamic limits, and to the rate of convergence to such limits. We say that a local observable $A \in \mathcal{L}(H_\Lambda)$ is supported in a subset $Q \subset \Lambda$ if $A$ can be seen also as an element of $\mathcal{L}(H_Q)$. The proof of the next Theorems 1.4 and 1.5 relies on the estimates of correlations, (Theorem 1.3), and also, directly on the decomposition of $\psi_\Lambda$ (Theorem 1.2).

**Theorem 1.4.** If the interaction satisfies $(H_\varepsilon)$, if $ht \leq T_0$ (defined in (1.8)), and if $t$ is small enough, for each local observable $A$, the following thermodynamic limit exists:

$$\omega_t(A) = \lim_{n \to +\infty} E_{\Lambda_n,t}(A) \quad \Lambda_n = \{-n, \ldots, n\}^d.$$  \hspace{1cm} (1.19)

Moreover, if $\varepsilon < \delta < 1$, there exists $t_1(\varepsilon, \delta)$ and a function $K(h,t,\varepsilon,\delta,N)$ such that, if $ht \leq T_0$, if $t \leq t_1(\varepsilon, \delta)$ and if $\Lambda_n$ contains the support of $A$,

$$|\omega_t(A) - E_{\Lambda_n,t}(A)| \leq K(h,t,\varepsilon,\delta,|\text{supp}(A)|) \|A\| \delta^{\text{dist}(\text{supp}(A),\Lambda_n)}.$$ \hspace{1cm} (1.20)

Theorems 1.3 and 1.4 can be applied to prove some properties of a state of the $C^*$ algebra $\mathcal{A}$ associated to the family of Hilbert spaces $H_\Lambda$. Let us recall (see B. Simon [24], Section II.1, or Bratteli–Robinson [9]), that, if $\Lambda_1 \subset \Lambda_2$, we have a natural identification of $\mathcal{L}(H_{\Lambda_1})$ as a subspace of $\mathcal{L}(H_{\Lambda_2})$, and $\mathcal{A}$ is the closure of the union of (equivalence classes of) all the $\mathcal{L}(H_{\Lambda})$. Then, for each $h$ and $t$ such that Theorem 1.4 can be applied, the limit in (1.19) defines a state on $\mathcal{A}$, still denoted by $\omega_t$. Theorem 1.3 proves that this state has the mixing property:

$$\lim_{|u| \to \infty} \left[ \omega_t(A \circ \tau_u B) - \omega_t(A)\omega_t(B) \right] = 0 \quad \forall A, B \in \mathcal{A}$$

if $ht < T_0$ and if $t$ is small enough. Here $\tau_u$ is the natural translation by a vector $u \in \mathbb{Z}^d$. For this application, it is useful that the condition of validity of Theorem 1.3 does not depend on the number of elements of the supports. In this application to the mixing property, invariance by translation is needed, and we assume that $B_{\lambda,\mu}$ depends only on $\lambda - \mu$.

In the second application, we consider the mean value, not of a local observable, but of the mean energy per site.

**Theorem 1.5.** If $(H_\varepsilon)$ is satisfied, and if $ht \leq T_0$, the following limit exists:

$$E(t) = \lim_{n \to +\infty} \frac{1}{|\Lambda_n|} X_{\Lambda_n}(t), \quad X_\Lambda(t) = \frac{\partial}{\partial t} \ln Z_\Lambda(t),$$ \hspace{1cm} (1.21)

where $Z_\Lambda(t)$ is defined in (1.2) and $\Lambda_n$ in (1.19). We can write:

$$\left| E(t) - \frac{1}{|\Lambda_n|} X_{\Lambda_n}(t) \right| \leq \frac{K(t,h)}{n}. \hspace{1cm} (1.22)$$

The constant $K(t,h)$ is bounded on each compact set of $\{(h,t), \, h > 0, t > 0, \, ht < T_0\}$. 

By the last statement, \( E(t) \) is a continuous function of \( t \) in the domain in which the theorem is applicable: in other words, there is no first order phase transition, if \( t = \beta \) is small enough.

Theorem 1.1 is proved in Section 2. In Section 3, we introduce the family of operators \( T_Q \), which can be defined for any function on \((\mathbb{R}^p)^\Lambda\), but are especially applied to the function \( \psi_{\Lambda} \) for Theorem 1.2. The estimations of \( T_Q \psi_{\Lambda} \) are proved in Section 4. In Sections 5 and 6, we prove some technical lemmas, which are used when \( \Lambda \) is the union of two disjoint subsets \( \Lambda_1 \) and \( \Lambda_2 \). The lemmas of Section 5 are used for the proof of Theorem 1.3, and those of Section 6 for Theorems 1.4 and 1.5. In Sections 7, we introduce the Mayer type decomposition, using Theorem 1.2, and then we prove the point b) of Theorem 1.3 (multiplicative case). Section 8 is devoted to the end of the proof of Theorem 1.3, and Section 9 to the proof of Theorems 1.4 and 1.5.

This article is an improved version of an earlier unpublished manuscript [3].

2. Proof of Theorem 1.1

The heat kernel \( U_{\Lambda}(x, y, t) \) satisfies

\[
\frac{\partial U_{\Lambda}}{\partial t} - \frac{h^2}{2} \Delta x U_{\Lambda} + V_{\Lambda}(x) U_{\Lambda} = 0 \quad \text{for } t > 0.
\]

Therefore, if \( U_{\Lambda} \) is written as in (1.9), the function \( \psi_{\Lambda} \) in \((\mathbb{R}^p)^\Lambda\) appearing in this expression must satisfy the Cauchy problem:

\[
\frac{\partial \psi_{\Lambda}}{\partial t} + \frac{x - y}{t} \cdot \nabla x \psi_{\Lambda} - \frac{h^2}{2} \Delta x \psi_{\Lambda} = V_{\Lambda}(x) - \frac{h^2}{2} |\nabla x \psi_{\Lambda}|^2 \tag{2.1}
\]

\[
\psi_{\Lambda}(x, y, 0, h) = 0 \tag{2.2}
\]

This section is devoted to the study of this Cauchy problem. We shall use a maximum principle for the linearized equation of (2.1), and more generally, for operators \( L_a \) in \((\mathbb{R}^p)^\Lambda \times [0, T] \) \((\Lambda \in \mathbb{Z}^d, T > 0)\) of the following form:

\[
(L_a u) = \frac{\partial u}{\partial t} + \frac{x - y}{t} \cdot \nabla u - \frac{h^2}{2} \Delta u + \sum_{\mu \in \Lambda} (a_{\mu}(x, t) \cdot \nabla x_{\mu} u), \tag{2.3}
\]

where \( a = (a_{\lambda})_{\lambda \in \Lambda} \) is a family of continuous and bounded functions in \((\mathbb{R}^p)^\Lambda \times [0, T] \), and \( y \in (\mathbb{R}^p)^\Lambda \). Since there is a drift term with unbounded coefficients, and since this maximum principle will be used again, it may be useful to give a precise statement:

**Proposition 2.1.** Let \( y \in (\mathbb{R}^p)^\Lambda \), let \( a = (a_{\lambda})_{\lambda \in \Lambda} \) be a family of continuous bounded functions \( a_{\lambda}(x, t) \) \((\lambda \in \Lambda)\) in \((\mathbb{R}^p)^\Lambda \times [0, T] \) \((T > 0)\), taking values in \( \mathbb{R}^p \). Let \( u \) be a function in \( C((\mathbb{R}^p)^\Lambda \times [0, T]) \cap C^2((\mathbb{R}^p)^\Lambda \times [0, T]) \) such that \( u \) and \( \nabla x_{\mu} u \) \((\lambda \in \Lambda)\) are bounded in \((\mathbb{R}^p)^\Lambda \times [0, T] \) and \( u(x, 0) = 0 \). Assume that the function \( f = L_a u \), defined by (2.3), \((\text{where } h > 0)\), is bounded. Then we have, for each \( t_0 \) and \( t \) \((0 \leq t_0 \leq t \leq T)\),

\[
\|u(., t)\| \leq \|u(., t_0)\| + \int_{t_0}^{t} \|L_a u(., s)\| \, ds.
\]
Proof. Let $\chi \in C^\infty((\mathbb{R}^p)^\Lambda)$ be a real-valued function with $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$. For $R \geq 0$, set $\chi_R(x, y, t) = \chi\left(\frac{|x-y|}{R}\right)$, $x \in (\mathbb{R}^p)^\Lambda$, $t > 0$.
If $0 < t_0 < t$, $y$ fixed, and $R > 0$, the standard maximum principle, applied in $(\mathbb{R}^p)^\Lambda \times [t_0, t]$ with bounded coefficients in the first order terms, gives
$$\|\chi_R u(. , t)\| \leq (\|\chi_R u(. , t_0)\| + \int_{t_0}^{t} \|L_a(\chi_R u)(. , s)\| \, ds).$$
An explicit computation of $L_a(\chi_R u)(. , s)$ when $\chi_R$ is of the previous form shows that, when $R \to +\infty$, $L_a(\chi_R u)(. , s) \to L_a u$. The proposition follows (we may also let $t_0 \to 0$).

Now, we give a result of global existence for the solution $\psi_\Lambda$ of the Cauchy problem (2.1), (2.2), with global bounds for all derivatives of this function. However, at this step, all the bounds, excepted for the first order derivatives, may still depend on the number of sites $|\Lambda|$.

Proposition 2.2. Under the hypotheses of Theorem 1.1, there exists a unique global classical solution $\psi_\Lambda(x, y, t)$ to (2.1), (2.2), which is $C^\infty$ on $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda \times [0, +\infty]$. Moreover, we have the estimations (1.10) for the first order derivatives, with $M_1(\varepsilon)$ defined in (1.7). For each $T > 0$ and $h > 0$, all the derivatives of order $\geq 2$ with respect to $x$ and $y$ of $\psi_\Lambda$ are bounded on $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda \times [0, T]$ (with bounds which may depend, at this step, on $\Lambda$, $T$ and $h$).

The proof of this proposition is a modification, of the proof of a similar result (see [2]) concerning a non linear heat equation. Let us only explain the differences, which are due to the term $\frac{x-y}{t}$.

Step 1. In order to solve (2.1), (2.2) by a fixed point theorem, we use the explicit solution of the following Cauchy problem
$$\begin{cases}
\frac{\partial u}{\partial t} + \frac{x-y}{t} \nabla_x u - \frac{h^2}{2} \Delta_x u = f(x, t) & (t > t_0), \\
u(x, y, t_0) = \Phi(x)
\end{cases}$$
(2.4)
The solution is given by:
$$u(x, y, t) = [G_h(y, t_0, t)\Phi](x) + \int_{t_0}^{t} [G_h(y, s, t)f(., s)](x) \, ds$$
where $G_h(y, s, t) (0 < s < t)$ denotes the operator with integral kernel $(x, x') \to G_h(x, x', y, s, t)$, where:
$$G_h(x, x', y, s, t) = \left(\frac{a(s, t)}{2\pi h^2}\right)^{\frac{|\Lambda|}{2}} e^{-\frac{a(s, t)|x-x'|^2}{2h^2}}$$
$$a(s, t) = \frac{t}{s(t-s)} \quad m(x, y, s, t) = \left(1 - \frac{s}{t}\right)y + \frac{s}{t}x.$$
We shall use the following property of this kernel, where $C > 0$ is independent of all the parameters:

$$
\int_{(\mathbb{R}^p)^\Lambda} |G_h^{\lambda}(x, x', y, s, t)| \, dx' \leq \frac{C}{h^{\sqrt{s}/(t-s)}} \tag{2.5}
$$

where $G_h^{\lambda} = \nabla_{x_\lambda} G_h$. The similar integral, with $G_h$ instead of $G_h^{\lambda}$, is equal to 1.

**Step 2. Local solution (in time) of a Cauchy problem for (2.1).** Now, we want to find a solution of (2.1) in an interval $[t_0, t_0 + \tau]$ ($t_0 \geq 0$) with, instead of (2.2), an initial data at time $t_0$:

$$
\psi_\lambda(x, y, t_0) = \varphi(x) \tag{2.6}
$$

with a given $\varphi$, with bounded derivatives, (with $\varphi = 0$ if $t_0 = 0$). For any $t_0$, and $\varphi$, we have to control the length $\tau$ of the interval in which we can solve the Cauchy problem. Let us prove that this length can be taken on the following form:

$$
\tau = (1 + t_0^2)^{-1}F \left( \frac{1}{t_0^{1/2}} |||\nabla_\varphi||| \right) \quad |||\nabla_\varphi||| = \sup_{\lambda \in \Lambda} |||\nabla_{x_\lambda} \varphi|||_{\infty} \tag{2.7}
$$

where $F$ depends also on $h$, $|\Lambda|$ and on the parameters appearing like $M_1(\varepsilon)$ in the hypothesis ($H_\varepsilon$). Here $y$ is a parameter, and all the estimations are uniform with respect to $y$ and we shall omit it. If $\psi_\lambda$ satisfies (2.1) and (2.6) in $[t_0, t_0 + \tau]$, the function $U_\lambda(x, t) = t\nabla_{x_\lambda} \psi_\lambda$ satisfies (2.4), with the following RHS and initial data:

$$
f_\lambda = t\nabla_{x_\lambda} \left[ V_\lambda(x) - \frac{h^2}{2} |\nabla_{x_\lambda} \psi_\lambda|^2 \right], \quad \Phi_\lambda = t_0 \nabla_{x_\lambda} \varphi.
$$

We can reduce these equations to an integral system of equations, using the above operator $G_h(y, s, t)$ and the operator $G_h^{(\lambda)}(y, s, t)$ with integral kernel $G_h^{\lambda}$. We remark that: $G_h^{(\lambda)}(y, s, t) = \frac{1}{2} G_h(y, s, t) \nabla_{x_\lambda}$. The two families of functions $U = (U_\lambda)_{\lambda \in \Lambda}$ and $\Phi = (\Phi_\lambda)_{\lambda \in \Lambda}$ defined above must satisfy:

$$
U = S\Phi + TU \tag{2.8}
$$

where $S\Phi$ and $TU$ are the families of functions defined by:

$$
(S\Phi)_{\lambda}(\cdot, t) = G_h(y, t_0, t)\Phi_\lambda + \int_{t_0}^{t} s G_h(y, s, t) (\nabla_{x_\lambda} V_\lambda) ds
$$

$$
(TU)_{\lambda}(\cdot, t) = -\frac{h^2}{2} \int_{t_0}^{t} \int_{s}^{t} \frac{f_{\lambda}}{s^2} G_h^{(\lambda)}(y, s, t) \sum_{\mu \in \Lambda} |U_\mu(\cdot, s)|^2 ds.
$$

In order to solve these integral equations, we denote, for each interval $I = [t_0, t_0 + \tau]$, by $E_1(I)$ the space of families of functions $U = (U_\lambda)_{\lambda \in \Lambda}$, continuous on $(\mathbb{R}^p)^\Lambda \times I$, such that:

$$
||U||_{1, I} = \sup_{t \in I} t^{-2} |||U(\cdot, t)||| < \infty, \quad |||U(\cdot, t)||| = \sup_{\lambda \in \Lambda} |||U_\lambda(\cdot, t)|||_{\infty},
$$
and we denote by $B_1(I, r)$ the closed ball in $E_1(I)$ with radius $r$ and centered at the origin. Now, we shall choose $\tau$ and $r$. First, we choose

$\tau > 0, of the form (2.7), such that the RHS of (2.8) is a contractive map in $B_1([t_0, t_0 + \tau], r)$. Then, choosing $\tau$ possibly smaller, but still depending on the same parameters, we see that $U_{\Lambda}$ has bounded first order derivatives, and that the function $\psi_{\Lambda}$ defined by:

$$\psi_{\Lambda}(., y, t) = G_{h}(y, t_0, t) \varphi + \int_{t_0}^{t} G_{h}(y, s, t) \left[ V_{\Lambda}(.) - \frac{h^2}{2s^2} \sum_{\lambda \in \Lambda} |U_{\Lambda}(., s)|^2 \right] ds$$

satisfies (2.1) and (2.6) in $\mathbb{R}^p \times [t_0, t_0 + \tau]$, and we have, for each $\Lambda$:

$$L_{a}(t \nabla_{x_{\lambda}} \psi_{\Lambda}) = t \partial_{x_{\lambda}} V_{\Lambda} \quad (2.9)$$

where $L_{a}$ is the operator defined in (2.3) with $a_{\mu}(x, t) = \frac{h^2}{2} \nabla_{x_{\mu}} \psi_{\Lambda}(x, y, t)$.

**Step 3: Global solution of the Cauchy problem.** The maximum principle, (Proposition 2.1), applied to (2.9), shows that the solution constructed in the second step in an interval $[t_0, t_0 + \tau]$ satisfies:

$$|\nabla_{x_{\lambda}} \psi_{\Lambda}(x, y, t)| \leq \frac{t_0}{t} |\nabla_{x_{\lambda}} \varphi(x)| + \frac{t^2 - t_0^2}{2t} M_{1}(\varepsilon).$$

Since the length $\tau$ of the interval has the form (2.7), we can iterate the above process to obtain a global solution (in space and time) for the Cauchy problem (2.1), (2.2), satisfying (1.10). We can also solve in a similar way the integral equations satisfied by its higher order derivatives, including the derivatives with respect to $x$ and $y$. Thus, we see that they are bounded when $t$ belongs to a bounded set. □

The function $U_{\Lambda}$ defined in (1.9), with this function $\psi_{\Lambda}$, is the integral kernel of the operator $e^{-tH_{\Lambda}}$. Excepted for the first order derivatives, the bounds given by the previous proof are not uniform with respect to the set $\Lambda$. The next step will be the proof of such uniform bounds for the second order derivatives. For that, we have to choose a norm $N_{2}$ on the second differential of any function $f$ defined on $(\mathbb{R}^p)^{\Lambda}$. We set, for $\varepsilon \in [0, 1]$:  

$$N_{2}(d^2 f(x), \varepsilon) = \sup_{\lambda \in \Lambda} \sum_{\mu \in \Lambda} \frac{|\nabla_{x_{\lambda}} \nabla_{x_{\mu}} f(x)|}{\varepsilon^{[\lambda - \mu]}} , \quad (2.10)$$

(with the euclidian norm on $\mathbb{R}^p$.) We denote by $\| \|$ the $L^{\infty}$ norm. The norm $N_{2}$, and the norms $N_{m}$ of (2.12), appear in Sjöstrand [25]. The norm $\|N_{2}(\psi_{\Lambda}(., y, t), \varepsilon)\|$ is well defined for each $\Lambda$, $y$ and $t$, by Proposition 2.2. The norm $\|N_{2}(d^2 V_{\Lambda}, \varepsilon)\|$ is bounded, independently of $\Lambda$, by the constant $M_{2}(\varepsilon)$ of (1.7).
Proposition 2.3. If \(ht \leq T_0 = \|N_2(d^2V_A(\cdot), \varepsilon)\|^{-1/2}\), we have, for all \(y \in (\mathbb{R}^p)^\Lambda\),
\[
\|N_2(d^2\psi_A(\cdot, y, t), \varepsilon)\| \leq 2t\|N_2(d^2V_A(\cdot), \varepsilon)\| = 2tM_2(\varepsilon) .
\tag{2.11}
\]

Proof. For each \(\lambda \in \Lambda\), for each \(X\) in \(\mathbb{R}^p\), for each sequence \((Y_\mu)_{\mu \in \Lambda}\) of vectors \(Y_\mu\) in \((\mathbb{R}^p)^\Lambda\), and for each \(y \in (\mathbb{R}^p)^\Lambda\), the function
\[
\varphi(x, t) = \sum_{\mu \in \Lambda} \frac{(X, \nabla_{x_\lambda})(Y_\mu, \nabla_{x_\lambda})\psi_A(..., t, h)}{e^{(\lambda - \mu)}}
\]
satisfies the equation \(L_\mu(t^2 \varphi) = t^2F - h^2t^2G\), where \(L_\mu\) is defined in (2.3) with, here, \(a_\lambda = h^2\nabla_{x_\lambda}\psi_A\) and:
\[
F(x) = \sum_{\mu \in \Lambda} \frac{(X, \nabla_{x_\lambda})(Y_\mu, \nabla_{x_\lambda})V_\Lambda}{e^{(\lambda - \mu)}}
\]
\[
G(x) = \sum_{(\mu, \nu) \in \Lambda^2} \frac{\langle \nabla_{x_\mu}(X, \nabla_{x_\lambda})\psi_A(x, y, t), \nabla_{x_\nu}(Y_\mu, \nabla_{x_\lambda})\psi_A(x, y, t) \rangle}{e^{(\lambda - \mu)}} .
\]

We have, by the definition (2.10) of the norm \(N_2\):
\[
|F(x)| \leq N_2(d^2V_A(x), \varepsilon) |X| \sup_{\mu \in \Lambda} |Y_\mu| ,
\]
\[
|G(x)| \leq N_2(d^2\psi_A(x, y, t), \varepsilon)^2 |X| \sup_{\mu \in \Lambda} |Y_\mu| .
\]

Therefore, by the maximum principle, (Proposition 2.1), applied to the function \(u = t^2\varphi\), and to the operator \(L_\mu\), with \(\mu_0 = 0\), taking the sup on all vectors \(X\) and sequences \(Y_\mu\), and on all points \(x\), we have:
\[
t^2\|N_2(d^2\psi_A(\cdot, y, t), \varepsilon)\| \leq \int_0^t |s^2\|N_2(d^2V_A(\cdot), \varepsilon)\| + h^2s^2\|N_2(d^2\psi_A(\cdot, y, s), \varepsilon)\|^2| ds .
\]

Then Proposition 2.3 follows easily, by a quadratic analogue of the Gronwall lemma. \(\square\)

Higher order derivatives with respect to \(x\). Now, we need a norm \(N_m\) for the higher order differential \(d^mf(x)\) of a function on \((\mathbb{R}^p)^\Lambda\). We set:
\[
N_m(d^mf(x), \varepsilon) = \sup_{(\lambda_1, \ldots, \lambda_m-1) \in \Lambda^{m-1}} \frac{\|\nabla_{x_{\lambda_1}} \cdots \nabla_{x_{\lambda_m-1}} \nabla_{x_{\lambda_m}} f(x)\|}{e^{diam\{\lambda_1, \ldots, \lambda_m-1, \lambda_m\}}} .
\tag{2.12}
\]

In [25], a family of functions, depending on \(\Lambda\), is called a \(0-\) standard function with exponential weight if \(N_m(d^mf(x), \varepsilon)\) is bounded, with bounds independent of \(\Lambda\). The definition of [25] is more complicated, since \(\ell^p\) norms are used, not only \(\ell^4\) and \(\ell^\infty\) norms like here. Again, \(\|N_m(d^mf\psi_A(\cdot, t), \varepsilon)\|\) is well defined by Proposition 2.2, and \(\|N_m(d^mV_A, \varepsilon)\|\) is bounded, independently of \(\Lambda\), if the hypothesis \((H_\varepsilon)\) is satisfied. Now, we can prove, by induction on \(m \geq 2\), that, if \((H_\varepsilon)\) is satisfied, we have
\[
(P_m) \quad \|N_m(d^m\psi_A(\cdot, t), \varepsilon)\| \leq tK_m(\varepsilon) \quad \text{if} \quad ht \leq T_0 ,
\]
where $K_m(\varepsilon)$ is independent of $\Lambda$. For each sequence $(\lambda^{(1)}, \ldots, \lambda^{(m-1)})$ of points in $\Lambda$, for each vectors $X_j$ ($1 \leq j \leq m$) and $Y_\mu$ ($\mu \in \Lambda$) in $\mathbb{R}^p$, we shall estimate the function $t^m \varphi$, where

$$
\varphi(x, t) = \sum_{\mu \in \Lambda} \frac{(X_1, \nabla x_{\lambda^1}) \cdots (X_{m-1}, \nabla x_{\lambda^{m-1}})(Y_\mu, \nabla x_{\mu})\psi_{\Lambda}}{\varepsilon^\text{diam}(\Lambda_{1}, \ldots, \Lambda_{m-1}, \mu)}.
$$

By differentiating (2.1), we see that this his function satisfies an equation of the type $L_a(t^m \varphi) = t^m F$ where $L_a$ is defined in (2.3) with $a_\mu = h^2 \nabla_{x_\mu} \psi_{\Lambda}$, and $F$ satisfies

$$
|F(x, y, t)| \leq \left[ \sup_{\mu \in \Lambda} |Y_\mu| \right] \prod_{j=1}^{m-1} |X_j| \left[ N_m(d^m V_{\Lambda}(x), \varepsilon) + C h^2 \sum_{k=2}^{m} N_k \left( d_k^m (\psi_{\Lambda}(x, y, t), \varepsilon) \right) N_{m+2-k}^{m+2-k} (\psi_{\Lambda}(x, y, t), \varepsilon) \right].
$$

We can apply the maximum principle (Proposition 2.1) to the operator $L_a$ as before. Taking the sup over all the sequences of vectors $X_j$ and $Y_\mu$, using the hypothesis ($H_\varepsilon$) for $V_{\Lambda}$, the induction hypothesis ($P_{m-1}$) for the terms with $2 \leq k < m$, and Proposition 2.3 for $k = 2$ or $k = m$, we obtain, if $ht \leq T_0$:

$$
\|N_m(d^m \psi_{\Lambda}(\ldots, t), \varepsilon)\| \leq (t + h^2 2^k) K_m(\varepsilon) + 4 C h^2 M_2(\varepsilon) \int_0^t N_m(d^m \psi_{\Lambda}(\ldots, s), \varepsilon) ds.
$$

Then property ($P_m$) follows by the usual Gronwall’s lemma, (without more conditions on $h$ and $t$).

**End of the proof of Theorem 1.1.** Then, all derivatives with respect to $x$ only are bounded as it is claimed in (1.11). Since $e^{-tH_\Lambda}$ is self-adjoint, we have also bounds for the derivatives with respect to $y$ only. For the mixed derivatives, we need a new induction. For instance, to estimate the matrix $\nabla_{x_\mu} \nabla_{y_\nu} \psi_{\Lambda}$, we see that, for $X$ and $Y$ in $\mathbb{R}^p$, the function $\varphi(x, t) = (X, \nabla x_{\lambda})(Y, \nabla y_{\nu}) \psi_{\Lambda}(x, y, t)$ satisfies $L_a(t^m \varphi) = F$, where $L_a$ is defined in (2.3) and

$$
F = (X, \nabla x_{\lambda})(Y, \nabla y_{\nu}) \psi_{\Lambda} - h^2 \sum_{\nu \in \Lambda} \langle \nabla x_{\nu} (Y, \nabla y_{\nu}) \psi_{\Lambda}, \nabla x_{\lambda} (X, \nabla x_{\lambda}) \psi_{\Lambda} \rangle.
$$

We use the estimations, already proven, of the derivatives with respect to $x$ only, we apply Proposition 2.1, and we apply again the usual Gronwall lemma. The new induction follows the same ideas, and leads to the proof of Theorem 1.1.

### 3. Decomposition of a function on the lattice

In this section, we shall define a decomposition of any function $f$ on $(\mathbb{R}^p)^L$ (where $L = \mathbb{Z}^d$) as a sum of functions $T_Q f$ associated to the boxes of $L$. We shall give two variants of this algorithm: the first one is simpler and works only for the restriction of our integral kernel to the diagonal, and the second one is useful for stronger
estimated (for example, for trace norms). In the next section, we shall apply this
decomposition to the function $\psi_\Lambda$ of Theorem 1.1, and prove estimates for each
term of the decomposition.

We shall associate, to each function $f$ and to each box $Q$, a function $T_Q f$. If $Q$
is a box in $L = \mathbb{Z}^d$, we denote by $\operatorname{Int}(Q)$ the set of non empty boxes $Q'$ ($Q' \subseteq Q$)
which are either $Q$ itself, or obtained by removing in $Q$ some faces. We denote by
$m(Q, Q')$ the number of faces of $Q$ which are removed in $Q'$ ($0 \leq m(Q, Q') < 2d$).
If $Q$ is any set of $L = \mathbb{Z}^d$, we define a map $\pi_Q : (\mathbb{R}^p)^L \to (\mathbb{R}^p)^L$ by:

$$
(\pi_Qx)_\lambda = \begin{cases} 
  x_\lambda & \text{if } \lambda \in Q \\
  0 & \text{if } \lambda \notin Q
\end{cases}.
$$

(3.1)

Then, we define the function $T_Q f$ by:

$$(T_Q f)(x) = \sum_{Q' \in \operatorname{Int}(Q)} (-1)^{m(Q, Q')} \left[ f(\pi_{Q'} x) - f(0) \right].$$

(3.2)

Clearly, $T_Q f$ is supported in $Q$. If $f$ has a finite support, let $\Lambda$ be a box containing
the support of $f$. Then we have:

$$
 f(x) - f(0) = \sum_{Q \subseteq \Lambda} (T_Q f)(x),
$$

(3.3)

where the sum is taken over all the non empty boxes $Q$ contained in $\Lambda$, including
the points. This equality follows from the definition of $T_Q f$ and from the following
remark, for each box $P$ contained in $\Lambda$:

$$
\sum_{Q \subseteq \Lambda, P \in \operatorname{Int}(Q)} (-1)^{m(Q, P)} = \begin{cases} 
  1 & \text{if } P = \Lambda \\
  0 & \text{if } P \neq \Lambda
\end{cases}.
$$

In order to estimate $T_Q f$, we shall use, rather than the derivatives of $f$, some
operators of translation, with the following notations. For each $u \in (\mathbb{R}^p)^L$, we set:

$$
(S_u f)(x) = f(x + u) - f(x) \quad \sigma(u) = \{ \lambda \in \mathbb{Z}^d, u_\lambda \neq 0 \}.
$$

(3.4)

For each box $Q$, of the form $Q = \prod_{j=1}^d [a_j, b_j]$, with $a_j \leq b_j$, we denote the different
“faces” of $Q$ by:

$$
B_k(Q) = \{ \lambda \in Q, \lambda_k = a_k \}, \quad B_k^+(Q) = \{ \lambda \in Q, \lambda_k = b_k \}.
$$

For each $k \leq d$ such that $a_k < b_k$, we can write $Q = [a_k, b_k] \times \tilde{Q}$, where $\tilde{Q}$ is a box
in $\mathbb{Z}^{d-1}$. For each box $P \in \operatorname{Int}(\tilde{Q})$, let us set:

$$
P_+ = [a_k, b_k] \times P, \quad P_- = [a_k, b_k] \times P, \quad P_0 = [a_k, b_k] \times P.
$$

With these notations, we can write:

$$
T_Q f(x) = \sum_{P \in \operatorname{Int}(\tilde{Q})} (-1)^{m(\tilde{Q}, P)} \left[ f(\pi_{P_+} x) - f(\pi_{P_-} x) - f(\pi_{P_0} x) + f(\pi_P x) \right].
$$
We can write this equality, using the previous operators $S_u$:

$$f(\pi p_x) - f(\pi p_x) - f(\pi p_x) + f(\pi p_x) = (S_{u}, f)(\pi p_x)$$

(3.5)

$$u_p = -\pi B^{(k)}(Q)^{-1} P x$$

$$u_p = -\pi B^{(k)}(Q)^{-1} P x$$

It follows that

$$\| T_Q f \| \leq 4^d \sup_{\sigma(\sigma) \subset B^{(k)}(Q)} \| S_u S_\sigma f \| .$$

We have similar estimates for the derivatives $\nabla x T_Q f$, if $\lambda$ is neither in $B^{(k)}_+(Q)$, nor in $B^{(k)}_+(Q)$. If $\lambda$ is, for example, in $B^{(k)}_+(Q)$, we can use, instead of (3.5):

$$f(\pi p_x) - f(\pi p_x) - f(\pi p_x) + f(\pi p_x) = (S_{u}, f)(\pi p_x) - (S_{\pi}, f)(\pi p_x)$$

and we get, if $\lambda$ is not in $B^{(k)}_+(Q)$:

$$\| \nabla x, T_Q f \| \leq 4^d \sup_{\sigma(\sigma) \subset B^{(k)}_+(Q)} \| S_u \nabla x, f \| .$$

If $a_k = b_k$, we cannot apply that, but we can write, in all the cases:

$$\| \nabla x, T_Q f \| \leq 4^d \| \nabla x, f \| .$$

It is possible to define similar projections by replacing in (3.2) $\pi_\lambda$ by other projections. The heat kernel and our function $\psi, \psi_\lambda$ are defined in $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda$, but we shall only use translations by vectors in the diagonal. Therefore, we shall replace $\pi_\lambda$ by the following projector:

$$(\Pi_\lambda(x, y)) = \begin{cases} (x, y) & \text{if } \lambda \in Q \\ (0, y - x) & \text{if } \lambda \notin Q \end{cases} .$$

(3.6)

(We could make another choice, preserving self-adjointness). For each function $f(x, y)$ on $(\mathbb{R}^p)^L \times (\mathbb{R}^p)^L$, we define $T_Q f$ as before, with a slight modification:

$$(T_Q f)(x, y) = \sum_{Q' \in \mathcal{N}(Q)} (-1)^{m(Q, Q')} \left[ f(\Pi_{Q'}(x, y)) - f(0, y - x) \right] .$$

(3.7)

Then the operator $S_u$ is replaced by the following one:

$$(S_{u})(x, y) = f(x + u, y + u) - f(x, y) \quad \forall (x, y) \in (\mathbb{R}^p)^L \times (\mathbb{R}^p)^L .$$

(3.8)

We verify easily for later use the properties listed in the next proposition.

**Proposition 3.1.** With the previous notations, for each function $f$ in $C^\infty((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda)$, we have:

i) $T_Q f$ depends only on the variables $x_\lambda$ and $y_\lambda$ such that $\lambda \in Q$, and on the variables $x_\lambda - y_\lambda$ such that $\lambda \notin Q$. If $f$ is supported in a finite box $\Lambda$, we have:

$$f(x, y) - f(0, y - x) = \sum_{Q \subseteq \Lambda} (T_Q f)(x, y)$$

(3.9)
ii) If \( f \) depends in a smooth way on a parameter \( \theta \), we have \( T_Q(\frac{\partial f}{\partial \theta}) = \frac{\partial (T_Q f)}{\partial \theta} \).

iii) If \( \lambda^{(1)}, \ldots, \lambda^{(m)} \) is a finite sequence of points in \( \Lambda \), we have:

\[
\|\nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} T_Q f\| \leq 4^d \|\nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} f\|. \tag{3.10}
\]

iv) If moreover, none of the points \( \lambda^{(j)} \) is in the face \( B^{(k)}_+(Q) \), we have

\[
\|\nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} T_Q f\| \leq 4^d \sup_{\sigma(u) \subset B^{(k)}_+(Q)} \|\nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} S_u f\|. \tag{3.11}
\]

v) If \( a_k \neq b_k \), and if the points \( \lambda^{(j)} \) are neither in \( B^{(2)}_+(Q) \), nor in \( B^{(2)}_-(Q) \) we have

\[
\|\nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} T_Q f\| \leq 4^d \sup_{\sigma(u) \subset B^{(2)}_+(Q)} \|\nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} S_u S_v f\|. \tag{3.12}
\]

4. Proof of Theorem 1.2, (points 1 to 3)

We define the functions \( T_Q \psi _\Lambda \) by (3.7), where \( f \) is replaced by the function \( \psi _\Lambda \) of Theorem 1.1. Then the properties 1 and 2 of Theorem 1.2 follow from Proposition 3.1. For the proof of the point 3, we shall modify the norm \( N_m(d^m f(x), \epsilon) \) of Section 2, and now, it will depend also on one subset \( E \) of \( \mathbb{Z}^d \), or sometimes on two subsets \( E \) and \( F \). We set, for each \( E_1, \ldots, E_m \) which are either subsets of \( \mathbb{Z}^d \), or points of \( \mathbb{Z}^d \),

\[
D(E_1, \ldots, E_m) = \sup_{j,k \leq m} \text{dist}(E_j, E_k). \tag{4.1}
\]

If \( f \) is a \( C^\infty \) function on \( (\mathbb{R}^d)^\Lambda \times (\mathbb{R}^d)^\Lambda \), we set, for each \( m \geq 2 \), for each point \( (x, y) \), for each \( \epsilon \) in \([0, 1]\), and for each subsets \( E \) and \( F \) of \( \mathbb{Z}^d \):

\[
N_m(d^m f(x, y), \epsilon, E, F) = \sup_{(\lambda^{(1)}, \ldots, \lambda^{(m-1)}) \in \Lambda^{m-1}} \sum_{\mu \in \Lambda} \left| \nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m-1)}} \nabla_{\mu} f(x, y) \right|. \tag{4.2}
\]

If \( m = 1 \), there is no sup in (4.2), and the sum is taken on all \( \lambda \in \Lambda \). We also set, if \( m \geq 1 \):

\[
N_m^\infty(d^m f(x, y), \epsilon, E, F) = \sup_{(\lambda^{(1)}, \ldots, \lambda^{(m)}) \in \Lambda^m} \left| \nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} f(x, y) \right|. \tag{4.3}
\]

We define similar norms with only one set \( E \). When no set \( E \) or \( F \) appears, the first norm \( N_m \) is the same as in Section 2 if \( m \geq 2 \). For \( m = 1 \), the sup in Section 2 (without sets \( E \) and \( F \)) is replaced here by a sum (when there is at least one set). The second norm \( N_m^\infty \) will be used in Section 6. We still denote by \( \| \| \) the \( L^\infty \) norm. With these notations, the point 3 of Theorem 1.2 will follow from the next lemma.
Lemma 4.1. If \((H_2)\) is satisfied \((0 < \varepsilon < 1)\), for each integer \(m \geq 1\), there exists \(K_m(\varepsilon) > 0\) such that, for each finite box \(\Lambda\) of \(\mathbb{Z}^d\), for each vectors \(u\) and \(v\) in \((\mathbb{R}^p)^\Lambda\), the functions \(S_u \psi\Lambda\) and \(S_u S_v \psi\Lambda\) defined as in (3.8), (with the function \(\psi\Lambda\) of Theorem 1.1), satisfy, if \(ht \leq T_0\), (defined in (1.7) and (1.8)):

\[
\|N_m(d^m(S_u \psi\Lambda)(\cdot, t, \varepsilon, \sigma(u)))\| \leq tK_m(\varepsilon)\|\sigma(u)\|,
\]

(4.4)

\[
\|N_m(d^m(S_u S_v \psi\Lambda)(\cdot, t, \varepsilon, \sigma(u), \sigma(v)))\| \leq tK_m(\varepsilon)\|\sigma(u)\|\|\sigma(v)\|,
\]

(4.5)

where the support \(\sigma(u)\) is defined in (3.4). If \(\sigma(u) \cap \sigma(v) = \emptyset\), we can write also:

\[
|S_u S_v \psi\Lambda(x, y, t)| \leq tK_0(\varepsilon)e^{\text{dist}(\sigma(u), \sigma(v))}\|\sigma(u)\|\|\sigma(v)\|.
\]

(4.6)

Since an argument of the proof will be used again twice in Section 6, we state it as a lemma.

Lemma 4.2. We consider a real valued function \(A(x, y, t)\), \(C^\infty\) on \((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda \times [0, \infty]\). We assume that there is \(\varepsilon\) in \([0, 1]\) such that, for each \(m \geq 1\), we can write, for some constant \(K_m > 0\):

\[
\|N_m(d^m A(\cdot, t, \varepsilon))\| \leq tK_m,
\]

(4.7)

where \(N_m\) is the norm of Section 2. We denote by \(L_\Lambda\) the differential operator of (2.3), with \(a_\Lambda = h^2 \nabla x_\Lambda A\). We consider also a smooth, real valued function \(\varphi(x, y, t)\), such that \(\varphi(y, x, t) = \varphi(x, y, t)\) and \(\varphi(x, y, 0) = 0\). We assume that there are two subsets \(E\) and \(F\) of \(\mathbb{Z}^d\) such that, for each \(m \geq 1\),

\[
\|N_m(d^m(L_\Lambda \varphi)(\cdot, t, \varepsilon, E, F))\| \leq K_m.
\]

(4.8)

Then, if \(ht\) is bounded, we can write, with another \(K_m\), for each \(m \geq 1\):

\[
\|N_m(d^m \varphi(\cdot, t, \varepsilon, E, F))\| \leq tK_m.
\]

(4.9)

We have the same result with only one set \(E\). We also have the same result if we replace, for all \(m\), the norm \(N_m\) by the norm \(N_m(\infty)\), both in the hypothesis (4.8) and in the conclusion (4.9).

The proof is like the one in Section 2, but simpler since we have no more quadratic Gronwall lemma, but the usual one. (In Section 2, we had \(A = \psi\Lambda\) and \(\varphi\) was a derivative of \(A\). We had to prove, at the same time, bounds for \(A\) and for \(\varphi\), but now the bounds for \(A\) are already available.)

Proof of Lemma 4.1. For each \(u\) in \((\mathbb{R}^p)^\Lambda\), let \(S_u\) be the operator defined in (3.8). Since \(\psi\Lambda\) satisfies (2.1), the function \(f = S_u \psi\Lambda\) satisfies, as a function of \(x\) and \(t\), while \(y\) and \(u\) are fixed, the equation \(L_\Lambda(f) = S_u V_\Lambda\), where \(L_\Lambda\) is the differential operator defined in (2.3), with:

\[
a_\Lambda(x, y, t) = \frac{h^2}{2} \nabla x_\Lambda \left[ \psi\Lambda(x + u, y + u, t) + \psi\Lambda(x, y, t) \right].
\]

(4.10)

By the hypothesis \((H_2)\), we can write, for each \(m \geq 1\), with some constant \(K_m(\varepsilon)\) independent of \(u\) and \(\Lambda\):

\[
\|N_m(d^m(S_u V_\Lambda)(\cdot, t, \varepsilon, \sigma(u)))\| \leq K_m(\varepsilon)\|\sigma(u)\|.
\]
Then, (4.4) follows from Lemma 4.2. For the proof of (4.5), we remark that, if \( f = S_u\psi_\Lambda \) satisfies \( L_a f = S_u V_\Lambda \), where \( a \) is defined in (4.10), the new function \( g = S_u S_v \psi_\Lambda = S_v f \) satisfies \( L_a g = G \), with:
\[
b_\Lambda(x, y, t) = a_\Lambda(x + v, y + v, t) \quad G = S_u S_v V_\Lambda - \sum_{\Lambda \in \Lambda} (S_v a_\Lambda) (\nabla_{x, y} S_u \psi_\Lambda).
\]

By the hypothesis \((H_\varepsilon)\), and by (4.4), for each \( m \geq 1 \), we can write, if \( ht \leq T_0 \):
\[
\| N_m (d^m(G)(\cdot, \cdot, t), \varepsilon, \sigma(u), \sigma(v)) \| \leq K_m(\varepsilon) \| \sigma(u) \| \| \sigma(v) \|.
\]
Then (4.5) follows from Lemma 4.2. If \( m = 0 \), we remark that, if \( \sigma(u) \cap \sigma(v) = \emptyset \), the unbounded self-interaction term disappear in \( S_u S_v \psi_\Lambda \), and therefore
\[
\| S_u S_v V_\Lambda \| \leq K(\varepsilon) e^{\text{dist}(\sigma(u), \sigma(v)) \inf(\| \sigma(u) \|, \| \sigma(v) \|)}.
\]
Then (4.6) follows from Proposition 2.1.

**Proof of the point 3 of Theorem 1.2.** If \( m = 0 \) and \( \text{diam}(Q) \neq 0 \), the bound of the point 3 follows from the point v) of Proposition 3.1 and (4.6), remarking that \( |\mathcal{B}_\pm^{(j)}(Q)| \leq (1 + \text{diam}(Q))^{d-1} \). If \( m \geq 1 \), according to the relative position of the points \( \lambda^{(j)} \) and the box \( Q \), we apply either the point v) of Proposition 3.1 and (4.5), or the point iv) and (4.4), or the point iii) and Theorem 1.1.

**5. Splitting the set \( \Lambda \): Case of a small subset**

For applications to the decay of correlations, (Section 8), the set \( \Lambda \) will be a large box, containing a smaller set \( E \), (the union of the two supports of the observables), and we have to dissociate the two sets \( E \) and \( \Lambda \setminus E \). If \( Z_\Lambda(t) \) is the partition function of \((1.2)\), the next proposition will be useful, for example, to compare \( Z_\Lambda(t) \) and \( Z_{\Lambda \setminus E}(t) \), and to estimate the trace norm of the partial trace (with respect to \( E \)), of some operators.

For each operator \( K \) with an integral kernel \( K(x, y) \) in the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \times \mathbb{R}^d \), and for each subset \( E \subset \Lambda \), we shall use in Section 7 the partial trace of \( K \) with respect to \( E \), and we shall estimate the trace norm of this partial trace, using integrals of the following type. We denote by \( \text{Diag}(\Lambda, E) \) the set of points \((x, y)\) in \((\mathbb{R}^d)^2 \times (\mathbb{R}^d)^2\) such that \( x_{\Lambda \setminus E} = y_{\Lambda \setminus E} \), this set being endowed with the measure \( d\nu_E(x, y) = dx dy d\nu_{E \setminus \Lambda}(x, y) \). Then we define, for every \( m' \) and \( \mu \):
\[
I_{E, m', \mu}(K) = \int_{\text{Diag}(\Lambda, E)} (1 + |x_E| + |y_E|)^{m'} e^{\mu|x_E - y_E|} |K(x, y)| d\nu_E(x, y).
\]
and for each integer \( m \), we set also:
\[
\|K\|_{E, m, m', \mu} = \sup_{\lambda^{(1)}, \ldots, \lambda^{(k)} \in \mathbb{R}^k} I_{m', \mu}(\nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(k)}} K),
\]
where the trace norm of the partial trace of \( K \) can be estimated by such a norm. The next proposition is useful to estimate this kind of norm, applied to the heat kernel \( U_\Lambda(t) \).
Proposition 5.1. With these notations, we can write, for each \( m' \) and \( \mu \), for some constant \( M \) depending on \( |E|, t, h, \varepsilon, m' \) and \( \mu \),
\[
\| U_\Lambda(t) \|_{E, \varepsilon, m', \mu} \leq M \ Z_\Lambda(t).
\]

This proposition is a direct consequence of (1.9) and of the following proposition, also used for the point 4 of Theorem 1.2.

If \( E \subset \Lambda \subset \mathbb{Z}^d \), we denote by \( \psi_{\Lambda \setminus E} \) the function defined by Theorem 1.1, for the set \( \Lambda \setminus E \). It can be seen also as a function on \((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda \times \mathbb{R}_+\), depending only on \( x_{\Lambda \setminus E}, y_{\Lambda \setminus E}, \) and \( t \). For each point \( \lambda \in E \), we set \( A_\lambda(x) = A(x_{\lambda}) \), where \( A \) is the self-interaction term in the definition (1.4), and we denote by \( A_\lambda(x, y) \) the mean value of \( A_\lambda \) on the segment between \( x \) and \( y \), as in (1.16).

Proposition 5.2. With these notations, there exists \( K(\varepsilon) > 0 \), independent of the two sets \( E \) and \( \Lambda \) \((E \subset \Lambda \subset \mathbb{Z}^d)\), such that:
\[
\left\| \psi_{\lambda}(., t) - \left[ t \sum_{\lambda \in E} A_\lambda + \psi_{\Lambda \setminus E}(., t) \right] \right\| \leq |E| K(\varepsilon)(t + h^2 t^2). \tag{5.2}
\]

Proof. First step. Let us denote by \( V_{\text{disc}} \) the potential defined as if all the points of \( E \) had no interaction with the others points in \( \Lambda \), and no interactions between themselves:
\[
V_{\text{disc}} = \sum_{\lambda \in \Lambda} A_\lambda + \sum_{\lambda, \mu \in \Lambda \setminus E} B_{\lambda, \mu}.
\]
For each \( \theta \in [0, 1] \), let us set \( V_{\Lambda, \theta} = V_{\text{disc}} + \theta(V_\Lambda - V_{\text{disc}}) \). Let us denote by \( \psi_{\Lambda, \theta} \) the solution of the Cauchy problem (2.1), (2.2), where \( V_\Lambda \) is replaced by \( V_{\Lambda, \theta} \).

We see easily that:
\[
\psi_{\Lambda, \theta, 0} = \sum_{\lambda \in E} \psi_{\lambda}(x_{\lambda}, y_{\lambda}, t) + \psi_{\Lambda \setminus E}(x_{\Lambda \setminus E}, y_{\Lambda \setminus E}, t) \quad \psi_{\Lambda, \theta, 1} = \psi_{\Lambda}, \tag{5.3}
\]
where \( \psi_{\lambda} \) is the function defined by Theorem 1.1 for the set reduced to the point \( \{\lambda\} \), and to the potential \( V_{\lambda} = A_\lambda \) reduced to the self-interaction term, according to (1.4). If we differentiate with respect to \( \theta \) the non linear equation like (2.1), satisfied by \( \psi_{\Lambda, \theta} \) (with \( V_{\Lambda, \theta} \) in the RHS), we see that \( L \frac{\partial \psi_{\Lambda, \theta}}{\partial \theta} = F \), where \( L \) is the differential operator of (2.3), with \( a_\theta = h^2 \nabla_x \psi_{\Lambda, \theta} \) and with \( F = V_{\text{disc}} - V_\Lambda \). By the hypothesis \( (H_\varepsilon) \), we can write: \( \| V_\Lambda - V_{\text{disc}} \| \leq K(\varepsilon)|E| \). Therefore, by Proposition 2.1, we can write \( \| \frac{\partial \psi_{\lambda}(., t)}{\partial \theta} \| \leq t K(\varepsilon)|E| \). Therefore we have, with another \( K(\varepsilon) \),
\[
\| \psi_{\Lambda} - \psi_{\Lambda, \theta, 0} \| \leq t |E| K(\varepsilon). \tag{5.4}
\]
Second step. Now, we shall compare \( \psi_{\lambda} \) and its semi-classical approximation \( \psi_{\lambda}^{(0)}(x, y, t) = tA(x_{\lambda}, y_{\lambda}) \). By a direct computation, we see that
\[
\frac{\partial \psi_{\lambda}^{(0)}}{\partial t} + \frac{x - y}{t} \cdot \nabla_x \psi_{\lambda}^{(0)} = A_\lambda.
\]
and therefore that the function \( \varphi = \psi_{(\lambda)} - \psi_{(\lambda)}^{(0)} \) satisfies the equation \( L_0(\varphi) = F \) where \( L_0 \) is the operator defined in (2.3, with \( a = 0 \) and \( F = \frac{h^2}{2}(\Delta_x \psi_{(\lambda)}^{(0)}) - |\nabla_x \psi_{(\lambda)}^{(0)}|^2) \). By Theorem 1.1, we can write, if \( ht \leq T_0 \), \( \|F(.,.,t)\| \leq K(\varepsilon)h^2(t^2 + t^3) \).

By Proposition 2.1, it follows that:

\[
\|\psi_{(\lambda)} - \psi_{(\lambda)}^{(0)}(.,.,t)\| \leq K(\varepsilon)h^2(t^2 + t^3) \quad \text{if} \quad ht \leq T_0
\]  

(5.5)

where \( T_0 \) is the constant of (1.8) and (1.7). Proposition 5.2 follows from (5.3), (5.4) and (5.5).

**Proof of the point (1.15) of Theorem 1.2.** We apply Proposition 5.2 with \( E \) reduced to a single point \( \lambda \). Then, we apply (3.10) with \( m = 0, Q = \{\lambda\} \), and \( f \) being the function in the LHS of (5.2). Thus we obtain:

\[
\|T_{(\lambda)}\psi_{\Lambda}(.,.,t) - tT_{(\lambda)}\tilde{A}_\lambda - T_{(\lambda)}\psi_{\Lambda\setminus(\lambda)}(.,.,t)\| \leq K(\varepsilon)(t + h^2t^2).
\]

Using the definition (3.7) of \( T_{(\lambda)} \), we see that

\[
T_{(\lambda)}\psi_{\Lambda\setminus(\lambda)} = 0 \quad T_{(\lambda)}\tilde{A}_\lambda(x,y) = \tilde{A}(x,\lambda) - \tilde{A}(0,y - x),
\]

and the bound (1.15) of Theorem 1.2 follows.

We may also apply Proposition 5.2 to \( E = \Lambda \), and we obtain that, if \( ht \leq T_0 \), the integral kernel \( U_{\Lambda}(x,y,t) \) is in \( S((\mathbb{R}^d)^{\Lambda} \times (\mathbb{R}^d)^{\Lambda}) \).

### 6. Splitting the box \( \Lambda \) by an hyperplane

This section will be used in the proof of Theorems 1.4 and 1.5. Let \( \Lambda \subset \mathbb{Z}^d \) be the union of two disjoints subsets \( \Lambda_1 \) and \( \Lambda_2 \), which are separated by an hyperplane \( \Sigma \), orthogonal to one of the vectors of the canonical basis. We may assume that, for some \( j \leq d \) and \( \sigma \in \mathbb{Z} \):

\[
\Lambda_1 \subset \{\lambda \in \mathbb{Z}^d, \lambda_j \leq \sigma\}, \quad \Lambda_2 \subset \{\lambda \in \mathbb{Z}^d, \lambda_j \geq \sigma\}.
\]

(6.1)

Then we set \( \Sigma = \{\lambda \in \mathbb{Z}^d, \lambda_j = \sigma\} \), and, for each box \( Q \) of \( \mathbb{Z}^d \), we denote by \( \pi_{\Sigma}(Q) \) the orthogonal projection of \( Q \) on \( \Sigma \). Again we have, as in Section 5, to dissociate the two subsets \( \Lambda_1 \) and \( \Lambda_2 \), but now, we need estimates which are uniform with respect to both \( \Lambda_1 \) and \( \Lambda_2 \). We denote by \( V_{\text{inter}} \) the sum of the interactions between a point of \( \Lambda_1 \) and a point of \( \Lambda_2 \) in the definition (1.4) of \( V_\Lambda^\theta \):

\[
V_{\text{inter}}(x) = \sum_{\lambda \in \Lambda_1, \mu \in \Lambda_2} B_{\lambda,\mu}(x_\lambda, x_\mu).
\]

(6.2)

For each \( \theta \in [0,1] \), we set \( V_{\Lambda,\theta} = V_\Lambda - \theta V_{\text{inter}} \), and we denote by \( H_{\Lambda,\theta} \) the Hamiltonian defined as in (1.1), with \( V_\Lambda \) replaced by \( V_{\Lambda,\theta} \), and by \( \psi_{\Lambda,\theta} \) the function associated to this Hamiltonian by Theorem 1.1. We are now interested to the derivative of this function with respect to \( \theta \), and to the decomposition \( T_0 \frac{\partial \psi_{\Lambda}}{\partial \theta} \) defined in Section 3.
Proposition 6.1. With the previous notations, we can write, for some constant $C_m(\epsilon)$ independent of $\Lambda$, $\Lambda_1$ and $\Lambda_2$, for each box $Q \subseteq \Lambda$, if $ht \leq T_0$ and $\theta \in [0,1]$:
\[
\left\| \mathcal{N}_m^{\infty} \left( T_Q \frac{\partial \psi_{\Lambda,\theta}}{\partial \theta} (\ldots, t), \epsilon, \pi_{\Sigma} (Q) \right) \right\| \leq t C_m(\epsilon) \langle Q \rangle^d. \tag{6.3}
\]

Since $V_{\Lambda,\theta}$ is of the same type that $V_{\Lambda}$, and satisfies the same hypotheses, with bounds independent of $\theta$, (excepted the hypothesis (1.6), which is not needed for Theorems 1.1 and 1.2), the bounds given by Theorem 1.1, 1.2, and Lemma 4.1 for the function $\psi_{\Lambda,\theta}$ are also uniform in $\theta$. With this remark, the proof of Proposition 6.1 relies on the following lemma.

Lemma 6.2. For each integer $m \geq 1$, there exists $K_m(\epsilon) > 0$ such that, if $ht \leq T_0$, we have, for each points $\lambda^{(1)}, \ldots, \lambda^{(m)}$ in $\Lambda$:
\[
\left\| \nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} \frac{\partial \psi_{\Lambda,\theta}}{\partial \theta} \right\| \leq K_m(\epsilon) t \epsilon^{D(\lambda^{(1)}, \ldots, \lambda^{(m)}, \Sigma)}. \tag{6.4}
\]
Moreover, for each $u$ in $(\mathbb{R}^d)^\Lambda$ with support $\sigma(u)$ contained, either in $\Lambda_1$ or in $\Lambda_2$, we have the following bound, also valid for $m = 0$:
\[
\left\| \nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} S_u \frac{\partial \psi_{\Lambda,\theta}}{\partial \theta} \right\| \leq K_m(\epsilon) t \epsilon^{D(\lambda^{(1)}, \ldots, \lambda^{(m)}, \Sigma)} \leq \sigma(\theta), \tag{6.5}
\]
where $D(\lambda^{(1)}, \ldots, \lambda^{(m)}, \Sigma)$ is defined in (4.1).

Proof. Since $\psi_{\Lambda,\theta}$ satisfies (2.1), with $V_{\Lambda}$ replaced by $V_{\Lambda,\theta}$, it follows that $f = \frac{\partial \psi_{\Lambda,\theta}}{\partial \theta}$ satisfies $L_a f = -V_{\text{Inter}}$, where $L_a$ is the operator of (2.3), and $a_{\lambda} = h^2 \nabla_{x,\lambda} \psi_{\Lambda,\theta}$.

We can write, for each $m \geq 1$:
\[
\left\| \mathcal{N}_m^{\infty} (d^m V_{\text{Inter}}, \epsilon, \Sigma) \right\| \leq K_m(\epsilon).
\]

Then (6.4) follows from Lemma 4.2, with the $\mathcal{N}_m^{\infty}$ norm. Now, we apply the operator $S_u$ defined in (3.8) to $f = \frac{\partial \psi_{\Lambda,\theta}}{\partial \theta}$. Since $L_a f = -V_{\text{Inter}}$, it follows that the function $g = S_u f$ satisfies $L_a g = G$, where still $a_{\lambda} = h^2 \nabla_{x,\lambda} \psi_{\Lambda,\theta}$ and:
\[
G(x, y, t) = -S_u V_{\text{Inter}} (x) - h^2 \sum_{\lambda \in \Lambda} \left( \nabla_{x,\lambda} S_u \psi_{\Lambda,\theta} (x + u, y + u, t) \right).
\]

In order to estimate this function, we use the form (6.2) of $V_{\text{Inter}}$, the hypothesis (H2), the bound (6.4), and Lemma 4 (bound (4.4)), applied to $\psi_{\Lambda,\theta}$ instead of $\psi_{\Lambda}$. We obtain, if $ht \leq T_0$:
\[
\left\| G(\ldots, t) \right\| \leq K_0(\epsilon) \epsilon^{\text{dist} (\sigma(u), \Sigma)} \sigma(\theta).
\]

By the same arguments, we can write, for each $m \geq 1$,
\[
\left\| \mathcal{N}_m^{\infty} (d^m G(\ldots, t), \epsilon, \sigma(u), \Sigma) \right\| \leq K_m(\epsilon) \sigma(u).
\]

Then, for $m = 0$, the estimation (6.5) follows from Proposition 2.1 and, for $m \geq 1$, it follows from Lemma 4.2 (with the $\mathcal{N}_m^{\infty}$ norm). \(\square\)
End of the proof of Proposition 6.1. We may assume that $\Lambda_1$, $\Lambda_2$ and $\Sigma$ satisfy (6.1), with $j = 1$. If the box $Q \subset \Lambda_1 \cup \Lambda_2$ is not disjoint from $\Sigma$, (6.3) follows from Theorem 1.2. Now, we may assume that $Q = \prod_{k=1}^{d}[a_k, b_k]$, with $\sigma < a_1 \leq b_1$ and $a_k \leq b_k$ for $k \geq 2$. Let $k \leq d$ such that the diameter of the set $Q \cup \pi_k(Q) \cup \{\lambda^{(1)} \cdots \lambda^{(m)}\}$ is the length of its $k$-th projection. If $k \neq 1$, the estimation (6.3) follows from Theorem 1.2 (with the potential $V_{\text{inter}}$). If $k = 1$, and $\lambda_1^{(k)} \neq b_1$ for all $k$, it follows from (6.5) and the point iv) of Proposition 3.1, applied with the set $B_1^+(Q)$. In the remaining case, it follows from (6.4) and point iii) of Proposition 3.1.

\[ \square \]

7. Representation of the quantum correlations

In this section, we shall give an expression of the correlation $\text{Cov}_{\Lambda,t}(A, B)$, (see (7.4)–(7.5)). Then, we shall prove Theorem 1.3, point b, i.e., when the local observables $A$ and $B$ are multiplications. In Section 8, we shall prove Theorem 1.3 in the general case. For the expression of the correlation, we shall use the tensor product of the heat kernel by itself (doubling of variables), and the action, on this kernel, of a finite group of symmetries.

1. Doubling of variables. We shall denote by $(x', x'')$ the variable of $(\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda}$. For each operator $A$ in $L^2((\mathbb{R}^p)^{\Lambda})$, we denote by $A'$ (resp. $A''$) the operator $A$, seen as an operator in $L^2((\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda})$, acting only on the variable $x'$ (resp. on $x''$). We denote by $\tilde{H}_{\Lambda}$ the operator in $(\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda}$ defined by $\tilde{H}_{\Lambda} = H_{\Lambda}' + H_{\Lambda}''$. We denote by $\text{Tr}_{\Lambda}(A)$ the trace of an operator $A \in \mathcal{L}(\tilde{H}_{\Lambda})$, where $\tilde{H}_{\Lambda} = L^2((\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda})$. Then, an easy computation shows that, if $A \in \mathcal{L}(H_{E_1})$ and $B \in \mathcal{L}(H_{E_2})$, where $E_1$ and $E_2$ are disjoint subsets, contained in a same box $\Lambda$, we have:

\[
\text{Cov}_{\Lambda,t}(A, B) = \frac{1}{2Z_{\Lambda}(t)} \text{Tr}_{\Lambda} \left( e^{-t\tilde{H}_{\Lambda}}(A' - A'')(B' - B'') \right) \quad (7.1)
\]

where $\tilde{Z}_{\Lambda}(t) = \text{Tr}_{\Lambda} \left( e^{-t\tilde{H}_{\Lambda}} \right)$.

2. Group of symmetries, and averaging. If $\psi_{\Lambda}$ is the function, associated by Theorem 1.1, to the operator $H_{\Lambda}$, and $\tilde{\psi}_{\Lambda}$ to the operator $\tilde{H}_{\Lambda}$, we have:

\[
\tilde{\psi}_{\Lambda}(X, t) = \psi_{\Lambda}(x, y, t) + \psi_{\Lambda}(x', y', t) \quad X = (x, y) = (x', y'). \quad (7.2)
\]

The function $\tilde{\psi}_{\Lambda}$ is invariant under the permutation of the variables $(x', y')$ and $(x'', y'')$. It will be also useful to permute $(x'_\lambda, y'_\lambda)$ and $(x''_\lambda, y''_\lambda)$ not everywhere, but only for $\lambda$ in a subset of $\Lambda$. We denote by $G_{\Lambda}(E_1, E_2)$ the set of maps $\sigma : \Lambda \rightarrow \sigma_{\Lambda}$ of $\Lambda$ in the group of substitutions $S_2$, which are constant on $E_1$ and on $E_2$, where $E_1$ and $E_2$ are disjoint subsets of $\Lambda$. This group has a natural action $X \rightarrow \sigma X$ on $(\mathbb{R}^{2p})^{\Lambda}$ and on $(\mathbb{R}^{4p})^{\Lambda}$. If $X$ is like in (7.2), we may have $(\sigma X)_\lambda = X_{\lambda}$ or
\((\sigma X)_\lambda = (x''_\lambda, x'_\lambda, y'_\lambda, y''_\lambda)\). If \(T\) is in \(L(\mathcal{H}_\Lambda)\), with integral kernel \(K\), let us denote by \(T(\sigma)\) the transformed operator, with integral kernel \(K(\sigma)(X) = K(\sigma X)\). For each set \(F \subset \Lambda\), we define \(\tau_F\) by:

\[
(\tau_F(x', x'', y', y''))_\lambda = \begin{cases} (x''_\lambda, x'_\lambda, y'_\lambda, y''_\lambda) & \text{if } \lambda \in F \\
(x''_\lambda, x'_\lambda, y'_\lambda, y''_\lambda) & \text{if } \lambda \notin F.
\end{cases}
\]

(7.3)

For each \(\sigma \in G_\Lambda(E_1, E_2)\), we denote by \(\text{sgn}_1(\sigma)\) the signature (in \([-1, 1]\)) of \(\sigma(\lambda)\) when \(\lambda\) is any point of \(E_j\) (\(j = 1, 2\)), and we set \(\text{sgn}(\sigma) = \text{sgn}_1(\sigma)\text{sgn}_2(\sigma)\). With these notations, we see, by easy computations, that:

\[
\text{Cov}_{\Lambda, t}(A, B) = \frac{1}{2Z_\Lambda(t)} \text{Tr}_A (W(t)(A' - A'')(B' - B''))
\]

where

\[
W(t) = \frac{1}{|G|} \sum_{\sigma \in G} \text{sgn}(\sigma)(e^{-t\lambda_\sigma}) G = G_\Lambda(E_1, E_2).
\]

(7.4)

Here \((A' - A'')(B' - B'')\) is an operator in \(L(\mathcal{H}_{E_1 \cup E_2})\), and we omit the tensor product with the identity on \(\Lambda \setminus (E_1 \cup E_2)\).

With the notations of B. Simon [24], if \(E \subset \Lambda \subset \mathbb{Z}^d\), we shall use the operator of partial trace \(\text{Tr}^A_E\), from \(L_1(\mathcal{H}_\Lambda)\) (the space of trace class operators) to \(L_1(\mathcal{H}_E)\), such that, for each \(W \in \mathcal{L}(\mathcal{H}_\Lambda)\), and for each operator \(T \in \mathcal{L}(\mathcal{H}_E)\), we have:

\[
\text{Tr}_A(\text{Tr}_E(W)) = \text{Tr}_E(T \text{Tr}^A_E(W)).
\]

(7.5)

With these notations, we have:

\[
\text{Cov}_{\Lambda, t}(A, B) = \frac{1}{2Z_\Lambda(t)} \text{Tr}_{E_1 \cup E_2} \left( (A' - A'')(B' - B'') \text{Tr}^A_{E_1 \cup E_2}(W(t)) \right).
\]

(7.6)

For each set \(E \subset \mathbb{Z}^d\), let us denote by \(\| \|_{\text{Tr}_E}^T\) the trace norm of an operator in \(\mathcal{H}_E\). Then Theorem 1.3 will follow from the next proposition, (where we write only what is needed for the points a) and b)):

**Proposition 7.1.** With the notations and hypotheses of Theorem 1.3, if \(ht \leq T_0\), \(\varepsilon < \delta < 1\), and if \(t\) is smaller than some constant \(T_1(\varepsilon, \delta) > 0\), the operator \(W(t)\) defined in (7.4) and its integral kernel \(W(X, t)\) satisfy:

\[
\|\|_{\text{Tr}_{E_1 \cup E_2}}^T(W(t))\|_{\text{Tr}_{E_1 \cup E_2}}^T \leq M(|E_1 \cup E_2|, t, h, \varepsilon, \delta) \tilde{Z}_\Lambda(t) \delta_{\text{dist}(E_1, E_2)},
\]

(7.7)

and, if \(\text{Diag}(\Lambda)\) is the diagonal of \((\mathbb{R}^{2p})^\Lambda \times (\mathbb{R}^{2p})^\Lambda\), with its natural measure \(\text{d}v(X)\)

\[
\int_{\text{Diag}(\Lambda)} |W(X, t)| \, \text{d}v(X) \leq t \inf(|E_1|, |E_2|) \, N(\varepsilon, \delta) \tilde{Z}_\Lambda(t) \delta_{\text{dist}(E_1, E_2)}.
\]

In this proof of this proposition, we shall use suitable decompositions of the heat kernel \(U_\Lambda(x, y, t)\) of \(e^{-t\hat{H}_\Lambda}\), and, therefore, of the kernel \(W(x, y, t)\). Each time a function \(\varphi\) is written as a sum of terms \(\varphi_Q\) associated, for example, to the boxes \(Q\) contained in a box \(\Lambda\), the Mayer decomposition is a simple way to write the exponential \(e^{-\varphi}\) as a sum of terms associated to the set of boxes \(\Gamma \in \mathcal{P}(\text{Box}(\Lambda))\), where \(\text{Box}(\Lambda)\) is the set of boxes in \(\Lambda\), not reduced to single points. This technique
is often used for classical spin systems in a lattice (see B. Simon [24]). Let us describe this technique in our situation.

3. Mayer decomposition. For each box $Q$, let $T_Q$ be the operator defined by (3.7) with $\mathbb{R}^p$ replaced by $\mathbb{R}^{2p}$. We apply this operator to the function $\tilde{\psi}_\Lambda(., t)$ defined in (7.2). For each box $Q \subset \Lambda$ which is not a single point, we set

$$M_Q(t) = \sup_{X \in (\mathbb{R}^{2p})^{\Lambda}} (T_Q \tilde{\psi}_\Lambda)(X, t) \quad f_Q(X, t) = e^{M_Q(t) - (T_Q \tilde{\psi}_\Lambda)(X, t)} - 1.$$  

(7.8)

For each box reduced to a single point $\lambda$, we use another notation, and we set:

$$f\{\lambda\}(X, t) = e^{- (T\{\lambda\} \tilde{\psi}_\Lambda)(X, t)}.$$  

(7.9)

We denote by $\tilde{U}_\Lambda^{(0)}(X, t)$ the heat kernel for the free Laplacian in $(\mathbb{R}^{2p})^{\Lambda}$, and we set:

$$\Phi_0(X, t) = \tilde{U}_\Lambda^{(0)}(X, t) e^{- \sum_{Q \in \text{Box}(\Lambda)} M_Q(t)}.$$  

(7.10)

In order to expand the last product, we write, for each set of boxes $\Gamma \in \mathcal{P}(\text{Box}(\Lambda))$:

$$K_\Gamma(X, t) = \Phi_0(X, t) \prod_{\lambda \in \Lambda} f\{\lambda\}(X, t) \prod_{Q \subset \Gamma} f_Q(X, t).$$  

(7.11)

We agree that, if $\Gamma$ is the empty set, the last product is 1. We have $K_\Gamma \geq 0$. We denote by $T_\Gamma(t)$ the operator with integral kernel $K_\Gamma(X, t)$. With these notations, we can write the heat kernel in the following form:

$$\tilde{U}_\Lambda(X, t) = \sum_{\Gamma \in \mathcal{P}(\text{Box}(\Lambda))} K_\Gamma(X, t), \quad e^{-t \tilde{H}_\Lambda} = \sum_{\Gamma \subseteq \text{Box}(\Lambda)} K_\Gamma(t).$$  

(7.12)

We can also write the mean operator $W(t)$ of (7.4), used for the correlation, in the following form, setting $G = G_\Lambda(E_1, E_2)$:

$$W(t) = \sum_{\Gamma \subseteq \text{Box}(\Lambda)} W_\Gamma(X, t) \quad W_\Gamma(t) = \frac{1}{|G|} \sum_{\sigma \in G} (\text{sgn}(\sigma)) K_\Gamma^{(\sigma)}(t).$$  

(7.13)

Among the set of boxes appearing in the Mayer decomposition, some of them, called polymers, play an important role.

4. Polymers, and sets $\mathcal{C}(E_1, E_2)$ and $\mathcal{NC}(E_1, E_2)$. According to the terminology of B. Simon [24], we call polymer a finite sequence of boxes $(Q_1, \ldots, Q_k)$, not reduced to single points, such that $Q_j \cap Q_{j+1} \neq \emptyset$ $(1 \leq j \leq k - 1)$. We say that a set $\Gamma \in \mathcal{P}(\text{Box}(\Lambda))$ is in $\mathcal{C}(E_1, E_2)$ if $E_1$ and $E_2$ are connected by $\Gamma$, i.e., if $\Gamma$ contains a polymer, with its first box intersecting $E_1$, and its last box intersecting $E_2$. We say that $\Gamma$ is in $\mathcal{NC}(E_1, E_2)$ in the opposite case.
All these tools will be used together in Section 8 for the proof of Proposition 7.1, and therefore of Theorem 1.3. Since the case of two observables which are multiplications is much simpler, let us prove the point b) of Theorem 1.3 now.

The next two lemmas will show the role of the sets \( \mathcal{C}(E_1, E_2) \) and \( \mathcal{N} \mathcal{C}(E_1, E_2) \).

**Lemma 7.2.** a) There exists a function \( a(t, \varepsilon) \) of the form \( a(t, \varepsilon) = t K(\varepsilon) e^{t K(\varepsilon)} \), such that, for each polymer \( \Pi = (Q_1, \ldots, Q_k) \), we have:

\[
\prod_{j=1}^{k} |f_{Q_j}(X, t)| \leq N(\Pi, \varepsilon, a(t, \varepsilon)) ,
\]

where, for each \( T \), we set:

\[
N(\Pi, \varepsilon, T) = \prod_{j=1}^{k} T e^{\text{diam}(Q_j)} (\text{diam}(Q_j))^{2d} .
\]

b) For each disjoint finite sets \( E_1 \) and \( E_2 \), for each \( \delta \) such that \( 0 < \varepsilon < \delta < 1 \), there exists \( T_1(\varepsilon, \delta) \) and \( K(\varepsilon, \delta) \) such that, if \( T < T_1(\varepsilon, \delta) \),

\[
\sum_{\Pi \in \text{Pol}(E_1, E_2)} N(\Pi, \varepsilon, T) \leq T \inf(|E_1|, |E_2|) K(\varepsilon, \delta) \delta^{\text{dist}(E_1, E_2)} ,
\]

where \( \text{Pol}(E_1, E_2) \) is the set of polymers connecting \( E_1 \) and \( E_2 \), (the first box intersecting \( E_1 \), and the last one \( E_2 \)).

**Proof.** By (7.8) and by the point 3 of Theorem 1.2, we can write, if \( ht \leq T_0 \),

\[
0 \leq f_Q(X, t) \leq t K_0(\varepsilon) e^{\text{diam}(Q)} (\text{diam}(Q))^{2d} e^{t K_0(\varepsilon)} ,
\]

and (7.14) follows easily. Let us prove the last point, assuming that \( |E_1| \leq |E_2| \).

For each polymer \( \Pi = (Q_1, \ldots, Q_p) \), let \( L(\Pi) \) be the sum of the diameters of the boxes \( Q_j \). We remark that the number of boxes with diameter \( R \geq 1 \) intersecting a given box of diameter \( R_0 \geq 0 \) is at most \( (R_0 + 1)^{2d}(R + 1)^{2d} \). If we take the sum of \( N(\Pi, \varepsilon, T) \) for all polymers starting from \( E_1 \), with a given total length \( L \), it follows that:

\[
\sum_{\Pi \in \text{Pol}(E_1, E_2)} N(\Pi, \varepsilon, T) \leq |E_1| \sum_{R_1 + \cdots + R_p = L} (1 + R_1)^{6d} \cdots (1 + R_p)^{6d} e^{L^p T^p} .
\]

We remember that the number of ordered sequences \( (R_1, \ldots, R_p) \) such that \( R_1 + \cdots + R_p = L \), and \( R_j \geq 1 \), is less than \( C_{L-1}^{p-1} \), and we set \( \Phi(t) = \sup_{R > 0} (1 + R)^{6d} t^R \) for each \( t \in [0, 1] \). If \( 0 < \varepsilon < \gamma \), it follows that:

\[
\sum_{\Pi \in \text{Pol}(E_1, E_2)} N(\Pi, \varepsilon, T) \leq |E_1| T \Phi(\varepsilon/\gamma) \gamma^L \left( 1 + T \Phi(\varepsilon/\gamma) \right)^{L-1} .
\]

We remark that, if \( \Pi \) is a polymer connecting \( E_1 \) and \( E_2 \), we have \( L(\Pi) \geq \text{dist}(E_1, E_2) \). If \( \varepsilon < \delta < 1 \), we apply (7.18) with \( \gamma = \sqrt{\varepsilon \delta} \). There exists \( T_1(\varepsilon, \delta) \).
such that, if $T < T_1(\epsilon, \delta)$, we have $\gamma(1 + T\Phi(\epsilon/\gamma)) \leq \delta$. With this condition, we have:

$$\sum_{\Pi \in \text{Pol}(E_1, E_2)} N(\Pi, \epsilon, T) \leq |E_1| T \Phi((\epsilon/\delta)^{1/2}) \sum_{L = \text{dist}(E_1, E_2)} \delta^L.$$ 

The lemma is proved. \hfill \Box

**Lemma 7.3.** If $\Gamma \in \mathcal{NC}(E_1, E_2)$, if $ht \leq T_0$, we have

$$\sum_{\sigma \in \text{G}_1(E_1, E_2)} (\text{sgn}(\sigma)) K_\Gamma(\sigma)(X, t) = 0 \quad \forall X = (x', x'', x', x'') \in \text{Diag}(\Lambda)$$

where $\text{Diag}(\Lambda)$ is the diagonal of $\mathbb{R}^\Lambda \times (\mathbb{R}^\Lambda)^\Lambda$.

**Proof.** Let $\Gamma \in \mathcal{NC}(E_1, E_2)$. For $k = 1, 2$, let us denote by $E_k$ the set of points $\lambda \in \Lambda$ which are, either in $E_k$, or connected to $E_k$ by a polymer in $\Gamma$. By Theorem 1.2, the function $f_Q$, when it is restricted to the diagonal, depends only on the variables $x_k$. By (7.2) and (7.9), it is also invariant when all the variables $(x', y')$ and $(x'', y'')$ are permuted. By a combination of these remarks, for each $\sigma \in \Gamma$, $f_Q$, restricted to the diagonal, is invariant by the map $\sigma_1 = \tau_{E_1}$ defined in (7.3), and therefore $K_\Gamma(\sigma)(X, t) = K_\Gamma(X, t)$ if $X$ is in the diagonal, and the lemma follows easily. \hfill \Box

**Proof of Theorem 1.3 (point b).** We can use the sum (7.13) for $W(X, t)$. When $X$ is in the diagonal $(x' = y', x'' = y'')$, the contribution of the terms $\Gamma \in \mathcal{NC}(E_1, E_2)$ in this sum (7.13) vanishes by Lemma 7.3. For each $\Gamma \in \mathcal{C}(E_1, E_2)$, we can choose a polymer $\Pi = \text{Pol}(E_1, E_2)$ contained in $\Gamma$. Then, we have, by Lemma 7.2, for each $\sigma$ in $G = G_\Lambda(E_1, E_2)$:

$$\sum_{\Gamma \in \mathcal{C}(E_1, E_2)} |K_\Gamma(\sigma)(X, t)| \leq \sum_{\Pi \in \text{Pol}(E_1, E_2)} N(\Pi, \epsilon, \mu) \sum_{\Gamma' \subseteq \text{Box}(\Lambda)} |K_{\Gamma'}(\sigma)(X, t)|$$

where $\mu = tK_0(\epsilon)e^{K_0(\epsilon)}$. We sum over $\sigma \in G$ and integrate on the diagonal. Applying the positivity of $K_\Gamma$ and (7.12), we obtain:

$$\int_{(\mathbb{R}^\Lambda)^\Lambda \times (\mathbb{R}^\Lambda)^\Lambda} |W(x', x'', x', x'', t)|dx'dx'' \leq \tilde{Z}_\Lambda(t) \sum_{\Pi \in \text{Pol}(E_1, E_2)} N(\Pi, \epsilon, a(t, \epsilon)).$$

By the last point of Lemma 7.2, the estimate (7.7), and the point b) of Theorem 1.3 follow. \hfill \Box

**8. Estimation of the quantum correlations in the general case**

Now, we shall prove (7.6), and therefore Theorem 1.3 in the general case. If the integral kernel of $W(t)$, (defined in (7.4)), is denoted by $W(x, y, t)$, the integral kernel of the partial trace $\text{Tr}_E^2 W(t)$ ($E = E_1 \cup E_2$) is

$$K(x_E, y_E, t) = \int_{(\mathbb{R}^{2\Lambda})^\Lambda} W(x_E, x_{\Lambda\setminus E}, y_E, x_{\Lambda\setminus E}, t)dx_{\Lambda\setminus E} \quad E = E_1 \cup E_2. \quad (8.1)$$
We shall estimate the trace norm of this partial trace, using the norm \( \| \cdot \|_{E,m,m',\mu} \) defined in (5.1).

**Proposition 8.1 (Main estimate).** With the notations and hypotheses of Proposition 7.1, if \( 0 < \varepsilon < \delta < 1 \), if \( h t \leq T_0 \), and if \( t \) is smaller than some constant \( T_1(\varepsilon, \delta) \), we have, for some constants \( m, m' \), for some function \( \mu(t) \) and for some function \( M = M([E_1 \cup E_2], t, h, \varepsilon, \delta) \):

\[
\| \text{Tr}^A_{E_1 \cup E_2} (W(t)) \|_{E_1 \cup E_2} \leq M \delta^{\text{dist}(E_1, E_2)} \sum_{\Gamma \subseteq \text{Box}(\Lambda)} \| K_{\Gamma}(t) \|_{E,m,m',\mu(t)}. \tag{8.2}
\]

Almost all this section is devoted to the proof of this proposition. Classically, there exist \( C > 0 \), \( m > 0 \) and \( m' > 0 \), depending on \( |E| \), such that, for each operator \( K \) with integral kernel in the Schwartz space, we have:

\[
\| \text{Tr}^A_E(K) \|_{E \leq C} \leq K \|_{E,m,m',0}. \tag{8.3}
\]

For each \( \Gamma \in \mathcal{NC}(E_1, E_2) \), let us denote by \( H_{\Gamma} \) the subgroup of all \( \sigma \in G_{\Lambda}(E_1, E_2) \) which are constant on each connected component of \( \Gamma \). Instead of the operators \( W_{\Gamma}(t) \) defined in (7.13), it will be easier to estimate the following ones:

\[
K_{\Gamma}^{\Lambda^\varepsilon}(t) = \frac{1}{|H_{\Gamma}|} \sum_{\sigma \in H_{\Gamma}} \text{sgn}(\sigma) K_{\Gamma}^{(\sigma)}(t). \tag{8.4}
\]

We denote by \( K_{\Gamma}^{\Lambda^\varepsilon}(X, t) \) the integral kernel of this operator. We see easily that:

\[
W_{\Gamma}(t) = \frac{1}{|G|} \sum_{\sigma \in G} (\text{sgn}(\sigma)) [K_{\Gamma}^{\Lambda^\varepsilon}]^{(\sigma)}(t).
\]

It follows easily, setting \( E = E_1 \cup E_2 \), that, for each set of boxes \( \Gamma \):

\[
\| \text{Tr}^A_E(W_{\Gamma}(t)) \|_{E \leq C} \leq K_{\Gamma}^{\Lambda^\varepsilon}(t) \|_{E,m,m',0} \leq C \| K_{\Gamma}(t) \|_{E,m,m',0}. \tag{8.5}
\]

For the set of boxes \( \Gamma \in \mathcal{C}(E_1, E_2) \), we use only a modification of the estimates (7.19) and (7.16), with also bounds for the derivatives of the functions. Without writing all the details, we obtain:

\[
\sum_{\Gamma \in \mathcal{C}(E_1, E_2)} \| K_{\Gamma}(t) \|_{E,m,m',0} \leq M \delta^{\text{dist}(E_1, E_2)} \sum_{\Gamma \subseteq \text{Box}(\Lambda)} \| K_{\Gamma}(t) \|_{E,m,m',\mu(t)}. \tag{8.6}
\]

where \( M \) depends on \( |E_1 \cup E_2|, t, h, \varepsilon, \) and \( \delta \). For the set of boxes \( \Gamma \in \mathcal{NC}(E_1, E_2) \), we shall estimate \( \| K_{\Gamma}^{\Lambda^\varepsilon}(t) \|_{m,m',\mu} \) for that, we need a study of \( K_{\Gamma}^{\Lambda^\varepsilon}(X, t) \) only in the following set, where \( E = E_1 \cup E_2 \):

\[
\text{Diag}(\Lambda, E) = \left\{ (x, y) \in (\mathbb{R}^{2p})^\Lambda \times (\mathbb{R}^{2p})^\Lambda, \ x_\lambda = y_\lambda \ \forall \lambda \notin E \right\}. \tag{8.7}
\]

We shall write \( f \sim g \) if \( f(X, t) = g(X, t) \) for all \( X \) in \( \text{Diag}(\Lambda, E_1 \cup E_2) \). Proposition 8.1 will follow from the above inequalities and from:
Proposition 8.2. For each $\Gamma$ and $\Gamma'$ such that $\Gamma \in \mathcal{NC}(E_1,E_2)$ and $\Gamma' \subseteq \Gamma$, we shall find a function $B_{\Gamma,\Gamma'}(X,t)$ such that, if $ht \leq T_0$:

$$K_{\Gamma}^{\Lambda t}(X,t) \sim \sum_{\Gamma' \subseteq \Gamma} B_{\Gamma,\Gamma'}(X,t)K_{\Gamma'}(X,t).$$  \hspace{1cm} (8.8)

For each $\Gamma' \in \mathcal{NC}(E_1,E_2)$, the function $S_{\Gamma'}$ defined by:

$$S_{\Gamma'}(X,t) = \sum_{\Gamma' \in \mathcal{NC}(E_1, E_2)} |B_{\Gamma,\Gamma'}(X,t)|$$  \hspace{1cm} (8.9)

satisfies, if $0 < \varepsilon < \delta$, $ht \leq T_0$, if $t$ is smaller than some constant $T_1(\varepsilon, \delta)$, and if $X \in \text{Diag}(\Lambda, E_1 \cup E_2)$,

$$|S_{\Gamma'}(X,t)| \leq M e^{\mu(t)}|x_E - y_E| \delta^{\text{dist}(E_1, E_2)},$$  \hspace{1cm} (8.10)

where $M$ and $\mu$ some functions, depending on $t$, $h$, $\varepsilon$, $\delta$ and $|E_1 \cup E_2|$, but independent of $\Lambda$ containing $E_1$ and $E_2$. We have similar estimates for the derivatives of $B_{\Gamma,\Gamma'}(X,t)$ with respect to $x_E$ and $y_E$. The constants in the inequalities depend on the order of derivation, but not the condition of validity $t < T_1(\varepsilon, \delta)$.

The proof of this proposition will need two more lemmas. First, let us introduce some functions which are bounded (in Lemma 8.3) like the RHS of (8.10).

Then, we shall write $K_{\Gamma}^{\Lambda t}(X,t)$, at least on the set $\text{Diag}(\Lambda, E_1 \cup E_2)$, as a polynomial expression of such functions.

For each $\Gamma \in \mathcal{NC}(E_1, E_2)$, we denote by $E_k$ ($k = 1, 2$), the set of points $\lambda \in \Lambda$ which are, either in $E_k$, or connected to $E_k$ by a polymer in $\Gamma$. The maps $\tau_{E_k}$ defined in (7.5) will be denoted by $\sigma_k$. By applying the operators $\sigma_1$ and $\sigma_2$ to the functions $f_Q$ of (7.8), or to the functions $f_\lambda$ of (7.9), we define the following functions $U_Q$, $V_Q$, $W_Q$, $U_\lambda$, $V_\lambda$, and $W_\lambda$, by:

$$U_Q = f_Q^{(\sigma_1)} - f_Q,$$  \hspace{1cm} (8.11)

$$V_Q = f_Q^{(\sigma_2)} - f_Q,$$  \hspace{1cm} (8.12)

$$W_Q := f_Q - f_Q^{(\sigma_1)} - f_Q^{(\sigma_2)} + f_Q^{(\sigma_1 \sigma_2)},$$  \hspace{1cm} (8.13)

$$W_\lambda := [f_\lambda - f_\lambda^{(\sigma_1)} - f_\lambda^{(\sigma_2)} + f_\lambda^{(\sigma_1 \sigma_2)}]f_\lambda^{-1}.$$  \hspace{1cm} (8.14)

The definition of these functions depends on the set $\Gamma$ since $\sigma_1$ and $\sigma_2$ depend on it. For the estimations of these functions, we shall use the following ones, where $X = (x, y) = (x', y'), Q$ is a box, $\langle Q \rangle$ is defined in (1.14), and $T > 0$:

$$M(Q, X) = \langle Q \rangle^{2d} \sum_{\alpha \in E_1 \cup E_2} |x_\alpha - y_\alpha| e^{\text{diam}(Q \cup \{\alpha\})},$$  \hspace{1cm} (8.15)

$$N(Q, X, \varepsilon, T) = T \sum_{\alpha \in E_1 \beta \in E_2} |x_\alpha - y_\alpha||x_\beta - y_\beta| e^{\text{diam}(Q \cup \{\alpha, \beta\})} \langle Q \rangle^{4d}.$$  \hspace{1cm} (8.16)
Lemma 8.3. With these notations, we can write, for each $\Gamma \in \mathcal{NC}(E_1, E_2)$, for each boxes $P$ and $Q$ such that $P$ is disjoint from $\hat{E}_1$ and $Q$ is disjoint from $\hat{E}_2$, if $X \in \text{Diag}(\Lambda, E_1 \cup E_2)$, and $ht \leq T_0$:

$$|U_P(X, t)| \leq a(t, \varepsilon)M(P, X, \varepsilon) \quad |V_Q(X, t)| \leq a(t, \varepsilon)M(Q, X, \varepsilon),$$  \hspace{1cm} (8.16)

where $a(t, \varepsilon) = tK(\varepsilon)e^{tK(\varepsilon)}$, ($K(\varepsilon)$ being independent of $\Lambda, E_1$ and $E_2$). If $Q$ is disjoint from $\hat{E}_1$ and $\hat{E}_2$, we have:

$$|W_Q(X, t)| \leq N(Q, X, \varepsilon, a(t, \varepsilon)).$$  \hspace{1cm} (8.17)

We have also similar estimations for points. If $\Phi_0$ is defined in (7.10), there exists a function $\Delta_0$ such that:

$$\Phi_0^{(\sigma_1)} - \Phi_0 \sim \Phi_0 \Delta_0 \quad |\Delta_0(X, t)| \leq tK(\varepsilon)e^{\text{dist}(E_1, E_2)}e^{tK(\varepsilon)}|x_E - y_{E_1}|.$$  \hspace{1cm} (8.18)

There exists $K(\varepsilon)$ such that, for each $T > 0$, for each finite set $\Lambda$,

$$\sum_{E \subset \Lambda} \prod_{\lambda \in E} TM(\lambda, X, \varepsilon) + \sum_{E \subset \text{Box}(\Lambda)} \prod_{E \subset Q} TM(Q, X, \varepsilon) \leq e^{tK(\varepsilon)}|x_E - y_{E_1}|.$$  \hspace{1cm} (8.19)

Proof. When it is restricted to $\text{Diag}(\Lambda, E_1 \cup E_2)$, $f_P$ depends only on $x_{E_1} - y_{E_1}$, $x_{E_2} - y_{E_2}$, and $x_P$. If $P$ is disjoint from $\hat{E}_1$, the map $\sigma_1 = \tau_{E_1}$ has the same effect, on $f_P$ restricted to $\text{Diag}(\Lambda, E_1 \cup E_2)$, as the permutation $\tau_{E_1}$. Then the estimation of $U_P$ follows from Theorem 1.2. For the functions associated to points, we need also the last statement of Theorem 1.2, which shows that the function $T(\lambda)\tilde{\psi}_\lambda(., t)$, up to an error $O(t + h^2)$, is equal to a function which depends only on $x_\lambda$ and $y_\lambda$, and is invariant by $\sigma_1$ and $\sigma_2$. By the form (7.10) of $\Phi_0$, by the form (7.2) of $\psi_\lambda$, the equality in (8.18) will be satisfied if we choose $\Delta_0 = e^\varrho - 1$, with

$$g(X, t) = \frac{1}{2} \left[ \tilde{\psi}_\lambda(0, y - x, t) + \tilde{\psi}_\lambda^{(\sigma_1, \sigma_2)}(0, y - x, t) - \tilde{\psi}_\lambda^{(\sigma_1)}(0, y - x, t) - \tilde{\psi}_\lambda^{(\sigma_2)}(0, y - x, t) \right].$$

The inequality of (8.18) follows from Theorem 1.1 if $ht \leq T_0$. The last inequality (8.19) is a consequence of the following:

$$\sum_{\lambda \in \mathbb{Z}^d} M(\lambda, X, \varepsilon) + \sum_{Q \in \text{Box}(\mathbb{Z}^d)} M(Q, X, \varepsilon) \leq K(\varepsilon)|x_E - y_{E_1}|.$$  \hspace{1cm} \square

Among the functions defined in (8.11)-(8.13), only the functions $N(Q, X, \varepsilon, T)$ and its analogue for points have a good rate of decay, when $\text{dist}(E_1, E_2)$ is large. They are used to estimate the functions $W_Q$ and $W_\lambda$. Beside these functions, we need also other functions, associated to polymers. If $\Pi = (Q_1, \ldots, Q_k)$ ($k \geq 2$) is a polymer, which does not connect $E_1$ and $E_2$, we have no information on the sum of the lengths of its boxes, and the product $f_{Q_1} \ldots f_{Q_k}$ has not necessarily the good rate of decay. However, we shall see that, if $U_Q$ and $V_Q$ are defined in (8.11) and (8.12), the functions $U_Q, f_Q, \ldots, f_{Q_k-1}, V_{Q_k}$ has a good rate of decay, like in (8.10), in terms of $\text{dist}(E_1, E_2)$. In order to make more precise this idea, let us define the functions, used for the estimations.
If \( \Pi = (Q_1, \ldots, Q_k) \) \((k \geq 2)\) is a polymer, we set, for each points \( \alpha, \beta \in E_1 \) and for each \( T > 0 \):

\[
N_{\alpha, \beta}(\Pi, \varepsilon, T) = T^k \varepsilon^{d_{\text{diam}}(Q_1 \cup \{\alpha\}) + d_{\text{diam}}(Q_k \cup \{\beta\})} \prod_{j=2}^{k-1} \varepsilon^{d_{\text{diam}}(Q_j)} \prod_{j=1}^{k} (Q_j)^{3d}. \tag{8.20}
\]

Then we set:

\[
N(\Pi, X, \varepsilon, T) = \sum_{\alpha \in E_1, \beta \in E_2} (1 + |x_\alpha - y_\alpha|)(1 + |x_\beta - y_\beta|)N_{\alpha, \beta}(\Pi, \varepsilon, T). \tag{8.21}
\]

**Lemma 8.4.** If \( \Pi = (Q_1, \ldots, Q_k) \) is a polymer, all its boxes belonging to \( \Gamma \in N\mathcal{C}(E_1, E_2) \), we can write, with some function \( a(t, \varepsilon) \):

a) If \( \Pi \) is starting from \( E_1 \) \((\text{i.e., } Q_1 \cap E_1 = \emptyset)\), and if \( \mu \in Q_k \),

\[
|f_{Q_1} \cdots f_{Q_{k-1}} V_{Q_k}(X, t)| + |f_{Q_1} \cdots f_{Q_{k-1}} f_{Q_k}(X, t)V_{\mu}(X, t)| \leq N(\Pi, X, \varepsilon, a(t, \varepsilon)). \tag{8.22}
\]

b) We have similar results if \( \Pi \) starts from \( E_2 \).

c) If all the boxes \( Q_1, \ldots, Q_k \) of \( \Pi \) are disjoint from \( \hat{E}_1 \) and \( \hat{E}_2 \), if \( \lambda \in Q_1 \) and \( \mu \in Q_k \), we can write:

\[
|U_{Q_1} \cdots V_{Q_k}(X, t)| + |U_{\lambda} f_{Q_1} \cdots V_{Q_k}(X, t)| \leq N(\Pi, X, \varepsilon, a(t, \varepsilon)) \tag{8.23}
\]

\[
|U_{Q_1} \cdots f_{Q_{k-1}} V_{\mu}(X, t)| + |U_{\lambda} f_{Q_1} \cdots f_{Q_{k-1}} V_{\mu}(X, t)| \leq N(\Pi, X, \varepsilon, a(t, \varepsilon)) \tag{8.24}
\]

The factors which are not written in these products are the \( f_{Q_j} \). In the points a), b) and c), we have similar estimations for the derivatives.

d) We can write also, if \( 0 < \varepsilon < \delta < 1 \) and \( t \leq T_1(\varepsilon, \delta) \):

\[
\sum_\Pi N(\Pi, X, \varepsilon, T) \leq K(T, \varepsilon, \delta, |E|)(1 + |x_E - y_{E}|)^2 \delta^{\text{dist}(E_1, E_2)} \tag{8.25}
\]

where the sum is taken on all the polymers in \( \mathbb{Z}^d \), and \( N(\Pi, X, \varepsilon, T) \) is defined in (8.21) if \( \Pi \) has at least two boxes, and in (8.15) if \( \Pi \) is reduced to a single box \( Q \).

**Proof.** The points a), b) and c) follow from Lemma 8.2. With our hypotheses, if \( \Pi \) is starting from \( E_1 \), all its boxes \( Q \) satisfy \( Q \cap \hat{E}_2 = \emptyset \), and we can apply (8.16) for \( V_{Q_k} \). For the last point, given two boxes \( Q \) and \( Q' \), we first make a summation over all polymers connecting \( Q \) and \( Q' \). Then we make a sum over the boxes \( Q \) and \( Q' \). For the first sum, we apply Lemma 7.2 with \( E_1 \) and \( E_2 \) replaced by \( Q \) and \( Q' \). For the second summation, we use the following inequality, if \( 0 < \varepsilon < \delta \) and \( \delta_1 = \sqrt{\delta} \):

\[
\sum_{Q, Q' \in \text{Box}(\mathbb{Z}^d)} \langle Q \rangle^{4d} \langle Q' \rangle^{4d} \delta_1^L(\alpha, Q, Q', \beta) \leq C(\varepsilon, \delta) \delta^{1-|\alpha - \beta|}, \tag{8.26}
\]

where we set, for every boxes \( Q \) and \( Q' \), for every points or sets \( E \) and \( E' \):

\[
L(E, Q, Q', E') = \text{diam}(Q \cup E) + \text{dist}(Q, Q') + \text{diam}(Q' \cup E').
\]

\( \square \)
Proof of Proposition 8.2. Step 1. Generators of $H_{\Gamma}$. For each set of boxes $A$, let us denote by $A_{\text{pct}}$ the corresponding set of points. We shall denote $\tau_A$ and $\tau_{\lambda}$ instead of $\tau_{A_{\text{pct}}}$ and $\tau_{\{\lambda\}}$ the operators defined like in (7.3). Let $\text{Comp}(\Gamma)$ be the set of connected components $A$ of $\Gamma$ such that $A_{\text{pct}}$ is disjoint from $E_1 \cup E_2$, and therefore from $\hat{E}_1 \cup \hat{E}_2$. We set $\text{Ext}(\Gamma) = \Lambda \setminus (\Gamma_{\text{pct}} \cup E_1 \cup E_2)$. The group $H_{\Gamma}$ is generated by the $\tau_A$ ($A \in \text{Comp}(\Gamma)$), by the already introduced $\tau_{\{\lambda\}}$ ($\lambda \in \text{Ext}(\Gamma)$), and by the elements $\sigma_k = \tau_{\hat{E}_k}$ ($k = 1, 2$). Therefore

$$|H_{\Gamma}| = 2^{(|\text{Comp}(\Gamma)|+|\text{Ext}(\Gamma)|+2)}.$$

The generators of the group $H_{\Gamma}$, listed above, have a different action on the factors of the product (7.11) defining $K_{\Gamma}$. Let us distinguish them. For each connected component $A$ in $\text{Comp}(\Gamma)$, let:

$$F_A = \prod_{Q \in A} f_Q \prod_{\lambda \in A_{\text{pct}}} f_{\lambda}.$$

For the connected components containing a box intersecting $E_1$, or $E_2$, we set:

$$\Phi_k(X,t) = \prod_{Q \in \Gamma \setminus C(E_k)} f_Q \prod_{\lambda \in \hat{E}_k} f_{\lambda} (1 \leq k \leq 2).$$

Then, using also the function $\Phi_0$ of (7.10), we can write:

$$K_{\Gamma} = \Phi_0 \Phi_1 \Phi_2 \prod_{A \in \text{Comp}(\Gamma)} F_A \prod_{\lambda \in \text{Ext}(\Gamma)} f_{\lambda}.$$

By Theorem 1.2, for each box $Q$, the function $f_Q(\cdot, t)$, restricted to $\text{Diag}(\Lambda, E_1 \cup E_2)$, depends only on the variables $x_{E_1} - y_{E_1}$, $x_{E_2} - y_{E_2}$, $x_Q$ and $y_Q$. The equality (7.2) shows that this function is invariant when all the variables $(x', y')$ and $(x'', y'')$ are permuted. From these two remarks, some properties follow for the functions defined above. We can write, if $A$ and $B$ are in $\text{Comp}(\Gamma)$, $A \neq B$, if $\lambda$ and $\mu$ are in $\text{Ext}(\Gamma)$, $\lambda \neq \mu$:

$$F_A^{(\tau_B)} \sim F_A \quad F_A^{(\tau_D)} \sim F_A^{(\tau_{\sigma_2})} \quad F_A^{(\tau_2)} \sim F_A \quad f_{\lambda}^{(\tau_2)} \sim f_{\lambda}$$

$$f_{\lambda}^{(\tau_2)} \sim f_{\lambda} \quad \Phi_k^{(\tau_2)} \sim \Phi_k^{(\tau_2)} \sim \Phi_k \quad (0 \leq k \leq 2) \quad \Phi_1^{(\sigma_1)} \sim \Phi_1^{(\sigma_2)}, \text{ etc.} \ldots$$

Let us denote by $H_{\Gamma}^+$ the subgroup of $H_{\Gamma}$ generated by the $\tau_A$ ($A \in \text{Comp}(\Gamma)$) and $\tau_{\lambda}$ ($\lambda \in \text{Ext}(\Gamma)$). By the above equalities and similar ones, we can write:

$$K_{\Gamma}^{+} \sim 2K_{\Gamma}^{+} \quad K_{\Gamma}^{+}(X,t) = \frac{1}{|H_{\Gamma}|} \sum_{\sigma \in H_{\Gamma}^+} K_{\Gamma}^{(\sigma)}(X,t).$$
We have also:

\[ K_\Gamma^+ \sim \frac{1}{|\Gamma|} \Phi_0 \Phi_1 \Phi_2 \prod_{\Lambda \in \text{Comp}(\Gamma)} \left[ F_\Lambda + F_\Lambda^{(\sigma_1, \sigma_2)} \right] \prod_{\lambda \in \text{Ext}(\Gamma)} \left[ f_\lambda + f_\lambda^{(\sigma_1, \sigma_2)} \right], \]  

(8.29)

\[ (K_\Gamma^+)^{(\sigma_1)} \sim \frac{1}{|\Gamma|} \Phi_0^{(\sigma_1)} \Phi_1^{(\sigma_2)} \Phi_2^{(\sigma_1)} \prod_{\Lambda \in \text{Comp}(\Gamma)} \left[ F_\Lambda^{(\sigma_1)} + F_\Lambda^{(\sigma_2)} \right] \prod_{\lambda \in \text{Ext}(\Gamma)} \left[ f_\lambda^{(\sigma_1)} + f_\lambda^{(\sigma_2)} \right] \]  

(8.30)

**Step 2. Polynomial expression of \( K_\Gamma^{\lambda V} \).** Now, we shall write the difference between (8.29) and (8.30) as a polynomial expression of functions that are bounded like in (8.10). We remember that:

\[ f_Q^{(\sigma_1)} = f_Q + U_Q, \quad f_Q^{(\sigma_2)} = f_Q + V_Q, \quad f_Q^{(\sigma_1, \sigma_2)} = f_Q + U_Q + V_Q + W_Q. \]

According to the notations (8.11)–(8.18), we have \( f_\lambda^{(\sigma_1)} = f_\lambda (1 + U_\lambda) \), etc. . . and \( \Phi_0^{(\sigma_1)} \sim (1 + \Delta_0) \Phi_0 \). Thus we can write

\[ K_\Gamma^{\lambda V} \sim G_\Gamma \Phi_0 \prod_{\lambda \in \Lambda} f_\lambda, \]

where \( G_\Gamma \) is a polynomial expression of the functions \( f_Q, U_Q, V_Q \) and \( W_Q \) (\( Q \in \text{Box}(\Lambda) \)), of the functions \( U_\Lambda, V_\Lambda \) and \( W_\Lambda \) (\( \lambda \in \Lambda \)), and of the function \( \Delta_0 \). Let us describe more carefully this polynomial. Let \( I \) be the set of partitions of \( \Gamma \) in four subsets \( F, U, V, W \). Let \( I_p \) be the set of triples \( (U^p, V^p, W^p) \) such that \( U^p, V^p, W^p \) are disjoint subsets of \( \Lambda \). Let \( J = I \times I_p \times \{0, 1\} \). Each element \( j \in J \) will be written \( j = (F_j, \ldots, W_j^p, m_j) \). Thus we can write:

\[ K_\Gamma^{\lambda V} \sim \sum_{j \in J} c_j G_\Gamma^{[j]} \quad G_\Gamma^{[j]} = \Phi_0 \Delta_0^{m_j} \prod_{Q \in F_j} f_Q \prod_{Q \in U_j} U_Q \cdots \prod_{\lambda \in W_j^p} W_\Lambda \prod_{\lambda \in \Lambda} f_\lambda \]  

(8.31)

where the coefficient \( c_j \) are constant, and uniformly bounded. Moreover, we have \( c_j = 0 \) unless one of the following three conditions (A), (B) or (C) is satisfied.

\[ (A) \quad W_j \neq \emptyset \quad \text{or} \quad W_j^p \neq \emptyset \quad \text{or} \quad m_j = 1 \]

\[ (B) \quad (V_j)^\text{pct} \cap \widehat{E}_1 \neq \emptyset \quad \text{or} \quad V_j^p \cap \widehat{E}_1 \neq \emptyset \quad \text{or} \quad (U_j)^\text{pct} \cap \widehat{E}_2 \neq \emptyset \quad \text{or} \quad U_j^p \cap \widehat{E}_2 \neq \emptyset \]

\[ (C) \text{ There exists a connected component } A \in \text{Comp}(\Gamma) \text{ such that, in one hand, } A \cap U_j \neq \emptyset, \text{ or } A^\text{pct} \cap U_j^p \neq \emptyset, \text{ and in the other hand, } A \cap V_j \neq \emptyset, \text{ or } A^\text{pct} \cap V_j^p \neq \emptyset. \]

Moreover, we have also \( c_j = 0 \) unless all the following condition are satisfied, and also the similar ones for the sets of points:

\[ (U_j)^\text{pct} \cap \widehat{E}_1 = \emptyset \quad (V_j)^\text{pct} \cap \widehat{E}_1 = \emptyset \quad (V_j)^\text{pct} \cap (\widehat{E}_1 \cup \widehat{E}_2) = \emptyset. \]
Step 3. Construction of \( B_{\Gamma, \Gamma'} \). For each \( j \in J \) such that \( c_j \neq 0 \), we shall write the term \( G^{(j)}_{\Gamma} \) in (8.31) in the form:

\[
G^{(j)}_{\Gamma} = B_j K_{\Gamma, j}
\]

where \( K_{\Gamma, j} \) is defined as in (7.11), with \( \Gamma \) replaced by some subset \( \Gamma_j \). The function \( B_j \) will have the rate of decay of (8.10) when \( \text{dist}(E_1, E_2) \) is large, because the product defining \( B_j \) will contain, either \( \Delta_0 \), or a function \( W_{\lambda} \), or a function, like those of Lemma 8.4, corresponding to a polymer \( \Pi_j \). In order to apply Lemma 8.4, we shall need sometimes one point \( \lambda_j \), or two. We shall denote by \( X_j \) the set of points needed for the application of Lemma 8.4: this set has 0, 1 or 2 points. Let us define \( \Gamma_j, \Pi_j \) and \( X_j \) in all the cases A, B and C.

If \( j \) satisfies (A), let \( \Gamma_j = F_j \), and let \( B_j \) be defined by (8.32). In the first case of (A), the polymer \( \Pi_j \) is reduced to single box \( Q \) chosen in \( W_j \), and \( X_j = \emptyset \).

In the second case, \( \Pi_j = \emptyset \), and for \( X_j \), we choose one point in \( W_j \). In the third case, \( \Pi_j = \emptyset \), and \( X_j = \emptyset \).

If \( j \) satisfies the first condition of (B), let \( Q \) be a box in \( V_j \) such that \( Q \cap \tilde{E}_1 \neq \emptyset \). There is a polymer \( \Pi_j \), starting in \( E_1 \), the last box of which being \( Q \). Let \( \Gamma_j = F_j \setminus \Pi_j \) be the set of the boxes in \( F_j \), excepted those we took from \( F_j \) to construct the polymer \( \Pi_j \). Let \( B_j \) be defined by (8.32), and \( X_j = \emptyset \). If \( V_j \cap \tilde{E}_1 \neq \emptyset \), let \( \mu_j \) be a point in this set. There is a polymer \( \Pi_j \), starting from \( E_1 \), such that \( \mu_j \) is in its last box. We define \( \Gamma_j \) and \( B_j \) as before, but \( X_j = \{ \mu_j \} \). We proceed in the same way in the other cases of (B).

Now, let \( j \) satisfying one of the conditions of (C), for example such that, for some connected component \( A \in \text{Comp}(\Gamma) \), we have \( A \cap U_j \neq \emptyset \) and \( A_{\text{net}} \cap V_j \neq \emptyset \). Let \( P \) be a box in the first set, and \( \mu_j \) be a point in the second one. Since they are in the same connected component, there is a polymer \( \Pi_j \), whose first box is \( P \), and such that \( \mu_j \) is in its last box. We define \( \Gamma_j \) and \( B_j \) as before, but \( X_j = \{ \mu_j \} \). We proceed in the same way in the other cases of (B).

Now for all \( j \), we have defined \( \Gamma_j \) and \( B_j \) such that (8.32) is satisfied. If, for some \( j \in J \), we are in several cases, we make a choice. For each \( \Gamma' \subseteq \Gamma \), we set

\[
B_{\Gamma, \Gamma'} = \sum_{j \in J} c_j B_j
\]

and (8.8) is satisfied.

Step 4. Estimation of \( B_{\Gamma, \Gamma'} \). When a polymer \( \Pi_j \) is used when applying Lemma 8.4, with a set \( \chi_j \subset \Lambda \) with 0, 1 or 2 points, we can write, by Lemmas 8.3 and 8.4:

\[
|B_j(X, t)| \leq N(\Pi_j, X, \varepsilon, a(t, \varepsilon))
\]

\[
\ldots \prod_{Q \in (U_j \cup V_j \cup W_j) \setminus \Pi_j} a(t, \varepsilon) M(Q, X, \varepsilon) \ldots \prod_{\lambda \in (U_j' \cup V_j' \cup W_j') \setminus \chi_j} a(t, \varepsilon) M(\lambda, X, \varepsilon).
\]

In order to estimate \( B_{\Gamma, \Gamma'} \), following (8.33), we make first a summation on all the sets \( \chi_j \) corresponding to a same polymer \( \Pi_j \): this gives only a change in the power...
of \((Q)\) in the definition of \(N(\Pi, \ldots)\). Then we sum on all the sets \(U_{\Pi}^p, V_{\Pi}^p\) and \(W_{\Pi}^p\), applying the last point of Lemma 8.3. We obtain, proceeding in a similar way for the terms without polymer, setting \(T = a(t, \varepsilon)\):

\[
|B_{\Gamma'}(X, t)| \leq e^{T|x|} |\mathcal{B}_{\Gamma}, \Gamma'|(X, t) \leq e^{x T} \sum_{\Pi \in \mathcal{Pol}_{\Gamma} \setminus \Gamma'} N(\Pi, X, \varepsilon, T) \prod_{Q \in \Gamma' \cup \Pi} TM_Q(X, \varepsilon).
\]

Therefore, applying the last points of Lemmas 8.3 and 8.4, we see that the function \(S_{\Gamma'}\) defined in (8.9) satisfies (8.10). We shall not write the details for the derivatives of \(B_{\Gamma'}\).

End of the proof of Proposition 8.1.

It remains to give an analogue of (8.6) for sets of boxes in \(\mathcal{N}_C(E_1, E_2)\). By Proposition 8.2, with the similar estimates for derivatives, we can write, for some constants \(m_1\) and \(m'_1\):

\[
\sum_{\Gamma \subseteq \mathcal{Box}(\Lambda)} \|K_{\Gamma}(t)\|_{E, m, m'_1, 0} \leq M \delta_{\text{dist}(E_1, E_2)} \sum_{\Gamma \subseteq \mathcal{Box}(\Lambda)} \|K_{\Gamma}(t)\|_{E, m_1, m'_1, \mu_1(t)}.
\]

where \(M\) and \(\mu_1\) depend on \(|E_1 \cup E_2|\), \(t\), \(h\), \(\varepsilon\), and \(\delta\). By (8.5), (8.6) and (7.13), Proposition 8.1 is proved.

The estimate (7.6) of Proposition 7.1, and therefore the point a) of Theorem 1.1, are proven. The points b) and d) were proved in Section 7, and the point c), (estimation of \(K_{\text{op}, \text{fc}}(E_1, E_2, t, h)\)), needs a small modification in the proof of the point a): we need a partial trace \(\text{Tr}_{E_1}^{\Lambda}\) instead of \(\text{Tr}_{E_1 \cup E_2}^{\Lambda}\). Thus we obtain, in the estimation, a constant depending on \(|E_1|\) instead of \(|E_1 \cup E_2|\). Theorem 1.3 is proved.

9. Proof of Theorems 1.4 and 1.5

For the proof of Theorem 1.4, we consider a box \(\Lambda\) in \(\mathbb{Z}^d\), which is the union of two disjoint boxes \(\Lambda_1\) and \(\Lambda_2\), separated by an hyperplane \(\Sigma\), like in (6.1). We consider a local observable \(A\), supported in one of these sets, for example \(\Lambda_1\). We shall estimate the difference between \(E_{\Lambda, t}(A)\), (the mean value of \(A\), defined
by (1.2), when $A$ is seen as an operator in $\mathcal{H}_\Lambda$, and $E_{\Lambda_1,t}(A)$, (the analogue for $\Lambda_1$). Theorem 1.4 will be an easy consequence of the next Proposition, since $\Lambda_{m+n}$ is obtain from $\Lambda_n$ (defined in (1.19)) by applying $2d$ times this procedure of enlarging.

**Proposition 9.1.** With the above notations, if the interaction satisfies $(H_\varepsilon)$, if $\varepsilon < \delta < 1$, there exists $T_1(\varepsilon, \delta)$ and a function $K(t, h, \varepsilon, \delta, N)$ such that, if $ht < T_0$ and $t < T_1(\varepsilon, \delta)$:

$$|E_{\Lambda_1,t}(A) - E_{\Lambda_1,t}(A)| \leq K(t, h, \varepsilon, \delta, |\text{supp}(A)|) \delta^{\text{dist}(\text{supp}(A), \Sigma)} \|A\|.$$  

(9.1)

**Proof.** For each $\theta \in [0,1]$, we shall use the potential $V_{\Lambda,\theta} = V_{\Lambda} - \theta V_{\text{inter}}$, (where $V_{\text{inter}}$ is defined in (6.2)), the corresponding Hamiltonian $H_{\Lambda,\theta}$, the heat kernel $U_{\Lambda,\theta}$, the function $\psi_{\Lambda,\theta}$ of Theorem 1.1, and the corresponding correlation $\text{Cov}_{\Lambda,\theta}$ of two operators, and the mean value $E_{\Lambda,\theta}(A)$. We denote by $Z_{\Lambda,\theta}(t)$ the trace of the operator $e^{-tH_{\Lambda,\theta}}$. If $A$ is supported in $\Lambda_1$, we have, with these notations:

$$E_{\Lambda_1,t}(A) = E_{\Lambda_1,t}(A) = \int_0^1 \partial_\theta E_{\Lambda_1,t}(A)d\theta.$$  

(9.2)

Let us calculate $\partial_\theta E_{\Lambda_1,t}(A)$. We set, for each $\theta \in [0,1]$, $R_\theta(x,y,t) = \partial_\theta \psi_{\Lambda,\theta}(x,y,t) - \partial_\theta \psi_{\Lambda,\theta}(x,t,x)$.

$$\partial_\theta E_{\Lambda_1,t}(A) = Z_{\Lambda,\theta}(t)^{-1}\text{Tr}\left(Op(U_{\Lambda,\theta}(...,t)R_\theta(...,t)) \circ A\right) + \frac{1}{2} \sum_{Q \subseteq \Lambda} \text{Cov}_{\Lambda_1,t}(A, T_Q \partial_\theta \psi_{\Lambda,\theta}(...,t))$$  

(9.3)

where we identify a function and the operator of multiplication by this function. Using (8.3), with the norm defined in (5.1), we can write, with $E = \text{supp}(A)$:

$$\left|\text{Tr}\left(Op(U_{\Lambda,\theta}(...,t)R_\theta) \circ A\right)\right| \leq K(|E| \|A\| \|Op(U_{\Lambda,\theta}(...,t)R_\theta)\|_{m,m',0})$$

with $m$ and $m'$ depending on $|E|$. By the point (6.4) of Lemma 6.2, we can write, for each points $\lambda^{(1)}, \ldots \lambda^{(m)}$ of $E$, for each $(x,y)$

$$|\nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} R_\theta(x,y,t)| \leq tK(\varepsilon) \varepsilon^{\text{dist}(E,\Sigma)}.$$  

(9.4)

Therefore,

$$\|Op(U_{\Lambda,\theta}(...,t)R_\theta\|_{m,m'+0} \leq tK(\varepsilon) \varepsilon^{\text{dist}(E,\Sigma)} \|U_{\Lambda,\theta}(...,t)\|_{m,m'+1,0}.$$  

(9.5)

By Proposition 5.1, applied to the subset $E$, we can write:

$$\|U_{\Lambda,\theta}(...,t)\|_{m,m'+1,0} \leq K(t,h,\varepsilon,|E|)Z_{\Lambda,\theta}(t).$$

Thus we have a bound for the first term of (9.4):

$$\left|\text{Tr}\left(Op(U_{\Lambda,\theta}(...,t)R_\theta) \circ A\right)\right| \leq K(t,h,\varepsilon,|E|) \|A\| \|Z_{\Lambda,\theta}(t)\| \varepsilon^{\text{dist}(E,\Sigma)}.$$  

(9.6)
Now, we shall estimate the second term in (9.4). By Proposition 6.1, we can write:
\[
||T_Q \partial_t \varphi_\theta(., t)|| \leq tK(\varepsilon)(Q)^{d\delta} \text{diam}(Q, \pi_\Sigma(Q)),
\]
where \( \pi_\Sigma(Q) \) is the orthogonal projection of \( Q \) on the hyperplane \( \Sigma \), which separates \( \Lambda_1 \) and \( \Lambda_2 \). If \( \varepsilon < \delta < 1 \), we can apply Theorem 1.3, point \( e \), for the correlation between an operator and a function, with \( \delta \) replaced by \( \sqrt{\varepsilon \delta} \). Thus we can write:
\[
|\text{Cov}_{\Lambda, t, \theta}(A, T_Q \partial_t \varphi_\theta(., t))| \leq K(t, h, \varepsilon, \delta, |E|) \cdot \|A\| \cdot \|T_Q \partial_t \varphi_\theta(., t)\| \cdot (\varepsilon \delta)^{\delta \text{dist}(E, Q)}.
\]
With the same relations between \( \varepsilon, \delta_1 \) and \( \delta \), we have:
\[
\sum_{Q \subseteq \mathbb{R}^d} (\varepsilon \delta)^{\frac{1}{2}\text{dist}(E, Q) + \text{diam}(Q, \pi_\Sigma(Q))} \leq K(\varepsilon, \delta) \delta^{\text{dist}(E, \Sigma)}.
\]
By (9.5)–(9.7), Proposition 9.1 is proved, and Theorem 1.4 follows easily.

Theorem 1.5, about the mean energy per site, will follow from the next proposition. The mean energy \( X_\Lambda(t) \) for the set \( \Lambda \) is defined in (1.22).

**Proposition 9.2.** For each box \( \Lambda \) of \( \mathbb{Z}^d \), split into two boxes \( \Lambda_1 \) and \( \Lambda_2 \), separated by an hyperplane \( \Sigma \) as in (6.1), for each \( t > 0 \) and \( h > 0 \) such that \( ht < T_0 \) and \( t \) is small enough,
\[
|X_\Lambda(t) - X_{\Lambda_1}(t) - X_{\Lambda_2}(t)| \leq K(t) |\Lambda_\perp|
\]
where \( \Lambda_\perp = \pi_\Sigma(\Lambda) \), and \( \pi_\Sigma \) is the orthogonal projection on \( \Sigma \).

**Proof.** For each \( \theta \in [0, 1] \), let \( X_\Lambda(\theta, t) \) be the mean energy, for the set \( \Lambda \), but with \( V_\Lambda \) replaced by the potential \( V_{\Lambda, \theta} = V_\Lambda - \theta V_{\text{inter}} \), where \( V_{\text{inter}} \) is defined in (6.2). Thus,
\[
X_\Lambda(t) - X_{\Lambda_1}(t) - X_{\Lambda_2}(t) = \int_0^1 \partial_\theta X_\Lambda(\theta, t) d\theta.
\]
We use again the function \( \varphi_\theta \) defined in (9.3). By computations, similar to those of Proposition 9.1, we find that:
\[
\partial_\theta X_\Lambda, \theta(t) = E_{\Lambda, t, \theta} \left( \frac{\partial^2 \varphi_\theta(., t)}{\partial t \partial \theta} \right) + \frac{1}{2} \text{Cov}_{\Lambda, t, \theta} \left( \frac{\partial \varphi_\theta}{\partial t}, \frac{\partial \varphi_\theta}{\partial \theta} \right).
\]

**Estimation of the first term in (9.10).** Following (2.1), with \( V_\Lambda \) replaced by \( V_{\Lambda, \theta} \), we have:
\[
\frac{\partial \varphi_\Lambda(x, t)}{\partial t} = \frac{h^2}{2} (\Delta_x \varphi_\Lambda(., t)) - \frac{h^2}{2} \nabla_x^2 \varphi_\Lambda(., t) + V_{\Lambda, \theta}(x).
\]
We may differentiate this equation with respect to \( \theta \), and estimate \(|\partial_\theta \partial_\theta \varphi_\Lambda(x, t)|\), using bounds for all the terms. Using (6.2), (6.1) and the hypothesis \( (H_z) \), we remark that:
\[
|\partial_\theta V_{\Lambda, \theta}(x)| = |V_{\text{inter}}(x)| \leq K \sum_{\Lambda \in \Lambda_1, \mu \in \Lambda_2} \varepsilon^{\mu - |\Lambda_\perp|} \leq K(\varepsilon) |\Lambda_\perp|.
\]
By Lemma 6.2 for $m = 1$ and $m = 2$, we can write, if $ht \leq T_0$,
\[ |\nabla_x \partial_y \psi_0(x, x, t)| + |(\Delta_x \partial_y \psi_0)(x, x, t)| \leq tK(\varepsilon)\varepsilon^{\text{dist}(\Lambda, \Sigma)}.
\]
We obtain:
\[ |\partial_t \partial_y \phi_\theta(x, x, t)| \leq K(\varepsilon)h^2(t + t^2) \varepsilon^{\text{dist}(\Lambda, \Sigma)} \leq K(\varepsilon)h^2(t + t^2)|A_\perp|
\]
and therefore
\[ |E_{\Lambda, t, \theta} \left( \frac{\partial^2 \phi_\theta(\cdot, t)}{\partial \theta^2} \right)| \leq K(\varepsilon, t, h)|A_\perp|.
\]

**Estimation of the second term in (9.10).** For the function $\partial_y \phi_\theta$, we use the operators $T_Q$ of Section 3, which give the decomposition (3.9):
\[ \frac{\partial \phi_\theta}{\partial \theta}(x, t) = \frac{\partial \phi_\theta}{\partial \theta}(0, t) + \sum_{Q \subseteq \Lambda} T_Q \frac{\partial \phi_\theta}{\partial \theta}(x, t).
\]

For the derivative $\partial_t \phi_\theta$, we use (9.11), and we use the decomposition (3.9) given by the operators $T_Q$ for the first and the last terms of (9.11), and for each function $\nabla_x \psi_\Lambda, \theta$. Then, we estimate, using Theorem 1.3, the correlations between all the terms in the expression of $\partial_t \phi_\theta$ and those in the expression of $\partial_y \phi_\theta$. Let us give only the details for one of the terms. If $\varepsilon < \delta < 1$, we apply Theorem 1.3 (in the multiplicative case) with $\delta$ replaced by $\sqrt{\varepsilon \delta}$. We obtain, for each boxes $Q$ and $Q'$ and for each point $\lambda$:
\[ |\text{Cov}_{\Lambda, t, \theta}(\Delta_x T_Q \psi_\Lambda, \theta)(\cdot, \cdot, t), T_Q \partial_y \phi_\theta(\cdot, t))| \leq K(\varepsilon, \delta, h)\varepsilon^{\text{dist}(Q, Q')}(\Delta_x T_Q \psi_\Lambda, \theta)(\cdot, \cdot, t), T_Q \partial_y \phi_\theta(\cdot, t))| \leq K(\varepsilon, \delta, h)|A_\perp|.
\]

We use, for the first factor, the estimate given by Theorem 1.2:
\[ |(\Delta_x T_Q \psi_\Lambda, \theta)(x, x, t)| \leq tK(\varepsilon)|Q'|^{2\varepsilon}\varepsilon^{\text{dist}(Q', \Lambda)}.
\]

For the last factor, Proposition 6.1 gives the estimate recalled in (9.6). Then, we take the sum of these inequalities, for all boxes $Q$ and $Q'$ and for all points $\lambda$, and we remark, using again (8.26) and the function $L$ defined after (8.26), that:
\[ \sum_{\lambda \in \Lambda} \sum_{Q \subseteq \Lambda} \sum_{Q' \subseteq \Lambda} \varepsilon^{\Delta}(\lambda, Q, Q', \varepsilon)(Q') \leq K(\varepsilon, \delta)|A_\perp|.
\]

All the other terms can be estimated in the same way. Thus we obtain, choosing for example $\delta = \sqrt{\varepsilon}$:
\[ |\text{Cov}_{\Lambda, t, \theta}(\frac{\partial \phi_\theta}{\partial t}, \frac{\partial \phi_\theta}{\partial \theta})| \leq K(t, h, \varepsilon)|A_\perp|.
\]

The proposition is proved, and Theorem 1.5 follows by the same arguments as Sjöstrand [25], Section 8, p. 45–46. \( \square \)
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