LIMIT THEOREMS FOR PERSISTENCE DIAGRAMS

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The persistent homology of a stationary point process on \( \mathbb{R}^N \) is studied in this paper. As a generalization of continuum percolation theory, we study higher dimensional topological features of the point process such as loops, cavities, etc. in a multiscale way. The key ingredient is the persistence diagram, which is an expression of the persistent homology. We prove the strong law of large numbers for persistence diagrams as the window size tends to infinity and give a sufficient condition for the limiting persistence diagram to have the full support. We also discuss a central limit theorem for persistent Betti numbers.

1. Introduction.

1.1. Background. The prototype of this work dates back to the random geometric graphs. In those original settings, a set \( V \) of points is randomly scattered in a space according to some probability distribution, and a graph with the vertices \( V \) is constructed by assigning edges whose distances are less than a certain threshold value \( r \geq 0 \). Then, some characteristic features in the graph such as connected components and loops are broadly and thoroughly studied (see e.g. [26]). Furthermore, the random geometric graphs provide mathematical models for applications such as mobile wireless networks [21, 23], epidemics [30], and so on.

Recently, the concept of random topology has emerged and rapidly grown as a higher dimensional generalization of random graphs [2, 19]. One of the simple models studied in random topology is simplicial complex, which is given by a collection of subsets closed under inclusion. Obviously, a graph is regarded as a one dimensional simplicial complex consisting of singletons as vertices and doubletons as edges.

In geometric settings, a simplicial complex is built over randomly distributed points in a space by a certain rule respecting the nearness of multiple points, like random geometric graphs. Two standard simplicial complex

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models constructed from the points are Čech complexes and Rips complexes, which are also determined by a threshold value $r$ measuring the nearness of points. Then, in such an extended geometric object, it is natural to study higher dimensional topological features such as cavities (2 dim.) and more general $q$-dimensional holes, beyond connected components (0 dim.) and loops (1 dim.).

In algebraic topology, $q$-dimensional holes are usually characterized by using the so-called homology. Here, the $q$th homology of a simplicial complex is given by a vector space and its dimension is called the Betti number which counts the number of $q$-dimensional holes. Hence, in the setting of random simplicial complexes, the Betti numbers become random variables through a random point configuration, and studying the asymptotic behaviors of the randomized Betti numbers is a significant problem for understanding global topological structures embedded in the random simplicial complexes (e.g., [7, 18, 25, 32, 33]).

On the other hand, another type of generalizations has been recently attracting much attention in applied topology. In that setting, we are interested in how persistent the holes are for changing the threshold parameter $r \in \mathbb{R}$. Namely, we deal with one parameter filtration of simplicial complexes obtained by increasing the parameter $r$ and characterize robust or noisy holes in that filtration. The persistent homology [9, 34] is a tool invented for this purpose, and especially, its expression called persistence diagram is now applied to a wide variety of applied areas (see e.g., [3, 10, 13, 24, 31]). From this point of view, there have been some works on a functional of persistence diagram, called lifetime sum or total persistence, for random complexes (that are not geometric in the sense above) such as Linial-Meshulam processes and random cubical complexes (e.g., [14, 15, 16]).

Therefore, it is natural to further extend the results on random geometric simplicial complexes to this generality, and the purpose of this paper is to show several these extensions. In particular, we are interested in asymptotic behaviors of persistence diagrams themselves defined on stationary point processes. These subjects are mathematically meaningful in its own right, but are also interesting for practical applications. For example, the paper [13] studies topological and geometric structures of atomic configurations in glass materials by comparing persistence diagrams with those of disordered states. By regarding atomic configurations in disordered states as random point processes, further understanding of those persistence diagrams will be useful for characterizing geometry and topology of glass materials, which is one of the important research topics in current physics.
1.2. Prior work. Let \( \Phi \) be a stationary point process on \( \mathbb{R}^N \) with all finite moments, i.e.,
\[
\mathbb{E} [\Phi(A)^k] < \infty, \quad \text{for all bounded Borel sets } A \text{ and any } k = 1, 2, \ldots.
\]
Here \( \Phi(A) \) denotes the number of points in \( A \). For simplicity, we always assume that \( \Phi \) is simple, i.e.,
\[
\mathbb{P}(\Phi(\{x\}) \leq 1 \text{ for every } x \in \mathbb{R}^N) = 1.
\]
We denote by \( \Phi_{\Lambda_L} \) the restriction of \( \Phi \) on \( \Lambda_L = [-\frac{L}{2}, \frac{L}{2})^N \).

Let \( C(\Phi_{\Lambda_L}, r) \) be the \( \check{C}ech \) complex built over the points \( \Phi_{\Lambda_L} \) with a parameter \( r > 0 \) (see Section 2.1 for the definition). Limiting behaviors of the Betti numbers \( \beta_q(C(\Phi_{\Lambda_L}, r)) \) \( (q = 1, 2, \ldots, N - 1) \) have been widely investigated [32, 33]. Among them, we here restate the most related results to this paper.

**Theorem 1.1** ([33, Lemma 3.3 and Theorem 3.5]). Assume that \( \Phi \) is a stationary point process on \( \mathbb{R}^N \) having all finite moments. Then there exists a constant \( \hat{\beta}_q \geq 0 \) such that
\[
\frac{\mathbb{E}[\beta_q(C(\Phi_{\Lambda_L}, r))]}{L^N} \to \hat{\beta}_q \text{ as } L \to \infty.
\]
In addition, if \( \Phi \) is ergodic, then
\[
\frac{\beta_q(C(\Phi_{\Lambda_L}, r))}{L^N} \to \hat{\beta}_q \text{ almost surely as } L \to \infty.
\]

**Theorem 1.2** ([33, Theorem 4.7]). Assume that \( \Phi \) is a homogeneous Poisson point process on \( \mathbb{R}^N \) with unit intensity. Then, there exists a constant \( \sigma_q^2 > 0 \) such that
\[
\frac{\beta_q(C(\Phi_{\Lambda_L}, r)) - \mathbb{E}[\beta_q(C(\Phi_{\Lambda_L}, r))]}{L^{N/2}} \xrightarrow{d} \mathcal{N}(0, \sigma_q^2) \text{ as } L \to \infty.
\]
Here \( \mathcal{N}(\mu, \sigma^2) \) denotes the normal distribution with mean \( \mu \) and variance \( \sigma^2 \), and \( \xrightarrow{d} \) denotes the convergence in distribution of random variables.

The purpose of this paper is to extend Theorem 1.1 to the setting on persistence diagrams and Theorem 1.2 to persistent Betti numbers.
1.3. Main results. In this paper, we study the following simplicial complex model for the point process $\Phi$ which is a generalization of the Čech complex and the Rips complex.

Let $\mathcal{F}(\mathbb{R}^N)$ be the collection of all finite (non-empty) subsets in $\mathbb{R}^N$. We can identify $\mathcal{F}(\mathbb{R}^N)$ with the set $\sqcup_{k=1}^{\infty}(\mathbb{R}^N)^k/\sim$, where $\sim$ is the equivalence relation induced by permutations of coordinates. For a function $f$ on $\mathcal{F}(\mathbb{R}^N)$, there exists a permutation invariant function $\tilde{f}_k$ on $(\mathbb{R}^N)^k$ for each $k \geq 1$ such that $f(\{x_1, \ldots, x_k\}) = \tilde{f}_k(x_1, \ldots, x_k)$. We say that $f$ is measurable if so is $\tilde{f}_k$ on $(\mathbb{R}^N)^k$ for each $k \geq 1$.

Let $\kappa: \mathcal{F}(\mathbb{R}^N) \to [0, \infty]$ be a measurable function satisfying

(K1) $0 \leq \kappa(\sigma) \leq \kappa(\tau)$, if $\sigma$ is a subset of $\tau$;
(K2) $\kappa$ is translation invariant, i.e., $\kappa(\sigma + x) = \kappa(\sigma)$ for any $x \in \mathbb{R}^N$, where $\sigma + x := \{y + x : y \in \sigma\}$;
(K3) there is an increasing function $\rho: [0, \infty] \to [0, \infty]$ with $\rho(t) < \infty$ for $t < \infty$ such that

$$\|x - y\| \leq \rho(\kappa(\{x, y\})),$$

where $\|x\|$ denotes the Euclidean norm in $\mathbb{R}^N$.

Without loss of generality, we can assume $\kappa(\{x\}) = 0$ because of the translation invariance.

**Example 1.3.** Two important examples of $\kappa$ which we have in mind are

\begin{align*}
(1.2) \quad & \kappa_C(\{x_0, x_1, \ldots, x_q\}) = \inf_{w \in \mathbb{R}^N} \max_{0 \leq i \leq q} \|x_i - w\|, \\
(1.3) \quad & \kappa_R(\{x_0, x_1, \ldots, x_q\}) = \max_{0 \leq i < j \leq q} \frac{\|x_i - x_j\|}{2},
\end{align*}

which define the Čech filtration $\mathcal{C}(\Phi) = \{C(\Phi, t)\}_{t \geq 0}$ and the Rips filtration $\mathcal{R}(\Phi) = \{R(\Phi, t)\}_{t \geq 0}$, respectively. Both $\kappa$’s satisfy Assumption (K3) with $\rho(t) = 2t$. See also Section 2.1 for these filtrations.

Given such a function $\kappa$, we construct a filtration $\mathcal{K}(\Xi) = \{K(\Xi, t) : 0 \leq t < \infty\}$ of simplicial complexes from a finite point configuration $\Xi \in \mathcal{F}(\mathbb{R}^N)$ by

\begin{equation}
(1.4) \quad K(\Xi, t) = \{\sigma \subseteq \Xi : \kappa(\sigma) \leq t\},
\end{equation}

i.e., $\kappa(\sigma)$ is the birth time of a simplex $\sigma$ in the filtration $\mathcal{K}(\Xi)$. Although we do not explicitly show the dependence on $\kappa$ in the notation $\mathcal{K}(\Xi)$ because the function $\kappa$ is fixed in the paper, we here call it the $\kappa$-filtration over $\Xi$. 
For the filtration $\mathcal{K}(\Xi)$, we denote its $q$th persistence diagram by

$$D_q(\Xi) = \{(b_i, d_i) \in \Delta : i = 1, \ldots, n_q\},$$

which is given by a multiset on $\Delta = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < y \leq \infty\}$ determined from the unique decomposition of the persistent homology (see (2.2) for the definition). The pair $(b_i, d_i)$ indicates the persistence of the $i$th homology class, i.e., it appears at $b_i$ and disappears at $d_i$, and $d_i = \infty$ means that the $i$th homology class persists forever.

In this paper, we deal with the persistence diagram $D_q(\Xi)$ as the counting measure

$$\xi_q(\Xi) = \sum_{(b_i, d_i) \in D_q(\Xi)} \delta_{(b_i, d_i)},$$

rather than as a multiset, where $\delta_{(x,y)}$ is the Dirac measure at $(x, y) \in \mathbb{R}^2$.

For each $L > 0$, we define a random filtration built over the points $\Phi_{\Lambda_L}$ and denote it by $\mathcal{K}(\Phi_{\Lambda_L}) = \{K(\Phi_{\Lambda_L}, t)\}_{t \geq 0}$. We write $\xi_{q,L}$ for the point process $\xi_q(\Phi_{\Lambda_L})$ and $E[\xi_{q,L}]$ for its mean measure (see Section 3 for the precise definition of mean measure).

**Example 1.4.** The top three panels in Figure 1 show point processes with negative (Ginibre), zero (Poisson), and positive (Poisson cluster) correlations, respectively. All point processes consist of 1,000,000 points with the density $1/2\pi$, and only restricted areas of them are visualized. The bottom shows the corresponding normalized persistence diagrams $\xi_{1,L}/L^2$ of the Čech filtrations applied to the above, respectively.

One of the main results in this paper is as follows.

**Theorem 1.5.** Assume that $\Phi$ is a stationary point process on $\mathbb{R}^N$ having all finite moments. Then there exists a unique Radon measure $\nu_q$ on $\Delta$ such that

$$\frac{1}{L^N}E[\xi_{q,L}] \xrightarrow{v} \nu_q \text{ as } L \to \infty.$$  \hfill (1.5)

Here $\xrightarrow{v}$ denotes the vague convergence of measures on $\Delta$. In addition, if $\Phi$ is ergodic, then almost surely,

$$\frac{1}{L^N}\xi_{q,L} \xrightarrow{v} \nu_q \text{ as } L \to \infty.$$  \hfill (1.6)
We call the limiting Radon measure $\nu_q$ the $q$th persistence diagram of a stationary ergodic point process $\Phi$. In non-ergodic case, by using the ergodic decomposition (cf. [11]), the right-hand side in (1.6) is replaced by the random measure $\nu_{q,\omega}$ which is measurable with respect to the translation invariant $\sigma$-field $\mathcal{I}$ defined in Section 3.

**Remark 1.6.** The set $\Delta$ is topologically the same as the triangle

$$\{(x, y) \in \mathbb{R}^2 : 0 \leq x < y \leq 1\}$$

with open boundary $\partial \Delta = \{(x, x) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$. Although we do not consider the mass on $\partial \Delta$, intuitively speaking, the (virtual) mass on $\partial \Delta$ comes from configurations of special forms such as three vertices of a right triangle. With the vague convergence, we do not see the mass escaping towards the boundary $\partial \Delta$ in the limit $L \to \infty$. In applications, the mass appearing near the boundary is often considered to be fragile under perturbation.

The limiting measure $\nu_q$ may be trivial. Indeed, for Čech complexes, $\nu_q = 0$ for $q \geq N$. This is just because there is no configuration in $\mathbb{R}^N$ that realizes

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**Figure 1.** Top: Point processes with negative (Ginibre), zero (Poisson), and positive (Poisson cluster) correlations. In these three point processes, the number of points and the density are set to be 1,000,000 and $1/2\pi$, respectively. Bottom: The normalized persistence diagrams $\xi_{1,L}/L^2$ of the above.
qth homology class for \( q \geq N \). In order to characterize the support of \( \nu_q \), we introduce the notion of realizability of a point in a persistence diagram.

**Definition 1.7.** We say that a point \((b, d) \in \Delta\) is realizable by \( \Xi \in \mathcal{F}(\mathbb{R}^N)\) in the qth persistent homology if \((b, d)\) is contained in the qth persistence diagram of the \( \kappa \)-filtration over \( \Xi \), i.e., \( \xi_q(\Xi)(\{(b, d)\}) \geq 1 \). If such \( \Xi \) exists for \((b, d)\), we call \((b, d)\) a realizable point. We denote by \( R_q = R_q(\kappa) \) the set of all realizable points in the qth persistent homology of the \( \kappa \)-filtration.

**Example 1.8.** For \( \alpha > 0 \) and \( \sigma \in \mathcal{F}(\mathbb{R}^N) \), we define \( \alpha \sigma \in \mathcal{F}(\mathbb{R}^N) \) by 
\[
\alpha \sigma = \{ \alpha x \in \mathbb{R}^N : x \in \sigma \}.
\]
It is easy to see that if \( \kappa \) is homogeneous in the sense that \( \kappa(\alpha \sigma) = \alpha \kappa(\sigma) \) for every \( \alpha > 0 \) and \( \sigma \in \mathcal{F}(\mathbb{R}^N) \), then \( R_q(\kappa) \) forms a cone in \( \Delta \). Since both \( \kappa_C \) and \( \kappa_R \) given in Example 1.3 are homogeneous, we can see that \( R_q(\kappa_C) \) and \( R_q(\kappa_R) \) are cones for every \( q \geq 0 \).

In particular, for Čech complexes, we have

\[
R_q(\kappa_C) = \begin{cases} \{0\} \times (0, \infty], & \text{if } q = 0, \\ \{(b, d) : 0 < b < d < \infty\}, & \text{if } q = 1, 2, \ldots, N - 1, \\ \emptyset, & \text{if } q = N, N + 1, \ldots. \end{cases}
\]

It is clear that \( \text{supp} \nu_q \subset R_q(\kappa) \). Indeed, if \( x \not\in \overline{R_q(\kappa)} \), there exists \( \varepsilon > 0 \) such that \( \xi_{q,L}(B_\varepsilon(x)) = 0 \), where \( B_\varepsilon(x) \) is the open \( \varepsilon \)-neighborhood of \( x \). It follows from the vague convergence (1.5) that \( \nu_q(B_\varepsilon(x)) = 0 \). Therefore \( x \not\in \text{supp} \nu_q \). In Theorem 4.7, we give sufficient conditions for a point in \( R_q(\kappa) \) to be in the support of \( \nu_q \). The following result, as a consequence of that general theorem, states that \( \text{supp} \nu_q \) coincides with \( \overline{R_q(\kappa)} \) under conditions that \( \kappa \) is Lipschitz continuous and all local densities of the point process \( \Phi \) are almost surely positive with respect to the Lebesgue measures.

**Theorem 1.9.** Let \( \Phi \) be a stationary point process on \( \mathbb{R}^N \) and \( \Theta \) its probability distribution. Assume that for every compact set \( \Lambda \subset \mathbb{R}^N \), the restriction \( \Theta|_\Lambda \) on \( \Lambda \) is absolutely continuous with respect to \( \Pi|_\Lambda \) and the Radon-Nikodym density \( d\Theta|_\Lambda/d\Pi|_\Lambda \) is strictly positive \( \Pi|_\Lambda \)-almost surely, where \( \Pi \) is the distribution of a homogeneous Poisson point process on \( \mathbb{R}^N \). In addition, assume that \( \kappa \) on \( \mathcal{F}(\mathbb{R}^N) \) is Lipschitz continuous with respect to the Hausdorff distance. Then, \( \text{supp} \nu_q = \overline{R_q(\kappa)} \).

**Example 1.10.** Homogeneous Poisson point processes, Gibbs measures and Ginibre point processes satisfy the assumption in Theorem 1.9. Thus if \( \kappa \) is Lipschitz continuous with respect to the Hausdorff distance, then \( \text{supp} \nu_q = \overline{R_q(\kappa)} \) for such point processes. In particular, for Čech filtrations,
supp \( \nu_q = \Delta, (q = 1, \ldots, N - 1) \). On the other hand, the shifted lattice considered in Example 4.3 does not satisfy the assumption and supp \( \nu_q \) for the Čech filtration turns out to be a singleton.

For the proof of Theorem 1.5, we exploit a general theory of Radon measures for the vague convergence (cf. [1, 20]). In particular, we show that the convergence of the values of measures on the class \( \{ A_{r,s} = [0, r] \times (s, \infty] : 0 \leq r \leq s < \infty \} \) is enough to ensure the vague convergence of random measures in Theorem 1.5. The value of \( \xi_{q,L} \) on \( A_{r,s} \) is nothing but the persistent Betti number

\[
\beta_{r,s}^r(\mathbb{K}(\Phi_{A_{r,s}})) = \xi_{q,L}([0, r] \times (s, \infty]) = |\{(b_i, d_i) : 0 \leq b_i \leq r \leq s < d_i\}|.
\]

Here \( |A| \) denotes the cardinality of a finite set \( A \). Later, \( |A| \) is also used to denote the Lebesgue measure of a set \( A \in \mathbb{R}^N \). The meaning is clear from the context. Hence Theorem 1.5 follows from the following strong law of large numbers for persistent Betti numbers.

**Theorem 1.11.** Assume that \( \Phi \) is a stationary point process having all finite moments. Then for any \( 0 \leq r \leq s < \infty \), there exists a constant \( \hat{\beta}_{r,s}^r \) such that

\[
\frac{\mathbb{E}[\beta_{r,s}^r(\mathbb{K}(\Phi_{A_{r,s}}))]}{L^N} \to \hat{\beta}_{r,s}^r \quad \text{as} \quad L \to \infty.
\]

In addition, if \( \Phi \) is ergodic, then

\[
\frac{\beta_{r,s}^r(\mathbb{K}(\Phi_{A_{r,s}}))}{L^N} \to \hat{\beta}_{r,s}^r \quad \text{almost surely as} \quad L \to \infty.
\]

Note that, for \( r = s \), the persistent Betti number becomes the usual Betti number, i.e., \( \beta_{r}^r(\mathbb{K}(\Phi_{A_{r,s}})) = \beta_{r}(\mathbb{K}(\Phi_{A_{r,s}}), r) \). Hence, this result is a generalization of Lemma 3.3 and Theorem 3.5 in [33].

For Poisson point processes, we also generalize the central limit theorem in [33] for Betti numbers to persistent Betti numbers as follows.

**Theorem 1.12.** Let \( \Phi \) be a homogeneous Poisson point process on \( \mathbb{R}^N \) with unit intensity. Then for any \( 0 \leq r \leq s < \infty \), there exists a constant \( \sigma_{r,s}^2 = \sigma_{r,s}^2(q) \) such that

\[
\frac{\beta_{r,s}^r(\mathbb{K}(\Phi_{A_{r,s}})) - \mathbb{E}[\beta_{r,s}^r(\mathbb{K}(\Phi_{A_{r,s}}))]}{L^{N/2}} \xrightarrow{d} \mathcal{N}(0, \sigma_{r,s}^2) \quad \text{as} \quad L \to \infty.
\]

We remark that the proof of the central limit theorem for (usual) Betti numbers in [33] uses the Mayer-Vietoris exact sequence to estimate the effect
of one point adding on the Betti number. However, in the setting of persistent homology, although we can obtain the Mayer-Vietoris exact sequence for each parameter $r$, we do not have the exactness property with regard to the parameter change. Hence, the same technique may not be applicable to the case of persistent Betti numbers. Instead, we give an alternative (and elementary) proof for the generalization. Remark also that by establishing the strong stabilization, the central limit theorem for Betti numbers of Čech complexes built over binomial point processes is also established in [33]. In this case, the positivity of the limiting variance is also proved under a certain condition on radius parameter $r$. The positivity problem for the limiting variance is left open in case of persistent Betti numbers of general $\kappa$-complexes built over homogeneous Poisson point processes.

The organization of this paper is given as follows. Necessary concepts and properties of persistent homology and random measures are explained in Section 2 and Section 3, respectively. Theorem 4.7 which characterizes the support of limiting persistence diagrams is stated and proved in Section 4.3. The proofs of Theorems 1.5, 1.9, 1.11 and 1.12 are given in Sections 4.2, 4.3, 4.1 and 5 in order. In Section 6, we summarize the conclusions of the paper and show some future works.

2. Geometric models and persistent homology. In this section, we assume fundamental properties about simplicial complexes and their homology. For details, the reader may refer to Appendix B or [8, 12].

2.1. Geometric models for point processes. Let $\kappa: \mathcal{F}(\mathbb{R}^N) \to [0, \infty]$ be a function satisfying the three conditions explained in Section 1, where $\mathcal{F}(\mathbb{R}^N)$ is the collection of all finite subsets in $\mathbb{R}^N$. For such a function $\kappa$, the $\kappa$-filtration $\mathcal{K}(\Phi) = \{K(\Phi, t)\}_{t \geq 0}$ can be defined in the same way as in (1.4) for an infinite point configuration (or a point process) $\Phi \subset \mathbb{R}^N$ as well as for a finite point process.

We remark that all vertices (i.e., 0-simplices) exist at $t = 0$. Also, all simplices in $K(\Phi, t)$ possessing a point $x$ must lie in the ball $\bar{B}_{\rho(t)}(x)$ since $\{x, x_1, \ldots, x_q\} \in K(\Phi, t)$ with Assumption (K3) implies that $\|x - x_i\| \leq \rho(\kappa(x, x_i)) \leq \rho(t)$ for all $i$. Here $B_r(x) = \{y \in \mathbb{R}^N : \|y - x\| \leq r\}$ is the closure of $B_r(x)$ which denotes the open ball of radius $r$ centered at $x$. Hence, for each parameter $t$, the presence of simplices containing $x$ is localized in $\bar{B}_{\rho(t)}(x)$.

This geometric model includes some of the standard models studied in random topology. For instance, the Čech complex $C(\Phi, t)$ is a simplicial
complex with the vertex set $\Phi$ and, for each parameter $t$, it is defined by

$$\sigma = \{x_0, \ldots, x_q\} \in C(\Phi, t) \iff \bigcap_{i=0}^{q} \bar{B}_t(x_i) \neq \emptyset$$

for $q$-simplices. Similarly, the Rips complex $R(\Phi, t)$ with a parameter $t$ is defined by

$$\sigma = \{x_0, \ldots, x_q\} \in R(\Phi, t) \iff \bar{B}_t(x_i) \cap \bar{B}_t(x_j) \neq \emptyset \text{ for } 0 \leq i < j \leq q.$$ 

It is clear that these geometric models are generated by the functions given in Example 1.3. We note that $R(\Phi, t/2) \subset C(\Phi, t) \subset R(\Phi, t)$ since $\kappa_R \leq \kappa_C \leq 2\kappa_R$.

In Example 1.8, we showed the set $R_q(\kappa_C)$ of the realizable points in (1.7) for the Čech filtration. We are going to give a proof of that fact here. The cases $q = 0$ and $q \geq N$ are easily derived. For $q = 1, \ldots, N-1$, we will show that any birth-death pair $(b, d)$ with $0 < b < d < \infty$ is realizable by explicitly constructing the points $\Xi \in \mathcal{F}(\mathbb{R}^N)$ realizing $(b, d)$. Indeed, let $S_d^q \subset \mathbb{R}^N$ be a $q$-dimensional sphere with radius $d$ and $z_0$ be any point in $S_d^q$. We choose points $\Xi_+$ on $S_d^q \cap \partial B_b(z_0)$ and $\Xi_-$ on $S_d^q \setminus \bar{B}_b(z_0)$ such that

1. $S_d^q \cap \bar{B}_b(z_0) \subset \bigcup_{x \in \Xi_+} B_b(x)$;
2. $\bigcup_{x \in \Xi_-} B_r(x)$ covers $S_d^q \setminus \bar{B}_b(z_0)$ earlier than $r = b$;
3. $\bigcup_{x \in \Xi_-} B_r(x)$ provides the generator of $q$-dimensional homology corresponding to $S_d^q$, where $\Xi = \Xi_+ \cup \Xi_-$.

Then the birth-death pair of the generator $\bigcup_{x \in \Xi} B_r(x)$ is exactly $(b, d)$.

### 2.2. Persistent homology

Let $\mathbb{K} = \{K_r : r \geq 0\}$ be a (right continuous) filtration of simplicial complexes, i.e., $K_r \subset K_s$ for $r \leq s$ and $K_r = \bigcap_{r<s} K_s$. In this paper, the homology $H_q(K)$ of a simplicial complex $K$ is defined on an arbitrary field $F$. For $r \leq s$, we denote the linear map on homologies induced from the inclusion $K_r \hookrightarrow K_s$ by $t_r^s : H_q(K_r) \to H_q(K_s)$. The $q$th persistent homology $H_q(\mathbb{K}) = (H_q(K_r), t_r^s)$ of $\mathbb{K}$ is defined by the family of homologies $\{H_q(K_r) : r \geq 0\}$ and the induced linear maps $t_r^s$ for all $r \leq s$.

A homological critical value of $H_q(\mathbb{K})$ is a number $r > 0$ such that the linear map $t_{r-\varepsilon}^{r+\varepsilon} : H_q(K_{r-\varepsilon}) \to H_q(K_{r+\varepsilon})$ is not isomorphic for any sufficiently small $\varepsilon > 0$. The persistent homology $H_q(\mathbb{K})$ is said to be tame if $\dim H_q(K_r) < \infty$ for any $r \geq 0$ and the number of homological critical values is finite. A tame persistent homology $H_q(\mathbb{K})$ has a nice decomposition property:
Theorem 2.1 ([34]). Assume that $H_q(\mathbb{K})$ is a tame persistent homology. Then, there uniquely exist indices $p \in \mathbb{Z}_{\geq 0}$ and $b_i, d_i \in \mathbb{R}_{\geq 0} \sqcup \{\infty\}$ with $b_i < d_i, i = 1, 2, \ldots, p$, such that the following isomorphism holds:

$$H_q(\mathbb{K}) \simeq \bigoplus_{i=1}^{p} I(b_i, d_i).$$

(2.1)

Here, $I(b_i, d_i) = (U_r, f^s_r)$ consists of a family of vector spaces

$$U_r = \begin{cases} F, & b_i \leq r < d_i, \\ 0, & \text{otherwise,} \end{cases}$$

and the identity map $f^s_r = \text{id}_F$ for $b_i \leq r \leq s < d_i$.

Each summand $I(b_i, d_i)$ in (2.1) is called a generator of the persistent homology and $(b_i, d_i)$ is called its birth-death pair. From the unique decomposition in Theorem 2.1, we define the $q$th persistence diagram as a multiset in $\mathbb{R}^2_{\geq 0}$,

$$(2.2) \quad D_q(\mathbb{K}) = \{(b_i, d_i) \in \mathbb{R}^2_{\geq 0} : i = 1, \ldots, p\}.$$ 

By denoting the multiplicity of the point $(b, d)$ in (2.2) by $m_{b,d} \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, we can also express the decomposition (2.1) as

$$H_q(\mathbb{K}) \simeq \bigoplus_{(b,d)} I(b, d)^{m_{b,d}}.$$ 

Later, we identify a persistence diagram $D_q(\mathbb{K})$ as an integer-valued Radon measure $\xi = \sum_{(b,d)} m_{b,d} \delta_{(b,d)}$ rather than as a multiset.

Although our target object $\mathbb{K}(\Phi)$ is built over infinite points, all persistent homologies studied in this paper are defined on the geometric models with finite points. Hence, the persistent homology becomes tame, and the persistence diagrams are well-defined.

Example 2.2. In Figure 2, two (1-dim)cycles appear at times 1 and 2 and disappear at times 3 and 4. The representation corresponding to $H_1(\mathbb{K})$ is given as

$$0 \to F(c_1 + c_2) \to F(c_1) \oplus F(c_2) \to F(c_1) \oplus F(c_2)/F(c_1) \to F(c_1) \oplus F(c_2)/F(c_1) \oplus F(c_2) \simeq 0,$$
where \( c_1 = \langle 12 \rangle + \langle 23 \rangle + \langle 31 \rangle \) and \( c_2 = \langle 13 \rangle + \langle 34 \rangle + \langle 41 \rangle \) and each arrow is the linear map induced by inclusion. As pairs of birth-death times, we have \((1, 4)\) and \((2, 3)\) since the decomposition of the representation is given by

\[
H_1(\mathbb{K}) = (0 \to F(c_1 + c_2) \to F(c_1 + c_2) \to F(c_1 + c_2) \to 0) \\
\oplus (0 \to 0 \to F(c_1) \to 0 \to 0).
\]

**Remark 2.3.** More generally, a persistence module \( U = (U_a, f^b_a) \) on \( \mathbb{K}_{\geq 0} \) is defined by a sequence of general vector spaces \( U_a, a \geq 0 \), and linear maps \( f^b_a : U_a \to U_b \) for \( a \leq b \) satisfying \( f^c_b = f^c_a \circ f^a_b \). Under the same definition of the tameness, we can similarly define its persistence diagrams.

**Remark 2.4.** There is another definition of persistent homology as graded modules over a monoid ring for the continuous parameter (resp. a polynomial ring for the discrete parameter). See, for example, [14].

**Remark 2.5.** The persistent homology \( H_q(\mathbb{K}) \) defined over the whole \( \Phi \) is not tame in general while \( H_q(\mathbb{K}_L) \) defined over a restriction \( \Phi_{\Lambda_L} \) is tame. Theorem 1.5 informally says that

\[
\frac{1}{L^N} H_q(\mathbb{K}_L) \xrightarrow{\sim} H_q(\mathbb{K}) = \int_{\Delta}^\oplus I(x, y) \nu_q (dxdy),
\]

where \( \mathbb{K}_L = \{K(\Phi_{\Lambda_L}, t)\}_{t \geq 0} \), where \( \int_{\Delta}^\oplus \) denotes the direct integral of interval representations (cf. [29]).

**Remark 2.6.** In our paper, we use the persistence diagram for representing topological information obtained from filtrations. People sometimes use the so-called barcode representation in which each persistence interval \( I(b, d) \) is represented as a barcode \([b, d]\) (cf. [32]). We consider the marginal measure of persistence diagram on death times (also on birth times), i.e., the induced measure \( \xi^{(\text{death})} \) obtained from a measure \( \xi \) on \( \Delta \) by the projection \( \Delta \ni (x, y) \mapsto y \in [0, \infty) \). The marginal measure \( \xi_{q, L}^{(\text{death})} \)
of a persistence diagram $\xi_{q,L}$ induces a (scaled) right-continuous step function $f_{q,L}(t) = L^{-N}\xi^{(\text{death})}_{q,L}([0,t])$, which corresponds to the one obtained by simulation in [32]. The function $f_{q,L}(t)$ is also expected to converge to a limit $f_{q,\infty}(t)$ as $L \to \infty$, however, it does not necessarily coincide with $f_q(t) := \nu^{(\text{death})}_q([0,t])$ because of the mass escaping to $\partial \Delta$.

2.3. Persistent Betti numbers. For a filtration $\mathbb{K}$, the $(r,s)$-persistent Betti number [9] is defined by

$$\beta_{q}^{r,s}(\mathbb{K}) = \text{rank} \frac{Z_q(K_r)}{Z_q(K_r) \cap B_q(K_s)} \quad (r \leq s),$$

where $Z_q(K_r)$ and $B_q(K_r)$ are the $q$th cycle group and boundary group, respectively. We remark that this is equal to the rank of $i^s_r: H_q(K_r) \to H_q(K_s)$, because

$$\text{im } i^s_r \simeq \frac{Z_q(K_r)}{B_q(K_r)} \simeq \frac{Z_q(K_r)}{Z_q(K_r) \cap B_q(K_s)}.$$ 

Thus, from the decomposition of the persistent homology, we have

$$\beta_{q}^{r,s}(\mathbb{K}) = \sum_{b \leq r, d > s} m_{b,d}.$$ 

This means that the $(r,s)$-persistent Betti number $\beta_{q}^{r,s}(\mathbb{K})$ counts the number of birth-death pairs in the persistence diagram $D_q(\mathbb{K})$ located in the gray region of Figure 3.

![Figure 3](image_url)
Lemma 2.7. Let $U = (U_a, f_a^b)$ be a persistence module on $\overline{\mathbb{R}}_{\geq 0}$ and let $V = (V_a, g_a^b)$ be its truncation on the interval $[r, s]$, meaning that

$$V_a = \begin{cases} U_r, & a \leq r, \\ U_a, & r \leq a \leq s, \\ U_s, & a \geq s, \end{cases}$$

$$g_a^b = \begin{cases} f_a^b, & r \leq a \leq b \leq s, \\ f_a^s, & r \leq a \leq s \leq b, \\ f_b^s, & a \leq r \leq b \leq s, \\ f_b^r, & a \leq r \leq s \leq b. \end{cases}$$

For interval decompositions $U \simeq \oplus I(b, d)^{m_{b, d}}$ and $V \simeq \oplus I(b, d)^{n_{b, d}}$, let

$$\beta_{r, s}(U) = \sum_{b \leq r, d > s} m_{b, d}, \quad \beta^{0, \infty}(V) = \sum_{b=0, d=\infty} n_{b, d}.$$ 

Then $\beta_{r, s}(U) = \beta^{0, \infty}(V)$.

Proof. This is because $\beta_{r, s}(U) = \text{rank } f_r^s = \beta^{0, \infty}(V)$. \qed

Here, we recall the following basic facts in linear algebra for later use.

Lemma 2.8. Let $A, B, U, V$ be subspaces of a vector space satisfying $A \subset U$ and $B \subset V$. Then,

$$\text{rank } \frac{U \cap V}{A \cap B} \leq \text{rank } \frac{U}{A} + \text{rank } \frac{V}{B},$$

Proof. Let us consider a surjective map

$$f : \frac{U \cap V}{A \cap B} \rightarrow \frac{U \cap V}{A \cap V}, \quad [c] \mapsto [c].$$

Since the kernel of the map is $\ker f = \frac{A \cap V}{A \cap B}$, we have

$$\frac{U \cap V}{A \cap B} \simeq \frac{U \cap V}{A \cap V}.$$

This leads to the conclusion that

$$\text{rank } \frac{U \cap V}{A \cap B} = \text{rank } \frac{A \cap V}{A \cap B} + \text{rank } \frac{U \cap V}{A \cap V} \leq \text{rank } \frac{V}{B} + \text{rank } \frac{U}{A}.$$

Here, the last inequality has been derived from the injectivity of the following maps

$$\phi : \frac{A \cap V}{A \cap B} \rightarrow \frac{V}{B}, \quad [c] \mapsto [c],$$

$$\psi : \frac{U \cap V}{A \cap V} \rightarrow \frac{U}{A}, \quad [c] \mapsto [c].$$

The proof is complete. \qed
Lemmas 2.9. Let $D = [A \; B]$ be a matrix composed by submatrices $A$ and $B$. Let $\ell$ be the number of columns in $B$. Then,

$$\text{rank } D \leq \text{rank } A + \ell, \quad \text{dim ker } D \leq \text{dim ker } A + \ell.$$  

Proof. Let $B = [b_1 \cdots b_\ell]$, where $b_i$ is the $i$th column vector of $B$, and set $D^{(i)} = [A \; b_1 \cdots b_i]$. Then, for each $i$, we have

$$\text{rank } D^{(i)} \leq \text{rank } D^{(i-1)} + 1, \quad \text{dim ker } D^{(i)} \leq \text{dim ker } D^{(i-1)} + 1.$$  

Hence, in total, we have the desired inequalities. 

Now, we show a basic estimate on the persistent Betti number for nested filtrations $\mathbb{K} \subset \tilde{\mathbb{K}}$. First, we note the following property.

Lemma 2.10. Let $\mathbb{K}$ be a filtration. For a fixed $a > 0$, let $\tilde{\mathbb{K}} = \{ \tilde{K}_t : t \geq 0 \}$ be a filtration given by

$$\tilde{K}_t = \begin{cases} 
K_t, & t < a, \\
K_t \cup \sigma, & t \geq a,
\end{cases}$$

where $\sigma$ is a new simplex added on $K_a$. Then, $\beta^{r,s}_q(\tilde{\mathbb{K}}) = \beta^{r,s}_q(\mathbb{K})$ for $\dim \sigma \neq q, q + 1$. For $\dim \sigma = q, q + 1$,

$$|\beta^{r,s}_q(\tilde{\mathbb{K}}) - \beta^{r,s}_q(\mathbb{K})| \leq \begin{cases} 
0, & \tilde{K}_r = K_r \text{ and } \tilde{K}_s = K_s, \\
1, & \text{otherwise.}
\end{cases}$$

Proof. We first note that

$$\beta^{r,s}_q(\tilde{\mathbb{K}}) - \beta^{r,s}_q(\mathbb{K}) = \text{rank } Z_q(\tilde{K}_r) - \text{rank } Z_q(\tilde{K}_r) \cap B_q(\tilde{K}_s)$$

$$- \left( \text{rank } Z_q(K_r) - \text{rank } Z_q(K_r) \cap B_q(K_s) \right)$$

$$= \text{rank } \frac{Z_q(\tilde{K}_r)}{Z_q(\tilde{K}_r) \cap B_q(\tilde{K}_s)} - \text{rank } \frac{Z_q(K_r)}{Z_q(K_r) \cap B_q(K_s)}.$$  

Hence, the statement is trivial for $\dim \sigma \neq q, q + 1$. Furthermore, when $\tilde{K}_r = K_r$ and $\tilde{K}_s = K_s$, we also have $\beta^{r,s}_q(\tilde{\mathbb{K}}) = \beta^{r,s}_q(\mathbb{K})$.

Let $\dim \sigma = q$. Then, it follows from Lemma 2.9 that

$$\text{rank } \frac{Z_q(\tilde{K}_r)}{Z_q(\tilde{K}_r)} = \text{rank } Z_q(\tilde{K}_r) - \text{rank } Z_q(K_r) = 0 \text{ or } 1,$$

$$\text{rank } \frac{B_q(\tilde{K}_s)}{B_q(\tilde{K}_s)} = 0.$$
Also, from Lemma 2.8, we have
\[
0 \leq \frac{Z_q(\tilde{K}_r) \cap B_q(\tilde{K}_s)}{Z_q(K_r) \cap B_q(K_s)} = \frac{Z_q(\tilde{K}_r)}{Z_q(K_r)} + \frac{B_q(\tilde{K}_s)}{B_q(K_s)} \leq \frac{Z_q(\tilde{K}_r)}{Z_q(K_r)}.
\]
Therefore, \(|\beta^{r,s}(\tilde{K}) - \beta^{r,s}(K)| \leq 1. The statement for \(\dim \sigma = q + 1\) is similarly proved.

**Lemma 2.11.** Let \(K = \{K_t\}_{t \geq 0}\) and \(\tilde{K} = \{\tilde{K}_t\}_{t \geq 0}\) be filtrations with \(K_t \subset \tilde{K}_t\) for \(t \geq 0\). Then,
\[
|\beta^{r,s}_q(\tilde{K}) - \beta^{r,s}_q(K)| \leq \sum_{j=q,q+1} (|\tilde{K}_{s,j} \backslash K_{s,j}| + |\{\sigma \in K_{s,j} \backslash K_{r,j} : \tilde{t}_\sigma \leq r\}|),
\]
where \(\tilde{K}_{t,j}\) (or \(K_{t,j}\)) is the set of \(j\)-simplices in \(\tilde{K}_t\) (or \(K_t\)), and \(\tilde{t}_\sigma\) (or \(t_\sigma\)) is the birth time of \(\sigma\) in the filtration \(\tilde{K}\) (or \(K\)).

**Proof.** We first decompose \(\tilde{K}_s \backslash K_r = Y \cup Y^c\) by
\[
Y = (\tilde{K}_s \backslash K_s) \cup \{\sigma \in K_s \backslash K_r : \tilde{t}_\sigma \leq r\}, \quad Y^c = \{\sigma \in K_s \backslash K_r : r < \tilde{t}_\sigma \leq t_\sigma\}.
\]
We use the same notation \(Y_j\) for the set of \(j\)-simplices in \(Y\). For the simplices in \(\tilde{K}_s \backslash K_r = \{\sigma_i\}_{i=1}^L\), we assign their indices so that the birth times are in increasing order \(\tilde{t}_{\sigma_1} \leq \cdots \leq \tilde{t}_{\sigma_L}\) and \(K_r \cup \{\sigma_1, \ldots, \sigma_L\}\) becomes a simplicial complex for each \(\ell\). We note that \(\tilde{t}_\sigma \leq t_\sigma\). Furthermore, it suffices to consider the truncations of \(\tilde{K}\) and \(K\) on \([r, s]\) from Lemma 2.7.

Now, we inductively construct a sequence of filtrations \(K = K^0 \subset K^1 \subset \cdots \subset K^L = \tilde{K}\). The filtration \(K^0 = \{K^t_1 : t \geq 0\}\) is given by adding a simplex \(\sigma_i\) to \(K^t_{i-1}\) at \(t_{\sigma_i}\), i.e.,
\[
K^i_t = \begin{cases} K^{i-1}_t, & t < \tilde{t}_{\sigma_i}, \\ K^{i-1}_t \cup \{\sigma_i\}, & t \geq \tilde{t}_{\sigma_i}. \end{cases}
\]
Then, it follows from Lemma 2.10 that \(|\beta^{r,s}_q(K^i) - \beta^{r,s}_q(K^{i-1})| \leq 1\) for \(\sigma_i \in Y\), since \(K^i_t \neq K^{i-1}_t\) or \(K^i_t \neq K^{i-1}_t\) holds. On the other hand, Lemma 2.10 implies \(\beta^{r,s}_q(K^i) = \beta^{r,s}_q(K^{i-1})\) for \(\sigma \in Y^c\). Therefore,
\[
|\beta^{r,s}_q(\tilde{K}) - \beta^{r,s}_q(K)| \leq \sum_{i=1}^L |\beta^{r,s}_q(K^i) - \beta^{r,s}_q(K^{i-1})| \leq |Y_q| + |Y_{q+1}|,
\]
which completes the proof of Lemma 2.11.

**Remark 2.12.** Let \(\Phi, \tilde{\Phi} \in \mathcal{F}(R^N)\) with \(\Phi \subset \tilde{\Phi}\), and \(t_\sigma\) and \(\tilde{t}_\sigma\) be the birth times of the simplex \(\sigma\) in the \(\kappa\)-filtrations \(K(\Phi)\) and \(K(\tilde{\Phi})\), respectively. Then, it is obvious that \(\tilde{t}_\sigma = t_\sigma\) if \(\sigma \subset \Phi \subset \tilde{\Phi}\). Hence, for the estimate \(|\beta^{r,s}_q(K(\tilde{\Phi})) - \beta^{r,s}_q(K(\Phi))|\), the second term obtained in Lemma 2.11 does not appear under this setting.
3. General theory of random measures. Let \( S \) be a locally compact Hausdorff space with countable basis and \( \mathcal{S} \) be the Borel \( \sigma \)-algebra on \( S \). It is well known that \( S \) is a Polish space, i.e., a complete separable metrizable space. If needed, we take a metric \( \rho \) which makes \( S \) complete and separable. We denote by \( \mathcal{B}(S) \) the ring of all relatively compact sets in \( S \). A measure \( \mu \) on \((S,\mathcal{S})\) is said to be a Radon measure if \( \mu(B) < \infty \) for every \( B \in \mathcal{B}(S) \).

Let \( \mathcal{M}(S) \) be the set of all Radon measures on \((S,\mathcal{S})\) and \( \mathcal{M}(S) \) be the \( \sigma \)-algebra generated by the mappings \( \mathcal{M}(S) \ni \mu \mapsto \mu(B) \in [0,\infty) \) for every \( B \in \mathcal{B}(S) \).

We say that a sequence \( \{\mu_n\}_{n \geq 1} \subset \mathcal{M}(S) \) converges to \( \mu \in \mathcal{M}(S) \) vaguely (or in the vague topology) if \( \langle \mu_n, f \rangle \to \langle \mu, f \rangle \) for every continuous function \( f \) with compact support, where \( \langle \mu, f \rangle = \int_{S} f(x) d\mu(x) \). In this case, we write \( \mu_n \xrightarrow{v} \mu \). The space \( \mathcal{M}(S) \) equipped with the vague topology again becomes a Polish space and its Borel \( \sigma \)-algebra coincides with \( \mathcal{M}(S) \).

We denote by \( \mathcal{N}(S) \) the subset in \( \mathcal{M}(S) \) of all integer-valued Radon measures on \( S \). Each element in \( \mathcal{N}(S) \) can be expressed as a sum of delta measures, i.e., \( \mu = \sum \delta_{x_i} \in \mathcal{N}(S) \). We note that the set \( \mathcal{N}(S) \) is a closed subset of \( \mathcal{M}(S) \) in the vague topology.

An \( \mathcal{M}(S) \)-valued (resp. \( \mathcal{N}(S) \)-valued) random variable \( \xi = \xi_\omega \) on a probability space \((\Omega,\mathcal{F},\mathbb{P})\) is called a random measure (resp. point process) on \( S \). If \( \lambda_1(A) := \mathbb{E}[\xi(A)] < \infty \) for all \( A \in \mathcal{B}(S) \), then \( \lambda_1 \) defines a Radon measure and is referred to as the mean measure, or the intensity measure of \( \xi \). Sometimes we denote it by \( \mathbb{E}[\xi] \).

In this paper, two kinds of point processes will appear. One is point processes on \( \mathbb{R}^N \) as spatial point data and the other is point processes on \( \Delta = \{(x,y) \in \mathbb{R}^2 : 0 \leq x < y \leq \infty\} \) as persistence diagrams. The former will be denoted by the upper case letters like \( \Phi \) and the latter by the lower case letters like \( \xi \).

The point process \( \Phi \) on \( \mathbb{R}^N \) is called stationary, if the distribution \( \mathbb{P}\Phi^{-1} \) is invariant under translations, i.e., \( \mathbb{P}\Phi_x^{-1} = \mathbb{P}\Phi^{-1} \) for any \( x \in \mathbb{R}^N \), where \( \Phi_x \) is the translated point process defined by \( \Phi_x(B) = \Phi(B-x) \) for \( B \in \mathcal{B}(\mathbb{R}^N) \). For \( A \subset \mathcal{M}(\mathbb{R}^N) \), let \( A_x = \{\mu_x : \mu \in A\} \) be a set of translated measures defined by \( \mu_x(B) = \mu(B-x) \). Given a point process \( \Phi \), let \( \mathcal{I} \) be the translation invariant \( \sigma \)-field in \( \mathcal{N}(\mathbb{R}^N) \), i.e., the class of subsets \( I \subset \mathcal{N}(\mathbb{R}^N) \) satisfying

\[
\mathbb{P}\Phi^{-1}((I \setminus I_x) \cup (I_x \setminus I)) = 0
\]

for all \( x \in \mathbb{R}^N \). Then, \( \Phi \) is called ergodic if \( \mathcal{I} \) is trivial, that is, for every \( I \in \mathcal{I}, \mathbb{P}\Phi^{-1}(I) \in \{0,1\} \).

From now on and until the end of this section, we fix a space \( S \) and write \( \mathcal{B} \) and \( \mathcal{M} \) for \( \mathcal{B}(S) \) and \( \mathcal{M}(S) \), respectively. For a subset \( A \subset S \), we denote
by \( \partial A \) and \( A^\circ \) the boundary and interior of \( A \), respectively. For a measure \( \mu \in \mathcal{M} \), let \( \mathcal{B}_\mu := \{ B \in \mathcal{B} : \mu(\partial B) = 0 \} \) be the class of relatively compact continuity sets of \( \mu \).

**Lemma 3.1 ([20, 15.7.2]).** Let \( \mu, \mu_1, \mu_2, \ldots \in \mathcal{M} \). Then the following statements are equivalent:

(i) \( \mu_n \overset{v}{\to} \mu \);
(ii) \( \mu_n(B) \to \mu(B) \) for all \( B \in \mathcal{B}_\mu \);
(iii) \( \limsup_{n \to \infty} \mu_n(F) \leq \mu(F) \) and \( \liminf_{n \to \infty} \mu_n(G) \geq \mu(G) \) for all closed \( F \in \mathcal{B} \) and open \( G \in \mathcal{B} \).

**Lemma 3.2 ([20, 15.7.5]).** A subset \( C \) in \( \mathcal{M} \) is relatively compact in the vague topology iff

\[
\sup_{\mu \in C} \mu(B) < \infty \text{ for every } B \in \mathcal{B}.
\]

A class \( \mathcal{A} \subset \mathcal{B} \) is called a convergence-determining class (for vague convergence) if for every \( \mu \in \mathcal{M} \) and every sequence \( \{ \mu_n \} \subset \mathcal{M} \), the condition

\[
\mu_n(A) \to \mu(A) \text{ for all } A \in \mathcal{A} \cap \mathcal{B}_\mu
\]

implies the vague convergence \( \mu_n \overset{v}{\to} \mu \). A class \( \mathcal{A}_\mu \subset \mathcal{B}_\mu \) is called a convergence-determining class for \( \mu \) if for any sequence \( \{ \mu_n \} \subset \mathcal{M} \), the condition

\[
\mu_n(A) \to \mu(A) \text{ as } n \to \infty \text{ for all } A \in \mathcal{A}_\mu,
\]

implies that \( \mu_n \overset{v}{\to} \mu \). By definition, a class \( \mathcal{A} \) is a convergence-determining class if and only if for any \( \mu \in \mathcal{M} \), \( \mathcal{A}_\mu = \mathcal{A} \cap \mathcal{B}_\mu \) is a convergence-determining class for \( \mu \).

We say that a class \( \mathcal{C} \) has the finite covering property if any subset \( B \in \mathcal{B} \) can be covered by a finite union of \( \mathcal{C} \)-sets.

**Lemma 3.3.** Let \( \mathcal{A} \) be a convergence-determining class with finite covering property. Let \( \{ \mu_n \} \) be a sequence of measures in \( \mathcal{M} \). If \( \mu_n(A) \) converges to a finite limit for any \( A \in \mathcal{A} \), then there exists a measure \( \mu \) to which the sequence \( \{ \mu_n \} \) converges vaguely.

**Proof.** For any relatively compact set \( B \in \mathcal{B} \), we can find a finite cover \( \{ A_i \}_{i=1}^m \subset \mathcal{A} \) of \( B \) so that

\[
\limsup_{n \to \infty} \mu_n(B) \leq \limsup_{n \to \infty} \mu_n(\bigcup_{i=1}^m A_i) \leq \lim_{n \to \infty} \sum_{i=1}^m \mu_n(A_i) < \infty.
\]
Therefore the sequence \( \{ \mu_n \}_{n \geq 1} \) is relatively compact by Lemma 3.2, and hence, there is a subsequence \( \mu_{n_k} \) and \( \mu \in \mathcal{M} \) such that \( \mu_{n_k} \overset{v}{\rightarrow} \mu \), i.e., \( \mu_{n_k}(A) \rightarrow \mu(A) \) for every \( A \in \mathcal{B}_\mu \). This together with the assumption implies that \( \mu_n(A) \rightarrow \mu(A) \) for every \( A \in \mathcal{A} \cap \mathcal{B}_\mu \). Consequently, \( \mu_n \) converges to \( \mu \) vaguely from the definition of convergence-determining class. The proof is complete.

**Proposition 3.4.** Let \( \mathcal{A} \) be a convergence-determining class with finite covering property and the property that for every \( \mu \in \mathcal{M} \), it contains a countable convergence-determining class for \( \mu \). Let \( \{\xi_n\} \) be a sequence of random measures on \( S \), i.e., a sequence of \( \mathcal{M} \)-valued random variables. Assume that for every \( A \in \mathcal{A} \), there exists \( c_A \in [0, \infty) \) such that \( E[\xi_n(A)] \rightarrow c_A \) as \( n \rightarrow \infty \). Then, there exists a unique measure \( \mu \in \mathcal{M} \) such that the mean measure \( E[\xi_n] \) converges vaguely to \( \mu \) and \( \mu(A) = c_A \) for \( A \in \mathcal{A} \cap \mathcal{B}_\mu \).

Assume further that for every \( A \in \mathcal{A} \),

\[
\xi_n(A) \rightarrow c_A \text{ almost surely as } n \rightarrow \infty.
\]

Then \( \{\xi_n\} \) converges vaguely to \( \mu \) almost surely.

**Proof.** Note that we implicitly assume that \( E[\xi_n(A)] < \infty \) for all \( A \in \mathcal{A} \) and all \( n \). Then it follows from the finite covering property that the mean measure \( E[\xi_n] \) exists for all \( n \). By Lemma 3.3, there exists a unique measure \( \mu \in \mathcal{M} \) such that \( E[\xi_n] \) converges vaguely to \( \mu \) as \( n \rightarrow \infty \), and hence \( \mu(A) = c_A \) for \( A \in \mathcal{A} \cap \mathcal{B}_\mu \).

Now let \( \mathcal{A}_\mu \subset \mathcal{A} \) be a countable convergence-determining class for \( \mu \). Then, almost surely

\[
\xi_n(A) \rightarrow \mu(A) \text{ as } n \rightarrow \infty, \text{ for all } A \in \mathcal{A}_\mu,
\]

which implies that the sequence \( \{\xi_n\} \) converges vaguely to \( \mu \) almost surely.

The proof is complete.

**4. Convergence of persistence diagrams.**

4.1. **Proof of Theorem 1.11.** Let \( \Phi \) be a stationary point process on \( \mathbb{R}^N \) having all finite moments. Let \( F_q(\Phi, r) \) be the number of \( q \)-simplices in \( K(\Phi, r) \) and \( \tilde{F}_q(\Phi, r; A) \) be the number of \( q \)-simplices in \( K(\Phi, r) \) with at least one vertex in \( A \subset \mathbb{R}^N \). Recall that every \( q \)-simplex in \( K(\Phi, r) \) containing \( x \) must lie in the closed ball \( \bar{B}_{\mu(r)}(x) \). Therefore similar to [33, Lemma 3.1], there exists a constant \( C_{q,r} \) such that

\[
E[F_q(\Phi, r)] \leq E[\tilde{F}_q(\Phi, r; A)] \leq C_{q,r}|A|
\]
for all bounded Borel sets $A$, where $|A|$ is the Lebesgue measure of $A$.

We divide $\Lambda_{mM}$ into $m^N$ rectangles that are congruent to $\Lambda_M$ and write as follows

$$\Lambda_{mM} = \bigcup_{i=1}^{m^N} (\Lambda_M + c_i),$$

where $c_i$ is the center of the $i$th rectangle. We compare $\mathbb{K}(\Phi_{\Lambda_{mM}})$ with a smaller filtration $\mathbb{K}(\Phi_{\Lambda_{mM}}) := \bigcup_{i=1}^{m^N} \mathbb{K}(\Phi_{\Lambda_M+c_i}).$

Let $\psi(L) = \mathbb{E}[\beta_q^{r,s}(\mathbb{K}(\Phi_{\Lambda_L}))]$ for $r \leq s$. By Lemma 2.11, we have

$$|\beta_q^{r,s}(\mathbb{K}(\Phi_{\Lambda_{mM}})) - \beta_q^{r,s}(\mathbb{K}(\Phi_{\Lambda_{mM}}))| \leq \sum_{j=q}^{q+1} \sum_{i=1}^{m^N} F_j(\Phi_{\partial\Lambda_M}([\rho(s)] + c_i, s)).$$

Here, for $A \subset \mathbb{R}^N$, we write $A^{(r)} = \{x \in \mathbb{R}^N : \inf_{y \in A} \|x - y\| \leq r\}$. Since $\mathbb{E}[F_j(\Phi_{\partial\Lambda_M}([\rho(s)] + c_i, s)) = O((\partial\Lambda_M([\rho(s)] + c_i, s)) = O(M^{N-1})$ as $M \to \infty$, we have

$$\psi(mM) = \frac{\psi(M)}{M^N} + O(M^{-1}).$$

Moreover, for $L > L'$,

$$|\beta_q^{r,s}(\mathbb{K}(\Phi_{\Lambda_L})) - \beta_q^{r,s}(\mathbb{K}(\Phi_{\Lambda_{L'}}))| \leq \sum_{j=q}^{q+1} F_j(\Phi_{\Lambda_L}, s; \Lambda_L \setminus \Lambda_{L'})$$

and

$$\mathbb{E}[F_j(\Phi_{\Lambda_L}, s; \Lambda_L \setminus \Lambda_{L'})] = O((\Lambda_L \setminus \Lambda_{L'})^{N-1}).$$

Then, for fixed $M > 0$, taking $m \in \mathbb{N}$ such that $mM \leq L < (m+1)M$, we see that

$$\frac{\psi(L)}{L^N} = \frac{\psi(mM)}{(mM)^N} + O(ML^{-1}).$$

It follows from (4.2) and (4.3) that $\{L^{-N}\psi(L)\}_{L \geq 1}$ is a Cauchy sequence by taking sufficient large $M$ first and then $L$, which completes the first part of the proof.

Let us assume now that $\Phi$ is ergodic. Since the arguments are similar to those in the proof of Theorem 3.5 in [33], we only sketch main ideas. By the multi-dimensional ergodic theorem, we see that almost surely as $m \to \infty$,

$$\frac{1}{mN}\beta_q^{r,s}(\mathbb{K}(\Phi_{\Lambda_{mM}})) = \frac{1}{mN} \sum_{i=1}^{m^N} \beta_q^{r,s}(\mathbb{K}(\Phi_{\Lambda_M+c_i})) \to \mathbb{E}[\beta_q^{r,s}(\mathbb{K}(\Phi_{\Lambda_M}))].$$
\[ \frac{1}{m^N} \sum_{i=1}^{m^N} F_j(\Phi(\partial_{\Lambda_M}(\rho(s))+\epsilon_i), s) \to \mathbb{E}[F_j(\Phi(\partial_{\Lambda_M}(\rho(s)), s)] = O(M^{N-1}). \]

Remark here that the above equations hold for all except a countable set of \( M \) (cf. [28, Theorem 1]). Therefore, it follows from (4.1) that

\[ \limsup_{m \to \infty} \frac{1}{(mM)^N} \beta_{q}^{r,s}(\mathbb{K}(\Phi_{\Lambda_M})) \leq \frac{1}{M} \mathbb{E}[\beta_{q}^{r,s}(\mathbb{K}(\Phi_{\Lambda_M})) + O(M^{-1}). \]

The rest of the proof is similar to the last step in the first part by noting that the following laws of large numbers for \( F_j(\Phi_{\Lambda_{L}}, s), j = q, q + 1 \), hold (cf. [33, Lemma 3.2]),

\[ \frac{F_j(\Phi_{\Lambda_{L}}, s)}{L^j} \to \hat{F}_j(s) \text{ almost surely as } L \to \infty. \]

This completes the second part of the proof. \( \square \)

**Corollary 4.1.** Let \( \Phi \) be a stationary point process on \( \mathbb{R}^N \) having all finite moments, and \( \xi_{q,L} \) be the point process on \( \Delta \) corresponding to the \( q \)th persistence diagram for \( \mathbb{K}(\Phi_{\Lambda_L}) \). Then, for every rectangle of the form \( R = (r_1, r_2] \times (s_1, s_2], [0, r_1] \times (s_1, s_2] \subset \Delta \), there exists a constant \( C_R \in [0, \infty) \) such that

\[ \frac{1}{L^N} \mathbb{E}[\xi_{q,L}(R)] \to C_R \text{ as } L \to \infty. \]

In addition, if \( \Phi \) is ergodic, then

\[ \frac{1}{L^N} \mathbb{E}[\xi_{q,L}(R)] \to C_R \text{ almost surely as } L \to \infty. \]

**Proof.** It is a direct consequence of Theorem 1.11 because for \( R = (r_1, r_2] \times (s_1, s_2] \),

\[ \xi_{q,L}(R) = \beta_{q}^{r_2,s_1}(\mathbb{K}(\Phi_{\Lambda_L})) - \beta_{q}^{r_2,s_2}(\mathbb{K}(\Phi_{\Lambda_L})) + \beta_{q}^{r_1,s_2}(\mathbb{K}(\Phi_{\Lambda_L})) - \beta_{q}^{r_1,s_1}(\mathbb{K}(\Phi_{\Lambda_L})), \]

and for \( R = [0, r_1] \times (s_1, s_2], \)

\[ \xi_{q,L}(R) = \beta_{q}^{r_1,s_1}(\mathbb{K}(\Phi_{\Lambda_L})) - \beta_{q}^{r_1,s_2}(\mathbb{K}(\Phi_{\Lambda_L})). \] \( \square \)
4.2. Proof of Theorem 1.5. Let \( S = \Delta = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < y \leq \infty\} \).

Set
\[
\mathcal{A} = \{(r_1, r_2] \times (s_1, s_2], [0, r_1] \times (s_1, s_2] \subset \Delta : 0 \leq r_1 < r_2 \leq s_1 < s_2 \leq \infty\}.
\]

We will show in Corollary A.3 that \( \mathcal{A} \) is a convergence-determining class which satisfies the condition in Proposition 3.4. Theorem 1.5 then follows from Proposition 3.4 and Corollary 4.1.

DEFINITION 4.2. We call the limiting Radon measure \( \nu_q \in \mathcal{M}(\Delta) \) in Theorem 1.5 the \( q \)-th persistence diagram for a stationary ergodic point process \( \Phi \) on \( \mathbb{R}^N \).

EXAMPLE 4.3. Let \( \Phi \) be a randomly shifted \( \mathbb{Z}^N \)-lattice with intensity 1, i.e., \( \Phi = \mathbb{Z}^N + U \), where \( U \) is a uniform random variable on the unit cube \([0, 1]^N\). Then, \( \Phi \) is a stationary ergodic point process in \( \mathbb{R}^N \). We compute the limiting persistence diagram \( \nu_q \) of the \( \check{\text{C}} \)ech filtration \( \check{C}(\Phi) = \{C(\Phi, r)\}_{r \geq 0} \) for \( q = 1, 2, \ldots, N - 1 \).

For this purpose, we introduce a filtration \( \check{C}(L) = \{\check{C}(L, r)\}_{r \geq 0} \) of cubical complexes by
\[
\check{C}(L, r) = \left\{ \begin{array}{ll}
\text{CL}(L, q), & \frac{\sqrt{q}}{2} \leq r < \frac{\sqrt{q} + 1}{2}, \\
\text{CL}(L, N), & r \geq \frac{\sqrt{N}}{2},
\end{array} \right.
\]
where \( \text{CL}(L, N) \) is the cubical complex consisting of all the elementary cubes in \([0, L] \times \cdots \times [0, L] \subset \mathbb{R}^N\), and \( \text{CL}(L, q) \) is the \( q \)-dimensional skeleton of \( \text{CL}(L, N) \). Here a cube \( Q = I_1 \times \cdots \times I_N \subset \mathbb{R}^N \) consisting of \( I_k = [a, a] \) or \( I_k = [a, a + 1] \) for some \( a \in \mathbb{Z} \) is called an elementary cube [17]. From the stationarity of \( \Phi \) and the homotopy equivalence between \( \check{C}(L, r) \) and \( C(\mathbb{Z}^N \cap [0, L]^N, r) \), it suffices to compute the persistence diagram by using the filtration \( \check{C}(L) \). We also note that \((\sqrt{q}/2, \sqrt{q} + 1/2)\) is the only birth-death pair for the \( q \)-th persistence diagram. Therefore, all we need to verify is the multiplicity of that pair with respect to \( L \).

The Euler-Poincaré formula for \( X = \text{CL}(L, q) \) is given by
\[
(4.4) \quad \sum_{k=0}^{q} (-1)^k |X_k| = \sum_{k=0}^{q} (-1)^k \beta_k(X).
\]
The number \( |X_k| \) of \( k \)-cells in \( X \) is given by (see, e.g., [15])
\[
|X_k| = \sum_{p = k}^{N} \binom{p}{k} S_p(L),
\]
where $S_p(x_1, \ldots, x_N)$ is the elementary symmetric polynomial of degree $p$ and $S_p(L)$ is an abbreviation for $S_p(L, \ldots, L)$. On the other hand, since $X$ is homotopy equivalent to a wedge sum of $q$-spheres, we have $\beta_0 = 1$ and $\beta_k = 0$ for $k = 1, \ldots, q - 1$. Then, it follows from (4.4) that

$$
\beta_q(X) = \sum_{k=0}^{q} (-1)^{k+q} \sum_{p=k}^{N} \binom{p}{k} S_p(L) + (-1)^{q+1},
$$

and hence

$$
\frac{\beta_q(X)}{L^N} = \sum_{k=0}^{q} (-1)^{k+q} \binom{N}{k} + O(L^{-1}) = \binom{N - 1}{q} + O(L^{-1}).
$$

Therefore, the limiting persistence diagram is given by

$$
\nu_q = \binom{N - 1}{q} \delta(\sqrt{q}/2, \sqrt{q+1}/2).
$$

4.3. The support of $\nu_q$. In this section, we give some sufficient conditions both on $\kappa$ and $\Phi$ to ensure the positivity of the limiting measure $\nu_q$. We use the following stability result on persistence diagrams of $\kappa$-filtrations (cf. [5, 6]). Here, the persistence diagram of the $\kappa$-filtration on $\Xi \in \mathcal{F}(\mathbb{R}^N)$ is simply denoted by $D_q(\Xi)$.

**Lemma 4.4.** Assume that $\kappa$ is Lipschitz continuous with respect to the Hausdorff distance, i.e., there exists a constant $c_\kappa$ such that

$$
|\kappa(\sigma) - \kappa(\sigma')| \leq c_\kappa d_H(\sigma, \sigma')
$$

for $\sigma, \sigma' \in \mathcal{F}(\mathbb{R}^N)$. Then, for $\Xi, \Xi' \in \mathcal{F}(\mathbb{R}^N)$,

$$
d_B(D_q(\Xi), D_q(\Xi')) \leq c_\kappa d_H(\Xi, \Xi').
$$

Here $d_B$ and $d_H$ denote the bottleneck distance and the Hausdorff distance, respectively.

See Appendix C for the detail, where we recall definitions of $d_B$ and $d_H$ and give a proof of a generalization of this lemma.

Next we introduce the notion of *marker* which is a finite point configuration for finding a specified point in $\Delta$. 

Definition 4.5. Let \( \Lambda \) be a bounded Borel set in \( \mathbb{R}^N \) and \((b, d) \in \Delta\). We say that \( \Xi \in \mathcal{F}(\mathbb{R}^N) \) is a \((b, d)\)-marker in \( \Lambda \) for the \( q \)th persistent homology (\( \text{PH}_q \)) if (i) \( \Xi \subset \Lambda \) and (ii) for any \( \Phi \in \mathcal{F}(\mathbb{R}^N) \)
\[
\xi_q(\Phi_{\Lambda^c} \cup \Xi)(\{(b, d)\}) \geq \xi_q(\Phi_{\Lambda^c})(\{(b, d)\}) + 1.
\]
Here \( \Lambda^c \) denotes the complement of \( \Lambda \) in \( \mathbb{R}^N \). For a subset \( A \subset \Delta \), we also say that \( \Xi \) is an \( A \)-marker in \( \Lambda \) if there exists \((b, d) \in A\) such that \( \Xi \) is a \((b, d)\)-marker in \( \Lambda \).

Example 4.6. (i) Assume that a point \((b, d) \in \Delta\) is realizable by \( \Xi \in \mathcal{F}(\mathbb{R}^N) \). Then, there exists \( M_0 > 0 \) such that \( \Xi \) is a \((b, d)\)-marker in \( \Lambda_M \) for any \( M \geq M_0 \) because \( \Xi \) is enough isolated from \( \Lambda_M^c \) for sufficiently large \( M \).

(ii) We note that \((1/2, \sqrt{2}/2) \in \Delta\) is realized in \( \text{PH}_1 \) by \( \{(0, 0), (1, 0), (0, 1), (1, 1)\} \in \mathcal{F}(\mathbb{R}^2) \) in the \( \check{\text{C}} \)ech or Rips filtration. It is easy to see that for each \( c \in \Lambda_1 \)
\[
\Xi_M = \sum_{x \in \mathbb{Z}^2 \cap \Lambda_M} \delta_{x+c} \text{ is a } (1/2, \sqrt{2}/2)\text{-marker in } \Lambda_M \text{ for any sufficiently large } M, \text{ for example, } M = 3.
\]

Theorem 4.7. Let
\[
A_{q,\varepsilon,(b,d)} := \bigcup_{M=1}^{\infty} \{ \Phi_{\Lambda_M} \text{ is a } B_\varepsilon((b,d))\text{-marker in } \Lambda_M \text{ for } \text{PH}_q \}
\]
and
\[
S_{q,\varepsilon} := \{(b, d) \in \Delta : \mathbb{P}(A_{q,\varepsilon,(b,d)}) > 0\}, \quad S_q := \bigcap_{\varepsilon > 0} S_{q,\varepsilon}.
\]
Then, \( S_q \subset \text{supp } \nu_q \).

Before proving Theorem 4.7, we give a lower bound for \( \nu_q \).

Lemma 4.8. For a closed set \( A \subset \Delta \),
\[
\nu_q(A) \geq \frac{1}{MN} \mathbb{P}(\Phi_{\Lambda_M} \text{ is an } A\text{-marker in } \Lambda_M \text{ for } \text{PH}_q).
\]

Proof. Let \( \Lambda \) be a bounded Borel set in \( \mathbb{R}^N \) and \((b, d) \in \Delta\). If one could find disjoint subsets \( \Lambda^{(1)}, \ldots, \Lambda^{(k)} \subset \Lambda \) such that \( \Phi_{\Lambda^{(i)}} \) is a \((b, d)\)-marker in \( \Lambda^{(i)} \) for each \( i \), then
\[
\xi_q(\Phi_{\Lambda})(\{(b, d)\}) \geq k.
\]
Indeed, by using (4.5) successively, we have
\[
\xi_q(\Phi_\Lambda((b,d))) \geq \xi_q(\Phi_\Lambda(\cup_{j=1}^k \Lambda^{(j)}))((b,d)) + k \geq k.
\]

For \(L > M > 0\) and \(m = \lfloor L/M \rfloor\), we claim that
\[
(4.8) \quad \xi_q(\Phi_{\Lambda_L})(A) \geq \sum_{i=1}^{m^N} \mathbf{1}_{\{\Phi_{\Lambda_M+c_i} \text{ is an } A\text{-marker in } \Lambda_M + c_i \text{ for PH}_q\}},
\]
where \(c_i \in \Lambda_L, i = 1, 2, \ldots, m^N\) are chosen so that \(\Lambda_L \supseteq \cup_{i=1}^{m^N} (\Lambda_M + c_i)\).
If the right-hand side of (4.8) is equal to \(k\), we have disjoint subsets \(I_j \subset \{c_1, c_2, \ldots, c_{m^N}\}, j = 1, 2, \ldots, J\) with \(\sum_{j=1}^J |I_j| = k\) such that for every \(j = 1, 2, \ldots, J\), \(\Phi_{\Lambda_M+c_i} \text{ is a } (b_j, d_j)\)-marker in \(\Lambda_M + c_i\) for \(\text{PH}_q\) for any \(c \in I_j\).
Here \((b_j, d_j) \in A, j = 1, 2, \ldots, J\) are all distinct. From (4.7), we have
\[
\xi_q(\Phi_{\Lambda_L})(A) \geq \sum_{j=1}^J \xi_q(\Phi_{\Lambda_L})((b_j, d_j)) \geq \sum_{j=1}^J |I_j| = k,
\]
which implies (4.8).

For a closed set \(A \subset \Delta\), from (4.8), we obtain
\[
\nu_q(A) \geq \limsup_{L \to \infty} \frac{1}{L^N} \mathbb{E}[\xi_q(\Phi_{\Lambda_L})](A)
\geq \limsup_{L \to \infty} \frac{1}{L^N} \sum_{i=1}^{m^N} \mathbb{P}(\Phi_{\Lambda_M+c_i} \text{ is an } A\text{-marker in } \Lambda_M + c_i \text{ for PH}_q)
= \frac{1}{M^N} \mathbb{P}(\Phi_{\Lambda_M} \text{ is an } A\text{-marker in } \Lambda_M \text{ for PH}_q).
\]
This completes the proof. \(\square\)

**Proof of Theorem 4.7.** If \((b,d) \in S_q\), then for every \(\varepsilon > 0\), there exists \(M = M_\varepsilon \in \mathbb{N}\) such that
\[
\mathbb{P}(\Phi_{\Lambda_M} \text{ is a } B_{\varepsilon,(b,d)}\text{-marker in } \Lambda_M \text{ for PH}_q) > 0.
\]
From (4.6), we see that \(\nu_q(\tilde{B}_{\varepsilon,(b,d)}) > 0\) for any \(\varepsilon > 0\), which implies \((b,d) \in \text{supp } \nu_q\). Therefore, \(S_q \subseteq \text{supp } \nu_q\). \(\square\)

For a bounded set \(\Lambda \subset \mathbb{R}^N\), the restriction \(\mathfrak{R}(\Lambda)\) of \(\mathfrak{R}(\mathbb{R}^N)\) on \(\Lambda\) can be identified with \(\bigcup_{k=0}^\infty \Lambda^k/\sim\), where \(\sim\) is the equivalence relation induced by
permutations on coordinates. Let $\Pi$ be the probability distribution of homogeneous Poisson point process with unit intensity. It is clear that the local densities, which are sometimes called Janossy densities, of the restriction of $\Pi$ on $\Lambda$ is given by

\[
\Pi|_{\Lambda} = \begin{cases} 
e^{-|\Lambda|} \frac{1}{k!} dx_1 dx_2 \cdots dx_k & \text{on } \Lambda^k, \\ 
e^{-|\Lambda|} & \text{on } \Lambda^0 = \emptyset. \end{cases}
\]

For a probability measure $\Theta$ on $\mathcal{M}(\mathbb{R}^N)$, if $\Theta|_{\Lambda}$ is absolutely continuous with respect to $\Pi|_{\Lambda}$ for a bounded set $\Lambda$, then $\Theta|_{\Lambda}$ is absolutely continuous with respect to the Lebesgue measure on each $\Lambda^k$ for every $k$; thus the Radon-Nikodym density $d\Theta|_{\Lambda}/d\Pi|_{\Lambda}$ is defined a.e. on $\Lambda^k$ for every $k$.

**Proof of Theorem 1.9.** Assume that $(b,d) \in R_q$ and it is realizable by $\{y_1, \ldots, y_m\}$. From continuity of persistence diagram in Lemma 4.4, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\xi_q(\{z_1, \ldots, z_m\})(B_\varepsilon((b,d))) \geq 1$ for any $(z_1, \ldots, z_m) \in B_\delta(y_1) \times \cdots \times B_\delta(y_m)$. From Example 4.6(i), there exists $M \in \mathbb{N}$ such that any $\{z_1, \ldots, z_m\}$ is a $B_\varepsilon((b,d))$-marker in $\Lambda_M$. Hence, we see that

\[
\mathbb{P}(\Phi_{\Lambda_M} \text{ is a } B_\varepsilon((b,d))-\text{marker in } \Lambda_M \text{ for } PH_q) \\
\geq \Theta|_{\Lambda_M}(\cap_{i=1}^m \{\Phi(B_\delta(y_i) = 1)\}) \cap \{\Phi(\Lambda_M \setminus \cup_{i=1}^m B_\delta(y_i)) = 0\} \\
= \frac{e^{-|\Lambda_M|}}{m!} \int_{B_\delta(y_1) \times \cdots \times B_\delta(y_m)} f_{\Lambda_M}(z_1, \ldots, z_m) dz_1 \cdots dz_m \\
> 0,
\]

where $f_{\Lambda_M} = d\Theta|_{\Lambda_M}/d\Pi|_{\Lambda_M}$. Hence, $R_q \subset S_q \subset \text{supp } \nu_q$ by Theorem 4.7. Since $\text{supp } \nu_q \subset \overline{R_q}$ as mentioned after Example 1.8, we conclude that $\text{supp } \nu_q = \overline{R_q}$. \hfill $\square$

**5. Central limit theorem for persistent Betti numbers.** In this section, let $\Phi = \mathcal{P}$ be a homogeneous Poisson point process with unit intensity, and we prove Theorem 1.12. The idea is to apply a result in [27] which shows a central limit theorem for a certain class of functionals defined on Poisson point processes.

We here summarize necessary properties for functionals to achieve the central limit theorem. First of all, let us consider a sequence $\{W_n\}$ of Borel subsets in $\mathbb{R}^N$ satisfying the following conditions:

(A1) $|W_n| = n$ for all $n \in \mathbb{N}$;

(A2) $\bigcup_{n \geq 1} \bigcap_{m \geq n} W_m = \mathbb{R}^N$;
(A3) \( \lim_{n \to \infty} |(\partial W_n)(r)|/n = 0 \) for all \( r > 0 \);

(A4) there exists a constant \( \gamma > 0 \) such that \( \text{diam}(W_n) \leq \gamma n^\gamma \).

Given such a sequence, let \( W = W(\{W_n\}) \) be the collection of all subsets \( A \) in \( \mathbb{R}^N \) of the form \( A = W_n + x \) for some \( W_n \) in the sequence and some point \( x \in \mathbb{R}^N \).

Let \( H \) be a real-valued functional defined on \( \mathcal{F}(\mathbb{R}^N) \). The functional \( H \) is said to be translation invariant if it satisfies \( H(\mathcal{X} + y) = H(\mathcal{X}) \) for any \( \mathcal{X} \in \mathcal{F}(\mathbb{R}^N) \) and \( y \in \mathbb{R}^N \). Let \( D_0 \) be the add one cost function

\[
D_0H(\mathcal{X}) = H(\mathcal{X} \cup \{0\}) - H(\mathcal{X}), \quad \mathcal{X} \in \mathcal{F}(\mathbb{R}^N),
\]

which is the increment in \( H \) caused by inserting a point at the origin. The functional \( H \) is weakly stabilizing on \( W \) if there exists a random variable \( D(\infty) \) such that \( D_0H(\mathcal{P}_A_n) \xrightarrow{a.s.} D(\infty) \) as \( n \to \infty \) for any sequence \( \{A_n \in W\}_{n \geq 1} \) tending to \( \mathbb{R}^N \). The Poisson bounded moment condition on \( W \) is given by

\[
\sup_{0 \in A \in W} \mathbb{E}[(D_0H(\mathcal{P}A))^4] < \infty.
\]

Then, we restate Theorem 3.1 in [27] in the following form.

**Lemma 5.1 ([27, Theorem 3.1]).** Let \( H \) be a real-valued functional defined on \( \mathcal{F}(\mathbb{R}^N) \). Assume that \( H \) is translation invariant and weakly stabilizing on \( W \), and satisfies the Poisson bounded moment condition. Then, there exists a constant \( \sigma^2 \in [0, \infty) \) such that \( n^{-1} \text{Var}[H(\mathcal{P}W_n)] \to \sigma^2 \) and

\[
\frac{H(\mathcal{P}W_n) - \mathbb{E}[H(\mathcal{P}W_n)]}{n^{1/2}} \xrightarrow{d} \mathcal{N}(0, \sigma^2) \text{ as } n \to \infty.
\]

By using Lemma 5.1, we prove the following theorem.

**Theorem 5.2.** Let \( \Phi = \mathcal{P} \) be a homogeneous Poisson point process with unit intensity. Assume that the sequence \( \{W_n\} \) satisfies (A1)–(A4). Then for any \( 0 \leq r \leq s < \infty \),

\[
\frac{\beta_q^{r,s}(\mathbb{K}(\mathcal{P}W_n)) - \mathbb{E}[\beta_q^{r,s}(\mathbb{K}(\mathcal{P}W_n))]}{n^{1/2}} \xrightarrow{d} \mathcal{N}(0, \sigma_{r,s}^2) \text{ as } n \to \infty.
\]

In particular, Theorem 1.12 is derived from this theorem by taking \( W_n = \Lambda_{L_n} \) with \( L_n = n^{1/N} \).

For the proof of Theorem 5.2, the essential part is to show the weak stabilization of the persistent Betti number \( \beta_q^{r,s}(\mathbb{K}(\cdot)) \) as a functional on \( \mathcal{F}(\mathbb{R}^N) \), on which we focus below.
We remark that, for almost surely, the Poisson point process $P$ consists of infinite points in $\mathbb{R}^N$ which do not have accumulation points. In view of this property, we first show a stabilization of persistent Betti numbers in the following deterministic setting.

**Lemma 5.3.** Let $P$ be a set of points in $\mathbb{R}^N$ without accumulation points. Then, for each fixed $r \leq s$, there exist constants $D_\infty$ and $R > 0$ such that

$$D_0\beta_{r,s}^q(\mathbb{K}(P_{\bar{B}_a(0)})) = D_\infty$$

for all $a \geq R$.

**Proof.** Let $P' = P \cup \{0\}$. Let $K_{r,a} = K(P_{\bar{B}_a(0)}, r)$ be the simplicial complex defined on $P_{\bar{B}_a(0)}$ with parameter $r$, and similarly let $K'_{r,a} = K(P'_{\bar{B}_a(0)}, r)$.

From the definition (2.3), $D_0\beta_{r,s}^q(\mathbb{K}(P_{\bar{B}_a(0)}))$ can be expressed as

$$D_0\beta_{r,s}^q(\mathbb{K}(P_{\bar{B}_a(0)})) = \text{rank} Z_q(K'_{r,a}) - \text{rank} Z_q(K_{r,a})$$

Hence, it suffices to show the stabilization with respect to $a$ for rank $Z_q(K_{r,a})$ and rank($Z_q(K_{r,a}) \cap B_q(K_{s,a})$) separately.

Let us study rank $Z_q(K_{r,a})$. Since the rank takes non-negative integer values, we show the bounded and the non-decreasing properties. First of all, note that $K_{r,a} \subset K'_{r,a}$, and hence $Z_q(K_{r,a}) \subset Z_q(K'_{r,a})$. Let us express $K'_{r,a}$ as a disjoint union $K'_{r,a} = K_{r,a} \cup K^0_{r,a}$, where $K^0_{r,a}$ is the set of simplices having the point 0, and let $K_{r,a,q} = \{\sigma \in (K'_{r,a})_q : 0 \in \sigma\}$.

Let $\partial_q$ and $\partial'_q$ be the $q$th boundary maps on $K_{r,a}$ and $K'_{r,a}$, respectively. Then, we can obtain the following block matrix form

$$\partial'_q = \begin{bmatrix} M_{1,\rho} & 0 \\ M_{2,\rho} & \partial_{q,a} \end{bmatrix},$$

where the first columns and rows are arranged by the simplices in $K^0_{r,a,q}$ and $K^0_{r,a,q-1}$, and the second columns and rows correspond to the simplices in $K_{r,a}$. Recall that any simplex $\sigma \in K(P, r)$ containing the point 0 is included in $\bar{B}_{\rho(r)}(0)$. Hence, the set $K^0_{r,a,q}$ becomes independent of $a$ for $a \geq \rho(r)$, which
we denote by $K_{r,s,a}^0$. From this observation and Lemma 2.9 applied to the matrix form (5.1), we have

$$\text{rank } Z_q(K_{r,a}') - \text{rank } Z_q(K_{r,a}) \leq |K_{r,a,q}^0| = |K_{r,s,a}^0|,$$

which gives the boundedness.

In order to show the non-decreasing property, let us consider a homomorphism defined by

$$f: \frac{Z_q(K_{r,a}')}{Z_q(K_{r,a})} \ni [c] \mapsto [c] \in \frac{Z_q(K_{r,a}')}{Z_q(K_{r,a})}$$

for $a_1 \leq a_2$. This map is well-defined because $Z_q(K_{r,a}') \subset Z_q(K_{r,a})$ and $Z_q(K_{r,a}') \subset Z_q(K_{r,a})$ hold. Suppose that $f[c] = 0$. Then, the cycle $c \in Z_q(K_{r,a}')$ is in $Z_q(K_{r,a})$. It means that the $q$-simplices consisting of $c$ do not contain the point 0, and hence $c \in Z_q(K_{r,a})$. This shows that the map $f$ is injective. From this observation, we have the inequality

$$\text{rank } Z_q(K_{r,a}')/Z_q(K_{r,a}) \leq \text{rank } Z_q(K_{r,a}']/Z_q(K_{r,a}),$$

which leads to the desired non-decreasing property. This finishes the proof of the stabilization of rank $Z_q(K_{r,a})$.

Let us study the stabilization of $\text{rank}(Z_q(K_{r,a}) \cap B_q(K_{s,a}))$. The strategy is basically the same as above. It follows from Lemma 2.8 that

$$\text{rank } \frac{Z_q(K_{r,a}') \cap B_q(K_{s,a}')}{Z_q(K_{r,a}) \cap B_q(K_{s,a})} \leq \text{rank } \frac{Z_q(K_{r,a}')}{Z_q(K_{r,a})} + \text{rank } \frac{B_q(K_{s,a}')}{B_q(K_{s,a})}.$$

Then, from the same reasoning used in rank $Z_q(K_{r,a})$, we have the stabilization $|K_{s,a,q+1}^0| = |K_{s,s,q+1}^0|$ for large $a$. Hence, we have the boundedness

$$\text{rank } Z_q(K_{r,a}') \cap B_q(K_{s,a}') - \text{rank } Z_q(K_{r,a}) \cap B_q(K_{s,a}) \leq |K_{r,s,a}^0| + |K_{s,s,a}^0|. $$

Similarly, we can show the injectivity of the map

$$f: \frac{Z_q(K_{r,a}') \cap B_q(K_{s,a}')}{Z_q(K_{r,a}) \cap B_q(K_{s,a})} \rightarrow \frac{Z_q(K_{r,a}')}{Z_q(K_{r,a})} \ni [c] \mapsto [c]$$

from which the non-decreasing property follows. This completes the proof of the lemma.

**Proposition 5.4.** The functional $\beta_{q,s}^r(K(\cdot))$ is weakly stabilizing.
Proof. Let $R > 0$ be chosen as in Lemma 5.3 and let $\{A_n \in W\}_{n \geq 1}$ be a sequence tending to $\mathbb{R}^N$. Then, there exists $n_0 \in \mathbb{N}$ such that $B_R(0) \subset A_n$ for all $n \geq n_0$.

For $n \geq n_0$, let us set $L_{r,n} = K(P_{A_n}, r)$. Then, since $A_n$ is bounded, there exists $a > R$ such that

$$B_R(0) \subset A_n \subset B_a(0).$$

Then, as in the same way used for showing the injectivity in the proof of Lemma 5.3, we can show

$$Z_q(K'_{r,R}) \subset Z_q(L'_{r,n}) \subset Z_q(K'_{r,a}),$$

where $K_{r,a} = K(P_{B_a(0)}, r)$ as before. Since the ranks of $Z_q(K'_{r,R})/Z_q(K_{r,R})$ and $Z_q(K'_{r,a})/Z_q(K_{r,a})$ are equal, for all $n \geq n_0$

$$\text{rank } Z_q(K'_{r,R}) - \text{rank } Z_q(K_{r,R}) = \text{rank } Z_q(L'_{r,n}) - \text{rank } Z_q(L_{r,n}).$$

We can also show that $\text{rank } Z_q(L'_{r,n}) \cap B_q(L'_{s,n}) - \text{rank } Z_q(L_{r,n}) \cap B_q(L_{s,n})$ is invariant for $n \geq n_0$ in a similar manner. This completes the proof.

Proof of Theorem 5.2. For fixed $r \leq s$, we regard the persistent Betti number $\beta_{q,s}^r(K(\cdot))$ as a functional on $\mathcal{F}(\mathbb{R}^N)$, and check the three conditions stated in Lemma 5.1. First, the translation invariance is obvious, because $\kappa$ is translation invariant. Next, let us consider the Poisson bounded moment condition on $W$. We note the following estimate:

$$|D_0 \beta_{q,s}^r(K(P_A))| = |\beta_{q,s}^r(K(P_A \cup \{0\})) - \beta_{q,s}^r(K(P_A))|$$

$$\leq \sum_{j=q,q+1} |K_j(P_A \cup \{0\}, s) \setminus K_j(P_A, s)|$$

$$\leq \sum_{j=q,q+1} F_j(P_{B_{\rho(s)}(0)}, s) < \infty.$$

Here, the second inequality follows from Lemma 2.11, and the boundedness of the last expression is shown by the finite moments of the Poisson point process on $B_{\rho(s)}(0)$. We showed the weak stabilization in Proposition 5.4. The proof of Theorem 5.2 is now complete.

6. Conclusions. In this paper, we studied a convergence of persistence diagrams and persistent Betti numbers for stationary point processes, and a central limit theorem of persistent Betti numbers for homogeneous Poisson point process. Several important problems are still yet to be solved.
1. We showed the existence of limiting persistence diagram for simplicial complexes built over stationary ergodic point processes. Such convergence results can be expected for more general random simplicial/cell complexes studied in [14, 15]. It would also be important to investigate the rate of convergence from the statistical and computational point of view.

2. Attractiveness/repulsiveness of point processes are reflected on persistence diagrams (see Figure 1). For example, the mass of the limiting persistence diagram $\nu_q$ for negatively correlated point process seems to become more concentrated than that for positively correlated point process.

3. The moments of the limiting persistence diagram $\int_\Delta |y - x|^n \nu_q(\text{d}x\text{d}y)$, should be studied. Other properties of limiting persistence diagrams such as continuity, absolute continuity/singularity, comparison etc. should also be investigated thoroughly for practical purposes (cf. [22, 13]).

4. The central limit theorem for persistent Betti numbers (even for usual Betti numbers) is only proved for Poisson point processes. It could be extended to more general stationary point processes. We also expect that a scaled persistence diagram converges to a Gaussian field on $\Delta$.

APPENDIX A: CONVERGENCE-DETERMINING CLASS FOR VAGUE CONVERGENCE

We provide a sufficient condition for a class of $\mathcal{B}$-sets to be a convergence determining class for vague convergence. We use the same notations as in Section 3. Assume that a class $\mathcal{A} \subset \mathcal{B}$ is closed under finite intersections. Let us define

$$\mathcal{R}(\mathcal{A}) = \left\{ \bigcup_{\text{finite}} A_i : A_i \in \mathcal{A} \right\}.$$

Then $\mathcal{R}(\mathcal{A})$ is closed under both finite intersections and finite unions. Furthermore, if $\mu_n(A) \to \mu(A)$ for all $A \in \mathcal{A}$, then so does for all $A \in \mathcal{R}(\mathcal{A})$, because

$$\mu\left(\bigcup_{i=1}^m A_i\right) = \sum_{i} \mu(A_i) - \sum_{i \neq j} \mu(A_i \cap A_j) + \cdots + (-1)^{m-1} \mu\left(\bigcap_{i=1}^m A_i\right).$$

LEMMA A.1. Assume that a class $\mathcal{A}$ is closed under finite intersections, and that

(i) each open set $G \in \mathcal{B}$ is a countable union of $\mathcal{R}(\mathcal{A})$-sets, and
(ii) each closed set $F \in \mathcal{B}$ is a countable intersection of $\mathcal{R}(\mathcal{A})$-sets.

If $\mu_n(A) \to \mu(A)$ for all $A \in \mathcal{A}$, then $\mu_n$ converges vaguely to $\mu$. In particular, the class $\mathcal{A}$ is a convergence-determining class for $\mu$ provided that $\mathcal{A} \subset \mathcal{B}_\mu$.

**Proof.** Let $G \in \mathcal{B}$ be an open set. By assumption, there are sets $A_i \in \mathcal{R}(\mathcal{A})$ such that

$$G = \bigcup_{i=1}^{\infty} A_i.$$ 

Given $\varepsilon > 0$, choose an $m$ such that

$$\mu(\bigcup_{i=1}^{m} A_i) > \mu(G) - \varepsilon.$$ 

Then we have

$$\mu(G) - \varepsilon < \mu(\bigcup_{i=1}^{m} A_i) = \lim_{n \to \infty} \mu_n(\bigcup_{i=1}^{m} A_i) \leq \liminf_{n \to \infty} \mu_n(G).$$

Since $\varepsilon$ is arbitrary, we get

$$\mu(G) \leq \liminf_{n \to \infty} \mu_n(G).$$

Now for a closed set $F \in \mathcal{B}$, take $A_i \in \mathcal{R}(\mathcal{A})$ such that

$$F = \bigcap_{i=1}^{\infty} A_i.$$ 

Since $A_i \in \mathcal{B}$, for given $\varepsilon > 0$, we can choose $m$ large enough such that

$$\mu(F) + \varepsilon > \mu(\bigcap_{i=1}^{m} A_i).$$

Then, it follows from $\bigcap_{i=1}^{m} A_i \in \mathcal{R}(\mathcal{A})$ that

$$\mu(F) + \varepsilon > \mu(\bigcap_{i=1}^{m} A_i) = \lim_{n \to \infty} \mu_n(\bigcap_{i=1}^{m} A_i) \geq \limsup_{n \to \infty} \mu_n(F).$$

Letting $\varepsilon \to 0$, we get

$$\mu(F) \geq \limsup_{n \to \infty} \mu_n(F).$$

Therefore, the conclusion follows from Lemma 3.1. \qed
For given $\mathcal{A}$, let $\mathcal{A}_{x,\varepsilon}$ be the class of $\mathcal{A}$-sets satisfying $x \in A^o \subset A \subset B_\varepsilon(x)$, where $A^o$ is the interior of $A$. Let $\partial \mathcal{A}_{x,\varepsilon}$ be the class of their boundaries, i.e., $\partial \mathcal{A}_{x,\varepsilon} = \{ \partial A : A \in \mathcal{A}_{x,\varepsilon} \}$.

The following theorem gives a sufficient condition for a class $\mathcal{A}$ to be a convergence-determining class for vague convergence of Radon measures (see Theorem 2.4 in [1] for an analogous result on weak convergence of probability measures).

**Theorem A.2.** Suppose that $\mathcal{A}$ is closed under finite intersections and, for each $x \in S$ and $\varepsilon > 0$, $\partial \mathcal{A}_{x,\varepsilon}$ contains either $\emptyset$ or uncountably many disjoint sets. Then, $\mathcal{A}$ is a convergence-determining class. Moreover, for any measure $\mu \in \mathcal{M}$, $\mathcal{A}$ contains a countable convergence-determining class for $\mu$.

**Proof.** Fix an arbitrary $\mu \in \mathcal{M}$, and let $\mathcal{A}_\mu = \mathcal{A} \cap B_\mu$ be the class of $\mu$-continuity sets in $\mathcal{A}$. Since

$$\partial (A \cap B) \subset (\partial A) \cup (\partial B),$$

$\mathcal{A}_\mu$ is again closed under finite intersections.

Let $G \in B$ be an open set. For $x \in G$, choose $\varepsilon > 0$ such that $B_\varepsilon(x) \subset G$. By the assumption, if $\partial \mathcal{A}_{x,\varepsilon}$ does not contain $\emptyset$, then it must contain uncountably many disjoint sets. Hence, in either case, $\partial \mathcal{A}_{x,\varepsilon}$ contains a set $A_x$ of $\mu$-measure 0, or $A_x \in \mathcal{A}_\mu$. Therefore, $G$ can be written as

$$G = \bigcup_{x \in G} A_x^o = \bigcup_{x \in G} A_x.$$

Since $S$ is a separable metric space, there is a countable subcollection $\{A^o_{x_i}\}$ of $\{A^o_x : x \in G\}$ which covers $G$, namely,

$$G = \bigcup_{i=1}^\infty A^o_{x_i}.$$

Let $\{G_i\}_{i=1}^\infty$ be a countable basis of $S$. For each $i$, we have just shown that there are countable sets $\{A_{i,j}\}_{j=1}^\infty \subset \mathcal{A}_\mu$ such that

$$G_i = \bigcup_{j=1}^\infty A^o_{i,j} = \bigcup_{j=1}^\infty A_{i,j}.$$

Set

$$\mathcal{A}'_\mu = \{ \cap_{\text{finite } A_{i,j}} \}.$$
Then $\mathcal{A}_\mu' \subset \mathcal{A}_\mu$ is countable and closed under finite intersections. The remaining task is to show that $\mathcal{A}_\mu'$ satisfies the two conditions in Lemma A.1. The condition for open sets is clear from the construction of $\mathcal{A}_\mu'$.

Next, let $F \in \mathcal{B}$ be a closed (thus compact) set. For each $\varepsilon > 0$, let $F(\varepsilon) = \{ x \in S : d(x, F) = \inf_{y \in F} \rho(x, y) \leq \varepsilon \}$.

Then $F = \bigcap_{p=1}^{\infty} F^{(\frac{1}{p})}$. We claim that, for each $\varepsilon > 0$, there exist $m = m(\varepsilon)$ and a collection of sets $\{ C_k \}_{k=1}^m \subset \mathcal{A}_\mu'$ such that

$$F \subset \bigcup_{k=1}^m C_k \subset F(\varepsilon).$$

Indeed, for each $x \in F$, there is a pair $(i_x, j_x)$ such that $x \in A_{i_x,j_x}^o \subset A_{i_x,j_x} \subset G_{i_x} \subset B_\varepsilon(x)$. Let $C_x = A_{i_x,j_x}$. Then

$$F \subset \bigcup_{x \in F} C_x^o.$$  

Since $F$ is compact, there is a finite collection $\{ C_{x_k}^o \}_{k=1}^m$ such that

$$F \subset \bigcup_{k=1}^m C_{x_k}^o.$$  

Finally, note that $C_{x_k}^o \subset C_{x_k} \subset F(\varepsilon)$, we have

$$F \subset \bigcup_{k=1}^m C_{x_k} \subset F(\varepsilon).$$

Therefore, the condition for closed sets in Lemma A.1 is satisfied, which completes the proof of Theorem A.2.

**Corollary A.3.** The class

$$\mathcal{A} \equiv \{ (r_1, r_2) \times (s_1, s_2), [0, r_2) \times (s_1, s_2] \subset \Delta : 0 \leq r_1 \leq r_2 \leq s_1 \leq s_2 \leq \infty \}$$

satisfies the conditions of Proposition 3.4, namely, for any measure $\mu$, it contains a countable convergence determining class for $\mu$.

**Proof.** It suffices to check the conditions in Theorem A.2. It is clear that $\mathcal{A}$ is closed under finite intersection and $\partial \mathcal{A}_{x,\varepsilon}$ contains countably many disjoint sets for any $x \in \Delta$ and $\varepsilon > 0$. 

\[\blacksquare\]
APPENDIX B: SIMPLICIAL COMPLEX AND HOMOLOGY

B.1. Simplicial complex. We first introduce a combinatorial object called simplicial complex. Let $P = \{1, \ldots, n\}$ be a finite set (not necessary to be points in a metric space). A simplicial complex with the vertex set $P$ is defined by a collection $K$ of subsets in $P$ satisfying the following properties:

(i) $\{i\} \in K$ for $i = 1, \ldots, n$, and
(ii) if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$.

Each subset $\sigma$ with $q + 1$ vertices is called a $q$-simplex. We denote the set of $q$-simplices by $K_q$. A subcollection $T \subset K$ which also becomes a simplicial complex is called a subcomplex of $K$.

Example B.1. Figure 4 shows two polyhedra of simplicial complexes

$$K = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\},$$
$$T = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$  

![Fig 4. The polyhedra of the simplicial complexes K (left) and T (right).](image)

B.2. Homology. The procedure to define homology is summarized as follows:

1. Given a simplicial complex $K$, build a chain complex $C_*(K)$. This is an algebraization of $K$ characterizing the boundary.
2. Define homology by quotienting out certain subspaces in $C_*(K)$ characterized by the boundary.

We begin with the procedure 1 by assigning orientations on simplices. When we deal with a $q$-simplex $\sigma = \{i_0, \ldots, i_q\}$ as an ordered set, there are $(q + 1)!$ orderings on $\sigma$. For $q > 0$, we define an equivalence relation $i_{j_0}, \ldots, i_{j_q} \sim i_{\ell_0}, \ldots, i_{\ell_q}$ on two orderings of $\sigma$ such that they are mapped to each other by even permutations. By definition, two equivalence classes exist, and each of them is called an oriented simplex. An oriented simplex is denoted by $\langle i_{j_0}, \ldots, i_{j_q}\rangle$, and its opposite orientation is expressed by adding
the minus $-\langle i_0, \ldots, i_q \rangle$. We write $\langle \sigma \rangle = \langle i_0, \ldots, i_q \rangle$ for the equivalence class including $i_0 < \cdots < i_q$. For $q = 0$, we suppose that we have only one orientation for each vertex.

Let $\mathbf{F}$ be a field. We construct a $\mathbf{F}$-vector space $C_q(K)$ as

$$C_q(K) = \text{Span}_\mathbf{F}\{\langle \sigma \rangle | \sigma \in K_q\}$$

for $K_q \neq \emptyset$ and $C_q(K) = 0$ for $K_q = \emptyset$. Here, $\text{Span}_\mathbf{F}(A)$ for a set $A$ is a vector space over $\mathbf{F}$ such that the elements of $A$ formally form a basis of the vector space. Furthermore, we define a linear map called the boundary map $\partial_q : C_q(K) \to C_{q-1}(K)$ by the linear extension of

$$(B.1) \quad \partial_q(i_0, \ldots, i_q) = \sum_{\ell=0}^q (-1)^\ell \langle i_0, \ldots, \widehat{i_\ell}, \ldots, i_q \rangle,$$

where $\widehat{i_\ell}$ means the removal of the vertex $i_\ell$. We can regard the linear map $\partial_q$ as algebraically capturing the $(q-1)$-dimensional boundary of a $q$-dimensional object.

For example, the image of the 2-simplex $\langle \sigma \rangle = \langle 1, 2, 3 \rangle$ is given by $\partial_2(\sigma) = \langle 2, 3 \rangle - \langle 1, 3 \rangle + \langle 1, 2 \rangle$, which is the boundary of $\sigma$ (see Figure 4).

In practice, by arranging some orderings of the oriented $q$- and $(q-1)$-simplices, we can represent the boundary map as a matrix

$$M_q = (M_{\sigma, \tau})_{\sigma \in K_{q-1}, \tau \in K_q}$$

with the entry $M_{\sigma, \tau} = 0, \pm 1$ given by the coefficient in (B.1). For the simplicial complex $K$ in Example B.1, the matrix representations $M_1$ and $M_2$ of the boundary maps are given by

$$(B.2) \quad M_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Here the 1-simplices (resp. 0-simplices) are ordered by $\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 3 \rangle$ (resp. $\langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle$).

We call a sequence of the vector spaces and linear maps

$$\cdots \longrightarrow C_{q+1}(K) \xrightarrow{\partial_{q+1}} C_q(K) \xrightarrow{\partial_q} C_{q-1}(K) \longrightarrow \cdots$$

the chain complex $C_\ast(K)$ of $K$. As an easy exercise, we can show $\partial_q \circ \partial_{q+1} = 0$ for every $q$. Hence, the subspaces $Z_q(K) = \ker \partial_q$ and $B_q(K) = \text{im} \partial_{q+1}$
satisfy $B_q(K) \subset Z_q(K)$. Then, the $q$th (simplicial) homology is defined by taking the quotient space

$$H_q(K) = Z_q(K)/B_q(K).$$

Intuitively, the dimension of $H_q(K)$ counts the number of $q$-dimensional holes in $K$ and each generator of the vector space $H_q(K)$ corresponds to these holes. We remark that the homology as a vector space is independent of the orientations of simplices.

For a subcomplex $T$ of $K$, the inclusion map $\iota: T \hookrightarrow K$ naturally induces a linear map in homology $\iota_q: H_q(T) \rightarrow H_q(K)$. Namely, an element $[c] \in H_q(T)$ is mapped to $[c] \in H_q(K)$, where the equivalence class $[c]$ is taken in each vector space.

For example, the simplicial complex $K$ in Example B.1 has

$$Z_1(K) = \text{Span}_\mathbb{F}[1 1 -1]^T = B_1(K)$$

from (B.2). Hence $H_1(K) = 0$, meaning that there are no 1-dimensional hole (ring) in $K$. On the other hand, since $Z_1(T) = Z_1(K)$ and $B_1(T) = 0$, we have $H_1(T) \simeq \mathbb{F}$, meaning that $T$ consists of one ring. Hence, the induced linear map $\iota_1: H_1(T) \rightarrow H_1(K)$ means that the ring in $T$ disappears in $K$ under $T \hookrightarrow K$.

**APPENDIX C: CONTINUITY OF PERSISTENCE DIAGRAMS OF $\kappa$-COMPLEXES**

We give a stability result for persistence diagrams of $\kappa$-filtrations which extends the stability result obtained in [5]. The notation used here follows the paper [5]. We first recall the definition of the Hausdorff distance and the bottleneck distance. The Hausdorff distance $d_H$ on $\mathcal{F}(\mathbb{R}^N)$ for $\sigma, \sigma' \in \mathcal{F}(\mathbb{R}^N)$ is given by

$$d_H(\sigma, \sigma') = \max \left\{ \max_{x \in \sigma} \inf_{x' \in \sigma'} \|x - x'\|, \max_{x' \in \sigma'} \inf_{x \in \sigma} \|x - x'\| \right\}.$$ 

We define the $\ell_\infty$-metric on $\Delta$ by $d_\infty((b_1, d_1), (b_2, d_2)) = \max(|b_1 - b_2|, |d_1 - d_2|)$, where $\infty - \infty = 0$. For $(b, d) \in \Delta$, we define $d_\infty((b, d), \partial \Delta) = d - b$. For finite multisets $X$ and $Y$ in $\Delta$, a partial matching between $X$ and $Y$ is a subset $M \subset X \times Y$ such that for every $x \in X$ there is at most one $y \in Y$ such that $(x, y) \in M$ and for every $y \in Y$ there is at most one $x \in X$ such that $(x, y) \in M$. An $x \in X$ (resp. $y \in Y$) is unmatched if there is no $y \in Y$ (resp. $x \in X$) such that $(x, y) \in M$. We say that a partial matching $M$ is
δ-matching if \( d_\infty(x, y) \leq \delta \) for every \((x, y) \in M\), \( d_\infty(x, \partial \Delta) \leq \delta \) if \( x \in X \) is unmatched, and \( d_\infty(y, \partial \Delta) \leq \delta \) if \( y \in Y \) is unmatched.

The bottleneck distance is defined as follows

\[
d_B(X,Y) := \inf\{\delta > 0 : \text{there exists a } \delta \text{-matching between } X \text{ and } Y\}.
\]

For \( \Xi, \Xi' \in \mathcal{F}(\mathbb{R}^N) \) and \( \kappa, \kappa' : \mathcal{F}(\mathbb{R}^N) \to [0, \infty] \), we define two complexes

\[
\mathcal{K}_\kappa(\Xi) = \{ K_\kappa(\Xi, t) \}_{t \geq 0}, \quad \mathcal{K}_{\kappa'}(\Xi') = \{ K_{\kappa'}(\Xi', t) \}_{t \geq 0}.
\]

Let \( C \) be a correspondence between \( \Xi \) and \( \Xi' \), i.e., \( C \subset \Xi \times \Xi' \) such that \( p_1(C) = \Xi \) and \( p_2(C) = \Xi' \), where \( p_i \) is the projection onto the \( i \)-th coordinate for \( i = 1, 2 \). We define the transpose \( C^T \) of \( C \), which is also a correspondence, by

\[
C^T := \{ (x', x) \in \Xi' \times \Xi : (x, x') \in C \}.
\]

A correspondence \( C \) defines a map from \( 2^\Xi \setminus \emptyset \) to \( 2^{\Xi'} \setminus \emptyset \) as

\[
C(\sigma) = \{ x' \in \Xi' : (x, x') \in C, x \in \sigma \}.
\]

The distortion of \( C \) is defined as

\[
dis(C) := \max \left\{ \sup_{\sigma \subset \Xi} |\kappa(\sigma) - \kappa'(C(\sigma))|, \sup_{\sigma' \subset \Xi'} |\kappa(C^T(\sigma')) - \kappa'(\sigma')| \right\}.
\]

**Lemma C.1.** If \( dis(C) \leq \varepsilon \), then \( H_q(\mathcal{K}_\kappa(\Xi)) \) and \( H_q(\mathcal{K}_{\kappa'}(\Xi')) \) are \( \varepsilon \)-interleaving.

**Proof.** Assume that \( \sigma \in K_\kappa(\Xi, t) \) and \( \kappa(\sigma) \leq t \). Then it follows from \( |\kappa(\sigma) - \kappa'(C(\sigma))| \leq \varepsilon \) that

\[
\kappa'(\sigma') \leq \kappa'(C(\sigma)) \leq \kappa(\sigma) + \varepsilon \leq t + \varepsilon \text{ for any } \sigma' \subset C(\sigma),
\]

which implies \( \sigma' \in K_{\kappa'}(\Xi', t + \varepsilon) \), and hence \( C \) is \( \varepsilon \)-simplicial from \( \mathcal{K}_\kappa(\Xi) \) to \( \mathcal{K}_{\kappa'}(\Xi') \). Symmetrically, \( C^T \) is also \( \varepsilon \)-simplicial. Therefore, the conclusion follows from Proposition 4.2 in [5].

Let us define

\[
S((\kappa, \Xi), (\kappa', \Xi')) := \sup_{\sigma \subset \Xi, \sigma' \subset \Xi'} |\kappa(\sigma) - \kappa'(\sigma')|.
\]

We remark that \( S((\kappa, \Xi), (\kappa', \Xi')) = \| \kappa - \kappa' \|_\infty \text{ if } \Xi = \Xi' \).
**Lemma C.2.** Let $C$ denote the correspondence defined by
\[ C = \{ (x, x') \in \Xi \times \Xi' : \|x - x'\| \leq d_H(\Xi, \Xi') \}. \]
Then,
\[ \text{dis}(C) \leq S((\kappa, \Xi), (\kappa', \Xi')). \]

**Proof.** We easily see that
\[ \sup_{\sigma \subset \Xi} d_H(\sigma, C(\sigma)) \leq d_H(\Xi, \Xi') \]
and
\[ \sup_{\sigma' \subset \Xi'} d_H(C^T(\sigma'), \sigma') \leq d_H(\Xi, \Xi') \]
which implies the assertion. \( \square \)

For $D_q(\kappa, \Xi) = D_q(\mathbb{K}_\kappa(\Xi))$ and $D_q(\kappa', \Xi') = D_q(\mathbb{K}_{\kappa'}(\Xi'))$, we obtain the following continuity result.

**Theorem C.3.**
\[ (C.2) \quad d_B(D_q(\kappa, \Xi), D_q(\kappa', \Xi')) \leq S((\kappa, \Xi), (\kappa', \Xi')). \]

**Proof.** It follows from Lemma C.2 and Lemma C.1 that $H_q(\mathbb{K}_\kappa(\Xi))$ and $H_q(\mathbb{K}_{\kappa'}(\Xi'))$ are $S((\kappa, \Xi), (\kappa', \Xi'))$-interleaving. Therefore, we obtain (C.2) from [4]. \( \square \)

**Corollary C.4.** Suppose that $\kappa$ is Lipschitz continuous with respect to $d_H$, i.e., there exists a constant $\gamma > 0$ such that
\[ |\kappa(\sigma) - \kappa(\sigma')| \leq \gamma d_H(\sigma, \sigma') \]
for $\sigma, \sigma' \in \mathcal{F}(\mathbb{R}^N)$. Then,
\[ (C.3) \quad d_B(D_q(\kappa, \Xi), D_q(\kappa, \Xi')) \leq \gamma d_H(\Xi, \Xi'). \]

**Proof.** From the assumption and (C.1), we see that
\[ S((\kappa, \Xi), (\kappa, \Xi')) \leq \gamma d_H(\Xi, \Xi'). \]
Therefore, (C.3) follows. \( \square \)

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