On linear stability of shear flows of an ideal stratified fluid: research methods and new results

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Abstract. The results, that obtained by the spectral method with use of integral relations for the problem of linear stability of steady-state shear plane-parallel flows of an inviscid stratified incompressible fluid in the gravity field with respect to plane perturbations in the Boussinesq approximation and without it, are specified, complemented and developed by the most powerful analytical method of the modern mathematical theory of hydrodynamic stability – the second (or direct) Lyapunov method. In both case, the new analytical method made it possible to prove that given steady-state flows of stratified fluid are absolutely unstable in theoretical sense with respect to small plane perturbations and to obtain the sufficient conditions for practical linear instability of considered flows. The illustrative analytical examples of given steady-state flows and small plane perturbations as normal waves imposed on them are constructed. Using the asymptotic method, it is proved that constructed perturbations grow in time irrespective of the fact whether the Miles-Howard and the Miles-type theorems are valid or not.

1. Introduction

The buoyancy effect on the inertial stability/instability of fluid flows is considered: the problem of linear stability of steady-state shear plane-parallel flows of an inviscid stratified incompressible fluid in the gravity field with and without the Boussinesq approximation is investigated. The statement and numerous studies of this problem are included in many monographs on hydrodynamics [1, 2] and geophysical hydrodynamics [3, 4, 5]. Resulting conditions of linear stability or instability of the considered fluid flows are widely used as conditions for the appearance of a laminar-turbulent transition in the processes observed on the ocean surface and in the atmosphere, also in aviation and hydraulic engineering.

The major conditions for the linear stability of considered fluid flows were obtained by the spectral method with use of integral relations [1, 6, 7, 8]. This method includes the use of integral relations and relations derived from the boundary value problems for differential equations with variable coefficients. In this method, the equivalence between differential and integral statements has not been proved, causing that not all possible perturbations are actually investigated. Moreover, except for states of rest given in work [9], it is impossible to find the conditions of theoretical stability for any steady-state flows of studied fluid with respect to small plane perturbations by using energy reasons. All this implies that there is a linear absolute
instability for studied fluid flows. The main purpose of this paper is to verify this assumption of linear absolute instability with new analytical method [10] and by comparing with the results obtained early by the spectral method with use of integral relations.

2. The direct Lyapunov method (in the Boussinesq approximation)

We consider the nonsteady-state plane flows of an inviscid stratified incompressible fluid in the gap between two steady-state impermeable solid parallel unbounded surfaces in the gravity field in the Boussinesq approximation [11]. This approximation allows us to disregard the changes in density (which affect inertia), rather than in weight (or bouyancy) of the fluid [3]. In the Cartesian coordinate system \((x, y)\) these flows are characterized by evolutionary solutions to an initial-boundary value problem of the form [3, 7, 6, 8]

\[
\begin{align*}
\tilde{\rho}_0 Du &= -p_x, \quad \tilde{\rho}_0 Dv = -p_y - \rho g, \quad D\rho = 0, \quad u_x + v_y = 0 \text{ in } \tau; \quad v = 0 \text{ on } \partial\tau; \\
u(x, y, 0) &= u_0(x, y), \quad \nu(x, y, 0) = v_0(x, y), \quad D \equiv \partial/\partial t + u\partial/\partial x + v\partial/\partial y, \\
\tau &\equiv \{(x, y) : -\infty < x < +\infty, 0 < y < H\}, \quad \partial\tau \equiv \{(x, y) : -\infty < x < +\infty; y = 0, H\},
\end{align*}
\]

where \(\tilde{\rho}_0 \equiv \text{const} > 0\) is the average fluid density; \(p\) and \(\rho\) are the pressure and density perturbations; \(0, g \equiv \text{const} > 0\) is the acceleration due to gravity, \((u, v)\) is the fluid velocity field. Initial values \((u_0, v_0)\) of fluid velocity make the fourth and fifth relation of problem (1) into identity.

Initial-boundary value problem (1) has exact steady-state solutions

\[
\rho = \rho_0(y), \quad u = U(y), \quad \nu = 0, \quad p = P(y) \equiv p_0 - g \int_0^y \rho_0(y_1) \, dy_1,
\]

where \(\rho_0, U\) are arbitrary functions of coordinate \(y\), \(\rho_0\) is an additive constant.

We linearize the mixed problem (1) in the neighbourhood of exact steady-state solutions (2)

\[
\begin{align*}
\tilde{\rho}_0(u'_t + U_u'x + v'\frac{dU}{dy}) &= -p'_x, \quad \tilde{\rho}_0(v'_t + U_v'x) = -p'_y - \rho' g, \\
\rho' + U\rho'_x + v'\frac{d\rho_0}{dy} &= 0, \quad u'_x + v'_y = 0 \quad \text{in } \tau; \\
v' &= 0 \quad \text{on } \partial\tau; \quad u'(x, y, 0) = u'_0(x, y), \quad v'(x, y, 0) = v'_0(x, y),
\end{align*}
\]

where \(u', v', \rho', \rho'\) are small plane perturbations of velocity \(u, v\), density \(\rho\) and pressure \(p\).

In order to demonstrate a instability of stationary solution (2) of mixed problem (1) with respect to small plane perturbations (3) we need at least one of these perturbations, but with the exponential time growth. The search for such perturbations is carried out below in subclass of plane flows of form Lagrangian displacements field \((\xi_1, \xi_2)\) [11, 12]:

\[
\xi_{1t} = u' - U\xi_{1x} + \xi_2 \frac{dU}{dy}, \quad \xi_{2t} = v' - U\xi_{2x}.
\]

It can be shown that the differential inequality holds for small plane perturbations (3), (4) of stationary flows (2)

\[
M \equiv \int_{-\infty}^{+\infty} \int_0^H \tilde{\rho}_0(\xi_1^2 + \xi_2^2) \, dydx : \frac{d^2M}{dt^2} - 2\lambda \frac{dM}{dt} + 2(\lambda^2 + \alpha)M \geq 0
\]

with a parameter \(\lambda\) and the constant \(\alpha \equiv (g/\tilde{\rho}_0) \max_{0 \leq y \leq H} | d\rho_0/\partial y | > 0.\)
Proposition 1. If \( \lambda > 0 \) and the following additional conditions
\[
M \left( \frac{\pi n}{2 \sqrt{\lambda^2 + 2 \alpha}} \right) > 0; \quad n = 0, 1, 2, \ldots, \quad \frac{dM}{dt} \left( \frac{\pi n}{2 \sqrt{\lambda^2 + 2 \alpha}} \right) \geq 2 \left( \frac{\lambda + \alpha}{\lambda} \right) M \left( \frac{\pi n}{2 \sqrt{\lambda^2 + 2 \alpha}} \right);
\]
\[
M \left( \frac{\pi n}{2 \sqrt{\lambda^2 + 2 \alpha}} \right) \equiv M(0) \exp \left( \frac{\pi n \lambda}{2 \sqrt{\lambda^2 + 2 \alpha}} \right), \quad \frac{dM}{dt} \left( \frac{\pi n}{2 \sqrt{\lambda^2 + 2 \alpha}} \right) \equiv \frac{dM}{dt}(0) \exp \left( \frac{\pi n \lambda}{2 \sqrt{\lambda^2 + 2 \alpha}} \right),
\]
are satisfied along with (5) the following a priori exponential lower estimate is obtained
\[
M(t) \geq C \exp(\lambda t), \quad C \equiv \text{const} > 0.
\]
The proof of Proposition 1 are constructed according to allowability standards for procedure of integrating differential inequality and it can be found in the papers of the authors [10, 13].

Thus, according to the Lyapunov definition of instability [14], stationary flows (2) are absolutely theoretical unstable with respect to small plane perturbations (3), (4). Moreover, inequalities of relations systems (6) can be interpreted as sufficient conditions for practical linear instability [15] of steady-state flows (2), but for small plane perturbations (3), (4) in the form of normal waves – as necessary and sufficient ones.

3. Spectral methods using integral relations (in the Boussinesq approximation)
The finding result of absolute instability is compared with the well-known result of spectral theory on the instability of considered shear flows (2), obtained earlier by the method of integral relations for small plane perturbations (3) as normal waves, – with the Miles-Howard theorem [6, 7].

The small plane perturbations (3) of stationary flows (2) as normal waves with a complex phase velocity \( c \equiv c_r + ic_i \) and a wave number \( k > 0 \) are considered. According to the Miles-Howard theorem [6, 7], the exponentially growing small plane perturbations arise if and only if the local Richardson number
\[
\text{Ri} \equiv -\frac{g}{\rho_0} \left( \frac{dU}{dy} \right)^{-2} < 1/4
\]
at least at one point in the fluid flow domain \( \tau \).

However, the authors in the work [11] proved that small plane perturbations (3) in the form of normal waves with an amplitude containing a countable set of branches are not covered by the Miles-Howard theorem [6, 7] and the following proposition holds.

Proposition 2. In the Boussinesq approximation, the reverse inequality (7) for local Richardson number in domain \( \tau \) of the fluid flow is the necessary and sufficient condition for the stability of exact steady-state solutions (2) to mixed problem (1) with respect to one incomplete nonclosed subclass of small plane perturbations (3), (4) as normal waves.

The validity of Proposition 2 indicates that no contradictions between the Miles-Howard theorem [6, 7] and linear instability of steady-state flows (2) are exist.

Below, we construct an example of exact solutions (2) to mixed problem (1) and the imposed on them small perturbations (3) as normal waves. Asymptotic behaviour of these perturbations indicates that the subclass small plane perturbations (3) of considered steady-state flows (2) which are not covered by the Miles-Howard theorem [6, 7] is not empty set.

We consider the steady-state flows form of
\[
U \equiv by + b_1; \quad b, b_1 > 0; \quad \rho_0 \equiv a - \frac{\rho_0}{g} y.
\]
Here, \( b, b_1 \) and \( a \) are constants.
For the small plane perturbations (3) in the form of normal waves of steady-state fluid flows
(2), (8), we obtain the dispersion function $DJ(c, k)$ and the dispersion relation $DJ(c, k) = 0$ of
forms

$$DJ(c, k) ≡ J_ν\left(\frac{k}{b}[c - b_1]\right)Y_ν\left(\frac{k}{b}[c - b_1 - bH]\right) - Y_ν\left(\frac{k}{b}[c - b_1]\right)J_ν\left(\frac{k}{b}[c - b_1 - bH]\right) = 0,$$

(9)

where $J_ν(ik[c - by - b_1]/b)$, $Y_ν(ik[c - by - b_1]/b)$ are transcendental Bessel functions of order $ν ≡ \sqrt{1/4 - 1/b^2}$, which have a countable set of branches [16]. Therefore, this functions will produce counterexamples to the Miles-Howard theorem [6, 7].

Dispersion relation (9) is too complicated, and its roots $c(k)$ for any order $ν$ cannot be found explicitly using analytical methods. However, using the rules of analytical operations with asymptotic expansions of the Bessel functions $J_ν(z)$, $Y_ν(z)$ that are valid for a fixed order $ν$ and for large modulus of argument $z$ [17, 18],

- it can be proved

**Proposition 3.** Every cylindrical function is uniquely determined by its asymptotic for a fixed order and large modulus of its argument;

- it can be obtained that the zero approximation of the dispersion function $DJ(c, k)$ (9) has no roots, the first approximation of the dispersion function $DJ(c, k)$ (9) has roots, but they are real, and the second approximation of the dispersion function $DJ(c, k)$ (9) has complex roots in the form

$$c_{1,2,3,4} = b_1 + \frac{bH}{2} \pm \frac{1}{2k\sqrt{2}} \left(2b^2H^2k^2 + kH \coth(kH) - 1\right) \pm$$

$$\pm i \coth(kH)\left[(1/b^4 + 4/b^2 + 16b^2H^2 + 8k^2H^2)\tanh^2(kH) + \right.$$

$$\left. +4(1/b^2 + 4)kH\tanh(kH) - 4k^2H^2\right]^{1/2} \right)^{1/2}.$$

(10)

Among complex roots $c_{1,2,3,4}$ (10), one root always has a positive imaginary part, regardless of which value of the local Richardson Ri (7) number is. Therefore, the second approximation of the dispersion function $DJ(c, k)$ (9) has the desired complex root with a positive imaginary part. Then, according to **Proposition 3**, it follows that the exact dispersion relation (9) also has the complex solution with a positive imaginary part, regardless of which value of the local Richardson Ri (7) number is.

4. **The direct Lyapunov method (without the Boussinesq approximation)**

The same fluid flows are considered, only without the Boussinesq approximation. In this case, the density field of the fluid is not divided into the average density and its perturbations. Then these flows are characterized by a problem in the form of (1) with exact solutions in the form (2), the small plane perturbations imposed on this exact solutions are now solutions to the linearized problem (3) in which the average density $\hat{\rho}_0$ and its perturbations $\rho$ have to be replaced by the density field $\rho$, and the perturbations of the pressure field $p$ now simply mean the pressure field.

Following Section 2, growing small plane perturbations (3) are sought in the subclass (4). In this subclass, the differential inequality holds for the auxiliary functional (5), but now with $\hat{\rho}_0 ≡ ρ$, and $α ≡ g \max_{0≤y≤H} |(d\hat{ρ}_0/dy)\hat{ρ}_0^{-1}| > 0$. Therefore, an a priori exponential lower estimate can be constructed (according to **Proposition 1**), which indicates the absolute linear instability of the considered steady-state flows (2) without the Boussinesq approximation.
5. Spectral methods (without the Boussinesq approximation)

According to the Miles-type theorem [2], the exponentially growing small plane perturbations arise if and only if the local Richardson number

$$\text{Ri} \equiv -\frac{g}{\rho_0} \frac{d \rho_0}{d y} \left( \frac{d U}{d y} \right)^{-2} < 1/4$$

at least at one point in the fluid flow domain $\tau$.

Along the lines of reasoning of Section 3, the authors proved that small plane perturbations (3) imposed on steady-state flows (2) without the Boussinesq approximation in the form of normal waves with an amplitude containing a countable set of branches are not covered by the Miles-type theorem [2], and Proposition 2 holds for local Richardson number as (11) [11].

It remains to show that the subclass of small plane perturbations as normal waves which are not covered by the Miles-type theorem [2] is not empty set.

We consider steady-state fluid flows in the form

$$U \equiv b y + b_1; b, b_1 > 0; \rho_0 \equiv b \exp(-m y), m \equiv \text{const} > 0. \quad (12)$$

The dispersion function and the dispersion relation for small plane perturbations as normal waves of steady-state flows (2), (12) take the following form

$$DW(c, k) \equiv M_{\frac{m^*}{k^*}, \nu}(k^*[c + b_1])W_{\frac{m^*}{k^*}, \nu}(k^*[c + bH + b_1]) -$$

$$-W_{\frac{m^*}{k^*}, \nu}(k^*[c + b_1])M_{\frac{m^*}{k^*}, \nu}(k^*[c + bH + b_1]) = 0,$$

where $M_{\frac{m^*}{k^*}, \nu}(y)$ and $W_{\frac{m^*}{k^*}, \nu}(y)$ are transcendental Whittaker functions [19] of order $(\frac{m^*}{k^*}, \nu)$, in which $m^* \equiv m/b, k^* \equiv \sqrt{m^{*2} + 4k^2/b^2}, \nu = \sqrt{1/4 - mg/b^2}$.

The transcendental Whittaker functions have a countable set of branches [18], similarly to the transcendental Bessel functions. Therefore, the Whittaker functions will generate counterexamples to the Miles-type theorem [2].

In analogy with dispersion relation (9), the dispersion relation (13) is also not susceptible to the fact that its roots can be found explicitly by any exact analytical methods. However, approximate solutions to dispersion relation (9) can be found, using the well-known asymptotic expansions for the Whittaker functions $M_{\frac{m^*}{k^*}, \nu}(z), W_{\frac{m^*}{k^*}, \nu}(z)$ that are admitted for fixed orders $\frac{m^*}{k^*}, \nu$ and for large modulus of argument $z$ [17]. So, using the rules of analytical operations with this asymptotic expansions for the Whittaker functions $M_{\frac{m^*}{k^*}, \nu}(z), W_{\frac{m^*}{k^*}, \nu}(z)$ [18],

- it can be proved

Proposition 4. Every solution $w(z)$ of the Whittaker equation

$$\frac{d^2 w}{dz^2} + \left( \frac{1}{z} - \frac{1}{4} + \frac{1/4 - \mu^2}{z^2} \right) w = 0$$

is uniquely determined by its asymptotic behaviour as $|z| \to \infty$;

- it can be showed that the zero approximation of the dispersion function $DW(c, k)$ (12) has complex roots in the form

$$c_r = b_1 - bH \frac{\exp(k^{*2}bH/m^*) \cos(\frac{znk^*}{m^*}) - 1}{1 - 2 \exp(k^{*2}bH/m^*) \cos(\frac{znk^*}{m^*}) + \exp(\frac{k^{*2}bH}{m^*})},$$

$$c_i = bH \frac{\exp(k^{*2}bH/m^*) \sin(\frac{znk^*}{m^*})}{1 - 2 \exp(k^{*2}bH/m^*) \cos(\frac{znk^*}{m^*}) + \exp(\frac{k^{*2}bH}{m^*})}, n \in \mathbb{Z}, \quad (14)$$
The expression (14) indicates that among the complex solutions to the zero approximate dispersion relation (13) there necessarily exists a countable set of solutions with a positive imaginary part (regardless of the truth/falsity of the necessary and sufficient condition of linear instability for the local Richardson number (11), the Miles-type theorem [2]). Then, according to Proposition 4, it follows that among solutions to the exact dispersion relation (13) there also necessarily exists the desired complex solution with a positive imaginary part.

6. Conclusion

In this paper, using the direct Lyapunov method, we have proved that steady-state plane-parallel shear flows of an inviscid stratified incompressible fluid in channel in the gravity field are absolutely unstable in theoretical sense with respect to small plane perturbations with and without the Boussinesq approximation. The sufficient conditions for practical linear instability of these flows have been obtained, and for small plane perturbations in the form of normal waves these conditions are the necessary and sufficient ones.

For both cases, the boundaries of applicability of the well-known necessary conditions for linear instability (the Miles-Howard and the Miles-type theorems), previously obtained by the spectral method using integral relations, have been clearly specified. It has been found that the Miles-Howard and the Miles-type theorems are inherently sufficient and necessary statements with respect to certain incomplete nonclosed subclasses of considered small perturbations. We have constructed illustrative analytical examples of given steady-state flows and small plane perturbations as normal waves imposed on them. Using the asymptotic method, we have proved that constructed perturbations grow in time irrespective of the fact whether the Miles-Howard and the Miles-type theorems are valid or not. Therefore, the results obtained earlier by other authors using the method of integral relations for problems of linear stability of steady-state plane-parallel shear flows of an ideal stratified incompressible fluid require a strict description of the specified partial classes of small plane perturbations because otherwise these results can be wrong.

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